Spiders and their Kin ($K_n$)

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December 11, 2018

Abstract

In this paper we consider a host of properties of chromatic symmetric functions for different graphs in what we call the phylum arthropoda, namely generalized spider graphs and horseshoe crab graphs. Properties of chromatic symmetric functions for specific graph classes have long been studied. One of the fundamental questions is whether a chromatic symmetric function uniquely determines a tree. This question was posed first by Stanley in 1995 and it remains an open problem, although it has been answered in the affirmative for a number of special classes of trees including caterpillars and spiders. Here we show the result holds for generalized spiders (line graphs of spiders), thereby extending the work of Martin, Morin and Wagner. A second fundamental question is whether a chromatic symmetric function is e-positive. Here we establish that certain classes of generalized spiders (generalized nets) are not e-positive, and we use yet another class of generalized spiders to construct a counterexample to a conjecture involving the e-positivity of claw-free, $P_4$-sparse graphs, showing that Tsujie’s result on the e-positivity of claw-free, $P_4$-free graphs cannot be extended to this set of graphs. Finally, we show that another type of “arthropods,” the horseshoe crab graphs (a class of natural unit interval graphs) are e-positive. This generalizes the work of Gebhard and Sagan and Cho and Huh.

1 Introduction

In the quest to establish properties of chromatic symmetric functions, researchers have considered several individual families of graphs, e.g. spiders, caterpillars, squids, and crabs [18]; lollipops and lariats [4]. In this spirit we explore two members of the phylum arthropoda, considering generalized spiders and horseshoe crabs (no relation to the crabs of [18]). In graph theoretic terms these amount to cliques with pendant vertices, attached paths, or some other type of protrusion. In this way these graphs generalize both the lollipops ($K_n$ with one path attached) and nets ($K_3$ with three pendant vertices).

Chromatic symmetric functions, defined by Richard Stanley [23] in 1995, are a generalization of the chromatic polynomial and are a graph invariant that provides information about the structure of the graph, including the number of vertices, edges, and possible acyclic orientations. In 1993 Stanley and Stembridge [25] conjectured that the chromatic symmetric functions of claw-free incomparability graphs can be written as a linear combination of elements of the elementary basis with nonnegative

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coefficients, a property called \(e\)-positivity. This led to a great deal of investigation into classes of \(e\)-positive graphs (by abuse of notation we say a graph is \(e\)-positive if its chromatic symmetric function is \(e\)-positive). Additionally, Stanley [23], in his 1995 paper, conjectured that the chromatic symmetric function distinguishes non-isomorphic trees. Since then Gasharov [6] proved that claw-free incomparability graphs are \(s\)-positive. Regarding the stronger condition of \(e\)-positivity, Gebhard and Sagan [7] and Dahlberg and van Willigenburg [11], have proved that \(k\)-chains and lollipop graphs are \(e\)-positive. Later Dahlberg, Foley, and van Willigenburg [5] gave three infinite classes of graphs that are not \(e\)-positive. Furthermore Guay-Paquet [10] reduced the class of graphs that need to be considered in order to prove the conjecture to natural unit interval graphs.

For uniqueness, Martin, Morrin, and Wagner [18] have established that particular trees known as spiders and caterpillars are uniquely determined by their chromatic symmetric function. Tsujie [26] also proved that trivially perfect graphs are uniquely determined by their chromatic symmetric function.

There has also been work involving the quasisymmetric refinement of the chromatic symmetric function, which was introduced by Shareshian and Wachs [22]. This led to a generalization of some of the aforementioned conjectures and also opened the door to new proof techniques. For example Cho and Huh [3] used injections in order to prove \(e\)-positivity of certain classes of natural unit interval graphs. Harada and Precup [12] have found that these functions have connections to Hessenberg varieties and Pawlowski [20] has studied chromatic symmetric functions through the group algebra of the symmetric group.

This paper starts by outlining some basic definitions and formulas from graph and symmetric function theory in Section 2. In Section 3 we take Stanley’s basic example of a claw-free, non-\(e\)-positive graph, known as the net (see Figure 2a), generalize the graph to create an infinite family, and give an explicit formula for the chromatic symmetric function in terms of the elementary symmetric functions, thereby proving it is not \(e\)-positive. We also use this result to show that the class of claw-free, \(P_4\)-free graphs, which have been proven to be \(e\)-positive by Tsujie in [26], cannot be further extended to claw-free, \(P_4\)-sparse graphs. In Section 4 we switch our focus to a question about uniqueness to show that an even further generalization of these graphs is uniquely determined by its chromatic symmetric function. Hence we establish a new class of uniquely distinguishable non-tree graphs. To do so we rely on several existing results on the information encoded in the chromatic symmetric function. Finally in Section 5 we look at the chromatic quasisymmetric function of natural unit interval graphs. Concerning this we prove \(e\)-positivity in the quasisymmetric sense for a certain class of graphs we call horseshoe crab graphs. In order to do so we use and generalize the method of weight preserving injections introduced by Cho and Huh. The last section, Section 6, looks to future work.

## 2 Background and Notation

We will start by defining a few basic graph theoretic concepts and some algebraic terms. A graph, \(G\), is a set of vertices, \(V\), and a set of edges, \(E \subseteq V \times V\) (formally, \(G = (V, E)\)). A tree is an acyclic connected graph. A coloring of a graph \(G\) is a function,

\[ \kappa : V \to \mathbb{N}. \]

A coloring is considered proper, if \(\kappa(u) \neq \kappa(v)\), where \(u\) and \(v\) are connected vertices. A spider, sometimes referred to as a starlike tree, is a tree with exactly one unique vertex with degree greater than two.

Given a graph \(G\), let \(H\) be the graph with vertices corresponding to the edges of \(G\) and edges between vertices which correspond to edges in \(G\) that share a vertex. This graph, \(H\), is called the line graph of \(G\). See an example in Figure 1.

Given any graph, \(G\), we can define an induced subgraph, \(H\), as the graph with vertex set \(V' \subset V\) and edge set

\[ \tilde{E} = \left\{ \{i,j\} \mid \{i,j\} \in \binom{V'}{2} \cap E \right\}. \]
A graph is said to be \textbf{H-free} if it does not contain $H$ as an induced subgraph. In this subject area it is of particular interest to study claw-free graphs, where a \textit{claw} is defined as the complete bipartite graph $K_{1,3}$. Let $P$ be a partially ordered set. We define the \textit{incomparability graph} of $P$ as the graph with vertex set $V = P$ and edge set $E$, where $\{x, y\} \in E$ if $x$ and $y$ are incomparable. A poset is said to be \textbf{(a+b)-free}, if it does not contain an induced disjoint union of chains of length $a$ and $b$. Note that the conditions claw-free and incomparability graph can also be expressed as being an incomparability graph of a $(3+1)$-free poset. Another very important class of graphs are \textbf{natural unit interval graphs}, which are precisely incomparability graphs of $(2+2)$ and $(3+1)$-free posets. For further definitions and information see Section 5.

