LONG TIME DECAY AND ASYMPTOTICS FOR THE COMPLEX MKDV EQUATION

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Abstract. We study the asymptotics of the complex modified Korteweg-de Vries equation
\[ \partial_t u + \partial_x^3 u = \pm |u|^2 \partial_x u \]
In the real valued case, it is known that solutions with small, localized initial data exhibit modified scattering for \(|x| \geq t^{1/3}\), and behave self-similarly for \(|x| \leq t^{1/3}\). We prove that the same asymptotics hold for complex mKdV. The major difficulty in the complex case is that the nonlinearity cannot be expressed as a derivative, which prevents us from using the scaling vector field to get control in weighted \(L^2\) spaces. Instead, we must argue carefully about how wave packets at different frequencies interact in physical space while exploiting cancellations to prevent a loss of derivatives. A key ingredient in our argument is the decomposition \(u = S + w\), where \(S\) is a self-similar solution with the same mean as \(u\) and \(w\) is a remainder that has better decay.

1. Introduction

We study the complex modified Korteweg-de Vries (mKdV) equation
\[ \partial_t u + \partial_x^3 u = \pm |u|^2 \partial_x u \] (1)
This equation appears as a model in nonlinear optics, where it models higher order corrections for waves travelling in a nonlinear medium [2, 32, 46]. It also describes higher order effects in vortex filament evolution [14]. The complex mKdV equation is completely integrable, and has infinitely many conserved quantities, the first few of which are the momentum, angular twist, and energy [11]:
\[ P(u) = \int |u|^2 \, dx, \quad W(u) = \int |u|^2 \arg(u)_x \, dx \quad E(u) = \int \frac{1}{2} |\partial_x u|^2 \mp \frac{1}{4} |u|^4 \, dx \]

1.1. Known results. The smoothing and maximal function estimates developed by Kenig, Ponce, and Vega in [40, 41] can be used to show that (1) is locally wellposed in \(H^s, s \geq 1/4\). For real initial data, Colliander, Keel, Staffilani, Takaoka, and Tao show in [7] that these solution exist globally, and global wellposedness for real initial data at the \(s = 1/4\) endpoint was shown by Kishimoto in [43] and independently by Guo in [23]. If we require uniformly continuous dependence on the initial data, then the \(s = 1/4\) endpoint is sharp, see [42] for the focusing case and [5] for the defocusing case. Local wellposedness has also been show in the weighted Sobolev spaces \(H^s \cap |x|^{-m}L^2\) for \(s \geq 2m, m \in \mathbb{Z}^+\) in [39] and in \(H^s \cap |x|^{-m}L^2\) for \(s \geq 1/4, s \geq 2m\) in [13]. For the equation set on the torus, wellposedness was shown by Chapouto in a range of Fourier-Lebesgue spaces [31]. Relaxing the requirement of uniformly continuous dependence on the initial data, Harrop-Griffiths, Killip, and Visan used the complete integrability of the equation to prove a weaker form of wellposedness in \(H^s\) for \(s > -1/2\) in [28]. They also show that for \(s \leq -1/2\), the equation exhibits instantaneous norm inflation, so no wellposedness result is possible. Outside the scale of \(H^s\) spaces, Grunrock proved in [21] that the equation is locally wellposed for real-valued initial data in the spaces \(\tilde{H}_r^s\) defined by the norms \(\|u\|_{\tilde{H}_r^s} := \|\langle \xi \rangle \hat{u}\|_{L^r}\) for the parameter range \(\frac{4}{3} < r \leq 2, s \geq \frac{1}{2} - \frac{1}{2r}\). The parameter range was
later improved by Grünrock and Herr in [22] to $1 < r \leq 2$, $s \geq \frac{1}{2} - \frac{1}{2r}$, which has the scaling-critical space $\tilde{H}^0_1$ as the (exclude) endpoint.

The long time asymptotics of the real-valued mKdV equation equation have received a great deal of study. The first complete results were given in [11], where Deift and Zhou used the complete integrability of the equations to obtain asymptotic formulas using the inverse scattering transform. The first results not depending on complete integrability were derived by Hayashi and Naumkin in [30,31], where it was shown that solutions starting with small, localized data decay at the linear rate and exhibit Painlevé asymptotics in the self-similar region $|x| \leq t^{1/3}$. These results were extended to proving modified scattering in the region $x \leq -t^{-1/3}$ and more rapid decay in the region $x \geq t^{-1/3}$ by Harrop-Griffiths in [27] using the method of testing with wave packets developed by Ifrim and Tataru in [34,35]. These results were also proved independently by Germain, Pusateri, and Rousset in [20] using the method of space time resonances, and it was further shown that solitons are stable under small, localized perturbations, and that for long times the perturbation has the same asymptotics as in the small data case. More recently, Correia, Côte, and Vega extended these results in [10] by allowing the solution to have a jump discontinuity at 0 in Fourier space, which corresponds to studying the dynamics of vortex filaments with corners, see [45]. Except for the partial results given in [33] using inverse scattering techniques, the asymptotics of (1) do not appear to have been studied.

Asymptotic results for the mKdV equation hinge on the decay properties of solutions to the linear equation

\begin{equation}
\begin{aligned}
\partial_t u + \partial_x^3 u &= 0 \\
u(t=0) &= u_0
\end{aligned}
\end{equation}

The fundamental solution is given in terms of the Airy function by

\[ F(x,t) = (3t)^{-1/3} \text{Ai}((3t)^{-1/3}x) \]

If $u_0$ is localized and regular, then for $t \geq 1$ we have

\[ |u(x,t)| \lesssim t^{-1/3} (x/t^{1/3})^{-1/4}, \quad |\partial_x u(x,t)| \lesssim t^{-2/3} (x/t^{1/3})^{1/4} \]

In particular, $|u||\partial_x u| \lesssim t^{-1}$, which suggests the problem (1) (like its real-valued counterpart), should be critical with respect to scattering.

1.2. Main results. We will consider the equation with data prescribed at $t = 1$:

\begin{equation}
\begin{aligned}
\partial_t u + \partial_x^3 u &= \pm|u|^2 \partial_x u \\
u(t=1) &= e^{-\partial_x^3} u_*
\end{aligned}
\end{equation}

We will prove the following result:

**Theorem 1.** Suppose $u_* \in H^2$. There exists an $\epsilon_0 > 0$ such that if $\epsilon < \epsilon_0$, and $u_*$ satisfies

\[ \|\hat{u}_*\|_{L^\infty} + \|xu_*\|_{L^2} \leq \epsilon \]

then the solution $u$ to (3) exists on $[1, \infty)$ and has the following asymptotics:

For $x \geq t^{1/3}$, we have rapid decay of the form

\[ |u(x,t)| \lesssim \epsilon t^{-1/3} (xt^{-1/3})^{-3/4} \]
For \( x \leq -t^{-1/3} \), we have modified scattering

\[
\begin{align*}
  u(x,t) &= \frac{1}{\sqrt{12t\xi_0}} \sum_{\nu \in \{1, -1\}} \exp \left( -2\nu it\xi_0^3 + \nu \frac{\tau}{4} \pm i\nu \int_1^t \frac{|\hat{f}(\nu\xi_0,s)|^2}{s} ds \right) \hat{f}_\infty(\nu\xi_0) \\
  &\quad + O(\epsilon t^{-1/3}(xt^{-1/3})^{-9/28})
\end{align*}
\]

(6)

where \( \xi_0 = \sqrt{-\frac{x}{3t}} \) and \( \hat{f}_\infty \) a bounded function of \( \xi \).

For \( |x| \leq t^{1/3+4\beta} \), we have self-similar behavior

\[
  u(x,t) = S(x,t;\alpha) + O(\epsilon t^{-1/3-\beta})
\]

(7)

where \( \alpha \) is some complex number with \( |\alpha| \lesssim \epsilon \), \( \beta = \frac{1}{6} - C\epsilon^2 \) for some constant \( C \), and \( S \) is a self-similar solution; that is \( S(x,t;\alpha) = t^{-1/3}\sigma(x/t^{1/3};\alpha) \), where \( \sigma \) is a bounded solution of

\[
\left\{ \begin{array}{l}
  \sigma'' - \frac{1}{3} x \sigma = \pm \frac{1}{3} |\sigma|^2 \sigma \\
  \hat{\sigma}(0) = \alpha
\end{array} \right.
\]

(8)

Remark 1. Equation (8) is nothing more than a complex, phase-rotation invariant version of the Painlevé II equation,

\[
\left\{ \begin{array}{l}
  \tau'' - x\tau = \pm \tau^3 \\
  \hat{\tau}(0) = \alpha \in \mathbb{R}
\end{array} \right.
\]

(9)

It is known (see \[9, 12, 29\]) that (9) has a unique bounded solution for \( |\alpha| < 1 \), and this fact will be used in Section 4 to prove that (8) has a unique, bounded solution.

Remark 2. Note that the assumption that \( u_* \in H^2 \) is only used to give local wellposedness for the equation (using the \( H^2 \cap x^{-1}L^2 \) theory from \[29\]). In particular, it plays no role in the a priori estimates which give us the asymptotics, and we do not need any smallness assumption on the \( H^2 \) norm of \( u_* \).

If we could prove local wellposedness for (3) with \( u_* \in \mathcal{F}L^\infty \cap x^{-1}L^2 \), then we could drop the requirement that \( u_* \in H^2 \) entirely, since our arguments would then imply that the local solution can be extended to a global one. However, proving local wellposedness in this space is not straightforward: the quasilinear behavior of the problem appears to preclude the use of a fixed-point argument, and smooth functions are not dense in \( \mathcal{F}L^\infty \), which makes compactness arguments more complicated. It is possible that local existence could be proved by arguing along the lines of \[10\]; however, nontrivial modifications outside the scope of this paper would be needed to account for the different algebraic structure of the nonlinearity in the complex case.

It might appear somewhat unnatural to prescribe initial data in this form. However, by combining Theorem 1 with the weighted local wellposedness result in \[39\] we obtain a result with initial conditions given at \( t = 0 \):

**Corollary 2.** Let \( u \) solve

\[
\left\{ \begin{array}{l}
  \partial_t u + \partial_x^3 u = \pm |u|^2 \partial_x u \\
  u(t=0) = u_0
\end{array} \right.
\]

Then, there exists an \( \epsilon_0 > 0 \) such that for all \( \epsilon < \epsilon_0 \), if

\[
\|xu_0\|_{L^2} + \|u_0\|_{H^2} \leq \epsilon
\]

then the solution \( u \) has the same asymptotics as in Theorem 1.
Proof. By [39] Theorem 8.1, for $\epsilon$ small enough there exists a local solution $u \in C([0, 1], H^2 \cap |x|^{-1} L^2)$ with $\sup_{0 \leq t \leq 1} \| u(t) \|_{H^2} + \| x u(t) \|_{L^2} \lesssim \epsilon$. Now, let $u_* = e^{it\partial_x^3} u(1)$. Since the linear propagator is unitary on $L^2$ Sobolev spaces, $u_* \in H^2$. Moreover, by using the identity $xe^{it\partial_x^3} = e^{it\partial_x^3}(x - 3t\partial_x^2)$, we see that

$$
\| (x) u_* \|_{L^2} \lesssim \| u_* \|_{L^2} + \| xe^{it\partial_x^3} u(1) \|_{L^2} \lesssim \| u(1) \|_{L^2} + \| (x - 3\partial_x^2) u(1) \|_{L^2} \lesssim \epsilon
$$

This controls (1) by the Sobolev-Morrey embedding, so Theorem 11 gives the result. \qed

Remark 3. A discussed in Remark 2, the role of the $H^2$ hypothesis in Corollary 2 is largely to allow us to use the weighted local wellposedness theory of [39]. In this case, however, it is much less clear that we could obtain a wellposedness theorem in the scaling critical space $FL^\infty \cap x^{-1} L^2$ because the dispersive decay estimates degenerate at $t = 0$. Even in the real-valued case, very little is known about wellposedness on $[0, 1]$ with initial data in $FL^\infty \cap x^{-1} L^2$: see [10].

1.3. Main difficulties. The main difficulty for complex mKdV over real valued mKdV is the unfavorable location of the derivative in the nonlinearity, which creates significant obstacles for the proof.

The first difficulty comes when we try to control $Lu$ in weighted spaces. Our argument requires us to control $Lu$ in $L^2$, where $L = e^{-it\partial_x^3} x e^{it\partial_x^3} = x - 3t\partial_x^2$. Prior works for real mKdV estimate $Lu$ by relating it to the scaling transform of $u$:

$$
\Lambda u = (1 + x\partial_x + 3t\partial_t)
$$

via the identity

$$
Lu = \partial_x^{-1} \Lambda u + 3tu^3
$$

which holds for solutions of real mKdV. Since the nonlinearity for real mKdV can be written as a derivative, we can integrate the equation for $\Lambda u$ and perform an energy estimate to get control of $\partial_x^{-1} \Lambda u$. This strategy fails completely for complex mKdV: the relationship between $Lu$ and $\Lambda u$ now reads

$$
Lu = \partial_x^{-1} \Lambda u + 3t\partial_x^{-1} (|u|^2 \partial_x u)
$$

and the last term cannot be bounded in $L^2$. Thus, we must use a different approach.

Our argument, very roughly speaking, amounts to performing an energy estimate on $Lu$ directly. The fact that we have a derivative in the nonlinearity means we must exploit some cancellation (via integration by parts) to avoid having to estimate terms containing a derivative, allowing us to obtain global bounds using only dispersive estimates. In the complex valued case, the derivative structure is less favorable, and the dispersive estimates for $u$ are only strong enough to prove $|\partial_t \hat{f}| \lesssim \epsilon^3 t^{-1}$, which is insufficient to prove global bounds.

\[4\]
To overcome this second difficulty, we perform a modulation argument. We write \( u = S + w \), where \( S \) is a self-similar solution to (1) satisfying \( \dot{S}(0) = \hat{u}(0) \) and \( w \) is a remainder. This decomposition is advantageous: the term \( S \) is essentially real-valued (up to a complex phase rotation), so we can use the Leibniz rule to write \( |S|^2 \partial_x S = \frac{1}{2} \partial_x(|S|^2 S) \). Moreover, the Fourier space estimates of Correia, Côte, and Vega in [9] can be combined with our estimates on the linear propagator to show that \( S \) obeys the same decay estimates as \( u \). In particular, \( S \) and \( u \) have matched asymptotics for \( |x| \lesssim t^{1/3} \) (which corresponds to \( |\xi| \lesssim t^{-1/3} \) in frequency space), so low-frequency projections of \( w \) obey stronger decay bounds than either \( u \) or \( S \). By writing

\[
\partial_t \hat{f}(0, t) = \pm \frac{1}{\sqrt{2\pi}} \int |u|^2 \partial_x u \, dx = \frac{1}{\sqrt{2\pi}} \int |u|^2 \partial_x u - |S|^2 \partial_x S \, dx = \pm \frac{1}{\sqrt{2\pi}} \int |u|^2 \partial_x w + 2\mathcal{R}(uw) \partial_x S \, dx
\]

and using this improved decay, we can prove that \( \partial_t \hat{f}(0, t) \) decays at an integrable rate, which allows us to show that \( \hat{f}(\xi, t) \) is bounded for \( |\xi| \lesssim t^{-1/3} \). As a bonus, we immediately get the asymptotics \( u \approx S \) for \( |x| \lesssim t^{1/3} \).

This argument has some similarities with the one used by Hayashi and Naumkin in [30]: indeed, the estimates we find for \( w \) are largely identical to theirs. However, our argument differs from theirs in three key regards. First, we cannot estimate on \( Lw \) using the scaling vector field \( \Lambda \), so we instead must use the method of space-time resonances to perform an energy estimate on \( Lw \) directly. Second, since the mean \( \hat{u}(0, t) \) varies in time, we must modulate in time rather than subtracting a fixed self-similar solution. Finally, our argument is different in terms of how the estimates on \( w \) fit into the proof. In [30], the estimates on \( w \) are performed after the solution has been shown to decay at the linear rate for all time, and are only necessary to obtain the asymptotics in the self-similar region. In our work, on the other hand, the estimates on \( w \) are necessary in order to prove that the solution \( u \) decays at the linear rate globally in time.

1.4. Plan of the proof.

1.4.1. Overview of the space-time resonance method. To prove Theorem 1 we will work within the framework of the method of space-time resonances. This method, first developed by Germain, Shatah, and Masmoudi in [18] and independently by Gustafson, Nakanishi, and Tsai in [24], has been used since then to derive improved estimates of existence and asymptotics for a variety of equations with dispersive character; see [5][6][7][10][25][36][38]. The method begins by rewriting the nonlinear equation for \( u \) in terms of the profile \( f(t) = e^{it\partial_x^3}u(t) \):

\[
\begin{array}{l}
\partial_t f = \pm e^{it\partial_x^3} \left( |e^{-it\partial_x^3} f|^2 \partial_x e^{-it\partial_x^3} f \right) \\
(\text{for } t = 1) = u_*
\end{array}
\]

Which can be re-written in mild form as

\[
\hat{f}(\xi, t) = \hat{u}_* + \frac{i}{2\pi} \int_1^t \int e^{i\xi \phi} \hat{f}(\eta, t) f(-\sigma, t) (\xi - \eta - \sigma) \hat{f}(\xi - \eta - \sigma, t) \, d\eta d\sigma \, ds
\]

where \( \phi \) is the phase associated with the four wave mixing by the cubic nonlinearity:

\[
\phi(\xi, \eta, \sigma) = \xi^3 - (\xi - \eta - \sigma)^3 - \eta^3 - \sigma^3 = 3(\eta + \sigma)(\xi - \eta)(\xi - \sigma)
\]

Roughly speaking, we would like to show that the change in \( f \) (given by the integral in (11)) is small in some norm that gives us the required decay estimates for \( u \). Heuristically, if we imagine that \( \hat{f} \) is a smooth bump function, then the integral term in (11) will be dominated by the stationary points of the phase where \( \nabla_{\xi, \eta, \sigma}(s\phi(\xi, \eta, \sigma)) = 0 \). The points where \( \phi = 0 \) corresponds to a resonance (in the
classical sense of the term) in the nonlinear interaction between plane waves of frequencies $\xi - \eta - \sigma$, $\eta$, and $-\sigma$.

For dispersive PDEs, it is more natural to think in terms of wave packets instead of plane waves. A wave packet at frequency $\xi$ is a bump function which travels at the group velocity, which for complex mKdV is $v_\xi = 3\xi^2$. Clearly, wave packets can interact over large timescales only if they have the same group velocity, and the condition $\nabla_{\eta,\sigma} \phi = 0$ is precisely what is required for three wave-packets at frequencies $\xi - \eta - \sigma$, $\eta$, and $-\sigma$ to have the same group velocities. See [15] for an expository overview of the method.

In our application, we will see in Section 3 the decay we want in Theorem 1 follows from the estimate $\sup_{t \geq 1} \|f\|_X \lesssim \epsilon$, where the $X$ norm is defined by

$$\|f\|_X = \|\hat{f}(t)\|_{L^\infty} + t^{-1/6} \|xf(t)\|_{L^2}$$

Note that this norm is scale invariant. In our argument, we use a bootstrap argument to show that this norm is small for all time.

1.4.2. Step 1: Stationary phase estimate for high frequencies. Let us first consider the $L^\infty$ bound for $\hat{f}(\xi)$. Since this amounts to a pointwise bound, it is natural to consider $\xi$ fixed. The stationary points $\nabla_{\eta,\sigma} \phi = 0$ are then given by

$$(\eta_1, \sigma_1) = (\xi, \xi)$$

$$(\eta_1, \sigma_1) = (\xi, -\xi)$$

$$(\eta_1, \sigma_1) = (-\xi, \xi)$$

$$(\eta_4, \sigma_4) = (\xi/3, \xi/3)$$

A formal stationary phase calculation then shows that

$$\partial_t \hat{f}(\xi, t) = \pm \frac{i \text{sgn } \xi}{6t} |\hat{f}(\xi, t)|^2 \hat{f}(\xi, t) + \epsilon e^{it\phi/3} \frac{\text{sgn } \xi}{t} |\hat{f}(\xi/3, t)|^2 \hat{f}(\xi/3, t) + \{\text{error}\}$$

where $c$ is some constant whose exact value is unimportant. Since the second term has a highly oscillatory phase, we expect that it will not be relevant on timescales $t \gtrsim |\xi|^{-3}$. Similarly, we expect the error term to be higher order, and hence not to contribute significantly to the asymptotics. After discarding the oscillatory term and the error term, we are left with a Hamiltonian ODE, which we can integrate explicitly to find that

$$\hat{f}(\xi, t) \approx \exp \left( \frac{\pm i}{6} \int_1^t \frac{|\hat{f}(\xi, s)|^2}{s} ds \right) f_\infty(\xi)$$

for some bounded function $f_\infty$.

1.4.3. Step 2: Modulation analysis in the self-similar region. The above argument only applies for frequencies $|\xi| \gtrsim t^{-1/3}$. For smaller frequencies, there is not enough oscillation to neglect the oscillating term, and the error term in the stationary phase expansion becomes unacceptably large due to the coalescence of the stationary points $(\eta_i, \sigma_i)$ as $\xi \to 0$. Using the embedding $H^1(dx) \to C^{0,1/2}$ and making the bootstrap assumption that $\|\partial_x \hat{f}\|_{L^2} = \|xf\|_{L^2} \lesssim \epsilon t^{1/6}$, we see that the problem of controlling low frequencies reduces to understanding the behavior of the zero Fourier mode. In the real-valued case, $\hat{f}(0, t)$ is conserved by the flow (and hence the low frequency bounds are immediate), but in the complex valued case,

$$\partial_t \hat{f}(0, t) = \partial_t \hat{u}(0, t) = \pm \frac{1}{\sqrt{2\pi}} \int |u|^2 \partial_x u \, dx$$
which is not zero in general.

The main difficulty for $|\xi| \lesssim t^{-1/3}$ is that the low-frequency component of $u$ evolves in a genuinely nonlinear manner. By analogy with the real-valued problem, we expect $u$ to exhibit self-similar asymptotics for $|x| \lesssim t^{1/3}$, which corresponds to the low frequency range $|\xi| \lesssim t^{-1/3}$. Thus, we will attempt to show the behavior of $u$ at low frequencies is approximately self-similar. If $S(x, t) = t^{-1/3} \sigma (xt^{-1/3})$ is a self-similar solution of (1), then $\sigma$ satisfies the third order ODE

$$\partial_x^3 \sigma - \frac{1}{3} \partial_x (x \sigma) = |\sigma|^2 \sigma_x$$

It can be seen that $\sigma$ has a constant phase, so $|\sigma|^2 \sigma_x = e^{-2i \arg \sigma} \sigma^2 \sigma_x = \frac{1}{3} \partial_x (|\sigma|^2 \sigma)$. Thus, we can reduce to the second order ODE (3). Since the mean of $u$ changes in time, we will also need to modulate the mean of the self-similar solution. This leads us to impose the condition $\partial_x \hat{f}(0, t)$ to the contribution from the self-similar region to obtain

$$\partial_x \hat{f}(0, t) = \pm \frac{1}{\sqrt{2\pi}} \int |u|^2 \partial_x u - |S|^2 \partial_x S \, dx = \pm \frac{1}{\sqrt{2\pi}} \int |u|^2 \partial_x w + (w \overline{w} + \overline{w} w) \partial_x S \, dx$$

where $w = u - S$.

1.4.4. Step 3: Weighted bounds for $w$. We now consider the difference $w$ in more detail. By definition, $w$ has mean zero, and so does $g = e^{i\xi^2} w$. In [30], it is shown that for real valued mKdV, $g$ obeys better weighted $L^2$ estimates, and our argument will show that $\|xg\|_{L^2} \lesssim \epsilon t^{1/6 - \beta}$ for $\beta$ as in Theorem 1. Assuming $g$ obeys these estimates, we find that $w$ has better dispersive decay than $u$, which allows us to prove a bound

$$|\partial_x \hat{f}(0, t)| \lesssim \epsilon^3 t^{-1 - \beta}$$

Integrating in time gives the boundedness for low frequencies, and shows that $u(x, t) \approx S(x, t; \alpha)$ for $|x| \lesssim t^{1/3}$ and $t$ large, where $\alpha = \lim_{t \to \infty} \hat{f}(0, t)$.

Using the self-similar scaling and (3), it can be seen that $S$ satisfies $\|LS\|_{L^2} \sim \epsilon^3 t^{1/6}$. Thus, $\|xf\|_{L^2} \leq \|LS\|_{L^2} + \|xg\|_{L^2}$ so the bound $\|xg\|_{L^2} \lesssim \epsilon t^{1/6}$ also follows from the improved $L^2$ bound for $xg$. In the real-valued case, it is possible to use the scaling vector field to control this term, but the non-divergence form of the nonlinearity in (1) precludes this argument. Instead, we will use a more direct argument. Since $F(xg) = -i \partial_x \hat{g}$, by Planchar’s theorem it suffices to prove bounds on $\partial_x \hat{g}$ in $L^2$. We find that

$$\partial_x \hat{g} = \mp \frac{s}{2\pi} \int \partial_x \phi e^{i\phi} \hat{f}(\eta) (\xi - \eta - \sigma) \hat{g}(-\xi + \eta + \sigma) f(-\sigma) \, d\eta d\sigma$$

$$+ \frac{1}{2\pi} \int e^{i\phi} \hat{f}(\eta) (\xi - \eta - \sigma) \partial_x \hat{g}(\xi - \eta - \sigma) f(-\sigma) \, d\eta d\sigma + \{\text{other terms}\} \quad (14)$$

The second term appears concerning, since in physical space it can be written as $\pm e^{-i\xi^2} (|u|^2 \partial_x Lw)$ and our estimates do not allow us to control derivatives of $Lw$. However, an energy estimate shows that this term is actually harmless:

$$\frac{1}{2} \partial_t \|xg\|_{L^2}^2 = \pm \int |u|^2 \partial_x |Lw|^2 \, dx + \{\text{other terms}\} = \mp \int \partial_x |u|^2 |Lw|^2 \, dx + \{\text{other terms}\}$$
and \( \int |u|^2 |Lw|^2 \, dx \lesssim e^{2t-1} \|xy\|_{L^2}^2 \), which is consistent with the slow growth of \( xg \) in \( L^2 \).

Thus, it only remains to control the first term in (14). We do this by considering the space-time resonance structure of \( \phi \), together with cancellations coming from the \( \partial_\xi \phi \) multiplier. The derivatives of \( \phi \) are

\[
\begin{align*}
\partial_\xi \phi &= 3(\eta + \sigma) (2\xi - \eta - \sigma) \\
\partial_\eta \phi &= 3(\xi - \sigma) (\xi - 2\eta - \sigma) \\
\partial_\sigma \phi &= 3(\xi - \eta) (\xi - \eta - 2\sigma)
\end{align*}
\]  

Based on (12) and (15), we introduce the space-time resonant sets

\[
\begin{align*}
\mathcal{T} &= \{ \xi = \eta \} \cup \{ \xi = \sigma \} \\
\mathcal{S} &= \{ \eta = \sigma = \xi/3 \} \\
\mathcal{R} &= \mathcal{S} \cap \mathcal{T} \\
&= \{(0,0,0)\}
\end{align*}
\]

where \( \mathcal{T} \) is the set of time resonances (where \( \phi \) vanishes to higher order than \( \partial_\xi \phi \)), \( \mathcal{S} \) is the set of space resonances (where \( \nabla_{\eta,\sigma} \phi \) vanishes to higher order than \( \partial_\xi \phi \)), and the set of space-time resonances \( \mathcal{R} = \mathcal{T} \cap \mathcal{S} \).

Away from the set \( \mathcal{T} \) of time resonances the quotient \( \frac{\partial_\xi \phi}{\nabla_{\eta,\sigma} \phi} \) is bounded and we may integrate by parts in the time variable. This is akin to the normal form transformation method introduced by Shatah in [47]. In particular, it transforms the cubic nonlinearity into a quintic nonlinearity, which gives us more decay and leads to better bounds.

On the other hand, outside of the set \( \mathcal{S} \) of space resonances, \( \frac{\partial_\xi \phi}{\nabla_{\eta,\sigma} \phi} \) is bounded, and we can integrate by parts using the relation \( \frac{\nabla_{\eta,\sigma} \phi}{\nabla_{\eta,\sigma} \phi} \cdot \nabla_{\eta,\sigma} e^{i\xi \phi} = e^{i\xi \phi} \) to gain a power of \( s^{-1} \). This is similar in spirit to the vector field method developed by Klainerman in [44]. In principle, this integration could result in a loss of derivatives when the space weight and the derivative fall on the same term. In practice, however, we only need to apply this integration by parts in a small neighborhood of \( \mathcal{T} \), where it can be seen that \( |\xi - \eta - \sigma| \lesssim \max(|\eta|, |\sigma|) \). Roughly speaking, this inequality will allow us to move the derivative from the term with an \( x \) weight to an unweighted term. Besides averting the aforementioned derivative loss, this has the added benefit of permitting us to use a form of the improved bilinear estimates, giving us enough decay to close the estimate.

Finally, in a small (time-dependent) neighborhood of the space-time resonant set \( \mathcal{R} \), we can integrate crudely using the volume bounds in Fourier space and the Hölder bound \( \tilde{g}(\xi) \lesssim e^{t^{1/6-\beta} |\xi|^{1/2}} \) together with the \( L^\infty \) bound for \( \tilde{f} \) to bound the contribution from \( \mathcal{R} \).

