Interval-Valued Fermatean Hesitant Fuzzy Sets and Infectious Diseases Application

Murat Kiriçi (mkirisci@hotmail.com)
İstanbul Üniversitesi-Cerrahpaşa  https://orcid.org/0000-0003-4938-5207

Necip Şimşek
İstanbul Commerce University: Istanbul Ticaret Universitesi

Research Article

Keywords: Interval-valued Fermatean hesitant fuzzy set, Fermatean hesitant fuzzy aggregation operator, infectious disease, multiple attribute group decision-making

Posted Date: January 26th, 2022

DOI: https://doi.org/10.21203/rs.3.rs-1273874/v1

License: This work is licensed under a Creative Commons Attribution 4.0 International License.
Read Full License
Interval-Valued Fermatean Hesitant Fuzzy Sets and Infectious Diseases Application

immediate

Article Info

Keywords: Interval-valued Fermatean hesitant fuzzy set; Fermatean hesitant fuzzy aggregation operator; infectious disease; multiple attribute group decision-making

2010 AMS:

Received:
Accepted:
Available online:

Abstract

The Hesitant Fuzzy Set, which is a generalization of fuzzy sets, is an important tool in dealing with the difficulties that arise in determining the membership of an element to a set when there is doubt between several different values in decision-making problems. In this study, Fermatean hesitant fuzzy set is given to ensure to operate the conditions in which professionals evaluate an alternative in probable membership values and non-membership values. Aggregation operators of newly defined sets are defined to implement to multi-attributed group decision-making problems. The main properties of the new sets were examined. A new score function and accuracy function are given to compare two interval-valued numbers. Finally, a numeric example exactly demonstrates the feasibility, practicality, and effectiveness of the offered technique.

1. Introduction

The reasoning and decision-making (DM) processes of people in the face of daily events are studied by many disciplines, including psychology, philosophy, cognitive science, and artificial intelligence. These processes are generally tried to be described based on various mathematical and statistical models. In this process, the problem of decision-making arises. DM is defined as the operation of selecting one or more of the alternative forms of behavior faced by a person or an institution in order to achieve a specific goal. Research shows that while it is sufficient to make many daily decisions intuitively, this path alone is not enough for complex and vital decisions. Multi-Attribute Decision Making (MADM) refers to the decision-making process in discrete situations where the alternatives examined in the decision problem are finite and clearly defined. In MADM problems, the alternatives are a predetermined number. MADM approaches are frequently used in decision problems such as choosing among alternatives, ranking, and comparing alternatives. They are frequently preferred methods in that they allow quick decision-making without requiring heavy mathematical operations and using a package program. There is only one purpose in the MADM method. The aim is to determine the most ideal (most benefit, least cost) alternative for the decision problem. For the example problem above, the purpose of the decision problem can be expressed as "determination of the most suitable supplier alternative".

Group decision making (GDM) is about using the unified wisdom and experience of those involved in the group to make decisions that are likely to provide affirmative benefits. One of the key advantages of GDM is its potential to involve people from different backgrounds and thought processes so that the issues facing the group can be explored from a wider range of perspectives. Individuals want to overcome the difficulties they face to reach their goals. Sometimes this task becomes so large and complex that the individual can’t solve it alone. In such cases, it is a more rational approach to making decisions by using group power. Whether working around a desk or dispersed in digital environments, the synergy that emerges as a group is an important tool in improving decisions and solving problems. Thus, individuals achieve some of their needs and goals that they cannot achieve alone through groups. Thus, although group members have their own thoughts and motivations, when they want to solve a problem, the problem will no longer be the process of choosing the best option according to a single decision-maker. The resulting group decision-making process involves the conflicts of different interest groups, different goals, and objectives, different criteria, political behaviour, etc. would be expanded to take into account. At this point, the final solution is not left to the initiative of a single decision maker, that is, the responsibility of all decision makers occurs.

In general, uncertainty is the situation in which a given event may have different consequences and there is no information about the probabilities of those consequences. Therefore, uncertainty is a very important notion for the DM process. It is not easy to know the probabilities of events happening in real-life. Therefore, the DM process occurs under uncertainty. Fuzzy logic theory [36] proposes a strong logical inference structure in the face of uncertain and imprecise knowledge. Fuzzy logic theory gives computers the ability to process people’s linguistic data and work using people’s experiences. While gaining this ability, it uses symbolic expressions instead of numerical expressions. These symbolic expressions are called fuzzy sets (FS). It is understood that the elements of fuzzy sets are actually decision variables containing probability states. Instead of probability values of possibilities, fuzzy sets arise by assigning membership degrees to each of them objectively.

Email addresses and ORCID numbers: xxxxxx@xxxx.xxx, xxxxxxx@xxxx.xxx
In the FS $A$, MD of the set is $\zeta_A$, while the degree of not belonging is $1 - \zeta_A$. Therefore, $MD + ND = 1$. However, this situation is insufficient to explain the uncertainty in some problems. For this reason, Atanassov [1] proposed the IFS theory. IFSs consist of MD and ND ($MD + ND \leq 1$). Yager [32] defined PFS as a more general and more comprehensive set than IFS. PFS is defined as $MD^2 + ND^2 \leq 1$. There is an extensive diversity of studies on FS, IFS, and PFS such as [2]-[5], [9]-[12], [18], [19], [26], [31], [33], [34], [38].

Yager [35] introduced the q-step orthopair fuzzy set. The basic rule in this set theory is that the sum of MD with ND should not be greater than 1. Based on this idea, Senapati and Yager [21] introduced the Fermatean fuzzy set (FFS) and examined its basic features. In [22], Fermatean arithmetic mean, division, and subtraction which are new transactions for FFS, are defined and some of their properties are examined. In [23], new weighted aggregated operators related to FFSs are defined. [13] have defined Fermatean fuzzy soft set (FFSS) and entropy measures. Shahzadi and Akram [24] offered a new decision support algorithm with respect to the FFSS and defined the new aggregated operators. Garg et al. [6] new FFS type aggregated operators were defined by utilizing the t-norm and t-conorm.

The FS notion was generalized to the HFS notion by Torra [27]. This new set of the FS can handle the situations that the complexity in building the MD does not get up from a margin of error or a certain probability distribution of the probable values, however, originates from hesitation among a few several values [37]. Hence the HFS can more precisely reflect the people’s hesitation in stating their preferences over objects, compared to the FS and its other generalizations. Later, HFS and IFS were combined to obtain a new HFS which is called IHFS [20]. The fundamental notion is to form the situation in which instead of a individual MD and ND, human beings hesitate among a set of MD and ND and they require to symbolize such a hesitation. In [39], the notion of a dual HFS was improved and was given some properties. As an extension of the dual IIVHFS, the HIVIFS approach was given [16]. In [17], the notion of HIFS to GDM problems using fuzzy cross-entropy was applied. The FFHS was initially given by Khan et al. [7]. PHFS compensates the case that the sum of its MDs is less than 1. The Fermatean hesitant fuzzy set has been defined by Kirisci [15].

This work is dedicated to extending FHFSs to IVFHF and improving $\text{MADM}$ processes to IVFHF environments by aggregation operators. Score functions and accuracy functions are defined. The basic properties are studied together with the definition of IVFHF. An algorithm is given by introducing the scenario describing the idea of $\text{MADM}$ in IVFHF environments. A medical application showing the feasibility and applicability of the offered technique is given.

2. Preliminaries

Throughout the paper, $U$, as the initial universe set, respectively will be denoted.

For $\zeta_F : U \rightarrow [0, 1]$ and $\eta_F : U \rightarrow [0, 1]$, the FFS $\mathcal{F}$ is indicated by $\mathcal{F} = \{(u, \zeta_F (u), \eta_F (u)) : u \in U\}$. For the FFS, the condition $0 \leq \zeta_F (u) + \eta_F (u) \leq 1$ holds.

The ID of $u$ to $\mathcal{F}$ is described as $\theta_F (u) = \sqrt{1 - (\zeta_F^2 (u) + \eta_F^2 (u))}$, for any FFS $\mathcal{F}$ and $u \in U$.

For FFSs $\mathcal{F}_1 = \{\zeta_{F_1}, \eta_{F_1}\}$ and $\mathcal{F}_2 = \{\zeta_{F_2}, \eta_{F_2}\}$, some operations as follows [21]:

i. $\mathcal{F}_1 \cap \mathcal{F}_2 = \{\min\{\zeta_{F_1}, \zeta_{F_2}\}, \max\{\eta_{F_1}, \eta_{F_2}\}\}$;
ii. $\mathcal{F}_1 \cup \mathcal{F}_2 = \{\max\{\zeta_{F_1}, \zeta_{F_2}\}, \min\{\eta_{F_1}, \eta_{F_2}\}\}$;
iii. $\mathcal{F}' = \{\eta_F, \zeta_F\}$;
iv. $\mathcal{F}_1 \boxplus \mathcal{F}_2 = \left(\sqrt{\zeta_{F_1}^3 + \zeta_{F_2}^3 - \zeta_{F_1} \cdot \zeta_{F_2}}, \eta_{F_1} \cdot \eta_{F_2}\right)$;
v. $\mathcal{F}_1 \boxminus \mathcal{F}_2 = \left(\zeta_{F_1}, \sqrt{\eta_{F_1}^3 + \eta_{F_2}^3 - \eta_{F_1} \cdot \eta_{F_2}}\right)$;
vi. $\alpha \mathcal{F} = \left(\zeta_{F_1}^\alpha, \eta_{F_1}^\alpha\right)$;
vii. $\mathcal{F}^\alpha = \left(\zeta_{F_1}, \eta_{F_1}^\alpha\right)$.

The properties of complement of FFS as follows [21]:

i. $(\mathcal{F}_1 \cap \mathcal{F}_2)^c = \mathcal{F}_1^c \cup \mathcal{F}_2^c$;
ii. $(\mathcal{F}_1 \cup \mathcal{F}_2)^c = \mathcal{F}_1^c \cap \mathcal{F}_2^c$;
iii. $(\mathcal{F}_1 \boxplus \mathcal{F}_2)^c = \mathcal{F}_1^c \boxminus \mathcal{F}_2^c$;
iv. $(\mathcal{F}_1 \boxminus \mathcal{F}_2)^c = \mathcal{F}_1^c \boxplus \mathcal{F}_2^c$;
v. $\alpha(\mathcal{F})^c = (\mathcal{F}^c)\alpha$;
vii. $\left((\mathcal{F})^\alpha\right)^c = (\mathcal{F})^\alpha$.

Definition 2.1. [21] Choose a FFS $\mathcal{F} = \{\zeta_F, \eta_F\}$. For FFS $\mathcal{F}$,

$$SF = \zeta_F^2 - \eta_F^2.$$  

(2.1)

is said to be a score function.

The function SF is in $[-1, 1]$.

