Brill-Noether duality for moduli spaces of sheaves on K3 surfaces
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1 Introduction

Hilbert schemes of points are among the simplest moduli spaces of sheaves on an algebraic surface $S$. Compactified relative Picards over a linear system of curves in $S$ may be considered as moduli space of sheaves with pure one-dimensional support. The latter is complicated in comparison. For example, the punctual Hilbert schemes are always smooth while the compactified relative Picard is rarely smooth. This work was motivated by a desire to understand (resolve) the birational Abel-Jacobi isomorphism

$$S^g \leftrightarrow \cdots \leftrightarrow \text{Pic}^g_{|O_S(1)|}$$

between the punctual Hilbert scheme $S^g$ of a symplectic surface with a linear system $|O_S(1)|$ of curves of genus $g$ and the relative compactified Picard $\text{Pic}^g_{|O_S(1)|}$. The latter is a completely integrable hamiltonian system [Hur, Mu1]. Of particular interest is the case where $S$ is the cotangent bundle of a smooth algebraic curve and $\text{Pic}^g_{|O_S(1)|}$ "is" a

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Hitchin system. The recursive nature of the geometry involved forced us to consider a more general duality among the moduli spaces of sheaves of arbitrary rank on the symplectic surface. However, due to the complexity of the birational isomorphisms, we consider here only the simplest setup; that of a K3 surface \( S \) and the collection of moduli spaces of sheaves whose first Chern class is minimal (Condition 6 Section 5.1). The structure, both local and global, of the birational Abel-Jacobi isomorphism turns out to be surprisingly beautiful. Local analytically, the birational isomorphism is modeled after two dual Springer resolutions

\[
T^*G(t, H) \to \overline{\mathcal{N}}^t \leftarrow T^*G(t, H^*)
\]

of the closure of the nilpotent orbit in \( \text{End}(H) \) of square-zero matrices of rank \( t \). We proceed to describe the global structure (Theorems 1 and 2 below).

A projective polarized K3 surface is a simply connected surface \( S \) with a trivial canonical bundle and a choice of an ample line bundle \( O_S(1) \). There is a sequence of 19-dimensional irreducible moduli spaces of polarized K3 surface parametrized by the genus \( g \geq 2 \) of a curve in the linear system \( |O_S(1)| \). The linear system \( |O_S(1)| \) is \( g \)-dimensional and the line bundle \( O_S(1) \) gives rise to a morphism \( \varphi: S \to \mathbb{P}^g \) which is an embedding as a surface of degree \( 2g - 2 \) if \( g > 2 \) and a double cover of \( \mathbb{P}^2 \) branched along a sextic if \( g = 2 \).

Consider first the generic case of a K3 surface with a cyclic Picard group generated by \( O_S(1) \). A theorem of Mukai implies that all moduli spaces of Gieseker-Simpson stable sheaves on \( S \), which happen to be compact, are smooth projective hyperkahler varieties (they admit an algebraic symplectic structure) \([\text{Mu1}]\). Compactness of the moduli space is automatic for the collection of components parametrizing sheaves whose rank \( r \), determinant \( O_S(d) \), and Euler characteristic \( \chi \) satisfy \( \gcd(r, d, \chi) = 1 \) (these correspond to primitive vectors in the Mukai lattice introduce below). In particular, compactness of the stable locus holds for the relative compactified Picards over the linear system \( |O_S(1)| \) (union of compactified Picards of all curves in the linear system). Here, the fact that all curves in the linear system \( |O_S(1)| \) are reduced and irreducible is crucial. The polarized weight 2 Hodge structure of a moduli space of stable sheaves is identified as a sub-quotient of the Mukai-lattice. This is the cohomology lattice \( \tilde{H}(S, \mathbb{Z}) := H^0(S, \mathbb{Z}) \oplus H^2(S, \mathbb{Z}) \oplus H^4(S, \mathbb{Z}) \) endowed with the non-degenerate symmetric pairing

\[
\langle (r', c'_1, s'), (r'', c''_1, s'') \rangle := c'_1 \cdot c''_1 - r's'' - r''s'.
\]

The Euler characteristic of a coherent sheaf \( F \) on \( S \) of rank \( r \) and Chern classes \( c_1, c_2 \) is \( \chi(F) = 2r + \frac{(c_1)^2}{2} - c_2 \). Following Mukai, we associate to \( F \) its Mukai vector \( v(F) := ch(F) \sqrt{Td(S)} = (r, c_1, s) \) where \( s = \chi(F) - r \). \( \text{Pic}(S) \) acts on the Mukai lattice by tensorization

\[
O_S(d) \otimes (r, c_1, s) = (r, c_1 + rd, s + \frac{1}{2}O_S(d) \cdot (2c_1 + rO_S(d))).
\]

The dimension of the moduli space \( \mathcal{M}(v) \) of stable sheaves with Mukai vector \( v \) is \( \langle v, v \rangle + 2 = (c_1)^2 - 2rs + 2 \). Conjecturally, the weight 2 Hodge structure of a smooth projective
moduli space $\mathcal{M}(v)$ is isomorphic to $v^\perp$ if $\dim(\mathcal{M}(v)) > 2$ and to $v^\perp/\mathbb{Z}\cdot v$ if $\dim(\mathcal{M}(v)) = 2$ (see [Mu2, 10, 12] for many known cases). Above, $v^\perp$ is the orthogonal hyperplane with respect to the Mukai pairing.

The collection of non-empty moduli spaces consists of the points in the “hyperboloid”

$$\mathcal{V} := \{v = (r, \mathcal{O}_S(d), s) \mid \frac{1}{2} \dim \mathcal{M}(v) = 1 + d^2(g - 1) - rs \geq 0\}$$

in $\mathbb{Z}^3$ satisfying the additional condition that the rank $r$ is $\geq 0$. $\mathcal{V}$ is symmetric with respect to $r$ and $s$ but the last constraint is not. Let us illustrate this lack of symmetry in the plane $c_1 = \mathcal{O}_S(1)$. We get the planar region $\mathcal{H} := \mathcal{V} \cap (c_1 = 1) := \{v = (r, \mathcal{O}_S(1), s) \mid \frac{1}{2} \dim \mathcal{M}(v) = g - rs \geq 0\}$ bounded by a hyperbola (see Figure 1). The symmetry is restored via Brill-Noether theory. There is a natural $\mathbb{Z}/2 \times \mathbb{Z}/2$ action by a group of isometries of the Mukai lattice. The hyperbola $\mathcal{H}$ is invariant and we get three involutions $\sigma$, $\tau$ and $\tau \circ \sigma$ where $\sigma$ and $\tau$ are reflections with respect to the hyperplanes $r - s = 0$ and $r + s = 0$.

![Figure 1: The region in the Mukai lattice of non-empty moduli spaces with $c_1 = \mathcal{O}_S(1)$.](image)

The Hilbert scheme $S^{[n]}$ of length $n$ zero dimensional subschemes is represented by
$M(1,0,1-n)$ as well as by any Pic(S)-translate, hence also by $M(1,1,g-n)$. The relative compactified Picard of degree $i$ over the linear system $|O_S(1)|$ is the moduli space $J^i := M(0,1,i+1-g)$. Observe that the birational isomorphism between $J^g$ and $S^{[g]}$, realized by the Abel-Jacobi map, is a lift of the involution $\sigma(1,1,0) = (0,1,1)$. The local Torelli theorem holds for the weight 2 Hodge structure of projective hyperk"ahler varieties [B1]. Hence, given a universal isometry of the Mukai lattice, it is natural to ask if it “lifts” to a birational transformation on the level of moduli spaces. Tyurin observed that this is indeed the case for $\sigma$ and moduli spaces parametrized by vectors in the first quadrant of the region $\mathcal{H}$ (see [Ty3] (4.11) and Theorem 4.1). In [G-H] the reflection $\tau(1,1,-2) = (2,1,-1)$ was lifted to a birational isomorphism $S^{[g+2]} \leftrightarrow M(2,1,-1)$.

**Stratified Mukai elementary transformations:** An *elementary Mukai transformation* is a birational transformation $(M, P) \leftrightarrow (W, P^*)$ between a symplectic variety $M$ containing a smooth codimension $n$ subvariety $P$ which is a $\mathbb{P}^n$-bundle $P \to Y$. The blow-up of $M$ along $P$ admits a second ruling, the contraction of which results in a symplectic variety $W$ containing the dual bundle $[Mu1]$. In Section 2 Theorem 6 we construct a *stratified analogue of a Mukai elementary transformation* - a birational transformation of a symplectic variety $M$ admitting a stratification with a highly recursive structure, which we call a *dualizable stratified collection* (see [12] and Definition 3). The birational transformation produces another symplectic variety $W$ and has the affect of “replacing” each stratum in $M$ - a grassmannian bundle - by the dual grassmannian bundle. The base of each grassmannian bundle is itself the dense open stratum in a smaller dualizable collection. The elementary transformation is a duality; when applied twice it recovers the original variety. In Section 2.1 the simplest example is introduced: A Springer resolution of the closure of a square-zero nilpotent (co)adjoint orbit in $\mathfrak{gl}_n$ is related to the dual Springer resolution by a stratified elementary transformation.

We use the Brill-Noether stratification [3] of the moduli spaces in $\mathcal{H}$, and prove:

**Theorem 1** Given a Mukai vector $v$ in $\mathcal{H}$ with negative rank, define the moduli space $M(v)$ to be identical to $M(\sigma \circ \tau(v))$. Then, the $\mathbb{Z}/2 \times \mathbb{Z}/2$ symmetry of $\mathcal{H}$, as a set of Mukai vectors, lifts to an action by stratified elementary transformations on the level of moduli spaces. Consequently, we obtain a *resolution* of the birational isomorphisms $M(v) \leftrightarrow M(\sigma(v))$ and $M(v) \leftrightarrow M(\tau(v))$ as a sequence of blow-ups along smooth sub-varieties, followed by a dual sequence of blow-downs.

For a more detailed statement, see Theorem 20. The formal identification $M(v) = M(\sigma \circ \tau(v))$ is well defined because the only vectors in $\mathcal{H}$, for which both $v$ and $\sigma \circ \tau(v)$ have non-negative rank, are Mukai vectors with $r = 0$ and $c_1 = O_S(1)$. They correspond to the compactified Jacobians over $|O_S(1)|$ and $\sigma \circ \tau$ takes the Mukai vector of $J^g$ to that of $J^{3g-2-n}$. These two moduli spaces are naturally isomorphic (see [Le1] Theorem 5.7). More conceptually, since $\sigma$ and $\tau$ lift to stratified elementary transformations with respect to the same (Brill-Noether) stratification, they coincide as birational transformations of $M(\tau(v))$. Hence, $\sigma \circ \tau[M(v)] = \tau \circ \tau[M(v)] = M(v)$. When the rank of $v$ is negative,
\( \mathcal{M}(v) \) can be also described as a moduli space of (equivalence classes) of complexes of sheaves.

The general definition of a dualizable stratified collection is illustrated in the context of moduli spaces of sheaves. Given an integer \( t \), denote by \( \vec{t} \) the Mukai vector \((t, 0, t)\) of the trivial rank \( t \) bundle. Given a Mukai vector \( v \) in \( \mathcal{H} \), let \( \mu(v) \), the distance of \( v \) from the boundary of \( \mathcal{H} \), be

\[
\mu(v) := \begin{cases} 
\max \{ t : v + \vec{t} \in \mathcal{H}, \ t \in \mathbb{Z}_{\geq 0} \}, & \text{if } \chi(v) \geq 0 \\
\max \{ t : v - \vec{t} \in \mathcal{H}, \ t \in \mathbb{Z}_{\geq 0} \}, & \text{if } \chi(v) \leq 0.
\end{cases}
\]  

Then \( \mu(v) + 1 \) is the length of the Brill-Noether stratification. For example, on the line \( r = 0 \) of compactified relative Jacobians in \( \mathcal{H} \), \( \mu(v) + 1 \) is equal to the usual length of the Brill-Noether stratification of a Petri-generic curve, \( \mu(J_{g-1+n}) = \mu(0, 1, n) = \max \{ 0, -n+[\sqrt{n^2+4g}] \} \).

Choose, for example, \( v \in \mathcal{H} \) with non-negative Euler characteristic \( \chi(v) = r + s \geq 0 \). If \( \mu(v) > 0 \) then \( \mu(v + \vec{1}) = \mu(v) - 1 \). The square \((\mu(v)+1) \times (\mu(v)+1)\) upper triangular matrix

\[
\begin{array}{cccc}
\mathcal{M}(v) & \subset & \mathcal{M}(v)^1 & \subset & \cdots & \subset & \mathcal{M}(v)^t & \subset & \cdots & \subset & \mathcal{M}(v)^\mu \\
\downarrow & & & & & & & & & & & \\
\mathcal{M}(v+\vec{1}) & \subset & \mathcal{M}(v+\vec{1})^1 & \subset & \cdots & \subset & \mathcal{M}(v+\vec{1})^{t-1} & \subset & \cdots & \subset & \mathcal{M}(v+\vec{1})^{\mu-1} \\
\downarrow & & & & & & & & & & & \\
\mathcal{M}(v+\vec{2}) & \subset & \cdots & \subset & \mathcal{M}(v+\vec{2})^{t-2} & \subset & \cdots & \subset & \mathcal{M}(v+\vec{2})^{\mu-2} \\
& & & & & & & & & & \vdots \\
& & & & & & & & & & \downarrow \\
& & & & & & & & & & \mathcal{M}(v+\vec{\mu})
\end{array}
\]  

is a stratified dualizable collection. If \( \chi(v) \leq 0 \), replace \( \mathcal{M}(v+\vec{1})^t \) by \( \mathcal{M}(v-\vec{1})^t \) in the matrix (2) to obtain the analogous stratified dualizable collection (here, even if we start with \( v \) satisfying rank(\( v \)) \( \geq 0 \), the Mukai vector \( v - \vec{i} \) may have negative rank and we use the convention of Theorem [3]). The diagonal entries are symplectic projective moduli spaces. Each row is the Brill-Noether stratification of the diagonal entry. When \( \chi \geq 0 \), we set

\[
\mathcal{M}(v)^t := \{ F \in \mathcal{M}(v) \mid h^1(F) \geq t \}
\]  

and when \( \chi \leq 0 \) we use \( h^0 \) instead. Every space \( \mathcal{M}(v+\vec{t})^{t-i} \) in the \( t \)-th column admits a dominant rational morphism to the diagonal symplectic entry \( \mathcal{M}(v+\vec{t}) \) which is regular away from the smaller stratum and realizes

\[
\mathcal{M}(v+\vec{t})^{t-i} \setminus \mathcal{M}(v+\vec{t})^{t-i+1} \rightarrow \mathcal{M}(v+\vec{t}) \setminus \mathcal{M}(v+\vec{t})^{1}
\]
as a Grassmannian bundle. The fiber over \( E \in \mathcal{M}(v+\vec{t}) \setminus \mathcal{M}(v+\vec{t})^{1} \) is \( G(t-i, H^0(E)) \).

As a subvariety of \( \mathcal{M}(v) \), the codimension of each Grassmannian bundle is equal to the
dimension of the Grassmannian fiber. Hence, the projectivized normal bundle of each Grassmannian bundle is isomorphic to its relative projectivized cotangent bundle. The latter is a bundle of homomorphisms and admits a canonical determinantal stratification. The stratified elementary transformation is performed by blowing up the matrix (2) column by column from right to left. The key to the success of the recursion is the fact that the intersection of the strict transform of the Brill-Noether stratification with a new exceptional divisor (which is fibered by projectivized cotangent bundles of grassmannians) is precisely the determinantal stratification of the latter. The existence of two dual sequences of blow-downs on the top iterated blow-up can be seen already on the level of the projectivised cotangent bundle of a single grassmannian: Given a vector space $H$, the top iterated blow-up of $\mathbb{P}^*G(t, H)$ is naturally isomorphic to that of $\mathbb{P}^*G(t, H^*)$.

The top iterated blow-up $B[1]\mathcal{M}_S(v)$ is a particularly nice compactification of the open dense $GL(H)$-orbit in $\mathbb{P}^*G(t, H)$. It admits an interpretation as a moduli space of complete collineations and its cohomology ring is well understood [BDP]. Complete collineations and complete quadrics play an important role in enumerative geometry [Lak1]. They received a modern treatment in [Van, Lak2, K1, Th2].

The top iterated blow-up $B[1]\mathcal{M}_S(v)$ is a coarse moduli space of complete stable sheaves. A complete sheaf $(F, \rho_2, \rho_3, \ldots, \rho_k)$ consists of a sheaf $F$ on $S$ and a sequence of non-zero homomorphisms, up to a scalar factor, defined recursively by

\begin{align*}
\rho_2 & : H^0(F) \to H^1(F) \\
\rho_{i+1} & : \ker(\rho_i) \to \text{coker}(\rho_i), \quad 2 \leq i \leq k - 1,
\end{align*}

such that the last homomorphism $\rho_k$ is surjective if $\chi(F) \geq 0$ and injective if $\chi(F) \leq 0$. The sequence is empty if either $H^0(F)$ or $H^1(F)$ vanishes. The sequence $\{\rho_2, \rho_3, \ldots, \rho_k\}$ should be viewed as a truncation of a complete collineation $\{\rho_1, \rho_2, \ldots, \rho_k\}$. The latter behaves well when the sheaf $F$ varies in a flat family. A choice of a section $\gamma$ of a sufficiently ample line bundle on $S$ gives rise to a homomorphism $\rho_1 : V_0(F) \to V_1(F)$ between vector spaces which ranks depend only on the Mukai vector $v(F)$ (see (70)). The kernel of $\rho_1$ is $H^0(F)$ and cokernel is $H^1(F)$. The notion of families of complete collineations is subtle but well understood [Lak2, K1]. It leads to the notion of families of complete sheaves once we fix a section $\gamma$ as above. Thaddeus’ work on complete collineations [Th2] combined with Theorem 1 suggests that the stratified elementary transformations in Theorem 1 come from a variation of Geometric Invariant Theory quotients in the sense of [DH, Th1].

The moduli space of complete sheaves $B[1]\mathcal{M}_S(v)$ is instrumental in studying the intersection theory on moduli spaces. Indeed, it plays a central role in the proof of Theorem 2. This is not surprising, considering the important role played by complete collineations in classical enumerative geometry. $B[1]\mathcal{M}_S(v)$ is different in general from the closure of the graph of the birational isomorphism in $\mathcal{M}(v) \times \mathcal{M}(\sigma(v))$. The latter is the moduli space of coherent systems $G^0(\chi(v), \mathcal{M}(\sigma(v)))$ discussed in section 5.4. The two are equal only in the case of a Mukai elementary transformation, i.e., when $\mu(v) \leq 1$. 

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While both moduli spaces are useful in the proof of Theorem 1, it is the moduli space of complete sheaves which admits the full recursive structure which makes transparent the analogy with dual Springer resolutions of nilpotent orbits.

**A correspondence inducing isomorphism of cohomology rings:**

Birational hyperkahler varieties \( M', M'' \) are especially similar (under mild conditions which are satisfied by our stratified elementary transformations): There exists a family \( M \rightarrow D^* \) of hyperkahler varieties over the punctured disk with two extensions \( M', M'' \) to smooth families over the disk with special fibers \( M', M'' \) respectively \( \text{[Huy]} \). Consequently, the Hodge structure, the cohomology ring structure and all continuous invariants of \( M' \) and \( M'' \) coincide. The Hodge conjecture suggests the existence of a correspondence (an algebraic cycle in \( M' \times M'' \)) which induces the isomorphism of Hodge structures. This correspondence contains the closure of the graph of the birational isomorphism as a component. It is easy to see that there are other components. For example, we saw that \( J_g \) and \( S[S] \) are birational \( \sigma(J_g) = S[S] \) (see Figure 1). The \( g \)-th symmetric product \( C[S] \) of a smooth member (curve) \( C \) of \( |O_S(1)| \), is a subvariety in \( S[S] \). The birational image of \( C[S] \) in \( J_g \) is the Jacobian \( J_g \) which has self intersection 0. However, the self intersection of \( C[S] \) in \( S[S] \) is \( (2g-2) \). Recently, progress has been made in understanding the ring structure of the cohomology of the Hilbert schemes \( \text{ES, Leh} \). In order to translate this to an understanding of the cup-product in the cohomology of birational components such as \( \sigma(S[S]) \), one needs to compute explicitly the isomorphism \( H^*(S[S], \mathbb{Z}) \rightarrow H^*(S[S], \mathbb{Z}) \).

**Theorem 2** Let \( \{ M(i^j) \} \leftrightarrow \{ W(i^j) \} \) be a stratified elementary transformation between two stratified dualizable collections associated with irreducible projective symplectic varieties \( M = M(0)^0 \) and \( W = W(0)^0 \). Then, the natural isomorphism \( H^*(M, \mathbb{Z}) \cong H^*(W, \mathbb{Z}) \) is induced by a correspondence which, as a cycle \( \Delta \) in \( M \times W \), is a sum

\[
\Delta = \Delta_0 + \sum_{t=1}^\mu \Delta_t
\]

where \( \Delta_0 \) is the closure of the graph of the birational transformation \( M \leftrightarrow W \) and \( \Delta_t \) is the closure of the fiber product \( M^t \times_{M} W^t \) of the two grassmannian bundles.

Note that the dimension of \( \Delta_t \) is equal to \( \dim(M) \). The Theorem is proven in Section 4.2. A few applications of Theorem 2 are discussed in Section 5.2.

**Auto-equivalences of the derived category:**

Let \( S \) be a K3 surface and consider the group \( G \) of Hodge isometries of the Mukai lattice \( \tilde{H}(S, \mathbb{Z}) \) of \( S \). These are integral isometries which send \( H^{2,0}(S) \) to itself when the lattice is complexified. Any Hodge-isometry \( \phi \) leaves the algebraic sublattice \( \tilde{H}_{Alg}(S, \mathbb{Z}) \) invariant. If \( \phi \) restricts to the identity automorphism of \( \tilde{H}_{Alg}(S, \mathbb{Z}) \) then it is induced by an automorphism of \( S \) (the Torelli theorem). It follows that \( G \) fits in the exact sequence:

\[
0 \rightarrow \text{Aut}(S, \text{Pic}(S)) \rightarrow G \rightarrow \text{Aut}(\tilde{H}_{Alg}(S, \mathbb{Z})) \rightarrow 0. \]
Let $G_{\text{Der}}$ be the group of (covariant and contravariant) auto-equivalences of the bounded derived category of $S$. We have a map

$$
\eta : D^b(S) &\to \tilde{H}(S,\mathbb{Z}) \\
(F_i, \partial_i) &\mapsto \sum_{i} (-1)^{i} v(F_i)
$$

sending an object represented by a complex $(F_i, \partial_i)$ to the Mukai vector of the associated class in K-theory, i.e., to the alternating sum of the Mukai vectors of its coherent sheaves. Notice that $\eta$ is equivariant with respect to the cyclic subgroup $\langle T \rangle \subset G_{\text{Der}}$ generated by the translation auto-equivalence $T : D^b(S) \to D^b(S)$ once we send $T$ to $-Id \in G$. Orlov proved that there is a surjective homomorphism

$$
G_{\text{Der}}/\langle T^2 \rangle &\to G.
$$

Any auto-equivalence $\Phi$ of the derived category $D^b(S)$ induces a Hodge isometry $\phi$ of $\tilde{H}(S,\mathbb{Z})$ and $\eta$ is $(\Phi, \phi)$-equivariant ([Or], Theorem 2.2). Moreover, any Hodge isometry can be lifted to an auto-equivalence of the derived category $D^b(S)$ ([Or], proof of Theorem 3.13). For example,

1. Tensorization by a line bundle $\gamma$ on $S$ is induced by the Fourier-Mukai functor

   $$
   R\pi_2_* \mathcal{E} \otimes \pi_1^*(\cdot) : D^b(S) \to D^b(S) \text{ associated to the sheaf } \mathcal{E} \text{ on } S \times S \text{ where } \mathcal{E} \text{ is supported on the diagonal as the line bundle } \gamma.
   $$

2. The Hodge-isometry $-\tau$ ($\tau$ as in Theorem [1]) is induced by the Fourier-Mukai functor associated to the ideal sheaf of the diagonal.

3. The Hodge isometry $-\sigma \circ \tau$ is induced by the contravariant involutive functor

   $$
   R\mathcal{H}om_S(\cdot, \mathcal{O}_S) : D^b(S) \to D^b(S)^{op}.
   $$

Assume that the K3 surface $S$ has a cyclic Picard group and let us introduce yet a third group $G_{\text{bir}}$. Let

$$
\mathcal{M}_S(\bullet) := \cup \{ \mathcal{M}_S(v) : v \in \tilde{H}_{\text{Alg}}(S,\mathbb{Z}) \text{ is a primitive Mukai vector, } \langle v, v \rangle \geq -2 \}
$$

be the disjoint union of moduli spaces of stable sheaves on $S$ with primitive algebraic Mukai vectors of non-negative dimension. Note that these moduli spaces are all smooth and compact. We use the convention that, if $\text{rank}(v) \leq 0$, then $\mathcal{M}_S(v)$ is equal to $\mathcal{M}_S(-v)$. The group $G_{\text{bir}}$ is defined to be the group of all possible lifts of elements of $G$ to birational automorphisms of $\mathcal{M}_S(\bullet)$. A natural question arises:

**Question:** Does the homomorphism ([1]) factor through $G_{\text{bir}}$?

The Fourier-Mukai functor lifting an isometry to an auto-equivalence sends, in general, a stable sheaf to the class of a complex supported at more than one degree. Nevertheless, there seems to exist, at least, a surjective homomorphism $G_{\text{bir}} \to G$. Clearly, $\text{Pic}_S$ and
Aut(S) lift to $G_{\text{bir}}$. The element $-Id$ lifts, by definition. Theorem 1 suggests that the Hodge isometries $\sigma$ and $\tau$ lift to $G_{\text{bir}}$. In the Theorem the lift is carried out only for the collection with $c_1 = O_S(1)$, but our definitions of $\sigma$ and $\tau$ on the Zariski open Brill-Noether stratum seem to extend to birational isomorphisms for other values of $c_1$. If indeed $\sigma$ and $\tau$ lift, then the image of $G_{\text{bir}} \to G$ has at most a finite index (and is surjective if $g = 2$). This follows from the exactness of (5) and the fact that $\{-Id, \sigma, \tau\}$ and $\text{Pic}_S$ generate a finite index subgroup $\Gamma$ of the group of isometries of the rank 3 lattice $H_{\text{Alg}}(S, \mathbb{Z})$. If $g = 2$, $\Gamma$ is the whole group, but in general it is a proper subgroup.

For example, if $g = 7$ the isometry
\[
\begin{pmatrix}
  r \\
  d \\
  s
\end{pmatrix} \mapsto \begin{pmatrix}
  2 & 12 & 3 \\
  1 & 5 & 1 \\
  3 & 12 & 2
\end{pmatrix} \begin{pmatrix}
  r \\
  d \\
  s
\end{pmatrix}
\]
is not in the subgroup $\Gamma$ because any isometry in $\Gamma$ takes $(0, 0, 1)$ to a vector $(r, d, s)$ such that $(g - 1)$ divides exactly one of the two integers $\{r, s\}$ and is relatively prime to the other.

A much harder problem would be to resolve the birational isomorphisms in $G_{\text{bir}}$. It is natural to try to generalize our results to other reflections of the Mukai lattice. The group of isometries of the rank 3 lattice $H_{\text{Alg}}(S, \mathbb{Z})$ contains a finite index subgroup $W \subset G$ generated by reflections with respect to Mukai vectors with $\langle v, v \rangle = \pm 2$. In fact, setting $\sigma' := O(1) \circ \sigma \circ O(-1)$ and $\tau' := O(1) \circ \tau \circ O(-1)$ we have the equality
\[
O(2) = \sigma' \tau' \sigma \tau \in W.
\]

Our results suggest a relationship between dual pairs of hyperkahler resolutions of singularities and reflections in $G_{\text{bir}}$. While at the level $c_1 = O(1)$ the reflections $\sigma$ and $\tau$ correspond to Springer resolutions of a nilpotent orbit with a simple (well ordered) stratification, it seems that for other values of $c_1$ more complicated singularities will arise. It would be interesting, for example, to interpret the resolution $\mathcal{M}_S(1, 0, 1-n) \cong S^{[n]} \to \text{Sym}^n S$ of the symmetric product as a special case of a lift of some reflection $(\tau ?)$ to $G_{\text{bir}}$.

The rest of the paper is organized as follows. Sections 2, 3, and 4 are devoted to the general study of stratified elementary transformations. In section 2 we construct the stratified elementary transformations. Section 3 contains the background information on determinantal varieties needed for the proof of the construction. In section 4 we complete the proof of the construction. We also prove Theorem 2 identifying the correspondence inducing the cohomology ring isomorphism. Section 5 is devoted to our main example, the moduli spaces of sheaves on a K3 surface. The organization of section 5 is described at the beginning of that section.

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2 Stratified Elementary Transformations

In section 2.1 we introduce the prototypical example of the symplectic birational isomorphisms considered in this paper. In section 2.2 we introduce a global analogue for projective symplectic varieties.

