Borcherds Algebras and $\mathcal{N} = 4$ Topological Amplitudes

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Abstract

The perturbative spectrum of BPS-states in the $E_8 \times E_8$ heterotic string theory compactified on $T^2$ is analysed. We show that the space of BPS-states forms a representation of a certain Borcherds algebra $\mathcal{G}$ which we construct explicitly using an auxiliary conformal field theory. The denominator formula of an extension $\mathcal{G}_{\text{ext}} \supset \mathcal{G}$ of this algebra is then found to appear in a certain heterotic one-loop $\mathcal{N} = 4$ topological string amplitude. Our construction thus gives an $\mathcal{N} = 4$ realisation of the idea envisioned by Harvey and Moore, namely that the ‘algebra of BPS-states’ controls the threshold corrections in the heterotic string.
1 Introduction

The study of BPS-states has played a prominent role in developing our current understanding of string theory. Quantities which only receive contributions from BPS-states (‘threshold corrections’) are protected under variations of the string coupling and are therefore ideal for probing strong-weak dualities and non-perturbative effects in string theory. Degeneracies of BPS-states (or rather BPS-indices) are moreover closely related to interesting mathematical structures, such as the counting of rational curves and special Lagrangian submanifolds in Calabi-Yau manifolds, or (generalised) Donaldson-Thomas invariants. The BPS-index is locally constant as a function of the moduli, but may jump at codimension one submanifolds, known as walls of marginal stability, on which bound states of BPS-states may decay or recombine [1, 2]. The behaviour of the BPS-index when crossing such walls has been the subject of intense research (see, e.g. [3, 4, 5, 6, 7, 8, 9, 10]), leading to interesting wall-crossing formulae with a broad range of applications in mathematics and physics.

In two insightful papers [11, 12] (see also [13] for a nice overview), Harvey and Moore argued that the space of BPS-states in string theory forms an algebra, and they provided evidence that this ‘algebra of BPS-states’ should be related to a (generalised) Borcherds-Kac-Moody (BKM) algebra [14]. In particular, they found that certain threshold corrections in heterotic $\mathcal{N} = 2$ compactifications can be written as infinite product representations of automorphic forms on the Grassmannian $SO(2, 2+n)/(SO(2) \times SO(2+n))$, which, through the work of Borcherds [21], are in turn related to denominator formulae for BKM-algebras. Although these results are intriguing and suggestive, a direct connection between these infinite product formulae and the algebra of BPS-states has not yet been established.

In the context of type II Calabi-Yau compactifications, the algebra of BPS-states was further analysed in [22, 23]. In [22] the vertex algebra realisation of the BPS-algebra was developed more explicitly, and it was, in particular, shown how a BKM-algebra appears as a certain subalgebra of the full algebra of BPS-states. A different point of view was taken in [23], where the description of D-brane states in terms of quivers was exploited. In this context, the algebra of BPS-states was related to quiver representations, and in certain specific examples this analysis revealed the BPS-algebra appearing as an affine Kac-Moody algebra attached to the D-brane quiver. However, the relation between the algebra of BPS-states and threshold corrections remained obscure. Recently it has also been suggested that the correct mathematical framework for analysing the algebra of BPS-states in Calabi-Yau compactifications is through the so called ‘cohomological Hall algebra’ [24].

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1The theory of (super-) BKM-algebras has made its appearance in many different corners of string theory in the past (see, e.g. [15, 16, 17, 18, 19, 20]).
A seemingly different development started with the work of Dijkgraaf, Verlinde and Verlinde [25], who conjectured that the degeneracies of (non-perturbative) dyonic 1/4 BPS-states in $\mathcal{N} = 4$ heterotic compactifications are encoded in the Fourier coefficients of a certain Siegel modular form, known as the 'Igusa cusp form'. As had earlier been shown by Gritsenko and Nikulin [26, 27], this Igusa cusp form has an infinite product representation which relates it to the denominator formula of a certain rank 3 (super) BKM-algebra, denoted $\mathfrak{g}_{1,\mathrm{II}}$ in [26]. In this way, the degeneracies of dyons also become related to the root multiplicities of the associated BKM-algebra. The physical role of the algebra $\mathfrak{g}_{1,\mathrm{II}}$ was further clarified in [5] (see also [28, 29, 30, 31, 32]), where it was shown that the wall-crossing behaviour of the dyon spectrum is controlled by the hyperbolic Weyl group $W(\mathfrak{g}_{1,\mathrm{II}})$ of this BKM-algebra.

The purpose of the present work is to elucidate the relation between the algebra of BPS-states and the infinite product formulas occurring in certain BPS couplings in heterotic string theory. Our analysis is inspired by the results of [11], but, as will become clear below, the approach we take is slightly different. We focus our attention on the spectrum of the $E_8 \times E_8$ heterotic string compactified on a six-torus $T^6$. For simplicity, we further assume that the torus splits according to $T^6 = T^4 \times T^2$, and take the large volume limit of the $T^4$. The Narain moduli space of the theory is therefore given by the arithmetic coset $SO(2, 18; \mathbb{Z}) \backslash SO(2, 18)/(SO(2) \times SO(18))$, where $SO(2, 18; \mathbb{Z})$ is the U-duality group which leaves the lattice of BPS-charges invariant. The spectrum of perturbative BPS-states corresponds to taking the right-moving sector of the heterotic string to be in its ground state, while allowing for arbitrary excitations in the left-moving sector. We show that the space of BPS-states forms a representation of a certain BKM-algebra $\mathcal{G}$, which we explicitly construct using an auxiliary bosonic conformal field theory. The BKM-algebra has root lattice $\Pi^{1,1} \oplus \Lambda_{\mathrm{ts}} \oplus \Lambda_{\mathrm{ts}}$ and can therefore be understood as a Borcherds extension of the Lorentzian Kac-Moody algebra $(\mathfrak{e}_8 \oplus \mathfrak{e}_8)^{++}$.

The auxiliary CFT construction actually leads to a slightly bigger Borcherds algebra $\mathcal{G}_{\text{ext}}$, that contains $\mathcal{G}$ as a natural subalgebra, $\mathcal{G} \subset \mathcal{G}_{\text{ext}}$. The extended algebra $\mathcal{G}_{\text{ext}}$ is based on the root lattice $\Pi^{1,1} \oplus \Lambda_{\mathrm{ts}} \oplus \Lambda_{\mathrm{ts}} \oplus \Lambda_{\mathrm{ts}}$. While it does not directly act on the space of BPS-states, it turns out to be relevant for understanding the algebraic structure of threshold corrections. To make this precise, we consider a particular class of BPS-saturated $g$-loop amplitudes $\mathcal{F}_g$ in type II string theory compactified on $K3 \times T^2$ which are captured by correlation functions of the $\mathcal{N} = 4$ topological string [33]. For any $g$, the dual amplitudes in heterotic string theory compactified on $T^6$ receive contributions at all loop orders in (heterotic) perturbation theory. However, the leading contribution in the heterotic weak coupling limit is a one-loop expression which is therefore amenable to an analysis similar to that of Harvey and Moore in the $\mathcal{N} = 2$ setting [11, 12]. Mathematically, heterotic one-loop amplitudes fall into the
category of so-called ’singular theta correspondences’, as analysed in detail by Borcherds [21, 34].

As a consequence of the 1/2 BPS nature of the corresponding effective couplings in supergravity, supersymmetric Ward identities predict that \( F_g \) satisfies particular second order differential equations, referred to as harmonicity equations [33, 35] (see also [36, 37, 38]). In string theory these equations get modified by anomalous world-sheet boundary terms which signal non-analytic contributions at the quantum level. This modification can thus be understood as an \( \mathcal{N} = 4 \) analogue of the holomorphic anomaly of the \( \mathcal{N} = 2 \) B-model topological string [39, 40]. A peculiar feature of the amplitudes \( F_g \) that appear in our context is that one can isolate an anomaly free part \( F_{\text{analy}} \) which is analytic and plays an analogous role as the threshold corrections of [11, 12]. We evaluate the integral \( F_{\text{analy}} \) explicitly using the method of orbits [41] (or in mathematical parlance, the Rankin-Selberg method [42]).

In order to make contact with the BKM-algebras \( \mathcal{G} \) and \( \mathcal{G}_{\text{ext}} \) discussed above, we first analyse the complex codimension one submanifolds where the integral \( F_{\text{analy}} \) develops singularities. They include the walls of the fundamental Weyl chamber of the double extension \( (\epsilon_8 \oplus \epsilon_8)^{++} \). This is similar to what was found for the non-perturbative 1/4 BPS dyon spectrum in [5]. Hence, the Weyl group of the Borcherds algebra \( \mathcal{G} \) controls the singularity structure of \( F_{\text{analy}}^1 \). We further show that the analytic integral \( F_{\text{analy}}^1 \) can be written as a specialised denominator identity based on the algebra \( \mathcal{G}_{\text{ext}} \), where the Cartan angles corresponding to the extra \( \epsilon_8 \) are set to zero. Our construction thus gives an explicit \( \mathcal{N} = 4 \) realisation of the idea envisioned by Harvey and Moore, namely that the algebra of BPS-states controls the threshold corrections in the heterotic string.

The paper is organised as follows. In section 2 we discuss aspects of the perturbative BPS-spectrum of the heterotic string on \( T^6 \), and introduce the BKM-algebra \( \mathcal{G} \) and its extension \( \mathcal{G}_{\text{ext}} \), that will play a key role in the remainder of the paper. In section 3 we recall the structure of the \( \mathcal{N} = 4 \) topological amplitudes \( F_g \), with emphasis on the role of harmonicity. We show, in particular, how to single out the anomaly free part \( F_{\text{analy}} \) of the full amplitude \( F_g \). In section 4 we restrict the analysis to the large volume limit of \( T^4 \subset T^6 \), and analyse in detail the singularity structure of the resulting integral as a function on the Narain moduli space \( SO(2,18)/(SO(2) \times SO(18)) \). In particular, we show that the rational quadratic divisors (introduced by Borcherds in [21]) coincide with the walls of the fundamental Weyl chamber of the Lorentzian Kac-Moody algebra \( (\epsilon_8 \oplus \epsilon_8)^{++} \). Section 5 is then devoted to evaluating the integral \( F_{\text{analy}}^1 \) explicitly. We show how to write part of the result in terms

\[\text{We note that similar structures related to hyperbolic Kac-Moody algebras also play a crucial role in a very different physical situation, namely the study of gravity close to a cosmological singularity [13, 14, 15, 16]. See appendix A.1 for a review of extensions of semisimple Lie algebras.}\]
of an infinite product, which we identify with (a restriction of) the denominator formula of
the BKM-algebra $\mathcal{G}_{\text{ext}}$. The paper includes two appendices. In appendix A we introduce
some relevant background on (in)finite-dimensional Lie algebras. In particular, we describe
double extensions of both simple and semisimple finite Lie algebras. We also discuss general
aspects of Borcherds-Kac-Moody algebras. In appendix B we collect some details on the
calculation of the integral presented in section 5.

\section{BPS-States and Borcherds-Kac-Moody Algebras}

Let us begin by studying the $1/2$ BPS-states in the heterotic theory. Inspired by \cite{11, 12}
(see also \cite{22}) we want to show that they form a representation of a certain Borcherds-Kac-
Moody (BKM) algebra, which we shall construct using conformal field theory methods. The
denominator formula of a closely related BKM algebra will later play a role in the topological
amplitudes that will be calculated in sections 3.3 and 5. In the following we shall consider
the $E_8 \times E_8$ theory compactified on $T^6$.

\subsection{BPS States in Narain Compactifications}

The classical moduli space for heterotic string theory on $T^6$ is described by the coset space
\begin{equation}
\mathcal{M} = \left( SL(2, \mathbb{R})/U(1) \right) \times \left( SO(6, 22)/(SO(6) \times SO(22)) \right),
\end{equation}
where the first factor encodes the heterotic ‘axio-dilaton’, while the second factor accounts
for the Narain moduli of the torus. The perturbative spectrum of the heterotic string consists
of the states that are created from a momentum ground state labelled by $(p^L, \vec{p}; p^R, \bar{p})$
by the action of the oscillators. Here the compactified (and internal) momenta take values in
the Narain lattice
\begin{equation}
(p^L, p^R) \in \Gamma_{6,22},
\end{equation}
while $\bar{p}$ describes the momentum in the uncompactified 4-dimensional theory. The Narain
lattice is invariant under the T-duality group $SO(6, 22; \mathbb{Z})$, and thus the quantum moduli
space is the quotient of (2.1) by this arithmetic group.

We shall mainly work in the covariant formulation, where $\vec{p} \in \mathbb{R}^{3,1}$. The physical states
of the heterotic string (in the NS sector, say) have to be annihilated by $L_n, n > 0$, and $G_r,
r > 0$, as well as $\bar{L}_n, n > 0$\footnote{In our conventions, the left-movers are ‘supersymmetric’, while the right-movers are ‘bosonic’. The right-movers are denoted by a bar.}. In addition, they have to satisfy the level matching and mass
shell conditions
\[
\begin{align*}
\frac{1}{2} & = \frac{1}{2}(p^L)^2 + \frac{1}{2}p^R + N_L \\
1 & = \frac{1}{2}(p^R)^2 + \frac{1}{2}p^L + N_R ,
\end{align*}
\]
where \(N_L\) and \(N_R\) are the left- and right-moving excitation numbers. The mass of a physical state is determined via
\[
M^2 = (N_L - \frac{1}{2}) + \frac{1}{2}(p^L)^2 .
\]
Since the compactification of heterotic string theory on \(T^6\) preserves \(N = 4\) supersymmetry, the massive \(BPS\)-states come in two classes: \(1/2\) BPS-states associated with short multiplets, and \(1/4\) BPS-states associated with intermediate multiplets. As has been discussed in [47] all \(1/4\) BPS-states are non-perturbative, and only the \(1/2\) BPS-states are perturbative. For the latter we have in addition to (2.3) that \(N_L = \frac{1}{2}\), and thus the mass is simply
\[
M_{BPS}^2 = \frac{1}{2}(p^L)^2 .
\]
For these states we can subtract the two equations in (2.3) from one another and obtain
\[
1 = -\frac{1}{2}(p^L,p^R)^2 + N_R ,
\]
where \((p^L,p^R)^2 = (p^L)^2 - (p^R)^2\) is the inner product in \(\Gamma^{6,22}\).

