TINGLEY’S PROBLEMS ON UNIFORM ALGEBRAS

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ABSTRACT. We prove that a surjective isometry between the unit spheres of two uniform algebras is extended to a surjective real-linear isometry between the uniform algebras. It provides the first positive solution for Tingley’s problem on a Banach space of analytic functions.

1. INTRODUCTION

Tingley’s problem asks the extensibility of a surjective isometry between the unit spheres of two Banach spaces. In 1987, Tingley [66] proposed the problem. Since then a lot of work has been done trying to solve this. Despite its simple statement, Tingley’s problem is a hard problem which remains unsolved for many Banach spaces. No counterexample is known. A Banach space \( E \) satisfies the Mazur-Ulam property if a surjective isometry between the unit spheres of \( E \) and any Banach space is extensible to a surjective isometry between \( E \) and the given Banach space (cf. [4]). Due to [75, p.730] Ding was the first to consider Tingley’s problem between different type spaces [9].

Wang [68] seems to be the first to solve Tingley’s problem between specific spaces. He dealt with \( C_0(\Omega) \), the space of all complex-valued continuous functions on a locally compact Hausdorff space \( \Omega \) which vanish at infinity. A considerable number of interesting results have shown that Tingley’s problem has a positive solution in \( \ell^p \) spaces, \( L^p \) spaces, \( C_0(\Omega) \) spaces, certain finite dimensional Banach spaces, and so on [7, 8, 10, 11, 13, 15, 16, 26, 27, 28, 29, 30, 34, 53, 54, 56, 58, 59, 62, 67, 68, 69, 70, 71, 72, 74, 76].

Results on the classical Banach space satisfying the Mazur-Ulam property is a little bit reduced. Including the space of real null sequences, the space of all bounded real-valued functions on a discrete set, the space of all real-valued continuous functions on a compact Hausdorff space, the space of all \( p \)-integrable or essentially bounded real-valued functions on a \( \sigma \)-finite measure space for \( 1 \leq p < \infty \) are studied [1, 3, 4, 5, 9, 12, 14, 17, 18, 32].

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In [60] Tan, Huang and Liu introduced the notion of generalized lush spaces and local GL spaces and proved that every local GL space satisfies the Mazur-Ulam property. The main result in [33] probably provides the first example of a complex Banach space satisfying the Mazur-Ulam property (cf. [49]).

Besides the large list of classical Banach spaces giving positive solutions, a series of papers by Tanaka [63, 64, 65] on the algebra of complex matrices, a finite dimensional $C^*$-algebra and a finite von Neumann algebra opened up another direction of study on Tingley’s problem. After [5, 19, 20, 21, 22, 24, 35, 44, 51] new achievements by Mori and Ozawa [45] and Fernández-Polo and Peralta [23] prove that the Mazur-Ulam property is satisfied by unital $C^*$-algebras [45], and weakly compact JB*-triples [23].

Nagy proved a variant of Tingley’s problem in [47, Theorem] and conjectured that the similar conclusion holds for every complex Hilbert space. Peralta [48, Theorem 6.10] completed a proof of Nagy’s conjecture and proposed a more general problem (cf. [50]). Combining with a problem posed by Mori for noncommutative $L^p$-spaces [44] Problem 6.3 Leung, Ng and Wong [37, Problem 1] extended a problem of Peralta for ordered Banach spaces (see also [36]). They provide a results on surjective isometry between the positive unit sphere of a function space $L^p(\mu)$ or $C(X)$ [37].

The reader is referred to the surveys [13, 48, 75] for details about Tingley’s problem.

In this paper we add a positive solution of Tingley’s problem. We prove that a surjective isometry between the spheres of two uniform algebras is extended to a surjective real-linear isometry. Let $X$ be a compact Hausdorff space and $C(X)$ a Banach algebra of all complex-valued continuous functions on $X$. A closed subalgebra of $C(X)$ which contains constants and separates the points of $X$ is called a uniform algebra on $X$. A uniform algebra is called a function algebra in [2], which is the name of other object in some literatures recently. To avoid a confusion, we use the terminology “uniform algebras”. Typical examples of a uniform algebra consist of analytic functions of one and several complex-variables such as the disk algebra, the polydisk algebra, the ball algebra. The algebra of all bounded analytic functions $H^\infty(D)$ on a certain domain $D$ is considered as a uniform algebra on the maximal ideal space. In fact, Theorem 2.1 provides the first positive solution for Tingley’s problem on a Banach space of analytic functions.

Further information about uniform algebras, see [2].
2. SURJECTIVE ISOMETRIES BETWEEN THE UNIT SPHERES OF
UNIFORM ALGEBRAS

Throughout the paper $A$ and $B$ are uniform algebras on compact Hausdorff spaces $X$ and $Y$ respectively. The norm of a uniform algebra is denoted by $\| \cdot \|$. We denote the unit sphere of $A$ (resp. $B$) by $S(A)$ (resp. $S(B)$). The map $T : S(A) \to S(B)$ is always a surjective isometry in this paper. We do not apply the notation of the Gelfand transform. The Gelfand transform of $f \in A$ is also denoted by $\hat{f}$. We may assume $f \in A$ is also defined on the maximal ideal space $M_A$ of $A$ without confusion.

Our main result is the following.

**Theorem 2.1.** Let $A$ and $B$ be uniform algebras on compact Hausdorff spaces $X$ and $Y$ respectively. Suppose that $T : S(A) \to S(B)$ is a surjective isometry. Then $T$ is extended to a real-linear surjective isometry from $A$ onto $B$. In particular, $|T| = 1$ on $M_B$, and there exists a homeomorphism $\Psi : M_B \to M_A$ and possibly empty disjoint closed and open subsets $M_B^+$ of $M_B$ such that a surjective real-linear isometry $\tilde{T} : A \to B$ defined by

$$\tilde{T} f = T 1 \times \begin{cases} f \circ \Psi & \text{on } M_B^+ \\ f \circ \Psi & \text{on } M_B^- \end{cases}$$

for every $f \in A$ extends $T$; i.e. $\tilde{T}|S(A) = T$.

In the following sections we prepare several lemmata to prove Theorem 2.1. We exhibit a proof of the theorem in section 7.

3. MAXIMAL CONVEX SETS

We upgrade Lemma 3.3 in [63].

**Lemma 3.1.** Let $E$ be a Banach space. Suppose that $F$ is a maximal convex subset of the unit sphere $S(E)$ of $E$. Then there exists an extreme point $\varphi$ of the closed unit ball $B(E^*)$ of $E^*$ such that $\varphi^{-1}(1) \cap S(E) = F$.

**Proof.** By [63] Lemma 3.3] there exists $\chi$ in the unit sphere $S(E^*)$ of the dual space $E^*$ of $E$ such that $\chi^{-1}(1) \cap S(E) = F$. We show that $\chi$ can be chosen so that it is an extreme point of the unit ball $B(E^*)$ of $E^*$. Put $\Phi = \{ \chi \in S(E^*) : \chi^{-1}(1) \cap S(E) = F \}$. It is routine work to show that $\Phi$ is a non-empty closed convex subset of $B(E^*)$. Then the Krein-Milman theorem asserts that there is an extreme point $\varphi$ in $\Phi$. We prove that $\varphi$ is also an extreme point in $B(E^*)$. Suppose that $\varphi = (\varphi_1 + \varphi_2)/2$ for $\varphi_1, \varphi_2 \in B(E^*)$. Then

$$1 = \varphi(f) = (\varphi_1(f) + \varphi_2(f))/2$$
and

$$|\varphi_1(f)| \leq 1, |\varphi_2(f)| \leq 1$$

assert that $\varphi_1(f) = \varphi_2(f) = 1$ for every $f \in F$. Hence we have $\varphi_1, \varphi_2 \in \Phi$. As $\varphi$ is an extreme point of $\Phi$, we see that $\varphi_1 = \varphi_2$, so that $\varphi$ is an extreme point of $B(E^*)$ and $\varphi^{-1}(1) \cap S(E) = F$. \hfill $\square$

We show that a maximal convex subset of $S(A)$ for a uniform algebra $A$ has a specific form. We say that a non-empty closed subset $K$ of $Y$ is a peak set for $B$ if $K = f^{-1}(1)$ for a function $f \in B$ such that $\|f\| = 1$ and $|f|^{-1}(1) = f^{-1}(1)$. Such a function $f$ is called a peaking function for $K$. An intersection of peak sets is called a weak peak set (peak set in the weak sense in [2]). When a peak set (resp. weak peak set) is a singleton, the unique element in $K$ is called a peak point (resp. weak peak point). The Choquet boundary of $A$ (resp. $B$) is denoted by $\text{Ch}(A)$ (resp. $\text{Ch}(B)$). It is known that $\text{Ch}(A)$ (resp. $\text{Ch}(B)$) consists of the weak peak points and $\text{Ch}(A)$ (resp. $\text{Ch}(B)$) is a boundary for $A$ (resp. $B$). (cf. [2] Theorems 2.2.9, 2.3.4)

For a uniform algebra every maximal convex subset of $S(A)$ (resp. $S(B)$) is represented as $\{f \in S(A) : f(x) = \lambda\}$ (resp. $\{f \in S(B) : f(x) = \bar{\lambda}\}$) for $x \in \text{Ch}(A)$ (resp. $\text{Ch}(B)$) and a unimodular complex number $\lambda$.

