Nonlocal elliptic problems with nonlinear argument transformations near the points of conjugation

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Abstract

We consider elliptic equations of order $2m$ in a domain $G \subset \mathbb{R}^n$ with nonlocal conditions that connect the values of the unknown function and its derivatives on $(n - 1)$-dimensional submanifolds $\Upsilon_i$ (where $\bigcup_i \Upsilon_i = \partial G$) with the values on $\omega_{is}(\Upsilon_i) \subset G$. Nonlocal elliptic problems in dihedral angles arise as model problems near the conjugation points $g \in \Upsilon_i \cap \Upsilon_j \neq \emptyset$, $i \neq j$. We study the case where the transformations $\omega_{is}$ correspond to nonlinear transformations in the model problems. It is proved that the operator of the problem remains Fredholm and its index does not change as we pass from linear argument transformations to nonlinear ones.

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Introduction

The first mathematicians who studied ordinary differential equations with nonlocal conditions were Sommerfeld [1], Tamarkin [2], Picone [3]. In 1932, Carleman [4] considered the problem of finding a holomorphic function in a bounded domain $G$, satisfying the following condition: the value of the unknown function at each point $x$ of the boundary is connected with the value at $\omega(x)$, where $\omega(\omega(x)) = x$, $\omega(\partial G) = \partial G$. Such a statement of the problem originated further investigations of

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nonlocal elliptic problems with the shifts mapping the boundary onto itself. In 1969, Bitsadze and Samarskii [5] considered essentially different type of nonlocal problems. They studied the Laplace equation in a bounded domain $G$ with the boundary-value condition connecting the values of the unknown function on a manifold $\mathcal{Y}_1 \subset \partial G$ with the values on some manifold inside $G$; on the set $\partial G \setminus \mathcal{Y}_1$ the Dirichlet condition was imposed. In a general case, such a problem was formulated as an unsolved one.

The most difficult situation in the theory of nonlocal problems is that where the support of nonlocal terms intersects with the boundary of domain. We consider the following example. Let $G \subset \mathbb{R}^n$ $(n \geq 2)$ be a bounded domain with the boundary $\partial G = \mathcal{Y}_1 \cup \mathcal{Y}_2 \cup \mathcal{K}_1$, where $\mathcal{Y}_i$ are smooth open (in the topology of $\partial G$) $(n - 1)$-dimensional $C^\infty$-manifolds, $\mathcal{K}_1 = \overline{\mathcal{Y}_1} \cap \overline{\mathcal{Y}_2}$ is an $(n - 2)$-dimensional connected $C^\infty$-manifold without a boundary. (If $n = 2$, then $\mathcal{K}_1 = \{g_1, g_2\}$, where $g_1$, $g_2$ are the ends of the curves $\overline{\mathcal{Y}_1}$, $\overline{\mathcal{Y}_2}$.) Suppose that, in a neighborhood of each point $g \in \mathcal{K}_1$, the domain $G$ is diffeomorphic to some $n$-dimensional dihedral angle (plain angle if $n = 2$).

In the domain $G$, we consider the nonlocal problem

\begin{align*}
\Delta u &= f_0(y) \quad (y \in G), \quad (0.1) \\
 u|_{\mathcal{Y}_i} - b_i u(\omega_i(y))|_{\mathcal{Y}_i} &= 0 \quad (i = 1, 2). \quad (0.2)
\end{align*}

Here $b_1, b_2 \in \mathbb{R}$; $\omega_i$ is an infinitely differentiable transformation mapping some neighborhood $O_i$ of the manifold $\mathcal{Y}_i$ onto the set $\omega(O_i)$ so that $\omega_i(\mathcal{Y}_i) \subset G$, $\omega_i(\mathcal{Y}_i) \cap \partial G \neq \emptyset$, see figures 0.1.a and 0.1.b.

Figure 0.1: The domain $G$ with the boundary $\partial G = \overline{\mathcal{Y}_1} \cup \overline{\mathcal{Y}_2}$ for $n = 2$.

Problems of type (0.1), (0.2) were considered by many mathematicians (see [6, 7, 8] and others). The most complete theory for such problems is developed by Skubachevskii and his pupils [9, 10, 11, 12, 13, 14]. In particular, Fredholm solvability of higher-order elliptic equations with general nonlocal conditions is proved, asymptotics for solutions near the points of conjugation of nonlocal conditions is established, smoothness of solutions is studied. It is shown [15] that the index of nonlocal problem is equal to the index of the corresponding local one if the support of nonlocal terms does not intersect with the points of conjugation (see Fig. 0.1.a with $g_1$ and $g_2$ being the points of conjugation). Otherwise (see Fig. 0.1.b), this is not true.

Properties of nonlocal problems in bounded domains are essentially determined by properties of model nonlocal problems in dihedral (plain if $n = 2$) angles $\Omega = \{x = (y, z) \in \mathbb{R}^n : b' < \varphi < b''\}$. The main result of this approach is that the index of nonlocal problem is equal to the index of the corresponding local one if the support of nonlocal terms does not intersect with the points of conjugation (see Fig. 0.1.a with $g_1$ and $g_2$ being the points of conjugation). Otherwise (see Fig. 0.1.b), this is not true.
corresponding to the points of conjugation of nonlocal conditions \( ((φ, r) \) are polar coordinates of \( y ))

Until now [9, 10, 11], it was studied the case where the transformations \( ω_{is} \) corresponded to linear transformations (i.e., compositions of rotation and expansion in \( y \)-plane) in model problems. However, such a restriction is quite unnatural in applications. Let us explain this on examples. Problem of type \((0.1), (0.2)\) is a mathematical model for some plasma process in a bounded domain [16]. Nonlocal conditions connect the plasma temperature on the boundary of the domain with the temperature inside the domain and at other points of the boundary.

Another important application arises in the theory of diffusion processes. Such processes describe, for example, the Brownian motion of a particle in the membrane \( G ⊂ \mathbb{R}^n \). It is known [17, 18, 19] that every diffusion process generates some Feller semigroup. By virtue of the Hille–Iosida theorem, the investigation of this semigroup may be reduced to the study of an elliptic operator with boundary-value conditions containing an integral over \( \bar{G} \) with respect to a non-negative Borel measure [20]. In the most difficult case where the measure is atomic, nonlocal conditions assume the form \((0.2)\). Their probabilistic sense is as follows: once the particle gets to a point \( y ∈ \Upsilon_i \), it either jumps to the point \( ω_i(y) \) with probability \( b_i (0 ≤ b_i ≤ 1) \) or “dies” with probability \( 1 − b_i \) (in this case, the process terminates). In general, both in the plasma theory and theory of diffusion processes, nonlinear argument transformations appear.

Let us mention one more application of nonlocal problems. In the monograph [21], it is shown that in some cases a boundary-value problem for elliptic differential-difference equation (in particular, arising in modern aircraft technology and modelling sandwich shells and plates [22, 21]) can be reduced to an elliptic equation with nonlocal conditions on shifts of the boundary. Thus, we again obtain nonlinear transformations. (These transformations are linear only if the boundary of domain coincides, on certain sets, with \((n − 1)\)-dimensional hyperplanes.)

Other applications and references to papers devoted to nonlocal problems can be found in [21].

In this paper, we consider an elliptic \( 2m \)-order equation in a domain \( G ⊂ \mathbb{R}^n \) with nonlocal conditions connecting the values of the unknown function and its derivatives on \((n − 1)\)-dimensional manifolds \( Υ_i \) (where \( \bigcup_{i} \bar{Υ}_i = \partial G \)) with the values on \( ω_{is}(Υ_i) ⊂ G \). As we mentioned before, the essential difficulties arise in the case where the support of nonlocal terms \( \bigcup_{i,s} \bar{ω}_{is}(Υ_i) \) intersects with the boundary of domain. In this situation, the generalized solutions may have power singularities near some set [9]. (For example, in case of problem \((0.1), (0.2)\), these singularities may appear near the points \( g_1 \) and \( g_2 \).) Therefore, it is natural to consider such problems in weighted spaces. This allows one to investigate higher-order elliptic equations with general nonlocal conditions. We study the case where the transformations \( ω_{is} \) correspond to nonlinear transformations in model problems. It turns out that the problem with nonlinear transformation is neither a small nor compact perturbation of the corresponding local problem. Nevertheless, we show that, when passing from linear transformations to nonlinear ones, the operator of the problem remains Fredholm and its index does not change.

Notice that a more general structure of the conjugation points and nonlocal terms for second-order elliptic equations with nonlocal perturbations of the Dirichlet problem was considered in [8]. This also justifies the importance of nonlinear transformations \( ω_{is} \). From our point of view, the advantage of the approach suggested is that it allows us to study \( 2m \)-order elliptic equations with general boundary-value conditions, nonlocal perturbations of which may be arbitrary large. On the other hand, this approach also allows us to investigate the asymptotic behavior of solutions near the conjugation points [9, 14].

Our paper is organized as follows. In § [11] we consider the statement of the problem and
discuss the conditions imposed on the argument transformations in nonlocal terms. Ibidem, we introduce basic functional spaces (Sobolev spaces with a weight) and obtain model problems in dihedral and plain angles. In §2 we give an example of nonlocal problem with nonlinear argument transformation and show that the operator corresponding to this problem is neither a small nor compact perturbation of the operator corresponding to the problem with linearized transformations. In §3 we study some properties of nonlinear transformations near the points of conjugation of nonlocal conditions and prove a number of lemmas which are used in §4 for getting a priori estimates of solutions. In §5 we construct a right regularizer, which, being combined with the a priori estimate, guarantees the Fredholm solvability of the nonlocal problem. Finally, in §6 we show that the index of the problem with nonlinear argument transformations is equal to that of the problem with the transformations linearized near the points of conjugation of nonlocal conditions.

1. Statement of the problem in a bounded domain

1. Let $G \subset \mathbb{R}^n$ $(n \geq 2)$ be a bounded domain with the boundary $\partial G = \bigcup_{i=1}^{N_0} \tilde{\mathcal{Y}}_i$, where $\mathcal{Y}_i$ are smooth open (in the topology of $\partial G$) $(n-1)$-dimensional $C^\infty$-manifolds. We assume that, in a neighborhood of each point $g \in \partial G \setminus \bigcup_{i=1}^{N_0} \mathcal{Y}_i$, the domain $G$ is diffeomorphic to some $n$-dimensional dihedral (plain if $n = 2$) angle $\Omega = \{x = (y, z) \in \mathbb{R}^n : 0 < b' < \varphi < b'' < 2\pi, \ z \in \mathbb{R}^{n-2}\}$, where $(\varphi, r)$ are polar coordinates of $y$.

We denote by $P(x, D)$ and $B_{i\mu s}(x, D)$ differential operators of order $2m$ and $m_{i\mu}$ respectively with complex-valued $C^\infty$-coefficients $(i = 1, \ldots, N_0; \mu = 1, \ldots, m; s = 0, \ldots, S_i)$. Let the operators $P(x, D)$ and $B_{i\mu 0}(x, D)$ satisfy the following conditions (see, for example, [23, Chapter 2, §1]).

**Condition 1.1.** For all $x \in \tilde{G}$, the operator $P(x, D)$ is properly elliptic.

**Condition 1.2.** For all $i = 1, \ldots, N_0$ and $x \in \tilde{\mathcal{Y}}_i$, the system $\{B_{i\mu 0}(x, D)\}_{\mu=1}^m$ covers the operator $P(x, D)$.

Let $\omega_{is}$ $(i = 1, \ldots, N_0; s = 1, \ldots, S_i)$ be an infinitely differentiable transformation mapping some neighborhood $\mathcal{O}_i$ of the manifold $\mathcal{Y}_i$ onto the manifold $\omega_{is}(\mathcal{O}_i)$ so that $\omega_{is}(\mathcal{Y}_i) \subset G$. We assume that the set

$$
\mathcal{K} = \left( \bigcup_{i} (\bar{\mathcal{Y}}_i \setminus \mathcal{Y}_i) \right) \cup \left( \bigcup_{i, s} \omega_{is}(\bar{\mathcal{Y}}_i \setminus \mathcal{Y}_i) \right) \cup \left( \bigcup_{j, p} \bigcup_{i, s} \omega_{jp}(\bar{\omega}_{is}(\bar{\mathcal{Y}}_i \setminus \mathcal{Y}_i) \cap \mathcal{Y}_j) \right)
$$

can be represented in the form $\mathcal{K} = \bigcup_{j=1}^3 \mathcal{K}_j$, where

$$
\mathcal{K}_1 = \bigcup_{p=1}^{N_3} \mathcal{K}_{1p} = \partial G \setminus \bigcup_{i=1}^{N_0} \mathcal{Y}_i, \quad \mathcal{K}_2 = \bigcup_{p=1}^{N_2} \mathcal{K}_{2p} \subset \bigcup_{i=1}^{N_0} \mathcal{Y}_i, \quad \mathcal{K}_3 = \bigcup_{p=1}^{N_3} \mathcal{K}_{3p} \subset G. \quad (1.1)
$$

Here $\mathcal{K}_{jp}$ are disjoint $(n-2)$-dimensional connected $C^\infty$-manifolds without a boundary (points if $n = 2$).
We consider the nonlocal boundary-value problem

\[ P(x, D)u = f_0(x) \quad (x \in G), \]  

\[ B_{i\mu}(x, D)u \equiv \sum_{s=0}^{S_i} (B_{i\mu s}(x, D)u)(\omega_{is}(x))|_{\gamma_i} = g_{i\mu}(x) \quad (x \in \gamma_i; \ i = 1, \ldots, N_0; \ \mu = 1, \ldots, m), \]

where \( (B_{i\mu s}(x, D)u)(\omega_{is}(x)) = B_{i\mu s}(x', D_{x'})u(x')|_{x' = \omega_{is}(x)}, \omega_{i0}(x) \equiv x. \)

**Example 1.1.** Let us consider problem \( (0.1), (0.2) \) in two-dimensional case, with the transformations \( \omega_i \) corresponding to Fig. 1.1. Then we have \( K_1 = \{g_1, g_2\}, K_2 = \{\omega_1(g_2)\}, \)

\[ K_3 = \{\omega_2(g_2), \omega_1(\omega_1(g_2))\}. \]

In [9], it is shown that the solutions for problem \( (1.2), (1.3) \) may have power singularities near the points of the set \( K_1 \). Therefore, it is natural to consider problem \( (1.2), (1.3) \) in weighted spaces. We introduce the space \( H_{b\nu}^l(Q) \) as a completion of the set \( C_0^\infty(\bar{Q} \setminus M) \) with respect to the norm

\[ \|u\|_{H_{b\nu}^l(Q)} = \left( \sum_{|\alpha| \leq l} \int_Q \rho^{2(b-l+|\alpha|)}|D^\alpha u|^2 dx \right)^{1/2}. \]

Here \( Q \) is the domain \( G \), angle \( \Omega \), or \( \mathbb{R}^n; M = K_1 \) if \( Q = G \) and \( M = \{x = (y, z) \in \mathbb{R}^n : y = 0, z \in \mathbb{R}^{n-2}\} \) if \( Q = \Omega \) or \( Q = \mathbb{R}^n; C_0^\infty(\bar{Q} \setminus M) \) is the set of infinitely differentiable functions with compact supports being subsets of \( \bar{Q} \setminus M; l \geq 0 \) is an integer; \( b \in \mathbb{R}; \rho = \rho(x) \in C^\infty(\mathbb{R}^n \setminus K_1) \) is a function\(^1\) satisfying \( c_1 \text{dist}(x, K_1) \leq \rho(x) \leq c_2 \text{dist}(x, K_1) \ (x \in G, c_1, c_2 > 0, \text{dist}(x, K_1) \) denotes

\(^1\)The existence of the function \( \rho(x) \) follows from Theorem 2 [24, Chapter 6, § 2].
the distance from $x$ to $K_1$) if $Q = G$ and $\rho(x) = |y|$ if $Q = \Omega$ or $Q = \mathbb{R}^n$. For $l \geq 1$, we denote by $H^{l-1/2}_b(\Upsilon)$ the space of traces on a smooth $(n - 1)$-dimensional manifold $\Upsilon \subset Q$ with the norm

$$\|\psi\|_{H^{l-1/2}_b(\Upsilon)} = \inf \|u\|_{H^l_b(Q)} \quad (u \in H^l_b(Q) : u|_\Upsilon = \psi).$$

We assume that $l + 2m - m_{i\mu} - 1 \geq 0$ for all $i, \mu$ and introduce the following bounded operator corresponding to nonlocal problem (1.2), (1.3):

$$L = \{ P(x, D), B_{i\mu}(x, D) \} : H^{l+2m}_b(G) \to H^l_b(G, \Upsilon) = H^l_b(G) \times \prod_{i=1}^{N_0} \prod_{\mu=1}^m H^{l+2m-m_{i\mu}-1/2}_b(\Upsilon_i).$$

From now on (unless the contrary is specified), we suppose that $b > l + 2m - 1$.

Let us explain the restriction on the exponent $b$. Suppose that the transformation $\omega_{is}$ takes a point $g \in \bar{\Upsilon}_i \cap K_1$ to the point $\omega_{is}(g)$ so that $\omega_{is}(g) \in K_2$ or $\omega_{is}(g) \in K_3$. Since the function $u(x)$ belongs to the Sobolev space $W^{l+2m}_2$ near the point $\omega_{is}(g)$, the function $u(\omega_{is}(x))$ belongs to the Sobolev space $W^{l+2m}_2$ near the point $g$. However, if $b \leq l + 2m - 1$, the function $u(\omega_{is}(x))$ does not belong (in general) to the weighted space $H^{l+2m}_b$. Therefore, the trace $(B_{i\mu}(x, D)u)(\omega_{is}(x))|_{\Upsilon_i}$ may not belong to the weighted space $H^{l+2m-m_{i\mu}-1/2}_b(\Upsilon_i)$, so the operator $L$ is not well defined. But if $b > l + 2m - 1$, then, by virtue of Lemma 5.2 [12], $W^{l+2m}_2(G) \subset H^{l+2m}_b(G)$. Thus, in this case, the operator $L$ is well defined.

Notice that, in two-dimensional case, problem (1.2), (1.3) can be considered in weighted spaces with arbitrary exponent $b$ (see [9]). To this end, one should impose some consistency conditions (generated by the transformations $\omega_{is}$); namely, one must assume that the solutions $u$ as well as the right-hand side $\{f_0, g_{ip}\}$ belong to the corresponding weighted spaces not only near the set $K_1$ but also near $K_2$ and $K_3$. One the one hand, this situation is in detail considered in [9] (where the problems with transformations linear near $K_1$ are studied). On the other hand, the changes described have nothing to do with the transformations $\omega_{is}$ near $K_1$. So, in two-dimensional case, we will omit the proofs of corresponding results concerning arbitrary values of $b$ (see the end of § 5).

