Liouville integrability and superintegrability of a generalized Lotka–Volterra system and its Kahan discretization

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Abstract
We prove the Liouville and superintegrability of a generalized Lotka–Volterra system and its Kahan discretization.

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1. Introduction
The Kac–van Moerbeke system is a prime example of an integrable system, described by the differential equations
\[ \dot{x}_i = x_i(x_{i+1} - x_{i-1}), \quad (i = 1, \ldots, n), \quad (1.1) \]
where \( x_0 = x_{n+1} = 0 \). It was first introduced and studied, together with some of its generalizations, by Lotka to model oscillating chemical reactions and by Volterra to describe population evolution in a hierarchical system of competing species (see [15, 21]). By now, many generalizations of (1.1) have been introduced and studied, often from the point of (Liouville or algebraic) integrability [2, 3, 9, 10], Lie theory [3, 6], Lie symmetries [2, 11, 12], but also in relation with other integrable systems [8, 16, 20]. In our recent study [20], a natural generalization of (1.1) came up in the study of a class of multi-sums of products: we considered the system

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we showed its Liouville integrability and superintegrability and we used it to show the Liouville integrability and superintegrability (or non-commutative integrability) of the Hamiltonian system defined by the above-mentioned class of functions. The system (1.2) has a Hamiltonian structure, described by the Hamiltonian function and Poisson structure, which are respectively given by

\[ H = \sum_{i=1}^{n} x_i, \quad \{x_i, x_j\} = x_i x_j, \quad (i < j). \]  

(1.3)

We consider in the present paper the case of a general linear Hamiltonian

\[ H = \sum_{i=1}^{n} a_i x_i, \]  

(1.4)

with the Poisson structure still given by (1.3). The differential equations which describe this Hamiltonian system are given by

\[ \dot{x}_i = x_i \left( \sum_{j=i}^{n} a_{ij} x_j - \sum_{j<i} a_{ij} x_j \right), \quad (i = 1, \ldots, n). \]  

(1.5)

When all the parameters \( a_i \) are different from zero, a trivial rescaling (which preserves the Poisson structure) leads us back to (1.3), so the novelty of our study is mainly concerned with the case where at least one (but not all!) of the parameters \( a_i \) is zero, though all results below are also valid in case all the parameters \( a_i \) are different from zero. By explicitly exhibiting a set of \((n+1)/2\) involutive (Poisson commuting) rational functions, which are shown to be functionally independent, we show that (1.4) is Liouville integrable (theorems 2.3 and 2.4). We also exhibit \( n - 1 \) functionally independent first integrals, thereby showing that (1.5) is superintegrable (theorem 2.5). Finally, we construct for any initial conditions explicit solutions of (1.5) (proposition 2.6).

In section 3 we study the Kahan discretization (see [5, 13, 17, 18] and the references therein) of (1.5), which we explicitly describe (proposition 3.1). We also show that the map defined by the Kahan discretization is a Poisson map (proposition 3.2). Upon comparing the latter map with the solutions to the continuous system (1.5), we prove that the Kahan map is a time advance map for this Hamiltonian system, and we derive from it that the discrete system is both Liouville and superintegrable, with the same first integrals as the continuous system (proposition 3.4 and corollary 3.3).

We finish the paper with some comments and perspectives for future work (section 4).

2. A generalized Lotka–Volterra system

Let \( n \) be an arbitrary positive integer. We consider on \( \mathbb{R}^n \) the generalized Lotka–Volterra system

\[ \dot{x}_i = x_i \sum_{j=1}^{n} A_{ij} x_j, \quad (i = 1, \ldots, n), \]  

(2.1)
where \( A \) is the square matrix

\[
A = \begin{pmatrix}
0 & a_2 & a_3 & \cdots & a_n \\
-a_1 & 0 & a_3 & \cdots & a_n \\
\vdots & \vdots & \vdots & & \vdots \\
-a_1 & a_2 & 0 & \cdots & a_n \\
-a_1 & a_2 & a_3 & \cdots & 0
\end{pmatrix},
\]

(2.2)

and \( (a_1, \ldots, a_n) \in \mathbb{R}^n \setminus \{ (0, \ldots, 0) \} \). Like most Lotka–Volterra system, it has a linear function as Hamiltonian, to wit \( H = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n \); the corresponding (quadratic) Poisson structure is defined by the brackets \( \{ x_i, x_j \} = x_i x_j \), for \( 1 \leq i < j \leq n \). The following elementary lemma, which will play a key rôle in the proof of theorem 2.3 below, shows that rescaling the parameters \( a_i \) by non-zero constants leads to isomorphic Hamiltonian systems.

**Lemma 2.1.** Let \( c_1, \ldots, c_n \) be arbitrary non-zero real constants. Then the linear change of coordinates \( x_i \mapsto x_i/c_i \) transforms the generalized Lotka–Volterra system with parameters \( a_1, \ldots, a_n \) into the generalized Lotka–Volterra system with parameters \( a_1 c_1, \ldots, a_n c_n \).

**Proof.** Let \( y_i = x_i/c_i \). Then \( \{ y_i, y_j \} = (x_i, x_j)/(c_i c_j) = \frac{x_i x_j}{c_i c_j} = y_i y_j \), for any \( i < j \), which shows that the change of variables preserves the Poisson structure. Clearly, in terms of the new variables, the Hamiltonian reads \( H = a_1 c_1 y_1 + \cdots + a_n c_n y_n \), which is the Hamiltonian of the generalized Lotka–Volterra system with constants \( a_i c_i \).

