ON THE MAXIMAL FUNCTION ASSOCIATED TO THE LACUNARY SPHERICAL MEANS ON THE HEISENBERG GROUP

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Dedicated to the memory of Eli Stein

Abstract. We investigate the $L^p$ boundedness of the lacunary maximal function $A_r f$ associated to the spherical means on the Heisenberg group. By suitable adaptation of an approach of M. Lacey in the Euclidean case, we obtain sparse bounds for these maximal functions, which lead to new weighted estimates. In order to prove the result, several properties of the spherical means have to be accomplished, namely, the $L^p$ improving property of the operator $A_r f$ and a continuity property of the difference $A_r f - \tau_y A_r f$, where $\tau_y f(x) = f(xy^{-1})$ is the right translation operator.

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1. Introduction and main results

A celebrated theorem of Stein [19] proved in 1976 says that the spherical maximal function $M$ defined by

$$Mf(x) = \sup_{r>0} |f * \sigma_r(x)| = \sup_{r>0} \left| \int_{|y|=r} f(x-y) d\sigma_r(y) \right|$$

is bounded on $L^p(\mathbb{R}^n)$, $n \geq 2$, if and only if $p > n/(n-1)$. Here $\sigma_r$ stands for the normalised surface measure on the sphere $S_r = \{ x \in \mathbb{R}^n : |x| = r \}$ in $\mathbb{R}^n$. The case $n = 2$ was proved later by Bourgain [3]. As opposed to this, in 1979, C. P. Calderón [4] proved that the lacunary maximal function

$$M_{\text{lac}}f(x) = \sup_{j \in \mathbb{Z}} \left| \int_{|y|=2^j} f(x-y) d\sigma_2^j(y) \right|$$

is bounded on $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$ for any $n \geq 1$. In a recent article, Lacey [11] has revisited the spherical maximal function. Using a new approach he has managed to prove certain sparse bounds for these maximal functions which has led him to obtain new weighted norm inequalities. Our main goal in this paper is to adapt the method of Lacey to prove an analogue of Calderón’s theorem in the context of certain spherical means on the Heisenberg group, and deduce weighted inequalities as immediate consequences.

Let $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$ be the $(2n + 1)$-dimensional Heisenberg group with the group law

$$(z, t)(w, s) = (z + w, t + s + \frac{1}{2} \text{Im} z \cdot \overline{w}).$$

Given a function $f$ on $\mathbb{H}^n$, consider the spherical means

$$A_r f(z, t) := f * \mu_r(z, t) = \int_{|w|=r} f(z-w, t - \frac{1}{2} \text{Im} z \cdot \overline{w}) \, d\mu_r(w)$$

where $\mu_r$ is the normalised surface measure on the sphere $S_r = \{ (z, 0) : |z| = r \}$ in $\mathbb{H}^n$. The maximal function associated to these spherical means was first studied by Nevo and Thangavelu in [17]. Later, improving the results in [17], Narayanan and Thangavelu [16] and Müller and Seeger [15], independently, proved the following sharp maximal theorem: the full maximal function $Mf(z, t) = \sup_{r>0} |A_r f(z, t)|$ is bounded on $L^p(\mathbb{H}^n)$, $n \geq 2$ if and only if $p > (2n)/(2n - 1)$. In this work we consider the lacunary maximal function

$$M_{\text{lac}}f(z, t) = \sup_{j \in \mathbb{Z}} |A_{\delta^j} f(z, t)|, \quad \delta > 0,$$

associated to the spherical means and prove the following result.

**Theorem 1.1.** Assume that $n \geq 2$. Then for any $0 < \delta < \frac{1}{96}$ the associated lacunary maximal function $M_{\text{lac}}$ is bounded on $L^p(\mathbb{H}^n)$ for any $1 < p < \infty$.

Actually we can take any $\delta$ in Theorem 1.1. For example, we can take $\delta = 2$. In our result we are taking $\delta < \frac{1}{96}$ not because the proof requires the restriction, but because we want to keep the proof simple, see more explanation after the statement of Lemma 4.3.

We remark that another kind of spherical maximal function on the Heisenberg group has been considered by Cowling. In [6] he has studied the maximal function associated to the spherical means taken over genuine Heisenberg spheres, i.e., averages taken over spheres defined in terms of a homogeneous norm on $\mathbb{H}^n$. It would be interesting to see if lacunary maximal functions associated these spherical means also have better mapping properties. We remark in passing that the spherical means studied in [17] [16] [15] (and hence in this paper)
are more singular than the one studied in [6] as these means are supported on codimension two submanifolds.

Theorem 1.2 as well as certain weighted versions, are easy consequences of the sparse bound in Theorem 1.2, which is the main result of this paper. Before stating the result let us set up the notation. As in the case of \( \mathbb{R}^n \), there is a notion of dyadic grids on \( \mathbb{H}^n \), the members of which are called (dyadic) cubes. A collection of cubes \( S \) in \( \mathbb{H}^n \) is said to be \( \eta \)-sparse if there are sets \( \{ E_S : S \in S \} \) which are pairwise disjoint and satisfy \( |E_S| > \eta |S| \) for all \( S \in S \). For any cube \( Q \) and \( 1 < p < \infty \), we define

\[
\langle f \rangle_{Q,p} := \left( \frac{1}{|Q|} \int_Q |f(x)|^p \, dx \right)^{1/p}, \quad \langle f \rangle_{Q} := \frac{1}{|Q|} \int_Q |f(x)| \, dx.
\]

In the above \( x = (z, t) \in \mathbb{H}^n \) and \( dx = dzdt \) is the Lebesgue measure on \( \mathbb{C}^n \times \mathbb{R} \) which incidentally is the Haar measure on the Heisenberg group. Following Lacey [11], by the term \((p, q)\)-sparse form we mean the following:

\[
\Lambda_{S,p,q}(f, g) = \sum_{S \in S} |S| \langle f \rangle_{S,p} \langle g \rangle_{S,q}.
\]

**Theorem 1.2.** Assume \( n \geq 2 \) and fix \( 0 < \delta < \frac{1}{96} \). Let \( 1 < p, q < \infty \) be such that \((\frac{1}{p}, \frac{1}{q})\) belongs to the interior of the triangle joining the points \((0, 1), (1, 0)\) and \((\frac{n}{n+1}, \frac{n}{n+1})\). Then for any pair of compactly supported bounded functions \((f, g)\) there exists a \((p, q)\)-sparse form such that \( \langle M_{\mathrm{lac}} f, g \rangle \leq C \Lambda_{S,p,q}(f, g) \).

In proving the corresponding result for the spherical means on \( \mathbb{R}^n \), Lacey [11] has made use of two important properties of the spherical means, namely, the \( L^p \) improving property of the operator \( S_r f = f \ast \sigma_r \) for a fixed \( r \), and the continuity property of the difference \( S_r f - \tau_y S_r f \) where \( \tau_y f(x) = f(x - y) \) is the translation operator.

A remark is in order. In order to keep the shape of our main results analogous to the ones related to the lacunary maximal function in \( \mathbb{R}^n \), we decided to restrict the range of \((p, q)\) to the same regions as in the Euclidean case. Nevertheless, enhanced results for the lacunary maximal function in \( \mathbb{H}^n \) are obtained (although we cannot say anything about the sharpness of such results) and we will also state them throughout the paper, see Subsections 2.1, 3.1, 4.1 and 5.1. In particular, a sharpened version of Theorem 1.2 is given in Theorem 4.9.

In the next section we establish \( L^p - L^q \) estimates for our spherical means \( A_r f \) on the Heisenberg group. In Section 3 we prove the continuity property of \( A_r f - A_r \tau_y f \), where now \( \tau_y f(x) = f(xy^{-1}) \) is the right translation operator. In Section 4 we establish the sparse bound and finally in the last section we deduce weighted boundedness properties of the lacunary maximal function.

2. **The \( L^p \) Improving Property of the Spherical Mean Value Operator**

The observation that the spherical mean value operator \( S_r f := f \ast \sigma_r \) on \( \mathbb{R}^n \) is a Fourier multiplier plays an important role in every work dealing with the spherical maximal function. In fact, we know that

\[
f \ast \sigma_r(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i\xi \cdot x} \hat{f}(\xi) \frac{J_{n/2-1}(r|\xi|)}{(r|\xi|)^{n/2-1}} \, d\xi
\]

where \( J_{n/2-1} \) is the Bessel function of order \( n/2 - 1 \). As Bessel functions \( J_\alpha \) are defined even for complex values of \( \alpha \) the above allows one to embed \( S_r f \) in an analytic family of
operators and Stein’s analytic interpolation theorem comes in handy in studying the spherical
maximal function. The same technique was employed by Strichartz [21] who studied the \( L^p \)
improving properties of \( S_r \). For the spherical means on the Heisenberg group we do have
such a representation if we replace the Euclidean Fourier transform by the group Fourier
transform on \( \mathbb{H}^n \).

For the group \( \mathbb{H}^n \) we have a family of irreducible unitary representations \( \pi_\lambda \) indexed by
non-zero reals \( \lambda \) and realised on \( L^2(\mathbb{R}^n) \). The action of \( \pi_\lambda(z,t) \) on \( L^2(\mathbb{R}^n) \) is explicitly given
by
\[
\pi_\lambda(z,t)\varphi(\xi) = e^{i\lambda t} e^{i\lambda (x \cdot \xi + \frac{1}{2} x \cdot y)} \varphi(\xi + y)
\]
where \( \varphi \in L^2(\mathbb{R}^n) \) and \( z = x + iy \). By the theorem of Stone and von Neumann, which
classifies all the irreducible unitary representations of \( \mathbb{H}^n \), combined with the fact that the
Plancherel measure for \( \mathbb{H}^n \) is supported only on the infinite dimensional representations, it
is enough to consider the following operator valued function known as the group Fourier
transform of a given function \( f \) on \( \mathbb{H}^n 
\[
\widehat{f}(\lambda) = \int_{\mathbb{H}^n} f(z,t) \pi_\lambda(z,t) \, dz \, dt.
\]
The above is well defined, e.g., when \( f \in L^1(\mathbb{H}^n) \) and for each \( \lambda \neq 0 \), \( \widehat{f}(\lambda) \) is a bounded
linear operator on \( L^2(\mathbb{R}^n) \). By the theorem of Stone and von Neumann, which
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linear operator on \( L^2(\mathbb{R}^n) \). Observe that the above definition makes sense even if we replace
\( f \) by a finite Borel measure \( \mu \). In particular, \( \widehat{\mu_r}(\lambda) \) are well defined bounded operators on
\( L^2(\mathbb{R}^n) \) which can be described explicitly. Combined with the fact that \( \hat{f} \ast \hat{g}(\lambda) = \hat{f}(\lambda)\hat{g}(\lambda) \)
we obtain \( \widehat{A_r f}(\lambda) = \hat{f}(\lambda)\widehat{\mu_r}(\lambda) \).