2.1 Dual resolutions of the closure of a nilpotent orbit

Let $H$ be a vector space of dimension $h$, $t \leq h/2$ an integer and consider the natural morphism $\pi_1 : T^*G(t,H) \rightarrow \text{End}(H)$ onto the closure $\overline{N^t}$ in $\text{End}(H)$ of the nilpotent orbit $N^t$ of square-zero nilpotent elements of rank $t$. The natural isomorphism $\text{End}(H) \cong \text{End}(H^*)$ provides another resolution

$$T^*G(t,H) \xrightarrow{\pi_1} \overline{N^t} \xleftarrow{\pi_2} T^*G(t,H^*).$$  \hfill (7)

There is a simultaneous deformation of (7) over $\mathbb{C}$ which smooths $\overline{N^t}$ away from the special fiber and deforms $\pi_i$ to isomorphisms. The cotangent bundle of $G(t,H)$ is isomorphic, as a bundle of Lie subalgebras of $\text{End}(H_G(t,H))$, to the nilpotent radical of the corresponding parabolic subalgebra $P_G(t,H) \subset \text{End}(H_G(t,H))$. We have a unique non-trivial extension

$$0 \rightarrow T^*_G(t,H) \rightarrow E(H) \rightarrow \mathcal{O}_G(t,H) \rightarrow 0.$$  \hfill (8)

A non-zero section $\gamma$ of $\mathcal{O}_G(t,H)$ determines a symplectic $T^*_G(t,H)$ torsor $X_\gamma$ embedded in $E(H)$. $X_\gamma$ does not have any section. Hence the lagrangian zero-section of $T^*_G(t,H)$ does not deform. The vector bundle $E(H)$ admits an embedding as a bundle of subalgebras in $\text{End}(H_G(t,H))$. Each point in $G(t,H)$ determines a decomposition of the Levi factor of the parabolic subalgebra (with ranks $t$ and $h-t$). $E(H)$ is the subalgebra of $P_G(t,H)$ which projects to the center in the first factor and to zero in the second factor. In other words, $E(H)$ is the subalgebra of endomorphisms leaving the subbundle $\tau(H)$ invariant, whose image in $\text{End}(\tau(H))$ is a scalar multiple of the identity and whose image in $\text{End}(q(H))$ is zero. The morphism $E(H) \rightarrow \text{End}(H)$ restricts to an embedding of $X_\gamma$, $\gamma \neq 0$, as a smooth closed orbit.

The birational isomorphism (7) is a special case of the setup of Section 2.2. Consider the square $(t+1) \times (t+1)$ upper triangular matrix whose rows are the determinantal stratification of $T^*G(k,H)$, $0 \leq k \leq t$, indexed as follows. $T^*G(k,H)$ is isomorphic to the homomorphism bundle $\mathcal{H}om(q_k, \tau_k)$ from the tautological quotient bundle to the tautological sub-bundle. The generic point corresponds to a surjective homomorphism. $T^*G(k,H)^i$ denotes the locus with $i$-dimensional cokernel.
that the morphism to admits a rational morphism to the symplectic diagonal entry in that column. Observe and is hence a Grassmannian fibration. In a similar fashion, every entry in each column Flag over Flag 

\[
\begin{pmatrix}
T^*G(t, H) \supset T^*G(t, H) \supset T^*G(t-1, H) \supset T^*G(t-1, H) \supset \cdots \supset T^*G(0, H) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{N}^t \supset \mathcal{N}^{t-1} \supset \cdots \supset \{0\}
\end{pmatrix}
\]

Above, $T^*G(0, H)$ is a point. For the sake of compatibility with the notation in the following sections, set $n := h-2t+1$ and $\mu = t$. We have a natural $G(r, n+2r-1)$-bundle:

\[ f_{t,r} : [T^*G(t, H)^r \setminus T^*G(t, H)^{r+1}] \to [T^*G(t-r, H) \setminus T^*G(t-r, H)^1], \quad 0 \leq r \leq \mu. \]  

$[T^*G(t, H)^r \setminus T^*G(t, H)^{r+1}]$ is isomorphic to the Zariski open stratum in the homomorphism bundle $\mathcal{H}om(q_{h-t+r}, \tau_{t-r})$ over $Flag(t-r, t, h-t+r, H)$. $[T^*G(t-r, H) \setminus T^*G(t-r, H)^{1}]$ is isomorphic to the Zariski open stratum in the homomorphism bundle $\mathcal{H}om(q_{h-t+r}, \tau_{t-r})$ over $Flag(t-r, h-t+r, H)$. $Flag(t-r, t, h-t+r, H)$ is isomorphic to the bundle $G(r, \tau_{h-t+r}/\tau_{t-r})$ over $Flag(t-r, h-t+r, H)$. The morphism (10) is the pullback of the morphism $Flag(t-r, t, h-t+r, H) \to Flag(t-r, h-t+r, H)$ and is hence a Grassmannian fibration. In a similar fashion, every entry in each column admits a rational morphism to the symplectic diagonal entry in that column. Observe that the morphism to $\mathcal{N}^t$ contracts these grassmannian fibrations.

In this paper we show that the birational isomorphisms (7) describe the local structure of the birational isomorphisms of moduli spaces of sheaves in Theorem (20). This fact depends heavily on the condition (6) imposed on the first Chern class. It is plausible that there is a generalization of the structure of birational isomorphisms of moduli spaces of sheaves with a more general first Chern class would also admit a group theoretic model. There is a generalization of the above construction valid for a nilpotent orbit $\mathcal{N}_\eta$ in End($H$) associated to any partition $\eta := (p_1 \geq p_2 \geq \cdots \geq p_k)$ of $h$. In the general case there are many Springer resolutions of the closure $\overset{\sim}{\mathcal{N}}_\eta$ but they come in dual pairs. Given a partition $\eta$ we get a Young diagram whose $i$-th row consists of $p_i$ boxes. The dual partition $\tilde{\eta} = (\tilde{p}_1 \geq \tilde{p}_2 \geq \cdots \geq \tilde{p}_m)$ is defined setting $\tilde{p}_i$ as the length of the $i$-th column of the diagram of $\eta$. Choose a permutation $\theta \in Sym_m$ and set $n(\theta) := (n_1 \leq n_2 \leq \cdots \leq n_{m-1})$ where

\[ n_j := \sum_{i=1}^{j} \tilde{p}_{\theta(i)}. \]
Then the cotangent bundle $T^*\text{Flag}(n_1, \ldots, n_{m-1}, H)$ is a resolution of the closure $\overline{\mathcal{N}_\eta}$ where $\mathcal{N}_\eta$ is the nilpotent orbit of matrices whose Jordan canonical form has $k$ blocks and the $i$-th block is a regular nilpotent $p_i \times p_i$ matrix. The analogue of (7) is then

$$T^*\text{Flag}(n_1, \ldots, n_{m-1}, H) \to \overline{\mathcal{N}_\eta} \leftarrow T^*\text{Flag}(h-n_{m-1}, \ldots, h-n_1, H). \quad (11)$$

The recursive structure of a Flag variety as a Grassmannian bundle over a smaller flag variety suggests an inductive way to reduce the study of (11) to that of (7). It seems likely that the work of Thaddeus [Th2] could be generalized to exhibit the collection of birational varieties $\mathbb{P}T^*\text{Flag}(n(\theta), H), \theta \in \text{Sym}_m$, as different geometric invariant theory quotients associated to polarizations in different faces of a convex polytope. The torus acting is likely to be the center of the Levi factor of the parabolic subgroups in $SL(H)$.

### 2.2 Construction of stratified transformations

Determinantal stratifications of symplectic varieties tend to have a recursive structure. Roughly, the recursive structure arises by taking the quotient of each stratum by the null-foliation of the symplectic form. In nice situations, these foliations are Grassmannian bundles. Dualizing them gives rise to a birational symplectic variety. We formalize the recursive setup in this section.

Let $\mu$ be a non-negative integer, $X(r), 0 \leq r \leq \mu$ a collection of smooth projective symplectic varieties and

$$X(r) = X(r)^0 \supset X(r)^1 \supset \ldots \supset X(r)^{\mu-r} \supset X(r)^{\mu+1-r} = \emptyset$$

a flag of closed subschemes. We denote $X(0)$ also by $X$ and assume that $X$ is connected. It is convenient to arrange the data in an upper triangular $(\mu+1) \times (\mu+1)$-matrix with symplectic diagonal entries:

$$
\begin{array}{cccc}
X(0) & \supset & X(0)^1 & \supset \cdots \supset X(0)^{t} & \cdots & \supset X(0)^{\mu} \\
\downarrow & & \downarrow & & \downarrow & \\
X(1) & \supset & X(1)^1 & \supset \cdots & \supset X(1)^{\mu-1} \\
\downarrow & & & & \downarrow & \\
X(2) & \supset & \cdots & \supset X(2)^{\mu-2} \\
& & & \vdots & \\
& & & \downarrow & \\
& & & X(\mu) & \\
\end{array}
$$

We will assume below (in Condition 2) that every entry $X(i)^t$ admits a rational morphism to the symplectic diagonal entry $X(i+t)$ in the same column. This morphism is regular away from $X(i)^{t+1}$ and realizes

$$X(i)^t \setminus X(i)^{t+1} \to X(i+t) \setminus X(i+t)^1$$
as a Grassmannian bundle.

We will sometimes denote the general X(r) by M and we set \( \mu(M) := \mu - r \), \( M^t := X(r)^t \) and \( M(r') := X(r + r') \). Denote by \( B^k M^t \), \( t < k \leq \mu(M) \) the blow-up of \( M^t \) along the subscheme \( M^k \). We introduce the following notation for iterated blow-ups:

\[
\begin{align*}
  k &= \mu \
  &\text{Set } B^{[\mu(M)]} M^t := B^{\mu(M)} M^t, \quad t < \mu(M). \\
  t < k &\text{ Define recursively } B^{[k]} M^t, \quad t < k \leq \mu(M) \text{ to be the blow-up of } B^{[k+1]} M^t \text{ along } B^{[k]} M^k. \text{ Note that we used here recursively the base change property of blowing-up to conclude that } B^{[k+1]} M^k \text{ is isomorphic to the strict transform of } M^k \text{ in } B^{[k+1]} M^t \text{ for all } t \leq k. \\
  t = k &\text{ We further denote by } B^{[k]} M^k \text{ the exceptional divisor in } B^{[k]} M \text{ corresponding to } M^k. \\
  t > k &\text{ Let } B^{[k]} M^t, \quad t > k \text{ be the strict transform in } B^{[k]} M \text{ of the Cartier divisor } B^{[k+1]} M^t. \text{ If } \mu \geq 1, \text{ denote by } n(M) \text{ the codimension of } M^1 \text{ in } M.
\end{align*}
\]

**Condition 1** \( \dim(M(r)) = \dim(M) - 2r[n(M) + r - 1] \).

**Condition 2** *(Quotient by the null foliation of the symplectic form)* There exists a morphism

\[
  f_{r,t} : B^{[t+1]} X(r)^t \rightarrow X(r + t). \tag{13}
\]

When using the notation \( M = X(r) \) we denote \( f_{r,t} \) by \( f_t \). The morphism \( f_t \) lifts to a smooth projective morphism

\[
  \tilde{f}_t : B^{[t+1]} M^t \rightarrow B^{[1]} M(t) \tag{14}
\]

realizing \( B^{[t+1]} M^t \) as a \( G(\mathbb{P}^{t-1}, \mathbb{P}^{n(M)+2t-2}) \)-bundle.

**Lemma 3** \( B^{[t+1]} X(r)^t \) is smooth. In particular, \( X(r)^t \setminus X(r)^{t+1} \) is smooth.

Note that the Lemma implies that each of the iterated blow-ups is a sequence of blow-ups of smooth varieties along smooth subvarieties.

**Proof:** (of lemma 3) The proof is by descending induction on \( r + t \). If \( r + t = \mu \) then \( B^{\mu-r+1} X(r)^{\mu-r} = X(r)^{\mu-r} \) and Condition 2 implies that \( B^{[\mu-r+1]} X(r)^{\mu-r} \) is a Grassmannian-bundle over \( B^{[1]} X(\mu) = X(\mu) \). Since \( X(\mu) \) is smooth, so is \( B^{\mu-r+1} X(r) \).

Induction step: Condition 2 implies that \( B^{[t+1]} X(r)^t \) is a Grassmannian bundle over \( B^{[1]} X(r + t) \). It suffices to prove that \( B^{[1]} X(r + t) \) is smooth. \( B^{[1]} X(r + t) \) is the blow-up of \( B^{[2]} X(r + t) \) along \( B^{[2]} X(r + t)^1 \). By the induction hypothesis, \( B^{[2]} X(r + t)^1 \) is smooth. Hence, so is \( B^{[1]} X(r + t) \). \( \square \)
Conditions 3 and 2 imply the equality
\[ n(X(r)) = n + 2r. \] (15)

A short calculation shows that we have an equality
\[ \text{codim}(M^k, M) = \text{codim}(M^t, M) + \text{codim}(M(t)^{k-t}, M(t)), \quad \text{for } t \leq k \leq \mu(M). \] (16)

Observe that for \( t = 1 \) we get that \( \tilde{f}_1 : B^{[2]}M^1 \to B^{[1]}M(1) \) is a \( \mathbb{P}^{n(M)} \)-bundle. Similarly,
\[ \tilde{f}_{r,1} : B^{[2]}X(r)^1 \to B^{[1]}X(r+1) \]
is a \( \mathbb{P}^{n+2r} \)-bundle. For \( 1 \leq r \leq \mu \), we denote the \( \mathbb{P}^{n+2r-2} \)-bundle over \( B^{[1]}X(r) \) by \( \mathbb{P}W_{B^{[1]}X(r)} \)
\[ \mathbb{P}W_{B^{[1]}X(r)} := B^{[2]}X(r-1)^1. \] (17)

We do not assume that there exists a global vector bundle \( W_{B^{[1]}X(r)} \) on \( B^{[1]}X(r) \) whose projectivization is \( \mathbb{P}W_{B^{[1]}X(r)} \). We will continue this abuse of notation and denote \( G(\mathbb{P}^{k-1}, \mathbb{P}W_{B^{[1]}X(r)}) \) by \( G(k, W_{B^{[1]}X(r)}) \). In section 5 \( X(r) \) will be a moduli space of sheaves on a K3 surface \( S \) and over \( B^{[1]}X(r) \) we may or may not have a vector bundle \( W_{B^{[1]}X(r)} \) whose fiber over a sheaf \( F \) is a subspace of \( H^0(S, F) \). The existence of \( W_{B^{[1]}X(r)} \) depends on the existence of a universal sheaf. Nevertheless, \( \mathbb{P}W_{B^{[1]}X(r)} \) exists and is canonical.

**Condition 3** The \( G(\mathbb{P}^{k-1}, \mathbb{P}^{n(M)+2r-2}) \)-bundle \((\mathbb{4})\) over \( B^{[1]}M(t) \) is identified with the Grassmannian bundle \( G(t, W_{B^{[1]}M(t)}) \).

The identification in Condition 3 introduces on \( B^{[t+1]}M^t \) natural projectivised sub and quotient bundles of \( \tilde{f}_t^*(\mathbb{P}W_{B^{[1]}M(t)}) \) which we denote by \( \mathbb{P}T_{B^{[t+1]}M^t} \) and \( \mathbb{P}Q_{B^{[t+1]}M^t} \). Denote the relative cotangent bundle of the Grassmannian bundle \( G(t, W_{B^{[1]}M(t)}) \) by \( (\Omega^1G)(t, W_{B^{[1]}M(t)}) \) and its projectivization by \( (\mathbb{P}\Omega^1G)(t, W_{B^{[1]}M(t)}) \).

**Condition 4** The morphisms \( f_t \) and \( \tilde{f}_t \) are compatible with the stratifications. In other words we have equality of Cartier divisors on \( B^{[t+1]}M^t \):
\[ [B^{[t+1]}M^t \cap B^{[t+1]}M^k] = \tilde{f}_t^{-1} (B^{[1]}M(t)^{k-t}), \quad \text{for } t + 1 \leq k \leq \mu(M). \]
(Compare Condition 3 with Theorem 7 part 4).

The conditions above imply the following (compare with Theorem 7 part 2).

**Lemma 4** The symplectic structure of \( M \) induces a canonical isomorphism
\[ \phi : N_{B^{[t]}M^t-1/B^{[t]}M} \cong \Omega^1_{f_t^{-1}} \left( - \sum_{k=t}^{\mu(M)} B^{[t]}M^k \right) \] (18)
between the normal bundle of \( B^{[t]}M^t-1 \) in \( B^{[t]}M \) and the twist of the relative cotangent bundle of the Grassmannian bundle \( \tilde{f}_t^{-1} : B^{[t]}M^t-1 \to B^{[1]}M(t-1) \).
Proof: The case $t = 1$ is trivial as $\tilde{f}_0$ is the identity. Assume from now on that $t \geq 2$. The symplectic structure $\sigma_M$ of $M$ pulls back to a degenerate 2-form on $B^{[t]} M$ and restricts to a 2-form $\sigma_{B^{[t]} M^{t-1}}$ on $B^{[t]} M^{t-1}$. As $\tilde{f}_{t-1} : B^{[t]} M^{t-1} \to B^{[1]} M(t-1)$ is a Grassmannian bundle, the 2-form $\sigma_{B^{[t]} M^{t-1}}$ vanishes on each fiber and hence it is a pullback of a 2-form on $B^{[1]} M(t-1)$. Contraction with $\sigma_M$ induces a sheaf homomorphism

$$\phi : N_{B^{[t]} M^{t-1}/B^{[t]} M} \to \Omega^1_{\tilde{f}_{t-1}}.$$ 

It is easy to check that $\phi$ vanishes along $(B^{[t]} M^{t-1}) \cap (B^{[t]} M^k)$ for $k \geq t$. Hence, $\phi$ in (18) is an isomorphism if and only if

$$\det(N_{B^{[t]} M^{t-1}/B^{[t]} M}^*) \otimes \omega_{\tilde{f}_{t-1}} \otimes O_{B^{[1]} M^{t-1}} \left( -\text{codim}(M^{t-1}, M) \cdot \sum_{k=t}^{\mu(M)} B^{[t]} M^k \right)$$

is the trivial line-bundle. The two short exact sequences

$$0 \to T_{B^{[t]} M^{t-1}} \to (T_{B^{[t]} M})_{B^{[t]} M^{t-1}} \to N_{B^{[t]} M^{t-1}/B^{[t]} M} \to 0, \quad \text{and}$$

$$0 \to T_{\tilde{f}_{t-1}} \to T_{B^{[t]} M^{t-1}} \to \tilde{f}_{t-1}^* (T_{B^{[1]} M(t-1)}) \to 0$$

imply that we have an isomorphism

$$\det(N_{B^{[t]} M^{t-1}/B^{[t]} M}^*) \otimes \omega_{\tilde{f}_{t-1}} \cong (\omega_{B^{[t]} M})^*_{B^{[t]} M^{t-1}} \otimes \tilde{f}_{t-1}^* \left( \omega_{B^{[1]} M(t-1)}^* \right). \quad (19)$$

Since $M$ and $M(t-1)$ are symplectic, we have the isomorphisms

$$\omega_{B^{[t]} M} \cong O_{B^{[t]} M} \left( \sum_{k=t}^{\mu(M)} \text{codim}(M^k, M) - 1 \right) \cdot B^{[t]} M^k,$$

and

$$\omega_{B^{[t]} M(t-1)} \cong O_{B^{[1]} M(t-1)} \left( \sum_{k=1}^{\mu(M(t-1))} \text{codim}(M(t-1)^k, M(t-1)) - 1 \right) \cdot B^{[t]} M(t-1)^k.$$ 

Condition 4 and equation (13) imply that the right hand side of (19) is the line-bundle

$$O_{B^{[t]} M^{t-1}} \left( \text{codim}(M^{t-1}, M) \cdot \sum_{k=t}^{\mu(M)} B^{[t]} M^k \right).$$

This completes the proof of Lemma 4. \hfill \Box

As a consequence of the lemma we see that $\pi_t : B^{[t]} M^t \to B^{[t+1]} M^t$ is identified with the projectivised homomorphism bundle $\mathbb{P} \text{Hom}(q_{B^{[t+1]} M^t}; \tau_{B^{[t+1]} M^t})$. 

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In fact, the following all denote the same space:
\[
B^{[t]} M^t = \mathbb{P} \text{Hom}(q_{B^{[t+1]} M^t}; \tau_{B^{[t+1]} M^t}).
\] (20)

Consequently, \(B^{[t]} M^t\) admits two canonical stratifications. One stratification is induced by that of \(B^{[t]} M\), namely,
\[
B^{[t]} M^t \cap \{ B^{[t]} M \supset B^{[t]} M^1 \supset \ldots \supset B^{[t]} M^{t-1} \}.
\]

The other stratification is the determinantal stratification indexed by the nullity of the homomorphisms
\[
\mathbb{P} \text{Hom}(q, \tau)^k := \{ \varphi \mid \min[\text{nullity}(\varphi), \text{nullity}(\varphi^*)] \geq k \}. \quad (21)
\]
The inequality \(\text{rank}(q_{B^{[t+1]} M^t}) = n(M) + t - 1 \geq t = \text{rank}(\tau_{B^{[t+1]} M^t})\) translates (21) to:
\[
\mathbb{P} \text{Hom}(q_{B^{[t+1]} M^t}; \tau_{B^{[t+1]} M^t})^k := \{ \varphi \mid \text{nullity}(\varphi) \geq k + n(M) - 1 \}, \ 0 \leq k \leq t - 1. \quad (22)
\]

**Condition 5** The two stratifications coincide scheme theoretically:
\[
\mathbb{P} \text{Hom}(q_{B^{[t+1]} M^t}; \tau_{B^{[t+1]} M^t})^k = (B^{[t]} M^t \cap B^{[t]} M^k), \ 0 \leq k \leq t - 1.
\]
(Compare Condition 4 with Theorem 7 part 5).

**Definition 5** A collection of stratified quasi-projective schemes
\[
\{ X(r)^t, f_{r,t} \mid 0 \leq r \leq \mu, \ 0 \leq t \leq \mu - r \}
\] satisfying all the conditions above will be called a symplectically dualizable stratified collection or a dualizable collection for short.

Note that if the collection (22) is dualizable, then the sub-collection obtained by deleting the top row is also dualizable.

**Theorem 6** Given a symplectically dualizable stratified collection \(\{ X(r)^t, f_{r,t} \}\) with \(X = X(0)^0\), there exists another dualizable collection \(\{ Y(r)^t, f'_{r,t} \}\) with \(Y = Y(0)^0\), \(n(Y) = n(X)\) and \(\mu(Y) = \mu(X)\) satisfying

1. The full iterated blow-ups are isomorphic
\[
\tilde{q}_r : B^{[1]} X(r) \xrightarrow{\sim} B^{[1]} Y(r), \ 0 \leq r \leq \mu.
\]

These isomorphisms restrict to isomorphisms on the exceptional divisors
\[
\tilde{q}_{r,t} : B^{[1]} X(r)^t \xrightarrow{\sim} B^{[1]} Y(r)^t, \ 0 \leq t \leq \mu - r.
\]
2. The two $\mathbb{P}^{n+2r-2}$-bundles $\mathbb{P}W_{B^1X(r)}$ and $\mathbb{P}W_{B^1Y(r)}$

\[
\tilde{f}_{r-1,1} : B^{[2]}X(r-1)^1 \to B^{[1]}X(r)
\]

\[
\tilde{f}_{r-1,1}^r : B^{[2]}Y(r-1)^1 \to B^{[1]}Y(r)
\]

are dual. In other words, $\bar{q}_r$ lifts to an isomorphism

\[
\begin{align*}
\bar{q}_r : \mathbb{P}W_{B^1X(r)} & \cong \mathbb{P}W_{B^1Y(r)}, \\
\end{align*}
\]

(24)

3. We have a commutative diagram of isomorphisms (for $1 \leq t \leq \mu - r$)

\[
\begin{array}{ccc}
B^{[1]}X(r)^t & \cong & (B^{[1]}\mathbb{P}^1G)(t, W_{B^1X(r+t)}) \\
\downarrow \bar{q}_{r,t} & & \downarrow \cong \\
B^{[1]}Y(r)^t & \cong & (B^{[1]}\mathbb{P}^1G)(t, W_{B^1Y(r+t)})
\end{array}
\]

where the right vertical isomorphism is the relative transposition isomorphism (see section 3.2).

We will refer to the collection \{$Y(r)^t$, $f'_{r,t}$\} as the dual collection. The proof of Theorem 6 is carried out in section 4.

3 Determinantal Varieties - Background

3.1 Blowing up determinantal ideals

We recall Vainsencher’s results about blowing up determinantal ideals. Let $V_0$ and $V_1$ be vector bundles over a scheme $M$ of finite type over $\mathbb{C}$. Given a vector bundle $E$ over $M$ we denote the tautological exact sequence over the Grassmanian bundle $G(i, E)$ by

\[
0 \to \tau_i(E) \to E_{G(i, E)} \to q_i(E) \to 0.
\]

Given another vector bundle $F$ we denote by $\mathcal{H}om(q_i(E), \tau_j(F))$ the corresponding vector bundle over $G(i, E) \times_M G(j, F)$. We index the determinantal loci in $\mathbb{P}\mathcal{H}om(V_0, V_1)$ as in (21) in terms of co-rank. (Inxeding by rank would simplify the notation in Theorem 7 but would be cumbersome in the context of Brill-Noether stratifications of moduli spaces of sheaves.) Denote the intersection $B^{[i]}\mathbb{P}\mathcal{H}om(V_0, V_1)^j \cap B^{[i]}\mathbb{P}\mathcal{H}om(V_0, V_1)^k$ by $B^{[i]}\mathbb{P}\mathcal{H}om(V_0, V_1)^{j \cap k}$.

Theorem 7 ([Vain] Theorem 2.4)
1. $B^{[i+1]} \mathbb{P} \text{Hom}(V_0, V_1)^i = B^{[i+1]} \mathbb{P} \text{Hom}(q_{r_0-r_1+i}(V_0), \tau_{r_1-i}(V_1))$ for $0 \leq i < r_1$.

2. Let $\mathcal{I}_i$ be the product of the invertible ideals $TB^{[i+1]} \mathbb{P} \text{Hom}(V_0, V_1)_k$, $k = i+1, \ldots, r_1 - 1$. Then the conormal sheaf $N^*_i$ of $B^{[i+1]} \mathbb{P} \text{Hom}(V_0, V_1)$ in $B^{[i+1]} \mathbb{P} \text{Hom}(V_0, V_1)$ is isomorphic to $\mathcal{I}_i \cdot \mathcal{O}(1) \otimes \text{Hom}(\tau_{r_0-r_1+i}(V_0), q_{r_1-i}(V_1))$.

3. Let $Z(i)$ be $G(r_0 - i, V_0) \times_M G(i, V_1)$. For $i \leq k$, $B^{[i]} \mathbb{P} \text{Hom}(V_0, V_1)_k = B^{[i]} \mathbb{P} \text{Hom}(\tau_{r_0-r_1+k}(V_0), q_{r_1-k}(V_1)) \times Z(r_1-k) B^{[1]} \mathbb{P} \text{Hom}(q_{r_0-r_1+k}(V_0), \tau_{r_1-k}(V_1))$. Moreover, under this identification, we have $N^*_{B^{[k]} \mathbb{P} \text{Hom}(V_0, V_1)} = \mathcal{O}(1) \otimes \mathcal{I}^{-1}$.

4. For $k > i \geq 0$, $B^{[i+1]} \mathbb{P} \text{Hom}(V_0, V_1)^{k\cap i} = B^{[i+2]} \mathbb{P} \text{Hom}(q_{r_0-r_1+i}(V_0), \tau_{r_1-i}(V_1))^{k-i} = B^{[i+1]} \mathbb{P} \text{Hom}(\tau_{r_0-r_1+k}(V_0), q_{r_1-k}(V_1))^i \times Z(r_1-k) B^{[2]} \mathbb{P} \text{Hom}(q_{r_0-r_1+k}(V_0), \tau_{r_1-k}(V_1))$.

5. $B^{[i]} \mathbb{P} \text{Hom}(V_0, V_1)^{k\cap i} = \mathbb{P} \text{Hom}(\tau_{r_0-r_1+i}(V_0), q_{r_1-i}(V_1))^i \times Z(r_1-i) B^{[1]} \mathbb{P} \text{Hom}(q_{r_0-r_1+i}(V_0), \tau_{r_1-i}(V_1))$ for $i > k$.

6. $(TB^{[i]} \mathbb{P} \text{Hom}(V_0, V_1)^k) \cdot \mathcal{O}_{B^{[i]} \mathbb{P} \text{Hom}(V_0, V_1)} = (TB^{[i]} \mathbb{P} \text{Hom}(V_0, V_1)^k) \cdot (TB^{[i]} \mathbb{P} \text{Hom}(V_0, V_1)^i)_{i+1}$ for $i > k$.

7. For $k > j$, $B^{[j-i]} \mathbb{P} \text{Hom}(V_0, V_1)^{k\cap j} = B^{[j-i]} \mathbb{P} \text{Hom}(\tau_{r_0-r_1+j}(V_0), q_{r_1-j}(V_1)) \times Z(j) B^{[2]} \mathbb{P} \text{Hom}(q_{r_0-r_1+j}(V_0), \tau_{r_1-j}(V_1))^{k-j} = B^{[j-i]} \mathbb{P} \text{Hom}(\tau_{r_0-r_1+k}(V_0), q_{r_1-k}(V_1))^j \times Z(r_1-k) B^{[2]} \mathbb{P} \text{Hom}(q_{r_0-r_1+k}(V_0), \tau_{r_1-k}(V_1))$.

