The finite representation property for representable residuated semigroups

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Abstract. In this paper, we show that the class of representable residuated semigroups, residuated semigroups on binary relations, has the finite representation property. That is, every representable residuated semigroup is isomorphic to some algebra over a finite base. This result gives a positive solution to Problem 19.17 from the monograph by Hirsch and Hodkinson [15].

Keywords: Relation algebras · The finite representation property · Residuated semigroups

1 Introduction

Relation algebras are the kind of Boolean algebras with operators representing algebras of binary relations [20]. One often emphasise the following two classes of relation algebras. The first class called RA consists of algebras the signature of which is \{ R, 0, 1, +, −, ;, −, 1 \} obeying the certain axioms that we define precisely above. The second class called RRA, the class of representable relation algebras, consists of algebras isomorphic to set relation algebras. RRA is clearly a subclass of RA, but the converse inclusion does not hold. That is, there exist relation algebras having no representation as set relation algebras [23]. Moreover, the class RRA is not finitely axiomatisable in contrast to RA [27]. The problem of determining whether a given finite relation algebra is representable is undecidable, see [14].

Under these circumstances, one may conclude that relation algebras are quite badly behaved. The study of relation algebras reducts is mostly motivated by such “bad behaviour” in order to avoid these restrictions and determine the possible reasons for them.

There are several results on reducts of relation algebras having no finite axiomatisation such as ordered monoids [12], distributive residuated lattices [2], join semilattice ordered semigroups [3], algebras the signature of which contains composition, meet, and converse [18], etc.

On the other hand, such classes as representable residuated semigroups and monoids [2], and ordered domain algebras [17] are finitely axiomatisable. There
are also plenty of subsignatures the question of finite axiomatisability for which remains open, see, e. g., [3].

The other direction we emphasise is related to the finite representation for subsignatures of the relation algebra signature. The finite representation property claims that a finite representable relation algebra is isomorphic to some representable relation algebra over a finite base. The investigation is of interest to study such aspects as decidability of membership of $R(\tau)$ and recursivity of the class consisting of all finite representable $\tau$-structures [11], where $\tau$ is a subsignature of $\{R, 0, 1, +, -, ;, -\}$. The examples of the class having the finite representation property are some classes of algebras [11] [17] [26], the subsignature of which contains the domain and range operators. The other classes of structures having the finite representation property have been studied by Andréka, Hodkinson, and Németi, see [1].

There are subsignatures $\tau$ such that the class $R(\tau)$ has no the finite representation property, for instance, \{;\}, see [25]. In general, (un)decidability of determining whether a finite relation algebra has a finite representation is an open question [15] Problem 18.18. The authors' conjecture claims that this problem might have a negative solution.

In this paper, we consider reducts of relation algebras the signature of which consists of composition, residuals, and the binary relation symbol denoting partial ordering, that is, the class of representable residuated semigroups. We show that $R(\; , \; \backslash, /, \leq)$ has the finite representation property. As result, Problem 19.17 of [15] has a positive solution. We also note that this result implies of membership decidability of $R(\; , \; \backslash, /, \leq)$ for finite structures and, moreover, the class of finite representable residuated semigroups is recursive.

## 2 Preliminaries

We recall the basic definitions [15] [24].

**Definition 1.** A relation algebra is an algebra $R = \langle R, 0, 1, +, -, ;, \backsim, 1 \rangle$ such that $(R, 0, 1, +, -)$ is a Boolean algebra and the following equations hold, for each $a, b, c \in R$:

1. $a; (b; c) = (a; b); c$,
2. $(a + b); c = (a; c) + (b; c)$,
3. $a; 1 = a$,
4. $a\backsim\backsim = a$,
5. $(a + b)\backsim = a\backsim + b\backsim$,
6. $(a; b)\backsim = b\backsim; a\backsim$,
7. $a\backsim; (a; b) \leq -b$.

where $a \leq b$ iff $a + b = b$. RA is the class of all relation algebras.