2. Now we consider the structure of the transformations $\omega_{is}$ near the set $K_1$ in more detail. We denote by $\omega_{is}^{+1}$ the transformation $\omega_{is} : \Omega_i \to \omega_{is}(\Omega_i)$ and by $\omega_{is}^{-1} : \omega_{is}(\Omega_i) \to \Omega_i$ the transformation being inverse to $\omega_{is}$. Consider a point $g \in K_1$. The set of all points $\omega_{ip_1}^{+1}(\ldots \omega_{ij_{s_1}}^{+1}(g)) \in K_1$ ($1 \leq s_j \leq S_j, \ j = 1, \ldots, p$) (that is, points which can be obtained by consecutive applying to the point $g$ the transformations $\omega_{ip_1}^{+1}$ or $\omega_{ij_{s_1}}^{-1}$ taking the points from $K_1$ to those from $K_1$) is called an orbit of $g \in K_1$ and denoted by $\text{Orb}(g)$.

We introduce the set $S_{i\mu} = \{0 \leq s \leq S_i : \omega_{is}(\bar{\Upsilon}_i) \cap K_1 \neq \emptyset\}$. Evidently, $0 \in S_{i\mu}$. Let the following conditions hold.

**Condition 1.3.** For each $g \in K_1$

(a) the set $\text{Orb}(g)$ consists of finitely many points $g^j$ ($j = 1, \ldots, N = N(g)$); (b) for the points $g^j$, there are neighborhoods

$$\hat{V}(g^j) \subset V(g^j) \subset \mathbb{R}^n \setminus \left\{ \bigcup_{i, s} \omega_{is}(\bar{\Upsilon}_i) \cup K_2 \cup K_3 \right\} \quad (s \notin S_{i\mu})$$

such that (I) $V(g^j) \cap V(g^k) = \emptyset$ ($j \neq k$) and (II) if $g^j \in \bar{\Upsilon}_i$ and $\omega_{is}(g^j) = g^j$, then $V(g^j) \subset \Omega_i$ and $\omega_{is}(\hat{V}(g^j)) \subset V(g^k)$. 


Condition 1.4. For each \( g \in \mathcal{K}_j \) and \( j = 1, \ldots, N(g) \), there is a non-degenerate smooth transformation \( x \mapsto x'(g, j) \) mapping \( \mathcal{V}(g^j) \) \((\mathcal{V}(g^j))\) onto a neighborhood of the origin \( \mathcal{V}_j(0) \) \((\mathcal{V}_j(0))\) so that

(a) the images of the sets \( G \cap \mathcal{V}(g^j) \) \((G \cap \mathcal{V}(g^j))\) and \( \Upsilon_i \cap \mathcal{V}(g^j) \) \((\Upsilon_i \cap \mathcal{V}(g^j))\) are respectively the intersection of the dihedral angle \( \Omega_j = \{ x = (y, z) \in \mathbb{R}^n : 0 < b'_j < \varphi < b''_j < 2\pi, \ z \in \mathbb{R}^{n-2} \} \) with \( \mathcal{V}_j(0) \) \((\mathcal{V}_j(0))\) and the intersection of the side of the angle \( \Omega_j \) with \( \mathcal{V}_j(0) \) \((\mathcal{V}_j(0))\); 

(b) for \( x \in \mathcal{V}(g^j) \), the transformation \( \omega_{is}(x) \) \((\omega_{is}(x))\) in new coordinates has the form \( (y', z') \mapsto (\omega'_{is}(y', z'), z') \), where \( \omega'_{is}(y', z') = G'_{is} y' + o(|x'|) \) \((G'_{is} y' + o(|x'|))\) with \( G'_{is} \) being the operator of rotation by an angle \( \varphi'_{is} \) and expansion \( \chi'_{is} > 0 \) times in \( y' \)-plane; moreover, we assume that \( \omega'_{is}(0, z) \equiv 0 \); 

(c) in new coordinates, the operator \( G''_{is} \) \((G''_{is})\) maps the side of the corresponding angle \( \Omega_j \) \((\Omega_j)\) onto an \((n-1)\)-dimensional half-plane being strictly inside an angle \( \Omega_k \) \((\Omega_k)\) \((k = k(i, s) \text{ and } j \text{ can be different})\).

Conditions 1.3 and 1.4 are analogous to those in [9,11], where the transformations linear near \( \mathcal{K}_1 \) \((\mathcal{K}_1)\) \((\text{and arbitrary outside a neighborhood of } \mathcal{K}_1)\) are studied.

Condition 1.3 (a) is in a sense analogous to Carleman’s condition [4], which is used in the theory of nonlocal problems with transformations mapping the boundary of domain onto itself.

Condition 1.4 in particular, means that if \( g \in \omega_{is}(\bar{\Upsilon}_i \setminus \Upsilon_i) \cap \bar{\Upsilon}_j \cap \mathcal{K}_1 \neq \emptyset \), then the surfaces \( \omega_{is}(\bar{\Upsilon}_i) \) \((\omega_{is}(\bar{\Upsilon}_i))\) and \( \bar{\Upsilon}_j \) have different tangent planes at the point \( g \). The requirement that \( \omega'_{is}(0, z) \equiv 0 \) is necessary for representation 1.1 to be possible. If \( \omega_{is}(\bar{\Upsilon}_i \setminus \Upsilon_i) \subset \mathcal{G} \setminus \mathcal{K}_1 \), then, like in [9,11], we have no restrictions on a geometrical structure of \( \omega_{is}(\bar{\Upsilon}_i) \) near \( \partial \mathcal{G} \).

Remark 1.1. One can consider the more general case where, for \( x \in \bar{\mathcal{V}}(g^j) \), the transformation \( \omega_{is}(x) \) \((\omega_{is}(x))\) in new coordinates has the form \( (y', z') \mapsto (\omega''_{is}(y', z'), \omega''_{is}(y', z')) \), where \( \omega''_{is}(y', z') \) is the same as before, \( \omega''_{is}(y', z') = z' + o(|x'|) \), \( \omega''_{is}(0, z') \equiv z' \) \((\text{the latter guarantees that item (a) in Condition 1.3 holds})\). However, for simplicity, we study the transformations described in Condition 1.4.

3. Let us write model problems corresponding to the points of \( \mathcal{K}_1 \).

We fix a point \( g \in \mathcal{K}_1 \). Let \( \text{supp } u \subset \left( \bigcup_{j=1}^{N(g)} \mathcal{V}(g^j) \right) \cap \mathcal{G} \). We denote the function \( u(x) \) for \( x \in \mathcal{V}(g^j) \cap \mathcal{G} \) by \( u_j(x) \). If \( g^j \in \bar{\Upsilon}_i \), \( x \in \bar{\mathcal{V}}(g^j) \), \( \omega_{is}(x) \in \mathcal{V}(g^k) \), then we denote \( u(\omega_{is}(x)) \) by \( u_k(\omega_{is}(x)) \). Clearly, \( u(\omega_{i0}(x)) \equiv u(x) \equiv u_j(x) \). Now nonlocal problem (1.2), (1.3) assumes the form

\[
\mathbf{P}(x, D)u_j = f_0(x) \quad (x \in \bar{\mathcal{V}}(g^j) \cap \mathcal{G}),
\]

\[
\sum_{s \in S_{i1}} (B_{is}(x, D)u_k)(\omega_{is}(x))|_{\Upsilon_i} = g_{i\mu}(x)
\]

\((x \in \bar{\mathcal{V}}(g^j) \cap \Upsilon_i; \ i \in \{1 \leq i \leq N_0 : \bar{\mathcal{V}}(g^j) \cap \Upsilon_i \neq \emptyset\}; j = 1, \ldots, N = N(g); \mu = 1, \ldots, m\).

By virtue of Condition 1.4 in new coordinates the linear part \( G'_{is} \) of the transformation \( \omega'_{is} \) maps one of the sides of the \( \Omega_j \) \((j = j(i))\) onto an \((n-1)\)-dimensional half-plane being strictly inside \( \Omega_k \) \((k = k(i, s) \text{ and } j \text{ can be different})\). We denote all these \((n-1)\)-dimensional half-planes by \( \Gamma_{k1}, \ldots, \Gamma_{k,R_k} \subset \Omega_k \). \((\text{If none of the sides of the angles } \Omega_1, \ldots, \Omega_N \text{ is mapped inside } \Omega_k, \text{ we put } R_k = 1.) \) We also denote \( b_{k1} = b'_k, b_{k,R_k+1} = b''_k \). Then the sets

\[
\Gamma_{k\sigma} = \{ x = (y, z) \in \mathbb{R}^n : \varphi = b_{k\sigma}, \ z \in \mathbb{R}^{n-2} \} \quad (\sigma = 1, R_k + 1)
\]
are the sides of $\Omega_k$, while the half-planes $\Gamma_{kq}$ have the forms

$$
\Gamma_{kq} = \{ x = (y, z) \in \mathbb{R}^n : \varphi = b_{kq}, z \in \mathbb{R}^{n-2} \} \quad (q = 2, \ldots, R_k),
$$

where $0 < b_{k1} < \ldots < b_{k,R_k+1} < 2\pi$.

Let us introduce the function $U_j(x') = u_j(x'(x'))$ and denote $x'$ again by $x$. Then, by virtue of Conditions 1.3 and 1.4, problem (1.2), (1.3) eventually assumes the form

$$
\mathcal{P}_j(x, D_y, D_z)U_j = f_j(x) \quad (x \in \Omega_j),
$$

$$
\mathcal{B}_{j\sigma\mu}(x, D_y, D_z)U = \mathcal{B}_{j\sigma\mu}(x, D_y, D_z)U|_{\Gamma_j} + \sum_{k, q, s} (\mathcal{B}_{j\sigma\mu kq}(x, D_y, D_z)U_k)(\omega'_{j\sigma kq}(y, z), z)|_{\Gamma_j} = g_{j\sigma\mu}(x) \quad (x \in \Gamma_j).
$$

Here (and further, until the contrary is indicated) $j, k = 1, \ldots, N; \sigma = 1, R_j + 1; q = 2, \ldots, R_k; \mu = 1, \ldots, m; s = 1, \ldots, S_{j\sigma kq}$; $\mathcal{P}_j(x, D_y, D_z)$, $\mathcal{B}_{j\sigma\mu}(x, D_y, D_z)$, and $\mathcal{B}_{j\sigma\mu kq}(x, D_y, D_z)$ are operators of order $2m$, $m_{j\sigma\mu}$, and $m_{j\sigma\mu}$ respectively with variable $C^\infty$-coefficients; $\omega'_{j\sigma kq}(y, z) = \mathcal{G}_{j\sigma kq}(y, z)$ with $\mathcal{G}_{j\sigma kq}$ being the operator of rotation by an angle $\varphi_{j\sigma kq}$ and expansion $\chi_{j\sigma kq} > 0$ times in $y$-plane; furthermore, $\omega_{j\sigma kq}(0, z) \equiv 0, b_{k1} < \beta_{j\sigma} + \varphi_{j\sigma kq} = b_{kq} < b_{k,R_k+1}$.

Let us define the spaces of vector-functions:

$$
\mathcal{H}_b^{l+2m,N}(\Omega) = \prod_j \mathcal{H}_b^{l+2m}(\Omega_j), \quad \mathcal{H}_b^{l,N}(\Omega, \Gamma) = \prod_j \mathcal{H}_b^{l}(\Omega_j, \Gamma_j),
$$

$$
\mathcal{H}_b^{l}(\Omega_j, \Gamma_j) = \mathcal{H}_b^{l}(\Omega_j) \times \prod_{\sigma, \mu} \mathcal{H}_b^{l+2m-m_{j\sigma\mu}-1/2}(\Gamma_j).
$$

We introduce the bounded operators

$$
\mathcal{L}^\omega = \{ \mathcal{P}_j(D_y, D_z), \mathcal{B}_{j\sigma\mu}^\omega(D_y, D_z) \} : \mathcal{H}_b^{l+2m,N}(\Omega) \rightarrow \mathcal{H}_b^{l,N}(\Omega, \Gamma),
$$

$$
\mathcal{L}^G = \{ \mathcal{P}_j(D_y, D_z), \mathcal{B}_{j\sigma\mu}^G(D_y, D_z) \} : \mathcal{H}_b^{l+2m,N}(\Omega) \rightarrow \mathcal{H}_b^{l,N}(\Omega, \Gamma).
$$

Here

$$
\mathcal{B}_{j\sigma\mu}^\omega(D_y, D_z)U = B_{j\sigma\mu}(D_y, D_z)U|_{\Gamma_j} + \sum_{k, q, s} (B_{j\sigma\mu kq}(D_y, D_z)U_k)(\omega_{j\sigma kq}(y, z), z)|_{\Gamma_j},
$$

$$
\mathcal{B}_{j\sigma\mu}^G(D_y, D_z)U = B_{j\sigma\mu}(D_y, D_z)U|_{\Gamma_j} + \sum_{k, q, s} (B_{j\sigma\mu kq}(D_y, D_z)U_k)(\mathcal{G}_{j\sigma kq}(y, z), z)|_{\Gamma_j}
$$

with $\mathcal{P}_j(D_y, D_z)$, $B_{j\sigma\mu}(D_y, D_z)$, and $B_{j\sigma\mu kq}(D_y, D_z)$ being the principal homogeneous parts of the operators $\mathcal{P}_j(0, D_y, D_z)$, $B_{j\sigma\mu}(0, D_y, D_z)$, and $B_{j\sigma\mu kq}(0, D_y, D_z)$ respectively.

In what follows, we will write, for short, $\mathcal{P}_j$, $B_{j\sigma\mu}$, $B_{j\sigma\mu kq}$, $\mathcal{B}_{j\sigma\mu}^\omega$, and $\mathcal{B}_{j\sigma\mu}^G$ instead of $\mathcal{P}_j(D_y, D_z)$, $B_{j\sigma\mu}(D_y, D_z)$, $B_{j\sigma\mu kq}(D_y, D_z)$, $\mathcal{B}_{j\sigma\mu}^\omega(D_y, D_z)$, and $\mathcal{B}_{j\sigma\mu}^G(D_y, D_z)$ respectively.

Notice that the operator $\mathcal{B}_{j\sigma\mu}^\omega$ contains nonlocal terms with nonlinear transformations $\omega_{j\sigma kq}'$ while the operator $\mathcal{B}_{j\sigma\mu}^G$ with linear ones $\mathcal{G}_{j\sigma kq}$. Thus, the operators $\mathcal{L}^\omega$ and $\mathcal{L}^G$ correspond to model problems with nonlinear and linearized transformations respectively.

\footnote{In what follows, we consider functions $U_k$ with compact supports concentrated in a neighborhood of the origin and such that $(\omega_{j\sigma kq}'(y, z), z) \in \Omega_k$ for $x \in \text{supp } U_k$. This guarantees that the operators $\mathcal{B}_{j\sigma\mu}^\omega(D_y, D_z)$ are well defined.}
As we mentioned before, the problem with transformations linear near \( K_1 \) was studied in [9, 10, 11]. In particular, its Fredholm solvability was investigated. In §2 of the present paper, we will show that the operator \( \mathcal{L}^G \) is neither a small nor compact perturbation of \( \mathcal{L}^G \) even if the functions \( U \) with arbitrary small supports are considered. That is why, to prove the Fredholm solvability of problem (1.2), (1.3) with nonlinear transformations, we have to obtain anew a priori estimates under which the operator \( L \) show that the operator \( L \) and its Fredholm solvability was investigated. In §3, where \( K = \{ y \in \mathbb{R}^2 : 0 < b' < \varphi < b'' < 2\pi \} \), we introduce the space \( E_b^l(K) \) as a completion of \( C_0^\infty(K) \) with respect to the norm

\[
\| u \|_{E_b^l(K)} = \left( \sum_{|\alpha| \leq l} \int_K |y|^{2b}(|y|^{2(|\alpha|-l)} + 1)|D^\alpha_y u(y)|^2 \, dy \right)^{1/2}.
\]

For \( l \geq 1 \), we denote by \( E_b^{l-1/2}(\gamma) \) the space of traces on a ray \( \gamma \subset K \) with the norm

\[
\| \psi \|_{E_b^{l-1/2}(\gamma)} = \inf \{ u \|_{E_b^l(K)} : u|_{\gamma} = \psi \}.
\]

One can find the constructive definitions of the trace spaces \( H_b^{l-1/2}(\gamma) \) and \( E_b^{l-1/2}(\gamma) \), equivalent to the above, in [25, §1].

We introduce the spaces of vector-functions

\[
E_b^{l+2m,N}(K) = \prod_j E_b^{l+2m}(K_j), \quad \mathcal{E}_b^{l,N}(K, \gamma) = \prod_j \mathcal{E}_b^l(K_j, \gamma_j),
\]

\[
\mathcal{E}_b^l(K_j, \gamma_j) = E_b^l(K_j) \times \prod_\sigma \prod_\mu E_b^{l+2m-m_{\sigma \mu}-1/2}(\gamma_{j\sigma}),
\]

where \( K_j = \{ y \in \mathbb{R}^2 : b_j1 < \varphi < b_{j,R_j+1} \} \), \( \gamma_{j\sigma} = \{ y \in \mathbb{R}^2 : \varphi = b_{j\sigma} \} \).

We consider the bounded operator

\[
\mathcal{L}^G(\theta) = \{ P_j(D_y, \theta), B_{j\sigma\mu}(D_y, \theta) \} : E_b^{l+2m,N}(K) \to \mathcal{E}_b^{l,N}(K, \gamma),
\]

where \( \theta \) is an arbitrary point of the unit sphere \( S^{n-3} = \{ \theta \in \mathbb{R}^{n-2} : |\theta| = 1 \} \).

