ON SPINORS, STRINGS, INTEGRABLE MODELS AND DECOMPOSED YANG-MILLS THEORY

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This paper deals with various interrelations between strings and surfaces in three dimensional ambient space, two dimensional integrable models and two dimensional and four dimensional decomposed SU(2) Yang-Mills theories. Initially, a spinor version of the Frenet equation is introduced in order to describe the differential geometry of static three dimensional string-like structures. Then its relation to the structure of the su(2) Lie algebra valued Maurer-Cartan one-form is presented; while by introducing time evolution of the string a Lax pair is obtained, as an integrability condition. In addition, it is show how the Lax pair of the integrable nonlinear Schrödinger equation becomes embedded into the Lax pair of the time extended spinor Frenet equation and it is described how a spinor based projection operator formalism can be used to construct the conserved quantities, in the case of the nonlinear Schrödinger equation. Then the Lax pair structure of the time extended spinor Frenet equation is related to properties of flat connections in a two dimensional decomposed SU(2) Yang-Mills theory. In addition, the connection between the decomposed Yang-Mills and the Gauß-Codazzi equation that describes surfaces in three dimensional ambient space is presented. In that context the relation between isothermic surfaces and integrable models is discussed. Finally, the utility of the Cartan approach to differential geometry is considered. In particular, the similarities between the Cartan formalism and the structure of both two dimensional and four dimensional decomposed SU(2) Yang-Mills theories are discussed, while the description of two dimensional integrable models as embedded structures in the four dimensional decomposed SU(2) Yang-Mills theory are presented.

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I. INTRODUCTION

The immersion of a string in the three dimensional Euclidean space \( \mathbb{R}^3 \) is a classic subject in differential geometry \( [2, 4] \). Strings have no intrinsic geometry. They differ from each other only in the way they twist and bend in the ambient space. The Frenet equation constructs a string in \( \mathbb{R}^3 \) entirely from the knowledge of this extrinsic geometry. The solution describes a dreibein field i.e. a SO(3) matrix transports along the string, in terms of its local curvature and torsion; one of the dreibein fields is tangent to the string, while the other two constitute a zweibein on the normal planes of the string.

The embedding of a two dimensional Riemann surface in \( \mathbb{R}^3 \) is an equally classic subject. It is governed by the Gauß-Codazzi equation \( [2, 4] \). This equation relates closely to the concept of integrability. Recall that, in the case of an embedded pseudosphere \( [3] \), the integrable sine-Gordon equation first appeared as a decomposition of the Gauß-Codazzi equation. Various other models with integrable dynamics have been subsequently investigated from this perspective \( [2, 4] \). Of contemporary interest are different decomposed versions of the Gauß-Codazzi equation that describe isothermic surfaces, and their relations to known integrable models in two dimensions \( [4, 10-14] \). The connection between integrable models and loop groups should also be mentioned \( [13] \).

However, most Riemann surfaces are neither pseudospheres nor isothermic manifolds. In the general case, the description of a Riemann surface embedded in \( \mathbb{R}^3 \) continues to remain beyond the realm of the traditional theory of integrable systems.

Here we are interested in the generic time evolution of a string in three dimensions. Its time evolution sweeps a Riemann surface that is embedded in \( \mathbb{R}^3 \). Thus the Hamiltonian dynamics should fundamentally relate to the Gauß-Codazzi equation. In fact, the Frenet equation that describes the string at a fixed time, corresponds to an auxiliary linear problem in the theory of integrable models. Moreover, both the nonlinear Schrödinger equation and the modified KdV equation have been extensively studied, in connection of the motion of vortex filaments and other regular string-like structures in three space dimensions \( [15-22] \).

Even thought not explicitly addressed here, among the motivations of the present work is to develop a Hamil-
tonian dynamics that describes proteins modeled as discrete piecewise linear polygonal chains. A biologically active protein is in a space filling collapsed phase i.e. it’s geometry is fractal. Thus standard techniques of embedding theory do not apply. Novel techniques need to be developed. The present article is a step towards the pertinent formalism, it addresses regular differentiable strings with no fractal affiliation, but using formalism that hopefully enables the description of dynamical fractal string-like objects [24]-[26].

Universality arguments allege that only the Polyakov action [27] should remain relevant in the high energy limit of a (relativistic) string. However, at lower energies the corrections can not be ignored. Such an example is the extrinsic curvature term [28]. Here we shall be interested in systematic inclusion of such additional corrections, that describe the bending and twisting motions of the string. In particular, the relativistic case and the low energy case is considered when the dynamics becomes subject to the Galilean invariance [29]. This is because the corresponding energy functions have many applications to Physics. Examples include vortex dynamics in superconductors; fluids and cosmic strings [30]-[38]; polymers, proteins, and their folding dynamics [23]-[26].

Moreover, beyond Physics, the Frenet equation has numerous applications. Examples include robotics, computer graphics and virtual reality, aeronautics and astronautics [39], [40]. In these applications the “gimbal lock”, a coordinate singularity in Euler angles, often appears as a nuisance that needs to be overcome. For this, in lieu of the dreibein Frenet frame description, a quaternionic formulation is commonly preferred to describe how rotation matrices become transported along strings and trajectories.

Similarly, the isothermal surfaces and, more generally, isothermal (conformal) coordinate representations of embedded Riemann surfaces are studied extensively; also, from the point of view of three dimensional visualization [11]. There is a wide range of applications such as: structural mechanics, architecture, industrial design and so forth [11]. In this setting, a quaternionic representation of rotations is similarly often advantageous [12].

In this paper a different approach is introduced and develop in order to describe strings, their time evolution, and the two dimensional Riemann surfaces that are swept by this time evolution. Instead of the classic dreibein or the more flexible quaternionic realization of the Frenet equation, a string in $\mathbb{R}^3$ is represented by a two component complex spinor. Then the dreibein Frenet equation becomes a two component spinor Frenet equation describing the dynamics of a spinor along the string. This spinor-based representation of a string has various conceptual and technical advantages over both the conventional dreibein approach and its quaternionic modernization. Note that despite its simplicity and apparent advantages, a spinor representation of the Frenet equation has been studied only sparsely [42], [43].

As an example of the spinor representation of the Frenet equation we show the relation between the time evolution of the spinor (that describes a string) to the integrable hierarchy of the nonlinear Schrödinger equation [44]. In particular, it is explained how a Lax pair that emerges as a consistency condition of the time dependent extension of the spinor Frenet equation, can be chosen so that it coincides with the $\text{su}(2)$ Lie algebra valued Lax pair of the NLSE. That way the conserved charges of the NLSE hierarchy can be taken as Hamiltonians, governing the time evolution and computing the energy of a string. Then the spinor description into a projection operator formalism is presented. This kind of operator formalism has been previous utilized extensively, to analyze the integrable $\mathbb{CP}^N$ models [45]-[47]. Finally, it is shown how the conserved charges of the NLSE hierarchy appear from this formalism.

Next it is shown that the Lax pair emerging from the spinor description relates in a natural fashion to the flat connection of two dimensional SU(2) Yang-Mills theory. Then this relation is studied in terms of a decomposed description of the Yang-Mills theory [48], [49]. In addition, the spinor representation of the Gauß-Godazzi equation and its relation with the decomposed Yang-Mills theory is presented. Finally, the occurrence of the NLSE equation and the Liouville equation when the metric tensor and the second fundamental form are decomposed in a manner that parallels the decomposition of the Yang-Mills connection, is presented.

We conclude by introducing the Cartan geometry [50]-[52]. This provides a natural framework for combining the concept of integrability with strings, two dimensional surfaces, and Yang-Mills theories. In particular, a decomposed representation of the four dimensional SU(2) Yang-Mills theory [54], [55] is investigated and is shown that it is a universal theory that governs various two dimensional integrable models as embedded structures. In particular, in terms of several examples, is is shown that two dimensional integrable models can be embedded in the structure of the decomposed D=4 Yang-Mills.

II. CLASSIC FRENET EQUATION

A. The equation

We start with a review of the classic Frenet equation [1], [2] that describes the geometry of a class $C^3$ differentiable string $\mathbf{x}(z)$ in $\mathbb{R}^3$. The parameter $z \in [0, L]$ is generic and $L$ is the length of the string in $\mathbb{R}^3$ defined by

$$L = \int_0^L dz \sqrt{\mathbf{x}'_z \cdot \mathbf{x}'_z} \equiv \int_0^L dz \sqrt{g}.$$

Note that this is the time independent part of the Nambu-Goto action. Let us assume, for simplicity, that there are no inflection points i.e. points where the curvature of the string vanishes. An inflection point is not
generic, a simple inflection point can always be removed by a small generic deformation of the string.

The string can then be globally framed as follows: The unit length tangent vector

\[ \mathbf{t} = \frac{1}{\sqrt{g}} \frac{d\mathbf{x}(z)}{dz} = \frac{1}{||x_z||} x_z \]  

(1)
is orthogonal to the unit length bi-normal vector

\[ \mathbf{b} = \frac{x_z \times x_{zzz}}{||x_z \times x_{zzz}||} \]

while the unit length normal vector is given by

\[ \mathbf{n} = \mathbf{b} \times \mathbf{t}. \]

Then, the three vectors \((\mathbf{n}, \mathbf{b}, \mathbf{t})\) form the right-handed orthonormal Frenet frame, at each point of the string.

The Frenet equation relates the frames at different points along the string \([1], [2]\). Explicitly, it is of the form

\[ \frac{d}{dz} \begin{pmatrix} \mathbf{n} \\ \mathbf{b} \\ \mathbf{t} \end{pmatrix} = \sqrt{g} \begin{pmatrix} 0 & \tau & -\kappa \\ -\tau & 0 & 0 \\ \kappa & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{n} \\ \mathbf{b} \\ \mathbf{t} \end{pmatrix} \]  

(2)

where

\[ \kappa(z) = \frac{||x_z \times x_{zz}||}{||x_z||^3} \]  

(3)
is the curvature of the string on the osculating plane that is spanned by \(\mathbf{t}\) and \(\mathbf{n}\), and

\[ \tau(z) = \frac{(x_z \times x_{zz}) \cdot x_{zzz}}{||x_z \times x_{zzz}||^2} \]  

(4)
is the torsion of the string.

In this case, if the local scale factor \(\sqrt{g}\) is determined, and the curvature and the torsion are known, the frames can be constructed by solving (2). In addition, the string can be constructed by solving (1). Note that, this solution is unique up to rigid translations and rotations of the string.

Thus, the Frenet equation construct reparametrization invariant energy functions in terms of the curvature and the torsion of the string.

In the following let us denote by \(s \in [0, L]\) the arc-length parameter, while \(z\) denotes a generic parametrization. The arc-length parameter \(s\) measures the length along the string in terms of the distance scale of the three dimensional ambient space \(\mathbb{R}^3\). The change of variables from a generic parameter \(z\) to the arc-length parameter \(s\) is

\[ s(z) = \int_0^z ||x_\zeta(\tilde{z})|| d\tilde{z}. \]

Accordingly, we consider the effects of infinitesimal local diffeomorphisms along the string, obtained by deforming \(s\) as follows

\[ s \rightarrow z = s + \epsilon(s). \]  

(5)

Here \(\epsilon(s)\) is an arbitrary infinitesimally small function such that

\[ \epsilon(0) = \epsilon(L) = 0 = \epsilon_z(0) = \epsilon_z(L). \]

The Lie algebra of diffeomorphisms \([1]\) of a line segment in \(\mathbb{R}^1\) is the classical Virasoro (Witt) algebra.

Next we define a function \(f(s)\) with support on the string to have a weight \(h\) akin to the conformal weight, if \(f(s)\) transforms according to

\[ \delta f(s) = -\left( \frac{d}{ds} + h \epsilon_s \right) f(s) \]  

(6)

under the infinitesimal diffeomorphism \([4]\). Since the three dimensional geometric shape of the string in \(\mathbb{R}^3\) does not depend on the way how it has been parametrized, the embedding \(x(z)\) transforms as a scalar \(i.e.\) it has weight \(h = 0\) under reparametrizations. Similarly, the curvature \([3]\) and the torsion \([4]\) are scalars under reparametrizations. Infinitesimally,

\[ \delta \kappa(s) = -\epsilon(s) \frac{dk}{ds} = -\epsilon \kappa_s, \]

\[ \delta \tau(s) = -\epsilon(s) \frac{d\tau}{ds} = -\epsilon \tau_s. \]

For the arc-length parameter given by

\[ ||x_\zeta|| = ||x'|| = 1 \]

the Frenet equation becomes \([1], [3], [39], [40]\)

\[ \frac{d}{ds} \begin{pmatrix} \mathbf{n} \\ \mathbf{b} \\ \mathbf{t} \end{pmatrix} = \begin{pmatrix} 0 & \tau & -\kappa \\ -\tau & 0 & 0 \\ \kappa & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{n} \\ \mathbf{b} \\ \mathbf{t} \end{pmatrix}. \]  

(7)

Note that the framing of the string by the Frenet dreibein is not unique, \(i.e.\) there are many ways to frame a string \([56], [39], [40]\).

Instead of the zweibein \((\mathbf{n}, \mathbf{b})\) any two mutually orthogonal vectors \((\mathbf{e}_1, \mathbf{e}_2)\) on the normal planes, \(i.e.\) planes that are perpendicular to the tangent vectors \(\mathbf{t}\), can be chosen. Such a generic frame is related to the Frenet zweibein by a local SO(2) frame rotation around the tangent vector \(\mathbf{t}(s)\) of the from

\[ \begin{pmatrix} \mathbf{n} \\ \mathbf{b} \end{pmatrix} \rightarrow \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix} = \begin{pmatrix} \cos \eta(s) & -\sin \eta(s) \\ \sin \eta(s) & \cos \eta(s) \end{pmatrix} \begin{pmatrix} \mathbf{n} \\ \mathbf{b} \end{pmatrix}. \]  

(8)

For the Frenet equation this yields

\[ \frac{d}{ds} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix} = \begin{pmatrix} 0 & (\tau - \eta') \\ -\kappa \cos \eta & \kappa \sin \eta \end{pmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix}. \]  

(9)

Next let us introduce the three matrices

\[ T_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad T_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \]
that determine the canonical adjoint representation of \( \mathfrak{so}(3) \) Lie algebra,

\[
[T_a, T_b] = \epsilon_{abc} T_c.
\]

Then the action of the SO(2) frame rotation on \( \kappa \) and \( \tau \) is described by

\[
\kappa T_2 \to e^{\eta T_3} (\kappa T_2) e^{-\eta T_3} = \kappa \left( \cos \eta T^2 + \sin \eta T^1 \right), \quad (10)
\]

\[
\tau T_3 \to (\tau - \eta') T_3. \quad (11)
\]

Observe that (10) and (11) has the same format as the SO(2) \( \simeq U(1) \) gauge transformation of an Abelian Higgs multiplet. The curvature \( \kappa \) is like a complex valued “Higgs field” with a real part that coincides with the \( T^2 \) components on (10), and an imaginary part that coincides with the \( T^1 \) component (or \textit{vice versa}). Finally, the torsion \( \tau \) transforms like the U(1) “gauge field” of the multiplet.

Let us conclude, by pointing out that the choice

\[
\eta(s) = \int_0^s \tau(\hat{s}) d\hat{s}
\]

that brings about the “\( \tau = 0 \)” gauge yields the parallel transport framing \[59\]. Unlike the Frenet framing that can not be defined at an inflection point of the string, the parallel transport framing can be defined continuously and unambiguously through inflection points and straight segments \[59\].

**B. Time evolution**

Let us proceed with the generic frame (10) and (11) by setting

\[
\tau_r = \tau - \eta', \quad \kappa_n = \kappa \sin \eta, \quad \kappa_g = \kappa \cos \eta. \quad (13)
\]

Then, in the form of a Darboux \textit{trièdre}, the Frenet equation (9) becomes

\[
\frac{d}{ds} \begin{pmatrix} \mathbf{x} \\ \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{t} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & \tau_r & -\kappa_g & 0 \\ 0 & -\tau_r & 0 & -\kappa_n \\ 0 & \kappa_g & \kappa_n & 0 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{t} \end{pmatrix} = R_s \begin{pmatrix} x \\ e_1 \\ e_2 \\ t \end{pmatrix}. \quad (14)
\]

Accordingly, let us assume that the string \( \mathbf{x}(s) \) lie on a surface \( \mathcal{S} \) which is embedded in \( \mathbb{R}^3 \). This surface is a putative world-sheet of the string, swept by its time evolution. Then, the vector \( \mathbf{e}_2 \) can be geometrically interpreted as the unit normal of the surface, so that \( \mathbf{t} \) and \( \mathbf{e}_1 \) span its tangent plane at the point \( \mathbf{x}(s) \). Also, \( \kappa_g \) is the geodesic curvature; \( \kappa_n \) is the normal curvature; and \( \tau_r \) is the relative (geodesic) torsion of the string on \( \mathcal{S} \). **Remark:** Different choices of \( \eta \) in (9) and (13) correspond to different choices of the surface \( \mathcal{S} \). The surface becomes uniquely determined only once the time evolution of \( \mathbf{x}(s) \) is specified.

