Reduction of Stokes-Dirac structures and gauge symmetry in port-Hamiltonian systems

Marko Seslija*, Arjan van der Schaft†, and Jacquelien M.A. Scherpen*

*Department of Discrete Technology and Production Automation, Faculty of Mathematics and Natural Sciences, University of Groningen, Nijenborgh 4, 9747 AG Groningen, The Netherlands, e-mail: {M.Seslija, J.M.A.Scherpen}@rug.nl
†Johann Bernoulli Institute for Mathematics and Computer Science, University of Groningen, Nijenborgh 9, 9747 AG Groningen, The Netherlands, e-mail: A.J.van.der.Schaft@rug.nl

May 5, 2012

Abstract

Stokes-Dirac structures are infinite-dimensional Dirac structures defined in terms of differential forms on a smooth manifold with boundary. These Dirac structures lay down a geometric framework for the formulation of Hamiltonian systems with a nonzero boundary energy flow. Simplicial triangulation of the underlying manifold leads to the so-called simplicial Dirac structures, discrete analogues of Stokes-Dirac structures, and thus provides a natural framework for deriving finite-dimensional port-Hamiltonian systems that emulate their infinite-dimensional counterparts. The port-Hamiltonian systems defined with respect to Stokes-Dirac and simplicial Dirac structures exhibit gauge and a discrete gauge symmetry, respectively. In this paper, employing Poisson reduction we offer a unified technique for the symmetry reduction of a generalized canonical infinite-dimensional Dirac structure to the Poisson structure associated with Stokes-Dirac structures and of a fine-dimensional Dirac structure to simplicial Dirac structures. We demonstrate this Poisson scheme on a physical example of the vibrating string.

Keywords: Port-Hamiltonian systems, Poisson structures, Dirac structures, distributed-parameter systems, symmetry reduction

1 Introduction

Geometric structures behind a variety of physical systems stemming from mechanics, electromagnetism and chemistry exhibit a remarkable unity enunciated by Dirac structures. The open dynamical systems defined with respect to these structures belong to the class of so-called port-Hamiltonian systems. These systems arise naturally from the energy-based modeling. Apart from offering a geometric content of Hamiltonian systems, Dirac structures supply a framework for modeling port-Hamiltonian systems as interconnected
and constrained systems. From a network-modeling perspective, this means that port-
Hamiltonian systems can be reticulated into a set of energy-storing elements, a set of
energy-dissipating elements, and a set of energy port by which the interconnection of
these blocks and environment is modeled. It is well-known that such a modeling strategy
also utilizes control synthesis for these systems.

The port-Hamiltonian formalism transcends the lumped-parameter scenario and has
been successfully applied to study of a number of distributed-parameter systems [7, 4]. The
centrepiece of the efforts concerning infinite-dimensional case is the Stokes-Dirac structure.
The canonical Stokes-Dirac structure is an infinite-dimensional Dirac structure defined
in terms of differential forms on a smooth manifold with boundary. The Hamiltonian
equations associated to this Dirac structure allow for non-zero energy exchange through
the boundary.

Although the differential operator in the Stokes-Dirac structure, in the presence of
nonzero boundary conditions, is not skew-symmetric, it is possible to associate a (pseudo-
)Poisson structure to the Stokes-Dirac structure [7]. In the absence of algebraic constraints,
the Stokes-Dirac structure specializes to a Poisson structure [3], and as such it can be
derived through symmetry reduction from a canonical Dirac structure on the phase space
[10]. How to conduct this reduction for the Poisson structure associated to the Stokes-
Dirac structure on a manifold with boundary is the central theme of this paper.

**Contribution and outline.** This paper very closely follows [10]. The reduction
scheme we are dealing with is the one from [10], the only difference being in that we con-
sider slightly augmented spaces in order to account for the behaviours associated with the
boundary. The perspective as well as the notation in Section 2 are taken verbatim from
[10], but now for the generalized Dirac structures that allow for the formulation of open
Hamiltonian systems. The proposed Poisson reduction is firstly applied in the reduction
of a generalized canonical Dirac structure to the Poisson structure associated with the
Stokes-Dirac structure. In the context of dynamics, the canonical port-Hamiltonian sys-
tems are those defined as in [5, 6], now only in the context of differential forms, while the
reduced port-Hamiltonian systems are exactly those presented in [7]. In the final section we
demonstrate how this reduction applies to the Poisson reduction of the port-Hamiltonian
systems on discrete manifolds [8, 9].

2 Dirac structures and reduction

Dirac structures were originally developed in [2, 3] as a generalization of symplectic and
Poisson structures. The formalism of Dirac structure was employed as the geometric
notion underpinning generalized power-conserving interconnections and thus allowing the
Hamiltonian formulation of interconnected and constrained dynamical systems.

Let $Q$ be a manifold and define a pairing on $TQ \oplus T^*Q$ given by

$$
\langle (v, \alpha), (w, \beta) \rangle = \frac{1}{2}(\alpha(w) + \beta(v)).
$$

For a subspace $D$ of $TQ \oplus T^*Q$, we define the orthogonal complement $D^\perp$ as the space
of all $(v, \alpha)$ such that $\langle (v, \alpha), (w, \beta) \rangle = 0$ for all $(w, \beta)$. A **Dirac structure** is then a
subbundle $D$ of $TQ \oplus T^*Q$ which satisfies $D = D^\perp$. 

2
The notion of Dirac structures just entertained is suitable for the formulation of closed Hamiltonian systems, however, our aim is a treatment of open Hamiltonian systems in such a way that some of the external variables remain free port variables.

Let $F$ be a linear vector space of external flows, with dual the space $F^*$ of external efforts. We deal with Dirac structures on the product space $Q \times F$. The pairing on $(TQ \times F) \oplus (T^*Q \times F)$ is given by

$$\langle (v, f), (\alpha, e) \rangle = \frac{1}{2} (\alpha(w) + e(\tilde{f}) + \beta(v) + \tilde{e}(f)).$$

(2.1)

A generalized Dirac structure $D$ is a subbundle of $(TQ \times F) \oplus (T^*Q \times F)$ which is maximally isotropic under (2.1).

