SEMILINEAR NONLOCAL ELLIPTIC EQUATIONS WITH CRITICAL AND SUPERCRITICAL EXPONENTS

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ABSTRACT. We study the problem

\[ \begin{cases}
(-\Delta)^s u = u^p - u^q & \text{in } \mathbb{R}^N, \\
u \in H^s(\mathbb{R}^N) \cap L^{q+1}(\mathbb{R}^N), \\
u > 0 & \text{in } \mathbb{R}^N,
\end{cases} \]

where \( s \in (0, 1) \) is a fixed parameter, \((-\Delta)^s\) is the fractional Laplacian in \( \mathbb{R}^N \), \( q > p \geq \frac{N+2s}{N-2s} \) and \( N > 2s \). For every \( s \in (0, 1) \), we establish regularity results of solutions of above equation (whenever solution exists) and we show that every solution is a classical solution. Next, we derive certain decay estimate of solutions and the gradient of solutions at infinity for all \( s \in (0, 1) \). Using those decay estimates, we prove Pohozaev type identity in \( \mathbb{R}^N \) and we show that the above problem does not have any solution when \( p = \frac{N+2s}{N-2s} \). We also discuss radial symmetry and decreasing property of the solution and prove that when \( p > \frac{N+2s}{N-2s} \), the above problem admits a solution. Moreover, if we consider the above equation in a bounded domain with Dirichlet boundary condition, we prove that it admits a solution for every \( p \geq \frac{N+2s}{N-2s} \) and every solution is a classical solution.

1. Introduction. In this paper, we consider the following problem:

\[ \begin{cases}
(-\Delta)^s u = u^p - u^q & \text{in } \mathbb{R}^N, \\
u \in H^s(\mathbb{R}^N) \cap L^{q+1}(\mathbb{R}^N), \\
u > 0 & \text{in } \mathbb{R}^N,
\end{cases} \]  \quad (1.1)

and

\[ \begin{cases}
(-\Delta)^s u = u^p - u^q & \text{in } \Omega, \\
u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \\
u > 0 & \text{in } \Omega, \\
u \in H^s(\Omega) \cap L^{q+1}(\Omega),
\end{cases} \]  \quad (1.2)
where \( s \in (0, 1) \) is fixed, \((-\Delta)^s\) denotes the fractional Laplace operator defined, up to a normalization factors, as

\[
-(-\Delta)^s u(x) = \frac{c_{N,s}}{2} \int_{\mathbb{R}^N} \frac{u(x + y) - 2u(x) + u(x - y)}{|y|^{N + 2s}} dy, \quad x \in \mathbb{R}^N, \tag{1.3}
\]

where

\[
c_{N,s} := \frac{2^{2s} s \Gamma(\frac{N}{2} + s)}{\pi^{\frac{N}{2}} \Gamma(1 - s)}.
\]

In (1.1) and (1.2), \( q > p \geq 2^* = 2^{\frac{N+2s}{N-2s}} \) and \( N > 2s \). In (1.2), \( \Omega \) is a bounded subset of \( \mathbb{R}^N \) with smooth boundary.

We denote by \( H^s(\Omega) \) the usual fractional Sobolev space endowed with the so-called Gagliardo norm

\[
\|g\|_{H^s(\Omega)} = \|g\|_{L^2(\Omega)} + \left( \int_{\Omega \times \Omega} \frac{|g(x) - g(y)|^2}{|x - y|^{N + 2s}} dxdy \right)^{1/2}. \tag{1.4}
\]

For further details on the fractional Sobolev spaces we refer to [22] and the references therein. Note that in problem (1.2), Dirichlet boundary data is given in \( \mathbb{R}^N \setminus \Omega \) and not simply on \( \partial \Omega \). Therefore, for the Dirichlet boundary value problem in the bounded domain, we need to introduce a new functional space \( X_0 \), which, in our opinion, is the suitable space to work with.

\[
X_0 := \{v \in H^s(\mathbb{R}^N) : v = 0 \quad \text{in} \quad \mathbb{R}^N \setminus \Omega \}. \tag{1.5}
\]

By [31, Lemma 6 and 7], it follows that

\[
\|v\|_{X_0} = \left( \int_Q \frac{|v(x) - v(y)|^2}{|x - y|^{N + 2s}} dxdy \right)^{1/2}, \tag{1.6}
\]

where \( Q = \mathbb{R}^{2N} \setminus (\Omega^c \times \Omega^c) \), is a norm on \( X_0 \) and \( (X_0, \|\cdot\|_{X_0}) \) is a Hilbert space, with the inner product

\[
\langle u, v \rangle_{X_0} = \int_Q \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N + 2s}} dxdy.
\]

Observe that, norms in (1.4) and (1.6) are not same, since \( \Omega \times \Omega \) is strictly contained in \( Q \). Clearly, the integral in (1.6) can be extended to whole \( \mathbb{R}^{2N} \) as \( v = 0 \) in \( \mathbb{R}^N \setminus \Omega \).

It is well known that the embedding \( X_0 \hookrightarrow L^r(\mathbb{R}^N) \) is compact, for any \( r \in [1, 2^*) \) (see [31, Lemma 8]) and \( X_0 \hookrightarrow L^{2^*}(\mathbb{R}^N) \) is continuous (see [30, Lemma 9]).

We set

\[
\|u\|^2_{\hat{H}^s(\mathbb{R}^N)} := \frac{c_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N + 2s}} dxdy,
\]

and we define \( \hat{H}^s(\mathbb{R}^N) \) as the completion of \( C_0^\infty(\mathbb{R}^N) \) w.r.t. the norm \( \|u\|_{\hat{H}^s(\mathbb{R}^N)} \) (see \[11\] and \[23\]).

**Definition 1.1.** We say that \( u \in \hat{H}^s(\mathbb{R}^N) \cap L^{2^+}(\mathbb{R}^N) \) is a weak solution of Eq. (1.1), if \( u > 0 \) in \( \mathbb{R}^N \) and for every \( \varphi \in \hat{H}^s(\mathbb{R}^N) \),

\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{N + 2s}} dxdy = \int_{\mathbb{R}^N} u^p \varphi \ dx - \int_{\mathbb{R}^N} u^q \varphi \ dx
\]

or equivalently,

\[
\int_{\mathbb{R}^N} (-\Delta)^{\frac{1}{2}} u (-\Delta)^{\frac{1}{2}} \varphi = \int_{\mathbb{R}^N} u^p \varphi \ dx - \int_{\mathbb{R}^N} u^q \varphi \ dx.
\]
Similarly, when $\Omega$ is a bounded domain, we say $u \in X_0 \cap L^{q+1}(\Omega)$ is a weak solution of Eq. (1.2) if $u > 0$ in $\Omega$ and for every $\varphi \in X_0$, the above integral expression holds.

**Definition 1.2.** A positive function $u \in C(\mathbb{R}^N)$ is said to be a classical solution of

$$(-\Delta)^s u = f(u) \quad \text{in} \quad \mathbb{R}^N,$$  

(1.7)

if $(-\Delta)^s u$ can be written as (1.3) and (1.7) is satisfied pointwise in all $\mathbb{R}^N$.

In recent years, a great deal of attention has been devoted to fractional and non-local operators of elliptic type. One of the main reasons comes from the fact that this operator naturally arises in several physical phenomenon like flames propagation and chemical reaction of liquids, population dynamics, geophysical fluid dynamics, mathematical finance etc (see [1, 9, 33, 34] and the references therein).

When $s = 1$, it follows by celebrated Pohozaev identity that (1.1) does not have any solution when $p = 2^* - 1$ and $q > p$. In this paper we prove this result for all $s \in (0, 1)$ by establishing the Pohozaev identity in $\mathbb{R}^N$ for the equation (1.1). We recall that (1.1) has an equivalent formulation by Caffarelli-Silvestre harmonic extension method in $\mathbb{R}^{N+1}_+$. For spectral fractional laplace equation in bounded domain, some Pohozaev type identities were proved in [5, 6, 7]. In [13], Fall and Weth have proved some nonexistence results associated with the problem $(-\Delta)^s u = f(x, u)$ in $\Omega$ and $u = 0$ in $\mathbb{R}^N \setminus \Omega$ by applying method of moving spheres.

Very recently Ros-Oton and Serra [27, Theorem 1.1] have proved Pohozaev identity by direct method for the bounded solution of Dirichlet boundary value problem. More precisely they have proved the following:

Let $u$ be a bounded solution of

$$
\begin{cases}
(-\Delta)^s u = f(u) & \text{in} \quad \Omega, \\
u = 0 & \text{in} \quad \mathbb{R}^N \setminus \Omega,
\end{cases}
$$

(1.8)

where $\Omega$ is a bounded $C^{1,1}$ domain in $\mathbb{R}^N$, $f$ is locally Lipschitz and $\delta(x) = \text{dist}(x, \partial \Omega)$. Then $u$ satisfies the following identity:

$$(2s - N) \int_\Omega u f(u) \, dx + 2N \int_\Omega F(u) \, dx = \Gamma(1 + s)^2 \int_{\partial \Omega} \left(\frac{u}{\delta(\nu)}\right)^2 (x \cdot \nu) dS,$$

where $F(t) = \int_0^t f$ and $\nu$ is the unit outward normal to $\partial \Omega$ at $x$ and $\Gamma$ is the Gamma function. For nonexistence result with general integro-differential operator we cite [28].

To apply the technique of [27] in the case of $\Omega = \mathbb{R}^N$, one needs to know decay estimate of $u$ and $\nabla u$ at infinity. In [27], Ros-Oton and Serra have remarked that assuming certain decay condition of $u$ and $\nabla u$, one can show that $(-\Delta)^s u = u^p$ in $\mathbb{R}^N$ does not have any nontrivial solution for $p > \frac{N+2s}{N-2s}$. In this article for (1.1) we first establish decay estimate of $u$ and $\nabla u$ at infinity and then using that we prove Pohozaev identity for the solution of (1.1) for all $s \in (0, 1)$ and consequently we deduce the nonexistence of nontrivial solution when $p = 2^* - 1$. In the appendix, using harmonic extension method in the spirit of Cabrè and Cinti [6], we give an alternative proof of Pohozaev identity in $\mathbb{R}^N$ for the equation of the form

$$(-\Delta)^s u = f(u) \quad \text{in} \quad \mathbb{R}^N,$$

where $u \in \dot{H}^s(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and $f \in C^2$. The interesting fact about this proof is that, here we do not require decay estimate of $u$ and $\nabla u$ at infinity as we use suitable cut-off function and in limit we take that cut-off function approaches to 1.
On the contrary to the nonexistence result for $p = 2^* - 1$, we show using constrained minimization method that Eq. (1.1) admits a positive solution when $p > 2^* - 1$. Moreover, we study the qualitative properties of solution. More precisely, using Moser iteration technique we prove that any solution, of (1.1) is in $L^\infty(\mathbb{R}^N)$ and we establish decay estimate of $u$ and $\nabla u$. Then using the Schauder estimate from [25] and the $L^\infty$ bound that we establish, we show that $u \in C^\infty(\mathbb{R}^N)$ if both $p$ and $q$ are integer and $C^{2ks+2s}(\mathbb{R}^N)$, where $k$ is the largest integer satisfying $\lfloor 2ks \rfloor < p$ if $p \notin \mathbb{N}$ and $\lfloor 2ks \rfloor < q$ if $p \in \mathbb{N}$ but $q \notin \mathbb{N}$, where $\lfloor 2ks \rfloor$ denotes the greatest integer less than equal to $2ks$.

Moreover, we study the qualitative properties of solution. More precisely, using Theorem 1.7.

