Breakdown of the Chiral Luttinger Liquid in One-Dimension.

A. F. Ho$^1$ and P. Coleman$^1$

$^1$Serin Laboratory, Rutgers University, P.O. Box 849, Piscataway, New Jersey 08855-0849

Abstract

We have developed a fermionic boot-strap method to solve a class of chiral one dimensional fermion model which cannot be tackled by bosonization. Using this scheme, we show that Luttinger liquid behavior in a gas of four interacting chiral Majorana fermions is highly sensitive to the velocity degeneracy. Upon changing the velocity of a one chiral fermion, a sharp bound (or anti-bound) state splits off from the original Luttinger liquid continuum, cutting off the X-ray singularity to form a broad incoherent excitation with a lifetime that grows linearly with frequency.
I. INTRODUCTION

The anomalous normal state behavior discovered in the cuprate superconductors has stimulated enormous interest in the possibility of new kinds of electronic fluid that might provide an alternative to Fermi liquid behavior. The classic model for non-Fermi liquid behavior is provided by the one-dimensional electron gas, where the generic fixed point behavior is a Luttinger Liquid. Luttinger liquid behavior in one dimension is associated with two special features: a the Fermi surface which consists of just two points $\pm k_f$ where the electrons interact very strongly and second, the special kinematics of one dimension, whereby energy and momentum conservation impose a single constraint on scattering processes near the Fermi surface, which gives rise to a qualitative enhancement in scattering phase space. The combination of these two factors cause the electron to lose its eigen-state status to the collective spin and charge density bosonic modes.

In this paper, we introduce a generalization of the Tomonaga Luttinger model, written

\[ H = \int dx \left\{ -i \sum_{a=0}^{3} v_a \Psi^{(a)}(x) \partial_x \Psi^{(a)}(x) + g \Psi^{(0)}(x)\Psi^{(1)}(x)\Psi^{(2)}(x)\Psi^{(3)}(x) \right\}, \]

where the $\Psi^{(a)}$ ($a = (0, 1, 2, 3)$) represent four real (Majorana) fermions such that $\Psi^{(a)}(x) = \Psi^{(a\dagger)}(x)$. The fermions are chiral (right-movers,say): this is a crucial property that ensures the system stays gapless, and allows for exact solutions in a number of cases. In the special case where the velocities of each of these modes are degenerate, this model has an $O(4)$ symmetry, and corresponds to the chiral Luttinger liquid. In this case, the four Majorana modes can loosely be associated with the spin up and spin down electron and hole excitations of the Fermi surface.

We shall show that once we break the degeneracy of the velocities by changing the velocity of a single Majorana model, a qualitatively new type of behavior develops. The model where only three of the velocities are equal, so that $v_{1,2,3} = v \neq v_0$, has an $O(3)$ symmetry. This model is actually motivated from two rather disparate sources:

- It is inspired in part by the transport phenomenology of the cuprates, which suggests...
FIG. 1. Schematic diagram showing the evolution of the spectral weight as we introduce velocity difference to the fermions. Inset indicates the bare spectral function, without interactions.

that electrons near the Fermi surface might divide up into two Majorana modes with different scattering rates and dispersion. To date, this kind of behavior has only been realized in impurity models, and their infinite dimensional generalization. We shall show that by breaking the velocity degeneracy of the original chiral Luttinger model, we obtain a one-dimensional realization of this behavior, where a sharp Majorana mode intimately co-exists with an incoherent continuum of excitations, reminiscent of the higher dimensional phenomenology.

- Frahm et. al. have recently proposed that the low energy Hamiltonian of an integrable spin-1 Heisenberg chain doped with mobile spin-1/2 holes is given by Eqn.1, with one Majorana fermion $\Psi^{(0)}$ describing a slow moving excitation coming from the dopant, interacting with three rapidly-moving Majorana fermions that describe the spin-1 excitations of the spin-chain. Such doped spin-chain models may be relevant to certain experimental systems such as $Y_{2-x}Ca_xBaNiO_5$.

