Existence Theory for the Boussinesq Equation in Modulation Spaces

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Abstract
In this paper we study the Cauchy problem for the generalized Boussinesq equation with initial data in modulation spaces \( M_{p,q}^{s}(\mathbb{R}^n) \), \( n \geq 1 \). After a decomposition of the Boussinesq equation in a 2 \times 2-nonlinear system, we obtain the existence of global and local solutions in several classes of functions with values in \( M_{p,q}^{s} \times D^{-1}JM_{p,q}^{s} \)-spaces for suitable \( p, q \) and \( s \), including the special case \( p = 2, q = 1 \) and \( s = 0 \). Finally, we prove some results of scattering and asymptotic stability in the framework of modulation spaces.

Keywords Boussinesq equation · Modulation spaces · Local and global solutions · Scattering · Asymptotic stability

Mathematics Subject Classification 35Q53 · 35A01 · 47J35 · 35B40 · 35B35

1 Introduction
We consider the initial value problem associated to the generalized Boussinesq equation

\[
\begin{align*}
\frac{\partial^2}{\partial t^2} u - \Delta u + \Delta^2 u + \Delta f(u) &= 0, & (x,t) \in \mathbb{R}^{n+1}, \\
u(x,0) &= u_0(x), & x \in \mathbb{R}^n, \\
\partial_t u(x,0) &= \phi(x) = \Delta v_0(x), & x \in \mathbb{R}^n,
\end{align*}
\] (1.1)

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where \( u : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \) is the unknown, \( u_0, v_0 : \mathbb{R}^n \to \mathbb{R} \) are given functions denoting the initial data and the nonlinear term is \( f(u) = u^\lambda \), for some \( 1 < \lambda < \infty \). Equation (1.1) is physically relevant in the modelling of shallow water waves, ion sound waves in a plasma, the dynamics of stretched string, and other physical phenomena (Cho and Ozawa 2007; Peregrine 1972). The IVP (1.1) is formally equivalent to the following system

\[
\begin{align*}
\partial_t u &= \Delta v, \\
\partial_t v &= u - \Delta u - f(u), \\
u(x, 0) &= u_0(x), \quad v(x, 0) &= v_0(x),
\end{align*}
\]

From Duhamel’s principle, the Cauchy problem associated to system (1.2) is equivalent to the integral equation

\[
[u(t), v(t)] = B(t)[u_0, v_0] - \int_0^t B(t - \tau)[0, f(u(\tau))]d\tau,
\]

where \( B \) is the solution of the linear problem associated to (1.2). More exactly, for initial data \([u_0, v_0] \) and \( t \in \mathbb{R} \), we have

\[
B(t)[u_0, v_0] = \int_{\mathbb{R}^n} e^{i \cdot \cdot \cdot \cdot} \left[ \widehat{\mathcal{B}_1(t)u_0}(\xi) + \widehat{\mathcal{B}_2(t)v_0}(\xi), \widehat{\mathcal{B}_3(t)u_0}(\xi) + \widehat{\mathcal{B}_1(t)v_0}(\xi) \right] d\xi,
\]

where \( \mathcal{B}_1(t) \), \( \mathcal{B}_2(t) \) and \( \mathcal{B}_3(t) \) are the multiplier operators with symbols \( \cos(t|\xi|\langle \xi \rangle) \), \( -|\xi|\langle \xi \rangle^{-1} \sin(t|\xi|\langle \xi \rangle) \) and \( |\xi|^{-1} \langle \xi \rangle \sin(t|\xi|\langle \xi \rangle) \) respectively, and \( \langle \xi \rangle = (1 + |\xi|^2)^{1/2} \).

Note that

\[
B(t)[u_0, v_0] = [B_1(t)u_0 + B_2(t)v_0, B_3(t)u_0 + B_1(t)v_0].
\]

Several authors have analyzed the local and global existence, and long time asymptotic behavior of solutions for (1.1) (cf. Bona and Sachs 1988; Cho and Ozawa 2007; Farah 2008, 2009a, b; Ferreira 2011; Kishimoto 2013; Kishimoto and Tsugawa 2010; Linares 1993; Liu 1997; Muñoz et al. 2018; Tsutsumi and Matahashi 1991 and references therein). In particular, in Bona and Sachs (1988), considering the 1D case, the authors decomposed (1.1) in the following system

\[
\begin{align*}
\partial_t u &= \partial_x v, \\
\partial_t v &= \partial_x (u - \partial_x u - f(u)), \\
u(x, 0) &= u_0(x), \quad v(x, 0) &= v_0(x),
\end{align*}
\]

and analyzed the local well-posedness with initial data \( u_0 \in H^{s+2}(\mathbb{R}), \phi = (v_0)_x, v_0 \in H^{s+1}(\mathbb{R}), s > 0 \) and \( f \) smooth. Existence results for \( f(u) = |u|^{\lambda-1}u \) and initial data \( u_0 \in H^1(\mathbb{R}), \phi = (v_0)_{xx}, v_0 \in H^1(\mathbb{R}) \), were obtained by Tsutsumi and Matahashi (1991). For the same nonlinearity \( f(u) \), Linares (1993) proved the local well-posedness with either \([u_0, \phi] = [u_0, (v_0)_x] \in L^2(\mathbb{R}) \times H^{-1}(\mathbb{R}) \) and \( 1 < \lambda < 5 \), or \([u_0, \phi] = [u_0, (v_0)_x] \in H^1(\mathbb{R}) \times L^2(\mathbb{R}) \) and \( 1 < \lambda < \infty \). The results in Linares (1993) were extended by Farah (2009a) for the \( n \)-dimensional case. Some local well-posedness results with initial data \([u_0, \phi] = [u_0, (v_0)_x] \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R}) \) and

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Results of asymptotic behavior have been addressed by Cho and Ozawa (2007), Farah (2008) and Liu (1997), in the framework of $H^s_p$ and $B^s_{p,2}$-spaces for $s > 0$ and $1 < p < 2$. More exactly, in Liu (1997) the author analyzed the existence of solutions in one-dimensional case for $f(u) = |u|^{\lambda-1}u$, $\lambda > 5$, and small initial data $[u_0, v_0] \in (H^1(\mathbb{R}) \times L^2(\mathbb{R})) \cap X^s_0$, where $X^s_0 := L^r_5 \times L^r_{s-1}$, being $L^r_s$ the Bessel potential space with potential $J^s = (1 - \partial_{xx})^{s/2}$. In the same paper, a scattering result for small perturbations was obtained. The results in Liu (1997) were extended in Cho and Ozawa (2007) to the case $n \geq 1$ in the framework of Besov $B^s_{p,2}$-spaces. The initial data considered in Cho and Ozawa (2007) belongs to a subset of $B^{s+n\delta}_{s+1,2} \times \Omega(B^{s+n\delta}_{s+1,2})$, $\delta = 1 - \frac{2}{s+1}$ and $\Omega(\psi) = |\xi| \langle \xi \rangle \hat{\psi}$. Using the formulation (1.2), in Farah (2008) was analyzed the reciprocal problem of the existence of solutions of (1.2) with initial data in modulation spaces.

Long time behavior and scattering theory results, in the $L^{(p,\infty)}$-framework, were also obtained in Ferreira (2011). Previous initial data classes satisfy the following embedding relations

$$B^{s}_{p,1} \subset H^s_p \subset B^{s}_{p,\infty} \subset L^{(q,\infty)}, \quad \text{for } s \geq 0, \quad \text{and } \frac{1}{q} = \frac{1}{p} - \frac{s}{n}.$$  

Motivated by the previous references, in this paper we study the local and global existence of solutions of (1.2) with initial data in modulation spaces $M^{s}_{p,q}(\mathbb{R}^n)$. Modulation spaces were introduced by Feichtinger (1983), prompted by the idea of measuring the smoothness classes of functions or distributions. Since their introduction, modulation spaces have become canonical for both time-frequency and phase-space analysis (Chaichenets et al. 2017). Wang and Hudzik (2007) gave an equivalent definition of modulation spaces by using the frequency-uniform-decomposition operators. In the same work, the existence of global solutions for nonlinear Schrödinger and Klein-Gordon equations in modulation spaces were analyzed. After then, several studies on nonlinear PDEs in the framework of modulation spaces have been addressed (cf. Chaichenets et al. 2017; Huang et al. 2016; Iwabuchi 2010; Ruzhansky et al. 2012; Wang and Huang 2007 and references therein). In this context, the contribution of this paper is to develop the existence and long time asymptotic behavior of the solutions for the generalized Boussinesq Eq. (1.1) with initial data in modulation spaces $M^{s_1}_{p,q}(\mathbb{R}^n)$. Our results provide a new class of initial data with regularity lower than $H^s$, $H^s_p$ and $B^s_p$ for large $s$; this is consequence of the embeddings $H^{s_1}_p \subset M^{s_2}_{p,q}$ and $B^{s_1}_p \subset M^{s_2}_{p,q}$, provided $s_1 > s + n\nu_1(p, q)$, for some positive
value $\nu_1(p, q)$ as described below (see Remark 1.7), complementing the previous existence results in energy spaces $H^s$ obtained in Farah (2008, 2009a, b). We also prove some results of scattering and asymptotic stability in the setting of modulation spaces. In order to get this aims, first we derive some linear estimates for the one parameter group $B$ introduced in (1.4) (cf. Sect. 2), which constitutes an additional contribution of this paper.

Before stating our main results, we review some definitions and notations related to the modulation spaces $M^s_{p,q}(\mathbb{R}^n)$ (see for instance Wang and Hudzik 2007). Let $Q_k$ be the unit close cube with center at $k$. Consider $\rho \in \mathcal{S}(\mathbb{R}^n)$ such that $\rho : \mathbb{R}^n \to [0, 1]$ is a radial smooth bump function satisfying $\rho(\xi) = 1$ for $|\xi| \leq \frac{\sqrt{n}}{2}$ and $\rho(\xi) = 0$ for $|\xi| \geq \sqrt{n}$. Let $\rho_k(\xi) = (\rho(\xi - k), k \in \mathbb{Z}^n$, a translation of $\rho$. It holds that $\rho_k(\xi) = 1$ in $Q_k$, and thus, $\sum_{k \in \mathbb{Z}^n} \rho_k(\xi) \geq 1$ for all $\xi \in \mathbb{R}^n$. Let

$$\sigma_k(\xi) = \rho_k(\xi) \left( \sum_{l \in \mathbb{Z}^n} \rho_l(\xi) \right)^{-1}, \quad k \in \mathbb{Z}^n, \xi \in \mathbb{R}^n.$$ 

Then, the sequence $\{\sigma_k\}_{k \in \mathbb{Z}^n}$ verifies the following properties:

$$|\sigma_k(\xi)| \geq C, \quad \forall \xi \in Q_k,$$

$$\text{supp}(\sigma_k) \subset \{\xi : |\xi - k| \leq \sqrt{n}\},$$

$$\sum_{k \in \mathbb{Z}^n} \sigma_k(\xi) = 1, \quad \forall \xi \in \mathbb{R}^n,$$

$$|D^\theta \sigma_k(\xi)| \leq C_m, \quad \forall \xi \in \mathbb{R}^n, \quad |\theta| \leq m.$$

We consider the frequency-uniform decomposition operators $\square_k := \mathcal{F}^{-1} \sigma_k \mathcal{F}, k \in \mathbb{Z}^n$. Then, for $s \in \mathbb{R}, 1 \leq p, q \leq \infty$, the modulations spaces $M^s_{p,q}$, are Banach spaces defined as (cf. Wang and Hudzik 2007):

$$M^s_{p,q} := \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \| f \|_{M^s_{p,q}} < \infty \right\},$$

where

$$\| f \|_{M^s_{p,q}} = \left\{ \begin{array}{ll}
(\sum_{k \in \mathbb{Z}^n} (1 + |k|)^s q \| \square_k f \|_p^q)^{1/q}, & \text{for } 1 \leq q < \infty, \\
\sup_{k \in \mathbb{Z}^n} (1 + |k|)^s \| \square_k f \|_p, & \text{for } q = \infty.
\end{array} \right.$$

For simplicity, we will write $M^0_{p,q}(\mathbb{R}^n) = M_{p,q}(\mathbb{R}^n)$. Properties on $M^s_{p,q}$, including embeddings in other known function spaces, can be found in Wang and Hudzik (2007).

