Control of underactuated marine crafts with matched disturbances

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ABSTRACT
Stabilisation of marine craft in the presence of disturbances is important to many offshore activities. The process of stabilising a marine craft using the actuators is known as dynamic positioning, and is challenging for underactuated craft. In our work, we develop a robust dynamic positioning controller for an underactuated marine craft that ensures input-to-state-stability with respect to matched disturbances. Using the strict Lyapunov function approach of Malisoff and Mazenc [(2009). Constructions of strict Lyapunov functions. Springer-Verlag], we show that the proposed controller ensures uniform global asymptotic stability of the desired equilibrium when there are no disturbances. We also show that the closed-loop system is input-to-state stable with respect to matched environmental force and moment disturbances. Finally, we illustrate the performance of the proposed controller through simulations.

1. Introduction
Dynamic positioning is the task of stabilising a marine craft to a set position in the presence of environmental disturbances using the vessel’s actuators (Sørensen, 2011). This stable positioning is important for certain offshore applications such as cable-laying, diving, and inter-ship transfers (Fossen & Perez, 2009). In general, crafts performing dynamic positioning tasks are fully actuated, however, there has been interest in control designs for underactuated crafts, see for example Aguiar and Pascoal (2007); Do et al. (2002a); Panagou and Kyriakopoulos (2014); Pettersen et al. (2004); Xie et al. (2017). These operations can be dramatically affected by the presence of environmental disturbances acting on the marine craft. The disturbances can be separated into two main forces (Fossen, 2011). The quickly-varying wave forces that act on the vessel at the frequency of the waves, and the slowly-varying or wave-drift forces that arise from long-term ocean currents and wind. These environmental disturbances can be characterised by their frequency spectrum as either a stochastic process (St. Denis & Pierson, 1953) or, as used in offshore engineering applications, as a discrete-frequency representation (Faltinsen, 1990).

Early dynamic positioning controllers for fully actuated vessels used traditional control techniques such as PID controllers (Sørensen, 2011). Balchen et al. (1980) introduced the use of a Kalman filter to estimate and subsequently ignore the oscillatory wave component in the stabilisation controller. This was coupled with integral action to compensate the slowly-varying current disturbance to stabilise the vessel to a neighbourhood of the desired position. The use of Kalman filters was combined with linear controllers, such as the LQG feedback controller, to improve performance (Sørensen et al., 1996). Loria et al. (2000) investigated the use of observer design to address measurement noise in the positions and reconstruct the unmeasured velocity states. Zhang and Wu (2020) recently proposed a dynamic positioning and tracking fixed-time control for fully actuated vessels with output constraints. For a comprehensive discussion on the history of dynamic positioning, particularly for fully actuated systems, we refer the reader to the survey by Sørensen (2011).

As shown by Do et al. (2002b), the presence of small non-vanishing disturbances can produce unstable trajectories, and thus robustness properties of the control system against disturbances is desirable. To maintain stability the controller needs to be designed both to reach the stability point and compensate for environmental disturbances. A property of interest for this class of stabilisation problems is the input-to-state stability (ISS) property of the closed-loop system with respect to the disturbances. This property ensures that the states remain bounded in a region around the equilibrium point provided that the input disturbances are bounded. Also, the states will converge to the equilibrium as the disturbances vanish (Sonntag, 2008). In addition, ISS controllers might allow the use of the separation principle to directly estimate and compensate for disturbances. As highlighted in Edwards et al. (2000), since the disturbances and stabilisation controllers are time-varying it is difficult to ensure that the states of the closed-loop system will remain bounded. The underactuated characteristic of marine craft considered in this work together with the presence of disturbances poses a challenging control design problem.

Stabilisation and tracking of marine craft in the presence of disturbances is particularly difficult in the case of underactuated systems (Pettersen & Egeland, 1996). Not only does a degree of freedom lack actuation, most commonly the sway dynamic in marine craft, but also the fact that Brockett’s necessary condition for stabilisation using smooth
controllers is not satisfied make mandatory the use of time-varying or non-smooth controllers (Brockett, 1983). Pettersen and Egeland (1996) considered the use of time-varying reference signals and designed a continuous controller to ensure that the closed loop trajectories converge to the desired equilibrium. This approach was improved in Pettersen and Fossen (2000) with the inclusion of integral action to compensate for constant disturbances. The authors showed that their controller exponentially stabilises the position and orientation of the vessel but noted that the state trajectories oscillate around the stationary point due to the presence of environmental forces. The controller of Bertin et al. (2000) uses feedback linearisation to stabilise the vessel in the direction of the environmental disturbance. A slow integrator is also used to cancel the constant disturbance and residual steady-state error. Panagou and Kyriakopoulos (2014) adopts a different approach with switching logic to ensure that the vessel approaches and remains inside a bounded set around the equilibrium then converges to the desired position once inside the set. The authors show that the vehicle is practically stable to disturbances with the orientation bounded for certain periods. A more recent paper from Zhang and Wu (2015) also used a switching based controller to focus on the problem of a non-symmetrical ship where the mass matrix is non-diagonal. To counter the computational demands of more complex controllers, Xie et al. (2017) proposes a simple controller to stabilise the north and east positions of an underactuated vehicle. However the proposed controller, while simple, lacks the stabilisation of the orientation.

The tracking problem aims at ensuring convergence of the state vector to a desired reference signal that describes the desired trajectories of the vehicle. Do et al. (2002a) utilised a virtual ship to generate a possible trajectory and designed a controller to converge to the virtual trajectory. This method allows the use of yaw reference signals that are zero or converge to zero. A different approach was taken in Do et al. (2003) where a linear course with constant forward velocity was considered, such that the surge motion is not included in sway-yaw dynamics. This enabled the authors to use a coordinate transformation of only the sway and yaw coordinates with the objective of stabilising the sway and yaw motion. Further improvements by Do (2016) included the rejection of earth fixed stochastic environmental disturbances set up as a Weiner process and ensure path tracking in a probability sense. A very recent approach with forward prediction is the use of a model predictive controller by Liang et al. (2020) that considered state and actuation constraints. We also note the use of discontinuous sliding mode controller with robustness to model errors (Perera & Soares, 2012) or with obstacle avoidance (Keymasi Khalaji & Tourajizadeh, 2020). Other approaches consider the issue of non-diagonal mass matrices with the stabilisation of a cascaded yaw and yaw rate subsystem (Xie et al., 2018). Also Wang and Su (2019) developed an observer for asymmetric vehicles to compensate for unknown vessel dynamics.

