UNROLLED QUANTUM $\mathfrak{sl}_2$ AND THE MULTI-VARIABLE ALEXANDER POLYNOMIAL

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ABSTRACT. J. Murakami showed that the multi-variable Alexander polynomial can be obtained from a link by a modified Turaev type construction. Using Jiang’s characterization of the Conway Potential Function, we work directly with skein relations to give a concise proof that the multi-variable Alexander polynomial is a quantum invariant of unrolled restricted quantum $\mathfrak{sl}_2$ at a primitive fourth root of unity.

1. INTRODUCTION

The multi-variable Alexander polynomial, as an invariant of oriented knots and links with labeling, is defined up to a unit and can be obtained through a combinatorial algorithm given by Alexander [Ale28]. The Conway Potential Function (CPF) defines a standard representative of the multi-variable Alexander polynomial. J. Murakami shows in [Mur93] that the CPF can be obtained from a Turaev-type state model [Tur88]. In [Mur92], it is shown that the $R$-matrix used in this state model coincides with the $R$-matrix of (unrolled) restricted quantum $\mathfrak{sl}_2$ at a primitive fourth root of unity for a particular family of representations. Using topological arguments, Jiang [Jia16] proves the CPF of colored links is determined uniquely by the five skein relations seen in Figure 1 below. The goal of this paper is to give a concise proof that the CPF is a quantum $\mathfrak{sl}_2$ invariant by working directly with the skein relations.

Let $\zeta$ be a primitive fourth root of unity. Denote the unrolled restricted quantum group at $q = \zeta$ by $\mathcal{U}^\zeta_H(\mathfrak{sl}_2)$. Let $V(t)$ be the one-parameter family of two dimensional highest-weight representations of $\mathcal{U}^\zeta_H(\mathfrak{sl}_2)$. The tensor category generated by these representations are the objects of a colored braid groupoid. The braid group action on $\bigotimes_{i=1}^n V(t_i)$ is determined by the quantum group $R$-matrix. Thus, each braid $b \in B_n$ determines an isomorphism

$$
\bigotimes_{i=1}^n V(t_i) \xrightarrow{\sim} \bigotimes_{i=1}^n V(t_{\pi(i)}),
$$

with $\pi$ the image of $b$ in the symmetric group $S_n$ given by the underlying permutation. Thus, given a braid representative of a link whose components are labeled by $t_i$, we have an isomorphism of quantum group representations. A polynomial invariant is given by taking quantum partial trace over $n-1$ tensor factors. We prove that this invariant is the CPF up to a factor of $\pm(t_i - t_i^{-1})^{-1}$ for some $i$. In other words, we have the following theorem.

**Theorem 1.1.** The link invariant determined by the representations $V(t)$ of $\mathcal{U}^\zeta_H(\mathfrak{sl}_2)$ is the Conway Potential Function.
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2. Unrolled Restricted Quantum $\mathfrak{sl}_2$ and Quantum Invariants

We begin by recalling the unrolled restricted quantum group $\mathcal{U}_\zeta^H(\mathfrak{sl}_2)$ at a primitive fourth root of unity $\zeta$, and refer the reader to [Oht02] for additional details. We will assume $\zeta = e^{i\pi/2}$, but its conjugate may be used just as well.

**Definition 2.1.** Let $\mathcal{U}_\zeta^H(\mathfrak{sl}_2)$ be the $\mathbb{C}$-algebra generated by $E, F, K, K^{-1}, H$ under the relations:

\[
KK^{-1} = K^{-1}K = 1 \quad KE = -EK \quad KF = -FK \quad [E, F] = \frac{K - K^{-1}}{\zeta - \zeta^{-1}} \quad (2)
\]

\[
HK = KH \quad [H, E] = 2E \quad [H, F] = -2F \quad E^2 = F^2 = 0. \quad (3)
\]

**Definition 2.2.** Fix $t \in \mathbb{C}^\times$. Let $(\rho_t, V(t))$ be the two dimensional representation of $\mathcal{U}_\zeta^H(\mathfrak{sl}_2)$, expressed in the standard basis $\langle v_0, Fv_0 \rangle$ as:

\[
\rho_t(F) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \rho_t(E) = \frac{t - t^{-1}}{\zeta - \zeta^{-1}} \begin{bmatrix} 0 & 1 \\ \zeta & 0 \end{bmatrix}, \quad \rho_t(K) = \begin{bmatrix} t & 0 \\ 0 & -t \end{bmatrix}, \quad \rho_t(H) = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha - 2 \end{bmatrix} \quad (4)
\]

for some $\alpha$ such that $t = \zeta^\alpha = e^{\pi \alpha/2}$.

