Ultimate Physical Limits to the Growth of Operator Complexity

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In an isolated system, the time evolution of a given observable in the Heisenberg picture can be efficiently represented in Krylov space. In this representation, an initial operator becomes increasingly complex as time goes by, a feature that can be quantified by the Krylov complexity. We introduce a fundamental and universal limit to the growth of the Krylov complexity by formulating a Robertson uncertainty relation, involving the Krylov complexity operator and the Liouvillian, as generator of time evolution. We further show the conditions for this bound to be saturated and illustrate its validity in the paradigmatic models of quantum chaos.

1 Main

Quantum speed limits impose fundamental constraints on the pace at which a physical process can unfold. Since their conception \cite{1, 2}, they have been formulated as bounds on the minimal time at which a distance between quantum states can be traversed. The freedom in the choice of the distance can be used to sharpen the discrimination between quantum states, and with it, the notion of the speed of evolution \cite{3, 4}. Additional efforts have been devoted to exploring the role of underlying dynamics, generalizing early results from isolated systems to open \cite{5, 6, 7, 8} and classical processes \cite{9, 10, 11}. The resulting speed limits have become a useful tool in various branches of physics, ranging from information processing \cite{12} to many-body physics \cite{13, 14, 15}, quantum control \cite{16, 17, 18} and quantum metrology \cite{19, 20}. However, it has become apparent that traditional QSL are too conservative in estimating the relevant time scale in many processes, such as thermalization \cite{21}. This has motivated the development of speed limits suited for specific measures and observables \cite{22, 23}, as in the pioneering work by Mandelstam and Tamm \cite{1}. In this sense, certain speed limits follow from generalized uncertainty relations such as those derived by Heisenberg and Robertson \cite{24}.

In parallel with the study of QSL, quantifying the complexity of a physical process is a central task for the advancement of fundamental physics and quantum technologies. As pointed out by LLoyd, QSL limit the computational complexity of physical processes \cite{25}. The circuit complexity of a quantum state \cite{26}, defined as the number of elementary operations required to generate it from a reference state, can be characterized in terms of conventional QSL \cite{27, 28, 29, 30}. A complementary approach for many-body quantum systems focuses on the buildup of complexity in the time-evolution of an initial local observable, known as operator growth \cite{31, 32, 33, 34, 35}. The intuition is that simple operators unitarily evolve into increasingly complex ones. Quantum information initially encoded in a few degrees of freedom is thus scrambled over the system in the course of evolution, making it impossible to recover
it through local measurements and giving rise to thermalization. The unambiguous description of this scrambling process remains an open problem. One possible way is to probe it via an out-of-time-ordered correlator [36, 37, 38] that may be used to identify an analog of the Lyapunov exponent, providing a connection with classical chaos, e.g., the butterfly effect. Such quantum Lyapunov exponent obeys a universal upper bound [38], which helps refine the notion of maximal chaos, is saturated by black holes, and is further tied to the eigenstate thermalization hypothesis [39, 40]. A related approach, which we shall pursue in this work, is to study the dynamical evolution of operators in Krylov space, exploited in numerical techniques such as the recursion method [41]. In this context, operator growth is quantified by the so-called Krylov complexity, a measure of the delocalization of the time-dependent operator in the Krylov basis [42, 43, 44, 45]. The authors of [42] made a conjecture on the universal operator growth, namely, that Krylov complexity can grow at most exponentially, and it does so in maximally chaotic systems. Remarkably, its growth rate upper bounds the Lyapunov exponent, establishing a connection with the bound on out-of-time-ordered correlators [38].

We characterize the growth of Krylov complexity by deriving a fundamental limit on its rate of change and by studying analytically the conditions under which this bound is saturated. Our results show that saturation, which is also found to correspond to a particular notion of minimum uncertainty, occurs whenever the dynamical evolution of the system has the underlying structure of a three-dimensional complexity algebra, which was introduced by [46]. In this setting, the unitary evolution of an operator can be represented as the displacement of generalized coherent states [46], which display classical-like behavior [47]. As demonstrated in several paradigmatic examples, the saturation of the growth rate may be possible in some chaotic systems, but quantum chaos is not required for it.

### 1.1 Quantum dynamics in Krylov space.

Consider an isolated quantum system in which the time evolution of an observable $\mathcal{O}$ is generated by a time-independent Hamiltonian $\hat{H}$ according to the Heisenberg equation of motion $\partial_t \mathcal{O}(t) = i[H, \mathcal{O}(t)]$. The solution to this equation with the initial condition $\mathcal{O}(0) = \mathcal{O}$ is given by $\mathcal{O}(t) = e^{itH}\mathcal{O}e^{-itH}$. In terms of the Liouvillian superoperator given by $\mathcal{L} = [\hat{H}, \cdot]$, the Taylor expansion of the time-evolving observable $\mathcal{O}(t) = \sum_{n=0}^{\infty} \frac{i^n}{n!} \mathcal{L}^n \mathcal{O}$ shows that its dynamics is contained in the complex linear span of the operators $\{\mathcal{L}^n \mathcal{O}\}_{n=0}^{\infty}$. This span is completely determined by the Hamiltonian and the initial observable and is known as the Krylov space.

From now on, we consider the restriction of each operator and superoperator to the Krylov space. To highlight the vector space structure, we make use of the braket notation $|\mathcal{A}\rangle$ when expressing an operator $\mathcal{A}$ in an equation. Following [46], we choose to equip the Krylov space with the inner product $(A|B) = \langle e^{\beta H/2} A \rangle \langle e^{-\beta H/2} B \rangle$. The bracket $\langle \cdot \rangle$ here denotes the thermal expectation value with respect to the thermal state $e^{-\beta H}/Z$. With this choice of inner product, one can show that the operators $\mathcal{O}$ and $\mathcal{L}\mathcal{O}$ will be orthogonal. Let $b_0 = ||\mathcal{O}||$ and $b_1 = ||\mathcal{L}\mathcal{O}||$ where $||\cdot||$ is the norm induced by the inner product.

By starting from the normalized operators $\mathcal{O}_0 = \mathcal{O}/b_0$ and $\mathcal{O}_1 = \mathcal{L}\mathcal{O}/b_1$, we can construct an orthonormal basis $\{\mathcal{O}_n\}_{n=0}^{D-1}$ for the Krylov space by applying the Lanczos algorithm. This algorithm works as follows: given the first $n + 1$ basis vectors, one constructs the orthogonal vector $|\mathcal{A}_{n+1}\rangle = \mathcal{L}|\mathcal{O}_n\rangle - b_n|\mathcal{O}_{n-1}\rangle$, where $b_n = ||\mathcal{A}_n||$ and then normalizes it to obtain $|\mathcal{O}_{n+1}\rangle$. We call the constructed basis the Krylov basis. It is possible that the Krylov dimension $D$ is infinite, in which case the Lanczos algorithm never halts. We like to point out that the recursion formula can take a more general form if we consider more general inner products. With our chosen inner-product however, the action of the Liouvillian on the Krylov basis takes a very nice form $\mathcal{L}|\mathcal{O}_n\rangle = b_{n+1}|\mathcal{O}_{n+1}\rangle + b_n|\mathcal{O}_{n-1}\rangle$. As pointed out in [46], this motivates one to consider abstract raising and lowering operators that we denote by $\mathcal{L}_+$ and $\mathcal{L}_-$, respectively. Their action on the Krylov basis is given by $\mathcal{L}_+|\mathcal{O}_n\rangle = b_{n+1}|\mathcal{O}_{n+1}\rangle$ and $\mathcal{L}_-|\mathcal{O}_n\rangle = b_n|\mathcal{O}_{n-1}\rangle$. The Liouvillian can then be expressed as their sum.

