BOUND NON OF THE SOLUTIONS TO A KIND OF PARABOLIC SYSTEMS

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Abstract. We deal with nonlinear systems of parabolic type satisfying componentwise structural conditions. The nonlinear terms are Carathéodory maps having controlled growth with respect to the solution and the gradient and the data are in anisotropic Lebesgue spaces. Under these assumptions we obtain essential boundedness of the weak solutions.

1. Introduction

This paper aims the boundedness of the weak solutions to the following nonlinear divergence form systems

\[ u_t - \text{div} A(x, t, u, Du) + b(x, t, u, Du) = 0 \quad (x, t) \in Q_T \quad (1.1) \]

where \( \Omega \subset \mathbb{R}^n \), \( n \geq 1 \), is a bounded domain, \( Q_T = \Omega \times (0, T) \), \( T > 0 \), is a cylinder in \( \mathbb{R}^n \times \mathbb{R}_+ \) and the nonlinear operators

\[
A(x, t, u, z): Q_T \times \mathbb{R}^N \times \mathbb{R}^{N \times n} \to \mathbb{R}^{N \times n} \\
b(x, t, u, z): Q_T \times \mathbb{R}^N \times \mathbb{R}^{N \times n} \to \mathbb{R}^N
\]

where \( A = \{A^i_j\}_{1 \leq i, j \leq n} \) and \( b = (b^1, \cdots, b^N) \) are Carathéodory maps, that is, these are measurable in \( (x, t) \in Q_T \) for every \( (u, z) \in \mathbb{R}^N \times \mathbb{R}^{N \times n} \) and continuous in \( (u, z) \) for almost all \( (a.a.) \) \( (x, t) \in Q_T \).

Concerning linear divergence form equations of elliptic type, it is well known by the result of De Giorgi and Nash \cite{11, 20} that any solution is locally Hölder continuous assuming that the leading coefficients are only \( L^\infty \). Almost ten years were necessary to show, via a counterexample, that this is not true for divergence form elliptic systems under the same minimal conditions on the coefficients because of the lack of maximum principle (cf. \cite{12}). Notice that Giusti and Miranda in \cite{16}

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proved that the De Giorgi-Nash result does not hold for quasi-linear systems, even if the coefficients are analytic functions.

In general, in order to obtain essential boundedness of the solutions, it is necessary to add some more restrictions on the structure of the operators considered. The simplest example is given by systems in diagonal form, or the so-called \textit{decoupled} systems.

\textbf{Example 1.1.} Let $u : \Omega \to \mathbb{R}^N$ be a weak solution of $\text{div} \left( A(x, Du) \right) = 0$ in $\Omega$ with

$$
A^\alpha_i (x, Du) = \sum_{j=1}^{n} \sum_{\beta=1}^{N} \delta_{\alpha \beta} A^\alpha_{ij} (x) D_j u^\beta
$$

where $\delta_{\alpha \beta}$ is the Kronecker delta. Then each component $u^\alpha$ solves a single elliptic equation and $\sup_\Omega u^\alpha \leq \sup_{\partial \Omega} u^\alpha$ for each $\alpha = 1, \ldots, N$.

As it concerns nonlinear elliptic type systems, in order to study the regularity properties of the operator, more structural conditions are required. For instance, divergence form equations

$$
\text{div} \left( A(x, u, Du) \right) = b(x, u, Du)
$$

have been studied under various componentwise conditions.

In [21] Nečas and Stará analyzed quasi-linear systems which are diagonal for large values of $u^\alpha$. Precisely, for each $\alpha = 1, \ldots, N$, it is assumed that

$$
0 < \theta_\alpha \leq u^\alpha \implies A^\alpha_i (x, u, Du) = \sum_{j=1}^{n} \sum_{\beta=1}^{N} \delta_{\alpha \beta} A^\alpha_{ij} (x, u) D_j u^\beta
$$

where the operator $\{A^\alpha_{ij}\}_{i \leq j \leq n}^{1 \leq \alpha \leq N}$ is supposed to be bounded and elliptic. Then each solution of the system $\text{div} \left( A(x, u, Du) \right) = 0$ verifies

$$
\sup_\Omega u^\alpha \leq \max \{ \theta_\alpha; \sup_{\partial \Omega} u^\alpha \}.
$$

In [19] Leonetti and Petricca analyzed nonlinear elliptic systems

$$
-\text{div} \left( A(x, u, Du) \right) + b(x, u, Du) = f(x) \quad x \in \Omega
$$

where they impose componentwise coercivity condition on the principal part and positivity of the lower order terms operator for large values of $u^\alpha$ that is there exists $\theta_\alpha > 0$ such that for each $u^\alpha \geq \theta_\alpha$, $\alpha = 1, \ldots, N$ one has

$$
\begin{align*}
\nu |z^\alpha|^p - M_\alpha & \leq \sum_{i=1}^{n} A^\alpha_i (x, u, z) z_i^\alpha \quad x \in \Omega \\
0 & \leq b^\alpha (x, u, z).
\end{align*}
$$

(1.3)
A bound for \( \|u\|_{L^\infty(\Omega, \mathbb{R}^N)} \) has been obtained under that hypotheses when the \( x \)-behavior of the nonlinear terms \( A(x, u, Du) \) and \( b(x, u, Du) \) is controlled in Lebesgue (19) and Morrey (13, 23, 25, 26) spaces.

Some more results concerning boundedness and regularity of the solutions for nonlinear elliptic and parabolic operators can be found in [4, 5, 6, 7, 8, 9, 10, 14, 15, 17, 22, 24].

As it concerns non-coercive equations, we can mention the works of Boccardo [3] that treat nonlinear elliptic equations and [1, 2] in case of non-coercive nonlinear equations in unbounded domain.

