Axisymmetric force-free magnetosphere in the exterior of a neutron star II: Maximum storage and open field energies

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ABSTRACT
A magnetar’s magnetosphere gradually evolves by the injection of energy and helicity from the interior. Axisymmetric static solutions for a relativistic force-free magnetosphere with a power-law current model are numerically obtained. They provide information about the configurations in which the stored energy is large. The energy along a sequence of equilibria increases and becomes sufficient to open the magnetic field. A magnetic flux rope, in which a large amount of toroidal field is confined, is formed in the vicinity of the star, for states exceeding the open field energy. These states are energetically metastable, and the excess energy may be ejected as a magnetar outburst.

Key words: stars: magnetars – stars: neutron – stars: magnetic fields

1 INTRODUCTION
Solar flares \((E = 10^{30} - 10^{32}\text{erg})\) are closely related to the Sun’s magnetic field. The flares often give rise to large coronal mass ejections, in which stored magnetic energy is suddenly converted to kinetic energy and radiation. Giant flares \((E = 10^{44} - 10^{46}\text{erg})\) observed in magnetars are widely believed to be analogous, but enormously scaled up (Lyutikov 2003, 2006; Beloborodov & Thompson 2007). The flare energy is a part of the total magnetic energy \((\sim 10^{47}(B_0/10^{14}\text{G})^2(R/12\text{km})^3)\).

At smaller energy scales, magnetars also exhibit highly variable bursting activity in the X-/gamma-ray band. This activity and persistent X-ray emission are powered by the rearrangement and dissipation of ultra-strong magnetic fields with strengths above \(B_0 = 10^{14}\text{G}\) (e.g., Turolla et al. 2015; Kaspi & Beloborodov 2017, for recent review).

The magnetic force is much larger than any other forces in the magnetar magnetosphere, and so the force-free approximation may be applicable. The magnetosphere is twisted by the current flowing in it and the presence of a toroidal field component is an obvious difference from potential magnetic fields in a vacuum. The structure changes as the result of the transfer of currents and helicities from the interior of the star. A quasi-steady shearing motion at the base of a magnetic field twists the exterior field, and the stored energy increases at the same time. When a state exceeds a threshold, the energy is abruptly released on a dynamical timescale, leading to energetic flares. Magnetic energy also builds up as a natural product of helicity accumulation. The interior itself evolves on a secular timescale by Hall drift, ambipolar diffusion, or some other mechanism (e.g., Goldreich & Reisenegger 1992; Hollerbach & Rüdiger 2004; Kojima & Kisaka 2012; Viganò et al. 2013; Gourgouliatos & Cumming 2014; Wood & Hollerbach 2015), and is affected by the exterior through the boundary. The interior and exterior are, therefore, coupled to each other. Recently, Akgün et al. (2017) modeled this time-dependent coupled system. Their evolution model shows that there is no equilibrium solution for the force-free magnetosphere on timescales of the order of thousands of years. This suggests an outburst during that time. Equilibrium solutions of twisted magnetospheres have been considered by a number of authors. For example, the magnetosphere models for a magnetar have been numerically constructed as a part of entire magnetic field structure from stellar core to the exterior (Glampedakis et al. 2014; Fujisawa & Kisaka 2014; Pili et al. 2015, 2017). Akgün et al. (2016) studied the effect of a covering current-free region on a twisted magnetosphere. The spacetime outside the magnetar is assumed to be flat in most of these works except for the work of Pili et al. (2015, 2017).

Treatment in flat spacetime seems to be reasonable as the lowest order approximation: a priori, the correction is expected to be not so large, since the relativistic factor is of order \(G_NM/(Rc^2) \sim 0.2 - 0.3\) in neutron stars.

