Cosmic microwave background with Brans-Dicke Gravity: I. Covariant Formulation

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In the covariant cosmological perturbation theory, a 1+3 decomposition ensures that all variables in the frame-independent equations are covariant, gauge-invariant and have clear physical interpretations. We develop this formalism in the case of Brans-Dicke gravity, and apply this method to the calculation of cosmic microwave background (CMB) anisotropy and large scale structures (LSS). We modify the publicly available Boltzmann code CAMB to calculate numerically the evolution of the background and adiabatic perturbations, and obtain the temperature and polarization spectra of the Brans-Dicke theory for both scalar and tensor mode, the tensor mode result for the Brans-Dicke gravity are obtained numerically for the first time. We first present our theoretical formalism in detail, then explicitly describe the techniques used in modifying the CAMB code. These techniques are also very useful to other gravity models. At last, we compare the CMB and LSS spectra in Brans-Dicke theory with those in the standard general relativity theory. Constraints on Brans-Dicke model with current observational data is presented in a companion paper (paper II).

I. INTRODUCTION

The Jordan-Fierz-Brans-Dicke theory (hereafter the Brans-Dicke theory for simplicity) is a natural alternative and a simple generalization of Einstein's general relativity theory, it is also the simplest example of a scalar-tensor theory of gravity. In the Brans-Dicke theory, the purely metric coupling of matter with gravity is preserved, thus ensuring the universality of free fall (equivalence principle) and the constancy of all non-gravitational constants. From early on, testing the Brans-Dicke theory with CMB anisotropy has been considered. However, the usual approach is to use a metric-based and gauge-dependent method, i.e. making the calculation with a particular gauge, e.g., Ref. [22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41]. The method also has been applied to problems in CMB physics. Instead of using the components of metric as basic variables, the covariant formalism performs a 1+3 split of the Bianchi and Ricci identities, using the kinematic quantities, energy-momentum tensors of the fluid(s) and the gravito-electromagnetic parts of the Weyl tensor to study how perturbations evolve. The most notable advantage of this method is that the covariant variables have transparent physical definitions, which ensures that predictions are always straightforward to interpret physically. Other advantages include the unified treatment of scalar, vector and tensor modes, a systematic linearization procedure which can be extended to consider higher-order effects (this means the covariant variables are exactly gauge-invariant, independent of any perturbative expansion), and the ability to linearize about a variety of background models, e.g. either the FLRW or the Bianchi models.

A pioneering work in applying the covariant approach to Brans-Dicke theory is Ref. [22], in which a conformal transformation was performed, and calculation was done in the Einstein frame. More recently, Ref. [49, 50] chose the effective fluid frame, implying $D_a \phi = 0$ and $\omega_{ab} = 0$, i.e. their foliation selects vorticity-free spacelike hypersurfaces in which $\phi = \text{const}$, hence greatly simplifies the calculations.

The aim of this paper is to construct a full set of co-

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variant and gauge-invariant linearized equations to calculate angular power spectra of CMB temperature and polarization anisotropies in the cold dark matter frame, and to show that the covariant method will lead to a clear, mathematically well-defined description of the evolution of density perturbations. In a companion paper [1] (hereafter denoted paper II), we shall apply the formalism developed in this paper to the latest CMB and large scale structure data to obtain constraint on the Brans-Dicke parameter.

In §2, we briefly review the Brans-Dicke theory and its background cosmological evolution. The formalism of covariant perturbation theory is presented in §3, and the numerical implementation and the results are discussed in §4. Finally, we summarize and conclude in §5.

Throughout this paper we adopt the metric signature (+ − − −). Our conventions for the Riemann and Ricci tensor are fixed by \[ [\nabla_a, \nabla_b]u_c = R_{abcd}u^d, \] where \( \nabla_a \) denotes the usual covariant derivative, and \( R_{ab} \equiv R_{acbd} \). We use \( \partial_a \) to represent ordinary derivative. The spatially projected part of the covariant derivative is denoted by \( D_a \). The index notation \( A_1 \ldots A_l \) refers to the index string \( a_1 \ldots a_l \). Round brackets around indices denote symmetrization on the indices enclosed, square brackets denote anti-symmetrization, and angled brackets denote the projected symmetric and tracefree (PSTF) part. We adopt \( \kappa = 8\pi G \) and use units with \( \hbar = c = k_B = 1 \) throughout. In the numerical work we use Mpc as unit for distance.

II. THE BRANS-DICKE THEORY AND BACKGROUND COSMOLOGY

The Brans-Dicke theory is a prototype of the scalar-tensor theory of gravity. One of its original motivations is to realize Mach’s principle of inertia [2, 3, 6]. It introduced a new degree of freedom of the gravitational interaction in the form of a scalar field non-minimally coupled to the geometry. The action for the Brans-Dicke theory in the usual (Jordan) frame is

\[
S = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left[ -\phi R + \frac{\omega}{\phi} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi \right] + S^{(m)},
\]

where \( \phi \) is the Brans-Dicke field, \( \omega \) is a dimensionless parameter, and \( S^{(m)} \) is the action for the ordinary matter fields \( S^{(m)} = \int d^4x \sqrt{-g} \mathcal{L}^{(m)} \). Matter is not directly coupled to \( \phi \), in the sense that the Lagrangian density \( \mathcal{L}^{(m)} \) does not depend on \( \phi \). For convenience, we also define a dimensionless field

\[
\varphi = G\phi,
\]

where \( G \) is the Newtonian gravitational constant measured today. The Einstein field equations are then generalized to

\[
G_{\mu\nu} = \frac{8\pi G}{\varphi} T_{\mu\nu}^{(m)} + \frac{\omega}{\varphi^2} \left( \nabla_\mu \varphi \nabla_\nu \varphi - \frac{1}{2} g_{\mu\nu} \nabla_\lambda \varphi \nabla^\lambda \varphi \right) + \frac{1}{\varphi} \left( \nabla_\mu \nabla_\nu \varphi - g_{\mu\nu} \nabla_\lambda \varphi \nabla^\lambda \varphi \right),
\]

where \( T_{\mu\nu}^{(m)} \) is the stress tensor for all other matter except for the Brans-Dicke field, and it satisfies the energy-momentum conservation equation, \( \nabla^\mu T_{\mu\nu}^{(m)} = 0 \). The equation of motion for \( \varphi \) is

\[

\nabla_a \nabla^a \varphi = \frac{\kappa}{2\omega + 3} T^{(m)}_{\mu\nu} \nabla^\mu \varphi,
\]

where \( T_{\mu\nu}^{(m)} = T^{(m)}_{\mu\nu} \) is the trace of the energy-momentum tensor. The action \( \mathcal{L}^{I} \) and the field equation \( \varphi \) suggests that the Brans-Dicke field \( \phi \) plays the role of the inverse of the gravitational coupling, \( G_{eff}(\varphi) = \frac{1}{\varphi} = \frac{\kappa}{\omega} \), which becomes a function of the spacetime point.

