Communication capacity of mixed quantum $t$ designs

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We operationally introduce mixed quantum $t$ designs as the most general arbitrary-rank extension of projective quantum $t$ designs which preserves indistinguishability from the uniform distribution for $t$ copies. First, we derive upper bounds on the classical communication capacity of any mixed $t$ design measurement, for $t \in [1, 5]$. Second, we explicitly compute the classical communication capacity of several mixed $t$ design measurements, including the depolarized version of: any qubit and qutrit symmetric, informationally complete (SIC) measurement and complete mutually unbiased bases (MUB), the qubit icosahedral measurement, the Hoggar SIC measurement, any anti-SIC (where each element is proportional to the projector on the subspace orthogonal to one of the elements of the original SIC), and the uniform distribution over pure effects.

I. INTRODUCTION

Arguably, the class of quantum configurations with the broadest application in any communication protocol is that of projective $t$ designs [1, 2], including e.g. symmetric, informationally complete (SIC) measurements [3–8] and mutually unbiased bases (MUB) [9–13]. Projective $t$ designs can informally be defined by the properties of i) being indistinguishable from the uniform distribution up to $t$ copies, and ii) comprising only pure elements. While indistinguishability from the uniform distribution is an operational restriction of the protocol, introduced e.g. to ensure that no information leaks to an adversary, purity is a purely mathematical idealization bound to be lost in the presence of noise.

Previous generalizations of projective $t$ designs to the arbitrary-rank case were either limited to specific subsets, such as SICs or MUBs [14–18], or relaxed the operational property of indistinguishability from the uniform distribution given $t$ copies [19]. In this work, we introduce the class of mixed $t$ designs as the most general extension of projective $t$ designs that preserves the operational property of indistinguishability from the uniform distribution up to $t$ copies, while relaxing the mathematical assumption of pure elements.

In any communication scenario, such as quantum key distribution [20] or locking [21] of information in quantum states, a relevant figure of merit is the classical communication capacity, namely the maximum amount of information that can reliably be extracted per use, in the asymptotic limit. More precisely, the celebrated Shannon’s coding theorem [22] guarantees that, for any communication channel, there exist an encoding and a decoding such that the amount of information that can be transmitted with null error probability is equal to the channel capacity, and also that no better performance can be achieved.

For measurements, the capacity was proven to be equivalent to its single-shot analogy, i.e. the informational power, introduced in Ref. [23]. Therein, the simple case of SIC measurement of a qubit was solved. Subsequently, considerable interest arose for computing the capacity of several symmetric measurements [24–32]. With very limited exceptions (e.g. mirror symmetric measurements, introduced in Ref. [23]), mixed $t$ designs introduced here (with $t \geq 2$, where our results provide novel insights) encompass all the classes of non-trivial measurements for which the classical capacity is known, and extend such classes by adding e.g. the isotropically noisy version of projective $t$ designs.

Here, we introduce mixed $t$ designs as the most general class of quantum configurations that are indistinguishable from the uniform distribution given $t$ copies, and discuss some of their basic properties. First, we derive upper bounds on the classical communication capacity of any mixed $t$ design measurement, for $t \in [1, 5]$. Second, we explicitly compute the classical communication capacity of several mixed $t$ design measurements, including the depolarized version of: any qubit and qutrit SIC measurement and complete MUB, the qubit icosahedral measurement, the Hoggar SIC measurement, any anti-SIC (where each element is proportional to the projector on the subspace orthogonal to one of the elements of the original SIC), and the uniform distribution over pure effects.

This work is structured as follows. In Sec. II A we introduce mixed $t$ designs and derive some of their basic properties. In Sec. II B we discuss some fundamental facts about the communication capacity of quantum measurements. In Sec. II C we present our main results about the capacity of mixed $t$ designs measurements. Finally, we summarize our results and discuss some outlooks in Sec. III.
II. CLASSICAL CAPACITY OF MIXED \( t \) DESIGNS

A. Mixed \( t \) designs

Let us first recall some standard definitions and facts from quantum theory \[33\]. Let \( \mathcal{H} \) be a (finite) \( d \)-dimensional Hilbert space, \( \mathcal{L}(\mathcal{H}) \) be the space of linear operators on \( \mathcal{H} \) and \( \mathcal{L}_+(\mathcal{H}) \) be the set of all the positive-semidefinite operators on \( \mathcal{H} \). The most general quantum state in \( \mathcal{H} \) is described by a density matrix, namely a trace-one operator in \( \mathcal{L}_+(\mathcal{H}) \). Adopting the Dirac notation, any pure state \( \phi \) is also denoted by \( |\phi\rangle \) (thus \( \phi \) corresponds to the projector on \( |\phi\rangle \)). Let \( (\Lambda, \Sigma) \) be a Borel locally compact metric space equipped with a Borel measure \( \mu \) (In particular, one can think of a finite set with the uniform measure or a locally compact group with the Haar measure, e.g., the group \( U(d) \) of unitaries in \( \mathcal{L}(\mathcal{H}) \)). The most general quantum preparation in \( \mathcal{H} \) is described by a quantum ensemble, namely a measurable function \( \rho : A \ni x \mapsto \rho_x \in \mathcal{L}_+(\mathcal{H}) \), where \( \rho_x \) is a quantum state, and \( x \mapsto \rho_x \) is a probability density (with respect to \( \mu \)). The most general quantum measurement in \( \mathcal{H} \) is described by a positive operator-valued measure (POVM). In particular, a POVM can be defined by a measurable function \( \pi : A \ni y \mapsto \pi_y \in \mathcal{L}_+(\mathcal{H}) \), where \( \pi_y \) is a quantum state, \( y \mapsto p_y \) is a probability density (with respect to \( \mu \)), and \( d \int d\mu(y)p_y\pi_y = \mathbb{1} \), the identity operator in \( \mathcal{L}(\mathcal{H}) \).

