A pair of Calabi-Yau manifolds from a two parameter non-Abelian gauged linear sigma model

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Abstract

We construct and study a two parameter gauged linear sigma model with gauge group \((U(1)^2 \times O(2))/\mathbb{Z}_2\) that has a dual model with gauge group \((U(1)^2 \times SO(4))/\mathbb{Z}_2\). The model has two geometric phases, three hybrid phases and one phase whose character is unknown. One of the geometric phases is strongly coupled and the other is weakly coupled, where strong versus weak is exchanged under the duality. They correspond to two Calabi-Yau manifolds with \((h^{1,1}, h^{2,1}) = (2, 24)\) which are birationally inequivalent but are expected to be derived equivalent. A region of the discriminant locus in the space of Fayet-Iliopoulos-theta parameters supports a mixed Coulomb-confining branch which is mapped to a mixed Coulomb-Higgs branch in the dual model.
1 Introduction

Gauged linear sigma models (GLSMs) \cite{1} have been useful tools in the construction and analysis of two-dimensional (2,2) superconformal field theories that can be used for supersymmetric string compactifications. The model has two classes of coupling constants that descend to exactly marginal parameters of the superconformal field theory — the superpotential couplings and the Fayet-Iliopoulos (FI) - theta parameters. The space of FI parameters is decomposed into chambers called the “phases” according to the pattern of gauge symmetry breaking, and the low energy theory in each phase has its own character. For example, many models have geometric phases where the gauge symmetry is completely Higgsed and the low energy theory is a non-linear sigma model with Calabi-Yau target space.

In the early days, a class of models with Abelian gauge groups has been studied extensively, partly because they have geometric phases corresponding to complete intersection Calabi-Yaus in toric varieties, for which a body of mathematical results are available \cite{2}. Also, mirror symmetry is well understood when the gauge group is Abelian \cite{3,4}. More recently, GLSMs with non-Abelian gauge groups have started to be considered. Unlike in Abelian models, non-Abelian theories may have “strongly coupled phases”, where continuous subgroups of the gauge group remain unbroken, and yet massless charged matter exists. In such a phase, the classical analysis is not reliable and it is in general difficult to understand the nature of the low energy theory. However, it is sometimes possible to obtain relevant results in the strongly coupled gauge sector \cite{5,6} with which we can understand the low energy behaviour of the models. We may end up with a non-linear sigma model with a Calabi-Yau target space, in a way quite different from the classical Higgs mechanism.

In \cite{5,8}, non-Abelian GLSMs with such strongly coupled and yet geometric phases have been constructed and studied. The Calabi-Yau manifolds that appear are some kind of determinantal varieties in (weighted) projective spaces. These models also have the standard weakly coupled geometric phases where different Calabi-Yau manifolds appear. When two Calabi-Yau manifolds, say $X$ and $Y$, appear in two different regimes of a common FI-theta
parameter space, a number of interesting conclusions can be drawn. For example, the mirrors of $X$ and $Y$ must be in the same complex deformation family and in particular the Gromov-Witten theories of $X$ and $Y$ must be governed by the same Picard-Fuchs system. Also, the topological B-models of $X$ and $Y$ must be equivalent and in particular the derived categories of coherent sheaves on $X$ and $Y$ must be equivalent: $D^b_{Coh}(X) \cong D^b_{Coh}(Y)$. The consequences apply to all GLSMs, including Abelian ones, but the distinguished feature of the models in [5–8] with weakly and strongly coupled phases is that $X$ and $Y$ are birationally inequivalent. In fact, these works were partly motivated by such mathematical results and in return have impact on mathematics as well.

The GLSMs in [5–8] are all “one parameter models” in the sense that they have a single FI-theta parameter and the resulting Calabi-Yau manifolds have Picard number one. A natural task then is to generalize them to “multiparameter models”. In fact, there are natural targets — we take the gauge group to be of the form

$$G = \frac{U(1)^L \times H}{\Gamma}$$

where $H$ is a symplectic or (special) orthogonal group and $\Gamma$ is a discrete subgroup of $U(1)^L \times H$, and the matter consisting of a number of $H$-singlets and $H$-fundamentals with various charges under $U(1)^L$. Indeed, all the models in [5–8] are of this type with $L = 1$. It is possible that such a generalization will yield a systematic construction of a large number of Calabi-Yau manifolds, in the same way as one parameter Calabi-Yau hypersurfaces in weighted projective spaces are generalized to complete intersection Calabi-Yaus in toric varieties.

In this paper, we make a modest first step toward generalization — to construct one explicit example of this form and study it in as much detail as possible. If possible, we would like to find a model with a weakly coupled geometric phase as well as a strongly coupled geometric phase. After some trials, we found a simple model with such properties. It is a two parameter model with gauge group

$$G = \frac{U(1) \times U(1) \times O(2)}{\{(\pm 1, \pm 1, \pm 1_2)\}}.$$  \hspace{1cm} (1.2)

It has a dual model with gauge group

$$\tilde{G} = \frac{U(1) \times U(1) \times SO(4)}{\{(\pm 1, \pm 1, \pm 1_4)\}}.$$  \hspace{1cm} (1.3)

It can be regarded as a two parameter generalization of the model [6] for Hosono-Takagi’s Calabi-Yau pair [10,11]. The model has six phases as depicted in Figure 1. There are two geometric phases corresponding to Calabi-Yau threefolds $X$ and $\tilde{Y}$, three hybrid phases and one phase whose character is unknown. In the original model (gauge group $G$), the geometric phase of $X$ and the hybrid phases are weakly coupled. The geometric phase of $\tilde{Y}$ is strongly

1The works [5, 6] and [8] are motivated by such pairs of Calabi-Yau threefolds found by Rødland [9], Hosono-Takagi [10,11] and Miura [12,13], respectively. The models in [7] realize the Pfaffian Calabi-Yau threefolds listed in [14] along with another “new” determinantal Calabi-Yau in the strongly coupled phases, and have hybrid models in the weakly coupled phases. (Such a manifold-hybrid pair had also been found in [15] in an Abelian GLSM which has a non-Abelian dual [2].) Recently, another pair of Calabi-Yau threefolds of Picard number one was found by Ito et al [16,17], which begs for a physics understanding. In the other direction, the works [5,6] motivated the proofs [18–21] of the derived equivalence. The duality [6] motivated to establish its categorical counterpart [22]. Also, the work [7] presents predictions on derived equivalences.
Figure 1: The phases of the model

coupled in the original model but weakly coupled in the dual (gauge group $\tilde{G}$). The remaining
phase is strongly coupled in both the original and dual models, and that is why we refer to it
as “unknown”.

The Calabi-Yau manifold $X$ is a free $\mathbb{Z}_2$ quotient of a complete intersection of hypersurfaces
in an eight dimensional toric variety. In particular, it is not simply connected, $|\pi_1(X)| \in 2\mathbb{Z}$. It has Hodge numbers $(h^{1,1}, h^{2,1}) = (2, 24)$. On the other hand, the Calabi-Yau manifold $\tilde{Y}$ is a $\mathbb{Z}_2$ cover of a symmetric determinantal variety in a four dimensional toric variety. It can also be realized as a free $G_C$ quotient of an open part of an affine variety. It is simply connected, $\pi_1(\tilde{Y}) = \{1\}$. We have not computed its Hodge numbers yet, but we obtained $h^{1,1} - h^{2,1} = -22$, consistent with $(h^{1,1}, h^{2,1}) = (2, 24)$. Note that

$$\pi_1(X) \not\cong \pi_1(\tilde{Y}).$$

Since the fundamental group is a birational invariant, $X$ and $\tilde{Y}$ cannot be birationally equiva-

tent. However, since they appear in two regimes of a common FI-theta parameter space, they must have equivalent derived categories,

$$D^b_{coh}(X) \cong D^b_{coh}((\tilde{Y}).$$

Thus, $X$ and $\tilde{Y}$ must be another example of a birationally inequivalent but derived equivalent
pair of Calabi-Yau manifolds.

As in any GLSM, the FI-theta parameter space has a discriminant locus that supports
a non-compact flat direction in the scalar component of the vector multiplet, such as the
Coulomb branch or a mixed Coulomb-Higgs branch. When projected to the FI parameter
space, it descends to the phase boundaries in the asymptotic directions. One interesting
feature of the present model is that there is a region of the discriminant locus that supports
a branch where the effective theory includes a strongly coupled gauge sector, in addition to a free
Maxwell theory. The horizontal phase boundary between the geometric phase of $\tilde{Y}$ and the
hybrid phase above (see Figure 1) lifts to a region of a discriminant component that supports
such a “mixed Coulomb-confining branch” in the original model, while it supports a mixed
Coulomb-Higgs branch in the dual. The other regime of the same component, descending to the boundary between the geometric phase of $X$ and the hybrid phase below, supports a standard mixed Coulomb-Higgs branch in the original but a mixed Coulomb-confining branch in the dual.

Mirror symmetry for non-abelian GLSMs is still an open problem. Nevertheless it is sometimes possible to construct a mirror for the Calabi-Yaus that arise as phases of non-abelian GLSMs. In our example we are in the lucky situation that the Calabi-Yau in one of the phases, $X$, is a free quotient of a complete intersection in a toric variety. Therefore it is possible to construct its mirror $X^\vee$ by standard methods. In fact, the calculation of the mirror is completely analogous to the mirror construction by Hosono and Takagi of their Calabi-Yaus [10]. This allows us to determine the Picard-Fuchs operators and the Gromov-Witten invariants associated to $X$. Once we have the Picard-Fuchs operators we are also able to extract some information about the other phases.

We would like to note that multiparameter GLSMs with non-Abelian gauge groups were studied also in a nice work [23]. All the phases in these models are weakly coupled. In a subclass of models called “linear PAX”, the phases are all geometric as well and correspond to Calabi-Yau manifolds which are mutually birationally equivalent. The authors of [23] worked out three two-parameter examples of this class with $(h^{1,1}, h^{2,1}) = (2, 52), (2, 34), (2, 52)$.

The rest of the paper is organized as follows. In Section 2 we recall some basic properties of the GLSM. After summarizing the defining data, we give an overview over the different types of phases that can occur in non-Abelian GLSMs. We also discuss how to extract the information about the discriminants from the GLSM. In Section 3 we introduce our model and its dual. The remaining sections contain a detailed analysis of this GLSM. In Section 4 we discuss the phases, focusing in particular on the geometric ones. We compute the topological characteristics of $X$ and $\tilde{Y}$, where for the latter we make use of the duality. We furthermore give a brief analysis of the other phases. In Section 5 we identify the discriminant locus of the FI-theta parameters by determining the Coulomb and mixed Coulomb-Higgs branches of the GLSM and its dual. The complement of the discriminant in the FI-theta parameter space determines the “Kähler moduli space” $\mathcal{M}_K$, i.e. the space of exactly marginal twisted chiral parameters of the infra-red superconformal field theories (SCFTs). In this section we also discuss the mixed Coulomb-confining branch mentioned above. In Section 6 we discuss the regularity condition for the GLSM superpotential such that the Higgs branch is compact. We find one condition, Condition (C), that works in all phases. It determines the “complex moduli space” $\mathcal{M}_C$, i.e. the space of exactly marginal chiral parameters of the infra-red SCFTs. We derive some important consequences of Condition (C) that are used in Section 4 for the analysis of each phase. In Section 7 we compute the mirror of $X$ and then determine the Picard-Fuchs operators and the Gromov-Witten invariants of $X$. With the Picard-Fuchs operators at hand, we are also able to extract the Gromov-Witten invariants of $\tilde{Y}$ in the strongly coupled phase, up to normalization. We end with some outlook on future directions of research in Section 8. Further details on the mirror symmetry calculations can be found in the appendix.

2 Basics of GLSMs

In this section we recall some basic properties of GLSMs which we will need for the discussion of our model. See [1, 2, 5, 6] for more details.
2.1 The data

To specify a GLSM, we choose a gauge group $G$, a matter representation $V$, a superpotential $W$, and a twisted superpotential $\tilde{W}$. We assume that $G$ is a compact Lie group and $\rho_V : G \to GL(V)$ is a faithful complex representation. The superpotential $W$ is a $G$-invariant polynomial of the scalar component $\phi$ of the matter chiral multiplet which takes values in $V$. The twisted superpotential $\tilde{W}$ is a $G$-invariant polynomial of the scalar component $\sigma$ of the vector multiplet which takes values in the complexified Lie algebra $g_C$ of $G$. We also choose $G$-invariant (hermitian) inner products on $V$ and $i g$. The hermitian inner product on $V$ determines the moment map $\mu : V \to i g^*$ and the inner product on $i g$ is parametrized by the gauge coupling constants $e$.

We are interested in models with vector and axial $U(1)$ R-symmetries with charge integrality. A vector $U(1)$ R-symmetry exists when we can assign R-charges on $\phi$, given by $R \in \text{End}(V)G$, under which the superpotential $W(\phi)$ has R-charge 2. It has charge integrality when $e^{i \pi R} = \rho_V(J)$ for some $J \in G$. The axial R-charge of $\sigma$ must be 2 and it is a symmetry of the classical system when $\tilde{W}(\sigma)$ is linear. It is anomaly free under the Calabi-Yau condition $\rho_V : G \to SL(V)$. In the Abelian case this reduces to the well-known condition that the gauge charges sum up to zero. Under these conditions, the theory is expected to flow in the infra-red limit to an SCFT of central charge $\hat{c} = \text{tr}_V(1-R) - \dim G$, with spectral flows between Ramond and Neveu-Schwarz sectors. In the following we will only consider such GLSMs.

The linear twisted superpotential is written as

$$\tilde{W}(\sigma) = -\langle t, \sigma \rangle,$$  \hspace{1cm} (2.1)

for a $t \in g_C^*G$ which can be decomposed as $t = \zeta - i \theta$ into the Fayet-Iliopoulos (FI) parameters $\zeta \in i g^*G$ and the $\theta$-angles $\theta \in i g^*G$. The theta angles are subject to appropriate periodicities.

A discrete theta angle and/or discrete torsion must be specified when $\text{Tors}(\pi_1(G))$ and/or $\pi_0(G)$ are non-trivial.

2.2 Phases

The classical vacua of the theory are determined by the zeroes of the potential

$$U = \frac{1}{8e^2} |\sigma, \bar{\sigma}|^2 + \frac{1}{2} (|\phi|^2 + |\bar{\phi}|^2) + \frac{e^2}{2} (\mu(\phi) - \zeta)^2 + |dW(\phi)|^2.$$  \hspace{1cm} (2.2)

The first term constrains $\sigma$ to take values in a Cartan subalgebra $t_C \subset g_C$. The last two terms, depending only on $\phi$, are called the D-term and the F-term respectively, and yield the D-term equations

$$\mu(\phi) - \zeta = 0,$$  \hspace{1cm} (2.3)

and the F-term equations

$$dW(\phi) = 0.$$  \hspace{1cm} (2.4)

Depending on the value of $\zeta$, some components of $\phi$ are forced to be non-zero by the D-term equations. This breaks some part of the gauge group, and hence the components of $\sigma$ corresponding to the broken generators are forced to vanish. The pattern of gauge symmetry
breaking by the solutions of the D-term and F-term equations divides the FI-parameter space into chambers, known as phases of the GLSM. The nature of the low energy physics in the different phases can be quite different.

In the interior of a phase, typically, and always when the gauge group is Abelian, the gauge symmetry is broken to a finite subgroup, and all components of $\sigma$ are forced to vanish. The continuous part of the gauge group is completely Higgsed and we can study the physics reliably by the classical analysis. In such a case, the space of solutions to the D-term equations alone modulo the gauge group action, which is the symplectic quotient $\mu^{-1}(\zeta)/G$, is a smooth manifold or an orbifold. It can also be described as the complex quotient

$$\mu^{-1}(\zeta)/G \simeq (V - F_\zeta)/G_C,$$

where $F_\zeta \subset V$ is the locus of $\phi \in V$ whose $G_C$-orbit does not hit $\mu^{-1}(\zeta)$, called the deleted set. The superpotential $W$ induces a holomorphic function $W_\zeta$ on $\mu^{-1}(\zeta)/G$, and the space of classical vacua is the critical locus of this function,

$$dW^{-1}(0) \cap \mu^{-1}(\zeta)/G = \text{Crit}(W_\zeta).$$

If $W_\zeta$ is a Bott-Morse function on $\mu^{-1}(\zeta)/G$, all modes transverse to Crit($W_\zeta$) are massive and can be integrated out. The theory reduces at low energies to the non-linear sigma model with target Crit($W_\zeta$). This target space is a Calabi-Yau manifold (or a Calabi-Yau orbifold) whose Kähler and B-field classes are determined by $\zeta$ and $\theta$. Such a phase is referred to as a geometric (or orbifold) phase. If $W_\zeta$ has a single isolated critical point, the low energy theory is the Landau-Ginzburg model or an orbifold thereof. Such a phase is referred to as a Landau-Ginzburg phase. If neither of the above holds for $(\mu^{-1}(\zeta)/G, W_\zeta)$, the phase is referred to as a hybrid phase.

On the interface between different phases, some of the solutions $\phi$ to the D-term and F-term equations leave continuous subgroups of the gauge group unbroken. Accordingly, $\sigma$ can take arbitrary values in the Cartan subalgebra of the unbroken gauge group. That is, we have a non-compact flat direction in the effective target space, called the Coulomb branch. To be precise, when some of the gauge group is broken by $\phi$, we shall call it mixed Coulomb-Higgs branch.

### 2.3 Strongly coupled phases

When the gauge group is non-Abelian, we may have a phase in which some of the solutions $\phi$ to the D-term and F-term equations leave continuous subgroups of the gauge group unbroken, and yet $\sigma$ for the unbroken gauge group cannot take large values. Then, the classical analysis is invalid to understand the nature of the low energy theory. Such a strongly coupled phase is very difficult to study in general.

In [5, 6], some useful results were obtained to deal with such a strongly coupled phase. In particular a two-dimensional analog of Seiberg duality has been identified. (See [24] for a recent discussion on an important point on such results.) When it is applied to a GLSM, strongly coupled phases in the model are sometimes mapped to weakly coupled phases in the dual, where the gauge symmetry is broken to finite subgroups and the classical analysis can be reliably used to understand the low energy physics.
2.4 Discriminants

We are primarily interested in regular models which have a discrete spectrum when formulated on $\mathbb{R} \times S^1$. This is ensured when the effective target space is compact. But this is not always the case — when the parameters of the theory are fine tuned to a special locus called the discriminant, a non-compact flat direction emerges in the effective target space. There are two classes of parameters, the chiral parameters in $W$ and the twisted chiral parameters in $\tilde{W}$, i.e., the FI-theta parameters. On the discriminant locus of the chiral parameters, the Higgs branch becomes non-compact. It is exactly determined by the classical analysis. On the discriminant locus of the FI-theta parameters a Coulomb branch or a mixed Coulomb-Higgs branch emerges. As discussed above, Coulomb or mixed branches appear at the phase boundaries. However, as we will discuss below, we should take into account the effect of the theta angles and quantum corrections to find the exact location.

Let us first determine the precise locus which supports a pure Coulomb branch where the discriminant, a non-compact flat direction emerges in the effective target space. There are two classes of parameters, the chiral parameters in $W$ and the twisted chiral parameters in $\tilde{W}$. This criterion of the (ir)regularity of the superpotential couplings was pointed out in [25, 26], and will significantly simplify our analysis.

Let us next determine the locus that supports a mixed Coulomb-Higgs branch where the FI-theta parameters. On such a branch, all the $T$-charged matter fields are heavy and should be integrated out, along with the $W$-bosons. This generates an effective twisted superpotential for such a $\sigma$:

$$\tilde{W}_{\text{eff}}(\sigma) = \tilde{W}(\sigma) + \pi i \sum_{\alpha > 0} \langle \alpha, \sigma \rangle - \sum_{Q} \langle Q, \sigma \rangle (\log \langle Q, \sigma \rangle - 1).$$

The sums of the second and the third terms are over the positive roots of $G$ and the weights of the representation $\rho_P$, respectively. This determines the effective FI-theta parameters $t_{\text{eff}}(\sigma) = -d\tilde{W}_{\text{eff}}(\sigma)$ which enter into the effective potential [1, 27] as

$$U_{\text{eff}} = \min_{n \in P} \frac{e_{\text{eff}}^2}{2} |t_{\text{eff}}(\sigma) + 2\pi in|^2,$$

where $P$ is the weight lattice of $T$. The vacuum equation is therefore

$$t_{\text{eff}}(\sigma) \equiv 0 \mod 2\pi i P.$$

By the Calabi-Yau condition, if $\sigma_*$ is such a vacuum, then its arbitrary complex multiple is also a vacuum. That is, the non-compact Coulomb branch exists if and only if the FI-theta parameter $t$ allows such a solution. In fact, equation [24] provides a parametric representation of the discriminant locus which supports the pure Coulomb branch.

Let us next determine the locus that supports a mixed Coulomb-Higgs branch where the gauge symmetry is broken to a non-maximal torus $T_L \subset G$ and $\sigma$ takes large values $\sigma_L$ in $t_{L, \mathbb{C}} = \text{Lie}(T_L)_\mathbb{C}$. On such a branch, one has to divide the matter fields $\phi = (\hat{\phi}, \hat{\phi})$ into those which receive mass by $\sigma_L$ (hatted) and those which do not (dotted). Similarly, we divide $\sigma$ into $(\hat{\sigma}, \sigma_L, \hat{\sigma})$, where $\hat{\sigma}$ receive mass by $\sigma_L$ and $(\hat{\sigma}, \sigma_L)$ do not, i.e., are in $\mathfrak{t}_{L, \mathbb{C}} = \text{Lie}(C^*_L)_\mathbb{C}$ where $C_L \subset G$ is the centralizer of $T_L$. Integrating out the hatted fields, we obtain the effective theory of the matter fields $\phi$ and the gauge group $C_L$ with the effective twisted superpotential $\tilde{W}_{\text{eff}, c_L}(\hat{\sigma}, \sigma_L)$ given by the same formula as [24] except that we only sum over the hatted roots and weights. The classical potential of the effective theory is as follows:

$$U = \frac{1}{8(e_{\text{eff}}^L)^2} |[\hat{\sigma}, \hat{\sigma}]|^2 + \frac{1}{2} \left( |\hat{\phi}|^2 + |\hat{\phi}|^2 \right) + \frac{(e_{\text{eff}}^L)^2}{2} \left( \mu^L(\hat{\phi}) - e_{\text{eff}}^L(\hat{\sigma}, \sigma_L) \right)^2 + |d\tilde{W}(\hat{\phi})|^2.$$

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2 This criterion of the (ir)regularity of the superpotential couplings was pointed out in [25, 26], and will significantly simplify our analysis.
Here $\mu^{\text{eff}}(\dot{\phi})$ is the restriction of $\mu(\dot{\phi})$ to $i\mathcal{L}$, $\zeta^{\text{eff}}$ is the real part of $t^{\text{eff}} = -d\dot{\mathcal{W}}_{\text{eff}}$, and $\dot{\mathcal{W}}$ is the restriction of $W$ to $\dot{\phi}$. We have the mixed branch when there is a solution to the vacuum equations which breaks $C_L$ to $T_L$, that is, $\dot{\sigma} = 0$ is forced by $\dot{\phi}$ where $(\dot{\phi}, \sigma_L)$ satisfy

$$t^{\text{eff}}(0, \sigma_L) \equiv 0 \mod 2\pi P_L, \quad \mu^{\text{eff}}(\dot{\phi}) = \zeta^{\text{eff}}(0, \sigma_L), \quad d\dot{\mathcal{W}}(\dot{\phi}) = 0,$$

with $P_L$ being the weight lattice of $T_L$. Again, the equation (2.11) provides a parametric representation of the discriminant locus which supports the mixed Coulomb-Higgs branch, under the condition that (2.12) has a solution.

It is also possible to have a mixed branch where the effective theory at large $\sigma_L$ is strongly coupled in the same sense as in Section 2.3 — the effective gauge group $C_L$ has an unbroken subgroup bigger than $T_L$ and yet $\dot{\sigma}$ cannot have large values. The results of [5,6] can again be of help in such a situation. In this paper, we shall indeed find in that way a discriminant locus supporting such a branch which may be called a “mixed Coulomb-confining branch”.

