The sharp Poincaré–Sobolev type inequalities in the hyperbolic spaces $\mathbb{H}^n$

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Abstract

In this note, we establish a $L^p$-version of the Poincaré–Sobolev inequalities in the hyperbolic spaces $\mathbb{H}^n$. The interest of this result is that it relates both the Poincaré (or Hardy) inequality and the Sobolev inequality with the sharp constant in $\mathbb{H}^n$. Our approach is based on the comparison of the $L^p$-norm of gradient of the symmetric decreasing rearrangement of a function in both the hyperbolic space and the Euclidean space, and the sharp Sobolev inequalities in Euclidean spaces. This approach also gives the proof of the Poincaré–Gagliardo–Nirenberg and Poincaré–Morrey–Sobolev inequalities in the hyperbolic spaces $\mathbb{H}^n$. Finally, we discuss several other Sobolev inequalities in the hyperbolic spaces $\mathbb{H}^n$ which generalize the inequalities due to Mugelli and Talenti in $\mathbb{H}^2$.

1 Introduction

Given $n \geq 2$, let $\mathbb{H}^n$ denote the hyperbolic space of dimension $n$. We will use the Poincaré ball model for the hyperbolic space $\mathbb{H}^n$, i.e., a unit ball $B^n$ with center at origin of $\mathbb{R}^n$ equipped with the metric $g(x) = \frac{1}{(1-|x|^2)^2} \sum_{i=1}^n dx_i^2$. The corresponding Riemannian volume element is $dV = \frac{1}{(1-|x|^2)^n} dx$ and for a measurable set $E \subset \mathbb{H}^n$, we denote by $V(E) = \int_E dV$. Our main result of this note states as follows.

**Theorem 1.1.** Let $n \geq 4$ and $\frac{2n}{n-1} \leq p < n$. Then for any $u \in W^{1,p}(\mathbb{H}^n)$ it holds

$$
\int_{\mathbb{B}^n} |\nabla_g u|^p g^{1-p} dV - \left(\frac{n-1}{p}\right)^p \int_{\mathbb{B}^n} |u|^p dV \geq S(n,p) \left(\int_{\mathbb{B}^n} |u|^\frac{np}{n-p} dV\right)^{\frac{n-p}{n}},
$$

(1.1)

where $\nabla_g = (\frac{1-|x|^2}{2})^2 \nabla$ denotes the hyperbolic gradient, $|\nabla_g u|_g = \sqrt{g(\nabla_g u, \nabla_g u)}$ and $S(n,p)$ is the best constant in the $L^p$–Sobolev inequality in $\mathbb{R}^n$ (see, e.g., [1, 33]). Furthermore, equality holds true in (1.1) if and only if $u \equiv 0$.

The most interest of the inequality (1.1) is that it connects both the sharp Poincaré (or Hardy) inequality and the sharp Sobolev inequality in the hyperbolic space $\mathbb{H}^n$. Let $n \geq 2$ and $p > 1$, the sharp Poincaré inequality asserts that

$$
\int_{\mathbb{B}^n} |\nabla_g u|^p g^{1-p} dV \geq \left(\frac{n-1}{p}\right)^p \int_{\mathbb{B}^n} |u|^p dV, \quad u \in C_0^\infty(\mathbb{B}^n).
$$

(1.2)

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The constant \((\frac{2n-1}{p})^p\) is sharp and is never attained. This leaves a room for several improvements of the inequality \((1.2)\). Notice that the non achievement of sharp constant does not always imply improvement (e.g., Hardy operator in the Euclidean space \(\mathbb{R}^n\), \(n \geq 2\)). However, in the hyperbolic space, the operator \(-\Delta_{p,\mathbb{H}^n} - (\frac{2n-1}{p})^p = \text{div}(|\nabla u|^p - 2\nabla u) - (\frac{2n-1}{p})^p\) is subcritical, hence improvement is possible. For examples, the reader can consult the papers \([6-8]\) for the improvements of \((1.2)\) by adding the remainder terms concerning to Hardy weights, i.e., the inequalities of the form

\[
\int_{\mathbb{H}^n} |\nabla_g u|^p_g dV - \frac{(n-1)^2}{4} \int_{\mathbb{H}^n} |u|^2 dV \geq C \int_{\mathbb{H}^n} W|u|^p dV,
\]

for some constant \(C\) and the weight \(W\) satisfying some appropriate conditions. For the case \(p = 2\), Mancini and Sandeep \([26]\) proved the following Poincaré–Sobolev inequalities in \(\mathbb{H}^n\) with \(n \geq 3\)

\[
\int_{\mathbb{H}^n} |\nabla_{g,\mathbb{H}^n} u|^2 dV - \frac{(n-1)^2}{4} \int_{\mathbb{H}^n} |u|^2 dV \geq C_n \left( \int_{\mathbb{H}^n} |u|^\frac{2p}{p-2} dV \right)^\frac{p}{2}, \quad u \in C_0^\infty(\mathbb{H}^n),
\]

where \(2 < q \leq \frac{2n}{n-2}\) and \(C\) is constant. The inequality \((1.3)\) is equivalent to the Hardy–Sobolev–Maz’ya inequality on the half spaces (see \([29, \text{Section 2.1.6}]\)). Especially, in the case \(q = \frac{2n}{n-2}\), we get

\[
\int_{\mathbb{H}^n} |\nabla_{g,\mathbb{H}^n} u|^2 dV - \frac{(n-1)^2}{4} \int_{\mathbb{H}^n} |u|^2 dV \geq C_n \left( \int_{\mathbb{H}^n} |u|^\frac{2p}{p-2} dV \right)^\frac{p}{2}, \quad u \in C_0^\infty(\mathbb{H}^n),
\]

where \(C_n\) denotes the sharp constant for which \((1.4)\) holds. It was shown by Tertikas and Tintarev \([35]\) that if \(n \geq 4\) then \(C_n\) is attained. Using test function, they show that \(C_n < S(n, 2)\) where \(S(n, 2)\) denotes the sharp constant in the \(L^2\)-Sobolev inequality in \(\mathbb{R}^n\). More surprisingly, Benguria, Frank and Loss \([5]\) proved that \(C_3 = S(3, 2)\) and \(C_3\) is not attained. The non attainment of \(C_3\) was also proved by Sandeep ad Mancini \([26]\) by a different method. We refer the reader to \([24]\) for the Hardy–Sobolev–Maz’ya inequalities of kind \((1.4)\) for higher order derivatives. Therefore, the inequality \((1.1)\) can be seen as an \(L^p\) analogue of the result of Benguria, Frank and Loss on the Hardy–Sobolev–Maz’ya inequality in \(\mathbb{H}^3\).