Some control designs consider the trajectory tracking and the dynamic positioning (or regulation) problems simultaneously. Do et al. (2002b) designed the first controller for an underactuated marine craft to solve both control problems. Their controller forced the vessel to follow the reference trajectory from a virtual ship, whose dynamics is a copy of the original ship dynamics. This work was later extended by including a disturbance observer to estimate the disturbance from the Earth frame (Do, 2010). Another example by Aguiai and Pascoal (2007), designs the controller to track to desired waypoints, with dynamic positioning at the final waypoint. In note of environmental disturbances, the desired orientation of the final waypoint is dropped to align the x-axis with the direction of current. The authors discuss how if this is not done, the craft will diverge from the target position or oscillate in a neighbourhood of the target (Aguiai & Pascoal, 2007). In a more recent example, Li (2019) used coordinate transforms such that the unactuated sway dynamic could be controlled through virtual signals, then computed the actual control signal through backstepping. This approach allowed the authors to solve both the tracking and positioning problems for vessels with non-diagonal inertia matrix and input saturation.

Of particular interest is the approach of Mazenc et al. (2002) with experimental results reported in Pettersen et al. (2004). In their work, the use of strict Lyapunov functions allowed for the development of a uniform global asymptotic stable controller that could stabilise a marine craft to a desired equilibrium. As noted in the experimental results, the performance of the controller was limited by modelling errors and environmental disturbances (Pettersen et al., 2004). The disturbances were not considered in the control design and no robust properties of the closed-loop respect to disturbances are shown. Further, as concluded in Pettersen et al. (2004), future investigation of stabilisation controllers should aim to include disturbances.

In our work, we extend previous results by explicitly considering an underactuated marine craft under the action of matched disturbances, such as those arising from wave and currents at sea. We design a time-varying controller that ensures uniform global asymptotic stability of the origin, and ISS of the closed-loop system with respect to environmental disturbances acting in the actuated channels. Following the approach of Malisoff and Mazenc (2009), we modify the yaw reference signals such that the yaw converges to the origin instead of continuously maintaining a persistent excitation. This choice introduces convergence while ensuring stabilisation. The choice is different from that of Mazenc et al. (2002), as the reference is only time dependent, not time and state dependent. This results in less complicated controller design.

This paper addresses ISS of a closed-loop underactuated system to matched disturbances, which has not yet been shown in the literature. This work is a step towards the more challenging problem of showing ISS respect to both matched and unmatched disturbances, and allows for future combination with a disturbance rejection technique under a separation principle argument.

The paper is structured as follows. We first introduce the standard marine system dynamics and formulate our problem in Section 2. We present the control design and show the stability properties of the closed-loop in Section 3. In Section 4, we illustrate the controller performance affected by disturbances via simulation, and provide concluding remarks in Section 5.
2. Problem formulation

2.1 Marine craft model for dynamic positioning

We consider the standard three degrees of freedom model for an underactuated surface marine craft (Fossen, 2011)

\[ \dot{x} = \cos(\psi) u - \sin(\psi) v \]
\[ \dot{y} = \sin(\psi) u + \cos(\psi) v \]
\[ \dot{\psi} = r \]
\[ \dot{u} = \frac{m_{22}}{m_{11}} vr - \frac{d_{11}}{m_{11}} u + \frac{1}{m_{11}} \tau_1 + \xi_1 \]
\[ \dot{v} = -\frac{m_{11}}{m_{22}} ur - \frac{d_{22}}{m_{22}} v \]
\[ \dot{r} = \frac{m_{11} - m_{22}}{m_{33}} uv - \frac{d_{33}}{m_{33}} r + \frac{1}{m_{33}} \tau_3 + \xi_3 \]

where \( x, y, \psi, u, v, r \) are the forward, sideways and heading positions, the surge and sway velocities, and yaw rate, respectively. The parameters \( m_{ij}, d_{ij}, \) with \( i = \{1,2,3\} \), represent the mass and damping coefficients, \( \tau_1, \tau_3 \) are the control inputs in the surge and yaw, and \( \xi_1 \) and \( \xi_3 \) are the matched disturbances.

2.2 Control objective

In this paper, we consider the control design of an underactuated marine craft that ensures the following properties:

\( O1. \) Consider the scenario where there are no disturbances. The control objective is to design the control laws \( \tau_1 \) and \( \tau_3 \) to ensure that the origin of the system (1), i.e. \( \{x, y, \psi, u, v, r\} = 0 \), is a uniform global asymptotically stable (UGAS) equilibrium point.

\( O2. \) Consider the scenario where there are matched disturbances \( \xi_1 \) and \( \xi_3 \). The control objective is to design control laws \( \tau_1 \) and \( \tau_3 \) such that the system (1) is input to state stable (ISS) with respect to the matched disturbances \( \xi_1 \) and \( \xi_3 \).

3. Control design

The system (1) does not satisfy the Brockett condition for stabilisation (Brockett, 1983). Thus it is not possible to achieve the control objective with a control law that is a smooth continuous function of the states (Pettersen & Egeland, 1996). Then, the control objective can be achieved by using either a mode switching controller or a time-dependent controller. In our work, we follow the latter approach and we introduce desired reference signals that converge to the origin, and design our controller to track those signals, and thus achieve the control objective.