The complement of the discriminant descends to the space of exactly marginal parameters of the infra-red SCFTs, the “Kähler moduli space” $M_K$ for the FI-theta parameters and the “complex moduli space” $M_C$ for the chiral parameters. It is known that the moduli space of 2d (2,2) SCFTs is a direct product $M_K \times M_C$ (see [28] for a fine point and references therein) possibly up to a discrete identification. This means that the discriminant locus for the superpotential couplings should not depend on the phases, even though the analysis itself depends on them. Unlike in Abelian models where a general argument exists for the independence [25,26], in non-Abelian models with strongly coupled phases, a quite non-trivial work has to be done in each model, to the best of our knowledge and ability. (See [18], [6,7] and also Section 6.)

3 The Model

In this section, we introduce the GLSM we study in this paper, and describe the classical phase structure. We also describe the dual model.

3.1 GLSM data and classical phases

The gauge group of the model is

$$G = \frac{U(1)_1 \times U(1)_2 \times O(2)}{\{(\pm 1, \pm 1, \pm 1)\}}.$$

By $O(2)$ we mean $O(2)_+$ as defined in [6], but we will omit the subscript here. The chiral matter consists of six $O(2)$ singlets $p^I$ and five $O(2)$ doublets $x_I$ which are charged under $U(1)_1 \times U(1)_2$ as in the following table:

| $U(1)_1$ | $p^{1..4}$ | $p^5$ | $p^6$ | $x_{1..4}$ | $x_5$ | FI |
|-----------|-------------|-------|-------|------------|-------|-----|
| $U(1)_2$  | -2 | -1 | 1 | 1 | 0 | $\zeta_1$ |
| $U(1)_2$  | 0 | -1 | -1 | 0 | 1 | $\zeta_2$ |
| $O(2)$    | 1 | 1 | 1 | □ | □ | - |
We see that the group $G$ as in (3.1) acts effectively on these variables. We denote by $\zeta_1, \zeta_2$ the FI-parameters associated to the two $U(1)$s. The superpotential is

$$W = \sum_{I,J=1}^{5} S^{IJ}(p)(x_I x_J).$$

The entries of the symmetric matrix $S^{IJ}(p)$ are determined by gauge invariance:

$$S^{ij}(p) = \sum_{k=1}^{4} S^{ij}_k p^k,$$

$$S^{5j}(p) = S^{5j}_5 p^5 + \sum_{k=1}^{4} S^{5j}_k p^k p^6,$$

$$S^{55}(p) = S^{55}_5 p^5 p^6 + \sum_{k=1}^{4} S^{55}_k p^k p^6.$$

The model has a vector $U(1)$ R-symmetry with the R-charges $2$ for $p^{1..5}$ and $0$ for $p^6, x_{1..5}$, or equivalently, $0$ for $p^{1..6}$ and $1$ for $x_{1..5}$. The model also has an axial $U(1)$ R-symmetry since each gauge transformation has determinant $1$. Both satisfy charge integrality. Therefore, the model is expected to flow to a family of superconformal field theories of central charge $c = 5(1 - 2) + 11 - 3$ (or $6 - 3 = 3$ that can be used as string backgrounds with spacetime supersymmetry. The number of Kähler parameters of the family is $2$ since there are two FI-theta parameters, while the number of complex parameters is $24$ (see Section 6.7 for the count). In fact, these are the full numbers of exactly marginal Kähler and complex parameters, since we shall see that there are two geometric phases and the Hodge numbers of one of the Calabi-Yau manifolds are computed to be $(h^{1,1}, h^{2,1}) = (2, 24)$.

It is sometimes convenient to use different parametrizations of the fields and the group elements. Let us put $u_I := x_I + i x_I^2$ and $v_I := x_I^1 - i x_I^2$. They have charge $1$ and $-1$, respectively, under the identity component $SO(2) \cong U(1)$ of $O(2)$ and are exchanged under diag$(1, -1) \in O(2)$. The charge table for these variables is

|       | $p^{1..4}$ | $p^5$ | $p^6$ | $u_{1..4}$ | $v_5$ | $v_{1..4}$ | $v_5$ | FI    |
|-------|------------|-------|-------|------------|-------|------------|-------|-------|
| $U(1)_1$ | $-2$ | $-1$ | $1$ | $0$ | $1$ | $0$ | $1$ | $\zeta_1$ |
| $U(1)_2$ | $0$ | $-1$ | $-1$ | $0$ | $0$ | $1$ | $1$ | $\zeta_2$ |
| $SO(2)$ | $0$ | $0$ | $0$ | $1$ | $1$ | $-1$ | $-1$ | $-$ |

The identity component $G_0$ of $G$ is isomorphic to $U(1)^3$ via

$$U(1)_1 \times U(1)_2 \times SO(2) / \{(\pm 1, \pm 1, \pm 12)\} \rightarrow \{z_1, z_2, h\} \rightarrow (z_1^2, z_1 z_2, z_1 h).$$

The $U(1)_0 \times U(1)_3 \times U(1)_4$ charges of the fields are

|       | $p^{1..4}$ | $p^5$ | $p^6$ | $u_{1..4}$ | $v_5$ | $v_{1..4}$ | $v_5$ | FI    |
|-------|------------|-------|-------|------------|-------|------------|-------|-------|
| $U(1)_0$ | $-1$ | $0$ | $0$ | $1$ | $0$ | $-1$ | $1$ | $\zeta_1$ |
| $U(1)_3$ | $0$ | $-1$ | $-1$ | $0$ | $1$ | $0$ | $1$ | $\zeta_2$ |
| $U(1)_4$ | $0$ | $0$ | $0$ | $1$ | $1$ | $-1$ | $-1$ | $-$ |

\[^3\text{We choose the convention } i, j, k, \ldots \in \{1, \ldots, 4\}, I, J, K, \ldots \in \{1, \ldots, 5\}. \text{ Furthermore we use the shorthand notation } |u_{1,2,3}|^2 = |u_1|^2 + |u_2|^2 + |u_3|^2 \text{ or } u_{1,2,3} = 0 \text{ for } u_1 = u_2 = u_3 = 0, \text{ etc.}\]
Figure 2: Classical phases. The non-Abelian D-term leads to an extra phase boundary.

To determine the classical vacua we have to solve the D-term and F-term equations. The D-term equations can be read off for instance from (3.7)
\[ |u_1| + |v_5| - |p^5| = \zeta_1, \]  
\[ |u_5| + |v_1| - |p^6| = \zeta_2, \]  
\[ |u_1| + |u_5| - |v_1| - |v_5| = 0. \]

The last one is the $O(2)$ D-term equation and does not come with an FI parameter. The F-term equations are
\[ S^i_k u_i v_j + S^5_j p^6 (u_5 v_j + v_5 u_j) + S^{55}_k (p^6)^2 u_5 v_5 = 0, \quad k = 1, \ldots, 4, \]  
\[ S^5_j (u_5 v_j + v_5 u_j) + S^{56} p^6 u_5 v_5 = 0, \]  
\[ S^{55}_k p^k (u_5 v_j + v_5 u_j) + (S^{55}_5 p^5 + 2S^{55}_k p^k) u_5 v_5 = 0, \]  
\[ S^{IJ}_I (p) u_J = S^{IJ}_I (p) v_J = 0, \quad I = 1, \ldots, 5. \]

The phase structure is determined by the pattern of gauge symmetry breaking by the classical vacua. In the Abelian theory with $W = 0$, the phase boundary is spanned by the charge vectors of the matter fields, and coincides with the secondary fan of the associated toric variety. When the superpotential $W$ is turned on, the F-term equations may lift some of these phase boundaries. The D-term equations associated to the non-Abelian factors of the gauge group may also alter the structure of the phase diagram. We indeed find a phase boundary associated to the non-Abelian D-term in our example. The phase diagram is depicted in figure 2. Let us confirm that there is indeed an additional phase boundary at $\zeta_1 = \zeta_2 =: \zeta > 0$. In this case there are solutions to the D-term and F-term equations where all fields vanish except $|u_1| = |v_5| = \sqrt{\zeta}$ (resp. $|u_5| = |v_1| = \sqrt{\zeta}$) for which a $U(1)$ subgroup of elements with $z_1 = z_2^{-1} = h^{-1}$ (resp. $z_1 = z_2^{-1} = h$) is unbroken.

The unbroken gauge groups at the classical vacua are all finite in phases $I_+, I_-, II$ and $V$. Thus, these are weakly coupled phases where a simple classical analysis is enough to identify
the low energy physics. We shall see that $I_+$ is a geometric phase while $I_-$, II and V are hybrid LG/sigma model phases. In phase IV, the unbroken gauge group is $O(2) \subset G$ at each classical vacuum, while in phase III, it ranges from the identity to $O(2)$ depending on the vacuum. These are strongly coupled phases where the classical analysis is invalid. To see what we get, we may employ the result of [6]. We shall see that IV is a geometric phase while we are unable to find the nature of the low energy physics of phase III. Once we have determined the Picard-Fuchs operator, we will provide further evidence by computing the monodromy matrices around the limiting points — only phases $I_+$ and IV have maximally unipotent monodromy.

### 3.2 The dual model

One of the results of [6] which can be useful is the 2d Seiberg duality. Employing that, we obtain the dual of our GLSM. It has gauge group

$$\tilde{G} = \frac{U(1)_1 \times U(1)_2 \times SO(4)}{\{(\pm 1, \pm 1, \pm 1, \pm 1)\}}$$

(3.17)

the matter content

|        | $p^{1-4}$ | $p^5$ | $p^6$ | $\tilde{x}^{1-4}$ | $\tilde{x}^5$ | $s_{ij}$ | $s_{i5}$ | $s_{55}$ | Fl |
|--------|-----------|-------|-------|-------------------|---------------|----------|----------|----------|----|
| $U(1)_1$ | $-2$     | $-1$ | $1$   | $-1$             | $0$           | $2$      | $1$      | $0$      | $\tilde{\zeta}_1$ |
| $U(1)_2$ | $0$     | $-1$ | $-1$  | $0$             | $-1$          | $0$      | $1$      | $2$      | $\tilde{\zeta}_2$ |
| $SO(4)$  | $1$      | $1$  | $1$   | $\square$        | $\square$     | $1$      | $1$      | $1$      | $-$ |

(3.18)

and the superpotential

$$W = \sum_{IJ} s_{IJ}(\tilde{x}^I \tilde{x}^J) + \sum_{IJ} S^{IJ}(p)s_{IJ},$$

(3.19)

where the variables $(s_{IJ})_{I,J=1}^{5}$ form a $5 \times 5$ symmetric matrix. The dual model also have vector and axial $U(1)$ R-symmetries with charge integrality. The vector R-charges of the fields are 2 for $p^{1-5}$, 1 for $\tilde{x}^I$ and 0 for $p^6$ and $s_{IJ}$, or equivalently, 2 for $s_{IJ}$ and 0 for all others.

We obtain exactly the same phase structure as in the original model, i.e., that of Figure 2. Note that the “new” phase boundary $\tilde{\zeta}_1 = \tilde{\zeta}_2 > 0$ between $I_+$ and $I_-$ is just the ray spanned by the charge vector of $s_{i5}$. What is interesting here is that the ray $\tilde{\zeta}_1 = 0$ and $\tilde{\zeta}_2 < 0$ spanned by the charge vector of $\tilde{x}^5$ is not a phase boundary. This is because there is no vacuum where only $\tilde{x}^5$ is nonzero; the F-term equation implies $(\tilde{x}^5 \tilde{x}^5) = 0$ but this is not compatible with the $SO(4)$ D-term equations if $\tilde{x}^5 \neq 0$.

In phase IV, the dual model is weakly coupled and we will indeed see, most decisively in this way, that the theory reduces to a Calabi-Yau sigma model. On the other hand, in phase III, the dual model also has classical vacua with continuous unbroken gauge symmetry, and we are unable to find the nature of the low energy theory.

### 4 The Phases

In this section, we provide a description of each phase, with a detailed discussion of phases $I_+$ and IV. In particular, we describe the topology of the Calabi-Yau manifolds that appear in these two phases. Throughout this section, we assume Condition (C) described in Section 5.
which is a genericity condition on the coefficients \( S_k^{ij} \), \( i, j \), of the superpotential (3.3). A lot of details associated with Condition (C) will be explained in Section 6. Here we just state the results.

### 4.1 Phase I

In phase I, where \( \zeta_1 > \zeta_2 > 0 \), the D-term equations forbid

\[
F_I = \{u_5 = v_5 = 0\} \cup \{p^6 = u_{1..4} = 0\} \cup \{p^6 = v_{1..4} = 0\} \cup \{u_{1..5} = 0\} \cup \{v_{1..5} = 0\}. \tag{4.1}
\]

The identity component \( G_0 \) of the gauge group acts freely on the space of solutions to the D-term equations, and the quotient defines a smooth non-compact toric variety \( V_{1+} = (V - F_{1+})/\mathbb{G}_0 \). Under Condition (C), the superpotential \( W = \sum_{I=1}^5 p^I S_I(u, v, p^6) \) is a Bott-Morse function on \( V_{1+} \) with the critical point set

\[
p^I = S_I(u, v, p^6) = 0, \quad I = 1, \ldots, 5. \tag{4.2}
\]

Under the same condition (C), the critical locus \( \tilde{X} \) is a simply connected smooth Calabi-Yau manifold on which the component group \( \mathbb{Z}_2 = G/G_0 \) acts freely. Hence, the theory reduces at low energies to the non-linear sigma model whose target space is the free quotient \( X = \tilde{X}/\mathbb{Z}_2 \), which is a smooth Calabi-Yau manifold with \( |\pi_1(X)| = 2|\pi_1(\tilde{X})| \).

In what follows, we describe the topology of the Calabi-Yau manifolds \( \tilde{X} \) and \( X \). Let \( \mathbb{P}_{1+} \) be the smooth compact toric variety of dimension 8 obtained by setting \( p^4 = \cdots = p^5 = 0 \) in \( V_{1+} \), or more directly as the quotient

\[
\mathbb{P}_{1+} = (V_{2,1+} - F_{1+})/\mathbb{G}_0 \tag{4.3}
\]

where \( V_{2,1+} \) is the space of \( (u, v, p^6) \). The weights of the variables under \( G_0 = U(1)_0 \times U(1)_3 \times U(1)_4 \) can be found in (3.3). Then, \( \tilde{X} \) is the complete intersection of the five hypersurfaces \( S_1(u, v, p^6) = \cdots = S_5(u, v, p^6) = 0 \) in \( \mathbb{P}_{1+} \). The group \( \mathbb{Z}_2 = G/G_0 \) acts on \( \tilde{X} \subset \mathbb{P}_{1+} \) via the exchange \( u \leftrightarrow v \) of the variables and the involution \( (g_0, g_3, g_4) \mapsto (g_0, g_3, g_0 g_4^{-1}) \) on the group \( G_0 \).

Let us first describe the topology of the toric variety \( \mathbb{P} = \mathbb{P}_{1+} \). We denote by \( H_0 \) the divisor class of the line bundle associated with the charge 1 representation of \( U(1)_a \) \( (a = 0, 3, 4) \). It is simpler to work with the combination

\[
x := H_4, \quad y := H_0 - H_4, \quad z := H_3. \tag{4.4}
\]

on which the \( \mathbb{Z}_2 \) acts as \( x \leftrightarrow y, z \rightarrow z \). The classes of the homogeneous coordinates are

\[
p^6 : H_0 - H_3 = x + y - z, \quad u_{1..4} : H_4 = x, \quad v_5 : H_0 + H_3 + H_4 = z - y, \quad v_{1..4} : H_0 - H_4 = y, \quad v_5 : H_3 - H_4 = z - x.
\]

Since the deleted set is \( \{1, 1\} \), we see that the relations among these classes are

\[
(z - y)(z - x) = 0, \\
(x + y - z)x^4 = (x + y - z)y^4 = 0, \tag{4.5}
\]

\[
x^4(z - y) = y^4(z - x) = 0.
\]

Since there is exactly one point with \( u_{1..4} = v_{1..4} = 0 \),

\[
\int_p x^4y^4 = 1. \tag{4.6}
\]
Finally the Chern class of $\mathbb{P}$ is
\[
c(\mathbb{P}) = (1 + x + y - z)(1 + x)^4(1 + z - y)(1 + y)^4(1 + z - x).
\]
(4.7)

It is a simple exercise to see that non-zero Hodge numbers of $\mathbb{P}$ are
\[
h^{0,0}, \ldots, h^{8,8} = 1, 3, 5, 7, 9, 7, 5, 3, 1,
\]
and hence $\chi(\mathbb{P}) = 41$, which can be checked with $\int_{\mathbb{P}} c_8(\mathbb{P}) = 41$.

Next we analyze the topology of $\tilde{X} \subset \mathbb{P}$ defined by the zero of a section of $\mathcal{O}(H_0)^{\oplus 4} \oplus \mathcal{O}(H_3)$. The intersection numbers in $\tilde{X}$ can be found from $\int_{\tilde{X}} \eta = \int_{\mathbb{P}} (x + y)^4 z \eta$ for a class $\eta \in H^6(\mathbb{P})$:
\[
x^3 = y^3 = 5, \quad x^2 y = xy^2 = 10, \quad x^2 z = y^2 z = 11, \quad xyz = 14,
\]
\[
xz^2 = yz^2 = 15, \quad z^3 = 16.
\]
The Chern class is $c(\tilde{X}) = c(\mathbb{P})/((1 + x + y)^4(1 + z))$, i.e., $c_4(\tilde{X}) = 0$ and
\[
c_2(\tilde{X}) = -x^2 - y^2 + (x + y)z + 4xy, \quad c_3(\tilde{X}) = -3x^2 y - 3xy^2 - 2xyz.
\]
(4.8)

In particular,
\[
c_2(\tilde{X}) \cdot x = c_2(\tilde{X}) \cdot y = 50, \quad c_2(\tilde{X}) \cdot z = 64, \quad \chi(\tilde{X}) = -88.
\]
(4.9)

Finally we can compute the topology of $X = \tilde{X}/\mathbb{Z}_2$. Note that $\mathbb{Z}_2$ acts on the classes $x, y, z$ on $\tilde{X}$ as $x \mapsto y$, $z \mapsto z$. The generating divisor classes of $X$ are $x + y = H_0$ and $z = H_3$, and the intersection numbers can be found from $\int_X \eta = \frac{1}{2} \int_{\tilde{X}} \pi^* \eta$ where $\pi : \tilde{X} \to X$ is the projection map:
\[
H_0^3 = 35, \quad H_0^2 H_3 = 25, \quad H_0 H_3^2 = 15, \quad H_3^3 = 8,
\]
(4.10)
\[
c_2(X) \cdot H_0 = 50, \quad c_2(X) \cdot H_3 = 32,
\]
(4.11)
and
\[
\chi(X) = -\frac{88}{2} = -44.
\]
(4.12)
Since $h^{1,1}(X) = 2$, the above Euler number tells us that $h^{2,1}(X) = 24$.

In ([29]) two Calabi-Yaus with $(h^{1,1}, h^{2,1}) = (2, 24)$, constructed as free quotients of complete intersections in toric ambient spaces, are listed. The geometry of these examples looks different to ours. However, due to redundancies in the description of complete intersection Calabi-Yaus one still may have the same Calabi-Yau. It would be interesting to know the other topological characteristics of the Calabi-Yaus of [29] in order to see whether there is a match.

### 4.2 Phase IV

In phase IV, where $\zeta_1 < \zeta_2 < 0$, the D-term equations forbid $(p^{1,4}, u_5, v_5) = 0$, $p^{1,5} = 0$ and $p^{5,6} = 0$. This alone would allow the possibility to have $p^{1,4} = 0$ but $(u_5, v_5) \neq 0$ and $p^5 \neq 0$, but that is eliminated by the F-term equations. Indeed, $S(p)u = S(p)v = 0$ imply in this case $S_5^5 p^5 u_5 = S_5^5 p^5 v_5 = 0$, and we know from Condition (C) in Section 3 that $S_5^5 \neq 0$. Thus,
this enforces $u_5 = v_5 = 0$, in contradiction to $(u_5, v_5) \neq 0$. Therefore, the deleted set in this phase is

$$F_{IV} = \{p^{1, \ldots, 4} = 0\} \cup \{p^{5, 6} = 0\}. \quad (4.13)$$

Under Condition (C), the vacuum equations also imply $u = v = 0$. The unbroken gauge group is $O(2) \subset G$ at every classical vacuum — we are in a strongly coupled phase. The manifold of classical vacua is

$$\mathbb{P}_{IV} = (V_{BIV} - F_{IV})/\mathbb{C}_0^* \times \mathbb{C}_{3}^*,$$  

where $V_{BIV}$ is the space of $(p^1, \ldots, p^6)$ and $\mathbb{C}_0^* \times \mathbb{C}_{3}^*$ is the complexification of the group $U(1)_0 \times U(1)_3$, with the weights given in (3.3). It is a smooth compact toric variety of dimension 4.

We can proceed as in the analysis [6] of the strongly coupled phase of Hosono-Takagi model. First, we may try to work in the Born-Oppenheimer approximation, where we first "solve" the $O(2)$ sector for a fixed value of $p$ and then consider the fluctuation in the $p$ field. Under Condition (C), the $5 \times 5$ mass matrix $S(p)$ is at least 3 as long as $p$ is away from the deleted set $F_{IV}$. Let $Y$ and $C$ be the loci of $[p] \in \mathbb{P}_{IV}$ where rank $S(p) \leq 4$ and rank $S(p) = 3$, respectively. $Y \subset \mathbb{P}_{IV}$ is a hypersurface (a three-fold) with an $A_1$ singularity along $C$ (a curve). The result of the $O(2)$ gauge theory in [6] implies that the low energy theory must be the non-linear sigma model whose the target space is a double cover of $\mathbb{C}^2/\mathbb{Z}_2$. From this analysis, however, it is not clear whether such a double cover exists globally. A natural construction of the cover is provided by the dual model, as we described in Section 6.2.

In the dual model, phase IV corresponds to $\zeta_1 < \zeta_2 < 0$. In this phase, as will be shown later in Section 6.2, the vacuum equations require $p$ to be away from the deleted set $F_{IV}$ in (4.13), breaking the gauge group $\tilde{G}$ to the subgroup $\{(\pm 1, \pm 1) \times SO(4))/\{(\pm 1, \pm 1, \pm 1, 1)\} \cong SO(4)$. The equations also imply $s_{IJ} = 0$ for all $I, J$, under Condition (C) as proven in Section 6.2.3. The remaining F-term equations are

$$S^{IJ}(p) + (\tilde{x}^I \tilde{x}^J) = 0 \quad \forall I, J = 1, \ldots, 5. \quad (4.15)$$

Since $S(p)$ for $p \not\in F_{IV}$ has at least rank 3 under (C), $\tilde{x}^I_n$ obeying these equations must always have rank 3 or more, completely breaking the residual gauge group $SO(4)$. Thus, we are in a weakly coupled phase. The vacuum manifold is

$$\tilde{Y} = \left\{ (p, \tilde{x}) \mid p \not\in F_{IV}, \text{ SO}(4) \text{ stability}, \quad (4.15) \right\}/\tilde{G}_C. \quad (4.16)$$

All modes transverse to this are massive, and the theory reduces at low energies to the non-linear sigma model with this target space. Since the equations (4.15) imply that $S(p)$ has rank 4 or less, $[(p, \tilde{x})] \rightarrow [p]$ defines a map $\pi : \tilde{Y} \rightarrow Y$. It is surjective — two to one over $Y - C$ and one to one over $C$ — and behaves as the quotient map $\mathbb{C}^2 \rightarrow \mathbb{C}^2/\mathbb{Z}_2$ in the directions transverse to $C$. Thus, $\tilde{Y}$ realizes the wanted double cover of $Y$ that unfolds the $A_1$ singularity along $C$.

### 4.2.1 Topology of $\tilde{Y}$

Let us study the topology of the Calabi-Yau threefold $\tilde{Y}$ by employing the method discussed in Appendix D of [7]. As stated above, it is a $\mathbb{Z}_2$ cover of a hypersurface $Y = \{\det S(p) = 0\}$ of $\mathbb{P}_{IV}$ unfolding the $A_1$ singularity along the curve $C = \{\text{rank } S(p) = 3\} \subset Y$. Let us describe
the topology of the four dimensional toric variety $P = P_{IV}^{4.14}$. We denote by $H_a$ the divisor class of the line bundle associated with the charge $-1$ representation of $C_a^\ast$ ($a = 0, 3$).