The iterated blow-up $B^{[k+i]} \mathbb{P} \text{Hom}(V_0, V_1)^k$ has several beautiful modular interpretations: as a space of complete collineations, as a closure in the fiber product of Grassmannian bundles $[\text{Vain}, \text{Lak}_2, \text{KT}]$, and three new modular interpretations discovered in $[\text{Th}_2]$.

### 3.2 Transpositions

Let $H$ be an $h$-dimensional vector space. The cotangent bundle $T^*G(t, H)$ of the Grassmannian of $t$-dimensional subspaces of $H$ is isomorphic to the homomorphism bundle $\text{Hom}(q, \tau)$. We obtain a natural determinantal stratification of $\mathbb{P} T^*G(t, H)$. Similarly, we have a natural determinantal stratification of $\mathbb{P} T^*G(t, H^*)$. Assume that $h \geq 2t$. We construct in this section a canonical isomorphism

$$B^{[1]} \mathbb{P} T^*G(t, H) \cong B^{[1]} \mathbb{P} T^*G(t, H^*)$$

between the two iterated blow-ups. Notice that $\text{Flag}(t, h-t, H)$ and $\text{Flag}(t, h-t, H^*)$ are isomorphic and under this identification we have the isomorphisms $\tau_i(H) \cong q_{h-i}^*(H^*)$ and $\tau_h(H) \cong q_{h-i}^*(H^*)$. The birational transposition isomorphism “factors” through

$$\mathbb{P} T^*G(t, H) \leftarrow \mathbb{P} (\tau_i(H) \otimes \tau_i(H^*))_{\text{Flag}(t, h-t, H)} \rightarrow \mathbb{P} T^*G(t, H^*) . \quad (25)$$
The whole construction works in the relative setting in which $H$ is a vector bundle over a base $Z$. For each pair of integers $(t, j)$ satisfying $0 \leq t \leq \frac{h}{2}$ and $0 \leq j \leq t + 1$, we get

$$(B^{[j]} \mathbb{P}^1 G)(t, H) \to G(t, H) \to Z.$$ 

For $j = t$ and $j = t + 1$ we set

$$(B^{[t]} \mathbb{P}^1 G)(t, H) = (\mathbb{P}^1 G)(t, H), \quad \text{and}$$

$$(B^{[t+1]} \mathbb{P}^1 G)(t, H) = G(t, H).$$

For $0 \leq j \leq t - 1$, $(B^{[j]} \mathbb{P}^1 G)(t, H)$ is the iterated blow-up of the determinantal stratification of $(\mathbb{P}^1 G)(t, H)$.

**Proposition 8** $(B^{[1]} \mathbb{P}^1 G)(t, H) \cong B^{[1]} \mathbb{P} \left( [\tau_t(H^*) \otimes \tau_t(H)]_{Flag(t, h-t, H)} \right) \cong (B^{[1]} \mathbb{P}^1 G)(t, H^*)$.

**Proof:** Theorem 7 part 1 (with $i = 0$, $V_0 = q_t(H)$, and $V_1 = \tau_t(H)$) implies the identity

$$B^{[1]} \mathbb{P} \text{Hom}(q_t(H), \tau_t(H)) = B^{[1]} \mathbb{P} \text{Hom}(q_{h-2t}(q_t(H)), \tau_t(H)).$$

But $\text{Flag}(t, h-t, H)$ is isomorphic to $G(h-2t, q_t(H))$. \hfill \Box

We denote by $W_{(B^{[1]} \mathbb{P}^1 G)(t, H)}$ the rank $h-2t$ bundle over the full iterated blow-up $(B^{[1]} \mathbb{P}^1 G)(t, H)$ obtained by pulling back the bundle $\tau_{h-t}/\tau_t$ from $\text{Flag}(t, h-t, H)$.

Given a third integer $i$ satisfying $0 \leq i \leq t - 1$ we get the variety $(B^{[i]} \mathbb{P}^1 G)(t, H)^i$ which is the strict transform of the strata $(\mathbb{P}^1 G)(t, H)^i$ under the iterated blow-up. The following identity will be useful in section 4:

**Lemma 9** There is a natural isomorphism for $j \leq i + 1$

$$(B^{[j]} \mathbb{P}^1 G)(t, H)^i \cong (B^{[i]} \mathbb{P}^1 G)(i, W_{(B^{[1]} \mathbb{P}^1 G)(t-i, H)}).$$

In particular, for $j = i$ and $j = i + 1$ we have a commutative diagram in which the horizontal morphisms are isomorphisms

$$(B^{[i]} \mathbb{P}^1 G)(t, H)^i \xrightarrow{\cong} (\mathbb{P}^1 G)(i, W_{(B^{[1]} \mathbb{P}^1 G)(t-i, H)})$$

$$\begin{array}{c}
\downarrow \beta_{i+1} \\
(B^{[i+1]} \mathbb{P}^1 G)(t, H)^i \xrightarrow{\cong} G(i, W_{(B^{[1]} \mathbb{P}^1 G)(t-i, H)}).
\end{array}$$

\[26\]
Proof: Theorem 7 part 3 and the identity $G(t-i, \tau_i(H)) \times_Z G(h-2t+i, q_t(H)) = \text{Flag}(t-i, t, h-t+i, H)$, imply that $(B[i] \mathcal{P} \Omega^1 G)(t, H)$ is isomorphic to $B[i] \mathcal{P} \text{Hom}(\tau_{h-2t+i}(q_t(H)), q_t^{-1}(\tau_i(H))) \times_{\text{Flag}(t-i, t, h-4t+i, H)} B[i] \mathcal{P} \text{Hom}(q_{h-4t+i}(H), \tau_{t-i}(H))$. The latter is $B[i] \mathcal{P} \Omega^1 G \left( i, W_{B[i] \mathcal{P} \text{Hom}(q_{h-4t+i}(H), \tau_{t-i}(H))} \right)$. Theorem 7 part 3 (with $i = 0$) identifies $B[i] \mathcal{P} \text{Hom}(q_{h-4t+i}(H), \tau_{t-i}(H))$ with $(B[i] \mathcal{P} \Omega^1 G)(t-i, H)$. \hfill $\square$

Theorem 7 and Lemma 9 imply that the stratified collection

$$P \Omega^1 G(t, H) \supset P \Omega^1 G(t, H)^1 \supset \cdots \supset P \Omega^1 G(t, H)^{t-1}$$

is dualizable in a sense analogous to that of Definition 5. Proposition 8 identifies the dual collection as the one associated to $H^*$. Note, however, that the diagonal entries of (27) are (fibrations over the base scheme $Z$ of) contact (rather than symplectic) varieties.

Denote by $\tau'_G(t, H^*)$ and $q'_G(t, H^*)$ the tautological sub- and quotient bundles. Let $h := h_{(P \Omega^1 G)(t, H)}$ (resp. $h' := h_{(P \Omega^1 G)(t, H^*)}$) be the tautological line sub-bundle of $\beta^*_t \text{Hom}(q, \tau)$ (resp. $\beta^*_t \text{Hom}(q', \tau')$).

Lemma 10 Over $(B[i] \mathcal{P} \Omega^1 G)(t, H)$ we have a natural isomorphism

$$\beta^* h_{(P \Omega^1 G)(t, H)} \cong \beta^* h_{(P \Omega^1 G)(t, H^*)}.$$  

Proof: The bundle $\tau'_t \otimes \tau_t$ over $\text{Flag}(t, h-t, H)$ is a subbundle of the pullback of both $\text{Hom}(q, \tau)$ and $\text{Hom}(q', \tau')$. Working over the three spaces in (25) we see that both $h$ and $h'$ pullback to the tautological line subbundle of the pullback of $\tau'_t \otimes \tau_t$ to $\mathbb{P}[\tau'_t \otimes \tau_t]$. \hfill $\square$

The proof of Theorem 7 will require an extension of Proposition 8. If the base scheme $Z$ is a point, the cotangent bundle of $G(t, H)$ admits a non-trivial extension $E(H)$ (see [8]). The complement $[P E(H) \setminus (P \Omega^1 G)(t, H)]$ parametrizes $C^*$-orbits of idempotents and maps isomorphically onto the Zariski open subset in $G(t, H) \times G(h-t, H)$ of decompositions of $H$. We have a relative analogue of the extension (8) over any base $Z$. Over $G(t, H) \times_Z G(h-t, H)$ we have a canonical homomorphism

$$\alpha : \tau_t(H) \to q_{h-t}(H).$$

(28)

It is the composition of the injection $\tau_t(H) \hookrightarrow H$ with the projection $j : H \to q_{h-t}(H)$. If the pair $\tau_t(H)$ and $\tau_{h-t}(H)$ provides a decomposition of $H$, then $\alpha$ is an isomorphism and
\((j \circ \alpha^{-1})\) is the projection to \(\tau_t(H)\). The restriction of \(\alpha\) to the flag variety vanishes. The section \(\alpha\) is transversal to the (affine) determinantal stratification of the affine bundle \(\mathcal{H}om(\tau_t(H), q_{ht}(H))\). We get a determinantal stratification of \(G(t, H) \times_Z G(h-t, H)\).

**Proposition 11** There is a canonical isomorphism

\[
B^{[1]} \mathbb{P}E(H) \cong B^{[1]}[G(t, H) \times_Z G(t, H^*)] \cong B^{[1]} \mathbb{P}E(H^*).
\]

**Proof:** \(B^{[1]} \mathbb{P}E(H)\) is smooth because the stratification of \(\mathbb{P}E(H)\) is supported on its divisor \(\mathbb{P}T^*G(t, H)\). \(B^{[1]}[G(t, H) \times_Z G(t, H^*)]\) is smooth because each determinantal stratum in \(G(t, H) \times_Z G(t, H^*)\) is smooth away from the next stratum and has the expected dimension. If we regard \(\mathbb{P}E(H)\) as a subbundle of \(\mathbb{P}[H^* \otimes \tau_t]\), then Theorem 7 part 1 implies that there is a regular morphism from \(B^{[1]} \mathbb{P}E(H)\) to \(G(t, H) \times_Z G(t, H^*)\). An inductive application of the universal property of blowing-up implies that the morphism lifts to \(B^{[1]}[G(t, H) \times_Z G(t, H^*)]\). It is easy to check that the lift is an isomorphism away from codimension two. By purity of the ramification locus, the lift must be an isomorphism. \(\Box\)

### 3.3 The Petri map

In this section we assume that \(M\) is a smooth algebraic variety. Let

\[
e : V_0 \to V_1
\]

be a homomorphism of vector bundles over \(M\). Denote by \(r_i\) the rank of \(V_i\) and Assume that \(e\) is generically surjective (in particular the inequality \(r_0 \geq r_1\)). Let

\[
M = M^0 \supset M^1 \supset \cdots \supset M^\mu \supset \emptyset
\]

be the determinantal stratification. Over \(M^t \setminus M^{t+1}\) we have two vector bundles \(\ker(e|_{M^t \setminus M^{t+1}})\) and \(\coker(e|_{M^t \setminus M^{t+1}})\).

**Lemma 12** [F, ACGH]

1. There exists a canonical homomorphism of \(O_{M^t \setminus M^{t+1}}\)-modules

\[
\phi : TM \otimes \ker(e|_{M^t \setminus M^{t+1}}) \to \coker(e|_{M^t \setminus M^{t+1}}).
\]

Given a point \(f\) in the fiber of \(\ker(e|_{M^t \setminus M^{t+1}})\) at \(x \in M^t \setminus M^{t+1}\) and a tangent vector \(\xi \in T_x M|_{M^t \setminus M^{t+1}}\), their image \(\phi(\xi \otimes f)\) is the first order infinitesimal obstruction to extending \(f\) as a section of \(\ker(e)\) in the direction of \(\xi\).
2. If $M^t$ has the expected dimension and
\[ G(r_0 - r_1 + t, M^t) := \{(x, W) \mid W \in G(r_0 - r_1 + t, \ker(e_x))\} \]
is smooth, then the tangent cone of $M^t$ at $x \in M^k \setminus M^{k+1}$, $k \geq t$, is the subscheme of $T_xM$ of zeroes of the section $k+1-t \wedge \Phi_x$. Here we interpret $\phi_x$ as a section $\Phi_x$ of the vector bundle $\Hom(\ker(e_x), \coker(e_x)) \otimes O_{T_xM}$ over $T_xM$.

3. As a section of
\[ \ker(e_{|M^t\setminus M^{t+1}}^*}) \otimes \coker(e_{|M^t\setminus M^{t+1}}^*)^* \to T^*M_{|M^t\setminus M^{t+1}}, \]
$\phi$ is equal to the section obtained from the dual homomorphism $e^* : V_1^* \to V_0^*$.

Part 2 of the Lemma indicates that the expected dimension of $M^t$ is
\[ \rho(t) := \dim(M) - t(r_0 - r_1 + t). \] (32)

### 3.4 Blowing up the smallest stratum

Let
\[ \beta_\mu : B^\mu M^t \to M^t, \ 0 \leq t \leq \mu - 1, \] (33)
be the blow-up of $M^t$ along $M^\mu$. Pulling back (29) we get the homomorphism
\[ \beta_\mu^*(e) : \beta_\mu^*V_0 \to \beta_\mu^*V_1. \] (34)

It defines a determinantal stratification
\[ B^\mu M = \beta_\mu^{-1}M^0 \supset \beta_\mu^{-1}M^1 \supset \cdots \supset \beta_\mu^{-1}M^{\mu} \supset \emptyset \]
which is the total-transform of the stratification (30).

Next we perform an elementary transformation of (34) whose determinantal stratification is the strict-transform of (30). Let
\[ B^\mu V_0 := \beta_\mu^*V_0 \]
\[ B^\mu V_1 := \ker(\beta_\mu^*V_1 \to \beta_\mu^*\coker(e_{|M^\mu})). \]
$E := \beta_\mu^{-1}M^\mu$ is a Cartier divisor on $B^\mu M$ and $\beta_\mu^*(e)$ has constant rank over $E$. Thus, $B^\mu V_1$ is locally free. We get a natural homomorphism
\[ B^\mu(e) : B^\mu V_0 \to B^\mu V_1. \] (35)

It defines a determinantal stratification
\[ B^\mu M = B^\mu M^0 \supset B^\mu M^1 \supset \cdots \supset B^\mu M^{\mu-1} \supset (B^\mu M)^\mu(B^\mu e) \supset \emptyset. \] (36)

If $\mu = 1$ and $n(M) = 1$ then $M^1$ is already a divisor on $M$. In that case $\beta_\mu : B^\mu M \to M$ is the identity, however $B^\mu V_1$ is different from $V_1$ and $B^\mu(e)$ is different from $e$. 

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Lemma 13 The subschemes $B^\mu M^t$, $0 \leq t \leq \mu - 1$ in the stratification (30) are the strict transforms of the subschemes $M^t$ in the stratification (30).

In general, the subscheme $(B^\mu M)^\mu(B^\mu e)$ may or may not be empty.

Proof: The Lemma holds for a general determinantal stratification over a scheme of finite type over $\mathbb{C}$ (without assuming smoothness). The Lemma is a reformulation of Theorem 7 part 6. The homomorphism $e$ fits in a commutative diagram:

$$
\begin{array}{ccc}
B^\mu M & \xrightarrow{\widetilde{P}(e)} & B^\mu \mathbb{P}\text{Hom}(V_0, V_1) \\
\beta \downarrow & & \downarrow \\
M & \xrightarrow{P(e)} & \mathbb{P}\text{Hom}(V_0, V_1)
\end{array}
$$

(37)

Above, $P(e)$ is a closed immersion and its image is disjoint from the loci $\mathbb{P}\text{Hom}(V_0, V_1)^i$ for all $i > \mu$. The section $e$ is the pullback of the tautological section $\tilde{e}$. The morphism $\widetilde{P}(e)$ is the canonical closed immersion (see [Ha] Corollary 7.15 page 165). The determinantal ideal-sheaves on $M$ and and their strict and total transforms on $B^\mu M$ are inverse images of the corresponding ideals on $\mathbb{P}\text{Hom}(V_0, V_1)$ and $B^\mu \mathbb{P}\text{Hom}(V_0, V_1)$ via the closed immersions $P(e)$ and $\widetilde{P}(e)$. Hence, it suffices to prove the Lemma with $M$ equal to the complement of $\mathbb{P}\text{Hom}(V_0, V_1)^{\mu+1}$ in the bundle $\mathbb{P}\text{Hom}(V_0, V_1)$ (renaming $V_1 \otimes O(1)$ by $V_1$ and $\tilde{e}$ by $e$).

The determinantal locus $M^t$ is the zero subscheme of $r_1 - t + 1 \wedge e$. Its ideal sheaf $\mathcal{I}_{M^t}$ is the image of

$$
(r_1 - t + 1 \wedge V_0) \otimes (r_1 - t + 1 \wedge V_1)^* \xrightarrow{r_1 - t + 1 \wedge e} \mathcal{O}_M.
$$

The ideal of the total transform $\beta^{-1}\mathcal{I}_{M^t}$ is the image of $r_1 - t + 1 \wedge \beta^*(e)$. Let $\mathcal{I}_t(B^\mu e)$ be the ideal sheaf which is the image of $r_1 - t + 1 \wedge B^\mu e$ in $\mathcal{O}_{B^\mu M}$ and let $\mathcal{I}_{B^\mu M^t}$ be the ideal of the strict transform $B^\mu M^t$ of $M^t$. We need to prove the equality $\mathcal{I}_{B^\mu M^t} = \mathcal{I}_t(B^\mu e)$. The equality follows from the two equalities

$$
\beta^{-1}\mathcal{I}_{M^t} = \mathcal{I}_t(B^\mu e) \cdot (\mathcal{I}_E)^{\mu-t+1} \quad \text{and} \quad \beta^{-1}\mathcal{I}_{M^t} = \mathcal{I}_{B^\mu M^t} \cdot (\mathcal{I}_E)^{\mu-t+1}.
$$

The first equality follows from a short local consideration. The second equality follows from Theorem 7 part 6. \hfill $\square$

Denote by

$$
E^t := E \cap B^\mu M^t, \quad 0 \leq t \leq \mu,
$$

the induced stratification of $E$. We describe below the determinantal stratification (10) of the exceptional divisor under the assumption that the smallest determinantal locus
$M^\mu$ is smooth and of the expected dimension. We will see that the determinantal stratification (36) of the blown-up homomorphism $B^\mu(e)$ is shorter than the determinantal stratification of $e$. In other words, $(B^\mu M)^\mu(B^\mu e)$ is empty.

Assume that $M$ is smooth and the lowest strata $M^\mu$ of (30) is smooth of the expected codimension $\mu(r_0 - r_1 + \mu)$. By Lemma 12. The Petri map of $e$ is an isomorphism

$$\phi : \text{Hom}(\ker(e|_{M^\mu}), \text{coker}(e|_{M^\mu})) \xrightarrow{\cong} N^*_{M^\mu/M}. \quad (38)$$

In particular, the exceptional divisor

$$E := \beta^{-1}_\mu M^\mu$$

in $B^\mu M$ is isomorphic to the projectivized homomorphism bundle

$$E \cong \mathbb{P}\text{Hom}(\ker(e|_{M^\mu}), \text{coker}(e|_{M^\mu})). \quad (39)$$

The exceptional divisor $E$ admits two determinantal stratifications:

$$E \cap \{B^\mu M = B^\mu M^0 \supset B^\mu M^1 \supset \cdots \supset B^\mu M^{\mu-1} \supset (B^\mu M)^\mu(B^\mu e)\} \quad \text{and} \quad (40)$$

$$E^k := \{\eta \mid \text{nullity}(\eta) \geq k + r_0 - r_1\}. \quad (41)$$

The equality of the two stratifications (40) and (41) is proven in Theorem 7 part 5 (see also Lemma 14 part 6 below). Denote by $\mathcal{O}_{\mathbb{P}\text{Hom}(\ker(e|_{M^\mu}), \text{coker}(e|_{M^\mu}))}(1)$ the relative ample line-bundle and let

$$\eta : \beta^*_\mu \ker(e|_{M^\mu}) \to (\beta^*_\mu \text{coker}(e|_{M^\mu}))(1) \quad (42)$$

be the tautological homomorphism.

**Lemma 14** 1. The two line bundles $\mathcal{O}_{\mathbb{P}\text{Hom}(\ker(e|_{M^\mu}), \text{coker}(e|_{M^\mu}))}(1)$ and $\mathcal{O}_E(-E)$ are canonically isomorphic.

2. The composition

$$\beta^*V_{0|E} \xrightarrow{B^\mu e} B^\mu V_{1|E} \to \text{Im}(\beta^*e|_E) \cong \beta^*V_{0|E}/\ker(\beta^*e|_E)$$

is the natural projection.

3. The restriction of $B^\mu(e)$ to the subbundle $\beta^*\ker(e|_{M^\mu})$

$$B^\mu(e)|_{\ker(\eta)} : \beta^*\ker(e|_{M^\mu}) \rightarrow \beta^*\text{coker}(e|_{M^\mu})(-E) \quad (43)$$

is equal to the composition of the Petri map with the codifferential of $\beta$

$$\beta^*\ker(e|_{M^\mu}) \xrightarrow{\beta^*\phi} \beta^*\text{coker}(e|_{M^\mu}) \otimes N^*_{M^\mu/M} \xrightarrow{d^*\beta} \beta^*\text{coker}(e|_{M^\mu})(-E).$$

We used above the identification

$$\mathcal{O}_E(-E) \cong N^*_{E/B^\mu M}. \quad 24$$
4. The pullback $\beta^* \phi$ of the Petri map $\phi$ of $e$ is related to $\eta$ by the commutative diagram:

$$
\begin{array}{c}
\beta^* \ker(e_{|_{M^\mu}}) \otimes \beta^* \text{coker}(e_{|_{M^\mu}})^* \xrightarrow{\beta^* \eta} \beta^* \left[ N_{M^\mu/M}^* \right] \\
\downarrow \eta \\
\mathcal{O}_E(-E).
\end{array}
$$

5. Over $E$ we have a canonical identifications

$$
coker(B^\mu e_{|_E}) = \text{coker} \left[ \eta : \ker(\beta^* e_{|_E}) \to \text{coker}(\beta^* e_{|_E}) \otimes (-E) \right],
$$

$$
\ker(B^\mu e_{|_E}) = \ker \left[ \eta : \ker(\beta^* e_{|_E}) \to \text{coker}(\beta^* e_{|_E}) \otimes (-E) \right].
$$

6. The two stratifications (41) and (40) of $E$ coincide

$$
\mathcal{P}\text{Hom(ker}(e_{|_{M^\mu}}), \text{coker}(e_{|_{M^\mu}}))^{\dagger} = E^t.
$$

In particular, $(B^\mu M)^\mu(B^\mu e)$ is empty and $B^\mu(e)$ has rank $\geq \text{rank}(V_1) + 1 - \mu$ throughout $B^\mu M$.

**Proof:** [4] The line-bundle $N_{E/B^\mu M}$ is the tautological line-sub-bundle of $\beta^* N_{M^\mu/M}$, while the line-bundle $\mathcal{O}_{\mathcal{P}\text{Hom}(\ker(e_{|_{M^\mu}}), \text{coker}(e_{|_{M^\mu}}))}(-1)$ is the tautological line-sub-bundle of $\beta^* \text{Hom}(\ker(e_{|_{M^\mu}}), \text{coker}(e_{|_{M^\mu}}))$. We assumed that the two vector bundles $N_{M^\mu/M}$ and $\text{Hom}(\ker(e_{|_{M^\mu}}), \text{coker}(e_{|_{M^\mu}}))$ are canonically isomorphic. Hence, so do the two tautological line-sub-bundles of their pull-backs.

[3] is clear.

[4] Notice first that $B^\mu V_{1|_E}$ is an extension

$$
0 \to \beta^* \text{coker}(e_{|_{M^\mu}})(-E) \to B^\mu V_{1|_E} \to \beta^* \text{Im}(e_{|_{M^\mu}}) \to 0.
$$

Clearly, $B^\mu(e)$ maps $\beta^* \ker(e_{|_{M^\mu}})$ to $\beta^* \text{coker}(e_{|_{M^\mu}})(-E)$. Tracing through the definition of $\phi$ shows the desired equality.

[4] $\eta^*$ is the embedding of $N_{E/B^\mu M}^*$ as a line-sub-bundle of $\beta^* N_{M^\mu/M}$ via $d^* \beta$.

[4] Follows immediately from parts [2], [3] and [4].

[3] Part [4] implies that, over $E$, the ranks of $\beta^* e$ and $B^\mu(e)$ are related by

$$
\text{rank}(B^\mu e) = \text{rank}(\beta^* e) + \text{rank}(\eta).
$$

Hence the two stratifications coincide set theoretically. Moreover, the equality of the cokernels of $\eta$ and $B^\mu(e)_{|_E}$ implies that both stratifications are the determinantal stratifications of a locally free presentation of the same sheaf on $E$.  

\[ \square \]
Our construction of the elementary transform $B^{\mu}e : B^{\mu}V_0 \to B^{\mu}V_1$ was not symmetric in $V_0$ and $V_1$. If we start instead with the dual homomorphism $e^* : V_1^* \to V_0^*$, its blow-up

$$B^{\mu}(e^*) : B^{\mu}V_1^* \to B^{\mu}V_0^*$$

maps the pullback $B^{\mu}V_1^* := \beta^*(V_1^*)$ to the elementary transform $B^{\mu}V_0^*$ of $\beta^*(V_0^*)$. Combining Lemma 14 part 5 with its analogue for $B^{\mu}(e^*)$ restores the symmetry:

**Corollary 15**

1. We have canonical isomorphisms:

$$\ker \left( \left. B^{\mu}(e^*) \right|_{B^{\mu}M \setminus \left( B^{\mu}M + 1 \right)} \right) \cong \coker \left( \left. B^{\mu}(e) \right|_{B^{\mu}M \setminus \left( B^{\mu}M + 1 \right)} \right)^* (-E),$$

$$\coker \left( \left. B^{\mu}(e^*) \right|_{B^{\mu}M \setminus \left( B^{\mu}M + 1 \right)} \right) \cong \ker \left( \left. B^{\mu}(e) \right|_{B^{\mu}M \setminus \left( B^{\mu}M + 1 \right)} \right)^* (-E).$$

Moreover, the sheaves $\coker((Be^*)^*(-E))$ and $\coker(B^{\mu}(e))$ are isomorphic globally over $B^{\mu}M$. Similarly, the sheaves $\coker((Be^*)^*(E))$ and $\coker((Be)^*)$ are isomorphic globally over $B^{\mu}M$.

2. The Petri maps of $B^{\mu}(e)$ and $B^{\mu}(e^*)$ are equal

$$\phi_e = \phi_{e^*} : \ker \left( \left. (B^{\mu}e) \right|_{B^{\mu}M \setminus \left( B^{\mu}M + 1 \right)} \right) \otimes \coker \left( \left. (B^{\mu}e) \right|_{B^{\mu}M \setminus \left( B^{\mu}M + 1 \right)} \right)^* \to T^*B^{\mu}M \mid_{B^{\mu}M \setminus \left( B^{\mu}M + 1 \right)}.$$

**Proof:** We have an injective sheaf homomorphism of short exact sequences:

$$0 \to (\beta^*V_1)^* \to (B^{\mu}V_1)^* \to \left[ \beta^* \coker(e_{|M'}) \right]^* (E) \to 0$$

$$0 \xrightarrow{B(e^*)} B^{\mu}(V_0^*) \xrightarrow{B(e)^*} \beta^*V_0^* \xrightarrow{\eta^*} \beta^* \ker(e_{|M'})^* \to 0.$$

We get the quotient short exact sequence

$$0 \to \coker(B(e^*)) \to \coker((Be)^*) \to \coker(\eta^*) \to 0.$$

Lemma 14 part 3 (applied to $e^*$) implies that the surjective homomorphism factors through an isomorphism

$$\coker((Be)^*) \cong \coker(\eta^*).$$

Hence, $\coker(B(e^*))$ is isomorphic to $\coker((Be)^*) \otimes \mathcal{O}_{B^{\mu}M}(-E)$ as sheaves over $B^{\mu}M$.

By Lemma 12 part 3 it suffices to prove that the Petri maps of $B^{\mu}(e)$ and $B^{\mu}(e^*)$ are equal. Part 1 implies that the two Petri maps arise from two locally free presentations of the same sheaf (up to a twist by the line bundle $\mathcal{O}_{B^{\mu}M}(E)$).
4 Construction of the dual collection

We have collected the background material on determinantal varieties necessary for the proofs of Theorems 3 and 2. We prove Theorem 3 is section 4.2.

4.1 Proof of Theorem 6

The main facts needed for the proof are the existence of the relative transposition isomorphism (Proposition 8) and Lemma’s 9 and 10. Lemma 10 implies that the normal bundle of the exceptional divisor $B^{[k]}Y(r)^k$ (which is to be blown-down next) restricts as $\mathcal{O}(-1)$ to the fibers of the ruling of $B^{[k]}Y(r)^k$. It follows that we can blow-down $B^{[k]}Y(r)$ along $B^{[k]}Y(r)^k$. Lemma 9 is needed in order to identify the affect of this blow-down on the remaining exceptional divisors.

