Tame and wild theorem for the category of filtered by standard modules

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Abstract
We introduce the notion of interlaced weak ditalgebras and apply reduction procedures to their module categories to prove a tame-wild dichotomy for the category $F(\Delta)$ of $\Delta$-filtered modules for an arbitrary finite homological system $(P, \leq, \{\Delta_i\}_{i \in P})$. This includes the case of standardly stratified algebras. Moreover, in the tame case, we show that given a fixed dimension $d$, for every $d$-dimensional indecomposable module $M \in F(\Delta)$, with the only possible exception of those lying in a finite number of isomorphism classes, the module $M$ coincides with its Auslander-Reiten translate in $F(\Delta)$. Our proofs rely on the equivalence of $F(\Delta)$ with the module category of some special type of ditalgebra.

1 Introduction

Denote by $k$ a fixed ground field, which will be assumed to be algebraically closed all over this work. Whenever we consider a $k$-algebra or a bimodule, we always assume that the field $k$ acts centrally on them. Given an algebra $\Lambda$, we denote by $\Lambda$-Mod the category of left $\Lambda$-modules, and by $\Lambda$-mod its full subcategory of finitely generated modules.

We recall that a preordered set $(P, \leq)$ is a non-empty set $P$ equipped with a reflexive and transitive relation $\leq$. Two elements $i, j \in P$ are equivalent iff $i \leq j$ and $j \leq i$. In this case, we write $i \sim j$. We denote with $\overline{P} = P/\sim$ the set of equivalence classes of $P$ modulo the equivalence relation $\sim$. For any $i \in P$, denote by $\overline{i}$ its equivalence class. Then, $\overline{P}$ is a partially ordered set with the relation defined by $\overline{i} \leq \overline{j}$ iff $i \leq j$.

We recall the following terminology from [15] and [7].

Definition 1.1. Given a finite-dimensional $k$-algebra $\Lambda$, a (finite) homological system $(P, \leq, \{\Delta_i\}_{i \in P})$ for $\Lambda$ consists of a finite preordered set $(P, \leq)$ and a

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\end{itemize}
family of pairwise non-isomorphic indecomposable finite-dimensional \( \Lambda \)-modules \( \{ \Delta_i \}_{i \in \mathcal{P}} \) satisfying the following two conditions:

1. \( \text{Hom}_\Lambda(\Delta_i, \Delta_j) \neq 0 \) implies \( i \leq j \);
2. \( \text{Ext}^1_\Lambda(\Delta_i, \Delta_j) \neq 0 \) implies \( i \leq j \) and \( i \not\sim j \).

We write \( \Delta := \{ \Delta_i \mid i \in \mathcal{P} \} \) and denote by \( \mathcal{F}(\Delta) \) the full subcategory of \( \Lambda\text{-mod} \) consisting of the trivial module and all those \( M \in \Lambda\text{-mod} \) which admit a \( \Delta \)-filtration, that is a filtration of submodules

\[
0 = M_t \subseteq M_{t-1} \subseteq \cdots \subseteq M_1 \subseteq M_0 = M
\]

such that \( M_j/M_{j+1} \) is isomorphic to some module in \( \Delta \), for each \( j \in [0, t-1] \).

A finite homological system \( \mathcal{H} = (\mathcal{P}, \leq, \{ \Delta_i \}_{i \in \mathcal{P}}) \) for a finite-dimensional \( k \)-algebra \( \Lambda \) is called \emph{admissible} if \( \Lambda \in \mathcal{F}(\Delta) \) and the number of isoclasses of indecomposable projective \( \Lambda \)-modules coincides with the cardinality of \( \mathcal{P} \).

In this case, the modules in the family \( \Delta \) are determined up to isomorphism and are known as the \emph{standard modules}; \( \Lambda \) is called a \emph{pre-standardly stratified algebra}, see [7](2.11).

We recall also that a pre-standardly stratified algebra \( \Lambda \), with preordered index set \( (\mathcal{P}, \leq) \), is called \emph{standardly stratified} if \( (\mathcal{P}, \leq) \) is a partial order. A standardly stratified algebra \( \Lambda \), equipped with the poset \( (\mathcal{P}, \leq) \), is \emph{quasi-hereditary} iff \( \text{End}_\Lambda(\Delta_i) \cong k \), for each \( i \in \mathcal{P} \).

In the following, when we say that \emph{almost every} object in a class \( \mathcal{M} \) of objects in a given category satisfies some property, we mean that every object in \( \mathcal{M} \) has this property, with the possible exception of those lying in a finite union of isoclasses of \( \mathcal{M} \).

**Definition 1.2.** Let \( \Lambda \) be a finite-dimensional algebra and \( \mathcal{C} \) a full subcategory of \( \Lambda\text{-mod} \) closed under direct summands and direct sums. Then,

1. The category \( \mathcal{C} \) is called \emph{tame} iff, for each dimension \( d \), there are rational algebras \( \Gamma_1, \ldots, \Gamma_{t_d} \) and bimodules \( Z_1, \ldots, Z_{t_d} \), where each \( Z_i \) is a \( \Lambda-\Gamma_i \)-bimodule, which is free of finite rank as a \( \Gamma_i \)-module, such that almost every indecomposable \( d \)-dimensional module \( M \) in \( \mathcal{C} \) is of the form \( M \cong Z_i \otimes_{\Gamma_i} S \), for some \( i \) and some simple \( \Gamma_i \)-module \( S \).

2. The category \( \mathcal{C} \) is called \emph{strictly tame} iff, for each dimension \( d \), there are rational algebras \( \Gamma_1, \ldots, \Gamma_{t_d} \) and bimodules \( Z_1, \ldots, Z_{t_d} \), where each \( Z_i \) is a \( \Lambda-\Gamma_i \)-bimodule, which is free of finite rank as a \( \Gamma_i \)-module, such that almost every indecomposable \( d \)-dimensional module \( M \) in \( \mathcal{C} \) is of the form \( M \cong Z_i \otimes_{\Gamma_i} N \), for some \( i \) and some indecomposable \( \Gamma_i \)-module \( N \). Moreover, for each \( i \in [1, t_d] \), the functor \( Z_i \otimes_{\Gamma_i} - : \Gamma_i\text{-mod} \rightarrow \Lambda\text{-mod} \) preserves isoclasses and indecomposables, and its image lies within \( \mathcal{C} \).

3. The category \( \mathcal{C} \) is \emph{wild} iff there is a \( \Lambda-k(x,y) \)-bimodule \( Z \), free of finite rank as a right \( k(x,y) \)-module, such that \( Z \otimes_{k(x,y)} - : k(x,y)\text{-mod} \rightarrow \mathcal{C} \) preserves indecomposables and isomorphism classes.
The preceding definitions are very natural generalizations of the usual definitions of tameness and wildness for finite-dimensional algebras over algebraically closed fields. This notion of wildness for subcategories of modules was already considered explicitly in [20](§14.2) and [21]. The notion of tameness in the first item, for particular cases, was somehow considered in [22] and [8]. Precise definitions were given in [14](2.9), but here we do not require that \( C \) is of infinite representation type to be of tame type.

It is not hard to see that strict tameness implies tameness.

Given a general homological system \((\mathcal{P}, \leq, \{\Delta_i\}_{i \in \mathcal{P}})\) for a finite-dimensional algebra \( \Lambda \), it is well known that the category \( \mathcal{F}(\Delta) \) of \( \Delta \)-filtered modules is a subcategory of \( \Lambda \)-mod closed under direct sums and direct summands, see [15](3.16) and [16]. The question of whether a tame and wild dichotomy theorem holds for the category \( \mathcal{F}(\Delta) \) of filtered by standard modules for a general quasi-hereditary algebra \( \Lambda \) is very natural and was explicitly raised in [14]. In 2017, we uploaded to arXiv an affirmative answer to this question, see [5]. This article is a revised version of that one, where some of the arguments are simplified, but furthermore, we generalize it to prove the tame-wild dichotomy for \( \mathcal{F}(\Delta) \) not only for the case of pre-standardly stratified algebras, but for arbitrary homological systems. We prove the following.

**Theorem 1.3.** Assume that the ground field \( k \) is algebraically closed, let \( \Lambda \) be a finite-dimensional \( k \)-algebra and \((\mathcal{P}, \leq, \{\Delta_i\}_{i \in \mathcal{P}})\) any homological system for \( \Lambda \). Then, the category of \( \Delta \)-filtered modules \( \mathcal{F}(\Delta) \) is either tame or wild, but not both. Moreover, the category \( \mathcal{F}(\Delta) \) is tame iff it is strictly tame.

An important precedent to our result is [8], where Th. Brüstle, S. Koenig, and V. Mazorchuk prove a dichotomy result in a special case of quasi-hereditary algebra by explicitly classifying the tame and wild subcategories of filtered modules. D. Simson presented in [22] another tame-wild dichotomy result for an interesting subcategory of modules over a commutative uniserial algebra.

By a well known result of C.M. Ringel, for a general homological system \((\mathcal{P}, \leq, \{\Delta_i\}_{i \in \mathcal{P}})\), the category \( \mathcal{F}(\Delta) \) admits almost split sequences, see [17]. Whenever we refer here to almost split sequences in \( \mathcal{F}(\Delta) \), we mean precisely in the sense of Ringel. We will prove the following statement, which is similar to a theorem of Crawley-Boevey for the category \( \Lambda \)-mod, when \( \Lambda \) is an arbitrary finite-dimensional tame algebra over an algebraically closed field, see [11].

**Theorem 1.4.** Assume that the ground field \( k \) is algebraically closed, let \( \Lambda \) be a finite-dimensional \( k \)-algebra and \((\mathcal{P}, \leq, \{\Delta_i\}_{i \in \mathcal{P}})\) any homological system for \( \Lambda \). If the category of \( \Delta \)-filtered modules \( \mathcal{F}(\Delta) \) is tame, then, for each \( d \in \mathbb{N} \), almost every \( d \)-dimensional indecomposable module \( M \in \mathcal{F}(\Delta) \) admits an almost split sequence in \( \mathcal{F}(\Delta) \) of the form \( 0 \longrightarrow M \longrightarrow E \longrightarrow M \longrightarrow 0 \).

In the quasi-hereditary case, Ziting Zeng has proved in [24](6.5) that the second Brauer-Thrall conjecture holds for \( \mathcal{F}(\Delta) \). That is, if \( \mathcal{F}(\Delta) \) is of infinite representation type, it admits infinitely many non-isomorphic \( d \)-dimensional indecomposables, for infinitely many \( d \in \mathbb{N} \). This makes the preceding result
particularly interesting. The problem of existence of some special type of generic modules for the category $\mathcal{F}(\Delta)$ in the tame representation type case has been addressed in [6].

In the case of admissible homological systems, the proofs of our main results rely on the existence of an equivalence from $\mathcal{F}(\Delta)$ to the module category of a suitable bocs with relations (or, equivalently, a ditalgebra with relations) which is constructed for pre-standardly stratified algebras in [7]. This construction generalizes the one given by S. Koenig, J. Külshammer, and S. Ovsienko in [13], for the particular, but of cardinal importance, case of quasi-hereditary algebras.

The bocs with relations mentioned before can be presented as a quotient of an interlaced weak ditalgebra $(A, I)$, see [7](5.22), a notion which will be defined rigorously in §2 and studied in the subsequent ones. Moreover, it will be a quotient of a $P$-oriented interlaced weak ditalgebra $(A, I)$ with special geometric features related to $P$, which we describe below.

**Definition 1.5.** A biquiver $\mathcal{B}$ is a triple $\mathcal{B} = (P, B_0, B_1)$ formed by a finite set $P$ of points, a finite set $B_0$ of full arrows, and a finite set $B_1$ of dashed arrows. Each arrow $\alpha \in B_0 \cup B_1$ has a starting point $s(\alpha) \in P$ and a terminal point $t(\alpha) \in P$.

Given a biquiver $\mathcal{B}$ with $n$ points, we consider the $k$-algebra $R$ defined as the product $R = k \times k \times \cdots \times k$ of $n$ copies of the ground field $k$. Denote by $e_i$ the idempotent of $R$ with 1 in its $i$th coordinate and 0 in the others. Then, we consider the vector space $W_0$ (resp. $W_1$) as the vector space with basis $B_0$ (resp. $B_1$). Set $W := W_0 \oplus W_1$. If we define $e_j \alpha e_i = \delta_{i,s(\alpha)} \delta_{t(\alpha), j} \alpha$, we get a natural structure of $R$-bimodule on the space $W$, and $W = W_0 \oplus W_1$ is an $R$-bimodule decomposition. Then, we have the graded tensor algebra $T = T_R(W)$, with $[T]_0 = T_R(W_0)$ and $[T]_1 = [T]_0 W_1 [T]_0$, which is called the tensor algebra of the biquiver $\mathcal{B}$.

Notice that $T$ can be identified with the path algebra $k\mathcal{B}$ of the biquiver $\mathcal{B}$, with underlying vector space with basis the set of paths (of any kind of arrows) of $\mathcal{B}$ (including one trivial path for each point $i \in P$). Each primitive idempotent $e_i$ of $R$ is identified with the trivial path at the point $i$. The subspace $[k\mathcal{B}]_u$ of $k\mathcal{B}$ of homogeneous elements of degree $u$ consists of the linear combinations of paths containing exactly $u$ dashed arrows.

**Definition 1.6.** Let $\mathcal{P} = (\mathcal{P}, \leq)$ be a finite preordered set and $\mathcal{B}$ a biquiver with set of points $\mathcal{P}$, then we say that $\mathcal{B}$ is $\mathcal{P}$-oriented iff

1. $i \leq j$, whenever there is a dashed arrow from $i$ to $j$;
2. $\bar{i} < \bar{j}$, whenever there is a solid arrow from $i$ to $j$.

A weak ditalgebra $\mathcal{A} = (T, \delta)$ will be called a $\mathcal{P}$-oriented weak ditalgebra iff its underlying tensor algebra $T$ is the tensor algebra of a $\mathcal{P}$-oriented biquiver.

A biquiver $\mathcal{B}$ is called directed iff it admits no oriented (non-trivial) cycle (composed by any kind of arrows).
Given a finite preordered set $\mathcal{P}$, a biquiver $\mathcal{B}$ is $\mathcal{P}$-quasi-directed iff the only (non trivial) oriented cycles (composed of any kind of arrows) consist of dashed arrows between points in the same $\sim$ class. In case $\mathcal{B}$ is $\mathcal{P}$-quasi-directed where $\mathcal{P}$ is a partially ordered set, then the only non trivial oriented cycles consist of dashed loops. Thus, $\mathcal{P}$-oriented biquivers are $\mathcal{P}$-quasi-directed, but not necessarily directed.

Our proofs, for admissible homological systems, rest on the following two theorems. For the sake of precision, let us recall from [4] that a ditalgebra $Q = (T, \delta)$ is a pair where $T$ is a differential tensor algebra with differential $\delta$.

We denote by $Q_0 := [T]_0$ the subalgebra of $T$ formed by the elements with degree zero and by $V := [T]_1$ the $Q$-$Q$-subbimodule of $T$ formed by the elements with degree one. The category of $Q$-modules, denoted by $Q\text{-Mod}$, has objects the $Q$-modules and a morphism $f : M \rightarrow N$ in $Q\text{-Mod}$ is a pair $f = (f^0, f^1)$ with $f^0 \in \text{Hom}_k(M, N)$ and $f^1 \in \text{Hom}_Q(V, \text{Hom}_k(M, N))$ such that $qf^0(m) = f^0(qm) + f^1(\delta(q))(m)$, for all $q \in Q$ and $m \in M$. The composition of $Q\text{-Mod}$ is defined using again the differential $\delta$, see [4](2.2). Given $M \in Q\text{-Mod}$, we denote by $\text{End}_Q(M)$ the endomorphism algebra of $M$ in $Q\text{-Mod}$. By definition, the right algebra of $Q$ is $\Gamma := \text{End}_Q(Q\text{-mod})$. We have the inclusion $\phi : Q \rightarrow \Gamma$, with $\phi(q) = (\rho_q, 0)$, where $\rho_q$ denotes the right multiplication by $q$ in $Q$. Thus $\Gamma$ is a $Q$-$Q$-bimodule. We denote by $\mathcal{I}(\Gamma)$ the full subcategory of $\Gamma\text{-mod}$ with objects $M$ of the form $M \cong \Gamma \otimes_Q N$, for some $N \in Q\text{-mod}$, and call it the category of induced modules. In the following statement we collect some of the main results obtained in [7] for pre-standardly stratified algebras, generalizing those of [13] for quasi-hereditary algebras, which play an essential role in our proofs of (1.3) and (1.4).

The ditalgebra $Q$ appearing in the next theorem is a quotient of a $\mathcal{P}$-oriented weak ditalgebra $A = (T, \delta)$. Thus, $A$ has a $\mathcal{P}$-quasi-directed biquiver. In the quasi-hereditary case, we furthermore know that it is a directed biquiver.

**Theorem 1.7.** Let $k$ be an algebraically closed field and $\Lambda$ a finite-dimensional $k$-algebra with an admissible homological system $(\mathcal{P}, \leq, \{\Delta_i\}_{i \in \mathcal{P}})$. Then,

1. There is a ditalgebra $Q$, quotient of a $\mathcal{P}$-oriented weak ditalgebra, such that $\Lambda$ is Morita equivalent to the right algebra $\Gamma$ of $Q$. In this situation, the right $Q$-module $\Gamma$ is finite-dimensional, projective, and the following holds:

2. Denote by $\{S_i\}_{i \in \mathcal{P}}$ the non-isomorphic simple $Q$-modules. Then, the algebra $\Gamma$ has the admissible homological system $(\mathcal{P}, \leq, \{\Delta'_i\}_{i \in \mathcal{P}})$, with

$$\Delta'_i = \Gamma \otimes_Q S_i, \text{ for each } i \in \mathcal{P}.$$  

3. There is an exact full and faithful functor $F : Q\text{-mod} \rightarrow \Gamma\text{-mod}$, which restricts to an equivalence of categories $F : Q\text{-mod} \rightarrow \mathcal{I}(\Gamma)$, satisfying

$$F(M) \cong \Gamma \otimes_Q M, \text{ for each } M \in Q\text{-mod}.$$
4. We have $\mathcal{I}(\Gamma) = \mathcal{F}(\Delta')$. So, the equivalence $F$ restricts to an equivalence of categories $F : \mathcal{Q}\text{-}mod \longrightarrow \mathcal{F}(\Delta')$, such that $\Delta'_i = F(S_i)$, for all $i \in \mathcal{P}$.

5. There is a Morita equivalence $\Theta : \Lambda\text{-}Mod \longrightarrow \Gamma\text{-}Mod$ such that $\Theta(\Delta_i) \cong \Delta'_i$, for all $i \in \mathcal{P}$.

6. There is an equivalence of categories $K : F(\Delta) \longrightarrow \mathcal{Q}\text{-}mod$ which maps short exact sequences onto conflations and $K(\Delta_i) \cong S_i$, for each $i \in \mathcal{P}$.

The preceding statements and their proofs are scattered throughout [7]: The construction of $\mathcal{Q}$ or, equivalently, of the $\mathcal{P}$-oriented interlaced weak ditalgebra $(\mathcal{A}(\Delta), I(\Delta))$, is done in [7]§(see (5.22),(13.1),(13.2)]; the fact that the ideal $I(\Delta)$ is triangular is verified in [7](10.10); for 2, see [7](13.7); for 3, see [7](12.11), and, to recall the exact structure of $\mathcal{Q}$-mod, see [4]§6 or [7]§11; for 4, see [7](13.8); for 5, see [7](13.9); item 6 follows from [7](11.12).

**Remark 1.8.** The equivalence $\Theta : \Lambda\text{-}Mod \longrightarrow \Gamma\text{-}Mod$ mentioned in the last theorem restricts to an equivalence of categories $\Theta_i : \mathcal{F}(\Delta) \longrightarrow \mathcal{F}(\Delta')$ given by a tensor product by some progenerator and its quasi-inverse $\Theta'_i : \mathcal{F}(\Delta') \longrightarrow \mathcal{F}(\Delta)$ has the same form. It follows that the category $\mathcal{F}(\Delta)$ is tame (resp. strictly tame, or wild) iff the category $\mathcal{F}(\Delta')$ is tame (resp. strictly tame, or wild). The argument that justifies this statement is similar to the one justifying that Morita equivalent finite-dimensional algebras are simultaneously tame (or strictly tame, or wild).

The word ditalgebra used before is an acronym for differential tensor algebra (also refered to as semi-free differential graded algebra). Their study was first introduced by the Kiev School in representation theory, see [18] and [19], and is handled in [13] and [14] with the equivalent formulation in terms of bocses.

The ditalgebra $\mathcal{Q}$ appearing in the last theorem is a special kind of quotient of a $\mathcal{P}$-oriented triangular weak $k$-ditalgebra, which will be discussed in the next sections. We will prove the following statement, which applies to them.

**Theorem 1.9.** Assume that the ground field $k$ is algebraically closed and that $\mathcal{P}$ is a finite preordered set. Then, every quotient $\mathcal{A}/J$ of a $\mathcal{P}$-oriented triangular weak $k$-ditalgebra $\mathcal{A} = (T, \delta)$, by an ideal $J$ of $\mathcal{A}$ generated by an ideal $I$ of the algebra $A = [T]_0$ (such that $I$ is contained in the radical of $\mathcal{A}$, is interlaced with $\mathcal{A}$ and is $\mathcal{A}$-triangular), is either tame or wild, but not both.

The strategy for the proof of the preceding theorem will be the following. We work in the general context of triangular interlaced weak ditalgebras $(\mathcal{A}, I)$, described in the first sections §2–§5, and then we will concentrate on the $\mathcal{P}$-oriented case. In this case, we will first reduce the study of the $(\mathcal{A}, I)$-modules with dimension bounded by some natural number $d$, to the corresponding study for stellar interlaced weak ditalgebras, then we reduce this problem to the corresponding problem for seminested ditalgebras and, finally, we apply the known theory which reduces this problem to the corresponding problem for minimal ditalgebras (developed by Drozd and Crawley-Boevey in [12] and [11]).
The proofs of our main results in the case of general homological systems follow from the corresponding statements for admissible homological systems and from the fact, proved by Mendoza, Sáenz, and Xi in [15], that the category $\mathcal{F}(\Delta)$ associated to a general homological system is equivalent as an exact category to the category $\mathcal{F}(\Theta)$ associated to an appropriate admissible homological system. The last section of this article is devoted to this.

The reader is referred to [4] as a general reference for the terminology not explained here and for a basic introduction to the subject. In particular, we use intensively reduction functors associated with admissible modules, a construction carefully studied in [4]. In this article, we do not introduce essentially new types of reductions for ditalgebras, but, since we consider weak ditalgebras with relations, we have to carefully monitor how the ideal of relations is modified by each classical reduction procedure.

We use freely the basic terminology of unital rings and modules employed in [1], but we accept the possibility that in a unital ring $A$ we have $1 = 0$, so $A$-$\text{Mod}$ has only trivial modules.

2 Interlaced weak ditalgebras and modules

In this section we introduce the notions of “weak ditalgebras $\mathcal{A}$”, of “interlaced weak ditalgebras $(\mathcal{A},I)$”, and their “module category $(\mathcal{A},I)$-$\text{Mod}$”. The first one is a trivial generalization of the notion of ditalgebra and the second one is a useful way to handle some special kind of ditalgebras with relations and their module categories.

**Definition 2.1.** Given a graded $k$-algebra $T = \bigoplus_{i \geq 0} [T]_i$, a derivation on $T$ is a linear map $\delta : T \longrightarrow T$ such that $\delta([T]_i) \subseteq [T]_{i+1}$, for all $i$, and such that for any homogeneous elements $a, b \in T$, the following Leibniz rule holds:

$$\delta(ab) = \delta(a)b + (-1)^{\deg(a)}a\delta(b).$$

The derivation $\delta$ is called a differential if $\delta^2 = 0$.

**Definition 2.2.** Given a $k$-algebra $A$ and an $A$-$A$-bimodule $V$, we have the tensor algebra $T = T_A(V)$ with canonical grading $[T]_i = V \otimes_A [T]_i$, for $i \in \mathbb{N} \cup \{0\}$. Thus, $[T]_0 = A$ and $[T]_1 = V$. A graded tensor algebra (or $t$-algebra $T$) is a graded algebra isomorphic to some $T_A(V)$. We often identify them.

A weak ditalgebra $\mathcal{A}$ is a pair $\mathcal{A} = (T, \delta)$, where $T$ is a graded tensor algebra and $\delta$ is a derivation on $T$. A morphism of weak ditalgebras $\eta : \mathcal{A} \longrightarrow \mathcal{A}'$ is a morphism of graded algebras $\eta : T \longrightarrow T'$ such that $\delta'\eta = \eta\delta$.

A weak ditalgebra $(T, \delta)$ is a ditalgebra if $\delta$ is a differential.

**Notation 2.3.** Given a weak ditalgebra $\mathcal{A} = (T_A(V), \delta)$, consider for any pair of $A$-modules $M$ and $N$ the set $U(M, N)$ defined as the collection of pairs $f = (f^0, f^1)$, with $f^0 \in \text{Hom}_k(M, N)$ and $f^1 \in \text{Hom}_{A-A}(V, \text{Hom}_k(M, N))$ such that, for any $a \in A$ and $m \in M$, the following holds

$$af^0[m] = f^0[am] + f^1(\delta(a))[m].$$
Given \( f^1 \in \text{Hom}_{A-A}(V, \text{Hom}_k(M, N)) \) and \( g^1 \in \text{Hom}_{A-A}(V, \text{Hom}_k(N, L)) \), we consider the morphism \( g^1 \ast f^1 \in \text{Hom}_{A-A}(V \otimes^2, \text{Hom}_k(M, L)) \) defined, for any \( \sum_j u_j \otimes v_j \in V \otimes_A V \), by
\[
(g^1 \ast f^1)(\sum_j u_j \otimes v_j) = \sum_j g^1(u_j)f^1(v_j) : M \longrightarrow L.
\]

**Lemma 2.4.** With the preceding notations, assume that \( f = (f^0, f^1) \in U(M, N) \) and \( g = (g^0, g^1) \in U(N, L) \) satisfy that \( (g^1 \ast f^1)(\delta^2(a)) = 0 \), for any \( a \in A \), then \( gf := (g^0f^0, (gf)^1) \in U(M, L) \), where
\[
(gf)^1(v) := g^0f^1(v) + g^1(v)f^0 + (g^1 \ast f^1)(\delta(v)), \quad v \in V.
\]

**Proof.** For \( a \in A, v \in V \), and \( m \in M \), we have
\[
(gf)^1(\alpha v)[m] = g^0f^1(\alpha v)[m] + g^1(\alpha v)f^0[m] + (g^1 \ast f^1)(\delta(\alpha v))[m]
\]
\[
= g^0(\alpha f^1(v))[m] + ag^1(v)f^0[m] + (g^1 \ast f^1)(\delta(a)v + a\delta(v))[m]
\]
\[
= ag^0f^1(v)[m] + ag^1(v)f^0[m] + a(g^1 \ast f^1)(\delta(v))[m]
\]
\[
= a(gf)^1(v)[m]
\]
and
\[
(gf)^1(\alpha a)[m] = g^0f^1(\alpha a)[m] + g^1(\alpha a)f^0[m] + (g^1 \ast f^1)(\delta(\alpha a))[m]
\]
\[
= g^0(\alpha f^1(a))[m] + (g^1 \ast f^1)(\delta(a)v - v\delta(a))[m]
\]
\[
= g^0f^1(v)[m] + g^0f^0[am] + g^1(\delta(v))f^0[am]
\]
\[
= ((gf)^1(v)a)[m],
\]
so \((gf)^1\) is a morphism of \( A \)-\( A \)-bimodules. Moreover,
\[
a(gf)^0[m] = a(g^0f^0[m])
\]
\[
= g^0(\alpha f^0[m]) + g^1(\delta(a))f^0[m]
\]
\[
= g^0f^0[am] + f^1(\delta(a))[m] + g^1(\delta(a))f^0[m]
\]
\[
= (gf)^0[am] + (gf)^1(\delta(a))[m] - (g^1 \ast f^1)(\delta^2(a))[m]
\]
and, since \((g^1 \ast f^1)(\delta^2(a)) = 0\), we obtain \( gf \in U(M, L) \), as claimed. \(\square\)

**Definition 2.5.** Let \( A = (T, \delta) \) be a weak ditalgebra, with \( T = T_3(V) \), and \( I \) an ideal of \( A \). The weak ditalgebra \( A \) is called **interlaced with the ideal** \( I \) if
\begin{enumerate}
\item \( \delta^2(A) \subseteq I[T]_2 + VIV + [T]_2I \), and
\item \( \delta^2(V) \subseteq I[T]_3 + VIT[T]_2 + [T]_2IV + [T]_3I \).
\end{enumerate}
Equivalently, \( \delta^2(A) \) and \( \delta^2(V) \) are contained in the ideal of \( T \) generated by \( I \).

The pair \((A, I)\) is called then an **interlaced weak ditalgebra**.

**Proposition 2.6.** Assume that \((A, I)\) is an interlaced weak ditalgebra. Then, we construct a category \((A, I)\)-\text{Mod} as follows: The class of objects of \((A, I)\)-\text{Mod} is the class of left \( A/I \)-modules \( M \); given the \( A/I \)-modules \( M \) and \( N \), the set of morphisms from \( M \) to \( N \) in \((A, I)\)-\text{Mod} is, by definition, \( \text{Hom}_{(A, I)}(M, N) := \)
second condition on the interlaced weak ditalgebra \((A, \delta)\) zero because
\[
\sum_i h^1(v_i) f^1(v'_i) + h^1(v_i) g^1(v'_i) + h^1(v_i) g^0 f^0 + \sum_i h^1(v_i) g^1(v'_i) f^0 + \sum_i h^1(v_i) g^1(v'_i) f^1(v'_i) + h^1(v_i) g^0 f^0
\]
and
\[
((hg)f)^1(v) = (hg)^0 f^1(v) + h^0 f^1(v) f^0 + \sum_i (h^0 g^1(v_i) f^1(v'_i) + h^1(v_i) g^1(v'_i) f^0 + \sum_i h^1(v_i) g^1(v'_i) f^1(v'_i).
\]
The difference \(\Delta = \sum_i h^1(u_{i,j}) g^0 f^1(u'_i) - \sum_i h^1(v_i) g^1(v'_i) f^1 (u'_i)\) is zero because \(\delta^2(v) = \delta(\sum_i v_i v'_i) = \sum_i u_{i,j} v_{i,j} f^1(v_i) - \sum_i v_i w_{i,s} u_{i,s} v'_i\) and, by the second condition on the interlaced weak ditalgebra \((A, I)\), we obtain \(\Delta[m] = 0\), for \(m \in M\).

Given an interlaced weak ditalgebra \((A, I)\), we will denote by \((A, I)\)-mod the full subcategory of \((A, I)\)-Mod formed by its finite-dimensional objects.

### 3 Quotients of weak ditalgebras

In this section, we extend the concept of ideal for ditalgebras, see [4]§8, to the more general case of weak ditalgebras and we relate the category \((A, I)\)-Mod, for an interlaced weak ditalgebra \((A, I)\), to the category \(A/I\)-Mod of modules over the ditalgebra \(A/I\), where \(I\) is the ideal of \(A\) generated by \(I\).