In view of the deleted set $4.13$, we find that these classes are related as

$$H_0^4 = 0, \quad H_3(H_3 - H_0) = 0.$$

(4.17)

Since there is one point with $p^1 = p^2 = p^3 = p^5 = 0$,

$$\int_{P} H_3^0 H_3 = 1.$$

(4.18)

The non-zero Hodge numbers are $h^{0,0}, \ldots, h^{4,4} = 1, 2, 2, 2, 1$. It is simply connected: $\pi_1(P) = \{1\}$.

We first resolve the $A_1$ singularity of $Y$ along $C$ by inserting $P(Ker S(p))$, which is a $P_1$ over $C$ and one point elsewhere on $Y$. Since $S(p)$ can be regarded as the bundle map $E \to E^\ast(H_0)$ for $E = O^4 \oplus O(H_0 - H_3)$, the resolution is given by

$$Z = \{ [(p, x)] \in P(E) \mid S(p)x = 0 \}.$$

(4.19)

The forgetful map $Z \to Y$ is one to one over $Y - C$ and is a $P_1$ bundle $D$ over $C$. We introduce $\tilde{Z}$ as the fiber product

$$\tilde{Z} \longrightarrow Z \subset P(E)$$

$$\downarrow \quad \quad \quad \downarrow$$

$$\tilde{Y} \longrightarrow Y \subset P$$

(4.20)

We shall denote the pre-images of $C \subset Y$ and $D \subset Z$ in $\tilde{Y}$ and $\tilde{Z}$ by the same symbols. Since $\tilde{Z} - D$ is an unramified double cover over $Z - D$ by the upper-horizontal arrow of $4.20$ and is mapped isomorphically onto $\tilde{Y} - C$ by the left-vertical arrow, we find the following relations among the Euler numbers

$$\chi(\tilde{Z} - D) = \begin{cases} 2\chi(Z - D) = 2\chi(Z) - 2\chi(D) \\ \chi(\tilde{Y} - C) = \chi(\tilde{Y}) - \chi(C). \end{cases}$$

(4.21)

Note also that $\chi(D) = 2\chi(C)$ as $D$ is a $P_1$ bundle on $C$. Combining these, we find

$$\chi(\tilde{Y}) = 2\chi(Z) - 3\chi(C).$$

(4.22)

The ambient space $P(E)$ is realized as the vacuum manifold in a certain phase of the GLSM with gauge group $G_{aux} = U(1)_0 \times U(1)_3 \times U(1)_5$, matter content

| $p^{1..4}$ $p^5$ $p^6$ $x_{1..4}$ $x_5$ | FI |
|-------------------------|-------------------------|
| $U(1)_0$ | $-1$ | $0$ | $1$ | $0$ | $-1$ | $\zeta^0$ |
| $U(1)_3$ | $0$ | $-1$ | $-1$ | $0$ | $1$ | $\zeta^3$ |
| $U(1)_5$ | $0$ | $0$ | $0$ | $-1$ | $-1$ | $\zeta^5$ |

(4.23)

and vanishing superpotential. The relevant phase is $\zeta^3 < 0$, $\zeta^5 < \zeta^0 < 0$ where the deleted set is

$$F_{aux} = \{ p^{1..4} = 0 \} \cup \{ p^{5,6} = 0 \} \cup \{ x_{1..5} = 0 \}.$$  

(4.24)
We denote by $h_a$ the divisor class of the line bundle associated with the charge $-1$ representation of $U(1)_a$ $(a = 0, 3, 5)$. In view of the deleted set $F_{aux}$, we see that the relations of these classes are

$$h_0^4 = 0, \quad h_3(h_0 - h_3) = 0, \quad h_5^4(h_5 + h_0 - h_3) = 0. \quad (4.25)$$

Since the zeroes of $p^1, p^2, p^3, p^5, x_1...4$ intersect transversally at one point,

$$\int_{\mathbb{P}(E)} h_0^3 h_3 h_5^3 = 1. \quad (4.26)$$

Finally the Chern class is

$$c(\mathbb{P}(E)) = (1 + h_0)^4 (1 + h_3)(1 - h_0 + h_3) (1 + h_5)^4 (1 + h_5 + h_0 - h_3). \quad (4.27)$$

This determines the topology of $\mathbb{P}(E)$. It follows from (4.25) that the non-zero Hodge numbers are $h^{0,0}, \ldots, h^{8,8} = 1, 3, 5, 7, 8, 7, 5, 3, 1$ and in particular the Euler number is 40. Also, (4.27) yields $\int_{\mathbb{P}(E)} c_8(\mathbb{P}(E)) = 40$ as well.

To determine the class of the divisor $D \subset Z$, we consider the diagram of vector bundles on $\mathbb{P}(E)$,

$$0 \rightarrow \mathcal{O}(-h_3) \rightarrow \varpi^*E \rightarrow \mathcal{F} \rightarrow 0 \\
0 \rightarrow \mathcal{F}^*(h_0) \rightarrow \varpi^*E^*(h_0) \rightarrow \mathcal{O}(h_0 + h_5) \rightarrow 0 \quad (4.28)$$

The upper line is the tautological exact sequence on the fiber of $\varpi : \mathbb{P}(E) \rightarrow \mathbb{P}$, and the lower line is its dual tensored with $\mathcal{O}(h_0)$. Note that $\det \mathcal{F} = \mathcal{O}(h_0 - h_3 + h_5)$. The submanifold $Z$ is the locus where the map from $\mathcal{O}(-h_3)$ to $\varpi^*E^*(h_0)$ vanishes. By the symmetry of $S(p)$, it is the same as the locus where the map from $\varpi^*E$ to $\mathcal{O}(h_0 + h_5)$ vanishes. On this locus, there is a map from $\mathcal{F}$ to $\mathcal{F}^*(h_0)$ and $D \subset Z$ is where that degenerates. That is, $D$ is the zero of a section of $\det(\mathcal{F}^*(h_0)) \otimes (\det \mathcal{F})^{-1} = (\det \mathcal{F})^{-2}(4h_0) = \mathcal{O}(2h_0 + 2h_3 - 2h_5)$ on $Z$, which means

$$[D] = (2h_0 + 2h_3 - 2h_5)|_Z. \quad (4.29)$$

Now, we are ready to compute the Euler numbers of $Z, D, C$ and hence of $\tilde{Y}$. Since $Z \subset \mathbb{P}(E)$ is the zero of $S(p)x$, which is a section of the vector bundle $\mathcal{G} = \mathcal{O}(h_0 + h_5) \oplus \mathcal{O}(h_3 + h_5)$, its Euler number is

$$\chi(Z) = \int_{\mathbb{P}(E)} c_{top}(\mathcal{G}) \cdot \frac{c(\mathbb{P}(E))}{c(\mathcal{G})} = -88. \quad (4.30)$$

Since $D \subset \mathbb{P}(E)$ is the zero of a section of $\mathcal{G} \oplus \mathcal{O}(2h_0 + 2h_3 - 2h_5)$, its Euler number is

$$\chi(D) = \int_{\mathbb{P}(E)} c_{top}(\mathcal{G})(2h_0 + 2h_3 - 2h_5) \cdot \frac{c(\mathbb{P}(E))}{c(\mathcal{G})(1 + 2h_0 + 2h_3 - 2h_5)} = -88. \quad (4.31)$$

Thus, $\chi(C) = \chi(D)/2 = -44$ (the curve $C$ has genus 23). Applying (4.22), we find

$$\chi(\tilde{Y}) = 2(-88) - 3(-44) = -44, \quad (4.32)$$

which means $h^{2,1}(\tilde{Y}) - h^{1,1}(\tilde{Y}) = 22$. This is consistent with $h^{2,1}(\tilde{Y}) = 24$ and $h^{1,1}(\tilde{Y}) = 2$, that follows physically from the result in phase $I_+$ by the deformation invariance of the Hodge diamond of RR ground states.
Let us try to compute the intersection numbers. Put $M_a := \pi^*(H_a|_Y)$ for $a = 0, 3$. Using $\int_{\tilde{Y}} \pi^*\eta = 2 \int_Y \eta$ for $\eta \in H^0(Y)$ and (4.17)-(4.18), we find

$$M_0^3 = 4, \quad M_0^2 M_3 = M_0 M_3^2 = M_3^3 = 10.$$  \hspace{1cm} (4.33)

Using the Riemann-Roch formula $\chi(M_a) = \frac{1}{2}c_2(\tilde{Y}) \cdot M_a + \frac{1}{3!}M_a^3$ and assuming $\chi(M_a) = h^0(\mathbb{P}, H_a)$, which is 4 for $a = 0$ and 5 for $a = 3$, we find

$$c_2(\tilde{Y}) \cdot M_0 = 40, \quad c_2(\tilde{Y}) \cdot M_3 = 40.$$  \hspace{1cm} (4.34)

$\tilde{Y}$ is simply connected. To show this, we first note that $Y \subset \mathbb{P}$ is the zero of $\det S(p)$, which is a section of the line bundle $O(3H_0 + 2H_3)$ on $\mathbb{P}$. And this line bundle is very ample, that is, the map $\mathbb{P} \to \mathbb{P}(H^0(O(3H_0 + 2H_3))^*) \cong \mathbb{P}^{110}$ is a smooth embedding, as one can explicitly see. By the Lefschetz hyperplane theorem, we find

$$\pi_1(Y) \cong \pi_1(\mathbb{P}) = \{1\}. \hspace{1cm} (4.35)$$

Next, we apply the van Kampen theorem for $\tilde{Y} = (\tilde{Y} - C) \cup \tilde{U}_C$ and $Y = (Y - C) \cup U_C$, where $\tilde{U}_C$ and $U_C$ are tubular neighborhoods of $C$ in $\tilde{Y}$ and $Y$. Writing $\tilde{S}_C = (\tilde{Y} - C) \cap \tilde{U}_C$ and $S_C = (Y - C) \cap U_C$, the theorem reads

$$\pi_1(\tilde{Y}) \cong \pi_1(\tilde{Y} - C) \ast \pi_1(C), \quad \pi_1(Y) \cong \pi_1(Y - C) \ast \pi_1(C). \hspace{1cm} (4.36)$$

Note that $\tilde{S}_C$ and $S_C$ are homotopy equivalent to $S^3$ and $S^3/\mathbb{Z}_2$ bundles on $C$, and hence we have exact sequences

$$1 \to \pi_1(\tilde{S}_C) \to \pi_1(C) \to 1, \quad \mathbb{Z}_2 \to \pi_1(S_C) \to \pi_1(C) \to 1.$$  \hspace{1cm} (4.37)

From (4.36) and (4.37), we find

$$\pi_1(\tilde{Y}) \cong \pi_1(\tilde{Y} - C), \quad \pi_1(Y) \cong \frac{\pi_1(Y - C)}{\text{Im(Ker} (\pi_1(S_C) \to \pi_1(C)))}. \hspace{1cm} (4.38)$$

Since $\tilde{Y} - C$ is a smooth double cover of $Y - C$, we have $|\pi_1(Y - C)| = 2|\pi_1(\tilde{Y} - C)|$, and note also that $|\pi_1(Y)| = 1$ by (4.35). By the second isomorphism of (4.38), we find

$$1 = \frac{2|\pi_1(\tilde{Y} - C)|}{|\text{Im(Ker} (\pi_1(S_C) \to \pi_1(C)))|}.$$  \hspace{1cm} (4.39)

By the second exact sequence of (4.37), the denominator is at most 2. The only possibility is that the denominator is indeed 2 and that $|\pi_1(\tilde{Y} - C)| = 1$, that is, $\pi_1(\tilde{Y} - C) = \{1\}$. By the first isomorphism of (4.38), we obtain

$$\pi_1(\tilde{Y}) = \{1\}. \hspace{1cm} (4.40)$$

This is what we wanted to show. As a corollary of this, we also obtain $H_1(\tilde{Y}, \mathbb{Z}) = \{0\}$ and $h^{1,0}(\tilde{Y}) = h^{2,0}(\til{Y}) = 0$. 

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4.2.2 Proof of $p \notin F_{IV}$ in the dual model

Here we show that the vacuum equations of the dual model also require $p \notin F_{IV}$ in phase IV, that is,

$$p^{5,6} \neq 0 \quad \text{and} \quad p^{1,4} \neq 0. \quad (4.41)$$

We recall that the D-term equations are

$$-2|p^{1,4}|^2 - |p|^2 - |\bar{x}^{1,4}|^2 + 2|s_{ij}|^2 + |s_{i5}|^2 = \zeta_1,$$  
$$-|p|^2 - |p^{5,5}|^2 - |\bar{x}^{5,5}|^2 + 2|s_{55}|^2 = \zeta_2,$$  
$$\sum_{I=1}^{5} (\bar{x}^I_{a} x^I_{b} - \bar{x}^I_{b} x^I_{a}) = 0, \quad a, b = 1, \ldots, 4, \quad (4.44)$$

and the F-term equations are

$$S^i_k s_{ij} + S^j_k p^6 s_{5j} + S^{55}_k (p^6)^2 s_{55} = 0, \quad k = 1, \ldots, 4, \quad (4.45)$$
$$S^5_k s_{5j} + S^{55}_k p^6 s_{55} = 0, \quad (4.46)$$
$$S^i_k p^5 s_{5j} + (S^{55}_k p^5 + 2 S^{55}_k p^6) s_{55} = 0, \quad (4.47)$$
$$S^{ij}_k (p) + (\bar{x}^I x^I) = 0, \quad I, J = 1, \ldots, 5, \quad (4.48)$$
$$s_{Ij} \bar{x}^J = 0, \quad I = 1, \ldots, 5. \quad (4.49)$$

In phase IV where $\zeta_1 < \zeta_2 < 0$, the equations (4.42) and (4.43) require

$$(p^{5,6}, \bar{x}^5) \neq 0, \quad (p^{1,4}, \bar{x}^{1,4}, s_{55}) \neq 0, \quad (p^{1,4}, \bar{x}^{1,4}) \neq 0. \quad (4.50)$$

Suppose $p^{5,6} = 0$. Then, (4.50) implies $\bar{x}^5 \neq 0$, and the F-term equations imply that $(\bar{x}^I \bar{x}^J) = 0$ for all $I$. Since $(\bar{x}^5 \bar{x}^5) = 0$, we can find a real orthogonal frame with respect to which $\bar{x}^5 = (c_5, c_5 i, 0, 0)^T$ for some $c_5 \neq 0$, and the other orthogonality means that $\bar{x}^J = (c_j, c_j i, *, *)$ for some $c_j \in \mathbb{C}$ for $j = 1, \ldots, 4$. Then, the $SO(4)$ D-term equation with $(a, b) = (1, 2)$ reads

$$0 = \sum_{I=1}^{5} \bar{c}_I \cdot (c_I) - \sum_{I=1}^{5} c_I \cdot \bar{c}_I = 2i \sum_{I=1}^{5} |c_I|^2$$  
$$\sum_{I=1}^{5} \bar{c}_I \cdot (c_I) = 2i \sum_{I=1}^{5} |c_I|^2 \quad (4.51)$$

which is impossible since $c_5 \neq 0$. This proves that $p^{5,6} \neq 0$.

Suppose $p^{1,4} = 0$. Then the F-term equations would imply $(\bar{x}^I \bar{x}^J) = 0$ for $i, j = 1, \ldots, 4$. We would also have $p^5 \neq 0$. To show this, let us suppose otherwise, i.e. $p^5 = 0$ in addition to $p^{1,4}$. Then, the last of (4.50) would mean that $\bar{x}^{1,4} \neq 0$. At the same time, (4.48) would mean $(\bar{x}^I \bar{x}^J) = 0$ but then there is no other solution to the $SO(4)$ D-term equations than $\bar{x}^J = 0$, in contradiction to $\bar{x}^{1,4} \neq 0$. Recall that $S^{5i}_k \neq 0$ follows from (C). We may assume that $S^{5i}_k = c d_1$ for $c \neq 0$ by a change of coordinates if necessary, so that (4.48) reads

$$(\bar{x}^I \bar{x}^J) = 0, \quad (\bar{x}^I \bar{x}^5) = -c d_1^2 p^5 (\neq 0). \quad (4.52)$$

Then, we can find a real orthonormal frame with respect to which

$$\bar{x}^1 = (c_1, c_1 i, 0, 0)^T,$$
$$\bar{x}^j = (c_j, c_j i, d_j, d_j i)^T \quad \text{for} \quad j = 2, 3, 4,$$
$$\bar{x}^5 = (a_5, b_5, c_5, d_5)^T. \quad (4.53)$$
with
\[ c_1(a_5 + i b_5) \neq 0, \]  
\[ c_j(a_5 + i b_5) + d_j(c_5 + i d_5) = 0 \quad \text{for } j = 2, 3, 4. \]  
\[ (4.54) \]

The F-term equations \((4.49)\) then imply
\[
\sum_{j=1}^{4} s_{Ij} c_j + s_{I5} a_5 = i \sum_{j=1}^{4} s_{Ij} c_j + s_{I5} b_5 = 0
\]
\[ (4.56) \]
from which it follows that \(s_{I5}(a_5 + i b_5) = 0\). Since we know that \(a_5 + i b_5 \neq 0\) from \((4.54)\), this means that \(s_{I5} = 0\). In particular, \(s_{55} = 0\). Then, the difference of \((4.42)\) and \((4.43)\) reads
\[
2|p^6|^2 - |\bar{x}^1...4|^2 + |\bar{x}^5|^2 + 2|s_{ij}|^2 = \bar{\zeta}_1 - \bar{\zeta}_2 < 0,
\]
\[ (4.57) \]
which implies
\[
|\bar{x}^1...4|^2 > |\bar{x}^5|^2.
\]
\[ (4.58) \]
On the other hand, the \(SO(4)\) D-term equations \((4.44)\) for \((a, b) = (1, 2)\) and \((3, 4)\) read
\[
2i \sum_{j=1}^{4} |c_j|^2 + \bar{c}_5 b_5 - \overline{b_5} a_5 = 0,
\]
\[
2i \sum_{j=2}^{4} |d_j|^2 + \bar{c}_5 d_5 - \overline{d_5} c_5 = 0,
\]
\[ (4.59) \]
which imply
\[
|\bar{x}^1...4|^2 \leq |\bar{x}^5|^2.
\]
\[ (4.60) \]
\((4.58)\) and \((4.60)\) contradict. This completes the proof that \(p^{1...4} \neq 0\).

4.3 Phase I

In phase I, where \(\zeta_2 > \zeta_1 > 0\), the D-term equations forbid
\[
F_{1-} = \{u_{1...4} = v_{1...4} = p^6 = 0\} \cup \{u_5 = p^{1...4} = 0\} \cup \{v_5 = p^{1...4} = 0\}
\]
\[
\cup \{u_5 = v_5 = 0\} \cup \{u_{1...5} = 0\} \cup \{v_{1...5} = 0\}.
\]
\[ (4.61) \]
The identity component \(G_0\) of the gauge group acts freely on the space of solutions to the D-term equations, and the quotient defines a smooth non-compact toric variety \(V_{1-} = (V - F_{1-})/G_0\). Under Condition (C), we may impose the constraint
\[
p^5 = S_5(u, v, p^6) = 0.
\]
\[ (4.62) \]

Let \(X_{1-}\) be the \(\mathbb{Z}_2 = G/G_0\) quotient of the locus \((4.62)\) in \(V_{1-}\). The theory reduces at low energies to the hybrid LG/sigma model with target \(X_{1-}\) and the superpotential \(W_{1-}\) induced from \(W\).

The critical locus of \(W_{1-}\) is the union of components \(Z, C_1, \ldots, C_{10}\) where \(Z\) is a Calabi-Yau threefold (quotient of the intersection of four quadrics in \(\mathbb{P}^7\) by a free involution) and the \(C_i\) are rational curves. \(Z\) has ten conifold points at which the \(C_i\)’s intersect. The \(U(1)_V\) R-symmetry acts trivially on \(Z\) but non-trivially on the \(C_i\)’s. This suggests that the model is a bad hybrid (or pseudo hybrid) in the sense of [23].
4.4 Phase II

In phase II where $\zeta_2 < 0, \zeta_1 + \zeta_2 > 0$, the D-term equations forbid

$$F_{\Pi} = \{p^{5,6} = 0\} \cup \{p^{6} = u_{1...4} = 0\} \cup \{p^{5} = v_{1...4} = 0\} \cup \{u_{1...5} = 0\} \cup \{v_{1...5} = 0\}. \quad (4.63)$$

The group $G_0$ acts freely on the space of solutions to the D-term equations, and the quotient defines a smooth non-compact toric variety $V_{\Pi} = (V - F_{\Pi})/G_0\mathbb{C}$. Under Condition (C), we may impose the constraint

$$p^{1...4} = S_{1...4}(u, v, p^6) = 0. \quad (4.64)$$

Let $X_{\Pi}$ be the $\mathbb{Z}_2 = G/G_0$ quotient of the locus (4.67) in $V_{\Pi}$. The theory reduces at low energies to the hybrid model with target $X_{\Pi}$ and the superpotential $W_{\Pi}$ induced from $W$.

The critical locus of $W_{\Pi}$ is the union of components $Z', C'_1, \ldots, C'_6$ where $Z'$ is a Calabi-Yau threefold ($\mathbb{Z}_2$-quotient of the intersection of five symmetric bilinears in $\mathbb{P}^4 \times \mathbb{P}^4$) and the $C'_i$ are rational curves. $Z'$ has six conifold points at which the $C'_i$'s intersect. The $U(1)_V$ R-symmetry acts trivially on $Z'$ but non-trivially on the $C'_i$'s. This suggests that the model is a bad hybrid.

4.5 Phase III

In phase III where $\zeta_1 + \zeta_2 < 0, \zeta_1 - \zeta_2 > 0$, the D-term equations forbid

$$F_{\Pi} = \{p^{5,6} = 0\} \cup \{p^{1...5} = 0\} \cup \{u_{1...4} = p^6 = 0\} \cup \{v_{1...4} = p^6 = 0\}. \quad (4.65)$$

It is possible to have $u = v = 0$ at which the F-term equations are all satisfied. At this locus, the $O(2)$ subgroup of the gauge group is unbroken. There is also a locus, such as $p^{1...4} = u_5 = v_5 = 0, p^5 \neq 0, u_{1...4} \neq 0, v_{1...4} \neq 0$, which satisfies the F-term equations if $S_k^j u_4 v_j = 0, S_5^j u_j = S_5^j v_j = 0$. There the gauge group is completely broken. Therefore, there is a mixture of strongly coupled vacua and weakly coupled vacua, and it is not easy to tell what the low energy theory is.

The dual theory cannot be of help. The weakly coupled vacua in the original theory correspond to strongly coupled vacua in the dual. Indeed, for $p^{1...4} = 0$ and $p^5 \neq 0$, the matrix $S(p)$ is of the form

$$S(p) = \begin{pmatrix} 0_{4 \times 4} & S_5^j p^5 \\ S_5^j p^5 & S_5^j p^5 p^6 \end{pmatrix} \quad (4.66)$$

and has rank 2. It is possible to find vacua with such $p$’s, and in such a vacuum ($\tilde{x}^I_{\Pi}$) has rank 2. They are strongly coupled vacua with an unbroken $SO(2)$ subgroup of $SO(4) \subset \tilde{G}$.

To summarize, we are unable to find the nature of the low energy theory in phase III.

4.6 Phase V

In phase V where $\zeta_1 < 0, \zeta_2 > 0$, the D-term equations forbid

$$F_{\Pi} = \{p^{1...5} = 0\} \cup \{u_5 = v_5 = 0\} \cup \{u_{1...5} = 0\} \cup \{v_{1...5} = 0\} \cup \{p^{1...4} = u_5 = 0\} \cup \{p^{1...4} = v_5 = 0\}. \quad (4.67)$$

The group $G_0$ acts freely on the space of solutions to the D-term equations except at the locus $p^5 = p^6 = u_{1...4} = v_{1...4} = 0$ with the stabilizer $\{(1, \pm 1, \pm 1)\} \cong \mathbb{Z}_2$, and the quotient
\( V_V = (V - F_V) / G_0 \) is a non-compact toric orbifold. Under Condition (C), we may impose the constraint \([1, 62]\). Let \( X_V \) be the \( \mathbb{Z}_2 = G / G_0 \) quotient of the locus \([1, 62]\) in \( V_V \). The theory reduces at low energies to the hybrid model with target \( X_V \) and the superpotential \( W_V \) induced from \( W \).