**Proposition 16** We construct, recursively with respect to $k$:

1. Schemes $B^{[k]}Y(r)^t$, parametrized by integers $(r, k, t)$ in the ranges $0 \leq r \leq \mu(X)$, $1 \leq k \leq \mu(X(r)) + 1$ and $0 \leq t \leq \mu(X(r))$. We denote $B^{[\mu(X(r)) + 1]}Y(r)^t$ also by $Y(r)^t$ and $Y(r)^0$ by $Y(r)$.

2. Morphisms

\[ \tilde{f}_{r,k-1} : B^{[k]}Y(r)^{k-1} \to B^{[k]}Y(r + k - 1), \]

and

3. blow-down morphisms

\[ \beta_{k-1}^{r} : B^{[k-1]}Y(r)^t \to B^{[k]}Y(r)^t, \quad 1 \leq k \leq \mu(X(r)) + 1, \]

satisfying

1. $B^{[1]}Y(r)^t := B^{[1]}X(r)^t$, for $0 \leq r \leq \mu(X(r))$ and $0 \leq t \leq \mu(X(r))$. We define $\mathbb{P}W_{B^{[1]}Y(j)}$ to be the dual $\mathbb{P}W^*_{B^{[1]}X(j)}$ of $\mathbb{P}W_{B^{[1]}Y(j)}$.

2. $B^{[k]}Y(r)$ is a smooth variety.

3. $B^{[k]}Y(r)^t$ is the scheme theoretic image of $B^{[k-1]}Y(r)^t$ via

\[ \beta_{k-1}^{r} : B^{[k-1]}Y(r)^t \to B^{[k]}Y(r). \]

4. If $k \leq t$, then $B^{[k]}Y(r)^t$ is a smooth divisor in $B^{[k]}Y(r)$.

5. The canonical line-bundle of $B^{[k]}Y(r)$ is given by

\[ \omega_{B^{[k]}Y(r)} \cong \mathcal{O}_{B^{[k]}Y(r)} \left( \sum_{t=k}^{\mu(X(r))} \left[ \text{codim}(X(r)^t, X(r)) - 1 \right] \cdot B^{[k]}Y(r)^t \right). \quad (45) \]
6. If \( t \geq k \) then we have an isomorphism

\[
B^{[k]}Y(r)^t \cong (B^{[k]}[\mathbb{P}\Omega^1 G](t, W_{B^1[Y(r+t)]}))
\]

(46)

inducing isomorphisms of the exceptional divisors in \( B^{[k]}Y(r)^t \)

\[
B^{[k]}Y(r)^{t\cup j} \cong (B^{[k]}[\mathbb{P}\Omega^1 G](t, W_{B^1[Y(r+t+j)]}^j)) \quad \text{for } j \geq k.
\]

(47)

Moreover, we have a commutative diagram of blow-down morphisms

\[
\begin{array}{ccc}
B^{[k-1]}Y(r) & \xrightarrow{\beta'_k} & B^{[k]}Y(r) \\
\cup & \uparrow & \uparrow \\
B^{[k-1]}Y(r)^t & \cong & B^{[k]}Y(r)^t \\
\downarrow & \equiv & \downarrow \\
(B^{[k-1]}[\mathbb{P}\Omega^1 G](t, W_{B^1[Y(r+t)]})) & \xrightarrow{\beta'_k} & (B^{[k]}[\mathbb{P}\Omega^1 G](t, W_{B^1[Y(r+t)]})) \\
\downarrow & \equiv & \downarrow \\
B^1[Y(r+t)] & \cong & B^1[Y(r+t)].
\end{array}
\]

(48)

If \( t = k - 1 \), then we have an isomorphism

\[
B^{[k]}Y(r)^{k-1} \cong G(k-1, W_{B^1[Y(r+k-1)]}^1).
\]

(49)

7. For \( t \geq k \), the normal bundle \( N_{B^{[k]}Y(r)^t/B^{[k]}Y(r)} \) is isomorphic to

\[
\beta'^* \left\{ h_{B^{[k]}Y(r)^t} \otimes (\tilde{f}'_{r,t} \circ \beta'_t)^* \mathcal{O}_{B^1[X(r+t)]} \left( - \sum_{i=1}^{\mu(X(r+t))} B^1[X(r+t)^i] \right) \right\}.
\]

(50)

Above, the line bundle

\[
h_{B^{[k]}Y(r)^t}
\]

(51)
on \( B^{[k]}Y(r)^t \) is the tautological sub-bundle of

\[
(\beta'_t)^* \left( \text{Hom}(q_{B^{[k+1]}Y(r)^t}, \tau_{B^{[k+1]}Y(r)^t}) \right)
\]

and \( \mathbb{P}\tau_{B^{[k+1]}Y(r)^t}, \mathbb{P}q_{B^{[k+1]}Y(r)^t} \) are the sub- and quotient bundles of \( (\tilde{f}'_{r,t})^* (\mathbb{P}W_{B^1[Y(r+t)]}^1) \). (Here \( B^{[k+1]}Y(r)^t \) involves the abuse of notation introduced in Remark 17). Note that although only the projectivization of \( W_{B^1[Y(r+t)]} \) is defined, the vector bundle \( \text{Hom}(q_{B^{[k+1]}Y(r)^t}, \tau_{B^{[k+1]}Y(r)^t}) \) is well defined.
Remark 17 The varieties $B^{[k]}Y(r)^t$, being subvarieties of $B^{[k]}Y(r)$, are not defined until $B^{[k]}Y(r)$ is constructed. Nevertheless, we will sometimes (ab)use the notation $B^{[k]}Y(r)^t$, with $t \geq k - 1$, even before $B^{[k]}Y(r)$ is constructed. In that case, $B^{[k]}Y(r)^t$ will only refer to the variety $(B^{[k]}\mathbb{P}^{1}G)(t, W_{B^{[1]}Y(r+t)})$ without claiming yet that the latter is a subvariety of $B^{[k]}Y(r)$. The isomorphism between $(B^{[k]}\mathbb{P}^{1}G)(t, W_{B^{[1]}Y(r+t)})$ and $B^{[k]}Y(r)^t$ is proven once $B^{[k]}Y(r)$ is constructed. (see (18) and (49)).

Let us first compute the normal-bundles to the exceptional divisors in the iterated blow-ups of $X(r)$.

Lemma 18 Assume that $i \leq t$. Let $\beta : B^{[i]}X(r)^t \to B^{[i]}X(r)^t$ be the iterated blow-up. It has been identified as the iterated blow-up

$$\beta : (B^{[i]}\mathbb{P}^{1}G)(t,W_{B^{[1]}X(r+t)}) \to (\mathbb{P}^{1}G)(t,W_{B^{[1]}X(r+t)}).$$

Moreover, $\beta_t : B^{[i]}X(r)^t \to B^{[i+1]}X(r)^t$ is the bundle map $(\mathbb{P}^{1}G)(t,W_{B^{[1]}X(r+t)}) \to G(t,W_{B^{[1]}X(r+t)})$ and $f_{r,t}$ is the bundle map $f_{r,t} : G(t,W_{B^{[1]}X(r+t)}) \to B^{[i]}X(r + t)$. Let $\beta' : (B^{[i]}\mathbb{P}^{1}G)(t,W_{B^{[1]}Y(r+t)}) \to (\mathbb{P}^{1}G)(t,W_{B^{[1]}Y(r+t)})$

(resp. $\beta'_t, f'_{r,t}$) be the analogous morphism with respect to the dual vector bundle $W_{B^{[1]}Y(r+t)}$. Denote by $h_{B^{[i]}X(r)^t}$ the tautological line sub-bundle analogous to (51). Then, we have the following isomorphisms:

1. $N_{B^{[i]}X(r)^t}/B^{[i]}X(r)$ is isomorphic to

$$\beta^* \left\{ h_{B^{[i]}X(r)^t} \otimes (f_{r,t} \circ \beta_t)^* \left( \mathcal{O}_{B^{[i]}X(r+t)} \left[ - \sum_{k=1}^{\mu(X(r+t))} B^{[1]}X(r + t)^k \right] \right) \right\}$$

2. Over the top iterated blow-up, $N_{B^{[i]}Y(r)^t}/B^{[i]}Y(r)$ (which is $N_{B^{[i]}X(r)^t}/B^{[i]}X(r)$) is also isomorphic to

$$\beta'^* \left\{ h_{B^{[i]}Y(r)^t} \otimes (f'_{r,t} \circ \beta'_t)^* \left( \mathcal{O}_{B^{[i]}Y(r+t)} \left[ - \sum_{k=1}^{\mu(X(r+t))} B^{[1]}Y(r + t)^k \right] \right) \right\}.$$

Note that $h_{B^{[i]}Y(r)^t}$ stands for $h_{(\mathbb{P}^{1}G)(t,W_{B^{[1]}X(r+t)})}$ (see Remark (17)).

Proof: [1] The line bundle $N_{B^{[i]}X(r)^t}/B^{[i]}X(r)$ is the tautological line sub-bundle of the vector bundle $\beta_t^* \left( N_{B^{[i+1]}X(r)^t}/B^{[i+1]}X(r) \right)$ and Lemma [4] identifies the latter as the homomorphism bundle

$$\beta_t^* \left[ \text{Hom}(q_{B^{[k+1]}X(r)^t}, 7_{B^{[k+1]}X(r)^t}) \left[ - \sum_{k=t+1}^{\mu(X(r))} B^{[k]}X(r)^k \right] \right].$$

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Hence, $N_{B^{[i]}X(r)^t/B^{[i]}X(r)}$ is isomorphic to $[h_{B^{[i]}X(r)^t} \otimes \left( - \sum_{k=t+1}^{\mu(X(r))} B^{[i]}X(r)^k \right)]$. The morphism $\tilde{f}_{r,t}$ is compatible with respect to the stratifications up to a shift of indices by $t$ (Condition [3]). The case $i = t$ of the lemma follows. The case $1 \leq i < t$ is an immediate consequence of the case $i = t$ since the normal line-bundle $N_{B^{[i]}X(r)^t/B^{[i]}X(r)}$ is simply $\mathcal{O}_{B^{[i]}X(r)^t}(B^{[i]}X(r)^t)$.

[2] Follows immediately from part [1] and the isomorphism $\beta^*(h_{B^{[i]}X(r)^t}) \cong (\beta')^*(h_{B^{[i]}Y(r)^t})$ which is proven in Lemma [14].

**Proof of Proposition [16]:** The proof is by induction on $k$.

The case $k = 1$: The morphisms $\tilde{f}_{r,0}$ and $\beta'_0$ are the identity morphisms. Properties [1] and [3] are definitions. Properties [2], [4], and [5] are clear since $B^{[1]}Y(r)^t = B^{[1]}X(r)^t$. In Property [3] the isomorphism (10) is the transposition isomorphism when $k = 1$ since $\mathbb{P}W_{B^{[1]}Y(r+t)}$ is $\mathbb{P}W^*_{B^{[1]}X(r+t)}$ (see Proposition [3]).

The commutativity of Diagram (48) is clear since $\beta'_0$ is the identity morphism. Property [4] is verified in Lemma [18].

**Induction Step:** Assume that $B^{[j]}Y(r)^t$, $\tilde{f}_{r,j-1}$, and $\beta'_{j-1}$ were defined for $j \leq k$ and that they satisfy the properties of the proposition. We need to construct $B^{[k+1]}Y(r)^t$, $\tilde{f}_{r,k} : B^{[k+1]}Y(r)^k \to B^{[1]}Y(r + k)$, and $\beta'_k : B^{[k]}Y(r)^t \to B^{[k+1]}Y(r)^t$ and prove that they satisfy the properties of the proposition. We first construct the blowing down of $B^{[k]}Y(r)$ along the exceptional divisor $B^{[k]}Y(r)^k$ (see (54) below). By the induction hypothesis (property [3]), $B^{[k]}Y(r)^k$ is isomorphic to $(\mathbb{P}\Omega^1G)(k, W_{B^{[1]}Y(r+k)})$. Define

\[ \beta'_k : B^{[k]}Y(r)^k \to G(k, W_{B^{[1]}Y(r+k)}) \quad \text{and} \quad \tag{52} \]

\[ \tilde{f}_{r,k} : G(k, W_{B^{[1]}Y(r+k)}) \to B^{[1]}Y(r + k) \quad \text{as the natural projections. (We will prove below that $B^{[k+1]}Y(r)^k$ is isomorphic to $G(k, W_{B^{[1]}Y(r+k)})$).} \]

Property [3] and the induction hypothesis imply that $N_{B^{[k]}Y(r)^k}$ is isomorphic to

\[ h_{B^{[k]}Y(r)^k} \otimes (\tilde{f}_{r,k} \circ \beta'_k)^* \left[ \mathcal{O}_{B^{[1]}Y(r+k)} \left( - \sum_{i=1}^{\mu(X(r+k))} B^{[1]}Y(r + k)^i \right) \right]. \]

In particular, $N_{B^{[k]}Y(r)^k}$ restricts as $\mathcal{O}(-1)$ to each fiber of $\beta'_k$ (which is a projective space $\mathbb{P}^{\text{codim}(X(r)^k, X(r) - 1)}$). Hence, there exists a smooth projective variety $B^{[k+1]}Y(r)$ and a morphism

\[ \beta'_k : B^{[k]}Y(r) \to B^{[k+1]}Y(r) \quad \text{(54)} \]

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which is an isomorphism away from \(B^{[k]}Y(r)^k\) and contracts the fibers of (52) (see [\(\mathbb{P}\mathbb{N}\)]). We get that the image \(B^{[k+1]}Y(r)^k\) of \(B^{[k]}Y(r)^k\) is isomorphic to \(G(k, W_{B^{[1]}Y(r+k)})\). Properties \(\mathbb{P}\mathbb{P}\) and \(\mathbb{P}\mathbb{P}\) follow.

**Verification of Property \(\mathbb{P}\mathbb{P}\):** The induction hypothesis provides the isomorphisms
\[
B^{[k]}Y(r)^t \cong (B^{[k]}[\mathbb{P}\Omega^1 G](t, W_{B^{[1]}Y(r+t)}), \ t \geq k \text{ and } B^{[k]}Y(r)^{k\cap t} \cong (B^{[k]}[\mathbb{P}\Omega^1 G](t, W_{B^{[1]}Y(r+t)}), \text{ for } t > k
\]
(see (47) and (46)). Lemma 9 implies that the following diagram is commutative

\[
\begin{array}{ccc}
B^{[k]}Y(r)^k & \xrightarrow{\cong} & (\mathbb{P}\Omega^1 G)(k, W_{B^{[1]}Y(r+k)}) \\
\cup \uparrow & & \cup \uparrow \\
B^{[k]}Y(r)^{k\cap t} & \xrightarrow{\cong} & (\mathbb{P}\Omega^1 G)(k, W_{B^{[1]}Y(r+k)}|_{B^{[1]}Y(r+k)^{t\cap k}}) \\
\Downarrow & & \Downarrow \\
B^{[k]}Y(r)^{k\cap t} & \xrightarrow{\cong} & (B^{[k]}[\mathbb{P}\Omega^1 G](t, W_{B^{[1]}Y(r+t)}^k) \\
\cap \Downarrow & & \cap \Downarrow \\
B^{[k]}Y(r)^t & \xrightarrow{\cong} & (B^{[k]}[\mathbb{P}\Omega^1 G](t, W_{B^{[1]}Y(r+t)}^k) \\
\end{array}
\]

(Diagram (24) with \(\mathbb{P}H = \mathbb{P}W_{B^{[1]}Y(r+t)}\) is the middle right-hand square in diagram (55)). The morphism \(\delta_k\) is precisely the restriction of \(\beta^\prime_k\) defined above (54). We see that both \(\beta^\prime_k : B^{[k]}Y(r)^t \to B^{[k+1]}Y(r)^t\) and

\[
\alpha_k : B^{[k]}Y(r)^t \to (B^{[k+1]}[\mathbb{P}\Omega^1 G](t, W_{B^{[1]}Y(r+t)}^k) \]

(see diagram (55)) are isomorphisms away from the exceptional divisor \(B^{[k]}Y(r)^{k\cap t}\) and both are the blow-down morphisms determined by the same ruling of \(B^{[k]}Y(r)^{k\cap t}\), namely, the contraction \(\gamma_k\) in diagram (55). Hence, the scheme theoretic image \(B^{[k+1]}Y(r)^t\) of \(B^{[k]}Y(r)^t\) is isomorphic to \((B^{[k+1]}[\mathbb{P}\Omega^1 G](t, W_{B^{[1]}Y(r+t)}\)) and diagram (55) is commutative also for \(\beta^\prime_k\). Note also that the right hand column of diagram (55) consists of the spaces \(B^{[k+1]}Y(r)^t\) (top node), \(B^{[k+1]}Y(r)^{k\cap t}\) (middle two), and \(B^{[k+1]}Y(r)^t\) (bottom node). The isomorphism (17) follows also for the \(k + 1\) case.

**Verification of Property \(\mathbb{P}\mathbb{P}\):** Follows immediately from (56).

**Verification of Property \(\mathbb{P}\mathbb{P}\):** The morphism \(\beta^\prime_k\) in (54) is the blow-up of \(B^{[k+1]}Y(r)\) along \(B^{[k+1]}Y(r)^k\). Hence, the canonical line bundle \(\omega_{B^{[k]}Y(r)}\) of \(B^{[k]}Y(r)\) is

\[
[(\beta^\prime_k)^*(\omega_{B^{[k+1]}Y(r)})] \left( + \mbox{codim}(X(r)^k, X(r)) - 1 \right) \cdot B^{[k]}Y(r)^k.
\]

By the induction hypothesis, \(\omega_{B^{[k]}Y(r)}\) is identified in (45). The injectivity of the pullback homomorphism

\[
(\beta^\prime_k)^* : \text{Pic}(B^{[k+1]}Y(r)) \to \text{Pic}(B^{[k]}Y(r))
\]

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implies that $\omega_{B^{(k+1)}Y(r)}$ is identified by (43) replacing $k$ by $k+1$.

**Verification of Property 7.** We need to prove that the normal line-bundle $N_{B^{(k)}Y(r)}$, i.e., $\mathcal{O}_{B^{(k)}Y(r)}(B^{(k)}Y(r)^t)$, is a pullback $\nu'(k)A$ of the line bundle $A$ on $B^{(i)}Y(r)^t$ specified in (50). Here, $B^{(i)}Y(r)^t$ denotes the space $(\mathbb{P}^{\Omega^1G})_t$ (see Remark 17). We know that $\mathcal{O}_{B^{(k)}Y(r)}$ pulls back to $\mathcal{O}_{B^{(1)}Y(r)}$ on $B^{(1)}Y(r)$. In step $k = 1$ it was shown that $\mathcal{O}_{B^{(1)}Y(r)}$ is the pullback $\nu'(1)A$. Thus, both $\nu'(k)A$ and $\mathcal{O}_{B^{(k)}Y(r)}$ pull back to the same line bundle on $B^{(1)}Y(r)$. Property 7 follows from the injectivity of the pullback homomorphism.

This completes the proof of Proposition 16. $\square$

Property 3 of Proposition 16 implies that $Y(r) := B^{(\mu(X(r)) + 1)}Y(r)$ has a trivial canonical line-bundle. Hence, $Y(r)$ is symplectic. This completes the proof of Theorem 6.

### 4.2 Isomorphism of cohomology rings

We prove Theorem 2 in this section. In the case of a Mukai elementary transformation, the theorem follows easily from the work of Huybrechts ([Huy] Theorem 3.4). The stratified case is analogous. The main ingredients of the argument are:

1. (A stratified version of Huybrechts’ trick) The stratified elementary transformation

   $$M \leftarrow B^{[1]}M \cong B^{[1]}W \rightarrow W$$

   can be extended to a suitably chosen one-parameter deformation $\mathcal{M} \rightarrow T$ of the hyperkahler variety $M$. It has the affect of replacing the special fiber $M$ by its dual $W$.

2. Proposition 11 is used to relate the fiber products of dual Grassmannian bundles to exceptional divisors in the blown-up family $B^{[1]}\mathcal{M}$.

Choose a smooth family $\mathcal{M} \rightarrow T$ of projective symplectic varieties over a smooth irreducible one-dimensional base $T$ whose fiber over a point $0 \in T$ is $M$. The extension class $\epsilon \in H^1(M, TM)$ of

$$0 \rightarrow TM \rightarrow T\mathcal{M} \rightarrow \mathcal{O}_M \rightarrow 0$$

is mapped via the symplectic structure to a $(1, 1)$-class $\epsilon'$ in $H^1(M, T^*M)$. Assume that the restriction of $\epsilon'$ to a Grassmannian-fiber in every stratum $[M^t \setminus M^{t+1}] \rightarrow [M(t) \setminus M(t+1)]$ does not vanish. The assumption on the class $\epsilon$ implies that the normal bundle $N_{\mathcal{M}^t}$ of $[M^t \setminus M^{t+1}]$ in $\mathcal{M}$ restricts to every Grassmannian-fiber $G(t, n + 2t - 1)$ as the non-trivial extension (8). (Compare with the deformation considered in section 2.1). In the
complex analytic category, any twistor deformation provides such a family because \( \epsilon' \) is a Kahler class. Such an algebraic deformation exists by the work of Beauville \([B1]\) and the transversality of the following two hyperplanes in the complex moduli of deformations of \( M \): 1) the kernel of the homomorphism \( H^1(M, TM) \to H^{1,1}(M) \to H^{1,1}(G(t, n+2t-1)) \) and 2) the tangent space to the algebraic deformations of the polarized projective variety \((M, O_M(1))\).

We regard the stratification of \( M \) also as a stratification of \( M_\mu \).

The analogue of Theorem 6 holds. In other words, the top iterated blow-up \( B^{[1]}M \) admits a dual sequence of blow-downs resulting in a family \( W \) in which the special fiber \( W \) is the dual of \( M \). The proof is identical once we replace the transposition isomorphism in Proposition 8 by the extended transposition isomorphism in Proposition 11. Denote by \( \tilde{f} : B^{[1]}M \to B^{[1]}W \) the natural isomorphism,

\[ \begin{align*}
\Gamma(\tilde{f}) \quad &\text{its graph in } B^{[1]}M \times_T B^{[1]}W, \\
\Gamma(f) \quad &\text{the closure of the graph of the birational isomorphism in } M \times_T W, \\
D^t \quad &\text{the image of the exceptional divisor } B^{[1]}M^t \text{ under the isomorphism } B^{[1]}M \to \Gamma(\tilde{f}), \\
D^0 \quad &\text{the graph of the isomorphism } \tilde{f}_0 : B^{[1]}M \to B^{[1]}W.
\end{align*} \]

Proposition 11 identifies \( D^t \) as the top-iterated blow-up of the fiber-product of dual grassmannian bundles over \( B^{[1]}M(t) \)

\[ D^t \cong B^{[1]}[G(t, W_{B^{[1]}M(t)}) \times_{B^{[1]}M(t)} G(t, W^*_t B^{[2]}M(t))]. \]

Above we used the notation of Condition 3 in Section 2. \( D^0 \) is embedded in the fiber of \( \Gamma(\tilde{f}) \) over \( 0 \in T \). The fiber of \( \Gamma(\tilde{f}) \) over \( 0 \in T \) is reduced and its irreducible components are \( \{D^t\}_{t=0}^\mu \) because \( \Gamma(\tilde{f}) \) is isomorphic to \( B^{[1]}M \). Since \( T \) is one-dimensional, both \( \Gamma(\tilde{f}) \) and \( \Gamma(f) \) are irreducible varieties flat over \( T \). We have the blow-down morphisms

\[ \begin{align*}
\beta \times \beta : B^{[1]}M \times_T B^{[1]}W &\to M \times_T W \quad \text{and} \\
\beta \times \beta : \Gamma(\tilde{f}) &\to \Gamma(f).
\end{align*} \]

The set-theoretic image of \( \Gamma(\tilde{f}) \) via \( (56) \) is \( \Gamma(f) \). Proposition 11 implies that, set theoretically, \( (\tilde{f}') \) is bijective away from codimension-two in \( \Gamma(f) \). The morphism \( \Gamma(\tilde{f}) \to T \) factors through \( \Gamma(f) \to T \). Hence, in order to prove that the differential of \( \Gamma(\tilde{f}) \to M \times_T W \) is injective away from a codimension two locus in \( \Gamma(f) \), it suffices to prove that the differential of \( D^t \to M \times_T W \) is injective away from a codimension one locus in \( [M \times W] \cap \Gamma(f) \).

This follows again from Proposition 11. In the notation of Theorem 6, we conclude that \( (\tilde{f}') \) is an isomorphism away from codimension-two in \( \Gamma(f) \) and it maps the divisor \( D^t \) birationally onto \( \Delta_t \). It follows that \( \sum_{t=0}^\mu \Delta_t \) represents the class in the middle dimension Chow group of \( M \times W \) of the fiber of \( \Gamma(f) \) over \( 0 \in T \). This completes the proof of Theorem 6.
5 Brill-Noether duality for moduli spaces of sheaves

In section 5.1 we restate more precisely Theorem 1, that σ and τ lift to stratified elementary transformations (Theorem 20). Section 5.2 describes the automorphisms of cohomology rings which arise when Theorems 1 and 2 are applied to the self-dual moduli spaces. In sections 5.3, 5.4 and 5.5 we collect facts, mostly known, about stability of sheaves, Le Potier’s moduli spaces of coherent systems, and Brill-Noether loci. In section 5.6 we prove that two resolutions of a Brill-Noether stratum $M^{S}(v)$ are isomorphic (Theorem 33). When applied to the whole moduli, Theorem 33 implies that two descriptions of the closure of the graph of the birational isomorphism corresponding to τ are indeed isomorphic. The analogous result for σ is carried out in section 5.8. In section 5.7 we prove that the collection of moduli spaces, endowed with their Brill-Noether stratifications, is a dualizable collection in the sense of Definition 5. In particular, Theorem 2 applies once the dual collection is identified. The rigorous identification of the dual collection is proven in section 5.9.

5.1 Dualizable collections

Let $S$ be a K3 surface, $H$ an ample line bundle on $S$. Given a coherent sheaf $F$, its Hilbert polynomial is defined by

$$P_{F}(n) := \chi (F \otimes H^{n}) := h^{0}(F(n)) - h^{1}(F(n)) + h^{2}(F(n)).$$

The Hilbert polynomial of a coherent sheaf $F$ on $S$ of rank $r \geq 0$ and pure $d$-dimensional support is:

$$P_{F}(n) = \frac{l_{0}}{d!} n^{d} + \frac{l_{1}}{(d-1)!} n^{d-1} + \ldots + l_{d}(F) := \left( \frac{r}{2} H^{2} \right) n^{2} + (H \cdot c_{1}(F)) n + \frac{1}{2} \left( c_{1}(F)^{2} - 2c_{2}(F) \right) + 2r.$$

Note that if $r > 0$ then

$$l_{0}(F) := rH^{2} \quad \text{and} \quad l_{1}(F) := H \cdot c_{1}(F),$$

while if $r = 0$ and $d = 1$ then

$$l_{0}(F) := H \cdot c_{1}(F) \quad \text{and} \quad l_{1}(F) := \frac{1}{2} \left( c_{1}(F)^{2} - 2c_{2}(F) \right) + 2r.$$

If $p$ and $q$ are two polynomials with real coefficients, we say that $p \succ q$ (resp. $p \succeq q$) if $p(n) > q(n)$ (resp. $p(n) \geq q(n)$) for all $n$ sufficiently large.

**Definition 19**

1. A coherent sheaf $F$ on $S$ is called $H$-semi-stable (resp. $H$-stable) if it has support of pure dimension $d$ and any non-trivial subsheaf $F' \subset F$, $F' \neq (0)$, $F' \neq F$ satisfies

$$\frac{P_{F'}}{l_{0}(F')} \lesssim \frac{P_{F}}{l_{0}(F)} \quad (\text{resp.} \prec).$$
2. A coherent sheaf \( F \) on \( S \) is called \( H \)-slope-semi-stable if it has support of pure dimension \( d \geq 1 \) and for any non-trivial subsheaf \( F' \) we have
\[
\frac{l_1(F')}{l_0(F')} \leq \frac{l_1(F)}{l_0(F)}.
\]

If \( d = 1 \), \( H \)-slope-stability is defined using above a strict inequality. If \( d = 2 \), \( H \)-slope-stability is defined using above a strict inequality and considering only non-trivial subsheaves \( F' \) of lower rank.

Observe that \( H \)-slope-stability implies \( H \)-stability and \( H \)-semi-stability implies \( H \)-slope-semi-stability.

Denote by \( \mathcal{M}_S(v) \) the moduli space of \( H \)-semistable sheaves on \( S \) with support of pure dimension \( \geq 1 \) and Mukai vector \( v \). (If \( r := \text{rank}(v) \) is larger than 0, we consider only torsion free sheaves. If \( r = 0 \) we consider sheaves with pure 1-dimensional support). \( \mathcal{M}_S(v) \) is a projective scheme \([\text{Gr}], \text{Sim1}\). If furthermore, all sheaves parametrized by \( \mathcal{M}_S(v) \) are \( H \)-stable then, if non-empty, \( \mathcal{M}_S(v) \) is smooth of dimension
\[
d(v) := 2 + \langle v, v \rangle
\]
\([\text{Mn1}]\).

Fix a line bundle \( \mathcal{L} \) on \( S \). Assume

**Condition 6**

1. \( \mathcal{L} \) is an effective cartier divisor with minimal degree
\[
0 < c_1(\mathcal{L}) \cdot H = \min\{H \cdot C' \mid C' \text{ is an effective divisor}\}.
\]
In particular, all curves in the linear system \( |\mathcal{L}| \) are reduced and irreducible.