If $G$ has a vertex set $V = \{v_1, v_2, v_3, ..., v_n\}$, then the \textit{chromatic symmetric function} of $G$ is defined as follows

$$X_G = \sum_\kappa x_{\kappa(v_1)} x_{\kappa(v_2)} x_{\kappa(v_3)} \cdots x_{\kappa(v_n)},$$

where the sum is over all proper colorings $\kappa$.

Now consider a \textit{partition}, a sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \lambda_3, ..., \lambda_\ell)$ such that $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq ... \geq \lambda_\ell$. If $\sum_{i=1}^\ell \lambda_i = n$ we say $\lambda$ is a partition of $n$ and denote it $\lambda \vdash n$. The \textit{conjugate} of $\lambda$ is defined as the partition $\lambda' = (\lambda'_1, ..., \lambda'_\ell)$ where $\lambda'_i = |\{j : \lambda_j \geq i\}|$.

Using partitions we can index symmetric functions, and next we define some of the classical types of symmetric function. All of these functions form bases of the ring of symmetric functions, $\Lambda$, and we will consider these bases later in this paper. For more information about $\Lambda$ and symmetric functions in general, see [17] and [21].

The \textbf{monomial symmetric function} corresponding to $\lambda$ is

$$m_\lambda = \sum x_{i_1}^{\lambda_1} x_{i_2}^{\lambda_2} \cdots x_{i_\ell}^{\lambda_\ell}.$$ 

Other very important symmetric functions, namely the \textbf{elementary} and \textbf{power sum symmetric functions}, are given by:

$$e_n = m_1^n = \sum_{i_1 < i_2 < \cdots < i_n} x_{i_1} x_{i_2} \cdots x_{i_n},$$

$$p_n = m_{(n)} = \sum_{i \geq 1} x_i^n.$$

For a partition $\lambda \vdash n$ we define

$$e_\lambda = e_{\lambda_1} \cdot \cdots \cdot e_{\lambda_\ell}.$$
and \( p_\lambda \) is defined analogously. With the elementary symmetric functions the Schur function for a given partition \( \lambda = (\lambda_1, \ldots, \lambda_\ell) \) can be obtained by the dual Jacobi-Trudi identity as follows:

\[
s_\lambda = \det(e_{\lambda'_i-j}),
\]

where \( i, j \in [\ell] \).

Note that given a partition \( \lambda = (\lambda_1, \lambda_2, \lambda_3, ..., \lambda_\ell) \), we say a proper coloring is of type \( \lambda \), if the number of vertices of each color written in descending order equals \( \lambda \).

A graph is said to be \( e \)-positive if its corresponding chromatic symmetric function can be written as a linear combination of \( e \)-basis elements with positive coefficients.

The importance of the \( p \)-basis can be seen in the following formula. Due to Stanley [23], we get the expansion

\[
X_G = \sum_{S \subseteq E} (-1)^{|S|} p_{\lambda(S)},
\]

where \( \lambda(S) \) denotes the partition of \( |V| \) whose parts correspond to the size of the connected components of the subgraph of \( G \) with vertex set \( V \) and edge set \( S \). Now let \( G \) and \( H \) be graphs and \( G \sqcup H \) denote the disjoint union of them, then:

\[
X_{G \sqcup H} = X_G \cdot X_H.
\]

Of especial relevance for Section 5 is the quasisymmetric refinement of the chromatic symmetric function, which was first stated by Shareshian and Wachs in [22]. Let \( G = (V, E) \) be a graph. Then the chromatic quasisymmetric function is given by:

\[
\tilde{X}_G(t) = \sum_\kappa t^{\text{asc}(\kappa)} x_{\kappa(v_1)} x_{\kappa(v_2)} \cdots x_{\kappa(v_n)},
\]

where the sum is again over all proper colorings \( \kappa \) and

\[
\text{asc}(\kappa) = \{(x, y) \in E : x < y \text{ and } \kappa(x) < \kappa(y)\}.
\]

If \( t = 1 \), we get the chromatic symmetric function. Consider again graphs \( G, H \) and their disjoint union \( G \sqcup H \). Then following equation holds:

\[
\tilde{X}_{G \sqcup H}(t) = \tilde{X}_G(t) \cdot \tilde{X}_H(t).
\]

For further definitions, information on which properties directly transfer from chromatic symmetric functions, and additional results on chromatic quasisymmetric functions, see Section 5, [3], and [22].

### 3 E-Positivity of Generalized Nets

A prominent counterexample, stated by Stanley [23], of a claw-free non \( e \)-positive graph is the net (Figure 2a). Originally Stanley presented this counterexample to exemplify that the condition of being an incomparability graph is necessary for the Stanley and Stembridge [25] conjecture. We extend this idea to an infinite family of claw-free graphs which are not \( e \)-positive.

**Definition 1.** A generalized net is a clique of size \( n \in \mathbb{N}, n \geq 3 \) (called body) with three additional vertices (called satellites), every one of which is connected to different vertices in the body via one edge.

An example of a generalized net can be seen in Figure 2b.

**Theorem 1.** The chromatic symmetric function in the \( e \)-basis for a generalized net \( G \) with a body of size \( n, n \geq 3 \) is:
In particular, generalized nets are not e-positive.

Proof. Given a generalized net, $W$, on $n + 3$ vertices consider all possible proper colorings. Now let a proper coloring of $W$ be of type $(\lambda_1, \lambda_2, ..., \lambda_r, 1^k)$, where $\lambda_i > 1$ for all $i \in [r]$ and

$$\sum_{i=1}^{r} \lambda_i + k = n + 3.$$  

Notice that all vertices inside the body must have different colors, so at least $n$ colors are required. This gives the condition

$$r + k \geq n.$$  

Next, since all $\lambda_i$ are strictly greater than 1,

$$\sum_{i=1}^{r} \lambda_i \geq 2r.$$  

Combining these facts:

$$\begin{cases}
\sum_{i=1}^{r} \lambda_i + k = n + 3 \\
\sum_{i=1}^{r} \lambda_i \geq 2r \\
r + k \geq n 
\end{cases} \Rightarrow \begin{cases}
n - k \geq 2r - 3 \\
r \geq n - k 
\end{cases} \Rightarrow 3 \geq r \Rightarrow \sum_{i=1}^{r} \lambda_i = n - k + 3 \Rightarrow r + 3 \geq \sum_{i=1}^{r} \lambda_i \geq 2r$$  

Considering each $r$ separately and using the restrictions listed, all possible types of proper colorings are given by:

$$(1^{n+3}), (2, 1^{n+1}), (3, 1^n), (4, 1^{n-1}), (2, 2, 1^{n-1}), (3, 2, 1^{n-2}), (2, 2, 2, 1^{n-3}).$$
This means that the chromatic symmetric function of generalized nets, written in terms of monomials, includes only terms corresponding to the types listed above. Furthermore, the coefficients in this linear combination count the possible colorings of the type given by the indexing partition. The coefficients can now be computed by counting all proper colorings.