For background cosmology, we treat the ordinary matter as the perfect fluid with the energy density \( \rho \) and pressure \( P \),

\[
T_{\mu\nu}^{(m)} = (\rho + P)u_\mu u_\nu - Pg_{\mu\nu}.
\]

The equations describing the background evolution are

\[
\rho' + 3H(\rho + P) = 0,
\]

\[
H^2 = \frac{\kappa S^2}{3\varphi} \rho + \frac{\omega}{6} \left( \frac{\varphi'}{\varphi} \right)^2 - \frac{\mathcal{H}^2}{\varphi},
\]

\[
\varphi'' + 2\mathcal{H}\varphi' = \frac{\kappa S^2}{2\omega + 3}(\rho - 3P),
\]

where the prime denotes derivative with respect to conformal time \( \eta \), \( S \) is the scale factor, and \( \mathcal{H} = S'/S \). General relativity is recovered in the limits

\[
\varphi' \rightarrow 0, \quad \varphi'' \rightarrow 0.
\]

To recover the value Newton’s gravitational constant today which is determined by Cavendish type experiments, we also require that the present day value of \( \varphi \) is given by

\[
\varphi_0 = \frac{2\omega + 4}{2\omega + 3}.
\]

III. PERTURBATION THEORY

A. The 1+3 covariant decomposition

The main idea of the 1 + 3 decomposition is to make space-time splits of physical quantities with respect to the 4-velocity \( u^\mu \) of an observer. There are many possible choices of the frame, for example, the CMB frame in
which the dipole of CMB anisotropy vanishes, or the local rest-frame of the matter. These frames are generally assumed to coincide when averaged on sufficiently large scale. Here it will be convenient to choose \( u^a \) to coincide with the velocity of the CDM component, since \( u^a \) is then geodesic, and acceleration vanishes. From the 4-velocity \( u^a \), we could construct a projection tensor \( h_{ab} \) into the space perpendicular to \( u^a \) (the instantaneous rest space of observers whose 4-velocity is \( u^a \)):

\[
h_{ab} \equiv g_{ab} - u_a u_b, \tag{11}\]

where \( g_{ab} \) is the metric of the spacetime. Since \( h_{ab} \) is a projection tensor, it can be used to obtain covariant tensor perpendicular to \( u^a \), and it satisfies

\[
h_{ab} = h_{(ab)}, \quad u^a h_{ab} = 0, \quad h^a_d h_{cb} = h_{ab}, \quad h^a_a = 3. \tag{12}\]

With the timelike 4-velocity \( u_a \) and its tensor counterpart \( h_{ab} \), one can decompose a spacetime quantity into irreducible timelike and spacelike parts. For example, we can use \( u_a \) to define the covariant time derivatives of a tensor \( T^{b...c}_{d...e} \):

\[
\dot{T}^{b...c}_{d...e} = u^a \nabla_a T^{b...c}_{d...e}, \tag{13}\]

furthermore, we can exploit the projection tensor \( h_{ab} \) to define a spatial covariant derivative \( D^a \) which returns a tensor which is orthogonal to \( u^a \) on every index:

\[
D^a T^{b...c}_{d...e} = h^a_b h^b_r \cdots h^a_d h^d_i \cdots h^a_e \nabla^p T_p^{r...s}_{t...u}, \tag{14}\]

If the velocity field \( u^a \) has vanishing vorticity, \( D^a \) reduces to the covariant derivative in the hypersurfaces orthogonal to \( u^a \). The projected symmetric tracefree (PSTF) parts of vectors and rank-2 tensors are

\[
V(a) = h^b_a V_b, \tag{15}\]

\[
T_{(ab)} = h_{db} h^b_d T_{cd} = h_{db} h^b_d T_{cd} - \frac{1}{3} h_{cd} T_{cd} h_{ab}. \tag{16}\]

One can also define a volume element for the observer’s instantaneous rest space:

\[
\varepsilon_{abc} = \eta_{abcd} u^d = \varepsilon_{[abc]}, \tag{17}\]

where \( \eta_{abcd} \) is the 4-dimensional volume element (\( \eta_{abcd} = \eta_{[abcd]} \)). Note that \( D_a h_{ab} = 0 = D_a \varepsilon_{abc}. \) The skew part of a projected rank-2 tensor is spatially dual to the projected vector \( T_a = \frac{1}{2} \varepsilon_{abc} T^{[bc]} \), and any projected second-rank tensor has the irreducible covariant decomposition

\[
T_{ab} = \frac{1}{3} T h_{ab} + \varepsilon_{abc} T^c + T_{(ab)}, \tag{18}\]

where \( T = T_{cd} u^d \) is the spatial trace. In the 1+3 covariant formalism, all quantities are either scalars, projected vectors or PSTF tensors. The covariant decomposition of velocity gradient are

\[
\nabla_a u_b = D_a u_b + u_a A_b, \tag{19}\]

\[
D_a u_b = \omega_{ab} + \sigma_{ab} + \frac{1}{3} \rho h_{ab}, \tag{20}\]

where \( \sigma_{ab} = D_{(a} u_{b)} \) is the shear tensor which satisfies \( \sigma_{ab} = \sigma_{(ab)} \), \( \sigma^a_a = 0 \) and \( u^a \sigma_{ab} = 0 \); \( \omega_{ab} = D_{[a} u_{b]} \) is the vorticity tensor, which satisfies \( \omega_{ab} = \omega_{[ab]} \) and \( u^a \omega_{ab} = 0 \). One can also define the vorticity vector \( \omega_a = \varepsilon_{abc} u^b \omega^c / 2 \) (with \( \omega_a = \varepsilon_{abc} \omega^c \)). The scalar \( \theta \equiv \nabla^a u_a = D^a u_a = 3 H \) is the volume expansion rate, \( H \) is the local Hubble parameter; and \( A_a \equiv u^b \nabla_b u_a = \dot{u}_a \) is the acceleration, which satisfies \( u^a A_a = 0 \). We note that the tensor \( D_a u_b \) describes the relative motion of neighbouring observers. The volume scalar \( \theta \) determines the average separation between two neighbouring observers. The effect of the vorticity is to change the orientation of a given fluid element without modifying its volume or shape, therefore it describes the rotation of matter flow. Finally, the shear describes the distortion of matter flow, it changes the shape while leaving the volume unaffected [30].

Gauge-invariant quantities can be constructed from scalar variables by taking their projected gradients. The comoving fractional projected gradient of the density field \( \rho^{(i)} \) of a species \( i \) is the key quantity of covariant method [21],

\[
X^{(i)}_{a} \equiv \frac{S}{\rho^{(i)}} D_a \rho^{(i)}, \tag{21}\]

which describes the density variation between two neighbouring fundamental observers. The comoving spatial gradient of the expansion rate orthogonal to the fluid flow is

\[
Z_a \equiv S D_a \theta, \tag{22}\]

which describes perturbations in the expansion. These quantities are in principle observable, characterizing inhomogeneity in a covariant way, and vanishes in the FLRW limit.

The matter stress-energy tensor \( T^{(m)}_{ab} \) can be decomposed irreducibly with respect to \( u^a \) as follows:

\[
T^{(m)}_{ab} \equiv \rho u_a u_b + 2u_a (q_b) - P h_{ab} + \sigma_{ab}, \tag{23}\]

where \( \rho \equiv T^{(m)}_{ab} u^a u^b \) is the density of matter measured by an observer moving with 4-velocity \( u^a \), \( q_a \equiv h^b_a T_{bc}^{(m)} u^c \) is the relativistic momentum density or heat (i.e. energy) flux and is orthogonal to \( u^a \), \( P \equiv -h^{ab} T_{ab}^{(m)} / 3 \) is the isotropic pressure, and the projected symmetric traceless tensor \( \sigma_{ab} \equiv T^{(m)}_{<ab>} \) is the anisotropic stress, which is also orthogonal to \( u^a \). The quantities \( \rho, P, q_a, \sigma_{ab} \) are referred to as dynamical quantities and \( \sigma_{ab}, \omega_{ab}, \theta, A_a \).
as kinematical quantities. In the FLRW limit, isotropy restricts $T^{(m)}_{ab}$ to the perfect-fluid form, so that the heat flux $q_a$ and anisotropic stress $\pi_{ab}$ must vanish.

The remaining first-order gauge-invariant variables which we need are derived from the Weyl tensor $C_{abcd}$, which is associated to the long-range gravitational field and vanishes in an exact FLRW universe due to isotropy. In analogy to the electromagnetic field, the Weyl tensor can be split into electric and magnetic parts, denoted by $E_{ab}$ and $B_{ab}$ respectively. They are both symmetric traceless tensors and orthogonal to $u^a$,

$$E_{ab} \equiv C_{acbd}u^c u^d = E_{<ab>},$$

$$B_{ab} \equiv -^*C_{acbd}u^c u^d = -\frac{1}{2} \varepsilon^e_a \varepsilon^f_b C_{efbd} u^d = B_{<ab>}.$$  \hspace{1cm} (24)

Here $^*$ denotes the dual, $^*C_{abcd} = \frac{1}{2} \eta_{ac} \varepsilon^f_b C_{efbd}$.

