An Abstract Stabilization Method with Applications to Nonlinear Incompressible Elasticity

Qingguo Hong, Liu Chunmei, Jinchao Xu
Department of Mathematics, The Pennsylvania State University,
University Park, PA 16802, USA
e-mail: huq11@psu.edu
College of Science, Hunan University of Science and Engineering,
Yongzhou 425199, Hunan, People’s Republic of China
e-mail: liuchunmei0629@163.com
Department of Mathematics, Pennsylvania State University,
University Park, PA 16802, USA
e-mail: xu@math.psu.edu

Abstract

In this paper, we propose and analyze an abstract stabilized mixed finite element framework that can be applied to nonlinear incompressible elasticity problems. In the abstract stabilized framework, we prove that any mixed finite element method that satisfies the discrete inf-sup condition can be modified so that it is stable and optimal convergent as long as the mixed continuous problem is stable. Furthermore, we apply the abstract stabilized framework to nonlinear incompressible elasticity problems and present numerical experiments to verify the theoretical results.

Keywords: nonlinear incompressible problem; mixed finite methods; stability

MR (2000) Subject Classifications: 65N30

1 Introduction

Many finite element schemes perform very well in terms of both accuracy and stability for linear elastic problems (see [1, 2, 3, 4]), including in highly constrained situations such as incompressible cases. However, it is well-known that though such schemes can be extended to nonlinear elasticity
cases (for instance, large deformation elasticity problems) the same level of
stability is by no means guaranteed (see [3, 6, 7, 8]).

In [9, 10], a stabilized discontinuous Galerkin method for nonlinear com-
pressible elasticity was introduced. Because the stabilization term adapts to
the solution of the problem by locally changing the size of a penalty term on
the appearance of discontinuities, it is called an adaptive stabilization stra-
gy. This same adaptive stabilization strategy can also be found in [11].
Although all three papers discuss compressible and nearly incompressible
nonlinear elasticity problems, they do not address the exactly incompress-
ible problems.

In [12], an isogeometric “stream function” formulation was used to ex-
actly enforce the linearized incompressibility constraint. In fact, the exact
satisfaction of the linearized incompressibility constraint leads to a numeri-
cal approximation of the stability range that converges to the exact one (i.e.,
the one for the continuum problem). As the stream function is given by a
fourth-order PDE, the high regularity of the NURBS shape functions must
be used there and the stream function in 3 dimensions is not unique.

Brezzi, Fortin, and Marini [13] presented a modified mixed formulation
for second-order elliptic equations and linear elasticity problems that auto-
matically satisfied the coercivity condition on the discrete level. Using the
same technique and applying stable Stokes elements to a modified formula-
tion for Darcy-Stokes-Brinkman models, Xie, Xu and Xue [14] obtained a
uniformly stable method.

In this paper, motivated by the modified formulation used in [13, 14]
for second-order equations and the Darcy-Stokes-Brinkman models respec-
tively, we devise a stabilization strategy for the classical mixed finite method
designed for Stokes equations and obtain a modified method for nonlinear
incompressible elasticity. As indicated in [3], a usual mixed finite element
method that is stable for Stokes equations such as the MINI element is not
stable for nonlinear incompressible elasticity even though the continuous
problem is stable. We note that, for the usual mixed finite method, the
unphysical instability is caused by the fact that the discrete solution is not
exactly divergence-free. Hence, we rewrite the continuous problem. Based
on that we design a stabilization strategy involving the divergence of the dis-
crete solution. We prove that the modified mixed finite element method can
remove the unphysical instability and lead to optimal convergence. Hence,
all stable Stokes elements can also be made stable for discrete nonlinear
elasticity problems.

The rest of the paper is organized as follows. In section 2 we present
the finite strain incompressible elasticity problem and obtain the linearized
formulation of the nonlinear elasticity problem. In section 3 we propose the stabilization strategy and derive the modified mixed finite element method in an abstract framework, and in section 4 we apply this modified method to the nonlinear incompressible elasticity problem. Furthermore, we obtain an optimal approximation of the modified finite element method. In section 5, we give two simple examples and the corresponding numerical experiments to verify the stability and accuracy of the modified finite element method. We present our conclusions in section 6.

2 Motivation: The finite strain incompressible elasticity problem

To study the finite strain incompressible elasticity problem, we adopt what is known as the material description in this paper. Given a reference configuration $\Omega \subset \mathbb{R}^d$ ($1 \leq d \leq 3$) for a d-dimensional bounded material body $B$, the deformation of $B$ can be described by the map $\tilde{\phi} : \Omega \rightarrow \mathbb{R}^d$ defined by

$$\tilde{\phi}(X) = X + \tilde{u}(X),$$

where $X = (X_1, \cdots, X_d)$ denotes the coordinates of a material point in the reference configuration and $\tilde{u}(X)$ represents the corresponding displacement vector. Following the standard notation (see [12]), we introduce the deformation gradient $\tilde{F}$ and the right Cauchy–Green deformation tensor $\tilde{C}$ by setting

$$\tilde{F} = I + \nabla \tilde{u}, \quad \tilde{C} = \tilde{F}^T \tilde{F},$$

(2.1)

where $I$ is the second-order identity tensor and $\nabla$ is the gradient operator with respect to the coordinate $X$.