2. The base locus of \( |\mathcal{L}| \) is either empty or zero-dimensional.

3. The generic curve in \( |\mathcal{L}| \) is smooth.

4. \( H^1(S, \mathcal{L}) = 0 \).

If, for example, \( \text{Pic}(S) \) is \( \mathbb{Z} \cdot \mathcal{O}_S(1) \) and \( \mathcal{O}_S(1) \) is very-ample, we can take \( \mathcal{L} \) to be \( \mathcal{O}_S(1) \). Note that ampleness of \( \mathcal{L} \) is not assumed. If, for example, \( \pi : S \rightarrow \mathbb{P}^1 \) is an elliptic K3 with Picard number 2 and without multiple fibers, then we can take \( \mathcal{L} \) to be \( \pi^* \mathcal{O}_{\mathbb{P}^1}(1) \).

The arithmetic genus of a curve in \( |\mathcal{L}| \) is \( g := \frac{1}{2}[c_1(\mathcal{L})]^2 + 2 \). The Hilbert scheme \( S^{[d]} \) is naturally isomorphic to the moduli space \( \mathcal{M}_S(1, \mathcal{L}, g-d) \)
\[
S^{[d]} \cong \mathcal{M}_S(1, \mathcal{L}, g-d).
\]

Simply associate to a subscheme \( D \) the \( \mathcal{L} \)-twist of its ideal sheaf \( \mathcal{L}(-D) := \mathcal{I}_{S,D} \otimes \mathcal{L} \).
Denote by $\mathcal{C} \subset S \times |\mathcal{L}|$ the universal curve. The compactified relative Picard $J^d_{\mathcal{C}} \to |\mathcal{L}|$ of degree $d$ is the moduli space

$$J^d_{\mathcal{C}} \cong \mathcal{M}_S(0, \mathcal{L}, d+1-g).$$

Observe that the Mukai vector $(1, \mathcal{L}, 0)$ of $S[\mathcal{L}]$ is the reflection of the Mukai vector $(0, \mathcal{L}, 1)$ of $J^d_{\mathcal{C}}$.

Let $X = S[\mathcal{L}]$ and $X(r) := \mathcal{M}_S(r, \mathcal{L}, r-1)$. Consider the Brill-Noether stratification

$$\mathcal{M}_S(r, \mathcal{L}, r-1)^t := \{ F \mid h^1(F) \geq t \}.$$

Our goal is to prove that the collection $\{X(r)^t\}_{r=1}^\mu$ is dualizable with $n(X) = 2$ and

$$\mu := \mu(X) := \max\{ r \mid d(r, \mathcal{L}, r-1) \geq 0 \} = \max\{ r \mid r(r-1) \leq g \}.$$

Its dual collection is $\{Y(r)^t\}_{r=1}^\mu$ where $Y := J^g_\mathcal{C}$, $Y(r) := \mathcal{M}_S(r-1, \mathcal{L}, r)$, and

$$\mathcal{M}_S(r-1, \mathcal{L}, r)^t := \{ F \mid h^1(F) \geq t \}.$$

More generally we consider the planar hyperbola $H$ in Figure 4 whose lattice points represent Mukai vectors $v$ with $c_1(v) = \mathcal{L}$.

**Theorem 20** Let $v$ be a Mukai vector with $c_1(v) = \mathcal{L}$. Then the collection $\{\mathcal{M}_S(v)\}$ associated to $\mathcal{M}_S(v)$ is a dualizable collection (Definition 3) with $\mu(\mathcal{M}_S(v))$ given by (3) and $n(\mathcal{M}_S(v)) = |\chi(v)| + 1$. Its dual collection is the one associated to the Mukai vector $\sigma(v)$ (or, equivalently, $\mathcal{M}_S(\sigma(v))$) where $\sigma$ and $\tau$ are the reflections defined in Figure 4.

The theorem is proven in sections 5.7, 5.8, and 5.9. In section 5.7 Corollary 37 we prove that the collection $\{\mathcal{M}_S(v)\}$ associated to $\mathcal{M}_S(v)$ is dualizable. In section 5.9 we prove that the collections associated to $\mathcal{M}_S(v)$ and $\mathcal{M}_S(\sigma(v))$ (or equivalently $\mathcal{M}_S(\tau(v))$) are indeed dual (Proposition 44).

**Example 21** Take $v = (1, \mathcal{L}, b)$ in Theorem 20. We get that $S[\mathcal{L}]$ and $\mathcal{M}_S(b, \mathcal{L}, 1)$ are related by a stratified Mukai elementary transformation. In particular, when $b = 0$ we get that $S[\mathcal{L}]$ and $J^g_{\mathcal{C}}$ are related by a stratified Mukai elementary transformation. If the genus is in the range $2 \leq g \leq 5$, then $\mu = 1$ and the elementary transformation is Mukai’s. If $6 \leq g \leq 11$, then $\mu = 2$. When the genus is 6 we get a Lagrangian $G(2, 5)$ in both $S[\mathcal{L}]$ and $J^g_{\mathcal{C}}$. The degree of the composition morphism $G(2, 5) \hookrightarrow J^g_{\mathcal{C}} \to |\mathcal{O}_S(1)| \cong \mathbb{P}^6$ is equal to 5, the cardinality of $W^2_6$ on a generic curve. More generally, whenever $4g+1$ is a perfect square, $\mu$ is equal to $\frac{-1+\sqrt{4g+1}}{2}$ and we have a Lagrangian $G(\mu, 2\mu+1)$ in $J^g_{\mathcal{C}}$ mapping to $\mathbb{P}^g$ via a finite morphism whose degree is the cardinality $g! \prod_{i=0}^{\mu} \frac{i!}{(\mu + i)!}$ of $W^\mu_g$ on a generic curve (use Castelnuovo’s formula, Theorem (1.3) of [ACGH]). The dual Lagrangian Grassmannian $G(\mu, 2\mu+1)$ in $S[\mathcal{L}]$ parametrizes length $g$ subschemes spanning a $\mathbb{P}^{g-1-\mu}$ in $\mathbb{P}^g$. 36
Example 22 Consider the positive rays on the $r$ and $s$ axis in the hyperbola $\mathcal{H}$ in Figure 4. Theorem 20 implies that $J^{b+g-1}_g$ and $M_S(b, L, 0)$, $b \geq 0$, are related by a stratified Mukai elementary transformation. In particular, when $b = 0$ we get that the relative Brill-Noether loci $\{W^r_{g-1} \mid -1 \leq r \leq \mu - 1\}$ constitute a dualizable collection. In this case the collection is self-dual (see section 5.2).

In the course of proving Theorem 20 we will consider also the moduli spaces of coherent systems $G^0(\chi(v), M_S(v))$ and their analogue $G_1(\chi(v), M_S(v))$. They are introduced in Section 5.4. It is instructive to compare the moduli spaces involved to their analogues in the example of dual Springer resolutions of section 2.1:

$$
\begin{align*}
B^{[1]} M_S(v) & \cong B^{[1]} M_S(\sigma(v)) \quad & B^{[1]} T^* G(t, H) & \cong B^{[1]} T^* G(t, H^*) \\
G^0(\chi(v), M_S(v)) & \cong G^0(\chi(v), M_S(\sigma(v))) \quad & \text{Hom}(q_{h-t}(H), \tau(H)) & \cong \text{Hom}(q_{h-t}(H^*), \tau(H^*)) \\
M_S(v) & \quad \quad & \quad \quad & \\
\bar{M} & \quad \quad & \quad \quad & \bar{N}^T \\
\text{Morphisms} & \quad \quad & \quad \quad & \\
\text{Inclusions} & \quad \quad & \quad \quad & \\
\end{align*}
$$

If $\tau(v)$ is considered and $\chi(v) \geq 0$, replace $G^0(\chi(v), M_S(\sigma(v)))$ by $G_1(\chi(v), M_S(\tau(v)))$. Above, Hom$(q_{h-t}(H), \tau(H))$ and Hom$(q_{h-t}(H^*), \tau(H^*))$ are the natural vector bundles over Flag$(t, h-t, H)$ and Flag$(t, h-t, H^*)$ respectively. As subvarieties of the product $T^* G(t, H) \times T^* G(t, H^*)$ both are isomorphic to the closure of the graph of the birational isomorphism.

The existence of the morphism $B^{[1]} M_S(v) \to G^0(\chi(v), M_S(v))$ is proven in Proposition 36. The isomorphism $G^0(\chi(v), M_S(v)) \cong G^0(\chi(v), M_S(\sigma(v)))$ is constructed in Theorem 39. The isomorphism $G^0(\chi(v), M_S(v)) \cong G_1(\chi(v), M_S(\tau(v)))$ is constructed in Theorem 33. Theorem 21 establishes the isomorphism $B^{[1]} M_S(v) \cong B^{[1]} M_S(\sigma(v))$.

Pursuing the analogy with Springer resolutions, there should exist a singular moduli space $\bar{M} := \bar{M}_S([v])$, analogous to the closure of a square-zero nilpotent coadjoint orbit. $\bar{M}_S([v])$ is associated to each $\mathbb{Z}/2 \times \mathbb{Z}/2$ orbit $[v]$ of a Mukai vector $v$ in $\mathcal{H}$. $\bar{M}_S([v])$ should parametrize equivalence classes of stable sheaves with respect to the following equivalence relation: If $\chi(F) \geq 0$, given any $t$-dimensional subspace $U \subset \text{Ext}^1(F, \mathcal{O}_S)$ corresponding to an extension

$$
0 \to U^* \otimes \mathcal{O}_S \to E \to F \to 0
$$

of $F$ by a trivial vector bundle, we identify $F$ with the formal difference $E - \mathcal{O}_S^\otimes_t$. Forgetting the extension data has the effect of contracting the Grassmannian $G(t, H^0(E))$ in $M_S(v)$ where $v = v(F)$. Similarly, if $\chi(F) \leq 0$, given any rank $t$ trivial subsheaf

$$
0 \to U \otimes \mathcal{O}_S \to F \to Q \to 0,
$$

37
we identify $F$ with the formal sum $Q + \mathcal{O}_{S}^\oplus$, forgetting the extension data. For example, when the K3 surface $S$ is elliptic and $L$ is the class of an elliptic fiber, $J^0 := \mathcal{M}_S(0, L, 0)$ is a K3 with an ordinary double point.

It seems plausible that $\bar{\mathcal{M}}_S([v])$ can be constructed in general by using Geometric Invariant Theory. The existence of $\bar{\mathcal{M}}_S([v])$ would also follow from a better understanding of the determinantal line bundles on $\mathcal{M}_S(v)$. The restriction homomorphism $\text{Pic}_{\mathcal{M}_S(v)} \to \mathbb{P}^{\chi(v)+1}$, to a fiber in the first stratum, has an infinite cyclic kernel. We need to know that there exist line bundles in this infinite cyclic kernel which are generated by global sections and give rise to a contraction of the Grassmannian-fibrations of the Brill-Noether strata

$$\mathcal{M}_S(v) \longrightarrow \bar{\mathcal{M}}_S([v]).$$

Theorem 20 and the above prediction provide a “contraction” of the hyperbola $H$ in Figure 1 to its diagonal and the two semi-diagonals of moduli spaces with Euler characteristic $\chi(v)$ equal to 0 or $ \pm 1$: Let $v = (r, L, s)$, assume that $r+s$ is even, and set $v' := v - \frac{1}{2}(r+s)$. Then $\bar{\mathcal{M}}_S(v)$ is embedded in $\mathcal{M}_S(v')$ as a contracted Brill-Noether locus. If $\chi(v) = r+s$ is odd, set $v' = v - \frac{1}{2}(r+s \pm 1)$ and get an embedding in a semi-diagonal entry.

### 5.2 Self-dual moduli spaces

There are two classes of self-dual moduli spaces: If the Euler characteristic $\chi(v) = r+s$ vanishes, then $\mathcal{M}_S(v)$ is $\tau$-self-dual. If $r=s$ then $\mathcal{M}_S(v)$ is $\sigma$-self-dual.

If $\mathcal{M}_S(v)$ is $\tau$-self-dual, then the birational isomorphism $\tau : \mathcal{M}_S(v) \to \mathcal{M}_S(v)$ is, by definition, the identity (see Theorem 20). The Grassmannian fibrations involved all have fibers of type $G(a, \mathbb{C}^{2a})$, $a \geq 1$, which are canonically isomorphic to their dual $G(a, (\mathbb{C}^{2a})^*)$. Nevertheless, Theorem 2 is non-trivial in this case. It states that the cycle $\Gamma(id) + \sum_{t=1}^{u(v)} \Delta_t$ in $\mathcal{M}_S(v) \times \mathcal{M}_S(v)$ induces an automorphism of order two of the ring $H^*(\mathcal{M}_S(v), \mathbb{Z})$. In particular, we get the identity

$$\left( \sum_{t=1}^{u(v)} \Delta_t \right)^2 = -2 \cdot \left( \sum_{t=1}^{u(v)} \Delta_t \right)$$

analogous to the reflections of a K3 lattice induced by a $(-2)$-curve. If, for example, $\mu(v) = 1$, then the divisor $\Theta := \mathcal{M}_S(v)^1$ is a $\mathbb{P}^1$-fibration over $\mathcal{M}_S(v+1)$. The intersection of $\Theta$ with a $\mathbb{P}^1$-fiber is $-2$ and the endomorphism $\Delta_1$ is $-2$ times a projection from $H^*(\mathcal{M}_S(v), \mathbb{Z})$ onto the image of

$$H^*(\mathcal{M}_S(v+1), \mathbb{Z}) \to H^{*+2}(\mathcal{M}_S(v), \mathbb{Z}).$$

When $\mathcal{M}_S(v)$ is $\sigma$-self-dual, the birational isomorphism $\sigma : \mathcal{M}_S(v) \to \mathcal{M}_S(v)$ is of order two (see Theorem 39 and Example 40). It is a regular automorphism only in the
case of $J^g = M_S(0, L, 0)$ or when $\mu(v) = 0$. Regardless of the regularity of $\sigma$, Theorem 2 implies that the cycle $\Gamma(\sigma) + \sum_{t=1}^{\mu(v)} \Delta_t$ in $M_S(v) \times M_S(v)$ induces an automorphism of order two of the ring $H^*(M_S(v), \mathbb{Z})$. In particular, we get the identity

$$\Gamma(\sigma)^2 + 2\Gamma(\sigma) \circ \left( \sum_{t=1}^{\mu(v)} \Delta_t \right) + \left( \sum_{t=1}^{\mu(v)} \Delta_t \right)^2 \equiv \Gamma(id). \quad (58)$$

Consider for example the Hilbert scheme $S^{[3]}$ of a K3 of genus $g = 4$. $S$ is the intersection of a cubic and a quadric $Q$ in $\mathbb{P}^4$. Then $v = (1, L, 1)$, $\mu(v) = 1$, and $(S^{[3]})^1$ is a lagrangian $\mathbb{P}^3$ parametrizing collinear subschemes. Clearly, we have a morphism from $(S^{[3]})^1$ to the variety $F(Q)$ of lines on $Q$. Since $S$ does not contain a line, $(S^{[3]})^1$ is isomorphic to $F(Q)$. $(S^{[3]})^1$ is also isomorphic to $\mathbb{P}H^0(E)$ where $E$ is the unique stable vector bundle with Mukai vector $(2, \mathcal{O}_S(1), 2)$. Theorem 39 implies that we have an exact sequence

$$0 \to E^* \to H^0(E) \otimes \mathcal{O}_S \to E \to 0.$$

Hence, $\mathbb{P}H^0(E)$ is canonically isomorphic to its dual $\mathbb{P}H^0(E)^*$. The cycle $\Delta_1$ is the product $\mathbb{P}H^0(E) \times \mathbb{P}H^0(E)^*$ and the endomorphism

$$\Delta_1 : H^*(S^{[3]}, \mathbb{Z}) \to H^*(S^{[3]}, \mathbb{Z})$$

is the projection onto the line spanned by $\{[\mathbb{P}^3]\}$ sending a class $\alpha$ in $H^6(S^{[3]}, \mathbb{Z})$ to $(\alpha \cup [\mathbb{P}^3]) \cdot [\mathbb{P}^3]$. Combining (58) with the equality

$$\Delta_1([\mathbb{P}^3]) = c_3(T^* \mathbb{P}^3) \cdot [\mathbb{P}^3] = -4 \cdot [\mathbb{P}^3],$$

we get that $\Gamma(\sigma)([\mathbb{P}^3])$ is $a \cdot [\mathbb{P}^3]$ where $a$ is 3 or 5. Indeed, $a = 3$. The proof of Theorem 2 shows that $\Gamma(\sigma)$ is isomorphic to the graph of a regular automorphism $\tilde{\sigma}$ of $B^{[1]}S^{[3]}$ which restricts to $\mathbb{P}T^* \mathbb{P}H^0(E) = Flag(1, 3, H^0(E)) \subset [\mathbb{P}H^0(E) \times \mathbb{P}H^0(E)^*]$ as the involution interchanging the two factors (using the above identification of the two factors). Using the “Key formula” (Proposition 6.7 page 114 in [1]), we get the equality

$$\Gamma(\sigma)([\mathbb{P}^3]) = (\beta_* \tilde{\sigma}_* \beta^*)([\mathbb{P}^3]) = 3[\mathbb{P}^3].$$

### 5.3 Stability criteria

We will need the following Lemma which, though a slight strengthening of the statement of Lemma 1.3, is actually proven there.

**Lemma 23** [Laz] Let $v = (r, A, s)$ be a Mukai vector with $r \geq 1$. Assume that $|A|$ is not empty and $A$ satisfies the minimality condition [part 1]. Let $F$ be a torsion-free sheaf with $v(F) = v$. Assume further that $s \geq 1$ or that $h^0(F) \geq r + 1$. Then The following are equivalent:
1. \( F \) is \( H \)-stable.

2. \( F \) is \( H \)-slope-stable.

3. \( F \) satisfies all the following conditions
   \( (a) \) \( H^2(F) = (0) \),
   \( (b) \) \( F \) is generated by its global sections away from a zero-dimensional subscheme.

4. Same as 3 except that we replace 3b by \( (b') \) \( F \) is generated away from a zero-dimensional subscheme by any \( r + 1 \) dimensional subspace of \( H^0(F) \).

**Remark 24** If we drop both the assumption that \( s \geq 1 \) and the assumption that \( h^0(F) \geq r + 1 \) then the following relations hold: \( \mathbb{2} \Leftrightarrow \mathbb{1} \Leftrightarrow \mathbb{2} \).

**Proof:** (of Lemma 23)

\( \mathbb{1} \Leftrightarrow \mathbb{2} \): The assumption on the linear system \( |A| \) implies that \( H \)-slope-semi-stability for \( F \) is equivalent to \( H \)-slope-stability. Hence \( H \)-stability is equivalent to \( H \)-slope-stability.

\( \mathbb{4} \Rightarrow \mathbb{3} \): If \( s \geq 1 \) then the Euler characteristic of \( F \) satisfies \( \chi_F = r + s \geq r + 1 \). Hence, \( h^0(F) \geq r + 1 \).

\( \mathbb{2} \Rightarrow \mathbb{4} \): If \( F \) is slope-stable then \( \text{Hom}(F, \mathcal{O}_S) \) vanishes and hence, by Serre’s Duality, \( H^2(F) = 0 \).

\( \mathbb{3} \Rightarrow \mathbb{2} \): Let \( U \subset H^0(F) \) be a subspace of dimension \( \geq r + 1 \). Let \( F' \) be the subsheaf generated by the global sections in \( U \). Since \( h^0(F') \) is larger than the rank \( r' \) of \( F' \), \( F' \) can not be the trivial rank \( r' \) bundle. Since \( F' \) is generated by its global sections, Lemma 26 implies that \( c_1(F') \) is represented by an effective (non-zero) divisor \( C' \subset S \). Condition 3 part 1 implies that \( c_1(A) \cdot H \leq C' \cdot H \). \( H \)-slope-stability of \( F \) implies that \( r = r' \). If \( r = r' \) and \( F/F' \) has a one-dimensional support, then \( c_1(F/F') \) is represented by an effective divisor \( C'' \) and \( c_1(F) = C'' + C'' \) contradicting the integrality of all curves in \( |A| \). We conclude that \( F/F' \) has zero-dimensional support and \( \mathbb{3} \) holds.

\( \mathbb{3} \Rightarrow \mathbb{2} \): Let \( F' \subset F \) be a non-trivial rank \( r' \) subsheaf, \( 0 < r' < r \). Let \( F''_0 \) be the quotient \( F/F' \) and \( F'' := F''_0/T(F''_0) \) where \( T(F''_0) \) is the torsion subsheaf of \( F''_0 \). Since \( \text{Hom}(F, \mathcal{O}_S) \cong H^2(F)^* \) vanishes, while the projection \( F \rightarrow F'' \) is surjective, we conclude that \( F'' \) is not the trivial vector bundle. Lemma 26 implies that \( c_1(F'') \) is represented by an effective (non-zero) curve \( C'' \). \( c_1(T(F''_0)) \) is represented by an effective (possibly zero) divisor \( D'' \), the 1-dimensional components of the support of \( T(F''_0) \). The minimality condition 3 part 1 implies the inequality

\[
    c_1(F''_0) \cdot H = (C'' + D'') \cdot H \geq c_1(A) \cdot H.
\]
We get the inequality
\[ c_1(F') \cdot H = (c_1(A) - c_1(F''_0)) \cdot H \leq 0 \]
which implies that \( F' \) does not slope-destabilize \( F \)
\[
\frac{l_1(F')}{l_0(F')} \leq \frac{c_1(F') \cdot H}{r'H^2} < 0 < \frac{c_1(F) \cdot H}{rH^2} = \frac{l_1(F)}{l_0(F)}.
\]
Hence \( F \) is \( H \)-slope-stable. \(\square\)

Lemma 25

1. Let \( U \) be a vector space,
\[
0 \to U \otimes \mathcal{O}_S \to F \to Q \to 0
\]
(59)
a short exact sequence of coherent sheaves on \( S \) such that the line-bundle \( \mathcal{A} := \det(F) = \det(Q) \) satisfies the minimality condition \(\mathcal{A} \) part \(\mathcal{B}\). Then the following are equivalent:

(a) \( F \) is \( H \)-stable.

(b) \( Q \) is \( H \)-stable (with support of pure dimension 1 or 2) and the homomorphism \( U^* \to \text{Ext}^1(Q, \mathcal{O}_S) \) is injective.

2. If stability holds in part \(\mathcal{B}\) then there are canonical exact sequences
\[
0 \to U \to H^0(F) \to H^0(Q) \to 0,
\]
(60)
\[
0 \to H^1(F) \to H^1(Q) \to U \otimes H^0(S) \to 0
\]
(61)
and \( h^2(Q) = h^2(F) = 0 \).

3. Let \( F \) be an \( H \)-stable sheaf of rank \( r \), \( \det(F) \) as in condition \(\mathcal{A} \) part \(\mathcal{B}\) and \( U \subset H^0(F) \) a subspace of dimension \( r' \leq r \). Then the natural sheaf homomorphism \( i : U \otimes \mathcal{O}_S \to F \) is injective and the quotient \( Q \) is \( H \)-stable.

4. Let \( F' \) be a subsheaf of an \( H \)-stable sheaf \( F \) of rank \( 0 < r' < r \) with \( c_1(F) \) as in condition \(\mathcal{B} \) part \(\mathcal{C}\). Then \( F' \) is generated by its global sections away from a zero-dimensional subscheme if and only if \( F' \) is isomorphic to the trivial rank \( r' \) bundle.

Proof:

[\(\mathcal{H}\)] The proof is by induction on \( r = \text{dim}(U) \).

Case \( r = 1 \): Let \( \epsilon \in \text{Ext}^1(Q, \mathcal{O}_S) \) be the extension class of
\[
0 \to \mathcal{O}_S \to F \to Q \to 0.
\]
(62)
Assume that $Q$ is $H$-stable and $\epsilon$ does not vanish. We need to prove that $F$ is torsion-free and $H$-stable. Clearly, the support of every subsheaf of $F$ has dimension at least 1. Assume that $T \subset F$ is a destabilizing subsheaf, $H \cdot c_1(T) \geq H \cdot c_1(F)$. Denote by $\bar{T}$ its image in $Q$. Clearly $T$ cannot be a subsheaf of $O_S$. Hence, $H \cdot c_1(\bar{T}) \geq H \cdot c_1(T) \geq H \cdot c_1(F)$. Slope-stability of $Q$ implies that $(Q/\bar{T})$ is supported on a zero-dimensional subscheme. Hence $\text{Ext}^1(Q/\bar{T}, O_S)$ vanishes and $\text{Ext}^1(Q, O_S) \hookrightarrow \text{Ext}^1(\bar{T}, O_S)$ is injective. Let

$$0 \to O_S \to F' \to \bar{T} \to 0$$

be the pullback extension. Then $F'$ is a subsheaf of $F$ and $F/F'$ is isomorphic to $Q/\bar{T}$. Since every subsheaf of $F$ has dimension at least 1, $F$ is torsion free (resp. $H$-stable) if and only if $F'$ is. Thus, we may assume that $T \to Q$ is surjective and $\bar{T} = Q$. Thus $F$ is torsion free if and only if $T$ is. If $T$ is torsion free then it is not a destabilizing subsheaf of $E$ since their ranks are equal. If the torsion subsheaf $\tau \subset T$ is non-trivial then $\tau$ has pure 1-dimensional support. Hence $F$ is torsion free if and only if $T$ is. If $T$ is torsion free then it is not a destabilizing subsheaf of $E$ since their ranks are equal. If the torsion subsheaf $\tau \subset T$ is non-trivial then $\tau$ has pure 1-dimensional support. Clearly $\tau \cap O_S = (0)$ and $\tau$ embeds as a subsheaf $\bar{\tau}$ of $Q$. Hence the image of $\epsilon$ in $\text{Ext}^1(\bar{\tau}, O_S)$ vanishes. This contradicts the non-vanishing of $\epsilon$ because $Q/\bar{\tau}$ has zero-dimensional support and thus the homomorphism $\text{Ext}^1(Q, O_S) \hookrightarrow \text{Ext}^1(\bar{\tau}, O_S)$ is injective.

Assume that $F$ is $H$-stable (and hence torsion free of rank $\geq 1$). Then the extension (62) is non-trivial. We first prove that $Q$ has support of pure dimension 1 or 2. Suppose $\bar{T} \subset Q$ is a subsheaf with zero-dimensional support. Let $T \subset F$ be its inverse image in $F$. Then the extension $0 \to O_S \to T \to \bar{T} \to 0$ splits. This contradicts the $H$-stability of $F$. Suppose now that $\bar{T} \subset Q$ is a subsheaf with one-dimensional support but $Q$ has a two dimensional support. Then $H \cdot c_1(\bar{T}) > 0$ and the inverse image $T$ of $\bar{T}$ is a destabilizing subsheaf of $F$. We conclude that $Q$ has support of pure dimension 1 or 2. If $\text{supp}(Q)$ is a curve $C$, then $\det(Q) = O_S(C)$, $C$ is an integral curve and $Q$ is a rank 1 torsion free sheaf on $C$. Hence $Q$ is $H$-stable. If $Q$ is torsion free then it must be $H$-slope-stable because the inverse image $T$ of any destabilizing subsheaf $\bar{T} \subset Q$ would destabilize $F$. This completes the proof of the case $r = 1$.

Induction step: Choose a line $U_1 \subset U$ and let $\bar{U}$ be the quotient. Let $\bar{F}$ be the quotient $F/(U_1 \otimes O_S)$. We get two exact sequences

$$0 \to \bar{U} \otimes O_S \to \bar{F} \to Q \to 0, \quad (63)$$
$$0 \to U_1 \otimes O_S \to F \to \bar{F} \to 0. \quad (64)$$

Denote by $\bar{\epsilon} : \bar{U}^* \to \text{Ext}^1(Q, O_S)$ the extension class of (63) and by $\epsilon_1 : U_1^* \to \text{Ext}^1(\bar{F}, O_S)$ the class of (64). Then

$F$ is $H$-stable $\iff$ (case $r = 1$)

$\bar{F}$ is $H$-stable and $\epsilon_1 \neq 0$ $\iff$ (case $r = \dim(U) - 1$)
$Q$ is $H$-stable, $\bar{\epsilon}$ is injective and $\epsilon_1 \neq 0 \iff$
$Q$ is $H$-stable and $\epsilon : U^* \to \text{Ext}^1(Q, \mathcal{O}_S)$ is injective.

This completes the proof of part I.

2) The exact sequences (33) and (61) are part of the long exact cohomology sequence associated to (23).

3) Proof by induction on $r'$. The case $r' = 1$ is clear (injectivity is obvious and the stability of $Q$ is proven in part I).

Induction step: Choose a line $U_1 \subset \mathcal{U}$ and consider the extensions (63) and (64). $\bar{F}$ is stable because $F$ is (use part I). By the induction hypothesis the sheaf homomorphism $\bar{U} \otimes \mathcal{O}_S \to \bar{F}$ is injective. Hence $U \otimes \mathcal{O}_S \to F$ is also injective. The stability of $Q$ follows from part I.