We define the functional
\[ F(v, \Omega) = \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} \, dx \, dy + \frac{1}{q+1} \int_{\Omega} |v|^{q+1} \, dx. \] (1.11)

Define,
\[ \mathcal{K} := \inf \left\{ F(v, \mathbb{R}^N) : v \in \dot{H}^s(\mathbb{R}^N) \cap L^{q+1}(\mathbb{R}^N), \int_{\mathbb{R}^N} |v|^{q+1} \, dx = 1 \right\}. \] (1.12)

Theorem 1.7. Let $s \in (0,1)$ and $q > p > 2^* - 1$. Then $\mathcal{K}$ in (1.12) is achieved by a radially decreasing function $u \in \dot{H}^s(\mathbb{R}^N) \cap L^{q+1}(\mathbb{R}^N)$ and Eq. (1.1) admits a nonnegative solution. Furthermore, if $q \geq (p-1) \frac{N}{2s} - 1$, then Eq. (1.1) admits a positive solution.
When Ω is a smooth bounded domain, we define
\[ S_{Ω} := \inf \left\{ F(v, Ω) : v \in X_{0} ∩ L^{q+1}(Ω), \int_{Ω} |u|^{p+1} dx = 1 \right\}. \] (1.13)

**Theorem 1.8.** Let s ∈ (0, 1) and q > p ≥ 2* − 1. Then S_{Ω} in (1.13) is achieved by a function u ∈ X_{0} ∩ L^{q+1}(Ω). Furthermore, there exists a constant λ > 0, such that u satisfies
\[ \begin{cases} (-Δ)^{s}u = λ|u|^{p-1}u - |u|^{q-1}u & \text{in } Ω, \\ u = 0 & \text{in } R^{N} \setminus Ω. \end{cases} \] (1.14)
Furthermore, if p ≥ 2* − 1 and q > (p - 1)\( \frac{N}{2p} \) - 1, then Eq.(1.14) admits a positive solution.

Note that the scaled function U = \( \lambda^{\frac{q}{p-q}} \) u satisfies the equation
\[ (-Δ)^{s}U = U^{p} - c^{*}U^{q}, \quad c^{*} = λ^{\frac{q}{p-q}}. \] (1.15)

We organise the paper as follows. In section 2, we recall equivalent formulation of (1.1) by the Caffarelli-Silvestre [8] associated extension problem-a local PDE in \( R^{N+1} \) and we also recall Schauder estimate for the nonlocal equation proved by Ros-Oton and Serra [25]. In Section 3, we establish u ∈ L^{∞}(R^{N}), decay estimate of solution and the gradient of solution at infinity. Section 4 deals with the proof of nonexistence result in \( R^{N} \) when p = 2* - 1. In section 5, we show that any solution of (1.1) is radially symmetric and strictly decreasing about some point in \( R^{N} \). While in section 6, we prove existence of solution to (1.1) for p > 2* - 1 and to (1.2) when Ω is bounded and p ≥ 2* - 1.

**Notations:** Throughout this paper we use the notation \( C^{β}(R^{N}) \), with β > 0 to refer the space \( C^{k,β}(R^{N}) \), where k is the greatest integer such that k < β and \( β' = β - k \). According to this, \( \|u\|_{C^{β}(R^{N})} \) denotes the following seminorm
\[ [u]_{C^{β}(R^{N})} = [u]_{C^{k,β}(R^{N})} = \sup_{x,y \in R^{N}, x \neq y} \frac{|D^{k}u(x) - D^{k}u(y)|}{|x - y|^β}. \]
Throughout this paper, C denotes the generic constant, which may vary from line to line and n denotes the unit outward normal.

**2. Preliminaries.** In this section we recall the other useful representation of fractional laplacian \((-Δ)^{s}\), which we will use to prove decay estimate of solution at infinity. Using the celebrated Caffarelli and Silvestre extension method, (see [8]), fractional laplacian \((-Δ)^{s}\) can be seen as a trace class operator (see [8, 15, 2]). Let u ∈ \( \dot{H}^{s}(R^{N}) \) be a solution of (1.1). Define w := E_{s}(u) be its s— harmonic extension to the upper half space \( R^{N+1}_{+} \), that is, there is a solution to the following problem:
\[ \begin{cases} \text{div}(y^{1-2s} \nabla w) = 0 & \text{in } R^{N+1}_{+}, \\ w = u & \text{on } R^{N} \times \{y = 0\}. \end{cases} \] (2.1)

Define the space \( X^{2s}(R^{N+1}) := \text{closure of } C^{∞}_{0}(R^{N+1}) \) w.r.t. the following norm
\[ \|w\|_{2s} = \|w\|_{X^{2s}(R^{N+1})} := \left( \frac{k_{2s}}{k_{2s}} \int_{R^{N+1}_{+}} y^{1-2s} |\nabla w|^{2} dx dy \right)^{\frac{1}{2}}, \]
where \( k_{2s} = \frac{Γ(\frac{1}{s})}{Γ(1-s)}(\frac{1}{(1-s)})^{\frac{1}{2}} \) is a normalizing constant, chosen in such a way that the extension operator \( E_{s} : \dot{H}^{s}(R^{N}) \rightarrow X^{2s}(R^{N+1}) \) is an isometry (up to constants),
that is, \( \|E_s u\|_{2s} = \|u\|_{H^s(\mathbb{R}^N)} = |(-\Delta)^s u|_{L^2(\mathbb{R}^N)} \). (see [11]). Conversely, for a function \( w \in X^{2s}(\mathbb{R}^{N+1}_+) \), we denote its trace on \( \mathbb{R}^N \times \{y = 0\} \) as:

\[
\text{Tr}(w) := w(x,0).
\]

This trace operator satisfies:

\[
\|w(.,0)\|_{H^s(\mathbb{R}^N)} = \|\text{Tr}(w)\|_{H^s(\mathbb{R}^N)} \leq \|w\|_{2s}.
\]  \(2.2\)

Consequently,

\[
\left( \int_{\mathbb{R}^N} |u(x)|^{2^*} \, dx \right)^{\frac{1}{2^*}} \leq S(N,s) \int_{\mathbb{R}^{N+1}_+} y_1^{1-2s}|\nabla w(x,y)|^2 \, dx \, dy. \tag{2.3}
\]

Inequality (2.3) is called the trace inequality. We note that \( H^1(\mathbb{R}^{N+1}_+, y^{1-2s}) \), up to a normalizing factor, is isometric to \( X^{2s}(\mathbb{R}^{N+1}_+) \) (see [15]). In [8], it is shown that \( E_s(u) \) satisfies the following:

\[
(-\Delta)^s u(x) = \frac{\partial w}{\partial \nu^{2s}} := -k_{2s} \lim_{y \to 0^+} y_1^{1-2s} \frac{\partial w}{\partial y}(x,y).
\]

With this above representation, (2.1) can be rewritten as:

\[
\begin{cases}
\text{div}(y^{1-2s} \nabla w) = 0 & \text{in } \mathbb{R}^{N+1}, \\
\frac{\partial w}{\partial \nu^{2s}} = w^p(.,0) - w^q(.,0) & \text{on } \mathbb{R}^N.
\end{cases} \tag{2.4}
\]

A function \( w \in X^{2s}(\mathbb{R}^{N+1}_+) \) is said to be a weak solution to (2.4) if for all \( \varphi \in X^{2s}(\mathbb{R}^{N+1}_+) \), we have

\[
k_{2s} \int_{\mathbb{R}^{N+1}_+} y_1^{1-2s} \nabla w \nabla \varphi \, dx \, dy = \int_{\mathbb{R}^N} w^p(x,0) \varphi(x,0) \, dx - \int_{\mathbb{R}^N} w^q(x,0) \varphi(x,0) \, dx.
\]  \(2.5\)

Note that for any weak solution \( w \in X^{2s}(\mathbb{R}^{N+1}_+) \) to (2.4), the function \( u := \text{Tr}(w) = w(.,0) \in H^s(\mathbb{R}^N) \) is a weak solution to (1.1).

Next, we recall Schauder estimate for the nonlocal equation by Ros-Oton and Serra [25].

**Theorem 2.1** (Ros-Oton and Serra, [25]). Let \( s \in (0,1) \) and \( u \) be any bounded weak solution to

\[
(-\Delta)^s u = f \quad \text{in } B_1(0).
\]

Then,

(a) If \( u \in L^\infty(\mathbb{R}^N) \) and \( f \in L^\infty(B_1(0)) \),

\[
\|u\|_{C^{2s}(B_{\frac{1}{2}}(0))} \leq C(\|u\|_{L^\infty(\mathbb{R}^N)} + \|f\|_{L^\infty(B_1(0))}) \quad \text{if } s \neq \frac{1}{2}
\]

and

\[
\|u\|_{C^{2s-\epsilon}(B_{\frac{1}{2}}(0))} \leq C(\|u\|_{L^\infty(\mathbb{R}^N)} + \|f\|_{L^\infty(B_1(0))}) \quad \text{if } s = \frac{1}{2},
\]

for all \( \epsilon > 0 \).

(b) If \( f \in C^\alpha(B_1(0)) \) and \( u \in C^\alpha(\mathbb{R}^N) \) for some \( \alpha > 0 \), then

\[
\|u\|_{C^{\alpha + 2s}(B_{\frac{1}{2}}(0))} \leq C(\|u\|_{C^\alpha(\mathbb{R}^N)} + \|f\|_{C^\alpha(B_1(0))}),
\]

whenever \( \alpha + 2s \) is not an integer. The constant \( C \) depends only on \( N, s, \alpha, \epsilon \).
We conclude this section by recalling some weighted embedding results from Tan and Xiong [32]. For this, we introduce the following notations

\[ Q_R = B_R \times [0, R) \subset \mathbb{R}^{N+1}, \]

where \( B_R \) is a ball in \( \mathbb{R}^N \) with radius \( R \) and centered at origin. Note that, \( B_R \times \{0\} \subset Q_R \). We define,

\[ H(Q_R, y^{1-2s}) := \left\{ U \in H^1(Q_R) : \int_{Q_R} y^{1-2s}(U^2 + |\nabla U|^2) \, dx \, dy < \infty \right\} \]

and \( X^s_0(Q_R) \) is the closure of \( C^\infty_0(Q_R) \) with respect to the norm

\[ ||w||_{X^s_0(Q_R)} = \left( \int_{Q_R} y^{1-2s}|\nabla w|^2 \, dx \, dy \right)^\frac{1}{2}. \]

We note that, \( s \in (0, 1) \) implies the weight \( y^{1-2s} \) belongs to the Muckenhoupt class \( A_2 \) (see [21]) which consists of all non-negative functions \( w \) on \( \mathbb{R}^{N+1} \) satisfying for some constant \( C \), the estimate

\[ \sup_B \left( \frac{1}{|B|} \int_B w \, dx \right) \left( \frac{1}{|B|} \int_B w^{-1} \, dx \right) \leq C, \]

where the supremum is taken over all balls \( B \) in \( \mathbb{R}^{N+1} \).

**Lemma 2.2.** Let \( f \in X^s_0(Q_R) \). Then there exists constant \( C \) and \( \delta > 0 \) depending only on \( N \) and \( s \) such that for any \( 1 \leq k \leq \frac{N+1}{n} + \delta \),

\[ \left( \int_{Q_R} y^{1-2s}|f|^{2k} \, dx \, dy \right)^\frac{1}{2k} \leq C(R) \left( \int_{Q_R} y^{1-2s}|\nabla f|^2 \, dx \, dy \right)^\frac{1}{2}. \]

**Proof.** It is known from [32, Lemma 2.1] that the lemma holds for \( f \in C^1_c(Q_R) \) (also see [12]). For general \( f \), the lemma can be easily proved applying density argument and Fatou’s lemma. \( \square \)

**Lemma 2.3.** Let \( f \in X^s_0(Q_R) \). Then there exists a positive constant \( \delta \) depending only on \( N \) and \( s \) such that

\[ \int_{B_R \times \{y=0\}} |f|^2 \, dx \leq \epsilon \int_{Q_R} y^{1-2s}|\nabla f|^2 \, dx \, dy + \frac{C(R)}{\epsilon^\delta} \int_{Q_R} y^{1-2s}|f|^{2} \, dx \, dy, \]

for any \( \epsilon > 0 \).