We introduce in the following some quantities that characterize ensembles and POVMs. For any measurable function \( \chi \) from \( A \) to positive semidefinite operators \( x \mapsto \chi_x := \nu p_x \chi_x \in \mathcal{L}(\mathcal{H}) \) such that \( \nu \) is a normalization, \( x \mapsto p_x \) is a probability density and \( \chi_x \) a quantum state, we call
\[
\mu_k(\chi) := \int d\mu(x)p_x \text{Tr}[\chi_x^k],
\]
the \( k \)-th moment of \( \chi \). For any pure state \( \phi \in \mathcal{L}(\mathcal{H}) \), we call
\[
\gamma_k(\chi, \phi) := \int d\mu(x)p_x \langle (\phi | \chi_x | \phi \rangle)^k,
\]
the index of coincidence.

Operationally, a projective quantum \( t \) design ensemble is an ensemble \( x \mapsto \phi_x := p_x \phi_x \) of pure states that cannot be discriminated from the uniformly distributed ensemble of pure states – other than by trivial guessing – when given \( t \) copies. Analogously, a projective quantum \( t \) design POVM is operationally defined as a POVM \( y \mapsto \pi_y := q_y \pi_y \) such that the ensemble \( \phi_y := \text{Tr}_2[\Phi^+(\mathbb{1} \otimes \pi_y)] = \frac{1}{\sqrt{\pi_{\mathcal{H}}}} \text{Tr} \text{steered on a maximally entangled state } |\Phi^+\rangle := \frac{1}{\sqrt{d}} \sum_{i,j} |i,i\rangle \) is a projective quantum \( t \) design ensemble. More generally, a projective quantum \( t \) design can be defined as a measurable function \( \phi \) from \( A \) to rank-one positive semidefinite operators \( x \mapsto \phi_x := \nu p_x \phi_x \) such that
\[
\int d\mu(x)p_x \phi_x^{\otimes t} = \int dg |\phi_g\rangle(\phi_g)^{\otimes t},
\]
where \( dg \) denotes the uniform (Haar) probability measure over the group \( U(d) \) of unitaries in \( \mathcal{L}(\mathcal{H}) \), see \[31\].

We introduce mixed \( t \) designs as a generalization of projective \( t \) designs that relax the constraint of being rank-one projects while preserving indistinguishability from the uniform distribution given \( t \) copies. Operationally, we define mixed \( t \) design ensembles \( x \mapsto \rho_x := p_x \rho_x \) as ensembles of (possibly) mixed states that cannot be distinguished from the uniform distribution given \( t \) copies. Analogously, we operationally define mixed \( t \) design POVMs as POVMs \( y \mapsto \pi_y \) such that the ensemble \( \rho_y := \text{Tr}_2[\Phi^+(\mathbb{1} \otimes \pi_y)] = \frac{1}{\sqrt{\pi_{\mathcal{H}}}} \text{steered on a maximally entangled state } |\Phi^+\rangle := \frac{1}{\sqrt{d}} \sum_{i,j} |i,i\rangle \) is a mixed \( t \) design ensemble. More generally, we define mixed quantum \( t \) designs as follows.

Definition 1 (Mixed \( t \) design). We call a mixed quantum \( t \) design any measurable function \( \chi \) from \( A \) to positive semidefinite operators \( x \mapsto \chi_x := \nu p_x \chi_x \), where \( x \mapsto p_x \) is a probability density (with respect to \( \mu \)) if
\[
\int d\mu(x)p_x \chi_x^{\otimes t} := \int dg(U_g \chi_g^{\otimes t}),
\]
where \( dg \) is the Haar probability measure on \( U(d) \) and \( \chi \) is some unit-trace positive-semidefinite operator.

In the following, for brevity we refer to mixed \( t \) designs simply as \( t \) designs. Let us prove some important properties of \( t \) designs. First, notice that if \( \chi \) is a \( t \) design, also \( \chi^T \) is a \( t \) design, and thus the operational interpretation of \( t \) design POVMs immediately follows. Notice also that by partial tracing Eq. \[16\] it immediately follows that any \( t \) design is also a \( k \) design for any \( 1 \leq k \leq t \), and therefore in particular
\[
\int d\mu(x)p_x \chi_x = \frac{1}{d},
\]
for any \( t \) design \( \chi \).