- \((1^{n+3})\): All \(n + 3\) vertices must be colored differently (Figure 3). This gives \((n + 3)!\) colorings.

![Figure 3: All vertices are colored differently.](image)

- \((2, 1^{n+1})\): There are two possibilities for this type: the same colored vertices are both satellites, or one is a satellite and the second one is in the body. The first case (Figure 4a) gives \(3(n + 1)!\) colorings and the second one (Figure 4b) gives \(3(n - 1)(n + 1)!\); Giving \(3n(n + 1)!\) in total.

  ![Figure 4: Two vertices share a color](image)

- \((3, 1^{n})\): Two of the three vertices with the same color must be satellites, but there are two possible choices for positioning the third vertex: in the body, or on the remaining satellite. In the first case (Figure 5a) there are three possible pairs of satellites and \(n - 2\) places for the vertex in the body. So in total, including the trivial coloring of the rest, \(3(n - 2)n!\) options. The second case is trivial, (Figure 5b) as it can only vary in the body which gives \(n!\) possibilities, resulting in a total of \((3n - 5)n!\) colorings.

- \((4, 1^{n-1})\): The only way to achieve this type is to use the same color for all of the satellites and one vertex in the body which is not connected to any satellites and color all the other vertices in different colors (Figure 6). There are \((n - 3)\) places for the fourth vertex of the same color (the one in the body) and \((n - 1)!\) possible ways to color the rest. This gives \((n - 3)(n - 1)!\) colorings.
(a) Two of the vertices that share a color are in satellites and one is in the body.

(b) All three vertices that share a color are in satellites.

Figure 5: Three vertices share a color

Figure 6: All satellites, and one vertex in the body, are colored the same color.

- \( (2, 2, 1^{n-1}) \): First possible case: one color appears twice in the body and another appears once on a satellite and once in the body (Figure 7a). There are two choices for the repeated color appearing in the body twice, three choices for the satellite, \( n - 1 \) choices for the fourth vertex in the body, and then \( (n - 1)! \) possibilities for the trivial coloring of the rest. This gives in total: \( 6(n - 1)(n - 1)! \).

The second case is when both colors appear once in the body and once on a satellite. It contains three variants. The first case is illustrated in Figure 7b and is the case when two vertices connected to satellites are colored with one of the repeated colors. There are three choices of “blue” color, a symmetry between the “red” and “green” colors, and the trivial coloring of the rest. This gives in total: \( 6(n - 1)! \) colorings. The second variant is illustrated in Figure 7c and is the case when one vertex connected to a satellite is colored with one of the repeated colors. This case has 12 degrees of freedom from arranging the satellites and the choice whether “red” or “green” is free in the body, and \( (n - 2)(n - 1)! \) choices in the body. This gives in total: \( 12(n - 2)(n - 1)! \) colorings. Finally, the last case, shown in Figure 7d is the case when none of the vertices connected to a satellite is colored with one of the repeated colors, and this case gives \( 6(n - 2)(n - 3)(n - 1)! \), where 6 comes from different choices on the satellites, and \( (n - 2)(n - 3)(n - 1)! \) comes from the body. Combining these options gives \( 6(n^2 - 2n + 2)(n - 1)! \) colorings.

- \( (3, 2, 1^{n-2}) \): There is only one way to split multiply appearing colors between the vertices of the body and the satellites. However, there are two cases to consider: the vertex in the body of the color that appears three times can be on a connecting vertex, or not (Figure 8). The first case gives \( 3(n - 1)! \) colorings. The second one gives \( 3(n - 3)(n - 2)(n - 2)! \). In total, \( 3(n^2 - 4n + 5)(n - 2)! \).

- \( (2, 2, 2, 1^{n-3}) \): Split the case such that 0, 1, 2, or 3 colors are on the connecting vertices, which gives already 4 different possibilities. In the first case (Figure 9a), there are 6 choices for how to color the satellites, and then there is the choice of which non-connecting vertices will be colored in the same colors as satellites. This gives \( (n - 3)(n - 4)(n - 5)(n - 3)! \) colorings.
(a) One repeated color appears twice in the body and the other repeated color appears once on a satellite and once in the body.

(b) Both repeated colors appear once in the body and once on a satellite: case one.

(c) Both repeated colors appear once in the body and once on a satellite: case two.

(d) Both repeated colors appear once in the body and once on a satellite: case three.

Figure 7: Two pairs of vertices share a color.

The second case differs by picking one of the colors and placing it on a connecting vertex (Figure 9b). It gives 3 choices of colors and 2 possible places for this color instead of $(n - 5)$. Adjusting accordingly gives $36(n - 3)(n - 4)(n - 3)!$ colorings.

In the third case (Figure 9c), there are 6 ways to color satellites, 3 choices of color, and 3 ways to place the pair of colors on connecting vertices. This gives $54(n - 3)(n - 3)!$ colorings.

In the last case (Figure 9d), our only choices are the color on the satellites (6 choices), the orientation for coloring the connecting vertices (2 choices), and coloring the remaining vertices ($(n - 3)!$ choices). This leads to $12(n - 3)!$ colorings. These facts lead to $6(n^3 - 6n^2 + 14n - 13)(n - 3)!$ colorings.

Combining these results gives:

$$X_G = (n + 3)!m_{(1,1,1,\ldots,1)} + 3n(n + 1)!m_{(2,1,1,\ldots,1)} + 6(n^2 - 2n + 2)(n - 1)!m_{(2,2,1,\ldots,1)} +$$
$$+ 6(n^3 - 6n^2 + 14n - 13)(n - 3)!m_{(2,2,2,1,\ldots,1)} + (3n - 5)n!m_{(3,1,1,\ldots,1)} +$$
$$+ 3(n^2 - 4n + 5)(n - 2)!m_{(3,2,1,\ldots,1)} + (n - 3)(n - 1)!m_{(4,1,1,\ldots,1)}$$

Now a change of basis from monomials to elementary symmetric functions is required. By looking at the expression of monomial symmetric functions in terms of elementary symmetric functions it can be
of five vertices in \( G \) nets we can show that Tsujie’s e-positive nature. Tsujie \[26\] proved that (claw,

Extension of (claw, \( P_4 \))-free Graphs

Tsujie \[20\] proved that (claw, \( P_4 \))-free graphs are e-positive. With our now established class of generalized nets we can show that Tsujie’s e-positivity result can not be further extended by looking at claw-free \( P_4 \)-sparse graphs. \( P_4 \)-sparse graphs were introduced by Ho` ang \[13\]. A graph \( G \) is \( P_4 \)-sparse if for every set of five vertices in \( G \) there exists at most one induced \( P_4 \). These graphs are well-studied as a generalization of the \( P_4 \)-free graphs, and researchers have proved several structure theorems for them \[11, 5, 10, 14, 13\]. As a further connection to our phylum arthropoda, note that Ho` ang also proved that a certain class of \( P_4 \)-sparse graphs are spiders \[13\].

