Multi-Scalar Production At Large Center-Of-Mass Energy

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Abstract
In quantum field theory, the probability of producing scalar particles grows factorially as a function of the number of the particles produced. This poses a problem theoretically, in maintaining unitarity, and is counter-intuitive phenomenologically. The factorial growth is a byproduct of the perturbation theory. Nevertheless, it has been recently proposed that the factorial growth might be observable in the future 100 TeV hadron collider. We collect some of the calculations that had been done in regards to this problem so far. We then find the ratio $\sigma_n/\sigma_{\text{total}}$ by calculating the number of scalar jets one would observe at high center-of-mass energies. We will present our results for $\phi^3$ theory in four and six space-time dimensions, $\phi^4$ and the broken theories in four spacetime dimensions.

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1 Introduction

It has been known for a while that the production of many-particle states at high energies can become a factorial function of the final number of particles. This problem drew attention in the Electro-Weak theory when it was found that the B+L-violating cross-section, via instanton-like processes, become large at high energies with large number of bosons in the final state [1, 2]. Motivated by this discovery, the effect was analyzed in simpler models i.e. scalar theories to find whether the ever-increasing cross-section is actually physical and what is the upper bound on the cross-section without violating unitarity [3, 4]. Multiple methods were used in order to find either the amplitude or the cross-section with many bosons in the final state. Using generating function method, Brown found the exact amplitude at threshold for any number of final states [5]. The method were applied to the loops and summing the loop corrections to all order [6–9]. Other methods includes, but are not limited to, Recursion relations [10, 11] Coherent state formalism [12, 13], instanton calculation (Lipatov method) [14–16], and Functional Shrödinger equation [17, 18].

Most of these results are from before LHC, where we did not observe any B-L violation. In the past few years, however, perhaps because the 100 TeV collider is getting closer to reality, the multi-particle cross-section, in particular multi-Higgs cross-section, has received some new attention [13, 20–24]. According to [23], the factorial growth is in fact physical and observable in experiment. Consequently, they foreseen new phenomenon in high energy colliders (of order 100 TeV or so) with many-particles signature. This phenomenon has been dubbed “Higgsplosion”. They propose that the unitarity can be preserved through a mechanism called “Higgspersion”. Moreover, it is claimed that these mechanisms can solve the Hierarchy problem.

Whether the Higgsplosion is physical and the calculation is conclusive had been subject of debate [25–28]: The inclusion of the heavy fermion loops can change the amplitude at high multiplicity of particles, unitarity might not be actually preserved in this mechanism, etc. We also believe that there is a great weight given to the perturbative calculation which in our opinion should not be trusted at the point where the amplitude becomes large (thought, it is claimed that non-perturbative calculation valid for any \( \lambda n \) also gives factorial growth).

Most of the calculations referred above are based on the multi-scalar cross-section near threshold, i.e. when the final particles are non-relativistic.
and are at fixed angle with respect to each other, hence free of collinear
dergences. This will confine these results to a small corner of the phase-
space. What we will try to do in this paper is to focuses on that region of
phase-space where there are enhancements in the amplitude due to collinear
dergence’s. That is, we want to address the phenomenology of multi-scalar
by finding the cross-section for producing \( n \) scalar jets. This is done using Jet
Generating Functional (JGF) \[29, 30\], which satisfies an equation analogous
to Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) equation \[31, 32\].
These calculations are semi-non-perturbative in nature, and as a result we
expect them to be accurate in the large multiplicity limit.

The main object in our calculation is the Sudakov form factor (SF) \[33,
34\], which is the probability that a particle would not split into two or more
particles, i.e. the probability that it stays distinct, given the resolution of
the experiment. We start with a particle at energy of order \( t \approx \mathcal{O}^2 \), where
\( \mathcal{O} \) is the hard scale, e.g. the center-of-mass energy, and an IR regulator \( t_0 \),
which can be the minimum opening angle required to resolve two lines, i.e.
\( t_0 = t_\delta^2 \), or, as we will discuss further below, in the massive theory, if the
opening angle is not too large, it is \( t_0 = m^2 \). Given these two scales, the
Sudakov factor, \( \Delta \), is of the form

\[
\Delta(t, t_0) = \exp \left( -\int_{t_0}^{t} dz \mathcal{P}(z, t') dt' \right).
\] (1)

where \( \mathcal{P} \) is the Altarelli-Parisi (AP) splitting function and \( z \) is the energy
fraction carried by the daughter particle.

We will find the AP function and SF for the cubic theory in four and six
spacetime dimensions; and for the quartic theory, broken and un-broken, in
four dimensions. We will then use these to write a JGF for these theories
and use it to find the jet rates.

Given that the previous calculation is relevant to fixed angle between
particles and for particles carrying small amount of energies, our result is
not in one to one correspondence with the perturbative calculation that leads
to the factorial growth. However, our result does not show any sign of the
factorial growth in the relativistic limit, while if we believe that the factorial
growth should become physical in one point of phase-space, there is no reason
to believe that it would not happen in other regions.

In section 2, we briefly review the perturbative and semi-classical/non-
perturbative methods that had been used in the past in regards to multi-
scalar cross-sections. We also review the phenomenology of the Higgspllosion,
where it is claimed that the multi-Higgs production become unsuppressed due to the factorial growth and observable in the future colliders. In section 3, we discuss the jet generating function method and apply it to the scalar theories in six and four spacetime dimensions. In section 4 we plot the most likely number of final particles in the quartic and cubic theories in the process off-shell $\phi^* \rightarrow n\phi$, and in section 5 we compare to fix-order numerical calculation.

2  ◦  Review of the Past Calculations

2.1  Generating Function Method

Brown [5] has utilized an elegant and simple method for finding the amplitude for an off-shell scalar to produce $n$ on-shell particles, when the particles are at threshold. The idea is that when the final particles are at threshold, the generating function of the tree amplitudes is the classical solutions of the equation of motion in the presence of a source term, and using the classical solution one finds the amplitude as the coefficient of the series in the source. In the unbroken $\phi^4$ theory,

$$\mathcal{L} = \frac{1}{2} (\partial \phi)^2 - \frac{1}{2} m^2 \phi^2 - \frac{1}{4!} \lambda \phi^4,$$

the classical solution is,

$$\phi_{cl} = \frac{z}{1 - (\lambda/48m^2)z^2}, \text{ unbroken sym.}$$

where $z$ is proportional to the source. The amplitudes are found by differentiating with respect to $z$ and setting $z = 0$. We find,

$$A_n^{tree} = \langle n|\phi_d|0\rangle = n! \left(\frac{\lambda}{48m^2}\right)^{n-1} \text{, unbroken sym.}$$

with $n = 3, 5, \ldots$. Consequently, the cross-section, once multiplied by $1/n!$ for accounting for the $n$ identical bosons in the final state, will grow factorially.
In the broken theory\(^1\), one finds \(^6\)
\[
\phi_{cl} = \frac{z}{1 - z\sqrt{\lambda/12m^2}}, \quad \text{broken sym.} \tag{6}
\]
Here the amplitude will be
\[
A_{n}^{\text{tree}} = n!\left(\frac{\lambda}{24m^2}\right)^{\frac{n-1}{2}}, \quad \text{broken sym.} \tag{7}
\]
Again, with a factorial growth. Loop correction does not alleviate the problem. In \(^6, 7\), using Brown’s method, and in \([10]\) by using recursion relation between the one loop diagrams, thy had been calculated \(^2\). In both cases one needs to choose the counter-terms appropriately, otherwise no analytic solution can be found. For the unbroken theory, the leading \(n\) dependence of generating function has been found to the first order in perturbation theory to be
\[
\phi_{cl} = \frac{z}{1 - (\lambda/48m^2)z^2}\left(1 - \frac{\lambda}{4}B\frac{(\lambda/48m^2)^2z^4}{(1 - (\lambda/48m^2)z^2)^2}\right), \quad \text{unbroken sym.} \tag{8}
\]
where \(B\) is given by
\[
B = \frac{\sqrt{3}}{2\pi^2}\left(\ln \frac{2 + \sqrt{3}}{2 - \sqrt{3}} - i\pi\right). \tag{9}
\]
The amplitude is modified to
\[A_n = A_n^{\text{tree}}\left(1 - \frac{\lambda B}{32}(3 - 2n + n^2)\right) \quad \text{unbroken sym.} \tag{10}\]
The fact that the loop correction vanishes for \(n = 3\) is the result of the subtraction scheme used.
For the broken case the results are as follow
\[
\phi_{cl} = \frac{z}{1 - z\sqrt{\lambda/12m^2}}\left(1 - \frac{\lambda^{3/2}z}{48\pi m(1 - z\sqrt{\lambda/12m^2})^2}\right) \quad \text{broken sym.} \tag{11}
\]
\(^1\)The solution is not unique, one also finds that
\[
\phi_{cl} = \phi_0\frac{1 + z/2\phi_0}{1 - z/2\phi_0}. \tag{5}\]
with \(\phi_0 = \sqrt{3!m^2/\lambda}\) works too \([5]\).
\(^2\)For a review, see appendix A where we calculate the loop correction for the cubic theory in six spacetime dimensions.
2.2 Exponentiation and The *Holly Grail* Function

Later on, from analyzing the singularities of the generating function for higher order corrections, it was shown that these loop corrections exponentiate for \( \lambda n < 1 \) \[8\]. If we write \( A_n = A_n^{\text{tree}} A_1^n \) with \( A_n^{\text{tree}} \) given in (4) and (7), we have

\[
A_1^n = \exp \left[ - \lambda n^2 \frac{1}{64\pi^2} \left( \ln(7 + 4\sqrt{3}) - i\pi \right) \right] \quad \text{unbroken sym.}
\]

\[
A_1^n = \exp \left[ \lambda n^2 \frac{\sqrt{3}}{8\pi} \right] \quad \text{broken sym.}
\]

These results had led to the conclusion that the cross-section can be written as the exponential of a function of \( \lambda n \), and dubbed the Holy Grail function:

\[
\sigma_n \propto e^{nF(\lambda n)} \quad \text{(12)}
\]

where the Holy Grail function, to first order, is

\[
F = \log \frac{\lambda n}{48} - 1 - \lambda n \frac{2Re[B]}{32\pi^2} \quad \text{unbroken sym.} \quad \text{(13)}
\]

\[
F = \log \frac{\lambda n}{24} - 1 + \lambda n \frac{\sqrt{3}}{4\pi} \quad \text{broken sym.} \quad \text{(14)}
\]

The higher loop corrections will be of order \( (\lambda n)^2 \) or greater. We can see that the tree calculations are a good approximation as long as \( \lambda n < 1 \). For \( \lambda n \approx 1 \), the perturbation series will blow up and we need to consider non-perturbative approaches which we shall discuss in the next sections.

