On VT-harmonic maps

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Abstract
VT-harmonic maps generalize the standard harmonic maps, with respect to the structure of both domain and target. These can be manifolds with natural connections other than the Levi-Civita connection of Riemannian geometry, like Hermitian, affine or Weyl manifolds. The standard harmonic map semilinear elliptic system is augmented by a term coming from a vector field \( V \) on the domain and another term arising from a 2-tensor \( T \) on the target. In fact, this geometric structure then also includes other geometrically defined maps, for instance magnetic harmonic maps. In this paper, we treat VT-harmonic maps and their parabolic analogues with PDE tools. We establish a Jäger–Kaul type maximum principle for these maps. Using this maximum principle, we prove an existence theorem for the Dirichlet problem for VT-harmonic maps. As applications, we obtain results on Weyl/affine/Hermitian harmonic maps between Weyl/affine/Hermitian manifolds, as well as on magnetic harmonic maps from two-dimensional domains. We also derive gradient estimates and obtain existence results for such maps from noncompact complete manifolds.

Keywords VT-harmonic maps · Maximum principle · Uniqueness · Existence

Mathematics Subject Classification 58E20 · 53C43

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1 Introduction

Let \((M^m, g)\) be a compact Riemannian manifolds with nonempty boundary \(\partial M\) and \((N^n, \tilde{g})\) a complete Riemannian manifold without boundary. Let \(d : N \times N \to \mathbb{R}\) be the distance function on \(N\) and \(B_{(1+\sigma)}R(p) := \{ q \in N : d(p, q) \leq (1 + \sigma)R \}\) a regular ball in \(N\), that is, disjoint from the cut locus of its center \(p\) and of radius \((1 + \sigma)R < \frac{\pi}{2\sqrt{\kappa}}\), where \(\kappa = \max\{0, \sup_{B_{(1+\sigma)}R(p)} K_N\}\) and \(\sup_{B_{(1+\sigma)}R(p)} K_N\) is an upper bound of the sectional curvature \(K\) of \(N\) on \(B_{(1+\sigma)}R(p)\), and \(\sigma > 0\) is any given constant.

Let \(V \in \Gamma(TM), T \in \Gamma(\otimes^{1,2}TN)\). We call a map \(u : M \to N\) a VT-harmonic map if \(u\) satisfies

\[
\tau(u) + du(V) + \text{Tr}_g T(du, du) = 0, \tag{1.1}
\]

where \(\tau(u) = \text{tr} D^2 u\) is the tension field of the map \(u\). This is a generalization the notion of a \(V\)-harmonic map that has been studied in recent years as a common framework including Hermitian, affine and Weyl harmonic maps into Riemannian manifolds, that is, the domain possessed a connection different from the Levi-Civita connection, but the target was a Riemannian manifold with its Levi-Civita connection. This generalized the standard harmonic map system \(\tau(u) = 0\) to a system of the form \(\tau(u) + du(V) = 0\) with a vector field \(V\) on the domain. Here, we want to consider targets that are of the same type as the domain. That leads to the system (1.1) with an additional term arising from a 2-tensor \(T\) on the target. As this new term \(\text{Tr}_g T(du, du)\), in contrast to the term \(du(V)\), is analytically of the same weight as the elliptic operator \(\tau(u)\) (which includes the Laplace–Beltrami operator of the domain), this makes the analysis more difficult and subtle. This is the problem that we are addressing in this paper.

In local coordinates \(\{x^\alpha\}\) on \(M\) and \(\{y^j\}\) on \(N\), respectively, we can write (1.1) as

\[
\Delta_M u^i + \Gamma^i_{jk}(u) \frac{\partial u^j}{\partial x^\alpha} \frac{\partial u^k}{\partial x^\beta} g^{\alpha\beta} + V^\alpha \frac{\partial u^i}{\partial x^\alpha} + T^i_{jk}(u) \frac{\partial u^j}{\partial x^\alpha} \frac{\partial u^k}{\partial x^\beta} g^{\alpha\beta} = 0, \tag{1.2}
\]

where \(\Delta_M\) is the Laplacian on \((M, g)\), \(\Gamma^i_{jk}\) stands for the Christoffel symbols of \((N, \tilde{g})\), \(V := V^\alpha \frac{\partial}{\partial x^\alpha}\) and \(T := T^i_{jk} \frac{\partial}{\partial y^j} \otimes dy^i \otimes dy^k\). This is a second-order semilinear elliptic system on the manifold \((M, g)\).

As is already the case for \(V\)-harmonic maps, in general, (1.1) is neither in divergence form, nor has a variational structure. Chen et al. [5] established a Jäger–Kaul type maximum principle for \(V\)-harmonic maps by using the method of [8], and combining this with the continuity method, the existence of \(V\)-harmonic maps into a regular ball could be proved. Therefore, it is natural to ask whether a maximum principle holds for VT-harmonic maps. However, the case of VT-harmonic maps is harder to deal with than \(V\)-harmonic maps since we now have an additional quadratic term arising from the tensor \(T\). Due to this additional structure, the construction of the elliptic operator in [5] is no longer valid in our case. To overcome this difficulty, we use another construction as in [7] to compensate this term and obtain the following maximum principle for VT-harmonic maps:

**Theorem 1** Let \(u_1, u_2 \in C^0(M, N)\) be two VT-harmonic maps into a geodesic ball \(B_R(p)\). For appropriate \(\sigma\) and \(R\), there exists a constant \(C_0\) depending only on \(\kappa, \sigma, R\) and the geometry of \(N\), such that if

\[
\max |\nabla T| + \max |T| \leq C_0, \tag{1.3}
\]

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then the function $\Theta : M \rightarrow \mathbb{R}$ defined by

$$\Theta := \frac{q_\kappa^\xi (\rho)}{(q_\kappa((1 + \sigma)R) - q_\kappa(\rho_1))^2 \cdot (q_\kappa((1 + \sigma)R) - q_\kappa(\rho_2))^2}$$

(1.4)

satisfies the maximum principle, namely

$$\max_M \Theta \leq \max_{\partial M} \Theta.$$

Here the expression of $q_\kappa$ is given in Sect. 2, and $\rho := d(u_1, u_2)$, $\rho_i := d(p, u_i), i = 1, 2$.

In particular, if $u_1 = u_2$ on the boundary $\partial M$, then $u_1 \equiv u_2$ on $M$.

**Remark** The explicit expression of the constant $C_0$ in the above and in the subsequent results can be seen in (3.5). Importantly, $C_0 \rightarrow \infty$ for $R \rightarrow 0$. Thus, we can also satisfy the condition on $T$ by making the target ball sufficiently small.

For the heat flow of $VT$-harmonic maps, an analogous result holds. For $T > 0$, we set

$$M_T := M \times [0, T]$$

and denote the parabolic boundary of $M_T$ by

$$\partial_p M_T := (M \times \{0\}) \cup (\partial M \times [0, T]).$$

We consider the heat flow of $VT$-harmonic maps

$$\partial_t u = \tau(u) + du(V) + \text{Tr}_g T(du, du)$$

(1.5)

and have

**Theorem 2** Let $u_1, u_2 \in C^0(M, N)$ be two solutions of heat flow Eq. (1.5) for $VT$-harmonic maps into a geodesic ball $B_R(p)$. For appropriate $\sigma$ and $R$, there exists a constant $C_0$ depending only on $\kappa, \sigma, R$ and the geometry of $N$, such that if

$$\max |\nabla T| + \max |T| \leq C_0,$$

then the function $\Theta : M_T \rightarrow \mathbb{R}$ defined by (1.4) with $M$ replaced by $M_T$ satisfies the maximum principle:

$$\max_{M_T} \Theta \leq \max_{\partial_p M_T} \Theta.$$

In particular, if $u_1 = u_2$ on the boundary $\partial_p M_T$, then $u_1 \equiv u_2$ on $M_T$.

As an application of the above maximum principle, we obtain the existence of $VT$-harmonic maps into a geodesic ball.

**Theorem 3** Let $M, N, V, T, B_R(p)$ be as in Theorem 2. Suppose $u_0 \in H^{2,q}(M, N)(q > m)$ with $u_0(M) \subset B_R(p)$. For appropriate $\sigma$ and $R$, there exists a constant $C_0$ depending only on $\kappa, \sigma, R$ and the geometry of $N$, such that if

$$\max |\nabla T| + \max |T| \leq C_0,$$

then the initial boundary value problem

$$\left\{ \begin{array}{l}
\partial_t u = \tau(u) + du(V) + \text{Tr}_g T(du, du), \\
u - u_0 \in H^{2,q}_0(M, N), \quad u(0) = u_0, \quad u(M \times [0, \infty)) \subset B_R(p),
\end{array} \right.$$  

(1.6)

satisfies the maximum principle, namely

$$\max_M \Theta \leq \max_{\partial M} \Theta.$$
admits a unique global solution $u$ which subconverges to a unique solution $u \in H^{2,q}(M, N)$ of the Dirichlet problem

\[
\begin{cases}
  \tau(u) + d u(V) + \text{Tr}_g T(d u, d u) = 0, \\
  u - u_0 \in H_0^{2,q}(M, N),
\end{cases}
\]

such that $u(M) \subset B_R(p)$.

Furthermore, based on Theorem 3, we shall also establish the existence of $VT$-harmonic maps from complete noncompact Riemannian manifolds by using a gradient estimate and the compact exhaustion method.

**Theorem 4** Let $(M^n, g)$ be a complete noncompact Riemannian manifold and $(N^m, \tilde{g})$ be a complete Riemannian manifold with sectional curvature bounded above by a positive constant $\kappa$. Let $B_R(p)$ be a geodesic ball with radius $R < \frac{\pi}{2(1+\sigma)\sqrt{\kappa}}$ and $u_0 : M \to N$ a smooth map with $u_0(M) \subset B_R(p)$. Suppose $\|V\|_{L^\infty} < +\infty$.

For appropriate $\sigma$ and $R$, there exists a constant $C'_0$ depending only on $\kappa, \sigma, R$ and the geometry of $N$, such that if

$$\max |\nabla T| + \max |T| \leq C'_0,$$

then there exists a $VT$-harmonic map $u \in C^\infty(M, N)$ homotopic to $u_0$ such that $u(M) \subset B_R(p)$.