In analogy with (5) let us introduce a U(1) rotation with an angle \( \chi \) of the Darboux \textit{trièdre} with the rotation around the normal vector \( \mathbf{e}_2 \) of the surface \( \mathcal{S} \). Thus, it rotates the zweibein of the tangent plane as follows

\[
\begin{pmatrix} e_1 \\ e_2 \\ t \end{pmatrix} \quad \xrightarrow{\text{U(1)}} \quad \left( \begin{pmatrix} \cos \chi & -\sin \chi \\ \sin \chi & \cos \chi \end{pmatrix} \right) \begin{pmatrix} e_1 \\ e_2 \\ t \end{pmatrix}.
\]

Then the Frenet equation (14) transforms to

\[
\frac{d}{ds} \begin{pmatrix} \mathbf{x} \\ \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{t} \end{pmatrix} \quad \xrightarrow{\mathcal{R}_s} \quad \mathcal{R}_s \begin{pmatrix} \mathbf{x} \\ \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{t} \end{pmatrix}
\]

where the \((ij)\) elements of the matrix \( \mathcal{R}_s \) are given by

\[
\begin{align*}
\mathcal{R}_s(12) &= -\sin \chi \\
\mathcal{R}_s(14) &= \cos \chi \\
\mathcal{R}_s(23) &= -\mathcal{R}_s(32) = \tau_r \cos \chi - \kappa_n \sin \chi \\
\mathcal{R}_s(34) &= -\mathcal{R}_s(43) = -\tau_r \sin \chi - \kappa_n \cos \chi \\
\mathcal{R}_s(24) &= -\mathcal{R}_s(42) = -\left(\kappa_g - \chi'\right),
\end{align*}
\]

\textit{i.e.} the transformation law of the Abelian Higgs multiplet. However, in this context the “Higgs field” has as its real and imaginary components the normal curvature \( \kappa_n \) and the geodesic torsion \( \tau_r \), respectively; while the geodesic curvature \( \kappa_g \) is like the “gauge field”.

Note that after the rotation by \( \chi \), the tangent of the string points in the direction is equal to

\[
\mathbf{x}'(s) = \cos \chi \mathbf{t} - \sin \chi \mathbf{e}_1
\]

while, in analogy with (12), in the frame where

\[
\chi(s) = \int_0^s \kappa_g(\hat{s}) d\hat{s}
\]

the geodesic curvature vanishes. Thus, in this frame the direction of the string coincides with that of a geodesic on \( \mathcal{S} \).

The preceding proposes us to consider the consequences of extending \( \mathbf{x}(s) \) into a one parameter family of strings \( \mathbf{x}(s,t) \); where \( t \) is the time so that \( \mathbf{x}(s,t) \) determines a surface which is the world sheet \( \mathcal{S} \) of the string. Then the time evolution transports the Frenet frames (7) along the direction \( \mathbf{x}(s,t) \) on the surface. By completeness of the dreibein, the time evolution is governed by an equation of the form

\[
\frac{d}{dt} \begin{pmatrix} \mathbf{n} \\ \mathbf{t} \end{pmatrix} = \mathcal{R}_t \begin{pmatrix} \mathbf{n} \\ \mathbf{t} \end{pmatrix}, \quad (15)
\]
where $R_t$ takes values in the $\mathfrak{so}(3)$ Lie algebra,

$$
R_t = uT_1 + vT_2 + wT_3 \equiv \begin{pmatrix} 0 & -w & v \\ w & 0 & -u \\ -v & u & 0 \end{pmatrix}.
$$

(16)

Here

$$u = \mathbf{i} \cdot \mathbf{b},$$

$$v = \mathbf{n} \cdot \mathbf{t},$$

$$w = \mathbf{b} \cdot \mathbf{n}.$$

Thus $(\mathbf{n}, \mathbf{b}, \mathbf{t})$ becomes extended into an orthonormal dreibein, at each point $x(s, t)$ of the surface.

For strings $x(s, t)$ that are of class $C^3$, the linear problem (14) and (15) is integrable since the ordering of derivatives commute

$$\frac{\partial}{\partial s} \frac{\partial}{\partial t} = \frac{\partial}{\partial t} \frac{\partial}{\partial s},$$

leading to the zero curvature condition

$$\mathcal{F}_{ts} \equiv \partial_t R_s - \partial_s R_t + [R_s, R_t] = 0,$$

(17)

which corresponds to the Gauß-Godazzi equation that describes the embedding of the surface $x(s, t)$ in the ambient space $\mathbb{R}^3$ [57, 58]. Moreover, we recognize in (17) the $\mathfrak{so}(3)$ Lax pair structure.

Indeed, the Frenet equation together with the time evolution equation (15), admits an interpretation as an auxiliary linear problem that has a fundamental role in the theory of integrable models [44].

Let us conclude by showing how the (focusing) integrable nonlinear Schrödinger equation

$$\frac{1}{2} \partial_t q = \partial_s q + \frac{1}{2} |q|^2 q$$

(18)

for $q(s, t)$ complex field, can be obtained from (17) in the case of a string that moves under the influence of the local induction approximation (LIA) [59–61]. Recall that, the LIA emerges from the Euler equations as the leading order self-induced inextensional motion of a vortex filament in an irrotational fluid [33].

Let us start with the expansion [62–63]

$$\frac{\partial x(s, t)}{\partial t} \equiv \dot{x} = v_1 \mathbf{t} + v_2 \mathbf{n} + v_3 \mathbf{b}$$

that follows from the completeness

$$\partial_s \partial_t x = \partial_t \partial_s x = \mathbf{i}.$$

Then the Frenet equation (17) take the form

$$u = \nu \tau + \nu'$$

$$v = -\nu \kappa - \nu' + v \tau$$

$$w = \frac{1}{\kappa} [\tau (\nu \kappa + \nu' - v \tau) + (v \tau + \nu')]$$

(19)

However, the local induction approximation

$$\dot{x} = \kappa \mathbf{b} \equiv \nu \mathbf{b}$$

simplifies (19) into

$$u = \kappa'$$

$$v = \kappa \tau$$

$$w = \tau^2 - \frac{1}{\kappa} \kappa'.$$

By demanding that the Lax pair $(R_s, R_t)$ obeys (17) i.e. derivatives commute, the following conditions are derived

$$\dot{k} = -(\kappa \tau)' - \kappa' \tau$$

$$\dot{\tau} = \left(\frac{\kappa'' \tau^2}{\kappa^2 - \kappa'^2}\right)' + \kappa' \kappa.$$

(20)

Finally, by introducing the Hasimoto variable (18)

$$q(s, t) = \kappa(s, t) e^{i \int \tau(u, t) du}$$

the zero curvature condition (20) becomes equivalent to (18).

III. SPINOR FRENET EQUATION

In this section, the classic Frenet equation of a string is described in terms of spinors. Let us start by introducing a two complex component spinor along the string i.e.

$$\psi : [0, L] \to \mathbb{C}^2 ; \quad \psi = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}.$$

(21)

In addition, the conjugate spinor $\bar{\psi}$ is obtained by acting the “charge conjugation” operation $C$ on $\psi$:

$$C \psi = -i \sigma_2 \bar{\psi}^* = \bar{\psi},$$

(22)

where $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ are the standard Pauli matrices. In terms of the spinor components, the charge conjugation amounts to

$$z_\alpha \xrightarrow{C} -\epsilon_{\alpha \beta} z_\beta^*$$

where $\epsilon_{12} = 1$ and $\epsilon_{13} = -\epsilon_{32}$. Also, note that

$$C^2 = -\mathbb{I}.$$

Next, we introduce the normalization condition

$$\psi^\dagger \psi \equiv <\psi, \psi> = 1 = <\bar{\psi}, \bar{\psi}> = \bar{\psi}^\dagger \bar{\psi}$$

(23)

so that $\psi$ maps the string onto a complex two-sphere; in accordance with the orthonormality condition

$$<\psi, \bar{\psi}> = 0.$$

Thus, the following completeness exists

$$|\psi > <\psi| + |\bar{\psi} > <\bar{\psi}| = \psi_\alpha \psi^\dagger_\beta + \psi_\alpha^\dagger \psi_\beta = \mathbb{I}_{\alpha \beta}$$

leading to the differential equation

$$\partial_s \psi = <\psi, \partial_s \psi > \psi + <\bar{\psi}, \partial_s \bar{\psi} > \bar{\psi}.$$

(24)
By comparing with (1) the following correspondence occurs

\[
\kappa_c = \kappa_g + i\kappa_n = 2 < \bar{\psi}, \partial_s \psi >
\]

(25)

and the real valued torsion as

\[
\tau_r = 2i < \bar{\psi}, \partial_s \psi >
\]

(26)

equation (24) transforms to

\[
\partial_s \psi = \frac{i}{2} \tau_r \psi + \frac{\kappa_c}{2} \bar{\psi},
\]

(27)

the so-called spinor Frenet equation. So, for given \((\kappa_c, \tau_r)\), the spinor \(\psi\) can be constructed along the string. Additionally, from the spinor \(\psi\) the unit length vector takes the form

\[
t = < \bar{\psi}, \partial_s \psi > = - < \bar{\psi}, \partial_s \bar{\psi} >
\]

(28)

can be obtained in order to evaluate the string \(x(s)\), by identifying \(s\) as the proper-length parameter and by integrating

\[
\frac{dx}{ds} = t(s) \equiv < \psi, \partial_s \psi > .
\]

(29)

Thus, the string is determined uniquely up to a global rotation and a global translation.

The normalization condition (23) can be relaxed. By assuming

\[
< \psi, \psi > = < \bar{\psi}, \bar{\psi} > = \sqrt{g(s)}
\]

the unit length vector takes the form

\[
t = \frac{1}{\sqrt{g(s)}} < \psi, \partial_s \psi > .
\]

By comparing with (1) the following correspondence occurs

\[
\sqrt{g} \simeq ||x_s|| = \sqrt{g}.
\]

Thus, re-defining the normalization of the spinors by

\[
|\psi| \rightarrow (g)^{-1/4} |\psi|
\]

which effectively sends \(\sqrt{\lambda} \rightarrow 1\), corresponds to the choice of arc-length parameterization along the string.

**IV. FRENET FRAMES FROM SPINORS**

In what follows, the relation between the classic version and the spinor version of the Frenet equation is studied in detail. Consider the local U(1) rotation

\[
\psi \rightarrow e^{i\varphi} \psi, \quad \bar{\psi} \rightarrow e^{-i\varphi} \bar{\psi}.
\]

(30)

Then the complex curvature defined in (25) becomes

\[
\kappa_c \rightarrow e^{2i\varphi} \kappa_c
\]

(31)

while the torsion (26) transforms to

\[
\tau_r \rightarrow \tau_r - 2 \partial_s \varphi.
\]

(32)

In this case, both the vector \(t\) defined by (28) and the string \(x(s)\) obtained from (29) remain invariant under the U(1) rotation (30)-(32).

In addition of \(t\), let us introduce the two complex vectors

\[
e_+ = \frac{1}{2}(e_1 + ie_2) = \frac{1}{2} < \psi, \partial_s \psi > \equiv \frac{1}{2} \psi^\dagger \sigma \psi
\]

(33)

\[
e_- = \frac{1}{2}(e_1 - ie_2) = \frac{1}{2} < \psi, \partial_s \bar{\psi} > \equiv \frac{1}{2} \psi^\dagger \bar{\sigma} \bar{\psi}
\]

(34)

where \(e_1\) and \(e_2\) are the real and imaginary parts of \(e_\pm\), respectively. A direct computation shows that

\[
e_i \cdot e_j = \delta_{ij}, \quad e_i \cdot t = 0
\]

and

\[
(e_1 \times e_2) \cdot t = 1.
\]

Thus \((e_1, e_2, t)\) is a right-handed orthonormal system. In particular, \((e_1, e_2)\) span the normal planes of the string that is obtained by integration of (29).

The local U(1) rotation (30) brings about the frame rotation

\[
e_\pm \rightarrow e^{\pm 2i\varphi} e_\pm
\]

and a direct computation using the definitions (25), (26), (33) and (34) leads to the following system

\[
\partial_s e_+ = -i \tau_r e_+ - \frac{i}{2} \kappa_c t
\]

(35)

\[
\partial_s t = 4(\kappa_c e_+ + \kappa_n^* e_-) .
\]

This coincides with the general frame (Darboux) equation (1) when \(\kappa_c\) is given by (25). Note that different choices of \(\varphi\) correspond to different choices of embedding surfaces.

To complete the interpretation of the Frenet frames in terms of spinors, introduce the following (local) coordinate representation of the components of spinor (21):

\[
z_1 = \cos \frac{\vartheta}{2} e^{i\phi_1}, \quad z_2 = \sin \frac{\vartheta}{2} e^{i\phi_2}.
\]

(36)

Then the tangent vector of the string (28) using (36) gives

\[
t = \begin{pmatrix}
\cos \phi_- \sin \vartheta \\
\sin \phi_- \sin \vartheta \\
\cos \vartheta
\end{pmatrix}
\]

(37)

where

\[
\phi_\pm = \phi_2 \pm \phi_1 .
\]
In addition, by introducing the unit vectors
\[
\mathbf{u} = \begin{pmatrix} \cos \phi_- \cos \vartheta \\ \sin \phi_- \cos \vartheta \\ -\sin \vartheta \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} -\sin \phi_- \\ \cos \phi_- \\ 0 \end{pmatrix}
\] (38)
and substituting (36) in (33) and (34) one obtains
\[
\mathbf{e}_\pm = \frac{1}{2} e^{\pm i\phi_+} (\mathbf{u} \pm i\mathbf{v}).
\] (39)
Furthermore, (37) becomes
\[
\partial_s \mathbf{t} = \partial_s \vartheta \mathbf{u} + \sin \vartheta \partial_s \phi_- \mathbf{v}.
\]
while, similarly to (7), one gets
\[
(\partial_s \vartheta) \mathbf{u} + \sin \vartheta \partial_s \phi_- \mathbf{v} = \kappa \mathbf{n}.
\] (40)
Here \(\kappa\) is the (geometric Frenet) curvature and \(\mathbf{n}\) is the normal vector of the string while,
\[
\kappa^2 = \kappa_c \kappa_c^* = \kappa_\vartheta^2 + \kappa_\psi^2 = (\partial_s \vartheta)^2 + \sin^2 \vartheta (\partial_s \phi_-)^2.
\]
On the other hand, using the complex curvature (25) in the parametrization (36) leads to
\[
\kappa_c = 2 (z_1 \partial_s z_2 - z_2 \partial_s z_1) = e^{i\phi_+} (\partial_s \vartheta + i \sin \vartheta \partial_s \phi_-).
\] (41)
Due to (26) and (35), the Frenet framing of the string is specified by demanding that \(\kappa_c\) is real. Thus,
\[
<\ddot{\psi}, \partial_s \psi > - <\dot{\psi}, \partial_s \ddot{\psi}> = 0.
\]
In terms of (36) this reads
\[
\sin \phi_- \frac{d\vartheta}{ds} + \cos \phi_- \sin \vartheta \frac{d\phi_-}{ds} = 0.
\]
The U(1) rotation (30) with the choice
\[
2\varphi = -\phi_+
\] (42)
implies that (41) transforms to
\[
\kappa_c \rightarrow \partial_s \vartheta + i \sin \vartheta \partial_s \phi_-.
\]
and due to the choice of the phase \(\phi_+\) one obtains that
\[
\kappa_g \simeq \partial_s \vartheta, \quad \kappa_n \simeq \sin \vartheta \partial_s \phi_-.
\]
This can be interpreted as follows: Take \(\vartheta(s)\) and \(\phi_-(s)\) to be the coordinates on a two-sphere \(S^2_R\) with radius \(R\). Assume this sphere osculates the string at the point \(\mathbf{x}(s)\), in such a manner that the tangent of the string becomes parallel with the tangent of a great circle of the sphere. At the point of contact the value of \(R\) then coincides with the inverse of the curvature \(\kappa(s)\) of the string. However, the great circle is a geodesic of \(S^2_R\) and \(s\) is the proper length parameter so
\[
\frac{1}{R^2} = (\partial_s \vartheta)^2 + \sin^2 \vartheta (\partial_s \phi_-)^2 = \kappa^2.
\]
By orienting the osculating two-sphere so that the osculating great circle coincides with the equator \(\vartheta = \pi/2\) of the sphere, the vectors (38) coincide with the Frenet frame normal and bi-normal vectors, respectively. In particular, along the equator \(\vartheta = \pi/2\) (is constant) so (40) implies
\[
\partial_s \phi_+ \mathbf{u} = \kappa \mathbf{n} = \frac{1}{R} \mathbf{n} \equiv \kappa \mathbf{n}
\] (43)
since \(\kappa_g = 0\) for a geodesic.
In addition, the torsion (26)
\[
\frac{1}{2i} \tau_r = <\psi, \partial_s \psi > = z_1^* \partial_s z_1 + z_2^* \partial_s z_2
\]
in terms of the parametrization (36) becomes
\[
\tau_r = \cos \vartheta \partial_s \phi_- - \partial_s \phi_+
\] (44)
while the U(1) rotation (12) gives
\[
\tau_r = \cos \vartheta \partial_s \phi_-.
\]
Thus, for a path along the equator (43) there is no torsion. With this choice of phase \(\varphi\), the torsion \(\tau_r\) coincides with the U(1) invariant “super-current”
\[
\frac{1}{2} \tau_r = \frac{1}{2i} \left\{ \frac{z_1^* \partial_s z_1 - z_1 \partial_s z_1^*}{z_1^* z_1} + \frac{z_2^* \partial_s z_2 - z_2 \partial_s z_2^*}{z_2^* z_2} \right\}.
\] (45)
As a consequence it appears natural to identify \(j\) with the Frenet frame torsion \(\tau\) of the string.

