Super $w_\infty$ 3-algebra

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Abstract

We investigate the super high-order Virasoro 3-algebra. By applying the appropriate scaling limits on the generators, we obtain the super $w_\infty$ 3-algebra which satisfies the generalized fundamental identity condition. We also define a super Nambu-Poisson bracket which satisfies the generalized skewsymmetry, Leibniz rule and fundamental identity. By means of this super Nambu-Poisson bracket, the realization of the super $w_\infty$ 3-algebra is presented.

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1 Introduction

Since Nambu [1] first proposed 3-bracket for the generalized Hamiltonian dynamics, 3-algebras have attracted much interest from physical and mathematical points of view. Recently with the development of string theory, it is found that 3-algebras have the important applications in M-theory, such as multiple M2-branes [2]-[6] and Nambu-Poisson M5-brane theory [7]-[9]. 3-algebras can be realized in two ways, i.e., classical Nambu and quantal brackets. These two brackets are defined by means of multi-variable Jacobians and antisymmetrized products of three linear operators, respectively. The properties of various 3-algebras have been widely investigated [9]-[12]. The Virasoro algebra is an infinite-dimensional algebra which plays an important role in conformal field theory. The centerless Virasoro 3-algebra, i.e., Virasoro-Witt 3-algebra has been constructed in the literature [13]-[15]. The \( W_\infty \) algebra is the higher-spin extensions of the Virasoro algebra [16]. Quite recently by applying a double scaling limits on the generators of the \( W_\infty \) algebra, Chakrabortty et al.[17] constructed a \( w_\infty \) 3-algebra which satisfies the fundamental identity (FI) condition of 3-algebra.

The supersymmetric generalizations of 3-algebras are of general interest [18]-[23]. Recently Sakakibara [21] constructed the super Nambu-Poisson algebra and demonstrated its connection with the generalized Batalin-Vilkovisky algebra. Soroka et al. proposed the Grassmann-odd Nambu brackets and investigated their properties in their serial paper [22][23]. The supersymmetric generalizations of the Virasoro and \( w \) algebras have been well investigated. As to their super 3-algebras, to our best knowledge, it has not been reported so far in the existing literature. The propose of this paper is to present the super \( w_\infty \) 3-algebra which satisfies the generalized FI condition.

This paper is organized as follows. In the next section, we investigate the super high-order Virasoro (SHOV) algebra and its 3-algebra. Then by applying an appropriate scaling limits on the generators of the SHOV algebra, we give the super \( w_\infty \) 3-algebra. In section 3, we define the super Nambu-Poisson bracket to realize the super \( w_\infty \) 3-algebra. We end this paper with the concluding remarks in section 4.

2 SHOV algebra and super \( w_\infty \) 3-algebra

The generators of SHOV algebra are given by

\[
L_i^m = (-1)^i \lambda_i^{m+i} z^{m+i} \partial_z^i,
\]

\[
\bar{L}_i^m = (-1)^i \lambda_i^{m+i} \theta \partial_\theta^i,
\]

\[
h_\alpha^{i+\frac{1}{2}} = (-1)^{i+1} \lambda_\alpha^{i+\frac{1}{2}} z^{r+\alpha} \partial_z^\alpha,
\]

\[
\bar{h}_\alpha^{i+\frac{1}{2}} = (-1)^{i+1} \lambda_\alpha^{i+\frac{1}{2}} z^{r+\alpha} \partial_\theta^\alpha,
\]

where \( i, \alpha \in \mathbb{Z}_+, m, r \in \mathbb{Z} \) and \( \lambda \) is an arbitrary parameter.

Their communication relations are

\[
[L_i^m, L_j^n] = \sum_{p=0}^{i} (-1)^p \lambda^{p-j} C_i^p B_{i+j}^p L_{m+n}^{i+j-p} - \sum_{p=0}^{j} (-1)^p \lambda^{p-i} C_j^p B_{i+j}^p L_{m+n}^{i+j-p},
\]

\[
[L_i^m, \bar{L}_j^n] = \sum_{p=0}^{i} (-1)^p \lambda^{p-j} C_i^p B_{i+j}^p \bar{L}_{m+n}^{i+j-p} - \sum_{p=0}^{j} (-1)^p \lambda^{p-i} C_j^p B_{i+j}^p \bar{L}_{m+n}^{i+j-p},
\]