## 2 Preliminaries

Let us first give some notations:

\[
s_\kappa(t) := \begin{cases} 
  t & \kappa = 0 \\
  \frac{1}{\sqrt{\kappa}} \sin \sqrt{\kappa} t & \kappa > 0,
\end{cases} \quad q_\kappa(t) := \begin{cases} 
  \frac{t^2}{2} & \kappa = 0 \\
  \frac{1}{\kappa} (1 - \cos \sqrt{\kappa} t) & \kappa > 0.
\end{cases}
\]

\[
a_\kappa(t) := \begin{cases} 
  0 & \text{if } t = 0 \\
  \frac{1 - s'_\kappa(t)}{s_\kappa(t)} & \text{if } t > 0,
\end{cases} \quad b_\kappa(t) := \begin{cases} 
  0 & \text{if } t = 0 \\
  \frac{1 - s'_\kappa(t)}{2s_\kappa(t)} \left( 1 + \frac{t}{s_\kappa(t)} \right) & \text{if } t > 0.
\end{cases}
\]

In local coordinates $\{x^a\}$ on $M$ and $\{y^i\}$ on $N$, respectively, the energy density of $u$ is

\[
e(u) := g^{ab} \tilde{g}_{ij}(u(x)) \frac{\partial u^i}{\partial x^a} \frac{\partial u^j}{\partial x^b}.
\]

Assume the metric of $N$ satisfies:

$$0 < \tilde{\lambda}(y)(\delta_{ij}) \leq (\tilde{g}_{ij}(y)) \leq \tilde{\Lambda}(y)(\delta_{ij}), \quad \forall y \in N.$$

Denote $\lambda := \min_N \tilde{\lambda}$ and $\Lambda := \max_N \tilde{\Lambda}$

$\forall y_1, y_2 \in B_R(p)$, there exists a unit speed geodesic $\gamma : [0, \rho] \to B_R(p) \subset N$ with $\gamma(0) = y_1, \gamma(\rho) = y_2$, where $\rho = \text{dist}(y_1, y_2)$. For any $v_j \in T_{y_j}N$, $j = 1, 2$, let $X$ be the unique Jacobi field along $\gamma$ with $X(0) = v_1, X(\rho) = v_2$. Then, we define a pseudo-distance

\[
\delta(v_1, v_2) := \begin{cases} 
  \left( \rho \int_0^\rho |\dot{X}|^2 \right)^{\frac{1}{2}} & \text{if } \rho > 0, \\
  |v_1 - v_2| & \text{if } \rho = 0.
\end{cases}
\]
Another pseudo-distance is given by
\[ \delta_0(v_1, v_2) := |v_1 - \tilde{v}_2|, \]
where \( \tilde{v}_2 \in T_{y_1}N \) stands for the vector obtained by parallel displacement of \( v_2 \in T_{y_2}N \) along \( \gamma \). Let \( L(T_xM, T_yN) \) be the space of all linear maps from \( T_xM \) to \( T_yN \). The pseudo-distance \( \delta \) on the tangent bundle can be extended to a pseudo-distance on the fibers, that is, for \( q_1, q_2 \in \bigcup_{y \in BR(p)} L(T_xM, T_yN) \) (disjoint union), we define their pseudo-distance as
\[ \delta(q_1, q_2) := \left( \sum_{\alpha=1}^m \delta^2(q_1(e_\alpha), q_2(e_\alpha)) \right)^{\frac{1}{2}}, \]
where \( \{e_1, \ldots, e_m\} \) is an orthonormal base for \( T_xM \).

We have the following relationship between these two pseudo-distances:

**Lemma 1** ([4]) There is a positive constant \( C \) depending only on \( BR(p) \) and the geometry of \( N \) such that for any \( y_j \in BR(p) \) and \( v_j \in T_{y_j}N \), we have
\[ \delta_0^2(v_1, v_2) - C(|v_1|^2 + |v_2|^2)\rho^2 \leq \delta^2(v_1, v_2) \leq \delta_0^2(v_1, v_2) + C(|v_1|^2 + |v_2|^2)\rho^2. \]

**Remark 1** In fact, by the proof in [4] and using a well-known expression of the curvature operator (see, e.g., Lemma 4.3.3 in [12]), it is not hard to see that if the sectional curvature \( K \) on \( BR(p) \) satisfies \( \theta \leq K|_{BR(p)} \leq \kappa \) for a constant \( \theta < 0 \), then the constant \( C \) can be expressed as \( 14(\kappa - \theta) \).

The following estimates will also be important for us:

**Lemma 2** ([7]) Let \( (M, g) \) be a compact Riemannian manifolds with nonempty boundary \( \partial M \) and \( (N, \tilde{g}) \) a complete manifold without boundary and \( BR(p) \) a regular ball in \( N \). Let
\[ g_1 := q_\kappa \circ d(p, \cdot) : BR(p) \to \mathbb{R}, \]
\[ h := q_\frac{\kappa}{4} \circ d : BR(p) \times BR(p) \to \mathbb{R}. \]

Then,
\[ \nabla^2 g_1(u, u) \geq s'_k(\tau)|u|^2 \] (2.1)
hold for \( u \in T_xN, x \in BR(p) \) and \( \tau := d(p, x) \).

\[ \nabla^2 h(v, v) \geq -s'_{\frac{\kappa}{4}}(\rho)a_k(\rho) \sum_{i=1}^2 |v_i|^2 \] (2.2)
and
\[ \nabla^2 h(v, v) \geq s'_\frac{\kappa}{2}(\rho)\delta^2(v_1, v_2) - s'_{\frac{\kappa}{4}}(\rho)b_k(\rho) \sum_{i=1}^2 |v_i|^2 \] (2.3)
holds for \( v = v_1 \oplus v_2, v_j \in T_{y_j}N, y_j \in BR(p), j = 1, 2, \rho = \text{dist}(y_1, y_2). \)
3 The maximum principle

Proof of Theorem 1  Let
\[ \psi(x) := q_{\frac{1}{2}} \circ d(u_1(x), u_2(x)), \]
\[ \psi_i(x) := q_k \circ d(p, u_i(x)), \quad i = 1, 2, \]
\[ \Phi(x) := \frac{1}{2} \sum_{i=1}^{2} \omega \circ \psi_i(x), \quad \text{where} \quad \omega(t) := -\log(q_k((1 + \sigma)R) - t). \]

We consider the operator
\[ L_V(\cdot) := e^{\Phi} \cdot \text{div}(e^{\Phi} \nabla \cdot) + e^{-\Phi} \cdot V(\cdot). \]

By direct computation, we obtain
\[ L_V(\Theta) = L_V(e^{\Phi} \cdot \psi) = \Delta \psi + \psi(\Delta \Phi - |\nabla \Phi|^2) + \psi V(\Phi) + V(\psi) = \Delta_V \psi + \psi(\Delta_V \Phi - |\nabla \Phi|^2). \]

Define \( U, U_1, U_2 : M \to N \times N \) by
\[ U(x) := (u_1(x), u_2(x)), \quad U_i(x) := (p, u_i(x)), \quad i = 1, 2. \]

Let \( v := \frac{\kappa}{T}, h := q_v \circ d, \phi := q_\kappa \circ d, \) then \( \psi = h \circ U, \psi_i = \phi \circ U_i. \)

For any \( x \in M, \) we let \( \tilde{\gamma} \) be the unique geodesic connecting \( u_1(x) \) and \( u_2(x). \) Choosing a parallel orthonormal frame \( \{E_i(t)\} \) along \( \tilde{\gamma} \) with \( E_1 = \tilde{\gamma}', \) and a local orthonormal frame \( \{e_\alpha\}_{\alpha=1}^m \) around \( x, \) assuming that \( \frac{\partial}{\partial y^\alpha} := a^i_\alpha E_j, \) we have
\[ \delta_0(T(du_1(e_\alpha), du_1(e_\alpha)), T(du_2(e_\alpha), du_2(e_\alpha))) \]
\[ = \delta_0 \left( (u_1)_\alpha^i (u_1)_\alpha^j T^r_{ij}(u_1) \frac{\partial}{\partial y^r}(u_1), (u_2)_\alpha^i (u_2)_\alpha^j T^r_{ij}(u_2) \frac{\partial}{\partial y^r}(u_2) \right) \]
\[ = \delta_0 \left( (u_1)_\alpha^i (u_1)_\alpha^j T^r_{ij}(u_1) a^\mu_\alpha (u_1) E_\mu(u_1), (u_2)_\alpha^i (u_2)_\alpha^j T^r_{ij}(u_2) a^\mu_\alpha (u_2) E_\mu(u_2) \right) \]
\[ \leq \sum_\mu \left| T^r_{ij}(u_1) a^\mu_\alpha (u_1)(u_1)_\alpha^i (u_1)_\alpha^j - T^r_{ij}(u_2) a^\mu_\alpha (u_2)(u_2)_\alpha^i (u_2)_\alpha^j \right| \]
\[ \leq \sum_\mu \left| (T^r_{ij}(u_1) - T^r_{ij}(u_2)) a^\mu_\alpha (u_1)(u_1)_\alpha^i (u_1)_\alpha^j \right| \]
\[ + \sum_\mu \left| T^r_{ij}(u_2) (a^\mu_\alpha (u_1) - a^\mu_\alpha (u_2))(u_1)_\alpha^i (u_1)_\alpha^j \right| \]
\[ + \sum_\mu \left| T^r_{ij}(u_2) a^\mu_\alpha (u_2)(u_1)_\alpha^i (u_1)_\alpha^j \left( (u_1)_\alpha^i (u_1)_\alpha^j - (u_2)_\alpha^i (u_2)_\alpha^j \right) \right| \]
\[ + \sum_\mu \left| T^r_{ij}(u_2) a^\mu_\alpha (u_2)(u_2)_\alpha^i (u_2)_\alpha^j \left( (u_1)_\alpha^i (u_1)_\alpha^j - (u_2)_\alpha^i (u_2)_\alpha^j \right) \right|. \]

Denote \( A = (a^i_\alpha), \) then \( AA^T = G := (\tilde{g}_{ik}) \). Since
\[ \sum_{i,k} |a^i_\alpha|^2 = \|A\|^2 = \text{tr}(AA^T) = \text{tr}(G) \leq n\Lambda, \]
\[ \left| (u_1)_\alpha^i (u_1)_\alpha^j \right| \leq \frac{1}{\lambda} \|du_1\|^2, \]

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we then obtain

\[
\delta_0(T(du_1(e_\alpha), du_1(e_\alpha)), T(du_2(e_\alpha), du_2(e_\alpha)))
\leq \sum_{\mu=1}^{n} \left( \frac{\sqrt{n\Lambda}}{\lambda} \rho \max |\nabla T||du_1|^{2} + \tilde{C}_1 \frac{\rho}{\lambda} \max |T||du_1|^{2} \right. \\
+ \frac{\sqrt{n\Lambda}}{\lambda} \max |T|(|du_1| + |du_2||du_1 - du_2|) \\
= \frac{n\sqrt{n\Lambda}}{\lambda} \rho \max |\nabla T||du_1|^{2} + \left( \frac{n\tilde{C}_1}{\lambda} + \sqrt{C} \right) \rho \max |T| \sum_{i=1}^{2} |du_i|^{2} \\
+ \frac{n\sqrt{n\Lambda}}{\lambda} \max |T|(|du_1| + |du_2||du_1 - du_2|),
\]

where \( \tilde{C}_1 > 0 \) is a constant depending only on the bound of \( (da^\mu_\nu) \) on \( B_{\tilde{z}_\nu}(p) \). By Lemma 1, we get

\[
\delta(T(du_1(e_\alpha), du_1(e_\alpha)), T(du_2(e_\alpha), du_2(e_\alpha)))
\leq \delta_0(T(du_1(e_\alpha), du_1(e_\alpha)), T(du_2(e_\alpha), du_2(e_\alpha))) + \sqrt{C} \rho (|T(du_1(e_\alpha), du_1(e_\alpha))| + |T(du_2(e_\alpha), du_2(e_\alpha))|) \\
\leq \frac{n\sqrt{n\Lambda}}{\lambda} \rho \max |\nabla T| \sum_{i=1}^{2} |du_i|^{2} \left( \frac{n\tilde{C}_1}{\lambda} + \sqrt{C} \right) \rho \max |T| \sum_{i=1}^{2} |du_i|^{2} \\
+ \frac{n\sqrt{n\Lambda}}{\lambda} \max |T|(|du_1| + |du_2||du_1 - du_2|),
\]

where \( C = 14(\kappa - \theta) \), and the constant \( \theta \) is a lower bound of the sectional curvature of \( N \) on \( B_{\tilde{z}_\nu}(p) \). The Cauchy inequality implies that

\[
\frac{n\sqrt{n\Lambda}}{\lambda} \max |T||du_i||du_1 - du_2|s_v(\rho) \leq \frac{\varepsilon_1 n^2}{2} |du_1 - du_2|^2 \\
+ \frac{n\Lambda}{2\varepsilon_1 \lambda^2} s_v^2(\rho) \max |T|^2 |du_i|^2.
\]