**Proof.** If \( f \in C^1_c(Q_R) \), then the lemma holds (see [32, Lemma 2.3]). For \( f \in X^s_0(Q_R) \), there exists \( f_\in \in C^\infty_0(Q_R) \) such that \( f_n \to f \) in \( ||.||_{X^s_0(Q_R)} \) and for \( f_n \), we have

\[ \int_{B_R \times \{y=0\}} |f_n|^2 \, dx \leq \epsilon \int_{Q_R} y^{1-2s}|\nabla f_n|^2 \, dx \, dy + \frac{C(R)}{\epsilon^\delta} \int_{Q_R} y^{1-2s}|f_n|^{2} \, dx \, dy, \quad (2.6) \]

for any \( \epsilon > 0 \). Clearly the 1st integral on RHS converges to \( \int_{Q_R} y^{1-2s}|\nabla f|^2 \, dx \, dy \).

Thanks to Lemma 2.2, it follows that the embedding \( X^s_0(Q_R) \hookrightarrow L^2(Q_R, y^{1-2s}) \) is continuous. Therefore, we can also pass to the limit in the 2nd integral of the RHS. On the other hand, using the trace embedding result, we can also pass to the limit on LHS. Hence, the lemma follows. \( \square \)
3. \(L^\infty\) estimate and decay estimates. Proof of Theorem 1.3

Proof. Case 1: Suppose \(\Omega = \mathbb{R}^N\).

Let \(u\) be an arbitrary weak solution of Eq. (1.1). We first prove that \(u \in L^\infty(\mathbb{R}^N)\) by Moser iterative technique (see, for example [17, 32]). From Section 2, we know that \(w(x, y)\), the \(s\)-harmonic extension of \(u\), is a solution of (2.4).

Let \(B_r\) denote the ball in \(\mathbb{R}^N\) of radius \(r\) and centered at origin. We define

\[
Q_r = B_r \times [0, r).
\]

Set \(\tilde{w} = w^+ + 1\) and for \(L > 1\), define

\[
w_L = \begin{cases} 
\tilde{w} & \text{if } w < L \\
1 + L & \text{if } w \geq L.
\end{cases}
\]

For \(t > 1\), we choose the test function \(\varphi\) in (2.5) as follows:

\[
\varphi(x, y) = \eta^2(x, y)(\tilde{w}(x, y)w_L^{2(t-1)}(x, y) - 1), \tag{3.1}
\]

where \(\eta \in C_0^\infty(Q_r)\) with \(0 \leq \eta \leq 1\), \(\eta = 1\) in \(Q_r\), \(0 < r < R \leq 1\) and \(|\nabla \eta| \leq \frac{2}{R-r}\).

Note that \(\varphi \in X^{2s}(\mathbb{R}^{N+1}_+)\). Using this test function, we obtain from (2.5)

\[
k_{2s} \int_{\mathbb{R}^{N+1}_+} y^{1-2s} \nabla w(x, y) \nabla \left( \eta^2(x, y)(\tilde{w}(x, y)w_L^{2(t-1)}(x, y) - 1) \right) \, dx dy \\
= \int_{\mathbb{R}^N} \left( w^p(x, 0) - w^q(x, 0) \right) \eta^2(x, 0)(\tilde{w}(x, 0)w_L^{2(t-1)}(x, 0) - 1) \, dx. \tag{3.2}
\]

Direct calculation yields

\[
\nabla (\eta^2 (\tilde{w}w_L^{2(t-1)} - 1)) = 2\eta (\tilde{w}w_L^{2(t-1)} - 1) \nabla \eta + \eta^2 w_L^{2(t-1)} \nabla \tilde{w} + 2(t-1) \eta^2 \tilde{w}w_L^{2(t-1)} - 1 \nabla w_L. \tag{3.3}
\]

Here we observe that on the set \(\{ w < 0 \}\), we have \(\varphi = 0\) and \(\nabla \varphi = 0\). Thus (3.2) remains same if we change the domain of integration to \(\{ w \geq 0 \}\). Therefore, in the support of the integrand \(\nabla w = \nabla \tilde{w}\). As a result, substituting (3.3) into (3.2), it follows

\[
k_{2s} \int_{\mathbb{R}^{N+1}_+} y^{1-2s} \left( 2\eta (\tilde{w}w_L^{2(t-1)} - 1) \nabla \eta \nabla \tilde{w} + \eta^2 w_L^{2(t-1)} \nabla \tilde{w}w_L \nabla w \right) \, dx dy \\
\leq \int_{\mathbb{R}^N} \eta^2(x, 0)w^p(x, 0)\tilde{w}(x, 0)w_L^{2(t-1)}(x, 0) \, dx.
\]

Notice that in the support of the integrand of second integral on the LHS \(\nabla \tilde{w} = \nabla w\) and in the third integral \(w_L = \tilde{w}\), \(\nabla w_L = \nabla w\). Hence the above expression reduces to

\[
k_{2s} \int_{\mathbb{R}^{N+1}_+} y^{1-2s} \left( 2\eta (\tilde{w}w_L^{2(t-1)} - 1) \nabla \eta \nabla \tilde{w} + \eta^2 w_L^{2(t-1)} |\nabla \tilde{w}|^2 + 2(t-1) \eta^2 w_L^{2(t-1)} |\nabla w_L|^2 \right) \, dx dy \\
\leq \int_{\mathbb{R}^N} \eta^2(x, 0)\tilde{w}^{p+1}(x, 0)w_L^{2(t-1)}(x, 0) \, dx, \tag{3.4}
\]

where for the RHS, we have used the fact that \(w \leq \tilde{w}\).
Using Young’s inequality we have,

\[ |2\eta(\bar{w}w_L^{2(t-1)} - 1)\nabla \eta \nabla \bar{w}| \leq \frac{1}{2} \eta^2 w_L^{2(t-1)}|\nabla \bar{w}|^2 + 2\bar{w}^2 w_L^{2(t-1)}|\nabla \eta|^2. \]  

(3.5)

Using (3.5), from (3.4) we obtain,

\[
\frac{k_2s}{2} \int_{\mathbb{R}^{N+1}} y^{1-2s} \left(|\nabla \bar{w}|^2 + (t-1)|\nabla w_L|^2\right) \eta^2 w_L^{2(t-1)}(x,y)dx\,dy \\
\leq 2k_2s \int_{\mathbb{R}^{N+1}} y^{1-2s} \bar{w}^2 w_L^{2(t-1)}|\nabla \eta|^2(x,y)dx\,dy \\
+ \int_{\mathbb{R}^N} \bar{w}^{p+1} w_L^{2(t-1)} \eta^2(x,0)dx. \tag{3.6}
\]

As \( t > 1 \) and \( \nabla w_L = 0 \) for \( w \geq L \), it is not difficult to observe that,

\[
\int_{\mathbb{R}^{N+1}} y^{1-2s} |\nabla(\eta \bar{w} w_L^{t-1})|^2 dx\,dy \\
\leq 3 \int_{\mathbb{R}^{N+1}} y^{1-2s} \left(\bar{w}^2 w_L^{2(t-1)}|\nabla \eta|^2 + \eta^2 w_L^{2(t-1)}|\nabla \bar{w}|^2 + (t-1)^2 \eta^2 w_L^{2(t-1)}|\nabla w_L|^2\right) dx\,dy \\
\leq 3t \int_{\mathbb{R}^{N+1}} y^{1-2s} \bar{w}^2 w_L^{2(t-1)}|\nabla \eta|^2 dx\,dy \\
+ 3t \int_{\mathbb{R}^{N+1}} y^{1-2s} \left(|\nabla \bar{w}|^2 + (t-1)|\nabla w_L|^2\right) \eta^2 w_L^{2(t-1)} dx\,dy. \tag{3.7}
\]

Combining (3.7) and (3.6), we have

\[
k_2s \int_{\mathbb{R}^{N+1}} y^{1-2s} |\nabla(\eta \bar{w} w_L^{t-1})|^2 dx\,dy \\
\leq 3tk_2s \int_{\mathbb{R}^{N+1}} y^{1-2s} \bar{w}^2 w_L^{2(t-1)}|\nabla \eta|^2 dx\,dy \\
+ 3t \left\{4k_2s \int_{\mathbb{R}^{N+1}} y^{1-2s} \bar{w}^2 w_L^{2(t-1)}|\nabla \eta|^2(x,y)dx\,dy + 2 \int_{\mathbb{R}^N} \bar{w}^{p+1} w_L^{2(t-1)} \eta^2(x,0)dx\right\}. \tag{3.8}
\]

For \( p \geq 2^* - 1 \), choose \( \alpha > 1 \) as follows:

\[
\frac{N}{2s} < \alpha < \frac{q+1}{p-1}. \tag{3.9}
\]

Note that for \( p = 2^* - 1 \) the interval \( \left(\frac{N}{2s}, \frac{q+1}{p-1}\right) \) is always a nonempty set. On the other hand, as \( q > (p-1)\frac{N}{2} - 1 \), it follows \( \left(\frac{N}{2s}, \frac{q+1}{p-1}\right) \neq \emptyset \), when \( p > 2^* - 1 \). From (3.9) we have,

\[
(p-1)\alpha < q + 1 \quad \text{and} \quad 2 < \frac{2\alpha}{\alpha - 1} < 2^*. \]

As \( \text{supp}(\eta(\cdot, 0)) \subset B_R \) and \( w(x,0) = u \in L^{q+1}(\mathbb{R}^N) \), it follows \( \bar{w}(\cdot, 0) = w^+(x,0) + 1 = u + 1 \in L^{q+1}(B_1) \). This along with the fact that \( \text{supp} \eta \subset Q_R \), where \( R < 1 \),
we obtain
\[
\int_{\mathbb{R}^N} \tilde{w}^{p+1} w_L^{2(t-1)} \eta^2(x,0) dx = \int_{B_1} \tilde{w}^{p+1} w_L^{2(t-1)} \eta^2(x,0) dx \\
= \int_{B_1} |\eta \tilde{w} w_L^{(t-1)}(x,0)|^2 \tilde{w}^{p-1}(x,0) dx \\
\leq \left( \int_{B_1} \tilde{w}^{\alpha(p-1)}(x,0) dx \right)^{\frac{1}{\alpha}} \left( \int_{B_R} |\eta \tilde{w} w_L^{(t-1)}| \tilde{w}^{\alpha} (x,0) dx \right)^{\frac{\alpha+1}{\alpha}} \\
\leq C \|\eta \tilde{w} w_L^{(t-1)}\|_{2 \frac{2\alpha}{2\alpha-1}} \left( B_R \right). \tag{3.10}
\]

By interpolation inequality,
\[
\|\eta \tilde{w} w_L^{(t-1)}\|_{2 \frac{2\alpha}{2\alpha-1}} \left( B_R \right) \leq \|\eta \tilde{w} w_L^{(t-1)}\|_{2 \frac{2\alpha}{2\alpha-1}} \left( B_R \right) \|\eta \tilde{w} w_L^{(t-1)}\|_{2(1-\theta)} \left( B_R \right), \tag{3.11}
\]
where \( \theta \) is determined by
\[
\frac{\alpha - 1}{2\alpha} = \frac{\theta}{2} + \frac{1 - \theta}{2^*}. \tag{3.12}
\]

