Abstract
We give a unified method to derive the strong convergence rate of the backward Euler scheme for monotone SDEs in $L^p(\Omega)$-norm, with general $p \geq 4$. The results are applied to the backward Euler scheme of SODEs with polynomial growth coefficients. We also generalize the argument to the Galerkin-based backward Euler scheme of SPDEs with polynomial growth coefficients driven by multiplicative trace-class noise.

Keywords Monotone SODEs · Stochastic Allen–Cahn equation · Monotone SPDEs · Backward Euler scheme · $L^p(\Omega)$-convergence rate

Mathematics Subject Classification 60H35 · 60H10 · 60H15

1 Introduction
There is a general theory on strong error estimations for SODEs and SPDEs with Lipschitz coefficients; see, e.g., the monographs [19, 22] and the references therein. For SODEs with one-sided Lipschitz continuous coefficients which grow super-linearly, the classical Euler–Maruyama (EM) scheme is known to diverge [15]. In particular, the authors in [15] obtained the divergence of $p$-moments for the EM scheme for all $p \in [1, \infty)$. Recently, this divergence result is generalized to the exponential and linear-implicit Euler–Maruyama schemes of second-order parabolic SPDEs driven by white noise [2]. Based on this type of divergence result, one has to use explicit or implicit strategies to design convergent numerical schemes for SDEs with polynomial growth coefficients.
For SODEs with polynomial growth coefficients, there are kinds of literature using explicit strategy including adaptive time step size technique \[10, 18, 23\] and tamed or truncated argument \[6, 13, 16, 30\] based on renormalized-increments to the scheme. Besides, implicit strategy including backward Euler (BE) scheme or its split-step version \[1, 12\] and stochastic \( \theta \) scheme \[33\] is also investigated. These explicit or implicit strategies are generalized to monotone SPDEs with polynomial growth coefficients; see, e.g., \[3, 4, 17, 32\] using tamed or truncated argument, \[11, 20, 21, 26–29\] using BE scheme or its split-step version; see also \[5\] using time splitting scheme. We also refer to \[7–9, 14\] where the authors used exponential integrability of both the exact and numerical solutions to SPDEs with non-Lipschitz continuous coefficients, including the 2D stochastic Navier–Stokes equations, 1D stochastic nonlinear Schrödinger equation, 1D Cahn–Hilliard–Cook equation, and 1D stochastic Burgers equation.

When using explicit strategy, some of the previous results got the convergence rate in \( L^p(\Omega) \)-norm with \( p \geq 2 \) in the additive noise case. For implicit strategy, most of the previous results derived the mean-square convergence rate, i.e., the convergence rate in \( L^p(\Omega) \)-norm with \( p = 2 \). One of the main reasons why it is not a straightforward consequence of \( L^p(\Omega) \)-convergence rate for general \( p \) is that it is not easy to derive the uniform boundedness of the \( p \)-moment of the implicit scheme in the general multiplicative noise case; see \[28\] for \( L^p(\Omega) \)-convergence rate of an implicit scheme for a special monotone SPDE (the stochastic Allen–Cahn equation) driven by additive noise.

The main aim of the present paper is to establish the strong convergence rate of the BE scheme \eqref{be} for monotone SODEs and Galerkin-based BE schemes \eqref{gbe} and \eqref{gbe2} for monotone SPDEs in \( L^p(\Omega) \)-norm with general \( p \geq 4 \). We point out that the convergence rate in almost surely (a.s.) sense, as another kind of important convergence rate, follows immediately from the \( L^p(\Omega) \)-convergence rate for \( p > 2 \) (see Corollary 2.1). Our first step is to derive the uniform boundedness of the \( p \)-moment of the BE scheme (see Proposition 3.1) in Sect. 2. Then we give the \( L^p(\Omega) \)-error representation between the (restricted) exact solution and the BE scheme (see Theorem 2.1). With the polynomial growth condition, we obtain the desired \( L^p(\Omega) \)-convergence rate which is exactly the same as in the Lipschitz case (see Theorem 2.2). The proofs of these \( L^p(\Omega) \)-error representation and convergence rate are given in Sect. 3. These arguments and results are extended in the last section to monotone SPDEs.

2 Preliminaries and main results

2.1 Preliminaries

Let \( T \in (0, \infty) \) be fixed. Consider the SODE

\[ dX_t = b(X_t)dt + \sigma(X_t)dW_t, \quad t \in (0, T], \]

\[ X(0) = X_0, \quad (2.1) \]

driven by an \( \mathbb{R}^m \)-valued Wiener process \( W \) on a filtered probability space \( (\Omega, \mathcal{F}, \mathbb{F} := \{ \mathcal{F}_t \}_{t \geq 0}, \mathbb{P}) \), where \( b : \mathbb{R}^n \rightarrow \mathbb{R}^n \) and \( \sigma : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times m} \) are measurable functions,
and $X_0$ is an $\mathbb{R}^n$-valued $\mathcal{F}_0$-measurable random variable. Our main conditions on the coefficients $b$ and $\sigma$ in Eq. (2.1) are the following monotone and Lipschitz assumptions.

**Assumption 2.1** There exist $L_b \in \mathbb{R}$ and $L_\sigma \geq 0$ such that for all $x, y \in \mathbb{R}^n$,

\[
\langle b(x) - b(y), x - y \rangle \leq L_b |x - y|^2, \quad (A1)
\]

\[
|\sigma(x) - \sigma(y)|_{HS} \leq L_\sigma |x - y|. \quad (B1)
\]

In the above and throughout Sects. 2 and 3, $| \cdot |$ denotes the Euclidean norm on $\mathbb{R}^n$ and $| \cdot |_{HS}$ denotes the Hilbert–Schmidt norm on $\mathbb{R}^{n \times m}$: $|\sigma|_{HS} = (\sum_{i=1}^n \sum_{j=1}^m \sigma_{ij}^2)^{1/2}$.