**Definition 3.1.** Assume that \(A = (T, \delta)\) is a weak ditalgebra, where \(T = T_A(V)\). An ideal \(J\) of \(A\) is an ideal \(J\) of \(T\) such that

1. \(J \cap A\) and \(J \cap V\) together generate the ideal \(J\) of \(T\);
2. \(\delta(J) \subseteq J\).
In this case, $J$ is a homogeneous ideal of $T = T_A(V)$, and is a proper ideal of $A$ iff $J \cap A$ is a proper ideal of $T$. Thus, $T/J$ is a graded $k$-algebra. The first condition guarantees that the algebra $T/J$ can be identified with the tensor algebra $T_{A(J\cap A)}((V/(J\cap V)))$, see $[4]$ (8.22); the second one implies that $\delta$ induces a derivation $\overrightarrow{\delta}$ on $T/J$. Thus, the pair $\mathcal{A}/J := (T/J, \overrightarrow{\delta})$ is a weak ditalgebra: the quotient of the weak ditalgebra $\mathcal{A}$ by the ideal $J$. The canonical projection $\eta : T \longrightarrow T/J$ is a morphism of weak ditalgebras $\eta : \mathcal{A} \longrightarrow \mathcal{A}/J$.

**Definition 3.2.** Let $\mathcal{A} = (T, \delta)$ be a weak ditalgebra with $T = T_A(V)$. Assume that $I$ is an ideal of $A$. Then,

1. We say that $I$ is a triangular ideal of $\mathcal{A}$ iff there is a sequence of $k$-subspaces $0 = H_0 \subseteq H_1 \subseteq \cdots \subseteq H_t = I$ such that $\delta(H_i) \subseteq AH_{i-1}V + VH_{i-1}A$, for all $i \in [1, t]$.

2. We say that $I$ is an $A$-balanced ideal of $\mathcal{A}$ iff $\delta(I) \subseteq IV + VI$.

Clearly, every $\mathcal{A}$-triangular ideal of $A$ is an $A$-balanced ideal of $A$.

**Remark 3.3.** Let $\mathcal{A} = (T, \delta)$ be a weak ditalgebra with $T = T_A(V)$. Assume that $I$ is an ideal of $A$ and that $0 = H_0 \subseteq H_1 \subseteq \cdots \subseteq H_t \subseteq I$ is a vector space filtration of $I$ such that $\delta(H_i) \subseteq AH_{i-1}V + VH_{i-1}A$, for all $i \in [1, t]$, and the ideal of $A$ generated by $H_1$ is $I$, then $0 = H_0 \subseteq H_1 \subseteq \cdots \subseteq H_t \subseteq H_{t+1} = I$ is a triangular filtration of $I$ as in the preceding definition. Indeed, the ideal $I$ is generated as a vector space by elements of the form $a[h]b$, with $a, b \in A$ and $h \in H_1$. Then, we have $\delta(a[h]b) = \delta(a)hb + a\delta(h)b + ah\delta(b)$, where $\delta(a)hb \in VH_1A$, $ah\delta(b) \in AH_1V$, and $a\delta(h)b \in a(AH_{i-1}V + VH_{i-1}A)b \subseteq AH_1V + VH_1A$. Hence, the definition of a triangular ideal given in $[5,2]$ coincides with the definition of $[4]$ (8.22).

**Lemma 3.4.** Assume that $\mathcal{A} = (T, \delta)$ is a weak ditalgebra, with $T = T_A(V)$, and that $I$ is some ideal of $A$. Consider the $A$-$A$-subbimodule $I_V = IV + \delta(I) + VI$ of $V$ and assume that we are in one of the following two cases:

1. $A$ is interlaced with the ideal $I$

2. $I$ is an $A$-balanced ideal.

Then, the ideal $J$ of $T$ generated by $I$ and $I_V$ is an ideal of the weak ditalgebra $\mathcal{A}$. In particular, $I = J \cap A$ and $I_V = J \cap V$. If $I$ is $A$-balanced, then $I_V = IV + VI$.

**Proof.** By $[4]$ (8.3 and 8.5), we have $J \cap A = I$ and $J \cap V = I_V$. Since $\delta(I) \subseteq I_V \subseteq J$ and $\delta^2(I) \subseteq J$, by the Leibniz rule, we have $\delta(I_V) \subseteq J$. Then, using again the Leibniz rule, we get $\delta(J) \subseteq J$.

**Definition 3.5.** In the preceding situation, we say that $J$ is the ideal of the weak ditalgebra $\mathcal{A}$ generated by the ideal $I$ of $A$. 

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Remark 3.6. Let $\mathcal{A} = (T, \delta) = (T_{\mathcal{A}}(V), \delta)$ be a weak ditalgebra and $I$ an ideal of $A$. Assume that we are in the situation of [4] and let $J$ be the ideal of $\mathcal{A}$ generated by $I$. Write $\overline{A} = A/I$, $\overline{V} := V/(IV + \delta(I) + VI)$ and consider the canonical morphism $\pi : T \longrightarrow T_{\mathcal{A}}(\overline{V})$ induced by the projections $\pi_0 : A \longrightarrow \overline{A}$ and $\pi_1 : V \longrightarrow \overline{V}$. From [4](8.4), we obtain an isomorphism $\overline{\pi} : T_{\mathcal{A}}(\overline{V}) \longrightarrow T/J$ such that $\overline{\pi}\pi = \eta$, where $\eta : A \longrightarrow A/J$ is the canonical projection to the quotient weak ditalgebra $A/J$. With the help of this isomorphism, we can transfer the derivation of $A/J$ onto a derivation $\overline{\delta}$ on $T_{\mathcal{A}}(\overline{V})$. Notice that $\overline{\delta}$ can also be obtained by [4](1.8) from the linear maps $\overline{\delta}_0 : A \longrightarrow [T]_1$ and $\overline{\delta}_1 : \overline{V} \longrightarrow [T]_2$ induced by the composition $\pi\delta$, when restricted respectively to $A$ and $V$. We write $\overline{\mathcal{A}} := (T, \overline{\delta})$ and we identify it with $A/J$.

Lemma 3.7. Let $\mathcal{A} = (T, \delta) = (T_{\mathcal{A}}(V), \delta)$ be a weak ditalgebra interlaced with an ideal $I$ of $A$, and let $J$ be the ideal of $\mathcal{A}$ generated by $I$. Then, the quotient $\mathcal{A}/J$ is a ditalgebra and we have

$$A/I\text{-Mod} \xrightarrow{L_{\mathcal{A}/J}} \mathcal{A}/J\text{-Mod} \xleftarrow{\Psi} (A, I)\text{-Mod},$$

where $\mathcal{A}/J$-Mod is the category of $\mathcal{A}/J$-modules, $L_{\mathcal{A}/J}$ is the canonical embedding defined by $L_{\mathcal{A}/J}(M) = M$, for any $A/I$-module, and $L_{\mathcal{A}/J}(f^0) = (f^0, 0)$, for any morphism $f^0$ of $A/I$-Mod, and $\Psi$ is an isomorphism of categories. The composition $L_{(A,I)} := \Psi^{-1}L_{\mathcal{A}/J}$ is also called the canonical embedding.

Proof. Since $\mathcal{A}$ is interlaced with $I$, we have $\delta^2(T) \subseteq J$, thus $\overline{\delta^2} = 0$ and $\mathcal{A}/J$ is a ditalgebra.

By definition, $\Psi$ is the identity on objects. It maps a morphism $(f^0, f^1) \in U(M, N)$ onto the pair $\Psi((f^0, f^1)) = (f^0, f^1)$, where $f^1 : V \longrightarrow \text{Hom}_k(M, N)$ is the morphism induced by $f^1 : V \longrightarrow \text{Hom}_k(M, N)$, which exists because $IM = 0$ and $IN = 0$, so $f^1(IV + \delta(I) + VI) = 0$. Consider the canonical projections $\pi_0 : A \longrightarrow \overline{A}$ and $\pi_1 : V \longrightarrow \overline{V}$, write $\overline{\pi} := \pi_1(\pi)$ and $\overline{\pi} = \pi_0(\delta(a))$, for $a \in A$. Then, $(f^0, f^1) \in U(M, N)$ if $a f^0(m) = f^0(a_m) + f^1(\delta(a))(m)$, for $a \in A$ and $m \in M$, or, equivalently $\overline{\pi} f^0(m) = f^0(\overline{\pi} m) + f^1(\overline{\pi} \delta(\pi))(m)$, for $\overline{\pi} \in \overline{A}$ and $m \in M$.

Notice that the preceding argument provides an alternative proof, to the one given before, of the fact that $(A, I)$-Mod is a category.

Remark 3.8. Assume that $\mathcal{A} = (T_{\mathcal{A}}(V), \delta)$ is a weak ditalgebra, that $I$ is an $\mathcal{A}$-balanced ideal of $A$, and denote by $J$ the ideal of $\mathcal{A}$ generated by $I$. Then, the following are equivalent statements:

1. $\mathcal{A}$ is interlaced with $I$;
2. $\delta^2(A) \subseteq J$ and $\delta^2(V) \subseteq J$.
3. \( \delta^2(T) \subseteq J \).

For instance, assuming 2, we know that \( \delta^2(A) \subseteq T IT + T IV + V I V \); so \( \delta^2(A) \subseteq V^2 I + V I V + IV^2 + I V V + V I V \). But \( I V = V I + IV \), because \( I \) is \( A \)-balanced, so \( \delta^2(A) \subseteq V^2 I + V I V + IV^2 \); similarly, we have that \( \delta^2(V) \subseteq V^3 I + V^2 IV + V I V + IV^3 \), and 1 holds.

**Definition 3.9.** A morphism \( \phi : (A, I) \longrightarrow (A', I') \) of interlaced weak ditalgebras is a morphism of weak ditalgebras \( \phi : A \longrightarrow A' \) such that \( \phi(I) \subseteq I' \).

The following is easy to show (see [1](2.4)).

**Lemma 3.10.** Any morphism of interlaced weak ditalgebras \( \phi : (A, I) \longrightarrow (A', I') \) induces, by restriction, a functor \( F_\phi : (A', I')-\text{Mod} \longrightarrow (A, I)-\text{Mod} \). Consider the restrictions \( \phi_0 : A \longrightarrow A' \) and \( \phi_1 : V \longrightarrow V' \) of \( \phi \). Then, for \( M \in (A', I')-\text{Mod} \), the \( A \)-module \( F_\phi(M) \) is obtained by restriction of scalars through the map \( \phi_0 \); for \( f = (f^0, f^1) \in \text{Hom}_{(A', I')}(M, N) \), we have \( F_\phi(f) = (f^0, f^1 \phi_1) \).

If \( \phi \) is surjective, then \( F_\phi \) is faithful. Moreover, if \( \psi : (A', I') \longrightarrow (A'', I'') \) is another morphism of interlaced weak ditalgebras, then \( F_{\phi \psi} = F_\phi F_\psi \).

**Remark 3.11.** The equivalent notions of *normal bocs* \( \mathcal{B} \) and *ditalgebra* \( \mathcal{A} \), and their categories of modules, are discussed in detail in [4](3.3), see also [14](3.4). There, we can see that the coassociativity of the comultiplication \( \mu \) of a normal bocs \( \mathcal{B} = (A, U, \mu, \epsilon) \) implies that the derivation \( \delta \) of the corresponding ditalgebra \( \mathcal{A} = (T_A(V), \delta) \) indeed satisfies \( \delta^2 = 0 \).

If \( (A, I) \) is an interlaced weak ditalgebra with underlying weak ditalgebra \( A = (T, \delta) \), we have the quotient ditalgebra \( \overline{A} = (T/J, \bar{\delta}) \), where \( J \) is the ideal of \( A \) generated by \( I \) and \( \bar{\delta} \) is the differential induced by the derivation \( \delta \) on \( T/J \), as remarked before in [3](6).

Now, given an interlaced weak ditalgebra \( (A, I) \), one could consider the corresponding *interlaced weak normal bocs* \( (\mathcal{B}, I) \), where \( I \) is the given ideal of \( A = [T]_0 \) and \( \mathcal{B} = (A, U, \mu, \epsilon) \) is constructed as in [4](3.3), using the derivation \( \delta \) of \( A \). This would be a *weak normal bocs* since the comultiplication \( \mu \) is not necessarily coassociative. Nevertheless, we would have a normal bocs \( \mathcal{B} = (\overline{A}, U, \bar{\mu}, \bar{\epsilon}) \), where \( \overline{A} = A/I, \bar{U} = U/(IU + UI) \), and \( \bar{\mu}, \bar{\epsilon} \) are induced by \( \mu, \epsilon \), respectively. The normal bocs \( \overline{\mathcal{B}} \) corresponds to the ditalgebra \( \overline{A} \).

We find it more desirable to work with a derivation \( \delta \) (which does not necessarily satisfies \( \delta^2 = 0 \)) than to deal with a not necessarily coassociative comultiplication.

### 4 Layered weak ditalgebras

In this section we consider some sufficient conditions on a ditalgebra with relations \( \mathcal{A} = (\overline{T}, \bar{\delta}) \) which allow us to lift its differential \( \delta \) to a derivation of an interlaced weak ditalgebra \( (A, I) \) in such a way that \( \mathcal{A} = A/J \), where \( J \) is the ideal of \( A \) generated by an \( A \)-balanced ideal \( I \) of \( A \).
Definition 4.1. A $t$-algebra $T$ has layer $(R, W)$ iff $R$ is a $k$-algebra and $W$ an $R$-$R$-bimodule equipped with an $R$-$R$-bimodule decomposition $W_0 \oplus W_1$ such that $R \cup W_0 \subseteq [T]_0$, $W_1 \subseteq [T]_1$, and $T$ is freely generated by the pair $(R, W)$, see Definition 4.1.

In this case, we have isomorphisms of algebras $T \cong T_R(W)$ and $A \cong T_R(W_0)$, and an isomorphism of $A$-$A$-bimodules $V \cong A \otimes_R W_1 \otimes_R A$ which we shall consider as identifications.

A weak ditalgebra $A = (T, \delta)$ has layer $(R, W)$ iff $T$ admits the layer $(R, W)$ and, moreover, $\delta(R) = 0$.

The following lemma can be proved as in Definition 4.1.

Lemma 4.2. Assume that $T$ is a $t$-algebra with layer $(R, W)$. Suppose that $\delta_0 : W_0 \rightarrow [T]_1$ and $\delta_1 : W_1 \rightarrow [T]_2$ are morphisms of $R$-$R$-bimodules. Then, there is a unique derivation $\delta : T \rightarrow T$, extending $\delta_0$ and $\delta_1$, such that $A = (T, \delta)$ is a weak ditalgebra with layer $(R, W)$.

The following lemma is probably known, but we did not find a reference.

Lemma 4.3. Let $W_0$ and $W_1$ be $R$-$R$-bimodules, where $R$ is a semisimple $k$-algebra, let $I$ be an ideal of $A := T_R(W_0)$, and set $\widehat{A} := A/I$. Consider the canonical projection $\pi : A \rightarrow \widehat{A}$. Then, the kernel of the morphism

$$\pi \otimes 1 \otimes \pi : A \otimes_R W_1 \otimes_R A \rightarrow \widehat{A} \otimes_R W_1 \otimes_R \widehat{A}$$

is $I \otimes_R W_1 \otimes R A + A \otimes_R W_1 \otimes R I$.

Proof. By assumption, we have the exact sequences

$$0 \rightarrow I \otimes_R W_1 \otimes R A \xrightarrow{1 \otimes 1 \otimes 1} A \otimes_R W_1 \otimes R A \xrightarrow{\pi \otimes 1 \otimes 1} \widehat{A} \otimes_R W_1 \otimes_R A \rightarrow 0$$

and

$$0 \rightarrow \widehat{A} \otimes_R W_1 \otimes R I \xrightarrow{1 \otimes 1 \otimes i} \widehat{A} \otimes_R W_1 \otimes R A \xrightarrow{1 \otimes 1 \otimes \pi} \widehat{A} \otimes_R W_1 \otimes_R \widehat{A} \rightarrow 0.$$
1. The pair \( A = (T, \delta) \) is a weak ditalgebra with layer \((R, W)\);

2. The canonical maps \( \pi_0 : A \rightarrow \hat{A} \) and \( \pi_1 : = \pi_0 \otimes 1 \otimes \pi_0 : V \rightarrow \hat{V} \) induce a morphism of weak ditalgebras \( \pi : A \rightarrow \hat{A} \);

3. The kernel of \( \pi_1 \) is precisely \( \text{Ker} \pi_1 = IV + VI \);

4. The ideal \( I \) of \( A \) is \( A \)-balanced and \( \text{Ker} \pi \) coincides with the ideal \( J \) of \( A \) generated by \( I \). The weak ditalgebra \( A \) is interlaced with \( I \) and \( A \cong A/J \).

Proof. It is clear that \( \pi_0 \) and \( \pi_1 \) induce a surjective morphism of \( k \)-algebras \( \pi : T \rightarrow \hat{T} \). Since \( \hat{\delta}([R + I]/I) = 0 \), we obtain that \( \hat{\delta} \) is a morphism of \( R-R \)-bimodules. Since \( k \) is perfect, \( R \otimes_k R \) is also semisimple and \( W_0 \) and \( W_1 \) are projective \( R-R \)-bimodules. Thus, we have the following commutative diagrams of \( R-R \)-bimodules

\[
\begin{array}{ccc}
W_0 & \xrightarrow{\pi_1} & \hat{A} \\
\downarrow \delta_0 & & \downarrow \hat{\delta} \\
V & \xrightarrow{\pi_1} & \hat{V} \\
\end{array}
\quad
\begin{array}{ccc}
W_1 & \xrightarrow{\pi_1} & \hat{V} \\
\downarrow \delta_1 & & \downarrow \hat{\delta} \\
V \otimes_A V & \xrightarrow{\pi_1} & \hat{V} \otimes_A \hat{V} \\
\end{array}
\quad
0.
\]

Then, by (4.2), we can extend these maps to a derivation \( \delta : T \rightarrow \hat{T} \) in such a way that (1) and (2) are satisfied. Item (3) follows from (4.3). The ideal \( I \) of \( A \) is weak ditalgebra because \( \pi \delta(I) = \hat{\delta} \pi(I) = 0 \), so \( \delta(I) \subseteq \text{Ker} \pi_1 \). From (3.4), we know that \( J \) is an ideal of \( A \).

We have \( \hat{A} = A/I = A \) and, by (3), the map \( \pi_1 \) induces an isomorphism \( \pi_1 : \hat{V} = V/(IV + VI) \rightarrow \hat{V} \), see (3.6). They determine an isomorphism of graded algebras \( \pi : \hat{T} = T_A(\hat{V}) \rightarrow T = T_A(\hat{V}) \). Thus \( \pi \) essentially coincides with the canonical projection \( T \rightarrow T/J \), once we have made the identification described in (3.6). Thus, the kernel of \( \pi \) is \( J \) and \( \hat{A} \cong A/J \). Finally, \( \pi \delta^2(T) = \hat{\delta}^2 \pi(T) = 0 \), so \( \delta^2(T) \subseteq J \) and, from (3.8), \( A \) is interlaced with \( I \).

5 Triangular interlaced weak ditalgebras

In this section we give a brief summary of the basic properties of the category of \((A, I)\)-modules which follow from triangularity conditions on the layer of \((A, I)\). We transfer terminology and basic properties from the case of triangular ditalgebras, see §5.

**Definition 5.1.** Assume that \( A = (T, \delta) \) is a weak ditalgebra with layer \((R, W)\). We say that this layer is triangular if

1. There is a filtration of \( R-R \)-subbimodules \( 0 = W_0^0 \subseteq W_1^0 \subseteq \cdots \subseteq W_r^r = W_0 \) such that \( \delta(W_i^{i+1}) \subseteq A_i W_1 A_i \), for all \( i \in [0, r-1] \), where \( A_i \) denotes the \( R \)-subalgebra \( A \) generated by \( W_0^i \).

2. There is a filtration of \( R-R \)-subbimodules \( 0 = W_1^0 \subseteq W_1^1 \subseteq \cdots \subseteq W_s^s = W_1 \) such that \( \delta(W_i^{i+1}) \subseteq AW_i A W_i^1 A \), for all \( i \in [0, s-1] \).
The layer is called additive triangular iff each $W^i_0$ is a direct summand of $W^{i+1}_0$, for all $i$. $A$ is called additive triangular if it has an additive triangular layer.

**Definition 5.2.** Assume that $(A,I)$ is an interlaced weak ditalgebra with layer $(R,W)$, then $(A,I)$ is called a triangular interlaced weak ditalgebra iff $(R,W)$ is triangular, as in the last definition, and $I$ is an $A$-triangular ideal of $A$, as in (3.2). Given a preordered set $P$, a triangular interlaced weak ditalgebra $(A,I)$ will be called $P$-oriented iff the underlying weak ditalgebra $A$ is $P$-oriented, see (1.6).

**Proposition 5.3.** Let $(A,I)$ be a triangular interlaced weak ditalgebra with layer $(R,W)$ and adopt the notation of (5.2). Assume that, for all $i \in [1,r]$, the algebra $A_i$ is freely generated by the pair $(R,W)$ and that the product map

$$A_i \otimes_R W_1 \otimes_R A_i \longrightarrow A_i W_1 A_i$$

is an isomorphism (this is the case if the layer of $A$ is additive triangular). Suppose that $M$ and $N$ are $R$-modules, $f^0 \in \text{Hom}_R(M,N)$ is an isomorphism, and $f^1 \in \text{Hom}_{R,-}(W_1, \text{Hom}_k(M,N))$. Then,

1. If $N$ is an $A$-module with underlying $R$-module $N$ and $IN = 0$, then there is an $A$-module structure on $M$, with underlying $R$-module $M$, such that $IM = 0$ and $(f^0,f^1) \in \text{Hom}_{A,(I)}(M,N)$.

2. If $M$ is an $A$-module with underlying $R$-module $M$ and $IM = 0$, then there is an $A$-module structure on $N$, with underlying $R$-module $N$, such that $IN = 0$ and $(f^0,f^1) \in \text{Hom}_{A,(I)}(M,N)$.

**Proof.** This is similar to the proof of (4)(5.3). For instance, for (1) we take a morphism of $R$-modules $g^0 : N \longrightarrow M$ with $f^0g^0 = 1_N$. Using the triangular filtration of $W_0$, we construct inductively morphisms $\phi_j : W^0_0 \otimes_R M \longrightarrow M$ of $R$-modules satisfying $\phi_j(w \otimes m) = g^0[wf^0(m)] - g^0[f^1(\delta(w))](m)]$, for $w \in W^0_0$ and $m \in M$, where the extension of $f^1$ to a morphism $A_{j-1}W_1A_{j-1} \longrightarrow \text{Hom}_k(M,N)$ is denoted with the same symbol $f^1$. Then, $\phi_r$ gives to $M$ a structure of an $A$-module and from its recipe we derive the formula

$$f^0(wm) = f^0g^0(wf^0(m) - f^1(\delta(w))[m]) = w f^0(m) - f^1(\delta(w))[m],$$

for any $w \in W_0$. Then the same identity holds for $w \in A$.

Now, adopt the notation of (5.2). So, given $a_1 \in H_1$, we get $f^0[a_1m] = a_1f^0(m) - f^1(0)[m] = 0$, and the injectivity of $f^0$ implies that $a_1m = 0$. If we assume that $a_jm = 0$, for $a_j \in H_j$ and $m \in M$, then for $a_{j+1} \in H_{j+1}$ we have $f^0(a_{j+1}m) = a_{j+1}f^0(m) - f^1(\delta(a_{j+1}))[m] = 0$, because $\delta(a_{j+1}) \in AH_jV + VH_jA$. Thus, $IM = 0$.

**Definition 5.4.** A triangular interlaced weak ditalgebra $(A,I)$ with layer $(R,W)$ is called a Roiter interlaced weak ditalgebra iff the following property is satisfied: for any isomorphism $f^0$ of $R$-modules $f^0 : M \longrightarrow N$ and any $f^1 \in \text{Hom}_{R,-}(W_1, \text{Hom}_k(M,N))$, if one of $M$ or $N$ has a structure of a left $A/I$-module, then the other one admits also a structure of a left $A/I$-module such that $(f^0,f^1) \in \text{Hom}_{(A,I)}(M,N)$.
Notice that, by [5,3], any $\mathcal{P}$-oriented triangular interlaced weak ditalgebra $(A,I)$ is a Roiter interlaced weak ditalgebra.

The following three statements, and their proofs, are similar to (5.7), (5.8), and (5.12) of [4]. With a slightly different language, these results are proved in [13] for the case of directed interlaced weak ditalgebras, see the following section.

**Proposition 5.5.** Let $(A,I)$ be a Roiter interlaced weak ditalgebra with layer $(R,W)$. Suppose that $f = (f^0,f^1) \in \text{Hom}_{(A,I)}(M,N)$. Then,

1. If $f^0$ is a retraction in $R$-$\text{Mod}$, then there is a morphism $h : M' \to M$ in $(A,I)$-$\text{Mod}$ such that $h^0$ is an isomorphism and $(fh)^1 = 0$;
2. If $f^0$ is a section in $R$-$\text{Mod}$, then there is a morphism $g : N \to N'$ in $(A,I)$-$\text{Mod}$ such that $g^0$ is an isomorphism and $(gf)^1 = 0$.

**Corollary 5.6.** Let $(A,I)$ be a Roiter interlaced weak ditalgebra and suppose that $f$ is a morphism in $(A,I)$-$\text{Mod}$. Then, $f$ is an isomorphism if and only if $f^0$ is bijective.

**Proposition 5.7.** Let $(A,I)$ be a Roiter interlaced weak ditalgebra, then idempotents split in $(A,I)$-$\text{Mod}$. That is, for any idempotent $e \in \text{End}_{(A,I)}(M)$, there is an isomorphism $h : M_1 \oplus M_2 \to M$ in $(A,I)$-$\text{Mod}$ such that

$$h^{-1}eh = \begin{pmatrix} M_1 & 0 \\ 0 & 0 \end{pmatrix}.$$

6 Reductions and wildness

In this section, we adapt standard reductions for ditalgebras, as presented in [4], to the case of triangular interlaced weak ditalgebras. By the end of this section, we relate them to the notion of wildness.

**Lemma 6.1.** Let $(A,I)$ be a triangular interlaced weak ditalgebra, where $A = (T,\delta)$ admits the triangular layer $(R,W)$. Assume that we have a surjective morphism of $k$-algebras $\phi : R \to R'$, $R'$-$R'$-bimodules $W'_0$ and $W'_1$, and surjective morphisms of $R$-$R'$-bimodules $\phi_r : W_r \to W'_r$, for $r \in \{0,1\}$. Consider the layered $t$-algebra $T' := T_{R'}(W'_0 \oplus W'_1)$. Then, the algebra $A' := T_{R'}(W'_0)$ can be identified with $[T']_0$ and $V' := A' \otimes_{R'} W'_1 \otimes_{R'} A'$ can be identified with $[T']_1$. Consider also the morphism of graded $k$-algebras $\phi : T \to T'$ determined by $\phi_0$, $\phi_1$, and $\phi_1$. Finally, assume that we have commutative diagrams

$$
\begin{array}{ccc}
W_0 & \xrightarrow{\delta} & V \\
\downarrow \phi_0 & & \downarrow \phi_1 \\
W'_0 & \xrightarrow{\delta'_0} & V'
\end{array}
\quad
\begin{array}{ccc}
W_1 & \xrightarrow{\delta} & [T]_2 \\
\downarrow \phi_1 & & \downarrow \phi_{[T]_2} \\
W'_0 & \xrightarrow{\delta'_1} & [T']_2
\end{array}
$$

where $\delta'_0$ and $\delta'_1$ are morphisms of $R'$-$R'$-bimodules. Then,

1. There is a derivation $\delta' : T' \to T'$ extending $\delta'_0$ and $\delta'_1$ with $\delta' \phi = \phi \delta$.
2. $\mathcal{A}' = (T', \delta')$ is a weak ditalgebra interlaced with the ideal $I' := \phi(I)$ of $\mathcal{A}'$;

3. $(\mathcal{A}', I')$ is a triangular interlaced weak ditalgebra with triangular layer $(R', W'_0 \oplus W'_1)$;

4. $\phi : (\mathcal{A}, I) \rightarrow (\mathcal{A}', I')$ is a morphism of interlaced weak ditalgebras and the induced functor $F_\phi : (\mathcal{A}', I')$-Mod $\rightarrow (\mathcal{A}, I)$-Mod is faithful and preserves dimension;

5. The image of $F_\phi$ consists of the modules in $(\mathcal{A}, I)$-Mod annihilated by the ideal $K_0 := \text{Ker}(\phi_A : A \rightarrow A')$ of $A$;

6. Set $K_1 := \text{Ker}(\phi_V : V \rightarrow V')$ and assume that $K_1 = K_0V + VK_0 + \delta(K_0)$, then the functor $F_0$ is full;

7. If $K = \text{Ker} \phi, K_0 = \text{Ker} \phi_A, and K_1 = \text{Ker} \phi_V$, as before, we have:

   (a) $K_0 = A \text{Ker} \phi_A + A \text{Ker} \phi_0A$;

   (b) $K = T \text{Ker} \phi_T + T \text{Ker} \phi_0T + T \text{Ker} \phi_1T$, and the ideal $K$ of $T$ is generated by $K_0$ and $K_1$.

Proof. The existence of $\phi$ follows from the universal property of the tensor algebra $T$. (1) follows from (12) and the Leibniz rule. (2): Since $I$ is triangular, it generates an ideal $J \subseteq \mathcal{A}$ using the surjectivity of $\phi$, we get that $\phi(I)$ generates the ideal $\phi(J)$ of $\mathcal{A}'$. Thus $(\delta')^2(T') = \phi(\delta^2(T)) \subseteq \phi(J)$, and $\mathcal{A}'$ is interlaced with $I'$. (3): It is easy to see that the image under $\phi$ of the filtrations provided by the triangularity of the layer $(R, W_0 \oplus W_1)$ of $\mathcal{A}$ give triangular filtrations for the layer $(R', W'_0 \oplus W'_1)$ of $\mathcal{A}'$. Similarly, the image under $\phi$ of the filtration provided by the $A$-trianglearity of the ideal $I$ gives an $\mathcal{A}'$-triangular filtration for $I'$. (4): Clearly, $\phi$ is faithful and preserves dimension.

(5) Notice that for $M \in (\mathcal{A}', I')$-Mod, we have $K_0F_\phi(M) = 0$. If $N \in (\mathcal{A}, I)$-Mod is such that $K_0N = 0$, then $N$ has a canonical structure of an $\mathcal{A}'$-module, let us denote such $\mathcal{A}'$-module by $N'$, then $N$ is the $A$-module obtained from $N'$ by restriction through $\phi : A \rightarrow \mathcal{A}'$. Moreover, $I'N' = 0$ because $IN = 0$ and $I' = \phi(I)$, thus $F_\phi(N') = N$.

