A characterization for the defect of rank one valued field extensions

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Abstract

In this paper, we present a characterization for the defect of a simple algebraic extension of rank one valued fields using the key polynomials that define the valuation. As a particular example, this gives the classification of defect extensions of degree $p$ as dependent or independent presented by Kuhlmann.

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1 INTRODUCTION

Let $(L/K, ν)$ be a finite valued field extension. Suppose that $L = K(η)$ for some $η ∈ L$ and let $g$ be the minimal polynomial of $η$ over $K$. We will consider the valuation $ν$ on $K[x]$ with support $gK[x]$ defined by $ν$ and $η$. Namely, for any $f ∈ K[x]$, we set $ν(f) := ν(f(η))$.

Fix an extension $\overline{ν}$ of $ν$ to $K[x]$, where $\overline{K}$ is a fixed algebraic closure of $K$. For each $f ∈ K[x]$, we define

$$\epsilon(f) := \max\{\overline{ν}(x - a) | a \text{ is a root of } f\}.$$ 

By [7, Remark 3.2], the value $\epsilon(f)$ does not depend on the extension $\overline{ν}$ of $ν$. A monic polynomial $Q ∈ K[x]$ is called a key polynomial for $ν$ if

$$\deg(f) < \deg(Q) \implies \epsilon(f) < \epsilon(Q) \text{ for all } f ∈ K[x].$$

Let $νL$ be the value group of $ν$ and denote by $Γ$ the divisible closure of $νL$. For $n ∈ \mathbb{N}$, we denote by $Ψ_n$ the set of all the key polynomials for $ν$ of degree $n$. We will say that $Ψ_n$ does not have a maximum or that $Ψ_n$ is bounded in $Γ$ if the same property is satisfied for $ν(Ψ_n)$. A key polynomial

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for $\Psi_n$ is a key polynomial for $\nu$ of the smallest degree larger than $n$. We denote by $\text{KP}(\Psi_n)$ the set of all the key polynomials for $\Psi_n$. If $\Psi_n$ does not have a maximum, then any key polynomial for $\Psi_n$ will be called a limit key polynomial for $\Psi_n$. In this case, we say that $\Psi_n$ is a plateau for $\nu$.

For polynomials $f, q \in K[x]$, there exist uniquely determined $f_0, \ldots, f_r \in K[x]$ with $\deg(f_i) < \deg(q)$ for every $i$, $0 \leq i \leq r$, such that

$$f = f_0 + f_1 q + \cdots + f_r q^r.$$ 

This expression is called the $q$-expansion of $f$. For any key polynomial $Q$ for $\nu$ and $f \in K[x]$, we will denote by

$$f = a_{Q_0}(f) + a_{Q_1}(f)Q + \cdots + a_{Q_r}(f)Q^r$$

the $Q$-expansion of $f$. We set

$$L_Q(f) = \{i \in \mathbb{N}_0 \mid a_{Q_i}(f) \neq 0\},$$

that is, the set of indices of the nonzero monomials in the $Q$-expansion of $f$. We define the truncation of $\nu$ at $Q$ as

$$\nu_Q(f) = \min_{i \in L_Q(f)} \{\nu(a_{Q_i}(f)Q^i)\}.$$ 

This mapping is a valuation ([9, Proposition 2.6]).

For $n \in \mathbb{N}$, $n < \deg(g)$, such that $\Psi_n \neq \emptyset$ the fact that $\nu(g) = \infty$ implies that $\Psi_n$ admits a key polynomial $F$. In particular, if $Q \in \Psi_n$, then $\nu_Q \rightarrow \nu_F$ is an augmentation ([8, Theorems 6.1 and 6.2]). Moreover, this augmentation is a limit augmentation if and only if $\Psi_n$ does not have a maximum. Hence, we can define the defect $d(\Psi_n)$ of $\Psi_n$ by $d(\nu_Q \rightarrow \nu_F)$ (see more details in Section 2).

We will denote by $p$ the characteristic exponent of $Kv$. The main goal of this paper is to prove the following result.

**Theorem 1.1.** Assume that $d(L/K, \nu) = p^d$ and that $\text{rk}(\nu) = 1$. Then there exist uniquely determined $d_1, \ldots, d_{r-1} \in \mathbb{N}$, $d_r \in \mathbb{N}_0$, and for every $i$, $1 \leq i < r$, a uniquely determined subset $I_i \subseteq \{0, \ldots, d_i - 1\}$ such that the following hold.

(i) $d = d_1 + \cdots + d_r$.

(ii) There exist $n_1, \ldots, n_r \in \mathbb{N}$ with $n_1 < n_2 < \cdots < n_r$ such that $\Psi_{n_i}$, $1 \leq i \leq r$, are all the plateaus for $\nu$.

(iii) For every $i$, $1 \leq i \leq r$, we have $d(\Psi_{n_i}) = p^{d_i}$.

For each $i$, $1 \leq i < r$, and every limit key polynomial $F$ for $\Psi_{n_i}$, there exists $Q_i \in \Psi_{n_i}$ such that for every $Q \in \Psi_{n_i}$ with $\nu(Q) \geq \nu(Q_i)$, we have:

(iv) $p^{d_i} : = \{p^j \mid j \in I_i\} \subseteq L_Q(F)$; and
(v) \[
a_{Q_0}(F) + \sum_{j \in I_i} a_{Q_{p_j}}(F) Q_{p_j}^j + Q_{p_i}^{q_i}
\] (1)

is a limit key polynomial for \( \Psi_{n_i} \).