1.4.5. Organization of the paper. The plan of the rest of the paper is as follows: In Section 2 we present some notation that will be used throughout the paper, and give some conventions and results about pseudoproduct operators. In Section 3 we will give decay estimates for the linear equation. In Section 4, we will consider the self-similar solution \( S \), and derive estimates which will be necessary for the later analysis. In Section 5, we will show how Theorem 1 reduces to a bootstrap argument on \( \|f\|_X \) and \( |\partial_t f(0,t)| \). In the following sections, we close the bootstrap. Section 6 is devoted to proving that under the bootstrap assumptions, \( \|xy\|_{L^2} \lesssim e^{t^{1/6-\beta}} \) using space-time resonances and a Grönonwall argument. We then close the bootstrap by verifying that \( |\partial_t \tilde{f}| \) obeys the required decay (and hence that the low frequencies of \( \tilde{f} \) are bounded) in Section 7, and by showing that at high frequencies \( \tilde{f} \) is bounded and undergoes the required logarithmic phase rotation.
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2. Preliminaries

2.1. Notation and basic inequalities. We will make use of the Japanese bracket notation

\[ \langle x \rangle := \sqrt{1 + x^2} \]

If \( X \) and \( Y \) are two quantities which we wish to compare, but we want to suppress constant factors, we will write

- \( X \lesssim Y \) if \( X \leq CY \) for some \( C > 0 \),
- \( X \sim Y \) if \( X \lesssim Y \) and \( Y \lesssim X \),
- \( X \ll Y \) if \( X \leq cY \), where \( c \) is a small constant, the exact value of which depends on the context.

If we want to allow the implicit constant to depend on some parameters \( P_1, P_2, \ldots, P_n \), then we will write

- \( X \lesssim_{P_1, P_2, \ldots, P_n} Y \),
- \( X \sim_{P_1, P_2, \ldots, P_n} Y \),
- \( X \ll_{P_1, P_2, \ldots, P_n} Y \),

respectively.

We use the Fourier transform convention

\[ \mathcal{F} f(\xi) = \hat{f}(\xi) := \frac{1}{\sqrt{2\pi}} \int f(x) e^{-ix\xi} \, dx \]

with the inverse transformation

\[ \mathcal{F}^{-1}(\xi) = \hat{f}(x) := \frac{1}{\sqrt{2\pi}} \int f(x) e^{ix\xi} \, dx \]

Under this convention, multiplication and convolution are linked by

\[ \mathcal{F}(fg)(\xi) = \frac{1}{\sqrt{2\pi}} \hat{f}(\xi) \hat{g}(\xi) \]

where

\[ f \ast g(x) = \int f(x-y)g(y) \, dy = \int f(y)g(x-y) \, dy \]

Using the Fourier transform, we can generalize the notion of differential operators to define Fourier multiplication operators. A Fourier multiplication operator with symbol \( m : \mathbb{R} \to \mathbb{C} \) is given by

\[ m(D)f(x) = \mathcal{F}^{-1}((m(\xi)\hat{f}(\xi))(x)) \]

An especially important family of Fourier multipliers is the Littlewood-Paley projectors. Let \( \psi \in C^\infty(\mathbb{R}) \) be a function supported on \( B_2(0) \) which is identically zero on \( B_{1/2}(0) \) satisfying

\[ \sum_{j \in \mathbb{Z}} \psi(2^j \xi) = 1 \]

for all \( \xi \neq 0 \). Then, we define the Littlewood-Paley projectors as

\[ P_j = \psi_j(D) = \psi \left( \frac{D}{2^j} \right) \]

and define \( P_j^+ \) and \( P_j^- \) to be the projectors to positive and negative frequencies, respectively:

\[ P_j^+ = \psi_j(D)1_{D>0}, \quad P_j^- = \psi_j(D)1_{D<0}, \]
We write
\[ P_{\leq j} = \sum_{k \leq j} P_k, \quad P_{\geq j} = \sum_{k \geq j} P_k, \quad P_{[j_1, j_2]} = \sum_{j_1 \leq k \leq j_2} P_k \]
with \( P_{< j} \) and \( P_{> j} \) being defined similarly. We also define
\[ P_{\leq j} = \sum_{k \leq j+10} P_k, \quad P_{\ll j} = \sum_{k < j+10} P_k, \quad P_{\sim j} = \sum_{j-10 \leq k \leq j+10} P_k \]
In our arguments, we will use the (time-dependent) frequency projector given by
\[ Q_j = \begin{cases} P_j & 2^j > t^{-1/3} \\ P_{\leq j} & 2^{j-1} < t^{-1/3} \leq 2^j \\ 0 & \text{else} \end{cases} \]

**Remark 4.** The choice to cut-off at \( 2^j \sim t^{-1/3} \) reflects the fact that the uncertainty principle for complex mKdV makes it useless to distinguish wave-packets with frequencies \( \ll t^{-1/3}, \) since these waves move too slowly to have dispersed appreciably over this timescale.

To denote the projector to low frequencies, we will sometimes write
\[ Q_{\leq \log t^{-1/3}} = \psi_{\leq \log t^{-1/3}}(D) := P_{\leq j} \]
where \( j \in \mathbb{Z} \) is given by \( 2^{j-1} < t^{-1/3} \leq 2^j, \) so that
\[ \text{Id} = Q_{\leq \log t^{-1/3}} + \sum_{2^j > t^{-1/3}} Q_j \]
As with the projectors \( P_j, \) we define
\[ Q_{\leq j} = \sum_{k \leq j} Q_k, \quad Q_{\geq j} = \sum_{k \geq j} Q_k, \quad Q_{[j_1, j_2]} = \sum_{j_1 \leq k \leq j_2} Q_k \]
with \( Q_{< j} \) and \( Q_{> j} \) being defined analogously, and
\[ Q_{\leq j} = Q_{\leq j+10}, \quad Q_{\sim j} = Q_{[j-10, j+10]}, \quad Q_{\ll j} = \chi_{< j-10} \]
All of these projectors are bounded from \( L^p(\mathbb{R}) \to L^p(\mathbb{R}), \) and moreover
\[ \sum_{j \in \mathbb{Z}} \|P_j f\|_{L^2}^2 \sim \sum_{j \in \mathbb{Z}} \|Q_j f\|_{L^2}^2 \sim \|f\|_{L^2}^2 \]
To complement the \( Q_j, \) it will be useful to consider time-dependent functions \( \chi_j \) with the property that if \( f \) is a bump function localized in space near 0, then \( e^{-it\partial^2_x} Q_j f \) will be localized near the support of \( \chi_j \) (up to more rapidly decaying tails). To do this, we define
\[ \chi_j(x; t) = \begin{cases} \chi(x/(2^{2j})) & 2^j > t^{-1/3} \\ \sum_{2^k \leq t^{-1/3}} \chi(x/(2^{2k})) & 2^j \leq t^{-1/3} < 2^{j+1} \\ 0 & 2^{j+1} \leq t^{-1/3} \end{cases} \]
where \( \chi \) is a non-negative bump function localized in the region \( |x| \approx 1 \) such that \( \sum_{j=\infty}^\infty \chi(x) = 1 \) for all \( x \neq 0. \) As with the Fourier projectors, we define
\[ \chi_{\leq j} = \sum_{k \leq j} \chi_k, \quad \chi_{\ll j} = \sum_{k < j} \chi_k, \quad \chi_{[j_1, j_2]} = \sum_{j_1 \leq k \leq j_2} \chi_k \]
with \( \chi_{>j} \) and \( \chi_{\geq j} \) defined in analogously, and
\[
\chi_{\leq j} = \chi_{\leq j+10}, \quad \chi_{<j} = \chi_{<j-10}, \quad \chi_{=j} = \chi_{[j-10,j+10]}
\]
and similarly for \( \chi_{>j} \) and \( \chi_{\geq j} \).

Note that if \( f \) is an \( L^2 \) function, the families \( \{ \chi_k f \}_{2^k \geq t^{-1/3}} \), \( \{ Q_j f \}_{2^j \geq t^{-1/3}} \) and \( \{ \chi_k Q_j f \}_{2^k, 2^j \geq t^{-1/3}} \) are almost orthogonal, which implies that
\[
\| f \|_{L^2}^2 \sim \sum_{2^k \geq t^{-1/3}} \| \chi_k f \|_{L^2}^2 \sim \sum_{2^j \geq t^{-1/3}} \| Q_j f \|_{L^2}^2 \sim \sum_{2^k, 2^j \geq t^{-1/3}} \| \chi_k Q_j f \|_{L^2}^2 \tag{16}
\]

We also recall the following bound (which expresses the pseudolocality of the projectors \( Q_j \)): For \( \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \),
\[
\|(Q_{\leq j} f) g\|_{L^p} \lesssim N (2^j d(\text{supp}(f), \text{supp}(g)))^{-N} \| f \|_{L^{p_1}} \| g \|_{L^{p_2}} \tag{17}
\]
which can be obtained by writing \( Q_{\leq j} f = \tilde{Q}_{\leq j} \ast f \) and noting that \( \tilde{Q}_j \) is a rapidly decreasing function. In particular, if the supports of \( f \) and \( g \) are separated by a distance much larger than \( 2^{-j} \), the term on the right in \( (17) \) will be small.

**Remark 5.** We will often apply the estimate \( (17) \) (and its multilinear generalization) as follows: Taking \( g = \chi \) for some cut-off function \( \chi : \mathbb{R} \to [0,1] \), and choosing \( \tilde{\chi} \) to be a larger cut-off (i.e. \( \text{supp}(1 - \tilde{\chi}) \subset \text{supp} \chi \)), we find that
\[
\| \chi Q_{\leq j} ((1 - \tilde{\chi}) F) \|_{L^p} \lesssim N (2^j R)^{-N} \| F \|_{L^p} \tag{18}
\]
Using this, we find that
\[
\| \chi Q_{\leq j} F \|_{L^p} \lesssim N \| \tilde{\chi} F \|_{L^p} + (2^j R)^{-N} \| F \|_{L^p}
\]
which can be interpreted as a sort of commutator estimate if \( \tilde{\chi} F \) satisfies better \( L^p \) estimate than \( F \), allowing us to commute the frequency projector \( Q_{\leq j} \) with physical space localization up to an error of size \( (2^j R)^{-N} \| F \|_{L^p} \), which is favorable if \( 2^j R \gg 1 \).

**Remark 6.** By writing \( Q_j = Q_{\leq j} - Q_{\leq j-1} \), we see that the same bounds hold true if \( Q_{\leq j} \) is replaced by \( Q_j \).

Furthermore, if \( f \) has mean 0 (i.e. \( \hat{f}(0) = 0 \)), then taking Fourier transforms and applying Hardy’s inequality (see \( \text{[26]} \)), we find that
\[
\| f \|_{H^{-1}}^2 := \sum_{2^j} 2^{-2j} \| P_j f \|_{L^2}^2 \lesssim \| f \|_{L^2}^2 \tag{19}
\]

2.2. Multilinear harmonic analysis. For a symbol \( m : \mathbb{R}^3 \to \mathbb{C} \), we define the trilinear pseudoproduct operator \( T_m \) by
\[
FT_m(f,g,h)(\xi) = \frac{1}{2\pi} \int m(\xi, \eta, \sigma) \hat{f}(\eta) \hat{g}(\xi - \eta - \sigma) \hat{h}(\sigma) \, d\eta d\sigma
\]
In particular, \( T_1(f,g,h)(x) = (fgh)(x) \), so \( T_m \) can be thought of as a generalized product. If the symbol \( m \) is sufficiently well-behaved, we can show that the pseudoproduct \( T_m(f,g,h) \) obeys H"{o}lder-type bounds (see also \( \text{[6]} \)):

**Theorem 3.** Suppose \( m \) is a symbol with \( \tilde{m} \in L^1 \), and \( \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = \frac{1}{p} \). Then,
\[
\| T_m(f,g,h) \|_{L^p} \lesssim \| \tilde{m} \|_{L^1} \| f \|_{L^{p_1}} \| g \|_{L^{p_2}} \| h \|_{L^{p_3}} \tag{20}
\]
Proof. By inverting the Fourier transform, we find
\[ T_m(f, g, h)(x) = \frac{1}{(2\pi)^{N/2}} \int \hat{m}(y, z, w) f(x - y - z) g(x - y) h(x - y - w) \, dydzdw \]
which yields the result by Young’s inequality. \( \square \)

Remark 7. In our analysis, we will often consider symbols \( m \) which are supported on a region of volume \( O(2^{3j}) \) and satisfy the symbol bounds
\[ |\partial_{\xi, \eta, \sigma}^\alpha m(\xi, \eta, \sigma)| \lesssim 2^{-|\alpha|} \]
For such symbols,
\[ |\hat{m}(y, z, w)| \lesssim_N \frac{2^j}{(1 + |2^j y|)^N} \frac{2^j}{(1 + |2^j z|)^N} \frac{2^j}{(1 + |2^j w|)^N} \]
which shows that \( m \) satisfies the hypotheses of Theorem 3.

We also need a pseudolocality property for pseudoproduct operators given in the following lemma:

Lemma 4. Suppose that \( f, g, h \) are functions, and suppose that \( \text{supp } f \) and \( \text{supp } h \) are separated by a distance \( R \). Let \( m \) be a symbol supported on \( |\xi| + |\eta| + |\sigma| \leq 2^{j+5} \) and satisfying the symbol bounds
\[ |\partial_{\xi, \eta, \sigma}^\alpha m(\xi, \eta, \sigma)| \lesssim 2^{-|\alpha|} \]
Then, for \( \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = \frac{1}{p} \), we have
\[ \|T_{m_j}(f, g, h)\|_{L^p} \lesssim_N (2^j R)^{-N} \|f\|_{L^{p_1}} \|g\|_{L^{p_2}} \|h\|_{L^{p_3}} \tag{21} \]
Similarly, if \( \text{supp } f \) and \( \text{supp } h \) are separated by a distance \( R \), then
\[ \|kT_{m_j}(f, g, h)\|_{L^p} \lesssim_N (2^j R)^{-N} \|f\|_{L^{p_1}} \|g\|_{L^{p_2}} \|h\|_{L^{p_3}} \|k\|_{L^{p_4}} \tag{22} \]
\( \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} + \frac{1}{p_4} = \frac{1}{p} \), and analogous results apply if we interchange the roles of \( f \), \( g \), and \( h \).

Proof. We begin by proving the bound (21). Inverting the Fourier transform in the definition of \( T_{m_j} \), we find that
\[ T_{m_j}(f, g, h) = \frac{1}{(2\pi)^{3/2}} \int m_j(\xi, \eta, \sigma) \hat{f}(\sigma) \hat{g}(\xi - \eta - \sigma) \hat{h}(\sigma) \, d\xi d\eta d\sigma \]
\[ = \int K_j(y, z, w) f(x - y) g(x - z) h(x - w) \, dydzdw \]
where
\[ K_j(y, z, w) = \frac{1}{(2\pi)^{3}} \int m_j(\xi, \eta, \sigma) e^{i(\xi - \eta - \sigma) y} e^{i\eta z} e^{i\sigma w} \, d\xi d\eta d\sigma \]
Since \( m_j \) is smooth and supported on a region of size \( O(2^{3j}) \), the same reasoning as in Remark 7 gives us the bound
\[ |K_j(y, z, w)| \lesssim_N (2^j y, 2^j z, 2^j w)^{-N - 10} \]
so \( K_j \in L^1_{y, z, w} \). Using the support hypothesis for \( f \) and \( g \), we find that
\[ \|T_{m_j}(f, g, h)\|_{L^p} \lesssim_N \int 2^{3j} (2^j y, 2^j z, 2^j w)^{-N - 10} \|f(x - y) g(x - z) h(x - w)\|_{L^p_y} \, dydzdw \]
\[ \lesssim_N \int_{|y-z| \geq R} 2^{3j} (2^j y, 2^j z, 2^j w)^{-N - 10} \|f(x - y) g(x - z) h(x - w)\|_{L^p_y} \, dydzdw \]
\[ \lesssim_N (2^j R)^{-N} \|f\|_{L^{p_1}} \|g\|_{L^{p_2}} \|h\|_{L^{p_3}} \]
To prove (22), we note that
\[ \|kT_m(f, g, h)\|_{L^p} = \left\| k(x) \int K_j(y, z, w) f(x - y)g(x - z)h(x - w) \, dydzdw \right\|_{L^p} \]
\[ \leq \int_{|y| \geq R} 2^{3j} \langle (2^j y, 2^j z, 2^j w) \rangle^{-n - 10} \| f(x - y)g(x - z)h(x - w)k(x) \|_{L^p} \, dydzdw \]
\[ \lesssim (2^j R)^{-N} \| f \|_{L^p} \| g \|_{L^p} \| h \|_{L^p} \| k \|_{L^p}. \]

\[ \square \]

3. Linear and multilinear estimates

3.1. The linear estimate. We begin by proving a linear estimate for the Airy propagator. Define the spaces \( X_j \) for \( 2^j \geq t^{-1/3} \) by the norm
\[ \| f \|_{X_j} := \| \overline{Q_{\sim j}} f \|_{L^\infty} + t^{-1/6} \| xQ_{\sim j} f \|_{L^2} \]
and note that \( \| f \|^2_{X_j} \lesssim \| f \|^2_X \), where \( X \) is the norm defined in (13).

**Lemma 5.** Let \( u(x, t) = e^{-it\xi_0^2} f(x, t) \). For \( 2^j > t^{-1/3} \), we have the pointwise estimate
\[ P^\pm_j u(x, t) = \frac{1}{\sqrt{2\pi t \xi_0}} e^{\frac{2\pi it t}{L^j}} f(\pm \xi_0) \mathbb{1}_{x < 0} \]
\[ + O \left( t^{-1/3} \left( 2^j t^{1/3} \right)^{-9/14} \| x < 0 \chi_{\sim j} (x, t) \| f \|_{X_j} + t^{-1/3} \right) \| f \|_{X_j} \]
where \( \xi_0 = \sqrt{\frac{4}{t}} \). Moreover, we have the estimate
\[ |Q_{\leq \log t^{-1/3}} u(x, t)| \lesssim t^{-1/3} (1 + t^{2/3} \xi_0^{-2})^{-1} \| f \|_{X_{\leq \log t^{-1/3}}} \]
so for \( p \in [4, \infty] \),
\[ \| Q_j u \|_{L^p} \lesssim t^{-\frac{3}{4} + \frac{1}{2} j} (\frac{3}{2} - \frac{1}{2})^j \| f \|_{X_j} \]
In particular, if \( p > 4 \),
\[ \| u \|_{L^p} \lesssim t^{-\frac{3}{4} + \frac{1}{2} j} \| f \|_{X} \]

**Proof.** The estimate (26) follows directly from (23) and (24), and (27) follows from (26) and (26) since
\[ \| u \|_{L^p} \lesssim \| Q_{\leq \log t^{-1/3}} u \|_{L^p} + \sum_{2^j > t^{-1/3}} \| Q_j u \|_{L^p} \]
so it only remains to prove (24) and (25).

To prove the low frequency estimate (25), it suffices to show that \( |Q_{\leq \log t^{-1/3}} u| \lesssim \min(t^{-1/3}, t^{-1} \xi_0^{-2}) \).

The bound \( |Q_{\leq \log t^{-1/3}} u| \lesssim t^{-1/3} \) follows from the \( L^\infty \) estimate for \( \hat{f} \) and the Hausdorff-Young inequality, so it suffices to consider the case \( |\xi_0| \gg t^{-1/3} \). In this case, writing
\[ Q_{\leq \log t^{-1/3}} u(x, t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \psi_{\leq \log t^{-1/3}} (\xi) e^{it\xi} \hat{f}(\xi) \, d\xi \]
for \( \phi(\xi) = \frac{\xi}{t} \xi + \xi^3 = \xi^3 - 3\xi^2 \xi \), the assumption \( |\xi_0| \gg t^{-1/3} \) implies that \( |\partial_\xi \phi| \sim |\xi_0|^2 \) on the support of \( \psi_{\leq \log t^{-1/3}} \hat{f} \), so integration by parts yields

\[
\begin{align*}
|Q_{\leq \log t^{-1/3}} u(x, t)| &\leq \frac{1}{t} \int_{\mathbb{R}} \left| \partial_\xi \left( \frac{1}{\partial_\xi \phi} \right) \right| \left| \psi_{\leq \log t^{-1/3}} \hat{f}(\xi) \right| d\xi \\
&\quad + \frac{1}{t} \int_{\mathbb{R}} \left| \psi_{\leq \log t^{-1/3}} \hat{f}(\xi) \right| d\xi \\
&\leq t^{-1} |\xi_0|^{-2} \left( t^{-1/3} |\xi_0|^{-1} \| Q_{\leq \log t^{-1/3}} \hat{f} \|_{L^\infty} + t^{-1/6} \| xQ_{\leq \log t^{-1/3}} \hat{f} \|_{L^2} \right) \\
&\leq t^{-1} |\xi_0|^{-2} \| f \|_{X_{t^{-1/3}}}
\end{align*}
\]

as required.

We now turn to the estimate (24). We consider the estimate for \( P^+_j u \): the estimate for \( P^-_j u \) is similar. As before, we write

\[
P^+_j u(x, t) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \psi_j^+(\xi) e^{it\phi(\xi)} \hat{f}(\xi) \, d\xi
\]

We distinguish between three cases depending on the relative sizes of \( 2^j \) and \( |\xi_0| \) and the sign of \( x \).

**Case** \( |\xi_0| < 2^{-10} j \). In this case, \( |\partial_\xi \phi| \sim 2^j \), and integration by parts gives

\[
\begin{align*}
|P^+_j u(x, t)| &\leq \frac{1}{t} \int_0^\infty \left| \partial_\xi \left( \frac{\psi_j^+(\xi)}{\partial_\xi \phi} \right) \right| \left| \psi_j^+(\xi) \hat{f}(\xi) \right| d\xi \\
&\quad + \frac{1}{t} \int_0^\infty \left| \psi_j^+(\xi) \frac{1}{\partial_\xi \phi} \partial_\xi \left( \psi_j^+(\xi) \hat{f}(\xi) \right) \right| d\xi
\end{align*}
\]

For (28a), we observe that \( \frac{\psi^+(\xi)}{\partial_\xi \phi} \) has size \( O(2^{-3j}) \) and is supported on a region of size \( O(2^j) \), so using Hardy’s inequality (19) yields

\[
28a \quad \lesssim t^{-1} 2^{-5/2} \| Q_{\sim j} \hat{f} \|_{L^2} \lesssim t^{-1} 2^{-3/2} \| Q_{\sim j} \hat{f} \|_{H^{-1}} \lesssim t^{-1/3} \left( t^{1/3} 2^j \right)^{-3/2} \| f \|_{X_j}
\]

since \( 2^j \gtrsim t^{1/3} \). Similarly, \( \frac{\psi^+(\xi)}{\partial_\xi \phi} \) has size \( O(2^{-2j}) \) and is supported on a region of size \( O(2^j) \), so

\[
28b \quad \lesssim t^{-1} 2^{-3/2} \| Q_{\sim j} x f \|_{L^2} \lesssim t^{-1/3} \left( 2^j t^{1/3} \right)^{-3/2} \| f \|_{X_j}
\]

as required.

**Case** \( |\xi_0| > 2^{j+10} \). In this case, \( |\partial_\xi \phi| \sim \xi_0^2 \), and a quick calculation gives that

\[
\begin{align*}
\left\| \frac{\psi^+_j (\xi)}{\partial_\xi \phi} \right\|_{L^2} &\lesssim \frac{1}{\xi_0^2 2^{j/2}} \\
\left\| \frac{\psi^+_j (\xi)}{\partial_\xi \phi} \right\|_{L^2} &\lesssim \frac{2^{j/2}}{\xi_0^2}
\end{align*}
\]

(29)
Arguing as in the previous case and integrating by parts, we find that

\[
|P_j^+ u(x,t)| \lesssim \frac{1}{t} \int_0^\infty \left| \frac{\partial_k \psi_j^+ (\xi)}{\partial_k \phi(\xi)} \right| |\psi_{j-} \hat{f}(\xi)| \, d\xi + \frac{1}{t} \int_0^\infty \left| \frac{\psi_j^+(\xi)}{\partial_k \phi(\xi)} \right| |\psi_{j-} \partial_k \hat{f}(\xi)| \, d\xi \tag{30a}
\]

Using the bounds (29) and argue as in the case \( |\xi_0| < 2^{j-10} \) to find that

\[
|P_j^+ u(x,t)| \lesssim t^{-1} \xi_0^{-2} 2^{-j/2} \| Q_{-j} \hat{f} \|_{L^2} \leq t^{-1} \xi_0^{-2} 2^{-j/2} \| Q_{-j} \hat{f} \|_{H^{-1}} \leq t^{-1/3} \left( t^{1/3} 2^{-3/4} \xi_0^{-4/3} \right)^{-3/2} \| f \|_{X_j} \tag{30b}
\]

which yields the desired bound for \( P_j^+ u(x,t) \).

**Case** \( x > 0 \), \( 2^{j-10} \leq |\xi_0| \leq 2^{j+10} \). Here, \( |\partial_k \phi| \sim \xi_0^{-2} \), and the estimate is identical to the previous case.

**Case** \( x < 0 \), \( 2^{j-10} \leq |\xi_0| \leq 2^{j+10} \). Now, \( \partial_k \phi \) vanishes at \( \xi = \xi_0 \), so we must employ the method of stationary phase. Write

\[
P_j^+ u(x,t) = \sum_{\ell = \ell_0}^{j+10} I_{j,\ell}
\]

where

\[
I_{j,\ell} = \frac{1}{\sqrt{2\pi}} \int_0^\infty \psi_j^+(\xi) \psi_{\ell} (\xi - \xi_0) e^{it\phi(\xi)} Q_{-j} \hat{f}(\xi) \, d\xi, \quad \ell > \ell_0
\]

\[
I_{j,\ell_0} = \frac{1}{\sqrt{2\pi}} \int_0^\infty \psi_j^+(\xi) \psi_{\ell_0} (\xi - \xi_0) e^{it\phi(\xi)} Q_{-j} \hat{f}(\xi) \, d\xi
\]

and \( \ell_0 \) is chosen such that \( 2^{\ell_0} \sim t^{-1/3} (2^j t^{1/3})^{-3/7} \), where the implicit constant is chosen small enough (independent of \( t \) and \( j \)) to ensure that \( \psi_j^+(\xi) \psi_{\ell_0} (\xi - \xi_0) = \psi_{\ell_0} (\xi - \xi_0) \), which is always possible since \( 2^k \geq t^{-1/3} \geq 2^\ell \).

For the \( I_{j,\ell} \) factors with \( \ell > \ell_0 \), we have that \( |\partial_k \phi| \sim 2^\ell 2^j \). Integrating by parts, we find that

\[
|I_{j,\ell}| \lesssim \frac{1}{t} \int_0^\infty \left| \frac{\partial_k \psi_j^+ (\xi)}{\partial_k \phi(\xi)} \right| |\psi_{j-} (\xi) \hat{f}(\xi)| \, d\xi + \frac{1}{t} \int_0^\infty \left| \frac{\psi_j^+(\xi)}{\partial_k \phi(\xi)} \right| |\psi_{j-} (\xi) \partial_k \hat{f}(\xi)| \, d\xi
\]

\[
\lesssim t^{-1} \left( 2^{-j} 2^{-\ell} \| Q_{-j} \hat{f} \|_{L^\infty} + 2^{-j} 2^{-\ell/2} \| \partial_k Q_{-j} \hat{f} \|_{L^2} \right)
\]
Finally, rescaling and using the classical theorem of nonstationary phase gives
\[
\sum_{\ell > \ell_0} |I_{j, \ell}| \lesssim t^{-1} \left( 2^{-j/2} \| Q_{-j} f \|_{L^\infty} + 2^{-j/2} \| \partial_x Q_{-j} \|_{L^2} \right)
\lesssim \left( t^{-1} 2^{-j/2} + t^{-5/6} 2^{-j/2} \right) \| f \|_{X_j}
\]  
(31)

For the \( I_{j, \ell_0} \) term, we write
\[
I_{j, \ell_0} = \frac{1}{\sqrt{2\pi}} \int_0^\infty \psi_{\ell_0}(\xi - \xi_0) e^{it\phi(\xi)} \left( \psi_j^+ (\xi) \tilde{Q}_{-j} \tilde{f} (\xi) - \psi_j^+ (\xi_0) \tilde{Q}_{-j} \tilde{f} (\xi_0) \right) d\xi
\]  
(32a)
\[
+ \frac{1}{\sqrt{2\pi}} \psi_j^+ (\xi_0) \tilde{f} (\xi_0) \int_0^\infty \psi_{\ell_0}(\xi - \xi_0) \left( e^{it\phi(\xi)} - e^{6it\xi_0(\xi - \xi_0)^2 - 2it\xi_0^3} \right) d\xi
\]  
(32b)
\[
+ \frac{1}{\sqrt{2\pi}} \psi_j^+ (\xi_0) \tilde{f} (\xi_0) e^{-2it\xi_0^3} \int_0^\infty \psi_{\ell_0}(\xi - \xi_0) e^{6it\xi_0 \xi^2} d\xi
\]  
(32c)

For the first term, we note that
\[
\left| \psi_j^+ (\xi) \tilde{Q}_{-j} \tilde{f} (\xi) - \psi_j^+ (\xi_0) \tilde{Q}_{-j} \tilde{f} (\xi_0) \right| \leq \left| \psi_j^+ (\xi) - \psi_j^+ (\xi_0) \right| \| \tilde{Q}_{-j} \tilde{f} \|_{L^\infty} + \left| \tilde{Q}_{-j} \tilde{f} (\xi) - \tilde{Q}_{-j} \tilde{f} (\xi_0) \right|
\lesssim \left( \frac{|\xi - \xi_0|}{2} + t^{1/6} |\xi - \xi_0|^{1/2} \right) \| f \|_{X_j}
\]

where the last line follows from the Sobolev-Morrey embedding \( H^1 \to C^{0,1/2} \). Using this bound, we find that
\[ \left| \psi_j^+ (\xi) \tilde{Q}_{-j} \tilde{f} (\xi) - \psi_j^+ (\xi_0) \tilde{Q}_{-j} \tilde{f} (\xi_0) \right| \lesssim \left( 2^{3/2} \xi_0^{1/6} + 2^{-j/2} \xi_0^{1/2} \right) \| f \|_{X_j} \]

For the second term, we observe that
\[
\phi(\xi) = -2\xi_0^3 + 6\xi_0(\xi - \xi_0)^2 + O((\xi - \xi_0)^3)
\]
so
\[
\left| \psi_j^+ (\xi) \tilde{f} (\xi_0) \right| \lesssim \| \psi_j^+ (\xi) \tilde{f} \|_{L^\infty} \int_0^\infty \psi_{\ell_0}(\xi - \xi_0) \left| e^{it\phi(\xi_0)^3} - 1 \right| d\xi
\lesssim t^{2\xi_0^3} \| f \|_{X_k}
\]

Finally, rescaling and using the classical theorem of nonstationary phase gives
\[ \left| \psi_j^+ (\xi_0) \tilde{f} (\xi_0) e^{-2it\xi_0^3} \right| \lesssim t^{-2\xi_0^3 + i\frac{\pi}{4}} + O \left( t^{-3/2} 2^{-2\xi_0^3} 2^{-3/2} \right) \]

Collecting the terms (31) and (32a) and recalling the definition of \( \ell_0 \), we find that
\[
u(t, x) = \frac{\psi_j^+ (\xi_0) \tilde{f} (\xi_0)}{\sqrt{12\xi_0^3}} e^{-2it\xi_0^3 + i\frac{\pi}{4}} + O \left( t^{-1/3} (2\ell_0^{1/3})^{-9/14} \| f \|_{X_j} \right)
\]

As a corollary of the above estimate, we obtain an improved bilinear decay estimate:
Corollary 6. If \( f, g \in X \), then
\[
|e^{-i\theta_3^2} f e^{-i\theta_3^2} \partial_x g| \lesssim t^{-1} \|f\|_X \|g\|_X
\]

Remark 8. Although Corollary 7 does not apply directly to pseudoproducts, we will see in Section 6 that Lemma 5 can be used together with the pseudolocality of pseudoproducts given in Lemma 4 to give bounds of the same type for pseudoproducts involving a \( \partial_x u \) term.