**Proposition 1.** There are infinitely many claw-free, \( P_4 \)-sparse graphs that are not e-positive.

**Proof.** It is easy to see that generalized nets claw-free. It is also easy to see that generalized nets are \( P_4 \)-sparse because for any five vertices chosen, there are only two relevant cases. On one hand, if all

### Figure 8: Three vertices share a color

(a) Repeated color appears on the satellites and once on a connecting vertex in the body.

(b) Repeated color appears once on a satellite and once on a non-connective vertex in the body.

checked easily that:

\[
m_{(1^{n+3})} = e_{(n+3)}
\]

\[
m_{(2,1^{n+1})} = e_{(n+2,1)} - (n + 3)e_{(n+3)}
\]

\[
m_{(2,1,1,\ldots,1)} = e_{(n+1,2)} - (n - 1)e_{(n+2,1)}
\]

\[
m_{(2,2,1,\ldots,1)} = e_{(n,3)} - (n - 1)e_{(n+1,2)} + \frac{(n-2)(n+1)}{2}e_{(n+2,1)} -
\]

\[
- \frac{(n-2)(n-1)(n+3)}{6}e_{(n+3)}
\]

\[
m_{(3,1,1,\ldots,1)} = e_{(n+1,1,1)} - 2e_{(n+2,1)} - e_{(n+3)} + (n + 3)e_{(n,3)}
\]

\[
m_{(3,2,1,\ldots,1)} = e_{(n,2,1)} - 3e_{(n,3)} - ne_{(n+1,1,1)} + 2(n - 1)e_{(n+1,2)} +
\]

\[
+ (2n + 1)e_{(n+2,1)} - (n + 3)(n - 1)e_{(n+3)}
\]

\[
m_{(4,1,1,\ldots,1)} = e_{(n,1,1,1,1)} - 3e_{(n,2,1)} + 3e_{(n,3)} - e_{(n+1,1,1,1)} +
\]

\[
+ 2e_{(n+1,2)} + e_{(n+2,1)} - (n + 3)e_{(n,3)}
\]

Now using these expressions, the explicit formula for the chromatic symmetric function in e-basis for a generalized net with a body of size \( n \) is:

\[
X_G = (n + 3)(n - 1)e_{(n+3)} + 3(n^2 - 3)(n - 2)e_{(n+2,1)} +
\]

\[
+ 6(n - 1)(n - 3)e_{(n+1,2)} + 3(n^2 - 2n - 1)(n - 2)e_{(n+1,1,1)} +
\]

\[
+ 6(n - 2)e_{(n,2,1)} - 6(n - 3)e_{(n,3)} + (n - 3)(n - 1)e_{(n,1,1,1)}
\]

\[\square\]
three satellites are chosen then the two remaining vertices are in the clique. Here there exists at least one isolated vertex in the subgraph and therefore there can be at most one $P_4$. On the other hand, if three or more vertices are part of the clique, the induced subgraph either contains a clique of at least size four and hence can not contain an induced $P_4$ or it is isomorphic to a triangle with two additional edges to pendant vertices and thus contains exactly one $P_4$. So the class of generalized nets gives an infinite family of claw-free, $P_4$-sparse graphs that are not $e$-positive.

\[ \square \]

## 4 Uniqueness of Generalized Spiders

As mentioned before a natural question to ask is whether the chromatic symmetric function of a graph determines the graph up to isomorphism. We now give a class of graphs, which are not trees, that can be distinguished by their chromatic symmetric function. We will see that this class is an even further generalization of generalized nets as we now allow multiple paths of arbitrary length instead of just three pendant vertices attached to the $K_n$.

**Definition 2.** A generalized spider is a clique of size $n \in \mathbb{N}, n \geq 3$, with paths of variable length attached such that each vertex in the clique has at most degree $n$. These paths of length $(m_1, \ldots, m_\lambda), \lambda \leq n$ are called legs.

The name generalized spiders originates in the fact that these graphs can also be constructed by replacing the unique vertex, $v$ with $\text{deg}(v) \geq 3$, of a spider by a clique of size $n \geq \text{deg}(v)$. It is important to note that this is not the only connection to spiders as the following remark shows.

**Proposition 2.** $G$ is a generalized spider with a clique of size $n \geq 3$ vertices if and only if $G$ is the line graph of some spider.
Proof. Let $S = (V,E)$ be a spider and let $v \in V$ be the unique vertex with $\text{deg}(v) \geq 3$. Forming the line graph of $S$ produces a clique of size $\text{deg}(v)$ as all adjacent edges share a vertex in $v$ and therefore are connected vertices in the line graph. It is easy to see that the line graph of a path of length $n$ is a path of length $n - 1$. So if $S$ has legs of length $(m_1,\ldots,m_\lambda)$ the corresponding generalized spider has legs of length $(m_1 - 1,\ldots,m_\lambda - 1)$. Legs of length 1 in $S$ are consequently just contributing to the size of the clique. Reversing the line graph construction (with the convention that for $n = 3$ the $K_3$ must be transformed to a claw not a $K_3$) shows that a generalized spider can be transformed to a spider. An illustration can be seen in Figure 10.

With this observation we can now use the uniqueness result regarding spiders to prove uniqueness of the chromatic symmetric function of their generalized counterparts.

**Theorem 2.** The chromatic symmetric function uniquely distinguishes spiders. [18]

We know that we can extract information about the graph from the chromatic symmetric function. Particularly, of interest to us, the number of $k$-matchings and the number of independent sets can be found. A $k$-matching is a set of size $k$ of independent edges, i.e. they do not share a vertex. An independent (or stable) set is defined similarly: it is a subset of vertices, $I$, of a graph $G$ such that no two vertices in $I$ are adjacent. These sets can be represented via polynomials. The matching polynomial $\mu(x)$ can be defined as follows:

$$\mu(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} m_k x^k,$$

where $m_k$ denotes the number of $k$-matchings. The value $m_k$ can be recovered from the chromatic symmetric function expanded in the $p$-basis stated in equation (1) and as $m_k$ defines $\mu(x)$, the matching polynomial can be recovered from the chromatic symmetric function. For more information see [19].