Moreover, if \( \Psi_{n_r} \) is bounded in \( \Gamma \), then we can also find a uniquely determined \( I_r \), and for \( F \in \Psi_{n_r} \) a polynomial \( Q_r \in \Psi_{n_r} \), satisfying (iv) and (v) (for \( i = r \)).

Theorem 1.1 can be seen as a generalization of the classification of defect extensions of degree \( p \) presented by Kuhlmann in [3] and extended by Kuhlmann and Rzepka in [4]. For a subset \( S \subseteq \Gamma \cup \{ \infty \} \), we define \( \overline{S} \) as the cut on \( \Gamma \) having the lower cut set given by

\[ \{ \gamma \in \Gamma \mid \exists s \in S \text{ with } \gamma \leq s \}. \]

Also, we define \( S^- \) as the cut on \( \Gamma \) having the lower cut set given by

\[ \{ \gamma \in \Gamma \mid \gamma < s \text{ for every } s \in S \}. \]

Suppose that \( vL = vK \). The distance of \( \eta \) to \( K \) is the cut

\[ \text{dist}(\eta, K) = \overline{\{v(\eta - b) \mid b \in K\}}. \]

In [3] and [4], the authors consider independent and dependent defect extensions in two cases. We will say that we are in the Artin–Schreier case if

\[
\begin{aligned}
L &= K(\eta) \text{ is an Artin–Schreier extension of } K, \\
\text{the minimal polynomial of } \eta \text{ over } K \text{ is } g = x^p - x - a; \text{ and } d(L/K, v) = p
\end{aligned}
\] (2)

We will say that we are in the Kummer case if

\[
\begin{aligned}
K \text{ contains a } p\text{th root of unity} \\
L &= K(\eta) \text{ is a Kummer extension of } K, \\
\text{the minimal polynomial of } \eta \text{ over } K \text{ is } g = x^p - a; \text{ and } v(a) = 0 \text{ and } d(L/K, v) = p
\end{aligned}
\] (3)

In the situation (2), we say that \( (L/K, v) \) is independent if

\[ \text{dist}(\eta, K) = H^- \text{ for some convex subgroup } H \text{ of } \Gamma. \]

Otherwise, it is called dependent. If (3) is satisfied, then we say that \( (L/K, v) \) is independent if

\[ \frac{v(p)}{p-1} + H^- \text{ for some convex subgroup } H \text{ of } \Gamma. \]

Otherwise, it is called dependent.
Proposition 1.2. Assume that either (2) or (3) is satisfied. Suppose that \( \text{rk}(\nu) = 1 \) and consider the valuation \( \nu \) on \( K[x] \) defined by \( v \) and \( \eta \). Then, in the notation of Theorem 1.1, we have \( r = 1 \) and \( d_1 = n_1 = 1 \). Moreover, \( \Psi_1 \) is bounded in \( \Gamma \) and

\[
I_1 = \emptyset \text{ if and only if } (L/K, \nu) \text{ is dependent.} \tag{4}
\]

Since the only possibilities for \( I_1 \) are \( \emptyset \) or \( \{0\} \), the condition (4) is equivalent to

\[
I_1 = \{0\} \text{ if and only if } (L/K, \nu) \text{ is independent.}
\]

The sets \( I_i \), appearing in Theorem 1.1, have a very explicit description. This description can be generalized even if \( \text{rk}(\nu) \neq 1 \). Namely, for a plateau \( \Psi_n \) and a limit key polynomial \( F \) for \( \Psi_n \), we consider the cut

\[
\delta_F = \{\nu_Q(F)\}_{Q \in \Psi_n}
\]
on \( \Gamma \). There exists \( D \in \mathbb{N} \) such that for every \( Q \in \Psi_n \), the \( Q \)-expansion of \( F \) is of the form

\[
F = a_{Q0}(F) + a_{Q1}(F)Q + \cdots + a_{QD}(F)Q^D.
\]

We define

\[
B(F) = \{b \in \{1, \ldots, D - 1\} \mid \nu(a_{Qb}(F)Q^b) \in \delta_F \text{ for every } Q \text{ with large enough value}\}.
\]

For a plateau \( \Psi_{n_i} \) and a limit key polynomial \( F \) for \( \Psi_{n_i} \) as in Theorem 1.1, the set \( I_i \) will be defined as the numbers \( j \) for which \( p_j \in B(F) \).

The next result is a generalization of Proposition 1.2 for rank greater than one.

Proposition 1.3. Assume that either (2) or (3) is satisfied. Consider the valuation \( \nu \) on \( K[x] \) defined by \( v \) and \( \eta \). Then, \( \delta_g < \infty^- \) and

\[
B(g) = \emptyset \iff (L/K, \nu) \text{ is dependent.} \tag{5}
\]

2 \hspace{1em} THE DEFECT OF AN AUGMENTATION

Let \( \mu \) be a valuation on \( K[x] \) with value group \( \Gamma_\mu \). The graded ring of \( \mu \) is defined as

\[
G_\mu := \bigoplus_{\gamma \in \Gamma_\mu} \{f \in K[x] \mid \mu(f) \geq \gamma\}/\{f \in K[x] \mid \mu(f) > \gamma\}.
\]

For \( h \in K[x] \) for which \( \mu(h) \neq \infty \), we define the initial form of \( h \) in \( G_\mu \) by

\[
\text{in}_\mu(h) := h + \{f \in K[x] \mid \mu(f) > \mu(h)\} \in G_\mu.
\]