Canonical Dirac structure on $TQ \oplus T^*Q$ is considered to be a symplectic structure. However, in this paper we shall deal with slightly different canonical Dirac structures. To that end, let the map $\sharp : T^*Q \times F^* \to TQ \times F$ induces the Poisson structure on $TQ \times F$. The graph of $\sharp$ given by

$$D_{T^*Q \times F^*} := \{ (\sharp(\alpha, e), (\alpha, e)) : \alpha \in T^*Q, e \in F^* \}$$

(2.2)

is a Dirac structure. If the mapping $\sharp$ is symplectic on $TQ$, that is if $\sharp(\alpha, 0) = 0$ implies $\alpha = 0$, the Dirac structure (2.2) is the generalized canonical Dirac structure.

There is a number of techniques for symmetry reduction of Dirac structures [1, 11]. The reduction considered in this paper is the Poisson reduction from [10]. For that purpose, let $G$ be a Lie group which acts on $Q$ from the right and assume that the quotient space $Q/G$ is again a manifold. Denote the action of $g \in G$ on $q \in Q$ by $q \cdot g$ and the induced actions of $g \in G$ on $TQ \times F$ and $T^*Q \times F^*$ by $(v, f) \cdot g$ and $(\alpha, e) \cdot g$, for $v \in TQ, f \in F, \alpha \in T^*Q$, and $e \in F^*$. The action on the $T^*Q \times F^*$ is defined by $\langle (\alpha, e) \cdot g, (v, f) \rangle = \langle (\alpha, e), (v, f) \rangle \cdot g^{-1}$. In what follows, we will focus mostly on the reduced cotangent bundle $(T^*Q \times F^*)/G$. In this paper, we will deal with the space denoted by $T^*Q/G \times F^*$.

Consider now the canonical Dirac structure on $T^*Q \times F^*$. Let $\sharp : T^*Q \times F^* \to TQ \times F$ be the map (2.2) used in the definition of $D_{T^*Q \times F^*}$. The reduced Dirac structure $D_{T^*Q/G \times F^*}$ on $T^*Q/G \times F^*$ can now be described as the graph of a reduced map $\underline{\sharp} : T^*(T^*Q/G \times F^*) \to T(T^*Q/G \times F^*)$ defined as follows.

Let $\pi_G : T^*Q \times F^* \to T^*Q/G \times F^*$ be the quotient map and consider an element $(\rho, \pi, \rho_b)$ in $T^*Q \times F^*$. The tangent map of $\pi_G$ at $(\rho, \pi, \rho_b)$ is denoted by $T_{(\rho, \pi, \rho_b)}\pi_G : T_{(\rho, \pi, \rho_b)}(T^*Q \times F^*) \to T_{(\rho, \pi, \rho_b)}(T^*Q/G \times F^*)$, and its dual by $T^*_{(\rho, \pi, \rho_b)}\pi_G : T^*_{\pi_G(\rho, \pi, \rho_b)}(T^*Q/G \times F^*) \to T^*_{\pi_G(\rho, \pi, \rho_b)}(T^*Q \times F^*)$. The reduced map $\underline{\sharp}$ now fits into the following extended commutative diagram

\[
\begin{array}{ccc}
T^*_{\pi_G(\rho, \pi, \rho_b)}(T^*Q/G \times F^*) & \xrightarrow{\underline{\sharp}} & T_{(\rho, \pi, \rho_b)}(T^*Q \times F^*) \\
\downarrow T^*_{(\rho, \pi, \rho_b)}\pi_G & & \downarrow T_{(\rho, \pi, \rho_b)}\pi_G \\
T^*_{\pi_G(\rho, \pi, \rho_b)}(T^*Q \times F^*) & \xrightarrow{\underline{\sharp}} & T_{\pi_G(\rho, \pi, \rho_b)}(T^*Q \times F^*).
\end{array}
\]
3 Constant Stokes-Dirac structures

Throughout this paper, let $M$ be an oriented $n$-dimensional smooth manifold with a smooth $(n-1)$-dimensional boundary $\partial M$ endowed with the induced orientation, representing the space of spatial variables. By $\Omega^k(M)$, $k = 0, 1, \ldots, n$, denote the space of exterior $k$-forms on $M$, and by $\Omega^k(\partial M)$, $k = 0, 1, \ldots, n-1$, the space of $k$-forms on $\partial M$. A natural non-degenerate pairing between $\rho \in \Omega^k(M)$ and $\sigma \in \Omega^{n-k}(M)$ is given by $\langle \sigma | \rho \rangle = \int_M \sigma \wedge \rho$. Likewise, the pairing on the boundary $\partial M$ between $\rho \in \Omega^k(\partial M)$ and $\sigma \in \Omega^{n-k-1}(\partial M)$ is given by $\langle \sigma | \rho \rangle = \int_{\partial M} \sigma \wedge \rho$ [7].

3.1 Stokes-Dirac structure

For any pair $p, q$ of positive integers satisfying $p + q = n + 1$, define the flow and effort linear spaces by

$$\mathcal{F}_{p,q} = \Omega^p(M) \times \Omega^q(M) \times \Omega^{n-p}(\partial M)$$
$$\mathcal{E}_{p,q} = \Omega^{n-p}(M) \times \Omega^{n-q}(M) \times \Omega^{n-q}(\partial M).$$

The bilinear form on the product space $\mathcal{F}_{p,q} \times \mathcal{E}_{p,q}$ is

$$\langle (f_1^p, f_2^p, f_3^p, e_1^q, e_2^q), (f_1^q, f_2^q, f_3^q, e_1^p, e_2^p) \rangle_{\mathcal{F}_{p,q} \times \mathcal{E}_{p,q}}$$

$$= \int_M e_1^p \wedge f_2^p + e_1^q \wedge f_2^q + e_2^p \wedge f_1^p + e_2^q \wedge f_1^q$$

$$+ \int_{\partial M} e_1^q \wedge f_2^q + e_2^q \wedge f_1^q.$$  \hspace{1cm} (3.1)

