Exact solution of position dependent mass Schrödinger equation by supersymmetric quantum mechanics

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A supersymmetric technique for the solution of the effective mass Schrödinger equation is proposed. Exact solutions of the Schrödinger equation corresponding to a number of potentials are obtained. The potentials are fully isospectral with the original potentials. The conditions for the shape invariance of the potentials are discussed.

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INTRODUCTION

The study of Schrödinger equation with a position dependent mass has been the subject of recent interest arising from the study of electronic properties of semiconductors, quantum dots, liquid crystals and nonuniform materials in which the carrier effective mass depends on the position [1]. The effective action for a particle with position-dependent mass has been calculated by background method [2] which in the context of quantum mechanics is described in the textbook [3]. By the way the method provides a way to keep manifest the reparametrization invariance of $\sigma$-models, and is therefore the preferred method for explicit calculations of the effective actions. Since the position-dependent mass Schrödinger equation is of considerable importance in condensed matter physics, we feel that it is necessary to study its solution systematically.

The supersymmetric (SUSY) method is very useful technique for exactly solvable potentials [4]. SUSY has been suggested to give encouraging results towards an understanding of the degeneracies in atoms and establish interesting atomic connections. Several authors have investigated the exact solution of Schrödinger equation with position-dependent mass using various techniques [5, 6, 7, 8, 9].

In this paper we develop the SUSY method to discuss the supersymmetry and shape invariance of the effective mass potentials. In section 2 we briefly review the concept of SUSY in quantum mechanics, then we present a modified SUSY that is applicable to the system with position-dependent mass. After presenting a general formulation of SUSY to obtain exact solution of the Schrödinger equation with position-dependent mass, we discuss the condition of shape invariance of the potentials, in section 3. Working with some explicit examples like harmonic oscillator, Coulomb and Morse family potentials we show that, in section 4, such a modification of SUSY is useful to study effective mass Hamiltonians. The conclusion is given in section 5.

EFFECTIVE MASS HAMILTONIAN AND THE SUSY METHOD

There are several ways to define the kinetic energy operator when the mass is a function of position. Since momentum and mass operators no longer commute, the generalization of the Hamiltonian is not trivial. We begin by defining a general Hermitian effective mass Hamiltonian which is proposed by von Roos [11],

$$H = \frac{1}{4} \left( m^\eta p m^\varepsilon p + m^\varepsilon p m^\eta p \right) + V(x)$$  \hspace{1cm} (1)

where $\eta + \varepsilon + \rho = -1$. The limit of the choice of the parameters $\eta, \varepsilon$ and $\rho$ is depends on the physical system. In fact the problem of choice of the parameters has been a long standing one in quantum mechanics [4]. We follow Morrow and Brownstein [12] who have shown that $\eta = \rho$, by comparing experimental results and/or exact solutions of the some simple models. Otherwise the wave function is unphysical. Using the restricted Hamiltonian from the $\eta = \rho$ constraint, we can write

$$H = \frac{1}{2} \left( m^{-\frac{1}{2}(\varepsilon+1)} p m^\varepsilon p m^{-\frac{1}{2}(\varepsilon+1)} \right) + V(x)$$  \hspace{1cm} (2)
Let us point out here that the Hamiltonian (2) includes the frequently used form of the Hamiltonians, in the literature \[13\], which can be expressed as
\[
H = \frac{1}{2} \left[ \frac{1}{m} \frac{\hat{p}^2}{\hat{p}} \right] + V(x) \quad \text{for} \quad \varepsilon = -1
\]
\[
H = \frac{1}{2} \left[ \frac{1}{\sqrt{m}} \frac{\hat{p}^2}{\hat{p}} \frac{1}{\sqrt{m}} \right] + V(x) \quad \text{for} \quad \varepsilon = 0
\]
\[
H = \frac{1}{4} \left[ \frac{1}{m} \hat{p}^2 + \frac{1}{m} \hat{p}^2 \right] + \frac{(\varepsilon - 1)^2 m^2}{8m^3} + \frac{\varepsilon m''}{4m^2} + V(x)
\]
Before going further we derive a general effective Hamiltonian for the case of position-dependent mass. Let us turn our attention to the Hamiltonian (1). Using the commutation relation
\[
[p, m^\alpha] = -i\hbar m^\alpha - 1
\]
on one can put the momenta to the right, the Hamiltonian (1) takes the form
\[
H = H_{eff} + V(x)
\]
The effective Hamiltonian \(H_{eff}\) is given by
\[
H_{eff} = \frac{\hat{p}^2}{2m} - \frac{i\hbar m'}{2m^2} \hat{p} - U(x)
\]
where
\[
U(x) = -\frac{\hbar^2}{4m^3} \left( 2(\varepsilon + \eta + \varepsilon \eta + \eta^2 + 1)m^2 - (\varepsilon + 1)mm'' \right).
\]
Note that the effective potential term \(U(x)\) can be eliminated by imposing the constraints over the parameters such that \(\varepsilon = -1\) and \(\eta = 0\). In this case the Schrödinger equation will not depend on the parameters.