Take the two FFSs $\mathcal{F}_1 = \{\zeta_{F_1}, \eta_{F_1}\}$ and $\mathcal{F}_2 = \{\zeta_{F_2}, \eta_{F_2}\}$. If the following condition (A) is hold, then it is called a natural quasi-ordering concerning the FFS [21]:
(A) \[ \mathcal{F}_1 \geq \mathcal{F}_2 \iff \zeta_{\mathcal{F}_1} \leq \eta_{\mathcal{F}_2}. \]

For the two FFSs \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \):

(a) \( SF_{\mathcal{F}_1} < SF_{\mathcal{F}_2} \Rightarrow \mathcal{F}_1 < \mathcal{F}_2 \),

(b) \( SF_{\mathcal{F}_1} > SF_{\mathcal{F}_2} \Rightarrow \mathcal{F}_1 > \mathcal{F}_2 \),

(c) \( SF_{\mathcal{F}_1} = SF_{\mathcal{F}_2} \Rightarrow \mathcal{F}_1 \sim \mathcal{F}_2 \).

**Definition 2.2.** [21] For a FFS \( \mathcal{F} = \{\zeta_{\mathcal{F}}, \eta_{\mathcal{F}}\} \),

\[ AF = \zeta_{\mathcal{F}}^2 + \eta_{\mathcal{F}}^2. \tag{2.2} \]

is said to be an accuracy function.

Then, \( AF \in [0, 1] \). Clearly, \( 0 \leq AF = \zeta_{\mathcal{F}}^2 + \eta_{\mathcal{F}}^2 \leq 1 \).

For the two FFSs \( \mathcal{F}_1 = (\zeta_{\mathcal{F}_1}, \eta_{\mathcal{F}_1}) \) and \( \mathcal{F}_2 = (\zeta_{\mathcal{F}_2}, \eta_{\mathcal{F}_2}) \):

(a) \( SF_{\mathcal{F}_1} < SF_{\mathcal{F}_2} \Rightarrow \mathcal{F}_1 < \mathcal{F}_2 \),

(b) \( SF_{\mathcal{F}_1} > SF_{\mathcal{F}_2} \Rightarrow \mathcal{F}_1 > \mathcal{F}_2 \),

(c) \( SF_{\mathcal{F}_1} = SF_{\mathcal{F}_2} \Rightarrow \mathcal{F}_1 \sim \mathcal{F}_2 \).

Then, \( \mathcal{F} \in [0, 1] \), and holds the following items:

(i) \( AF_{\mathcal{F}_1} < AF_{\mathcal{F}_2} \Rightarrow \mathcal{F}_1 < \mathcal{F}_2 \),

(ii) \( AF_{\mathcal{F}_1} > AF_{\mathcal{F}_2} \Rightarrow \mathcal{F}_1 > \mathcal{F}_2 \),

(iii) \( AF_{\mathcal{F}_1} = AF_{\mathcal{F}_2} \Rightarrow \mathcal{F}_1 = \mathcal{F}_2 \).

**Definition 2.3.** [28] The set

\[ \Gamma = \{(u, \tau_\Gamma(u)) : u \in U\} \tag{2.3} \]

is called HFS, where \( \tau_\Gamma(u) \) indicates the set of some values in unit interval, that is probable MD of \( u \in U \) to \( \Gamma \).

From now on, HFN will be used as \( \tau = \tau_\Gamma(u) \) throughout the paper.

**Definition 2.4.** The following operations are hold for three HFNs \( \tau, \tau_1, \tau_2 \):

1. \( \tau^c = \cup_{\delta \in \xi} (1 - \delta) \);
2. \( \tau_1 \cap \tau_2 = \cup_{\delta \in \xi_1, \delta_2 \in \xi_2} \min\{\delta_1, \delta_2\} \);
3. \( \tau_1 \cup \tau_2 = \cup_{\delta \in \xi_1, \delta_2 \in \xi_2} \max\{\delta_1, \delta_2\} \).

**Definition 2.5.** The set

\[ \Pi = \{(u, \zeta_{\Pi}(u), \eta_{\Pi}(u)) : u \in U\} \tag{2.4} \]

is called PHFS in \( U \), where \( \zeta_{\Pi}(u), \eta_{\Pi}(u) \) are functions from \( U \) to \([0, 1]\), showing a probable MD and ND of \( u \in U \) in \( \Pi \) respectively.

Further, for each element \( u \in U \):

(i.) \( \forall \eta_{\Pi}(u) \in \zeta_{\Pi}(u), \exists \eta_{\Pi}(x) \in \eta_{\Pi}(u), \) such that \( 0 \leq \tau_{\Pi}(u) + \tau_{\Pi}(x) \leq 1 \)

(ii.) \( \forall \eta_{\Pi}(u) \in \zeta_{\Pi}(u), \exists \eta_{\Pi}(u) \in \zeta_{\Pi}(u), \) such that \( 0 \leq \tau_{\Pi}(u) + \tau_{\Pi}(u) \leq 1 \).

**Definition 2.6.** Choose two interval-numbers \( \rho = [\rho^-, \rho^+] \) and \( \zeta = [\zeta^-, \zeta^+] \).

\[ P(\rho \triangleright \zeta) = \max\left\{1 - \left[\frac{\zeta^- - \rho^-}{J(\rho) - J(\zeta)}\right], 0\right\} \tag{2.5} \]

is stated as the likelihood of \( \rho \triangleright \zeta \), where \( J(\rho) = \rho^+ - \rho^- \) and \( J(\zeta) = \zeta^+ - \zeta^- \) and holds the following items:

(i.) \( 0 \leq P(\rho \triangleright \zeta) \leq 1 \),

(ii.) \( P(\rho \triangleright \zeta) = P(\zeta \triangleright \rho) = 1/2, \) if \( P(\rho \triangleright \zeta) = P(\zeta \triangleright \rho) \),

(iii.) \( P(\rho \triangleright \zeta) + P(\zeta \triangleright \rho) = 1 \).
3. New Hesitant Fuzzy Sets

In this section, IVFH will be introduced and its properties will be examined in order to get better results in preventing information loss and to increase the flexibility and applicability of decision-making techniques when dealing with qualitative information.

Definition 3.1.

\[ \mathcal{F} = \{(u, h_\mathcal{F}(u) : u \in U\} \tag{3.1} \]

is called an IVFH \( h_\mathcal{F} \) on \( U \), where

\[ h_\mathcal{F}(u) = \left\{ \left( \hat{\mathcal{F}}(u), \hat{\eta}_\mathcal{F}(u) \right) : \hat{\mathcal{F}}(u) = (\hat{\zeta}_1, \hat{\zeta}_2) \in D[0,1], \hat{\eta}_\mathcal{F}(u) = [\eta_{\mathcal{F}^-}, \eta_{\mathcal{F}^+}] \in D[0,1], (\hat{\zeta}_1)^3 + (\eta_{\mathcal{F}^-})^3 \leq 1 \right\}, \]

where \( \hat{\mathcal{F}}(u) \) is the possible Fermatean membership interval and \( \hat{\eta}_\mathcal{F}(u) \) is the possible Fermatean non-membership intervals of \( \mathcal{F} \). Throughout this article, \( Y \) will show the set of all IVFESs.

Apparently, if there is only one pair of intervals in \( h_\mathcal{F}(u) \), the IVFH converts into an IVFFS, if both \( \hat{\mathcal{F}}(u), \hat{\eta}_\mathcal{F}(u) \) converts one singleton, the IVFH may be viewed as an FHFS, if \( \hat{\zeta}_1^+ + \hat{\zeta}_2^+ \leq 1 \) the IVFH can be seen as an IIVFS, for each \( u \in U \).

For \( \hat{\mathcal{F}} = (\hat{\zeta}, \hat{\eta}) : \hat{\mathcal{F}} = [\zeta^-, \zeta^+], \hat{\eta} = [\eta^-, \eta^+] \), each pair \( \hat{\mathcal{F}} = h_\mathcal{F}(u) \) is said to be an interval-valued Fermatean hesitant fuzzy element (IVFHE).

Definition 3.2. Choose be three IVFESs \( \hat{\mathcal{F}}, \mathcal{F}_1, \mathcal{F}_2 \) and \( \alpha > 0 \), then \( \hat{\mathcal{F}}, \hat{\mathcal{F}}^\alpha, \alpha \mathcal{F}, \mathcal{F}_1 \boxplus \mathcal{F}_2, \mathcal{F}_1 \boxcap \mathcal{F}_2, \mathcal{F}_1 \boxdot \mathcal{F}_2, \mathcal{F}_1 \triangle \mathcal{F}_2, \mathcal{F}_1 \cap \mathcal{F}_2 \) are all IVFESs.

Proof. It is clear that \( \mathcal{F}^\alpha \) is an IVFES.

For any \( (\zeta, \eta) \in \mathcal{F} \), since \( (\zeta^+)^3 + (\eta^+)^3 \leq 1 \)

\[ ((\zeta^+)^3 + (\eta^+)^3)^\alpha = (\zeta^+)^3 + (\eta^+)^3 \leq 1 \]

So, \( \hat{\mathcal{F}}^\alpha \) is an IVFES. Similarly, \( \alpha \mathcal{F} \) is also an IVFES. As for \( \mathcal{F}_1 \boxplus \mathcal{F}_2 \),

\[ (\zeta_1^+)^3 + (\zeta_2^+)^3 - (\zeta_1^+)^3(\zeta_2^+)^3 + (\eta_1^+)^3(\eta_2^+)^3 \leq (\zeta_1^+)^3 + (\zeta_2^+)^3 - (\zeta_1^+)^3(\zeta_2^+)^3 + (1 - (\zeta_2^+)^3)(1 - (\zeta_2^+)^3) \]

\[ = (\zeta_1^+)^3 + (\zeta_2^+)^3 - (\zeta_1^+)^3(\zeta_2^+)^3 + 1 - (\zeta_2^+)^3 + (\zeta_2^+)^3(\zeta_2^+)^3 = 1 \]

So, \( \mathcal{F}_1 \boxplus \mathcal{F}_2 \) is an IVFES. Similarly, \( \mathcal{F}_1 \boxcap \mathcal{F}_2 \) is also an IVFES.

Assume that \( \zeta_1^+ < \zeta_2^+ \). Then,

\[ \max(\zeta_1^+, \zeta_2^+) = (\zeta_2^+)^3 \]

So, \( \mathcal{F}_1 \triangle \mathcal{F}_2 \) is an IVFES. Similarly, \( \mathcal{F}_1 \cap \mathcal{F}_2 \) is also an IVFES.