### 2.2 Eight Dimensions

In order to simplify the analysis we shall now consider the decompactification limit to eight dimensions. This is to say, we split \(T^6 = T^2 \times T^4\), and take the large-volume limit of the \(T^4\), effectively setting the \(T^4\) momenta to zero. This corresponds to restricting ourselves to momentum ground states in the even self-dual lattice \(\Gamma^{2,18}\) of signature \((2,18)\), where
\[
\Gamma^{6,22} = \Gamma^{2,18} \oplus \Gamma^{4,4} ,
\]
and \(\Gamma^{4,4}\) describes the momenta of the \(T^4\).

The elements in \(\Gamma^{2,18}\) characterise the momentum ground states of the heterotic \(E_8 \times E_8\) string, compactified on \(T^2\). The moduli space of such compactifications is described by the Kähler \((T)\) and complex \((U)\) structure moduli of \(T^2\), as well as by two real Wilson lines \(\vec{v}_a \in \mathbb{R}^{16}, \alpha = 1,2\). At any point in this moduli space, a general element of the momentum lattice \(\Gamma^{2,18}\) can be parametrised as \(x = (m_1, n_1; m_2, n_2; \vec{\ell})\), where \((m_1,m_2)\) and \((n_1,n_2)\) are the momentum and winding numbers along \(T^2\), while \(\vec{\ell} \in \Lambda_{E_8} \oplus \Lambda_{E_8}\). We will also use the notation \(\vec{\ell} = (\vec{\ell}_1, \vec{\ell}_2)\) with \(\vec{\ell}_{1,2} \in \Lambda_{E_8}\) respectively. The inner product on \(\Gamma^{2,18}\) is defined by
\[
\langle x|x' \rangle = -m_1n_1' - n_1m_1' - m_2n_2' - n_2m_2' + \vec{\ell} \cdot \vec{\ell}'
\]
\[= -m_1n_1' - n_1m_1' - m_2n_2' - n_2m_2' + \vec{\ell}_1 \cdot \vec{\ell}_1' + \vec{\ell}_2 \cdot \vec{\ell}_2' ,
\]
where the first four terms represent the Lorentzian inner product on $\Gamma^{2,2} \simeq \Pi^{1,1} \oplus \Pi^{1,1}$, and the last term is the standard Euclidean inner product inherited from $\mathbb{R}^{16} \supset \Lambda_{es} \oplus \Lambda_{es}$. For given $x = (m_1, n_1; m_2, n_2; \ell') \in \Gamma^{2,18}$, the actual internal momentum is then a vector in $\mathbb{R}^{16}$

$$P(x) = n_1 \bar{v}_1 + n_2 \bar{v}_2 + \ell'. \quad (2.9)$$

For the following it is useful to combine $\bar{v}_1$ and $\bar{v}_2$ into a complex Wilson line, $\bar{V} = \bar{v}_1 + i\bar{v}_2$. Sometimes we will also use the notation $\bar{V} = (\bar{V}_1, \bar{V}_2)$ with $\bar{V}_{1,2} \in \mathbb{C}^8$, respectively. We parametrise an arbitrary point in the moduli space by $y = (U, T; \bar{V}) \in C^{1,17}$, with inner product

$$(y|y') = -TU' - T'U + \bar{V} \cdot \bar{V}' = -TU' - T'U + \bar{V}_1 \cdot \bar{V}'_1 + \bar{V}_2 \cdot \bar{V}'_2, \quad (2.10)$$

which particularly implies $(y|y) = -2T U + \bar{V}^2$ for the norm of $y$. For the following it is also useful to define the map (see [11])

$$u : C^{1,17} \rightarrow C^{2,18}, \quad y = (U, T; \bar{V}) \mapsto u(y) = \left( U, T; \frac{(y|y)}{2}, 1; \bar{V} \right), \quad (2.11)$$

which associates to every element $y \in C^{1,17}$ a light-like vector $u(y) \in C^{2,18}$. Here the inner product on $C^{2,18}$ is defined by $\langle \cdot | \cdot \rangle$ as in (2.8). With this notation an arbitrary momentum state $x \in \Gamma^{2,18}$ parametrised by $x = (m_1, n_1; m_2, n_2; \ell')$ has

$$|p^L|^2 = -2 \frac{|\langle x|u(y) \rangle|^2}{(3y|3y)} = \frac{1}{(T_2 U_2 - \frac{1}{2} \bar{V}^2)} \left| m_2 + m_1 T + n_1 U + \frac{n_2}{2} (y|y) - \ell' \cdot \bar{V} \right|^2$$

$$\left( |p^R|^2 - |p^L|^2 \right) = \langle x|x \rangle = \ell^2 - 2m_1 n_1 - 2m_2 n_2 \equiv 2D, \quad (2.12)$$

where $3y = (U_2, T_2; 3\bar{V})$ is the imaginary part of $y$.

### 2.3 The BKM Algebra

Next we want to construct a Borcherds-Kac-Moody (BKM) algebra $\mathcal{G}$ that has a natural action on the space of BPS-states and that will be relevant for the topological amplitudes we shall calculate below. This algebra will arise as a subalgebra of a closely related algebra $\mathcal{G}_{ext}$ that we shall construct first.

Let us denote by $\Gamma^{1,17} \subset \Gamma^{2,18}$ the sublattice of the even self-dual lattice $\Gamma^{2,18}$ which is obtained by setting the momenta and windings along the second circle to zero, $m_2 = n_2 = 0$. The lattice $\Gamma^{1,17}$ is an even self-dual lattice of signature $(1, 17)$. We can adjoin the even self-dual root lattice $\Lambda_{es}$ of dimension 8 to it, and define the even self-dual lattice $\Gamma^{1,25}$ via

$$\Gamma^{1,25} = \Gamma^{1,17} \oplus \Lambda_{es}. \quad (2.13)$$
On $\Gamma^{1,25}$ we then consider an (auxiliary) chiral conformal field theory, whose ‘physical’ states $\psi$ are characterised by the property that they are annihilated by the Virasoro generators $L_n$ with $n > 0$, together with the mass-shell condition that the $L_0$ eigenvalue is one,

$$L_n \psi = 0 \quad n > 0 , \quad L_0 \psi = \psi \ .$$

(2.14)

As is familiar from string theory, these physical states form a Lie algebra, where one defines the bracket as

$$[\psi, \phi] = V_0(\psi) \phi \ .$$

(2.15)

Since $\psi$ is Virasoro primary and of conformal dimension $h = 1$, $[L_n, V_0(\psi)] = 0$, and thus the right hand side is again a physical state. Since the underlying momentum space has only one time-like direction, the resulting Lie algebra is in fact a Borcherds-Kac-Moody (BKM) algebra [18], and we denote it by $G_{ext}$.

We can label the elements of $\Gamma^{1,25}$ as $(m, n; \vec{\ell})$, where $(m, n)$ describes the component in $\Pi^{1,1}$ and $\vec{\ell} \in \Lambda_{ts} \oplus \Lambda_{ts} \oplus \Lambda_{ts}$. The mass-shell condition in the auxiliary conformal field theory is then

$$1 = N_{exc} + \frac{1}{2} \vec{\ell}^2 - mn \ ,$$

(2.16)

where $N_{exc}$ denotes the ‘excitation’ number in the 26 directions of the auxiliary chiral conformal field theory. Given $(m, n, \vec{\ell})$, this equation can be solved for $N_{exc}$, and thus we can directly determine the multiplicity $c_{ext}(m, n; \vec{\ell})$ of the roots corresponding to $(m, n; \vec{\ell})$. Their generating function is of the form

$$\sum_{N \geq -1} \sum_{\vec{\ell} \in \Lambda_{ts} \oplus \Lambda_{ts} \oplus \Lambda_{ts}} c_{ext}(N, \vec{\ell}) \bar{q}^N e^{2\pi i \vec{\ell} \cdot \vec{z}} = \frac{\Theta_{ts \oplus ts \oplus ts}(\bar{\tau}, \bar{z})}{\eta(\bar{q})^{24}} \ ,$$

(2.17)

where $N \equiv mn$ and $\Theta_g(\tau, z)$ is the usual theta series of the root lattice $\Lambda_g$. Here $q = e^{2\pi i \tau}$, and $\vec{z} = (\vec{z}_1, \vec{z}_2, \vec{z}_3)$ is a 24-dimensional vector in the weight space of $\xi_8 \oplus \xi_8 \oplus \xi_8$. Note that (2.18) is a weak Jacobi form, and consequently the Fourier coefficients $c_{ext}(N, \vec{\ell})$ only depend on $(N, \vec{\ell})$ through the combination $N - \vec{\ell} \cdot \vec{\ell}/2$ [19].

In the above construction the additional $\xi_8$ lattice in (2.13) was introduced by hand, and was not really crucial for the definition; in fact, the construction would have worked equally for any even lattice of dimension eight. It is therefore natural to restrict $G_{ext}$ to the subalgebra that is generated by those physical states for which the momenta actually lie in the sublattice $\Gamma^{1,17}$; the resulting subalgebra will be denoted by $\mathcal{G} \subset G_{ext}$. Note that $\mathcal{G}$ is indeed a consistent subalgebra since momentum is additive under the product in (2.15), and thus the bracket (2.15) closes on $\mathcal{G}$. The root multiplicities of $\mathcal{G}$ are described by

$$\sum_{N \geq -1} \sum_{\vec{\ell} \in \Lambda_{ts} \oplus \Lambda_{ts}} c(N, \vec{\ell}) \bar{q}^N e^{2\pi i \vec{\ell} \cdot \vec{z}} = \frac{\Theta_{ts \oplus ts}(\bar{\tau}, \bar{z})}{\eta(\bar{q})^{24}} \ ,$$

(2.18)
where \( \vec{z} = (\vec{z}_1, \vec{z}_2) \) is a 16-dimensional vector in the weight space of \( e_8 \oplus e_8 \).

By construction the Lie algebra \( \mathcal{G} \) is a Borcherds extension of \( g^{++} \), where \( g = e_8 \oplus e_8 \). Here \( g^{++} \) is the so-called double extension of the finite dimensional Lie algebra \( g \), which is a Lorentzian Kac-Moody algebra with root lattice

\[
\Gamma^{1,17} = \Pi^{1,1} \oplus \Lambda_{e_8} \oplus \Lambda_{e_8}, \quad (2.19)
\]

for more details about double extensions see appendix A.1. The definition of \( \mathcal{G} \) is similar to the construction of \( \mathcal{H}_0^{\text{mult}} \) in [12], see also [22].

To each BKM-algebra one can associate its denominator formula. This is obtained by restricting the standard Weyl-Kac-Borcherds character formula to the trivial representation, yielding an equivalence between an infinite sum over the Weyl group and an infinite product over the positive roots (see Appendix A.2 for some details). For our purposes it is the infinite product side of the denominator formula that will play a crucial role. Let \( \Delta_{\mathcal{G}_{\text{ext}}} \) be the root lattice of \( \mathcal{G}_{\text{ext}} \) and denote an arbitrary root by \( \alpha \). In this case the infinite product part of the general denominator formula (A.19) may be written as

\[
\Phi_{\mathcal{G}_{\text{ext}}}(\hat{y}) = \prod_{\alpha \in \Delta_{\mathcal{G}_{\text{ext}}}^+} \left( 1 - e^{2\pi i (\alpha | \hat{y})} c_{\text{ext}}(-\alpha^2/2) \right), \quad (2.20)
\]

where the multiplicity of the root \( \alpha \) is given by the Fourier coefficients in (2.17) via

\[
\text{mult} \alpha = c_{\text{ext}} \left( -\frac{1}{2} \alpha^2 \right) = c_{\text{ext}} \left( mn - \frac{1}{2} \vec{\ell} \cdot \vec{\ell} \right). \quad (2.21)
\]

The moduli vector \( \hat{y} \) is valued in \( \mathbb{C}^{1,25} \), and can be chosen as \( \hat{y} = (y, \vec{z}_3) \), where \( y = (U, T; \vec{V}) \) represents the standard Narain moduli and \( \vec{z}_3 \in \mathbb{C}^8 \) is a vector in the weight space of the auxiliary \( e_8 \) used in the construction of \( \mathcal{G}_{\text{ext}} \).

### 2.4 Action on the BPS-States

In the following we want to show that the BKM algebra \( \mathcal{G} \) plays a natural role in the description of the theory. While it is not in one-to-one correspondence with the space of BPS-states — and hence does not deserve the name ‘BPS algebra’ — it has a natural action on the space of BPS-states. To see this, we simply observe that to each BPS-state with \((m_2, n_2) = (0, 0)\) and with no oscillator excitation in the corresponding circle direction, we can associate an element \( \phi \) of the auxiliary conformal field theory associated to \( \Gamma^{1,25} \). Indeed, we ignore the left-moving oscillators (with \( N_L = \frac{1}{2} \)), and identify \((p^L, p^R)\) with an element \((m_1, n_1, \vec{l}) \in \Gamma^{1,17} \subset \Gamma^{1,25} \). Furthermore, we identify the 16 internal right-moving oscillators,
as well as the right-moving oscillator associated to the \((m_1, n_1)\) direction to the oscillators of the auxiliary conformal field theory corresponding to the 17 right-moving momenta in \(\Gamma^{1,17}\). The remaining eight right-moving oscillators of the BPS-state are finally identified with suitable oscillators corresponding to the \(\Lambda_8\) lattice of the auxiliary conformal field theory. Because of \((2.12)\), eq. \((2.6)\) can then be interpreted as the ‘physical mass-shell condition’ \((2.16)\) in the auxiliary conformal field theory.