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In a previous paper (Kojima 2017), however, we found that general relativistic effects are significant. The maximum energies stored in a current-flowing magnetosphere increases by a factor of a few times from the current-free dipole field energy in relativistic models. This contrasts with maximum excess energies of only a few tens of a percent in non-relativistic models. This large increase in relativistic models is related to the formation of a flux rope, an axially symmetric torus in the vicinity of the stellar surface, when the magnetic field structure is highly twisted. Curved spacetime helps to confine the torus. This energy that comes from non-potential magnetic fields is available for rapid release through a variety of mechanisms that may involve instabilities, loss of equilibrium, and/or reconnection. Is it possible to make a transition from a magnetosphere containing a detached magnetic flux to an open field corresponding to mass ejection? The problem is a dynamical one, but is here examined by comparing energies for two different configurations in topology. One is the energy of equilibrium model, for which magnetic field lines are closed and may contain magnetic flux rope. The other is the energy of open field configuration, for which all magnetic field lines are open and have the same surface condition as the equilibrium. When the energy of a state in a static sequence of models exceeds the open field one, then we may conclude that a transition to a dynamic state must occur.

This paper is organized as follows. We briefly discuss our model and relevant equations for a non-rotating force-free magnetosphere in a Schwarzschild spacetime in Section 2. We then numerically solve the so-called Grad–Shafranov equation assuming that the current function is given by a simple power-law model. The results are given in Section 3. Finally, our conclusions are given in Section 4. We use geometrical units of \( c = G_N = 1 \).

2 EQUATIONS

2.1 Magnetic fields

In this section, we briefly summarize our formalism. We consider the static magnetic configuration in Schwarzschild space-time for the solar flare model (Kojima et al. 2004; Zhang et al. 2006). Poloidal current flow is described by \( 4\pi \alpha j_\phi = \nabla \times (\alpha \vec{B}) = \nabla \times (\vec{e}_\phi/\alpha) \). The components in eq. (1) can be explicitly written as

\[
[B_\phi, B_\theta, B_\rho] = \left[ \frac{G_\phi}{\alpha}, -\frac{G_\theta}{\alpha}, \frac{G_\rho}{\alpha} \right].
\]

In the force-free magnetic field, the current function \( S \) should be a function of \( G \), and the global structure is determined by the so-called Grad-Shafranov equation:

\[
a^2 \frac{\partial}{\partial r} \left( \frac{1}{r^2} \frac{\partial G}{\partial r} \right) + \frac{\alpha^2}{r^2} \sin \theta \frac{\partial}{\partial \theta} \left( \frac{1}{\sin \theta} \frac{\partial G}{\partial \theta} \right) = \frac{1}{2} dS^2.
\]

In the numerical calculations, we adopt the following model for \( S(G) \):

\[
S = \left( \frac{1}{3} \right)^{1/2} G^3,
\]

where \( \gamma \) is constant, and the source term in eq.(3) is simply reduced to \(-\gamma G^5 \) (a power-law model with \( n = 5 \)). This model has been extensively studied in flat spacetime for the solar flare model (Flyer et al. 2004; Zhang et al. 2006, 2012). It is therefore easy to examine relativistic effects.

It is useful to show a solution in vacuum for eq.(3) with \( S = 0 \). The magnetic function \( G \) is expanded in terms of Legendre polynomials \( P_l(\theta) \):

\[
G(r, \theta) = -\sum_{l=1} g_l(r) \sin \theta \frac{dP_l(\theta)}{d\theta},
\]

and the radial functions \( g_l \) are given by an analytic function. For example, the radial function for a dipole (\( l = 1 \)) is

\[
g_1 = \frac{2}{8M^3} \left[ \ln \left( 1 - \frac{2M}{r} \right) + \frac{2M}{r^2} + \frac{2M^2}{r^3} \right] \approx \frac{B_0 R^3}{r} \left[ 1 + \frac{3M}{2r} + \frac{2M^2}{3b^2} + \cdots \right], \quad (M/r \ll 1)
\]

where \( B_0 \) is the typical field strength and \( R \) is the stellar surface radius. In this expression, the first expression is an exact solution and the second is its approximation in the weak gravity regime \( M/r \ll 1 \). The magnetic dipole moment \( \mu \) is given by \( \mu = B_0 R^2 \) from the asymptotic form of \( g_1 \) at infinity. The field strength at the surface pole is \( 2B_0 \) in a non-relativistic model, whereas it is larger by a factor of order \( O(M/R) \) in a relativistic model with the same dipole moment \( \mu \). In this paper, we
Figure 1. Magnetic field lines for a potential dipole (left panel) and an open field (right panel). In the top panels, the magnetic functions \( G_2(\theta)/g_1(R) \) and \( G_2(\theta)/g_1(R) \) at the surface are shown by a solid line. All dipolar field lines are closed in the left panel, whereas those of the open field extend from the surface to infinity. The radial component \( B_r \) is positive on the surface in the right panel. A realistic field is obtained by changing the sign of \( B_r \) in the southern hemisphere, and locating the current sheet for the split-monopole-like configuration on the equator, although this is not necessary for this work.

use as the normalization factor the field \( B_0 \), which is defined by the dipole moment \( \mu \), but does not denote the field strength \( 2B_r(R,0) \) at the surface pole except for the case \( M/R = 0 \).

