Cosmic Discordance: Detection of a modulation in the primordial fluctuation spectrum

Kiyotomo Ichiki$^{1,2}$, Ryo Nagata$^{1,3}$, and Jun’ichi Yokoyama$^1$
$^1$Research Center for the Early Universe, University of Tokyo, 7-3-1 Hongo, Bunkyo-ku, Tokyo 113-0033, Japan
$^2$Department of Physics and Astrophysics, Nagoya University, Nagoya 464-8602, Japan
$^3$Cosmophysics Group, IPNS, KEK, Tsukuba 305-0801, Japan
$^4$Institute for the Physics and Mathematics of the Universe, The University of Tokyo, Kashiwa 277-8568, Japan

As a test of the standard inflationary cosmology, which generically predicts nearly scale-invariant spectrum of primordial curvature fluctuations, we perform Markov-Chain Monte-Carlo analysis to search for possible modulations in the power spectrum and determine its shape together with the cosmological parameters using cosmic microwave background radiation data. By incorporating various three-parameter features on the simple power-law spectrum, we find an oscillatory modulation localized around the comoving wavenumber $k \simeq 0.009\text{Mpc}^{-1}$ at 99.995% confidence level which improves the log-likelihood as much as $-\Delta^2 \ln \mathcal{L} \equiv \Delta \chi^2_{\text{eff}} = -22$. This feature can be detected even if we use only the cross correlation between the temperature and the E-mode polarization anisotropies.

PACS numbers: 98.70.Vc, 95.30.-k, 98.80.Es

I. INTRODUCTION

In the standard inflationary cosmology $^1$, $^2$, $^3$, $^4$ the observed large-scale structures and the anisotropies in the cosmic microwave background (CMB) originate in tiny quantum fluctuations generated during the inflationary expansion stage in the very early Universe $^5$, $^6$, $^7$. Conventionally, the power spectrum of primordial curvature fluctuations, $P(k) \equiv \langle |R_k|^2 \rangle$, has been assumed to follow a simple functional form such as a power-law $A(k) \equiv \frac{4 \pi k^3}{(2 \pi)^3} P(k) = A(k/k_0)^{n_s-1}$ which is quantified by the amplitude $A$ and the spectral index $n_s$, and the previous statistical analyses of the power spectrum have mostly focused on these two parameters (and at best the scale dependence of $n_s$, the running, which may be important to distinguish the inflation model$^8$). It is true that the simplest class of inflation models predicts a power-law spectrum with $n_s$ close to unity $^5$, $^6$, $^7$, but we may not be able to identify the correct theory of the early Universe if we restrict our parameter space from the beginning. From the viewpoint of observational cosmology, the shape of the primordial fluctuation spectrum should be determined purely from the observational data without any theoretical prejudices.

Along this line of thought, several Markov-Chain Monte-Carlo (MCMC) analyses have been performed to search for possible deviation from the power law, but none has detected statistically much significant modulations so far $^9$, $^{10}$, $^{11}$, $^{12}$, $^{13}$, $^{14}$. To be more quantitative, the presence of a nontrivial feature may be decided in terms of Akaike’s Information Criterion (AIC) $^{15}$ which asserts that introduction of an additional fitting parameter is justified if and only if $\chi^2_{\text{eff}}$ improves more than $-2$ with it. Most of the previous analyses resulted in the improvement of $\Delta \chi^2_{\text{eff}}$ much less than that required by AIC, and others reached it only at a marginal level, so that it was hardly possible to claim the presence of a feature. It should be emphasized, however, that those analyses did not have sufficient resolution to detect spectral fine structure, because they tried to fit the primordial power spectrum in a broad range of scales in terms of a limited number of degrees of freedom using sparse sampling or some specific theoretical models. This lack of resolution was inevitable because exploring large dimensional parameter space is extremely time-consuming even with the help of MCMC analysis.

Recently, on the other hand, two of the present authors (RN and JY) performed reconstruction of primordial power spectrum from the angular power spectrum of the CMB temperature fluctuations, $C^{TT}_\ell$, of the five-year WMAP data (hereafter WMAP5) $^{16}$, $^{17}$ using two different non-parametric methods, namely, the cosmic inversion method $^{18}$, $^{19}$, $^{20}$, $^{21}$, $^{22}$, $^{23}$ and the maximum-likelihood reconstruction method $^{24}$, $^{25}$, $^{26}$. They have probed the primordial power spectrum with finer resolution than aforementioned MCMC analyses and found an anomalously large deviation from the best-fit power-law primordial spectrum of WMAP5 around the wavenumber $k \simeq 0.009\text{Mpc}^{-1}$.