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\[ [L^i_m, h^\alpha_r + \frac{1}{2}] = \sum_{p=0}^{i} (-1)^p \lambda^{p-\frac{1}{2}} C^p_i B^\alpha_{p+m+r} h^\alpha_{m+r} - \sum_{p=0}^{\alpha} (-1)^p \lambda^{p-\frac{1}{2}} C^p_{\alpha} B^m_{p} h^\alpha_{m+r} + i^\alpha h^\alpha_{m+r}, \]

\[ [L^i_m, \tilde{L}^j_n] = \sum_{p=0}^{i} (-1)^p \lambda^{p+\frac{1}{2}} C^p_i B^{n+j}_{p+m+n} - \sum_{p=0}^{j} (-1)^p \lambda^{p+\frac{1}{2}} C^p_j B^{m+i}_{p+j+n}, \]

\[ [L^i_m, \tilde{h}^\alpha_r + \frac{1}{2}] = \sum_{p=0}^{i} (-1)^p \lambda^{p+\frac{1}{2}} C^p_i B^\alpha_{p+m+r} h^\alpha_{m+r} + \tilde{h}^\alpha_{m+r}, \]

\[ [h^\alpha_r + \frac{1}{2}, \tilde{h}^\beta_s + \frac{1}{2}] = \sum_{p=0}^{\alpha} (-1)^p \lambda^{p+\frac{1}{2}} C^p_{\alpha} B^{\alpha+\beta}_{p+r+s} \tilde{L}^{\alpha+\beta}_{r+s} + \sum_{p=0}^{\beta} (-1)^p \lambda^{p+\frac{1}{2}} C^p_{\beta} B^{\alpha+\beta}_{p} \tilde{L}^{\alpha+\beta}_{r+s} - \sum_{p=0}^{\alpha} (-1)^p \lambda^{p+\frac{1}{2}} C^p_{\alpha} B^{\alpha+\beta}_{p} \tilde{L}^{\alpha+\beta}_{r+s}, \number{2} \]

where \( B^m_p = \begin{cases} n(n-1) \cdots (n-p+1), & p \leq n \\ 0, & p > n \end{cases} \), \( C^m_n = \frac{n(n-1) \cdots (n-m+1)}{m!} \) and the communication relation is defined by

\[ [f, g] = fg - (-1)^{|f||g|} gf, \number{3} \]

\(|f|\) and \(|g|\) are the parity of \(f\) and \(g\), respectively. A notational convention used frequently in the rest of this paper is that for any arbitrary \(h\), the symbol \(|h|\) appearing in the exponent of \((-1)\) is to be understood as the parity of \(h\). When \(\lambda = 1, \number{2}\) leads to the communication relations of SHOV algebra derived by Zha and Zhao \([24]\). Recently Chakrabortty et al. \([17]\) constructed a \(w_\infty\) 3-algebra by applying a double scaling limits on the generators of \(W_\infty\) algebra. In order to construct the super \(w_\infty\) 3-algebra, we introduce a parameter \(\lambda\) into the generators of SHOV algebra \([11]\). Not as done by Chakrabortty et al., we’ll construct the super \(w_\infty\) 3-algebra by taking a single scaling limit on the generators \([11]\). Therefore the parameter \(\lambda\) plays a crucial role in the following investigation.

Let us define a super 3-bracket as follows:

\[ [f, g, h] = [f, g]h + (-1)^{|f||g|+|h|}[g, h]f + (-1)^{|h||f|+|g|}[h, f]g, \number{4} \]

where the commutator [ , ] is defined by \([3]\). Substituting the generators \([11]\) into the generalized ternary commutator \([11]\), we may obtain the SHOV 3-algebra. Due to too many 3-algebra
relations, we only list some of them that will be used in the late discussion.

\[
[L^i_m, L^j_n, L^k_h] = \left( \sum_{p=0}^{i} C^p_i B^{n+j} - \sum_{p=0}^{j} C^p_j B^{m+i} \right) \sum_{q=0}^{i+j-p} C^q_{i+j-p} B^{k+h} + \sum_{p=0}^{j} C^p_j B^{k+h} \\
- \sum_{p=0}^{h} C^p_h B^{n+j} \sum_{q=0}^{j+h-p} C^q_{j+h-p} B^{m+i} + \left( \sum_{p=0}^{h} C^p_h B^{m+i} \right) \\
- \sum_{p=0}^{i} C^p_i B^{k+h} \sum_{q=0}^{h+i-p} C^q_{h+i-p} B^{n+j}(-1)^{p+q} \lambda^{p+q-1} L^{i+j+h-p-q}_{m+n+k},
\]