3.1 Reference signals

We first introduce the desired reference signals \( z_3^*(t) \) and \( r^*(t) \) for yaw and yaw rate respectively, the coordinate transformation

\[ z_1 = \cos(\psi)x + \sin(\psi)y \]
\[ z_2 = -\sin(\psi)x + \cos(\psi)y + 1/\rho_2 v \]
\[ z_3 = \psi - z_3^* \]
\[ z_4 = r - r^* \]

and the controller

\[ \tau_1 = m_{11} \alpha_1 - m_{22} \nu r + d_{11} u \]
\[ \tau_3 = m_{33} \alpha_3 - (m_{11} - m_{22}) \nu v + d_{33} r, \]

with \( \rho_1 = \frac{m_{11}}{m_{22}}, \rho_2 = \frac{d_{22}}{m_{22}}, \) and \( \alpha_1 \) and \( \alpha_3 \) as new control inputs. Then, the dynamics (1) can be written as follows:

\[ \dot{z}_1 = u + \left( \frac{z_2 - 1}{\rho_2} \right) (z_4 + r^*) \]
\[ \dot{z}_2 = -\left( \frac{z_1 + \rho_1}{\rho_2} u \right) (z_4 + r^*) \]
\[ \dot{z}_3 = (z_4 + r^*) - z_3^* \]
\[ \dot{u} = \alpha_1 + \xi_1 \]
\[ \dot{v} = -\rho_1 u (z_4 + r^*) - \rho_2 v \]
\[ \dot{z}_4 = \alpha_3 - r^* + \xi_3. \]

The coordinate transformation introduced here is a global diffeomorphism to render the system (1) into (4). This class of transformation that suppresses the trigonometric functions of the coordinates was used in Pettersen and Egeland (1996) to simplify the control design and obtain a continuous time-varying feedback law for the marine system dynamics. Another global diffeomorphism that removes coordinate dependence is introduced by Zhang and Wu (2015) with six new states, amongst other examples.

We follow the transform by Pettersen and Egeland (1996) of introducing \( z_1, z_2 \) and \( z_3 \), and combine with the second coordinate transform by Mazenc et al. (2002), to introduce \( v \) in the transform for \( z_2 \). This second transform ensures that \( v \) does not appear in the dynamic of \( z_2 \), which allows the construction of strict Lyapunov functions and subsequent control design. We also introduce the desired reference signals \( z_3^* \) and \( r^* \) by modifying the transform of \( z_3 \) and introducing \( z_4 \). Including the reference signals into the transform allows us to seek controllers to stabilise the origin of the transformed system, i.e. \( \{z_1, z_2, z_3, u, v, z_4\} = 0 \).

If the reference signals are selected to verify \( z_3^* \to 0 \) and \( r^* \to 0 \) as time goes to infinity, then our control objective can be achieved by finding \( \alpha_1 \) and \( \alpha_3 \) such that the origin of (4) is UGAS with no disturbance and the closed-loop is ISS with respect to the disturbances \( \xi_1 \) and \( \xi_3 \). To this end, we propose the reference signals

\[ r^* = -\epsilon \exp(-\lambda t) \sin(\omega t) \]
\[ z_3^* = \frac{\epsilon}{\lambda^2 + \omega^2} \exp(-\lambda t)(\lambda \sin(\omega t) + \omega \cos(\omega t)) \]

where \( \epsilon > 0, \lambda > 0 \) and \( \omega > 0 \) are positive design constants. We note that \( r^* = z_3^* \) and that both \( r^* \) and \( z_3^* \) converge to zero as time goes to infinity. Our choice of reference signals satisfies the persistence of excitation condition in the construction of Strict Lyapunov Functions using the work of Malisoff and Mazenc (2009), and converges to the origin according to the exponential decay parameter to achieve our original control objective. This choice of reference signal is similar to references for trajectory tracking (Do & Pan, 2009; Fossen, 2011; Malisoff & Mazenc, 2009; Pettersen et al., 2004).
In the first step of the control design, we now exploit the cascade structure of the dynamics (4) and design a control law for the subsystem $(z_1, z_2, z_3, v)$ considering $u$ and $z_4$ as the virtual control inputs $u_d$ and $z_{4d}$. In a second step of the design, we will compute the control inputs $\alpha_1$ and $\alpha_3$ using backstepping to ensure the stability properties of the complete system (4), which includes the dynamics of $u$ and $z_4$. That is, we concentrate first on the system

\[
\begin{align*}
\dot{z}_1 &= u_d + \left( z_2 - \frac{1}{\rho_2} v \right) [z_{4d} - \epsilon \exp(-\lambda t) \sin(\omega t)] \\
\dot{z}_2 &= -\left( z_1 + \frac{\rho_1}{\rho_2} u_d \right) [z_{4d} - \epsilon \exp(-\lambda t) \sin(\omega t)] \\
\dot{z}_3 &= z_{4d} \\
\dot{v} &= -\rho_1 u_d [z_{4d} - \epsilon \exp(-\lambda t) \sin(\omega t)] - \rho_2 v 
\end{align*}
\]

and then we compute the control inputs $\alpha_1$ and $\alpha_3$ to ensure asymptotic tracking of $u$ and $z_4$ to $u_d$ and $z_{4d}$ respectively, stability of the full closed-loop, and robust properties against disturbances.

### 3.2 Control for the reduced system

In this section, we propose the virtual controllers $u_d$ and $z_{4d}$ to stabilise (5). The controller and stability properties are presented in the following proposition.

**Proposition 3.1:** Consider the system (5) in closed-loop with the virtual controllers

\[
\begin{align*}
u_d &= -\frac{\rho_2}{\rho_1} z_1 + k_2 z_2 [z_{4d} - \epsilon \exp(-\lambda t) \sin(\omega t)], \\
z_{4d} &= -k_0 z_3, 
\end{align*}
\]

where $k_0 > 0$ and $k_2 > 0$.

Then, the equilibrium point $[z_1, z_2, z_3, v]^T = [0, 0, 0, 0]^T$ of the closed-loop system is uniform global asymptotically stable.

In our proof of Proposition 3.1, we use strict Lyapunov functions that have been constructed using the method developed by Malisoff and Mazenc (2009).