The Lie bracket \((2.15)\) now defines an action of \(\psi \in G\) on \(\phi\), and the image under this action is again associated with a BPS-state with the above properties. Thus we can define an action of \(G\) on the space of such BPS-states by letting \(\psi\) act trivially on the left-moving oscillators (which we ignored in mapping the BPS state to an element of the auxiliary conformal field theory).

It remains to show that this action can also be extended to the BPS-states for which \((m_2, n_2) \neq (0, 0)\), and that also have oscillators in the corresponding circle direction. As regards the oscillators, we define the action of \(\psi\) to be trivial on them. (This is consistent since \(\psi\) does not carry any momentum along this direction.) If \((m_2, n_2) \neq (0, 0)\), \((p^L, p^R)\) is no longer an element of \(\Gamma^{1,25}\), but the action of \(\psi\) can still be defined on it since both the momenta \((p^L, p^R)\) and the momentum associated to \(\psi\) are elements of the even self-dual lattice \(\Gamma^{2,18}\). Thus the corresponding fields are local relative to one another, and the contour integral that is implicit in \((2.15)\) is well-defined. Thus we can extend the action of \(\psi \in G\) to all BPS-states, thereby proving our claim.

We have therefore shown that the BKM algebra \(G\) acts naturally on the full space of perturbative BPS-states. The algebra itself, however, is only associated to a subspace of BPS-states, namely to those with momentum in \(\Gamma^{1,17}\) — in particular, we required that \((m_2, n_2) = (0, 0)\) in order to have only one time-like direction, leading to a BKM algebra. Furthermore, in defining the Lie algebra, we ignored the choice of polarisation for the left-movers. (This is similar to the construction in \([22]\).) As a consequence our construction does not define a ‘BPS-algebra’, i.e. it does not make the full space of BPS-states into a Lie algebra, as was originally envisaged in \([11]\), but only makes the space of BPS-states into a representation of a BKM.

### 3 \(\mathcal{N} = 4\) BPS Couplings and Differential Equations

In this section we introduce and review some relevant aspects of a particular class of topological \(\mathcal{N} = 4\) couplings \(\mathcal{F}_g\) in heterotic string theory compactified on \(T^6\) (see \([33, 35]\)). In the naive field-theory limit these couplings only receive contributions from perturbative 1/2
BPS-states. However, in string theory additional non-analytic terms appear as well. A key observation in our work is that one may use particular differential equations satisfied by $F_g$ – usually called ‘harmonicity equations’ – to isolate an analytic part $F^\text{analy}_g$. This represents the $\mathcal{N} = 4$ analogue of the ‘threshold corrections’ in $\mathcal{N} = 2$ theories. The corresponding coupling $F^\text{analy}_g$ will play a central role in the remainder of the paper.

### 3.1 Review of $\mathcal{N} = 4$ Topological Amplitudes

In [33, 35] (see also [38]) a particular class of $\mathcal{N} = 4$ topological string amplitudes has been discovered. These amplitudes appear at the $g$-loop level in type II string theory compactified on $K3 \times \mathbb{T}^2$, while their dual counterparts in heterotic string theory compactified on $\mathbb{T}^6$ start receiving contributions at the one-loop level. The latter expressions can be written as

$$F_g(y) = \int_{\mathbb{F}} \frac{d^2 \tau}{\eta^{24}} \frac{\tau^{2g-1}}{\tau_2} G_{g+1}(\tau, \bar{\tau}) \Theta_g^{(6,22)}(\tau, \bar{\tau}, y),$$

where the integral is over the fundamental domain $\mathbb{F}$ of $SL(2, \mathbb{Z})$. Moreover, the expression

$$\Theta_g^{(6,22)}(\tau, \bar{\tau}, y) = \begin{cases} 
\sum_{p \in \Gamma^{6,22}} (p_L^+) (2g-2) q^{\frac{1}{2}} |p|^2 \bar{q}^{\frac{1}{2}} |p_R|^2 & g > 1 \\
\sum_{p \in \Gamma^{6,22}, p \neq 0} q^{\frac{1}{2}} |p|^2 \bar{q}^{\frac{1}{2}} |p_R|^2 & g = 1 
\end{cases}$$

is a Siegel-Narain theta-function of the even unimodular lattice $\Gamma^{6,22}$ with momentum insertions $p_L^+$. Here the inner product on $\Gamma^{6,22}$ is defined as in equation (2.8), and

$$p_L^+(y) = \frac{1}{2} \epsilon^{ab} \bar{u}_I^a \bar{u}_J^b p_{IJ}^L(y), \quad \text{with} \quad u_I^{\pm a} \in SU(4) / S(U(2) \times U(2))$$

$$y \in SO(6,22) / SO(6) \times SO(22)$$

for a particular harmonic projection of the six left-moving lattice momenta [2,2] with the harmonic coordinates $u_I^{\pm a}$. We will use the notation that $I = 1, \ldots, 4$ is an index of $SU(4) \cong SO(6)$, while $a = 1, 2$ and $\dot{a} = 1, 2$ are indices of either of the two $SU(2)$'s, and the signs denote the charge with respect to the diagonal $U(1)$. Finally, the quantities $y_I^{L \dot{J}}$ (with $A = 1, \ldots, 22$ an index of $SO(22)$) span the moduli-space

$$\mathcal{M}_{(6,22)} = SO(6,22) / (SO(6) \times SO(22))$$

of the $\mathcal{N} = 4$ string compactification. Notice that for $g = 1$ we do not sum over $p = 0$ in the definition (3.2). The reason for this can be understood as follows. Since the amplitude $F_1$ has no explicit $p^L$-insertions it would receive contributions from $p = 0$. However, integrating this contribution over $\tau$ will diverge for any value of $y_I^{L \dot{J}}$, thereby rendering $F_1$ infinite. In order to avoid this singularity, we have chosen to regularise $F_1$ by performing the summation.
in (3.2) only over \( p \neq 0 \). This implies that the amplitude does not receive contributions from massless 1/2 BPS-states.

The object \( G_g(\tau, \bar{\tau}) \) in (3.1) is a weight \( 2(g+1) \) non-antiholomorphic modular form which can be obtained as the coefficient of \( \lambda^{2g} \) of the generating functional \([50]\) (see also \([51]\))

\[
G(\lambda, \tau, \bar{\tau}) = \left( \frac{2\pi i \lambda \bar{\eta}^3}{\theta(\lambda, \bar{\tau})} \right)^2 e^{-\frac{\pi \lambda^2}{2\tau}} .
\]

(3.5)
The modular form \( G_g \) can be written in terms of Eisenstein series as \([52]\)

\[
G_g(\tau, \bar{\tau}) = -S_g \left( \hat{\mathcal{E}}_2, \frac{1}{2} \mathcal{E}_4, \ldots, \frac{1}{2g} \mathcal{E}_{2g} \right) ,
\]

(3.6)
where \( S \) are the Schur-polynomials \( S_k(x_1, \ldots, x_k) = x_k + \cdots + x_k^k/(k!) \) and

\[
\mathcal{E}_{2k}(\tau) = 2\zeta(2k)E_{2k}(\tau)
\]

(3.7)
are the rescaled Eisenstein series of weight \( 2k \). Recall that \( \tilde{E}_2 \) is a ‘quasi-modular form’ \([56, 57]\), implying that it does not only transform with a weight under modular transformations but receives an additional anomalous shift-term. Following standard practice we have therefore introduced the quantity

\[
\hat{\mathcal{E}}_2(\tau, \bar{\tau}) = \frac{\pi^2}{3} \left( E_2(\bar{\tau}) - \frac{3}{\pi \tau_2} \right) ,
\]

(3.8)
which is an honest weight 2 modular form, but is non-antiholomorphic in \( \tau \).

In \([35]\) the amplitudes (3.1) were shown to compute BPS couplings in the string effective action, which in harmonic superspace take the form (see \([35]\) for further details)

\[
S = \int d^4 x \int d^4 u \int d^4 \theta^+ \int d^4 \bar{\theta}^- (K^{++}_\mu \bar{K}^{++,\mu})^{g+1} \mathcal{F}_g(Y_A^{++}, u) .
\]

(3.9)
Here \( K^{IJJ}_\mu \) is a particular super-descendant of the (linearised) \( \mathcal{N} = 4 \) supergravity multiplet, while \( Y_A^{IJJ} \) is a linearised \( \mathcal{N} = 4 \) vector-multiplet, whose lowest components \( y_A^{IJJ} \) form the moduli space \( \mathcal{M}_{(6,22)} \) of the \( \mathcal{N} = 4 \) string compactification.

### 3.2 Differential Equations for \( g > 1 \)

As was shown in \([35]\), for \( g > 1 \) the amplitudes (3.1) satisfy certain differential equations with respect to the moduli of the heterotic \( \mathcal{N} = 4 \) compactification. In particular

\[
e_{ab} \epsilon^{IJKL} \frac{\partial}{\partial \bar{u}^I_{+b}} D_{KL,A} \mathcal{F}_g = (2g-2)\bar{u}^I_{+a} D_{++A} \mathcal{F}_{g-1} \]

(3.10)
\[
(\epsilon^{IJKL} D_{IJ,A} D_{KL,B} + 4(g+1)\delta_{AB}) \mathcal{F}_g = 4D_{++A} D_{++B} \mathcal{F}_{g-1} ,
\]

(3.11)

\textsuperscript{4}See also \([53, 47, 54, 55]\) for further examples of heterotic one-loop amplitudes involving non-holomorphic integrands.
where $D_{++}, A$ are harmonic projections of the covariant derivatives $D_{ij,A}$ in the moduli space $M_{(6,22)}$. We will refer to these equations as the harmonicity and second order relation, respectively.

Note that in both equations the amplitude $F_{g-1}$ appears on the right hand side. As for the holomorphic anomaly equation (see e.g. [40]), we shall call these contributions anomalous. From the string effective action point of view they can be understood as arising via the violation of certain analyticity properties of the corresponding BPS-couplings. To understand this, we recall that the coupling (3.9) is half-BPS in the sense that the integrand is annihilated by half of the spinor-derivatives of the $\mathcal{N} = 4$ harmonic superspace (‘$G$-analyticity constraint’). For this to be true, however, it is essential that $F_{g-1}$ is a function of only a particular projection of the vector multiplets, namely $Y_{A}^{++} = \epsilon_{ab}u_{I}^{+}u_{J}^{+}Y_{A}^{IJ}$. As was explained in [35], this particular dependence leads to the differential equations (3.10) and (3.11), however, with the right hand side replaced by zero. The appearance of $F_{g-1}$ in the explicit string computation can therefore be understood as an anomalous violation of these analyticity constraints.

To understand how the anomalous terms in (3.10) and (3.11) arise from the string amplitude (3.1) we recall from [35]

\[
\epsilon_{ab}\epsilon_{IJKL} \frac{\partial}{\partial \hat{u}_{+b}} D_{KL,A} F_{g} = 4i(2g - 2)\bar{u}_{+a} \int \frac{d^2 \tau}{\eta^2} G_{g+1}(\tau, \bar{\tau}) \frac{\partial}{\partial \tau} \left[ \frac{\tau^{2g}}{2} \sum_{p \in \Gamma^{6,22}} (p_{+}^{L})^{2g-3} p_{A}^{R} q^{p_{L}^{R}} \bar{q}^{p_{L}^{L}} \right], \quad (3.12)
\]

and the second order equation

\[
(\epsilon^{IJKL} D_{IJ,AD} D_{KL,B} + 4(g + 1)\delta_{AB}) F_{g} = -32\pi i \int \frac{d^2 \tau}{\eta^2} G_{g+1}(\tau, \bar{\tau}) \frac{\partial}{\partial \tau} \left[ \frac{\tau^{2g+1}}{2} \sum_{p \in \Gamma^{6,22}} \left( p_{A}^{R} p_{B}^{R} - \frac{\delta_{AB}}{4\tau_2} \right) (p_{+}^{L})^{2g-2} q^{p_{L}^{R}} \bar{q}^{p_{L}^{L}} \right]. \quad (3.13)
\]

Notice that in both cases, since $g > 1$, after performing a partial integration, the boundary term vanishes and the anomaly stems from the contribution which is proportional to

\[
\frac{\partial}{\partial \tau} G_{g+1}(\tau, \bar{\tau}) = -i\frac{\pi}{2\tau_2} G_{g}(\tau, \bar{\tau}). \quad (3.14)
\]

Recalling the expression (3.6) for $G_{g}$ in terms of Schur polynomials, we deduce that the only source of non-antiholomorphicity is the explicit dependence on $\tau_2$ in $\hat{E}_{2}$. Therefore, we can
split

\[ G_g(\tau, \bar{\tau}) = G_g^{\text{analy}}(\bar{\tau}) + G_g^{\text{non-analy}}(\tau, \bar{\tau}) \]  

(3.15)

with the explicit expressions

\[ G_g^{\text{analy}}(\bar{\tau}) = -S_g\left(0, \frac{1}{2} \bar{E}_4, \ldots, \frac{1}{2g} \bar{E}_{2g}\right) \]  

(3.16)

\[ G_g^{\text{non-analy}}(\tau, \bar{\tau}) = -S_g\left(\hat{E}_2, \frac{1}{2} \bar{E}_4, \ldots, \frac{1}{2g} \bar{E}_{2g}\right) + S_g\left(0, \frac{1}{2} \bar{E}_4, \ldots, \frac{1}{2g} \bar{E}_{2g}\right) \]  

(3.17)

which both have weight \(2g\) under modular transformations. The anomaly of (3.10) and (3.11) (and therefore also the violation of G-analyticity) can now be fully (and uniquely) attributed to \(G_g^{\text{non-analy}}\). It is therefore consistent to define the purely analytic contribution to the amplitude as

\[ F_g^{\text{analy}}(y) = \int d^2 \tau \bar{\eta} \frac{\tau_2}{24} \left( \frac{1}{2} G_g^{\text{analy}}(\bar{\tau}) \Theta_g(6,22)(\tau, \bar{\tau}, y) \right) \]  

(3.18)

which yields a vanishing anomaly when inserted into the harmonicity and second order relation.

