INTRUSIVE AND NON-INTRUSIVE POLYNOMIAL CHAOS APPROXIMATIONS FOR A TWO-DIMENSIONAL STEADY STATE NAVIER-STOKES SYSTEM WITH RANDOM FORCING

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ABSTRACT. While convergence of polynomial chaos approximation for linear equations is relatively well understood, a lot less is known for non-linear equations. The paper investigates this convergence for a particular equation with quadratic nonlinearity.

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1. INTRODUCTION

There are two main ways to study an equation with random input. One way is to use deterministic tools for each particular realization of randomness; in what follows, we call it path-wise approach. An alternative, which we call random field approach, is to consider the random input as an additional independent variable in the equation, along with space and/or time.

Questions such as existence/uniqueness/regularity of the solution are often addressed with a combination of the two approaches; cf. [18, 19] for ordinary differential equations and [17, 24, 25] for equations with partial derivatives.

The difference between the two approaches becomes noticeable in numerical computations; see, for example, [13, 35]. Path-wise approach leads to repeated numerical solutions of the underlying equation for various realizations of the random input; a typical example is Monte Carlo simulations. In computational terms, this approach is non-intrusive, because no new numerical procedures are required to solve the equation compared to the deterministic case.

The random field approach reduces the problem to a fixed system of deterministic equations via a stochastic Galerkin approximation; in many cases, the result is a generalized polynomial chaos (gPC) expansion [35, Chapter 6]. In computational terms, this approach is intrusive, because the resulting system is more complicated than the original equation and requires different numerical procedures to obtain a solution.

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The stochastic collocation method [35, Chapter 7], with sampling at pre-determined realizations of the random input, somewhat bridges the gap between pure random sampling (Monte Carlo) and complete elimination of randomness (gPC). In computational terms, the method is non-intrusive [3, 31]. In this paper, we consider the discrete projection, or pseudo-spectral version of the method, when the sampled solution is used to approximate the coefficients in the chaos expansion via Gauss quadrature.

For many, although apparently not all [8], equations, various empirical studies [15, 28, 32, etc.] suggest that the stochastic Galerkin approximation method, with a fixed computational cost, can be a much more efficient way to study statistical properties of the solution than Monte Carlo or stochastic collocation methods. In the case of nonlinear equations, this experimental success has yet to be fully justified theoretically; for linear equations, the picture is rather clear; see, for example, [9, 20, 21] as well as [22, Chapter 5] and [30, Section 8.3].

Accordingly, our objective in this paper is to carry out a comparative theoretical analysis of an intrusive and a non-intrusive approximations for a particular nonlinear equation. Specifically, we consider the stationary Navier-Stokes system in a smooth bounded planar domain with zero boundary conditions and with randomness in the external force, and we establish a priori error bounds for both approximations.

The paper is organized as follows. Section 2 describes the model and introduces the necessary function spaces. Section 3 introduces the stochastic Galerkin approximation and gives the proof of convergence. Section 4 investigates a non-intrusive pseudo-spectral approximation. Section 5 puts the results in a broader context.

Throughout the paper, $G$ is a bounded domain in $\mathbb{R}^2$ with area $|G|$ and sufficiently regular (e.g. locally Lipschitz) boundary $\partial G$. We use the following convention with the notations of various function spaces and their elements: if $X$ denotes a space of scalar fields $f$ on $G$, then $X$ denotes the corresponding space of vector fields $f$, and $\mathbb{X}$ denotes the collection of $\mathbb{X}$-valued random elements $f$.

2. The Setting

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space such that the $L_2(\Omega)$ is a separable Hilbert space and has a complete orthogonal basis $\{\mathbb{P}_n, n \geq 0\}$: with
c(n) = \mathbb{E}\mathbb{P}_n^2,  \quad (2.1)

every $\zeta \in L_2(\Omega)$ can be represented as
\[ \zeta = \sum_{k \geq 0} \frac{\mathbb{E}(\zeta \mathbb{P}_k)}{c(k)} \mathbb{P}_k. \]

In what follows, we always assume that the basis $\{\mathbb{P}_n, n \geq 0\}$ has the following property: for every $m, n \geq 0$, there are finitely many real numbers $A_{m,n,l}$, $l \geq 0$, such that
\[ \mathbb{P}_m \mathbb{P}_n = \sum_{l \geq 0} A_{m,n,l} \mathbb{P}_l; \quad (2.2) \]
in that case
\[ A_{m,n,l} = \frac{\mathbb{E}(\mathcal{P}_m \mathcal{P}_n \mathcal{P}_l)}{c(l)}. \]

Property (2.2) holds when each \( \mathcal{P}_n \) is a polynomial or a tensor product of polynomials.

Denote by \( \mathcal{P}^N \) the orthogonal projection in \( L_2(\Omega) \) on the subspace spanned by \( \{ \mathcal{P}_k, \ k = 0, \ldots , N \} \).

Consider a steady-state Navier-Stokes system with random forcing in a bounded domain \( G \subset \mathbb{R}^2 \) with sufficiently regular boundary \( \partial G \):
\begin{equation}
\begin{aligned}
\nu \Delta u(x) &= (u \cdot \nabla)u + \nabla p(x) + f(x), \ x \in G, \\
\text{div} \ u(x) &= 0, \ x \in G, \ u|_{\partial G} = 0.
\end{aligned}
\end{equation}

In equation (2.3),
\begin{itemize}
  \item \( \nu > 0 \) is the kinematic viscosity coefficient, \( x = (x_1, x_2) \), \( \Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \) is the Laplace operator, and \( \nu \) is constant;
  \item \( u(x) = (u^1(x), u^2(x)) \) is the (unknown) velocity and
  \[ \text{div} \ u = \nabla \cdot u = \frac{\partial u^1}{\partial x_1} + \frac{\partial u^2}{\partial x_2}, \ (u \cdot \nabla)u^i = u^1 \frac{\partial u^i}{\partial x_1} + u^2 \frac{\partial u^i}{\partial x_2}, \ i = 1, 2; \]
  \item \( p = p(x) \) is the (unknown scalar) pressure and \( (\nabla p)^i = \frac{\partial p}{\partial x_i}, \ i = 1, 2; \)
  \item \( f \) is the random forcing.
\end{itemize}

Two standard references for the deterministic counterpart of (2.3) are [11, Chapter IX] and [33, Chapter II].