By using the formula (2.13) in [7], it follows that

\[
\langle (\nabla h) \circ U, -T(dU(e_\alpha), dU(e_\alpha)) \rangle \\
= -s_v(\rho) \langle (\nabla d) \circ U, T(dU(e_\alpha), dU(e_\alpha)) \rangle \\
= -s_v(\rho) \langle \tilde{e}_1(U) \oplus \tilde{e}_2(U), T(du_1(e_\alpha), du_1(e_\alpha)) \oplus T(du_2(e_\alpha), du_2(e_\alpha)) \rangle \\
\geq -s_v(\rho) \delta(T(du_1(e_\alpha), du_1(e_\alpha)), T(du_2(e_\alpha), du_2(e_\alpha))) \\
\geq -s_v(\rho) \left( \frac{n\sqrt{n\Lambda}}{\lambda} \rho \max |\nabla T| \sum_{i=1}^{2} |du_i|^{2} \left( \frac{n\tilde{C}_1}{\lambda} + \sqrt{C} \right) \rho \max |T| \sum_{i=1}^{2} |du_i|^{2} \right) \\
- \varepsilon_1 n^2 |du_1 - du_2|^2 - \frac{n\Lambda}{2\varepsilon_1 \lambda^2} s_v^2(\rho) \max |T|^2 \sum_{i=1}^{2} |du_i|^2,
\]
where \( \tilde{e}_1 = -\tilde{y}'(0), \tilde{e}_2 = \tilde{y}'(\rho). \) Since

\[
\delta_0^2 (du_1(e_\alpha), du_2(e_\alpha)) = \delta_0^2 \left( (u_1)_\alpha^i \frac{\partial}{\partial y^i} (u_1), (u_2)_\alpha^i \frac{\partial}{\partial y^i} (u_2) \right)
\]

\[
= \delta_0^2 \left( (u_1)_\alpha^i a_i^j (u_1) E_j (u_1), (u_2)_\alpha^i a_i^j (u_2) E_j (u_2) \right) = \sum_j \left[ (u_1)_\alpha^i a_i^j (u_1) - (u_2)_\alpha^i a_i^j (u_2) \right]^2
\]

\[
= \sum_j \left[ (u_1)_\alpha^i (a_i^j (u_1) - a_i^j (u_2)) + a_i^j (u_2)((u_1)_\alpha^i - (u_2)_\alpha^i) \right]^2
\]

\[
\geq \frac{1}{2} \sum_j \sum_{ik} \tilde{g}_{ik} ((u_1)_\alpha^i - (u_2)_\alpha^i)^2 - \sum_j \left[ (u_1)_\alpha^i (a_i^j (u_1) - a_i^j (u_2)) \right]^2
\]

\[
= \frac{1}{2} \sum_j \sum_{ik} \tilde{g}_{ik} ((u_1)_\alpha^i - (u_2)_\alpha^i)^2 - \sum_j \left[ (u_1)_\alpha^i (a_i^j (u_1) - a_i^j (u_2)) \right]^2
\]

\[
\geq \frac{n\lambda}{2} |(du_1 - du_2)(e_\alpha)|^2 - n\tilde{C}_1 \rho^2 |du_1(e_\alpha)|^2,
\]

then by Lemma 1, we have

\[
\delta^2 (du_1(e_\alpha), du_2(e_\alpha)) \geq \delta_0^2 (du_1(e_\alpha), du_2(e_\alpha)) - C \left(|du_1(e_\alpha)|^2 + |du_2(e_\alpha)|^2\right) \rho^2
\]

\[
\geq \frac{n\lambda}{2} |(du_1 - du_2)(e_\alpha)|^2 - (n\tilde{C}_1 + C) \rho^2 \sum_{i=1}^2 |du_i(e_\alpha)|^2.
\]

Namely,

\[
\delta^2 (du_1, du_2) \geq \frac{n\lambda}{2} |du_1 - du_2|^2 - (n\tilde{C}_1 + C) \rho^2 \sum_{i=1}^2 |du_i|^2.
\]

(3.1)

Therefore,

\[
\langle (\nabla h) \circ U, -T(du_1(e_\alpha), du_2(e_\alpha)) \rangle
\]

\[
\geq -s_v(\rho) \left\{ \frac{n\sqrt{n\lambda}}{\rho} - \rho \max |\nabla T| \sum_{i=1}^2 |du_i|^2 + \left( \frac{n\tilde{C}_1}{\rho} + \sqrt{C} \right) \max |T| \sum_{i=1}^2 |du_i|^2 \right\}
\]

\[
- \frac{2n\epsilon_1}{\lambda} \left[ \delta^2 (du_1, du_2) + (n\tilde{C}_1 + C) \rho^2 \sum_{i=1}^2 |du_i|^2 \right] - \frac{n\Delta}{2\epsilon_1^2 \lambda^2} s_v(\rho) \rho \max |T| \sum_{i=1}^2 |du_i|^2
\]

\[
= -s_v(\rho) \left\{ \frac{n\sqrt{n\lambda}}{\rho} \max |\nabla T| + \left( \frac{n\tilde{C}_1}{\rho} + \sqrt{C} \right) \max |T| + \frac{n\Delta}{2\epsilon_1^2 \lambda^2} \max |T|^2 \right\}^2 \sum_{i=1}^2 |du_i|^2
\]

\[
- \frac{2n\epsilon_1}{\lambda} \left[ \delta^2 (du_1, du_2) + (n\tilde{C}_1 + C) \rho^2 \sum_{i=1}^2 |du_i|^2 \right].
\]
The above inequality and (2.3) imply that
\[
\Delta V \psi = \Delta V (h \circ U) = \sum_{\alpha} \nabla^2 h(dU(e_{\alpha}), dU(e_{\alpha})) + ((\nabla h) \circ U, \tau_V(U)) - \sum_{\alpha} ((\nabla h) \circ U, -T(dU(e_{\alpha}), dU(e_{\alpha})))
\]
\[
\geq \left( s'_v(\rho) - \frac{2n\varepsilon_1}{\lambda} \right) \delta^2 (du_1, du_2) - s_v(\rho) b_\kappa(\rho) \sum_{i=1}^2 |du_i|^2
\]
\[
- s_v(\rho) \rho \left\{ \frac{n\sqrt{n\Lambda}}{\lambda} \max |\nabla T| + \left( \frac{n\tilde{C}_1}{\lambda} + \sqrt{C} \right) \max |T| \right\}
\]
\[
+ \frac{n\Lambda}{2\varepsilon_1 \lambda^2} \max |T|^2 \} \sum_{i=1}^2 |du_i|^2
\]
\[
- \frac{2n\varepsilon_1}{\lambda} (n\tilde{C}_1 + C) \rho^2 \sum_{i=1}^2 |du_i|^2.
\]
Choosing \( \varepsilon_1 = \frac{\lambda}{2n} \cos(\sqrt{\kappa} R) > 0 \), then we obtain
\[
\Delta V \psi \geq -s_v(\rho) b_\kappa(\rho) \sum_{i=1}^2 |du_i|^2 - \cos(\sqrt{\kappa} R) (n\tilde{C}_1 + C) \rho^2 \sum_{i=1}^2 |du_i|^2
\]
\[
- s_v(\rho) \rho \left\{ \frac{n\sqrt{n\Lambda}}{\lambda} \max |\nabla T| + \left( \frac{n\tilde{C}_1}{\lambda} + \sqrt{C} \right) \max |T| \right\}
\]
\[
+ \frac{n^2 \Lambda}{\lambda^3 \cos(\sqrt{\kappa} R)} \max |T|^2 \} \sum_{i=1}^2 |du_i|^2.
\]
It follows from (2.1) that
\[
\Delta V \psi_i = \Delta V (\phi \circ U_i) = \sum_{\alpha} \nabla^2 \phi(dU_i(e_{\alpha}), dU_i(e_{\alpha})) + ((\nabla \phi) \circ U_i, \tau_V(U_i))
\]
\[
\geq s'_k(\rho_i)|du_i|^2 + s_k(\rho_i)((\nabla d) \circ U_i, 0 \oplus (-T(dU_i(e_{\alpha}), dU_i(e_{\alpha}))))
\]
\[
\geq s'_k(\rho_i)|du_i|^2 - s_k(\rho_i) |T(dU_i(e_{\alpha}), dU_i(e_{\alpha}))|
\]
\[
\geq s'_k(\rho_i)|du_i|^2 - s_k(\rho_i) \max |T||du_i|^2.
\]
It is easy to check that
\[
\omega'' = \omega^2, \quad \omega' \circ \psi_i = \frac{1}{q_k((1 + \sigma) R) - q_k(\rho_i)}.
\]
Therefore,

\[ L_V(\Theta) = \Delta V \psi + \psi(\Delta V \Phi - |\nabla \Phi|^2) \]

\[ = \Delta V \psi + \frac{1}{2} \psi \sum_{i=1}^{2} (\omega' \circ \psi_i) \Delta V \psi_i + \sum_{i=1}^{2} \left[ \frac{1}{2} \psi(\omega'' \circ \psi_i) - \frac{1}{4} \psi(\omega' \circ \psi_i)^2 \right] |\nabla \psi_i|^2 \]

\[ = \Delta V \psi + \frac{1}{2} \psi \sum_{i=1}^{2} (\omega' \circ \psi_i) \Delta V \psi_i + \frac{\psi}{4} \sum_{i=1}^{2} (\omega' \circ \psi_i)^2 |\nabla \psi_i|^2 \]

\[ \geq - s_v(\rho)b_k(\rho) \sum_{i=1}^{2} |du_i|^2 - \cos(\sqrt{\kappa} R) (n\tilde{C}_1 + C) \rho^2 \sum_{i=1}^{2} |du_i|^2 \]

\[ - s_v(\rho) \rho \left\{ \frac{n\sqrt{n\Lambda}}{\lambda} \max |\nabla T| + \left( \frac{n\tilde{C}_1}{\lambda} + \sqrt{C} \right) \max |T| \right\} \]