For a homogeneous neo-Hookean material, for example, latex and rubber, we define (see [15]) the potential energy function as

$$\psi(\tilde{u}) = \frac{1}{2} \mu [I : \tilde{C} - d] - \mu \ln \tilde{J} + \frac{\lambda}{2} (\ln \tilde{J})^2,$$

(2.2)

where $\lambda$ and $\mu$ are positive constants, “:” represents the usual inner product for second-order tensors, and $\tilde{J} = \det \tilde{F}$ is the deformation gradient Jacobian.

When we introduce the pressure-like variable (or simply pressure) $\tilde{p} = \lambda \ln \tilde{J}$, the potential energy (2.2) can be equivalently written as the following function of $\tilde{u}$ and $\tilde{p}$ (still denoted as $\psi$ with a little abuse of notation):

$$\psi(\tilde{u}, \tilde{p}) = \frac{1}{2} \mu [I : \tilde{C} - d] - \mu \ln \tilde{J} + \tilde{p} \ln \tilde{J} - \frac{1}{2\lambda} (\tilde{p})^2.$$
When the body $B$ is subject to a given load $b = b(X)$ per unit volume in the reference configuration, the total elastic energy functional reads as

$$\Pi(\hat{u}, \hat{p}) = \int_\Omega \psi(\hat{u}, \hat{p}) - \int_\Omega b \cdot \hat{u}. \quad (2.3)$$

According to the standard variational principles, the equilibrium is derived by searching for critical points of $(2.3)$ in suitable admissible displacement and pressure spaces $\hat{V}$ and $\hat{Q}$. The corresponding Euler–Lagrange equations arising from $(2.3)$ lead to this solution: Find $(\hat{u}, \hat{p}) \in \hat{V} \times \hat{Q}$ such that

$$\left\{ \begin{array}{l}
\mu \int_\Omega \hat{F} : \nabla v + \int_\Omega (\hat{p} - \mu) \hat{F}^{-T} : \nabla v = \int_\Omega b \cdot v \quad \forall v \in V, \\
\int_\Omega (\ln \hat{J} - \frac{\hat{p}}{\lambda}) q = 0 \quad \forall q \in Q,
\end{array} \right. \quad (2.4)$$

where $V$ and $Q$ are the admissible variation spaces for the displacements and pressures, respectively. Also note that in $(2.4)$, the linearization of the deformation gradient Jacobian is

$$DJ(\hat{u})[v] = J(\hat{u})F(\hat{u})^{-T} : \nabla v = J(\hat{u})F(\hat{u}) : \nabla v \quad \forall v \in V.$$

We now focus on the case of an incompressible material, which corresponds to taking the limit $\lambda \to +\infty$ in $(2.4)$. Hence, our problem becomes: Find $(\hat{u}, \hat{p}) \in \hat{V} \times \hat{Q}$ such that

$$\left\{ \begin{array}{l}
\mu \int_\Omega \hat{F} : \nabla v + \int_\Omega (\hat{p} - \mu) \hat{F}^{-T} : \nabla v = \int_\Omega b \cdot v \quad \forall v \in V, \\
\int_\Omega q \ln \hat{J} = 0 \quad \forall q \in Q,
\end{array} \right. \quad (2.5)$$

or, in residual form: Find $(\hat{u}, \hat{p}) \in \hat{V} \times \hat{Q}$ such that

$$\left\{ \begin{array}{l}
\mathcal{R}_u((\hat{u}, \hat{p}), v) = 0 \quad \forall v \in V, \\
\mathcal{R}_p((\hat{u}, \hat{p}), q) = 0 \quad \forall q \in Q,
\end{array} \right. \quad (2.6)$$

where

$$\mathcal{R}_u((\hat{u}, \hat{p}), v) := \mu \int_\Omega \hat{F} : \nabla v + \int_\Omega (\hat{p} - \mu) \hat{F}^{-T} : \nabla v - \int_\Omega b \cdot v,$$

$$\mathcal{R}_p((\hat{u}, \hat{p}), q) := \int_\Omega q \ln \hat{J}. \quad (2.7)$$