In particular, the following embeddings hold:

(i) $M^{s_1}_{p_1,q_1} \subset M^{s_2}_{p_2,q_2}$, if $s_1 \geq s_2, 0 < p_1 \leq p_2, 0 < q_1 \leq q_2$,

(ii) $M^{s_1}_{p,q_1} \subset M^{s_2}_{p,q_2}$, if $q_1 > q_2, s_1 > s_2, s_1 - s_2 > n/q_2 - n/q_1$,

(iii) $M^{s_1}_{p,q} \subset L^\infty \cap L^p$, for $1 < p \leq \infty$,

(iv) For $0 < p, q \leq \infty$ and $s \in \mathbb{R}$, it holds that $B^{s+n/q}_{p,q} \subset M^s_{p,q}$.
(v) For $0 < p, q \leq \infty$, $s, \sigma \in \mathbb{R}$, the operator $(I - \Delta)^{\sigma/2} : M^s_{p,q} \to M^{s-\sigma}_{p,q}$ is an isomorphic mapping.

(vi) $B^s_{p,q} \subset M^s_{p,q}$ if and only if $s_1 \geq s_2 + n v_1(p, q)$.

(vii) $H^s_p \subset M^s_{p,q}$, if $s_1 > s_2 + n v_1(p, q)$, where

\[
v_1(p, q) = \begin{cases}
0, & \text{if } \left(\frac{1}{p}, \frac{1}{q}\right) \in \left(\frac{1}{p}, \frac{1}{q}\right) \subset [0, \infty)^2 : \frac{1}{q} \leq \frac{1}{p} \leq 1 - \frac{1}{p}, \\
\frac{1}{p} + \frac{1}{q} - 1, & \text{if } \left(\frac{1}{p}, \frac{1}{q}\right) \in \left(\frac{1}{p}, \frac{1}{q}\right) \subset [0, \infty)^2 : \frac{1}{p} \geq \frac{1}{q} \geq 1 - \frac{1}{p}, \\
-\frac{1}{p} + \frac{1}{q}, & \text{if } \left(\frac{1}{p}, \frac{1}{q}\right) \in \left(\frac{1}{p}, \frac{1}{q}\right) \subset [0, \infty)^2 : \frac{1}{q} \leq \frac{1}{p} \leq 1 - \frac{1}{p}.
\end{cases}
\]

Let $D^s = (-\Delta)^{s/2}$ and $J^s = (I - \Delta)^{s/2}$, for any $s \in \mathbb{R}$. Given the Banach space $X = L^\gamma(\mathbb{R}^n)$, we also consider the function spaces $l^q_X(X)$ and $l^{s,q}_X(X)$, where $s, q < \infty$, $s, q \in \mathbb{R}$, introduced in Wang and Hudzik (2007), which are defined as follows:

\[
l^q(\mathbb{R}^{n+1}) = \left\{ f \in \mathcal{F}(\mathbb{R}^{n+1}) : \|u\|_{l^q(\mathbb{R}^{n+1})} := \left( \sum_{k \in \mathbb{Z}^n} \|\Box_k f\|_X^q \right)^{1/q} < \infty \right\},
\]

\[
l^{s,q}(\mathbb{R}^{n+1}) = \left\{ f \in \mathcal{F}(\mathbb{R}^{n+1}) : \|u\|_{l^{s,q}(\mathbb{R}^{n+1})} := \left( \sum_{k \in \mathbb{Z}^n} (1 + |k|)^{sq} \|\Box_k f\|_X^q \right)^{1/q} < \infty \right\}.
\]

We consider the following time-dependent spaces on which we will establish our existence results. We denote by $L^\alpha_{a,s}$ the distribution-valued pairs $[u, v] : (-\infty, \infty) \to M^s_{p,q} \times D^{-1}J M^s_{p,q}$ with norm given by

\[
\|[u, v]\|_{L^\alpha_{a,s}} := \sup_{-\infty < t < \infty} (1 + |t|)^\alpha \left( \|u(t)\|_{M^s_{p,q}} + \|v(t)\|_{D^{-1}J M^s_{p,q}} \right),
\]  

(1.6)

where $\alpha = n \left(\frac{1}{2} - \frac{1}{p}\right)$, with $2 \leq p$, and $\|v(t)\|_{D^{-1}J M^s_{p,q}} := \|J^{-1}Dv(t)\|_{M^s_{p,q}}$. Since $J^{\sigma/2} : M^s_{p,q} \to M^{s-\sigma}_{p,q}$ is an isomorphic mapping, $[u, v] : (-\infty, \infty) \to M^s_{p,q} \times D^{-1}J M^s_{p,q}$ is equivalent to say that $[u, Dv] : (-\infty, \infty) \to M^s_{p,q} \times M^{s-1}_{p,q}$ and

\[
\|[u, v]\|_{L^\alpha_{a,s}} = \sup_{-\infty < t < \infty} (1 + |t|)^\alpha \left( \|u(t)\|_{M^s_{p,q}} + \|Dv(t)\|_{M^{s-1}_{p,q}} \right).
\]  

(1.7)

We also consider the space $L^T_{a,s}$ of the distribution-valued pairs $[u, v] : (-T, T) \to M^s_{p,q} \times D^{-1}J M^s_{p,q}$ with norm given by

\[
\|[u, v]\|_{L^\alpha_{a,s}} := \sup_{-T < t < T} (1 + |t|)^\alpha \left( \|u(t)\|_{M^s_{p,q}} + \|v(t)\|_{D^{-1}J M^s_{p,q}} \right).
\]  

(1.8)

Throughout this paper, $\lambda_0(n)$ corresponds to the positive root of the equation $n\lambda^2 - (n + 2)\lambda - 2 = 0$, that is, $\lambda_0(n) = \frac{n^2 + 12n + 4}{2n - 2}$. We also denote by $\lambda_1(n) = \frac{n + 2}{n - 2}$.
for $n \geq 3$ and $\lambda_1(n) = \infty$ if $n = 1, 2$. Now we are in position to establish the main results of this paper.

**Theorem 1.1** Let $\lambda$ a positive integer such that $\lambda > \lambda_0(n)$, $p = \lambda + 1$, $1 \leq q < 2$, and $n - \frac{n}{p} \leq s < \frac{n}{q}$. Suppose that $[u_0, v_0] \in M^{s}_{p', q} \times M^{s}_{p', q}$. There exists $\epsilon > 0$ small enough such that if $\|u_0\|_{M^{s}_{p', q}} + \|v_0\|_{M^{s}_{p', q}} \leq \epsilon$, the IVP (1.2) has a unique global mild solution $[u, v] \in L^\infty_{\alpha, s}$ satisfying $\|u, v\|_{L^\infty_{\alpha, s}} \leq 2\epsilon\overline{c}$, for some constant $\overline{c} > 0$ independent of the initial data. Moreover, the data-solution map $[u_0, v_0] \mapsto [u, v]$ from $M^{s}_{p', q} \times M^{s}_{p', q}$ into $L^\infty_{\alpha, s}$ is locally Lipschitz.

**Remark 1.2** System (1.2) has not a scaling relation which brings some disadvantages compared to related models such as the Schrödinger or semilinear heat equations with nonlinearities of type $f(u) = u^\lambda$ (Braze Silva et al. 2009; Ferreira and Villamizar-Roa 2006, 2012). However, as point out in Ferreira (2011), system (1.2) has an intrinsical scaling given by

$$[u, v] \mapsto [\rho u, \rho v] := \rho^{\frac{2}{3}}[u(\rho x, \rho^2 t), v(\rho x, \rho^2 t)], \quad \text{for } \rho > 0. \quad (1.9)$$

Thus, if $\beta = \frac{1-q}{\lambda - 1} > 0$, with $\lambda > 1$, $\alpha = n\left(\frac{1}{2} - \frac{1}{p}\right) < 1$, $p = \lambda + 1$, then $\beta$ is the unique one such that the norm $\|[u, v]\|_{H^\infty_{\beta, s}}$ defined by

$$\|[u, v]\|_{H^\infty_{\beta, s}} := \sup_{-\infty < t < \infty} |t|^\beta (\|u(t)\|_{M^{s}_{p, q}} + \|v(t)\|_{D^{-1}JM^{s}_{p, q}}), \quad (1.10)$$

is invariant by (1.9). Condition $\beta > 0$ is equivalent to $\lambda < \lambda_1(n)$. Comparing the norms $\|[u, v]\|_{H^\infty_{\beta, s}}$ and $\|[u, v]\|_{L^\infty_{\alpha, s}}$ it holds that $0 < \beta \leq \alpha$ if and only if $\lambda_0(n) \leq \lambda < \lambda_1(n)$. Consequently, it holds that $\|[u, v]\|_{H^\infty_{\beta, s}} \leq \|[u, v]\|_{L^\infty_{\alpha, s}}$ and $L^\infty_{\alpha, s} \subset H^\infty_{\beta, s}$, for $\lambda_0(n) \leq \lambda < \lambda_1(n)$. Observe that condition $\lambda_0(n) \leq \lambda < \lambda_1(n)$ implies $1 \leq n \leq 5$.

**Remark 1.3** Consider $p, q, s$ as in Theorem 1.1 and $\lambda_0(n) < \lambda < \lambda_1(n)$. There exists $\epsilon > 0$ such that if

$$\sup_{-\infty < t < \infty} |t|^\beta \|B_1(t)u_0\|_{M^{s}_{p, q}} + \sup_{-\infty < t < \infty} |t|^\beta \|B_2(t)v_0\|_{M^{s}_{p, q}} < \frac{\epsilon}{2}, \quad (1.11)$$

$$\sup_{-\infty < t < \infty} |t|^\beta \|B_3(t)u_0\|_{D^{-1}JM^{s}_{p, q}} + \sup_{-\infty < t < \infty} |t|^\beta \|B_1(t)v_0\|_{D^{-1}JM^{s}_{p, q}} < \frac{\epsilon}{2}, \quad (1.12)$$

then the IVP (1.2) has a unique global mild solution $[u, v] : (-\infty, \infty) \rightarrow M^{s}_{p, q} \times D^{-1}JM^{s}_{p, q}$ satisfying $\|[u, v]\|_{H^\infty_{\beta, s}} < \infty$ (see Remarks 2.3, 2.5, 3.2 and 3.4 below). Observe that condition $\lambda_0(n) < \lambda < \lambda_1(n)$ implies $1 \leq n \leq 5$.

Theorem 1.1 excludes the case $M_{2.1}$. Next theorem ensures the existence of global solution for initial data in $M_{2.1} \times M_{2.1}$. In this case, the relations $B_{2.1}^{n/2} \subset M_{2.1} \subset L^\infty \cap L^2$ hold.
Theorem 1.4 Let $n \geq 1, \lambda \in \mathbb{N}$ such that $\lambda > 1 + \frac{4}{n}$, and $p \in \left[2 + \frac{4}{n}, \lambda + 1\right] \cap \mathbb{N}$. Assume that $u_0, v_0 \in M_{2,1}$. There exists $\epsilon > 0$ small enough such that if $\|u_0\|_{M_{2,1}} \times \|v_0\|_{M_{2,1}} \leq \epsilon$, then (1.2) has a unique global solution

$$[u, v] \in [C(\mathbb{R}; M_{2,1}) \cap H^1_\alpha((L^p(\mathbb{R}; L^p))) \times C(\mathbb{R}; D^{-1}JM_{2,1})].$$

Theorem 1.1 provides the existence of global solution for $\lambda > \lambda_0(n)$. For $\lambda \leq \lambda_0(n)$ we are able to ensure a local in time solution. This is the content of next theorem.