We now discuss the boundary conditions needed to solve eq. (3) with (4). Along the polar axis, the magnetic function \( G \) should satisfy the regularity condition \( G = 0 \) at \( \theta = 0 \) and \( \pi \). Asymptotically \( (r \to \infty) \), the function should decrease as \( G \propto r^{-1} \). At the stellar surface \( r = R \), the magnetic function \( G \) is assumed to be a dipolar \((l = 1)\) field:

\[
G_S(\theta) \equiv G(R, \theta) = g_1(R) \sin^2 \theta,
\]  

where \( g_1(R) \) is given by eq. (6). The numerical method for solving the non-linear equation (3) with (4) is described in Kojima (2017).

2.2 Helicity and energy

Two integrals, magnetic helicity and energy, are useful to characterize equilibrium solutions of the magnetospheres. Magnetic helicity represents a global property of magnetic fields, and is obtained by integrating the product of two vectors, namely, \( \vec{A} \) and \( \vec{B} = \nabla \times \vec{A} \). A gauge-invariant quantity, \( H_R \), is defined by the difference of the magnetic helicity of a force-free field from that of a potential field with the same surface boundary condition. The total relative helicity in the exterior \((r \geq R)\) is given by

\[
H_R = \int_{r \geq R} \vec{A} \cdot \vec{B} \sqrt{g_3} d^3 x = 4 \pi \int_{r \geq R} \frac{G S \, d r d \theta}{\alpha^2 \sin \theta},
\]

where \( \sqrt{g_3} \equiv \alpha^{-1} r^2 \sin \theta \) is the determinant of the 3-dimensional space metric (Kojima 2017).

Magnetic energy stored in the force-free magnetosphere is also given by integrating over a 3-dimensional volume:

\[
E_{EM} = \int_{r \geq R} \frac{\alpha B^2}{8 \pi} \sqrt{g_3} d^3 x = \frac{1}{4} \int_{r \geq R} B^2 r^2 \sin \theta d r d \theta = \frac{1}{4} \int_{r \geq R} \left( \frac{\partial G}{\partial r} \right)^2 + \left( \frac{1}{r} \frac{\partial G}{\partial \theta} \right)^2 + \left( \frac{S}{\alpha} \right)^2 \frac{d r d \theta}{\sin \theta},
\]

In eq. (9), the factor \( \alpha \) in front of \( B^2 \) may be understood by considering the Maxwell equations in curved spacetime. Equivalently, the expression (9) can also be obtained by \( \int T_{ij} \sqrt{-g_3} d^3 x \) in terms of the energy momentum tensor \( T_{ij} \) and the determinant of the 4-dimensional spacetime metric \( \sqrt{-g_4} \) (Kojima 2017). The numerical results for \( H_R \) and \( E_{EM} \) in a force-free magnetosphere, which depend on the twist, will be given in the next section.

We here discuss the energy for two reference configurations. For a given dipolar field at the surface (7), the lowest energy state is given by the potential field. This energy is denoted by \( E_0 \), and is given by \( E_0 = B_0^2 R^3 / 3 \) for a dipolar potential field in flat spacetime (e.g., Low & Smith 1993). The value increases in relativistic models; for example, it has been numerically calculated as \( E_0 = 0.74 B_0^2 R^3 \) for a model with \( M/R = 0.25 \) (Kojima 2017).