We will use the following function spaces:
\begin{itemize}
  \item \( \mathcal{C}_0^\infty(G) \), the collection of infinitely differentiable real-valued functions on \( G \) with compact support in \( G \);
  \item \( \mathcal{D}(G) = \{ \varphi = (\varphi^1, \varphi^2), \ \varphi^i \in \mathcal{C}_0^\infty(G), \ i = 1, 2 : \text{div} \ \varphi = 0 \} \);
  \item \( L^r(G) \), \( 1 \leq r < +\infty \), the collection of measurable functions \( g \) on \( G \) such that
  \[ |g|_{L^r} = \left( \int_G |g(x)|^r \ dx \right)^{1/r} < \infty; \]
  for \( g, f \in L^2(G) \), we write
  \[ (f, g)_0 = \int_G f(x)g(x) \ dx; \]
  \item \( L^r(G) \), the collection of vector fields \( \mathbf{g} = (g^1, g^2) \) on \( G \) such that \( g^1, g^2 \in L^r(G) \), and endowed with norm
  \[ |\mathbf{g}|_{L^r} = \left( |g^1|_{L^r}^r + |g^2|_{L^r}^r \right)^{1/r}; \]
  for \( \mathbf{g}, \mathbf{f} \in L^2(G) \), we write
  \[ (\mathbf{f}, \mathbf{g})_0 = \int_G \left( f^1(x)g^1(x) + f^2(x)g^2(x) \right) \ dx; \]
\end{itemize}
II.3.5. We denote the corresponding duality by

\[ \langle g, h \rangle = (g^1, g^2) \]

such that

\[ \|g\|_{1,2} = (\mathbb{E} \|g\|_{L^2})^{1/2} < \infty; \]

- \( H_{-1,2}^0 (G) \), the completion of \( \mathcal{C}_0^\infty (G) \) with respect to the norm

\[
\|g\|_{-1,2} = \sup \left\{ \int_G g(x) \varphi(x) \, dx, \varphi \in H_{-1,2}^0 (G), \|\varphi\|_{1,2} \leq 1 \right\};
\]

- \( H_{0,1}^1 (G) \), the collection of vector fields \( g = (g^1, g^2) \) on \( G \) such that \( g^1, g^2 \in H_{0,1}^2 (G) \), and endowed with norm

\[
\|g\|_{1,2} = \left( |g^1|_{1,2}^2 + |g^2|_{1,2}^2 \right)^{1/2};
\]

for \( f, g \in H_{0,1}^{1,2} (G) \), we write

\[
(\nabla f, \nabla g)_0 = \sum_{i,j=1}^2 \int_G \left( \frac{\partial f^i(x)}{\partial x_j} \frac{\partial g^i(x)}{\partial x_j} \right) \, dx,
\]

so that

\[
\|g\|_{1,2}^2 = (\nabla g, \nabla g);
\]

- \( \mathbb{H}_{0,1}^{1,2} (G) = L^2 (\Omega; \mathbb{H}^{1,2}_{-1,2} (G)) \);

- \( \hat{H}_{0,1}^{1,2} (G) \), the completion of \( \mathcal{D}(G) \) with respect to the norm \( \| \cdot \|_{1,2}; \)

- \( \hat{H}_{0,1}^{1,2} (G) = L^2 (\Omega; \hat{H}^{1,2}_{-1,2} (G)) \);

- \( H_{0,1}^{1,2} (G) \), the collection of vector fields \( g = (g^1, g^2) \) such that \( g^1, g^2 \in H_{0,1}^{1,2} (G) \), and endowed with norm

\[
\|g\|_{1,2} = \left( |g^1|_{1,2}^2 + |g^2|_{1,2}^2 \right)^{1/2};
\]

- \( \mathbb{H}_{0,1}^{1,2} (G) = L^2 (\Omega; \mathbb{H}^{1,2}_{0,1} (G)) \).

The (Banach space) dual of \( H_{0,1}^{1,2} (G) \) is isomorphic to \( H_{0,1}^{1,2} (G) \); see [11, Theorem II.3.5]. We denote the corresponding duality by \( \langle f, g \rangle \), \( f \in H_{0,1}^{1,2} (g), g \in H_{0,1}^{1,2} (G) \). Similarly, the dual of \( H_{0,1}^{1,2} (G) \) is isomorphic to \( H_{0,1}^{1,2} (G) \) and the duality is denoted by \( \langle f, g \rangle \), \( f \in H_{0,1}^{1,2} (G), g \in H_{0,1}^{1,2} (G) \).
For \((u, v, w) \in H^{1,2}_0(G) \times H^{1,2}_0(G) \times H^{1,2}_0(G)\), we define the tri-linear form
\[
a(u, v, w) = ((u \cdot \nabla)v, w)_0;
\] (2.7)
similar to (2.4),
\[(u \cdot \nabla)v^i = u^1 \frac{\partial v^i}{\partial x_1} + u^2 \frac{\partial v^i}{\partial x_2}, \ i = 1, 2.
\]

**Lemma 2.1.** The trilinear form \(a\) has the following properties:

1. If \((u, v, w) \in H^{1,2}_0(G) \times H^{1,2}_0(G) \times H^{1,2}_0(G)\), then
   \[
   |a(u, v, w)| \leq \frac{\sqrt{|G|}}{2} |u|_{1,2} |v|_{1,2} |w|_{1,2};
   \] (2.8)
2. If \(u \in \hat{H}^{1,2}_0(G)\), then
   \[
a(u, v, v) = 0
   \] (2.10)
   and
   \[
a(u, v, w) = -a(u, w, v).
   \] (2.11)

For the proofs, see [11, Lemma IX.1.1] and [11, Lemma IX.2.1], respectively. Note that (2.8) follows from the Hölder inequality
\[
|a(u, v, w)| \leq |u|_{L^q} |v|_{1,2} |w|_{L^r}, \quad \frac{1}{q} + \frac{1}{r} = \frac{1}{2},
\] (2.12)
by taking \(q = r = 4\) and using a suitable embedding theorem; for other versions of (2.8) and (2.12), see, for example, [11, Exercise IX.2.1].

