Integration over matrix spaces with unique invariant measures

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Abstract

We present a method to calculate integrals over monomials of matrix elements with invariant measures in terms of Wick contractions. The method gives exact results for monomials of low order. For higher–order monomials, it leads to an error of order $1/N^\alpha$ where $N$ is the dimension of the matrix and where $\alpha$ is independent of the degree of the monomial. We give a lower bound on the integer $\alpha$ and show how $\alpha$ can be increased systematically. The method is particularly suited for symbolic computer calculation. Explicit results are given for $O(N), U(N)$ and for the circular orthogonal ensemble.

I. INTRODUCTION

The calculation of group integrals of monomials of matrix elements for compact Lie groups has a long tradition going back to Ullah and Porter \cite{1}. Their results were later extended \cite{2} to the space of symmetric unitary matrices endowed with Dyson’s invariant measure \cite{3}. The problem was nearly dormant for some years but was recently solved completely for the orthogonal group by recursion \cite{4}. Unfortunately, it seems that there is no easy generalization of this method to other groups. The method proposed in Ref. \cite{2}, on the other hand, is quite general but soon becomes rather cumbersome. Moreover, that method is not suited for computer–supported analytical work.

Aside from their immanent group–theoretical significance, group integrals over monomials of matrix elements for compact Lie groups are important in applications of random–matrix theory. This field has seen an explosive growth over the last decade and has become an important tool in a great variety of fields in physics, chemistry and related areas \cite{5}. This fact lends further urgency to the evaluation of the above–mentioned group integrals.

In this paper, we present and analyse a novel approach to the problem. We encountered the problem in the study of a random–matrix model for a class of chaotic systems (the semi–separable systems) \cite{6}. We found that we could use the invariants of the orthogonal group to construct a weight function. With the help of this function, it was possible to evaluate the group integrals in question by simple Wick contractions. We implemented the scheme on the computer and found that beyond the expected exact results for low–order monomials, monomials of higher order were also calculated correctly up to and including the sub–leading order in $1/N$ where $N$ is the dimension of the matrices under consideration.
We conjectured that this statement holds for monomials of any order. It is the purpose of
the present paper to extend and prove the conjecture and to explore the scope of its validity
beyond the orthogonal group.

The group integrals extend over a compact matrix space with a measure which is uniquely
determined by the underlying symmetry group. In order to reduce the computation to Wick
contractions, we consider an extended matrix space where all matrix elements are indepen-
dent Gaussian variables. In this space, all integrations trivially reduce to Wick contractions
and can easily be implemented in many programming languages. The constraints due to the
group structure are then introduced in an approximate fashion through a weight function w
appearing as factor in the integrand. This function is chosen in such a way that the integrals
yield the exact values for the lowest–order invariants of the group. It turns out that w is
not always positive and, thus, not a measure. This, however, is not a significant obstacle.

Candidates for our spaces are the orthogonal group O(N), the unitary group U(N) =
CUE(N), also known as the circular unitary ensemble, the circular orthogonal ensemble
of symmetric unitary matrices COE(N), and the circular symplectic ensemble which is
isomorphic to the unitary symplectic group CSE(N) = USP(2N). We note that COE(N) is
not a group — for all other cases we can use the Haar measure, while in this case we have
to use Dyson’s invariant measure.

We first present our arguments for the case of the orthogonal group. In Section II we
construct the weight function w from the invariants of the orthogonal group. We show that
the defining equations for w always have a unique solution. We give explicit expressions for
w in the simplest cases. We show that low–order monomials are calculated exactly using
Wick contraction. In Section III, we show that monomials of higher order are evaluated
correctly by Wick contraction, up to an error of order N^{−α}. We establish a lower bound for
the exponent α. In Section IV we extend our arguments to other matrix spaces. We give
explicit expressions for the weight functions for U(N) and for COE(N). The more involved
and less important case of CSE is only touched upon.