Almost equivalently we define the independence polynomial of $G$ as:

$$I(x) = \sum_{i=0}^{\lfloor n/2 \rfloor} \Phi_i x^i,$$

where $\Phi_i$ stands for the number of independent sets of size $i$. Note that $I(x)$ is sometimes referred to as stable set polynomial, see [24].

For the sake of completeness, we give the proof of how to recover the number of independent sets from the chromatic symmetric function.

**Lemma 1.** For a graph $G$ the number of independent sets of a certain size, $k$, can be recovered from its corresponding chromatic symmetric function.

**Proof.** Let $G = (V,E)$ with $|V| = n$ be a graph. If $k$ vertices in a proper coloring of $G$ have the same color, then they necessarily are an independent set. Since the coefficients of the chromatic symmetric function expanded in $m$-basis directly correspond to colorings, the coefficient of the term $m_{k,1^{n-k}}$ determines the
number of independent sets of size $k$ up to permutation of the colors. So by dividing the coefficient by $n!$ we get the number of independent sets of size $k$.

Finally we need some more graph theoretic results in order to conclude that the chromatic symmetric functions of generalized spiders are unique. As mentioned in [9], the characteristic polynomial of the adjacency matrix of trees is equal to their matching polynomial. Therefore the result in [16] that non-isomorphic spiders have unique characteristic polynomial gives us the uniqueness of the matching polynomial of spiders. Furthermore the matching polynomial of a graph is the independence polynomial of its line graph. This can easily be seen because all edges in the $k$-matching directly correspond to an independent set of the line graph. Hence the two polynomials coincide. Now we can finally state and prove the main theorem of this section.

**Theorem 3.** The chromatic symmetric function of generalized spiders is unique.

**Proof.** Let $G$ be a generalized spider. The independence polynomial $I(x)$ of $G$ is unique, because by Proposition 2, $G$ is the line graph of some spider with a unique matching polynomial. Therefore by Lemma 1, $X_G$ is uniquely determined.

5 Horseshoe Crab Graphs

As Guay-Paquet showed in [10] it is sufficient to show $e$-positivity for the incomparability graphs of $(2+2)$ and $(3+1)$-free posets in order to resolve Stanley’s $e$-positivity conjecture. As mentioned in Section 2 these graphs correspond exactly to the natural unit interval graphs (defined below). In the following section we give another class of natural unit interval graphs which are $e$-positive and in doing so enlarge the set of graphs which the conjecture concerns. These graphs we will call horseshoe crab graphs, an illustration can be seen in Figure 11b and a definition is in Definition 3. Note that in biology horseshoe crabs, like spiders, are phylum arthropoda thus are indeed spider kin. We will use our result on horseshoe crabs to give an even larger class of $e$-positive natural unit interval graphs, which leads to the main theorem (see Theorem 9) of this section.

The technique we use to prove $e$-positivity of our horseshoe crab graphs was first introduced by Cho and Huh in [3]. In our work with these graphs we also generalize their method of weight preserving injections in Lemma 2. In order to state and prove our result, we need to provide some further background. First we need to introduce natural unit interval orders. A natural unit interval order is a partial ordering $P$ on the elements of $[n]$, which is induced by a sequence $m = (m_1, m_2, m_3, ..., n,..., n)$ of positive non-decreasing integers. The sequence must satisfy the following condition:

$$\forall i \in [n - 1] : i \leq m_i \leq n.$$

Then the corresponding order relation $<_P$ is given by:

$$m_i < j \leq n \implies i <_P j.$$

The incomparability graph of $P(m)$ is called a natural unit interval graph. Using the sequence $m$ it is easy to characterize the edges of the incomparability graph since, for $i < j$:

$$i \text{ and } j \text{ are incomparable } \iff m_i \geq j.$$

In other words $i$ is incomparable to every $j$ such that $i \leq j \leq m_i$ and each $k$ such that $k < i \leq m_k$.

**Definition 3.** Given $m = (2, m_2, m_3, n, ..., n)$, the incomparability graph of $P(m)$ are called a horseshoe crab graph.

An example of the horseshoe crab graphs can be seen in Figure 11b. Recall now the definition of the chromatic quasisymmetric function as given in Section 2. For natural unit interval graphs, the chromatic quasisymmetric function has the following properties. This is Theorem 4.5 and Corollary 4.6 of [22].
Theorem 4. [22] Let $G = (V, E)$ be a natural unit interval graph. Then

$$\tilde{X}_G(t) \in \Lambda[t]$$

so the coefficients of $t^i$ are symmetric functions. Furthermore the coefficients form a palindromic sequences, i.e. $\tilde{X}_G(t) = t^{|E|} \tilde{X}_G(t^{-1})$.

We can now state a necessary and sufficient condition for $\tilde{X}_G(t)$ being $e$-positive. Let $G$ be a natural unit interval graph and

$$\tilde{X}_G(t) = \sum_{i=1}^{m} f_i t^i = \sum_{\lambda \vdash n} E_\lambda(t) e_\lambda, \quad f_i \in \Lambda.$$

Then $\tilde{X}_G(t)$ is $e$-positive if and only if the polynomial $E_\lambda(t)$ has non-negative coefficients for all $\lambda$. (See [3])

Schur positivity, or $s$-positivity, for natural unit interval graphs has been proven for $t = 1$ by Gasharov in [6]. Shareshian and Wachs extended this result for the quasisymmetric refinement in [22]. In order to state this result and also use it for proving $e$-positivity, we need to define $P$-tableaux, which were used by Gasharov in [6] in order to prove $s$-positivity of $(3+1)$-free posets.

Definition 4. Let $P$ be a poset on $n$ elements and $\lambda \vdash n$. A $P$-tableau of shape $\lambda$ is a filling of a Young diagram of shape $\lambda$ (in English notation) with elements of $P$ such that:

- Each element of $P$ appears exactly once
- The rows are strictly increasing ($a \in P$ appears to the left of $b \in P \implies a <_P b$)
- The columns are pairwise non-decreasing ($a \in P$ appears immediately above $b \in P \implies b \not<_P a$)

Let $G = (V, E)$ be the incomparability graph of $P$ and $T$ be a $P$-tableau. An edge $\{i, j\}$ with $i < j$ and $i$ appearing above $j$ in $T$ is called a $G$-inversion. The number of $G$-inversions is denoted by $\text{inv}_G(T)$ and referred to as the weight of the $P$-tableau. (Definition 6.1 in [22])

Equivalently $\text{inv}_G(T)$ can be described as the number of incomparable pairs $(a, b)$ with $a < b$ and $b$ appearing above $a$ in $T$. For example for the sequence $m = (2, 4, 6, 8, 8, 8)$ a possible $P$-tableau of shape $(3, 2, 1^3)$ and with weight 5 is shown in Figure 11a. The weight has been calculated as follows:

$$\text{inv}_G = |\{\{5, 4\}, \{6, 4\}, \{7, 4\}, \{7, 5\}, \{7, 6\}\}| = 5.$$

The corresponding natural unit interval graph can be seen in 11b. As it will be useful for a later proof, a general filling of a $P$-tableau is given in Figure 11c.