5. Let us write the operators \( P_j(D_y, 0), B_{j\sigma\mu}(D_y, 0) \) in polar coordinates: \( P_j(D_y, 0) = r^{-2m} \tilde{P}_j(\varphi, D_\varphi, r D_r) \), \( B_{j\sigma\mu}(D_y, 0) = r^{-m_{\sigma\mu}} \tilde{B}_{j\sigma\mu}(\varphi, D_\varphi, r D_r) \). We consider the analytic operator-valued function \( \tilde{\mathcal{L}}(\lambda) : W_2^{l+2m,N}(b_1, b_2) \to W_2^{l,N}[b_1, b_2] \) given by

\[
\tilde{\mathcal{L}}(\lambda) \tilde{U} = \{ \tilde{P}_j(\varphi, D_\varphi, \lambda) \tilde{U}_j, \tilde{B}_{j\sigma\mu}(\varphi, D_\varphi, \lambda) \tilde{U}_j(\varphi)|_{\varphi=b_{j\sigma}} + \sum_{k, q, s} e^{(i\lambda-m_{j\sigma\mu})\ln \chi_{j\sigma kqs}} \tilde{B}_{j\sigma\mu kqs}(\varphi, D_\varphi, \lambda) \tilde{U}_k(\varphi + \varphi_{j\sigma kqs})|_{\varphi=b_{j\sigma}} \},
\]
where

\[ W_2^{l+2m,N}(b_1, b_2) = \prod_j W_2^{l+2m}(b_{j1}, b_{j,R_j+1}), \quad W_2^l[b_1, b_2] = \prod_j W_2^l[b_{j1}, b_{j,R_j+1}], \]

\[ W_2^l[b_{j1}, b_{j,R_j+1}] = W_2^l(b_{j1}, b_{j,R_j+1}) \times \mathbb{C}^{2m}. \]

By virtue of Lemmas 2.1, 2.2 [10], there exists a finite-meromorphic operator-valued function \((\mathcal{L}^G)^{-1}(\lambda)\) such that \((\mathcal{L}^G)^{-1}(\lambda)\) is the inverse to \(\mathcal{L}^G(\lambda)\) if \(\lambda\) is not a pole of \((\mathcal{L}^G)^{-1}(\lambda)\); furthermore, for every pole \(\lambda_0\), there is a \(\delta > 0\) such that the set \(\{\lambda \in \mathbb{C} : 0 < |\text{Im}\, \lambda - \text{Im}\, \lambda_0| < \delta\}\) contains no poles of \((\mathcal{L}^G)^{-1}(\lambda)\).

If \(n = 2\), then, by Theorem 2.1 [10], the operator \(\mathcal{L}^G\) is an isomorphism if and only if the line \(\text{Im}\, \lambda = b + 1 - l - 2m\) contains no poles of \((\mathcal{L}^G)^{-1}(\lambda)\).

Suppose that \(n \geq 3\) and assume that the system \(\{B_{j\sigma\mu}(D_y, D_z)\}_{\mu=1}^m\) is normal on \(\Gamma_{j\sigma}\) and the orders \(m_{j\sigma\mu}\) of the operators \(B_{j\sigma\mu}(D_y, D_z)\) are less or equal to \(2m - 1\). In this case, by virtue of Theorem 9.1 [13], the operator \(\mathcal{L}^G(\theta)\) is Fredholm if and only if the line \(\text{Im}\, \lambda = b + 1 - l - 2m\) contains no poles of \((\mathcal{L}^G)^{-1}(\lambda)\). By Theorem 3.3 [10], if, in addition, \(\dim \ker (\mathcal{L}^G(\theta)) = \text{codim } \mathcal{R}(\mathcal{L}^G(\theta)) = 0\) for \(b\) replaced by \(b - l\), \(l\) replaced by \(0\), and all \(\theta \in S^{n-3}\), then the operator

\[ \mathcal{L}^G = \{\mathcal{P}_j(D_y, D_z), \mathcal{B}_{j\sigma\mu}(D_y, D_z)\} : H_b^{l+2m,N}(\Omega) \to H_b^l,N(\Omega, \Gamma) \]

is an isomorphism (see the corresponding example in [13, §10]). Notice that if \(\mathcal{L}^G\) is not an isomorphism, then \(\mathcal{L}^G(\theta)\) is not Fredholm (see Theorem 9.3 [13]).

Since the operators \(\mathcal{L}^\omega, \mathcal{L}^G, \mathcal{L}^G(\theta)\), and \(\mathcal{L}^G(\lambda)\) corresponding to problem \((1.4), (1.5)\) depend on the choice of \(g \in \mathcal{K}_1\), we denote them by \(\mathcal{L}^G_{\omega}, \mathcal{L}^G_g, \mathcal{L}^G_{\theta}, \mathcal{L}^G_{\lambda}\) respectively.

## 2 Example of nonlocal problem with nonlinear argument transformations

In this section, we show on a simple example that a problem with a transformation nonlinear in a neighborhood of \(\mathcal{K}_1\) is neither a small nor compact perturbation of the problem with the linearized transformation.

1. Let us assume for simplicity that problem \((1.2), (1.3)\) is considered in a plain domain. Let the model problem \((1.4), (1.5)\) corresponding to some point of \(\mathcal{K}_1\) have the form

\[ \Delta u = f(y) \quad (y \in K), \]

\[ u|_{\gamma_1} + u(\omega(y))|_{\gamma_1} = g_1(y) \quad (y \in \gamma_1), \]

\[ u|_{\gamma_2} = g_2(y) \quad (y \in \gamma_2). \]

Here \(K = \{y \in \mathbb{R}^2 : r > 0, |\varphi| < \pi/2\}\) is a plain angle (of opening \(\pi\)) with the sides \(\gamma_i = \{y \in \mathbb{R}^2 : r > 0, \varphi = (-1)^i\pi/2\} \quad (i = 1, 2)\). We suppose that \(\omega(y) = \mu(\mathcal{G}y)\), where \(\mathcal{G}\) is the operator of rotation by the angle \(\pi/2\) mapping \(\gamma_1\) onto a ray \(\gamma = \{y \in \mathbb{R}^2 : r > 0, \varphi = 0\}\);

\[ \mu : (y_1, y_2) \mapsto \left( \frac{y_1}{\sqrt{1 + y_1^2}}, \frac{y_2}{\sqrt{1 + y_1^2}} \right) \]

is an infinitely differentiable transformation mapping \(\gamma\) onto the curve \(\mu(\gamma)\), which is tangent to \(\gamma\) at the origin (see Fig. 2.1).
The operators $\mathcal{L}^\omega$, $\mathcal{L}^\varphi : H^{l+2}_b(K) \to H^l_b(K) \times \prod_{i=1}^2 H^{l+3/2}_b(\gamma_i)$ corresponding to the model problems with nonlinear and linearized transformations have the form

$$\mathcal{L}^\omega u = \{\triangle u, u|_{\gamma_1} + u(\omega'(y))|_{\gamma_1}, u|_{\gamma_2}\},$$

$$\mathcal{L}^\varphi u = \{\triangle u, u|_{\gamma_1} + u(\varphi y)|_{\gamma_1}, u|_{\gamma_2}\}.$$

Clearly, a non-zero component of the difference $\mathcal{L}^\varphi u - \mathcal{L}^\omega u$ is

$$u(\varphi y)|_{\gamma_1} - u(\omega'(y))|_{\gamma_1} = u(y)|_{\gamma} - u(\mu(y))|_{\gamma}.$$

We introduce the operator $A_\varepsilon : H^{l+2}_b(K) \to H^{l+3/2}_b(\gamma)$ with the domain $D(A_\varepsilon) = \{u \in H^{l+2}_b(K) : \text{supp } u \subset \{r < \varepsilon\} \cap \bar{K}\}$ given by

$$A_\varepsilon u(y) = u(y)|_{\gamma} - u(\mu(y))|_{\gamma}.$$

In this example, we prove that one cannot make the operator $A_\varepsilon$ small or compact, choosing sufficiently small $\varepsilon$. For simplicity, we show this in the case where $A_\varepsilon$ acts from $H^1_b(K)$ to $H^{1/2}_b(\gamma)$. The general case can be considered in the same way. We shall construct a sequence $u_\varepsilon \in D(A_\varepsilon)$, $\varepsilon \to 0$, such that

$$\|u_\varepsilon|_{\gamma} - u_\varepsilon(\mu(\cdot))|_{H^{1/2}_b(\gamma)} \geq c\|u_\varepsilon\|_{H^1_b(K)},$$

where $c > 0$ is independent of $\varepsilon$.

Let us write the restriction of $\mu$ on $\gamma$ in polar coordinates $(\varphi, r)$:

$$\mu|_{\gamma} : (0, r) \mapsto (\Phi(r), r),$$

where $\Phi(r) = \arctan r$. Clearly, $\Phi(0) = 0$, $\Phi(1) = \pi/4$, $\frac{1}{\sqrt{2}} \leq \frac{\Phi}{r}, \frac{\partial \Phi}{\partial r} \leq 1$ on $[0, 1]$.

Let us consider the transformation

$$\tilde{\mu} : (\varphi, r) \mapsto (\varphi + \Phi(r), r).$$
One can see that \( u(\mu(y))|_\gamma = u(\tilde{\mu}(y))|_\gamma \) since \( \mu|_\gamma = \tilde{\mu}|_\gamma \). Therefore, without loss of generality, we may assume that the transformation \( \mu \) is given by

\[
\mu : (\varphi, r) \mapsto (\varphi + \Phi(r), r).
\]

Notice that the norm of any function \( u \in H^1_h(K) \) written in polar coordinates is equivalent to

\[
\left( \sum_{|\alpha| \leq 1} \int_{-\pi/2}^{\pi/2} \int_{0}^{\infty} \left| r^{|\alpha|-1} (rD_r)^\alpha (D_\varphi)^\alpha u(\varphi, r) \right|^2 d\varphi dr \right)^{1/2}.
\]

Set \( r = e^{-t} \); then, in new coordinates, the transformation \( \mu \) assumes the form

\[
\mu : (\varphi, t) \mapsto (\varphi + \Phi(e^{-t}), t).
\]

Putting \( v(\varphi, t) = u(\varphi, e^{-t}) \), we see that the norm \( \| u \|_{H^1_h(K)} \) is equivalent to the norm

\[
\| v \|_{W^1_2(Q)} = \left( \sum_{|\alpha| \leq 1} \int_{-\pi/2}^{\pi/2} \int_{-\infty}^{\infty} e^{-2bt} |D_t^\alpha (D_\varphi)^\alpha v(\varphi, t)|^2 d\varphi dt \right)^{1/2},
\]

where \( Q = \{ t \in \mathbb{R}, |\varphi| < \pi/2 \} \) and \( W^1_{2,0}(Q) \) is the space with norm (2.1). Evidently, \( W^1_{2,0}(Q) \) coincides with the Sobolev space \( W^1_2(Q) \).

Since the norms \( \| v \|_{W^1_{2b}(Q)} \) and \( \| e^{-bt} v \|_{W^1_2(Q)} \) are equivalent, it suffices to study the case where \( b = 0 \). In what follows, we consider functions \( v(\varphi, t) \) with the support being a subset of the strip \( \{ |\varphi| \geq \pi/2 \} \). Putting \( v = 0 \) for \( |\varphi| \geq \pi/2 \), we obtain \( \| v \|_{W^1_2(Q)} = \| v \|_{W^1_2(\mathbb{R}^2)} \).

Thus, our task is reduced to constructing a sequence \( v_s \in W^1_2(\mathbb{R}^2) \) such that \( \text{supp} v_s \subset \{ t > 2s, |\varphi| < \pi/2 \} \) and

\[
\| v_s(0, t) - v_s(\Phi(e^{-t}), t) \|_{W^1_2(\mathbb{R}^2)} \geq c \| v_s \|_{W^1_2(\mathbb{R}^2)},
\]

where \( c > 0 \) is independent of \( s \).

To this end, we pass from variables \( (\varphi, t) \) to \( (\varphi, \tau) \): we introduce the sets

\[
Q_s = \left\{ |\theta| \leq \frac{\pi}{2}, 2s \leq \tau \leq 2s + 1 \right\}, \quad s = 0, 1, 2, \ldots,
\]

and put

\[
\varphi = F(\theta, \tau), \quad t = \tau.
\]

Here \( F(\theta, \tau) = \theta e^{2s} \Phi(e^{-\tau}) \) for \( (\theta, \tau) \in Q_s, s = 0, 1, 2, \ldots, \) and \( F(\theta, \tau) \) is extended onto \( \mathbb{R}^2 \setminus \bigcup_{s=0}^{\infty} Q_s \) so that the transformation (2.2) remains continuously differentiable with the Jacobian \( \frac{\partial F}{\partial \theta} \) such that

\[
0 < c_1 \leq \left| \frac{\partial F}{\partial \theta} \right| \leq c_2 \quad \text{on } \mathbb{R}^2.
\]

Such an extension does exist: indeed,

\[
\frac{\partial F}{\partial \theta} = e^{2s} \Phi(\tau), \quad \frac{\partial F}{\partial \tau} = -\theta e^{-\tau} + 2s \frac{d\Phi}{dr} \bigg|_{r=e^{-\tau}}, \quad (\theta, \tau) \in Q_s;
\]
therefore (by virtue of the above properties of $\Phi$), in $\bigcup_{s=0}^{\infty} Q_s$ the function $F(\theta, \tau)$ is continuously differentiable with respect to $\theta$ and $\tau$ and inequalities (2.3) hold.

One easily sees that, under change of variables (2.2), the segment $Q_s \cap \{\theta = 0\}$ is an image of the corresponding segment of the line $\{\varphi = 0\}$. Furthermore, the transformation $\mu$ on $Q_s$ has the form

$$\mu : (\theta, \tau) \mapsto (\theta + e^{-2s}, \tau), \quad (\theta, \tau) \in Q_s. \quad (2.4)$$

We consider functions $f, g \in C^\infty(\mathbb{R})$ such that $\text{supp} f \subset \{|\theta| < \frac{\pi}{2}\}$, $f(0) \neq f(1)$, $\text{supp} g \subset \{0 < \tau < 1\}$, $g(\tau) \neq 0$ and define the sequence $w_s(\theta, \tau) = f_s(\theta) g_s(\tau)$, where

$$f_s(\theta) = f(\theta e^{2s}), \quad g_s(\tau) = g((\tau - 2s) e^{2s}), \quad s = 0, 1, 2, \ldots$$

Clearly, $\text{supp} w_s \subset Q_s$ (see Fig. 2.2).

![Figure 2.2: The supports of $w_s$ are contained in the hatched domains.](image)

We have

$$\|w_s\|^2_{W_2^{1/2}(\mathbb{R}^2)} = \|f_s\|^2_{L_2(\mathbb{R})} \|g_s\|^2_{L_2(\mathbb{R})} + \left\| \frac{df_s}{d\theta} \right\|_{L_2(\mathbb{R})}^2 + \left\| \frac{dg_s}{d\tau} \right\|_{L_2(\mathbb{R})}^2 = e^{-4s} \|f\|^2_{L_2(\mathbb{R})} \|g\|^2_{L_2(\mathbb{R})} + \left\| \frac{df}{d\theta} \right\|_{L_2(\mathbb{R})}^2 + \left\| \frac{dg}{d\tau} \right\|_{L_2(\mathbb{R})}^2 \quad . \quad (2.5)$$

Analogously, using the fact that the norm in $W_2^{1/2}(\mathbb{R})$ is given by

$$\|g\|_{W_2^{1/2}(\mathbb{R})} = \left( \|g\|^2_{L_2(\mathbb{R})} + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{|g(\tau_1) - g(\tau_2)|^2}{|\tau_1 - \tau_2|^2} d\tau_1 d\tau_2 \right)^{1/2}$$
(see [26]) and the form (2.4) of the transformation $\mu$ in coordinates $(\theta, \tau)$, we get
\[ \|w_s|_{\theta=0} - w_s(\mu(\cdot))|_{\theta=0}\|_{W^{1/2}(\mathbb{R})}^2 = |f_s(0) - f_s(e^{-2s})|^2\|g_s\|_{W^{1/2}(\mathbb{R})}^2 \geq \]
\[ \geq |f(0) - f(1)|^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{|g(\tau_1) - g(\tau_2)|^2}{|\tau_1 - \tau_2|^2} d\tau_1 d\tau_2. \tag{2.6} \]

From (2.5) and (2.6), it follows that
\[ \|w_s|_{\theta=0} - w_s(\mu(\cdot))|_{\theta=0}\|_{W^{1/2}(\mathbb{R})}^2 \geq c\|w_s\|_{W^{1/2}(\mathbb{R}^2)}^2. \]

2. Using the sequence $w_s$, one can easily show that, for any $\varepsilon$, the operator $A_\varepsilon$ is not compact. Indeed, the sequence $w_s$ is bounded in $W^1_2(\mathbb{R}^2)$. However, one cannot choose from $w_s|_{\theta=0} - w_s(\mu(\cdot))|_{\theta=0}$ a subsequence convergent in $W^{1/2}_2(\mathbb{R})$, since, according to (2.6), for all natural $s \neq h$ the expression
\[ \|w_s|_{\theta=0} - w_s(\mu(\cdot))|_{\theta=0} - [w_h|_{\theta=0} - w_h(\mu(\cdot))|_{\theta=0}]\|_{W^{1/2}(\mathbb{R})} = \]
\[ = \|w_s|_{\theta=0} - w_s(\mu(\cdot))|_{\theta=0}\|_{W^{1/2}(\mathbb{R})} + \|w_h|_{\theta=0} - w_h(\mu(\cdot))|_{\theta=0}\|_{W^{1/2}(\mathbb{R})} \]
is bounded from below by a positive constant.

3 Argument transformations near the set $K_1$

From the results of § [2] it follows that, to prove the Fredholm solvability of the problem with transformations nonlinear near $K_1$, one has to obtain anew a priori estimates and construct the right regularizer. To this end, we start by studying some properties of the transformations $\omega_{is}$ near the set $K_1$.

We fix a point $g \in K_1$, make, for each $j = 1, \ldots, N = N(g)$, the change of variables $x \mapsto x^j(g, j)$, and consider the transformations $\omega_j^{(\varepsilon)}(y, z)$ for $(y, z) \in V_{\varepsilon_0}(0) = \{ x \in \mathbb{R}^n : |x| < \varepsilon_0 \}$. The number $\varepsilon_0$ is supposed to be small so that $V_{\varepsilon_0}(0) \subset \hat{V}_j(0)$, $j = 1, \ldots, N$. In the sequel, we shall impose some additional conditions on $\varepsilon_0$.

1. Before we proceed to study the transformations $\omega_{is}$, let us prove an auxiliary result, which will be used for proving a lemma on a representation of $\omega_{is}$ in polar coordinates (see Lemma 3.2).

**Lemma 3.1.** Let $h = h(r, z)$ be a function such that $|D_r^k D_z^\alpha h| \leq c_{k\alpha}$ for $r \geq 0$, $z \in \mathbb{R}^{n-2}$, $(r^2 + |z|^2)^{1/2} \leq \varepsilon_0$. Set $f(r, z) = r^{-1} h(r, z)$ for some $l \in \mathbb{N}$ and assume that $|f| \leq c$. Then $|D_r^k f| \leq c_k$ for $r \geq 0$, $z \in \mathbb{R}^{n-2}$, $(r^2 + |z|^2)^{1/2} \leq \varepsilon_0$, and any $k = 1, 2\ldots$

**Proof.** 1) First, we consider the case where $l = 1$, that is $f(r, z) = r^{-1} h(r, z)$. By Leibnitz’ formula, we have
\[ \frac{\partial^k f(r, z)}{\partial r^k} = \sum_{s=0}^{k} \frac{(-1)^s k!}{(k-s)!} r^{k-s} \frac{\partial^{k-s} h(r, z)}{\partial r^{k-s}}. \]
Expanding $\frac{\partial^{k-s} h}{\partial r^{k-s}}$ by the Taylor formula near $r = 0$ and using the boundedness of the derivatives of $h$, we obtain

$$
\frac{\partial^k f(r, z)}{\partial r^k} = \sum_{s=0}^{k} \frac{(-1)^s k!}{(k-s)! (s-p)!} \left[ \sum_{p=0}^{s} \frac{1}{p!} \frac{\partial^{k-s+p} h(0, z)}{\partial r^{k-s+p}} \right] r^p + \frac{\partial^{k+1} h}{\partial r^{k+1}}(\kappa_{rz} r, z) r^{s+1} = \sum_{s=0}^{k} \sum_{p=0}^{s} \frac{(-1)^s k!}{(k-s)! (s-p)!} \frac{\partial^{k-s+p} h}{\partial r^{k-s+p}}(0, z) r^{-s-1+p} + O(1),
$$

where $\kappa_{rz} \in (0, 1)$.