Recently there are studied nonlinear systems satisfying componentwise coercivity conditions [19].

In particular, we consider the following general nonlinear system

\[
\text{div } A(x, u, Du) = b(x, u, Du). \tag{1.4}
\]

In [19] the authors proves point-wise bounds for the components of the solutions. This can be done assuming componentwise coercivity condition on the main part of the system \( A \). The nonlinear system (1.2) is treated also in [25] where the conditions assumed on the operators \( A \) and \( b \) are expressed in terms of Morrey functions.

In what follows we use the standard notation

- \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n, (x, t) \in \mathbb{R}^n \times \mathbb{R}_+ \);
- \( B_\rho(x) = \{ y \in \mathbb{R}^n : |x - y| < \rho \} \) is a ball in \( \mathbb{R}^n \) centered in \( x \) and of radius \( \rho > 0 \), \( |B_\rho| \sim C\rho^n \);
- \( \Omega \subset \mathbb{R}^n, n > 2 \) is a bounded domain with a boundary \( \partial \Omega \), \( \overline{\Omega} \) is the closure of \( \Omega \), and \( |\Omega| \) stands for the Lebesgue measure of \( \Omega \);
- \( Q_T = \Omega \times (0, T) \) denotes a cylinder with base \( \Omega \) and high \( T > 0 \);
- \( S_T = \partial \Omega \times [0, T] \) is the lateral boundary of \( Q_T \);
- \( \partial Q_T = S_T \cup \Omega \times \{t = 0\} \) stands for the parabolic boundary of \( Q_T \), and \( \overline{Q}_T = \overline{\Omega} \times [0, T] \);
- \( \chi_D \) is the characteristic function of any set \( D \subset \mathbb{R}^{n+1} \);
- \( \mathbf{u} = (u^1, \ldots, u^N) \) is a vector function in \( \mathbb{R}^N, N \geq 2 \), where \( u^\alpha(x, t) : Q_T \rightarrow \mathbb{R}, \forall \alpha = 1, \ldots, N \);
- \( u^{\alpha, k} = \max\{u^\alpha(x, t) - k; 0\} \) for any \( k > 0 \) and all \( (x, t) \in Q_T \);
- \( D_iu^\alpha = \partial u^\alpha / \partial x_i \) and \( D\mathbf{u} = \{D_iu^\alpha\}_{i \leq n}^{\alpha \leq N} \in \mathbb{R}^{N \times n} \) is the vector of the space derivatives of \( \mathbf{u} \), \( D_iu^\alpha = u^\alpha_i = \partial u^\alpha / \partial t \);
- \( D_iu^{\alpha, k} = D_iu^\alpha \) and \( D_iu^{\alpha, k} = u^\alpha_i \) for all \( \alpha = 1, \cdots, N \);
- for any vector-function \( \mathbf{f} \in L^p(Q, \mathbb{R}^N), p > 1 \) we write \( \|\mathbf{f}\|_{L^p(Q)} \) instead of \( \|\mathbf{f}\|_{L^p(Q, \mathbb{R}^N)} \).
The letter $C$ denotes a positive constant depending on known quantities that can vary from one occurrence to another. In addition the summation convention on the repeated indexes is adopted.

2. Definitions and auxiliary results

We say that the domain $\Omega$ satisfies the (A) condition if there exists a positive constant $A_\Omega < 1$ such that, for each ball $B_\rho$ centered in some point $x \in \partial \Omega$ and $\rho \leq \text{diam} \Omega$, there holds

$$|\Omega \cap B_\rho| \leq (1 - A_\Omega)|B_\rho|.$$ (A)

For the function spaces that we are going to use we adopt the notations introduced in [18].

1. We say that $u = (u^1, \ldots, u^N) : Q_T \rightarrow \mathbb{R}^N$ belongs to $L^2(Q_T; \mathbb{R}^N)$ if $u^\alpha \in L^2(Q_T)$ for each $\alpha = 1, \ldots, N$ and

$$\|u\|_{2;Q_T} = \left( \int_{Q_T} |u|^2 \, dx \, dt \right)^\frac{1}{2} = \left( \int_0^T \int_{\Omega} \sum_{\alpha=1}^N |u^\alpha(x,t)|^2 \, dx \, dt \right)^\frac{1}{2}.$$

2. For any $q, r \geq 1$ we consider the anisotropic Lebesgue space $L^{q,r}(Q_T; \mathbb{R}^N)$ equipped with mixed norm

$$\|u\|_{q,r;Q_T} = \left[ \int_0^T \left( \int_{\Omega} \sum_{\alpha=1}^N |u^\alpha(x,t)|^q \, dx \right)^\frac{q}{r} \, dt \right]^\frac{1}{r}.$$

In the particular case $q = 2, r = \infty$, we have

$$\|u\|_{2,\infty;Q_T} = \text{esssup}_{t \in [0,T]} \|u(\cdot, t)\|_{2;\Omega}$$

and for $q = r$ we obtain the parabolic Lebesgue space $L^q(Q; \mathbb{R}^N)$.

3. $W^{1,0}_{2}(Q_T; \mathbb{R}^N)$ and $W^{1,1}_{2}(Q_T; \mathbb{R}^N)$ are the Banach spaces consisting of all $u \in L^2(Q_T; \mathbb{R}^N)$ for which the following norms are finite

$$\|u\|_{W^{1,0}_{2}(Q_T)} = \|u\|_{2;Q_T} + \|Du\|_{2;Q_T},$$

$$\|u\|_{W^{1,1}_{2}(Q_T)} = \|u\|_{2;Q_T} + \|Du\|_{2;Q_T} + \|u_t\|_{2;Q_T}.$$

The space $\dot{W}^{1,1}_{2}(Q_T; \mathbb{R}^N)$ is the closure of $C_0^\infty(Q_T; \mathbb{R}^N)$ with respect to the last norm.