The critical locus of \( W_V \) is the union of components \( Z'' \), \( C_1'' \ldots, C_{10}'' \) where \( Z'' \) is the \( \mathbb{Z}_2 = G / G_0 \) quotient of a weighted projective space \( \mathbb{P}_{[222]}^3 \) and the \( C_i'' \) are teardrops \( \mathbb{P}_{[21]}^3 \). \( Z'' \) intersects with each \( C_i'' \) at the \( \mathbb{Z}_2 \) point of the latter. The \( U(1)_V \) R-symmetry acts trivially on \( Z'' \) but non-trivially on the \( C_i'' \)'s. This suggests that the model is a bad hybrid.

5 Kähler Moduli

In this section and the next, we determine the regularity conditions of the parameters of the system. The criterion is that there is no non-compact flat direction in the effective target space \([1] \). In this section, we focus on the condition on the FI-theta parameters, \( t_1 \) and \( t_2 \). We shall identify the discriminant locus where there is a non-compact Coulomb branch or mixed Coulomb-Higgs branch \([1, 2] \). The complement is the “Kähler moduli space” \( \mathcal{M}_K \), that is, the space of exactly marginal twisted chiral parameters of the infra-red SCFTs. In Section \([1, 2] \), we shall revisit the same problem by examining the Picard-Fuchs operator.

5.1 The original model

It is convenient to work with the scalar components \((\sigma_0, \sigma_3, \sigma_4)\) of the vector multiplet for \( U(1)_0 \times U(1)_3 \times U(1)_4 \) since the corresponding theta angle has simple periodicity, i.e., \( (2\pi\mathbb{Z})^3 \).

They are related to the ones \((\sigma_1, \sigma_2, \sigma_h)\) for \( U(1)_1 \times U(1)_2 \times SO(2) \) via \([5, 8]\) as \( \sigma_0 = 2\sigma_1, \sigma_3 = \sigma_1 + \sigma_2 \) and \( \sigma_4 = \sigma_1 + \sigma_h \). The \( O(2) \) Weyl reflection \((\sigma_1, \sigma_2, \sigma_h) \mapsto (\sigma_1, \sigma_2, -\sigma_h)\) is given by \((\sigma_0, \sigma_3, \sigma_4) \mapsto (\sigma_0, \sigma_3, \sigma_0 - \sigma_4)\). If we write \( t_1 \) and \( t_2 \) for the FI-theta parameter of \( U(1)_1 \times U(1)_2 \), the tree level twisted superpotential is

\[
\tilde{W}_{\text{tree}} = -t_1 \sigma_1 - t_2 \sigma_2 = -\frac{t_1 - t_2}{2}\sigma_0 - t_2\sigma_3,
\]

where for this choice of gauge group \( \tilde{W}_{\Theta}(\sigma) = 0 \). The periodicity of these parameters is \((\frac{1}{2}, \frac{1}{2}, t_2) \equiv (\frac{1}{2}, \frac{1}{2}, t_2) + 2\pi i(n, m)\) for \( n \) and \( m \) integers. It would be more appropriate to use \( t_0 \) and \( t_3 \) defined by \( \tilde{W}_{\text{tree}} = -t_0\sigma_0 - t_3\sigma_3 \), but we informally use \( t_1 \) and \( t_2 \) as they are convenient to compare with the phase diagram \([2]\).

Since \( O(2) \) does not have any roots, the quantum corrections given in \([2, 7]\) come only from integrating out the massive matter fields. With that, the vacuum equations \( \partial_{\sigma_n} \tilde{W}_{\text{eff}} \equiv 0 \) (mod \( 2\pi i\mathbb{Z} \)) yield

\[
e^{-\frac{t_1 - t_2}{2}} = \frac{(\sigma_0 - \sigma_4)^4(\sigma_0 - \sigma_3)}{(-\sigma_0)^4(-\sigma_0 + \sigma_3 + \sigma_4)}, \quad e^{-t_2} = \frac{(-\sigma_0 + \sigma_3 + \sigma_4)(\sigma_3 - \sigma_4)}{(-\sigma_3)(\sigma_0 - \sigma_3)},
\]

\[
\frac{\sigma_4^4(-\sigma_0 + \sigma_3 + \sigma_4)}{(\sigma_0 - \sigma_4)^4(\sigma_3 - \sigma_4)} = 1.
\]

For \( y := \sigma_3 / \sigma_0 \) and \( z := \sigma_4 / \sigma_0 \), and the equations \([5, 8]\) read

\[
e^{-\frac{t_1 - t_2}{2}} = \frac{(1 - z)^4(1 - y)}{-1 + y + z}, \quad e^{-t_2} = \frac{(-1 + y + z)(y - z)}{-y(1 - y)}, \quad \frac{z^4(-1 + y + z)}{(1 - z)^4(y - z)} = 1.
\]
The last equation factorizes as

$$(2z - 1)(y - f(z)) = 0, \quad f(z) := \frac{z(1 - z)(z^2 - z + 1)}{2z^2 - 2z + 1}, \quad (5.4)$$

and there are two solutions: (i) $2z - 1 = 0$ and (ii) $y = f(z)$. For these we have

$$(i) \quad e^{-\frac{t_1 - t_2}{2}} = 2 - 3\frac{1 - y}{2y - 1}, \quad e^{-\frac{t_1 + t_2}{2}} = -2 - 3\frac{2y - 1}{y}, \quad (5.5)$$

$$(ii) \quad e^{-\frac{t_1 - t_2}{2}} = -y^2 + 3y - 1, \quad e^{-\frac{t_1 + t_2}{2}} = \frac{v^3}{1 - v}, \quad (5.6)$$

where we used the Weyl invariant $v = z(1 - z)$ in the latter. Equations (5.5) and (5.6) show the location of the singularity. This encodes an amoeba whose spines reproduce part of the classical phase boundaries. Let us see how this works explicitly. We start with the component (i). At $y = 1$ we have $e^{-\frac{t_1 - t_2}{2}} = 0, e^{-\frac{t_1 + t_2}{2}} = \text{const.}$ which implies $t_1 = -t_2 > 0$. This is the phase boundary separating phase II and III. For $y = \frac{1}{2}, e^{-\frac{t_1 - t_2}{2}} = \infty, e^{-\frac{t_1 + t_2}{2}} = 0$ and therefore $t_1 = 0, t_2 > 0$. This is the phase boundary between phase V and phase I. Finally, $y = 0$ yields the phase boundary $t_1 = t_2 < 0$ separating phases III and IV. On the component (ii) the roots of $y^2 - 3y + 1$ correspond to the phase boundary separating phases II and III. $y = 0$ corresponds to the phase boundary between phase I and I. Another end of the second component is at $v = \infty$ which supplies the phase boundary $t_1 < 0, t_2 = 0$ between phase IV and V. Finally, at $v = 1$ we get $e^{-\frac{t_1 - t_2}{2}} = \text{const.}, e^{-\frac{t_1 + t_2}{2}} = \infty$ and therefore $t_1 = t_2 < 0$. This is the boundary between phases III and IV.

The boundary between phases I and II is missing. In fact, there is an additional discriminant locus associated to the mixed Coulomb-Higgs branch where $U(1)_3$ is unbroken and $\sigma_3$ is arbitrarily large. On this branch, $u_5, v_5, p^6$ are heavy and should be integrated out. This yields the following effective FI-theta parameters

$$t^0_{\text{eff}}(\sigma) = \frac{t_1 - t_2}{2} + \log(\sigma_0 - \sigma_3) - \log(-\sigma_0 + \sigma_3 + \sigma_4)$$

$$= \frac{t_1 - t_2}{2} + \pi i + \log \left( \frac{1 - \sigma_0}{\sigma_3} \left( \frac{1 - \sigma_0}{\sigma_3} \right)^{-1} \right), \quad (5.7)$$

$$t^3_{\text{eff}}(\sigma) = t_2 + \log \left( \frac{1 - \sigma_0 - \sigma_4}{\sigma_3} \left( \frac{1 - \sigma_4}{\sigma_3} \right)^{-1} \right), \quad (5.8)$$

$$t^4_{\text{eff}}(\sigma) = \log \left( \frac{1 - \sigma_0 - \sigma_4}{\sigma_3} \left( \frac{1 - \sigma_4}{\sigma_3} \right)^{-1} \right). \quad (5.9)$$

The scalar potential of the effective theory is

$$U_{\text{eff}} = |\sigma_4 u_1 ... u_4|^2 + |(\sigma_0 - \sigma_4) v_1 ... v_4|^2 + |\sigma_3 p^1 ... p^4|^2$$

$$+ \frac{1}{2} \sum_{a,b} (e_{\text{eff}})^2_{ab} (\mu^a_{\text{eff}} - \zeta^a_{\text{eff}}(\sigma)) (\mu^b_{\text{eff}} - \zeta^b_{\text{eff}}(\sigma))$$

$$+ \sum_k \left( \sum_{i,j} S^i_k u_i v_j \right)^2 + \sum_j \left( \sum_{k,i} S^i_k p^k u_i \right)^2 + \sum_i \left( \sum_{k,j} S^i_k p^k v_j \right)^2. \quad (5.10)$$
Here $\zeta_{\text{eff}}^a(\sigma):=\Re t_{\text{eff}}^a(\sigma)$ and
\begin{align}
\mu_{\text{eff}}^0 &= |u_{1\ldots 4}|^2 - |p^{1\ldots 4}|^2, \\
\mu_{\text{eff}}^3 &= 0, \\
\mu_{\text{eff}}^4 &= |u_{1\ldots 4}|^2 - |v_{1\ldots 4}|^2. \tag{5.11}
\end{align}

There are also theta angles $\theta_{\text{eff}}^a(\sigma):=\Im t_{\text{eff}}^a(\sigma)$. When
\begin{equation}
(iii)_+ \quad e^{-t_2} = 1, \quad \zeta_1 \gg 0, \tag{5.12}
\end{equation}
the effective theory at arbitrarily large $\sigma_3$ has supersymmetric vacua in which $u_{1\ldots 4}$ and $v_{1\ldots 4}$ are both non-zero, breaking the gauge symmetry to $U(1)_3$ and forcing $\sigma_0 = \sigma_4 = 0$. That is, there is a non-compact mixed Coulomb-Higgs branch. Thus, we need to include (5.12) as a part of the discriminant locus, which accounts for the missing phase boundary. Since the discriminant locus must be an analytic subspace, we expect that the condition $\zeta_1 \gg 0$ can be removed. There are indeed supersymmetric vacua in the opposite regime $\zeta_1 \ll 0$ of the same line $e^{-t_2} = 1$, but all or most of them have $u_{1\ldots 4} = v_{1\ldots 4} = 0$, leaving the $O(2)$ subgroup also unbroken. The theory is strongly coupled and the classical analysis is not reliable. To be sure, for now we count only (5.12) as a part of the discriminant locus. We shall reconsider the other region $\zeta_1 \ll 0$ in the dual model in Sections 5.2 and 5.4 and directly in Section 5.4.

Apart from this, there are no further mixed branches. To confirm this, one has to systematically analyze all field configurations where a continuous subgroup of the gauge group is unbroken. To illustrate how a situation where there is no mixed branch manifests itself, we consider the situation where $U(1)_0$ is unbroken and $\sigma_0$ can become arbitrarily large. This happens when $p^{1\ldots 4}$, $p^5$, $u^5$ and $v_{1\ldots 4}$ are heavy and can be integrated out. In this case the scalar potential of the effective theory is
\begin{align}
U_{\text{eff}} &= |\sigma_3 p^5|^2 + |\sigma_4 u_{1\ldots 4}|^2 + |(\sigma_3 - \sigma_4) v_5|^2 \\
&\quad + \frac{1}{2} \sum_{a,b} (e_{\text{eff}})^2, ab \left( \mu_{\text{eff}}^a - \zeta_{\text{eff}}^a(\sigma) \right) \left( \mu_{\text{eff}}^b - \zeta_{\text{eff}}^b(\sigma) \right) \\
&\quad + \sum_j |S^j_3 u_j v_5|^2 + \sum_j |S^j_5 p^5 v_5|^2 + \sum_i |S^i_5 p^5 u_i|^2, \tag{5.13}
\end{align}
with
\begin{align}
\mu_{\text{eff}}^0 &= 0, \\
\mu_{\text{eff}}^3 &= -|p^5|^2 + |v_5|^2, \\
\mu_{\text{eff}}^4 &= |u_{1\ldots 4}|^2 - |v_5|^2 \tag{5.14}
\end{align}
and
\begin{align}
t_{\text{eff}}^0(\sigma) &= \frac{t_1 - t_2}{2} + \pi i + \log \left[ \left( 1 - \frac{\sigma_4}{\sigma_0} \right) \left( 1 - \frac{\sigma_3 + \sigma_4}{\sigma_0} \right)^{-1} \right], \\
t_{\text{eff}}^3(\sigma) &= t_2 + \pi i + \log \left[ \left( 1 - \frac{\sigma_3 + \sigma_4}{\sigma_0} \right) \left( 1 - \frac{\sigma_3}{\sigma_0} \right)^{-1} \right], \\
t_{\text{eff}}^4(\sigma) &= \pi i + \log \left[ \sigma_0^{-3} \left( 1 - \frac{\sigma_3 + \sigma_4}{\sigma_0} \right) \left( 1 - \frac{\sigma_4}{\sigma_0} \right)^{-4} \right]. \tag{5.15}
\end{align}
We are looking for a mixed branch where only \( U(1)_0 \) is unbroken, so that \( \sigma_3 = \sigma_4 = 0 \) while \( \sigma_0 \) can have arbitrary values. There the effective D-term equations reduce to
\[
0 = \zeta_1 - \zeta_2 \\
-|p^5|^2 + |v_5|^2 = \zeta_2 \\
|u_{1...4}|^2 - |v_5|^2 = \log \frac{1}{|\sigma_0|^3}. \tag{5.16}
\]

It is straightforward to show that it is not possible to get \( U_{\text{eff}} = 0 \) for arbitrary \( \sigma_0 \). By the first D-term equation, we have \( \zeta_1 = \zeta_2 \) and we distinguish between the two cases \( \zeta_2 > 0 \) and \( \zeta_2 < 0 \). If \( \zeta_2 > 0 \), by the second equation of (5.16), we have \( v_5 \neq 0 \), but then the F-term equations in the last line of (5.13) enforce \( p^5 = 0 \). Then the second D-term equation reduces to \( |v_5|^2 = \zeta_2 \) and the third one becomes
\[
|u_{1...4}|^2 - \zeta_2 = \log \frac{1}{|\sigma_0|^3}. \tag{5.17}
\]
For a fixed \( \zeta_2 \) it is impossible to satisfy this equation for an arbitrarily large \( \sigma_0 \). For \( \zeta_2 < 0 \) the second equation in (5.16) yields \( p^5 \neq 0 \). From the F-term equations it then follows that \( v_5 = 0 \). Then the last equation in (5.16) reduces to
\[
|u_{1...4}|^2 = \log \frac{1}{|\sigma_0|^3}, \tag{5.18}
\]
which cannot be satisfied for large \( \sigma_0 \). Hence we conclude that there is no mixed branch with unbroken \( U(1)_0 \).

A systematic analysis of all possibilities of unbroken \( U(1)s \) shows that there are no further mixed branches. Note that no work is necessary to show that there is no mixed branch with unbroken \( U(1)_4 \). These are the \( u_I \) and \( v_I \). The effective potential is the same as for an \( O(2) \)-theory with five fundamentals. In [6] it was shown that \( O(k) \)-theories with \( N \) fundamentals with trivial theta angle have no Coulomb branch if and only if \( N - k \) is odd. This is indeed the case here, and therefore there is no mixed branch with unbroken \( U(1)_4 \).

### 5.2 The dual model

Let us next identify the discriminant locus in the dual model. As a maximal torus of the gauge group \( \tilde{G} \), we take
\[
\frac{U(1)_1 \times U(1)_2 \times SO(2) \times SO(2)}{\{\pm 1, \pm 1, \pm 1, \pm 1\}} \cong U(1)_0 \times U(1)_3 \times U(1)_4 \times U(1)_5 \tag{5.19}
\]
We shall work with the fields \((\sigma_0, \sigma_1, \sigma_4, \sigma_5)\) corresponding to the group on the right hand side of (5.19). The \( SO(4) \) Weyl group acts on them as
\[
(\sigma_0, \sigma_3, \sigma_4, \sigma_5) \to (\sigma_0, \sigma_3, \sigma_5, \sigma_4), \ (\sigma_0, \sigma_3, \sigma_0 - \sigma_4, \sigma_0 - \sigma_5). \tag{5.20}
\]
In the Coulomb branch analysis, we disregard possible solutions to the vacuum equation at the fixed points\footnote{This is an empirical rule which can sometimes be supported by detailed argument \cite{5}. See \cite{24} for a recent proposal on this point.}

\begin{equation}
\sigma_4 = \sigma_5, \quad \sigma_4 + \sigma_5 = \sigma_0.
\end{equation}

The tree level twisted superpotential is

\begin{equation}
\bar{W}_{\text{tree}} = -\tilde{t}_1 \sigma_1 - \tilde{t}_2 \sigma_2 = -\frac{\tilde{t}_1 - \tilde{t}_2}{2} \sigma_0 - \tilde{t}_2 \sigma_3.
\end{equation}

The $U(1)_0 \times U(1)_3 \times U(1)_4 \times U(1)_5$ charges of the fields are

\begin{equation}
\begin{array}{cccccc}
p^4 \quad & (-1, 0, 0, 0) & u_i^{4-4} (-1, 0, 1, 0) & \tilde{u}_i \quad & (0, -1, 1, 0) & s_{ij} \quad (1, 0, 0, 0) \\
p^5 \quad & (0, -1, 0, 0) & u_i^{4-4} (0, 0, -1, 0) & \tilde{v}_i \quad & (1, -1, -1, 0) & s_{i5} \quad (0, 1, 0, 0) \\
p^6 \quad & (1, -1, 0, 0) & u_i^{4-4} (-1, 0, 0, 1) & w_i \quad & (0, -1, 0, 1) & s_{55} \quad (-1, 2, 0, 0) \\
\end{array}
\end{equation}

The equations that determine the full Coulomb branch are

\begin{equation}
e^{-\frac{\tilde{t}_1 - \tilde{t}_2}{2}} = \frac{\sigma_0^10(\sigma_0 - \sigma_3 - \sigma_4)(\sigma_0 - \sigma_3 - \sigma_5)(\sigma_0 - \sigma_3)}{(-\sigma_0 + \sigma_4)^4(-\sigma_0 + \sigma_5)^4(-\sigma_0 + 2\sigma_3)}
\end{equation}

\begin{equation}
e^{-\tilde{t}_2} = \frac{\sigma_4^3}{(-\sigma_0 + 2\sigma_3)^2}
\end{equation}

\begin{equation}
1 = \frac{(-\sigma_0 + \sigma_4)^4(-\sigma_3 + \sigma_4)}{(-\sigma_3)^4(\sigma_0 - \sigma_3 - \sigma_4)}
\end{equation}

For $y := \sigma_3/\sigma_0$, $z := \sigma_4/\sigma_0$, $w := \sigma_5/\sigma_0$, the $SO(4)$ Weyl group action is

\begin{equation}
(y, z, w) \rightarrow (y, w, z), \ (y, 1 - z, 1 - w),
\end{equation}

and the disregarded locus is

\begin{equation}
z = w, \quad z + w = 1.
\end{equation}

The last two equations of (5.24) read

\begin{equation}
(2z - 1)(y - f(z)) = (2w - 1)(y - f(w)) = 0,
\end{equation}

where $f(z)$ is as in (5.4). Note that

\begin{equation}
f(z) - f(w) = \frac{1}{2}(w - z)(z + w - 1) \left\\{ \frac{1}{(2z^2 - 2z + 1)(2w^2 - 2w + 1)} + 1 \right\}.
\end{equation}

Since $z = w$ and $z + w = 1$ are disregarded, we have either (i) $y = f(z)$ and $(2z^2 - 2z + 1)(2w^2 - 2w + 1) = -1$ or (ii) $y = f(z)$ and $2w = 1$, up to the Weyl group action. In these cases, we have (with $\tilde{v} := z(1 - z)$)

\begin{equation}
e^{-\frac{\tilde{t}_1 - \tilde{t}_2}{2}} = \frac{\tilde{v}^2 - 3\tilde{v} + 1}{2\tilde{v}^2 - 4\tilde{v} + 1}, \quad e^{-\frac{\tilde{t}_1 + \tilde{t}_2}{2}} = -\frac{2\tilde{v}^2 - 4\tilde{v} + 1}{\tilde{v}(1 - \tilde{v})},
\end{equation}

\begin{equation}
e^{-\frac{\tilde{t}_1 + \tilde{t}_2}{2}} = -\frac{2\tilde{v}^2 - 3\tilde{v} + 1}{(1 - 2\tilde{v})^2}, \quad e^{-\frac{\tilde{t}_1 + \tilde{t}_2}{2}} = 2\left(1 - \tilde{v}\right)^3 \frac{1}{\tilde{v}(1 - 2\tilde{v})^2}.
\end{equation}
There is an additional discriminant locus associated with a mixed Coulomb-Higgs branch with $U(1)_3$ unbroken. We look at the regime where $\sigma_3$ is large and all the matter fields other than $p^{1\ldots 4}, \tilde{x}^{1\ldots 4}$ and $s_{ij}$ are integrated out. When

$$e^{-\tilde{t}_2} = 4, \quad \tilde{\zeta}_1 \ll 0,$$  \hspace{1cm} (5.31)

the effective theory at arbitrarily large $\sigma_3$ has supersymmetric vacua. At most of them, non-zero values of the matter fields break the gauge symmetry to $U(1)_3$ and force $\sigma_0 = \sigma_{SO(4)} = 0$. That is, there is a non-compact mixed Coulomb-Higgs branch. Thus, we need to include (5.31) as a part of the discriminant. Again, we expect that the condition $\tilde{\zeta}_1 \ll 0$ can be removed. There are indeed supersymmetric vacua in the other regime of the line $e^{-\tilde{t}_2} = 4$, but all of them are strongly coupled. Therefore, we take only (5.31) for now.

5.3 Summary

To summarize, as the discriminant locus, we identified two complete components, (i) at (5.29) and (ii) at (5.30), plus one half-line (iii) at (5.12) in the original model, and two complete components, (i) at (5.29) and (ii) at (5.30), plus one half-line (iii) at (5.31) in the dual model. Under

$$e^{-\frac{t_1-t_2}{2}} = -2^{-3} e^{-\frac{t_1-t_2}{2}}, \quad e^{-\frac{t_1+t_2}{2}} = -2^{-5} e^{-\frac{t_1+t_2}{2}}.$$  \hspace{1cm} (5.32)

the complete components for the original and the dual are mapped to each other by

(i) $y = \frac{\tilde{\nu}(1 - \tilde{\nu})}{1 - 2\tilde{\nu}},$ \hspace{1cm} (5.33)

(ii) $v = \frac{1 - \tilde{\nu}}{1 - 2\tilde{\nu}},$ \hspace{1cm} (5.34)

while the two half-lines (iii)$_+$ and (iii)$_-$ are opposite regimes of a complete line

$$e^{-t_2} = 1 \iff e^{-\tilde{t}_2} = 4.$$  \hspace{1cm} (5.35)

This suggests that (5.32) is the map of the parameters under the duality, and that the complete line (5.35) is indeed one component of the discriminant locus. Altogether, the discriminant consists of the three complete components, (i), (ii) and (iii), as shown in Figure 3.

5.4 Some detail on the mixed branch

Let us describe the detail of the mixed branch supported at the component (iii) of the discriminant. We assume the following genericity condition on $S_{ij}^k$:

**Condition (C$_{mixed}$):** If $u_{1\ldots 4} \neq 0$ and $v_{1\ldots 4} \neq 0$ satisfy $S_{ij}^k u_i v_j = 0$, then the $4 \times 8$ matrix $(Su, Sv)$ has rank 4.