Whereas the O(4) model can be treated by bosonization, changing the velocity of a single Majorana fermion introduces non-linear terms into the bosonized Hamiltonian that preclude a separation in terms of Gaussian spin and charge degrees of freedom. We shall show, using a new fermionic approach to the problem, that when we break the degeneracy
of the velocities, the X-ray catastrophe associated with the Luttinger liquid behavior is cut-off by the velocity shift. The “horn-like” feature in the spectral weight of the Luttinger liquid is then split into a sharp bound (or anti-bound) state that co-exists with an incoherent spin-charge decoupled continuum, with a scattering rate linear in frequency. We summarize these results in the schematic diagram Fig.1.

II. METHOD: PHILOSOPHY

We now develop a fermionic scheme for solving both the $O(4)$ and the $O(3)$ versions of the chiral Majorana model. Our method is based on the observation that for these models, the skeleton self-energy, without vertex corrections, is exact, so that

$$\Sigma_a(x, \tau) = g^2 G_b(x, \tau) G_c(x, \tau) G_d(x, \tau),$$

where the $G_a$ are the Greens functions of the interacting system and $\{a, b, c, d\}$ is a cyclic permutation of $\{0, 1, 2, 3\}$. We can represent this result diagrammatically in Fig.2. These equations close with the usual relations:

$$\Sigma_a(k, \omega) = (i\omega - v_a k) - G_a(k, \omega)^{-1}. \quad (a = 0, 1, 2, 3)$$

These coupled equations (3,4) together define a boot-strap method to solve the problem. Our method bears marked similarity to the Non-Crossing Approximation used in solving various magnetic impurity models, but in this case case, no approximation is involved.

Provided that we have a minimal $O(3)$ symmetry, then the three current densities $j^a(x) = -i\epsilon_{abc} \Psi^{(b)}(x) \Psi^{(c)}(x) \ (a, b \in (1, 2, 3))$ are conserved. Following Dzyaloshinskii and Larkin
since charge and current are the same in a chiral model, the continuity equation guarantees that the N-point connected current-current correlation functions vanish for $N > 2$:

$$\langle j^a(1)j^a(2)\ldots j^a(N) \rangle_C = 0, \quad (N > 2) \quad (4)$$

For the non-interacting system, this result leads to the “loop cancellation theorem”, which means that if we take the amplitude associated with a closed fermion loop with $N > 2$ conserved current insertions, then the sum over all possible permutations $P$ of the space-time indices of the current operators must give zero. Dzyaloshinskii and Larkin used this fact to eliminate all diagrams that contain such closed loops, considerably simplifying the calculation of the vertex function and polarization bubbles.

We use the loop cancelation theorem in a new way, to show that the vertex corrections to the skeleton self-energy identically vanish. To illustrate the idea, consider the self-energy of the singlet Majorana mode in the O(3) model. The NC contributions to its self-energy are entirely constructed by combining loops with two current insertions, as shown in Fig 3(iv). These contributions are finite. Non-skeleton diagrams involve a sum of loops with more than two current insertions. For example, the fourth order diagram, as shown in Fig. 3, involves a loop with four current insertions, and these sum to give zero, leaving behind the single non-crossing contribution. This result can be generalized to all higher order graphs, showing that the self-energy $\Sigma_0$ of the singlet Majorana fermion is given by the skeleton self-energy diagram without vertex corrections. From this result, we can show that the full Kadanoff-Baym Free energy $F[G]$. The condition that $\delta F[G]/\delta G = 0$ generates the equations of motion for the self-energies. But in order that this generate the skeleton self-energy diagram for $\Sigma_0$, the Kadanoff-Baym Free energy Functional must truncate at the leading skeleton diagrams.

$$F = -k_B T \{ \text{Tr} \ln [G^{-1}] - \text{Tr} [\Sigma G] \} + \text{(5)}$$

By differentiating the Free-energy functional with respect to the Greens functions $G_{1,2,3}$ of
FIG. 3. Order $g^4$ diagrams for the $O(3)$ model, demonstrating how the closed-loop theorem works. Dotted lines indicate bare propagator for the singlet Majorana fermion $\Psi^{(0)}$. Full lines indicate bare propagator for the triplet Majorana fermions. Bold line highlights the loop with 2 or 4 current insertions.

the triplet Majorana fermions, we can then show that each triplet self-energies are also given by the non-crossing skeleton expression.