**Proof:** The closed-loop dynamics is obtained by using (6) and (7) in (5) as follows:

\[
\begin{align*}
\dot{z}_1 &= -\frac{\rho_2}{\rho_1} z_1 + \frac{1}{\rho_2} v \epsilon \exp(-\lambda t) \sin(\omega t) + h_1(t, v, z_2, z_3) \\
\dot{z}_2 &= -\frac{\rho_1}{\rho_2} k_2 z_2 [k_0 z_3 + \epsilon \exp(-\lambda t) \sin(\omega t)]^2 \\
\dot{z}_3 &= -k_0 z_3 \\
\dot{v} &= -\rho_2 v - \rho_2 z_1 \epsilon \exp(-\lambda t) \sin(\omega t) + h_2(t, z_1, z_2, z_3)
\end{align*}
\]

with

\[
\begin{align*}
h_1(t, v, z_2, z_3) &= -(1 + k_2) z_2 [k_0 z_3 + \epsilon \exp(-\lambda t) \sin(\omega t)] + \frac{1}{\rho_2} v k_0 z_3 \\
h_2(t, z_1, z_2, z_3) &= -\rho_2 k_0 z_1 z_3 - \rho_1 k_2 z_2 [k_0 z_3 + \epsilon \exp(-\lambda t) \sin(\omega t)]^2.
\end{align*}
\]

Now consider the following Lyapunov function:

\[
W_3(t, z_1, v, z_2, z_3) = W_1(z_1, v) + K_3(\epsilon, \omega) W_2(t, z_2, z_3)
\]

where

\[
\begin{align*}
W_1(z_1, v) &= \ln(W_1(z_1, v) + 1), \\
W_2(t, z_2, z_3) &= \exp \left( \frac{\rho_1 k_2}{\rho_2 k_0} k_0 e^2 [2\pi - \sin(2\omega t)] \right) \\
&\quad \times V_2(z_2, z_3) \exp[V_2(z_2, z_3)], \\
V_1(z_1, v) &= \frac{1}{2} \rho_2^2 z_1^2 + \frac{1}{2} v^2, \\
V_2(z_2, z_3) &= \frac{1}{2} k_1^2 z_2^2 + \frac{1}{2} k_4 z_3^2,
\end{align*}
\]

the coefficients $k_{13}, k_{14} > 0$ are free constants. The positive constant $K_3(\epsilon, \omega)$ is defined as

\[
K_3(\epsilon, \omega) = 16 \left( \frac{1}{k_{18}} K_1 + \frac{4}{k_{18}} \left( \frac{k_0}{\epsilon} \right)^4 \frac{k_{14}}{k_{14}} K_2(\epsilon) \right) \\
\times \frac{\rho_2}{\rho_1 k_0} 16 \frac{1}{k_1 k_2 k_0 e^2} \exp \left( \frac{\rho_1 k_2}{\rho_2 k_0} \frac{1}{k_0 e^2 [2\pi - 1]} \right)
\]

where $k_{13} k_{13} \geq 1, \bar{k}_{14} \bar{k}_{14} \geq 1$ and

\[
K_1 = \bar{k}_{14} k_0^2 \left( \frac{k_9}{k_5} + \frac{k_{11}}{k_6} \right), \\
K_2(\epsilon) = \bar{k}_{13}^2 \left( \frac{\rho_1^2 k_8}{2 k_5} (1 + k_2) \epsilon^2 + \frac{\bar{k}_{12} \rho_1^2 k_5^2}{2 k_8} \epsilon^2 \right).
\]

The Lyapunov functions $W_1$ and $W_2$, which are used to build the strict Lyapunov function $W_3$, were constructed from $V_1$ and $V_2$ following the strictification approach of Malisoff and Mazenc (2009) (see the appendix for a sketch of the construction). To simplify the presentation, we compute the time derivative of $W_3$ along the trajectories of (8) as $W_3 = W_1 + K_3 W_2$. We first consider $W_1(z_1, v)$ and compute its time derivative as follows:

\[
\dot{W}_1 = \frac{V_1(z_1, v)}{V_1(z_1, v) + 1} \\
\leq \left( \frac{\rho_1}{\rho_1^2} - \frac{\rho_2 k_0}{2 \pi} \frac{1}{V_1(z_1, v) + 1} \right)^2 + \frac{\rho_2^2}{V_1(z_1, v) + 1} \\
\times \frac{1}{k_{14}^2 V_1(z_1, v) + 1} h_1^2(t, z_1, z_2, z_3) \\
\leq k_{14} (1 + k_2)^2 z_2^2 (k_0 |z_3| + \epsilon)^2 + \frac{2 k_9 k_5^2}{\rho_2^2} z_3^2,
\]

where we have used Young’s inequality in the second line. Since $V_1$ is positive definite, algebraic manipulation shows that the last two terms in (13) can be bounded using

\[
\frac{h_1^2(t, v, z_2, z_3)}{V_1(z_1, v) + 1} \leq k_{14} (1 + k_2)^2 z_2^2 (k_0 |z_3| + \epsilon)^2 + \frac{2 k_9 k_5^2}{\rho_2^2} z_3^2,
\]
\[
\frac{h^2_2(t,z_1,z_2,z_3)}{V_1(z_1,v)} \leq 2k_11k_6^2z_3^2 + k_{12} \rho_1^2 k_2^2 z_3^2 (k_0|z_3| + \epsilon)^4,
\]

where \(k_0, k_9, k_{11}, k_{12} \geq 1\) are free parameters. Then, we can write the derivative as

\[
\dot{W}_1 \leq -\frac{\rho_1^2}{2\rho_1} (2\rho_2 - \rho_1 k_5) \frac{2\rho_2 - k_6}{V_1(z_1,v)} z_1^2 - \frac{1}{2} (2\rho_2 - k_6) V_1(z_1,v) v^2 \\
+ k_0^2 \left( \frac{k_9}{k_5} + \frac{k_{11}}{k_6} \right) \frac{1}{k_5} e^2 \left( \frac{k_2 k_9}{k_5} (1 + k_2)^2 \right) \\
+ k_{12} \rho_1^2 k_2^2 \left( \frac{k_9}{k_5} \right) \frac{1}{k_5} e^2 \left( \frac{k_2 k_9}{k_5} (1 + k_2)^2 \right) \\
\leq -\frac{\rho_1^2}{2\rho_1} (2\rho_2 - \rho_1 k_5) \frac{2\rho_2 - k_6}{V_1(z_1,v)} z_1^2 - \frac{1}{2} (2\rho_2 - k_6) V_1(z_1,v) v^2 \\
+ 2K_1 V_2(z_2,z_3) + 2K_2 (\epsilon) V_2(z_2,z_3) \\
\times \left( \frac{k_0}{k_5} \frac{1}{k_5} e^2 \left( \frac{k_2 k_9}{k_5} (1 + k_2)^2 \right) \right).
\]