Here $Su$ stands for the $4 \times 4$ matrix whose $(i, j)$th entry is $S_{ij}^k u_k$ (and similarly for $Sv$). This is different form Condition (C) for the regularity of the full theory. We assume (C$_{mixed}$) just for simplicity of the following discussion. Nothing is wrong even if it is violated — it is just that the mixed branch would be non-compact also in the Higgs direction.
Under (C mixed, that follows from the latter two of (5.38) implies that \( p \) write there is a map \( \tilde{\text{locus of rank}} \)

Under (C mixed), SO isomorphic to \( u \) quotient of \( \tilde{\text{gauge group}} \).

Thus, the mixed branch is the product of the Coulomb branch for \( U(1)_3 \) and the Higgs branch \( X_{\text{Higgs}} \) which is an Enriques surface.

For \( \zeta_1 \gg 0 \) (i.e. \( \zeta_1 \ll 0 \)), it is better to work in the dual model where the effective theory at large \( \sigma_3 \) has matter fields \( p^{1\ldots4}, u_{1\ldots4} \) and \( v_{1\ldots4} \). The D-term equations require that both \( u_{1\ldots4} \) and \( v_{1\ldots4} \) have non-zero values, breaking the gauge group to \( U(1)_0 \). Then, under (C mixed), the equations (5.38) enforce either \( u_{1\ldots4} = 0 \) or \( v_{1\ldots4} = 0 \), hence \( s_{jk} = 0 \). Thus, the vacuum manifold is

\[
\tilde{\text{Y}}_{\text{mixed}} = \left\{(p^{1\ldots4} \neq 0, \tilde{x}^{1\ldots4}) \mid \text{SO(4) stability, } S^{ij}(p) + (\tilde{x}^i \tilde{x}^j) = 0 \right\} / \tilde{\text{G}}_{-\mathcal{C}}.
\]

where \( \tilde{\text{G}}_{-\mathcal{C}} = (U(1)_1 \times \text{SO(4)})/(\{\pm 1, \pm 1_4\}) \). For the part where \( S^{ij}(p) \) has rank 4 and 3 the gauge group \( \tilde{\text{G}}_{-\mathcal{C}} \) acts freely, but at the points where the rank is exactly 2, there is a stabilizer isomorphic to \( \text{SO(2)} \). Thus, unfortunately, there are bad points in the quotient. Note that there is a map \( \tilde{\text{Y}}_{\text{mixed}} \rightarrow \mathbb{P}^3 \) that forgets \( \tilde{x}^{1\ldots4} \). It is a double cover that is ramified over the locus of rank \( \leq 3 \), i.e. \{det \( S^{ij}(p) = 0 \)\} which is a singular K3 surface.

A part of this can be seen also in the original model. The \( U(1)_0 \) D-term equation requires \( p^{1\ldots4} \neq 0 \), breaking \( U(1)_0 \) and forcing \( \sigma_0 = 0 \). Then, under (C mixed), the F-term equations
and the $U(1)_4$ D-term equation force $u_{1\ldots4} = v_{1\ldots4} = 0$. Thus, $\zeta_1 \ll 0$ is a strongly coupled phase where the $O(2)$ is unbroken. On the generic points of $[p] \in \mathbb{P}^2$ where $S^0(p)$ has rank 4, all $x_i$'s are massive. But there is a non-trivial twisted superpotential for the vector multiplet of $O(2)$, as shown in (5.9), and there is an isolated critical point at $\sigma_4 = 0$. Alternatively, the theory has a single $O(2)$ doublet $x_3$ with a large twisted mass $\sigma_3$. As shown in [6], such a theory has two massive vacua. (Recall that our $O(2)$ is $O_+^\prime(2)$, and see Eqn (3.15) in [6].) This matches with the picture obtained in the dual theory that the effective target space is a double cover of $\mathbb{P}^3$. Note that $U(1)_0$ is Higgsed, $U(1)_3$ is in Coulomb phase and $O(2)$ is confined. Thus, it is a mixed Higgs-Coulomb-confining branch in the original theory. Similarly, in the regime $\tilde{\zeta}_1 \gg 0$ on (iii), we have a mixed Higgs-Coulomb-confining branch in the dual theory. Note that the nature of the mixed branch is very different between the two opposite regimes, $\zeta_1 \gg 0$ and $\zeta_1 \ll 0$. Even the dimension of the effective target spaces are different — $X_{\text{mixed}}$ is an Enriques surface while $Y_{\text{mixed}}$ is a (singular) Calabi-Yau threefold. This is not an immediate problem since it is not that we have a family of Higgs/confining branch theories in isolation — they appear only in the large $\sigma_3$ regimes. But still, it is interesting to see that we can have such very different types of mixed branch theories on the same discriminant component.

### 6 Complex Moduli

In this section, we determine the regularity condition of the superpotential so that the Higgs branch is compact [25][26]. This is relevant to find the “complex moduli space” $\mathcal{M}_C$, that is, the space of exactly marginal chiral parameters of the infra-red SCFTs. Note that the conditions must be the same for all phases since the moduli space of SCFTs must be the direct product $\mathcal{M}_K \times \mathcal{M}_C$. Indeed, we shall find one condition, Condition (C), that works in all phases. The condition is found in phase $I_+$ straightforwardly, but it is very non-trivial to confirm that it also works in other phases (except in phase $I_-\ldots$). We shall also derive some consequences of Condition (C) which are used in earlier sections.

#### 6.1 Phase $I_+$

Recall that the superpotential can be written as $W = \sum_{i=1}^5 p^i S_i(u, v, p^6)$, with

$$
S_i(u, v, p^6) = S^i_{jk} u_j v_k + S^i_{5j} p^6 (u_5 v_j + v_5 u_j) + S^{55}_i (p^6)^2 u_5 v_5,
$$  

$$
S_5(u, v, p^6) = S^5_{5j} (u_5 v_j + v_5 u_j) + S^{55}_5 p^6 u_5 v_5.
$$  (6.1)

In phase $I_+$, the range of the fields $u, v, p^6$ is bounded by the D-term equations — they form the homogeneous coordinates of a compact space — provided that $p^1, \ldots, p^5$ are bounded. Hence, the only source of non-compactness comes from the fields $p^1, \ldots, p^5$. They enter into the F-term potential quadratically. Thus, the condition in phase $I_+$ is that the mass matrix of the fields $p^1, \ldots, p^5$ has full rank. That is,

**Condition (C):** If $(u, v, p^6) \not\in F_{I_+}$ solves the equations

$$
S_1(u, v, p^6) = \cdots = S_5(u, v, p^6) = 0,
$$  (6.2)

If $(C_{\text{mixed}})$ is violated, we may have special weakly coupled vacua where $u_{1\ldots4}$ and $v_{1\ldots4}$ are non-zero.
then the $5 \times (5 + 5 + 1) = 5 \times 11$ matrix

$$M := \left( \begin{array}{c} \frac{\partial S_i}{\partial u_j} \\ \frac{\partial S_i}{\partial v_j} \\ \frac{\partial S_i}{\partial p^6} \end{array} \right)$$

has rank 5.

This is an open condition: First, note that the matrix $M$ annihilates $(u, 0, 0)^T$ and $(0, v, 0)^T$ if $(u, v, p^6)$ solves (6.2). Thus the number of conditions that $M$ has rank 4 or less is $(5 - 4) \times (11 - 2 - 4) = 5$. This is generically impossible to satisfy since $\tilde{X}$ only has dimension 3.

Since (6.2) is the defining equation for $\tilde{X}$ and (6.3) is its first order differential, (C) is equivalent to the condition for smoothness of $\tilde{X}$. (This is always the case in the usual geometric phases.) It turns out that this condition also implies that the $\mathbb{Z}_2$ action on $\tilde{X}$ that exchanges $u$ and $v$ is free, so that the quotient $\tilde{X}/\mathbb{Z}_2 = X$ is also smooth. Let us prove this.

Suppose there is $(u, u, p^6) \not\in F_{14}$ that solves the equations (6.2). (We would like to show that $M$ would have rank 4 or less, in contradiction to Condition (C).) Note that, at this point, the first two $5 \times 5$ matrix factors of $M$ are identical and that they are of rank 4 or less since $\frac{\partial S_i}{\partial u_j} \cdot w^j = S_I = 0$ for a non-zero $u$. We are not done yet since there are also the last $5 \times 1$ entries $\frac{\partial S_i}{\partial p^6}$ in $M$. Here we note that

$$\left( \frac{\partial S_I}{\partial u_j} \right)_{u = v} \begin{pmatrix} -u_j \\ u_5 \end{pmatrix} = \begin{pmatrix} -S^{ijkl} u_k u_j - S^{5ij}_I p^6 u_5 u_j + S^{5k}_I u_k u_5 + S^{55}_I (p^6)^2 u_5^2 \\ -S^{5j}_I u_5 u_j + S^{5k}_I u_k u_5 + S^{55}_I p^6 u_5^2 \end{pmatrix}$$

$$= \begin{pmatrix} 2S^{5k}_I p^6 u_k u_5 + 2S^{55}_I (p^6)^2 u_5^2 \\ S^{55}_I p^6 u_5^2 \end{pmatrix} = p^6 \begin{pmatrix} \frac{\partial S_I}{\partial p^6} \end{pmatrix} \bigg|_{u = v},$$

where we used $S_i(u, u, p^6) = 0$ in the second equality. This means that, as long as $p^6 \neq 0$, the last $5 \times 1$ entries of $M$ is a linear combination of the first $5 \times 5$ entries of $M$, so that $M$ has rank 4 or less, in contradiction to Condition (C). When $p^6 = 0$, a separate discussion is needed. In this case, the equations are $S^{ijkl}_I u_k u_j = 0$ ($k = 1, \ldots, 4$) and $S^{5j}_I u_5 u_j = 0$. Note that $u_5 = 0$ is not allowed since $u_5 = v_5 = 0$ would be in the deleted set $F_{14}$. Thus, the latter equation is equivalent to $S^{5j}_I u_j = 0$. In this case,

$$\left( \frac{\partial S_I}{\partial u_j} \right) = \begin{pmatrix} S^{ijkl}_I 0 \\ S^{5j}_I 0 \end{pmatrix},$$

is of rank 3 or less since it annihilates the column vector $(u_j, 0)^T$. Since the last $5 \times 1$ entries contribute at most rank one, $M$ is of rank 4 or less, again in contradiction to Condition (C).

This completes the proof of the claim that $\tilde{X}$ misses the diagonal $u = v$.

### 6.2 Phase IV

In phase IV, the range of the fields $p^1, \ldots, p^6$ is bounded by the D-term equations — they form homogeneous coordinates of a compact space — provided that the fields $u$ and $v$ are bounded. We shall show that Condition (C) ensures that the vacuum equations in phase IV force $u$ and $v$ to vanish, removing the danger of non-compactness. In the dual model, phase IV is the usual geometric phase where the target space $\hat{Y}$ is defined by the equations $\partial_{s_{ij}} W = 0$. Hence the condition is that the mass matrix for the fields $s_{ij}$ is of full rank on $\hat{Y}$, or equivalently, smoothness of the variety $\hat{Y}$. We shall also see that this is ensured by Condition (C).

As a preparation, we derive two consequences of Condition (C).
6.2.1 Consequence 1: \( S_{5i}^j \neq 0 \)

First consequence of (C) is that \( S_{5i}^j \neq 0 \) for some \( i = 1, \ldots, 4 \). Indeed, if \( S_{5i}^j \) were all zero, \((u, v, p^6)\) with \( v_5 = p^6 = 0, S_{5i}^j u_i v_j = 0\), \( u_5 \neq 0\), \( u_{1\ldots4} \neq 0\) and \( v_{1\ldots4} \neq 0\) would solve the equations \((6.2)\) and

\[
M = \begin{pmatrix}
S_{i^k}^j v_k & 0 & 0 \\
0 & S_{i^j}^k u_k & 0 \\
0 & 0 & S_{i^k}^5 u_5 v_k
\end{pmatrix}.
\]

This matrix has rank 4 or less, in contradiction to Condition (C).

6.2.2 Consequence 2: \( \text{rank} S(p) \geq 3 \text{ for } p \not\in F_{IV} \)

The second consequence of (C) is that the matrix \( S(p) \) has at least rank 3 if \( p \) represents a point of \( F_{IV} \), i.e., if \( p \not\in F_{IV} \).

Suppose there is \( p_* \not\in F_{IV} \) such that \( S(p_*) \) has rank 2 or less. We first assume that \( p_* \) is non-zero. Then, on dimensional grounds, it is possible to find \( [(u, v, p^6)] \in \tilde{X} \) such that \( S(p_*)u = S(p_*)v = 0 \). There, we can show that \( (p_1^* \ldots, p_5^*) \cdot M(u, v, p^6) = 0 \). Indeed,

\[
(p_1^* \ldots, p_5^*) \cdot M = \left( S_{JK}^{JK}(p_*) v_K, S_{JK}^{JK}(p_*) u_K, \partial_{
u^5} S^{KL}(p_*) u_K v_L \right),
\]

and the last entry, which is

\[
\partial_{
u^5} S^{KL}(p_*) u_K v_L = p^5 S_j^k (u_5 v_k + v_5 u_k) + 2 p^j S_j^5 p_6 u_5 v_5 + p^5 S_{55}^5 u_5 v_5,
\]

also vanishes provided \( p_* \neq 0 \) since

\[
0 = u_5 S_j^5 (p_*) u_j + v_5 S_j^5 (p_*) u_j
\]

\[
= S_j^5 (p_*) u_5 v_k + S_{55}^5 (p_*) u_5 v_5 + S_j^5 (p_*) v_5 u_k + S_{55}^5 (p_*) u_5 v_5
\]

\[
= p^5 \partial_{
u^5} S^{KL}(p_*) u_K v_L + p^5 \left( S_{55}^5 (u_5 v_k + v_5 u_k) + S_{55}^5 p^6 u_5 v_5 \right). \tag{6.9}
\]

As \( (p_1^* \ldots, p_5^*) \neq 0 \), this means that \( M(u, v, p^6) \) has rank 4 or less, in contradiction to Condition (C).

We need a separate discussion for the case \( p_* = 0 \). Note that \( p_* \neq 0 \) since \( p_* \not\in F_{IV} \). In this case, \( S_j^i (p_*) = S_k^i p_* \), \( S_{5j}^i (p_*) = S_{55}^j p_* \) and \( S_{55}^i (p_*) = 0 \). Recall that \( S_{55}^i \neq 0 \) and hence \( S_{5j}^i (p_*) \neq 0 \). Then, we see that \( S(p_*) \) has rank 2 or less (actually rank exactly 2) if and only if

\[
S_k^i p_*^k = x_*^j S_{5j}^i + x_*^5 S_{55}^i \tag{6.10}
\]

for some \( x_*^i \). But we can show that presence of such \( p_*^i \) contradicts Condition (C). That is, we can find \( p_*^5, u, v \) such that \( [(u, v, 0)] \in \tilde{X} \) and that \( (p_1^5 \ldots, p_5^5) \cdot M(u, v, 0) = 0 \). Indeed, \( [(u, v, 0)] \in \tilde{X} \) means \( (u, v, 0) \not\in F_{IV} \) and

\[
\text{(a)} \ S_j^i u_i v_j = 0, \quad \text{(b)} \ S_{5j}^i (u_5 v_j + v_5 u_j) = 0,
\]

and \( (p_1^5 \ldots, p_5^5) \cdot M(u, v, 0) = 0 \) reads

\[
\begin{align*}
(1) \ x_5^j v_j + p_5^5 v_5 &= 0, \quad (2) \ S_{5j}^5 v_j &= 0, \\
(3) \ x_5^j u_j + p_5^5 u_5 &= 0, \quad (4) \ S_{5j}^5 u_j &= 0, \\
(5) \ p_5^5 S_{5j}^5 (u_5 v_j + v_5 u_j) + p_5^5 S_{55}^5 u_5 v_5 &= 0.
\end{align*}
\]
Note that one of the four equations in (a) follows from (2) and (4) provided (6.10) holds. Note also that (b) follows from (2) and (4) as well. The number of equations is therefore $3 + 5 = 8$. The number of variables is $11 - 3 = 8$ where 11 comes from $u_{1...5}, v_{1...5}, p_i^6$ and −3 comes from the gauge group action. Thus, there is a solution, and the contradiction against (C) is confirmed.

This completes the proof under (C) that $S(p)$ has rank at least 3 for $p \not\in F_{IV}$.

6.2.3 The proof — the original model

Suppose $(u, v, p)$ solves the vacuum equations in phase IV. In particular, $p \not\in F_{IV}$ and $(u, v, p)$ solves the F-term equations. In view of $W = \sum_{I=1}^{5} p^I S_I(u, v, p^6)$, we see that the F-term equations $\partial_{u_i} W = \partial_{v_i} W = \partial_{p_i} W = 0$ read $(p^1, \ldots, p^5) \cdot M = 0$, where $M$ is the $5 \times 11$ matrix defined by (6.3). Since $(p^1, \ldots, p^5) \not\equiv 0$ by $p \not\in F_{IV}$, this means that $M$ has rank 4 or less. Since $(u, v, p^5)$ solves the equations (6.2), this means by Condition (C) that $(u, v, p^5)$ has to land in $F_{I\lambda}$. We now show that $u = v = 0$ under (C), in each of the five components of $F_{I\lambda}$:

(i) $u_{1...5} = 0$: Then $v_{1...5} = 0$ by the (2) D-term equation.
(ii) $v_{1...5} = 0$: Then $u_{1...5} = 0$ for the same reason.
(iii) $u_{1...4} = p^6 = 0$: Then, $p^5 \neq 0$ by $p \not\in F_{IV}$. By the F-term equation,

$$0 = S(p)u = \begin{pmatrix} S_5^5 p^5 u_5 \\ 0 \end{pmatrix}. \quad (6.11)$$

Since $S_5^5 \neq 0$ (a consequence of (C)), this means $u_5 = 0$. Therefore, $u = 0$, and hence $v = 0$ by the (2) D-term equation.
(iv) $v_{1...4} = p^5 = 0$: Then $u = v = 0$ for the same reason.
(v) $u_5 = v_5 = 0$: Then, the equations are $S_k^{ij} u_i v_j = 0$, $S^I(p) u_j = S^I(p) v_j = 0$, i.e.,

$$S_k^{ij} u_i v_j = 0, \quad k = 1, \ldots, 4, \quad (6.12)$$

$$S_k^{ij} p^k u_j = S_k^{ij} p^k v_j = 0, \quad i = 1, \ldots, 4, \quad (6.13)$$

$$S_k^{ij} p^5 u_j + S_k^{ij} p^6 u_j + S_k^{ij} p^5 v_j + S_k^{ij} p^6 v_j = 0. \quad (6.14)$$

Suppose $u = v = 0$ fails. By the (2) D-term equation, this means that $u_i \neq 0$ and $v_j \neq 0$ for some $i, j \in \{1, \ldots, 4\}$. Let us now put $U = (u_1, \ldots, u_4, U_5)$ and $V = (v_1, \ldots, v_4, V_5)$ and ask if there is $(U, V, p_i^6) \not\in F_{I\lambda}$ solving (6.2), but $M$ has rank 4 or less. We may put $p_6 = 0$ and still have $(U, V, 0) \not\in F_{I\lambda}$ provided $(U_5, V_5) \neq (0, 0)$. Then, the first four equations in (6.2) are equivalent to (6.12) and the last one reads

$$S_5^{ij}(U_5v_j + V_5 u_j) = 0, \quad (6.15)$$

Also, by (6.13) the matrix $M$ at $(U, V, 0)$ satisfies $(p^1, \ldots, p^4, 0) \cdot M = 0$ provided

$$p^i S_5^{ij}(U_5v_j + V_5 u_j) = 0. \quad (6.16)$$

The equations (6.15) and (6.16) have a solution with $(U_5, V_5) \neq (0, 0)$ provided

$$\det \begin{pmatrix} S_5^{ij} v_j & S_5^{ij} u_j \\ S_5^{ij} p^i v_j & S_5^{ij} p^i u_j \end{pmatrix} = 0. \quad (6.17)$$

This is indeed the case since (6.14) has a solution with $(p^5, p^6) \neq (0, 0)$. Therefore, we are able to draw a contradiction to (C). This proves that $u = v = 0$. 

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6.2.4 The proof — the dual model

In phase IV, the range of the fields \((p, \tilde{x})\) is bounded by the D-term equations and a part of the F-term equations:

\[
S^{IJ}(p) + (\tilde{x}^I \tilde{x}^J) = 0.
\]

(6.18)

Therefore, the only source of non-compactness comes from the fields \(s_{IJ} = s_{JI}\) with \(I, J = 1, \ldots 5\). Since they enter into the F-term potential quadratically, the condition in phase IV is that the mass matrix has the full rank, or equivalently, the F-term equations force \(s_{IJ} = 0\). The F-term equations imply, if we use (6.18),

\[
S(p)^{IJ} s_{JK} = 0, \quad I, K = 1, \ldots 5.
\]

(6.19)

\[
\partial_{p^\alpha} S^{IJ}(p) s_{IJ} = 0, \quad \alpha = 1, \ldots 6.
\]

(6.20)

Since \(p \not\in F_{IV}\), the matrix \(S(p)\) has rank 3 or higher (a consequence of (C)). Then (6.19) requires that \(s_{JK}\) is of rank 2 or less and can be written as \(s_{IJ} = u_I v_J + v_I u_J\) for some \(u_I\)'s and \(v_J\)'s satisfying

\[
S(p)^{IJ} u_J = S(p)^{IJ} v_J = 0.
\]

(6.21)

The first five equations of (6.20) are nothing but the equations (6.2), and (6.21) together with the last of (6.20) is equivalent to \((p_1, \ldots, p_5) \cdot M = 0\). Since \((p_1, \ldots, p_5) \neq 0\) by \(p \not\in F_{IV}\), this means by Condition (C) that \((u, v, p_6)\) must land in \(F_{1+}\). Then, we can reuse most of the argument in the original model (Section 6.2.3), and show that \(s_{IJ} = 0\) is enforced under Condition (C). The \(O(2)\) D-term equation was important in the original model but cannot be used here. However, that is not necessary since we only need either \(u = 0\) or \(v = 0\) to conclude \(s_{IJ} = 0\): In components (i) and (ii), \(s_{IJ} = 0\) from the outset. In component (iii) (resp. (iv)), \(u = 0\) (resp. \(v = 0\)) is derived without the \(O(2)\) D-term equation. In component (v), \(u_i \neq 0\) and \(v_j \neq 0\) for some \(i, j \in \{1, \ldots, 4\}\) is the only non-trivial possibility to exclude. So again, no need for the \(O(2)\) D-term equation.

This must be equivalent to the condition for smoothness of \(\tilde{Y}\). Indeed, this can be seen explicitly. Extending the analysis in [6] for a similar problem, we see that the smoothness condition goes as follows:

Take a point \([p] \in C\) so that \(S(p)\) has rank 3. Then, the linear map \(\text{Sym}^2 \mathbb{C}^5 \to \mathbb{C}^6\) represented by the 6 \(\times\) 15 matrix \(N^{(i,j)}(p) = \partial_{p^\alpha} S^{IJ}(p)\) has maximal rank (= 3) when restricted to the subspace \(\text{Sym}^2 \ker S(p)\).

The conclusion part is obviously equivalent to “(6.19) and (6.20) require \(s = 0\).”

6.3 Phase I−

Under Condition (C), the vacuum equations in phase I− require

\[
p^5 = 0.
\]

(6.22)

To show this, suppose \(p^5 \neq 0\). Then, by Condition (C), \((u, v, p^6)\) must land in \(F_{1+}\). In view of the deleted set \(\{1.41\}\), the only possibility is \(p^5 = v_{1.4} = 0, u_{1.4} \neq 0, v_5 \neq 0\) or \(u \leftrightarrow v\) exchanged case). But then the F-term equations include \(S_{5}^{5} p^5 v_5 = 0\) which is impossible since \(S_{5}^{5} \neq 0\) by (C). This proves that \(p^5 = 0\). Since \(p^5\) enters into the superpotential at
most linearly, this means that we may impose $p^5 = S_5(u, v, p^6) = 0$ without losing massless degrees of freedom.