**Remark 2.1** The conditions (A1) and (B1) yield the following coercivity condition for any $\varepsilon > 0$ and $x, y \in \mathbb{R}^n$:

\[
\langle b(x), x \rangle \leq (L_b + \varepsilon)|x|^2 + \frac{1}{4\varepsilon}|b(0)|^2, \quad (2.2)
\]

\[
|\sigma(x)|_{HS} \leq L_\sigma |x| + |\sigma(0)|_{HS}, \quad (2.3)
\]

and thus the following monotone and coercivity conditions with $L := 2L_b + L_\sigma^2$ and some constants $\alpha, \beta \geq 0$ hold for all $x, y \in \mathbb{R}^n$:

\[
2\langle b(x) - b(y), x - y \rangle + |\sigma(x) - \sigma(y)|_{HS}^2 \leq L|x - y|^2,
\]

\[
2\langle b(x), x \rangle + |\sigma(x)|_{HS}^2 \leq \alpha + \beta|x|^2.
\]

Therefore, Eq. (2.1) possesses a unique $\mathbb{F}$-adapted solution $\{X_t\}_{t \in [0, T]}$ with continuous sample paths (see [25, Theorem 3.1.1]). Moreover, if $\mathbb{E}|X_0|^p < \infty$ for some $p \geq 2$, then there exists $C$ such that (see [12, Lemma 3.2])

\[
\mathbb{E}\sup_{t \in [0, T]} |X_t|^p \leq C(1 + \mathbb{E}|X_0|^p). \quad (2.4)
\]

To introduce the numerical scheme of Eq. (2.1), we set $\mathbb{N}_+ \ni M > T$ and $\tau := T/M \in (0, 1)$. Denote $\mathbb{Z}_M := \{0, 1, \ldots, M\}$, $\mathbb{Z}_M^* := \{1, \ldots, M\}$, and $\delta_k W := W(t_{k+1}) - W(t_k)$ with $t_k := k\tau$, $k \in \mathbb{Z}_{M-1}$, and $t_M = T$. Under (A1) and $L_b\tau < 1$, the following backward Euler (BE) scheme is (a.s.) solvable (see, e.g., [31, Theorem C.2]):

\[
Y_{k+1}^M = Y_k^M + b(Y_{k+1}^M)\tau + \sigma(Y_k^M)\delta_k W, \quad k \in \mathbb{Z}_{M-1}; \quad Y_0^M = X_0. \quad (2.5)
\]

### 2.2 Main results

Now we are in the position to present our main results. Our first result is the following representation of the $L^p(\Omega)$-error estimate between the restricted exact solution $\{X_k : k \in \mathbb{Z}_M\}$ of Eq. (2.1) and the backward Euler scheme $\{Y_k^M : k \in \mathbb{Z}_M\}$ defined in (2.5). In what follows $C$ denotes a generic constant which would be different in each appearance but always independent of the discretization parameters.
Theorem 2.1 Let Assumption 2.1 hold for some $L_b \in \mathbb{R}$ and $L_{\sigma} \geq 0$. Assume that $\mathbb{E}|X_0|^p < \infty$ for some $p \geq 4$, then for any $M \in \mathbb{N}_+$ such that $2L_b \tau < 1$, there exists $C$ such that

$$\mathbb{E} \sup_{k \in \mathbb{Z}_M} |X_{tk} - Y_k^M|^p \leq C \sum_{i=0}^{M-1} \int_{t_i}^{t_{i+1}} (\mathbb{E}|b(X_r) - b(X_{t_{i+1}})|^p + \mathbb{E}|X_r - X_{t_i}|^p)dr. \quad (2.6)$$

With the following polynomial growth conditions on the drift function $b$, we have the same $L^p(\Omega)$-convergence rates between the BE scheme (2.5) and the (restricted) exact solution of Eq. (2.1) as in the Lipschitz case.

Assumption 2.2 There exist $\tilde{L}_b \geq 0$ and $q \geq 1$ such that for $x, y \in \mathbb{R}^n$,

$$|b(x) - b(y)| \leq \tilde{L}_b (1 + |x|^{q-1} + |y|^{q-1})|x - y|. \quad (A2)$$

Remark 2.2 A motivating example of $b$ such that Assumption 2.1 holds true is a polynomial of odd order $q$ with a negative leading coefficient, perturbed by a Lipschitz continuous function.

Theorem 2.2 Let Assumptions 2.1 and 2.2 hold for some $L_b \in \mathbb{R}$, $\tilde{L}_b$, $L_{\sigma} \geq 0$, and $q \geq 1$. Assume that $\mathbb{E}|X_0|^{p(2q-1)} < \infty$ for some $p \geq 4$, then for any $M \in \mathbb{N}_+$ such that $2L_b \tau < 1$, there exists $C$ such that

$$\mathbb{E} \sup_{k \in \mathbb{Z}_M} |X_{tk} - Y_k^M|^p \leq C (1 + \mathbb{E}|X_0|^{p(2q-1)})^{p/2}. \quad (2.7)$$

Remark 2.3 As pointed out in Remark 3.1, the error representation (2.6) and the error estimate (2.7) are also valid when $p = 2$ under the $l^\infty(\mathbb{Z}_M; L^2(\Omega))$-norm. The estimate (2.7) with $p = 2$ under the $l^\infty(\mathbb{Z}_M; L^2(\Omega))$-norm had been shown firstly in [12]; see also [33] where the authors used another type of representations for $\sup_{k \in \mathbb{Z}_M} \mathbb{E}|X_{tk} - Y_k^M|^2$.

Remark 2.4 There is a possible generalization to SODEs with exponential growth drifts for some $\tilde{L}_b$, $\tilde{L}_b \geq 0$:

$$|b(x) - b(y)| \leq \tilde{L}_b \left(1 + e^{\tilde{L}_b|x|} + e^{\tilde{L}_b|y|}\right)|x - y|, \quad x, y \in \mathbb{R}^n.$$

A motivating example is $b(x) = e^{-x}$, $x \in \mathbb{R}$, perturbed by an odd polynomial with a negative leading coefficient and a Lipschitz continuous function. Using the representation (2.6) and previous exponential growth condition, we have

$$\mathbb{E} \sup_{k \in \mathbb{Z}_M} |X_{tk} - Y_k^M|^p \leq C \left(1 + \sup_{t \in [0,T]} \mathbb{E} \exp \left(2p\tilde{L}_b|X_t|\right)\right)^{p/2}. $$

\( \Box \) Springer
It is not clear whether one can apply the exponential integrability criteria in [7, 8, 14] to derive the boundedness of \( \sup_{t \in [0, T]} \mathbb{E}\exp(2pL_b|X_t|) \) depending on certain exponential moments of \( X_0 \). Once this boundedness is established, we obtain the \( L^p(\Omega) \)-error estimate: \( \mathbb{E}\sup_{k \in \mathbb{Z}_M} |X_{tk} - Y^M_k|^p \leq C \tau^{p/2}. \)

As a consequence of the above \( L^p(\Omega) \)-convergence rate (2.7) and Borel–Cantelli lemma, we have the following a.s. convergence rate.

**Corollary 2.1** Let Assumptions 2.1 and 2.2 hold for some \( L_b \in \mathbb{R}, \tilde{L}_b, L_\sigma \geq 0 \), and \( q \geq 1 \). Assume that \( \mathbb{E}|X_0|^{p(2q-1)} < \infty \) for some \( p \geq 4 \), then for any \( M \in \mathbb{N}_+ \) such that \( 2L_b\tau < 1 \) and any \( \alpha \in (0, 1/2 - 1/p) \), there exists a random variable \( K \) with bounded \( p \)-moments such that

\[
\sup_{k \in \mathbb{Z}_M} |X_{tk} - Y^M_k| \leq K \tau^\alpha \text{ a.s.} \tag{2.8}
\]

Moreover, if \( \mathbb{E}|X_0|^p < \infty \) for any \( p \geq 4 \), then (2.8) holds for any \( \alpha \in (0, 1/2) \).