(6) Assume that $(f^0, f^1) \in \text{Hom}_{\mathcal{A},I}(F_\phi(M), F_\phi(N))$. Take $a \in K_0$, then from the identity $af^0(m) = f^0(am) + f^1(\delta(a))[m]$ for each $m \in M$, we get $f^1(\delta(a)) = 0$. Thus $f^1(K_0V + VK_0 + \delta(K_0)) = 0$. The morphism of $A$-$A$-bimodules $f^1 : V = \text{Hom}_k(F_\phi(M), F_\phi(N))$ induces a morphism of $A$-$A$-bimodules $f^1 : V/(K_0V + VK_0 + \delta(K_0)) \rightarrow \text{Hom}_k(M, N)$. Let us denote by $\phi : V/(K_0V + VK_0 + \delta(K_0)) \rightarrow V'$ the isomorphism given by our assumption on $K_1$. Thus, $g^1 := f^1 \phi^{-1} : V' \rightarrow \text{Hom}_k(F_\phi(M), F_\phi(N))$ is a morphism of $A$-$A$-bimodules, which is also a morphism of $A'$-$A'$-bimodules $g^1 : V' \rightarrow \text{Hom}_k(M, N)$ with $g^1 \phi_V = f^1$. Given $a \in A$, we have

\[
\phi(a)f^0(m) = af^0(m) = f^0(am) + f^1(\delta(a))[m] = f^0(\phi(a)m) + g^1(\phi(\delta(a)))[m] = f^0(\phi(a)m) + g^1(\phi(\delta(a)))[m].
\]
Thus \((f^0, g^1) \in \text{Hom}(A', I')(M, N)\) and \(F_{\phi}(f^0, g^1) = (f^0, f^1)\).

(7): We have that \(\text{Ker} \phi_0\) is an ideal of \(R\) and \(\text{Ker} \phi_0\) is an \(R\)-\(R\)-subbimodule of \(W_0\) such that \(\text{Ker} \phi_0 \oplus \text{Ker} \phi_0 \subseteq \text{Ker} \phi_0\). Then, by \([4](8.3)\), \(N_0 := A\text{Ker} \phi_0 A + A\text{Ker} \phi_0 A\) is an ideal of \(A = T_R(W_0)\) which is \((R, W_0)\)-compatible. Then, by \([4](8.4)\), we have a commutative square

\[
\begin{array}{ccc}
A & \xrightarrow{\nu_A} & A/N_0 \\
T_R(W_0) & \xrightarrow{\phi} & T'_R(W'_0) \\
\end{array}
\]

where \(\nu_A\) is the canonical projection and \(\phi\) is an isomorphism. So \(N_0 = K_0\).

Similarly, since \(\text{Ker} \phi_0 W + \text{Ker} \phi_1 \subseteq \text{Ker} \phi_0 + \text{Ker} \phi_1\), we have that \(N := T\text{Ker} \phi_0 T + T\text{Ker} \phi_0 + \text{Ker} \phi_1 T\) is an ideal of \(T = T_R(W)\) which is \((R, W)\)-compatible. Then, we have a commutative square

\[
\begin{array}{ccc}
T & \xrightarrow{\nu_T} & T/N \\
T_R(W) & \xrightarrow{\phi} & T'_R(W'_0) \\
\end{array}
\]

where \(\phi\) is an isomorphism. So \(N = K\), and \(K\) is generated by \(K_0\) and \(K_1\).

In the following construction, we say that \((A^d, I^d)\) is obtained from \((A, I)\) by deletion of the idempotent \(1 - e\).

**Proposition 6.2** (deletion of idempotents). Assume that \((A, I)\) is a triangular interlaced weak ditalgebra with layer \((R, W)\). Assume that \(e \in R\) is a central non-trivial idempotent of \(R\). Consider the canonical projections \(\phi_r : R \rightarrow eRe\), and \(\phi_r : W_r \rightarrow eWe\), for \(r \in \{0, 1\}\). Set \(T^d := T_{RRe}(eWe)\). Then, as in the first paragraph of the last lemma, we have a morphism of graded \(k\)-algebras \(\phi : T \rightarrow T^d\) and we have the ideal \(I^d = \phi(I)\) of \(A^d\). Then, there is a triangular interlaced weak ditalgebra \((A^d, I^d)\) with layer \((R^d, W^d)\) where \(R^d = eRe\), \(W^d_0 = eW_0 e\), and \(W^d_1 = eW_1 e\). The morphism \(\phi : (A, I) \rightarrow (A^d, I^d)\) of interlaced weak ditalgebras induces a full and faithful functor \(F^d : (A^d, I^d)\)-Mod \(\rightarrow (A, I)\)-Mod whose image consists of the objects annihilated by \(1 - e\).

**Proof.** We can identify \(W^d_r\) with \(W_r / ((1 - e)W_r + W_r (1 - e))\), for \(r \in \{0, 1\}\). Since \(\delta\) is a morphism of \(R\)-\(R\)-bimodules, there are commutative diagrams

\[
\begin{array}{ccc}
W_0 & \xrightarrow{\delta} & A \otimes_R W_1 \otimes_R A \\
\downarrow \phi_0 & & \downarrow \phi \\
W^d_0 & \xrightarrow{\delta^d} & A^d \otimes_{R^d} W^d_1 \otimes_{R^d} A^d \\
\end{array}
\]

\[
\begin{array}{ccc}
W^d_1 & \xrightarrow{\delta^d} & A^d \otimes_{R^d} W^d_1 \otimes_{R^d} A \otimes_{R^d} W^d_1 \otimes_{R^d} A \\
\downarrow \phi_2 & & \downarrow \phi \\
W^d_1 & \xrightarrow{\delta^d} & A^d \otimes_{R^d} W^d_1 \otimes_{R^d} A^d \otimes_{R^d} W^d_1 \otimes_{R^d} A^d \\
\end{array}
\]

indeed \(\phi \delta [(1 - e)W_r + W_r (1 - e)] = 0\), for \(r \in \{0, 1\}\). Then, from \([4](6.1)\), we have a derivation \(\delta^d : T^d \rightarrow T^d\) such that \(A^d = (T^d, \delta^d)\) is a weak ditalgebra with layer
\( (R^d, W^d) \), and \( \phi : (A, I) \rightarrow (A^d, I^d) \) is a morphism of triangular interlaced weak ditalgebras. Moreover, an object in \( (A, I) \)-Mod is annihilated by \( K_0 = \ker \phi \cdot A = A(1 - e)A \) if \( (1 - e)M = 0 \). Recall that \( K_1 = \ker(\phi_V : V \rightarrow V^d) \). Since \( \delta(K_0) \subseteq K_0V + VK_0 \), by (3.4), the ideal \( J \) of \( T \) generated by \( K_0 \) and \( K_0V + VK_0 \) is an ideal of \( \mathcal{A} \). But, from (6.1), we get \( K = \ker \phi = J \), so \( K_1 = K \cap V = K_0V + VK_0 \), and again from the last lemma, \( F_\phi \) is full and faithful.

\[ \square \]

**Proposition 6.3** (regularization). Assume that \( (A, I) \) is a triangular interlaced weak ditalgebra with layer \((R, W)\). Assume that we have \( R-R \)-bimodule decompositions \( W_0 = W'_0 \odot W''_0 \) and \( W_1 = \delta(W'_0) \odot W''_1 \), set \( W'' := W'_0 \odot W''_0 \). Consider the identity map \( \phi : R \rightarrow R \), the canonical projections \( \phi_j : W_j \rightarrow W'_j \), for \( j \in \{0, 1\} \), and the tensor algebra \( T' = T_R(W'') \). Then, we have a morphism of graded algebras \( \phi : T \rightarrow T' \) and the ideal \( I' = \phi(I) \) of \( A' \). Then, there is a triangular interlaced weak ditalgebra \( (A', I') \) with layer \( (R', W') \), where \( R' = R, W'_0 = W'_0 \), and \( W'_1 = W''_1 \). The morphism \( \phi : (A, I) \rightarrow (A', I') \) of interlaced weak ditalgebras induces a full and faithful functor \( F' : (A', I')-\text{Mod} \rightarrow (A, I)-\text{Mod} \). Moreover, if \( (A, I) \) is a Roiter interlaced weak ditalgebra, then \( M \in (A, I)-\text{Mod} \) is isomorphic to an object in the image of \( F' \) if \( \ker \delta \cap W_0 \) annihilates \( M \). In particular, if this intersection is zero, \( F' \) is an equivalence of categories.

**Proof.** It is clear that the morphisms \( \delta'_0 := \phi_V \delta|_{W''_0} : W'_0 \rightarrow V' \) and \( \delta'_1 := \phi_{|T_1} \delta|_{W''_1} : W'_1 \rightarrow [T']_2 \) provide commutative squares as in the statement of (6.1), so we can apply this result to obtain a derivation \( \delta'' \) on \( T'' \) such that \( \phi : (A, I) \rightarrow (A', I') \) is a morphism of interlaced weak ditalgebras and \( A' \) admits the layer \((R, W'')\). Moreover, we know that the induced functor \( F' : (A', I')-\text{Mod} \rightarrow (A, I)-\text{Mod} \) is faithful.

Here, the kernel of \( \phi_{|A} : A \rightarrow A' \) is \( K_0 = AW'_0A \). From (6.1), we know that \( K = \ker \phi = TW'_0T + T\delta(W'_0)T \). By (3.4), the ideal \( J \) of \( T \) generated by \( K_0 \) and \( K_0V + \delta(K_0) + VK_0 \) is an ideal of \( \mathcal{A} \). Since \( K = J \), the kernel of \( \phi_V : V \rightarrow V' \) is \( K_1 = K \cap V = K_0V + \delta(K_0) + VK_0 \). From (6.1), we obtain that \( F' \) is a full functor.

Finally, take \( M \in (A, I)-\text{Mod} \) such that \( (\ker \delta \cap W'_0)M = 0 \). Consider the action map \( \psi \in \text{Hom}_{R-R}(W'_0, \text{Hom}_{R}(M, M)) \) of \( W'_0 \) on the \( A \)-module \( M \) and denote by \( \psi' \) its restriction to \( W'_0 \). The condition on \( M \) implies that we can factor \( \psi' \) through \( \delta \), so there is a morphism of \( R-R \)-bimodules \( f_1^1 : \delta(W''_0) \rightarrow \text{Hom}_{R}(M, M) \) such that \( \psi' = f_1^1 \delta \). Consider the morphism of \( R-R \)-bimodules \( f^1 := (f_1^1, 0) : \delta(W'_0) \oplus W''_0 \rightarrow \text{Hom}_{R}(M, M) \). If \( (A, I) \) is a Roiter interlaced weak ditalgebra, we obtain an isomorphism \( (1_M, f^1) : M \rightarrow M \) in \( (A, I)-\text{Mod} \). We claim that \( M \) is in the image of \( F' \). In order to apply (6.1), we want to show that \( M \) is annihilated by \( K_0 \), or equivalently by \( W'_0 \), and is therefore in the image of \( F' \). Take \( w \in W'_0, m \in M \), and denote by \( w \cdot m \) the action corresponding to the structure of the \( A \)-module \( M \), then

\[ w \cdot m = wm - f^1(\delta(w))[m] = wm - f^1(\delta(w))[m] = wm - wm = 0. \]

\[ \square \]
Lemma 6.4 (factoring out a direct summand of \( W_0 \)). Let \((\mathcal{A}, I)\) be a triangular interlaced weak ditalgebra, where \( \mathcal{A} = (T, \delta) \) admits the triangular layer \((R, W)\).

Assume that there is a decomposition of \( R\)-\( R\)-bimodules \( W_0 = W'_0 \oplus W''_0 \), such that \( W'_0 \subseteq I \) and \( \delta(W'_0) \subseteq AW'_0 V + WV'_0 A \). Set \( T^q = T_R(W^q) \), where \( W'_0^q = W'_0 \), \( W''_0^q = W''_0 \), and \( W_1^q = W_1 \). Then, there is a derivation \( \delta^q \) on \( T^q \) such that \( \mathcal{A}^q := (T^q, \delta^q) \) is a weak ditalgebra with triangular layer \((R, W^q)\). The \( t\)-algebra \( A^q = T_R(W''_0') \) can be identified with the quotient algebra \( A/I' \), where \( I' = AW'_0 A \) is the ideal of \( A \) generated by \( W'_0 \), so we can consider \( I' := I/I' \) as an ideal of \( A^q \). Then \((\mathcal{A}^q, I')\) is a triangular interlaced weak ditalgebra, and there is a morphism of interlaced weak ditalgebras \( \phi : (\mathcal{A}, I) \longrightarrow (\mathcal{A}^q, I') \), with \( \phi(I) = I^q \), which induces an equivalence of categories

\[
\mathcal{F}^q := F_{\phi} : (\mathcal{A}^q, I')-\text{Mod} \longrightarrow (\mathcal{A}, I)-\text{Mod}.
\]

**Proof.** Consider the identity map \( \phi_0 : R \longrightarrow R \), the canonical projection \( \phi_0 : W_0 \longrightarrow W_0^q \), the identity map \( \phi_1 : W_1 \longrightarrow W_1^q \), and the induced morphism of graded algebras \( \phi : T \longrightarrow T^q \). Consider the algebras \( A = T_R(W_0) \) and \( A^q = T_R(W''_0') \), and the \( A\)-\( A\)-bimodules \( V = [T]_1 = AW_1 A \) and \( V^q = [T^q]_1 = A^q W_1 A^q \). Then, we have commutative squares

\[
\begin{array}{ccc}
W_0 & \rightarrow & V \\
\downarrow \phi_0 & & \downarrow \phi_U \\
W'_0 & \rightarrow & V^q \\
\delta^q_0 & & \delta^q_1 \\
\end{array}
\quad
\begin{array}{ccc}
W_1 & \rightarrow & [T]_2 \\
\downarrow \phi_1 & & \downarrow \phi_{[T]_2} \\
W'_1 & \rightarrow & [T^q]_2 \\
\delta^q_1 & & \delta^q_2 \\
\end{array}
\]

where \( \delta \) is the derivation of \( \mathcal{A} \), and \( \delta^q_0 := \phi_U \delta_U W^q \) and \( \delta^q_1 := \phi_{[T]_2} \delta_{[T]_2} \) are morphisms of \( R\)-\( R\)-bimodules. Then we can apply (6.1) to this situation to obtain a derivation \( \delta^q \) on \( T^q \) such that \( \phi : (\mathcal{A}, I) \longrightarrow (\mathcal{A}^q, I') \) is a morphism of interlaced weak ditalgebras, where \( I^q = \phi(I) \), and \((\mathcal{A}^q, I')\) admits the triangular layer \((R, W^q)\). Moreover, we have and induced faithful functor

\[
\mathcal{F}^q := F_{\phi} : (\mathcal{A}^q, I')-\text{Mod} \longrightarrow (\mathcal{A}, I)-\text{Mod}.
\]

Using the definition of \( \phi \), by (6.1)\,(7), we find that the kernel of the restriction \( \phi_{[T]} : A \longrightarrow A^q \) is \( K_0 = I' \), and that the kernel of \( \phi : T = T_R(W) \longrightarrow T_R(W^q) = T^q \) is the ideal \( K \) of \( T \) generated by \( W_0 \). Our assumption on \( \delta(W'_0) \) implies that \( \delta(K') \subseteq IV + VI' \). Thus the ideal \( K \) coincides with the ideal \( J' \) of \( T \) generated by \( I' \) and \( I'V + VI' \). This implies, see (3.4), that \( K_0 = K \cap A = I' \) and \( K_1 = K \cap V = I'V + VI' \), where \( K_1 \) is the kernel of the restriction map \( \phi_{[V]} : V \longrightarrow V^q \). Therefore, by (6.1)\,(5&6), the functor \( \mathcal{F}^q \) is full and dense. \( \square \)

Lemma 6.5 (absorption). Let \((\mathcal{A}, I)\) be a triangular interlaced weak ditalgebra, where \( \mathcal{A} = (T, \delta) \) admits the triangular layer \((R, W)\). Assume that there is a decomposition of \( R\)-\( R\)-bimodules \( W_0 = W'_0 \oplus W''_0 \) with \( \delta(W'_0) = 0 \). Then, we can consider another layer \((R^a, W^a)\) for the same weak ditalgebra \( \mathcal{A} \), where \( R^a \) is the subalgebra of \( T \) generated by \( R \) and \( W'_0 \), thus we can identify it with \( T_R(W'_0) \), \( W'_0 = R^a W''_0 R^a \), and \( W'_1 = R^a W_1 R^a \). We denote by \( \mathcal{A}^a \) the same ditalgebra \( \mathcal{A} \) equipped with its new layer \((R^a, W^a)\); in particular, we have \( \delta^a = \delta \). We
say that $A^\alpha$ is obtained from $A$ by absorption of the bimodule $W_0^\alpha$. The layer $(R^\alpha, W^\alpha)$ is triangular and we obtain a triangular interlaced weak ditalgebra $(A^\alpha, I^\alpha)$, with $I^\alpha = I$. The identity is a morphism of interlaced weak ditalgebras $\phi : (A, I) \longrightarrow (A^\alpha, I^\alpha)$, which induces an isomorphism of categories

$$P^\alpha := F_\phi : (A^\alpha, I^\alpha)\text{-Mod} \longrightarrow (A, I)\text{-Mod}.$$  

Proof. It is easy to show, see [4](8.20).

In [4]§12, the construction of a new ditalgebra $A^X$, for a given ditalgebra $A$ and an admissible module $X$ is detailed. The following statement claim is that, given a layered weak ditalgebra $A = (T, \delta)$, the construction of a layered weak ditalgebra $A^X$ can be made in a very similar way. We reproduce here its main constituents for terminology and precision purposes, and we restrict a little the generality of the construction in [4] for the sake of simplicity.

**Proposition 6.6.** Assume that $A = (T, \delta)$ is a weak ditalgebra with layer $(R, W)$ such that there is an $R$-$R$-bimodule decomposition $W_0 = W_0^0 \oplus W_0^\prime$ with $\delta(W_0^0) = 0$. Suppose that $X$ is an admissible $B$-module, where $B = T_R(W_0^0)$. This means that the algebra $\Gamma = \text{End}_B(X)^{op}$ admits a splitting $\Gamma = S \oplus P$, where $S$ is a subalgebra of $\Gamma$, $P$ is an ideal of $\Gamma$, the direct sum is an $S$-$S$-bimodule decomposition of $\Gamma$, and the right $S$-modules $X$ and $P$ are finitely generated projective. By $(x_i, \nu_i)_{i \in I}$ and $(p_j, \gamma_j)_{j \in J}$ we denote a finite dual basis of the right $S$-modules $X$ and $P$, respectively.

There is a comultiplication $\mu : P^* \longrightarrow P^* \otimes_S P^*$ induced by the multiplication of $P$, which is coassociative (see [4](11.1)). For $\gamma \in P^*$, we have

$$\mu(\gamma) = \sum_{i,j \in J} \gamma(p_ip_j)\nu_j \otimes \nu_i.$$  

The action of $P$ on $X$ induces (see [4](10–11)) a morphism of $S$-$S$-bimodules $\lambda : X^* \longrightarrow P^* \otimes_S X^*$ and a morphism of $R$-$S$-bimodules $\rho : X \longrightarrow X \otimes_S P^*$ with

$$\lambda(\nu) = \sum_{i,j \in J} \nu(x_i p_j)\gamma_j \otimes \nu_i \text{ and } \rho(x) = \sum_{j \in J} x p_j \otimes \gamma_j.$$  

Set $W_0^X = BW_0^\prime B$ and $W_1^X = BW_1 B$. Recall that we can identify $A = T_R(W_0^0)$ with $T_B(W_0^0)$ and $T$ with $T_B(W_1^X)$, where $W = W_0^0 \oplus W_1^X$, see [3](12.2).

We have the $S$-$S$-bimodules

$$W_0^X = X^* \otimes_B W_0^X \otimes X = X^* \otimes_R W_0^X \otimes_R X$$  

and

$$W_1^X = (X^* \otimes_B W_1^X) \otimes_R X = (X^* \otimes_R W_1) \otimes_R X \otimes P^*.$$  

Consider the tensor algebra $T^X = T_S(W^X)$, where $W^X = W_0^X \oplus W_1^X$. Following [4](12.8), observe that for $\nu \in X^*$ and $x \in X$, there is a linear map

$$\sigma_{\nu, x} : T \longrightarrow T^X$$
such that $\sigma_{\nu,x}(b) = \nu(bx)$, for $b \in \mathcal{B}$, and given $w_1, w_2, \ldots, w_n \in \mathcal{W}$, we have $\sigma_{\nu,x}(w_1 w_2 \cdots w_n)$ is given by

$$
\sum_{i_1, i_2, \ldots, i_n} \nu \otimes w_1 \otimes x_{i_1} \otimes \nu_{i_1} \otimes w_2 \otimes x_{i_2} \otimes \nu_{i_2} \otimes \cdots \otimes x_{i_{n-1}} \otimes \nu_{i_{n-1}} \otimes w_n \otimes x.
$$

There is a derivation $\delta^X$ on $T^X$ determined by $\delta^X(\gamma) = \mu(\gamma)$, for $\gamma \in P^*$ and $w \in \mathcal{W}_0 \cup \mathcal{W}_1$, $\nu \in X^*$, and $x \in X$.

Then, $A^X = (T^X, \delta^X)$ is a weak ditalgebra. If the layer $(R,W)$ of $A$ is triangular and $X$ is a triangular admissible $B$-module, as in [4](14.6), then the layer $(S,W^X)$ of $A^X$ is also triangular.

**Proof.** This is similar to the first steps in the proof of [4](12.9). For the triangularity statement, choose appropriate dual basis for $X$ and $P$ as in [4](14.7) and then follow the proof of [4](14.10), where we can assume that $W'_0$ belongs to the triangular filtration of $W_0$, so $B$ is an initial subalgebra of $A$.

**Proposition 6.7** (reduction with an admissible module $X$). Assume that $(\mathcal{A},I)$ is a triangular interlaced weak ditalgebra, where $\mathcal{A} = (T,\delta)$ is a weak ditalgebra with layer $(R,W)$ such that there is an $R$-$R$-bimodule decomposition $W_0 = W'_0 \oplus W''_0$ with $\delta(W'_0) = 0$. Suppose that $X$ is a triangular admissible $B$-module, where $B = T_R(W'_0)$. As usual, denote by $A^X = [T^X]_0$. Consider the ideal $I^X$ of $A^X$ generated by the elements $\sigma_{\nu,x}(h)$, for $\nu \in X^*$, $h \in I$, and $x \in X$. Then,

1. The pair $(A^X, I^X)$ is a triangular interlaced weak ditalgebra;
2. There is a functor $F^X : (A^X, I^X)$-Mod $\rightarrow (\mathcal{A}, I)$-Mod such that for $M \in (A^X, I^X)$-Mod, the underlying B-module of $F^X(M)$ is $X \otimes_S M$ and

$$a \cdot (x \otimes m) = \sum_i x_i \otimes \sigma_{\nu,x}(a)m,$$

for $a \in A$, $x \in X$, and $m \in M$. Moreover, given the morphism $f = (f^0, f^1) \in \text{Hom}_{(A^X, I^X)}(M,N)$, we have $F^X(f) = (F^X(f)^0, F^X(f)^1)$ given by

$$F^X(f)^0(x \otimes m) = x \otimes f^0(m) + \sum_j x_p \otimes f^1(\gamma_j)[m]$$

$$F^X(f)^1(v|x \otimes m) = \sum_i x_i \otimes f^1(\sigma_{\nu,x}(v))[m]$$

for $v \in V$, $x \in X$, and $m \in M$.

3. There is a constant $c_X \in \mathbb{N}$ such that, for any $M \in (A^X, I^X)$-mod, we have $\dim_k F^X(M) \leq c_X \dim_k M$.

4. For any $N \in (\mathcal{A}, I)$-mod which is isomorphic as a $B$-module to some $B$-module of the form $X \otimes_S M$, for some $M \in S$-Mod, we have $N \cong F^X(M)$ in $(\mathcal{A}, I)$-mod, for some $M \in (A^X, I^X)$-mod.
Proof. We choose appropriate dual basis for the right $S$-modules $P$ and $X$ as in $[4](14.7)$.

(1): We show first that $I^X$ is an $A^X$-triangular ideal. By assumption, we have a vector space filtration $0 = H_0 \subseteq H_1 \subseteq \cdots \subseteq H_\ell = I$ such that $\delta(H_i) \subseteq A H_{i-1} V + V H_{i-1} A$, for each $i \in \{1, \ell\}$. Then, we have the vector space filtration

$$0 = H_0^X \subseteq H_1^X \subseteq \cdots \subseteq H_{2\ell X(i+1)}^X \subseteq I^X,$$

where each space $H_m^X$ is generated by the set

$$\{\sigma_{\nu,x}(h) \mid \nu \in X^*, h \in I, x \in X, \text{ with } \mathfrak{h}(\nu) + 2\ell_X \mathfrak{h}(h) + \mathfrak{h}(x) \leq m\},$$

and the heights are taken relative to the filtrations of $X^*$, $I$, and $X$. Then, we can see that the computations of $[4]§14$ also verify that the filtration of $I^X$ makes $I^X$ an $A^X$-triangular ideal. See also $[4]§14$ and the solution to Exercise (14.11) of $[4]$.

In order to see that $A^X$ is interlaced with $I^X$, we need to show the inclusion $(\delta^X)^2(T^X) \subseteq J^X$, where $J^X$ is the ideal of $A^X$ generated by $I^X$. For this, it will be enough to show that $(\delta^X)^2(W_0^X) \subseteq J^X$ and $(\delta^X)^2(W_1^X) \subseteq J^X$. Equivalently, we need to show that $(\delta^X)^2(\sigma_{\nu,x}(w)) \in J^X$, for any $\nu \in X^*$, $w \in W_0 \cup W_1$, and $x \in X$; and that $(\delta^X)^2(\gamma) \in J^X$, for any $\gamma \in P^*$. The argument is essentially contained in the proof of $[4](12.9)$, where, under the assumption $\delta^2 = 0$ it is proved that $(\delta^X)^2 = 0$. In our situation here, it is again easy to see that $(\delta^X)^2(\gamma) = 0$, for $\gamma \in P^*$, so we look at the other case, where we have to calculate $(\delta^X)^2(\sigma_{\nu,x}(w))$. The same computations given there, show that

$$(\delta^X)^2(\sigma_{\nu,x}(w)) = \sigma_{\nu,x}(\delta^2(w)).$$

Since $\delta^2(w) \in J$, where $J$ is the ideal of $A$ generated by $I$, from $[4](12.8)(3)$, we obtain $\sigma_{\nu,x}(\delta^2(w)) \in \sigma_{\nu,x}(I[T]_2 + V I V + [T]_2 I) \subseteq I^X [T^X]_2 + V^X I^X V^X + [T^X]_2 I^X$, for $w \in W_0$, and $\sigma_{\nu,x}(\delta^2(w)) \in \sigma_{\nu,x}(I[T]_3 + V I V I V + V[T]_3 I) \subseteq I^X [T^X]_3 + V^X I^X [T^X]_2 + [T^X]_2 I^X V^X + [T^X]_3 I^X$, for $w \in W_1$.

(2): For the existence of $F^X : (A^X, I^X)$-Mod $\longrightarrow (A, I)$-Mod, we need to have in mind the construction of the functor $F^X$ in $[4](12.10)$, where given an $A^X$-module $M$, a structure of an $A$-module can be defined on $X \otimes_S M$ by the recipe

$$a \cdot (x \otimes m) = \sum_i x_i \otimes \sigma_{\nu,x}(a)m$$

for $a \in A$, $x \in X$, and $m \in M$. From this formula, we get that $I^X M = 0$ implies that $I(X \otimes_S M) = 0$. Then, the computations in the proof of $[4](12.10)$ show that there is a functor $F^X : (A^X, I^X)$-Mod $\longrightarrow (A, I)$-Mod, with $F^X(M) = X \otimes_S M$.

(3): Denote by $c_X$ the cardinality of the dual basis of the right $S$-module $X$. Then, for $M \in (A^X, I^X)$-mod, we have

$$\dim_k F^X(M) = \dim_k X \otimes_S M \leq \dim_k (S^{cx} \otimes_S M) \leq \dim_k M^{cx} \leq c_X \dim_k M.$$
(4): Finally, we verify the “density claim” for the functor \( F^X \). If \( N \in (\mathcal{A}, I)-\text{Mod} \) and \( \varphi : N \longrightarrow X \otimes_S M \) is an isomorphism of \( B \)-modules, we can transfer the structure of \( B \)-module of \( N \) onto \( X \otimes_S M \) through \( \varphi \) and obtain an object \( X \otimes_S M \in \mathcal{A}-\text{Mod} \) with underlying \( B \)-module \( X \otimes_S M \) and such that \( N \cong X \otimes_S M \) in \((\mathcal{A}, I)-\text{Mod}\). Now, recall from \( \mathcal{A} \)(16.1), that given an \( A \)-module with underlying \( B \)-module of the form \( X \otimes_S M \), where \( M \) is an \( S \)-module, a structure of \( A^X \)-module can be defined on \( M \) by the formula

\[
(\nu \otimes w \otimes x) \cdot m = \sigma(\epsilon \otimes 1)[\nu \otimes w \circ (x \otimes m)]
\]

where \( \nu \in X^*, x \in X, w \in \mathcal{W}_0, \epsilon : X^* \otimes_B X \longrightarrow S \) is the evaluation map determined by \( \epsilon(\nu \otimes x) = \nu(x) \), and \( \sigma : S \otimes_S M \longrightarrow M \) is the product map; here, \( \circ \) denotes the given \( A \)-module structure on \( X \otimes_S M \). Let us show by induction on \( n \) that, for \( w_1, \ldots, w_n \in \mathcal{W}_0, x \in X, \nu \in X^*, \) and \( m \in M \), we have

\[
\sigma_{\nu,x}(w_1w_2\cdots w_n) \cdot m = \sigma(\epsilon \otimes 1)[\nu \otimes w_1w_2\cdots w_n \circ (x \otimes m)].
\]

Suppose that \( n > 1 \), write \( w = w_1 \) and \( t = w_2 \cdots w_n \), and assume that the statement holds for \( n - 1 \). Assume that \( t \circ (x \otimes m) = \sum x_i \otimes m_s \). Then, applying the induction hypothesis and \( \mathcal{A} \)(12.8)(3), we have

\[
\sigma_{\nu,x}(wt) \cdot m = \sum_i \sigma_{\nu,x_i}(w) \sigma_{\nu,x_i}(t) \cdot m
= \sum_i \sigma_{\nu,x_i}(w) \cdot \sigma(\epsilon \otimes 1)[\nu \circ t \circ (x \otimes m)]
= \sum_i \sigma_{\nu,x_i}(w \otimes x_i) \cdot \nu(x_i) m_s
= \sum_i \sigma(\epsilon \otimes 1)[\nu \otimes w \circ (\sum x_i \otimes \nu(x_i) m_s)]
= \sum_s \sigma(\epsilon \otimes 1)[\nu \otimes w \circ (x_s \otimes m_s)]
= \sigma(\epsilon \otimes 1)[\nu \otimes wt \circ (x \otimes m)].
\]

Hence \( \sigma_{\nu,x}(a)m = \sigma(\epsilon \otimes 1)[\nu \otimes a \circ (x \otimes m)] \), for \( a \in A \). Then, from \( I(X \otimes_S M) = 0 \) it follows that \( I^X M = 0 \). From \( \mathcal{A} \)(16.1), we obtain that the \((A^X, I^X)\)-module \( M \) satisfies that \( F^X(M) = X \otimes_S M \cong N \) in \((\mathcal{A}, I)-\text{Mod}\). \( \square \)

Notice that the existence of some \( 0 \neq M \in (\mathcal{A}, I)-\text{Mod} \) as in \( \mathcal{A} \)(4) implies that \( I^X \) is a proper ideal of \( A^X \), because \((A^X, I^X)\)-Mod can not be trivial.