On the other hand, the inequality \((1.1)\) can be seen as a concrete example in the hyperbolic space of the hyperbolic program on the sharp Sobolev inequality in Riemannian manifolds \([17]\). Let \((M, g)\) be a complete Riemannian manifold of dimension \(n \geq 2\). We denote by \(H^{1,p}(M)\) the completion of \(C_0^\infty(M)\) under the norm \(\|u\|_{H^{1,p}} = (\|\nabla u\|_{L^p(M)} + \|u\|_{L^p(M)})^{1/p}\). We wonder to know that for \(\theta \in [1, p]\), is there a constant \(B\) such that

\[
S(n, p)^\theta \|u\|_{L^p,M}^{\theta} \leq \|\nabla u\|_{L^p(M)}^{\theta} + B \|u\|_{L^p(M)}^{\theta} \quad (I_{p,\text{opt}}^\theta)
\]

for any \(u \in H^{1,p}(M)\)? In the case of complete compact Riemannian manifolds, it was proved by Hebey and Vaugon \([22, 23]\), by Druet \([16]\) and by Aubin and Li \([3]\) that \(I_{p,\text{opt}}^\theta\) holds for \(\theta = \min\{2, p\}\). This solves a long standing conjecture due to Aubin \([1]\). We refer the reader to the original article by Aubin \([1]\) or to the book by Hebey \([21]\) or the paper by Druet and Hebey \([17]\) for a complete survey on the compact Riemannian manifolds. In the case of complete non-compact Riemannian manifolds, there is several results in which \(I_{p,\text{opt}}^\theta\) is valid. For example, Aubin, Druet and Hebey \([2]\) proved that \(I_{p,\text{opt}}^\theta\) holds for any \(1 \leq p < n\) with \(B = 0\) on the Cartan–Hadamard manifolds (i.e., complete simply connected Riemannian manifold) satisfying Cartan–Hadamard conjecture. In particular, \(I_{p,\text{opt}}^\theta\) is valid in the hyperbolic spaces for any \(1 \leq p < n\). Since the inequality \((1.1)\) relates both the sharp Poincaré and sharp Sobolev inequalities, then the constants in \((1.1)\) are sharp and can not be improved. Hence, the \((1.1)\) gives an example in which the sharp second constant \(B\) can be explicitly computed. We refer to \([21, \text{Theorem 7.7}]\) for some other examples in the case \(p = 2\). Note that, in the hyperbolic space \(\mathbb{H}^n\), the following inequality holds

\[
S(n, 2)^2 \left( \int_{\mathbb{H}^n} |u|^\frac{2n}{n-2} dV \right)^\frac{1}{n-2} \leq \int_{\mathbb{H}^n} |\nabla_{g,\mathbb{H}^n} u|^2 dV - \frac{n(n-2)}{4} \int_{\mathbb{H}^n} |u|^2 dV.
\]
The constant \( n(n-2)/4 \) is sharp when \( n \geq 4 \). By the result of Benguria, Frank and Loss, this constant is not sharp when \( n = 3 \). In this case, the sharp constant is \( (n-1)^2/4 = 1 \). By this observation, we can not hope the valid of (1.1) for any \( p \in [1, n) \). We will see below that (1.1) follows by a pointwise estimate for which the condition \( \frac{2n}{n-1} \leq p < n \) is sharp. However, in the case \( n = 3 \), we have \( \frac{2n}{n-1} = 3 > 2 \). Hence, the condition \( p \geq \frac{2n}{n-1} \) may be not optimal for the valid of (1.1). So, it is more interesting if we can find the sharp \( p_0 \in [1, n) \) such that (1.1) holds for \( p \in [p_0, n) \).

Let us explain briefly the method used in the proof of Theorem 1.1. Our proof lies heavily on the symmetric non-increasing rearrangement arguments. More precisely, for any function \( u \in W^{1,p}(\mathbb{H}^n) \) we define a function \( u^\# \) which is non-increasing rearrangement function of \( u \) (see the precise definition in Section 2 below). From this \( u^\# \) we define two new functions \( u_1^\# \) on \( \mathbb{H}^n \) and \( u_2^\# \) on \( \mathbb{R}^n \) by \( u_1^\#(x) = u^\#(V(B_g(0, \rho(x)))) \), \( x \in \mathbb{B}^n \) where \( \rho(x) = \ln \frac{1+|x|}{|x|} \) denotes the geodesic distance from \( x \) to \( 0 \), and \( B_g(0, r) \) denotes the open geodesic ball center at \( 0 \) and radius \( r > 0 \) in \( \mathbb{H}^n \), and \( u_2^\#(x) = u^\#(\sigma_n(|x|^n)) \), \( x \in \mathbb{R}^n \) where \( \sigma_n \) denotes the volume of unit ball in \( \mathbb{R}^n \), respectively.

The functions \( u_1^\# \) and \( u_2^\# \) has the same decreasing rearrangement function (which is \( u^\# \)), then \( \|u_1^\#\|_{L^p(\mathbb{H}^n)} = \|u_2^\#\|_{L^p(\mathbb{R}^n)} = \|u\|_{L^p(\mathbb{H}^n)} \) for any \( q \geq 1 \). The key in our proof is a result which compares \( \|\nabla u_1^\#\|_{L^p(\mathbb{H}^n)} \) and \( \|\nabla u_2^\#\|_{L^p(\mathbb{R}^n)} \). Indeed, we will show that

\[
\|\nabla u_1^\#\|_{L^p(\mathbb{H}^n)} - \|\nabla u_2^\#\|_{L^p(\mathbb{R}^n)} \geq \left( \frac{n-1}{p} \right)^p \|u\|_{L^p(\mathbb{H}^n)}^p.
\]

Using the sharp Sobolev inequality in \( \mathbb{R}^n \) and the Pólya–Szegő principle in \( \mathbb{H}^n \), we obtain the inequality (1.1).

The approach to prove Theorem 1.1 above also yields the proofs for the following Poincaré–Gagliardo–Nirenberg and Poincaré–Morrey–Sobolev inequalities in the hyperbolic space \( \mathbb{H}^n \).

**Theorem 1.2.** Let \( n \geq 4 \) and \( \frac{2n}{n-1} \leq p < n \) and \( \alpha \in (0, \frac{n}{n-p}) \), \( \alpha \neq 1 \). Then for any \( u \in C_0^\infty(\mathbb{H}^n) \), the following inequalities holds.