Now let us show that the Higgs branch is compact. We consider the cases $p^{1 \ldots 4} = 0$ and $p^{1 \ldots 4} \neq 0$ separately.

$p^{1 \ldots 4} = 0$: Possibly non-zero fields are $u, v, p^6$, but their charges under $U(1)_1 \times U(1)_2$ lie strictly inside a half space of the charge lattice. Therefore, their values are bounded by the $U(1)_1 \times U(1)_2$ D-term equations.

$p^{1 \ldots 4} \neq 0$: By (C), we must have $p^6 = v_{1 \ldots 4} = 0$, $u_{1 \ldots 4} \neq 0$, $v_5 \neq 0$ (or $u \leftrightarrow v$). Possibly non-zero fields are then $p^{1 \ldots 4}$, $u_{1 \ldots 4}$, $u_5$, and their charges under the $U(1)$ subgroup $\{(z^{-1}, z^2, z)\} \subset U(1)_0 \times U(1)_3 \times U(1)_4$ are all positive $(1, 1, 4, 1 \text{ respectively})$. Thus, their values are bounded by the corresponding D-term equation.

With a little more work, we can also find what the Higgs branch is. For $p^{1 \ldots 4} = 0$, we must have $u_5 \neq 0$ and $v_5 \neq 0$ which breaks $G_0$ to the subgroup $\{(z^2, z, z)\} \cong U(1)$. Under this $U(1)$, the possibly non-zero fields $p^6, u_{1 \ldots 4}, v_{1 \ldots 4}$ all have charge 1, defining a $\mathbb{P}^8$, or a $\mathbb{P}^7$ if we take into account $S_5(u, v, p^6) = 0$ which is linear in these variables. The non-trivial F-term equations $S_1(u, v, p^6) = \cdots = S_4(u, v, p^6) = 0$ are quadratic in these variables. Thus, we have the intersection of four quadrics in $\mathbb{P}^7$ (a Calabi-Yau threefold). $\mathbb{Z}_2 = G/G_0$ acts freely on it. Let $Z$ be the quotient. $Z$ has a conifold singularity at $p^6 = v_{1 \ldots 4} = 0$, $u_{1 \ldots 4} = u_{1 \ldots 4}^*$ where $u_{1 \ldots 4}^* \neq 0$ satisfy the following equations for some $p_{*}^{1 \ldots 4} \neq 0$:

\begin{align}
S_5^{ij} u_j^* &= 0, \\
p_{*}^k S_k^{ij} u_j^* &= 0, \quad i = 1, \ldots, 4, \\
p_{*}^k S_k^{ij} u_j^* &= 0.
\end{align}

(6.23)

Up to scaling, there are ten such $(p_{*}^{1 \ldots 4}, u_{1 \ldots 4}^*)$. For $p^{1 \ldots 4} \neq 0$, we have $p^6 = v_{1 \ldots 4} = 0$, $u_{1 \ldots 4} \neq 0$, $v_5 \neq 0$, and the non-trivial F-term equations are nothing but (6.23). Thus, there are ten isolated solutions for $(p^{1 \ldots 4}, u_{1 \ldots 4})$ up to scale. The non-zero values of $v_5$ and $u_{1 \ldots 4}$ break $G_0$ to $\{(z^{-1}, 1, 1)\} \cong U(1)$ under which the remaining fields $p^{1 \ldots 4}$ and $u_5$ both have charge 1. Thus, we have $\mathbb{P}^1$ minus one point with $p^{1 \ldots 4} = 0$. The deleted point is nothing but one of the ten singular points of $Z$. In conclusion, the Higgs branch is a singular Calabi-Yau threefold $Z$ and ten rational curves rooted at ten conifold points of $Z$. The behaviour of the superpotential near the roots is

$$W \sim p(x_1 x_2 + x_3 x_4).$$

(6.24)

Indeed, the equation $dW = 0$ reads

$$x_1 x_2 + x_3 x_4 = 0, \quad p x_1 = p x_2 = p x_3 = p x_4 = 0,$$

(6.25)

and $\text{Crit}(W)$ is the union of $\{p = 0, x_1 x_2 + x_3 x_4 = 0\}$ (a conifold) and $\{p \text{ free, } x_1 = x_2 = x_3 = x_4 = 0\}$ (a line) which touch each other at the origin.

6.4 Phase II

Under Condition (C), the vacuum equations in phase II require

$$p^{1 \ldots 4} = 0.$$

(6.26)
To show this, suppose $p^{1\cdots4} \neq 0$. Since $p^{5\cdots6} \neq 0$, we have $p^{1\cdots6} \notin F_{IV}$. Then, by the analysis of phase IV, this implies under Condition (C) that $u = v = 0$, but that is forbidden. This proves that $p^{1\cdots4} = 0$. Since $p^{1\cdots4}$ enters into the superpotential at most linearly, this means that we may impose $p^{1\cdots4} = S_{1\cdots4}(u, v, p^6) = 0$ without losing massless degrees of freedom.

Let us show that the Higgs branch is compact. We consider the cases $p^5 = 0$ and $p^5 \neq 0$ separately.

$p^5 = 0$: Possibly non-zero fields are $u, v, p^6$, and their values are bounded by the D-term equations.

$p^5 \neq 0$: Then, by Condition (C), $(u, v, p^6)$ must land in $F_{IV}$. In view of the deleted set, we must have $u_5 = v_5 = 0$. Then, possibly non-zero fields are $p^5, p^6, u_{1\cdots4}, v_{1\cdots4}$, and their values are bounded by the D-term equations.

With a little more work, we can also find what the Higgs branch is. For $p^5 = 0$, we must have $p^6 \neq 0$ which breaks $G_0$ to the subgroup $\{(zw, zw, w)\} \cong U(1) \times U(1)$. Under this, the possibly non-zero fields $u_{1\cdots5}$ and $v_{1\cdots5}$ have charge $(0,1)$ and $(1,0)$, defining $\mathbb{P}^4 \times \mathbb{P}^4$. The non-trivial F-term equations $S_1(u, v, p^6) = \cdots = S_5(u, v, p^6) = 0$ are of degree $(1,1)$ and define a Calabi-Yau threefold. Let $Z'$ be its quotient by $\mathbb{Z}_2 = G/G_0$. It has a conifold singularity at $u_5 = v_5 = 0$ and $u_{1\cdots4} \neq 0$ such that

$$S_k^{ij} u_i v_j = 0, \quad k = 1, \ldots, 4,$$

$$S_5^{ij} u_i v_j = 0.$$  \hspace{1cm} (6.27)

Up to scaling, there are six such $(u, v)$. For $p^5 \neq 0$, we have $u_5 = v_5 = 0, u_{1\cdots4} \neq 0 v_{1\cdots4} \neq 0$, and the non-trivial F-term equations are nothing but (6.27). Thus, there are six isolated solutions up to scale. The non-zero values of $u_{1\cdots4}$ and $v_{1\cdots4}$ break $G_0$ to $U(1)_3$ under which the remaining fields $p^5$ and $p^6$ both have charge $-1$. Thus, we have $\mathbb{P}^4$ minus one point with $p^5 = 0$. The deleted point is nothing but one of the six singular points of $Z'$. Thus, the Higgs branch is a singular Calabi-Yau threefold $Z'$ and six rational curves rooted at six singular conifold points of $Z'$. The behaviour of the superpotential near the roots is as in (6.24).

### 6.5 Phase III

Let us show that the Higgs branch in phase III is compact under Condition (C). We consider the cases $p^{1\cdots4} = 0$ and $p^{1\cdots4} \neq 0$ separately.

$p^{1\cdots4} = 0$: In view of the deleted set, we must have $p^5 \neq 0$. On the other hand, the F-term equations require $S_5^{ij} p^5 u_5 = S_5^{ij} p^6 v_5 = 0$. Since (C) implies $S_5^{ij} \neq 0$, we must have $u_5 = v_5 = 0$. Possibly non-zero fields are thus $p^5, p^6, u_{1\cdots4}, v_{1\cdots4}$, and their values are bounded by the D-term equations.

$p^{1\cdots4} \neq 0$: Since $p^{1\cdots6} \neq 0$, we have $p^{1\cdots6} \notin F_{IV}$. Then, by the analysis in phase IV, this implies under Condition (C) that $u = v = 0$. Possibly non-zero fields are thus $p^{1\cdots6}$ and their values are bounded by the D-term equations.

### 6.6 Phase V

Under Condition (C), the vacuum equations in phase V require

(i) $p^{1\cdots4} \neq 0$,  
(ii) $p^5 = p^6 = 0$,  
(iii) $u_{1\cdots4} = 0$ or $v_{1\cdots4} = 0$. 

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Indeed, in view of the deleted set \([4.67]\), \(p^{1..4} = 0\) would imply \(p^5 \neq 0\), \(u_5 \neq 0\) and \(v_5 \neq 0\) but that is inconsistent with the F-term equations \(S_5^5 p^5 u_5 = S_5^5 p^5 v_5 = 0\) as we know \(S_5^5 \neq 0\) from (C). This establishes (i). Then, if we assume \(p^{5..6} = 0\), we have \(p^{1..6} \notin F_{1..6}\). By the analysis in phase IV, this implies under (C) that \(u = v = 0\), but that is forbidden \([4.67]\). This establishes (ii). To show (iii), suppose both \(u_{1..4}\) and \(v_{1..4}\) are non-zero. Then, since \(u_5 = v_5 = 0\) is a part of the deleted set \([4.67]\), this means that \((u, v, p^6) = 0\) \(\notin F_{1..4}\). Then, by (C), we must have \(p^{1..5} = 0\), but that is inconsistent with \(p^{1..4} \neq 0\). This proves (iii). Since \(p^5\) enters into the superpotential at most linearly, (ii) means that we may impose \(p^5 = S_5 (u, v, p^6) = 0\) without losing massless degrees of freedom.

Let us show that the Higgs branch is compact. In view of (iii) and the symmetry, we may assume \(v_{1..4} = 0\). Possibly non-zero fields are then \(p^{1..4}, u_{1..4}, u_5, v_5\). Their values are bounded by the D-term equation for \(U(1) \cong \{(z^{-1}, z, z)\} \subset G_0\) (see the analysis in phase I.).

We can also determine what the Higgs branch is. For \(u_{1..4} = v_{1..4} = 0\), the F-term equations are all satisfied (recall \(p^5 = p^6 = 0\)). In view of the deleted set \([4.67]\), we must have \(u_5 \neq 0\) and \(v_5 \neq 0\) which break \(G_0\) to the \(U(1)\) subgroup \(\{(z, z, z)\}\). Under this, the remaining fields \(p^{1..4}\) have charge \(-2\), defining the weighted projective space \(\mathbb{P}^3_{[2222]}\). For \(u_{1..4} = 0\) and \(v_{1..4} = 0\), we have \(v_5 \neq 0\), and non-trivial F-term equations are the same as \((6.28)\). Thus, we have ten isolated pairs for \((p^{1..4}, u_{1..4})\) up to scale. The non-zero values of \(p^{1..4}\) and \(v_5\) break \(G_0\) to \(\{1, z, z\} \cong U(1)\) under which the remaining two fields \(u_{1..4}\) and \(u_5\) have charge 1 and 2 respectively. Thus we have the weighted projective line (or the teardrop) \(\mathbb{P}^1_{[12]}\) minus the \(Z_2\) point \(u_{1..4} = 0\). The deleted point is nothing but a point of \(\mathbb{P}^3_{[2222]} / Z_2\). Thus, the Higgs branch is the union of a \(\mathbb{P}^3_{[2222]} / Z_2\) and ten teardrops.

6.7 The Moduli Space

Transformations of variables that commute with the gauge symmetry are

\[
\begin{align*}
  x_1 &\to a^i_1 x_j + c_i x_5 p^6, & x_5 &\to a^5_5 x_5, \\
  p^i &\to b^i_j p^j, & p^5 &\to b^5_j p^j + b_5 p^6, & p^6 &\to b^6_6 p^6,
\end{align*}
\]

where \((a^i_j), (b^i_j), (b^5_5), (b^6_6)\) are invertible. They induce the following transformations of the space \(S\) of coefficients \((S^i_j, \ldots)\)

\[
\begin{align*}
  S^i_j &\to S^m_i a^i_m b^j_k, \\
  S^5_j &\to S^5_m a^5_m b^j_5, \\
  S^i_k &\to S^m_i a^5_m b^j_k b^j_6 + S^5_m a^5_m b^j_k + S^m_i b^j_k c_i, \\
  S^5_5 &\to S^5_5 (a^5_5 b^j_5 b^j_6 + 2 S^5 a^5_5 b^j_5 c_j), \\
  S^5_5 &\to S^i_5 (a^5_5 b^j_5 b^j_6)^2 + S^5_5 (a^5_5 b^j_5 b^j_6 + 2 S^i_5 a^5_5 b^j_5 c_j + 2 S^5 a^5_5 b^j_5 b^j_6 + 2 S^i_5 a^5_5 b^j_5 b^j_6)
\end{align*}
\]

The transformations from the complexified gauge group,

\[
\begin{align*}
  a^i_j &= \lambda_1 b^i_j, & a^5_j &= \lambda_2, \\
  b^i_j &= \lambda_1^{-2} b^j_j, & b^5_5 &= \lambda^{-1}_1 \lambda_2^{-1}, & b_j &= 0, & b^6_6 &= \lambda_1 \lambda_2^{-1}, & c_j &= 0,
\end{align*}
\]

form the subgroup that acts trivially on \(S\). Let \(G\) be the effective group of transformations, and let \(S_{(C)} \subset S\) be the subset consisting of \((S^i_j, \ldots)\) that satisfies Condition (C). Then, the
complex moduli space is the quotient
\[ \mathcal{M}_C = S(C)/G. \] (6.30)

Note that the dimension of \( S \) is 40 + 4 + 1 + 16 + 1 + 4 + 1 + 4 − 2 = 43 − 2 = 41. Thus, \( \mathcal{M}_C \) has dimension 65 − 41 = 24, in agreement with the number of complex moduli of \( X \) and \( \tilde{Y} \).

We expect that \( G \) acts on \( S(C) \) without continuous stabilizer. Indeed, a point with continuous stabilizer would correspond to a continuous symmetry of either \( X \) or \( \tilde{Y} \), but that would be impossible since both \( X \) and \( \tilde{Y} \) have \( h^{2,0} = 0 \). Therefore, the quotient (6.30) should be a good one. It would be nicer to show this mathematically.

7 Mirror Symmetry

Mirror symmetry for Calabi-Yaus which are not complete intersections in toric ambient spaces is still mostly an open problem. However, since phase I\(^+\) is a free \( \mathbb{Z}_2 \)-quotient of a complete intersection in a toric variety it is possible to work with well-established methods of toric geometry and mirror symmetry. In the following we will recompute the topological data of this Calabi-Yau, determine its mirror and the Picard-Fuchs operator and compute the Yukawa couplings and Gromov-Witten invariants. Once we have the Picard-Fuchs operator we are also able to compute the Gromov-Witten invariants in phase IV, up to normalization.

7.1 Phase I\(^+\)

7.1.1 Toric analysis and topological data

The present example is of particular interest not only because it has two geometric phases but also because we are able to compute the mirror in phase I\(^+\) using toric geometry. Recall that we denote the two-parameter model by \( X \) and the three-parameter model by \( \tilde{X} \). Their respective mirrors are denoted by \( X^\vee \) and \( \tilde{X}^\vee \). The analysis relies heavily on the machinery of toric geometry and the toric mirror construction by Batyrev and Borisov \[30,31\]. The main references for toric mirror symmetry of complete intersection Calabi-Yaus are \[32,33\] and the book by Cox and Katz \[34\]. A nice exposition focusing on complete intersections can also be found in \[35\], mirror symmetry for free quotients has been discussed for instance in \[36\].

The complexity of the calculations requires the use of specialized computer programs, most importantly the toric geometry package PALP \[37,38\].

As can be seen already from (3.7) and (3.9) the relation to a toric three-parameter model stems from the fact that the maximal torus of \( O(2) \) is \( SO(2) \simeq U(1) \). In order to reveal some more properties of the toric ambient space, we make a change of basis. Starting from (3.7), we choose the following linear combinations of the charge vectors:

\[ \{Q_{U(1)_1}, Q_{U(1)_2}, Q_{SO(2)}\} \rightarrow \{Q_{U(1)_1} + Q_{U(1)_2} + Q_{SO(2)}, Q_{U(1)_2}, Q_{U(1)_1} + Q_{SO(2)}\}. \]

The first of the new charge vectors shows that there is a \( \mathbb{Z}_2 \) fixed point since all charges are either 0 or 2. Modding out the \( \mathbb{Z}_2 \) simply means dividing the charge vector by two. Further subtracting this new vector from the second and the third vector we arrive at the following:

| \( p^1 \ldots p^4 \) | \( p^5 \) | \( p^6 \) | \( u_{1\ldots4} \) | \( u_5 \) | \( v_{1\ldots4} \) | \( v_5 \) | \( FI \) |
|---|---|---|---|---|---|---|---|
| −1 | −1 | 0 | 1 | 1 | 0 | 0 | \((\zeta_1 + \zeta_2)/2\) |
| −2 | −1 | 1 | 1 | 0 | 1 | 0 | \(\zeta_1\) |
| −1 | −1 | 0 | 0 | 0 | 1 | 1 | \((\zeta_1 + \zeta_2)/2\) |

(7.1)
Let us focus on phase I$_+$ where $p^1 = \ldots = p^5 = 0$. The charges of the remaining fields are all positive and define a toric ambient space given by $\mathbb{P}^8$ with two $\mathbb{P}^4$s blown up in orthogonal directions. This geometry is smooth. A codimension 5 complete intersection as given by the F-terms of the GLSM in this phase is a three-parameter Calabi-Yau with the three parameters corresponding to the volumes of $\mathbb{P}^8$ and the two $\mathbb{P}^4$s. As one can read off from the FI-parameters, the volumes of the two $\mathbb{P}^4$s get identified. There is a $\mathbb{Z}_2$ that exchanges the two $\mathbb{P}^4$s. This confirms again that $h^{1,1}(X) = 2$ and that phase I$_+$ is free $\mathbb{Z}_2$-quotient of a complete intersection of codimension five of in the ambient space defined by (7.1).

In the following we will also require the Kähler cone, and its dual, the Mori cone, which is given by a particular basis. We give this for later reference:

\begin{equation}
\begin{array}{ccccccc}
p^1 & p^5 & p^6 & u_{1\ldots 4} & u_5 & v_{1\ldots 4} & v_5 \\
-1 & 1 & 0 & 1 & 0 & 0 & -1 \\
0 & -1 & -1 & 0 & 1 & 0 & 1 \\
-1 & 1 & 0 & 0 & -1 & 1 & 0 \\
\end{array}
\begin{array}{c}
FI \\
(\zeta_1 - \zeta_2)/2 \\
\zeta_2 \\
(\zeta_1 - \zeta_2)/2 \\
\end{array}
\end{equation}

The phase diagram of the three-parameter model (with arbitrary FI parameters), which coincides with the secondary fan of the associated toric variety, is depicted on the left side of figure 4. In [39] this toric model associated with a non-Abelian GLSM was called Cartan model. Identifying the FI parameters as in the charge table amounts to projecting into the $x-z$-plane. While the $p$-fields are already in the plane the $u$- and $v$-fields combine into the fundamentals $x$ along the dashed blue lines. From this picture one can also clearly see the appearance of the extra phase boundary that separates phases I$_-$ and I$_+$: the dashed blue lines connecting the charge vectors of $u_{1\ldots 4}$ with $v_5$ (resp. $v_{1\ldots 4}$ with $u_5$) project onto the extra phase boundary. This combines into the information encoded in the non-Abelian D-term.

Given the data of the three-parameter model $\tilde{X}$ it is possible to determine the mirror $\tilde{X}^\vee$. 

Figure 4: Phases of the two-parameter model from a three-parameter complete intersection.
Mirror symmetry for complete intersections in toric varieties is connected to the following data associated to lattice polytopes in dual integer lattices, called the M- and the N-lattice:

\[ \Delta = \Delta_1 + \ldots + \Delta_r \quad \Delta^o = \langle \nabla_1, \ldots, \nabla_r \rangle_{\text{conv}} \]

\[ \nabla^o = \langle \Delta_1, \ldots, \Delta_r \rangle_{\text{conv}} \quad \nabla = \nabla_1 + \ldots + \nabla_r \]

(7.3)

Here \( r \) is the codimension of the Calabi-Yau and the defining equations \( f_i = 0 \) are sections of \( \mathcal{O}(\Delta_i) \). The decomposition of the M-lattice polytope \( \Delta \subset M_\mathbb{R} \) into a Minkowski sum \( \Delta = \Delta_1 + \ldots + \Delta_r \) is dual to a nef (numerically effective) partition of the vertices of a reflexive polytope \( \nabla \subset N_\mathbb{R} \) such that the convex hulls \( \langle \nabla_i \rangle_{\text{conv}} \) of the respective vertices and \( 0 \in N \) only intersect at the origin. The hypersurface equations are then given by

\[ f_m = \sum_{w_k \in \Delta_m} c_k^m \prod_{n=1}^r \prod_{\nu_i \in \nabla_n} x_i^{(\nu_i, w_k) + \delta_{nm}}. \]

(7.4)

Mirror symmetry is realized by exchanging the M- and N-lattices. The software package PALP provides the routine \texttt{nef.x} which computes the polytopes and the nef partitions for a complete intersection Calabi-Yau from the weight matrix given by the three \( U(1) \)-charges of the GLSM. The resulting M-lattice polytope which describes the ambient space has 41 vertices and 12740 points, the N-lattice polytope has 11 vertices plus the interior point at the origin. Explicitly, the vertices of \( \nabla \in N \) are:

\[
\begin{array}{cccccccccccc}
\nu_1 & \nu_2 & \nu_3 & \nu_4 & \nu_5 & \nu_6 & \nu_7 & \nu_8 & \nu_9 & \nu_{10} & \nu_{11} \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
-1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & -1 \\
-1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 \\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\
\end{array}
\]

(7.5)

The columns above the horizontal line are the vertices \( \nu_i \in \mathbb{Z}^8 \) of the polytope in the eight-dimensional ambient space. The three lines below the horizontal line denote the linear relations \( \sum a_i \nu_i = 0 \) between the vertices. This coincides with the basis \( \{\mathbb{Z}\} \) of the \( U(1) \)s in the GLSM. Each of the eight-dimensional vertices is associated to a toric divisor \( D_i \). The information about the complete intersection is encoded in the nef partition. For this example there are 241, 16 of which cannot be related through symmetries of the polytope. The one we are looking for is

\[ E = \{(D_1, D_5), (D_3, D_6), (D_4, D_7), (D_9, D_{11}), (D_2, D_8, D_{10})\}. \]

(7.6)

The corresponding hypersurfaces have degrees \{\((2, 1, 1), (2, 1, 1), (2, 1, 1), (2, 1, 1), (1, 1, 1)\)\} which is exactly what we have for the complete intersection in phase I_+.
We can read off the Laurent polynomials defining the mirror from the vertices of the N-lattice polytope via $f_r = \sum_{E_r} a_i x^{E_r}$, where $E_r$ are the elements of the nef-partition, each of which also contains the origin. Therefore we get for our example:

$$
\begin{align*}
    f_1 &= 1 + \alpha_1 \frac{x_7}{x_1x_2x_3x_8} + \alpha_5 x_4 \\
    f_2 &= 1 + \alpha_3 x_2 + \alpha_6 x_5 \\
    f_3 &= 1 + \alpha_4 x_3 + \alpha_7 x_6 \\
    f_4 &= 1 + \alpha_9 x_1 + \alpha_{11} \frac{1}{x_4x_5x_6x_7} \\
    f_5 &= 1 + \alpha_2 x_8 + \alpha_8 \frac{x_8}{x_7^2} + \alpha_{10} x_7,
\end{align*}
$$

(7.7)

where the constants $\alpha_i$ redundantly encode the complex structure parameters on the mirror. For the mirror symmetry calculations it is not necessary to match the variables $x_i$ with $u_l, v_l, p^6$ but it is useful to confirm that one can recover the equations for the complete intersection that was obtained from the GLSM. This can be done by calculating the dual of the Gorenstein cone associated to the nef-partition above. In the M-lattice, four elements of the nef partition have 25 points and one has nine points. This corresponds to the number of monomials of the defining equation of the complete intersection in phase I$. Computing the hypersurface equations using the polytope data as in (7.4) one can indeed recover the defining equations of the complete intersection.

