Jet formation as a result of magnetic flux tube–Kerr black hole interaction

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Abstract

Relativistic magnetohydrodynamics can be reformulated in terms of magnetic flux tubes which turn out to obey equations of non-linear strings. The string approach is applied to study how a test flux tube falls into a Kerr black hole. Analytical treatment and numerical simulations show that the leading portion of the falling tube loses angular momentum and energy as the string brakes, and to compensate for this loss, momentum and energy has to be generated to conserve energy and momentum for the tube. Inside the ergosphere the energy of the leading part can be negative, and the rest of the tube then extracts energy from the hole. Increasing centrifugal forces eject the part of the tube with extra positive energy from the ergosphere after a time producing a relativistic jet.

I. INTRODUCTION

The mechanism by which energy is extracted from a black hole and astrophysical jet forms is a key problem in our understanding of phenomena such as active galactic nuclear, quasars, and probably gamma ray bursts. The present paper considers an alternative approach to the famous Blanford-Znajek process [1]. This is described in [2,3], which is an extension of the Penrose mechanism [4] for extracting energy from a rotating black hole by a relativistic
string. It was shown that the falling magnetic flux tube can effectively generate negative energy, but the parts of the tube with both negative and compensated positive energy are localized inside the ergosphere and almost does not show up outside. The aim of this paper is to present examples of how a falling flux tube can also produce a relativistic jet.

II. FORMULATION OF RMHD IN TERMS OF STRINGS

An appropriate formulation for studying the behaviour of flux tubes can be achieved through the introduction of Lagrangian coordinates \[2,3\] into the following relativistic magnetohydrodynamic (RMHD) equations \[\text{\cite{1}}\]

\[
\begin{align*}
\nabla_i \rho u^i &= 0, \\
\nabla_i T^{ik} &= 0, \\
\nabla_i (h^i u^k - h^k u^i) &= 0.
\end{align*}
\]

Here \(u^i\) is the time-like vector of the 4-velocity, \(u^i u_i = 1\), \(h^i = *F^{ik} u_k\) is the space-like 4-vector of the magnetic field, \(h^i h_i < 0\), \(*F^{ik}\) is the dual tensor of the electromagnetic field, and \(T^{ij}\) is the energy-momentum tensor:

\[
T^{ij} = Q u^i u^j - P g^{ij} - \frac{1}{4\pi} h^i h^j,
\]

where

\[
P \equiv p - \frac{1}{8\pi} h^k h_k, \quad Q \equiv p + \varepsilon - \frac{1}{4\pi} h^k h_k.
\]

Here \(p\) is the plasma pressure, \(P\) is the total (plasma plus magnetic) pressure, \(\varepsilon\) is the internal energy, and \(g_{ik}\) is the metric tensor with signature \((1, -1, -1, -1)\).

We can find a function \(q\) such that \(\nabla_i q h^i = 0\), then using \(\text{\cite{2,3}}\) the Maxwell equation \(\text{\cite{2,3}}\) can be rewritten in the form of a Lie derivative

\[
\frac{h^i}{\rho} \nabla_i \frac{u^k}{q} = \frac{u^i}{q} \nabla_i \frac{h^k}{\rho},
\]
and we can therefore introduce Lagrangian coordinates \( \tau, \alpha \) such that

\[
x^i_\tau \equiv \frac{\partial x^i}{\partial \tau} = \frac{u^i}{q}, \quad x^i_\alpha \equiv \frac{\partial x^i}{\partial \alpha} = \frac{h^i}{\rho}
\]  

(2.7)

with new coordinate vectors \( u^i/q, h^i/\rho \) tracing the trajectory of a fluid element and the magnetic field in a flux tube. Since the introduction of these coordinates relies on the frozen-in property of the plasma, we will refer to them as frozen-in coordinates. The mass coordinate \( \alpha \) along the relativistic flux tube has the sense of a mass of the plasma for a tube with unit flux in the proper system of reference. The second coordinate \( \tau \), the string time, is not any more Lagrangian or proper time, but is just a time-like parameter which traces the flux tube in the space-time of general relativity. The functions \( x^i(\tau, \alpha) \) sweep the 2D worldsheet in the space-time continuum which consists of trajectories of the fluid elements for \( \alpha = \text{const} \), and the magnetic field lines for \( \tau = \text{const} \).