The operators \( \widehat{\mu_r}(\lambda) \) turn out to be functions of the Hermite operator \( H(\lambda) = -\Delta + \lambda^2 |x|^2 \).
Indeed, if the spectral decomposition of \( H(\lambda) \) is written as
\[
H(\lambda) = \sum_{k=0}^{\infty} (2k + n)|\lambda| P_k(\lambda)
\]
where \( P_k(\lambda) \) are the Hermite projection operators, then (see [23, Proposition 4.1])
\[
\widehat{\mu_r}(\lambda) = \sum_{k=0}^{\infty} \psi_k^n(\sqrt{|\lambda| r}) P_k(\lambda),
\]
where for any \( \delta > -1 \) the normalised Laguerre functions are defined by
\[
\psi_k^\delta(r) = \frac{\Gamma(k+1)\Gamma(\delta+1)}{\Gamma(k+\delta+1)} L_k^\delta \left( \frac{1}{2} r^2 \right) e^{-\frac{1}{2} r^2}.
\]
In the above definition \( L_k^\delta(r) \) stands for the Laguerre polynomials of type \( \delta \). Thus we have
the relation
\[
\widehat{A_r f}(\lambda) = \hat{f}(\lambda) \sum_{k=0}^{\infty} \psi_k^n(\sqrt{|\lambda| r}) P_k(\lambda),
\]
which is the analogue of \( (2.1) \) in our situation. Thus, as in the Euclidean case, the spherical
mean value operators \( A_r \) are (right) Fourier multipliers on the Heisenberg group. We now
proceed to rewrite \( (2.4) \) in terms of Laguerre expansions, which is more suitable for defining
an analytic family of operators containing the spherical means.
The irreducible unitary representations $\pi_\lambda$ admit the factorisation $\pi_\lambda(z, t) = e^{i\lambda t}\pi_\lambda(z, 0)$ and hence we can write the Fourier transform as

$$\hat{f}(\lambda) = \int_{\mathbb{C}^n} f^\lambda(z)\pi_\lambda(z, 0)\,dz,$$

where for a function $f$ on $\mathbb{H}^n$, $f^\lambda(z)$ stands for the partial inverse Fourier transform

$$f^\lambda(z) = \int_{-\infty}^{\infty} e^{i\lambda t} f(z, t)\,dt.$$

We now make use of the special Hermite expansion of the function $f^\lambda$, which can be put in a compact form as follows. Let $\varphi_k^\lambda(z) = L_k^\lambda_n\left(\left(\frac{1}{2}|\lambda||z|^2\right)e^{-\frac{1}{2}|z|^2}\right)$ stand for the Laguerre functions of type $(n - 1)$ on $\mathbb{C}^n$. The $\lambda$-twisted convolution $f^\lambda \ast_\lambda \varphi_k^\lambda(z)$ is then defined by

$$f^\lambda \ast_\lambda \varphi_k^\lambda(z) = \int_{\mathbb{C}^n} f^\lambda(z - w)\varphi_k^\lambda(w) e^{i\frac{1}{2}Im|w|^2} \,dw.$$

It is well known that one has the expansion (see [25, Chapter 3, proof of Theorem 3.5.6])

$$f^\lambda(z) = (2\pi)^{-n}|\lambda|^n \sum_{k=0}^{\infty} f^\lambda \ast_\lambda \varphi_k^\lambda(z),$$

which leads to the formula (see [25, Theorem 2.1.1])

$$f(z, t) = (2\pi)^{-n-1} \int_{-\infty}^{\infty} e^{-i\lambda t} \left(\sum_{k=0}^{\infty} f^\lambda \ast_\lambda \varphi_k^\lambda(z)\right)|\lambda|^n d\lambda.$$

Applying this to $f \ast \mu_r$ we have the formula

$$f \ast \mu_r(z, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\lambda t} f^\lambda \ast_\lambda \mu_r(z) \,d\lambda$$

where we used the fact that $(f \ast \mu_r)^\lambda(z) = f^\lambda \ast_\lambda \mu_r(z)$. It can be shown that [23, Theorem 4.1], [17, Proof of Proposition 6.1],

$$f^\lambda \ast_\lambda \mu_r(z) = (2\pi)^{-n}|\lambda|^n \sum_{k=0}^{\infty} \frac{k!(n - 1)!}{(k + n - 1)!} \varphi_k^\lambda(r)f^\lambda \ast_\lambda \varphi_k^\lambda(z),$$

leading to the expansion (see [17, 16])

(2.5) \quad \quad A_r f(z, t) = (2\pi)^{-n-1} \int_{-\infty}^{\infty} e^{-i\lambda t} \left(\sum_{k=0}^{\infty} \psi_k^{n-1}(\sqrt{|\lambda|} r)f^\lambda \ast_\lambda \varphi_k^\lambda(z)\right)|\lambda|^n d\lambda.

By replacing $\psi_k^{n-1}$ by $\psi_k^\delta$ we get the family of operators taking $f$ into

$$(2\pi)^{-n-1} \int_{-\infty}^{\infty} e^{-i\lambda t} \psi_k^\delta(\sqrt{|\lambda|} r)f^\lambda \ast_\lambda \varphi_k^\lambda(z)|\lambda|^n d\lambda.$$

We make use of these operators in studying the $L^p$ improving properties of the spherical mean value operator.

In what follows we require sharp estimates on the normalised Laguerre functions given in (2.3). It is convenient to express $\psi_k^\delta(r)$ in terms of the standard Laguerre functions

$$L_k^\delta(r) = \left(\frac{\Gamma(k + 1)\Gamma(\delta + 1)}{\Gamma(k + \delta + 1)}\right)^{\frac{1}{2}} L_k(r)e^{-\frac{1}{2}r^2}\delta/2$$
which form an orthonormal system in $L^2((0, \infty), dr)$. In terms of $L^\delta_k(r)$, we have
\[
\psi_k^\delta(r) = 2^\delta \left( \frac{\Gamma(k + 1) \Gamma(\delta + 1)}{\Gamma(k + \delta + 1)} \right)^{\frac{1}{2}} r^{-\delta} L^\delta_k \left( \frac{1}{2} r^2 \right).
\]

Asymptotic properties of $L^\delta_k(r)$ are well known in the literature and we have the following result, see [24, Lemma 1.5.3] (actually, the estimates in Lemma 2.1 below are sharp, see [13, Section 2] and [14, Section 7]).

**Lemma 2.1** ([24]). For $\delta > -1$, we have the following:
\[
|L^\delta_k(r)| \leq C \left\{ \begin{array}{ll}
(kr)^{\delta/2}, & 0 \leq r \leq \frac{1}{k} \\
(kr)^{-\frac{1}{2}}, & \frac{1}{k} \leq r \leq \frac{k}{2} \\
k^{-\frac{1}{2}}(k^{\frac{3}{4}} + |k - r|)^{-\frac{1}{4}}, & \frac{k}{2} \leq r \leq \frac{3k}{2} \\
e^{-\gamma r}, & r \geq \frac{3k}{2},
\end{array} \right.
\]
where $\gamma > 0$ is a fixed constant.

We can now rewrite the above estimates of $L^\delta_k$ in terms of estimates for the normalised Laguerre functions $\psi_k^\delta$.

**Lemma 2.2.** For any $\delta \geq -\frac{1}{3}$, we have the uniform estimates
\[
\sup_k |\psi_k^\delta(\sqrt{|\lambda|})| \leq C \begin{cases} 
1, & \text{if } |\lambda| \leq 1 \\
|\lambda|^{-\delta-\frac{1}{2}}, & \text{if } |\lambda| > 1.
\end{cases}
\]

**Proof.** Since $\frac{\Gamma(k+1)\Gamma(\delta+1)}{\Gamma(k+\delta+1)} \leq C k^{-\delta}$ we need to bound $(k|\lambda|)^{-\delta/2} L^\delta_k \left( \frac{1}{2} |\lambda| \right)$ for $|\lambda| \geq 1$. When $1 \leq \frac{1}{2} |\lambda| \leq \frac{k}{2}$ we have the estimate
\[
|\psi_k^\delta(\sqrt{|\lambda|})| \leq C (k|\lambda|)^{-\delta/2 - 1/4}.
\]
From here, since $\delta + \frac{1}{2} \geq 0$, $\lambda^2 \leq k|\lambda|$, we get
\[
|\psi_k^\delta(\sqrt{|\lambda|})| \leq C |\lambda|^{-\delta - 1/2}.
\]
When $\frac{k}{2} \leq \frac{1}{2} |\lambda| \leq \frac{3k}{2}$, $|\lambda|$ is comparable to $k$ and hence we have
\[
|\psi_k^\delta(\sqrt{|\lambda|})| \leq C (k|\lambda|)^{-\delta/2} k^{-\frac{1}{2}} k^{-\frac{1}{2}} \leq C |\lambda|^{-\delta - \frac{1}{2}}.
\]
On the region $|\lambda| \geq \frac{3k}{2}$ we have exponential decay. Finally, the estimate $\sup_k |\psi_k^\delta(\sqrt{|\lambda|})| \leq C$ for $0 \leq |\lambda| \leq 1$ is immediate, in view of Lemma 2.1. With this we prove the lemma.

**Lemma 2.3.** For any $\delta \geq \frac{1}{2}$ and $|\lambda| \geq 1$ we have
\[
\sup_k (k|\lambda|)^{\frac{1}{2}} |\psi_k^\delta(\sqrt{|\lambda|})| \leq C |\lambda|^{-\delta + \frac{3}{2}}.
\]

**Proof.** As in the proof of Lemma 2.2, in the range $1 \leq \frac{1}{2} |\lambda| \leq \frac{k}{2}$,
\[
(k|\lambda|)^{\frac{1}{2}} |\psi_k^\delta(\sqrt{|\lambda|})| \leq C (k|\lambda|)^{-\delta/2 + 1/4} \leq C |\lambda|^{-\delta + \frac{1}{2}}
\]
as $\delta \geq \frac{1}{2}$. When $\frac{k}{2} \leq \frac{1}{2} |\lambda| \leq \frac{3k}{2}$, as before
\[
(k|\lambda|)^{\frac{1}{2}} |\psi_k^\delta(\sqrt{|\lambda|})| \leq C |\lambda|^{-\delta + 1 - \frac{1}{4}} \leq C |\lambda|^{-\delta + \frac{3}{2}}.
\]
The Laguerre functions $\psi_k^\beta$ can be defined for all values of $\delta > -1$, even for complex $\delta$ with $\text{Re}\delta > -1$ and we would like to use this fact to embed $A_1$ into an analytic family of operators. With the analytic interpolation in mind we define

$$A^\beta f(z, t) = (2\pi)^{-n-1} \int_{-\infty}^{\infty} e^{-i\lambda t} \left( \sum_{k=0}^{\infty} \psi_k^{\beta+n-1}(\sqrt{|\lambda|} f^\lambda *_\lambda \varphi_k^\lambda(z)) \right) |\lambda|^n d\lambda,$$

for $\text{Re}(\beta + n - 1) > -1$. Note that for $\beta = 0$ we recover $A_1$, thus $A_1 = A^0$. We will use the following relation between Laguerre polynomials of different types in order to express $A^\beta$ in terms of $A_1$ (see [18, (2.19.2.12)])

$$L_k^{\mu+\nu}(r) = \frac{\Gamma(k+\mu+\nu+1)}{\Gamma(\nu)\Gamma(k+\mu+1)} \int_0^1 t^\mu(1-t)^{\nu-1} L_k^\mu(rt) dt,$$

valid for $\text{Re} \mu > -1$ and $\text{Re} \nu > 0$. We define

$$P_r f(z, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\lambda t} e^{-\frac{1}{4}|\lambda|^2} f^\lambda(z) d\lambda$$

to be the Poisson integral of $f$ in the $t$-variable. We see that for $\text{Re} \beta > 0$, $A^\beta$ is given by the following representation.

**Lemma 2.4.** Let $\text{Re} \beta > 0$. The operator $A^\beta$ is given by the formula

$$A^\beta f(z, t) = 2 \frac{\Gamma(\beta+n)}{\Gamma(\beta)\Gamma(n)} \int_0^1 r^{2n-1}(1-r^2)^{\beta-1} P_{1-r^2} f * \mu_r(z, t) dr.$$

**Proof.** In view of (2.6), it is enough to verify

$$2 \frac{\Gamma(\beta+n)}{\Gamma(\beta)\Gamma(n)} \int_0^1 r^{2n-1}(1-r^2)^{\beta-1} P_{1-r^2} f * \mu_r(z, t) dr$$

$$= (2\pi)^{-n-1} \int_{-\infty}^{\infty} e^{-i\lambda t} \left( \sum_{k=0}^{\infty} \psi_k^{\beta+n-1}(\sqrt{|\lambda|} f^\lambda *_\lambda \varphi_k^\lambda(z)) \right) |\lambda|^n d\lambda.$$

Note that the left hand side of the above equation is well defined only for $\text{Re} \beta > 0$ whereas the right hand side makes sense for all $\text{Re} \beta > -n$. We can thus think of the right hand side as an analytic continuation of the left hand side. In view of (2.8), the Poisson integral $P_r f$ of $f$ in the $t$-variable can be written as

$$(P_r f)^\lambda(z) = e^{-\frac{1}{4}|\lambda|^2} f^\lambda(z).$$

Then, by (2.5) we consider the equation

$$P_{1-r^2} f * \mu_r(z, t) = (2\pi)^{-n-1} \sum_{k=0}^{\infty} \int_{-\infty}^{\infty} e^{-i\lambda t} \psi_k^{n-1}(\sqrt{|\lambda|} f^\lambda *_\lambda \varphi_k^\lambda(z)) |\lambda|^n d\lambda.$$

Integrating the above equation against $r^{2n-1}(1-r^2)^{\beta-1} dr$, we obtain

$$\int_0^1 r^{2n-1}(1-r^2)^{\beta-1} P_{1-r^2} f * \mu_r(z, t) dr = (2\pi)^{-n-1} \sum_{k=0}^{\infty} \int_{-\infty}^{\infty} e^{-i\lambda t} \rho_k(\sqrt{|\lambda|}) f^\lambda *_\lambda \varphi_k^\lambda(z) |\lambda|^n d\lambda,$$

where

$$\rho_k(\sqrt{|\lambda|}) = \int_0^1 r^{2n-1}(1-r^2)^{\beta-1} \psi_k^{n-1}(\sqrt{|\lambda|} f^\lambda *_\lambda \varphi_k^\lambda(z)) |\lambda|^n d\lambda.$$

(2.9)
Recalling the definition of \( \psi_{k}^{n-1} \) given in (2.3) we have
\[
\rho_{k}(\sqrt{|\lambda|}) = \frac{\Gamma(k+1)\Gamma(n)}{\Gamma(k+n)} \int_{0}^{1} r^{2n-1}(1 - r^2)^{\beta-1} L_{k}^{n-1}\left(\frac{1}{2}r^2|\lambda|\right) e^{-\frac{1}{2}|\lambda|} dr.
\]
We now use the formula (2.7). First we make a change of variables \( t \to s^2 \) and then choose \( \mu = n - 1 \) and \( \nu = \beta \), so that
\[
(2.10) \quad \rho_{k}(\sqrt{|\lambda|}) = \frac{\Gamma(k+1)\Gamma(n)}{\Gamma(k+n)} \frac{\Gamma(\beta)\Gamma(k+n+\beta)}{2 \Gamma(\beta+n)} e^{-\frac{1}{2}|\lambda|} L_{k}^{n+\beta-1}\left(\frac{|\lambda|}{2}\right) = \frac{1}{2} \frac{\Gamma(\beta)\Gamma(n)}{\Gamma(\beta+n)} \psi_{k}^{n+\beta-1}(\sqrt{|\lambda|}).
\]
The proof is complete. \( \square \)

In particular, from the computations in the proof of Lemma 2.4, we infer the following identity.