### 3.3 Differential Equations for \(g = 1\)

In this paper we will mostly be interested in the amplitude (3.1) for \(g = 1\); the case \(g = 1\) is somewhat more subtle and requires special care. First of all, the right hand side of the harmonicity relation (3.10) vanishes, which reflects the fact that \(F_1\) is independent of the harmonic variables \(\bar{u}_{\pm a}\), as can also be seen from the effective action coupling (3.9). Focusing on the remaining second order relation (3.11), we notice that for \(g = 1\) a partial integration of (3.13) will also produce a non-trivial boundary contribution at \(\tau_2 \to \infty\) for those points of the lattice for which \(p_L = 0\). Explicitly we find

\[ (\epsilon^{IJKL} D_I J A D_{K L B} + 8 \delta_{AB}) F_1 \]

\[ = -32\pi i \lim_{\tau_2 \to \infty} \int d^2 \tau \bar{\eta} \tau_2^3 G_2(\tau, \bar{\tau}) \sum_{p \in \Gamma^g, 22 \neq 0} \left( p^R A p^R B - \frac{\delta_{AB}}{4\pi \tau_2} \right) q_2^{1/2} |p^L|^2 q_2^{1/2} |p^R|^2 \]

\[ + 16\pi^2 \int d^2 \tau \bar{\eta} \tau_2^3 \left( \frac{\partial}{\partial \tau} G_2^{\text{non-analy}}(\tau, \bar{\tau}) \right) \sum_{p \in \Gamma^g, 22 \neq 0} \left( p^R A p^R B - \frac{\delta_{AB}}{4\pi \tau_2} \right) q_2^{1/2} |p^L|^2 q_2^{1/2} |p^R|^2 . \]  

(3.19)

The last line arises by the same mechanism as just discussed for the case \(g > 1\), and it will vanish if we restrict \(F_1\) to its analytic part \(F_1^{\text{analy}}\). The first line, however, is an additional
contribution for which we can write
\[
(\epsilon^{IJKL} D_{I,J,A} D_{K,L,B} + 8 \delta_{AB}) \mathcal{F}_1^{\text{analy}} = 8i \delta_{AB} \lim_{\tau_2 \to \infty} \int_\mathcal{D} \frac{d\tau_1}{\eta^{24}} \tau_2^2 G_2^{\text{analy}}(\tau) \sum_{p \in \mathcal{I}_6, p \neq 0} q^D e^{-\pi \tau_2 |p|^2},
\]
where we have extended the definitions (3.15) and (3.18) to the case \( g = 1 \), and \( D \) was defined in (2.12). At a generic point in moduli space (i.e. for generic \( y_{IJ}^{(I)} \)) the limit will simply vanish. Therefore we can (as in the case of \( g > 1 \)) attribute the anomaly completely to the term
\[
G_2^{\text{non-analy}}(\tau, \bar{\tau}) = -S_2 \left( \tilde{E}_2, \frac{1}{2} \bar{E}_4 \right) + S_2 \left( 0, \frac{1}{2} \bar{E}_4 \right).
\]

4  Singularities of the Analytic Integral

In this section we shall study explicitly the analytic part of the genus one topological amplitude \( \mathcal{F}_{g=1} \) discussed in the previous section. We shall analyse in detail the structure of the singularities of the integral as a function of the moduli \( y = (U, T; \vec{V}) \in \mathbb{C}^{1,17} \). This will reveal the first piece of evidence for a relation to the BKM algebra \( G \) introduced in section 2. In particular, we shall show that at least certain singularities are associated with Weyl reflections of the Lorentzian Kac-Moody algebra \( (\mathfrak{e}_8 \oplus \mathfrak{e}_8)^{++} \) whose Borcherds lift is \( G \).

4.1 General Analysis of the Singularities in Eight Dimensions

In order to make contact with the discussion in section 2 we will consider the internal manifold to be factorised as \( \mathbb{T}^6 = \mathbb{T}^4 \times \mathbb{T}^2 \), and take the large volume limit of \( \mathbb{T}^4 \). On the level of the integral these assumptions mean that the Siegel-Narain theta function of the original \( \Gamma^{6,22} \) Narain-lattice will be decomposed as
\[
\frac{G_2(\tau, \bar{\tau}) \tau_2^2}{\eta^{24}} \Theta_{g=1}^{(6,22)} \sim \text{Vol} \frac{G_2(\tau, \bar{\tau})}{\eta^{24}} \Theta_{g=1}^{(2,18)}(\tau, \bar{\tau}, y),
\]
where \( \text{Vol} \) is the volume of \( \mathbb{T}^4 \) and \( \Theta_{g=1}^{(2,18)}(\tau, \bar{\tau}, y) \) is the corresponding Siegel-Narain theta function for the lattice \( \Gamma^{2,18} \). Furthermore, the contribution from the analytic part of \( G_2(\tau, \bar{\tau}) \)

\footnote{As we will see in the following section 4 at very particular points in the moduli space, (3.20) may diverge for \( p_L = 0 \) and \( D = 1 \). At these particular points the differential equation will only be consistent after some additional proper regularisation. For determining the analytic part of the integral, however, we will assume to work at a generic point in the moduli space.}
can be rewritten as
\[ \frac{C_{\text{analy}}(\tau, \bar{\tau})}{\eta^{24}(\bar{\tau})} = -\zeta(4) \frac{E_4(\bar{\tau})}{\eta(\bar{\tau})^{24}} \equiv -\zeta(4)P(\tau). \] (4.2)

We shall drop the irrelevant overall factor of \(-\zeta(4)\) from now on and write the integral \(F_{\text{analy}}^1\) as
\[ F_{\text{analy}}^1 = \int_\mathcal{F} d^2 \tau \frac{P(\bar{\tau})}{\tau_2} \sum_{p \in \Gamma_{2,18}, p \neq 0} q^D e^{-\pi \tau_2 |p^L|^2}, \] (4.3)

where \(D = \frac{1}{2} (|p^R|^2 - |p^L|^2)\) was introduced in (2.12), and \(p = (m_1, n_1; m_2, n_2; \vec{\ell})\) with \(\vec{\ell} = (\vec{\ell}_1, \vec{\ell}_2) \in \Lambda_{ts} \oplus \Lambda_{ts}\). To understand the mechanism by which a singularity might occur in (4.3) (see also [21, 11, 34, 58, 59, 60, 5] for related discussions), we observe that in the standard fundamental domain \(\mathcal{F}\) of \(SL(2, \mathbb{Z})\), the upper boundary of the \(\tau_2\) integration is at infinity. At a generic point in moduli space, the integrand is exponentially suppressed as \(\tau_2 \to \infty\) due to the exponentials of the Narain momenta. However, at particular points in moduli space this damping might fail, thus leaving an unregulated integral which ultimately leads to a logarithmic divergence. A necessary condition for such a divergence to appear is
\[ |p^L| = 0, \] (4.4)

since then the suppression induced by the factor \(e^{-\pi \tau_2 |p^L|^2}\) in (4.3) is absent. However, condition (4.4) alone is not sufficient since the integrand also involves a power series in \(\bar{q}^m\), and for \(|p^L| = 0\), the \(\tau_1\)-integral picks out the constant term, \(\bar{q}^0\). Fourier expanding the integrand
\[ \mathcal{P}(\bar{\tau}) \sum_{p \in \Gamma_{2,18}, p \neq 0, p^L = 0} q^{|p^R|^2} = \sum_{p \in \Gamma_{2,18}, p \neq 0, p^L = 0} \sum_{n = -1}^{\infty} d(n) \bar{q}^{n+D} \] (4.5)

we therefore only encounter a divergence if \(d(-D) \neq 0\) for some vector in the sum over \(\Gamma_{2,18}\). In the following we shall focus on those terms for which \(D = 1\), i.e. the terms that arise from the \(\bar{q}^{-1}\) term of \(1/\eta(\bar{\tau})^{24}\) in (1.2); these singularities will be directly related to the Weyl reflections in \(\mathcal{G}\). Given the construction of section 2, the other singularities (with \(D < 1\)) should then have an interpretation in terms of \(\mathcal{G}_{\text{ext}}\) — this takes into account the overall \(E_4(\bar{\tau}) = \Theta_{\text{es}}(\bar{\tau})\) factor in (1.2).

In order to study this problem, let us fix a \(p \in \Gamma_{2,18}\) with \(\langle p|p \rangle = 2\) (so that \(D = 1\)), and ask for which values of the moduli \(y = (U, T; \vec{V})\) its contribution to the sum leads to a divergence, i.e. for which \(y\) we have \(p^L = 0\). Because of (2.12), we have the equivalence
\[ p^L = 0 \iff \langle p|u(y) \rangle = 0. \] (4.6)
The actual moduli space is parametrised by $y$, where we identify $y \sim y'$ if $u(y) = Au(y')$ for $A \in SO(2, 18; \mathbb{Z})$; this is the familiar T-duality action. Next we observe that the inner product (4.6) is invariant under this T-duality action, i.e.

$$\langle Ap|Au(y) \rangle = \langle p|u(y) \rangle \quad \text{for all } A \in SO(2, 18; \mathbb{Z}).$$

(4.7)

Thus if (4.6) is satisfied for $p = p_1$ at $y = y_1$, and we consider $p_2 = Ap_1$ with $A \in SO(2, 18; \mathbb{Z})$, then (4.6) vanishes for $p = p_2$ at $y = y_2 \sim y_1$ since $u(y_2) = Au(y_1)$. It is therefore sufficient to consider one representative of $p$ for each $SO(2, 18; \mathbb{Z})$-orbit.

It was shown in [61] (see also [62]) that for every $p \in \Gamma_{2, 18}$ with $\langle p|p \rangle = 2$, there exists an $SO(2, 18; \mathbb{Z})$ transformation that maps it to $\hat{p} = Ap \in \Gamma_{1, 17}$, i.e.

$$\hat{p} = (m_1, n_1; m_2 = 0, n_2 = 0; \vec{\ell}) .$$

(4.8)

It is therefore sufficient to restrict ourselves to such vectors. For those the analysis of $\hat{p}^L = 0$ is now straightforward since (2.12) implies that $\hat{p}^L = 0$ is equivalent to

$$m_1 T + n_1 U - \vec{\ell} \cdot \vec{V} = 0 ,$$

(4.9)

while the constraint $\langle p|p \rangle = 2$ leads to

$$\vec{\ell}^2 - 2m_1n_1 = \vec{\ell}_1^2 + \vec{\ell}_2^2 - 2m_1n_1 = 2 .$$

(4.10)

This condition now has a nice Lie algebraic interpretation: since $\hat{p} \in \Gamma_{1, 17} = \Pi^{1,1} \oplus \Lambda_{\epsilon_8} \oplus \Lambda_{\epsilon_8}$, we can think of $\hat{p}$ as an element of the root lattice of the double extension $g^{++}$ of $g = \mathfrak{e}_8 \oplus \mathfrak{e}_8$. The constraint $\langle \hat{p}|\hat{p} \rangle = (\alpha|\alpha) = 2$ implies that $\alpha = \hat{p}$ is a real root of $g^{++}$, and the condition (4.6) is then equivalent to the statement that $u(y)$ is a fixed point of the Weyl reflection $w_\alpha \in W(g^{++})$ with respect to the root $\alpha$. Here $w_\alpha$ acts on the moduli vector $y = (U, T; \vec{V}) \in \mathbb{C}^{1,17}$ as

$$w_\alpha : y \mapsto y - (y|\alpha) \alpha ,$$

(4.11)

where $(\cdot|\cdot)$ is the inner product on the lattice $\Lambda_{g^{++}}$,

$$(\alpha|\alpha) = -2m_1n_1 + \vec{\ell} \cdot \vec{\ell} = -2m_1n_1 + \vec{\ell}_1^2 + \vec{\ell}_2^2 ,$$

(4.12)

which is the one inherited from $\langle \cdot|\cdot \rangle$ on $\Gamma^{2,18}$. Note that for vectors of the type $\hat{p}$ (for which $m_2 = n_2 = 0$) it does not matter whether we take the inner product with $y$ or $u(y)$. Thus we conclude that singularities of the BPS integral occur precisely at the fixed points of the Weyl group $W(g^{++})$. 