II. THE WEIGHT FUNCTION FOR O(N)

As explained in Section I, we start with a space of real matrices M. The elements are
taken as independent Gaussian–distributed variables with zero mean and identical variances.
In other words, our measure dμM for integration is the product of the differentials of all matrix
elements times \( N \exp\{-N \text{trace}(MM^T)\} \) where N is a normalization factor. We recall that
N is the dimension of the matrices M. We are interested in values of N ≫ 1. The measure
is invariant under right or left multiplication of M with any orthogonal matrix. To restrict
the integration to the orthogonal group, we could think of multiplying dμM with a set of delta
functions expressing all the constraints due to orthonormality. This is clearly impractical.
Instead, we modify the measure by multiplying dμM with a weight function wκ. This function
is chosen in such a way that all orthogonal invariants up to and including order 2κ are exactly
reproduced when we use wκ as a weight function under the integrals over M. We note in
passing that our present notation differs from that of Ref. [6]. Our index κ equals two times
the index k used there.

To determine wκ, we consider all invariants \( I_j(O) \) up to order 2κ in the matrix elements
O of the orthogonal group. Here, j is a running index. We recall that all such invariants are
even in the $O$’s. In every invariant $I_j(O)$, we replace $O$ by $M$ to obtain $I_j(M)$. We write $w$ as a linear combination of all the $I_j(M)$’s up to order $2\kappa$ in $M$. The coefficients of the linear combination are determined by the requirement that the average of every $I_j(M)$, calculated by integration over $d\mu_g$ with weight function $w$, yields the same result as integration of $I_j(O)$ over the orthogonal group. We recall that with $n$ a positive integer, the invariants of the orthogonal group are given by expressions of the form $\text{tr}\{(O^T)^n\}$, or by products of such expressions. We accordingly write the quantities $I_j(M)$ in the form

$$I^{(k)}_k(M) = \prod_{k_i} \text{tr}\{(MM^T)^{k_i}\}. \tag{1}$$

Here $2k$ denotes the degree of $I$ in $M$, and $k = (k_1, k_2, \ldots)$ is a partition of $k$ into positive integers $k_i \geq 1$ with $k_1 + k_2 + \ldots = k$. Without loss of generality we require that $k_1 \geq k_2 \geq \ldots$. The weight function $w$ is now written as

$$w(M) = a^{(\kappa)}_0 + \sum_{k=1}^{\kappa} \sum_{k} a^{(\kappa)}_k I^{(k)}_k(M). \tag{2}$$

The sum on the right–hand side of Eq. (3) extends over a complete set of linearly independent invariants up to order $2\kappa$ in $M$.

We determine the coefficients $a^{(\kappa)}_k$ from the conditions of orthonormality. More precisely, we require that the relations

$$\int d\mu_g w(M) = 1,$$

$$\int d\mu_g w(M)(MM^T)_{i_1j_1} = \delta_{i_1j_1},$$

$$\int d\mu_g w(M)(MM^T)_{i_1j_1}(MM^T)_{i_2j_2} = \delta_{i_1j_1}\delta_{i_2j_2},$$

$$\ldots$$

$$\int d\mu_g w(M)(MM^T)_{i_1j_1}(MM^T)_{i_2j_2} \cdots (MM^T)_{i_nj_n} = \delta_{i_1j_1}\delta_{i_2j_2} \times \ldots \times \delta_{i_nj_n} \tag{3}$$

be fulfilled identically. Relations of the form (3) hold for any value of $\kappa$ for the orthogonal group but must be imposed for the integration over the matrices $M$.

Eqs. (3) determine the coefficients $a^{(\kappa)}_k$ uniquely. To show this, we take traces over these equations in such a way that the integrals on the left–hand sides take the form $\int d\mu_g w(M)I^{(k)}_k(M)$. The resulting set of equations has the form