4) By Lemma 26 either $F'$ is the trivial rank $r'$ bundle, or $c_1(F')$ is represented by an effective (or zero) divisor. In that case, $c_1(U) = 0$ if and only if $U$ is a trivial vector bundle. \[ \square \]

**Lemma 26** [Laz] Let $U$ be a torsion-free sheaf on a smooth projective surface. If $U$ is generated by its global sections away from a zero-dimensional subscheme, then $c_1(U)$ is represented by an effective (or zero) divisor. In that case, $c_1(U) = 0$ if and only if $U$ is a trivial vector bundle.

### 5.4 Coherent systems

Consider the coarse moduli space $G^0(t, \mathcal{M}_S(v))$ parametrizing pairs $(F, U)$ consisting of an $H$-stable sheaf $F$ with $\nu(F) = v$ and a $t$-dimensional subspace $U$ of $H^0(S, F)$. Le Potier constructed this moduli space as a projective scheme coarsely representing a functor in [Le1] Theorem 4.12. Le Potier’s semi-stability condition is more relaxed and does not imply the stability of $F$. Nevertheless, $G^0(t, \mathcal{M}_S(v))$ embeds as a Zariski open subset in the stable locus of Le Potier’s moduli.

**Remark 27** Let $t$ be a positive integer and set $v' := v + (t, 0, t)$. It should be easy to check that $G^0(\chi(v'), \mathcal{M}_S(v'))$ is also a union of components of the Hilbert scheme parametrizing subvarieties in $\mathcal{M}_S(v)$ isomorphic to the Grassmannian $G(\chi(v) + t, \chi(v) + 2t)$.

A family of such pairs over a Noetherian scheme $G$ is a pair $(\mathcal{F}, \tau)$. $\mathcal{F}$ is a sheaf over $G \times S$, flat over $S$, such that its restriction $\mathcal{F}|_{S_g}$ is stable with Mukai vector $v$ over every closed point $g$ in $G$. The family of vector spaces of sections is encoded by a rank $t$ locally free $\mathcal{O}_G$-module $\tau$ in the following way: the dual $\tau^*$ is a quotient of the relative Ext sheaf $\text{Ext}^2_p(\mathcal{F}, \omega_p)$ where $p : G \times S \to G$ is the projection and $\omega_p$ is the relative dualizing sheaf. Here we use 1) Serre’s Duality $\text{Ext}^2_S(F_g, \omega_S) \cong H^0(S, F_g)^*$, and 2) the base change theorem for relative Ext sheaves ([Lan] Theorem 1.4) which implies that the natural homomorphism $\text{Ext}^2_p(\mathcal{F}, \omega_p)|_g \to \text{Ext}^2_S(F_g, \omega_S)$ is surjective for all closed points $g$ in $G$ (use the vanishing of $\text{Ext}^3_S(F_g, \omega_S)$).
The constructions in section 5.6 require the following descriptions of \( \tau \) as a subsheaf of \( p_*\mathcal{F} \).

**Lemma 28** Let \( \mathcal{F} \) be a flat family over \( G \) of stable sheaves with Mukai vector \( v \) and \( \tau \) a locally free \( \mathcal{O}_G \)-module. The following data are equivalent:

1. A surjective homomorphism \( \mathcal{E}xt^2_p(\mathcal{F}, \omega_p) \rightarrow \tau^* \).

2. An injective homomorphism \( \iota : \tau \rightarrow p_*\mathcal{F} \) satisfying the property that its restriction \( \tau_g : \tau_g \rightarrow (p_*\mathcal{F})_g \) is injective for all closed points \( g \) in \( G \).

**Proof:** \( \boxed{1} \Rightarrow \boxed{2} \) Choose a very ample line bundle \( \mathcal{O}_S(n) \) on \( S \) with the property that the sheaf \( \mathcal{E}xt^1_p(\mathcal{F}(n), \omega_p) \) vanishes for \( i = 0, 1 \), is locally free for \( i = 2 \), and the sheaf \( p_*\mathcal{F}(n) \) is locally free. A choice of a section of \( \mathcal{O}_S(n) \) yields the long exact sequence of relative Ext sheaves associated to the short exact sequence

\[
0 \rightarrow \mathcal{F} \hookrightarrow \mathcal{F}(n) \rightarrow \mathcal{Q} \rightarrow 0.
\]

Since \( \mathcal{E}xt^3_p(\mathcal{Q}, \omega_p) \) vanishes, the homomorphism \( \mathcal{E}xt^2_p(\mathcal{F}(n), \omega_p) \rightarrow \mathcal{E}xt^2_p(\mathcal{F}, \omega_p) \) is surjective. Hence, \( \tau \) embeds in \( \mathcal{E}xt^2_p(\mathcal{F}(n), \omega_p)^* \) as a subsheaf of the image of \( \mathcal{E}xt^2_p(\mathcal{F}, \omega_p)^* \). Similarly, the homomorphism \( p_*\mathcal{F} \rightarrow p_*\mathcal{F}(n) \) is injective. Serre’s Duality identifies \( \mathcal{E}xt^2_p(\mathcal{F}(n), \omega_p)^* \) with \( p_*\mathcal{F}(n) \) and exhibits \( \tau \) as a subsheaf of \( p_*\mathcal{F}(n) \) which is contained in \( p_*\mathcal{F} \). The surjectivity of the composition

\[
\mathcal{E}xt^2_p(\mathcal{F}(n), \omega_p)|_g \rightarrow \mathcal{E}xt^2_p(\mathcal{F}, \omega_p)|_g \rightarrow \tau^*|_g
\]

implies that the composition

\[
\tau_g \rightarrow (p_*\mathcal{F})|_g \rightarrow (p_*\mathcal{F}(n))|_g
\]

is injective for all closed points \( g \) in \( G \). It follows that \( \tau_g \rightarrow (p_*\mathcal{F})|_g \) is injective.

\( \boxed{2} \Rightarrow \boxed{1} \) The proof is similar. \( \square \)

We will study also the coarse moduli space \( G_1(t, \mathcal{M}_S(v)) \) of pairs \( (\mathcal{F}, U) \) consisting of an \( H \)-stable sheaf \( \mathcal{F} \) with \( v(\mathcal{F}) = v \) and a \( t \)-dimensional subspace \( U \) of \( \text{Ext}^1(\mathcal{F}, \mathcal{O}_S) \). The corresponding functor (from Noetherian schemes to sets) associates to a Noetherian scheme \( G \) the set of equivalence classes of pairs \( (\mathcal{F}, \tau) \) as in the case of coherent systems except that \( \tau^* \) is a quotient of the higher direct image sheaf \( R^1_p(\mathcal{F} \otimes \omega_p) \). Here we use the vanishing of \( R^2_p(\mathcal{F} \otimes \omega_p) \) and the base change theorem for cohomology (Theorem III.12.11) to conclude that the natural homomorphism \( R^1_p(\mathcal{F} \otimes \omega_p)|_g \rightarrow H^1(S, F_g \otimes \omega_S) \) is surjective for all closed points \( g \) in \( G \). An analogue of Lemma 28 translates the data of a surjective homomorphism \( R^1_p(\mathcal{F} \otimes \omega_p) \rightarrow \tau^* \) to the data of a homomorphism \( \tau \hookrightarrow \mathcal{E}xt^1_p(\mathcal{F}, \mathcal{O}_{G \times S}) \) injective in each fiber. The existence of \( G_1(t, \mathcal{M}_S(v)) \) was proven in [Le1] Theorem 5.6 in the case where \( \text{rank}(v) = 0 \), i.e., when the sheaves have pure
one-dimensional support. For a general Mukai vector with \( c_1(v) \) satisfying Condition 3 part Ⅲ, the existence of \( G_1(t, \mathcal{M}_S(v)) \) as a projective scheme follows from the proof of Theorem 33 where the functor is shown to be equivalent to the functor represented by \( G^0(t, \mathcal{M}_S(v + \vec{t})) \).

The following Lemma will be needed in the proof of Theorem 33.

**Lemma 29** ([Maj] Application 2 page 150)

Let \( G \) be a Noetherian scheme and \( \tau \to E \to Q \to 0 \) a right exact sequence of coherent sheaves over \( G \). Assume that \( E \) is flat over \( G \). Then the following are equivalent:

1. \( u \) is injective and \( Q \) is flat over \( G \),
2. The restriction \( u_{|_g} : \tau_{|_g} \to E_{|_g} \) of \( u \) to the fiber over every closed point \( g \) in \( G \) is injective.

### 5.4.1 Universal properties

Fix a Mukai vector \( v \) with \( c_1(v) = \mathcal{L} \). Choose a very ample line bundle \( \mathcal{O}_S(n) \) on \( S \) satisfying the property: The higher cohomologies \( H^i(S, F) \), \( i > 0 \), vanish for every stable sheaf \( F \) on \( S \) with Mukai vector \( v \). Fix also a non-zero section \( \gamma \) of \( \mathcal{O}_S(n) \). We get a functorial injective homomorphism \( \otimes \gamma : \mathcal{F} \to \mathcal{F}(n) \) for every family of sheaves on \( S \). If a universal sheaf \( \mathcal{F}_v \) exists over \( S \times \mathcal{M}_S(v) \), then the pushforward \( p_* \mathcal{F}_v(n) \) is a vector bundle on \( \mathcal{M}_S(v) \) and we get a projective bundle \( \mathbb{P}(p_* \mathcal{F}_v(n)) \) over \( \mathcal{M}_S(v) \). The projective bundle exists even if \( \mathcal{F}_v \) does not. We abuse notation and denote this universal projective bundle by \( \mathbb{P}(p_* \mathcal{F}_v(n)) \) (it is locally trivial in the étale topology). Over \( G^0(t, \mathcal{M}_S(v)) \) we have a tautological \( \mathbb{P}^{t-1} \) bundle which is a subbundle

\[
\mathbb{P}^t := \mathbb{P}^t_{G^0(t, \mathcal{M}_S(v))} \subset \pi^* \mathbb{P}(p_* \mathcal{F}_v(n))
\]

of the pullback of \( \mathbb{P}(p_* \mathcal{F}_v(n)) \) to \( G^0(t, \mathcal{M}_S(v)) \).

**Proposition 30** The moduli space \( G^0(k, \mathcal{M}_S(v)) \) satisfies the following universal property. Assume we are given

1. a scheme \( T \) of finite type, a Zariski open covering \( \{ T_\alpha \} \), finite surjective étale morphisms \( \tilde{T}_\alpha \to T_\alpha \), a family \( \mathcal{E}_\alpha \), flat over \( \tilde{T}_\alpha \), of stable sheaves on \( \tilde{T}_\alpha \times S \) with Mukai vector \( v \),
2. a rank \( k \) locally free sheaf \( \mathcal{W}_\alpha \) on \( \tilde{T}_\alpha \), and
3. a homomorphism of \( \mathcal{O}_{\tilde{T}_\alpha} \)-modules

\[
i_\alpha : \mathcal{W}_\alpha \to p_* \mathcal{E}_\alpha
\]

which is injective on each fiber,
satisfying the compatibility condition: the pull-back of \((E, i)\) to \(\tilde{T}_\alpha \times_T \tilde{T}_\beta\) is equivalent, as a coherent system, to the pull-back of \((E, i)\).

Then there exists a unique pair \((\kappa, \{\delta_\alpha\})\) consisting of a morphism

\[
\kappa : T \to G^0(k, \mathcal{M}_S(v))
\]

and a collection of isomorphisms

\[
\delta_\alpha : \mathbb{P}W_{\tilde{T}_\alpha} \xrightarrow{\cong} (\kappa^*\mathbb{P}T)|_{\tilde{T}_\alpha}
\]

such that the composition of \(\kappa\) with the forgetful morphism to \(\mathcal{M}_S(v)\) is the classifying morphism of \(E\) and the following diagrams commute:

\[
\begin{array}{ccc}
\mathbb{P}(p_*E_\alpha(n)) & \cong & \kappa^*\mathbb{P}(p_*F_v(n))|_{\tilde{T}_\alpha} \\
\uparrow i\otimes\gamma & & \uparrow \\
\mathbb{P}W_{\tilde{T}_\alpha} & \xrightarrow{\delta_\alpha} & (\kappa^*\mathbb{P}T)|_{\tilde{T}_\alpha}
\end{array}
\]

(65)

Note that the fact that \(G^0(k, \mathcal{M}_S(v))\) is a coarse moduli scheme implies additional universal properties (see [GIT] Definition 7.4).

**Proof:** Since \(G^0(k, \mathcal{M}_S(v))\) is a coarse moduli space, we get a unique collection \(\{(\kappa_\alpha, \delta_\alpha)\}\) where \(\kappa_\alpha : \tilde{T}_\alpha \to G^0(k, \mathcal{M}_S(v))\) is the classifying morphism. The compatibility condition implies that the two compositions

\[
\tilde{T}_\alpha \times_T \tilde{T}_\beta \to \tilde{T}_\alpha \xrightarrow{\kappa_\alpha} G^0(k, \mathcal{M}_S(v)) \quad \text{and} \quad \tilde{T}_\alpha \times_T \tilde{T}_\beta \to \tilde{T}_\beta \xrightarrow{\kappa_\beta} G^0(k, \mathcal{M}_S(v))
\]

are equal. This implies that the collection \(\{\kappa_\alpha\}\) descends and patches to a global morphism \(\kappa\) (see [Mi] Chapter I Theorem 2.17). \(\square\)

**Proposition 31** The universal property of \(G_1(k, \mathcal{M}_S(v))\), analogous to that in Proposition 30, holds as well.

### 5.5 Construction of the Brill-Noether loci

We review the construction of the Brill-Noether loci in order to set up the notation used throughout the rest of the paper. Assume for simplicity of notation that there exists a universal sheaf \(F\) over \(\mathcal{M}_S(v) \times S\). We indicate in Remark 32 the changes necessary if a universal sheaf exists only locally. We carry out the construction in the case \(\chi(v) \geq 0\). (The case \(\chi(v) < 0\) is identical, but for a shift by \(\chi(v)\) of the index \(t\) in \(\mathcal{M}_S(v)^t\)).

Choose a section \(\gamma\) of \(H^\otimes n\) for \(n\) sufficiently large so that \(H^i(F(n))\) vanishes for \(i > 0\) and for all \(F\) in \(\mathcal{M}_S(v)\). Let \(\Gamma \in |H^n|\) be the zero divisor of \(\gamma\). Consider the short exact sequence of sheaves over \(\mathcal{M}_S(v) \times S\)
Denote by \( \rho : \mathcal{M}_S(v) \times S \to \mathcal{M}_S(v) \) the projection. We get the long exact sequence of the right-derived functor

\[
0 \to p_* \mathcal{F} \to p_* \mathcal{F}(n) \to p_* \left( \mathcal{F}(n)_{|\Gamma \times \mathcal{M}_S(v)} \right) \to R^1_{p_*} \mathcal{F} \to 0.
\] (68)

Note that \( H \)-slope-stability of all the sheaves parametrized by \( \mathcal{M}_S(v) \) implies that \( R^2_{p_*} \mathcal{F} \) vanishes. Hence, so does \( R^1_{p_*} \left( \mathcal{F}(n)_{|\Gamma \times \mathcal{M}_S(v)} \right) \). Clearly, \( R^2_{p_*} \left( \mathcal{F}(n)_{|\Gamma \times \mathcal{M}_S(v)} \right) \) vanishes as well. Thus \( V_1 := p_* \left( \mathcal{F}(n)_{|\Gamma \times \mathcal{M}_S(v)} \right) \) is a locally free sheaf of rank \( (rH^2)n^2 + (H \cdot \mathcal{L})^n + r(1 - g_r) \). As the genus \( g_r \) equals \( \frac{2n^2H^2}{2} \) the rank is \( \left( \frac{rH^2}{2} \right)n^2 + (H \cdot \mathcal{L})^n \). Let \( V_0 \) be \( p_* \mathcal{F}(n) \) and

\[
\rho : V_0 \to V_1
\] (69)

the restriction map. \( V_0 \) is locally free of rank

\[
\text{rank}(V_0) = \left( \frac{rH^2}{2} \right)n^2 + (H \cdot \mathcal{L})^n + \chi(v) = \text{rank}(V_1) + \chi(v).
\]

In our new notation, \( \text{(68)} \) becomes

\[
0 \to p_* \mathcal{F} \to V_0 \xrightarrow{\rho} V_1 \to R^1_{p_*} \mathcal{F} \to 0.
\] (70)

Truncating the leftmost sheaf in \( \text{(70)} \) we get a locally free presentation of \( R^1_{p_*} \mathcal{F} \). Observe that Corollary \( \text{[44]} \) implies that \( \rho \) is generically surjective if \( \chi(v) \geq 0 \).

Define the determinantal loci \( \mathcal{M}_S(v)^t \) for \( t \geq 0 \) as the subscheme of \( \mathcal{M}_S(v) \) which is

the zero locus of \( \wedge^{\text{rank}(V_1)+1-t} \rho \). Note that \( \text{(69)} \) is a locally-free presentation of \( R^1_{p_*} \mathcal{F} \). A standard argument shows that the subscheme structure of \( \mathcal{M}_S(v)^t \) is independent of the choices of \( \mathcal{F}, n \) and \( \gamma \) (being independent of the locally-free presentation, it is independent of \( n \) and \( \gamma \). Any other choice \( \mathcal{F}' \) of a universal sheaf would result in \( R^1_{p_*} \mathcal{F}' \) which is a twist of \( R^1_{p_*} \mathcal{F} \) by a line bundle on \( \mathcal{M}_S(v) \). Hence \( \mathcal{M}_S(v)^t \) is independent of \( \mathcal{F} \). Note that \( \mathcal{M}_S(v)^t \) is supported by the set of sheaves \( F \) with \( h^1(F) \geq t \)

\[
\mathcal{M}_S(v)^t = \{ F | h^1(F) \geq t \}.
\]

Using the language of Fitting ideals (and their rank) one can show that the Brill-Noether loci \( \mathcal{M}_S(v)^t \) represent a functor (see \[ACGH\] Remark 3.2 page 179).

For future reference we note that if we take the relative \( \mathcal{E}xt_p(\bullet, \omega_S) \) of \( \text{(67)} \) we get the exact sequence of sheaves on \( \mathcal{M}_S(v) \)

\[
0 \to \mathcal{E}xt^1_p(\mathcal{F}, \omega_S) \to \mathcal{E}xt^2_p(\mathcal{F}(n)_{|\Gamma \times \mathcal{M}_S(v)}, \omega_S) \to \mathcal{E}xt^2_p(\mathcal{F}(n), \omega_S) \to \mathcal{E}xt^2_p(\mathcal{F}, \omega_S) \to 0.
\]
Serre’s Duality identifies it as the locally free presentation of $\mathcal{E}xt_p^2(\mathcal{F}, \omega_S)$ dual to \( (70) \)
\[
0 \to \mathcal{E}xt_p^1(\mathcal{F}, \omega_S) \to V_1^* \xrightarrow{f^*} V_0^* \to \mathcal{E}xt_p^2(\mathcal{F}, \omega_S) \to 0. \tag{71}
\]

**Remark 32** We do have a universal sheaf if $c_2(v) = g - 2$ or $g$ and $c_1(v) = L$ because in this case $s - r = \pm 1$ and hence $gcd(r, c_1(v)^2, s) = 1$. If $m := gcd(r, c_1(v)^2, s) > 1$ then we only have a global quasi-universal sheaf of similitude $m$ (see [Mu2] Appendix 2). Nevertheless, in general, we do have a universal sheaf locally (in the complex or étale topology) over $\mathcal{M}_S(v)$. It is possible to carry out the construction of the Brill-Noether loci using only the local existence of a universal sheaf. We carry out the constructions locally, show independence of the choice of the universal sheaf, and conclude that the constructions glue as a global algebraic object. In fact, the stability (and hence simplicity) of the sheaves parametrized by $\mathcal{M}_S(v)$ imply (as in [Mu2] Appendix 2) that the vector bundles $\mathcal{H}om^k(\Lambda V_0, \Lambda V_1)$ and their sections $\Lambda \rho$, $k \geq 1$, exist globally and depend canonically on $n$ and $\gamma$ even though the universal sheaf $\mathcal{F}$ and the vector bundles $V_0, V_1$ exist only locally.

### 5.6 Tyurin’s extension morphism

The determinantal loci $\mathcal{M}_S(v)^t$, the moduli spaces $G^0(t, \mathcal{M}_S(v))$, $G_1(t, \mathcal{M}_S(v))$ and the forgetful morphisms

\[
G_1(t, \mathcal{M}_S(v)) \to \mathcal{M}_S(v)^t,
\]
\[
G^0(\chi(v) + t, \mathcal{M}_S(v)) \to \mathcal{M}_S(v)^t
\]

were constructed for the relevant Mukai vectors. We extend results of Tyurin [Tv3] and construct an isomorphism

\[
\tilde{f} : G_1(t, \mathcal{M}_S(v)) \xrightarrow{\cong} G^0(t, \mathcal{M}_S(v'))
\]

where $v' = v + (t, 0, t)$.

Let $F$ be a sheaf of rank $r \geq 0$ and $V \subset \mathcal{E}xt^1(F, \mathcal{O}_S)$ a $t$-dimensional subspace. There exists a canonical rank $r + t$ sheaf $f(F, V)$ and an extension

\[
0 \to V^* \otimes \mathcal{O}_S \to f(F, V) \to F \to 0. \tag{72}
\]

Simply define $f(F, V)$ via the canonical class in $\mathcal{E}xt^1_S(F, V^* \otimes \mathcal{O}_S)$ using the isomorphism

\[
\mathcal{E}xt^1_S(F, V^* \otimes \mathcal{O}_S) \cong \mathcal{E}xt^1_S(F, \mathcal{O}_S) \otimes V^* \cong \text{Hom}[V, \mathcal{E}xt^1_S(F, \mathcal{O}_S)].
\]

It is easy to see that the Mukai vector of $f(F, V)$ is

\[
v(f(F, V)) = v(F) + (t, 0, t).
\]
The sheaf $f(F, V)$ is torsion-free and $H$-slope-stable if and only if $F$ is $H$-slope-stable (Lemma 23). Moreover, we have the exact sequences

$$0 \to V^* \to H^0(f(F, V)) \to H^0(F) \to 0 \quad \text{and}$$

$$0 \to H^1(f(F, V)) \to H^1(F) \to V^* \otimes H^0(S) \to 0. \quad (73)$$

**Theorem 33** Let $v = (r, L, s)$ be a Mukai vector with $r \geq 0$ and $t$ a positive integer. There is a natural isomorphism

$$\tilde{f} : G_1(t, \mathcal{M}_S(v)) \xrightarrow{\cong} G_0(t, \mathcal{M}_S(v'))$$

where $v' = v + (t, 0, t)$. The isomorphism $\tilde{f}$ is compatible with the Brill-Noether stratification with a shift of indices by $t$: For $k \geq t$, $\tilde{f}$ realizes $G_1(t, \mathcal{M}_S(v^k)) \backslash G_1(t, \mathcal{M}_S(v^{k+t}))$ as a $G(t, \chi(v') + k - t)$-bundle over $\mathcal{M}_S(v')^{k-t} \backslash \mathcal{M}_S(v')^{k+1-t}$.

**Proof:** We prove the equivalence of the two functors coarsely represented by $G_1(t, \mathcal{M}_S(v))$ and $G_0(t, \mathcal{M}_S(v'))$. Let $G$ be a Noetherian scheme, $\mathcal{F}_{v'}$ a sheaf on $G \times S$, flat over $S$, whose restriction to $S_g$ is a stable sheaf with Mukai vector $v'$ for all closed points $g$ in $G$. Let $\tau$ be a rank $t$ locally free $\mathcal{O}_G$-subsheaf of $p_*\mathcal{F}_{v'}$ satisfying the property that $\tau|_g \to (p_*\mathcal{F}_{v'})|_g$ is injective for all closed points $g$ in $G$. We get a short exact sequence of sheaves on $G \times S$:

$$0 \to p^*\tau \to \mathcal{F}_{v'} \to Q \to 0. \quad (76)$$

By Lemma 25 part 3, the restriction of (76) to $S_g$ is exact for all closed points $g$ in $G$. Lemma 25 implies that $Q$ is flat over $G$. Slope stability of the bundles $(\mathcal{F}_{v'})|_g$, $g \in G$, implies that the $\mathcal{O}_G$-module $p_*\mathcal{H}om(\mathcal{F}_{v'}, \mathcal{O}_{G \times S})$ vanishes and the following short exact sequence of $\mathcal{O}_G$-modules

$$0 \to \tau^* \to \mathcal{E xt}^1_{p_*}(Q, \mathcal{O}_{G \times S}) \to \mathcal{E xt}^1_{p_*}(\mathcal{F}_{v'}, \mathcal{O}_{G \times S}) \to 0$$

is part of the long exact sequence of relative extension sheaves.

Conversely, let $\mathcal{F}_v$ be a sheaf on $G \times S$, flat over $S$, whose restriction to $S_g$ is a stable sheaf with Mukai vector $v$ for all closed points $g$ in $G$. Denote by $\omega_p$ the relative dualizing sheaf $p^*\omega_S$. Let $\tau$ be a rank $t$ locally free $\mathcal{O}_G$-module, $\epsilon : \tau \to \mathcal{E xt}^1_{p_*}(\mathcal{F}_v, \omega_p)$ an injective homomorphism into the relative extension sheaf satisfying the property that $\tau|_g \to \mathcal{E xt}^1_{p_*}(\mathcal{F}_v, \omega_p)|_g$ is injective for all closed points $g$ in $G$. We get a section $\epsilon$ of $H^0(G, \mathcal{E xt}^1_{p_*}(\mathcal{F}_v, p^*\tau^* \otimes \omega_p))$. The Grothendieck spectral sequence

$$H^p(G, \mathcal{E xt}^q(\mathcal{F}_v, p^*\tau^* \otimes \omega_p)) \Rightarrow \mathcal{E xt}^{p+q}_{G \times S}(\mathcal{F}_v, p^*\tau^* \otimes \omega_p)$$

gives the exact sequence:

$$0 \to H^1(G, p_*\mathcal{H}om(\mathcal{F}_v, p^*\tau^* \otimes \omega_p)) \to \mathcal{E xt}^1_{G \times S}(\mathcal{F}_v, p^*\tau^* \otimes \omega_p) \to H^0(G, \mathcal{E xt}^1_{p_*}(\mathcal{F}_v, p^*\tau^* \otimes \omega_p))$$

$$\to H^2(G, p_*\mathcal{H}om(\mathcal{F}_v, p^*\tau^* \otimes \omega_p)).$$
But the sheaf $p_*\text{Hom}(F_v, p^*\tau^* \otimes \omega_p)$ vanishes because $F_v$ is a family of torsion free sheaves $F$ with vanishing $H^2(F)$. Hence we get an isomorphism

$$\text{Ext}^1_{G \times S}(F_v, p^*\tau^* \otimes \omega_p) \xrightarrow{\cong} H^0(G, \text{Ext}^1_p(F_v, p^*\tau^* \otimes \omega_p))$$

and the section $\epsilon$ determines an extension of sheaves on $G \times S$:

$$0 \to p^*\tau^* \otimes \omega_p \to \mathcal{E} \to F_v \to 0. \quad (77)$$

Since $p^*\tau^*$ and $F_v$ are flat over $G$, so is $\mathcal{E}$. Lemma 25 part 1 implies that $\mathcal{E}$ restricts to $S_g$ as a stable sheaf with Mukai vector $v'$ for all closed points $g$ in $G$.

The two constructions are inverse of each other modulo the equivalence of objects parametrized. In other words, given the data $\tau_0 \rightarrow p_*F_{v'}$ in the first construction and the data $\tau_1 \rightarrow \text{Ext}^1_{p_*}(F_v, \omega_p)$ in the second construction, there exists a line bundle $K$ on $G$ such that tensoring the short exact sequence (77) by $\tau$ we get the sequence (76). In particular, $\tau_0 \cong (\tau_1^* \otimes p_*\omega_p) \otimes K$ which is a relative version of Lemma 25 part 2 combined with the relative version of Serre’s Duality.

Finally we prove that the isomorphism $\tilde{f}$ in (73) is compatible with the Brill-Noether stratifications. By definition, these stratifications are the pull-back of the Brill-Noether stratifications on $M_S(v)$ and $M_S(v')$ via the forgetful morphism. Given a Noetherian scheme $G$ and data $F_{v'}, \tau_0$ as in (70) and $F_v, \tau_1$ as in (74) we get on $G \times S$ the short exact sequence

$$0 \to p^*\tau \to F_{v'} \to F_v \otimes p^*K \to 0 \quad (78)$$

for some line bundle $K$ on $G$ (where $\tau \cong \tau_0 \cong (\tau_1^* \otimes p_*\omega_p) \otimes K$). The long exact sequence of higher direct image brakes into two short exact sequences on $G$ one of which is

$$0 \to R^1_{p_*F_{v'}} \to (R^1_{p_*F_v}) \otimes K \xrightarrow{j} \tau \otimes \text{C} H^{0,2}(S) \to 0. \quad (79)$$

Since the quotient $\tau \otimes \text{C} H^{0,2}(S)$ is locally free of rank $t$, the determinantal stratification defined by the sheaf $R^1_{p_*F_{v'}}$ coincides with that of $(R^1_{p_*F_v}) \otimes K$ with indices shifted by $t$. To see that, choose a locally free presentation

$$A \xrightarrow{\xi} B \xrightarrow{\eta} (R^1_{p_*F_v}) \otimes K \to 0$$

with rank(A) $\geq$ rank(B). The kernel $B_1 := \ker(j \circ \eta)$ is locally free because $B$ and $\tau$ are and $(j \circ \eta)$ is surjective. Denote by $a$, $b$ and $b_1$ the corresponding ranks. The image $e(A)$ is a subsheaf of $B_1$ and we get a commutative diagram of locally free presentations (and short exact columns):

$$\begin{array}{cccccc}
A & \xrightarrow{e_1} & B_1 & \xrightarrow{\eta} & R^1_{p_*F_{v'}} & \to 0 \\
\downarrow & \uparrow & \downarrow & \uparrow & \downarrow & \\
A & \xrightarrow{e} & B & \xrightarrow{\eta} & (R^1_{p_*F_v}) \otimes K & \to 0 \\
\downarrow \xi & \uparrow \eta & \downarrow \tau & \uparrow j \eta & \downarrow j & \\
\quad & \tau & = & \tau & \quad & \\
\end{array}$$

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Corollary 34 \leq t \geq \mu(v) then \( \mathcal{M}_S(v)^t \) is empty. If \( 0 \leq t \leq \mu(v) \) then the codimension of \( \mathcal{M}_S(v)^t \) in \( \mathcal{M}_S(v) \) is \( (|\chi(v)| + t)t \).