**Lemma 3.2.** Let $A$ be a uniform algebra on $X$. Suppose that $F$ is a maximal convex subset of $S(A)$. Then there exists a Choquet boundary point $x \in \text{Ch}(A)$ and a unimodular complex number $\lambda$ such that $F = \{f \in S(A) : f(x) = \lambda\}$. Suppose conversely that $x \in \text{Ch}(A)$ and $\lambda$ is a unimodular complex number. Then $\{f \in S(A) : f(x) = \lambda\}$ is a maximal convex subset of $S(A)$.

**Proof.** Suppose that $F$ is a maximal convex subset of $S(A)$. By Lemma 3.1 there exists an extreme point $\varphi$ in $B(A^*)$ with $F = \varphi^{-1}(1) \cap S(A)$. It is well known that the extreme point in $B(A^*)$ is of the form $\gamma \delta_x$ for $x \in \text{Ch}(A)$ and a unimodular constant $\gamma$, where $\delta_x$ denotes the point evaluation at $x$ (cf. [25] Corollary 2.3.6 and Definition 2.3.7)). It follows that $F = \{f \in S(A) : f(x) = \lambda\}$, where $\lambda = \bar{\gamma}$.

Conversely suppose that $x \in \text{Ch}(A)$ and $\lambda$ is a unimodular constant and $F = \{f \in S(A) : f(x) = \lambda\}$. By a simple calculation we have that $F$ is a convex subset of $S(A)$. We prove that it is maximal. Applying Zorn’s lemma there exists a maximal convex subset $C$ of $S(A)$ with $F \subseteq C$. Then by the first part of the proof, there exist $x_0 \in \text{Ch}(A)$ and a unimodular constant $\lambda_0$ such that $C = \{f \in S(A) : f(x_0) = \lambda_0\}$. Thus we have $\{f \in S(A) : f(x) = \lambda\} \subset \{f \in S(A) : f(x_0) = \lambda_0\}$.

We prove $x_0 = x$. Suppose not. Since a Choquet boundary point is a weak peak point [2, Theorem 2.3.4], there exists $g \in A$ such that $g(x) = 1 = \|g\|$.
such that { \( f \in S(A) : f(x) = \lambda \) } and \( \lambda g \not\in \{ f \in S(A) : f(x) = \lambda_0 \} \), which is a contradiction. Thus we have \( x_0 = x \), and \( \lambda_0 = \lambda \) follows. Hence we have \( C = F \). \( \square \)

In the following we denote the maximal convex subsets of \( S(A) \) (resp. \( S(B) \)) by \( F_{x,\lambda} \) for \( x \in \text{Ch}(A) \) (resp. \( x \in \text{Ch}(B) \)) and a unimodular complex number \( \lambda \).

4. Maps between maximal convex sets

The following may be known for the readers who are familiar with uniform algebras. For the convenience of general readers we exhibit it with a short proof.

**Lemma 4.1.** Suppose that \( y_1, y_2 \in \text{Ch}(B) \) are different points. Then for any pair \( \mu_1, \mu_2 \) of unimodular complex numbers, there exists \( g \in S(B) \) such that \( g(y_1) = \mu_1 \) and \( g(y_2) = \mu_2 \).

**Proof.** First we prove that \( \{ y_1, y_2 \} \) is a weak peak set. Since \( y_1 \) and \( y_2 \) are weak peak points, there exist possibly infinitely many peak sets \( K_1^1 \) and \( K_2^2 \) such that \( \{ y_1 \} = \cap_i K_i^1 \) and \( \{ y_2 \} = \cap_i K_i^2 \). Let \( f_1^1 \) and \( f_2^2 \) be peaking functions for \( K_1^1 \) and \( K_2^2 \) respectively. Let \( (1 - f_1^1)^{\frac{1}{2}} \) and \( (1 - f_2^2)^{\frac{1}{2}} \) be the principal values. By a simple calculation we see that \( \exp(-(1 - f_1^1)^{\frac{1}{2}}(1 - f_2^2)^{\frac{1}{2}}) \) is a peaking function for \( K_1^1 \cup K_2^2 \). As \( \{ y_1, y_2 \} = \cap_i (K_i^1 \cup K_i^2) \) we conclude that \( \{ y_1, y_2 \} \) is a weak peak set.

Since \( B \) separates the points of \( Y \) and contains constant functions, it is straightforward to see that there exists \( h \in B \) with \( h(y_1) = \mu_1 \) and \( h(y_2) = \mu_2 \). If \( \| h \| = 1 \), then \( h \) is the desired function \( g \). Suppose that \( \| h \| > 1 \). Put \( L_0 = \{ y \in Y : |h(y)| \geq 1 + 1/2 \} \). For a positive integer \( n \), put \( L_n = \{ y \in Y : 1 + 1/2^n + 1 \leq |h(y)| \leq 1 + 1/2^n \} \). Then \( \cup_{n=0}^\infty L_n = \{ y \in Y : |h(y)| > 1 \} \). Since \( \{ y_1, y_2 \} \) is a weak peak set and \( Y \setminus L_0 \) is an open neighborhood of \( \{ y_1, y_2 \} \), there exists a peak set \( \{ y_1, y_2 \} \subset K_1^1 \cup K_2^2 \subset Y \setminus L_0 \) and a corresponding peaking function \( u_0 \in B \) for \( K_1^1 \cup K_2^2 \) such that \( |u_0| < 1/\| h \| \) on \( L_0 \). In the same way for each positive integer \( n \), there exists a peaking function \( u_n \in B \) which peaks on some peak set which contains \( \{ y_1, y_2 \} \) such that \( |u_n| < 1/(2^n + 1) \) on \( L_n \). Put \( u = u_0 \sum_{n=1}^\infty \frac{u_n}{2^n} \). Then we see that \( hu \) is the desired function \( g \) as follows. Since \( u = 1 \) on \( \{ y_1, y_2 \} \), we have \( hu(y_1) = \mu_1 \) and \( hu(y_2) = \mu_2 \). We prove that \( \| hu \| = 1 \). Let \( y \in L_0 \). Then

\[
|hu(y)| \leq |h(y)||u_0(y)| \leq 1.
\]
Let $y \in L_n$. Then

$$|hu(y)| \leq |h(y)| \left( \frac{\mu_n(y)}{2^n} + \sum_{k \neq n} \frac{1}{2^k} \right) \leq \left( 1 + \frac{1}{2^n} \right) \left( \frac{1}{2^n(2^n + 1)} + 1 - \frac{1}{2^n} \right) = 1.$$ 

Thus $|hu| \leq 1$ on $\cup_{n=0}^\infty L_n$. Let $y \in Y \setminus \cup_{n=0}^\infty L_n$. As $\cup_{n=0}^\infty L_n = \{ y \in Y : |h(y)| > 1 \}$ we have $|h(y)| \leq 1$, and so $|hu(y)| \leq 1$. It follows that $\|hu\| = 1$. Thus $hu$ is the desired function $g$. 

**Lemma 4.2.** There exists a bijection $\phi : \text{Ch}(A) \to \text{Ch}(B)$ and a map $\tau : \text{Ch}(A) \times \mathbb{T} \to \mathbb{T}$ such that

$$T(F_{x,\lambda}) = F_{\phi(x),\tau(x,\lambda)}$$

for every maximal convex subset $F_{x,\lambda}$ of $S(A)$. Hence $T$ defines a one-to-one correspondence between the set of all maximal convex subsets of $S(A)$ and $S(B)$.