The threshold limit corresponds to the limit where the kinetic energy of the final particles vanishes. Unlike the amplitudes found so far, the cross-sections at the threshold actually is zero, since there are no phase-space available. To find the cross-section slightly away from threshold, where we can still use the amplitudes, we will work in the approximation where all the final particles have the same average kinetic energy

\[
\epsilon = \frac{E - nm}{nm} \quad \text{(15)}
\]

where \( E \) is the energy of the incoming off-shell particle.
Since the amplitudes are space-time independent, the cross-section is approximately given by
\[ \sigma_n \approx \frac{1}{n!} |A_n|^2 \left( \frac{\epsilon^{3/2}}{3\pi} \right)^n \] (16)

As for the amplitudes, the energy dependence is found by recursion relations. For small energies one finds that [9]
\[ A_n = A_n^{\text{tree}} e^{-\frac{5}{6} n \epsilon} \] (17)

Integrating the square of the amplitude over the phase-space near threshold, the total contribution to the Holy Grail functions will be
\[ F = F_0 + f(\epsilon) \] (18)
\[ f(\epsilon) = \frac{3}{2} \left( \log \frac{\epsilon}{3\pi} + 1 \right) - \frac{17}{12} \epsilon \] (19)

Where \( F_0 \) is given by (13) and (14). This expression is valid for \( \lambda n < 1 \) and \( \epsilon \ll 1 \).

### 2.3 Semi-Classical and Non-Perturbative Methods

There has been considerable number of attempts to address the multi-scalar problem in 90s using different methods. For completeness, let us briefly discuss some of them and refer to the original works for further details.

- The Coherent State Formalism approach is based on the steepest decent method using coherent states in QFT and is similar to Landau WKB method in quantum mechanics. The application to multi-particle had been pioneered by Son [12] and more recently studied in more detail in [13]. The result for the loop corrections in the \( \lambda n \gg 1 \) is
\[ F = \log \lambda n - 1 + 0.85\sqrt{\lambda n} \quad \text{broken sym.} \] (20)

As far as the energy dependence is concerned, in the \( \epsilon \ll 1 \) limit, the next to leading correction is [9]
\[ f(\epsilon) = \frac{3}{2} \left( \log \frac{\epsilon}{3\pi} + 1 \right) - \frac{17}{12} \epsilon + \frac{1327}{432} \epsilon^2 \] (21)
Furthermore, using this method in $\epsilon \to \infty$ limit, Son [12] had shown that the cross-section satisfy the lower bound\(^3\)

$$\sigma_n > n! \left( \frac{\lambda}{48\pi^2} \right)^n$$  \hspace{1cm} (22)

- The Lipatov Method approach is based on analytically continuing to the negative values of $\lambda = -\lambda'$, where the potential will be inverted and all the amplitudes will acquire an imaginary part. One can write the real part of the amplitude for positive $\lambda$ in terms of the imaginary part for the negative $\lambda$, through the dispersion relation

$$A_n(p_i, \lambda) = \text{const.} + \frac{\lambda}{\pi} \int_0^\infty d\lambda' \frac{\text{Im}[A_n(p_i, \lambda')]}{(\lambda' + \lambda')\lambda'}$$ \hspace{1cm} (23)

For negative $\lambda$, the imaginary part of the amplitudes can be calculated using the instanton solution of the inverted potential [14]. The authors of [15, 16] have applied this idea to the amplitude $A_n(2 \to n - 2)$, in the scalar theory. Writing the amplitude as

$$A_n = \sum_l a_n^l \lambda^{n/2 - 1 + l}$$ \hspace{1cm} (24)

where $l$ is the loop order ($l = 0$ is the tree level). The coefficients $a_n^l$ can be calculated by expanding (23):

$$a_n^l = (-1)^{n/2 + l} \frac{1}{\pi} \int d\lambda' \frac{\text{Im}[A_n(p_i, \lambda')]}{\lambda'^{n/2 + l}}$$ \hspace{1cm} (25)

It is shown that in this method the expansion parameter is\(^4\)

$$\eta = \frac{n}{n + l}.$$ \hspace{1cm} (26)

And that the energy dependence of the amplitudes becomes important as $\eta \to 1$. Hence, since the expansion parameter should remain small,

\(^3\)Although this result is based on semi-classical calculation, it is in contradiction of what we claim to be the case.

\(^4\)Given that the integral (25) peaks at $\lambda' = \frac{16\pi^2}{\lambda'}$, the expansion parameter is $\eta \propto \lambda'n$, which is reminiscent of the expansion parameter found earlier.
the results in this method should be trusted when the final particles are near threshold.

It is shown that the factorial behavior shows up in this method as well:

\[ a_n^l \propto (l + n/2)! \quad l \gg 1, n = \mathcal{O}(1) \]  
\[ a_n^l \propto \Gamma(l + n + 3/2) \quad l \gg n \gg 1 \]  
\[ (27) \quad (28) \]

• Yet another attempt had been done using Functional Schrodinger Method. The most important outcome is that the cross-section should not grow at high multiplicity of the final particles. For example, it can be shown that the amplitude for \( n\lambda \gg 1 \), the cross-section decays exponentially\(^5\) [17, 18]

\[ \sigma_n \propto \exp\left(-\frac{\pi}{2}n\right). \]  
\[ (29) \]

\[ 2.4 \quad \text{Enhancement of Multi-Higgs Cross-Section via Higgsplson} \]

Finally, let us review the most recent work on the observation of multi-Higgs cross-section [20–24]. For observing the factorial growth in the experiment, one needs to find the energy dependence away from threshold limit. None of the method known gives a good approximation in large \( \epsilon \) limit, that is when the final particles are relativistic. It is, however, possible to extract the epsilon dependence at tree level using Monte Carlo simulation if we accept the ansatz that the cross-section will exponentiate into a holy grail form (12), specifically that the dependence on \( \lambda \) comes in \( \lambda n \) form[21].

The point is that, at tree level, the \( f(\epsilon) \) does not depend on \( \lambda \), and since the expansion parameter is \( \lambda n \), it will not depend on \( n \) either. Hence, if we look at the ratio of two consecutive cross-sections, we find that

\[ \log \sigma_{n+1}/\sigma_n = (n + 1)F_0(n + 1) - nF_0(n) + f(\epsilon) \]  
\[ (30) \]

Since \( F_0 \) is known, the authors of [21] used Madgraph [49] to find a fit for \( f(\epsilon) \) for \( n = 5 \). We have not redo their simulation; instead used a fit to their graph in figure 2 of [21] to display their result here to complete this section. We used the fit together with the expression in (20) for the broken theory (note that we do not have a non-perturbative equation for the unbroken theory), to

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\(^5\)This can also be shown in a quantum mechanical system [19].
produce Fig. 1 (compare to figure 6 in [20]; however, in this paper the authors used the perturbation result for the loop correction (14) which is not valid at large $\lambda n$). We can see that the cross-section becomes un-suppressed at finite values of $\epsilon$ before they fall down due to suppression from the phase-space.

It is important to note that the enhancement of the cross-section at $\mathcal{O}(100) \text{ TeV}$ energies and low number of final particles (what is suggested to look for in Multi-Higgs papers) is not so much a result of the factorial growth but the fact that the correction to the tree result had come with a positive sign in (20). If we naively just use the tree result, we would have an enhancement at much higher energies $\approx \mathcal{O}(10^4) \text{ TeV}$.

Furthermore, the cross-section becomes un-suppressed at values of $\epsilon$ for which the final particles are relativistic. As explained earlier, we will use jet generating function method to sum the contribution for producing $n$ jets.


3.1 Overview

Jet generating functional (GF) [29, 30] is a functional of a probing function, \( u(p_i) \), where \( p_i \) is the momentum of \( i \)'th particle, such that when expanding in \( u \), the coefficients are the exclusive particle cross-sections. Here, we will use the simplified version where we had integrated over the momentums of the particles to obtain jets. So that the coefficients are jet rate and we have a function instead of a functional.

Calling the generating function \( \Phi \), by definition the jet rates are

\[
R_n = \frac{\sigma_{n\text{-jets}}}{\sigma_{\text{total}}} = \frac{1}{n!} \left( \frac{\partial}{\partial u} \right)^n \Phi[u, t] \bigg|_{u=0}.
\]

where \( t \) is the energy of the incoming particle that produces jets.

One can also use this function to find the average multiplicity number as follow

\[
\bar{n} = \sum nR_n = \frac{\partial}{\partial u} \Phi[u, t] \bigg|_{u=1}.
\]

In our understanding, there is no real derivation of the generating functional from first principles. We had found that Chang and Lau [35], and later Taylor [36], were probably the first ones to apply the idea to \( \phi^3 \) theory in six spacetime dimensions. Later, Kinoshi and others [37] used Altarelli-Parisi evolution equation [31, 32, 39, 40] to “derive” an equation for gluon and quark jet GF, mainly based on jet evolution from a perturbation understanding.

For now, let us work with a cubic theory, whether QED or QCD, or \( \phi^3 \) theory. The main equation for a JGF can be written as follow

\[
\Phi[u, t] = \Phi[u, t_0] \Delta(t, t_0)
\]

\[
+ \int_{t_0}^{t} dt' dz \frac{\partial P(z, t')}{\partial z \partial t'} \frac{\Delta(t, t_0)}{\Delta(t', t_0)} \Phi[u, z^2t'] \Phi[u, (1-z)^2t'].
\]
The $\Delta(t, t_0)$, hence, is the probability of not splitting given the two scales, $t_0$, and $t$. It is called the Sudakov form factor [33, 34]. The second term on the right hand side is the sum (turned into an integral) of probabilities of the line splits into two other jets at scale $t'$. The $\mathcal{P}$ function is the probability weight of splitting into two lines with energy fractions $z$ and $1 - z$. The fraction $\Delta(t, t_0)/\Delta(t', t_0) \approx \Delta(t, t')$ is the probability that the line had not splatted before splitting at scale $t'$.