In particular, with \(w = u - v\),
\[
a(u, u, w) - a(v, v, w) = a(w, u, w) + a(v, w, w),
\]
so that, if \(v \in \hat{H}^{1,2}_0(G)\), then (2.8) and (2.10) imply
\[
|a(u, u, u - v) - a(v, v, u - v)| \leq \frac{\sqrt{|G|}}{2} |u - v|_{1,2}^2 |u|_{1,2}.
\] (2.13)

Similar to [11, Definition IX.1.1], we have

**Definition 2.2.** Let \(f \in H^{1,2}_0(G)\). A random vector field \(u \in \hat{H}^{1,2}_0(G)\) is called a solution to (2.3) if, for every \(\phi \in D(G)\),
\[
P\left(\nu (\nabla u, \nabla \phi)_0 + a(u, u, \phi) = -\langle f, \phi \rangle_1\right) = 1,
\] (2.14)
where \(a\) is the tri-linear form (2.7).

By applying [11, Lemma IX.1.2] to (2.3) with a particular realization of \(\xi\), we get the following result.
Lemma 2.3. If \( f \in H^{-1,2}_0(G) \), \( u \in \hat{H}^{1,2}_0(G) \), and (2.14) holds, then there exists a \( p \in L^2(G) \) with \( P(\int_G p(x) \, dx = 0) = 1 \) such that, for every \( \varphi \in H^1_0(G) \),

\[
P(\nu (\nabla u, \nabla \varphi)_0 + a(u, u, \varphi) = (p, \text{div} \varphi)_0 - \langle f, \varphi \rangle_1) = 1.
\] (2.15)

Similarly, applying [11, Theorems IX.2.1 and IX.3.2] to (2.3), we get the basic existence and uniqueness result.

Theorem 2.4. If \( f \in H^{-1,2}_0(G) \), then, with probability one, equation (2.3) has a solution and

\[
P(\nu |u|_{1,2} \leq |f|_{-1,2}) = 1.
\]

If, in addition, there exists a non-random \( \theta \in (0, 1) \) such that

\[
P( |f|_{-1,2} \leq \frac{2\theta \nu^2}{\sqrt{|G|}}) = 1,
\] (2.16)

then the solution is unique and satisfies

\[
P( |u|_{1,2} \leq \frac{2\nu \theta}{\sqrt{|G|}}) = 1.
\] (2.17)

Intuitively, once we know the velocity field \( u \), we should be able to recover pressure \( p \) from the original equation (2.3). Lemma 2.3 confirms this intuition; see also [33, Proposition I.1.1]. As a result, in what follows, we only consider the function \( u \).

Sometimes it is convenient to work with alternative characterizations of the solution of (2.3).

Proposition 2.5. Let \( f \in H^{-1,2}_0(G) \), \( u \in \hat{H}^{1,2}_0(G) \), and let \( a \) be the tri-linear form (2.7). Then \( u \) is a solution to (2.3) if and only of, for every \( w \in \hat{H}^{1,2}_0(G) \),

\[
P\left( \nu (\nabla u, \nabla w)_0 + a(u, u, w) = -\langle f, w \rangle_1 \right) = 1,
\] (2.18)

or

\[
\nu E(\nabla u, \nabla w)_0 + E a(u, u, w) = -E \langle f, w \rangle_1.
\] (2.19)

Proof. By construction,

\( (2.14) \Rightarrow (2.18) \Rightarrow (2.19). \)

To establish \( (2.19) \Rightarrow (2.14) \), take \( w = \varphi \zeta \) with \( \varphi \in \mathcal{D}(G) \) and a bounded random variable \( \zeta \). \( \square \)

Corollary 2.6. Let \( f, g \in H^{-1,2}_0(G) \) and let \( u, v \in \hat{H}^{1,2}_0(G) \) be the corresponding solutions of (2.3). If (2.16) holds, then

\[
P\left( |u - v|_{1,2} \leq \frac{|f - g|_{-1,2}}{\nu(1 - \theta)} \right) = 1.
\] (2.20)
Proof. By (2.18), we have, with probability one,
\[ \nu \left( \nabla (u - v), \nabla w \right)_0 + a(u, u, w) - a(v, v, w) = - \langle f - g, w \rangle_1. \]  
(2.21)

Taking \( w = u - v \) and using (2.13), we re-write (2.21) as
\[ \nu |u - v|^2_{1,2} - \frac{\sqrt{|G|}}{2} |u - v|^2_{1,2} |u|_{1,2} \leq \langle f - g, u - v \rangle_1 \leq |f - g|_{-1,2} |u - v|_{1,2}, \]
and then (2.20) follows from (2.17).

\[ \square \]

3. Analysis of the Stochastic Galerkin Approximation

For an integer \( N \geq 1 \), consider the equation
\[ \nu \Delta v_N = \mathcal{P}^N \left( (v_N \cdot \nabla)v_N \right) + \nabla p_N + \mathcal{P}^N f, \]
(3.1)
\[ \text{div } v_N = 0, \quad v_N |_{\partial G} = 0. \]

Recall that \( f \) is the random forcing of the form
\[ f(x) = f(\xi, x) = (f^1(\xi, x), f^2(\xi, x)) \]
with a suitable (known) non-random vector field \( f \) and a random variable \( \xi \), and \( \mathcal{P}^N \) is the orthogonal projection in \( L^2(\Omega, \mathcal{F}_\xi, \mathbb{P}) \) on the subspace spanned by the first \( N \) orthogonal polynomials corresponding to the distribution of \( \xi \).

Similar to Definition 2.2, we have

**Definition 3.1.** Given \( f \in H^{-1,2}_0(G) \), a random vector field \( v_N \in \mathcal{P}^N \left( \widetilde{H}^{1,2}_0(G) \right) \) is called a solution of (3.1) if, for every \( \varphi \in D(G) \),
\[ \mathbb{P} \left( \nu \left( \nabla v_N, \nabla \varphi \right)_0 + \mathcal{P}^N a(v_N, v_N, \varphi) = - \langle \mathcal{P}^N f, \varphi \rangle_1 \right) = 1. \]  
(3.2)

We call \( v_N \) a polynomial chaos approximation of the solution \( u \) of equation (2.3).

Similar to (2.18) and (2.19), we will establish two alternative characterizations of the solution of (3.1).