$$\int d\mu_g w(M)I^{(k)}_k(M) = B^{(k)}_k \tag{4}$$

where the coefficients $B^{(k)}_k$ are given by powers of $N$, with $N$ the dimension of the matrices $M$. Recalling Eq. (3), we see that Eqs. (4) constitute a set of linear equations for the coefficients $a^{(\kappa)}_k$. There are obviously as many equations as there are coefficients $a^{(\kappa)}_k$. We conclude that Eqs. (4) possess a unique solution unless the determinant of the matrix $C$ with elements $C^{(k_1k_2)}_{k_1k_2} = \int d\mu_g I^{(k_1)}_{k_1}I^{(k_2)}_{k_2}$ vanishes. But if $\text{det}(C^{(k_1k_2)}) = 0$, there exists a nontrivial solution $b^{(k)}_{k_2}$ of the homogeneous equation $\sum_{k_2} C^{(k_1k_2)}_{k_1k_2} b^{(k)}_{k_2} = 0$. The existence of this solution implies that we also have $\sum_{k_1,k_2} \sum_{k_1,k_2} C^{(k_1k_2)}_{k_1k_2} b^{(k)}_{k_1} C^{(k_1k_2)}_{k_1k_2} b^{(k)}_{k_2} = 0$. Recalling the definition of the
matrix $C$, we observe that the last relation can be written as
\[ \int d\mu g \sum_{k_1} I_{k_1}^{(k_1)} b_{k_1}^{(k_1)} = 0. \]
But the integrand in the last expression is positive semidefinite and does not vanish identically. Therefore, it is not possible that $\det(C^{(k_1 k_2)})$ vanishes, and the solution of Eqs. (1) exists and is unique. This solution also solves Eqs. (2). To see this, let us assume the contrary and focus attention on the second of Eqs. (2). (The argument is easily extended to the entire set of Eqs. (3)). Inserting the solution of Eqs. (1) into the left-hand side of that equation yields on the right-hand side the terms $\delta_{i_1 j_1} + A_{i_1 j_1}$ where the matrix $A$ is both traceless and invariant under every orthogonal transformation. This implies $A = 0$, in contradiction to the assumption that we did not find a solution of the second of Eqs. (2).

Eqs. (3) imply that the integrals over all polynomials of degree $n \leq 2\kappa$ in $M$ are equal to the corresponding expressions for $O(N)$. To see this, it suffices to consider the integral over an arbitrary monomial of degree $n$. It is obvious that the integral vanishes unless $n$ is even, $n = 2k$. We write the monomial as $M^{(n)} = M_{i_1 j_1} M_{i_2 j_2} \ldots M_{i_n j_n}$. The integral over $M^{(n)}$ is obviously invariant under right or left multiplication with any orthogonal transformation. Therefore, the integral over $M^{(n)}$ must be a linear combination of invariants multiplied by a suitable set of Kronecker deltas in the indices $i_1, \ldots, i_n$ and $j_1, \ldots, j_n$. By construction the invariants have the same values as in $O(N)$.

Inspection shows that the weight function $w_0 = 1$ fulfills the second of Eqs. (3) automatically. Thus, $w_1 = w_0$ and, therefore, $a_0^{(1)} = 1$, $a_1^{(1)} = 0$. The first nontrivial condition is, therefore, the one appearing in line 3 of Eqs. (3). This condition (and all that follow below it) is violated by $w_0$. We now give the explicit results for the first few weight functions $w_\kappa$. These were obtained with the help of the Mathematica program. For $\kappa = 2$, we find

\begin{align*}
a_0^{(2)} &= 1 - \frac{N^2}{4} \\
a_1^{(2)} &= \frac{N}{2} \\
a_2^{(2)} &= -\frac{N^3}{4(-1 + N)(2 + N)} \\
a_{11}^{(2)} &= \frac{N^2}{4(-1 + N)(2 + N)}.
\end{align*}

For $\kappa = 3$, we have

\begin{align*}
a_0^{(3)} &= 1 - \frac{7N^2}{12} \\
a_1^{(3)} &= \frac{3N}{2} \\
a_2^{(3)} &= -\frac{5N^3}{4(-1 + N)(2 + N)} \\
a_{11}^{(3)} &= \frac{5N^2}{4(-1 + N)(2 + N)} \\
a_3^{(3)} &= \frac{N^5}{3(-2 + N)(-1 + N)(2 + N)(4 + N)} \\
a_{21}^{(3)} &= -\frac{N^4}{(-2 + N)(-1 + N)(2 + N)(4 + N)}
\end{align*}
\[ a_{111}^{(3)} = \frac{2N^3}{3(-2+N)(-1+N)(2+N)(4+N)}. \]  