**Theorem 3.1 (Stokes-Dirac structure [7]).** Given linear spaces $\mathcal{F}_{p,q}$ and $\mathcal{E}_{p,q}$, and the bilinear form $\langle \cdot, \cdot \rangle$, define the following linear subspace $\mathcal{D}$ of $\mathcal{F}_{p,q} \times \mathcal{E}_{p,q}$

$$\mathcal{D} = \{ (f_p, f_q, f_b, e_p, e_q, e_b) \in \mathcal{F}_{p,q} \times \mathcal{E}_{p,q} \mid$$

$$\begin{pmatrix} f_p \\ f_q \\ f_b \\ e_p \\ e_q \\ e_b \end{pmatrix} = \begin{pmatrix} 0 & (-1)^{p+1}d & e_p \\ d & 0 & e_q \\ tr & 0 & (-1)^{n-q}tr \end{pmatrix} \begin{pmatrix} e_p \\ e_q \end{pmatrix} \},$$

(3.2)

where $d$ is the exterior derivative and $tr$ stands for a trace on the boundary $\partial M$. Then $\mathcal{D} = \mathcal{D}^\perp$, that is, $\mathcal{D}$ is a Dirac structure.

It is possible to associate a Poisson structure to the Stokes-Dirac structure $\mathcal{D}$. Here we just sketch the essence and refer the reader to [7].

The space of admissible efforts is $\mathcal{E}_{\text{adm}} := \{ e \in \mathcal{E}_{p,q} \mid \exists f \in \mathcal{F}_{p,q} \text{ such that } (f, e) \in \mathcal{D} \}$. The set of admissible mappings $\mathcal{K}_{\text{adm}} := \{ k : \mathcal{F}_{p,q} \rightarrow \mathbb{R} \mid \forall a \in \mathcal{F}_{p,q}, \exists e(k, a) \in \mathcal{E}_{\text{adm}} \text{ such that } \forall a \in \mathcal{F}_{p,q} k(a + \partial a) = k(a) + \langle e(k, a) | \partial a \rangle + O(\partial a) \}$. The set $\mathcal{K}_{\text{adm}}$ consists of those functions $k : \Omega^p(M) \times \Omega^q(M) \times \Omega^{n-p}(\partial M) \rightarrow \mathbb{R}$ whose derivatives $\delta k(z) = (\delta_p k(z), \delta_q k(z), \delta_b k(z)) \in \Omega^{n-p}(M) \times \Omega^{n-q}(M) \times \Omega^{n-q}(\partial M)$ satisfy $\delta_b k(z) = \int_{\partial M} \delta k(z) \wedge \rho$. 


\[-(1)^n\text{tr}(\delta_y k(z)).\] The Poisson bracket on \(\mathcal{K}_{\text{adm}}\) is given as

\[
\{k^1, k^2\}_\mathcal{D} = \int_M ((\delta_p k^1) \wedge (-1)^r d((\delta_y k^2) + (\delta_y k^1) \wedge d(\delta_y k^2)))
- \int_{\partial M} ((-1)^{q-1}(\delta_y k^1) \wedge (\delta_p k^2)).
\]

Using Stokes’ theorem, it follows that the bracket is skew-symmetric and that it satisfies the Jacobi identity: \(\{\{k^1, k^2\}_\mathcal{D}, k^3\}_\mathcal{D} + \{\{k^1, k^3\}_\mathcal{D}, k^2\}_\mathcal{D} + \{\{k^2, k^3\}_\mathcal{D}, k^1\}_\mathcal{D} = 0\) for all \(k^i \in \mathcal{K}_{\text{adm}}\).

In this paper we will exclusively be dealing with Poisson and associated Poisson structures.

### 3.2 Simplicial Dirac structures

In the discrete setting, the smooth manifold \(M\) is replaced by an \(n\)-dimensional well-centered oriented manifold-like simplicial complex \(K\) \([8, 9]\). The flow and the effort spaces will be the spaces of complementary primal and dual forms. The elements of these two spaces are paired via the discrete primal-dual wedge product. Let

\[
\mathcal{F}^d_{p,q} = \Omega^p_d(\star K) \times \Omega^q_d(K) \times \Omega^{n-p}_d(\partial(K))
\]

\[
\mathcal{E}^d_{p,q} = \Omega^{n-p}_d(K) \times \Omega^{n-q}_d(\star K) \times \Omega^{q}_d(\partial(\star K)).
\]

The primal-dual wedge product ensures a bijective relation between the primal and dual forms, between the flows and efforts. A natural discrete mirror of the bilinear form (3.1) is a symmetric pairing on the product space \(\mathcal{F}^d_{p,q} \times \mathcal{E}^d_{p,q}\) defined by

\[
\langle\langle f_p, f_q, e_p, e_q, f^1_p, f^1_q, e^1_p, e^1_q, f^2_p, f^2_q, e^2_p, e^2_q \rangle\rangle_d
\]

\[
= \langle e^1_p \wedge f^1_p + e^1_q \wedge f^1_q + e^2_p \wedge f^2_p + e^2_q \wedge f^2_q, K \rangle
+ \langle e^1_b \wedge f^1_b + e^2_b \wedge f^2_b, \partial K \rangle.
\]

A discrete analogue of the Stokes-Dirac structure is the finite-dimensional Dirac structure constructed in the following theorem \([8]\).

**Theorem 3.2 (Simplicial Dirac structure [8]).** Given linear spaces \(\mathcal{F}^d_{p,q}\) and \(\mathcal{E}^d_{p,q}\), and the bilinear form \(\langle\langle \cdot, \cdot \rangle\rangle_d\). The linear subspace \(\mathcal{D}_d \subset \mathcal{F}^d_{p,q} \times \mathcal{E}^d_{p,q}\) defined by

\[
\begin{pmatrix}
\hat{f}_p \\
\hat{f}_q
\end{pmatrix}
= \begin{pmatrix}
0 & (-1)^r d^{n-q}_p \\
d^{n-p} & 0
\end{pmatrix}
\begin{pmatrix}
e_p \\
e_q
\end{pmatrix}
+ (-1)^r \begin{pmatrix}
d^{n-q}_b \\
0
\end{pmatrix} \hat{e}_b,
\]

\[
f_b = (-1)^p \text{tr}^{n-p} e_p,
\]

with \(r = pq + 1\), is a Dirac structure with respect to the pairing \(\langle\langle \cdot, \cdot \rangle\rangle_d\).
The operators \( \mathbf{d}^{n-p} \) is the discrete exterior operator mapping \( \Omega_d^{n-p}(K) \) to \( \Omega_d^p(K) \), and \( \mathbf{d}_b^{n-q} \) is the dual discrete exterior derivative. Note that since \( \mathbf{d}^{n-q} = (-1)^q (\mathbf{d}^{n-p})^* \) and \( \mathbf{d}_b^{n-q} = (-1)^{n-p} (\mathbf{d}_b^{n-p})^* \), the structure (3.4) is in fact a Poisson structure on the state space \( \Omega_d^k(\mathfrak{g}K) \times \Omega_d^k(K) \).