If \( u \in H^{1,2}_0(G) \), \( v \in H^{1,2}_0(G) \), and \( w \in \mathcal{P}^N \left( \widetilde{H}^{1,2}_0(G) \right) \), then \( \mathcal{P}^N w = w \) and therefore
\[ \mathbb{E} \left( \mathcal{P}^N (u \cdot \nabla)v, w \right) = \mathbb{E} \left( (u \cdot \nabla)v, \mathcal{P}^N w \right) = \mathbb{E} a(u, v, w). \]  
(3.3)

As a result, direct computations lead to the first alternative characterization of the solution of (3.1).

**Proposition 3.2.** A random vector field \( v_N \in \mathcal{P}^N \left( \widetilde{H}^{1,2}_0(G) \right) \) is a solution of (3.1) if and only if, for every \( w \in \mathcal{P}^N \left( \widetilde{H}^{1,2}_0(G) \right) \),
\[ \nu \mathbb{E} \left( \nabla v_N, \nabla w \right) + \mathbb{E} a(v_N, v_N, w) = - \langle f, w \rangle_1. \]  
(3.4)
In particular, if a solution \( v_N \) exists, then, taking \( w = v_N \) and using (2.10), we find
\[
\nu |v_N|_{H_0^{1,2}} \leq |f|_{H_0^{1,2}}. \tag{3.5}
\]
Equality (3.4) shows that \( v_N \) is indeed a stochastic Galerkin approximation of \( u \).

To derive yet another form of (3.1), start by writing
\[
v_N = \sum_{l=0}^{N} v^l_N \mathcal{P}_l, \quad \mathcal{P}^N f = \sum_{l=0}^{N} f^l \mathcal{P}_l.
\]
Then, using the numbers \( A_{m,k;l} \) defined in (2.2), we compute
\[
\left( (v_N \cdot \nabla) v_N \right) = \sum_{m,k=0}^{N} (v^m_N \cdot \nabla) v^k_N \mathcal{P}_m \mathcal{P}_k = \sum_{m,k=0}^{N} (v^m_N \cdot \nabla) v^k_N \sum_{l=0}^{m+k} A_{m,k;l} \mathcal{P}_l
\]
\[
= 2^N \sum_{l=0}^{N} \left( \sum_{m,n=0}^{N} A_{m,k;l} (v^m_N \cdot \nabla) v^k_N \right) \mathcal{P}_l,
\]
that is,
\[
\mathcal{P}^N \left( (v_N \cdot \nabla) v_N \right) = \sum_{l=0}^{N} \left( \sum_{m,n=0}^{N} A_{m,k;l} (v^m_N \cdot \nabla) v^k_N \right) \mathcal{P}_l.
\]
As a result, (3.1) is equivalent to the following system of equations for the non-random vector functions \( v^l_N, \, l = 0, \ldots, N \):
\[
\nu \Delta v^l_N = \sum_{m,n=0}^{N} A_{m,k;l} (v^m_N \cdot \nabla) v^k_N + \mathcal{P}^N p^l + f^l. \tag{3.7}
\]
This system is more complicated than (2.3) and will require more sophisticated numerical procedures to compute a solution, whence the term “intrusive” in connection with stochastic Galerkin approximation.

For example, if \( \xi \) is a standard normal random variable and \( \mathcal{F} = \sigma(\xi), \, f(x) = f(\xi, x) = (f^1(\xi, x), f^2(\xi, x)) \) for a non-random vector field \( f \), then \( \mathcal{P}_n = H_n(\xi) \), where
\[
H_n(x) = (-1)^n e^{x^2/2} \frac{d^n e^{-x^2/2}}{dx^n}
\]
is \( n \)-th Hermite polynomial, \( c(n) = n! \), and
\[
\mathcal{P}_m \mathcal{P}_n = \sum_{k=0}^{\min(m,n)} \frac{m! \ n!}{(m-k)! \ (n-k)! \ k!} Q_{m+n-2k}^k\ (cf. \ [35, Formula (6.7)])); after some algebraic manipulations, (3.7) becomes
\[
\nu \Delta v^l_N = \sum_{n=0}^{N} \frac{1}{n!} \sum_{k+n \leq N, \ m+n \leq N} \frac{(k+n)! \ (m+n)!}{k! \ m!} (v^k_N \cdot \nabla) v^{m+n}_N + \mathcal{P}^N p^l + f^l. \tag{3.8}
\]
Combining (3.7) with Proposition 3.2, we get the second alternative characterization of the solution of (3.1).

**Proposition 3.3.** A collection of functions \( v^l_N, l = 0, \ldots, N \), with each \( v^l_N \in \hat{H}^{1,2}_0(G) \), is a solution of (3.7) if and only if, for every collection of functions \( \{ w^l, l = 0, \ldots, N \} \), \( w^l \in D(G) \), the following equality holds:

\[
\begin{align*}
\nu & \sum_{l=0}^{N} c(l) \left( \nabla v^l_N, \nabla w^l \right) + \sum_{l=0}^{N} c(l) \sum_{m,n=0}^{N} A_{m,k;l} a(v^m_N, v^m_N, w^l) \\
&= -\sum_{l=0}^{N} c(l) \left( f^l, w^l \right)_{\mathcal{H}^{1,2}}, \quad c(l) = \mathbb{E} \mathbb{P}^2_l.
\end{align*}
\] (3.9)

Given \( \bar{u}, \bar{v}, \bar{w} \) in \( (H^{1,2}_0(G))^{N+1} \), with \( \bar{u} = (u^0, \ldots, u^N) \) and similarly for \( \bar{v}, \bar{w} \), define

\[
\mathcal{A}(\bar{u}, \bar{v}, \bar{w}) = \sum_{l=0}^{N} c(l) \sum_{m,n=0}^{N} A_{m,k;l} a(u^k, v^m, w^l). \] (3.10)

Then we can re-write (3.9) as

\[
\begin{align*}
\nu & \sum_{l=0}^{N} c(l) \left( \nabla v^l_N, \nabla w^l \right) + \mathcal{A}(\bar{v}_N, \bar{v}_N, \bar{w}) = -\sum_{l=0}^{N} c(l) \left( f^l, w^l \right)_{\mathcal{H}^{1,2}}.
\end{align*}
\] (3.11)

Furthermore, given \( \bar{u}, \bar{v}, \bar{w} \) in \( (H^{1,2}_0(G))^{N+1} \), define

\[
\begin{align*}
u & = \sum_{k=0}^{N} u^k \mathcal{P}_k, \quad v = \sum_{k=0}^{N} v^k \mathcal{P}_k, \quad w = \sum_{k=0}^{N} w^k \mathcal{P}_k.
\end{align*}
\]

Then equality (3.3) implies

\[
\mathcal{A}(\bar{u}, \bar{v}, \bar{w}) = \mathbb{E} a(u, v, w). \] (3.12)

In particular, by (2.10),

\[
\mathcal{A}(\bar{u}, \bar{v}, \bar{v}) = 0 \] (3.13)

provided \( u^k \in \hat{H}^{1,2}_0(G) \) for all \( k = 0, \ldots, N \).