*Electronic address: yokoyama@resceu.s.u-tokyo.ac.jp
or $kd \simeq 124$ where $d = 1.43 \times 10^4 \text{Mpc}$ is the distance to the last scattering surface. This scale corresponds to the multipole $\ell \simeq 120$ and the length scale $r \simeq 710 \text{Mpc}$.

In their reconstruction procedure, however, they had to fix values of the cosmological parameters and adopt an *ad hoc* smoothing prior to ensure sensible reconstruction. The latter tends to discard a large part of the information on the fine structure which may have affected the significance of the claimed spectral feature itself. Furthermore although the reconstructed power spectrum exhibited oscillatory features with both peaks and dips, the result of decomposition of the continuous reconstructed curve to statistically independent band-powers, which is necessary to discuss their statistical significance, revealed only a $3.3\sigma$ peak but no dips $^{24}$.

The purpose of this paper is to focus on the fine structures of the spectral shape. We use a forward analysis implemented by MCMC simulation which circumvents the above-mentioned difficulties peculiar to the inverse mapping approach. We also overcome the lack of resolution, which was a very serious problem of previous MCMC analyses as mentioned above, by searching for and concentrating on the most prominent localized feature imprinted on the otherwise power-law spectrum.

## II. METHOD

First we prepare various model power spectra in which a localized spectral feature is imprinted on the power-law with three fitting parameters. We then calculate the temperature-temperature (TT) angular power spectrum and the cross correlation between the temperature and the E-mode polarization (TE) to find the best-fit values of these parameters as well as those of other cosmological parameters in the $\Lambda$CDM model using the CosmoMC$^{27}$ code.

We start with the following three types of three-parameter modulations.

**$\Lambda$-type:** a $\Lambda$-shaped peak is introduced at $k = k_*$ and connected to a power-law $A(k) = A(k/k_0)^{n_* - 1}$ by straight lines in the log $k – A(k)$ plane. The location of the peak, $k_*$, its width, and height are additional fitting parameters. Here $k_0 \equiv 0.002 \text{Mpc}^{-1}$ is the pivot scale.

**$\nu^3$-type:** a peak at $k = k_*$ and a dip at $k < k_*$ with the same amplitude of deviation from the standard power-law are connected by piecewise straight lines to $A(k) = A(k/k_0)^{n_* - 1}$ in the log $k – A(k)$ plane. Again the height and width of the peak are the parameters characterizing modulation besides the location $k_*$. **$S$-type:** peaks and dips are characterized by the following smooth function.

$$A(k) = A(k/k_0)^{n_* - 1} + B(k/k_0)^{n_* - 1} \exp[-(k - k_*)^2/\kappa^2] \cos[\pi(k - k_*)/\kappa]$$

where $k_*$, $B$, and $\kappa$ are additional fitting parameters.

We assume flat priors for the additional parameters to cover the parameter range as broad as possible. For the $S$-type modulation, for example, we take $10^{10}B = [0, 10^3]$ and $10^4\kappa = [1, 10] \text{Mpc}^{-1}$. As for the location of the feature, $k_*$, we started our calculation allowing it to vary in the full range of observationally accessible domain with a sufficient accuracy so that we could detect a modulation localized anywhere in $k$ space. We have confirmed, however, that in the range of the wavenumber accessible by current observation the most prominent feature is located at $k_*d \simeq 124$ as was found in the reconstruction analysis $^{24}$ together with other possible modulations whose statistical significance is smaller. In fact, if we allowed $k_*$ to vary beyond $k_*d > 135$ and/or $k_*d < 115$, other features would also contribute to MCMC analysis, albeit with little significance, in addition to the most prominent one at $k_*d \simeq 124$. Then our three-parameter models would no longer be a good parametrization and the MCMC calculation would not converge properly. We therefore decided to limit the range of $k_*$ to $k_*d = [115, 135]$ in the final simulations in order to calculate the probability distribution of the amplitude of the most prominent modulation, $B$, properly.