Our task is now to discuss the solution of the Hamiltonian (2) in the framework of SUSY quantum mechanics. Let us take a look at the SUSY quantum mechanics for the standard Schrödinger equation. The algebra of SUSY satisfies the following commutation relations:
\[
\{ Q^+, Q^- \} = H, \quad [Q^\pm, H] = 0, \quad \{ Q^\pm, Q^\pm \} = 0
\]
The supercharges \(Q^\pm\) are defined as
\[
Q^\pm = B^\mp \sigma^\pm, \quad B^\pm = \frac{1}{\sqrt{2}} \left( \pm \frac{d}{dx} + \Phi(x) \right)
\]
where \(\sigma^\pm\) are Pauli matrices and \(\Phi(x)\) is a superpotential. We may construct a supersymmetric quantum mechanical system by defining the Hamiltonians such that the relations in (8) holds,
\[
H_\pm = B^\mp B^\pm = -\frac{1}{2} \frac{d^2}{dx^2} + V_\pm(x)
\]
The partner potentials \(V_\pm(x)\) are related to the superpotential \(\Phi(x)\) by
\[
V_\pm(x) = \frac{1}{2} \left( \Phi^2(x) \pm \Phi'(x) \right)
\]
The Hamiltonians \(H_+\) and \(H_\) possess the same eigenvalues except for the zero energy ground state. The zero-energy eigenstate belongs to the \(H_\) system, and supersymmetry of quantum system is said to be good SUSY if the ground state energy of \(H_\) (or \(H_\)) vanishes. In the other case SUSY is said to be broken. For good SUSY the ground state of \(H_\) is given by
\[
\psi_0^-(x) = C \exp \left( -\int \Phi(x) dx \right)
\]
where $C$ is normalization constant. The potentials are shape invariant \[14\], that is $V_+(x)$ has the same functional form as $V_-(x)$ but different parameters except for an additive constant:

$$V_+(x; a_0) = R(a_0) + V_-(x; a_1)$$  \hspace{1cm} (13)

where $a_0$ and $a_1$ stand for the potential parameters in the supersymmetric partner potentials, and $R(a_0)$ is a constant. This property permits an immediate analytical determination of eigenvalues and eigenfunctions. The eigenvalues and eigenfunctions of the Hamiltonians $H_+$ and $H_-$ are related by

$$E_0^+ = 0, \quad E_1^- = R(a_0), \quad E_{n+1}^- = E_n^+, \quad E_n^- = \sum_{k=0}^{n-1} R(a_k), \quad \psi_n^-(x; a_0) = B^+(x; a_0)\psi_{n-1}^-(x; a_1)$$  \hspace{1cm} (14a, 14b)

In the following we shall modify the standard SUSY technique to the systems with position-dependent mass. Since the mass is a function of the position, the supersymmetric operators include mass term. It will be shown that the following form of the operators are appropriate to study the Hamiltonian \[2\],

$$A^+ = -\frac{1}{\sqrt{2}} \left[ m^{-\frac{1}{2}(\epsilon+1)} \frac{d}{dx} m^{\frac{1}{2}} \right] + \frac{W(x)}{\sqrt{2m}} \quad \text{ (15a)}$$

$$A^- = \frac{1}{\sqrt{2}} \left[ m^{\frac{1}{2}} \frac{d}{dx} m^{-\frac{1}{2}(\epsilon+1)} \right] + \frac{W(x)}{\sqrt{2m}} \quad \text{ (15b)}$$

where $W(x)$ is the superpotential and $m$ depends on the position. It can be checked that the supersymmetry relations in \[8\] and \[9\] are satisfied when $B^\pm$ are replaced by $A^\pm$. Note that the operator $\frac{d}{dx} m^\alpha$ read as follows:

$$\frac{d}{dx} m^\alpha = m^\alpha \frac{d}{dx} + \alpha m^{\alpha-1} \frac{dm}{dx} \quad \text{ (16)}$$