We now use (3.9) to establish a basic solvability result for equation (3.1).

**Theorem 3.4.** For every \( f \in \mathbb{H}^{-1,2}_0(G) \) and \( N \geq 1 \), equation (3.1) has a solution \( v_N \) and

\[
|v_N|_{H^{1,2}_0} \leq \frac{|f|_{\mathbb{H}^{-1,2}_0}}{\nu}. \] (3.14)

The solution is unique if there exists a non-random number \( \varepsilon_N \in (0, 1) \) such that

\[
\mathbb{P}\left( |v_N|_{1,2} \leq \frac{2\nu(1 - \varepsilon_N)}{\sqrt{|G|}} \right) = 1. \] (3.15)
Proof. For $M \geq 1$ and $l = 0, \ldots, N$, define
\[ v^l_{M,N} = \sum_{k=0}^{M} z^{k,l}_{M,N} h_k, \tag{3.16} \]
where $z^{k,l}_{M,N} \in \mathbb{R}$ and the functions $h_k$ have the following properties:

1. $h_k \in \hat{H}^{1,2}_0(G), k \geq 0$;
2. Finite linear combinations of $h_k$ are dense in the space $\hat{H}^{1,2}_0(G)$;
3. $|h_k|^{L^2} = 1$, $(h_k, h_m) = 0$, $k \neq m$.

A possible choice is the normalized eigenfunctions of the Stokes operator [33, Section I.2.6].

Also, we will use the notations
\[ \bar{v}_{M,N} = (v^0_{M,N}, \ldots, v^N_{M,N}), \quad v_{M,N} = \sum_{l=0}^{N} v^l_{M,N} \bar{v}_l. \]

Consider the system of equations
\[ \nu \left( \nabla v^l_{M,N}, \nabla h_k \right) + \sum_{m,n=0}^{N} A_{m,n,l} \left( (v^m_{M,N}, \nabla) v^n_{M,N}, h_k \right) + (f^l, h_k)_{1,1} = 0, \tag{3.17} \]

$k = 0, \ldots, M$, $l = 0, \ldots, N$; with in mind, we think of (3.17) as a system of equations for the numbers $z^{k,l}_{M,N}$.

To show that (3.17) has a solution for every $M \geq 1$, we introduce the following notations:
\[ \bar{z} = (z^{0,0}_{M,N}, \ldots, z^M_{M,N}, z^0_{M,N}, \ldots, z^M_{M,N}, z^0_{M,N}, \ldots, z^M_{M,N}), \]
\[ Q_{k,l}(\bar{z}) = \nu \left( \nabla v^l_{M,N}, \nabla h_k \right) + \sum_{m,n=0}^{N} A_{m,n,l} \left( (v^m_{M,N}, \nabla) v^n_{M,N}, h_k \right) + (f^l, h_k)_{1,1}, \]
\[ F(\bar{z}) = \sum_{k,l} c(l) Q_{k,l}(\bar{z}) z^{k,l}_{M,N}. \]

Combining (3.16), (3.11), and (3.13),
\[ F(\bar{z}) = \nu \sum_{l=0}^{N} c(l) |v^l_{M,N}|^2_{1,2} + 2(\bar{v}_{M,N}, \bar{v}_{M,N}, \bar{v}_{M,N}) + \sum_{l=0}^{N} c(l) \left( f^l, v^l_{M,N} \right)_{1,1} \]
\[ = \nu \sum_{l=0}^{N} c(l) |v^l_{M,N}|^2_{1,2} + \sum_{l=0}^{N} c(l) \left( f^l, v^l_{M,N} \right)_{1,1}. \tag{3.18} \]

It follows that
\[ F(\bar{z}) \geq 0 \]
if
\[ \frac{2\nu^2}{|G|} \sum_{k,l} c(l) |z^{k,l}_{M,N}|^2 = |f|^2_{H^{1,2}_0}. \]
Indeed, by the Cauchy-Schwarz inequality,
\[ F(\bar{z}) \geq \|v_{M,N}\|_{H_0^{1,2}}^2 \left( \nu \|v_{M,N}\|_{H_0^{1,2}} - \|f\|_{H_0^{-1,2}} \right), \]
whereas the Poincaré inequality (2.5) implies
\[ \|v_{M,N}\|_{H_0^{1,2}}^2 \geq \frac{2}{|G|} \sum_{k,l} c(l) |z_{M,N}^{k,l}|^2. \]

By a multi-dimensional version of the intermediate value theorem [11, Lemma IX.3.1], we conclude that there exists a \( \bar{z}^* \) with
\[ \nu \sqrt{\|G\|} \sum_{k,l} c(l) |z_{M,N}^{k,l}|^2 \leq \|f\|_{H_0^{-1,2}}^2 \]
such that \( Q_{k,l}(\bar{z}^*) = 0 \) for all \( k,l \). In other words, we now have existence of solution of (3.17) for every \( M \geq 1 \). Moreover, by (3.18) and the Cauchy-Schwarz inequality, the solution satisfies
\[ \|v_{M,N}\|_{H_0^{1,2}} \leq \frac{|f|_{H_0^{-1,2}}}{\nu}, \]
which, together with the compactness of the embedding \( H_0^{1,2}(G) \subset L^2(G) \), allows us to pass to the limit \( M \to \infty \) and get a solution of (3.9) in the same way as in [11, pp. 600–601].