3 \( L^p \)-convergence rate of BE schemes of monotone SODEs

3.1 Stability of backward Euler scheme

We begin with the following moments’ stability of the above BE scheme (2.5).

**Proposition 3.1** Let Assumption 2.1 hold for some \( L_b \in \mathbb{R} \) and \( L_\sigma \geq 0 \) and \( \{Y^M_k : k \in \mathbb{Z}_M\} \) be the solution of the BE scheme (2.5). Assume that \( \mathbb{E}|X_0|^p < \infty \) for some \( p \geq 4 \), then for any \( M \in \mathbb{N}_+ \) such that \( 2L_b\tau < 1 \), there exists \( C \) such that

\[
\mathbb{E}\sup_{k \in \mathbb{Z}_M} |Y^M_k|^p \leq C(1 + \mathbb{E}|X_0|^p). \tag{3.1}
\]

**Proof** For simplicity, set \( b_{k+1} = b(Y^M_{k+1}) \) and \( \sigma_k = \sigma(Y^M_k) \). Testing (2.5) with \( Y^M_{k+1} \) and using the elementary equality

\[
\langle \alpha - \beta, \alpha \rangle = \frac{1}{2}(\|\alpha\|^2 - \|\beta\|^2) + \frac{1}{2}\|\alpha - \beta\|^2 \tag{3.2}
\]

for \( \alpha, \beta \in \mathbb{R}^n \), we have

\[
\frac{1}{2}(\|Y^M_{k+1}\|^2 - \|Y^M_k\|^2) + \frac{1}{2}|Y^M_{k+1} - Y^M_k|^2 = (Y^M_{k+1}, b_{k+1})\tau + (Y^M_{k+1} - Y^M_k, \sigma_k\delta_k W) + (Y^M_k, \sigma_k\delta_k W).
\]
By the conditions (2.2)–(2.3) and Cauchy–Schwarz inequality, there exists $C$ such that

$$
\frac{1}{2} (|Y_{k+1}^M|^2 - |Y_k^M|^2) + \frac{1}{2} |Y_{k+1}^M - Y_k^M|^2 \\
\leq C (\tau + |\delta_k W|^2) + (L_b + \varepsilon)|Y_{k+1}^M|^2 \tau + \frac{1}{2} |Y_{k+1}^M - Y_k^M|^2 \\
+ L^2_\sigma |Y_k^M|^2 |\delta_k W|^2 + \langle Y_k^M, \sigma_k \delta_k W \rangle.
$$

It follows that

$$
(1 - 2(L_b + \varepsilon)\tau)|Y_{k+1}^M|^2 \\
\leq C (\tau + |\delta_k W|^2) + (1 + 2L^2_\sigma |\delta_k W|^2)|Y_k^M|^2 \\
+ 2 \langle Y_k^M, \sigma_k \delta_k W \rangle,
$$

and thus for $k \in \mathbb{Z}_M^*$,

$$
(1 - 2(L_b + \varepsilon)\tau)|Y_k^M|^2 \\
\leq C + C \sum_{i=0}^{k-1} |\delta_i W|^2 + \sum_{i=0}^{k-1} (2(L_b + \varepsilon)\tau + 2L^2_\sigma |\delta_i W|^2)|Y_i^M|^2 \\
+ 2 \sum_{i=0}^{k-1} \langle Y_i^M, \sigma_i \delta_i W \rangle.
$$

Since $2L_b \tau < 1$, one can choose $\varepsilon > 0$ such that $1 - 2(L_b + \varepsilon)\tau \geq C_0$ for some $C_0 \in (0, 1)$. Then

$$
\mathbb{E} \sup_{k \in \mathbb{Z}_M^*} |Y_k^M|^p \leq C + C \mathbb{E} \left( \sum_{i=0}^{M-1} |\delta_i W|^2 \right)^{p/2} + C \mathbb{E} \left( \sum_{i=0}^{M-1} (\tau + |\delta_i W|^2)|Y_i^M|^2 \right)^{p/2} \\
+ C \mathbb{E} \sup_{k \in \mathbb{Z}_M^*} \left| \sum_{i=0}^{k-1} \langle Y_i^M, \sigma_i \delta_i W \rangle \right|^{p/2}.
$$

Using the elementary inequality

$$
\left( \sum_{i=0}^{M-1} a_i \right)^\gamma \leq M^{\gamma-1} \sum_{i=0}^{M-1} a_i^\gamma, \quad \forall \gamma \geq 1, \ a_i \geq 0,
$$

(3.3)
and the independence between $\delta_i W$ and $Y^M_i$, we derive
\[
\mathbb{E}\left(\sum_{i=0}^{M-1} |\delta_i W|^2\right)^{p/2} \leq M^{p/2-1} \sum_{i=0}^{M-1} \mathbb{E}|\delta_i W|^p \leq C
\]
\[
\mathbb{E}\left(\sum_{i=0}^{M-1} (\tau + |\delta_i W|^2)|Y^M_i|^2\right)^{p/2} \leq \left(\sum_{i=0}^{M-1} \| (\tau + |\delta_i W|^2)|Y^M_i|^2 \|_{L^p(\Omega)}\right)^{p/2}
\]
\[
\leq C \tau^{p/2} \left(\sum_{i=0}^{M-1} \|Y^M_i\|_{L^p(\Omega)}^2\right)^{p/2} \leq C \tau \sum_{i=0}^{M-1} \mathbb{E}|Y^M_i|^p.
\]

The same argument, in combination with the discrete Burkholder–Davis–Gundy inequality $\| \sum_{i=0}^{m} Z_i \|_{L^p(\Omega)} \leq C(\sum_{i=0}^{m} \|Z_i\|_{L^{p/2}(\Omega)}^{2})^{1/2}$ which holds true for any discrete martingale $\{Z_m\}_{m \in \mathbb{Z}_M}$ with bounded $p/2 \geq 2$-moments (see, e.g., [27, Lemma 2.2]), the condition (2.3), and Young inequality, yields
\[
\mathbb{E}\left(\sum_{i=0}^{k-1} \langle Y^M_i, \sigma_i \delta_1 W \rangle\right)^{p/2} \leq C \left(\sum_{i=0}^{M-1} \| \langle Y^M_i, \sigma_i \delta_1 W \rangle \|_{L^{p/2}(\Omega)}^2\right)^{p/4}
\]
\[
\leq C + C \tau \sum_{i=0}^{M-1} \mathbb{E}|Y^M_i|^p, \quad \forall \ p \geq 4.
\]

Combining the above three inequalities implies
\[
\mathbb{E}\sup_{k \in \mathbb{Z}_M} |Y^M_k|^p \leq C + \mathbb{E}|X_0|^p + C \tau \sum_{i=0}^{M-1} \mathbb{E}|Y^M_i|^p.
\]

We conclude (3.1) by the above inequality and Grönwall lemma. \hfill \Box

**Remark 3.1** When $p = 2$, the expectation of the stochastic term $\langle Y^M_i, \sigma_i \delta_1 W \rangle$ in the proof of Proposition 3.1 vanishes, so we can use the same argument as in the proof of Proposition 3.1 to get the following type of second moment’s estimation:
\[
\sup_{k \in \mathbb{Z}_M} \mathbb{E}|Y^M_k|^2 \leq C(1 + \mathbb{E}|X_0|^2),
\]
provided (A1)–(B1) hold and $\mathbb{E}|X_0|^2 < \infty$.