As an application of the lemma, we have that when the parameters \( a_i \) are all non-zero, we can rescale them all to 1, and (2.1) becomes (1.2) (which is system (3.5) in [20]). In this case, the matrix \( A \) is skew-symmetric and so (2.1) is a genuine Lotka–Volterra system, whose Liouville and superintegrability have extensively been studied in [20]. When some of the parameters \( a_i \) are zero, we get new (non-isomorphic) systems. As we will show in this section, all these systems are Liouville and superintegrable.

For the study of the general case, it is convenient to introduce the functions \( v_i = a_1 x_1 + \cdots + a_i x_i \), for \( i = 1, \ldots, n \); we also set \( v_0 = 0 \). In terms of these functions, \( H = v_n \) and the system (2.1) can equivalently be written as

\[
\dot{x}_i = x_i (H - v_i - v_{i-1}), \quad (i = 1, \ldots, n).
\]

(2.3)

For \( i < j \), one has \( \{ v_i, x_j \} = v_i x_j \), and so the Poisson brackets of the functions \( v_i \) are given by

\[
\{ v_i, v_j \} = v_i (v_j - v_i), \quad (i < j).
\]

(2.4)

In particular, remembering that \( H = v_n \),

\[
\dot{v}_i = \{ v_i, H \} = v_i (H - v_i),
\]

(2.5)

for \( i = 1, \ldots, n \). If \( a_1 a_2 \cdots a_n \neq 0 \), the functions \( v_i \) define new coordinates on \( \mathbb{R}^n \), since then \( x_k = (v_k - v_{k-1})/a_k \) for \( k = 1, \ldots, n \); moreover, the system (2.1) totally decouples in terms of these coordinates since it takes the simple form \( \dot{v}_i = v_i (H - v_i) \), for \( i = 1, \ldots, n \). However, the functions \( v_i \) do not define coordinates when at least one of the \( a_k \) is zero, because if \( a_k = 0 \) then \( v_k = v_{k-1} \).
With a view to proving Liouville integrability, we define for \( k = 1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor \) the functions
\[
J_k := \frac{x_1 x_3 \ldots x_{2k-1}}{x_2 x_4 \ldots x_{2k}},
\]
and for \( k = 1, \ldots, \left\lceil \frac{n+1}{2} \right\rceil \) the functions
\[
F_k := \begin{cases} 
\frac{x_{2k-1} x_{2k+1} x_{2k+3} \ldots x_n}{x_{2k} x_{2k+2} \ldots x_{n-1}} & \text{if } n \text{ is odd,} \\
\frac{x_{2k} x_{2k+2} x_{2k+4} \ldots x_{n-1}}{x_{2k+1} x_{2k+3} \ldots x_{n-1}} & \text{if } n \text{ is even.}
\end{cases}
\]

Notice that \( F_{(n+1)/2} = v_n = H \), the Hamiltonian (1.4). For odd \( n \), we also introduce the function
\[
C := \frac{x_1 x_3 \ldots x_n}{x_2 x_4 \ldots x_{n-1}}.
\]

**Proposition 2.2.** For any \( k, l \in \left\{ 1, \ldots, \left\lceil \frac{n}{2} \right\rceil \right\} \),
\[
\{J_k, J_l\} = \{F_k, F_l\} = \{F_k, H\} = 0.
\]
Moreover, when \( n \) is odd, \( C \) is a Casimir function of the Poisson bracket \( \{\cdot, \cdot\} \).

**Proof.** First, we notice that for any \( k = 1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor \)
\[
x_i \frac{\partial I_k}{\partial x_i} = \begin{cases} 
(-1)^{i+1} I_k & \text{for } 1 \leq i \leq 2k, \\
0 & \text{for } 2k < i \leq n.
\end{cases}
\]
It follows that, for \( k < l \in \left\{ 1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor \right\} \), we have
\[
\{J_k, J_l\} = \sum_{1 \leq i < j \leq n} x_i x_j \left( \frac{\partial J_k}{\partial x_j} \frac{\partial J_l}{\partial x_i} - \frac{\partial J_k}{\partial x_i} \frac{\partial J_l}{\partial x_j} \right)
= \sum_{1 \leq i < j \leq 2k} \left[ (-1)^{i+1} J_k (-1)^{j+1} J_l - (-1)^{j+1} J_k (-1)^{i+1} J_l \right]
+ \sum_{1 \leq i < j \leq 2k} (-1)^{i+1} J_k (-1)^{j+1} J_l = 0.
\]
This proves the first equality of (2.8). We show the two other equalities of (2.8) for even \( n \). To do this, it suffices to prove that \( \{F_k, I_l\} = 0 \) for \( 1 \leq k < l \leq n/2 \) since \( F_{n/2} = H \). We set \( F_{1} = \nu_{\Delta} \), i.e., we define \( I_k \) by
\[
I_k := \frac{x_{2k+2} x_{2k+4} \ldots x_n}{x_{2k+1} x_{2k+3} \ldots x_{n-1}}.
\]
As in (2.9), we have that
\[
x_i \frac{\partial I_k}{\partial x_i} = \begin{cases} 
0 & \text{for } 1 \leq i \leq 2k, \\
(-1)^{i} I_k & \text{for } 2k < i \leq n,
\end{cases}
\]
from which it follows, as above, that \( \{I_k, I_l\} = 0 \) and that \( \{I_k, J_l\} = 0 \) for all \( k, l \in \{1, \ldots, n/2\} \). Also, for any \( j \in \{1, \ldots, n\} \),
\[ \{ k, x_j \} = \sum_{i=1}^{n} \frac{\partial l_k}{\partial x_i} (x_i, x_j) = \left( \sum_{i \leq j} x_i \frac{\partial l_k}{\partial x_i} - \sum_{j < i \leq n} x_i \frac{\partial l_k}{\partial x_i} \right) x_j \]