Applying Young’s inequality, (3.11) yields
\[
\|\eta \tilde{w} w_L^{(t-1)}\|_{2 \frac{2\alpha}{2\alpha-1}} \left( B_R \right) \leq C(s, \alpha, N) \varepsilon^2 \|\eta \tilde{w} w_L^{(t-1)}\|_{2^*(\mathbb{R}^N)}^2 \\
+ C(\alpha, s, N) \varepsilon^{-\frac{2(1-s)}{s}} \|\eta \tilde{w} w_L^{(t-1)}\|^2_{L^2(B_R)}. \tag{3.13}
\]

Therefore, using Sobolev Trace inequality (2.2) and the value of \( \theta \) from (3.12), we have
\[
\|\eta \tilde{w} w_L^{(t-1)}\|_{2 \frac{2\alpha}{2\alpha-1}} \left( B_R \right) \leq C(s, \alpha, N) \varepsilon^2 \int_{\mathbb{R}^{N+1}} y^{1-2s} |\nabla \left( \eta \tilde{w} w_L^{(t-1)} \right) |^2 dx dy \\
+ C(\alpha, s, N) \varepsilon^{-\frac{2N}{2s}} \int_{B_R} |\eta \tilde{w} w_L^{(t-1)}(x,0)|^2 dx. \tag{3.14}
\]

Thanks to Lemma 2.3, for \( \delta > 0 \) we have
\[
\int_{B_R} |\eta \tilde{w} w_L^{(t-1)}(x,0)|^2 dx = \int_{B_1} |\eta \tilde{w} w_L^{(t-1)}(x,0)|^2 dx \\
\leq \delta \int_{Q_{s'}} y^{1-2s} |\nabla \left( \eta \tilde{w} w_L^{(t-1)} \right) |^2 dx dy + \frac{C}{\delta^3} \int_{Q_{s'}} y^{1-2s} |\eta \tilde{w} w_L^{(t-1)}|^2 dx dy, \tag{3.15}
\]
where \( \beta = \frac{s'+1}{s'-1} \), with some \( 1 < s' < \frac{1}{1-s} \). Substituting (3.15) in (3.14) and then (3.14) in (3.10) yields
\[
\int_{\mathbb{R}^N} \tilde{w}^{p+1} w_L^{2(t-1)} \eta^2(x,0) dx \leq C(s, \alpha, N) \varepsilon^2 \int_{\mathbb{R}^{N+1}} y^{1-2s} |\nabla \left( \eta \tilde{w} w_L^{(t-1)} \right) |^2 dx dy \\
+ C(\alpha, s, N) \varepsilon^{-\frac{2N}{2s}} \delta \int_{\mathbb{R}^{N+1}} y^{1-2s} |\nabla \left( \eta \tilde{w} w_L^{(t-1)} \right) |^2 dx dy \\
+ C(\alpha, s, N) \varepsilon^{-\frac{2N}{2s}} \frac{1}{\delta^3} \int_{\mathbb{R}^{N+1}} y^{1-2s} |\eta \tilde{w} w_L^{(t-1)}|^2 dx dy. \tag{3.16}
\]
Consequently, substituting (3.16) in (3.8), we obtain

\[
\int_{R^{N+1}} y^{1-2s} |\nabla (\eta \bar{w} w_L^{-1})|^2 \, dx \, dy \leq C \int_{R^{N+1}} y^{1-2s} \bar{w}^2 w_L^{2(t-1)} |\nabla \eta|^2 \, dx \, dy
\]
\[
+ C \left( \varepsilon^2 + \varepsilon^{-\frac{2N}{2N-s-N}} \delta \right) \int_{R^{N+1}} y^{1-2s} |\nabla \left( \eta \bar{w} w_L^{(t-1)} \right)|^2 \, dx \, dy
\]
\[
+ C \varepsilon^{-\frac{2N}{2N-s-N}} \delta^{-\beta} \int_{R^{N+1}} y^{1-2s} |\eta \bar{w} w_L^{(t-1)}|^2 \, dx \, dy. \quad (3.17)
\]

Choose

\[
\varepsilon = \frac{1}{2\sqrt{Ct}} \quad \text{and} \quad \delta = \frac{\varepsilon^{\frac{2N}{2N-s-N}}}{4Ct}.
\]

Hence, from (3.17), a direct calculation yields

\[
\frac{1}{2} \int_{R^{N+1}} y^{1-2s} |\nabla (\eta \bar{w} w_L^{-1})|^2 \, dx \, dy
\]
\[
\leq C \int_{R^{N+1}} y^{1-2s} \bar{w}^2 w_L^{2(t-1)} |\nabla \eta|^2 \, dx \, dy + C \varepsilon^{\frac{2N}{2N-s-N}} \int_{R^{N+1}} y^{1-2s} |\eta \bar{w} w_L^{(t-1)}|^2 \, dx \, dy
\]
\[
\leq C \gamma \int_{R^{N+1}} y^{1-2s} (\eta^2 + |\nabla \eta|^2) \bar{w}^2 w_L^{2(t-1)} \, dx \, dy. \quad (3.18)
\]

where \( \gamma = \frac{2s(\beta + 1)}{2N-s-N} \). Applying Sobolev inequality (see Lemma 2.2), we obtain from (3.18)

\[
\left( \int_{Q_1} y^{1-2s} |\nabla (\eta \bar{w} w_L^{-1})|^2 \, dx \, dy \right)^\frac{1}{2} \leq C \int_{Q_1} y^{1-2s} |\nabla (\eta \bar{w} w_L^{-1})|^2 \, dx \, dy
\]
\[
\leq C \gamma \int_{Q_1} y^{1-2s} (\eta^2 + |\nabla \eta|^2) \bar{w}^2 w_L^{2(t-1)} \, dx \, dy,
\]

where \( \chi = \frac{N+1}{N} > 1 \). Now using the fact that \( 0 < r < R < 1 \), \( \eta = 1 \) in \( Q_r \), \( |\nabla \eta| \leq \frac{2}{r} \) and supp \( \eta = Q_R \), we get

\[
\left( \int_{Q_r} y^{1-2s} \bar{w}^2 w_L^{2(t-1)} \, dx \, dy \right)^\frac{1}{2} \leq \frac{C \gamma}{(R-r)^2} \int_{Q_R} y^{1-2s} \bar{w}^2 w_L^{2(t-1)} \, dx \, dy.
\]

As \( w_L \leq \bar{w} \), the above expression yields,

\[
\left( \int_{Q_r} y^{1-2s} \bar{w}^{2t} \, dx \, dy \right)^\frac{1}{2} \leq \frac{C \gamma}{(R-r)^2} \int_{Q_R} y^{1-2s} \bar{w}^{2t} \, dx \, dy,
\]

provided the right-hand side is bounded. Passing to the limit \( L \to \infty \) via Fatou’s lemma we obtain

\[
\left( \int_{Q_r} y^{1-2s} \bar{w}^{2t} \, dx \, dy \right)^\frac{1}{2} \leq \frac{C \gamma}{(R-r)^2} \int_{Q_R} y^{1-2s} \bar{w}^{2t} \, dx \, dy,
\]

that is,

\[
\left( \int_{Q_r} y^{1-2s} \bar{w}^{2t} \, dx \, dy \right)^\frac{1}{2t} \leq \left( \frac{C \gamma}{(R-r)^2} \right)^\frac{1}{2} \left( \int_{Q_R} y^{1-2s} \bar{w}^{2t} \, dx \, dy \right)^\frac{1}{2t}. \quad (3.19)
\]
Now we iterate the above relation. We take \( t_i = \chi^i \) and \( r_i = \frac{1}{2} + \frac{1}{2^i} \) for \( i = 0, 1, 2, \ldots \). Note that \( t_i = \chi t_{i-1} \), \( r_{i-1} - r_i = \frac{1}{2^{i-1}} \). Hence from (3.19), with \( t = t_i \), \( r = r_i \), \( R = r_{i-1} \), we have

\[
\left( \int_{Q_{r_i}} y^{1-2s} \bar{w}^{2t+1} \, dx \, dy \right)^{\frac{1}{2t+1}} \leq C_i \left( \int_{Q_{r_{i-1}}} y^{1-2s} \bar{w}^{2t} \, dx \, dy \right)^{\frac{1}{2t}}, \quad i = 0, 1, 2, \ldots,
\]

where \( C \) depend only on \( N, s, p, q \). Hence, by iteration we have

\[
\left( \int_{Q_{r_i}} y^{1-2s} \bar{w}^{2t+1} \, dx \, dy \right)^{\frac{1}{2t+1}} \leq C_i \sum_{x_i} \left( \int_{Q_{r_0}} y^{1-2s} \bar{w}^{2t_0} \, dx \, dy \right)^{\frac{1}{2t_0}}, \quad i = 0, 1, 2, \ldots,
\]

Letting \( i \to \infty \) we have

\[
\sup_{Q_{\frac{1}{2}}} \bar{w} \leq C|\bar{w}|_{L^2(Q_{1},y^{1-2s})},
\]

which in turn implies

\[
\sup_{B_{\frac{1}{2}}} u = \sup_{B_{\frac{1}{2}}} w^+ \leq \sup_{Q_{\frac{1}{2}}} w^+ \leq C\|w\|_{L^2(Q_{1},y^{1-2s})}.
\]

Hence, \( u \in L^\infty(B_{\frac{1}{2}}(0)) \). Translating the equation, similarly it follows that \( u \in L^\infty_{loc}(\mathbb{R}^N) \).

To show the \( L^\infty \) bound at infinity, we define the Kelvin transform of \( u \) by the function \( \tilde{u} \) as follows:

\[
\tilde{u}(x) = \frac{1}{|x|^{N-2s}(x^2)} u\left( \frac{x}{|x|^2} \right), \quad x \in \mathbb{R}^N \setminus \{0\}.
\]

It follows from [26, Proposition A.1],

\[
(-\Delta)^s \tilde{u}(x) = \frac{1}{|x|^{N+2s}} (-\Delta)^s u\left( \frac{x}{|x|^2} \right).
\]

Thus

\[
(-\Delta)^s \tilde{u}(x) = \frac{1}{|x|^{N+2s}} \left( u^p\left( \frac{x}{|x|^2} \right) - u^q\left( \frac{x}{|x|^2} \right) \right) = \frac{1}{|x|^{N+2s}} \left( |x|^{p(N-2s)} \tilde{u}^p(x) - |x|^{q(N-2s)} \tilde{u}^q(x) \right).
\]

This implies \( \tilde{u} \) satisfies the following equation

\[
\begin{cases}
(-\Delta)^s \tilde{u} = |x|^{p(N-2s)-(N+2s)} \tilde{u}^p - |x|^{q(N-2s)-(N+2s)} \tilde{u}^q & \text{in } \mathbb{R}^N, \\
\tilde{u} \in H^s(\mathbb{R}^N) \cap L^{q+1}(\mathbb{R}^N, |x|^{(N-2s)(q+1)-2N}), \\
\tilde{u} > 0 & \text{in } \mathbb{R}^N.
\end{cases}
\]

That is,

\[
(-\Delta)^s \tilde{u} = f(x, \tilde{u}) \quad \text{in } \mathbb{R}^N,
\]

where

\[
f(x, \tilde{u}) := |x|^{p(N-2s)-(N+2s)} \tilde{u}^p - |x|^{q(N-2s)-(N+2s)} \tilde{u}^q.
\]

Since \( q > p \geq \frac{N+2s}{N-2s} \), we get \( (-\Delta)^s \tilde{u} \leq \tilde{u}^p \) in \( (B_1(0)) \). Applying the Moser iteration technique along the same line of arguments as above with a suitable modification,
we get $\sup_{B_{\rho}(0)} \tilde{u} \leq C$, for some $\rho > 0$ and $C$ is a positive constant. This in turn implies,

$$u(x) \leq \frac{C}{|x|^{N-2s}}, \quad |x| > R_0,$$

(3.24)

for some large $R_0$. Hence, $u \in L^\infty(\mathbb{R}^N)$. As a consequence $\tilde{u} \in L^\infty(B_1(0))$. Applying Theorem 2.1, it follows that $\tilde{u} \in C(B_{\frac{1}{2}}(0))$.