To establish uniqueness, let \( v_N \) be the solution of (3.9) satisfying (3.15) and let \( \tilde{v} \) the difference between \( v_N \) and any other possible solution of (3.9). By (3.4),
\[ \nu \mathbb{E}(\nabla \tilde{v}, \nabla w)_0 + \mathbb{E}(v_N, \tilde{v}, w) - \mathbb{E}(\tilde{v}, v_N, w) = 0. \]

Because \( P_N \tilde{v} = \bar{v} \) and \( \bar{v} \in H_0^{1,2}(G) \), we can put \( w = \bar{v} \) in (3.19) and then use (2.10) and (2.11) to conclude that
\[ \nu \mathbb{E}|\tilde{v}|_{1,2}^2 + \mathbb{E}a(\bar{v}, v_N, \tilde{v}) = 0, \]
which, together with (2.8) implies
\[ \mathbb{E} \left( |\tilde{v}|_{1,2}^2 \left( \nu - \frac{\sqrt{|G|}}{2} |v_N|_{1,2} \right) \right) \leq 0. \]

If (3.15) holds, then
\[ \nu - \frac{\sqrt{|G|}}{2} |v_N|_{1,2} \geq \nu \varepsilon_N > 0, \]
and (3.20) is only possible when \( \mathbb{P}(|\tilde{v}|_{1,2}^2 = 0) = 1 \), that is, when \( v_N \) is the unique solution of (3.9).

Similar to (2.16), we need (3.15) to guarantee uniqueness of the stochastic Gelerkin approximation. In fact, without (2.16), uniqueness can fail for the original equation (2.3) [11, Section IX.2]. Even though system of equations (3.9) has been successfully used for numerical simulations [14, 32], it is not immediately clear how condition (3.15) can be verified.

The following theorem is the first key result of the paper and shows that, under (2.16) and (3.15), stochastic Gelerkin approximation \( v_N \) is indeed an approximation of the
orthogonal projection $P^N u$. In particular, if $\varepsilon_N$ does not depend on $N$, then stochastic Galerkin approximation is asymptotically equivalent to the orthogonal projection, in the sense that, as $N \to \infty$, both converge to the true solution at the same rate.

**Theorem 3.5.** Assume that (2.16) holds so that (2.3) has a unique solution $u$, and let $v_N = v_N(\xi)$ be the unique solution of (3.1) satisfying (3.15). Then

$$|P^N u - v_N|_{H_0^{1,2}} \leq \frac{\theta + 1 - \varepsilon_N}{\varepsilon_N} |u - P^N u|_{H_0^{1,2}}, \quad (3.21)$$

$$|u - v_N|_{H_0^{1,2}} \leq \left( 1 + \frac{\theta + 1 - \varepsilon_N}{\varepsilon_N} \right) |u - P^N u|_{H_0^{1,2}}. \quad (3.22)$$

**Proof.** To make the formulas shorter, we write

$$u^N = P^N u, \quad v_N = u^N - v_N, \quad \text{and} \quad w^N = P^N w \quad \text{for} \quad w \in \tilde{H}_0^{1,2}(G)$$

Using (2.19),

$$\nu E(\nabla u^N, \nabla w^N)_0 + E(a(u, u, w^N)) = -E(f, w^N)_1,$$

and, after subtracting (3.4),

$$\nu E(\nabla u^N, \nabla w^N)_0 + E(a(u, u, w^N)) - E(a(v_N, v_N, w^N)) = 0.$$

Next,

$$a(u, u, w^N) - a(v_N, v_N, w^N) = a(u, u - v_N, w^N) + a(u - v_N, v_N, w^N)$$

$$= a(u, u - v^N, w^N) + a(u, u^N - v_N, w^N)$$

$$+ a(u - u^N, v_N, w^N) + a(u^N - v_N, v_N, w^N).$$

Taking $w^N = u_N$ leads to

$$\nu E|u_N|_{H_0^{1,2}}^2 + E(a(u_N, v_N, u_N)) + E(a(u, u - u_N, u_N)) + E(a(u - u^N, v_N, u_N)) = 0.$$ 

Then (2.8), (2.17), (3.15), and the Cauchy-Schwarz inequality [for expectations] imply

$$\nu \varepsilon_N |u_N|_{H_0^{1,2}}^2 \leq \nu(\theta + 1 - \varepsilon_N)|u_N|_{H_0^{1,2}}|u^N|_{H_0^{1,2}}.$$

We now get (3.21), and then, by triangle inequality, (3.22). \qed

4. A Non-Intrusive Approximation Using Gauss Quadrature

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a random variable $\xi$ and let $\mathcal{F}_\xi$ be the $\mathbb{P}$-completion of the sigma algebra generated by $\xi$. We assume that the moment generating function $\lambda \mapsto E e^{\lambda \xi}$ is defined in some neighborhood of $\lambda = 0$. Under this assumption, given a collection $\{P_n, n \geq 0\}$ of orthogonal polynomials corresponding to the distribution of $\xi$, the collection of random variables

$$\mathcal{P}_n = P_n(\xi), \quad n \geq 0,$$

is an orthogonal basis in $L_2(\Omega, \mathcal{F}_\xi, \mathbb{P})$. Denote by $P^N$ the orthogonal projection in $L_2(\Omega, \mathcal{F}_\xi, \mathbb{P})$ on the subspace spanned by $\{\mathcal{P}_k, k = 0, \ldots, N\}$. Let

$$c(n) = E \|\mathcal{P}_n\|^2,$$
so that, for every $\zeta \in L_2(\Omega, \mathcal{F}_\xi, \mathbb{P})$, 
\[
\zeta = \sum_{k \geq 0} \frac{\mathbb{E}(\zeta \psi_k)}{c(k)} \psi_k.
\]

In this section, we assume that the random forcing in equation (2.3) has a special form
\[
f(x) = f(\xi, x) = (f^1(\xi, x), f^2(\xi, x)),
\]
where $f$ is a non-random vector field. If $u = u(\xi)$ is a solution of (2.3) corresponding to the particular realization of $\xi$ and $u^N = P^N u$, then
\[
u(\xi) \approx u^N(\xi), \quad u^N(\xi) = \sum_{k=0}^{N} \frac{\mathbb{P}(\xi)}{c(k)} u_k, \quad u_k = \mathbb{E}(u \psi_k).
\]

To compute the coefficients $u_k$, $k = 0, \ldots, N$, we use the Gauss quadrature approximation $u_k \approx u_k^{(N)}$, where
\[
u_k^{(N)} = \sum_{j=1}^{N+1} w_{j,N} u(\xi_{j,N}) \psi_k(\xi_{j,N}),
\]

$\xi_{j,N}$, $j = 1, \ldots, N+1$, are the roots of $P_{N+1}$, and $w_{j,N}$ are the corresponding weights; cf. [12, Section 1.4]. The resulting discrete projection or pseudo-spectral approximation,
\[
u^{(N)}(\xi) = \sum_{k=0}^{N} \frac{\psi_k(\xi)}{c(k)} u_k^{(N)},
\]
requires the solution $u(\xi_{j,N})$ of (2.3) for $N+1$ distinct values of $\xi$.