We had not yet defined the variable $t$. It is claimed, and usually taken to be,

$$t = \frac{k^2_T}{z^2(1 - z)^2}, \quad (35)$$

where $K_T$ is the transverse momentum of daughter jets. For a two-body decay, it is

$$t = Q^2(1 - \cos^2 \theta), \quad (36)$$

where the angle is between the decayed particles. It is claimed that the generating function that satisfies (33), with the so defined $t$, correctly sums the divergent logs for jets in $K_T$-algorithms [38].

In this paper, we will use $t = K^2_T$. Given the limits we take and approximation we make, there will be not much difference, and we will make note where there will be. Furthermore, we will argue that in the massive theory in $t_0 \to 0$ limit, where $t_0$ is the resolution of the experiment, in the final expression we can change $t_0$ to mass and $t$ to $Q^2$, the hard scale energy (As had been done in the earlier versions of the generating functional [37]).

In QCD, (33) gives the jet rate which phenomenologically gives different scaling for abelian and non-abelian splittings: Poisson pattern and staircase pattern respectively, that can be examined in experiment [41]. Our main goals would be to find $\mathcal{P}$ and $\Delta$ in the scalar theory, and plug them into (33) and use (31) to find the jet rates.

### 3.2 $\phi^3$ Theory In Six Spacetime Dimensions

The cubic scalar theory in six dimensions provide a good working ground for analyzing the multi-scalar problem. This is due to two facts. First, unlike in four dimensions, the coupling is dimensionless\(^6\). And, secondly, each particle

\(^6\)The theory is asymptotically free, which is why it used to be a toy model for gluons.
can split into two particles, which makes finding the splitting functions and Sudakov factor easier than in quartic theory where each particle can only split into 3 particles in perturbation theory.

The Lagrangian is given by

$$\mathcal{L} = \frac{1}{2} (\partial \phi)^2 - m^2 \phi^2 - \frac{g}{3!} \phi^3 + \frac{1}{2} \delta_Z (\partial \phi)^2 - \frac{1}{2} \delta_m \phi^2 - \frac{1}{3!} \delta g \phi^3 - \delta \tau \phi. \quad (37)$$

The multi-particle amplitude via the classical generating function method used by Brown is [11]

$$\phi_{cl} = \frac{z}{(1 - \frac{\lambda}{12} z)^2} \phi^3 \text{in 6D} \quad (38)$$

This gives the amplitude at tree level,

$$A_{1\to n} = nn! \left( \frac{\lambda}{12 m^2} \right)^{n-1} \phi^3 \text{in 6D} \quad (39)$$

where we have restored the mass. Again, we see a factorial growth which is expected since the tree graphs amplitude basically counts the number of the Feynman diagrams.

We calculated the loop correction in appendix A. The result is

$$A_n = \frac{d^n}{dx^n} (\phi_0 + \phi_1) \bigg|_{x=0} = nn! \left( \frac{g}{12} \right)^{n-1} \left[ 1 + 3 g^2 (n^2 + 3n + 2) \right] \phi^3 \text{in 6D} \quad (40)$$

Interestingly, we find that the conjecture that the expansion is in $g n$ (here $(g n)^2$), holds here as well.

### 3.2.1 IR Divergences, AP Function, and The Sudakov Form Factor

Under Bloch-Nordsieck theorem [42] and Kinoshita-Lee-Nauenberg theorem [43], in QED and QCD respectively, the mass singularities cancel in the total cross-section between the virtual contribution and the radiation contribution [44]. This phenomenon happens for the scalar theories too. Here we will follow Srednicki [45] who had shown the cancelation for the scalar cubic theory in six dimensions and we will show it for the cubic and quartic theories in four dimensions below.
Following Srednicki, for the exclusive process $\phi\phi \rightarrow \phi\phi$, in the $m \rightarrow 0$ limit, we need to use MS scheme instead of un-shell scheme. In this scheme and in the mass-less limit, the 1PI loop corrections do not depend on mass after renormalization:

$$M = R^2 M_0 \left[ 1 - \frac{11}{12} \frac{g^2}{(4\pi)^3} \left( \log \frac{s}{\mu^2} + \mathcal{O}(m) \right) + \ldots \right]$$ \hspace{1cm} (41)

In this scheme, however, the residue of the two-point function at the mass pole is not one. The residue, $R$, comes from LSZ formula, with each leg contributing $\sqrt{R}$. We have

$$R = \frac{1}{1 - \Pi'(m_{\text{phy}})} \approx \frac{1}{1 - \Pi'(m)} \approx 1 - \frac{1}{12} \frac{g^2}{(4\pi)^3} \log \frac{\mu^2}{m^2}$$ \hspace{1cm} (42)

where prime is differentiation with respect to $p^2$. Altogether, we find,

$$M = M_0 \left[ 1 - \frac{g^2}{(4\pi)^3} \left( \frac{11}{12} \log \frac{s}{\mu^2} + \frac{1}{6} \log \frac{\mu^2}{m^2} + \mathcal{O}(m) \right) + \ldots \right]$$ \hspace{1cm} (43)

These logs would not sum by renormalization of the coupling constant; we cannot remove them by changing $\mu$. They cancel when we add to the cross-section the radiations off of the legs. The radiation contribution to the cross-section is

$$\frac{1}{2!} \int \frac{d^6p_a d^6p_b}{(p_a^2 - m^2)^2}$$ \hspace{1cm} (44)

where the line $a$ has splatted to $b$ and $c$ as shown in Fig. 2. A delta function is added at this point to turn one of the phase-space integrals into the integral over $p_a$, so that we can factor out the cross-section of one less particle process. Carrying out the integral under the assumptions $m^2 \ll Q^2$ and $m^2/Q^2 < \delta^2$, where $\delta$ is the resolution angle, we find

$$\frac{g^2}{2(4\pi)^3} \int_0^1 z(1-z)dz \int_0^\delta \frac{\theta^3 d\theta}{(\theta^2 + (m^2/E_a^2)f(z))^2} = \frac{g^2}{12(4\pi)^3} \log \frac{\delta^2 E_a^2}{m^2} + \ldots$$ \hspace{1cm} (45)

where $f(z) = (1 - z + z^2)/(z - z^2)^2$. There is no soft divergence, just the collinear divergence. Hence the single log. This is just what we need for
canceling the log from the virtual diagrams and turning it into a log of $\delta$. Using $E_a^2 = s/4$, we find

$$\sigma = \left(1 + \frac{g^2}{12(4\pi)^3} \log \frac{\delta^2 E_a^2}{m^2}\right)^4 |M|^2 \quad (46)$$

$$= M_0^2 \left[1 - \frac{g^2}{(4\pi)^3} \left(\frac{3}{2} \log \frac{s}{\mu^2} + \frac{1}{3} \log \frac{1}{\delta^2} O(m)\right) + \ldots\right] \quad (47)$$

The $\frac{g^2}{12(4\pi)^3} \log \delta^2$ can be interpreted as the probability of a leg not splitting. The fact that it blows up instead of reaching 1 as $\delta \to 0$, is a sign of the fact that we need to re-sum the perturbation series. This is usual practice in QED and QCD, and can be shown for cubic scalar in six dimensions too, by summing further radiations of the leg, or purely on probabilistic grounds [44]. The result is the modulation of the cross-section with the Sudakov form factor

$$\Delta = \exp \left[-\frac{g^2}{12(4\pi)^3} \log[1/\delta^2]\right] \quad (48)$$

for each external leg. The forth power of Sudakov form factor, when expanded, would give the log in (45).

The remainder of the integral in (45) is nothing but the probability of a line splitting. Integrated over the range of $\theta$ where the opening angle is bigger than the resolution, we find

$$\frac{g^2}{2(4\pi)^3} \int z(1-z)dz \int_0^1 \frac{\theta^3 d\theta}{(\theta^2 + (m^2/E_a^2)f(x))^2} \approx \frac{g^2}{12(4\pi)^3} \log \frac{1}{\delta^2} \quad (49)$$

Let us note that these results are correct for $m \ll Q$, i.e. in the massless limit. What if $\frac{m^2}{Q^2} > \delta^2$? In this limit, the integral in (45) vanishes as $\delta \to 0$,
and the radiation probability becomes \( \approx \frac{g^2}{12(4\pi)^3} \log \frac{Q^2}{m^2} \). In the massless limit, the Sudakov form factor sums the radiation contribution and virtual corrections in such a way to give the probability of not splitting given some jet definition (loosely, we can change \( \delta \) with other jet definition, for example jet mass or trust). So we expect that for \( \delta < m/Q \), it becomes

\[
\Delta = \exp \left[ -\frac{g^2}{12(4\pi)^3} \log \frac{Q^2}{m^2} \right]. \tag{50}
\]

This can actually be shown in Soft Collinear Effective Theory [46], that in a massive theory the Sudakov form factor sums the IR divergent logs. In other words, we can think of the mass as the IR regulator, since we find the same expression by changing \( \delta \) to \( m \).

From now on, we will set \( m = 0 \) whenever possible, and use the symbol \( t_0 \) for the IR scale. If \( m/Q > \delta \), \( t_0 = m^2 \); if \( m/Q < \delta \), \( t_0 = \delta^2 Q^2 \). We will also use the symbol \( t \) for \( Q^2 \) interchangeably. We also note that in both limits, we still have \( m^2/Q^2 \ll 1 \), or \( t_0/t \ll 1 \).