Thus there exists $C_1 > 0$ such that $\tilde{u} > C_1$ in $(B_{\frac{1}{2}}(0))$, which in turn implies $u(x) > \frac{C_1}{|x|^{N-2s}}$, for $|x| > R$. This along with (3.24), yields (1.9).

**Case 2:** $\Omega$ is a bounded domain.

Arguing along the same line with minor modifications, it can be shown that $u \in L^\infty(\Omega)$. Therefore the conclusion follows as $u = 0$ in $\mathbb{R}^N \setminus \Omega$.

**Proof of Theorem 1.4.** (i) From Theorem 1.3, we know any solution $u$ of Eq.(1.1) is in $L^\infty(\mathbb{R}^N)$. Therefore, we have

$$(-\Delta)^s u = f(u), \quad f(u) := u^p - u^q \in L^\infty(\mathbb{R}^N).$$

(3.25)

As a result, applying Theorem 2.1(a), we obtain

$$||u||_{C^{2s}(B_{\frac{1}{2}}(0))} \leq C(||u||_{L^\infty(\mathbb{R}^N)} + ||f(u)||_{L^\infty(\mathbb{R}^N)})$$

\begin{align*}
&\leq C(||u||_{L^\infty(\mathbb{R}^N)} + ||f(u)||_{L^\infty(\mathbb{R}^N)}) \quad \text{if } s \neq \frac{1}{2}, \\
&\leq C(||u||_{L^\infty(\mathbb{R}^N)} + ||f(u)||_{L^\infty(\mathbb{R}^N)}) \quad \text{if } s = \frac{1}{2},
\end{align*}

(3.26) (3.27)

for all $\varepsilon > 0$. Here the constants $C$ are independent of $u$, but may depend on radius $\frac{1}{2}$ and centre 0. Since the equation is invariant under translation, translating the equation, we obtain

$$||u||_{C^{2s}(B_{\frac{1}{2}}(y))} \leq C(||u||_{L^\infty(\mathbb{R}^N)} + ||f(u)||_{L^\infty(\mathbb{R}^N)})$$

\begin{align*}
&\leq C(1 + ||u||_{L^\infty(\mathbb{R}^N)})^q \quad \text{when } s \neq \frac{1}{2}, \\
&\leq C(1 + ||u||_{L^\infty(\mathbb{R}^N)})^q \quad \text{when } s = \frac{1}{2},
\end{align*}

(3.28) (3.29)

Note that in (3.28) and (3.29) constants $C$ are same as in (3.26) and (3.27) respectively. Thus, in (3.28) and (3.29) constants do not depend on $y$. This implies $u \in C^{2s}(\mathbb{R}^N)$ when $s \neq \frac{1}{2}$ and in $C^{2s-\varepsilon}(\mathbb{R}^N)$, when $s = \frac{1}{2}$. Hence, $f(u) \in C^{2s}(\mathbb{R}^N)$ when $s \neq \frac{1}{2}$ and in $C^{2s-\varepsilon}(\mathbb{R}^N)$, when $s = \frac{1}{2}$. Therefore, applying Theorem 2.1(b), we have

$$||u||_{C^{4s}(B_{\frac{1}{2}}(0))} \leq C(||u||_{C^{2s}(\mathbb{R}^N)} + ||f(u)||_{C^{2s}(B_{\frac{1}{2}}(0))})$$

\begin{align*}
&\leq C(||u||_{C^{2s}(\mathbb{R}^N)} + ||f(u)||_{C^{2s}(\mathbb{R}^N)}) \\
&\leq C(1 + ||u||_{L^\infty(\mathbb{R}^N)})^{2q} \quad \text{if } s \neq \frac{1}{2}, \frac{1}{2}, \frac{3}{4}.
\end{align*}

(3.30)
Similarly,
\[ \|u\|_{C^{4s-\varepsilon}(B_{12}(0))} \leq C(\|u\|_{C^{2s-\varepsilon}(\mathbb{R}^N)} + \|f(u)\|_{C^{2s-\varepsilon}(B_1(0))}) \]
\[ \leq C(1 + \|u\|_{L^\infty(\mathbb{R}^N)})^{2q} \quad \text{if} \quad s = \frac{1}{2} \quad \text{and} \quad 4s - \varepsilon \notin \mathbb{N}. \quad (3.31) \]

Arguing as before, we can show that \( u \in C^{4s}(\mathbb{R}^N) \) when \( s \neq \frac{1}{2} \) and in \( C^{4s}(\mathbb{R}^N) \), when \( s = \frac{1}{2} \). We can repeat this argument to improve the regularity \( C^{\infty}(\mathbb{R}^N) \) if both \( p \) and \( q \) are integer and \( C^{2k} \mathbb{R}^N \), where \( k \) is the largest integer satisfying \( |2k| < p \) if \( p \notin \mathbb{N} \) and \( |2k| < q \) if \( p \in \mathbb{N} \) but \( q \notin \mathbb{N} \), where \( |2k| \) denotes the greatest integer less than equal to \( 2k \).

(ii) Suppose, \( u \) is an arbitrary solution of \((1.2)\), then by Theorem 1.3, \( u \in L^\infty(\mathbb{R}^N) \) and thus \( f(u) = u^p - u^q \in L^\infty(\mathbb{R}^N) \). Consequently, by [26, Proposition 1.1], it follows \( u \in C^\infty(\mathbb{R}^N) \). Since \( q, p > 1 \), we have \( f(u) \in C^{2\alpha}_{0loc}(\mathbb{R}^N) \). Therefore by Theorem 2.1(ii), \( u \in C^{2\alpha}_{0loc}(\mathbb{R}^N) \) for some \( \alpha \in (0, 1) \). \( \square \)

**Proposition 1.** Let \( p, q, s \) are as in Theorem 1.3. If \( u \) is any nonnegative weak solution of Eq. \((1.1)\) or \((1.2)\), then \( u \) is a classical solution.

**Proof.** **Case 1:** Let \( u \) be a weak solution of \((1.1)\).

First, we show that \((-\Delta)^s u(x)\) can be defined as in \((1.3)\). Using \( u \in L^\infty(\mathbb{R}^N) \), we see that
\[ \left| \int_{\mathbb{R}^N \setminus \overline{B_{12}(0)}} \frac{u(x + y) - 2u(x) + u(x - y)}{|y|^{N+2s}} dy \right| \leq C \int_{\mathbb{R}^N \setminus \overline{B_{12}(0)}} \frac{dy}{|y|^{N+2s}} < \infty. \]

On the other hand, by Theorem 1.4, \( u \in C^{2\alpha}_{0loc}(\mathbb{R}^N) \) for some \( \alpha \in (0, 1) \), it follows that \( \left| \int_{\overline{B_{12}(0)}} \frac{u(x + y) - 2u(x) + u(x - y)}{|y|^{N+2s}} dy \right| < \infty \). Hence \((-\Delta)^s u(x)\) is defined pointwise.

Next, we show that the Eq. \((1.1)\) is satisfied in pointwise sense. \( u \) is a weak solution implies
\[ \int_{\mathbb{R}^N} (-\Delta)^s u(-\Delta)^s \varphi dx = \int_{\mathbb{R}^N} u^p \varphi dx - \int_{\mathbb{R}^N} u^q \varphi dx \quad \forall \varphi \in C^\infty_0(\mathbb{R}^N). \]

This in turn implies
\[ \int_{\mathbb{R}^N} \varphi(-\Delta)^s u dx = \int_{\mathbb{R}^N} u^p \varphi dx - \int_{\mathbb{R}^N} u^q \varphi dx \quad \forall \varphi \in C^\infty_0(\mathbb{R}^N). \]

Therefore, \((-\Delta)^s u = u^p - u^q \) in \( \mathbb{R}^N \) almost everywhere and \( u \in C^{2\alpha}(\mathbb{R}^N) \) implies \((-\Delta)^s u(x) = u^p(x) - u^q(x) \) \( \forall x \in \mathbb{R}^N \).

Hence, \( u \) is a classical solution of \((1.1)\).

**Case 2:** Suppose \( u \) is a weak solution of \((1.2)\). Then applying Theorem 1.3 and Theorem 1.4, we can show as in Case 1 that \((-\Delta)^s u(x)\) can be defined in pointwise sense.

Now we are left to show that \((1.2)\) is satisfied in pointwise sense. Towards this goal, we define
\[ f(u) = u^p - u^q, \quad u_\varepsilon := u * \rho_\varepsilon \quad \text{and} \quad f_\varepsilon := f(u) * \rho_\varepsilon, \]
where \( \rho_\varepsilon \) is the standard mollifier. Namely, we take \( \rho_\varepsilon = \varepsilon^{-N} \rho(\varepsilon \cdot) \) where \( \rho \in C^\infty_0(\mathbb{R}^N) \) with \( 0 \leq \rho \leq 1 \), \( \text{supp} \rho \subseteq \{ |x| \leq 1 \} \) and \( \int_{\mathbb{R}^N} \rho dx = 1 \).
Then \( u_\varepsilon, f_\varepsilon \in C^\infty \). Proceeding along the same line as in the proof of [29, Proposition 5], we can show that, for \( \varepsilon > 0 \) small enough it holds
\[
(-\Delta)^s u_\varepsilon = f_\varepsilon \quad \text{in} \quad U,
\] (3.32)
in the classical sense, where \( U \) is any arbitrary subset of \( \Omega \) with \( U \subset \subset \Omega \). Moreover, it is easy to note that \( u_\varepsilon \to u \) and \( f_\varepsilon \to f \) locally uniformly and
\[
||u_\varepsilon||_{L^\infty(B_1(0))} \leq ||u||_{L^\infty(\mathbb{R}^N)} \quad \text{and} \quad ||f_\varepsilon||_{L^\infty(B_1(0))} \leq C||u||_{L^\infty(\mathbb{R}^N)}.
\]
Taking the limit \( \varepsilon \to 0 \) on both the sides of (3.32) and using the regularity estimate of \( u_\varepsilon \) from theorem 1.4, we obtain,
\[
\lim_{\varepsilon \to 0} \int_{\mathbb{R}^N} \frac{u_\varepsilon(x + y) - 2u_\varepsilon(x) + u_\varepsilon(x - y)}{|y|^{N+2s}} dy = f(u).
\]
Using the arguments used before, it is not difficult to check that LHS of above relation converges to \((-\Delta)^s u\) as \( \varepsilon \to 0 \) and hence the result follows. \( \square \)