The tensor category generated by the representations $\{V(t) : t \in \mathbb{C}^\times\}$ are the objects of a colored braid groupoid. The braiding, which is defined in terms of $H$, can be normalized to only depend on $\zeta^\alpha$. Therefore, the precise value of $\alpha$ will not be important for our discussion.

To assign a linear map to a tangle, we use the convention that an upward pointing strand is the identity on $V(t)$ and a downward pointing strand is the identity on $V(t)^*$. Recall evaluation and coevaluation on the tensor products, which allow us to define a quantum partial trace on representations. We have the following duality maps on $V(t)$ associated to oriented “cups” and “caps”:

\[
\begin{aligned}
\bigcirc \sim ev_t \in Hom(V(t)^* \otimes V(t), \mathbb{C}) & \quad \bigtriangleup \sim coev_t \in Hom(\mathbb{C}, V(t) \otimes V(t)^*) \\
\bigtriangledown \sim e\overline{v}_t \in Hom(V(t) \otimes V(t)^*, \mathbb{C}) & \quad \bigtriangleup \sim \overline{coev}_t \in Hom(\mathbb{C}, V(t)^* \otimes V(t)).
\end{aligned}
\]

These maps satisfy the duality (zig-zag) relations

\[
(id_{V(t)} \otimes ev_t)(coev_t \otimes id_{V(t)}) = id_{V(t)} = (e\overline{v}_t \otimes id_{V(t)})(id_{V(t)} \otimes \overline{coev}_t) \quad (7)
\]

and

\[
(ev_t \otimes id_{V(t)^*})(id_{V(t)^*} \otimes coev_t) = id_{V(t)^*} = (id_{V(t)^*} \otimes e\overline{v}_t)(\overline{coev}_t \otimes id_{V(t)^*}). \quad (8)
\]

Given any basis $(e_i)$ of $V(t)$ and a corresponding dual basis $(e_i^*)$, the above maps are defined as

\[
\begin{aligned}
\begin{aligned}
ev_t(e_i^* \otimes e_j) &= e_i^*(e_j), \\
coev_t(1) &= \sum_i e_i \otimes e_i^*.
\end{aligned}
\end{aligned}
\]

\[
\begin{aligned}
\begin{aligned}
e\overline{v}_t(e_i \otimes e_j^*) &= e_j^*(K^{-1}e_i), \\
\overline{coev}_t(1) &= \sum_i e_i^* \otimes Ke_i.
\end{aligned}
\end{aligned}
\]

\[
\begin{aligned}
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ev_t(e_i^* \otimes e_j) &= e_i^*(e_j), \\
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e\overline{v}_t(e_i \otimes e_j^*) &= e_j^*(K^{-1}e_i), \\
\overline{coev}_t(1) &= \sum_i e_i^* \otimes Ke_i.
\end{aligned}
\end{aligned}
\]
and do not depend on the choice of basis. The standard basis of \( V(t) \) extends to a basis of the tensor product \( V(t) \otimes V(s) \):

\[
\langle v_0 \otimes v_0, v_0 \otimes Fv_0, Fv_0 \otimes v_0, Fv_0 \otimes Fv_0 \rangle.
\]  

(11)

A braiding on tensor representations \( V(t) \otimes V(s) \) is given by a linear map

\[
R : V(t) \otimes V(s) \to V(s) \otimes V(t)
\]  

(12)

called the \( R \)-matrix, which corresponds to right-over-left crossings. In [Oht02], the \( R \)-matrix is given by the following equation,

\[
R = \sigma \circ \rho_t \otimes \rho_s \left( \zeta^{H \otimes H/2}(1 \otimes 1 + 2\zeta E \otimes F) \right).
\]  