V. MAURER-CARTAN RELATIONS

Let us start by noting that (11) can be interpreted as the pull-back of the complex valued one-form
\[
\kappa_c = i \mathbf{b}_- = e^{i\varphi_-} (d\vartheta + i \sin \vartheta d\phi_-)
\] (46)
to the surface \(\mathcal{S}\) that is determined by \(\mathbf{x}(s,t)\). Similarly, (43) can be interpreted as the pull-back of the real valued one-form
\[
\tau_r = -a = \cos \vartheta d\phi_- - d\phi_+
\] (47)
to this surface. Equation (17) can be identified as the Dirac monopole connection one-form, by interpreting \((\vartheta, \phi_-)\) as the local coordinates of the base \(S^2\) and \(\phi_+\) as the coordinate of the U(1) fibre.
Let \(g\) be the SU(2) matrix defined as
\[
g = \left( \begin{array}{c} z_1 \\ z_2 \end{array} \right)
\] (48)
such that
\[
g \mathbf{g}^{-1} = \mathbf{t} \cdot \mathbf{e}\equiv \dot{\mathbf{t}}
\] (49)
\[
g \mathbf{g}^{-1} = \frac{1}{2} g (\sigma_1 \pm i \sigma_2) g^{-1} = \mathbf{e}_\pm \cdot \sigma = \dot{\mathbf{e}}_\pm.
\] (50)
Then, the matrices \((\mathbf{e}_\pm, \mathbf{t})\) obey the \(su(2)\) Lie algebra
\[ [\mathbf{t}, \mathbf{e}_\pm] = \pm 2\mathbf{e}_\pm, \quad [\mathbf{e}_+, \mathbf{e}_-] = \mathbf{t}. \] (51)

Note that the matrix \(\mathbf{t}\) is defined up to a \(h \in U(1) \subset SU(2)\) multiplication of \(g\) from the right. I.e. for
\[ g \xrightarrow{h} gh, \quad h = e^{i\varphi}\sigma_3 \] (52)
one obtains that
\[ \mathbf{t} \xrightarrow{h} gh\sigma_3 h^{-1} g^{-1} = \mathbf{t}. \] (53)

In addition,
\[ \mathbf{e}_\pm \xrightarrow{h} gh\sigma_\pm h^{-1} g^{-1} = e^{\pm 2i\varphi}\mathbf{e}_\pm. \] (54)

Note that (54) coincides in form with (39), by identifying \(\phi_+ \simeq 2\varphi\).

Next, let us arrange the components of the vectors \((\mathbf{e}_\pm, \mathbf{t})\) into elements of a SO(3) matrix i.e.
\[ \mathcal{O}_a^i \overset{\text{def}}{=} (e_1^a, e_2^a, t^a) \]
which relates the local basis of the Lie algebra at unity to the Lie algebra basis at the point \(g\) on \(SU(2)\) since
\[ (\sigma_\pm, \sigma_3) \xrightarrow{g} \mathcal{O}_{ia}^a \sigma_a = (\mathbf{e}_\pm, \mathbf{t}). \]

But \(\mathcal{O}_{ia}^a\) define the components of the dreibein, thus the corresponding Levi-Civita connection one-form \(\omega_{ij}\) is subject to the Cartan structure equation
\[ d\mathcal{O}_i + \omega_{ij}\mathcal{O}_j = 0. \]

In terms of \((\mathbf{e}_\pm, \mathbf{t})\) this becomes
\[ de_\pm = <\mathbf{t}, de_\pm > t + 2<\mathbf{e}_\pm, de_\pm > e_\pm \\
dt = -2<\mathbf{t}, de_+ > e_- - 2<\mathbf{t}, de_- > e_+ . \]

By identifying
\[ a = -2i<\mathbf{e}_+, de_+ > = -i \text{Tr}(\sigma_3 g^{-1}dg) = -i \text{Tr}(\sigma_3 R) \] (55)
\[ b_\pm = \pm 2i<\mathbf{e}_+, dt >= -2i \text{Tr}(\sigma_\pm g^{-1}dg) = -2i \text{Tr}(\sigma_\pm R) \] (56)

where
\[ R = g^{-1}dg \] (57)
is the connection one-form, one gets the (right) Maurer-Cartan form
\[ dR + [R, R] = 0. \] (58)

In terms of the local coordinates (39) we confirm that (50) and (53) coincide with the one-forms (40) and (47), respectively. In particular, the U(1) rotation (54) and (53) reproduced (31) and (32) since
\[ b_\pm \xrightarrow{h} b_\pm e^{\pm 2i\varphi} b_\pm \]
\[ a \xrightarrow{h} a + 2d\varphi. \] (59)

Thus \(a\) and \(b_\pm\) can be combined into the single \(SU(2)\) Maurer-Cartan form \(R\) by setting
\[ -2i R = -2i g^{-1}dg = a \sigma_3 + b_+ \sigma_+ + b_- \sigma_- . \] (60)

Alternatively, in terms of the (left) Maurer-Cartan form
\[ L = dg g^{-1} = gRg^{-1} \] (61)
one gets
\[ dL + [L, L] = 0 \]
where
\[ -2i Lu = a \mathbf{t} + b_+ \mathbf{e}_+ + b_- \mathbf{e}_- = a \mathbf{t} + \frac{1}{2i} [d\mathbf{t}, \mathbf{t}] . \] (62)

### VI. LAX PAIR

#### A. Non-linear Schrödinger equation (NLSE)

In this section, the construction of the Lax pair for the nonlinear Schrödinger equation (44)
\[ \frac{1}{i} \partial_t q = \partial_{ss}q - 2\lambda|q|^2q \] (63)
is reviewed and then its connection with the Frenet equation is presented.

The NLSE Hamiltonian is
\[ H_3 = \int ds \left( \left| \frac{\partial q}{\partial s} \right|^2 + \lambda |q|^4 \right) \] (64)

where \(q\) is a complex variable. The Poisson brackets that determine the time evolution are
\[ \{q(s), q(s')\} = \{\bar{q}(s), \bar{q}(s')\} = 0 \]
\[ \{q(s), \bar{q}(s')\} = i\delta(s-s'). \]

The NLSE is an integrable equation and its Lax pair \((U, V)\) is defined by
\[ U = U_0 + \xi U_1 \] (65)
where \(\xi\) is a complex spectral parameter and
\[ U_0 = \sqrt{X} \begin{pmatrix} 0 & \bar{q} \\ q & 0 \end{pmatrix} = \sqrt{X}(\bar{q} \sigma_+ + q \sigma_-) \] (66)
\[ U_1 = \frac{1}{2i} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_3. \] (67)
In addition, \[ V = V_0 + \xi V_1 + \xi^2 V_2 \]
where
\[
V_0 = i\sqrt{\lambda} \left( \sqrt{\lambda}q^2 \frac{\partial}{\partial q} - \sqrt{\lambda} \right)
\]
\[
= i\lambda q^2 \sigma_3 - i\sqrt{\lambda} (\partial_s q \sigma_+ - \partial_t q \sigma_-)
\]
and
\[
V_1 = -U_0, \quad V_2 = -U_1.
\]
The integrability of NLSE is an outcome of the overdeterminacy of the auxiliary linear equations
\[
\frac{\partial}{\partial s} \left( \frac{f_1}{f_2} \right) = U(s,t,\lambda) \left( \frac{f_1}{f_2} \right) \quad (70)
\]
\[
\frac{\partial}{\partial t} \left( \frac{f_1}{f_2} \right) = V(s,t,\lambda) \left( \frac{f_1}{f_2} \right)
\]
due to the compatibility condition
\[
\frac{\partial}{\partial s} \frac{\partial}{\partial t} \left( \frac{f_1}{f_2} \right) = \frac{\partial}{\partial t} \frac{\partial}{\partial s} \left( \frac{f_1}{f_2} \right)
\]
provided that the Lax pair \((U,V)\) comprises the components of a flat SU(2) connection
\[
F_{ts} = \partial_t U - \partial_s V + [U,V] = 0. \quad (72)
\]
See also (17).

To sum up, when the explicit representations (60)-(69) are substituted in (72) the NLSE (63) occurs; while when \(q\) obeys the NLSE, the Lax pair \((U,V)\) is a flat SU(2) connection.

B. Majorana realization of Frenet equation

Next, the relation between the Lax pair \((U,V)\) of the NLSE and the spinor Frenet equations is studied. To do so the spinor Frenet equation is combined with its conjugate equation
\[
\partial_s \bar{\psi} = \frac{i}{2} \tau_r \bar{\psi} - \frac{\kappa^*_r}{2} \psi \quad (73)
\]
into a single equation. This can be done by merging the two spinors into a four-component (conjugate) spinor of the form
\[
\Psi = \begin{pmatrix} \bar{\psi} \\ \psi \end{pmatrix}.
\]
This four-spinor is subject to the Majorana condition: \(i.e.\) under the charge conjugation \((22)\) transforms as
\[
\mathcal{C} \Psi = \begin{pmatrix} \bar{\psi} \\ \psi \end{pmatrix} = i\sigma_2 \Psi \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \bar{\psi} \\ \psi \end{pmatrix}.
\]
Here the Pauli matrices \(\sigma_\alpha\) act in the two dimensional space of the spinor components of \(\Psi\). In terms of the Majorana spinor \(\Psi\), the spinor Frenet equations (27) combine to
\[
\partial_s \Psi = \frac{i}{2} (\tau_r \sigma_3 - i\kappa^*_r \sigma^+ + i\kappa_r \sigma^-) \Psi.
\]
By using the relations (16), (17) and the identification Maurer-Cartan one-form (60) this becomes
\[
(\partial_s + g^{-1} \partial_t g) \Psi \equiv (\partial_s + R_s) \Psi = 0 \quad (75)
\]
where \(R_s\) denotes the \(s\)-component of the right Maurer-Cartan form (57).

Observe that in terms of the SU(2) transformed spinor
\[
\Psi_g = g^{-1} \Psi \quad (76)
\]
the equation (75) changes into
\[
\partial_s \Psi_g = 0.
\]
Thus the spinor \(\Psi_g(s)\) is constant along the string \(x(s)\).

On the tangent vector (28) the SU(2) rotation (76) acts as follows
\[
t = \begin{pmatrix} \cos \phi_r \sin \theta_r & \sin \phi_r \sin \theta_r & \cos \theta_r \\ -\sin \phi_r \sin \theta_r & \cos \phi_r \sin \theta_r & -\cos \theta_r \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.
\]
This means that (76) determines a spatial SO(3) frame rotation, which at each point \(x(s)\) of the string orients the Cartesian coordinates so that the tangent vector points along the positive \(z\)-axis. Similarly, the vectors (38) that span the normal plane of the string transform like
\[
u + i\nu  \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}.
\]
Here the choice of \(\varphi(s)\) specifies a frame on the normal planes. If the local frame is chosen so that the direction of the vector \(\mathbf{u}\) in (77) at each point coincides with the direction of the Frenet frame normal vector along the string, the continuum version of the co-moving framing is obtained [24].

C. Lax pair for the Frenet equation

Following [15], a one parameter family of string \(x(s,t)\) is considered with \(t\) being time so that \(x(s,t)\) describes the time evolution of the string. By completeness, in terms of the Majorana spinor (24), its time evolution is described by the equation
\[
(\partial_t + R_t) \Psi = 0. \quad (78)
\]
Here \(R_t(s,t)\) is SU(2) Lie algebra valued \(i.e.\) it is a linear combination of Pauli matrices \(\sigma_\alpha\) of the form
\[
R_t(s,t) = \alpha(s,t) \sigma_1 + \beta(s,t) \sigma_2 + \gamma(s,t) \sigma_3
\]
where \((\alpha, \beta, \gamma)\) are coefficients.

In analogy with \((70)\) and \((71)\) the linear system \((75)\) and \((78)\) is over-determined. Its integrability yields to the zero-curvature condition

\[
F_{ts} = \partial_t R_s - \partial_s R_t + [R_s, R_t] = 0 \quad (79)
\]

implying that \((R_s, R_t)\) is a flat two dimensional SU(2) connection one-form. Note that \((R_s, R_t)\) can be considered as the Lax pair of the spinor Frenet equation; \textit{i.e.} the Gauß-Codazzi equation that governs the embedding of the surface \(x(s, t)\) in the ambient \(\mathbb{R}^3\).

### D. Relation between NLSE and Frenet Lax pairs

In what follows, solution of equation \((79)\) is constructed which relates the string and the NLSE. Due to \((46)\) and \((47)\) the relative torsion and the complex curvature can be combined into the (putative) right Maurer-Cartan form as

\[
-2iR_s = \tau_r \sigma_3 + i\kappa_c \sigma_+ - i\kappa^*_c \sigma_- \quad (80)
\]

Next introduce the \(U(1)\) transformation \textit{a.k.a.} frame rotation of the string \((52)\) which acts on \(R_s\) by sending

\[
\kappa_c \rightarrow e^{-2i\varphi} \kappa_c, \quad \tau_r \rightarrow \tau_r + 2\partial_s \varphi. \quad (81)
\]

Here \(\varphi\) is chosen

\[
\varphi = -\frac{1}{2} \int_0^s ds' \{\tau_r(s') + \xi\} \quad (82)
\]

such that the first term implements the gauge transformation \textit{i.e.} frame rotation from the generic Darboux frame to the parallel transport frame \((12)\); while \(\xi\) is the putative complex spectral parameter.

Then the \(U(1)\) transformation \((51)\) sends

\[
R_s \rightarrow R^h_s \quad (80)
\]

where

\[
R^h_s = \frac{\xi}{2i} \sigma^3 - \frac{\kappa_c}{2} e^{\int \tau_r(s') + \xi} \sigma_+ + \frac{\kappa^*_c}{2} e^{-\int \tau_r(s') + \xi} \sigma_-.
\]

By introducing the Hasimoto variable

\[
\sqrt{\lambda} \bar{q} = -\frac{\kappa_c}{2} e^{\int \tau_r(s') + \xi} \quad (83)
\]

one obtains

\[
R^h_s = \frac{\xi}{2i} \sigma^3 + \sqrt{\lambda} \bar{q} \sigma_+ + \sqrt{\lambda} \bar{q} \sigma_- \quad (84)
\]

\textit{i.e.} the \(U\) Lax operator of the NLSE \((69)\).

In addition, the Lax operator \(R_t\) can be obtained, by simply identifying it with the \(V_0\) Lax operator of the NLSE \((65)\), that is,

\[
R_t = i\lambda|q|^2 \sigma_3 = i\sqrt{\lambda}(\partial_s \bar{q} \sigma_+ - \partial_s q \sigma_-).
\]

Thus, the integrability condition \((79)\) is satisfied for \(q\) being a solution of the NLSE \((65)\). Therefore, an \textit{embedding} of the NLSE Lax pair in the Lax pair of the spinor Frenet equation is derived. In particular, the NLSE determines the time evolution of the string.

We conclude, by pointing out that the complex spectral parameter \(\xi\) can be interpreted as a parameter for a family of loops in the group manifold, determined by

\[
g(s) \mapsto g(\xi, s)
\]

with

\[
R(\xi, s) = g^{-1}(\xi, s) dg(\xi, s).
\]

The real part of \(\xi\) parameterizes a gauge transformation \textit{i.e.} a rotation of the parallel transport frame, that proceeds linearly in the arc-length parameter \(s\). The imaginary part of \(\xi\) corresponds to a Weyl transformation \((6)\) \textit{i.e.} a rescaling of the string that similarly proceeds linearly in the arc-length parameter.

### VII. STRING HAMILTONIANS

Once a Lax pair of an integrable model is known, the conserved charges can be constructed. The procedure is standard, it utilizes an expansion in the spectral parameter and has been studied in detail in \((44)\). In particular, an infinite number of string Hamiltonians can be constructed from the Lax pair \((R_s, R_t)\) of the spinor Frenet equation (obtained from the NLSE Lax pair).

More generally, \textit{any} SU(2) Lax pair of a one dimensional integrable model can be utilized to construct a time evolution of a string. To do so, equations \((79)\) and \((80)\) can be used to identify \((R_s, R_t)\) in the spinor Frenet equations \((76)\) and \((78)\) in terms of the Lax pair of the given integrable model, similar to the NLSE case.

In what follows it is shown how string Hamiltonians in the NLSE hierarchy can be obtained alternatively, using a formalism of projection operators; introduced originally in the integrable \(\mathbb{CP}^N\) models. Note that the spinorial approach to strings engages locally the structure of \(S^2 \times S^1\). Since \(S^2 \simeq \mathbb{CP}^1\), the connection between the projection operator formalism of the \(\mathbb{CP}^1\) model and the spinor Frenet equation can be explored.

#### A. Projection operators

Let us start with the component representations \((21)\) and \((22)\). Using the matrix \(g \in SU(2)\) of \((18)\) one can easily show that

\[
\psi = g \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \bar{\psi} = g \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\]

Then, the projection operator can be defined as

\[
P = |\psi><\psi| \equiv \psi \otimes \bar{\psi} = g \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} g^{-1}.
\]
which has been utilized widely, in the context of \( \mathbb{C}P^1 \) (more generally \( \mathbb{C}P^N \) model \[43\-17\].