To simplify the formulas, it is convenient to introduce the square matrix $\mathfrak{M} = (\mathfrak{M}_{kj}, k = 0, \ldots, N, j = 1, \ldots, N+1)$, with
\[
\mathfrak{M}_{kj} = w_{j,N} \psi_k(\xi_{j,N}).
\]

Then (4.2) becomes
\[
u_k^{(N)} = \sum_{j=1}^{N+1} \mathfrak{M}_{kj} u(\xi_{j,N}).
\]

The basic property of the Gauss quadrature is that the equality
\[
\mathbb{E} h(\xi) = \sum_{j=1}^{N+1} w_{j,N} h(\xi_{j,N})
\]
holds for all functions $h = h(\xi)$ that are polynomials in $\xi$ of degree at most $2N+1$; cf. [12, Theorem 1.45]. In particular, for every $k, m = 0, \ldots, N$,
\[
\sum_{j=1}^{N+1} \mathfrak{M}_{kj} \psi_m(\xi_{j,N}) = \sum_{j=1}^{N+1} w_{j,N} \psi_k(\xi_{j,N}) \psi_m(\xi_{j,N}) = \mathbb{E}(\psi_k \psi_m) = \begin{cases} c(k) > 0, & \text{if } k = m, \\
0, & \text{if } k \neq m,
\end{cases}
\]
which means that the matrix $\mathfrak{M}$ is non-singular.
With the above choice of the sampling points $\xi_{j,N}$, the discrete projection (4.3) is equivalent to interpolation:

**Proposition 4.1.** The equality

$$u(\xi_{j,N}) = u^{(N)}(\xi_{j,N})$$

holds for all $j = 1, \ldots, N + 1$.

**Proof.** Equality (4.3) implies that $u^{(N)}$ is a polynomial in $\xi$ of order at most $N$, so that each product $u^{(N)} p_k$, $k = 0, \ldots, N$, is a polynomial in $\xi$ or order at most $2N$. Then

$$E(u^{(N)} p_k) = \sum_{j=1}^{N+1} W_{kj} u^{(N)}(\xi_{j,N}), \quad k = 0, \ldots, N.$$  

(4.6)

On the other hand, (4.3) also implies

$$E(u^{(N)} p_k) = u_k^{(N)},$$

(4.7)

and then (4.5) follows from (4.4) and non-degeneracy of the matrix $W$. □

The following theorem is the second key result of the paper and gives an upper bound on the approximation error $E|u - u^{(N)}|_{1,2}$. Recall that $u^N = P^N u$.

**Theorem 4.2.** Define

$$\delta_N = \sup_{\xi} |u(\xi) - u^N(\xi)|_{1,2}.$$  

(4.8)

Then

$$E|u - u^{(N)}|_{1,2}^2 \leq E|u - u^N|_{1,2}^2 + N(\delta_N)^2.$$  

(4.9)

**Proof.** By orthogonality,

$$E|u - u^{(N)}|_{1,2}^2 = E|u - u^N|_{1,2}^2 + E|u^N - u^{(N)}|_{1,2}^2$$

$$= E|u - u^N|_{1,2}^2 + \sum_{k=0}^{N} \frac{|u_k - u_k^{(N)}|_{1,2}^2}{c(k)}.$$  

(4.10)

Combining (4.2), (4.6), and (4.7) results in

$$u_k - u_k^{(N)} = \sum_{j=1}^{N+1} w_{j,N}(u^N(\xi_{j,N}) - u(\xi_{j,N})) p_k(\xi_{j,N})$$

or, using the Cauchy-Schwarz inequality and $w_{j,N} > 0$,

$$|u_k - u_k^{(N)}|_{1,2}^2 \leq \left(\sum_{j=1}^{N+1} w_{j,N}|u^N(\xi_{j,N}) - u(\xi_{j,N})|_{1,2}^2\right) \left(\sum_{j=1}^{N+1} w_{j,N} p_k^2(\xi_{j,N})\right).$$

Properties of the Gauss quadrature imply

$$\sum_{j=1}^{N+1} w_{j,N} p_k^2(\xi_{j,N}) = E p_k^2 = c(k), \quad k = 0, \ldots, N,$$

and

$$\sum_{j=1}^{N+1} w_{j,N} = 1.$$
whereas (1.8) implies
\[ \sum_{j=1}^{N+1} w_{j,N} |u^N(\xi_{j,N}) - u(\xi_{j,N})|^2 \leq (\delta_N)^2 \sum_{j=1}^{N+1} w_{j,N}, \]
As a result,
\[ |u_k - u_k^{(N)}|^2 \leq (\delta_N)^2 c(k), \]
and (4.9) follows from (4.10).

□

Of course, \( E|u - u^N|^2 \leq (\delta_N)^2 \), leading to a somewhat weaker form of (4.9):
\[ E|u - u^{(N)}|^2 \leq (\delta_N)^2 (1 + N). \]

Remark 4.3. Both intrusive and non-intrusive approximations require an \( L_\infty \)-bound, either in the form of (3.15) or (1.8), to establish an \( L_2 \)-bound on the approximation error; for (4.9) to be useful, one additionally needs to establish
\[ \lim_{N \to \infty} \sqrt{N} \delta_N = 0. \] (4.11)
On the one hand, condition (4.11) is easier to verify than condition (3.15). On the other hand, under condition (3.15), the error bound (3.22) can be better than (4.9), and this difference can become even more pronounced as the stochastic dimension of the problem (the number of independent random variables in the input) grows.

The proof of Theorem 4.2 does not use the fact that \( u \) solves (2.3). This additional information about \( u \), as well as the properties of the random variable \( \xi \) and the function \( f(x) = f(\xi, x) \), are necessary to establish (4.11).