Using the definitions and approximation above, we can find the probability density of one line splitting into two lines now, called the Altarelli-Parisi function:

\[
\frac{\partial P}{\partial z \partial t'} = \frac{g^2}{2(4\pi)^3} \frac{z(1-z)}{t'} \tag{51}
\]

Knowing the Alterali-Parisi, we can form the Sudakov form factor. It is the exponential of the AP function:

\[
\Delta(t, t_0) = \exp \left[ -\int_0^1 dz \int_{t_0}^t dt' \frac{\partial P}{\partial z \partial t'} \right]. \tag{52}
\]

### 3.2.2 Jet Generating Function

We now have all the ingredients for writing the differential equation for the generating function. For the cubic theory it is

\[
\Phi_{(3)}[t] = u \Delta(t, t_0) + \frac{1}{2(4\pi)^3} \int_{t_0}^t \int_{t_0}^{t'} dt'' \frac{\Delta(t', t_0)}{\Delta(t', t_0)} \int dz \frac{g^2 z(1-z)}{t'} \Phi_{(3)}[z^2 t''] \Phi_{(3)}[(1-z)^2 t''] \tag{53}
\]
Noting that the distribution in $z$ is fairly even, we disregard the $z$-dependence of the functions under integral to find the simplified equation,

$$\Phi(3)[t] = u\Delta(t, t_0)$$  \hspace{1cm} (54)

$$+ \frac{1}{2(4\pi)^3} \int_{t_0}^{t} dt' \frac{\Delta(t, t_0)}{\Delta(t', t_0)} \int dz \frac{g^2 z(1-z)}{t'} \Phi(3)[t'] \Phi(3)[t'].$$  \hspace{1cm} (55)

Now, differentiation with respect to $t$ from both sides gives

$$\frac{\Phi(3)[t]}{dt} = \frac{1}{\Delta} \frac{d\Delta}{dt} \left( \Phi(3)[t] - \Phi^2(3)[t] \right).$$  \hspace{1cm} (56)

Using the boundary condition, $\Phi[t_0] = u$, we find

$$\Phi(3)[t] = \frac{u}{u + (1-u)\Delta^{-1}}.$$  \hspace{1cm} (57)

This is in agreement with Taylor [36] and Kinoshi [37]. This also is the JGF for a gluon in pure Yang-Mills theory, that is without splitting of the quark and anti-quarks into gluons and vice versa, which leads to staircase pattern for the gluons [41] that had been checked in experiment.

Using (57) and (31), we find that

$$R_n = \Delta(1 - \Delta)^{n-1}.$$  \hspace{1cm} (58)

Using (32), we can find the average jet multiplicity

$$\bar{n} = \frac{\partial}{\partial u} \Phi \bigg|_{u=1} = \Delta^{-1}.$$  \hspace{1cm} (59)

As $g \to 0$, we have $\bar{n} \to 1$, which means that there are no splittings; on the other hand $\bar{n}$ increases with $Q^2/m^2$, or decrease with $\delta$, as expected.

### 3.3 $\phi^4$ Theory In Four Spacetime Dimensions

#### 3.3.1 Cancelation of IR Divergences

We now turn to the $\phi^4$ theory

$$\mathcal{L} = \frac{1}{2} (\partial \phi)^2 - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!} \phi^4.$$  \hspace{1cm} (60)
We want to analyze the scattering process $X \rightarrow \phi \rightarrow 3\phi$ - where $X$ can be, for example, $e^+e^-$, and the intermediate $\phi$ is off-shell. The infrared divergences should cancel between the virtual diagram of the $X \rightarrow 3\phi$ process against the divergences in $X \rightarrow 5\phi$ process. The virtual diagrams are at two loops level. One might suspect that the “fish diagrams” would contribute as well. However, the fish diagram does not contain any IR divergence. This should be the case since the contribution of $X \rightarrow 5\phi$ process to the total cross-section is at $\lambda^4$ order, while, if the fish diagram has IR divergence, there would be a $\lambda^3$ order contribution to the total cross-section (from the interference of the fish diagram with the tree diagram).

Checking the IR cancelation in the total cross-section for the process $X \rightarrow 3\phi$ requires calculating 5-body phase-space. We instead check the cancelation of the mass-divergence of the 2 loop diagram in $2\phi \rightarrow 2\phi$ scattering (Fig. 3) via the Srednicki method.

![Figure 3: $\phi\phi \rightarrow \phi\phi$.](image)

We want to find the 1PI four-point function in the $m^2 \rightarrow 0$ limit. The virtual diagrams are shown in Fig. 4.

![Figure 4: Types of virtual corrections to $\phi\phi \rightarrow \phi\phi$.](image)

As note in [45], the correct renormalization scheme to use in zero mass limit is the MS scheme. In this scheme, the residue at the physical mass pole is not unity and we have a factor of $\sqrt{R}$ multiplying the cross-section for each external leg coming from LSZ formula. Where

$$R = \frac{1}{1 - \Pi'(p^2 = m^2)}.$$  (61)
In this scheme, the leg loop correction, i.e. the sunset diagram (Fig. 5), is not absorbed in the counter term and needs to be added to the cross-section.

Figure 5: The “sunset” diagram.

The “fish” diagrams with the counter term added, in $\overline{\text{MS}}$ scheme, in the $m \to 0$ limit, gives

$$M(1) = -\frac{i\lambda^2}{2(4\pi)^2} \sum_{A=s,t,u} \left( \log \frac{A}{\mu^2} - 2 \right)$$

(62)

where $\mu$ is the renormalization scale and $s$, $t$, and $u$ are the Peskin parameters.

The double “fish” diagrams, in the same limit, gives

$$M(2) = -\frac{i\lambda^2}{4(4\pi)^4} \sum_{A=s,t,u} \left( \log \frac{A}{\mu^2} - 2 \right)^2$$

(63)

The “vertex correction” diagrams give

$$M(3) = -\frac{i\lambda^3}{(4\pi)^4} \sum_{A=s,t,u} \left( \frac{1}{2} \log^2 \frac{A}{\mu^2} - 3 \log \frac{A}{\mu^2} + \frac{11}{2} \right)$$

(64)

The “sunset” diagram gives

$$R = \frac{1}{1 - \Pi'(m^2)} \approx 1 + \Pi'(m^2) = 1 + \frac{\lambda^2}{12(4\pi)^4} \log \frac{m^2}{\mu^2} + \text{const.} \times \mathcal{O}(\lambda^2)$$

(65)

The cross-section is proportional to

$$R^2 |M|^2 = \lambda^2 R^2 \left| 1 + M(1) + M(2) + M(3) \right|^2.$$  

(66)

We note that in $\overline{\text{MS}}$ scheme, all that survive in $m \to 0$ limits are the log of $m$ in $R$. The amplitude depends on the external momentum and $\mu$ (the
cross-section does not depend on $\mu$ by the running of $\lambda$ but not on the mass. This is interesting since we can now see that IR divergences should be at $\lambda^2$ order since there is no field renormalization in the quartic theory at one loop level. Furthermore, since $R$ only depends on $\log m$, we expect that there are no double logs at this order for the quartic theory either.

To find the physical amplitude in the $m \to 0$ limit, we need to add to (66) the probability of the legs not split given resolution angle $\delta$.

We start with a line, labeled $a$, splitting into 3 other lines, $b$, $c$, and $d$, as shown in Fig. 6.

![Figure 6: $\phi \to 3\phi$ splitting. Momentums are flowing to the right.](image)

The splitting is proportional to

$$\frac{1}{3!} \int \frac{\lambda^2}{(p_a^2 - m^2)^2} d^4p_b d^4p_c d^4p_d$$  \hspace{1cm} (67)

we multiply this by

$$1 = \int d^4p_a 2E_a(2\pi)^3\delta(\vec{p}_a - \vec{p}_b - \vec{p}_c - \vec{p}_d).$$  \hspace{1cm} (68)

and isolate the $d^4p_a$ integral to be absorbed into the cross-section with two less legs. We find

$$\frac{\lambda^2}{48(2\pi)^6} \int \frac{E_a}{E_bE_cE_d(p_a^2 - m^2)^2} \sin \theta_b d\phi_b \sin \theta_c d\phi_c E_b^2 E_c^2 dE_b dE_c$$  \hspace{1cm} (69)

We can simplify this integral by changing the variables to: $\theta, \alpha, x$, and $y$. Where $\theta$ is the angle between $b$ and $c$; $\alpha$ is the angle between $d$ and the sum of $b$ and $c$; and $x$ and $y$ are defined as follow

$$E_b = (1 - x)(1 - y)E_a$$  \hspace{1cm} (70)

$$E_c = x(1 - y)E_a$$  \hspace{1cm} (71)

$$E_d = yE_a$$  \hspace{1cm} (72)
The integral becomes
\[
\frac{\lambda^2}{48(2\pi)^4} \int_0^1 dx \int_0^1 dy \int_0^\delta \theta d\theta \int_0^\delta \alpha d\alpha \frac{-x(1-x)(1-y)}{y} \times \\
\left( \theta^2 + \frac{x(1-x)}{y} \alpha^2 + \frac{1}{2yx(1-x)} E_a^2 \right)^{-2}
\] (73)

We first integrate over \(\alpha\) and \(\theta\) using \textit{mathematica}. We then expand in \(m\) and carry out the rest of the integrals to find
\[
\frac{\lambda^2}{24(4\pi)^4} \log \frac{E_a^2 \delta^2}{m^2}. \tag{74}
\]

Adding four of these to (66) exactly cancels the mass divergence that comes from \(R^2\):
\[
R^2 + 4 \times \frac{\lambda^2}{24(4\pi)^4} \log \frac{E_a^2 \delta^2}{m^2} = 1 + \frac{\lambda^2}{6(4\pi)^4} \log \frac{\delta^2 E_a^2}{\mu^2} \tag{75}
\]

### 3.3.2 AP Function and The Sudakov Form Factor

It is not as straight forward to find an expression for the AP function as it was in the cubic theory in six dimension (or as it is in QED and QCD). It is mainly because we cannot separate the integrals over the angles from those of the energies. The AP function is usually written in terms of energy fractions of the daughter partons and their transverse momentums. Using these, in appendix C, we have found that the AP function, in the massless theory, is
\[
P(x, z, t, t') = \frac{\lambda^2}{16(2\pi)^4} \frac{xz(1-x-z)[x(1-x)t + z(1-z)t']}{\left[ (x(1-x)t + z(1-z)t')^2 - 4x^2z^2tt' \right]^{3/2}} \tag{76}
\]

where \(x = E_b/E_a\) and \(z = E_c/E_a\). The integration with these choice of variables is not easy. But, from our other choice of variables in (73), we know that it gives
\[
\int P(x, z, t, t')dzdxdt = \frac{\lambda^2}{24(4\pi)^4} \log \frac{E_a^2 \delta^2}{m^2}. \tag{77}
\]
Knowing the AP function, we find the Sudakov factor:

$$\Delta(t, t_0) = \exp \left[ -\int P(x, z, t, t') dz dx dt dt' \right]$$

(78)

$$= \exp \left[ -\frac{\lambda^2}{24(4\pi)^4} \log \frac{t}{t_0} \right] \phi^4 \text{ in 4D}$$

(79)

where $t$ is the large scale where a hard process starts.