\[ + \frac{n^2 \Lambda}{\lambda^3 \cos(\sqrt{\kappa} R)} \max |T|^2 \sum_{i=1}^{2} |du_i|^2 \]

\[ + \frac{1}{2} \psi \sum_{i=1}^{2} q_k((1 + \sigma) R) - q_k(\rho_i) \left[ s_k'(\rho_i) |du_i|^2 - s_k(\rho_i) \max |T| |du_i|^2 \right] \]

\[ \geq \psi \sum_{i=1}^{2} \left\{ \frac{s_k'(\rho_i)}{2(q_k((1 + \sigma) R) - q_k(\rho_i))} - \frac{b_k(\rho)_s_v(\rho)}{q_v(\rho)} \right\} \]

\[ - s_v(\rho) \rho \left\{ \frac{n\sqrt{n\Lambda}}{\lambda} \max |\nabla T| + \left( \frac{n\tilde{C}_1}{\lambda} + \sqrt{C} \right) \max |T| \right\} \]

\[ + \frac{n^2 \Lambda}{\lambda^3 \cos(\sqrt{\kappa} R)} \max |T|^2 \sum_{i=1}^{2} |du_i|^2 \]

\[ + \frac{2n\sqrt{n\Lambda}}{\lambda} \max |\nabla T| + \left( \frac{2n\tilde{C}_1}{\lambda} + 2\sqrt{C} \right) \max |T| \]

\[ + \frac{2n^2 \Lambda}{\lambda^3 \cos(\sqrt{\kappa} R)} \max |T|^2 \sum_{i=1}^{2} |du_i|^2 \]

\[ - \frac{\kappa R^2}{\lambda^2 \cos(\sqrt{\kappa} R) - \cos((1 + \sigma) \sqrt{\kappa} R)} \max |T| \}

where we have used the fact that \( \frac{s_k(\rho)}{q_v(\rho)} \) is nonincreasing in \((0, 2R]\) and \( \frac{\rho^2}{q_v(\rho)} \) is increasing in \((0, 2R]\). Direct computation gives us

\[ \frac{b_k(\rho)_s_v(\rho)}{q_v(\rho)} = \frac{\kappa}{4} (1 + \frac{1}{s_k'(\rho)})(1 + \frac{\rho}{s_k(\rho)}) =: \alpha(\rho), \]
and $\alpha(t)$ is increasing in $[0, 2R]$ and $\beta_R(t) = \frac{s'_k(t)}{2(q_k((1+\sigma)R-q_k(t))}$ is increasing in $[0, R]$. Hence, we obtain

$$\frac{s'_k(\rho)}{2(q_k((1+\sigma)R-q_k(\rho)))} - \frac{b_k(\rho)s_v(\rho)}{q_v(\rho)} \geq \frac{1}{2q_k((1+\sigma)R)} - \frac{\kappa}{4} \left(1 + \frac{1}{s'_k(2R)}\right) \left(1 + \frac{2R}{s_k(2R)}\right)$$

$$= \frac{\kappa}{2[1 - \cos((1+\sigma)\sqrt{k}R)]} - \frac{\kappa[1 + \cos(\sqrt{k}R)][2\sqrt{k}R + \sin(2\sqrt{k}R)]}{4 \cos(\sqrt{k}R) \sin(2\sqrt{k}R)}.$$

It follows that

$$\mathcal{L}_V(\Theta) \geq \psi \sum_{i=1}^{2} \left\{ \frac{\kappa}{2[1 - \cos((1+\sigma)\sqrt{k}R)]} - \frac{\kappa[1 + \cos(\sqrt{k}R)][2\sqrt{k}R + \sin(2\sqrt{k}R)]}{4 \cos(\sqrt{k}R) \sin(2\sqrt{k}R)} \right\} \max |T|$$

$$- \frac{2n\sqrt{\Lambda}}{\lambda} \max |\nabla T|$$

$$- \left(\frac{2n\tilde{C}_1}{\lambda} + 2\sqrt{C} + \frac{\sqrt{k} \sin(\sqrt{k}R)}{2 \left[\cos(\sqrt{k}R) - \cos((1+\sigma)\sqrt{k}R)\right]}\right) \max |T|$$

$$- \frac{2n^2\Lambda}{\lambda^3 \cos(\sqrt{k}R)} \max |T|^2 - \frac{(n\tilde{C}_1 + C) \kappa R^2 \cos(\sqrt{k}R)}{1 - \cos(\sqrt{k}R)} |du_i|^2.$$

Clearly, by choosing appropriate $\sigma$ and $R$, we obtain

$$\frac{\kappa}{2[1 - \cos((1+\sigma)\sqrt{k}R)]} - \frac{\kappa[1 + \cos(\sqrt{k}R)][2\sqrt{k}R + \sin(2\sqrt{k}R)]}{4 \cos(\sqrt{k}R) \sin(2\sqrt{k}R)} > 0.$$

Hence, if

$$\frac{2n\sqrt{\Lambda}}{\lambda} \max |\nabla T| + \left(\frac{2n\tilde{C}_1}{\lambda} + 2\sqrt{C} + \frac{\sqrt{k} \sin(\sqrt{k}R)}{2 \left[\cos(\sqrt{k}R) - \cos((1+\sigma)\sqrt{k}R)\right]}\right) \max |T|$$

$$+ \frac{2n^2\Lambda}{\lambda^3 \cos(\sqrt{k}R)} \max |T|^2$$

$$\leq \frac{\kappa}{2[1 - \cos((1+\sigma)\sqrt{k}R)]} - \frac{\kappa[1 + \cos(\sqrt{k}R)][2\sqrt{k}R + \sin(2\sqrt{k}R)]}{4 \cos(\sqrt{k}R) \sin(2\sqrt{k}R)}$$

$$- \frac{(n\tilde{C}_1 + C) \kappa R^2 \cos(\sqrt{k}R)}{1 - \cos(\sqrt{k}R)},$$

then we have

$$\mathcal{L}_V(\Theta) \geq 0.$$

(For $\sqrt{k} R \to 0$, we use the Taylor expansions of $\sin$ and $\cos$ to obtain positive values on the right-hand side of (3.5).) It is easy to see that there exists a constant $C_0$ depending only on $\kappa, \sigma, R$ and the geometry of $N$, so that if

$$\max |\nabla T| + \max |T| \leq C_0,$$  

then

$$\mathcal{L}_V(\Theta) \geq 0.$$
then (3.5) holds true; consequently, $\mathcal{L}_V(\Theta) \geq 0$. Applying the ordinary maximum principle, we have

$$\max_{M} \Theta \leq \max_{\partial M} \Theta.$$  

\[\square\]

**Proof of Theorem 2** We consider a parabolic operator of the form

$$\tilde{L}_V := \mathcal{L}_V - e^{-\Phi} \partial_t.$$  

By using

$$(\Delta_V - \partial_t) \psi = (\Delta_V - \partial_t) (h \circ U) = \sum_{\alpha} \nabla^2 h(dU(e_\alpha), dU(e_\alpha))$$

and

$$(\Delta_V - \partial_t) \psi_i = (\Delta_V - \partial_t) (\phi \circ U_i) = \sum_{\alpha} \nabla^2 \phi(dU_i(e_\alpha), dU_i(e_\alpha))$$

as in the proof of Theorem 1, we can conclude that $\tilde{L}_V(\Theta) \geq 0$ on $M_T$. From the parabolic maximum principle, we have

$$\max_{M_T} \Theta \leq \max_{\partial_p M_T} \Theta.$$  

**4 Existence results**

Using the maximum principle obtained in the last section, we shall prove the existence of solutions of the Dirichlet problem for $VT$–harmonic maps.

**Proof of Theorem 3** Let us choose normal coordinates $\{y^i\}_{i=1,2,...,n}$ centered at $p$, then any $VT$–harmonic map $u : M \to B_R(p) \subset N$ can be written as

$$u = (u^1, \ldots, u^n) \in (H^{2,q}(M))^n$$

which satisfies the elliptic system

$$\Delta_M u^i + \Gamma_{jk}^i(u) \frac{\partial u^i}{\partial x^\alpha} \frac{\partial u^k}{\partial x^\beta} g^{\alpha\beta} + V^\alpha \frac{\partial u^i}{\partial x^\alpha} + T_{jk}^i(u) \frac{\partial u^j}{\partial x^\alpha} \frac{\partial u^k}{\partial x^\beta} g^{\alpha\beta} = 0, \ i = 1, 2, \ldots, n.$$  

For simplicity of notation, we write it in a concise form

$$\Delta u + \Gamma(du, du) + du(V) + Tr_g T(du, du) = 0.$$  

Now we consider the initial boundary value problem for the heat flow of $VT$–harmonic maps

$$\left\{
\begin{array}{l}
\partial_t u = \Delta u + \Gamma(du, du) + du(V) + Tr_g T(du, du), \\
u - u_0 \in H^{2,q}_0(M, N), \ u(0) = u_0, \\
u(M \times [0, +\infty)) \subset B_R(p).
\end{array}\right.$$  

\[\square\]
As in the proof of Theorem 3 in [5], by a continuity method that rests on the maximum principle Theorem 2, we can conclude the global existence of a solution \( u(x, t) \) of the above flow (4.1). This solution satisfies
\[
\|u(\cdot, t)\|_{1+\alpha} \leq C(q, M, V, T, N, u_0, R), \quad \forall t \in (0, +\infty)
\]
for some \( \alpha > 0 \). Consequently, by the parabolic regularity theory, we have the uniform estimate
\[
\|u\|_{C^{1+\alpha, 2+\alpha}(M)} \leq C.
\] (4.2)

For \( u_1(x, t) = u(x, t), u_2(x, t) = u(x, t + \sigma_1), \sigma_1 > 0, \forall (x, t) \in M \times (0, +\infty) \), as in the proof of Theorem 2, the function \( \Theta_1 \) satisfies
\[
\begin{cases}
(\Delta - \partial_t)(\frac{\Theta_1}{\sigma_1}) + (V - 2\nabla \Phi, \nabla(\frac{\Theta_1}{\sigma_1})) \geq 0, \\
\Theta_1|_{\partial M} = 0.
\end{cases}
\]

By the ordinary maximum principle for functions, it follows that (see pp.178–179 in [17])
\[
\left(\frac{\Theta_1}{\sigma_1}\right) \leq C(t - t_0)^{-k}, \quad \forall t \geq t_0
\]
for any positive integer \( k \) and some \( t_0 > 0 \). Letting \( \sigma_1 \to 0 \), then we obtain \( |u_t| \to 0 \) as \( t \to +\infty \), from which together with (4.2), we have \( u \) subconverges to a \( VT \)-harmonic map \( u_\infty \) satisfying (1.7) and \( u_\infty(M) \subset B_R(p) \).