For the radiation field, we can make a 1+3 covariant decomposition of the photon 4-momentum as

$$p^a = E(u^a + e^a),$$

where $E = p^a u_a$ is the energy of the photon. $e^a$ describes the propagation direction of photon in the instantaneous rest space of the observer. The observer can introduce a pair of orthogonal polarization vectors $(e_1)^a$ and $(e_2)^a$, which are perpendicular to $u^a$ and $e^a$, to form a right-handed orthonormal tetrad $\{u^a, (e_1)^a, (e_2)^a, e^a\}$ at the observation point. The (screen) projection tensor is defined as

$$\mathcal{H}_{ab} = g_{ab} - u_a u_b + e_a e_b,$$  \hspace{1cm} (27)

which is perpendicular to both $u^a$ and $e^a$, and satisfies $\mathcal{H}^a_b (e_1)^b = (e_1)^a$.

Using the polarization basis vectors, the observer can decompose an arbitrary radiation field into Stokes parameters $I(E, e^a), Q(E, e^a), U(E, e^a)$ and $V(E, e^a)$ [22]. Therefore one can introduce a second-rank transverse polarization tensor $P_{ab}(E, e^c)$

$$P_{ab}(e_i)^a (e_j)^b = \frac{1}{2} \left( I + Q \begin{array}{c} U \ \ V \end{array} \begin{array}{c} -V \ \ U - Q \end{array} \right),$$  \hspace{1cm} (28)

for $i$ and $j$ = 1, 2, and we have omitted the arguments $E$ and $e^a$. $P_{ab} \propto E^3 \mathcal{H}_c^e \mathcal{H}_d^f f_{cd}$, where $f_{cd}$ is photon distribution function. Decomposing $P_{ab}(E, e^c)$ into its irreducible components, one obtains

$$P_{ab}(E, e^d) = -\frac{1}{2} I(E, e^d) \mathcal{H}_{ab} + P_{ab}(E, e^d)$$

$$+ \frac{1}{2} V(E, e^d) \varepsilon_{abc} e^c,$$  \hspace{1cm} (29)

where the linear polarization tensor $P_{ab}(E, e^d)$ satisfies

$$P_{ab}(e_i)^a (e_j)^b = \frac{1}{2} \left( Q \begin{array}{c} U \ \ -Q \end{array} \right).$$  \hspace{1cm} (30)

It is convenient to define energy-integrated multipole for total intensity brightness and the electric part of the linear polarization:

$$I_{Ai} = \int_0^\infty dE \int d\Omega \ I(E, e^c) e_{<Ai>},$$

$$E_{Ai} = M_i^2 \int_0^\infty dE \int d\Omega \ e_{<A-i>}(E, e^c)$$  \hspace{1cm} (32)

where $e_{Ai} = e_{a} e_{b} e_{c} \cdots e_{i}$. In the FLR W limit, isotropy restricts $T_{\alpha\beta}$ to the perfect-fluid form, so the heat flux $q_a$ and anisotropic stress $\pi_{ab}$ must vanish.

B. The linearized perturbation equations

In the 1+3 covariant approach, the fundamental quantities are not the metric, which is gauge-dependent, but the kinematic quantities of the fluid, namely the shear $\sigma_{ab}$, the vorticity $\omega_{ab}$, the volume expansion rate $\theta$ and the acceleration $A_a$, the energy-momentum of matter and the gravito-electromagnetic parts of the Weyl tensor. The fundamental equations governing these quantities are the Bianchi identities and the Ricci identities. The Riemann tensor in these equations is expressed in terms of $E_{ab}$, $B_{ab}$ and the Ricci tensor $R_{ab}$. The modified Einstein equation connects the Ricci tensor to the matter energy-momentum tensor. In the following, we have linearized all the perturbation equations. We should also note that the definitions of covariant variables do not assume any linearization, and exact equations can be found for their evolution.

The first set of equations are derived from the Ricci identities for the vector field $u^a$, i.e.

$$2 \nabla_\alpha [u^\beta \nabla u^\gamma] = R_{abcd} u^d.$$  \hspace{1cm} (33)

Substituting the 4-velocity gradient and the decomposition of the Riemann tensor, and separating out the time-like projected part into the trace, the symmetric trace-free and the skew symmetric parts, we obtain three propagation equations. The first propagation equation is the Raychaudhuri equation,

$$\dot{\theta} + \frac{1}{3} \theta^2 - D^\alpha u_\alpha + \frac{\kappa}{2} \rho (\rho + 3p) +$$

$$\frac{1}{2} \left( 2\dot{\omega} \dot{\varphi} - \frac{1}{\varphi} D_\alpha D^\alpha \varphi + \theta \dot{\varphi} + \frac{3}{\varphi} \ddot{\varphi} \right) = 0,$$  \hspace{1cm} (34)

which is the key equation of gravitational collapse, accounting for the time evolution of $\theta$. The second is the vorticity propagation equation,

$$\dot{\omega}_{ab} - D_\alpha \hat{u}_{ab} + \frac{\kappa}{3} \ell_{ab} = 0.$$  \hspace{1cm} (35)

The last one is the shear propagation equation,

$$\dot{\sigma}_{ab} + \frac{2}{3} \theta \sigma_{ab} - D_{<a} \hat{u}_{b>} = E_{ab} + \frac{\kappa}{2} \pi_{ab}$$

$$+ \frac{1}{2\varphi} D_{<a} D_{b>} \varphi + \frac{1}{2\varphi} \dot{\varphi} \sigma_{ab} = 0,$$  \hspace{1cm} (36)
which describes the evolution of kinematical anisotropies. It shows that the tidal gravitational field $E_{ab}$ and the anisotropic stress $\pi_{ab}$ would induce shear directly, and the shear will change the spatial inhomogeneity of the expansion through the constraint equations (37).

The propagation equations are complemented by three constraint equations, which are spacelike components of Eq. (35). The first is the shear constraint,

$$D^b\omega_{ab} + D^b\sigma_{ab} - \frac{2}{3} D_a\theta - \kappa \varphi q_{a} - \omega \frac{\dot{\varphi}}{\varphi^2} D_a\varphi = 0,$$

which shows the relation between the momentum flux $q_a$, the shear $\sigma_{ab}$ and the spatial inhomogeneity of the expansion. The second constraint equation is the vorticity divergence identity,

$$D^c(\varepsilon_{abc}\omega^{bc}) = 0.$$  

(38)

The propagation equations are the $B$-equation,

$$B_{ab} + (D^c\omega_{d(a} + D^c\sigma_{d(a)}\eta_{b)c})d u^c = 0,$$

which shows that the magnetic Weyl tensor can be constructed from the vorticity tensor and the shear tensor. With this last equation $B_{ab}$ may be eliminated from some equations in favor of the vorticity and the shear.

So far we have only discussed propagation and constraint equations for the kinematical quantities. The second set of equations arises from the Bianchi identities of the Riemann tensor,

$$\nabla_c R_{cd}ab = 0,$$

which gives constraint on the curvature tensor and leads to the Bianchi identities for Weyl tensor after contracting once,

$$\nabla^d C_{abcd} = \nabla[b]R_{a|c} + \frac{1}{6} g_{c[b} \nabla_a] R.$$  

(41)

The 1+3 splitting of the once contracted Bianchi identities leads to two propagation and two constraint equations which are similar in form to the Maxwell field equations in an expanding universe, governing the evolution of the long range gravitational field. The first propagation equation is the $E$-equation,

$$\dot{E}_{ab} + \theta E_{ab} + D^c B_{d(a}\eta_{b)c}d u^c + \frac{\kappa}{6 \varphi} [3(\rho + P)\sigma_{ab} +$$

$$3D_{<a}q_{b> - 3\pi_{ab} - \theta \pi_{ab}] + \frac{1}{2} \sigma_{ab}(\omega + \frac{3}{2} \frac{\dot{\varphi}}{\varphi^2} -$$

$$\frac{1}{6} \varphi D_\mu \varphi + \frac{1}{2} \varphi D_{<a}D_{b>}\varphi$$

$$+ \frac{1}{2} \frac{\dot{\varphi}}{\varphi} E_{ab} + \frac{3}{4} \kappa \frac{\dot{\varphi}}{\varphi^2} \pi_{ab} = 0,$$

and the second propagation is the $B$-equation

$$\dot{B}_{ab} + (\theta + \frac{\dot{\varphi}}{2\varphi}) B_{ab} - [D^c E_{d(a} + \kappa \frac{2}{\varphi} D^c \pi_{d(a} +$$

$$\frac{1}{2} \frac{\dot{\varphi}}{\varphi} D_{d(a} \varphi - \frac{1}{6} \frac{\dot{\varphi}}{\varphi} D_{d(a} \varphi D_{b)(c} \eta_{b)c}d u^c = 0.$$  