Now we briefly discuss an additional condition on the admissible \( B \)-module \( X \) which guarantees that \( F^X \) is full and faithful.

**Remark 6.8.** Assume that \( \mathcal{A}' = A^X_{\mathbf{a}, \mathbf{z}} \) is an interlaced weak ditalgebra with layer \((R', W')\), obtained by applying successively a finite sequence of reductions of type \( z_1, \ldots, z_t \in \{a, d, r, q, X\} \) from an interlaced weak ditalgebra \( \mathcal{A} \) with layer \((R, W)\), where \( R \) is a finite product of fields. Then, the \( R'-R' \)-bimodule \( W' \) is projective.

Indeed, if \( \mathcal{A}^z \) is the interlaced weak ditalgebra with layer \((R^z, W^z)\), obtained by a reduction of type \( z \in \{a, d, r, q, X\} \) from an interlaced weak ditalgebra \( \mathcal{A} \) with layer \((R, W)\), where \( W \) is a projective \( R-R \)-bimodule, it is not hard to verify that the \( R^z-R^z \)-bimodule \( W^z \) is projective.

The layers of the weak ditalgebras \( \mathcal{A} \) appearing in our arguments from §6 to the end of this work are typically pairs \((R, W)\), where \( R \) is a minimal algebra, as in \( \mathcal{A} \)(12), and \( W \) is a projective \( R-R \)-bimodule.
**Remark 6.9.** Assume that $B = T_R(W'_0)$ is a tensor algebra and let $X$ be an admissible $B$-module. Thus, we have a splitting $\Gamma = \text{End}_B(X)^\text{op} = S \oplus P$, as in (6.6). We have the ditalgebra $(B, 0)^X = (T_S(P^*), \delta)$, where $\delta$ is the differential determined by the comultiplication $\mu : P^* \rightarrow P^* \otimes_S P^*$. Recall from [4], that the admissible $B$-module $X$ is called *complete* if the functor

$$F^X : (B, 0)^X\text{-Mod} \rightarrow B\text{-Mod}$$

is full and faithful. From (17.4), (17.5), and (17.11) of [4], we know that we obtain complete admissible $B$-modules $X$ in the following cases:

1. $X$ is a finite direct sum of non-isomorphic finite-dimensional indecomposables in $B$-mod;
2. $X$ is the $B$-module obtained from the regular $S$-module $S$ by restriction through a given epimorphism of $k$-algebras $\phi : B \rightarrow S$;
3. $X = X_1 \oplus X_2$, where $X_1$ and $X_2$ are complete triangular admissible $B$-modules such that $\text{Hom}_B(I_{X_1}, I_{X_2}) = 0$, for $i \neq j$, and $I_X$ denotes the class of $B$-modules of the form $X_i \otimes_S N$, for some $N \in S$-Mod.

All the complete admissible $B$-modules we shall consider in this paper are constructed using 1, 2, and 3.

**Proposition 6.10.** Assume that $B = T_R(W'_0)$ and $X$ is a complete admissible $B$-module. Then, for any ideal $I_0$ of $B$, the functor $F^X$ induces a full and faithful functor

$$F^X : ((B, 0)^X, I^X_0)\text{-Mod} \rightarrow B/I_0\text{-Mod}.$$

**Proof.** By definition, the category $((B, 0)^X, I^X_0)\text{-Mod}$ coincides with the full subcategory of $(B, 0)^X\text{-Mod}$ formed by the $S$-modules $M$ such that $I^X_0 M = 0$. We have the commutative diagram

$$
\begin{array}{ccc}
((B, 0)^X, I^X_0)\text{-Mod} & \xrightarrow{F^X} & B/I_0\text{-Mod} \\
\downarrow & & \downarrow \\
(B, 0)^X\text{-Mod} & \xrightarrow{F^X} & B\text{-Mod},
\end{array}
$$

where the vertical functors, as well as $F^X$, are full and faithful functors. Therefore, the functor $F^X$ is also full and faithful.

**Proposition 6.11.** Under the assumptions of (6.7), consider the ideal $I_0 := B \cap I$ of $B = T_R(W'_0)$. Then, we can consider the triangular interlaced weak ditalgebra $(B, I_0)$, where $B = (B, 0)$. Then, for any complete admissible $B$-module $X$, we have a full and faithful functor

$$F^X : (B^X, I^X_0)\text{-Mod} \rightarrow (B/I_0)\text{-Mod} = (B/I_0)\text{-Mod},$$

and a full and faithful functor $F^X : (A, I^X)\text{-Mod} \rightarrow (A, I)\text{-Mod}$. 

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Proof. The proof of the fact that $F^X$ is full and faithful, for any complete admissible $B$-module $X$, is similar to the proof of [4](13.5), now using $F^X$. \square

**Definition 6.12.** An interlaced weak ditalgebra $(A, I)$ over the field $k$ is called **wild** iff there is an $A/I$-$k\langle x, y \rangle$-bimodule $Z$, which is free of finite rank as a right $k\langle x, y \rangle$-module, such that the composition functor

$$k\langle x, y \rangle\text{-Mod} \xrightarrow{Z \otimes_{k\langle x, y \rangle} -} A/I\text{-Mod} \xrightarrow{L(A, I)} (A, I)\text{-Mod}$$

preserves isomorphism classes of indecomposables. Here $L_{(A, I)}$ denotes the canonical embedding functor mapping each morphism $f^0$ onto $(f^0, 0)$. In this case, we say that $Z$ **produces the wildness of** $(A, I)$.

The following statement is just [4](22.7) rewritten for interlaced weak ditalgebras. The proof given there works here too.

**Lemma 6.13.** Assume that $H : (A', I')$-Mod$\longrightarrow(A, I)$-Mod is a functor obtained as a finite composition of functors of type $F^X$, for some admissible module $X$, or $F_\phi$, for some morphism $\phi : (A, I) \longrightarrow (A', I')$ of interlaced weak ditalgebras. Then, $H$ induces by restriction a functor $\overline{H}$ which makes the right square of the following diagram commutative. If $D$ is any $k$-algebra and $Z$ is an $(A'/I')$-$D$-bimodule, then $H(Z)$ is an $(A/I)$-$D$-bimodule and the first square in the following diagram commutes up to isomorphism

$$\begin{array}{ccc}
D\text{-Mod} & \xrightarrow{Z \otimes_D -} & (A'/I')\text{-Mod} \\
\| & & \downarrow \overline{H} \\
D\text{-Mod} & \xrightarrow{H(Z) \otimes_D -} & (A/I)\text{-Mod}
\end{array}
\quad \begin{array}{ccc}
 & \xrightarrow{L_{(A', I')}} & (A', I')\text{-Mod} \\
 & \downarrow H & \\
 & \xrightarrow{L_{(A, I)}} & (A, I)\text{-Mod}
\end{array}
$$

If $Z$ is a projective right $D$-module, so is $H(Z)$. In the particular case $D = A'/I'$, we get that $\overline{H} \cong H(A'/I') \otimes_{(A'/I')}$ is exact and preserves direct sums.

**Proposition 6.14.** Assume that a triangular interlaced weak ditalgebra $(A^z, I^z)$ is obtained from a Roiter interlaced weak ditalgebra $(A, I)$ by some of the procedures described in this section, that is $z \in \{d, r, q, a, X\}$. Then, $(A^z, I^z)$ is a Roiter interlaced weak ditalgebra. The associated full and faithful functor

$$F^z : (A^z, I^z)\text{-Mod} \longrightarrow (A, I)\text{-Mod}$$

preserves isomorphism classes and indecomposables. Moreover, if $(A^z, I^z)$ is wild (with wildness produced by an $(A^z/I^z)$-$k\langle x, y \rangle$-bimodule $Z$) then so is $(A, I)$ (with wildness produced by the $(A/I)$-$k\langle x, y \rangle$-bimodule $F^z(Z)$).

Proof. The fact that $(A^z, I^z)$ is a Roiter interlaced weak ditalgebra can be proved in a similar way as in the proof of the same fact for ditalgebras as in (9.3) and (16.3) of [4]. The last statement follows from the preceding lemma, as in (22.8) and (22.10) of [4]. In case $z = X$, we use [6.7](4). \square
7 Stellar weak ditalgebras

In this section we shall see that for a special kind of interlaced weak ditalgebras \((A, I)\), the stellar ones, we can apply reduction procedures and reach after a finite number of steps a seminested ditalgebra, as in \([3](23.5)\). The corresponding reduction functor covers finite-dimensional \((A, I)\)-modules with dimension bounded by some number \(d \in \mathbb{N}\).

**Definition 7.1.** Recall that a minimal algebra \(R\) is a finite product of algebras \(R = \prod_{i \in P} R_i\), where each \(R_i\) is either a rational \(k\)-algebra or is isomorphic to the field \(k\). So, we have a decomposition of the unit of \(R\) as a sum \(1 = \sum_{i \in P} e_i\) of primitive orthogonal idempotents. Assume \(A\) is a triangular weak ditalgebra with layer \((R, W)\), where \(R\) is a minimal algebra. Then, the points of \(A\) are the mentioned idempotents (or the set \(P\) of their subscripts). A multiple source \(\Omega\) of \(A\) is a subset \(\Omega\) of \(P\) such that, if we define \(e_\Omega := \sum_{\omega \in \Omega} e_\omega\), the following are satisfied: \(e_\Omega W_0 = 0\), \(e_\Omega W_1 = e_\Omega W_1 e_\Omega\), and \(Re_\omega = ke_\omega\), for all \(\omega \in \Omega\).

The elements of \(\Omega\), or the corresponding family of idempotents, \(\{e_\omega\}_{\omega \in \Omega}\) are called the sources of the given multiple source \(\Omega\).

**Definition 7.2.** A stellar weak ditalgebra \(A\) is a weak ditalgebra with triangular layer \((R, W)\), with \(R\) a minimal algebra and \(W\) a projective \(R\)-\(R\)-bimodule, such that there is a multiple source \(\Omega\) of \(A\), and \(W_0 = W_0 e_\Omega\). For each \(\omega \in \Omega\), the algebra \(T_{Re_\omega}(W_0 e_\omega)\) is called the star of \(A\) with center \(e_\omega\).

Notice that if \(A\) is a stellar weak ditalgebra with layer \((R, W)\), then we have \(A = R \oplus W_0\).

The following elementary lemma, where \(R_h\) denotes the localization of the ring \(R\) with respect to the element \(h \in R\), will be useful to us.

**Lemma 7.3.** Given a principal ideal domain \(R\) and a finitely generated \(R\)-module \(U\) with a filtration in \(R\)-mod

\[
0 = U_0 \subseteq U_1 \subseteq \cdots \subseteq U_\ell = U,
\]

there is \(h \in R\) such that \(R_h \otimes_R U\) is a free \(R_h\)-module and

\[
0 = R_h \otimes_R U_0 \subseteq R_h \otimes_R U_1 \subseteq \cdots \subseteq R_h \otimes_R U_\ell = R_h \otimes_R U\]

is an additive filtration of \(R_h \otimes_R U\) in \(R_h\)-mod, that is each term of the filtration is a direct summand of the next one.

**Proof.** Step 1: The case \(\ell = 1\). Here, we just have to show that for any \(U \in R\)-mod there is \(h \in R\) such that \(R_h \otimes_R U\) is a free \(R_h\)-module.

This is a standard procedure, we have an exact sequence in \(R\)-mod

\[
R^s \xrightarrow{H} R^r \longrightarrow U \longrightarrow 0,
\]

with \(H\) a matrix in \(R^{r \times s} \subseteq K^{r \times s}\), where \(K\) is the field of fractions of \(R\). Then, there are invertible matrices \(P\) and \(Q\) with coefficients in \(K\) such that

\[
PHQ = \begin{pmatrix} I_d & 0 \\ 0 & 0 \end{pmatrix},
\]

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where \( d \) is the rank of \( H \). Consider \( h \in R \) such that \( P \) and \( Q \) have entries in \( R_h \), then \( R_h \otimes_R U = \text{Coker}(1_{R_h} \otimes H) \cong R_h^{d} \) is free in \( R_h \text{-mod} \).

**Step 2:** The case \( \ell = 2 \). We take \( U' \subseteq U \) and consider the following exact sequence in \( R \text{-mod} \)

\[
0 \rightarrow U' \rightarrow U \rightarrow U/U' \rightarrow 0.
\]

Then, apply the first step to obtain \( h \in R \) with \( R_h \otimes_R (U \oplus U' \oplus [U/U']) \) free in \( R_h \text{-mod} \). So, \( R_h \otimes_R U', R_h \otimes_R U, \) and \( R_h \otimes_R [U/U'] \) are free modules too. The exact sequence obtained from the preceding one by tensoring by \( R_h \) splits and we are done.

**Step 3:** The general case. This is an easy induction.

**Remark 7.4.** Notice that, in the last lemma, whenever \( U \) is not a torsion \( R \text{-module} \), we have \( R_h \otimes_R U \neq 0 \).

Moreover, given a finite sequence of finitely generated \( R \text{-modules} \) \( U^1, \ldots, U^t \) with filtrations in \( R \text{-mod} \)

\[
0 = U^t_0 \subseteq U^t_1 \subseteq \cdots \subseteq U^t_t = U^t
\]

with \( t \in [1, m] \), there is \( h \in R \), such that after tensoring by the same \( R_h \) the preceding filtrations, we obtain additive filtrations with free terms in \( R_h \text{-mod} \)

\[
0 = R_h \otimes_R U^t_0 \subseteq R_h \otimes_R U^t_1 \subseteq \cdots \subseteq R_h \otimes_R U^t_t = R_h \otimes_R U^t.
\]

**Lemma 7.5** (multiple unravelling with stars). Let \((A, I)\) be a triangular interlaced weak ditalgebra with triangular layer \((R, W)\). Suppose that \(A\) is stellar with stars centers \(\{e_\omega\}_{\omega \in \Omega}\). We have \( R = \prod_{i \in \mathcal{P}} R_i e_i \), where each \(R_i e_i\) is either isomorphic to \(k\) or to some rational algebra. We can assume that \(\mathcal{P} = J \uplus J'\) where \(R e_i = k e_i\), for \(i \in J\), and \(R e_j = R e_j = k[e_j],\) for \(j \in J'\). Notice that, \(\Omega \subseteq J\).

Take \(d \in \mathbb{N}\) and non-zero elements \(h_j \in R_j,\) for \(j \in J'\). Then, there is complete triangular admissible \(R\)-module \(X\) such that \((A^X, I^X)\) is a triangular interlaced weak ditalgebra with triangular layer \((S, W^X)\), where \(S\) is a minimal algebra of the form

\[
S = \left[ \prod_{\omega \in \mathcal{P}} k f_\omega \right] \times \left[ \prod_{j \in J'} e_j(R_j) h_j \right] \times \prod_{j \in J} k e_j.
\]

Where \(\{f_\omega\}_{\omega \in \mathcal{P}}\) is a new finite family of primitive idempotents of \(S\). The weak ditalgebra \(A^X\) is stellar with stars centers \(\{e_\omega\}_{\omega \in \Omega}\). We have that \(I^X \subseteq W^X_0\), whenever \(I \subseteq W_0\). Moreover, there is a full and faithful functor

\[
F^X: (A^X, I^X)\text{-Mod} \longrightarrow (A, I)\text{-Mod}
\]

such that for any \(M \in (A, I)\text{-Mod}\) with \(\dim_k M \leq d\), there is some \(N \in (A^X, I^X)\text{-Mod}\) with \(F^X(N) \cong M\).
Since $X$ with triangular layer $(S,W)$ is complete triangular admissible with filtration $X$ where $W$ is full and faithful. As remarked in (6.8), we have the splitting $\text{End}_R(X) = S \oplus P$, where

$$X = Z \bigoplus \left( \bigoplus_{j \in J'} e_j(R_j)_{h_j} \right) \bigoplus Re,$$

Then, we have the splitting $\text{End}_R(X)^{\text{op}} = S \oplus P$, where

$$f_s \in \text{End}_R(X)^{\text{op}}$$

is the idempotent corresponding to the indecomposable direct summand $Z_s$ of $X$, and $P = \text{radEnd}_R(Z)^{\text{op}}$.

Consider the filtration of $P$ given by its powers

$$0 = P^{(i+1)}(\ell_p) \subseteq P^{(i+2)}(\ell_p) \subseteq \ldots \subseteq P^{(\ell_p)} = P,$$

thus $P^{(i)}(P^{(j)}) \subseteq P^{(i+j)}$ for all $i, j \in [1, \ell_p]$ with $i + j \leq \ell_p$, and $P^{(i)}(P^{(j)}) = 0$, otherwise. It determines a filtration of the $R$-module $Z$

$$0 = Z_0 \subseteq Z_1 \subseteq \ldots \subseteq Z_{\ell_p} = Z,$$

with $Z_j P \subseteq Z_{j-1}$, for all $j \in [1, \ell_p]$. From (6.9), we know that the $R$-module $X$ is complete triangular admissible with filtration

$$0 = X_0 \subseteq X_1 \subseteq \ldots \subseteq X_{\ell_p} = X,$$

where $X_1 = \left( \bigoplus_{j \in J'} e_j(R_j)_{h_j} \right) \oplus Re \oplus Z_1$, $X_2 = X_1 \oplus Z_2$, ..., $X_{\ell_p} = X_1 \oplus Z_{\ell_p}$.

Then, by (6.7), we have a triangular interlaced weak ditalgebra $(A^X, I^X)$ with triangular layer $(S, W^X)$ given by

$$W^X_0 = X^* \otimes_R W_0 \otimes_R X \quad \text{and} \quad W^X_1 = (X^* \otimes_R W_1 \otimes_R X) \oplus P^*.$$

Since $X$ is complete, by (6.11), the functor $F^X : (A^X, I^X)\text{-}\text{Mod} \longrightarrow (A, I)\text{-}\text{Mod}$ is full and faithful. As remarked in (6.8), $W^X$ is a projective $S-S$-bimodule.

Since $e_\alpha W_0 = 0$, for $\omega \in \Omega$, we have

$$e_\omega[W^X_0] = e_\omega[X^*] \otimes_R W_0 \otimes_R X = e_\omega k \otimes_R W_0 \otimes_R X = e_\omega k \otimes e_\omega W_0 \otimes_R X = 0.$$

Thus, $e_\alpha(W^X_0) = 0$. Moreover, since $e_\alpha P^* = 0$ and $e_\alpha W_1 = e_\alpha W_1 e_\alpha$, we have:

$$e_\alpha W^X_1 = \bigoplus_{\omega \in \Omega} e_\omega(X^*) \otimes_R W_1 \otimes_R X$$

$$= \bigoplus_{\omega \in \Omega} ke_\omega \otimes_R W_1 \otimes_R X$$

$$= \bigoplus_{\omega \omega' \in \Omega} ke_\omega \otimes_R e_\omega W_1 e_{\omega'} \otimes_R X$$

$$= \bigoplus_{\omega \omega' \in \Omega} ke_\omega \otimes_R e_\omega W_1 e_{\omega'} \otimes_R ke_{\omega'}$$

$$= \bigoplus_{\omega \omega' \in \Omega} e_\omega(X^*) \otimes_R W_1 \otimes_R X e_{\omega'} = e_\alpha W^X_1 e_\alpha.$$
So, $\Omega$ is a multiple source in $A^X$. Since $W_0 = W_0e_\alpha$, we also have
\[
W_0^X e_\alpha = X^* \otimes_R W_0 \otimes_R X e_\alpha = \bigoplus_{\omega \in \Omega} X^* \otimes_R W_0 \otimes_R ke_\omega = \bigoplus_{\omega \in \Omega} X^* \otimes_R W_0 e_\omega \otimes_R X = W_0^X.
\]
So $A^X$ is a stellar weak ditalgebra with stars centers $\{e_\omega\}_{\omega \in \Omega}$.

Let us show that if $I \subseteq W_0$, then $I^X \subseteq W_0^X$. More precisely, that $I^X = A^X (X^* \otimes_R I \otimes_R X)A^X = X^* \otimes_R I \otimes_R X$, where $X^* \otimes_R I \otimes_R X$ denotes the image of the canonical map $X^* \otimes_R I \otimes_R X \rightarrow X^* \otimes_R W_0 \otimes_R X$. Indeed, by assumption we have $W_0 = W_0 e_\alpha = \oplus \omega W_0 e_\omega$ and also $e_\alpha W_0 = 0$, so for $h \in I \subseteq W_0$, we have $h = \sum_\omega he_\omega$. Then, given generators $\nu \otimes h \otimes y \in X^* \otimes_R I \otimes_R X$ and $\nu' \otimes w_0 \otimes y' \in X^* \otimes_R W_0 \otimes_R X$, we have $(\nu \otimes h \otimes y)(\nu' \otimes w_0 \otimes y') = \sum_\omega (\nu \otimes he_\omega \otimes y)(\nu' \otimes w_0 \otimes y') \in \sum_\omega (X^* \otimes_R I \otimes_R ke_\omega)(e_\omega k \otimes_R W_0 \otimes_R X) = 0$. Similarly, we obtain that $(\nu' \otimes w_0 \otimes y') (\nu \otimes h \otimes y) \in \sum_\omega (X^* \otimes_R W_0 \otimes_R ke_\omega)(e_\omega k \otimes_R I \otimes_R X) = 0$.

Now, take any $M \in (A, I)-Mod$ with $\dim_k M \leq d$. Let us show that there is some $N \in (A^X, I^X)-Mod$ with $F^X(N) \cong M$. By (6.7)(4), it will be enough to see that there is some left $S$-module $N_0$ such that $M \cong X \otimes_S N_0$ as $R$-modules. Since $M$ satisfies that $\dim_k e'M \leq \dim_k M \leq d$, we know that $M = e'M \oplus eM$, and $e'M \cong \bigoplus_{s \in J'} n_s Z_s \oplus M_e$, where every $h_j$ acts invertibly on $e_j M_e$. Then, we can consider the left $S$-module
\[
N_0 = \bigoplus_{s \in J'} n_s k f_s \bigoplus M_e \bigoplus eM.
\]
Hence, $M \cong X \otimes_S N_0$ as $R$-modules, and there is $N \in (A^X, I^X)-Mod$ with $F^X(N) \cong M$.

**Lemma 7.6.** Let $(A, I)$ be a triangular interlaced weak ditalgebra with triangular layer $(R, W)$. Suppose that $A$ is stellar. Moreover, assume that $I \subseteq W_0$ and take any $d \in \mathbb{N}$. Then, there is a triangular interlaced weak ditalgebra $(A^X, I^X)$ with triangular layer $(S, W^X)$, such that $A^X$ is stellar and $I^X$ is a direct summand of $W_0^X$ as an $S$-$S$-bimodule. Moreover, there is a full and faithful functor
\[
F^X : (A^X, I^X)-Mod \rightarrow (A, I)-Mod
\]
such that for any $M \in (A, I)-Mod$ with $\dim_k M \leq d$, there is some $N \in (A^X, I^X)-Mod$ with $F^X(N) \cong M$.

**Proof.** We have $R = \prod_{i \in \mathcal{P}} Re_i$, where each $Re_i$ is either isomorphic to $k$ or to some rational algebra. We can assume that $\mathcal{P} = J \cup J'$ where $Re_i = ke_i$, for $i \in J$, and $Re_j = R_je_j$ with $R_j = k[x]_{g_j}$, for $j \in J'$. Then we get $R = Re \times Re'$, where $e = \sum_{i \in J} e_i$ and $e' = \sum_{j \in J'} e_j$. Notice that, if $\{e_\omega\}_{\omega \in \Omega}$ are the centers of the stars of $A$, we have $\Omega \subseteq J$.

By assumption $I \subseteq W_0$. For each $j \in J'$ and $\omega \in \Omega$, the bimodule $e_je_0 e_\omega = e_je_0 e_\omega$ is an $R_j = R_j \otimes_k k$-module. From (7.3), there is some $h_j \in R_j$ such that, for all $\omega \in \Omega$, we have that $(R_j)_{h_j} \otimes R_j e_je_0 e_\omega$ is a free $(R_j)_{h_j}$-module and $(R_j)_{h_j} \otimes R_j e_je_0 e_\omega$ is a direct summand of $(R_j)_{h_j} \otimes R_j e_je_0 e_\omega$. So
we obtain basis \( \mathcal{B}_I(j, \omega) \subset \mathcal{B}_0(j, \omega) \) of the free \((R_j)_h\)-modules \((R_j)_h \otimes_R e_j I e_\omega\) and \((R_j)_h \otimes_R e_j W_0 e_\omega\).

Now, we can apply \([7.3]\) to the parameters \( d \) and \( \{h_j\}_{j \in J'} \) to obtain the weak interlaced ditalgebra \((A^X, I X)\) with layer \((S, W^X)\) as specified in that statement, and we have a full and faithful functor with the wanted properties.

So we only have to show that \( I X \) is a direct summand of \( W^X_0 \) as \( S\)-\( S\)-bimodules. For this, we keep the notation of the proof of \([7.5]\). Since \( I \subset W_0 \), we already know that \( I X \subset W^X_0 \), \( e_\omega W^X_0 = 0 \), \( W^X_0 = W^X_{\epsilon_{\Omega}} \), and \( e_\omega W^X_1 = e_\omega W^X_{\epsilon_{\Omega}} \). Thus, we get \( I X = I X e_\alpha \). For \( j \in J' \) and \( \omega \in \Omega \), we have

\[
e_j[W^X_0] e_\omega = e_j[X^*] \otimes_R W_0 \otimes_R X e_\omega = e_j(R_j)_h \otimes_R W_0 \otimes_R ke_\omega \cong e_j(R_j)_h \otimes R W_0 e_\omega.
\]

Since each \( e_j(R_j)_h \)-module filtration \( e_j[I X] e_\omega \subset e_j[W^X_0] e_\omega \) is isomorphic to the additive filtration \( e_j(R_j)_h \otimes_R I e_\omega \subset e_j(R_j)_h \otimes_R W_0 e_\omega \) of free modules described before, we can find bases \( \mathcal{B}_{I X}(j, \omega) \subset \mathcal{B}_0^X(j, \omega) \) of \( e_j[I X] e_\omega \) and \( e_j[W^X_0] e_\omega \). If \( j \in J \), we choose any vector space basis \( \mathcal{B}_{I X}(j, \omega) \subset \mathcal{B}_0^X(j, \omega) \) of \( e_j[I X] e_\omega \) and \( e_j[W^X_0] e_\omega \), respectively; similarly, we choose any vector space basis \( \mathcal{B}_{I X}(s, \omega) \subset \mathcal{B}_0^X(s, \omega) \) of \( f_s I X e_\omega \) and \( f_s W^X_0 e_\omega \), respectively, for any \( s \in J'' \). Thus, \( I X e_\omega \)

is a direct summand of \( W^X_0 e_\omega \) as \( S\)-\( S\)-bimodules, because they are respectively freely generated by the sets

\[
\bigcup_{j \in J} \mathcal{B}_{I X}(j, \omega) \bigcup_{s \in J''} \mathcal{B}_{I X}(s, \omega) \subset \bigcup_{j \in J} \mathcal{B}_0^X(j, \omega) \bigcup_{s \in J''} \mathcal{B}_0^X(s, \omega).
\]

It follows that \( I X = \bigoplus_{\omega \in \Omega} I X e_\omega \) is a direct summand of \( W^X_0 \) as \( S\)-\( S\)-bimodules, and we are done.

\[\square\]

**Proposition 7.7.** Let \((A, I)\) be triangular interlaced weak ditalgebra with triangular layer \((R, W)\). Moreover, assume that \( A \) is stellar and that there is a decomposition of \( R\)-\( R\)-bimodules \( W_0 = W'_0 \oplus W''_0 \), where \( W'_0 \) generates the ideal \( I \) of \( A \), and \( \delta(W''_0) \subset IV + VI \). Set \( W'_1 := W'_0 \), \( W''_1 := W'_1 \), \( W'' := W''_0 \oplus W''_1 \), and \( T'^I := T_R(W''_0) \). Then, there is a derivation \( \delta^I \) on \( T'^I \) such that \( A'^I := (T'^I, \delta^I) \) is a weak ditalgebra and \((A'^I, 0)\) is a triangular interlaced weak ditalgebra with \( A'^I \) stellar and triangular layer \((R, W'^I)\). Moreover, there is a morphism \( \phi : (A, I) \longrightarrow (A'^I, 0) \) of interlaced weak ditalgebras which induces an equivalence of categories

\[
F'^I = F_0 : (A'^I, 0) \rightarrow \text{-Mod} \longrightarrow (A, I) \rightarrow \text{-Mod}.
\]

**Proof.** It follows from \([6.8]\), and form \([6.4]\) with \( I = I' \).

\[\square\]

Notice that the following statement is about ditalgebras, not weak ditalgebras, so we can use freely the terminology and results of \([4]\). Anyway, we extend the terminology to this context. A directed element of a given \( R\)-\( R\)-bimodule, where \( R \) is a minimal algebra, is any non-zero element \( w \) such that \( w = e_j w e_i \), for some primitive idempotents \( e_i, e_j \) of \( R \).

**Definition 7.8.** Let \( A \) be a layered weak ditalgebra. Then,
1. A layer \((R, W)\) of \(\mathcal{A}\) is called seminested iff \(R\) is a minimal \(k\)-algebra, the layer \((R, W)\) is triangular, the \(R-R\)-bimodule \(W_1\) is freely generated by a finite directed subset \(\mathbb{B}_1\) of \(W_1\), the \(R-R\)-bimodule filtration \(\mathcal{F}(W_0) = \{W_0^j\}_{j=0}^\infty\) of \(W_0\) is freely generated by a set filtration \(\mathcal{F}(\mathbb{B}_0) = \{\mathbb{B}_0^j\}_{j=0}^\infty\) of some finite directed subset \(\mathbb{B}_0\) of \(W_0\). See [4](23.2);

2. A layered weak ditalgebra \(\mathcal{A}\) is called seminested iff its layer is seminested.

A weak seminested ditalgebra \(\mathcal{A}\) is called minimal iff its layer \((R, W)\) is seminested with \(W_0 = 0\).

**Lemma 7.9.** Let \(\mathcal{A}\) be a stellar ditalgebra with triangular layer \((R, W)\) and such that \(W_1\) is freely generated by some finite directed subset. Then, given any \(d \in \mathbb{N}\), there is a stellar seminested ditalgebra \(\mathcal{A}'\) and a full and faithful functor

\[
F : \mathcal{A}'-\text{Mod} \longrightarrow \mathcal{A}-\text{Mod}
\]

such that for any \(M \in \mathcal{A}-\text{Mod}\) with \(\dim_k M \leq d\), there is some \(N \in \mathcal{A}'-\text{Mod}\) with \(F(N) \cong M\).