We now derive the linearization of problem $(2.5)$ around a generic point $(\hat{u}, \hat{p})$. Observing that

$$D\hat{F}(\hat{u})[u] = -\hat{F}^{-T}(\nabla u)^T \hat{F}^{-T} \quad \forall u \in V,$$
we easily get the problem for the infinitesimal increment \((u, p)\): Find \((u, p) \in V \times Q\) such that
\[
\begin{cases}
\tilde{a}(u, v) + b(v, p) = -\mathcal{R}_v((\hat{u}, \hat{p}), v) \quad \forall v \in V, \\
\tilde{b}(u, q) = -\mathcal{R}_p((\hat{u}, \hat{p}), q) \quad \forall q \in Q,
\end{cases}
\] (2.8)
where
\[
\begin{align*}
\tilde{a}(u, v) &= \mu \int_{\Omega} \nabla u : \nabla v + \int_{\Omega} (\mu - \hat{p})(\hat{F}^{-1} \nabla u)^T : \hat{F}^{-1} \nabla v, \\
\tilde{b}(u, q) &= \int_{\Omega} q\hat{F}^{-T} : \nabla u.
\end{align*}
\] (2.9)

**Remark 2.1** Since problem (2.9) is the linearization of problem (2.5), it can be interpreted as the generic step of a Newton-like iteration procedure for the solution of the nonlinear problem (2.5).

**Remark 2.2** Taking \((\hat{u}, \hat{p}) = (0, 0)\) in (2.9), we immediately recover the classical linear incompressible elasticity problem for small deformations; i.e., we find \((u, p) \in V \times Q\) such that
\[
\begin{cases}
2\mu \int_{\Omega} \epsilon(u) : \epsilon(v) + \int_{\Omega} \rho \text{div} v = \int_{\Omega} b \cdot v \quad \forall v \in V, \\
\int_{\Omega} q \text{div} u = 0 \quad \forall q \in Q,
\end{cases}
\] (2.10)
where \(\epsilon(u) = \frac{(\nabla u)^T + \nabla u}{2}\) denotes the symmetric gradient operator.

We now note that the Piola identity \(\text{div}(J\hat{F}^{-T}) = 0\) and \(J = 1\) give \(\text{div}\hat{F}^{-T} = 0\). Hence, we have
\[
\text{div}(\hat{F}^{-T} \nabla u) = \text{div}\hat{F}^{-T} \cdot u + \hat{F}^{-T} : \nabla u = \hat{F}^{-T} : \nabla u.
\]
Let \(\hat{F}^{-T} \nabla u = w\), we then consider the following problem: Find \((w, p) \in V \times Q\) such that
\[
\begin{cases}
a(w, v) + b(v, p) = -\mathcal{R}_v((\hat{u}, \hat{p}), v) \quad \forall v \in V, \\
b(w, q) = -\mathcal{R}_p((\hat{u}, \hat{p}), q) \quad \forall q \in Q,
\end{cases}
\] (2.11)
where
\[
\begin{align*}
a(w, v) &= \tilde{a}(\hat{F}w, \hat{F}v) = \mu \int_{\Omega} \nabla(\hat{F}w) : \nabla(\hat{F}v) + \\
&\quad \int_{\Omega} (\mu - \hat{p})(\hat{F}^{-1} \nabla(\hat{F}w))^T : \hat{F}^{-1} \nabla(\hat{F}v), \\
b(w, p) &= \int_{\Omega} p \text{div} w.
\end{align*}
\] (2.12)
Let \( V_h \subset V \) and \( Q_h \subset Q \) be the compatible finite element spaces. The discrete problem of (2.11) reads: Find \((w_h,p_h) \in V_h \times Q_h\) such that
\[
\begin{aligned}
\{ & a(w_h,v_h) + b(v_h,p_h) = -\mathcal{R}_u((\hat{u},\hat{p}),v_h) \forall v_h \in V_h, \\
& b(w_h,q_h) = -\mathcal{R}_p((\hat{u},\hat{p}),q_h) \forall q_h \in Q_h,
\}
\tag{2.13}
\end{aligned}
\]
where \( a(\cdot,\cdot) \) and \( b(\cdot,\cdot) \) are defined by (2.12).

In some cases, it is possible that the continuous problem (2.11) is stable (or well-posed) but that the discrete problem (2.13) is not (see [5]). In such cases, we say that there is an unphysical instability caused by the discretization.

3 Abstract stabilization strategy for mixed approximation

As shown at the end of the previous section, it is necessary to establish stable discretization when the continuous problem (2.11) is stable. So in this section, we propose an abstract stability strategy framework for the mixed approximation.