The simplicial Dirac structure (3.4) is used as \textit{terminus a quo} for the geometric formulation of spatially discrete port-Hamiltonian systems [9].

4 Reduction of Stokes-Dirac structure

The configuration manifold is a vector space \( Q := \Omega^k(M) \) with the tangent bundle \( TQ = Q \times Q \) and the cotangent bundle \( T^*Q = Q \times Q^* \), where \( Q^* = \Omega^{n-k}(M) \). The space of the boundary flows \( F \) will be an admissible subset of \( \Omega^{n-k-1}(\partial M) \), while the space of the boundary efforts is \( E := F^* = \Omega^k(\partial M) \).

The tangent bundle \( T(T^*Q \times F^*) \) is isomorphic to \( (Q \times Q^* \times F^*) \times (Q \times Q^* \times F^*) \), with a typical element denoted by \( (\rho, \pi, \rho_b, \dot{\rho}, \pi_b) \), while \( T^*(T^*Q \times F^*) = (Q \times Q^* \times F^*) \times (Q \times Q^* \times F^* \times F^*) \), with a typical element denoted by \( (\rho, \pi, \rho_b, e_\rho, e_\pi, e_b) \). For the duality pairing between \( T(T^*Q \times F^*) \) and \( T^*(T^*Q \times F^*) \) we chose

\[
\langle (\rho, \pi, \rho_b, e_\rho, e_\pi, e_b), (\rho, \pi, \rho_b, \dot{\rho}, \pi_b) \rangle = \int_M (e_\rho \wedge \dot{\rho} + e_\pi \wedge \dot{\pi}) + \int_{\partial M} (e_b \wedge \dot{\rho} + e_b \wedge \text{tr} \dot{\rho}).
\]

(4.1)

The choice for this non-degenerate pairing will become clear later on.

4.1 The symmetry group

Let \( G \) be an Abelian group of \((k-1)\)-forms. For any \( \alpha \in G \) and \( \rho \in Q \), the group \( G \) action on \( Q \) is

\[
\rho \cdot \alpha = \rho + d\alpha.
\]

(4.2)

This action of gauge group lifts to \( TQ \times F \) and \( T^*Q \times F^* \) as \( (\rho, \dot{\rho}, e_b) \cdot \alpha = (\rho + d\alpha, \dot{\rho}, e_b) \) and \( (\rho, \pi, \rho_b) \cdot \alpha = (\rho + d\alpha, \rho_b, \pi) \) for \( \alpha \in G \), \( (\rho, \dot{\rho}, e_b) \in TQ \times F \) and \( (\rho, \pi, \rho_b) \in T^*Q \times F^* \).

The elements of \( Q/G \) are equivalence classes \([\rho]\) of \( k \)-forms up to exact forms, so that the exterior differential determines a well-defined map from \( Q/G \) to \( d\Omega^k \), given by \( [\rho] \mapsto d\rho \). If the \( k \)-th cohomology of \( M \) vanishes, we have \( Q/G = d\Omega^k \). Consequently, the quotient \( (T^*Q/G \times F^*) \) is isomorphic to \( Q/G \times Q^* \times F^* \), or explicitly

\[
(T^*Q \times F^*)/G = d\Omega^k(M) \times \Omega^{n-k}(M) \times \Omega^k(\partial M).
\]

The quotient map denoted as \( \pi_G : T^*Q \times F^* \rightarrow (T^*Q)/G \times F^* \) is given by

\[
\pi_G(\rho, \pi, \rho_b) = (d\rho, \pi, \rho_b).
\]

(4.3)

Let a representative element of \( T^*Q/G \times F^* \) be \( (\dot{\rho}, \dot{\pi}, \dot{\rho}_b) \), with \( \dot{\rho} \in d\Omega^k(M), \dot{\pi} \in \Omega^{n-k}(M) \) and \( \dot{\rho}_b \in \Omega^k(\partial M) \). Elements of \( T(T^*Q/G \times F^*) \) will be denoted by \( (\dot{\rho}, \dot{\pi}, \dot{\rho}_b, \dot{\pi}, \dot{\rho}_b) \), while the elements of \( T^*(T^*Q/G \times F^*) \) will be denoted by \( (\dot{\rho}, \dot{\pi}, \dot{\rho}_b, e_\rho, e_\pi, e_b) \). For the duality pairing, we use

\[
\langle (\dot{\rho}, \dot{\pi}, \dot{\rho}_b, e_\rho, e_\pi, e_b), (\dot{\rho}, \dot{\pi}, \dot{\rho}_b, \dot{\pi}, \dot{\rho}_b) \rangle = \int_M (\dot{e}_\rho \wedge \dot{\rho} + \dot{e}_\pi \wedge \dot{\pi}) + \int_{\partial M} \dot{e}_b \wedge \dot{\rho}_b.
\]

(4.4)
Whenever the base point \((\hat{\rho}, \hat{\pi}, \hat{\rho}_b)\) is clear from the context, we will denote \((\hat{\rho}, \hat{\pi}, \hat{\rho}_b, \hat{\rho}, \hat{\pi}, \hat{\rho}_b)\) simply by \((\hat{\rho}, \hat{\pi}, \hat{\rho}_b)\), and similarly for \((\hat{\rho}, \hat{\pi}, \hat{\rho}_b, \hat{\varepsilon}_\rho, \hat{\varepsilon}_\pi, \hat{\varepsilon}_b)\).