Another important criterion is the open field energy \( E_{open} \). The open field configuration is demonstrated in Fig. 1.
Suppose that initially closed magnetic field lines of a force-free magnetosphere are stretched out to infinity by some artificial means, keeping the same boundary condition. Additional energy is necessary to open it. When the energy $E_{EM}$ of a force-free magnetosphere is less than $E_{open}$, opening is difficult. When $E_{EM} > E_{open}$, an open field configuration is energetically preferable. An abrupt transition to the open field may be related to the mass ejection in flares. It is therefore important to examine whether or not there exists a state with $E_{EM} > E_{open}$.

The calculation of $E_{open}$ has been discussed previously (e.g., Low & Smith 1993). Here, we briefly summarize the procedure. We modify the boundary condition (7) at the surface as

$$G_2^\ast(\theta) = G_2(\theta) \quad (0 \leq \theta \leq \pi/2).$$

$$G_2^\ast(\theta) = 2g_1(R) - G_2(\theta) \quad (\pi/2 < \theta \leq \pi).$$

Note that $G_2(\pi/2) = g_1(R)$ and $G_2^\ast(\theta)$ is a continuous function on the whole range $0 \leq \theta \leq \pi$. The functions $G_2(\theta)$ and $G_2^\ast(\theta)$ are displayed in the top panels of Fig.1. By solving eq.(3) with $S = 0$ and surface boundary condition $G_2(\theta)$, we have a dipolar potential field as shown in the left panel of Fig.1. By replacing the boundary condition with $G_2^\ast(\theta)$, an open field solution $G_{open}$ is obtained, as shown in the right panel of Fig.1. The boundary condition $G_2^\ast(\theta)$ is a monopolar magnetic field; that is, radial component $B_r = |B_r(\theta)|$ is one-way direction at the surface, so that all the field lines extend to infinity. The desired solution is obtained by taking this unphysical magnetic field and reversing its direction only on those lines in the southern hemisphere ($\pi/2 < \theta \leq \pi$). The magnetic energy is unchanged by this sign-flipping, and may be calculated for the solution $G_{open}$. The result is $E_{open}/E_0 = 1.66$ for a dipolar field in flat spacetime (e.g., Low & Smith 1993; Flyer et al. 2004).

The open field is strict poloidal, with $B_\phi = 0$, although the force-free field is twisted with $B_\theta \neq 0$. An finite twist is assumed to propagate to infinity along open field lines. The field necessarily includes a current sheet on the equator, which separates the regions of opposite magnetic polarity.

It is instructive to approximate magnetic function $G_{open}(r, \theta)$ as a monopole solution $G_M(r, \theta) \equiv g_1(R)(1 - \cos \theta)$. The function $G_2^\ast(\theta)$ is very close to $G_2(R, \theta)$ at the surface, but has some higher multi-poles with small amplitudes. The magnetic energy $E_{monopole}$ is calculated as $g_1(R)^2/(2R)$, which is reduced to $E_{monopole}/E_0 = 1.5$ in flat spacetime. The open field energy $E_{open}$ contains 16% contribution from higher multi-poles.

### 2.3 Virial

Here we derive some useful relations concerning total magnetic energy. We multiply eq. (3) by $F \partial G/\partial r$, where $F$ is an arbitrary function of $r$, and integrate over the space outside a radius $R$. Using integration by parts, we have the identity:

$$\frac{1}{4}\int_0^\pi \int_0^\infty \alpha r^2 \frac{dE}{dr} (B_r^2 + B_\theta^2 + B_\phi^2) \, dr \sin \theta d\theta = \frac{1}{4}\int_0^\pi \left[\frac{\alpha \gamma^2 F}{\gamma^2 - 1} r^2 \frac{\partial B_r^2}{\partial r} \right]_{r=R} \sin \theta d\theta + \frac{1}{4}\int_R^\infty \int_0^\pi \frac{dF}{dr} \left(\frac{\alpha^2 \gamma^2 F^2}{\gamma^2 - 1}\right) B_r^2 \, dr \sin \theta d\theta,$$

where we have used the components (2) of magnetic fields, and assumed that $\dot{B}$ approaches zero at infinity. As a first application, we consider this formula in flat spacetime by setting $\alpha = 1$. By choosing $F = r$, the left hand side in eq.(11) is reduced to the magnetic energy $E_{EM}$ stored in the exterior $r \geq R$, and the volume integral part of the right hand side vanishes. Thus, the magnetic energy is expressed by the surface term, that is, the virial theorem (Chandrasekhar 1961; Flyer et al. 2004):

$$E_{EM} = \frac{1}{4}\int_0^\pi \left[\frac{\alpha \gamma^2 r^2 (B_r^2 - B_\theta^2 - B_\phi^2)}{\gamma^2 - 1}\right]_{r=R} \sin \theta d\theta. \quad (12)$$