\[
[L^i_m, L^j_n, \tilde{L}^k_h] = \left( \sum_{p=0}^{i} C^p_i B^{n+j} - \sum_{p=0}^{j} C^p_j B^{m+i} \right) \sum_{q=0}^{i+j-p} C^q_{i+j-p} B^{k+h} + \sum_{p=0}^{j} C^p_j B^{k+h} \\
- \sum_{p=0}^{h} C^p_h B^{n+j} \sum_{q=0}^{j+h-p} C^q_{j+h-p} B^{m+i} + \left( \sum_{p=0}^{h} C^p_h B^{m+i} \right) \\
- \sum_{p=0}^{i} C^p_i B^{k+h} \sum_{q=0}^{h+i-p} C^q_{h+i-p} B^{n+j}(-1)^{p+q} \lambda^{p+q-1} \tilde{L}^{i+j+h-p-q}_{m+n+k},
\]

\[
[L^i_m, L^j_n, h^{\alpha+\frac{1}{2}}_r] = \left( \sum_{p=0}^{i} C^p_i B^{n+j} - \sum_{p=0}^{j} C^p_j B^{m+i} \right) \sum_{q=0}^{i+j-p} C^q_{i+j-p} B^{r+\alpha} + \sum_{p=0}^{j} C^p_j B^{r+\alpha} \\
- \sum_{p=0}^{\alpha} C^p_\alpha B^{n+j} \sum_{q=0}^{j+\alpha-p} C^q_{j+\alpha-p} B^{m+i} + \left( \sum_{p=0}^{\alpha} C^p_\alpha B^{m+i} \right) \\
- \sum_{p=0}^{i} C^p_i B^{r+\alpha} \sum_{q=0}^{\alpha+i-p} C^q_{\alpha+i-p} B^{n+j}(-1)^{p+q} \lambda^{p+q-1} h^{i+j+\alpha-p-q+\frac{1}{2}}_{m+n+r},
\]

\[
[L^i_m, L^j_n, \tilde{h}^{\alpha+\frac{1}{2}}_r] = \left( \sum_{p=0}^{i} C^p_i B^{n+j} - \sum_{p=0}^{j} C^p_j B^{m+i} \right) \sum_{q=0}^{i+j-p} C^q_{i+j-p} B^{r+\alpha} + \sum_{p=0}^{j} C^p_j B^{r+\alpha} \\
- \sum_{p=0}^{\alpha} C^p_\alpha B^{n+j} \sum_{q=0}^{j+\alpha-p} C^q_{j+\alpha-p} B^{m+i} + \left( \sum_{p=0}^{\alpha} C^p_\alpha B^{m+i} \right) \\
- \sum_{p=0}^{i} C^p_i B^{r+\alpha} \sum_{q=0}^{\alpha+i-p} C^q_{\alpha+i-p} B^{n+j}(-1)^{p+q} \lambda^{p+q-1} \tilde{h}^{i+j+\alpha-p-q+\frac{1}{2}}_{m+n+r},
\]
\[
[L_{m}^{i}, h_{r}^{\alpha+{\frac{1}{2}}}, h_{s}^{\beta+{\frac{1}{2}}}] = \sum_{p=0}^{i} C_{p}^{q} B_{m}^{p+\alpha} - \sum_{q=0}^{\alpha+\beta-p} C_{q}^{r} B_{m}^{q+\alpha} + \sum_{p=0}^{i} C_{p}^{q} B_{m}^{i+\alpha-p} B_{q}^{\alpha+\beta-p-q} + \sum_{p=0}^{i} C_{p}^{q} B_{m}^{i+\alpha-p} B_{q}^{\alpha+\beta-p-q} - \sum_{p=0}^{i} C_{p}^{q} B_{m}^{i+\alpha-p} B_{q}^{\alpha+\beta-p-q}.
\]