In order to check the stability of our result with respect to other features localized at different wavenumbers we ran MCMC analysis incorporating another feature beyond $k_*d > 135$ at the same time with three more fitting parameters. As a result we found that the inclusion of additional parameters to incorporate another feature does not affect the posterior distributions of the original three parameters for the most prominent feature at $k_*d \simeq 124$, which implies that we can treat the relevant feature independently from other possible features at different wavenumbers.

Based on the above observations we have also adopted another model to fit the feature around the most prominent modulation at $k_*d \simeq 124$ as follows.

**$W$-type:** three-step modulation on a power law with the functional form,

$$A(k) = A(k/k_0)^{n_* - 1} + B_\alpha \theta(kd - 114)\theta(122 - kd)$$

$$+ B_\beta \theta(kd - 122)\theta(126 - kd) + B_\gamma \theta(kd - 126)\theta(134 - kd),$$

where $B$, $B_\alpha$, and $B_\beta$ are fitting parameters.
Among the four models we use, the $\Lambda$-type modulation contains only an extra power so that it enhances the dispersion of fluctuations compared with the background power-law spectrum for the same value of $A$. In the other three types of models, both excess and deficit are incorporated so that they do not necessarily affect overall normalization of fluctuations. In these three models, we put $A(k) = 0$ whenever $A(k)$ takes a negative value. In the course of MCMC calculations, we have found a degeneracy between the width and the height of modulation in $\Lambda$- and $V_\Lambda$-type models and they tend to give a larger modulation amplitude with a smaller width. Such a tendency is undesirable from the physical point of view because $A(k)$ should be positive definite. Hence we fixed the width of modulation to $\Delta \ln k = 0.043$ in these two models which is the best-fit value in our first trial run and this choice is consistent with the result of the $S$-type model where the width $\kappa$ is treated as a free fitting parameter.

III. RESULTS

The result of MCMC calculations is summarized in Table 1. First by incorporating an excess with the $\Lambda$-type model, the effective $\chi^2_{\text{eff}} = -2 \ln L$, where $L$ is the likelihood function, improves by $\Delta \chi^2_{\text{eff}} = -6.5$ with three extra parameters added to the standard power-law $\Lambda$CDM model. To this end this model satisfies the AIC. Inspecting the details of $\Delta \chi^2_{\text{eff}}$, however, we find that the improvement of $-6.5$ is entirely due to the better fit to the TE data. Although the fit to TT data improves around $\ell \approx 120$, it gets worse for $\ell \lesssim 100$. This is because the transfer function from $P(k)$ to $C^\ell_{TT}$ is non-vanishing for $\ell \lesssim kd$ so that an excessive power around $kd = k_* d$ increases all $C^\ell_{TT}$ for $\ell \lesssim 120$ to affect overall normalization in an unwanted manner.

Therefore even if it was not detected in a band-power analysis [26], we should take neighboring dips observed in the reconstructed curve as in the other three models, where $\chi^2_{\text{eff}}$ improves as much as $\Delta \chi^2_{\text{eff}} = -16 \sim -22$ with the same numbers of extra parameters. We can regard it extremely significant and interpret that the result strongly suggests a sharp and strong oscillatory deviation from a power-law around $k = k_*$. The marginalized 1D distributions of the parameter $B$ made by MCMC calculations are shown in figure 1. Based on the posterior statistical distribution of the amplitude of spectral modulation, we find that the pure power-law model with $B = 0$ is $4.0\sigma$ ($4.1\sigma$) away from the mean value of $B$ for $S$-type ($V_\Lambda$-type) model, respectively.

We also examined the probability to find such a large deviation by accumulating more samples to explore the tail of the posterior probability distribution for the $S$-type model which improved $\chi^2_{\text{eff}}$ the best among the models we adopted. We have found 304 samples in the smallest bin of the modulation parameter with $B < 10^{-10} \approx A/23$, from
negative values in the calculation instead of setting $A$ generally induce shifts in the estimated values of the cosmological parameters compared with the standard power-law larger than the expected errors in the Planck mission. It has been already shown that globally non-power-law spectra in Table 1. Although the shifts are smaller than the observational errors of five-year WMAP data, some of them are larger than the expected errors in the Planck mission. Therefore this may be regarded as an independent support to our discovery of a non-power-law feature in the primordial power spectrum.