**Remark 3.2** The $p$-moment’s estimation with $p \in (2, 4)$ is unknown. In our argument, to apply the elementary inequality (3.3) and discrete Burkholder–Davis–Gundy inequality the restriction $p \geq 4$ is needed in the first and second inequalities of (3.4), respectively. This is the main reason why our results (e.g., Theorems 2.1, 2.2, and 4.1) hold true only for $p = 2$ and $p \geq 4$.\[ Springer]
3.2 Proof of Theorem 2.1

For $k \in \mathbb{Z}_M$, denote $e_k := X_{t_k} - Y^M_k$. It is clear that for all $k \in \mathbb{Z}_{M-1}$,

$$X_{t_{k+1}} = X_{t_k} + \int_{t_k}^{t_{k+1}} b(X_r)dr + \int_{t_k}^{t_{k+1}} \sigma(X_r)dW_r.$$ 

The above equality and (2.5) yield that

$$e_{k+1} - e_k = \int_{t_k}^{t_{k+1}} (b(X_r) - b(X_{t_{k+1}}))dr + \int_{t_k}^{t_{k+1}} (\sigma(X_r) - \sigma(X_{t_k}))dW_r + (b(X_{t_{k+1}}) - b(Y^M_{k+1}))\tau + (\sigma(X_{t_k}) - \sigma(Y^M_k))\delta_k W.$$ 

Testing with $e_{k+1}$ and using the identity (3.2), we have

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$$\left(\frac{1}{2}|e_{k+1}|^2 - |e_k|^2\right) + \frac{1}{2}|e_{k+1} - e_k|^2 = \langle e_{k+1}, \int_{t_k}^{t_{k+1}} (b(X_r) - b(X_{t_{k+1}}))dr \rangle + \langle e_{k+1}, b(X_{t_{k+1}}) - b(Y^M_{k+1})\rangle \tau$$

$$+ \langle e_{k+1}, \int_{t_k}^{t_{k+1}} (\sigma(X_r) - \sigma(Y^M_k))dW_r \rangle. \quad (3.5)$$

By Cauchy–Schwarz inequality and the condition (A1), we get

$$\left\langle e_{k+1}, \int_{t_k}^{t_{k+1}} (b(X_r) - b(X_{t_{k+1}}))dr \right\rangle + \langle e_{k+1}, b(X_{t_{k+1}}) - b(Y^M_{k+1})\rangle \tau$$

$$\leq (L_b + \varepsilon)\tau|e_{k+1}|^2 + \frac{1}{4\varepsilon} \int_{t_k}^{t_{k+1}} |b(X_r) - b(X_{t_{k+1}})|^2 dr, \quad \forall \varepsilon > 0,$$

$$= \langle e_{k+1}, \int_{t_k}^{t_{k+1}} (\sigma(X_r) - \sigma(Y^M_k))dW_r \rangle$$

$$= \left\langle e_{k+1} - e_k, \int_{t_k}^{t_{k+1}} (\sigma(X_r) - \sigma(Y^M_k))dW_r \right\rangle + \left\langle e_k, \int_{t_k}^{t_{k+1}} (\sigma(X_r) - \sigma(Y^M_k))dW_r \right\rangle$$

$$\leq \frac{1}{2}|e_{k+1} - e_k|^2 + \int_{t_k}^{t_{k+1}} (\sigma(X_r) - \sigma(X_{t_k}))dW_r \right|^2 + L^2_\sigma |e_k|^2|\delta_k W|^2$$

$$+ \left\langle e_k, \int_{t_k}^{t_{k+1}} (\sigma(X_r) - \sigma(Y^M_k))dW_r \right\rangle.$$ 

Substituting the above inequalities into (3.5), we obtain

$$\left(1 - 2(L_b + \varepsilon)\tau\right)|e_{k+1}|^2 - (1 + 2L^2_\sigma|\delta_k W|^2)|e_k|^2$$

$$\leq \frac{1}{2\varepsilon} \int_{t_k}^{t_{k+1}} |b(X_r) - b(X_{t_{k+1}})|^2 dr + 2 \int_{t_k}^{t_{k+1}} (\sigma(X_r) - \sigma(X_{t_k}))dW_r \right|^2 + 2R_k.$$
where $R_k := \langle e_k, \int_{t_k}^{t_{k+1}} (\sigma(X_r) - \sigma(Y^{M}_{k}'))dW_r \rangle$. Then for $k = 1, \cdots, M$,

\[
(1 - 2(L_b + \varepsilon)\tau)|e_k|^2 \leq 2 \sum_{i=0}^{k-1} ((L_b + \varepsilon)\tau + L^2_\sigma |\delta_i W|^2)|e_i|^2 + 2 \sum_{i=0}^{k-1} R_i \\
+ \frac{1}{2\varepsilon} \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} |b(X_r) - b(X_{t_{i+1}})|^2 dr + 2 \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} (\sigma(X_r) - \sigma(X_{t_i}))dW_r \right)^2.
\]

(3.6)

For any $M \in \mathbb{N}_+$ such that $2L_b \tau < 1$, one can choose $\varepsilon > 0$ such that $1 - 2(L_b + \varepsilon)\tau \geq C_0$ for some $C_0 \in (0, 1)$. For any $p \geq 2$, the inequality (3.6) yields the existence of $C$ such that

\[
\mathbb{E} \sup_{k \in \mathbb{Z}_M} |e_k|^p \\
\leq C \mathbb{E} \left( \sum_{i=0}^{M-1} (\tau + |\delta_i W|^2)|e_i|^2 \right)^{p/2} + C \mathbb{E} \left( \sum_{i=0}^{M-1} \int_{t_i}^{t_{i+1}} |b(X_r) - b(X_{t_{i+1}})|^2 dr \right)^{p/2} \\
+ C \mathbb{E} \left( \sum_{i=0}^{M-1} \int_{t_i}^{t_{i+1}} (\sigma(X_r) - \sigma(X_{t_i}))dW_r \right)^{2} + C \mathbb{E} \sup_{k \in \mathbb{Z}_M} \left| \sum_{i=0}^{k-1} R_i \right|^{p/2}.
\]