Let us take the scaling limit \( \lambda \to 0 \) and for convenience denote the generators with the same notations for this and other kinds of limit throughout this paper, then from (5), we obtain the following super \( w_{\infty} \) 3-algebra:

\[
[L_{m}^{i}, L_{n}^{j}, L_{k}^{k}] = (h(n-m) + j(m-k) + i(k-n))L_{m+n+k}^{i+j+h-1},
\]

\[
[L_{m}^{i}, L_{n}^{j}, \bar{L}_{k}^{k}] = -[L_{m}^{i}, \bar{L}_{k}^{k}, L_{n}^{j}] = [\bar{L}_{k}^{k}, L_{m}^{i}, L_{n}^{j}] = (h(n-m) + j(m-k) + i(k-n))L_{m+n+k}^{i+j+h-1},
\]

\[
[L_{m}^{i}, L_{n}^{j}, h_{r}^{\alpha+{\frac{1}{2}}}] = -[L_{m}^{i}, h_{r}^{\alpha+{\frac{1}{2}}}, L_{n}^{j}] = [h_{r}^{\alpha+{\frac{1}{2}}}, L_{m}^{i}, L_{n}^{j}] = (\alpha(n-m) + j(m-r) + i(r-n))L_{m+n+r}^{i+j+\alpha-1+{\frac{1}{2}}},
\]

\[
[L_{m}^{i}, L_{n}^{j}, \bar{h}_{r}^{\alpha+{\frac{1}{2}}}] = -[L_{m}^{i}, \bar{h}_{r}^{\alpha+{\frac{1}{2}}}, L_{n}^{j}] = [\bar{h}_{r}^{\alpha+{\frac{1}{2}}}, L_{m}^{i}, L_{n}^{j}] = (\alpha(n-m) + j(m-r) + i(r-n))L_{m+n+r}^{i+j+\alpha-1+{\frac{1}{2}}},
\]

\[
[L_{m}^{i}, h_{r}^{\alpha+{\frac{1}{2}}}, \bar{h}_{s}^{\beta+{\frac{1}{2}}}] = [L_{m}^{i}, \bar{h}_{s}^{\beta+{\frac{1}{2}}}, h_{r}^{\alpha+{\frac{1}{2}}}] = [h_{r}^{\alpha+{\frac{1}{2}}}, \bar{h}_{s}^{\beta+{\frac{1}{2}}}, L_{m}^{i}] = [\bar{h}_{s}^{\beta+{\frac{1}{2}}}, h_{r}^{\alpha+{\frac{1}{2}}}, L_{m}^{i}]
\]

\[
= -[\bar{h}_{s}^{\beta+{\frac{1}{2}}}, L_{m}^{i}, \bar{h}_{r}^{\alpha+{\frac{1}{2}}}]
\]

\[
= \sum_{p=0}^{i} C_{p}^{q} B_{m}^{i+\alpha-p} B_{q}^{\alpha+\beta-p-q} - \sum_{p=0}^{i} C_{p}^{q} B_{m}^{i+\alpha-p} B_{q}^{\alpha+\beta-p-q} - [\bar{h}_{s}^{\beta+{\frac{1}{2}}}, L_{m}^{i}, \bar{h}_{r}^{\alpha+{\frac{1}{2}}}]
\]

\[
= (i(r-s) + \alpha(s-m) + \beta(m-r))L_{m+r+s}^{i+j+\alpha+\beta-1},
\]

with all other 3-brackets vanishing. It should be pointed out that under this kind of scaling limit we have the non-null 3-algebra (6), but (7) becomes the null algebra. Note that the first 3-algebraic relation in (7) is the so-called \( w_{\infty} \) 3-algebra derived in Ref. [17]. It is known that this \( w_{\infty} \) 3-algebra satisfies the usually FI condition. As to the super \( w_{\infty} \) 3-algebra (6), it should satisfy the generalized FI condition due to the involution of fermionic generators. We find that the super \( w_{\infty} \) 3-algebra (6) satisfies the following generalized FI condition:

\[
[A, B, [C, D, E]] = [[A, B, C], D, E] + (-1)^{(|A|+|B|)(|C|)}[C, [A, B, D], E] + (-1)^{(|A|+|B|)(|C|+|D|)}[C, D, [A, B, E]],
\]
3 Super Nambu-Poisson bracket

For the graded communicating bracket (3), it is well-known that the classical limit is given by

\[ \{ , \} = \lim_{\hbar \to 0} \frac{1}{\hbar^2} [ , ] . \]  

(8)