4. $V_2(Q_T; \mathbb{R}^N)$ is a subspace of $W^{1,0}_{2}(Q_T; \mathbb{R}^N)$ such that

$$\|u\|_{V_2(Q_T)} = \|u\|_{2,\infty;Q_T} + \|Du\|_{2;Q_T} < \infty.$$ (2.1)

Namely,

$$V_2(Q_T; \mathbb{R}^N) = L^\infty(0, T; L^2(\Omega; \mathbb{R}^N)) \cap L^2(0, T; W^1_2(\Omega; \mathbb{R}^N)).$$
endowed with the norm (2.1).  
5. $V_{2}^{1,0}(Q_T; \mathbb{R}^N)$ consists of all $u \in V_2(Q_T; \mathbb{R}^N)$ continuous in $t$ with respect to the norm of $L^2(Q_T; \mathbb{R}^N)$, that is
\[
\lim_{\Delta t \to 0} \|u(\cdot, t + \Delta t) - u(\cdot, t)\|_{2, \Omega} = 0, \tag{2.2}
\]
endowed with the norm (2.1). As usual, we denote by $\overset{\circ}{V}_2(Q_T; \mathbb{R}^N)$ the closure of $C_0^\infty(Q_T; \mathbb{R}^n)$ with respect to the norm (2.1).

For reader’s convenience, we recall the Hölder inequality on anisotropic Lebesgue spaces and some of its modifications (cf. [18, Ch. II]). Precisely, for any exponents $q \geq q_1 \geq 1$, $r \geq r_1 \geq 1$, there hold
\[
\left| \int_{Q_T} u(x, t) v(x, t) \, dx \, dt \right| \leq \|u\|_{q, r; Q_T} \|v\|_{\frac{q}{r} - \frac{q_1}{r_1}; Q_T}, \tag{2.3}
\]
and
\[
\|uv\|_{q_1, r_1; Q_T} \leq \|u\|_{q, r; Q_T} \|v\|_{\frac{q_1}{r_1} \frac{r_1}{q_1}; Q_T}. \tag{2.4}
\]

Now, we give also some properties of the spaces $V_2(Q_T)$ and $\overset{\circ}{V}_2(Q_T)$.

Let $u \in \overset{\circ}{V}_2(Q_T)$, then from the Gagliardo–Nirenberg interpolation inequality for the $\overset{\circ}{W}^{1,2}(\Omega)$-functions there exists a constant $C_q = C_q(n, q)$ such that
\[
\|u(\cdot, t)\|_{q, \Omega} \leq C_q \|u(\cdot, t)\|_{2, \Omega}^{\frac{1}{2} - \frac{\alpha}{2}} \|Du(\cdot, t)\|_{2, \Omega}^{\frac{\alpha}{2}} \quad \text{for a.a. } t \in [0, T],
\]
with $\alpha = \frac{n}{2} - \frac{n}{q}$, and where $q$ is as in (2.6). Hence, we have the following imbedding type inequality in $\overset{\circ}{V}_2(Q_T)$
\[
\|u\|_{q, r; Q_T} \leq C_q \|u\|_{2, \infty; Q_T}^{\frac{1}{2} - \frac{\alpha}{r}} \|Du\|_{2, \infty; Q_T}^{\frac{\alpha}{r}}.
\]
Taking $\alpha = 2/r$, where $r$ is as in (2.6), we get a series of inequalities and the first one is
\[
\|u\|_{q, r; Q_T} \leq C_q \|u\|_{2, \infty; Q_T}^{\frac{1}{2} - \frac{2}{r}} \|Du\|_{2; Q_T}^{\frac{2}{r}} \tag{2.5}
\]
where $\frac{1}{r} + \frac{n}{2q} = \frac{n}{4}$, and $r$ and $q$ satisfy
\[
\begin{cases}
  r \in [2, \infty] & q \in [2, \frac{2n}{n-2}], & \text{for } n \geq 3 \\
  r \in (2, \infty) & q \in [2, \infty), & \text{for } n = 2 \\
  r \in [4, \infty] & q \in [2, \infty], & \text{for } n = 1.
\end{cases} \tag{2.6}
\]
Estimating the right-hand side of (2.5) by Young’s inequality
\[
a b \leq \frac{2}{r} a^{\frac{r}{2}} + \frac{r - 2}{r} b^{\frac{r}{r-2}}, \quad r > 2
\]
we obtain the second inequality
\[ \|u\|_{q,r;Q_T} \leq C_q \frac{2}{r} \|Du\|_{2;Q_T} + C_q \frac{r-2}{r} \|u\|_{2,\infty;Q_T} \leq \beta \|u\|_{V^2(Q_T)} \] (2.7)
where \( \beta = \beta(n, q, r) \) and \( q \) and \( r \) are as in (2.6).