**Definition 2.** A proper relation algebra (or, a set relation algebra) is an algebra $R = \langle R, 0, 1, \cup, -, ;, \backsim, 1 \rangle$ such that $R \subseteq P(W)$, where $W \subseteq X \times X$ is an equivalence relation on a base set; $0 = \emptyset$; $1 = W$; $\cup$ and $-$ are set-theoretic union
and complement respectively; \(;\) is relation composition, \(\sim\) is relation converse, \(1\) is the identity relation restricted to \(W\), that is:

1. \(a; b = \{(x, z) \mid \exists y \langle x, y \rangle \in a \land \langle y, z \rangle \in b\}\)
2. \(a^{-} = \{(x, y) \mid \langle y, x \rangle \in a\}\)
3. \(1 = \{(x, y) \mid x = y\}\)

**PRA** is the class of all proper relation algebras. **Rs** is the class of all relation set algebras, proper relation algebra with a diagonal subrelation as the identity. **RRA** is the class of all representable relation algebras, that is, the closure of **PRA** under isomorphic copies. That is, \(\text{RRA} = \text{IPRA}\).

Note that the (quasi)equational theories of these classes coincide, that is

\(\text{IPRA} = \text{RRA} = \text{SPRs}\)

where **SPRs** is the closure of **Rs** under subalgebras (S) and direct products (P).

We will use the following notation due to, for example, [10]. Let \(\tau\) be a subsets of operations definable in RA. \(R(\tau)\) is the class of subalgebras of \(\tau\)-subreducts of algebras belonging to **Rs**.

A \(\tau\)-structure is **representable** if it is isomorphic to some algebra of relations of \(\tau\)-signature. A representable finite \(\tau\)-structure has a **finite representation over a finite base** if it is isomorphic to some finite representable over a finite base. \(R(\tau)\) has the finite representation property if every \(A \in R(\tau)\) has a finite representation over a finite base.

One may express residuals in every \(R \in RA\) as follows using Boolean negation, inversion, and composition as follows:

1. \(a \setminus b = -(a^{-}; -b)\)
2. \(a/b = -(-a; b^{-})\)

These residuals have the following explicit definition in \(R \in PRA\):

1. \(a \setminus b = \{(x, y) \mid \forall z \langle z, x \rangle \in a \Rightarrow \langle z, y \rangle \in b\}\)
2. \(a/b = \{(x, y) \mid \forall z \langle y, z \rangle \in b \Rightarrow \langle x, z \rangle \in a\}\)

One may visualise the composition and residual operations in **PRA** with the following triangles:

In this paper, the underlying signature \(\tau\) is \(\{;\setminus, /, \leq\}\).
3 The main result

Let us discuss the $\mathbf{R}(;\,\setminus,\,/,\,\leq)$ class closely. Let us introduce the notion of a residuated semigroup. Historically, residuated structures were introduced by Krull to study ideals of rings [21]. Such algebras further has been considered within semantic questions of substructural logics, see [19].

**Definition 3.** A residuated semigroup is an algebra $\mathcal{A} = \langle A, ;, \setminus, /, \leq \rangle$ such that $\langle A, ;, \leq \rangle$ is an partially ordered residuated semigroup and $\setminus, /$ are binary operations satisfying the following condition:

$$b \leq a \setminus c \iff a \leq c \setminus b$$

$\mathbf{RS}$ is the class of all residuated semigroups.

The logic of such structures is the Lambek calculus [22] allowing one to characterise inference in categorial grammars, the equivalent version of context-free grammars [30]. Such formalisms as the Lambek calculus might be considered from a dynamic perspective as well, see [34]. One may define the Lambek calculus as the following (cut-free) sequent calculus:

**Definition 4.**

$$\varphi \to \varphi \stackrel{ax}{\rightarrow}$$

$$\Gamma \to \psi \quad \Delta, \psi, \Theta \to \theta$$

$$\Delta, \Gamma, \varphi \setminus \psi, \Theta \to \theta \quad \setminus \rightarrow$$

$$\varphi, \Pi \to \psi \quad \Pi \to \varphi \setminus \psi \quad \rightarrow \setminus$$

$$\Gamma \to \varphi \quad \Delta, \psi, \Theta \to \theta$$

$$\Delta, \psi / \varphi, \Gamma, \Theta \to \theta \quad / \rightarrow$$

$$\Pi, \varphi \to \psi \quad \Pi \to \varphi / \psi \quad \rightarrow /$$

$$\Gamma, \varphi, \psi, \Delta \to \theta$$

$$\Delta, \psi, \Theta \to \theta \quad \bullet \rightarrow$$

$$\rightarrow \Delta, \varphi \bullet \psi, \Delta \to \theta$$

$$\Gamma, \Delta \to \varphi \bullet \psi \quad \rightarrow$$

Let us define the class $\mathbf{R}(;\,\setminus,\,/,\,\leq)$ explicitly:

**Definition 5.** Let $A$ be a set of binary relations on some base set $W$ such that $R = \cup A$ is transitive and $\{ x, y | xRy \} = W$. A relation residuated semigroup is an algebra $\mathcal{A} = \langle A, ;, \setminus, /, \leq \rangle$ where for each $a, b \in A$

1. $a; b = \{ (x, z) | \exists y \in W \ (x, y) \in a \land (y, z) \in b \}$,
2. $a \setminus b = \{ (x, y) | \forall z \in W \ ((z, x) \in a \Rightarrow (z, y) \in b) \}$,
3. $a/b = \{ (x, y) | \forall z \in W \ ((y, z) \in b \Rightarrow (x, z) \in a) \}$,
4. $a \leq b$ iff $a \subseteq b$.

A residuated semigroup is called representable if it is isomorphic to some algebra belonging to $\mathbf{R}(;\,\setminus,\,/,\,\leq)$. $\mathbf{RRS} = I(\mathbf{R}(;\,\setminus,\,/,\,\leq))$ is the class of all representable residuated semigroups, where $I$ is the closure under isomorphic copies.
Definition 6. Let \( \tau = \{ ; , \backslash , \leq \} \), \( \mathcal{A} \) a \( \tau \)-structure, and \( X \) a base set. An interpretation \( R \) over a base \( X \) maps every \( a \in \mathcal{A} \) to a binary relation \( a^R \subseteq X \times X \). A representation of \( \mathcal{A} \) is an interpretation \( R \) satisfying the following conditions:

1. \( a \leq b \) iff \( a^R \subseteq a^R \)
2. \( (a; b)^R = \{(x, y) \mid \exists z \in X (x, z) \in a^R \land (z, y) \in b^R \} = a^R \land b^R \)
3. \( (a \backslash b)^R = \{(x, y) \mid \forall z \in X ((z, x) \in a^R \Rightarrow (z, y) \in b^R) \} = a^R \backslash b^R \)
4. \( (a/b)^R = \{(x, y) \mid \forall z \in X ((y, z) \in a^R \Rightarrow (x, z) \in b^R) \} = a^R / b^R \)

Andrěka and Mikulás proved the following representation theorem for \( R S \) in [2] that implies relational completeness of the Lambek calculus, the logic of \( R S \). The following fact has been proved in the fashion of step-by-step representation, see this paper to learn more about step by step representations [13].

Theorem 1. \( RS = RRS \).

This fact also claims that the theory of \( R(\; , \backslash , \leq) \) is finitely axiomatisable since the universal theories of \( RS \) and \( RRS \) coincide and the class of all residuated semigroup is obviously finitely axiomatisable. Pentus also proved that the Lambek calculus NP-complete reducing this problem to SAT [31]. Concerning this result, we also conclude that the universal theory of \( RRS \) is NP-complete.

One may rephrase the result of Theorem 1 as \( \mathcal{A} \) is representable iff \( \mathcal{A} \) is a residuated semigroup. Thus, it is sufficient to show that any finite residuated semigroup has a representation over a finite base in order to show that \( R(\; , \backslash , \leq) \) has the finite representation property.