(i) If \( \alpha > 1 \), then we have

\[
\|u\|_{L^{\alpha p-1}(\mathbb{H}^n)} \leq GN(n, p, \alpha) \left( \|\nabla u\|_{L^p(\mathbb{H}^n)}^p - \left( \frac{n-1}{p} \right)^p \|u\|_{L^p(\mathbb{H}^n)}^p \right)^{\frac{\alpha}{p}} \|u\|_{L^{r(\alpha-1)}(\mathbb{H}^n)}^{1-\theta},
\]

with \( \theta = \frac{\alpha(n-1)}{\alpha p - (\alpha p + 1 - \alpha)(n-p)} \).

(ii) If \( \alpha \in (0, 1) \), then we have

\[
\|u\|_{L^{\alpha p-1}(\mathbb{H}^n)} \leq GN(n, p, \alpha) \left( \|\nabla u\|_{L^p(\mathbb{H}^n)}^p - \left( \frac{n-1}{p} \right)^p \|u\|_{L^p(\mathbb{H}^n)}^p \right)^{\frac{\alpha}{p}} \|u\|_{L^{r(\alpha-1)}(\mathbb{H}^n)}^{1-\theta},
\]

with \( \theta = \frac{\alpha(n-1)}{(\alpha p + 1 - \alpha)(n-\alpha(n-p))} \).

The constant \( G(n, p, \alpha) \) which appears in (1.6) and (1.7) denotes the sharp constant in the Gagliardo–Nirenberg inequality in \( \mathbb{H}^n \) (see, e.g., [13–15]).

Suppose that \( n \geq 2 \) and \( p > n \). Then for any function \( u \in C_0^\infty(\mathbb{H}^n) \), it holds

\[
\|u\|_{L^p(\mathbb{H}^n)} \leq b_{n, p} V(\text{supp } u) \frac{\pi n}{n-1} \left( \int_{\mathbb{H}^n} |\nabla u|^p dV - \left( \frac{n-1}{p} \right)^p \int_{\mathbb{H}^n} |u|^p dV \right),
\]

where \( \text{supp } u \) denotes the support of the function \( u \), and \( b_{n, p} \) is the sharp constant in the Morrey–Sobolev inequality in \( \mathbb{R}^n \) (see, e.g., [34]).

Similar to (1.1), the inequalities (1.6), (1.7) and (1.8) relate both the sharp Poincaré inequality and the sharp Gagliardo–Nirenberg and the sharp Morrey–Sobolev inequalities in the hyperbolic spaces \( \mathbb{H}^n \), so they can not be improved on the constants. The inequality (1.1) is a special case.
of (1.6) with $\alpha = \frac{n}{n-p}$. The case $p = n$ is not included in Theorems 1.1 and 1.2. In this situation, there are some Hardy–Moser–Trudinger type inequalities (see, e.g., [25, 28, 31, 32, 36]). We refer the reader to the papers [9–12] for more information about the Gagliardo–Nirenberg inequality in the compact Riemannian manifolds.

The rest of this paper is organized as follows. In Section 2, we recall some basic facts about the symmetric decreasing rearrangement of function in the hyperbolic space $\mathbb{H}^n$ and prove an important result relating the symmetric decreasing rearrangement of function both in hyperbolic space and Euclidean space (see Theorem 2.2 below). Section 3 is devoted to prove Theorems 1.1 and 1.2. In section 4, we discuss some related Sobolev inequalities in hyperbolic space which generalize the inequalities due to Mugelli and Talenti in $\mathbb{H}^2$ to higher dimension.

2 Symmetric decreasing rearrangements

It is now known that the symmetrization argument works well in the setting of the hyperbolic spaces $\mathbb{H}^n$ (see, e.g., [4] for a reference on this technique). Let us recall some facts about the rearrangement in the hyperbolic spaces. Let $u: \mathbb{H}^n \to \mathbb{R}$ be a function such that

$$\text{Vol}_g(\{x \in \mathbb{H}^n : |u(x)| > t\}) = \int_{\{x \in \mathbb{H}^n : |u(x)| > t\}} dV < \infty, \quad \forall t > 0.$$  

For such a function $u$, its distribution function, denoted by $\mu_u$, is defined by

$$\mu_u(t) = V\{x \in \mathbb{H}^n : |u(x)| > t\}, \quad t > 0.$$  

The function $(0, \infty) \ni t \rightarrow \mu_u(t)$ is non-increasing and right-continuous. Then the decreasing rearrangement function $u^*$ of $u$ is defined by

$$u^*(t) = \sup\{s > 0 : \mu_u(s) > t\}.$$  

Note that the function $(0, \infty) \ni t \rightarrow u^*(t)$ is non-increasing. We now define the symmetric decreasing rearrangement function $u_g^*$ of $u$ by

$$u_g^*(x) = u^*(V(B_g(0, \rho(x)))), \quad x \in \mathbb{H}^n. \quad (2.1)$$

We also define a function $u_g^x$ on $\mathbb{H}^n$ by

$$u_g^x(x) = u^*(\sigma_n|x|^n), \quad x \in \mathbb{H}^n, \quad (2.2)$$

where $\sigma_n$ denotes the volume of unit ball in $\mathbb{R}^n$. Since $u$, $u_g^*$ and $u_g^x$ has the same non-increasing rearrangement function (which is $u^*$), then we have

$$\int_{\mathbb{H}^n} \Phi(|u|)dV = \int_{\mathbb{H}^n} \Phi(u_g^*)dV = \int_{\mathbb{H}^n} \Phi(u_g^x)dV = \int_0^\infty \Phi(u^*(t))dt, \quad (2.3)$$

for any increasing function $\Phi: [0, \infty) \to [0, \infty)$ with $\Phi(0) = 0$. This equality is a consequence of layer cake representation. Moreover, by Pólya–Szegő principle, we have

$$\int_{\mathbb{H}^n} |\nabla_g u_g^*|^p dV \leq \int_{\mathbb{H}^n} |\nabla u|^p dV. \quad (2.4)$$

Our next aim is to compare $\|\nabla_g u_g^*\|_{L^p(\mathbb{H}^n)}$ and $\|\nabla u_g^x\|_{L^p(\mathbb{R}^n)}$. For simplifying notation, we denote $v = u^*$. By a straightforward computation, we have

$$\int_{\mathbb{R}^n} |\nabla u_g^x|^p dx = (n\sigma_n)^p \int_0^\infty |v'(s)|^p \left(\frac{s}{\sigma_n}\right)^{(n-1)p} ds. \quad (2.5)$$
Note that
\[ V(B(0, \rho(x))) = n\sigma_n \int_0^{\rho(x)} (\sinh t)^{n-1} dt = \sigma_n \Phi(\rho(x)), \]
where
\[ \Phi(t) = n \int_0^t (\sinh s)^{n-1} ds. \quad (2.6) \]