**Proof.** By [4 Lemma 5.1 ii)] the surjective isometry $T$ gives a one-to-one correspondence between the set of all maximal convex sets. Let $F_{x,\lambda}$ be a maximal convex subset of $S(A)$. Then Lemma 5.1 ii) of Cheng and Dong we have that $TF_{x,\lambda}$ is a maximal convex subset of $S(B)$. Hence by Lemma 3.2 we obtain $\phi(x, \lambda) \in \text{Ch}(B)$ and a unimodular complex number $\tau(x, \lambda)$ such that

$$TF_{x,\lambda} = F_{\phi(x,\lambda),\tau(x,\lambda)}.$$ 

We prove that $\phi(x, \lambda)$ does not depend on the second term $\lambda$. Suppose that there exists $x \in \text{Ch}(A)$ and complex numbers $\lambda$ and $\lambda'$ of unit moduli such that $\phi(x, \lambda) \neq \phi(x, \lambda')$. Then by Lemma 4.1 there exists $g \in S(B)$ such that

$$g(\phi(x, \lambda)) = \tau(x, \lambda), \quad g(\phi(x, \lambda')) = \tau(x, \lambda').$$

This means that $g \in F_{\phi(x,\lambda),\tau(x,\lambda)} = TF_{x,\lambda}$ and $g \in F_{\phi(x,\lambda'),\tau(x,\lambda')} = TF_{x,\lambda'}$. Then there exists $f_1 \in F_{x,\lambda}$ and $f_2 \in F_{x,\lambda'}$ such that $Tf_1 = Tf_2 = g$. As $T$ is an injection, $f_1 = f_2$. But it is impossible since $F_{x,\lambda} \cap F_{x,\lambda'} = \emptyset$. We have proved that $\phi(x, \lambda)$ is independent of $\lambda$. Thus we write $\phi(x)$ instead of $\phi(x, \lambda)$ and the conclusion holds.

Applying the same argument for $T^{-1}$ we get $\phi' : \text{Ch}(B) \to \text{Ch}(A)$ and $\tau' : \text{Ch}(B) \times \mathbb{T} \to \mathbb{T}$ such that $T^{-1}F_{y,\lambda} = F_{\phi'(y),\tau'(y,\lambda)}$. Then we have

$$F_{x,\lambda} = T^{-1}(TF_{x,\lambda}) = T^{-1}F_{\phi(x),\tau(x,\lambda)} = F_{\phi'(\phi(x)),\tau'(\phi(x),\tau(x,\lambda))}$$

and

$$F_{y,\lambda} = T(T^{-1}F_{y,\lambda}) = TF_{\phi'(y),\tau'(y,\lambda)} = F_{\phi'(\phi'(y)),\tau'(\phi'(y),\tau'(y,\lambda))}.$$
Therefore \( x = \phi' \circ \phi(x) \) for every \( x \in \text{Ch}(A) \) and \( y = \phi \circ \phi'(y) \) for every \( y \in \text{Ch}(B) \) hold. Hence \( \phi \) is a bijection. \( \square \)

**Lemma 4.3.** Let \( \text{Ch}(A)_+ = \{ x \in \text{Ch}(A) : \frac{\tau(x,i)}{\tau(x,1)} = i \} \) and \( \text{Ch}(A)_- = \{ x \in \text{Ch}(A) : \frac{\tau(x,i)}{\tau(x,1)} = \bar{i} \} \). Then \( \text{Ch}(A)_+ \cup \text{Ch}(A)_- = \text{Ch}(A) \). For every unimodular complex number \( \lambda \) we have

\[
\tau(x, \lambda) = \begin{cases} 
\lambda \tau(x,1), & x \in \text{Ch}(A)_+, \\
\bar{\lambda} \tau(x,1), & x \in \text{Ch}(A)_-. 
\end{cases}
\]

**Proof.** Let \( x \in \text{Ch}(A) \) and \( \lambda \in \mathbb{T} \). Since \( |\tau(\cdot, \cdot)| = 1 \) on \( \text{Ch}(A) \times \mathbb{T} \) we have

(1) \[ |1 - \lambda| = ||1 - \lambda|| = ||T(1) - T(\lambda)|| = ||\tau(\cdot, 1) - \tau(\cdot, \lambda)|| \]

\[ \geq |\tau(x, 1) - \tau(x, \lambda)| = \left| 1 - \frac{\tau(x, \lambda)}{\tau(x, 1)} \right| . \]

Since \( \text{Ch}(A) \) is a boundary, we have \( \{ \gamma \} = \cap_{x \in \text{Ch}(A)} \mathbb{F}_x, \gamma \) for every unimodular constant \( \gamma \in \mathbb{A} \). Then by Proposition 2.3 in [44] we have that

(2) \[ \{ T(-\mu) \} = T(\cap_{x \in \text{Ch}(A)} \mathbb{F}_x, -\mu) = \cap_{x \in \text{Ch}(A)} T(\mathbb{F}_x, -\mu) \]

\[ = \cap_{x \in \text{Ch}(A)} T(-\mathbb{F}_x, \mu) = \cap_{x \in \text{Ch}(A)} (-T(\mathbb{F}_x, \mu)) \]

\[ = - \cap_{x \in \text{Ch}(A)} T(\mathbb{F}_x, \mu) = -T(\cap_{x \in \text{Ch}(A)} \mathbb{F}_x, \mu) = \{ -T(\mu) \}, \]

hence \( T(-\mu) = -T(\mu) \) for every unimodular constant \( \mu \). Applying (2) for \( \mu = 1 \) we have

(3) \[ |1 - \lambda| = ||1 + \lambda|| = ||-1 + \lambda|| = ||-T(-1) + T(\lambda)|| \]

\[ = ||T(1) + T(\lambda)|| = ||\tau(\cdot, 1) + \tau(\cdot, \lambda)|| \geq |\tau(x, 1) + \tau(x, \lambda)| = \left| 1 - \frac{\tau(x, \lambda)}{\tau(x, 1)} \right| . \]

Applying Yoko-Kuwagata for (1) and (3) to get \( \frac{\tau(x, \lambda)}{\tau(x, 1)} = \lambda \) or \( \bar{\lambda} \). In particular, \( \frac{\tau(x, i)}{\tau(x, 1)} = i \) or \( \bar{i} \). Thus \( \text{Ch}(A)_+ \cup \text{Ch}(A)_- = \text{Ch}(A) \).

We prove that \( \frac{\tau(x, \lambda)}{\tau(x, 1)} = \lambda \) if \( x \in \text{Ch}(A)_+ \). Let \( x \in \text{Ch}(A)_+ \) and \( \lambda \in \mathbb{T} \). Nothing is needed to prove if \( \lambda \) is a real number as \( \lambda = \bar{\lambda} \). We may suppose that \( \lambda \) is not a real number, hence the imaginary part \( \text{Im} \lambda \) is positive or negative. We consider the case of \( \text{Im} \lambda > 0 \). We have

\[ |i - \lambda| = ||i - \lambda|| = ||T(i) - T(\lambda)|| = ||\tau(\cdot, i) - \tau(\cdot, \lambda)|| \]

\[ \geq |\tau(x, i) - \tau(x, \lambda)| = \left| \frac{\tau(x, i)}{\tau(x, 1)} - \frac{\tau(x, \lambda)}{\tau(x, 1)} \right| = \left| i - \frac{\tau(x, \lambda)}{\tau(x, 1)} \right|. \]
Since \( \frac{\tau(x,\lambda)}{\tau(x,1)} = \lambda \) or \( \lambda \), and \( \text{Im} \lambda > 0 \), the inequality \(|i - \lambda| \geq |i - \frac{\tau(x,\lambda)}{\tau(x,1)}|\)
asserts that \( \frac{\tau(x,\lambda)}{\tau(x,1)} = \lambda \). Next we consider the case of \( \text{Im} \lambda < 0 \). Applying (2) for \( \mu = i \), by a similar calculation in (3) we get
\[
| - i - \lambda | = \| T(-i) - T(\lambda) \| = \| T(i) - T(\lambda) \| = \| - \tau(i,i) - \tau(\cdot,\lambda) \| \geq \begin{vmatrix} - \tau(x,i) - \tau(x,\lambda) \end{vmatrix} = \begin{vmatrix} - i - \frac{\tau(x,\lambda)}{\tau(x,1)} \end{vmatrix}.
\]
As \( \text{Im} \lambda < 0 \), we have \( \frac{\tau(x,\lambda)}{\tau(x,1)} = \lambda \).