**Proof:** Assume first that \( \chi(v) \) is non-negative. The proof is by induction on \( d(v) \) where \( v \) ranges in the set \( \{ v_0 + t \cdot \bar{t} \mid t \in \mathbb{Z} \text{ and } \chi(v_0 + t \cdot \bar{t}) \geq 0 \} \) for some fixed vector \( v_0 \). Notice that \( d(v_0 + t \cdot \bar{t}) \) is a decreasing function of \( t \) on this set. If \( d(v) \) is negative, \( \mathcal{M}_S(v)^t \) is empty for any integer \( t \geq 0 \). If \( d(v) \) is non-negative and \( t = 0 \) then \( \mathcal{M}_S(v)^0 = \mathcal{M}_S(v) \) is non-empty by Lemma 33. If \( d(v + (1, 0, 1)) \) is negative but \( d(v) \) is non-negative then \( \mu(v) = 0 \). Hence, the first step of the induction involves \( v \) with \( \mu(v) = 0 \) for which Corollary 34 holds. Assume that the statement holds for all \( v'' \) with \( d(v'') < d(v) \). Note that \( G_1(t, \mathcal{M}_S(v)^t) \setminus G_1(t, \mathcal{M}_S(v)^{t+1}) \) is isomorphic to \( \mathcal{M}_S(v)^t \setminus \mathcal{M}_S(v)^{t+1} \). Set \( v' := v + t \cdot \bar{t} \) with \( t \geq 1 \). The induction hypothesis implies that either \( \mathcal{M}_S(v') \) is empty, or \( \mathcal{M}_S(v')^1 \) is a proper (possibly empty) subscheme of \( \mathcal{M}_S(v') \). By definition, if \( t \leq \mu(v) \), then \( \mu(v') = \mu(v) - t \). If \( t > \mu(v) \) then the induction hypothesis implies that \( \mathcal{M}_S(v') \) is empty. Hence, Theorem 33 implies that \( \mathcal{M}_S(v')^t \) is also empty. If \( t \leq \mu(v) \) then the induction hypothesis implies that \( \mathcal{M}_S(v') \) is not empty and Theorem 33 implies that the dimension \( \dim[G_1(t, \mathcal{M}_S(v)^t) \setminus G_1(t, \mathcal{M}_S(v)^{t+1})] \) is equal to

\[
d(\nu) - (t(r + s + t) + d(\nu')) = t(r + s + t).
\]

Thus, the codimension of \( \mathcal{M}_S(v)^t \) in \( \mathcal{M}_S(v) \) is

\[
d(\nu) - (t(r + s + t) + d(\nu')) = t(r + s + t).
\]

The case of negative Euler characteristic \( \chi(v) \) can be reduced to the non-negative Euler characteristic case via the isomorphism

\[
[\mathcal{M}_S(v)^t \setminus \mathcal{M}_S(v)^{t+1}] = [G_1(t - \chi(v), \mathcal{M}_S(v)) \setminus G_1(t - \chi(v), \mathcal{M}_S(v)^{t+1})] \cong [G^0(t - \chi(v), \mathcal{M}_S(v + t \cdot \bar{t} - \chi(v))) \setminus G^0(t - \chi(v), \mathcal{M}_S(v + t \cdot \bar{t} - \chi(v))^1)].
\]

The first equality is a definition. The second isomorphism follows from Theorem 33. The Mukai vector \( v + t \cdot \bar{t} - \chi(v) \) has Euler characteristic \( 2t - \chi(v) \) which is non-negative. \( \square \)
Lemma 35  Let \(v = (r, \mathcal{L}, s)\) be a Mukai vector. If \(d(v)\) is non-negative then \(\mathcal{M}_S(v)\) is non-empty.

**Proof:** We may assume that \(r\) is non-negative (otherwise, replace \(v\) by \((\sigma \circ \tau)(v) := (-r, \mathcal{L}, -s)\)). Let \(v' := (0, \mathcal{L}, s - r)\). Note that \(\mathcal{M}_S(v')\) is the compactified relative Jacobian \(J^{s-1+r-\nu}_{\mathcal{L}}\). Condition 8 Part 3 assures us the existence of a smooth curve in \(|\mathcal{L}|\). Use the non-emptiness results from classical Brill-Noether theory for a smooth curve in \(|\mathcal{L}|\) to conclude that \(G_1(r, \mathcal{M}_S(v'))\) is not empty if \(d(v) = 2g - 2rs\) is non-negative. Then use Theorem 33 to conclude that \(G^0(r, \mathcal{M}_S(v))\) is non-empty. \(\square\)

### 5.7 Dualizability

Let \(v = (a, \mathcal{L}, b)\) be a Mukai vector as in Theorem 24. We prove in this section that, if \(\chi(v) \geq 0\), the collection

\[\{ \mathcal{M}_S(v + \vec{r})^t \mid 0 \leq r \leq \mu(v) \}\]  

(80)

is a dualizable collection (Definition 1). Similarly, for \(\chi(v) \leq 0\) the collection with \(\mathcal{M}_S(v - \vec{r})^t\) replaced by \(\mathcal{M}_S(v - \vec{r})^t\) is dualizable.

Assume first that \(\chi(v)\) and \(\text{rank}(v)\) are non-negative. See Remark 13 for the general case. We also assume, for simplicity of notation, that we can fix universal sheaves \(\mathcal{F}_{v'}\) over the moduli spaces \(\mathcal{M}_S(v')\) in our collection. In Remark 38 we indicate the changes necessary when the universal sheaves exist only locally. We obtain the homomorphisms \(\rho_{v'}\) (70), the iterated blow-ups \(\beta : B^{|k|} \mathcal{M}_S(v') \to \mathcal{M}_S(v')\), and the elementary transforms

\[B^{|k|}(\rho_{v'}) : B^{|k|}V_{v',0} \to B^{|k|}V_{v',1}.\]  

(81)

Recall that \(B^{|k|}V_{v',0}\) is simply the pullback of \(V_{v',0}\) by the iterated blow-up, while \(B^{|k|}V_{v',1}\) is the result of an iteration of two operations: pullback via a single blow-up morphism followed by a Hecke-transformation along the exceptional divisor (see (32)).

**Proposition 36**

1. \(B^{|k+1|}(\rho_{v'}) : B^{|k+1|}V_{v',0} \to B^{|k+1|}V_{v',1}\) has constant rank over \(B^{|k+1|} \mathcal{M}_S(v')^k\), \(0 \leq k \leq \mu(v')\). The kernel of its restriction

\[W^k_{v'} := \ker \left( (B^{|k+1|} \rho_{v'})_{|\mathcal{B}^{|k+1|} \mathcal{M}_S(v')^k} \right)\]

is a rank \(k + \chi(v')\)-vector bundle over \(B^{|k+1|} \mathcal{M}_S(v')^k\) which is a sub-sheaf of the pullback \(\beta^* \left( \mathcal{F}_{v'} \right)_{|\mathcal{M}_S(v')^k}\).

2. The cokernel \(U^k_{v'}\) of the restriction of \(B^{|k+1|}(\rho_{v'})\) to \(B^{|k+1|} \mathcal{M}_S(v')^k\) is a rank \(k\) vector bundle

\[U^k_{v'} := \text{coker} \left( (B^{|k+1|} \rho_{v'})_{|\mathcal{B}^{|k+1|} \mathcal{M}_S(v')^k} \right)\]
over $B^{[k+1]}M_S(v')^k$. Its dual $(U^k_{v'})^*$ is a subsheaf of the twisted pullback

$$
\beta^* \left( \mathcal{E}xt^1_{B^k} \left( (F_{v'})_{M_S(v'), k}, \omega_S \right) \right) + \sum_{i=k+1}^{\mu(v')} B^{[k+1]}M_S(v')^i.
$$

(82)

3. $B^{[1]}(\rho_{v'}) : B^{[1]}V_{v', 0} \rightarrow B^{[1]}V_{v', 1}$ is surjective. The kernel $W_{v'} := W^0_{v'}$ of its restriction is a rank $\chi(v')$-vector bundle over $B^{[1]}M_S(v')$.

4. The Grassmannian bundle $G(k, W_{v'})$ over $B^{[1]}M_S(v')$ satisfies the universal property analogous to that of $G^0(k, M_S(v'))$ for the restricted class of those families $(E, W_T)$ over $T$ as in Proposition \[30\] which satisfy the additional property:

The inverse image $(\kappa^{-1}L_{M_S(v')}) \cdot O_T$ of the ideal sheaves $L_{M_S(v')}$ of all the Brill-Noether loci with $1 \leq t \leq \mu(v')$ via the classifying morphism $\kappa : T \rightarrow M_S(v')$ are all invertible sheaves of ideals on $T$.

5. $B^{[k+1]}M_S(v')^k$ satisfies the universal property analogous to that of $G_1(k, M_S(v'))$ for the restricted class of those families $(E, U_T)$ over $T$ as in Proposition \[31\] which satisfy the additional property:

The inverse image $(\kappa^{-1}L_{M_S(v')}) \cdot O_T$ of the ideal sheaves $L_{M_S(v')}$ of all the Brill-Noether loci with $k + 1 \leq t \leq \mu(v')$ via the classifying morphism $\kappa : T \rightarrow M_S(v')$ are all all invertible sheaves of ideals on $T$.

6. There is a canonical isomorphism

$$
f_{v', k}^\sim : B^{[k+1]}M_S(v')^k \cong G(k, W_{v'+\bar{k}}).
$$

(83)

Denote by

$$
f_{v', k}^\sim : B^{[k+1]}M_S(v')^k \rightarrow B^{[1]}M_S(v' + \bar{k})
$$

the composition of $f_{v', k}$ with the projection to $B^{[1]}M_S(v' + \bar{k})$.

7. $B^{[k+1]}M_S(v')^k$ is smooth, non-empty and of codimension $k \cdot (\lfloor \chi(v') \rfloor + k)$, for $0 \leq k \leq \mu(v')$.

8. Let $\tau^k_{v'}$ and $q^k_{v'}$ be the universal sub- and quotient bundles of $f_{v', k}^\sim(W_{v'+\bar{k}})$ (which are vector bundles over $B^{[k+1]}M_S(v')^k$). There exists a line bundle $K$ on $B^{[k+1]}M_S(v')^k$ and isomorphisms

$$
\tau^k_{v'} \left( - \sum_{j=k+1}^{\mu(v')} B^{[k+1]}M_S(v')^j \right) \cong U^k_{v'} \otimes K
$$

(84)

$$
q^k_{v'} \cong W^k_{v'} \otimes K.
$$

(85)
9. The isomorphism

\[ N_{B^{[k+1]}\mathcal{M}_S(v')/B^{[k+1]}\mathcal{M}_S(v')} \xrightarrow{j_{\mathcal{M}_S(v')}} \hom(q^k_{v'}, \tau^k_{v'}) \otimes \mathcal{O}_{B^{[k+1]}\mathcal{M}_S(v')} \left( -\sum_{i=k+1}^{\mu(v')} B^{[k+1]}\mathcal{M}_S(v')^i \right) \]

induced by the symplectic structure of \( \mathcal{M}_S(v') \) (see Lemma 4) is conjugated via (84) and (85) to the dual of the Petri map of \( B^{[k+1]}(\rho_v) \)

\[ \phi_{v',k} : \hom(U^k_{v'}, W^k_{v'}) \xrightarrow{\cong} N^*_{B^{[k+1]}\mathcal{M}_S(v')/B^{[k+1]}\mathcal{M}_S(v')} \]  

**Proof:** The proof is by ascending induction on the pair \((\mu(v'), \mu(v') - k)\) with respect to the usual order: \((a, b) > (a', b')\) if \(a > a'\) or \(a = a'\) and \(b > b'\). Note that for a fixed Mukai vector \(v\), the map

\[(r, k) \mapsto (\mu(v + \vec{r}), \mu(v + \vec{r}) - k)\]

is strictly decreasing in both \(r\) and \(k\). For each fixed \(v\) the subset

\[\{(r, k) \mid v + \vec{r} \in \mathcal{H} \text{ and } k \leq \mu(v + \vec{r})\}\]

of the hyperbola in Figure [1] is a finite set.

The case \((\mu(v), \mu(v) - k) = (0, 0)\). In this case \(B^{[1]}\mathcal{M}_S(v)^0 = \mathcal{M}_S(v)\) and the Brill-Noether stratification is trivial: \(h^1(F) = 0\) for every sheaf parametrized by \(\mathcal{M}_S(v)\). \(B^{[1]}\rho_v = \rho_v\) is surjective, \(U^0_v\) is the zero bundle. Both \(f_{v,0}\) and \(\tilde{f}_{v,0}\) are the identity.

The induction step: Assume that the proposition (and the dualizability conditions) hold for all \((v', k')\) with \((\mu(v'), \mu(v') - k') < (\mu(v), \mu(v) - k)\).

[1] By the induction hypothesis \(B^{[k+2]}(\rho_v)\) has constant rank over \(B^{[k+2]}(\mathcal{M}_S(v)^{k+1})\) and \(B^{[k+2]}(\mathcal{M}_S(v)^{k+1})\) is smooth. Corollary [3] implies that the generic rank on every component of \(B^{[k+1]}(\mathcal{M}_S(v)^k)\) is \(k + \chi(v)\). Lemma [2] part [3] implies that \(B^{[k+1]}(\rho_v)\) has constant rank over \(B^{[k+1]}(\mathcal{M}_S(v)^k)\).

[2] The only non-obvious statement is that \((U^k_v)^*\) is a subsheaf of the \(\mathcal{E}xt^1_B\)-sheaf (82).

Using Corollary [3] inductively, we get the isomorphisms

\[ U^k_v := \coker(B^{[k+1]}\rho_v) \cong \coker((B^{k+1}(B^{[1]}\rho_v)^*)^*) (-B^{[k+1]}\mathcal{M}_S(v)^{k+1}) \cong \ldots \]

\[ \cong \coker((B^{[1]}\rho_v)^*) (-\sum_{i=k+1}^{\mu(v)} B^{[k+1]}\mathcal{M}_S(v)^i) \]

It remain to show that

\[ \left( \coker((B^{[k+1]}\rho_v)^*) \right)^\ast \cong \ker((B^{[k+1]}\rho_v)^*) \]
is a subsheaf of $\beta^* \left( \text{Ext}_{p_*}^1(F_v, \omega_S)|_{\mathcal{M}_S(v)^k} \right)$. Part 3 implies that $(B^{[k+1]}\rho^*_v)$ has constant rank over $B^{[k+1]}\mathcal{M}_S(v)^k$ and the kernel of its restriction is a subsheaf of $\beta^* \left( \ker(\rho^*_v)|_{\mathcal{M}_S(v)^k} \right)$.

Serre’s Duality (71) identifies $\ker(\rho^*_v)|_{\mathcal{M}_S(v)^k}$ with $\text{Ext}_{p_*}^1(F_v, \omega_S)|_{\mathcal{M}_S(v)^k}$.

3) This is a special case of part 1.

4) We prove the analogue of Proposition 30 in the case of a single family $\mathcal{E}$ of stable sheaves with Mukai vector $v$ and $W_T$ a rank $k$ locally free subsheaf of $\pi_T^* \mathcal{E}$. The patching argument needed in the general case is identical to that in the proof of Proposition 30. Assume that the classifying morphism $\kappa : T \to \mathcal{M}_S(v)$ of $\mathcal{E}$ pulls back all Brill-Noether loci to Cartier divisors on $T$. Let $\tilde{\kappa} : T \to G^0(k, \mathcal{M}_S(v))$ be the classifying morphism of $(\mathcal{E}, W_T)$. We claim that there exists a natural lift $\kappa^{[1]} : T \to B^{[1]}\mathcal{M}_S(v)$ of $\kappa$. $\kappa^{[1]}$ exists by the universal property of blowing-up and induction. There is a subtlety in the $i$-th step of the induction: we need to check that if the inverse image in $T$ of $\mathcal{I}_{B^{[i+1]}\mathcal{M}_S(v)^t}$, $t \leq i$, via $\kappa^{[i+1]}$ is invertible on $T$, then the inverse image of its strict transform $\mathcal{I}_{B^{[i]}\mathcal{M}_S(v)^t}$ via $\kappa^{[i]}$ is also invertible on $T$. This follows from the comparison between the strict transform and the inverse image via the blow-up map in Theorem 9 part 1. The existence of $\kappa^{[1]}$ follows. We have a canonical homomorphism (up to a scalar factor)

$$\pi_T^* \mathcal{E} \otimes K \xrightarrow{\cong} \kappa^*(p_* F_v)$$

for some line bundle $K$ on $T$. Hence we have a canonical injective bundle homomorphism $W_T \otimes K \hookrightarrow \kappa^* V_0$ obtained as a composition

$$W_T \otimes K \hookrightarrow (\pi_T^* \mathcal{E}) \otimes K \cong \kappa^*(p_* F_v) \hookrightarrow \kappa^* V_0$$

(see (73)). In particular, we have a canonical morphism to the Grassmannian bundle $T \to G(k, V_0)$ over $\mathcal{M}_S(v)$, namely the composition

$$T \xrightarrow{\tilde{\kappa}} G(k, \mathcal{M}_S(v)) \hookrightarrow G(k, V_0).$$

But $\kappa^* V_0$ is isomorphic to $(\kappa^{[1]})^*(B^{[1]} V_0)$. Hence we get a canonical morphism

$$\alpha : T \to G(k, B^{[1]} V_0)$$

to the Grassmannian bundle over $B^{[1]}\mathcal{M}_S(v)$. We need to show that $\alpha$ factors through $G(k, W_v)$. By definition of $\tilde{\kappa}$, the composition (88) factors through $(\tilde{\kappa})^* \tau_{G^0(k, \mathcal{M}_S(v))}$. We need to show that $(\tilde{\kappa})^* \tau_{G^0(k, \mathcal{M}_S(v))}$ is also in the kernel of

$$B^{[1]}(\kappa^*(\rho_v)) : B^{[1]} \kappa^* (V_0) \to B^{[1]} \kappa^* (V_1)$$

(89)

(the kernel of (89) is canonically isomorphic to $(\kappa^{[1]})^* W_v$). $B^{[1]} \kappa^* (V_0)$ is equal to $\kappa^*(V_0)$ and $B^{[1]} \kappa^* (V_1)$ is, by definition, a sub-sheaf of $\kappa^*(V_1)$ which contains the image of $\kappa^*(\rho_v)$. 55
Hence, the sheaf theoretic kernel of (89) is the same as that of $\kappa^*(\rho_v)$. We conclude that indeed $\alpha$ factors through $G(k, W_v^k)$.

3) Part 2 implies the equality $B^{[k+1]}M_s(v)^k = G(k, U^k_v)$. The rest of the proof is analogous to that of Part 4.

3) The statement is a tautology for $k = 0$. Assume that $k \geq 1$. The induction hypothesis implies that $G(k, W_{v+\tilde{k}})$ is smooth. By part 2 for $(v, k)$ and the universal property of $G_1(k, M_S(v))$ (Proposition 31) there is a canonical classifying morphism

$$\alpha : B^{[k+1]}M_S(v)^k \to G_1(k, M_S(v))$$

compatible with the Brill-Noether stratifications. Theorem 33 implies that we have a canonical isomorphism

$$f : G_1(k, M_S(v)) \cong G^0(k, M_S(v + \tilde{k})) \quad (90)$$

compatible (up to a shift by $k$) with the Brill-Noether stratification. We claim that the composition $f \circ \alpha$ lifts to morphism

$$\tilde{f}_{v,k} : B^{[k+1]}M_s(v)^k \to G^0(k, W_{v+\tilde{k}}).$$

Indeed, this would follow from the universal property of $G^0(k, W_{v+\tilde{k}})$ (part 4 of the Proposition) provided that we check that $f \circ \alpha$ pulls back all the Brill-Noether strata to Cartier divisors on $B^{[k+1]}M_s(v)^k$. By the compatibility of $f$ and $\alpha$ with respect to the (total-transform of the) Brill-Noether stratifications, all we need to check is that the subschemes $B^{[k+1]}M_s(v)^k, t \geq k + 1$ are all Cartier divisors in $B^{[k+1]}M_s(v)^k$. This is indeed the case.

Conversely, $G(k, W_{v+\tilde{k}})$ admits a morphism to $G(k, M_S(v + \tilde{k}))$

$$g : G(k, W_{v+\tilde{k}}) \to G(k, M_S(v + \tilde{k})).$$

The existence of $g$ follows from the universal property of $G(k, M_S(v + \tilde{k}))$ (Proposition 30). The morphism $g$ is compatible with the Brill-Noether stratification. Hence, the composition

$$f^{-1} \circ g : G(k, W_{v+\tilde{k}}) \to G_1(k, M_S(v))$$

is compatible with the Brill-Noether stratification (up to a shift by $k$). We claim that $f^{-1} \circ g$ lifts to a morphism

$$\tilde{g}_{v,k} : G(k, W_{v+\tilde{k}}) \to B^{[k+1]}M_s(v)^k.$$
(total-transform of the) Brill-Noether stratifications, all we need to check is that the subschemes $G^0(k, W_{v+k})$, $1 \leq t \leq \mu(v) - k$ are all Cartier divisors in $G(k, W_{v+k})$. This is indeed the case.

Both $f_{v,k}$ and $g_{v,k}$ are birational isomorphisms and $f_{v,k} \circ g_{v,k}$, $g_{v,k} \circ f_{v,k}$ are regular morphisms which are generically the identity. $G(k, W_{v+k})$ is smooth by the induction hypothesis. Hence $f_{v,k} \circ g_{v,k}$ is the identity. By properness, both $f_{v,k}$ and $g_{v,k}$ are surjective. Hence, $f_{v,k}$ is also injective. Since $G(k, W_{v+k})$ is smooth, $f_{v,k}$ must be an isomorphism (by Zariski’s Main Theorem).

The smoothness and codimension formula follow immediately from the induction hypothesis and part (i). The non-emptiness follows from Corollary [34].

(8) We identify $B^{[k+1]}M_S(v)^k$ with $G^0(k, W_{v+k})$. We denote the pullback of the universal sheaves to $B^{[k+1]}M_S(v)^k \times S$ by the same notation $F_v$ and $F_{v+k}$. Consider the two exact sequences (76) and (77) over $G$ where $G$ denotes both $G^0(k, M_S(v+k))$ and $G_1(k, M_S(v))$. Pulling back (76) and (77) to $B^{[k+1]}M_S(v)^k \times S$ via the classifying morphisms of the families $[F_{v+k}, \tau_v^k]$ and $[F_v, (U_v^k)^*(−\sum_{i=k+1}^{\mu(v)} B^{[k+1]}M_S(v)^i)]$ we get the two short exact sequences:

$$0 \rightarrow p^*\tau_v^k \rightarrow F_{v+k} \rightarrow Q \rightarrow 0,$$

and

$$0 \rightarrow p^*U_v^k \left(\sum_{i=k+1}^{\mu(v)} B^{[k+1]}M_S(v)^i\right) \rightarrow E \rightarrow F_v \rightarrow 0. \quad (92)$$

We already know that there exists a line bundle $K$ on $G_1(k, M_S(v))$ such that $(\mathcal{O}_G) \otimes K$ is isomorphic to (74). Hence the pullback of this line bundle to $B^{[k+1]}M_S(v)^k$ (denoted also by $K$) gives the isomorphism between the short exact sequences $(\mathcal{O}_S) \otimes K$ and (71). In particular, we get the isomorphism (83).

Next we construct the isomorphism (83). Pushing forward $(\mathcal{O}_S) \otimes K$ and (71) via the projection $p : B^{[k+1]}M_S(v)^k \times S \rightarrow B^{[k+1]}M_S(v)^k$ we get the short exact sequence

$$0 \rightarrow \tau_v^k \rightarrow p_*F_{v+k} \rightarrow (p_*F_v) \otimes K \rightarrow 0. \quad (93)$$

Both $q_v^k$ and $W_v^k \otimes K$ are subsheaves of $(p_*F_v) \otimes K$ of maximal rank. At a generic point (in the dense open subset $B^{[k+1]}M_S(v)^k \setminus \cup_{i=k+1}^{\mu(v)} B^{[k+1]}M_S(v)^i$) all three sheaves are equal.

Moreover, $(p_*F_v) \otimes K$ is a subsheaf of the vector bundle $p_*F_v(n) \otimes K = B^{[k+1]}V_v(n) \otimes K$ (see (70)). By part (ii), $W_v^k \otimes K$ is a subbundle of $B^{[k+1]}V_v(n) \otimes K$. The short exact sequence (83) implies that $q_v^k$ is also a subbundle of $p_*F_v(n) \otimes K$ (the injective sheaf homomorphism is also injective on each fiber). These two subbundles of $B^{[k+1]}V_v(n) \otimes K$ are equal on a dense open subset. We get two sections of the Grassmannian bundle $G(\chi(v) + k, B^{[k+1]}V_v \otimes K) \rightarrow B^{[k+1]}M_S(v)^k$ which are equal on a dense open subset. Smoothness of $B^{[k+1]}M_S(v)^k$ implies that they are globally equal.
It suffices to prove the statement over $\mathcal{M}_S(v)^k \setminus \mathcal{M}_S(v)^{k+1}$. Let $F$ be such a sheaf and

$$0 \to H^1(F) \otimes \omega_S \hookrightarrow E \to F \to 0$$

the natural extension. It is known that the Petri map $\phi: H^1(F)^* \otimes H^0(F) \to T_{[F]}^* \mathcal{M}_S(v)$ becomes the Yoneda product under the natural identifications $H^1(F)^* \cong \text{Ext}^1(F, \omega_S)$, $H^0(F) \cong \text{Hom}(\omega_S, F \otimes \omega_S)$, and $T_{[F]}^* \mathcal{M}_S(v) \cong \text{Ext}^1(F, F \otimes \omega_S)$. Let $W \in G(k, H^0(E))$ represent the point $i(H^1(F) \otimes H^0(\omega_S))$ and

$$\psi: T_W G(k, H^0(E)) \to T_{[F]}^* \mathcal{M}_S(v)$$

the differential of the natural embedding. It is easy to verify that $\psi$ is also induced by the Yoneda product under the natural identification of $T_W G(k, H^0(E))$ with $\text{Ext}^1(F, \mathcal{O}_S) \otimes H^0(F)$ and $T_{[F]}^* \mathcal{M}_S(v)$ with $\text{Ext}^1(F, F)$. It follows that the following diagram commutes

$$\begin{array}{ccc}
\text{Ext}^1(F, \omega_S) \otimes H^0(F) & \xrightarrow{\phi} & \text{Ext}^1(F, F \otimes \omega_S) \\
\uparrow & & \uparrow \\
\text{Ext}^1(F, \mathcal{O}_S) \otimes H^0(F) & \xrightarrow{\psi} & \text{Ext}^1(F, F)
\end{array}$$

where the vertical isomorphisms are induces by cup-product with the symplectic structure of $S$. Part 3 follows since the right vertical isomorphism is the one induced by the symplectic structure on $\mathcal{M}_S(v)$.

\[\square\]

**Corollary 37** The collection $\{\mathcal{M}_S(v + \epsilon \cdot \vec{r})^i \mid 0 \leq r \leq \mu(v)\}$ is dualizable. Above $\epsilon$ is 1 if $\chi(v)$ is positive, $-1$ if $\chi(v)$ is negative and either one if $\chi(v) = 0$.

**Proof:** Dualizability condition 4 follows from Mukai’s dimension formula $\dim \mathcal{M}_S(v) = 2 + (\langle v, v \rangle)$. Conditions 2 and 3 are verified in part 6 of Proposition 36 (where $P_W B^{(k), \mathcal{M}_S(v + t)}$ is denoted by $P_W B^{(k)}$). The compatibility of the two stratifications in Condition 4 is proven exactly as in the last part of the proof of Theorem 33. Condition 5 follows from Lemma 14 part 3 and part 3 of Proposition 36.

\[\square\]

**Remark 38** In the absence of global universal sheaves, the statement of Proposition 36 should be modified as follows: The vector bundles $B^{(k)}V_{\nu, 0}$, $B^{(k)}V_{\nu, 1}$, $W^{k}_\nu$, $U^{k}_\nu$ exist only locally, but the corresponding projective bundles exist globally. The vector bundles $\mathcal{H}om(B^{(k)}V_{\nu, 0}, B^{(k)}V_{\nu, 1})$ and their sections $B^{(k)}(\rho_{\nu})$ in (34) exist globally (see Remark 32). The statement involving equation (32) is valid only locally. The vector bundles $\mathcal{H}om(q^{k}_\nu, \tau^{k}_\nu)$ in (36) and $\mathcal{H}om(W^{k}_\nu, U^{k}_\nu)$ in (37) exist globally. Equations (34) and (35) should be replaced by their projectivization but the simultaneous conjugation by one and the inverse of the other in part 3 of the proposition is a global isomorphism of vector bundles.