![Figure 11](image-url)

Figure 11: Sequence $m = (2, 4, 6, 8, 8, 8)$

With this definitions we can now state a very important result which is Theorem 6.3 in [22].
Theorem 5. For a natural unit interval order $P$ the chromatic quasisymmetric function of its incomparability graph $G$ has the following expansion:

$$
\tilde{X}_G(t) = \sum_{T} t^{\text{inv}_G(T)} s_{\lambda(T)},
$$

where the sum is over all $P$-tableaux and $\lambda(T)$ denotes the shape of $T$. Hence $\tilde{X}_G(t)$ is $s$-positive.

From this formula, the corresponding expansions in the $e$-basis can be found using the dual Jacobi-Trudi identity. Furthermore Shareshian and Wachs proved positivity of a certain coefficient by giving an explicit formula, which leads to the following theorem which is Corollary 7.2 in [22].

Theorem 6. Let $G$ be a natural unit interval graph on $n$ elements and $\tilde{X}_G(t) = \sum_{\lambda \vdash n} E_{\lambda}(t)e_{\lambda}$. Then $E_{\lambda}(t)$ is a positive polynomial in $t$.

Having established all these results, we can now use the technique of weight preserving injections to prove $e$-positivity of a class of natural unit interval orders corresponding to a certain sequence. This leads on from the work of Cho and Huh [3] who established a number of significant results including the following expansions by Theorem 5, where we use $\text{inv}_G(t)$ to denote the incomparability graph of $G$.

Remark 1. Let $m = (m_1, m_2, m_3, \ldots, m_n)$. For the incomparability graph $G$ of the natural unit interval order $P(m)$, $\tilde{X}_G(t)$ is $e$-positive. Therefore $X_G$ is $e$-positive. [23]

Let $m = (r, m_2, m_3, \ldots, m_n)$ and $G$ be the incomparability graph of $P(m)$. Then $\tilde{X}_G(t)$ and thus $X_G$ is $e$-positive.

Theorem 7. Let $m = (2, m_2, m_3, \ldots, n)$ and $G$ be the incomparability graph of $P(m)$ (i.e. a horseshoe crab graph). Then $\tilde{X}_G(t)$ and thus $X_G$ is $e$-positive.

Proof. Let $G$ denote the incomparability graph of $P(m)$ on $n$ elements with $m = (2, m_2, m_3, \ldots, n)$. As the only non-maximal elements of $P(m)$ are 1, 2, and 3, the possible shapes of legal $P$-tableaux can be found quite easily. Since a maximal chain in $P(m)$ has 3 elements and the maximal number of independent 2-chains is 3, the only possible shapes are:

$$(1^n), (2, 1^{n-2}), (2^2, 1^{n-4}), (3, 1^{n-6}), (3, 2, 1^{n-5}).$$

This leads to following expansion of the chromatic quasisymmetric function by Theorem 5 where we use $S$ to denote the appropriate coefficient:

$$
\tilde{X}_G(t) = S_{1^n}(t)s_{1^n} + S_{2,1^{n-2}}(t)s_{2,1^{n-2}} + S_{2,1^{n-4}}(t)s_{2,1^{n-4}} + S_{3,1^{n-6}}(t)s_{3,1^{n-6}} + S_{3,2,1^{n-5}}(t)s_{3,2,1^{n-5}}.
$$

By applying dual Jacobi-Trudi identity, the coefficients for elementary symmetric functions can be calculated:

$$
E_{n}(t) = S_{1^n}(t) + S_{3,1^{n-3}}(t) - S_{2,1^{n-2}}(t) \quad (1)
$$

$$
E_{n-1,1}(t) = S_{2,1^{n-2}}(t) + S_{3,2,1^{n-5}}(t) - S_{2,1^{n-3}}(t) - S_{3,1^{n-3}}(t) \quad (2)
$$

$$
E_{n-2,2}(t) = S_{2,1^{n-4}}(t) - S_{2,1^{n-3}}(t) - S_{3,1^{n-3}}(t) \quad (3)
$$

$$
E_{n,-2,1}(t) = S_{3,1^{n-3}}(t) - S_{2,1^{n-6}}(t) \quad (4)
$$

$$
E_{n-3,3}(t) = S_{2,1^{n-6}}(t) - S_{3,2,1^{n-5}}(t) \quad (5)
$$

$$
E_{n-3,2,1}(t) = S_{2,1^{n-7}}(t) \quad (6)
$$

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In order to prove the \( e \)-positivity of \( G \) it is sufficient to show that the coefficients of all the polynomials \( E_\lambda(t) \) are positive. By Theorem [4] \( E_n(t) \) is a positive polynomial. Furthermore since by Theorem [5]

\[
S_\lambda(t) = s_\lambda \cdot \sum_{T \in \mathcal{T}_\lambda} t^{\operatorname{inv}(T)},
\]

where \( \mathcal{T}_\lambda \) denotes the set of \( P \)-tableaux of shape \( \lambda \), the coefficient \( E_{n-3,2,1}(t) \) in (6) is trivially positive. For the remaining coefficients, positivity can be shown by the use of weight preserving injections. Those will be defined for the individual cases and the notation of the general \( P \)-tableau in Figure 11c will be used throughout:

- **\( E_{n-1,1}(t) \):** As seen in (2) and (7), if there exists a weight preserving injection

\[
\phi : \mathcal{T}_{2^2,1^{n-4}} \cup \mathcal{T}_{3,1^{n-3}} \to \mathcal{T}_{2,1^{n-2}} \cup \mathcal{T}_{3,2,1^{n-5}},
\]

the positivity of the coefficients of all \( t^i \) in \( E_{n-1,1}(t) \) is shown. Before an explicit injection is given, it is useful to establish structural properties of the \( P \)-tableaux. As mentioned before 1, 2, and 3 are the only comparable elements in \( P(m) \). Furthermore 1 \( \not<_{P} 2 \), so a maximal chain can only start with 1, 3 and the smallest legal element to come after 3 is \( m_3 + 1 \). The 2-chains can start with either 1, 2, or 3. One intuitive way to map the \( P \)-tableaux of shape \((3, 1^{n-3})\) is moving the \( a_{1,3} \) element to the first column exactly above the first element it is incomparable with. This rule does not affect the weight and also produces a legal \( P \)-tableau of shape \((2, 1^{n-2})\) as the pairwise non-decreasing rule is never violated. The same method is successful for \( \mathcal{T}_{2^2,1^{n-4}} \) by moving the \( a_{2,2} \) element. Unfortunately this leads to some overlap (i.e. a tableau from \( 3, 1n - 3 \) and one from \( 2^2, 1^{n-4} \) being mapped to the same image) as seen in Figure 12. Note that this only happens when \( a_{2,1} = 2 \) in the \((3, 1^{n-3})\) case as otherwise the first element of the second row will not be a 2, which is never possible in the \((2^2, 1^{n-4})\) case. This issue is solved by mapping the \( P \)-tableaux of shape \((3, 1^{n-3})\) with \( a_{2,1} = 2 \) differently. There are two cases, which have to be considered. They are illustrated in Figure 13.