Let \( (L/K, \nu) \) be a simple algebraic valued field extension (not necessarily of rank one) and take \( \eta \in L \) such that \( L = K(\eta) \). Consider the corresponding valuation \( \nu \) on \( K[x] \) defined by \( v \) and \( \eta \). For
a key polynomial $Q$ for $\nu$, we can consider the graded ring of $\nu_Q$ which we denote by $G_Q$ (instead of $G_{\nu_Q}$). For $f \in K[x]$, with $\nu_Q(f) \neq \infty$, we denote $\text{in}_Q(f) := \text{in}_{\nu_Q}(f)$. Let

$$R_Q := \langle \{\text{in}_Q(f) \mid \deg(f) < \deg(Q)\} \rangle$$

and $y_Q := \text{in}_Q(Q) \in G_Q$. This means that $R_Q$ is the abelian subgroup of $G_Q$ generated by the initial forms of polynomials of degree smaller than $\deg(Q)$.

**Proposition 2.1** [8, Proposition 4.5]. The set $R_Q$ is a subring of $G_Q$, $y_Q$ is transcendental over $R_Q$ and

$$G_Q = R_Q[y_Q].$$

In view of the previous proposition, for every $f \in K[x]$, with $\nu_Q(f) \neq \infty$, we can define the degree of $f$ with respect to $Q$ as the degree of $\text{in}_Q(f)$ with respect to $y_Q$, that is,

$$\deg_Q(f) := \deg_{y_Q}(\text{in}_Q(f)).$$

For $n \in \mathbb{N}$, suppose that $\Psi_n$ does not have a maximum and that $\Psi_n$ admits a limit key polynomial $F$. For any $Q \in \Psi_n$, by [8, Theorem 6.2], $\nu_Q \rightarrow \nu_F$ is a limit augmentation. Hence, we can define the defect of $\Psi_n$ (denoted by $d(\Psi_n)$) as the defect of $\nu_Q \rightarrow \nu_F$ as in [6, Definition 6.2]. Namely,

$$d(\Psi_n) := \lim_{Q' \in \Psi_n, \nu(Q') \geq \nu(Q)} \{\deg_{Q'}(F)\}.$$ 

**Theorem 2.2.** Let $(L/K, \nu)$ be a simple algebraic valued field extension and fix a generator $\eta$ of $L|K$. Consider the corresponding valuation $\nu$ on $K[x]$ defined by $\nu$ and $\eta$. Let $n_1, \ldots, n_r \in \mathbb{N}$ be all the natural numbers $n$ for which $\Psi_n$ is a plateau. Then

$$d(L/K, \nu) = \prod_{i=1}^{r} d(\Psi_{n_i}).$$

Moreover, if $\text{rk}(\nu) = 1$, then for every $i$, $1 \leq i \leq r$, for which $\Psi_{n_i}$ is bounded in $\Gamma$ we have

$$d(\Psi_{n_i}) = \frac{\deg(F)}{\deg(Q)} \text{ for every } Q \in \Psi_{n_i} \text{ and every } F \in KP(\Psi_{n_i}). \quad (6)$$

**Proof.** Let $\{m_1, \ldots, m_s\}$ be the set of natural numbers $m$ for which $\Psi_m$ is nonempty. For $j$, $1 \leq j \leq s$, if $\Psi_{m_j}$ has a maximum, then we choose $Q_j \in \Psi_{m_j}$ such that $\nu(Q_j)$ is the maximum. If $\Psi_{m_j}$ does not have a maximum (i.e., $m_j \in \{n_1, \ldots, n_r\}$), then we choose any $Q_j \in \Psi_{m_j}$. It follows from [8, Theorems 6.1 and 6.2] that

$$\nu_{Q_1} \rightarrow \nu_{Q_2} \rightarrow \cdots \nu_{Q_s} = \nu.$$
satisfies the conditions of [6, Theorem 6.14], and consequently
\[ d(L/K, v) = \prod_{j=1}^{s-1} d\left( \nu_{Q_j} \to \nu_{Q_{j+1}} \right) = \prod_{i=1}^{r} d(\Psi_{n_i}). \]

The last equality holds because \( d\left( \nu_{Q_j} \to \nu_{Q_{j+1}} \right) = 1 \) if the augmentation is ordinary ([6, Lemma 6.3]) and by the definition of \( d(\Psi_{n_i}) \).

If \( rk(v) = 1 \), then (6) follows from [6, Corollary 7.7].

\[ \square \]

Remark 2.3. Observe that as a consequence of (6), we deduce that, in this case, \( d(\Psi_{n_i}) > 1 \).