Note that the function \( \Phi : [0, \infty) \to [0, \infty) \) is a diffeomorphism, strictly increasing with \( \Phi(0) = 0 \) and \( \lim_{t \to \infty} \Phi(t) = \infty. \) The gradient of \( V(B_g(0, \rho(x))) \) is then given by
\[ \nabla_g V(B(x_0, \rho(x))) = n\sigma_n (\sinh \rho(x))^{n-1} \nabla_g \rho(x). \]
Since \( |\nabla \rho(x)|_g = 1 \) for \( x \neq 0, \) then we get
\[
\int_{\mathbb{B}^n} |\nabla_g u_g^p|^p dV = \int_{\mathbb{B}^n} |v'(V(B_g(0, \rho(x))))|^p (n(\sinh(\rho(x)))^{n-1})^p dV \\
= n\sigma_n \int_0^\infty |v'(V(B_g(0, t)))|^p (n\sigma_n (\sinh(t))^{n-1})^p (\sinh t)^{n-1} dt \\
= (n\sigma_n)^p \int_0^\infty |v'(V(B_g(0, t)))|^p (\sinh t)^{(n-1)} n\sigma_n (\sinh t)^{n-1} dt.
\]

Making the change of variable \( s = V(B_g(0, t)) = \sigma_n \Phi(t) \) or \( t = \Phi^{-1}(\frac{s}{\sigma_n}), \) we have \( ds = n\sigma_n (\sinh t)^{n-1} dt \) and
\[
\int_{\mathbb{B}^n} |\nabla_g u_g^p|^p dV = (n\sigma_n)^p \int_0^\infty |v'(s)|^p \left( \sinh \Phi^{-1} \left( \frac{s}{\sigma_n} \right) \right)^{p(n-1)} ds. \quad (2.7)
\]

Let us define the function \( k_{n,p} \) on \([0, \infty)\) by
\[ k_{n,p}(s) = (\sinh \Phi^{-1}(s))^{p(n-1)} - \frac{p(n-1)}{n} s. \]
We then obtain from (2.5) and (2.7) that
\[
\int_{\mathbb{B}^n} |\nabla_g u_g^p|^p dV = \int_{\mathbb{R}^n} |\nabla u_k^p|^p dx + (n\sigma_n)^p \int_0^\infty |v'(s)|^p k_{n,p} \left( \frac{s}{\sigma_n} \right) ds. \quad (2.8)
\]

To proceed, we next find an estimate for \( k_{n,p} \) from below. In fact, we have the following results.

**Lemma 2.1.** It holds
\[ k_{n,p}(s) \geq \left( \frac{n-1}{n} \right)^p s^p, \quad s \geq 0, \quad (2.9) \]
for any \( p \geq 2 \) if \( n = 2, \) and for any \( p \geq \frac{2n}{n-1} \) if \( n \geq 3. \)

**Proof.** It is enough to prove that
\[ F_{n,p}(t) = k_{n,p}(\Phi(t)) - \left( \frac{n-1}{n} \right)^p (\Phi(t))^p \geq 0, \quad t \geq 0, \quad (2.10) \]
for any \( p \geq 2 \) if \( n = 2, \) and for any \( p \geq \frac{2n}{n-1} \) if \( n \geq 3. \)

If \( n = 2, \) we have \( \Phi(t) = 2(\cosh t - 1), \) and
\[ F_{2,p}(t) = \left( \frac{\Phi(t)^2}{4} + \Phi(t) \right)^\frac{p}{2} - \Phi(t)^\frac{p}{2} - \frac{1}{2p} \Phi(t)^p \geq 0, \]
for any \( t \geq 0 \) if \( p \geq 2. \)
Suppose that $n \geq 3$. Differentiating the function $F_{n,p}$ we get

\[ F'_{n,p}(t) = p(n - 1)(\sinh t)^{p(n-1)-1} \cosh t - p(n - 1)(\sinh t)^{n-1} \Phi(t) \frac{p(n-1)-1}{n-1} \]

\[ - \left( \frac{n-1}{n} \right)^p p(n-1)(\sinh t)^{n-1} \Phi(t)^{p-1} \]

\[ = p(n - 1)(\sinh t)^{n-1} \left( \sinh t)^{p(n-1)-n} \cosh t - \Phi(t) \frac{p(n-1)-1}{n-1} \right) \]

\[ - \left( \frac{n-1}{n} \right)^p \Phi(t)^{p-1} \]

\[ =: p(n - 1)(\sinh t)^{n-1} G_{n,p}(t). \]

We continue differentiating the function $G_{n,p}$ to obtain

\[ G'_{n,p}(t) = (p(n - 1) - n)(\sinh t)^{p(n-1)-n} \cosh t) + (\sinh t)^{(p-1)(n-1)} \]

\[ - (p(n - 1) - n)(\sinh t)^{n-1} \Phi(t) \frac{p(n-1)-2}{n-1} \left( \frac{n-1}{n} \right)^p \Phi(t)^{p-2} \]

\[ = (p(n - 1) - n)(\sinh t)^{n-1} \left( \sinh t)^{(p-2)(n-1)} \right) \]

\[ - \left( \frac{n-1}{n} \right)^p \Phi(t)^{p-2} \]

\[ + (p(n - 1) - n)(\sinh t)^{n-1} \left( \sinh t)^{(p-1)-2n} - \Phi(t) \frac{p(n-1)-2}{n-1} \right). \]

It is easy to see that

\[ \Phi(t) = n \int_0^t (\sinh s)^{n-1} ds < n \int_0^t (\sinh s)^{n-1} \cosh s ds = (\sinh t)^n, \quad t > 0, \]

and

\[ \Phi(t) = n \int_0^t (\sinh s)^{n-1} ds < n \int_0^t (\sinh s)^{n-2} \cosh s ds = \frac{n}{n-1}(\sinh t)^{n-1}, \quad t > 0. \]

Plugging these previous estimates into the expression of $G'_{n,p}$, we get that $G'_{n,p}(t) > 0$ for any $t > 0$. This implies $G_{n,p}(t) > G_{n,p}(0) = 0$ for any $t > 0$, or equivalently $F'_{n,p}(t) > 0$ for any $t > 0$. Consequently, $F_{n,p}(t) > F(0) = 0$ for any $t > 0$. This completes our proof.