Theorem 1.5 Assume that $\lambda$ is a positive integer such that $1 < \lambda \leq \lambda_0(n)$, $p = \lambda + 1$, $1 \leq q < \infty$, $n - \frac{n}{q} \leq s < \frac{n}{q}$. Then, if $[u_0, v_0] \in M_{p',q}^s \times M_{p',q}^s$, there exists $0 < T < \infty$ such that the IVP (1.2) has a unique local mild solution $[u, v] \in L^T_{\alpha,s}$. Moreover, the data-solution map $[u_0, v_0] \mapsto [u, v]$ from $M_{p',q}^s \times M_{p',q}^s$ into $L^T_{\alpha,s}$ is locally Lipschitz.

Remark 1.6 Observe that restriction $1 < \lambda \leq \lambda_0(n)$ implies $1 \leq n \leq 3$.

Remark 1.7 The initial data class in Theorems 1.1, 1.4 and 1.5 is larger than the $H^s_p$ and $B^s_{p,q}$, provided $s_1$ be large enough. This is consequence of the embeddings $H^s_p \subset M^s_{p,q}$, and $B^s_{p,q} \subset M^s_{p,q}$, provided $s_1 > s_2 + n\nu_1(p, q)$. On the other hand, Theorems 1.1 and 1.4 remain true if we replace the time interval $(-\infty, \infty)$ by the compact interval $[-T, T]$ throughout their statements, which gives a new class for local existence (see Farah 2008, 2009a, b).

Now we establish a result on the scattering theory which describes the asymptotic behavior of solutions for the Boussinesq system in the framework of modulation spaces. We find an initial data $[u_0^\pm, v_0^\pm]$ such that the solution $[u^\pm, v^\pm]$ of the linear problem associated to the system (1.2), with initial data $[u_0^\pm, v_0^\pm]$, describes the asymptotic behavior of the global solution provided by Theorem 1.1. This is the content of the next theorem.

Theorem 1.8 (Scattering) Assume the conditions on Theorem 1.1, and let $[u, v]$ be the solution of (1.1) provided by Theorem 1.1 with data $[u_0, v_0] \in M_{p',q}^s \times M_{p',q}^s$. Then there exists $[u_0^\pm, v_0^\pm] \in L^\infty_{\alpha,s}$ such that

$$\|\beta(t) - u^\pm(t), v(t) - v^\pm(t)\|_{M_{p',q}^s \times D^{-1}JM_{p,q}^s} = O(|t|^{1-\alpha}), \quad \text{as } t \to \pm \infty,$$

(1.13)

where $[u^\pm(t), v^\pm(t)], [u^-(t), v^-(t)]$ stand for the unique global mild solutions of the linear problem associated to (1.2) with initial data $[u_0^+, v_0^+]$ and $[u_0^-, v_0^-]$, respectively.

Theorem 1.9 (Asymptotic stability) Assume the conditions on Theorem 1.1, and let $[u, v], [ar{u}, \bar{v}]$ be the solutions of (1.1) provided by Theorem 1.1 with data $[u_0, v_0], [\bar{u}_0, \bar{v}_0] \in M_{p',q}^s \times M_{p',q}^s$, respectively. Then

$$\lim_{|t| \to \infty} (1 + |t|)^\alpha \|\beta(t)[u_0 - \bar{u}_0, v_0 - \bar{v}_0]\|_{M_{p,q}^s \times D^{-1}JM_{p,q}^s} = 0.$$
if and only if
\[
\lim_{|t| \to \infty} (1 + |t|)^\alpha \|[u(t) - \tilde{u}(t), v(t) - \tilde{v}(t)]\|_{M_{p,q}^s \times D^{-1} M_{p,q}^s} = 0.
\]

2 Linear and Nonlinear Estimates

In this section we establish some linear estimates for the group \( B \) defined in (1.4). We start denoting
\[
l_2(t) g := J^{-1} DB_3(t) g = [\sin(t|\xi|\langle\xi\rangle)\hat{g}(\xi)]^\vee.
\]

**Lemma 2.1** Let \( 2 \leq p < \infty \) and \( i = 1, 2, 3 \). Then for all \( g \in L^p \) it holds
\[
\|B_i(t) g\|_{L^p} \leq c|t|^{-\alpha} \|g\|_{L^{p'}},
\]
\[
\|l_2(t) g\|_{L^p} \leq c|t|^{-\alpha} \|g\|_{L^{p'}}.
\]
with \( \frac{1}{p} + \frac{1}{p'} = 1 \) and \( \alpha = n \left( \frac{1}{2} - \frac{1}{p} \right) \).

**Proof** The estimate for \( B_i, i = 1, 2, 3 \), can be found in Farah (2008), Lemma 3.8, Inequality (21) [see also Linares and Scialom (1995) (Theorem 2.1) and Ferreira (2011), Lemma 2.1]. In order to prove the estimate for \( l_2 \) notice that the symbol \( \langle\xi\rangle^{-1}|\xi| \) of the operator \( J^{-1} D \) defines a multiplier in \( L^p \) and therefore it is continuous (cf. Stein 1970); therefore,
\[
\|l_2(t) g\|_{L^p} = \|J^{-1} DB_3(t) g\|_{L^p} \leq c\|B_3(t) g\|_{L^p} \leq c|t|^{-\alpha} \|g\|_{L^{p'}}.
\]

\( \square \)

**Lemma 2.2** Let \( 2 \leq p < \infty, 0 < q < \infty, i = 1, 2 \) and \( s \in \mathbb{R} \). Then, for all \( g \in M_{p,q}^s \) it holds
\[
\|B_i(t) g\|_{M_{p,q}^s} \leq c(1 + |t|)^{-\alpha} \|g\|_{M_{p,q}^s} \tag{2.1}
\]
with \( \frac{1}{p} + \frac{1}{p'} = 1 \) and \( \alpha = n \left( \frac{1}{2} - \frac{1}{p} \right) \).

**Proof** From Lemma 2.1 and since \( \Box_k \) and \( B_i(t) \) commute, we have
\[
\|\Box_k B_i(t) g\|_{L^p} \leq c|t|^{-\alpha} \|\Box_k g\|_{L^{p'}} \leq c|t|^{-\alpha} \sum_{l \in \Lambda} \|\Box_k+l g\|_{L^{p'}}, \tag{2.2}
\]
where \( \Lambda = \{l \in \mathbb{Z}^n : B(l, \sqrt{2n}) \cap B(0, \sqrt{2n}) \neq \emptyset \} \). We also have used that
\[
\|\Box_k f\|_{L^{p_2}} \leq c \sum_{l \in \Lambda} \|\Box_{k+l} f\|_{L^{p_1}},
\]
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for all $0 < p_1 \leq p_2 \leq \infty$ [see for instance Wang et al. (2006)]. From Hölder’s inequality we have that

$$\| □_k B_i(t)g \|_{L^p} = \| \mathcal{F}^{-1} \sum_{l \in \Lambda} \sigma_k(\xi) \sigma_{k+l}(\xi) \hat{B}_i(\xi) \hat{g}(\xi) \|_{L^p} \leq c \sum_{l \in \Lambda} \| \sigma_k(\xi) \sigma_{k+l}(\xi) \hat{B}_i(\xi) \hat{g}(\xi) \|_{L^{p'}} \leq c \| \sigma_k(\xi) \hat{g}(\xi) \|_{L^{r'}} \leq c \| □_k g \|_{L^{r'}} \leq c \sum_{l \in \Lambda} \| □_{k+l} g \|_{L^{p'}} ,$$

where $1 \leq r \leq 2$. Therefore,

$$\| □_k B_i(t)g \|_{L^p} \leq c \sum_{l \in \Lambda} \| □_{k+l} g \|_{L^{p'}} . \quad (2.3)$$

From (2.2), (2.3) and considering the cases $|t| \leq 1$ and $|t| > 1$, we arrive at

$$\| □_k B_i(t)g \|_{L^p} \leq c(1 + |t|)^{-\alpha} \sum_{l \in \Lambda} \| □_{k+l} g \|_{L^{p'}} . \quad (2.4)$$

Multiplying by $(1 + |k|)^s$ and then taking the $l^q$-norm in both sides of (2.4), we immediately obtain the desired result. \[\square\]

**Remark 2.3** Let $2 \leq p < \infty$, $i = 1, 2$ and $s \in \mathbb{R}$. Then, since $(1 + |t|)^{-\alpha} \leq |t|^{-\alpha}$, Lemma 2.2 implies that

$$\| B_i(t)g \|_{M_{p,q}^s} \leq c |t|^{-\alpha} \| g \|_{M_{p'}^{s},q} \quad (2.5)$$

for all $g \in M_{p',q}^s$, with $\frac{1}{p} + \frac{1}{p'} = 1$ and $\alpha = n \left( \frac{1}{2} - \frac{1}{p} \right)$.

In a similar way to the proof of Lemma 2.2, we also obtain the next result.

**Lemma 2.4** Let $2 \leq p < \infty$, $0 < q \leq \infty$, $i = 1, 3$ and $s \in \mathbb{R}$. Then, for all $g \in M_{p',q}^s$ it holds

$$\| B_i(t)g \|_{D^{-1} J M_{p,q}^s} \leq c(1 + |t|)^{-\alpha} \| g \|_{M_{p',q}^s} \quad (2.6)$$

with $\frac{1}{p} + \frac{1}{p'} = 1$ and $\alpha = n \left( \frac{1}{2} - \frac{1}{p} \right)$.

**Remark 2.5** Let $2 \leq p \leq \infty$, $i = 1, 3$ and $s \in \mathbb{R}$. Then, since $(1 + |t|)^{-\alpha} \leq |t|^{-\alpha}$, Lemma 2.4 implies that

$$\| B_i(t)g \|_{D^{-1} J M_{p,q}^s} \leq c |t|^{-\alpha} \| g \|_{M_{p',q}^s} \quad (2.7)$$

for all $g \in M_{p',q}^s$, with $\frac{1}{p} + \frac{1}{p'} = 1$ and $\alpha = n \left( \frac{1}{2} - \frac{1}{p} \right)$.
**Remark 2.6** The right-hand side of (2.5)–(2.7) have a singularity at $t = 0$. In estimates (2.1)–(2.6) the singularity is removed but preserving the decay at $t = \infty$.