We also obtain the following result relating to the decay of \( \chi_{k-C} u \):

Corollary 7. Let \( C \) be a positive integer. Then, if \( f \in X \) and \( u = e^{-i\theta_3^2} f \),
\[
\|\chi_{\geq k-C} u\|_{L^\infty} \lesssim C^{-1/2} \|f\|_X
\]

It will also be important later to have decay estimates for \( e^{-i\theta_3^2} g \) when \( \hat{g}(0) = 0 \). We record them here:

Corollary 8. Suppose \( \hat{g}(0) = 0 \) and \( xg \in L^2 \). Then, if \( w = e^{-i\theta_3^2} g \), we have the bounds
\[
|Q_j w| \lesssim \left( t^{-1/2} \chi_{\sim j} + t^{-5/6} 2^{j/2} \left( 2^j + 2^{-j/3} |\xi_0|^{4/3} \right)^{-3/2} \right) \|Q_{[j-30, j+30]}(xg)\|_{L^2}
\]
and
\[
\|\chi_k w\|_{L^\infty} \lesssim t^{-1/2} \|Q_{[k-40, k+40]}(xg)\|_{L^2} + t^{-5/6} 2^{-k} \|xg\|_{L^2}
\]

Proof. By the Morrey-Sobolev embedding \( \dot{H}^1 \to C^{1/2} \), we have that for \( 2^j \gtrsim t^{-1/3} \),
\[
\|g\|_{X_j} = \|F(Q_{[j-10, j+10]} g)\|_{L^\infty} + t^{-1/6} \|xQ_{[j-10, j+10]} g\|_{L^2}
\lesssim (1 + t^{-1/6} 2^{-j/2}) \|F(Q_{[j-20, j+20]} g)\|_{L^\infty} + t^{-1/6} \|Q_{[j-20, j+20]}(xg)\|_{L^2}
\lesssim 2^{j/2} \|Q_{[j-30, j+30]}(xg)\|_{L^2}
\]
so applying Lemma 5 gives (33). To prove the localized bound (34), we note that (33) implies that
\[
\|\chi_k w\|_{L^\infty} \lesssim \|\chi_k Q_j w\|_{L^\infty} + \|\chi_k \chi_j Q_{\leq k} w\|_{L^\infty} + \|\chi_k Q_{> k} w\|_{L^\infty}
\lesssim \sum_{|\ell| \leq 10} \|Q_\ell w\|_{L^\infty} + t^{-5/6} \sum_{\ell < k-10} \|\chi_k Q_\ell w\|_{L^\infty} + \sum_{\ell > k+10} \|\chi_k Q_\ell w\|_{L^\infty}
\lesssim \sum_{|\ell| \leq 10} t^{-1/2} \|Q_{[\ell-30, \ell+30]}(xg)\|_{L^2} + t^{-5/6} \sum_{\ell < k-10} 2^{\ell-2k} \|xg\|_{L^2} + \sum_{\ell > k+10} 2^{-\ell} \|xg\|_{L^2}
\lesssim t^{-1/2} \|Q_{[40, 40]}(xg)\|_{L^2} + t^{-5/6} 2^{-k} \|xg\|_{L^2}
\]
as required.

3.2. Bounds for cubic terms. Since the complex mKdV equation has a cubic nonlinearity, we will naturally find ourselves dealing with frequency-localized terms of the form \( Q_j(|u|^2 \partial_x u) \) and the like. In this section, we collect some basic bounds for these terms for later reference.
Let $u_i = e^{-i\omega f_i}$ for $i = 1, 2, 3$ and $f_i \in X$. We begin by considering $\chi_{\geq j-10}Q_{\sim j}(u_1u_2u_3)$. Using \cite{stewart09} to control the error in commuting the physical and Fourier localization operators, we find that

$$
\|\chi_{\geq j-10}Q_{\sim j}(u_1u_2u_3)\|_{L^\infty} \lesssim \|\chi_{\geq j-10}Q_{\sim j}(u_1u_2u_3)\|_{L^\infty} + \|\chi_{\geq j-10}Q_{\sim j}(u_1u_2u_3)\|_{L^\infty}
\lesssim \|\chi_{\geq j-10}u_1\|_{L^\infty}\|\chi_{\geq j-10}u_2\|_{L^\infty}\|\chi_{\geq j-10}u_3\|_{L^\infty}
+ (2^{3j})^{-1}\|u_1\|_{L^\infty}\|u_2\|_{L^\infty}\|u_3\|_{L^\infty}
\lesssim c_t^{-3/2}2^{-3/2j}\prod_{i=1}^3\|f_i\|_X
$$

Arguing in the same manner, we find that

$$
\|\chi_{\geq j-10}Q_{\sim j}(u_1u_2u_3)\|_{L^\infty} \lesssim c_t^{-3/2}2^{-3/2j}\|f_1\|_X\|f_2\|_X\|f_3\|_X
$$

Moreover, by using the bilinear decay estimate given in Corollary \cite{stewart09} we can obtain analogous bounds for terms containing a derivative:

$$
\|\chi_{\geq j-10}Q_{\sim j}(u_1\partial_x u_2u_3)\|_{L^\infty} \lesssim c_t^{-3/2}2^{-3/2j}\prod_{i=1}^3\|f_i\|_X
$$

In the bounds \cite{stewart09}, the frequency localization in some sense plays no role, in that the bounds would remain true if we removed the frequency projection operator. The situation changes when we consider the cubic $Q_{\sim j}(u_1u_2u_3)$ in the region $|x| \ll t^{2j}$, since the frequency localization eliminates the worst contribution in this region. A straightforward paraproduct decomposition yields

$$
Q_{\sim j}(u_1u_2u_3) = Q_{\sim j}\left(u_{1, <j-20}u_{2, <j-20}u_{3, <j-20} + u_{1, [j-20, j+20]}u_{2, <j-20}u_{3, <j-20}
+ \sum_{\ell \geq j-20} u_{1, \ell}u_{2, [\ell-20, \ell+20]}u_{3, <\ell-20} + \sum_{\ell \geq j-20} u_{1, \ell}u_{2, [\ell-20, \ell+20]}u_{3, [\ell-20, \ell+20]}
\right)
+ \{\text{similar terms}\}
$$

Note that the term $Q_{\sim j}(u_{1, <j-20}u_{2, <j-20}u_{3, <j-20})$ vanishes. If $k < j - 30$, we find, after using \cite{stewart09} to commute the physical- and frequency-space localizations and applying Lemma \cite{stewart09},

$$
\|\chi_kQ_{\sim j}(u_{1, [j-20, j+20]}u_{2, <j-20}u_{3, <j-20})\|_{L^\infty} \lesssim \left(t^{-11/6}2^{-3/2j-k} + (t^{2k+j})^{-1}t^{-7/6}2^{-j/2}\right)\prod_{i=1}^3\|f_i\|_X
$$

$$
\|\chi_kQ_{\sim j}(u_{1, \ell}u_{2, [\ell-20, \ell+20]}u_{3, <\ell-20})\|_{L^\infty} \lesssim \left(t^{-4/3}2^{-3\ell-k/2} + (t^{2k+j})^{-1}t^{-4/3}2^{-\ell}\right)\prod_{i=1}^3\|f_i\|_X
$$

$$
\|\chi_kQ_{\sim j}(u_{1, \ell}u_{2, [\ell-20, \ell+20]}u_{3, [\ell-20, \ell+20]})\|_{L^\infty} \lesssim \left(t^{-5/2}2^{-9/2\ell} + (t^{2k+j})^{-1}t^{-3/2}2^{-3/2\ell}\right)\prod_{i=1}^3\|f_i\|_X
$$
Summing in $\ell$, we find that
\[
\|\chi_k Q_j (u_1 u_2 u_3)\|_{L^\infty} \lesssim t^{-11/6} 2^{-3/2j} 2^{-k/6} \prod_{i=1}^3 \|f_i\|_X
\] (38)

Summing over $k < j - 30$ and combining the result with (35), we obtain the bound
\[
\|Q_j (u_1 u_2 u_3)\|_{L^\infty} \lesssim t^{-3/2} 2^{-3/2j} 2^{-k/6} \prod_{i=1}^3 \|f_i\|_X
\] (39)

By a similar argument, we find that for $k < j - 30$,
\[
\|\chi_k Q_j (u_1 \partial_x u_2 u_3)\|_{L^\infty} \lesssim t^{-11/6} 2^{-j/2} 2^{-k/6} \prod_{i=1}^3 \|f_i\|_X
\] (40)

so
\[
\|Q_j (u_1 \partial_x u_2 u_3)\|_{L^\infty} \lesssim t^{-3/2} 2^{-j/2} \prod_{i=1}^3 \|f_i\|_X
\] (41)

4. Bounds for the self-similar term

As discussed in the introduction, the self-similar solutions to (1) that are of interest to us are given by
\[ S(x, t; p) = t^{-1/3} \sigma(t^{-1/3} x; p), \]
where $\sigma$ solves a phase-rotation invariant version of the Painlevé II equation:
\[
\begin{align*}
\sigma''(x; p) &= \frac{1}{3} x \sigma(x; p) + \frac{1}{3} |\sigma(x; p)|^2 \sigma(x; p) \\
\sigma(0; p) &= p 
\end{align*}
\] (42)

From the phase invariance of the equation, we see that if $\theta = \arg p$, then $\sigma(x; p) = e^{i\theta} \sigma(x; e^{-i\theta} p)$. Let $\Phi = e^{-i\theta} \sigma$. Then, by [9], we have that
\[
\|\Phi\|_{L^\infty} + \|x \Phi\|_{L^2} \lesssim |p|
\]

It follows that $h(x, t; p) := e^{i\theta} S(x, t; p) = t^{-1/3} \Phi(t^{-1/3} x; p)$ satisfies $\|h\|_X \lesssim p$, so $S$ satisfies Lemma 5 (cf. [12][29]).

From (12) and the scaling of $S$, a simple calculation shows that
\[
LS = \pm t |S|^2 S \tag{43}
\]

In particular, this gives us the $L^2$ bound $\|LS\|_{L^2} \lesssim t \|S\|_{L^\infty} \lesssim |p|^{3/2} t^{1/6}.$

Since our argument involves modulating around $S$, we need $\sigma$ to have some smoothness in the $p$ parameter. Since the nonlinearity of (12) is not analytic, $\sigma$ is not complex differentiable in $p$. Instead, we will show that $\sigma$ is differentiable as a function of $\Re p$ and $\Im p$. In particular, for $p : \mathbb{R} \to \mathbb{C}$ differentiable, we write
\[
\partial_x \sigma(x; p(s)) = \partial_{\Re p} \sigma \Re p'(s) + [\partial_{\Im p} \sigma] \Im p'(s) =: D_p \sigma p'(s)
\] (44)

Changing to polar coordinates $p = re^{i\theta}$, we find that
\[
\begin{align*}
\partial_r \sigma(x; re^{i\theta}) &= e^{i\theta} \partial_r \sigma(x; r) \\
\partial_{\theta} \sigma(x; re^{i\theta}) &= ie^{i\theta} \sigma(x; r) = i\sigma(x, re^{i\theta})
\end{align*}
\] (45)
In particular, \( \partial_0 S = -t^{-1/3} \partial_0 \sigma(xt^{-1/3}, re^{i\theta}) = iS \) satisfies the same bounds as the self-similar solution \( S \). Since \( S = e^{-i0 \partial_0 \sigma} \), and \( \|h\|_\infty \lesssim |p| \), we have

\[
\|\partial_0 S\|_{L^2} = \|\partial_0 S\|_{L^2} \lesssim |p| t^{-1/3 + 1/3q}
\]

and

\[
|Q_{-j} \partial_0 S| = |Q_{-j} S| \lesssim t^{-1/2} 2^{-j/2} \chi_{-j} + t^{-5/6} \left( 2^j + 2^{-j/3} |\xi_0(x)|^{4/3} \right)^{-3/2}
\]

where \( \xi_0 = \xi_0(x) \) is as in Lemma 5. To understand \( \partial_0 S \), we recall that by 9 Theorem 1] the profile \( \Phi \) satisfies

\[
\tilde{\Phi}(\xi; r) = \chi(\xi)e^{ia \ln |\xi|} \left( A + B e^{2a \ln |\xi|} e^{-i \frac{3}{\xi^3}} \right) + z(\xi; r)
\]

where \( \chi \) is a cut-off function supported on \( |\xi| \geq 1 \), \( A, B \), and \( a \) are real-valued and have a Lipschitz dependence on \( r \) (at least for \( r \) sufficiently small), and \( z \) is some function which has a Lipschitz dependence on \( r \) with respect to the norm

\[
\|z\|_2 := \|z(\xi)(\xi)\|_{L^\infty} + \|z'(\xi)(\xi)\|_{L^\infty}
\]

for \( k \in \left( \frac{1}{2}, \frac{3}{2} \right) \). From this, we can see that the worst term in \( \partial_0 \tilde{\Phi} \) occurs when the derivative hits the logarithmic oscillating phase, so

\[
|\partial_0 \tilde{\Phi}(\xi; r)| \lesssim \ln(2 + |\xi|)
\]

Similarly, differentiating in \( \xi \), we find that

\[
|\partial_\xi \partial_0 \tilde{\Phi}(\xi)| \lesssim \frac{\ln 2 + |\xi|}{\xi^3} + O \left( \frac{\xi^{-3/2}}{\xi^3} \right)
\]

Recalling that \( h(x, t; p) = t^{-1/3} \tilde{\Phi}(t^{-1/3} x; p) \), we see that

\[
\|\partial_0 h\|_{L^\infty} = \|Q_{-j} \partial_0 h\|_{L^\infty} + t^{-1/6}\|x(Q_{-j} \partial_0 h)\|_{L^2} = \|Q_{-j} \tilde{\Phi}\|_{L^\infty} + \|\partial_0 Q_{-j} \tilde{\Phi}\|_{L^2} \lesssim \ln(2 + t^{-1/3} 2^j)
\]

In particular, applying Lemma 5, we find that

\[
|Q_{-j} \partial_0 S| \lesssim t^{-1/2} 2^{-j/2} \ln(2 + t^{-1/3} 2^j) \chi_{-j} + t^{-5/6} \left( 2^j + 2^{-j/3} |\xi_0(x)|^{4/3} \right)^{-3/2}
\]

where \( \xi_0 = \xi_0(x) \) is as in Lemma 5. Combining this estimate with the one for \( \partial_0 S \), we find that

\[
\|Q_{-j} D_p S\|_{L^\infty} = \frac{1}{|p|} \|Q_{-j} \partial_0 S\|_{L^\infty} + \|Q_{-j} \partial_0 S\|_{L^\infty} \leq t^{-1/2} 2^{-j/2} \ln(2 + t^{-1/3} 2^j)
\]

and, for \( 4 < q < \infty \)

\[
\|Q_{-j} D_p S\|_{L^q} \lesssim t^{-\frac{1}{2} + \frac{1}{2q}} 2^{-\frac{j}{2} + \frac{3}{2}} \ln(2 + t^{-1/3} 2^j)
\]

which can be summed to give

\[
\|D_p S\|_{L^q} \lesssim t^{-\frac{1}{2} + \frac{1}{2q}}
\]

To perform the weighted \( L^2 \) estimates, we also need a bound on \( LD_p S \) in \( L^2 \). As above, we have

\[
\|LD_p S\|_{L^2} = t^{1/6} \|LD_p \sigma\|_{L^2} \lesssim t^{1/6} \frac{1}{|p|} \|(3\partial_x^2 - x) \partial_0 \sigma\|_{L^2} + t^{1/6} \|3\partial_x^2 - x\|_{L^2} \|\partial_0 \sigma\|_{L^2}
\]

Since \( \partial_0 \sigma = i\sigma \), we have from 12 that

\[
\frac{1}{p} \|L \partial_0 \sigma\|_{L^2} = \frac{1}{|p|} \|(3\partial_x - x) \sigma\|_{L^2} = \frac{1}{|p|} \|\sigma\|_{L^2}^2 \lesssim |p|^2
\]
and the problem reduces to finding bounds on \((3\partial^2_x - x)\partial_x \sigma (x; r)\), where \(r\) is real. Differentiating shows that \(\partial_x \sigma (x; r)\) satisfies \((3\partial^2_x - x)\partial_x \sigma = \pm \sigma^2 \partial_x \sigma\). Using the \(L^p\) estimates for \(\sigma\) and \(\partial_x \sigma\) (which follow immediately from the estimates for \(S\) and \(\partial_x S\) given above), we obtain the bound
\[
\| (\partial^2_x - x) \partial_x \sigma \|_{L^2} \lesssim \| \sigma \|_{L^6}^2 \| \partial_x \sigma \|_{L^6} \lesssim \varepsilon^2
\]
and so
\[
\| LD_{p}S\|_{L^2} \lesssim t^{1/6} \varepsilon^2
\]

5. Reduction of the main theorem to profile estimates. Let
\[
w(x, t) = u(x, t) - S(x, t; \tilde{u}(0, t)),
\]
and define \(g = e^{i\theta x} w\). The remainder of the paper will be devoted to proving the nonlinear bounds on \(f\) and \(g\) given in the following theorem:

**Theorem 9.** Suppose \(u_0 \in H^2\) and \(\| u_0 \|_{L^\infty} + \| xu_0 \|_{L^2} \leq \varepsilon\). Then, for \(\varepsilon\) sufficiently small, the solution \(u\) to \(\text{Equation (3)}\) is global, and the following bounds hold for all \(t \in [1, \infty)\)
\[
\| xg(t) \|_{L^2} \lesssim \varepsilon t^{1/6 - \beta}
\]
\[
\| f(t) \|_{L^\infty} \lesssim \varepsilon
\]
where \(\beta = \beta(\varepsilon) = \frac{1}{6} - C \varepsilon^2\) for some constant \(C\). Moreover, there exists a bounded function \(f_\infty(\xi)\) such that
\[
f_\infty(\xi) = \exp \left( \pm \frac{i}{6} \int_{1}^{t} \frac{\| f(\xi, s) \|^2}{s} ds \right) f_\infty(\xi) + O(\varepsilon^3 (t^{-1/3}|\xi|)^{-1/4})
\]
and
\[
|\partial_t \tilde{u}(0, t)| \lesssim \varepsilon^3 t^{-1 - \beta}
\]
Assuming Theorem 9, we can prove Theorem 1.

**Proof of Theorem 1.** Since \(Lu = Lw + LS\), \(\| xg \|_{L^2} \lesssim \| xg \|_{L^2} + \| LS \|_{L^2}\). From (13) and (50), we see that \(\| xg \|_{L^2} \lesssim \varepsilon t^{1/6}\), which combined with (51) shows that \(f \in X\), which is enough to show (5) using the rapid decay given by Lemma 6 for \(x < -t^{-1/3}\). Moreover, by using (12) and Lemma 5 we can establish (6).

It only remains to verify that the asymptotics for \(|x| \lesssim t^{1/3+4\beta}\) are given by (7). By Corollary 8 and the hypothesis (50), \(\| w \|_{L^\infty} \lesssim t^{-1/2} \| xg \| \lesssim t^{-1/3-\beta}\), so in the region \(|x| \lesssim t^{1/3+4\beta}\)
\[
u(x, t) = S(x, t; \tilde{u}(0, t)) + O(\varepsilon t^{-1/3-\beta})
\]
Now, since \(t^{-1-\beta}\) is integrable over \([1, \infty)\), (53) implies that \(\alpha = \lim_{t \to \infty} \tilde{u}(0, t)\) has size \(O(\varepsilon)\) and satisfies \(|\tilde{u}(0, t) - \alpha| = O(\varepsilon^3 t^{-\beta})\). Using (10) to bound the terms \(\| Q_{j}D_{p}S \|_{L^\infty}\), we find that
\[
\| S(x, t; \tilde{u}(0, t)) - S(x, t; \alpha) \|_{L^\infty} \lesssim \sum_{2^j \geq t^{-1/3}} \| Q_{j}D_{p}S \|_{L^\infty} |\tilde{u}(0, t) - \alpha| \lesssim \varepsilon^3 t^{-1/3 - \beta}
\]
which can be combined with (54) to obtain (7). □
5.2. Plan of the proof of Theorem [9]. To prove Theorem [9] we use a bootstrap argument. From [3], we have that [3] is locally wellposed in the space $H^2 \cap x^{-1}L^2$, so for some $\tau > 1$ there exists a solution $u$ on $[1, \tau]$ satisfying $\|\langle x \rangle u(t)\|_2 + \|\partial_x^2 u(t)\|_2 \leq 10\epsilon$. It follows that

$$
\|\hat{f}(t)\|_2 + \|x f\|_2 \leq \|\hat{u}\|_2 + \|Lu\|_2 \\
\leq 10\epsilon
$$

Thus, there exists a $T > 1$ such that the following hypotheses hold:

$$
\sup_{1 \leq t \leq T} \left( \|\hat{f}(t)\|_2 + t^{-1/6} \|xf(t)\|_2 \right) \leq CM\epsilon, \\
\sup_{1 \leq t \leq T} |\partial_t \hat{u}(0, t)| \leq CM^3 \epsilon t^{-1-\beta}
$$

(BH)

where $M \gg 1$ is a fixed constant, the exact value of which is unimportant, and $C$ is an arbitrary but fixed constant. Let us fix an $\epsilon_0 \ll M^{-3/2}$, and suppose that $\epsilon \leq \epsilon_0$. Using (BH), we prove in Section [6] that $\|x g\|_2 \leq C\epsilon t^{1/6-\beta}$. Then, using this bound on $x g$ and the bootstrap, we verify that $|\partial_t \hat{u}(0, t)| \leq CM^3 \epsilon t^{-1-\beta}$ in Section [7] and that $\|\hat{f}(t)\|_2 \leq C\epsilon$ in Section [8]. These results imply that in fact we have the improved bounds

$$
\sup_{1 \leq t \leq T} \left( \|\hat{f}(t)\|_2 + t^{-1/6} \|xf(t)\|_2 \right) \leq C\epsilon, \\
\sup_{1 \leq t \leq T} |\partial_t \hat{u}(0, t)| \leq CM^2 \epsilon t^{-1-\beta}
$$

(BH+)

Moreover, a simple energy estimate shows that

$$
\frac{d}{dt} \|\partial_x^2 u\|_2^2 \lesssim \|u\|_2 \|\partial_x^2 u\|_2 \|\partial_x^2 u\|_2 \lesssim M^2 \epsilon^2 t^{-1}\|\partial_x^2 u\|_2^2
$$

so, by Grönwall’s inequality, $\|\partial_x^2 u\|_2$ grows at a polynomial rate in time. By using the $L^\infty$ bound on $\hat{f}$ to control low frequencies, we see that $\|u\|_{H^2}$ does not blow up at time $T$, so we can use local wellposedness to extend to a solution on a longer time interval $[1, T']$, and assume that the bootstrap bounds (BH) hold up to time $T'$. By a standard continuity argument, this shows that the estimates (BH) hold for all time. Moreover, in the course of proving the $FL^\infty$ bound in Section [8] we obtain (52), which proves Theorem [9] (and hence Theorem [1] as well).

6. The weighted energy estimate

In this section, we will show that $\|x g\|_2 \lesssim \epsilon t^{1/6-\beta}$, where $\beta$ is as in Theorem [9]. To establish this bound, we show that the bootstrap hypotheses imply

$$
\|x g(t)\|_2^2 \leq \|x u_\ast\|_2^2 + \int_1^t \left[M^2 \epsilon^2 s^{-1} \|x g(s)\|_2^2 + M^2 \epsilon^3 s^{-5/6-\beta}\|x g(s)\|_2\right] ds
$$

(55)

for all $t \leq T$. By adding a factor of $\epsilon^2 t^{1/3-2\beta}$, (55) implies that

$$
\|x g(t)\|_2^2 + \epsilon^2 t^{1/3-2\beta} \lesssim \|x u_\ast\|_2^2 + \epsilon^2 + \int_1^t \left[M^2 \epsilon^2 s^{-1} \|x g(s)\|_2^2 + \epsilon^2 s^{1/3-2\beta}\right] ds
$$

(56)

(recall that by the definition of $\beta$, $\frac{1}{6} - \beta = O(M^{-2})$). Applying Grönwall’s inequality to (55) for $t \in [1, T]$, we obtain the desired bound for $\|x g(t)\|_2$. To prove (55), we write the inequality in
differential form using the expansion
\[ x\partial_t g = x e^{i\partial_x^3} (\partial_t + \partial_x^3)(u - S) \]
\[ = \pm x e^{i\partial_x^3} \left( |u|^2 \partial_x w - |S|^2 \partial_x S - D_p S \partial_t \hat{u}(0,t) \right) \]
\[ = \pm x e^{i\partial_x^3} \left( |u|^2 \partial_x w + (\bar{w}u + uw) \partial_x S \right) - x e^{i\partial_x^3} D_p S \partial_t \hat{u}(0,t) \]
\[ = \pm x e^{i\partial_x^3} \left( |e^{-i\partial_x^3} f|^2 (e^{-i\partial_x^3} \partial_x g) + \left( e^{-i\partial_x^3} f e^{-i\partial_x^3} g + e^{-i\partial_x^3} f e^{-i\partial_x^3} g \right) (e^{-i\partial_x^3} \partial_x h) \right) \]
\[ - e^{i\partial_x^3} \partial_x e^{-i\partial_x^3} D_p S \partial_t \hat{u}(0,t) \]
\[ = \mp \mathcal{F}^{-1} \frac{1}{2\pi} \partial_x \int e^{it\phi}(\xi - \eta - \sigma) \left( \hat{f}(\eta) \overline{\hat{\varphi}(\xi - \eta - \sigma)} + \left( \hat{\varphi}(\xi) \overline{\hat{\varphi}(\sigma)} + \hat{\varphi}(\eta) \overline{\hat{f}(\sigma)} \right) \hat{h}(\xi - \eta - \sigma) \right) d\eta d\sigma \]
\[ - e^{i\partial_x^3} LD_p S \partial_t \hat{u}(0,t) \]
\[ = \mp i t (T_{\partial_x \varphi e^{i\sigma \theta}} (f, \partial_x g, J) + T_{\partial_x \varphi e^{i\sigma \theta}} (f, \partial_x h, \overline{J}) + T_{\partial_x \varphi e^{i\sigma \theta}} (g, \partial_x h, \overline{J})) \]
\[ + e^{i\partial_x^3} \left( |u|^2 w + (\bar{w}u + uw) S \right) \]
\[ + e^{i\partial_x^3} \left( |u|^2 \partial_x e^{-i\partial_x^3} (xy) + (\bar{w}u + uw) \partial_x e^{-i\partial_x^3} (xh) \right) \]
\[ - e^{i\partial_x^3} LD_p S \partial_t \hat{u}(0,t) \]

where \( h = e^{i\partial_x^3} S \). Thus,
\[ \frac{1}{2} \partial_t \| xg \|_{L^2}^2 = \Re (xg, \partial_t xg) \]
\[ = \mp i 3 \langle xg, T_{\partial_x \varphi e^{i\sigma \theta}} (f, \partial_x g, J) + T_{\partial_x \varphi e^{i\sigma \theta}} (f, \partial_x h, \overline{J}) + T_{\partial_x \varphi e^{i\sigma \theta}} (g, \partial_x h, \overline{J}) \rangle \]  
\[ + \Re (e^{-i\partial_x^3} xg, |u|^2 w + (\bar{w}u + uw) S) \]  
\[ + \Re (e^{-i\partial_x^3} xg, |u|^2 \partial_x e^{-i\partial_x^3} (xy)) \]  
\[ + \Re (e^{-i\partial_x^3} xg, (\bar{w}u + uw) \partial_x e^{-i\partial_x^3} (xh))) \]  
\[ - \Re (e^{-i\partial_x^3} xg, LD_p S \partial_t \hat{u}(0,t)) \]  

We will first show that the terms \([58b]-[58d]\) decay pointwise in time:
\[ \| [58b] \| + \| [58c] \| + \| [58d] \| \lesssim M^2 t^{-1} \| xg \|_{L^2} \]
which is compatible with \([55]\) after integrating in time. For \([58d]\), we note that by Corollary \([7]\) and the bootstrap hypotheses,
\[ \| \chi_k u \|_{L^\infty} \lesssim t^{-1/2} \| f \|_{X} \lesssim M^k t^{-1/2} \]
Similarly, using Corollary \([8]\)
\[ \| \chi_k w \|_{L^2} \lesssim \| \chi_{-k} \|_{L^2} \| \chi_k w \|_{L^\infty} \lesssim 2^k \left( \| Q_{-k} (xy) \|_{L^2} + t^{-1/2} \| xg \|_{L^2} \right) \]
By using these bounds together with almost orthogonality in space, we find that

\[ 0 \leq \|xg\|_{L^2} 2^{k_t} \left( \sum_{2^k \geq 1^{-1/3}} \| \chi_k (|u|^2 w) \|_{L^2} \right) + \{ \text{similar terms} \} \]

\[ \leq \|xg\|_{L^2} 2^{k_t} \left( \sum_{2^k \geq 1^{-1/3}} (\| \chi_{|w|} \|_{L^2}^2 - \| \chi_{|w|} \|_{L^2}) \right)^{1/2} + \{ \text{similar terms} \} \]

\[ \leq M^2 \epsilon^2 t^{-1} \|xg\|_{L^2}^2 \sum_{2^k \geq 1^{-1/3}} \| Q_{|w|} xg \|_{L^2}^2 + t^{-2/3} 2^{-2k} \|xg\|_{L^2} \] + \{ \text{similar terms} \}

\[ \leq M^2 \epsilon^2 t^{-1} \|xg\|_{L^2}^2 \]

For (58c), we integrate by parts and use Corollary 6 to find that

\[ \|58c\| = \left| \int \text{Re}(e^{-i\partial_x xg} \partial_x e^{-i\partial_x xg}) |u|^2 dx \right| \]

\[ = \frac{1}{2} \left| \int \partial_x |u|^2 e^{i\partial_x xg} dx \right| \]

\[ \leq M^2 \epsilon^2 t^{-1} \|xg\|_{L^2}^2 \]

For (58d), we use equation (43) to write $LS = \mp t |S|^2 \partial_x S$. Then, since $h \in X$, we can use the bilinear bound given in Corollary 6 and an arguments similar to the one for (58b) give us the bound

\[ \|58d\| \leq \|xg\|_{L^2} \|uw|S|^2 \partial_x S \|_{L^2} \]

\[ \leq M^2 \epsilon^2 \|xg\|_{L^2} \|uwS\|_{L^2} \]

\[ \leq M^2 \epsilon^2 \|xg\|_{L^2} \left( \sum_{2^k \geq 1^{-1/3}} \| \chi_k (uwS) \|_{L^2} \right) \]

\[ \leq M^2 \epsilon^2 \|xg\|_{L^2} \left( \sum_{2^k \geq 1^{-1/3}} (\| \chi_k u \|_{L^2} \| \chi_k S \|_{L^2} \| \chi_k w \|_{L^2}) \right)^{1/2} \]

\[ \leq M^2 \epsilon^4 t^{-1} \|xg\|_{L^2} \left( \sum_{2^k \geq 1^{-1/3}} \| Q_{|w|} xg \|_{L^2}^2 + t^{-2/3} 2^{-2k} \|xg\|_{L^2} \right)^{1/2} \]

\[ \leq M^4 \epsilon^4 t^{-1} \|xg\|_{L^2}^2 \]

which is better than required, since the smallness assumption on $\epsilon$ implies that $M^2 \epsilon^2 \ll 1$.