and using (2.10) we derive that

\[
\{ k, x_j \} = \begin{cases} 0 & \text{for } j \leq 2k, \\ -l_k x_j & \text{for } 2k < j, \end{cases} \quad \text{and } \{ l, v_j \}
\]

\[
\begin{cases} 0 & \text{for } j \leq 2k, \\ -l_k (v_j - v_{2k}) & \text{for } 2k < j. \end{cases}
\]

(2.11)

It follows from (2.4) and (2.11) that, for any \( k < l \leq n/2 \),

\[
\{ F_k, F_l \} = \{ v_{2k} l_k, v_{2l} l_l \} = v_{2k} l_k \{ l_k, v_{2l} \} + v_{2l} l_l \{ v_{2k}, l_l \} + l_k l_l \{ v_{2k}, v_{2l} \}
\]

\[ = -v_{2k} l_k (v_{2l} - v_{2k}) + 0 + v_{2k} l_k (v_{2l} - v_{2k}) = 0. \]

This shows the second half of (2.8) for \( n \) even; for \( n \) odd, the proof is very similar (in this case, \( H = F_{n+1}/2 \) and one proves as above that \( \{ F_k, F_l \} = 0 \) for \( 1 \leq k < l \leq (n+1)/2 \).

Finally we show that \( C \) is a Casimir function (when \( n \) is odd). For \( j = 1, \ldots, n \),

\[
\{ C, x_j \} = \sum_{i=1}^{n} \frac{\partial C}{\partial x_i} (x_i, x_j) = \left( \sum_{i \leq j} x_i \frac{\partial C}{\partial x_i} - \sum_{j < i \leq n} x_i \frac{\partial C}{\partial x_i} \right) x_j
\]

\[ = \sum_{1 \leq i < j} (-1)^{i+j} C x_j - \sum_{j < i \leq n} (-1)^{i+j} C x_j = 0, \]

which shows our claim. \( \square \)

**Theorem 2.3.** Suppose that \( n \) is even. Let \( \ell \) denote the smallest integer such that \( a_{\ell+1} = 0 \) (in particular, \( \ell = 0 \) when \( a_1 = 0 \)) and let \( \lambda := \left[ \frac{\ell}{2} \right] \). The \( \frac{n}{2} \) functions \( J_1, J_2, \ldots, J_\lambda, H, F_{\lambda+1}, F_{\lambda+2}, \ldots, F_{n-1} \) are pairwise in involution and functionally independent, hence they define a Liouville integrable system on \( (\mathbb{R}^n, \{ \cdot, \cdot \}) \).

**Proof.** We know already from proposition 2.2 that the functions \( J_k \) are pairwise in involution, and also the functions \( F_l \) (recall that \( F_{n/2} = H \)). We show that \( \{ J_k, F_l \} = 0 \) for \( k = 1, \ldots, \lambda \) and \( l = \lambda + 1, \ldots, \frac{n}{2} \). To do this, we use the following analog of (2.11), which is easily obtained from (2.9):

\[
\{ J_k, v_j \} = \begin{cases} J_k v_j & \text{for } j \leq 2k, \\ J_k v_{2k} & \text{for } 2k < j. \end{cases}
\]

It follows that, for the above values of \( k, l \), which satisfy \( k \leq \lambda < l \), one has \( \{ J_k, v_{2l} \} = J_k v_{2l} = 0 \) (the last equality follows from \( 2k \leq 2\lambda < \ell \) and \( v_j = 0 \) for \( i < \ell \)), and so

\[
\{ J_k, F_l \} = \{ J_k, v_{2l} l_l \} = v_{2l} \{ J_k, l_l \} + l_l \{ J_k, v_{2l} \} = 0;
\]

in the last step we also used that the functions \( l_l \) and \( J_l \) are in involution (see the proof of proposition 2.2). This shows that the \( \frac{n}{2} \) functions

\[
J_1, J_2, \ldots, J_\lambda, H, F_{\lambda+1}, F_{\lambda+2}, \ldots, F_{n-1}
\]

(2.12)

are pairwise in involution.