III. METHOD-DETAILS

We now apply the Non-Crossing approach (NCA) to the $O(3)$, using the limiting case of the $O(4)$ model to check the validity of our results. Our equations are dramatically simplified by seeking solutions to the NCA equations which satisfy a scaling form

$$G_a(x, \tau) = \frac{1}{2\pi i x} G_a(\tau/ix), \quad (6)$$

This form is motivated by the observation that chirality prevents space from acquiring an anomalous dimension. Under a Fourier Transform, this scaling function is self dual,

$$\frac{1}{2\pi i x} G_a(\tau/ix) \xrightarrow{F.T.} \frac{1}{i\omega} G_a(k/i\omega), \quad (7)$$

where the same function $G_a$ appears on both sides of the Fourier transform pair. Inserting the scaling form into Eqn. (4) and Fourier transforming the resulting expression, we obtain
\[ \Sigma_a(x, \tau) = -\frac{1}{2\pi i x} \frac{d^2}{du^2} [1 - v_a u - 1/G_a(u)]_{u=i\tau/x}, \]  
\quad \text{(8)}

Since the scaling form for the bare Green function is \(1/G^0_a(u) = 1 - v_a u\), it does not contribute to the self-energy. Taking equation (3) together with equation (8), we obtain the following differential equations:

\[ \frac{d^2}{du^2} [G_a(u)]^{-1} = -(g/2\pi)^2 G_b(u) G_c(u) G_d(u) \]  
\quad \text{(9)}

where \(\{a, b, c, d\}\) are cyclic permutations of \(\{0, 1, 2, 3\}\). The boundary conditions are:

\[ G_a(0) = 1, \]  
\quad \text{(10)}

\[ G'_a(0) = v_a. \]  
\quad \text{(11)}

derived from the physical requirement that at high frequencies, the fermions are free particles, moving with the bare velocity \(v_a\). For the \(O(4)\) model, where \(G_a(u) \equiv G(u) \ (a = 0, 3)\), eqs. (9-11) reduce to a single differential equation,

\[ \frac{d^2}{du^2} [G(u)]^{-1} = -(g/2\pi)^2 [G(u)]^3 \]  
\quad \text{(12)}

for which the solution is

\[ G(x, \tau) = \frac{1}{2\pi i x} [1 - v_+/i\tau]^{-1/2} [1 - v_- i\tau]^{-1/2} \]  
\quad \text{(13)}

where \(v_\pm = v \pm (g/2\pi)\) and \(v\) is the bare velocity. Identical results can be obtained by bosonization. This provides an important consistency check on our results, confirming that the non-crossing self-energy is exact for the Tomonaga Luttinger model.

\textbf{IV. RESULTS FOR } O(3) \text{ MODEL.}

As is well known, in the \(O(4)\) model, the spectral weight has two square-root singularity at the renormalized velocities \(v_+, v_-\) (x-ray edge catastrophe), with a continuum in between (Fig.4).\v

We now show that if \(\Delta v = v - v_0\) is finite, the X-ray edge catastrophe is partially eliminated. If \(v_0 < v\), we find that low velocity “Horn”, originally with velocity \(v_-\), develops
a sharp bound-state pole in the singlet channel, and a broad incoherent excitation in the triplet channel with a lifetime growing linearly in energy. If \( v_0 > v \), the high velocity “horn” splits off a singlet anti-bound-state and the triplet channel develops a high-velocity incoherent excitation. (Fig.5). The sharp bound-state in the singlet channel develops once a velocity difference is introduced, because energy and momentum conservation now provide distinct constraints to scattering (unlike in the \( O(4) \) model), leading to much less phase space for \( \Psi^{(0)} \) to decay into.

To see this, we must analyze the the more general coupled differential equations for the \( O(3) \) case, which take the form

\[
\frac{d^2}{du^2} G^{-1}_3 = -(g/2\pi)^2 (G_3)^2 G_0, \\
\frac{d^2}{du^2} G^{-1}_0 = -(g/2\pi)^2 (G_3)^3.
\] (14)