4.2 The reduced Dirac structure

The generalized canonical Dirac structure is a Poisson structure induced by the linear mapping \(\sharp : T^*(T^*Q \times F^*) \to T(T^*Q \times F^*)\) given by

\[
\sharp(\rho, \pi, \rho_b, \varepsilon_\rho, \varepsilon_\pi, \varepsilon_b) = (\rho, \pi, \rho_b, \varepsilon_\rho, \varepsilon_\pi, -(\varepsilon_\rho - \varepsilon_\pi)\varepsilon_b).
\] (4.5)

In order to obtain the reduced Poisson structure from the canonical Dirac structure (4.5), we need to specify what are the operators \(T\pi_G\) and \(T^*\pi_G\) in the diagram (2.3). The space \(F\) is the set of admissible forms \(\Omega^{n-k-1}(\partial M)\) that are the traces of \((d\Omega^k)^*\), as will be made clear in Lemma 4.1. Consider an element \((\rho, \pi, \rho_b) \in T^*Q \times F^*\), and we recall that \(\pi_G(\rho, \pi, \rho_b) = (d\rho, \pi, \rho_b)\). Let \(T\rho, \pi, \rho_b)\pi_G : T\rho, \pi, \rho_b)T^*(Q^* \times F^*) \to T\rho, \pi, \rho_b)T^*(Q^* / G \times F^*)\) be the tangent map to \(\pi_G\) at \((\rho, \pi, \rho_b)\) and consider the adjoint map \(T^*\rho, \pi, \rho_b)\pi_G : T^*(Q^* / G \times F^*) \to T^*(\rho, \pi, \rho_b)T^*(Q \times F^*)\).

**Lemma 4.1.** The tangent and cotangent maps \(T\rho, \pi, \rho_b)\pi_G\) and \(T^*\rho, \pi, \rho_b)\pi_G\) are given by

\[
T\rho, \pi, \rho_b)\pi_G(\rho, \pi, \rho_b, \hat{\rho}, \hat{\pi}, \hat{\rho}_b) = (d\rho, \pi, \rho_b, d\hat{\rho}, \hat{\pi}, \hat{\rho}_b)
\] (4.6)

and

\[
T^*\rho, \pi, \rho_b)\pi_G(d\rho, \pi, \rho_b, \varepsilon_\rho, \varepsilon_\pi, -(\varepsilon_\rho - \varepsilon_\pi)\varepsilon_b)
= (\rho, \pi, \rho_b, -(\varepsilon_\rho - \varepsilon_\pi)\varepsilon_b)
\] (4.7)

**Proof.** The expression (4.6) for \(T\rho, \pi, \rho_b)\pi_G\) follows from (6.3). To prove (4.7), we let \((\hat{\rho}, \hat{\pi}, \hat{\rho}_b) \in T\rho, \pi, \rho_b)T^*(Q \times F^*)\) and consider

\[
\left\langle T^*(\rho, \pi, \rho_b)\pi_G(e_\rho, e_\pi, -(\varepsilon_\rho - \varepsilon_\pi)\varepsilon_b, \hat{\rho}, \hat{\pi}, \hat{\rho}_b)\right\rangle
= \left\langle (e_\rho, e_\pi, -(\varepsilon_\rho - \varepsilon_\pi)\varepsilon_b), (d\hat{\rho}, \hat{\pi}, \hat{\rho}_b)\right\rangle
\]

Applying Stokes’ theorem, we have

\[
\left\langle (\hat{\rho}, \hat{\pi}, -(\varepsilon_\rho - \varepsilon_\pi)\varepsilon_b), (d\hat{\rho}, \hat{\pi}, \hat{\rho}_b)\right\rangle
= \int_M (e_\rho \wedge d\hat{\rho} + e_\pi \wedge \hat{\pi})
+ \int_M (\varepsilon_\rho \wedge \rho_b) - \int_M (\varepsilon_\rho \wedge \varepsilon_b)
\]

Thus, \(T^*\rho, \pi, \rho_b)\pi_G(e_\rho, e_\pi, -(\varepsilon_\rho - \varepsilon_\pi)\varepsilon_b) = (\varepsilon_\rho - \varepsilon_\pi)\varepsilon_b)\).
As in the case of a boundaryless manifold [10], the reduced Poisson structure in (2.3) is given by

\[ [\sharp](d\rho, \pi, \rho_b) = T^*(\rho, \pi, \rho_b) \pi G \circ [\sharp] \circ T^*(d\rho, \pi, \rho_b) \pi G \]

for all \((d\rho, \pi, \rho_b) \in T^*Q/G \times F^*\).

**Theorem 4.2.** The reduced Poisson structure is given by

\[ [\sharp](\bar{\rho}, \bar{\pi}, -(-1)^{n-k} \text{tr} \bar{\rho}) = (d\bar{\pi}, (-(-1)^{n(k+1)}d\bar{\rho}, -\text{tr} \bar{\pi}) \] (4.8)

**Relation to the Stokes-Dirac structure.** The matrix form of the reduced Poisson structure is

\[
\begin{pmatrix}
\dot{\bar{\rho}} \\
\dot{\bar{\pi}} \\
\dot{\rho}_b
\end{pmatrix}
= \begin{pmatrix}
0 & d & 0 \\
-(-1)^{n(k+1)}d & 0 & 0 \\
0 & -\text{tr} & 0
\end{pmatrix}
\begin{pmatrix}
\bar{\rho} \\
\bar{\pi} \\
(-1)^{n-k} \text{tr} \bar{\rho}
\end{pmatrix}.
\] (4.9)

The sign convention in (4.9) and [7] is not the same. To match the signs we introduce new flow variables \(f_p, f_q, f_b\) and effort variables \(e_p, e_q, e_b\) defined as \(e_p = \bar{\rho}, e_q = (-(-1)^r \bar{\pi}, f_p = \hat{\pi}, f_q = (-(-1)^{n(k+1)+1} \hat{\pi}, f_b = -(-1)^r \hat{\rho}_b\), where \(p = k + 1, q = n - k, \) and \(r = pq + 1.\) With this choice of signs, (4.9) becomes

\[
\begin{pmatrix}
f_p \\
f_q
\end{pmatrix}
= \begin{pmatrix}
0 & (-1)^r d \\
d & 0
\end{pmatrix}
\begin{pmatrix}
e_p \\
e_q
\end{pmatrix}
\] (4.10)

\[ f_b = \text{tr} e_q + (-1)^{n-k} \text{tr} \bar{\rho}_b.\]

Here, it is important to point out that the boundary effort \(e_b\), unlike in the case of the Stokes-Dirac structure, does not follow from the associate Poisson structure, but rather belongs to the set of admissible derivatives of the flow restricted to the boundary.