We now show that these functions are functionally independent. We first do this when all \( a_i \) are zero, except for \( a_{\ell+1} \) which we may suppose to be equal to 1; then \( v_j = x_{\ell+1} = H \) for
\( i > \ell \) and \( v_i = 0 \) for \( i \leq \ell \). The Jacobian matrix of the above functions (2.12) with respect to \( x_1, \ldots, x_n \) (in that order) is easily seen to have the following block form:

\[
\text{Jac} = \begin{pmatrix}
A & 0 & 0 \\
0 & 1 & 0 \\
0 & * & B
\end{pmatrix},
\]

where \( A \) has size \( \lambda \times \ell \) and \( B \) has size \( \left( \frac{n}{2} - \lambda - 1 \right) \times (n - \ell - 1) \). We show that this matrix has full rank \( n/2 \) (which is equal to the number of rows of \( \text{Jac} \)). To do this, it is sufficient to show that \( A \) has full rank \( \lambda \) and that \( B \) has full rank \( n/2 - \lambda - 1 \) (the value of the column vector * is irrelevant). Consider the square submatrix \( A' \) of \( A \) consisting only of its even-numbered columns. For \( k < \ell \) we have \( A'_{kj} = \partial x_{2k}/\partial x_{2j} = 0 \), since \( J_k \) only depends on \( x_1, \ldots, x_{2k} \). It follows that \( A' \) is a lower triangular matrix. Moreover, \( A'_{jk} = A_{k,2k} = \partial x_{2}/\partial x_{2k} = 0 \), hence \( A' \) is non-singular. This shows that rank \( (A) = \text{rank} (A') = \lambda \). Similarly, we extract from \( B \) a square submatrix \( B' \) by selecting from \( B \) its even-numbered (respectively odd-numbered) columns when \( \ell \) is even (respectively odd). For \( k > \ell \) we have \( B'_{kj} = \partial x_{2\lambda+1+2j}/\partial x_{2\lambda+1+2k} = \partial (x_{2\lambda+1+2k} - x_{2\lambda+1+2k})/\partial x_{2\lambda+1+2j} = 0 \), since \( J_{2k} \) is independent of \( x_1, \ldots, x_{2\lambda+2k} \). However, \( B'_{kj} = x_{2\lambda+1+2k}/\partial x_{2\lambda+1+2j} = 0 \), because \( J_{2k} \) does depend on \( x_{2\lambda+1+2k} \). This shows that \( B' \) is a non-singular upper triangular matrix, hence rank \( (B) = \text{rank} (B') = n/2 - \lambda - 1 \). We have thereby shown that if \( H = x_{\ell+1} \), then the \( n/2 \) functions in (2.12) are functionally independent; since the rank of the Poisson structure \( \{ \cdot, \cdot \} \) is \( n \), we have shown Liouville integrability in this case.

We now consider the general case, where several of the \( a_i \) may be non-zero. We may still suppose that \( a_{\ell+1} = 1 \); as above, \( a_1 = \ldots = a_{\ell} = 0 \). Let us view \( a_{\ell+2}, \ldots, a_n \) as arbitrary parameters and consider the matrix

\[
\text{Jac}' := \begin{pmatrix}
A' & 0 & 0 \\
0 & 1 & 0 \\
0 & * & B'
\end{pmatrix},
\]

where \( A' \) and \( B' \) are square matrices which are constructed as in the previous paragraph. It depends polynomially on the parameters \( a_{\ell+2}, \ldots, a_n \) and we have shown that the determinant of \( \text{Jac}' \) is non-zero when we set all the parameters \( a_{\ell+2}, \ldots, a_n \) equal to zero. By continuity, the determinant remains non-zero when the parameters \( a_{\ell+2}, \ldots, a_n \) are sufficiently close to zero, which proves that the \( n/2 \) functions in (2.12) are functionally independent for such values of the parameters. In view of lemma 2.1, any non-zero rescaling of the parameters leads to isomorphic systems, so for any values of \( a_{\ell+2}, \ldots, a_n \), the functions in (2.12) are functionally independent. This shows Liouville integrability for any values of the parameters \( a_1, \ldots, a_n \). \( \square \)

When \( n \) is odd, the rank of the Poisson structure \( \{ \cdot, \cdot \} \) is \( n - 1 \), so for Liouville integrability we need \((n + 1)/2\) functionally independent functions in involution. Recall from proposition 2.2 that in this case \( C \) is a Casimir function. The Liouville integrability is in this case given by the following theorem, whose proof is omitted because it is very similar to the proof of theorem 2.3.

**Theorem 2.4.** Suppose that \( n \) is odd. As before, let \( \ell \) denote the smallest integer such that \( a_{\ell+1} = 0 \) and let \( \lambda := \left\lfloor \frac{\ell}{2} \right\rfloor \). The \( \frac{n+1}{2} \) functions \( J_1, J_2, \ldots, J_{\lambda}, H, F_{\lambda+2}, F_{\lambda+3}, \ldots, F_{n-1}, C \) are
pairwise in involution and functionally independent, hence define a Liouville integrable system on $(\mathbb{R}^n, \{\cdot, \cdot\})$. We show in the following theorem that the Hamiltonian vector field defined by $H$ is also superintegrable.

**Theorem 2.5.** The Hamiltonian system (1.5) has $n-1$ functionally independent first integrals, hence is superintegrable.

**Proof.** We denote, as before, by $\ell$ the smallest integer such that $a_{\ell+1} = 0$ (in particular, $\ell = 0$ when $a_1 \neq 0$). Suppose first that $a_{\ell+1}$ is the only $a_i$ which is different from zero; by a simple rescaling, we may assume $a_{\ell+1} = 1$, so that $H = x_{\ell+1}$. Then the equations of motion (1.5) take the following simple form:

$$
\dot{x}_i = \begin{cases} 
  x_i H & i \leq \ell, \\
  0 & i = \ell + 1, \\
  -x_i H & i > \ell + 1.
\end{cases}
$$

(2.13)

When $\ell = 0$, a complete set of $n-1$ independent first integrals of (2.13) is given by $H = x_1$ and $x_i/x_2$, ($i = 3, \ldots, n$). When $\ell \neq 0$, we can take besides the Hamiltonian $H = x_{\ell+1}$ the functions $x_i/x_1$, ($i = 2, \ldots, \ell$) and $x_3x_i$, ($i = \ell + 2, \ldots, n$).