(43)

This pair of equations for electric and magnetic parts of the Weyl tensor would give rise wavelike behavior for its propagation: if we take the time derivative of the $E$-equation, commuting the time and spatial derivatives of $B$ term and substituting from the $\dot{B}$-equation to eliminate $B$, we would obtain a $\vec{E}$ term and a double spatial derivatives term, which together give the wave operator acting on $E$: similarly we can obtain a wave equation for $B$ by taking time derivative of the $B$-equation. These waves are also subjected to two constraint equations, which emerge from the spacelike component of the decomposed Eq. (41). The first constraint is

$$D^b E_{ab} - \frac{\kappa}{6 \varphi} (2D_a \rho + 3q_{a} + 3D_b \pi_{ab}) + \frac{2}{3} \rho \frac{D_a \varphi}{\varphi^2} -$$

$$\frac{\kappa}{2} \frac{\dot{\varphi}}{\varphi^2} q_{a} - (\omega + \frac{1}{2} \frac{\dot{\varphi}}{\varphi^2} \frac{4}{3} \theta D_a \varphi + (D_a \varphi) + \dot{u} \varphi = 0.$$  

(44)

This is the div $E$ equation, with the source term given by the spatial gradient of energy density. It can be regarded as a vector analogue of the Newtonian Poisson equation, and shows that the vector modes will result in a non-zero divergence of $E_{ab}$, and hence a non-zero gravitational E-field. The second constraint equation is

$$D^b B_{ab} - \frac{\kappa}{2 \varphi} [3(\rho + P)\eta_{ab} + h^{cd} u^b \omega_{cd} + h_{abcd} u^b D^c q^d] -$$

$$\frac{1}{2} \left[ (\omega - \frac{\dot{\varphi}}{\varphi^2}) - \frac{1}{3} \frac{\dot{\varphi}}{\varphi^2} D_d \varphi - \frac{\theta}{3} \frac{\dot{\varphi}}{\varphi^2} \right] \eta_{ab}u^c \omega_{cd} +$$

$$h_{abcd}u^b \left( \omega \frac{\dot{\varphi}}{\varphi^2} D^c D^d \varphi + \frac{1}{3} \frac{\dot{\varphi}}{\varphi^2} (D^c D^d \varphi) +$$

$$\frac{\theta}{3} \frac{\dot{\varphi}}{\varphi^2} D^c D^d \varphi + \frac{\dot{\varphi}}{\varphi^2} D^c u^d \right] = 0.$$  

(45)

This is the div $B$ equation, with the fluid vorticity serving as source term. It shows that the vector modes will result in non-zero divergence of $B_{ab}$, and hence a non-zero gravitational B-field. The above equations are remarkably similar to the Maxwell equations of the electromagnetism, so we have chosen to use $E_{ab}$ and $B_{ab}$ as the symbols.

The last set of equations arises from the twice-contracted Bianchi identities. Projecting parallel and orthogonal to $u^c$, we obtain two propagation equations,

$$\dot{\rho} + \theta(\rho + P) + D_a q^a = 0,$$

$$\dot{q}_a + \frac{4}{3} \theta q_a + (\rho + P) \dot{u}_a + D^b \pi_{ab} - D_a P = 0,$$

(47)
respectively. For perfect fluids, these reduce to
\[
\dot{\rho} + \theta (\rho + P) = 0 , \tag{48}
\]
\[
(\rho + P) \dot{u}_a - D_a P = 0 , \tag{49}
\]
which are the energy conservation equation and momentum conservation equation respectively.

The background field equation for Brans-Dicke field is given in Eq.$(57)$. The first order covariant and gauge-invariant perturbation variable of the Brans-Dicke field is defined as the spatial derivative of the Brans-Dicke field,
\[
V_a \equiv S D_a \varphi . \tag{50}
\]

Taking the covariant spatial derivative of Eq.$(57)$, commuting the spatial and time derivatives of $V$ term, we could obtain the first order perturbation equation for Brans-Dicke field after linearization,
\[
V''_a + 2 \mathcal{H} V_a + 2 S Z_a \varphi + 2 S^2 D_a D_b V_b = \frac{\kappa S^2}{3 + 2 \omega} \sum_i (1 - 3 c_s^{(i)} ) (\rho^{(i)} X^{(i)} , \tag{51}
\]

where the upper index $(i)$ labels the particle species.

In the absence of rotation, $\omega_{ab} = 0$, one can define a global 3-dimensional spacelike hypersurfaces that are everywhere orthogonal to $\dot{u}^a$. This 3-surfaces is meshed by the instantaneous rest space of comoving observers. The geometry of the hypersurfaces is determined by the 3-Riemann tensor defined by
\[
[D_a, D_b] u_c = (3) R_{abcd} u^d , \tag{52}
\]

which is similar to the definition of Riemann tensor $R_{abcd}$ but with a conventional opposite sign. The relationship between $(3) R_{abcd}$ and $R_{abdc}$ is
\[
(3) R_{abcd} = - h_a^\nu h_b^\mu h_c^\lambda h_d^\sigma R_{\mu \nu \lambda \sigma} - v_{ac} v_{bd} + v_{ad} v_{bc} = (3) R_{[ab][cd]} , \tag{53}
\]

where $v_{ab} = D_b u_a$ is the relative flow tensor between two neighbouring observers. In analogy to 4-dimension, the projected Ricci tensor and Ricci scalar are defined by
\[
(3) R_{ab} = (3) R_{acbd} h^{cd} = (3) R^{\,\,\,\,c}_{\,\,\,\,a} , \tag{54}
\]

and
\[
(3) R = (3) R_{ab} h^{ab} . \tag{55}
\]

The $(3) R_{ab}$ is determined by the Gauss-Codacci formula
\[
(3) R_{ab} = \frac{1}{3} (3) R h_{ab} - \frac{1}{3} \theta \sigma_{ab} - \frac{\kappa}{2} \pi_{ab} + E_{ab} , \tag{56}
\]

where
\[
(3) R = 2 (\kappa \rho - \frac{1}{3} \dot{g}^2) . \tag{57}
\]

The Eq.$(57)$ is also the generalized Friedmann equation, showing how the matter tensor determines the 3-space average curvature.

The last first-order covariant and gauge-invariant variables can be obtain from the spatial derivative of the projected Ricci scalar,
\[
\eta_a = \frac{1}{2} S D_a (3) R , \tag{58}
\]

after a tedious calculation, we obtain
\[
\eta_a = \kappa \rho X_a - \kappa \rho V_a - \frac{1}{S} (2 \mathcal{H} + \frac{\dot{\varphi}'}{\varphi^2} ) Z_a + \frac{1}{S^2} (\omega \frac{\dot{\varphi}'}{\varphi^2} - 3 \mathcal{H} ) (V_a - \mathcal{H} V_a)
+ (\omega + 3) \frac{1}{S} \frac{\dot{\varphi}'}{\varphi^2} \mathcal{H} V_a + \frac{1}{S} (\omega \frac{\dot{\varphi}'}{\varphi^2} - 3 \mathcal{H} ) W_a
- \frac{\omega \dot{\varphi}'}{S^2 \varphi^2} V_a - \frac{1}{\varphi} D_a D_b \nu^{\nu} - \frac{3}{S} \mathcal{H} \frac{V_a}{\varphi} . \tag{59}
\]

C. Mode expansion in spherical harmonics

In the linear perturbation theory it is convenient to expand the $O(\epsilon)$ variables in harmonic modes, since it splits the perturbations into scalar, vector or tensor modes and decouples the temporal and spatial dependencies of the 1+3 equations. This converts the constraint equations into algebraic relations and the propagation equations into ordinary differential equations along the flow lines. In this paper we focus on the scalar and tensor perturbation modes, since the vector modes would decay in an expanding universe in the absence of sources such as topological defects.