One more notion that we need is that of the Steklov average. For a function \( \zeta(x, t) \in \overset{\circ}{V}^1_2(Q_{[h,T]}) \) that is zero for \( t \leq 0 \) and \( t \geq T - h \), we define the Steklov average in the future as
\[ \zeta_h(x, t) = \frac{1}{h} \int_{t-h}^{t+h} \zeta(x, \tau) \, d\tau \] (2.8)
and the Steklov average in the past as
\[ \zeta_h(x, t) = \frac{1}{h} \int_{t}^{t-h} \zeta(x, \tau) \, d\tau. \] (2.9)

We notice that, for each fixed \( h \), \( \zeta_h(x, t) \) is defined in \( Q_{T-h} = \overline{\Omega} \times [0, T-h] \) and \( \zeta_h(x, t) \) is defined in \( Q_{[-h,T]} = \overline{\Omega} \times [-h, T] \), where \( 0 < h < T \). It is well known that if \( \zeta \in L^{q,r}(Q_T) \) with \( q, r \geq 1 \), then \( \zeta_h \) approximates \( \zeta \) with respect to the norm in \( L^{q,r} \). Furthermore, if \( \zeta \in \overset{\circ}{V}^1_2(Q_T) \), then \( \zeta_h \in \overset{\circ}{W}^{1,1}_2(Q_{T-h}) \) and \( \zeta_h \) approximates \( \zeta \) with respect to the norm in \( V_2 \) (cf. [18, Ch. II]). Moreover, if we take \( \zeta \in \overset{\circ}{V}^1_2(Q_T) \), then \( \zeta_h \in \overset{\circ}{W}^{1,1}_2(Q_{T-h}) \).

We recall that the Steklov averages satisfy these two properties:
\[ (\zeta_h)_t = (\zeta)_\tau \]
\[ \int_0^T u(x, t) \zeta_h(x, t) \, dt = \int_0^{T-h} u_h(x, t) \zeta(x, t) \, dt. \] (2.10)

The last equation holds for any square summable function, such that \( \zeta(x, t) = 0 \) for \( t \leq 0 \) and for \( t \geq T - h \).

The following result permits to give a total estimate for the maximum of the modulus of the solutions in the whole domain of the definition. Let \( u \) in \( V_2(Q_T) \). Denote by \( u^k = \max\{u(x, t) - k; 0\} \), \( k > 0 \) for all \( (x, t) \in Q_T \) and define the set
\[ A^k(t) = \{x \in \Omega : u(x, t) > k, t \in [0, T]\}. \]

**Theorem 2.1** ([18], Ch. II, Theorem 6.1). Suppose that \( \|u\|_{\infty;S_T} \leq M_0 \), \( M_0 \geq 0 \), and
\[ \|u^k\|_{V^1_2(Q_T)} \leq Ck \mu_{\frac{1+r}{r}}(k) \] (2.11)
hold for \( k \geq M_0 \), where \( \mu(k) = \int_0^T |A_k(t)|^{\frac{\varepsilon}{r}} dt \), \( q \) and \( r \) as in (2.6) and \( 0 < \varepsilon < 1 \). Then

\[
\|u\|_{\infty; Q_T} \leq 2M_0 \left[ 1 + 2^{\frac{2\varepsilon}{r}} + \frac{1}{2} \beta c \right] T^{1 + \frac{\varepsilon}{r}} |\Omega|^{1 + \frac{\varepsilon}{r}}
\]

where \( \beta \) is the constant from (2.7) and \( C \) is an arbitrary constant.

3. Statement of the problem

Recall that we are going to study the boundedness of the solutions to the system

\[
u_t^\alpha - \sum_{i=1}^n D_i \left( A_i^\alpha(x, t, u, Du) \right) + b^\alpha(x, t, u, Du) = 0 \quad (3.1)
\]

in \( Q_T \) with \( \alpha = 1, \ldots, N \).

The vector function \( u \in V_2^{1,0}(Q_T; \mathbb{R}^N) \) is a weak solution of (3.1) if, for any \( 0 \leq t_0 \leq t_1 \leq T \) and each \( \alpha = 1, \ldots, N \) it holds

\[
- \int_{t_0}^{t_1} \int_\Omega u^\alpha(x, t) \eta_t^\alpha(x, t) \, dx \, dt + \int_{t_0}^{t_1} \int_\Omega \left\{ \sum_{i=1}^n A_i^\alpha(x, t, u, Du) \right\} D_i \eta^\alpha(x, t) \, dx \, dt + \int_{t_0}^{t_1} \int_\Omega b^\alpha(x, t, u, Du) \eta^\alpha(x, t) \, dx \, dt = 0
\]

for any \( \eta(x, t) \in \tilde{W}_2^{1,1}(Q_T; \mathbb{R}^N) \).

Now, we take as test function in (3.2)

\[
\zeta^\alpha_h = \frac{1}{h} \int_{t-h}^{t} \zeta^\alpha(x, \tau) \, d\tau,
\]

where \( \zeta^\alpha \in \tilde{V}_2^{1,0}(Q_{T-h}) \). Then, (3.2) becomes

\[
- \int_{t_0}^{t_1} \int_\Omega u^\alpha(x, t) \left( \zeta^\alpha_h \right)_t(x, t) \, dx \, dt + \int_{t_0}^{t_1} \int_\Omega \left\{ \sum_{i=1}^n A_i^\alpha(x, t, u, Du) \right\} D_i \zeta^\alpha_h(x, t) \, dx \, dt + \int_{t_0}^{t_1} \int_\Omega b^\alpha(x, t, u, Du) \zeta^\alpha_h(x, t) \, dx \, dt + \int_{t_0}^{t_1} \int_\Omega u^\alpha(x, t) \zeta^\alpha_h(x, t) \, dx \, dt = 0
\]