17
4.2 Explicit Singular Loci

Let us describe the relevant Weyl group $\mathcal{W}(\mathfrak{g}^{++})$ and its singularities more explicitly. The Lie algebra $\mathfrak{g}^{++}$ is the double extension of $\mathfrak{g} = \mathfrak{e}_8 \oplus \mathfrak{e}_8$. As explained in appendix \[A.1\] the construction of $\mathfrak{g}^{++}$ naturally involves an ‘auxiliary’ algebra $\tilde{\mathfrak{g}}^{++}$, from which the physically relevant double extension $\mathfrak{g}^{++}$ is obtained by taking the quotient $\tilde{\mathfrak{g}}^{++}/\mathfrak{r}$ by the center $\mathfrak{r}$. Let us begin by introducing a basis of simple roots $\tilde{\alpha}_I$ for $\tilde{\mathfrak{g}}^{++}$

\[
\tilde{\alpha}_{-1} = (1, -1; 0; \vec{0}) \quad \tilde{\alpha}_0 = (-1, 0; -\vec{\theta}; \vec{0}) \quad \tilde{\alpha}_0^{(2)} = (0, 0; -\vec{\theta}; \vec{0}) \quad \tilde{\alpha}_0^{(2)} = (0, 0; \vec{0}; \vec{e}_i)
\]

(4.13)

where $\vec{e}_i$ is a basis of simple roots $(i = 1, \ldots, 8)$ for $\mathfrak{e}_8$, and $\vec{\theta}$ the corresponding highest root. The roots (4.13) define an overcomplete basis for the root lattice $\Lambda_{\mathfrak{g}^{++}} = \Pi^{1,1} \oplus \Lambda_{\mathfrak{e}_8} \oplus \Lambda_{\mathfrak{e}_8}$. In fact, there is one relation (generating the center $\mathfrak{r}$ of $\tilde{\mathfrak{g}}^{++}$, see appendix \[A.1\] which we may use to express $\tilde{\alpha}_0^{(2)}$ in terms of the other roots

\[
\tilde{\alpha}_0^{(2)} = \tilde{\alpha}_0^{(1)} + \sum_{i=1}^8 (\vec{\theta} \cdot \tilde{f}^i) (\tilde{\alpha}_i^{(1)} - \tilde{\alpha}_i^{(2)}) \quad .
\]

(4.14)

Here $\tilde{f}^i$ are the fundamental weights of $\mathfrak{e}_8$. Hence, we can then write for any $\alpha \in \Lambda_{\mathfrak{g}^{++}}$

\[
\alpha = x_{-1} \tilde{\alpha}_{-1} + x_0 \tilde{\alpha}_0^{(1)} + \sum_{i=1}^8 \left( x_i^{(1)} \tilde{\alpha}_i^{(1)} + x_i^{(2)} \tilde{\alpha}_i^{(2)} \right) ,
\]

(4.15)

with integer coefficients $(x_{-1}, x_0; x_i^{(1)}, x_i^{(2)})$. Using the same inner product as in (2.10) we find that the product between $\alpha$ and a moduli vector $y = (U, T; \tilde{V}_{(1)}, \tilde{V}_{(2)})$ reads

\[
(\alpha | y) = x_{-1} T + (x_0 - x_{-1}) U - x_0 (\vec{\theta} \cdot \tilde{V}_{(1)}) + \sum_{i=1}^8 \left[ x_i^{(1)} (\vec{e}_i \cdot \tilde{V}_{(1)}) + x_i^{(2)} (\vec{e}_i \cdot \tilde{V}_{(2)}) \right] .
\]

(4.16)

Thus there are 18 linearly independent singular divisors

\[
\tilde{\mathcal{D}}_{-1} = \{ y \in \mathcal{M}_{2,18} \mid (y | \tilde{\alpha}_{-1}) = U - T = 0 \} \\
\tilde{\mathcal{D}}_0 = \{ y \in \mathcal{M}_{2,18} \mid (y | \tilde{\alpha}_0^{(1)}) = T - \vec{\theta} \cdot \tilde{V}_{(1)} = 0 \} \\
\tilde{\mathcal{D}}_i^{(a)} = \{ y \in \mathcal{M}_{2,18} \mid (y | \tilde{\alpha}_i^{(a)}) = \vec{e}_i \cdot \tilde{V}_{(a)} = 0 \} \quad a = 1, 2 \\
i = 1, \ldots, 8 ,
\]

(4.17)

which we will sometimes collectively denote by $\tilde{\mathcal{D}}_I$ with $I = -1, \ldots, 16$. The divisor $\tilde{\mathcal{D}}_{-1}$ is independent of the Wilson line, and thus the corresponding singularity of the integral cannot be removed by a shift of $\tilde{V}$. In fact, this is exactly the locus of enhanced gauge symmetry which was for example discussed in [53, 111, 33].
For a given point $y$ in the moduli space $\mathbb{C}^{1,17}$ the divisors $\tilde{D}_I$ represent the ‘dominant’ walls of the complexified Weyl chamber, in the sense that all other walls lie ‘behind’ this set of walls. If we restrict the moduli to the fundamental Weyl chamber

$$C_C = \{ y \in \mathbb{C}^{1,17} \mid \tilde{D}_I \geq 0 \}, \quad \text{(4.18)}$$

the only singularities appear at the boundary of $C_C$. Note that since we are working with the complexified Weyl chamber, eq. (4.18) should be understood as providing separate conditions on the real and imaginary parts of the moduli vector $y = (U,T;\vec{V})$.

5 BPS Amplitude and Denominator Identity

Our next aim is to evaluate the 1-loop integral (4.3) explicitly. As we shall see, the analytic part $F_{g=1}^{\text{analy}}$ can be related to the infinite product side of the denominator formula for the Borcherds algebra $G_{\text{ext}}$ which contains the BPS symmetry algebra $G$ as a subalgebra.

5.1 Torus Integral

We now use the methods developed in [41] and further extended in [11] (see also [58, 64, 65, 59]) to tackle the $\tau$ world-sheet torus integral. The moduli dependence is described by the Siegel-Narain theta function of the lattice $\Gamma^{2,18}$

$$\Theta^{(2,18)}(\tau, \bar{\tau}; y) = \sum_{x \neq (0,0,0,0)} q^{\frac{1}{2} |x|_2} e^{2\pi \tau \frac{|x|_2^2}{|x|_2}} e^{2\pi i \bar{\tau} \frac{|x|_2}{|x|_2}} e^{2\pi i (x \cdot y)} e^{2\pi i (\bar{x} \cdot \bar{y})} , \quad \text{(5.1)}$$

for which we shall use the same notation as in section 2 and parametrise the summation by $x = (m_1,n_1; m_2,n_2; \vec{\ell})$ with $\vec{\ell} \in \Lambda_{8} \oplus \Lambda_{8}$. We can perform a Poisson resummation on the indices $m_{1,2}$

$$\Theta^{(2,18)} = \sum_{(p_1,n_1;p_2,n_2) \in \Lambda_{8} \oplus \Lambda_{8}} \int_{-\infty}^{\infty} du_{1,2} q^{\frac{1}{2} \bar{\vec{\ell}} \cdot \vec{\ell}} e^{2\pi i (p_1 u_1 + p_2 u_2) - \frac{\pi Y^2}{\tau_2} u_1 u_1 U - \frac{\pi n_2}{\tau_2} (\vec{V} \cdot \vec{V})} e^{2\pi i (u_1 U + n_1 T - \frac{\pi}{Y^2} (y \cdot y) + \bar{\vec{V}} \cdot \vec{V})} .$$

Both $u$-integrals are of Gaussian type and can therefore be performed using elementary methods. Thus we get (see also [41, 11, 58, 64, 65, 59])

$$F_{g=1}^{\text{analy}} = \frac{d^2 \tau}{\tau_2^2} \frac{Y}{U_2} P(\bar{\tau}) \sum_{(p_1,n_1;p_2,n_2) \in \Lambda_{8} \oplus \Lambda_{8}} q^{\frac{1}{2} \bar{\vec{\ell}} \cdot \vec{\ell}} e^{2\pi i \frac{n_2}{Y^2} (\bar{\vec{V}}^2 - \vec{V}^2) - \frac{\pi Y^2}{\tau_2} (n_1 + n_2 U) \Lambda} e^{2\pi i (\bar{\vec{V}} \cdot \vec{V})^2 (n_1 + n_2 U) \Lambda} , \quad \text{(5.2)}$$
where \( P(\bar{\tau}) \) was defined in (4.2),

\[
A = \begin{pmatrix} n_1 & -p_1 \\ n_2 & p_2 \end{pmatrix}, \quad A = (1, U) A \begin{pmatrix} \bar{\tau} \\ 1 \end{pmatrix}, \quad \bar{A} = (1, \bar{U}) A \begin{pmatrix} \bar{\tau} \\ 1 \end{pmatrix},
\]

(5.3)

and

\[
Y = (3y|\bar{3}y) \quad \vec{z} = \frac{i}{2U_2} (\bar{V} \bar{A} - \bar{V} \mathcal{A}) .
\]

(5.4)

For computing the \( \tau \)-integration it is convenient to introduce the Fourier expansion

\[
P(\bar{\tau}) \sum_{\vec{\ell} \in \Lambda_e \oplus \Lambda_e} e^{2\pi i \vec{\ell} \cdot \vec{z}} = \sum_{n=-1}^{\infty} \sum_{\vec{\ell} \in \Lambda_e \oplus \Lambda_e} \bar{c}_{\text{ext}}(n - \frac{1}{2} \vec{\ell}^2) \bar{q}^n e^{2\pi i \vec{\ell} \cdot \vec{z}},
\]

(5.5)

where we have used again that the left hand side is a weak Jacobi form and thus the Fourier coefficients only depend on \((n, \vec{\ell})\) through the combination \(n - \frac{1}{2} \vec{\ell}^2\). Note that the coefficients agree precisely with those appearing in the Fourier expansion (2.17) after restricting one of the theta series to its associated ‘theta constant’, i.e.

\[
\sum_{n=-1}^{\infty} \sum_{\vec{\ell} \in \Lambda_e \oplus \Lambda_e} \bar{c}_{\text{ext}}(n - \frac{1}{2} \vec{\ell}^2) \bar{q}^n e^{2\pi i \vec{\ell} \cdot \vec{z}} = \Theta_{\text{ext}}(\bar{q}, \vec{z}) \Theta_{\text{ext}}(\bar{q}, \vec{0}) / \eta(\bar{q})^{24},
\]

(5.6)

where \( \Theta_{\text{ext}}(\bar{q}, \vec{0}) = E_4(\bar{\tau}) \). The coefficients \( \bar{c}_{\text{ext}} \) can therefore be expressed in terms of the \( c_{\text{ext}} \) as

\[
\bar{c}_{\text{ext}}(n - \frac{1}{2} \vec{\ell}^2) = \sum_{\vec{\alpha} \in \Lambda_e} c_{\text{ext}}(n, \vec{\ell}, \vec{\alpha}) ,
\]

(5.7)

and thus can be interpreted as ‘averages’ over the root lattice of \( e_8 \). The first few terms are explicitly

\[
\bar{c}_{\text{ext}}(-1) = 1, \quad \bar{c}_{\text{ext}}(0) = 264, \quad \bar{c}_{\text{ext}}(1) = 8244, \quad \bar{c}_{\text{ext}}(2) = 139520,
\]

(5.8)

and \( \bar{c}_{\text{ext}}(n) = 0 \) for \( n < -1 \).

With these preparations out of the way we can now compute the \( \tau \)-integral in (5.2) following [41, 11, 66, 58, 54, 64]. Using modular invariance of the integrand, we can trade a modular transformation \( \tau \mapsto a\tau + b \) for a transformation of the matrix

\[
A \mapsto A \begin{pmatrix} a & b \\ c & d \end{pmatrix}.
\]

(5.9)

This allows us to extend the domain of integration to images of \( \mathbb{F} \) under \( SL(2, \mathbb{Z}) \), while simultaneously restricting the summation over \( A \) to inequivalent \( SL(2, \mathbb{Z}) \)-orbits. As was first discussed in [41], there are three inequivalent contributions

\[
\mathcal{F}_{g=1}^{\text{analy}} = \mathcal{I}_0^{\text{analy}} + \mathcal{I}_{ND}^{\text{analy}} + \mathcal{I}_D^{\text{analy}},
\]

(5.10)
corresponding to three different classes of representatives of the matrix $A$, called the zero, non-degenerate and degenerate orbits, respectively. The computation of each of these separately is rather tedious but follows quite closely \[41, 11, 58, 64\]. We have relegated these calculations to appendix B, and the final result is

$$F^{\text{analy}}_{g=1} = \sum_{\vec{\ell} \in \Lambda_{\epsilon_8}^+ \oplus \Lambda_{\epsilon_8}^+} \left[ \frac{2\pi Y}{3U_2} \left( \tilde{c}_{\text{ext}}(0, \vec{\ell}) - 24\tilde{c}_{\text{ext}}(-1, \vec{\ell}) \right) + 2\log \left| 1 - e^{2\pi i \vec{\ell} \cdot \vec{V}} \right| \right]^{c_{\text{ext}}(0, \vec{\ell})} + 2\log \left| 1 - e^{2\pi i (n' \vec{r} + \vec{\ell} \cdot \vec{V})} \right|^{c_{\text{ext}}(0, \vec{\ell})}$$

$$+ \tilde{c}_{\text{ext}}(0, \vec{0}) \left( \frac{\pi U_2}{3} - \ln Y + K \right) + 2\log \prod_{n=1}^{\infty} \left| 1 - e^{2\pi inU} \right| \tilde{c}_{\text{ext}}(0, \vec{0})$$

$$+ \frac{2U_2}{3\pi} + \frac{2\pi}{U_2} (\vec{\ell} \cdot \Im \vec{V}) \left( (\vec{\ell} \cdot \Im \vec{V}) + U_2 \right),$$  \hspace{1cm} (5.11)

where $K = \gamma_E - 1 - \ln \frac{8\pi}{3\sqrt{3}}$, with $\gamma_E$ being the Euler-Mascheroni constant. Furthermore, we have introduced the shorthand notation for the modified scalar-product

$$\vec{\ell} \cdot \Im \vec{V} = \ell \cdot \Re \vec{V} + i \left| \vec{\ell} \cdot \Im \vec{V} \right|.$$  \hspace{1cm} (5.12)

Here we have decided to work in a chamber of the moduli space where

$$\Im \vec{V} \in \left( \Lambda_{\epsilon_8}^+ \oplus \Lambda_{\epsilon_8}^+ \right) \otimes \mathbb{C},$$  \hspace{1cm} (5.13)

such that the only contribution to the degenerate orbit with $\vec{\ell} \neq \vec{0}$ comes from vectors $\vec{\ell} = (\vec{\ell}_1, \vec{\ell}_2)$ where either $\vec{\ell}_1$ or $\vec{\ell}_2$ is a simple root of $\epsilon_8$ such that $\vec{\ell} \cdot \vec{\ell} = 2$.