1. Scalar mode

For scalar perturbations we expand in the scalar eigenfunctions $Q^{(k)}$ of the generalized Helmholtz equation
\[
S^2 D^a D_a Q^{(k)} = k^2 Q^{(k)} \tag{60}
\]
at zero order. They are defined so as to be constant along flow lines, i.e. independent of proper time $\dot{Q}^{(k)} = O(\epsilon)$, and orthogonal to the fluid 4-velocity $u^a$.

For the $l$-th multipoles of the radiation anisotropy and polarization, we expand in the rank-1 PSTF tensor, $Q^{(k)}_{A_l}$, derived from the scalar harmonics with
\[
Q^{(k)}_{A_l} = \left( \frac{S}{k} \right)^l D_{<a_1 \ldots a_l>} Q^{(k)} , \tag{61}
\]

where the index notation $A_l$ denotes the index string $a_1 \ldots a_l$. The recursion relation for the $Q^{(k)}_{A_l}$,
\[
Q^{(k)}_{A_l} = \frac{S}{k} D_{<a_l} Q^{(k)}_{A_{l-1}} \tag{62}
\]
follows directly from Eq. (61). The factor of \((S/k)^4\) in the definition of the \(Q^{(k)}_{A_1}\) ensures that \(\dot{Q}^{(k)}_{A_1}\) is zero at zero-order. The \(Q^{(k)}_{A_1}\) also satisfies some other zero-order properties,

\[
u^{a_{i}Q^{(k)}_{A_1}} = 0, \quad h^{a_{i}a_{j}Q^{(k)}_{A_1}} = 0. \tag{63}
\]

We also have the following differential relations which can be derived from Eqs. (60) and (62):

\[
\begin{align*}
D^a Q^{(k)}_{a_1 \ldots a_i} & = \frac{l}{2l-1} S \left[ 1 - \frac{(l-1)k^2}{k^2} \right] Q^{(k)}_{a_2 \ldots a_i}, \tag{64} \\
D^2 Q^{(k)}_{a_1 \ldots a_i} & = \frac{k^2}{S^2} \left[ 1 - l(l+1) \frac{K}{k^2} \right] Q^{(k)}_{a_1 \ldots a_i}. \tag{65}
\end{align*}
\]

Now we can expand the gauge-invariant variable in the following dimensionless harmonic coefficients:

\[
\begin{align*}
X^{(i)}_a & = \sum_k k X^{(i)}_a Q^{(k)}_a, \tag{66} \\
Z_a & = \sum_k k^2 Z_k Q^{(k)}_a, \tag{67} \\
q^{(i)}_a & = \rho^{(i)} \sum_k q^{(i)}_k Q^{(k)}_a, \tag{68} \\
v^{(i)}_a & = \sum_k v^{(i)}_k Q^{(k)}_a, \tag{69} \\
\pi^{(i)}_{ab} & = \rho^{(i)} \sum_k \pi^{(i)}_k Q^{(k)}_{ab}, \tag{70} \\
E_{ab} & = \sum_k k^2 \Phi_k Q^{(k)}_{ab}, \tag{71} \\
\sigma_{ab} & = \sum_k k \sigma_k Q^{(k)}_{ab}, \tag{72} \\
A_a & = \sum_k k W_k Q^{(k)}_a, \tag{73} \\
V_a & = \sum_k k V_k Q^{(k)}_a, \tag{74} \\
\eta_a & = \sum_k k^3 \eta_k Q^{(k)}_a, \tag{75} \\
I_{A_i} & = \rho \gamma \sum_k I^{(i)}_k Q^{(k)}_{A_i}, \tag{76} \\
E_{A_i} & = \rho \gamma \sum_k \epsilon^{(i)}_k Q^{(k)}_{A_i}, \tag{77}
\end{align*}
\]

where the upper index \((i)\) labels the particle species. The scalar expansion coefficients, such as \(X^{(i)}_a\), are first-order gauge-invariant variables, and their spatial gradients are second-order, for example \(D^2 X^{(i)}_a = O(2)\). In the covariant and gauge-invariant approach, we characterize scalar perturbations by requiring that the vorticity and the magnetic part of the Weyl tensor be at most second-order. Demanding \(\omega_{ab} = O(2)\) ensures that density gradients are not from kinematic effects due to vorticity, and setting \(B_{ab} = O(2)\) ensures that gravitational waves are excluded to the first order.

To obtain the scalar equations for the scalar expansion coefficients, one could substitute the harmonic expansions of the covariant variables into the propagation and constraint equations given in the section above. Here we will consider only the adiabatic modes. For the \((i)\) fluid,

\[
D^a P^{(i)} = c^{(i)}_s D^a \rho^{(i)}, \tag{78}
\]

where \(c^{(i)}_s\) is the adiabatic sound speed of the \((i)\) fluid. For the spatial gradients of the total density \(X_k\), we find

\[
X'_k + \frac{3H}{\rho} \sum_i \rho^{(i)} X^{(i)}_k + k \left[ 1 + P^{(i)} \right] Z_k + q_k = 0. \tag{79}
\]

For the individual fluid of the \((i)\) species, the propagation equation satisfies

\[
X^{(i)}_k + 3H(c^{(i)}_s)^2 - \frac{P^{(i)}}{\rho^{(i)}} X^{(i)}_k + k \left[ 1 + \frac{P^{(i)}}{\rho^{(i)}} \right] Z_k + q^{(i)}_k = 0. \tag{80}
\]

For the heat fluxes, we have

\[
q^{(i)}_k + \frac{\gamma}{3} (1 - \frac{1}{\gamma}) X^{(i)}_k + (1 + \frac{P^{(i)}}{\rho^{(i)}}) k W_k + \frac{2}{3} k (1 - \frac{3k}{k^2}) \pi^{(i)}_k - k c^{(i)}_s X^{(i)}_k = 0. \tag{81}
\]

The heat flux for each fluid component is often given by \(q^{(i)}_k = (\rho^{(i)} + P^{(i)}) v^{(i)}_k\), so we can derive the propagation equations for \(v^{(i)}_k\) from Eq. (81).

We also can obtain the time evolution of the spatial gradient of the expansion

\[
Z'_k + HZ_k + \frac{W_k}{k} (3H - 3H^2 - k^2) + \frac{1}{k^2} \frac{k^2}{2} \sum_i (1 + 3c^{(i)}_s)^2 \rho^{(i)} X^{(i)}_k + \frac{1}{2k} \left[ V_k \left[ -4 \omega \frac{\varphi^2}{\varphi^2} + 3 \frac{\varphi''}{\varphi^2} - \frac{k^2}{k^2} (\rho^{(i)} + 3P) + \frac{k^2}{k^2} \right] + 4 \omega \frac{\varphi''}{\varphi^2} V'_k + 3 \frac{V''}{\varphi} + k Z_k \frac{\varphi'}{\varphi} + W_k (4 \omega \frac{\varphi^2}{\varphi^2} + 6 \frac{\varphi''}{\varphi^2} - 3H \frac{\varphi'}{\varphi} + 3 \frac{\varphi'}{\varphi} W'_k) = 0. \tag{82}
\]