Proposition 3.3. Let \( \hat{\mathcal{F}}, \hat{\mathcal{F}}_1, \hat{\mathcal{F}}_2 \) be four IVFESs and \( \alpha, \alpha_1, \alpha_2 > 0 \), then

i. \( \mathcal{F}_1 \boxplus \mathcal{F}_2 = \mathcal{F}_1 \boxplus \mathcal{F}_2 \)

ii. \( \mathcal{F}_1 \boxplus \mathcal{F}_2 = \mathcal{F}_2 \boxplus \mathcal{F}_1 \)

iii. \( \alpha \mathcal{F}_1 \boxplus \alpha \mathcal{F}_2 = \alpha \mathcal{F}_1 \boxplus \mathcal{F}_2 \)

iv. \( \mathcal{F}_1 \triangle \mathcal{F}_2 = (\mathcal{F}_1 \triangle \mathcal{F}_2)^\alpha \)

\[ \square \]
\[ v. \alpha_1 \tilde{\mathcal{F}} \boxplus \alpha_2 \tilde{\mathcal{F}} = (\alpha_1 + \alpha_2) \tilde{\mathcal{F}}, \]
\[ vi. \tilde{\mathcal{F}}_{u_1} \boxplus \tilde{\mathcal{F}}_{u_2} = (\tilde{\mathcal{F}}_{u_1} + \tilde{\mathcal{F}}_{u_2}). \]

**Proof.** (i.) Based on Definition 3.2, item (i) and (ii) are obvious. Now let's prove item (iii):
\[
\alpha \tilde{\mathcal{F}}_1 \boxplus \alpha \tilde{\mathcal{F}}_2 = \left\{ \left( \left[ \sqrt{1 - (1 - (\xi^*_1)^3)^a} \right], \left[ \sqrt{1 - (1 - (\xi^*_2)^3)^a} \right] \right), \left( \eta_2^+, \eta_2^- \right) : \left( \xi_2^+, \xi_2^- \right) \in \tilde{\mathcal{F}}_2 \right\}
\]
\[
\boxplus \left\{ \left( \left[ \sqrt{1 - (1 - (\xi^*_1)^3)^a} \right], \left[ \sqrt{1 - (1 - (\xi^*_2)^3)^a} \right] \right), \left( \eta_2^+, \eta_2^- \right) : \left( \xi_2^+, \xi_2^- \right) \in \tilde{\mathcal{F}}_2 \right\}
\]
\[
= \left\{ \left( \left[ \sqrt{1 - (1 - (\xi^*_1)^3)^a} \right], \left[ \sqrt{1 - (1 - (\xi^*_2)^3)^a} \right] \right), \left( \eta_2^+, \eta_2^- \right) : \left( \xi_2^+, \xi_2^- \right) \in \tilde{\mathcal{F}}_2, i = 1, 2 \right\},
\]
and
\[
\alpha \left( \tilde{\mathcal{F}}_1 \boxplus \tilde{\mathcal{F}}_2 \right) = \alpha \left\{ \left( \left[ \sqrt{1 - (1 - (\xi^*_1)^3)^a} \right], \left[ \sqrt{1 - (1 - (\xi^*_2)^3)^a} \right] \right), \left( \eta_2^+, \eta_2^- \right) : \left( \xi_2^+, \xi_2^- \right) \in \tilde{\mathcal{F}}_2 \right\}
\]
\[
= \alpha \left\{ \left( \left[ \sqrt{1 - (1 - (\xi^*_1)^3)^a} \right], \left[ \sqrt{1 - (1 - (\xi^*_2)^3)^a} \right] \right), \left( \eta_2^+, \eta_2^- \right) : \left( \xi_2^+, \xi_2^- \right) \in \tilde{\mathcal{F}}_2, i = 1, 2 \right\},
\]
So \( \alpha \tilde{\mathcal{F}}_1 \boxplus \alpha \tilde{\mathcal{F}}_2 = \alpha \left( \tilde{\mathcal{F}}_1 \boxplus \tilde{\mathcal{F}}_2 \right) \) holds. Similarly, we have \( \tilde{\mathcal{F}}_1 \boxplus \tilde{\mathcal{F}}_2 = \tilde{\mathcal{F}}_1 \boxplus \tilde{\mathcal{F}}_2 \). \( \Box \)

**Proposition 3.5.** Let \( \tilde{\mathcal{F}}, \tilde{\mathcal{F}}_1, \tilde{\mathcal{F}}_2, \tilde{\mathcal{F}}_3 \) be four IVFPs and \( \alpha, \alpha_1, \alpha_2 > 0 \), then
\[ i. (\tilde{\mathcal{F}}^C)^a = (\alpha \tilde{\mathcal{F}})^C, \]
\[ ii. \alpha \tilde{\mathcal{F}} = (\alpha \tilde{\mathcal{F}})^C, \]
\[ iii. (\tilde{\mathcal{F}}^C_1 \boxplus (\tilde{\mathcal{F}}_2)^C) = (\tilde{\mathcal{F}}_1 \boxplus \tilde{\mathcal{F}}_2)^C, \]
\[ iv. (\tilde{\mathcal{F}}_1)^C \boxplus (\tilde{\mathcal{F}}_2)^C = (\tilde{\mathcal{F}}_1 \boxplus \tilde{\mathcal{F}}_2)^C, \]
\[ v. (\tilde{\mathcal{F}}_1 \boxplus \tilde{\mathcal{F}}_2) \boxplus \tilde{\mathcal{F}}_3 = \tilde{\mathcal{F}}_1 \boxplus (\tilde{\mathcal{F}}_2 \boxplus \tilde{\mathcal{F}}_3) \]
\[ vi. (\tilde{\mathcal{F}}_1 \boxplus \tilde{\mathcal{F}}_2) \boxplus \tilde{\mathcal{F}}_3 = (\tilde{\mathcal{F}}_1 \boxplus \tilde{\mathcal{F}}_2) \boxplus \tilde{\mathcal{F}}_3 \]

**Proof.**

\[
(\tilde{\mathcal{F}}^C)^a = \left\{ \left( \left[ \eta^-, \eta^+ \right], \left[ \sqrt{1 - (1 - (\xi^*_1)^3)^a} \right], \left[ \sqrt{1 - (1 - (\xi^*_2)^3)^a} \right] \right) : \left( \xi_2^+, \xi_2^- \right) \in \tilde{\mathcal{F}} \right\}
\]
\[= \left\{ \left( \left[ \sqrt{1 - (1 - (\xi^*_1)^3)^a} \right], \left[ \sqrt{1 - (1 - (\xi^*_2)^3)^a} \right] \right), \left( \eta_2^+, \eta_2^- \right) : \left( \xi_2^+, \xi_2^- \right) \in \tilde{\mathcal{F}} \right\}
\]
\[= \alpha (\tilde{\mathcal{F}})^C.
\]
\[
\tilde{\mathcal{F}}_1 \boxplus \tilde{\mathcal{F}}_2 = \left\{ \left( \left[ \eta_1^-, \eta_1^+ \right], \left[ \xi_1^*, \xi_1^+ \right] \right) : \left( \xi_1^+, \xi_1^- \right) \in \tilde{\mathcal{F}}_1 \right\} \boxplus \left\{ \left( \left[ \eta_2^-, \eta_2^+ \right], \left[ \xi_2^*, \xi_2^+ \right] \right) : \left( \xi_2^+, \xi_2^- \right) \in \tilde{\mathcal{F}}_2 \right\}
\]
\[= \left\{ \left( \left[ \sqrt{1 - (1 - (\xi^*_1)^3)^a} \right] \right), \left[ \sqrt{1 - (1 - (\xi^*_2)^3)^a} \right] \right) : \left( \xi_2^+, \xi_2^- \right) \in \tilde{\mathcal{F}}_2, i = 1, 2 \right\},
\]
\[= \left\{ \left( \left[ \xi_1^*, \xi_2^* \right], \left[ \eta_1^-, \eta_1^+ \right], \left[ \eta_2^-, \eta_2^+ \right] \right) : \left( \xi_2^+, \xi_2^- \right) \in \tilde{\mathcal{F}}_2, i = 1, 2 \right\},
\]
\[= \left( \tilde{\mathcal{F}}_1 \boxplus \tilde{\mathcal{F}}_2 \right)^C.
\]
\[
(\tilde{\mathcal{F}}_1 \boxplus \tilde{\mathcal{F}}_2) \boxplus \tilde{\mathcal{F}}_3 = \left\{ \left( \left[ \sqrt{1 - (1 - (\xi^*_1)^3)^a} \right] \right), \left[ \sqrt{1 - (1 - (\xi^*_2)^3)^a} \right] \right) : \left( \xi_2^+, \xi_2^- \right) \in \tilde{\mathcal{F}}_2, \left( \xi_1^+, \xi_1^- \right) \in \tilde{\mathcal{F}}_1 \right\}
\]
\[= \left\{ \left( \left[ \sqrt{1 - (1 - (\xi^*_1)^3)^a} \right] \right), \left[ \sqrt{1 - (1 - (\xi^*_2)^3)^a} \right] \right) : \left( \xi_2^+, \xi_2^- \right) \in \tilde{\mathcal{F}}_2, i = 1, 2, 3 \right\},
\]
\[= \left\{ \left( \left[ \sqrt{1 - (1 - (\xi^*_1)^3)^a} \right] \right), \left[ \sqrt{1 - (1 - (\xi^*_2)^3)^a} \right] \right) : \left( \xi_2^+, \xi_2^- \right) \in \tilde{\mathcal{F}}_2, i = 1, 2, 3 \right\},
\]
\[= \left( \tilde{\mathcal{F}}_1 \boxplus (\tilde{\mathcal{F}}_2 \boxplus \tilde{\mathcal{F}}_3) \right) = \left( \tilde{\mathcal{F}}_1 \boxplus \tilde{\mathcal{F}}_2 \right) \boxplus \tilde{\mathcal{F}}_3.
\]
Proposition 3.6. Let \( \hat{\mathcal{F}}, \hat{\mathcal{F}}_1, \hat{\mathcal{F}}_2, \hat{\mathcal{F}}_3 \) be four IVFEs and \( \alpha, \alpha_1, \alpha_2 > 0 \), then

i. \( \hat{\mathcal{F}}_1 \cup \hat{\mathcal{F}}_2 = \hat{\mathcal{F}}_2 \cup \hat{\mathcal{F}}_1 \),
ii. \( \hat{\mathcal{F}}_1 \cap \hat{\mathcal{F}}_2 = \hat{\mathcal{F}}_2 \cap \hat{\mathcal{F}}_1 \),
iii. \( \alpha \hat{\mathcal{F}}_1 \cup \alpha \hat{\mathcal{F}}_2 = \alpha (\hat{\mathcal{F}}_1 \cup \hat{\mathcal{F}}_2) \),
iv. \( (\hat{\mathcal{F}}_1)^\alpha \cap (\hat{\mathcal{F}}_2)^\alpha = (\hat{\mathcal{F}}_1 \cap \hat{\mathcal{F}}_2)^\alpha \),
v. \( (\hat{\mathcal{F}}_1 \cup \hat{\mathcal{F}}_2) \cap \hat{\mathcal{F}}_3 = \hat{\mathcal{F}}_1 \cap (\hat{\mathcal{F}}_2 \cap \hat{\mathcal{F}}_3) \),
vi. \( \hat{\mathcal{F}}_1 \cup (\hat{\mathcal{F}}_2 \cap \hat{\mathcal{F}}_3) = (\hat{\mathcal{F}}_1 \cup \hat{\mathcal{F}}_2) \cap \hat{\mathcal{F}}_3 \).