It is remarkable that the pointwise estimate (2.9) is sharp in $p$. Indeed, if $n = 2$ and $p \in (1, 2)$ then a reversed estimate of (2.9) holds. Suppose that $n \geq 3$ and $p < \frac{2n}{n-1}$ we next show that a reversed estimate of (2.9) holds for $s$ large enough. Indeed, suppose that $n \geq 4$, we have

\[ \Phi(t) = n^{2-1-n} \int_0^t (e^s - e^{-s})^{n-1} ds = \frac{n^{2-1-n}}{n-1} e^{(n-1)t} \left( 1 - \frac{(n-1)^2}{n-3} e^{-2t} + o(e^{-2t}) \right), \]

as $t \to \infty$. Consequently

\[ \Phi(t)^{\frac{p(n-1)}{n}} = \left( \frac{n^{2-1-n}}{n-1} \right)^{\frac{p(n-1)}{n}} e^{\frac{p(n-1)^2}{n} t} \left( 1 - \frac{p(n-1)^2}{n(n+1)} e^{-2t} + o(e^{-2t}) \right), \]

and

\[ \Phi(t)^p = \left( \frac{n^{2-1-n}}{n-1} \right)^p e^{p(n-1)t} \left( 1 - \frac{p(n-1)^2}{n(n-3)} e^{-2t} + o(e^{-2t}) \right). \]
as \( t \to \infty \). Note that
\[
(sinh \, t)^{p(n-1)} = 2^{p(n-1)}e^{p(n-1)t} \left(1 - p(n - 1)e^{-2t} + o(e^{-2t})\right),
\]
as \( t \to \infty \). Therefore
\[
F_{n,p}(t) = (sinh \, t)^{p(n-1)} - \Phi(t)^{\frac{n(n-1)}{n}} - \left(\frac{n-1}{n}\right)^p \Phi(t)^p
= 2^{p(n-1)}e^{p(n-1)t-2t} \left(\frac{2p(n - 1)}{n - 3} - \left(\frac{n^2-1}{n - 1}\right)\right)^{\frac{n(n-1)}{n}} e^{2t - \frac{n(n-1)}{n}t} + o(1)
\]
as \( t \to \infty \). If \( p < \frac{2n}{n-1} \) we then have \( \frac{n(n-1)}{n} < 2 \), and \( F_{n,p}(t) < 0 \) for \( t > 0 \) large enough. Suppose \( n = 3 \), we have
\[
\Phi(t) = \frac{3}{8}(e^{2t} - e^{-2t} - 4t) = \frac{3}{8}e^{2t}(1 - 4te^{-2t} + o(e^{-2t})),
\]
as \( t \to \infty \). Hence
\[
\Phi(t)^p = \frac{3p}{8}e^{2pt} (1 - 4pt e^{-2t} + o(e^{-2t})),
\]
and
\[
\Phi(t)^{\frac{2p}{3}} = \left(\frac{3}{8}\right)^{\frac{2p}{3}} e^{\frac{4pt}{3}} (1 - \frac{8pt}{3} e^{-2t} + o(te^{-2t})),
\]
as \( t \to \infty \). Evidently,
\[
(sinh \, t)^{2p} = 2^{-2p}e^{2pt}(1 + o(te^{-2t}))
\]
as \( t \to \infty \). Consequently, we get
\[
F_{3,p}(t) = \frac{1}{4p}e^{2(p-1)t} \left(4p - 4^{p-1} \left(\frac{3}{8}\right)^{\frac{2p}{3}} t^{-1}e^{2t} - \frac{2p}{3} + o(1)\right),
\]
as \( t \to \infty \). Since \( p < 3 \), then we have \( 2 - \frac{2p}{3} > 0 \) and hence \( F_{3,p}(t) < 0 \) for \( t > 0 \) large enough.

Combining (2.9) and (2.8) together, we arrive
\[
\int_{\mathbb{R}^n} |\nabla g u_g|^p g dV \geq \int_{\mathbb{R}^n} |\nabla u_c|^p dx + (n-1)^p \int_0^\infty |v'(s)|^p s^p ds. \tag{2.11}
\]
Making the change of function \( w(s) = v(s) s^\frac{1}{p} \) or equivalently \( v(s) = w(s) s^{-\frac{1}{p}} \). Differentiating the function \( v \), we have
\[
v'(s) = w'(s) s^{-\frac{1}{p}} - \frac{1}{p} w(s) s^{-\frac{1}{p} - 1}.
\]
We can readily check that if \( a - b \leq 0, b \geq 0 \) and \( p \geq 2 \) then
\[
|a - b|^p \geq |a|^p + |b|^p - pab^{p-1}
\]
Since \( v' \leq 0 \), applying the previous inequality we get
\[
\int_0^\infty |v'(s)|^p s^p ds \geq \int_0^\infty |w'(s)|^p s^{p-1} ds + \frac{1}{p^p} \int_0^\infty w(s)^p s^{-1} ds - p^{2-p} \int_0^\infty w'(s)w(s)^{p-1} ds
= \int_0^\infty |w'(s)|^p s^{p-1} ds + \frac{1}{p^p} \int_0^\infty v(s)^p ds,
\]
here we use integration by parts. Plugging this estimate into (2.11) we get
\[
\int_{\mathbb{R}^n} |\nabla g u_g|^p g dV \geq \int_{\mathbb{R}^n} |\nabla u_c|^p dx + \frac{(n-1)^p}{p^p} \int_0^\infty |v|^p ds + (n-1)^p \int_0^\infty |(v(s)s^\frac{1}{p})'|^p s^{p-1} ds. \tag{2.12}
\]
Since $v = u^*$ is non-increasing rearrangement function of $u_0^*$, then
\[
\int_0^\infty |v|^p ds = \int_{\mathbb{R}^n} |u_0^*|^p dV.
\] (2.13)
Plugging (2.13) into (2.12), we obtain the main result of this section as follows,

**Theorem 2.2.** Let $p \geq 2$ if $n = 2$ and $p \geq \frac{2n}{n-1}$ if $n \geq 3$. It holds
\[
\int_{\mathbb{R}^n} |\nabla u|^p dV - \left( \frac{n-1}{p} \right)^p \int_{\mathbb{R}^n} |u|^p dV \geq \int_{\mathbb{R}^n} |\nabla u_0^*|^p dx.
\] (2.14)

Theorem 2.2 was proved in [32] in the case $p = n$ as a key to establish several improved Moser–Trudinger type inequalities in the hyperbolic space.

### 3 Proof of Theorems 1.1 and 1.2

In this section, we provide the proof of Theorem 1.1 and Theorem 1.2. Our proof uses Theorem 2.2 above and the known inequalities in the Euclidean spaces such as the sharp Sobolev, Gagliardo–Nirenberg and Morrey–Sobolev inequalities. Let us recall them here. The sharp Sobolev inequality in the euclidean space was independently proved by Aubin and Talenti [1,33] and has the form
\[
S(n, p) \|u\|_{L^{p^*}(\mathbb{R}^n)} \leq \|\nabla u\|_{L^p(\mathbb{R}^n)}, \quad u \in C_0^\infty(\mathbb{R}^n),
\] (3.1)
for $p \in (1, n)$, $p^* = \frac{n p}{n-p}$ and the sharp constant $S(n, p)$ is given by
\[
S(n, p) = \left[ \frac{1}{n} \left( \frac{n(p-1)}{n-p} \right)^{1-\frac{1}{p}} \frac{\Gamma(n)}{\Gamma\left(\frac{n}{2}\right)\Gamma(n+1-\frac{n}{2})} \right]^{\frac{1}{p}},
\]
where $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt, x > 0$ denotes the usual Gamma function. The family of extremal functions is determined uniquely by the function $u(x) = (1 + |x|^{\frac{2}{n-p}})^{-\frac{n-p}{n}}$ up to a translation, dilation and multiplying by constant.