### 5.2 Denominator Formula

We shall now analyse the result (5.11) in a little more detail. In the following we shall entirely focus on the logarithmic terms. Most of the non-logarithmic terms contribute to the Weyl vector $\rho$, appearing in the exponential prefactor of the denominator formula (A.19) and ensure that the whole denominator formula has good modular properties under $SL(2, \mathbb{Z})$ \[21, 33\]. Since these terms will not be of relevance for our present analysis we will suppress them in the following.

The relevant part of (5.11) can then be written as

$$F^{\text{analy}}_1(y) \sim \log \| \Phi(y) \|^2 + \cdots ,$$  \hspace{1cm} (5.14)
where we have defined

$$
\Phi(y) = \prod_{(r,n';\vec{\ell}) > 0} \left( 1 - e^{2\pi i (rT + n'U + \vec{\ell}V)} \right)^{\bar{c}_{\text{ext}}(n'r-\vec{\ell}\vec{\ell}/2)} .
$$

(5.15)

Furthermore, the range of the product \((r,n';\vec{\ell}) > 0\) is

$$
n'\bar{r} - \frac{1}{2} \vec{\ell} \cdot \vec{\ell} \geq -1 \quad \text{and} \quad \left\{ \begin{array}{ll}
  r > 0, & n' \in \mathbb{Z}, \vec{\ell} \in \Lambda_{e_8} \oplus \Lambda_{e_8} \\
  r = 0, & n' > 0, \vec{\ell} \in \Lambda_{e_8} \oplus \Lambda_{e_8} \\
  r = n' = 0, & \vec{\ell} \in (\Lambda_{e_8} \oplus \Lambda_{e_8})^+ ,
\end{array} \right.
$$

(5.16)

where \((\Lambda_{e_8} \oplus \Lambda_{e_8})^+\) denotes the positive part of the root lattice \(\Lambda_{e_8} \oplus \Lambda_{e_8}\). The norm \(\| \cdot \|^2\) in (5.15) takes into account that there are contributions with \((r,n';\vec{\ell}) > 0\) and contributions with \((r,n';\vec{\ell}) < 0\). These conditions can be shown to characterise the positive roots in \(\Lambda_{e_8}^+\) with norm 2 [67].

The key observation is now that we can identify (5.15) with a restriction of the denominator formula \(\Phi_{G_{\text{ext}}}^{\hat{y}}\) for \(G_{\text{ext}}\) in (2.20), where we set the weight vector \(\vec{z}_3 = 0\) to zero and drop the terms that vanish in this limit, i.e.

$$
\Phi(y) = \lim_{\vec{z}_3 \to \vec{0}} \Phi_{G_{\text{ext}}}^{\hat{y}}|_{\text{reg}} = \prod_{\alpha \in \Delta_{G_{\text{ext}}}^+} (1 - e^{2\pi i (\alpha|y)})^{\bar{c}_{\text{ext}}(-\alpha^2/2)} .
$$

(5.17)

Here the prime at the product means that we ignore the roots that lie entirely within the additional \(e_8\) root lattice in (2.13) — these give vanishing contributions since then \((\alpha|y) = 0\).

Thus \(\mathcal{F}_1^{\text{analy}}\) is directly related to the denominator formula for the Borcherds algebra \(G_{\text{ext}}\) that was constructed explicitly in section 2 using an auxiliary conformal field theory. The restriction to \(\vec{z}_3 = 0\) mirrors the fact that the additional root lattice in (2.13) was added by hand and does not play a role for the symmetry algebra \(G\) of the BPS spectrum. The underlying physical reason for this restriction is that we have been studying the problem in eight dimensions, i.e. we have set some of the moduli (describing compactification along the additional directions) to special values. Thus we do not ‘see’ the full root lattice of the underlying symmetry algebra. It would be interesting to understand in more detail the algebraic structure that arises for amplitudes on \(\mathbb{T}^n\) with \(n > 2\), see also section 6.

---

\(^{6}\)Alternatively, one can use the philosophy of Borcherds-Gritsenko-Nikulin [21, 34, 26, 27] to interpret (5.15) as the denominator formula of an ‘automorphic correction’ of \(g^{++}\), where \(g = e_8 \oplus e_8\). This defines another Borcherds-Kac-Moody algebra \(G(g^{++})\), whose root multiplicities are directly defined by the coefficients \(\bar{c}_{\text{ext}}\) in (5.7). This alternative point of view will be pursued in [67].

22
6 Discussion and Conclusions

In this paper we have constructed a Borcherds-Kac-Moody (BKM) algebra \( \mathcal{G} \) that acts on the perturbative BPS-states of the heterotic string theory on \( T^2 \). The Lie algebra \( \mathcal{G} \) plays in many ways the role of a ‘BPS-algebra’, and its lattice of real roots coincides with that of the Lorentzian algebra \((e_8 \oplus e_8)^{++}\). We have shown that \( \mathcal{G} \), as well as the closely related BKM algebra \( \mathcal{G}_{\text{ext}} \supset \mathcal{G} \), are relevant for the description of \( \mathcal{N} = 4 \) threshold corrections. More specifically, we have analysed a certain class of one-loop \( \mathcal{N} = 4 \) topological amplitudes \( F_g \) in heterotic string theory compactified on \( T^6 \). Upon splitting \( T^6 = T^4 \times T^2 \) and taking the large volume limit of \( T^4 \), we have shown that the analytic part of the simplest amplitude \( F_{g=1}^{\text{analy}} \) has an infinite product form which can be identified with a certain restriction of the denominator formula for the BKM-algebra \( \mathcal{G}_{\text{ext}} \). Furthermore, we have demonstrated that the singularities of this amplitude are (partially) controlled by the Weyl group of \( \mathcal{G} \).

It would be interesting to extend the analysis beyond the case of \( T^2 \), and evaluate the integral (3.1) for the full Narain lattice of the six-torus \( T^6 \). For compactifications on \( T^6 \) the associated algebra would be constructed from an indefinite Kac-Moody algebra of signature \((5,21)\), for which the auxiliary CFT has multiple temporal directions which complicates the description of its physical states. Moreover, the Narain moduli space \( SO(6,22)/(SO(6) \times SO(22)) \) is no longer a hermitian symmetric domain and it is therefore unclear whether the theta correspondence affords an infinite product representation which can be related to a denominator formula [34] (see also [68, 59] for related discussions).

Furthermore, we have neglected a detailed analysis of the non-analytic part of \( F_1 \), i.e. the part of the amplitude which contributes to the ‘harmonic anomaly’. One of the main complications here is that the result of the integral involves sums over polylogarithms of order \( r > 1 \) (see e.g. [11, 52]), which in particular cannot be written as infinite products. One might speculate that these terms can be recast as an expansion in terms of characters of irreducible highest weight representations of the BKM-algebra \( \mathcal{G}_{\text{ext}} \). To this end it might be useful to interpret the full \( F_1 \) as a ‘generalised prepotential’, along the lines of [68].

Our results should have a dual type IIA interpretation, which ought to shed light on the geometric meaning of the Fourier coefficients of the Jacobi forms in (5.5). It is useful to look at the \( \mathcal{N} = 2 \) situation for guidance. In this case, heterotic string theory on \( K3 \times T^2 \) is dual to type II strings on a K3-fibered Calabi-Yau threefold [69], and the Fourier coefficients of the modular forms entering into the theta correspondence on the heterotic side become identified with the Gopakumar-Vafa invariants [70, 71] on the type IIA side, thus explaining

\[ \text{Somewhat similar speculations have been offered in the context of non-perturbative dyonic BPS-states in [5].} \]
their integrality [11, 72, 52, 73]. In the $\mathcal{N} = 4$ setting, on the other hand, heterotic string theory on $T^6$ is dual to type II string theory on $K3 \times T^2$, and it is therefore natural to speculate that the Fourier coefficients (5.5), extended to the full amplitude (3.1) on $T^6$, are related to some topological invariants of K3-surfaces. To this end, it would be necessary to generalise the Gopakumar-Vafa analysis to the $\mathcal{N} = 4$ situation, as discussed in [75].

Finally, let us offer some further speculations as to the geometric role of the $\mathcal{N} = 4$ amplitudes $\mathcal{F}_1$. It is well-known that the one-loop amplitude of the $\mathcal{N} = 2$ B-model topological string on a Calabi-Yau threefold can be written as a weighted product of Ray-Singer torsions [40], thereby capturing information about the spectrum of Laplacians on the complex structure moduli space of Calabi-Yau threefolds. It would be interesting if a similar interpretation exists for the $\mathcal{N} = 4$ amplitudes considered here, possibly related to determinants of Laplacians on the moduli spaces of K3-surfaces as in [76, 77]. We hope to return to these and related issues in future work.

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A Infinite Dimensional Lie Algebras

In this appendix we shall describe some of the relevant background material about infinite-dimensional Lie algebras. We begin by discussing double extensions of (semi-)simple finite Lie algebras, which are in particular Lorentzian subalgebras of Borcherds-Kac-Moody algebras. We also review some general aspects of Borcherds-Kac-Moody algebras.

A.1 Double Extensions of Finite Dimensional Lie Algebras

In this section we shall briefly sketch the affine and hyperbolic extensions of finite-dimensional Lie algebras. We begin by discussing the case when the underlying finite dimensional Lie algebra $\mathfrak{g}$ is simple and generalise later to semisimple Lie algebras.
Extensions of Simple Lie Algebras

Let $\Lambda_\mathfrak{g}$ be the root lattice of the finite-dimensional simple Lie algebra $\mathfrak{g}$. A basis of $\Lambda_\mathfrak{g}$ is described by the positive simple roots $\alpha_i$, $i = 1, \ldots, r$, where $r$ is the rank of the Lie algebra $\mathfrak{g}$. Every finite-dimensional simple Lie algebra $\mathfrak{g}$ has a highest root $\theta \in \Delta_+$ such that $\theta + \alpha_i \notin \Delta$ is not a root.

We want to extend the root lattice of $\mathfrak{g}$ by a sublattice of $\Pi^{1,1}$, the even unimodular lattice of dimension 2. We denote the standard basis of $\Pi^{1,1}$ by $\{\beta_1, \beta_2\}$, where we have the inner products

\[
(\beta_1|\beta_2) = 1, \quad (\beta_1|\beta_1) = (\beta_2|\beta_2) = 0, \quad (\beta_1|\alpha_i) = (\beta_2|\alpha_i) = 0, \quad \forall i = 1, \ldots, r.
\] (A.1)

Here the last identities mean that $\Pi^{1,1}$ is orthogonal to $\Lambda_\mathfrak{g}$.

We now append the simple positive roots $\alpha_i$, $i = 1, \ldots, r$ by the simple roots (see [78] and [79])

\[
\alpha_0 = \beta_1 - \theta \quad \text{(affine root)}
\] (A.2)

\[
\alpha_{-1} = -\beta_1 - \beta_2 \quad \text{(hyperbolic root)}.
\] (A.3)

The inner product matrices of this new set of simple roots define the Cartan matrices of the affine extension $\mathfrak{g}^+$ and hyperbolic extension $\mathfrak{g}^{++}$ of $\mathfrak{g}$ respectively,

\[
C^\mathfrak{g}^+_ab = 2 \frac{(\alpha_a|\alpha_b)}{(\alpha_a|\alpha_a)} \quad a, b = 0, \ldots, r
\] (A.4)

\[
C^\mathfrak{g}^{++}IJ = 2 \frac{(\alpha_I|\alpha_J)}{(\alpha_I|\alpha_I)} \quad I, J = -1, \ldots, r.
\] (A.5)

The corresponding root lattices are given by

\[
\Lambda_\mathfrak{g}^+ = \sum_{a=0}^r \mathbb{Z} \alpha_a \subset \Lambda_\mathfrak{g} \oplus \Pi^{1,1} \quad \text{and} \quad \Lambda_\mathfrak{g}^{++} = \sum_{I=-1}^r \mathbb{Z} \alpha_I = \Lambda_\mathfrak{g} \oplus \Pi^{1,1}.
\] (A.6)