Using the inequality (3.3), the independence between $\delta_i W$ and $e_i$, Hölder and Burkholder–Davis–Gundy inequalities, and the condition (B1), we derive

\[
\mathbb{E} \sup_{k \in \mathbb{Z}_M} |e_k|^p \leq C \tau \sum_{i=0}^{M-1} \mathbb{E}|e_i|^p + C \sum_{i=0}^{M-1} \int_{t_i}^{t_{i+1}} \mathbb{E}|b(X_r) - b(X_{t_{i+1}})|^p dr \\
+ C \sum_{i=0}^{M-1} \int_{t_i}^{t_{i+1}} \mathbb{E}|X_r - X_{t_i}|^p dr + C \mathbb{E} \sup_{k \in \mathbb{Z}_M} \left| \sum_{i=0}^{k-1} R_i \right|^{p/2}.
\]

(3.7)

Finally, using both the discrete and continuous Burkholder–Davis–Gundy inequalities, the inequality (3.3), Hölder and Cauchy–Schwarz inequalities, and the condition (B1), we have

\[
\mathbb{E} \sup_{k \in \mathbb{Z}_M} \left| \sum_{i=0}^{k-1} R_i \right|^{p/2} \leq C \left( \sum_{i=0}^{M-1} \left\langle \int_{t_i}^{t_{i+1}} (e_i, (\sigma(X_r) - \sigma(Y^{M}_{i}'))dW_r \right\rangle \right)^{2} L^{p/2} \\
\leq C \left( \sum_{i=0}^{M-1} \left\langle \int_{t_i}^{t_{i+1}} (e_i, (\sigma(X_r) - \sigma(Y^{M}_{i}))dW_r \right\rangle \right)^{2} L^{p/2} \\
\leq CM^{p/4-1} \sum_{i=0}^{M-1} \left( \int_{t_i}^{t_{i+1}} \left\| e_i, (\sigma(X_r) - \sigma(Y^{M}_{i})) \right\|_{L^{p/2}}^2 dr \right)^{p/4}
\]

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\[ \leq C M^{p/4-1} \tau^{p/4-1} \sum_{i=0}^{M-1} \int_{t_i}^{t_{i+1}} \| (e_i, \sigma(X_r) - \sigma(Y^M_i)) \|^2 \| \mu \|^2 dr \]

\[ \leq C \sum_{i=0}^{M-1} \int_{t_i}^{t_{i+1}} \left( \mathbb{E}|e_i|^p + \mathbb{E}|X_r - X_{t_i}|^p \right) dr \]

\[ \leq C \tau \sum_{i=0}^{M-1} \mathbb{E}|e_i|^p + C \sum_{i=0}^{M-1} \int_{t_i}^{t_{i+1}} \mathbb{E}|X_r - X_{t_i}|^p dr. \]

The above two inequalities, in combination with Grönwall inequality, complete the proof of Theorem 2.1.

### 3.3 Proof of Theorem 2.2

We begin with the following Hölder continuity of the exact solution in the whole time interval.

**Lemma 3.1** Let Assumptions 2.1 and 2.2 hold for some \( L_b \in \mathbb{R}, \tilde{L}_b, L_\sigma \geq 0, \) and \( q \geq 1. \) Assume that \( \mathbb{E}|X_0|^{pq} < \infty \) for some \( p \geq 2, \) then there exists a constant \( C \) such that

\[ \mathbb{E}|X_t - X_s|^p \leq C (1 + \mathbb{E}|X_0|^{pq})|t-s|^{p/2}, \quad t, s \in [0, T]. \]  

(3.8)

**Proof** Without loss of generality, we assume \( 0 \leq s \leq t \leq T. \) Then

\[ X_t - X_s = \int_s^t b(X_r)dr + \int_s^t \sigma(X_r)dW_r. \]

Using Burkholder–Davis–Gundy and Hölder inequalities and the conditions (A2) and (2.3), we get

\[ \mathbb{E}|X_t - X_s|^p \leq C(t-s)^{p-1} \int_s^t \mathbb{E}|b(X_r)|^p dr + C(t-s)^{p/2-1} \int_s^t \mathbb{E}|\sigma(X_r)|_{H}^p dr \]

\[ \leq C(t-s)^p \sup_{t \in [0,T]} \mathbb{E}|b(X_t)|^p + C(t-s)^{p/2} \sup_{t \in [0,T]} \mathbb{E}|\sigma(X_t)|_{H}^p \]

\[ \leq C(t-s)^{p/2} (1 + \sup_{t \in [0,T]} \mathbb{E}|X_t|^{pq}). \]

Then we get (3.8) by the above estimate and the estimate (2.4). \( \square \)

Now we can give the proof of Theorem 2.2. By (2.6), (A2), the Hölder inequality

\[ \mathbb{E}|Z_1|^p |Z_2|^p \leq (\mathbb{E}|Z_1|^{p(2q-1)/q)}^{q/(2q-1)} (\mathbb{E}|Z_2|^{p(2q-1)/q-1)}^{q-1)/(2q-1)} , \]

\[ \mathcal{S} \text{ Springer} \]
with \( Z_1 = X_r - X_{t_{i+1}} \) and \( Z_2 = 1 + |X_r|^{q-1} + |X_{t_{i+1}}|^{q-1} \), and the Hölder estimate (3.8), we have

\[
\mathbb{E} \sup_{k \in \mathbb{Z}_M} |X_{t_k} - Y_k^M|^p
\leq C \sum_{i=0}^{M-1} \int_{t_i}^{t_{i+1}} \mathbb{E}|b(X_r) - b(X_{t_i+1})|^p + \mathbb{E}|X_r - X_{t_i}|^p \, dr
\leq C(1 + \sup_{t \in [0,T]} \mathbb{E}|X_t|^p)^{\frac{q-1}{2q-1}} \left( \sup_{t \neq s} \frac{\mathbb{E}|X_t - X_s|^p}{|t-s|^{p(2q-1)/q}} \right)^{\frac{2q}{2q-1}} \tau^{p/2}
\leq C(1 + \mathbb{E}|X_0|^p)^{\frac{p(2q-1)}{2q-1}} \tau^{p/2},
\]

which shows (2.7) and we complete the proof of Theorem 2.2.