### 5 Symmetry in port-Hamiltonian systems

Let \(t \mapsto (\alpha, \dot{\alpha}) \in \Omega^k(M) \times \Omega^{n-k}(M)\) be a time function, and let the Hamiltonian be

\[ H(\alpha, \dot{\alpha}) = \int_M \mathcal{H}(d\alpha, \dot{\alpha}) \] .

It follows that at any time instance \(t \in \mathbb{R}\)

\[
\frac{dH}{dt} = \int_M \delta H \wedge \frac{\partial \alpha}{\partial t} + \frac{\partial H}{\partial \dot{\alpha}} \wedge \frac{\partial \alpha}{\partial t} + \int_{\partial M} \frac{\partial \mathcal{H}}{\partial \alpha} \wedge \frac{\partial \dot{\alpha}}{\partial t}.
\]

The differential forms \(\delta \mathcal{H}, \delta \dot{\alpha}\) represent the generalized velocities of the energy variables \(\alpha, \dot{\alpha}.\) The connection with the canonical Dirac structure is made by setting the flows

\[ \dot{\alpha} = -\frac{\partial \alpha}{\partial t}, \quad \dot{\pi} = -\frac{\partial \pi}{\partial t}.\]
and the efforts
\[ e_ρ = \frac{δH}{δα_ρ}, \quad e_π = \frac{δH}{δα_π}. \]

The canonical distributed-parameter port-Hamiltonian system on an \( n \)-dimensional manifold, with the state space \( Ω^k(M) \times Ω^{n-k}(M) \), the Hamiltonian \( H \) and the canonical Dirac structure (4.5), is given as

\[ (−\frac{∂α}{∂t} \dot{ρ}, \dot{π}) =\begin{pmatrix} 0 & 1 -\frac{1}{k} \left(\frac{δH}{δα_ρ} \right) \\ -\frac{1}{k} \left(\frac{δH}{δα_π} \right) \end{pmatrix} \begin{pmatrix} 0 \\ -\frac{1}{k} \left(\frac{δH}{δα_π} \right) \end{pmatrix}. \]

Proposition 5.1. For the port-Hamiltonian system (5.1) the following property

\[ \frac{dH}{dt} = \int_{\partial M} e_b \wedge f_b \]

expresses the fact that the increase in energy on the domain \( M \) is equal to the power supplied to the system through the boundary \( \partial M \).

5.1 The reduced port-Hamiltonian systems

The Hamiltonian \( H \) is invariant if a spatially independent \( k \)-form is added to \( α_ρ, \) thus the Poisson reduction is applicable. Let the reduced field be \( \bar{α}_ρ := dα_ρ \), then the reduced Hamiltonian is

\[ H(\bar{α}_ρ, α_π) = \int_M H(\bar{α}_ρ, α_π). \]

The port-Hamiltonian system with respect to the reduced Poisson structure is

\[ (−\frac{∂\bar{α}_ρ}{∂t} \dot{ρ}, \dot{π}) =\begin{pmatrix} 0 & 1 -\frac{1}{k} \left(\frac{δH}{δα_ρ} \right) \\ -\frac{1}{k} \left(\frac{δH}{δα_π} \right) \end{pmatrix} \begin{pmatrix} 0 \\ -\frac{1}{k} \left(\frac{δH}{δα_π} \right) \end{pmatrix}. \]

This is precisely the port-Hamiltonian system given in [7].

We will show how the general considerations of the reduction of port-Hamiltonian systems apply to a physical example of the vibrating string.

5.2 Vibrating string

Consider an elastic string of length \( l \), elasticity modulus \( T \), and mass density \( μ \), subject to traction forces at its ends. The underlying manifold is the segment \( M = [0, l] \subset \mathbb{R} \), with coordinate \( z \).

Under the assumption of linear elasticity, the Hamiltonian is given by

\[ H(u, p) = \int_M H(u, p) = \frac{1}{2} \int_M (μ^{-1} p \wedge \star p + Tdu \wedge \star du), \]
where \( p \in \Omega^1(M) \) is the momentum conjugate to the displacement \( u \in \Omega^0(M) \), and \( * \) is the Hodge star.

The canonical Hamiltonian equations are

\[
\left( \frac{\partial u}{\partial t}, \frac{\partial p}{\partial t} \right) = \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) \left( \begin{array}{c} \frac{\delta H}{\delta u} \\ \frac{\delta H}{\delta p} \end{array} \right)
\]

\[
f_b = \text{tr}(\mu^{-1} p)
\]

\[
e_b = \text{tr} \left( \frac{\partial \mathcal{H}}{\partial (du)} \right),
\]

or component-wise

\[
\frac{\partial u}{\partial t} = \mu^{-1} p
\]

\[
\frac{\partial p}{\partial t} = d(*T du)
\]

\[
f_b = \text{tr}(\mu^{-1} p)
\]

\[
e_b = \text{tr}(\mu^{-1} p).
\]

The Hamiltonian formulation (5.3) is identical to the formulation of the heavy chain system in [6].