The moments up to \( t \) of any \( t \) design \( \chi \) are given by
\[
\mu_k(\chi) = \text{Tr}[\chi^k].
\]
for any \( 1 \leq k \leq t \). This uniquely identifies \( \chi \) up to unitaries if \( t \geq d \). Equation \[22\] can be readily verified by first multiplying Eq. \[11\] by the shift operator \( S_t := \sum_{i_1,...,i_t} \otimes^t_{j=1} |i_j\rangle \langle i_j| \) that replaces the state of \( \mathcal{H}_{\otimes t} \) with \( \mathcal{H}_1 \) (here \( \otimes \) denotes sum modulus \( t \) and \( \{i_j\}_{j=1}^d \) is an orthonormal basis of space \( \mathcal{H}_1 \)), and then tracing using the property of the shift operator that \( \text{Tr}[S_t V^{\otimes t}] = \text{Tr}[V] \) for any \( t \) and any Hermitian operator \( V \).

For quantum \( t \) designs \( \chi \), it follows from the definition that for any \( 1 \leq k \leq t \) the index of coincidence \( \gamma_k \) is
independent of \( \phi \) and is given by
\[
\gamma_k(\chi) = \left( \frac{d + k - 1}{k} \right)^{-1} \text{Tr}[P_k \hat{\chi}^{\otimes k}]
\]
\[
= \frac{1}{(d + k - 1)!} B_k(x_1, \ldots, x_k),
\]
where \( P_k \) denotes the projector over the symmetric subspace of \( \mathcal{H}^{\otimes k} \), \( x_i := (i - 1)! \text{Tr}[\hat{\chi}^{\otimes i}] \), and \( B_k(x_1, \ldots, x_k) \) is the complete exponential Bell polynomial [8] given by
\[
B_k(x_1, \ldots, x_k) = \sum_{j=1}^k B_{k,j}(x_1, x_2, \ldots, x_{k-j+1}),
\]
where
\[
B_{k,j}(x_1, x_2, \ldots, x_{k-j+1}) = \frac{k!}{i_1! \cdots i_{k-j+1}!} x_1^{i_1} \cdots x_{k-j+1}^{i_{k-j+1}},
\]
and the sum is over all sequences \( i_1, i_2, \ldots, i_{k-j+1} \) such that
\[
\begin{align*}
(i_1 + i_2 + \cdots + i_{k-j+1}) &= j, \\
i_1 + 2i_2 + 3i_3 + \cdots + (k-j+1)i_{k-j+1} &= k.
\end{align*}
\]
The second equality in Eq. (3) follows from a lengthy but straightforward calculation. Therefore explicitly for \( k \in [1, 5] \) one has
\[
\begin{align*}
\gamma_1 &= \frac{1}{2}, \\
\gamma_2 &= \frac{1 + \mu_s}{d(d+1)}, \\
\gamma_3 &= \frac{1 + 3\mu_s + 2\mu_p}{d(d+1)(d+2)}, \\
\gamma_4 &= \frac{1 + 6\mu_s + 3\mu_p^2 + 8\mu_s^3 + 6\mu_p^3}{d(d+1)(d+2)(d+3)}, \\
\gamma_5 &= \frac{1 + 10\mu_s + 15\mu_p + 20\mu_s^2 + 30\mu_s^3 + 20\mu_s^3\mu_p + 24\mu_p^4}{d(d+1)(d+2)(d+3)(d+4)}.
\end{align*}
\]

Relevant classes of \( t \) designs include symmetric, informationally complete (SIC) POVMs [3, 4, 5] and mutually unbiased bases (MUB) [6, 7] as well as their mixed generalizations given by symmetric informationally complete measurements (SIM) [14, 15, 16] and mutually unbiased measurements (MUM) [14]. Despite their importance, the existence of SICs and MUBs is still a matter of conjecture [5]. For SICs, existence has been proved analytically for dimensions 2 – 15 and 17, 19, 24, 35, and 48 [3, 4, 5, 8, 9, 10, 11], and numerically for dimensions 2 – 67 [12]. For MUBs, existence is questioned for a dimension as small as 6 [11, 12]. On the other hand, it is known that MUBs exist in infinitely many dimensions, i.e. all dimensions being prime powers [3, 5]. Other than in the aforementioned communication protocols, SICs and MUBs have applications in quantum tomography [3, 5], uncertainty relations [57], and in foundational problems [38, 42].

Any function \( \chi \) from \( \{0, 1, \ldots, d^2 - 1\} \) to positive semi-definite operators \( x \mapsto \chi_x := \frac{1}{d} |\chi_x\rangle \langle \chi_x| \) such that
\[
|\langle \chi_x | \chi_{x'} \rangle|^2 = \delta_{x,x'} + (1 - \delta_{x,x'}) \frac{1}{d + 1},
\]
for any \( x \) and \( x' \) defines a set \( \{\chi_x\}_x \) which we call a symmetric informationally complete (SIC) set. Any measurable function \( \chi \) from \( \{0, 1, \ldots, d\} \times \{0, 1, \ldots, d - 1\} \) to positive semidefinite operators \( (x,y) \mapsto \chi_{x,y} := \frac{1}{d(d+1)} |\chi_{x,y}\rangle \langle \chi_{x,y}| \) such that
\[
|\langle \chi_{x,y} | \chi_{x',y'} \rangle|^2 = \delta_{x,x'}\delta_{y,y'} + (1 - \delta_{x,x'}) \frac{1}{d},
\]
for any \( x, y, x', y' \) and \( y' \) defines a set \( \{\chi_{x,y}\}_{x,y} \) which we call a mutually unbiased set. For simplicity, we use the notation \( \chi_{x,y} = \chi_{dx+y} \).