(3.3)
where \(0 \leq t_0 \leq t_1 \leq T - h\). By integration by parts and (2.10) the first term in (3.3) becomes

\[
- \int_{t_0}^{t_1} \int_{\Omega} u^\alpha(x, t) (\xi_{\alpha}^h_t \times t) \, dx \, dt = - \int_{t_0}^{t_1} \int_{\Omega} u^\alpha(x, t) (\xi_{\alpha}^h) \, dx \, dt
\]

\[
= - \int_{t_0}^{t_1} \int_{\Omega} u^\alpha_h(x, t) \xi_{\alpha}^h(x, t) \, dx \, dt
\]

\[
= - \int_{t_0}^{t_1} \int_{\Omega} u^\alpha_h(x, t) \xi_{\alpha}^h(x, t) \bigg|_{t_0}^{t_1} \, dx + \int_{t_0}^{t_1} \int_{\Omega} u^\alpha_h(x, t) \xi_{\alpha}^h(x, t) \, dx \, dt
\]

\[
= - \int_{t_0}^{t_1} \int_{\Omega} u^\alpha(x, t) \xi_{\alpha}^h(x, t) \bigg|_{t_0}^{t_1} \, dx + \int_{t_0}^{t_1} \int_{\Omega} u^\alpha_h(x, t) \xi_{\alpha}^h(x, t) \, dx \, dt,
\]

where \(u^\alpha_h\) is the Steklov average of \(u^\alpha\) defined true the (2.8). Hence,

\[
\int_{t_0}^{t_1} \int_{\Omega} \left[ u^\alpha_h(x, t) \xi_{\alpha}^h(x, t) + \sum_{i=1}^{n} A^\alpha_i(x, t, u, Du) \xi_{\alpha}^h(x, t) \right] \, dx \, dt = 0.
\]

Let us take \(\xi_{\alpha}^h(x, t) = \xi^2(x, t) \max\{u^\alpha_h(x, t) - k; 0\} = \xi^2 u^\alpha_{h,k}\) where \(\xi(x, t)\) is an arbitrary nonnegative continuous piecewise-smooth function that is equal to zero on the lateral boundary \(S_T\). Keeping in mind that

\[
D_t u^\alpha_{h,k} = D_t u^\alpha, \quad D_i u^\alpha_{h,k} = D_i u^\alpha
\]

we transform the first term in (3.4) as follows

\[
\int_{t_0}^{t_1} \int_{\Omega} u^\alpha_h u^\alpha_{h,k} \xi^2 \, dx \, dt = \frac{1}{2} \int_{t_0}^{t_1} \left( u^\alpha_{h,k} \right)^2 \xi^2(x, t) \bigg|_{t_0}^{t_1} \, dx
\]

\[
- \int_{t_0}^{t_1} \int_{\Omega} \left( u^\alpha_{h,k} \right)^2 \xi(x, t) \xi_t(x, t) \, dx \, dt.
\]

Now, we pass to the limit in (3.4) as \(h \to 0\), obtaining

\[
\frac{1}{2} \left\| u^\alpha_{h,k} \cdot t \xi \cdot t \right\|_{2, \Omega}^2 \bigg|_{t_0}^{t_1} + \int_{t_0}^{t_1} \int_{\Omega} \left[ - \left( u^\alpha_{h,k} \right)^2 \xi(x, t) \xi_t(x, t) \right.
\]

\[
+ \sum_{i=1}^{n} A^\alpha_i(x, t, u, Du) \xi_{\alpha}^h(x, t) \right] dx \, dt = 0.
\]

The validity of the limit in all of the terms follows by the properties of the Steklov average (see [18]).
In what follows, we assume that the weak solution $u$ of (3.1) belongs to $L^{q_1,r_1}(Q_T,\mathbb{R}^N)$ where the exponents $q_1, r_1 > 1$ and the other parameters to be used in the sequel, satisfy

$$\begin{cases}
  n \geq 2 \\
  (\delta - 2) \left( \frac{1}{r_1} + \frac{n}{2q_1} \right) < 1, \\
  \delta > 2, \\
  \frac{1}{r_2} + \frac{n}{2q_2} < 1, \\
  q_1, r_2 \in \left( \frac{n}{2}, \infty \right), \\
  r_1 \delta - 2, r_2 \in (1, \infty] \\
  \psi \in L^{q_2,r_2}(Q_T) \\
  q_2 \in \left( \frac{n}{2}, \infty \right), r_2 \in (1, \infty]
\end{cases} \tag{3.6}$$

for a.a. $(x,t) \in Q_T$ and all $(u,z) \in \mathbb{R}^N \times \mathbb{R}^{N \times n}$.

$$(H_1)$$ Controlled growth conditions: there exist a positive constant $\Lambda$ and functions $\varphi_1 \in L^2(Q_T)$ and $\varphi_2 \in L^{q_0,r_0}(Q_T)$, with $r_0,q_0 \geq 1$, $\frac{1}{r_0} + \frac{n}{2q_0} = 1 + \frac{n}{4}$, such that

$$|A(x,t,u,z)| \leq \Lambda \left( \varphi_1(x,t) + |u|^{\frac{n+2}{n}} + |z| \right) \tag{3.7}$$

$$|b(x,t,u,z)| \leq \Lambda \left( \varphi_2(x,t) + |u|^{1 + \frac{4}{n}} + |z|^{\frac{n+4}{n+2}} \right) \tag{3.8}$$

for a.a. $(x,t) \in Q_T$ and all $(u,z) \in \mathbb{R}^N \times \mathbb{R}^{N \times n}$.

$(H_2)$ Componentwise coercivity of the differential operator: there exist positive constants $\nu, \mu$ and $k_0$ such that for all $\alpha \in \{1, \ldots, N\}$ and for $|u^\alpha| \geq k_0$, it holds

$$\sum_{i=1}^{n} A_i^\alpha(x,t,u,z) z_i^\alpha \geq \nu |z^\alpha|^2 - \mu |u^\alpha|^\delta - |u^\alpha|^2 \psi(x,t) \tag{3.9}$$

for a.a. $(x,t) \in Q_T$.