Let us take the generators of SHOV algebra as follows:

\[
\begin{align*}
L^i_m &= (-i\hbar)^i z^{m+i} \partial_z^i, \\
\tilde{L}^i_m &= (-i\hbar)^{i+2} z^{m+i} \partial_\theta \partial_z^i, \\
\bar{h}_r^{\alpha+1/2} &= (-i\hbar)^{\alpha+1} z^{r+\alpha} \partial_\theta \partial_z^\alpha, \\
\tilde{h}_r^{\alpha+1/2} &= (-i\hbar)^{\alpha+1} z^{r+\alpha} \partial_\theta \partial_z^\alpha, \\
\end{align*}
\]

(9)

where \( i = \sqrt{-1} \) and \( \hbar \) is introduced in the above generators. Substituting the generators (9) into (3) and taking the classical limit (11), we obtain the classical super \( \omega_\infty \) algebra

\[
\begin{align*}
\{ L^i_m, L^j_n \} &= (mj - ni)L^i_{m+j+n-1}, \\
\{ L^i_m, \tilde{L}^j_n \} &= (mj - ni)\tilde{L}^i_{m+j+n-1}, \\
\{ L^i_m, \bar{h}_r^{\alpha+1/2} \} &= (m\alpha - r\alpha)\bar{h}_r^{i+m+j-\alpha-1/2}, \\
\{ L^i_m, \tilde{h}_r^{\alpha+1/2} \} &= (m\alpha - r\alpha)\tilde{h}_r^{i+m+j-\alpha-1/2}, \\
\{ \tilde{L}^i_m, \bar{h}_r^{\alpha+1/2} \} &= \{ \tilde{L}^i_m, \tilde{h}_r^{\alpha+1/2} \} = \{ L^i_m, L^j_n \} = 0, \\
\{ \bar{h}_r^{\alpha+1/2}, \tilde{h}_s^{\beta+1/2} \} &= (sa - r\beta)\bar{h}_s^{i+m+j-\alpha-\beta-1}. \\
\end{align*}
\]

(10)

This super \( \omega_\infty \) algebra is different with ones derived by Pope and Shen [25].

Let us turn to discuss the case of 3-bracket. For the 3-bracket defined by (4), we require the following classical limit to be hold [10]:

\[ \{ , , \} = \lim_{\hbar \to 0} \frac{1}{\hbar^3} [ , , ] . \]

(11)

where \( \{ , , \} \) should be understood as the super Nambu-Poisson bracket.

Substituting the generators (9) into (4) and taking the classical limit (11), we obtain the classical super \( \omega_\infty \) 3-algebra

\[
\begin{align*}
\{ L^i_m, L^j_n, L^k_n \} &= (h(n - m) + j(m - k) + i(k - n))L^i_{m+n+k-1} , \\
\{ L^i_m, L^j_n, \tilde{L}^k_n \} &= -\{ L^i_m, \tilde{L}^j_n, L^k_n \} = \{ \tilde{L}^k_n, L^i_m, L^j_n \} \\
&= (h(n - m) + j(m - k) + i(k - n))\tilde{L}^i_{m+n+k-1} , \\
\{ L^i_m, L^j_n, \bar{h}_r^{\alpha+1/2} \} &= -\{ L^i_m, \bar{h}_r^{\alpha+1/2}, L^j_n \} = \{ \bar{h}_r^{\alpha+1/2}, L^i_m, L^j_n \} \\
&= (\alpha(n - m) + j(m - r) + i(r - n))\bar{h}_r^{i+j+\alpha-1+1/2} , \\
\{ L^i_m, L^j_n, \tilde{h}_r^{\alpha+1/2} \} &= -\{ L^i_m, \tilde{h}_r^{\alpha+1/2}, L^j_n \} = \{ \tilde{h}_r^{\alpha+1/2}, L^i_m, L^j_n \} \\
&= (\alpha(n - m) + j(m - r) + i(r - n))\tilde{h}_r^{i+j+\alpha-1+1/2} , \\
\end{align*}
\]
The limit relation (11) can be regarded as the analog of (8). Takhtajan [10] investigated the fact that the limit relation (8) establishes the relation between classical and quantum mechanics. The algebra and non-null super Nambu-Poisson bracket:

\[ \{ f, g, h \} = \frac{\partial f}{\partial x_1} \frac{\partial g}{\partial x_2} \frac{\partial h}{\partial x_3} = \sum_{\sigma} (-1)^{\varepsilon(\sigma)} \frac{\partial f}{\partial x_{\sigma(1)}} \frac{\partial g}{\partial x_{\sigma(2)}} \frac{\partial h}{\partial x_{\sigma(3)}}, \]

for the super Nambu-Poisson variables does not be introduced in (13). By means of (13), it is easy to prove that the following generalization of skew-symmetric properties are satisfied:

\[
\begin{align*}
\{ g, f, h \} &= (-1)^{1 + |f||g|} \{ f, g, h \}, \\
\{ f, h, g \} &= (-1)^{1 + |f||h|} \{ f, g, h \}, \\
\{ h, g, f \} &= (-1)^{1 + |g||h| + |f||h|} \{ f, g, h \}. & (14)
\end{align*}
\]