### 3 THE SUBSET I

For this section, we assume that \( rk(v) = 1 \), so we can suppose that \( vL \subseteq \mathbb{R} \). Fix \( n \in \mathbb{N} \) for which \( \Psi_n \) does not have a maximum and is bounded in \( \Gamma \). Throughout this section, we will fix a limit key polynomial \( F \) for \( \Psi_n \). We denote
\[ K[x]_n := \{ f \in K[x] | \deg(f) < n \}. \]

Write \( F = L(Q) \) for some \( L(X) \in K[x]_n[X] \) and denote \( D := \deg_X(L) \). The polynomial \( L(X) \) depends on \( Q \) and can be obtained from the \( Q \)-expansion of \( F \). Namely,
\[ L(X) = a_{Q0}(F) + a_{Q1}(F)X + \cdots + a_{QD}(F)X^D. \]

Set
\[ B = \lim_{Q \in \Psi_n} \nu(Q) \text{ and } \overline{B} = \lim_{Q \in \Psi_n} \nu_Q(F). \]

Take \( Q_0 \in \Psi_n \) and choose \( Q \in \Psi_n \) such that
\[ \varepsilon(Q) - \varepsilon(Q_0) > D(B - \nu(Q)). \tag{7} \]

By [9, Proposition 2.10 (iii)], we deduce that for \( Q, Q' \in \Psi_n \), we have \( \nu(Q') > \nu(Q) \) if and only if \( \varepsilon(Q') > \varepsilon(Q) \). Hence, \( Q \in \Psi_n \) as in (7) exists because the left-hand side grows and the right-hand side tends to zero when \( \nu(Q) \) grows.

Remark 3.1. We can deduce from [1, Proposition 4.1] that \( \overline{B} = D \cdot B \) and \( \nu_Q(F) = D \cdot \nu(Q) \) for \( Q \) with large enough value. Hence, condition (7) is equivalent to
\[ \varepsilon(Q) - \varepsilon(Q_0) > \overline{B} - \nu_Q(F). \tag{8} \]

We will consider the ring \( K(x)[X] \) where \( X \) is an indeterminate and let \( \partial_i \) denote the \( i \)-th Hasse derivative with respect to \( X \). Then, for every \( l(X) \in K(x)[X] \) and \( a, b \in K(x) \), we have the Taylor
expansion

\[ l(b) = l(a) + \sum_{i=1}^{\deg l} \partial_i l(a)(b - a)^i. \]

For simplicity of notation, we will take a well-ordered family \( \{Q_\rho\}_{\rho < \lambda} \) whose values \( \gamma_\rho := \nu(Q_\rho) \) are larger than \( \nu(Q) \) and form a cofinal family in \( \nu(\Psi_n) \). For each \( \rho < \lambda \), set \( h_\rho = Q - Q_\rho \). In particular, we can consider the Taylor expansion of \( F \) with respect to \( h_\rho \):

\[ F = L(h_\rho) + \sum_{i=1}^{D} \partial_i L(h_\rho) Q_\rho^i. \]  

(9)

For simplicity of notation, we will denote \( \nu_\rho := \nu_{Q_\rho} \) for every \( \rho < \lambda \).

**Lemma 3.2** [10, Corollary 2.5]. If \( \deg(f) < \deg(F) \), \( f = l(Q) \) for some \( l(X) \in K[x]_n[X] \), then there exists \( \rho \) such that

\[ \nu(l(h_\sigma)) = \nu(f) = \nu_\sigma(f) = \nu(a_\sigma(0)) \]

for every \( \sigma, \rho < \sigma < \lambda \).

For each \( i, 1 \leq i \leq D \), the polynomial \( \partial_i L(Q) \) has degree smaller than \( \deg(F) \). Hence, by Lemma 3.2, there exists \( \rho_0 < \lambda \) such that

\[ \beta_i := \nu(\partial_i L(Q)) = \nu(\partial_i L(h_\rho)) \]  

for every \( \rho, \rho_0 \leq \rho < \lambda \).

Moreover, by [2, Lemma 4], we can take \( \rho_0 \) so large that for every \( j, k, 1 \leq j < k \leq D \), we have

\[ \beta_j + j \gamma_\rho \neq \beta_k + k \gamma_\rho \]  

for every \( \rho_0 \leq \rho < \lambda \).

(11)

From now on, we will only consider \( \rho \) (and consequently \( \sigma \) and \( \theta \) appearing below), such that (10) and (11) are always satisfied (i.e., \( \min\{\rho, \sigma, \theta\} > \rho_0 \)).

For each \( \rho < \lambda \), denote

\[ F = a_{\rho_0}(F) + a_{\rho_1}(F)Q_\rho + \cdots + a_{\rho_r}(F)Q_\rho^D \]

the \( Q_\rho \)-expansion of \( F \).

**Lemma 3.3** [10, Lemma 4.2]. Fix \( \rho < \lambda \) and for each \( i, 0 \leq i \leq D \), set \( b_{\rho i} := a_{\rho_0}(\partial_i L(h_\rho)) \). Then

\[ \nu_\rho(\partial_i L(h_\rho) - b_{\rho i}) + iv(Q_\rho) > B. \]

**Corollary 3.4.** With the notation above, we have

\[ \nu(a_{\rho i}(F)Q_\rho^i) > B \iff \beta_i + i \gamma_\rho > B. \]
Proof. By the Taylor expansion of $F$ with respect to $h_\rho$, we have

$$a_\rho(F) = b_\rho + G,$$

where

$$G = \sum_{j < l} a_{\rho l-j}(\partial_j L(h_\rho)) = \sum_{j < l} a_{\rho l-j}(\partial_j L(h_\rho) - b_{\rho j}).$$

Hence, the result follows trivially from Lemma 3.3.

Denote by $J_\rho(F)$ the set

$$J_\rho(F) = \{j \in \{1, \ldots, D\} \mid \nu\left(a_{\rho j}(F)Q_j^l\right) > B\}.$$

Corollary 3.5. For $i \notin J_\rho(F)$, we have

$$\nu(a_\rho(F)) = \beta_i.$$

Proof. It follows again from the definition of $J_\rho(F)$ and Lemma 3.3.