Substituting the covariant harmonic expansion into Eq. (59), and then taking the time derivative of this equation, we obtain the evolution of the spatial gradient of the
3-curvature scalar:

\[ k^2 \eta_k = -X_k [S^2 \kappa (\rho + 3P) \frac{\mathcal{H}}{\rho} + S^2 \kappa \rho \frac{\phi'}{\phi^2}] + \]

\[ S^2 \kappa \rho \frac{X_k'}{\phi} + V_k \left( S^2 \kappa (\rho + 3P) \frac{\mathcal{H}}{\rho^2} + \frac{S^2 \kappa (\rho - P) \phi'}{\rho^3} \right) + 3 \mathcal{H} \frac{\phi''}{\phi^2} - 6 \omega \phi' \phi' - 2 \omega \phi'^2 + 2 \omega \phi' \phi' + k^2 \frac{\phi'}{\phi^2} \]

\[ + \mathcal{V}_k \left( \frac{S^2 \kappa}{2 \phi'^2} (3P - \rho) + \omega \frac{3 \phi''}{\phi^2} - 2 \omega \phi'^2 + 6 \mathcal{H} \frac{\phi'}{\phi^2} \right) \]

\[ - \frac{k^2}{\phi} \left( \frac{\mathcal{V}_k' (\omega \frac{\phi'}{\phi^2} - 3 \mathcal{H} \frac{\phi'^2}{\phi^2}) + W_k (2 \omega \frac{\phi'^2}{\phi^2} - 2 \omega \phi' \phi' - 3 \mathcal{H} \frac{\phi'}{\phi^2} \right) \]

\[ kZ_k (2 \mathcal{H} + \frac{\phi'}{\phi^2} - \frac{\phi'^2}{\phi^2}) - kZ'_k (2 \mathcal{H} + \frac{\phi'}{\phi^2} \right) \]

As mentioned by Ref. [51], solving the propagation equation of \( \eta_k \) avoids the numerical instability problem in isocurvature modes when we work in the CDM frame.

From the shear propagation equation \([38]\), the propagation equation for \( \sigma_k \) becomes

\[ \sigma'_k + \mathcal{H} \sigma_k - kW_k + k \Phi_k + \frac{S^2 \kappa}{2 \phi} \rho \pi_k + k \mathcal{V}_k \frac{\phi'}{\phi^2} \]

\[ 1 \frac{\phi'}{\phi^2} \sigma_k = 0 \right]. \] (84)

From the Div E equation \([43]\), we could obtain the \( \Phi_k \) equation,

\[ 2 \frac{k^3}{3} (1 - \frac{3K}{K^2}) \Phi_k - k \frac{\kappa \rho}{S} [X_k + (1 - \frac{3K}{K^2}) \pi_k] - \]

\[ 3 \mathcal{H} \kappa \rho \frac{\phi'}{\phi^2} q_k + 2 \frac{k^3}{3} \frac{\phi'}{\phi^2} \frac{V_k}{\phi^2} - 3 \frac{1}{2} \frac{\phi'}{\phi^2} \kappa \rho q_k - \]

\[ (\omega + \frac{3}{2}) \frac{k^3}{S} \frac{\phi'}{\phi^2} (\mathcal{V}_k + 3 \mathcal{H} \mathcal{V}_k + W_k \frac{\phi'}{\phi^2}) = 0 \right]. \] (85)

The algebraic equation of \( \sigma_k \) can be derived from the shear constraint equation \([44]\)

\[ \frac{3}{2} k^2 [Z_k - \sigma_k (1 - \frac{3K}{K^2})] + \frac{S^2 \kappa}{k} \rho q_k + \frac{\phi'}{\phi^2} \mathcal{V}_k \]

\[ + \frac{1}{\phi} (\mathcal{V}_k' - \mathcal{H} \mathcal{V}_k) + \frac{\phi'}{\phi^2} W_k = 0 \right]. \] (86)

From the first-order perturbation equation for the Brans-Dicke field \([51]\), we could derive the quadratic differential equation of \( \mathcal{V}_k \)

\[ \mathcal{V}_k'' + 2 \mathcal{H} \mathcal{V}_k' + kZ_k \phi' + k^2 \mathcal{V}_k \]

\[ \frac{3 S^2 \kappa}{2} \sum_{i} (1 - 3 c_s^2) \rho (i) X^{(i)} \right]. \] (87)

The variables \( X_k, q_k \) and \( \pi_k \) (without upper index \( i \) ) refer to variables of the total matter, and can be expressed as

\[ \rho X_k = \rho X_k^{(\gamma)} + \rho X_k^{(\nu)} + \rho X_k^{(b)} + \rho X_k^{(c)} \]

\[ \rho q_k = \rho q_k^{(\gamma)} + \rho q_k^{(\nu)} + \rho q_k^{(b)} + \rho q_k^{(c)} \]

\[ \rho \pi_k = \rho \pi_k^{(\gamma)} + \rho \pi_k^{(\nu)} \] (90)

2. Tensor mode

For tensor modes, we expand the first order perturbation variables in the rank-2, zero-order PSTF tensor eigenfunctions \( Q^{(k)}_{ab} \) of the comoving Laplacian,

\[ S^2 \mathcal{D}^a \mathcal{D}_b Q^{(k)}_{ab} = k^2 Q^{(k)}_{ab} \right]. \] (91)

Similar to the case of scalar modes, this equation holds at the zero-order. The tensor harmonics are transverse, orthogonal to \( u^a \), and constant along the integral curves of \( u^a \):

\[ D^a Q^{(k)}_{ab} = 0, \quad u^a Q^{(k)}_{ab} = 0, \quad \bar{Q}^{(k)}_{ab} = 0 \right]. \] (92)

They can also be classified as having electric parity (denoted by \( Q^{(k)}_{ab} \)) or magnetic parity (denoted by \( \bar{Q}^{(k)}_{ab} \)). These two parity harmonics are related by a curl:

\[ \text{curl} Q^{(k)}_{ab} = \frac{k}{S} \sqrt{1 + \frac{3K}{k^2}} \bar{Q}^{(k)}_{ab} \right], \] (93)

\[ \text{curl} \bar{Q}^{(k)}_{ab} = \frac{k}{S} \sqrt{1 + \frac{3K}{k^2}} Q^{(k)}_{ab} \right]. \] (94)

For tensor mode perturbations, the vorticity and all gauge-invariant vectors vanish at the first order, i.e. \( \omega_{ab}, X_a, Z_a, q_{ab}, A_a, V_a, \eta_a \) all equal to zero \([46, 52]\). The rest rank-2 gauge-invariant tensors are constrained to be transverse:

\[ (3) \nabla^a E_{ab} = 0, \quad (3) \nabla^a B_{ab} = 0, \]

\[ (3) \nabla^a \sigma_{ab} = 0, \quad (3) \nabla^a \pi_{ab} = 0 \right]. \] (95)

And they can be expanded in electric and magnetic parity tensor harmonics:

\[ E_{ab} = \sum_k \frac{k^2}{S^2} (E_k Q^{(k)}_{ab} + \bar{E}_k \bar{Q}^{(k)}_{ab}) \right], \] (96)

\[ B_{ab} = \sum_k \frac{k^2}{S^2} (B_k Q^{(k)}_{ab} + \bar{B}_k \bar{Q}^{(k)}_{ab}) \right], \] (97)

\[ \sigma_{ab} = \sum_k \frac{k}{S} (\sigma_k Q^{(k)}_{ab} + \bar{\sigma}_k \bar{Q}^{(k)}_{ab}) \right], \] (98)

\[ \pi_{ab}^{(i)} = \rho (i) \sum_k (\pi_k^{(i)} Q^{(k)}_{ab} + \bar{\pi}_k^{(i)} \bar{Q}^{(k)}_{ab}) \right]. \] (99)
Substituting these into equations in section 3B we obtain the $E_k$ and $\sigma_k$ propagation equations:

\[
\begin{align*}
 k^2 E_k + k^2 E_k (\mathcal{H} + \frac{1}{2} \frac{\varphi'}{\varphi}) - k^3 (1 + 3 \frac{K}{k^2}) \sigma_k + \\
 \frac{\kappa S^2}{2\varphi} (\rho + P) k \sigma_k + \frac{\varphi'^2}{2\rho_0} (\omega + \frac{3}{2}) k \sigma_k + \frac{k S^2}{2} \rho \kappa k \mathcal{H} + \\
 \frac{3}{2} \frac{\kappa S^2}{2} \rho \kappa_k \mathcal{H} + \frac{3}{4} \frac{\varphi'}{\varphi} k S^2 \rho \kappa_k - \frac{1}{2\varphi} k S^2 \rho \kappa_k = 0, \\
 \end{align*}
\]

\[
\sigma_k' = -\mathcal{H} \sigma_k - k E_k - \frac{\kappa S^2}{2k} \rho \kappa_k \frac{1}{\varphi} \frac{\varphi'}{\varphi} - \frac{1}{2} \frac{\varphi'}{\varphi} \sigma_k. \tag{101}
\]