Noting that we can bound the quartic term by \(V_2^2(z_2,z_3)\) and a constant, we write a new bound for \(W_1\)

\[
\dot{W}_1 \leq -\frac{\rho_1^2}{2\rho_1} (2\rho_2 - \rho_1 k_5) \frac{2\rho_2 - k_6}{V_1(z_1,v)} z_1^2 - \frac{1}{2} (2\rho_2 - k_6) V_1(z_1,v) v^2 \\
+ 2K_1 V_2(z_2,z_3) + 16K_2 (\epsilon) V_2(z_2,z_3) \\
\times \left( \frac{4}{\rho_1^2} k_0^2 k_4 (V_2(z_2,z_3) + 1) \right). \tag{14}
\]

Note that the first two terms in (14) form a negative definite function of \(z_1\) and \(v\) provided that \(2\rho_2 - \rho_1 k_5 > 0\) and \(2\rho_2 - k_6 > 0\).

We now compute the time derivative of \(W_3\), which can be bounded as follows (see appendix for further details):

\[
\dot{W}_3 \leq -\frac{\rho_1^2}{2\rho_1} (2\rho_2 - \rho_1 k_5) \frac{2\rho_2 - k_6}{V_1(z_1,v)} z_1^2 - \frac{1}{2} (2\rho_2 - k_6) V_1(z_1,v) v^2 + 2K_1 \\
\times V_2(z_2,z_3) \\
+ 16K_2 (\epsilon) V_2(z_2,z_3) \left( \frac{4}{\epsilon^4} k_0^2 k_4^2 (V_2(z_2,z_3) + 1) \right) \\
+ K_3 (\epsilon, \omega) \left[ -\frac{\rho_1 k_2}{\rho_2} \frac{1}{k_0^2} \frac{1}{\epsilon^2} \frac{1}{\omega^2} \right] \\
\times \left( \frac{4}{\rho_1^2} k_0^2 k_4 (V_2(z_2,z_3) + 1) \right). \tag{15}
\]

Utilising (14) and (15), we can readily compute the time derivative of \(W_3\) as follows:

\[
\dot{W}_3 \leq -\frac{\rho_1^2}{2\rho_1} (2\rho_2 - \rho_1 k_5) \frac{2\rho_2 - k_6}{V_1(z_1,v)} z_1^2 - \frac{1}{2} (2\rho_2 - k_6) V_1(z_1,v) v^2 + 2K_1 \\
\times V_2(z_2,z_3) \\
+ 16K_2 (\epsilon) V_2(z_2,z_3) \left( \frac{4}{\epsilon^4} k_0^2 k_4^2 (V_2(z_2,z_3) + 1) \right) \\
+ K_3 (\epsilon, \omega) \left[ -\frac{\rho_1 k_2}{\rho_2} \frac{1}{k_0^2} \frac{1}{\epsilon^2} \frac{1}{\omega^2} \right] \\
\times \left( \frac{4}{\rho_1^2} k_0^2 k_4 (V_2(z_2,z_3) + 1) \right). \tag{16}
\]

Notice that since \(\Sigma\) is a positive definite function, then there exists a \(K_\infty\) function \(\sigma\) such that

\[
W_3 \leq -\sigma ((z_1, v, z_2, z_3)). \tag{17}
\]

We can conclude from (17) that \(W_3\) is a strict Lyapunov function of the closed loop, and thus the origin of the closed-loop system (8) is uniform global asymptotically stable (Malisoff & Mazenc, 2009, Lemma 2.1), which completes the proof. \(\blacksquare\)

### 3.3 Control of the complete system

In this section, we design the control inputs \(\alpha_1\) and \(\alpha_3\) to stabilise the full system (4). We follow the backstepping approach, define the error signals \(\bar{u} = v - u_d\) and \(\bar{z}_4 = z_4 - z_{4d}\) and re-write the dynamics (4) in the form

\[
\dot{z}_1 = u_d + \left( z_2 - \frac{1}{\rho_2^2} \right) (z_{4d} + r^*) + R_1 \\
\dot{z}_2 = -\left( z_1 + \frac{\rho_1}{\rho_2^2} u_d \right) (z_{4d} + r^*) + R_2 \\
\dot{z}_3 = z_{4d} + R_3 \dot{\bar{u}} = -\rho_1 u_d (z_{4d} + r^*) - \rho_2 v + R_4 \\
\dot{\bar{z}}_4 = \alpha_3 + R_6 + \xi_1 \\
\dot{\bar{z}}_4 = \alpha_3 + R_6 + \xi_3 \tag{18}
\]

where

\[
R_1 = \bar{u} + \left( z_2 - \frac{1}{\rho_2^2} \right) \bar{z}_4 \\
R_2 = -z_1 \bar{z}_4 - \frac{\rho_1}{\rho_2} (\bar{u} \bar{z}_4 + \bar{u} z_{4d} + \bar{u} r^* + u_d \bar{z}_4)
\]
The following proposition presents the controller $\alpha_1$ and $\alpha_3$, and the properties of the system in closed-loop with the controller.