It is further convenient to introduce the real-valued functions $\varphi_n(t)$, which appear in the expansion of $\mathcal{O}(t)$ as $|\mathcal{O}(t)\rangle = \frac{1}{\sqrt{Z}} \sum_{n=0}^{D-1} i^n \varphi_n(t)|\mathcal{O}_n\rangle$. We will refer to these functions as the amplitudes of the observable. These amplitudes evolve according to the recursion relation $\partial_t \varphi_k(t) = b_{k-1} \varphi_{k-1}(t) + b_k \varphi_{k+1}(t)$. Thinking of the Krylov basis vectors as forming the sites of a one dimensional lattice, $\varphi_n$ can be interpreted as a hopping amplitude, see, e.g., [42, 43]. In this sense, we can think of $\mathcal{O}$ as a one dimensional discrete wave function that is initially localized and then spreads out over the lattice as time evolves. An increase in the population of the sites further away from the origin reflects a greater increase of
complexity of the observable. In order to quantify this, it is natural to consider the Krylov complexity of \( \mathcal{O}(t) \), defined to be
\[
K(t) = \sum_{n=0}^{D-1} n|\varphi_n(t)|^2.
\] (1)

The main task of our work is to bound the growth of Krylov complexity. Due to unitary dynamics, the norm of the evolution is preserved and the Krylov complexity is unchanged if we normalize our states. We will, therefore, without loss of generality, consider \( \mathcal{O} \) to be normalized. By introducing the complexity operator \( \mathcal{K} = \sum_{n=1}^{D} n|\mathcal{O}_n(\mathcal{O})| \) it is possible to express Krylov complexity as the “expectation value” of \( \mathcal{K} \) with respect to \( \mathcal{O}(t) \). More precisely, if \( \langle \mathcal{K} \rangle_t \equiv \langle \mathcal{O}(t)|\mathcal{K}\mathcal{O}(t) \rangle \) then \( K(t) = \langle \mathcal{K} \rangle_t \).

2 Dispersion bound on Krylov complexity

If the Krylov space forms an inner product space in which \( \mathcal{A} \) and \( \mathcal{B} \) are self-adjoint superoperators, then there ought to exist a Robertson uncertainty relation given by \( \Delta \mathcal{A} \Delta \mathcal{B} \geq \frac{1}{2} |[\mathcal{A}, \mathcal{B}]| \), where \( \Delta \mathcal{A} = \sqrt{\langle \mathcal{A}^2 \rangle - \langle \mathcal{A} \rangle^2} \) is the dispersion of \( \mathcal{A} \) with respect to some state \( |\mathcal{A}\rangle \). When the Krylov dimension is infinite it is necessary that \( |\mathcal{A}\rangle \) is contained in the intersection between the domains of \( \mathcal{A} \mathcal{B} \) and \( \mathcal{B} \mathcal{A} \), otherwise the inequality might not hold [48]. Letting \( \mathcal{A} = \mathcal{O}(t), \mathcal{A} = \mathcal{L}, \mathcal{B} = \mathcal{K} \) and noting that \( \Delta \mathcal{L} = b_1 \), we can rewrite the uncertainty relation as
\[
|\partial_t K(t)| \leq 2b_1 \Delta K.
\] (2)

In other words, the growth of Krylov complexity is upper bounded by a constant times the dispersion of the complexity operator. By defining a characteristic time-scale \( \tau_K = \Delta K/|\partial_t K(t)| \), one obtains \( \tau_K b_1 \geq 1/2 \) which takes the form of a Mandelstam-Tamm bound, and emphasizes the role of \( b_1 = ||\mathcal{L}\mathcal{O}|| \) as a norm of the generator of evolution in Krylov space. To avoid confusion with the uncertainty relation for observables, we will refer to this bound as the dispersion bound. The reader might wonder whether one could obtain a tighter bound than (2) by considering the more general Schrödinger uncertainty relation. The answer is no and an explicit evaluation of the extra term given by the anti-commutator shows that this identically vanishes and thus the two uncertainty relations give the same bound, as shown in [49].

It is not self-evident that saturation of the dispersion bound can be achieved for unitary dynamics of the observable. First of all, as further discussed below, the unitary adjoint actions on \( \mathcal{O} \) form a proper subset of the set of unitary superoperators. Secondly, there are very specific relations between \( \mathcal{L}, \mathcal{O} \) and \( \mathcal{K} \) that needs to hold, namely, the Liouvillian is required to be tridiagonal in the eigenbasis of the complexity operator and the initial state of the observable is required to be parallel to the eigenvector with the lowest eigenvalue. These restrictions give us fewer degrees of freedom to consider in contrast to the question of when the Robertson uncertainty relation is saturated in general. Interestingly, the dispersion bound is saturated for a class of reference models. Before considering these examples, we discuss how the saturation of the dispersion bound can be given a geometrical interpretation. We note that the bound is saturated if and only if the corresponding curve in the projective Krylov space moves along the gradient of the Krylov complexity. In other words, the dispersion bound is saturated if and only if the dynamics is directed along the direction that maximizes the local growth of complexity. The only exception involves extremal points in which any direction away from the extremal point leads to saturation. This is indeed the case for \( t = 0 \). To simplify, the geometrical arguments that we will use are based on the assumption that the Krylov space is of finite dimension. However, the results could potentially be extended to apply to infinite-dimensional Krylov spaces as well.

The projective Krylov space can be identified with the manifold of all rank one orthogonal projections on Krylov space. We can express the Krylov complexity of an element \( \varphi \) of this manifold to be given by \( K(\varphi) = \text{tr}(\mathcal{K}\varphi) \). If \( \varphi(t) = |\mathcal{O}(t)\rangle \langle \mathcal{O}(t) | \), then \( K(\varphi(t)) \) is just the Krylov complexity of our system as defined above. The tangent vectors of the projective Krylov space at a point \( \varphi \) are identified with the superoperators \( i[\mathcal{A}, \varphi] \), where \( \mathcal{A} \) can be any self-adjoint superoperator. It is natural to equip the projective space with the Fubini-Study metric \( g_\mathcal{K} \) which, for any pair of tangent vectors \( X_\mathcal{A} = i[\mathcal{A}, \varphi] \) and \( X_\mathcal{B} = i[\mathcal{B}, \varphi] \) acts according to \( g_\mathcal{K}(X_\mathcal{A}, X_\mathcal{B}) = \frac{1}{2} \text{tr}(X_\mathcal{A} X_\mathcal{B}) \). The differential of Krylov complexity, denoted by \( dK \), can be used together with the metric to define the gradient of Krylov complexity. It follows from the theory of differential geometry that the gradient of Krylov complexity at \( \varphi \), denoted by \( \nabla K(\varphi) \), is the unique vector satisfying the expression \( g_\mathcal{K}(\nabla K(\varphi), X) = dK(X) \) for all tangent vectors.
X at $\varphi \ [50]$. It can be checked that the gradient must then be given by $\nabla K(\varphi) = 2\{\mathcal{K} - \langle \mathcal{K} \rangle, \varphi\}$, where the curly brackets denotes the anti-commutator. The change of Krylov complexity along the curve $\varphi(t)$, generated by the Liouvillian, is given by $\partial_t K(t) = g_{\mathcal{K}}(\nabla K(\varphi(t)), X(t))$, where $X(t) = i[\mathcal{L}, \varphi(t)]$. If we let $\|\cdot\|_b$ denote the norm induced by the Fubini-Study metric then applying Cauchy-Schwarz on the right-hand side gives us the inequality

$$|\partial_t K(t)| \leq \|\nabla K(\varphi(t))\|_b \|\partial_t \varphi(t)\|_b.$$  

(3)

The right-hand side of this inequality is exactly $2b_1 \Delta \mathcal{K}$ and we note that it is saturated if and only if the tangent vector of $\varphi(t)$ is parallel to the gradient of Krylov complexity. We also note that the gradient at time zero is equal to the zero vector and so the dispersion bound is always initially saturated.