IV. NUMERICAL IMPLEMENTATION

We carry out our numerical study by modifying the CAMB code. The original CAMB code, written by Antony Lewis and Anthony Challinor\cite{CAMB}, is a Fortran 90 program which calculates CMB anisotropies in the standard Einstein general relativity, by solving the Boltzmann-Einstein equations for various components in the Universe. Most of the equations to be solved are in the file \textit{equations.f90}, which can be modified conveniently. The background evolution equation $dr/da$ is written in function \textit{dtauda}, and it can be modified for different background. The Boltzmann-Einstein equation group is listed in the functions \textit{fderiv} (scalar mode for flat Universe), \textit{fderiv(t)} (tense mode for non-flat Universe) and \textit{fderiv(t)} (tense mode for non-flat Universe). This equation group includes the propagation equations of scalar factor $S$, the 3-Ricci scalar perturbation $\eta$, the cold dark matter perturbation $X_c$, the baryon perturbations $X_b$ and $\nu_b$, photon multipole moments, and neutrino multipole moments in the covariant approach. The CAMB code uses the Runge-Kutta method (subroutine \textit{dverk} in file \textit{subroutines.f90}) to solve these equations. To speed up the calculation, the line-of-sight integration method first developed by Seljak and Zaldarriaga\cite{SeljakZaldarriaga} is used: the differential equation for photon temperature perturbation is integrated along the l.o.s. to obtain $\delta T/T$. The multipoles today is a definite integral of source term multiplied by the spherical Bessel functions from early time to today. The source term of scalar perturbation at a given time for a given wavenumber is encoded in subroutine \textit{output}. The subroutine evolves the perturbation equations and does the integration in \textit{cambmain.f90}. The main routine for running CAMB is wrapped in file \textit{camb.f90}. Using these equations, we modify the code for calculation in the Brans-Dicke theory. The most important three parts of modifications are: the background evolution, the Boltzmann-Einstein differential equations, and the source term in the line-of-sight integration.

For the background evolution, we implement the procedure described in the appendix of Ref.\cite{Ref}. To satisfy the end point condition Eq.\tag{10}, we start from an epoch which is deemed early enough. We then evolve the model forwards (to avoid numerical instability we do not evolve backwards) to obtain the $\varphi$ value today. The procedure is repeated with a Brent algorithm (see e.g. \ref{55}) to find the initial value of $\varphi$ at that epoch. In doing this we set $\varphi' = 0$ and $\mathcal{V}_k = \mathcal{V}_k = 0$ at the initial point. The initial condition $\varphi' = 0$ can be justified by Eq.\tag{3}: in the radiation dominated era, the R.H.S. of Eq.\tag{3}, $\rho - 3P$ is negligible compared with other terms, then

\[
\varphi' = c_1 + c_2 S^{-2}. \tag{102}
\]

This mean that any initial velocity quickly dies out in a few Hubble times and approaches a terminal velocity $c_1$, this velocity is constrained by nucleosynthesis, so it should be very small. The initial condition $\mathcal{V}_k = \mathcal{V}_k = 0$ is the simplest choice which matches the requirement of Eq.\tag{57}. Initial perturbations in $\varphi$ are damped during the radiation dominated era, so the choice of the initial condition of $\mathcal{V}_k$ have little impact on CMB anisotropy in the adiabatic perturbation case.

To realize the background evolution described above, we write a separate module. The function of this module is that, for a given value of $\varphi$ today which is determined by Brans-Dicke parameter $\omega$, we first find out the initial value of $\varphi$ at sufficiently early time which can evolve the given value of $\varphi$ today, and then we could calculate $\varphi$ and $\varphi'$ at each scale factor $S$ and store them into arrays for interpolation in subsequent process. Therefore, if one want to use $\varphi$ and $\varphi'$ in the code, just simply use this module first.

To be consistent with modified Friedmann equation \tag{7}, in the code we define the critical density as

\[
\rho_{cr} = \frac{3\varphi_0}{\kappa} H_0^2, \tag{103}
\]

where $H_0$ is the hubble parameter today and $\varphi_0$ is given in Eq.\tag{11}. This definition differs from the conventional one by an additional factor $\varphi$. The definition of the fractional density is the same as the traditional one: $\Omega_V^{(i)} = \rho_0^{(i)}/\rho_{cr}$. Because $\varphi'$ approximately vanishes today (c.f. Fig.\tag{5}), from Eq.\tag{7}, we find $\Omega_{total} \simeq 1$ for the flat geometry. This definition is convenient in studying the non-flat universe. We also should note that the difference with the traditional one is very small, in most case, less than 1%, because $\varphi_0 = 1.001$ when $\omega = 50$.

In this work we adopt the cosmological constant as dark energy, this is equivalent to set the potential of the Brans-Dicke field to a constant. The more general case of dark energy, this is equivalent to set the potential of the Brans-Dicke field to a constant. The more general case of dark energy, this is equivalent to set the potential of the Brans-Dicke field to a constant.

For the the Boltzmann-Einstein differential equations, we modified the scale factor evolution equation and $\eta_k$
propagation equation according to the Eqs. (7) and (8) respectively in functions _fderins_ and _fderivst_ in equations.90. Some other complementary equations, such as the constraint equations, have also been modified correspondingly.

To speed up the calculation, the CAMB code integrates the system of differential equations by using the line-of-sight integration method, first developed by Seljak and Zaldarriaga for the CMBFAST code. In this method, the multipole moment of photon intensity \( I_{k}^{(l)} \) could be express as:

\[
I_{k}^{(l)} = 4 \int_{0}^{\eta_0} d\eta \, S e^{-\tau} \left\{ \left( k \sigma_k + \frac{1}{4} n_e \sigma_T \kappa_2^{-1} \frac{3}{4} I_{k}^{(2)} \right) + \frac{9}{2} \xi_k^{(2)} \right\} \times \left[ \frac{1}{3} \Phi_k^{(l)}(x) + \frac{1}{k^2 r^2} \frac{d^2}{dx^2} \Phi_k^{(l)}(x) \right] + \sigma_T v_k \frac{1}{k r} \frac{d}{dx} \Phi_k^{(l)}(x)
\]

\[
- \left[ \frac{1}{3} S \xi_k^{(0)} - \frac{1}{4} n_e \sigma_T I_{k}^{(0)} \right] \Phi_k^{(l)}(x) \right\}
\]

where \( \eta_0 \) is the conformal time today, \( x = (\eta_0 - \eta) / r \), \( r = 1 / \sqrt{K} \), and \( \tau \) is the zero-order optical depth back to \( x \). Here,

\[
\Phi_k^{(l)}(x) = \frac{l!}{(l - \nu)!} \frac{j_l(x)}{x^\nu},
\]

are the ultra-spherical Bessel functions, \( \kappa_2 = (1 - 3K/k^2)^{1/2} \), and \( \xi_k^{(2)} \) is the quadrupole of the E-like polarization of the CMB photons. After integration by parts, one could eliminate the derivatives of ultra-spherical Bessel functions and write temperature anisotropies as a time integral over a geometrical term \( \Phi_k^{(l)}(x) \) and a source term:

\[
I_{k}^{(l)} = 4 \int_{0}^{\eta_0} d\eta \, \Phi_k^{(l)}(x) \times S
\]

where the source term is given by

\[
S = \frac{1}{12k^2 \kappa_2} \left[ 12 k \sigma_k g'(\eta) \kappa_2 + 24 k \sigma_k g(\eta) \kappa_2 + 12 k \sigma_k g'(\eta) \kappa_2 + 3 g'(\eta) \kappa_2 + 6 g'(\eta) \kappa_2 + 3 g(\eta) \kappa_2' + 12 k \kappa_2 g'(\eta) \nu_k + 12 k \kappa_2 g(\eta) \nu_k + 4 k^3 \sigma_k e^{-\tau} \kappa_2 + k^2 g(\eta) \kappa_2 - 4 k^2 e^{-\tau} \xi_k^{(0)} + 3 k^2 g(\eta) \kappa_2 \right] ,
\]

in which

\[
\kappa_k = \frac{3}{4} I_{k}^{(2)} + \frac{9}{2} \xi_k^{(2)} ,
\]

\[
g(\eta) = -\tau' e^{-\tau} = n_e \sigma_T S e^{-\tau} ,
\]

and \( \xi_k^{(0)} \) is the visibility function. Using the first order derivative perturbation equations described in Section \( \sigma_k' \) and \( \sigma_k'' \) in the source terms could be further expanded to the zeroth and first order derivative terms which are expressed in variables used in the output subroutine in the CAMB code.