(6)

For \( \kappa = 4 \), we have

\[
\begin{align*}
    a_0^{(4)} &= 1 - \frac{23N^2}{24} + \frac{N^4}{32}, \\
    a_1^{(4)} &= 3N - \frac{N^3}{8}, \\
    a_2^{(4)} &= \frac{-60N^3 + N^5}{16(-1+N)(2+N)}, \\
    a_{11}^{(4)} &= \frac{56N^2 + 2N^3 + N^4}{16(-1+N)(2+N)}, \\
    a_3^{(4)} &= \frac{7N^5}{3(-2+N)(-1+N)(2+N)(4+N)}, \\
    a_{21}^{(4)} &= \frac{-48N^4 - 2N^5 - N^6}{8(-2+N)(-1+N)(2+N)(4+N)}, \\
    a_{111}^{(4)} &= \frac{88N^3 + 6N^4 + 3N^5}{24(-2+N)(-1+N)(2+N)(4+N)}, \\
    a_4^{(4)} &= \frac{-N^7(6+5N)}{8(-3+N)(-2+N)(-1+N)(1+N)(2+N)(4+N)(6+N)}, \\
    a_{31}^{(4)} &= \frac{-N^6(6+5N)}{2(-3+N)(-2+N)(-1+N)(1+N)(2+N)(4+N)(6+N)}, \\
    a_{22}^{(4)} &= \frac{-N^7(18+5N+N^2)}{32(-3+N)(-2+N)(-1+N)(1+N)(2+N)(4+N)(6+N)}, \\
    a_{211}^{(4)} &= \frac{-N^5(72+78N+5N^2+N^3)}{16(-3+N)(-2+N)(-1+N)(1+N)(2+N)(4+N)(6+N)}, \\
    a_{1111}^{(4)} &= \frac{-N^4(72+78N+5N^2+N^3)}{32(-3+N)(-2+N)(-1+N)(1+N)(2+N)(4+N)(6+N)}. \\
\end{align*}
\]  

(7)

We note that with increasing \( \kappa \), the expressions become rather involved. Moreover, the coefficients \( a_k^{(\kappa)} \) with the same lower indices \( k \) change with \( \kappa \).

### III. MONOMIALS OF HIGHER ORDER FOR \( O(N) \): THE \( 1/N \) EXPANSION

We have seen that integrals over all polynomials of degree \( n \leq 2\kappa \) have the same values as for \( O(N) \). What about polynomials of higher order? Again, it suffices to consider monomials \( \mathcal{M}^{(k)} \) of even degree \( 2k \) with \( k > \kappa \). We show that the integral over \( \mathcal{M}^{(k)} \) coincides with the result for \( O(N) \) up to terms of order \( N^{-\alpha} \) where the integer exponent \( \alpha \) is positive and independent of \( k \). More precisely, we show that for \( k > \kappa \), we have

\[ N^k \int d\mu g w_{\kappa}(M) \prod_{\nu=1}^{2k} M_{i\nu j\nu} = N^k \int dh_{O(N)} \prod_{\nu=1}^{2k} O_{i\nu j\nu} + O(1/N^\alpha) \text{ where } \alpha \geq [\kappa/2] + 1. \]  

(8)
Here $\lfloor \kappa/2 \rfloor$ indicates the integer part of $\kappa/2$, and $dh_{O(N)}$ denotes the Haar measure for integration over $O(N)$. We note that the factors $N^k$ in front of the integrals normalise the $N$-dependence so that these terms are (at most) of order 1. Another equivalent form of Eq. (8) is obtained by summing over pairs of indices $j_1 = j_2, j_3 = j_4, \ldots$. This removes the factors $N^k$ and yields

$$\int d\mu_g w_\kappa(M) \prod_{\nu=1}^k (MM^T)_{i_\nu l_\nu} = \int dh_{O(N)} \prod_{\nu=1}^k (OO^T)_{i_\nu l_\nu} + \mathcal{O}(1/N^\alpha) \text{ where } \alpha \geq \lfloor \kappa/2 \rfloor + 1.$$  

(9)

The equivalence of Eq. (8) and Eq. (9) follows from the fact the matrix $C$ discussed above, if defined with respect to properly scaled monomials, does not depend on $N$. The remainder of this Section is devoted to proving Eq. (9).