Let \( V \) and \( Q \) each be a Hilbert space, and assume that
\[
a : V \times V \to \mathbb{R}, \quad b : V \times Q \to \mathbb{R}
\]
are continuous bilinear forms. Let \( f \in V' \) and \( g \in Q' \). We denote both the dual pairing of \( V \) with \( V' \) and that of \( Q \) with \( Q' \) by \( \langle \cdot, \cdot \rangle \). We consider the following problem: Find \((w,p) \in V \times Q\) such that
\[
\begin{aligned}
\{ & a(w,v) + b(v,p) = \langle f,v \rangle \forall v \in V, \\
b(w,q) = \langle g,q \rangle \forall q \in Q.
\}
\tag{3.1}
\end{aligned}
\]
We associate a mapping \( B \) with the form \( b \):
\[
B : V \to Q' : \langle Bw,q \rangle = b(w,q) \forall q \in Q.
\]
Define
\[
K = \ker(B) := \{ v \in V : b(v,q) = 0 \forall q \in Q \}.
\]
As is well-known, the continuous problem (3.1) is well-posed under the following assumptions:

- \( A_1 \) : the inf-sup condition, i.e., it holds that
\[
\inf \sup_{q \in Q \atop \|q\|_Q} \frac{b(v,q)}{\|v\|_V \|q\|_Q} \geq \beta > 0,
\tag{3.2}
\]
where \( \| \cdot \|_V \) and \( \| \cdot \|_Q \) are norms on the spaces \( V \) and \( Q \), respectively;
A2: the coercivity on the kernel space, i.e., it holds that
\[ a(v, v) \geq \alpha \|v\|_V^2 \quad \forall \ v \in K, \text{ for some } \alpha > 0. \tag{3.3} \]

Now we rewrite the problem (3.1) in an equivalent form as follows: Find \((w, p) \in V \times Q\) such that
\[
\begin{cases}
A(w, v) + b(v, p) = \langle f, v \rangle + M(g, Bv)_{Q'} \quad \forall v \in V, \\
b(w, q) = \langle g, q \rangle \quad \forall q \in Q,
\end{cases}
\tag{3.4}
\]
where \(A(w, v) = a(w, v) + M(Bw, Bv)_{Q'}\), \((\cdot, \cdot)_{Q'}\) denotes the inner product on \(Q'\), and \(M\) is a parameter to be chosen properly.

Let \(V_h \subset V\) and \(Q_h \subset Q\) be finite dimensional subspaces of \(V\) and \(Q\) such that the following assumption is satisfied:

\(A3\): discrete inf-sup condition holds that
\[
\inf_{q_h \in Q_h} \sup_{v_h \in V_h} \frac{b(v_h, q_h)}{\|v_h\|_V \|q_h\|_Q} \geq \beta_1 > 0,
\tag{3.5}
\]
where \(\beta_1\) is independent of \(h\).

We consider the discretization of the problem (3.4) as follows: Find \((w_h, p_h) \in V_h \times Q_h\) such that
\[
\begin{cases}
A(w_h, v_h) + b(v_h, p_h) = \langle f, v_h \rangle + M(g, Bv_h)_{Q'} \quad \forall v_h \in V_h, \\
b(w_h, q_h) = \langle g, q_h \rangle \quad \forall q_h \in Q_h.
\end{cases}
\tag{3.6}
\]

**Remark 3.1** Note that when \(M = 0\), (3.6) becomes the classical mixed finite element method. We call (3.6) the modified mixed finite element.

**Lemma 3.1** (see [16]) \(A1\) is equivalent to the following condition, the operator \(B: V^\perp \to Q'\) is an isomorphism, and
\[
\|Bv\|_{Q'} \geq \beta \|v\|_V \quad \forall v \in V^\perp.
\tag{3.7}
\]

**Theorem 3.1** Under the assumptions \(A1\) and \(A2\), there is a constant \(M_0\) such that the discrete problem (3.6) is stable for any \(M \geq M_0\) in the sense that there exists a positive constant \(\alpha_1\) such that
\[
A(v_h, v_h) \geq \alpha_1 \|v_h\|_V^2 \quad \forall v_h \in V_h.
\tag{3.8}
\]
Proof For any \( v_h \in V_h \), noting that \( V_h \subset V \), we have \( v_h = \varphi + \varphi^\perp \), where \( \varphi \in K \), \( \varphi^\perp \in K^\perp \) and \( K^\perp \) denotes the orthogonal of \( K \) in \( V \). By Lemma 3.1 we have

\[
A(v_h, v_h) = a(v_h, v_h) + M(Bv_h, Bv_h)_{Q'}
\]

\[
= a(\varphi + \varphi^\perp, \varphi + \varphi^\perp) + M(B\varphi^\perp, B\varphi^\perp)_{Q'}
\]

\[
\geq a(\varphi, \varphi) + a(\varphi, \varphi^\perp) + a(\varphi^\perp, \varphi) + a(\varphi^\perp, \varphi^\perp) + M\beta^2\|\varphi^\perp\|_V.
\]

By assumption \( A_2 \), the continuity of \( a(\cdot, \cdot) \) and the Cauchy inequality, we obtain that

\[
A(v_h, v_h) \geq \alpha \|\varphi\|_V^2 - C_1 \|\varphi\|_V \|\varphi^\perp\|_V + C_2 \|\varphi^\perp\|_V + M\beta^2 \|\varphi^\perp\|_V^2
\]

\[
\geq (\alpha - \varepsilon C_1) \|\varphi\|_V^2 - (C_2 + \frac{C_1}{\varepsilon}) \|\varphi^\perp\|_V + M\beta^2 \|\varphi^\perp\|_V^2
\]

\[
\geq (\alpha - \varepsilon C_1) \|\varphi\|_V^2 + (M\beta^2 - (C_2 + \frac{C_1}{\varepsilon})) \|\varphi^\perp\|_V^2.
\]