Now that we have the toric data of the three-parameter model, we can determine its topological characteristics and the intersection ring and compare with the results of section 4.1. The Mori generators encode the linear relations in the intersection ring and can be determined by an algorithm which requires a maximal star triangulation of the N-lattice polytope. In our example the only non-vertex of the polytope is the origin. Therefore there is only one such triangulation. The Mori generators of a complete intersection of codimension $r$ are of the form $l^{(a)} = \langle l^{(a)}_{0,1}, \dots, l^{(a)}_{0,r}; l^{(a)}_{1}, \dots, l^{(a)}_{n} \rangle$ with $a = 1, \dots, h^{1,1}(\tilde{X})$ and $\sum_{m=1}^{r} l^{(a)}_{0,m} + \sum_{i=1}^{n} l^{(a)}_{i} = 0$. Using the $\text{mori\_x}$ routine of PALP and the information about the degrees of the complete intersections one gets the following result for the generators of the Mori cone:

$$
\begin{align*}
    l^{(1)} &= ( -1, -1, -1, -1, 0; 0, 1, 0, 0, 1, 1, 1, -1, 0, 0, 1 ) \\
    l^{(2)} &= ( 0, 0, 0, 0, -1; 0, -1, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0 ) \\
    l^{(3)} &= ( -1, -1, -1, -1, 0; 1, 1, 1, 1, 0, 0, 0, 0, 0, 1, -1, 0 )
\end{align*}
$$

(7.8)

The entries to the right of the semicolon coincide with the basis $\{1, 2\}$. The entries to the left of the semicolon encode the hypersurface degrees of the complete intersection in the given basis.

The intersection ring of the ambient variety has the form $\mathbb{Z}[D_1, \ldots, D_n]/\langle I_{\text{lin}} + I_{\text{non-lin}} \rangle$, where the linear relations $I_{\text{lin}}$ can be read off from the Mori generators and the non-linear relations $I_{\text{non-lin}}$ are encoded in the Stanley-Reisner ideal. By adjunction, one gets the intersection ring of the complete intersection by modding out by the hypersurface ideals encoded in the nef partition which we denote by $I_{\text{CY}}$. For the present example the linear relations are

$$
\begin{align*}
    I_{\text{lin}} &= \{D_1 - D_9, D_2 - D_{11} + D_{10}, D_3 - D_9, D_4 - D_9, D_5 - D_{11}, D_6 - D_{11}, \\
    &\quad \quad D_7 - D_{11}, D_8 + D_{11} - D_9 - D_{10} \}.
\end{align*}
$$

(7.9)
The Stanley-Reisner ideal can be obtained by using the PALP program \texttt{mori.x}:

\[ I_{SR} = \{ D_8 D_{10}, D_5 D_6 D_7 D_{11}, D_2 D_5 D_6 D_7 D_{11}, D_1 D_3 D_4 D_8 D_9, D_1 D_2 D_3 D_4 D_9 \}. \]  
(7.10)

The ideal of the complete intersection is

\[ I_{CY} = \prod_r \sum_{i \in E_r} D_{i,r} = (D_1 + D_5)(D_3 + D_6)(D_4 + D_7)(D_9 + D_{11})(D_2 + D_8 + D_{10}). \]  
(7.11)

We choose the basis

\[ J_1 = D_9 \quad J_2 = D_{10} - D_9 \quad J_3 = D_{11}. \]  
(7.12)

Given this data, we obtain the following triple intersection numbers of the three-parameter Calabi-Yau $\tilde{X}$:

\[ \{ J_1^3 = J_2^3 = J_3^3, J_1 J_2^2 = J_2 J_3^2 = 11, J_1 J_2^2 J_3 = 10, J_1 J_2 J_3 = 15, J_1 J_2 J_3 = 14, J_2^3 = 16 \}. \]  
(7.13)

This coincides with (1.8) computed in section 4.11 under the identification $x = J_1, y = J_3, z = J_2$. The Chern class of the complete intersection can be computed using the formula

\[ c(\tilde{X}) = \prod_{i=1}^n (1 + D_i) / \prod_{m=1}^r (1 + \sum_{j \in E_r} D_{j,m}). \]  
(7.14)

This indeed yields the results for second Chern class and the Euler number of $\tilde{X}$ we have already computed. The calculation of the topological characteristics of $X$ proceeds exactly like in section 4.11 and we will not repeat it here. In the following we will only slightly alter our notation and write $H_0 \equiv \tilde{J}_1 = J_1 - J_3$, $H_2 \equiv \tilde{J}_2$ so that we have

\[ \tilde{J}_1^3 = 35, \quad \tilde{J}_1^2 \tilde{J}_2 = 25, \quad \tilde{J}_1 \tilde{J}_2^2 = 15, \quad J_2^3 = 8, \]  
(7.15)

and

\[ c_2 \cdot \tilde{J}_1 = 50 \quad c_2 \cdot J_2 = 32. \]  
(7.16)

### 7.1.2 Picard-Fuchs equations and discriminant

For the mirror calculation we have to go to the large complex structure limit. This information is encoded in the Mori cone. The complex structure moduli on the mirror at the large complex structure limit are $z_a = (-1)^l \prod_i \prod_m \prod_l a_i^{(a)}$, where we insert (7.8). Therefore we identify

\[ z_1 = \frac{\alpha_2 \alpha_3 \alpha_6 \alpha_7 \alpha_{11}}{\alpha_8}, \quad z_2 = -\frac{\alpha_8 \alpha_{10}}{\alpha_2}, \quad z_3 = \frac{\alpha_1 \alpha_2 \alpha_3 \alpha_4 \alpha_9}{\alpha_{10}}. \]  
(7.17)

For this choice of moduli, modding out by the freely acting $\mathbb{Z}_2$ amounts to setting $z_1 = z_3$.

Using the toric information we can explicitly compute the fundamental period $\omega_0(z_1, z_2, z_3)$ of $\tilde{X}^\vee$. The Picard-Fuchs operators of $X^\vee$ are then determined by making an ansatz for a differential operator and requiring that the fundamental period is annihilated. Once the Picard-Fuchs system is identified one can employ the mirror symmetry machinery to determine the B-model Yukawa couplings and the mirror map. With help of the intersection data, we then compute the normalized Yukawa couplings in phase $I_4$. 

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Given the Laurent polynomials \((7.21)\) of the complete intersection of the mirror and the moduli at the large complex structure limit \((7.17)\), the fundamental period of \(X^\vee\) is given by the residue integral

\[
\varpi_0(X^\vee) = \frac{1}{(2\pi i)^{11}} \int_\gamma \prod_i \frac{dx_i}{x_i^{a_i} f_1 f_2 f_3 f_4 f_5},
\]

where \(\gamma\) is a suitably chosen cycle. The result is

\[
\varpi_0(X^\vee) = \sum_{a,b,c} (a + c) 4 \begin{pmatrix} b & a - b + c \\ a & -a + b \end{pmatrix} z_1^a z_2^b z_3^c.
\]

The sum is such that \(a - b + c \geq 0\), \(-a + b \geq 0\) and \(b - c \geq 0\). The calculation is very similar to the mirror symmetry calculation of \([10]\) and we give details in appendix A.1. The expression is symmetric under the exchange of \(z_1\) and \(z_3\). The fundamental period of the two-parameter model \(X^\vee\) is then obtained by setting \(z_3 = z_1\). The first few terms in the expansion are

\[
\varpi_0(X^\vee) = 1 + 2z_1 z_2 + 16z_1^2 z_2^2 + 34z_1^3 z_2^3 + \ldots.
\]

Next we make an ansatz for the Picard-Fuchs system and impose the condition that the Picard-Fuchs operators annihilate \(\varpi_0(X^\vee)\). As expected for a two-parameter model we find one degree two and one degree three operator

\[
\begin{align*}
\mathcal{L}_1 &= 5\theta_1^2 - 19\theta_2\theta_1 + 20\theta_2^2 + z_1 z_2 \left(-12\theta_1^2 + 12\theta_1\theta_2 - 12\theta_2 + 12\theta_1\right) \\
&+ z_1 \left(5\theta_1^2 - 4\theta_2\theta_1 + 5\theta_1 - 12\theta_2 - 10\theta_2\right) + z_2 \left(-\theta_1^2 - 19\theta_2\theta_1 + 20\theta_1 + 20\theta_2^2 + 20\theta_2\right) \\
&+ z_1 z_2 \left(-11\theta_1^2 + 8\theta_2\theta_1 - 11\theta_1 - 24\theta_2 - 22\theta_2 - 12\right), \\
\mathcal{L}_2 &= 3\theta_1^2 - 13\theta_2\theta_1^2 + 20\theta_2^2 \theta_1 - 10\theta_2^3 \\
&+ z_1 \left(3\theta_1^2 - 4\theta_2\theta_1^2 + 6\theta_1^2 - 4\theta_2^2 \theta_1 - 10\theta_2\theta_1 + 3\theta_1 - 4\theta_2 - 6\theta_2\right) \\
&+ z_2 \left(-10\theta_1^2 + 20\theta_1\theta_2^2 - 20\theta_2^2 - 10\theta_2\theta_2 + 30\theta_1 \theta_2 - 10\theta_2 - 10\theta_2 + 10\theta_1\right) \\
&+ z_1 z_2 \left(4\theta_2\theta_1^2 + 4\theta_1^2 - 4\theta_2\theta_1 + 4\theta_1 - 4\theta_2 - 4\theta_2\right),
\end{align*}
\]

where \(\theta_i = \frac{\partial}{\partial z_i}\). The power series expansions of the remaining periods can be easily determined from the Picard-Fuchs operators. There are two linearly independent log-solutions, two \(\log^2\) solutions and one \(\log^3\) solution. The first few terms of the expansions are given in appendix A.2.

Given the two Picard-Fuchs operators we can also calculate the Gauss-Manin system and the monodromy matrix. For a suitable basis \(\Pi\) of \(H^3(X^\vee)\) the Gauss-Manin system reads

\[
\theta_i \Pi = M_{i\Pi} \quad i = 1, 2.
\]

We choose \(\Pi = (\int \Omega, \theta_1 \int \Omega, \theta_2 \int \Omega, \theta_1^2 \int \Omega, \theta_1 \theta_2 \int \Omega, \theta_1^3 \int \Omega)\), where \(\Omega\) is the holomorphic three-form of \(X^\vee\). The matrices \(M_{1,2}\) can be obtained from the Picard-Fuchs equations and their derivatives. Evaluated at \(z_1 = 0\) and \(z_2 = 0\), respectively, and transformed into their Jordan normal forms, one obtains \((1/(2\pi i))\) times the logarithms of the monodromy matrices \(T = e^{2\pi i M}\), where \(M = a_i M_i\) for \(a_i > 0\) \([40]\). Note that the relation between the connection

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matrix $M$ and the monodromy matrix $T$ only works if the eigenvalues of $M_i(0)$ are zero. In phase $I_+$ we get for the Jordan normal forms:

$$M_1|_{z_1=0} \sim M_2|_{z_2=0} \sim \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$  \hspace{1cm} (7.24)

This shows that the monodromy in phase $I_+$ is maximally unipotent.

### 7.1.3 Yukawa couplings and Gromov-Witten invariants

Now we have all the ingredients to compute the Gromov-Witten invariants of $X$. There are several methods available. We will choose the following, as described for instance in section 5.6 of [34]. We first compute the B-model Yukawa couplings $\kappa_{z_1 z_2 z_3} = \int_X \Omega \wedge \nabla_{\theta_i} \nabla_{\theta_j} \nabla_{\theta_k} \Omega$ up to normalization. Here $\nabla$ denotes the covariant derivative with respect to the Gauss-Manin connection. A further ingredient is the mirror map. With the two log-solutions $\varpi_{1,1}$ and $\varpi_{1,2}$ (cf. appendix A.2), it is

$$t_1(z_1, z_2) = \frac{\varpi_{1,1}}{\varpi_0}, \quad t_2(z_1, z_2) = \frac{\varpi_{1,2}}{\varpi_0}. \hspace{1cm} (7.25)$$

Using the inverse mirror map $q_i = e^{2\pi i t_i(z)} = z_i + \ldots$ we can extract from that the normalized Yukawa couplings in the A-model as follows

$$\kappa_{t_i, t_j, t_k} = \frac{(2\pi i)^3}{\varpi_0^2} \left( \frac{t_i \partial z_l}{z_l \partial t_i} \right) \left( \frac{t_j \partial z_m}{z_m \partial t_j} \right) \left( \frac{t_k \partial z_n}{z_n \partial t_k} \right) \kappa_{z_l z_m z_n}. \hspace{1cm} (7.26)$$

From this, we can read off the A-model Yukawa couplings which have the following expansion in terms of the Gromov-Witten invariants $n_\beta$:

$$\kappa_{t_i, t_j, t_k} = \int_X J_{i} \cdot J_{j} \cdot J_{k} + \sum_{\beta \neq 0} n_\beta \frac{q^\beta}{1 - q^\beta} \int_\beta J_i \int_\beta J_j \int_\beta J_k, \hspace{1cm} (7.27)$$

where $J_i, J_j, J_k \in H^2(X, \mathbb{Z})$ and $\beta$ is the homology class of a rational curve in $X$. For a particular Yukawa coupling in the two-parameter case, say $\kappa_{t_1 t_2 t_2}$, this looks as follows. Choosing $\beta = aJ_1 + bJ_2$ one gets

$$\kappa_{t_1 t_2 t_2} = \int_X J_{i} \cdot J_{j} \cdot J_{j} + \sum_{(a,b) \neq (0,0)} n_{a,b} \frac{a^2 b q_1^a q_2^b}{1 - q_1^a q_2^b}. \hspace{1cm} (7.28)$$

The first term is the triple intersection number of divisors Poincaré dual to the $J_i$, which, by abuse of notation, we also call $J_i$. Fixing one of the intersection numbers corresponds to choosing the normalization of (one of) the Yukawa couplings. This information is not contained in the solutions of the Picard-Fuchs equations but can be fixed by the topological data of $X$. 

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The first step is to use the Picard-Fuchs equations and their derivatives to express the four Yukawa couplings in the B-model in terms of a single one. One finds for example

\[ \kappa_{\gamma_1,\gamma_2} = \frac{(32(5z_2^2 + 6z_2 + 1)z_2^3 - 4(35z_2^2 + 172z_2 - 23)z_2^3 + (-20z_2^2 + 185z_2 + 85)z_1 + 5(7z_2 + 5))}{(z_2 + 1)(64(z_2 + 1)z_2^3 - 20(z_2 - 9)z_2^3 - 20(z_2 - 8)z_2 + 35)}\kappa_{2z_1,\gamma_1}. \]  

(7.29)

Using the fact that the Yukawa couplings and their derivatives satisfy linear differential equations [32, 34], we obtain the remaining Yukawa coupling by using

\[ \int X^\gamma \Omega \wedge \nabla^4_{\gamma_1} \Omega = 2\theta_1 \kappa_{2z_1,\gamma_1}. \]  

(7.30)

Once again using the Picard-Fuchs equations we get the Yukawa coupling up to an integration constant \( c \)

\[ \kappa_{2z_1,\gamma_1} = \frac{c(-4z_1(z_1(-35z_2 + 16z_2 + 1) + 45) - 5(z_2 - 8)) - 35)}{(32z_1(8z_1 + 1)z_2 - 1)(z_1(11z_2 + z_1 (z_1 + 1)^2 - z_2(z_2 + 14) + 3) + 3) + 1}. \]  

(7.31)

The other Yukawa couplings are

\[ \kappa_{\gamma_1,\gamma_2} = \frac{c(-32(z_2 + 1)(5z_2 + 3z_2 + 172z_2 - 23)z_2^3 + 5(z_2(4z_2 - 37) - 17)z_1 - 5(7z_2 + 5))}{(z_2 + 1)(32z_1(8z_1 + 1)z_2 - 1)(z_1(11z_2 + z_1 (z_1 + 1)^2 - z_2(z_2 + 14) + 3) + 3) + 1}. \]  

\[ \kappa_{\gamma_1,\gamma_2} = \frac{c(-16z_1 + 5)(z_1 + 1)^2 + 20z_1((7 - 8z_1)z_1 + 1)z_2^3 + 5(z_1(16z_2 + 73) - 46) - 7)z_2}{(z_2 + 1)(32z_1(8z_1 + 1)z_2 - 1)(z_1(11z_2 + z_1 (z_1 + 1)^2 - z_2(z_2 + 14) + 3) + 3) + 1}. \]  

\[ \kappa_{\gamma_2,\gamma_2} = \frac{c(-8(z_1 + 1)^3 + 20z_1((7 - 8z_1)z_1 + 1)z_2^3 + 5(z_1(56z_2^2 + 38z_1 - 54) - 7)z_2^2 - 5(z_1(2z_1(8z_1 + 7) + 51) + 8)z_2)}{(z_2 + 1)^2(32z_1(8z_1 + 1)z_2 - 1)(z_1(11z_2 + z_1 (z_1 + 1)^2 - z_2(z_2 + 14) + 3) + 3) + 1}. \]  

(7.32)

The Yukawa couplings are of the form \( \frac{p(z)}{q(z)} \), where \( p, q \) are polynomials in \( z_1, z_2 \) and \( \Delta \) is the discriminant

\[ \Delta = (z_2 + 1)(32z_1(8z_1 + 1)z_2 - 1)(z_1(11z_2 + z_1 (z_1 + 1)^2 - z_2(z_2 + 14) + 3) + 3) + 1. \]  

(7.33)

As expected from the analysis of the GLSM it has three components. They match with the Coulomb branch analysis. The first factor \(-z_2 = 1\) obviously corresponds to the mixed branch with \( t_2 = 0 \). Using the parametric expressions [53] from the Coulomb branch analysis and the identification \( z_1 = e^{-\frac{t_1-t_2}{2}}, z_2 = -e^{-t_2} \) we can identify the second factor of (7.33) with branch (i) and the third factor with branch (ii) as discussed in section [41].

The mirror map is easily computed from the periods. The calculation of its inverse requires inverting series in two variables. This is efficiently implemented in the Mathematica package INSTANTON [11]. Computing the A-model Yukawa couplings and matching the classical term with the triple intersection numbers we find that the integration constant \( c \) has to be set to \( c = 1 \). Since we only need one triple intersection number for the normalization, the appearance of the other three is a non-trivial check. The instanton numbers for low degrees
This completes our discussion of phase $I_+$.  

\subsection*{7.2 Phase IV}

Phase IV is harder to analyze since the powerful machinery of toric geometry is no longer at our disposal for this determinantal variety. Since we have the Picard-Fuchs operator we can still obtain a lot of information about this phase.

At first, we can confirm that phase IV is indeed geometric in the sense that the limiting point in the moduli space is a point of maximally unipotent monodromy. In phase $I_+$ we have chosen the coordinates $z_1, z_2$. The coordinates in the other phases can be read off from the the phase diagram in figure 2 by simply transforming from one coordinate patch to the other. Up to a numerical factors $\rho, \sigma$, we have to make the following change of coordinates to get to phase IV:

$$z_{1,IV} = \frac{\sigma}{z_1}, \quad z_{2,IV} = \frac{\rho}{z_2}. \quad (7.35)$$

In order to compute the Jordan forms of the Gauss-Manin connections in the other phases, one can either transform the Picard-Fuchs operators or the connection matrices themselves. After rescaling the holomorphic threeform $\Omega \rightarrow z_1 z_2 \Omega$, one finds

$$M_1|_{z_{1,IV}=0} \sim \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad M_2|_{z_{2,IV}=0} \sim \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (7.36)$$

This confirms that the monodromy is maximally unipotent in phase IV.

In the other phases relating the Gauss-Manin connection to the monodromy is not as simple, because in some cases the difference between eigenvalues is an integer. While a transformation to zero eigenvalues always exists \cite{10,12} it is hard to find in practice. In these cases one can also check that the solutions of the Picard-Fuchs equations are not as expected for a large complex structure limit. Therefore we conclude that only $z_1 = z_2 = 0$ and $z_{1,IV} = z_{2,IV} = 0$ correspond to large complex structure points and all the other phases are of hybrid type. For completeness, we list the Jordan normal forms of the connection matrices of the other phases in appendix \ref{app3}. 

\begin{center}
\begin{tabular}{|c|c|c|c|c|c|}
\hline
$d_1$ & $d_2$ & 0 & 1 & 2 & 3 & 4 & 5 \\
\hline
0 & - & 3 & 0 & 0 & 0 & 0 & 0 \\
1 & 10 & 40 & 0 & 0 & 0 & 0 \\
2 & 0 & 185 & 140 & 0 & 0 & 0 \\
3 & 0 & 45 & 1150 & 280 & 0 & 0 \\
4 & 0 & -20 & 3210 & 10875 & 1260 & 0 \\
5 & 0 & 3 & 640 & 62428 & 80912 & 4592 \\
\hline
\end{tabular}
\end{center}
Using (7.35) the Picard-Fuchs operators in phase IV are

\[
\mathcal{L}_{1}^{IV} = -120 \rho^2 \sigma^2 + 120 \rho \sigma \rho^2 + z_2 (5 \theta^2 - 40 \rho^2 \sigma \theta_1 + \sigma \theta_1 - 12 \theta_2 \sigma + 18 \theta_2 \sigma - 6 \sigma) \\
+ z_1 z_2^2 (5 \theta^2_1 - 19 \theta_2 \theta_1 - 9 \theta_1 + 20 \theta^2_2 + 21 \theta_2 + 6) \\
+ z_2 (-11 \rho \sigma \theta^2_1 + 86 \rho \sigma \theta_1 - 3 \rho \sigma \theta_1 - 24 \theta^2_2 \rho \sigma - 18 \theta_2 \rho \sigma - 6 \rho \sigma) \\
+ z_1 z_2 (-\rho \theta^2_2 - 19 \theta_2 \rho \theta_1 - \rho \theta_1 + 20 \theta^2_2 \rho + \theta_2 \rho) \\
\]

\[
\mathcal{L}_{2}^{IV} = -4 \rho \theta \rho \sigma \theta^2_1 + 4 \rho^2 \rho \sigma \theta_1 + z_1 (10 \rho \theta^2_3 - 20 \rho_1 \rho^2_2 - 10 \rho^2_2 \rho \theta_2 + 10 \rho_1 \rho \theta_2) \\
+ z_2 (-3 \sigma \theta^2_1 + 4 \theta_2 \sigma \theta^3_1 + \sigma \theta^2_1 + 4 \theta^2_3 \sigma \theta_1 + 6 \theta_2 \sigma \theta_1 + 2 \sigma \theta_1) \\
+ z_1 z_2 (-3 \theta^2_2 + 13 \theta_2 \theta^2_1 + 4 \theta^2_1 - 20 \theta^2_2 \theta_1 - 14 \theta_2 \theta_1 - 3 \theta_1 + 10 \theta^3_2 + 10 \theta^2_2 + 3 \theta_2). \\
(7.37)
\]

For the sake of readability we have written \( z_{1,IV} \equiv z_1, \theta_{1,IV} \equiv \theta_1 \). Up to order 3, the expansion of the fundamental period is

\[
\omega_0(Y) = 1 - \frac{z_2}{2 \rho} + \frac{z_1 z_2^2}{2 \rho \sigma} + \frac{3 z_2^3}{8 \rho^2} - \frac{21 z_1 z_2^2}{16 \rho^2 \sigma} - \frac{5 z_3^2}{16 \rho^3} + \ldots \\
(7.38)
\]

We can also compute the Yukawa couplings, depending on the parameters \( \rho, \sigma \), and including an integration constant \( c \). The result is

\[
\kappa_{1,1}^{IV} = \frac{c (4 \rho (-16 \rho^2 + 35 \rho \sigma + 5 \sigma^2) - z_2 (64 \rho^4 + 180 \rho^2 \sigma z_1 + 160 \rho \sigma z_1^2 + 35 \sigma^2))}{(32 \rho \sigma (8 \rho + z_1 - z_1^2 \rho) \rho^2 \sigma^2 (\sigma - z_1) + \rho \sigma z_2 (2x^2 - 14 \sigma z_1 + 11 z_1^2) + z_2^2 (\sigma + z_1)^2)} \\
\kappa_{1,2}^{IV} = \frac{c (20 \rho \sigma (\sigma - z_1) (8 \rho + z_1 + z_2 (192 \rho^3 - 688 \rho^2 z_1 + 185 \rho \sigma z_1^2 + 35 \sigma^2) + z_2^2 (\sigma + z_1) (4 \rho + 5 \sigma z_1) (8 \rho + 5 \sigma z_1))}{(\rho + z_2) (32 \rho \sigma (8 \rho + z_1 - z_1^2 \rho) \rho^2 \sigma^2 (\sigma - z_1) + \rho \sigma z_2 (2x^2 - 14 \sigma z_1 + 11 z_1^2) + z_2^2 (\sigma + z_1)^2)} \\
\kappa_{2,1}^{IV} = \frac{c (20 \rho \sigma (\sigma - z_1) (8 \rho + z_1 + z_2 (192 \rho^3 - 688 \rho^2 z_1 + 185 \rho \sigma z_1^2 + 35 \sigma^2) + z_2^2 (\sigma + z_1) (4 \rho + 5 \sigma z_1) (8 \rho + 5 \sigma z_1))}{(\rho + z_2) (32 \rho \sigma (8 \rho + z_1 - z_1^2 \rho) \rho^2 \sigma^2 (\sigma - z_1) + \rho \sigma z_2 (2x^2 - 14 \sigma z_1 + 11 z_1^2) + z_2^2 (\sigma + z_1)^2)} \\
\kappa_{2,2}^{IV} = \frac{c (20 \rho \sigma (\sigma - z_1) (8 \rho + z_1 + z_2 (192 \rho^3 - 688 \rho^2 z_1 + 185 \rho \sigma z_1^2 + 35 \sigma^2) + z_2^2 (\sigma + z_1) (4 \rho + 5 \sigma z_1) (8 \rho + 5 \sigma z_1))}{(\rho + z_2) (32 \rho \sigma (8 \rho + z_1 - z_1^2 \rho) \rho^2 \sigma^2 (\sigma - z_1) + \rho \sigma z_2 (2x^2 - 14 \sigma z_1 + 11 z_1^2) + z_2^2 (\sigma + z_1)^2)} \\
(7.39)
\]

From the classical limit \( z_1 = z_2 = 0 \) we can extract the triple intersection numbers

\[
J_{1,1}^{IV} = -\frac{c}{4 \rho^2 \sigma^2} \\
J_{1,1}^{IV} J_{1,1}^{IV} = J_{1,1}^{IV} J_{2,1}^{IV} = J_{1,1}^{IV} J_{2,1}^{IV} = -\frac{5 c}{8 \rho^2 \sigma^2}. \\
(7.40)
\]

We note that in each of the intersection numbers the same combination of the unknown constants appears. Therefore they are determined up to an overall factor. Comparing with the topological analysis of phase IV in section 4.2, we find agreement if we set \( \frac{c}{\rho \sigma} = -16 \).