With the Schauder and higher regularity estimates, we can improve Theorem 3 to the following

**Theorem 5** Let \( M, N, V, T, B_R(p) \) be as in Theorem 1. Suppose \( u_0 \in C^0(M, N) \) with \( u_0(M) \subset B_R(p) \). For appropriate \( \sigma \) and \( R \), there exists a constant \( C_0 \) depending only on \( \kappa, \sigma, R \) and the geometry of \( N \), such that if
\[
\max |\nabla T| + \max |T| \leq C_0,
\]
then the Dirichlet problem
\[
\begin{cases}
\tau(u) + du(V) + \text{Tr}_g T(du, du) = 0, \\
u|_{\partial M} = u_0|_{\partial M}
\end{cases}
\]
admits a unique solution \( u \in C^\infty(M, N) \cap C^0(\overline{M}, N) \) such that \( u(M) \subset B_R(p) \).

## 5 Applications

### 5.1 Weyl harmonic maps (c.f. [14])

Let \( (M, [g], ^W \nabla) \) be a Weyl manifold. According to the definition, there exists a 1-form \( \Theta \) such that \( ^W g = \Theta \otimes g \) for any \( g \in [g] \). Equivalently, \( ^W \nabla \) is defined by
\[
^W \nabla_X Y = \nabla_X Y - \frac{1}{2}\Theta(X)Y - \frac{1}{2}\Theta(Y)X + \frac{1}{2}g(X, Y)\Theta^2, \quad \forall X, Y \in \Gamma(TM),
\]
where $\nabla$ is the Levi-Civita connection and $\Theta^\sigma$ the vector field dual to $\Theta$ w.r.t. $g$. Let $\Gamma^\gamma_{\alpha\beta}$, $W\Gamma^\gamma_{\alpha\beta}$ be the Christoffel symbols corresponding to $\nabla$ and $W\nabla$, respectively.

Let $(N, [g], W\tilde{\nabla})$ be also a Weyl manifold, and correspondingly, we denote by $\tilde{\Theta}$ the $1$-form, and $\Gamma^\gamma_{ij}$, $W\Gamma^\gamma_{ij}$ are the Christoffel symbols for the Levi-Civita connection $\tilde{\nabla}$ and Weyl connection $W\nabla$, respectively. Let $u : (M, [g], W\nabla) \to (N, [\tilde{g}], W\tilde{\nabla})$ be the usual smooth map.

Let

$$V := (\Gamma^\gamma_{\alpha\beta} - W\Gamma^\gamma_{\alpha\beta})g^{ab} \frac{\partial}{\partial x^\gamma},$$

$$T(\tilde{X}, \tilde{Y}) := -\frac{1}{2}\tilde{\Theta}(\tilde{X})\tilde{Y} - \frac{1}{2}\tilde{\Theta}(\tilde{Y})\tilde{X} + \frac{1}{2}g(\tilde{X}, \tilde{Y})\tilde{\Theta}^\nu, \quad \forall \tilde{X}, \tilde{Y} \in \Gamma(TN).$$

Then, we have

$$\tau(g, W\nabla, W\tilde{\nabla}) = g^{\alpha\beta}(u^k_{\alpha\beta} + W\tilde{\Gamma}^k_{ij}u^i_\alpha u^j_\beta - W\Gamma^\gamma_{\alpha\beta}u^\gamma_\sigma)\partial_jk$$

$$= g^{\alpha\beta}(u^k_{\alpha\beta} + \tilde{\Gamma}^k_{ij}u^i_\alpha u^j_\beta + T^k_{ij}u^i_\alpha u^j_\beta - \Gamma^\gamma_{\alpha\beta}u^\gamma_\sigma) + V^\alpha u^\beta_\sigma \partial_jk$$

$$= \tau(g, \nabla, \tilde{\nabla}) + du(V) + Tr_g(du, du).$$

Hence,

$$\tau(g, W\nabla, W\tilde{\nabla}) = 0 \iff \tau(u) + du(V) + Tr_g(du, du) = 0.$$

**Corollary 1** Let $(M, [g], W\nabla)$ be a compact Weyl manifold with nonempty boundary $\partial M$ and $(N, [\tilde{g}], W\tilde{\nabla})$ a complete Weyl manifold with sectional curvature bounded from above by $\kappa \geq 0$. Let $u_0 : M \to N$ be a continuous map with $u_0(M) \subset B_R(p)$, a geodesic ball with radius $R < \frac{\pi}{2(1+\kappa)^{\frac{1}{2}}\sqrt{\kappa}}$. For appropriate $\sigma$ and $R$, there exists a constant $C_0$ depending only on $\kappa$, $\sigma$, $R$ and the geometry of $N$, such that if

$$\max|\nabla\tilde{\Theta}| + \max|\tilde{\Theta}| \leq C_0,$$

then there exists a unique Weyl harmonic map $u : M \to B_R(p) \subset N$ with $u = u_0$ on $\partial M$.

### 5.2 Affine harmonic maps (c.f. [9,10])

Let $(M, g, \tilde{\nabla})$, $(N, h, \nabla')$ both be affine manifolds, where $\tilde{\nabla}$ is a global flat and torsion-free connection on $M$ and $\nabla'$ is a torsion-free connection on $N$. Then, we have

$$\tau(g, \tilde{\nabla}, \nabla') = g^{\alpha\beta}(u^k_{\alpha\beta} + \tilde{\Gamma}^k_{ij}u^i_\alpha u^j_\beta)\partial_jk,$$

where $\tilde{\Gamma}^k_{ij}$ are the Christoffel symbols of $\nabla'$.

Regarding $(M, g)$ and $(N, h)$ as Riemannian manifolds, let $\Gamma^\gamma_{\alpha\beta}$ and $\Gamma^i_{jk}$ be the Christoffel symbols of the Levi-Civita connections $\nabla$ and $\nabla'$ of $(M, g)$ and $(N, h)$, respectively. We then have the usual tension field

$$\tau(g, \nabla, \nabla') = g^{\alpha\beta}(u^k_{\alpha\beta} - \Gamma^\gamma_{\alpha\beta}u^\gamma_\sigma + \Gamma^i_{jk}u^i_\alpha u^j_\beta)\partial_jk.$$

Let

$$V := g^{\alpha\beta}\Gamma^\gamma_{\alpha\beta}\partial_\gamma,$$

$$T^k_{ij} := \tilde{\Gamma}^k_{ij} - \Gamma^k_{ij}.$$
Then, we have
\[ \tau(g, \tilde{\nabla}, \tilde{\nabla}') = \tau(g, \nabla, \nabla') + du(V) + \text{Tr}_g(du, du). \]

Therefore,
\[ \tau(g, \tilde{\nabla}, \tilde{\nabla}') = 0 \quad \text{iff} \quad \tau(u) + du(V) + \text{Tr}_g T(du, du) = 0. \]

**Corollary 2** Let \((M, g, \tilde{\nabla})\) be a compact affine manifold with nonempty boundary \(\partial M\) and \((N, h, \tilde{\nabla}')\) a complete affine manifold with sectional curvature bounded from above by \(\kappa \geq 0\), where \(\tilde{\nabla}\) is a global flat and torsion-free connection on \(M\) and \(\tilde{\nabla}'\) is a torsion-free connection on \(N\). Let \(u_0 : M \to N\) be a continuous map with \(u_0(M) \subset B_R(p)\), a geodesic ball with radius \(R < \frac{\pi}{2(1+\sigma)\sqrt{\kappa}}\). Denote \(T^k_{ij} := \tilde{\Gamma}^k_{ij} - \Gamma^k_{ij}\), where \(\tilde{\Gamma}^k_{ij}\) and \(\Gamma^k_{ij}\) stand for the Christoffel symbols of \(\tilde{\nabla}'\) and \(\nabla'\), respectively. For appropriate \(\sigma\) and \(R\), there exists a constant \(C_0\) depending only on \(\kappa, \sigma, R\) and the geometry of \(N\), such that if
\[ \max |\nabla T| + \max |T| \leq C_0, \]
then there exists a unique affine harmonic map \(u : M \to B_R(p) \subset N\) with \(u = u_0\) on \(\partial M\).

### 5.3 Hermitian harmonic maps (c.f. [11,15])

Let \((M^m, g, \tilde{\nabla})\), \((N^n, h, \tilde{\nabla}')\) are both Hermitian manifolds, where \(\tilde{\nabla}\) and \(\tilde{\nabla}'\) are holomorphic torsion-free connections on \(M\) and \(N\), respectively. Direct calculation gives us
\[ \tau(g, \tilde{\nabla}, \tilde{\nabla}') = g^{\alpha\bar{\beta}} \left( \frac{\partial^2 u^i}{\partial z^\alpha \partial \bar{z}^\beta} + \Gamma^i_{jk} \frac{\partial u^j}{\partial z^\alpha} \frac{\partial u^k}{\partial \bar{z}^\beta} \right) \frac{\partial}{\partial w^i} + g^{\alpha\bar{\beta}} \left( \frac{\partial^2 u^i}{\partial \bar{z}^\alpha \partial \bar{z}^\beta} + \Gamma^i_{jk} \frac{\partial u^j}{\partial \bar{z}^\alpha} \frac{\partial u^k}{\partial \bar{z}^\beta} \right) \frac{\partial}{\partial \bar{w}^i}, \]
where \(\Gamma^i_{jk}\) are the Christoffel symbols of \(\tilde{\nabla}'\).

Let \(J\) be the almost complex structure, and \(\{e_A\} = \{e_1, \ldots, e_m, Je_1, \ldots, Je_m\}\) a local basis of \(M\). Let \(\nabla, \nabla'\) be the Levi-Civita connections on \(M\) and \(N\), respectively, and \(\Gamma^i_{jk}\) the Christoffel symbols of \(\nabla'\). Set
\[ V := \tilde{\nabla}e_A e_A - \nabla e_A e_A \quad \text{and} \quad T^i_{jk} := \Gamma^i_{jk} - \Gamma^i_{jk}, \]
then we have
\[ \tau(g, \tilde{\nabla}, \tilde{\nabla}') = \tau(u) + du(V) + \text{Tr}_g T(du, du). \]

Namely,
\[ \tau(g, \tilde{\nabla}, \tilde{\nabla}') = 0 \quad \text{iff} \quad \tau(u) + du(V) + \text{Tr}_g T(du, du) = 0. \]

**Corollary 3** Let \((M^m, g, \tilde{\nabla})\) be a compact Hermitian manifold with nonempty boundary \(\partial M\) and \((N^n, h, \tilde{\nabla}')\) a complete Hermitian manifold with sectional curvature bounded from above by \(\kappa \geq 0\), where \(\tilde{\nabla}\) and \(\tilde{\nabla}'\) are holomorphic torsion-free connections on \(M\) and \(N\), respectively. Let \(u_0 : M \to N\) be a continuous map with \(u_0(M) \subset B_R(p)\), a geodesic ball with radius \(R < \frac{\pi}{2(1+\sigma)\sqrt{\kappa}}\). Denote \(T^k_{ij} := \Gamma^k_{ij} - \Gamma^k_{ij}\), where \(\Gamma^k_{ij}\) and \(\Gamma^k_{ij}\) stand for the Christoffel symbols of \(\tilde{\nabla}'\) and \(\nabla'\), respectively. For appropriate \(\sigma\) and \(R\), there exists a constant \(C_0\) depending only on \(\kappa, \sigma, R\) and the geometry of \(N\), such that if
\[ \max |\nabla T| + \max |T| \leq C_0, \]
then there exists a unique Hermitian harmonic map \(u : M \to B_R(p) \subset N\) with \(u = u_0\) on \(\partial M\).
5.4 Magnetic harmonic maps

We now consider a case that, in contrast to the previous ones, does not arise from a structure different from the Riemannian, but from an additional structure on a Riemannian manifold.