Let $p \in (1, n)$ and $\alpha \in (0, \frac{n}{n-p})$, $\alpha \neq 1$. The sharp Gagliardo–Nirenberg inequalities in $\mathbb{R}^n$ was established by Del Pino and Dolbeault [14,15] and has the forms:

(i) for $\alpha > 1$,
\[
\|u\|_{L^{q^*}(\mathbb{R}^n)} \leq GN(n, p, \alpha) \|\nabla u\|_{L^p(\mathbb{R}^n)}^\alpha \|u\|_{L^{q^*}(\mathbb{R}^n)}^{1-\alpha}, \quad u \in C_0^\infty(\mathbb{R}^n),
\] (3.2)
with $\theta = \frac{n(\alpha-1)}{\alpha(n+1-\alpha)(n-p)}$, the sharp constant $GN(n, p, \alpha)$ is given by
\[
GN(n, p, \alpha) = \left( \frac{q-p}{p\sqrt{\pi}} \right)^\theta \left( \frac{pq}{n(q-p)} \right)^\frac{\delta}{p} \left( \frac{\Gamma\left(q\frac{n-1}{p}\right)}{\Gamma\left(\frac{q-1}{p}\right)\Gamma(n\frac{n-1}{p}+1)} \right)^\frac{1}{p},
\]
with $q = \alpha(p-1)+1$, $\delta = np - (n-p)q$, and an extremal functions is given the function $u(x) = (1+|x|^{\frac{2}{n-\alpha}})^{-\frac{1}{n-\alpha}}$.

(ii) for $\alpha < 1$,
\[
\|u\|_{L^\alpha(\mathbb{R}^n)} \leq GN(n, p, \alpha) \|\nabla u\|_{L^p(\mathbb{R}^n)}^\alpha \|u\|_{L^{\alpha^*}(\mathbb{R}^n)}^{1-\alpha}, \quad u \in C_0^\infty(\mathbb{R}^n),
\] (3.3)
with $\theta = \frac{n(1-\alpha)}{(n+1-\alpha)(\alpha(n-p))}$, the sharp constant $GN(n, p, \alpha)$ is given by
\[
GN(n, p, \alpha) = \left( \frac{p-q}{p\sqrt{\pi}} \right)^\theta \left( \frac{pq}{n(p-q)} \right)^\frac{\delta}{p} \left( \frac{\Gamma\left(q\frac{p-1}{p-q}\right)}{\Gamma\left(\frac{q-1}{p-q}\right)\Gamma(n\frac{p-1}{p-q}+1)} \right)^\frac{1}{p},
\]
with \( q = \alpha(p-1)+1, \delta = np-q(n-p) > 0, \) and an extremal functions is given the function 

\[ u(x) = (1 - |x|^\frac{q}{p-n})^+, \]

where \( a_+ = \max\{a, 0\} \) denotes the positive part of a number \( a \).

We refer the reader to the paper of Cordero-Erausquin, Nazaret and Villani \([13]\) for a completely different proof of the sharp Sobolev inequality \((3.1)\) and the sharp Gagliardo–Nirenberg inequality \((3.2)\) and \((3.3)\) by using the mass transportation method.

Finally, we recall the sharp Morrey–Sobolev inequality in \( \mathbb{R}^n \). Given \( p > n \), then for any function \( u \in C_0^\infty(\mathbb{R}^n) \), the following inequality holds

\[ \|u\|_{L^\infty(\mathbb{R}^n)} \leq b_{n,p} \Vol(\text{supp } u)^\frac{p-n}{p} \|\nabla u\|_{L^p(\mathbb{R}^n)}, \tag{3.4} \]

here \( \Vol \) denotes the Lebesgue measure of any measurable subset of \( \mathbb{R}^n \), the sharp constant \( b_{n,p} \) is given by

\[ b_{n,p} = n^{-\frac{1}{p}} \sigma_n^{-\frac{1}{n}} \left( \frac{p-1}{p-n} \right)^{\frac{1}{n}}, \]

and an extremal function is given by \( u(x) = (1 - |x|^\frac{q}{p-n})^+ \). For more about this inequality, the reader may consult \([34]\).

Let us go to prove Theorem 1.1 and Theorem 1.2.

**Proof of Theorem 1.1.** Suppose \( u \) is a function in \( W^{1,p}(\mathbb{H}^n) \). Let us define two new functions \( u^g \) and \( u^e \) by \((2.1)\) and \((2.2)\) respectively. Theorem 2.2 implies

\[ \|\nabla g u^g\|_{L^p(\mathbb{H}^n)} = \left( \frac{n-1}{p} \right)^{\frac{p}{p}} \|u_g\|_{L^p(\mathbb{H}^n)} \geq \|\nabla u^g\|_{L^p(\mathbb{H}^n)}, \]

for any \( p \geq 2 \) if \( n = 2 \), and for any \( p \geq \frac{2n}{n-1} \) if \( n \geq 3 \). Note that \( \|u_g\|_{L^p(\mathbb{H}^n)} = \|u\|_{L^p(\mathbb{H}^n)} \). Hence, applying Pólya–Szegő principle \((2.4)\) and equality \((2.3)\), we get

\[ \|\nabla g u\|_{L^p(\mathbb{H}^n)} = \left( \frac{n-1}{p} \right)^{\frac{p}{p}} \|u\|_{L^p(\mathbb{H}^n)} \geq \|\nabla u^e\|_{L^p(\mathbb{H}^n)}. \tag{3.5} \]

Suppose that \( n \geq 4 \) and \( \frac{2n}{n-1} \leq p < n \). Using the sharp Sobolev inequality \((3.1)\) for \( u^e \) and using the equality \( \|u^e\|_{L^p(\mathbb{H}^n)} = \|u\|_{L^p(\mathbb{H}^n)} \), we obtain the desired inequality \((1.1)\).