**Corollary 2.5.** Let \( \operatorname{Re} \beta > 0 \) and \( \alpha > -1 \). Then, for \( t > 0 \),
\[
\psi_{k}^{\alpha+\beta}(t) = 2 \frac{\Gamma(\beta+\alpha+1)}{\Gamma(\beta)\Gamma(\alpha+1)} \int_{0}^{1} s^\alpha (1 - s)^{\beta-1} \psi_{k}^{\beta}(t\sqrt{s}) e^{-\frac{1}{2}t^2(1-s)} ds.
\]

**Proof.** The identity follows from (2.9) and (2.10), after a change of variable. \( \square \)

We slightly modify the family in Lemma 2.4 and define a new family \( T_{\beta} \). The modification becomes necessary since we want our family to have some \( L^p \) improving property for large values of \( \beta \). The original operator \( \mathcal{A}^\beta \) remains as convolution with a distribution supported on \( \mathbb{C}^n \times \{0\} \) however large \( \beta \) is. This is in sharp contrast with the Euclidean case, see [21]. As we will see below the modified family of operators \( T_{\beta} \) has a better behaviour for \( \beta \geq 1 \). Let \( k_{\beta}(t) = \frac{1}{\Gamma(\beta)} t^{\beta-1} e^{-t}, \operatorname{Re} \beta > 0 \), which defines a family of distributions on \( \mathbb{R} \) and \( \lim_{\beta \to 0} k_{\beta}(t) = \delta_0 \), the Dirac distribution at 0. Given a function \( f \) on \( \mathbb{H}^n \) and \( \varphi \) on \( \mathbb{R} \) we use the notation \( f *_{3} \varphi \) to stand for the convolution in the central variable:
\[
f *_{3} \varphi(z, t) = \int_{-\infty}^{\infty} f(z, t - s) \varphi(s) ds.
\]
Thus we note that \( P_{1-r^2} f(z, t) = f *_{3} p_{1-r^2}(z, t) \) where \( p_{1-r^2} \) is the usual Poisson kernel in the one dimensional variable \( t \), associated to \( P_{1-r^2} \). In fact, \( p_{r}(t) \) is defined by the relation \( \int_{-\infty}^{\infty} e^{i\lambda t} p_{r}(dt) = e^{-\frac{1}{2}r^2|\lambda|} \) and it is explicitly known: \( p_{r}(t) = cr(r^2 + 16t^2)^{-1} \) for some constant \( c > 0 \), see for example [20]. With the above notation we define the new family by
\[
T_{\beta} f (z, t) = \frac{\Gamma(\beta+n)}{\Gamma(\beta)\Gamma(n)} \int_{0}^{1} r^{2n-1}(1 - r^2)^{\beta-1} P_{1-r^2}(f *_{3} k_{\beta}) * \mu_{r}(z, t) dr.
\]
In other words
\[
T_{\beta} f = \mathcal{A}^\beta(f *_{3} k_{\beta}).
\]

**Lemma 2.6.** The operator \( T_{\beta} f \) has the explicit expansion
\[
T_{\beta} f (z, t) = (2\pi)^{-n-1} \int_{-\infty}^{\infty} e^{-i\lambda t}(1 - i\lambda)^{-\beta} \left(\sum_{k=0}^{\infty} \psi_{k}^{\beta+n-1}(\sqrt{|\lambda|}) f *_{3} \varphi_{k}(z)\right) |\lambda|^n d\lambda.
\]

**Proof.** The statement follows from Lemma 2.4 (2.6), and from the fact
\[
\int_{-\infty}^{\infty} e^{i\lambda t} k_{\beta}(t) dt = \frac{1}{\Gamma(\beta)} \int_{0}^{\infty} e^{i\lambda t} t^{\beta-1} e^{-t} dt = (1 - i\lambda)^{-\beta}.
\]
This can be verified by looking at the function
\[ F(\beta, z) = \frac{1}{\Gamma(\beta)} \int_0^\infty t^{\beta-1} e^{-tz} dt \]
defined and holomorphic for \( \text{Re} \beta > 0, \text{Re} z > 0 \). Indeed, when \( z, \) with \( \text{Re} z > 0, \) is fixed, we have the relation \( F(\beta, z) = z F(\beta + 1, z) \) which allows us to holomorphically extend \( F(\beta, z) \) in the \( \beta \) variable. It is clear that when \( z > 0, F(\beta, z) = z^{-\beta}, \) which allows us to conclude that the Fourier transform of \( k_\beta \) at \( \lambda \) is given by \((1-i\lambda)^{-\beta},\) as claimed. \( \square \)

We will show that when \( \beta = 1 + i\gamma, T_\beta \) is bounded from \( L^p(\mathbb{H}^n) \) into \( L^\infty(\mathbb{H}^n) \) for any \( p > 1, \) and for certain negative values of \( \beta, T_\beta \) is bounded on \( L^2(\mathbb{H}^n) \). We can then use analytic interpolation to obtain a result for \( T_0 = A_0 = A_1. \)

**Proposition 2.7.** For any \( \delta > 0, \gamma \in \mathbb{R} \)
\[ ||T_{1+i\gamma} f||_\infty \leq C_1(\gamma) ||f||_{1+\delta}, \]
where \( C_1(\gamma) \) is of admissible growth.

**Proof.** Without loss of generality we can assume that \( f \geq 0. \) For \( \beta = 1 + i\gamma \) it follows that
\[ |T_{1+i\gamma} f(z, t)| \leq \frac{|\Gamma(1 + i\gamma + n)|}{|\Gamma(1 + i\gamma)|^2 \Gamma(n)} \int_0^1 r^{2n-1} P_{1-r^2}(f *_3 \varphi) * \mu_r(z, t) dr \]
where \( \varphi(t) = e^{-t} \chi_{(0, \infty)}(t). \) Since \( \varphi \geq 0 \) it follows that
\[ P_{1-r^2}(f *_3 \varphi) = \varphi *_3 p_{1-r^2} *_3 f \leq \varphi *_3 \Lambda f \]
where \( \Lambda f \) is the Hardy–Littlewood maximal function in the \( t \)-variable. In proving the above we have used the well known fact that \( \sup_{r>0} |P_r g(t)| \leq CA g(t) \) for any \( g \) on \( \mathbb{R}. \) Thus we have the estimate
\[ |T_{1+i\gamma} f(z, t)| \leq C_1(\gamma) \int_0^1 (\Lambda f *_3 \varphi) * \mu_r(z, t) r^{2n-1} dr. \]

Now we make the following observation: Suppose \( K(z, t) = k(|z|) \varphi(t). \) Then
\[ f * K(z, t) = \int_0^\infty (f *_3 \varphi) * \mu_r(z, t) k(r) r^{2n-1} dr, \]
which can be verified by recalling the definition of the spherical means \( f * \mu_r(z, t) \) in (1.1) and integrating in polar coordinates. This gives us
\[ |T_{1+i\gamma} f(z, t)| \leq C_1(\gamma) \Lambda f * K(z, t) \]
where \( K(z, t) = \chi_{|z| \leq 1}(z) \varphi(t). \) As \( \Lambda f \in L^{1+\delta}(\mathbb{H}^n) \) and \( K \in L^q(\mathbb{H}^n) \) for any \( q \geq 1, \) by Hölder we get
\[ ||T_{1+i\gamma} f||_\infty \leq C_1(\gamma) ||\Lambda f||_{1+\delta} \leq C_1(\gamma) ||f||_{1+\delta}. \]
\( \square \)

In the next proposition we show that \( T_\beta \) is bounded on \( L^2(\mathbb{H}^n) \) for some \( \beta < 0. \) It is possible to sharpen the following result, see Subsection 2.1, but for the sake of simplicity (and to mimic the corresponding Euclidean result), we consider only the case \( \text{Re} \beta \geq -\frac{(n-1)}{2}. \)

**Proposition 2.8.** Assume that \( n \geq 1 \) and \( \beta \geq -\frac{(n-1)}{2}. \) Then for any \( \gamma \in \mathbb{R} \)
\[ ||T_{\beta+i\gamma} f||_2 \leq C_2(\gamma) ||f||_2. \]
Proof. We only have to check that the functions
\[(1 + \lambda^2)^{-\beta/2} |\psi_k^{\beta+i\gamma+n-1}(\sqrt{\lambda})| \leq C_2(\gamma)\]
where $C_2(\gamma)$ is independent of $K$ and $\lambda$. When $\gamma = 0$, it follows from the estimates of Lemma 2.2 that
\[(1 + \lambda^2)^{-\beta/2} |\psi_k^{\beta+n-1}(\sqrt{\lambda})| \leq C|\lambda|^{-\beta}|\lambda|^{-\beta-(n-1) - \frac{1}{2}}\]
for $|\lambda| \geq 1$, which is clearly bounded for $\beta \geq -\frac{n+1}{2}$ (actually, it is bounded for $\beta \geq -\frac{n}{2} + \frac{1}{2}$, so it is for $\beta \geq -\frac{n}{2} + \frac{1}{2}$). For $\gamma \neq 0$ we can express $\psi_k^{\beta+i\gamma+n-1}(\sqrt{\lambda})$ in terms of $\psi_k^{\beta-i\gamma+n-1}(\sqrt{\lambda})$ for a small enough $\varepsilon > 0$ and obtain the same estimate. Indeed, by Corollary 2.5 and using the asymptotic formula $|\Gamma(\mu + iv)| \sim \sqrt{2\pi |v|}^{\mu-1/2} e^{-\pi|v|/2}$, as $v \to \infty$ (see for instance [20, p. 281 bottom note])
\[|\psi_k^{\beta+i\gamma+n-1}(\sqrt{\lambda})| = \frac{2 |\Gamma(\beta + i\gamma + n)|}{|\Gamma(\varepsilon + i\gamma)\Gamma(\beta - \varepsilon + n)|}
\times \int_0^1 s^{\beta-\varepsilon+n-1}(1 - s)^{\varepsilon+i\gamma-1} |\psi_k^{\beta-\varepsilon+n-1}(\sqrt{\lambda}s)| e^{-\frac{1}{2}\lambda(1-s)} ds \]
\[\leq \frac{|\gamma|^{\beta+n-1/2}}{|\gamma|^{\varepsilon-1/2}} \int_0^1 s^{\beta-\varepsilon+n-1}(1 - s)^{\varepsilon+i\gamma-1} |\psi_k^{\beta-\varepsilon+n-1}(\sqrt{\lambda}s)| e^{-\frac{1}{2}\lambda(1-s)} ds ,
\]
where the constant depends on $\beta$. Now, by the estimate for $\psi_k^\beta$ in Lemma 2.2 and the integrability of the function $s^{\beta-\varepsilon+n-1}(1 - s)^{\varepsilon+i\gamma-1}$ we have
\[(1 + \lambda^2)^{-\beta/2} |\psi_k^{\beta+i\gamma+n-1}(\sqrt{\lambda})| \leq C|\lambda|^{-\beta}|\lambda|^{-\beta-(n-1) - \frac{1}{2}}|\lambda|-\frac{1}{2} - \frac{1}{2} .\]
For $|\lambda| \geq 1$, the above is bounded for $\beta \geq -\frac{n+1}{2}$ (actually, it is bounded for $\beta - \varepsilon \geq -\frac{n}{2} + \frac{1}{2}$ with $\varepsilon$ small enough, so it is for $\beta \geq -\frac{n}{2} + \frac{1}{2}$). The proof is complete. \hfill \qed

Theorem 2.9. Assume that $n \geq 2$. Then $A_1 : L^p(\mathbb{H}^n) \to L^{n+1}(\mathbb{H}^n)$ for any $\frac{n+1}{n} < p < (n+1)$.