Putting $p' = s - p$ in the last sum and denoting $p'$ again by $p$, we get

$$
\frac{\partial^k f(r, z)}{\partial r^k} = \sum_{s=0}^{k} \sum_{p=0}^{s} \frac{(-1)^s k!}{(k-s)! (s-p)!} \frac{\partial^{k-p} h}{\partial r^{k-p}}(0, z) r^{-p-1} + O(1).
$$

Write the coefficient $a_p(z)$ at $r^{-p-1}$ on the right-hand side of the last identity:

$$
a_p(z) = \frac{\partial^{k-p} h(0, z)}{\partial r^{k-p}} \sum_{s=0}^{k} \frac{(-1)^s k!}{(k-s)! (s-p)!} = \frac{\partial^{k-p} h(0, z)}{\partial r^{k-p}} (0, z) (-1)^p \sum_{s=0}^{k-p} k(k-1) \cdots (k-(s+p)+1) \frac{1}{s!} (-1)^s, \quad p = 0, \ldots, k.
$$

Since $|r^{-1} h(r, z)| \leq c$ by assumption, we have $h(0, z) \equiv 0$; therefore, $a_k(z) \equiv 0$. On the other hand, notice that, for $0 \leq p < k$, we have

$$
0 = \frac{d^p}{dt^p}(t+1)^k \bigg|_{t=0} = \left( \sum_{s=0}^{k-p} k(k-1) \cdots (k-(s+p)+1) \frac{1}{s!} t^s \right) \bigg|_{t=1} = \sum_{s=0}^{k-p} k(k-1) \cdots (k-(s+p)+1) \frac{1}{s!} (-1)^s.
$$

Thus, $a_p(z) \equiv 0$ for all $p = 0, \ldots, k$, and the lemma is proved for $l = 1$.

2) For $l \geq 2$, we use the mathematical induction method. Let the lemma be true for $l = 1, \ldots, l_1 - 1$. We claim that it is true for $l = l_1$. We have $f = r^{-1} f_1$, where $f_1 = r^{-(l_1-1)} h$. Since $|f| \leq c$, it follows that $|f_1| \leq c$, and, therefore, by the inductive assumption (for $l = l_1 - 1$) the estimate $|D^k_r D^\alpha_z f_1| \leq c_{\alpha} k$ holds. Applying the inductive assumption once more (now, for $l = 1$), we get the conclusion of the lemma for $r^{-1} f_1$, that is, for $f = r^{-1} h$.

Now let us proceed to investigate the transformations $\omega_i$. The following lemma describes the structure of $\omega_i'_{j \sigma k q s}$ in cylindrical coordinates. Such a representation turns out to be convenient for the study of nonlocal problems in weighted spaces.

**Lemma 3.2.** For sufficiently small $\varepsilon_0$, the transformation $\omega_i'_{j \sigma k q s}(y, z)|_{r_j \sigma \cap W_0(0)}$ can be represented in polar coordinates in the form

$$
(b_{j \sigma}, r) \mapsto (b_{k q} + \Phi_{j \sigma k q s}(r, z), \chi_{j \sigma k q s} r + R_{j \sigma k q s}(r, z)) \quad \text{for} \ (r^2 + |z|^2)^{1/2} \leq \varepsilon_0,
$$

where
where $\Phi_{j\sigma kqs}(r, z)$, $R_{j\sigma kqs}(r, z)$ are infinitely differentiable functions such that

$$|\Phi_{j\sigma kqs}| \leq c_{\varepsilon_0}, \quad |R_{j\sigma kqs}| \leq c_{\varepsilon_0}r,$$  \hspace{1cm} (3.2)

$$|D^k_1 D^\alpha_2 \Phi_{j\sigma kqs}| \leq c_k, \quad |D^k_1 D^\alpha_2 (R_{j\sigma kqs}/r)| \leq c_k.$$. \hspace{1cm} (3.3)

Here $k + |\alpha| \geq 1$; $c$, $c_k > 0$ are independent of $\varepsilon_0$.

Proof. Let $\omega_{j\sigma kqs}(y, z) = (\omega_{j\sigma kqs}(y, z), \omega_{j\sigma kqs}(y, z))$. By condition 1.4 we have $\omega_{j\sigma kqs}(0, z) \equiv 0$ ($i = 1, 2$); therefore, the Taylor formula near $r = 0$ implies

$$\omega_{j\sigma kqs}(r \cos b_\sigma, r \sin b_\sigma, z) = \left( \frac{\partial \omega_{j\sigma kqs}}{\partial y_1}(0, z) \cos b_\sigma + \frac{\partial \omega_{j\sigma kqs}}{\partial y_2}(0, z) \sin b_\sigma \right) r + O(r^2). \hspace{1cm} (3.4)$$

Here $O(r^2)$ is a function with absolute values majorized by $cr^2$, where $c$ is independent of $r$ and $z$. (To verify this, one should write the remainder of the Taylor formula in Lagrange’s form and use smoothness of $\omega_{j\sigma kqs}$.) Expanding $\frac{\partial \omega_{j\sigma kqs}}{\partial y_1}(0, z)$ and $\frac{\partial \omega_{j\sigma kqs}}{\partial y_2}(0, z)$ by the Taylor formula near $z = 0$, from (3.4) we obtain

$$\omega_{j\sigma kqs} = \left( \frac{\partial \omega_{j\sigma kqs}}{\partial y_1}(0) \cos b_\sigma + \frac{\partial \omega_{j\sigma kqs}}{\partial y_2}(0) \sin b_\sigma \right) r + O(|z|) + O(r^2). \hspace{1cm} (3.5)$$

Notice that $\frac{\partial \omega_{j\sigma kqs}}{\partial y_1}(0) \cos b_\sigma + \frac{\partial \omega_{j\sigma kqs}}{\partial y_2}(0) \sin b_\sigma$ and $\frac{\partial \omega_{j\sigma kqs}^2}{\partial y_1}(0) \cos b_\sigma + \frac{\partial \omega_{j\sigma kqs}^2}{\partial y_2}(0) \sin b_\sigma$ are not simultaneously equal to zero. (This follows from non-degeneracy of the Jacobian of the transformation $(y, z) \mapsto (\omega_{j\sigma kqs}(y, z), z)$ at the origin.) For definiteness, we assume that

$$\frac{\partial \omega_{j\sigma kqs}}{\partial y_1}(0) \cos b_\sigma + \frac{\partial \omega_{j\sigma kqs}}{\partial y_2}(0) \sin b_\sigma \neq 0. \hspace{1cm} (3.6)$$

Hence, by virtue of (3.3),

$$\omega_{j\sigma kqs}^1 \neq 0 \quad \text{for} \quad (r^2 + |z|^2)^{1/2} \leq \varepsilon_0 \hspace{1cm} (3.7)$$

with $\varepsilon_0$ small enough, and the transformation $\omega_{j\sigma kqs}(r, z)$ in polar coordinates has the form

$$(b_\sigma, r) \mapsto \left( \arctan \frac{\omega_{j\sigma kqs}^2}{\omega_{j\sigma kqs}^1} + \pi l, \left( \sum_{i=1}^2 (\omega_{j\sigma kqs}^i)^2 \right)^{1/2} \right), \hspace{1cm} (3.8)$$

where $l = 0$ if $\omega_{j\sigma kqs}^1 > 0$ and $\omega_{j\sigma kqs}^2 \geq 0$, $l = 1$ if $\omega_{j\sigma kqs}^1 < 0$, $l = 2$ if $\omega_{j\sigma kqs}^1 > 0$ and $\omega_{j\sigma kqs}^2 < 0$.

From (3.3) and the Taylor formula, it follows that

$$\arctan \frac{\omega_{j\sigma kqs}^2}{\omega_{j\sigma kqs}^1} = \arctan \frac{\frac{\partial \omega_{j\sigma kqs}^2}{\partial y_1}(0) \cos b_\sigma + \frac{\partial \omega_{j\sigma kqs}^2}{\partial y_2}(0) \sin b_\sigma}{\frac{\partial \omega_{j\sigma kqs}^1}{\partial y_1}(0) \cos b_\sigma + \frac{\partial \omega_{j\sigma kqs}^1}{\partial y_2}(0) \sin b_\sigma} + O(|z|) + O(r),$$

$$\sqrt{\sum_{i=1}^2 (\omega_{j\sigma kqs}^i)^2} = \sqrt{\sum_{i=1}^2 \left( \frac{\partial \omega_{j\sigma kqs}^i}{\partial y_1}(0) \cos b_\sigma + \frac{\partial \omega_{j\sigma kqs}^i}{\partial y_2}(0) \sin b_\sigma \right)^2} + O(|z|) + O(r^2).$$
Setting 

\[ b_{kq} = \arctan \left( \frac{\partial \omega^2_{j\sigma(kq)}(0)}{\partial y_1} \cos b_{j\sigma} + \frac{\partial \omega^2_{j\sigma(kq)}(0)}{\partial y_2} \sin b_{j\sigma} + \pi l \right), \]

we get formula (3.1) and inequalities (3.2).

Let us prove the first inequality in (3.3). By (3.7), we have \( |\frac{\omega^2_{j\sigma(kq)}}{\omega^1_{j\sigma(kq)}}| \leq c \) for \( (r^2 + |z|^2)^{1/2} \leq \varepsilon_0 \). Therefore, by virtue of (3.1) and (3.8), it suffices to prove that the derivatives \( D_r^k D^\alpha_z \omega^2_{j\sigma(kq)} \) are bounded. Clearly, we have

\[ \frac{\omega^2_{j\sigma(kq)}}{\omega^1_{j\sigma(kq)}} = \frac{r^{-1} \omega^2_{j\sigma(kq)}}{r^{-1} \omega^1_{j\sigma(kq)}}. \]

From (3.5) and (3.6), it follows that \( r^{-1} \omega^1_{j\sigma(kq)} \neq 0 \) for \( (r^2 + |z|^2)^{1/2} \leq \varepsilon_0 \). Hence, it suffices to prove that

\[ |D_r^k D^\alpha_z (r^{-1} \omega^i_{j\sigma(kq)})| = |D_r^k (r^{-1} D^\alpha_z \omega^i_{j\sigma(kq)})| \leq c_{k\alpha}, \quad i = 1, 2. \]

But the function \( D^\alpha_z \omega^i_{j\sigma(kq)} \) is infinitely differentiable for \( (r^2 + |z|^2)^{1/2} \leq \varepsilon_0 \); furthermore, since \( \omega^i_{j\sigma(kq)}(0, z) \equiv 0 \), we have \( D^\alpha_z \omega^i_{j\sigma(kq)} = O(r) \). Therefore, \( |r^{-1} D^\alpha_z \omega^i_{j\sigma(kq)}| \leq c_\alpha \). Now the conclusion of the lemma follows from Lemma 3.1.

Similarly, one can prove the second inequality in (3.3). From (3.1) and (3.8), it follows that

\[ \frac{R_{j\sigma(kq)}(r, z)}{r} = \sqrt{\sum_{i=1}^{2} \frac{(\omega^i_{j\sigma(kq)})^2}{r^2}} - \chi_{j\sigma(kq)}. \]

By virtue of (3.5) and (3.6), we obtain \( \sum_{i=1}^{2} (\omega^i_{j\sigma(kq)})^2/r^2 \neq 0 \) for \( (r^2 + |z|^2)^{1/2} \leq \varepsilon_0 \); therefore, it suffices to prove that

\[ \left| D_r^k D^\alpha_z \sum_{i=1}^{2} (\omega^i_{j\sigma(kq)})^2/r^2 \right| \leq c_{k\alpha}. \]

But the function \( D^\alpha_z \sum_{i=1}^{2} (\omega^i_{j\sigma(kq)})^2 \) is infinitely differentiable for \( (r^2 + |z|^2)^{1/2} \leq \varepsilon_0 \); furthermore, since \( \omega^i_{j\sigma(kq)}(0, z) \equiv 0 \), we have \( D^\alpha_z \sum_{i=1}^{2} (\omega^i_{j\sigma(kq)})^2 = O(r^2) \). Hence, \( \left| D^\alpha_z \sum_{i=1}^{2} (\omega^i_{j\sigma(kq)})^2/r^2 \right| \leq c_\alpha \), and the conclusion of the lemma again follows from Lemma 3.1.

2. Denote \( \delta = \min\{b_{jq+1} - b_{jq}\}/2 \) (\( j = 1, \ldots, N; \quad q = 1, \ldots, R_j \)), \( d_1 = \min\{1, \chi_{j\sigma(kq)}\}/2 \), \( d_2 = 2 \max\{1, \chi_{j\sigma(kq)}\} \). Let \( \varepsilon_0 \) be so small that

\[ |\Phi_{j\sigma(kq)}| \leq \delta/2, \quad |R_{j\sigma(kq)}| \leq \chi_{j\sigma(kq)}r/2 \quad \text{for} \quad (r^2 + |z|^2)^{1/2} \leq \varepsilon_0/d_1. \]  

The existence of such an \( \varepsilon_0 \) follows from Lemma 3.2.
We introduce infinitely differentiable functions \( \zeta_{j\sigma,i}(\varphi), \zeta_{kq,i}(\varphi) \) such that
\[
\zeta_{j\sigma,i}(\varphi) = 1 \text{ for } |b_{j\sigma} - \varphi| \leq \delta/2^{i+1}, \quad \zeta_{j\sigma,i}(\varphi) = 0 \text{ for } |b_{j\sigma} - \varphi| \geq \delta/2^{i},
\]
(3.10)
i = 0, \ldots, 4. Clearly, \( \zeta_{kq,i}(\varphi) = 1 \) for \( |b_{kq} - \varphi| \leq \delta/2^{i+1} \), \( \zeta_{kq,i}(\varphi) = 0 \) for \( |b_{kq} - \varphi| \geq \delta/2^{i} \).

Let us consider the transformation \( \tilde{\omega}'_{j\sigma kq}(y, z) \) that are given in polar coordinates by
\[
(\varphi, r) \mapsto (\varphi + \varphi'_{j\sigma kq} + \Phi'_{j\sigma kq}(r, z), \chi_{j\sigma kq} r + R_{j\sigma kq}(r, z)).
\]
(3.11)
By virtue of Lemma 3.2, we have \( \tilde{\omega}'_{j\sigma kq}(y, z) \) in polar coordinates has the form
\[
(\varphi, r) \mapsto (\varphi + \Phi'(r, z), r + R'(r, z)),
\]
(3.12)
where \( \Phi'(r, z) = \Phi_{j\sigma kq}(\chi^{-1}_{j\sigma kq} r, z), R'(r, z) = R_{j\sigma kq}(\chi^{-1}_{j\sigma kq} r, z). \) It is easy to see that \( \Phi'_{j\sigma kq} \) and \( R'_{j\sigma kq} \) also satisfy inequalities (3.2), (3.3).

**Lemma 3.3.** For sufficiently small \( \varepsilon_0 \) and any \( W \in H^l_b(\Omega_k) \) with \( W \in \bar{\Omega}_k \cap \mathcal{V}_{\varepsilon_0}(0) \) we have \( \zeta_{kq,0} W \in H^l_b(\Omega_k) \) and
\[
\| \zeta_{kq,1} W \|_{H^l_b(\Omega_k)} \leq c \| W \|_{H^l_b(\Omega_k)},
\]
where \( c > 0 \) is independent of \( W \) and \( \varepsilon_0 \).

**Proof.** In the proof, we shall use the following obvious assertion:
\[
W \in H^l_b(\Omega_k) \iff D^a W \in H^0_{b+|a|+l}(\Omega_k), \ |a| \leq l.
\]
(3.13)
From formula (3.12) and inequalities (3.9), it follows that the transformation (3.12) maps \( \mathcal{V}_{\varepsilon_0}(0) \cap \{ x : |\varphi - b_{kq} | < \delta \} \cap \Omega_k \) into \( \Omega_k \) for \( q = 2, \ldots, R_k \). Furthermore, inequalities (3.2) and (3.3) imply that, for small \( \varepsilon_0 \), the absolute value of the Jacobian of transformation (3.12) is bounded and does not vanish in \( \mathcal{V}_{\varepsilon_0}(0) \cap \{ x : |\varphi - b_{kq} | < \delta \} \cap \Omega_k \). This proves the lemma for \( l = 0 \) and \( \zeta_{kq,0} \) substituted for \( \zeta_{kq,1} \).

Let us consider functions \( \zeta_{kq,0}^p \in C^\infty_0(\mathbb{R}) \) \( (p = 0, \ldots, l) \) such that \( \zeta_{kq,0}^0 = \zeta_{kq,0}, \zeta_{kq,0}^1 = \zeta_{kq,1} \), and \( \zeta_{kq,0}^{-1}(\varphi) = 1 \) for \( \varphi \in \supp \zeta_{kq,0}^p \). Let us assume that the lemma is true for \( l = p - 1 \) and \( \zeta_{kq,0}^{l-1} \) substituted for \( \zeta_{kq,1} \). We claim that it is true for \( l = p \) and \( \zeta_{kq,0}^p \) substituted for \( \zeta_{kq,1} \). Indeed, let \( W \in H^p_b(\Omega_k) \); then
\[
\frac{1}{r} \frac{\partial W}{\partial r} = \frac{\partial W}{\partial \varphi} \cdot (1 + \frac{R'_{j\sigma kq}}{r}),
\]
\[
\frac{\partial W}{\partial r} = \frac{\partial W}{\partial \varphi} \cdot (1 + \frac{R'_{j\sigma kq}}{r}) \cdot \frac{\partial \varphi'}{r} + \frac{\partial W}{\partial r} \cdot (1 + \frac{R'_{j\sigma kq}}{r}),
\]
(3.14)
Therefore, by the inductive assumption, we have
\[
\zeta_{kq,0}^{p-1} \frac{1}{r} \frac{\partial W}{\partial r} \in H^{p-1}_b(\Omega_k), \zeta_{kq,0}^p \frac{\partial W}{\partial \varphi} \in H^{p-1}_b(\Omega_k).
\]
From this, relations
\[
\frac{\partial W}{\partial r} = \frac{\partial W}{\partial \varphi} \cdot (1 + \frac{R'_{j\sigma kq}}{r}) \cdot \frac{\partial \varphi'}{r} + \frac{\partial W}{\partial r} \cdot (1 + \frac{R'_{j\sigma kq}}{r}),
\]
(3.14)
inequalities (3.2), (3.3), and Lemma 2.1 [27], we get
\[
\zeta_{kq,0}^{p-1} \frac{1}{r} \frac{\partial \hat{W}}{\partial \varphi}, \quad \zeta_{kq,0}^{p-1} \frac{\partial \hat{W}}{\partial r}, \quad \zeta_{kq,0}^{p-1} \frac{\partial \hat{W}}{\partial z_\xi} \in H_b^{p-1}(\Omega_k). \tag{3.15}
\]
Furthermore, the relation \( W \in H_b^p(\Omega_k) \), embedding \( H_b^p(\Omega_k) \subset H_b^{0}(\Omega_k) \), and the conclusion of the lemma for \( l = 0 \) imply \( \zeta_{kq,0}^p \hat{W} \in H_b^{0}(\Omega_k) \). From this, (3.13), and (3.15), it follows that \( D^\alpha (\zeta_{kq,0}^p \hat{W}) \in H_b^{0+|\alpha|-p}(\Omega_k), |\alpha| \leq p \). Once more using (3.13), we complete the proof.