**Proposition 3.2:** Consider the system (18) in closed-loop with the controller

$$
\alpha_1 = -[k_1 + f_1(t, z_1, z_2, z_3, v, \ddot{\psi}, \dot{\theta})]\ddot{\psi} - R_5 - \frac{\partial W_3}{\partial z_1}
+ \left(\frac{\partial W_3}{\partial \nu} + \frac{\partial W_3}{\partial v}\right) [z_{4d} - \epsilon \exp(-\lambda t) \sin(\omega t)]
$$

(19)

$$
\alpha_3 = -[k_2 + f_2(t, z_1, z_2, z_3, v, \ddot{\psi}, \dot{\theta})]z_3 - R_6
- \frac{\partial W_3}{\partial z_1} (z_2 - \frac{1}{\rho_2} v) + \frac{\partial W_3}{\partial z_2} (z_1 + \frac{\rho_1}{\rho_2} (\ddot{u} + u_d))
- \frac{\partial W_3}{\partial v} + \frac{\partial W_3}{\partial \nu} \rho_1 (\ddot{u} + u_d)
$$

(20)

where $k_1, k_2 > 0$ are tuning parameters, $f_1$ and $f_2$ are positive definite functions of the states to be chosen, and

$$W_4 = W_3(t, z_1, z_2, z_3, v) + \frac{1}{2}(\ddot{u}^2 + z_3^2).$$

Then, the closed-loop system has the following properties:

(i) The equilibrium point $[z_1, z_2, z_3, v, \ddot{\psi}, \dot{\theta}, \ddot{\theta}]^T = [0, 0, 0, 0, 0, 0]^T$ of the closed-loop system is uniform global asymptotically stable, when there are no disturbances (i.e. $\xi_1 = 0$ and $\xi_3 = 0$).

(ii) The closed-loop system is ISS with respect to the disturbances $\xi_1$ and $\xi_3$.

**Remark 3.1:** The control input of the ship $\tau_1$ and $\tau_3$ can be obtained by replacing $\alpha_1$ and $\alpha_3$ with (19) and (20) in (2) and (3) above.

**Proof:** We use $W_4$ as a candidate Lyapunov function and compute its time derivative along the trajectories of the closed-loop to obtain

$$\dot{W}_4 \leq -\Sigma(z_1, v, z_2, z_3) + \frac{\partial W_3}{\partial z_1} R_1 + \frac{\partial W_3}{\partial z_2} R_2 + \frac{\partial W_3}{\partial z_3} R_3
+ \frac{\partial W_3}{\partial v} R_4 + \ddot{u}(\alpha_1 + R_5 + \xi_1) + \dot{\theta}_4(\alpha_3 + R_6 + \xi_3).$$

Note when computing $\dot{W}_3(t, z_1, z_2, z_3, v)$ using the chain rule, the time derivatives of $z_1, z_2, z_3$, and $v$, see (18), can be separated into the terms appearing in (5), plus $R_1, R_2, R_3$, and $R_4$, related to the error in the virtual control signal. Considering just the terms from (5), we are able to utilise Proposition 3.1 and the result (16).

Now, using the controller (19) and (20) and Young’s inequality we obtain

$$\dot{W}_4 \leq -\Sigma(z_1, v, z_2, z_3) + \ddot{u}\xi_1 + \ddot{\theta}_4\xi_3
- (k_1 + f_1(t, z_1, z_2, z_3, v, \ddot{\psi}, \dot{\theta}))\ddot{\psi}^2
-k_2 + f_2(t, z_1, z_2, z_3, v, \ddot{\psi}, \dot{\theta}))z_3^2
\leq -\Sigma(z_1, v, z_2, z_3) + \frac{1}{2k_{d_1}}\ddot{\theta}_1^2 + \frac{1}{2k_{d_3}}\ddot{\theta}_3^2
-k_1 + f_1 - k_{d_1}/2)\ddot{\psi}^2 - (k_2 + f_2 - k_{d_3}/2)z_3^2.$$

We define

$$\dot{\Sigma}(z_1, v, z_2, z_3, \ddot{\psi}, \dot{\theta}) = \Sigma(z_1, v, z_2, z_3)
+ (k_1 + f_1 - k_{d_1}/2)\ddot{\psi}^2 + (k_2 + f_2 - k_{d_3}/2)z_3^2,
\dot{\gamma}(\xi_1, \xi_3) = \frac{1}{2k_{d_1}}\ddot{\theta}_1^2 + \frac{1}{2k_{d_3}}\ddot{\theta}_3^2,$$

which are positive definite functions for some small $k_{d_1}$ and $k_{d_3}$. Then, there exist two $\kappa_\infty$ functions $\ddot{\theta}$ and $\dot{\gamma}$ that verify

$$\dot{W}_4 \leq -\ddot{\theta}(z_1, v, z_2, z_3, \ddot{\psi}, \dot{\theta}) + \ddot{\gamma}(\xi_1, \xi_3).$$

(21)

Clearly, from (21) and under the assumption that there are no disturbances, then $W_4$ is a strict Lyapunov function of the closed-loop, and thus its origin is uniform global asymptotically stable, which shows property (i). Also, as (21) holds true, then the $W_4$ is an ISS Lyapunov function (Edwards et al., 2000), and consequently the system (18) in closed-loop with (19) and (20) is ISS with respect to the matched disturbances $\xi_1$ and $\xi_3$, which shows property (ii).

Property (i) of Proposition 3.2 gives Uniform Global Asymptotic Stability for the transformed coordinates, which implies that the same property also holds for the original coordinates. This satisfies O1 of our Control Objective. Property (ii) of Proposition 3.2 gives ISS with respect to the matched disturbances $\xi_1$ and $\xi_3$ which implies the same property also holds for the original coordinates. This satisfies O2 of our Control Objective.

**4. Simulation example**

In this section, we simulate an unactuated marine craft described by the dynamics (1) in closed-loop with the proposed controller (2), (3), (19) and (20). We compare the results with the controllers proposed in Xie et al. (2017) and Do et al. (2002b). We consider three scenarios. In the first scenario, we consider no disturbances, in the second we consider matched disturbances, and as a test of robustness in the third scenario we consider the effect of parameter uncertainty.