1. \( 2 <_{P} a_{3,1} \): This case can easily be solved by moving \( a_{3,1} \) next to the 2, giving a tableau in \( \mathcal{T}_{3,2,1^{n-5}} \).
2. \( 2 \not<_{P} a_{3,1} \): Instead of dropping the element \( a_{1,3} \), the 3 can be dropped directly under the 1. It is important to note that this method produces a legal \( P \)-tableau of shape \((2, 1^{n-2})\) as the pairwise non-decreasing condition is guaranteed by the fact that \( 2 \not<_{P} 3 \) and since 3 and \( a_{1,3} \) are comparable, dropping the 3 does not add any inversions. Moreover by the definition of \( P(m) \), \( 2 \not<_{P} a_{3,1} \) implies \( 3 \not<_{P} a_{3,1} \). Hence the resulting \( P \)-tableau is not in the image \( \phi(T_{2^2,1^{n-4}}) \). It will also not be in the image of \( P \)-tableaux of shape \((3, 1^{n-3})\) with \( a_{2,1} \neq 2 \) as the second element of the first row will not be a 3.

Now consider \( P \)-tableaux \( T_1, T_2 \) with \( \phi(T_1) = \phi(T_2) \). If \( \phi(T_1) \in \mathcal{T}_{3,2,1^{n-5}} \) injectivity can be seen easily, since the original \( P \)-tableau can be recovered by simply dropping the element next to the 2, underneath the 2, it follows that \( T_1 = T_2 \). The other case, \( \phi(T_1) \in \mathcal{T}_{2,1^{n-2}} \), has already been established before.
Note that in the image of $P$ chain in a few cases need to be considered. For $T \in \mathcal{P}_3$ a straightforward mapping rule is moving the $a_{3,2}$ element exactly above the $a_{4,1}$ element. Since 1, 2 and 3 are the only possible ways to start a 2-chain in $P(m)$, this does not violate being pairwise non-decreasing along the columns. Furthermore it is trivially weight preserving as the relative positions of the incomparable elements do not change. Note that in the image of $T_2^3$-tableaux under $\xi$, $(a_{1,1}, a_{2,1}, a_{3,1})$ forms a certain permutation of $(1, 2, 3)$. Now consider $P$-tableaux of shape $(3, 1^{n-3})$. They are mapped as follows:

- $a_{2,1} \neq 2, a_{3,1} \neq 2 \land 3 \not\prec_P a_{2,1}$: By moving the $a_{2,1}$ up next to the 1 and dropping the 3, $a_{1,3}$ constellation to second row a $P$-tableau of shape $(2^2, 1^{n-4})$ is obtained. Note that $(3, a_{2,1})$ adds a $G$-inversion, but since $3 \not\prec_P a_{2,1}$ and $3 \prec_P a_{1,3} \implies a_{2,1} < a_{1,3}$, it cancels.

- $a_{2,1} \neq 2, a_{3,1} \neq 2 \land 3 \prec_P a_{2,1}$: This case can be solved by simply moving the 3 in front of the $a_{2,1}$ element. This does not add or subtract any $G$-inversions because $3 \prec_P a_{2,1}$. Therefore it is not an inversion. It is important to mention that these two cases cannot appear in the image of $T_2^3$-tableaux as $a_{2,1} \neq 2$ and $a_{3,1} \neq 2$. This does not overlap with the previous case as the second element in the first row of the image is never a 2, respecting injectivity.

- $a_{3,1} = 2$: As $P$-tableaux need to be pairwise non-decreasing along the columns, the following holds: $2 \not\prec_P a_{2,1}$. This implies $3 \not\prec_P a_{2,1}$ and the same rule as in the previous case can be used. Unfortunately this leads again to overlap, which is illustrated in Figure 14a. This can be resolved by mapping the affected $P$-tableaux of shape $(2^3, 1^{n-6})$ differently. That is, moving the 3 next to the 1, the 2 next to the $a_{2,2}$ and dropping $a_{2,1}$ directly underneath the 2, as shown in Figure 14b. Injectivity is fine for these shifted tableaux as they are the only ones with a 3 as the second element of the first row. Since the overlap only occurs when $2 \not\prec_P a_{1,2}$ and $3 \not\prec_P a_{1,2}$, it is implied that, $a_{1,2} < a_{2,2}$. Therefore one weight is lost by the $(3, a_{1,2})$ inversion and one is gained by the $(a_{1,2}, a_{2,2})$ inversion.

- $a_{2,1} = 2$: $P$-tableaux of this form will be mapped as seen in Figure 14c. Weight is preserved as the $(2, 1)$ inversion is gained and the $(3, 2)$ inversion is lost. Since $3 \prec_P a_{1,3}$ it does not contribute any weight. Injectivity is respected as none of the other cases have a 2 as the first element of the first row.

- $E_{n-2,12}(t)$: Recalling equation (1), $T_{3,2}^3$ needs to be injected into $T_{3,1}$. As mentioned before the $P$-tableaux of shape $(3, 2, 1^{n-5})$ have a lot of structure due to the fact that 1, 2 and 3 are the only non-maximal elements of $P(m)$. Therefore a very straightforward weight-preserving injection $\eta$ is illustrated in Figure 15. Note that the image is a legal $P$-tableau as $a_{2,2}, a_{3,1} > 4$.
(a) Overlapping $P$-tableaux

1 $a_{1,2}$
3 $a_{2,2}$
2 $a_{3,2}$

(b) Solution

1 3 $a_{2,2}$

Figure 14: Overlap issues of $\xi$

Figure 15: Part of the map $\xi$

and hence $a_{3,1} \not\prec_P a_{2,2}$. Furthermore as the relative positions of the elements did not change, the number of inversions is the same for both $P$-tableaux. For injectivity consider $T_1, T_2 \in T_{3,2,1,n-5}$ with $\eta(T_1) = \eta(T_2)$. Since the original $P$-tableau can be recovered by moving the element directly underneath the 2 up, it follows that $T_1 = T_2$. Thus $\eta$ is injective.