Next we prove that \( \frac{\tau(x,\lambda)}{\tau(x,1)} = \lambda \) if \( x \in \text{Ch}(A)_- \). The proof is similar to that for the case of \( x \in \text{Ch}(A)_+ \). Let \( x \in \text{Ch}(A)_- \) and \( \lambda \in \mathbb{T} \). As before we may assume that \( \text{Im} \lambda > 0 \) or \( \text{Im} \lambda < 0 \). Suppose that \( \text{Im} \lambda > 0 \). Then we have
\[
| i - \lambda | = \| T(i) - T(\lambda) \| \geq \begin{vmatrix} \tau(x,i) - \tau(x,\lambda) \begin{vmatrix} \tau(x,1) \tau(x,1) \end{vmatrix} = \begin{vmatrix} - i - \frac{\tau(x,\lambda)}{\tau(x,1)} \end{vmatrix}.
\]
As \( \text{Im} \lambda > 0 \) we obtain \( \frac{\tau(x,\lambda)}{\tau(x,1)} = \lambda \). Suppose next that \( \text{Im} \lambda < 0 \). Applying (2) to get
\[
| - i - \lambda | = \| T(-i) - T(\lambda) \| = \| - T(i) - T(\lambda) \| \geq \begin{vmatrix} - \tau(x,i) - \tau(x,\lambda) \end{vmatrix} = \begin{vmatrix} i - \frac{\tau(x,\lambda)}{\tau(x,1)} \end{vmatrix}.
\]
As \( \text{Im} \lambda < 0 \) we obtain \( \frac{\tau(x,\lambda)}{\tau(x,1)} = \lambda \).

5. An additive Bishop’s lemma

Let \( R \) be a closed solid rhombus in the complex plane \( \mathbb{C} \) whose four vertices are \( 0, \frac{\sqrt{3}}{2} \exp(\pm \frac{2\pi}{3} i), 1 \). Let \( \tilde{R} \) be a closed solid hexagon in \( \mathbb{C} \) whose six vertices are \( 0, \frac{1}{3} \exp(\pm \frac{2\pi}{3} i), \frac{\sqrt{3}}{2} \exp(\pm \frac{2\pi}{3} i), 1 \). The following Lemma holds.

**Lemma 5.1.** The following two statements hold.

(i) If \( z \in \tilde{R} \), then \(|2z - 1| \leq 1\).

(ii) If \( z, w \in R \), then \( zw \in \tilde{R} \).

**Proof.** It is trivial that (i) holds. We prove (ii). Since \( R \) is the convex hull of the vertices \( 0, \frac{\sqrt{3}}{2} \exp(\pm \frac{2\pi}{3} i) \) and \( 1 \), the set \( \{zw : z, w \in R\} \) is the convex hull of the products of these vertices: \( 0, \frac{1}{3}, \frac{\sqrt{3}}{2} \exp(\pm \frac{2\pi}{3} i), \frac{1}{3} \exp(\pm \frac{2\pi}{3} i) \) and \( 1 \), which coincides with \( \tilde{R} \).

\( \square \)
In the following
\[ D = \{ z \in \mathbb{C} : |z| < 1 \}, \quad \overline{D} = \{ z \in \mathbb{C} : |z| \leq 1 \} \]

**Lemma 5.2.** Let \( x \in \text{Ch}(A) \) and \( G \) a open neighborhood of \( x \) in \( X \), \( \delta > 0 \). There exists \( u \in A \) which satisfies the following.

\( 1 \) \( 1 = u(x) = \| u \|, u^{-1}(1) = \| u \|^{-1}(1) \),
\( 2 \) \( \| u \| < \delta \) on \( X \setminus G \),
\( 3 \) \( u(X) \subset R \).

**Proof.** Let \( \pi_0 : \overline{D} \to R \) be a homeomorphism such that \( \pi_0|D \) is an analytic map from \( D \) onto the interior of \( R \). Such a homeomorphism exists by the well known theorem of Carathéodory (cf. [52]). In fact \( \pi_0 \) is the homeomorphic extension of the Riemann map from the interior of \( D \) onto \( D \). We may assume that \( \pi_0(1) = 1 \) and \( \pi_0(-1) = 0 \). Put \( \Omega = \pi_0^{-1}(R \cap \{ z \in \mathbb{C} : |z| < \delta \}) \). Choose \( 0 \leq r < 1 \) so that \( \pi_0(\{ z \in \mathbb{C} : |z| < 1/2 \}) \subset \Omega \), where \( \pi_0(z) = \frac{z - r}{1 - rz} \). It is possible if \( |1 - r| \) is sufficiently small. Put \( \pi = \pi_0 \circ \pi_0 \). Choose a peaking function \( f \) such that \( 1 = f(x) = \| f \|, f^{-1}(1) = \| f \|^{-1}(1) \), \( | f(x) | < 1/2 \) on \( X \setminus G \). Put \( \alpha = \pi \circ f \). As \( \pi \) is approximated by analytic polynomials on \( \overline{D} \), we have \( u \in A \). It is automatic that (1), (2) and (3) hold. \( \square \)

Let \( x \in \text{Ch}(A) \). Put \( P_x = \{ f \in S(A) : f(x) = 1, f^{-1}(1) = | f^{-1}(1) | \} \). In fact, the set \( P_x \) is the set of all peaking functions in \( A \) which peaks at \( x \). The following is an additive Bishop’s lemma.

**Lemma 5.3.** Let \( f \in S(A), x \in \text{Ch}(A) \). Put \( f(x) = \alpha \). For any \( 0 < r < 1 \) there exists \( u_r \in P_x \) such that \( g_{r+} = \left( \frac{\alpha}{|\alpha|} - r\alpha \right) u_r + rf \) and \( g_{r-} = \left( \frac{-\alpha}{|\alpha|} - r\alpha \right) u_r + rf \) are elements in \( S(A) \) with \( g_{r+}(x) = \frac{\alpha}{|\alpha|}, g_{r-}(x) = \frac{-\alpha}{|\alpha|} \). Here \( \frac{\alpha}{|\alpha|} \) reads 1 if \( \alpha = 0 \).

**Proof.** Let \( 0 < r < 1 \) and \( 0 < \varepsilon < 1 - r|\alpha| \). Put
\[ F_0 = \{ y \in X : |r\alpha - rf(y)| \geq \varepsilon/4 \}. \]
For a positive integer \( n \) put
\[ F_n = \{ y \in X : \varepsilon/2^{n+2} \leq |r\alpha - rf(y)| \leq \varepsilon/2^{n+1} \}. \]
Then by Lemma 5.2 there exists \( u_0 \in P_x \) such that \( u_0(X) \subset R \) and \( |u_0| < \frac{1-r}{1+r|\alpha|} \) on \( F_0 \). For every positive integer \( n \) there exists \( u_n \in P_x \) such that \( u_n(X) \subset R \) and \( |u_n| < \frac{1}{2^n r} \) on \( F_n \). Put \( u_r = u_0 \sum_{n=1}^{\infty} \frac{u_n}{2^n} \). Note that the sum uniformly converges since \( \| u_n \| = 1 \). Then \( u_r \) is the desired function. We prove this.

As \( u_n(X) \subset R \) for all positive integer \( n \) and \( R \) is a convex set containing \( 0 \), we have that \( \left( \sum \frac{u_n}{2^n} \right) (X) \subset R \). Since \( u_0(X) \subset R \), we obtain that \( u_r(X) \subset R \) and \( \| 1 - 2u_r \| \leq 1 \) by Lemma 5.1. We prove \( |g_{r+}(y)| \leq 1 \) and \( |g_{r-}(y)| \leq 1 \).
for any $y \in X$ in three cases: 1) $y \in F_0$; 2) $y \in F_n$ for some positive integer $n$; 3) $y \in X \setminus (U_{n=1}^\infty F_n) \cup F_0$.