Suppose \( u \in W^{1,p}(\mathbb{H}^n) \) such that the equality in \((1.1)\) holds for \( u \). Let \( v = u^* \) the decreasing rearrangement function of \( u \) on \([0,\infty)\), and define \( u^g \) and \( u^e \) by \((2.1)\) and \((2.2)\) respectively. Since the equality in \((1.1)\) holds for \( u \), we must have \( \|\nabla g u\|_{L^p(\mathbb{H}^n)} = \|\nabla g u^g\|_{L^p(\mathbb{H}^n)} \) and

\[ \|\nabla g u^g\|_{L^p(\mathbb{H}^n)} = \left( \frac{n-1}{p} \right)^{\frac{p}{p}} \|u^g\|_{L^p(\mathbb{H}^n)} = \|\nabla u^e\|_{L^p(\mathbb{H}^n)}. \]

From the proof of Theorem 2.2, we see that the second condition implies

\[ \int_0^\infty |(v(s)s^\frac{1}{p})'|^p s^{p-1} ds = 0. \]

Thus, we have \( v(s) = cs^\frac{1}{p} \) for some constant \( c \in \mathbb{R} \). However, \( \int_0^\infty v(s)^p ds < \infty \) which forces \( c = 0 \). This finishes our proof of Theorem 1.1. \(\blacksquare\)

**Proof of Theorem 1.2.** The proof of Theorem 1.2 is similar with the one of Theorem 1.1. Suppose that \( n \geq 4 \) and \( \frac{2n}{n-1} \leq p < n \). By \((3.5)\), we can apply the sharp Gagliardo–Nirenberg inequalities \((3.2)\) and \((3.3)\) for function \( u^g \) to derive the desired inequalities \((1.6)\) and \((1.7)\) as done for the inequality \((1.1)\), respectively.

Suppose that \( n \geq 2 \) and \( p > n \). We note that \((3.5)\) still holds under this condition. We now can apply the sharp Morrey–Sobolev inequality \((3.4)\) for \( u^g \) to yield the inequality \((1.8)\) with remark that \( \|u^g\|_{L^\infty(\mathbb{H}^n)} = \|u\|_{L^\infty(\mathbb{H}^n)} \) and \( \Vol(\text{supp } u^g) = V(\text{supp } u) \). \(\blacksquare\)
We conclude this section by a remark in the case $\alpha \rightarrow 1^+$ of the inequality (1.6). Taking the limit as done in [15], we obtain the following Poincaré–Sobolev logarithmic inequality in $\mathbb{H}^n$ which is an extension of the optimal Euclidean $L^p$–Sobolev logarithmic inequality [14, 15] to the hyperbolic spaces. Suppose $u \in W^{1,p}(\mathbb{H}^n)$ with $\|u\|_{L^p(\mathbb{H}^n)} = 1$, it holds
\[
\int_{\mathbb{H}^n} |u|^p \ln(|u|^p) \, dV \leq \frac{n}{p} \ln \left( \mathcal{L}_{n,p} \int_{\mathbb{H}^n} \left( |\nabla_g u|^p - \left( \frac{n-1}{n} \right) |u|^p \right) \, dV \right)
\]  
for any $n \geq 4$ and $\frac{n}{n-1} \leq p < n$ with the constant $\mathcal{L}_{n,p}$ is given by
\[
\mathcal{L}_{n,p} = \frac{p}{n} \left( \frac{p-1}{e} \right)^{p-1} \pi^{\frac{p}{2}} \left[ \frac{\Gamma\left( \frac{n}{2} + 1 \right)}{\Gamma(n \frac{p}{p-1} + 1)} \right]^{\frac{p}{2}}.
\]

4 Other Sobolev inequalities in the hyperbolic spaces

In this section, we establish several Sobolev inequalities in the hyperbolic spaces $\mathbb{H}^n$. These inequalities generalize the results of Mugelli and Talenti [30] in $\mathbb{H}^2$ to higher dimensional spaces. The main results of this section read as follows.

**Theorem 4.1.** Let $n \geq 2$ and $p \in [1, \infty)$. Then for any function $u \in W^{1,p}(\mathbb{H}^n)$, the following inequalities holds.

(i) If $p = 1$ then
\[
(n-1)^n \left( \int_{\mathbb{H}^n} |u| \, dV \right)^n + S(n,1)^n \left( \int_{\mathbb{H}^n} |u|^{\frac{2n}{n-1}} \, dV \right)^{n-1} \leq \left( \int_{\mathbb{H}^n} |\nabla_g u|^p \, dV \right)^n.
\]

(ii) If $1 < p < n$ then
\[
\left( \frac{n-1}{p} \right)^n \left( \int_{\mathbb{H}^n} |u|^p \, dV \right)^{\frac{n}{p}} + S(n,p)^n \left( \int_{\mathbb{H}^n} |u|^{\frac{2n}{n-1}} \, dV \right)^{\frac{n-1}{p}} \leq \left( \int_{\mathbb{H}^n} |\nabla_g u|^p \, dV \right)^{\frac{p}{n}}.
\]

(iii) If $n < p < \infty$ then
\[
\sup_{x \in \mathbb{H}^n} |u(x)| \leq C(n,p) \left( \int_{\mathbb{H}^n} |\nabla_g u|^p \, dV \right)^{\frac{1}{p}},
\]
with
\[
C(n,p) = (2^{n-1}n\sigma_n)^{\frac{1}{p}} \left( \frac{\Gamma\left( \frac{n-1}{2(p-1)} \right) \Gamma\left( \frac{p-1}{p-1} \right)}{\Gamma\left( \frac{n-1}{2(p-1)} \right)} \right)^{\frac{n-1}{p}}.
\]

Furthermore, the inequality holds in (4.3) if $u(x) = v(V(B_p(0,\rho(x))))$ with
\[
v(r) = c \int_r^{\infty} \left( \sinh \Phi^{-1} \left( \frac{s}{\sigma_n} \right) \right)^{-\frac{p(n-1)}{p-1}} \, ds.
\]

Obviously, $(a+b)^\alpha \leq a^\alpha + b^\alpha$ for any $\alpha \in [0,1]$ and $a,b \geq 0$. As a consequence, the inequality (4.2) is weaker than the inequality (1.1). However, the inequality (4.2) is valid for any $n \geq 2$ and $1 < p < n$.