An explicit way to construct families of mixed \( t \) designs is by the affine combination of any mixed \( t \) design \( x \mapsto \chi_x \) with the maximally mixed operator as follows:
\[
\mathcal{D}_\lambda(\chi_x) = \lambda \chi_x + (1 - \lambda) \mathbb{I} \text{Tr}[\chi_x] \frac{d}{d+1},
\]
for any \( \lambda \) such that \( \mathcal{D}_\lambda(\chi_x) \geq 0 \) \( \forall x \), as follows by applying \( \mathcal{D}_\lambda^{\otimes k} \) to both sides of Eq. (1) and noticing that the map \( \mathcal{D}_\lambda \) commutes with any unitary channel. More precisely, let \( \chi_x := \sum_{x,k} \alpha_{x,k} |\phi_{x,k}\rangle \langle \phi_{x,k}| \) be a spectral decomposition of \( \chi_x \) and let \( \alpha_{\text{max}} := \max_{x,k} \alpha_{x,k} \) and \( \alpha_{\text{min}} := \min_{x,k} \alpha_{x,k} \). Then by direct inspection it follows that \( \mathcal{D}_\lambda(\chi_x) \geq 0 \) if and only if
\[
\lambda \in \left[ \frac{1}{1 - d\alpha_{\text{max}}} \leq \frac{1}{1 - d}, \frac{1}{1 - d\alpha_{\text{min}}} \geq 1 \right].
\]
The linear map \( \mathcal{D}_\lambda \) corresponds to a depolarizing channel if and only if \( \lambda \in [0, 1] \). Then it follows that the depolarized version of any \( t \) design is a \( t \) design. Furthermore, for any \( t \) design \( \chi_x \), we define the corresponding anti-\( t \) design as \( \mathcal{D}_{(1-d\alpha_{\text{max}})-1}(\chi) \). Notice that due to Eq. (1) any anti-\( t \) design is a \( t \) design. Finally one has the following simple relation between the moments of any \( t \) design \( \chi_x \) and those of its depolarized version \( \mathcal{D}_\lambda(\chi_x) \):
\[
\mu_k(\mathcal{D}_\lambda(\chi_x)) = \sum_{n=0}^k \binom{k}{n} \lambda^n \left( \frac{1 - \lambda}{d} \right)^{k-n} \mu_n(\chi).
\]

## B. Communication capacity

Let us first recall some standard definitions and facts from information theory [13, 14]. Intuitively, a means of quantifying the distinctiveness of two given probability densities \( p \) and \( q \) on \( A \) (with respect to \( \mu \)) is given by the relative entropy \( D(\mu || q) \), also known as the Kullback-Leibler divergence, defined as
\[
D(\mu || q) := \int d\mu(x) p_x \ln \frac{p_x}{q_x}.
\]
A measure of the correlation between any two given random variables \( X \) and \( Y \) distributed according to the joint
probability density \( p_{(X,Y)} \), is given by the mutual information \( I(X;Y) \) defined as
\[
I(X;Y) := \text{D} \left( p_{(X,Y)} \left\| p_X p_Y \right. \right),
\]
where \( p_X \) and \( p_Y \) are the marginal probability distributions. For any ensemble \( \rho \) and POVM \( \pi \) we denote with \( I(\rho, \pi) \) the mutual information \( I(X;Y) \) between random variables \( X \) and \( Y \) distributed according to \( (x,y) \mapsto p_{x,y} := \text{Tr}[p_x \pi_y] \).

Following Ref. [25], we define the capacity of any POVM as follows.

**Definition 2** (Unassisted classical capacity of POVMs).
The unassisted classical capacity of any POVM \( \pi \) is given by
\[
C(\pi) := \lim_{t \to \infty} \frac{1}{t} \max_{\rho} I(\rho, \pi),
\]
where \( \pi^{\otimes t} \) stands for the POVM \( y \mapsto \pi_y^{\otimes t} \) and the maximum is over ensembles \( \rho \).

The operational interpretation of Definition 2 is provided by Shannon’s noisy-channel coding theorem [22], which proves that the capacity represents the maximum amount of information that can be reliably conveyed through POVM per use of the device, in the asymptotic limit.

Its explicit computation is in general very challenging. To proceed, let us first notice that the problem can be simplified as follows
\[
C(\pi) = W(\pi) := \max_{\rho} I(\rho, \pi),
\]
where \( W(\pi) \) is the informational power [22, 32] of POVM \( \pi \), and its additivity has been proved in Ref. [23].

A further simplification in the calculation of \( W(\pi) \) follows from the fact that, without loss of generality, the maximum in Eq. (5) can be taken over pure ensembles, as proved in Ref. [23].

We now introduce the following important preliminary result.