**Proof.** We proceed as in the proof of (7.6). We have \(R = \prod_{i \in \mathcal{P}} R_i e_i\), where each \(R_i e_i\) is either isomorphic to \(k\) or to some rational algebra. We can assume that \(\mathcal{P} = J \cup J'\) where \(R e_j = k e_j\), for \(j \in J\), and \(R e_j = R_j e_j\) with \(R_j = k[x]_{\mathbb{Z}_j}\), for \(j \in J'\). Thus, if \(\{e_\omega\}_{\omega \in \Omega}\) denote the centers of the stars, \(\Omega \subseteq J\). Then we get \(R = Re \times R'\), where \(e = \sum_{j \in J} e_j\) and \(e' = \sum_{j \in J'} e_j\).

From the triangularity of \((R, W)\), we have an \(R-R\)-bimodule filtration

\[
0 = W_0^0 \subseteq W_0^1 \subseteq \cdots \subseteq W_0^d = W_0.
\]

For each \(j \in J'\) and \(\omega \in \Omega\), the bimodule \(e_j W_0^0 e_\omega\) is an \(R_j = R_j \otimes_k k\)-module. Then, by (7.6), we can find \(h_j \in R_j\) such that, for all \(\omega \in \Omega\), the \((R_j)_{h_j}\)-module filtration

\[
0 = (R_j)_{h_j} \otimes_R W_0^0 e_\omega \subseteq (R_j)_{h_j} \otimes_R W_0^1 e_\omega \subseteq \cdots \subseteq (R_j)_{h_j} \otimes_R W_0^d e_\omega = (R_j)_{h_j} \otimes_R W_0 e_\omega
\]

of the free \((R_j)_{h_j}\)-module \((R_j)_{h_j} \otimes_R e_j W_0^0 e_\omega\) is additive. Then, we obtain set filtrations

\[
0 = \mathbb{B}_0^0(j, \omega) \subseteq \mathbb{B}_0^1(j, \omega) \subseteq \cdots \subseteq \mathbb{B}_0^d(j, \omega) = \mathbb{B}_0(j, \omega)
\]

of basis \(\mathbb{B}_0^i(j, \omega)\) of the free \((R_j)_{h_j}\)-modules \((R_j)_{h_j} \otimes_R e_j W_0^i e_\omega\).

Fix \(d \in \mathbb{N}\) and apply (7.5) to the weak ditalgebra \((A, 0)\) and the family \(\{h_j\}_{j \in J'}\) to obtain the weak interlaced ditalgebra \((A^X, I^X)\) with layer \((S, W^X)\) as specified in that statement, and we have the full and faithful functor with the wanted properties. Moreover, we know that \(e_\alpha W_0^X = 0\), \(e_\alpha W_1^X = e_\alpha W_1^X e_\alpha\), and \(W_0^X = W_0^X e_\alpha\).

With the notation used in (7.5), we only have to show that the layer \((S, W^X)\) is seminested. In the proof of [4](14.10), the triangular filtrations of \(W_0^X\) and \(W_1^X\) are exhibited. The typical term \([W_0^X]^m\) of the triangular filtration of the \(S-S\)-bimodule \(W_0^X\), defined for \(m \in [0, 2f_X(\ell_0 + 1)]\), is the sum

\[
[W_0^X]^m = \sum_{r+2f_Xs+t \leq m} [X^s]_r \otimes_R W_0^r \otimes_R X_t,
\]

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where each summand $[X^*]_r \otimes_R W_0^* \otimes_R X_t$ denotes the image of the canonical map $[X^*]_r \otimes_R W_0^* \otimes_R X_t \longrightarrow X^* \otimes_R W_0 \otimes_R X$. Here, $X_t$ denotes the term $t$ of the filtration of $X$ specified in the proof of \([7.3]\), and $[X^*]_r$ denotes the term $r$ of the corresponding dual filtration of $X^*$, see \([4]\). For each $j \in J'$ and $\omega \in \Omega$, we have isomorphisms of $(R_j)_{h_j}$-modules

$$e_j W_0 X e_\omega = e_j X^* \otimes_R W_0 \otimes_R X e_\omega = e_j (R_j)_{h_j} \otimes_R W_0 \otimes_R k e_\omega \cong e_j (R_j)_{h_j} \otimes_R W_0 e_\omega$$

For $r \neq \ell_X$, we have $e_j [X^*]_r = 0$; and $e_j [X^*]_{\ell_X} = e_j X^* = e_j (R_j)_{h_j}$, so

$$e_j X^* \otimes_R W_0^* \otimes_R X e_\omega = e_j (R_j)_{h_j} \otimes_R W_0^* \otimes_R k e_\omega \cong e_j (R_j)_{h_j} \otimes_R W_0^* e_\omega,$$

for each $s$, $t$ with $\ell_X + 2\ell_X s + t \leq m$. They determine the following commutative squares

$$\bigoplus_{s,t} e_j X^* \otimes_R W_0^* \otimes_R X e_\omega \xrightarrow{\sigma_m} \bigoplus_{s,t} e_j (R_j)_{h_j} \otimes_R W_0^* e_\omega$$

where $\sigma_m$ and $\sigma_j$ are isomorphisms and the vertical arrows are the canonical maps. Then, the restriction of $\sigma_j$ to the image of the canonical maps gives an isomorphism of free $(R_j)_{h_j}$-modules, so an isomorphism of freely generated $S$-$S$-bimodules,

$$\bar{\sigma}_m : e_j [W_0 X]^m e_\omega \longrightarrow e_j (R_j)_{h_j} \otimes_R W_0^* e_\omega,$$

for a suitable $s(m)$ independent of $\omega$. Clearly, $s(m) \leq s(m + 1)$. This implies that from the additivity of the filtration of the $S$-$S$-bimodule $e_j (R_j)_{h_j} \otimes_R W_0 e_\omega$ that we constructed before we can derive the additivity of the filtration

$$0 = e_j [W_0 X]_0 e_\omega \subseteq e_j [W_0 X]^1 e_\omega \subseteq \cdots \subseteq e_j [W_0 X]^m e_\omega \subseteq \cdots \subseteq e_j W_0 X e_\omega.$$

Since each one of these $S$-$S$-bimodules $e_j [W_0 X]^m e_\omega$ is freely generated, we can find a filtration of bases, where each $(\mathbb{B}_0^X)(j, \omega)^m$ freely generates the $S$-$S$-bimodule $e_j [W_0 X]^m e_\omega$, as follows:

$$0 = (\mathbb{B}_0^X)(j, \omega)^0 \subseteq (\mathbb{B}_0^X)(j, \omega)^1 \subseteq \cdots \subseteq (\mathbb{B}_0^X)(j, \omega)^m \subseteq \cdots \subseteq (\mathbb{B}_0^X)(j, \omega).$$

These filtrations can be completed to a filtration of sets, where each $(\mathbb{B}_0^X)(\omega)^m$ freely generates the $S$-$S$-bimodule $W_0^X e_\omega$, as follows

$$0 = (\mathbb{B}_0^X)(\omega)^0 \subseteq (\mathbb{B}_0^X)(\omega)^1 \subseteq \cdots \subseteq (\mathbb{B}_0^X)(\omega)^m \subseteq \cdots \subseteq (\mathbb{B}_0^X)(\omega),$$

where $\mathbb{B}_0^X(\omega)$ freely generates the $S$-$S$-bimodule $W_0^X e_\omega$. This completion is done by suitable choices of vector space basis of each $e_j [W_0 X]^m e_\omega$, for $j \in J$, and of each $f_s [W_0 X]^m e_\omega$, for $s \in J''$, as in the proof of \([7.3]\). Taking the union over $\omega \in \Omega$, we get a set filtration $\mathcal{F}(\mathbb{B}_0^X)$ of a directed subset $\mathbb{B}_0^X$ of $W_0^X$ which freely generates it.

It follows that $\mathcal{A}^X$ is a seminested ditalgebra. Indeed, we already know that $W_1^X$ is freely generated because $W_1$ is so. \(\square\)
Lemma 7.10. Let $A$ be a stellar triangular ditalgebra. Then, given any $d \in \mathbb{N}$, there is a stellar ditalgebra $A'$ with triangular layer $(R', W')$, such that $W_1'$ is freely generated by some finite directed subset, and a full and faithful functor

$$F : A' \text{-Mod} \longrightarrow A \text{-Mod}$$

such that for any $M \in A \text{-Mod}$ with $\dim_k M \leq d$, there is some $N \in A' \text{-Mod}$ with $F(N) \cong M$.

Proof. It is similar to the proof of the preceding lemma: if the triangular layer of $A$ is $(R, W)$, we “localize” some of the factors of $R$ in order to transform simultaneously every $R_j$-module $e_j W_1 e_\omega$ into a free module $(R_j) h_j \otimes R_j e_j W_1 e_\omega$, for $\omega \in \Omega$. □

Theorem 7.11. Let $(A, I)$ be a triangular interlaced weak ditalgebra with $A$ stellar, triangular layer $(R, W)$, and stars centers $\{e_\omega\}_{\omega \in \Omega}$. Suppose that $e_\omega \notin I$, for all $\omega \in \Omega$, and take any $d \in \mathbb{N}$. Then, there is a seminested ditalgebra $A'$ and a composition of full and faithful reduction functors

$$F : (A', 0) \text{-Mod} \longrightarrow (A, I) \text{-Mod}$$

such that any $M \in (A, I) \text{-Mod}$ with $\dim_k M \leq d$ is of the form $F(N) \cong M$, for some $N \in (A', 0) \text{-Mod}$.

Proof. Since $e_\omega W_0 = 0$ and $W_0 = W_0 e_\omega$, we know that $A = R \oplus W_0$ and $V = W_1 \oplus (W_1 \otimes R W_0) \oplus W_0 e_\omega W_1 e_\omega$. In fact, the following formula holds

$$(*) \quad I = (I \cap R) \oplus (I \cap W_0).$$

Indeed, we have $R = \prod_{i \in P} R e_i$, and $I = \bigoplus_{i \in P \setminus \Omega} I e_i \oplus \bigoplus_{\omega \in \Omega} I e_\omega$. For $i \notin \Omega$, since $W_0 e_i = 0$ and $I \subseteq R \oplus W_0$, we have $I e_i = I \cap R e_i = (I \cap R) e_i$. Now, take $\omega \in \Omega$ and $a \in I e_\omega$, so $a = r + \nu$, with $r = re_\omega \in Re_\omega$ and $\nu = ve_\omega \in W_0 e_\omega$. If $r \neq 0$, then $e_\omega W_0 = 0$, we get $ce_\omega = re_\omega + e_\omega v = e_\omega a \in I$, with $0 \neq c \in k$. So, $e_\omega \in I$, a contradiction. Thus we get $I e_\omega = I \cap W_0 e_\omega = (I \cap W_0) e_\omega$ and the formula $(*)$ holds.

Case 1: There is some $j \in P \setminus \Omega$ with $e_j \in I \cap R$.

Fix such idempotent $e_j$. Thus, we have $e_j M = 0$, for each $M \in (A, I) \text{-Mod}$. Then, we can consider the interlaced weak ditalgebra $(A^d, I^d)$ obtained by deletion of the idempotent $e_j$ and the corresponding full and faithful functor $F^d : (A^d, I^d) \text{-Mod} \longrightarrow (A, I) \text{-Mod}$, where $R^d$ has less factors than $R$ and $F^d$ is dense. Moreover, $(A^d, I^d)$ is a stellar weak ditalgebra with stars centers $\{e_\omega\}_{\omega \in \Omega}$ and $e_\omega \notin I^d$, for all $\omega \in \Omega$. We can repeat this process a finite number of times, if necessary, and so we can assume that $R \cap I$ contains no primitive idempotent of $R$. For the rest of this proof, we assume that.

Case 2: $I \cap R \neq 0$. 

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We will reduce this Case 2 to the following Case 3. Since $I_0 := I \cap R \neq 0$, we have the following non-empty set of indexes
\[ S = \{s \in P \mid \Omega \mid I_0e_s \neq 0\}. \]
For $s \in S$, since $e_s \notin I$, we know that $0 \neq I_0e_s$ is a proper ideal of $Re_s$.

Define $e = \sum_{s \in S} e_s$ and $f = 1 - e - e_\alpha$. So $1 = f + e + e_\alpha$ is a decomposition of $1 \in R$ as a sum of orthogonal idempotents. We have the ideal $I_0 \leq Re$ and the quotient algebra $Re/I_0 \cong \prod_{s \in S} Re_s/I_0e_s$ has finite representation type. Let $\{Z_t\}_{t \in J}$ be a complete (finite) system of representatives of the isoclasses of the indecomposable $Re/I_0$-modules. Take $Z = \bigoplus_{t \in J} Z_t$ and consider the following $R$-module
\[ X = Z \oplus Rf \oplus Re_\alpha, \]
where $Z$ is an $R$-module by restriction through the surjective morphism of algebras $R \longrightarrow Re/I_0$. So, we have $I_0X = 0$. We have the splitting $\Gamma = \End_R(X)^{op} = S \oplus P$, where
\[ S = \prod_{t \in J} kf_t \times Rf \times \prod_{\omega \in \Omega} ke_\omega, \]
$P = \rad \End_R(Z)^{op}$, and $f_t$ is the idempotent corresponding to $Z_t$ in the given decomposition of $X$. Thus $S$ is a subalgebra of $\Gamma$, the given decomposition of $\Gamma$ is an $S$-$S$-bimodule decomposition, and $X$ and $P$ are finitely generated projective right $S$-modules. In fact $X$ is a triangular admissible $R$-module and we can form the triangular interlaced weak ditalgebra $(\mathcal{A}^X, I^X)$ with layer $(S, W_0^X \oplus W_1^X)$, where
\[ W_0^X = X^* \otimes_R W_0 \otimes_R X \quad \text{and} \quad W_1^X = (X^* \otimes_R W_1 \otimes_R X) \oplus P^*, \]
as in (6.7). By (6.11), the admissible $R$-module $X$ is complete and we have a full and faithful functor
\[ F^X : (\mathcal{A}^X, I^X)\text{-Mod} \longrightarrow (\mathcal{A}, I)\text{-Mod}. \]
Let us show that $F^X$ covers every finite-dimensional $(\mathcal{A}, I)$-module. Recall from (6.7) that $M \in (\mathcal{A}, I)$-Mod is of the form $M \cong F^X(N)$ for some $N \in (\mathcal{A}^X, I^X)$-Mod iff the underlying $R$-module of $M$ is isomorphic to $X \otimes_S N_0$, for some $S$-module $N_0$. Assume that $M \in (\mathcal{A}, I)$-Mod is finite-dimensional. We have that $I_0fM = 0$, so $fM$ is a left $Re/I_0$-module and, therefore, it has the form $fM \cong \bigoplus_{t \in J} n_tZ_t$. Then, we have the left $S$-module
\[ N_0 = \bigoplus_{t \in J} n_t(kf_t) \bigoplus fM \bigoplus e_\alpha M \]
which satisfies that $M = fM \oplus eM \oplus e_\alpha M \cong X \otimes_S N_0$, as left $R$-modules. Hence $F^X$ covers $M$, and we have verified the announced property of $F^X$.

Now, let us examine more closely the triangular interlaced weak ditalgebra $(\mathcal{A}^X, I^X)$, which is stellar with stars centers $\{e_\omega\}_{\omega \in \Omega}$ because we have $e_\alpha W_0^X =
0, $W_0^X e_\omega = W_0^X$, $e_\omega W_1^X = e_\omega W_1^X e_\omega$, and $S e_\omega = k e_\omega$, for $\omega \in \Omega$. We claim that $I^X \subseteq W_0^X$, so we are reduced to the situation examined in Case 3, and the composition of $F^X$ with the functor constructed in that case for the weak ditalgebra $(A^X, I^X)$ and a given $d$ gives us the functor we were looking for $(A, I)$ and $d$.

Indeed, we have $I^X \subseteq W_0^X$, because by the formula $(\ast)$, the ideal $I^X$ is generated by the elements of the form $\sigma_{\nu,x}(h)$, with $h \in I_0 \cup (I \cap W_0)$. If $h \in I_0$, we have $\sigma_{\nu,x}(hx) = 0$, because $I_0 X = 0$. If $h \in I \cap W_0$, then $\sigma_{\nu,x}(h) = \nu \otimes h \otimes x \in W_0^X$. So the ideal $I^X$ is generated by elements in $W_0^X$, which implies that $I^X \subseteq W_0^X$, because $W_0^X e_\omega = W_0^X$ and $e_\omega W_0^X = 0$.

Case 3: $I \cap R = 0$.

From the formula $(\ast)$, we get $I \subseteq W_0$. After an application of (7.9), we may assume that $I$ is in fact a direct summand of $W_0$. Then, we can apply (7.7) to obtain a triangular interlaced weak ditalgebra $(A', 0)$ and an equivalence of categories $F : (A', 0)\text{-Mod} \longrightarrow (A, I)\text{-Mod}$ which preserves dimensions. Now, take $d \in \mathbb{N}$, and apply first (7.10) and then (7.9) to obtain a seminested ditalgebra $A''$ and a full and faithful functor $G : (A'', 0)\text{-Mod} \longrightarrow (A', 0)\text{-Mod}$ such that any $L \in (A', 0)\text{-Mod}$ with $\dim_k L \leq d$ is of the form $G(N) \cong L$, for some $N \in (A'', 0)\text{-Mod}$. Hence, if $M \in (A, I)\text{-Mod}$ satisfies $\dim_k M \leq d$, we know the existence of $L \in (A', 0)\text{-Mod}$ with $F(L) \cong M$ and $\dim_k L \leq \dim_k M \leq d$. Therefore, there is some $N \in (A'', 0)\text{-Mod}$ with $FG(N) \cong M$. \hfill \Box

8 Reductions with sources and restrictions

In this section we go back to the reduction procedures and functors described in §7, but in more specific situations which consider a triangular interlaced weak ditalgebra $(A, I)$ with a fixed multiple source $\Omega$. In this case, the interlaced weak ditalgebra obtained from the preceding one by deletion of the idempotent $e_\omega$, admits a realization $(A^\odot, I^\odot)$ within the original weak ditalgebra $(A, I)$, which permits the definition of a restriction functor $\text{Res} : (A, I)\text{-Mod} \longrightarrow (A^\odot, I^\odot)\text{-Mod}$.

We show how the reduction procedures and functors are compatible with this construction and their module categories, respectively.

**Definition 8.1.** Assume that $(A, I)$ is a triangular interlaced weak ditalgebra with layer $(R, W)$ where $R$ is a minimal algebra. Assume that $1 = \sum_{i \in \mathcal{P}} e_i$ is the decomposition of the unit of $R$ as sum of primitive orthogonal idempotents. Then, the subset $\Omega \subseteq \mathcal{P}$ is called a multiple source of $(A, I)$ iff $\Omega$ is a multiple source of $A$, as in (7.1), and $e_\omega \notin I$, for all $\omega \in \Omega$. We have the associated multiple source idempotent $e_{\Omega} = \sum_{\omega \in \Omega} e_\omega$.

From now on, the layers $(R, W)$ of the triangular interlaced weak ditalgebras we consider always have an $R$ which is a minimal algebra.

**Lemma 8.2** (suppression of the multiple source idempotent). Assume that $(A, I)$ is a triangular interlaced weak ditalgebra with a fixed multiple source $\Omega$.
Let \((R, W)\) be the triangular layer of \(A = (T, \delta)\). Define \(f := 1 - e_\Omega\), where \(e_\Omega = \sum_{\alpha \in \Omega} e_\alpha\). Consider the tensor algebra \(T^:\!\!: = f T f \cong T_{Rf} fW\), which is a vector subspace of \(T = T_R(W)\) invariant under \(\delta\). Denote by \(\delta^\circ\) the restriction of \(\delta\) to \(T^\circ\). Then, we obtain a new weak ditalgebra \(A^\circ = (T^\circ, \delta^\circ)\) with layer \((R^\circ, W^\circ) := (Rf, fWf)\), which is also triangular.

Notice that \(f T f\) is an algebra with unit \(f\) and we have a surjective morphism of algebras \(\phi : T \longrightarrow f T f = T^\circ\) given by \(\phi(t) = f t f\), for \(t \in T\). Observe also that \(\phi(t) = \delta^\circ \phi(t)\), for \(t \in T\).

The space \(I^\circ = f I f\) is an ideal of \(A^\circ := [T^\circ]_0\) and \((A^\circ, I^\circ)\) is a triangular interlaced weak ditalgebra, which is a Roiter interlaced weak ditalgebra whenever \((A, I)\) is so.

Moreover, we have the restriction functor \(\text{Res} : (A, I)\text{-Mod} \longrightarrow (A^\circ, I^\circ)\text{-Mod}\) such that \(\text{Res}(M) = f M\) and, for \(h = (h^0, h^1) \in \text{Hom}_k(A, I)(M, N)\), its restriction is defined by \(\text{Res}(h^0, h^1) = (\text{Res}(h^0), \text{Res}(h^1))\), where \(\text{Res}(h^0) : f M \longrightarrow f N\) is the restriction of the map \(h^0\) and \(\text{Res}(h^1) : f V f \longrightarrow \text{Hom}_k(f M, f N)\) is obtained from the restriction of \(h^1 : V \longrightarrow \text{Hom}_k(M, N)\).

**Proof.** Notice that \(W = f W \oplus e_{\Omega} W I e_{\Omega}\), which implies the “convexity property” \(fW W \cdots W f = f W f W f \cdots f W f\), and \(T^\circ \cong T_{Rf} fW\). The triangularity of the layer \((Rf, fWf)\) for \(T^\circ\) clearly follows from the triangularity of the layer \((R, W)\) of \(T\). The rest of the statement is also clear. 

**Remark 8.3.** With the notation of the preceding statement, notice that, up to isomorphism, the weak interlaced ditalgebra \((A^\circ, I^\circ)\) is obtained from \((A, I)\) by deletion of the multiple source idempotent \(e_\Omega\), as in [6.2]. We adopt here this special notation and terminology in order to stress the particularities derived from the properties of the multiple source idempotent \(e_\Omega\). Notably, the convexity property, that is, with the preceding notation, we have that \(f g h f = f g f h f\), for any \(g, h \in T\), where \(T\) is the underlying tensor algebra of \(A\). This permits to realize the interlaced weak ditalgebra obtained by deletion of the idempotent \(e_\Omega\) within \((A, I)\) and to define the restriction functor \(\text{Res}\). This construction plays an essential role in our arguments.

From [6.13], we have that, if \((A, I)\) is not wild, neither is \((A^\circ, I^\circ)\).

The following series of lemmas guarantees that, given a triangular interlaced weak ditalgebra \((A, I)\) with multiple source \(\Omega\), whenever we can perform a reduction of type \(z \in \{d, r, q, a, X\}\) on \((A^\circ, I^\circ)\), then we can perform a corresponding reduction of the same type \(z\) on \((A, I)\). Moreover, important relations between their module categories are established. We start with the following general remark which will then be particularized to the special cases \(z \in \{d, r, q\}\).

**Remark 8.4.** Assume that \((A, I)\) is a triangular interlaced weak ditalgebra with multiple source \(\Omega\). Let \((R, W)\) be the triangular layer of \(A = (T, \delta)\). Since \(R\) is a minimal algebra, we have \(R = \prod_{\nu \in \Omega} R e_{\nu}\). Assume that \((A', I')\) is a triangular interlaced weak ditalgebra with layer \((R', W')\), and that \(\phi : (A, I) \longrightarrow (A', I')\) is a morphism of interlaced weak ditalgebras as in [6.1](4), such that: \(\mathcal{P} = D \oplus C\), where \(D\) may be empty, \(\Omega \subseteq C\), and the minimal algebra \(R'\) is \(R e_{\mathcal{C}}\), where
$e_C := \sum_{i \in C} e_i \in R$. Moreover, assume that $\phi : R \rightarrow R'$ is the canonical projection. Defining $e'_i := \phi_b(e_i)$, for $i \in C$, we obtain $R' = \prod_{i \in C} R'e'_i$, with $R'e'_i \cong R_i$. We will furthermore assume that $\phi(I) = I'$.

Consider the idempotents $e_\alpha = \sum_{\alpha \in \Omega} e_i \in R$ and its image $e'_\alpha = \phi_b(e_\alpha) = \sum_{\alpha \in \Omega} e'_i \in R'$, as well as $f_\alpha = 1 - e_\alpha \in R$ and its image $f'_\alpha = \phi_b(f_\alpha) = 1 - e'_\alpha \in R'$. Then, $\Omega \subseteq C$ is also a multiple source of $(A', I')$.

Indeed, we clearly have that $e'_\alpha W_0 = 0$ and $e'_\alpha W_1 = e'_\alpha W_1 e'_\alpha$. If we had $e'_\omega \in I'$, for some $\omega \in \Omega$, since $\phi(I) = I'$ and $e_\omega W_0 = 0$, we would get $e_\omega \in I$, which is not the case.

So, we can consider the triangular interlaced weak ditalgebras $(A^{\circ}, I^{\circ})$ and $(A^{\circ'}, I^{\circ'})$ obtained, respectively, from $(A, I)$ and $(A', I')$ by deletion of the multiple source idempotents $e_\alpha$ and $e'_\alpha$. By definition, we have $(A^{\circ}, I^{\circ}) = ((f_\alpha T_R(W)f_\alpha), \delta^{\circ}), f_\alpha I_{f_\alpha}$ and $(A^{\circ'}, I^{\circ'}) = ((f'_\alpha T_R(W')f'_\alpha), \delta^{\circ'}), f'_\alpha I'_{f'_\alpha})$.

It is easily seen that the given morphism $\phi : (A, I) \rightarrow (A', I')$ induces, by restriction, a morphism of interlaced weak ditalgebras $\phi^{\circ} : (A^{\circ}, I^{\circ}) \rightarrow (A^{\circ'}, I^{\circ'})$ and that we have the following commutative diagram of functors:

$$
\begin{array}{ccc}
(A', I')-\text{Mod} & \xrightarrow{F_\phi} & (A, I)-\text{Mod} \\
\downarrow \text{Res} & & \downarrow \text{Res} \\
(A^{\circ'}, I^{\circ'})-\text{Mod} & \xrightarrow{F_{\phi^{\circ}}} & (A^{\circ}, I^{\circ})-\text{Mod}.
\end{array}
$$

Lemma 8.5 (deletion of idempotents with multiple source). Assume that $(A, I)$ is a triangular interlaced weak ditalgebra with multiple source $\Omega$. Let $(R, W)$ be the triangular layer of $A = (T, \delta)$, with $R = \prod_{i \in P} R_i$. Assume that $P = \Omega \cup C \cup D$, a disjoint union of non-empty subsets. Thus we have the decomposition $1 = e_\alpha + e_C + e_D$, a sum of orthogonal idempotents of $R$. Then, we can form the triangular interlaced weak ditalgebras $(A^d, I^d)$ obtained from $(A, I)$ by deletion of the idempotent $e_D$, as in [7, 9]. Thus $A^d = (T^d, \delta^d)$, where $T^d = T_{R^d}(W^d)$, with $R^d = R f_D$, $W^d = f_D W_0 f_D \oplus f_D W_1 f_D$, and $f_D := 1 - e_D \in R$. Moreover, $I^d = \phi(I)$, where $\phi : (A, I) \rightarrow (A^d, I^d)$ is the morphism of triangular interlaced weak algebras determined by the canonical projections $\phi_b : R \rightarrow R f_D$, $\phi_b : W_0 \rightarrow f_D W_0 f_D$, and $\phi_b : W_1 \rightarrow f_D W_1 f_D$. From [8, 4], we have that $\Omega$ is a multiple source of $(A^d, I^d)$. So, we can consider the triangular interlaced weak ditalgebras $(A^{\circ d}, I^{\circ d})$ and $(A^{\circ d'}, I^{\circ d'})$ obtained respectively from $(A, I)$ and $(A^d, I^d)$ by suppression of the multiple source idempotents $e_\alpha \in R$ and $e_\alpha \in R f_D$, respectively.

We can also consider the triangular interlaced weak ditalgebra $(A^{\circ d}, I^{\circ d})$, obtained from $(A^{\circ}, I^{\circ})$ by deletion of the idempotent $e_D \in R^d = R f_\Omega$, where $f_\Omega = 1 - e_\Omega \in R$. We have the corresponding morphism of triangular interlaced weak algebras $\phi^{\circ d} : (A^{\circ}, I^{\circ}) \rightarrow (A^{\circ d}, I^{\circ d})$ determined by the canonical projections $\phi^{\circ d} : R^{\circ} \rightarrow R^{\circ} f_\Omega$, $\phi^{\circ d}_0 : W_0^{\circ} \rightarrow f W_0^{\circ} f$, and $\phi^{\circ d}_1 : W_1^{\circ} \rightarrow f W_1^{\circ} f$, where $f = f_\Omega - e_\Omega \in R^\circ$. Then, we have $(A^{\circ d}, f^{\circ d}) = (A^{\circ d}, I^{\circ d})$, $\phi^\circ = \phi^d$, and the commutative diagram of functors

$$
\begin{array}{ccc}
(A^d, I^d)-\text{Mod} & \xrightarrow{F^d} & (A, I)-\text{Mod} \\
\downarrow \text{Res} & & \downarrow \text{Res} \\
(A^{\circ d}, I^{\circ d})-\text{Mod} & \xrightarrow{F^{\circ d}} & (A^{\circ}, I^{\circ})-\text{Mod}.
\end{array}
$$
where $F^d = F_{\phi}$ and $F^{\ominus d} := F_{\phi^d} = F_{\phi_{\ominus d}}$ are the corresponding reduction functors. Moreover, if $M \in (A, I)$-Mod is such that $\text{Res}(M) \cong F^{\ominus d}(N')$, for some $N' \in (A^{d \ominus}, I^{d \ominus})$-Mod, then $M \cong F^d(N)$, for some $N \in (A^d, I^d)$-Mod.

Proof. The relevant idempotents in the construction of $A^{\ominus d} = (T^{\ominus d}, \delta^{\ominus d})$ and $A^{d \ominus} = (T^{d \ominus}, \delta^{d \ominus})$ are $f_D, f_0, f$, as defined above, and $f' := f_D - e_0 \in R^{\ominus d} = Rf_0$. Observe that their product in $R$ satisfies $f_Df' = e_C = f f_0$. Therefore, we get $R^{d \ominus} = Rf_Df' = Rf_0f = R^{\ominus d}$ and $W_i^{d \ominus} = f_i f_D W_i f_D f' = f_i f_0 W_i f_0 f = W_i^{\ominus d}$, for $i \in \{0, 1\}$. Hence, we have $T^{d \ominus} = T_{R^{\ominus d}}(W^{d \ominus}) = T_{R^{d \ominus}}(W^{\ominus d}) = T^{d \ominus}$. Observe that the restrictions of the map $\phi^{\ominus} : T^{\ominus} \longrightarrow T^{d \ominus}$ to $R^{\ominus}, W^{\ominus}$, and $W^{d \ominus}$ coincide, respectively, with the maps $\phi_R^d, \phi_0^d$ and $\phi_d^d$ which determine $\phi^d$. Thus, we get $\phi^{\ominus} = \phi^d$, as claimed.