In the general case, we partition the set $\{1, 2, \ldots, n\}$ into three subsets ($A$ or $C$ may be empty):

$$
A := \{1, 2, \ldots, \ell\}, \\
B := \{i | a_i \neq 0\}, \\
C := \{i | i > \ell + 1 \text{ and } a_i = 0\}.
$$

Since we have treated the case $\#B = 1$, we may henceforth assume that $\#B \geq 2$. Notice that each function $v_i$ (and in particular $H$) depends only on the variables $x_i$ with $i \in B$. It follows that the differential equations (2.3),

$$
\dot{x}_i = x_i (H - v_i - v_{i-1}), \quad (i \in B),
$$

involve only the variables $x_j$ with $j \in B$, so they form a subsystem which is the same as the original system, but now of dimension $m := \#B$, and with all parameters $a_i$, $i \in B$ different from zero. As explained above (see lemma 2.1 and the remarks which follow its proof) this subsystem is by a simple rescaling isomorphic to the system (1.2), for which we know from [20] that it is superintegrable, with $m - 1$ first integrals which we denote here by $G_1, \ldots, G_{m-1}$. We do not need here the precise formulas for these functions, but only the fact that they depend only on the variables $x_j$ with $j \in B$; this obvious fact implies that the functions $G_1, \ldots, G_{m-1}$ are first integrals of the full system (1.5) as well. Consider, for $i \in A \cup C$ the following rational function:

$$
K_i = \begin{cases} 
  \frac{(H - a_{\ell+1}x_{\ell+1})x_i}{x_{\ell+1}}, & i \in A, \\
  \frac{(H - a_{\ell+1}x_{\ell+1})v_i^2}{x_i x_{\ell+1}}, & i \in C.
\end{cases}
$$
Notice that $H - a_{t+1}x_{t+1}$ is different from zero, because $\#B \geq 2$. For $i \in A$, we have that

$$(\ln K_i) = (\ln (H - a_{t+1}x_{t+1})) + (\ln (x_i/x_{t+1}))$$

$$= - \frac{a_{t+1}x_{t+1}}{H - a_{t+1}x_{t+1}} + a_{t+1}x_{t+1} = 0.$$ 

Indeed, $\dot{x}_{t+1} = x_{t+1}(H - v_{t+1} - v_i) = x_{t+1}(H - a_{t+1}x_{t+1})$. Similarly, for $i \in C$, we have from (2.3) and (2.5) that

$$(\ln K_i) = (\ln (H - a_{t+1}x_{t+1})) + 2(\ln v_i) - (\ln (x_i/x_{t+1}))$$

$$= - a_{t+1}x_{t+1} + 2(H - v_i) - (H - 2v_i) - (H - a_{t+1}x_{t+1}) = 0.$$ 

This shows that the $n - 1$ functions $G_1, \ldots, G_{n-1}$ and $K_i$, $i \in A \cup C$, are first integrals of (1.5). Recall that the functionally independent functions $G_1, \ldots, G_{n-1}$ depend on $x_i$ with $i \in B$ only and notice that for $i \in A \cup C$ the variable $x_i$ appears only in $K_i$. It follows that these $n - 1$ first integrals of (1.5) are functionally independent, hence (1.5) is superintegrable.

Finally, we compute the solution $x(t)$ of (2.1) which corresponds to any given initial condition $x^{(0)} = (x_1^{(0)}, \ldots, x_n^{(0)})$. We also introduce the derived functions $v_i(t) = a_i x_i(t) + \cdots + a_n x_n(t)$, for $i = 1, \ldots, n$. We denote by $h_0$ the value of the Hamiltonian $H$ at the initial condition $x^{(0)}$ and we denote $v_i^{(0)} := v_i(0)$. It follows from (2.3) and (2.5) that we need to solve

$$\frac{dx_i}{dt}(t) = x_i(t)(h_0 - v_i(t) - v_{i-1}(t)), \quad (i = 1, \ldots, n), \quad (2.14)$$

where

$$\frac{dv_i}{dt}(t) = v_i(t)(h_0 - v_i(t)), \quad (i = 1, \ldots, n). \quad (2.15)$$

When $v_i^{(0)} = 0$, the latter equation has $v_i(t) = 0$ as its unique solution; otherwise (2.15) is easily integrated by a separation of variables, giving

$$v_i(t) = \frac{1}{h_0} + C_i e^{-h_0 t}, \quad \text{or} \quad v_i(t) = \frac{1}{t + C'_i}, \quad (2.16)$$

depending on whether $h_0 \neq 0$ or $h_0 = 0$. The integrating constants $C_i$ and $C'_i$ are computed from $v_i(0) = v_i^{(0)}$, which leads to

$$C_i = \frac{1}{v_i^{(0)}} - \frac{1}{h_0}, \quad \text{and} \quad C'_i = \frac{1}{v_i^{(0)}}.$$ 