For the last coefficient $E_{n-3,3}(t)$ we need to establish some additional results in order to use the method of injections. First of all we define Schur-unimodality for quasisymmetric functions.

**Definition 5.** Let $G$ be a graph and $	ilde{X}_G(t) = \sum_{i=0}^m f_i t^i$ for $f_i \in \Lambda$. $	ilde{X}_G(t)$ is said to be $s$-unimodal if $f_{i+1} - f_i$ is $s$-positive whenever $0 \leq i \leq \frac{m - 1}{2}$.

The next theorem gives a powerful result involving Schur-unimodality. The result that leads to this was first conjectured by Shareshian and Wachs in [22], then proved by Brosnan and Chow in [2] and later by Guay-Paquet [11] using a different method.

**Theorem 8.** [2] [11] Let $G$ be a natural unit interval graph. Then $\tilde{X}_G(t)$ is $s$-unimodal.

With this theorem and the palindromicity result stated in Theorem 4 we can now generalize the method of weight preserving injections.

**Lemma 2.** If there exists an injection $\psi: \mathcal{T}_{\lambda_1} \to \mathcal{T}_{\lambda_2}$ and a constant $c \in \mathbb{Z}$ such that:

$$\forall T \in \mathcal{T}_{\lambda_1} : \text{inv}_G(\psi(T)) = \text{inv}_G(T) + c.$$ 

Then the coefficients of $S_{\lambda_2}$ dominate the coefficients of $S_{\lambda_1}$ or equivalently there are at least as many $P$-tableaux of shape $\lambda_2$ as there are of shape $\lambda_1$ for any particular weight.
Proof. Let $S_{\lambda_1}^{(j)}$ be the coefficient for the term $t^{j}s_{\lambda_1}$. To show that $S_{\lambda_1}^{(j)} < S_{\lambda_2}^{(j)}$, notice that the weight shifting injection shows $S_{\lambda_1}^{(j)} < S_{\lambda_2}^{(j+c)}$. First consider the case that $c \geq 0$. By palindromicity the terms are symmetrical so it is enough to prove it for the second half of the terms, that is to show

$$j \geq \frac{|E|}{2} \implies S_{\lambda_1}^{(j)} < S_{\lambda_2}^{(j)}.$$  

Since

$$S_{\lambda_1}^{(j)} < S_{\lambda_2}^{(j+c)}$$

and $j > \frac{|E|}{2}$, palindromicity and Schur-unimodality result in

$$S_{\lambda_2}^{(j+c)} < S_{\lambda_2}^{(j)},$$

as desired.

If $c$ is negative take $j \leq \frac{|E|}{2}$ and get

$$S_{\lambda_1}^{(j)} < S_{\lambda_2}^{(j+c)} < S_{\lambda_2}^{(j)}$$

by Schur-unimodality.

Using Lemma 5 we can now finish the proof of Theorem 7.

Continuation of the proof of Theorem 7. The last necessary step in order to prove e-positivity of $\tilde{X}_G(t)$ is showing that the polynomial $E_{n-3,3}(t)$ only has positive coefficients. By Lemma 5 and equation (5) it is sufficient to show that there exists an injection

$$\psi: \mathcal{T}_{3,2,1n-5} \to \mathcal{T}_{2,1n-6}$$

with a constant weight shift. The details of $\psi$ are illustrated in Figure 17. As seen easily this injection adds exactly one weight, since only the $(1,2)$ inversion is added. Furthermore for any $T \in \psi(\mathcal{T}_{3,2,1n-5})$ the pre-image can easily be recovered by moving the 1 in front of the 3. Therefore $\psi$ is injective.

Hence $\tilde{X}_G(t)$ is $e$-positive.

The result of Theorem 7 can even be further extended by combining it with already known $e$-positivity results summarized by Remark 1. With all this preparation the broadest theorem of this section can be established and proved.
Theorem 9. Let \( m = (m_1, m_2, m_3, n, \ldots, n) \) and \( G \) be the incomparability graph of \( P(m) \). Then \( \tilde{X}_G(t) \) is \( e \)-positive.

Proof. Let \( G \) denote the incomparability graph of \( P(m) \). The sequence \( m = (m_1, m_2, m_3, n, \ldots, n) \) can be split into three different cases in respect to the value of \( m_1 \):

- \( m_1 = 1 \): Then 1 is comparable with every element in \([n]\) and therefore contributes a disconnected vertex in the incomparability graph of \( P(m) \). Since the chromatic (quasi)symmetric function of a single vertex is given by \( e_1 \), it is \( e \)-positive. The other part of the incomparability graph is isomorphic to the incomparability graph \( H \) induced by the sequence \((m_2, m_3, n−1, \ldots, n−1)\), this is known to be \( e \)-positive by Remark 1. So by equation (2)

\[
\tilde{X}_G(t) = e_1 \cdot \tilde{X}_H(t)
\]

and is therefore \( e \)-positive.

- \( m_1 = 2 \): This case is proven to be \( e \)-positive by Theorem 7.

- \( m_1 \geq 3 \): This case can be reduced to \((r, m_2, m_3, \ldots, m_r, n, \ldots, n)\) by setting \( m_i = n \) for \( 4 \leq i \leq r \) and so it is \( e \)-positive.

6 Looking Forward

The work of Shareshian and Wachs (22) and Cho and Huh (3) establishes that unit interval graphs on \( n \) vertices with cliques of size \( n−2 \) are \( e \)-positive. This is the first step in extending that result to unit interval graphs with cliques of size \( n−3 \). The remaining cases are, up to isomorphism, induced by sequences of the form

\[
(m_1, m_2, n−1, \ldots, n−1, n, \ldots, n),
\]

where \( 1 \leq m_1 \leq m_2 \leq n−1 \). More simply put they are precisely those induced by sequences with \( n−1 \) as the third element. An immediate observation is that this case has the same permissible tableaux shapes as Theorem 7. So it would have the same set of injections. This suggests it might be possible in some cases to use similar or even identical injections as those developed here, for instance the injection used for \( E_{n−2,2} \) works again without modification.
7 Acknowledgements

The authors thank Owen Merkel for suggesting the problem \(e\)-positivity of generalized nets, and Chính Hoàng for suggesting the problem of \(e\)-positivity of claw-free, \(P_4\)-sparse graphs. This work was supported by the Canadian Tri-Council Research Support Fund. The author A.M.F. was supported by an NSERC Discovery Grant. This research was conducted at the Fields Institute, Toronto, Canada as part of the 2018 Fields Undergraduate Summer Research Program and was funded by that program.

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