The energy balance for the vibrating string is

\[
\frac{dH}{dt} = \int_M \delta H \wedge \frac{\partial u}{\partial t} + \frac{\delta H}{\partial p} \wedge \frac{\partial p}{\partial t} + \int_{\partial M} \frac{\partial \mathcal{H}}{\partial (du)} \wedge \frac{\partial u}{\partial t}
\]

\[
= \int_M -d(*T du) \wedge \mu^{-1} p + \mu^{-1} p \wedge d(*T du)
\]

\[
+ \int_{\partial M} \mu^{-1} p \wedge *T du
\]

\[
= \int_{\partial M} \mu^{-1} p \wedge *T du = \int_{\partial M} e_b \wedge f_b.
\]

The Hamiltonian is invariant if a time function is added to \( u \). The potential energy can be expressed in terms of the strain \( \alpha = du \) so that the reduced Hamiltonian is given by

\[
H_r(\alpha, p) = \int_M \mathcal{H}_r(u, p) = \frac{1}{2} \int_M (\mu^{-1} p \wedge \mu^{-1} p \wedge T \alpha \wedge \alpha).
\]

The Hamiltonian equations of the vibrating string now read as

\[
\left( \frac{\partial \alpha}{\partial t}, \frac{\partial p}{\partial t} \right) = \left( \begin{array}{cc} 0 & d \\ d & 0 \end{array} \right) \left( \begin{array}{c} \frac{\delta H_r}{\delta \alpha} \\ \frac{\delta H_r}{\delta p} \end{array} \right)
\]

\[
\left( \begin{array}{c} f_b \\ e_b \end{array} \right) = \text{tr} \left( \begin{array}{c} \frac{\partial \mathcal{H}_r}{\partial (du)} \\ \frac{\partial \mathcal{H}_r}{\partial (d\alpha)} \end{array} \right),
\]

These are the equations that correspond to the formulation of the vibration string system with respect to the Stokes-Dirac structure as is given in [7].
6 Symmetry reduction in discrete setting

In the discrete world, the configuration space is the set of primal discrete forms $Q = \Omega^k(K)$ with the dual $Q^* = \Omega^{n-k}(\ast_i K)$. The space of the boundary efforts is $E = F^* = \Omega^k(\partial(K))$, and the space of the boundary flows is $F = \Omega^{n-k-1}(\partial(\ast K))$.

For the duality pairing between $T(T^*Q \times F^*)$ and $T^*(T^*Q \times F^*)$ we choose

$$
\langle (\rho, \pi, \rho_b, e_\rho, e_\pi, e_b), (\rho, \pi, \rho_b, \dot{\rho}, \dot{\pi}, \dot{\rho}_b) \rangle = \int_M (e_\rho \wedge \dot{\rho} + e_\pi \wedge \dot{\pi}) + \int_{\partial M} e_b \wedge \dot{\rho}_b, \quad (6.1)
$$

where $\wedge$ is the primal-dual wedge product.

The generalized canonical Dirac structure is a Poisson structure induced by the linear mapping $\sharp : T^*(T^*Q \times F^*) \rightarrow T(T^*Q \times F^*)$ given by

$$
\sharp(\rho, \pi, \rho_b, e_\rho, e_\pi, e_b) = (\rho, \pi, \rho_b, e_\rho, e_\pi, -(1)^{k(n-k)}(e_\rho + d_b^{n-k-1}e_b), -tr^k e_\pi). \quad (6.2)
$$

The group $G$ that acts on $Q$ is described by the following action

$$
\alpha \cdot \rho = \rho + d^{k-1}\alpha
$$

for $\alpha \in G$ and $\rho \in Q$, where $d^{k-1}$ is the discrete exterior derivative.

The quotient is $(T^*Q/G \times F^*) = d^k\Omega^k(K) \times \Omega^{n-k}(\ast_i K) \times \Omega^k(\partial(K))$.

As in the continuous setting, the quotient map denoted as $\pi_G : T^*Q \times F^* \rightarrow (T^*Q)/G \times F^*$ is given by

$$
\pi_G(\rho, \pi, \rho_b) = (d^k\rho, \pi, \rho_b). \quad (6.3)
$$

For the duality pairing between $T^*(T^*Q/G \times F^*)$ and $T(T^*Q/G \times F^*)$, we take

$$
\langle (\bar{\rho}, \bar{\pi}, \bar{\rho}_b, e_\rho, \bar{e}_\pi, \bar{e}_b), (\bar{\rho}, \bar{\pi}, \bar{\rho}_b, \dot{\bar{\rho}}, \dot{\bar{\pi}}, \dot{\bar{\rho}}_b) \rangle = \int_M (\bar{e}_\rho \wedge \dot{\bar{\rho}} + \bar{e}_\pi \wedge \dot{\bar{\pi}}) + \int_{\partial M} \bar{e}_b \wedge \dot{\bar{\rho}}_b.
$$

As before, whenever the base point $(\bar{\rho}, \bar{\pi}, \bar{\rho}_b)$ is clear, we will denote $(\bar{\rho}, \bar{\pi}, \bar{\rho}_b, \dot{\bar{\rho}}, \dot{\bar{\pi}}, \dot{\bar{\rho}}_b)$ simply by $(\dot{\bar{\rho}}, \dot{\bar{\pi}}, \dot{\bar{\rho}}_b)$, and similarly for $(\dot{\bar{\rho}}, \dot{\bar{\pi}}, \dot{\bar{\rho}_b})$.

Lemma 6.1. The tangent and cotangent maps $T_{(\rho, \pi, \rho_b)}\pi_G$ and $T^*_{(\rho, \pi, \rho_b)}\pi_G$ are given by

$$
T_{(\rho, \pi, \rho_b)}\pi_G(\rho, \pi, \rho_b, \dot{\rho}, \dot{\pi}, \dot{\rho}_b) = (d^k\rho, \pi, \rho_b, d^k\dot{\rho}, \dot{\pi}, \dot{\rho}_b) \quad (6.4)
$$

and

$$
T^*_{(\rho, \pi, \rho_b)}\pi_G(d^k\rho, \pi, \rho_b, \dot{\rho}, \dot{\pi}, \dot{\rho}_b) = (\rho, \pi, \rho_b, (-1)^{n-k}d^{n-k-1}_i e_\rho, e_\pi, e_b). \quad (6.5)
$$

Theorem 6.2 (Reduced simplicial Dirac structure). The reduced simplicial Poisson structure is given by $[\sharp](\dot{\bar{e}}_\rho, \dot{\bar{e}}_\pi, \dot{\bar{e}}_b) = (d^k\dot{\bar{e}}_\pi, -(1)^{n(k+1)}(-1)^{n-k}d^{n-k-1}_i \dot{\bar{e}}_\rho + d^{n-k-1}_b \dot{\bar{e}}_b), -tr^k \dot{\bar{e}}_\pi)$. 