$(H_3)$ Componentwise sign condition: for $|u^\alpha| \geq k_0$ we have

$$- b^\alpha(x,t,u,z) u^\alpha \leq |u^\alpha|^2 \psi(x,t) + \mu |u^\alpha|^\delta + \Lambda |z^\alpha|^2 \tag{3.10}$$
for a.a. \((x, t) \in Q_T\) and for all \(z \in \mathbb{R}^{N \times n}\). The function \(\psi\) is as in (3.6).

If \(\|u\|_{\infty; S_T} \leq M_0\), we take \(k \geq \max\{k_0, M_0\}\) and observe that \(u^{\alpha, k}\) is equal to zero on the boundary \(\partial Q_T\). Thus, in (3.5) we can choose \(t_0 = 0\) and \(\xi = 1\), and hence \(\xi_t = 0\), that gives
\[
\frac{1}{2} \|u^{\alpha, k}(\cdot, t_1)\|_{2, A^{\alpha, k}(t_1)}^2 + \int_{Q^{\alpha, k}_{t_1}} \left[ \sum_{i=1}^n A^\alpha_i(x, t, u, Du) D_i u^{\alpha, k} + b^\alpha(x, t, u, Du) u^{\alpha, k} \right] dxdt = 0,
\]
(3.11)
where the integration is taken on the sets
\[
A^{\alpha, k}(t_1) = \left\{ x \in \Omega : u^\alpha(x, t_1) > k, \ t_1 \in [0, T] \right\}
\]
\[
Q^{\alpha, k}_{t_1} = \left\{ (x, t) \in \Omega \times [0, t_1] : u^\alpha(x, t) > k \right\}.
\]

4. Main result

The conditions imposed above permit us to obtain a component-wise maximum principle and, hence, also essential boundedness of the solution of (3.1).

**Theorem 4.1.** Let \(u \in V^{1,0}_2(Q_T, \mathbb{R}^N) \cap L^{q_1}(Q_T, \mathbb{R}^N)\) be a weak solution of the system (3.1) which is essentially bounded on the boundary \(\partial Q_T\). Assume that the terms \(A(x, t, u, z)\) and \(b(x, t, u, z)\) satisfy conditions (3.6)-(3.10) with \(\nu - \Lambda > 0\). Then
\[
\|u^\alpha\|_{\infty; Q_T} \leq K
\]
with a constant \(K\) depending on known quantities and \(\|u^\alpha\|_{q_1, r_1; Q_T}\). If we take \(\mu = 0\) in (3.9)-(3.10), then the constant \(K\) does not depend on the last norm.

**Proof.** By (3.11), taking into account \(\frac{u^{\alpha, k}}{u^\alpha} < 1\), (3.9) and (3.10), we get
\[
\frac{1}{2} \|u^{\alpha, k}\|_{2, A^{\alpha, k}(t_1)}^2 + \int_{Q^{\alpha, k}_{t_1}} \left( \nu |Du^\alpha|^2 - \mu |u^\alpha|^\delta - |u^\alpha|^2 \psi \right) dxdt
\leq - \int_{Q^{\alpha, k}_{t_1}} b^\alpha(x, t, u, Du) u^\alpha(x, t) \frac{u^{\alpha, k}(x, t)}{u^\alpha(x, t)} dxdt
\leq \int_{Q^{\alpha, k}_{t_1}} \left( |u^\alpha|^2 \psi + \mu |u^\alpha|^\delta + \Lambda |Du^\alpha|^2 \right) dxdt.
\]
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Hence
\[ \frac{1}{2} \| u^{\alpha,k} \|_{2,A^\alpha,k(t_1)}^2 + (\nu - \Lambda) \int_{Q^\alpha,k_{t_1}} |Du^{\alpha}|^2 \, dxdt \]
\[ \leq 2 \int_{Q^\alpha,k_{t_1}} (\mu |u^{\alpha}|^\delta + |u^{\alpha}|^2 \psi) \, dxdt. \tag{4.3} \]

Keeping in mind that \( \nu - \Lambda > 0 \), we obtain
\[ \| u^{\alpha,k} \|_{2,A^\alpha,k(t_1)}^2 + \| Du^{\alpha} \|_{2,Q^\alpha,k_{t_1}}^2 \leq C \int_{Q^\alpha,k_{t_1}} (|u^{\alpha}|^\delta + |u^{\alpha}|^2 \psi) \, dxdt, \tag{4.4} \]
where \( C \) depends on \( \mu \) and \( \nu - \Lambda \).

Let us estimate the integrals in the right-hand side. For the first one, using the inequalities in §2, we obtain
\[ I_1 = \int_{Q^\alpha,k_{t_1}} |u^{\alpha}|^{\delta - 2} \cdot |u^{\alpha}|^2 \, dxdt \]
\[ \leq \left[ \int_0^{t_1} \left( \int_{A^\alpha,k(t_1)} |u^{\alpha}|^{q_1} \, dx \right)^{\frac{r_1}{q_1}} \frac{\delta - 2}{r_1} \right. \]
\[ \times \left[ \int_0^{t_1} \left( \int_{A^\alpha,k(t_1)} |u^{\alpha}|^{q_1} \, dx \right)^{\frac{r_1}{q_1}} \frac{\delta - 2}{r_1} \right]^{\frac{r_1}{q_1}} \]
\[ = \| u^{\alpha} \|_{q_1,r_1,Q^\alpha,k_{t_1}}^{\delta - 2} \| u^{\alpha} \|_{2,Q^\alpha,k_{t_1}}^2 \]
\[ \leq 2 \| u^{\alpha} \|_{q_1,r_1,Q^\alpha,k_{t_1}}^{\delta - 2} \left[ \| u^{\alpha} - k \|_{2,Q^\alpha,k_{t_1}}^2 \right. \]
\[ + k^2 \left( \int_0^{t_1} (|A^{\alpha,k}(t)|)^{\frac{q_1}{r_1}} \, dt \right)^{\frac{2}{r_1}}. \] \tag{4.5}