Define the unitary orbit of $\mathcal{O}$ to be the set of all points $[U^t \mathcal{O} U]/[U \mathcal{O} U^t]$, where $U$ is a unitary operator. We emphasize that this is a proper subset of the projective Krylov space which is the set of all points $U(t) \mathcal{O}(t) [U(t)]$, where $U$ is a unitary superoperator. The gradient we have considered is with respect to the manifold of the projective Krylov space and it is therefore not obvious that this gradient will ever be tangential to the unitary orbit of $\mathcal{O}$. We have already seen however that the gradient is indeed tangential to the unitary orbit at time zero. Indeed, what we will discuss next is that there exists Liouvillians of the form $\mathcal{L} = \mathcal{L}_+ + \mathcal{L}_-$ for which the tangent of the generated path will be parallel with the gradient for all times.

### 3 Saturation of the dispersion bound

Define the superoperator $\mathcal{B} = \mathcal{L}_+ - \mathcal{L}_-$. Following [46], we consider their simplicity hypothesis: namely, the assumption that $\mathcal{L}$, $\mathcal{B}$ and the commutator $\tilde{\mathcal{K}} = [\mathcal{L}, \mathcal{B}]$ close an algebra with respect to the Lie bracket. As we show in [49], this forces $\tilde{\mathcal{K}}$ to be related to the complexity operator via $\tilde{\mathcal{K}} = \alpha \mathcal{K} + \gamma$, where $\gamma$ is a positive number and $\alpha$ is a real number satisfying the condition $\alpha \geq 0$ for infinite Krylov dimension and $\alpha = -\frac{\pi^2}{M-1}$ for finite Krylov dimension. Moreover, the only possible closure of the algebra is given by the commutation relations

$$[\mathcal{L}, \mathcal{B}] = \tilde{\mathcal{K}}, \quad [\tilde{\mathcal{K}}, \mathcal{L}] = \alpha \mathcal{B}, \quad [\tilde{\mathcal{K}}, \mathcal{B}] = \alpha \mathcal{L}.$$  

(4)

Given this algebra, the evolving observable can be interpreted as a curve of generalized coherent states evolving according to the displacement operator $D(\xi) = e^{\xi \mathcal{L}_+ - \bar{\xi} \mathcal{L}_-}$, where $\xi = it$. Moreover, the initial state is the highest weight state of the representation, which is annihilated by $\mathcal{L}_-$ by construction. Coherent states can be viewed as the states closest to the classical ones in the sense that they typically minimize an uncertainty relation. It is for example known that coherent states of the Harmonic oscillator saturate the Robertson uncertainty relation for the pair of observables of position and momentum. Building on this intuition, we could expect that the dispersion bound is saturated for the simplicity hypothesis. It turns out that this intuition is indeed correct. In fact, as we shown in [49], the dispersion bound is saturated if and only if the simplicity hypothesis holds. Another equivalent statement to the dispersion bound being saturated is that the complexity solves the differential equation $\partial_t^2 K(t) = \alpha K(t) + \gamma$ with the conditions that $K(0) = 0$ and $K(-t) = K(t)$, where $\alpha$ and $\gamma$ have the same conditions as explained above. For finite Krylov dimension, this has the consequence that saturation of the dispersion bound is equivalent to the complexity growing according to $K(t) = (D-1) \sin^2 \omega t$, where $\omega = \sqrt{\frac{2\pi}{M-1}}$. For infinite Krylov dimension we get two possible ways for the complexity to grow: for $\alpha > 0$ we have $K(t) = \frac{2\pi}{\alpha} \sinh^2 \frac{\sqrt{\alpha} t}{M}$ and for $\alpha = 0$ we have $K(t) = \frac{2}{\sqrt{\pi}} t^2$. Reference examples maximizing the Krylov-complexity growth rate at all times are discussed in [49]. One such example with $\alpha > 1$ is the Sachdev-Ye-Kitaev (SYK) model [51, 37], a paradigm of quantum chaos. However, the saturation of the bound does not require quantum chaos and can indeed be achieved by a single qubit [49].

### 4 Krylov complexity in generic systems

We next discuss the Krylov complexity growth in generic systems not fulfilling the simplicity hypothesis. As shown in [49], the dispersion bound is saturated if and only if the Lanczos coefficients grow according
to

\[ b_n = \sqrt{\frac{1}{4} \alpha n (n-1) + \frac{1}{2} \gamma n}. \]  

(5)

where \( \alpha \) and \( \gamma \) are the same coefficients as considered above. For large \( n \), this dependence captures the linear growth \( b_n = \sqrt{\alpha n} \) conjectured by Parker et al. to maximize the Krylov complexity growth, expected in chaotic systems [42]. We can use this fact to estimate when and at what time scale a generic system deviates from the bound. By expanding Krylov complexity up to fourth order we find that

\[ K(t) = b_1^2 t^2 + \frac{1}{6} b_1^4 (2b_2^2 - b_3 t^4 + O(t^6)). \]

Since we can always find a value on \( \alpha \) and \( \gamma \) such that \( b_1 \) and \( b_2 \) satisfies (5), we conclude that the bound (2) is saturated up to third order in time. By expanding Krylov complexity up to sixth order, we find that the Lanczos coefficient \( b_3 \) will appear in the last term and since we are not guaranteed to be able to find a value on \( \alpha \) and \( \gamma \) such that \( b_1, b_2 \) and \( b_3 \) satisfies (5), we conclude that the system can only start deviating from the bound (2) as a result from fifth order terms. We can estimate this time scale by finding the value of \( t \) for which the third order coefficient of \( \partial_t K(t) \) is equal to its fifth order coefficient. We will call this time the deviation time, denoted by \( \tau_d \), and it is explicitly given by

\[ \tau_d = \sqrt{\frac{\frac{1}{6} b_1^4 (2b_2^2 - b_3)}{\frac{1}{20} b_1^2 (b_2^2 + b_3^2) - \frac{1}{6} b_2^2 (b_1^2 + b_2^2 + b_3^2) + \frac{1}{2} b_2^2 b_3^2}}. \]

(6)

To get an understanding of the complexity growth in a generic setting, we next illustrate the Krylov dynamics of a system described by a random matrix Hamiltonian. Specifically, we consider the Krylov