1) Suppose that $y \in F_0$. We have $|u_r(y)| \leq |u_0(y)| < \frac{1-r}{1+r|\alpha|}$. As $\frac{|\alpha|}{|\alpha|} - r|\alpha| = 1 - r|\alpha|$ and $\frac{|\alpha|}{|\alpha|} - r|\alpha| = 1 + r|\alpha|$, we see that

$$|g_{r+}(y)| \leq (1 - r|\alpha|) \frac{1-r}{1+r|\alpha|} + r|f(y)| < 1,$$

$$|g_{r-}(y)| \leq (1 + r|\alpha|) \frac{1-r}{1+r|\alpha|} + r|f(y)| \leq 1$$

since $\|f\| = 1$.

2) Suppose that $y \in F_n$ for some positive integer $n$. Then

$$|u_r(y)| \leq \sum_{m=1}^\infty \frac{|u_m(y)|}{2^m} = \sum_{m \neq n} \frac{|u_m(y)|}{2^m} + \frac{|u_n(y)|}{2^n} \leq \sum_{m \neq n} \frac{1}{2^m} + \frac{1}{2^{n+1}2^{n+1}} = 1 - \frac{1}{2^n} + \frac{1}{2^{n+2n+1}} < 1.$$

Then

$$|g_{r+}(y)| \leq \left| \left( \frac{\alpha}{|\alpha|} - r|\alpha| \right) u_r(y) + r\alpha \right| + |rf(y) - r\alpha|$$

$$\leq (1 - r|\alpha|)|u_r(y)| + r|\alpha| + \frac{\epsilon}{2^n} + 1$$

$$\leq (1 - r|\alpha|) \left( 1 - \frac{1}{2^n} + \frac{1}{2^{n+1}} \right) + r|\alpha| + \frac{\epsilon}{2^n}$$

as $0 < \epsilon < 1 - r|\alpha|$

$$\leq (1 - r|\alpha|) \left( 1 - \frac{1}{2^n} + \frac{1}{2^{n+1}} + \frac{1}{2^{n+1}} \right) + r|\alpha|$$

$$< (1 - r|\alpha|) + r|\alpha| = 1$$

We also have

$$|g_{r-}(y)| \leq \left| \left( \frac{-\alpha}{|\alpha|} + r|\alpha| \right) u_r(y) - 2r\alpha u_r(y) + r\alpha \right| + |rf(y) - r\alpha|$$

$$\leq \left| \frac{-\alpha}{|\alpha|} + r\alpha \right| |u_r(y)| + r|\alpha| |1 - 2u_r(y)| + \frac{\epsilon}{2^n}$$

as $\|1 - 2u_r\| \leq 1$ and $0 < \epsilon < 1 - r|\alpha|$ we have

$$\leq (1 - r|\alpha|) \left( 1 - \frac{1}{2^n} + \frac{1}{2^{n+1}} + \frac{1}{2^{n+1}} \right) + r|\alpha| < 1$$
3) Suppose that \( y \in X \setminus (\bigcup_{n=1}^{\infty} F_n) \cup F_0 \). In this case \( rf(y) = r\alpha \). We have
\[
|g_{r+}(y)| \leq (1 - r|\alpha|)|u_r(y)| + r|\alpha| \leq 1
\]
since \( ||u_r|| = 1 \). We also have
\[
|g_{r-}(y)| = \left| \left( \frac{-\alpha}{|\alpha|} + r\alpha \right) u_r(y) - 2r\alpha u_r(y) + r\alpha \right|
\leq (1 - r|\alpha|)|u_r(y)| + r|\alpha||2u_r(y) - 1| \leq 1
\]
As \( g_{r+}(x) = \frac{\alpha}{|\alpha|} \) and \( g_{r-}(x) = \frac{-\alpha}{|\alpha|} \) since \( u_r(x) = 1 \), we observe by 1), 2) and 3) that \( ||g_{r+}|| = ||g_{r-}|| = 1 \), hence \( g_{r+}, g_{r-} \in S(A) \).

\[\square\]

### 6. A Form of \( Tf \) on the Choquet Boundary

**Definition 6.1.** Let \( x \in \text{Ch}(A) \) and \( \alpha \) a complex number with \( |\alpha| \leq 1 \). A subset \( M_{x,\alpha} \) of \( S(A) \) is defined as
\[
M_{x,\alpha} = \{ f \in S(A) : d(f, F_{x,\frac{\alpha}{|\alpha|}}) = 1 - |\alpha|, \ d(f, F_{x,\frac{-\alpha}{|\alpha|}}) = 1 + |\alpha| \},
\]
where \( d(f, F) = \inf \{ \|f - g\| : g \in F \} \) and \( \frac{\alpha}{|\alpha|} \) reads 1 if \( \alpha = 0 \).

The set \( M_{x,\alpha} \) corresponds to \( E_\lambda \), appeared in [45].

**Lemma 6.2.** For every pair \( x \in \text{Ch}(A) \) and a unimodular complex number \( \lambda \), the equality
\[
d(f, F_{x,\lambda}) = d(Tf, F_{\phi(x), \tau(x,\lambda)}).
\]
holds for every \( f \in S(A) \).

**Proof.** As \( T \) is an isometry we have \( d(f, g) = d(Tf, Tg) \) for every \( g \in F_{x,\lambda} \). Thus \( d(f, F_{x,\lambda}) = d(Tf, TF_{x,\lambda}) \). By Lemma 4.2 we have \( d(f, F_{x,\lambda}) = d(Tf, F_{\phi(x), \tau(x,\lambda)}) \). \( \square \)

**Lemma 6.3.** For every pair \( x \in \text{Ch}(A) \) and a complex number \( \alpha \) with \( |\alpha| \leq 1 \) we have \( M_{x,\alpha} = \{ f \in S(A) : f(x) = \alpha \} \).

**Proof.** Suppose that \( f \in M_{x,\alpha} \). For every \( h \in F_{x,\frac{\alpha}{|\alpha|}} \) we have
\[
\left| f(x) - \frac{\alpha}{|\alpha|} \right| = |f(x) - h(x)| \leq ||f - h||.
\]
Hence
\[
(4) \quad \left| f(x) - \frac{\alpha}{|\alpha|} \right| \leq d(f, F_{x,\frac{\alpha}{|\alpha|}}) = 1 - |\alpha|.
\]
For every \( g \in F_{x,\frac{-\alpha}{|\alpha|}} \) we have
\[
\left| f(x) - \frac{-\alpha}{|\alpha|} \right| = |f(x) - g(x)| \leq ||f - g||.
\]
Hence
\[
\left| f(x) - \frac{-\alpha}{|\alpha|} \right| \leq d(f, F_{x, \frac{\alpha}{|\alpha|}}) = 1 + |\alpha|.
\]

By (4) and (5) we see that \( f(x) = \alpha \).

Suppose that \( f \in S(A) \) and \( f(x) = \alpha \). For any \( 0 < r < 1 \), functions
\[
g_{r+} = \left( \frac{\alpha}{|\alpha|} - r\alpha \right) u_r + rf \in F_{x, \frac{\alpha}{|\alpha|}}
\]
and
\[
g_{r-} = \left( -\frac{\alpha}{|\alpha|} - r\alpha \right) u_r + rf \in F_{x, \frac{-\alpha}{|\alpha|}}
\]
in Lemma 5.3 satisfies that \( g_{r+}, g_{r-} \in S(A) \) and \( g_{r+}(x) = \frac{\alpha}{|\alpha|} \) and \( g_{r-}(x) = \frac{-\alpha}{|\alpha|} \). Thus \( g_{r+} \in F_{x, \frac{\alpha}{|\alpha|}} \) and \( g_{r-} \in F_{x, \frac{-\alpha}{|\alpha|}} \). We have
\[
\|g_{r+} - f\| \leq (1 - r|\alpha|)\|u_r\| + \|rf - f\| = (1 - r|\alpha|) + 1 - r
\]
and
\[
\|g_{r-} - f\| \leq (1 + r|\alpha|)\|u_r\| + \|rf - f\| = (1 + r|\alpha|) + 1 - r.
\]
As \( r \) is arbitrary number of \( 0 < r < 1 \), we have \( d(f, F_{x, \frac{\alpha}{|\alpha|}}) \leq 1 - |\alpha| \) and \( d(f, F_{x, \frac{-\alpha}{|\alpha|}}) \leq 1 + |\alpha| \). On the other hand,
\[
1 - |\alpha| = \left| \alpha - \frac{\alpha}{|\alpha|} \right| = |f(x) - h(x)| \leq \|f - h\|
\]
for any \( h \in F_{x, \frac{\alpha}{|\alpha|}} \) and
\[
1 + |\alpha| = \left| \alpha - \frac{-\alpha}{|\alpha|} \right| = |f(x) - h'(x)| \leq \|f - h'\|
\]
for any \( h' \in F_{x, \frac{-\alpha}{|\alpha|}} \). It follows that \( d(f, F_{x, \frac{\alpha}{|\alpha|}}) = 1 - |\alpha| \) and \( d(f, F_{x, \frac{-\alpha}{|\alpha|}}) = 1 + |\alpha| \). Hence we get \( f \in M_{x, \alpha} \).