To control (58e), we observe that by (49) and (49)

\[ \|LD_p S\hat{u}(0,t)\|_{L^2} \leq M^5 \epsilon^5 t^{-5/6 - \beta} \]

so

\[ \|58e\| \leq M^5 \epsilon^5 t^{-5/6 - \beta} \|xg\|_{L^2} \]

which is better than the bound required by (55) since $\epsilon \ll M^{-3/2}$.
It only remains to control \(6.1\). We will re-write this term to exploit the space-time resonance structure of the phase \(\phi\). Recall that the space-time resonances are
\[
\mathcal{T} = \{\xi = \eta\} \cup \{\xi = \sigma\}
\]
\[
\mathcal{S} = \{\eta = \sigma = \xi/3\}
\]
\[
\mathcal{R} = \{(0, 0, 0)\}
\]
Let \(\chi^S, \chi^T, \chi^R\) be a smooth partition of unity such that \(\chi^S\) and \(\chi^T\) are 0-homogeneous outside a ball of radius 2 and vanish within a ball of radius 1, and such that \(\chi^S\) is supported on the set
\[
\mathcal{T} = \{(\xi, \eta, \sigma): \xi \in [1 - c, 1 + c] \text{ or } \xi \in [1 - c, 1 + c]\}
\]
where \(c \ll 1\) is a small constant. (Note that \(\mathcal{T} \cap \mathcal{S}\) is empty when \(c\) is small), and \(\chi^T\) is supported away from \(\mathcal{T}\). Define \(\chi^* \equiv \chi^* (t^{1/3})\) for \(\bullet = \mathcal{S}, \mathcal{T}, \mathcal{R}\).

**Remark 9.** The assumption (60) could be relaxed: it is most important that \(|\xi - \eta - \sigma| \lesssim \max\{|\eta|, |\sigma|\}\), since this allows us to move derivatives from the middle term to more favorable locations (and in particular prevents a loss of derivatives in the term (60)). However, (60) is technically helpful, since it lets us conclude that on the support of \(\chi^S\), if \(|\xi| \sim |\eta| \sim |\sigma| \sim 2^j\), then \(|\xi - \eta - \sigma| \sim 2^j\) (since if \(|\xi - \eta - \sigma| \leq c, then |\xi - \eta| \ll 2^j\), and similarly if \(|\xi - \eta - \sigma| \leq c\). This restriction eliminates the term \(T_{\mathcal{S}} (p_{\sim j}, q_{\ll j}, r_{\sim j})\) from (67), reducing the number of terms we have to estimate in Section 6.2.

If we write
\[
T_{\partial_t \phi^e i\xi \phi} = T_{\partial_t \phi^e i\xi \phi} \chi^S_t + T_{\partial_t \phi^e i\xi \phi} \chi^R_t
\]
then we can naturally write (68a) as
\[
(68a) = \pm t3 \langle xg, T_{\partial_t \phi^e i\xi \phi} \chi^S_t (f, \partial_x g, \overline{f}) + T_{\partial_t \phi^e i\xi \phi} \chi^R_t (g, \partial_x h, \overline{f})\rangle
\]
\[
\pm t3 \langle xg, T_{\partial_t \phi^e i\xi \phi} \chi^S_t (f, \partial_x g, \overline{f}) + T_{\partial_t \phi^e i\xi \phi} \chi^R_t (f, \partial_x h, \overline{f})\rangle
\]
\[
\pm t3 \langle xg, T_{\partial_t \phi^e i\xi \phi} \chi^S_t (f, \partial_x g, \overline{f}) + T_{\partial_t \phi^e i\xi \phi} \chi^R_t (g, \partial_x h, \overline{f})\rangle
\]
Thus, it suffices to show that the space-time resonant and space non-resonant terms satisfy the estimate
\[
|6(2a)|, |6(2b)| \lesssim M^2 \epsilon^2 \|xg\|^2_{L^2} t^{-1}
\]
pointwise in time, and that
\[
\left| \int_1^t ds \lesssim \|u_x\|^2_{L^2} + \int_1^t \left[ M^2 \epsilon^2 s^{-1} \|xg(s)\|^2_{L^2} + M^2 \epsilon^3 s^{-5/6 - \beta} \|xg\|_{L^2} \right] ds + M^2 \epsilon^2 \|xg\|^2_{L^2}
\]

6.1. **The space-time resonant multiplier.** Since the space-time resonant set is a single point, we can control its contribution using the \(L^\infty\) bounds on \(\hat{g}, \hat{f}\) and \(\hat{S}\). Define
\[
m_t^R = i(\xi - \eta - \sigma) \chi^R_t \partial_t \phi^e i\xi \phi
\]
Then, \( m^R_t \) is of size \( O(t^{-1}) \) and is supported in a region \(|\xi| + |\eta| + |\sigma| \lesssim t^{-1/3} \), so

\[
\| \mathcal{N}^S(f, g, h) \|_{L^2} \lesssim M^2 t^{-2} \||xg||^2_{L^2}
\]
as required.

### 6.2. The space non-resonant multiplier.

The terms supported away from the space resonant set in frequency space we handle using integration by parts in \( \eta \) and \( \sigma \). To ease notation, let us write

\[
\mathcal{N}^S(f, g, h) = tT_{\partial_\xi \phi \mathcal{M}^S_{\nabla} \partial_\xi g, \mathcal{M}^S_{\nabla} \partial_\xi h, \mathcal{M}^S_{\nabla} \partial_\xi h} + tT_{\partial_\xi \phi \mathcal{M}^S_{\nabla} \partial_\xi g, \partial_\xi h, \mathcal{M}^S_{\nabla} \partial_\xi h} + tT_{\partial_\xi \phi \mathcal{M}^S_{\nabla} \partial_\xi g, \partial_\xi g, \mathcal{M}^S_{\nabla} \partial_\xi h}
\]

Then, the desired bound for (62) follows from showing that

\[
\| \mathcal{N}^S(f, g, h) \|_{L^2} \lesssim M^2 t^{-1} \||xg||^2_{L^2}
\]

(64)

By writing \( u = S + w \), we see that \( \mathcal{N}^S \) contains a term \( tT_{\partial_\xi \phi \mathcal{M}^S_{\nabla} \partial_\xi g, \partial_\xi g, \mathcal{M}^S_{\nabla} \partial_\xi h} \). Integrating by parts in frequency gives

\[
tT_{\partial_\xi \phi \mathcal{M}^S_{\nabla} \partial_\xi g, \partial_\xi g, \mathcal{M}^S_{\nabla} \partial_\xi h} = \int t \partial_\xi \phi \mathcal{M}^S_{\nabla} e^{it\phi} (\xi - \eta - \sigma)\hat{g}(\xi - \eta - \sigma) \hat{h}(\eta) \hat{h}(-\sigma) d\eta d\sigma
\]

\[
= -\int e^{it\phi} \nabla_{\eta, \sigma} \cdot \left( e^{it\phi} \nabla_{\eta, \sigma} \phi \mathcal{M}^S_{\nabla} \partial_\xi g, \partial_\xi g, \mathcal{M}^S_{\nabla} \partial_\xi h \right) d\eta d\sigma
\]

\[
= -e^{-it\xi} T_{\partial_\xi \phi \mathcal{M}^S_{\nabla} \partial_\xi g, \partial_\xi g, \mathcal{M}^S_{\nabla} \partial_\xi h} (S, w, \overline{S})
\]

\[
= -e^{-it\xi} T_{\partial_\xi \phi \mathcal{M}^S_{\nabla} \partial_\xi g, \partial_\xi g, \mathcal{M}^S_{\nabla} \partial_\xi h} (S, \partial_\xi w, L\overline{S})
\]

\[
+ e^{-it\xi} T_{\partial_\xi \phi \mathcal{M}^S_{\nabla} \partial_\xi g, \partial_\xi g, \mathcal{M}^S_{\nabla} \partial_\xi h} (S, L\overline{S}, \overline{S})
\]

\[
+ \{ \text{similar terms} \}
\]

(65)

where \( m^S_{\xi} \) is the vector-valued symbol \( m^S_{\xi} = \frac{\partial_\xi \phi}{\nabla_{\eta, \sigma} \phi} \nabla_{\eta, \sigma} \phi \mathcal{M}^S_{\nabla} \partial_\xi g, \partial_\xi g, \mathcal{M}^S_{\nabla} \partial_\xi h \), and \( m^S_{\xi}, m^S_{\eta}, m^S_{\sigma} \) are its components. Similar expressions hold for the other terms in \( \mathcal{N}^S \). Note that each symbol \( m \) occurring after the last equality in (65) satisfies Coifman-Meyer type bounds

\[
|(|\xi| + |\eta| + |\sigma|)^{1-\alpha} \partial^\alpha \hat{m}| \lesssim 1
\]
and is supported on \( \{ \xi \sim \eta \gtrsim t^{-1/3} \} \cup \{ \xi \sim \sigma \gtrsim t^{-1/3} \} \). Thus, writing \( m_j^S = \psi \left( \frac{\xi^2 + \eta^2 + \sigma^2}{2j} \right) m \) where \( m \) is a generic symbol occurring in (65), we can write

\[
N^S(f, g, h) = \sum_{2^j \gtrsim t^{-1/3}} T_{m_j^S}(S, w, \mathbf{3})
\]

(66a)

\[+ T_{m_j^S}(S, \partial_x Lw, \mathbf{3}) \]

(66b)

\[+ T_{m_j^S}(S, \partial_x w, \mathbf{LS}) \]

(66c)

\[+ T_{m_j^S}(S, S, \mathbf{w}) \]

(66d)

\[+ T_{m_j^S}(u, \partial_x LS, \mathbf{w}) \]

(66e)

\[+ T_{m_j^S}(u, \partial_x S, \mathbf{Lw}) \]

(66f)

\[\{ \text{similar or easier terms} \} \]

where the symbol represented by \( m_j^S \) is allowed to change from line to line. Here, \( \{ \text{similar terms} \} \) denotes pseudoproducts like (66a)–(66f) but with the first and last argument of the pseudoproduct permuted, which can be handled analogously. The \( \{ \text{easier terms} \} \) denote two types of pseudoproduct terms:

(a) terms containing an \( Lu \) factor, which can be controlled by writing \( Lu = LS + Lw \) and modifying the following arguments using the fact that the other \( w \) factor has better dispersive decay than \( S \) and \( u \), and

(b) terms like (66a)–(66c) where one or both of the \( S \) factors is replaced by a \( w \), which are similarly handled using the improved dispersive decay of \( w \).

We now turn to the task of deriving estimates for (66a)–(66f). Since \( m_j^S \) is supported on the annular region \( |\xi| + |\eta| + |\sigma| \sim 2^j \) with the restriction that \( |\xi| \) is comparable to either \( |\eta| \) or \( |\sigma| \), we can decompose any pseudoproduct \( T_{m_j^S} \) as

\[T_{m_j^S}(p, q, r) = T_{m_j^S}(p \ll_j, q \ll_j, r \sim_j) + T_{m_j^S}(p \sim_j, q \ll_j, r \ll_j)
\]

+ \( Q \sim_j T_{m_j^S}(p \sim_j, q \sim_j, r \sim_j) \) (67)

(The term \( T_{m_j^S}(p \sim_j, q \ll_j, r \sim_j) \) does not appear in the expansion because of the support assumption (60): see Remark 9). In particular, the restriction \( \xi \sim \eta \) or \( \xi \sim \sigma \) means that \( |\xi - \eta - \sigma| \lesssim \max\{|\eta|, |\sigma|\} \), which helps us control the derivative.

We now consider each term (66a)–(66f) in turn and use the division (67) and the decay estimates to obtain the bound (64).

6.2.1. The bound for (66a): From (67), we can write

\[T_{m_j^S}(S, w, \mathbf{3}) = T_{m_j^S}(S \ll_j, w \ll_j, \mathbf{3} \sim_j)
\]

(68a)

\[+ Q \sim_j T_{m_j^S}(S \sim_j, w \sim_j, \mathbf{3} \sim_j)
\]

(68b)

\[\{ \text{similar terms} \} \]
For (68b), we use Theorem 3 together with Hardy’s inequality and the decay bounds on \( S \) to obtain

\[
\left\| \sum_{2^j \geq t^{-1/3}} Q_{\sim j} T_m^\delta (S_{\sim j}, w_{\sim j}, \overline{S}_{\sim j}) \right\|_{L^2} \lesssim \left( \sum_{2^j \geq t^{-1/3}} \| T_m^\delta (S_{\sim j}, w_{\sim j}, \overline{S}_{\sim j}) \|_{L^2} ^2 \right)^{1/2}
\]

\[
\lesssim \left( \sum_{2^j \geq t^{-1/3}} \| S_{\sim j} \|_{L^\infty} \| w_{\sim j} \|_{L^2} ^2 \right)^{1/2}
\]

\[
\lesssim M^2 \epsilon^2 t^{-1} \left( \sum_{2^j \geq t^{-1/3}} 2^{-2j} \| w_{\sim j} \|_{L^2} ^2 \right)^{1/2}
\]

\[
\lesssim M^2 \epsilon^2 t^{-1} \| xg \|_{L^2}
\]

as required.

For (68a), we introduce a further dyadic decomposition in space to write

\[
T_m^\delta (S_{\ll j}, w_{\ll j}, \overline{S}_{\ll j}) = \sum_{2^k \geq t^{-1/3}} T_m^\delta (S_{\ll j}, w_{\ll j}, \chi_k \overline{S}_{\ll j})
\]

\[
= T_m^\delta (S_{\ll j}, w_{\ll j}, (1 - \chi_{[j-20,j+20]}) \overline{S}_{\ll j}) + T_m^\delta (S_{\ll j}, w_{\ll j}, \chi_{[j-20,j+20]} \overline{S}_{\ll j})
\]

(69a)

(69b)

Now, since \( S = e^{-t\partial^2} h \) and \( \| h \|_X \lesssim M \epsilon \), Lemma 5 implies that

\[
\| (1 - \chi_{[j-20,j+20]}) S_{\sim j} \|_{L^\infty} \lesssim M \epsilon t^{-5/6} 2^{-3/2j}
\]

so

\[
\| (69a) \|_{L^2} \lesssim \| S_{\ll j} \|_{L^\infty} \| (1 - \chi_{[j-20,j+20]} S_{\sim j} \|_{L^\infty} \| w_{\ll j} \|_{L^2} \lesssim M^2 \epsilon^2 t^{-7/6} 2^{-j/2} \| w \|_{H^{-1}} \lesssim M^2 \epsilon^2 t^{-7/6} 2^{-j/2} \| xg \|_{L^2}
\]

which is sufficient. Turning to the second term (69b), we write

\[
(69b) = \chi_{[j-30,j+30]} T_m^\delta (\chi_{[j-20,j+20]} S_{\ll j}, \chi_{[j-30,j+30]} w_{\ll j}, \chi_{[j-20,j+20]} \overline{S}_{\ll j})
\]

\[
+ \sum_{|k-j| \leq 20} (1 - \chi_{[j-30,j+30]} ) T_m^\delta (S_{\ll j}, w_{\ll j}, \chi_{[j-20,j+20]} \overline{S}_{\ll j})
\]

\[
+ \chi_{[j-30,j+30]} T_m^\delta (1 - \chi_{[j-30,j+30]} S_{\ll j}, w_{\ll j}, \chi_{[j-20,j+20]} \overline{S}_{\ll j})
\]

\[
+ \chi_{[j-30,j+30]} T_m^\delta (\chi_{[j-30,j+30]} S_{\ll j}, (1 - \chi_{[j-30,j+30]} ) w_{\ll j}, \chi_{[j-20,j+20]} \overline{S}_{\ll j})
\]

(70a)

(70b)

(70c)

(70d)

The subterms (70a) – (70d) are non-pseudolocal in the sense of Lemma 8, so they satisfy the bound

\[
\| (70b) \|_{L^2} + \| (70c) \|_{L^2} + \| (70d) \|_{L^2} \lesssim \sum_{|k-j| \leq 20} (t^2)^{-(k+1)} \| S_{\ll j} \|_{L^\infty} \| w_{\ll j} \|_{L^2} \| S_{\sim j} \|_{L^\infty}
\]

\[
\lesssim M^2 \epsilon^2 t^{-11/6} 2^{-5/2j} \| xg \|_{L^2}
\]
which gives the required bound after summing in $j$. To bound the remaining contribution from (70a), we observe that Corollary 8 implies that
\[ \| \chi_{[j-30,j+30]} \|_{L^2} \lesssim \| \chi_{[j-40,j+40]} \|_{L^2} \lesssim \| \chi_{[j-30,j+30]} \|_{L^2} \lesssim 2^k \left( \| Q_{[k-70,k+70]} (xg) \|_{L^2} + t^{-1/3} 2^{-k} \| xg \|_{L^2} \right) \]

Moreover, by Corollary 4, we have the bound
\[ \| \chi_{[j-30,j+30]} \|_{L^\infty} \lesssim M^2 \epsilon^2 t^{-1/2} 2^{-j/2} \]

Thus, using almost orthogonality, we find that
\[
\left\| \sum_j T_{m^2} (\chi_{[j-30,j+30]} S_{\ll j}, \chi_{[j-30,j+30]} w_{\ll j}, \chi_{[j-20,j+20]} S_{\sim j}) \right\|_{L^2} \lesssim \left( \sum_{2^j \geq t^{-1/3}} \left( \| \chi_{[j-30,j+30]} S_{\ll j} \|_{L^\infty} \| \chi_{[j-30,j+30]} w_{\ll j} \|_{L^2} \| S_{\sim j} \|_{L^\infty} \right)^2 \right)^{1/2} \lesssim M^2 \epsilon^2 t^{-1} \left( \sum_{2^j \geq t^{-1/3}} \left( \| Q_{[k-70,k+70]} (xg) \|_{L^2} + t^{-1/3} 2^{-k} \| xg \|_{L^2} \right)^2 \right)^{1/2} \lesssim M^2 \epsilon^2 t^{-1} \| xg \|_{L^2} \]

Collecting the bounds for (66a) and (68a), we find
\[ \left\| \sum_{2^j \geq t^{-1/3}} T_{m^2} \right\|_{L^2} \lesssim M^2 \epsilon^2 t^{-1} \| xg \|_{L^2} \]

as required.

6.2.2. The bound for (66b). The estimates for (66b) are analogous to those for (66a) once we use the bounds
\[ \sum_{2^j \geq t^{-1/3}} 2^{-2j} \| \partial_x (Lw)_j \|_{L^2} \lesssim \| xg \|_{L^2} \]

and
\[ \| \partial_x (Lw)_{\ll j} \|_{L^2} \lesssim 2^j \| xg \|_{L^2} \]

in place of the Hardy-type bounds on $w$.

6.2.3. The bound for (66c). Applying (67), we find
\[
T_{m^2} (S_{\ll j}, \partial_x w_{\ll j}, (LS)_{\sim j}) = T_{m^2} (S_{\ll j}, \partial_x w_{\ll j}, (LS)_{\sim j}) + T_{m^2} (S_{\sim j}, \partial_x w_{\ll j}, (LS)_{\ll j}) + Q_{\sim j} T_{m^2} (S_{\sim j}, \partial_x w_{\sim j}, (LS)_{\sim j}) \] (71a)
\[
T_{m^2} (S_{\sim j}, \partial_x w_{\sim j}, (LS)_{\ll j}) \] (71b)
\[
T_{m^2} (S_{\sim j}, \partial_x w_{\sim j}, (LS)_{\sim j}) \] (71c)
Let us first consider \((71c)\). Recall that from \((13)\) we know that \(LS = t|S|^2S\), so, the refined cubic estimate \((39)\) implies that
\[
\|(LS)_{\sim j}\|_{L^\infty} \lesssim M^3 \epsilon^3 t^{-1/2} 2^{-j/2}
\]
Thus, using almost orthogonality, we find that
\[
\left\| \sum_j (71c) \right\|_{L^2} \lesssim \left( \sum_{2j \gtrsim t^{-1/3}} \left\| T_m \phi(S_{\sim j}, \partial_x w_{\sim j}, (LS)_{\sim j}) \right\|_{L^2}^2 \right)^{1/2}
\]
\[
\lesssim \left( \sum_{2j \gtrsim t^{-1/3}} \|S_{\sim j}\|_{L^\infty}^2 \|\partial_x w_{\sim j}\|_{L^2}^2 \|(LS)_{\sim j}\|_{L^\infty} \right)^{1/2}
\]
\[
\lesssim M^4 \epsilon^4 \left( \sum_{2j \gtrsim t^{-1/3}} 2^{-4j} \|\partial_x w_{\sim j}\|_{L^2}^2 \right)^{1/2}
\]
\[
\lesssim M^4 \epsilon^4 t^{-1} \|w\|_{H^{-1}}
\]
\[
\lesssim M^4 \epsilon^4 t^{-1} \|xg\|_{L^2}
\]
where on the last line we have used Hardy’s inequality. For \((71b)\), we divide dyadically in space to obtain
\[
T_m \phi(S_{\ll j}, \partial_x w_{\ll j}, (LS)_{\sim j}) = T_m \phi(\chi_{\geq j} S_{\ll j}, \partial_x w_{\ll j}, (LS)_{\sim j})
\]
\[
+ \sum_{k < j} T_m \phi(\chi_k S_{\ll j}, \partial_x w_{\ll j}, (LS)_{\sim j})
\]
\[
+ \sum_{j-40 < k < j} T_m \phi(\chi_k S_{\ll j}, \partial_x w_{\ll j}, (LS)_{\sim j})
\]
For the first term, observe that by Lemma \(8\)
\[
\|\chi_{\geq j} S_{\ll j}\|_{L^\infty} \lesssim M t^{5/6} 2^{-3/2j}
\]
so
\[
\left\| \sum_{2j \gtrsim t^{-1/3}} T_m \phi(\chi_{\geq j} S_{\ll j}, \partial_x w_{\ll j}, (LS)_{\sim j}) \right\|_{L^2} \lesssim \left\| \chi_{\geq j} S_{\ll j} \right\|_{L^\infty} \|\partial_x w_{\ll j}\|_{L^2} \|(LS)_{\sim j}\|_{L^\infty}
\]
\[
\lesssim M^4 \epsilon^4 t^{-4/3} 2^{-3j} \|\partial_x w_{\ll j}\|_{L^2}
\]
\[
\lesssim M^4 \epsilon^4 t^{-4/3} 2^{-3j} \|w\|_{H^{-1}}
\]
\[
\lesssim M^4 \epsilon^4 t^{-4/3} 2^{-3j} \|xg\|_{L^2}
\]
By summing in \(j\) and recalling that \(\epsilon \ll M^{-3/2}\), we see that this bound is sufficient.
For the second term, we perform the further division
\[
\left(72b\right) = \sum_{k < j} T_m \phi(\chi_k S_{\ll j}, \chi_{\sim k} \partial_x w_{\ll j}, \chi_{\sim k} (LS)_{\sim j})
\]
\[
+ \sum_{k < j} T_m \phi(\chi_k S_{\ll j}, (1 - \chi_{\sim k}) \partial_x w_{\ll j}, (LS)_{\sim j})
\]
\[
+ \sum_{j-40 < k < j} T_m \phi(\chi_k S_{\ll j}, \chi_{\sim k} \partial_x w_{\ll j}, (1 - \chi_{\sim k}) (LS)_{\sim j})
\]
Lemma 4 applies to the pseudoproducts in the terms (73b) and (73c), yielding the bound
\[ \| \text{(73b)} \|_{L^2} + \| \text{(73c)} \|_{L^2} \lesssim \sum_{t^{-1/3} \lesssim 2^j \lesssim 2^{j-40}} (2^{2k+j})^{-1} \| S_{\leq j} \|_{L^\infty} \| \partial_x w_{\leq j} \|_{L^2} \| (LS)_{\sim j} \|_{L^\infty} \]
\[ \lesssim \sum_{t^{-1/3} \lesssim 2^k \lesssim 2^{j-40}} M^4 e^4 t^{-11/6} 2^{-2k/3} \| w \|_{H^{-1}} \]
\[ \lesssim M^4 e^4 t^{-7/6} 2^{-j/2} \| xg \|_{L^2} \]
which is sufficient. To bound the remaining term (73a), we begin by deriving a bound for \( \chi_{\sim k} \partial_x w_{\leq j} \).
By using the identity \( x \partial_x g_t = \partial_x (xg_t) - g_t \), we see that
\[ \| x \partial_x g_t \|_{L^2} \lesssim \| \partial_x (xg_t) \|_{L^2} + \| g_t \|_{L^2} \lesssim 2^t \| Q_{[t-10, t+10]}(xg) \|_{L^2} \]
Thus, we can apply Corollary 8 with \( g \) replaced by \( \partial_x g_{\sim \ell} \) to obtain
\[ \| \partial_x w_{\leq j} \| \lesssim 2^t \left( t^{-1/2} \chi_{\sim \ell} + t^{-5/6} 2^{\ell/2} \left( 2^t + 2^{-\ell/2} [\xi_0]^{4/3} \right)^{-3/2} \right) \| Q_{[\ell-40, \ell+40]}(xg) \|_{L^2} \]
It follows that
\[ \| \chi_{\sim k} \partial_x w_{\leq j} \|_{L^2} \lesssim \chi_{[k-20, k+20]} \| L^2 \sum_{\ell \leq j+10} \| \chi_{\sim k} \partial_x w_{\ell} \|_{L^\infty} \]
\[ \lesssim \frac{1}{2^k} \left( \sum_{t^{-1/3} \lesssim 2^k \lesssim 2^{j-20}} t^{-5/6} 2^{\ell-k/3} \| xg \|_{L^2} + \sum_{|\ell-k| \leq 20} t^{-1/2} 2^t \| Q_{[\ell-40, \ell+40]}(xg) \|_{L^2} \right) \]
\[ \lesssim 2^{2k} \| Q_{[k-60, k+60]}(xg) \|_{L^2} + t^{-1/3} 2^k \| xg \|_{L^2} \]
In particular, since \( 2^k \gtrsim t^{-1/3} \),
\[ \| \chi_{\sim k} \partial_x w_{\leq j} \|_{L^2} \lesssim 2^{2k} \| xg \|_{L^2} \]
In addition, Corollary 7 gives us the bound
\[ \| \chi_{k} S_{\leq j} \|_{L^2} \lesssim M e t^{-1/3} 2^{-k/2} \]
Thus, using (38) to estimate \( \chi_{\sim k} (LS)_{\sim j} \) in \( L^\infty \), we find that
\[ \| \text{(73a)} \|_{L^2} \lesssim \sum_{t^{-1/3} \lesssim 2^k \lesssim 2^{j-40}} \| \chi_{k} S_{\leq j} \|_{L^\infty} \| \chi_{\sim k} \partial_x w_{\leq j} \|_{L^2} \| \chi_{\sim k} (LS)_{\sim j} \|_{L^\infty} \]
\[ \lesssim \sum_{t^{-1/3} \lesssim 2^k \lesssim 2^{j-40}} M^4 e^4 t^{-4/3} 2^{k/2} \| xg \|_{L^2} \]
\[ \lesssim M^4 e^4 t^{-4/3} 2^{-j/2} \| xg \|_{L^2} \]
To bound (72c), we write

\[
(72c) = \sum_{j-40 \leq k < j} \chi_k T_m (\chi_k S_{k\ll j}, \chi_k \partial_x w_{\ll j}, (LS)_{\ll j})
\]

(75a)

\[
+ \sum_{j-40 \leq k < j} (1 - \chi_k) T_m (\chi_k S_{k\ll j}, \chi_k \partial_x w_{\ll j}, (LS)_{\ll j})
\]

(75b)

\[
+ \sum_{j-40 \leq k < j} T_m (\chi_k S_{k\ll j}, (1 - \chi_k) \partial_x w_{\ll j}, (LS)_{\ll j})
\]

(75c)

The terms (75b) and (75c) can be handled using Lemma 4 in the same manner as (73b) and (75c). For the term (75a), we can use the bound (74) to find that

\[
\left\| \sum_{2^k \geq t \geq 1/3} \sum_{j-40 \leq k < j} \chi_k T_m (\chi_k S_{k\ll j}, \chi_k \partial_x w_{\ll j}, (LS)_{\ll j}) \right\|_{L^2} \lesssim M^4 \epsilon t^{-1/2}
\]

\[
\lesssim \sum_{2^k \geq t \geq 1/3} \sum_{j-40 \leq k < j} \left\| T_m (\chi_k S_{k\ll j}, \chi_k \partial_x w_{\ll j}, (LS)_{\ll j}) \right\|_{L^2}^{2/3}
\]

\[
\lesssim M^4 \epsilon t^{-1/2} \left( \sum_{2^k \geq t \geq 1/3} \left( \sum_{j-40 \leq k < j} 2^{-k/2-3j/2} \left\| \chi_k \partial_x w_{\ll j} \right\|_{L^2} \right) \right)^{1/2}
\]

\[
\lesssim M^4 \epsilon t^{-1/2} \left( \sum_{2^k \geq t \geq 1/3} \left( \sum_{j-40 \leq k < j} \left\| Q_{[k-60,k+60]} \chi_k \partial_x w_{\ll j} \right\|_{L^2} + t^{-1} \right) \right)^{1/2}
\]

Combining the estimates for (72a)–(72c) and recalling that $\epsilon \ll M^{-3/2}$ now gives the required bound for (71a).