Since the following relations are valid in the new coordinates \( \tau, \alpha \):

\[
\frac{u^i}{q} \nabla_i = \frac{\partial}{\partial \tau}, \quad \frac{h^i}{\rho} \nabla_i = \frac{\partial}{\partial \alpha},
\]  

(2.8)

the energy-momentum equation (2.2) can be rearranged to form a set of string equations:

\[
-\frac{\partial}{\partial \tau} \left( \frac{Qq}{\rho} x^i_\tau \right) - \frac{Qq}{\rho} \Gamma_{ik}^l x^l_\tau x^k_\tau
+ \frac{\partial}{\partial \alpha} \left( \frac{\rho}{4\pi q} x^l_\alpha \right) + \frac{\rho}{4\pi q} \Gamma_{ik}^l x^i_\alpha x^k_\alpha = -\frac{g^{il}}{\rho q} \frac{\partial P}{\partial x^i},
\]  

(2.9)

where \( \Gamma_{ik}^l \) is the Christoffel symbol. Generally speaking, the total pressure \( P(x^i) \) is unknown in advance, but fortunately there are a set of problems when \( P(x^i) \) can be considered as a given function. In this case a test flux tube may represent MHD flow as a whole. It is known that characteristics of the string equations (2.9) are Alfvén and slow waves. Hence, the fast wave which is also characteristic of the general MHD system of equations (2.1-2.3), is left out in the string approach. The physical reason for this is clear: the fast wave is produced by gradients in the total pressure, but \( P(x^i) \) is fixed in equations (2.9), and there is no driving mechanism for fast waves.
Therefore, MHD problems which involve significant variations of total pressure such as gasdynamic-type explosion, are unlikely to be applicable using the string equations (2.9). But processes of accumulation and relaxation of Maxwellian tensions can very often be described by the string approach since the total pressure does not seem to vary appreciably, like in Alfvén wave, for example. Hence, $P(x^i)$ may be considered as a given function or at least may be determined by a perturbation method. As a result, the general 4D RMHD problems can be reduced to investigation of the behaviour of a 2D test flux tube/string. Such a method has been successfully applied to problems in magnetospheric [7], solar [8] and astrophysical [3] plasmas.

III. FLUX TUBE – KERR BLACK HOLE COUPLING

The Kerr metric is given in Boyer-Lindquist coordinates by [6]

$$
\begin{align*}
(ds)^2 &= (1 - \frac{2Mr}{\Sigma})dt^2 - \frac{\Sigma}{\Delta}dr^2 - \Sigma d\theta^2 - \\
&\quad (r^2 + a^2 + \frac{2Mr^2}{\Sigma} \sin^2 \theta) \sin^2 \theta d\phi^2 + \frac{4Mr}{\Sigma} \sin^2 \theta d\phi dt,
\end{align*}
$$

(3.1)

where

$$
\Delta = r^2 - 2Mr + a^2, \quad \Sigma = r^2 + a^2 \cos^2 \theta.
$$

(3.2)

Here $M$ and $a$ are the mass and specific angular momentum of the hole, respectively, and the units are chosen such that $c = 1$, $G = 1$.