A very convenient way to discuss these equations is to map them onto a central force problem of a fictitious particle. If we write \( \vec{r} = (G_3^{-1}, G_0^{-1}) \), \( \vec{F} = -(gG_3/2\pi)^2(G_0, G_3) \), we notice that these equations can be cast in the form \( \vec{F} = \vec{F} \). By inspection, \( \vec{r} \times \vec{F} = 0 \), so the force is radial and we immediately deduce that the “angular momentum”, \( \vec{r} \times \vec{r} = -\Delta v \) is a constant. If we use polar co-ordinates, \( (G_3^{-1}, G_0^{-1}) = r(\cos \theta, \sin \theta) \) the equations for the Green-function resemble the motion of a Fictitious particle under the influence of an anisotropic central force:

\[
\ddot{r} - \frac{\Delta v^2}{r^3} = -(g/2\pi)^2 \frac{1}{r^3 \cos^3 \theta \sin \theta} \\
r^2 \dot{\theta} = -\Delta v,
\] (15)

with boundary condition \( r(0) = \sqrt{2}, \theta(0) = \pi/4 \). In these equations the velocity difference \( \Delta v = v - v_0 \) provides a the repulsive centrifugal force. The “particle” starts out at \( r_0 = (1, 1) \). So long as the “particle” falls directly into the origin, both \( G_3 \) and \( G_0 \) diverge with X-ray singularities. However, once \( \Delta v \) is finite, the orbit no longer passes through the origin, thereby eliminating the associated X-ray singularity in the spectral function. The quantity \( \zeta = (v - v_0)/g \) plays the role of a coupling constant, and approximate analytic solutions are possible in the limiting cases of small and large \( \zeta \).
Suppose first, that $\Delta v > 0$. In this case, $\theta \to 0$ at some finite “time” $u = \tau_0$, at which $r = C$ and $\dot{\theta} = -\Delta v/C^2$. For $u \sim u_0$, it follows that $(r, \theta) = (C, C\theta(u - u_0))$, from which we can read off the following asymptotic behavior

$$G_0(u)^{-1} \sim C,$$

$$G_0(u)^{-1} \sim \frac{\Delta v\tau_0}{C}(1 - u\tau_0^{-1}),$$

so that

$$G_0 \sim Z/(1 - v_0^* u)$$

where

$$Z = C/\Delta v\tau_0, \quad v_0^* = \tau_0^{-1}. \quad (19)$$

Thus when one Majorana fermion move faster than the others, an anti-bound state with spectral weight $Z$, moving with velocity $v_0^*$, splits off above the continuum. For $\Delta v >> \frac{g}{2\pi}$ interactions can be ignored, so $v_0^* \to v_0$, and $Z \to 1^-$. For $\Delta v << \frac{g}{2\pi}$, the “motion” of the fictitious particle emulates that of the $O(4)$ model until the angle $\theta$ approaches zero. We may estimate $\tau_0$ and $C$ by setting

$$\pi/4 = \int_0^{\tau_0} \frac{\Delta v}{\tilde{r}^2(u)} du, \quad C \approx \tilde{r}(\tau_0)$$

where $\tilde{r} = [2(1 - v_+ u)(1 - v_- u)]^{1/2}$ is the solution to the $O(4)$ case. This estimate gives

$$v_0^* = v_+ + \frac{g}{\pi} e^{-\frac{g^2}{2\Delta v}},$$

$$Z = \left(\frac{\sqrt{2}g}{\pi\Delta v}\right) e^{-\frac{g^2}{2\Delta v}} \quad (21)$$

indicating that the formation of the sharp anti-bound-state is non-perturbative in the velocity difference. These results can be generalized to the case where $\Delta v < 0$ by replacing $v_+ \to v_-, g \to -g$.

We have carried out numerical solutions of the differential equations (14) for intermediate values of the coupling constant $\zeta$. The equations were integrated from the initial boundary conditions using a standard adaptive integration routine. These results are summarized in Fig. 5. and Fig.6.
FIG. 4. Spectral weight of O(4) model

FIG. 5. Spectral weight of the O(3) model. For clarity, we have shifted up the curves for various momenta by 8 units.
FIG. 6. Quasi-particle weight $z$ of $\Psi^{(0)}$ in the $O(3)$ model.

V. DISCUSSION AND CONCLUSION

In summary, we have demonstrated that in a system of interacting chiral fermions with different velocities, due to restriction of the scattering phase space, the single Majorana fermion with extremal velocity splits off from the Luttinger continuum to form a sharp bound state or anti-bound state, leading to a system with two qualitatively distinct spectral peaks and scattering rates. This is a significant departure from the Luttinger liquid scenario and demonstrating a new class of of one-dimensional fixed point behavior.