![Graphical representation of Krylov complexity growth](image-url)
complexity of an ensemble $E(H)$ of random matrix Hamiltonians, as a paradigm of quantum chaos [52]. We sample the Hamiltonian matrices $H$ from the Gaussian Orthogonal Ensemble $\text{GOE}(d)$, where $d$ is the dimension of the Hilbert space. We then calculate the Lanczos coefficients $\{b_n\}$ with partial re-orthogonalization [53, 44]. Specifically, we consider samples of real matrices $H = (X + X^\top)/2$, where all elements $x \in \mathbb{R}$ of $X$ are pseudo-randomly generated with probability measure given by the Gaussian, $\exp(-x^2/(2\sigma^2))/(\sigma\sqrt{2\pi})$. In order to study the general behaviour of Lanczos coefficients, we choose as initial observable operator a normalized constant vector in the eigenbasis of the Liouvillian $|O\rangle = (1/d, 1/d, \ldots, 1/d)^T$. However, the following results do not depend strongly on the choice $O$, provided it is sparse. Figure 1 a) shows the squares of the Lanczos coefficients for a single realization and the average $\langle \{b_n\} \rangle_{E(H)}$ over 100 different Hamiltonians of dimension $d = 32$, sampled from $\text{GOE}(d)$ with standard deviation $\sigma = 1$. The time-dependent amplitudes are found by solving the recursion relation and exhibit diffusion-like dynamics on the Krylov basis, shown for a single realization in Figure 1 b). The corresponding time evolution of Krylov complexity and its growth rate are shown in panels c) and d), respectively. (top black line) over the two sides of the bound of equation (2) (blue solid, red dashed lines). Hamiltonians sampled from $\text{GOE}(d)$ behave as a generic system, given that the Lanczos coefficients do not, in general, grow according to (5) as shown in 1 a). As a result, the growth rate starts deviating from the dispersion bound around the time scale $\tau_d$ in Eq. (6), indicated by the vertical line in 1 c) and d). In short, while GOE Hamiltonians provide a useful paradigm in the description of quantum chaotic systems, the dynamics generated by them does not maximize the growth of Krylov complexity.

Summary and discussions.— Employing the Krylov complexity, we have established the ultimate limits to operator growth in isolated quantum systems. Specifically, we have introduced a fundamental dispersion bound that governs the growth rate of Krylov complexity and which plays the role of a Mandelstam-Tamm uncertainty relation in operator space. This bound is saturated by quantum systems in which the Liouvillian governing the time evolution fulfills a simplicity algebra. The latter arises naturally in certain quantum chaotic systems, such as the SYK model. However, other paradigmatic instances of quantum chaotic systems, such as random-matrix Hamiltonians, do not maximize the growth of Krylov complexity. Indeed, we have shown that the saturation of the bound does not require quantum chaos and can be achieved, e.g., by a single qubit. Our results shed new light on the notion of maximal complexity growth, elucidating the conditions on the underlying dynamics and the signatures of quantum chaos required to achieve it. We thus expect them to find broad applications in nonequilibrium phenomena, quantum simulation of many-body systems, numerical methods, quantum chaos and black hole physics.

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Appendices

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A Vanishing of the anticommutator contribution in the Robertson uncertainty relation for $K$ and $L$

We establish a universal feature of Krylov complexity, valid for any physical system: namely, that its anticommutator with the Liouvillian $L$ has vanishing expectation value over the evolved operator $|O(t)\rangle$.

The relevance of this result relies on the fact that this quantity enters the Shrödinger uncertainty principle for the two operators $K$ and $L$

\[ 4(\Delta K \Delta L)^2 \geq |\langle O(t) | [K, L] | O(t) \rangle|^2 \]

\[ + |\langle O(t) | [K, L] | O(t) \rangle|^2, \]

from which one can bound the complexity rate $\dot{K}$. We have that

\[ \langle O(t) | [K, L] | O(t) \rangle = 2i \text{Im} \langle O(t) | KL | O(t) \rangle \]

and

\[ \langle O(t) | [K, L] | O(t) \rangle = 2 \text{Re} \langle O(t) | KL | O(t) \rangle, \]

where

\[ KL = \sum_{n=0}^{D-1} b_{n+1} |n|O_n)(O_{n+1} + (n+1)O_{n+1})(O_n|. \]

Let us now demonstrate that the anticommutator term in Eq. (9) is identically zero. By using expanding $|O(t)\rangle$ over the Krylov basis we obtain

\[ \langle O(t) | KL | O(t) \rangle = \sum_{m,n,k=0}^{D-1} (-i)^{m+k} \phi_m \phi_k b_{n+1}(n \delta_{mn} \delta_{n+1,k} \]

\[ + (n+1) \delta_{m,n+1} \delta_{nk} \). \]

(11)
which, by performing the sums over $k$ and over $n$, yields

$$\langle \mathcal{O}(t)|[\mathcal{K},\mathcal{L}]|\mathcal{O}(t)\rangle = i \sum_{m=0}^{D-1} m\phi_m (\phi_{m+1} b_{m+1} - \phi_{m-1} b_m)$$

$$= -i \sum_{m=0}^{D-1} m\phi_m \phi_m.$$

Since the amplitudes $\phi_n$ and the coefficients $b_n$ are real quantities, comparing Eqs. (9) and (12) we immediately conclude that

$$\langle \mathcal{O}(t)|[\mathcal{K},\mathcal{L}]|\mathcal{O}(t)\rangle = 0 \ \forall t.$$  

Let us note that the key point to obtain this result is the fact that the Liouvillian connects only first neighbors states on the Krylov lattice, so that we are left with a purely imaginary phase $(-i)^m (i)^{m\pm 1} = \pm i$. It is this peculiar property that allows the Liouvillian to be interpreted as a sum of generalized ladder operators $\mathcal{L}_\pm$ [46]. However, let us point that here we are not making any assumption regarding the commutation rules between these operators: we are considering the structure of Krylov space in full generality.

Moreover, from Eq. (8) we immediately obtain the relation between the anticommutator $[\mathcal{K},\mathcal{L}]$ and the complexity rate $\dot{\mathcal{K}}$:

$$\langle \mathcal{O}(t)|[\mathcal{K},\mathcal{L}]|\mathcal{O}(t)\rangle = -2i \sum_{m=0}^{D-1} m\phi_m \phi_m = -i\dot{\mathcal{K}}.$$  

Therefore, the Schrödinger uncertainty relation (7) can be recast as the dispersion bound (2) on the growth of Krylov complexity:

$$|\partial_t K| \leq 2b_1 \Delta \mathcal{K}.$$  

## B On the closure of the complexity algebra

Here we show the proof that the only possible closure of the complexity algebra introduced by [46] is given by Eq. (4). The (anti-Hermitian) operator $\mathcal{B} = \mathcal{L}_+ - \mathcal{L}_- "conjugated"$ to the Liouvillian can be expanded in Krylov space as

$$\mathcal{B} = \sum_{n=0}^{D-1} b_{n+1} [\mathcal{O}_{n+1} (\mathcal{O}_n) - (\mathcal{O}_n) (\mathcal{O}_{n+1})],$$  

where it is understood that $b_0$ has to be replaced with 0. Let us now investigate the conditions under which $\mathcal{L}$, $\mathcal{B}$ and $\mathcal{K}$ form a closed algebra with respect to the operation $[,]$: the so-called complexity algebra [46]. This happens if and only if the commutators $[\mathcal{L},\mathcal{K}]$ and $[\mathcal{B},\mathcal{K}]$ can be written as linear combinations of the operators $\mathcal{L}$, $\mathcal{B}$ and $\mathcal{K}$ themselves. These commutators can be expanded over the Krylov basis as follows:

$$[\mathcal{L},\mathcal{K}] = 2 \sum_{n=0}^{D-1} f(n) b_{n+1} [\mathcal{O}_{n+1} (\mathcal{O}_n) - (\mathcal{O}_n) (\mathcal{O}_{n+1})],$$  

$$[\mathcal{B},\mathcal{K}] = 2 \sum_{n=0}^{D-1} f(n) b_{n+1} [\mathcal{O}_{n+1} (\mathcal{O}_n) + (\mathcal{O}_n) (\mathcal{O}_{n+1})],$$

where we have defined

$$f(n) = b_{n+1}^2 - b_n^2 - (b_{n+2}^2 - b_{n+1}^2) = \frac{\mathcal{K}_{nn} - \mathcal{K}_{n+1,n+1}}{2}.$$
Now, it is clear that the commutator (18) between $\mathcal{L}$ and $\tilde{K}$ cannot contain any element of the complexity algebra other than $\mathcal{B} = \sum_{n=0}^{D-1} b_{n+1} |\mathcal{O}_{n+1}\rangle\langle\mathcal{O}_n| - |\mathcal{O}_n\rangle\langle\mathcal{O}_{n+1}|$, while the commutator (18) can only contain $\mathcal{L} = \sum_{n=0}^{D-1} b_{n+1} |\mathcal{O}_{n+1}\rangle\langle\mathcal{O}_n| + |\mathcal{O}_n\rangle\langle\mathcal{O}_{n+1}|$. Moreover, the only possibility for the algebra to be closed is that the discrete function $f(n)$ is a constant. By looking at Eq. (20), we conclude that $f(n)$ is constant if and only if
\[2(b_{n+1}^2 - b_n^2) = \alpha n + 2\gamma,\] for some constants $\alpha$ and $\gamma$ (the factors 2 are included for convenience). Again, $b_0$ has to be replaced with 0, so that Eq. (21) holds for $n \geq 1$, while $2b_1^2 = \alpha + 2\gamma$. Then, the function $f(n)$ takes the constant value $f = -\alpha/2$, so that the only possible closure of the complexity algebra is given by:
\[[\mathcal{L}, \mathcal{B}] = \tilde{K}, \quad [\tilde{K}, \mathcal{L}] = \alpha \mathcal{B}, \quad [\tilde{K}, \mathcal{B}] = \alpha \mathcal{L}.\] (22)
Moreover, from Eq. (21) we immediately conclude that
\[\tilde{K} = \alpha \mathcal{K} + \gamma.\] (23)
Therefore, if $\alpha \neq 0$, the Krylov complexity is related to $\tilde{K}$ by a shift. Conversely, if $\alpha = 0$ there is no simple relation between the Krylov complexity and the operator $\tilde{K}$. In this case, $\tilde{K}$ is proportional to the identity and the complexity algebra reduces to the Heisenberg-Weyl algebra [47], being $[\mathcal{L}_+, \mathcal{L}_-] = \gamma \mathcal{I}$.

**C Explicit models**

In this appendix we introduce three dynamical models that, having the structure of a closed complexity algebra, display a maximal growth of complexity, in the sense that the complexity rate saturates the dispersion bound. Moreover, we show that quantum chaos, in the Hamiltonian sense, is not necessary to have maximal complexity growth. In particular, it is shown that the dynamics of simple solvable Hamiltonians can saturate our bound.

We first consider a finite-dimensional model, namely the $SU(2)$ algebra, and then turn to the infinite-dimensional case, which allows us to comment on the famous conjecture by Parker et al. [42]. In particular, we show that our notion of maximal complexity growth is more general than the one proposed in their work, as the latter represents a special case of the former.

**C.1 SU(2) algebra**

Let us start with the $SU(2)$ algebra $[J_+, J_-] = i\epsilon_{ijk} J_k$. That is, let us consider the dynamical evolution generated by the Liouvilian
\[\mathcal{L} = \omega (J_+ + J_-),\] (24)
where $J_\pm = J_1 \pm i J_2$ are the familiar $SU(2)$ ladder operators and $\mathcal{L}_\pm = \alpha J_\pm$. Now, the Krylov basis corresponds to the usual basis of the representation $j$: $|\mathcal{O}_n\rangle = |j, n\rangle$ with $-j \leq n \leq j$. Following [46], let us relabel the vectors with $n \rightarrow n + j$, so that $n = 0, \ldots, 2j$, the dimension of the Krylov space being equal to $2j + 1$. By construction, the initial operator $|\mathcal{O}_0\rangle$ is just the highest weight state $|j, -j\rangle$ and is annihilated by $J_-$. From the action of the ladder operators on the representation basis:
\[J_+ |j, -j + n\rangle = \sqrt{(n+1)(2j-n)} |j, -j + n + 1\rangle,\] (25)
\[J_- |j, -j + n\rangle = \sqrt{n(2j-n+1)} |j, -j + n - 1\rangle,\] (26)
being $\mathcal{L}_- |\mathcal{O}_n\rangle = b_n |\mathcal{O}_{n-1}\rangle$, we can read off the Lanczos coefficients:
\[b_n = \omega \sqrt{n(2j-n+1)}.\] (27)

The Heisenberg evolution of an operator can be understood as the displacement of a generalized coherent state [46]:
\[|\mathcal{O}(t)\rangle = e^{i\mathcal{L}t} |\mathcal{O}_0\rangle = D(\xi = i\omega t) |\mathcal{O}_0\rangle.\] (28)
Indeed, the displacement operator is defined as
\[D(\xi) = e^{\xi J_+ - \xi^* J_-}.\] (29)
The generalized coherent state $|\xi,j\rangle$ can be expanded over the spin basis $|j,-j+n\rangle$ as follows [46]:

$$
|\xi,j\rangle = (1 + |\xi|^2)^{-j} \sum_{n=0}^{2j} \xi^n \frac{\Gamma(2j+1)}{n!\Gamma(2j-n+1)} |j,-j+n\rangle.
$$

(30)

For this model it is convenient to use complex polar coordinates $\xi = e^{i\phi} \tan \theta$. Indeed, by replacing $\theta = \omega t$ and $\phi = \pi/2$ and by using the correspondence between the spin and the Krylov basis $|O_n\rangle = |j,-j+n\rangle$,

from above one can read the components of the operator wavefunction:

$$
\phi_n(t) = \tan^n(\omega t) \cos^{2j}(\omega t) \frac{\Gamma(2j+1)}{n!\Gamma(2j-n+1)},
$$

(31)

from which we can compute both the mean (i.e. the Krylov complexity) and the variance of the complexity operator $K$:

$$
K(t) = \sum_{n=0}^{2j} n \phi_n^2(t) = 2j \sin^2 \omega t,
$$

(32)

$$
\Delta K(t) = \sqrt{\sum_{n=0}^{2j} n^2 \phi_n^2(t) - K^2} = \sqrt{\frac{j}{2} \sin 2\omega t},
$$

(33)

Since $b_1 = \omega \sqrt{2j}$, one can check that $|\partial K| = 2b_1 \Delta K$ at any time $t$: that is, as expected from the closure of the 3-dimensional complexity algebra, the dispersion bound is identically saturated.