Extensions of Semisimple Lie Algebras

When the finite dimensional Lie algebra $\mathfrak{g}$ is semisimple, i.e. a direct sum of simple Lie algebras, the extension procedure requires a little bit more care. Let $\mathfrak{g}$ be a rank $r$ finite semisimple Lie algebra corresponding to a direct sum of $n$ simple subalgebras

\[
\mathfrak{g} = \bigoplus_{i=1}^n \mathfrak{g}_{(i)}.
\] (A.7)
It follows that each $g_i$ is an ideal in $g$. Double extensions of semisimple Lie algebras have been discussed previously in \[80\], and we shall recall the salient features from there. As before, we begin by constructing the affine extension $g^+$. This is now done in two steps. First we extend each individual summand of \[(A.7)\] into an affine Kac-Moody algebra
\[g^+_i = g_i[[t, t^{-1}]] \oplus \mathbb{C}c_i \oplus \mathbb{C}d_i,\] (A.8)
where $g_i[[t, t^{-1}]]$ is the loop algebra of $g_i$ with spectral parameter $t$, $c_i$ is the central generator, and $d_i$ is the so called 'derivation'. We recall from \[78\] that the derivation is needed in order to obtain a non-degenerate inner product $(\cdot | \cdot)$ on the Cartan subalgebra of $g^+_i$. We denote by $\tilde{g}^+$ the direct sum of all $g^+_i$
\[\tilde{g}^+ = \bigoplus_{i=1}^{n} g^+_i,\] (A.9)
which is a again an affine Kac-Moody algebra. By extending each summand of \[(A.7)\] in this way, the resulting root lattice, which is a sublattice of $\Lambda_g \oplus \bigoplus_{i=1}^{n} \Pi^{1,1}_i$, is clearly too big; in fact, as in the simple case treated in section \[A.1\] we are interested in constructing an affine extension $g^+$ whose root lattice $\Lambda_g$ is a sublattice of $\Lambda_g \oplus \Pi^{1,1}$. To achieve this, we can now take the quotient of $\tilde{g}^+$ by the $2(n-1)$-dimensional ideal generated by the elements $(c_1 - c_2), (c_2 - c_3), \ldots, (c_{n-1} - c_n)$ as well as the elements $(d_1 - d_2), (d_2 - d_3), \ldots, (d_{n-1} - d_n)$. We thus define the affine extension of $g$ as
\[g^+ = \tilde{g}^+ / \left( \bigoplus_{a=1}^{n-1} \mathbb{C}(c(a) - c(a+1)) \oplus \bigoplus_{a=1}^{n-1} \mathbb{C}(d(a) - d(a+1)) \right)\] (A.10)
where in the second line we have explicitly identified $c = c_1 = \cdots = c_n$ as well as $d = d_1 = \cdots = d_n$. Note that in contrast to $\tilde{g}^+$, the algebra $g^+$ is not a Kac-Moody algebra; it is however the physically relevant affine algebra in our context.\[9\]

The double extension $g^{++}$ is now obtained as before by promoting the derivation $d$ of \[(A.10)\] to a proper Cartan generator. The structure of the resulting algebra is most easily explained by first adding to $\tilde{g}^+$ a new node that attaches with a a single link to all the affine nodes of the individual summands of $\tilde{g}^+$; the resulting algebra will be denoted by $\tilde{g}^{++}$. The algebra $g^{++}$ is then obtained by dividing by a suitable ideal in $\tilde{g}^{++}$. To describe this ideal, we observe that the Cartan matrix associated with the Dynkin diagram of $\tilde{g}^{++}$ is indefinite.

---

8Special cases of this construction were also considered earlier in \[81\].
9Although the physical context is different, the reasons for singling out the algebra $(g(1) \oplus \cdots \oplus g(n))^+$ are similar to the analysis in \[80\].
of rank $r + 2$, and that it has one negative eigenvalue, as well as $n - 1$ zero eigenvalues (with the remaining eigenvalues all being positive). Let us denote the null eigenvectors as $u_a$, $a = 1, \ldots, n - 1$, with components $u_a(l)$, $l = 1, \ldots, r + n + 1$. We furthermore call the $r + n + 1$ Cartan generators of $\tilde{g}^{++}$ in the Chevalley basis $h_I$. It follows \cite{78} that the center $\mathfrak{r}$ of $\tilde{g}^{++}$ is $(n - 1)$-dimensional, and is generated by the elements

$$c_a = \sum_{l=1}^{r+n+1} u_a(l) h_l .$$

(A.11)

The double extension $g^{++}$ may then be defined as the quotient of $\tilde{g}^{++}$ by the center

$$g^{++} = \tilde{g}^{++} / \mathfrak{r} .$$

(A.12)

Again, we stress that the Lorentzian algebra so obtained is not a Kac-Moody algebra \cite{80}, but it is nevertheless the algebra that will be relevant in our context.

### A.2 Borcherds-Kac-Moody Algebras

Next we want to give a very brief introduction to Borcherds-Kac-Moody (BKM) algebras that were first introduced in \cite{14} (see also \cite{82, 83}). These algebras are also sometimes referred to as Generalised Kac-Moody algebras or GKM.

The BKM algebra $\mathcal{G}$ is characterised by a Cartan matrix $C$, which is now allowed to have infinite rank and is generically of indefinite signature. Let $\{h_I, e_I, f_I\}, I = 1, \ldots, \text{rank } \mathcal{G}$, be the set of Chevalley generators subject to the relations (no summation on repeated indices)

$$[h_I, e_J] = C_{IJ} e_J \quad [h_I, f_J] = -C_{IJ} f_J \quad [e_I, f_J] = h_{IJ}$$
$$\text{ad}_{e_I}^{1-C_{IJ}}(e_J) = 0 \quad \text{ad}_{f_I}^{1-C_{IJ}}(f_J) = 0 \quad \forall C_{IJ} = 2, \ I \neq J$$
$$[e_I, e_J] = 0 \quad [f_I, f_J] = 0 \quad \forall C_{IJ} \leq 0, \ C_{IJ} < 0, \ C_{IJ} = 0 .$$

(A.13)

As in the case of finite-dimensional Lie algebras, all generators of $\mathcal{G}$ can be obtained by applying repeated commutators \cite{78, 14}. Furthermore, the diagonal elements $h_I$ generate the Cartan subalgebra $\mathcal{H}$, while the $e_I$ and $f_I$ generate nilpotent subalgebras $\mathcal{N}^+$ and $\mathcal{N}^-$, respectively. Thus, also BKM-algebras exhibit a standard triangular decomposition

$$\mathcal{G} = \mathcal{N}^- \oplus \mathcal{H} \oplus \mathcal{N}^+ .$$

(A.14)

As for standard Kac-Moody algebras there is an invariant, non-degenerate symmetric bilinear form on $\mathcal{H}^*$, that we shall denote by $(\cdot | \cdot)$. However, the main difference relative to standard Kac-Moody algebras is that the diagonal entries of this inner product are not required to
be positive. Thus the simple roots of $\mathcal{G}$ come in two classes: real simple roots satisfying $(\alpha_I|\alpha_I) > 0$, and imaginary simple roots satisfying $(\alpha_I|\alpha_I) \leq 0$.

We denote by $\Delta$ the set of all roots. Generalising the usual terminology, a root is said to be positive (resp. negative) if it is a non-negative (resp. non-positive) integer linear combination of the simple roots. The set of roots thus splits again into a direct sum of positive and negative roots, $\Delta = \Delta^+ \oplus \Delta^-$. We also introduce the root lattice $\Lambda_G$ to be the integral span of all simple roots. It decomposes as $\Lambda_G = \Lambda^+_G \cup \Lambda^-_G$, where $\Lambda^+_G$ contains the non-negative integer linear combinations of the simple roots, and similarly for $\Lambda^-_G$.

The Weyl group $W(\mathcal{G})$ is the group of reflections in $\Lambda_G \otimes \mathbb{C}$ with respect to the real simple roots. In other words, upon denoting by $\alpha_I$, $I = 1, \ldots, n$ the real simple roots, $W(\mathcal{G})$ is generated by $n$ fundamental reflections

$$w_I : \alpha \mapsto \alpha - 2\frac{(\alpha|\alpha_I)}{(\alpha_I|\alpha_I)}\alpha_I, \quad \alpha \in \Lambda_G \otimes \mathbb{C}. \quad (A.15)$$

An additional important property of a BKM-algebra is the existence of a Weyl vector $\rho$, satisfying

$$(\rho|\alpha) \leq -\frac{1}{2}(\alpha|\alpha), \quad (A.16)$$

with equality if and only if $\alpha$ is a simple root. For any (integrable) lowest weight representation $R(\lambda)$ of $\mathcal{G}$ one further has the Weyl-Kac-Borcherds character formula \cite{78, 14}

$$\text{ch } R(\lambda) = \sum_{w \in W} \epsilon(w)w(S)e^{-\rho} \prod_{\alpha \in \Delta^+}(1 - e^\alpha)^{\text{mult } \alpha}, \quad (A.17)$$

where $\epsilon(w) = (-1)^{\ell(w)}$ with $\ell(w)$ the length of the Weyl element $w$ (see e.g. \cite{84}). This expression differs from the standard Weyl-Kac character formula by the factor $w(S)$ which contains a correction due to the imaginary simple roots \cite{14}

$$S = e^{\lambda + \rho} \sum_{\alpha \in \Lambda^+_G} \xi(\alpha)e^{\alpha}. \quad (A.18)$$

Here $\xi(\alpha) = (-1)^m$ if $\alpha$ is a sum of $m$ distinct pairwise orthogonal imaginary simple roots which are orthogonal to $\lambda$, and $\xi(\alpha) = 0$ otherwise. For our purposes we are interested in the simplest case of the trivial representation $\lambda = 0$, for which $\text{ch } R(\lambda) = 1$, and the character formula reduces to the so called denominator formula

$$\sum_{w \in W} \epsilon(w)w(S)e^{-\rho} = \prod_{\alpha \in \Delta^+}(1 - e^\alpha)^{\text{mult } \alpha}. \quad (A.19)$$

This formula relates a sum over the Weyl group $W(\mathcal{G})$ to an infinite product over all positive roots of $\mathcal{G}$.  

\textsuperscript{10}Notice that $n$ need not be finite. For the construction of BKM with an infinite number of real simple roots see e.g. \cite{32, 30}.
B One-Loop Integral in Terms of SL(2, Z)-Orbits

In this appendix we will explicitly evaluate the three different contributions in $[5,10]$, corresponding to the different inequivalent SL(2, Z) orbits.

B.1 The Zero Orbit

The contribution from $A = 0$ takes the form

$$I_0^{(\text{analy})} = \int_{\mathbb{H}} \frac{d^2 \tau}{\tau_2^2} \frac{Y}{U_2} \sum_{n \geq -1} \sum_{\ell \in \Lambda_e \otimes \Lambda_8} \bar{c}_{\text{ext}}(n, \ell) \bar{q}^n = \frac{2iY}{\pi U_2} \int_{\mathbb{H}} \frac{d^2 \tau}{\tau_2} \frac{\partial}{\partial \tau} \left[ \sum_{n \geq -1} \sum_{\ell \in \Lambda_e \otimes \Lambda_8} \bar{c}_{\text{ext}}(n, \ell) \bar{q}^n \right]. \quad (B.1)$$

Performing an integration by parts and using modular invariance of the integrand (see e.g. [85]), we can readily evaluate this integral to get

$$I_0^{(\text{analy})} = \frac{2 \pi Y}{3 U_2} \sum_{\ell \in \Lambda_e \otimes \Lambda_8} \left[ \bar{c}_{\text{ext}}(0, \ell) - 24 \bar{c}_{\text{ext}}(-1, \ell) \right]. \quad (B.2)$$

B.2 The Non-Degenerate Orbit

A representative matrix for the non-degenerate orbit can be taken to be

$$A = \begin{pmatrix} r & j \\ 0 & p \end{pmatrix} \quad \text{with} \quad \begin{cases} p \in \mathbb{Z} \neq 0 \\ r > j \geq 0 \end{cases}, \quad (B.3)$$

whereas the integration domain can be extended to the double-cover of the upper half-plane

$$I_{ND}^{(\text{analy})} = \frac{2Y}{U_2} \lim_{\tau_2 \to \infty} \left[ \sum_{n \geq -1} \sum_{\ell \in \Lambda_e \otimes \Lambda_8} \bar{c}_{\text{ext}}(n, \ell) \bar{q}^n \right] \quad \text{for} \quad \begin{cases} p \neq 0 \\ r > j \geq 0 \end{cases}, \quad (B.4)$$

Performing the coordinate transformation

$$\tau_1' = -r \tau_1 + j + p U_1 \quad \text{and} \quad A = \tau_1' + i(p U_2 + r \tau_2) \quad \tilde{A} = \tau_1' - i(p U_2 + r \tau_2)$$

we find that all the $j$-dependence of the integrand is in the factor $q^n = e^{-2 \pi \tau_2 n + 2 \pi i n (r_1' - p U_1)}$ stemming from the Fourier expansion (5.5). In this case, the summation over $j$ yields only
a non-vanishing result if \( n \) is a multiple of \( r \). We thus introduce \( n = n' r \), with \( n' \in \mathbb{Z} \) and obtain
\[
I_{ND}^{(\text{analy})} = - \frac{2 Y}{U_2} \int_{\mathbb{H}} d\tau' \frac{d^2 r'}{\tau_2^2} \sum_{\ell \in \Lambda_{\text{eq}} \oplus \Lambda_{\text{sk}}} \sum_{n' r \in \mathbb{Z} \atop p \in \mathbb{Z}} e^{-2 \pi n' \tau_2 - 2 \pi i \left( p U_1 + \xi \right) e^{2 \pi i \ell \cdot \vec{e}}} \tilde{c}_{\text{ext}}(n' r, \ell')
\]
\[
\times e^{-\frac{\pi Y}{U_2} \left( \tau_1^2 + (p U_2 + \tau_2)^2 \right) - 2 \pi i (T r_0 - \frac{2 n (\xi \cdot \vec{V})^2 r}{U_2}) (p U_2 + \tau_2)}
\]
\[
\times \tilde{c}_{\text{ext}}(n' r, \ell').
\] (B.5)

The \( \tau_1 \)-integral is now Gaussian and can be solved using elementary methods, leading to
\[
I_{ND}^{(\text{analy})} = - 2 \sqrt{Y} \int_0^\infty \frac{d \tau_2}{\tau_2} \sum_{\ell \in \Lambda_{\text{eq}} \oplus \Lambda_{\text{sk}}} \sum_{n' r \in \mathbb{Z} \atop p \in \mathbb{Z}} e^{-2 \pi n' \tau_2 - 2 \pi i \left( p U_1 - 2 \pi i p \tilde{R}(\vec{e} \cdot \vec{V}) \right) - 2 \pi i (\xi \cdot \vec{V})^2 r}{U_2} (p U_2 + \tau_2)
\]
\[
\times \tilde{c}_{\text{ext}}(n' r, \ell') e^{2 \pi i (\xi \cdot \vec{V}) \tau_2}{U_2} - \frac{\pi Y}{U_2} \left( p U_2 + \tau_2 \right) - 2 \pi i Tr p - \frac{\pi Y}{U_2} \left( p U_2 + \tau_2 \right) - 2 \pi i (\xi \cdot \vec{V})^2 r \frac{U_2}{U_2^2} \right)^{2}.
\] (B.6)