Noting that \( \|v_h\|_V^2 = \|\varphi\|_V^2 + \|\varphi^\perp\|_V^2 \) and choosing \( \varepsilon = \frac{\alpha}{2C_1}, M_0 = \frac{\alpha + C_2 + 2C^2}{\beta^2} \), we have

\[
A(v_h, v_h) \geq \alpha_1 (\|\varphi\|_V^2 + \|\varphi^\perp\|_V^2) = \alpha_1 \|v_h\|_V^2,
\]

which completes the proof.

Furthermore, noting that \( A(\cdot, \cdot) \) is continuous, by the classic Brezzi theory for the saddle point problem\[16, 17, 18\], we have

**Theorem 3.2** Let \( w, p \) be the solution of the problem \( (3.4) \), then under the assumptions \( A_1, A_2, A_3 \) and providing that \( M \) is sufficiently large, the discrete problem \( (3.6) \) has a unique solution \( (w_h, p_h) \in V_h \times Q_h \), which satisfies

\[
\|w - w_h\|_V + \|p - p_h\|_Q \leq C \left( \inf_{v_h \in V_h} \|w - v_h\|_V + \inf_{q_h \in Q_h} \|p - q_h\|_Q \right),
\]

where \( C \) is a constant depending on \( \alpha_1 \) and \( \beta_1 \).

### 4 Application to the nonlinear elasticity problem

We now apply the abstract stabilized framework proposed in the previous section to nonlinear incompressible elasticity problems. Suppose that \( \Omega \) is
connected, we consider the d-dimensional Dirichlet boundary value problem (other boundary value problems are similar) of (2.11), which means that 
\[ V = (H^1_0(\Omega))^d, Q = L^2_0(\Omega). \]
We choose \( V_h \) and \( Q_h \) as finite element spaces such that \( A_3 \) is satisfied. Furthermore, \( K = \{ v \in (H^1_0(\Omega))^d : \text{div} v = 0 \} \). We rewrite (2.11) as:

\[
\begin{align*}
    A(w, v) + b(v, p) &= -\mathcal{R}_a((\hat{u}, \hat{p}), v) - M\mathcal{R}_p((\hat{u}, \hat{p}), \text{div} v) \quad \forall v \in V, \\
    b(w, q) &= -\mathcal{R}_p((\hat{u}, \hat{p}), q) \quad \forall q \in Q,
\end{align*}
\]

where

\[
A(w, v) = a(w, v) + M(\text{div} w, \text{div} v), \ b(\cdot, \cdot) \text{ and } b(\cdot, \cdot) \text{ are defined by (2.12), } \langle \cdot, \cdot \rangle \text{ denotes the inner product on } L^2(\Omega), \text{ and } M \text{ is a parameter chosen properly.}
\]

The discretization of the problem (4.1) is as follows: Find \((w_h, p_h) \in V_h \times Q_h\) such that

\[
\begin{align*}
    A(w_h, v_h) + b(v_h, p_h) &= -\mathcal{R}_a((\hat{u}_h, \hat{p}_h), v_h) - M\mathcal{R}_p((\hat{u}_h, \hat{p}_h), \text{div} v_h) \quad \forall v_h \in V_h, \\
    b(w_h, q_h) &= -\mathcal{R}_p((\hat{u}_h, \hat{p}_h), q_h) \quad \forall q_h \in Q_h.
\end{align*}
\]

Lemma 4.1 ([19], Corollary 2.3) We have

\[ K^\perp = \{(-\Delta)^{-1}\text{grad} f : f \in L^2_0(\Omega)\}. \]

Let \( K^0 = \{ y \in (H^{-1}(\Omega))^d : \langle y, \phi \rangle = 0 \quad \forall \phi \in K \}, \) where \( \langle \cdot, \cdot \rangle \) denotes the duality paring between \((H^{-1}(\Omega))^d\) and \((H^1_0(\Omega))^d\).

Lemma 4.2 ([19], Corollary 2.4) Let \( \Omega \) be connected. Then

1. the operator \( \text{grad} \) is an isomorphism from \( L^2_0(\Omega) \) onto \( K^0 \), and
2. the operator \( \text{div} \) is an isomorphism from \( K^\perp \) onto \( L^2_0(\Omega) \).