Since $E_{EM} \geq 0$, we have an inequality for magnetic components at $r = R$:

$$\int_0^\pi \left[\frac{B_r^2}{r^2}\right]_{r=R} \sin \theta d\theta \geq \int_0^\pi \left[\frac{B_\theta^2}{r^2}\right]_{r=R} \sin \theta d\theta \geq \int_0^\pi \left[\frac{B_\phi^2}{r^2}\right]_{r=R} \sin \theta d\theta.$$

We consider a sequence of solutions with fixed boundary condition (7), which means that the radial component ($B_r \propto G_\phi$) is always fixed at the surface. Equation (13) constrains the toroidal component $B_\phi \propto \gamma^{1/2}$. Thus, there is a maximum of $\gamma$ (Flyer et al. 2004).

Extension to the relativistic case with $\alpha \neq 1$ needs a little care, since the left hand side with $F = r$ in eq.(11) is no longer $E_{EM}$. It differs by a factor $\alpha^2$ (See eq.(9)). Some calculations provide

$$E_{EM} = \frac{1}{4}\int_0^\pi \left[\frac{\alpha^2 \gamma^2 (B_r^2 - B_\theta^2 - B_\phi^2)}{\gamma^2 - 1}\right]_{r=R} \sin \theta d\theta + \frac{1}{4}\int_R^\infty \int_0^\pi \frac{\alpha^2 \gamma^2 F^2}{\gamma^2 - 1} (B_r^2 + B_\theta^2 + B_\phi^2) \, dr \sin \theta d\theta.$$

This was derived in Yu (2011). Here the volume integral is included in the expression for $E_{EM}$. Another expression for $E_{EM}$ is also possible. By choosing a tortoise coordinate $F = r_\ast (r + 2M \ln(r/2M - 1))$, which satisfies $dr_\ast/dr = \alpha^{-2}$, the left hand side in eq.(11) is reduced to $E_{EM}$, and is the equation can be written as

$$E_{EM} = \frac{1}{4}\int_0^\pi \left[\frac{\alpha^2 \gamma^2 r^2 (B_r^2 - B_\theta^2 - B_\phi^2)}{\gamma^2 - 1}\right]_{r=R} \sin \theta d\theta + \frac{1}{4}\int_R^\infty \int_0^\pi \left(\frac{r^2 - r_\ast^2 + 3M r_\ast}{r^2 - r_\ast^2 + 3M r_\ast}B_r^2 \sin \theta drd\theta. \quad (15)$$
The relativistic expressions (14) and (15) represent the fact that the amount of exterior magnetic energy is determined not only by the surface values but also by some volume integral, unlike in the non-relativistic case (12). The additional term is positive in reasonable stellar models, and the radial component $B_r^2$ is dominant there. The term is of order $E_{\text{EM}} \propto (M/R)$, and acts to non-linearly increase $E_{\text{EM}}$, when $M/R$ is not very small. That is, a correction of $E_{\text{EM}}$ further increases $E_{\text{EM}}$ itself. Thus, a state having large $E_{\text{EM}}$ is less sensitive to the surface boundary in a relativistic system. These expressions explain the properties of an interesting structure, a soliton-like magnetic flux rope, found in the numerical models.

Finally, if we choose $F = r/\alpha$, then the surface integral is given by a volume integral as

$$\frac{1}{4} \int_0^\infty \left[ \alpha^2 r^2 (B_r^2 - B_\theta^2 - B_\phi^2) \right] r \sin \theta d\theta = \frac{1}{4} \int_0^\infty \int_0^{2\pi} \alpha^{-1} r (r - 3M) (B_r^2 + B_\theta^2 + B_\phi^2) r dr \sin \theta d\theta. \quad (16)$$

As long as $R > 3M$, the right hand side is positive definite, so that we have the same relation (13) as in flat spacetime.