11
Port-Hamiltonian systems on a simplicial complex

The canonical port-Hamiltonian system with respect to the canonical Dirac structure is

\[-\frac{\partial \alpha_{\dot{\rho}}}{\partial t} = \frac{\partial H}{\partial \alpha_{\dot{\rho}}}(\alpha_{\dot{\rho}}, \alpha_{\dot{\pi}})\]

\[-\frac{\partial \alpha_{\dot{\pi}}}{\partial t} = (-1)^{k(n-k)} \left( \frac{\partial H}{\partial \alpha_{\dot{\rho}}}(\alpha_{\dot{\rho}}, \alpha_{\dot{\pi}}) + d^{n-k-1}_b \bar{b}_e b \right)\]

(6.6)

\[\dot{\rho}_b = -\text{tr} \left( k \frac{\partial H}{\partial \alpha_{\dot{\pi}}}(\alpha_{\dot{\rho}}, \alpha_{\dot{\pi}}) \right)\]

The rank of the underlying Poisson structure is the rank of the symplectic phase space \(\Omega^k(K) \times \Omega^{n-k}(K)\).

The canonical Hamiltonian \((\alpha_{\dot{\rho}}, \alpha_{\dot{\pi}}) \mapsto H(\alpha_{\dot{\rho}}, \alpha_{\dot{\pi}})\) can be expressed as

\[H(\alpha_{\dot{\rho}}, \alpha_{\dot{\pi}}) := H_r(d^k \bar{\alpha}_{\dot{\rho}}, \alpha_{\dot{\pi}})\]

(6.7)

The reduced port-Hamiltonian equations assume the following form

\[-\frac{\dot{\alpha}_{\dot{\rho}}}{\partial t} = -d^k \frac{\partial \alpha_{\dot{\rho}}}{\partial \alpha_{\dot{\pi}}} = d^k \frac{\partial H}{\partial \alpha_{\dot{\pi}}}(\alpha_{\dot{\rho}}, \alpha_{\dot{\pi}}) = d^k \frac{\partial H_r}{\partial \alpha_{\dot{\pi}}}(\bar{\alpha}_{\dot{\rho}}, \alpha_{\dot{\pi}})\]

\[-\frac{\partial \alpha_{\dot{\pi}}}{\partial t} = (-1)^{k(n-k)} \left( \frac{\partial H}{\partial \alpha_{\dot{\rho}}}(\alpha_{\dot{\rho}}, \alpha_{\dot{\pi}}) + d^{n-k-1}_b \bar{b}_e b \right)\]

\[= (-1)^{k(n-k)} \left( -d^{n-k-1}_i \bar{d}_i d^{n-k-1}_e \frac{\partial H_r}{\partial \alpha_{\dot{\rho}}}(\bar{\alpha}_{\dot{\rho}}, \alpha_{\dot{\pi}}) + d^{n-k-1}_b \bar{b}_e b \right)\]

\[\dot{\rho}_b = -\text{tr} \left( k \frac{\partial H}{\partial \alpha_{\dot{\pi}}}(\alpha_{\dot{\rho}}, \alpha_{\dot{\pi}}) \right) = -\text{tr} \left( k \frac{\partial H_r}{\partial \alpha_{\dot{\pi}}}(\bar{\alpha}_{\dot{\rho}}, \alpha_{\dot{\pi}}) \right)\]

This is precisely the port-Hamiltonian system on a simplicial manifold as presented in [8, 9].

7 Final remark

This paper addresses the issue of the symmetry reduction of the generalized canonical Dirac structure to the Poisson structure associated with the Stokes-Dirac structure. The open avenue for the future work is to find a reduction procedure that would directly lead to the Stokes-Dirac structure.

Acknowledgments

The first author expresses his warmest thanks to Joris Vankerschaver for his hospitality during the author’s stay in California.

References

[1] G. Blankenstein and A. J. van der Schaft, “Symmetry and reduction in implicit generalized Hamiltonian systems,” Rep. Math. Phys., vol. 47, no. 1, pp. 57–100, 2001.
[2] T. Courant, “Dirac manifolds,” Trans. American Math. Soc., 319, pp. 631–661, 1990.

[3] I. Dorfman, Dirac Structures and Integrability of Nonlinear Evolution Equations, John Wiley, Chichester, 1993.

[4] A. Macchelli, A. van der Schaft, and C. Melchiorri, Port hamiltonian formulation of infinite dimensional systems: Part I modeling, In Proc. 43rd IEEE Conf. Decision and Control (CDC), pp. 3762–3767, 2004.

[5] K. Schlacher, “Mathematical modeling for nonlinear control: a hamiltonian approach,” Mathematics and Computers in simulation, vol. 97, pp. 829–849, 2008.

[6] M. Schoberl, “On Casimir functionals for field theories in Port-Hamiltonian description for control purposes,” In Proc. of the 50th IEEE Conf. on Decision and Control and European Control Conference, Orlando, Florida, 2011.

[7] A.J. van der Schaft, B.M. Maschke, “Hamiltonian formulation of distributed-parameter systems with boundary energy flow”, Journal of Geometry and Physics, vol. 42, pp. 166–194, 2002.

[8] M. Seslija, J.M.A. Scherpen, A.J. van der Schaft, “Port-Hamiltonian systems on discrete manifolds,” MathMod 2012 – 7th Vienna International Conference on Mathematical Modelling, http://arxiv.org/abs/1201.5764, 2012.

[9] M. Seslija, A.J. van der Schaft, J.M.A. Scherpen, “Discrete Exterior Geometry Approach to Structure-Preserving Discretization of Distributed-Parameter Port-Hamiltonian Systems,” Journal of Geometry and Physics, Volume 62, Issue 6, June 2012, Pages 1509–153, http://arxiv.org/abs/1111.6403

[10] J. Vankerschaver, H. Yoshimura, and J.E. Marsden, “Stokes-Dirac structures through reduction of infinite-dimensional Dirac structures,” In Proc. 49th IEEE Conference on Decision and Control, Atlanta, USA, December 2010.

[11] H. Yoshimura and J. E. Marsden, “Reduction of Dirac structures and the Hamilton-Pontryagin principle,” Rep. Math. Phys., vol. 60, no. 3, pp. 381–426, 2007.