In addition, introduce the complementary projection operator
\[
\bar{P} = |\bar{\psi}\rangle \otimes \langle \bar{\psi}| \equiv \bar{\psi} \otimes \bar{\psi}^\dagger = g \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} g^{-1}.
\]
The projection operators satisfy the following identities
\[
\begin{align*}
\bar{P}^2 &= P, \\
\bar{P} P &= \bar{P}, \\
P \bar{P} &= \bar{P} P = 0, \\
P + \bar{P} &= \mathbb{I}.
\end{align*}
\]
Finally, introduce the nilpotent operators
\[
\begin{align*}
Q &= |\psi\rangle \otimes \langle \bar{\psi}| \equiv \psi \otimes \bar{\psi}^\dagger = g \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} g^{-1} \\
\bar{Q} &= |\bar{\psi}\rangle \otimes \langle \psi| \equiv \bar{\psi} \otimes \psi^\dagger = g \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} g^{-1}
\end{align*}
\]
where
\[
Q^2 = \bar{Q}^2 = 0.
\]
The \( Q \) and \( \bar{Q} \) exchange the spinors since
\[
\begin{align*}
Q \bar{\psi} &= \psi, \\
Q \psi &= 0, \\
\bar{Q} \bar{\psi} &= \bar{\psi}, \\
\bar{Q} \psi &= 0.
\end{align*}
\]
Also the projection and nilpotent operators are related through the relations
\[
\begin{align*}
\bar{Q} Q &= \bar{P}, \\
Q \bar{Q} &= P, \\
Q P &= \bar{P} Q = 0, \\
\bar{Q} \bar{P} &= P Q = Q.
\end{align*}
\]
In terms of the \( \mathfrak{su}(2) \) Lie algebra generators \[31\], one can set
\[
\begin{align*}
P - \bar{P} &\simeq \mathbf{1} \\
Q &\simeq \mathbf{e}_+ \\
\bar{Q} &\simeq \mathbf{e}_-
\end{align*}
\]
and note that the \( U(1) \) transformations \[81\] and \[82\] leads to
\[
\begin{align*}
Q &\xrightarrow{\varphi} e^{2i\varphi} Q \\
\bar{Q} &\xrightarrow{\varphi} e^{-2i\varphi} \bar{Q}
\end{align*}
\]
while the projection operators \( P \) and \( \bar{P} \) remain intact under the \( U(1) \) transformation.

Then the following relations between the derivatives of these operators and the left Maurer-Cartan form \[61\] exist
\[
\begin{align*}
(P - \bar{P})_s &= [L, P - \bar{P}] = \kappa_c \mathbf{e}_- + \kappa_c^* \mathbf{e}_+, \\
Q_s &= [L, Q] = -i\tau_r \mathbf{e}_+ - \frac{\kappa_c^*}{2} \mathbf{t}, \\
\bar{Q}_s &= [L, \bar{Q}] = i\tau_r \mathbf{e}_- - \frac{\kappa_c}{2} \mathbf{t}
\end{align*}
\]
which, due to \[85\-87\], can be identified as the spinor Frenet equation, in the dreibein Darboux format \[35\].

By defining the \( L \)-covariant derivative
\[
D_L = \partial_s - [L, \_]
\]
one can re-write the spinor Frenet equation compactly as
\[
\begin{align*}
D_L (P - \bar{P}) &= 0 \\
D_L Q &= 0 \\
D_L \bar{Q} &= 0.
\end{align*}
\]
Furthermore, since \( (\mathbf{e}_\pm, \mathbf{t}) \) span the \( \mathfrak{su}(2) \) Lie algebra, we \[90\-92\] can reverted so that \( L \) can be represented in terms of the operators \[85\-87\] as
\[
\begin{align*}
\frac{i}{4} [P - \bar{P}, (P - \bar{P})_s] + \frac{i}{2} [Q, \bar{Q}_s] + \frac{i}{2} [\bar{Q}, Q_s]
\end{align*}
\]
which follows: First observe that
\[
\begin{align*}
\hat{e}_+ \equiv a \mathbf{t} + b_+ \mathbf{e}_+ + b_- \mathbf{e}_-. \\
\hat{e}_- \equiv b \mathbf{t} + b_- \mathbf{e}_+ + b_+ \mathbf{e}_-
\end{align*}
\]
Since the left Maurer-Cartan form \( L \) relates to \[84\] by a combination of the gauge transformations \[81\], \[84\] and \[82\] one of the two Lax pair operators of NLSE is obtained.

The other Lax pair operator of NLSE is obtained as follows: First observe that
\[
P_{ss} = \frac{1}{2} [\hat{D}_s \kappa_c \bar{Q} + D_s \kappa_c \bar{Q} - \kappa_c^2 (P - \bar{P})]
\]
where the covariant derivative is given by
\[
D_s = \partial_s + i\tau_r. \\
\]
By introducing the \( U(1) \) gauge transformation \[59\], \[85\], \[80\], the operator \[95\] is gauge invariant. Thus, by choosing
\[
\varphi = -\frac{1}{2} \int^s \tau_r (s') ds' \\
\]
i.e. using the parallel transport frame, one gets the explicit representation
\[
P_{ss} = \frac{1}{2} \left[ \partial_s \kappa_c^* \mathbf{e}_+ + \partial_s \kappa_c \mathbf{e}_- - |\kappa_c|^2 \mathbf{t} \right].
\]

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Similar to the case of \([44]\), recalling \([43]\), the left Maurer-Cartan realization of the second NLSE Lax pair operator is obtained. Explicitly, by applying the U(1) transformation \([43]\) to \([44]\) and using the identification \([88]\) leads to

\[
\mathbb{P}^h \rightarrow \mathbb{P}^{h}
\]

where

\[
\mathbb{P}^{h} = -\frac{i}{2} \sqrt{\lambda} \partial_q \sigma_+ + \frac{i}{2} \sqrt{\lambda} q \sigma_- - \frac{1}{2} \lambda |q|^2 \sigma^3
\]

\[
\equiv \frac{i}{2} V_0
\]

i.e. equation \([88]\).

**B. Hamiltonians**

From the previous subsection, we conclude that the NLSE Lax operators and the conserved quantities in the NLSE hierarchy can be constructed in terms of the two operators \([43]\) and \([44]\). However, in generic Frenet frame, the Lax pair constitutes the two operators \([43]\) and \([45]\). Thus, the conserved charges can be expected to be combinations of the projection operators \(\mathbb{P}, \mathbb{P}\); the exchange operators \(Q, \bar{Q}\); and their (covariant) derivatives. Next, we proceed to elaborate on the relations, beyond the Lax pair.

An example of a familiar Hamiltonian density is the “number operator” in the NLSE hierarchy i.e. Heisenberg spin chain Hamiltonian

\[
\mathcal{H}_2 = \text{tr} \{ \mathbb{P} \mathbb{P} \} = \frac{1}{2} |\kappa_c|^2 = \frac{1}{2} \partial_s t \cdot \partial_s t.
\]  

Using \([27]\) and \([28]\), this can be further presented as

\[
\mathcal{H}_1 = \frac{1}{2} \left| (\partial_s + \frac{i}{2} \tau_r) \psi \right|^2 = \frac{1}{2} \left| (\partial_s - <\psi^\dagger, \partial_s \psi>) \psi \right|^2
\]

i.e. the Hamiltonian of the \(\mathbb{C} P^1\) model.

Another familiar example is the NLSE Hamiltonian given by

\[
\mathcal{H}_3 = \text{tr} \{ \mathbb{P} \mathbb{S} \mathbb{P} \} = \frac{1}{2} |D_s \kappa_c|^2 + |\kappa_c|^4.
\]  

The following manifestly U(1) gauge invariant “Lax” pair operators have been considered in the context of the \(\mathbb{C} P^1\) model \([47]\):

\[
L(\lambda) = 2 \frac{[\partial_s \mathbb{P}, \mathbb{P}]}{1 + \lambda} = \frac{\mathbb{L}}{1 + \lambda}
\]

\[
\bar{L}(\lambda) = 2 \frac{[\partial_s \bar{\mathbb{P}}, \bar{\mathbb{P}}]}{1 + \lambda} = \frac{\bar{\mathbb{L}}}{1 + \lambda}.
\]

In terms of the exchange operators

\[
\mathbb{L} = \kappa_c \bar{Q} - \kappa_c^* Q
\]

and using the properties of the projection and exchange operators, it can be shown that

\[
\frac{1}{2} \mathbb{L}^{2n} = (-1)^n |\kappa_c|^{2n} \mathbb{I}
\]

\[
\text{tr} (\mathbb{L}^{2n+1}) = 0.
\]

Thus not all conserved charges in the NLSE hierarchy can be obtained from \([93]\) and \([100]\). In particular, the NLSE Hamiltonian \([85]\) can not be presented in terms of polynomials of these operators which means that derivatives of \(\mathbb{L}\) also need to be introduced:

\[
L_s = D_s \kappa_c \bar{Q} - (D_s \kappa_c)^* Q.
\]

Then

\[
\frac{1}{2} \mathbb{L}_s^{2n} = (-1)^n |D_s \kappa_c|^{2n} \mathbb{I}
\]

\[
\text{tr} (\mathbb{L}_s^{2n+1}) = 0.
\]

In this case, using combinations of \(\mathbb{L}\) and \(\mathbb{L}_s\) various U(1) gauge invariant and conserved densities can be constructed.

For example, the NLSE Hamiltonian \([85]\) can also be presented as

\[
\mathcal{H}_3 \mathbb{I} = -\frac{1}{2} L_s^2 + \gamma \mathbb{L}^4.
\]

where \(\gamma\) is a parameter while the number operator \([97]\) becomes

\[
\mathcal{H}_1 \mathbb{I} = -\frac{1}{4} L^2.
\]

Furthermore, in analogy with \([99]\) by introducing the U(1) gauge invariant operator

\[
T = [L_s, L]
\]

the gauge invariant conserved momentum density of the NLSE hierarchy

\[
T = 8i \mathcal{H}_2 (\bar{\mathbb{P}} - \mathbb{P})
\]

is derived. Thus, the conserved momentum density can be presented as follows

\[
\mathcal{H}_2 \mathbb{I} = \frac{i}{8} \left\{ [\bar{\mathbb{P}} T - T \bar{\mathbb{P}}] \right\}
\]

\[
= -\frac{1}{4} \left\{ \tau_r |\kappa_c|^2 - \frac{1}{2} (\kappa_c \partial_s \kappa_c^* - \partial_s \kappa_c \kappa_c^*) \right\}.
\]

Recall that the U(1) invariant “super-current” that appears

\[
J = \tau_r - \frac{1}{2} \kappa_c \partial_s \kappa_c^* - \partial_s \kappa_c \kappa_c^* |\kappa_c|^2
\]

is akin to the super-current introduced in \([45]\).
The above construction does not exhaust the full set of conserved charges of the NLSE hierarchy. In particular, for the conserved helicity (which can not be derived from the standard NLSE Lax pair) one concludes that
\[ Q_s Q = -\frac{\kappa_c}{2} Q, \]
\[ P_s P = \frac{\kappa_c}{2} \bar{Q}, \]
leading to the equation
\[ [P, Q_s] = -i\tau_r Q. \]
That way the helicity density can be derived since
\[ H_{-2} = \{ \bar{Q}, Q_s \} = -i\tau_r I. \] (101)
Its integral is a conserved quantity in the NLSE hierarchy, and also invariant under the U(1) frame rotation [22] provided that
\[ \varphi(0) = \varphi(L) = 0. \]
It is notable that a gauge transformation such as [22], does not preserve helicity.
Finally, let us point out that the length of the string
\[ L = \int_0^L ds \sqrt{x \cdot \dot{x}} \equiv \int_0^L ds H_{-1} \]
also appears as a conserved charge in the NLSE hierarchy, albeit it can not be derived from the NLSE Lax pairs. Its density can be identified e.g. with
\[ H_{-1} \simeq P + \bar{P}. \]
Note that \( H_{-1} \) relates intimately to the Nambu action, and thus to the Polyakov action of a relativistic string. More generally, a combination of \( H_{-1} \) and \( H_1 \) introduced in [37] constitutes the essence of Polyakov’s rigid string action [23].

VIII. DECOMPOSING LAX PAIR

In what follows let us consider a generic Riemann surface in \( S \in \mathbb{R}^3 \). In the sequel, \( (u, v) \) denote generic coordinates on the Riemann surface. Two components of the su(2) Lie algebra valued right Maurer-Cartan form the following currents
\[ R_u = g^{-1}\partial_u g, \quad R_v = g^{-1}\partial_v g \]
which satisfy the Lax pair. However, for a flat connection the zero-curvature condition [23] becomes an identity, and there is no equation left to be satisfied. In order to construct a Lax pair yielding an integrable system, a Maurer-Cartan form needs to be deformed.

A. Abelian Higgs Model

Let us start with the following decomposition of a generic two-dimensional SU(2) Yang-Mills connection
\[ A^a = C_\alpha n^a + \epsilon^{abc} \partial_a n^b n^c + \rho_1 \partial_\alpha n^a + \rho_2 \epsilon^{abc} \partial_\alpha n^b n^c \] (102)
where \( \alpha = 1, 2 \) correspond to \( u, v \); \( C \) is a two-component field; and
\[ \rho = \rho_1 + i\rho_2 \]
is a complex field. Note that for \( \rho = 0 \) and \( C \) identified with [15], the decomposition (102) becomes a left Maurer-Cartan form, e.g. like (92). In particular, a nontrivial multiplet \( (C_\mu, \rho) \) specifies a deformation of the Maurer-Cartan form.

We first argue that (102) is a complete decomposition of the two dimensional SU(2) connection. To see that simply observe that in the l.h.s. of (102) there are six real valued field degrees of freedom. On the other hand, in the r.h.s. decomposition there are: two real field degrees of freedom in the complex field \( \rho \); two in the one-form \( C \); and two in the unit vector \( n^a \). Thus the number of field degrees of freedom matches. A detailed proof of the completeness is presented in [23].

Note that under the gauge transformation around the direction of \( n \)
\[ h = \exp\{i\theta n \cdot \sigma\} \]
the functional form (102) remains intact when redefining
\[ C_\alpha \xrightarrow{h} C_\alpha + 2\partial_\alpha \theta, \quad \rho \xrightarrow{h} e^{-2i\theta} \rho. \]
This follows due to the property of the decomposition under a gauge transformation. Thus \( (C, \rho) \) comprises an Abelian Higgs multiplet.

In accordance with [19] let us introduce a \( g \in SU(2) \) matrix defined by
\[ \hat{g} = g \sigma^3 g^{-1} \] (103)
so that the decomposition (102) can be written as
\[ A = g \left( C \sigma^3 + i R^{\text{diag}} + \rho_1 [R, \sigma^3] - i\rho_2 R^{\text{off}} \right) g^{-1} - i L. \] (104)
Here \( R^{\text{diag}} \) is the diagonal part, \( R^{\text{off}} \) is the off-diagonal part of the right Maurer-Cartan form \( R \) and, \( L \) is the left Maurer-Cartan form of \( g \). Thus the decomposed connection (104) is gauge equivalent to
\[ B = C \sigma^3 + i R^{\text{diag}} + \rho_1 [R, \sigma^3] - i\rho_2 R^{\text{off}}. \]
We remark that \( g \) is not unique, there is the U(1) latitude [22] and [52] which leaves (102) invariant under the following right multiplication
\[ g \to g e^{i\theta \sigma^3}. \]
Note that, using the local coordinate parametrization
\[ n = \begin{pmatrix} \cos \varphi \sin \theta \\ \sin \varphi \sin \theta \\ \cos \theta \end{pmatrix} \]
and introducing the notation \[ \text{[46]}, \] the off-diagonal components of \( B \) take the form
\[ B^{\pm} = (\rho_1 \pm i \rho_2) (d \vartheta \mp i \sin \vartheta d \varphi) \approx (\rho_1 \pm i \rho_2) B_{\pm}. \]
Thus \[ \text{[102]} \] is gauge equivalent to the deformation of the right Cartan form
\[ R \rightarrow A \]
where
\[ R = a \sigma^3 + b^+ \sigma^+ + b^- \sigma^- \]
\[ A = C \sigma^3 + \rho b^+ \sigma^+ + \rho^* b^- \sigma^- \equiv C \sigma^3 + B^+ \sigma^+ + B^- \sigma^- . \]

Then the action of the U(1) rotation on its components is of the form
\[ A \rightarrow e^{ig \vartheta^3} (C + 2d \vartheta) \sigma^3 \rightarrow e^{2g \vartheta} B^+ \sigma^+ + e^{-2g \vartheta} B^- \sigma^- . \]

By substituting \[ \text{[102]} \] in the curvature two-form
\[ F_{a \beta} = \partial_a A_{\beta} - \partial_{\beta} A_a + e^{abc} A_a A_{\beta}^c \]
one gets
\[ F_{a \beta}^\alpha = n^a (G_{a \beta} - [1 - \rho \rho^*] H_{a \beta}) + \{(\partial_a \rho_1 - C_{\alpha} \rho_2) \partial_{\beta} n^a - (\partial_{\beta} \rho_1 - C_{\beta} \rho_2) \partial_a n^a \} + e^{abc} n^a \{(\partial_a \rho_1 - C_{\alpha} \rho_2) \partial_{\beta} n^c - (\partial_{\beta} \rho_2 + C_{\beta} \rho_1) \partial_a n^c \} \]
where
\[ G_{a \beta} = \partial_a C_{\beta} - \partial_{\beta} C_a \]
\[ H_{a \beta} = \epsilon_{abc} n^a \partial_b n^c \]
and
\[ D_{a \rho} = (\partial_a + i C_a) \rho \]
\[ = (\partial_a \rho_1 - C_{\alpha} \rho_2) + i (\partial_a \rho_2 + C_{\alpha} \rho_1) . \]

But the curvature \[ \text{[103]} \] of the Lax pair of an integrable system must vanish
\[ F_{a \beta}^\alpha = 0 \]
which leads to the system of equations
\[ G_{a \beta} - (1 - \rho \rho^*) H_{a \beta} = 0 \]
\[ D_{a \rho} = 0 . \]

The above equations have the form of the Bogomolny equations \[ \text{[56]-[57]} \] for the energy function of the Abelian Higgs Model on a Riemann surface with metric
\[ g_{a \beta} = \int_S \sqrt{g} dudv \left\{ \frac{1}{4} G_{a \beta}^2 + |D_{a \rho}|^2 + (1 - |\rho|^2)^2 \right\} . \]

Vortex solutions to these equations in the background of a given Riemann surface have been studied extensively in \[ \text{[60]-[72]} \], to which we refer for explicit constructions.