In some sense, this new fixed point lies mid-way between the Luttinger and the Fermi liquid, and is closest in character to the Marginal Fermi liquid phenomenology originally introduced in the context of cuprate metals. Unlike the Luttinger liquid, here we have a sharp quasiparticle bound-state co-existing with an incoherent continuum. Like the Luttinger liquid however, there are two velocities which define the limits of the broad continuum of excitations. The qualitative picture of a sharp split-off peaks coexisting with broad features in the spectral weights is a consequence of the a phase space restriction relative to the Luttinger liquid.

Had we chosen to change two Majorana velocities at the same time, so that $v_0 = v_1$ and
$v_2 = v_3$, we would have reduced the symmetry still further, to an $O(2) \times O(2)$ symmetry. In this case, it is possible to bosonize the Hamiltonian, obtaining the results

$$G_3(x, \tau) = \frac{1}{2\pi i x} \left[ 1 - v_+ \tau / ix \right]^{-1/2+\gamma} \left[ 1 - v_- \tau / ix \right]^{-1/2-\gamma}$$

(22)

and $v_\pm = [v_0 + v_3 \pm \sqrt{(v_3 - v_0)^2 + (g/\pi)^2}] / 2$, $\gamma = (v_3 - v_0) / (2\sqrt{(v_3 - v_0)^2 + (g/\pi)^2})$. Curiously, this result may also be obtained by solving equations (9), even though, as far as we can see, the closed-loop cancellation is not sufficient in the case of the $O(2) \times O(2)$ model to cancel all vertex corrections. This suggests that a more general cancellation principle may be at work, and that the range of validity of our solution may extend beyond the $O(3)$ down to models with just $O(2)$ symmetry. At present we have not been able to prove this result.

Our work raises the question whether this kind of non-Fermi Liquid behavior might survive in dimensions higher than one. In higher dimensions energy conservation and momentum conservation are distinct constraints on scattering phase space, and for this reason the Fermi surface reverts from a Luttinger to a Fermi liquid in higher dimensions. By contrast, the $O(3)$ model can not be bosonized, and its unusual properties do not rely on the coincidence between momentum and energy conservation. For this reason, there is at least a grain of hope that this kind of behavior might be more robust in higher dimensions. We do in fact know that near infinite dimensions, two lifetimes behavior persists in the $O(3)$ model, but here, the thermodynamics near zero temperature is that of a Fermi Liquid. The case of small, but finite dimensions is however, still open.

Acknowledgment We should like to thank Natan Andrei, Thierry Giamarchi, Alexei Tsvelik and particularly Walter Metzner for discussions related to this work. This work was supported by NSF grant NSF DMR 96-14999.
REFERENCES

1 F.D.M. Haldane, J. Phys. C 14, 2585, (1981)

2 P. W. Anderson, Phys. Rev. Lett. 67, 2092 (1991), P. Coleman, A. J. Schofield, and A. M. Tsvelik, J. Phys. (Cond. Matt.) 8, 9985 (1996).

3 V. J. Emery and S. Kivelson, Phys. Rev. B46, 10812 (1992), P. Coleman, L. Ioffe and A. M. Tsvelik, Phys. Rev. B52, 6611 (1995).

4 A.F. Ho and P. Coleman, Phys. Rev. B 58, 4418 (1998)

5 H. Frahm, M.P. Pfannmüller and A.M. Tsevelik, Phys. Rev. Lett. 81, 2116 (1998)

6 J.F. di Tusa et.al., Phys. Rev. Lett. 73, 1857 (1994)

7 J. Voit, Phys. Rev.B 47, 6740 (1993)

8 N.E. Bickers, Rev. Mod. Phys. 59, 845 (1987)

9 I.E. Dzyaloshinskii and A.I. Larkin, Sov. Phys. JETP, 38, 202 (1974).

10 W. Metzner, C. Castellani and C. di Castro, Adv. Phys. 47 (1998).

11 P. Kopietz, J. Hermisson and K. Schönhammer, Phys. Rev. B 52, 10877 (1995).

12 C. M. Varma, P. B. Littlewood, S. Schmitt-Rink, E. Abrahams and A. E. Ruckenstein, Phys. Rev. Lett. 63, 1996, (1989).