Proposition 3.7. Let \( \hat{\mathcal{F}}, \hat{\mathcal{F}}_1, \hat{\mathcal{F}}_2, \hat{\mathcal{F}}_3 \) be four IVFEs, then

i. \( (\hat{\mathcal{F}}_1)^C \cup (\hat{\mathcal{F}}_2)^C = (\hat{\mathcal{F}}_2 \cap \hat{\mathcal{F}}_1)^C \),
ii. \( (\hat{\mathcal{F}}_1)^C \cap (\hat{\mathcal{F}}_2)^C = (\hat{\mathcal{F}}_2 \cup \hat{\mathcal{F}}_1)^C \).

Definition 3.8. Take an IVFE \( \hat{\mathcal{F}} = \{ (\xi, \eta): \hat{\xi} = [\xi^-, \xi^+], \hat{\eta} = [\eta^-, \eta^+] \} \).

\[
SC(\hat{\mathcal{F}}) = \frac{1}{2|\hat{\mathcal{F}}|} \sum_{(\xi, \eta) \in \hat{\mathcal{F}}} \left( \hat{\xi}^3 - \hat{\eta}^3 \right) = \left[ \frac{1}{2|\hat{\mathcal{F}}|} \sum_{(\xi, \eta) \in \hat{\mathcal{F}}} ((\xi^-)^3 - (\eta^-)^3), \frac{1}{2|\hat{\mathcal{F}}|} \sum_{(\xi, \eta) \in \hat{\mathcal{F}}} ((\xi^+)^3 - (\eta^+)^3) \right].
\]

is called the score function \( SC(\hat{\mathcal{F}}) \). Further,

\[
AF(\hat{\mathcal{F}}) = \frac{1}{2|\hat{\mathcal{F}}|} \sum_{(\xi, \eta) \in \hat{\mathcal{F}}} \left( \hat{\xi}^3 + \hat{\eta}^3 \right) = \left[ \frac{1}{2|\hat{\mathcal{F}}|} \sum_{(\xi, \eta) \in \hat{\mathcal{F}}} ((\xi^-)^3 + (\eta^-)^3), \frac{1}{2|\hat{\mathcal{F}}|} \sum_{(\xi, \eta) \in \hat{\mathcal{F}}} ((\xi^+)^3 + (\eta^+)^3) \right].
\]

is called the accuracy function \( AF(\hat{\mathcal{F}}) \).

Definition 3.9. Let \( \hat{\mathcal{F}}_1, \hat{\mathcal{F}}_2 \) be two IVFEs.

i. If \( P(SC(\hat{\mathcal{F}}_1) > SC(\hat{\mathcal{F}}_2)) < 1/2 \), then we say \( \hat{\mathcal{F}}_1 < \hat{\mathcal{F}}_2 \)
ii. If \( P(SC(\hat{\mathcal{F}}_1) > SC(\hat{\mathcal{F}}_2)) = 1/2 \), then

a) If \( P(AF(\hat{\mathcal{F}}_1) > AF(\hat{\mathcal{F}}_2)) < 1/2 \), we say \( \hat{\mathcal{F}}_1 < \hat{\mathcal{F}}_2 \),
b) If \( P(AF(\hat{\mathcal{F}}_1) > AF(\hat{\mathcal{F}}_2)) = 1/2 \), we say \( \hat{\mathcal{F}}_1 = \hat{\mathcal{F}}_2 \).

Proposition 3.10. Take two IVFEs \( \hat{\mathcal{F}}_1 = \{ (\xi_{ij}, \hat{\eta}_{ij}): \hat{\xi}_{ij} = [\xi_{ij}^-, \xi_{ij}^+], \hat{\eta}_{ij} = [\eta_{ij}^-, \eta_{ij}^+] \}, \hat{\mathcal{F}}_2 = \{ (\xi_{ij}, \hat{\eta}_{ij}): \hat{\xi}_{ij} = [\xi_{ij}^-, \xi_{ij}^+], \hat{\eta}_{ij} = [\eta_{ij}^-, \eta_{ij}^+] \}, \)

\( \hat{\mathcal{F}}_2 \{ (\xi_{ij}, \hat{\eta}_{ij}): \hat{\xi}_{ij} = [\xi_{ij}^-, \xi_{ij}^+], \hat{\eta}_{ij} = [\eta_{ij}^-, \eta_{ij}^+] \}, \hat{\mathcal{F}}_2 \{ (\xi_{ij}, \hat{\eta}_{ij}): \hat{\xi}_{ij} = [\xi_{ij}^-, \xi_{ij}^+], \hat{\eta}_{ij} = [\eta_{ij}^-, \eta_{ij}^+] \}, \hat{\mathcal{F}}_2 \{ (\xi_{ij}, \hat{\eta}_{ij}): \hat{\xi}_{ij} = [\xi_{ij}^-, \xi_{ij}^+], \hat{\eta}_{ij} = [\eta_{ij}^-, \eta_{ij}^+] \}, \)

\( \hat{\mathcal{F}}_2 \{ (\xi_{ij}, \hat{\eta}_{ij}): \hat{\xi}_{ij} = [\xi_{ij}^-, \xi_{ij}^+], \hat{\eta}_{ij} = [\eta_{ij}^-, \eta_{ij}^+] \}, \)

if for any \( \xi_{ij}^+ \leq \xi_{ij}^+ \leq \xi_{ij}^+ \), \( \eta_{ij}^+ \geq \eta_{ij}^+ \geq \eta_{ij}^+ \), then \( SC(\hat{\mathcal{F}}_1) \leq SC(\hat{\mathcal{F}}_2) \).

Proof. From Definition 3.8,

\[
SC(\hat{\mathcal{F}}_1) = \frac{1}{2m} \sum_{j=1}^{m} \left[ (\xi_{ij}^3 - (\eta_{ij}^3)^3 - (\eta_{ij}^3)^3) \right], \quad SC(\hat{\mathcal{F}}_2) = \frac{1}{2m} \sum_{j=1}^{m} \left[ (\xi_{ij}^3 - (\eta_{ij}^3)^3 - (\eta_{ij}^3)^3) \right].
\]

Suppose

\[
\mathcal{L} = \frac{\sum_{j=1}^{m} \left[ (\xi_{ij}^3 - (\eta_{ij}^3)^3 + (\eta_{ij}^3)^3 - (\xi_{ij}^3)^3) \right]}{\sum_{j=1}^{m} \left[ (\xi_{ij}^3 - (\eta_{ij}^3)^3 + (\eta_{ij}^3)^3 + (\eta_{ij}^3)^3 - (\eta_{ij}^3)^3)^3 \right]} \tag{3.2}
\]

Hence, we obtain \( P(SC(\hat{\mathcal{F}}_1) > SC(\hat{\mathcal{F}}_2)) = \max \{ 1 - \max \{ \mathcal{L}, 0 \}, 0 \} \). \( \xi_{ij} \leq \xi_{ij}^+, \xi_{ij}^+ \leq \xi_{ij}^+ \), \( \eta_{ij} \geq \eta_{ij}^+, \eta_{ij}^+ \geq \eta_{ij}^+ \), \( \xi_{ij}^+ \leq \eta_{ij}^+, \xi_{ij}^+ \geq \eta_{ij}^+ \), \( \xi_{ij}^+ \leq \eta_{ij}^+, \xi_{ij}^+ \geq \eta_{ij}^+ \), \( (\xi_{ij}^3)^3 - (\eta_{ij}^3)^3 - (\xi_{ij}^3)^3 \) \( \geq \) \( 0 \)

then \( 1/2 \leq \mathcal{L} \leq 1 \). So, \( 0 \leq -\max \{ \mathcal{L}, 0 \} = 1 - \mathcal{L} \leq 1/2 \), for any \( j = 1, 2, \cdots, m \). Therefore, \( P(SC(\hat{\mathcal{F}}_1) > SC(\hat{\mathcal{F}}_2)) = 1 - \mathcal{L} \leq 1/2 \). That means \( \hat{\mathcal{F}}_1 \leq SC(\hat{\mathcal{F}}_2) \) holds.
4. Aggregation Operators

Firstly IFWA, IFWG, GIFWA, and GIFWG operators will be given:

Choose \( \mathcal{F}_i (i = 1, 2, \ldots, n) \), \( \omega = (\omega_1, \omega_2, \ldots, \omega_n)^T \) as a composition of some IVFES and the weight vector of \( \mathcal{F}_i (\omega_i \in [0, 1], \sum_{i=1}^{n} \omega_i = 1 \), and \( \alpha > 0 \), respectively.

\[
\text{IFWA} (\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_n) = \omega_1 \mathcal{F}_1 \oplus \ldots \ominus \omega_n \mathcal{F}_n
\]

(4.1)

is called an IFWA operator, where \( \text{IFWA} : Y^n \to Y \).

\[
\text{IFWG} (\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_n) = \omega_1 \mathcal{F}_1 \boxplus \ldots \boxplus \omega_n \mathcal{F}_n
\]

(4.2)

is called IFWG operator, where \( \text{IFWG} : Y^n \to Y \).

Example 4.1. Choose the three IVFESs \( \mathcal{F}_1 = \{([0.6,0.8],[0.5,0.7])\}, \mathcal{F}_2 = \{([0.3,0.9],[0.5,0.8]),([0.4,0.7],[0.5,0.8])\}, \mathcal{F}_3 = \{([0.6,0.7],[0.5,0.8]),([0.3,0.7],[0.4,0.8]),([0.5,0.7],[0.5,0.6])\} \). Take the weight vector for these IVFESs as \( \omega = (0.23,0.37,0.40)^T \).

\[
\begin{align*}
\text{IFWA} & = \{ ([0.532,0.87],[0.5,0.776]), ([0.444,0.73],[0.457,0.776]), ([0.438,0.8],[0.46,0.7]), ([0.546,0.73],[0.5,0.776]), ([0.444,0.73],[0.46,0.776]), ([0.5,0.73],[0.457,0.7]) \} \\
\text{IFWG} & = \{ ([0.464,0.8],[0.5,0.782]), ([0.352,0.8],[0.46,0.782]), ([0.438,0.8],[0.466,0.717]), ([0.516,0.68],[0.5,0.782]), ([0.4,0.722],[0.466,0.782]), ([0.48,0.722],[0.5,0.74]) \}
\end{align*}
\]

Now, we calculate the score values: \( \text{SC(IFWA)} = [-0.155,0.176] \), \( \text{SC(IFWG)} = [-0.18,0.167] \) and so \( P(\text{SC(IFWG)} \leq \text{SC(IFWA)}) = 0.6023 \).