### 3.3.3 Jet Generating Function

We are now ready to write down the JGF for the quartic theory. In this case we need to take into account that a line splits into at least 3 other lines. It reads

$$\Phi_4(t) = \Delta(t, t_0) \Phi_4(t_0)$$

(80)

$$+ \int dx \int dz \int_{t_0}^{t} dt' \Delta(t, t') P(x, z, t') \times$$

$$\Phi_4(x^2 t') \Phi_4(z^2 t') \Phi_4((1 - x - z)^2 t')$$

(81)

(82)

Since, like the cubic theory, there is no IR divergence in the energies, and so the integrand is not concentrated around $x, z \approx 0, 1$, we again make the simplification of ignoring the energy dependence of the functions under the integral. Using $\Delta(t, t') = \Delta(t, t_0)/\Delta(t', t_0)$, and differentiating with respect to $t$ from both sides, we find

$$\frac{d\Phi_4(t)}{dt} = \frac{d\Delta(t, t_0)}{dt} (\Phi_4(t) - \Phi_3(t))$$

(83)

Solving this equation with the boundary condition $\Phi[t_0] = u$, we find

$$\Phi_4(t) = \frac{u}{\sqrt{u^2 + (1 - u^2) \Delta^2(t, t_0)}}$$

(84)

Using mathematica, we find that

$$R_n = f(n) \Delta \left( 1 - \Delta^2 \right)^{\frac{n-1}{2}} \quad n = \text{odd} \geq 3,$$

(85)

with $f(n)$ a slowly decreasing function of $n$ as shown in Fig. 7.
3.4 $\phi^3$ Theory In Four Spacetime Dimensions

We have presented the perturbation result for the broken theory in section 2. Before we discuss the jet rates in this theory, let us briefly look at the cubic theory in four dimensions. The Lagrangian is

$$\mathcal{L} = \frac{1}{2} (\partial \phi)^2 - \frac{1}{2} m^2 \phi^2 - \frac{g}{3!} \phi^3 \quad (86)$$

We can assume there exists a $\lambda \phi^4$ term with a negligible quartic coupling $\lambda \ll 0$ to avoid an unstable vacuum. Since in a general scalar theory, the cubic and the quartic couplings do not necessarily correlate, the IR divergences for radiations through cubic coupling should cancel independent of quartic coupling.

3.4.1 Cancelation of IR Divergences

To show the cancelation, for convenience, we set $m = 0$ and use dimensional regularization to regulate both the UV and the IR divergences [48]. Let us find the total cross-section of the process $e^+ e^- \to \phi \to X$, where $\phi$ is off-shell:

$$\sigma_{total} = R^2 \sigma_b + \sigma_v + \sigma_r. \quad (87)$$

Here $R$ is the field renormalization, $b$ stands for Born, $v$ for virtual, and $r$ for the real emission contributions. In the limit $m \to 0$, keeping $g$ fixed, IR divergences should cancel as shown in Fig. 8.
Since the final particles are identical, we need to divide the cross-sections by \( n! \), where \( n \) is the number of the final particles. Calling all the pre-factors that depend on the production of \( \phi \), \( A_b \) (in the example above, \( e^+e^- \to \phi \)), we have

\[
\sigma_b = A_b \frac{1}{2!} \int \Pi_2 |M_0|^2 \\
\sigma_v = A_b \frac{1}{2!} \int \Pi_2 |\delta M_v|^2 \\
\sigma_r = A_b \frac{1}{3!} \int \Pi_3 |M_R|^2
\]

The divergent part of the field strength tensor, \( R \), is proportional to

\[
\int \frac{d^4 k}{k^4} = \frac{1}{\epsilon} - \frac{1}{\epsilon'}.
\]

The \( \epsilon \) is due to IR divergence and \( \epsilon' \) due to UV divergence. Later on, both will cancel separately in the total cross-section. Bu, for convenience, instead of keeping both UV and IR regulators, we can instead set \( \epsilon = \epsilon' \) in [44], and have \( R = 1 \).

For the born cross-section we find

\[
\sigma_b = A_b \frac{g^2}{16\pi} \quad (92)
\]

There are two virtual diagrams, once interfere with the tree lever, give the next order correction in coupling constant. The first one is the vertex correction shown in Fig. 9.
We can write the amplitude as

\[ M_{v1} = -ig^3 \Gamma(3 - d/2) \frac{4\pi}{Q^2} Q^{2-d/2} \int dzdy \frac{1}{(-yz)^{3-d/2}} \]  

where \( d = 4 - \epsilon \), but \( \epsilon < 0 \) so that the integral converges.

We have

\[ \delta M^2 = M_0 M_{v1}^* + M_{v1}^* M_0 + M_0^* M_0 \]  

\[ = + \frac{g^4}{(4\pi)^2} 2\Gamma(3 - d/2) \frac{4\pi}{Q^2} Q^{2-d/2} \int dzdy \frac{1}{(-yz)^{3-d/2}} \]

The phase-space integral in \( d \) dimension is

\[ \int \Pi_2 = \left( \frac{4\pi}{Q^2} \right)^{2-d/2} \frac{1}{2^d \sqrt{\pi} \Gamma\left(\frac{d-1}{2}\right)} \]

We find

\[ \sigma_{v1} = A_0 \frac{1}{2} \int \Pi |\delta M_{v1}|^2 \]

\[ = A_0 \frac{g^4}{128\pi^3 Q^2} \left( \frac{4\pi e^{-\gamma_E}}{Q^2} \right)^{4-d} \left( - \frac{4}{\epsilon^2} - \frac{4}{\epsilon} - 4 + \frac{5\pi^2}{6} \right) \]

where \( \gamma_E \) is the Euler-Mascheroni constant. The other loop diagram is the vacuum diagram for the intermediate Higgs shown in Fig. 10.

The integral for this diagram is UV divergent but not IR divergent (unlike the vacuum diagrams of the final particles once we set \( p_1^2 = p_2^2 = 0 \)). The amplitude is

\[ M_{v2} = \frac{ig^3}{2(4\pi)^2 Q^2} \left( \frac{4\pi}{Q^2} \right)^{2-d/2} \Gamma(2-d/2) \int \frac{dx}{(x(x-1))^{2-d/2}} \]
where the factor of two in the first line is the symmetry factor. We have

$$\sigma_{v2} = A_b \frac{g^4}{128\pi^3 Q^2} \left( \frac{4\pi e^{-\gamma E}}{Q^2} \right)^{4-d} \left( -\frac{1}{\epsilon} - 2 \right)$$  \hspace{1cm} (101)$$

Altogether, the cross-section is

$$\sigma_v = \sigma_{v1} + \sigma_{v2} = A_b \frac{g^4}{128\pi^3 Q^2} \left( \frac{4\pi e^{-\gamma E}}{Q^2} \right)^{4-d} \left( -\frac{4}{\epsilon^2} - \frac{5}{\epsilon} - 6 + \frac{5\pi^2}{6} \right)$$  \hspace{1cm} (102)$$

For the real emission, we have three diagrams, shown in Fig. 11. The amplitude is given by

$$M_r = -ig^2 \frac{Q^2}{Q^2} \left( \frac{-1 + x_1 x_2 + (2 - x_1 - x_2)(x_1 + x_2)}{(1 - x_1)(1 - x_2)(x_1 + x_2 - 1)} \right)$$  \hspace{1cm} (103)$$

where \( x_i = \frac{2E_i}{Q} \), and \( x_1 + x_2 + x_3 = 2 \). The 3-body phase-space integral in dimensional regularization is

$$\int d\Pi_3 = \left( \frac{Q^2}{4\pi} \right)^{d-4} \frac{Q^2}{128\pi^3 \Gamma(d-2)} \times$$

$$\int_0^1 dx_1 \int_0^1 dx_2 \frac{1}{((1 - x_1)(1 - x_2)(x_1 + x_2 - 1))^{2-d/2}}$$  \hspace{1cm} (104)$$
A nice way of doing the integral is by changing the variable $x_2 = 1 - yx_1$, $0 < y < 1$. We find

$$
\sigma_r = A_b \frac{1}{3!} \int d\Pi_3 |M_R|^2 \\
= A_b \frac{g^4}{128\pi^3 Q^2} \left( \frac{4\pi e^{-\gamma_E}}{Q^2} \right)^{4-d} \left( \frac{4}{\epsilon^2} + \frac{5}{\epsilon} + \frac{9}{2} - \frac{5\pi^2}{6} \right)
$$

The sum of the two contributions to the total cross-sections is

$$
\sigma_v + \sigma_r = A_b \frac{g^4}{128\pi^3 Q^2} \left( \frac{4\pi e^{-\gamma_E}}{Q^2} \right)^{4-d} \left( -\frac{3}{2} \right)
$$

Hence, after setting $d = 4$,

$$
\sigma_{\text{total}} = A_b \frac{g^2}{16\pi} \left( 1 - \frac{3g^2}{16\pi^2 Q^2} \right),
$$

which is a finite number.

Although we have used dimensional regularization, but in the massive theory, in the limit $m \to 0$, the total cross-section should be the same (108).