Thus, we proved that the operator \( W \mapsto \zeta_{kq,0} \hat{W} \) is bounded in \( H_b^1(\Omega_k) \).

Lemma 3.4. For any \( W \in H_b^1(\Omega_k) \) with supp \( W \subset \Omega_k \cap \mathcal{V}_{\varepsilon_0}(0) \) and any multi-index \( \gamma, 1 \leq |\gamma| \leq l \), the following inequality holds:
\[
\| \zeta_{kq,2} D^\gamma \hat{W} - \zeta_{kq,2} \widehat{D^\gamma W} \|_{H_b^{l-|\gamma|}(\Omega_k)} \leq c \varepsilon_0 \| W \|_{H_b^1(\Omega_k)}, \tag{3.16}
\]
where \( q = 2, \ldots, R_k \); \( c > 0 \) is independent of \( W \) and \( \varepsilon_0 \).

**Proof.** We introduce functions \( \zeta_{kq,1}^p \in C_0^\infty(\mathbb{R}) \) (\( p = 1, \ldots, l \)) such that \( \zeta_{1}^{kq,1} = \zeta_{kq,1}, \zeta_{1}^{kq,2} = \zeta_{kq,2}, \) and \( \zeta_{kq,1}^{-1}(\varphi) = 1 \) for \( \varphi \in \text{supp} \zeta_{kq,1}^p \) (\( p = 2, \ldots, l \)).

Let \( |\gamma| = 1 \); then it suffices to prove inequality (3.16) for the case where the operator \( D^\gamma \) is replaced by \( \frac{1}{r} \frac{\partial}{\partial \varphi}, \frac{\partial}{\partial r}, \frac{\partial}{\partial z_\xi} \). Let us consider the operator \( \frac{1}{r} \frac{\partial}{\partial \varphi} \) (the other operators can be considered in the same way). Combining the first relation in (3.14) with Leibniz’ formula, we get
\[
\left\| \zeta_{kq,1}^{1} \frac{1}{r} \frac{\partial \hat{W}}{\partial \varphi} - \zeta_{kq,1}^{1} \frac{\partial \hat{W}}{\partial \varphi} \right\|_{H_b^{l-1}(\Omega_k)}^2 \leq \left\| \frac{1}{r} \frac{\partial \hat{W}}{\partial \varphi} \frac{R_{j\varphi k}}{r} \right\|_{H_b^{l-1}(\Omega_k)}^2 \leq k_1 \sum_{|\alpha| \leq l-1} \sum_{|\beta| \leq |\alpha|} \int \left( r^{2(b+|\alpha|-l+1)} \right) \left| D^\alpha \frac{R_{j\varphi k}}{r} \right| \left| D^\beta \left( \zeta_{kq,1}^{1} \frac{\partial \hat{W}}{\partial \varphi} \right) \right| \, dx.
\]
From this, the last inequality in (3.2), and the last inequality in (3.3), we obtain
\[
\left\| \zeta_{kq,1}^{1} \frac{1}{r} \frac{\partial \hat{W}}{\partial \varphi} - \zeta_{kq,1}^{1} \frac{\partial \hat{W}}{\partial \varphi} \right\|_{H_b^{l-1}(\Omega_k)}^2 \leq k_2 \varepsilon_0 \left\| \frac{1}{r} \frac{\partial \hat{W}}{\partial \varphi} \right\|_{H_b^{l-1}(\Omega_k)}^2. \tag{3.17}
\]
Estimate (3.17) and Lemma 3.3 prove the lemma for \( |\gamma| = 1 \) and \( \zeta_{kq,1}^p \) substituted for \( \zeta_{kq,2} \).

We assume that the lemma is true for \( 1 \leq |\gamma| \leq p - 1 \) and \( \zeta_{kq,1}^{p} \) substituted for \( \zeta_{kq,2} \). Let us prove that it is true for \( |\gamma| = p \) and \( \zeta_{kq,1}^{p} \) substituted for \( \zeta_{kq,2} \) (\( p \geq 2 \)). We have
\[
\| \zeta_{kq,1}^{p} D^\gamma \hat{W} - \zeta_{kq,1}^{p} \widehat{D^\gamma W} \|_{H_b^{l-|\gamma|}(\Omega_k)} \leq \| \zeta_{kq,1}^{p} D^{|\gamma|-1}(D^1 \hat{W}) - \zeta_{kq,1}^{p} \widehat{D^{|\gamma|-1}D^1 W} \|_{H_b^{l-|\gamma|}(\Omega_k)} + \| \zeta_{kq,1}^{p} D^{|\gamma|-1}D^1 \hat{W} - \zeta_{kq,1}^{p} \widehat{D^{|\gamma|-1}D^1 W} \|_{H_b^{l-|\gamma|}(\Omega_k)} + \| \zeta_{kq,1}^{p} D^{|\gamma|-1} \widehat{D^1 \hat{W}} - \zeta_{kq,1}^{p} \widehat{D^{|\gamma|-1}(D^1 W)} \|_{H_b^{l-|\gamma|}(\Omega_k)}.
\]
\footnote{Lemma 2.1 [27] (and Lemmas 2.2, 3.5, 3.6 [27], see below) is proved by Kondrat’ev for domains with angular or conical points. However, it is easy to see that it remains true for the domains with edges under consideration.}
where $D^{[\gamma]-1}$ and $D^1$ are some derivatives of order $|\gamma| - 1$ and 1 respectively. By the inductive assumption, for each of the two norms on the right-hand side of (3.18), the following estimates hold:

$$\|\zeta_{kq,1} D^{[\gamma]} \hat{W} - \zeta_{kq,1} D^{[\gamma]} W\|_{H^{-1}_{b}(\Omega_k)} \leq k_4 \xi_0 \|W\|_{H^1_{\beta}(\Omega_k)},$$

$$\|\zeta_{kq,1} D^{[\gamma]} \hat{W} - \zeta_{kq,1} D^{[\gamma]} W\|_{H^{-1}_{b}(\Omega_k)} \leq k_5 \xi_0 \|D^1 W\|_{H^{-1}_{b}(\Omega_k)} \leq k_6 \xi_0 \|W\|_{H^1_{\beta}(\Omega_k)}.$$

This and (3.18) imply the conclusion of the lemma. \qed

Notice that the multiplier $\varepsilon_0$ appears in (3.16) since the minuend and subtrahend both contain the same transformation $\omega_{j\sigma kqs}(G_{j\sigma kqs} y, z)$, but the minuend is the derivative $D^\gamma$ of the transformed function $\hat{W}$ while the subtrahend is the transformation of the derivative $D^\gamma W$.

**Lemma 3.5.** For any $U_k \in H^{l+2m}_{b}(\Omega_k)$ with $\text{supp } U_k \subset \bar{\Omega}_k \cap \mathcal{V}_\gamma(0)$, the following inequality holds:

$$\| (B_{j\sigma kqs} U_k) (G_{j\sigma kqs} y, z) |_{\Gamma_{\gamma}} \|_{H^{l+2m-j_{\sigma} \mu-1/2}_{b}(\Gamma_{\gamma})} \leq c(\varepsilon_0 \|U_k\|_{H^{l+2m}_{b}(\Omega_k)} + \|\zeta_{kq,3} \hat{U}_k - \zeta_{kq,3} U_k\|_{H^{l+2m}_{b}(\Omega_k)}), \quad (3.19)$$

where $c > 0$ is independent of $U$ and $\varepsilon_0$.

**Proof.** Using the boundedness of the trace operator in weighted spaces, we get

$$\| (B_{j\sigma kqs} U_k) (G_{j\sigma kqs} y, z) |_{\Gamma_{\gamma}} \|_{H^{l+2m-j_{\sigma} \mu-1/2}_{b}(\Gamma_{\gamma})} \leq \leq k_1 \|\zeta_{kq,4} B_{j\sigma kqs} U_k - \zeta_{kq,4} \hat{U}_k\|_{H^{l+2m-j_{\sigma} \mu}_{b}(\Omega_k)} \leq 0 \|\zeta_{kq,4} B_{j\sigma kqs} U_k - \zeta_{kq,4} \hat{U}_k\|_{H^{l+2m-j_{\sigma} \mu}_{b}(\Omega_k)} + \|\zeta_{kq,4} B_{j\sigma kqs} \hat{U}_k - \zeta_{kq,4} \hat{U}_k\|_{H^{l+2m-j_{\sigma} \mu}_{b}(\Omega_k)}). \quad (3.20)$$

Let us estimate the first norm on the right-hand side of (3.20) as follows:

$$\|\zeta_{kq,4} B_{j\sigma kqs} U_k - \zeta_{kq,4} \hat{U}_k\|_{H^{l+2m-j_{\sigma} \mu}_{b}(\Omega_k)} \leq \leq k_2 \|\zeta_{kq,3} \hat{U}_k - \zeta_{kq,3} U_k\|_{H^{l+2m}_{b}(\Omega_k)}, \quad (3.21)$$

The second norm on the right-hand side of (3.20) can be estimated with the help of Lemma 3.4

$$\|\zeta_{kq,4} B_{j\sigma kqs} \hat{U}_k - \zeta_{kq,4} \hat{U}_k\|_{H^{l+2m-j_{\sigma} \mu}_{b}(\Omega_k)} \leq \leq k_3 \xi_0 \|U_k\|_{H^{l+2m}_{b}(\Omega_k)}. \quad (3.22)$$

From (3.20) - (3.22), the conclusion of the lemma follows. \qed

Notice that the right-hand side of (3.19) contains the norm of the difference of the non-transformed function and the transformed one. To estimate such differences, we need the following result.

**Lemma 3.6.** For any $W \in H^{l+1}_{b+1}(\Omega_k)$ with $\text{supp } W \subset \bar{\Omega}_k \cap \mathcal{V}_\gamma(0)$, the following inequality holds:

$$\|\zeta_{kq,1} W - \zeta_{kq,1} \hat{W}\|_{H^{l}_{b}(\Omega_k)} \leq c \xi_0 \|W\|_{H^{l+1}_{b+1}(\Omega_k)}, \quad (3.23)$$

where $c > 0$ is independent of $W$ and $\xi_0$. 20
Proof. Writing the arguments of the functions $W$ and $\hat{W}$ in cylindrical coordinates, we obtain
\[
\|\zeta_{k q, 1} W - \zeta_{k q, 1} \hat{W}\|_{H^0_b(\Omega_k)} \leq \|\zeta_{k q, 1} W(\varphi, r, z) - \zeta_{k q, 1} W(\varphi + \Phi'_{j \sigma k q s}(r, z), r, z)\|_{H^0_b(\Omega_k)} + \\
+ \|\zeta_{k q, 1} W(\varphi + \Phi'_{j \sigma k q s}(r, z), r, z) - \zeta_{k q, 1} W(\varphi + \Phi'_{j \sigma k q s}(r, z), r + R'_{j \sigma k q s}(r, z), z)\|_{H^0_b(\Omega_k)}.
\]
Using the Schwartz inequality, we estimate the square of the first norm on the right-hand side of (3.24):
\[
\|\zeta_{k q, 1} W(\varphi, r, z) - \zeta_{k q, 1} W(\varphi + \Phi'_{j \sigma k q s}(r, z), r, z)\|^2_{H^0_b(\Omega_k)} = \\
\int_{\mathbb{R}^{n-2}} \int_{0}^{\infty} \int_{b_{k_1}}^{b_{k_2}} \int_{0}^{r_{2b}} \frac{\partial W}{\partial \varphi'} d\varphi' d\varphi \leq \\
\int_{\mathbb{R}^{n-2}} \int_{0}^{\infty} \int_{b_{k_1}}^{b_{k_2}} \int_{0}^{r_{2b}} \left|\zeta_{k q, 1}(\varphi + \Phi'_{j \sigma k q s}(r, z))\right|^2 \frac{\partial W}{\partial \varphi'} d\varphi' d\varphi.
\]
Taking into account the restrictions on the support of the functions $W$ and $\zeta_{k q, 1}$ and inequalities (3.9), we can change the order of integration with respect to $\varphi$ and $\varphi'$; as a result, using (3.2), we get
\[
\|\zeta_{k q, 1} W(\varphi, r, z) - \zeta_{k q, 1} W(\varphi + \Phi'_{j \sigma k q s}(r, z), r, z)\|^2_{H^0_b(\Omega_k)} \leq \\
\leq k_1 \int_{\mathbb{R}^{n-2}} \int_{0}^{\infty} \int_{b_{k_1}}^{b_{k_2}} \frac{1}{r} \left|\frac{\partial W}{\partial \varphi}\right|^2 d\varphi' d\varphi \leq \\
\leq k_2 \int_{\mathbb{R}^{n-2}} \int_{0}^{\infty} \int_{b_{k_1}}^{b_{k_2}} \frac{1}{r} \left|\frac{\partial W}{\partial \varphi}\right|^2 d\varphi' d\varphi \leq k_3 \varepsilon^2 \|W\|^2_{H^0_b(\Omega_k)}.
\]
Similarly, one can estimate the square of the second norm on the right-hand side of (3.24). □

Thus, the multiplier $\varepsilon_0$ appears in (3.23) if one increases the order of differentiation by 1. (The left-hand side of (3.23) contains the norm in $H^0_b(\Omega_k)$ while the right-hand side does in $H^1_{b+1}(\Omega_k)$.) This can be explained as follows: unlike in (3.16), in this case one estimates the difference of the two functions the first one of which does not contain a transformation while the second one does.

4 A priori estimates of solutions

In this section, we prove an a priori estimate for the operator $L$, which guarantees that its kernel is of finite dimension and its range is closed.

1. First, we prove an a priori estimate for functions with the support being a subset of some neighborhood of $K_1$. To this end, we will use the invertibility of the model operators $L^g_{y}, g \in K_1,$ with linear transformations as well as Lemmas 3.3 3.6. Then, in subsection 2 of this section, using the results of 11 and Lemma 5.2 12, we will obtain an a priori estimates for functions with the support in the whole of $G$.

We denote $O_\varepsilon(K_1) = \{ x \in \mathbb{R}^n : \text{dist}(x, K_1) < \varepsilon \}$. 

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Lemma 4.1. Let Conditions 1.1–1.4 hold and, for each \( g \in K_1 \), the operator \( L_g^G \) be an isomorphism. Then there is an \( \varepsilon, 0 < \varepsilon < \text{dist}(K_1, K_2 \cup K_3)/2 \), such that for all \( u \in \{ u \in H^{l+2m}_b(G) : \supp u \subset G \cap O_c(K_1) \} \) the following estimate holds:

\[
\|u\|_{H^{l+2m}_b(G)} \leq c(\|Lu\|_{H^l_b(G, \gamma)} + \|u\|_{H^{l-1-2m}_b(G)}),
\]

where \( c > 0 \) is independent of \( u \).

Using the unity partition method, Leibniz’ formula, Lemma 2.1 [27], and Lemma 1.2 [9], one can reduce the proof of Lemma 4.1 to the proof of the following result.

Lemma 4.2. Let the conditions of Lemma 4.1 hold. Then for each \( g \in K_1 \) there is an \( \varepsilon_0 = \varepsilon_0(g) > 0 \) such that for any \( U \in \{ U \in H^{l+2m,N}_b(\Omega) : \supp U_j \subset \Omega_j \cap \mathcal{V}_{\varepsilon_0}(0), \ j = 1, \ldots, N = N(g) \} \) the following inequality holds:

\[
\|U\|_{H^{l+2m,N}_b(\Omega)} \leq c \|L_g^U\|_{H^{l,N}_b(\Omega)},
\]

where \( \mathcal{V}_{\varepsilon_0}(0) = \{ x \in \mathbb{R}^n : |x| < \varepsilon_0 \} \), \( c > 0 \) is independent of \( U \).