Let us consider the 3-bracket \( \{ f_1 f_2, g, h \} \). By means of (13), we obtain

\[
\begin{align*}
\{ f_1 f_2, g, h \} &= \sum_{\sigma} (-1)^{\varepsilon(\sigma)} \left( \frac{\partial f_1}{\partial x_{\sigma(1)}} f_2 + f_1 \frac{\partial f_2}{\partial x_{\sigma(2)}} \right) \frac{\partial g}{\partial x_{\sigma(3)}} \frac{\partial h}{\partial x_{\sigma(3)}} \\
&= \sum_{\sigma} (-1)^{\varepsilon(\sigma)} (-1)^{|f_1||f_2|} f_1 \frac{\partial f_2}{\partial x_{\sigma(1)}} \frac{\partial g}{\partial x_{\sigma(2)}} \frac{\partial h}{\partial x_{\sigma(3)}} \\
&\quad + \sum_{\sigma} (-1)^{\varepsilon(\sigma)} f_1 \frac{\partial f_2}{\partial x_{\sigma(1)}} \frac{\partial g}{\partial x_{\sigma(2)}} \frac{\partial h}{\partial x_{\sigma(3)}} \\
&= f_1 \{ f_2, g, h \} + (-1)^{|f_1||f_2|} f_2 \{ f_1, g, h \}. & (15)
\end{align*}
\]

In a similar way, we have another two Leibniz rule relations

\[
\begin{align*}
\{ f, g_1 g_2, h \} &= (-1)^{|g_2||f|} g_1 \{ f, g_1, h \} + (-1)^{|g_1||f|} g_1 \{ f, g_2, h \} \\
&= (-1)^{|g_2||f|} g_1 \{ f, g_1, h \} g_2 + (-1)^{|g_1||f|} g_1 \{ f, g_2, h \}, & (16)
\end{align*}
\]
\[ \{f, g, h_1 h_2\} = \{f, g, h_1\} h_2 + (-1)^{|f||h_1|} h_1 \{f, g, h_2\} \\
= \{f, g, h_1\} h_2 + (-1)^{|h_1||h_2|} \{f, g, h_2\} h_1. \]

(17)

To discuss the fundamental identity, we first give the following relations by means of (13):

\[
\{A, B, \{C, D, E\}\} = \{A, B, \sum_{\sigma'} (-1)^{\varepsilon(\sigma') + \varepsilon(\sigma')} \frac{\partial C}{\partial x_{\sigma'(1)}} \frac{\partial D}{\partial x_{\sigma'(2)}} \frac{\partial E}{\partial x_{\sigma'(3)}} \} \\
= \sum_{\sigma, \sigma'} (-1)^{\varepsilon(\sigma) + \varepsilon(\sigma')} \frac{\partial A}{\partial x_{\sigma(1)}} \frac{\partial B}{\partial x_{\sigma(2)}} \frac{\partial^2 C}{\partial x_{\sigma(1)} \partial x_{\sigma(2)} \partial x_{\sigma(3)}} \frac{\partial D}{\partial x_{\sigma(2)}} \frac{\partial E}{\partial x_{\sigma(3)}} \\
+ \frac{\partial C}{\partial x_{\sigma(1)}} \frac{\partial^2 D}{\partial x_{\sigma(2)} \partial x_{\sigma(3)}} \frac{\partial E}{\partial x_{\sigma(3)}} + \frac{\partial C}{\partial x_{\sigma(1)}} \frac{\partial D}{\partial x_{\sigma(2)}} \frac{\partial^2 E}{\partial x_{\sigma(3)}},
\]