**Proof.** The part (i) follows from part (ii) by letting $p \downarrow 1$. We next prove part (ii). Let $u \in W^{1,p}(\mathbb{H}^n)$, we define two new functions $u_g^\#$ and $u_g^\circ$ by (2.1) and (2.2) respectively. Denote $v = u^\circ$.
and recall the function \( \Phi \) from (2.6). By (2.7), we have

\[
\int_{\mathbb{R}^n} |\nabla_y u_y|^p dV = (n\sigma_n)^p \int_0^\infty |v'(s)|^p \left( \sinh \Phi^{-1} \left( \frac{s}{\sigma_n} \right) \right)^{p(n-1)} ds \\
= (n\sigma_n)^p \int_0^\infty |v'(s)|^p \left( \left( \sinh \Phi^{-1} \left( \frac{s}{\sigma_n} \right) \right)^{n(n-1)} \right) \frac{ds}{s^{\frac{p-1}{p}}} \\
= (n\sigma_n)^p \int_0^\infty |v'(s)|^p \left( k_{n,n} \left( \frac{s}{\sigma_n} \right) + \left( \frac{s}{\sigma_n} \right)^{n-1} \right) \frac{ds}{s^{\frac{p-1}{p}}} \\
\geq (n\sigma_n)^p \int_0^\infty |v'(s)|^p \left( \frac{n-1}{n} \right)^n \left( \frac{s}{\sigma_n} \right)^n + \left( \frac{s}{\sigma_n} \right)^{n-1} \right) \frac{ds}{s^{\frac{p-1}{p}}},
\]

here we use Lemma 2.1 to bound \( k_{n,n} \) from below. It is easy to see that for \( \alpha \in (0,1) \), it holds

\[
(a + b)^\alpha = \sup_{t \in [0,1]} (t^{1-\alpha}a^{\alpha} + (1-t)^{1-\alpha)b^{\alpha}). \tag{4.5}
\]

Consequently, for any \( t \in [0,1] \), we get

\[
\int_{\mathbb{R}^n} |\nabla_y u_y|^p dV \geq t^{1-\frac{p}{n}} (n-1)^p \int_0^\infty |v'(s)|^p s^p ds + (1-t)^{1-\frac{p}{n}} (n\sigma_n)^p \int_0^\infty |v'(s)|^p \left( \frac{s}{\sigma_n} \right)^{p(n-1)} ds \\
\geq t^{1-\frac{p}{n}} (n-1)^p \int_0^\infty |v'(s)|^p ds + (1-t)^{1-\frac{p}{n}} \|\nabla u|_{L^p(\mathbb{R}^n)}^p \\
\geq t^{1-\frac{p}{n}} \frac{(n-1)^p}{p^n} \|u_y|^p_{L^p(\mathbb{R}^n)} + (1-t)^{1-\frac{p}{n}} S(n,p) \|u|^p_{L^p(\mathbb{R}^n)} \\
\] 

here we use the Hardy inequality in \([0, \infty)\) for the first inequality, and the sharp Sobolev inequality for the second inequality. Taking the supremum over \( t \in [0,1] \) and using again (4.5) and the fact \( \|u\|_{L^p(\mathbb{R}^n)} = \|u_y\|_{L^p(\mathbb{R}^n)} \) and \( \|u\|_{L^p^*(\mathbb{R}^n)} = \|u_y\|_{L^p^*(\mathbb{R}^n)} \) with \( p^* = \frac{np}{n-p} \), we obtain

\[
\int_{\mathbb{R}^n} |\nabla_y u_y|^p dV \geq \left[ \frac{n-1}{p} \|u\|_{L^p(\mathbb{R}^n)}^n + S(n,p)^n \|u\|_{L^{p^*}(\mathbb{R}^n)}^n \right] \frac{2}{n},
\]

which implies (4.2) by the Pólya–Szego principle in \( \mathbb{R}^n \).

We next prove part (iii). Denote

\[
l(s) = \left[ \sinh \Phi^{-1} \left( \frac{s}{\sigma_n} \right) \right]^{n-1}, \quad s \geq 0,
\]

where \( \Phi \) is defined by (2.6). It is easy to check that \( l(s) \sim s^{\frac{n-1}{2}} \) as \( s \to 0 \) and \( l(s) \sim s \) as \( s \to \infty \). Consequently, we get

\[
\int_0^\infty l(s)^{-\frac{p}{p-1}} ds < \infty.
\]

Moreover, making the change \( s = \sigma_n \Phi(t) \) with \( ds = n\sigma_n (\sinh t)^{n-1} dt \), we have

\[
\int_0^\infty l(s)^{-\frac{p}{p-1}} ds = \int_0^\infty \frac{\sinh \Phi^{-1} \left( \frac{s}{\sigma_n} \right)}{s^{\frac{p-1}{p}}} ds = n\sigma_n \int_0^\infty (\sinh t)^{-\frac{p-1}{p}} dt \\
= n\sigma_n 2^{\frac{n-p}{2p}} \frac{\Gamma \left( \frac{p-n}{p} \right) \Gamma \left( \frac{n-1}{2p-1} \right)}{\Gamma \left( \frac{2p-n}{2p-1} \right)},
\]

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Using the identity \( \Gamma(x)\Gamma(x + \frac{1}{2}) = 2^{1-2x}\Gamma(2x) \), we get
\[
\int_0^\infty s^{-\frac{p}{p-1}}l(s)s^\sigma ds = n\sigma n^{2-\frac{p}{p-1}}\frac{\Gamma\left(\frac{p-n}{2p-1}\right)\Gamma\left(\frac{n}{2p-1}\right)}{\Gamma\left(\frac{p+n}{2p-1}\right)}.
\] (4.6)

Denote \( v = u^* \), we have \( \lim_{s \to \infty} v(s) = 0 \), and hence
\[
v(r) = \int_r^\infty v'(s)ds = \int_r^\infty v'(s)l(s)s^{-1}ds.
\]

Thank to H"older inequality, we get
\[
\sup_{x \in \mathbb{R}^n} |u(x)| = v(0) \leq \left( \int_0^\infty |v'(s)|^p l(s)ds \right)^{\frac{1}{p}} \left( \int_0^\infty l(s)^{-\frac{p}{p-1}}ds \right)^{\frac{p-1}{p}}
\]
\[
= \frac{1}{n\sigma_n} \left( \int_0^\infty l(s)^{-\frac{p}{p-1}}ds \right)^{\frac{p-1}{p}} \left( (n\sigma_n)^p \int_0^\infty |v'(s)|^p l(s)ds \right)^{\frac{1}{p}}
\]
\[= C(n,p) \| \nabla u^*_\|_{L^p(\mathbb{R}^n)},
\]
here the second equality comes from (2.7) and (4.6). This proves (4.3).

To check the sharpness of \( C(n,p) \), we see that if \( u(x) = v(V(B_g(0,\rho(x)))) \) with \( v \) defined by (4.4), then \( u^* = v \). Hence \( \sup_{x \in \mathbb{R}^n} |u(x)| = v(0) \). Moreover, for such a choice of function \( v \), we have equality in the H"older inequality above. This proves the sharpness of \( C(n,p) \) and the equality holds for this function \( u \). \( \square \)

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