The proof goes through if we replace the global family $\mathcal{F}_v$ by an open covering of $\mathcal{M}_S(v)$ (in the complex or étale topology) and a local family over each open set. The main
point is that any choice of gluing transformations (a cochain over \( \mathcal{M}_S(v) \times S \)) fails only slightly to be a cocycle. The coboundary of any such choice of gluing transformations is necessarily the pullback of a Čech 2-coboundary for the sheaf \( \mathcal{O}_{\mathcal{M}_S(v)}^\times \) of invertible functions over \( \mathcal{M}_S(v) \). This follows from the simplicity of the sheaves involved (see [Mu2] Appendix 2).

5.8 Lazarsfeld’s reflection isomorphism

**Theorem 39** 1. There exists a natural isomorphism

\[
\tilde{q}_t : G^0(a + b + t, \mathcal{M}_S(a, \mathcal{L}, b)) \xrightarrow{\sim} G^0(a + b + t, \mathcal{M}_S(b + t, \mathcal{L}, a + t)),
\]

for all integers \( a, b, t \) satisfying \( a \geq 0 \) and \( b + t \geq 0 \). Equivalently, there exist natural isomorphisms

\[
G^0(k, \mathcal{M}_S(v)) \xrightarrow{\sim} G^0(k, \mathcal{M}_S(\sigma \circ \tau(v - \bar{k}))), \quad \text{for} \quad k \geq \text{rank}(v) \geq 0, \quad \text{and}
\]

\[
G^0(\chi(v), \mathcal{M}_S(v)) \xrightarrow{\sim} G^0(\chi(v), \mathcal{M}_S(\sigma(v))), \quad \text{for} \quad \chi(v) \geq \text{rank}(v) \geq 0.
\]

2. The compositions \( \tilde{q}_t \circ \tilde{q}_{-t} \) and \( \tilde{q}_{-t} \circ \tilde{q}_t \) are both the identity morphism.

**Example 40** Consider the case where \( v = (1, \mathcal{L}, g - d) \), \( 1 \leq d \leq 2g - 2 \), and \( k = 2 \). Then \( \mathcal{M}_S(v) = S^{[d]} \), \( G^0(2, S^{[d]}) \) parametrizes pairs consisting of a \( \mathbb{P}^{g-2} \) in \( \mathbb{P}^g \) containing a length \( d \) subscheme \( D \), while \( \mathcal{M}_S(\sigma \circ \tau(v - \bar{2})) = S^{[2g - d]} \). The \( \mathbb{P}^{g-2} \) intersects \( S \) in a length \( 2g - 2 \) subscheme \( \bar{D} \) and the isomorphism

\[
G^0(2, S^{[d]}) \cong G^0(2, S^{[2g - d]})
\]

maps a pair \( D \subset \mathbb{P}^{g-2} \) to the complementary pair \( D^\perp \subset \mathbb{P}^{g-2} \). If \( \bar{D} \) is reduced, then \( D^\perp \) is the set theoretic complement \( \bar{D} \setminus D \). For a general complete intersection \( D \), the dualizing sheaf \( \omega_{\bar{D}} \) is a free \( \mathcal{O}_{\bar{D}} \)-module of rank 1 and the ideal \( \mathcal{I}_{\bar{D}, D^\perp} \) of \( D^\perp \) as a subscheme of \( \bar{D} \) is the annihilator of \( \mathcal{I}_{\bar{D}, D} \otimes_{\mathcal{O}_{\bar{D}}} \omega_{\bar{D}} \) under the perfect local duality pairing \( H^0(\mathcal{O}_{\bar{D}}) \otimes H^0(\omega_{\bar{D}}) \rightarrow \mathbb{C} \).

Theorem 39 is geometrically intuitive also in the following special case:

**The case** \( (a, b) = (1, 0) \) and \( t = 0 \): Let \( \mathcal{C} \subset S \times |\mathcal{L}| \) be the universal curve. \( \mathcal{M}_S(1, \mathcal{L}, 0) \) is \( S^{[g]} \) and \( \mathcal{M}_S(0, \mathcal{L}, 1) \) is the relative Picard Pic\(^g(\mathcal{C}/|\mathcal{L}|) \). Both \( G(1, \mathcal{M}_S(1, \mathcal{L}, 0)) \) and \( G(1, \mathcal{M}_S(0, \mathcal{L}, 1)) \) are isomorphic to the universal relative Hilbert scheme \( \text{Hilb}_g(\mathcal{C}) \) of length \( g \) subschemes of curves in the linear system \( |\mathcal{L}| \).

\[
\begin{array}{ccc}
\text{Pic}_C^g & \xrightarrow{\text{hom}_C(\bullet, \mathcal{O}_C)} & \text{Pic}_C^{-g} \\
\text{Hilb}_g(\mathcal{C}) & \xrightarrow{a} & S^{[g]} \\
\end{array}
\]
Note that both $e$ and $a$ are surjective regular morphisms. The morphism $e : \text{Hilb}_g(C) \to S^{[g]}$ is a birational isomorphism since $g$ points in general position on $S$ determine a unique curve in the linear system $|L|$ passing through them. The morphism $a : \text{Hilb}_g(C/|L|) \to J^{-g}$ is the Abel-Jacobi map (see [A-K] for its construction). It is a birational isomorphism by the Abel-Jacobi theorem. Given an integral curve $C$ with planar singularities, the morphism $\text{Hom}_C(\bullet, \mathcal{O}_C)$ is an involution of the compactified Picard. Set theoretically, we need only the fact that a rank 1 torsion free sheaf $F$ on $C$ is a reflexive $\mathcal{O}_C$-module and $\chi(\text{Hom}_C(F, \omega_C)) = -\chi(F)$. This is a special case of a theorem of Serre which states that the functor $\text{Ext}^c(\bullet, \mathcal{O}_S)$ is exact and involutive on the category of Cohen-Macaulay modules of codimension $c$. Serre’s Theorem enters the picture via:

**Lemma 41** Let $F$ be an $\mathcal{O}_S$-module with pure one-dimensional support $C \subset S$. Then $F$ is a Cohen-Macaulay module and we have a natural isomorphism

$$\text{Hom}_S(F, \omega_C) \cong \text{Ext}^1_S(F, \mathcal{O}_S).$$

**Proof:** Consider the long exact sequence of extension sheaves obtained by taking $\text{Hom}_S(F, \bullet)$ of the short exact sequence

$$0 \to \mathcal{O}_S \to \mathcal{O}_S(C) \to \omega_C \to 0.$$ 

The homomorphism $\text{Ext}^1_S(F, \mathcal{O}_S) \to \text{Ext}^1_S(F, \mathcal{O}_S(C))$ vanishes since $C$ is also the support of $\text{Ext}^1_S(F, \mathcal{O}_S)$. Hence the connecting homomorphism is an isomorphism. \hfill $\square$

**Proof:** (of Theorem 39) The proof consists of two steps. In the first step we describe the bijection (94) set theoretically taking care of stability issues. In the second step we work in families and prove that the two functors coarsely represented by the two moduli spaces are equivalent.

**Step I** (stability) The set theoretic description of the bijection (94) is given below in (109) and (112). A stable sheaf $F \in \mathcal{M}_S(a, L, b)$ and a subspace $U \in G(a + b + t, H^0(F))$ determine the exact sequence:

$$0 \to E(\tilde{F}, U) \xrightarrow{i} U \otimes_C \mathcal{O}_S \xrightarrow{ex} \tilde{F} \to 0$$

where $\tilde{F}$ is the subsheaf of $F$ generated by $U$. Our assumptions imply that $a + b + t \geq a$. We have two cases which are slightly different: $a + b + t = a$ and $a + b + t > a$.

**The case** $a + b + t > a$: Lemma 23 implies that $U$ generates $F$ away from a zero-dimensional subscheme. Thus $F/\tilde{F}$ has a zero-dimensional support

$$0 \to \tilde{F} \xrightarrow{\epsilon} F \to F/\tilde{F} \to 0.$$ 

If $F$ is generated by $U$, we define $q(F, U)$ to be the vector bundle $E(F, U)^* \in \mathcal{M}_S(b + t, L, a + t)$. Otherwise, we proceed to define the stable torsion free sheaf $q(F, U)$ as a sub-sheaf of $E(\tilde{F}, U)^*$ (see (107)). We have the locally free presentations

$$0 \to F \xrightarrow{\iota} F^{**} \to F^{**}/F \to 0$$

and

$$0 \to \tilde{F} \xrightarrow{\iota} F^{**} \to F^{**}/\tilde{F} \to 0.$$
Note that \( E(\tilde{F},U) \) in (96) is locally free and
\[
0 \to E(\tilde{F},U) \xrightarrow{i} U \otimes_{\mathcal{O}_S} \mathcal{O}_S \xrightarrow{\epsilon_U} F^{**} \to F^{**}/\tilde{F} \to 0
\]
is a locally free resolution of \( F^{**}/\tilde{F} \). The Local Duality Theorem ([GH] page 693) implies that
\[
\mathcal{E}xt^1_S(F^{**}/F, \mathcal{O}_S) = \mathcal{E}xt^1_S(F/\tilde{F}, \mathcal{O}_S) = 0, \quad \mathcal{E}xt^2_S(F, \mathcal{O}_S) = 0, \quad \text{and, using (98),}
\]
\[
\mathcal{E}xt^2_S(F/\tilde{F}, \mathcal{O}_S) \cong \mathcal{E}xt^2_S(F^{**}/F, \mathcal{O}_S).
\]
Moreover, we have a perfect pairing
\[
H^0(\mathcal{E}xt^2(F/\tilde{F}, \omega_S)) \otimes_{\mathbb{C}} H^0(F/\tilde{F}) \to \mathbb{C}.
\]
Taking \( \mathcal{H}om(\bullet, \mathcal{O}_S) \) of (94) and using (101) and (102) we get the short exact piece of the long exact sequence of local Exts:
\[
0 \to \mathcal{E}xt^1_S(F, \mathcal{O}_S) \xrightarrow{\epsilon_U} \mathcal{E}xt^1_S(\tilde{F}, \mathcal{O}_S) \to \mathcal{E}xt^2_S(F/\tilde{F}, \mathcal{O}_S) \to 0.
\]
Taking \( \mathcal{H}om(\bullet, \mathcal{O}_S) \) of (96) we get the long exact sequence
\[
0 \to F^* \xrightarrow{\epsilon_U^*} U^* \otimes_{\mathcal{O}_S} \mathcal{O}_S \xrightarrow{i_U^*} E(\tilde{F},U)^* \xrightarrow{\eta_U} \mathcal{E}xt^1_S(\tilde{F}, \mathcal{O}_S) \to 0.
\]
Moding out (106) by \( F^* \) we get an extension class
\[
\epsilon(\tilde{F},U) \in \text{Ext}^1_S \left[ \mathcal{E}xt^1_S(\tilde{F}, \mathcal{O}_S), \text{coker}(\epsilon_U^*) \right].
\]
Pulling back (106) via the injective homomorphism \( \epsilon^* : \mathcal{E}xt^1_S(F, \mathcal{O}_S) \to \mathcal{E}xt^1_S(\tilde{F}, \mathcal{O}_S) \) we get the class
\[
\epsilon(F,U) \in \text{Ext}^1_S \left[ \mathcal{E}xt^1_S(F, \mathcal{O}_S), \text{coker}(\epsilon_U^*) \right]
\]
representing an exact sequence defining \( q(F,U) \)
\[
0 \to F^* \xrightarrow{\epsilon_U^*} U^* \otimes_{\mathcal{O}_S} \mathcal{O}_S \xrightarrow{i_U^*} q(F,U) \xrightarrow{\eta_U} \mathcal{E}xt^1_S(F, \mathcal{O}_S) \to 0.
\]
Moding out (106) by (107) and using the exactness of (105) we see that the sheaf \( q(F,U) \) fits in the short exact sequence
\[
0 \to q(F,U) \xrightarrow{\iota} E(\tilde{F},U)^* \to \mathcal{E}xt^2_S(F/\tilde{F}, \mathcal{O}_S) \to 0.
\]
By definition (96)
\[
c_2(E(\tilde{F},U)) = 2g - 2 - c_2(\tilde{F}) = 2g - 2 - c_2(F) + \text{length}(F/\tilde{F}).
\]
Local Duality \((104)\) and \((108)\) imply that \(c_2(q(F,U)) = 2g - 2 - c_2(F)\). Clearly, \(\text{rank}(q(F,U)) = b + t\) and \(\det(q(F,U)) = \mathcal{L}\). Hence, the Mukai vector of \(q(F,U)\) is

\[
v(q(F,U)) = (b + t, \mathcal{L}, a + t).
\]

It is easy to see that the dual sheaf does not have any global sections \(H^0(q(F,U^*) = H^0(E(\tilde{F}, U)) = 0\). Moreover, \(q(F,U)\) is generated by its global sections away from a zero-dimensional sub-scheme (see \((107)\) and \((103)\)). Lemma \((23)\) implies that \(q(F,U)\) is stable. As \(H^0(F^\ast)\) vanishes, the homomorphism \(i^\ast\) in \((107)\) embeds \(U^\ast\) in \(H^0(S, q(F,U))\). We get a pair \((q(F,U), W) \in C^0(a + b + t, \mathcal{M}(b + t, \mathcal{L}, a + t))\). We define \((104)\) by

\[
\tilde{q}_t(F,U) := (q(F,U), W).
\]

The case \((a,b) = (1,0)\) and \(t = 0\) revisited: This case is part of the more general case \(a + b + t = a\). We treat it separately as a warm-up. Let \((\mathcal{L}(-D), s)\) be a pair consisting of the \(\mathcal{L}\)-twisted ideal sheaf of a length \(g\) subscheme \(D \subset S\) and a section \(s \in \mathbb{P}H^0(S, \mathcal{L}(-D))\). The zero-locus of \(s\) determines, as a section of \(\mathcal{L}\), a curve \(C\) containing \(D\) as a subscheme. We get the triple \((C, \mathcal{L}, s)\) where \(s_1 \in \mathbb{P}\text{Ext}^1_0(\mathcal{I}_{C,D} \otimes \mathcal{L}, \mathcal{O}_S)\) is the extension class of

\[
0 \to \mathcal{O}_S \xrightarrow{s_0} \mathcal{L}(-D) \to \mathcal{I}_{C,D} \otimes \mathcal{L} \to 0.
\]

Serre’s Duality on \(S\) identifies \(\mathbb{P}\text{Ext}^1_0(\mathcal{I}_{C,D} \otimes \mathcal{L}, \mathcal{O}_S)\) with \(\mathbb{P}H^1(S, \mathcal{I}_{C,D} \otimes \mathcal{L})^\ast\). Serre’s Duality on \(C\) identifies the latter with \(\mathbb{P}H^0(C, \text{Hom}(\mathcal{I}_{C,D}, \mathcal{O}_C))\). Denote the corresponding section by \(s_0\). The triple \((C, \text{Hom}(\mathcal{I}_{C,D}, \mathcal{O}_C), s_0)\) is a point in \(G^0(1, \text{Pic}_S(C/|\mathcal{L}|))\).

Conversely, given a triple \((C, L, s_0)\) representing a point in \(G^0(1, \text{Pic}_S(C/|\mathcal{L}|))\) we use Serre’s Duality on \(S\) and on \(C\) to interpret \(s_0\) as an extension class \(s_1\) of \(\mathcal{O}_S\)-modules

\[
0 \to \mathcal{O}_S \hookrightarrow F \xrightarrow{j} \text{Hom}(L, \omega_C) \to 0.
\]

Lemma \((23)\) implies that \(F\), which is \(f(\text{Hom}(L, \omega_C), \text{span}(s_1))\), is a stable rank 1 torsion free sheaf. Composing \(j\) with the evaluation at \(s_0\) \(\text{Hom}(L, \omega_C) \xrightarrow{s_0} \omega_C\) we get a non-zero homomorphism

\[
\bar{s}_0 : F \to \omega_C \cong \mathcal{L}|_C.
\]

It is easy to check that \(\bar{s}_0\) maps to zero by the connecting homomorphism \(\delta\) of the long exact sequence

\[
0 = \text{Hom}(F, \mathcal{O}_S) \to \text{Hom}(F, \mathcal{L}) \to \text{Hom}(F, \mathcal{L}|_C) \xrightarrow{\delta} \text{Ext}^1(F, \mathcal{O}_S)
\]

(replace \((F,V)\) in \((74)\) by \((\text{Hom}(L, \omega_C), \text{span}(s_1))\) and apply Serre’s Duality). Hence \(\bar{s}_0\) determines a non-trivial homomorphism

\[
i : F \hookrightarrow \mathcal{L}.
\]
We recover an ideal sheaf of a length \( g \) subscheme \( D \subset S \) by setting \( \mathcal{I}_{S,D} \) to be the image of \( F \otimes \mathcal{L}^{-1} \xrightarrow{i} \mathcal{O}_S \).

**The case** \( a + b + t = a \): In that case, \( t = -b \) and \( \mathcal{M}_S(b + t, \mathcal{L}, a + t) \) is \( \mathcal{M}_S(0, \mathcal{L}, a - b) \) which is also \( \text{Pic}^{a-b+g-1}_C \). The evaluation homomorphism in (96) is injective and \( E(\tilde{F},U) \) is the zero sheaf. We have a short exact sequence

\[
0 \to U \otimes \mathcal{O}_S \to F \to Q \to 0. \tag{110}
\]

The quotient sheaf \( Q \) is a torsion \( \mathcal{O}_S \)-module which is supported, as a rank 1 torsion free sheaf, on a curve \( C \) in the linear system \( |\mathcal{L}| \) (see Lemma \[23\] Part [3]). In particular, \( Q \) represents a sheaf in \( \mathcal{M}_S(0, \mathcal{L}, b - a) \) which is \( \text{Pic}^{b-a+g-1}_C \). As in the case \( (a,b) = (1,0) \) and \( t = 0 \), we carry out the reflection on the curve \( C \) supporting \( Q \). We define

\[
q(F,U) := \mathcal{H}om_C(Q, \omega_C). \tag{111}
\]

The exact sequence (110) corresponds to an embedding \( i : U^* \hookrightarrow \text{Ext}^1_S(Q, \mathcal{O}_S) \). Serre’s Duality on \( S \) represents \( U \) as a quotient of \( H^1(S, Q \otimes \omega_S) \). Serre’s Duality on \( C \) represents \( U^* \) as a subset \( W \) of \( H^0(q(F,U)) \). We define

\[
\tilde{q}_b(F,U) := (q(F,U), W). \tag{112}
\]

Lemma \[41\] identifies the reflection (111) as an operation on \( \mathcal{O}_S \)-modules: we have an equality \( q(F,U) = \text{Ext}^1_S(Q, \mathcal{O}_S) \).

**The case** \( a + b + t > a \) **revisited:** The reflection (109), while canonical, is inconvenient to work with in families. Upon a choice of an \( a \)-dimensional subspace \( W \subset U \), we can express (109) as the conjugation of the reflection along a curve (112) by the extension isomorphism in Theorem \[33\]. We need (109) in order to prove that the conjugation is independent off the choice of \( W \).

Let \( W \) be an \( a \)-dimensional subspace of \( U \) and assume that \( a + b + t > a \). Then \( Q := F/(W \otimes \mathcal{O}_S) \) is a stable sheaf with pure one-dimensional support (depending on \( W \)). \( U/W \) is a subspace of \( H^0(Q) \) and \( W^* \) is a subspace of \( \text{Ext}^1_S(Q, \mathcal{O}_S) \). Consequently, \( W^* \) is a subspace of \( H^0(\mathcal{E}xt^1_S(Q, \mathcal{O}_S)) \) and \( (U/W) \) is a subspace of \( \text{Ext}^1_S[\mathcal{E}xt^1_S(Q, \mathcal{O}_S), \mathcal{O}_S] \). Theorem \[33\] implies that \( (U/W)^* \) is a subspace of global sections of the stable sheaf \( f[\mathcal{E}xt^1_S(Q, \mathcal{O}_S), U/W] \) extending \( \mathcal{E}xt^1_S(Q, \mathcal{O}_S) \) by \( (U/W)^* \otimes \mathcal{O}_S \). We get that \( U^* \) is isomorphic to the inverse image in \( H^0(f[\mathcal{E}xt^1_S(Q, \mathcal{O}_S), U/W]) \) of \( W^* \) under the quotient sheaf homomorphism \( f[\mathcal{E}xt^1_S(Q, \mathcal{O}_S), U/W] \to \mathcal{E}xt^1_S(Q, \mathcal{O}_S) \).

**Lemma 42** The sheaves \( q(F,U) \) and \( f(q(f^{-1}(F,W), U/W), (U/W)^*) \) are isomorphic and the isomorphism conjugates the embeddings of \( U^* \) as a subspace of global sections.
Proof: In the special case where $U$ generates $F$ and $F$ is locally free, the Lemma follows from the commutative diagram with short exact rows and columns

\[
\begin{array}{ccc}
0 & \to & (U/W)^* \otimes O_S \\
\downarrow & & \downarrow \\
F^* & \to & q(F,U) \\
\downarrow & & \downarrow \\
F^* & \to & \text{Ext}^1_S(Q,O_S)
\end{array}
\]

The proof of the general case is a laborious unwinding of the definition of the reflection (109).

Step II: (of the proof Theorem 33) We work out the relative version of the maps $\tilde{q}_i$ in (109) and (112). Consequently, the functors, which are coarsely represented by $G^0(k,\mathcal{M}_S(v))$ and $G^0(k,\mathcal{M}_S(\sigma \circ \tau(v-\bar{k})))$ are equivalent.

Set $r := \text{rank}(v)$. The case $r = k$ was proven by Le Potier (Theorem 5.7 in [Le1] when $k = r = 0$ and Theorem 5.12 when $k = r > 0$). The general case $k \geq r$ is a conjugate of the case $r = k = 0$ as in Lemma [12]. Assume given 1) a family $\mathcal{F}_v$ over $S \times T$ flat over a scheme $T$, 2) a locally free $O_T$-module $U$ of rank $k$ and 3) a homomorphism $i : U \hookrightarrow p_*(\mathcal{F}_v)$ injective on each fiber. Choose an open covering $\{T_j\}$ of $T$ and local sections of $G(r,U)$ corresponding to subbundles $W_j$ of $U_{T_j}$. By Theorem 33, we have an equivalence of functors $G^0(r,\mathcal{M}_S(v)) \cong G_1(r,\mathcal{M}_S(v-\bar{r}))$. We get a short exact sequence flat over each $T_j$

\[
0 \to W_j \hookrightarrow \mathcal{F}_v \to \mathcal{F}_{v-\bar{r}},j \to 0
\]

and homomorphisms, injective on each fiber,

\[
\begin{align*}
W_j & \hookrightarrow \text{Ext}^1_p(\mathcal{F}_{v-\bar{r}},j,\mathcal{O}_{T_j \times S}) & (113) \\
U/W_j & \hookrightarrow p_*\mathcal{F}_{v-\bar{r}},j. & (114)
\end{align*}
\]

Note that if $k > r$ the families $\mathcal{F}_{v-\bar{r}},j$ need not patch. (Even their support, which is of relative dimension 1, need not patch!) Applying Le Potier’s Theorem 5.7 we get families

\[
\mathcal{F}_{\sigma\tau(v-\bar{k}),j} := \text{Ext}^1_{T_j \times S}(\mathcal{F}_{v-\bar{r}},j,\mathcal{O}_{T_j \times S})
\]

flat over $T_j$ and the analogues of (113) and (114). Applying Theorem 33 once more we get families $\mathcal{F}_{\sigma\tau(v-\bar{k}),j}$ of rank $r - k$ stable sheaves and natural homomorphisms, injective on each fiber,

\[
t_j : (U^*)|_{T_j} \hookrightarrow p_*\mathcal{F}_{\sigma\tau(v-\bar{k}),j}.
\]

We claim that the the families $\mathcal{F}_{\sigma\tau(v-\bar{k}),j}$ and the homomorphisms $t_j$ patch naturally to a global family $\mathcal{F}_{\sigma\tau(v-\bar{k})}$ and a global homomorphism $t$. Indeed, Lemma 12 and the simplicity of the sheaves parametrized imply that the the sheaf $p_*\mathcal{H}om(\mathcal{F}_{\sigma\tau(v-\bar{k}),i},\mathcal{F}_{\sigma\tau(v-\bar{k}),j})$ is a line bundle on $T_i \cap T_j$ and it has a canonical invertible section $\phi_{i,j}$ satisfying $\phi_{i,j} \circ t_j = t_i$. The collection $\{\phi_{i,j}\}$ is a 1-cocycle which glues $\{\mathcal{F}_{\sigma\tau(v-\bar{k}),j}\}$ because its restriction to the invariant subsheaves $U^{*}_{T_j}$ is. This completes the proof of Theorem 39. 

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Remark 43 We can now indicate the modifications necessary in the statement and proof of the analogue of Proposition 36 in case $\text{rank}(v) \geq 0$ and $\chi \leq 0$. Recall that the Brill-Noether stratification is indexed now by $h^0$. In the statement of parts 1 and 2 the rank of $W_{v^+}$ is $k$ and the rank of $U_{v^+}$ is $k - \chi(v')$. In the statement of part 3 we need to replace $G(k, W_{v^+})$ by $G(k, U_{v^+})$. In this case, even though $\text{rank}(v)$ is non-negative, we may end up with a non-empty Brill-Noether locus $M_S(v)$ such that $v - \mathbf{k}$ has negative rank.

If $k > \text{rank}(v)$, we define $U_{v^-}$ to be $W^*_{\text{for}(v^-)}$ and $W_{v^-}$ to be $W^*_{\text{for}(v^-)}$. Also define $G_1(k, M_S(v - \mathbf{k}))$ to be $G_0(k, M_S(\sigma \circ \tau(v - \mathbf{k})))$ if $k > \text{rank}(v)$.

In the proof of part 6 we need to replace the isomorphism (90) by

\[ f : G_0(k, M_S(v)) \overset{\cong}{\rightarrow} G_1(k, M_S(v - \mathbf{k})). \]

If $k \leq \text{rank}(v)$, this isomorphism follows from Theorem 53. If $k > \text{rank}(v)$, the above definitions translate it to the isomorphism

\[ G_0(k, M_S(v)) \cong G_0(k, M_S(\sigma \circ \tau(v - \mathbf{k}))). \]

The latter isomorphism is precisely Theorem 59.

5.9 The collections are dual

We prove in this section that the two dualizable collections in Theorem 20 are dual to each other. This completes the proof of the Theorem.

Proposition 44 Let $v = (r, \mathcal{L}, s)$ be a Mukai vector and $\mathcal{L}$ a line bundle satisfying Condition 4. Then there exist natural isomorphisms

\[ \tilde{q} : B_1^1 M_S(v) \overset{\cong}{\rightarrow} B_1^1 M_S(\sigma(v)), \quad \text{and} \]

\[ \tilde{q} : B_1^1 M_S(v) \overset{\cong}{\rightarrow} B_1^1 M_S(\tau(v)) \]

(115) compatible with respect to the Brill-Noether loci.

Proof: Proposition 44 follows from the universal properties of the coarse moduli spaces involved. It suffices to construct the isomorphism (115) in case $v = (a, \mathcal{L}, b)$ and both $a$ and $b$ are non-negative integers. Proposition 30 part 4 produces the pair $(B_1^1 M_S(v), \mathbb{P}W_v)$. By the universal property of $G_0(\chi(v), M_S(v))$ we get a morphism

\[ B_1^1 M_S(v) \rightarrow G_0(\chi(v), M_S(v)) \]

(see Proposition 30). Similarly, we get a morphism

\[ B_1^1 M_S(\sigma(v)) \rightarrow G_0(\chi(\sigma(v)), M_S(\sigma(v))). \]
By Theorem 39, the two moduli spaces $G^0(\chi(v), \mathcal{M}_S(v))$ and $G^0(\chi(\sigma(v)), \mathcal{M}_S(\sigma(v)))$ are isomorphic. Part 4 of Proposition 36 implies that there exist morphisms

$$B^{[1]}\mathcal{M}_S(v) \to G(0, W_{\sigma(v)}) = B^{[1]}\mathcal{M}_S(\sigma(v)), \quad \text{and}$$
$$B^{[1]}\mathcal{M}_S(\sigma(v)) \to G(0, W_v) = B^{[1]}\mathcal{M}_S(v).$$

The composition of the two morphisms is generically the identity morphism (by Corollary 34). It must be globally the identity since $B^{[1]}\mathcal{M}_S(v)$ and $B^{[1]}\mathcal{M}_S(\sigma(v))$ are smooth (part 7 of Proposition 36).

It suffices to construct the isomorphism (116) in case $a \geq 0$ and $b \leq 0$. The construction is similar to that of (114). Simply use Theorem 33 instead of Theorem 39. $\square$

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