In order to check whether we have explored the tails of the distribution with sufficient accuracy, we have ran simulations 6279082 MCMC samples generated with temperature parameter $T = 1$ [28]. Hence the relative probability is

$$P(B < 1 \times 10^{-10}) = 4.8 \times 10^{-5}.$$  

In order to check whether we have explored the tails of the distribution with sufficient accuracy, we have ran simulations with different temperature parameters ($T = 1.5, 2$) in the MCMC analysis and confirmed that the probability converges to $P = (3.9 \times 10^{-5}, 5.6 \times 10^{-5})$, respectively. Therefore we conclude that the posterior probability for $B = 0$ to be the case is only $O(10^{-5})$.

We note that among the improvement of $\Delta \chi^2_{\text{eff}} = -22$ for the $S$-type spectrum, $-14$ is due to the improvement of the fit to the TT power spectrum, $C_{\ell}^{TT}$, while the remaining $-8$ is from the TE cross correlation, $C_{\ell}^{TE}$. Figures 2 and 3 depict $C_{\ell}^{TT}$ and $C_{\ell}^{TE}$ in the relevant range. It is intriguing that significant portion of the improvement of the likelihood comes from the TE data which was not used in the reconstruction approach at all [28]. Therefore this may be regarded as an independent support to our discovery of a non-power-law feature in the primordial power spectrum.

In order to pursue it, we have performed two distinct sets of MCMC calculations one using only TT data and the other TE data. In both cases the $S$-type modulation was used and the cosmological parameters have been fixed to the best-fit values for this model shown in Table 1. To ensure the stability of calculation we discarded any $A(k)$ reaching negative values in the calculation instead of setting $A(k) = 0$. As a result we have obtained the following values of the model parameters from TT and TE data, respectively.

\[
\begin{align*}
k^*_{TT} d &= 124.54_{-1.42}^{+1.64}, & k^*_{TE} d &= 123.95_{-3.24}^{+3.87}, \\
10^4 \kappa_{TT} &= 3.63_{-1.26}^{+1.52}, & 10^4 \kappa_{TE} &= 7.23_{-3.13}^{+2.77}, \\
B_{TT} &= (3.89_{-1.22}^{+1.38}) \times 10^{-9}, & B_{TE} &= (4.13_{-2.21}^{+1.17}) \times 10^{-9},
\end{align*}
\]

where the errors represent 95% upper and lower bounds. Here the location of the peak is determined in good shape in both cases and the results are in good agreement with each other. On the other hand, TE data fails to constrain the width parameter $\kappa$ and its upper bound is essentially determined by the prior cutoff. Finally as for the amplitude of modulation $B$, both datasets give similar posterior distributions and we find the relative frequency to find vanishingly small modulation is $O(10^{-3})$ in both cases. More precisely, we find

$$P(B_{TT} < 1 \times 10^{-10}) = 2 \times 10^{-3} \text{ and } P(B_{TE} < 1 \times 10^{-10}) = 1 \times 10^{-3}.$$

Thus we can conclude both TT data and TE data suggest the existence of a feature at the same wavenumber separately and mutually consistently.

Finally we note that such a feature also shifts the best-fit values of other cosmological parameters slightly as seen in Table 1. Although the shifts are smaller than the observational errors of five-year WMAP data, some of them are larger than the expected errors in the Planck mission. It has been already shown that globally non-power-law spectra generally induce shifts in the estimated values of the cosmological parameters compared with the standard power-law spectra.

![Angular power spectrum of CMB temperature anisotropy in the range 100 \(\leq \ell \leq 140\). Data points are from WMAP5 with smaller error bars indicating only the measurement errors. Green line is the result of best-fit power-law \(\Lambda\)CDM model, while the most wavy black curve and the less wavy magenta curve represent the results of $S$-type modulation and $\Lambda$-type modulation, respectively. The dotted curve is the angular power spectrum recalculated from $P(k)$ reconstructed by the maximum-likelihood reconstruction method [26].](image-url)
FIG. 3: Angular cross correlation of CMB temperature and E-mode polarization in the same range. Data points are the observational result of WMAP5. Nearly straight green line is the result of best-fit power-law ΛCDM model, while the black curve and the magenta curve represent the results of S-type modulation and Λ-type modulation, respectively.