**Lemma 1.** The capacity \( C(\pi) \) of any POVM \( y \mapsto dq_y \pi_y \) is upper bounded by
\[
C(\pi) \leq \max_{\phi} \text{D} \left( q^\phi \left\| q \right. \right),
\]
where \( q^\phi \) denotes the probability density \( y \mapsto dq_y \langle \phi \mid \pi_y \phi \rangle \) and \( \eta(x) := -x \ln x \). The inequality is tight if \( \text{I} / \text{d} \in \text{conv}(\{ \langle \phi \mid \pi_y \phi \rangle \}) \) where \( \{ \langle \phi \mid \pi_y \phi \rangle \} \) consists of the states that optimize Eq. (5) and \( \text{conv} \) denotes the convex hull.

**Proof.** For any pure ensemble \( x \mapsto \phi_x := p_x |\phi_x \rangle \langle \phi_x| \), let \( \sigma := \int \text{d} \mu(x) p_x \phi_x \) be the average state. Then one has
\[
I(\phi_x, \pi_y)
= d \int \text{d} \mu(x) \text{d} \mu(y) p_{x,y} \langle \phi_x \mid \pi_y \phi_x \rangle \ln \frac{\langle \phi_x \mid \pi_y \phi_x \rangle}{\text{Tr}[\sigma \pi_y]}.
\]
Since one has
\[
\ln \frac{\langle \phi_x \mid \pi_y \phi_x \rangle}{\text{Tr}[\sigma \pi_y]} = \ln \left( d \langle \phi_x \mid \pi_y \phi_x \rangle \right) - \ln \frac{\text{Tr}[\sigma \pi_y]}{q_y},
\]
it follows that
\[
I(\phi_x, \pi_y)
\leq d \int \text{d} \mu(x) \text{d} \mu(y) p_{x,y} \langle \phi_x \mid \pi_y \phi_x \rangle \ln \left( d \langle \phi_x \mid \pi_y \phi_x \rangle \right) - d \text{Tr}[\sigma \pi_y] \ln \frac{\text{Tr}[\sigma \pi_y]}{q_y}.
\]
From the non-negativity of the relative entropy one has
\[
I(\phi_x, \pi_y)
\leq \text{D} \left( d \langle \phi_x \mid \pi_y \phi_x \rangle \left\| p_{x,y} \right. \right) - \text{D} \left( d \langle \phi_x \mid \pi_y \phi_x \rangle \left\| q_y \right. \right)
\]
where the last inequality follows from upper bounding the average with the largest element. Notice that both inequalities are saturated iff \( \text{Tr}[\sigma \pi_y] = 1 \) which is fulfilled whenever there exists a probability density \( x \mapsto p_x \) such that \( \sigma = \text{I} / \text{d} \) and all the states \( \phi_x \) optimize Eq. (6).

Notice that, as a trivial consequence of Holevo’s theorem [44], the quantity \( \ln d - C(\pi) \), for which Lemma 1 provides a lower bound, can be interpreted as a measure of how suboptimal measurement \( \pi \) is for communication tasks.

The maximization of the communication capacity can now be significantly simplified in some cases. Indeed, let us set \( a := \min_{\psi, \phi} \langle \psi \mid \pi_y \psi \rangle \), \( b := \max_{\psi, \phi} \langle \psi \mid \pi_y \psi \rangle \), and let \( r : [a, b] \to \mathbb{R} \) be the Hermite interpolating polynomial of \( \eta \) such that the assumptions of Lemma 3 in the Appendix are fulfilled (and so \( r \) interpolates \( \eta \) from below). Then
\[
\int \text{d} \mu(y) p_{y,\eta} r(\langle \phi \mid \pi_y \phi \rangle) \geq \int \text{d} \mu(y) p_{y,\eta} r(\langle \phi \mid \pi_y \phi \rangle) =: Q(\phi),
\]
with equality if and only if \( \phi \) is such that all the points of interpolation \( x_i \) are of the form \( \langle \phi \mid \pi_y \phi \rangle \) (then necessarily the set of all overlaps \( \{ \langle \phi \mid \pi_y \phi \rangle \} \subseteq A \) has to be finite).

In consequence, we get
\[
C(\pi) \leq \ln d - \min_\phi Q(\phi).
\]
Let us now assume that we have some predictions about the state that minimizes the lhs of the inequality in Eq. 7 and let \( \phi_0 \) be the supposed minimizer. Then, if the points of interpolation are chosen to be \( \{ \langle \phi_0 \rangle \}_{y \in A} \), in order to show that \( \phi_0 \) is indeed a minimizer, it is enough to show that \( \phi_0 \) minimizes \( Q \). In particular, this becomes trivial whenever \( Q(\phi) \) is constant. In consequence, we get the equality in Eq. 8 exactly as in Lemma 1. The remainder of this work is devoted to the quantification of \( C(\pi) \) through the optimization of \( Q(\phi) \), for the specific case of \( t \) designs.

### C. Main results

Our first main result is an upper bound on the capacity \( C(\pi) \) of any \( t \) design POVM \( \pi \), as a function of the dimension and of the indices of coincidence \( \gamma_i(\chi) \) (or, equivalently, of the moments \( \mu_\chi(\pi) \), for \( t \in [1,5] \).