As an example, consider the random variable \( \xi \) that is uniform on \([-1, 1]\). Then \( \mathcal{P}_n = P_n(\xi) \), where \( P_n \) is \( n \)th Legendre polynomial; the standard normalization [2, equation (6.4.4.)] is \( P_n(1) = 1 \), and then
\[ c_n = \frac{1}{2} \int_{-1}^{1} P_n^2(x) \, dx = \frac{1}{2n + 1}. \]

Theorem 4.4. Assume that, in (4.11), the random variable \( \xi \) is uniform on \([-1, 1]\) and the function \( f \) is Lipschitz continuous as a function of \( \xi \); there exists a positive number \( C_f \) such that, for all \( \xi_1, \xi_2 \in [-1, 1] \),
\[ |f(\xi_1, \cdot) - f(\xi_2, \cdot)|_{-1,2} \leq C_f |\xi_1 - \xi_2|. \] (4.12)
If (2.16) holds and \( u = u(\xi) \) is the corresponding unique solution of (2.3), then
\[ \sup_{\xi} |u(\xi) - u^N(\xi)|_{1,2} \leq C N^{-3/4} \] (4.13)
for some \( C \) depending only on \( C_f, \nu, \) and \( \theta \). In particular, we have (4.11).

Proof. By (2.20),
\[ |u(\xi_1) - u(\xi_2)|_{1,2} \leq \frac{|f(\xi_1, \cdot) - f(\xi_2, \cdot)|_{-1,2}}{\nu(1 - \theta)} \leq \frac{C_f}{\nu(1 - \theta)} |\xi_1 - \xi_2|. \] (4.14)
For the rest of the proof, \( C \) denotes positive number depending only on \( C_f, \nu, \) and \( \theta \). The value of \( C \) can be different in different formulas.
Let $\mathcal{E}_N$ be the error of the best uniform approximation of $u$ by an element of $\Hat{H}^{1,2}_0(G)$ that is a polynomial of degree at most $N$ in $\xi$:

$$\mathcal{E}_N(u) = \inf \left( \max_{\xi \in [-1,1]} |u(\xi) - v(\xi)|_{1,2} \; : \; v \in \mathcal{P}^N(\Hat{H}^{1,2}_0(G)) \right).$$

Then

- Jackson’s Theorem [29, Theorem 1.4], together with (4.14), implies
  $$\mathcal{E}_N(u) \leq \frac{C}{N}; \quad (4.15)$$

- Combining (4.14) with [34, Theorem 2.1] yields
  $$\mathbb{E}|u - u^{(N)}|^2_{1,2} \leq \frac{C}{N^3}; \quad (4.16)$$

- Combining (4.15) and (4.16) with [5, Theorem 1 (p=2)] leads to (4.13) and completes the proof.

5. Summary and Discussion

Within the general framework of numerical analysis, this paper studies a priori error bounds, as opposed to a posteriori error analysis that requires some basic knowledge about convergence of the numerical procedure; cf. [1, Section 9.3]. Comparing the (intrusive) stochastic Galerkin approximation [Theorem 3.5] and a (non-intrusive) stochastic collocation/Gauss quadrature approximation [Theorem 4.2] for equation (2.3), we see that

- The intrusive approximation works for a broader class of random input and can, in principle, achieve an asymptotically optimal rate of convergence;
- The non-intrusive approximation is easier to study, both analytically and numerically.

The main technical difficulties to overcome when analyzing stochastic Galerkin approximation in general and when proving Theorem 3.5 in particular is related to the fact that, for a nonlinear equation,

$$\mathcal{P}^N v \neq v_N.$$

The two possible sources of non-linearity are (a) the structure of the underlying deterministic equation, and (b) the way the random perturbation enters the equation. For example, the heat equation

$$\frac{\partial v}{\partial t} = a \Delta v$$

with random $a$ is non-linear when it comes to polynomial chaos approximation. Another example is (2.3) with random $\nu$; see (5.2) below. This difficulty can be somewhat mitigated by replacing the usual product with Wick product [16, 20], which is a convolution operation $\odot$ such that $\mathfrak{P}_m \odot \mathfrak{P}_n = \alpha_{mn} \mathfrak{P}_{m+n}$, $\alpha_{mn} \in \mathbb{R}$ [27]; the price to pay is reduction of physical relevance of the resulting equations. Moreover, the analysis
is much more manageable for equations of the type (5.1), when the underlying
deterministic equation is linear \[6, 7\]. In fact, it is the “deterministic nonlinearity” that
leads to hard-to-verify conditions of the type (3.15).

While the setting in the paper, a stationary two-dimensional Navier-Stokes system
with zero boundary conditions and additive random perturbation, is intentionally
simple to isolate the effects of non-linearity (the convection term) on the stochastic
Galerkin approximation, some of the results are rather universal and can be used for
many other equations with a quadratic-type nonlinearity. The key is equality (3.6)
describing the product of two chaos expansions.

For example, (3.6) implies that equations (3.7) describe the stochastic Galerkin ap-
proximation for the stationary Navier-Stokes system in any number of dimensions
and with randomness in both boundary conditions and the external force; after mi-
nor modifications, time-dependent problems with a random initial condition will also
be covered. The number of random variables does not matter either, as long as the
corresponding orthogonal basis \(\\{P_n, n \geq 0\}\) can be constructed [27].

Similarly, (3.6) shows that the stochastic Galerkin approximation for the system (2.3)
with random viscosity will be

\[
\sum_{m,k=0}^{N} A_{m,k,l} \nu_k \Delta v_N^m = \sum_{m,k=0}^{N} A_{m,k,l} (v_N^k \cdot \nabla) v_N^m + \nabla p_N^l + f_l, \quad l = 0, \ldots, N, \quad (5.2)
\]

where we assume

\[
\nu = \sum_{k=0}^{\infty} \nu_k P_k.
\]

Equation (5.2) illustrates the effects of two sources of nonlinearity: the convection
term leads to the coupling of the functions \(v_N^m\) on the right-hand side, whereas random
viscosity leads to a similar coupling on the left-hand side. While not very different
from (3.7), analysis of (5.2) must be carried out from scratch and, for now, is left to
an interested reader.

To conclude, let us note that there are many equivalent ways to write Navier-Stokes
equations: even the basic velocity-pressure formulation (2.3) admits at least four
alternative forms [10, Section 5], not to mention alternative variables, such stream
function and vorticity [4, 23]. For the purpose of our investigation, it appears that
none of the alternatives will lead to any major simplifications, but, as reference [10]
suggests, one should keep those alternatives in mind for further analysis of various
approximations of (2.3).

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