The vessel parameters are $m_{11} = 19.0$, $m_{22} = 35.2$, $m_{33} = 4.2$, $d_{11} = 4.0$, $d_{22} = 10.0$, and $d_{33} = 1.0$ (Petterson et al., 2004), which implies that $\rho_1 = 0.5398$ and $\rho_2 = 0.2841$. These values are in SI units, see other examples of model parameters in Fossen (2011), Do (2010), and Panagou and Kyriakopoulos (2014). The initial conditions are $\bar{x}_0 = -2$, $y_0 = -1.94$, and $\psi_0 = 0.8$, with zero velocities.

In the simulations, our reference signal parameters are $\epsilon = 0.1$, $\lambda = 0.05$, and $\omega = 0.1$. These parameters were chosen such that the reference signal for yaw starts at the initial heading value, i.e. $z^*_\theta(0) = \psi(0)$. The control parameters are $k_0 = 10^{-2}$,
\( k_5 = \frac{c_1}{\rho_1}, k_6 = \rho_2, k_8 = 1, k_9 = 1, k_{11} = 1, k_{12} = 1, k_{13} = 10^{-3}, k_{15} = 10^3, k_{14} = 10^{-1}, k_{11} = 10^4, k_{17} = 1.589, k_{18} = 1, k_{19} = 1, k_{20} = 10 \). These parameters satisfy the bounds in Propositions 3.1 and 3.2. The controller nonlinear functions used are

\[
\begin{align*}
    f_1(t, z_1, z_2, z_3, v, \tilde{u}, \tilde{z}_4) &= \frac{10^2}{V_1 + 1}, \\
    f_2(t, z_1, z_2, z_3, v, \tilde{u}, \tilde{z}_4) &= \frac{256 \rho_2}{\rho_1 \epsilon^2 k_0 k_2} \left( K_1 \frac{K_2}{k_18} + \frac{1}{k_18} + \frac{4 k_4 k_2}{\epsilon^2 k_14} \right) K_2(\epsilon) \\
    &\times \exp \left( \frac{\rho_1 \epsilon^2 k_0 k_2}{32 \rho_2 \omega} \left( 1 - \sin(2\omega t) \right) \right) (V_2 + 1) \exp(V_2).
\end{align*}
\]

Inspiration for the nonlinear damping functions was drawn from the partial derivative terms in the controllers (19) and (20).

The parameters for the controller of Xie et al. (2017) are: \( k_1 = 0.15 \) and \( k_2 = -0.05 \). The parameters for the controller of Do et al. (2002b) are: \( c_1 = 3.7, c_2 = 2.5, k_1 = 0.689, k_2 = 1, k_3 = 0.2, k_4 = 5, k_5 = 2.2, \lambda_1 = 0.015, \lambda_2 = 0.005, \lambda_3 = 4 \). As noted in Do (2010), the controller in Do et al. (2002b) is complex to tune. However, we have tuned all controllers to the best of our ability via extensive simulation trials.

In all of the following figures, the solid blue line represents the simulation results of using the controller proposed in this article. For comparison purposes, we include the results obtained using the controller, proposed in Xie et al. (2017) and Do et al. (2002b), which are shown in black dash–dot line and pink dotted line, respectively.

### 4.1 Scenario 1. No disturbances

We first consider the case of no disturbances, i.e. \( \xi_1 = \xi_3 = 0 \). Figures 1 and 2 show the motion in surge and sway. Figure 3 shows the yaw angle and rate together with the desired trajectories, \( z^\star_3 \) and \( r^\star \), and Figure 4 shows the control force and moment \( \tau_1 \) and \( \tau_3 \) respectively.

The simulation of our proposed controller shows that the trajectories of surge and sway positions asymptotically converge to the origin, with their respective velocities also converging to zero in a smooth manner. Also, the yaw angle and rate follow their desired references. Both of the comparison controllers converge to the origin for the

![Figure 1](image1.png)

**Figure 1.** Time histories of positions in the no disturbance case. The response of the proposed controller is in solid blue line [---], the controller of Xie et al. (2017) in black dash–dot line [–], and the controller of Do et al. (2002b) in pink dotted line [····].

![Figure 2](image2.png)

**Figure 2.** Time histories of velocities in the no disturbance case: (a) velocity in surge, (b) velocity in sway.

![Figure 3](image3.png)

**Figure 3.** Time histories of yaw angle and yaw rate in the no disturbance case. The reference signals \( z^\star_3 \) and \( r^\star \) in dashed red lines [····] are used in the proposed controller. Note that the yaw angle and rate from our proposed controller directly follow the corresponding reference signals. (a) Yaw angle, (b) Yaw rate.
surge and sway positions as expected. However, the orientation does not converge using the simple controller of Xie et al. (2017), as it is not included in the control design.

### 4.2 Scenario 2. Disturbances

The second simulation considers the scenario where the vessel is under the action of (body-frame) disturbances

\[
\begin{align*}
\xi_1 &= \frac{1}{m_{11}} (-0.5 + 0.2 \cos(1.7t)) \\
\xi_3 &= \frac{1}{m_{33}} (-1 + 1.5 \cos(1.7t))
\end{align*}
\]

This simple disturbance form is commonly used in the literature (see e.g. Keymasi Khalaji & Tourajizadeh, 2020), and emulates environmental disturbances that can be separated into the quickly-varying wave-frequency motion, and the slowly-varying long-term disturbances from ocean currents and wind (Fossen, 2011). The simulation illustrates the ISS property of our proposed controller in closed-loop with respect to surge force disturbances and yaw moment disturbances.

Figures 5 and 6 show the motion in surge and sway. Figure 7 shows the yaw angle and rate together with the desired trajectories, \(z^*\) and \(r^*\), and Figure 8 shows the control force and moment \(\tau_1\) and \(\tau_3\) respectively.

The simulation shows that the motion in surge, sway and yaw under our proposed controller remains bounded as the disturbances are bounded, which is ensured by the ISS property.
of the closed loop. We also notice that the time histories of the states and the control inputs are sufficiently smooth.

The performance of the controller of Xie et al. (2017) is greatly deteriorated by the presence of disturbances as can be seen in Figures 5 – 8. Also, the sinusoidal disturbance produces significant sinusoidal responses and the orientation angle diverges.