**Lemma 2.7** Let $1 \leq p, p_1, p_2, \sigma, \sigma_1, \sigma_2 \leq \infty$. If $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \frac{1}{\sigma} = \frac{1}{\sigma_1} + \frac{1}{\sigma_2} - 1$ and $s \geq 0$, there exists $C > 0$ such that for any $u \in M_{p_1, \sigma_1}^{s}(\mathbb{R}^n)$ and $v \in M_{p_2, \sigma_2}^{s}(\mathbb{R}^n)$, it holds

$$
\|uv\|_{M_{p, \sigma}^{s}} \leq C \|u\|_{M_{p_1, \sigma_1}^{s}} \|v\|_{M_{p_2, \sigma_2}^{s}}.
$$

**Proof** From Remark 2.4 in Iwabuchi (2010), we have

$$
\|uv\|_{M_{p, \sigma}^{s}} \leq C \|u\|_{M_{p_1, \sigma_1}^{s}} \|v\|_{M_{p_2, \sigma_2}^{s}} + \|u\|_{M_{p_3, \sigma_3}^{0}} \|v\|_{M_{p_4, \sigma_4}^{s}},
$$

where

$$
\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4} \quad \text{and} \quad \frac{1}{\sigma} = \frac{1}{\sigma_1} + \frac{1}{\sigma_2} - 1 = \frac{1}{\sigma_3} + \frac{1}{\sigma_4} - 1.
$$

Taking $p_1 = p_3, p_2 = p_4, \sigma_1 = \sigma_3$ and $\sigma_2 = \sigma_4$, we arrive at

$$
\|uv\|_{M_{p, \sigma}^{s}} \leq C \|u\|_{M_{p_1, \sigma_1}^{s}} \|v\|_{M_{p_2, \sigma_2}^{s}} + \|u\|_{M_{p_1, \sigma_1}^{0}} \|v\|_{M_{p_2, \sigma_2}^{s}}, \quad (2.8)
$$

where

$$
\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \quad \text{and} \quad \frac{1}{\sigma} = \frac{1}{\sigma_1} + \frac{1}{\sigma_2} - 1.
$$

Using Proposition 2.1(ii) in Iwabuchi (2010) we get

$$
\|v\|_{M_{p_2, \sigma_2}^{0}} \leq \|v\|_{M_{p_2, \sigma_2}^{s}} \quad \text{and} \quad \|u\|_{M_{p_1, \sigma_1}^{0}} \leq \|u\|_{M_{p_1, \sigma_1}^{s}}. \quad (2.9)
$$

Combining (2.8) with (2.9) we obtain the desired result. \qed

With the aim of making the reading easier, we present three lemmas which allow us to deal with the nonlinearity $f(u) = u^\lambda$. The proof of the first two can be found in Iwabuchi (2010) [Proposition 2.7 (ii) and Corollary 2.9 (ii)] and the proof of the third one is in Wang and Hudzik (2007) (Lemma 8.2).

**Lemma 2.8** (Iwabuchi 2010) Let $1 \leq p, p_1, p_2 \leq \infty, 1 < \sigma, \sigma_1, \sigma_2 < \infty$. If $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \frac{1}{\sigma} - \frac{1}{\sigma_1} - \frac{1}{\sigma_2} + 1 \leq \frac{s}{n} < \frac{1}{\sigma}$, there exists $C > 0$, such that for any $u \in M_{p_1, \sigma_1}^{s}(\mathbb{R}^n)$ and $v \in M_{p_2, \sigma_2}^{s}(\mathbb{R}^n)$, it holds

$$
\|uv\|_{M_{p, \sigma}^{s}} \leq C \|u\|_{M_{p_1, \sigma_1}^{s}} \|v\|_{M_{p_2, \sigma_2}^{s}}.
$$
Lemma 2.9 (Iwabuchi 2010) Let $1 \leq q \leq \infty$, $p \in \mathbb{N}$, $0 < s < n/\nu$, and $1 \leq \mu, \nu < \infty$ satisfy

$$\frac{1}{\nu} - \frac{(p - 1)s}{n} \leq \frac{p}{\mu} - p + 1, \quad 1 \leq \mu \leq \nu.$$ 

Then, there exists $C > 0$ such that for any $u \in M^{s}_{p,q,\mu}(\mathbb{R}^{n})$, we have

$$\|u^{p}\|_{M^{\nu}_{q,\nu}} \leq C \|u\|_{M^{s}_{p,q,\mu}}^{p}.$$ 

Lemma 2.10 (Wang and Hudzik 2007) Let $1 \leq p, \gamma, \gamma_{i} \leq \infty$, satisfy

$$\frac{1}{p} = \frac{1}{p_{1}} + \frac{1}{p_{2}} + \cdots + \frac{1}{p_{m}}, \quad \frac{1}{\gamma'} = \frac{1}{\gamma_{1}} + \frac{1}{\gamma_{2}} + \cdots + \frac{1}{\gamma_{m}}.$$ 

Let $\alpha = 0$ and $q = 1$, or $\alpha > 0$ and $q' \alpha > nm$. Then we have

$$\|u_{1}u_{2} \cdots u_{m}\|_{l^{a,q}_{\square}(L^{\gamma'}(\mathbb{R};L^{p'}))} \lesssim \prod_{i=1}^{m} \|u_{i}\|_{l^{a}_{\square}(L^{\gamma_{i}}(\mathbb{R};L^{p_{i}}))}.$$ 

Now the aim is to obtain some Strichartz-type estimates in the spaces $l^{a,q}_{\square}(L^{\gamma}(\mathbb{R};L^{p}))$ for $B$. We first recall the following time-space estimates for a general dispersive semigroup due to Wang and Hudzik (2007), Propositions 5.1, 5.2 and 5.3. For that, let us consider $U(t)$ a dispersive semigroup given by $U(t) = \mathcal{F}^{-1}e^{itP(\xi)}\mathcal{F}$, where $P : \mathbb{R}^{n} \to \mathbb{R}$ is a real valued function. Assume that $U(t)$ satisfies the following estimate:

$$\|U(t)f\|_{M^{p,q}_{\square}} \lesssim (1 + |t|)^{-\delta} \|f\|_{M^{p',q}_{\square}}, \quad (2.10)$$

where $2 \leq p < \infty$, $1 \leq q < \infty$, $\delta = \delta(p) > 0$. Then, the following estimates hold.

Proposition 2.11 Let $U(t) = \mathcal{F}^{-1}e^{itP(\xi)}\mathcal{F}$, where $P : \mathbb{R}^{n} \to \mathbb{R}$ is a real valued function. If $U(t)$ satisfies $(2.10)$, where $2 \leq p < \infty$, $1 \leq q < \infty$, $\delta = \delta(p) > 0$, then for any $\gamma \geq \max\{2, 2/\delta\}$ it holds

$$\|U(t)g\|_{l^{a}_{\square}(L^{\gamma}(\mathbb{R};L^{p}))} \lesssim \|g\|_{M^{2,q}_{\square}}, \quad (2.11)$$

$$\left\| \int_{0}^{t} U(t - \tau)g(\cdot, \tau)d\tau \right\|_{l^{a}_{\square}(L^{\infty}(\mathbb{R};L^{2}))} \lesssim \|g\|_{l^{a}_{\square}(L^{\gamma'}(\mathbb{R};L^{p'}))}, \quad (2.12)$$

$$\left\| \int_{0}^{t} U(t - \tau)g(\cdot, \tau)d\tau \right\|_{l^{a}_{\square}(L^{\gamma}(\mathbb{R};L^{p}))} \lesssim \|g\|_{l^{a}_{\square}(L^{\gamma'}(\mathbb{R};L^{p'}))}. \quad (2.13)$$

As a direct consequence of Proposition 2.11 we have the following Strichartz-type estimates.
Proposition 2.12 Let $2 \leq p < \infty$, $1 \leq q < \infty$ and $\gamma \geq \max\{2, \gamma_p\}$, where

$$\frac{2}{\gamma_p} = n \left( \frac{1}{2} - \frac{1}{p} \right).$$

Then, for $i = 1, 2$ we have

$$\|B_i(t)g\|_{l_1^p(L^q(\mathbb{R}; L^p))} \leq \|g\|_{M_{2,q}^p}, \quad (2.14)$$

$$\|l_2(t)g\|_{l_1^p(L^q(\mathbb{R}; L^p))} \leq \|g\|_{M_{2,q}^p}, \quad (2.15)$$

$$\left\| \int_0^t B_i(t - \tau)g(\cdot, \tau)d\tau \right\|_{l_1^p(L^q(\mathbb{R}; L^p))} \leq \|g\|_{l_1^p(L^q(\mathbb{R}; L^p))}, \quad (2.16)$$

$$\left\| \int_0^t B_i(t - \tau)g(\cdot, \tau)d\tau \right\|_{l_1^p(L^q(\mathbb{R}; L^p))} \leq \|g\|_{l_1^p(L^q(\mathbb{R}; L^p))}, \quad (2.17)$$

Proof The proof of (2.14) follows from (2.11) by considering $U(t) = B_i(t)$ and using the estimate (2.1) which validates (2.10). On the other hand, using the estimate (2.7) we get

$$\|l_2(t)g\|_{M_{p,q}} = \|J^{-1}DB_3(t)g\|_{M_{p,q}} = \|B_3(t)g\|_{D^{-1}JM_{p,q}} \leq c(1 + |t|)^{-\alpha}\|g\|_{M_{p,q}}.$$

Therefore, (2.15) follows from (2.11) taking $U(t) = l_2(t)$. Finally, estimate (2.16) follows from (2.12) and estimate (2.17) follows from (2.13).

Proposition 2.13 Let $2 \leq p < \infty$, $1 \leq q < \infty$ and $\gamma \geq \max\{2, \gamma_p\}$, where

$$\frac{2}{\gamma_p} = n \left( \frac{1}{2} - \frac{1}{p} \right).$$

Then, for $i = 1, 3$, we have

$$\|B_i(t)g\|_{l_1^p(L^q(\mathbb{R}; D^{-1}JL^p))} \leq \|g\|_{M_{2,q}^p}, \quad (2.18)$$

$$\left\| \int_0^t B_1(t - \tau)g(\cdot, \tau)d\tau \right\|_{l_1^p(L^q(\mathbb{R}; D^{-1}JL^2))} \leq \|g\|_{l_1^p(L^q(\mathbb{R}; L^p))}, \quad (2.19)$$

$$\left\| \int_0^t B_1(t - \tau)g(\cdot, \tau)d\tau \right\|_{l_1^p(L^q(\mathbb{R}; D^{-1}JL^p))} \leq \|g\|_{l_1^p(L^q(\mathbb{R}; L^p))}, \quad (2.20)$$

Proof Notice that $\|g\|_{l_1^p(L^q(\mathbb{R}; D^{-1}JL^p))} := \|J^{-1}Dg\|_{l_1^p(L^q(\mathbb{R}; L^p))}$. Then, recalling that the symbol $\langle \xi \rangle^{-1}|\xi|$ of the operator $J^{-1}D$ defines a multiplier in $L^p$ and therefore it is continuous (cf. Stein 1970), and using Proposition 2.12, we have
\[ \|B_1(t)g\|_{L^q(L^\gamma(\mathbb{R}; D^{-1} JL^p)))} = \| J^{-1} DB_1(t)g\|_{L^q(L^\gamma(\mathbb{R}; L^p)))} \leq \| B_1(t)g\|_{L^q(L^\gamma(\mathbb{R}; L^p)))} \lesssim \|g\|_{M_{2,q}}. \]

Similarly, we obtain
\[ \|B_3(t)g\|_{L^q(L^\gamma(\mathbb{R}; D^{-1} JL^p)))} = \| J^{-1} DB_3(t)g\|_{L^q(L^\gamma(\mathbb{R}; L^p)))} = \|l_2(t)g\|_{L^q(L^\gamma(\mathbb{R}; L^p)))} \leq \|g\|_{M_{2,q}}. \]

Again, from Proposition 2.12 we have
\[ \left\| \int_0^t B_1(t - \tau)g(\cdot, \tau)d\tau \right\|_{L^q(L^\gamma(\mathbb{R}; D^{-1} JL^2)))} = \left\| \int_0^t J^{-1} DB_1(t - \tau)g(\cdot, \tau)d\tau \right\|_{L^q(L^\gamma(\mathbb{R}; L^2)))} \leq \left\| \int_0^t B_1(t - \tau)g(\cdot, \tau)d\tau \right\|_{L^q(L^\gamma(\mathbb{R}; L^2)))} \leq \|g\|_{L^q(L^\gamma(\mathbb{R}; L^p)))}. \]

By the \(L^p\)-continuity of \(J^{-1} D\) and Proposition 2.12, we obtain
\[ \left\| \int_0^t B_1(t - \tau)g(\cdot, \tau)d\tau \right\|_{L^q(L^\gamma(\mathbb{R}; D^{-1} JL^p)))} = \left\| \int_0^t J^{-1} DB_1(t - \tau)g(\cdot, \tau)d\tau \right\|_{L^q(L^\gamma(\mathbb{R}; L^p)))} \leq \left\| \int_0^t B_2(t - \tau)g(\cdot, \tau)d\tau \right\|_{L^q(L^\gamma(\mathbb{R}; L^p)))} \leq \|g\|_{L^q(L^\gamma(\mathbb{R}; L^p))).} \]