We now turn to (71b). Dividing dyadically in space, we can write

\[
(71b) = \sum_{k \leq j-30} T_m (S_{\sim j}, \partial_x w_{\ll j}, \chi_k (LS)_{\ll j})
\]

(76a)

\[
+ T_m (S_{\sim j}, \partial_x w_{\ll j}, \chi_{j-30,j+30} (LS)_{\ll j})
\]

(76b)

\[
+ \sum_{k > j+30} T_m (S_{\sim j}, \partial_x w_{\ll j}, \chi_k (LS)_{\ll j})
\]

(76c)
For (76a), we have
\[
(76a) = \sum_{k<j-30} T_{mj}^s \left( \chi_{\sim k} S_{\sim j}, \chi_{\sim k} \partial_x w_{\leq j}, \chi_k(LS)_{\ll j} \right) + \sum_{k<j-30} T_{mj}^s \left((1 - \chi_{\sim k}) S_{\sim j}, \partial_x w_{\leq j}, \chi_k(LS)_{\ll j} \right) + \sum_{k<j-30} T_{mj}^s \left(\chi_{\sim k} S_{\sim j}, (1 - \chi_{\sim k}) \partial_x w_{\leq j}, \chi_k(LS)_{\ll j} \right)
\]

For (77a), we have
\[
(77a) = \chi_{\sim k} S_{\sim j}, \chi_{\sim k} \partial_x w_{\leq j}, \chi_k(LS)_{\ll j}
\]

For (77b) and (77c), Lemma 4 immediately gives us the bound
\[
\| (77b) + (77c) \|_{L^2} \lesssim \sum_{k<j-30} (t2^{2k+j})^{-2} \| S_{\sim j} \|_{L^\infty} \| (LS)_{\ll j} \|_{L^\infty} \| \partial_x w_{\leq j} \|_{L^2}
\]
\[
\lesssim M^4 \epsilon^2 t^{-7/6} \sum_{t^{-1/3} < 2^{k-30} < t} 2^{-j/2-4k} \| w \|_{\dot{H}^{-1}}
\]
\[
\lesssim M^4 \epsilon^2 t^{-7/6} 2^{-j/2} \| xg \|_{L^2}
\]

which is sufficient. Turning to (78a), we see that over the range of summation,
\[
\| \chi_{\sim k} S_{\sim j} \|_{L^\infty} \lesssim M \epsilon t^{-5/6} 2^{-3/2j}
\]

Moreover, using (76a), we have that
\[
\| \chi_k(LS)_{\ll j} \|_{L^\infty} \lesssim M^4 \epsilon^4 t^{-1/2} 2^{-3/2k}
\]

so
\[
\| (76a) \|_{L^2} \lesssim \sum_{k<j-30} \| T_{mj}^s \left( \chi_{\sim k} S_{\sim j}, \chi_{\sim k} \partial_x w_{\leq j}, \chi_k(LS)_{\ll j} \right) \|_{L^2}
\]
\[
\lesssim \sum_{k<j-30} \| \chi_{\sim k} S_{\sim j} \|_{L^\infty} \| \chi_{\sim k} \partial_x w_{\leq j} \|_{L^2} \| \chi_k(LS)_{\ll j} \|_{L^\infty}
\]
\[
\lesssim M^4 \epsilon^4 t^{-4/3} 2^{-j} \| xg \|_{L^2}
\]

which is acceptable, completing the argument for (76a). Turning to (76b), we can write
\[
(76b) = \chi_{[j-40,j+40]} T_{mj}^s \left( S_{\sim j}, \chi_{[j-40,j+40]} \partial_x w_{\leq j}, \chi_{[j-30,j+30]}(LS)_{\ll j} \right) + (1 - \chi_{[j-40,j+40]}) T_{mj}^s \left( S_{\sim j}, \chi_{[j-40,j+40]} \partial_x w_{\leq j}, \chi_{[j-30,j+30]}(LS)_{\ll j} \right) + T_{mj}^s \left( S_{\sim j}, (1 - \chi_{[j-40,j+40]}) \partial_x w_{\leq j}, \chi_{[j-30,j+30]}(LS)_{\ll j} \right)
\]

The terms (78b) and (78c) can be controlled using Lemma 4
\[
\| (78b) + (78c) \|_{L^2} \lesssim (t2^{3j})^{-1} \| S_{\sim j} \|_{L^\infty} \| \partial_x w_{\leq j} \|_{L^2} \| (LS)_{\ll j} \|_{L^\infty}
\]
\[
\lesssim M^4 \epsilon^4 t^{-3/2} 2^{-3/2j} \| xg \|_{L^2}
\]
which is acceptable. For (78a), we use the bound (74) on \( \chi_{[j-40,j+40]} \partial_x w_{\leq j} \) together with almost orthogonality to obtain

\[
\left\| \sum_j (78a) \right\|_{L^2} \lesssim \left( \sum_{2j \geq 2^{-1/3}} \| T_{m_j} \chi_{[j-40,j+40]} \partial_x w_{\leq j}, \chi_{[j-30,j+30]} (L_S)_{\leq j} \|_{L^2}^2 \right)^{1/2}
\]

\[
\lesssim \left( \sum_{2j \geq 2^{-1/3}} \| T_{m_j} \chi_{[j-40,j+40]} \partial_x w_{\leq j}, \chi_{[j-30,j+30]} (L_S)_{\leq j} \|_{L^2}^2 \right)^{1/2}
\]

\[
\lesssim \left( \sum_{2j \geq 2^{-1/3}} \| S_{\leq j} \|_{L^\infty} \| \chi_{[j-40,j+40]} \partial_x w_{\leq j} \|_{L^2} \| \chi_{[j-30,j+30]} (L_S)_{\leq j} \|_{L^\infty} \right)^{1/2}
\]

\[
\lesssim M^4 \epsilon^4 t^{-1} \left( \sum_{2j \geq 2^{-1/3}} \left( \| \chi_j \|_{L^2}^2 + t^{-1/3} 2^{-j} \| \partial_x \|_{L^2} \right)^2 \right)^{1/2}
\]

\[
\lesssim M^4 \epsilon^4 t^{-1} \| \partial_x \|_{L^2}
\]

as required.

Finally, for (76c), we have

\[
(76c) = T_{m_j} \chi_{[j+20]} S_{\leq j}, \partial_x w_{\leq j}, \chi_{[j+30]} (L_S)_{\leq j}) \quad (79a)
\]

\[
+ T_{m_j} \chi_{[j-20]} S_{\leq j}, \partial_x w_{\leq j}, \chi_{[j+30]} (L_S)_{\leq j}) \quad (79b)
\]

The second term is easily controlled using Lemma 4

\[
\| (79b) \|_{L^2} \lesssim (2^{2j})^{-1} \| S_{\leq j} \|_{L^\infty} \| \partial_x w_{\leq j} \|_{L^2} \| (L_S)_{\leq j} \|_{L^\infty}
\]

\[
\lesssim M^4 \epsilon^4 t^{-3/2} 2^{-3/2} \| w \|_{H^{-1}}
\]

\[
\lesssim M^4 \epsilon^4 t^{-3/2} 2^{-3/2} \| \partial_x w \|_{L^2}
\]

which is acceptable. For the remaining term (79a), we see that

\[
\| \chi_{[j+30]} S_{\leq j} \|_{L^2} \lesssim M t^{-5/6} 2^{-3/2}
\]

by Lemma 5 so

\[
\| (79a) \|_{L^2} \lesssim \| \chi_{[j+30]} S_{\leq j} \|_{L^\infty} \| \partial_x w_{\leq j} \|_{L^2} \| \chi_{[j+30]} (L_S)_{\leq j} \|_{L^\infty}
\]

\[
\lesssim M^4 \epsilon^4 t^{-3/2} 2^{-j} \| w \|_{H^{-1}}
\]

\[
\lesssim M^4 \epsilon^4 t^{-3/2} 2^{-j} \| \partial_x w \|_{L^2}
\]

Collecting the bounds for (76a)–(76c), we see that

\[
\| (66c) \|_{L^2} \lesssim M^4 \epsilon^4 t^{-1} \| \partial_x \|_{L^2}
\]

which is better than required, since \( \epsilon \ll M^{-3/2} \).

**Remark 10.** In estimating (66a)–(66c), we frequently encountered terms of the form

\[
T_{m_j} (p, q, \chi_k f) = \chi_{k} T_{m_j} (\chi_{k} p, \chi_{k} q, \chi_{k} f) + \{\text{non-pseudolocal terms}\}
\]

where \{non-pseudolocal terms\} denotes terms which can be estimated using Lemma 4. The estimates for the non-pseudolocal remainder terms are routine: they do not require any refined linear or cubic
estimates and are insensitive to the exact structure of the space weights in the pseudoproducts. Thus, in the interest of the exposition, we will not estimate these non-pseudolocal remainders or write them explicitly in the following sections.

6.2.4. The bound for (66d). Here, we have

\[ T_{m_j}^\delta (S, S, \overline{w}) = Q_{\sim j} T_{m_j}^\delta (S_{\ll j}, S_{\ll j}, \overline{w}_{\sim j}) + T_{m_j}^\delta (S_{\ll j}, S_{\ll j}, \overline{w}_{\ll j}) \]

(80a)

\[ + T_{m_j}^\delta (S_{\ll j}, S_{\ll j}, \overline{w}_{\ll j}) + T_{m_j}^\delta (S_{\ll j}, S_{\ll j}, \overline{w}_{\ll j}) \]

(80b)

\[ + T_{m_j}^\delta (S_{\ll j}, S_{\ll j}, \overline{w}_{\ll j}) \]

(80c)

\[ + Q_{\sim j} T_{m_j}^\delta (S_{\sim j}, S_{\sim j}, \overline{w}_{\sim j}) \]

(80d)

\[ + Q_{\sim j} T_{m_j}^\delta (S_{\sim j}, S_{\sim j}, \overline{w}_{\sim j}) \]

(80e)

Notice that the terms \((80c)\) and \((80d)\) can be controlled in the same way as \((68a)\), and that the term \((80e)\) can be controlled in the same way as \((68b)\). Thus, it only remains to consider the contribution from the first two terms.

For \((80a)\), we write

\[ Q_{\sim j} T_{m_j}^\delta (S_{\sim j}, S_{\sim j}, \overline{w}_{\sim j}) \]

(81a)

\[ = Q_{\sim j} T_{m_j}^\delta (S_{\sim j}, S_{\sim j}, \overline{w}_{\sim j}) \]

(81b)

\[ + Q_{\sim j} T_{m_j}^\delta (S_{\sim j}, \chi_{\sim j} - 30 S_{\ll j}, \chi_{\sim j} - 20 \overline{w}_{\sim j}) \]

(81c)

\[ + \{ \text{non-pseudolocal terms} \} \]

The non-pseudolocal terms are easily handled (see Remark 10). For \((81a)\), observe that by Corollary 8

\[ \| \chi_{\leq j - 30 \overline{w}_{\sim j}} \|_{L^2} \lesssim \| \chi_{\leq j - 20} \|_{L^2} \| \chi_{\leq j - 30 \overline{w}_{\sim j}} \|_{L^\infty} \]

\[ \lesssim t^{-1/3} \| Q_{[j-40, j+40]} (xg) \|_{L^2} \]

(82)

Thus, using the dispersive bound \( \| S_{\ll j} \|_{L^\infty} \lesssim M t^{-1/3} \) together with the fact that the terms in \((81a)\) are almost orthogonal, we find that

\[ \left\| \sum_j [81a] \right\|_{L^2} \lesssim \left( \sum_{2^j \geq t^{-1/3}} \left\| T_{m_j}^\delta (\chi_{\leq j - 30 S_{\ll j}}, \chi_{\leq j - 20 S_{\ll j}}, \chi_{\sim j - 20 \overline{w}_{\sim j}}) \right\|_{L^2}^2 \right)^{1/2} \]

\[ \lesssim \left( \sum_{2^j \geq t^{-1/3}} \| S_{\ll j} \|_{L^\infty}^2 \| \chi_{\leq j - 30 \overline{w}_{\sim j}} \|_{L^2}^2 \right)^{1/2} \]

\[ \lesssim M^2 t^2 t^{-1} \left( \sum_{2^j \geq t^{-1/3}} \| Q_{[j-40, j+40]} (xg) \|_{L^2}^2 \right)^{1/2} \]

\[ \lesssim M^2 t^2 t^{-1} \| xg \|_{L^2} \]

as required.

For \((81b)\), we note that

\[ \| \chi_{\geq j - 30 S_{\ll j}} \|_{L^\infty} + \| \chi_{\geq j - 40 S_{\ll j}} \|_{L^\infty} \lesssim M t^{-1/2} 2^{-j/2} \]
by Corollary 7 so
\[
\left\| \sum_{j} w_{j} \right\|_{L^{2}} \lesssim \left( \sum_{2^{j} \geq t^{-1/3}} \left\| \sum_{j} T_{m_{j}^{g}} (\chi_{j} S_{\lesssim j}, \chi_{j-30} S_{\lesssim j}, \overline{w}_{\lesssim j}) \right\|_{L^{2}}^{2} \right)^{1/2}
\]
\[
\lesssim \left( \sum_{2^{j} \geq t^{-1/3}} \left\| \chi_{j} S_{\lesssim j} \right\|_{L^{\infty}}^{2} \left\| \chi_{j-30} S_{\lesssim j} \right\|_{L^{\infty}}^{2} \left\| w_{\lesssim j} \right\|_{L^{2}}^{2} \right)^{1/2}
\]
\[
\lesssim M^{2} c^{2} t^{-1} \left( \sum_{2^{j} \geq t^{-1/3}} \left\| w_{\lesssim j} \right\|_{H^{-1}}^{2} \right)^{1/2}
\]
\[
\lesssim M^{2} c^{2} t^{-1} \left\| xg \right\|_{L^{2}}
\]
which is sufficient.

Turning to (80b), we write
\[
(80b) = (81b)
\]
\[
(80b) = T_{m_{j}^{g}} ((1 - \chi_{j-20,j+20}) S_{\sim j}, S_{\sim j}, \overline{w}_{\lesssim j})
\]
\[
= T_{m_{j}^{g}} ((1 - \chi_{j-20,j+20}) S_{\sim j}, S_{\sim j}, \overline{w}_{\lesssim j}) + \chi_{j=30,j+30} T_{m_{j}^{g}} (\chi_{j=20,j+20}) S_{\sim j}, S_{\sim j}, \overline{w}_{\lesssim j}) \quad (83a)
\]
\[
+ \chi_{j=30,j+30} T_{m_{j}^{g}} (\chi_{j=20,j+20}) S_{\sim j}, S_{\sim j}, \chi_{j=30,j+30} \overline{w}_{\lesssim j}) \quad (83b)
\]

By Lemma 5
\[
\left\| (1 - \chi_{j=20,j+20}) S_{\sim j} \right\|_{L^{\infty}} \lesssim M t^{-5/6} 2^{-3/2j}
\]
which gives us the bound
\[
\left\| (83a) \right\|_{L^{2}} \lesssim \left\| (1 - \chi_{j=20,j+20}) S_{\sim j} \right\|_{L^{\infty}} \left\| S_{\sim j} \right\|_{L^{\infty}} \left\| w_{\lesssim j} \right\|_{L^{2}}
\]
\[
\lesssim M^{2} c^{2} t^{-4/3} 2^{-j} \left\| xg \right\|_{L^{2}}
\]
For (83b), we use Corollary 8 to obtain the bound
\[
\left\| \chi_{j=30,j+30} \overline{w}_{\lesssim j} \right\|_{L^{2}} \lesssim \left\| \chi_{j=30,j+30} \overline{w}_{\lesssim j} \right\|_{L^{\infty}} \left\| \chi_{j=20,j+20} \overline{w}_{\lesssim j} \right\|_{L^{2}}
\]
\[
\lesssim 2^{j} \left( \left\| Q_{j=30,j+30} (xg) \right\|_{L^{2}} + t^{-1/3} 2^{-j} \left\| xg \right\|_{L^{2}} \right)
\]
which then implies that
\[
\left\| \sum_{j} (83b) \right\|_{L^{2}} \lesssim \left( \sum_{2^{j} \geq t^{-1/3}} \left\| T_{m_{j}^{g}} (\chi_{j=20,j+20} S_{\sim j}, S_{\sim j}, \chi_{j=30,j+30} \overline{w}_{\lesssim j}) \right\|_{L^{2}}^{2} \right)^{1/2}
\]
\[
\lesssim \left( \sum_{2^{j} \geq t^{-1/3}} \left\| S_{\sim j} \right\|_{L^{\infty}}^{2} \left\| \chi_{j=30,j+30} \overline{w}_{\lesssim j} \right\|_{L^{2}}^{2} \right)^{1/2}
\]
\[
\lesssim M^{2} c^{2} t^{-1} \left( \sum_{2^{j} \geq t^{-1/3}} \left( \left\| Q_{j=30,j+30} (xg) \right\|_{L^{2}} + t^{-1/3} 2^{-j} \left\| xg \right\|_{L^{2}} \right)^{2} \right)^{1/2}
\]
\[
\lesssim M^{2} c^{2} t^{-1} \left\| xg \right\|_{L^{2}}
\]
as required.
The bound for (66d) now follows.

6.2.5. The bound for (66e). Here, we decompose the pseudoproduct as

\[ T_{m_j}^\phi(u, \partial_x LS, \overline{w}) = T_{m_j}^\phi(u_{< j}, \partial_x (LS)_{\approx j}, \overline{w}_{\approx j}) + Q_{\approx j} T_{m_j}^\phi(u_{< j}, \partial_x (LS)_{\ll j}, \overline{w}_{\ll j}) + T_{m_j}^\phi(u_{\approx j}, \partial_x (LS)_{\ll j}, \overline{w}_{\ll j}) + Q_{\ll j} T_{m_j}^\phi(u_{\approx j}, \partial_x (LS)_{\ll j}, \overline{w}_{\ll j}) \]

For (85a), we have

\[ T_{m_j}^\phi((\chi_{> j} + \chi_{[j-40, j]} + \chi_{< j-40}) u_{< j}, \partial_x (LS)_{\approx j}, \overline{w}_{\approx j}) \]

\[ = T_{m_j}^\phi(\chi_{> j} u_{< j}, \partial_x (LS)_{\approx j}, \overline{w}_{\approx j}) + \chi_{[j-50, j+10]} T_{m_j}^\phi(\chi_{[j-40,j]} u_{< j}, \partial_x (LS)_{\approx j}, \overline{w}_{\approx j}) + T_{m_j}^\phi(\chi_{< j-40} u_{< j}, \partial_x (LS)_{\ll j}, \chi_{< j-30} \overline{w}_{\ll j}) + \{ \text{non-pseudolocal terms} \} \]

Note that by (441),

\[ \| \partial_x (LS)_{\approx j} \|_{L^\infty} \lesssim M^3 \epsilon^3 t^{-1/2} 2^{-j/2} \]

and by Lemma 5,

\[ \| \chi_{> j} u_{< j} \|_{L^\infty} \lesssim M t^{-5/6} 2^{-3j/2} \]

so

\[ \| \chi_{[j-50,j+10]} T_{m_j}^\phi(\chi_{[j-40,j]} u_{< j}, \partial_x (LS)_{\approx j}, \overline{w}_{\approx j}) \|_{L^\infty} \lesssim M^4 \epsilon^4 t^{-4/3} 2^{-j} \| u \|_{H^{-1}} \]

\[ \lesssim M^4 \epsilon^4 t^{-4/3} 2^{-j} \| xg \|_{L^2} \]

Similarly, Corollary 7 implies that

\[ \| \chi_{[j-40,j]} u_{< j} \|_{L^\infty} \lesssim M t^{-1/2} 2^{-j/2} \]

and hence

\[ \| \sum_j (86c) \|_{L^2} \lesssim \left( \sum_{2^j \geq t^{-1/3}} \left\| T_{m_j}^\phi(\chi_{[j-40,j]} u_{< j}, \partial_x (LS)_{\ll j}, \overline{w}_{\ll j}) \right\|_{L^2}^2 \right)^{1/2} \]

\[ \lesssim M^4 \epsilon^4 t^{-1} \left( \sum_{2^j \geq t^{-1/3}} \left\| \chi_{[j-40,j]} u_{< j} \|_{L^\infty}^2 \right\|_{L^2} \right)^{1/2} \]

\[ \lesssim M^4 \epsilon^4 t^{-1} \left( \sum_{2^j \geq t^{-1/3}} \left\| w_{\ll j} \|_{H^{-1}}^2 \right\|_{L^2} \right)^{1/2} \]

\[ \lesssim M^4 \epsilon^4 t^{-1} \| xg \|_{L^2} \]

Finally, recalling (82), we find that

\[ \| \sum_j (86c) \|_{L^2} \lesssim \| u_{< j} \|_{L^\infty} \| \partial_x (LS)_{\ll j} \|_{L^\infty} \| \chi_{< j-30} w_{\ll j} \|_{L^2} \]

\[ \lesssim M^4 \epsilon^4 t^{-7/6} 2^{-j/2} \| xg \|_{L^2} \]
which gives the required bound for \(85a\).

Turning to \((85b)\), we have
\[
(85b) = Q_{j} T_{n}^j (u \ll_j, \partial_x (LS) \ll_j, (\chi_{< j-30} + \chi_{\geq j-30}) m_{\sim j})
\]
\[
= Q_{j} T_{n}^j (u \ll_j, \partial_x (LS) \ll_j, \chi_{< j-30} m_{\sim j})
\]  
\[
+ Q_{j} T_{n}^j (\chi_{\geq j-40} u \ll_j, \chi_{\geq j-40} \partial_x (LS) \ll_j, \chi_{\geq j-30} m_{\sim j}) \quad (87a)
\]
\[
+ \{\text{non-pseudolocal terms}\}
\]

For \((87a)\), we note that
\[
\| \partial_x (LS) \ll_j \|_{L^\infty} \leq M^3 e^3 t^{-1/3}
\]
so, using \((82)\) to control the contribution from \(\chi_{< j-30} w_{\sim j}\), we have
\[
\left\| \sum_{j} (87a) \right\|_{L^2} \lesssim \left( \sum_{2^j \geq t^{-1/3}} \left\| T_{n}^j (u \ll_j, \partial_x (LS) \ll_j, \chi_{< j-30} m_{\sim j}) \right\|_{L^2}^2 \right)^{1/2}
\]
\[
\lesssim \left( \sum_{2^j \geq t^{-1/3}} \left\| u \ll_j \right\|_{L^\infty} \left\| \partial_x (LS) \ll_j \right\|_{L^\infty} \left\| \chi_{< j-30} w_{\sim j} \right\|_{L^2} \right)^{1/2}
\]
\[
\lesssim M^4 e^4 t^{-1} \left( \sum_{2^j \geq t^{-1/3}} \left\| Q_{j} (\chi_{\geq j-40, j+40} (xg)) \right\|_{L^2}^2 \right)^{1/2}
\]
\[
\lesssim M^4 e^4 t^{-1} \| xg \|_{L^2}
\]

For \((87b)\), we have the bound
\[
\left\| \sum_{j} (87b) \right\|_{L^2} \lesssim \left( \sum_{2^j \geq t^{-1/3}} \left\| T_{n}^j (\chi_{\geq j-40} u \ll_j, \chi_{\geq j-40} \partial_x (LS) \ll_j, \chi_{\geq j-30} m_{\sim j}) \right\|_{L^2}^2 \right)^{1/2}
\]
\[
\lesssim \left( \sum_{2^j \geq t^{-1/3}} \left\| \chi_{\geq j-40} u \ll_j \right\|_{L^2}^2 \left\| \chi_{\geq j-40} \partial_x (LS) \ll_j \right\|_{L^2}^2 \left\| \chi_{\geq j-30} m_{\sim j} \right\|_{L^2} \right)^{1/2}
\]
\[
\lesssim M^4 e^4 t^{-1} \left( \sum_{2^j \geq t^{-1/3}} \left\| w_{\sim j} \right\|_{H^{-1}}^2 \right)^{1/2}
\]
\[
\lesssim M^4 e^4 t^{-1} \| xg \|_{L^2}
\]

where we have used Corollary \(\mathbf{4}\) to control \(\chi_{\geq j-40} u \ll_j\) and \((37)\) to control \(\chi_{\geq j-40} \partial_x (LS) \ll_j\). The bound for \((85b)\) now follows.

For \((85c)\), we have
\[
(85c) = T_{n}^j \left( (\chi_{[j-20, j+20]} + (1 - \chi_{[j-20, j+30]})) u_{\sim j}, \partial_x (LS) \ll_j, \overline{m} \ll_j \right)
\]
\[
= T_{n}^j \left( (1 - \chi_{[j-20, j+20]}) u_{\sim j}, \partial_x (LS) \ll_j, \overline{m} \ll_j \right)
\]
\[
+ \chi_{[j-30, j+30]} T_{n}^j \left( \chi_{[j-20, j+20]} u_{\sim j}, (j-30, j+30) \partial_x (LS) \ll_j, \chi_{[j-30, j+30]} \overline{m} \ll_j \right) \quad (88a)
\]
\[
+ \{\text{non-pseudolocal terms}\} \quad (88b)
\]
By Lemma 5
\[ \| (1 - \chi_{[j-20,j+20]}) u_{\sim j} \|_{L^\infty} \lesssim M t^{-5/6} 2^{-3/2j} \]
and (37) gives us the bound
\[ \| \partial_x (LS) \|_{L^\infty} \lesssim M^3 t^{-1/3} \]
so
\[ \| SSa \|_{L^2} \lesssim \| \chi_{[j-20,j+20]} u_{\sim j} \|_{L^\infty} \| \partial_x (LS) \|_{L^\infty} \| w_{\sim j} \|_{L^2} \]
\[ \lesssim M^4 t^{-7/6} 2^{-j/2} \|w\|_{\dot{H}^{-1}} \]
\[ \lesssim M^4 t^{-7/6} 2^{-j/2} \| xg \|_{L^2} \]
which is sufficient. Turning to (88b), we use (84) can be used to control \( \chi_{[j-30,j+30]} w_{\sim j} \) and (37) to control \( \chi_{[j-30,j+30]} \partial_x (LS) \|_{L^\infty} \), yielding
\[ \| \sum_j \| SSb \|_{L^2} \| \lesssim \left( \sum_{2^j \geq t^{-1/3}} \| T_{m_j^S} (\chi_{[j-20,j+20]} u_{\sim j}, \chi_{[j-30,j+30]} \partial_x (LS) \|_{L^\infty} \| \chi_{[j-30,j+30]} w_{\sim j} \|_{L^2} \right)^{1/2} \]
\[ \lesssim \left( \sum_{2^j \geq t^{-1/3}} \| u_{\sim j} \|_{L^\infty} \| \chi_{[j-30,j+30]} \partial_x (LS) \|_{L^\infty} \| \chi_{[j-30,j+30]} w_{\sim j} \|_{L^2} \right)^{1/2} \]
\[ \lesssim M^4 t^{-1} \left( \sum_{2^j \geq t^{-1/3}} \| Q_{[j-70,j+70]} (xg) \|_{L^2} + t^{-1/3} 2^{-j} \| xg \|_{L^2} \right)^{1/2} \]
\[ \lesssim M^4 t^{-1} \| xg \|_{L^2} \]
which completes the bound for (85c).