Let \((\Sigma^m, g)\) be an \(m\)-dimensional compact oriented Riemannian manifold with nonempty boundary, \((N, \tilde{g})\) a Riemannian manifold of dimension \(n\).

Let \(u : (\Sigma^m, g) \to (N, \tilde{g})\) be a map and \(Z = \Gamma(\text{Hom}(\Lambda^mTN, TN)) \cong \Gamma(\Lambda^mT^*N \otimes TN)\).

Consider the following system:

\[
\tau(u) + Z(du(e_1) \wedge \cdots \wedge du(e_m)) = 0, \tag{5.1}
\]

where \(\{e_1, \ldots, e_m\}\) is a positively oriented local orthonormal frame of \(\Sigma^m\). In string theory, it can be interpreted as the motion equation of an \((m-1)\)-brane under an extrinsic magnetic force (c.f. [13]). In [13], the author obtained the global existence of the heat flow in one-dimensional case.

Using a similar method as above, in the two-dimensional case, we can obtain the following

Theorem 6 Let \(u_1, u_2 \in C^0(\Sigma^2, N)\) be two magnetic harmonic maps into a geodesic ball \(B_R(p)\). For appropriate \(\sigma\) and \(R\), there exists a constant \(C_0\) depending only on \(\kappa, \sigma, R\) and the geometry of \(N\), such that if

\[
\max |\nabla Z| + \max |Z| \leq C_0,
\]

then the function \(\Theta : \Sigma^2 \to \mathbb{R}\) defined by

\[
\Theta := \frac{q_{\frac{\kappa}{2}}(\rho)}{(q_{\kappa}((1 + \sigma)R) - q_{\kappa}(\rho_1))^\frac{1}{2}} \cdot (q_{\kappa}((1 + \sigma)R) - q_{\kappa}(\rho_2))^\frac{1}{2} \tag{5.2}
\]

satisfies the maximum principle, namely

\[
\max_{\Sigma^2} \Theta \leq \max_{\partial \Sigma^2} \Theta.
\]

Here \(\rho := d(u_1, u_2), \rho_i := d(p, u_i), i = 1, 2\).

In particular, if \(u_1 = u_2\) on the boundary \(\partial \Sigma^2\), then \(u_1 \equiv u_2\) on \(\Sigma^2\).

For the heat flow of magnetic harmonic maps, an analogous result holds. For \(T > 0\), we set

\[
\Sigma_T^2 := \Sigma^2 \times [0, T]
\]

and denote the parabolic boundary of \(\Sigma_T^2\) by

\[
\partial_p \Sigma_T^2 := (\Sigma^2 \times \{0\}) \cup (\partial \Sigma^2 \times [0, T]).
\]

For the heat flow of magnetic harmonic maps

\[
\partial_t u = \tau(u) + Z(du(e_1) \wedge du(e_2)), \tag{5.3}
\]

we have

Theorem 7 Let \(u_1, u_2 \in C^0(\Sigma^2, N)\) be two solutions of heat flow Eq. (5.3) for magnetic harmonic maps into a geodesic ball \(B_R(p)\). For appropriate \(\sigma\) and \(R\), there exists a constant \(C_0\) depending only on \(\kappa, \sigma, R\) and the geometry of \(N\), such that if

\[
\max |\nabla Z| + \max |Z| \leq C_0,
\]
then the function $\Theta : \Sigma^2_T \to \mathbb{R}$ defined by (5.2) with $\Sigma^2$ replaced by $\Sigma^2_T$ satisfies the maximum principle:

$$\max_{\Sigma^2_T} \Theta \leq \max_{\partial_p \Sigma^2_T} \Theta.$$  

In particular, if $u_1 = u_2$ on the boundary $\partial_p \Sigma^2_T$, then $u_1 \equiv u_2$ on $\Sigma^2_T$.

As an application of the above maximum principle, we obtain the existence of magnetic harmonic maps into a geodesic ball.

**Theorem 8** Let $\Sigma^2, N, Z, B_R(p)$ be as in Theorem 7. Suppose $u_0 \in H^{2,q}(\Sigma^2, N)(q > 2)$ with $u_0(\Sigma^2) \subset B_R(p)$. For appropriate $\sigma$ and $R$, there exists a constant $C_0$ depending only on $\kappa, \sigma, R$ and the geometry of $N$, such that if

$$\max |\nabla Z| + \max |Z| \leq C_0,$$

then the initial boundary value problem

$$\begin{cases}
\partial_t u = \tau(u) + Z(du(e_1) \wedge du(e_2)), \\
u - u_0 \in H^{2,q}_0(\Sigma^2, N), \quad u(0) = u_0, \quad u(\Sigma^2 \times [0, \infty)) \subset B_R(p),
\end{cases}$$

admits a unique global solution $u$ which subconverges to a unique solution $u \in H^{2,q}(\Sigma^2, N)$ of the Dirichlet problem

$$\begin{cases}
\tau(u) + Z(du(e_1) \wedge du(e_2)) = 0, \\
u - u_0 \in H^{2,q}_0(\Sigma^2, N),
\end{cases}$$

such that $u(\Sigma^2) \subset B_R(p)$.

**Remark 2** In local coordinates $\{x^\alpha\}$ on $M$ and $\{y^j\}$ on $N$, respectively, the term $\text{Tr}_g T(du, du)$ can be written as $T^i_{jk}(u) \frac{\partial u^j}{\partial x^\alpha} \frac{\partial u^k}{\partial x^\beta} g^{\alpha\beta}$, where $T := T^i_{jk} \frac{\partial}{\partial y^i} \otimes dy^j \otimes dy^k$. Correspondingly, the term $Z(du(e_1) \wedge du(e_2))$ in (5.3) could be written as $(Z^i_{jk} - Z^i_{kj}) \frac{\partial u^j}{\partial x^\alpha} \frac{\partial u^k}{\partial x^\beta} g^{\alpha\beta}$. Using $Z^i_{jk} - Z^i_{kj}$ in place of $T^i_{jk}$ in the proof of the results for $VT$-harmonic maps, we can conclude the above theorems for magnetic harmonic maps.

### 6 VT-harmonic maps from complete manifolds into geodesic balls

In this section, we shall establish the existence of $VT$-harmonic maps from complete non-compact manifolds into geodesic balls in complete Riemannian manifolds with sectional curvature bounded above by a positive constant.

Before proving the existence theorem, we first give the following Bochner formula:

**Lemma 3** Let $(M^m, g)$ and $(N^n, \tilde{g})$ be Riemannian manifolds. Let $\text{Ric}_V := \text{Ric} - \frac{1}{2} L_V g$, where $\text{Ric}$ is the Ricci curvature of $M$ and $L_V$ is the Lie derivative. Suppose $u$ is a $VT$-harmonic map from $M$ to $N$, then
\[
\frac{1}{2} \Delta V e(u) = |\nabla du|^2 + \sum_{\alpha=1}^{m} \langle du(\text{Ric}_V(e_\alpha)), du(e_\alpha) \rangle
\]
\[
- \sum_{\alpha, \beta=1}^{m} R^N(du(e_\alpha), du(e_\beta), du(e_\alpha), du(e_\beta))
\]
\[
- \sum_{\alpha, \beta=1}^{m} ((\nabla_{e_\alpha} T)(du(e_\beta), du(e_\beta)), du(e_\alpha))
\]
\[
- \sum_{\alpha, \beta=1}^{m} (2T((\nabla_{e_\alpha} du)(e_\beta), du(e_\beta)), du(e_\alpha)),
\]
where \{e_\alpha\} is a local orthonormal frame of \(M\).

**Proof** By the proof of Proposition 1.3.5 in [18] (c.f. also in [1]), we have
\[
\frac{1}{2} \Delta e(u) = |\nabla du|^2 + \sum_{\alpha=1}^{m} \langle \nabla_{e_\alpha} \tau(u), du(e_\alpha) \rangle + \sum_{\alpha=1}^{m} \langle du(\text{Ric}(e_\alpha)), du(e_\alpha) \rangle
\]
\[
- \sum_{\alpha, \beta=1}^{m} R^N(du(e_\alpha), du(e_\beta), du(e_\alpha), du(e_\beta)).
\]

Let \{e_\alpha\} be a local orthonormal normal frame of \(M\) at the considered point. Since
\[
\sum_{\alpha=1}^{m} \langle \nabla_{e_\alpha} du(V), du(e_\alpha) \rangle = \sum_{\alpha=1}^{m} \langle (\nabla_{e_\alpha} du)(V) + du(\nabla_{e_\alpha} V), du(e_\alpha) \rangle
\]
\[
= \sum_{\alpha=1}^{m} \langle \nabla_V du(e_\alpha) + du(\nabla_{e_\alpha} V), du(e_\alpha) \rangle = \sum_{\alpha=1}^{m} \langle \nabla_V du(e_\alpha) + du(\nabla_{e_\alpha} V), du(e_\alpha) \rangle
\]
\[
= \frac{1}{2} |\nabla du|^2 + \sum_{\alpha, \beta=1}^{m} \langle \nabla_{e_\alpha} V, e_\beta \rangle \langle du(e_\beta), du(e_\alpha) \rangle
\]
\[
= \frac{1}{2} |\nabla du|^2 + \frac{1}{2} \sum_{\alpha, \beta=1}^{m} (L_V g)(e_\alpha, e_\beta) \langle du(e_\alpha), du(e_\beta) \rangle
\]
and
\[
\nabla_{e_\alpha} (\text{Tr}_V T(du, du)) = \nabla_{e_\alpha} (T(du(e_\beta), du(e_\beta)))
\]
\[
= (\nabla_{e_\alpha} T)(du(e_\beta), du(e_\beta)) + 2T((\nabla_{e_\alpha} du)(e_\beta), du(e_\beta)).
\]

Therefore, we get
\[
\frac{1}{2} \Delta e(u) = |\nabla du|^2 - \frac{1}{2} |\nabla du|^2 - \frac{1}{2} \sum_{\alpha, \beta=1}^{m} (L_V g)(e_\alpha, e_\beta) \langle du(e_\alpha), du(e_\beta) \rangle
\]
\[
- \sum_{\alpha, \beta=1}^{m} ((\nabla_{e_\alpha} T)(du(e_\beta), du(e_\beta)) + 2T((\nabla_{e_\alpha} du)(e_\beta), du(e_\beta)), du(e_\alpha))
\]
\[
+ \sum_{\alpha=1}^{m} \langle du(\text{Ric}(e_\alpha)), du(e_\alpha) \rangle - \sum_{\alpha, \beta=1}^{m} R^N(du(e_\alpha), du(e_\beta), du(e_\alpha), du(e_\beta)),
\]
which implies that (6.1) holds. □

Using the above Bochner formula and the estimate of $\Delta_V r$ in [6] (here $r$ denotes the distance function on $M$), we establish the gradient estimate for $V T$-harmonic maps.