**Proof of Theorem 1.5.** First, we observe that from Theorem 1.4, it follows \( u \) is differentiable as \( p > 1 \). Let \( R_0 \) be as in Theorem 1.3. For \( R > R_0 \), define \( v(x) = R^{N-2s}u(Rx) \). Then
\[
(-\Delta)^s v(x) = R^N (-\Delta)^s u(Rx) = R^N (u^p(Rx) - u^q(Rx)) = R^N - p(N-2s)u^p - R^N - q(N-2s)u^q.
\] (3.33)
From Theorem 1.3, we have \( |u(x)| \leq \frac{C}{|x|^{N-2s}} \) for \( |x| > R_0 \). Consequently, we get
\[
|v(x)| \leq \frac{C}{|x|^{N-2s}} \quad \text{for} \quad |x| > \frac{R_0}{R},
\] (3.34)
where \( C \) is independent of \( R \). Let \( A_1 := \{ 1 < |x| < 2 \} \) and \( x_0 \in A_1 \). Suppose \( r > 0 \) is such that \( B_{2r}(x_0) \subset A_1 \). We choose \( \eta \in C_0^\infty(\mathbb{R}^N) \) such that \( \eta = 1 \) in \( B_r(x_0) \) and \( \text{supp}\ \eta \subset B_{2r}(x_0) \). Clearly \( \eta v \in L^\infty(\mathbb{R}^N) \) and \( ||\eta v||_{L^\infty(\mathbb{R}^N)} \leq C_1 \), where \( C_1 \) is independent of \( R \). Moreover,
\[
(-\Delta)^s (\eta v) = (-\Delta)^s v + (-\Delta)^s ((\eta - 1)v).
\] (3.35)
Note that, for \( z \in B_r(x_0) \) we have
\[
(-\Delta)^s ((\eta - 1)v)(z) = c_{N,s} \int_{\mathbb{R}^N \setminus B_r(x_0)} \frac{-(\eta - 1)v(y)}{|z - y|^{N+2s}} dy.
\]
From this expression we obtain
\[
||(-\Delta)^s ((\eta - 1)v)||_{L^\infty(B_r(x_0))} \leq C \int_{\mathbb{R}^N} \frac{v(y)}{(1 + |y|)^{N+2s}} dy \leq C \int_{|y| > \frac{R_0}{R}} \frac{v(y)}{(1 + |y|)^{N+2s}} dy + C \int_{|y| \leq \frac{R_0}{R}} \frac{v(y)}{(1 + |y|)^{N+2s}} dy,
\] (3.36)
Now, using the definition of \( v \) and the fact that \( u \in L^\infty(\mathbb{R}^N) \), we get
\[
\int_{B_{R_0}(0)} \frac{v(y)}{(1 + |y|)^N} \, dy = R^{N-2s} \int_{B_{R_0}(0)} \frac{u(Ry)}{(1 + |y|)^N} \, dy
= CR^N \int_{B_{R_0}(0)} \frac{u(x) \, dx}{(R + |x|)^N}
\leq C \frac{R^N}{R^{N+2s}} |B_{R_0}(0)| < C',
\]
where \( C' \) is independent of \( R \) (since, \( R^{-2s} < 1 \)). On the other hand, using (3.34) we have
\[
\int_{|y| > \frac{R_0}{R}} \frac{v(y)}{(1 + |y|)^N} \, dy = C \int_{|y| > \frac{R_0}{R}} \frac{dy}{|y|^{N-2s}(1 + |y|)^N}
\leq C \frac{R^N}{R^{N-2s}(1 + |y|)^N}
\leq C \int_{B_1(0)} \frac{dy}{|y|^{N-2s}} + \int_{|y| > 1} \frac{dy}{|y|^{2N}}
\leq C,
\]
for some constant \( C > 0 \), which does not depend on \( R \). Plugging (3.37) and (3.38) into (3.36) we have
\[
||(-\Delta)^s((\eta - 1)u)||_{L^\infty(B_r(x_0))} < C,
\]
where \( C \) depends only on \( N, s, p, q, R_0 \). Furthermore, we observe that if \( z \in B_r(x_0) \subset A_1 \) then \( |Rz| > R > R_0 \) and thus \( |u(Rz)| < \frac{C}{|Rz|^{N-2s}} \). Consequently, from (3.33), it follows that
\[
|(-\Delta)^s v(z)| \leq R^N (u^p(Rz) + u^q(Rz)) \leq R^{N-p(N-2s)} + R^{N-q(N-2s)} < C.
\]
In the last inequality we have used the fact that \( N - p(N - 2s) < 0 \) and \( N - q(N - 2s) < 0 \), as \( q, p \geq 2^* - 1 \). Hence,
\[
||(-\Delta)^s v||_{L^\infty(B_r(x_0))} \leq C,
\]
where \( C \) is independent of \( R \). Combining (3.39) and (3.40) along with (3.35) yields
\[
||(-\Delta)^s(\eta v)||_{L^\infty(B_r(x_0))} < C,
\]
where \( C \) depends only on \( N, s, p, q, R_0 \). Consequently, using [26, Proposition 2.3] (see also [25]), we obtain
\[
||\eta v||_{C^\beta(B_{1/2}(x_0))} \leq C \quad \forall \beta \in (0, 2s),
\]
where \( C \) depends only on \( N, s, p, q, R_0 \). As a consequence,
\[
||v||_{C^\beta(B_{1/2}(x_0))} \leq C.
\]
Thus, thanks to [26, Corollary 2.4] we have
\[
||v||_{C^{\beta+2s}(B_{1/2}(x_0))} \leq C.
\]
We continue to apply this bootstrap argument and after a finitely many steps we have \( ||v||_{C^{\beta+kr}(B_{r_0}(x_0))} \leq C \), for some \( r_0 > 0 \) and \( \beta + ks > 1 \). This in turn implies
\[
||\nabla v||_{L^\infty(B_{r_0}(x_0))} \leq C.
\]
This further yields to
\[
||\nabla v||_{L^\infty(A_1)} \leq C,
\]
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where \( C \) depends only on \( N, s, p, q, R_0 \). Therefore, using the definition of \( v \), we obtain

\[
|\nabla u(Rx)| \leq \frac{C}{R^{N-2s+1}} \quad \text{for } 1 < |x| < 2.
\]

From the above expression, it is easy to deduce that

\[
|\nabla u(y)| \leq \frac{C}{|y|^{N-2s+1}} \quad \text{for } R < |y| < 2R.
\]

As \( R > R_0 \) was arbitrary we get

\[
|\nabla u(y)| \leq \frac{C}{|y|^{N-2s+1}} \quad \text{for } |y| > R,
\]

for some \( R \) large.

4. Pohozaev identity and nonexistence result.

\textbf{Proof of Theorem 1.6.} We prove this theorem by establishing Pohozaev identity in the spirit of Ros-Oton and Serra [27]. For \( \lambda > 0 \), define \( u_\lambda(x) = u(\lambda x) \). Multiplying the equation (1.1) by \( u_\lambda \) yields,

\[
\int_{\mathbb{R}^N} (u^p - u^q) u_\lambda \, dx = \int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} u(-\Delta)^{\frac{s}{2}} u_\lambda \, dx
\]

\[
= \lambda^s \int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} u(x)((-\Delta)^{\frac{s}{2}} u)(\lambda x) \, dx
\]

\[
= \lambda^s \int_{\mathbb{R}^N} w_\lambda \, dx, \quad (4.1)
\]

where, \( w(x) := (-\Delta)^{\frac{s}{2}} u(x) \) and \( w_\lambda(x) = w(\lambda x) \). With the change of variable \( x = \sqrt{\lambda} y \), we have

\[
\lambda^s \int_{\mathbb{R}^N} w_\lambda \, dx = \lambda^s \int_{\mathbb{R}^N} w(x)w(\lambda x) \, dx = \lambda^{\frac{N-2s}{2}} \int_{\mathbb{R}^N} w \sqrt{\lambda} w^{\frac{1}{N}} \, dy. \quad (4.2)
\]

Therefore,

\[
\int_{\mathbb{R}^N} (u^p - u^q) u_\lambda \, dx = \lambda^{\frac{N-2s}{2}} \int_{\mathbb{R}^N} w \sqrt{\lambda} w^{\frac{1}{N}} \, dy. \quad (4.3)
\]

Observe that using the decay estimate at infinity of \( u \) and \( \nabla u \) from Theorem 1.3 and Theorem 1.5 , we get \( \int_{\mathbb{R}^N} (u^p - u^q)(x \cdot \nabla u) \, dx \) is well defined and that integral can be written as \( \int_{\mathbb{R}^N} x \cdot \nabla \left( \frac{u^{p+1}}{p+1} - \frac{u^{q+1}}{q+1} \right) \, dx \). Again using the decay estimate of \( u \) from Theorem 1.3, we justify the following integration by parts

\[
- \frac{N}{p+1} \int_{\mathbb{R}^N} u^{p+1} \, dx + \frac{N}{q+1} \int_{\mathbb{R}^N} u^{q+1} \, dx = \int_{\mathbb{R}^N} x \cdot \nabla \left( \frac{u^{p+1}}{p+1} - \frac{u^{q+1}}{q+1} \right) \, dx. \quad (4.4)
\]
Thus, using (4.3) we simplify the LHS of above expression as follows:

\[
\text{LHS of (4.4)} = \int_{\mathbb{R}^N} (u^p - u^q)(x \cdot \nabla u) dx
\]

\[
= \frac{d}{d\lambda} \bigg|_{\lambda=1} \int_{\mathbb{R}^N} (u^p - u^q)u_\lambda dx
\]

\[
= \frac{d}{d\lambda} \bigg|_{\lambda=1} \left( \lambda^{-\frac{N-2s}{2}} \int_{\mathbb{R}^N} w^{(-\frac{N+2s}{2})} \right) dx.
\]

\[
= - \left( \frac{N-2s}{2} \right) \int_{\mathbb{R}^N} w^2 dx + \frac{d}{d\lambda} \bigg|_{\lambda=1} \int_{\mathbb{R}^N} w^{\frac{1}{N-2s}} dy
\]

\[
= - \left( \frac{N-2s}{2} \right) \|u\|^2_{H^s(\mathbb{R}^N)}. \tag{4.5}
\]

On the other hand, multiplying (1.1) by \(u\) we have,

\[
\|u\|^2_{H^s(\mathbb{R}^N)} = \int_{\mathbb{R}^N} (u^p + 1 - u^q + 1) dx.
\]

Combining this expression along with (4.5) we obtain the Pohozaev identity

\[
\left( \frac{N-2s}{2} - \frac{N}{p+1} \right) \int_{\mathbb{R}^N} u^{p+1} dx = \left( \frac{N-2s}{2} - \frac{N}{q+1} \right) \int_{\mathbb{R}^N} u^{q+1} dx.
\]

Clearly, from the above identity, it follows that (1.1) does not admit any solution when \(p = 2^* - 1\) and \(q > p\). This completes the theorem. \(\square\)

5. Symmetry and monotonically decreasing property.

**Theorem 5.1.** Let \(p, q, s\) are as in Theorem 1.3 and \(u\) be any solution of Eq. (1.1). Then \(u\) is radially symmetric and strictly decreasing about some point in \(\mathbb{R}^N\).

**Proof.** By Proposition 1, \(u\) is a classical solution of (1.1). Define \(f(u) = u^p - u^q\).

Then clearly \(f\) is locally Lipschitz.

**Claim:** There exists \(s_0, \gamma, C > 0\) such that

\[
\frac{f(v) - f(u)}{v - u} \leq C(u + v)^\gamma \quad \text{for all} \quad 0 < u < v < s_0.
\]

To see the claim,

\[
f(v) - f(u) = (v^p - u^p) - (v^q - u^q)
\]

\[
= p(\theta_1 v + (1 - \theta_1)u)^{p-1} (v - u) - q(\theta_1 v + (1 - \theta_1)u)^{q-1} (v - u),
\]

for some \(\theta_1, \theta_2 \in (0, 1)\). Thus, for \(0 < u < v\)

\[
\frac{f(v) - f(u)}{v - u} = p(\theta_1 v + (1 - \theta_1)u)^{p-1} - q(\theta_2 v + (1 - \theta_2)u)^{q-1}
\]

\[
\leq p(\theta_1 v + (1 - \theta_1)u)^{p-1}
\]

\[
\leq p(u + v)^{p-1}.
\]

Therefore, the claim holds with \(C = p\) and \(\gamma = p - 1\) and for any positive \(s_0\).