We assume that, for good SUSY the ground state wave function belongs to $H_-$ and is given by

$$\psi_0^-(x) = m^{\frac{1}{2}(\epsilon+1)} \exp \left(- \int W(x) dx \right) \quad \text{ (17)}$$

One can easily check that $A^- \psi_0^-(x) = 0$. The Hamiltonians of quantum systems with position-dependent mass take the form

$$H_- = A^+ A^- = -\frac{1}{2} \left[ m^{-\frac{1}{2}(\epsilon+1)} \frac{d}{dx} m^\epsilon \frac{d}{dx} m^{-\frac{1}{2}(\epsilon+1)} \right] + V_-(x) \quad \text{ (18a)}$$

$$H_+ = A^- A^+ = -\frac{1}{2} \left[ m^{\frac{1}{2}} \frac{d}{dx} m^{-(\epsilon+1)} \frac{d}{dx} m^{\frac{1}{2}} \right] + V_+(x) \quad \text{ (18b)}$$

where the partner potentials are given by

$$V_-(x) = \frac{1}{2m} \left( W^2(x) - W'(x) - \frac{\varepsilon m'}{m} W(x) \right) \quad \text{ (19a)}$$

$$V_+(x) = \frac{1}{2m} \left( W^2(x) + W'(x) - \varepsilon m W'(x) \right)$$

$$+ \frac{(2\varepsilon + 1)}{2m} \left( \frac{3 m'^2}{8 m^2} - \frac{1 m''}{4 m} \right) \quad \text{ (19b)}$$

It is obvious that the kinetic energy terms of the effective mass Hamiltonian \[2\] and $H_-$ are not identical. Therefore the shape-invariance condition \[8\] does not satisfy for the position-dependent mass system. If mass is constant it is easy to check that the physical quantities of the position-dependent mass system reduce to the physical quantities of the standard system.

In the following section we will discuss the exact solvability of the position-dependent mass Schrödinger equation by using the supersymmetric procedure given in \[8\] through \[19b\].
CONSTRUCTION OF THE SHAPE-INVARIENT POTENTIALS

In this section we present a method to construct shape invariant potentials. It is known that the potentials which satisfy shape invariance condition are exactly solvable. To illustrate the method we first analyze the construction of the shape invariant potentials for standard SUSY system. It is well known that the exactly solvable potentials can be categorized in two groups: the potentials that their eigenfunctions include Hypergeometric functions and the confluent hypergeometric functions. The first group consists of the Pöschl-Teller, Eckart, Hultén potentials etc. and the second group contains Harmonic oscillator, Coulomb and Morse potentials. The potentials in each group can be mapped onto each others by point canonical transformation. This property implies that the superpotential \( \Phi(x) \) may be expressed in two different forms for Natonzon class of potentials by considering operator transformation applied to the shape-invariant potentials [4].

The shape invariance condition for constant mass quantum mechanical systems is given by (13). Let us introduce the following general superpotential:

\[
\Phi(x; \lambda_1, \lambda_2) = \lambda_1 r' + \lambda_2 \frac{r'}{r} + \frac{r''}{2r} \tag{20}
\]

where \( \lambda_1 \) and \( \lambda_2 \) are arbitrary constants, and \( r = r(x) \). The harmonic oscillator, Coulomb and Morse type shape invariant potentials can be constructed by the appropriate choice of \( \lambda_1, \lambda_2 \), and \( r \). The parameters \( \lambda_1 \) and \( \lambda_2 \) can be determined from the relation (13), while \( r \) can be obtained from the following constraint:

\[
\left( q_0 + \frac{q_1}{r} + \frac{q_2}{r^2} \right) r'^2 = 1 \tag{21}
\]

where \( q_1, q_2 \) and \( q_3 \) are arbitrary constants. One can obtain the shape-invariant potentials when the parameters of (21) are chosen as:

\[
q_0 = q_2 = 0, \quad q_1 = 1, \quad r = \frac{1}{4}x^2, \quad \lambda_1 = 2\omega, \quad \lambda_2 = -\frac{\ell}{2} - \frac{3}{4}
\]

\[
\Phi(x; \ell) = \frac{(\ell + 1)}{x} + \omega x, \quad R(\ell) = \ell + \frac{3}{2}, \tag{22a}
\]