Finally, for (85d), we use almost orthogonality and (37) to conclude that
\[ \| \sum_j \| SSD \|_{L^2} \| \lesssim \left( \sum_{2^j \geq t^{-1/3}} \| u_{\sim j} \|_{L^\infty} \| \partial_x (LS)_{\sim j} \|_{L^\infty} \| w_{\sim j} \|_{L^2} \right)^{1/2} \]
\[ \lesssim M^4 t^{-1} \left( \sum_{2^j \geq t^{-1/3}} \| w_{\sim j} \|_{\dot{H}^{-1}} \right)^{1/2} \]
\[ \lesssim M^4 t^{-1} \| xg \|_{L^2} \]
as required.

6.2.6. The bound for (86a). Using the support condition for \( m_j^S \), we can decompose the each summand as
\[ T_{m_j^S} (u, \partial_x S, Lw) = T_{m_j^S} (u_{\sim j}, \partial_x S_{\leq j}, Lw) \]
\[ + T_{m_j^S} (u_{\ll j}, \partial_x S_{\ll j}, (Lw)_{\sim j}) \]
\[ + Q_{\sim j} T_{m_j^S} (u_{\ll j}, \partial_x S_{\ll j}, (Lw)_{\sim j}) \]
For (89a), we have

\[ T_m^s \left( (\chi_{[j-30,j+30]} + (1 - \chi_{[j-30,j+30]}) u_{\sim j}, \partial_x S_{\ll j}, \overline{Lw}) \right) \]

\[ = \chi_{[j-40,j+40]} T_m^s (\chi_{[j-30,j+30]} u_{\sim j}, \partial_x S_{\ll j}, \chi_{[j-40,j+40]} \overline{Lw}) \]  
\[ + T_m^s ((1 - \chi_{[j-30,j+30]}) u_{\sim j}, \partial_x S_{\ll j}, \overline{Lw}) \]  
\[ + \text{\{non-pseudolocal terms\}} \]  

(90a)

(90b)

For the first term, we use Lemma 5 to obtain the bounds

\[ \| u_{\sim j} \|_{L^\infty} \lesssim M \epsilon t^{-1/2} 2^{j/2} \]

\[ \| \partial_x S_{\ll j} \|_{L^\infty} \lesssim M \epsilon t^{-1/2} 2^{j/2} \]

so, by almost orthogonality,

\[ \left\| \sum_j (90a) \right\|_{L^2} \lesssim \left( \sum_{2^j \geq t^{-1/3}} \| T_m^s (\chi_{[j-30,j+30]} u_{\sim j}, \partial_x S_{\ll j}, \chi_{[j-40,j+40]} \overline{Lw}) \|_{L^2}^2 \right)^{1/2} \]

\[ \lesssim \left( \sum_{2^j \geq t^{-1/3}} \| u_{\sim j} \|_{L^\infty}^2 \| \partial_x S_{\ll j} \|_{L^2}^2 \| \chi_{[j-40,j+40]} \overline{Lw} \|_{L^2}^2 \right)^{1/2} \]

\[ \lesssim M^2 \epsilon^2 t^{-1} \| xg \|_{L^2} \]

as required. Turning to the second term, we note that

\[ \| (1 - \chi_{[j-30,j+30]}) u_{\sim j} \|_{L^\infty} \lesssim M \epsilon t^{-5/6} 2^{-3j} \]

so,

\[ \| (91b) \|_{L^2} \lesssim \| (1 - \chi_{[j-30,j+30]} u_{\sim j} \|_{L^\infty} \| \partial_x S_{\ll j} \|_{L^\infty} \| \overline{Lw} \|_{L^2} \]

\[ \lesssim M^2 \epsilon^2 t^{-4/3} 2^{-j} \| xg \|_{L^2} \]

which is acceptable, completing the argument for (89a).

We now consider (89b). Introducing a dyadic decomposition in space, we find that

\[ T_m^s \left( u_{\ll j}, (\chi_{[j-20,j+20]} + (1 - \chi_{[j-20,j+20]}) \partial_x S_{\sim j}, (\overline{Lw})_{\sim j}) \right) \]

\[ = T_m^s (u_{\ll j}, (1 - \chi_{[j-20,j+20]}) \partial_x S_{\sim j}, (\overline{Lw})_{\sim j}) \]  
\[ + \chi_{[j-30,j+30]} T_m^s (\chi_{[j-30,j+30]} u_{\ll j}, \chi_{[j-20,j+20]} \partial_x S_{\sim j}, (\overline{Lw})_{\sim j}) \]  
\[ + \text{\{non-pseudolocal terms\}} \]  

(91a)

(91b)

For (91a), we note that by Lemma 5

\[ \| (1 - \chi_{[j-20,j+20]}) \partial_x S_{\sim j} \|_{L^\infty} \lesssim M \epsilon t^{-5/6} 2^{-j/2} \]

so

\[ \| (91a) \|_{L^2} \lesssim \| u_{\ll j} \|_{L^\infty} \| (1 - \chi_{[j-20,j+20]}) \partial_x S_{\sim j} \|_{L^\infty} \| \overline{Lw} \|_{L^2} \]

\[ \lesssim M^2 \epsilon^2 t^{-7/6} 2^{-j/2} \| xg \|_{L^2} \]

which is sufficient. For (91b), we instead use the bound

\[ \| \partial_x S_{\sim j} \|_{L^\infty} \lesssim M \epsilon t^{-1/2} 2^{j/2} \]
which yields

$$\left\| \sum_{j} \right\|_{L^2}^{6.3.} \lesssim \left( \sum_{2^j \geq t^{-1/3}} \| T_{m_j} \chi_{[j-30,j+30]}(\xi_{\xi_{j}}, \chi_{[j-20,j+20]}(\partial_x S_{\sim j}, (Lw)_{\sim j}) \right) \|_{L^2}^{1/2}$$

$$\lesssim \left( \sum_{2^j \geq t^{-1/3}} \| \chi_{[j-30,j+30]}(\xi_{\xi_{j}}) \|_{L^\infty}^{2} \| \chi_{[j-20,j+20]}(\partial_x S_{\sim j}) \|_{L^\infty}^{2} \| (Lw)_{\sim j} \|_{L^2} \right) \|^{1/2}$$

$$\lesssim M^2 \epsilon^2 t^{-1} \left( \sum_{2^j \geq t^{-1/3}} \| (Lw)_{\sim j} \|_{L^2} \right)$$

which completes the bound for (89c).

For the final term (89c), we see immediately that

$$\left\| \sum_{j} \right\|_{L^2}^{6.3.} \lesssim \left( \sum_{2^j \geq t^{-1/3}} \| T_{m_j} \chi_{\sim j} \|_{L^\infty}^{2} \| (Lw)_{\sim j} \|_{L^\infty}^{2} \right) \|^{1/2}$$

$$\lesssim M^2 \epsilon^2 t^{-1} \| xg \|_{L^2}$$

which completes the argument for (66f).

6.3. The time non-resonant multiplier. We control the term (62c) by integrating by parts in s. Defining $m_s = \frac{i(\xi - \eta - \sigma) \partial_x \phi^{-1}}{\partial_x}$, we can integrate by parts to obtain

$$\int_{t}^{s} (62c) \, ds = \pm \Re \int_{1}^{t} (e^{-s0^2_2} xg, T_{m_s}(u, w, \bar{w}) + T_{m_s}(u, S, \bar{w}) + T_{m_s}(u, S, \bar{w})) ds$$

$$\int_{t}^{s} (62c) \, ds = \pm \Re \int_{1}^{t} (s(e^{-s0^2_2} xg, T_{m_s}(u, w, \bar{w}) + T_{m_s}(u, S, \bar{w}) + T_{m_s}(u, S, \bar{w})) ds$$

$$\int_{t}^{s} (62c) \, ds = \pm \Re \int_{1}^{t} (s(e^{-s0^2_2} xg, T_{m_s}(e^{-s0^2_2} \partial_x f, w, \bar{w}) + T_{m_s}(e^{-s0^2_2} \partial_x g) + T_{m_s}(e^{-s0^2_2} \partial_x h, \bar{w})$$

$$\int_{t}^{s} (62c) \, ds = \pm \Re \int_{1}^{t} \{ \text{similar terms} \}$$

Note that the $m_s$ satisfy Coifman-Meyer type bounds uniformly in time:

$$\| (\xi^2 + \eta^2 + \sigma^2)^{|\alpha|/2} \partial_x \phi_{\xi, \eta, \sigma m_t} \| \lesssim 1$$

(93)
Thus, by multiplying by the cut-off $\psi\left(\frac{x^2+y^2+\sigma^2}{2}\right)$ we can decompose pseudoproducts involving $m_t$ as

$$T_{m_t}(p, q, r) = \sum_{2^i \geq t^{-1/3}} T_{m_t}(p_{\leq i}, q_{\leq i}, r_{\leq i})$$

(94)

where $m_t$ stands for a generic symbol localized to $|\xi| + |\eta| + |\sigma| \sim 2^j$ and satisfying $|\partial^\alpha_{\xi,\eta,\sigma} m_t| \lesssim 2^{-|\alpha|}$. By Remark 7, all of the pseudoproducts on the right obey Hölder type bounds.

6.3.1. The bound for (82). We will first give the argument for the boundary $s = t$. Since none of the frequencies $\eta, \sigma$ and $\xi - \eta - \sigma$ play a distinguished role, and since $S$ and $u$ obey the same decay estimates, it suffices to prove to obtain bounds for $T_{m_t}(u, w, \overline{u})$. Using (94), we find that

$$T_{m_t}(u, w, \overline{u}) = \sum_{2^i \geq t^{-1/3}} T_{m_t}(u_{\sim i}, w_{\leq i}, \overline{u}_{\sim i})$$

(95a)

$$+ T_{m_t}(u_{\sim i}, w_{\leq i}, \overline{u}_{\sim i})$$

(95b)

$$+ T_{m_t}(u_{\leq i}, w_{\sim i}, \overline{u}_{\sim i})$$

(95c)

$$+ Q_{\sim i} T_{m_t}(u_{\leq i}, w_{\sim i}, \overline{u}_{\sim i})$$

(95d)

The terms (95b) and (95c) are controlled using the same arguments as for (80a):

$$\| (95a) \|_{L^2} + \| (95b) \|_{L^2} \lesssim M^2 e^2 t^{-1} \|xg\|_{L^2}$$

Similarly, using the same argument as for (80a), we find that

$$\| (95d) \|_{L^2} \lesssim M^2 e^2 t^{-1} \|xg\|_{L^2}$$

It only remains to bound (95a). Here, we have that

$$\| (95a) \|_{L^2} \lesssim \left( \sum_{2^i \geq t^{-1/3}} \| T_{m_t}(\chi_{|j-30, j+30|} u_{\sim i}, \chi_{|j-30, j+30|} w_{\leq i}, \overline{u}_{\sim i})\|_{L^2}^2 \right)^{1/2}$$

(96a)

$$\lesssim \left( \sum_{2^i \geq t^{-1/3}} \| u_{\sim i} \|_{L^2} \| \chi_{|j-30, j+30|} w_{\leq i} \|_{L^2}^2 \right)^{1/2}$$

(96b)

For (96a), observe that the same reasoning as in (54) gives us the bound

$$\| \chi_{|j-30, j+30|} w_{\leq i} \|_{L^2} \lesssim 2^j \left( \| Q_{|j-70, j+70|}(xg)\|_{L^2} + t^{-1/3} 2^{-j} \|xg\|_{L^2} \right)$$

(97)

so we can use almost orthogonality to obtain

$$\| (96a) \|_{L^2} \lesssim \left( \sum_{2^i \geq t^{-1/3}} \| T_{m_t}(\chi_{|j-20, j+20|} u_{\sim i}, \chi_{|j-30, j+30|} w_{\leq i}, \overline{u}_{\sim i})\|_{L^2}^2 \right)^{1/2}$$

$$\lesssim \left( \sum_{2^i \geq t^{-1/3}} \| u_{\sim i} \|_{L^2} \| \chi_{|j-30, j+30|} w_{\leq i} \|_{L^2}^2 \right)^{1/2}$$

$$\lesssim M^2 e^2 t^{-1} \left( \sum_{2^i \geq t^{-1/3}} \left( \| Q_{|j-70, j+70|}(xg)\|_{L^2} + t^{-1/3} 2^{-j} \|xg\|_{L^2} \right) \right)^{1/2}$$

$$\lesssim M^2 e^2 t^{-1} \|xg\|_{L^2}$$
Turning to (96b), we see that
\[
\| (1 - \chi_{[j-20,j+20]}) u_{\sim j} \|_{L^\infty} \lesssim M t^{-5/6} 2^{-3/2j}
\]
so
\[
\| (96b) \|_{L^2} \lesssim \sum_{2^j \geq t^{-1/3}} \| (1 - \chi_{[j-20,j+20]}) u_{\sim j} \|_{L^\infty} \| w_{\leq j} \|_{L^2} \| u_{\sim j} \|_{L^\infty}
\]
\[
\lesssim M^2 \epsilon^2 t^{-4/3} \sum_{2^j \geq t^{-1/3}} 2^{-j} \| u \|_{H^{-1}}
\]
\[
\lesssim M^2 \epsilon^2 t^{-1} \| xg \|_{L^2}
\]
Combining these estimates and using Cauchy-Schwarz, we see that
\[
|\Im \langle e^{-t \partial^3_x} x, T_m(u, w, \overline{w}) + T_m(w, S, \overline{S}) + T_m(w, S, \overline{\pi}) \rangle | \lesssim M^2 \epsilon^2 \| xg \|_{L^2}^2
\]
Repeating the above arguments at \( s = 1 \) and recalling that \( xg(1) = xu_\ast \), we see that
\[
\| (92a) \| \lesssim M^2 \epsilon^2 \| xg \|_{L^2}^2 + M^2 \epsilon^2 \| xu_\ast \|_{L^2}^2
\]
Since \( \epsilon \ll M^{-3/2} \), the first term can be absorbed into the left-hand side of (55), and the second term is better than required, since \( M^2 \epsilon^2 \ll 1 \).

6.3.2. The bound for (92b). By repeating the arguments from above, we have at once that
\[
\| (92b) \| \lesssim \int_1^t M^2 \epsilon^2 s^{-1} \| xg(s) \|_{L^2}^2 \, ds
\]
which is acceptable.

6.3.3. The bound for (92c). A simple computation shows that \( s \partial_s m_s \) also obeys symbols bounds of the form (93), so \( (92c) \) can be controlled in the same way as \( (92b) \).

6.3.4. The bound for (92d). By differentiating \( f, g, \) and \( h \) in time, we find that
\[
e^{-s \partial^3_x} \partial_s f = |u|^2 \partial_x u
\]
\[
e^{-s \partial^3_x} \partial_s h = |S|^2 \partial_x S + D_p S \partial_t \hat{u}(0, t)
\]
\[
e^{-s \partial^3_x} \partial_s g = |u|^2 \partial_x w + (u \overline{w} + u \overline{w}) \partial_x S - D_p S \partial_t \hat{u}(0, t)
\]
Thus, we can write
\[
\| (92d) \| \lesssim \int_1^t \| xg \|_{L^2} \| G(s) \|_{L^2} \, ds
\]
where
\[
G(s) = \sum_j T_{m_j}(|u|^2 \partial_x u), w, \overline{w})
\]
\[
+ T_{m_j}(u, D_p S \hat{u}(0, t), \overline{w})
\]
\[
+ T_{m_j}(u, (|u|^2 \partial_x w), \overline{w})
\]
\[
+ T_{m_j}(u, (u \partial_x S \overline{w}), \overline{w})
\]
\[
+ \{ \text{similar terms} \}
\]
Note that the desired bound on (92d) would follow if we could show that
\[
\| G(s) \|_{L^2} \lesssim M^2 \epsilon^2 s^{-2} \| xg \|_{L^2} + M^2 \epsilon^3 s^{-5/6 - \beta}
\]
so it suffices to bound the quantities \((98a)\)–\((98d)\) in \(L^4\). For \((98a)\), we can write

\[
T_{m_j} \left( \|u\|^2 \partial_x u \right)_{\leq j, w} = Q_j T_{m_j} \left( \|u\|^2 \partial_x u \right)_{\leq j, w} + T_{m_j} \left( \|u\|^2 \partial_x u \right)_{\leq j, \pi} + T_{m_j} \left( \|u\|^2 \partial_x u \right)_{\leq j, \chi} + T_{m_j} \left( \|u\|^2 \partial_x u \right)_{\leq j, \bar{\pi}} \tag{99a}
\]

The term \((99a)\) is similar to \((80a)\): we write

\[
(99a) = Q_j T_{m_j} \left( \chi_{\leq -30} \left( \|u\|^2 \partial_x u \right)_{\leq j, w} \chi_{\leq -20} w \right) \tag{99b}
\]

\[
+ Q_j T_{m_j} \left( \chi_{\geq -30} \left( \|u\|^2 \partial_x u \right)_{\leq j, w} \chi_{\geq -40} \bar{\pi} \right) \tag{99c}
\]

\[
+ \{\text{non-pseudolocal terms}\} \tag{99d}
\]

Using \((87)\) to control the cubic \((\|u\|^2 \partial_x u)_{\leq j}\) and \((82)\) to control \(\chi_{\leq -20} w \), we find that

\[
\left\| \sum_j (100a) \right\|_{L^2} \lesssim \left( \sum_{2^j \geq s-1/3} \| T_{m_j} \left( \chi_{\leq -30} \left( \|u\|^2 \partial_x u \right)_{\leq j} \chi_{\leq -20} w \right) \|_{L^2}^2 \right)^{1/2}
\]

\[
\lesssim \left( \sum_{2^j \geq s-1/3} \| \left( \|u\|^2 \partial_x u \right)_{\leq j} \|_{L^\infty}^2 \| \chi_{\leq -20} w \|_{L^2}^2 \| \bar{\pi} \|_{L^\infty}^2 \right)^{1/2}
\]

\[
\lesssim M^4 \epsilon^4 s^{-2} \left( \sum_{2^j \geq s-1/3} \| Q_{(j-60, j+60)} \|_{L^2}^2 \right)^{1/2}
\]

\[
\lesssim M^4 \epsilon^4 s^{-2} \| x g \|_{L^2}^2
\]
as required. Similarly, if we use \((87)\) to control \(\chi_{\geq -30} \left( \|u\|^2 \partial_x u \right)_{\leq j}\), we find that

\[
\left\| \sum_j (100a) \right\|_{L^2} \lesssim \left( \sum_{2^j \geq s-1/3} \| T_{m_j} \left( \chi_{\geq -30} \left( \|u\|^2 \partial_x u \right)_{\leq j} w \right) \|_{L^2}^2 \right)^{1/2}
\]

\[
\lesssim \left( \sum_{2^j \geq s-1/3} \| \chi_{\geq -30} \|_{L^\infty}^2 \| \chi_{\geq -40} \bar{\pi} \|_{L^\infty}^2 \right)^{1/2}
\]

\[
\lesssim M^4 \epsilon^4 s^{-2} \left( \sum_{2^j \geq s-1/3} \| w \|_1^2 \right)^{1/2}
\]

\[
\lesssim M^4 \epsilon^4 s^{-2} \| x g \|_{L^2}^2
\]

which gives us the required bound for \((99a)\).

Turning to \((99b)\), we find that

\[
(99b) = T_{m_j} \left( \|u\|^2 \partial_x u \right)_{\leq j, w} \left( \chi_{[j-30, j+30]} \right) \tag{101a}
\]

\[
+ \chi_{[j-40, j+40]} T_{m_j} \left( \|u\|^2 \partial_x u \right)_{\leq j, w} \chi_{[j-30, j+30]} \tag{101b}
\]

\[
+ \{\text{non-pseudolocal terms}\}
\]
For (101a), the linear and cubic dispersive estimates give us the bounds
\[
\| (101a) \|_{L^2} \lesssim \| (|u|^2 \partial_x u) \|_{L^\infty} \| w_{\sim j} \|_{L^2} \| (1 - \chi_{|j-30,j+30|}) \|_{L^\infty} \\
\lesssim M^4 \epsilon^4 s^{-13/6} 2^{-j/2} \| u \|_{\dot{H}^{-1}} \\
\lesssim M^4 \epsilon^4 s^{-13/6} 2^{-j/2} \| xg \|_{L^2}
\]
which gives the required bound after summing in \( j \). For the second term, we have
\[
\left\| \sum_{j} (101a) \right\|_{L^2} \lesssim \left( \sum_{2^j \geq s^{-1/3}} \| T_{m_j} (|u|^2 \partial_x u) \|_{L^2} \right)^{1/2} \\
\lesssim \left( \sum_{2^j \geq s^{-1/3}} \| \chi_{|j-40,j+40|} (|u|^2 \partial_x u) \|_{L^\infty} \| w_{\sim j} \|_{L^2} \| u_{\sim j} \|_{L^\infty} \right)^{1/2} \\
\lesssim M^4 \epsilon^4 s^{-2} \left( \sum_{2^j \geq s^{-1/3}} \| w_{\sim j} \|_{\dot{H}^{-1}} \right)^{1/2} \\
\lesssim M^4 \epsilon^4 s^{-2} \| xg \|_{L^2}
\]
which is sufficient, completing the bound for (99c).

Similarly, for (99c), we have that
\[
(99c) = T_{m_j} \left( (|u|^2 \partial_x u) \chi_{|j-30,j+30|} \right) \\
+ \chi_{|j-30,j+30|} T_{m_j} \chi_{|j-40,j+40|} \chi_{|j-30,j+30|} \chi_{|j-30,j+30|} \\
+ \{ \text{non-pseudolocal terms} \}
\]

The estimate for (102a) is analogous to the one for (101a). For (102b), we find
\[
\left\| \sum_{j} (102b) \right\|_{L^2} \lesssim \left( \sum_{2^j \geq s^{-1/3}} \| T_{m_j} \chi_{|j-40,j+40|} (|u|^2 \partial_x u) \|_{L^2} \right)^{1/2} \\
\lesssim \left( \sum_{2^j \geq s^{-1/3}} \| \chi_{|j-40,j+40|} (|u|^2 \partial_x u) \|_{L^\infty} \| \chi_{|j-40,j+40|} \chi_{|j-30,j+30|} \chi_{|j-30,j+30|} \|_{L^\infty} \right)^{1/2} \\
\lesssim M^4 \epsilon^4 s^{-2} \left( \sum_{2^j \geq s^{-1/3}} \left( \| Q_{|j-80,j+80|} (xg) \|_{L^2} + s^{-1/3} 2^{-j} \| xg \|_{L^2} \right)^2 \right)^{1/2} \\
\lesssim M^4 \epsilon^4 s^{-2} \| xg \|_{L^2}
\]
where we have used \( S_{31} \) to control \( \chi_{|j-40,j+40|} \). This finishes the bound for (99c).
Finally, for (99a), we find that
\[
\chi_{30}(u^2 \partial_x u)_{j-30} \chi_{k} w_{j} \chi_{m} w_{j} = \chi_{30}(u^2 \partial_x u)_{j-30} \chi_{k} w_{j} \chi_{m} w_{j} \chi_{30}(u^2 \partial_x u)_{j-30} \chi_{k} w_{j} \chi_{m} w_{j} 
\]
(103a)
\[
+ \chi_{j-40,j+40} T_m \chi_{j-30,j+30} (u^2 \partial_x u)_{j-30} w_{j}, \chi_{j-40,j+40} \chi_{m} w_{j} 
\]
(103b)
\[
+ T_m (\chi_{j+30} (u^2 \partial_x u)_{j-30} w_{j}, \chi_{j+30} \chi_{m} w_{j}) 
\]
(103c)
\[
+ \{\text{non-pseudolocal terms} \}
\]
For (103a), the refined cubic estimate (40) gives us the bound
\[
\| \chi_k (u^2 \partial_x u)_{j} \|_{L^\infty} \lesssim M^3 s^{-11/6} 2^{-j/2} -k
\]
and, by Corollary 8
\[
\| \chi_{30} w_{j} \|_{L^2} \lesssim \| \chi_{k-20,k+20} \|_{L^2} \| \chi_{30} w_{j} \|_{L^\infty} \lesssim 2^k \| xg \|_{L^2}
\]
so,
\[
\| (103a) \|_{L^2} \lesssim \sum_{k < j - 30} \| T_m \chi_k (u^2 \partial_x u)_{j-30} w_{j}, \chi_{m} w_{j} \|_{L^2} \lesssim \sum_{k < j - 30} \| \chi_k (u^2 \partial_x u)_{j-30} \|_{L^\infty} \| \chi_{30} w_{j} \|_{L^2} \| \chi_{30} w_{j} \|_{L^\infty} \lesssim M^4 s^{-11/6} 2^{-j} \| xg \|_{L^2}
\]
which is sufficient. Turning to (103b), we note that the same reasoning as in (84),
\[
\| \chi_{j-40,j+40} w_{j} \|_{L^2} \lesssim 2^j \left( \| Q_{j-80,j+80} (xg) \|_{L^2} + t^{-1/3} 2^{-j} \| xg \|_{L^2} \right)
\]
so we can use almost orthogonality to obtain
\[
\left\| \sum_j (103b) \right\|_{L^2} \lesssim \left( \sum_{2^j > s^{-1/3}} \| T_m \chi_{j-30,j+30} (u^2 \partial_x u)_{j-30} \|_{L^2} \| \chi_{j-40,j+40} \|_{L^2} \| \chi_{j-40,j+40} \|_{L^\infty} \right)^{1/2}
\]
\[
\lesssim \left( \sum_{2^j > s^{-1/3}} \| (u^2 \partial_x u)_{j-30} \|_{L^\infty} \| \chi_{j-40,j+40} \|_{L^2} \| \chi_{j-40,j+40} \|_{L^\infty} \right)^{1/2}
\]
\[
\lesssim M^4 s^{-2} \left( \sum_{2^j > s^{-1/3}} \left( \| Q_{j-80,j+80} (xg) \|_{L^2} + t^{-1/3} 2^{-j} \| xg \|_{L^2} \right)^2 \right)^{1/2}
\]
\[
\lesssim M^4 s^{-2} \| xg \|_{L^2}
\]
Finally, for (103c), we use the bound \| \chi_{j+20} w_{j} \|_{L^\infty} \lesssim M s^{-5/6} 2^{-3j} \] to conclude that
\[
\left\| (103c) \right\|_{L^2} \lesssim \| (u^2 \partial_x u)_{j-30} \|_{L^\infty} \| \chi_{j+20} w_{j} \|_{L^2} \chi_{j+20} w_{j} \|_{L^\infty} \lesssim M^4 s^{-4/3} 2^{-j} \| xg \|_{L^2}
\]
which is sufficient, completing the argument for (99a). Combining the estimates for (99a–99c), we obtain the required estimate for (98a).
We will also need refined estimates for \( \chi \) these bounds, we note that, after performing a Littlewood-Paley decomposition, we have

\[
\begin{align*}
T_{m_j} (u_{<j}, Q_{<j} D_p S \partial_t \hat{u}(0,t), \varpi_{<j}) &= T_{m_j} (u, Q_{<j} D_p S \partial_t \hat{u}(0,t), \varpi) \\
&+ T_{m_j} (u_{<j}, Q_{<j} D_p S \partial_t \hat{u}(0,t), \varpi_{<j})
\end{align*}
\]

(104a)

By taking the supremum over the dyadic regions, we find that

\[
\sup_{2^j \geq s-t^{1/3}} 2^{-j/6} \sum_{2^k \geq t^{1/3}} \ln(2 + t^{-1/3} 2^j) 2^{-j/6}
\]

\[
\lesssim M^5 \epsilon s^{17/9 - \beta} \lesssim M^5 \epsilon s^{-11/6 - \beta}
\]

which is better than required, since \( \epsilon \ll M^{-3/2} \). For (104b) and (104c), similar reasoning using (26) and (48) gives

\[
\left\| \sum_j (104b) + (104c) \right\|_{L^2} \lesssim M^5 \epsilon s^{-11/6 - \beta}
\]

which yields the desired bound for (98).