V. RESULTS

In Fig. we show the time evolution of the Brans-Dicke field \( \varphi \). For models with \( \omega > 0 \), the value of \( \varphi \) increases with time, whereas for models with \( \omega < 0 \), \( \varphi \) decreases with time. During the radiation dominated era, the variation of \( \varphi \) is very small, almost zero. When entering the matter dominated epoch, \( \varphi \) begins to increase or decrease. After dark energy begin to dominate, \( \varphi \) changes more rapidly. The time evolution of \( \varphi' \) is plotted in Fig. \( |\varphi'| \) reaches a terminal velocity in the radiation dominated era, and then begin to decay in the matter dominated epoch. In the dark energy dominated age, it will increase again, and its present day value for this model is of the order \( 10^{-6} \).
The effective Newtonian gravitational coupling $G_{\text{eff}}$ is the inverse of $\varphi$ in the unit of $G$. The time evolutions of $G_{\text{eff}}$ are shown in Fig. 3. We can see that $G_{\text{eff}}$ changes rapidly at low redshift, so it may not be reliable to use the Type Ia supernovae (SNe Ia) data to constrain the Brans-Dicke theory: the Chandrasekhar mass $M_{\text{Ch}} \propto G^{-3/2}$, so the variation of the gravitational coupling $G$ means that the peak luminosity of SNe, which is approximately proportional to the Chandrasekhar mass, may also change, making it not reliable as a standard candle.

The CMB temperature and polarization angular power spectra for the Brans-Dicke theories with $\omega = \infty$ (i.e. general relativity) and $\pm 100$ are plotted in Fig. 4. As can be seen, compared with the general relativity theory with the same cosmological parameters, both the location and height of the CMB acoustic peaks are changed. The Brans-Dicke model with a positive $\omega$ has broader and lower acoustic peaks for this set of parameters. As $|\omega|$ increases, the difference in CMB angular spectra between Brans-Dicke theory and general relativity diminishes. The difference is more apparent at large $l$ (small angular scale), so high resolution CMB data would be very useful in distinguishing the different models. From Fig. 4 it is also very clear that the polarization spectra have a strong discriminating power. With the higher angular resolution and polarization data which we expect in the nearby future, we should be able to lift the the degeneracy of parameters and place a more stringent constraint on the Brans-Dicke models.

We compared the CMB angular power spectra results with those obtained with our CMBFAST code in the synchronous gauge. We find that the difference in CMB power spectra is typically less than 1 percent, and is due primarily to the difference in the original (Einstein gravity) codes–for really making highly precise constraint on cosmological parameters with the CMB data, the precision of the CMB Boltzmann needs to be further improved. The new code of course has better program architecture and runs faster. Particularly, if one calculate $\partial C_l / \partial \omega$, which reflects the impact of the gravity model on the CMB angular power spectrum, the results of the two code agree with each other at high precision, as shown in Fig. 5. The result on $\Delta C_l = C_l(\omega = \infty) - C_l(\omega = 100)$ for TT, TE and EE correlations are very consistent in two codes, and the two curves are almost indiscernible in Fig. 5.

We also plot the CMB temperature and polarization spectra yielded by tensor modes in Fig. 6. The tensor-to-scalar ratio is set to 0.1. The primordial gravitational
FIG. 6: CMB temperature and polarization power spectra for the Brans-Dicke theories in tensor mode. The solid, dotted and dashed curves represent the Brans-Dicke model with $\omega = \infty$, 100 and $-100$ respectively. The tensor-to-scalar ratio $R$ is set to 0.1

wave produces large temperature fluctuations at the large scales, as well as a unique B mode polarization. In contrast to scalar modes, when compared with the result of general relativity, the height of the peaks are higher for positive $\omega$. Similar to the scalar mode, positive $\omega$ shifts the peaks to smaller scale. At both the very large scales and very small scales, the differences in spectra between the Brans-Dicke theory and the general relativity are very small, almost invisible, and the differences are only sensitive at $l \sim 80$.

Fig.7 shows the impact of Brans-Dicke field on the matter power spectra at $z=0$. For $\omega = 100$, the bend of the matter power spectrum occurs at short wavelengths, and there is thus more small-scale power, in agreement with the prediction of Ref. [65].

VI. CONCLUSION AND SUMMARY

Compared with Einstein’s general relativity, there is an additional scalar field coupled with the Ricci scalar in the Brans-Dicke gravity, which makes the perturbation theory more complicated. With a covariant 1+3 approach, we have developed a full set of covariant and gauge-invariant formalism for calculating the cosmic microwave background temperature and polarization anisotropies in the Brans-Dicke gravity. Instead of using the components of metric as basic variables, the covariant formalism performs a 1+3 split of the Bianchi and Ricci identities, using the kinematic quantities, energy-momentum tensors of the fluid(s) and the gravito-electromagnetic parts of the Weyl tensor to study how perturbations evolve. Adopting covariantly defined, gauge-invariant variables throughout ensures that in our discussion the gauge ambiguities is avoided, and all variables had a clear, physical interpretation. Since the definition of the covariant variables does not assume any linearization, exact equations can be found for their evolution, which can then be linearized around the chosen background model. Furthermore, unified treatment of scalar, vector and tensor modes do not require decomposing the different modes from beginning as done in the metric method. A price we have to pay is that with this method the calculation is more complicated.

We then calculate the CMB temperature and polarization spectra for the Brans-Dicke models using a modified CAMB code. In this paper we consider both the scalar modes and the tensor modes in adiabatic initial condition, and adopt $\varphi_0 = (2\omega + 4)/(2\omega + 3)$ at the current epoch and $\varphi' = 0$ at early time as initial condition of the Brans-Dicke field. Compared with the general-relativistic model with the same cosmological parameters, both the amplitude and the width of the acoustic peaks are different in the Brans-Dicke models. We find that the small scale spectra and the polarization spectra will provide a sensitive and vigorous constraint on the different Brans-Dicke models in scalar mode. For tensor modes, the largest difference in CMB spectra for various Brans-Dicke models are located at $l \sim 80$. The structure formation process in the Brans-Dicke theory is also studied. The matter power spectrum is shown in Fig.7.

For positive $\omega$ case, the bend of the matter power spectra occurs at shorter wavelengths, and there is thus more small-scale power compared with the General Relativity case.

Our numerical results confirmed the results obtained with particular guages (e.g. the synchronous gauge[15]). We also obtained for the first time the temperature and polarization spectra for tensor mode perturbations in the
Brans-Dicke theory. In the present code the speed and program structure are greatly improved, thus providing a more powerful and convenient tool for further studies. In paper II, we use the code and MCMC algorithm to derive the constraint on the Brans-Dicke parameter \( \omega \) with the latest CMB and LSS observational data.

As the final remark, the covariant approach and corresponding CMB code for the Brans-Dicke theory developed in this paper, together with the synchronous gauge approach and corresponding code developed in the previous paper [12], provide consistent, systematic and complete methods to study the Brans-Dicke theory. These methods and codes could be generalized to study more general scalar-tensor theory, as well as more complex initial condition, we plan to carry out such generalization in subsequent studies.

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