\[
q_1 = q_2 = 0, \quad q_0 = 1, \quad r = x, \quad \lambda_1 = \frac{Ze^2}{\ell + 1}, \quad \lambda_2 = (\ell + 1),
\]

\[
\Phi(x; \ell) = \frac{(\ell + 1)}{x} + \frac{Ze^2}{\ell + 1}, \quad R(\ell) = \frac{(Ze^2)^2}{2} \left( \frac{1}{(\ell + 1)^2} - \frac{1}{\ell^2} \right), \tag{22b}
\]

\[
q_0 = q_1 = 0, \quad q_2 = \frac{1}{\alpha^2}, \quad r = e^{\alpha x}, \quad \lambda_1 = \frac{b}{\alpha}, \quad \lambda_2 = a,
\]

\[
\Phi(x; a) = a(\frac{1}{2} + \ell) + be^{\alpha x}, \quad R(\ell) = a^2(\ell + 1). \tag{22c}
\]

The superpotentials (22a, 22b, 22c) are related with the shape invariant harmonic oscillator, Coulomb and Morse potentials, respectively. Similar procedure can be followed to obtain shape invariant family of potentials for the position-dependent mass quantum mechanical systems.

Let us express the shape-invariance condition for the position-dependent mass operators \( A^- \) and \( A^+ \):

\[
A^-(x, a_0)A^+(x, a_0) - A^+(x, a_1)A^-(x, a_1) = R(a_0). \tag{23}
\]

By substituting operators \( A^\pm \) in (23) we obtain the following shape-invariance condition:

\[
\frac{1}{2m^2} \left( W^2(x, a_0) - W^2(x, a_1) + W'(x, a_0) + W'(x, a_1) \right) = \frac{1}{2m^2} \left( (1 + \varepsilon)W(x, a_0) - \varepsilon W(x, a_1) \right) + \frac{(1 + 2\varepsilon)}{8m^2} \left( 3m^2 - 2mm'' \right) = R(a_0) \tag{24}
\]
An interesting feature of the SUSY quantum mechanics for the shape-invariant system satisfying the condition (24) is that the entire spectrum can be determined algebraically. It should be noted that the shape-invariance is not a general integrability condition. In this work we consider the shape-invariant potentials obtained from the following superpotential:

\[ W(x; \lambda_1; \lambda_2) = \lambda_1 r' + \lambda_2 \frac{r'}{r} + \frac{r''}{2r'} + \frac{\varepsilon m'}{2m} \quad (25) \]

with the condition

\[ (q_0 + \frac{q_1}{r} + \frac{q_2}{r^2}) r^2 = m \quad (26) \]

Using the analogy of standard SUSY method we obtain the following superpotentials (let \( u = \int \sqrt{m} dx \)):

\[ q_0 = q_2 = 0, \quad q_1 = 1, \quad r = \frac{1}{4} u^2, \]

\[ \lambda_1 = 2\omega, \quad \lambda_2 = \frac{\ell}{2} + \frac{1}{4}, \quad W(x; \ell) = \frac{(\ell + 1)\sqrt{m}}{u} + \omega \sqrt{ma} + \frac{(2\varepsilon + 1) m'}{4m}; \quad (27a) \]

\[ q_1 = q_2 = 0, \quad q_0 = 1, \quad r = u, \]

\[ \lambda_1 = \frac{Ze^2}{\ell + 1}, \quad \lambda_2 = -(\ell + 1), \quad W(x; \ell) = -\frac{(\ell + 1)\sqrt{m}}{u} + \frac{Ze^2 \sqrt{m}}{\ell + 1} + \frac{(2\varepsilon + 1) m'}{4m}; \quad (27b) \]

\[ q_0 = q_1 = 0, \quad q_2 = \frac{1}{\alpha^2}, \quad r = e^{\alpha u}, \]

\[ \lambda_1 = \frac{b}{\alpha}, \quad \lambda_2 = \frac{a}{\alpha} - \frac{1}{2}, \quad W(x; a; b) = (a + be^{\alpha u}) \sqrt{m} + \frac{(2\varepsilon + 1) m'}{4m}; \quad (27c) \]

Thus we have obtained superpotentials \( W(x) \) for the harmonic oscillator, Coulomb, and Morse family potentials. Note that the square root of the mass \( m \) should be integrable. In the following section we demonstrate our method on some explicit examples.