It is useful to introduce a few auxiliary concepts. We consider Gaussian integrals over monomials of $M$ without the weight function $w_\kappa$. We write for brevity

$$\int d\mu_g \prod_{\nu=1}^k (MM^T)_{i_\nu l_\nu} = \langle (MM^T)^k \rangle_g$$  

(10)

where the index $g$ indicates the purely Gaussian integration. To define the completely correlated part of this expression, we consider first the case $k = 2$. We use Wick contraction and have

$$\langle (MM^T)^2 \rangle_g = \langle (MM^T) \rangle_g \langle (MM^T) \rangle_g + \langle (MM^T)^2 \rangle_{gc}.$$  

(11)

The last term on the right-hand side of Eq. (11) is the completely correlated term. For the general case of arbitrary order $2k$, we define the correlated part $\langle (MM^T)^k \rangle_{gc}$ as that contribution to $\langle (MM^T)^k \rangle_g$ which cannot be written in the form of products of two or more factors, each of which is a complete Wick contraction of powers of $MM^T$. It is easy to see that

$$\langle (MM^T)^k \rangle_{gc} = \mathcal{O}\left(1/N^{k-1}\right).$$  

(12)

The linear increase with $k$ in inverse powers of $N$ in Eq. (12) is due to the fact that every Wick contraction which connects two $M$'s appearing in different factors $MM^T$ suppresses one summation index. Therefore, the correlated part $\langle (MM^T)^k \rangle_{gc}$ contributes the highest-order terms in $1/N$ to $\langle (MM^T)^k \rangle_g$.

We now consider integrals involving the weight function $w_\kappa$ and use the same notation,

$$\int d\mu_g w_\kappa \prod_{\nu=1}^k (MM^T)_{i_\nu l_\nu} = \langle w_\kappa (MM^T)^k \rangle_g.$$  

(13)

Again using Wick contraction, we define the correlated part $\langle w_\kappa (MM^T)^k \rangle_{gc}$ of this expression as that part which cannot be written as the product of two or more factors, each of which is a complete Wick contraction of powers of $MM^T$ and/or $w_\kappa$.

We proceed to show that in the equations relating the integral $\langle w_\kappa (MM^T)^k \rangle_g$ to the integral over the Haar measure, the leading correction term (lowest order in $1/N$) which does not cancel is given by
\[ \langle w_\kappa (MM^T)^k \rangle_{gc} = O \left( \frac{1}{N^{[(k+1)/2]}} \right) \quad \text{for} \quad 1 < k \leq \kappa . \]  

This relation is based upon the assumption that there is no accidental cancellation among the terms contributing to lowest order in $1/N$. Therefore, $[(k + 1)/2]$ actually constitutes a lower bound on the exponent of $1/N$.

To prove the relation (14), we rewrite the defining equations for $w_\kappa$, Eqs. (3), as follows. We consider the expression $\langle w_\kappa (MM^T)^k \rangle_g$ with $k$ integer and $k \leq \kappa$. We decompose this expression into correlated contributions. These originate from all partitions $k = (k_1, k_2, \ldots)$ of $k$ with $k_i \geq k_{i+1}$ and $\sum_i k_i = k$. We denote by $i_0$ the smallest index for which all $k_i$ with $i > i_0$ are equal to one. Then, we have

\[ \langle w_\kappa (MM^T)^k \rangle_g = \sum_k \prod_i \langle (MM^T)^{k_i} \rangle_{gc} + \sum_{k \neq 1^k} \sum_{i_0} \langle w_\kappa (MM^T)^{k_{i_0}} \rangle_{gc} \prod_{j \neq i} \langle (MM^T)^{k_j} \rangle_{gc} . \]  