We will now consider a test magnetic flux tube falling into a Kerr black hole (Figure 1a). We solve the string equations (2.9) numerically using TVD scheme similar to [3]. We normalize plasma density to its initial value $\rho_0$, length to the radius of the static limit surface in the equatorial plane $r_g$, velocity to the light speed $c$, time scale to $r_g/c$, magnetic field to $\sqrt{4\pi \rho_0 c}$, the plasma pressure and the energy density to $\rho_0 c^2$, $\alpha$ to $r_g \sqrt{\rho_0 / (4\pi c^2)}$. The number of grid points along the string $-20 < \alpha < 20$ is chosen to be 4000. Only the central part $-10 < \alpha < 10$ will be shown in Figures. The black hole is supposed to be near the extreme rotation $a = .99M$. 

4
We need to specify the total pressure in the string approach. The dependence $P(r) = \frac{c_P}{(r - r_h)^2}$ has been used in [3], where $c_P$ is a constant. In this case both negative and the most of positive energy was confined inside the ergosphere. Here we increase the pressure gradient $P(r) = \frac{c_P}{(r - r_h)^3}$ and as a result are able to observe ejecta from the hole.

It is convenient to chose an initial flux tube outside the ergosphere so that each element of the tube has no angular momentum. Hence, if there is no magnetic field, no tube element would interact with any other element, and as it falls into the black hole it would rotate with an angular velocity $\omega_0 = -g_{t \phi}/g_{\phi \phi}$ of fiducial observers [6].

Because of inhomogeneous rotation of the Kerr geometry, the flux tube becomes stretched and twisted (Figure 1a), and gravitational energy is partially converted into magnetic energy of the swirling tube. The strong magnetic field slows down rotation of the leading part of the flux tube as it falls, hence this part of the tube obtains negative momentum (Figure 2a). Angular momentum of the tube as a whole needs to be conserved, and to compensate this a corresponding positive angular momentum is generated (Figure 2a).

Simulations show that the stretching and twisting of the falling flux tube is most pronounced inside the ergosphere (Figure 1b) where $g_{tt} < 0$ and the energy of the central part of the tube with negative momentum turns out to be also negative (Figure 2b). Since the energy of the tube as a whole has to be conserved, the part of the tube with positive energy gains an energy greater than the initial tube energy (Figure 3). This is a variant of the Penrose process [4], but now we do not need to invoke the decay of particles, since just a single tube can extract energy from the hole.

The process of redistribution of angular momentum is continuous: the deeper the tube falls in the hole, the more the tube is stretched, the stronger the magnetic field gets, the slower the central part of the tube rotates, the more negative momentum is generated, and the more positive momentum is created (Figure 2b). It turns out that both parts with negative and most of the positive energy are localized at this first stage in the narrow layer near the event horizon [3]. Note that the part of the flux tube with positive momentum and with extra positive energy has to spin faster and faster. Eventually, increasing centrifugal
forces eject plasma from the ergosphere producing relativistic jet with the Lorentz factor \( \Gamma \sim 3 \) (Figures 1c, 2c). Note that the whole string remains uninterrupted and simple connected.

When the part of the tube with positive energy is ejected, the remaining part with negative energy can not leave the ergosphere. It continues spinning around the event horizon producing the spiral magnetic field and extracting rotational energy of the hole. During this quasistationary stage the rate of energy extraction is approximately equal to the negative energy creation rate and nearly twice as big as energy accretion rate (Figure 3).

IV. CONCLUSION

After submission of our paper we got to know the results of general RMHD numerical simulations modeled accretion of a plasma into a Kerr black hole \([9]\). The comparison of the results obtained with two different methods show that they are in a good agreement. This includes the stretching and twisting of magnetic flux tubes around the event horizon, generation of negative angular momentum and negative energy of the plasma near the event horizon, appearance of compensated positive momentum and positive energy. The problem under consideration is a stiff one, MHD parameters increase very fast inside the ergosphere, and at some stage errors of calculation started to grow exponentially. Therefore the authors of \([9]\) were able to model only early stage of plasma accretion into a rotating black hole and they had to stop calculations at \( t = 7r_g/c \). At this stage which roughly correspond to our Figure 1b, the plasma jet is not yet formed, instead they observed MHD wave which transferred electromagnetic energy outside the ergosphere. We also found this effect in our previous study \([3]\), but unfortunately it turns out that MHD wave extract energy at relatively low rate compared to the negative energy creation rate. The fact is that the infall velocity of the plasma is approximately equal to the speed of the MHD wave.