Given the expression of the Liouvillian in Krylov space, it is generally a difficult task to derive a corresponding Hamiltonian that generates the dynamics in the Hilbert space. In particular, the former contains less information than the latter and therefore many different Hamiltonians can give rise to the same dynamics in Krylov space. Moreover, one has not only to specify the Hamiltonian but also the initial operator $O_0$. Nevertheless, we find that the evolution of the operator $O_0 = \sigma_1 + \sigma_3$ under the single-qubit (two-level) Hamiltonian $H = \omega \sigma_3$, where $\sigma_i$ is the $i$-th Pauli matrix, is given in Krylov space by the representation $j = 1$ of the $SU(2)$ algebra. More precisely, by explicitly performing the Lanczos algorithm, which in this case involves only two steps, we find Lanczos coefficients $b_1 = b_2 = \omega \sqrt{2}$, which coincide with Eq. (27) for $j = 1$. We note that here the dimension of the Krylov space is $D = 3$, which is the maximum allowed for a Hilbert dimension $d = 2$, being $D \leq d^2 - d + 1$ [44]. This is achieved due to the choice made for the initial operator $O_0$, which has non-zero components along all the Liouvillian eigenspaces. If instead one starts with an initial operator $O_0 = \sigma_x$, the Krylov dimension shrinks to $D = 2$, in which case the bound is always trivially saturated, being the complexity algebra given by the representation $j = 1/2$ of $SU(2)$. From this example, we deduce that non-chaotic Hamiltonian can give rise to maximal complexity growth in Krylov space.

As a final remark, let us note that by considering a more general two-level Hamiltonian $H = 1 + \vec{v} \cdot \vec{\sigma}$ we can still obtain the same dynamics in Krylov space, i.e. representation $j = 1$ of $SU(2)$, provided that we tune the parameters $v_i$ in such a way that $b_1 = b_2$: if this condition does not hold, the bound cannot be saturated. More generally, we prove that in a Krylov space of dimension $D = 3$, the algebraic closure and thus the saturation of the bound is possible if and only if $b_1 = b_2$. This can be checked by explicitly computing the double commutator $[\mathcal{L}, [\mathcal{K}, \mathcal{B}]]$ and observing that it vanishes only if $b_1 = b_2$. Remarkably, this implies that the only possible complexity algebra for $D = 3$ is given, up to a multiplicative constant, by $SU(2)$ (in the $j = 1$ representation, compatibly with the space dimension). It could be interesting to investigate if this circumstance holds also for higher (finite) dimensions or if new algebraic structures arise as the Krylov dimension is increased.

### C.2 Heisenberg-Weyl algebra

Let us now consider the case of infinite-dimensional Krylov space. An emblematic example in which the bound is saturated is the one in which the dynamical evolution is given in terms of the Heisenberg-Weyl algebra $[a, a^\dagger] = 1$. In this case, the Liouvillian is given by

$$
\mathcal{L} = \omega(a^\dagger + a),
$$

(34)
and the generalized ladder operators $L_{\pm}$ are just the raising and lowering operators $a^\dagger$ and $a$, times the constant $\omega$. Here the initial operator $|O_0\rangle$ is represented as the vacuum state $|0\rangle$ and the Krylov basis corresponds to the usual basis constructed by acting with $a^\dagger$ on the vacuum:

$$|O_n\rangle = |n\rangle \frac{1}{\sqrt{n!}} (a^\dagger)^n |0\rangle,$$

that is, the eigenbasis of the number operator $a^\dagger a$, which coincides with the complexity operator $K$. We note that in this case the Krylov space has infinite dimension. From the well known relations

$$a^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle, \quad a|n\rangle = \sqrt{n} |n-1\rangle,$$

one can see that $b_n = \omega \sqrt{n}$. The time-evolved operator $|O(t)\rangle$ can be represented as the standard coherent state

$$|\xi\rangle = D(\xi) |0\rangle = e^{-|\xi|^2/2} \sum_{n=0}^{\infty} \frac{\xi^n}{\sqrt{n!}} |n\rangle$$

for $\xi = i\omega t$. Therefore, the components of the operator wavefunction are

$$\phi_n(t) = e^{-|\xi|^2/2} (\omega t)^n \sqrt{n!},$$

from which we can compute that

$$K(t) = (\Delta K)^2 = \omega^2 t^2,$$

We thus conclude that, being $b_1 = \omega$, the dispersion bound is always saturated: that is, $|\partial K| < 2b_1 \Delta K \forall t$. This model provides an example in which maximal complexity growth (in the sense of saturation of our bound) is achieved, while the conjecture by Parker et al. [42], i.e. linear growth of Lanczos coefficients, does not hold. We therefore see that the two notions of maximal complexity growth are not equivalent.

### C.3 SYK model

Finally, let us consider the celebrated prototype for quantum chaos: the SYK model of $N$ Majorana fermions with $q$-body interaction, given by the Hamiltonian

$$H_{SYK}^{(q)} = \frac{g}{2} \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq N} J_{i_1 \cdots i_k} \gamma_{i_1} \cdots \gamma_{i_k}.$$  

In the large-$N$ limit the model can be solved analytically and, for asymptotically large $q$, has been proven to obey the universal growth hypothesis by Parker et al. [42]: namely, the growth of the Lanczos coefficients is asymptotically linear in $n$, resulting in the exponential time-behaviour of Krylov complexity. More precisely, it can be shown that, in this limit, the SYK belongs to a family of exact solutions with Lanczos coefficients [42]

$$b_n = \omega \sqrt{n(n - 1 + \eta)}$$

and amplitudes

$$\phi_n(t) = \sqrt{\frac{(\eta)n}{n!}} \tanh^n(\omega t) \text{sech}^n(\omega t),$$

where $(\eta)n = \eta(\eta + 1) \cdots (\eta + n - 1)$ is the Pochhammer symbol. From these amplitudes one can extract the complexity $K(t) = \eta \sinh^2(\omega t)$, which, as expected from the asymptotic linear behaviour of the Lanczos coefficients, shows an asymptotic exponential growth. Remarkably, the linear growth of the Lanczos coefficients is a sufficient (but not necessary, as shown above) condition for the saturation of the dispersion bound on complexity, as shown in figure 2. This saturation is due to the presence of an underlying complexity algebra: indeed, by substituting the expression (41) for the Lanczos coefficients into Eq. (20), one can see that $f(n)$ becomes constant, meaning that the commutators (18) and (19) close the algebra. Indeed, one of the main results of our work is the proof that the closure of the complexity algebra is both a sufficient and a necessary condition for the dispersion bound to be saturated. For this particular family of solutions, the underlying algebra is that of $SL(2, \mathbb{R})$ [46].
D  Equivalence between the dispersion bound and the complexity algebra being closed

The right-hand side of the dispersion bound is equal to two times the norm of the vectors \((K - \langle K \rangle)^t |O(t)\) and \((L - \langle L \rangle)^t |O(t)\), while the left-hand side is obtained by applying the Cauchy-Schwarz inequality. From this, it is clear that the bound is saturated if and only if the two vectors are linearly dependent. In other words, the bound is saturated if and only if the vectors \((K - K)^t |O(t)\) and \(L^t |O(t)\) are linearly dependent, where we have chosen to suppress the time dependence of \(K\). What will follow is a series of steps proving that the complexity algebra being closed is both necessary and sufficient for the vectors \((K - K)^t |O(t)\) and \(L^t |O(t)\) to be linearly dependent. Said differently, the complexity algebra being closed is equivalent to the dispersion bound being saturated. When carrying out the proofs, we will use the convention that \(b_0 = 0\) and for finite Krylov dimension \(D\), we will also introduce \(b_D = 0\). For any superoperator \(M\) we will write \(M_{n,m} \equiv |O_n| M |O_m\), where \(M_{n,m}\) can be thought of as the entries of a matrix representing \(M\).