Corollary 3.6. If $\rho < \sigma$, then $J_\rho(F) \subseteq J_\sigma(F)$.

Proof. Follows trivially from Corollary 3.4.

Since $\{1, \ldots, D\}$ is finite, there exists $\rho_1, \rho_0 \leq \rho_1 < \lambda$ such that for every $\rho$, $\rho_1 \leq \rho < \lambda$, we have $J_\rho(F) = J_{\rho_1}(F)$. Set

$$B_n(F) = \{1, \ldots, D-1\} \setminus J_{\rho_1}(F).$$

For a subset $S$ of $\{1, \ldots, D-1\}$ and $\rho, \rho_1 \leq \rho < \lambda$, we denote by

$$F_{S, \rho} = a_{\rho 0}(F) + \sum_{s \in S} a_{\rho s}(F)Q_s^l + a_{\rho D}Q_s^l.$$

Proposition 3.7. Take $\rho < \lambda, \rho_1 \leq \rho < \lambda$, and $S \subseteq B_n(F)$. Then $F_{S, \rho}$ is a limit key polynomial for $\Psi_n$ if and only if $S = B_n(F)$.

Proof. Suppose that $S = B_n(F)$. Then, for every $Q \in \Psi_n$, with $\nu(Q) \geq \nu(Q_{\rho})$, we have

$$\nu_Q(F - F_{S, \rho}) = \min_{j \in J(F)} \left\{ \nu_Q\left(a_{\rho j}(F)Q_j^l\right) \right\} = \min_{j \in J(F)} \left\{ \nu\left(a_{\rho j}(F)Q_j^l\right) \right\} > B > \nu_Q(F).$$

Hence, $\nu_Q(F) = \nu_Q(F_{S, \rho})$. Since $\deg(F) = \deg(F_{S, \rho})$, we conclude that $F_{S, \rho}$ is also a limit key polynomial for $\Psi_n$. 


Suppose now that $S \subset B_n(F)$. For any $Q \in \Psi_n$ such that
\[ \nu_Q(F) > \beta_h + h \gamma \rho := \min_{k \in B_n(F) \setminus S} \{ \beta_k + k \rho \}, \]
we have (by (11))
\[ \nu_Q(F_{S, \rho} - F_{B_n(F), \rho}) = \beta_h + h \gamma \rho < \nu_Q(F_{B_n(F), \rho}). \]
Hence, $\nu_Q(F_{S, \rho}) = \beta_h + h \gamma \rho$, which implies that $F_{S, \rho}$ is not a limit key polynomial for $\Psi_n$. \hfill \Box

**Proposition 3.8.** Let $H$ be another limit key polynomial for $\Psi_n$. Then $B_n(F) = B_n(H)$.

**Proof.** Since both $F$ and $H$ are monic, the polynomial $h = H - F$ has degree smaller than $\deg(F)$. Hence, there exists $\theta < \lambda$ such that $\nu_{\sigma}(h) = \nu_{\theta}(h)$ for every $\sigma$, $\theta \leq \sigma < \lambda$. Since $\{\nu_{\rho}(F)\}_{\rho \leq \lambda}$ and $\{\nu_{\rho}(H)\}_{\rho \leq \lambda}$ are increasing, this implies that
\[ \nu_{\sigma}(h) \geq B \text{ and } \nu_{\sigma}(F) = \nu_{\sigma}(H) \text{ for every } \sigma, \theta \leq \sigma < \lambda. \tag{12} \]

Take $j \in B_n(H)$. This means that for every $\sigma$, $\theta < \sigma < \lambda$, we have $\nu\left(a_{\sigma j}(H) Q_{\sigma j}^i\right) < B$. Since $a_{\sigma j}(F) = a_{\sigma j}(H) + a_{\sigma j}(h)$, this and (12) imply that
\[ \nu\left(a_{\sigma j}(F) Q_{\sigma j}^i\right) = \nu\left(a_{\sigma j}(H) Q_{\sigma j}^i\right) < B. \]
Hence, $B_n(H) \subset B_n(F)$. The other inclusion follows by the symmetric argument. \hfill \Box

Since the set $B_n(F)$ does not depend on the choice of $F$, we will denote it by $B_n$.

When referring to a polynomial $Q \in \Psi_n$ with large enough value, we mean that
\[ Q \text{ satisfies (7), } \nu(Q) > \nu(Q_{\rho_0}) \text{ and } \nu(Q) > \nu(Q_{\rho_1}). \tag{13} \]

**Corollary 3.9.** For every key polynomial $F$ for $\Psi_n$ and every $\sigma$ for which $Q_{\sigma} \in \Psi_n$ satisfies (13), we have
\[ B_n \subset L_{Q_{\sigma}}(F). \]

**Proof.** Take $i \in B_n$. By Corollary 3.4 and Corollary 3.5, we have
\[ \nu(a_{\sigma i}(F) Q_{\sigma i}^j) = \beta_i + i \gamma \sigma < B. \]
In particular, $i \in L_{Q_{\sigma}}(F)$.
\hfill \Box

### 3.1 Geometric interpretation of $B_n$

For $F \in KP(\Psi_n)$ and $Q \in \Psi_n$, we denote by $\Delta_Q(F)$ the *Newton polygon of $F$ with respect to $Q$*. This is defined as the lowest part of the convex hull of
\[ \{(i, \nu(a_{Q i}(F))) \mid i \in \mathbb{N}_0\}. \]
In this example, $p \in B_n$ and $1, 2, p^{d-2}, p^d - 1 \notin B_n$.