The functions $v_i(t)$ in (2.16) have very simple primitives, to wit

$$\int v_i(t) dt = \ln \left( \frac{e^{h_0 t}}{h_0} + C_i \right), \quad \text{or} \quad \int v_i(t) dt = \ln (t + C'_i). \quad (2.17)$$

Substituted in (2.14), which we write now as $\frac{dx_i}{dt}(t) = h_0 - v_i(t) - v_{i-1}(t)$, we obtain by integration and by using the primitives (2.17) (or $\int v_i(t) dt = \text{constant in case } v_i^{(0)} = 0$) and the initial condition $x_i(0) = x_i^{(0)}$, the following result:

**Proposition 2.6.** The solution $x(t)$ of (2.1) which corresponds to the initial condition $x^{(0)} = (x_1^{(0)}, \ldots, x_n^{(0)})$ is given by...
where \( f(t) = \frac{e^{h_0} - 1}{(e^{h_0} + 1)h_0} = \frac{1}{h_0} \tanh \left( \frac{h_0}{2} \right) \) when \( h_0 \) (the value of \( H \) at \( x^{(0)} \)) is different from zero and \( f(t) = 1/2 \) otherwise. Also, \( v_i^{(0)} = a_1 x_1^{(0)} + \cdots + a_i x_i^{(0)} \).

Notice that when \( h_0 = 0 \), (2.18) can be rewritten as
\[
x_i(t) = \frac{x_i^{(0)} e^{h_0} h_0^2}{(h_0 + (e^{h_0} - 1)v_i^{(0)})(h_0 + (e^{h_0} - 1)v_i^{(0)})}, \quad (i = 1, \ldots, n).
\]

**Remark 2.7.** When several of the parameters \( a_i \) in the Hamiltonian function \( H \) are equal to zero, so that \( H \) is independent of the corresponding variables \( x_i \), the vector field (1.5) is a Hamiltonian vector field with respect to a family of compatible Poisson structures, always with the same Hamiltonian \( H \). Indeed, suppose that \( a_i = a_j = 0 \), with \( i < j \). Then, in the computation of the vector field \( \dot{x}_k = \{x_k, H\}, k = 1, \ldots, n \), the Poisson brackets \( \{x_i, x_j\} = -\{x_j, x_i\} \) are not used, so we may replace \( \{x_i, x_j\} = -\{x_j, x_i\} \) by an arbitrary function \( f_{ij} \) of \( x_1, \ldots, x_p \) without any effect on the vector field. However, in order for the new bracket to be a Poisson bracket, it has to satisfy the Jacobi identity, which puts several restrictions on the function \( f_{ij} \). One way to satisfy this restriction is to take \( f_{ij} = a_{ij} x_i x_j \), where \( a_{ij} \) is an arbitrary constant. In fact, replacing \( \{x_i, x_j\} = x_i x_j \) by \( \{x_i, x_j\} = a_{ij} x_i x_j \) for all \( i < j \) for which \( a_i = a_j = 0 \), the new brackets will still be of the general form \( \{x_i, x_j\} = b_{ij} x_i x_j \), known in the literature as diagonal brackets; such brackets are known to automatically satisfy the Jacobi identity [14, example 8.14] so they are Poisson brackets. Clearly, any linear combination of these diagonal Poisson brackets is again a diagonal Poisson bracket, hence all these brackets are compatible. The upshot is that when \( k \geq 2 \) parameters are equal to zero, then (1.5) has a multi-Hamiltonian structure: it is Hamiltonian with respect to a \( \left( \begin{array}{c} k \\ 2 \end{array} \right) \)-dimensional family of Poisson brackets.

### 3. The Kahan discretization

In this section we consider the Kahan discretization of the system (2.1). Let us recall quickly the construction of the Kahan discretization of a quadratic vector field \( \dot{x} = Q(x) \) (see e.g. [5]). Let \( \Phi_j(y, z) \) denote the symmetric bilinear form which is associated to the quadratic form \( Q \), and let \( \epsilon \) denote a positive parameter, which should be thought of as being small. Then the **Kahan discretization with step size** \( \epsilon \) is the map\(^5 \) \( x \mapsto \hat{x} \), implicitly defined by
\[
\hat{x}_i - x_i = \epsilon \Phi_j(x, \hat{x}). \tag{3.1}
\]
We refer to this map as the **Kahan map** (associated to \( \dot{x} = Q(x) \)). It is well known that the Kahan map preserves the linear integrals of the initial continuous system (quadratic vector field). So, in our case of the generalized Lotka–Volterra system, its Hamiltonian function \( H = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n \) is an invariant of the Kahan map. As we are going to show in this section the Kahan map (of this system) preserves the Poisson structure as well; we will also see in the next section that all constants of motion, in particular the ones that appear in theorems 2.3–2.5, are also invariants of the Kahan map.

\(^5\) When the map which is defined by the discretization is iterated, one often writes it as \( x_i^{(n)} \mapsto x_i^{(n+1)} \).
We begin with a lemma which provides an explicit formula for the Kahan discretization of the generalized Lotka–Volterra system.