Proof. For $\frac{n+1}{n} < p < (n+1)$ choose $\delta > 0$ such that $p = \frac{(n+1)(1+\delta)}{n+\delta}$, which is possible as $\frac{1}{n} < \frac{1+\delta}{n+\delta} < 1$. By considering the analytic family $T_\alpha(z)$ where $\alpha(z) = \frac{n+1}{2}(z - 1) + z$ with $z = u + iv$, in view of Propositions 2.7 and 2.8 and interpolation between the endpoints $\text{Re} z = 0$ and $\text{Re} z = 1$ we obtain
\[T_\alpha(u) : L^{p_u}(\mathbb{H}^n) \to L^{q_u}(\mathbb{H}^n)\]
where $\frac{1}{p_u} = \frac{1-u}{2} + \frac{u}{1+\delta}$ and $\frac{1}{q_u} = \frac{1-u}{2}$. The choice $u = \frac{n+1}{n+1}$ gives $q_u = n+1$ and $p_u = \frac{(n+1)(1+\delta)}{n+\delta} = p$. Since $\alpha(n+1) = 0$ we obtain the result. \hfill \qed

Remark 2.10. Observe the restriction on the dimension in Theorem 2.9, that comes into play due to the restriction (that we imposed, a bit artificially, for cosmetic reasons) on the parameter $\beta$ in Proposition 2.8. This is the only place in the $L^p$-improving estimates where the dimensional restriction arises, but we insist that we imposed that. Actually, the results we can obtain concerning $L^p$-improving estimates are sharp and valid for all the dimensions, see Subsection 2.1.

Corollary 2.11. Assume that $n \geq 2$. Then
\[A_1 : L^p(\mathbb{H}^n) \to L^q(\mathbb{H}^n)\]
whenever \( \left( \frac{1}{p}, \frac{1}{q} \right) \) lies in the interior of the triangle joining the points \((0, 0), (1, 1)\) and \(\left( \frac{n}{n+1}, \frac{1}{n+1} \right)\), as well as the straight line segment joining the points \((0, 0), (1, 1)\), see \(L'_n\) in Figure 1.

**Proof.** The result follows from Theorem 2.9 after applying Marcinkiewicz interpolation theorem with the obvious estimates \(\|A_1 f\|_1 \leq \|f\|_1\) and \(\|A_1 f\|_\infty \leq \|f\|_\infty\).

![Figure 1. Triangle \(L'_n\) shows the region for \(L^p - L^q\) estimates for \(A_1\). The dual triangle \(L_n\) is on the left.](image-url)

2.1. **A sharpened result.** As indicated in the proof of Proposition 2.8, we could state an enhanced result as follows.

**Proposition 2.12** (Proposition 2.8 sharpened). Assume that \(n \geq 1\) and \(\beta > -\frac{n}{2} + \frac{1}{3}\). Then for any \(\gamma \in \mathbb{R}\)

\[
\|T_{\beta+i\gamma} f\|_2 \leq C_2(\gamma) \|f\|_2.
\]

On the other hand, let us consider the following holomorphic function \(\alpha(z)\) on the strip \(\{z : 0 \leq \text{Re} z \leq 1\}\), given by \(\alpha(z) = \left( \frac{n}{2} - \frac{1}{3} \right)(z - 1 + \varepsilon) + z\). We have \(\alpha(0) = \left( -\frac{n}{2} + \frac{1}{3} \right)(1 - \varepsilon)\) and \(\alpha(1) = 1\). Then, \(T_{\alpha(z)}\) is an analytic family of linear operators and it was already shown that \(T_{1+i\gamma}\) is bounded from \(L^{1+\delta}(\mathbb{H}^n)\) to \(L^{\infty}(\mathbb{H}^n)\). Therefore, we can apply Stein’s interpolation theorem. Letting \(z = u + iv\), we have

\[
\alpha(z) = 0 \iff \left( \frac{n}{2} - \frac{1}{3} \right)(u - 1 + \varepsilon) + u = 0 \iff u = \frac{3n - 2}{3n + 4}(1 - \varepsilon).
\]

Since \(\varepsilon > 0\) is arbitrary, we obtain

\[T_{\alpha(u)} : L^{p_u}(\mathbb{H}^n) \to L^{q_u}(\mathbb{H}^n)\]

where

\[
\frac{1}{p_u} = \frac{3n + 1}{3n + 4} - \varepsilon \frac{3n - 2}{2(3n + 4)}, \quad \frac{1}{q_u} = \frac{3 + \frac{1}{2}(3n - 2)\varepsilon}{3n + 4}.
\]

This leads to the following result, the enhanced version of Theorem 2.9.
**Theorem 2.13** (Theorem 2.9 sharpened). Assume that $n \geq 1$ and $\varepsilon > 0$. Then $A_1 : L^p(\mathbb{H}^n) \to L^q(\mathbb{H}^n)$ for any $p, q$ such that

$$\frac{1}{p} = \frac{3n + 1}{3n + 4} - \frac{\varepsilon}{2(3n + 4)}, \quad \frac{1}{q} = \frac{3 + \frac{1}{2}(3n - 2)\varepsilon}{3n + 4}.$$ 

And we deduce the following corollary.

**Corollary 2.14** (Corollary 2.11 sharpened). Assume that $n \geq 1$. Then $A_1 : L^p(\mathbb{H}^n) \to L^q(\mathbb{H}^n)$ whenever $(\frac{1}{p}, \frac{1}{q})$ lies in the interior of the triangle joining the points $(0, 0), (1, 1)$ and $(\frac{3n + 1}{3n + 4}, \frac{3}{3n + 4})$, as well as the straight line segment joining the points $(0, 0), (1, 1)$, see $S'_n$ in Figure 2.

![Figure 2](image-url)  

**Figure 2.** Triangle $S'_n$ shows the region for sharpened $L^p - L^q$ estimates for $A_1$. The dual triangle $S_n$ is on the left.

### 3. The Continuity Property of the Spherical Mean Value Operator

In the work of Lacey [11] dealing with the lacunary spherical maximal function on $\mathbb{R}^n$, the continuity property of the spherical mean value operator plays a crucial role. In the case of the Heisenberg group we require the following continuity property.

**Proposition 3.1.** Assume that $n \geq 2$. Then for $y \in \mathbb{H}^n, |y| \leq 1$, we have

$$\|A_1 - A_1 \tau_y\|_{L^2 \to L^2} \leq C|y|$$

where $\tau_y f(x) = f(xy^{-1})$ is the right translation operator.

**Proof.** For $f \in L^2(\mathbb{H}^n)$ we estimate the $L^2$ norm of $A_1 f - A_1(\tau_y f)$ using Plancherel theorem for the Fourier transform on $\mathbb{H}^n$. Recall that $A_1 f(x) = f * \mu_1(x)$ so that $\hat{A_1} f(\lambda) = \hat{f}(\lambda) \hat{\mu}_1(\lambda)$, where $\hat{\mu}_1(\lambda)$ is explicitly given by

$$\hat{\mu}_1(\lambda) = \sum_{k=0}^{\infty} \psi_k^{n-1}(|\lambda|) P_k(\lambda).$$
We also have
\[ \hat{\tau}_y f(\lambda) = \int_{\mathbb{H}^n} f(xy^{-1}) \pi_\lambda(x) dx = \hat{f}(\lambda) \pi_\lambda(y). \]

Thus by the Plancherel theorem for the Fourier transform we have
\[ \| A_1 f - A_1(\tau_y f) \|_2^2 = c_n \int_{-\infty}^{\infty} \| \hat{f}(\lambda)(I - \pi_\lambda(y)) \hat{\mu}_1(\lambda) \|_{HS}^2 |\lambda|^n d\lambda. \]

Since the space of all Hilbert-Schmidt operators is a two sided ideal in the space of all bounded linear operators, it is enough to estimate the operator norm of \((I - \pi_\lambda(y)) \hat{\mu}_1(\lambda)\).

(For more about Hilbert-Schmidt operators see V. S. Sunder [22].) Again, \(\hat{\mu}_1(\lambda)\) is self adjoint and \(\pi_\lambda(y)^* = \pi_\lambda(y^{-1})\) and so we will estimate \(\hat{\mu}_1(\lambda)(I - \pi_\lambda(y))\).

We make use of the fact that for every \(\sigma \in U(n)\) there is a unitary operator \(\mu_\lambda(\sigma)\) acting on \(L^2(\mathbb{R}^n)\) such that \(\pi_\lambda(\sigma z, t) = \mu_\lambda(\sigma)^* \pi_\lambda(z, t) \mu_\lambda(\sigma)\) for all \((z, t) \in \mathbb{H}^n\). This follows from the well known Stone–von Neumann theorem which says that any irreducible unitary representation of the Heisenberg group which acts like \(e^{i\lambda t}I\) when restricted to the center is unitarily equivalent to \(\pi_\lambda\), see [8]. Actually, \(\mu_\lambda\) has an extension to a double cover of the symplectic group as a unitary representation and is called the metaplectic representation. Given \(y = (z, t) \in \mathbb{H}^n\) we can choose \(\sigma \in U(n)\) such that \(y = (|z|\sigma e_1, t)\) where \(e_1 = (1, 0, ..., 0)\). Thus
\[ \pi_\lambda(y) = \mu_\lambda(\sigma)^* \pi_\lambda(|z|e_1, t) \mu_\lambda(\sigma). \]

Also, it is well known that \(\mu_\lambda(\sigma)\) commutes with functions of the Hermite operator \(H(\lambda)\) given by \(H(\lambda) = -\Delta + \lambda^2 |x|^2\). Since \(\hat{\mu}_1(\lambda)\) is a function of \(H(\lambda)\) it follows that
\[ \hat{\mu}_1(\lambda)(I - \pi_\lambda(z, t)) = \mu_\lambda(\sigma)^* \hat{\mu}_1(\lambda)(I - \pi_\lambda(|z|e_1, t)) \mu_\lambda(\sigma). \]

Thus it is enough to estimate the operator norm of \(\hat{\mu}_1(\lambda)(I - \pi_\lambda(|z|e_1, t))\). In view of the factorisation \(\pi_\lambda(|z|e_1, t) = \pi_\lambda(|z|e_1, 0) \pi_\lambda(0, t)\) we have that
\[ I - \pi_\lambda(|z|e_1, t) = I - \pi_\lambda(|z|e_1, 0) \pi_\lambda(0, t) = (I - \pi_\lambda(0, t)) + (I - \pi_\lambda(|z|e_1, 0)) \pi_\lambda(0, t) \]
so it suffices to estimate the norms of \(\hat{\mu}_1(\lambda)(I - \pi_\lambda(0, t))\) and \(\hat{\mu}_1(\lambda)(I - \pi_\lambda(|z|e_1, 0)) \pi_\lambda(0, t)\) separately. Moreover, we only have to estimate them for \(|\lambda| \geq 1\) as they are uniformly bounded for \(|\lambda| \leq 1\).

Assuming \(|\lambda| \geq 1\) we have, in view of (2.2),
\[ \hat{\mu}_1(\lambda)(I - \pi_\lambda(0, t)) \varphi(\xi) = (1 - e^{i\lambda t}) \hat{\mu}_1(\lambda) \varphi(\xi), \quad \varphi \in L^2(\mathbb{R}^n). \]

By mean value theorem, the operator norm of \((1 - e^{i\lambda t}) \hat{\mu}_1(\lambda)\) is bounded by
\[ C|t||\lambda| \sup_k |\psi_k|^{-1}(\sqrt{|\lambda|}) \leq C|t||\lambda|^{-(n-1)+2/3} \]
where we have used the estimate in Lemma 2.2. Thus for \(n \geq 2\),
\[ \| \hat{\mu}_1(\lambda)(I - \pi_\lambda(0, t)) \|_{L^2 \to L^2} \leq C|t| \leq C|(z, t)|^2, \]
where \(|x| = |(z, t)| = (|z|^4 + t^2)^{1/4}\) is the Koranyi norm on \(\mathbb{H}^n\). In order to estimate \(\hat{\mu}_1(\lambda)(I - \pi_\lambda(|z|e_1, 0))\) we recall that
\[ \pi_\lambda(|z|e_1, 0) \varphi(\xi) = e^{i\lambda |z|\xi_1} \varphi(\xi), \quad \varphi \in L^2(\mathbb{R}^n). \]

Since we can write
\[ (1 - e^{i\lambda |z|\xi_1}) = -i\lambda |z|\xi_1 \int_0^1 e^{itu|z|\xi_1} dt = \lambda |z|\xi_1 m_\lambda(|z|, \xi) \]

with a bounded function $m_\lambda(|z|, \xi)$, it is enough to estimate the norm of the operator $|z|\hat{\mu}_1(\lambda)M_\lambda$ where $M_\lambda \varphi(\xi) = \lambda \xi_1 \varphi(\xi)$.