In $\mathbb{Q} \times \Gamma$. If $\Psi_n$ is bounded, then for large enough $Q$, the set $\Delta_Q(F)$ is the line segment connecting $(p^d, 0)$ and $(0, p^d \nu(Q))$ for $p^d = d(\Psi_n)$ (by [10, Proposition 3.2 and Lemma 4.2]). Consider the line $\pi$ passing through $(p^d, 0)$ and $(0, B)$. Since $B = p^d B$, this is the line with equation

$$\pi(y) = -B y + B.$$

**Lemma 3.10.** For $k \in \{1, \ldots, D - 1\}$, we have $k \in B_n$ if and only if $(k, \beta_k) \in \pi$.

**Proof.** By [10, Proposition 3.2 and Lemma 4.2], for every $\rho < \lambda$ with large enough value, we have $p^d \gamma_\rho \leq \beta_k + k \gamma_\rho$. Hence, $k \in B_n$ if and only if

$$p^d \gamma_\rho \leq \beta_k + k \gamma_\rho < B = p^d B$$

for every $\rho < \lambda$.

Taking the supremum of each of the expressions, this is equivalent to

$$p^d B \leq \beta_k + kB \leq p^d B.$$ 

This is equivalent to $\beta_k = -Bk + B$ and this happens if and only if $(k, \beta_k) \in \pi$. 

In Figure 1 below, we present the characterization of the set $B_n$ using Newton polygons described above. We consider $Q \in \Psi_n$ with large enough value and $F \in \text{KP}(\Psi_n)$. The Newton polygon $\Delta_Q(F)$ is represented in blue. The blue dots represent the points $(i, \nu(a_Q(F)))$. The line $\pi$ is represented in red.

### 4 PROOF OF THEOREM 1.1

**Proof of Theorem 1.1.** Since $\nu(g) = \infty$, there exist finitely many $n \in \mathbb{N}$ for which $\Psi_n \neq \emptyset$. Let $\{n_1, \ldots, n_r\}$ ($n_1 < \ldots < n_r$) be the set all the natural numbers for which $\Psi_n$ is a plateau. By Theorem 2.2, we have

$$d(\Psi_{n_1}) \mid d(L/K, v) = p^d.$$
Hence, for each \( i, 1 \leq i \leq r \), \( d(\Psi_{n_i}) = p^{d_i} \) for some \( d_i \in \mathbb{N}_0 \). The numbers \( d_1, \ldots, d_r \) are uniquely determined and \( d = d_1 + \cdots + d_r \). Moreover, since \( \text{rk}(v) = 1 \), the set \( \Psi_{n_i} \) is bounded for \( 1 \leq i < r \). By Remark 2.3, it follows from (6) that

\[
d(\Psi_{n_i}) = p^{d_i} > 1,
\]

and consequently, \( d_i > 0, 1 \leq i < r \).

For every \( i, 1 \leq i < r \), consider the set \( B_{n_i} \) constructed in the previous section. Set

\[
I_i := \{ j \in \mathbb{N}_0 \mid p^j \in B_{n_i} \}.
\]

By [10, Theorem 1.1], every element of \( B_{n_i} \) is a power of \( p \), that is, \( B_{n_i} = p^{I_i} \). If \( \Psi_{n_r} \) is bounded, then we also define \( I_r \) in the analogous way.

For each \( i \) such that \( \Psi_{n_i} \) is bounded, by Corollary 3.9, for every \( F \in \text{KP}(\Psi_{n_i}) \), take \( Q_i \in \Psi_{n_i} \) satisfying (13). For every \( Q \in \Psi_{n_i} \) with \( \nu(Q) \geq \nu(Q_i) \), we have \( p^j \subseteq L_Q(F) \) (by Corollary 3.9). Observe that \( a_{QD}(F) = 1 \) (by [5, Proposition 3.5]) and \( D = p^{d_i} \) (by Theorem 2.2). By Proposition 3.7,

\[
a_{Q_0}(F) + \sum_{j \in I_i} a_{Qp^j}(F)Q^{p^j} + Q^{p^{d_i}}
\]

is a limit key polynomial for \( \Psi_{n_i} \).

Take \( i, 1 \leq i < r \), such that \( \Psi_{n_i} \) is bounded. Suppose that \( I \) is any subset satisfying the conditions (iv) and (v) of Theorem 1.1. Since for every \( Q \) with large enough value,

\[
F_i := a_{Q_0}(F) + \sum_{j \in I_i} a_{Qp^j}(F)Q^{p^j} + Q^{p^{d_i}}
\]

is a limit key polynomial for \( \Psi_{n_i} \), we deduce from (iv) that \( I \subseteq I_i \) (because \( a_{Qp^j}(F_i) = 0 \) if \( j \not\in I_i \)). On the other hand, by Proposition 3.7, we cannot have \( I \not\subseteq I_i \). Hence, the set \( I_i \) is uniquely determined. This concludes the proof of Theorem 1.1.

\[\square\]

5 | DEFECT EXTENSIONS OF DEGREE \( p \)

5.1 | The rank one case

We will proceed with the proof of Proposition 1.2.