## 3 NUMERICAL RESULTS

A sequence of magnetospheres is numerically constructed for a fixed boundary condition (7) at the surface. We start with a potential field solution, and follow the change of structure by increasing the toroidal magnetic field for a fixed relativistic factor $M/R$. A simple method is increasing the parameter $\gamma$ in eq.(4). Numerical solutions are however limited by the method as discussed below: the higher energy branch of solutions cannot be obtained An alternative method, which is used for the same power law current model in flat spacetime (Flyer et al. 2004; Zhang et al. 2006, 2012), is increasing azimuthal flux or helicity as the degree of twist. The constant $\gamma$ is determined a posteriori. Thus, both magnetic energy $E_{\text{EM}}$ and relative helicity $H_R$ are a multi-valued function of $\gamma$. A similar method is used in a different model (Pili et al. 2015; Akgün et al. 2018), where the physical extent of a field line is specified first, and the correspondence of poloidal field strength is determined as the result.

The magnetic energy and the relative helicity for the models with $M/R = 0, 0.1, 0.2, 0.3$ are shown in Fig. 2. For a better understanding of the mechanism, the energy difference $\Delta E = E_{\text{EM}} - E_0$ is divided as $\Delta E = \Delta E_t + \Delta E_p$, into a toroidal component and a poloidal component. The general tendency is the same in all models. There is a maximum of $\gamma$, and there are two branches in the curves of $\Delta E_t$, $\Delta E_t$ and $H_R$, when we consider solutions as a function of $\gamma$. In the lower branch, an increase of $\Delta E_t$ is evident, whereas $\Delta E_p$ is almost zero. The toroidal energy $\Delta E_t$ monotonically increases with $\gamma$, since $B^2_{\phi|p} \propto \gamma$ by eq.(4). However, there is a certain limit to $B^2_{\phi|p}$ or $\gamma$ by eq.(13). After passing the turning point of $\gamma$, $\Delta E_p$ increases dramatically in the upper branch. This means that the poloidal field structure significantly changes from that of the potential field in order for a larger toroidal field to be supported. The curve of $\Delta E_p$ or $\Delta E_t$ in Fig. 2 no longer goes up, but curls into a limiting point with a further increase of twist. This behavior is similar to that often appearing near a critical point in nonlinear dynamics. Flyer et al. (2004); Zhang et al. (2006, 2012) have shown the detailed behavior of this current model in a flat spacetime. A careful treatment is necessary when changing the parameter near the endpoint. In this work, we do not resolve the endpoint of the sequence, because the maximum value of energy or helicity is unchanged even if we approach the termination more closely. The maximum of the ratios $\Delta E_p/E_0$ and $\Delta E_t/E_0$ increase with $M/R$. There is a qualitative difference between the model with $M/R = 0$ and that with $M/R = 0.3$. At the maximum, we have $\Delta E_t > \Delta E_p$ in the former, while $\Delta E_t < \Delta E_p$ in the latter. Near the endpoint, the ratio is $E_t/E_p = \Delta E_t/(E_0 + \Delta E_p) < 0.2-0.4$, that is, the poloidal field is always dominated for stable configurations.

Figure 3 shows the magnetic field structure of a highly twisted state, i.e., near the endpoint along a sequence for each

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**Figure 2.** Increase in magnetic energy $\Delta E/E_0$ from the potential dipole field is shown in the left scale and total relative helicity $H_N = H_R/(4\pi E_0 R)$ is shown in the right scale. The energy of the poloidal component is denoted by crosses, that of the toroidal component by asterisks, and helicity by squares. The horizontal axis denotes the dimensionless value $\gamma (B_0 R^2)^{3/2}$. From left to right, the panels show the results for $M/R = 0, 0.1, 0.2, 0.3$. The relativistic expressions (14) and (15) represent the fact that the amount of exterior magnetic energy is determined not only by the surface values but also by some volume integral, unlike in the non-relativistic case (12). The additional term is positive in reasonable stellar models, and the radial component $B_r^2$ is dominant there. The term is of order $E_{\text{EM}} \propto (M/R)$, and acts to non-linearly increase $E_{\text{EM}}$, when $M/R$ is not very small. That is, a correction of $E_{\text{EM}}$ further increases $E_{\text{EM}}$ itself. Thus, a state having large $E_{\text{EM}}$ is less sensitive to the surface boundary in a relativistic system. These expressions explain the properties of an interesting structure, a soliton-like magnetic flux rope, found in the numerical models.