Now, recall that the derivations $\delta^d$ and $\delta^{\ominus d}$ are uniquely determined by the commutativity of the squares

$$
\begin{array}{ccc}
W & \xrightarrow{\delta} & T \\
\downarrow \phi & & \downarrow \phi \\
W^{\ominus d} & \xrightarrow{\delta^{\ominus d}} & T^{d \ominus}
\end{array}
\quad
\begin{array}{ccc}
W^{\ominus} & \xrightarrow{\delta} & T^{\ominus} \\
\downarrow \phi & & \downarrow \phi \\
W^{d \ominus} & \xrightarrow{\delta^{d \ominus}} & T^{d \ominus}
\end{array}
$$

Since $\phi^{\ominus} = \phi^d$, applying $\ominus$ to the first diagram, we obtain the second one, with $\delta^{d \ominus} = \delta^{\ominus d}$. Thus, $A^{d \ominus} = (T^{d \ominus}, \delta^{d \ominus}) = (T^{\ominus d}, \delta^{\ominus d}) = A^{\ominus d}$. Since $\phi(f_0) = f'$, we have $f^{d \ominus} = \phi'(f_0) = f'$, $f'(f_0)f = f f_0 f = f^{d \ominus}$.

The commutativity of the diagram follows from (8.4). Finally, given $M \in (A, I)$-Mod, we know that $F^{\ominus d}(N') \cong \text{Res}(M)$, for some $N' \in (A^{d \ominus}, I^{d \ominus})$-Mod, means that $e''fM = 0$, so $e'M = 0$, and $M \cong F^d(N)$, for some $N \in (A^d, I^d)$-Mod. 

\[\square\]

Lemma 8.6 (regularization with multiple source). Assume that $(A, I)$ is a triangular interlaced weak ditalgebra with multiple source $\Omega$. Let $(R, W)$ be the triangular layer of $A = (T, \delta)$. Suppose that we have $R$-$R$-bimodule decompositions $W_0 = W_0'' \oplus W_0'$ and $W_1 = \delta(W_0'' \oplus W_0')$. Then, we can form the triangular interlaced weak ditalgebra $(A', I')$ obtained by regularization of the bimodule $W_0'$, as in (6.3). Thus $A' = (T', \delta')$ and $I' = \phi(I)$, where $T' = T_R(W')$, with $W' = W_0'' \oplus W_1'$, and $\phi : (A, I) \longrightarrow (A', I')$ is the morphism of triangular interlaced weak ditalgebras determined by the canonical projections $\phi_0 : R \longrightarrow R$, $\phi_0 : W_0'' \longrightarrow W_0''$, and $\phi_1 : W_1 \longrightarrow W_1'$. From (8.4), we have that $\Omega$ is a multiple source of $(A', I')$.

Assume furthermore that $W_0' e_0 = 0$. If we take $f = 1 - e_0$, we get $f W_0' f = W_0''$, $W_0'' = f W_0 f = W_0'' \oplus f W_0''$, $W_1 = \delta(W_0'' \oplus f W_0'')$, and $W_1' = f W_1 f = \delta(W_0'' \oplus f W_0'')$. So, we can consider the triangular interlaced weak ditalgebra $(A''^{\ominus r}, I^{\ominus r})$, obtained from $(A^{\ominus}, I^{\ominus})$ by regularization of the $R^{\ominus}$-$R^{\ominus}$-bimodule $W_0'$, as in (6.3). We have the corresponding morphism of triangular interlaced weak ditalgebras $\phi_r : (A^{\ominus}, I^{\ominus}) \longrightarrow (A''^{\ominus r}, I^{\ominus r})$ determined by the canonical projections $\phi_r^r : R^{\ominus r} \longrightarrow R^{\ominus}$, $\phi_0 : W_0'' \longrightarrow f W_0'' f = W_0''^{\ominus r}$, and $\phi_1 : W_1 \longrightarrow f W_1 f = W_1^{\ominus r}$. Then, we have...
where \( F^r = F_\phi \) and \( F^{r^r} = F_{\phi^r} \) are the associated reduction functors. Moreover, if \((A, I)\) is a Roiter interlaced weak ditalgebra and \( M \in (A, I)\)-Mod is such that \( \text{Res}(M) \cong F^{r^r}(N') \), for some \( N' \in (A^{r^r}, I^{r^r})\)-Mod, then \( M \cong F^r(N) \), for some \( N \in (A^r, I^r)\)-Mod.

**Proof.** We have \( R^{r^r} = R^r = Rf = R^{r^r}, W^{r^r}_0 = fW_0^r f = W^{r^r}_0, \) and \( W^{r^r}_1 = fW_1^r f = W^r_1 \). Hence, we get \( T^{r^r} = T^r \). As before, we notice that the restrictions of the map \( \phi^r : T^r \longrightarrow T^{r^r} \) to \( R^r, W^0_r \), and \( W^t_1 \) coincide, respectively, with the maps \( \delta_0^r, \delta_0^r \) and \( \phi^r_1 \) which determine \( \phi^r \). Thus, we get \( \phi^r = \phi^r \).

The same argument involving the commutative squares in the proof of [6.7], replacing \( d \) by \( r \), gives that \( \delta^{r^r} = \delta^r \). So, \( A^{r^r} \cong (T^{r^r}, \delta^{r^r}) = (T^r, \delta^r) = A^r \). Clearly, we have \( I^{r^r} = \delta^r((1f)I) = f(1f)I = f\phi(I)I = I^{r^r} \).

The commutativity of the diagram follows from [6.7]. If \( \text{Res}(M) \cong F^{r^r}(N') \), then \( (\text{Ker}(\delta^r \cap W^t_0)I)M = 0 \). So, \( (\text{Ker}(\phi^r \cap W^t_0)I)M = 0 \), because \( M = e_\mu M \oplus fM \) and \( W^t_0 e_\mu = 0 \). But \( \delta^{r^r}_{W^r_0} = \delta^r_{W^r_0} \) implies that \( \text{Ker}(\delta^r \cap W^t_0)I \cong \text{Ker}(\phi^r \cap W^t_0)I \). It follows that \( F^r(N) \cong M \), for some \( N \in (A^r, I^r)\)-Mod. \( \square \)

**Lemma 8.7** (factoring out a direct summand of \( W_0 \) with multiple source). Let \((A, I)\) be a triangular interlaced weak ditalgebra with multiple source \( \Omega \) and triangular layer \((R, W)\). Define \( f = 1 - e_\alpha \). Assume that there is a decomposition \( fW_0 f = W_0^0 \oplus W_0^r \) as \( Rf \) such \( \delta(W^r_0) \subseteq A^{r^r}W^r_0 fV + fV W^r_0 A \). Thus \( W_0 = W^0_0 \oplus (W^r_0 \oplus W_0 e_\alpha) \) is a decomposition of \( R-R \)-bimodules with \( W^0_0 \subseteq I \) and \( \delta(W^r_0) \subseteq AW^r_0 V + VW^r_0 A \). Then, we can form the triangular interlaced weak ditalgebra \((A^g, I^g)\), determined by this last decomposition, as in [6.4]. Thus \( A^g = (T^g, \delta^g) \) and \( I^g = \phi(I) \), where \( T^g = T_R(W^g) \), with \( W^g_0 = W^0_0 \oplus W_0 e_\alpha \) and \( W^g_1 = W_1 \), and \( \phi : (A, I) \longrightarrow (A^g, I^g) \) is the morphism determined by the canonical projections \( \phi_0 : R \longrightarrow R^g, \phi_0 : W_0 \longrightarrow W^g_0, \) and \( \phi_1 : W_1 \longrightarrow W^g_1 \). From [6.4], we have that \( \Omega \) is a multiple source in \((A^g, I^g)\).

We can also consider the triangular interlaced weak ditalgebra \((A^{r^g}, I^{r^g})\), obtained from \((A^{r^r}, I^{r^r})\) by factoring out the direct summand \( W^t_0 \) of \( R^r \) such \( \phi_0^r : R^{r^r} \longrightarrow R^{r^r}, \phi_0^r : W^r_0 \longrightarrow fW^r_0 f = W^r_0, \) and \( \phi_1^r : W^r_1 \longrightarrow fW^r_1 f = W^r_1 = 0 \). Then, we have \((A^{r^g}, I^{r^g}) = (A^{r^g}, I^{r^g})\), \( \phi^g = \phi^r \), and the commutative diagram of functors

\[
\begin{array}{ccc}
(A^g, I^g)\text{-Mod} & \xrightarrow{F^g} & (A, I)\text{-Mod} \\
\downarrow \text{Res} & & \downarrow \text{Res} \\
(A^{r^g}, I^{r^g})\text{-Mod} & \xrightarrow{F^{r^g}} & (A^{r^r}, I^{r^r})\text{-Mod},
\end{array}
\]

where \( F^g = F_\phi \) and \( F^{r^g} = F_{\phi^r} \) are the associated reduction functors.
Proof. We have $T^{\otimes q} = T_{Rf}(fWf)^{\otimes q} = T_{Rf}(W_0^u \oplus fW_1f) = T^{\otimes q}$. As in the preceding lemmas, the equality $\phi^{\otimes} = \phi^\ast$ is easy to show and, from this, again as before, we get $A^{\otimes q} = (T^{\otimes q}, \delta^{\otimes q}) = (T^{\otimes q}, \delta^{\otimes q}) = A^{\otimes q}$. Similarly, we have $I^{\otimes q} = \delta(fIf) = \phi(fIf) = f\phi(I)f = I^{\otimes q}$. The commutativity of the diagram follows from (8.4). □

Lemma 8.8 (absorption of a loop with multiple source). Let $(A, I)$ be a triangular interlaced weak ditalgebra with multiple source $\Omega$, where $A = (T, \delta)$ admits the triangular layer $(R, W)$, with $Re_i = ke_i$ for some $i \in \mathcal{P} \setminus \Omega$. Set $f = 1 - e_\omega$. Assume that there is a decomposition of $RfRf$-bimodules $fW_0f = W_0' \oplus W_0''$ with $W_0' \cong e_0R \otimes_k R e_0$ and $\delta(W_0') = 0$. Then, we can consider the triangular interlaced weak ditalgebra $(A^{\otimes q}, I^{\otimes q})$ and the triangular interlaced weak ditalgebra $(A^{\otimes q}, I^{\otimes q})$ obtained by the absorption of the $RfRf$-subbimodule $W_0'$, as in (6.4). We also have the decomposition of $RfRf$-bimodules $W_0 = W_0' \oplus (W_0'' \oplus W_0e_\omega)$, and we can consider the triangular interlaced weak ditalgebra $(A^\otimes, I^\otimes)$ obtained by absorption of the $RfRf$-subbimodule $W_0'$. Then, $\Omega$ is a multiple source in $(A^\otimes, I^\otimes)$ and we can consider the triangular interlaced weak ditalgebra $(A^{\otimes q}, I^{\otimes q})$. We have that $(A^{\otimes q}, I^{\otimes q}) = (A^\otimes, I^\otimes)$ and there is a commutative square of functors

$$
\begin{array}{ccc}
(A^\otimes, I^\otimes)\text{-Mod} & \xrightarrow{f^\otimes} & (A, I)\text{-Mod} \\
\downarrow \text{Res} & & \downarrow \text{Res} \\
(A^{\otimes q}, I^{\otimes q})\text{-Mod} & \xrightarrow{f^{\otimes q}} & (A^{\otimes q}, I^{\otimes q})\text{-Mod} \\
\end{array}
$$

Proof. Here $1 = \sum_{i \in \mathcal{P}} e_j$ is a decomposition of the unit of the minimal algebra $R^a \cong T_R(W_0')$ as a sum of orthogonal primitive idempotents, which contains the algebra $R$. The layer of $A^a$ is $(R^a, W^a)$ with $R^a e_i = k[x], W_0^a = R^a [W_0' \oplus W_0e_\omega]R^a$, and $W_0^a = R^a W_1^a R^a$. Thus, $e_0 W_0^a = e_0 R^a W_0' R^a + e_0 R^a W_0 R^a - 0$, and $e_0 W_0^a = e_0 R^a W_1 R^a = R^a e_0 W_1 R^a = e_0 R^a e_0 W_0 R^a$, for $\omega \in \Omega$, and we already know that $e_\omega \not\in I = I^q$. Thus, $\Omega$ is a multiple source in $(A^\otimes, I^\otimes)$.

Now, $fW_0f = fR^a W_0' R^a f \oplus fR^a(W_0 e_\omega)R^a f = fR^a W_0' R^a f$ and $fW_1f = fR^a W_1 R^a f$. Then, $(R^a f) = \prod_{j \in \mathcal{P} \setminus \Omega \cup \{\omega\}} R e_j R^a e_i = (R^a f)$, and we get

$$
T_{Rf}(fR^a W_0' R^af \oplus fR^a W_1 R^af) = T_{Rf}((R^a f)W_0'' (R^a f) \oplus (R^a f)W_1 (R^a f)).
$$

That is $T^{\otimes q} = T^{\otimes a}$. Moreover, $\delta^{\otimes} = (\delta^a)$, so $A^{\otimes q} = A^{\otimes a}$, and clearly $I^{\otimes q} = fI^q = I^{\otimes a}$. The rest of the proof is straightforward. □

Lemma 8.9 (reduction with an admissible module with multiple source). Assume that $(A, I)$ is a triangular interlaced weak ditalgebra with multiple source $\Omega$. Let $(R, W)$ be the triangular layer of $A = (T, \delta)$. Write $f = 1 - e_\omega$ and assume that we have a decomposition $fW_0f = W_0' \oplus W_0''$ of $RfRf$-bimodules with $\delta(W_0') = 0$. Thus we have the decomposition of $RfRf$-bimodules $W_0 = W_0' \oplus (W_0'' \oplus W_0 e_\omega)$, with $\delta(W_0') = 0$.

Consider the $k$-subalgebra $B^{\otimes} = T_{Rf}(W_0')$ of $T^{\otimes} = fTf$ and the $k$-subalgebra $B = T_R(W_0') = Re_\omega \times B^{\otimes}$ of $T$. Suppose that $X^{\otimes}$ is an complete triangular
admissible $B^\otimes$-module, thus we have a splitting $\Gamma^\otimes = \text{End}_{B^\otimes}(X^\otimes) = S^\otimes \oplus P^\otimes$, where $S^\otimes$ is a subalgebra of $\Gamma^\otimes$, $P^\otimes$ is an ideal of $\Gamma^\otimes$, and $X^\otimes$ and $P^\otimes$ are finitely generated projective right $S^\otimes$-modules.

Now, consider the $B$-module $X = \text{Re}_{\alpha} \oplus X^\otimes$, where $X^\otimes$ is considered as a $B$-module by restriction through the projection map $B \longrightarrow B^\otimes$. Then, we have a splitting $\Gamma = \text{End}_B(X)^{\text{op}} = S \oplus P$, where $S = \text{Re}_{\alpha} \times S^\otimes$ is a subalgebra of $\Gamma$, $P$ is an ideal of $\Gamma$, which is identified with the $S$-$S$-bimodule obtained from $P^\otimes$ through the projection map $S \longrightarrow S^\otimes$, and $X$ and $P$ are finitely generated projective right $S$-modules. Then, $X$ is a complete triangular admissible $B$-module, and we can form the triangular interlaced weak ditalgebra $(A^X, I^X)$ obtained from $(A, I)$ by reduction using the admissible $B$-module $X$, as in (6.7).

We have that $\Omega$ is a multiple source in $(A^X, I^X)$ and we can form the triangular interlaced weak ditalgebra $(A^{X\otimes}, I^{X\otimes})$. We can also form the triangular interlaced weak ditalgebra $((A^\otimes)^{X\otimes}, (I^\otimes)^{X\otimes})$ obtained from $(A^\otimes, I^\otimes)$ by reduction using the admissible $B^\otimes$-module $X^\otimes$.

There is a canonical isomorphism of interlaced weak ditalgebras between $(A^{X\otimes}, I^{X\otimes})$ and $((A^\otimes)^{X\otimes}, (I^\otimes)^{X\otimes})$, which allows us to identify them and we have the following commutative diagram of functors with full and faithful rows

\[
\begin{array}{ccc}
(A^X, I^X)\text{-Mod} & \xrightarrow{F^X} & (A, I)\text{-Mod} \\
\downarrow\text{Res} & & \downarrow\text{Res} \\
((A^\otimes)^{X\otimes}, (I^\otimes)^{X\otimes})\text{-Mod} & \xrightarrow{F^{X\otimes}} & (A^\otimes, I^\otimes)\text{-Mod}.
\end{array}
\]

Moreover, if $M \in (A, I)\text{-Mod}$ is such that $\text{Res}(M) \cong F^{X\otimes}(N')$, for some $N' \in ((A^\otimes)^{X\otimes}, (I^\otimes)^{X\otimes})\text{-Mod}$, then $M \cong F^X(N)$, for some $N \in (A^X, I^X)\text{-Mod}$.

**Proof.** Assume that $X^\otimes$ is complete triangular admissible $B^\otimes$-module. Then, directly from (6.11), we have the associated full and faithful functor $F^{X\otimes} : ((A^\otimes)^{X\otimes}, (I^\otimes)^{X\otimes})\text{-Mod} \longrightarrow (A^\otimes, I^\otimes)\text{-Mod}$, where $\Gamma^\otimes = \text{Iff}$.

We have the projection map $\pi : B \longrightarrow B^\otimes$, which enable us to consider $X^\otimes$ as a $B$-module, so we have the splitting

\[
\Gamma^\otimes := \text{End}_{B^\otimes}(X^\otimes)^{\text{op}} = \text{End}_B(X^\otimes)^{\text{op}} = S^\otimes \oplus P^\otimes,
\]

and $X^\otimes$ is a triangular admissible $B^\otimes$-module, relative to the given splitting of $\Gamma^\otimes$. We have $P^\otimes = \text{Hom}_{B^\otimes}(P^\otimes, S^\otimes)$ and the corresponding comultiplication $\mu^\otimes : P^\otimes \longrightarrow P^\otimes \otimes_{S^\otimes} P^\otimes$, which determines the structure of the ditalgebra $(B^\otimes, 0)^{X^\otimes} = (\text{End}_{B^\otimes}(P^\otimes), \delta(\mu^\otimes)) = (B, 0)^{X^\otimes}$, see (6.11). Since $B^\oplus X^\otimes$ is complete, the first row in the following commutative diagram is a full and faithful functor

\[
\begin{array}{ccc}
(B^\otimes, 0)^{X^\otimes}\text{-Mod} & \xrightarrow{F^{X\otimes}} & B^\otimes\text{-Mod} \\
\| & & \downarrow F^\otimes \\
(B, 0)^{X^\otimes}\text{-Mod} & \xrightarrow{F^X} & B\text{-Mod},
\end{array}
\]

so the second one is full and faithful and $X^\otimes$ is a complete admissible $B$-module.

Since we have a surjective morphism of algebras $B \longrightarrow \text{Re}_{\alpha}$ from (6.9)(2), the admissible $B$-module $\text{Re}_{\alpha}$ is complete. Since the $B$-module $X := \text{Re}_{\alpha} \oplus X^\otimes$
satisfies Hom_B(\mathcal{I}_{Re_\omega}, \mathcal{I}_X) = 0 \text{ and } \text{Hom}_B(\mathcal{I}_X, \mathcal{I}_{Re_\omega}) = 0$, by (5.9) (3), it is complete. So, the associated functor $F^X : (B, 0)^X\text{-Mod} \rightarrow B\text{-Mod}$ is full and faithful. We have the ideal $I_0 = B \cap I$ of $B$ and the associated full and faithful functors

$$F^X : (B, 0)^X, I_0^X)\text{-Mod} \rightarrow (B/I_0)\text{-Mod},$$

and $F^X : (A^X, I^X)\text{-Mod} \rightarrow (A, I)\text{-Mod}$, see (6.11).

In order to compare $((\mathcal{A}^\circ)^{X\circ}, (I^\circ)^{X\circ})$ and $((\mathcal{A}^X)^{\circ\circ}, (I^X)^{\circ\circ})$, in the following we recall their constituents. We need to consider the splitting $\Gamma = \text{End}_B(X) = S \oplus P$, where $S = Re_0 \times S^\circ$ and $P$ denotes the $S$-$S$-bimodule obtained from the $S^\circ$-$S^\circ$-bimodule $P$ by restriction through the projection map $S \rightarrow S^\circ$. Observe that $W_0 = fW_0 = fW_0f \oplus fW_0e_\alpha = W_0' \oplus (W_0'' \oplus W_0e_\alpha)$. The layer of the weak ditalgebra $\mathcal{A}^X$ is $(S, W^X)$, where

$$W^X_0 = X^* \otimes_R (W''_0 \oplus W_0e_\alpha) \otimes_R X \text{ and } W^X_1 = (X^* \otimes_RW_1 \otimes_RX) \oplus P^*,$$

where $X^* = \text{Hom}_S(X, S)$ and $P^* = \text{Hom}_S(P, S)$. Notice that the $S$-$S$-bimodule $P^*$ can be identified with the $S$-$S$-bimodule obtained from the $S^\circ$-$S^\circ$-bimodule $P^{0\circ} = \text{Hom}_S(S^\circ, S^\circ)$ by restriction through the projection map $S \rightarrow S^\circ$. Moreover, if we denote by $X^{0\circ} = \text{Hom}_S(X^\circ, S^\circ)$, then the $S$-$R$-bimodule $X^*$ can be identified with the direct sum $Re_0 \oplus X^{0\circ}$, where $X^{0\circ}$ is considered as an $S$-$R$-bimodule by restriction through the projections $S \rightarrow S^\circ$ and $R \rightarrow Rf$.

Hence,

$$W^X_0 = X^* \otimes_R (W''_0 \oplus W_0e_\alpha) \otimes_R X$$

and

$$W^X_1 = [X^* \otimes_RW_1 \otimes_RX] \oplus P^*$$

and

$$W^X_0 = [X^* \otimes_RW_1 \otimes_RX] \oplus P^*$$

The layer of $\mathcal{A}^X$ is $(S, W^X)$, where $W^X_0 = X^* \otimes_R (W''_0 \oplus W_0e_\alpha) \otimes_R X$. Let us show that $e_\omega W^X_0 = 0$ and $e_\omega W^X_1 = Re_\alpha \otimes_R e_\omega W_1 \otimes_R Re_\alpha = e_\omega W^X_1$. For this notice that $e_\omega W_0 = 0$ and $e_\omega \notin I$. Thus, $fI = I$. We claim that, for any $\omega \in \Omega$, $\nu \in X^*$, $x \in X$, and $h \in I$, we have that $e_\omega \sigma_{\nu,x}(h) = 0$. Indeed, for $h \in I = fI$, we have $h = fa$ with $a \in I \subseteq A = TR(W_0)$, thus $a = r + \gamma$, with $r \in R$ and $\gamma \in [\oplus_nW_0^{\langle n}} \in [\oplus_nW_0^{\langle n}}$, so $h = fr + \gamma$. Since, $e_\omega \sigma_{\nu,x}(fr) = e_\omega \nu(fr)$ with $fr \in X^\circ$, we get $e_\omega \nu(fr) = 0$. Moreover, from [4](12.8)(3), we get $e_\omega \sigma_{\nu,x}(f\gamma) \in e_\omega [\oplus_nW_0^{\langle n}} = 0$. Hence $e_\omega \sigma_{\nu,x}(h) = e_\omega \sigma_{\nu,x}(fr) + e_\omega \sigma_{\nu,x}(f\gamma) = 0$, as
Thus claimed. Now, if \( e_\omega \in I^X \), we get \( e_\omega = \sum_v (s_v + \gamma_v) \sigma_{\nu_v,x_v}(h_v) \zeta_v \), where \( s_v \in S^\circ \), \( \gamma_v \in [\oplus_n(W_0^X)^n] \), \( \zeta_v \in A^X \), \( \nu_v \in X^\ast \), \( x_v \in X \), and \( h_v \in I \). Thus, from our precedent claim, we obtain \( e_\omega = \sum_v s_v e_\omega \sigma_{\nu_v,x_v}(h_v) \zeta_v = 0 \), a contradiction. Thus \( e_\omega \not\in I^X \), for all \( w \in \Omega \).

Then, \( \Omega \) is a multiple source of \((A^X, I^X)\) and we have the interlaced weak di-
talgebra \((A^X, I^X)\) obtained by suppression of the multiple source idempotent \( e_\alpha \in S \). Here, we have \( T^{X^\circ} = T_{Sf}(f(W_0^X \oplus W_1^X)\hat{f}) \), where \( \hat{f} = 1 - e_\alpha \in S \).

We also have \( T^\circ = T_{Rf}(fWf) \) and \( (T^\circ)^{X^\circ} = T_{Sf}((fW_0f)^{X^\circ} \oplus (fW_1f)^{X^\circ}) \).

From the description given before for \( W_0^X \), we get:

\[
(fW_0f)^{X^\circ} = X^\ast \otimes_R W_0'' \otimes_R f^X \hat{f}.
\]

From the description given before for \( W_1^X \), we have \((W_1^\circ)^{X^\circ} = (fW_1f)^{X^\circ} = [X^\ast \otimes_R fW_1f \otimes_R X^\circ] \oplus P^{\circ} \otimes R = \hat{f}(X^\ast fW_1f)\hat{f} \).

Therefore, we get \((T^\circ)^{X^\circ} = T^{X^\circ} \).

The choose, naturally, the fixed dual basis for \( X^\ast \) which is obtained from the dual basis of \( X^\circ \) by adding, for each \( \omega \in \Omega \), the elements \( x_\omega := e_\omega \in X \) and \( \nu_\omega \in X^\ast \) given by \( \nu_\omega(e_\omega) = e_\omega \) and \( \nu_\omega(X^\circ + \sum_{w \neq w'} k e_{w'}) = 0 \). Then, we have the commutative squares

\[
\begin{array}{ccc}
X^\ast & \xrightarrow{\lambda} & P^\ast \otimes_S X^\ast \\
\uparrow \lambda & & \uparrow \lambda \\
X^{\circ\ast} & \xrightarrow{\hat{\lambda}} & P^{\circ\ast} \otimes_S X^{\circ\ast}
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
X & \xrightarrow{\rho} & X \otimes_S P^\ast \\
\uparrow \rho & & \uparrow \rho \\
X^\circ & \xrightarrow{\rho^\circ} & X^\circ \otimes_S P^{\circ\ast}
\end{array}
\]

where the superscript \( \circ \) has been attached to the maps associated to the ad-
missible \( B^\circ \)-module \( X^\circ \) and the vertical arrows denote inclusions. Indeed, this
follows immediately from the expressions for these maps given in (7.6) in terms
of the chosen dual basis. Similarly, for \( \nu \in X^{\circ\ast} \) and \( x \in X^\circ \), we have the commutative square

\[
\begin{array}{ccc}
T & \xrightarrow{\sigma_{\nu,x}} & T^X \\
\uparrow \sigma_{\nu,x} & & \uparrow \sigma_{\nu,x} \\
(T^\circ)^{X^\circ} & \xrightarrow{\sigma_{\nu,x}^\circ} & (T^\circ)^{X^\circ}
\end{array}
\]

where, again, the vertical arrows denote inclusions.

Then, for \( \nu \in X^{\circ\ast} \), \( x \in X^\circ \), and \( w \in B^\circ W''_0 B^\circ \cup B^\circ W_1 B^\circ \), we get

\[
\delta^{X^\circ}(\nu \otimes w \otimes x) = \lambda(\nu) \otimes w \otimes x + \sigma_{\nu,x}(\delta(w)) + (-1)^{\deg w + 1} \nu \otimes w \otimes \rho(x)
\]

and we obtain \( A^{X^\circ} = (T^{X^\circ}, \delta^{X^\circ}) = ((T^\circ)^{X^\circ}, (\delta^\circ)^{X^\circ}) = (A^\circ)^{X^\circ} \).

Now, let us prove that \((T^\circ)^{X^\circ} = (I^X)^{\circ} \), that is \((If)^{X^\circ} = \hat{f}I^X \hat{f} \). As before, since \( e_\Omega W_0^X = 0 \) and \( e_\omega \not\in I^X \), we obtain that \( \hat{f}I^X = I^X \). So we want to show the equality \( I^X \hat{f} = (If)^{X^\circ} \). Given a generator \( \sigma_{\nu,x}(hf) \) of \((If)^{X^\circ} \), with \( x \in X^\circ \), \( \nu \in X^{\circ\ast} \), and \( h \in I \), from (12.8)(3), we obtain \( \sigma_{\nu,x}(hf) = \sum_i \sigma_{\nu,x_i}(h)\sigma_{\nu,x_i}(f) = \sum_i \sigma_{\nu,x_i}(h)\nu_i(fx) \). But, since \( fx \in X^\circ \), we have \( \nu_i(fx) = \nu_i(fx) \hat{f} \). Thus \( \sigma_{\nu,x}(hf) \in I^X \hat{f} \), and \((If)^{X^\circ} \subseteq I^X \hat{f} \). Now, consider a generator
claimed. Suppose that $M$ is a differential $\delta$-algebra obtained (for some $N$) of the ideal $\sigma_A \in \mathcal{A}$, the reduction of $\sigma_A$ through the projection map $\nu_i(x) \in S_i$. Now, we can factor out from $(\mathcal{A}, I)$ the ideal $I_A = 0$, determined by the following data $\nu_i(x) = \sum_{i} \nu_i(x_i) \in (I_A)^X$.

From the preceding facts, we get $(\mathcal{A}^{X \otimes}, I^{X \otimes}) = ((\mathcal{A}^{X \otimes})^{X \otimes}, (I^{X \otimes})^{X \otimes})$, as claimed. Suppose that $M \in (\mathcal{A}, I)$-Mod is such that $\text{Res}(M) \cong F^{X \otimes}(N')$, for some $N' \in (\mathcal{A}^{X \otimes}, I^{X \otimes})$-Mod. So we have an isomorphism of left $B^{X \otimes}$-modules $fM \cong X \otimes S \otimes S \otimes N'$. But the $B$-module $M$ admits the decomposition $M = e_0M \oplus fM$ and $X \otimes S e_0M \cong \text{Re}_S \otimes S e_0M \cong e_0M$. Consider the $S$-module $N := e_0M \oplus N'$, where $N'$ is considered as an $S$-module by restriction through the projection map $S \longrightarrow S_n$. Therefore we have an isomorphism of left $B$-modules $X \otimes S_n N = X \otimes S (e_0M \oplus N') \cong (\text{Re}_S \otimes S e_0M) \oplus (X \otimes S \otimes S \otimes N') \cong e_0M \oplus fM$, which, by (8.7)(4), implies that $M \cong F^{X}(N)$, for some $N \in (\mathcal{A}^{X}, I^{X})$-Mod.