Finally, we can extract the Gromov-Witten invariants up to the three constants. The result for low degrees is

\[
\begin{array}{cccccc}
| d_1 | d_2 | 0 | 1 | 2 | 3 | 4 |
\hline
0 | - & 5c & 5c (4ρ^2 + 3) & c (5 - 8ρ^2) & 15c (16ρ^2 - 1) & 32768ρ^6 σ^4 & 256ρ^6 σ^4 & 16384ρ^4 σ^4 & 5c (8832ρ^4 σ^4 + 9201) & 524288ρ^4 σ^4 & 617225c \\
1 | 3c & 64ρ^4 σ^4 & 256ρ^4 σ^4 & 512ρ^4 σ^4 & 4c (504ρ^4 σ^4 - 12551) & 2885c & 2048ρ^4 σ^4 & 32768ρ^4 σ^4 & 1769472ρ^4 σ^4 & 5c (1285224ρ^4 σ^4 - 528741767) & 38870912ρ^4 σ^4 \\
2 | 3c & 4096ρ^4 σ^4 & 512ρ^4 σ^4 & 65536ρ^4 σ^4 & 5c (12032ρ^4 σ^4 - 626729) & 50145c & 32768ρ^4 σ^4 & 1048576ρ^4 σ^4 & 5c (1285224ρ^4 σ^4 - 528741767) & (7.41)
\end{array}
\]
We can try to make an educated guess for the choice of unknowns by looking for the minimal values of \( \{ \rho, \sigma, c \} \) such that the coefficients in the fundamental period positive integers and the Gromov-Witten invariants are integer. The most obvious choice compatible with our results of the triple intersection numbers seems to be
\[
\sigma = -\frac{1}{8}, \quad \rho = -\frac{1}{4}, \quad c = -\frac{1}{64}.
\] (7.42)

Considering the relation (5.32) between the FI-theta parameters of the GLSM and its dual, we observe that this choice of constants in (7.35) is consistent with the identification
\[
\frac{1}{z_{1,IV}} = e^{-\frac{(t_1-t_2)}{2}}, \quad \frac{1}{z_{2,IV}} = -e^{-t_2}.
\] (7.43)

Fixing the constants in this way gives the following Gromov-Witten invariants:

| \( d_1 \) \( d_2 \) | 0 | 1 | 2 | 3 | 4 |
|-----------------|---|---|---|---|---|
| 0               | - | 40| 10| 0 | 0 |
| 1               | 6 | 470|1380|1380|470|
| 2               | -6|120 |15630|92320|229880|
| 3               | 6 | -160|5620|928470|9875600|
| 4               | -12|360 |-9930|401160|82613940|

Since we do not know if our choice of the constants \( \rho, \sigma, c \) is the correct one, these numbers remain conjectural.

8 Summary and Outlook

In this paper, we constructed and studied a two parameter non-Abelian GLSM with six phases. Two phases are geometric, one weakly coupled and the other strongly coupled, and correspond to Calabi-Yau manifolds, \( X \) and \( \tilde{Y} \), which are birationally inequivalent but are expected to be derived equivalent. Three others are hybrid phases, described by pairs \( (X_\alpha, W_\alpha) \) of spaces and potentials, \( \alpha = I, II, V \). These are presumably bad hybrids because the vector \( U(1) \) R-symmetry acts non-trivially on Crit(\( W_\alpha \)). We were unable to find the character of the remaining phase since the original and the dual models are both strongly coupled.

Having constructed and analyzed one particular example in detail, the next obvious task is to explore more examples and try to systematize the analysis. This may lead to a novel systematic construction of Calabi-Yau varieties which parallels the systemic construction and classification of the complete intersection Calabi-Yaus in toric varieties. This would considerably expand our knowledge on the landscape of Calabi-Yau varieties, or more generally, of 2d \((2,2)\) SCFTs with charge integrality.

Of course, there are still many things that could be studied just for our model. Recently, techniques to compute supersymmetric partition functions of 2d \((2,2)\) gauge theories have been developed [43-52]. In the present paper, we limited our analysis to those that can be done with the “classical” methods, like undergraduate topology and classical mirror symmetry, but the new technology can tell us more. For example, by studying the sphere partition function [43-44,53-55], we expect to obtain more information about the hybrid phases as well as the mysterious phase where the classical analysis gave us no clue. In particular, we can
examine whether the distances to the limiting loci are finite, settling the question about the badness of the hybrids.

The hemisphere partition function [47–49] also has a good application. From general principles, we expect that the Calabi-Yau manifolds $X$ and $\tilde{Y}$ are derived equivalent (1.5) and that the equivalence depends on the homotopy class of paths in $\mathcal{M}_K$ that connect the two phases. But we would like to know what the equivalence is for each homotopy class. The solution to this problem in Abelian GLSMs [56] (completed and generalized in [57–59]) was found to be reproduced by analyzing the hemisphere partition function [47], and is being extended to the non-Abelian models of [5, 6] in [60]. When applied to our model, we expect to obtain equivalences not only between $\mathcal{D}^b_{\text{Coh}}(X)$ and $\mathcal{D}^b_{\text{Coh}}(\tilde{Y})$ but also with the categories of B-branes in the other phases, that is, the categories of matrix factorizations of the pairs $(X_\alpha, W_\alpha)$:

\[
\begin{array}{c}
\mathcal{D}^b_{\text{Coh}}(\tilde{Y}) \\
\cong \\
\mathcal{D}^b_{\text{Coh}}(X) \\
\cong \\
\mathcal{D}^b_{\text{Coh}}(Y) \quad ? \\
\cong \\
\mathcal{D}^b_{\text{Coh}}(\tilde{Y}) \\
\cong \\
\mathcal{D}^b_{\text{Coh}}(X) \\
\cong \\
\mathcal{D}^b_{\text{Coh}}(Y) \\
\cong \\
\mathcal{D}^b_{\text{Coh}}(\tilde{Y}) \\
\cong \\
\mathcal{D}^b_{\text{Coh}}(X) \\
\cong \\
\mathcal{D}^b_{\text{Coh}}(Y) \\
\cong \\
\mathcal{D}^b_{\text{Coh}}(\tilde{Y}) \\
\cong \\
\mathcal{D}^b_{\text{Coh}}(X) \\
\cong \\
\mathcal{D}^b_{\text{Coh}}(Y) \\
\cong \\
\mathcal{D}^b_{\text{Coh}}(\tilde{Y}) \\
\end{array}
\]

For many of the Calabi-Yau pairs that appear in the one parameter models, including those found by Rødland, Hosono-Takagi and Miura, it is known that the derived equivalences fit into the framework of “Homological Projective Duality” by A. Kuznetsov [61]. It would be interesting to see whether the derived equivalence of our $X$ and $\tilde{Y}$ in the two parameter model also fits into this framework. A preliminary discussion shows the appearance of some of the structures in the hybrid models. The study on this point may shed new light on the relation between the Homological Projective Duality and the gauge theory understanding of the equivalences that involves 2d Seiberg duality.

Non-birational but derived equivalent pairs of varieties with Picard number $\geq 2$ had been known for a long time. A “trivial” example is $B \times S$ and $B \times S'$ for some variety $B$ where $S$ and $S'$ are birationally inequivalent but derived equivalent K3 or Abelian surfaces, related by a Fourier-Mukai functor. We may obtain non-trivial Calabi-Yau examples if we consider manifolds with a structure of K3 or Abelian fibration and applying Fourier-Mukai transforms on the fibers. Indeed, such a pair $(X_1, X_2)$ was found in [62]. They are Abelian surface fibrations over $\mathbb{P}^1$ which are related by T-duality along the fibers. (They have $(h^{1,1}, h^{2,1}) = (2, 2)$. $X_1$ is simply connected and $X_2$ turns out to be the quotient of $X_1$ by a freely acting symmetry group $\mathbb{Z}_8 \times \mathbb{Z}_8$. Hence they cannot be birationally equivalent.) It would be interesting to see if our pair $(X, \tilde{Y})$ is or is not of this type.

We have not completed the analysis of the topology of $X$ and $\tilde{Y}$, due to lack of our ability to do so. Most importantly, we have not proved that the Hodge numbers of $\tilde{Y}$ are $(h^{1,1}, h^{2,1}) = (2, 24)$, although that must certainly be the case. Also, to compute the intersection numbers on $\tilde{Y}$, we needed to make some assumptions. These are obvious gaps in our analysis. Another important topological information is the topological K-theory. Our model is expected to flow to a family of SCFTs with $\hat{c} = 3$ and charge integrality that can be used as supersymmetric

\[\text{We thank A. Kuznetsov for showing his picture on our example.}\]

\[\text{We thank Y. Toda for informing us of such a construction and the example [62].}\]
backgrounds for Type II string theory. When we consider Type II string theory on a spacetime $X$, the D-brane (or Ramond-Ramond) charge is classified by the K-theory of $X$, of various relative types depending on the dimension of the objects [63]. For this reason, the K-theory of $X$ and $\tilde{Y}$ is important. Since the charge lattice must be stable under continuous deformation, we expect that the K-theory of $X$ and that of $\tilde{Y}$ are isomorphic. Indeed that seems to be the case under (1.5); according to [64,65], the (topological) K-theory is derived invariant.

It was shown in [66–68] that the torsion parts of the K-groups of a Calabi-Yau threefold $M$ (which are the information not obtained by the Hodge numbers) are given by $\text{Tors}(K^0(M)) = A(M) \oplus B(M)^*$ and $\text{Tors}(K^{-1}(M)) = A(M)^* \oplus B(M)$ where $A(M) = \text{Hom}(\pi_1(M), \mathbb{Q}/\mathbb{Z})$ and $B(M) = \text{Tors}(H^2(M, \mathbb{Z}))$ (the latter is called the Brauer group of $M$). Here for a finite Abelian group $\gamma$, we write $\gamma^* = \text{Hom}(\gamma, \mathbb{Q}/\mathbb{Z})$ for its dual. Thus, we need to compute the fundamental group and the Brauer group of $X$ and $\tilde{Y}$, of which we only know $\pi_1(\tilde{Y}) = \{1\}$ at this stage. A related problem is to determine the K-theory of non-geometric phases [69]. Recently, topological K-theory of dg-categories, such as the categories of B-branes for Landau-Ginzburg and hybrid models, has been defined [69,70]. It would be interesting to compute it in the hybrid phases of our model and check that they match with the results in the geometric phases. And it would be interesting to see if the construction [69,70] works directly in GLSMs.

It may also be interesting to further study mirror symmetry for our model. We have drawn all our conclusions on the mirror starting from the Calabi-Yau $X$ in phase I. One could also attempt to construct the mirror of the determinantal Calabi-Yau $\tilde{Y}$ in phase IV. Mirrors of Pfaffian Calabi-Yaus have been proposed in [9,14] partly based on [71] (see also [72]). It would be interesting to see if any of these constructions also work for $\tilde{Y}$, and also the determinantal Calabi-Yau constructed in [7]. Of course, understanding mirror symmetry for 2d (2,2) non-Abelian gauge theories more generally is an important problem.

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A Additional details on mirror symmetry

A.1 Evaluation of the fundamental period of $\tilde{X}^\vee$

Here we give further details on the evaluation of the fundamental period (7.19) of $\tilde{X}^\vee$. We introduce a short-hand notation for the Laurent polynomials (7.7):

$$f_1 = 1 + \alpha_1 \frac{x_7}{x_1x_2x_3x_7} + \alpha_5 x_4 := 1 + \beta_1 + \beta_5$$
$$f_2 = 1 + \alpha_3 x_2 + \alpha_6 x_5 := 1 + \beta_3 + \beta_6$$
$$f_3 = 1 + \alpha_4 x_3 + \alpha_7 x_6 := 1 + \beta_4 + \beta_7$$
$$f_4 = 1 + \alpha_9 x_1 + \alpha_{11} \frac{1}{x_4x_5x_6x_7} := 1 + \beta_9 + \beta_{11}$$
$$f_5 = 1 + \alpha_2 x_8 + \alpha_8 x_7 + \alpha_{10} x_7 := 1 + \beta_2 + \beta_8 + \beta_{10},$$

where the $\beta_i = \alpha_1 \frac{x_i}{x_1x_2x_3x_7}$, etc. This we insert into the residue formula for the fundamental period given by

$$\varpi_0(\tilde{Y}) = \frac{1}{(2\pi i)^{11}} \int \prod \frac{dx_i}{x_i} \frac{1}{f_1f_2f_3f_4f_5}.$$  \hspace{1cm} (A.2)

We rewrite the integrand as follows

$$K = \frac{1}{f_1f_2f_3f_4f_5} = \frac{1}{1 + \beta_1 + \beta_5 1 + \beta_3 + \beta_6 1 + \beta_4 + \beta_7 1 + \beta_9 + \beta_{11} 1 + \beta_2 + \beta_8 + \beta_{10}}$$

$$= \sum_{n_1=0}^{\infty} (-1)^{n_1} (\beta_1 + \beta_5)^{n_1} \sum_{n_2=0}^{\infty} (-1)^{n_2} (\beta_3 + \beta_6)^{n_2} \sum_{n_3=0}^{\infty} (-1)^{n_3} (\beta_4 + \beta_7)^{n_3} \times$$

$$\sum_{n_4=0}^{\infty} (-1)^{n_4} (\beta_9 + \beta_{11})^{n_4} \sum_{n_5=0}^{\infty} (-1)^{n_5} (\beta_2 + \beta_8 + \beta_{10})^{n_5}$$

$$= \sum_{n_1=0}^{\infty} (-1)^{n_1} \sum_{k_1=1}^{n_1} \left( \frac{n_1}{k_1} \right) \beta_1^{k_1} \beta_5^{n_1-k_1} \sum_{n_2=0}^{\infty} (-1)^{n_2} \sum_{k_2=1}^{n_2} \left( \frac{n_2}{k_2} \right) \beta_3^{k_2} \beta_6^{n_2-k_2} \times$$

$$\sum_{n_3=0}^{\infty} (-1)^{n_3} \sum_{k_3=1}^{n_3} \left( \frac{n_3}{k_3} \right) \beta_4^{k_3} \beta_7^{n_3-k_3} \times \sum_{n_4=0}^{\infty} \sum_{k_4=1}^{n_4} \left( \frac{n_4}{k_4} \right) \beta_9^{k_4} \beta_{11}^{n_4-k_4} \times$$

$$\sum_{n_5=0}^{\infty} \sum_{k_5=0}^{n_5} \sum_{l_5=0}^{n_5-k_5} \left( \frac{n_5}{k_5} \right) \beta_2^{k_5} \beta_8^{l_5} \beta_{10}^{n_5-k_5-l_5}.$$ \hspace{1cm} (A.3)

Only those products of the $\beta_i$ which are independent of $x_i$ contribute to the residue integral. Which monomials these are encoded in the Mori generators (7.8):

$$\frac{\beta_2 \beta_6 \beta_7 \beta_8}{\beta_8} = \frac{\alpha_2 \alpha_5 \alpha_6 \alpha_7}{\alpha_8} = z_1 \quad \frac{\beta_8 \beta_{10}}{\beta_2} = -z_2 \quad \frac{\beta_1 \beta_2 \beta_4 \beta_9}{\beta_{10}} = z_3$$ \hspace{1cm} (A.4)
The fundamental period is therefore a power series of the form
\[
\sum_{a,b,c \geq 0} c_{abc} z_1^a z_2^b z_3^c = \sum_{a,b,c} c_{abc} (-1)^b \beta_1^a \beta_2^b \beta_3^c \beta_4 \beta_5 \beta_6 \beta_7 \beta_8 \beta_9 \beta_{10} \beta_{11} \quad (A.5)
\]
Comparing coefficients with the integrand \( K \) above we find
\[
n_1 = n_2 = n_3 = n_4 = a + c \quad k_1 = k_2 = k_3 = k_4 = c
\]
\[
n_5 = b \quad k_5 = a - b + c \quad l_5 = -a + b
\]
\[
a - b + c \geq 0 \quad -a + b \geq 0 \quad b - c \geq 0 \quad (A.6)
\]
Inserting this into the residue integral, we arrive at the result (7.19) for the fundamental period.

A.2 Periods of \( X^0 \)

Here we give the first few terms of the power series expansion of the solutions to the Picard-Fuchs system (7.21). The first few terms of the series expansion of the two log-solutions are:
\[
\varpi_{1,1} = -\frac{1}{3} z_1^3 + 16 \log(z_1) z_2 z_1^2 + 15 z_2^2 z_1^3 + \frac{z_2^3}{2} + 2 z_2^2 z_1 + 2 \log(z_1) z_2 z_1 + 5 z_2 z_1 - z_1
\]
\[
+ \frac{z_2^3}{3} - \frac{z_2^3}{2} + \log(z_1) + z_2 + \ldots
\]
\[
\varpi_{1,2} = \frac{2z_2^3}{3} + 16 \log(z_2) z_2 z_1^2 + 34 z_2^2 z_1^3 - z_1^2 - 2 z_2^2 z_1 + 2 \log(z_2) z_2 z_1 + 2 z_1
\]
\[
- \frac{z_2^3}{3} + \frac{z_2^3}{2} + \log(z_2) - z_2 + \ldots \quad (A.7)
\]
The log² solutions have the following form:
\[
\varpi_{2,1} = \log^2(z_1) + \frac{10}{19} \log(z_2) \log(z_1) + \left( \frac{28z_2}{19} - \frac{18z_1}{19} \right) \log(z_1)
\]
\[
- \frac{32z_1}{19} + \log(z_2) \left( \frac{10z_2}{19} - \frac{10z_1}{19} \right) + \frac{18z_2}{19} + \ldots
\]
\[
\varpi_{2,2} = \log^2(z_2) + \frac{40}{19} \log(z_1) \log(z_2) + \left( \frac{36z_1}{19} + \frac{2z_2}{19} \right) \log(z_2) + \frac{100z_1}{19}
\]
\[
+ \log(z_1) \left( \frac{80z_1}{19} - \frac{40z_2}{19} \right) - \frac{42z_2}{19} + \ldots \quad (A.8)
\]
The log³ solution has the following expansion:
\[
\varpi_3 = \log^3(z_1) + \frac{15}{7} \log(z_2) \log^2(z_1) + \left( \frac{9z_1}{7} + \frac{6z_2}{7} \right) \log^2(z_1) + \frac{9}{7} \log^2(z_2) \log(z_1)
\]
\[
+ \frac{12}{7} \log(z_1) + \log(z_2) \left( \frac{6z_1}{7} + \frac{12z_2}{7} \right) \log(z_1) + \frac{8}{35} \log^3(z_2) - \frac{24z_1}{7}
\]
\[
+ \log^2(z_2) \left( \frac{3z_1}{35} + \frac{3z_2}{5} \right) + \frac{18}{35} \log(z_2) z_2 - \frac{36z_2}{35} + \ldots \quad (A.9)
\]
A.3 Gauss-Manin connection matrices

Up to numerical scaling factors the local coordinates in the various phases are expressed in terms of the coordinates of \( z_{1,2} \) of phase \( I_+ \) as follows

\[
\begin{align*}
z_{1,\text{II}} &= \frac{1}{z_1} \quad z_{2,\text{II}} = z_1 z_2^2 \\
z_{1,\text{III}} &= \frac{1}{\sqrt{z_1 z_2}} \quad z_{2,\text{III}} = \sqrt{z_1} \\
z_{1,\text{IV}} &= \frac{1}{z_1} \quad z_{2,\text{IV}} = \frac{1}{z_2} \\
z_{1,\text{V}} &= z_2 \quad z_{2,\text{V}} = \frac{1}{z_1 z_2} \\
z_{1,\text{I}^{-}} &= z_1 z_2 \quad z_{2,\text{I}^{-}} = \frac{1}{z_1}.
\end{align*}
\] (A.10)

We compute Jordan normal forms of the Gauss-Manin connection matrices evaluated at \( z_{i,*} = 0 \) by transforming the Picard-Fuchs system of phase \( I_+ \) (without rescaling the holomorphic three-form). The results for the phases other than \( I_+ \) and \( IV \) already given in the main text are

\[
\begin{align*}
M_{z_1,\text{II}} &= \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix} & M_{z_2,\text{II}} &= \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix} \\
M_{z_1,\text{III}} &= \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix} & M_{z_2,\text{III}} &= \begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix} \quad \text{(A.11)}
\end{align*}
\]

\[
\begin{align*}
M_{z_1,\text{V}} &= \begin{pmatrix}
\frac{1}{2} & 1 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{z_2} & 1 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{z_2} & 1 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{z_2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix} & M_{z_2,\text{V}} &= \begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix} \quad \text{(A.12)}
\end{align*}
\]

\[
\begin{align*}
M_{z_1,\text{I}^{-}} &= \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} & M_{z_2,\text{I}^{-}} &= \begin{pmatrix}
\frac{1}{2} & 1 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{z_2} & 1 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{z_2} & 1 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{z_2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix} \quad \text{(A.13)}
\end{align*}
\]
None of these matrices have a structure compatible with maximally unipotent monodromy. This, and explicitly solving the Picard-Fuchs equations in these phases, gives further evidence that phases I+ and IV are the only geometric phases.

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