Let $A(\lambda) = \frac{\partial}{\partial \xi_1} + |\lambda| \xi_1$ and $A(\lambda)^* = -\frac{\partial}{\partial \xi_1} + |\lambda| \xi_1$ be the annihilation and creation operators, so that we can express $M_\lambda$ as $M_\lambda = \frac{1}{2}(A(\lambda) + A(\lambda)^*)$. Thus it is enough to consider $|z|\hat{\mu}_1(\lambda)A(\lambda)$ and $|z|\hat{\mu}_1(\lambda)A(\lambda)^*$. Moreover as the Riesz transforms $H(\lambda)^{-1/2}A(\lambda)$ and $H(\lambda)^{-1/2}A(\lambda)^*$ are bounded on $L^2(\mathbb{R}^n)$ we only need to consider $|z|\hat{\mu}_1(\lambda)H(\lambda)^{1/2}$. But the operator norm of $\hat{\mu}_1(\lambda)H(\lambda)^{1/2}$ is given by $\sup_k ((2k + n)|\lambda|)^{1/2}|\psi_k^{-1}(\sqrt{\lambda})|$ which, in view of Lemma 2.3, is bounded by $C|\lambda|^{-(n-1)/2}$. Thus for $n \geq 2$ we obtain

$$
\|\hat{\mu}_1(\lambda)(I - \pi_\lambda(|z|e_1, 0))\|_{L^2 \to L^2} \leq C|z| \leq C|\langle z, t \rangle|.
$$

This completes the proof of the proposition. \hfill \Box

**Remark 3.2.** Observe that the result above is restricted to the case $n \geq 2$, and this due to the restriction on the available (and sharp!) estimates for the Laguerre functions in Lemmas 2.1, 2.2 and 2.3. We do not know whether there is a way to reach a restriction on the available (and sharp!) estimates for the Laguerre functions in Lemmas 2.1, 2.2 and 2.3.

**Corollary 3.3.** Assume that $n \geq 2$. Then for $y \in \mathbb{H}^n$, $|y| \leq 1$, and for $(\frac{1}{p}, \frac{1}{q})$ in the interior of the triangle joining the points $(0, 0), (1, 1)$ and $(\frac{n}{n+1}, \frac{1}{n+1})$, we have the inequalities

$$
\|A_1 - A_1\tau_y\|_{L^p \to L^q} \leq C|y|^\eta
$$

for some $0 < \eta < 1$, where $\tau_y f(x) = f(xy^{-1})$ is the right translation operator.

**Proof.** The result follows by Riesz-Thorin interpolation theorem, taking into account Corollary 2.1 and Proposition 3.1. \hfill \Box

We need a version of the inequality in Corollary 3.3 when $A_1$ is replaced by $A_r$. This can be easily achieved by making use of the following lemma which expresses $A_r$ in terms of $A_1$. Let $\delta_r \varphi(w, t) = \varphi(rw, r^2t)$ stand for the non-isotropic dilation on $\mathbb{H}^n$.

**Lemma 3.4.** For any $r > 0$ we have $A_r f = \delta_r^{-1} A_1 \delta_r f$.

**Proof.** This is just an easy verification. Starting from the expression in (2.5) we have

$$
A_r f(z, t) = (2\pi)^{-n-1} \int_{-\infty}^{\infty} e^{-i\lambda t} \left( \sum_{k=0}^{\infty} \psi_k^{n-1}(\sqrt{|\lambda|r}) f^{\lambda} \ast \varphi_k^\lambda(z) \right) |\lambda|^n d\lambda
$$

$$
= (2\pi)^{-n-1} r^{-2n-2} \int_{-\infty}^{\infty} e^{-i\frac{\lambda t}{r^2}} \left( \sum_{k=0}^{\infty} \psi_k^{n-1}(\sqrt{|\lambda|}) f^{\lambda/r^2} \ast \varphi_k^{\lambda/r^2}(z) \right) |\lambda|^n d\lambda.
$$

In view of the relation

$$
f^{\lambda/r^2}(rw) = \int_{-\infty}^{\infty} f(rw, t) e^{i\lambda/r^2 t} dt = r^2 \int_{-\infty}^{\infty} f(rw, r^2 t) e^{i\lambda t} dt
$$

we make the following simple computation:

$$
f^{\lambda/r^2} \ast \varphi_k^{\lambda/r^2}(z) = \int_{\mathbb{C}^n} f^{\lambda/r^2}(w) \varphi_k^{\lambda/r^2}(z - w) e^{-i\frac{\lambda}{r^2} \text{Im} z \cdot \bar{w}} dw
$$

$$
= \int_{\mathbb{C}^n} f^{\lambda/r^2}(rw) \varphi_k^{\lambda}(z/r - w) e^{-i\frac{\lambda}{r} \text{Im} \bar{w} \cdot r^{2n}} dw
$$

$$
= r^{2+2n} \int_{\mathbb{C}^n} (\delta_r f)^{(\lambda)}(w) \varphi_k^{\lambda}(z/r - w) e^{-i\frac{\lambda}{r} \text{Im} \bar{w} \cdot r^{2n}} dw
$$
\[ = r^{2+2n}(\delta_r f * \lambda \varphi_k^\lambda)(z/r). \]

Therefore, we have

\[ A_r f(z, t) = (2\pi)^{-n-1} \int_{-\infty}^{\infty} e^{-i \Delta t} \left( \sum_{k=0}^{\infty} \psi_k^{n-1}(\sqrt{|\lambda|})(\delta_r f * \lambda \varphi_k^\lambda)(z/r) \right) |\lambda|^{n} d\lambda = A_1(\delta_r f) \left( \frac{z}{r}, \frac{t}{r^2} \right), \]

which proves the stated result. \qed

**Corollary 3.5.** Assume that \( n \geq 2 \). Then for \( y \in \mathbb{H}^n, |y| \leq 1 \), and for \( \left( \frac{1}{p}, \frac{1}{q} \right) \) in the interior of the triangle joining the points \((0,0), (1,1)\) and \( \left( \frac{n+1}{n+1}, \frac{1}{n+1} \right) \), we have the inequality

\[ \| A_r - A_r \tau_y \|_{L^p \to L^q} \leq C r^{-\eta} |y|^{\eta} r^{(2n+2)(\frac{1}{p} - \frac{1}{q})} \]

for some \( \eta > 0 \).

**Proof.** Observe that \( \delta_r (\tau_y f) = \tau_{\delta_r^{-1} y} (\delta_r f) \), which follows from the fact that \( \delta_r : \mathbb{H}^n \to \mathbb{H}^n \) is an automorphism. The corollary follows from Corollary 3.3, Lemma 3.4 and the fact that \( \| \delta_r f \|_p = r^{-\frac{(2n+2)}{p}} \) for any \( 1 \leq p < \infty \). \qed

### 3.1. A sharpened continuity property.

By using Corollary 2.14 instead of Corollary 2.11 we could obtain a sharpened version of Corollary 3.3, so that we indeed can obtain the following.

**Corollary 3.6.** Assume that \( n \geq 2 \). Then for \( y \in \mathbb{H}^n, |y| \leq 1 \), and for \( \left( \frac{1}{p}, \frac{1}{q} \right) \) in the interior of the triangle joining the points \((0,0), (1,1)\) and \( \left( \frac{3n+1}{3n+1}, \frac{3}{3n+1} \right) \), we have the inequality

\[ \| A_r - A_r \tau_y \|_{L^p \to L^q} \leq C r^{-\eta} |y|^{\eta} r^{(2n+2)(\frac{1}{q} - \frac{1}{p})} \]

for some \( \eta > 0 \).

### 4. Sparse bounds

Our aim in this section is to prove the sparse bounds for the lacunary spherical maximal function stated in Theorem 1.2. In doing so we closely follow 11 with suitable modifications that are necessary since we are dealing with a non-commutative set up. We can equip \( \mathbb{H}^n \) with a metric induced by the Koranyi norm which makes it a homogeneous space. On such spaces there is a well defined notion of dyadic cubes and grids with properties similar to their counter parts in the Euclidean setting. However, we need to be careful with the metric we choose since the group is non-commutative.

Recall that the Koranyi norm on \( \mathbb{H}^n \) is defined by \( |x| = |(z,t)| = (|z|^4 + t^2)^{1/4} \) which is homogeneous of degree one with respect to the non-isotropic dilations. Since we are considering \( f * \mu_r \) it is necessary to work with the left invariant metric \( d_L(x,y) = |x^-y| = d_L(0,x^-y) \) instead of the standard metric \( d(x,y) = |xy|^{-1} = d(0,xy^{-1}) \), which is right invariant. The balls and cubes are then defined using \( d_L \). Thus \( B(a,r) = \{ x \in \mathbb{H}^n : |a^-x| < r \} \). With this definition we note that \( B(a,r) = a \cdot B(0,r) \), a fact which is crucial. This allows us to conclude that when \( f \) is supported in \( B(a,r) \) then \( f * \mu_s \) is supported in \( B(a,r+s) \). Indeed, as support of \( \mu_s \) is contained in \( B(0,s) \) we see that \( f * \mu_s \) is supported in \( B(a,r) \cdot B(0,s) \subset a \cdot B(0,r) \cdot B(0,s) \subset B(a,r+s) \).
Theorem 4.1. Let $\delta \in (0, 1)$ with $\delta \leq 1/96$. Then there exists a countable set of points $\{z_{\nu}^{k, \alpha} : \nu \in \mathcal{K}_k\}$, $k \in \mathbb{Z}$, $\alpha = 1, 2, \ldots, N$ and a finite number of dyadic systems $D^\alpha := \bigcup_{k \in \mathbb{Z}} D^\alpha_k$, $D^\alpha_k := \{Q^{k, \alpha}_\nu : \nu \in \mathcal{K}_k\}$ such that

1. For every $\alpha \in \{1, 2, \ldots, N\}$ and $k \in \mathbb{Z}$ we have
   i) $\mathbb{H}^n = \bigcup_{Q \in D^\alpha_k} Q$ (disjoint union).
   ii) $Q, P \in D^\alpha \Rightarrow Q \cap P \in \{\emptyset, P, Q\}$.
   iii) $Q^{k, \alpha}_\nu \in D^\alpha \Rightarrow B(z^{k, \alpha}_\nu, \frac{1}{12} \delta^k) \subseteq Q^{k, \alpha}_\nu \subseteq B(z^{k, \alpha}_\nu, 4\delta^k)$. In this situation $z^{k, \alpha}_\nu$ is called the center of the cube and the side length $\ell(Q^{k, \alpha}_\nu)$ is defined to be $\delta^k$.

2. For every ball $B = B(x, r)$, there exists a cube $Q_B \in \bigcup \alpha D^\alpha$ such that $B \subseteq Q_B$ and $\ell(Q_B) = \delta^{k-1}$, where $k$ is the unique integer such that $\delta^{k-1} < r \leq \delta^k$.

Proof. It follows from Theorem 4.1, the proof of Lemma 4.12, Remark 4.13 and Theorem 2.2 in [10], where the choices $c_0 = 1/4$ and $C_0 = 2$ in [10] Theorem 2.2] are made so that the property (2) holds (see [10] Lemma 4.10)).

We will first prove a lemma that is the analogue of [11] Lemma 2.3.

Lemma 4.2. Let $f$ and $g$ be supported on a cube $Q$ and let $\ell(Q) = r$. For $(\frac{1}{p}, \frac{1}{q})$ in the interior of the triangle joining the points $(0, 1), (1, 0)$ and $(\frac{n}{n+1}, \frac{n}{n+1})$, there holds

$$|\langle A_r f - A_r \tau_y f, g \rangle| \lesssim |y/r|^\eta |Q|^{\frac{1}{p} + \frac{1}{q}} (f)_{Q,p} (g)_{Q,q}, \quad |y| < r.$$  

Proof. Observe that continuity property holds for the pair $(\frac{1}{p}, \frac{1}{q})$. By Hölder’s inequality and Corollary 3.5 we have, for $|y| < r$,

$$|\langle A_r f - A_r \tau_y f, g \rangle| \leq \|A_r f - A_r \tau_y f\|_p \|g\|_q$$

$$\leq C_{r, (2n+2)(\frac{1}{q} - \frac{1}{p})}^2 |y|^\eta \|f\|_p \|g\|_q$$

$$= C_{r, (2n+2)(\frac{1}{q} - \frac{1}{p})}^2 |y|^\eta |Q|^\frac{1}{q} + \frac{1}{q} (f)_{Q,p} (g)_{Q,q}$$

$$\lesssim |Q|^\frac{1}{q} |y|^\eta |Q|^\frac{1}{q} + \frac{1}{q} |y|^\eta (f)_{Q,p} (g)_{Q,q}$$

$$\lesssim |Q|^\frac{1}{q} |y|^\eta (g)_{Q,p} (g)_{Q,q},$$

as $|Q|$ is comparable to $r^{2n+2}$.