| power law | Λ-type | χ^2-type | S-type | W-type | Δ_max | WMAP5 | Planck |
|-----------|--------|----------|--------|--------|-------|--------|--------|
| Ω_b       | 0.0438 | 0.0441   | 0.0443 | 0.0441 | 0.0444| 0.0006 | 0.0030 | 0.0003 |
| Ω_m       | 0.256  | 0.256    | 0.260  | 0.257  | 0.262 | 0.006  | 0.027  |
| Ω_Λ       | 0.744  | 0.744    | 0.740  | 0.743  | 0.738 | 0.006  | 0.015  | 0.009  |
| H_0       | 72.1   | 72.1     | 71.7   | 72.0   | 71.6  | 0.5    | 2.7    | 2.7    |
| 10^{10}A  | 23.88  | 23.24    | 23.51  | 23.34  | 23.90 | 0.54   | 1.12   |
| n_s       | 0.964  | 0.975    | 0.969  | 0.970  | 0.964 | 0.006  | 0.015  | 0.0045 |
| τ         | 0.0864 | 0.0879   | 0.0846 | 0.0835 | 0.0845| 0.0029 | 0.017  | 0.005  |
| Δχ^2_{eff}| 0      | −6.5     | −19    | −22    | −16   |
| k_0       | 124.5  | 124.4    | 124.5  |       |       |
| 10^{10}B  | 23.80  | 47.26    | 55.66  | 37.95  |

TABLE I: Cosmological parameters estimated from CosmoMC code using four types of modulated power spectra. Here Ω_b, Ω_m and Ω_Λ are fractions of cosmic energy density in baryons, matters and cosmological constant, respectively, H_0 is the Hubble parameter in unit of km/s/Mpc, and τ is the optical depth to the last scattering surface. For Λ- and χ^2-type models the parameter B stands for the difference between the peak amplitude and the power-law background value at k = k_0. Δχ^2_{eff} represents improvement of goodness-of-fit χ^2 eff for the best-fit model of each modulated spectrum from the most-updated best-fit power-law ΛCDM model. Δ_max is the maximum of the difference of each parameter between the power-law model and models with modulated spectra except for the Λ-type which does not improve the fit. Column WMAP5 (Planck) represents errors observed (expected) by five-year WMAP (Planck). Errors expected in Planck mission are taken from the “Blue book”. The mean value of κ in the S-type model is κ = 3.58 × 10^{-4}.

ΛCDM model [8, 10, 12, 29, 30] because there is a degeneracy between the global shape of P(k) and the cosmological parameters whose values are imprinted on the shape of the acoustic oscillation. Nevertheless it is intriguing that features localized in a narrow range of wavenumber as discussed here also changes their values at a non-negligible level for the next-generation analysis.

IV. CONCLUSION

In this paper we have found a spectral feature with a 4σ deviation from the best-fit power-law primordial power spectrum in a narrow range around k ≃ 0.009Mpc^{-1}, whose existence is suggested separately by both TT data and TE data from five-year WMAP data. This feature can be further tested and hopefully confirmed by forthcoming EE data from the Planck mission.

As emphasized above, the detected feature consists not only of a peak but also a dip. Since we cannot lower the amplitude of the power spectrum by any sources uncorrelated with the preexistent fluctuations, this means that the feature was created together with the bulk of the power-law spectrum, suggesting some nontrivial phenomenon during
inflation. At the moment, while a number of inflation models have been proposed to produce spectral modulation, we are not aware of any model that can realize a spectrum with such a localized feature. Hence efforts should be made toward construction of models that can account for the detected fine structures.

Currently inflation models are observationally constrained in terms of a small number of discrete parameters such as $n_s$, $dn_s/d\ln k$, the tensor-to-scalar ratio $r$, and a nonlinearity parameter $f_{NL}$. It is difficult to determine the underlying particle-physics model with these parameters alone, because there are huge degeneracies. If our finding is further confirmed, it can certainly serve as an important clue to single out the correct model. On the other hand, whether this feature is a result of some nontrivial physics or a realization of an extremely rare event, we must take it into account in cosmological parameter estimation in the analysis of forthcoming higher precision data.

Acknowledgments

We are grateful to François Bouchet, Eiichiro Komatsu, David Spergel, and Masahiro Takada for useful communications. This work was partially supported by JSPS Grant-in-Aid for Scientific Research No. 19340054(JY) and 21740177(KI), the Grant-in-Aid for Scientific Research on Innovative Areas No. 2111006(JY & RN), JSPS Core-to-Core program “International Research Network on Dark Energy”, and the Global Center of Excellence program at Nagoya University ”Quest for Fundamental Principles in the Universe: from Particles to the Solar System and the Cosmos” from the MEXT of Japan.

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