**4 \( L^p \)-convergence rate of Galerkin-based BE schemes of monotone SPDEs**

In the last section, we apply our main results, Theorems 2.1–2.2, to the following second-order SPDEs with polynomial growth coefficients:

\[
\frac{\partial X_t(\xi)}{\partial t} = \Delta X_t(\xi) + b(X_t(\xi)) + \sigma(X_t(\xi)) \frac{dW_t(\xi)}{dt}, \quad (t, \xi) \in (0, T] \times \mathcal{O}, \quad (4.1)
\]

under the homogeneous Dirichlet boundary condition \( X_t(\xi) = 0 \), \( (t, \xi) \in [0, T] \times \partial \mathcal{O} \), with the initial random variable \( X_0(\xi), \xi \in \mathcal{O} \), where \( \mathcal{O} \subset \mathbb{R}^d \) with \( d = 1, 2, 3 \) is an open, bounded set with piecewise smooth boundary.

For \( r \in [2, \infty) \) and \( \theta \in \mathbb{R} \), we set \( (L^r = L^r(\mathcal{O}), \| \cdot \|_{L^r}) \) and \( (\dot{H}^{-\theta} = \dot{H}^{-\theta}(\mathcal{O}), \| \cdot \|_{\dot{H}^{-\theta}}) \) the usual Lebesgue and Sobolev interpolation spaces, respectively. In particular, we denote \( H := L^2, V := H^1, \) and \( V^* := H^1 \). Denote the inner product and norm in \( H \) by \( \langle \cdot, \cdot \rangle \) and \( \| \cdot \| \), respectively. The norms in \( V \) and \( V^* \) and the dual between them are denoted by \( \| \cdot \|_1, \| \cdot \|_{-1}, \) and \( \| \langle \cdot, \cdot \rangle \|_{-1} \), respectively. Let \( Q \) be a self-adjoint and nonnegative definite linear operator on \( H \). Denote \( H_0 := Q^{1/2} H \) and denote by \( (\mathcal{L}_2^0 := HS(H_0; H), \| \cdot \|_{\mathcal{L}_2^0}) \) and \( (\mathcal{L}_2^1 := HS(H_0; V), \| \cdot \|_{\mathcal{L}_2^1}) \) the space of Hilbert–Schmidt operators from \( H_0 \) to \( H \) and \( V \), respectively. We also use \( \mathcal{L}(H_0, H) \) to denote the space of bounded linear operators \( H_0 \) to \( H \).

The driving process \( W \) in Eq. (4.1) is an infinite-dimensional \( H \)-valued \( Q \)-Wiener process, which has the Karhunen–Loève expansion

\[
W(t) = \sum_{k \in \mathbb{N}_+} \sqrt{\lambda_k} g_k \beta_k(t), \quad t \in [0, T].
\]

Here \( \{g_k\}_{k=1}^\infty \) forming an orthonormal basis of \( H \) are the eigenvectors of \( Q \) subject to the eigenvalues \( \{\lambda_k\}_{k=1}^\infty \), and \( \{\beta_k\}_{k=1}^\infty \) are mutually independent Brownian motions in…
\((\Omega, \mathcal{F}, \mathbb{P})\). We mainly focus on trace-class noise, i.e., \(Q\) is a trace-class operator, or equivalently, \(\sum_{k \in \mathbb{N}_+} \lambda_k < \infty\).

Denote by \(A\) the homogeneous Dirichlet Laplacian operator. For \(q \geq 1\), define by \(F : L^{(q+1)'} \to L^{q+1}\) the Nemytskii operator associated with \(b\):

\[
F(u)(\xi) := b(u(\xi)), \quad u \in V, \; \xi \in \mathcal{O},
\]

where \((q+1)'\) denotes the conjugation of \((q+1)\), i.e., \(1/(q+1)' + 1/(q+1) = 1\). Then it follows from the monotone condition (A1) in Assumption 2.1 that the operator \(F\) defined in (4.2) has a continuous extension from \(L^{q+1}\) to \(L^{(q+1)'}\) and satisfies

\[
L^{(q+1)'} \langle F(x) - F(y), x - y \rangle_{L^{q+1}} \leq L_b \|x - y\|^2, \quad x, y \in L^{q+1},
\]

where \(L^{(q+1)'} \langle \cdot, \cdot \rangle_{L^{q+1}}\) denotes the dual between \(L^{(q+1)'}\) and \(L^{q+1}\). In particular, if \(q \geq 1\) for \(d = 1, 2\) and \(q \in [1, 3]\) for \(d = 3\), one has the embeddings

\[
V \subset L^{2q} \subset L^{q+1} \subset H \subset L^{(q+1)'} \subset L^{(2q)'} \subset V^*,
\]

so that (4.3) leads to

\[
1 \langle u - v, F(u) - F(v) \rangle_{-1} \leq L_b \|u - v\|^2, \quad u, v \in V.
\]

Denote by \(G : H \to \mathcal{L}^0_2\) the Nemytskii operator associated with \(\sigma\):

\[
G(u)g_k(\xi) := \sigma(u(\xi))g_k(\xi), \quad u \in H, \; k \in \mathbb{N}_+, \; \xi \in \mathcal{O}.
\]

Then the SPDE (4.1) is equivalent to the stochastic evolution equation

\[
dX_t = (AX_t + F(X_t))dt + G(X_t)dW_t, \quad t \in (0, T],
\]

\[
X(0) = X_0.
\]

We focus on Galerkin-based BE schemes of Eq. (4.7). Let \(h \in (0, 1)\), \(\mathcal{T}_h\) be a regular family of partitions of \(\mathcal{O}\) with maximal length \(h\), and \(V_h \subset V\) be the space of continuous functions on \(\bar{\mathcal{O}}\) which are piecewise linear over \(\mathcal{T}_h\) and vanish on the boundary \(\partial \mathcal{O}\). Let \(A_h : V_h \to V_h\) and \(\mathcal{P}_h : V^* \to V_h\) be the discrete Laplacian and generalized orthogonal projection operators, respectively, defined by

\[
\langle A_h u^h, v^h \rangle = -\langle \nabla u^h, \nabla v^h \rangle, \quad u^h, v^h \in V_h,
\]

\[
\langle \mathcal{P}_h u, v^h \rangle = \langle v^h, u \rangle_{-1}, \quad u \in V^*, \; v^h \in V_h.
\]

Let \(N \in \mathbb{N}_+\) and \(V_N\) be the space spanned by the first \(N\)-eigenvectors of the Dirichlet Laplacian operator which vanish on the boundary \(\partial \mathcal{O}\). Similarly, one can define the spectral Galerkin approximate Laplacian and generalized orthogonal projection operators, respectively, as

\[
\langle A_N u^N, v^N \rangle = -\langle \nabla u^N, \nabla v^N \rangle, \quad u^N, v^N \in V_N.
\]
\[ \langle \mathcal{P}_N u, v^N \rangle_1 = \langle v^N, u \rangle_{-1}, \quad u \in V^*, \ v^N \in V_N. \]

Our main condition on the diffusion operator \( G \) defined in (4.6) is the following assumption.