Proposition 4.2. Let \( \mathcal{F}_i \) be a composition of IVFESs \( (i = 1, 2, \ldots, n) \), \( \omega_i > 0 \), \( \sum_{i=1}^{n} \omega_i = 1 \). Therefore,

\[
\text{IFWG} (\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_n) \leq \text{IFWA} (\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_n)
\]

(4.3)

Since \( \prod_{i=1}^{n} \omega_i \mathcal{F}_i \), for \( \mathcal{F} > 0 \), \( \omega_i > 0 \), and \( \sum_{i=1}^{n} \omega_i = 1 \) (the equality holds iff \( \mathcal{F}_1 = \mathcal{F}_2 = \ldots = \mathcal{F}_n \) [29], this proposition can be easily proved.

\[
\text{GIFWA}_\alpha (\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_n) = (\omega_1 \mathcal{F}_1^\alpha \oplus \ldots \ominus \omega_n \mathcal{F}_n^\alpha)^{1/\alpha} = \left\{ \left[ \frac{1}{\alpha} \left( 1 - \prod_{i=1}^{n} (1 - (\zeta_i^{-})^{3\alpha})^{\alpha} \right)^{1/\alpha} \right] \right\} \quad (4.4)
\]

is called a GIFWA operator, where \( \text{GIFWA}_\alpha : Y^n \to Y \).

\[
\text{GIFWG}_\alpha (\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_n) = \frac{1}{\alpha} (\alpha \mathcal{F}_1^\alpha \boxplus \ldots \boxplus \alpha \mathcal{F}_n^\alpha) = \left\{ \left[ \frac{1}{\alpha} \left( 1 - \prod_{i=1}^{n} (1 - (\zeta_i^{+})^{3\alpha})^{\alpha} \right)^{1/\alpha} \right] \right\} \quad (4.5)
\]

is called a GIFWG operator, where \( \text{GIFWG}_\alpha : Y^n \to Y \).
Example 4.3. Take the values of Example 4.1 and choose $\alpha = 4$. Then

$$
\text{GIFWA}_4 = \left\{ (\{0.531, 0.844\}, (0.844, 0.85)), (\{0.66, 0.85\}, (0.844, 0.85)) \right\}
$$

Now, we calculate the score values: $\text{SC}(\text{GIFWA}_4) = [-0.10571, 0.0511]$, $\text{SC}(\text{GIFWG}_4) = [-0.097, 0.205]$ and so $P(\text{SC}(\text{GIFWG}_\alpha) \leq \text{SC}(\text{GIFWA}_4)) = 0.6381$, $P(\text{SC}(\text{GIFWG}_\alpha) \leq \text{SC}(\text{IFWA})) = 0.642$.

Proposition 4.4. Let $\tilde{F}_i (i = 1, 2, \cdots , n)$ be a composition of IVFES, $\alpha > 0 \omega_i > 0$ and $\sum_{i = 1}^{n} \omega_i = 1$, then

$$
\text{IVWG}(\tilde{F}_i), \text{GIFVA}(\tilde{F}_i) \tag{4.6}
$$

Proposition 4.5. Let $\tilde{F}_i (i = 1, 2, \cdots , n)$ be a composition of IVFES, $\omega_i \geq \omega_1 > 0 \omega_i > 0$ and $\sum_{i = 1}^{n} \omega_i = 1$, then

$$
\text{GIFVA}_\alpha(\tilde{F}_i), \text{GIFVA}(\tilde{F}_i) \tag{4.7}
$$

IPOA, IPFG, GIFOA, and GIFOG operators will be defined:

Let $\tilde{F}_{\sigma(i)}$ be the $i$th largest of $\tilde{F}_i (i = 1, 2, \cdots , n)$, and $\kappa = (\kappa_1, \kappa_2, \cdots , \kappa_0)^T$ be the associated vector ($\kappa_i \in [0,1], \sum_{i = 1}^{n} \kappa_i = 1$, and $\alpha > 0$.

$$
\text{IPOA}(\tilde{F}_1, \tilde{F}_2, \cdots , \tilde{F}_n) = (\kappa_1 \tilde{F}_{\sigma(1)} \boxplus \cdots \boxplus \kappa_\alpha \tilde{F}_{\sigma(\alpha)}) \tag{4.8}
$$

is called an IPOA operator, where $\text{IPOA} : \mathbb{Y}^n \rightarrow \mathbb{Y}$.

$$
\text{IPOG}(\tilde{F}_1, \tilde{F}_2, \cdots , \tilde{F}_n) = (\kappa_1 \tilde{F}_{\sigma(1)} \boxdot \cdots \boxdot \kappa_\alpha \tilde{F}_{\sigma(\alpha)}) \tag{4.9}
$$

is called an IPFG operator, where $\text{IPOG} : \mathbb{Y}^n \rightarrow \mathbb{Y}$.

$$
\text{GIFOA}_\alpha(\tilde{F}_1, \tilde{F}_2, \cdots , \tilde{F}_n) = (\kappa_1 \tilde{F}_{\sigma(1)} \boxplus \cdots \boxplus \kappa_\alpha \tilde{F}_{\sigma(\alpha)})^{1/\alpha} = \left\{ \left[ 1 - \prod_{i = 1}^{n} \left( 1 - (\xi_i^+)^{3\alpha} \right)^\kappa \right]^{1/\alpha} \right\} \tag{4.10}
$$

is called a GIFOA operator, where $\text{GIFOA} : \mathbb{Y}^n \rightarrow \mathbb{Y}$ and $\tilde{F}_{\sigma(i)}$ is the largest $i$th of $\tilde{F}_k = \text{nna}_k \tilde{F}$ ($k = 1, 2, \cdots , n$).

$$
\text{GIFOG}_\alpha(\tilde{F}_1, \tilde{F}_2, \cdots , \tilde{F}_n) = \left( \frac{1}{\alpha} \left( \tilde{F}_{\sigma(1)} \boxdot \cdots \boxdot \tilde{F}_{\sigma(\alpha)} \right) \right)^{1/\alpha} \tag{4.11}
$$

is called a GIFOG operator, where $\text{GIFOG} : \mathbb{Y}^n \rightarrow \mathbb{Y}$ and $\tilde{F}_{\sigma(i)}$ is the largest $i$th of $\tilde{F}_k = \text{nna}_k \tilde{F}$ ($k = 1, 2, \cdots , n$).
Example 4.6. Take the values of Example 4.1 and choose \( \alpha = 4 \). Let the aggregation-associated vector be chosen as \( \kappa = (0.3, 0.38, 0.32)^T \), \( SC(\hat{F}_1) = [-0.0635, 0.1935], SC(\hat{F}_2) = [-0.23325, 0.2055], SC(\hat{F}_3) = [-0.2, 0.13] \). Then we obtain \( P(SC(\hat{F}_1) > SC(\hat{F}_3)) = 0.534 \) and \( P(SC(\hat{F}_2) > SC(\hat{F}_3)) = 0.6927 \). Then \( \hat{F}_{\alpha(1)} = \hat{F}_1, \hat{F}_{\alpha(2)} = \hat{F}_2, \hat{F}_{\alpha(3)} = \hat{F}_3 \).

\[
\text{IFOA} = \left\{ ([0.53, 0.83], [0.5, 0.77]), ([0.445, 0.83], [0.465, 0.77]), ([0.5, 0.83], [0.465, 0.7]), ([0.54, 0.74], [0.5, 0.77]), ([0.466, 0.74], [0.466, 0.77]), ([0.5, 0.74], [0.466, 0.7]) \right\} 
\]

\[
\text{IFOG} = \left\{ ([0.5, 0.776], [0.461, 0.8]), ([0.473, 0.745], [0.37, 0.8]), ([0.473, 0.724], [0.435, 0.8]), ([0.5, 0.776], [0.514, 0.74]), ([0.473, 0.776], [0.485, 0.73]), ([0.473, 0.724], [0.485, 0.73]) \right\} 
\]

\[
\text{GIFOA}_4 = \left\{ ([0.543, 0.85], [0.624, 0.825]), ([0.543, 0.85], [0.624, 0.825]), ([0.543, 0.854], [0.624, 0.825]), ([0.543, 0.74], [0.624, 0.825]), ([0.543, 0.74], [0.624, 0.825]), ([0.543, 0.74], [0.624, 0.825]) \right\} 
\]

\[
\text{GIFOG}_4 = \left\{ ([0.543, 0.9], [0.485, 0.75]), ([0.543, 0.9], [0.485, 0.75]), ([0.543, 0.9], [0.485, 0.75]), ([0.605, 0.87], [0.485, 0.75]), ([0.605, 0.87], [0.485, 0.75]), ([0.605, 0.87], [0.485, 0.75]) \right\} 
\]

Hence, we can obtain that \( SC(\text{IFOA}) = [-0.137, 0.197], SC(\text{IFOG}) = [-0.147, 0.165], SC(\text{GIFOA}_4) = [-0.101, 0.179], SC(\text{GIFOG}_4) = [-0.116, 0.307] \) and so \( P(SC(\text{IFOA}) \leq SC(\text{IFOG}) = 0.4538, P(SC(\text{IFOG}) \leq SC(\text{GIFOA}_4) = 0.6127 P(SC(\text{GIFOA}_4) \leq SC(\text{IFOA}) = 0.5405.

Finally \( \text{IFHA}, \text{IFHG}, \text{GIFHA}, \) and \( \text{GIFHG} \) operators will be defined as follows:

Let \( \hat{F}_i \) be a composition of some IVPF \( \alpha \), \( \omega \), \( \kappa \) are the weight vector and the associated vector of \( \hat{F}_i \), and \( \alpha > 0 \).

\[
\text{IFHA}(\hat{F}_1, \hat{F}_2, \ldots, \hat{F}_n) = \left\{ \left( \prod_{i=1}^{n} (1 - (\hat{\xi}_{\alpha(i)})^k)^{\kappa}, \sqrt{1 - \prod_{i=1}^{n} (1 - (\hat{\xi}_{\alpha(i)})^k)} \right), \left( \prod_{i=1}^{n} (\hat{\eta}_{\alpha(i)}^k, (\hat{\eta}_{\alpha(i)}^+)^k) \right) \right\} 
\]

is called an IFHA operator, where \( \text{IFHA} : Y^m \rightarrow Y \) and \( \hat{F}_{\alpha(i)} \) is the largest \( \alpha \)-th of \( \hat{F}_{\alpha(i)}, i = 1, 2, \ldots, n \).

\[
\text{IFHG}(\hat{F}_1, \hat{F}_2, \ldots, \hat{F}_n) = \left( \frac{\text{IFHA}(\hat{F}_1, \hat{F}_2, \ldots, \hat{F}_n)}{\kappa} \right)^{1/\alpha} \otimes \ldots \otimes \left( \frac{\text{IFHA}(\hat{F}_1, \hat{F}_2, \ldots, \hat{F}_n)}{\kappa} \right)^{1/\alpha}
\]

is called an IFHG operator, where \( \text{IFHG} : Y^m \rightarrow Y \) and \( \hat{F}_{\alpha(i)} \) is the largest \( \alpha \)-th of \( \hat{F}_{\alpha(i)} \), \( i = 1, 2, \ldots, n \).