D.1  Proving necessity

Linear dependence between \((K - K)^t |O(t)\) and \(L^t |O(t)\) is equivalent with linear dependence between \(e^{-itL}(K - K)e^{itL}|O\) and \(|O_1\). To simplify, we will use the notation \(L^n\) to mean \([L, \cdot] applied to \(K\) \(n\) times. By Taylor expanding the vector \(e^{-itL}(K - K)e^{itL}|O\) at \(t = 0\), we have that

\[
e^{-itL}(K - K)e^{itL}|O| = \left(-K + \sum_{n=1}^{\infty} \frac{1}{n!}(-i)^n L^n t^n\right)|O|.
\]

(43)

It is clear that \(L = L_+ - L_-\) while \(L^2 = 2[L_+, L_-] \) is diagonal in the Krylov basis with eigenvalues \((L^2)_{n,n} = -2(b_{n+1}^2 - b_n^2)\). Applying \([L, \cdot]\) once more, one finds that \(L^3\) consists only of a subdiagonal and superdiagonal with values given by \((L^3)_{n+1,n} = -(L^2)_{n,n+1} = -2b_n f(n)\), where \(f\) is the discrete function defined by \(f(n) = (b_{n+2}^2 - b_{n+1}^2) - (b_n^2 - b_{n+1}^2)\). By the \(k\)-diagonal of a matrix, we mean the diagonal of the matrix going top-left to bottom-right direction where \(k\) is an offset from the main diagonal. We use the convention that \(k = 0\) is the main diagonal while \(k = 1\) and \(k = -1\) are the superdiagonal and subdiagonal respectively, and so on. From the form of \(L^3\), it should be clear that \(k\)-diagonals of \(L^{n+3}\) for which \(|k| > 1 + n\) must only consist of zero-valued entries. Consequently, we must have that \((L^{n+4})_{n+m+2,m} = [L_+, L^{n+3}]_{n+m+2,m}\) which more explicitly can be written as the recursion relation \((L^{n+4})_{n+m+2,m} = b_{n+m+3} (L^{n+3})_{n+m+1,m} - b_{m+1} (L^{n+3})_{n+m+2,m+1}\). To simplify some notation, we will write \(L(n,m) \equiv (L^{n+4})_{n+m+2,m}\) and the recursion relation can then be written as \(L(n,m) = b_{n+m+2} L(n-1,m) - b_{m+1} L(n-1,m+1)\) for \(n > 0\).
We now observe that the following proposition must be true:

**Proposition 1.** The condition: \( L(n, 0) = 0 \ \forall \ \ 0 \leq n \leq D - 3 \), is a necessary condition for the vector \( e^{-it\mathcal{L}(\mathcal{K} - K)}e^{it\mathcal{L}(\mathcal{O})} \) to be linearly dependent of \( |\mathcal{O}_1\rangle \), and therefore, a necessary condition for the dispersion bound to be satisfied.

By applying \([\mathcal{L}, \cdot]\) to \( L(s) \), one finds that \( L(0, m) = 2b_{m+1}b_{m+2}g(m) \), where we have defined \( g(m) = f(m) - f(m + 1) \). We will show that the condition \( L(n, 0) = 0 \ \forall \ \ 0 \leq n \leq D - 3 \) is equivalent to the complexity algebra being closed. Together with Proposition 1, this would then prove that the algebra being closed is a necessary condition for saturation of the dispersion bound. In order to prove this however, we will first prove another proposition.

**Lemma 2.** Consider the discrete function \( L(n, m) \) where \( 0 \leq n \leq D - 3 \) and \( 0 \leq m \leq D - 1 \). The recursion relation \( L(n, m) = b_{n+m+2}L(n-1, m) - b_{m+1}L(n-1, m+1) \) together with the initial condition \( L(0, m) = 2b_{m+1}b_{m+2}g(m) \) implies:

\[
L(n, m) = 2 \prod_{j=1}^{n+m} b_{j} \sum_{k=0}^{n} (-1)^k \binom{n}{k} g(m + k).
\]  

(44)

**Proof.** We prove this by using mathematical induction. For the base case we have that

\[
L(1, m) = b_{m+3}L(0, m) - b_{m+1}L(0, m + 1)
= b_{m+3}(2b_{m+1}b_{m+2}g(m)) - b_{m+1}(2b_{m+2}b_{m+3}g(m + 1))
= 2b_{m+1}b_{m+2}b_{m+3}(g(m) - g(m + 1)).
\]  

(45)

For the inductive step we have

\[
L(n, m) = b_{n+m+2}L(n-1, m) - b_{m+1}L(n-1, m+1)
= 2 \prod_{j=n+m+2}^{n+1} b_{j} \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} g(m + k) - \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} g(m + 1 + k)
= 2 \prod_{j=n+m+2}^{n+1} b_{j} \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} g(m + k) + \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} g(m + k)
= 2 \prod_{j=n+m+2}^{n+1} b_{j} \left( g(m) + (-1)^n g(m + n) + \sum_{k=1}^{n-1} (-1)^k \binom{n-1}{k-1} + \binom{n-1}{k} \right) g(m + k)
= 2 \prod_{j=n+m+2}^{n+1} b_{j} \sum_{k=1}^{n} (-1)^k \binom{n}{k} g(m + k),
\]

where, in obtaining the second last line, we have made use of the binomial identity \( \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \).

**Corollary 3.** \( L(n, 0) = 2 \prod_{j=2}^{n} b_{j} \sum_{k=0}^{n} (-1)^k \binom{n}{k} g(k) \).

**Proposition 4.** \( L(n, 0) = 0 \ \forall \ \ 0 \leq n \leq D - 3 \Leftrightarrow g(n) = 0 \ \forall \ \ 0 \leq n \leq D - 3 \).

**Proof.** We can think of the set of functions \( g(n) \) as spanning a subset of \( \mathbb{R}^{D-3} \). It should then be clear from Corollary 3 that the set of functions \( L(n, 0) \) must then have the same span. This means that we can express each \( g(n) \) as a linear combination of the functions \( L(n, 0) \) or vice versa. Equating each function \( L(n, 0) \) (\( g(n) \)) with zero then results in \( g(n) = 0 \) (\( L(n, 0) = 0 \)) for all \( 0 \leq n \leq D - 3 \).

We are now ready to prove the following proposition:

**Proposition 5.** The saturation of the dispersion bound implies that the complexity algebra is closed.

**Proof.** We have that \( \mathcal{B} \equiv \mathcal{L} \) and \( \mathcal{K} \equiv \mathcal{L}^2 \) and the complexity algebra is closed per definition if and only if \( \mathcal{L}^3 = [\mathcal{L}, \mathcal{K}] \) can only be written as a linear combination of \( \mathcal{L} \), \( \mathcal{B} \) and \( \mathcal{K} \). It should be clear that this is possible if and only if \( f(n) = C \ \forall \ \ 0 \leq n \leq D - 2 \), where \( C \in \mathbb{R} \). This is clearly equivalent to the condition \( g(n) = 0 \ \forall \ \ 0 \leq n \leq D - 3 \), which together with Proposition 1 and 4 is implied by saturation of the dispersion bound.
D.2 Proving sufficiency