**Assumption 4.1** The operator \( G : H \to \mathcal{L}^0_2 \) defined in (4.6) is Lipschitz continuous, i.e., there exists a constant \( L_\sigma \geq 0 \) such that

\[ \|G(u) - G(v)\|_{\mathcal{L}^0_2} \leq L_\sigma \|u - v\|, \quad u, v \in H. \] (B2)

Moreover, \( G(\dot{H}^1) \subset \mathcal{L}^1_2 \) and there exists a constant \( \widetilde{L}_\sigma \geq 0 \) such that

\[ \|G(z)\|_{\mathcal{L}^1_2} \leq \widetilde{L}_\sigma (1 + \|z\|_1), \quad z \in V. \] (B3)

Under conditions (A1), (A2), (B2), and (B3), Eq. (4.1), or equivalently, Eq. (4.7) with initial datum \( X_0 \in V \) possesses a unique \( \mathbb{F} \)-adapted solution \( \{X_t\}_{t \in [0,T]} \) in \( V \) with continuous sample paths, see [24]. Moreover, we proved in [27, Theorem 3.1] the following moment’s estimation provided \( X_0 \in L^p(\Omega; V) \) with \( p \geq q + 1, \ q \geq 1 \) for \( d = 1, 2 \) and \( q \in [1, 3] \) for \( d = 3 \):

\[ \mathbb{E} \sup_{t \in [0,T]} \|X_t\|_1^p \leq C(1 + \mathbb{E}\|X_0\|_1^p). \] (4.8)

The Galerkin finite element-based BE scheme for Eq. (4.7) is to find a sequence of \( \mathbb{F} \)-adapted \( V_h \)-valued process \( \{X^h_m : m \in \mathbb{Z}_{M-1}\} \) such that

\[ X^h_{m+1} = X^h_m + \tau A_h X^h_m + \tau \mathcal{P}_h F(X^h_m) + \mathcal{P}_h G(X^h_m) \delta_m W, \quad m \in \mathbb{Z}_{M-1}, \] (4.9)

with the initial value \( X^h_0 = \mathcal{P}_h X_0 \). We call it finite element BE scheme. It is clear that the finite element BE scheme (4.9) is equivalent to the scheme

\[ X^h_{m+1} = S_{h,\tau} X^h_m + \tau S_{h,\tau} \mathcal{P}_h F(X^h_m) + S_{h,\tau} \mathcal{P}_h G(X^h_m) \delta_m W, \quad m \in \mathbb{Z}_{M-1}, \] (4.10)

with initial datum \( X^h_0 = \mathcal{P}_h X_0 \), where \( S_{h,\tau} := (\text{Id} - \tau A_h)^{-1} \) is a space-time approximation of the continuous semigroup \( S \) in one step. Iterating (4.10) for \( m \)-times, we obtain for \( m \in \mathbb{Z}_{M-1} \),

\[ X^h_{m+1} = S_{h,\tau}^{m+1} X^h_0 + \tau \sum_{i=0}^{m} S_{h,\tau}^{m+1-i} \mathcal{P}_h F(X^h_{i+1}) + \sum_{i=0}^{m} S_{h,\tau}^{m+1-i} \mathcal{P}_h G(X^h_i) \delta_i W. \] (4.11)
Similarly, the spectral Galerkin BE scheme for Eq. (4.7) is to find a sequence of $\mathbb{F}$-adapted $V_h$-valued process $\{X^N_m : m \in \mathbb{Z}_M\}$ such that

$$X^N_{m+1} = X^N_m + \tau A_N X^N_{m+1} + \tau \mathcal{P}_N F(X^N_{m+1}) + \mathcal{P}_N G(X^N_m) \delta_m W, \quad m \in \mathbb{Z}_{M-1},$$

(4.12)

with the initial value $X^N_0 = \mathcal{P}_N X_0$.

Using the idea in the SODEs setting in Sect. 3, we have the following $L^p(\Omega)$-convergence rate of the finite element BE scheme (4.9). Note that the case $p = 2$ had been shown in [27, Theorems 1.1 and 4.1].

**Theorem 4.1** Let (A1), (A2), (B2), and (B3) hold for some $L_b \in \mathbb{R}$, $\widehat{L}_b, L_\sigma, \widehat{L}_\sigma \geq 0$ and $q \geq 1$ for $d = 1, 2$ and $q \in [1, 3]$ for $d = 3$. Assume that $\mathbb{E}\|X_0\|_{1}^{p(q^2+q-1)} < \infty$ for some $p \geq \max\{4, q + 1\}$, then for any $M \in \mathbb{N}_+$ such that $2L_b \tau < 1$, there exists $C$ such that

$$\sup_{k \in \mathbb{Z}_M} \mathbb{E}\|X_{t_k} - X^h_k\|_1^p \leq C(1 + \mathbb{E}\|X_0\|_{1}^{p(q^2+q-1)})(h^p + \tau^{p/2}),$$

(4.13)

$$\sup_{k \in \mathbb{Z}_M} \mathbb{E}\|X_{t_k} - X^N_k\|_1^p \leq C(1 + \mathbb{E}\|X_0\|_{1}^{p(q^2+q-1)})(N^{-p/d} + \tau^{p/2}).$$

(4.14)

**Proof** It suffices to prove (4.13); similar arguments would immediately yield (4.14).

To this end, we introduce the auxiliary process

$$\tilde{X}^h_{m+1} = S_{h, \tau}^{m+1} X^0_0 + \sum_{i=0}^{m} \tau \sum_{i=0}^{m} S_{h, \tau}^{m+1-i} \mathcal{P}_h F(X_{t_i+1}) + \sum_{i=0}^{m} S_{h, \tau}^{m+1-i} \mathcal{P}_h G(X_{t_i}) \delta_i W,$$

(4.15)

for $m \in \mathbb{Z}_{M-1}$, where the terms $X^h_{t_i+1}$ and $X^h_{t_i}$ in the discrete deterministic and stochastic convolutions of (4.11) are replaced by $X_{t_i+1}$ and $X_{t_i}$, respectively. In view of the stability (4.8) of the exact solution $X$ to Eq. (4.1), the well-known uniform boundedness of the discrete semigroup $S_{h, \tau}$, and the conditions (A2) and (B3), it is not difficult to show that (see [27, (4.11)])

$$\sup_{k \in \mathbb{Z}_M} \mathbb{E}\|\tilde{X}^h_k\|_1^p \leq C(1 + \mathbb{E}\|X_0\|_{1}^{pq}).$$

(4.16)

In [27, Lemma 4.2], we had proved that

$$\sup_{k \in \mathbb{Z}_M} \mathbb{E}\|X_{t_k} - \tilde{X}^h_k\|_1^p \leq C(1 + \mathbb{E}\|X_0\|_{1}^{p(q^2-1)})(h^p + \tau^{p/2}).$$

(4.17)

It remains to estimate $\sup_{k \in \mathbb{Z}_M} \mathbb{E}\|\tilde{X}^h_k - X^h_k\|_p$. For $k \in \mathbb{Z}_M$, denote $e^h_k := \tilde{X}^h_k - X^h_k$.