**Theorem 1.** The classical capacity \( C(\pi) \) of any \( t \) design POVM \( \pi \) is upper bounded by \( C(\pi) \leq C_t \), where

\[
C_t = \begin{cases} 
C_1 & = \ln d \\
C_2 & = \ln d + \ln \frac{2\chi}{2}
\end{cases}
\]

\[
C_3 = \ln d + \frac{d(d-1)}{2(\gamma_1-\gamma_2)} \ln \frac{\gamma_1 - \gamma_2}{\gamma_3},
\]

\[
C_4 = \ln d + \ln 2
\]

\[
+ \frac{d(d-1)}{2(\gamma_1-\gamma_2)} \ln \frac{\gamma_1 - \gamma_2}{\gamma_3},
\]

\[
\frac{d(d-1)}{2(\gamma_1-\gamma_2)} \ln \frac{\gamma_1 - \gamma_2}{\gamma_3}
\]

with \( \Delta_4 = -3\gamma_2^2 \gamma_3 + 4\gamma_1 \gamma_3^3 + 4\gamma_2^2 \gamma_4 - 6\gamma_1 \gamma_2 \gamma_5 + \gamma_2^2 \gamma_4^2 \).

**Proof.** Let \( 0 = x_0 < x_1 < \ldots < x_{\frac{\chi}{2}} \leq 1 \) and let \( x_{\frac{\chi}{2}} = 1 \) iff \( t \) is odd. According to Lemma 1, the polynomial \( r(x) := \sum a_i x^i \) of degree at most \( t \) such that \( r(x_k) = \eta(x_k) \) for any \( k \in [0, \frac{\chi}{2}] \) and \( r'(x_k) = \eta'(x_k) \) for any \( k \in [1, \frac{\chi}{2}] \) is such that \( r(x) \leq \eta(x) \) for any \( x \in [0,1] \). Therefore Eq. 9 becomes

\[
C(\pi) \leq C_t := \ln d - d \sum_{i=1}^{t} a_i \gamma_i,
\]

where we used the fact that the index of coincidence \( \gamma_i \) is independent of the choice of \( \phi \) for \( i \leq t \).

In order to upper bound \( C(\pi) \), we express \( \{ a_i \} \) as a function of \( \{ x_k \} \) and minimize \( C_t \) over \( \{ x_k \} \). \( C_t \) is an immediate result as it does not involve any optimization. Upon denoting with \( \{ x_k \} \) the optimal solution one has

\[
\begin{align*}
x_1 & = \gamma_1, \\
x_2 & = \frac{\gamma_1 - \gamma_2}{\gamma_3}, \\
x_{1,2} & = \frac{\gamma_1 \gamma_3 - \gamma_2 \gamma_4 \sqrt{\Delta_4}}{2(\gamma_1 - \gamma_2)},
\end{align*}
\]

The expression for \( C_5 \) is too lengthy to be reproduced here, but the derivation goes along the same lines as that of \( C_4 \) with

\[
x_{1,2} = \frac{\gamma_2 \gamma_3 - \gamma_2^2 \gamma_4 + \gamma_2 \gamma_5 + \gamma_2 \gamma_7 - \gamma_2 \gamma_5 - \gamma_2 \gamma_7 \pm \sqrt{\Delta_5}}{2(\gamma_2 + \gamma_3 + \gamma_4)},
\]

and

\[
\Delta_5 = (\gamma_3(\gamma_3 - \gamma_4 + \gamma_4) + \gamma_4(\gamma_4 - \gamma_5 + \gamma_5)) \gamma_6 + 4 \left( \frac{\gamma_2 + \gamma_3 + \gamma_7}{\gamma_2} \right)^2 \\
\times \left( \gamma_3(\gamma_2 - \gamma_4) - \gamma_2 \gamma_6 \right)
\]

Notice that the optimization over \( \{ x_k \} \) in the proof of Theorem 1 is over \( \frac{\chi}{2} \) real parameters, and becomes cumbersome for \( t \) larger than 5, namely for 3 parameters or more.

Notice also that, as expected, the bounds in Theorem 1 reduce to those given in Ref. 31 for projective \( t \) designs, i.e. when \( \mu_\chi(\pi) = 1 \) for all \( k \in [1, t] \).

Our second main result is the derivation of the capacity \( C(D_\Delta(\pi)) \) for the depolarized version of several \( t \)-design POVMs: 2-dimensional SIC [23, 53, 54, 53] (tetrahedron), complete MUB [27, 51] (octahedron), or icosahedron [27, 51]; 3-dimensional SIC [28, 50, 51] or complete MUB [51, 53]; 8-dimensional Hoggar SIC [22]; \( d \)-dimensional anti-SIC or uniform rank-one [53] POVM.

**Theorem 2.** The classical capacity \( C(D_\Delta(\pi)) \) of the depolarized POVM \( \pi \), where \( \pi \) is a 2-dimensional SIC (tetrahedron), complete MUB (octahedron), or icosahedron, or 3-dimensional SIC or complete MUB, or 8-dimensional Hoggar SIC, or \( d \)-dimensional anti-SIC, or the uniform rank-one POVM is given by