Moreover, from Theorem 1.4, we have

\[
u(x) = O\left(\frac{1}{|x|^{N-2s}}\right) \quad \text{as} \quad |x| \to \infty.
\]
Since \( p \geq \frac{N+2s}{N-2s} \), it is easy to check that
\[
N - 2s > \max \left( \frac{2s}{\gamma}, \frac{N}{\gamma + 2} \right),
\]
where \( \gamma = p - 1 \), as found in the above claim. Hence, the theorem follows from [14, Theorem 1.2].

**Theorem 5.2.** Suppose \( \Omega \) is a smooth bounded convex domain, \( p, q, s \) are as in Theorem 1.3. Assume further that \( \Omega \) is convex in \( x_1 \) direction and symmetric w.r.t. to the hyperplane \( x_1 = 0 \). Let \( s \in (0, 1) \) and \( u \) be any solution of Eq. (1.2). Then \( u \) is symmetric w.r.t. \( x_1 \) and strictly decreasing in \( x_1 \) direction for \( x = (x_1, x') \in \Omega, x_1 > 0 \).

**Proof.** Follows from [13, Theorem 3.1] (also see [16, Cor. 1.2]).

**6. Existence results.**

**Lemma 6.1.** Let \( s \in (0, 1) \). If \( u \) is any radially symmetric decreasing function in \( H^s(\mathbb{R}^N) \), then
\[
u(|x|) \leq \frac{C}{|x|^{\frac{N-2s}{2}}}
\]

**Proof.** It is enough to show that if \( u \in \dot{H}^s(\mathbb{R}^N) \) with \( u(x) = u(|x|) \) and \( u(r_1) \leq u(r_2) \), when \( r_1 \geq r_2 \), then it holds \( u(R) \leq \frac{C}{R^{\frac{N-2s}{2}}} \) for any \( R > 0 \). To see this, we note that by Sobolev inequality we can write,
\[
\frac{1}{S} \|(-\Delta)^{\frac{s}{2}} u\|_{L^2(\mathbb{R}^N)} \geq \left( \int_{\mathbb{R}^N} |u(x)|^{2^*} \ dx \right)^{\frac{1}{2^*}} \\
\geq \left( \int_0^R \int_{\partial B_r} |u(r)|^{2^*} dS dr \right)^{\frac{1}{2^*}} \\
\geq u(R) \left( \int_0^R \omega_n r^{N-1} dr \right)^{\frac{1}{2^*}} \\
= \left( \frac{\omega_N}{N} \right)^{\frac{1}{2^*}} u(R) R^\frac{N}{2^*}.
\]

As \( u \in \dot{H}^s(\mathbb{R}^N) \) implies LHS is bounded above, the above inequality yields
\[
u(R) \leq \left( \frac{N}{\omega_N} \right)^{\frac{1}{2^*}} \frac{1}{S} \|(-\Delta)^{\frac{s}{2}} u\|_{L^2(\mathbb{R}^N)} R^{-\frac{N-2s}{2}} \leq CR^{-\frac{N-2s}{2}}.
\]

**Proof of Theorem 1.7.** We are going to work on the manifold
\[
\mathcal{N} = \left\{ u \in \dot{H}^s(\mathbb{R}^N) \cap L^{q+1}(\mathbb{R}^N) : \int_{\mathbb{R}^N} |u|^{q+1} dx = 1 \right\},
\]
and \( F(.) \) on \( \mathcal{N} \) reduces as
\[
F(u) = \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} dxdy + \frac{1}{q+1} \int_{\mathbb{R}^N} |u|^{q+1} dx.
\]
Let \( u_n \) be a minimizing sequence in \( \mathcal{N} \) such that
\[
F(u_n) \to K \quad \text{with} \quad \int_{\mathbb{R}^N} |u_n|^{p+1} dx = 1.
\]
Thus, \( \{u_n\} \) is a bounded sequence in \( \dot{H}^s(\mathbb{R}^N) \) and \( L^{q+1}(\mathbb{R}^N) \). Therefore, there exists \( u \in \dot{H}^s(\mathbb{R}^N) \) and \( L^{q+1}(\mathbb{R}^N) \) such that \( u_n \to u \) in \( \dot{H}^s(\mathbb{R}^N) \) and \( L^{q+1}(\mathbb{R}^N) \). Consequently \( u_n \to u \) pointwise almost everywhere.

Using symmetric rearrangement technique, without loss of generality, we can assume that \( u_n \) is radially symmetric and decreasing (see [24]). We claim that \( u_n \to u \) in \( L^{p+1}(\mathbb{R}^N) \).

To see the claim, we note that \( u_n^{p+1} \to u^{p+1} \) pointwise almost everywhere. Since \( \{u_n\} \) is uniformly bounded in \( L^{q+1}(\mathbb{R}^N) \), using Vitali’s convergence theorem, it is easy to check that \( \int_K |u_n|^{p+1} dx \to \int_K |u|^{p+1} dx \) for any compact set \( K \) in \( \mathbb{R}^N \) containing the origin. Furthermore, applying Lemma 6.1 it follows, \( \int_{\mathbb{R}^N \setminus K} |u_n|^{p+1} dx \) is very small and hence we have strong convergence. Moreover, \( \int_{\mathbb{R}^N} |u_n|^{p+1} dx = 1 \) implies \( \int_{\mathbb{R}^N} |u|^p dx = 1 \).

Now we show that \( K = F(u) \).

We note that \( u \to ||u||^2 \) is weakly lower semicontinuous. Using this fact along with Fatou’s lemma, we have
\[
\mathcal{K} = \lim_{n \to \infty} \left[ \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^2}{|x-y|^{N+2s}} dxdy + \frac{1}{q+1} \int_{\mathbb{R}^N} |u_n|^{q+1} dx \right]
\]
\[
= \lim_{n \to \infty} \left[ \frac{1}{2} ||u_n||^2 + \frac{1}{q+1} \int_{\mathbb{R}^N} |u_n|^{q+1} dx \right]
\]
\[
\geq \frac{1}{2} ||u||^2 + \frac{1}{q+1} \int_{\mathbb{R}^N} |u|^{q+1} dx
\]
\[
\geq F(u).
\]
This proves \( F(u) = \mathcal{K} \). Moreover, using the symmetric rearrangement technique via Polya-Szego inequality (see [24]), it is easy to check that \( u \) is nonnegative, radially symmetric and radially decreasing Applying the Lagrange multiplier rule, we obtain \( u \) satisfies
\[
-\Delta u + u^q = \lambda u^p,
\]
for some \( \lambda > 0 \). This in turn implies
\[
(-\Delta)^s u = \lambda u^p - u^q \quad \text{in} \quad \mathbb{R}^N.
\]
Finally, if \( q > (p-1) \frac{N}{2s} - 1 \), then we know that \( u \) is a classical solution. Therefore, if there exists \( x_0 \in \mathbb{R}^N \) such that \( u(x_0) = 0 \), that would imply \( (-\Delta)^s u(x_0) < 0 \) (since, \( u \) is a nontrivial solution). On the other hand, \( (\lambda u^p - u^q)(x_0) = 0 \) and that yields a contradiction. Hence \( u > 0 \) in \( \mathbb{R}^N \).

Furthermore, we observe that by setting \( v(x) = \lambda^{-\frac{1}{q-p}} u(\lambda^{-\frac{q-1}{2s(q-p)}} x) \), it holds
\[
(-\Delta)^s v = v^p - v^q \quad \text{in} \quad \mathbb{R}^N.
\]
Hence the theorem follows. \( \square \)

**Proof of Theorem 1.8.** We are going to work on the manifold
\[
\tilde{\mathcal{N}} = \left\{ u \in X_0 \cap L^{q+1}(\Omega) : \int_\Omega |u|^{p+1} = 1 \right\}.
\]
Then $F_\Omega$ reduces to
\[ F_\Omega(u) = \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} \, dx \, dy + \frac{1}{q+1} \int_{\Omega} |u|^{q+1} \, dx. \]

Let $u_n$ be a minimizing sequence in $\bar{N}$ such that $F_\Omega(u_n) \to S_\Omega$, then
\[ F(u_n) \to S_\Omega \text{ with } \int_{\Omega} |u_n|^{p+1} \, dx = 1. \]

Then $u_n$ is bounded in $X_0 \cap L^{p+1}(\Omega)$. Consequently, $u_n \rightharpoonup u$ on $H^s(\Omega)$ and $u_n \to u$ on $L^2(\Omega)$. As a result, $u_n \to u$ pointwise almost everywhere. By the interpolation inequality, we must have $u_n \to u$ on $L^{p+1}(\Omega)$. Hence, $\int_{\Omega} |u|^{p+1} \, dx = 1$.

Now we show that $S_\Omega = F_\Omega(u)$. Using Fatou's Lemma and the fact that $u \mapsto ||u||^2$ is weakly lower semicontinuous,
\[ S_\Omega = \lim_{n \to \infty} \left[ \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^2}{|x-y|^{N+2s}} \, dx \, dy + \frac{1}{q+1} \int_{\Omega} |u_n|^{q+1} \, dx \right] \geq \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} \, dx \, dy + \frac{1}{q+1} \int_{\Omega} |u|^{q+1} \, dx \]
\[ \geq F_\Omega(u). \]

By the Lagrange multiplier rule, we obtain $u$ satisfies
\[ (-\Delta)^s u + |u|^{q-1} u = \lambda |u|^{p-1} u. \]

Now we replace $\bar{N}$ by $\bar{N}_+ := \{ u \in X_0 \cap L^{p+1}(\Omega) : \int_{\Omega} (u^+)^{p+1} = 1 \}$, the functional $F_\Omega(.)$ by $F_\Omega(.)$ defined as follows
\[ \tilde{F}_\Omega(u) := \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} \, dx \, dy + \frac{1}{q+1} \int_{\Omega} (u^+)^{q+1} \, dx, \]
and $S_\Omega$ by $\tilde{S}_\Omega := \inf \{ F(v, \Omega) : v \in \bar{N}_+ \}$. Repeating the same argument as before (with a little modification), it can be easily shown that there exists $u \in X_0 \cap L^{p+1}(\Omega)$ which satisfies
\[ (-\Delta)^s u + (u^+)^q = \lambda (u^+)^p \quad \text{in} \quad \Omega. \quad (6.2) \]

Taking $u^-$ as the test function for (6.2) we obtain from Definition 1.1 that
\[ \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(u^-(x) - u^-(y))}{|x-y|^{N+2s}} \, dx \, dy = 0. \quad (6.3) \]
Furthermore,
\[ \text{LHS of (6.3)} = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u^+(x) - u^+(y))(u^-(x) - u^-(y))}{|x-y|^{N+2s}} \, dx \, dy \]
\[ = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u^+(x) - u^+(y))(u^+(x) - u^-(y))(u^- - u^-)(y))}{|x-y|^{N+2s}} \, dx \, dy \]
\[ = -u^-(x)u^+(y) - u^+(x)u^-(y) - ||u^-||^2 \]
\[ \leq - ||u^-||^2 \quad (6.4) \]

Hence, from (6.3) we obtain $u^- = 0$, i.e., $u \geq 0$. Moreover, since for $p \geq 2^* - 1$ and $q \geq (p-1)^\frac{N}{2} - 1$, Proposition 1 implies $u$ is a classical solution, applying maximum principle as in Theorem 1.7, we conclude $u > 0$ in $\Omega$. This completes the proof. \qed
Appendix A. In this section we give an alternative proof of Pohozaev identity in $\mathbb{R}^N$ for the following type of equations:

$$(-\Delta)^s u = f(u) \text{ in } \mathbb{R}^N,$$  \hspace{1cm} (A.1)

where $u \in \dot{H}^s(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and $f \in C^2$. Here we do not require the decay estimate of $u$ or $\nabla u$ at infinity.