Theorem 4.1 If \( a(\cdot, \cdot) \) defined by (2.12) satisfies

\[ a(v, v) \geq \alpha \|v\|^2_1 \quad \forall v \in K, \text{ for some } \alpha > 0, \]

the discrete problem (4.2) is stable in the sense that there exists a positive constant \( \alpha_1 \) such that

\[ A(v_h, v_h) \geq \alpha_1 \|v_h\|^2_1 \quad \forall v_h \in V_h. \]

Proof By the abstract framework, the proof is obvious. \(\blacksquare\)
Also from the abstract framework in the previous section, we can get an approximate accuracy result similar to (3.9).

**Theorem 4.2** Let \( w, p \) be the solution of the problem (2.11), then under the assumptions \( A_2 \) and \( A_3 \) and providing that \( M \) is sufficiently large, the discrete problem (4.2) has a unique solution \( (w_h, p_h) \in V_h \times Q_h \), which satisfies

\[
\|w - w_h\|_1 + \|p - p_h\|_0 \leq C \left( \inf_{v_h \in V_h} \|w - v_h\|_1 + \inf_{q_h \in Q_h} \|p - q_h\|_0 \right),
\]

where the constant \( C \) depends on \( \alpha_1, \beta_1 \).

5 Numerical experiment

In this section, we report our numerical experiments in regard to the performance of the stabilization strategy for two simple examples. We consider here simple problems using some mixed finite element formulations that are known to be optimal for Stokes equations. We first briefly present the finite element under consideration and then show the numerical results obtained using such element.

5.1 Two examples

In this subsection, we present two simple problems that will be used in subsection 5.3 to discuss the performance of the stabilization strategy proposed in Section 3. Using the usual Cartesian coordinates \((X, Y)\), we consider a square material body whose reference configuration is \( \Omega = (-1, 1) \times (-1, 1) \). We denote \( \Gamma = [-1, 1] \times \{1\} \) as the upper part of its boundary, while the remaining part of \( \partial \Omega \) is denoted with \( \Gamma_D \). The total energy is assumed to be as in (2.3), where the external loads are given by the vertical uniform body forces: \( b = \gamma f \), where \( f = (0, 1)\). The two problems differ in regard to the imposed boundary conditions. More precisely:

- **Problem 1.** We set clamped boundary conditions on \( \Gamma_D \), but traction-free boundary conditions on \( \Gamma \).
- **Problem 2.** We set vanishing normal displacements on \( \Gamma_D \), but traction-free boundary conditions on \( \Gamma \).

It is easy to see that both problems admit a trivial solution for every \( \gamma \in \mathbb{R} \), i.e. \((\hat{u}, \hat{p}) = (0, \gamma r)\), where \( r = r(X, Y) = 1 - Y \).
For the problems under investigation, the corresponding linearized problems (cf. (2.12)) can both be written as: Find \((w, p) \in V \times Q\) such that

\[
\begin{aligned}
2\mu \int_{\Omega} \epsilon(w) : \epsilon(v) - \gamma \int_{\Omega} r(\nabla w)^\top : \nabla(v) + \int_{\Omega} p \text{div} v &= \delta \gamma \int_{\Omega} f \cdot v, \quad \forall v \in V, \\
\int_{\Omega} q \text{div} w &= 0 \quad \forall q \in Q.
\end{aligned}
\]

(5.1)

where \(\delta \gamma\) is the increment of the parameter of \(\gamma\).

For these two different problems, the spaces \(V\) and \(Q\) are defined as follows:

- **Problem 1.** \(V = \{v \in H^1(\Omega)^2 : v|_{\Gamma_{D}} = 0\}; Q = L^2(\Omega)\).

- **Problem 2.** \(V = \{v \in H^1(\Omega)^2 : (v \cdot n)|_{\Gamma_{D}} = 0\}; Q = L^2(\Omega)\), where \(n\) denotes the outward normal vector.

The stable discrete formulation reads as follows (see (4.2)): Find \((w_h, p_h) \in V_h \times Q_h\) such that

\[
\begin{aligned}
A(w_h, v_h) + b(v_h, p_h) &= \delta \gamma \int_{\Omega} f \cdot v_h, \quad \forall v_h \in V_h, \\
b(w_h, q_h) &= 0, \quad \forall q_h \in Q_h.
\end{aligned}
\]

(5.2)

where \(A(w, v) = 2\mu \int_{\Omega} \epsilon(w) : \epsilon(v) - \gamma \int_{\Omega} r(\nabla w)^\top : \nabla(v) + M(\gamma) \int_{\Omega} \text{div} w \text{div} v; b(w, p) = \int_{\Omega} p \text{div} v\). Here \(M(\gamma)\) is a parameter chosen depending on \(\gamma\).

**Remark 5.1** For Problem 1, it has been theoretically proved in [5] that the continuous problem (5.1) is stable when \(\gamma < 3\mu\).