(18)

\[
\{\{A, B, C\}, D, E\} = \sum_{\sigma, \sigma'} (-1)^{\varepsilon(\sigma) + \varepsilon(\sigma')} \frac{\partial^2 C}{\partial x_{\sigma(1)} \partial x_{\sigma(2)} \partial x_{\sigma(3)}} \frac{\partial D}{\partial x_{\sigma(2)}} \frac{\partial E}{\partial x_{\sigma(3)}} + \frac{\partial A}{\partial x_{\sigma(1)}} \frac{\partial B}{\partial x_{\sigma(2)}} \frac{\partial D}{\partial x_{\sigma(3)}} \\
+ \frac{\partial^2 B}{\partial x_{\sigma(1)} \partial x_{\sigma(2)} \partial x_{\sigma(3)}} \frac{\partial E}{\partial x_{\sigma(3)}} \\
\{C, \{A, B, D\}, E\} = \sum_{\sigma, \sigma'} (-1)^{\varepsilon(\sigma) + \varepsilon(\sigma')} \frac{\partial C}{\partial x_{\sigma'(1)}} \frac{\partial D}{\partial x_{\sigma'(2)}} \frac{\partial^2 A}{\partial x_{\sigma'(1)} \partial x_{\sigma'(2)} \partial x_{\sigma'(3)}} \frac{\partial B}{\partial x_{\sigma'(2)}} \frac{\partial E}{\partial x_{\sigma'(3)}} \\
+ \frac{\partial A}{\partial x_{\sigma'(1)}} \frac{\partial^2 B}{\partial x_{\sigma'(2)} \partial x_{\sigma'(3)}} \frac{\partial E}{\partial x_{\sigma'(3)}} \\
\{C, D, \{A, B, E\}\} = \sum_{\sigma, \sigma'} (-1)^{\varepsilon(\sigma) + \varepsilon(\sigma')} \frac{\partial C}{\partial x_{\sigma'(1)}} \frac{\partial D}{\partial x_{\sigma'(2)}} \frac{\partial^2 A}{\partial x_{\sigma'(1)} \partial x_{\sigma'(2)} \partial x_{\sigma'(3)}} \frac{\partial B}{\partial x_{\sigma'(2)}} \frac{\partial E}{\partial x_{\sigma'(3)}} \\
+ \frac{\partial A}{\partial x_{\sigma'(1)}} \frac{\partial^2 B}{\partial x_{\sigma'(2)} \partial x_{\sigma'(3)}} \frac{\partial E}{\partial x_{\sigma'(3)}} + \frac{\partial A}{\partial x_{\sigma'(1)}} \frac{\partial B}{\partial x_{\sigma'(2)}} \frac{\partial^2 E}{\partial x_{\sigma'(3)}},
\]

(19)

(20)

(21)

Substituting equations (18)-(21) into (14), we find that the super Nambu-Poisson bracket (13) satisfies the generalized FI condition (17) with the substitution \([\ , \ , \ ] \to \{\ , \ , \ \}\).

Let us take

\[
L^i_m = \sqrt{2} \exp[i\frac{1}{2}x - 2my],
\]

\[
\bar{L}^i_m = \bar{\theta}_1 \theta_2 \sqrt{2} \exp[i\frac{1}{2}x - 2my],
\]

\[
h^\alpha_+ \frac{1}{2} = \theta_1 \sqrt{2} \exp[(\alpha - \frac{1}{2})x - 2ry],
\]

\[
h^{-\alpha + \frac{1}{2}} = -\bar{\theta}_2 \sqrt{2} \exp[(\alpha - \frac{1}{2})x - 2ry].
\]

(22)

It is noticed from (22) that \(L^i_m\) is the generator of \(w_\infty\) 3-algebra presented in (17). Substituting the generators (22) into (13), we obtain the classical super \(w_\infty\) 3-algebra (12).
4 Concluding Remarks

We have investigated the SHOV 3-algebra. By applying the appropriate scaling limits on the generators, we obtained the super $w_\infty$ 3-algebra. We also found that this super $w_\infty$ 3-algebra satisfies the generalized FI condition. To present the realization of the super $w_\infty$ 3-algebra, we defined a super Nambu-Poisson bracket which satisfies the generalized skewsymmetry, Leibniz rule and fundamental identity, but the derivative term with respect to the Grassmann variables are not involved in the 3-bracket. Moreover, we tried to introduce the Grassmann derivative term in the super Nambu-Poisson bracket such that the super $w_\infty$ 3-algebra can be realized. Unfortunately, we did not succeed in finding any one. Whether there exists this kind of super Nambu-Poisson bracket still deserves further study. Furthermore the application of this super $w_\infty$ 3-algebra in physics should be of interest.

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