**Example 8.10.** Let us start from a very simple interlaced weak ditalgebra $(\mathcal{A}, I)$, where $\mathcal{A}$ is the tensor algebra of the following biquiver (without dashed arrows and with trivial derivation) and ideal $I$ generated by the path $\beta \alpha$.

\[
\begin{array}{ccc}
A: & \alpha \rightarrow \beta & \beta \alpha = 0 \\
& 1 \rightarrow 2 & \\
& \lambda \rightarrow 3 & \\
0 & & \\
\end{array}
\]

In a first step, we apply the edge-reduction of $\beta$ to $(\mathcal{A}, I)$, which corresponds to the reduction of $(\mathcal{A}, I)$ by a suitable admissible module $X$ see [4](23.18). The interlaced weak ditalgebra obtained $(\mathcal{A}^{\beta}, I^{\beta})$ has the following biquiver with differential $\delta$ and ideal $I^{\beta}$ determined by the following data.

\[
\begin{array}{ccc}
A^{\beta}: & \delta (\alpha_1) = \xi \alpha_2, & \alpha_2 = 0 \\
& 1 \rightarrow 2 & \\
& \lambda \rightarrow \eta & \\
0 & & \\
\end{array}
\]

Now, we can factor out from $(\mathcal{A}^{\beta}, I^{\beta})$ the solid arrow $\alpha_2$, which belongs to the interlaced ideal $I^{\beta}$, to obtain the interlaced weak ditalgebra $(\mathcal{A}^{\beta q}, I^{\beta q})$, with $I^{\beta q} = 0$, determined by the following data.
Then, we can proceed to apply the edge-reduction of $\alpha_1$ to the last ditalgebra to obtain $(A^{\beta q \alpha_1}, I^{\beta q \alpha_1})$ given by the following data (with trivial ideal)

The biquiver of the ditalgebra obtained from the preceding one by supressing the source idempotent $e_0$ is the following minimal seminested ditalgebra

On the other hand, we can start from the original interlaced weak ditalgebra $(A, I)$ and suppress first the source idempotent $e_0$ to obtain

Then we can apply edge-reduction of the arrow $\beta$ to obtain
and then factor out the arrow $\alpha_2$ to get

\[
\mathcal{A}^{\otimes \beta q} : \begin{array}{c}
\alpha_1 \\
\downarrow \xi \\
1 \\
\downarrow 3' \\
\downarrow \eta \\
\downarrow 3
\end{array}
\quad \delta (\alpha_1) = 0
\]

Finally, if we apply edge-reduction of the arrow $\alpha_1$ to the preceding ditalgebra, we obtain

\[
\mathcal{A}^{\otimes \beta q \alpha_1} : \begin{array}{c}
1 \\
\downarrow \xi' \\
\downarrow \eta' \\
\downarrow 2' \\
\downarrow 3'
\end{array}
\]

which coincides with $\mathcal{A}^{\otimes \beta q \alpha_1}$.

9 Reduction to minimal ditalgebras

This section is devoted to the proof of the following theorem, our main result for non-wild $\mathcal{P}$-oriented interlaced weak ditalgebras $(\mathcal{A}, I)$, see (1.6). This result reduces the study of the $(\mathcal{A}, I)$-modules with dimension bounded by some $d \in \mathbb{N}$ to the study of modules over a minimal ditalgebra obtained from $(\mathcal{A}, I)$ by a finite number of reductions. This is the main step to the proof of the tame-wild dichotomy for this special type of weak interlaced ditalgebras $(\mathcal{A}, I)$.

Remark 9.1. Assume that $\mathcal{P}$ is a finite preordered set and that $\mathcal{A} = (T, \delta)$ is a $\mathcal{P}$-oriented weak ditalgebra with layer $(R, W)$, where $R = \prod_{i \in \mathcal{P}} k e_i$. Then, if $\Omega \in \mathcal{T}$ is any minimal element, we have that $\Omega \subseteq \mathcal{P}$ satisfies $e_\omega W_0 = 0$, $e_\omega W_1 = e_0 W_1 e_\omega$, and $Re_\omega = ke_\omega$, for all $\omega \in \Omega$. That is, $\Omega$ is a multiple source of $\mathcal{A}$.

Theorem 9.2. Assume that the ground field $k$ is algebraically closed and let $\mathcal{P}$ be a finite preordered set. Assume $(\mathcal{A}, I)$ is a $\mathcal{P}$-oriented triangular interlaced weak ditalgebra as in (5.2), where $I$ is an ideal of $\mathcal{A}$ contained in the radical of $\mathcal{A}$. Suppose that $(\mathcal{A}, I)$ is not wild and take $d \in \mathbb{N}$. Then, there is a finite sequence of reductions

\[
(\mathcal{A}, I) \hookrightarrow (\mathcal{A}^{z_1}, I^{z_1}) \hookrightarrow \cdots \hookrightarrow (\mathcal{A}^{z_1 z_2 \cdots z_t}, I^{z_1 z_2 \cdots z_t})
\]

of type $z_i \in \{a, d, r, q, X\}$ such that $\mathcal{A}^{z_1 z_2 \cdots z_t}$ is a minimal ditalgebra, we have $I^{z_1 z_2 \cdots z_t} = 0$, and almost every $(\mathcal{A}, I)$-module $M$ with $\dim_k M \leq d$ has the form $M \cong F^{z_1} \cdots F^{z_t}(N)$, for some $N \in (\mathcal{A}^{z_1 \cdots z_t}, I^{z_1 \cdots z_t})$-Mod.

Proof. Let $(R, W)$ be the layer of $\mathcal{A}$. By assumption $R = \prod_{i \in \mathcal{P}} ke_i$ is a product of fields. Since $I \subseteq \text{rad} \mathcal{A}$, we have that no idempotent $e_i$ belongs to $I$. 

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Since $\mathcal{A}$ is $\mathcal{P}$-oriented, if we choose a minimal element $\Omega$ in the poset $\overline{\mathcal{P}}$, we obtain a multiple source $\Omega \subseteq \mathcal{P}$ for the interlaced weak ditalgebra $(\mathcal{A}, I)$. So, $e_\Omega W_0 = 0$, $e_\Omega W_1 = e_\Omega W_1 e_\Omega$, and $Re_\omega = k e_\omega$ with $e_\omega \notin I$, for all $\omega \in \Omega$.

We will proceed by induction on the number of points of the interlaced weak ditalgebra $(\mathcal{A}, I)$. The base of the induction is clear: if there is only one point $i$ in $\mathcal{A}$, we have $\mathcal{P} = \Omega = \{i\}$ and $\mathcal{A}$ admits no solid arrows, so $\mathcal{A}$ is a minimal ditalgebra, and we are done. The same holds whenever $\mathcal{P} = \Omega$, so we can assume that this is not the case.

Assume $n \in \mathbb{N}$, with $n > 1$. Suppose that the theorem holds for any non-wild $\mathcal{P}'$-oriented triangular interlaced weak ditalgebra $(\mathcal{A}', I')$ with $m < n$ points, where $I'$ is an ideal of $\mathcal{A}'$ such that $I' \subseteq \text{rad} \mathcal{A}'$. Suppose that $(\mathcal{A}, I)$ is a non-wild $\mathcal{P}$-oriented triangular interlaced weak ditalgebra with multiple source $\Omega$ and $n$ points, where $I$ is an ideal of $\mathcal{A}$ with $I \subseteq \text{rad} \mathcal{A}$, and take $d \in \mathbb{N}$. Now, consider the non-empty subset $\mathcal{P}^\oplus = \mathcal{P} \setminus \Omega$ of $\mathcal{P}$ with the induced preorder and notice that the biquiver $\mathbb{B}^\oplus$ of $\mathcal{A}^\oplus$ is $\mathcal{P}^\oplus$-oriented. So $(\mathcal{A}^\oplus, I^\oplus)$ is a $\mathcal{P}^\oplus$-oriented non-wild triangular interlaced weak ditalgebra where $I^\oplus$ is an ideal of $\mathcal{A}^\oplus$ with $I^\oplus \subseteq \text{rad} \mathcal{A}^\oplus$, to which we can apply our induction hypothesis. So, for a fixed $d \in \mathbb{N}$, to obtain a finite sequence of reductions

$$(\mathcal{A}^\oplus, I^\oplus) \mapsto (\mathcal{A}^\circ_1, I^\circ_1) \mapsto \cdots \mapsto (\mathcal{A}^\circ_{k_1} \circ_2 \cdots \circ_{k_t}, I^\circ_{k_1} \circ_2 \cdots \circ_{k_t})$$

of type $z_i \in \{a, d, r, q, X\}$ such that $\mathcal{A}^\circ_{k_1} \circ_2 \cdots \circ_{k_t}$ is a minimal ditalgebra, we have $I^\circ_{k_1} \circ_2 \cdots \circ_{k_t} = 0$, and any $(\mathcal{A}^\oplus, I^\oplus)$-module $M''$ with $\dim_k M'' \leq d$ has the form $M'' \cong F_{z_1} \cdots F_{z_t}(N'')$, for some $N'' \in (\mathcal{A}^\circ_{k_1} \circ_2 \cdots \circ_{k_t}, I^\circ_{k_1} \circ_2 \cdots \circ_{k_t})$-$\text{Mod}$.

From the lemmas in $\S 5$ we can perform a corresponding finite sequence of reductions

$$(\mathcal{A}, I) \mapsto (\mathcal{A}^{z_1}, I^{z_1}) \mapsto \cdots \mapsto (\mathcal{A}^{z_1} \cdots z_t, I^{z_1} \cdots z_t)$$

of the same type $z_i \in \{a, d, r, q, X\}$ as those applied successively to $(\mathcal{A}^\oplus, I^\oplus)$, such that $(\mathcal{A}^{z_1} \cdots z_t, I^{z_1} \cdots z_t) = (\mathcal{A}^{z_1} \cdots z_t, I^{z_1} \cdots z_t)$ and there is a commutative diagram

$$
\begin{array}{ccc}
(A^{z_1} \cdots z_t, I^{z_1} \cdots z_t)-\text{Mod} & \xrightarrow{F_{z_t}} & (A, I)-\text{Mod} \\
\downarrow \text{Res} & & \downarrow \text{Res} \\
(A^{z_1} \cdots z_t, I^{z_1} \cdots z_t)-\text{Mod} & \xrightarrow{F_{z_t}} & (A^\oplus, I^\oplus)-\text{Mod}
\end{array}
$$

where $A^{z_1} \cdots z_t$ is a minimal ditalgebra, $I^{z_1} \cdots z_t = 0$, and any $M' \in (A^\oplus, I^\oplus)$-$\text{Mod}$ with $\dim_k M' \leq d$ is of the form $F_{z_1} \cdots F_{z_t}(N') \cong M'$. If $M \in (A, I)$-$\text{Mod}$ and $\dim_k M \leq d$, then $\dim_k \text{Res}(M) \leq d$, and $\text{Res}(M) \cong F_{z_1} \cdots F_{z_t}(N')$. Thus, for some $N \in (A^{z_1} \cdots z_t, I^{z_1} \cdots z_t)$-$\text{Mod}$, we have $F_{z_1} \cdots F_{z_t}(N) \cong M$. Moreover, $\dim_k N \leq d'$, for some fixed $d'$ which depends on $d$. For the sake of simplicity, set $(\mathcal{A}', I') := (A^{z_1} \cdots z_t, F_{z_1} \cdots z_t)$.

Here, we have that $\mathcal{A}^\oplus$ is a minimal ditalgebra, $I^\oplus = 0$, and $(\mathcal{A}', I')$ is a triangular interlaced weak ditalgebra with multiple source $\Omega$. If $(R', W')$ denotes the layer of $\mathcal{A}'$, say with $R' = \prod_{i \in \mathcal{P}'} R'e_i$, and we define $f = 1 - e_\Omega = \cdots$
$\sum_{j \in \mathcal{P} \setminus \Omega} e_j \in R'$, we know that $e_0 W_0' = 0$ and $f W_0' f = 0$, so $W_0' e_i = W_0'$. Then, we get that $(A', I')$ is a stellar triangular interlaced weak ditalgebra with stars centers $\{e_\omega\}_{\omega \in \Omega}$. Then, apply (10.11) to $d'$, to obtain a composition of full and faithful reduction functors

$$(A'', 0)\text{-Mod} \xrightarrow{F^{y_0}} \cdots \xrightarrow{F^{y_1}} (A', I')\text{-Mod}$$

such that $A''$ is a non-wild seminested ditalgebra and any $N \in (A', I')\text{-Mod}$ with \(\dim_k N \leq d'\) is of the form $F^{y_0} \cdots F^{y_s}(L) \cong N$ for some $L \in (A'', 0)\text{-Mod}$. Moreover, $\dim_k L \leq d''$, for some fixed $d''$ depending on $d'$. Then, from [4](28.22), there is a minimal ditalgebra $B$ and a composition of full and faithful reduction functors $F : B\text{-Mod} \rightarrow A''\text{-Mod}$ such that for any $A''\text{-module} L$ with $\dim_k L \leq d''$, there is a $B$-module $K$ such that $F(K) \cong L$.

Thus, given $M \in (A, I)\text{-Mod}$ with $\dim_k M \leq d$, we obtain

$$M \cong F^{z_1} \cdots F^{z_t} F^{y_1} \cdots F^{y_s} F(K).$$

\[\square\]

10 Tame and wild dichotomy

In this section we proceed to the proofs of our main results in the case of admissible homological systems. In order to be precise, we need first to adapt some known facts on tame ditalgebras to the case of interlaced weak ditalgebras.

The following lemma is probably known, but we include a proof for the sake of the reader.

**Lemma 10.1.** Let $\Lambda$ be a finite-dimensional algebra over an algebraically closed field. Let $C$ be a full subcategory of $\Lambda\text{-mod}$, closed under direct summands and direct sums. Then, the category $C$ can not be simultaneously tame and wild.

**Proof.** The proof is essentially the same proof given on page 366 of [4]. Consider the usual variety $\text{mod}_\Lambda (\mathcal{A})$ of $\Lambda$-modules with dimension vector $\mathcal{A}$, where the algebraic group $\mathcal{G}_{\mathcal{A}}$ acts in such a way that two $\mathcal{A}$-dimensional $\Lambda$-modules are isomorphic iff their corresponding points (denoted with the same symbols $M$ and $N$) satisfy that $N = hM$, for some $h \in \mathcal{G}_{\mathcal{A}}$. If $C$ is wild, there is a morphism of varieties $\varphi : k^2 = \text{mod}_k(x, y)(1) \rightarrow \text{mod}_\Lambda (\mathcal{A})$ induced by the functor $F : k(x, y)\text{-mod} \rightarrow \Lambda\text{-mod}$, with image in $C$, which is provided by the wildness of $C$ as in (1.2). Since $F$ preserves indecomposables and isomorphism classes, each $\varphi(\lambda, \mu)$ represents an indecomposable $\Lambda$-module in $C$ and $\varphi(\lambda, \mu) \notin \mathcal{G}_{\mathcal{A}} \varphi(\lambda', \mu')$, for all different pairs $(\lambda, \mu), (\lambda', \mu') \in k^2$. Consider, for each $\lambda \in k$, the curve $\varphi_\lambda : k \rightarrow \text{mod}_\Lambda (\mathcal{A})$, defined by $\varphi_\lambda(\mu) = \varphi(\lambda, \mu)$, for $\mu \in k$.

If $C$ is tame, there is a finite number of curves $\{\gamma_i : E_i \rightarrow \text{mod}_\Lambda (\mathcal{A})\}_{i=1}^m$, defined on cofinite subsets $E_i$ of $k$, such that every $\mathcal{A}$-dimensional indecomposable $\Lambda$-module in $C$ is represented by a point in $\bigcup_{i=1}^m \mathcal{G}_{\mathcal{A}} \gamma_i(E_i)$.

Then, for each $\lambda \in k$, $\varphi_\lambda(k) \subseteq \bigcup_{i=1}^m \mathcal{G}_{\mathcal{A}} \gamma_i(E_i)$. It follows that $\varphi_\lambda(D^\lambda) \subseteq \mathcal{G}_{\mathcal{A}} \gamma_i(E_i)$, for some cofinite subset $D^\lambda \subseteq k$ and some $i$ depending on $\lambda$. Then,
\(\gamma_i(E_i^\lambda) \subseteq \mathbb{G}_\varphi \varphi_i(D^\lambda)\), for some cofinite subset \(E_i^\lambda \subset E_i\). Since we are dealing with finitely many curves \(\gamma_1, \ldots, \gamma_m\), then there is \(\lambda' \neq \lambda\) such that \(\varphi_{\lambda'}(D^{\lambda'}) \subseteq \mathbb{G}_\varphi \varphi_i(E_i)\), for the same \(i\). Since \(E_i^\lambda\) is cofinite in \(E_i\) and \(D^{\lambda'}\) is infinite, there exists \(\mu \in D^{\lambda'}\) such that \(\varphi_{\lambda'}(\mu) \in \mathbb{G}_\varphi \varphi_i(E_i) \subseteq \mathbb{G}_\varphi \varphi_i(D^\lambda)\). This entails a contradiction. \(\square\)

**Definition 10.2.** An interlaced weak ditalgebra \((A, I)\) is called tame iff, for each \(d \in \mathbb{N}\), there is a finite collection \(\{(\Gamma_i, Z_i)\}_{i=1}^m\), where \(\Gamma_i = k[x]_{f_i}\) and \(Z_i\) is an \((A/I)-\Gamma_i\)-bimodule, which is free of finite rank as a right \(\Gamma_i\)-module, such that for every indecomposable \(M \in (A, I)\)-Mod with \(\dim_k M \leq d\), there are an \(i \in [1, m]\) and a simple \(\Gamma_i\)-module \(S\) such that \(Z_i \otimes_{\Gamma_i} S \cong M\) in \((A, I)\)-Mod.

The condition given in the last definition can be rearranged into an equivalent one, where the simple \(\Gamma_i\)-module \(S\) is replaced by an indecomposable \(\Gamma_i\)-module \(N\). See [1](27.2).

**Definition 10.3.** An interlaced weak ditalgebra \((A, I)\) is called strictly tame iff, for each \(d \in \mathbb{N}\), there is a finite collection \(\{(\Gamma_i, Z_i)\}_{i=1}^m\), where \(\Gamma_i = k[x]_{f_i}\) and \(Z_i\) is an \((A/I)-\Gamma_i\)-bimodule, which is free of finite rank as a right \(\Gamma_i\)-module, such that for almost every indecomposable \(M \in (A, I)\)-Mod with \(\dim_k M \leq d\), there are an \(i \in [1, m]\) and an indecomposable \(\Gamma_i\)-module \(N\) such that \(Z_i \otimes_{\Gamma_i} N \cong M\) in \((A, I)\)-Mod. Moreover, the functors

\[
\begin{align*}
\Gamma_i\text{-mod} & \xrightarrow{Z_i \otimes_{\Gamma_i} -} (A/I)\text{-mod} & \xrightarrow{L(A/I)} & (A, I)\text{-mod}
\end{align*}
\]

are required to preserve isoclasses and indecomposables.

It can be seen that in the last definition, we can replace the condition of being “free of finite rank right \(\Gamma_i\)-module”, by the apparently weaker one of being “finitely generated right \(\Gamma_i\)-module”, see [1](27.5).

The following result is just a more concise formulation of Theorem (1.9).

**Theorem 10.4.** Assume that the ground field is algebraically closed and that \((A, I)\) is a \(\mathcal{P}\)-oriented triangular interlaced weak ditalgebra where \(I\) is an ideal of \(A\) contained in the radical of \(A\). Then, \((A, I)\) is either tame or wild, but not both. Moreover, \((A, I)\) is tame iff it is strictly tame.

**Proof.** If \((A, I)\) was wild and tame simultaneously, we can easily adapt the geometric algebraic argument given in [1]§33 (pages 360–366) to get a contradiction: the condition \(d^2 = 0\) is irrelevant there (it is only used to have a well defined category of modules) and we can use \(A/I\) instead of \(A\) in that argument. Indeed, we already know that \((A, I)\) is a Roiter seminested weak ditalgebra.

If \((A, I)\) is not wild and \(d \in \mathbb{N}\), then we can apply [1.2] to obtain a finite sequence of reductions

\[
(A, I) \rightsquigarrow (A^{z_1}, I^{z_1}) \rightsquigarrow \cdots \rightsquigarrow (A^{z_1 z_2 \cdots z_t}, I^{z_1 z_2 \cdots z_t})
\]

of type \(z_t \in \{d, r, q, o, X\}\) such that \(I^{z_1 z_2 \cdots z_t} = 0\), \(\mathcal{B} = A^{z_1 \cdots z_t}\) is a minimal ditalgebra, and almost any \((A, I)\)-module \(M\) with \(\dim_k M \leq d\) has the form \(M \cong F(N)\), where \(F = F^{z_1} \cdots F^{z_t}\), for some \(N \in \mathcal{B}\)-Mod.
Given a point \( i \) of \( B \), we either have \( Be_i = ke_i \) or \( \Gamma_i = Be_i \neq ke_i \). The finite-dimensional indecomposable \( B \)-modules are of the form \( S_i = ke_i \), for \( i \) such that \( Be_i = ke_i \), or \( \Gamma_i/(x - \lambda)^t \), with \( \lambda \in k \) and \( t \in \mathbb{N} \), for any \( i \) such that \( \Gamma_i = Be_i \neq ke_i \).

Apply (6.13) to each \( \Gamma_i \) and \( F \) to obtain the following diagram which commutes up to isomorphism

\[
\begin{array}{cccc}
\Gamma_i\text{-Mod} & \xrightarrow{\Gamma_i \otimes_{\Gamma_i} -} & B\text{-Mod} & \xrightarrow{L_B} (B,0)\text{-Mod} \\
\Gamma_i\text{-Mod} & \xrightarrow{Z_i \otimes_{\Gamma_i} -} & (A/I)\text{-Mod} & \xrightarrow{L_{(A,I)}} (A,I)\text{-Mod}
\end{array}
\]

where \( Z_i := F(\Gamma_i) \). Since \( F \) is a composition of full and faithful functors, the composition functor \( FL_B(\Gamma_i \otimes_{\Gamma_i} -) \) preserves indecomposability and isoclasses, and the functor \( L_{(A,I)}(Z_i \otimes_{\Gamma_i} -) \) has the same properties. Moreover, \( Z_i \) is an \((A/I)-\Gamma_i\)-bimodule which is finitely generated projective by the right. But \( \Gamma_i \) is a principal ideal domain, so \( Z_i \) is a finitely generated free right \( \Gamma_i \)-module. If \( N \cong \Gamma_i/(x - \lambda)^t \), then \( L_{(A,I)}(Z_i \otimes_{\Gamma_i} N) \cong FL_B(\Gamma_i \otimes_{\Gamma_i} N) \cong M \). We have shown that \((A,I)\) is strictly tame. Finally, from the remark between definitions (10.2) and (10.3), we see that strict tameness implies tameness.

\[ \square \]

**Proof of Theorem (1.3) in the admissible homological system case.**

Consider the the category of \( \Delta \)-filtered modules \( \mathcal{F}(\Delta) \) for a finite-dimensional algebra \( A \) with admissible homological system \( (\mathcal{P}, \leq, \{ \Delta_i \}_{i \in \mathcal{P}}) \). From (10.1), we already know that \( \mathcal{F}(\Delta) \) can not be simultaneously tame and wild. By (1.8), it will be enough to show that \( \mathcal{F}(\Delta') \) is either tame or wild, and it is tame iff it is strictly tame.

The ditalgebra of \( \mathcal{Q} \) of (1.7) is the quotient \( \mathcal{Q} = A/J \), where \((A,I)\) is the \( \mathcal{P} \)-oriented triangular interlaced weak ditalgebra associated to the given admissible homological system, constructed in [7]§5, and \( J \) is the ideal of \( A \) generated by \( I \), see [7](5.22) and (3.6). As remarked in [7](13.1), we know that \( I \subseteq \text{rad} A \). Here, \( \mathcal{A} = (T, \delta) = (T_A(V), \delta) \) is a triangular weak ditalgebra and we can adopt the notation of (3.6). Thus \( \mathcal{Q} = \mathcal{A} = (\mathcal{T}, \mathcal{S}) \), with \( \mathcal{T} = T_{\mathcal{A}}(\mathcal{V}) \), where \( \mathcal{A} = A/I \) and \( \mathcal{V} = V/(IV + \delta(I) + VI) \).

From (1.7) there is an equivalence of categories

\[ F : \mathcal{A}\text{-mod} \longrightarrow \mathcal{F}(\Delta') \]

Moreover, \( F(M) \cong \mathcal{G} \otimes_{\mathcal{A}} M \), for each \( M \in \mathcal{A}\text{-mod}, \) where \( \mathcal{G} = \text{End}_{\mathcal{A}}(\mathcal{A})^{\text{op}} \) is the right algebra of \( \mathcal{A} \), and the right \( \mathcal{A} \)-module \( \mathcal{G} \) is finitely generated projective.

By (3.6), there is an equivalence \( G : \mathcal{A}\text{-mod} \longrightarrow (A,I)\text{-mod} \), which is the identity on objects. By (10.1), we know that \((A,I)\) is wild or strictly tame. It follows immediately that \( \mathcal{A} \) is wild or strictly tame.

If \( \mathcal{A} \) is wild, then we have the composition

\[ k(x,y)\text{-mod} \xrightarrow{Z \otimes_{k(x,y)} -} \mathcal{A}\text{-mod} \xrightarrow{L_{\mathcal{A}}} \mathcal{A}\text{-mod} \xrightarrow{F} \mathcal{F}(\Delta') \]

where \( Z \) realizes the wildness of \( \mathcal{A} \). If \( N \in k(x,y)\text{-mod} \) is indecomposable, so is

\[ F(Z \otimes_{k(x,y)} N) \cong \mathcal{G} \otimes_{\mathcal{A}} Z \otimes_{k(x,y)} N, \]
where $\Gamma \otimes A Z$ is a finitely generated projective right module. Clearly, the tensor product by the preceding bimodule preserves isoclasses. So, $F(\Delta')$ is wild.

Notice that, for any $N \in \mathcal{A}$-mod, we have $\dim_k N \leq \dim_k (\Gamma \otimes A N)$. Indeed, since $\Gamma \otimes A^{-} : \mathcal{A}$-mod $\longrightarrow \Gamma$-mod is an exact functor, if we have $\dim_k N = \sum_{i \in P} m_i$, where $m_i$ denotes the multiplicity of the simple $A$-module $S_i$ in the composition series of the $\mathcal{A}$-module $N$, then we also have that $\dim_k ((\Gamma \otimes A)S_i) = \sum_{i \in P} m_i \dim_k \Delta'_i$, because from (1.7)(2), we know that $\Delta'_i \cong \Gamma \otimes A S_i$.

Then, if $\mathcal{A}$ is strictly tame and $d \in \mathbb{N}$, we have the composition functors

$$
\Gamma_i-\text{mod} \xrightarrow{Z_i \otimes \Gamma_i -} \mathcal{A}$-mod \xrightarrow{L_{\mathcal{A}}} \mathcal{A}$-mod \xrightarrow{F} F(\Delta')
$$

where $Z_1, \ldots, Z_t$ are the bimodules parametrizing the indecomposable $\mathcal{A}$-modules $N$ with $\dim_k N \leq d$. Then, the bimodules $\Gamma \otimes A Z_1, \ldots, \Gamma \otimes A Z_t$ parametrize the indecomposable $\Gamma$-modules in $F(\Delta')$ with dimension at most $d$. Moreover, the functors

$$
\Gamma \otimes A Z_i \otimes \Gamma_i - : \Gamma_i-\text{mod} \longrightarrow F(\Delta')
$$

preserve isoclasses and indecomposables, for all $i \in [1, t]$. This finishes our proof.

\[\square\]

**Proof of Theorem (1.4) in the admissible homological system case.**

We assume that $(\mathcal{P}, \preceq, \{\Delta_i\}_{i \in P})$ is an admissible homological system for the finite-dimensional algebra $\Lambda$ and $F(\Delta)$ is tame. We proceed in four steps, with the notation of (1.7).

**Step 1:** If $E \xrightarrow{g} M$ is a minimal right almost split morphism in $\mathcal{I}(\Gamma)$ and

$$
0 \longrightarrow DtrM \xrightarrow{u} H \xrightarrow{v} M \longrightarrow 0
$$

is an almost split sequence in $\Gamma$-mod, then $\dim_k E \leq \dim_k (\Gamma \otimes H)$.

Indeed, recall that the category $\mathcal{I}(\Gamma)$ is the full subcategory of $\Gamma$-mod which consists of the $\Gamma$-modules of the form $\Gamma \otimes N$, for some $N \in \mathcal{Q}$-mod. Given $H \in \Gamma$-mod, we have the product map $\mu : \Gamma \otimes H \longrightarrow H$, and every morphism $g : Z \longrightarrow H$ of $\Gamma$-modules with $Z \in \mathcal{I}(\Gamma)$ factors through $\mu$. Then, given an almost split sequence in $\Gamma$-mod

$$
0 \longrightarrow DtrM \xrightarrow{u} H \xrightarrow{v} M \longrightarrow 0,
$$

we obtain that any non-retraction $h : Z \longrightarrow M$ in $\mathcal{I}(\Gamma)$ factors through $v$, say $h'v = h$, for some $h' : Z \longrightarrow H$; then, $h'$ factors through $\mu$. So the composition $v\mu : \Gamma \otimes H \longrightarrow M$ is a right almost split morphism in $\mathcal{I}(\Gamma)$. Since $g$ is minimal right almost split, we get $\dim_k E \leq \dim_k (\Gamma \otimes H)$.

**Step 2:** There is a constant $c_0 \in \mathbb{N}$ such that, for any almost split conflation $\zeta : \tau M \longrightarrow E \xrightarrow{g} M$ in $\mathcal{Q}$-mod, we have $\dim_k \tau M \leq \dim_k E \leq c_0 \times \dim_k M$.

Indeed, by (1.7)(3), the exact full and faithful functor $F : \mathcal{Q}$-mod $\longrightarrow \Gamma$-mod maps $\zeta$ on the exact sequence

$$
0 \longrightarrow F(\tau M) \xrightarrow{F(g)} FE \xrightarrow{F(g)} FM \longrightarrow 0.
$$

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Here, we have $FE \cong \Gamma \otimes_Q E$ and $FM \cong \Gamma \otimes_Q M$. So, the morphism $F(g)$ belongs to $\mathcal{I}(\Gamma)$ and is minimal right almost split in this category because the restriction $F: \mathcal{Q}\text{-mod} \longrightarrow \mathcal{I}(\Gamma)\text{-mod}$ is an equivalence and $g$ is minimal right almost split in $\mathcal{Q}\text{-mod}$.

Now, consider the almost split sequence in $\Gamma \otimes_Q M$:

$$0 \longrightarrow Dtr(\Gamma \otimes_Q M) \longrightarrow H \longrightarrow \Gamma \otimes_Q M \longrightarrow 0.$$  

By Step 1, since $F(g): \Gamma \otimes_Q E \longrightarrow \Gamma \otimes_Q M$ is minimal right almost split in $\mathcal{I}(\Gamma)$, we obtain $\dim_k(\Gamma \otimes_Q E) \leq \dim_k(\Gamma \otimes_Q H)$. From the general theory of almost split sequences, we also have that $\dim_k Dtr(\Gamma \otimes_Q M) \leq c \dim_k(\Gamma \otimes_Q M)$, for some $c \in \mathbb{N}$ which depends only on $\Gamma$. Therefore, we obtain

$$\dim_k H = \dim_k Dtr(\Gamma \otimes_Q M) + \dim_k(\Gamma \otimes_Q M) \leq (c + 1) \times \dim_k(\Gamma \otimes_Q M).$$

Hence, we get

$$\dim_k \tau M \leq \dim_k E \leq \dim_k(\Gamma \otimes_Q E) \leq \dim_k(\Gamma \otimes_Q H) \leq c_0 \times \dim_k M,$$

where $c_0 := (c + 1) \times (\dim_k \Gamma)^2$.