\textit{Proof.} Since \((L/K, v)\) is a defect extension of degree \( p \), it is immediate. In particular, \( \Psi_1 \) does not have a maximum and is bounded in \( \Gamma \). Since \( v_{x-b}(g) < \infty = v(g) \), the plateau \( \Psi_1 \) admits a limit key polynomial. Theorem 1.1 implies that \( g \) is a limit key polynomial for \( \Psi_1 \) (because for any limit key polynomial \( F \) for \( \Psi_1 \), we have \( p \leq \deg(F) \)).

Assume that (2) is satisfied. Since \( \text{rk}(v) = 1 \), we can assume that \( \Gamma \subseteq \mathbb{R} \). We set

\[
\gamma = \sup \{ v(\eta - b) \mid b \in K \} \in \mathbb{R}.
\]

(14)
Then, dist(\(\eta, K\)) = \(\gamma^-\). Since the only nontrivial convex subgroup of \(\Gamma\) is \(\{0\}\), \((L/K, v)\) is independent if and only if \(\gamma = 0\).

For each \(b \in K\), we have

\[ g = (x - b)^p - (x - b) + g(b). \]

Hence,

\[ v_{x-b}(g) = v(g(b)) = p \cdot v(x - b). \tag{15} \]

Set

\[ \delta = \sup\{v_{x-b}(g) \mid b \in K\} = \sup\{v_Q(g) \mid Q \in \Psi_1\}. \tag{16} \]

By (15) and (16), we conclude that \(\delta = p \cdot \gamma\). By definition of \(I_1\), we have \(0 \in I_1\) if and only if \(\delta = \gamma\) and this is satisfied if and only if \(\gamma = 0\).

Assume now that (3) is satisfied. Denote by \(\alpha = \frac{v(p)}{p-1} \in \Gamma\). For any \(b \in K\), we have

\[ g = (x - b)^p + pb(x - b)^{p-1} + \cdots + pb^{p-1}(x - b) + (b^p - a). \tag{17} \]

By [4, Proposition 3.7], we have \(\gamma \leq \alpha\). In particular, \(v_{x-b}(g) = p \cdot v(x - b)\) and consequently \(\delta = p \cdot \gamma\) for \(\gamma\) and \(\delta\) as in (14) and (16). Again dist(\(\eta, K\)) = \(\gamma^-\) and analogously to the Artin–Schreier case, the condition for being independent is satisfied if and only if \(\gamma = \alpha\). On the other hand, by (17), the condition \(0 \in I_1\) is equivalent to

\[ v(p) + \gamma = \delta = p \cdot \gamma \]

and this is equivalent to \(\gamma = \alpha\). This ends the proof of Proposition 1.2. \(\square\)

In what follows, we present the geometric description, as in Section 3.1, of each case. In Figures 2 and 3 below, we represent the geometric characterization of situations (2) and (3), respectively. The blue line represents the Newton polygon \(\Delta_{x-b}(g)\) for \(v(x - b)\) large enough. The red line represents the line \(\pi\) connecting \((0, \delta)\) and \((p, 0)\). This line has equation \(\pi(y) = -\gamma y + \delta\).
For the Artin–Schreier case, we consider the corresponding points that define $\Delta_{x-b}(g)$:

$$P_1 = (0, p \cdot \nu(x-b)), P_2 = (1, 0) \text{ and } P_3 = (p, 0).$$

In this case, $\gamma \leq 0$. One can see that $0 \in I_1$ (i.e., $P_2$ lies on $\pi$) if and only if $\gamma = 0$.

For the Kummer case, we consider the corresponding points that define $\Delta_{x-b}(g)$:

$$P_1 = (0, p \cdot \nu(x-b)), P_2 = (1, v(p)) \text{ and } P_3 = (p, 0).$$

In this case, $\gamma \leq \alpha$. One can see that $0 \in I_1$ (i.e., $P_2$ lies on $\pi$) if and only if $\gamma = \alpha$.

### 5.2 The higher rank case

For both cases, we set $\gamma = \text{dist}(\eta, K)$.

**Proof of Proposition 1.3.** Assume that (2) is satisfied. As before, for each $b \in K$, we deduce that $\nu(x-b) < 0$. Hence,

$$\nu_{x-b}(g) = p \cdot \nu(x-b) < \nu(x-b)$$

and consequently, $\delta_g \leq 0^- < \infty^-$. By [3, Proposition 4.2 and Lemma 2.14], $(L/K, \nu)$ is independent if and only if $p \cdot \gamma = \gamma$. In order to conclude the proof of Proposition 1.3 for this case, it is enough to show that $p \cdot \gamma \neq \gamma$ if and only if $B(g) = \emptyset$.

It follows from (18) that $p \cdot \gamma = \delta_g \leq \gamma$. Since the only possibility for $B(g)$ is $\{1\}$ or $\emptyset$, the condition $B(g) = \emptyset$ is equivalent to the existence of $b \in K$ such that

$$\nu(\eta - b) = \nu(x-b) > \delta_g = p \cdot \gamma.$$

This is, by definition, equivalent to $p \cdot \gamma < \gamma$.