A relativistic flux tube (i.e. string) is more simple object as the whole MHD flow. Thus, we were able to continue our calculations to later time \( \tau = 20r_g/c \) and to observe a real
plasma ejecta from the hole. We also had a similar problem with the code and we needed to enhance all factors stimulating plasma ejection from the ergosphere such as the angular momentum of the hole and the total pressure gradient. However, the Kerr black hole is so powerful object that for another distribution of the total pressure the jet seems to appear just a bit later anyway. Unfortunately we can not simulate this more realistic situation so far due to the code problem.

The physics of the energy extraction from a rotating black hole and formation of a cosmic jet is rather simple. In the spinning Kerr geometry the leading part of the falling flux tube progressively loses angular momentum and energy as the string/tube brakes, which leads to creation of negative energy inside the ergosphere. To conserve energy and angular momentum for the tube as a whole, the positive energy and angular momentum has to be generated for the trailing part of the tube. The MHD wave can not effectively remove generated positive energy from the ergosphere, and as a result the part of the tube with positive angular momentum has to rotate faster and faster in the course of time. Eventually the growing centrifugal forces push this part of the tube from the ergosphere producing plasma ejecta and extracting spin energy from the hole.

It is worth discussing the string model in relation to the Blandford-Znajek mechanism [1]. The string process is inherently time-dependent because it is based on differential rotation even for the quasi-stationary regime. It is well known that for a purely steady-state axisymmetric configuration, the flux surface must have rigid rotation [10] which is not the case for the string mechanism. There is only a tendency to rigid rotation, which is clear from the redistribution of the angular momentum, i.e., the leading part of the falling tube spins slower, the trailing part faster. But to establish exactly rigid rotation takes an infinite time, which can hardly happen in reality. Therefore, the string mechanism differs from the Blandford-Znajek one based on steady-state and axisymmetrical patterns. Besides, the Blandford-Znajek process needs a magnetic field embedded in the hole’s event horizon while the flux tube in the string approach does not reach the horizon at all in our simulations, which emphasizes the difference even more. However, the main idea of Blandford and Znajek
considering the magnetic forces as the most appropriate to couple the black hole’s spin to external matter is also valid for the string approach.

There is a direct extension of the theory described above to cosmic strings. Formally the equations of motion for ordinary cosmic strings generated by the Nambu-Goto action [11], and especially for superconducting cosmic strings, are the same as those for the flux tubes. Therefore, we can expect extraction of spin energy of the hole by a cosmic string, as it was first pointed out in [12].

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FIGURES

FIG. 1. Time evolution of the falling flux tube is shown for the initial position of the tube (dashed line) and the moment of encounter of the static limit surface (solid line)(a), for the beginning of the ejecta (b), and for the final time of simulation (c). Event horizon is depicted in the center of the pictures.

FIG. 2. Numerical results for a flux tube falling into a Kerr black hole as functions of the mass parameter $\alpha$. Distributions of the density of the angular momentum (a), of the density of the energy (b), and of the radial Boyer-Lindquist coordinate $r$ along the string are shown for the following string times: $\tau_1$ is the initial time, $\tau_2$ is the moment of encounter of the static limit, $\tau_3$ is the beginning of the negative energy creation, $\tau_4$ is the beginning of the ejecta, and $\tau_5$ is the final time of simulation.

FIG. 3. Behaviour of the positive (1), total (3), and negative (5) energy of the string in the course of string time. The curves (2) and (4) shows evolution of the energy of the part of the tube outside and inside the ergosphere, respectively.