The integral over \( \tau_2 \) is of Bessel-type for which we can use the identity
\[
\int_0^\infty \frac{d x}{x^{3/2}} e^{-a x - b / x} = \sqrt{\frac{\pi}{b}} e^{-2 \sqrt{a b}} \quad \text{for} \quad a > 0 \text{ and } b > 0.
\] (B.7)

In order to be able to use this relation, we need to specify the point in moduli space at which we are working. Without loss of generality, we will assume
\[
T_2 > 0 \quad \text{and} \quad U_2 > 0,
\] (B.8)
in which case we only have to distinguish the cases (i) \( \vec{e} \cdot (\Im \vec{V}) > 0 \) and (ii) \( \vec{e} \cdot (\Im \vec{V}) < 0 \). We will in the following explicitly treat case (i), while case (ii) will follow similarly. Splitting the summation over \( p \) into the pieces \( p > 0 \) and \( p < 0 \) we obtain
\[
I_{ND}^{(\text{analy})} = \sum_{\ell \in \Lambda_{\text{eq}} \oplus \Lambda_{\text{sk}}} \sum_{n' r \in \mathbb{Z} \atop p = 1} e^{2 \pi i p (r T + n' U + \xi \cdot \vec{V})} + e^{-2 \pi i p (r T + n' U + \xi \cdot \vec{V})} \right] \tilde{c}_{\text{ext}}(n' r, \ell').
\] (B.9)

Recalling the identity \( \sum_{i=1}^\infty \frac{x^i}{i} = \log(1 - x) \), we can perform the sum over \( p \) to obtain the result
\[
I_{ND}^{(\text{analy})} = 2 \log \prod_{\ell \in \Lambda_{\text{eq}} \oplus \Lambda_{\text{sk}}} \left| \log(1 - e^{2 \pi i (r T + n' U + \xi \cdot \vec{V})}) \right| \tilde{c}_{\text{ext}}(n' r, \ell').
\] (B.10)

Notice that the second term in (B.9) is just the complex conjugate of the first term, which explains the appearance of the absolute square in the final result. The result for the case (ii), i.e. \( \vec{e} \cdot (\Im \vec{V}) < 0 \) can be obtained by replacing \( (\Im \vec{V}) \) by \(- (\Im \vec{V}) \).
The last orbit to consider is the so-called degenerate orbit consisting of matrices with vanishing determinant. We can pick a representative $A$ to be of the form

$$A = \begin{pmatrix} 0 & j \\ 0 & p \end{pmatrix}$$

with

$$\begin{cases} (j, p) \neq (0, 0) \\ j, p \in \mathbb{Z} \end{cases}, \quad (B.11)$$

and we will integrate over the semi-infinite strip $\mathcal{S} = \{ \tau_1 \in [-1/2, 1/2], \tau_2 \in [0, \infty) \}$. The integral then becomes

$$I^{(analy)}_{D} = \frac{Y}{U_2} \int_{\mathcal{S}} d\tau_1 d\tau_2 \sum_{\vec{\ell} \in \Lambda_\mathbb{R} \oplus \Lambda_\mathbb{R}} \bar{c}_{\text{ext}}(n, \vec{\ell}) q^n e^{2\pi i \vec{\ell} \cdot \vec{z}} e^{-\frac{\pi Y}{\tau_2} |A|^2}. \quad (B.12)$$

Notice that in this case the only $\tau_1$-dependence comes from the factor of $\bar{q}^n$. The only non-vanishing contribution to the integral $\tau_1 \in [-1/2, 1/2]$ therefore comes from $n = 0$. For the remaining expression, also the $\tau_2$ integration can be performed by elementary methods such that we obtain

$$I^{(analy)}_{D} = \frac{U_2}{\pi} \sum_{\vec{\ell} \in \Lambda_\mathbb{R} \oplus \Lambda_\mathbb{R}} \bar{c}_{\text{ext}}(0, \vec{\ell}) e^{-\frac{2\pi i}{U_2} \vec{\ell} \cdot \vec{z}} e^{2\pi i [j(\Im \vec{V}) + p(U_1(\Re \vec{V}) - U_2(\Im \vec{V}))]} \quad (B.13)$$

For the remaining sum over $j$ and $p$ it turns out to be useful to split the summation into contributions for $\vec{\ell} = 0$ and $\vec{\ell} \neq 0$. The former contribution is identical to the one found in e.g. [86], and equals

$$I^{(analy)}_{D, \vec{\ell} = 0} = \frac{U_2}{\pi} \sum_{(j,p) \neq (0,0)} \frac{1}{|j + p U|^2} \bar{c}_{\text{ext}}(0, \vec{0}) \left( \frac{\pi U_2}{3} - \ln Y + \gamma_E - 1 - \ln \frac{8\pi}{3\sqrt{3}} \right) - \ln \prod_{n=1}^{\infty} \left| 1 - e^{2\pi inU} \bar{c}_{\text{ext}}(0, \vec{0}) \right|. \quad (B.14)$$

In order to calculate the contribution $\vec{\ell} \neq \vec{0}$ we first recall that $\bar{c}_{\text{ext}}(0, \vec{\ell}) = 0$ for $\vec{\ell} \cdot \vec{\ell} > 2$. Moreover, without loss of generality, we will assume to be working in a region in moduli space where

$$U_2 > 0 \quad \text{and} \quad U_2 > \left| \vec{\ell} \cdot (3\vec{V}) \right|. \quad (B.15)$$

In order to proceed, we have to distinguish two different contributions, namely (i) $\vec{\ell} \cdot (3\vec{V}) > 0$ and (ii) $\vec{\ell} \cdot (3\vec{V}) < 0$. In the following we will explicitly work out the first case and indicate the

\[\text{(We are using here the same regularisation as in equation (B.19) of [86]).}\]
result for the second, which can be obtained in a similar fashion. With these assumptions, we can write the sum over \( j \) and \( p \) as

\[
\begin{align*}
\mathcal{I}_{D, \ell \neq 0}^{(\text{analy})} &= \frac{U_2}{\pi} \sum_{\ell \in \Lambda_s \ominus \Lambda_s \begin{array}{c} \ell \in \Lambda_s \ominus \Lambda_s \end{array}} \left[ \sum_{j \neq 0} \frac{\bar{\epsilon}_{\text{ext}}(0, \tilde{\ell})}{j^2} e^{-\frac{2\pi i}{2} \bar{\ell} (\mathbb{V}) j} \right. \\
&+ \left. \sum_{p, j \in \tilde{\ell} \neq 0} \frac{\bar{\epsilon}_{\text{ext}}(0, \tilde{\ell})}{(j + pU_1)^2 + p^2U_2^2} e^{-\frac{2\pi i}{2} \bar{\ell} (\mathbb{V}) j - \frac{2\pi i}{2} \bar{\ell} [U_1 (\mathbb{V}) j - U_2 (\mathbb{W})]} \right]. 
\end{align*}
\] (B.16)

In order to calculate these sums, we use the relations (see e.g. [86])

\[
\sum_{j=1}^{\infty} \frac{\cos \theta j}{j^2} = \frac{\theta (\theta - 2\pi)}{4} + \frac{\pi^2}{6},
\] (B.17)

\[
\sum_{j=-\infty}^{\infty} \frac{e^{\theta j}}{(j + a_1)^2 + a_2^2} = \frac{\pi}{a_2} \left[ \frac{e^{-i\theta(a_1 + i a_2)}}{1 - e^{-2\pi i(a_1 + i a_2)}} + \frac{e^{-i\theta(a_1 - i a_2)}}{1 - e^{2\pi i(a_1 + i a_2)}} \right],
\] (B.18)

where the second identity holds for \( a_2 > 0 \) and \( 0 \leq \theta \leq 2\pi \). Using the first identity in the first term in (B.16), we find explicitly

\[
\sum_{j \neq 0} \frac{e^{-\frac{2\pi i}{2} j (\tilde{\ell} \cdot \mathbb{V})}}{j^2} = 2 \sum_{j=1}^{\infty} \frac{\cos \left( \frac{2\pi}{\ell} j (\tilde{\ell} \cdot \mathbb{V}) \right)}{j^2} = \frac{2\pi^2}{U_2} \left( \frac{\tilde{\ell} \cdot \mathbb{V}}{U_2} - 1 \right) + \frac{\pi^2}{3}.
\] (B.19)

Changing first \( j \rightarrow -j \) and \( p \rightarrow -p \) we can similarly treat the second term in (B.16) using the second relation (B.18), and thus obtain

\[
\sum_{p, j \in \tilde{\ell} \neq 0} \frac{e^{-\frac{2\pi i}{2} j (\tilde{\ell} \cdot \mathbb{V})}}{(j + pU_1)^2 + p^2U_2^2} e^{-\frac{2\pi i}{2} p \tilde{\ell} (U_1 (\mathbb{V}) j - U_2 (\mathbb{W}))} = \sum_{j=-\infty}^{\infty} \sum_{p=1}^{\infty} \left[ \frac{e^{-\frac{2\pi i}{2} p \tilde{\ell} (\mathbb{V}) U^2}}{(j + pU_1)^2 + p^2U_2^2} \sum_{p, j \in \tilde{\ell} \neq 0} \frac{e^{-\frac{2\pi i}{2} p \tilde{\ell} (U_1 (\mathbb{V}) j - U_2 (\mathbb{W}))}}{(j + pU_1)^2 + p^2U_2^2} \right] e^{\frac{2\pi i}{2} p \tilde{\ell} (U_1 (\mathbb{V}) j - U_2 (\mathbb{W}))}
\] (B.20)

\[
= \sum_{j=-\infty}^{\infty} \sum_{p=1}^{\infty} \left[ \frac{e^{-\frac{2\pi i}{2} p \tilde{\ell} (\mathbb{V}) U^2}}{1 - e^{-2\pi i pU^2}} \sum_{p, j \in \tilde{\ell} \neq 0} \frac{e^{-\frac{2\pi i}{2} p \tilde{\ell} (U_1 (\mathbb{V}) j - U_2 (\mathbb{W}))}}{1 - e^{-2\pi i pU^2}} \right] e^{\frac{2\pi i}{2} p \tilde{\ell} (U_1 (\mathbb{V}) j - U_2 (\mathbb{W}))}.
\]
Then we use the identity \( \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \) to simplify this to
\[
\sum_{p=1}^{\infty} \sum_{n=0}^{\infty} \frac{\pi}{p U_2} \left[ e^{2\pi ip(nU+\vec{e} \vec{V})} + e^{2\pi ip(\vec{e} \vec{V}-(n+1)U)} + e^{-2\pi ip(nU+\vec{e} \vec{V})} + e^{-2\pi ip(\vec{e} \vec{V}-(n+1)U)} \right]
\]
\[= \sum_{p=1}^{\infty} \frac{\pi}{p U_2} \left[ \sum_{n=1}^{\infty} \left( e^{2\pi ip(nU+\vec{e} \vec{V})} + e^{2\pi ip(\vec{e} \vec{V}-(n-1)U)} \right) \right] + \text{c.c.}
\]
\[= \frac{2\pi}{U_2} \left( \log \prod_{n=1}^{\infty} \left| 1 - e^{2\pi i(nU+\vec{e} \vec{V})} \right| + \log \left| 1 - e^{2\pi i\vec{e} \vec{V}} \right| \right) .
\]
(B.21)

Here we have used the explicit expression for the polylogarithms, \( \text{Li}_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n} \). Notice that the final expression in (B.21) is manifestly real. The expression for \( \vec{e} \cdot (\vec{3} \vec{V}) < 0 \) can be obtained in exactly the same manner and yields the same result with \( (\vec{3} \vec{V}) \) replaced by \( -(\vec{3} \vec{V}) \). The total contribution of the degenerate orbit is therefore given by
\[
\mathcal{I}_{D}^{(\text{analy})} = \bar{c}_{\text{ext}}(0,0) \left( \frac{\pi U_2}{3} - \ln Y + \gamma_E - 1 - \ln \frac{8\pi}{3 \sqrt{3}} \right) - \ln \prod_{n=1}^{\infty} \left| 1 - e^{2\pi i nU} \right|^{4\bar{c}_{\text{ext}}(0,0)}
\]
\[+ 2 \sum_{\vec{e} \in \Lambda_8 \otimes \Lambda_8, \vec{e} \neq \vec{0}} \left( \log \prod_{n=1}^{\infty} \left| 1 - e^{2\pi i(nU+\vec{e} \vec{V})} \right|^{\bar{c}_{\text{ext}}(0,\vec{e})} + \log \left| 1 - e^{2\pi i\vec{e} \vec{V}} \right|^{\bar{c}_{\text{ext}}(0,\vec{e})} \right)
\]
\[+ \frac{2U_2}{3\pi} + \frac{2\pi}{U_2} (\vec{e} \cdot \vec{3} \vec{V}) \left[ (\vec{e} \cdot \vec{3} \vec{V}) + U_2 \right] .
\]
(B.22)

Let us recall again that the contribution for \( \vec{e} \cdot (\vec{3} \vec{V}) < 0 \) can be obtained by replacing \( (\vec{3} \vec{V}) \) by \( -(\vec{3} \vec{V}) \). Putting everything together we then obtain (5.10).

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