Lemma 4.3. For $Q$ with $\ell(Q) = \delta^k$ we consider

$$V_Q = \{P \in D^\alpha_{k+3} : B(z_P, \delta^{k+1}) \subseteq Q\}.$$  

and define

$$A_Q f = A_{\delta^{k+2}} (f 1_{V_Q})$$

where $V_Q = \bigcup_{P \in V_Q} P$. Then for any $f$ supported in $Q$ the support of $A_Q f$ is also contained in $Q$. Moreover,

$$A_{\delta^{k+2}} f \leq \sum_{\alpha=1}^N \sum_{Q \in D^\alpha_k} A_Q (f).$$

We emphasize that we can take any $\delta$ in Lemma 4.3 (and in the rest of the paper), in particular we could take $\delta = 2$. In that case we have to do some modifications in defining $A_Q f$, where one has to use the fact that if $\delta < \frac{1}{96}$ then the number of points of the form $2^m$, $m \in \mathbb{Z}$, lying between $\delta^j$ and $\delta^{j+1}$, $j \in \mathbb{Z}$, does not depend on $j$. 
Proof of Lemma 4.3. Observe that for any \( x \in \mathbb{R}^n \) there exists \( P \in \mathcal{D}_{k+3}^1 \) such that \( x \in P \subseteq B(z_P, 4\delta^{k+3}) \). Then \( P \subseteq B(z_P, \delta^{k+1}) \subseteq Q \) for some \( Q \in \mathcal{D}_k^1 \), for some \( \alpha \). Therefore \( P \in V_Q \) and hence \( x \in V_Q \). This proves that \( \mathbb{R}^n = \bigcup_{\alpha=1}^N \bigcup_{Q \in \mathcal{D}_k^1} V_Q \) and hence we have \( f \leq \sum_{\alpha=1}^N \sum_{Q \in \mathcal{D}_k^1} f \chi_{V_Q} \) and consequently, \( A_{\delta^{k+2}} f \leq \sum_{\alpha=1}^N \sum_{Q \in \mathcal{D}_k^1} A_Q f \). It remains to be proved that \( A_Q f \) is supported in \( Q \). Now assume that \( \text{supp} f \subseteq Q \) and recall \( A_{\delta^{k+2}} f(x) = f \ast \mu_{\delta^{k+2}}(x) \). Then it is enough to show that \( \text{supp} A_{\delta^{k+2}}(f \chi_P) \subseteq B(z_P, \delta^{k+1}) \) for every \( P \in \mathcal{V}_Q \). Indeed,

\[
\text{supp}(f \chi_P) \ast \mu_{\delta^{k+2}} \subseteq \left( \text{supp}(f \chi_P) \right) \cdot (\text{supp} \mu_{\delta^{k+2}}) \subseteq z_P \cdot B(0, \delta^{k+2}) \cdot B(0, \delta^{k+2})
\]

which is contained in \( B(z_P, \delta^{k+1}) \subseteq Q \) by the definition of \( V_Q \). Observe that the above argument fails if we use balls defined by the standard right invariant metric. The lemma is proved.

In view of Lemma 4.3 it suffices to prove the sparse bound for each \( M_{\mathcal{D}^n} f = \sup_{Q \in \mathcal{D}^n} A_Q f \) for \( \alpha = 1, 2, \ldots, N \). Let us fix then \( \mathcal{D} = \mathcal{D}^n \). We will linearise the supremum. Let \( \mathcal{Q} \) be the collection of all dyadic subcubes of \( Q_0 \in \mathcal{D} \). Let us define

\[
E_Q := \{ x \in Q : A_Q f(x) \geq \frac{1}{2} \sup_{P \in Q} A_P f(x) \}
\]

for \( Q \in \mathcal{Q} \). Note that for any \( x \in \mathbb{R}^n \) there exists a \( Q \in \mathcal{Q} \) such that

\[
A_Q f(x) \geq \frac{1}{2} \sup_{P \in Q} A_P f(x)
\]

and hence \( x \in E_Q \). If we define \( B_Q = E_Q \setminus \bigcup_{Q' \supseteq Q} E_{Q'} \), then \( \{ B_Q : Q \in \mathcal{Q} \} \) are disjoint and also, \( \bigcup_{Q \in \mathcal{Q}} B_Q = \bigcup_{Q \in \mathcal{Q}} E_Q \). Let \( f, g > 0 \). Then

\[
\langle \sup_{Q \in \mathcal{Q}} A_Q f, g \rangle = \sum_{Q \in \mathcal{Q}} \left( \sup_{B_Q} A_Q f(x) g(x) \right) \int_{B_Q} A_Q f(x) g(x) \, dx
\]

\[
\leq 2 \sum_{Q \in \mathcal{Q}} \int_{B_Q} A_Q f(x) g(x) \, dx
\]

\[
\leq 2 \sum_{Q \in \mathcal{Q}} \int_{\mathbb{R}^n} A_Q f(x) g(x) 1_{B_Q}(x) \, dx
\]

\[
\leq 2 \sum_{Q \in \mathcal{Q}} \langle A_Q f, g 1_{B_Q} \rangle.
\]

Defining \( g_Q = g 1_{B_Q} \) we will deal with \( \sum_{Q \in \mathcal{Q}} \langle A_Q f, g_Q \rangle \).

Lemma 4.4. Let \( 1 < p, q < \infty \) be such that \( \left( \frac{1}{p}, \frac{1}{q} \right) \) in the interior of the triangle joining the points \( (0, 1), (1, 0) \) and \( \left( \frac{n}{n+1}, \frac{n}{n+1} \right) \). Let \( f = 1_F \) and let \( g \) be any bounded function supported in \( Q_0 \). Let \( C_0 > 1 \) be a constant and let \( \mathcal{Q} \) be a collection of dyadic subcubes of \( Q_0 \in \mathcal{D} \) for which the following holds

\[
\sup_{Q' \in \mathcal{Q}} \sup_{Q' \subseteq Q \subseteq Q_0} \frac{\langle f \rangle_{Q_p}}{\langle f \rangle_{Q_{0,p}}} < C_0.
\]

Then there holds

\[
\sum_{Q \in \mathcal{Q}} \langle A_Q f, g \rangle \lesssim |Q_0| \langle f \rangle_{Q_{0,p}} \langle g \rangle_{Q_{0,q}}.
\]
Thus we have above argument it is important that the balls are defined using the left invariant metric).

\[
\langle f \rangle_{Q,p} > 2C_0 \langle f \rangle_{Q_0,p}.
\]

Set \( f = g_1 + b_1 \), where

\[
b_1 = \sum_{P \in \mathcal{B}} (f - \langle f \rangle_P) 1_P = \sum_{k=s_0+1}^\infty \sum_{P \in \mathcal{B}(k)} (f - \langle f \rangle_P) 1_P =: \sum_{k=s_0+1}^\infty B_{1,k},
\]

where \( \ell(Q_0) = \delta^{s_0} \) and \( \mathcal{B}(k) = \{ P \in \mathcal{B} : \ell(P) = \delta^k \} \). Now

\[
\| \sum_{Q \in \mathcal{Q}} \langle A_Q f, g_Q \rangle \| \lesssim \sum_{Q \in \mathcal{Q}} \| A_Q g_1 \|_1 \lesssim |Q|.
\]

We now make the following useful observation. For all \( Q \in \mathcal{Q} \) and \( P \in \mathcal{B} \), if \( P \cap Q \neq \emptyset \) then \( P \) is properly contained in \( Q \). For otherwise, \( Q \subseteq P \) and by the assumption on \( \mathcal{Q} \), we get \( \langle f \rangle_{P,p} < C_0 \langle f \rangle_{Q_0,p} \). But this contradicts the Calderón–Zygmund decomposition since \( \langle f \rangle_{P,p} > 2C_0 \langle f \rangle_{Q_0,p} \). Therefore, for any \( Q \in \mathcal{Q} \) with \( \ell(Q) = \delta^s \) we have

\[
\langle A_Q b_1, g_Q \rangle = \sum_{k>s} \langle A_Q B_{1,k}, g_Q \rangle = \sum_{k=1}^\infty \langle A_Q B_{1,s+k}, g_Q \rangle
\]

and so

\[
\sum_{Q \in \mathcal{Q}} \langle A_Q b_1, g_Q \rangle \leq \sum_{k=1}^\infty \sum_{Q \in \mathcal{Q}} \langle A_Q B_{1,s+k}, g_Q \rangle.
\]

By making use of the mean zero property of \( b_1 \), we see that

\[
\langle A_Q B_{1,s+k}, g_Q \rangle = \langle B_{1,s+k}, A_Q^* g_Q \rangle
\]

\[
= \sum_{P \in B(s+k)} \left| \int_P A_Q^* g_Q(x) B_{1,s+k}(x) \, dx \right|
\]

\[
\leq \sum_{P \in B(s+k)} \frac{1}{|P|} \left| \int_P \int_P [A_Q^* g_Q(x) - A_Q^* g_Q(x')] B_{1,s+k}(x) \, dx \, dx' \right|.
\]

In the integral with respect to \( x' \) we make the change of variables \( x' = xy^{-1} \) and note that \( P^{-1} x \subseteq P^{-1} P \). Since \( P \subset B(z_p, 4\delta^{s+k}) = z_p \cdot B(0, 4\delta^{s+k}) \) it follows that \( P^{-1} \subset B(0, 4\delta^{s+k})z_p^{-1} \) and hence \( P^{-1} P \subset P_0 = B(0, 8\delta^{s+k}) \subset B(0, \delta^{s+k-1}) \) (observe that for the above argument it is important that the balls are defined using the left invariant metric). Thus we have

\[
\langle A_Q B_{1,s+k}, g_Q \rangle \leq \sum_{P \in B(s+k)} \frac{1}{|P|} \left| \int_{P^{-1} P} \int_P [A_Q^* g_Q(x) - \tau_y A_Q^* g_Q(x)] B_{1,s+k}(x) \, dx \, dy \right|
\]

\[
\lesssim \frac{1}{|P_0|} \int_{P_0} \int_Q g_Q(x)(A_Q - A_Q \tau_{y^{-1}}) B_{1,s+k}(x) \, dx \, dy.
\]
\[ \sum_{Q \in \mathcal{Q}} |Q| \langle B_{1,s+k} \rangle_{Q,p} \langle g \rangle_{Q,q} \leq |Q_0| \langle f \rangle_{Q_0,p} \langle g \rangle_{Q_0,q}, \]

for all \( k \geq 1 \) and for all \( 1 < p, q < \infty \) such that \((\frac{1}{p}, \frac{1}{q})\) are in the interior of the triangle joining the points \((0, 1), (1, 0)\) and \((1, 1)\) (including the segment joining \((0, 1)\) and \((1, 0)\), excluding the endpoints).

Let us fix the integer \( k \). From the definition and (4.1) it follows that we can dominate

\[ |B_{1,s+k}| \lesssim \langle f \rangle_{Q_0,p} \chi_{E_s} + \chi_{F_{1,s}}, \]

where \( E_s = E_{s,k} \) are pairwise disjoint sets in \( Q_0 \) as \( s \) varies, and \( F_{1,s} = F_{1,s,k} \) are pairwise disjoint sets in \( F_1 \). This produces two terms to control. For the first one, we will show that

\[ \langle f \rangle_{Q_0,p} \sum_{Q \in \mathcal{Q}} |Q| \langle 1_{E_s} \rangle_{Q,p} \langle g \rangle_{Q,q} \lesssim |Q_0| \langle f \rangle_{Q_0,p} \langle g \rangle_{Q_0,q}. \]

First we consider the case when \( 1/p + 1/q = 1 \), i.e. \( p = q' \), for \( 1 < p < \infty \).

\[ \sum_{Q \in \mathcal{Q}} |Q| \langle 1_{E_s} \rangle_{Q,p} \langle g \rangle_{Q,q} = \sum_{Q \in \mathcal{Q}} \left( \int_{Q} 1_{E_s} \, dx \right)^{1/p} \left( \int_{Q} |g(x)|^{p'} \, 1_{B_Q} \, dx \right)^{1/p'} \]

\leq \left( \sum_{Q \in \mathcal{Q}} \int_{Q} 1_{E_s} \, dx \right)^{1/p} \left( \sum_{Q \in \mathcal{Q}} \int_{Q} |g(x)|^{p'} \, 1_{B_Q} \, dx \right)^{1/p'}.