**Proposition 3.1.** The Kahan discretization with step size $2\varepsilon$ of the system (2.1) is the rational map $K : (x_1, \ldots, x_n) \mapsto (\tilde{x}_1, \ldots, \tilde{x}_n)$, given by

$$
\tilde{x}_i = x_i \frac{(1 - \varepsilon H)(1 + \varepsilon)}{(1 - \varepsilon H + 2\varepsilon v_{i-1})(1 - \varepsilon H + 2\varepsilon v_i)}, \quad (i = 1, \ldots, n). \tag{3.2}
$$

**Proof.** Let us write $\tilde{v}_j = a_i \tilde{x}_i + \cdots + a_j \tilde{x}_j$, in analogy with the functions $v_j$. According to (3.1), the Kahan discretization of (2.3) (which is equivalent to (2.1)) is given by

$$
\tilde{x}_i - x_i = \varepsilon x_i (H - \tilde{v}_i - \tilde{v}_{i-1}) + \varepsilon \tilde{x}_i (H - v_i - v_{i-1}), \quad (i = 1, \ldots, n), \tag{3.3}
$$

where we have used that $H$ is invariant ($\tilde{H} = H$). Summing up these equations, multiplied by $a_i$, for $i = 1, \ldots, j$, we get

$$
\tilde{v}_j - v_j = \varepsilon (v_j H + \tilde{v}_j H - \delta_j), \tag{3.4}
$$

where $\delta_j$ is given by

$$
\delta_j := \sum_{i=1}^{j} a_i x_i (\tilde{v}_i + v_{i-1}) + \sum_{i=1}^{j} a_i \tilde{x}_i (v_i + v_{i-1}) = 2v_j \tilde{v}_j.
$$

The last equality can be proven by an easy recursion on $j$: on the one hand,

$$
\delta_1 = a_i x_0 \tilde{x}_0 + a_i \tilde{x}_0 v_0 = 2v_1 \tilde{v}_1,
$$

while on the other hand

$$
\delta_{j+1} - \delta_j = 2a_{j+1} v_{j+1} \tilde{v}_j + 2a_{j+1} \tilde{x}_{j+1} v_j + 2a_{j+1} \tilde{x}_{j+1} \tilde{v}_j + 2a_{j+1} \tilde{x}_{j+1} \tilde{x}_{j+1},
$$

and so

$$
\delta_{j+1} = 2a_{j+1} x_{j+1} \tilde{v}_j + 2a_{j+1} \tilde{x}_{j+1} v_j + 2a_{j+1} \tilde{x}_{j+1} \tilde{v}_j + 2a_{j+1} \tilde{x}_{j+1} \tilde{x}_{j+1} + 2v_j \tilde{v}_j
$$

$$
= 2(v_j + a_{j+1} v_{j+1})(\tilde{v}_{j+1} + a_{j+1} \tilde{x}_{j+1}) = 2v_{j+1} \tilde{v}_{j+1}.
$$

Solving (3.4) (with $\delta_j = 2v_j \tilde{v}_j$) linearly for $\tilde{v}_j$ we get

$$
\tilde{v}_j = v_j \frac{1 + \varepsilon H}{1 - \varepsilon H + 2\varepsilon v_j}. \tag{3.5}
$$

Substituting this into (3.3) leads to

$$
\tilde{x}_i - x_i = \varepsilon x_i \left( H - v_i \frac{1 + \varepsilon H}{1 - \varepsilon H + 2\varepsilon v_i} - v_{i-1} \frac{1 + \varepsilon H}{1 - \varepsilon H + 2\varepsilon v_{i-1}} \right) + \varepsilon \tilde{x}_i (H - v_i - v_{i-1}),
$$

which can be solved linearly for $\tilde{x}_i$. It yields the formula (3.2). \qed

**Proposition 3.2.** The Kahan map $K$, given by (3.2), is a Poisson map with respect to the Poisson bracket $\{\cdot, \cdot\}$.

**Proof.** Recall that the Poisson bracket $\{\cdot, \cdot\}$ is given by $\{x_i, x_j\} = x_i x_j$, for $1 \leq i < j \leq n$. Therefore, we need to show that $\{\tilde{x}_i, \tilde{x}_j\} = \tilde{x}_i \tilde{x}_j$, for $1 \leq i < j \leq n$. We set, for $k = 1, \ldots, n,$
so that \( \tilde{\xi}_k = A_k / B_k \). Then

\[
\{\xi_i, \xi_j\} = \frac{A_i A_j \{B_i, B_j\} - A_i B_j \{A_i, B_j\} - B_i A_j \{A_i, B_j\} + B_i B_j \{A_i, A_j\}}{B_i^2 B_j^2}.
\]

The Poisson brackets in the right-hand side of this equation can be computed using besides (2.4) the following formulas:

\[
\{x_i, H\} = x_i (H - v_i - v_{i-1}), \quad \{x_i, v_j\} = \begin{cases} 
x_i (v_j - v_i - v_{j-1}) & \text{for } i \leq j, \\
v_i v_j & \text{for } i > j.
\end{cases}
\]

After some computation, it leads to

\[
\{\tilde{\xi}_i, \tilde{\xi}_j\} = \frac{(1 - \epsilon H^2)^2 x_i x_j}{(1 - \epsilon H + 2 v_{i-1})(1 - \epsilon H + 2 v_i)(1 - \epsilon H + 2 v_{j-1})(1 - \epsilon H + 2 v_j)} = \tilde{\xi}_i \tilde{\xi}_j,
\]

as was to be shown. \( \square \)