\[
\text{GIFHA}_q(\hat{F}_1, \hat{F}_2, \ldots, \hat{F}_n) = \left[ \left( \prod_{i=1}^{n} (1 - (\hat{\xi}_{\alpha(i)})^{\lambda \alpha}) \right)^{\kappa}, \left( \prod_{i=1}^{n} (1 - (\hat{\eta}_{\alpha(i)})^{\lambda \alpha}) \right)^{\kappa} \right] = \left[ \left( \prod_{i=1}^{n} (1 - (\hat{\xi}_{\alpha(i)})^{\lambda \alpha}) \right)^{1/\alpha}, \left( \prod_{i=1}^{n} (1 - (\hat{\eta}_{\alpha(i)})^{\lambda \alpha}) \right)^{1/\alpha} \right], \left( \hat{\xi}_{\alpha(i)}, \hat{\eta}_{\alpha(i)} \right) \in \hat{F}_{\alpha(i)}, i = 1, 2, \ldots, n \}
\]

is called a GIFHA operator, where \( \text{GIFHA} : Y^m \rightarrow Y \) and \( \hat{F}_{\alpha(i)} \) is the largest \( \alpha \)-th of \( \hat{F}_k \rightarrow \hat{F}_{\alpha(i)} \), \( k = 1, 2, \ldots, n \).

\[
\text{GIFGH}_q(\hat{F}_1, \hat{F}_2, \ldots, \hat{F}_n) = \frac{1}{\alpha} \left[ \left( \alpha \hat{F}_{\alpha(i)} \right)^{\kappa}, \left( \alpha \hat{F}_{\alpha(i)} \right)^{\kappa} \right] = \left[ \left( \prod_{i=1}^{n} (1 - (\hat{\xi}_{\alpha(i)})^{\lambda \alpha}) \right)^{1/\alpha}, \left( \prod_{i=1}^{n} (1 - (\hat{\eta}_{\alpha(i)})^{\lambda \alpha}) \right)^{1/\alpha} \right], \left( \hat{\xi}_{\alpha(i)}, \hat{\eta}_{\alpha(i)} \right) \in \hat{F}_{\alpha(i)}, i = 1, 2, \ldots, n \}
\]
is called a GIFHG operator, where GIFHG : \( \mathbb{Y} \rightarrow \mathbb{Y} \) and \( \check{H}_{\sigma(i)} \) is the largest \( \check{H} \)th of \( \check{H}_{k} = (\check{H})^{n \times n} \). (k = 1, 2, \ldots, n).

**Example 4.7.** Take the values of Example 4.1 and choose \( \alpha = 4 \). Let \( \kappa = (0.3, 0.38, 0.32)^{T} \) be the aggregation-associated vector.

\[
\begin{align*}
\check{F}_{1} &= (4 \times 0.23), \check{F}_{1} = \{([0.585, 0.785], [0.53, 0.72])\}, \\
\check{F}_{2} &= (4 \times 0.37), \check{F}_{2} = \{([0.341, 0.95], [0.36, 0.72]), ([0.453, 0.774], [0.36, 0.72])\} \\
\check{F}_{3} &= (4 \times 0.40), \check{F}_{3} = \{([0.68, 0.79], [0.33, 0.7]), ([0.35, 0.79], [0.231, 0.7]), ([0.58, 0.79], [0.231, 0.442])\}, \\
\check{F}_{1} &= (\check{F}_{1})^{(4 \times 0.23)} = \{([0.625, 0.814], [0.49, 0.68])\} \\
\check{F}_{2} &= (\check{F}_{2})^{(4 \times 0.37)} = \{([0.17, 0.86], [0.564, 0.87]), ([0.26, 0.59], [0.564, 0.87])\} \\
\check{F}_{3} &= (\check{F}_{3})^{(4 \times 0.40)} = \{([0.441, 0.565], [0.577, 0.88]), ([0.146, 0.565], [0.464, 0.88]), ([0.33, 0.565], [0.464, 0.686])\}
\end{align*}
\]

\[
\begin{align*}
\text{SC}(\check{F}_{1}) &= [-0.0864, 0.167], \quad \text{SC}(\check{F}_{2}) = [-0.08325, -0.06825], \quad \text{SC}(\check{F}_{3}) = [-0.0365, 0.24], \\
\text{SC}(\check{F}_{1}) &= [0.26, 0.2033], \quad \text{SC}(\check{F}_{2}) = [-0.02, 0.21], \quad \text{SC}(\check{F}_{3}) = [-0.324, 0.11].
\end{align*}
\]

Using the Definition 3.9, \( P(\text{SC}(\check{F}_{1}) \leq \text{SC}(\check{F}_{2})) = 1 \); \( P(\text{SC}(\check{F}_{2}) \leq \text{SC}(\check{F}_{3})) = 0.822 \); \( P(\text{SC}(\check{F}_{3}) \leq \text{SC}(\check{F}_{2})) = 0.809 \) and \( P(\text{SC}(\check{F}_{2}) \leq \text{SC}(\check{F}_{1})) = 1. \). So, we get \( \check{F}_{\sigma(1)} = \check{F}_{3}, \check{F}_{\sigma(2)} = \check{F}_{2}, \check{F}_{\sigma(3)} = \check{F}_{1}, \check{F}_{\sigma(1)} = \check{F}_{1}, \check{F}_{\sigma(2)} = \check{F}_{2}, \check{F}_{\sigma(3)} = \check{F}_{3}. \)

Then,

\[
\begin{align*}
\text{IFHA} &= \left\{ ([0.57, 0.882], [0.4, 0.714]), ([0.4513, 0.882], [0.35, 0.714]), ([0.52, 0.882], [0.35, 0.616]), ([0.586, 0.783], [0.4, 0.714]), \\
&\quad ([0.481, 0.783], [0.351, 0.714]), ([0.542, 0.783], [0.351, 0.616]) \right\} \\
\text{IFHG} &= \left\{ ([0.5, 0.846], [0.426, 0.714]), ([0.41, 0.846], [0.414, 0.714]), ([0.4753, 0.846], [0.414, 0.67]), ([0.555, 0.712], [0.564, 0.74]), \\
&\quad ([0.455, 0.712], [0.558, 0.74]), ([0.53, 0.712], [0.558, 0.7]) \right\} \\
\text{GIFHA}_{4} &= \left\{ ([0.626, 0.9], [0.9616, 0.86]), ([0.53, 0.9], [0.9617, 0.86]), ([0.56, 0.9], [0.9617, 0.871]), ([0.63, 0.77], [0.9616, 0.86]), \\
&\quad ([0.532, 0.77], [0.9617, 0.86]), ([0.561, 0.77], [0.9617, 0.871]) \right\} \\
\text{GIFHG}_{4} &= \left\{ ([0.91, 0.65], [0.48, 0.714]), ([0.95, 0.65], [0.479, 0.714]), ([0.94, 0.65], [0.479, 0.7]), ([0.91, 0.8], [0.48, 0.714]), \\
&\quad ([0.95, 0.8], [0.479, 0.714]), ([0.94, 0.8], [0.479, 0.7]) \right\}
\end{align*}
\]

**5. Medical Decision-Making Application**

**IVFHN scenarios description:**

MADHM is a technique to evaluate the most convenient alternative according to professional person decisions sustaining offered attributes and to construct and solve the planning and judgemental issues. In these types of problems, professionals usually decide according to different research techniques and different structures in their background knowledge. For this reason, MADHM is very important for the optimal model that can be acquired with the highest level of consensus of professionals. Here, we suggest a new MADHM technique in accordance with the IVAGOs. Furthermore, the ranking of integrated IVFHNs is carried out by taking up seriously the SC. The certain circumstances of the MADHM technique are demonstrated in the following.

The sets of alternatives, attributes and professional persons denoted by \( A = \{A_{i} : i = 1, 2, \cdots, m\} \), \( K = \{K_{j} : j = 1, 2, \cdots, n\} \), \( P = \{P_{k} : k = 1, 2, \cdots, l\} \).

\( \check{F}_{ij} = (\check{F}_{ij}^{(k)}, (\check{H}_{ij}^{(k)})) \), \( i = 1, 2, \cdots, m; j = 1, 2, \cdots, n \) is an IVFE served by the professional person \( P_{k} \), in which \( \check{F}_{ij}^{(k)} \) points out the probable membership intervals that the alternative \( A_{i} \) satisfies the attribute \( K_{j} \) and \( (\check{H}_{ij}^{(k)}) \) points out the probable non-membership intervals that the alternative \( A_{i} \) satisfies the attribute \( K_{j} \), then let \( D^{(k)} = (\check{F}_{ij}^{(k)})_{m \times n} \) be shown as the IVFHM of the \( k \)th professional person.

For MAGDM, we can say that the larger the value of the attribute, the benefit attribute, and the smaller the value of the attribute, the cost attribute. Therefore, convert the cost attribute values into the benefit attribute values and normalize the IVFHM \( D^{(k)} = (\check{F}_{ij}^{(k)})_{m \times n} \) into the corresponding IVFHM \( E^{(k)} = (\check{F}_{ij}^{(k)})_{m \times n} \).
Based on these considerations, a new technique was constructed for IVFMH in IVFHF environments. The algorithm for this technique is as follows:

**Algorithm:**

1. **Step 1:** Form the IVFMH $D^{(k)} = \left( \tilde{F}^{(k)}_{ij} \right)_{m \times n}$ and convert $D^{(k)}$ into the $E^{(k)} = \left( \hat{F}^{(k)}_{ij} \right)_{m \times n}$.

2. **Step 2:** For $\rho = (\rho_1, \rho_2, \ldots, \rho_l)^T$ is the weight vector of professionals $k_i (k = 1, 2, \ldots, l)$, employ the GIFHA (or GIFHG) operator to collected all IVFMHs $E^{(k)} = \left( \hat{F}^{(k)}_{ij} \right)_{m \times n}$, and the certain operation is as follows:

   \[
   \tilde{F}_{ij} = \text{GIFHA}_\alpha(\tilde{F}^{(1)}_{ij}, \tilde{F}^{(2)}_{ij}, \ldots, \tilde{F}^{(n)}_{ij}) = \left\{ \begin{array}{l}
   \left( \frac{1}{\alpha} \sum_{k=1}^{\alpha} \frac{1}{(1 - \left( \tilde{F}^{(k)}_{ij} \right)^{\rho_k})} \right)^{1/\alpha}, \\
   \left( \frac{1}{\alpha} \sum_{k=1}^{\alpha} \frac{1}{(1 - \left( \tilde{F}^{(k)}_{ij} \right)^{\rho_k})} \right)^{1/\alpha}, \\
   \end{array} \right. (5.1)
   \]

3. **Step 3:** For the associated weight vector $\kappa = (\kappa_1, \kappa_2, \ldots, \kappa_m)^T$, utilize the GIFHA (or GIFHG) operator to collect all the preference values $\tilde{F}_{i}$ all over. The certain operation is as follows:

   \[
   \tilde{F}_{i} = \text{GIFHA}_\alpha(\tilde{F}_1, \tilde{F}_2, \ldots, \tilde{F}_m) = \left\{ \begin{array}{l}
   \left( \frac{1}{\alpha} \sum_{j=1}^{\alpha} \frac{1}{(1 - \left( \tilde{F}^{(j)}_{i} \right)^{\kappa_j})} \right)^{1/\alpha}, \\
   \left( \frac{1}{\alpha} \sum_{j=1}^{\alpha} \frac{1}{(1 - \left( \tilde{F}^{(j)}_{i} \right)^{\kappa_j})} \right)^{1/\alpha}, \\
   \end{array} \right. (5.3)
   \]

Here, $\omega$ is the weight vector of the attributes $K_j$. Hence, $\tilde{F}_{i}$ (or $\tilde{F}_{i}$) can be defined as

\[
\tilde{F}_{ij} = (n \times \omega_j) \otimes \tilde{F}_{ij} = \left\{ \begin{array}{l}
\left( \frac{1}{\alpha} \sum_{j=1}^{\alpha} \frac{1}{(1 - \left( \tilde{F}^{(j)}_{ij} \right)^{\kappa_j})} \right)^{1/\alpha}, \\
\left( \frac{1}{\alpha} \sum_{j=1}^{\alpha} \frac{1}{(1 - \left( \tilde{F}^{(j)}_{ij} \right)^{\kappa_j})} \right)^{1/\alpha}, \\
\end{array} \right. (5.5)
\]
**Step 4:** Calculate the score values $SC(\tilde{F}_i)$ and the accuracy values $AF(\tilde{F}_i)$.