Here it suffices to make the following remarks: Assume that \( x(u, v) \) describes the points on the surface of \( S \in \mathbb{R}^3 \), and that the two tangent vector fields \( \partial_u x \) and \( \partial_v x \) are linearly independent. Let us identify \( n \) as the normal vector of \( S \)
\[ n = \frac{x_u \times x_v}{||x_u \times x_v||} \]
which defines the Gauß map
\[ n : S \mapsto S^2 . \]

Then
\[ \partial_u n \times \partial_v n = \frac{1}{2} R \partial_u x \times \partial_v x = K \partial_u x \times \partial_v x \]
where \( R \) is the scalar curvature which is twice the Gaussian curvature \( K \) of \( S \).

Next let us introduce the zweibein field \( e^i_\alpha \),
\[ \delta_{ij} e^i_\alpha e^j_\beta = g_{\alpha \beta} = \delta_{\mu \nu} \partial_\alpha x^\mu \partial_\beta x^\nu \]
where \( g_{\alpha \beta} \) is the induced metric. For the spin connection \( \omega^i_{\alpha \beta} \) first define
\[ \omega^i_\alpha = \frac{1}{\sqrt{g}} \delta_{ij} e^j_\alpha e^i_\beta \partial_\beta e^j_\gamma \partial_\gamma \epsilon^j_\beta \]
and then set
\[ \omega^i_{\alpha \beta} = \omega^i_\alpha e^i_\beta . \]

Note that then the Gauß map, the spin connection and the scalar curvature of \( S \) are related since
\[ \frac{1}{2} \epsilon_{abc} e^a_{\alpha \beta} \partial_\alpha n^b \partial_\beta n^c = e^a_{\alpha \beta} \partial_\alpha \omega^c_\beta = \frac{1}{2} \sqrt{g} R . \]

Now by considering the combined Weyl transformation and SO(2) rotation such that
\[ e^i_\alpha \mapsto e^i_\alpha \epsilon^j_\alpha \left( \delta^j_\alpha \cos \frac{\theta}{2} - \epsilon^j_\alpha \sin \frac{\theta}{2} \right) \]
the metric tensor, the spin connection and the scalar curvature of \( S \) transform to
\[ g_{\alpha \beta} \mapsto e^\phi g_{\alpha \beta} \]
\[ \omega_\alpha \mapsto \omega_\alpha + \sqrt{g} \epsilon^\alpha_\beta \partial_\beta \phi + \partial_\alpha \theta \]
\[ R \mapsto e^{-\phi} (R + \Delta g \phi) \]
where
\[ \Delta g = -\frac{1}{\sqrt{g}} \partial_\alpha (\sqrt{g} g^{\alpha \beta} \partial_\beta) \]
is the Laplacian on \( S \).
Finally, let us write the first equation (109) as
\[ G = \frac{1}{2} (1 - p\bar{p}) \sqrt{g} R \] (113)
and assume that the surface $S$ is closed. By integrating (113) over the surface one gets
\[ \text{Ch}_1[C] \overset{\text{def}}{=} \frac{1}{2\pi} \int_S G \]
\[ = \frac{1}{4\pi} \int_S \sqrt{g} R - \frac{1}{4\pi} \int_S p\bar{p} \sqrt{g} R \]
\[ \overset{\text{def}}{=} \chi - \frac{1}{4\pi} \int_S p\bar{p} \sqrt{g} R \] (114)
where $\text{Ch}_1[C]$ is the first Chern character of $G$ and $\chi$ is the Euler character of $S$. Note that although for a compact surface with no boundary each is an integer, in the case of e.g. hyperbolic manifolds this does not need to be the case.

The effect of the Weyl transformation (112) in the relation
\[ \frac{1}{4\pi} \int_S \rho\bar{p} \sqrt{g} R = \chi - \text{Ch}_1 \]
leaves intact the two quantities in the r.h.s. while the l.h.s. quantity becomes
\[ -\frac{1}{4\pi} \int_S \Delta_\phi (\rho\bar{p}) \phi \]
which, for general $\phi$, vanishes only if $\rho\bar{p}$ is harmonic.

On the other hand, in terms of the complex coordinate $z = u + iv$ the second equation (109) becomes
\[ D_z \rho = (\partial_z + iC_z) \rho = 0 \]
implying that the compination
\[ \rho e^{-i\int C_z} \]
is (anti-)holomorphic.

In general, the product of two holomorphic functions is not harmonic. However, examples of non-trivial solutions can be obtained, with constant $|\rho| \neq 1$. E.g. for $S$ being a punctured Riemann surface; or for $S$ being a hyperbolic Riemann surface; or, more generally, for $S$ being a Riemann surface that is not simply connected.

In such cases, there are generically loops for which the line integral
\[ \oint d\tilde{C} \]
does not need to vanish, and the solution to the first equation (109) is
\[ C = (1 - |\rho|^2) d^{-1} H. \]
For example, if $\rho$ vanishes on $S$ the one-form $C$ is like the Dirac monopole field and (102) coincides with (102).

B. Surfaces in $\mathbb{R}^3$

Next the relations between the two dimensional decomposed Yang-Mills theory and the Gauß-Godazzi equations is considered.

Let $x(u, v)$ describe the points of a Riemann surface $S$ in $\mathbb{R}^3$, in terms of the local coordinates $(u, v)$ on $S$. Then the Gauß-Codazzi and the Weingarten equation are given by
\[ x_{\alpha\beta} = \Gamma^\gamma_{\alpha\beta} x_\gamma + h_{\alpha\beta} n \] (115)
\[ n_\alpha = -h_{\alpha\beta} g^{\beta\gamma} x_\gamma = -h_{\alpha\beta} x_\beta, \] (116)
respectively. Here
\[ \Gamma^\alpha_{\beta\gamma} = \frac{1}{2} g^{\alpha\delta} (g_{\beta\gamma, \delta} + g_{\delta\gamma, \beta} - g_{\beta\delta, \gamma}) \]
are the Christoffel symbols, $n$ is the normal vector to the surface and $g_{\alpha\beta}$ is taken to be the (induced) metric
\[ g_{\alpha\beta} = \partial_\alpha x \cdot \partial_\beta x. \]
Finally, $h_{\alpha\beta}$ are the components of the second fundamental form
\[ h_{\alpha\beta} = h_{\beta\alpha} = \left( \begin{array}{cc} x_{uv} \cdot n & x_{uv} \cdot n \\ x_{u} \cdot n & x_{v} \cdot n \end{array} \right). \]

Note that equations (115) and (116) can be combined into the matrix equation
\[ \partial_\alpha \left( \begin{array}{c} x \\ u \\ v \\ n \end{array} \right) = \left( \begin{array}{cccc} 0 & \delta_{uu} & \delta_{uv} & 0 \\ 0 & \Gamma_{uu} & \Gamma_{uv} & h_{u} \\ 0 & \Gamma_{uv} & \Gamma_{vv} & h_{v} \\ 0 & -h_{u} & -h_{v} & 0 \end{array} \right) \left( \begin{array}{c} x \\ u \\ v \\ n \end{array} \right). \] (117)

In what follows, let us assume that at each point $x(u, v)$ of the surface we have an orthonormal frame $(e_1, e_2, n)$ in $\mathbb{R}^3$, where $n$ is the Gauß map (i.e. a unit normal vector).

For example, by choosing the coordinates $(u, v)$ so that the vectors $e_1$ and $e_2$ point into the principal directions
\[ e_1 = \frac{x_u}{||x_u||} \] (118)
\[ e_2 = \frac{x_v}{||x_v||} \] (119)
\[ n = e_1 \times e_2 \] (120)
the corresponding integral curves are the lines of curvature.

Thus (117) shows that a generic orthonormal frame in $\mathbb{R}^3$ is transported along the surface by the following equation
\[ d \left( \begin{array}{c} x \\ e_1 \\ e_2 \\ n \end{array} \right) = \left( \begin{array}{cccc} 0 & \omega_1 & \omega_2 & 0 \\ 0 & 0 & \omega_1 & 0 \\ 0 & \omega_2 & 0 & \omega_2 \\ 0 & \omega_3 & \omega_3 & 0 \end{array} \right) \left( \begin{array}{c} x \\ e_1 \\ e_2 \\ n \end{array} \right), \] (121)
where $d$ is the exterior derivative and the $\omega$’s are one-forms.

Then the integrability of $\omega_{\alpha\beta}$ implies that acting on $\Omega_{\pm}$ the exterior derivative must remain nilpotent, that is,

$$d^2 = 0$$

leading to the following structure equations

$$\begin{align*}
d\omega_1 - \omega_2 \wedge \omega_2^1 &= 0 = d\omega_2 - \omega_1 \wedge \omega_1^2 \quad (125) \\
d\omega_1 \wedge \omega_1^3 + \omega_2 \wedge \omega_2^3 &= 0. \quad (126)
\end{align*}$$

In fact, equation (122) is the Gauß equation; while equation (123) and (124) are the Codazzi equations.

In addition the following two equations (also) occur

$$\begin{align*}
d\omega_1 - \omega_2 \wedge \omega_2^1 &= 0 = d\omega_2 - \omega_1 \wedge \omega_1^2 \\
d\omega_1 \wedge \omega_1^3 + \omega_2 \wedge \omega_2^3 &= 0. \quad (126)
\end{align*}$$

Then equations (122), (125) determine the surface $S$ since: A solution to the Gauß-Codazzi equations specifies the frames $(\mathbf e_1, \mathbf e_2, \mathbf n)$ at each point on the surface; while the one-forms $\omega_1$ and $\omega_2$ can be constructed from (125) and (126), in terms of a solution to the Gauß-Codazzi equations. That way the surface $\mathbf x(u, v)$ is obtained, up to translations and rotations in $\mathbb R^3$, by integrating the equations (113) and (114), that is,

$$\begin{align*}
\frac{\partial \mathbf x}{\partial u} &= \omega_1 u \mathbf e_1 + \omega_2 u \mathbf e_2 \\
\frac{\partial \mathbf x}{\partial v} &= \omega_1 v \mathbf e_1 + \omega_2 v \mathbf e_2.
\end{align*}$$

The effect of a local frame rotation in $(\mathbf e_1, \mathbf e_2)$ around the normal vector $\mathbf n$ by an angle $\chi$ corresponds to a rotation of the one-forms:

$$\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \rightarrow \begin{pmatrix} \cos \chi & -\sin \chi \\ \sin \chi & \cos \chi \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \quad (127)$$

However, from (125) one obtains

$$\omega_2^1 = -\omega_1^2 \rightarrow \omega_1^2 + d\chi$$

i.e. $\omega_1^2$ transforms like the U(1) connection $C$ in (107), with $\chi = 2\theta$, under the frame rotations. Also, (125) and (124) imply that

$$\begin{pmatrix} \omega_3^1 \\ \omega_3^2 \end{pmatrix} \rightarrow \begin{pmatrix} \cos \chi & \sin \chi \\ -\sin \chi & \cos \chi \end{pmatrix} \begin{pmatrix} \omega_3^1 \\ \omega_3^2 \end{pmatrix}$$

thus

$$\Omega _{\pm} = \omega_3^1 \pm i \omega_3^2 \rightarrow e^{\pm i \chi} \Omega _{\pm}$$

transforms like the complex one-form $B_{\pm}$, in (107), under the U(1) rotation.

To sum up note that $(\omega_2^1, \Omega _{\pm}) \sim (C, B_{\pm})$ can be interpreted as a two dimensional SU(2) multiplet, under the SO(2)~U(1) frame rotation (127) around the normal vector $\mathbf n$.

Next note that the relation (126) remains invariant under this U(1) rotation. Moreover, by choosing $(u, v)$ to be the coordinates in the principal directions at the point $\mathbf x(u, v)$ one gets

$$\omega_3^1 = \kappa_1 \omega_1, \quad \omega_3^2 = \kappa_2 \omega_2$$

where $\kappa_1$ and $\kappa_2$ are the two principal curvatures of the surface; this is akin to choosing the Frenet frames in the case of strings. Then, the relation (122) becomes

$$d\mathbf F \sim d\omega_2 = K \omega_3^1 \wedge \omega_3^2$$

i.e. (111), implying that we have fully recovered the formalism of decomposed SU(2) Yang-Mills theory/Abelian Higgs Model in the case of generic two-dimensional Riemann surfaces (for details see section VIII.A.).

Finally, observe that

$$\kappa_1 + i \kappa_2 \simeq \rho_1 + i \rho_2.$$ 

This completes the identification of the structure of decomposed two dimensional SU(2) Yang-Mills theory, and the Gauß-Godazzi construction of surfaces.

### C. Spinors and surfaces

Next we proceed to transcribe (121) into a spinorial form, following Section III. By introducing the spinor (21), the conjugate spinor (22) and following (161), the spinor is related to the normal vector $\mathbf n$ since

$$\mathbf n = \langle \psi, \sigma \psi \rangle = -\langle \bar\psi, \sigma \bar\psi \rangle.$$ 

Note that the spinor $\psi$ is akin to the Gauß map, it defines a mapping from the surface $S$ to a complex two-sphere. Then for the tangent vectors $(\mathbf e_1, \mathbf e_2)$ of the surface $S$ following (33) and (34) one gets

$$\mathbf e_1 = \frac{1}{2} \langle \bar\psi, \sigma \bar\psi \rangle, \quad \mathbf e_2 = \frac{1}{2} \langle \psi, \sigma \psi \rangle.$$ 

The spinor version of (121), in terms of (24) (by suppressing the equation for $d\mathbf x$), becomes

$$d\psi = d \begin{pmatrix} -\bar\psi \\ \psi \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} -\bar\psi \\ \psi \end{pmatrix}$$

where the one-form valued matrix is defined as

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \langle \bar\psi, d\bar\psi \rangle & -\langle \bar\psi, d\psi \rangle \\ -\langle \psi, d\bar\psi \rangle & \langle \psi, d\psi \rangle \end{pmatrix}.$$ 

This yields to the relations

$$\begin{align*}
\omega_2^1 &= -2 \text{Im} \langle \psi, d\psi \rangle \\
\omega_3^1 &= 2 \text{Re} \langle \psi, d\psi \rangle \\
\omega_3^2 &= 2 \text{Im} \langle \bar\psi, d\psi \rangle.
\end{align*}$$

(128) (129) (130)
i.e. the spinor version of the Gauß-Godazzi equations, derived by direct substitution of (128)-(130) in (122)-(124).

Let us conclude by pointing out that in terms of the components (21) one obtains

\[
\omega^1_2 = i[z_1^* dz_1 - z_1 dz_1^* + z_2^* dz_2 - z_2 dz_2^*]
\]

i.e. the standard composite U(1) gauge field in the CP\(^1\) model. Recall that, in terms of (36)

\[
\omega^1_2 = \cos \theta d\phi_+ - d\phi_-
\]

one can recognize the structure of the Dirac monopole connection. Similarly, for \(\Omega\) one gets

\[
\omega^1_3 + i\omega^2_3 = 2(z_1 dz_2 - z_2 dz_1) = \frac{1}{2}(\mp i d\theta - \sin \theta d\phi_-).
\]

**D. NLSE as a surface in \(\mathbb{R}^3\)**

In order to construct a Lax pair that yields an integrable system, a deformation of the Maurer-Cartan form (128)-(130) needs to be introduced. In what follows, as an example it is described how the NLSE equation and the Liouville equation arises as a decomposed deformation.

In [7-8] several examples of integrable systems that describe surfaces in \(\mathbb{R}^3\) have been presented. In the case of NLSE, let us start with the induced metric

\[
g_{\alpha\beta} dw^\alpha dw^\beta = (du)^2 + \kappa^2 (dv)^2 = \partial_\alpha x \cdot \partial_\beta x dw^\alpha dw^\beta
\]

and choose the second fundamental form to be

\[
h_{\alpha\beta} = \left( \frac{\kappa}{-\lambda \kappa} - \frac{\kappa^3}{2} - \kappa \eta \right).
\]

Here \((\kappa, \lambda, \eta)\) are three functions that decompose the second fundamental form, and need to be specified. Following (118) and (119) and using the induced metric one can identify as

\[
e_1 = \frac{x_u}{\|x_u\|} = x_u
\]

\[
e_2 = \frac{x_v}{\|x_v\|} = \frac{1}{\kappa} x_v.
\]

Then comparing (117) and (121) one computes that

\[
\omega^1_1 = \kappa_u dv
\]

\[
\omega^1_3 = \lambda u du - \kappa dv
\]

\[
\omega^2_3 = \lambda du - \left( \frac{1}{2} \kappa^2 - \eta \right) dv.
\]

By substituting this in the Gauß and Codazzi equations (122)-(124) one obtains

\[
\kappa_v = -\lambda_u \kappa - 2\lambda \kappa_u
\]

\[
\eta = \frac{1}{\kappa} \left[ \kappa_{uu} - \lambda^2 \kappa + \frac{\kappa^3}{2} \right].
\]

where, when identifying

\[
\lambda = \partial_u \Phi \quad \eta = \partial_v \Phi
\]

and

\[
\Phi(u, v) = \int^v du' \tau(u', v)
\]

the NLSE equation (21) is recovered by taking the derivative of (132) with respect to \(v\).

**E. Isothermic surfaces and integrable models**

A correspondence between two-dimensional isothermic manifolds and integrable models has been pointed out in [6-7]. In particular it has been shown that isothermic surfaces have a one-parameter family i.e. a pencil of flat connections. Thus there exists a putative one-parameter family of Lax pairs, with the parameter corresponding to the spectral parameter of integrable hierarchy.