Finally, if we choose $F = r/\alpha$, then the surface integral is given by a volume integral as

$$\frac{1}{4} \int_0^\infty \left[ \alpha^2 r^2 (B_r^2 - B_\theta^2 - B_\phi^2) \right] r \sin \theta d\theta = \frac{1}{4} \int_0^\infty \int_0^{2\pi} \alpha^{-1} r (r - 3M) (B_r^2 + B_\theta^2 + B_\phi^2) r dr \sin \theta d\theta. \quad (16)$$

As long as $R > 3M$, the right hand side is positive definite, so that we have the same relation (13) as in flat spacetime.
Figure 3. Contour lines of the magnetic function $G$ and color contours of normalized $B_\phi (\geq 0)$ in the $r-\theta$ plane. The structure corresponds to the model of maximum energy in for the sequences with $M/R = 0$ (left panel) and $M/R = 0.3$ (right panel).

model, at which the stored energy is a maximum. We compare the model for $M/R = 0$ with that for $M/R = 0.3$. In the figure, we show the magnetic function $G$ by contour lines, and the toroidal component $B_\phi$ in the $r-\theta$ plane by colors. Only the interior part is shown, since the field outside approaches a vacuum solution due to $S \approx G^3 \to 0$, and so the outer part does not change. Magnetic lines are stretched toward the exterior by a strong twist. The maximum of the toroidal magnetic field $B_\phi$ is located near the surface for the model with $M/R = 0$. The topology of the magnetic function for the model $M/R = 0.3$ is different. There are loops of field lines around the center $(r/R, \theta) = (1.2, \pi/2)$, and the maximum of $B_\phi$ occurs there. The structure represents a flux rope braided by toroidal and poloidal magnetic fields in three-dimensional space. The magnetic flux is likely to expand, but general relativistic effects suppress the expansion, and allow a larger amount of magnetic energy to be stored at the same time. It should be noted that the similar flux-rope structure was also found in previous results in literature. For example, Pili et al. (2015) obtained it in the exterior model of a neutron star, by using a different current model and numerical method. In their numerical method, the radial extent of current-flowing field lines is specified to calculate a static solution. When the region extends to several times the stellar radius, then a remarkable flux-rope can be seen. Akgün et al. (2018) applied the similar method to a model in flat space-time, and obtained it. A direct comparison is difficult due to the differences in both models. However, by comparing our models with different relativistic factor, the flux-rope formation is not inherent in general relativity, but is sustained by the effect.

Figure 4 shows three energies, $E_0$, $E_{\text{max}}$ and $E_{\text{open}}$ normalized by $B_\phi^2 R^4 (= \mu^2 / R^4)$, as a function of the relativistic factor $M/R$. The potential field energy $E_0$ is the minimum, and the maximum $E_{\text{max}}$ is calculated along a sequence of force-free magnetosphere models. As inferred from eq.(6), $E_0 \propto g_1 (R^2)$ increases with $M/R$ in our normalization for fixed magnetic dipole moment. Figure 4 shows the open field energy $E_{\text{open}}$ also increases with $M/R$. The ratio $E_{\text{open}}/E_0$ depends less on the normalization, but it also slightly increases. For example, $E_{\text{open}}/E_0 = 1.66$ at $M/R = 0$ (Low & Smith 1993; Flyer et al. 2004) and it increases to 2.05 at $M/R = 0.3$. The increase of $(E_{\text{open}} - E_0)/E_0$ means a large load energy is required to open field, and it seems to be more difficult to make the transition from closed to open configurations in more relativistic system. However, the curve of maximum energy $E_{\text{max}}$ with $M/R$ is steeper, as shown in Fig. 4. Thus, a state with $E_{\text{RM}} > E_{\text{open}}$ is realized in a relativistic system with $M/R > 0.2$. The steep increase is closely related to the formation of a detached flux rope. The excess energy $E_{\text{EM}} - E_{\text{open}}$ is released with the flux rope eruption. The maximum energy is, for example, $(E_{\text{max}} - E_{\text{open}})/E_0 = 0.33$ for the model with $M/R = 0.25$ and 1.19 for the model with $M/R = 0.3$.