On one hand, from the disjointness of \( B_Q \),

\[ \sum_{Q \in \mathcal{Q}} \int_{Q} |g(x)|^{p'} \, 1_{B_Q} \, dx = \int_{\bigcup_{Q \in \mathcal{Q}} B_Q} |g(x)|^{p'} \, dx \leq \left( \frac{1}{|Q_0|} \int_{Q_0} |g(x)|^{p'} \, dx \right) |Q_0| = |Q_0| \langle g \rangle_{Q_0,q}^{p'}.

On the other hand, as \( E_s \cap Q \) are disjoing subsets of \( Q_0 \) we finally obtain

\[ \sum_{Q \in \mathcal{Q}} \int_{Q} 1_{E_s} \, dx = \sum_{Q \in \mathcal{Q}} |E_s \cap Q| \leq |Q_0|. \]

Thus the required inequality (4.3) is proved in the case \( 1/p + 1/q = 1 \). In the case \( 1/p + 1/q = 1 + \tau > 1 \), set \( 1/p = 1/p - \tau \). Then, \( 1/p + 1/q = 1 \), and \( p < p' \), so that

\[ \langle 1_{E_s} \rangle_{Q,p} \langle g \rangle_{Q,q} \lesssim \langle 1_{E_s} \rangle_{Q,p} \langle g \rangle_{Q,q}. \]

Then, (4.4) follows from the previous case since \( 1/p + 1/q' = 1 \).

Concerning the second term, we will show that

\[ \sum_{Q \in \mathcal{Q}} |Q| \langle 1_{F_{1,s}} \rangle_{Q,p} \langle g \rangle_{Q,q} \lesssim |Q_0| \langle f \rangle_{Q_0,p} \langle g \rangle_{Q_0,q}. \]
Again, the inequality holds in the case of $1/p + 1/q = 1$. For $1/p + 1/q = 1 + \tau > 1$, we define $\tilde{p}$ as above. By using the stopping condition (4.2) we have then
\[
\langle f, g \rangle_{Q, \tilde{p}} \lesssim \langle f, g \rangle_{Q, \tilde{p}, 0} + \langle f_{\tilde{p}}, g_{\tilde{p}} \rangle_{Q, \tilde{p}, 0}.
\]
From this and by using the previous case, since $1/\tilde{p} + 1/q = 1$, we can conclude (4.5), and therefore (4.3). The proof is complete. \hfill \Box

Let us proceed to prove Theorem 1.2. We will state it also here, for the sake of the reading.

**Theorem 4.5.** Assume $n \geq 2$ and fix $0 < \delta < \frac{1}{96}$. Let $1 < p, q < \infty$ be such that $(\frac{1}{p}, \frac{1}{q})$ belongs to the interior of the triangle joining the points $(0, 1), (1, 0)$ and $(\frac{n}{n+1}, \frac{n}{n+1})$. Then for any pair of compactly supported bounded functions $(f, g)$ there exists a $(p, q)$-sparse form such that $\langle M_{\text{lac}}f, g \rangle \lesssim C\Lambda_{S, p, q}(f, g)$.

**Proof.** Fix a dyadic grid $D$ and consider the maximal function
\[
M_D f(x) = \sup_{Q \in D} |A_Q f(x)|.
\]
We can assume that $f \geq 0$ and supported in $Q_0$ so that $A_Q f = 0$ for all large enough cubes. According to this, we will therefore prove the sparse bound for the maximal function
\[
M_{D \cap Q_0} f(x) = \sup_{Q \in D} |A_Q f(x)|.
\]
From this, it follows that $M_{\text{lac}}$ is bounded by the sum of a finite number of sparse forms. But it is known that there exists one universal dominating sparse form (see for instance [12, Lemma 4.7] and [5, Proposition 2.1]). Namely, given $f, g$, there is a constant $C > 1$ and sparse family of dyadic cubes $S_0$ so that $\sup_{S \in S_0} \Lambda_{S, p, q}(f, g) \lesssim C\Lambda_{S_0, p, q}(f, g)$. This fact, proved in the Euclidean setting, is also valid in our case and we will not enter into details. Therefore, the claimed sparse bound holds.

As explained above, by linearising the supremum it is enough to prove the sparse bound for the sum
\[
\sum_{Q \in D \cap Q_0} \langle A_Q f, g 1_{B_Q} \rangle
\]
for the collection of pairwise disjoint $B_Q \subset Q$ described just before Lemma 4.4.

Given $1 < p, q < \infty$ so that the $L^p$ improving and continuity properties of the spherical means hold for $(\frac{1}{p}, \frac{1}{q})$ (i.e., Corollaries 2.11 and 3.5 hold), we have to produce a sparse family $S$ of subcubes of $Q_0$ such that
\[
\langle M_{D \cap Q_0} f, g \rangle \lesssim \sum_{Q \in D \cap Q_0} \langle A_Q f, g 1_{B_Q} \rangle \lesssim C \sum_{S \in S} |S| \langle f \rangle_{S, p} \langle g \rangle_{S, q}
\]
where for each $S \in S$, there exists $F_S \subset S$ with $|F_S| \geq \frac{1}{2}|S|$.

We first prove (4.6) when $f$ is the characteristic function of a set $F \subset Q_0$. Consider the collection $E_{Q_0}$ of maximal children $P < Q_0$ for which
\[
\langle f \rangle_{P, p} > 2 \langle f \rangle_{Q_0, p}.
\]
Let $E_{Q_0} = \cup_{P \in E_{Q_0}}$. For a suitable choice of $c_n > 1$ we can arrange $|E_{Q_0}| < \frac{1}{2}|Q_0|$. We let $F_{Q_0} = Q_0 \setminus E_{Q_0}$ so that $|F_{Q_0}| \geq \frac{1}{2}|Q_0|$. We define
\[
Q_0 = \{Q \in D \cap Q_0 : Q \cap E_{Q_0} = \emptyset\}.
\]
Note that when $Q \in Q_0$ then $\langle f \rangle_{Q,p} \leq 2\langle f \rangle_{Q_0,p}$. For otherwise, if $\langle f \rangle_{Q,p} > 2\langle f \rangle_{Q_0,p}$ then there exists $P \in E_{Q_0}$ such that $P \supset Q$, which is a contradiction. For the same reason, if $Q' \in Q_0$ and $Q' \subset Q \subset Q_0$ then $\langle f \rangle_{Q,p} \leq 2\langle f \rangle_{Q_0,p}$. Thus

$$\sup_{Q' \in Q_0} \sup_{Q' \subset Q \subset Q_0} \langle f \rangle_{Q,p} \leq 2\langle f \rangle_{Q_0,p}.$$  

Note that for any $Q \in D \cap Q_0$, either $Q \in Q_0$ or $Q \subset P$ for some $P \in E_{Q_0}$. Thus

$$\sum_{Q \in D \cap Q_0} \langle A_Q f, g 1_{B_Q} \rangle = \sum_{Q \in Q_0} \langle A_Q f, g 1_{B_Q} \rangle + \sum_{P \in E_{Q_0}, Q \subset P} \sum_{Q' \in Q_0} \langle A_Q f, g 1_{B_Q} \rangle$$

for any $Q \in Q_0$, $Q \subset F_{Q_0}$ and hence

$$\sum_{Q \in Q_0} \langle A_Q f, g 1_{B_Q} \rangle = \sum_{Q \in Q_0} \langle A_Q f, g 1_{F_{Q_0}} 1_{B_Q} \rangle.$$  

Applying Lemma 4.4 we obtain

$$\sum_{Q \in Q_0} \langle A_Q f, g 1_{B_Q} \rangle \leq C|Q_0| \langle f \rangle_{Q_0,p} \langle g 1_{F_{Q_0}} \rangle_{Q_0,q}.$$  

Let $\{P_j\}$ be an enumeration of the cubes in $E_{Q_0}$. Then the second sum above is given by

$$\sum_{j=1}^{\infty} \sum_{Q \in F_{P_j} \cap D} \langle A_Q f, g 1_{B_Q} \rangle.$$  

For each $j$ we can repeat the above argument recursively. Putting everything together we get a sparse collection $S$ for which

$$\sum_{Q \in D \cap Q_0} \langle A_Q f, g 1_{B_Q} \rangle \leq C \sum_{S \in S} |S| \langle f \rangle_{S,p} \langle g 1_{F_{S}} \rangle_{S,q}.$$  

This proves the result when $f = 1_F$. We pause for a moment to remark that we have actually proved a sparse domination stronger than the one stated in the theorem. However, we are not able to prove such a result for general $f$.

Now we prove the theorem for any bounded $f \geq 0$ supported in $Q_0$. We start as in the case of $f = 1_F$ but now we define $Q_0$ using stopping conditions on both $f$ and $g$. Thus we let $E_{Q_0}$ stand for the collection of maximal subcubes $P$ of $Q_0$ for which either $\langle f \rangle_{P,p} > 2\langle f \rangle_{Q_0,p}$ or $\langle g \rangle_{P,q} > 2\langle g \rangle_{Q_0,q}$. As before, we define $E_{Q_0} = \cup_{P \in E_{Q_0}}$ and $F_{Q_0} = Q_0 \setminus E_{Q_0}$ so that $|F_{Q_0}| \geq \frac{1}{2}|Q_0|$. We let

$$Q_0 = \{ Q \in D \cap Q_0 : Q \cap E_{Q_0} = \emptyset \}.$$  

Then it follows that

$$\sup_{Q' \in Q_0} \sup_{Q' \subset Q \subset Q_0} \langle f \rangle_{Q,p} \leq 2\langle f \rangle_{Q_0,p}$$  

and

$$\sup_{Q' \in Q_0} \sup_{Q' \subset Q \subset Q_0} \langle g \rangle_{Q,q} \leq 2\langle g \rangle_{Q_0,q}.$$  

If we can show that

$$\sum_{Q \in Q_0} \langle A_Q f, g 1_{B_Q} \rangle \leq C|Q_0| \langle f \rangle_{Q_0,p} \langle g \rangle_{Q_0,q}$$  

(4.9)
for some $\rho > p$, then we can proceed as in the case of $f = 1_F$ to get the sparse domination
\[
\langle M_D f, g \rangle \leq C \sum_{S \in \mathcal{S}} |S| \langle f \rangle_{S,p} \langle g \rangle_{S,q}.
\]

In order to prove (4.9) we make use of the sparse domination already proved for $f = 1_F$. Defining $E_m = \{ x \in Q_0 : 2^m \leq f(x) \leq 2^{m+1} \}$ and $f_m = f 1_{E_m}$ we have the decomposition $f = \sum_m f_m$ (since $f$ is bounded it follows that $E_m = \emptyset$ for all $m \geq m_0$ for some $m_0 \in \mathbb{Z}$). By applying the sparse domination to $1_{E_m}$ we obtain the following:
\[
\sum_{Q \in \mathcal{Q}_0} \langle A_Q f_m, g 1_{B_Q} \rangle \leq 2^{m+1} \sum_{Q \in \mathcal{Q}_0} \langle A_Q 1_{E_m}, g 1_{B_Q} \rangle = 2^{m+1} \sum_{Q \in \mathcal{Q}_0} \langle A_Q 1_{E_m}, g 1_{F_{Q_0}} 1_{B_Q} \rangle \leq 2^{m+1} \sum_{Q \in \mathcal{Q}_0 \cap \mathcal{D}} \langle A_Q 1_{E_m}, g 1_{F_{Q_0}} 1_{B_Q} \rangle \leq C 2^{m+1} \sum_{S \in \mathcal{S}_m} |S| \langle 1_{E_m} \rangle_{S,p} \langle g 1_{F_{Q_0}} \rangle_{S,q},
\]
where in the last three lines we used that for any $Q \in \mathcal{Q}_0, Q \subset F_{Q_0}$, (4.7) and (4.8). In the above sum, $\langle g 1_{F_{Q_0}} \rangle_{S,q} = 0$ unless $S \cap F_{Q_0} \neq \emptyset$. If $S \subset F_{Q_0}$ then by the definition of $Q_0$ in (4.7) it follows that $S \in \mathcal{Q}_0$ and
\[
\langle g 1_{F_{Q_0}} \rangle_{S,q} \leq \langle g \rangle_{S,q} \leq c_n \langle g \rangle_{Q_0,q}.
\]
If $S \cap F_{Q_0} \neq \emptyset$ as well as $S \cap E_{Q_0} \neq \emptyset$ then for some $P \in \mathcal{E}_{Q_0}, P \subset S$. But then by the maximality of $P$ we have
\[
\langle g 1_{F_{Q_0}} \rangle_{S,q} \leq \langle g \rangle_{S,q} \leq 2 \langle g \rangle_{Q_0,q}.
\]
Using this we obtain
\[
\sum_{Q \in \mathcal{Q}_0} \langle A_Q f_m, g 1_{B_Q} \rangle \leq C 2^{m+1} \langle g \rangle_{Q_0,q} \sum_{S \in \mathcal{S}_m} |S| \langle 1_{E_m} \rangle_{S,p}.
\]
By Lemma 4.8 we get
\[
\sum_{Q \in \mathcal{Q}_0} \langle A_Q f_m, g 1_{B_Q} \rangle \leq C 2^{m+1} \langle g \rangle_{Q_0,q} \langle 1_{E_m} \rangle_{Q_0,p_1} |Q_0|
\]
for some $p_1 > p$. As $f = \sum_m f_m$ it follows that
\[
\sum_{Q \in \mathcal{Q}_0} \langle A_Q f, g 1_{B_Q} \rangle \leq C \langle g \rangle_{Q_0,q} |Q_0| \sum_m 2^m \langle 1_{E_m} \rangle_{Q_0,p_1}.
\]

We now claim that (see Lemma 4.7 below)
\[
(4.10) \quad \sum_m 2^m \langle 1_{E_m} \rangle_{Q_0,p_1} \leq C \| f \|_{L^{p_1,1}(Q_0,d\mu)}
\]
where $L^{p,1}(Q_0,d\mu)$ stands for the Lorentz space defined on the measure space $(Q_0,d\mu)$, $d\mu = \frac{1}{|Q_0|} dx$. We also know that on a probability space, the $L^{p,1}(Q_0,d\mu)$ norm is dominated by the $L^p(Q_0,d\mu)$ norm for any $\rho > p_1$ (Lemma 4.6). Using these two results we see that
\[
\sum_{Q \in \mathcal{Q}_0} \langle A_Q f, g 1_{B_Q} \rangle \leq C \langle g \rangle_{Q_0,q} |Q_0| \langle f \rangle_{Q_0,p}.
\]
Hence (4.9) is proved and thus completes the proof of Theorem 4.5.