As we pointed out in the proof of Proposition 5, the complexity algebra being closed is equivalent with \( f(n) = C \forall 0 \leq n \leq D - 2 \), where \( C \in \mathbb{R} \). We note that

\[
 f(n) = C \forall 0 \leq n \leq D - 2 \iff 2(b_{n+1}^2 - b_n^2) = \alpha n + \gamma \ \forall 0 \leq n \leq D - 1 \tag{47}
\]

\[
 \iff b_n = \sqrt{\frac{1}{4} \alpha n(n-1) + \frac{1}{2} \gamma n + \delta} \ \forall 0 \leq n \leq D, \tag{48}
\]

where \( \alpha, \gamma \) and \( \delta \) are real constants and \( C = \frac{1}{4} \alpha \). We stress that (48) holds under the convention that \( b_0 = 0 \), and we note that this implies that \( \delta = 0 \) and so we must have

\[
 b_n = \sqrt{\frac{1}{4} \alpha n(n-1) + \frac{1}{2} \gamma n} \ \forall 1 \leq n \leq D. \tag{49}
\]

The right hand side of the equivalence sign in (47) is equivalent to \( \bar{K} = \alpha K + \gamma \). Consequently, the closed complexity algebra is entirely determined by the commutation relations \( [K, L] = B, [K, \mathcal{B}] = \mathcal{L} \) and \( [\mathcal{L}, \mathcal{B}] = \alpha K + \gamma \).

**Lemma 6.** The complexity algebra being closed implies that \( \mathbb{L}^{2n} = (-1)^n \frac{1}{\sqrt{\alpha}}(\sqrt{\alpha})^{2n} (\alpha K + \gamma) \) and \( \mathbb{L}^{2n+1} = (-1)^{n+1} \frac{1}{\sqrt{\alpha}}(\sqrt{\alpha})^{2n+1} B \) when \( \alpha \neq 0 \) and \( \mathbb{L}^2 = -\mathcal{B}, \mathbb{L}^3 = -\gamma \mathbb{L} \) and \( \mathbb{L}^n = 0 \) for \( n > 2 \) when \( \alpha = 0 \).

**Proof.** The case for \( \alpha = 0 \) is trivial while for \( \alpha \neq 0 \) we will use mathematical induction. For the base case we have \( \mathbb{L}^2 = [\mathcal{L}, [\mathcal{L}, K]] = -[\mathcal{L}, \mathcal{B}] = -\alpha (\alpha K + \gamma) \) and \( \mathbb{L}^3 = [\mathcal{L}, -\alpha (\alpha K + \gamma)] = \alpha \mathbb{B} \). For the inductive step, we have \( \mathbb{L}^{2n} = [\mathcal{L}, [\mathcal{L}, \mathbb{L}^{2(n-1)}]] = (-1)^n \frac{1}{\sqrt{\alpha}}(\sqrt{\alpha})^{2(n-1)}[\mathcal{L}, [\mathcal{L}, K]] = (-1)^n \frac{1}{\sqrt{\alpha}}(\sqrt{\alpha})^{2n} (\alpha K + \gamma) \) and \( \mathbb{L}^{2n+1} = [\mathcal{L}, \mathbb{L}^{2n}] = (-1)^{n+1} \frac{1}{\sqrt{\alpha}}(\sqrt{\alpha})^{2n+1} B \).

**Proposition 7.** The complexity algebra being closed implies that the dispersion bound is saturated.

**Proof.** By Lemma 6 we have \( \mathbb{L}^{2n} = (-1)^n \frac{1}{\sqrt{\alpha}}(\sqrt{\alpha})^{2n} (\alpha K + \gamma) \) and \( \mathbb{L}^{2n+1} = (-1)^{n+1} \frac{1}{\sqrt{\alpha}}(\sqrt{\alpha})^{2n+1} B \) when \( \alpha \neq 0 \). By substituting these into the Taylor expansion of \( e^{-it\mathcal{L}}(K - K)e^{it\mathcal{L}}\mathcal{O} \), we have

\[
 e^{-it\mathcal{L}}(K - K)e^{it\mathcal{L}}\mathcal{O} = \left( -K + \sum_{n=1}^{\infty} \frac{1}{n!}(-i)^n \mathbb{L}^n t^n \right)\mathcal{O} = \left( -K + \sum_{n=1}^{\infty} \frac{1}{n!}(-i)^n \mathbb{L}^n t^n \right)\mathcal{O} = \left( -K + \frac{\gamma}{\alpha} \left( \sum_{n=1}^{\infty} \frac{1}{n!}((\sqrt{\alpha} t)^{2n} + \frac{1}{\sqrt{\alpha}} iB \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}((\sqrt{\alpha} t)^{2n+1})\right)\mathcal{O} = \left( -K + \frac{\gamma}{\alpha} (\cosh \sqrt{\alpha} t - 1) + \frac{1}{\sqrt{\alpha}} iB \sinh \sqrt{\alpha} t\right)\mathcal{O}. \tag{50}\right.
\]

Since \( B\mathcal{O} = b_1\mathcal{O}_1 \), it follows from the definition of Krylov complexity that the first two terms in the expression above must cancel. We thus have that

\[
 e^{-it\mathcal{L}}(K - K)e^{it\mathcal{L}}\mathcal{O} = \frac{b_1}{\sqrt{\alpha}} \sinh \sqrt{\alpha} t\mathcal{O}_1. \tag{51}\]

When \( \alpha = 0 \), one has that \( \mathbb{L} = -\mathcal{B}, \mathbb{L}^2 = -\gamma \) and \( \mathbb{L}^n = 0 \) for \( n > 2 \). Substituting these into the Taylor expansion, one finds that

\[
 e^{-it\mathcal{L}}(K - K)e^{it\mathcal{L}}\mathcal{O} = \left( -K - iB t - \frac{1}{2} L^2 t^2 \right)\mathcal{O} = \left( -K + \frac{\gamma}{2} t^2 + iB t\right)\mathcal{O} = ib_1\mathcal{O}_1. \tag{52}\]

We thus have that the algebra being closed is a sufficient requirement for saturating the dispersion bound. \( \blacksquare \)
The proofs of Proposition 5 and 7 leads to the conclusion that saturation of the dispersion bound is equivalent with the complexity algebra being closed.

**Remark 8.** We would like to point out that equation (50) and (52) shows that the general solution for Krylov complexity, whenever the dispersion bound is saturated, is given by $K(t) = \frac{2\gamma}{\alpha} \sinh^2 \frac{\sqrt{\alpha} t}{2}$ when $\alpha > 0$, $K(t) = -\frac{2\gamma}{\alpha} \sin^2 \frac{\sqrt{-\alpha} t}{2}$ when $\alpha < 0$ and $K(t) = \frac{\gamma}{2} t^2$ when $\alpha = 0$.

**Remark 9.** The requirement that $b_n \geq 0$ for all $n$ implies that $\gamma \geq 0$ and $\alpha \geq -\frac{2}{n-1} \gamma$ for all $n$. In the infinite dimensional case we see that this implies that $\alpha \geq 0$. In the finite-dimensional case, the condition $b_D = 0$ implies that $\alpha = -\frac{2}{D-1} \gamma$ and so the solution of Krylov complexity only depends on $\gamma$, namely $K(t) = (D-1) \sin^2 \sqrt{\frac{\gamma}{2(D-1)}} t$. If we put $\omega = \sqrt{\frac{\gamma}{2(D-1)}}$ then $K(t) = (D-1) \sin^2 \omega t$ and the Lanczos coefficients grow according to $b_n = \omega \sqrt{n(D-n)}$ for the finite dimensional case.