(13)

Here, \( \sigma \) is the tensor swap and

\[
\zeta^{H \otimes H/2}(w_1 \otimes w_2) = \zeta^{\lambda_1 \lambda_2/2}(w_1 \otimes w_2)
\]  

(14)

for \( H \) weight vectors \( w_1 \) and \( w_2 \) of weight \( \lambda_1 \) and \( \lambda_2 \). In the standard basis,

\[
R = \begin{bmatrix}
\zeta^{\alpha \beta/2} & 0 & 0 & 0 \\
0 & 0 & \zeta^{(a-2)\beta/2} & 0 \\
0 & \zeta^{a(\beta-2)/2} & \zeta^{a(\beta-2)/2}(t - t^{-1}) & 0 \\
0 & 0 & 0 & \zeta^{(a-2)(\beta-2)/2}
\end{bmatrix}
\]  

(15)

\[
= \zeta^{a(\beta-2)/2} \begin{bmatrix}
\zeta^{\alpha} & 0 & 0 & 0 \\
0 & 0 & \zeta^{\alpha - \beta} & 0 \\
0 & 1 & (t - t^{-1}) & 0 \\
0 & 0 & 0 & -\zeta^{-\beta}
\end{bmatrix}
\]  

(16)

\[
= \zeta^{a(\beta-2)/2} \begin{bmatrix}
t & 0 & 0 & 0 \\
0 & 0 & ts^{-1} & 0 \\
0 & 1 & t^{-1} & 0 \\
0 & 0 & 0 & -s^{-1}
\end{bmatrix}
\]  

(17)

We define \( R_t \) to be \( \zeta^{-a(\beta-2)/2}R_t \), a braiding on \( V(t) \otimes V(s) \), which depends only on \( t \) and \( s \). The matrix presented here is equivalent to the one given in [Oht02]. There it is proven that we obtain an invariant of 1-tangles by associating each of them to an endomorphism of \( V(t) \). Given a link \( L \) we label each component by a formal parameter \( t_i \). Let \( T_i \) be a 1-tangle such that the closure of \( T_i \) is \( L \) and all but the \( i \)-th component of \( L \) is closed in \( T_i \). The endomorphism of \( V(t_i) \) associated to \( T_i \) is of the form \( c_i \cdot id_{V(t_i)} \) for some \( c_i \in \mathbb{C}(t) \). If \( L \) has multiple components, then a choice was made in selecting \( T_i \) as a 1-tangle representative for \( L \). It can be shown that \( c_i/(t_i - t_i^{-1}) \) is an invariant of multi-component links, independent of the choice of 1-tangle representative. We claim that this invariant determines the multi-variable Alexander Polynomial.

3. Proof of Theorem 1.1

As noted earlier we use the bottom-to-top convention for assigning maps on oriented tangles. This convention is the opposite from the one used in [Jia16, Mur93, Oht02]. The orientations of all arrows are reversed, but yield the same results. In this section, we verify each equation in Figure 1 holds. This will prove Theorem 1.1.
Proof of (II). Relation (II) is a generalization of the single variable Alexander polynomial skein relation, which relates the crossing maps to the identity map. Jiang characterizes both the (II) and (III) relations in the context of braids, and so it is enough to prove the equalities only for the diagrams shown. Relation (II) translates into the equation
\[
R_{st} R_{ts} + (R_{st} R_{ts})^{-1} = \left( ts + \frac{1}{ts} \right) \cdot id_{V(t)} \otimes id_{V(s)}.
\] (18)

Then as
\[
R_{st} R_{ts} = \begin{bmatrix} st & 0 & 0 & 0 \\ 0 & s & s - \frac{s}{t^2} & 0 \\ 0 & s - \frac{1}{s} + \frac{1}{st} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{st} \end{bmatrix}
\]
and \((R_{st} R_{ts})^{-1}\) determines the map
\[
\begin{bmatrix} 1 & 0 & 0 & 0 \\ st & 0 & 0 & 0 \\ 0 & st - \frac{s}{t} + \frac{1}{st} & s - \frac{s}{t^2} & 0 \\ 0 & \frac{1}{s} - s & \frac{s}{t} & 0 \end{bmatrix},
\]
the relation is seen to hold. □