The limit of integration of the 3-body phase-space (104) is given by the expressions below:

$$
x_2^{\text{max}} = 1 + \frac{b}{x_1(1 - x_1)} + \mathcal{O}(b^2)
$$

$$
x_2^{\text{min}} = 1 - x_1 - 3b + \frac{b}{x_1(1 - x_1)} + \mathcal{O}(b^2)
$$

$$
x_1^{\text{max}} = 1 - 4b
$$

$$
x_1^{\text{min}} = 2\sqrt{b} - b
$$

These will specify the region covered with the blob (Dalitz diagram) as shown in Fig. 12
We will skip the calculation here. We point out that in the massive theory, we have learned that in the \( \bar{\text{MS}} \) scheme, the mass singularity comes from the residue of the mass pole. It is given by

\[
R^{-1} = 1 + \Pi'(m_p^2)
\]  

where as usual prime is differentiation with respect to \( p^2 \) and we have

\[
\Pi = \frac{g^2}{(4\pi)^2} \frac{\Gamma(\epsilon/2)}{\Gamma(1-\epsilon/2)} \int_0^1 dx \frac{1}{(m^2 + p^2 x(x-1))^{\epsilon/2}}
\]  

with \( d = 4 - \epsilon \). \( R \) is UV finite and we can set \( \epsilon = 0 \). We find that

\[
R^{-1} = 1 + \frac{g^2}{(4\pi)^2} \left( -1 + \frac{2\pi}{3\sqrt{3}} \right)
\]

3.4.2 AP Function and The Sudakov Form Factor

The calculation of AP function is the same as in the previous sections. The splitting shown in Fig. 2 gives
\[
\frac{1}{2!} \int \frac{g^2}{(p_a^2 - m^2)^2} d^4 p_b d^4 p_c.
\]  

(116)

Writing the integral over in terms of the \( \theta \) and \( z \). We have

\[
\frac{g^2}{(4\pi)^2} \int dz \int_0^\delta \frac{d\theta}{Q^2 z (1 - z) \left( \theta^2 + m^2/Q^2 f(z) \right)^2},
\]

where \( \delta \) is our angular resolution, and

\[
f(z) = \frac{1 - z + z^2}{(z - z^2)^2}.
\]

(117)

Integrating over the \( \theta \), we find

\[
\frac{1}{2z(1 - z)} \left( \frac{1}{m^2 f(z)} - \frac{1}{Q^2 \delta^2 + m^2 f(z)} \right)
\]

(118)

At this point we need to choose whether \( \delta^2 \) is less than or bigger than \( m^2/Q^2 \).

If we fix \( \delta^2 < m^2/Q^2 \), the expression is finite as \( \delta \to 0 \). In fact this term

vanishes for \( \delta = 0 \), which means that these diagrams do not contribute to

the jets with fewer legs.

However, if we take the limit \( m^2/Q^2 \to 0 \) and \( \delta \to 0 \), while keeping

\( \delta^2 > m^2/Q^2 \), the above expression would become singular both in \( m^2/Q^2 \)

and \( \delta^2 \). Integrating over \( z \) first and then expanding (119) in \( m^2/Q^2 \), we find

\[
\frac{g^2}{2(4\pi)^2} \left[ \left( -1 + \frac{2\pi}{3\sqrt{3}} \right) \frac{1}{m^2} + \frac{1}{Q^2 \delta^2} + \mathcal{O}(m^2) \right].
\]

(119)

(120)

As expected, two times this expression (for each leg) cancels the \( m^2 \) di-

vergence of \( R \) in expression (115).

Following our arguments of the previous sections, we want to write the

AP function by setting mass to zero and regulated the integral of \( t \) with

\( \max[\delta^2, m^2/Q^2] \). However, we should be careful since we cannot set \( m = 0 \) in

(117). It gives a wrong result since \( f(z) \) has poles at \( z = 0,1 \). We can find

the AP function by changing the variable to \( k_T \), it gives (see appendix B for

another derivation)

\[
P(z,t) = \frac{g^2}{(4\pi)^2} \frac{z(1 - z)}{(t + m^2(1 - z + z^2))^2}
\]

(121)

(122)

28
Now, setting $m = 0$ we find

$$P = \frac{g^2}{(4\pi)^2} \frac{z(1-z)}{t^2}$$  \hspace{1cm} (123)

$$P = \chi^2 = \frac{\chi^2}{24(4\pi)^4} \log \frac{t}{t_0}$$  \hspace{1cm} (124)

Our result matches the result in [47]. The power divergence, called *ultra-collinear* divergence, has been discussed in that paper.

Sudakov form factor is the exponentiation of (123) [44]:

$$\Delta(t, t_0) = \exp \left[ - \int_{t_0}^{t} dt' \int dz P(z, t') \right]$$

$$\approx \exp \left[ - \frac{g^2}{6(4\pi)^2 t_0} \phi^3 \text{ in 4D} \right]$$  \hspace{1cm} (125)

### 3.4.3 Jet Generating Function

The equation for the generating function does not change with the dimension. Hence, equation (53) for the cubic theory in six dimensions is good here as well. We again find (57):

$$\Phi_{(3)}[t] = \frac{u}{u + (1-u)\Delta^{-1}}$$  \hspace{1cm} (126)

with Sudakov factor given by (125). All the analysis follows as in six dimensions.

### 3.5 The Broken Theory In Four Spacetime Dimension

Now, let us consider the presence of both cubic and quartic interactions.

$$\mathcal{L} = \frac{1}{2} (\partial \phi)^2 - \frac{1}{2} m^2 \phi^2 - \frac{g}{3!} \phi^3 - \frac{\lambda}{4!} \phi^4$$  \hspace{1cm} (127)

We have shown that the IR divergences for each coupling cancel independently and there. The Sudakov form factor, by definition, is the multiplication of the Sudakov form factors for the cubic and quartic couplings,

$$\Delta(t, t_0) = \Delta_{\phi^3}(t, t_0) \Delta_{\phi^4}(t, t_0)$$  \hspace{1cm} (128)

$$= \exp \left[ - \frac{g^2}{6(4\pi)^2 t_0} - \frac{\chi^2}{24(4\pi)^4} \log \frac{t}{t_0} \right].$$  \hspace{1cm} (129)
If we look at the Sudakov factor as a function of $t_0$, we can see that as we decrease $t_0$ the cubic term dominates. But we know that at some point $t_0/t$ drops below the resolution angle $\delta^2$ and we need to swap $t_0$ with $t\delta^2$. The cubic term becomes proportional to $1/t$ and quartic term becomes a constant (a function of $\delta^2$). Hence, either as a function of $t_0$ or $t$, as $t_0/t$ decrease, eventually the quartic term dominates. However, because of the log divergence vs power divergence, and because of the extra $1/(4\pi^2)$, this happens at an extremely small $t_0/t$. For all realistic purposes, the cubic term is the dominant one which tells us to a good approximation we can ignore the quartic term in the Sudakov factor and also the quartic splitting.

Turning to the broke theory in 4D, we have

$$L = \frac{1}{2}(\partial h)^2 - \frac{1}{2}m_h^2 h^2 - \sqrt{\frac{\lambda_h}{2}} m_h h^3 - \frac{1}{4}\lambda_h h^4$$  \hspace{1cm} (130)

And we have for the Sudakov factor

$$\Delta_h(t, t_0) = \Delta_{\phi^4}\Delta_{\phi^4}$$

$$= \exp \left[ - \frac{3\lambda_h m_h^2}{(4\pi)^2 t_0} - \frac{3\lambda_h^2}{2(4\pi)^4} \log \frac{t}{t_0} \right]$$ \hspace{1cm} \text{broken } \phi^4 \text{ in 4D}  \hspace{1cm} (131)

The generating function is given by

$$\Phi_h[t] = \Delta_h(t, t_0)\Phi_h[t_0]$$

$$+ \int_{t_0}^t dt' \Delta_{\phi^4}(t, t') \int dz \mathcal{P}_{\phi^4}(z, t') \Phi_h[z^2 t'] \Phi_h[(1 - z)^2 t']$$

$$+ \int dx \int dz \int_{t_0}^t dt' \Delta_{\phi^4}(t, t') \mathcal{P}_{\phi^4}(x, z, t') \times$$

$$\Phi_h[x^2 t'] \Phi_h[z^2 t'] \Phi_h[(1 - x - z)^2 t']$$ \hspace{1cm} (132)

As discussed above, we will ignore the quartic interaction and only consider $1 \to 2$ splitting (we have checked this approximation numerically). The generating function is given by,

$$\Phi_h[t] \approx \Delta_h(t, t_0)\Phi_h[t_0] + \int_{t_0}^t dt' \Delta_h(t, t') \int dz \mathcal{P}_h(z, t') \Phi_h[z^2 t'] \Phi_h[(1 - z)^2 t']$$ \hspace{1cm} (133)

The solution is identical to the cubic theory in six dimensions (57) with substitution of the sudakov factor.
4 **Off-shell** $\phi^* \to n\phi$ Process.

The generating function corresponds to the evolution of a highly relativistic and approximately on-shell particle. Hence, to apply this method to the process $\phi^* \to n\phi$, where the $\phi^*$ is highly off-shell, we approximate the JGF for this process as

$$
\Phi_{\phi^*_\to n\phi}[t] = \Phi_{(3)}[t/4]^2 + \ldots, \tag{134}
$$

for the cubic theory and

$$
\Phi_{\phi^*_\to n\phi}[t] = \Phi_{(4)}[t/9]^3 + \ldots, \tag{135}
$$

for the quartic theory. The next terms will be of order $O(\lambda^2)$ and higher with additional Sudakov factor. The last expression also works for the unbroken theory since the 2-body cross section for this theory is suppressed compared to the 3-body cross-section.

![Figure 13: $f(n)$ in equation (137).](image)

For the cubic and quartic theories, the jet rate becomes

$$
R_n^{(3)} = (n - 1)\Delta^2(1 - \Delta)^{n-2}, \tag{136}
$$

$$
R_n^{(4)} = f(n)n\Delta^3(1 - \Delta^2)^{n+3}, \tag{137}
$$

with $f(n)$ given in Fig. 13.
In Fig. 14, we have plotted few of the jet rates as a function of $-\log[\Delta]$. We can see that the multi-scalar rates start to dominate when

$$-\log[\Delta] > 1$$  \hfill (138)
Table 1: Number of Feynman diagrams for the process $gg \to \phi^* \to n\phi$ for different theories.

| $\phi^3$ | $\phi^4$ | Broken Theory |
|----------|----------|---------------|
| n | # of diagrams | n | # of diagrams | n | # of diagrams |
| 2 | 1 | 3 | 1 | 2 | 1 |
| 3 | 3 | 5 | 10 | 3 | 4 |
| 4 | 15 | 7 | 280 | 4 | 25 |
| 5 | 105 | 9 | 15400 | 5 | 220 |
| 6 | 945 | 6 | 2485 |
| 7 | 10395 | 7 | |

\[ \frac{Q^2}{m^2} > \exp \left[ \frac{12(4\pi)^3}{g^2} \right]. \quad (139) \]

This is an extremely large number for any perturbative value of the coupling constant. Similar situation holds for the other theories. Hence, almost always we will see two jets in any high energy process involving only scalar particles.