An easy comparison of the solution (2.18) to the continuous system and the Kahan map (3.2) shows that the Kahan map is a time advance map for the continuous system, hence preserves all integral curves of the continuous system and so all constants of motion of the continuous system are invariants for the Kahan map. Precisely, let \( x^{(0)} = (x_1^{(0)}, \ldots, x_n^{(0)}) \) be any point of \( \mathbb{R}^n \) and let \( \epsilon \in \mathbb{R} \) be small but positive. As above, the value of \( H \) at \( x^{(0)} \) is denoted by \( h_0 \). Let \( t_0 \) denote the unique solution to the equation \( f(t_0) = \epsilon \), where \( f(t) \) is the function given in proposition 2.6. With these notations, (2.18) and (3.2) imply that \( x_i(t_0) = \tilde{x}_i^{(0)} \). It leads, in view of theorems 2.3 and 2.4, to the following corollary:

**Corollary 3.3.** The Kahan discretization (3.2) is Liouville integrable, with invariants given in theorem 2.3 (respectively theorem 2.4) when \( n \) is even (respectively when \( n \) is odd). It is also superintegrable, with invariants given in theorem 2.5.

Let us denote the \( k \)th iterate of the Kahan map (3.2) starting from the initial condition \( x^{(0)} = (x_1^{(0)}, \ldots, x_n^{(0)}) \) by \( x^{(k)} \). Then the relation between the solutions to the continuous system and the Kahan map can be written as \( x_i(t_k) = x_i^{(k)} \). Now notice that \( t_k \) depends only on \( x^{(0)} \) through \( h_0 \); this implies that the restriction of \( K \) to the integral curve through \( x^{(0)} \) is the time \( t_k \) flow of the continuous system (restricted to the integral curve through \( x^{(0)} \)). Thus, \( x^{(2)} \) is obtained from \( x^{(1)} \) by the time \( t_1 \) flow, and hence from \( x^{(0)} \) by the time \( 2t_1 \) flow, \( x^{(2)} = x(2t_1) \); more generally, \( x^{(m)} \) is obtained from \( x^{(0)} \) by the time \( mt_k \) flow, \( x^{(m)} = x(mt_k) \). It leads to the following proposition.

**Proposition 3.4.** The solution of the discrete system

\[
\tilde{x}_i = x_i \frac{(1 - \epsilon H)(1 + \epsilon H)}{(1 - \epsilon H + 2 v_{i-1})(1 - \epsilon H + 2 v_i)} , \quad (i = 1, \ldots, n)
\]
with \( H = \sum a_i x_i \) and initial condition \( x^{(0)} \) is given by

\[
x_i^{(m)} = x_i^{(0)} - \frac{a_i}{h_0 + v_i^{(0)}} \left( \left( \frac{1 + \theta a_i}{1 - \theta a_i} \right)^m h_0^2 - 1 \right)
\]

(3.7)

When \( h_0 \) (the value of \( H \) at \( x^{(0)} \)) is different from zero. When \( h_0 = 0 \),

\[
x_i^{(m)} = \frac{1}{1 + 2m v_i^{(0)} \left( 1 + 2m v_i^{(0)} \right)}.
\]

(3.8)

**Proof.** In view of proposition 2.6,

\[
x_i^{(m)} = x_i^{(0)} - \frac{a_i}{h_0 + v_i^{(0)}} \left( \left( \frac{1 + \theta a_i}{1 - \theta a_i} \right)^m h_0^2 - 1 \right).
\]

(3.9)

When \( h_0 = 0 \), it follows easily from \( f(t) = \frac{e^{\theta a_i} - 1}{(e^{\theta a_i} + 1)h_0} \) and \( f(\epsilon t) = \epsilon \) that \( e^{\theta a_i} = \frac{1 + h_0 \epsilon}{1 - h_0 \epsilon} \). In turn, we can compute \( f(\epsilon t) \) from it, namely

\[
f(\epsilon t) = \frac{1}{h_0} e^{\theta a_i t h_0} - 1 = \frac{1}{h_0} \left( \frac{1 + h_0 \epsilon}{1 - h_0 \epsilon} \right)^m - 1.
\]

(3.10)

It now suffices to substitute (3.10) in (3.9) and to simplify the resulting expression to obtain (3.7). When \( h_0 = 0 \), we have that \( f(\epsilon t) = m \epsilon \), since \( f(t) = t/2 \). Substituted in (3.9) (with \( h_0 = 0 \)), we get at once (3.8).

4. Conclusion

We presented a new class of generalized Lotka–Volterra systems which are, together with their Kahan discretizations, Liouville integrable and superintegrable, and we provided their explicit solutions. Since linear Hamiltonians are always preserved under Kahan discretization and since the Poisson structure that we used is quadratic, it is natural to ask which quadratic Poisson structures on \( \mathbb{R}^n \) are preserved by the Kahan discretization of every Hamiltonian vector field with linear Hamiltonian; in view of what we have shown, the Poisson structure defined by the brackets \( \{ x_i, x_j \} \equiv x_i x_j \), for \( 1 \leq i < j \leq n \), belongs to this class. The Hamiltonian systems which are defined by them would then be good candidates for being Liouville integrable and/or superintegrable. In view of the recent developments in discretization of polynomial vector fields by polarization [4], similar questions can also be considered for higher degree polynomial Hamiltonian vector fields.

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