where \( q_1 = \frac{2q_1}{q_1 - (\delta - 2)}, r_1 = \frac{2r_1}{r_1 - (\delta - 2)} \). We notice that
\[ \begin{cases} 2 \leq r_1 < \infty, & 2 \leq q_1 < \frac{2n}{n - 2} \quad \text{for } n \geq 2 \\ 2 < r_1 < \infty, & 2 \leq q_1 \leq \infty \quad \text{for } n = 1 \end{cases} \] \tag{4.6}

and \( \frac{1}{r_1} + \frac{n}{2q_1} > \frac{n}{4} \). Hence we can find an opportune \( 0 < \kappa_1 < 1 \), such that \( \frac{1}{r_1} + \frac{n}{2q_1} = \frac{n}{4}(1 + \kappa_1) \). In case of \( n = 2 \) we consider \( q_1 \in [2, \infty) \).

Introduce
\[ \hat{q}_1 = q_1 (1 + \kappa_1) \quad \text{and} \quad \hat{r}_1 = r_1 (1 + \kappa_1). \]
Direct calculations show that \( \hat{q}_1 \) and \( \hat{r}_1 \) belong to the intervals (2.6) and by (2.4) we obtain

\[
\|u_{\alpha,k}\|_{q_1, \alpha, k, Q_1}^2, r_1, \alpha, k, Q_1 \leq \|u_{\alpha,k}\|_{q_1, \alpha, k, Q_1}^2 \left( t_1^{1} \left| \Omega \right|^{\frac{1}{q_1}} \right)^{2\kappa_1}. \tag{4.7}
\]

In the case \( q_1 = \delta - 2 \), which is possible if \( n = 1 \), we obtain \( q_1 = \infty \) and hence also \( \hat{q}_1 = \infty \). Then (4.7) becomes

\[
\|u_{\alpha,k}\|_{\infty, q_1, \alpha, k, Q_1}^2 \leq \|u_{\alpha,k}\|_{\infty, \hat{q}_1, \hat{r}_1, Q_1}^2 \left( t_1^{1} \left| \Omega \right|^{\frac{1}{q_1}} \right)^{2\kappa_1}. \tag{4.8}
\]

This permits us to apply (2.7). In the case \( n \geq 2 \) we obtain

\[
I_1 \leq 2\|u^\alpha\|_{\delta-2, q_1, \alpha, k, Q_1}^2 \left[ \beta^2\|u_{\alpha,k}\|_{V_2^{1,0}(Q_1)}^2 \right] \left( t_1^{1} \left| \Omega \right|^{\frac{1}{q_1}} \right)^{2\kappa_1} \tag{4.9}
\]

where \( \beta = \beta(n, \overline{q}_1, q_1) \) and \( \overline{q}_1 = \overline{r}_1 = \kappa_1 + 1 \).

If \( q_1 = \delta - 2 \), in view of (4.8), we have

\[
I_1 \leq 2\|u^\alpha\|_{\delta-2, q_1, \alpha, k, Q_1}^2 \left[ \beta^2\|u_{\alpha,k}\|_{V_2^{1,0}(Q_1)}^2 \right] \left( t_1^{1} \left| \Omega \right|^{\frac{1}{q_1}} \right)^{2\kappa_1} + k^2 \left( \int_0^{t_1} |A_{\alpha,k}(t)| \frac{\hat{r}_1}{q_1} dt \right)^{2(\kappa_1 + 1)} \tag{4.10}
\]

We trate \( I_2 \) analogously. By the Hölder inequality (2.3), we obtain

\[
I_2 = \int_{Q_1} |u^\alpha|^2 \psi \, dx \, dt \leq \|\psi\|_{q_2, r_2, Q_1} \|u^\alpha\|_{q_2, r_2, Q_1}^2 \tag{4.11}
\]
where \( \bar{q}_2 = \frac{2q_2}{q_2 - 1}, \bar{r}_2 = \frac{2r_2}{r_2 - 1} \). We notice that

\[
\begin{align*}
&\left\{ \begin{array}{l}
\frac{1}{\bar{r}_2} + \frac{n}{2\bar{q}_2} > \frac{n}{4}
\end{array} \right.
\end{align*}
\]

\[
\begin{align*}
&\begin{cases}
2 \leq \bar{q}_2 < \frac{2n}{n - 2}, & 2 \leq \bar{r}_2 < \infty, \text{ for } n \geq 2
\end{cases}
\end{align*}
\]

\[
\begin{align*}
&\begin{cases}
2 \leq \bar{q}_2 \leq \infty, & 2 < \bar{r}_2 < \infty, \text{ for } n = 1.
\end{cases}
\end{align*}
\]

As before, we can find \( 0 < \kappa_2 < 1 \) such that

\[
\bar{q}_2 = \frac{1}{\bar{r}_2} + \frac{n}{2\bar{q}_2} = \frac{n}{4}(1 + \kappa_2) \text{ and } \hat{q}_2 = \frac{1}{\bar{r}_2}(1 + \kappa_2) \text{ and } \hat{r}_2 = \frac{1}{\bar{r}_2}(1 + \kappa_2) \text{ verify the conditions } (2.6). \]