\[
\begin{align*}
C_{\text{tetra}} & = \ln 2 - \frac{\eta(\frac{1+\sqrt{5}}{2}) + \eta(\frac{1-\sqrt{5}}{2})}{4}, \\
C_{\text{octa}} & = \ln 2 - \frac{\eta(\frac{1+\sqrt{5}}{2}) + \eta(\frac{1-\sqrt{5}}{2})}{4}, \\
C_{\text{icosah}} & = \ln 2 - \frac{\eta(\frac{1+\sqrt{5}}{2}) + \eta(\frac{1-\sqrt{5}}{2})}{4} + \frac{\eta(\frac{1+\sqrt{5}}{2}) + \eta(\frac{1-\sqrt{5}}{2})}{4}, \\
C_{\text{SIC}} & = \ln 3 - \eta(\frac{1+\sqrt{5}}{2}) - 2\eta(\frac{1+\sqrt{5}}{2}) - 2\eta(\frac{1+\sqrt{5}}{2}), \\
C_{\text{Hoggar}} & = \ln 8 - \frac{\eta(\frac{1+\sqrt{5}}{2}) + \eta(\frac{1-\sqrt{5}}{2})}{4}, \\
C_{\text{anti-SIC}} & = \ln d - \frac{\eta(\frac{1+\sqrt{5}}{2}) + \eta(\frac{1-\sqrt{5}}{2})}{4}, \\
C_{\text{uniform}} & = \ln(1-\lambda) + \lambda + \frac{\lambda^2}{d} F_1(1,1-d+2,\frac{d-1}{d},1-\lambda),
\end{align*}
\]

where \( 2F_1 \) denotes the hypergeometric function [14]. Moreover, the optimal ensembles for the depolarized versions of \( \pi \) are exactly the same as for \( \pi \) and they all average to the maximally mixed state.

**Proof.** It is well known that all the rank-one POVMs \( \pi \) included in the theorem are projective \( t \) designs, with \( t = 3 \) if \( \pi \) is a 2-dimensional complete MUB, \( t = 5 \) if \( \pi \) is an icosahedron, \( t = \infty \) if \( \pi \) is the uniform distribution, and \( t = 2 \) otherwise. It is easy to observe that the \( d \)-dimensional anti-SIC is a mixed \( 2 \)-design. Thus, due to Eq. 11, \( D_\Delta(\pi) \) is also a \( t \) design.

Let us first discuss the case where \( t \) is finite. Let us recall what the optimal ensembles are for POVMs, \( \pi \), as
are exactly \( \lambda x_k + (1 - \lambda) \frac{1}{2} \times m_k \)
where \( x_1 \leq \cdots \leq x_t \), \( x_1 = 0 \) and \( x_t = 1 \) whenever \( t \) is odd. Here the notation \( x_k \times m_k \) denotes the value \( x_k \) with multiplicity \( m_k \). Moreover, in all these cases there are exactly \( \lceil \frac{t}{2} \rceil + 1 \) different values of \( x_k \), thus \( t := \lceil \frac{t}{2} \rceil \).

Therefore one has

\[
\langle \phi | D_\lambda(\tilde{\pi}) | \phi \rangle = \begin{cases} 
\lambda x_0 + (1 - \lambda) \frac{1}{2} \times m_0, \\
\ldots \\
\lambda x_{\lceil \frac{t}{2} \rceil} + (1 - \lambda) \frac{1}{2} \times m_{\lceil \frac{t}{2} \rceil}.
\end{cases}
\]

Notice that

\[
\begin{cases}
\lambda x_1 + (1 - \lambda) \frac{1}{2} := \min_{\phi,y} \text{Tr}[\phi D_\lambda(\tilde{\pi}_y)], \\
\lambda x_{\lceil \frac{t}{2} \rceil} + (1 - \lambda) \frac{1}{2} := \max_{\phi,y} \text{Tr}[\phi D_\lambda(\tilde{\pi}_y)]
\end{cases}
\]

if \( t \) is odd.

Therefore, according to Lemma 8, the Hermite interpolating polynomial \( r \) such that \( r(x_k) = \eta(x_k) \) for any \( k \in [0,\lceil \frac{t}{2} \rceil] \) and \( r'(x_k) = \eta'(x_k) \) for any \( k \in [1,\lceil \frac{t-1}{2} \rceil] \) is such that \( \deg r \leq t \) and \( r(x) \leq \eta(x) \) for any \( x \in [\frac{t}{2}, (1 - \lambda) \frac{t}{2}] \). Therefore, the average of this polynomial has to be constant and according to our remarks at the end of Sec. IIB the maximum in Eq. 6 is attained by \( \phi \).

Let us now discuss the case \( t = \infty \). The statement follows by expanding \( \eta(x) = -x \ln x \) in Lemma 1 in a Taylor series around \( \lambda + \frac{1}{2} \lambda \), applying the binomial theorem, replacing \( x = \lambda | \langle \phi | U_y | \phi \rangle |^2 + \frac{1}{2} - \frac{1}{2} \), and using the identity \( \int \text{d}g | \langle \phi | U_y | \phi \rangle |^2 \)k \( = \binom{d+k-1}{k}^{-1} \). Notice that the series for \( \eta(x) \) has radius of convergence \( \lambda + \frac{1}{2} \lambda \), which is also equal to \( b := \max_{\phi} \langle \psi | D_\lambda(\pi_y) | \psi \rangle \), therefore \( \eta(x) \) is equivalent to its series in \( [a,b] \).

Notice that, as expected, the capacities in Theorem 2 reduce to those given in Refs. 23, 27, 32, 45 for projective \( t \) designs when one takes the limit \( \lambda \rightarrow 1 \).