**Step 5:** Obtain the priority of the alternatives $A_i$ by ranking $SC(\tilde{F}_i)$.

**Example 5.1.** Let's choose the $A_i$ ($i = 1, 2, 3$) as the set of alternatives made up of hospital management system software. Denote the set $P_k$ ($k = 1, 2, 3$) three physicians and the set $K$ three criteria. The first criterion is “price”, which is the cost type. The second and third criterion are “speed” and “efficiency” respectively, which are the benefit type. The weight vector of the physicians is $\rho = (0.18, 0.52, 0.3)^T$. The weight vector of the criterion is $\omega = (0.5, 0.2, 0.3)^T$. Suppose that physicians provide their own IVFHM $D(k) = \left(\tilde{F}_{ij}\right)_{m \times n}$ (Tables 1-3), where $\tilde{F}_{ij}^{(k)}$ is an IVFHM offered by the professionals $P_k$ (Tables 4-7).

**Step 1.** Convert the matrix $D(k)$ into the matrix $E(k) = \left(\tilde{F}_{ij}^{(k)}\right)_{m \times n}$ (Tables 4-6).

**Step 2:** Assume that $\alpha = 5$ and employ the GIFHA operator to collect the 3 IVFHM $E(k) = \left(\tilde{F}_{ij}^{(k)}\right)_{3 \times 3}$ ($k = 1, 2, 3$) into the composition IVFHM $E = \left(\tilde{F}_{ij}\right)_{3 \times 3}$ (Table 7).

**Step 3:** For associated weighting vector $\kappa = (0.23, 0.58, 0.19)^T$, collect all the preference values $\tilde{F}_{ij}$ ($j = 1, 2, 3$) in the ith line of $E$ based on the GIFHA operators.

**Step 4:** Calculate the three score values: $SC(\tilde{F}_1) = [0.3017, 0.3280]$, $SC(\tilde{F}_2) = [0.2453, 0.3376]$, $SC(\tilde{F}_3) = [0.2746, 0.3112]$.

**Step 5:** Obtain the priority of the alternatives by ranking the score functions. Then, the ranking order $A_1 > A_3 > A_2$. So the optimal scheme is $A_1$.

| Table 1: IVFHM for First Physician |
|-----------------------------------|
| $K_1$                            | $K_2$                            | $K_3$                            |
| $A_1$                            | $\{(0.5, 0.6), [0.4, 0.5]\}$    | $\{(0.3, 0.4), [0.6, 0.9]\}$    | $\{(0.5, 0.6), [0.4, 0.5]\},$ |
|                                  | $\{(0.7, 0.8), [0.3, 0.4]\}$    | $\{(0.7, 0.8), [0.3, 0.4]\}$    | $\{(0.7, 0.8), [0.2, 0.3]\}$ |
| $A_2$                            | $\{(0.2, 0.4), [0.5, 0.6]\}$    | $\{(0.5, 0.6), [0.5, 0.5]\}$    | $\{(0.7, 0.8), [0.2, 0.2]\}$ |
|                                  | $\{(0.1, 0.2), [0.7, 0.8]\}$    | $\{(0.7, 0.8), [0.1, 0.2]\}$    | $\{(0.6, 0.8), [0.2, 0.4]\}$ |
| $A_3$                            | $\{(0.6, 0.8), [0.1, 0.1]\}$    | $\{(0.5, 0.9), [0.2, 0.1]\}$    | $\{(0.4, 0.6), [0.6, 0.6]\}$ |

| Table 2: IVFHM for Second Physician |
|-------------------------------------|
| $K_1$                              | $K_2$                            | $K_3$                            |
| $A_1$                              | $\{(0.6, 0.8), [0.1, 0.1]\}$    | $\{(0.5, 0.7), [0.4, 0.4]\}$    | $\{(0.7, 0.7), [0.3, 0.4]\}$ |
|                                  | $\{(0.7, 0.8), [0.1, 0.2]\}$    | $\{(0.6, 0.8), [0.3, 0.5]\}$    | $\{(0.7, 0.7), [0.3, 0.4]\}$ |
| $A_2$                              | $\{(0.2, 0.3), [0.5, 0.6]\}$    | $\{(0.7, 0.7), [0.2, 0.3]\}$    | $\{(0.6, 0.8), [0.2, 0.4]\}$ |
|                                  | $\{(0.1, 0.2), [0.7, 0.8]\}$    | $\{(0.6, 0.8), [0.2, 0.4]\}$    | $\{(0.6, 0.8), [0.2, 0.4]\}$ |
| $A_3$                              | $\{(0.3, 0.5), [0.6, 0.7]\}$    | $\{(0.6, 0.8), [0.2, 0.3]\}$    | $\{(0.2, 0.3), [0.5, 0.8]\}$ |
|                                  | $\{(0.5, 0.9), [0.1, 0.1]\}$    | $\{(0.3, 0.4), [0.6, 0.7]\}$    | $\{(0.3, 0.4), [0.6, 0.7]\}$ |

| Table 3: IVFHM for Third Physician |
|------------------------------------|
| $K_1$                              | $K_2$                            | $K_3$                            |
| $A_1$                              | $\{(0.4, 0.4), [0.6, 0.6]\}$    | $\{(0.3, 0.4), [0.5, 0.6]\}$    | $\{(0.7, 0.8), [0.2, 0.3]\}$ |
|                                  | $\{(0.2, 0.3), [0.7, 0.8]\}$    | $\{(0.7, 0.8), [0.2, 0.3]\}$    | $\{(0.7, 0.8), [0.2, 0.3]\}$ |
| $A_2$                              | $\{(0.5, 0.6), [0.3, 0.4]\}$    | $\{(0.1, 0.2), [0.7, 0.8]\}$    | $\{(0.3, 0.4), [0.6, 0.6]\}$ |
|                                  | $\{(0.5, 0.5), [0.5, 0.6]\}$    | $\{(0.2, 0.3), [0.7, 0.8]\}$    | $\{(0.3, 0.4), [0.6, 0.7]\}$ |
| $A_3$                              | $\{(0.7, 0.8), [0.2, 0.2]\}$    | $\{(0.3, 0.6), [0.4, 0.4]\}$    | $\{(0.7, 0.8), [0.1, 0.1]\}$ |
|                                  | $\{(0.6, 0.7), [0.1, 0.2]\}$    | $\{(0.4, 0.5), [0.4, 0.5]\}$    | $\{(0.3, 0.5), [0.4, 0.5]\}$ |

This example can also be solved with the GIFHA operator instead of GIFHG. In addition, new solution values can be reached by changing the $\alpha$ values.
| Table 4: Normalized IVFHM for First Physician |
|-------|-------|-------|
|       | $K_1$                          | $K_2$                           | $K_3$                           |
| $A_1$ | $\{(0.4, 0.5), (0.5, 0.6)\}$ | $\{(0.3, 0.4), (0.6, 0.9)\}$   | $\{(0.5, 0.6), (0.4, 0.5), (0.7, 0.8), (0.3, 0.4), (0.7, 0.8), (0.2, 0.3)\}$ |
| $A_2$ | $\{(0.7, 0.7), (0.1, 0.3), (0.5, 0.6), (0.2, 0.4)\}$ | $\{(0.5, 0.6), (0.5, 0.5), (0.7, 0.8), (0.1, 0.2), (0.6, 0.8), (0.2, 0.4)\}$ | $\{(0.7, 0.8), (0.2, 0.2)\}$ |
| $A_3$ | $\{(0.7, 0.7), (0.3, 0.4), (0.5, 0.8), (0.1, 0.3)\}$ | $\{(0.5, 0.9), (0.2, 0.1)\}$   | $\{(0.4, 0.6), (0.6, 0.6)\}$   |

| Table 5: Normalized IVFHM for Second Physician |
|-------|-------|-------|
|       | $K_1$                          | $K_2$                           | $K_3$                           |
| $A_1$ | $\{(0.1, 0.1), (0.6, 0.8), (0.1, 0.2), (0.7, 0.8), (0.1, 0.2), (0.6, 0.9)\}$ | $\{(0.5, 0.7), (0.4, 0.4), (0.6, 0.8), (0.3, 0.5)\}$ | $\{(0.7, 0.7), (0.3, 0.4)\}$ |
| $A_2$ | $\{(0.2, 0.3), (0.7, 0.8)\}$   | $\{(0.7, 0.7), (0.2, 0.3)\}$   | $\{(0.6, 0.8), (0.2, 0.4), (0.6, 0.8), (0.2, 0.4)\}$ |
| $A_3$ | $\{(0.6, 0.7), (0.3, 0.5)\}$   | $\{(0.6, 0.8), (0.2, 0.3), (0.5, 0.9), (0.1, 0.1)\}$ | $\{(0.3, 0.4), (0.6, 0.7)\}$   |

| Table 6: Normalized IVFHM for Third Physician |
|-------|-------|-------|
|       | $K_1$                          | $K_2$                           | $K_3$                           |
| $A_1$ | $\{(0.6, 0.6), (0.4, 0.4)\}$   | $\{(0.3, 0.4), (0.5, 0.6), (0.2, 0.3), (0.7, 0.8)\}$ | $\{(0.7, 0.8), (0.2, 0.3)\}$   |
| $A_2$ | $\{(0.3, 0.4), (0.5, 0.6), (0.5, 0.6), (0.5, 0.5)\}$ | $\{(0.1, 0.2), (0.7, 0.8)\}$   | $\{(0.3, 0.4), (0.6, 0.6), (0.2, 0.3), (0.7, 0.8), (0.3, 0.4), (0.6, 0.7)\}$ |
| $A_3$ | $\{(0.2, 0.2), (0.7, 0.8), (0.1, 0.2), (0.6, 0.7)\}$ | $\{(0.3, 0.6), (0.4, 0.4), (0.4, 0.5), (0.3, 0.5), (0.4, 0.5)\}$ | $\{(0.7, 0.8), (0.1, 0.1)\}$   |