This permits us to apply (2.4), obtaining

\[
\| u^{\alpha,k} \|_{q_2,r_2;Q_{t_1}}^2 \leq \| u^{\alpha,k} \|_{q_2,\hat{q}_2;Q_{t_1}}^2 \| X^{\alpha,k} \|_{\frac{1}{\hat{q}_2},\frac{1}{\hat{r}_2};Q_{t_1}}^2 \leq \| u^{\alpha,k} \|_{q_2,\hat{q}_2;Q_{t_1}}^2 \left( \frac{1}{t_1^{\frac{1}{\hat{q}_2}}} |\Omega|^{\frac{1}{\hat{r}_2}} \right)^{2\kappa_2}.
\]

Hence, using (2.7), we get

\[
I_2 \leq 2\| \psi \|_{q_2,\hat{q}_2;Q_T} \left\{ \beta^2 \| u^{\alpha,k} \|_{1,0(Q_{t_1})}^2 \left( \frac{1}{t_1^{\frac{1}{\hat{q}_2}}} |\Omega|^{\frac{1}{\hat{r}_2}} \right)^{2\kappa_2} \right.
\]

\[
+ k^2 \left[ \int_0^{t_1} |A^{\alpha,k}(t)|^{\hat{q}_2} \, dt \right]^{2(\kappa_2 + 1)} \hat{r}_2^{\kappa_2} \right\}. \tag{4.12}
\]

In case of \( q_2 = 1 \) and hence \( \bar{q}_2 = \infty \), which is possible if \( n = 1 \), we obtain

\[
I_2 \leq 2\| \psi \|_{q_2,\hat{q}_2;Q_T} \left( \beta^2 \| u^{\alpha,k} \|_{1,0(Q_{t_1})}^2 \left( \frac{1}{t_1^{\frac{1}{\hat{q}_2}}} |\Omega|^{\frac{1}{\hat{r}_2}} \right)^{2\kappa_2} + k^2 \right). \tag{4.13}
\]
Combine (4.9), (4.12) and (4.4), we obtain
\[
\|u^{α,k}\|_{V^{1,0}_2(Q^{α,k}_{t_1})}^2 \\
\leq C \left\{ \|u^{α,k}\|_{V^{1,0}_2(Q^{α,k}_{t_1})}^2 \left[ \beta^2 \|u^{α}\|_{H^{δ−2}_2,Ω}^2 \left( t_1^{-1} |Ω|^{1/α} \right)^{2α_1} \\
+ \|\tilde{u}\|_{q_2,r_2,Q_2} \left( t_1^{-2} |Ω|^{1/2} \right)^{2α_2} \right] \\
+ \|u^{α}\|_{q_1,r_1,Ω}^2 k^2 \left( \int_0^{t_1} |A^{α,k}(t)|^{\frac{\tilde{q}_1}{q_1}} dt \right)^{\frac{2(2k+1)}{r_1}} \right\}
\]
(4.14)

Taking \( t_1 \) such small that
\[
C\beta^2 \left( \|u^{α}\|_{H^{δ−2}_2,Ω}^2 \left( t_1^{-1} |Ω|^{1/α} \right)^{2α_1} + \|\tilde{u}\|_{q_2,r_2,Q_T} \left( t_1^{-2} |Ω|^{1/2} \right)^{2α_2} \right) < \frac{1}{2},
\]
we obtain
\[
\|u^{α,k}\|_{V^{1,0}_2(Q^{α,k}_{t_1})}^2 \\
\leq C k^2 \left\{ \|u^{α}\|_{H^{δ−2}_2,q_1,r_1,K^{α,k}} \left( \int_0^{t_1} |A^{α,k}(t)|^{\frac{\tilde{q}_1}{q_1}} dt \right)^{\frac{2(2k+1)}{r_1}} \\
+ \|\tilde{u}\|_{q_2,r_2,Q_T} \left( \int_0^{t_1} |A^{α,k}(t)|^{\frac{\tilde{q}_2}{q_2}} dt \right)^{\frac{2(2k+1)}{r_2}} \right\}.
\]
(4.15)

Analogously in case of \( n = 1 \).

Since (4.15) is equivalent to (2.11) in Theorem 2.1 we deduce that \( u^{α} \) is essentially bounded from above in a cylinder with high \( t_1 \) which is small enough. In order to extend this result in the whole cylinder we consider \( u^{α} \) successively in \( Q_2 = Ω \times (t_1, 2t_1) \), \( Q_3 = Ω \times (2t_1, 3t_1) \) and so on, covering in such way the whole cylinder \( Q_T \).

Moreover, it is possible to estimate \( u^{α}(x,t) \) from below. To this aim, we apply the result just obtained to the function \( \tilde{u}^{α}(x,t) = -u^{α}(x,t) \) which verifies equations similar to \( u^{α}(x,t) \), i.e.
\[
\tilde{u}^{α}_t - \text{div} \tilde{A}^{α}(x,t, \tilde{u}, D\tilde{u}) + \tilde{b}^{α}(x,t, \tilde{u}, D\tilde{u}) = 0 \quad (x,t) \in Q_T,
\]
where
\[
\tilde{A}^{α}(x,t, \tilde{u}, D\tilde{u}) = -A^{α}(x,t, -\tilde{u}, -D\tilde{u})
\]
and
\[ \tilde{b}^\alpha(x, t, \tilde{u}, D\tilde{u}) = -b^\alpha(x, t, -\tilde{u}, -D\tilde{u}) \]
satisfy the conditions (3.6)-(3.10). Thus, by Theorem 2.1 we obtain
\[ \|u^\alpha\|_{\infty; Q_T} \leq K, \]
where \( K = K(M_0, k_0, q_i, r_i, n, \Omega, |\Omega|, T, \|u^\alpha\|_{q_i, r_i; Q_T}) \).
\[ \square \]

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