This finish the proof of the proposition. \(\square\)

### 3 Proofs

In this section we prove the results established in Sect 1. We start estimating the nonlinear part of (1.3). For this, let us define the operators
\[
\Phi_1[u(t), v(t)] = B_1(t)u_0 + B_2(t)v_0 - \int_0^t B_2(t - \tau) f(u(\tau))d\tau,
\]
\[
\Phi_2[u(t), v(t)] = B_3(t)u_0 + B_1(t)v_0 - \int_0^t B_1(t - \tau) f(u(\tau))d\tau,
\]
and \(\Phi[u, v] = [\Phi_1[u, v], \Phi_2[u, v]].\)
Proposition 3.1 Assume that $\lambda > 1$ is a positive integer, $1 \leq q < 2$, $p = \lambda + 1$, and $n - \frac{n}{q} \leq s < \frac{n}{q}$, and $\lambda > \lambda_0(n)$. There exists a constant $C_1 > 0$ such that

$$\|\Phi[u, v] - \Phi[\tilde{u}, \tilde{v}]\|_{L^\infty_{\alpha, s}} \leq C_1 \sup_{-\infty < t < \infty} (1 + |t|)^{\alpha} \|u - \tilde{u}(t)\|_{M^s_{p, q}}$$

$$\times \sup_{-\infty < t < \infty} (1 + |t|)^{\alpha(\lambda - 1)} \sum_{k=1}^{\lambda} \|u(t)\|_{M^s_{p, q}}^{\lambda - k} \|\tilde{u}(t)\|_{M^s_{p, q}}^{k - 1},$$

for all $[u, v], [\tilde{u}, \tilde{v}] \in L^\infty_{\alpha, s}$, with $u(0) = \tilde{u}(0), v(0) = \tilde{v}(0)$.

Proof Without loss of generality we assume $t > 0$. First suppose that $1 < q < 2$. Since $n - \frac{n}{q} \leq s < \frac{n}{q}$, from Lemmas 2.2, 2.4, 2.8, 2.9, we obtain

$$\|\Phi_1[u(t), v(t)] - \Phi_1[\tilde{u}(t), \tilde{v}(t)]\|_{M^s_{p, q}} + \|\Phi_2[u(t), v(t)] - \Phi_2[\tilde{u}(t), \tilde{v}(t)]\|_{D^{-1}JM^s_{p, q}}$$

$$\leq \int_0^t \|B_2(t - \tau)(f(u) - f(\tilde{u}))\|_{M^s_{p, q}} d\tau + \int_0^t \|B_1(t - \tau)(f(u) - f(\tilde{u}))\|_{D^{-1}JM^s_{p, q}} d\tau$$

$$\leq \int_0^t (1 + |t - \tau|)^{-\alpha} \left\|u - \tilde{u}\right\|_{M^s_{p, q}} \left(\sum_{k=1}^{\lambda} \|u^{\lambda - k}\|_{M^s_{p, q}} \|\tilde{u}^{k - 1}\|_{M^s_{p, q}}^\prime\right) d\tau$$

$$\leq \int_0^t (1 + |t - \tau|)^{-\alpha} \|u - \tilde{u}\|_{M^s_{p, q}} \sum_{k=1}^{\lambda} \|u\|_{M^s_{p, q}}^{\lambda - k} \|\tilde{u}\|_{M^s_{p, q}}^{k - 1} d\tau$$

$$\leq \sup_{0 < t < \infty} (1 + t)^{\alpha} \|u - \tilde{u}\|_{M^s_{p, q}} \sup_{0 < t < \infty} (1 + t)^{\alpha(\lambda - 1)} \sum_{k=1}^{\lambda} \|u\|_{M^s_{p, q}}^{\lambda - k} \|\tilde{u}\|_{M^s_{p, q}}^{k - 1} d\tau$$

$$\times \int_0^t (1 + t - \tau)^{-\alpha} (1 + \tau)^{-\alpha\lambda} d\tau.$$  \hspace{1cm} (3.2)

Since $\lambda > \lambda_0(n)$, then $\alpha\lambda > 1$; thus, it follows that

$$\int_0^{t/2} (1 + t - \tau)^{-\alpha} (1 + \tau)^{-\alpha\lambda} d\tau \leq (1 + t)^{-\alpha} \int_0^{t/2} (1 + \tau)^{-\alpha\lambda} d\tau \leq (1 + t)^{-\alpha}. $$  \hspace{1cm} (3.3)

Also, it is straightforward to get

$$\int_0^t (1 + t - \tau)^{-\alpha} (1 + \tau)^{-\alpha\lambda} d\tau = \int_0^{t/2} (1 + \tau)^{-\alpha} (1 + t - \tau)^{-\alpha\lambda} d\tau$$

$$\leq (1 + t)^{-\alpha\lambda} \int_0^{t/2} (1 + \tau)^{-\alpha} d\tau \leq (1 + t)^{-\alpha}. $$  \hspace{1cm} (3.4)
Indeed, for $0 < \alpha < 1$, since $\alpha \lambda > 1$, it holds
\[
(1 + t)^{-\alpha \lambda} \int_0^{t/2} (1 + \tau)^{-\alpha} \, d\tau = \frac{(1 + t)^{-\alpha \lambda}}{1 - \alpha} \left[ \left(1 + \frac{t}{2}\right)^{1-\alpha} - 1 \right] \leq (1 + t)^{-\alpha}.
\]

For $\alpha > 1$, it holds
\[
(1 + t)^{-\alpha \lambda} \int_0^{t/2} (1 + \tau)^{-\alpha} \, d\tau = \frac{(1 + t)^{-\alpha \lambda}}{\alpha - 1} \left[ \left(1 + \frac{t}{2}\right)^{1-\alpha} - 1 \right] \leq (1 + t)^{-\alpha}.
\]

If $\alpha = 1$ and $\lambda \geq 2$, it holds
\[
(1 + t)^{-\lambda} \int_0^{t/2} (1 + \tau)^{-\alpha} \, d\tau \leq (1 + t)^{-1} \frac{\ln(1 + t)}{(1 + t)^{\lambda-1}} \leq (1 + t)^{-1} (1 + t)^{2-\lambda} \leq (1 + t)^{-1}.
\]

From (3.2)–(3.4) we conclude the proof of (3.1) for $1 < q < 2$. The proof in the case $q = 1$ is similar, just using Lemma 2.7 instead of Lemma 2.8. □

Remark 3.2 Assume $p, q$ and $s$ as in Proposition 3.1. If $\lambda$ is a positive integer such that $\lambda_0(n) < \lambda < \lambda_1(n)$, then there exists a constant $C_2 > 0$ such that
\[
\|\Phi[u, v] - \Phi[\tilde{u}, \tilde{v}]\|_{H^{s, \infty}} \leq C_2 \sup_{-\infty < t < \infty} |t|^\beta \|u - \tilde{u}\|_{M^s_{p, q}}
\times \sup_{-\infty < t < \infty} |t|^\beta(\lambda - 1) \sum_{k=1}^\lambda \|u(t)\|_{M^s_{p, q}}^{\lambda - k} \|\tilde{u}(t)\|_{M^s_{p, q}}^{k-1} \tag{3.5}
\]

Indeed, using Remarks 2.3, 2.5, Lemmas 2.7, 2.8, 2.9, and following the proof of Proposition 3.1 we get
\[
\|\Phi_1[u(t), v(t)] - \Phi_1[\tilde{u}(t), \tilde{v}(t)]\|_{M^s_{p, q}} + \|\Phi_2[u(t), v(t)] - \Phi_2[\tilde{u}(t), \tilde{v}(t)]\|_{D^{-1}JM^s_{p, q}}
\leq \int_0^t (t - \tau)^{-\alpha} \|u - \tilde{u}\|_{M^s_{p, q}} \sum_{k=1}^\lambda \|u\|_{M^s_{p, q}}^{\lambda - k} \|\tilde{u}\|_{M^s_{p, q}}^{k-1} \, d\tau
\leq \sup_{0 < t < \infty} |t|^\beta \|u - \tilde{u}\|_{M^s_{p, q}} \sup_{0 < t < \infty} |t|^\beta(\lambda - 1) \sum_{k=1}^\lambda \|u(t)\|_{M^s_{p, q}}^{\lambda - k} \|\tilde{u}(t)\|_{M^s_{p, q}}^{k-1}
\times \int_0^t (t - \tau)^{-\alpha \tau^{-\beta \lambda}} \, d\tau. \tag{3.6}
\]

Since $1 - \beta(\lambda - 1) - \alpha = 0$, multiplying by $t^\beta$ and taking the supremum over $\mathbb{R}$ we conclude (3.5). Notice the the condition $\lambda_0(n) < \lambda < \lambda_1(n)$ is due the integrability of the beta function in (3.6). Observe that condition $\lambda_0(n) < \lambda < \lambda_1(n)$ implies $1 \leq n \leq 5$. 

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Proposition 3.3 Assume that \( \lambda > 1 \) is a positive integer, \( 1 \leq q < 2 \), \( p = \lambda + 1 \) and \( n - \frac{n}{q} \leq s < \frac{n}{q} \). Then there exists a constant \( C_3 > 0 \) such that

\[
\| \Phi[u, v] - \Phi[\tilde{u}, \tilde{v}] \|_{\mathcal{L}^T_{\alpha, s}} \leq C_3 T \sup_{-T < t < T} (1 + |t|)^{\alpha} \| (u - \tilde{u})(t) \|_{M^s_{p, q}}
\]

\[
\times \sup_{-T < t < T} (1 + |t|)^{\alpha(\lambda - 1)} \sum_{k=1}^{\lambda} \| u(t) \|_{M^s_{p, q}}^{\lambda - k} \| \tilde{u}(t) \|_{M_{p, q}}^{k-1},
\]

for all \([u, v], [\tilde{u}, \tilde{v}] \in \mathcal{L}^T_{\alpha, s}\), with \( u(0) = \tilde{u}(0), v(0) = \tilde{v}(0) \).