Let us introduce isothermal coordinates so that the induced metric admits the conformal form

\[
g^{\alpha\beta} du^\alpha dw^\beta = \partial_\alpha x \cdot \partial_\beta x du^\alpha du^\beta
\]

and decompose the second fundamental form as

\[
h^{\alpha\beta} = \left( \frac{H e^{2\Phi} + \bar{Q} - \bar{Q}}{Q - \bar{Q}} \right)
\]

Accordingly,

\[
g^{\alpha\beta} h^{\alpha\beta} = \kappa_1 + \kappa_2 = 2H
\]

i.e. \(H\) is the mean curvature; and \(Q\) is the Hopf differential. For an isothermal manifold the Hopf differential is real valued, so that akin to the metric tensor, the second fundamental form becomes diagonal

\[
Q - \bar{Q} = 0.
\]

Again, by following (118) and (119) and using the induced metric one can identify

\[
e_1 = \frac{x_u}{\|x_u\|} = e^{-\Phi} x_u
\]

\[
e_2 = \frac{x_v}{\|x_v\|} = e^{-\Phi} x_v.
\]

Then by comparing (117) and (121) one computes

\[
\omega^1_1 = -\frac{1}{2} \phi_u du + \frac{1}{2} \phi_v dv
\]

\[
\omega^1_3 = -\left[ e^{-\Phi} H + e^{\Phi} Q \right]
\]

\[
\omega^2_3 = -\left[ e^{-\Phi} H - e^{\Phi} Q \right] dv.
\]

Finally by substituting this in the Gauß and Codazzi equations (122)-(124) one obtains

\[
\phi_{uu} + \phi_{uv} + e^{-2\Phi} H^2 - e^{2\Phi} Q^2 = 0.
\]

Moreover, in the case of a constant mean curvature surface for which the Hopf differential is also constant (and \(H = Q\)), the sinh-Gordon equation is derived while if \(Q = 0\) the Liouville equation is obtained.
IX. CARTAN GEOMETRY, SURFACES IN $\mathbb{R}^2$ AND TWO DIMENSIONAL YANG-MILLS THEORY

In this section the Cartan geometry \[50, 53\] is used since it provides a unified framework to describe string, surfaces, integrable models and decomposed Yang-Mills theories.

A. Cartan geometry

In the Cartan approach in order to describe a $k$-dimensional Riemannian manifold $\mathcal{X}$ one utilizes a model space, i.e., a tangent space manifold $\mathcal{M}$ that can be more elaborate than the Euclidean $\mathbb{R}^k$ used in the Riemannian approach.

Let us assume that the model space $\mathcal{M}$ is a $k$-dimensional homogeneous coset

$$\mathcal{M} \simeq G/H.$$  

In particular, consider

$$\mathcal{M} \simeq SU(2)/U(1) \sim S^2.$$  

Next consider the following general approach: The total space $G$ is a Lie group that acts transitively on $\mathcal{M}$ while the gauge group $H$ is the stabilizer subgroup. Accordingly the Lie algebra $\mathfrak{g}$ of $G$ is resolved into the vector space sum, that is,

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}.$$  

Here $\mathfrak{h}$ is the Lie algebra of $H$, and $\mathfrak{m}$ spans the linear tangent space of $\mathcal{M}$. Then the Lie algebra $\mathfrak{g}$ has the following structure,

$$[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h},$$  

$$[\mathfrak{h}, \mathfrak{m}] \subseteq \mathfrak{m},$$  

$$[\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{h} + \mathfrak{m}.$$  

In addition, consider a Yang-Mills connection one-form $A$ with gauge group $G$, defined on the Riemannian manifold $\mathcal{X}$. Then the Lie algebra $\mathfrak{g}$ valued connection can be decomposed into a linear combination as

$$A = \omega + e$$  

where $\omega$ is the spin connection of $\mathcal{X}$, i.e., the projection of $A$ onto $\mathfrak{h}$. The co-frame $e$ of $\mathcal{X}$ is the projection of $A$ onto $\mathfrak{m}$.

By substituting the decomposition \[135\] in the curvature two-form one gets

$$F = dA + A \wedge A = F_h + F_m$$  

where $F_h$ is the projection of $F$ onto $\mathfrak{h}$

$$F_h = R + (e \wedge e)|_h = d\omega + \omega \wedge \omega + (e \wedge e)|_h$$ \[137\]  

while $F_m$ the projection onto $\mathfrak{m}$

$$F_m = T + (e \wedge e)|_m = de + \omega \wedge (e \wedge e)|_m.$$ \[138\]

Note that \[137\] and \[138\] correspond to the Cartan structure relations when we identify $\mathsf{R}$ as the Riemann-Cartan curvature two-form and $T$ as the torsion two-form of the manifold $\mathcal{X}$.

In addition, \[137\] and \[138\] are covariant under gauge transformations in the subgroup $H$. In particular, the gauge transformation $h \in H$ acts as follows

$$\omega \to h^{-1}\omega h + h^{-1}dh$$  

$$e \to h^{-1}eh$$  

$$F \to h^{-1}Fh.$$ \[139\]

Then the Yang-Mills action on $\mathcal{X}$ becomes decomposed into

$$S = \frac{1}{4} \int \text{tr} (F \wedge \ast F)$$  

$$= \frac{1}{4} \int \left\{ 2 R \wedge \ast (e \wedge e)|_h + (e \wedge e) \wedge \ast (e \wedge e) \right\}.$$ \[140\]

Note that we tacitly assume that the Killing metric is non-degenerate, with $\mathfrak{h}$ and $\mathfrak{m}$ mutually orthogonal. We also note that the first term of \[140\] has the functional form of the Einstein-Hilbert action in the Palatini formalism, while the second term is akin to the cosmological constant contribution.

Explicitly, if $f_{\alpha\beta}^a$ for $\alpha, \beta, \ldots = 1, \ldots, \text{dim}(\mathfrak{g})$ are the Lie algebra structure constants of $\mathfrak{g}$ and $r, s, \ldots = 1, \ldots, \text{dim}(\mathfrak{m})$ label the subspace $\mathfrak{m}$ and $a, b, \ldots = \text{dim}(\mathfrak{m})+1, \ldots, \text{dim}(\mathfrak{g})$ label the remaining $\mathfrak{h}$, the relations \[137\] and \[138\] become

$$F_h^a = d\omega^a + f_{\alpha\beta}^a \omega^\beta \wedge \omega^\alpha + f_{aB}^a e^B \wedge e^B$$  

$$F_m^a = de^a + f_{ar}^a \omega^r \wedge e^r + f_{rt}^s e^r \wedge e^t.$$  

Moreover, when $A$ being the $\mathfrak{g}$ valued Maurer-Cartan form

$$A = g^{-1}dg$$

the total curvature vanishes, and the local geometry of $\mathcal{X}$ coincides with that of the model space $\mathcal{M}$. In this case from \[137\] the curvature two-form $R$ of $\mathcal{X}$ can be obtained since

$$F_h = 0 \Rightarrow R \equiv d\omega + \omega \wedge \omega = -(e \wedge e)|_h.$$  

This implies in particular that $\mathcal{X}$ is locally conformally flat. Similarly, from \[138\] the components of the torsion two-form $T$ can be derived since

$$F_m = 0 \Rightarrow T \equiv de + \omega \wedge e = -(e \wedge e)|_m.$$
Next consider the specific case: \( G = SU(2) \). The model space is \( \mathcal{M} = S^2 \), while the Hopf fibration

\[
SU(2)/U(1) \simeq S^3/S^1 \sim S^2
\]

coincides with the Dirac monopole bundle in \( \mathbb{R}^3 \). As before, we choose the SU(2) Lie algebra generators to be the standard Pauli matrices, and introduce the complex combinations \( \sigma^\pm \) defined in \([50]\). In this representation,

\[
\text{su}(2) \simeq \text{u}(1) \oplus T S^2
\]

with \( T S^2 \) the tangent bundle of \( S^2 \), the Hopf fibration structure of the SU(2) Lie algebra is manifest since

\[
[u(1), T S^2] \subseteq T S^2
\]

\[
[T S^2, T S^2] \subseteq u(1)
\]

\[
[u(1), u(1)] \subseteq u(1)
\]

where \( \sigma^3 \) in \( u(1) \) and \( \sigma^\pm \) the basis for \( T S^2 \). Then the SU(2) valued connection one-form \( A \) is on the manifold \( X \) (which we shall specify in more detail in the sequel). In line with \([135]\) we decompose \( A \) as

\[
A = \omega \sigma^3 + e \sigma^+ + \bar{e} \sigma^- = \begin{pmatrix} \omega & 0 \\ 0 & -\omega \end{pmatrix} + \begin{pmatrix} e & 0 \\ 0 & -\bar{e} \end{pmatrix}
\]

where the off-diagonal part \( e \) of \( A \) in a complex basis are represented by the holomorphic polarization of the SU(2) Lie algebra. Then, the Cartan curvature two-form \( F \) decomposes as

\[
F = dA + A \wedge A = \begin{pmatrix} d\omega - e \wedge \bar{e} & 0 \\ 0 & -d\omega + e \wedge \bar{e} \end{pmatrix} + \begin{pmatrix} 0 & de + 2\omega \wedge e \\ -d\bar{e} - 2\bar{e} \wedge \bar{e} & 0 \end{pmatrix}.
\]

(141)

**B. Two dimensional Yang-Mills**

Finally, let us relate the previous construction to that of decomposed Yang-Mills, presented in section VIII. Assume that the manifold \( X \) is a Riemann surface in \( \mathbb{R}^3 \), implying that \([141]\) vanishes. Then the vanishing of the \( \sigma^3 \sim u(1) \) component of \( F \) implies that the curvature is

\[
\mathcal{R} = d\omega = e \wedge \bar{e}.
\]

(142)

Similarly, the \( \sigma^\pm \sim TS^2 \) component states that the torsion vanishes, \( i.e. \)

\[
\mathcal{T} = de + \omega \wedge e = 0.
\]

In the Cartan geometry, these two equations are the Bogomolny equations of \([109]\).

In addition, the integral

\[
\frac{1}{4\pi} \int \mathcal{R} = \frac{1}{4\pi} \int e \wedge \bar{e}
\]

over a surface in \( \mathbb{R}^3 \) yields the Gauß-Bonnet formula that computes its Euler characteristic; see equation \([114]\) and the subsequent analysis.

For example in order to use explicit formulas, let us introduce \( g \in SU(2) \) to be of the form

\[
g = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}
\]

where, motivated by the Gauß map \([106]\) and \([111]\), set

\[
\alpha = \cos \frac{\vartheta}{2} e^{i\psi} \equiv \cos \frac{\vartheta}{2} e^{i\psi^+}
\]

\[
\beta = \sin \frac{\vartheta}{2} e^{-i\psi} \equiv \sin \frac{\vartheta}{2} e^{i\psi^-}.
\]

In this case \((\vartheta, \psi^\pm)\) can be identified as the angles that parametrize the two hemispheres of \( S^2 \in \mathbb{R}^3 \) of the Gauß map of the surface \( X \), possibly with \( n \)-fold covering of the ensuing sphere. The integer \( n \) counts the number of \( 2\pi \) circulations in \( \psi^\pm \).

Then the Maurer-Cartan form, see \([10] \), \([17] \) and \([106]\), becomes

\[
\omega = \frac{1}{2} (\cos \vartheta \, d\psi + d\phi) \quad (143)
\]

\[
e = \frac{1}{2} e^{-i\varphi} (i\vartheta d\beta + \sin \vartheta \, d\psi) \quad (144)
\]

Thus the standard Dirac (Hopf) monopole bundle in \( \mathbb{R}^3 \) has been recovered, with \( \omega \) being the monopole connection, while the monopole number coincides with \( n \). Moreover, the \( r.h.s. \) of \((142)\) becomes

\[
e \wedge \bar{e} = -\frac{i}{2} \sin \vartheta d\vartheta \wedge \psi
\]

i.e. the standard functional form of volume two-form on the two-sphere \( S^2 \) of the Gauß map.

**X. D=4 YANG-MILLS AND EMBEDDED TWO DIMENSIONAL INTEGRABLE MODELS**

In this section, the four dimensional SU(2) Yang-Mills theory and its embedded two dimensional models is studied. In particular, the two dimensional embedding structures are related to strings, their dynamics, and Riemann surfaces in \( \mathbb{R}^3 \). The technique used is a Kaluza-Klein reduction in combination of field variable elimination akin to the Hopf differential \([83]\).

Let us consider a four dimensional manifold that has the product structure \( \mathcal{M} \times \mathcal{N} \) where \( \mathcal{M} \) and \( \mathcal{N} \) are both two dimensional. The manifold \( \mathcal{M} \) is the base manifold. When needed, it is equipped with the local coordinates \((x_1, x_2) \sim (u, v)\) which correspond to the space-time coordinates of the field variables in the putative two dimensional integrable field theory. The Kaluza-Klein reduction takes place with respect to the two dimensional auxiliary manifold \( \mathcal{N} \) fibered over \( \mathcal{M} \) (or \( \text{vice versa} \)).
Then the imposed requirement on $\mathcal{M}$ and $\mathcal{N}$ is that the four dimensional product manifold $\mathcal{M} \times \mathcal{N}$ is locally conformally flat. Remark: This geometric structure will be imposed upon us in the sequel, by the structure of the decomposed four dimensional Yang-Mills theory.

- Examples with compact $\mathcal{M}$ include Riemann surfaces with genus $g \geq 1$, that are fibered by the two-dimensional torus $\mathcal{N} = \mathbb{T}^2 = S^1 \times S^1$.
- Examples of $\mathcal{M}$ with hyperbolic geometry include warped products of the Poincaré half-plane $\mathbb{H}^2$ fibered with manifolds such as $\mathbb{R}^2$, $S^2$, $\mathbb{T}^2$, etc.
- Note that neither the product manifold $S^2 \times S^2$ nor the product manifold $S^2 \times \mathbb{R}$ is locally conformally flat.

### A. Yang-Mills Field

Let us start by using the holomorphic basis (50) of the $\mathfrak{su}(2)$ Lie algebra, i.e. the following expansion of the connection one-form $A$:

$$A = A^3_\mu \sigma^3 dx^\mu + X^+\mu \sigma^- dx^\mu + X^-\mu \sigma^+ dx^\mu \quad (145)$$

where

$$X^\pm_\mu = A^1_\mu \pm i A^2_\mu.$$  

Similarly, to the two dimensional Cartan’s formalism select the $SU(2)$ gauge group to be the total space $G$; acting on the gauge field in the usual fashion

$$A \to gAg^{-1} + 2i gdg^{-1}. $$

The gauge group $H$ is chosen to be the diagonal Cartan subgroup $H \simeq U_C(1) \in SU(2)$. In particular, by setting

$$h = e^{i \omega X^3} \in U_C(1)$$

one finds that the component $A^3_\mu \sim A_\mu$ transforms as a $U_C(1)$ gauge field

$$\delta h A_\mu = \partial_\mu \omega$$

while the off-diagonal $X^\pm_\mu$ gives

$$\delta h X^\pm_\mu = e^{\mp i \omega} X^\pm_\mu.$$ 

In parallel with (139) the Yang-Mills field strength tensor

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + e^{abc} A^b_\mu A^c_\nu \quad (a = 1, 2, 3)$$

decomposes as follows:

The diagonal Cartan component gives

$$F^3_{\mu\nu} = F_{\mu\nu} + P_{\mu\nu} \sim F_h$$

$$= \partial_\mu A_\nu - \partial_\nu A_\mu + \frac{i}{2} (X^\mu_+ X^\nu_- - X^\nu_+ X^\mu_-) \quad (146)$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$P_{\mu\nu} = \frac{i}{2} (X^\mu_+ X^\nu_- - X^\nu_+ X^\mu_-). \quad (147)$$

Finally, the off-diagonal components becomes

$$F^\pm_{\mu\nu} = F^1_{\mu\nu} \pm i F^2_{\mu\nu} \sim F_h$$

$$= (\partial_\mu \pm i A_\mu) X^\pm_\nu - (\partial_\nu \pm i A_\nu) X^\pm_\mu$$

$$= D^\pm_\mu X^\pm_\nu - D^\pm_\nu X^\pm_\mu.$$  

#### 1. Grassmanian structure

The antisymmetric tensor $P_{\mu\nu}$ in (147) obeys the condition

$$P_{12}P_{34} - P_{13}P_{24} + P_{23}P_{14} = 0.$$

In the context of projective geometry, this is the relation that defines the Plücker coordinates of the Klein quadric. The Klein quadric describes the embedding of the real Grassmannian $G(4, 2)$ in the five dimensional projective space $\mathbb{RP}^5$ as a degree four hypersurface. This Grassmannian is the four dimensional manifold of two dimensional planes that are embedded in $\mathbb{R}^4$. It coincides with the homogeneous space

$$G(4, 2) \sim \frac{SO(4)}{SO(2) \times SO(2)} \simeq S^2 \times S^2.$$  

Let us describe a two-plane in $\mathbb{R}^4$ with an orthonormal zweibein

$$e_\alpha^a e^\beta = \delta^a_\beta \quad (\alpha, \beta = 1, 2)$$

or in terms of the complex base,

$$e_\mu = \frac{1}{\sqrt{2}} (e_1^\mu \pm i e_2^\mu). \quad (148)$$

Note that

$$e_\mu e^\mu = 0, \quad e_\mu e^\mu = 1.$$  

Since $e_\mu$ spans a two dimensional plane in four dimensions, an additional complex zweibein can be introduced

$$m_\mu = \frac{1}{\sqrt{2}} (m_1^\mu \pm i m_2^\mu).$$

That way an orthonormal basis that spans the entire $\mathbb{R}^4$ is obtained. That is,

$$<e, e> = <m, m> = <e, m> = 0$$

$$<e, e> = <m, m> = 1.$$  

Next by combining the $(e_\mu^a, m_\mu^a)$ into a $SO(4)$ valued $4 \times 4$ zweibein matrix

$$\varepsilon^\kappa_\mu = (e_1^\mu, e_2^\mu, m_1^\mu, m_2^\mu)$$

the ensuing $SO(4) \simeq SU(2) \times SU(2)$ Cartan equation

$$d\varepsilon^\kappa + \omega^\kappa_\tau \varepsilon^\tau = 0.$$
gives the components of the Levi-Civita connection one-form \( \omega^\kappa_\tau \) in terms of \( e \) and \( m \). That is,

\[
de \, e = \langle \bar{e}, de \rangle e + \langle \bar{m}, de \rangle m + \langle m, de \rangle \bar{m}, \\
dm \, e = \langle \bar{m}, dm \rangle m + \langle \bar{e}, dm \rangle e + \langle e, dm \rangle \bar{e}.
\]