Proof. Using the invertibility of \( L_g^U \) and Lemma 3.5 for all \( U \in H^{l+2m,N}_b(\Omega) \) with \( \supp U_j \subset \Omega_j \cap \mathcal{V}_{\varepsilon_0}(0) \) we get

\[
\|U\|_{H^{l+2m,N}_b(\Omega)} \leq k_1 \|L_g^U\|_{H^{l,N}_b(\Omega)} \leq k_2(\|L_g^U\|_{H^{l,N}_b(\Omega)} + \varepsilon_0\|U\|_{H^{l+2m,N}_b(\Omega)} + \sum_{k=1}^{N} \sum_{q=2}^{R_k} \|\zeta_{kq,3}U_k - \zeta_{kq,3}\hat{U}_k\|_{H^{l+2m}_b(\Omega_k)}). \tag{4.1}
\]

Let us estimate the last norm in (4.1). By Theorem 4.1 [25], we have

\[
\|\zeta_{kq,3}U_k - \zeta_{kq,3}\hat{U}_k\|_{H^{l+2m}(\Omega_k)} \leq k_3(\|P_k(\zeta_{kq,3}U_k - \zeta_{kq,3}\hat{U}_k)\|_{H^l_b(\Omega_k)} + \|\zeta_{kq,3}U_k - \zeta_{kq,3}\hat{U}_k\|_{H^{l-1-2m}_b(\Omega_k)}). \tag{4.2}
\]

From Lemma 3.6 and the continuity of the embedding \( H^{l+2m}_b(\Omega_k) \subset H^{l-1-2m}_b(\Omega_k) \), it follow that

\[
\|\zeta_{kq,3}U_k - \zeta_{kq,3}\hat{U}_k\|_{H^{l-1-2m}_b(\Omega_k)} \leq k_4\varepsilon_0\|U_k\|_{H^{l+2m}_b(\Omega_k)}. \tag{4.3}
\]

To estimate the first norm on the right-hand side of (4.2), we apply Leibniz’ formula and Lemmas 3.3 and 3.4.

\[
\|P_k(\zeta_{kq,3}U_k - \zeta_{kq,3}\hat{U}_k)\|_{H^l_b(\Omega_k)} \leq k_5(\|\zeta_{kq,3}P_kU_k\|_{H^l_b(\Omega_k)} + \|\zeta_{kq,3}P_k\hat{U}_k\|_{H^l_b(\Omega_k)}) + \sum_{|\beta| \leq 2m-1} \sum_{|\gamma| = 2m-|\beta|} \|D^\gamma \zeta_{kq,3}D^\beta U_k - D^\gamma \zeta_{kq,3}D^\beta \hat{U}_k\|_{H^l_b(\Omega_k)} \leq k_6(\|P_kU_k\|_{H^l_b(\Omega_k)} + \varepsilon_0\|U_k\|_{H^{l+2m}_b(\Omega_k)} + \sum_{|\beta| \leq 2m-1} \sum_{|\gamma| = 2m-|\beta|} \|D^\gamma \zeta_{kq,3}D^\beta U_k - D^\gamma \zeta_{kq,3}D^\beta \hat{U}_k\|_{H^l_b(\Omega_k)}). \tag{4.4}
\]

\[\text{In subsection 5 of §11 one can find necessary and sufficient condition under which } L^G_g \text{ is an isomorphism.}\]
Since \(|D^\gamma \zeta_{kq,3}| \leq k\gamma r^{-|\gamma|}|\zeta_{kq,2}|\), it follows that

\[
\sum_{|\beta| \leq 2m-1} \sum_{|\gamma| = 2m-|\beta|} \|D^\gamma \zeta_{kq,3}D^\beta U_k - D^\gamma \zeta_{kq,3}D^\beta \hat{U}_k\|_{H^l_0(\Omega_k)} \leq
\]

\[
\leq k_8 \sum_{|\alpha| \leq l+2m-1} \|\zeta_{kq,2}D^\alpha U_k - \zeta_{kq,2}D^\alpha \hat{U}_k\|_{H^0_{b+|\alpha|-l-2m}(\Omega_k)} \leq
\]

\[
\leq k_9 \sum_{|\alpha| \leq l+2m-1} \left\{ \|\zeta_{kq,2}D^\alpha U_k - \zeta_{kq,2}D^\alpha \hat{U}_k\|_{H^0_{b+|\alpha|-l-2m}(\Omega_k)} + \|\zeta_{kq,2}D^\alpha U_k - \zeta_{kq,2}D^\alpha \hat{U}_k\|_{H^0_{b+|\alpha|-l-2m}(\Omega_k)} \right\}. \quad (4.5)
\]

Using Lemma 3.3 and the continuity of the embedding \(H^{l+2m}_b(\Omega_k) \subset H^{1+|\alpha|}_{b+1+|\alpha|-l-2m}(\Omega_k)\) for \(|\alpha| \leq l + 2m - 1\), we obtain

\[
\|\zeta_{kq,2}D^\alpha U_k - \zeta_{kq,2}D^\alpha \hat{U}_k\|_{H^0_{b+|\alpha|-l-2m}(\Omega_k)} \leq
\]

\[
\leq k_{10}\varepsilon_0 \|D^\alpha U_k\|_{H^{1+|\alpha|}_{b+1+|\alpha|-l-2m}(\Omega_k)} \leq k_{11}\varepsilon_0 \|U_k\|_{H^{l+2m}_b(\Omega_k)}. \quad (4.6)
\]

Similarly, from Lemma 3.4, it follows that

\[
\|\zeta_{kq,2}D^\alpha U_k - \zeta_{kq,2}D^\alpha \hat{U}_k\|_{H^0_{b+|\alpha|-l-2m}(\Omega_k)} \leq k_{12}\varepsilon_0 \|U_k\|_{H^{l+2m+2}_b(\Omega_k)}. \quad (4.7)
\]

Now the conclusion of the lemma follows from (4.1)–(4.7) with sufficiently small \(\varepsilon_0\). \(\square\)

2. Repeating the proof of Theorem 2.1 \([1]\) and taking into account Lemma 5.2 \([12]\), from Lemma 4.1 of the present work and Lemmas 2.4 and 2.5 of \([11]\), we deduce the following result.

**Theorem 4.1.** Let the conditions of Lemma 4.1 hold and \(b > l + 2m - 1\). Then, for all \(u \in H^{l+2m}_b(G)\), the following estimate holds:

\[
\|u\|_{H^{l+2m}_b(G)} \leq c(\|Lu\|_{H^1_0(G, \gamma)} + \|u\|_{H^0_{b+1-l-2m}(G)}), \quad (4.8)
\]

where \(c > 0\) is independent of \(u\).

By virtue of the compactness of the embedding \(H^{l+2m}_b(G) \subset H^{0}_{b+1-l-2m}(G)\) (see Lemma 3.5 \([27]\)), from Theorem 4.1 it follows that the operator \(L\) has a finite-dimensional kernel and a closed range.

## 5 Construction of right regularizer

In this section, we construct a right regularizer for \(L\), which, being combined with Theorem 4.1 allows us to prove the Fredholm solvability of nonlocal boundary-value problem \((1.2), (1.3)\).

1. To begin with, we consider the case where the supports of functions are subsets of a neighborhood of \(K_1\). In this situation, we will use the invertibility of the operators \(L^*_g\), \(g \in K_1\), with linear transformations as well as some special constructions “compensating” the nonlinearity in the argument transformations. Then, in subsection 2 of this section, using the results of \([11]\) and Lemma 5.2 \([12]\), we will construct the right regularizer in the whole of \(G\).

First of all, let us prove the following auxiliary result.
Lemma 5.1. Let $H$, $H_1$, and $H_2$ be Hilbert spaces, $\mathcal{A} : H \to H_1$ a linear bounded operator, $\mathcal{T}_0 : H \to H_2$ a linear compact operator. Suppose that, for some $\varepsilon$, $c > 0$ and all $f \in H$, the following inequality holds:
\[
\|\mathcal{A}f\|_{H_1} \leq \varepsilon \|f\|_H + c\|\mathcal{T}_0 f\|_{H_2}.
\] (5.1)

Then there are bounded operators $\mathcal{M}$, $\mathcal{F} : H \to H_1$ such that
\[
\mathcal{A} = \mathcal{M} + \mathcal{F},
\]
where $\|\mathcal{M}\| \leq 2\varepsilon$ and the operator $\mathcal{F}$ is finite-dimensional.

Proof. As is well known (see, e.g., [28, Chapter 5, Section 85]), any compact operator is the limit of a uniformly convergent sequence of finite-dimensional operators. Therefore, there are bounded operators $\mathcal{M}_0$, $\mathcal{F}_0 : H \to H_2$ such that $\mathcal{T}_0 = \mathcal{M}_0 + \mathcal{F}_0$, $\|\mathcal{M}_0\| \leq c^{-1}\varepsilon$, and $\mathcal{F}_0$ is finite-dimensional. From this and (5.1), it follows that
\[
\|\mathcal{A}f\|_{H_1} \leq 2\varepsilon\|f\|_H + c\|\mathcal{F}_0 f\|_{H_2} \quad \text{for all } f \in H. \tag{5.2}
\]

We denote by $\ker(\mathcal{F}_0)^ot$ the orthogonal supplement in $H$ to the kernel of $\mathcal{F}_0$. Since the finite-dimensional operator $\mathcal{F}_0$ maps $\ker(\mathcal{F}_0)^ot$ onto its range in a one-to-one manner, it follows that the subspace $\ker(\mathcal{F}_0)^ot$ is of finite dimension. Let $I$ denote the unity operator in $H$ and $P_0$ the orthogonal projector onto $\ker(\mathcal{F}_0)^ot$. Obviously, $\mathcal{A}P_0 : H \to H_1$ is a finite-dimensional operator. Furthermore, since $I - P_0$ is the orthogonal projector onto $\ker(\mathcal{F}_0)$, it follows that $\mathcal{F}_0(I - P_0) = 0$. Therefore, substituting in (5.2) the function $(I - P_0)f$ for $f$, we get
\[
\|\mathcal{A}(I - P_0)f\|_{H_1} \leq 2\varepsilon\|(I - P_0)f\|_H \leq 2\varepsilon\|f\|_H \quad \text{for all } f \in H.
\]

Denoting $\mathcal{M} = \mathcal{A}(I - P_0)$ and $\mathcal{F} = \mathcal{A}P_0$ completes the proof. \hfill \Box

Now we proceed to construct the right regularizer.

Lemma 5.2. Let the conditions of Lemma 4.1 hold. Then, for all sufficiently small $\varepsilon$, $0 < \varepsilon < \text{dist}({\mathcal{K}_1}, {\mathcal{K}_2}\cup{\mathcal{K}_3})/2$, there are bounded operators $\mathcal{R}_1$, $\mathcal{M}_1$, and a compact operator $\mathcal{T}_1$ acting from $\{f \in \mathcal{H}_b^1(G, \Gamma) : \text{supp } f \subset G \cap \mathcal{O}_\varepsilon(\mathcal{K}_1)\}$ to $\mathcal{H}_b^{1+2m}(G)$, $\mathcal{H}_b^1(G, \Gamma)$, and $\mathcal{H}_b^1(G, \Gamma)$ respectively and such that
\[
LR_1 f = f + \mathcal{M}_1 f + \mathcal{T}_1 f,
\]
\[
\|\mathcal{M}_1 f\|_{\mathcal{H}_b^1(G, \Gamma)} \leq c\varepsilon\|f\|_{\mathcal{H}_b^1(G, \Gamma)}. \quad \text{Here } c > 0 \text{ is independent of } \varepsilon \text{ and } f.
\]

Using the unity partition method, Leibniz’ formula, and Lemma 2.1 [27], one can reduce the proof of Lemma 5.2 to the proof of the following result.

Lemma 5.3. Let the conditions of Lemma 4.1 hold. Then, for each $g \in \mathcal{K}_1$ and all sufficiently small $\varepsilon_1 = \varepsilon_1(g) > 0$, there are bounded operators $\mathcal{R}_g$, $\mathcal{M}_g$, and a compact operator $\mathcal{T}_g$ acting from $\{f \in \mathcal{H}_b^1(\Omega, \Gamma) : \text{supp } f \subset \mathcal{V}_{\varepsilon_1}(0)\}$ to $\mathcal{H}_b^{1+2m,N}(\Omega)$, $\mathcal{H}_b^1(\Omega, \Gamma)$, and $\mathcal{H}_b^1(\Omega, \Gamma)$ respectively and such that
\[
\mathcal{L}_g^{\varepsilon_1} \mathcal{R}_g f = f + \mathcal{M}_g f + \mathcal{T}_g f,
\] (5.3)
\[
\|\mathcal{M}_g f\|_{\mathcal{H}_b^1(G, \Gamma)} \leq c\varepsilon_1\|f\|_{\mathcal{H}_b^1(G, \Gamma)}. \quad \text{Here } c > 0 \text{ is independent of } \varepsilon_1 \text{ and } f.
Proof. 1) As before, we denote \(d_1 = \min\{1, \chi_{j \sigma k \mu s}\}/2, d_2 = 2\max\{1, \chi_{j \sigma k \mu s}\}\). We choose \(\varepsilon_1 < d_1\varepsilon_0/4\), where \(\varepsilon_0\) is defined in Lemma 4.2. We introduce a function \(\psi_{\varepsilon_1}(x) = \psi(x/\varepsilon_1)\), where \(\psi \in C^\infty(\mathbb{R}^n), \psi(x) = 1\) for \(|x| \leq 1, \psi(x) = 0\) for \(|x| \geq 2\). It is obvious that \(\psi_{\varepsilon_1} \in C^\infty(\mathbb{R}^n)\), \(\psi_{\varepsilon_1}(x) = 1\) for \(|x| \leq \varepsilon_1, \psi_{\varepsilon_1}(x) = 0\) for \(|x| \geq 2\varepsilon_1\). Since \(|D^\alpha \psi_{\varepsilon_1}| \leq c_\alpha r^{-|\alpha|}\), from Lemma 2.1 [27] it follows that

\[
\|\psi_{\varepsilon_1} v\|_{H_{b}^{l+2m}(\Omega_k)} \leq c\|v\|_{H_{b}^{l+2m}(\Omega_k)} \quad \text{for all} \quad v \in H_{b}^{l+2m}(\Omega_k),
\]

where \(c > 0\) is independent of \(\varepsilon_1\). Moreover, we assume that \(\psi_{\varepsilon_1}\), being written in cylindrical coordinates, does not depend on \(\varphi\).

Put \(f_0 = \{f_j\}, g = \{g_{j \sigma \mu}\}, \{f_0, g\} = \{f_j, g_{j \sigma \mu}\}\).

By assumption, the operator \(L_g^\alpha : H_{b}^{l+2m,N}(\Omega) \to H_{b}^{l,N}(\Omega, \Gamma)\) has a bounded inverse \((L_g^\alpha)^{-1} : H_{b}^{l,N}(\Omega, \Gamma) \to H_{b}^{l+2m,N}(\Omega)\). Therefore, we can introduce the operators

\[
R_1 : H_{b}^{l,N}(\Omega) \to H_{b}^{l+2m,N}(\Omega), \quad R_2 : H_{b}^{l,N}(\Omega, \Gamma) \to H_{b}^{l+2m,N}(\Omega)
\]

given by

\[
R_1 f_0 = \psi_{\varepsilon_1}(L_g^\alpha)^{-1} \{f_0, 0\}, \quad R_2 g = \psi_{\varepsilon_1}(L_g^\alpha)^{-1} \{0, g\},
\]

where \(H_{b}^{l,N}(\Gamma) = \prod_{j, \sigma, \mu} H_{b}^{l+2m-m_{j \sigma \mu}-1/2}(\Gamma_{j \sigma})\). Thus, the supports of \(R_1 f_0\) and \(R_2 g\) are subsets of the ball of radius \(2\varepsilon_1\) centered at the origin.

Let us introduce the operators

\[
P : H_{b}^{l+2m,N}(\Omega) \to H_{b}^{l,N}(\Omega),
B^\alpha, B^\sigma : H_{b}^{l+2m,N}(\Omega) \to H_{b}^{l,N}(\Gamma)
\]
given by

\[
P U = \{P_j U_j\}, \quad B^\alpha U = \{B_{j \sigma \mu}^\alpha U\}, \quad B^\sigma U = \{B_{j \sigma \mu}^\sigma U\}.
\]

Now we establish a relation between the operators \(P, B^\alpha, B^\sigma\) and \(R_1, R_2\). To this end, we will use the following well-known property of weighted spaces (see Lemma 3.5 [27]): (star) the embedding operator from \(\{v \in H_{b}^{l+1}(\Omega_j) : \text{supp} v \subset \mathcal{V}_d(0), d > 0\}\) into \(H_{b}^{l}(\Omega_j)\) is compact.

From Leibniz’ formula, the boundedness of \(\text{supp } \psi_{\varepsilon_1}\), and property (star), it follows that

\[
P R_1 f_0 = \psi_{\varepsilon_1} f_0 + T_1 f_0, \quad P R_2 g = T_2 g,
\]

where \(T_1 : H_{b}^{l,N}(\Omega) \to H_{b}^{l,N}(\Omega)\) and \(T_2 : H_{b}^{l,N}(\Gamma) \to H_{b}^{l,N}(\Omega)\) are compact operators. Similarly,

\[
B^\alpha R_2 g = \psi_{\varepsilon_1} g + \left\{ \sum_{k, q, s} (\psi_{\varepsilon_1}(\chi_{j \sigma k \mu s}x) - \psi_{\varepsilon_1}(x))(B_{j \sigma k \mu s}[L_g^\alpha]^{-1} \{0, g\})_k(\mathcal{G}_{j \sigma k \mu s} y, z)|_{\Gamma_{j \sigma}} \right\} + T_3 g,
\]

where \(T_3\) is a compact operator in \(H_{b}^{l,N}(\Gamma)\); here and in what follows, we denote by \([\cdot]_k\) the \(k\)th component of an \(N\)-dimensional vector and by \(\{\ldots\}\) a vector with the components defined by the indices \(j, \sigma, \mu\).

Let us show that each term in the sum in (5.6) is a compact operator. Let \(\zeta_{k \sigma \mu}^s, \zeta_{k \sigma \mu}^t\) be the functions defined by formulas (3.10). We also introduce the functions \(\hat{\psi}_0, \hat{\psi}_1 \in C_0^\infty(\mathbb{R}^n)\) such that

\[
\hat{\psi}_1(x) = 1 \text{ for } 2d_1 \varepsilon_1 \leq |x| \leq d_2 \varepsilon_1, \hat{\psi}_1(x) = 0 \text{ outside } d_1 \varepsilon_1 \leq |x| \leq 2d_2 \varepsilon_1,
\]
Then, by virtue of the boundedness of the trace operator in weighted spaces, we have
\[
\| (\psi_{\varepsilon_1}(x) - \psi_{\varepsilon_1}(x)) (B_{j_0 \mu k} q^k \mathcal{L}^g_1)^{-1} \{0, g\} \|_{H_b^{l+2m-\frac{m}{2}-1/2} (\Gamma_{j, \alpha})} \leq k_2 \zeta_{k_1, 2} (\psi_{\varepsilon_1}(x) - \psi_{\varepsilon_1}(x)) B_{j_0 \mu k} q^k \mathcal{L}^g_1)^{-1} \{0, g\} \|_{H_b^{l+2m-\frac{m}{2}-1/2} (\Omega_k)} \leq k_3 \| \zeta_{k_1, 1} \psi_{\varepsilon_1}(x) (\mathcal{L}^g_1)^{-1} \{0, g\} \|_{H_b^{l+2m} (\Omega_k)}. (5.7)
\]
Since the support of \( \hat{\psi}_1 \) is bounded and does not intersect with the origin and \( \zeta_{k_1} \) vanishes near the sides of the angle \( \Omega_k \), we can apply Theorem 5.1 [23, Chapter 2]. Then, using the relation \( \mathcal{P}_k (\mathcal{L}^g_1)^{-1} \{0, g\} = 0 \) from (5.7), we get
\[
\| (\psi_{\varepsilon_1}(x) - \psi_{\varepsilon_1}(x)) (B_{j_0 \mu k} q^k \mathcal{L}^g_1)^{-1} \{0, g\} \|_{H_b^{l+2m-\frac{m}{2}-1/2} (\Gamma_{j, \alpha})} \leq k_1 \| \psi_{\varepsilon_1}(x) (\mathcal{L}^g_1)^{-1} \{0, g\} \|_{H_b^{l+2m} (\Omega_k)}.
\]
Since the support of \( \hat{\psi}_0 \) is bounded, from the last inequality and property (*) it follows that
\[
\left\{ \sum_{k, q, s} (\psi_{\varepsilon_1}(x) - \psi_{\varepsilon_1}(x)) (B_{j_0 \mu k} q^k \mathcal{L}^g_1)^{-1} \{0, g\} \|_{H_b^{l+2m-\frac{m}{2}-1/2} (\Gamma_{j, \alpha})} \right\}
\]
is a compact operator acting in \( \mathcal{H}_b^{l, N} (\Gamma) \). Combining this with (5.6) yields
\[
\mathcal{B}^\omega \mathcal{R}_2 g = \psi_{\varepsilon_1} g + \mathcal{T}_4 g, (5.8)
\]
where \( \mathcal{T}_4 \) is a compact operator acting in \( \mathcal{H}_b^{l, N} (\Gamma) \).