Let us now consider the term (98). The estimates for (98) require us to obtain bounds for cubic expressions of the form \(|u|^2 \partial_x w\). Since \( w \not\in X \), these bounds do not immediately follow from the work in Section 3.2 so we instead argue directly. By applying Corollary 8 to the terms \( \partial_x Q_j w \), we can see that

\[
|Q_j \partial_x w| \lesssim \left( s^{-1/2} 2^{j} \chi_{<j} + s^{-5/6} 2^{3j/2} \left( 2^{j/2} + 2^{-j/3} |\xi_0|^{4/3} \right)^{-3/2} \right) \|Q_{|j-30,j+30|}(xg)\|_{L^2}
\]

(105)

so

\[
\|\chi_k \partial_x w\|_{L^\infty} \lesssim \sum_{2^j \geq s-t^{1/3}} \|\chi_k Q_j \partial_x w\|_{L^\infty}
\]

(106)

\[
\lesssim s^{-1/2} 2^{j/2} \left( \|Q_{|k-40,k+40|}(xg)\|_{L^2} + s^{-1/3} 2^{-k} \|xg\|_{L^2} \right)
\]

Since \( \|\chi_k u\|_{L^\infty} \lesssim M s^{-1/2} 2^{-k/2} \), we see that

\[
\|\chi_k |u|^2 \partial_x w\|_{L^\infty} \lesssim M^2 s^{-3/2} \left( \|Q_{|k-40,k+40|}(xg)\|_{L^2} + s^{-1/3} 2^{-k}\|xg\|_{L^2} \right)
\]

(107)

By taking the supremum over the dyadic regions, we find that

\[
\| |u|^2 \partial_x w\|_{L^\infty} \lesssim \sup_k \|\chi_k |u|^2 \partial_x w\|_{L^\infty}
\]

(108)

\[
\lesssim M^2 \epsilon^2 2^{-3/2} \|xg\|_{L^2}
\]

We will also need refined estimates for \( \chi_k |u|^2 \partial_x w\) when \( k < j - 30 \) in the spirit of (35). To obtain these bounds, we note that, after performing a Littlewood-Paley decomposition, we have

\[
\chi_k (|u|^2 \partial_x w)_{<j} = \chi_k \sum_{k-j} \chi_k (|u|^2 \partial_x w)_{<j} + \{\text{non-pseudolocal terms}\}
\]

\[
\lesssim M^2 \epsilon^2 2^{-3/2} \|xg\|_{L^2}
\]
Here, \{\text{better}\} denotes terms which decay more rapidly on the support of \(\chi_k\) and thus can be neglected. Using (103), we see that
\[
\|\chi_k u_{<j-20}^2 \partial_x w_{j-20,j+20}\|_{L^\infty} \lesssim M^2 c^2 s^{-11/6} 2^{-k} \|Q_{[j-40,j+40]}(xg)\|_{L^2}
\]
and
\[
\|\chi_k u_{<j-20} \partial_x w_{j-20}\|_{L^\infty} \lesssim M^2 c^2 s^{-11/6} 2^{k/2-3/2} \|xg\|_{L^2}
\]
so
\[
\|\chi_k (|u|^2 \partial_x w)_{<j}\|_{L^\infty} \lesssim M^2 c^2 s^{-11/6} 2^{-k} \left(\|Q_{[j-40,j+40]}(xg)\|_{L^2} + 2^{3/2(k-j)} \|xg\|_{L^2}\right) 
\] (109)
Finally, we record some \(L^4\) bounds for \(u\), all of which are easily derived from Lemma [5]
\[
\|u_{<j}\|_{L^4} + \|\chi_{[j-C,j+C]} u_{<j}\|_{L^4} + \|\chi_{[j-C,j+C]} u_{<j}\|_{L^4} \lesssim C M \epsilon s^{-1/4}
\]
(110)
\[
\|u_{<j}\|_{L^4} + \|\chi_{[j-20,j+20]} u_{<j}\|_{L^4} \lesssim C M \epsilon s^{-7/12} 2^{-j}
\]
Now, let us write
\[
T_m \partial_j (u, |u|^2 \partial_x w, u) = T_m \partial_j (u_{<j}, |u|^2 \partial_x w, u_{<j}) + T_m \partial_j (u_{<j}, |u|^2 \partial_x w, u_{<j}) + Q_{<j} T_m \partial_j (u_{<j}, (|u|^2 \partial_x w)_{<j}, u_{<j})
\]
For the term (111a), we write
\[
(111a) = \chi_{[j-40,j+40]} T_m \partial_j (u_{<j}, \chi_{[j-40,j+40]} |u|^2 \partial_x w, \chi_{[j-30,j+30]} u_{<j}) + T_m \partial_j (1 - \chi_{[j-20,j+20]} u_{<j}, |u|^2 \partial_x w, (1 - \chi_{[j-30,j+30]} u_{<j}) + \{\text{non-pseudolocal terms}\}
\]
For the subterm (112a), the \(L^4\) estimates (110) together with (108) imply
\[
\|112a\|_{L^2} \lesssim (1 - \chi_{[j-20,j+20]} u_{<j}\|_{L^4} \|u|^2 \partial_x w\|_{L^\infty} \|u_{<j}\|_{L^4} \lesssim M^4 c^4 s^{-13/6} 2^{-j/2} \|xg\|_{L^2}
\]
which is acceptable. Turning to the subterm (112a), we use (107) to find that
\[
\sum_j_{L^2} \lesssim \left( \sum_{2j \geq s^{-1/3}} \|u_{<j}\|_{L^4} \|\chi_{[j-40,j+40]} u_{<j}\|_{L^2} \|\chi_{[j-30,j+30]} u_{<j}\|_{L^2} \right)^{1/2}
\]
\[
\lesssim M^4 c^4 s^{-2} \left( \sum_{2j \geq s^{-1/3}} \|Q_{[k-80,k+80]}(xg)\|_{L^2} + s^{-1/3} 2^{-k} \|xg\|_{L^2} \right)^{1/2}
\]
\[
\lesssim M^4 c^4 s^{-2} \|xg\|_{L^2}
\]
which completes the bound for (111a). The bound for (111b) is identical, since none of the estimates from (110) change if we replace $u_{\lesssim J}$ with $u_{\lesssim j}$. For (111c), we have

\[
(111c) = Q_{\sim j} \sum_{k<j-30} T_m^{(7)} (\chi_{\sim k} u_{\ll j}, \chi_{\sim k} u_{\ll j}) + Q_{\sim j} T_m^{(7)} (\chi_{[j-40,j+40]} u_{\ll j}, \chi_{[j-30,j+30]} (|u|^2 \partial_x w)_{\sim j}, \chi_{[j-40,j+40]} u_{\ll j}) + \{\text{non-pseudolocal terms}\}
\]

For the first subterm, we use (109) and (110) to conclude that

\[
(113a) \quad \left( \sum_{j \geq s^{1/3}} \left( \sum_{k<j-30} T_m^{(7)} (\chi_{\sim k} u_{\ll j}, \chi_{\sim k} u_{\ll j}) \right)^2 \right)^{1/2} \lesssim M^4 \epsilon^4 s^{-7/3} \left( \sum_{2 \geq s^{1/3}} \sum_{s-1/3 < k < 2s} 2^{-k} \| Q_{[j-40,j+40]} (xg) \|_{L^2} + 2^{k/2 - 3/2j} \| xg \|_{L^2} \right)^{1/2} \lesssim M^4 \epsilon^4 s^{-2} \| xg \|_{L^2}
\]

as required. For the second term, we use the bound (107) to control $\chi_{[j-30,j+30]} |u|^2 \partial_x w$, yielding

\[
\left( \sum_{j \geq s^{1/3}} \left( \sum_{2 \geq s^{1/3}} \| \chi_{[j-40,j+40]} u_{\ll j} \|_{L^4}^4 \| \chi_{[j-30,j+30]} (|u|^2 \partial_x w)_{\sim j} \|_{L^\infty}^2 \right)^{1/2} \lesssim M^4 \epsilon^4 s^{-2} \left( \sum_{2 \geq s^{1/3}} \left( \| Q_{[j-70,j+70]} (xg) \|_{L^2} + s^{-1/3} 2^{-j} \| xg \|_{L^2} \right)^2 \right)^{1/2} \lesssim M^4 \epsilon^4 s^{-2} \| xg \|_{L^2}
\]

Finally, for the last term, we use the $L^\infty$ bound for $|u|^2 \partial_x w$ together with the improved bounds for $\chi_{[j-40,j+40]} u_{\ll j}$ given in (110) to conclude that

\[
\left( \sum_{j \geq s^{1/3}} \left( \sum_{2 \geq s^{1/3}} \| \chi_{[j-40,j+40]} u_{\ll j} \|_{L^4}^4 \| \chi_{[j-30,j+30]} (|u|^2 \partial_x w)_{\sim j} \|_{L^\infty}^2 \right)^{1/2} \lesssim M^4 \epsilon^4 s^{-8/3} 2^{-2j} \| xg \|_{L^2}
\]

which is sufficient, completing the bound for (111c).

The argument for (110d) is quite similar. We have that

\[
T_m \left( u_{\lesssim j}, u_{\ll j} \partial_x S \ll j \right) = T_m \left( u_{\lesssim j}, u_{\ll j} \partial_x S \ll j \right) + T_m \left( u_{\ll j}, u_{\ll j} \partial_x S \ll j \right) + Q_{\sim j} T_m \left( u_{\ll j}, (u \partial_x S)_{\lesssim j} \right)
\]
Now, the decay estimates given in Lemma 6 and Corollary 8 immediately imply that
\[ \| \chi_k u_\partial S \|_{L^\infty} \lesssim M^2 \epsilon^2 s^{-3/2} \left( \| Q_{[k-40,k+40]}(xg) \|_{L^2} + s^{-1/3} 2^{-k} \| xy \|_{L^2} \right) \]
and
\[ \| u_\partial S \|_{L^\infty} \lesssim M^2 \epsilon^2 s^{-3/2} \| xy \|_{L^2} \]
which are analogous to the estimates (1093) and (1103). Thus, we can control (114a) and (114b) by the same arguments used for (111a) and (111b), respectively. Moreover, we see that for \( k < j - 30 \), we can write
\[ \chi_k (u_\partial S \sim_j = \chi_k Q \sim_j \chi_k \left( u_{<j-20} \partial S_{<j-20} w_{<j-20} + \{ \text{better} \} \right) \]
\[ + \chi_k Q \sim_j \chi_k \left( u_{<j-20} \partial S_{<j-20} + u_{<j-20,j+20} \partial S_{<j-20} w_{<j-20} + \{ \text{better} \} \right) \]
\[ + \{ \text{non-pseudoplocal terms} \} \]
where \{ \text{better} \} denotes terms which decay faster on the support of \( \chi_k \). For the first term, Corollary 6 and Corollary 8 give the bound
\[ \| \chi_k u_{<j-20} \partial S_{<j-20} w_{<j-20} \|_{L^\infty} \lesssim M^4 \epsilon^4 s^{-11/6} 2^{-j} \| Q_{[j-50,j+50]}(xg) \|_{L^2} \]
and similarly, using Lemma 6,
\[ \| \chi_k u_{<j-20} \partial S_{[j-20,j+20]} w_{<j-20} \|_{L^\infty} \lesssim M^4 \epsilon^4 s^{-11/6} 2^{-k/2-j/2} \left( \| Q_{[k-40,k+40]}(xg) \|_{L^2} + s^{-1/3} 2^{-k} \| xy \|_{L^2} \right) \]
with an identical estimate holding for \( u_{[j-20,j+20]} \partial S_{<j-20} w_{<j-20} \). Thus,
\[ \| \chi_k (u_\partial S \sim_j \|_{L^\infty} \lesssim M^4 \epsilon^4 s^{-11/6} 2^{-k/2-j/2} \| Q_{[k-40,k+40]}(xg) \|_{L^2} + s^{-1/3} 2^{-3k/2-j/2} \| xy \|_{L^2} \]
\[ + 2^{-j} \| Q_{[j-50,j+50]}(xg) \|_{L^2} \]
which is better than (1109), since \( k < j - 30 \). Thus, the term (114c) can be controlled in the same manner as (111c), which completes the argument for (986). Combining all these bounds, we find that
\[ \| G(s) \|_{L^2} \lesssim M^4 \epsilon^4 s^{-2} \| xy \|_{L^2} + M^5 \epsilon^5 t^{-11/6 - \beta} \]
which is better than required, since \( \epsilon \ll M^{-3/2} \).

6.3.5. The bound for (92a). To control the term (92a), we must take advantage of cancellations. To simplify notation, let use write
\[ T(s) = T_m(w, u) + T_m(u, S, \overline{w}) + T_m(w, S, \overline{w}) \]
Expanding the derivative using (57) gives
\[ (92a) = \pm \Re \int_1^{t} s \left\{ \frac{1}{2} \| u_\partial Lw \|_{L^2} + \Re \int_1^{t} s \left\{ \| u_\partial \|_{L^2} + 2 \Re(u \overline{w}) \right. \]
If we expand $\partial_x \phi e^{is\phi}$ using (61) and observe that $i(\xi - \eta - \sigma) \partial_x \phi x^T = \phi m_s$, we can write

$$\tag{92a} = \pm \Im \int_1^t s^2 \left< e^{-s\partial_x^2} (T_{\phi m_s e^{is\phi}}(f, g) \overline{T}) + T_{\phi m_s e^{is\phi}}(f, h, \overline{g}) + T_{\phi m_s e^{is\phi}}(g, h, \overline{T}) \right> \, ds$$

$$\tag{115a} \mp \Re \int_1^t s \langle |u|^2 \partial_x Lw, T(s) \rangle \, ds$$

$$\tag{115b} \mp \Re \int_1^t s \langle H(s), T(s) \rangle \, ds$$

$$\tag{115c} \mp \Re \int_1^t s \langle LD_p S \partial_s \varepsilon(0, s), T(s) \rangle \, ds$$

where

$$H(s) = e^{-s\partial_x^2} (T_{\partial_x \phi e^{is\phi} x} (f, \partial_x g, \overline{T}) + T_{\partial_x \phi e^{is\phi} x} (f, \partial_x h, \overline{g}) + T_{\partial_x \phi e^{is\phi} x} (g, \partial_x h, \overline{T})$$

$$+ e^{-s\partial_x^2} (T_{\partial_x \phi e^{is\phi} x} (f, \partial_x g, \overline{T}) + T_{\partial_x \phi e^{is\phi} x} (f, \partial_x h, \overline{g}) + T_{\partial_x \phi e^{is\phi} x} (g, \partial_x h, \overline{T})$$

$$+ |u|^2 w + 2 \Re(u\overline{w}) S + 2 \Re(u\overline{w}) L S$$

Let us first consider the term (115c). Note that the previous arguments allow us to bound each term in $H(s)$ in $L^2$ pointwise in time, yielding the bound

$$\|H(s)\|_{L^2} \lesssim M^2 e^2 s^{-1} \|xg(s)\|_{L^2}$$

Similarly, the bound obtained for (92a) in Section 6.3.1 allows us to conclude that

$$\|T(s)\|_{L^2} \lesssim M^2 e^2 s^{-1} \|xg(s)\|_{L^2}$$

so,

$$\tag{115b} \lesssim \int_1^t M^4 e^4 s^{-1} \|xg(s)\|_{L^2}^2 \, ds$$

which is better than required. Similar reasoning using (69) shows that

$$\tag{115c} \lesssim \int_1^t M^7 e^7 s^{-5/6-\beta} \|xg(s)\|_{L^2} \, ds$$

which is again better than required.

Turning to the term (115d), we note that

$$\left< |u|^2 \partial_x Lw, T(s) \right> = - \left< Lw, \partial_x (|u|^2 T(s)) \right>$$

so the desired bound will follow immediately if we can show that

$$\|\partial_x (|u|^2 (T(s)))\|_{L^2} \lesssim M^4 e^4 s^{-2} \|xg\|_{L^2}$$

Using Corollary 6 and the estimates in Section 6.3.1, we see that

$$\|\partial_x |u|^2 T(s)\|_{L^2} \lesssim M^4 e^4 s^{-2} \|xg\|_{L^2}$$

so it suffices to prove that

$$\|u|^2 \partial_x T(s)\|_{L^2} \lesssim M^4 e^4 s^{-2} \|xg\|_{L^2}$$
We will show how to obtain the bound for $|u|^2 \partial_x T_{m_j}(u, w, \pi)$: the bounds for the other terms are similar. Recalling that $m_x = \frac{i(\xi - \eta - \sigma)\partial_x \phi x}{\phi}$, we can write

$$
|u|^2 \partial_x T_{m_j}(u, w, \pi) = \sum_{2^j \geq s^{-1/3}} |u|^2 T_{m_j}^\tau(u, \partial_x w, \pi)
$$

$$
= \sum_{2^j \geq s^{-1/3}} |u|^2 T_{m_j}^\tau(u_{\sim j}, \partial_x w_{\lesssim j}, \pi)
$$

$$
+ |u|^2 T_{m_j}^\tau(u_{\ll j}, \partial_x w_{\ll j}, \pi_{\ll j})
$$

$$
+ |u|^2 Q_{\sim j} T_{m_j}^\tau(u_{\ll j}, \partial_x w_{\sim j}, \pi_{\ll j})
$$

(116a)

For (116a), we note that

$$
|u|^2 T_{m_j}^\tau(u_{\sim j}, \partial_x w_{\lesssim j}, \pi) = \sum_{k \leq j - 30} \chi_k |u|^2 T_{m_j}^\tau(\chi_{\sim k} u_{\sim j}, \chi_{\sim k} \partial_x w_{\lesssim j}, \chi_{\sim k} \pi)
$$

$$
+ \chi_{j - 30, j + 30}[|u|^2 T_{m_j}^\tau(u_{\sim j}, \chi_{[j - 40, j + 40]} \partial_x w_{\lesssim j}, \chi_{[j - 40, j + 40]} \pi)]
$$

$$
+ \chi_{j + 30}[|u|^2 T_{m_j}^\tau(u_{\gg j}, \chi_{j + 20} \partial_x w_{\lesssim j}, \chi_{j + 20} \pi)]
$$

$$
+ \{\text{non-pseudolocal terms}\}
$$

(117a)

For the first sub-term, we use (106) to control $\chi_{\sim k} \partial_x w$ and find that

$$
\left\|\left[\sum_j \chi_{k \leq j - 30} |u|^2 T_{m_j}^\tau(\chi_{\sim k} u_{\sim j}, \chi_{\sim k} \partial_x w_{\lesssim j}, \chi_{\sim k} \pi)\right]\right\|_{L^2} \lesssim \sum_{k \leq j - 30} \left\|\chi_k \left\|\chi_{\sim k} u_{\sim j}\right\|_{L^2} \left\|\chi_{\sim k} \partial_x w_{\lesssim j}\right\|_{L^\infty}\right\|
$$

$$
\lesssim M^2 \epsilon^4 s^{-7/3} \sum_{k \leq j - 30} 2^{k/2 - 3/2j} \|xg\|_{L^2}
$$

$$
\lesssim M^2 \epsilon^4 s^{-7/3} \sum_{k \leq j - 30} 2^{k/2 - 3/2j} \|xg\|_{L^2}
$$

(117b)

which is sufficient. For the second sub-term, we use almost orthogonality to conclude that

$$
\left\|\sum_j \chi_{j - 30, j + 30}[|u|^2 T_{m_j}^\tau(u_{\sim j}, \chi_{[j - 40, j + 40]} \partial_x w_{\lesssim j}, \chi_{[j - 40, j + 40]} \pi)]\right\|_{L^2} \lesssim \left(\sum_j \left\|\chi_{j - 30, j + 30}[|u|^2 T_{m_j}^\tau(u_{\sim j}, \chi_{[j - 40, j + 40]} \partial_x w_{\lesssim j}, \chi_{[j - 40, j + 40]} \pi)]^2\right\|_{L^2}^{1/2}
$$

$$
\lesssim \left(\sum_j \left\|\chi_{j - 30, j + 30}[|u|^2 T_{m_j}^\tau(u_{\sim j}, \chi_{[j - 40, j + 40]} \partial_x w_{\lesssim j}, \chi_{[j - 40, j + 40]} \pi)]^2\right\|_{L^2}^{1/2}
$$

$$
\lesssim \sum_j \left(\left\|\chi_{j - 30, j + 30}[|u|^2 T_{m_j}^\tau(u_{\sim j}, \chi_{[j - 40, j + 40]} \partial_x w_{\lesssim j}, \chi_{[j - 40, j + 40]} \pi)]^2\right\|_{L^2}^{1/2}
$$

$$
\lesssim \sum_j \left(\left\|\chi_{j - 30, j + 30}[|u|^2 T_{m_j}^\tau(u_{\sim j}, \chi_{[j - 40, j + 40]} \partial_x w_{\lesssim j}, \chi_{[j - 40, j + 40]} \pi)]^2\right\|_{L^2}^{1/2}
$$

$$
\lesssim M^2 \epsilon^4 s^{-2} \left\|xg\right\|_{L^2}
$$

(117c)

as required. For the final sub-term, we use (105) to conclude that $\chi_{j - 20} \partial_x w_{\lesssim j}$ decays in $L^\infty$ like $s^{-5/6}$, yielding

$$
\left\|\sum_j 2^{j}(\sum_j \chi_{j - 20}[|u|^2 T_{m_j}^\tau(u_{\sim j}, \chi_{j - 20} u_{\sim j}, \chi_{j - 20} \partial_x w_{\lesssim j}]\pi]\right\|_{L^2} \lesssim M^2 \epsilon^4 s^{-8/3} 2^{j} \|xg\|_{L^2}
$$

(117d)
which can be summed over \(2^j \geq s^{-1/3}\) to give the desired result. Collecting the bounds now gives the bound for (116a). Since \(u_{\ll j}\) satisfies better decay estimates that \(u\), we can bound (116b) in the same way.

For (116c), we write

\[
|u|^2 T_m^\tau (u_{\ll j}, \partial_x w_{\sim j}, \pi_{\ll j}) = \sum_{k<j-30} \chi_k |u|^2 Q_{j} T_m^\tau (\chi_{\sim k} u_{\ll j}, \chi_{\sim k} \partial_x w_{\sim j}, \chi_{\sim k} \pi_{\ll j}) \tag{118a}
\]

\[
+ \chi_{[j-30,j+30]} |u|^2 T_m^\tau (\chi_{[j-40,j+40]} u_{\ll j}, \partial_x w_{\sim j}, \chi_{[j-40,j+40]} \pi_{\ll j}) \tag{118b}
\]

\[
+ \chi_{> j+30} |u|^2 T_m^\tau (\chi_{> j+20} u_{\ll j}, \chi_{> j+20} \partial_x w_{\sim j}, \chi_{> j+20} \pi_{\ll j}) \tag{118c}
\]

\[
+ \{\text{non-pseudolocal terms}\}
\]

For the first term, we note that we can interchange the order of summation to obtain

\[
\left\| \sum_j (118a) \right\|_{L^2} = \left\| \sum_k \chi_k |u|^2 \sum_{j>k+30} Q_{j} T_m^\tau (\chi_{\sim k} u_{\ll j}, \chi_{\sim k} \partial_x w_{\sim j}, \chi_{\sim k} \pi_{\ll j}) \right\|_{L^2}
\]

Thus, using almost orthogonality together with the bounds (105) for \(\partial_x w_{\sim j}\) and the \(L^4\) bounds for \(u\) given in (110), we find that

\[
\left\| \sum_j (118a) \right\|_{L^2} \lesssim \sum_j \|\chi_{\sim k} u\|_{L^\infty} \left( \sum_{j>k+30} \|\chi_{\sim k} u_{\ll j}\|_{L^4}^4 \|\chi_{\sim k} \partial_x w_{\sim j}\|_{L^2}^2 \right)^{1/2}
\]

\[
\lesssim M^4 \epsilon^4 s^{-7/6} \sum_k 2^{-k} \left( \sum_{j>k+30} \|Q_{[j-40, j+40]} (xg)\|_{L^2}^2 \right)^{1/2}
\]

\[
\lesssim M^4 \epsilon^4 s^{-2} \|xg\|_{L^2}
\]

Turning to the next sub-term, we again use almost orthogonality to obtain

\[
\left\| \sum_j (118b) \right\|_{L^2} \lesssim \left( \sum_j \|\chi_{[j-30, j+30]} u\|_{L^2}^2 T_m^\tau (\chi_{[j-40, j+40]} u_{\ll j}, \partial_x w_{\sim j}, \chi_{[j-40, j+40]} \pi_{\ll j}) \right)^{1/2}
\]

\[
\lesssim \left( \sum_j \|\chi_{[j-40, j+40]} u\|_{L^2}^2 \|\chi_{[j-40, j+40]} u_{\ll j}\|_{L^\infty}^4 \|\partial_x w_{\sim j}\|_{L^2}^2 \right)^{1/2}
\]

\[
\lesssim M^4 \epsilon^4 s^{-2} \left( \sum_j \|w_{\sim j}\|_{H^{-1}}^2 \right)^{1/2}
\]

\[
\lesssim M^4 \epsilon^4 s^{-2} \|xg\|_{L^2}
\]

For the final sub-term, we use the \(L^4\) bound for \(\chi_{> j-20} u_{\ll j}\) given in (110) together with the improved decay for \(\chi_{> j-20} \partial_x w_{\sim j}\) given by (105) to find that

\[
\left\| (118c) \right\|_{L^2} \lesssim \|\chi_{> j-20} u\|_{L^2}^2 \|\chi_{> j-20} u_{\ll j}\|_{L^4}^4 \|\chi_{> j-20} \partial_x w_{\sim j}\|_{L^\infty}
\]

\[
\lesssim M^4 \epsilon^4 s^{-3} 2^{-3j} \|xg\|_{L^2}
\]

which gives the desired bound after summing. This completes the argument for (116c).
It only remains to bound the contribution from \((115a)\). Let \(T(s) = T_m(u, w, \overline{w}) + T_m(u, S, \overline{w}) + T_m(w, S, \overline{w})\). Then, we have at once that
\[
\frac{1}{2} \partial_s \|T(s)\|^2_{L_2} = \Re \left( \partial_s (T(s), T(s)) \right)
\]
We see that
\[
\frac{1}{2} \partial_s \|T(s)\|^2_{L_2} = \Re \left( e^{-is\beta^2} \partial_s e^{s\beta^2} T(s), T(s) \right) - \Re \left( e^{-is\beta^2} \partial_s e^{s\beta^2} T(s), T(s) \right)
\]
The second term on the last line vanishes because \(\partial^2_s\) is a skew-adjoint operator. Noting that
\[
e^{s\beta^2} T(s) = T_{m,e^{is\beta}}(f, g, \overline{f}) + T_{m,e^{is\beta}}(f, h, \overline{g}) + T_{m,e^{is\beta}}(g, h, \overline{f})
\]
we see that
\[
\frac{1}{2} \partial_s \|T(s)\|^2_{L_2} = \Re \left( T_{m,e^{is\beta}}(f, g, \overline{f}) + T_{m,e^{is\beta}}(f, h, \overline{g}) + T_{m,e^{is\beta}}(g, h, \overline{f}), (s) \right)
\]
we can use \((119)\) to write
\[
\text{(115a)} = \pm \Re \int_1^t \left( T_{m,e^{is\beta}}(f, g, \overline{f}) + T_{m,e^{is\beta}}(f, h, \overline{g}) + T_{m,e^{is\beta}}(g, h, \overline{f}), (s) \right) ds
\]
so we can use \((119)\) to write
\[
\frac{1}{2} \int_1^t s^2 \partial_s \|T(s)\|^2_{L_2} ds
\]
\[
\text{(120a)}
\]
\[
\pm \Re \int_1^t s^2 (F(s), T(s)) ds
\]
where
\[
F(s) = T_{m,e^{is\beta}}(f, g, \overline{f}) + T_{m,e^{is\beta}}(f, h, \overline{g}) + T_{m,e^{is\beta}}(g, h, \overline{f})
\]
\[
+ T_{m,e^{is\beta}}(\partial_s f, g, \overline{f}) + T_{m,e^{is\beta}}(\partial_s f, h, \overline{g}) + T_{m,e^{is\beta}}(\partial_s g, h, \overline{f})
\]
\[
+ T_{m,e^{is\beta}}(f, \partial_s g, \overline{f}) + T_{m,e^{is\beta}}(f, \partial_s h, \overline{g}) + T_{m,e^{is\beta}}(g, \partial_s h, \overline{f})
\]
\[
+ T_{m,e^{is\beta}}(f, g, \partial_s \overline{f}) + T_{m,e^{is\beta}}(f, h, \partial_s \overline{g}) + T_{m,e^{is\beta}}(g, h, \partial_s \overline{f})
\]
The arguments for used to bound \((92a) - (92d)\) give us the bound
\[
\|F(s)\|_{L_2} \lesssim M^2 e^2 s^{-2} \|xg\|_{L^2} + M^2 e^3 s^{-11/6 - \beta}
\]
so
\[
\text{(120b)}
\]
\[
\lesssim \int_1^t s^2 \|F(s)\|_{L_2} \|T(s)\|_{L_2} ds
\]
\[
\lesssim \int_1^t M^4 e^4 s^{-1} \|xg\|_{L^2}^2 + M^4 e^5 s^{-5/6 - \beta} \|xg\|_{L_2} ds
\]
as required. Turning to (120a), integration by parts shows that
\[
|120a| = \left| -\frac{1}{2} \epsilon^2 \norm{T(s)}_{L^2}^2 \right|_{s=1}^{s=t} + \int_1^t \norm{T(s)}_{L^2}^2 \, ds
\]
\[
\lesssim M^4 \epsilon^4 \norm{xg}^2_{L^2} + M^4 \epsilon^4 \norm{xu_s}_{L^2}^2 + \int_1^t M^4 \epsilon^4 s^{-1} \norm{xg(s)}^2_{L^2} \, ds
\]
which is better than required, completing the bound for (92a).

7. Zero-mode convergence

In this section, we prove (53). Note that \( \hat{f}(0, t) = \hat{u}(0, t) \) satisfies
\[
\partial_t \hat{u}(0, t) = \pm \frac{1}{\sqrt{2\pi}} \int |u|^2 \partial_x u \, dx
\]
Recall that we can write \( S(x, t; \hat{u}(0, t)) = \exp(i \arg(\hat{u}(0, t))) S(x, t; |\hat{u}(0, t)|) \). Since \( S(x, t; |\hat{u}(0, t)|) \) is real valued, it follows that \( |S|^2 \partial_x S = \frac{1}{3} \partial_x (|S|^2 S) \), and we can write
\[
\partial_t \hat{u}(0, t) = \pm \frac{1}{\sqrt{2\pi}} \int |u|^2 \partial_x u - |S|^2 \partial_x S \, dx
\]
\[
= \pm \frac{1}{\sqrt{2\pi}} \int |u|^2 \partial_x w + (u \overline{\sigma} + \overline{u} \sigma) \partial_x S \, dx
\]
We will focus on the bounding the contribution from the \( |u|^2 \partial_x w \) term, and explain the modifications to deal with the other terms. We can write this term as
\[
\int |u|^2 \partial_x w \, dx = \sum_{2^j \geq t^{-1/3}} I_j + \tilde{I}_j
\]
where
\[
I_j = \int u_j \overline{\nu}_{\leq j} \partial_x w \, dx
\]
\[
\tilde{I}_j = \int u_{<j} \overline{\nu}_{j} \partial_x w \, dx
\]
We will focus on the estimates for \( I_j \); the estimates for \( \tilde{I}_j \) are completely analogous. We first consider the case \( 2^j - 1 < t^{-1/3} \), so that \( Q_j = Q_{\leq j} = Q_{\log t^{-1/3}} \). We have that
\[
|I_j| = \left| \int |Q_{\leq \log t^{-1/3}} u|^2 Q_{\leq \log t^{-1/3}} \partial_x w \, dx \right|
\]
\[
\lesssim \norm{Q_{\log t^{-1/3}} u}_{L^2}^2 \norm{Q_{\log t^{-1/3}} \partial_x w}_{L^\infty}
\]
\[
\lesssim \left(t^{-1/6} \norm{f}_{L^\infty} \right)^2 \left(t^{-5/6} \norm{xg}_{L^2} \right)
\]
\[
\lesssim M^2 \epsilon^4 t^{-1 - \beta}
\]
where on the last line we have used the fact that \( \norm{xg}_{L^2} \lesssim \epsilon t^{1/6 - \beta} \), which was proved in Section 6.