Proof Without loss of generality we assume \( t > 0 \). First suppose that \( 1 < q < 2 \). Since \( n - \frac{n}{q} \leq s < \frac{n}{q} \), from Lemmas 2.2, 2.4, 2.8, 2.9, we obtain

\[
\| \Phi_1[u(t), v(t)] - \Phi_1[\tilde{u}(t), \tilde{v}(t)] \|_{M^s_{p, q}} + \| \Phi_2[u(t), v(t)] - \Phi_2[\tilde{u}(t), \tilde{v}(t)] \|_{D^{-1}JM^s_{p, q}}
\]

\[
\leq \int_0^T \| B_2(t - \tau)(f(u) - f(\tilde{u})) \|_{M^s_{p, q}} d\tau
\]

\[
+ \int_0^T \| B_1(t - \tau)(f(u) - f(\tilde{u})) \|_{D^{-1}JM^s_{p, q}} d\tau
\]

\[
\leq \int_0^T (1 + |t - \tau|)^{-\alpha} \left\| (u - \tilde{u}) \left( \sum_{k=1}^{\lambda} u^{\lambda - k} \tilde{u}^{k-1} \right) \right\|_{M^s_{p, q}} d\tau
\]

\[
\leq \int_0^T (1 + |t - \tau|)^{-\alpha} \| u - \tilde{u} \|_{M^s_{p, q}} \sum_{k=1}^{\lambda} \| u^{\lambda - k} \|_{M^s_{p, q}}^{\lambda - k} \| \tilde{u}^{k-1} \|_{M_{p, q}}^{k-1} d\tau
\]

\[
\leq \int_0^T (1 + |t - \tau|)^{-\alpha} \| u - \tilde{u} \|_{M^s_{p, q}} \sum_{k=1}^{\lambda} \| u \|_{M^s_{p, q}}^{\lambda - k} \| \tilde{u}^{k-1} \|_{M_{p, q}}^{k-1} d\tau
\]

\[
\leq \sup_{0 < t < T} (1 + t)^{\alpha} \| u - \tilde{u} \|_{M^s_{p, q}} \sup_{0 < t < T} (1 + t)^{\alpha(\lambda - 1)} \sum_{k=1}^{\lambda} \| u \|_{M^s_{p, q}}^{\lambda - k} \| \tilde{u}^{k-1} \|_{M_{p, q}}^{k-1}
\]

\[
\times \int_0^t (1 + t - \tau)^{-\alpha} (1 + \tau)^{-\alpha\lambda} d\tau.
\]

Since \( \alpha\lambda > 0 \), we obtain

\[
\int_0^{t/2} (1 + t - \tau)^{-\alpha} (1 + \tau)^{-\alpha\lambda} d\tau \leq (1 + t)^{-\alpha} \int_0^{t/2} (1 + \tau)^{-\alpha\lambda} d\tau \leq (1 + t)^{-\alpha} T.
\]

Using that \( \alpha > 0 \) and \( \alpha(\lambda - 1) > 0 \) we arrive at

\[\square\]
\[
\int_{t/2}^{t} (1 + t - \tau)^{-\alpha} (1 + \tau)^{-\alpha\lambda} \, d\tau = \int_{0}^{t/2} (1 + \tau)^{-\alpha} (1 + t - \tau)^{-\alpha\lambda} \, d\tau \\
\leq (1 + t)^{-\alpha\lambda} \int_{0}^{t/2} (1 + \tau)^{-\alpha} \, d\tau \\
\leq (1 + t)^{-\alpha} (1 + t)^{-\alpha(\lambda - 1)} \int_{0}^{t/2} (1 + \tau)^{-\alpha} \, d\tau \leq (1 + t)^{-\alpha T}.
\] (3.10)

From (3.8)–(3.10) we conclude the proof of (3.7) for \(1 < q \leq 2\). The proof in the case \(q = 1\) is the same, just using Lemma 2.7 instead of Lemma 2.8. \(\square\)

### 3.1 Proof of Theorem 1.1

We prove that the mapping \(\Phi[u, v] = [\Phi_1(u, v), \Phi_2(u, v)]\) defines a contraction in the metric space \(B_{2\tilde{c}\epsilon} = \{[u, v] \in \mathcal{L}_{a,s}^\infty : \|[u, v]\|_{\mathcal{L}_{a,s}^\infty} \leq 2\tilde{c}\epsilon\}\), for some \(\epsilon > 0\) and \(\tilde{c}\) is the sum of the constants appearing in (2.1) and (2.6). From Proposition 3.1 and applying the Young inequality in each term \(\|[u(t)]_{M_{p,q}^{k-1}} - \tilde{u}(t)\|_{M_{p,q}^{k-1}}, k = 1, \ldots, \lambda\), we have

\[
\|[\Phi[u, v] - \Phi[\tilde{u}, \tilde{v}]]\|_{\mathcal{L}_{a,s}^\infty} \leq C_1\|[u, v] - [\tilde{u}, \tilde{v}]\|_{\mathcal{L}_{a,s}^\infty} (\|[u, v]\|_{\mathcal{L}_{a,s}^\infty}^{k-1} + \|[\tilde{u}, \tilde{v}]\|_{\mathcal{L}_{a,s}^\infty}^{k-1}) \\
\leq 2^\lambda \epsilon^{\lambda - 1} C_1 \tilde{c} \|[u, v] - [\tilde{u}, \tilde{v}]\|_{\mathcal{L}_{a,s}^\infty}.
\] (3.11)

Using (3.11) with \([\tilde{u}, \tilde{v}] = [0, 0]\), Lemmas 2.2 and 2.4, we have

\[
\|[\Phi[u, v]]\|_{\mathcal{L}_{a,s}^\infty} \leq \sup_{0 < t < \infty} (1 + t)^\alpha \|B_1(t)u_0 + B_2(t)v_0\|_{M_{p,q}^s} \\
+ \sup_{0 < t < \infty} (1 + t)^\alpha \|B_3(t)u_0 + B_4(t)v_0\|_{M_{p,q}^s} + 2\epsilon \epsilon^{\lambda - 1} C_1 \tilde{c} \\
\leq \tilde{c} (\|u_0\|_{M_{p,q}^s} + \|v_0\|_{M_{p,q}^s}) + 2\epsilon \epsilon^{\lambda - 1} C_1 \tilde{c} \leq 2\tilde{c}\epsilon,
\] (3.12)

provided \(0 < \epsilon^{\lambda - 1} \leq \max\left\{\frac{1}{2\lambda C_1}, \frac{1}{2\lambda C_1}\right\}\) and \([u, v] \in B_{2\tilde{c}\epsilon}\). From (3.11) and (3.12) we get that \(\Phi\) is a contraction, which implies that the integral equation (1.3) has a unique solution \([u, v] \in \mathcal{L}_{a,s}^\infty\), satisfying \(\|[u, v]\|_{\mathcal{L}_{a,s}^\infty} \leq 2\tilde{c}\epsilon\). Finally, we prove that the data-solution map is locally Lipschitz. Let \([u, v], [\tilde{u}, \tilde{v}]\) global solutions of (1.3) with initial data \([u_0, v_0]\) and \([\tilde{u}_0, \tilde{v}_0]\) \(\in M_{p,q}^s\), respectively, satisfying \(\|[u, v]\|_{\mathcal{L}_{a,s}^\infty}, \|[\tilde{u}, \tilde{v}]\|_{\mathcal{L}_{a,s}^\infty} \leq 2\tilde{c}\epsilon\). Then, taking the difference between the integral equations and the \(\mathcal{L}_{a,s}^\infty\)-norm, we get

\[
\|[u - \tilde{u}, v - \tilde{v}]\|_{\mathcal{L}_{a,s}^\infty} \\
\leq \|B(t)[u_0 - \tilde{u}_0, v_0 - \tilde{v}_0]\|_{\mathcal{L}_{a,s}^\infty} \\
+ \left\| \int_{0}^{t} [B_2(t - \tau)(f(u) - f(\tilde{u})), B_4(t - \tau)(f(u) - f(\tilde{u}))] d\tau \right\|_{\mathcal{L}_{a,s}^\infty} \\
\leq \tilde{c} \|[u_0 - \tilde{u}_0, v_0 - \tilde{v}_0]\|_{M_{p,q}^s \times M_{p,q}^s} + 2\epsilon ^{\lambda - 1} C_1 \tilde{c} \|[u - \tilde{u}, v - \tilde{v}]\|_{\mathcal{L}_{a,s}^\infty}.
\] (3.13)
Therefore, since \(2^\lambda e^{\lambda-1} C_1 \tilde{c} < 1\), from (3.13) we conclude the result. \(\square\)

**Remark 3.4** Assume that \([u_0, v_0]\) satisfies (1.11)–(1.12). Then, from (3.5) with \([\tilde{u}, \tilde{v}] = [0, 0]\), we have

\[
\sup_{-\infty < t < \infty} |t|^{\beta} \|\Phi[u, v]\|_{M_{p, q}^s} \leq \sup_{-\infty < t < \infty} |t|^{\beta} \|B_1(t)u_0 + B_2(t)v_0\|_{M_{p, q}^s} + 2^\lambda e^\lambda C_2 \leq \epsilon + 2^\lambda e^\lambda C_2 \leq 2\epsilon,
\]

provided \(2^\lambda e^{\lambda-1} C_2 < 1\) and \([u, v] \in B_{2\epsilon}\). From (3.5) and (3.14) we get that \(\Phi\) is a contraction, which implies that the integral equation (1.3) has a unique solution \([u, v] \in H_{\beta, s}'\).

### 3.2 Proof of Theorem 1.4

We start by considering the following function space

\[
X = l_1^1(L^\infty(\mathbb{R}; L^2)) \cap l_1^1(L^p(\mathbb{R}; L^p)).
\]

Since \(\| \Box_k B_1(t)u_0 \|_{L^2} \leq \| \Box_k u_0 \|_{L^2}, \Box_k B_2(t)v_0 \|_{L^2} \leq \| \Box_k v_0 \|_{L^2}\), using Proposition 2.12 (with \(q = 1\)) and taking \(p\) a positive integer such that \(p \leq \lambda + 1\), we have

\[
\|\Phi_1[u, v]\|_{l_1^1(L^\infty(\mathbb{R}; L^2))} \leq \|B_1(t)u_0\|_{l_1^1(L^\infty(\mathbb{R}; L^2))} + \|B_2(t)v_0\|_{l_1^1(L^\infty(\mathbb{R}; L^2))} + \int_0^t B_2(t-\tau) f(u(\tau))d\tau \leq \|u_0\|_{M_{2, 1}} + \|v_0\|_{M_{2, 1}} + \|f(u)\|_{l_1^1(L^\infty(\mathbb{R}; L^2))} \leq \|u_0\|_{M_{2, 1}} + \|v_0\|_{M_{2, 1}} + \|u\|_{l_1^1(L^p(\mathbb{R}; L^p))} \leq (3.15)
\]

In the last inequality we use Lemma 2.10 with \(\alpha = 0, q = 1, p_i = \gamma_i = p\), for \(i = 1, 2, ..., p - 1\) and \(p_i = \gamma_i = \infty\), for \(i = p, p + 1, ..., \lambda + 1\). Note that the use of Lemma 2.10 forces \(\gamma = p\) in Proposition 2.12. The condition \(p \geq \max\{2, \gamma_p\}\) implies \(p \neq 2\), because \(\gamma_2 = \infty\). Therefore we get

\[
p \geq \gamma_p \iff \frac{2}{p} \leq \frac{2}{\gamma_p} = n \left(1 - \frac{1}{2} - \frac{1}{p}\right) = \frac{n(p-2)}{2p} \iff 2 + \frac{4}{n} \leq p. (3.16)
\]

Next, using the fact that \(\| \Box_k u \|_{\infty} \leq \| \Box_k u \|_2\), for all \(k \in \mathbb{Z}^n\), from (3.15) we obtain

\[
\|\Phi_1[u, v]\|_{l_1^1(L^\infty(\mathbb{R}; L^2))} \leq \|u_0\|_{M_{2, 1}} + \|v_0\|_{M_{2, 1}} + \|u\|_{l_1^1(L^p(\mathbb{R}; L^p))} \leq (3.15)
\]
Therefore,
\[ \| \Phi_1[u, v]\|_{L^1(L^\infty(\mathbb{R}; L^2))} \leq \| u_0 \|_{M_{2,1}} + \| v_0 \|_{M_{2,1}} + \| u \|_X^\gamma. \tag{3.17} \]

Now, again from Proposition 2.12, with \( \gamma = p \) and \( q = 1 \) [this choice implies the same condition (3.16)], and Lemma 2.10 we obtain
\[ \| \Phi_1[u, v]\|_{L^1(L^p(\mathbb{R}; L^p))} \leq \| u_0 \|_{M_{2,1}} + \| v_0 \|_{M_{2,1}} + \| u \|_X^\gamma. \tag{3.18} \]

In order to deal with the variable \( v \), we consider the space
\[ Y = l^1(L^\infty(\mathbb{R}; D^{-1} JL^2)). \]