In the first term on the right–hand side, we have used that $\langle w_\kappa \rangle_g = 1$. In the second term, we have used that $\langle w_\kappa (MM^T)^k \rangle_{gc} = 0$. Trivially, the second sum on the right–hand side of Eq. (15) does not extend over the partition $1^k = (1, 1, 1, \ldots)$ ($k$ terms unity). According to Eqs. (3), the expression in Eq. (17) equals $\langle ((MM^T)^g)^k \rangle$. This equals the contribution from the first sum on the right–hand side for the partition $1^k$. The remaining terms must vanish,

\[ \sum_k \prod_i \langle (MM^T)^{k_i} \rangle_{gc} + \sum_{k \neq 1^k} \sum_{i_0} \langle w_\kappa (MM^T)^{k_{i_0}} \rangle_{gc} \prod_{j \neq i} \langle (MM^T)^{k_j} \rangle_{gc} = 0 . \]  

Eq. (16) must hold for all values of $k$ with $k \leq \kappa$. To proceed, we observe that the partitions of $k$ can be grouped into classes as follows: Partitions within the same class carry the same number $p$ of $k_i$'s that have value unity. The classes are labeled by $p$, namely $C_p$. For instance, for $k = 6$, class $C_2$ contains the partitions $(4, 1, 1)$ and $(2, 2, 1, 1)$. In Eq. (16), we order the sum over $k$ by grouping together all partitions which belong to the same class. We show presently that each such contribution must vanish separately. Then, we have for every $p = 0, 1, \ldots, k - 2$ that

\[ \sum_{k \neq 1^k} \prod_{i_0} \langle (MM^T)^{k_{i_0}} \rangle_{gc} + \sum_{k \in C_p} \sum_{i_0} \langle w_\kappa (MM^T)^{k_{i_0}} \rangle_{gc} \prod_{j \neq i} \langle (MM^T)^{k_j} \rangle_{gc} = 0 . \]  

Eq. (17) follows directly from the facts that Eq. (16) holds for all $k \leq \kappa$, and that the contributions from class $C_p$ to a partition of $k$ are the same as the contributions of class $C_0$ to a partition of $k-p$, except for a string of Kronecker delta’s due to the factors $\langle ((MM^T)^g)^p \rangle$.

We are now in the position to prove the relation (14). We observe that in Eqs. (17), the term $\langle w_\kappa (MM^T)^k \rangle_{gc}$ appears only in the class $C_0$. Therefore, we have

\[ \langle w_\kappa (MM^T)^k \rangle_{gc} = - \sum_{k \in C_0} \prod_i \langle (MM^T)^{k_i} \rangle_{gc} \]

\[ - \sum_{k \in C_0, k \neq 1^k} \sum_{i_0} \langle w_\kappa (MM^T)^{k_{i_0}} \rangle_{gc} \prod_{j \neq i} \langle (MM^T)^{k_j} \rangle_{gc} . \]  

Using complete induction, i.e., assuming that the relation (14) holds for all values of $k'$ with $k' < k$, and using Eq. (12), we conclude from Eq. (18) that the relation (14) also holds for
\[ k' + 1 = k \]. The terms of lowest order in \( 1/N \) originate from partitions which have either the form \( (2, 2, 2, \ldots) \) (for even \( k \)) or \( (3, 2, 2, 2, \ldots) \) (for odd \( k \)). Again, we cannot rule out the occurrence of accidental cancellations which would increase the power \( [(k + 1)/2] \) in the relation (14).