Next let us identify as

\[
C = i \langle \bar{e}, de \rangle 
\]

the \( U_I(1) \) connection. Then by introducing a local frame rotation on the two-plane (\( e, \bar{e} \)) that sends

\[
e_\mu \rightarrow e^{-i\lambda} e_\mu
\]

leads to

\[
C \rightarrow C + d\lambda.
\]

Similarly, the dual connection

\[
Q = i \langle \bar{m}, dm \rangle
\]

transforms like a \( U(1) \) gauge field

\[
Q \rightarrow Q + d\chi
\]

under a rotation of the \( (m^1_\mu, m^2_\mu) \) frame that spans the (tangent plane of) \( N \), i.e.

\[
m \rightarrow e^{-i\chi} m.
\]

The remaining components of \( \omega^\kappa_\mu \)

\[
\Phi^+_\mu = \langle m, \partial_\mu e \rangle, \\
\Phi^-_\mu = \langle m, \partial_\mu \bar{e} \rangle
\]

together with their complex conjugates transform homogeneously under the rotations of the \( (e^1_\mu, e^2_\mu) \) and \( (m^1_\mu, m^2_\mu) \) basis vectors since

\[
\Phi^{\pm}_\mu \rightarrow e^{-i(\chi \mp \lambda)} \Phi^{\pm}_\mu.
\]

Finally, by combining the \( \omega^\kappa_\mu \) into the two \( SU(2) \) Lie-algebra valued one-forms,

\[
(Q + C) \sigma^3 + \frac{1}{2i} \Phi^+ \sigma^+ + \frac{1}{2i} (\Phi^+)^* \sigma^- \\
(Q - C) \sigma^3 + \frac{1}{2i} \Phi^- \sigma^+ + \frac{1}{2i} (\Phi^-)^* \sigma^-
\]

a direct computation reveals that

\[
d (Q \pm C) = \frac{i}{2} \Phi^{\pm} \wedge (\Phi^{\pm})^* \\
d\Phi^{\pm} = -i (Q \pm C) \wedge \Phi^{\pm}
\]

i.e. the \( SO(4) \simeq SU(2) \times SU(2) \) Maurer-Cartan structure equations.

2. Grassmannian electric-magnetic duality

Let us proceed, by defining the tensor

\[
H_{\mu\nu} = \frac{i}{2} (e_\mu \bar{e}_\nu - e_\nu \bar{e}_\mu)
\]

and introducing the electric (\( E_i \)) and magnetic (\( B_i \)) components of \( H_{\mu\nu} \),

\[
E_i = \frac{i}{2} (e_\mu \bar{e}_\nu - e_\nu \bar{e}_\mu), \\
B_i = \frac{i}{2} \epsilon_{ijk} e_j^* e_k^*.
\]

They are subject to the following properties

\[
\vec{E} \cdot \vec{B} = 0 \\
\vec{E} \cdot \vec{E} + \vec{B} \cdot \vec{B} = \frac{1}{4}.
\]

Next, by introducing the selfdual and anti-selfdual combinations

\[
\vec{s}_\pm = 2(\vec{B} \pm \vec{E})
\]

two independent unit vectors are obtained that determine two-spheres \( S^2 \) in \( \mathbb{R}^4 \).

Note that, using \( (154) \) and inverting \( (153) \) the zweibein \( e_\mu \) can be expressed as

\[
e_\mu = \frac{1}{2} e^{i\xi} \left( \sqrt{1 - \bar{s}_+ \cdot s_-}, \frac{s_+ \times s_- + i(s_- - s_+)}{\sqrt{1 - \bar{s}_+ \cdot s_-}} \right).
\]

Here the phase factor \( \xi \) that is not visible in \( H_{\mu\nu} \), is a section of the internal \( U_I(1) \) bundle. The definition

\[
\sin \eta = \frac{1}{\sqrt{2}} \sqrt{1 + \bar{s}_+ \cdot s_-} \quad \tau_{\text{eta}}
\]

leads to

\[
\vec{E} = \sin \eta \cdot \vec{p} \\
\vec{B} = \cos \eta \cdot \vec{r}
\]

where \( \vec{p} \) and \( \vec{r} \) are two orthogonal unit vectors. Together with their exterior product

\[
\vec{q} = \vec{r} \times \vec{p}
\]

one obtains an orthonormal triplet.

Also, the zweibein \( e_\mu \) take the form

\[
\frac{1}{\sqrt{2}} (e^1_\mu + i e^2_\mu) = \frac{e^{i\xi}}{\sqrt{2}} \left( \sin \eta \quad -i \bar{p} \cos \eta \cdot \vec{q} \right)
\]

\[
def \equiv \frac{e^{i\xi}}{\sqrt{2}} (e^1_\mu + i e^2_\mu)
\]

and the dual zweibein take the form

\[
m^1 = e^{i\delta} \left( \cos \eta \quad -i \vec{r} \sin \eta \cdot \vec{q} \right)
\]

\[
def \equiv e^{i\xi} (\bar{m}^1 + i \bar{m}^2).
\]
Here $\xi$ and $\delta$ are the phase for the frame rotation on the corresponding two plane.

Next, define a transformation $R$

$$R e_\mu = m_\mu$$

$$R e_\mu = \bar{m}_\mu$$

$$R^2 = I.$$ 

which is a duality transformation between the two planes that are spanned by $e_\mu$ and $m_\mu$ in $\mathbb{R}^4$, respectively. Then the action of $R$ for $H_{\mu\nu}$ coincides with the action of the Hodge duality

$$H_{\mu\nu} \xrightarrow{R} *H_{\mu\nu}$$

where

$$*H_{\mu\nu} = \frac{i}{2} (m_\mu \bar{m}_\nu - m_\nu \bar{m}_\mu).$$

On the electric and magnetic components of $H_{\mu\nu}$, the action of $R$ coincides with the electric-magnetic duality transformation

$$R \left( \begin{array}{c} B \\ E \end{array} \right) = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \left( \begin{array}{c} B \\ E \end{array} \right).$$

The two unit vectors (154) are eigenvectors of $R$ since

$$R \bar{s}_\pm = \pm \bar{s}_\pm.$$ 

Thus $m_\mu$ can be expressed in terms of $\bar{s}_\pm$, up to the overall phase, by simply acting with $R$ in (159), i.e.

$$m_\mu = \frac{1}{2} e^{i\phi} \left( \sqrt{1 + \bar{s}_+ \cdot \bar{s}_+} \cdot \bar{s}_+ - i (\bar{s}_- + \bar{s}_+) \right).$$

Finally, equations (150) and (151) lead to

$$C_{\mu\nu} = \partial_\mu C_{\nu} - \partial_\nu C_{\mu}$$

$$= \bar{s}_+ \cdot (\partial_\mu \bar{s}_+ \times \partial_\nu \bar{s}_+) + \bar{s}_- \cdot (\partial_\mu \bar{s}_- \times \partial_\nu \bar{s}_-) + \Sigma_{\mu\nu}(\lambda)$$

(157)

$$Q_{\mu\nu} = \partial_\mu Q_{\nu} - \partial_\nu Q_{\mu}$$

$$= \bar{s}_+ \cdot (\partial_\mu \bar{s}_+ \times \partial_\nu \bar{s}_+) - \bar{s}_- \cdot (\partial_\mu \bar{s}_- \times \partial_\nu \bar{s}_-) + \Sigma_{\mu\nu}(\delta)$$

(158)

with last terms in (157) and (158) akin to Dirac string contributions,

$$\Sigma_{\mu\nu}(\lambda) = [\partial_\mu, \partial_\nu] \lambda$$

$$\Sigma_{\mu\nu}(\delta) = [\partial_\mu, \partial_\nu] \delta.$$ 

Note that the first two $\bar{s}_\pm$ dependent terms in (154) and (158) are related to each other by the duality $R$.

In what follows, consider the definition (153). If $\bar{E} = 0$ then $\bar{s}_+ = \bar{s}_- = \bar{s}$ while

$$C_{\mu\nu} = 2 \bar{s} \cdot (\partial_\mu \bar{s} \times \partial_\nu \bar{s}) + \Sigma_{\mu\nu}(\lambda)$$

$$Q_{\mu\nu} = \Sigma_{\mu\nu}(\delta).$$

i.e. $C_{\mu}$ can be interpreted in terms of a connection for magnetic monopoles, and, similarly, $Q_{\mu}$ is a connection for magnetic strings. On the other hand, by setting $\bar{B} = 0$ in (154), then $\bar{s}_+ = \bar{s}_- = \bar{s}$ while

$$C_{\mu\nu} = \Sigma_{\mu\nu}(\lambda)$$

$$Q_{\mu\nu} = 2 \bar{s} \cdot (\partial_\mu \bar{s} \times \partial_\nu \bar{s}) + \Sigma_{\mu\nu}(\delta).$$

Thus $Q_{\mu\nu}$ is a connection for electric monopoles and $C_{\mu}$ is a connection for electric strings.

Let us conclude, by introducing the following explicit realization

$$\bar{s}_\pm = \left( \begin{array}{c} \cos \phi \pm \sin \phi \cos \theta \\ \sin \phi \pm \sin \phi \sin \theta \\ \cos \theta \end{array} \right)$$

implying that

$$C_{\mu} = - \cos \theta \epsilon_{\phi, \phi} - \cos \theta \epsilon_{\phi, \phi} + d\lambda$$

$$Q_{\mu} = - \epsilon_{\phi, \phi} \epsilon_{\phi, \phi} + \cos \theta \epsilon_{\phi, \phi} + d\delta.$$ 

Once more the functional forms of the Dirac monopoles are resolved, except that (now) both electric and magnetic monopoles occur; see (17) and (143). In addition, from (152) one obtains

$$\Phi^\pm = \mp i e^{\pm i\sigma} (d\theta^\pm \mp i \sin \theta \pm d\phi^\pm)$$

where $\sigma$ is a phase, coinciding with (17) and (14).

3. Grassmannian Decomposition of Yang-Mills Field

Initially, the fields $X_\mu^\pm$ in (146) are decomposed in accordance with the Grassmannian structure

$$X_\mu^\pm = A^1_\mu \pm i A^2_\mu = \psi_1 e_\mu + \psi_2 e_\mu.$$ 

(159)

Since (159) are components of the SU(2) connection, the Grassmannian structure becomes locally defined:

The $\psi_\alpha$ are two complex line bundles over $\mathbb{R}^4$ and $(e_\mu, e_\mu)$ spans a bundle of two dimensional planes. These planes are akin the tangent bundle of the manifold $M$. The Hodge dual $(m_\mu, \bar{m}_\mu)$ has a natural identification as the tangent bundle of the manifold $N$. Finally, the vector bundles (154) can be viewed as the corresponding (locally defined) Gauß maps.

Then, by substituting (159) into $P_{\mu\nu}$ one gets

$$P_{\mu\nu} = \frac{i}{2} \left( |\psi_1|^2 - |\psi_2|^2 \right) (e_\mu \bar{e}_\nu - e_\nu \bar{e}_\mu)$$

$$= \frac{i}{2} \rho^2 t_3 (e_\mu \bar{e}_\nu - e_\nu \bar{e}_\mu)$$

$$\equiv \rho^2 t_3 H_{\mu\nu}.$$ 

(160)

Also, by introducing the three component unit vector

$$t = \frac{1}{\rho^2} (\psi_1^* \psi_2^*) \hat{\sigma} \left( \begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right) = \left( \begin{array}{c} \cos \phi \sin \theta \\ \sin \phi \sin \theta \\ \cos \theta \end{array} \right)$$

(161)
where \((\psi_1, \psi_2)\) are the local coordinates
\[\begin{align*}
\psi_1 &= \rho e^{ie} \cos \frac{\theta}{2} e^{-i \phi/2} \\
\psi_2 &= \rho e^{ie} \sin \frac{\theta}{2} e^{i \phi/2},
\end{align*}\] (162)
\[\rho^2 = |\psi_1|^2 + |\psi_2|^2,\] (163)
the tensor \(H_{\mu\nu}\) satisfies the relation
\[H_{\mu\nu}H_{\mu\nu} = \frac{1}{2}\]
The decomposition (159) entails an internal \(U_I(1)\) symmetry, not visible to \(A_{\mu}^a\), of the form
\[U_I(1) : e_{\mu} \rightarrow e^{-i \lambda} e_{\mu}, \psi_1 \rightarrow e^{i \lambda} \psi_1, \psi_2 \rightarrow e^{-i \lambda} \psi_2.\] (164)
This is the local rotation (150) of the frames \((e_{\mu}^1, e_{\mu}^2)\) that coincides with the two dimensional plane of the Grassmannian. The ensuing connection is given by (149) since
\[C_{\mu} \rightarrow C_{\mu} + \partial_{\mu} \lambda.\]
Finally, on the unit vector (161) the internal \(U_I(1)\) transformation acts as
\[t_\pm = \frac{1}{2}(t_1 \pm it_2) \rightarrow e^{\pm 2i \lambda} t_\pm\] (165)
while the component \(t_3\) remains intact. Thus, \(t_3\) is the projection of \(t\) onto the normal of the Grassmannian two-plane \((e_{\mu}^1, e_{\mu}^2)\) towards the direction of a “Gauss map”.

Next let us define the \(U_C(1) \times U_I(1)\) covariant derivative \(D_\mu\) as
\[D_\mu \psi_1 = (\partial_\mu + i A_\mu - i C_\mu) \psi_1,\]
\[D_\mu \psi_2 = (\partial_\mu + i A_\mu + i C_\mu) \psi_1,\]
\[D_\mu e_\nu = (\partial_\mu + i C_\mu) e_\nu.\]
On the components of the vector \(t\) it acts in the following manner:
\[(D_\mu)_{\alpha\beta} = \delta_{\alpha\beta} \partial_\mu + 2 \epsilon_{\alpha\beta\gamma} C_{\gamma}.\]

In addition of the continuous \(U_I(1)\) transformation, the decomposition (161) introduces the following discrete \(Z_2\) symmetry, that is,
\[Z_2 : \begin{align*}
e_{\mu} &\rightarrow \bar{e}_{\mu} \\
\psi_1 &\rightarrow \psi_2, \quad \psi_2 \rightarrow \psi_1, \quad C_\mu \rightarrow -C_\mu
\end{align*}\]
while for the vector \(t\) one gets
\[Z_2 : \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} \rightarrow \begin{pmatrix} t_1 \\ t_2 \\ -t_3 \end{pmatrix}.\]
This changes the orientation on the two-plane of the Grassmannian spanned by \(e_\mu\). In terms of the angular variables (102) it corresponds to
\[Z_2 : \begin{align*}
\theta &\rightarrow \pi - \theta \\
\phi &\rightarrow 2\pi - \phi.
\end{align*}\]
Thus, we may eliminate the \(Z_2\) degeneracy by a restriction to the upper hemisphere \(\theta \in [0, \pi/2]\) of \(S^2\).

Note that under the \(Z_2\) transformation the Plücker coordinates \(P_{\mu\nu}\) of (147) remains intact. However, although under the continuous \(U_I(1)\) rotation \(H_{\mu\nu}\) is invariant, for \(H_{\mu\nu}\) in (160) one gets
\[Z_2 : H_{\mu\nu} \rightarrow -H_{\mu\nu}.\]
At regular points where \((e_\mu^1, e_\mu^2)\) determines a (co-)frame \(i.e.\) span two planes, the matrix \(H_{\mu\nu}\) is non-degenerate.

On the other hand, at points where \(H_{\mu\nu}\) becomes degenerate by setting either \(\rho = 0\) or \(\theta = \pi/2\) the \(P_{\mu\nu}\) in (160) remains regular.

Finally, the vector \(t\) by setting set \(\theta = \pi/2\) simplifies to
\[t \rightarrow \begin{pmatrix} \cos \phi \\ \sin \phi \\ 0 \end{pmatrix}.\]
This corresponds to the boundary of the upper hemisphere that was introduced in order to eliminate the \(Z_2\) degeneracy.

4. Grassmannian Decomposition of Yang-Mills Lagrangian

Recall that the Yang-Mills Lagrangian is defined as
\[L_{YM} = \frac{1}{4} (F_{\mu\nu})^2 + \frac{\zeta}{2} |D_{\lambda}^A X_+^\mu|^2.\]
The second term is a gauge fixing term with \(\zeta\) being a gauge parameter.