The results of Theorems 1 and 2 are represented in Figs. 1, 2, and 3 in the 2-, 3-, and 8-dimensional cases, respectively.

Finally, let us notice that the capacity \( C(\pi) \) of any measurement \( \pi \) can be expressed in terms of the well-known accessible information 14, 15, 16, 52 \( A(\rho) := \max_{\pi} I(\rho, \pi) \) of ensemble \( \rho \), where the maximum is over any POVM \( \pi \). The relation is given by \( C(\pi) = \max_{\pi} A(\sigma^{1/2} \pi \sigma^{1/2}) \), where by \( \sigma^{1/2} \pi \sigma^{1/2} \) we mean the ensemble \( y \mapsto \frac{1}{2} \sigma^{1/2} \pi \sigma^{1/2} \) and the maximum is taken over all states \( \sigma \). Therefore, in general one has \( W(\pi) \geq A(\frac{1}{2} \pi) \), and thus the upper bound on \( C(\pi) \) provided
in Thm. 1 also upper bounds the accessible information $A_{\frac{\pi}{2\pi}}$. Moreover, whenever the ensemble attaining the capacity for POVM $\pi$ averages to the maximally mixed state, one has that $A_{\frac{\pi}{2\pi}}$ the informational power $W(\pi) = A_{\frac{\pi}{2\pi}}$. Therefore, the expressions for the capacities in Thm. 2 also represent the accessible information $A_{\frac{\pi}{2\pi}}$ of the corresponding ensembles.

### III. CONCLUSIONS AND OUTLOOK

In this work we introduced mixed quantum $t$ designs as the most general arbitrary-rank extension of projective $t$ designs that preserves indistinguishability from the uniform distribution given $t$ copies. We addressed the problem of quantifying the communication capacity of mixed $t$ design measurements by deriving upper bounds on such a quantity for any $t \in [1, 5]$. We refined our results by providing a closed-form solution for the communication capacity of several mixed $t$ designs measurements, including the depolarized version of: any qubit and qutrit SIC and MUBs, any qubit icosahedral measurement, any Hoggar SIC, any anti-SIC, and the uniform distribution.

One might conjecture that the quantification of the communication capacity of mixed ($t = \infty$)-design measurements would shine new light on the problem of lower bounding the communication capacity of mixed quantum POVMs, in the same way as the quantification of the accessible information of projective ($t = \infty$)-designs allowed for the computation of a lower bound on the accessible information of ensembles of pure states.

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### Appendix A: Hermite interpolation

For the completeness of our reasoning we add a brief description of the optimization method based on Hermite interpolation, first introduced in this context and discussed in detail in Ref. [27]. Let us first recall the well-known formula for the Hermite interpolation error.

**Lemma 2.** Let $a \leq x_0 < \cdots < x_{m-1} \leq b$ be reals and let $\{j_i\}_{i=0}^{m-1}$ be positive integers. Let $\eta(x)$ be a real function with at least $t+1$ continuous derivatives on $[a, b]$ and let $r(x)$ be its Hermite interpolating polynomial, namely the polynomial of degree at most $t$ := $\sum_i j_i - 1$ that agrees with $\eta(x)$ at $x_i$ up to derivative of order $j_i - 1$ for all $i$, i.e. $r^{(k)}(x_i) = \eta^{(k)}(x_i), \quad 0 \leq k \leq j_i - 1, \quad 0 \leq i \leq m - 1$.

Then for any $x \in [a, b]$ there exists $x'$ such that $\min(x, x_0) < x' < \max(x, x_{m-1})$ and

$$\eta(x) - r(x) = \frac{\eta^{(t+1)}(x')}{(t+1)!} \prod_{i=0}^{m-1} (x - x_i)^{j_i}.$$  

**Proof.** See, e.g. Ref. [57].

In this work we set $\eta(x) := -x \ln(x)$. Next lemma provides sufficient conditions for the polynomial $r$ to interpolate $\eta$ from below.

**Lemma 3.** Let $m \geq 2$, $x_0 = a$ and $j_0 = 1$. Then whenever

a. $x_{m-1} = b$, $j_{m-1} = 1$ and $j_i = 2$ for $0 < i < m - 1$, or

b. $x_{m-1} < b$, and $j_i = 2$ for $0 < i$,

one has $\eta(x) - r(x) \geq 0$ for any $x \in [a, b]$.

**Proof.** The proof was first derived in Ref. [27]. We report it here for clarity.

a. In this case one has

$$\prod_{i=0}^{m-1} (x - x_i)^{j_i} = (x - a)(x - b) \prod_{i=0}^{m-1} (x - x_i)^2 \leq 0.$$  

Likewise, $t$ is odd and so $\frac{\eta^{(t+1)}(x')}{(t+1)!} \leq 0$, since all even derivatives of $\eta(x)$ are negative.
b. In this case one has
\[ \prod_{i=0}^{m-1} (x-x_i)^{j_i} = (x-a) \prod_{i=0}^{m-1} (x-x_i)^2 \geq 0. \]

Likewise, \( t \) is even and \( \frac{\eta^{(t+1)}(x)}{(t+1)!} \geq 0 \), since all odd (greater than 1) derivatives of \( \eta(x) \) are positive. \( \square \)

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