Using that \( \| \Box_k B_1(t) u_0 \|_{D^{-1} JL^2} \leq \| \Box_k u_0 \|_{L^2} \), \( \| \Box_k B_3(t) v_0 \|_{D^{-1} JL^2} \leq \| \Box_k v_0 \|_{L^2} \), Proposition 2.13 and Lemma 2.10, we obtain
\[ \| \Phi_2[u, v]\|_{L^1(L^\infty(\mathbb{R}; D^{-1} JL^2))} \]
\[ \leq \| B_3(t) u_0 \|_{l^1(L^\infty(\mathbb{R}; D^{-1} JL^2))} + \| B_1(t) v_0 \|_{l^1(L^\infty(\mathbb{R}; D^{-1} JL^2))} + \left\| \int_0^t \Box B_1(t - \tau)f(u(\tau))d\tau \right\|_{l^1(L^\infty(\mathbb{R}; D^{-1} JL^2))} \]
\[ \leq \| u_0 \|_{M_{2,1}} + \| v_0 \|_{M_{2,1}} + \| f(u) \|_{l^1(L^\infty(\mathbb{R}; D^{-1} JL^2))} \]
\[ \leq \| u_0 \|_{M_{2,1}} + \| v_0 \|_{M_{2,1}} + \| u \|_X^\gamma. \tag{3.19} \]

We prove that the mapping \( \Phi[u, v] = [\Phi_1(u, v), \Phi_2(u, v)] \) defines a contraction in the metric space \( B_{2\epsilon} = \{ u, v \} : \| u \|_X + \| v \|_Y \leq 2\epsilon \}, \) for some \( \epsilon > 0 \). Following (3.17), (3.18) and (3.19), and noting that \( u(0) = \tilde{u}(0), v(0) = \tilde{v}(0) \), we get
\[ \| \Phi[u, v] - \Phi[\tilde{u}, \tilde{v}]\|_{X \times Y} = \| \Phi_1[u, v] - \Phi_1[\tilde{u}, \tilde{v}]\|_X + \| \Phi_2[u, v] - \Phi_2[\tilde{u}, \tilde{v}]\|_Y \]
\[ = \left\| \int_0^t B_2(t - \tau)(f(u) - f(\tilde{u}))d\tau \right\|_{L^1(L^\infty(\mathbb{R}; L^2))} \]
\[ + \left\| \int_0^t B_2(t - \tau)(f(u) - f(\tilde{u}))d\tau \right\|_{L^1(L^p(\mathbb{R}; L^p))} \]
\[ + \left\| \int_0^t B_1(t - \tau)(f(u) - f(\tilde{u}))d\tau \right\|_{L^p(\mathbb{R}; L^p)} \]
\[ \leq C\| u - \tilde{u} \|_X (\| u \|_X^{\lambda-1} + \| \tilde{u} \|_X^{\lambda-1}) \]
\[ \leq C 2^\lambda \epsilon^{\lambda-1} \| u - \tilde{u} \|_X \| v - \tilde{v} \|_X \| Y. \]

Also,
\[ \| \Phi[u, v]\|_{X \times Y} \leq \| u_0 \|_{M_{2,1}} + \| v_0 \|_{M_{2,1}} + \| u \|_X^\gamma \leq \| u_0 \|_{M_{2,1}} + \| v_0 \|_{M_{2,1}} + C 2^\lambda \epsilon^\lambda \leq 2\epsilon. \]
provided $2^\lambda e^{\lambda-1} C < 1$. Thus, $\Phi$ defines a contraction, which implies that the integral Eq. (1.3) has a solution $[u, v] \in X \times Y$. Notice that $l_1^\Delta (L^\infty (\mathbb{R}; L^2)) \subset L^\infty (\mathbb{R}; M_{2,1})$ and $Y \subset L^\infty (\mathbb{R}; D^{-1} J M_{2,1})$. Thus, it remains to prove the time-continuity of the solution. For that, let $t_0 \in \mathbb{R}$ be fixed. We will show that

$$
\lim_{t \to t_0} \|[u(t), v(t)] - [u(t_0), v(t_0)]\|_{M_{2,1} \times D^{-1} J M_{2,1}} = 0.
$$

From the integral Eq. (1.3) we get

$$
u(t_0) = B_1(t_0)u_0 + B_2(t_0)v_0 - \int_0^{t_0} B_2(t_0 - \tau) f(u(\tau)) d\tau. \tag{3.20}$$

Then, taking the $M_{2,1}$-norm of the difference between the first equation of (1.3) and (3.20) we obtain

$$
\|u(t) - u(t_0)\|_{M_{2,1}} \leq \|[B_1(t) - B_1(t_0)]u_0\|_{M_{2,1}} + \|[B_2(t) - B_2(t_0)]v_0\|_{M_{2,1}} + \|I(t, t_0)\|_{M_{2,1}},
$$

where $I(t, t_0)$ is given by

$$
I(t, t_0) = \int_0^t B_2(t - \tau) f(u(\tau)) d\tau - \int_0^{t_0} B_2(t_0 - \tau) f(u(\tau)) d\tau.
$$

From Plancherel’s theorem we obtain,

$$
\|[B_1(t) - B_1(t_0)]u_0\|_{M_{2,1}} = \sum_{k \in \mathbb{Z}^n} \|[B_1(t) - B_1(t_0)]\underline{k} u_0\|_{L^2} = \sum_{k \in \mathbb{Z}^n} \|[\cos(t|\xi|\langle \xi \rangle) - \cos(t_0|\xi|\langle \xi \rangle)]\underline{k} u_0\|_{L^2}.
$$

Since $u_0 \in M_{2,1}$ and $t \mapsto \cos(t|\xi|\langle \xi \rangle)$ is a continuous function, the Lebesgue Dominated Convergence Theorem implies that

$$
\|[B_1(t) - B_1(t_0)]u_0\|_{M_{2,1}} \to 0, \quad \text{as} \quad t \to t_0. \tag{3.21}
$$

In a similar way, we arrive at

$$
\|[B_2(t) - B_2(t_0)]v_0\|_{M_{2,1}} \to 0, \quad \text{as} \quad t \to t_0.
$$

Now, we deal with $\|I(t, t_0)\|_{M_{2,1}}$. For that, notice that
\[ \left\| I(t, t_0) \right\|_{M_{2,1}} = \left\| (B_2(t) - B_2(t_0)) \int_0^t B_2(-\tau) f(u(\tau)) d\tau + \int_0^t B_2(t - \tau) f(u(\tau)) d\tau \right\|_{M_{2,1}} \]

\[ \leq \left\| (B_2(t) - B_2(t_0)) \int_0^t B_2(-\tau) f(u(\tau)) d\tau \right\|_{M_{2,1}} + \left\| \int_0^t B_2(t - \tau) f(u(\tau)) d\tau \right\|_{M_{2,1}} = I_1 + I_2. \]

Observe that

\[ \left\| \int_0^t B_2(-\tau) f(u(\tau)) d\tau \right\|_{M_{2,1}} = \left\| B_2(-t_0) \int_0^t B_2(t_0 - \tau) f(u(\tau)) d\tau \right\|_{M_{2,1}} \]

\[ \leq \left\| \int_0^t B_2(t_0 - \tau) f(u(\tau)) d\tau \right\|_{M_{2,1}} < \infty, \]

where the last bound holds from (3.20) since \( u(t_0) \in M_{2,1}. \) Therefore, analogously to (3.21) we obtain \( \lim_{t \to t_0} I_1 = 0. \) Now, in order to deal with \( I_2 \) we use Lemma 2.2 to get

\[ I_2 \leq \int_0^t \left\| B_2(t - \tau) f(u(\tau)) \right\|_{M_{2,1}} d\tau \leq \int_0^t \left\| f(u(\tau)) \right\|_{M_{2,1}} d\tau \leq \int_0^t \left\| u(\tau) \right\|_{\tilde{M}_{2,1}} \frac{\lambda}{2} d\tau \]

\[ \leq \int_0^t \left\| u(\tau) \right\|_{\tilde{M}_{2,1}} \left( L^\infty(\mathbb{R}; L^2) \right) d\tau \leq C |t - t_0| e^{\frac{\lambda}{2}} \to 0, \text{ as } t \to t_0. \]

In conclusion we have that \( \lim_{t \to t_0} \| u(t) - u(t_0) \|_{M_{2,1}} = 0. \) In a similar way we obtain that

\[ \lim_{t \to t_0} \| v(t) - v(t_0) \|_{D^{-1}J_{M_{2,1}}} = 0. \]

Thus we conclude the proof. \( \square \)

### 3.3 Proof of Theorem 1.5

Let \( R = \tilde{c}(\| u_0 \|_{M_{p,q}}^x + \| v_0 \|_{M_{p,q}}^x), \) where \( \tilde{c} \) is the sum of the constants appearing in (2.1) and (2.6). We consider the metric space \( B_{2R} = \{ [u, v] \in L^T_{a,s} : \| [u, v] \|_{L^T_{a,s}} \leq 2R \}. \) We prove that the mapping \( \Phi[u, v] = [\Phi_1(u, v), \Phi_2(u, v)] \) defines a contraction in the metric space \( B_{2R}, \) provided \( T > 0 \) be small enough. Take \( T > 0 \) such that \( C_3 R^{\alpha - 12^T} T < 1, \) where \( C_3 \) is the constant provided by Proposition 3.3. From Lemmas 2.2 and 2.4 and Proposition 3.3 we have

\[ \| \Phi[u, v] \|_{L^T_{a,s}} \leq \sup_{-T < t < T} \| t \|_{L^T_{a,s}} \| B_1(t) u_0 \|_{M_{p,q}} + \| B_2(t) v_0 \|_{M_{p,q}} \]

\[ + \sup_{-T < t < T} \| t \|_{L^T_{a,s}} \| B_3(t) u_0 \|_{D^{-1}J_{M_{p,q}}} + \| B_1(t) v_0 \|_{D^{-1}J_{M_{p,q}}} + C_3 T \| [u, v] \|_{L^T_{a,s}} \]
\[ \leq \tilde{c}(\|u_0\| M^s_{p',q} + \|v_0\| M^s_{p',q}) + C_3 T (2R)^\lambda \]
\[ \leq R + C_3 T (2R)^\lambda < 2R, \] (3.22)

for all \([u, v] \in \mathcal{B}_{2R}\) and therefore, \(\Phi(\mathcal{B}_{2R}) \subseteq \mathcal{B}_{2R}\). On the other hand, from Proposition 3.1, and applying the Young inequality in each term \(\|u(t)\|_{M^s_{p,q}} \|\tilde{u}(t)\|_{M^s_{p,q}}\), \(k = 1, \ldots, \lambda\), we have

\[ \|\Phi[u, v] - \Phi[\tilde{u}, \tilde{v}]\|_{L^s_{\alpha,s}} \leq C_3 T \|\|u, v\| - [\tilde{u}, \tilde{v}]\|_{L^s_{\alpha,s}} (\|u, v\|_{L^s_{\alpha,s}}^{\lambda - 1} + \|\tilde{u}, \tilde{v}\|_{L^s_{\alpha,s}}^{\lambda - 1}) \]
\[ \leq C_3 2^{\lambda - 1} R^{\lambda - 1} \|\|u, v\| - [\tilde{u}, \tilde{v}]\|_{L^s_{\alpha,s}}. \]
(3.23)

for all \([u, v], [\tilde{u}, \tilde{v}] \in \mathcal{B}_{2R}\). Since \(C_3 2^{\lambda} R^{\lambda - 1} T < 1\), from (3.22) and (3.23) we get that \(\Phi\) is a contraction, which implies that the integral Eq. (1.3) has a unique solution \([u, v] \in \mathcal{L}^s_{\alpha,s}\), satisfying \(\|\|u, v\|\|_{\mathcal{L}^s_{\alpha,s}} \leq 2R\). The proof that the data-solution map is locally Lipschitz follows as in the proof of this property in Theorem 1.1. Notice that the choice of \(T\) depends on the size of \(R\); thus, \(T\) can be arbitrary large provided \(R\) be small enough.