| Table 7: Normalized IVFHM |
|-------|-------|-------|
|       | $K_1$                          | $K_2$                           | $K_3$                           |
| $A_1$ | $\{(0.375, 0.766), (0.98, 0.94), (0.67, 0.766), (0.98, 0.94), (0.575, 0.865), (0.98, 0.94)\}$ | $\{(0.48, 0.67), (0.945, 0.779), (0.48, 0.67), (0.9, 0.766), (0.574, 0.766), (0.943, 0.78), (0.574, 0.766), (0.9, 0.766)\}$ | $\{(0.7, 0.75), (0.99, 0.97), (0.7, 0.77), (0.99, 0.98), (0.7, 0.77), (0.99, 0.98)\}$ |
| $A_2$ | $\{(0.67, 0.767), (0.91, 0.91), (0.67, 0.766), (0.91, 0.91), (0.67, 0.767), (0.91, 0.95), (0.67, 0.766), (0.91, 0.93)\}$ | $\{(0.67, 0.672), (0.9, 0.842), (0.684, 0.73), (0.9, 0.842), (0.672, 0.78), (0.9, 0.842)\}$ | $\{(0.64, 0.78), (0.94, 0.94), (0.64, 0.78), (0.9, 0.84), (0.64, 0.78), (0.94, 0.94), (0.64, 0.78), (0.9, 0.84), (0.64, 0.78), (0.94, 0.94)\}$ |
| $A_3$ | $\{(0.646, 0.739), (0.906, 0.88), (0.554, 0.646), (0.906, 0.88), (0.646, 0.739), (0.932, 0.848), (0.554, 0.647), (0.932, 0.848)\}$ | $\{(0.58, 0.83), (0.98, 0.98), (0.58, 0.83), (0.98, 0.966), (0.58, 0.83), (0.98, 0.97), (0.56, 0.81), (0.98, 0.98), (0.49, 0.81), (0.97, 0.97), (0.49, 0.81), (0.97, 0.97)\}$ | $\{(0.65, 0.74), (0.94, 0.82), (0.65, 0.74), (0.93, 0.89)\}$ |
6. Conclusion

This work expands IVHFSs and IVFFSs to IVFHs. The score and accuracy functions belonging to IVFHFSs are given. These functions are used for comparing the size of IVFEs according to the comparison of interval numbers. Various aggregation operators such as IFWA, IFWG, GIFWA, GIFWG, IFPA, IFPO, GIFPA, GIFPO, IFHA, IFHG, GIFHA, and GIFHG connected to IVFH information were built to be used in solving MAGDM problems. In order to account for the distinctions of the thought of all decision-makers in MAGDM problems, a new MAGDM approach based on IVFH has been presented. It has been demonstrated both theoretically and practically that our approach can handle MAGDM problems in a flexible and objective mode under IVHF environment.

Abbreviations

MD: membership degree
ND: non-membership degree
ID: degree of indeterminacy
FS: Fuzzy set
IFS: Intuitionistic fuzzy set
PFS: Pythagorean fuzzy set
HFS: hesitant fuzzy set
HFN: Hesitant fuzzy number
IHFS: intuitionistic hesitant fuzzy set
PFHS: Pythagorean hesitant fuzzy set
FSS: Fermatean fuzzy set
IFSS: Fermatean fuzzy soft set
IVHFS: interval-valued hesitant fuzzy set
IVVFS: interval-valued Fermatean fuzzy set
IVHF: Interval-valued Fermatean hesitant fuzzy set
IVHFHN: Interval-valued Fermatean Hesitant Fuzzy Number
IVAO: Interval-valued Fermatean Hesitant Fuzzy aggregation operators
IVFHM: Interval-valued Fermatean hesitant fuzzy decision matrix
IVFE: Interval-valued Fermatean hesitant fuzzy element
IFWA: IVFHF weighted averaging operator
IFWG: IVFHF weighted geometric operator
GIFWA: GIVFHF weighted averaging operator
GIFWG: GIVFHF weighted geometric operator
IFPA: IVFHF ordered weighted averaging operator
IFPO: IVFHF ordered weighted geometric operator
GIFPA: GIVFHF ordered weighted averaging operator
GIFPO: GIVFHF ordered weighted geometric operator
IFHA: IVFHF hybrid weighted averaging operator
IFHG: IVFHF hybrid weighted geometric operator
GIFHA: GIVFHF hybrid weighted averaging operator
GIFHG: GIVFHF hybrid weighted geometric operator.

References

[1] Atanassov K. Intuitionistic fuzzy sets. Fuzzy Sets and Systems, 1986;20:87–96.
[2] Feng F., Fujita H., Ali M.A., Yager R.R., Liu X. Another View on Generalized Intuitionistic Fuzzy Soft Sets and Related Multi-attribute Decision Making Methods. IEEE Transactions on Fuzzy Systems, 2018;27(3):474–488 DOI:10.1109/TFUZZ.2018.2860967.
[3] Garg H. A New Generalized Pythagorean Fuzzy Information Aggregation Using Einstein Operations and Its Application to Decision Making. Int J Intell Syst, 2016;31:886–920.
[4] Garg H. A novel correlation coefficients between Pythagorean fuzzy sets and its application to decision-making process. Int. J. Intell. Syst., 2016;31(12):1234–1252, DOI: 10.1002/int.21827
[5] Garg H, Some series of intuitionistic averaging aggregation operators. SpringerPlus 2016;5(1):999, DOI: 10.1186/s40064-016-2591-9
[6] Garg H., Shahzadi G., Akram M. Decision-Making Analysis Based on Fermatean Fuzzy Yager Aggregation Operators with Application in COVID-19 Testing Facility. Mathematical Problems in Engineering, 2020, Article ID 7279027, DOI: 10.1155/2020/7279027
[7] Khan M.S.A., Abdullah S., Ali A., Siddiqui, N., Amin, F. Pythagorean hesitant fuzzy sets and their application to group decision making with incomplete weight information Journal of Intelligent & Fuzzy Systems 2017;33:3971–3985, DOI:10.3233/JIFS-17811
[8] Kim, S.H., Ahn, B.S. Interactive group decision making procedure under incomplete information, European Journal of Operation Research 1999;116:498–507.
[9] Kirisci, M. Comparison the medical decision-making with Intuitionistic fuzzy parameterized fuzzy soft set and Riesz Summability, New Mathematics and Natural Computation, 2019;15: 351–359, doi: 10.1142/S1793005719500194.
[10] Kirisci, M., Simsek, N., Decision making method related to Pythagorean Fuzzy Soft Sets with infectious diseases application. Journal of King Saud University–Computer and Information Sciences, DOI: 10.1016/j.jksuci.2021.08.010
[11] Kirisci, M., Fermatean hesitant fuzzy sets with entropy measures, (under review).
[12] Kirisci, M., Simsek, N., Decision making method related to Pythagorean Fuzzy Soft Sets with infectious diseases application. Journal of King Saud University–Computer and Information Sciences, DOI: 10.1016/j.jksuci.2021.08.010
[13] Kirisci, M., Fermatean hesitant fuzzy sets with Medical Decision Making Application. DOI:10.21203/rs.3.rs-1151389/v1
[16] Peng, J.J., Wang J.Q., Wang, J., Chen, X.H. Multi-criteria decision-making approach with hesitant interval valued intuitionistic fuzzy set, The ScientificWorldJournal (2014), 22. Article ID 868515.

[17] Peng, J.J., Wang, J.Q., Wu, X.H., Zhang, H.Y., Chen, X.H. The fuzzy cross-entropy for intuitionistic hesitant fuzzy sets and their application in multi-criteria decision-making, International Journal of Systems Science 2015;46:2335–2350.

[18] Peng X., Yang Y., Song J., Jiang Y. Pythagorean fuzzy soft set and its application. Computer Engineering. 2015;41:224–229.

[19] Peng X., Selvachandran G. Pythagorean fuzzy set: state of the art and future directions. Artif. Intell. Rev. 2019;52:1873–1927. DOI:10.1007/s10462-017-9596-9

[20] Qian. G., Wang, H., Feng, X.Q. Generalized hesitant fuzzy sets and their application in decision support system, Knowledge-Based Systems 2013;37:357–365.

[21] Senapati, T., Yager, R.R. Fermatean fuzzy sets. Journal of Ambient Intelligence and Humanized Computing 2020;11:663–674.

[22] Senapati, T., Yager, R.R. Some new operations over Fermatean fuzzy numbers and application of Fermatean fuzzy WPM in multiple criteria decision making. Informatica 2019;30:391–412.

[23] Senapati, T., Yager R.R. Fermatean fuzzy weighted averaging/geometric operators and its application in multi-criteria decision-making methods. Engineering Applications of Artificial Intelligence, 2019;85:112–121. DOI:10.1016/j.engappai.2019.05.012

[24] Shahzadi, G., Akram, M. Group decision-making for the selection of an antivirus mask under Fermatean fuzzy soft information. Journal of Intelligent & Fuzzy Systems. 2021;40:1401-1416.

[25] Shui XZ, Li DQ. A possibility based method for priorities of interval judgment matrix. Chinese J Manage Sci. 2003;11(1):63-65.

[26] Smarandache, F. Neutrosophic set, a generalisation of the intuitionistic fuzzy sets. Inter. J. Pure Appl Math 2005;24:287–297.

[27] Torra, V. Hesitant fuzzy sets. International Journal of Intelligent Systems 2010;25:529–539.

[28] Xia, M., Xu, Z. Hesitant fuzzy information aggregation in decision making, International Journal of Approximate Reasoning 2011;52:395–407.

[29] Xu, Z.S. On consistency of the weighted geometric mean complex judgement matrix in AHP. Eur J Oper Res 2000;126:683–687

[30] Wang, Y.M. Using the method of maximizing deviations to make decision for multi-indices. System Engineering and Electronics 1998;7:24–26.

[31] Yager, R.R. Pythagorean fuzzy subsets. Proc Joint IFSA World Congress and NAFIPS Annual Meeting, Edmonton, Canada, 2013.

[32] Yager, R.R. Pythagorean membership grade in multicriteria decision making. IEEE Fuzzy Syst. 2014;22:958–965.

[33] Yager R.R. Multicriteria decision making with ordinal linguistic intuitionistic fuzzy sets for mobile apps, IEEE Trans. Fuzzy Syst., 2016, 24:590–599.

[34] Yager, R.R., Abbasov, A.M. Pythagorean membership grades, complex numbers and decision making. Int. J. Intell. Syst. 2013;28:436–452.

[35] Yager, R.R., Generalized orthopair fuzzy sets. IEEE Trans. Fuzzy Syst. 2017;25:1222–1230.

[36] Zadeh, L.A. Fuzzy sets. Inf Comp 1965;8:338–353.

[37] Zhu B., Xu Z. Some results for dual hesitant fuzzy sets. J Intell Fuzzy Syst. 2014;26(4):1657–1668.