Let us, for simplicity, choose the \(\zeta \rightarrow \infty\) gauge. This is the widely used Maximal Abelian Gauge (MAG)
\[(\partial_\mu \pm i C_\mu) X_+^\mu = D_{\lambda}^A X_+^\mu = 0.\]

When substituting the decomposition (161), (150) and (161) in the Lagrangian one gets
\[L_{YM} = \frac{1}{4} (F_{\mu\nu}^2 + 2 \rho^2 t_3 H_{\mu\nu})^2 - \frac{3}{8} \rho^4 + \frac{1}{2} |D_{\lambda}^A X_+^\mu|^2.\] (166)
Comparison with (136)-(138) leads to the following identifications terms,
\[
\frac{1}{4} \mathcal{R} \wedge \ast \mathcal{R} \sim \frac{1}{4} F_{\mu\nu}^2
\]
\[
\frac{1}{4} (e \wedge e) \wedge (e \wedge e) \sim \frac{1}{8} \rho^4
\]
\[
\frac{1}{2} \mathcal{R} \wedge (e \wedge e) \wedge \mathcal{R} \sim \rho^2 t_3 F_{\mu\nu} H_{\mu\nu}
\]
and
\[ \frac{1}{4} T \wedge * T \sim \frac{1}{2} |D^+_{\mu} X^\nu|^2. \] (167)

Note that the last term in (140) is absent.

### B. Torsion Term

Let us start by analyzing the terms in (166), one-by-one. First, note that the torsion term (167) is equal to
\[ |D^+_{\mu} X^\nu|^2 = |D_{\mu} \psi_1|^2 + |D_{\mu} \psi_2|^2 + \rho^2 |D_{\mu} e_\nu|^2 \]
\[ + \frac{1}{2} \rho^2 t_+ (\bar{D}_{\mu} e_\nu)^2 + \frac{1}{2} \rho^2 t_- (D_{\mu} e_\nu)^2. \]

(168)

Also, let us introduce the $U_C(1) \times U_I(1)$ invariant super-current
\[ 2i|\psi_1 \psi_2| J_{\mu} = \{ \psi_1 D_{\mu} \psi_1 - \psi_1 \bar{D}_\mu \psi_1 - \psi_2 D_{\mu} \psi_2 - \psi_2 \bar{D}_\mu \psi_2 \}
\[ \Leftrightarrow |\psi_1 \psi_2| J_{\mu} = \rho^2 (\partial_{\mu} \xi - t_3 K_{\mu}) \]

Note that $\xi$ comes from the explicit parametrization (102). Then the following *tHooft tensor structure exists
\[ T_{\mu \nu} = \partial_{\mu}(t_3 K_{\nu}) - \partial_{\nu}(t_3 K_{\mu}) \]
\[ = \partial_{\mu}(t_3 C_{\nu}) - \partial_{\nu}(t_3 C_{\mu}) - \frac{1}{2} t \cdot \partial_{\nu} t \times \partial_{\mu} t. \]

Next by introducing the linear combinations
\[ \left( \begin{array}{c} J_{\mu} \\ K_{\mu} \end{array} \right) = \left( \begin{array}{c} 2 \sqrt{1 + t_-} \\ 2 \sqrt{1 + t_-} \end{array} \right) \left( \begin{array}{c} J_{\mu} \\ K_{\mu} \end{array} \right) \]
\[ = \left( \begin{array}{cc} \sin \vartheta & \cos \vartheta \\ - \cos \vartheta & \sin \vartheta \end{array} \right) \left( \begin{array}{cc} J_{\mu} \\ K_{\mu} \end{array} \right) \]

one gets
\[ |D_{\mu} \psi_1|^2 + |D_{\mu} \psi_2|^2 = \frac{1}{2} (\partial_{\mu} \vartheta)^2 + \rho^2 \vartheta^2 (\partial_{\mu} \vartheta)^2 \]
\[ + \frac{\rho^2}{2} \sin^2 \vartheta (J_{\mu}^2 + K_{\mu}^2), \]

while the following two relations are satisfied
\[ |D_{\mu} t|^2 = (\partial_{\mu} \vartheta)^2 + (1 - t_3^2) K_{\mu}^2 \]
\[ = (\partial_{\mu} \vartheta)^2 + \sin^2 \vartheta K_{\mu}^2 \]
\[ F_{\mu \nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} \]
\[ = \partial_{\mu} J_{\nu} - \partial_{\nu} J_{\mu} + \frac{1}{2} [\partial_{\mu}, \partial_{\nu}] \xi. \] (169)

Finally, let us consider the last three terms in the torsion contribution (168). Using the real and imaginary components of (148), one gets
\[ \rho^2 |D_{\mu} e_\nu|^2 + \frac{1}{2} \rho^2 t_+(\bar{D}_{\mu} e_\nu)^2 + \frac{1}{2} \rho^2 t_-(D_{\mu} e_\nu)^2 = \]
\[ \frac{\rho^2}{2} g_{\alpha \beta} (\bar{D}_{\mu} e_\nu)^\alpha (\bar{D}_{\mu} e_\nu)^\beta, \quad a, b = 1, 2. \] (170)

Using the frame rotation (164) and (165) the covariant derivative and the metric tensor are defined by
\[ \hat{D}_{\mu}^{ab} = \delta^{ab} \partial_{\mu} + e^{ab}(\cos \vartheta J_{\mu} + \sin \vartheta K_{\mu}) \]
\[ g_{\alpha \beta} = \left( \begin{array}{cc} 1 + t_1 & t_2 \\ t_2 & 1 - t_1 \end{array} \right), \] (171)

while the Yang-Mills Lagrangian remains intact under this rotation.

Finally, by choosing
\[ 2\lambda = \phi - \frac{\pi}{2} \]

which sends $t_1 \to 0$ according to (165) and (164), the metric (171) becomes
\[ g_{\alpha \beta} \to \left( \begin{array}{cc} \sin \vartheta & 0 \\ 0 & 1 \end{array} \right) \]

i.e. the metric tensor of a pseudosphere with constant negative Gaussian curvature -2, in Chebyshev coordinates on the plane $\mathbb{R}^2$
\[ ds^2 = du_1^2 + du_2^2 + 2 \sin \vartheta du_1 du_2 \] (172)

when $\vartheta$ is a solution of the two-dimensional integrable sine-Gordon equation.

### C. Embedded Integrable Structures

Initially, let us express the first term of the Yang-Mills Lagrangian (166) in terms of the super-currents
\[ \frac{1}{4} (F_{\mu \nu} + 2 \rho^2 t_3 H_{\mu \nu})^2 = \frac{1}{4} \left( \partial_{\mu} J_{\nu} - \partial_{\nu} J_{\mu} + 2 \rho^2 t_3 H_{\mu \nu} \right)^2. \] (173)

For simplicity, we overlook the Dirac string contribution in (169). Also, let us point out the similarity between (173) and the first term in (108) since
\[ \partial_{\mu} J_{\nu} - \partial_{\nu} J_{\mu} + 2 \rho^2 t_3 H_{\mu \nu} \sim \partial_{\alpha} C_{\beta} - \partial_{\beta} C_{\alpha} - [1 - \bar{\rho} \rho] H_{\alpha \beta}. \]

Then the $H_{\mu \nu}$ contribution in (173) gives rise to
\[ \frac{1}{4} \rho^2_{\mu \nu} = \frac{1}{2} t_3 \rho^4 \]

which we combine with the middle term of (166). In addition, by removing all vector fields in the Yang-Mills Lagrangian; this is the analog of the steps in (133)-(154), the following structure is found to be embedded in the Yang-Mills Lagrangian
\[ \frac{\rho^2}{2} \left( (\partial_{\mu} \vartheta)^2 + \frac{\rho^2}{4} \cos^2 \vartheta \right). \]

Note that in the London limit where $\rho$ is constant, and with a straightforward Kaluza-Klein reduction with $\mathcal{N} \simeq \mathbb{R}^2$ so that $\mathbb{R}^4 \to \mathcal{M} \simeq \mathbb{R}^2$, the integrable sine-Gordon Lagrangian is obtained. Thus, we have identified
the integrable structure of the sine-Gordon hierarchy together with the ensuing pseudosphere geometry \[172\]. Note that, both are naturally embedded in the structure of the decomposed four dimensional Yang-Mills theory.

Consider now the first term in the r.h.s of \[170\] which can be written as

\[
\rho^2|D_\mu e_\nu|^2 = \frac{1}{2} \left( \partial_\mu \eta \right)^2 + \frac{1}{2} \left( \partial_\mu \tilde{p} \right)^2 + \frac{1}{2} \cos^2 \eta \left\{ (\partial_\mu \tilde{q})^2 - 2 < \tilde{p}, \partial_\mu \tilde{q} >^2 \right\}.
\]

Then by defining

\[
\begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} = \begin{pmatrix} \sin \eta & \cos \eta \\ -\cos \eta & \sin \eta \end{pmatrix} \begin{pmatrix} \tilde{p} \\ \tilde{r} \end{pmatrix}
\]

one gets

\[
\rho^2|D_\mu e_\nu|^2 = \frac{1}{4} \rho^2 \left| \partial_\mu (\tilde{u} + i \tilde{v}) \right|^2 = \frac{\rho^2}{2} \left[ \left( \partial_\mu \tilde{E} \right)^2 + \left( \partial_\mu \tilde{B} \right)^2 \right].
\]

In this form, the invariance under the electric-magnetic duality transformation is manifest.

By specifying \( \eta = 0 \) the purely magnetic contribution is obtained

\[
\rho^2|D_\mu e_\nu|^2 \xrightarrow{\eta \rightarrow 0} \frac{\rho^2}{2} \left( \partial_\mu \tilde{p} \right)^2
\]

which leads to the Heisenberg spin chain action in the vector field \( \tilde{r} \). Recall that the Heisenberg spin chain defines an integrable model in two dimensions, which is also a conserved charge in the NLSE hierarchy; see \[177\].

Similarly, by specifying \( \eta = \pi/2 \) the purely electric contribution is obtained

\[
\rho^2|D_\mu e_\nu|^2 \xrightarrow{\eta \rightarrow \pi/2} \frac{\rho^2}{2} \left( \partial_\mu \tilde{q} \right)^2
\]

which leads to the Heisenberg spin chain action in the vector field \( \tilde{q} \). Therefore, the Heisenberg spin chain and the ensuing NLSE hierarchy, is embedded in the decomposed representation of the four dimensional SU(2) Yang-Mills theory. Moreover, the two variants are related to each other by electric-magnetic duality.

The remaining contribution to \[170\] can also be presented in terms of the vectors \( \tilde{E}, \tilde{B} \). But these terms are also multiplied by the \( U_I(1) \) dependent components \( t_\pm \). Since the ground state is necessarily \( U_I(1) \) invariant and since \( t_\pm \) transform according to \[165\] the following conditions need to be imposed

\[
t_\pm = 0.
\]

Following \[165\] and by assuming the London limit \( \rho = 0 \) where the vector \( \tilde{r} \) acquires its ground state value leads to

\[
\partial_\mu \mathcal{J}_\nu - \partial_\nu \mathcal{J}_\mu + \frac{1}{2} [\partial_\mu, \partial_\nu] + 2 \rho^2 H_{\mu\nu} = 0
\]

\textit{i.e.} the complexified Heisenberg model \[174\].

To conclude, note that with a Kaluza-Klein reduction of \( \mathcal{M} \times \mathcal{N} \simeq \mathbb{R}^4 \rightarrow \mathbb{R}^2 \) and by specifying either to the electric \(( \tilde{B} = 0 )\) or to the magnetic \(( \tilde{E} = 0 )\) sector equation \[172\] is obtained; which corresponds to an embedded integrable NLSE hierarchy within the decomposed four dimensional SU(2) Yang-Mills theory.

### D. Conformal Geometry

Finally, we reveal the local conformal geometry of the four dimensional Yang-Mills theory, alluded to in the beginning of the present section \[53\], \[55\]. Interpret \[163\] as the scale of a (locally) conformally flat metric tensor

\[
G_{\mu\nu} = \left( \frac{\rho}{\Delta} \right)^2 \delta_{\mu\nu},
\]

where \( \Delta \) is a constant with dimensions of mass and the parameter \( \rho \) has dimensions of mass. Therefore, a dimensionful parameter is necessary for the components of the metric tensor in order to acquire the correct dimensionality.

Let us introduce the vierbein as usual,

\[
E^a_\mu = \delta_{ab} E^b_\mu
\]

so that

\[
E^a_\mu = \frac{\rho}{\Delta} \delta^a_\mu
E^a_\mu E^b_\nu = \delta^a_\nu.
\]

Then the Christoffel symbol of the metric \[166\] takes the form

\[
\Gamma^\mu_{\sigma\tau} = \frac{1}{2} G^{\mu\nu} \left( \partial_\nu G_{\sigma\nu} + \partial_\sigma G_{\nu\sigma} - \partial_\sigma G_{\nu\sigma} \right) = \frac{1}{4} \left( \delta^\mu_\sigma \delta^\nu_\tau + \delta^\nu_\sigma \delta^\mu_\tau - \delta^\mu_\sigma \delta^\nu_\tau \right) \partial_\nu \ln \sqrt{G}.
\]

where

\[
\sqrt{G} = \left( \frac{\rho}{\Delta} \right)^4.
\]

Also, the spin connection \( \omega^a_{\mu b} \) is obtained from

\[
\partial_\mu E^\nu_a + \Gamma^\nu_{\lambda\mu} E^\lambda_a - \omega^b_{\mu a} E^\nu_b = 0
\]

which gives

\[
\omega^a_{\mu b} = E^a_\nu \nabla_\mu E^\nu_b = E^a_\nu \left( \partial_\mu E^\nu_b + \Gamma^\nu_{\lambda\mu} E^{\lambda b} \right) = -E^b_\nu \nabla_\mu E^\nu_a = -E^b_\nu \left( \partial_\mu E^\nu_a - \Gamma^\nu_{\mu a} E^\lambda_a \right).
\]

Note that these relations define the action of the covariant derivative \( \nabla_\mu \) on the vector fields and the co-vector fields. Explicitly, the spin connection is

\[
\omega^a_{\mu b} = \frac{1}{4} \left\{ \delta^a_\mu \delta^\sigma_\nu - \delta^a_\nu \delta^\sigma_\mu \right\} \partial_\nu \ln \sqrt{G}.
\]
Next let us employ the vierbein $E_a^\mu$ and the complex Grassmannian zweibein (148) in order to introduce the following complex zweibein
\[
\mathfrak{e}_\mu = E_a^\mu e_a \\
\bar{\mathfrak{e}}_\mu = E_a^\mu \bar{e}_a.
\]
This zweibein is normalized w.r.t. the metric $G_{\mu\nu}$ according to
\[
G_{\mu\nu} \mathfrak{e}_\mu \bar{\mathfrak{e}}_\nu = 1 \\
G_{\mu\nu} \mathfrak{e}_\mu \mathfrak{e}_\nu = 0.
\]
Initially, we push forward the spin connection into
\[
\omega_\mu^{ab} \rightarrow \omega_\mu^{\lambda\nu} = E_a^\lambda \omega_\mu^{a} \bar{e}_\nu^b
\]
then introduce the covariantization of the internal $U_f(1)$ connection
\[
C_\mu = i{e^a}(\partial_\mu e_\sigma - \Gamma_\mu^{\lambda\sigma} e_\lambda + \omega_\mu^{\lambda\sigma} e_\lambda)
= i{e^a} \nabla_\mu e_\sigma + i\bar{e}^b \omega_\mu^{a\lambda} e_\lambda
\]
and finally, we twist the covariant derivative operator with (176) as
\[
\nabla^C_\mu = \nabla_\mu + iC_\mu.
\]
Finally, the Yang-Mills Lagrangian can be recast in a generally covariant format. This results to
\[
L_{YM} = \sqrt{G} \left\{ \frac{1}{4} G^{\mu\rho} G^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma} + \frac{\Delta^2}{12} R - \frac{3}{8} \Lambda^4 t_3^2 \\
+ \frac{\Delta^2}{2} (1 - t_3^2) G^{\mu\nu} J_{\mu} J_{\nu} + \frac{\Delta^2}{2} G^{\mu\nu} \nabla_\mu t \cdot \nabla_\nu t \\
+ \frac{\Delta^2}{4} g_{\alpha\beta} G^{\mu\nu} G^{\lambda\sigma} (D_\mu^{\sigma} e_\lambda) ^\alpha (D_\nu^{\kappa} e_\kappa) ^\beta \right\}
\]
where
\[
F_{\mu\nu} = \partial_\mu J_\nu - \partial_\nu J_\mu + \Delta t_3 H_{\mu\nu} \\
H_{\mu\nu} = \frac{i}{2} (\mathfrak{e}_\mu \bar{e}_\nu - \mathfrak{e}_\nu \bar{e}_\mu).
\]
Note that, the Lagrangian (177) has a manifestly covariant form, but by construction it presumes that the underlying manifold is locally conformally flat. Thus, the two manifolds $\mathcal{M}$ and $\mathcal{N}$ alluded to at the beginning of the present section have to be chosen so that the product $\mathcal{M} \times \mathcal{N}$ is locally conformally flat.

XI. CONCLUDING REMARK

In this paper a reformulation of the Frenet equation in order to describe three dimensional strings in terms of spinors is presented. In addition, it is shown that an extension of the spinor Frenet equation to include the time evolution of strings leads to a Maurer-Cartan structure, which is related to the Lax pair representation of two dimensional integrable models. Since the time evolution of a string determines a Riemann surface in $\mathbb{R}^3$, it has been shown that there is a direct connection between the spinor representation of Frenet equation and of the Gauß-Godazzi equations.

Finally, the relations between strings, surfaces and integrable models in terms of decomposed representations of both two dimensional and four dimensional SU(2) Yang-Mills theories has been studied in detail. This study indicates that the four dimensional decomposed Yang-Mills in combination with Cartan geometry, provides a unifying framework to describe string dynamics, the structure of Riemann surfaces, and two dimensional integrable models.

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