**Theorem 9** Let $(M^m, g)$ be a complete noncompact Riemannian manifold with

$$\text{Ric}_V := \text{Ric} - \frac{1}{2} L_V g \geq -A,$$

where $A \geq 0$ is a constant, $\text{Ric}$ is the Ricci curvature of $M$ and $L_V$ is the Lie derivative. Let $(N^n, \tilde{g})$ be a complete Riemannian manifold with sectional curvature bounded from above by a positive constant $\kappa$. Let $u : M \rightarrow N$ be a $V T$-harmonic map such that $u(M) \subset B_{\tilde{R}}(p)$, where $B_{\tilde{R}}(p)$ is a regular ball in $N$, i.e., disjoint from the cut locus of $p$ and $\tilde{R} < \frac{\pi}{\sqrt{\kappa}}$. Suppose $\|V\|_{L^\infty} < +\infty$, $\|T\|_{L^\infty} < +\infty$, $\|\nabla T\|_{L^\infty} < +\infty$ and

$$\left(1 + (m + 1)^2 - \frac{1}{(m + 1)^2}\right) \|T\|_{L^\infty}^2 + \frac{\sqrt{\kappa}}{\cos(\sqrt{\kappa} \tilde{R})} \|T\|_{L^\infty} < \frac{\kappa}{\min\{m, n\}}. \quad (6.2)$$

Then, we have

$$|\nabla u| \leq C_6 (\sqrt{A} + 1),$$

where $C_6 > 0$ is a constant depending only on $m, n, \kappa, \tilde{R}, V, T$.

**Proof** Let $r, \rho$ be the respective distance functions on $M$ and $N$ from some fixed points $\tilde{p} \in M$, $p \in N$. Let $B_a(\tilde{p})$ be a geodesic ball of radius $a$ around $\tilde{p}$. Define $\varphi := \cos(\sqrt{\kappa} \rho)$. Then, the Hessian comparison theorem implies

$$\text{Hess}^N(\varphi) \leq -\kappa (\cos(\sqrt{\kappa} \rho)) \tilde{g}. \quad (6.3)$$

Define $f : B_a(\tilde{p}) \rightarrow \mathbb{R}$ by

$$f := (a^2 - r^2) \frac{|\nabla u|_\varphi}{\varphi \circ u}.$$

Denote $\psi := \frac{|\nabla u|_\varphi}{\varphi \circ u}$. Clearly, $f$ achieves its maximum at some interior point of $B_a(\tilde{p})$, say $q$. WLOG, we assume that $\nabla u(q) \neq 0$. Then, from

$$\nabla f(q) = 0, \quad \Delta_V f(q) \leq 0,$$

we obtain at $q$:

$$\frac{\nabla r^2}{a^2 - r^2} = \frac{\nabla \psi}{\psi}, \quad \frac{\Delta_V \psi}{\psi} \geq \frac{\Delta_V r^2}{a^2 - r^2} - \frac{2(\nabla r^2, \nabla \psi)}{(a^2 - r^2)^2} \leq 0. \quad (6.4)$$

It follows from the above two inequalities that

$$\frac{\Delta_V \psi}{\psi} \geq \frac{\Delta_V r^2}{a^2 - r^2} - \frac{2|\nabla r|^2}{(a^2 - r^2)^2} \leq 0. \quad (6.5)$$

By formula (2.4) in [6] (see also [16]), we have

$$\Delta_V r^2 = 2r \Delta_V r + 2|\nabla r|^2 \leq 2r(A(r - r_0) + \tilde{C}_0) + 2, \quad (6.6)$$

where $C_6 > 0$ is a constant depending only on $m, n, \kappa, \tilde{R}, V, T$. □
where $r_0 > 0$ is a sufficiently small constant and \( \tilde{C}_0 := \max_{\partial B_{r_0}(\tilde{p})} |\nabla V| r \).

Let \( \{e_\alpha\} \) be a local orthonormal frame field of \( M \) and \( s \) the rank of \( u \) at the point. We shall compute in normal coordinates at the considered point of \( N \). By Newton’s inequality, we get

\[
\sum_{\alpha, \beta} R^N(du(e_\alpha), du(e_\beta), du(e_\alpha), du(e_\beta)) = \sum_{\alpha \neq \beta} R^N(du(e_\alpha), du(e_\beta), du(e_\alpha), du(e_\beta))
\]

\[
\leq 2\kappa \sum_{1 \leq \alpha < \beta \leq s} \left( \sum_i (u^i_\alpha)^2 \right) \left( \sum_j (u^j_\beta)^2 \right) \leq 2\kappa \cdot \frac{s(s-1)}{2} \cdot \frac{1}{s^2} \left( \sum_{\alpha, i} (u^i_\alpha)^2 \right)^2
\]

\[
= \frac{s-1}{s} \kappa |du|^4 \leq \frac{s_0 - 1}{s_0} \kappa |du|^4,
\]

where we have used the fact that \( s_0 := \min\{m, n\} \geq s \) in the third " \leq ".

The Cauchy–Schwarz inequality gives us

\[
|\langle \nabla_v T(du(e_\beta), du(e_\beta), du(e_\alpha)) \rangle| \leq \|\nabla T\|_{L^\infty} |du|^2 \cdot |du|
\]

\[
\leq \epsilon_2 |e(u)| + \frac{1}{2\epsilon_2} \|\nabla T\|_{L^\infty}^2 |e(u)|^2,
\]

\[
|\langle 2T((\nabla_v du)(e_\beta), du(e_\beta), du(e_\alpha)) \rangle| \leq 2\|T\|_{L^\infty} |\nabla du| |du| \cdot |du| \leq \epsilon_3 |\nabla du|^2
\]

\[
+ \frac{1}{\epsilon_3} \|T\|_{L^\infty}^2 |e(u)|^2.
\]

The formula (3.12) in [2] (see also [3]) implies that for any \( \epsilon > 0 \)

\[
|\nabla du|^2 \geq \frac{1-\epsilon}{m-1} |\tau(u)|^2 + \left( \frac{m}{m-1} - \frac{1}{(m-1)\epsilon} \right) |\nabla \sqrt{e(u)}|^2.
\]

Choosing \( \epsilon = m \), then we have

\[
|\nabla du|^2 \geq -|\tau(u)|^2 + \left( 1 + \frac{1}{m} \right) |\nabla \sqrt{e(u)}|^2.
\]

By the \( VT \)-harmonic map equation (1.1), it is easy to see that

\[
|\tau(u)|^2 \leq e(u) \|V\|_{L^\infty}^2 + e(u)^2 \|T\|_{L^\infty}^2.
\]

Hence, from the Bochner formula (6.1), we obtain

\[
\frac{1}{2} \Delta_v e(u) \geq \left( -\epsilon_3 \|V\|_{L^\infty}^2 - A - \epsilon_2 \right) e(u) + \left( 1 - \epsilon_3 \right) \left( 1 + \frac{1}{m} \right) |\nabla \sqrt{e(u)}|^2
\]

\[
+ \left( (1 - \epsilon_3) \|T\|_{L^\infty}^2 - \kappa \left( 1 - \frac{1}{s_0} \right) - \frac{1}{4\epsilon_2} \|\nabla T\|_{L^\infty}^2 - \frac{1}{\epsilon_3} \|T\|_{L^\infty}^2 \right) e(u)^2.
\]

Since

\[
\frac{1}{2} \Delta_v e(u) = \frac{1}{2} \Delta_v |\nabla u|^2 = |\nabla |\nabla u||^2 + |\nabla u| \Delta_v |\nabla u|,
\]

therefore

\[
\Delta_v |\nabla u| \geq \left[ (1 - \epsilon_3) \left( 1 + \frac{1}{m} \right) - 1 \right] \frac{|\nabla |\nabla u||^2}{|\nabla u|} + \left( (1 - \epsilon_3) \|V\|_{L^\infty}^2 - A - \epsilon_2 \right) |\nabla u|
\]

\[
+ \left[ (1 - \epsilon_3) \|T\|_{L^\infty}^2 - \kappa \left( 1 - \frac{1}{s_0} \right) - \frac{1}{4\epsilon_2} \|\nabla T\|_{L^\infty}^2 - \frac{1}{\epsilon_3} \|T\|_{L^\infty}^2 \right] |\nabla u|^3.
\]
Choose $\varepsilon_3 = \frac{1}{(m+1)^2}$, then

$$
\Delta V |\nabla u| \geq \frac{1}{m+1} |\nabla \nabla u|^2 + \left[ -\left( 1 - \frac{1}{(m+1)^2} \right) \|V\|^2_{L^\infty} - A - \varepsilon_2 \right] |\nabla u| \\
+ \left[ -\left( 1 - \frac{1}{(m+1)^2} \right) \|T\|^2_{L^\infty} - \kappa \left( 1 - \frac{1}{s_0} \right) \right]
- \frac{1}{4\varepsilon_2} \|\nabla T\|^2_{L^\infty} - (m+1)^2 \|T\|^2_{L^\infty} |\nabla u|^3.
$$

(6.8)

For simplicity in the following computation, we denote

$$
C_1 := \frac{1}{m+1}, \quad C_2 := -\left( 1 - \frac{1}{(m+1)^2} \right) \|V\|^2_{L^\infty} - A - \varepsilon_2, \\
C_3 := -\left( 1 - \frac{1}{(m+1)^2} \right) \|T\|^2_{L^\infty} - \kappa \left( 1 - \frac{1}{s_0} \right) \frac{1}{4\varepsilon_2} \|\nabla T\|^2_{L^\infty} - (m+1)^2 \|T\|^2_{L^\infty}.
$$

By direct calculation, we have

$$
\nabla \psi = \frac{\nabla |\nabla u|}{\phi \circ u} - \frac{|\nabla u| \nabla (\phi \circ u)}{(\phi \circ u)^2}, \\
\Delta V \psi = \frac{\Delta V |\nabla u|}{\phi \circ u} - \frac{|\nabla u| \Delta V (\phi \circ u)}{(\phi \circ u)^2} - \frac{2}{\phi \circ u} \langle \nabla \psi, \nabla (\phi \circ u) \rangle
\geq C_1 \frac{|\nabla |\nabla u||^2_{L^\infty}}{(\phi \circ u)|\nabla u|} + C_2 \frac{|\nabla u|}{\phi \circ u} + C_3 \frac{|\nabla u|^3}{\phi \circ u} - \frac{\psi \Delta V (\phi \circ u)}{\phi \circ u} - \frac{2 \langle \nabla \psi, \nabla (\phi \circ u) \rangle}{\phi \circ u}.
$$