**Theorem A.1.** Let $u \in \dot{H}^s(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ be a positive solution of (A.1) and $F(u) \in L^1(\mathbb{R}^N)$. Then

$$(N - 2s) \int_{\mathbb{R}^N} u f(u) \, dx = 2N \int_{\mathbb{R}^N} F(u) \, dx,$$

where $F(u) = \int_0^u f(t) \, dt$.

**Proof.** We prove this theorem using the harmonic extension method introduced in Section 2.

Let $u$ be a nontrivial positive solution of (A.1). Suppose, $w$ is the harmonic extension of $u$. Then $w$ is a solution of

$$\begin{cases}
\text{div}(y^{1-2s} \nabla w) = 0 & \text{in } \mathbb{R}^{N+1}_+,
\frac{\partial w}{\partial y^{2s}} = f(w(\cdot,0)) & \text{on } \mathbb{R}^N,
\end{cases}$$  \hspace{1cm} (A.2)

(see (2.4)). For $r > 0$, we define $B_r$ to be the ball in $\mathbb{R}^{N+1}$, that is,

$$B_r := \{(x,y) \in \mathbb{R}^{N+1} : |(x,y)| < r\}.$$

Define

$$B_r^+ = B_r \cap \mathbb{R}^{N+1}_+$$

and

$$Q_r = B_r^+ \cup (B_r \cap (\mathbb{R}^N \times \{0\})).$$

Let $\varphi \in C_0^\infty(\mathbb{R}^{N+1})$ with $0 \leq \varphi \leq 1$, $\varphi = 1$ in $B_1$, $\varphi$ has support in $B_2$ and $|\nabla \varphi| \leq 2$.

For $R > 0$, define

$$\psi_R(x,y) = \psi \left( \frac{(x,y)}{R} \right), \quad \text{where} \quad \psi = \varphi|_{\mathbb{R}^{N+1}_+}.$$

Multiplying (A.2) by $((x,y) \cdot \nabla w)\psi_R$ and integrating in $\mathbb{R}^{N+1}_+$ we have,

$$\int_{Q_{2R}} \text{div}(y^{1-2s} \nabla w) \left[ ((x,y) \cdot \nabla w) \psi_R \right] dxdy = 0.$$  \hspace{1cm} (A.3)

Then integration by parts yields

$$\int_{Q_{2R}} y^{1-2s} \nabla w \nabla \left[ ((x,y) \cdot \nabla w) \psi_R \right] dxdy$$

$$= \int_{\partial Q_{2R}} y^{1-2s} (\nabla w \cdot \nu) \left[ ((x,y) \cdot \nabla w) \psi_R \right] dS$$

$$= - \lim_{y \to 0^+} \int_{B_{2R} \cap (\mathbb{R}^N \times \{y\})} y^{1-2s} \frac{\partial w}{\partial y}(x,y) ((x,y) \cdot \nabla w) \psi_R \, dx$$

$$= k_{2s}^{-1} \int_{B_{2R} \cap (\mathbb{R}^N \times \{0\})} (x \cdot \nabla_x u) \psi_R \frac{\partial w}{\partial y^{2s}} \, dx$$  \hspace{1cm} (A.4)
where $k_{2s}$ is as defined in Section 2. In the above steps we have used the fact that $\psi_R = 0$ on $\partial B_{2R}$. From (A.2), we know $\frac{\partial w}{\partial \nu} = f(w(x,0))$ on $\mathbb{R}^N$. Therefore, RHS of (A.4) simplifies as

$$\text{RHS of (A.4)} = k_{2s}^{-1} \int_{B_{2R} \cap (\mathbb{R}^N \times \{0\})} (x \cdot \nabla_x w) f(w) \psi_R dx$$

$$= k_{2s}^{-1} \int_{B_{2R} \cap (\mathbb{R}^N \times \{0\})} (x \cdot \nabla_x F(w)) \psi_R dx$$

$$= -Nk_{2s}^{-1} \int_{B_{2R} \cap (\mathbb{R}^N \times \{0\})} F(w) \psi_R dx - k_{2s}^{-1} \int_{B_{2R} \cap (\mathbb{R}^N \times \{0\})} F(w)(x \cdot \nabla_x \psi_R) dx. \quad (A.5)$$

Since $w(x,0) = u(x)$, from (A.2), we find $F(w) = F(u)$ on $\mathbb{R}^N$. Moreover, $|\nabla \psi_R| \leq \frac{1}{R}$. Hence, the 2nd integral on RHS of (A.5) can be written as

$$\int_{B_{2R} \cap (\mathbb{R}^N \times \{0\})} F(w)(x \cdot \nabla_x \psi_R) dx \leq C \int_{(B_{2R} \setminus B_R) \cap (\mathbb{R}^N \times \{0\})} F(u) \frac{|x|}{R} dx$$

$$\leq C \int_{(B_{2R} \setminus B_R) \cap (\mathbb{R}^N \times \{0\})} F(u) dx, \quad (A.6)$$

which converges to 0 as $R \to \infty$ (since, $F(u) \in L^1(\mathbb{R}^N)$). As a result,

$$\lim_{R \to \infty} \text{RHS of (A.4)} = -Nk_{2s}^{-1} \int_{\mathbb{R}^N} F(u) dx \quad (A.7)$$

Next, we like to simplify LHS of (A.4). Towards this aim, let us first simplify the term $\nabla w \nabla \left[(x,y) \cdot \nabla w \psi_R\right]$.

$$\nabla w \nabla \left[(x,y) \cdot \nabla w \psi_R\right] = \nabla w \cdot \nabla \psi_R((x,y) \cdot \nabla w) + \nabla w \cdot \nabla ((x,y) \cdot \nabla w) \psi_R. \quad (A.8)$$

By doing a straightforward computation, we further simplify the 2nd term on the RHS of above expression as below:

$$\nabla w \cdot \nabla ((x,y) \cdot \nabla w) \psi_R = \left[|\nabla w|^2 + \frac{1}{2} \sum_{j=1}^{N} \frac{\partial}{\partial x_j} (|\nabla w|^2)x_j + \frac{1}{2} \frac{\partial}{\partial y} (|\nabla w|^2)y\right] \psi_R.$$

Substituting back this expression into (A.8) and then plugging (A.8) into LHS of (A.4), we obtain

$$\int_{Q_{2R}} y^{1-2s} \nabla w \nabla \left[(x,y) \cdot \nabla w \psi_R\right] dx dy$$

$$= \int_{Q_{2R}} y^{1-2s} |\nabla w|^2 \psi_R dx dy + \frac{1}{2} \int_{Q_{2R}} y^{1-2s} \left(\sum_{j=1}^{N} \frac{\partial}{\partial x_j} (|\nabla w|^2)x_j\right) \psi_R dx dy$$

$$+ \frac{1}{2} \int_{Q_{2R}} y^{2-2s} \left(\frac{\partial}{\partial y} (|\nabla w|^2)\right) \psi_R dx dy + \int_{Q_{2R}} y^{1-2s} \nabla w \cdot \nabla \psi_R((x,y) \cdot \nabla w) dx dy. \quad (A.9)$$
Performing integration by parts on RHS of above expression, followed by simple computation yields,

\[
\int_{Q_{2R}} y^{1-2s}\nabla w \nabla [(x, y) \cdot \nabla \psi_R] \, dx \, dy
\]

\[= - \left( \frac{N - 2s}{2} \right) \int_{Q_{2R}} y^{1-2s}|\nabla w|^2 \psi_R \, dx \, dy - \frac{1}{2} \int_{Q_{2R}} y^{1-2s}|\nabla w|^2 (x, y) \cdot \nabla \psi_R \, dx \, dy
\]

\[+ \frac{1}{2} \int_{\partial Q_{2R}} y^{1-2s}|\nabla w|^2 ((x, y) \cdot \mathbf{n}) \psi_R \, dS + \int_{Q_{2R}} \nabla w \cdot \nabla \psi_R ((x, y) \cdot \nabla w) \, dx \, dy. \tag{A.10}
\]

Note that,

\[\left| \int_{Q_{2R}} y^{1-2s}|\nabla w|^2 ((x, y) \cdot \nabla \psi_R) \, dx \, dy \right| \leq C \int_{Q_{2R} \setminus Q_R} y^{1-2s}|\nabla w|^2 \frac{|(x, y)|}{R} \, dx \, dy
\]

\[\leq C \int_{Q_{2R} \setminus Q_R} y^{1-2s}|\nabla w|^2 \, dx \, dy
\]

\[\to 0 \quad \text{as} \quad R \to \infty. \tag{A.11}
\]

Similarly,

\[\int_{Q_{2R}} y^{1-2s}(\nabla w \cdot \nabla \psi_R) ((x, y) \cdot \nabla w) \, dx \, dy \to 0 \quad \text{as} \quad R \to \infty. \tag{A.12}
\]

Since \(\psi_R = 0\) on \(\partial B_{2R}\), we have,

\[\frac{1}{2} \int_{\partial Q_{2R}} y^{1-2s}|\nabla w|^2 ((x, y) \cdot \mathbf{n}) \psi_R \, dS = - \lim_{y \to 0} \frac{1}{2} \int_{B_{2R} \cap (\mathbb{R}^N \times \{0\})} y^{2-2s}|\nabla w|^2 \psi_R \, dS
\]

\[= 0 \tag{A.13}
\]

Combining (A.11), (A.12) and (A.13) with (A.10) we obtain

\[\lim_{R \to \infty} \int_{Q_{2R}} y^{1-2s}\nabla w \nabla [(x, y) \cdot \nabla \psi_R] \, dx \, dy = - \left( \frac{N - 2s}{2} \right) \int_{\mathbb{R}^N_+} y^{1-2s}|\nabla w|^2 \, dx \, dy. \tag{A.14}
\]

Thus, (A.14) and (A.7) along with (A.4), yields

\[- \left( \frac{N - 2s}{2} \right) \int_{\mathbb{R}^N_+} y^{1-2s}|\nabla w|^2 \, dx \, dy = - N k_{2s}^{-1} \int_{\mathbb{R}^N} F(u) \, dx. \tag{A.15}
\]

We multiply (A.2) by \(w \psi_R\) and then integrate by parts. Since \(\psi_R = 0\) on \(\partial B_{2R}\) and

\[- \lim_{y \to 0} y^{1-2s} \frac{\partial w}{\partial y} = k_{2s}^{-1} \frac{\partial w}{\partial y} \, w \psi_R \, dx
\]

\[= k_{2s}^{-1} \int_{B_{2R} \cap (\mathbb{R}^N \times \{0\})} \frac{\partial w}{\partial y} \, w \psi_R \, dx
\]

\[= k_{2s}^{-1} \int_{B_{2R} \cap (\mathbb{R}^N \times \{0\})} f(u) \, w \psi_R \, dx. \tag{A.16}
\]

Therefore,

\[\lim_{R \to \infty} \text{RHS of (A.16)} = k_{2s}^{-1} \int_{\mathbb{R}^N} u f(u) \, dx. \tag{A.17}
\]
Proceeding same as before we show that
\[
\lim_{R \to \infty} \text{LHS of (A.16)} = \int_{\mathbb{R}^{N+1}_+} y^{1-2s} |\nabla w|^2 \, dx \, dy.
\]  
(A.18)

Consequently,
\[
\int_{\mathbb{R}^{N+1}_+} y^{1-2s} |\nabla w|^2 \, dx \, dy = k_{2s}^{-1} \int_{\mathbb{R}^N} f(u) \, dx.
\]  
(A.19)

Substituting (A.19) into (A.15), we have
\[
(N - 2s) \int_{\mathbb{R}^N} f(u) \, dx = N \int_{\mathbb{R}^N} F(u) \, dx,
\]
which completes the proof.

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