Proof of (III). Relation (III) is the most complicated relation to check, which is a condition on linear maps
\[
M_1, M_2, M_3 : V(t_i) \otimes V(t_j) \otimes V(t_k) \to V(t_i) \otimes V(t_j) \otimes V(t_k).
\] (19)

We describe each of the three terms and their relation here:
\[
M_1 = (R_{st} \otimes id_{V(u)}) (id_{V(s)} \otimes R_{ut}) (R^{-1}_{su} \otimes id_{V(t)}) + (R^{-1}_{ts} \otimes id_{V(u)}) (id_{V(s)} \otimes R_{tu}^{-1}) (R_{su} \otimes id_{V(t)})
\]
\[
M_2 = (R^{-1}_{ts} \otimes id_{V(u)}) (id_{V(s)} \otimes R_{ut}) (R_{us} \otimes id_{V(t)}) + (R_{st} \otimes id_{V(u)}) (id_{V(s)} \otimes R_{tu}^{-1}) (R^{-1}_{su} \otimes id_{V(t)})
\]
\[
M_3 = (R_{st} \otimes id_{V(u)}) (id_{V(s)} \otimes R_{ut}) (R_{us} \otimes id_{V(t)}) + (R^{-1}_{ts} \otimes id_{V(u)}) (id_{V(s)} \otimes R_{tu}^{-1}) (R^{-1}_{su} \otimes id_{V(t)})
\]
\[
\left( \frac{1}{ts} - ts \right) M_1 + \left( s u - \frac{1}{s u} \right) M_2 + \left( \frac{t}{u} - \frac{u}{t} \right) M_3 = 0.
\] (20)

The matrices \(M_1, M_2, \) and \(M_3\) are given below. It follows from a calculation that they satisfy equation (20). □

Proof of (IO). The map defined by the circle colored by the representation \(V(t)\) is the quantum dimension of \(V(t)\). Here,
\[
e v \circ c o e v = \tilde{e} v \circ c o e v = 0.
\] (21)

Therefore, this invariant assigns a link which has a disjoint unknot the value zero. For the same reason, the endomorphism determined by any closed tangle is associated with the zero map. □

Proof of (Φ). As a \(V(t)\) endomorphism, equation (Φ) determines the map
\[
(id_{V(t)} \otimes \tilde{e} v_{V(s)}) \cdot ((R_{st} R_{ts})^{-1} \otimes id_{V(s)}) \cdot (id_{V(t)} \otimes c o e v_{V(s)}) = (t - t^{-1}) id_{V(t)}.
\] (22)

Proof of (H). Cutting either string in the Hopf link produces the 1-tangle seen in (Φ). Thus, the \(V(t_i)\) endomorphism associated to the Hopf link having cut the component is \((t_i - t_i^{-1}) id_{V(t_i)}\). Since a color was chosen, we must divide by \(t_i - t_i^{-1}\), and so the multi-variable Alexander invariant is 1. This proves the theorem. □
\( \nabla \left( \begin{array}{cc} i & j \\ \hline & \hline \end{array} \right) + \nabla \left( \begin{array}{cc} i & j \\ \hline & \hline \end{array} \right) = (t_it_j + t_j^{-1}t_i^{-1})\nabla \left( \begin{array}{cc} i & j \\ \hline & \hline \end{array} \right) \)  \hspace{1cm} (II)

\[
(t_i^{-1}t_j^{-1} - t_it_j) \left\{ \nabla \left( \begin{array}{cc} i & j & k \\ \hline & \hline & \hline \end{array} \right) + \nabla \left( \begin{array}{cc} i & j & k \\ \hline & \hline & \hline \end{array} \right) \right\} + (t_j^{-1}t_k^{-1} - t_jt_k) \left\{ \nabla \left( \begin{array}{cc} i & j & k \\ \hline & \hline & \hline \end{array} \right) + \nabla \left( \begin{array}{cc} i & j & k \\ \hline & \hline & \hline \end{array} \right) \right\} = 0 \quad (III)
\]

\[
\nabla \left( \begin{array}{cc} i & i \\ \hline & \hline \end{array} \right) = 0 \quad (IO)
\]

\[
\nabla \left( \begin{array}{cc} j & i \\ \hline & \hline \end{array} \right) = (t_i - t_i^{-1})\nabla \left( \begin{array}{cc} i & i \\ \hline & \hline \end{array} \right) \quad (\Phi)
\]

\[
\nabla \left( \begin{array}{cc} i & j \\ \hline & \hline \end{array} \right) = 1 \quad (H)
\]