\[ \square \]

**Lemma 6.4.** For every \( f \in S(A) \) we have
\[
Tf(\phi(x)) = \tau(x, 1) \times \begin{cases} f(x), & x \in \text{Ch}(A)_+ \\ \overline{f(x)}, & x \in \text{Ch}(A)_- \end{cases}
\]
Proof. Let \( x \in \text{Ch}(A) \) and \( |\alpha| \leq 1 \). As \( T(S(A)) = S(B) \) we have by Lemma 6.2 that

\[
TM_{x,\alpha} = \{ g \in S(B) : d(g, F_{\phi(x)}(x, \tau(x, |\alpha|))) = 1 - |\alpha|, \ d(g, F_{\phi(x)}(x, \tau(x, |\alpha|))) = 1 + |\alpha| \},
\]

then by Lemma 4.3

\[
\begin{cases}
\{ g : d(g, F_{\phi(x)}(x, \alpha \tau(x, 1))) = 1 - |\alpha|, \ d(g, F_{\phi(x)}(x, \alpha \tau(x, 1))) = 1 + |\alpha| \}, & x \in \text{Ch}(A) + \\
\{ g : d(g, F_{\phi(x)}(x, \bar{\alpha} \tau(x, 1))) = 1 - |\alpha|, \ d(g, F_{\phi(x)}(x, \bar{\alpha} \tau(x, 1))) = 1 + |\alpha| \}, & x \in \text{Ch}(A) - \\
M_{\phi(x), \alpha \tau(x, 1)}, & x \in \text{Ch}(A) + \\
M_{\phi(x), \bar{\alpha} \tau(x, 1)}, & x \in \text{Ch}(A) -
\end{cases}
\]

applying Lemma 6.3 for \( M_{\phi(x), \beta} \subset S(B) \) for \( \beta = \alpha \tau(x, 1) \), \( \bar{\alpha} \tau(x, 1) \) we have

\[
\begin{cases}
\{ g \in S(B) : g(\phi(x)) = \alpha \tau(x, 1) \}, & x \in \text{Ch}(A) + \\
\{ g \in S(B) : g(\phi(x)) = \bar{\alpha} \tau(x, 1) \}, & x \in \text{Ch}(A) -.
\end{cases}
\]

Let \( f \in S(A) \). Put \( \alpha = f(x) \). Then \( f \in M_{x,\alpha} \) by Lemma 6.3. By the above we have \( Tf(\phi(x)) = \alpha \tau(x, 1) \) if \( x \in \text{Ch}(A) + \) and \( Tf(\phi(x)) = \bar{\alpha} \tau(x, 1) \) if \( x \in \text{Ch}(A) - \). As \( f(x) = \alpha \) we have the conclusion. \( \qed \)

7. Proof of Theorem 2.1

Proof of Theorem 2.1. By Lemma 4.2 the map \( \phi : \text{Ch}(A) \to \text{Ch}(B) \) is a bijection. Denote the inverse of \( \phi \) by \( \psi : \text{Ch}(B) \to \text{Ch}(A) \). Then by Lemma 6.4 we have

\[
Tf(y) = \tau(\psi(y), 1) \times \begin{cases} f \circ \psi(y), & y \in \phi(\text{Ch}(A)_+) \\ f \circ \psi(y), & y \in \phi(\text{Ch}(A)_-) \end{cases}
\]

for every \( f \in S(A) \). Note that since \( \phi : \text{Ch}(A) \to \text{Ch}(B) \) is a bijection we have \( \phi(\text{Ch}(A)_+) \cap \phi(\text{Ch}(A)_-) = \emptyset \) and \( \phi(\text{Ch}(A)_+) \cup \phi(\text{Ch}(A)_-) = \text{Ch}(B) \).

We prove that \( T1 \) is invertible and \( |T1| = 1 \) on \( MB \). Since \( T \) is a surjection, there exists \( f_1 \in S(A) \) with \( Tf_1 = 1 \). By (6) we have

\[
T1(y) = \tau(\psi(y), 1), \quad y \in \text{Ch}(B),
\]

and

\[
1 = Tf_1(y) = \tau(\psi(y), 1) \times \begin{cases} f_1 \circ \psi(y), & y \in \phi(\text{Ch}(A)_+) \\ f_1 \circ \psi(y), & y \in \phi(\text{Ch}(A)_-) \end{cases},
\]

and

\[
T(f_1)^2(y) = \tau(\psi(y), 1) \times \begin{cases} (f_1)^2 \circ \psi(y), & y \in \phi(\text{Ch}(A)_+) \\ (f_1)^2 \circ \psi(y), & y \in \phi(\text{Ch}(A)_-) \end{cases},
\]
Combining (7), (8) and (9) we get
\[ 1 = (T f_1)^2 = T T_1 T_1 (f_1^2) \] on Ch(B).

Since the Choquet boundary is a uniqueness set for B, we obtain
\[ 1 = T T_1 T_1 (f_1^2) \] on M_B.

Thus T_1 is invertible in B. On the other hand, |T(1)| = 1 on Ch(B) induces that |(T_1)^{-1}| = 1 on Ch(B). Thus |T(1)| ≤ 1 and |((T_1)^{-1})| ≤ 1 on M_B. It follows that |T 1| = 1 on M_B.

Define T_1 : A → B by
\[
T_1 f = \overline{T T_1} \times \begin{cases} 0, & f = 0 \\ \| f \| T \left( \frac{f}{\| f \|} \right), & f \neq 0. \end{cases}
\]

By a simple calculation T_1 is a bijection since T is. It is easy to see that T_1 is an extension of \( \overline{T T} \), hence T T_1 is an extension of T. By (6) and (7) we infer that
\[
T_1 f = \begin{cases} f \circ \psi & \text{on } \phi(Ch(A)_+) \\ \frac{f}{\psi} & \text{on } \phi(Ch(A)_-). \end{cases}
\]

As Ch(B) is a uniqueness set for B, we infer that T_1 is a real-linear algebra-isomorphism from A onto B. Then by [31, Theorem 2.1] there exist a homeomorphism \( \Psi : M_B \rightarrow M_A \), possibly empty disjoint closed and open subsets M_B_+ and M_B_- of M_B with M_B_+ ∪ M_B_- = M_B such that
\[
T_1 f = \begin{cases} f \circ \Psi & \text{on } M_B_+ \\ \frac{f}{\Psi} & \text{on } M_B_- \end{cases}
\]
for every f ∈ A. It follows that
\[
T T_1 T_1 f = T_1 \times \begin{cases} f \circ \Psi & \text{on } M_B_+ \\ \frac{f}{\Psi} & \text{on } M_B_- \end{cases}
\]
for every f ∈ A is a surjective real-linear isometry from A onto B since |T_1| = 1 on M_B. As is already pointed out that T T_1 T_1 is an extension of T we conclude the proof.

\[\square\]

8. Remarks

We close the paper with a few remarks. As the first one we conjecture that a uniform algebra satisfies the Mazur-Ulam property. A surjective isometry between the unit spheres of uniform algebras are represented by a “weighted composition operator”, it is possible because a surjective real-linear isometry between uniform algebras is represented as in this form. In fact, the point of the proof of Theorem 2.1 is to show the form of the given isometry.
between the unit spheres. In general, a surjective real-linear isometry between a uniform algebra and a Banach space of continuous functions is not expected to have the form of a “weighted composition operator”. We do not know how to prove the conjecture.