Figure 5 shows total magnetic energy along an increasing sequence of relative helicity for a model with $M/R = 0.25$. The magnetic function is also shown by contours for four representative states. They are characterized, in increasing energy, by a
potential field, the maximum of $\gamma$, an energy equal to the open field case $E_{EM} = E_{open}$ and the maximum energy $E_{max}$. A flux rope is evident after passing the turnover of $\gamma$ (i.e., in the upper branch in Fig. 2).

4 CONCLUSION

In this paper, we have studied energy storage in a relativistic force-free magnetosphere with power-law current model. Total magnetic energy increases as the helicity increases in axially symmetric equilibria. An evolution scenario of twisting magnetospheres is constructed through a quasi-static sequence of equilibrium states. That is, the magnetosphere over a long timescale gradually changes so as to accumulate magnetic helicity. The helicity stored in magnetosphere decreases only in a dynamical process. In general, total helicity is conserved as far as the ideal MHD condition $\vec{E} \cdot \vec{B} = 0$ holds. A catastrophic change, for which the acceleration field $\vec{E} \cdot \vec{B} \neq 0$ should be relevant, may be an outburst.

It is interesting to note that larger energy and helicity are capable of being stored in a relativistic magnetosphere than in a non-relativistic one. The energy at the endpoint along our equilibrium sequence with $M/R \geq 0.2$ exceeds the open field energy. This means that the high-energy states with $E_{EM} > E_{open}$ are metastable. A transition to a lower energy state is associated with the eruption of a magnetic flux rope. This is observed as a magnetar flare. It is, however, not clear at the moment how much energy is ejected. It depends on the stability of the high-energy states. That is, the excess $E_{EM} - E_{open}$ is almost zero when instability sets in soon after reaching a state with $E_{EM} = E_{open}$. On the other hand, the amount of energy increases, when the high-energy state is more stable and energy is built up before a bursting event. Such a problem requires a dynamical method for its solution (e.g., Li et al. 2012; Parfrey et al. 2013; Kojima & Kato 2014, as resistive simulation in flat spacetime), which is beyond the scope of the quasi-equilibrium approach used here.

The present paper as well as similar studies (e.g., Flyer et al. 2004; Wolfson et al. 2007; Akgiin et al. 2016) are useful to explore and describe conditions that result in equilibrium solutions containing substantial energy. By combining these works, it is evident that a large amount of the energy stored is related with flux rope formation in the vicinity of the surface. The maximum energy of detached configuration exceeds the open field energy, so that a transition to the lower energy state is possible. In the dynamical transition, a flux rope may be ejected. There are at least three elements studied so far that increase the energy stored in a force-free magnetosphere. Two are related to the current model, so that we, for convenience, assume the power law form $(B_0^2 \propto S^2 \propto G^{-n+1}$ (see eq. (4).) As the power index $n$ increases, the distribution of the toroidal magnetic field becomes steeper. The flux rope is formed due to strong confinement and the maximum energy increases. When the index $n$ is larger than 9, the energy exceeds the dipolar open-field energy by a few percent (Flyer et al. 2004). The second important element is covering by an external current-free magnetic field. The model can be described as $S^2 \propto (G - G_c)^{n+1}$ for $G \geq G_c$, while $S = 0$ for $G < G_c$. The cut-off means that current flowing is spatially limited. The interior non-potential field is held down, and energy storage is enhanced. For example, Wolfson et al. (2007) found that the maximum excess energy is $13\%$ of the dipolar potential field energy. The third element is confinement by curved space-time, considered here and in a previous paper. General relativistic effects suppress the outward eruption of magnetic flux, and relativistic models are capable of
of storing significantly more energy than the corresponding potential energy. The excess is 30% for\( M/R = 0.25 \) and 119% for\( M/R = 0.3 \).

The maximum of the buildup energy also depends on other factors, such as the magnetic field at the surface (Wolfson et al. 2012). At moment, it is not clear which factors are important, since we do not know the correct current model and surface condition of a magnetar. However, it is a relativistic object, so general relativistic effects should be taken into account in any model.

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