The Cauchy–Schwarz implies that

$$
- \frac{2 \langle \nabla \psi, \nabla (\phi \circ u) \rangle}{\phi \circ u}
= -(2 - 2C_1) \frac{\langle \nabla \psi, \nabla (\phi \circ u) \rangle}{\phi \circ u} - 2C_1 \frac{\langle \nabla \psi, \nabla (\phi \circ u) \rangle}{\phi \circ u}
= -(2 - 2C_1) \frac{\langle \nabla \psi, \nabla (\phi \circ u) \rangle}{\phi \circ u} - 2C_1 \frac{\langle \nabla |\nabla u|, \nabla (\phi \circ u) \rangle}{(\phi \circ u)^2} + 2C_1 \frac{|\nabla (\phi \circ u)|^2 |\nabla u|}{(\phi \circ u)^3}
\geq -(2 - 2C_1) \frac{\langle \nabla \psi, \nabla (\phi \circ u) \rangle}{\phi \circ u} - C_1 \frac{|\nabla |\nabla u||^2_{L^\infty}}{(\phi \circ u)|\nabla u|} + C_1 \frac{|\nabla (\phi \circ u)|^2 |\nabla u|}{(\phi \circ u)^3}.
$$

From the above two inequalities and (6.4), we get

$$
\frac{\Delta V \psi}{\psi} \geq C_2 + C_3 |\nabla u|^2 - \frac{\Delta V (\phi \circ u)}{\phi \circ u} + C_1 \frac{|\nabla (\phi \circ u)|^2}{(\phi \circ u)^2} - (2 - 2C_1) \frac{\langle \nabla r^2, \nabla (\phi \circ u) \rangle}{(a^2 - r^2)(\phi \circ u)}.
$$

(6.9)

Since

$$
\frac{\langle \nabla r^2, \nabla (\phi \circ u) \rangle}{(a^2 - r^2)(\phi \circ u)} \leq \frac{2r |\nabla (\phi \circ u)| \cdot |\nabla r|}{(a^2 - r^2)(\phi \circ u)} \leq \frac{2r |\nabla (\phi \circ u)|}{(a^2 - r^2)(\phi \circ u)} \leq \frac{2\sqrt{r} |\nabla u|}{(a^2 - r^2)(\phi \circ u)}.
$$

(6.10)
and from (6.3),
\[
\Delta_{\mathcal{V}}(\varphi \circ u) = g^{\alpha \beta} \nabla^2 \varphi(\partial_{\alpha} u, \partial_{\beta} u) + \langle (\nabla \varphi) \circ u, \tau(u) + du(V) \rangle \\
= g^{\alpha \beta} u_{\chi^\alpha} u_{\chi^\beta} \text{Hess}^{N}(\varphi) \left( \frac{\partial}{\partial y^\alpha}, \frac{\partial}{\partial y^\beta} \right) + \langle (\nabla \varphi) \circ u, -\text{Tr}_g (du, du) \rangle \\
\leq (-\kappa \cos(\sqrt{k} \rho) + \sqrt{k} \|T\|_{L^\infty}) |\nabla u|^2.
\] (6.11)

Therefore, from (6.6), (6.7), (6.9)–(6.11), we obtain
\[
C_2 + \left( C_3 + \kappa - \frac{\sqrt{k} \|T\|_{L^\infty}}{\cos(\sqrt{k} \rho) \circ u} \right) |\nabla u|^2 - \frac{4(1 - C_1) \sqrt{k} r}{(a^2 - r^2)(\varphi \circ u)} |\nabla u| \leq 0.
\] (6.12)

Since the condition (6.2) tells us
\[
\left( 1 + (m + 1)^2 - \frac{1}{(m + 1)^2} \right) \|T\|_{L^\infty}^2 + \frac{\sqrt{k}}{\cos(\sqrt{k} R)} \|T\|_{L^\infty} < \frac{\kappa}{s_0},
\] there exists a constant \( \varepsilon_0 > 0 \), such that
\[
\left( 1 + (m + 1)^2 - \frac{1}{(m + 1)^2} \right) \|T\|_{L^\infty}^2 + \frac{\sqrt{k}}{\cos(\sqrt{k} R)} \|T\|_{L^\infty} + \varepsilon_0 < \frac{\kappa}{s_0}.
\]

Choosing \( \varepsilon_2 = \frac{\|\nabla T\|_{L^\infty}^2 + 1}{4\varepsilon_0} \), then we have
\[
C_3 + \kappa - \frac{\sqrt{k} \|T\|_{L^\infty}}{\cos(\sqrt{k} \rho) \circ u} = -\left( 1 - \frac{1}{(m + 1)^2} \right) \|T\|_{L^\infty}^2 - \kappa \left( 1 - \frac{1}{s_0} \right)
\]
\[
- \frac{1}{4\varepsilon_2} \|\nabla T\|_{L^\infty}^2 - (m + 1)^2 \|T\|_{L^\infty}^2 + \kappa - \frac{\sqrt{k} \|T\|_{L^\infty}}{\cos(\sqrt{k} \rho) \circ u}
\]
\[
\geq \frac{s_0}{\kappa} - \left( 1 - \frac{1}{(m + 1)^2} \right) \|T\|_{L^\infty}^2 - \frac{1}{4\varepsilon_2} \|\nabla T\|_{L^\infty}^2 - (m + 1)^2 \|T\|_{L^\infty}^2 - \frac{\sqrt{k} \|T\|_{L^\infty}}{\cos(\sqrt{k} R)}
\]
\[
= \frac{s_0}{\kappa} - \left[ \left( 1 + (m + 1)^2 - \frac{1}{(m + 1)^2} \right) \|T\|_{L^\infty}^2 + \frac{1}{4\varepsilon_2} \|\nabla T\|_{L^\infty}^2 + \frac{\sqrt{k} \|T\|_{L^\infty}}{\cos(\sqrt{k} R)} \right]
\]
\[
> \frac{s_0}{\kappa} - \left[ \left( 1 + (m + 1)^2 - \frac{1}{(m + 1)^2} \right) \|T\|_{L^\infty}^2 + \varepsilon_0 + \frac{\sqrt{k} \|T\|_{L^\infty}}{\cos(\sqrt{k} R)} \right] =: C_4 > 0.
\]

Therefore, it follows from (6.12) that
\[
C_4 |\nabla u|^2 - \frac{4(1 - C_1) \sqrt{k} r}{(a^2 - r^2)(\varphi \circ u)} |\nabla u| \leq \left( \frac{2Ar^2 - 2Ar_0r + 2C_0r + 2}{a^2 - r^2} + \frac{8r^2}{(a^2 - r^2)^2} - C_2 \right)
\]
\[
\leq 0.
\]

Note the elementary fact that if \( ax^2 - bx - c \leq 0 \) with \( a, b, c \) all positive, then
\[
x \leq \frac{b}{a} + \frac{c}{a}.
\]

Hence, at the point \( q \),
\[
|\nabla u| \leq \frac{4(1 - C_1) \sqrt{k} r}{C_4(a^2 - r^2) \cos(\sqrt{k} R)} + \sqrt{\frac{1}{C_4} \left( \frac{2Ar^2 - 2Ar_0r + 2C_0r + 2}{a^2 - r^2} + \frac{8r^2}{(a^2 - r^2)^2} - C_2 \right)}.
\]
From this, we can derive the upper bound of $f$, and it is easy to conclude that at every point of $B_{\tilde{v}}(\tilde{p})$, we have
\[
|\nabla u| \leq C_5 \left( \sqrt{A + \frac{\|\nabla T\|_{L^\infty}^2 + 1}{4\varepsilon_0}} + \left( 1 - \frac{1}{(m+1)^2} \right) \|V\|_{L^\infty} + \frac{1}{a} + \frac{1}{\sqrt{a}} \right)
\]
(6.13)

\[
\leq C_6 \left( \sqrt{A + 1} + \frac{1}{a} + \frac{1}{\sqrt{a}} \right),
\]
where $C_6 > 0$ is a constant depending only on $m, n, \kappa, \tilde{R}, V, T$.

For any fixed $x \in M$, letting $a \to \infty$ in (6.13), we obtain $|\nabla u| \leq C_6(\sqrt{A} + 1)$.

\[\square\]

**Proof of Theorem 4** We first choose a constant $C_0'$ depending only on $\kappa, \sigma, R$ and the geometry of $N$, such that if
\[
\max |\nabla T| + \max |T| \leq C_0',
\]
then both the condition (4.3) in Theorem 5 and the condition (6.2) in Theorem 9 hold. Let $\{\Omega_i\}$ be a compact exhaustion of $M$. By Theorem 5, we have a sequence of maps $\{u_i\}$ which solve the Dirichlet problem
\[
\begin{cases}
\tau(u_i) + du_i(V) + \text{Tr}_g T(du_i, du_i) = 0, \\
u_i|_{\partial \Omega_i} = u_0|_{\partial \Omega_i}, \\
u_i \text{ homotopic to } u_0 \text{ rel. } \partial \Omega_i,
\end{cases}
\]
where $u_i \in C^\infty(\Omega_i, N) \cap C^0(\overline{\Omega_i}, N)$ such that $u_i(\Omega_i) \subset B_R(p)$.

For any compact set $K \subset M$, there exists an integer $i_0 > 0$, such that $K \subset \Omega_i$ for $i > i_0$. Then, by (6.8),
\[
\Delta_V |\nabla u_i| \geq \frac{1}{m+1} \frac{|\nabla|\nabla u_i| |}{|\nabla u_i|} + \left[ -\left( 1 - \frac{1}{(m+1)^2} \right) \|V\|_{L^\infty} - \tilde{A} - \frac{\|\nabla T\|_{L^\infty}^2 + 1}{4\varepsilon_0} \right] |\nabla u_i| + \left[ -\left( 1 + (m + 1)^2 - \frac{1}{(m+1)^2} \right) \|T\|_{L^\infty}^2 - \kappa \left( 1 - \frac{1}{s_0} \right) - \varepsilon_0 \right] |\nabla u_i|^3.
\]
(6.14)

where $\tilde{A}$ is a positive constant depending only on the bounds for Ricci curvature of $K$ and $\|V\|_{C^1(K)}$.

Since $K$ is compact, there exist finitely many such geodesic balls $\{B_{\alpha_j}(\tilde{p}_j)\}_{j=1}^{k_0} \subset M$, such that $\bigcup_{j=1}^{k_0} B_{\alpha_j}(\tilde{p}_j) \supset K$. Hence, for any $q \in K$, there is a geodesic ball, say $B_{\alpha_{j_0}}(\tilde{p}_{j_0})$ ($1 \leq j_0 \leq k_0$), containing $q$. Then, by Theorem 9, we can conclude that
\[
|\nabla u_i|(q) \leq C_{j_0} \sqrt{\tilde{A} + 1}.
\]
Hence,
\[
\sup_K |\nabla u_i| \leq \max_{1 \leq j_0 \leq k_0} \left\{ C_{j_0} \sqrt{\tilde{A} + 1} \right\} =: C_8,
\]
where $C_8$ is a positive constant independent of $i$. Then, by the standard elliptic theory, $u_i$ subconverges to a $VT$-harmonic map $u \in C^\infty(M, N)$ with $u(M) \subset B_R(p)$ and $u$ is homotopic to $u_0$. \[\square\]
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