The second remark concerns Tingley’s problem on a Banach space of analytic functions. As we have already pointed out that Theorem 2.1 provides the first positive solution for Tingley’s problem on a Banach space of analytic functions. It is interesting to study Tingley’s problem on several Banach spaces of analytic functions. The form of a surjective complex-linear isometry on the Hardy space $H^1(D)$ on the open unit disk is derived by a theorem of deLeeuw, Rudin and Wermer [6] (cf. [46]) on the isometry between uniform algebras. The point of the proof is to show that the isometry $U : H^1(D) \rightarrow H^1(D)$ essentially preserve the range of functions. It reminds us that if we can prove that the range of the function is essentially preserved by the isometry $T : S(H^1(D)) \rightarrow S(H^1(D))$, we have a chance to solve Tingley’s problem for $H^1(D)$.

As a final remark we encourage researches on Tingley’s problem on several Banach algebras of continuous functions. Comparing with the theorem of Wang [69] on the Banach algebra of $C(X)$ it is interesting to study a surjective isometry on the unit sphere of a Banach space or algebra of Lipschitz functions.

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REFERENCES

[1] J. Becerra Guerrero, The Mazur-Ulam property in $\ell_\infty$-sum and $c_0$-sum of strictly convex Banach spaces, J. Math. Anal. Appl. 489 (2020), 124166, 13pp doi:10.1016/j.jmaa.2020.124166
[2] A. Browder, Introduction to function algebras, W. A. Benjamin, Inc., New York-Amsterdam 1969 Xii+273 pp
[3] J. Cabello Sánchez, A reflection on Tingley’s problem and some applications, J. Math. Anal. Appl. 476 (2019), 319–336 doi:10.1016/j.jmaa.2019.03.041
[4] L. Cheng and Y. Dong, On a generalized Mazur-Ulam question: extension of isometries between unit spheres of Banach spaces, J. Math. Anal. Appl. 377 (2011), 464–470 doi:10.1016/j.jmaa.2010.11.025
[5] M. Cueto-Avellaned and A. M. Peralta, On the Mazur-Ulam property for the space of Hilbert-space-valued continuous functions, J. Math. Anal. Appl. 479 (2019), 875–902 doi:10.1016/j.jmaa.2019.06.056
[6] K. deLeeuw, W. Rudin and J. Wermer, The isometries of some function spaces, Proc. Amer. Math. Soc. 11 (1960), 694–698
[7] G. G. Ding The 1-Lipschitz mapping between the unit spheres of two Hilbert spaces can be extended to a real linear isometry of the whole space, Sci. China Ser. A 45 (2002), 479–483
[8] G. G. Ding, *The isometric extension problem in the unit spheres of $l^p(\Gamma)$ ($p > 1$) type spaces*, Sci. China Ser. A *46* (2003), 333–338

[9] G. G. Ding, *On extension of isometries between unit spaces of $E$ and $C(\Omega)$*, Acta Math. Sin. (Engl. Ser.) *19* (2003), 793–800

[10] G. G. Ding, *The representation theorem of onto isometric mappings between two unit spheres of $l^\infty$-type spaces and the application to the isometric extension problem*, Sci China Ser. A *47* (2004), 722–729

[11] G. G. Ding, *The representation theorem of onto isometric mappings between two unit spheres of $l^1(\Gamma)$ type spaces and the application to the isometric extension problem*, Acta Math. Sin. (Engl. Ser.) *20* (2004), 1089–1094 doi:10.1007/s10114-004-0447-7

[12] G. G. Ding, *The isometric extension of the into mapping from a $L^\infty(\Gamma)$-type space to some Banach space*, Illinois J. Math. *51* (2007), 445–453

[13] G. G. Ding, *On isometric extension problem between two unit spheres*, Sci. Chin. Ser. A *52* (2009), 2069–2083 doi:10.1007/s11425-009-0156-x

[14] G. G. Ding and J. Z. Li, *Sharp corner points and isometric extension problem in Banach spaces*, J. Math. Anal. Appl. *405* (2013), 297–309 doi:10.1016/j.jmaa.2013.04002

[15] G. G. Ding and J. Z. Li, *Isometries between unit spheres of the $c^\infty$-sum of strictly convex normed spaces*, Bull. Aust. Math. Soc. *88* (2013), 369–375 doi:10.1017/S000497271300018X

[16] G. G. Ding and J. Z. Li, *Isometric extension problem between strictly convex two-dimensional normed spaces*, Acta Math. Sin. (Engl. Ser.) *35* (2019), 513–518 doi:10.1007/s10114-019-7509-3

[17] X. N. Fang, J. H. Wang, *Extension of isometries between the unit spheres of normed space $E$ and $C(\Omega)$*, Acta Math. Sin. (Engl. Ser.) *22* (2006), 1819–1824 doi:10.1007/s10114-005-0725-z

[18] X. N. Fang, J. H. Wang, *Extension of isometries on the unit sphere of $l^p(\Gamma)$ space*, Sci. China Math. *53* (2010), 1085–1096

[19] F. J. Fernández-Polo, J. J. Garcés, A. M. Peralta and I. Villanueva, *Tingley’s problem for spaces of trace class operators*, Linear Algebra Appl. *529* (2017), 294–323 doi:10.1016/j.laa.2017.04.024

[20] F. J. Fernández-Polo, E. Jordá and A. M. Peralta, *Tingley’s problem for $p$-Schantzen von Neumann classes*, J. Spectr. Theory *10* (2020), 809–841 doi:10.4171/JST/313

[21] F. J. Fernández-Polo and A. M. Peralta, *On the extension of isometries between the unit spheres of a $C^*$-algebra and $B(H)$*, Trans. Amer. Math. Soc. Ser. B *5* (2018), 63–80 doi:10.1090/btran/21

[22] F. J. Fernández-Polo and A. M. Peralta, *On the extension of isometries between the unit spheres of von Neumann algebras*, J. Math. Anal. Appl. *466* (2018), 127–143 doi:10.1016/j.jmaa.2018.05.062

[23] F. J. Fernández-Polo and A. M. Peralta, *Low rank compact operators and Tingley’s problem*, Adv. Math. *338* (2018), 1–40 doi:10.1016/j.aim.2018.08.018

[24] F. J. Fernández-Polo and A. M. Peralta, *Tingley’s problem through the facial structure of an atomic JBW*-triples*, J. Math. Anal. Appl. *455* (2017), 750–760 doi:10.1016/j.jmaa.2017.06.002

[25] R. J. Fleming and J. E. Jamison, *Isometries on Banach spaces: function spaces*, Chapman & Hall/CRC Monographs and Surveys in Pure and Applied Mathematics, 129. Chapman & Hall/CRC, Boca Raton, FL, 2003. x+197 pp. ISBN: 1-58488-040-6
[26] X. Fu, *The problem of isometric extension in the unit sphere of the space $s_p(\alpha,H)$*, Banach J. Math. Anal. 8 (2014), 179–189

[27] X. Fu, *On isometric extension in the spaces $s_n(H)$*, J. Funct. Spaces 2014, Art. ID 678407, 4pp doi:10.1155/2014/678407

[28] X. Fu, *The problem of isometric extension on the unit sphere of the space $s_p(\alpha)$*, Nonlinear Anal. 74 (2011), 733–738 doi:10.1016/j.na.2010.08.053

[29] X. Fu, *The problem of isometric extension in the unit sphere of the spaces $s_p(H)$*, J. Funct. Spaces 2014, Art. ID 678407, 4pp doi:10.1155/2014/678407

[30] X. Fu and S. Stević, *The problem of isometric extension on the unit sphere of the space $l_\cap l_p(H)$ for $0 < p < 1$*, Ann. Funct. Anal. 6 (2015), 87–95 doi:10.15352/afa/06-3-8

[31] J. Gao, *Isometries between the unit spheres of $L_\beta^\gamma$-sum of strictly convex normed spaces*, Quaest. Math. 33 (2010), 497–505 doi:10.2989/QM.2010.541988

[32] O. Hatori and T. Miura, *Real linear isometries between function algebras. II*, Cent. Eur. J. Math. 11 (2013), 1838–1842 doi:10.2478/s11533-013-0282-0

[33] F. X. Hong, *The isometric extension of the into mapping from the unit sphere $S(E)$ to $S(l_\infty(\Gamma))$*, Acta Math. Sin. (Engl. Ser.) 24 (2008), 1475–1482 doi:10.1007/s10114-008-7286-x

[34] A. Jiménez-Vargas, A. Morales-Campoy, A. M. Peralta and M. I. Ramírez, *The Mazur-Ulam property for the space of complex null sequences*, Linear Multilinear Algebra 67 (2019), 799–816 doi:10.1080/03081087.2018.1433625