From (4.15), it is clear that

$$\tilde{X}^h_{m+1} = \tilde{X}^h_m + \tau A_h \tilde{X}^h_{m+1} + \tau \mathcal{P}_h F(X_{t_{m+1}}) + \mathcal{P}_h G(X_{t_m}) \delta_m W.$$
Then by (4.9) and the above equality, we have

\[ e_{k+1}^h - e_k^h = \tau \Delta e_{k+1}^h + \tau (F(X_{t_{k+1}}) - F(X_{k+1}^h)) + (G(X_t) - G(X_k^h))\delta_k W. \tag{4.18} \]

Testing with \( e_{k+1}^h \) and using the inequality (3.2) which holds also for all \( \alpha, \beta \in H \) and the conditions (4.5) and (B2), we obtain

\[
\frac{1}{2} (\|e_{k+1}^h\|^2 - \|e_k^h\|^2) + \frac{1}{2} \|e_{k+1}^h - e_k^h\|^2 + \|\nabla e_{k+1}^h\|^2 \tau
\]

\[= \langle e_{k+1}^h, F(X_{t_{k+1}}) - F(X_{k+1}^h) \rangle + \tau (e_{k+1}^h, F(X_{k+1}^h) - F(X_{k+1})) - \tau \langle e_{k+1}^h, (G(X_t) - G(X_k^h))\delta_k W \rangle + \langle e_k^h, (G(X_t) - G(X_k^h))\delta_k W \rangle\]

\[\leq \|\nabla e_{k+1}^h\|^2 \tau + L_b \tau \|e_{k+1}^h\|^2 + \langle e_k^h, (G(X_t) - G(X_k^h))\delta_k W \rangle + C \tau \|F(X_{t_{k+1}}) - F(X_{k+1}^h)\|_1^2
\]

\[+ \frac{1}{2} \|e_{k+1}^h - e_k^h\|^2 + L_\alpha^2 \|X_t - \tilde{X}_k^h\|_2^2 \|\delta_k W\|^2 + L_\alpha^2 \|e_k^h\|^2 \|\delta_k W\|^2.\]

It follows that

\[(1 - 2L_b \tau)\|e_{k+1}^h\|^2 \leq (1 + 2L_\alpha^2 \|\delta_k W\|^2)\|e_k^h\|^2 + C \tau \|F(X_{t_{k+1}}) - F(X_{k+1}^h)\|_1^2
\]

\[+ 2L_\alpha^2 \|X_t - \tilde{X}_k^h\|_2^2 \|\delta_k W\|^2 + 2\langle e_k^h, (G(X_t) - G(X_k^h))\delta_k W \rangle.\]

Using the same arguments to derive (3.7) in the proof of Theorem 2.1, we get

\[
\sup_{k \in \mathbb{Z}_M} \mathbb{E} \|e_k^h\|^p \leq C \tau \sum_{i=0}^{M-1} \mathbb{E} \|e_i^h\|^p + C \tau^{p/2} \mathbb{E} \left( \sum_{i=0}^{M-1} \|F(X_{t_{i+1}}) - F(X_{k+1}^h)\|_1^2 \right)^{p/2}
\]

\[+ C \mathbb{E} \left( \sum_{i=0}^{M-1} \|X_t - \tilde{X}_i^h\|_2^2 \|\delta_k W\|^2 \right)^{p/2}
\]

\[+ C \mathbb{E} \left( \sum_{i=0}^{M-1} \langle e_i^h, (G(X_t) - G(X_i^h))\delta_i W \rangle \right)^{p/2}
\]

\[\leq C \tau \sum_{i=0}^{M-1} \mathbb{E} \|e_i^h\|^p + C \sup_{k \in \mathbb{Z}_M} \mathbb{E} \|F(X_{t_{k+1}}) - F(X_{k+1}^h)\|_1^p
\]

\[+ C \sup_{k \in \mathbb{Z}_M} \mathbb{E} \|X_t - \tilde{X}_k^h\|^p + C \mathbb{E} \left( \sum_{i=0}^{M-1} \langle e_i^h, (G(X_t) - G(X_i^h))\delta_i W \rangle \right)^{p/2}.\]
Applying discrete Burkholder–Davis–Gundy inequality as used in (3.4) and Cauchy–Schwarz inequality, we have
\[
\mathbb{E}\left|\sum_{i=0}^{M-1} (e_i^h, (G(X_t^h) - G(X_t^h))\delta_t W)\right|^{p/2} \leq C\tau \sum_{i=0}^{k-1} \mathbb{E}\|e_i^h\|^p + C \sup_{k \in \mathbb{Z}_M} \mathbb{E}\|X_{t_k} - \tilde{X}_k^h\|^p.
\]
Consequently, we get from the discrete Grönwall inequality that
\[
\sup_{k \in \mathbb{Z}_M} \mathbb{E}\|e_k^h\|^p \leq C \left( \sup_{k \in \mathbb{Z}_M} \mathbb{E}\|X_{t_k} - \tilde{X}_k^h\|^p + \sup_{k \in \mathbb{Z}_M} \mathbb{E}\|F(X_{t_k}) - F(\tilde{X}_k^h)\|^p \right).
\] (4.19)

It follows from the embedding (4.4) and the condition (A2) that
\[
\|F(u) - F(v)\|_{-1} \leq C(1 + \|u\|_{L_2q}^{q-1} + \|v\|_{L_2q}^{q-1})\|u - v\|, \quad u, v \in V.
\] (4.20)

Then (4.19) implies
\[
\sup_{k \in \mathbb{Z}_M} \mathbb{E}\|e_k^h\|^p \leq C \sup_{k \in \mathbb{Z}_M} \mathbb{E}\left[ (1 + \|X_{t_k}\|_1^{p(q-1)} + \|\tilde{X}_k^h\|_1^{p(q-1)})\|X_{t_k} - \tilde{X}_k^h\|^p \right]
\leq C \left( 1 + \sup_{t \in [0,T]} \mathbb{E}\|X_t\|_1^{p(q^2+q-1)} + \sup_{k \in \mathbb{Z}_M} \mathbb{E}\|\tilde{X}_k^h\|_1^{p(q^2+q-1)} \right)^{\frac{q^2+q-1}{q^2+q-1}}
\times \left( \sup_{k \in \mathbb{Z}_M} \mathbb{E}\|X_{t_k} - \tilde{X}_k^h\|_1^{p(q^2+q-1)} \right)^{\frac{2q-1}{2q^2+q-1}}.
\]

We conclude (4.13) by the above estimate, the stability (4.8) and (4.16), and the error estimate (4.17). □

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