Figure 1. Jiang’s CPF Skein Relations. The labels \( i, j, k \) in the diagrams are used to denote \( t_i, t_j, t_k \).

\[
M_1 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{u^2}{t^3} - \frac{u^2}{t} - \frac{su^2}{st} & \frac{u^2}{t} - \frac{su^2}{st} & 0 & \frac{su}{t} + \frac{u}{st} & 0 & 0 & 0 & 0 \\
0 & \frac{su^2}{t^3} - \frac{su^2}{t} & \frac{u^2}{t} + \frac{1}{t} & 0 & \frac{u}{t} - \frac{1}{tu} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & u \frac{1}{t^2} - u & 0 & 1 - \frac{1}{t^2} & -\frac{su^2}{t^2} - \frac{1}{s} & 0 & 0 \\
0 & \frac{1}{s} + \frac{su^2}{t^2} & 1 - \frac{1}{s^2} & 0 & u - \frac{1}{u} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & u \frac{1}{t} - u \frac{1}{ts^2} & 0 & -\frac{1}{t} - \frac{u^2}{t} & \frac{u^2}{st} - \frac{su^2}{t} & 0 & 0 \\
0 & 0 & 0 & -\frac{u}{st} - \frac{su}{t} & 0 & -\frac{su^2}{t} + \frac{s}{t} + \frac{u^2}{t} - \frac{1}{t} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{u}{t^2} - u
\end{bmatrix}
\]
\[
M_2 = \begin{bmatrix}
\frac{u^2}{t} + \frac{1}{t} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{u}{t^2} - u & \frac{u}{s} - su & 0 & \frac{u^2}{t^2} - s & 0 & 0 & 0 & 0 \\
0 & \frac{su}{t^2} - su & u + \frac{u}{t^2} & 0 & \frac{u^2}{t^2} - t^{-2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{u^2}{t^3} - \frac{u^2}{t} & 0 & \frac{u}{t} - \frac{u}{t^3} - \frac{u}{st} - \frac{su}{t} & 0 & 0 & 0 \\
0 & \frac{su}{t} + \frac{u}{st} - \frac{u}{t s^2} & 0 & \frac{u^2}{t} - 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{u^2}{t^2} - \frac{u^2}{s^2 t^2} & 0 & \frac{u}{t^2} - u & \frac{u}{s} - su & 0 & 0 \\
0 & 0 & 0 & -s - \frac{u^2}{t^2 s} & 0 & \frac{s}{u} - su & u - 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{t} - \frac{u^2}{t^3}
\end{bmatrix}
\]

\[
M_3 = \begin{bmatrix}
\frac{su^2}{t^2} + \frac{1}{t^2 s} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{u}{t^3 s} - \frac{u}{st} \frac{u}{s^2} - \frac{u}{t} & 0 & \frac{u^2}{t} + \frac{1}{t} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{u}{t^3} - \frac{u}{t} \frac{su}{t} + \frac{u}{st} & 0 & \frac{su^2}{t} - s & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{su^2}{t^2} - \frac{su^2}{t} & 0 & \frac{su}{t^2} - \frac{su}{t^2} & -\frac{u}{t^2} - u & 0 & 0 \\
0 & u + \frac{u}{t^2} & su - \frac{u}{s} & 0 & su^2 - s & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{su^2}{t} - \frac{u^2}{st} & 0 & \frac{u}{st} - \frac{su}{t} \frac{u}{ts^2} - \frac{u}{t} & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{1}{t} - \frac{u^2}{t} & 0 & \frac{1}{tu} - \frac{u}{t} \frac{u}{st} - \frac{1}{tus} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{t^2 s} - su^2
\end{bmatrix}
\]

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