Assume that (3) is satisfied and again denote by $\alpha = \frac{v(p)}{p-1} \in \Gamma$. By [4, Proposition 3.7], we have

$$\gamma \leq \alpha + H^-$$

for some convex subgroup $H$ of $\Gamma$ that does not contain $v(p)$. Let $H$ be the largest convex subgroup of $\Gamma$ with this property.
By (19) for every \( b \in K \) and every \( i, 1 \leq i < p \), we have \( \nu(x - b) < \frac{\nu(p)}{p-1} \). In particular,

\[
p \cdot \nu(x - b) < \nu(p) + i \nu(x - b) \quad \text{for every } i, 1 \leq i < p.
\]

Hence, \( \nu_{x-b}(g) = p \cdot \nu(x - b) \) and consequently \( \delta_g = p \cdot \gamma \leq (p \cdot \alpha)^- < \infty^- \). We also conclude that either \( B(g) \ni 1 \) or \( B(g) = \emptyset \).

For simplicity of notation, we will consider a well-ordered family \( \{b_\rho\}_{\rho < \lambda} \) in \( K \) such that \( \gamma_\rho := \nu(x - b_\rho) \) form a cofinal family in the lower cut set of \( \gamma \).

Suppose that \( B(g) = \emptyset \). We will show that there exists \( \varepsilon \in \Gamma, \varepsilon > H \) such that \( \alpha - \nu(x - c) > \varepsilon \) for every \( c \in K \). This will imply that

\[
\gamma - \alpha \leq (-\varepsilon)^- < H^- 
\]

and consequently, the extension is dependent. We assume (taking \( \gamma_\rho \) large enough) that for every \( \rho < \lambda \), we have

\[
\nu(p) + \gamma_\rho > p \cdot \gamma_\sigma \quad \text{for every } \sigma < \lambda.
\]

If there exist \( \rho, \sigma, \rho < \sigma < \lambda \) such that \( \varepsilon_0 := \gamma_\sigma - \gamma_\rho > H \), then for every \( \theta, \sigma < \theta < \lambda \), we have

\[
\gamma_\theta - \varepsilon_0 = \gamma_\theta - \gamma_\sigma + \gamma_\rho > \gamma_\rho > p \cdot \gamma_\theta - \nu(p).
\]

Hence,

\[
\alpha - \gamma_\theta > \frac{\varepsilon_0}{p-1}.
\]

Since \( \varepsilon_0 > H \) and \( H \) is a convex subgroup of \( \Gamma \), we deduce that \( \varepsilon := \frac{\varepsilon_0}{p-1} > H \).

Suppose that for every \( \rho, \sigma, \rho < \sigma < \lambda \), we have \( \gamma_\sigma - \gamma_\rho \not\in H \). Since \( H \) is convex, this implies that \( \gamma_\sigma - \gamma_\rho \in H \). Condition (19) implies that \( \alpha - \gamma_\rho > H \) for every \( \rho < \lambda \). Fix \( \rho < \lambda \) and set \( \epsilon = \frac{\alpha - \gamma_\rho}{2} \).

For every \( \sigma, \rho < \sigma < \lambda \), we have

\[
\alpha - \gamma_\sigma = \frac{\alpha - \gamma_\sigma}{2} + \frac{\alpha - \gamma_\sigma}{2} > \frac{\alpha - \gamma_\sigma}{2} = \epsilon + \frac{\gamma_\rho - \gamma_\sigma}{2}.
\]

We claim that \( \alpha - \gamma_\sigma > \epsilon \). Indeed, if this were not the case, then by (20), we would have

\[
0 \leq \epsilon - \alpha + \gamma_\sigma < \frac{\gamma_\sigma - \gamma_\rho}{2}.
\]

Since \( \frac{\gamma_\sigma - \gamma_\rho}{2} \in H \) (and \( H \) is convex), this would imply that \( \epsilon - \alpha + \gamma_\sigma \in H \). On the other hand, we have

\[
\epsilon - \alpha + \gamma_\sigma = \frac{\alpha - \gamma_\rho}{2} - \alpha + \gamma_\sigma = \frac{(\gamma_\sigma - \gamma_\rho)}{2} - \frac{(\alpha - \gamma_\sigma)}{2}.
\]

We would obtain that \( \alpha - \gamma_\sigma \in H \) and this is a contradiction to (19).
For the converse, assume that \((L/K, \nu)\) is dependent. Then there exists \(\epsilon > H\) such that
\[
\gamma - \alpha \leq \left(-\frac{\epsilon}{p-1}\right)^- < H^-.
\]
This implies that for every \(\rho < \lambda\), we have
\[
(p - 1) \cdot \gamma_{\rho} - \nu(p) < -\epsilon.
\]
Hence,
\[
\nu(p) + \gamma_{\rho} > p \cdot \gamma_{\rho} + \epsilon.
\]
Let \(\Gamma_1\) be the smallest convex subgroup of \(\Gamma\) for which (19) is not satisfied for \(H\) replaced by \(\Gamma_1\). In particular, \(\Gamma_1/H\) has rank one, \(\epsilon \in \Gamma_1 \setminus H\) and
\[
\nu(p) - (p - 1) \cdot \gamma_{\rho} \in \Gamma_1 \setminus H\text{ for }\rho\text{ large enough.}
\]
Taking infimum in \(\Gamma_1/H\), we deduce that there exists \(\rho < \lambda\) such that for every \(\sigma, \rho < \sigma < \lambda\), we have
\[
p \cdot (\gamma_{\sigma} - \gamma_{\rho}) < \epsilon.
\]
This and (21) imply that
\[
\nu(p) + \gamma_{\rho} > p \cdot \gamma_{\sigma}\text{ for every }\sigma, \rho < \sigma < \lambda.
\]
Hence, \(1 \notin B(g)\) and consequently \(B(g) = \emptyset\). This concludes the proof of Proposition 1.3. \(\square\)

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