It remains to prove Lemma 4.6 and the claim (4.10). The first one is a well known fact which we include here for the sake of completeness.

**Lemma 4.6.** On a probability space \((X, d\mu), L^p(X, d\mu) \subset L^{r,1}(X, d\mu)\) for \(p > r\).

**Proof.** Recall that the Lorentz spaces \(L^{p,q}(X, d\mu)\) are defined in terms of the Lorentz norms (see [9])

\[
\|f\|_{p,q} = \begin{cases} 
\left( \int_0^\infty \left( \frac{1}{t} f^*(t) \right)^q dt \right)^{1/q} & \text{if } q < \infty, \\
\sup_{t > 0} \frac{1}{t} f^*(t) & \text{if } q = \infty,
\end{cases}
\]

where \(f^*(t)\) stands for the non-decreasing rearrangement of \(f\). When \(f \in L^p(X, d\mu)\), as \(d\mu\) is a probability measure, we know that the distribution function \(df(s)\) of \(f\) is bounded by 1 and hence \(f^*(t) = 0\) for \(t \geq 1\). Now

\[
\|f\|_{L^{r,1}(X, d\mu)} = \int_0^\infty t^{\frac{1}{p}-1} f^*(t) dt = \int_0^1 t^{\frac{1}{r}-1} f^*(t) dt.
\]

By Hölder’s inequality

\[
\|f\|_{L^{r,1}(X, d\mu)} \leq \left( \int_0^1 t^{-\frac{p'}{r'}-1} dt \right)^{1/p'} \left( \int_0^1 f^*(t)^p dt \right)^{1/p} = C_{r,p} \left( \int_0^1 f^*(t)^p dt \right)^{1/p}
\]

where \(C_{r,p} < \infty\) since \(p' < r'\). This proves the claim since

\[
\left( \int_0^1 f^*(t)^p dt \right)^{\frac{1}{p}} = \|f\|_{L^p(X, d\mu)}.
\]

□

The claim (4.10) is the content of the next lemma.

**Lemma 4.7.** Let \(f = \sum_m f_m\), \(f_m = f 1_{E_m}\) where \(E_m = \{ x \in Q_0 : 2^m \leq |f(x)| \leq 2^{m+1} \}\). We consider the probability measure \(d\mu = |Q_0|^{-1} dx\) on \(X = Q_0\). Then for any \(r > 1\) we have

\[
\sum_m 2^m \langle 1_{E_m} \rangle_{Q_{0,r}} \leq C \|f\|_{L^{r,1}(Q_0, d\mu)}.
\]

**Proof.** We make use the following definition of the Lorentz norm in terms of \(df(s)\):

\[
\|f\|_{L^{r,1}(X, d\mu)} = \int_0^\infty df(s)^{\frac{1}{r}} ds.
\]

As \(df(s)\) is a decreasing function of \(s\) we have

\[
\|f\|_{L^{r,1}(X, d\mu)} = \sum_m \int_{2^m}^{2^{m+1}} df(s)^{\frac{1}{r}} ds \\
\geq \sum_m df(2^m)^{\frac{1}{r}} (2^{m+1} - 2^m) \\
= \frac{1}{2} \sum_m df(2^m)^{\frac{1}{r}} 2^m.
\]
As \( f_m = f 1_{E_m} \), it follows that \( \mu(E_m) = df(2^m) - df(2^{m+1}) \leq df(2^m) \) and consequently,
\[
\sum_m \mu(E_m) 2^m \leq \sum_m df(2^m) 2^m \leq 2\|f\|_{L^{1,1}(X,d\mu)}.
\]
This proves the lemma. \( \square \)

In proving Theorem 4.5 we have made use of the following lemma, which is proved in [11, Proposition 2.19]. We include a proof here for the convenience of the reader.

**Lemma 4.8** ([11]). Let \( S \) be a collection of sparse subcubes of a fixed dyadic cube \( Q_0 \) and let \( 1 \leq s < t < \infty \). Then, for a bounded function \( \phi \),
\[
\sum_{Q \in S} \langle \phi \rangle_{Q,s} |Q| \lesssim \langle \phi \rangle_{Q_0,t} |Q_0|.
\]

**Proof.** By sparsity,
\[
\sum_{Q \in S} \langle \phi \rangle_{Q,s} |Q| = \sum_{Q \in S} \langle \phi \rangle_{Q,s} |Q|^{1/t+1/t'} \leq \left( \sum_{Q \in S} \langle \phi \rangle_{Q,s} |Q| \right)^{1/t} \left( \sum_{Q \in S} |Q| \right)^{1/t'} \lesssim \left( \sum_{Q \in S} \langle \phi \rangle_{Q,s} |Q| \right)^{1/t} |Q_0|^{1/t'} \lesssim \|\phi 1_{Q_0} \|_{t} |Q_0|^{1/t'}.
\]
\( \square \)

4.1. **A sharpened sparse domination.** Although we have stated Theorem 4.5 for a slightly more restricted region \( L_n \), indeed the sparse domination holds for \( (\frac{1}{p}, \frac{1}{q}) \) in the interior of the triangle \( S_n \) (of course Lemma 4.4 holds also for the enlarged triangle since Lemma 4.2 does and so on). This means, in particular, that Theorem 4.5 is true for the closed triangle \( L_n \).

**Theorem 4.9.** Assume \( n \geq 2 \) and fix \( 0 < \delta < \frac{1}{96} \). Let \( 1 < p,q < \infty \) be such that \( (\frac{1}{p}, \frac{1}{q}) \) belongs to the interior of the triangle joining the points \( (0,1), (1,0) \) and \( (\frac{3n+1}{3n+4}, \frac{3n+1}{3n+4}) \). Then for any pair of compactly supported bounded functions \( (f,g) \) there exists a \( (p,q) \)-sparse form such that \( \langle M_{\text{inc}} f, g \rangle \leq C \Lambda_{S,p,q}(f,g) \).

5. **Boundedness properties**

Consequences inferred from sparse domination are well-known and have been studied in the literature. We refer to [1, Section 4] for an account of the same. In particular, sparse domination provides unweighted and weighted inequalities for the operators under consideration.

The strong boundedness is a result by now standard, see [7], also [11, Proposition 6.1]. Our Theorem 1.1 follows from Theorem 4.9 (or just Theorem 1.2) and Proposition 5.1.

**Proposition 5.1** ([7]). Let \( 1 \leq r < s' \leq \infty \). Then,
\[
\Lambda_{r,s}(f,g) \lesssim \|f\|_{L^p} \|g\|_{L^{p'}}, \quad r < p < s'.
\]
Once again for the sake of completeness we reproduce the proof which is quite simple: as the collection $S$ is sparse, we have

$$\Lambda_{r,s}(f,g) \leq C \sum_{S \in S} \int_{E_S} \langle f \rangle_{S,r} \langle g \rangle_{S,s} 1_{E_S} \, dx$$

where $E_S \subset S$ are disjoint with the property that $|E_S| \geq \eta |S|$. The above leads to the estimate

$$\Lambda_{r,s}(f,g) \leq C \int_{\mathbb{H}^n} (\Lambda |f|^r(x))^{1/r} (\Lambda |g|^s(x))^{1/s} \, dx$$

where $\Lambda h$ stands for the Hardy-Littlewood maximal function of $h$. In view of the boundedness of $\Lambda$, an application of Hölder’s inequality completes the proof of the proposition.

A weight $w$ is a nonnegative locally integrable function defined on $\mathbb{H}^n$. Given $1 < p < \infty$, the Muckenhoupt class of weights $A_p$ consists of all $w$ satisfying

$$[w]_{A_p} := \sup_Q \langle w \rangle_Q \langle \sigma \rangle_Q^{p-1} < \infty, \quad \sigma := w^{1-p'}$$

where the supremum is taken over all cubes $Q$ in $\mathbb{H}^n$. On the other hand, a weight $w$ is in the reverse Hölder class $RH_p$, $1 \leq p < \infty$, if

$$[w]_{RH_p} = \sup_Q \langle w \rangle_Q^{-1} \langle w \rangle_Q < \infty,$$

again the supremum taken over all cubes in $\mathbb{H}^n$.

The following theorem was shown in [2, Section 6].

**Theorem 5.2 ([2]).** Let $1 \leq p_0 < q_0' \leq \infty$. Then,

$$\Lambda_{p_0,q_0}(f,g) \leq \{ [w]_{A_{p_0/p_0}} \cdot [w]_{RH_{(q_0'/p_0)'}} \}^\alpha \| f \|_{L^p(w)} \| g \|_{L^{q_0'}(\sigma)}, \quad p_0 < p < q_0'$$

with $\alpha = \max \left\{ \frac{1}{p_0-1} - \frac{1}{q_0'-1}, 0 < \frac{1}{p_0} \leq \frac{n}{n+1}, \frac{n}{n+1} < \frac{1}{p_0} < 1 \right\}$.

In view of Theorem 5.2 and with the sharpened sparse domination in Theorem 4.9 at hand, but restricting ourselves to values of $(1/p, 1/q)$ on $L_n$, we can obtain the following corollary: it provides unprecedented weighted estimates for the lacunary maximal spherical means in $\mathbb{H}^n$.

**Corollary 5.3.** Let $n \geq 2$ and define

$$\frac{1}{\phi(1/p_0)} = \begin{cases} 1 - \frac{1}{n p_0}, & 0 < \frac{1}{p_0} \leq \frac{n}{n+1}, \\ n \left( 1 - \frac{1}{p_0} \right), & n \left( 1 - \frac{1}{p_0} \right) \frac{n}{n+1} < \frac{1}{p_0} < 1. \end{cases}$$

Then $M_{\text{lac}}$ is bounded on $L^p(w)$ for $w \in A_{p_0/p_0} \cap RH_{(\phi(1/p_0)/p)'},$ and all $1 < p_0 < p < (\phi(1/p_0))'$.

Quantitative weighted estimates could have been stated in Corollary 5.3, because by Theorem 4.9 we have the sparse domination in the closed triangle $L_n$.

**5.1. A sharpened weighted inequality.** Finally, we remark that an enhanced version of Corollary 5.3 with the range of $(1/p, 1/q)$ in the interior of $S_n$, might be also stated (see [11, Corollary 6.3] and [11, Corollary 4.2] for similar discussions).
Corollary 5.4. Let \( n \geq 2 \) and define
\[
\frac{1}{\phi(1/p_0)} = \begin{cases} 
1 - \frac{1}{p_0} \frac{3}{3n+1}, & 0 < \frac{1}{p_0} \leq \frac{3n+1}{3n+4}, \\
\frac{3n+1}{3n+4} \left(1 - \frac{1}{p_0}\right), & \frac{3n+1}{3n+4} < \frac{1}{p_0} < 1.
\end{cases}
\]

Then \( M_{lac} \) is bounded on \( L^p(w) \) for \( w \in A_{p/p_0} \cap \text{RH}_{\phi(1/p_0)/p} \) and all \( 1 < p_0 < p < (\phi(1/p_0))' \).

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