### 5.2 The MINI element

The considered scheme for (5.2) is the MINI element (see [18]). Let \(T_h\) be a triangular mesh of \(\Omega\) with the mesh size \(h\). For the discretization of the displacement field, we take

\[V_h = \{v_h \in V : v_h|_T \in P_1(T)^2 + B(T)^2 \forall T \in T_h\},\]

where \(P_1(T)\) is the space of linear functions on \(T\), and \(B(T)\) is the linear space generated by \(b_T\), the standard cubic bubble function on \(T\). For the pressure discretization, we take

\[Q_h = \{q_h \in H^1(\Omega) \cap Q : q_h|_T \in P_1(T) \forall T \in T_h\}.\]
5.3 Numerical results

We now study the stability performance of the discretized model problems by means of the modified mixed finite element formulations briefly described above. It has been theoretically proved and numerically verified in [5] that for Problem 1 the classical mixed finite element method is stable when \( \gamma < \mu \), but unstable when \( \gamma > \frac{3}{2} \mu \). In this numerical experiment, we will demonstrate that the modified mixed finite element method is stable for both Problem 1 and Problem 2 when \( -\infty < \gamma < 3 \mu \) (which is the stability range for the continuous case of Problem 1 in [5]) as predicated by our Theorem 3.1.

Noting Theorem 3.1, we study the eigenvalues of the matrix induced by the bilinear form \( A(\cdot, \cdot) \) for the problems under consideration. The first loads for which we find a negative eigenvalue are the critical ones. We start from \( \gamma = 0 \) for both positive and negative loading conditions, i.e., for \( \gamma < 0 \) and \( \gamma > 0 \). We indicate as the critical loads \( \gamma_{m,h} \) and \( \gamma_{M,h} \) the first load values for which we find a negative eigenvalue. A subsequent bisection-type procedure is used to increase the accuracy of the critical load detections. The corresponding nondimensional quantities are denoted with \( \tilde{\gamma}_{m,h} \) and \( \tilde{\gamma}_{M,h} \), respectively, where \( \tilde{\gamma} = \frac{\gamma L}{\mu} \). Here, \( L \) is some problem characteristic length, set equal to 1 for simplicity, consistent with the geometry of the model problems. If we do not detect any negative eigenvalue for extremely large values of the load multiplier (\( \tilde{\gamma} > 10^6 \)), we set \( \tilde{\gamma} = \infty \). Noting that the bound constant \( C_1 \) for the formulation \( a(\cdot, \cdot) \) is of order \( \gamma + 2 \mu \) and the stable constant of \( a(\cdot, \cdot) \) is of order \( \mu \), we can choose \( M_0 = m_1|\tilde{\gamma}| + m_2 \tilde{\gamma}^2 \) by the proof of Theorem 3.1. Hence, in the codes, we set \( \mu = 40, M(\gamma) = m_1|\tilde{\gamma}| + m_2 \tilde{\gamma}^2 \), then we adjust the parameter \( \tilde{\gamma}, m_1 \) and \( m_2 \) to investigate the stability performance. Furthermore, we take \( m_1 = 320, m_2 = 0 \) for Problem 1 and \( m_1 = 320, m_2 = 1.36 \) for Problem 2.

Table 1: Stability Limits for Problem 1

| Nodes | \( \tilde{\gamma}_{m,h} \) | \( \tilde{\gamma}_{M,h} \) |
|-------|-----------------|-----------------|
| 5 \times 5 | \( -\infty \) | \( +\infty \) |
| 9 \times 9 | \( -\infty \) | 14.50 |
| 17 \times 17 | \( -\infty \) | 8.25 |
| 33 \times 33 | \( -\infty \) | 7.13 |

To verify the convergence result (3.2), we set another data for (5.1) that \( f = (-e^x(1 - y), e^x)^T \) and the true solution is \( (w, p) = (0, \delta \gamma e^x(1 - y)) \).

Table 1 and Table 2 show that the stabilization strategy together with the
corresponding modified mixed finite element method proposed in this paper is effective. The stability performance can be improved obviously. In fact, these values of $\tilde{\gamma}_{M,h}$ are competitive to the “exact” values claimed in [12]. Furthermore, we can see that the modified mixed finite element method is also locking-free by verifying the convergence results (3.2), which are shown in Table 3 and Table 4. In fact, the classical mixed finite element method is unstable for the Problem 1 when $\tilde{\gamma} = 7.125$, and hence the convergence fails. However, the modified method performs well.