## 5. Comparison to Fix-order Calculation

For an arbitrary jet clustering algorithm, the divergent logs of the IR parameter does not necessarily exponentiate. In QCD the JADE algorithm is one example [50]. As explained earlier, in QCD it has been proposed and checked numerically and experimentally that in the $K_T$-Algorithm, the one we used here, the divergent logs do exponent.

We have further argued that we can change the resolution parameter to the mass. In this case the jet rates become the particle cross-sections and we can utilities MADGRAPH [49] to compute cross-section of $gg \to \phi^* \to \text{few } \phi$. We can expand the expressions for the cross-sections (136) and (137) to find the leading contributions. Since $\sum \sigma_n/\sigma_{\text{total}} = 1$, we know that,

\[ \sigma_2/\sigma_{\text{total}} = 1 - \sigma_3/\sigma_{\text{total}} - \sigma_4/\sigma_{\text{total}} - \ldots, \quad (140) \]

for the cubic theory (for the quartic theory we have to start from $\sigma_3$). Since $\sigma_2/\sigma_{\text{total}}$ is proportional to the second power of the sudakov factor, the leading terms of the cross-sections confirm the exponentiation of these logs. Note that
we interpret these logs in $\sigma_2/\sigma_{\text{total}}$ to come from the virtual corrections, and we shall not take them into account in our comparison to Madgraph since program computes the diagrams at tree-level.

Figure 15: Madgraph computation vs. Jet calculation of $gg \rightarrow \phi^* \rightarrow n\phi$ in the $\phi^4$ theory. From top to bottom: $n = 3, 5, \text{and} 7$.

Another point we shall make is that we expect the total cross-sections becomes independent of mass in the mass-less limit. We have shown this to the next leading order in all the theories above. To the first order, the total cross-section is equal to $\sigma_2$.

Figure 16: Madgraph computation vs. Jet calculation of $gg \rightarrow \phi^* \rightarrow n\phi$ in the $\phi^3$ theory.

(a) $g = 1/8$

(b) $g = 1/80$
We will use the model \textit{HEFT} and turn off cubic interactions for analyzing $\phi^4$ theory and, vise versa, turn off the quartic coupling to study $\phi^3$ theory. Due to the huge number of Feynman diagram, listed in Table. 1, it is not possible\textsuperscript{7} to compute more than few particles in the final state.

In Fig. 15, we have computed the cross-section as a function of the scalar mass while fixing the center-of-mass energy at 10 TeV. We can clearly see that our approximation becomes better at smaller $m$. While the number of the Feynman diagram grow factorially, our result matches the perturbation calculation.

In Fig. 16, we have repeated the calculation for the cubic theory (no quartic interaction). We again see that the approximation becomes better at smaller mass. However, here we see that for $g = 1/8$, at around $m = 0.01$ GeV, the prediction fails. At this point the ratio $g^2/(4\pi)^2 m^2$ becomes larger than 1, and we cannot trust our prediction at fix order. In Fig. 16b we have shown that this is the case for $g = 1/80$ as well. If we do add the sudakov factors, all the cross-sections will decrease rapidly at these points, as shown in Fig. 17.

\textsuperscript{7}In a reasonable amount of time using a home computer.
6 ○ Conclusion and Discussion

In summary, we have shown that the IR divergences cancel in scalars theories, permitting defining scalar jets. We found that there is only a single logarithm divergence in these theories corresponding to collinear divergent. Hence, we defined scalar jets with the opening angle $\delta$. We argued that for the massive theory, if $\delta$ is smaller than $m/Q$, where $Q$ is the hard scale, we can substitute $\delta$ with $m/Q$ in the Altarelli-Parisi function and consequently in the Sudakov factor.

We used the Sudakov factor to write an equation for the generating function of jet rates. The jet rates are given by equations (58) and (85), for the cubic theory and the quartic theory respectively. For the broken theory, we argued that the generating function is approximately the same as the cubic theory generating function.

The jet cross-sections are given by $\sigma_n = R_n \sigma_{\text{total}}$, where the total cross-section corresponds to fix $Q$ (note that $\sum_n R_n = 1$). We do not know what the total cross-section is, but for the purpose of comparing to the past results, it is enough to know the ratios of the jet cross-sections,

$$\frac{\sigma_{n+1}}{\sigma_n} = \frac{R_{n+1}}{R_n}. \quad (141)$$

Since our formalism is based on highly relativistic particles, however, we are not allowed to compare to the threshold result. If we set

$$Q = nm(1 + \epsilon) \quad (142)$$

where $\epsilon$ is the average kinetic energy of the final particles, the (??), for example, is valid in the limits $\epsilon \gg 1$ and $Q/m \gg 1^8$, while the threshold limit is at $\epsilon \ll 1$.

Nevertheless, we can state few things regarding multi-scalar cross-sections:

- Our result does not support the Holy-grail function. There is no dependence on $n\lambda$, but on $n$ and $\lambda$.

- Our result is in contradiction with the multi-Higgs proposal which states that the cross-section will become unsuppressed at high energy, specially since based on their work, the region where the final Higgs will become semi-relativist ($\epsilon \approx 10$) becomes unsuppressed as well.

---

8 This is especially important in a massive theory since the splittings that lead to production of massive particles can become suppressed
• A jet contains all the particles that can be fitted into the jet cone and jet energy. The fact that even the two jet cross-section is finite as $Q \to \infty$, indicates that the factorial divergence does not exist.

• We have found that while the final particles are relativistic, the cross-sections do not grow factorially. It is thus hard to believe that the dependence of the cross-section on the number of the final particles change as we change the energy of the particles. It is interesting to see why the semi-classical methods sometimes give the factorial divergent as well. It is also interesting to see why in the calculations that sum the leading virtual corrections to all order, the $\log m/Q$ does not show up.

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Loop Corrections to All Order at Threshold for $\phi^3$ Theory in Six Spacetime Dimensions

The one-loop correction to multi-scalar process in the cubic theory can be find using the methods in [6, 10, 11].

Let us define $a_i(n)$ as the amplitude of producing $n$ particle at $i$'th loop order. We have

$$\phi_0(x) = \sum \frac{ia_0(n)x^n}{n!(n^2 - 1)}$$  \hspace{1cm} (143)

$$\phi_1(x) = \sum \frac{ia_1(n)x^n}{n!(n^2 - 1)}$$  \hspace{1cm} (144)

We start by writing the one loop amplitude recursively via Fig. 18. We have

$$\frac{a_1(n)}{n!} = -ig \sum_{n_1,n_2} \frac{ia_1(n_1)ia_0(n_2)}{n_1!(n_1^2 - 1)n_2^2(n_2^2 - 1)}$$

$$-\frac{ig}{2} \int \frac{d^Dk}{(2\pi)^D} \frac{D(n,k)}{n!}$$

$$- iT_2 \frac{ia_0(n)}{n!(n^2 - 1)}$$

$$- iT_3 \frac{ia_0(n_1)ia_0(n_2)}{n_1!(n_1^2 - 1)n_2!(n_2^2 - 1)}$$  \hspace{1cm} (145)

Figure 18: The recursion relation for the one loop corrections.

Where $q = (1,0,0,0)$ and we have set the mass equal to unity in the denominators. The $T_3$ is the mass counter-term and the $T_2$ is the mass and
field counter-terms:

\[-iT_2 = -i \left( m^2 \delta_m + n^2 m^2 \delta_Z \right),\]
\[-iT_3 = -i \delta_g,\]

where from the usual renormalization we know that

\[
\delta_m = -\frac{1}{\epsilon} \frac{g^2}{(4\pi)^2} + \tag{146}
\]
\[
\delta_Z = \frac{1}{6\epsilon} \frac{g^2}{(4\pi)^2} + \tag{147}
\]
\[
\delta_g = -\frac{1}{\epsilon} \frac{g^3}{(4\pi)^3} \tag{148}
\]

The coefficient of the \(\delta_Z\) term is \(p^2 = (nm)^2\). Let us also define,

\[
f(x, k) = \sum \frac{-iD(n, k)x^n}{n!}. \tag{149}
\]

Multiplying (145) by \(x^n\) and summing over \(x\), we find an equation for \(\phi_1\):

\[
\left( x \frac{d}{dx} \left( x \frac{d}{dx} \right) - 1 \right) \phi_1(x) = g\phi_1(x)\phi_0(x) + \frac{g}{2} \int \frac{d^Dk}{(2\pi)^D} f(x, k)
+ m^2 \left[ \delta_m \phi_0(x) + \delta_Z \left( \phi_0(x) + \frac{g}{2} \phi_0^2 \right) \right]
+ \frac{g}{2} \delta_g \phi_0^2 \tag{150}
\]

The recursion relation for the propagator is given in Fig. 19 and is

\[
\frac{D(n, k)}{n!} = -\frac{i}{g} \frac{i}{(k + nq)^2 - 1 + i\epsilon} \sum \frac{ia(n_2)}{n_2!(n_2^2 - 1)} \frac{D(n_1, k)}{n_1!} \tag{151}
\]

Writing \(x = -\frac{12}{g} e^\tau\) and \(f(x, K) = ye^{-\epsilon\tau}\) and \(\epsilon = kq = k_0\) and \(\omega = \sqrt{k^2 + 1 - i\epsilon}\), the equation for \(y\) would be

\[
\left( \frac{d^2}{d\tau^2} - \omega^2 + \frac{3}{\cosh \tau/2} \right) y = e^{\epsilon\tau} \tag{152}
\]
With $u = e^\tau$, the solutions are

$$f_1 = \left(u^{-\omega}(1 + u)^{-3}\right) \left(3 - 27u + 27u^2 - 3u^3 - 11\omega + 27u\omega + 27u^2\omega - 11u^3\omega + 12\omega^2 + 12u\omega^2 - 12u^2\omega^2 - 12u^3\omega^2 - 4\omega^3 - 12u\omega^3 - 12u^2\omega^3 - 4u^3\omega^3\right)$$

$$f_2 = f_1(\omega \to -\omega)$$

The Wronskian is $W = 2\omega(9 - 49\omega^2 + 56\omega^4 - 16\omega^6)$. The solution for $f$ is

$$f(x, K) = \frac{e^{-\epsilon\tau}}{W} \left( f_1 \int d\epsilon f_2 + f_2 \int d\epsilon f_2 \right). \quad (155)$$