From this simulation, we remark that while our proposed controller is more complex, this increased complexity gives additional properties. We prove asymptotic convergence to the origin of the surge and sway positions under no disturbances, as does Xie et al. (2017), but we also show the convergence of the orientation. Additionally, we show ISS to matched disturbances.

The controller of Do et al. (2002b) had acceptable transient response. However, the disturbances produce undesirable oscillations in the yaw motion. Also, we observed a bias in the orientation angle, which increases with the magnitude of the disturbance. Also, the position in $y$ shows a non-zero error at steady state, which is produced by the action of the disturbances.

### 4.3 Scenario 3. Parameter uncertainty

Our third simulation considers the effect of uncertainty in the values of the model parameters used in the controller to test robustness. We vary the values of $m_{11}$, $m_{22}$, $m_{33}$, $d_{11}$, $d_{22}$, and $d_{33}$ in the controller by 10% to consider the parameter uncertainty of the vessel model. The parameter values used in the
controller are $m_{11} = 20.9$, $m_{22} = 31.68$, $m_{33} = 4.62$, $d_{11} = 3.6$, $d_{22} = 11$, and $d_{33} = 0.9$ in (2) and (3), which gives $\rho_1 = 0.6597$ and $\rho_2 = 0.3472$ in (19) and (20). We do not modify the values of $m_{ii}$ and $d_{ii}$ used in the vessel model (1).

Figures 9 and 10 show the motion in surge and sway. Figure 11 shows the yaw angle and rate together with the desired trajectories, $z^*_y$ and $r^*_y$, and Figure 12 shows the control force and moment $t_1$ and $t_3$ respectively.

The performance of our proposed controller is slightly reduced. However, the position in $x$ and $y$ and orientation of the vessel converges to the origin showing the robust properties of the controller. Reduced performance is seen for both comparison controllers by Xie et al. (2017) and Do et al. (2002b). We notice that in the latter the position in $y$ does not converge to zero.

5. Conclusion and future work

In this paper, we have developed a controller that stabilises the origin of an underactuated marine craft. We use strict Lyapunov functions to ensure that the origin is a uniform global asymptotic stable equilibrium of the closed-loop for the case with no disturbances and the closed-loop system is ISS with respect to matched environmental disturbances, which are force disturbances in surge and moment disturbances in yaw. The simulations of a case study illustrates the performance and stability properties of the proposed controller under different scenarios. It remains an open problem to develop a stabilising controller of this form for an underactuated marine craft that ensures ISS of the closed loop with respect to both matched and unmatched environmental disturbances.

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Figure 12. Time histories of control inputs in the model error case: (a) control in surge, (b) control in yaw.
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Appendix: Construction of the strict Lyapunov function

In the following, we briefly explain how we develop the strict Lyapunov function $W_1$. Consider the simple candidate Lyapunov function $V_1$ and compute its time derivative as follows:

$$
\dot{V}_1 = -\left(\frac{\rho_1}{\rho_2} - \frac{\rho_2 k_5}{2}\right) z_1^2 - \left(\rho_2 - \frac{k_6}{2}\right) v^2 + \frac{\rho_2^2}{2 K_5} h_1^2(t,v,z_2,z_3) + \frac{1}{2 K_6} h_2^2(t,z_1,z_2,z_3).
$$

Notice that $h_1$ and $h_2$ are functions of $v$ and $z_1$, respectively. The structure of these functions allows to find bounds for $h_1^2$ and $h_2^2$ that depends on $z_2$ and $z_3$ only if a term $V_1 + 1$ appears in the denominator. Then, we consider a new candidate Lyapunov function

$$
W_1(z_1,v) = \ln(V_1(z_1,v) + 1),
$$

and thus $W_1 = \frac{V_1(z_1,v)}{V_1(z_1,v) + 1}$ as desired.

Now we consider $V_2(z_2,z_3)$ defined in (12) and compute its time derivative as

$$
\dot{V}_2 = -\frac{\rho_1}{\rho_2} k_2 k_3 k_4 k_6 + \frac{\rho_1}{\rho_2} k_4 k_6 \sin^2(o(t)) - k_8 k_4 z_2^2
\leq -\frac{\rho_1 k_2}{\rho_2} k_6 e^{2 \sin^2(o(t))} \frac{V_2(z_2,z_3)}{2(1 + V_2(z_2,z_3))}
$$

where we bound $k_{14} = \frac{\rho_1}{\rho_2} k_2$, and $0 < k_6 \leq 1$. We notice that $V_2$ is a weak Lyapunov function but satisfies the form

$$
\dot{V}_2(z_2,z_3) \leq -a p(t) \frac{V_2(z_2,z_3)}{L(V_2(z_2,z_3))},
$$

where $a = \frac{\rho_1 K_2}{\rho_2} k_6 e^2$, $p(t) = \sin^2(o(t))$, and $L(m) = \frac{1}{2} (1 + m)$, which is required to apply the strictification method developed by Malisoff and Mazenc (2009, Section 6.4). Then, the strict Lyapunov function is

$$
W_2(t,z_1,z_3) = \exp(R(t)) k(V_2(z_2,z_3))
$$

where

$$
k(s) = \begin{cases} 
\exp \left( \int_{s-T}^{s} \frac{L(m)}{m} \, dm \right) & \text{if } s > 0, \\
0 & \text{if } s = 0
\end{cases}
$$

and

$$
R(t) = a \int_{t-T}^{t} \int_{s-T}^{s} p(m) \, dm \, ds
= \frac{1}{K_5} \frac{1}{\rho_2} k_6 e^2 \left[ 2 \pi - \sin(2o(t)) \right],
$$

from where we obtain the strict Lyapunov function (10). The full strict Lyapunov function $W_3(t,z_1,v,z_2,z_3)$ is built by combining $W_1(z_1,v)$ and $W_2(t,z_2,z_3)$, and choosing the positive constant $K_3(\epsilon, \omega)$, such that the positive $V_2$ terms from $W_1$ are cancelled by the negative $V_2$ terms from $W_2$. 