Having established the relation (14), we turn to the center piece of this Section, Eq. (9). We first consider the case \( k = \kappa + 1 \) and decompose the integral into correlated terms with contributions from all partitions of \( \kappa + 1 \). For these partitions, we write \( \mathbf{K} + 1 = (k_1, k_2, \ldots) \) with \( k_1 + k_2 + \ldots = \kappa + 1 \) and \( k_1 \geq k_2 \geq \ldots \). We use the notation introduced above. Then,

\[
\langle w_\kappa (MM^T)^{\kappa+1} \rangle_g = \sum_p \sum_{(\mathbf{K}+1) \in C_p} \prod_i \langle (MM^T)^{k_i} \rangle_{gc} \nonumber
\]

\[
+ \sum_p \sum_{(\mathbf{K}+1) \in C_p} \sum_i \langle w_\kappa (MM^T)^{k_i} \rangle_{gc} \prod_{j \neq i} \langle (MM^T)^{k_j} \rangle_{gc}. \tag{19}
\]

Eqs. (17) imply that all terms with \( p \neq 0 \) and \( p \neq \kappa + 1 \) vanish, and we are left with

\[
\langle w_\kappa (MM^T)^{\kappa+1} \rangle_g = \langle (MM^T)^{\kappa+1} \rangle_g + \sum_{(\mathbf{K}+1) \in C_0} \prod_i \langle (MM^T)^{k_i} \rangle_{gc} \nonumber
\]

\[
+ \sum_{(\mathbf{K}+1) \in C_0} \sum_i \langle w_\kappa (MM^T)^{k_i} \rangle_{gc} \prod_{j \neq i} \langle (MM^T)^{k_j} \rangle_{gc} \tag{20}.
\]

We use the same argument as in the previous paragraph and the result (14) and Eq. (12). We conclude that the contributions of lowest order in \( 1/N \) result from the partitions in class \( C_{\kappa+1} \). But these give exactly the same contributions as those for \( k = \kappa + 1 \) that were estimated in the last paragraph. This completes the proof of Eq. (9).

IV. THE UNITARY GROUP AND OTHER UNITARY ENSEMBLES

In this Section, we primarily address integrals over monomials of unitary matrices \( U \) with respect to the Haar measure of the unitary group. We start with Gaussian integrals over complex matrices \( M \). The real and imaginary parts of the matrix elements are independent and Gaussian-distributed. The integrals are again worked out using Wick contractions. For the weight function \( w_\kappa^u \), we write in analogy to Eq. (2)

\[
w_\kappa^u(M) = b_0^{(\kappa)} + \sum_{k=1}^{\kappa} \sum_k b_k^{(\kappa)} f_k^{(k)}(M). \tag{21}
\]

The invariants are defined as in Eq. (11) with \( MM^T \) replaced by \( MM^\dagger \) where \( \dagger \) stands for Hermitean conjugation. The upper index \( u \) refers to the unitary case. The values of the coefficients \( b_k^{(\kappa)} \) are, of course, not the same as for the orthogonal case. The arguments for the
existence and uniqueness of the solutions carry over without change, and again computer programs are available to perform the contractions and calculate the coefficients $b_k$. For $w_2^u$ and $w_4^u$ we find

$$
\begin{align*}
b_0^2 &= 1 - \frac{N^2}{2} \\
b_1^2 &= N \\
b_2^2 &= -\frac{N^3}{2(N - 1)(N + 1)} \\
b_{11}^2 &= \frac{N^2}{2(N - 1)(N + 1)} \, ,
\end{align*}
$$

and

$$
\begin{align*}
b_0^4 &= \frac{24 - 46 N^2 + 3 N^4}{24} \\
b_1^4 &= -\frac{(N (-12 + N^2))}{2} \\
b_2^4 &= \frac{N^3 (-30 + N^2)}{4 (-1 + N) (1 + N)} \\
b_{111}^4 &= \frac{N^2 (28 + N^2)}{4 (-1 + N) (1 + N)} \\
b_3^4 &= \frac{14 N^5}{3 (-2 + N) (-1 + N) (1 + N) (2 + N)} \\
b_{21}^4 &= \frac{-N^4 (24 + N^2)}{2 (-2 + N) (-1 + N) (1 + N) (2 + N)} \\
b_{1111}^4 &= \frac{N^3 (44 + 3 N^2)}{6 (-2 + N) (-1 + N) (1 + N) (2 + N)} \\
b_4^4 &= \frac{-5 N^7}{4 (-3 + N) (-2 + N) (-1 + N) (1 + N) (2 + N) (3 + N)} \\
b_{31}^4 &= \frac{5 N^6}{(-3 + N) (-2 + N) (-1 + N) (1 + N) (2 + N) (3 + N)} \\
b_{22}^4 &= \frac{N^6 (6 + N^2)}{8 (-3 + N) (-2 + N) (-1 + N) (1 + N) (2 + N) (3 + N)} \\
b_{211}^4 &= \frac{-N^5 (36 + N^2)}{4 (-3 + N) (-2 + N) (-1 + N) (1 + N) (2 + N) (3 + N)} \\
b_{1111}^4 &= \frac{N^4 (36 + N^2)}{8 (-3 + N) (-2 + N) (-1 + N) (1 + N) (2 + N) (3 + N)} \, .
\end{align*}
$$