Now, let us consider the case when \( 2^{j-1} \geq t^{-1/3} \). Then, we can write
\[
I_j = -i \int (\eta + \sigma) \hat{Q}_j(\eta) \hat{Q}_{\leq j}(\sigma) e^{it\phi(0, \eta, \sigma)} \hat{f}(\eta) \overline{f(-\sigma) \hat{g}(-\eta - \sigma)} \, d\eta d\sigma
\]
Similarly, defining $m$ we see that

$$I_j = \frac{1}{2\pi t} \int \frac{\nabla_{\eta,\sigma} \phi}{|\nabla_{\eta,\sigma} \phi|^2} \cdot \nabla_{\eta,\sigma} \left( (\eta + \sigma) \hat{Q}_j(\eta) \hat{Q}_{\leq j}(\sigma) \right) e^{i\phi} \hat{f}(\eta) \hat{f}(\sigma) \hat{g}(\eta - \sigma) \ d\eta d\sigma$$  \hspace{1cm} (121a)

$$+ \frac{1}{2\pi t} \int \frac{\partial_\eta \phi}{|\nabla_{\eta,\sigma} \phi|^2} (\eta + \sigma) \hat{Q}_j(\eta) \hat{Q}_{\leq j}(\sigma) e^{i\phi} \partial_\eta \hat{f}(\eta) \hat{f}(\sigma) \hat{g}(\eta + \sigma) \ d\eta d\sigma$$  \hspace{1cm} (121b)

$$+ \frac{1}{2\pi t} \int \frac{\partial_\sigma \phi}{|\nabla_{\eta,\sigma} \phi|^2} (\eta + \sigma) \hat{Q}_j(\eta) \hat{Q}_{\leq j}(\sigma) e^{i\phi} \hat{f}(\eta) \hat{f}(\sigma) \partial_\sigma \hat{g}(\eta + \sigma) \ d\eta d\sigma$$  \hspace{1cm} (121c)

+ \{ similar terms \}

We first consider (121a). We can write it as

$$m_j^1 = 2^{2j} \hat{Q}_{\leq j}(\xi) \frac{\nabla_{\eta,\sigma} \phi(0, \eta, \sigma)}{|\nabla_{\eta,\sigma} \phi(0, \eta, \sigma)|^2} (\eta + \sigma) \hat{Q}_j(\eta) \hat{Q}_{\leq j}(\sigma)$$

Observe that $m_j^1$ satisfies the symbol bounds

$$|\partial_\eta^{\alpha} \partial_\sigma^{\beta} m_j^1| \lesssim 2^{-|\alpha|j}, \quad |\text{supp } m_j^1| \lesssim 2^3 j$$

so by Remark [7] the pseudoproduct $T_m\langle \cdot, \cdot \rangle$ satisfies Hölder-type bounds uniformly in $j$. Thus, the Hausdorff-Young inequality gives us the bound

$$|121a| \lesssim t^{-1/2} ||T_{m_j}(u_{\leq j}, w_{\leq j}, \pi_{\leq j})||_{L^1}$$

$$\lesssim t^{-1/2} ||u_{\leq j}||_{L^\infty} ||u_{\leq j}||_{L^2} ||w_{\leq j}||_{L^2}$$

$$\lesssim M^2 \epsilon^3 t^{-4/3 - \beta} 2^{-j}$$

Similarly, defining

$$m_j^2 = 2^{j} \hat{Q}_{\leq j}(\xi) \frac{\nabla_{\eta,\sigma} \phi(0, \eta, \sigma)}{|\nabla_{\eta,\sigma} \phi(0, \eta, \sigma)|^2} (\eta + \sigma) \hat{Q}_j(\eta) \hat{Q}_{\leq j}(\sigma)$$

we see $m_j^2$ satisfies the same type of symbol bounds as $m_j^1$, so

$$|121b| = t^{-1/2 - j} \left| \hat{T}_{m_j^2}(u_{\leq j}, w_{\leq j}, \pi_{\leq j}) \right|$$

$$\lesssim t^{-1/2 - j} ||u_{\leq j}||_{L^\infty} ||u_{\leq j}||_{L^2} ||w_{\leq j}||_{L^2}$$

$$\lesssim M^2 \epsilon^3 t^{-4/3 - \beta} 2^{-j/2}$$

and

$$|121c| = t^{-1/2 - j} \left| \hat{T}_{m_j^2}(u_{\leq j}, (Lw)_{\leq j}, \pi_{\leq j}) \right|$$

$$\lesssim t^{-1/2 - j} ||u_{\leq j}||_{L^\infty} ||u_{\leq j}||_{L^2} ||xg||_{L^2}$$

$$\lesssim M^2 \epsilon^3 t^{-4/3 - \beta} 2^{-j}$$

Summing in $j$, we find that

$$\sum_{2^j \geq t^{-1/3}} |I_j| \lesssim M^2 \epsilon^3 t^{-1 - \beta}$$

Since the $\hat{I}_j$ satisfy identical bounds, we have shown the required bound for $\int |u|^2 \partial_x w \ dx$. To bound the term $\int (w \pi + w \pi) \partial_x S \ dx$, we simply observe that all the above estimates continue to hold if we replace a $u$ factor by $S$ and move the derivative from $w$ to $S$. 

8. The $L^\infty$ estimates

In this section, we will show how to control the growth of $\hat{f}(\xi, t)$. At times $t \geq |\xi|^{-3}$ (that is, when $|\xi| \geq t^{-1/3}$), $\hat{f}(\xi, t)$ essentially has ODE dynamics, which produce a logarithmic phase correction. For $|\xi| < t^{-1/3}$, we control the growth using the 1/2-Hölder bounds on $\hat{f}$ given by the weighted $L^2$ bound together with the estimate for $\hat{f}(0, t)$.

8.1. The low-frequency bounds. Using the Sobolev-Morrey embedding, we find that

$$|\hat{f}(\xi, t) - \hat{f}(\eta, t)| \lesssim |\xi - \eta|^{1/2} ||xf||_{L^2} \lesssim (\|xg\|_{L^2} + \|LS\|_{L^2}) \lesssim (e^{t^{1/6}} + M^3 s t^{1/6})|\xi - \eta|^{1/2}$$

Since $M^3 e^3 \ll 1$, this implies that $|\hat{f}(\xi, t) - \hat{f}(0, t)| \lesssim \epsilon$ for $|\xi| \lesssim t^{-1/3}$. Since $\hat{f}(0, t) = \hat{u}(0, t)$, and (123) implies that $\hat{u}(0, t)$ is bounded as $t \to \infty$, this gives us the estimate when $|\xi| < t^{-1/3}$.

8.2. The perturbed Hamiltonian dynamics. In this section, we show that $\hat{f}(\xi, t)$ satisfies a perturbed Hamiltonian ODE for each $\xi$, and as a consequence $||f||_{L^\infty}$ is uniformly bounded in time. In particular, we will show that for $|\xi| \geq t^{-1/3}$, $\hat{f}(\xi, t)$ satisfies

$$\partial_t \hat{f}(\xi, t) = \pm \frac{\text{sgn} \xi}{6t} |\hat{f}(\xi, t)|^2 \hat{f}(\xi, t) + ce^{i8/3g \xi^3} \text{sgn} \xi \hat{f}(\xi, t)^3 + R(\xi, t)$$

for some constant $c$, where $R(\xi, t) \lesssim M^3 e^3 t^{-1}(|\xi| t^{-1/3})^{-1/14}$. From this, it will follow that for $B(t, \xi)$ defined by

$$B(t, \xi) := \pm \frac{\text{sgn} \xi}{6} \int_1^t \frac{\hat{f}(\xi, s)^2}{s} \, ds$$

that $v = e^{itB} \hat{f}$ satisfies

$$\partial_t v = ce^{i8/3g \xi^3} e^{iB(t, \xi)} \text{sgn} \xi \hat{f}(\xi, t)^3 + R(\xi, t)$$

Let us consider $|v(t_1) - v(t_2)|$ for $\max(1, |\xi|^{-3}) \leq t_1 < t_2 \leq T$, where $T$ is the time given in the bootstrap argument. Integrating by parts, we find (omitting the $\xi$ factors in the argument):

$$\left| \int_{t_1}^{t_2} e^{i8/3g \xi^3} e^{iB(s, \xi)} \text{sgn} \xi \hat{f}(s)^3 \hat{f}(s) \, ds \right| \lesssim \int_{s=t_1}^{s=t_2} \left| \hat{f}(s)^3 \right| \frac{ds}{s} \leq 1 + II + III + IV$$

Using the definition of $B$ and the bound of $\hat{f}$, we see that

$$|I| + |II| + |III| \lesssim (M^3 e^3 + M^5 e^5) t^{-1} |\xi|^{-3}$$

which is sufficient, since $M^3 e^3 \ll 1$. Moreover, substituting the expression given in (122) for $\partial_x f(s)$, we find that

$$|IV| \lesssim \int_{t_1}^{t_2} M^2 e^2 \frac{s |\xi|^3}{s} \left( M^3 e^3 + R(s, \xi) \right) \, ds \lesssim M^5 e^5 t^{-1} |\xi|^{-3}$$

Taking $t_1 = \max(1, |\xi|^{-3})$ and observing $|v(\xi, t_1)| = |\hat{f}(\xi, t_1)| \lesssim \epsilon$ by Section 8.1, we see that for $t \in (t_1, T)$,

$$|\hat{f}(\xi, t)| = |v(\xi, t)| \lesssim \epsilon + M^3 e^3 t_1^{-1} |\xi|^{-3} \lesssim \epsilon$$

since $\epsilon \ll M^{-3/2}$. In particular, this closes the bootstrap for the $FL^\infty$ component of the $X$ norm.
It follows that we can take $t_2 = \infty$ in the argument above, so $v(\xi, t)$ converges as $t \to \infty$ for each fixed $\xi$. Thus, if we write $f_\infty(\xi) = \lim_{t \to \infty} v(\xi, t)$, we find that
\[
\hat{f}(\xi, t) = \exp \left( \pm \frac{i}{6} \int_1^t \frac{\hat{f}(\xi, s)}{s} \, ds \right) f_\infty(\xi) + O(M^3 e^3 (t^{-1/3} |\xi|)^{-1/14})
\]
so (3) holds. Thus, the proof of the main theorem will be complete once we verify (122).

8.3. The stationary phase estimate. We now prove (122). Note that we can write $\partial_t \hat{f}(\xi, t)$ as
\[
\partial_t \hat{f}(\xi, t) = \pm \frac{i}{2\pi} \int e^{-it\phi}(\xi - \eta - \sigma) \hat{f}(\eta) \hat{f}(\xi - \eta - \sigma) \, d\eta d\sigma \tag{125}
\]
The stationary points for the phase $\phi$ are given by
\[
(\eta, \sigma_1) = (\xi, \xi)
(\eta, \sigma_1) = (-\xi, \xi)
(\eta, \sigma_1) = (\xi, -\xi)
(\eta, \sigma_4) = (\xi/3, \xi/3)
\]
We will now divide the integral dyadically in $\eta$ and $\sigma$, and use stationary phase to estimate each piece. Defining $k$ to be the integer with $2^{k-1} < |\xi| \leq 2^k$, let us write
\[
\partial_t \hat{f}(\xi, t) = I_{lo} + I_{stat} + \sum_{k_1 > k + 10} \left( I_{k_1} + \bar{I}_{k_1} \right)
\]
where
\[
I_{lo} = \pm \frac{i}{2\pi} \int e^{-it\phi} \psi_{\leq k}(\eta) \psi_{\leq k}(\sigma)(\xi - \eta - \sigma) \hat{f}(\eta) \hat{f}(\xi - \eta - \sigma) \, d\eta d\sigma
\]
\[
I_{stat} = \pm \frac{i}{2\pi} \int e^{-it\phi} \left( \psi_{\leq k}(\eta) \psi_{< k}(\sigma) - \psi_{\leq k}(\eta) \psi_{< k}(\sigma) \right) \hat{f}(\eta) \hat{f}(\xi - \eta - \sigma) \, d\eta d\sigma
\]
\[
I_{k_1} = \pm \frac{i}{4\pi} \int e^{-it\phi}(\xi - \eta - \sigma) \psi_{< k_1}(\eta) \psi_{< k_1}(\sigma) \hat{f}(\eta) \hat{f}(\xi - \eta - \sigma) \, d\eta d\sigma
\]
\[
\bar{I}_{k_1} = \pm \frac{i}{4\pi} \int e^{-it\phi}(\xi - \eta - \sigma) \psi_{< k_1}(\eta) \psi_{< k_1}(\sigma) \hat{f}(\eta) \hat{f}(\xi - \eta - \sigma) \, d\eta d\sigma
\]
We will show how to estimate the terms $I_{lo}$, $I_{stat}$ and $I_{k_1}$: the $\bar{I}_{k_1}$ terms follow from similar reasoning.

8.3.1. The estimate for $I_{lo}$. Over the support of the integrand, $|\partial_\eta \phi|, |\partial_\sigma \phi| \sim 2^k$. Integrating by parts with respect to $\eta$ yields
\[
I_{lo} = \pm \frac{i}{2\pi} \int e^{it\phi} \frac{\xi - \eta - \sigma}{\partial_\eta \phi} \psi_{\leq k}(\eta) \psi_{\leq k}(\sigma) \partial_\eta \hat{f}(\xi - \eta - \sigma) \hat{f}(\xi - \eta - \sigma) \, d\eta d\sigma \tag{127a}
\]
\[
+ \frac{1}{4\pi} \int e^{it\phi} \partial_\eta \left( \frac{\xi - \eta - \sigma}{\partial_\eta \phi} \psi_{\leq k}(\eta) \psi_{< k}(\sigma) \right) \hat{f}(\eta) \hat{f}(\xi - \eta - \sigma) \hat{f}(\xi - \eta - \sigma) \, d\eta d\sigma \tag{127b}
\]
+ \{similar terms\}
\]
For the first term, we note that $m^1 = 2^k \xi - \eta - \sigma \partial_\sigma \phi \partial_\eta \psi_{\leq k}(\eta) \psi_{< k}(\xi)$ is a smooth symbol supported on $|\xi|, |\eta|, |\sigma| \lesssim 2^k$ and satisfies the Coifman-Meyer type symbol bounds
\[
|\partial_\xi^{\alpha} m^1| \lesssim_\alpha 2^{-k|\alpha|}
\]
Similarly, defining the symbol $m^2 = 2^{2k} \partial_\eta \left( \frac{\xi - \eta - \sigma}{\bar{\nu} \eta \phi} \psi_{\leq k}(\eta) \psi_{\leq k}(\sigma) \right) \psi_{\sim k}(\xi)$, we find that

$$|I_{10}| \lesssim M^3 t^{-3/2} \left( t^{1/2} \right)^{-1/2}$$

which is consistent with the estimate for the remainder term in (122).

8.3.2. The estimate for $I_{k_1}$. For these terms, $|\nabla \eta, \sigma| \sim 2^{2k_1}$. Integrating by parts using the identity $\frac{1}{\eta(\nabla \eta, \sigma) \cdot \nabla \eta, \sigma} \nabla \eta, \sigma \cdot \nabla \eta, \sigma e^{it\phi} = e^{it\phi}$, we find that

$$I_{k_1} = \frac{1}{2\pi t} \int \frac{e^{it\phi}}{|\nabla \eta, \sigma|^2} \psi_{k_1}(\eta) \psi_{\leq k_1}(\sigma) \partial_\eta \hat{f}(\eta) \hat{f}(\xi - \eta - \sigma) d\eta d\sigma$$

$$\lesssim \frac{1}{2\pi t} \int e^{it\phi} \nabla \eta, \sigma \cdot \left( \frac{\xi - \eta - \sigma}{|\nabla \eta, \sigma|^2} \psi_{k_1}(\eta) \psi_{\leq k_1}(\sigma) \right) \hat{f}(\eta) \hat{f}(\xi - \eta - \sigma) d\eta d\sigma$$

and observing that $m^1_{k_1}, m^2_{k_1}$ are satisfy the conditions given in Remark 7 uniformly in $k_1$, we find that

$$|I_{128a}| \lesssim t^{-1/2} \left| T_{m^1}(L_u, u_{\leq k_1}, u_{\leq k_1}) \right|_{L^2}$$

$$\lesssim t^{-1/2} \left| T_{m^1}(u_{\leq k_1}, u_{\leq k_1}, u_{\leq k_1}) \right|_{L^2}$$

$$\lesssim M^3 t^{-3/2}$$
An analogous argument holds for $\tilde{I}_{k_1}$, so summing over $k_1 > k + 10$, we find that

$$\left| \sum_{k_1 > k+10} I_{k_1} + \tilde{I}_{k_1} \right| \lesssim M^3 c^3 t^{-1} \left( t^{1/3+2k} \right)^{-1/2}$$

which allows us to treat these terms as remainders in \[(122)\].

8.3.3. The estimate for $I_{\text{stat}}$. The integral here contains the four stationary points given in \[(126)\]. Note that each of the stationary points are at a distance $\sim 2^k$ from each other. Using this, we can write

$$I_{\text{stat}} = \sum_{r=1}^{4} \sum_{2^k \lesssim 2^{r+1}} \left( J^{(r)}_{\ell} + \tilde{J}^{(r)}_{\ell} \right) + \{\text{remainder}\}$$

where

$$J^{(r)}_{\ell} = \pm \frac{i}{2\pi} \int e^{-it\phi} \left[ \psi_{\ell} \right] \left( \eta - \eta_r \right) \psi_{\ell} \left( \sigma - \sigma_r \right) \left( \xi - \eta - \sigma \right) f(\eta) \overline{f(\xi - \eta - \sigma)} \; d\eta d\sigma$$

$$\tilde{J}^{(r)}_{\ell} = \pm \frac{i}{2\pi} \int e^{-it\phi} \left[ \psi_{\ell} \right] \left( \eta - \eta_r \right) \psi_{\ell} \left( \sigma - \sigma_r \right) \left( \xi - \eta - \sigma \right) f(\eta) \overline{f(\xi - \eta - \sigma)} \; d\eta d\sigma$$

with

$$\psi_{\ell} = \begin{cases} \psi_{\ell} & \ell > \ell_0 \\ \psi_{\ell} & \ell = \ell_0 \\ 0 & \ell < \ell_0 \end{cases}$$

for a parameter $\ell_0$ defined such that $2^{\ell_0} \sim t^{-1/3} (t^{1/3+2k})^{-\gamma}$, where $\gamma > 0$ is a constant which will be specified later. The contribution from the remainder can be controlled using an argument similar to the one for $I_{k_1}$, so we will focus on controlling the contribution from the $J^{(r)}_{\ell}$ terms. There are two cases to consider: either $\ell = \ell_0$ or $\ell > \ell_0$.

Case $\ell > \ell_0$. We first consider the bound for $J^{(r)}_{\ell}$. Integrating by parts gives

$$J^{(r)}_{\ell} = \pm \frac{1}{2\pi t} \int e^{-it\phi} \left( \xi - \eta - \sigma \right) \nabla_{\eta,\sigma} \psi_{\ell} \left( \eta - \eta_r \right) \psi_{\ell} \left( \sigma - \sigma_r \right) \cdot \nabla_{\eta,\sigma} \left( \overline{f(\eta) f(\xi - \eta - \sigma)} \right) \; d\eta d\sigma$$

$$= \pm t^{-1} e^{it\xi^3} T_{m_1} (L_{\ell}, \tilde{P}_{\xi - \eta_r - \sigma_r} \lesssim t^{\ell} u, \mathcal{Q}_{\xi \ell} \mathcal{P})$$

$$+ t^{-1} e^{it\xi^3} T_{m_2} (\tilde{P}^{\eta - u}_{\xi - \eta_r + \sigma_r} \lesssim t^{\ell} u, \mathcal{Q}_{\xi \ell} \mathcal{P})$$

$$+ \{\text{similar terms}\} \quad (129)$$

where $\tilde{P}_{\xi \ell} = \psi_{\xi \ell + C}(D - \xi)$ for $\xi \in \mathbb{R}$, and the symbols $m_1^{\xi}$ and $m_2^{\xi}$ are given by

$$m_1^{\xi} = 2^\xi \left( \xi - \eta - \sigma \right) \frac{\partial_\phi}{\nabla_{\eta,\sigma} \phi} \psi_{\ell} \left( \eta - \eta_r \right) \psi_{\ell} \left( \sigma - \sigma_r \right) \psi_{\ell} \left( \xi - \xi_0 \right)$$

$$m_2^{\xi} = 2^\xi \nabla_{\eta,\sigma} \cdot \left( \xi - \eta - \sigma \right) \frac{\nabla_{\eta,\sigma} \phi}{\nabla_{\eta,\sigma} \phi} \psi_{\ell} \left( \xi - \xi_0 \right)$$

where $\xi_0$ is any point within a distance $\ll 2^\ell$ from $\xi$. It is clear that these symbols are supported on a region of volume $\sim 2^{2\ell}$. Moreover, over the support of the integral we have that

$$|\xi - \eta - \sigma| \lesssim 2^k, \quad |\nabla_{\eta,\sigma} \phi| \sim 2^{k+\ell}, \quad |\partial_\xi,\eta,\sigma \nabla_{\eta,\sigma} \phi| \lesssim 2^{(2-|\alpha|)k}$$
where for the last inequality we have used the fact that $2^\ell \ll 2^k$. Thus, we see that $m^1_\ell$ and $m^2_\ell$ obey the Coifman-Meyer type bounds

$$|\partial^\alpha_{\xi,\eta,\sigma}m^1_\ell| + |\partial^\alpha_{\xi,\eta,\sigma}m^2_\ell| \lesssim 2^{-|\alpha|\ell}$$

It follows from Remark 7 and the Hausdorff-Young inequality that

$$\|\mathcal{T}_n^\ell\|_{L^1} \lesssim t^{-1/2} \|P_{\leq 2^\ell}f\|_{L^2} \|P_{=k}u\|_{L^\infty} \lesssim M^3 2^{\gamma/2 - 2^{-\ell/2}}$$

and

$$\|\mathcal{T}_M^\ell\|_{L^1} \lesssim t^{-1/2} \|P_{\leq 2^\ell}f\|_{L^2} \|P_{=k}u\|_{L^\infty} \lesssim M^3 2^{\gamma/2 - 2^{-\ell/2}}$$

Summing over $\ell > \ell_0$ yields

$$\sum_{\ell \geq \ell_0} J^{(r)}_\ell \lesssim t^{-1}(1/2^k)^{-1/2}$$

A similar argument gives an identical bound for the $\tilde{J}^{(r)}_\ell$.

**Case** $\ell = \ell_0$. By performing the linear change of variables $\eta \to \eta_r$, $\sigma \to \sigma_r$, we obtain

$$J^{(r)}_{\ell_0} = \pm \frac{i}{2\pi} \int e^{-it\phi}\psi_{\leq \ell_0}(\eta)\psi_{\leq \ell_0}(\sigma)(\xi - \eta_r - \sigma_r - \eta - \sigma)F_r(\xi, \eta, \sigma) \, d\eta d\sigma$$

where

$$F_r(\xi, \eta, \sigma) = \hat{f}(\eta + \eta_r)\hat{f}(\xi - \eta_r - \sigma_r)\hat{f}(-\sigma - \sigma_r)$$

We can re-write this as

$$J^{(r)}_{\ell_0} = \pm \frac{i}{2\pi} \int e^{-it\phi}\psi_{\leq \ell_0}(\eta)\psi_{\leq \ell_0}(\sigma)(\xi - \eta_r - \sigma_r - \eta - \sigma)(F_r(\xi, \eta, \sigma) - F_r(\xi, \eta, \sigma)) \, d\eta d\sigma$$

(131a)

$$\pm \frac{i}{2\pi} \int F_r(\xi, 0, 0) e^{-it\phi}\psi_{\leq \ell_0}(\eta)\psi_{\leq \ell_0}(\sigma)(\eta + \sigma) \, d\eta d\sigma$$

(131b)

$$\pm \frac{i}{2\pi} \int F_r(\xi, 0, 0)(\xi - \eta_r - \sigma_r) e^{-it\phi}\psi_{\leq \ell_0}(\eta)\psi_{\leq \ell_0}(\sigma) \, d\eta d\sigma$$

(131c)

For (131a), we recall that the $L^2$ bound on $xf$ implies that $\hat{f}$ is $1/2$-Hölder, so

$$|F_r(\xi, \eta, \sigma) - F_r(\xi, 0, 0)| \lesssim M^3 2^{\gamma/2 - 1/2}$$

and

$$\|\mathcal{T}_n^\ell\|_{L^1} \lesssim M^3 2^{\gamma/2 - 1/2} \int \psi_{\leq \ell_0}(\eta)\psi_{\leq \ell_0}(\sigma)(\eta + \sigma + \eta_r + \sigma_r)(|\eta| + |\sigma|)^{1/2} \, d\eta d\sigma$$

and

$$\|\mathcal{T}_M^\ell\|_{L^1} \lesssim M^3 2^{\gamma/2 - 1/2} \int \psi_{\leq \ell_0}(\eta)\psi_{\leq \ell_0}(\sigma)(|\eta| + |\sigma|)^{1/2} \, d\eta d\sigma$$

$$\lesssim M^3 2^{\gamma/2 - 1/2} \left(\frac{1}{2^k}\right)^{1-5/2\gamma}$$
Similarly, the $L^\infty$ bound on $\hat{f}$ from (131) shows that $|F_r(\xi, 0, 0)| \lesssim M^3 e^3$, so

\[
|F_r(\xi, 0, 0)| \lesssim M^3 e^3 2^{2\ell_0} \lesssim M^3 e^3 r^{-1} \left( t^{1/3} 2^k \right)^{-3\gamma}
\]

The term $131c$ contains the leading order contribution to (122). We will extract this contribution using the method of stationary phase. By Taylor expansion, we find

\[\phi(\xi, \eta, \sigma + \sigma_r) = \phi_r + Q_r(\eta, \sigma) + O(|\eta|^3 + |\sigma|^3)\]

where $\phi_r = \phi(\xi, \eta, \sigma)$ and $Q_r$ is the quadratic form associated to the Hessian matrix $\text{Hess}_{\eta, \sigma} \phi(\xi, \eta, \sigma)$. Thus, $|e^{-it\phi} - e^{-it(\phi_r + Q_r(\eta, \sigma))}| \lesssim t(|\eta|^3 + |\sigma|^3)$, so

\[
\left| (\xi - \eta_r - \sigma_r)F_r(\xi, 0, 0) \int e^{-it\phi} - e^{-it(\phi_r + Q_r(\eta, \sigma))} \psi_{\leq \ell_0}(\eta)\psi_{\leq \ell_0}(\sigma) \, d\eta d\sigma \right| \lesssim M^3 e^3 2^{2\ell_0} t \ell_0 \lesssim M^3 e^3 \left( t^{1/3} 2^k \right)^{-1-5\gamma}
\]

By rescaling and using stationary phase, we find that

\[
\int e^{-itQ_r(\eta, \sigma)}\psi_{\leq \ell_0}(\eta)\psi_{\leq \ell_0}(\sigma) \, d\eta d\sigma = 2\pi t e^{i\frac{1}{2} \text{sign Hess}_{\eta, \sigma} \phi(\xi, \eta, \sigma)} \frac{t}{\sqrt{|\det \text{Hess}_{\eta, \sigma} \phi(\xi, \eta, \sigma)|}} + O(t^{-2} 2^{2\ell_0} 2^{-2k})
\]

where on the last line we have used the fact that $|\det \text{Hess}_{\eta, \sigma} \phi(\xi, \eta, \sigma)| \sim 2^{2k}$ to obtain the error term. Collecting all these calculations, we find that

\[
131c = \pm iF_r(\xi, 0, 0) \frac{(\xi - \eta_r - \sigma_r)e^{-it\phi_r} + i\frac{1}{2} \text{sign } Q_r}{t \sqrt{|\det Q_r|}} + O(M^3 e^3 r^{-1} \left[ (t^{1/3} 2^k)^{-1-5\gamma} + (t^{1/3} 2^k)^{-1-2\gamma} \right])
\]

Collecting the results for $131a$–$131c$ and simplifying using the definition of $\ell_0$, we find that

\[
J^{(r)}_{\ell_0} = \pm iF_r(\xi, 0, 0) \frac{(\xi - \eta_r - \sigma_r)e^{-it\phi_r} + i\frac{1}{2} \text{sign } Q_r}{t \sqrt{|\det Q_r|}} + O(M^3 e^3 r^{-1} \left[ (t^{1/3} 2^k)^{-1-5\gamma} + (t^{1/3} 2^k)^{-3\gamma} + (t^{1/3} 2^k)^{-1-2\gamma} \right])
\]

By direct calculation, for $r = 1, 2, 3$

\[
\phi_r = 0, \quad \det Q_r = -36\xi^2, \quad \text{sign } Q_r = 0
\]

\[
\phi_4 = 8/9\xi^3, \quad \det Q_4 = 12\xi^2, \quad \text{sign } Q_r = -2 \text{ sgn } \xi
\]

Thus, combining (132) with (130) and taking $\gamma = 3/7$, we find that

\[
\sum_{r=1}^{4} J^{(r)} = \pm \frac{i \text{ sgn } \xi}{6t} |\hat{f}(\xi, t)|^2 \hat{f}(\xi, t) \pm e^{it8/9\xi^3} \text{ sgn } \xi e^{-i\frac{2}{3}\text{ sgn } \xi} \frac{3\sqrt{12t}}{\text{ sgn } \xi} |\hat{f}(\xi/3, t)|^2 \hat{f}(\xi/3, t)
\]

\[+ O(M^3 e^3 t^{-1} (2^{k} t^{1/3})^{-1/14})\]

which concludes the proof of (122) and gives Theorem [4].
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