The $K_0$ integral in (150) gives a delta function and we find

$$\frac{g}{2} \int \frac{d^Dk}{(2\pi)^D} f(x, k) = \frac{g}{2} \int \frac{d^dK}{(2\pi)^d} \frac{f_1 f_2}{W}, \quad (156)$$

where $d = D - 1$. It is possible to write this expression entirely in terms of $\phi_0$. We have

$$\frac{g}{2} \int \frac{d^dK}{(2\pi)^d} \frac{1}{2\sqrt{1 + K^2}} \left[ \frac{1}{2} - \frac{g\phi_0}{3 + 4K^2} + \frac{5g^2\phi_0^2}{(24 + 32K^2)K^2} + \frac{25g^3\phi_0^2}{(15 + 8K^2 - 16K^4)24K^2} \right] \quad (157)$$

$$\quad + \frac{25g^3\phi_0^2}{(15 + 8K^2 - 16K^4)24K^2} \quad (158)$$

The first integral gives exactly the tadpole contribution. The divergent parts of the second and third terms are

$$\frac{g}{2} \int \frac{d^dK}{(2\pi)^d} \frac{1}{2\sqrt{1 + K^2}} \left[ \frac{1}{6 + 8K^2} \right] = \frac{g^2}{(4\pi)^3} \frac{5}{2\epsilon} + \text{const.} \quad (159)$$

$$\frac{g}{2} \int \frac{d^dK}{(2\pi)^d} \frac{1}{2\sqrt{1 + K^2}} \frac{g^3\phi_0^2}{(24 + 32K^2)K^2} = \frac{g^3}{(4\pi)^3} \frac{5}{6\epsilon} + \text{const.} \quad (160)$$
where \( d = 6 - \epsilon \) and \( A \) and \( B \) are some numerical constant. The divergent of the first integral is canceled by the \( \mathcal{O}(\phi_0) \) part of the counter-terms in (151). And the divergent of the second integral is canceled by the remaining part of \( \delta Z \) and \( \delta g \).

The equation for \( \phi_1 \) bowls down to

\[
\left( x \frac{d}{dx} \left( x \frac{d}{dx} \right) - 1 - g\phi_0 \right) \phi_1(x) = A g^2 \phi_0(x) + B g^3 \phi_0(x)^2 C g^4 \phi_0^3.
\]

(161)

It is customery to set the finite part of the counter-terms such \( A \) and \( B \) are zero [Agyres, Smitt]. We will do the same but we should warn that the solution with these coefficients might lead to \( \phi_1 = \log x + \ldots \) terms that are singular at \( x = 0 \).

With \( A = B = 0 \), we find that

\[
\phi_1(x) = \frac{18g^2 x}{(1 - \frac{g}{12} x)^4}
\]

(162)

We finally find the amplitude to the first order

\[
A_n = \frac{d^n}{dx^n}(\phi_0 + \phi_1)|_{x=0} = n n! \left( \frac{g}{12} \right)^{n-1} \left[ 1 + 3g^2 (n^2 + 3n + 2) \right]
\]

(163)

B \ ᵀ⁴ AP Function for \( \phi^3 \) Theory in Four Spacetime Dimensions

![Figure 20: \( \phi \rightarrow \phi\phi \). Momentums are flowing to the right. “c” is integrated over.](image)

We start by writing the cross-section for production of \( n + 1 \) particles in terms of the cross-section of production of \( n \) particle plus a split particles, labeled \( c \) (Fig. 20), that is radiated from one of the final legs. We have
\[ \sigma_{1\to n+c} = \text{flux factor} \times \frac{1}{n!} \int d\Pi d\Pi^c \ |M_{n+c}|^2 \]  
\[ = \text{flux factor} \times \frac{n}{n!} \int d\Pi d\Pi^c d\Pi_T^2 \mathcal{P}(z, k_T) |M_n|^2 \]  
\[ = \sigma_{1\to n} \int dz dk_T^2 \mathcal{P}_{\phi^3}(z, k_T) \]  
where \( c \) is assumed to be collinear to the leg labeled \( b \). We have defined \( z = E_b/E_c \). The \( \mathcal{P} \) is the splitting function \(^9\). The \( n \) in the second line comes from the fact that the \( n+1 \) phase-spaces decomposes to \( n \) regions, in each region \( c \) is collinear to one of the \( n \) legs, and each region gives the same factor.

We write the momentum as (called Sudakov decomposition \([30]\)),

\[ p_b = z p_a + \beta n + k_T \]  
\[ p_c = (1 - z) p_a - \beta n - k_T \]

so that \( p_a = p_b + p_c \). The vector \( n \) is an arbitrary vector perpendicular to \( k_T \), we choose it to be \((1, 0, 0, -1)\). The phase-space integral for particle \( c \) becomes

\[ d\Pi = \frac{dz dk_T dk_T d\phi d\beta}{(2\pi)^4} \times J(= E_a + p_{a3}) \times (2\pi) \delta(p_c^2 - m^2) \]

\[ = \frac{dz dk_T^2}{4(2\pi)^2(1 - z)} \]

where in the second line we have integrated over \( \beta \) and \( \phi \). We further have

\[ p_a^2 = p_b^2 + p_c^2 + 2p_b p_c = \frac{k_i^2}{z(1 - z)} + \frac{p_b^2}{(1 - z)} + \frac{p_c^2}{z}, \quad k_T, p_{b,c}^2 \ll E_{b,c} \]

Hence,

\[ \mathcal{P}_{\phi^3} = \frac{g^2}{4(2\pi)^2} \frac{1}{z(1 - z)} \frac{1}{(p_a^2 - m^2)^2}. \]

The extra \( 1/z \) in the first fraction comes from changing the phase-space factor.

\(^9\text{Contrary to what is usual, we have moved the } k_T \text{ dependence into the definition of } \mathcal{P} \text{ since in the case of massive particles it does not factor out.}\)
of particle $b$ to that of particle $a$ (Fig. 20). Using (171) we find

$$\mathcal{P}_{\phi^4} = \frac{g^2}{16\pi^2} \frac{z(1-z)}{\left(k_T^2 + z p_b^2 + (1-z)p_c^2 - z(1-z)m^2 \right)^2}$$

(173)

$$= \frac{g^2}{16\pi^2} \frac{z(1-z)}{\left(k_T^2 + m^2(1-z + z^2) \right)^2}$$

(174)

$$= \frac{g^2}{16\pi^2} \frac{z(1-z)}{\left(k_T^2 + m^2 \beta^2 \right)^2}$$

(175)

C ○ AP Function for $\phi^4$ Theory in Four Spacetime Dimensions

The cross-section is

$$\sigma_{1\rightarrow n+c+d} = \text{flux factor} \times \int d\Pi^f d\Pi^c d\Pi^d \ |M_{n+c+d}|^2$$

(176)

$$= \text{flux factor} \times \int d\Pi^f dz dx dk_{Tc}^2 dk_{Td}^2 \mathcal{P}_{\phi^4}(z, x, k_{Tc}, k_{Td}) |M_n|^2$$

(177)

$$= \sigma_{1\rightarrow n} \int dz dx dk_{Tc}^2 dk_{Td}^2 \mathcal{P}_{\phi^4}(z, x, k_{Tc}, k_{Td})$$

(178)

where $z = E_c/E_a$ and $x = E_d/E_a$. The Sudakov decomposition of momentum is given by

$$p_b = (1 - z - x)p_a + (\beta + \alpha)n + K_{Tc} - K_{Td}$$

(179)

$$p_c = zp_a - \alpha n - K_{Tc}$$

(180)

$$p_d = xp_a - \beta n + K_{Td}$$

(181)

so that $p_a = p_b + p_c + p_d$. The vector $n$ is chosen such that it is perpendicular to $K_{Tc,d}$ and $n^2 = 0$. Choosing it to be $(1,0,0,-1)$, gives $n.p_a \approx 2E_a$, assuming that $p_a$ is highly boosted. The phase-space integrals become

$$d\Pi^b d\Pi^c d\Pi^d = d\Pi^a \frac{dz dz' dk_{Tc}^2 dk_{Td}^2 d\phi_c d\phi_d}{16(2\pi)^6 E_a^2 z x (1 - z - x)}$$

(182)

We can find $\beta$ and $\alpha$ by imposing on-shell condition for particles $c$ and $d$. Assuming that all the particles are massless, these conditions give

$$2\alpha n.p_a = zp_a^2 - K_{Tc}^2/z \quad \text{and} \quad 2\beta n.p_a = xp_a^2 - K_{Td}^2/x$$

(183)
Using these equations and on-shellness of the final particles, we arrive at

\[ p_a^2 = \frac{1}{zx(1-z-x)} \left[ z(1-z)k^2_{T_c} + x(1-x)k^2_{T_d} - 2zxk_{T_c}k_{T_d} \cos \phi \right] \quad (184) \]

where \( \phi \) is the angle between \( K_{T_c} \) and \( K_{T_d} \). Noting that \( M_{n+c+d} = \frac{\lambda}{p_a^2} M_\alpha \), and furthermore integrating over the azimuthal angels, we find the splitting function to be

\[
P_{\phi^4}(z, x, k_{T_c}, k_{T_d}) = \frac{\lambda^2}{16(2\pi)^3} \frac{zx(1-z-x)[z(1-z)k^2_{T_c} + x(1-x)k^2_{T_d}]}{\left[ (z(1-z)k^2_{T_c} + x(1-x)k^2_{T_d})^2 - 4zx^2k^2_{T_c}k^2_{T_d} \right]^{3/2}} \quad (185)\]
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