Finally, from (5.8), we obtain the formula for the composition \( \mathcal{B}^\omega \mathcal{R}_2 \):
\[
\mathcal{B}^\omega \mathcal{R}_2 g = \psi_{\varepsilon_1} g + \mathcal{T}_4 g + \left\{ \sum_{k, q, s} (B_{j_0 \mu k} q^k \mathcal{L}^g_1)^{-1} \{0, g\} \|_{H_b^{l+2m-\frac{m}{2}-1/2} (\Gamma_{j, \alpha})} \right\}. (5.9)
\]

2) Let us introduce the operator \( \mathcal{R}_g : \mathcal{H}_b^{l, N} (\Omega, \Gamma) \rightarrow H_b^{l+2m, N} (\Omega) \) given by
\[
\mathcal{R}_g \{ f_0, g \} = \mathcal{R}_1 f_0 - \mathcal{R}_2^\omega \mathcal{R}_1 f_0 + \mathcal{R}_2 g.
\]
Here \( \mathcal{R}_2 : \mathcal{H}_b^{l, N} (\Gamma) \rightarrow H_b^{l+2m, N} (\Omega) \) is the bounded operator given by
\[
\mathcal{R}_2 g = \psi_{\varepsilon_1} (d_1 x/2) (\mathcal{L}^g_1)^{-1} \{0, g\}.
\]
Similarly to (5.5) and (5.9), one can prove that
\[
\mathcal{P} \mathcal{R}_2 g = \mathcal{T}_2^\omega g, (5.10)
\]
\[
\mathcal{B}^\omega \mathcal{R}_2 g = \psi_{\varepsilon_1} (d_1 x/2) g + \mathcal{T}_4^\omega g + \left\{ \sum_{k, q, s} (B_{j_0 \mu k} q^k \mathcal{L}^g_1)^{-1} \{0, g\} \|_{H_b^{l+2m-\frac{m}{2}-1/2} (\Gamma_{j, \alpha})} \right\}. (5.11)
\]
where $T_2', T_3'$ are compact operators acting in the same spaces as the operators $T_2, T_3$ do.

Let us show that the operator $\mathcal{R}_g$ satisfies relation (5.3). From (5.3) and (5.10), it follows that

$$\mathcal{P}\mathcal{R}_g\{f_0, g\} = \psi_{\varepsilon_1}f_0 + T_5\{f_0, g\},$$

(5.12)

where $T_5 : \mathcal{H}_b^{l,N}(\Omega, \Gamma) \to H_b^{l,N}(\Omega)$ is a compact operator.

Taking into account that $\psi_{\varepsilon_1}(d_1 x/2)B^2\mathcal{R}_1f_0 \equiv B^2\mathcal{R}_1f_0$ and using (5.11), we derive

$$B^2\mathcal{R}_g\{f_0, g\} = B^2\mathcal{R}_1f_0 - B^2\mathcal{R}_1B^2\mathcal{R}_1f_0 + B^2\mathcal{R}_2g =$$

$$= -T_4'\mathcal{B}^2\mathcal{R}_1f_0 - \left\{ \sum_{k, q, s} \left( (B_{j\sigma qkqs}R_2B^2\mathcal{R}_1f_0)_k(\omega'_{j\sigma qkqs}(y, z), z)\right|_{r_{j\sigma}} -
\right.

$$
$$- (B_{j\sigma qkqs}R_2B^2\mathcal{R}_1f_0)_k(\omega'_{j\sigma qkqs}(y, z), z)\right|_{r_{j\sigma}}
\right\} + \mathcal{B}^2\mathcal{R}_2g.
$$

From this, using (5.9), we obtain

$$B^2\mathcal{R}_g g = \psi_{\varepsilon_1}g + T_6\{f_0, g\} +$$

$$+ \left\{ \sum_{k, q, s} \left( (B_{j\sigma qkqs}R_2g)_k(\omega'_{j\sigma qkqs}(y, z), z)\right|_{r_{j\sigma}} - (B_{j\sigma qkqs}R_2g)_k(\omega'_{j\sigma qkqs}(y, z), z)\right|_{r_{j\sigma}}
\right\} -$$

$$- \left\{ \sum_{k, q, s} \left( (B_{j\sigma qkqs}R_2B^2\mathcal{R}_1f_0)_k(\omega'_{j\sigma qkqs}(y, z), z)\right|_{r_{j\sigma}} -
\right.

$$
$$- (B_{j\sigma qkqs}R_2B^2\mathcal{R}_1f_0)_k(\omega'_{j\sigma qkqs}(y, z), z)\right|_{r_{j\sigma}}
\right\}, \quad (5.13)
$$

where $T_6 : \mathcal{H}_b^{l,N}(\Omega, \Gamma) \to \mathcal{H}_b^{l,N}(\Gamma)$ is a compact operator.

Let us consider the terms of the first sum on the right-hand side of (5.13). By Lemma 3.5, we have

$$\|(B_{j\sigma qkqs}R_2g)_k(\omega'_{j\sigma qkqs}(y, z), z)\right|_{r_{j\sigma}} -$$

$$- (B_{j\sigma qkqs}R_2g)_k(\omega'_{j\sigma qkqs}(y, z), z)\right|_{r_{j\sigma}} \leq k_5(\varepsilon_1||[R_2g]||_{H_b^{l+2m}(\Omega)} +$$

$$+ \|(\zeta_{qk3}R_2g) - \zeta_{qk3}\mathcal{R}_2g\||_{H_b^{l+2m}(\Omega)}) \quad (5.14)
$$

From inequalities (4.2) – (4.7) for the function $U_k = [R_2g]_k$, inequality (5.14), and the second relation in (5.3), we obtain

$$\|(B_{j\sigma qkqs}R_2g)_k(\omega'_{j\sigma qkqs}(y, z), z)\right|_{r_{j\sigma}} -$$

$$- (B_{j\sigma qkqs}R_2g)_k(\omega'_{j\sigma qkqs}(y, z), z)\right|_{r_{j\sigma}} \leq k_6(\varepsilon_1||[R_2g]||_{H_b^{l+2m}(\Omega)} +$$

$$+ \mathcal{P}_k[R_2g]||_{H_b^{l}(\Omega)}) =$$

$$= k_6(\varepsilon_1||[\mathcal{L}_g^\varepsilon]^{-1}(0, g)||_{H_b^{l+2m}(\Omega)} + ||T_2g||_{H_b^{l}(\Omega)}) \quad (5.15)
$$

This, being combined with inequality (5.4) and the boundedness of the operator $(\mathcal{L}_g^\varepsilon)^{-1} : \mathcal{H}_b^{l,N}(\Omega, \Gamma) \to H_b^{l+2m,N}(\Omega)$, finally implies

$$\|(B_{j\sigma qkqs}R_2g)_k(\omega'_{j\sigma qkqs}(y, z), z)\right|_{r_{j\sigma}} -$$

$$- (B_{j\sigma qkqs}R_2g)_k(\omega'_{j\sigma qkqs}(y, z), z)\right|_{r_{j\sigma}} \leq$$

$$\leq k_7(\varepsilon_1||g||_{H_b^{l,N}(\Gamma)} + ||T_2g||_{H_b^{l}(\Omega)}) \quad (5.15)
$$
Therefore, by Lemma 5.1, we have
\[
(B_{j\sigma \mu k q s} [\mathcal{R}_2 g]_k)(\omega'_{j\sigma k q s}(y, z), z)|_{r_{j\sigma}} - (B_{j\sigma \mu k q s} [\mathcal{R}_2 g]_k) (\mathcal{G}_{j\sigma k q s} y, z)|_{r_{j\sigma}} = \mathcal{M}_{j\sigma \mu k q s} g + \mathcal{F}_{j\sigma \mu k q s} g
\]
with the operators
\[
\mathcal{M}_{j\sigma \mu k q s}, \mathcal{F}_{j\sigma \mu k q s} : H^l_b(\Gamma) \to H^{l+2m-m_{j\sigma}}_{b-1/2}(\Gamma_{j\sigma})
\]
such that \(\| \mathcal{M}_{j\sigma \mu k q s} \| \leq 2k_7\varepsilon_1\) and the operator \(\mathcal{F}_{j\sigma \mu k q s}\) is finite-dimensional.

Analogously, one can prove that each term of the second sum on the right-hand side of (5.13) can be represented as the sum of an operator with small norm and a compact one. From this, (5.13), and (5.12), choosing \(\text{supp} \{ f_0, g \} \subset \mathcal{V}_{\varepsilon_1}(0)\), we get the conclusion of the lemma. \(\square\)

2. Now we can prove that, under certain conditions, the operator \(L : H^{l+2m}_b(G) \to H^l_b(G, \Upsilon)\) is Fredholm.

Theorem 5.1. Let the conditions of Lemma 4.1 hold and \(b > l + 2m - 1\). Then the operator \(L : H^{l+2m}_b(G) \to H^l_b(G, \Upsilon)\) is Fredholm.

Proof. By virtue of Theorem 4.1 of the present paper and Theorems 7.1, 15.2 [29], it suffices to construct a right regularizer \(R\) for \(L\).

Repeating the arguments of [11, § 3] and taking into account Lemma 5.2 [12], from Lemma 5.2 of the present paper we deduce the existence of bounded operators
\[
R' : H^l_b(G, \Upsilon) \to H^{l+2m}_b(G),
\]
\[
M, T : H^l_b(G, \Upsilon) \to H^l_b(G, \Upsilon)
\]
such that
\[
LR' = I + M + T,
\]
where \(\|M\| < 1\) and the operator \(T\) is compact. Since \(\|M\| < 1\), it follows that the operator \(I + M\) has a bounded inverse. Obviously, the operator \(R = R'(I + M)^{-1}\) is a right regularizer for \(L\). \(\square\)

3. Until now, we assumed that \(b > l + 2m - 1\). In this subsection, using results of [9], we study the case where \(b\) is arbitrary but \(n = 2\). As mentioned before, if \(b\) is arbitrary, we have to consider solutions and right-hand sides of the nonlocal problem as functions with power singularities not only near the set \(K_1\) but also near \(K_2\) and \(K_3\). This corresponds to the consistency conditions (see §[1]).

Thus, let \(n = 2\). We introduce the space \(\tilde{H}^l_b(G)\) as the completion of \(C^\infty_0(G \setminus K)\) with respect to the norm
\[
\|u\|_{\tilde{H}^l_b(G)} = \left( \sum_{|\alpha| \leq l} \int_G \tilde{\rho}^{2(b-l+|\alpha|)} |D^\alpha u|^2 dy \right)^{1/2},
\]
where \(\tilde{\rho} = \rho(y) = \text{dist}(y, K)\) (cf. §[1]). For \(l \geq 1\), we denote by \(\tilde{H}^{l-1/2}_{b-1/2}(\Upsilon)\) the space of traces on a smooth curve \(\Upsilon \subset G\) with the norm
\[
\|\psi\|_{\tilde{H}^{l-1/2}_{b-1/2}(\Upsilon)} = \inf \|u\|_{\tilde{H}^l_b(G)} \quad (u \in \tilde{H}^l_b(G) : u|_{\Upsilon} = \psi).
\]

We assume that the following condition holds.
**Condition 5.1.** If \( g \in \mathcal{K}_3 \cap \omega_{is}(\Upsilon_i) \neq \emptyset \), then \( \omega_{is}^{-1}(g) \in \mathcal{K} \).

The fulfillment of Condition 5.1 guarantees that the set of points in which the consistency condition must be imposed is finite. If Condition 5.1 fails, then the consecutive shifts of the set \( \mathcal{K}_1 \) (under the transformations \( \omega_{is} \) and \( \omega_{is}^{-1} \)) may form an infinite set, which should be used instead of \( \mathcal{K} \) in the definition of weighted spaces.

In this subsection, we consider the following bounded operator corresponding to problem (1.2), (1.3):

\[
\mathbf{L} = \{ \mathbf{P}(y, D), \mathbf{B}_{1\mu}(y, D) \} : \tilde{H}_{b}^{l+2m}(G) \to \tilde{H}_{b}^{l}(G) \times \prod_{i=1}^{N_{a}} \prod_{\mu=1}^{m_{i}} \tilde{H}_{b}^{l+2m-\mu - 1/2}(\Upsilon_{i}), \quad b \in \mathbb{R}.
\]

Since solutions and right-hand sides of the nonlocal problem may now have power singularities near the points of \( \mathcal{K}_2 \) and \( \mathcal{K}_3 \), we have to consider the model problems corresponding to these points in weighted spaces but not in the Sobolev spaces.

We fix a point \( g \in \mathcal{K}_2 \cup \mathcal{K}_3 \). Let \( y \mapsto y'(g) \) be a non-degenerate infinitely differentiable argument transformation mapping some neighborhood \( \mathcal{V}(g) \) of the point \( g \) onto a neighborhood \( \mathcal{V}_y(0) \) of the origin, so that the point \( g \) maps to the origin. We denote by \( \mathcal{P}(D_{y}), \mathbf{B}_{i\mu 0}(D_{y}) \) the principal homogeneous parts of the operators \( \mathbf{P}(g, D), \mathbf{B}_{i\mu 0}(g, D) \) written in new coordinates \( y' = y'(g) \) (with after-denoting \( y' \) by \( y \)). Now we write the operators \( \mathcal{P}(D_{y}), \mathbf{B}_{i\mu 0}(D_{y}) \) in polar coordinates:

\[
\mathcal{P}(D_{y}) = r^{-2m} \tilde{\mathcal{P}}(\varphi, D_{\varphi}, rD_{r}), \quad \mathbf{B}_{i\mu 0}(D_{y}) = r^{-m_{i} \mu} \tilde{\mathbf{B}}_{i\mu 0}(\varphi, D_{\varphi}, rD_{r}).
\]

If \( g \in \mathcal{K}_2 \), then \( g \in \Upsilon_i \) for some \( i = i(g) \). By virtue of the smoothness of \( \Upsilon_i \), in a sufficiently small neighborhood \( \mathcal{V}(g) \) of \( g \) there is a non-degenerate infinitely smooth argument transformation \( y \mapsto y'(g) \) mapping \( \mathcal{V}(g) \cap G \) onto the intersection of the half-plane \( \mathbb{R}_+^2 = \{ y : |\varphi| < \pi/2 \} \) with a neighborhood of \( \mathcal{V}_{y}(0) \). Let us introduce the bounded operator

\[
\mathcal{L}_{g} : H_{b}^{l+2m}(K_{\pi/2}) \to H_{b}^{l}(K_{\pi/2}) \times \prod_{j=1}^{2} \prod_{\mu=1}^{m} H_{b}^{l+2m-\mu - 1/2}(\gamma_{j})
\]

given by

\[
\mathcal{L}_{g}U = \{ \mathcal{P}(D_{y})U, \mathbf{B}_{i\mu 0}(D_{y})U \}_{\gamma_{j}},
\]

where \( K_{\pi/2} = \{ y : |\varphi| < \pi/2 \} \), \( \gamma_{j} = \{ y : \varphi = (-1)^{j}\pi/2 \} \), \( j = 1, 2 \). We also introduce the bounded operator

\[
\tilde{\mathcal{L}}_{g}(\lambda) : W_{2}^{l+2m}(-\pi/2, \pi/2) \to W_{2}^{l}(-\pi/2, \pi/2) \times C^{2m}
\]

given by

\[
\tilde{\mathcal{L}}_{g}(\lambda)U = \{ \tilde{\mathcal{P}}(\varphi, D_{\varphi}, \lambda)U(\varphi), \tilde{\mathbf{B}}_{i\mu 0}(\varphi, D_{\varphi}, \lambda)\tilde{U}(\varphi) \}_{\varphi = (-1)^{j}\pi/2}, \quad j = 1, 2.
\]

If \( g \in \mathcal{K}_3 \), we introduce the bounded operator

\[
\mathcal{L}_{g} = \mathcal{P}(D_{y}) : H_{b}^{l+2m}(\mathbb{R}^2) \to H_{b}^{l}(\mathbb{R}^2).
\]

Let us also introduce the bounded operator

\[
\tilde{\mathcal{L}}_{g}(\lambda) = \tilde{\mathcal{P}}(\varphi, D_{\varphi}, \lambda) : W_{2,2\pi}^{l+2m}(0, 2\pi) \to W_{2,2\pi}^{l}(0, 2\pi),
\]

\(^{5}\text{Notice that equation (1.2) is now considered in } G \setminus \mathcal{K}_3 \text{ but not in the whole of } G.\)
where $W^t_{2, 2\pi}(0, \ 2\pi)$ is the closure of the set of infinitely differentiable $2\pi$-periodic functions in $W^t_{2}(0, \ 2\pi)$.

From [27 § 1] and [9 § 1], it follows that for each $g \in \mathcal{K}_3 \cup \mathcal{K}_3$ there is a finite-meromorphic operator-valued function $\mathcal{L}_g^{-1}(\lambda)$ such that (I) its poles, maybe with the exception of finitely many of them, belong to a double angle of opening $< \pi$, containing the imaginary axis, and (II) for a $\lambda$ which is not a pole of $\mathcal{L}_g^{-1}(\lambda)$, the operator $\mathcal{L}_g^{-1}(\lambda)$ is the bounded inverse for $\mathcal{L}_g(\lambda)$.

From Theorem 1.1 [27] and results of [9 § 1], it follows that the operator $\mathcal{L}_g$ is an isomorphism if and only if the line $\text{Im} \lambda = b + 1 - l - 2m$ contains no poles of $\mathcal{L}_g^{-1}(\lambda)$.

Theorem 5.2. Let Conditions 1.1–1.4 and 5.1 hold. Suppose that $b \in \mathbb{R}$ is such that for all $g \in \mathcal{K}_1$ the operator $\mathcal{L}_g^0$ is an isomorphism and for all $g \in \mathcal{K}_2 \cup \mathcal{K}_3$ the operator $\mathcal{L}_g$ is an isomorphism.

Then the operator $L: \mathcal{H}^{1+2m}_b(G) \to \mathcal{H}^l_b(G, \ Y)$ is Fredholm.