**EXAMPLES**

In this section we discuss the exact solution of the Schrödinger equation for which the particle mass is given by

\[ m = \left( \frac{\delta + x^2}{1 + x^2} \right)^2 \quad (28) \]

As we mentioned before the the spectrum of the position-dependent mass systems and constant mass systems are identical. For the given mass the function \( u \) takes the form

\[ u = \int \sqrt{m} dx = x + (\delta - 1) \arctan x \quad (29) \]

**Harmonic oscillator family potential**

The superpotential for the harmonic oscillator potential is given in (27a). Substitution of (27a) into equations (19a) and (19b) leads to

\[ V_-(x; \ell) = \frac{\ell(\ell + 1)}{2u^2} + \frac{1}{2} \omega u^2 - \frac{1}{2} (2\ell + 3)\omega + V_m \]

\[ V_+(x; \ell) = \frac{\ell(\ell + 3)}{2u^2} + \frac{1}{2} \omega u^2 - \frac{1}{2} (2\ell + 1)\omega + V_m \quad (30) \]
where $V_m$ is given by

$$V_m = \frac{(\varepsilon(2 - \varepsilon) + 5/4)m^2}{8m^3} - \frac{(\varepsilon + 1/2)m''}{4m^2} \quad (31)$$

For the mass given in (28), $V_m$ takes the form

$$V_m = \frac{(\delta - 1)(2\varepsilon + 1) \left[-\delta + (2 - 2\varepsilon + 2\varepsilon\delta)x^2 + 3x^4\right]}{2(\delta + x^2)^4} \quad (32)$$

The spectrum of the mass dependent harmonic oscillator family potentials is the same as the standard harmonic oscillator potential, and it is given by

$$E = 2n\omega \quad (33)$$

Therefore one may obtain isospectral potentials with different masses. The reason for this is that the SUSY quantum mechanical procedure given here affects only the potential and leaves the position-dependent mass unchanged. In the supersymmetric formulation the oscillator potential is singular with singularity at $x = 0$.

**Coulomb family potentials**

Similarly one can obtain Coulomb family potential by substituting the superpotential (27b) into equations (19a) and (19b):

$$V_-(x; \ell) = \frac{\ell(\ell + 1)}{2a^2} - \frac{Ze^2}{u} + \frac{Z^2e^4}{2(\ell + 1)^2} + V_m$$

$$V_+(x; \ell) = \frac{(\ell + 1)(\ell + 2)}{2u^2} - \frac{Ze^2}{u} + \frac{Z^2e^4}{2(\ell + 1)^2} + V_m \quad (34)$$

The functions $u$ and $V_m$ are the same as defined in (29) and (31). Using the standard procedure one can obtain the following eigenvalues:

$$E = \frac{Z^2e^4}{2(\ell + 1)^2} - \frac{Z^2e^4}{2(\ell + n + 2)^2} \quad (35)$$

which is the same as the constant mass Coulomb potential.

**Morse family potentials**

The final example is the construction of the Morse family potential. Using the standard procedure we can obtain the following potentials:

$$V_-(x; a) = \frac{b}{2} (2a - \alpha) e^{\alpha u} + \frac{b^2}{2} e^{2\alpha u} + a^2 + V_m$$

$$V_+(x; a) = \frac{b}{2} (2a + \alpha) e^{\alpha u} + \frac{b^2}{2} e^{2\alpha u} + a^2 + V_m \quad (36)$$

with the eigenvalues

$$E = \frac{1}{2} \left( a^2 - (a + n\alpha)^2 \right) \quad (37)$$

The potential exhibits the spectrum of the Morse oscillator potential.

It also can be shown that when $\delta = 1$ the potentials (30, 34, 36) reduce to the standard harmonic oscillator, Coulomb, and Morse potentials, respectively.
CONCLUSION

In this paper we have discussed the exact solution of the position-dependent mass Schrödinger equation by using SUSY quantum mechanical method. We have shown that both, Schrödinger equations with different masses and potentials can exactly be isospectral. Isospectral potentials have identical spectra, and are “self isospectral” in the sense that the potentials have identical shape\textsuperscript{15}. These potentials have been constructed by linear transformation of the ladder operators, in the constant mass SUSY system\textsuperscript{16}. Detailed properties of isospectral potentials have been discussed in\textsuperscript{17}.

Finally we would like to mention that the method discussed here can be generalized for the Pöschl-Teller, Eckart, etc. potentials. The systems with position dependent mass are relevant in many areas of physics, such as nuclear physics, hetero-structures, and inhomogeneous crystals, that is quickly developing. It is hoped that the formalism developed here can be used for the treatment of the quantum mechanical problems with position-dependent masses.

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