The arguments determining the leading contribution to the $1/N$ expansion are the same ones as for the orthogonal group. Thus, we have to increase $\kappa$ by two to improve the error by one order in $1/N$ in the calculation of monomials of high order. This is the reason for our not giving $w_3^u$ but only $w_4^u$ which yields correct values for the integrals up to order $1/N^2$.

For COE and CSE, the situation is slightly more complicated. The constraints on the matrices are not expressible in a simple way in terms of products as done in Eqs. (3). It
seems, therefore, most convenient to limit the space of independent matrix elements from the outset. In the case of the COE this is fairly simple: The symmetry reduces the number of independent complex matrix elements to \( N(N + 1)/2 \). Therefore, we consider Gaussian averages in the space of complex symmetric matrices \( S = S^T \). We accordingly have a contraction rule with two terms,

\[
\int d\mu_g(S) S_{ij}^* S_{kl} = \frac{1}{N} (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) .
\]  

(24)

The invariants are now defined in terms of \( SS^* \). Introducing the rule (24) into our program we can again calculate \( w_\kappa^c \). As an example we the coefficients \( c_k^2 \) for \( w_2^c \),

\[
c_0^2 = 1 - \frac{N(N + 1)}{4}
\]

\[
c_1^2 = \frac{N + 1}{2}
\]

\[
c_2^2 = -\frac{(N + 1)^3}{4N(N + 3)}
\]

\[
c_{11}^2 = \frac{(N + 1)^2}{4N(N + 3)} .
\]

(25)

In addition, the following subtle point must be considered. In Section I we have used the invariance of the Haar measure to justify that contraction with \( w_\kappa \) gives exact results for all polynomials up to order \( 2\kappa \) in the matrix elements. In the present case we have no invariance group and by consequence no Haar measure. On the other hand Dyson’s measure with respect to which we integrate, is also totally defined by an invariance group albeit a smaller one than that of \( U(N) \). The important point is that again the measure is uniquely defined by a linear group of transformations. The orthogonality conditions resulting from the unitarity of the matrices are the same and symmetry is taken into account explicitly in the contractions. Therefore all arguments again go through and indeed inspection of the results obtained by our code with those obtained in Ref. [2] shows agreement.

The case of CSE is simpler because it involves an invariance group, but more complicated because the matrices are symplectic. Two ways seem open to address this case. We might include the symplectic property from the outset in the contraction rules, or we might introduce this property as a constraint in the expression for the weight function. While both ways seem possible it is not clear which one is easier to follow. In view of the fact that CSE is of minor importance for practical applications, we have left this problem open. It is clear, however, that it can be tackled along the same lines.

V. CONCLUSIONS

We have presented a systematic way to calculate integrals over monomials of matrix elements for compact matrix groups and for other matrix ensembles whose measure is defined uniquely by an invariance group, such as the circular orthogonal ensemble of unitary symmetric matrices. This method gives exact results for monomials of low order. For higher–order monomials, it leads to an error of order \( 1/N^\alpha \) which is independent of the degree of
the monomial. We have given a lower bound on the integer $\alpha$, and we have shown how $\alpha$ can be increased systematically. The method is particularly suited for symbolic computer calculation. Codes are available for $O(N)$, $U(N)$ as well as for the circular orthogonal ensemble in Mathematica and in C from one of the authors (T.P.).

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