Proof. Notice that Lemmas 4.1 and 5.2 are true for any $b \in \mathbb{R}$ for which the operators $\mathcal{L}_g^0$, $g \in \mathcal{K}_1$, are isomorphisms. Therefore, using Lemmas 4.1 and 5.2 analogously to the proof of Theorem 3.4 [9], we can obtain an a priori estimate (4.8) (in the spaces $\mathcal{H}^l_b(\cdot)$) and construct a right regularizer. \hfill \square

6 Index stability for nonlocal elliptic problems

In this section, we study an influence of the transformations $\omega_{is}$ upon the index of nonlocal elliptic problems. We show that the index of the problem is determined by the linear part of the transformations $\omega_{is}$ in a neighborhood of $\mathcal{K}_1$. Notice that, in the case where the support $\bigcup \omega_{is}(\bar{Y}_i)$ of nonlocal terms does not intersect with the set $\mathcal{K}_1$ consisting of the points of conjugation of nonlocal conditions, the index stability for the corresponding problem was proved in [15].

1. Parallel to problem (1.2), (1.3), we consider the following problem:

$$P(x, \ D)u = f_0(x) \quad (x \in G),$$

$$\hat{B}_{i\mu}(x, \ D)u \equiv \sum_{s=0}^{\hat{S}_i} \langle \hat{B}_{i\mu s}(x, \ D)u \rangle (\hat{\omega}_{is}(x))_{|\tau_i} = g_{i\mu}(x) \quad (x \in \bar{Y}_i; \ i = 1, \ldots, N_0; \ \mu = 1, \ldots, m).$$

Here $P(x, \ D)$, $\hat{B}_{i\mu 0}(x, \ D) = B_{i\mu 0}(x, \ D)$ are the same 6 differential operators as those in § 11 $\hat{B}_{i\mu s}(x, \ D)$ ($s = 1, \ldots, \hat{S}_i$) are some differential operators of orders $m_{i\mu}$ with complex-valued $C^\infty$-coefficients; $\hat{\omega}_{is}$ ($i = 1, \ldots, N_0; \ s = 1, \ldots, \hat{S}_i$) are infinitely differentiable non-degenerate transformations mapping some neighborhood $O_i$ of the manifold $\bar{Y}_i$ onto $\hat{\omega}_{is}(O_i)$ so that $\hat{\omega}_{is}(\bar{Y}_i) \subset G; \ \omega_{i0}(x) \equiv x$. We assume that the set

$$\hat{\mathcal{K}} = \left\{ \bigcup_i (\bar{Y}_i \setminus Y_i) \right\} \cup \left\{ \bigcup_{i,s} \hat{\omega}_{is}(\bar{Y}_i \setminus Y_i) \right\} \cup \left\{ \bigcup_{p, i,s} \hat{\omega}_{jp}(\hat{\omega}_{is}(\bar{Y}_i \setminus Y_i) \cap Y_j) \right\}$$

6It suffices that only the principal homogeneous parts of the operators $P(x, \ D)$ and $\hat{B}_{i\mu 0}(x, \ D)$ from this section and those from § 11 coincide. But, for simplicity, we assume that junior terms of the corresponding operators also coincide.
can be represented in the form \( \hat{\mathcal{K}} = \bigcup_{j=1}^{3} \bigcup_{p=1}^{N_j} \hat{\mathcal{K}}_{jp} \), where

\[
\hat{\mathcal{K}}_1 = \bigcup_{p=1}^{N_1} \hat{\mathcal{K}}_{1p} = \partial G \setminus \bigcup_{i=1}^{N_0} \mathcal{Y}_i, \quad \hat{\mathcal{K}}_2 = \bigcup_{p=1}^{N_2} \hat{\mathcal{K}}_{2p} \subset \bigcup_{i=1}^{N_0} \mathcal{Y}_i, \quad \hat{\mathcal{K}}_3 = \bigcup_{p=1}^{N_3} \hat{\mathcal{K}}_{3p} \subset G
\]

(cf. (1.1)). Here \( \hat{\mathcal{K}}_{jp} \) are disjoint \((n - 2)\)-dimensional \(C^\infty\)-manifolds without a boundary (points if \( n = 2 \)); moreover, \( N_1 = N_1, \hat{\mathcal{K}}_{1p} = \mathcal{K}_{1p}, p = 1, \ldots, N_1 \).

Let the transformations \( \hat{\omega}_{is} \) satisfy Conditions 1.3 and 1.4. Furthermore, we assume that the operators \( \hat{\mathcal{B}}_{i_s}(x, D) \) and the transformations \( \hat{\omega}_{is} \) \((s = 1, \ldots, \hat{\ell}_i)\) are such that for each \( g \in \hat{\mathcal{K}}_1 = \mathcal{K}_1 \) the operator \( \mathcal{L}_g^\omega \) (which is defined similarly to the operator \( \mathcal{L}_g^\omega \) from § 1) equals the operator \( \mathcal{L}_g^\omega \) defined in § 1.

Thus, \( \hat{\omega}_{is} \) is a linear part of \( \omega_{is} \) in a neighborhood of \( \mathcal{K}_1 \).

We introduce the bounded operator corresponding to nonlocal problem (6.1), (6.2):

\[
\hat{\mathbf{L}} = \{ \mathbf{P}(x, D), \hat{\mathcal{B}}_{i_s}(x, D) \} : H_b^{l+2m}(G) \rightarrow \mathcal{H}_b^l(G, \mathcal{Y}).
\]

**Theorem 6.1.** Let the conditions of Lemma 4.2 hold and \( b > l + 2m - 1 \). Then the operators \( \mathbf{L}, \hat{\mathbf{L}} : H_b^{l+2m}(G) \rightarrow \mathcal{H}_b^l(G, \mathcal{Y}) \) are Fredholm and \( \text{ind} \mathbf{L} = \text{ind} \hat{\mathbf{L}} \).

**Proof.** We consider the operator \( \mathbf{L}_t : H_b^{l+2m}(G) \rightarrow \mathcal{H}_b^l(G, \mathcal{Y}) \) given by

\[
\mathbf{L}_t u = \{ \mathbf{P}(x, D)u, \hat{\mathcal{B}}_{i_s}(x, D) + t(\hat{\mathcal{B}}_{i_s}(x, D) - \mathcal{B}_{i_s}(x, D)) \}.
\]

Obviously, \( \mathbf{L}_0 = \mathbf{L}, \mathbf{L}_1 = \hat{\mathbf{L}} \).

In a neighborhood of \( \mathcal{K}_1 \), the transformations \( \omega_{is} \) and \( \hat{\omega}_{is} \) coincide up to infinitesimals; therefore, by Theorem 5.1 the operators \( \mathbf{L}_t \) are Fredholm for all \( t \). Furthermore, for all \( t_0 \) and \( t \), we have

\[
\| \mathbf{L}_t u - \mathbf{L}_{t_0} u \|_{\mathcal{H}_b^l(G, \mathcal{Y})} \leq k_{t_0} |t - t_0| \| u \|_{H_b^{l+2m}(G)},
\]

where \( k_{t_0} > 0 \) is independent of \( t \in [0, 1] \). Hence, by Theorem 16.2 \([29]\), we have \( \text{ind} \mathbf{L}_t = \text{ind} \mathbf{L}_{t_0} \) for all \( t \) from some small neighborhood of \( t_0 \). These neighborhoods cover the segment \([0, 1]\). Choosing a finite subcovering, we get \( \text{ind} \mathbf{L} = \text{ind} \mathbf{L}_0 = \text{ind} \mathbf{L}_1 = \text{ind} \hat{\mathbf{L}}. \]

Analogously to the above, using Theorem 5.2 instead of Theorem 5.1 one can prove the index stability for nonlocal problem (1.2), (1.3) in the case where \( n = 2, b \in \mathbb{R} \).

Let us suppose that \( \hat{N}_j = N_j, \hat{\mathcal{K}}_{jp} = \mathcal{K}_{jp}, j = 1, 2, 3, p = 1, \ldots, N_j \).

**Theorem 6.2.** Let the conditions of Theorem 5.2 hold. Then the operators \( \mathbf{L}, \hat{\mathbf{L}} : \bar{H}_b^{l+2m}(G) \rightarrow \mathcal{H}_b^l(G, \mathcal{Y}) \) are Fredholm and \( \text{ind} \mathbf{L} = \text{ind} \hat{\mathbf{L}} \).

2. In this subsection, we present another proof of Theorem 6.2 based upon ideas of \([15]\). (Notice that, using Lemma 5.2 \([12]\), one can similarly prove Theorem 5.1.) The proof given below is more complicated; however it makes clear the phenomenon—why index of the operator is completely determined by the linear part of the transformations \( \omega_{is} \) in a neighborhood of \( \mathcal{K}_1 \). We show that if the operators \( \mathbf{L} \) and \( \hat{\mathbf{L}} \) are both Fredholm, then the restriction of their difference to the kernel \( \text{ker} (\mathbf{P}) \subset \bar{H}_b^{l+2m}(G) \) of the operator \( \mathbf{P} = \mathbf{P}(y, D) \) (we remind that \( x = y \) if \( n = 2 \)) can be “reduced” to the sum of an operator with an arbitrary small norm and an operator the
square of which is compact. The first operator appears at the expense of the nonlinear part of the transformations $\omega_{i,n}$ near $\mathcal{K}_1$ while the second one appears at the expense of transformations originating the sets $\mathcal{K}_2$ and $\mathcal{K}_3$ (see § 2). Notice that this “reduction” does not contradict the example of § 2 since the “reduction” procedure contains projecting to the subspace $\text{ker } (\mathbf{P})$ of infinite codimension. By the same reason, the considerations below do not prove that the operator $\hat{\mathbf{L}}$ is Fredholm whenever $\mathbf{L}$ is Fredholm (or vice versa). The only thing they imply is that $\text{ind } \mathbf{L} = \text{ind } \hat{\mathbf{L}}$ whenever we are a priori aware of $\mathbf{L}$ and $\hat{\mathbf{L}}$ being both Fredholm.

Thus, let us proceed to the alternative proof of Theorem 6.2.

1) We introduce the operators

$$
\mathbf{B}, \hat{\mathbf{B}} : \tilde{\mathcal{H}}^{l+2m}_b(G) \to \tilde{\mathcal{H}}^l_b(\partial G) = \prod_{i=1}^{N_0} \prod_{\mu=1}^m \tilde{\mathcal{H}}^{l+2m-\mu-1/2}_b(\gamma_i)
$$

given by $\mathbf{B} = \{\mathbf{B}_{i\mu}(y, D)\}$, $\hat{\mathbf{B}} = \{\hat{\mathbf{B}}_{i\mu}(y, D)\}$. We denote by $\mathbf{C}$, $\hat{\mathbf{C}}$ the restrictions of the operators $\mathbf{B}$, $\hat{\mathbf{B}}$ to the subspace $\text{ker } (\mathbf{P}) \subset \tilde{\mathcal{H}}^{l+2m}_b(G)$. By Theorem 5.1 the operators $\mathbf{L}$, $\hat{\mathbf{L}}$ are Fredholm. Therefore, by virtue of Lemma 1.1 [15], the operators $\mathbf{C}$, $\hat{\mathbf{C}}$ are also Fredholm. Now, to prove Theorem 6.2 it suffices to show that $\text{ind } \mathbf{C} = \text{ind } \hat{\mathbf{C}}$.

2) We denote by $\mathbf{C}_1$, $\hat{\mathbf{C}}_1$ the restrictions of $\mathbf{C}$, $\hat{\mathbf{C}}$ to the subspace $\text{ker } (\mathbf{C})^\perp \subset \text{ker } (\mathbf{P})$. It is obvious that $\mathbf{C}_1 = \mathbf{C}\mathbf{I}_0$, $\hat{\mathbf{C}}_1 = \hat{\mathbf{C}}\mathbf{I}_0$, where $\mathbf{I}_0 : \text{ker } (\mathbf{C})^\perp \to \text{ker } (\mathbf{P})$ is the operator of embedding of $\text{ker } (\mathbf{C})^\perp$ into $\text{ker } (\mathbf{P})$. Clearly, we have $\dim \text{ker } (\mathbf{I}_0) = 0$, $\text{codim } \mathcal{R}(\mathbf{I}_0) = \dim \text{ker } (\mathbf{C}) = m_0 < \infty$. Therefore, from Theorem 12.2 [29], it follows that

$$
\text{ind } \mathbf{C}_1 = \text{ind } \mathbf{C} + \text{ind } \mathbf{I}_0 = \text{ind } \mathbf{C} - m_0,
$$

$$
\text{ind } \hat{\mathbf{C}}_1 = \text{ind } \hat{\mathbf{C}} + \text{ind } \mathbf{I}_0 = \text{ind } \hat{\mathbf{C}} - m_0.
$$

Thus, it suffices to prove that $\text{ind } \mathbf{C}_1 = \text{ind } \hat{\mathbf{C}}_1$.

3) We denote by $\mathbf{P}_\perp$ the operator that orthogonally projects $\tilde{\mathcal{H}}^l_b(\partial G)$ onto $\mathcal{R}(\mathbf{C}_1)^\perp$. Since $\text{codim } \mathcal{R}(\mathbf{C}_1) < \infty$, it follows that the operator $\mathbf{P}_\perp$ is finite-dimensional. Therefore, we have

$$
\text{ind } \hat{\mathbf{C}}_1 = \text{ind } (\mathbf{C}_1 + (\mathbf{I} - \mathbf{P}_\perp)(\hat{\mathbf{C}}_1 - \mathbf{C}_1)).
$$

Hence, it suffices to prove that

$$
\text{ind } \mathbf{C}_1 = \text{ind } (\mathbf{C}_1 + (\mathbf{I} - \mathbf{P}_\perp)(\hat{\mathbf{C}}_1 - \mathbf{C}_1)).
$$

Since $\mathbf{C}_1 u, \mathbf{C}_1 u + (\mathbf{I} - \mathbf{P}_\perp)(\hat{\mathbf{C}}_1 - \mathbf{C}_1)u \in \mathcal{R}(\mathbf{C}_1)$ for $u \in \text{ker } (\mathbf{C})^\perp$, we may regard $\mathbf{C}_1$, $\mathbf{C}_1 + (\mathbf{I} - \mathbf{P}_\perp)(\hat{\mathbf{C}}_1 - \mathbf{C}_1)$ as the operators acting from $\text{ker } (\mathbf{C})^\perp$ into $\mathcal{R}(\mathbf{C}_1)$. In this case, the indices of these operators increase the same number $m_1 = \text{codim } \mathcal{R}(\mathbf{C}_1)$.

Evidently, the operator $\mathbf{C}_1 : \text{ker } (\mathbf{C})^\perp \to \mathcal{R}(\mathbf{C}_1)$ has the bound inverse $\mathbf{R}_1 = (\mathbf{C}_1)^{-1} : \mathcal{R}(\mathbf{C}_1) \to \text{ker } (\mathbf{C})^\perp$ and $\text{ind } \mathbf{C}_1 = 0$. By Theorem 12.2 [29], we have

$$
\text{ind } (\mathbf{C}_1 + (\mathbf{I} - \mathbf{P}_\perp)(\hat{\mathbf{C}}_1 - \mathbf{C}_1)) = \text{ind } (\mathbf{I} + \mathbf{R}_1(\mathbf{I} - \mathbf{P}_\perp)(\hat{\mathbf{C}}_1 - \mathbf{C}_1)).
$$

It remains to show that $\text{ind } (\mathbf{I} + \mathbf{R}_1(\mathbf{I} - \mathbf{P}_\perp)(\hat{\mathbf{C}}_1 - \mathbf{C}_1)) = 0$.

4) Let us introduce a function $\psi_\varepsilon \in C^\infty_0(\mathbb{R}^2)$ such that $\psi_\varepsilon(y) = 1$ for $y \in \mathcal{O}_{\varepsilon/2}(\mathcal{K})$, $\psi_\varepsilon(y) = 0$ for $y \notin \mathcal{O}_{\varepsilon}(\mathcal{K})$, and

$$
|D^\alpha \psi_\varepsilon(y)| \leq k_\alpha(\tilde{\rho}(y))^{-|\alpha|} \quad (y \in \mathcal{O}_{\varepsilon}(\mathcal{K})),
$$

(6.3)
where $k_\alpha > 0$ is independent of $\varepsilon$.

We consider the operators $A_1, A_2 : \ker (C) \to \ker (C) \cap \ker (B)$ given by

\[
A_1 u = R_1 (I - P_\perp) (\hat{B} - B) \psi u, \\
A_2 u = R_1 (I - P_\perp) (\hat{B} - B) (1 - \psi) u.
\]

It is clear that $I + A_1 + A_2 = I + R_1 (I - P_\perp) (\hat{C}^1 - C^1)$. Since the support of $(1 - \psi) u$ does not intersect with the origin, it follows from the proof of Theorem 3.1 [15] that the operator $(A_2)^2$ is compact.

Let us study the operator $A_1$. Since the operator $R_1 (I - P_\perp)$ is bounded, it follows that

\[
\|A_1 u\|_{\tilde{H}^{l+2m} (G)} \leq c \|(\hat{B} - B) \psi u\|_{\tilde{H}^{l} (\partial G)}.
\]

From this, using the unity partition method and estimates (4.2)–(4.7), followed by (6.3), we obtain

\[
\|A_1 u\|_{\tilde{H}^{l+2m} (G)} \leq c_1 (\varepsilon) \|\psi u\|_{\tilde{H}^{l+2m} (G)} + \|\hat{P} \psi u\|_{\tilde{H}^{l} (G)} + k_1 (\varepsilon) \|u\|_{\tilde{H}^{l+2m-1} (G)} \leq c_2 (\varepsilon) \|u\|_{\tilde{H}^{l+2m} (G)} + \|\hat{P} \psi u\|_{\tilde{H}^{l} (G)} + k_1 (\varepsilon) \|u\|_{\tilde{H}^{l+2m-1} (G)},
\]

(6.4)

Since $u \in \ker (P)$, from (6.4) and Leibniz’ formula, we get

\[
\|A_1 u\|_{\tilde{H}^{l+2m} (G)} \leq c_2 \varepsilon \|u\|_{\tilde{H}^{l+2m} (G)} + k_2 (\varepsilon) \|u\|_{\tilde{H}^{l+2m-1} (G)},
\]

(6.5)

where $c_2$ is independent of $\varepsilon$. From (6.5), the compactness of the embedding $\tilde{H}^{l+2m} (G) \subset \tilde{H}^{l+2m-1} (G)$, and Lemma 5.1 it follows that $A_1 = M_1 + F_1$, where $\|M_1\| \leq 2c_2 \varepsilon$ and the operator $F_1$ is finite-dimensional.

Thus, we have $R_1 (I - P_\perp)(\hat{C}^1 - C^1) = M_1 + F_1 + A_2$. Therefore, choosing sufficiently small $\varepsilon$, we obtain from Theorems 15.4 and 16.2 [29] that $\text{ind} (I + R_1 (I - P_\perp)(\hat{C}^1 - C^1)) = 0$.

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