### 3.4 Proof of Theorem 1.8

We only prove (1.13) in the case \(t \to \infty\). The case \(t \to -\infty\) follows analogously. We define

\[ [u_0^+, v_0^+] = [u_0, v_0] - \int_0^\infty B(-\tau)[0, \ell(u(\tau))]d\tau. \]

Let \([u^+, v^+] = B(t)[u_0^+, v_0^+]\) the solution of the linear problem associated to (1.2), that is,

\[
\begin{align*}
\partial_t u^+ &= \Delta v^+, \\
\partial_t v^+ &= u^+ - \Delta u^+, \\
u^+(x, 0) &= u_0^+(x), \\
v^+(x, 0) &= v_0^+(x).
\end{align*}
\]

The pair \([u^+, v^+]\) can be expressed as

\[ [u^+, v^+] = B(t)[u_0, v_0] - \int_0^\infty B(t - \tau)[0, f(u(\tau))]d\tau. \]
(3.24)

Taking the difference between (1.3) and (3.24) and computing the \(M^s_{p,q} \times D^{-1} J M^s_{p,q}\)-norm we get

\[ \|\|u(t) - u^+(t), v(t) - v^+(t)\|\|_{M^s_{p,q} \times D^{-1} J M^s_{p,q}} \]
\[ = \left\| \int_t^\infty B(t - \tau)[0, f(u(\tau))]d\tau \right\|_{M^s_{p,q} \times D^{-1} J M^s_{p,q}} \]
\[ \leq \int_t^\infty (1 + |t - \tau|)^{-\sigma} \|u(\tau)\|_{M^s_{p,q}} d\tau \]
Let \( t \) be the hypothesis we have that \( \lim_{t \to \infty} (1 + t)^\alpha \| u(t) \|_{\mathcal{M}^{s}_{p,q}} \)\)

\[ \leq \int_{t}^{\infty} (1 + |t - \tau|)^{-\alpha} (1 + \tau)^{-\alpha \lambda} d\tau \]

\[ \leq Ct^{1 - \alpha \lambda}. \]

### 3.5 Proof of Theorem 1.9

Let \([u, v], [\tilde{u}, \tilde{v}]\) be two solutions of (1.1) with data \([u_0, v_0], [\tilde{u}_0, \tilde{v}_0]\) \(\in \mathcal{M}^{s}_{p,q} \times \mathcal{M}^{s}_{p,q}\), respectively. We assume only the case \( t > 0 \); the case \( t < 0 \) can be addressed analogously. Taking the difference between the integral equations (1.3) and computing the \( \mathcal{M}^{s}_{p,q} \times D^{-1}J \mathcal{M}^{s}_{p,q}\)-norm we get

\[
\| [u(t) - \tilde{u}(t), v(t) - \tilde{v}(t)] \|_{\mathcal{M}^{s}_{p,q} \times D^{-1}J \mathcal{M}^{s}_{p,q}} \\
\leq \| B(t)[u_0 - \tilde{u}_0, v_0 - \tilde{v}_0] \|_{\mathcal{M}^{s}_{p,q} \times D^{-1}J \mathcal{M}^{s}_{p,q}} \\
+ \left\| \int_{0}^{t} B(t - \tau)[0, f(u) - f(\tilde{u})] d\tau \right\|_{\mathcal{M}^{s}_{p,q} \times D^{-1}J \mathcal{M}^{s}_{p,q}} := J_1 + J_2. \tag{3.25}
\]

Multiplying (3.25) by \((1 + t)^\alpha\) it holds

\[
(1 + t)^\alpha \| [u(t) - \tilde{u}(t), v(t) - \tilde{v}(t)] \|_{\mathcal{M}^{s}_{p,q} \times D^{-1}J \mathcal{M}^{s}_{p,q}} \leq (1 + t)^\alpha J_1 + (1 + t)^\alpha J_2. \tag{3.26}
\]

From the hypothesis we have that \( \lim_{t \to \infty} (1 + t)^\alpha J_1 = 0 \). Working as in Proposition 3.1, and using that \([u, v], [\tilde{u}, \tilde{v}] \) \(\in B_{2e} \subset \mathcal{L}^{\alpha}_{s,\infty}\), we bound \( J_2 \) as follows

\[
J_2 \leq \int_{0}^{t} (1 + |t - \tau|)^{-\alpha} \| u - \tilde{u} \|_{\mathcal{M}^{s}_{p,q}} \sum_{k=1}^{\lambda} \| u \|_{\mathcal{M}^{s}_{p,q}}^{k - 1} \| \tilde{u} \|_{\mathcal{M}^{s}_{p,q}} d\tau \\
\leq 2^\lambda e^{\lambda - 1} \int_{0}^{t} (1 + |t - \tau|)^{-\alpha} (1 + \tau)^{-\alpha(\lambda - 1)} \| [u(\tau) - \tilde{u}(\tau), v(\tau) - \tilde{v}(\tau)] \|_{\mathcal{M}^{s}_{p,q} \times D^{-1}J \mathcal{M}^{s}_{p,q}} d\tau. \tag{3.27}
\]

Let us denote by

\[
H := \limsup_{t \to \infty} (1 + t)^\alpha \| [u(t) - \tilde{u}(t), v(t) - \tilde{v}(t)] \|_{\mathcal{M}^{s}_{p,q} \times D^{-1}J \mathcal{M}^{s}_{p,q}} \\
= \limsup_{k \to \infty} \limsup_{t \geq k} (1 + t)^\alpha \| [u(t) - \tilde{u}(t), v(t) - \tilde{v}(t)] \|_{\mathcal{M}^{s}_{p,q} \times D^{-1}J \mathcal{M}^{s}_{p,q}}.
\]
From Theorem 1.1, $H < \infty$. On the other hand, by (3.27) it holds

$$
(1 + t)^\alpha J_2 \leq (1 + t)^\alpha 2^\lambda \varepsilon^{\lambda - 1} \int_0^{t/2} (1 + (t - \tau))^{-\alpha}(1 + \tau)^{-\alpha(\lambda - 1)}
$$

\[
\| [u - \tilde{u}, v - \tilde{v}] \|_{M^{p,q}_p \times D^{-1} J M^{s}_{p,q}} d\tau
\]

\[
+ (1 + t)^\alpha 2^\lambda \varepsilon^{\lambda - 1} \int_{t/2}^t (1 + (t - \tau))^{-\alpha}(1 + \tau)^{-\alpha(\lambda - 1)}
\]

\[
\| [u - \tilde{u}, v - \tilde{v}] \|_{M^{p,q}_p \times D^{-1} J M^{s}_{p,q}} d\tau
\]

\[
:= 2^\lambda \varepsilon^{\lambda - 1}(M_1 + M_2).
\]

Then,

$$
M_1 \leq \int_0^{t/2} (1 + \tau)^{-\alpha(\lambda - 1)} \| [u(\tau) - \tilde{u}(\tau), v(\tau) - \tilde{v}(\tau)] \|_{M^{p,q}_p \times D^{-1} J M^{s}_{p,q}} d\tau
$$

\[
= \int_0^{1/2} (1 + t\omega)^{-\alpha(\lambda - 1)} t \| [u(t\omega) - \tilde{u}(t\omega), v(t\omega) - \tilde{v}(t\omega)] \|_{M^{p,q}_p \times D^{-1} J M^{s}_{p,q}} d\omega.
\]

Therefore,

$$
\sup_{t \geq k} M_1 \leq \int_0^{1/2} \sup_{t \geq k} (1 + t\omega)^{-\alpha(\lambda - 1)} t \| [u(t\omega) - \tilde{u}(t\omega), v(t\omega) - \tilde{v}(t\omega)] \|_{M^{p,q}_p \times D^{-1} J M^{s}_{p,q}} d\omega.
$$

For $\omega \in (0, 1/2)$ we have

$$
\lim_{k \to \infty} \sup_{t \geq k} (1 + t\omega)^{\alpha} \| [u(t\omega) - \tilde{u}(t\omega), v(t\omega) - \tilde{v}(t\omega)] \|_{M^{p,q}_p \times D^{-1} J M^{s}_{p,q}} = H,
$$

and, for each $k \geq 0$,

$$
\sup_{t \geq k} (1 + t\omega)^{-\alpha(\lambda - 1)} t \| [u(t\omega) - \tilde{u}(t\omega), v(t\omega) - \tilde{v}(t\omega)] \|_{M^{p,q}_p \times D^{-1} J M^{s}_{p,q}}
$$

\[
= \sup_{t \geq k} t (1 + t\omega)^{-\alpha\lambda} t (1 + t\omega)^{\alpha} \| [u(t\omega) - \tilde{u}(t\omega), v(t\omega) - \tilde{v}(t\omega)] \|_{M^{p,q}_p \times D^{-1} J M^{s}_{p,q}}
\]

\[
\leq C 2^\lambda \varepsilon^{\lambda - 1}.
\]

Then, by the Dominated Convergence Theorem it holds that

$$
\lim_{t \to \infty} \sup_{t \geq k} M_1 \leq 2^\lambda \varepsilon^{\lambda - 1} H.
$$
\[ M_2 \leq \sup_{t/2 < \tau < t} (1 + \tau)^\alpha \|[u(\tau) - \tilde{u}(\tau), v(\tau) - \tilde{v}(\tau)]\|_{M^s_{p,q} \times D^{-1} J M^s_{p,q}} \]
\[ \times (1 + t)^\alpha \int_{t/2}^t (1 + (t - \tau))^{-\alpha} (1 + \tau)^{-\alpha \lambda} d\tau \]
\[ \leq \sup_{t/2 < \tau < t} (1 + \tau)^\alpha \|[u(\tau) - \tilde{u}(\tau), v(\tau) - \tilde{v}(\tau)]\|_{M^s_{p,q} \times D^{-1} J M^s_{p,q}} \]
For all \( t \geq k \),
\[ \sup_{t/2 < \tau < t} (1 + \tau)^\alpha \|[u(\tau) - \tilde{u}(\tau), v(\tau) - \tilde{v}(\tau)]\|_{M^s_{p,q} \times D^{-1} J M^s_{p,q}} \]
\[ \leq \sup_{k/2 < \tau} (1 + \tau)^{-\alpha} \|[u(\tau) - \tilde{u}(\tau), v(\tau) - \tilde{v}(\tau)]\|_{M^s_{p,q} \times D^{-1} J M^s_{p,q}} , \]
which implies that
\[ \limsup_{t \to \infty} M_2 \leq H. \quad (3.30) \]

Thus, from (3.28)–(3.30), recalling that \( 2^\lambda e^{\lambda - 1} < 1 \) and taking the \( \limsup_{t \to \infty} \) in (3.26) we get
\[ \lim_{t \to \infty} (1 + t)^\alpha \|[u(t) - \tilde{u}(t), v(t) - \tilde{v}(t)]\|_{M^s_{p,q} \times D^{-1} J M^s_{p,q}} = 0. \]

Now, we will prove the converse proposition. First, notice that from Theorem 1.1
\[ K := \left( \|[u, v]\|_{L^\infty_{a,s}} \right)^{\lambda - 1} + \left( \|[\tilde{u}, \tilde{v}]\|_{L^\infty_{a,s}} \right)^{\lambda - 1} < \infty. \]

Then we get,
\[ \limsup_{t \to \infty} (1 + t)^\alpha \|B(t)[u_0 - \tilde{u}_0, v_0 - \tilde{v}_0]\|_{M^s_{p,q} \times D^{-1} J M^s_{p,q}} \]
\[ \leq \limsup_{t \to \infty} (1 + t)^\alpha \|[u(t) - \tilde{u}(t), v(t) - \tilde{v}(t)]\|_{M^s_{p,q} \times D^{-1} J M^s_{p,q}} \]
\[ + \limsup_{t \to \infty} (1 + t)^\alpha \left\| \int_0^t B(t - \tau)[0, f(u) - f(\tilde{u})]d\tau \right\|_{M^s_{p,q} \times D^{-1} J M^s_{p,q}} \]
\[ \leq 0 + CKH = 0, \]
because \( H = 0 \) by hypothesis. Thus we conclude the proof of Theorem 1.9. \( \Box \)

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