**Step 3:** For any positive integer $d$, almost every $d$-dimensional indecomposable $M$ in $\mathcal{Q}\text{-mod}$ satisfies that $\tau M \cong M$.

It is well known that the category $\mathcal{Q}\text{-mod}$ has almost split sequences, see [3], [10], and [9]. The strategy for this Step is the same used by Crawley-Boevey in [11]. Fix $d \in \mathbb{N}$ and let $\zeta: \tau M \overset{f}{\longrightarrow} E \overset{g}{\longrightarrow} M$ be an almost split conflation in $\mathcal{Q}\text{-mod}$ with $\dim_k M \leq d$. Since $\mathcal{F}(\Delta)$ is not wild, neither is $\mathcal{F}(\Delta')$ nor $\mathcal{Q}\text{-mod}$. Then, from [9], if we consider the number $d_0 := c_0 d$, there is a minimal seminested ditalgebra $\mathcal{B}$ and a full and faithful functor $F': \mathcal{B}\text{-mod} \longrightarrow \mathcal{Q}\text{-mod}$, such that almost every $N \in \mathcal{Q}\text{-mod}$ with $\dim_k N \leq d_0$ is of the form $F'(N') \cong N$, for some $N' \in \mathcal{B}\text{-mod}$. This is the case for almost every $M$, $E$, and $\tau M$. So $M \cong F'(M')$, $\tau M \cong F'(N')$, and $E \cong F'(E')$. Since $F'$ is full, we have $f': N' \longrightarrow E'$ and $g': E' \longrightarrow M'$ such that $F'(f') = f$ and $F'(g') = g$. Since $F'$ is full and faithful, and $\zeta$ is an almost split conflation, it is not hard to see that $\zeta': N' \overset{f'}{\longrightarrow} E' \overset{g'}{\longrightarrow} M'$ is a conflation and, in fact, that it is an almost split conflation in $\mathcal{B}\text{-mod}$. But the almost split conflations in $\mathcal{B}\text{-mod}$ are well known, see [4](32.3). Then, we can apply Crawley-Boevey’s analysis to show that, after discarding a finite number of possibilities for the indecomposable $M \in \mathcal{Q}\text{-mod}$, we have that $\tau M \cong M$. For details, see the proof of [4](32.6).

**Step 4:** Final argument.

From [17](6), we have an equivalence of categories $K: \mathcal{F}(\Delta) \longrightarrow \mathcal{Q}\text{-mod}$ which maps short exact sequences onto conflations and $K(\Delta_i) \cong S_i$, for each $i \in \mathcal{P}$. So $K$ maps almost split sequences of $\mathcal{F}(\Delta)$ on almost split conflations of $\mathcal{Q}\text{-mod}$, and $\tau KM = K\tau M$, for all indecomposable $M \in \mathcal{F}(\Delta)$ at which an almost split sequence ends.

For $M \in \mathcal{F}(\Delta)$, we can denote by $m_i$ the multiplicity of $\Delta_i$ as a composition factor in the $\Delta$-filtration of $M$, thus $\dim_k M = \sum_{i \in \mathcal{P}} m_i \dim_k \Delta_i$. Since $K$
is exact, we get \( \dim_k KM = \sum_{i \in P} m_i \dim_k K(\Delta_i) = \sum_{i \in P} m_i \). So, we have \( \dim_k KM \leq \dim_k M \).

Then, given \( d \in \mathbb{N} \), for all indecomposable \( M \in \mathcal{F}(\Delta) \) with \( \dim_k M \leq d \), such that an almost split sequence ends at \( M \), we also have that \( KM \) is indecomposable with \( \dim_k KM \leq d \). From Step 3, for almost all such indecomposable \( KM \), we have \( \tau KM \cong KM \). It follows that, for almost all such \( M \), we have \( \tau M \cong M \).

The equivalence \( \Theta : \Lambda\text{-Mod} \longrightarrow \Gamma\text{-Mod} \) of (1.7)(5) induces an equivalence of categories \( \Theta' : \mathcal{F}(\Delta) \longrightarrow \mathcal{F}(\Delta') \) which preserves and reflects almost split sequences. Thus for almost all indecomposable \( M \in \mathcal{F}(\Delta) \) there is an almost split sequence ending at \( M \) iff the same property holds for \( \mathcal{F}(\Delta') \). We will see now that the last case occurs, and from this fact we obtain what we wanted to show for \( \mathcal{F}(\Delta) \). Indeed, look again to the equivalence of categories of (1.7)(4):

\[
F : \text{mod-}Q \longrightarrow \mathcal{I}(\Gamma)
\]

such that composed with the inclusion into \( \Gamma\text{-mod} \) gives an exact functor. If \( \zeta : \tau M \xrightarrow{f} E \xrightarrow{g} M \) is an almost split conflation in \( \text{mod-}Q \) then its image \( F\zeta : F\tau M \longrightarrow F\Delta \longrightarrow FM \) under \( F \) is an exact sequence in \( \Gamma\text{-mod} \) with all its terms in \( \mathcal{F}(\Delta') = \mathcal{I}(\Gamma) \); moreover, the morphism \( F(f) \) is minimal left almost split in \( \mathcal{I}(\Gamma) \) and \( F(g) \) is minimal right almost split in \( \mathcal{I}(\Gamma) \), because \( F \) is an equivalence.

We know, see (12.4), that \( \text{mod-}Q \) admits only finitely many indecomposable projectives relative to its exact structure. Thus, with the exception of finitely many indecomposables \( M \), every indecomposable \( M \in \text{mod-}Q \) admits an almost split conflation ending at \( M \), see for instance (7.18). This implies that \( \mathcal{F}(\Delta') \) has the same property.

\[ \square \]

11 The case of general homological systems

This section is devoted to the extension of the tame-wild dichotomy for \( \mathcal{F}(\Delta) \) to the general case of an arbitrary homological system. This extension is a consequence of our theorem (1.3) for the admissible homological case and the following theorem (11.5), due to Mendoza, Sáenz, and Xi. The version we present here is a reformulation of some of their results in (15)§3. We will briefly recall some of their arguments, in their dual form, for the sake of the reader.

Fix a general homological system \( (\mathcal{P}, \leq, \{\Delta_i\}_{i \in \mathcal{P}}) \) for a finite-dimensional algebra \( \Lambda \). The following statement is crucial in the proof of (11.5), its dual is verified in the first part of the proof of (15)(3.12).

**Lemma 11.1.** For each \( i \in \mathcal{P} \), there is an exact sequence

\[
0 \longrightarrow V_i \longrightarrow U_i \longrightarrow \Delta_i \longrightarrow 0
\]

such that \( U_i \) is an indecomposable \( \mathcal{F}(\Delta) \)-projective and \( V_i \in \mathcal{F}(\Delta) \) has a \( \Delta \)-filtration with factors of the form \( \Delta_j \) with \( j > i \).
We fix a family of special exact sequences, as provided by the last lemma, for the rest of this section.

**Lemma 11.2.** For each \( M \in \mathcal{F}(\Delta) \), there is an exact sequence

\[
0 \longrightarrow K \longrightarrow W \longrightarrow M \longrightarrow 0
\]

in \( \mathcal{F}(\Delta) \), such that \( W \cong \bigoplus_{i \in P} n_i U_i \). Moreover, each \( n_i \) is the multiplicity of the factor \( \Delta_i \) in any \( \Delta \)-filtration of \( M \).

The family \( \{U_i\}_{i \in P} \) is a complete set of representatives of the isomorphism classes of the indecomposable \( \mathcal{F}(\Delta) \)-projective modules.

**Proof.** Consider the commutative diagram

\[
\begin{array}{cccccc}
0 & 0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow \\
0 & K_L & K_M & K_N & 0 \\
\downarrow & \downarrow & \downarrow \\
0 & W_L & W_L \oplus W_N & W_N & 0 \\
\downarrow & \downarrow & \downarrow \\
0 & L & M & N & 0 \\
\downarrow & \downarrow & \downarrow \\
0 & 0 & 0 & 0
\end{array}
\]

where the lower row is any exact sequence in \( \mathcal{F}(\Delta) \), the first column is an exact sequence in \( \mathcal{F}(\Delta) \) where \( W_L \) is \( \mathcal{F}(\Delta) \)-projective, the third column is an exact sequence in \( \mathcal{F}(\Delta) \) where \( W_N \) is \( \mathcal{F}(\Delta) \)-projective, and the central row is constructed as in the horseshoe lemma. So, all the rows and columns are exact. Therefore, the exact sequence in the central column belongs to \( \mathcal{F}(\Delta) \) and \( W_L \oplus W_N \) is \( \mathcal{F}(\Delta) \)-projective.

With the preceding argument, using the exact sequences of (11.1), it can be shown that each \( M \in \mathcal{F}(\Delta) \) admits an exact sequence as described in the statement of this lemma. We recall, from [10], that the multiplicity of the factors in a \( \Delta \)-filtration of \( M \) is independent of the filtration.

It follows that any indecomposable \( \mathcal{F}(\Delta) \)-projective is a direct summand of some \( \bigoplus_{i \in P} n_i U_i \), so it is isomorphic to some \( U_i \). Finally, the fact that the family \( \{U_i\}_{i \in P} \) consists of pairwise non-isomorphic modules is proved in [15](3.7). \( \square \)

**Definition 11.3.** Define \( U := \bigoplus_{i \in P} U_i \) and \( \Gamma := \text{End}_\Lambda(U)^{op} \). Moreover, consider the family \( \{\Theta_i\}_{i \in P} \) of \( \Gamma \)-modules given by \( \Theta_i := \text{Hom}_\Lambda(U, \Delta_i) \), for \( i \in P \).

In the following, we review part of the proof of (11.5): that \( (\mathcal{P}, \leq, \{\Theta_i\}_{i \in P}) \) is an admissible homological system such that \( \mathcal{F}(\Delta) \) is equivalent to \( \mathcal{F}(\Theta) \) as exact categories. The quasi-inverse equivalences will be realized by the appropriate restrictions of the functors \( H := \text{Hom}_\Lambda(U, -) : \Lambda\text{-mod} \longrightarrow \Gamma\text{-mod} \) and \( T := U \otimes_\Gamma - : \Gamma\text{-mod} \longrightarrow \Lambda\text{-mod} \). We start with the following.

**Lemma 11.4.** The functor \( H \) restricts to a full and faithful exact functor

\[
H : \mathcal{F}(\Delta) \longrightarrow \Gamma\text{-mod}.
\]
Proof. Consider the full subcategory $\text{add}(U)$ of $\Lambda\text{-mod}$ formed by the direct summands of finite direct sums of $U$. Then, from [2](II.2.1), we know that $H$ restricts to an equivalence of categories $\text{add}(U)\longrightarrow \text{add}(\Gamma)$, where $\text{add}(\Gamma)$ coincides with the full subcategory of $\Gamma\text{-mod}$ formed by the finitely generated projectives. Moreover, for each $W \in \text{add}(U)$ and $N \in \Lambda\text{-mod}$, the same functor $H$ restricts to an isomorphism $\text{Hom}_\Lambda(W,N)\longrightarrow \text{Hom}_{\Gamma}(H(W),H(N))$.

Given $M \in \mathcal{F}(\Delta)$, from [1](II.2) we derive the existence of an exact sequence of the form

$$W'\longrightarrow W\longrightarrow \pi\longrightarrow M\longrightarrow 0,$$

where $W, W' \in \text{add}(U)$. This sequence is obtained by splicing the exact sequence $0\longrightarrow K\longrightarrow W\longrightarrow M\longrightarrow 0$ produced for $M$ by this lemma, with the exact sequence $0\longrightarrow K'\longrightarrow W'\longrightarrow K\longrightarrow 0$ produced for $K$ by this lemma. Since $U$ is $\mathcal{F}(\Delta)$-projective, applying $H$, we obtain an exact sequence

$$H(W')\longrightarrow H(W)\longrightarrow \pi\longrightarrow H(M)\longrightarrow 0.$$

Then, for any $N \in \Lambda\text{-mod}$, we have the commutative diagram

$$
\begin{array}{cccccc}
0 & \to & \text{Hom}_\Lambda(M, N) & \to & \text{Hom}_\Lambda(W, N) & \to & \text{Hom}_\Lambda(W', N) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & \text{Hom}_\Gamma(H(M), H(N)) & \to & \text{Hom}_\Gamma(H(W), H(N)) & \to & \text{Hom}_\Gamma(H(W'), H(N))
\end{array}
$$

where the second and third vertical arrows are isomorphisms, then so is the first one. This shows that the restriction of $H$ to the category $\mathcal{F}(\Delta)$ is full and faithful. This restriction is exact because $U$ is an $\mathcal{F}(\Delta)$-projective module.  

\textbf{Theorem 11.5.} Given a homological system $(\mathcal{P}, \leq, \{\Delta_i\}_{i \in \mathcal{P}})$ for $\Lambda$, consider the algebra $\Gamma = \text{End}_\Lambda(U)^{op}$ as before and the triple $(\mathcal{P}, \leq, \{\Theta_i\}_{i \in \mathcal{P}})$, where $\Theta_i = \text{Hom}_\Lambda(U, \Delta_i)$, for all $i \in \mathcal{P}$. Then, $(\mathcal{P}, \leq, \{\Theta_i\}_{i \in \mathcal{P}})$ is an admissible homological system for $\Gamma$ and the functors $H = \text{Hom}_\Lambda(U, -)$ and $T = U \otimes_\Gamma -$ induce quasi-inverse equivalences of exact categories between $\mathcal{F}(\Delta)$ and $\mathcal{F}(\Theta)$.

\textbf{Proof.} With the preceding notations, we consider the following natural transformations $\alpha : U \otimes_\Gamma H \longrightarrow \text{id}_{\Lambda\text{-mod}}$ and $\beta : \text{id}_{\Gamma\text{-mod}} \longrightarrow H(U \otimes_\Gamma -)$ given by

1. For $M \in \Lambda\text{-mod}$, the morphism $\alpha_M : U \otimes_\Gamma H(M) \longrightarrow M$ has the recipe $\alpha_M(u \otimes g) = g(u)$.
2. For $N \in \Gamma\text{-mod}$, the morphism $\beta_N : N \longrightarrow H(U \otimes_\Gamma N)$ has the recipe $\beta_N(n)[u] = u \otimes n$.

The notation $\mathcal{F}(\Theta)$ makes sense for any family of modules. So it will be enough to show that we have quasi-inverse exact equivalences

$$\mathcal{F}(\Delta)\xrightarrow{H} \mathcal{F}(\Theta) \quad \text{and} \quad \mathcal{F}(\Theta)\xrightarrow{T} \mathcal{F}(\Delta),$$

since this clearly implies that $(\mathcal{P}, \leq, \{\Theta_i\}_{i \in \mathcal{P}})$ is a homological system. It is admissible because $\Gamma = \bigoplus_{i \in \mathcal{P}} \Theta_i \in \mathcal{F}(\Theta)$.  

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**Step 1.** \( \alpha_M \) is an isomorphism, for all \( M \in F(\Delta) \).

Since \( \alpha_U \) is the composition \( U \otimes_\Gamma H(U) = U \otimes_\Gamma \Gamma \cong U \), we know that \( \alpha_U \) is an isomorphism. It follows that \( \alpha_W \) is an isomorphism, for each \( W \in \text{add}(U) \).

For a given \( M \in F(\Delta) \), consider an exact sequence

\[
W' \rightarrowtail W \rightarrowtail M \rightarrowtail 0,
\]

as before, in the proof of (11.4), with \( W, W' \in \text{add}(U) \). Then, we have a commutative diagram

\[
\begin{array}{cccc}
U \otimes_\Gamma H(W') & \to & U \otimes_\Gamma H(W) & \to & U \otimes_\Gamma H(M) & \to & 0 \\
\downarrow \alpha_{W'} & & \downarrow \alpha_W & & \downarrow \alpha_M & & \\
W' & \to & W & \to & M & \to & 0
\end{array}
\]

with exact rows, and \( \alpha_N \) is an isomorphism.

**Step 2.** The functor \( T = U \otimes_\Gamma - \) restricts to an exact functor \( F(\Theta) \rightarrow F(\Delta) \).

Given one of the fixed special exact sequences \( 0 \rightarrowtail V_i \rightarrowtail U_i \rightarrowtail \Delta_i \rightarrowtail 0 \) given by (11.1), we can apply the functor \( H \) to this sequence and obtain an exact sequence \( 0 \rightarrowtail H(V_i) \rightarrowtail H(U_i) \rightarrowtail H(\Delta_i) \rightarrowtail 0 \), thanks to (11.4). The middle term of this exact sequence is projective in \( \Gamma\text{-mod} \), so from the long homology sequence associated to this sequence, we obtain an exact sequence

\[
0 \rightarrow \text{Tor}^\Gamma_1(U, \Theta_i) \rightarrow U \otimes_\Gamma H(V_i) \rightarrow U \otimes_\Gamma H(U_i) \rightarrow U \otimes_\Gamma H(\Delta_i) \rightarrow 0.
\]

Comparing this sequence with the original special exact sequence using the isomorphism \( \alpha \), we obtain that \( \text{Tor}^\Gamma_1(U, \Theta_i) = 0 \). So this holds for all \( i \in P \). A simple induction argument shows that \( \text{Tor}^\Gamma_1(U, N) = 0 \), for any \( N \in F(\Theta) \).

Now, given an exact sequence \( 0 \rightarrow \Theta_j \rightarrow N \rightarrow \Theta_i \rightarrow 0 \) in \( \Gamma\text{-mod} \), since \( \text{Tor}^\Gamma_1(U, \Theta_i) = 0 \), we get an exact sequence

\[
0 \rightarrow U \otimes_\Gamma \Theta_j \rightarrow U \otimes_\Gamma N \rightarrow U \otimes_\Gamma \Theta_i \rightarrow 0.
\]

Thus, we have the diagram

\[
\begin{array}{cccc}
0 & \rightarrow & U \otimes_\Gamma \Theta_j & \rightarrow & U \otimes_\Gamma N & \rightarrow & U \otimes_\Gamma \Theta_i & \rightarrow & 0 \\
\downarrow \alpha_{\Delta_j} & & \downarrow \text{id} & & \downarrow \alpha_{\Delta_i} & & \\
0 & \rightarrow & \Delta_j & \rightarrow & U \otimes_\Gamma N & \rightarrow & \Delta_i & \rightarrow & 0,
\end{array}
\]

where the vertical arrows are isomorphisms and the middle morphisms of the lower row are defined so that the diagram commutes. Since the first row is exact, so is the second one and \( U \otimes_\Gamma N \in F(\Delta) \).

A simple induction extends the preceding argument to show that \( U \otimes_\Gamma N \in F(\Delta) \) for any \( N \in F(\Theta) \). Moreover, if \( 0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0 \) is an exact sequence in \( F(\Theta) \), using that \( \text{Tor}^\Gamma_1(U, N) = 0 \), we obtain that the sequence \( 0 \rightarrow U \otimes_\Gamma L \rightarrow U \otimes_\Gamma M \rightarrow U \otimes_\Gamma N \rightarrow 0 \) is exact.

**Step 3.** \( \beta_N \) is an isomorphism, for all \( N \in F(\Theta) \).
For $i \in \mathcal{P}$, we have that the following composition is the identity map

$$\Theta_i = H(\Delta_i) \xrightarrow{\beta_{\Theta_i}} H(U \otimes_{\Gamma} H(\Delta_i)) \xrightarrow{(\alpha_{\Delta_i})_*} H(\Delta_i) = \Theta_i.$$ 

So, we know that $\beta_{\Theta_i}$ is an isomorphism for all $i \in \mathcal{P}$. Again, if we consider an exact sequence of the form $0 \rightarrow \Theta_j \rightarrow N \rightarrow \Theta_i \rightarrow 0$ in $\Gamma$-mod, we have a commutative diagram with exact rows

$$
\begin{array}{cccccc}
0 & \rightarrow & \Theta_j & \rightarrow & N & \rightarrow & \Theta_i & \rightarrow & 0, \\
0 & \rightarrow & H(U \otimes_{\Gamma} \Theta_j) & \rightarrow & H(U \otimes_{\Gamma} N) & \rightarrow & H(U \otimes_{\Gamma} \Theta_i) & \rightarrow & 0.
\end{array}
$$

So $\beta_N$ is an isomorphism. A simple induction, using the preceding argument shows that $\beta_N$ is an isomorphism, for each $N \in \mathcal{F}(\Theta)$.

In summary, we have shown that we can restrict the functors $H$ and $T$ to functors $H : \mathcal{F}(\Delta) \rightarrow \mathcal{F}(\Theta)$ and $T : \mathcal{F}(\Theta) \rightarrow \mathcal{F}(\Delta)$ and that these restrictions are exact. Moreover, the natural transformations $\alpha$ and $\beta$ when restricted appropriately give rise to isomorphisms of functors $\alpha : TH \rightarrow \text{id}_{\mathcal{F}(\Delta)}$ and $\beta : \text{id}_{\mathcal{F}(\Theta)} \rightarrow HT$. So, the restrictions $H$ and $T$ are quasi-inverse equivalences between the exact categories $\mathcal{F}(\Delta)$ and $\mathcal{F}(\Theta)$.

Finally, we can proceed to the following.

**Proof of Theorem (1.3) in the general homological system case.**

Consider the algebra $\Gamma$, the admissible homological system $(\mathcal{P}, \leq, \{\Theta_i\}_{i \in \mathcal{P}})$ for $\Gamma$, and the equivalence functor $U \otimes_{\Gamma} - : \mathcal{F}(\Theta) \rightarrow \mathcal{F}(\Delta)$ as in (1.3).

From (11.1), we already know that $\mathcal{F}(\Delta)$ can not be simultaneously tame and wild. From (1.3), we know that $\mathcal{F}(\Theta)$ is tame or wild.

If $\mathcal{F}(\Theta)$ is wild, there is a $\Gamma$-$k(x, y)$-bimodule $Z$, free of finite rank as a right $k(x, y)$-module, such that the functor $Z \otimes_{k(x, y)} - : k(x, y)$-mod $\rightarrow \mathcal{F}(\Theta)$ preserves indecomposables and isomorphism classes. The $\Lambda$-$k(x, y)$-bimodule $Z := U \otimes_{\Gamma} Z$ is finitely generated by the right and $Z \otimes_{k(x, y)} - : k(x, y)$-mod $\rightarrow \mathcal{F}(\Delta)$ preserves indecomposables and isomorphism classes. The bimodule $Z$ can be converted into a new $\Lambda$-$k(x, y)$-bimodule $\hat{Z}$, which is free of finite rank by the right, and such that $\hat{Z} \otimes_{k(x, y)} - : k(x, y)$-mod $\rightarrow \mathcal{F}(\Delta)$ preserves indecomposables and isomorphism classes, see (12), (11)§6, or (4)(22.17).

Now, assume that $\mathcal{F}(\Theta)$ is tame. In order to show that the tameness of $\mathcal{F}(\Theta)$ transfers to $\mathcal{F}(\Delta)$, it is convenient to notice the following.

Let us call a *multiplicity vector* $m = \{m_i\}_{i \in \mathcal{P}}$ any sequence with non-negative integer entries and define $\|m\|_{\Delta} = \sum_i m_i \dim_k \Delta_i$. Notice that, for any $d \in \mathbb{N}$, there are only finitely many multiplicity vectors with $\|m\|_{\Delta} = d$. Any $M \in \mathcal{F}(\Delta)$ has a multiplicity vector $m_M = \{m_i\}_{i \in \mathcal{P}}$, where $m_i$ is the multiplicity of $\Delta_i$ as a factor in any $\Delta$-filtration of $M$, which satisfies that $\|m\|_{\Delta} = \dim_k M$.

Thus, we can parametrize almost every indecomposable $\Lambda$-module in $\mathcal{F}(\Delta)$ with dimension $d$, for all $d \in \mathbb{N}$, if we can parametrize almost every indecomposable $\Lambda$-module in $\mathcal{F}(\Delta)$ with multiplicity vector $m_M = m$, for all $m$ with $\|m\|_{\Delta} = d$. 

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Then, in order to show that $\mathcal{F}(\Delta)$ is tame, it will be enough to show that almost every indecomposable $\Lambda$-module in $\mathcal{F}(\Delta)$ with given multiplicity vector $m = \{m_i\}_{i \in \mathcal{P}}$ can be parametrized with a finite family of $\Lambda$-$\Gamma_i$-bimodules. So fix any such multiplicity vector $m$. Since $\mathcal{F}(\Theta)$ is tame, there are rational algebras $\Gamma_1, \ldots, \Gamma_t$ and $\Gamma$-$\Gamma_i$-bimodules $Z_1, \ldots, Z_t$ finitely generated and free by the right, such that every indecomposable $\Gamma$-module $N$ in $\mathcal{F}(\Theta)$ with multiplicity vector $m_N = m$ is of the form $N \cong Z_i \otimes_{\Gamma_i} S$, for some $i$ and some simple $\Gamma_i$-module $S$.

Now take any indecomposable $M \in \mathcal{F}(\Delta)$ with multiplicity vector $m$, then there is some indecomposable $N \in \mathcal{F}(\Theta)$ with $M \cong U \otimes_{\Gamma} N$. Since $U \otimes_{\Gamma} -$ is exact and $U \otimes_{\Gamma} \Theta_i \cong \Delta_i$, for all $i \in \mathcal{P}$, the multiplicity vectors of $N$ and $M$ coincide: $m_N = m_M = m$. Then, for almost every such module $M$, there is an index $i$ and a simple $\Gamma_i$-module $S$ such that $M \cong U \otimes_{\Gamma} N \cong U \otimes_{\Gamma} Z_i \otimes_{\Gamma_i} S$. Now, the $\Lambda$-$\Gamma_i$-bimodule $\tilde{Z}_i := U \otimes_{\Gamma} Z_i$ is finitely generated by the right. As usual, we can replace each one of these bimodules $\tilde{Z}_i$ by new $\Lambda$-$\Gamma_i$-bimodules $\hat{Z}_i$ which are finitely generated and free by the right and which parametrize almost every $\Lambda$-module parametrized by $\tilde{Z}_i$. So, $\mathcal{F}(\Delta)$ is also tame.

The proof of theorem (1.4) in the general homological system case is similar to the preceding one: It is reduced to the admissible homological system case using multiplicity vectors, which are preserved by the exact equivalence $U \otimes_{\Gamma} -$ : $\mathcal{F}(\Theta) \longrightarrow \mathcal{F}(\Delta)$ given by theorem (11.5).

**Remark 11.6.** In [23], H. Treffinger constructs a family of examples of non-admissible homological systems for finite-dimensional algebras $\Lambda$ with size which can be arbitrarily large in comparison to the number of isomorphism classes of simple $\Lambda$-modules. The preceding theorem applies to $\mathcal{F}(\Delta)$ for each one of these homological systems.

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**References**

[1] F.W. Anderson, K.R. Fuller. *Rings and Categories of Modules*. Springer Graduate Texts in Math. 13, Springer-Verlag, New-York, Heidelberg, Berlin, 1973.

[2] M. Auslander, I. Reiten, S.O. Smalø. Representation Theory of Artin Algebras. Cambridge Studies in Advanced Mathematics 36. Cambridge University Press 1995.

[3] R. Bautista, M. Kleiner. *Almost split sequences for relatively projective modules*. Journal of Algebra 135 (1990) 19–56.
[4] R. Bautista, L. Salmerón, and R. Zuazua. *Differential Tensor Algebras and their Module Categories*. London Math. Soc. Lecture Note Series 362. Cambridge University Press, 2009.

[5] R. Bautista, E. Pérez, and L. Salmerón. *Tame and wild theorem for the category of filtered by standard modules for a quasi-hereditary algebra*. Preprint: arXiv: 1706.07386v1, June, 2017.

[6] R. Bautista, E. Pérez, and L. Salmerón. *Generic modules for the category of filtered by standard modules for a quasi-hereditary algebra*. Preprint: arXiv: 2110.08999v1, October, 2021.

[7] R. Bautista, E. Pérez, and L. Salmerón. *Homological systems and bocses*. Journal of Algebra 617 (2023) 192–274.

[8] Th. Brüstle, S. Koenig, and V. Mazorchuk. *The coinvariant algebra and representation types of blocks of category O*. Bull. London Math. Soc. 33 (2001) 669–681.

[9] W.L. Burt, *Almost split sequences and bocses*, 2005, Unpublished Monograph.

[10] W. L. Burt and M. C. R. Butler. *Almost split sequences for BOCSes*. Representations of Finite Dimensional Algebras, Canadian Math. Society Conference Proceedings Vol.11 (1991) 89–121.

[11] W. W. Crawley-Boevey. *On tame algebras and bocses*. Proc. London Math. Soc. (3) 56 (1988) 451–483.

[12] Yu. A. Drozd. *Tame and wild matrix problems*. Representations and quadratic forms. [Institute of Mathematics, Academy of Sciences, Ukrainian SSR, Kiev (1979) 39–74] Amer. Math. Soc. Transl. 128 (1986) 31–55.

[13] S. Koenig, J. Külshammer, and S. Ovsienko. *Quasi-hereditary algebras, exact Borel subalgebras, A∞-categories and boxes*. Adv. Math. 262 (2014) 546–592.

[14] J. Külshammer. *In the bocs seat: Quasi-hereditary algebras and representation type*. Representation Theory – Current Trends and Perspectives. EMS Ser. Congr. Rep. Eur. Math. Soc., Zürich 2017, 375–426.

[15] O. Mendoza, C. Sáenz, and Ch. Xi. *Homological systems in module categories over preordered sets*. Quarterly J. Math. 60, 1 (2009) 75–103.

[16] E. Pérez. *A note on homological systems*. Arch. Math. 117 (2021) 631–634.

[17] C.M. Ringel. *The category of modules with good filtrations over a quasi-hereditary algebra has almost split sequences*. Math. Z. 208 (1991) 209–223.
[18] A.V. Roiter and M.M. Kleiner. Representations of differential graded categories. Matrix problems, Inst. Mat. Akad. Nauk. Ukrain SSR, Kiev, 1977, 5–70. Abridged English version: Representations of algebras, Proc. Internat. Conf., Carleton Univ., Ottawa, Ont., 1974, Lecture Notes in Math. 488 (Springer, Berlin, 1975) 316–339.

[19] A.V. Roiter. Matrix problems and representations of bocses. Springer Lect. Notes in Math. 831, 1980, 288–324.

[20] D. Simson. Linear Representations of Partially Ordered Sets and Vector Space Categories. Algebra. Logic Appl., Vol. 4, Gordon & Breach Science Publishers, New York, 1992.

[21] D. Simson. On representation types of module subcategories and orders. Bulletin of the Polish Academy of Sciences, Mathematics, 41 (2) (1993) 78–93.

[22] D. Simson. Chain Categories of Modules and Subprojective Representations of Posets over Uniserial Algebras. The Rocky Mountain J. of Math. 32, (4) (2002) 1627–1650.

[23] H. Treffinger. The size of a stratifying system can be arbitrarily large Comptes Rendus Mathématique Vol. 361 (2023) 15–19.

[24] Ziting Zeng. On preprojective and preinjective partitions of a Δ-good module category. Journal of Algebra 285 (2005) 608–622.

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