### 6 Conclusions

Within the framework of finite elasticity for incompressible materials, it is well known that the classical mixed finite element discretization can sometimes be unstable even though the continuous problem is stable. In this paper, we reformulated the continuous problem and proposed an abstract stabilization strategy based on the new continuous formulation and obtained a modified mixed finite element method. We proved theoretically in Section 3 that for a sufficiently large $M$, the modified mixed finite element method is stable whenever the continuous problem is stable, and the method maintains the optimal convergence of the classical one. We verified by numerical experiments in Section 5 that the modified mixed finite element method is much more stable than the classical one and is also locking-free. The $M_0$ provided in Section 3 always overestimates the parameter used in practical problems. However, we can choose the parameter heuristically by analyz
Table 4: The convergence of \( \tilde{\gamma} = 3.23 \) for Problem 2

| Nodes   | \( \|p - p_h\|_0 \)   | \( \|w - w_h\|_1 \)   | order |
|---------|-----------------|-----------------|-------|
| 5 \times 5 | 6.0187 \times 10^{-2} | 7.5563 \times 10^{-6} | --   |
| 9 \times 9  | 1.5129 \times 10^{-2} | 1.6314 \times 10^{-6} | 2     |
| 17 \times 17 | 3.7867 \times 10^{-3} | 3.3616 \times 10^{-7} | 2     |
| 33 \times 33 | 9.4691 \times 10^{-4} | 7.7049 \times 10^{-8} | 2     |

ing the stability and continuity of continuous problems in the numerical experiments presented in Section 5.

References

[1] B. Reddy, J. Simo, Stability and convergence of a class of enhanced strain methods, SIAM J. Numer. Anal. 32 (6) (1995) 1705–1728.

[2] D. Braess, C. Carstensen, B. Reddy, Uniform convergence and a posteriori error estimators for the enhanced strain finite element method, Numer. Math. 96 (3) (2004) 461–479.

[3] P. Houston, D. Schotzau, T. Wihler, An hp-adaptive mixed discontinuous galerkin fem for nearly incompressible linear elasticity, Comput. Methods Appl. Mech. Engrg. 195 (25-28) (2006) 3224–3246.

[4] P. Hansbo, M. Larson, Discontinuous Galerkin methods for incompressible and nearly incompressible elasticity by Nitsche’s method, Comput. Methods Appl. Mech. Engrg. 191 (17-18) (2002) 1895–1908.

[5] F. Auricchio, L. Beirao da Veiga, C. Lovadina, A. Reali, A stability study of some mixed finite elements for large deformation elasticity problems, Comput. Methods Appl. Mech. Engrg. 194 (9-11) (2005) 1075–1092.

[6] P. Wriggers, S. Reese, A note on enhanced strain methods for large deformations, Comput. Methods Appl. Mech. Engrg. 135 (3-4) (1996) 201–209.

[7] C. Lovadina, F. Auricchio, On the enhanced strain technique for elasticity problems, Comput. & Structures 81 (8-11) (2003) 777–787.

[8] D. Pantuso, K. Bathe, On the stability of mixed finite elements in large strain analysis of incompressible solids, Finite Elem. Anal. Des. 28 (2) (1997) 83–104.
[9] A. Ten Eyck, F. Celiker, A. Lew, Adaptive stabilization of discontinuous Galerkin methods for nonlinear elasticity: Analytical estimates, Comput. Methods Appl. Mech. Engrg. 197 (33-40) (2008) 2989–3000.

[10] A. Ten Eyck, F. Celiker, A. Lew, Adaptive stabilization of discontinuous Galerkin methods for nonlinear elasticity: Motivation, formulation, and numerical examples, Comput. Methods Appl. Mech. Engrg. 197 (45-48) (2008) 3605–3622.

[11] A. Ten Eyck, F. Eyck, A. Lew, An adaptive stabilization strategy for enhanced strain methods in non-linear elasticity, Internat. J. Numer. Methods Engrg. 81 (11) (2010) 1387–1416.

[12] F. Auricchio, L. Beirão da Veiga, C. Lovadina, A. Reali, The importance of the exact satisfaction of the incompressibility constraint in nonlinear elasticity: mixed FEMs versus NURBS-based approximations, Comput. Methods Appl. Mech. Engrg. 199 (5-8) (2010) 314–323.

[13] F. Brezzi, M. Fortin, L. Marini, Mixed finite element methods with continuous stresses, Math. Models Methods Appl. Sci. 3 (1993) 275–287.

[14] X. Xie, J. Xu, G. Xue, Uniformly stable finite element methods for Darcy-Stokes-Brinkman models, J. Comput. Math. 26 (3) (2008) 437–455.

[15] J. Bonet, R. Wood, Nonlinear continuum mechanics for finite element analysis, Cambridge University Press, New York, 1997.

[16] F. Brezzi, M. Fortin, Mixed and hybrid finite element methods, Springer-Verlag, New York, 1991.

[17] F. Brezzi, On the existence, uniqueness and approximation of saddle-point problems arising from lagrangian multipliers, RAIRO Anal. Numer. 8 (2) (1974) 129–151.

[18] D. Arnold, F. Brezzi, M. Fortin, A stable finite element for the Stokes equations, Calcolo 21 (4) (1984) 337–344.

[19] V. Girault, P. Raviart, Finite element methods for Navier-Stokes equations: theory and algorithms, Springer-Verlag, Berlin-Heidelberg-New York-Tokyo, 1986.