[35] V. Kadets and M. Martín, *Extension of isometries between unit spheres of finite-dimensional polyhedral Banach spaces*, J. Math. Anal. Appl. 396 (2012), 441–447 doi:10.1016/j.jmaa.2012.06.031

[36] A. T. M. Lau, C. K. Ng and N. C. Wong, *Normal states are determined by their facial distances*, Bull. Lond. Math. Soc. 52 (2020), 505–514 doi:10.1112/blms.12344

[37] C. W. Leung, C. K. Ng and N. C. Wong, *Metric preserving bijections between positive spherical shells of non-commutative $L^p$-spaces*, J. Operator Theory 80 (2018), 429–452

[38] C. W. Leung, C. K. Ng and N. C. Wong, *On a variant of Tingley’s problem for some function spaces*, J. Math. Anal. Appl. 496 (2021) 124800 doi:10.1016/j.jmaa.2020.124800

[39] J. Z. Li, *Mazur-Ulam property of the sum of two strictly convex Banach spaces*, Bull. Aust. Math. Soc. 93 (2016), 473–485 doi:10.1017/S0004972715001215

[40] L. Li, *Isometries on the unit sphere of the $\ell^1$-sum of strictly convex normed spaces*, Ann. Funct. Anal. 7 (2016), 33–41 doi:10.1215/20088752-3163452

[41] R. Li and W. Ren, *On extension of isometries between unit spheres of $L^\infty$ and $E_\infty(G)$*, Quaest. Math. 31 (2008), 209–218 doi:10.2989/QM.2008.31.3.3.545

[42] R. Liu, *On extension of isometries between unit spheres of $L^\infty(\Gamma)$-type space and a Banach space $E_\infty$*, J. Math. Anal. Appl. 333 (2007), 959–970 doi:10.1016/j.jmaa.2006.11.044

[43] R. Liu, *A note on extension of isometric embedding from a Banach space $E$ into the universal space $l_\infty(\Gamma)$*, J. Math. Anal. Appl. 363 (2010), 220–224 doi:10.1016/j.jmaa.2009.08.033

[44] R. Liu and L. Zhang, *On extension of isometries and approximate isometries between unit spheres*, J. Math. Anal. Appl. 352 (2009), 749–761 doi:10.1016/j.jmaa.2008.11.034

[45] M. Mori, *Tingley’s problem through the facial structure of operator algebras*, J. Math. Anal. Appl. 466 (2018), 1281–1298 doi:10.1016/j.jmaa.2018.06.050

[46] M. Mori and N. Ozawa, *Mankiewicz’s theorem and the Mazur-Ulam property for $C^*$-algebras*, Studia Math. 250 (2020), 265–281 doi:10.4064/sm180727-14-11
[46] M. Nagasawa, *Isomorphisms between commutative Banach algebras with an application to rings of analytic functions*, Kōdai Math. Sem. Rep., 11 (1959), 182–188

[47] G. Nagy, *Isometries of spaces of normalized positive operators under the operator norm*, Publ. Math. Debrecen 92 (2018), 243–254 doi:10.5486/PMD.2018.7967

[48] A. M. Peralta, *A survey on Tingley’s problem for operator algebras*, Acta Sci. Math. (Szeged) 84 (2018), 81–123

[49] A. M. Peralta, *Extending surjective isometries defined on the unit sphere of $\ell_\infty(\Gamma)$*, Rev. Mat. Complut. 32 (2019), 99–114 doi:10.1007/s13163-018-0269-2

[50] A. M. Peralta and R. Tanaka, *A solution to Tingley’s problem for isometries between the unit spheres of compact $C^*$-algebras and JB*-triples*, Sci. China Math. 62 (2019), 553–568 doi:10.1007/s11425-017-9188-6

[51] Ch. Pomerenke, *Boundary behavior of conformal maps*, Grundlehren der Mathematischen Wissenschaften, 299 Springer-Verlag, Berlin, 1992, x+300 pp

[52] D. N. Tan, *Nonexpansive mappings on the unit spheres of some Banach spaces*, Bull. Aust. Math. Soc. 80 (2009), 139–146 doi:10.1017/S000497270900015X

[53] D. N. Tan, *Nonexpansive mappings and expansive mappings on the unit spheres of some $F$-spaces*, Bull. Aust. Math. Soc. 82 (2010), 22–30

[54] D. N. Tan, *Extension of isometries on unit spheres of $L_\infty$*, Taiwanese J. Math. 15 (2011), 819–827

[55] D. N. Tan, *On extension of isometries on the unit spheres of $L_p$-spaces for $0 < p \leq 1$*, Nonlinear Anal. 74 (2011), 1959–1966 doi:10.1016/j.na.2011.07.035

[56] D. N. Tan, *Extension of isometries on the unit sphere of $L^p$ spaces*, Acta Math. Sin. (Engl. Ser.) 28 (2012), 1197–1208 doi:10.1007/s10114-011-0302-6

[57] D. N. Tan, *Some new properties and isometries on the unit spheres of generalized James spaces $J_p$*, J. Math. Anal. Appl. 393 (2012), 457–469 doi:10.1016/j.jmaa.2012.03.024

[58] D. N. Tan, *Isometries of the unit spheres of the Tsirelson space $T$ and the modified Tsirelson space $T_M$*, Houston J. Math. 38 (2012), 571–581

[59] D. Tan, X. Huang and R. Liu, *Generalized-lush spaces and the Mazur-Ulam property*, Studia Math. 219 (2013), 139–153 doi:10.4064/sm219-2-4

[60] R. Tanaka, *Symmetric absolute normalized norm on $\mathbb{R}^2$*, Acta Math. Sin. (Engl. Ser.) 30 (2014), 1324–1340 doi:10.1007/s10114-014-3491-y

[61] D. Tingley, *Isometries of the unit sphere of $C^*$-algebras, J. Math. Anal. Appl. 451 (2017), 319–326 doi:10.1016/j.jmaa.2017.02.013

[62] D. Tingley, *Isometries of the unit sphere*, Geom. Dedicata 22 (1987), 371–378

[63] J. Wang, *On extension of isometries between unit spheres of $AL_p$-spaces ($0 < p < \infty$)*, Proc. Amer. Math. Soc. 132 (2004), 2899–2909 doi:10.1090/S0002-9939-04-07482-9
[68] R. S. Wang, *Isometries between the unit spheres of \( C_0(\Omega) \) type spaces*, Acta Math. Sci. (English Ed.) 14 (1994), 82–89

[69] Risheng Wang, *Isometries of \( C_0^n(\Omega) \)*, Hokkaido Math. J. 25 (1996), 465–519 doi:10.14492/hokmj/1351516747

[70] Risheng Wang, *Isometries on the \( l^p \)-sum of \( C_0(\Omega, E) \) type spaces*, J. Math. Sci. Univ. Tokyo 3 (1996), 471–493

[71] R. D. Wang, *On the extension problem between separable smooth Banach spaces with RNP*, Quaest. Math. 34 (2011), 67–73 doi:10.2989/16073606.2011.570295

[72] Ruidong Wang, *On the isometric extension problem*, Quaest. Math. 36 (2013), 321–330 doi:10.2989/16073606.2013.779953

[73] Ruidong Wang and X. Huang, *The Mazur-Ulam property for two dimensional somewhere-flat spaces*, Linear Algebra Appl. 562 (2019), 55–62 doi:10.1016/j.laa.2018.09.024

[74] X. Yang, *On extension of isometries between unit spheres of \( L_p(\mu) \) and \( L_p(\mu,H) \) (\( 1 < p \neq 2, H \) is a Hilbert space)*, J. Math. Anal. Appl. 323 (2006), 985–992 doi:10.1016/j.jmaa.2005.11.013

[75] X. Yang and X. Zhao, *On the extension problems of isometric and nonexpansive mappings*, In: Mathematics without boundaries. Edited by Themistocles M. Rassias and Panos M. Pardalos, 725– Springer, New York, 2014

[76] J. Yi, X. Wang and Ruidong Wang, *Extension of isometries between the unit spheres of complex \( l^p(\Gamma) \) \( (p > 1) \) spaces*, Acta Math. Sci. Ser. B (Engl. Ser.) 30 (2014), 1324–1340 doi:10.1016/S0252-9602(14)60102-8

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