3.1 Quantales and quantic nuclei

A quantic nucleus is a closure operator on an ordered semigroup allowing one to define subalgebras. Such an operator is a generalisation of a well-known nucleus operator in locale theory, see, e. g., [4]. The following definition and the proposition below are given due to [9] and [32].

Definition 7. A quantic nucleus on a partially ordered semigroup \( \langle A, \; \leq \rangle \) is a map \( j : A \rightarrow A \) such that \( j \) a closure operator such that \( ja; jb \leq j(a; b) \).

Proposition 1. Let \( \mathcal{A} = \langle A, \; \leq \rangle \) be a partially ordered semigroup and \( j \) a quantic nucleus, then \( \mathcal{A}_j = \{ a \in A \mid ja = a \} \) is a partially ordered subgroup, where \( a; j b = j(a; b) \).

A quantale is complete lattice ordered semigroup initially introduced by Mulvey to provide a noncommutative generalisation of locales, study the spectra of \( C^* \)-algebras, and classify Penrose tilings, see [28] [29].

Definition 8. A quantale is an algebra \( Q = \langle Q, \; \vee \rangle \) such that \( Q = \langle Q, \vee \rangle \) is a complete lattice, \( \langle Q, ; \rangle \) is a semigroup, and the following conditions hold for each \( a \in Q \) and \( A \subseteq Q \):

1. \( a; \vee A = \vee \{ a; q \mid q \in A \} \)
Note that any quantale is a residuated semigroup as well. One may express residuals uniquely with suprema and products as follows:

1. $a \backslash b = \bigvee \{ c \mid a \leq c \leq b \}$,
2. $a/b = \bigvee \{ c \mid b \leq c \leq a \}$.

One may embed any residuated semigroup into some quantale via the Dedekind-MacNeille completion (see, for example, [33]) as follows. According to Goldblatt [10], residuated semigroups have the following representation via quantic nuclei (a closure operator over an ordered semigroup) and the Galois connection:

**Theorem 2.** Every residuated semigroup has an isomorphic embedding to the subalgebra of some quantale.

**Proof.** We provide a proof sketch.

Let $A = \langle A, \leq, \vdash, / \rangle$ be a residuated semigroup.

Let $X \subseteq A$. We put $lX$ and $uX$ as the sets of lower and upper bounds of $X$ in $A$. We also put $mX = luX$. Note that the lower cone of an arbitrary $x$, $\downarrow x = \{ y \mid S \ni y \leq x \}$, is $m$-closed, that is, $m(\downarrow x) = \downarrow x$. Moreover, $m : P(A) \to P(A)$ is a closure operator and $\langle (P(A))_m, \subseteq \rangle$ (where $(P(A))_m = \{ X \in P(S) \mid mX = X \}$ ) is a complete lattice with $\bigvee_m X = m(\bigcup X)$ and $\bigwedge_m = \bigcap X$ [3]. Moreover, according to Proposition [4], $\langle (P(A))_m, \subseteq, \vdash_m \rangle$ is a subquantale of $\langle (P(S))_m, \subseteq, ;_m \rangle$, since $m$ is a quantic nucleus. Here $X; Y = \{ x; y \mid x \in X, y \in Y \}$ and $X;_m Y = m(X; Y)$.

We define a map $f_m : A \to (P(A))_m$ such that $f_m : a \mapsto \downarrow a$. Note that $f_m$ preserves residuals and all existing suprema, and, thus, $f_m$ is order-preserving.

In their turn, quantales have a relational representation. First of all, let us define a relational quantale. The notion of a relational quantale was introduced by Brown and Gurr to represent quantales as algebras of relations and study relational semantics of the full Lambek calculus, the logic of bounded residuated lattices, see [5] and [6].

**Definition 9.** Let $A$ be a set. A relational quantale on $A$ is an algebra $\langle R, \subseteq, ; \rangle$, where

1. $R \subseteq P(A \times A)$,
2. $\langle R, \subseteq \rangle$ is a complete join-semilattice,
3. $; :$ is a relational composition that respects all suprema in both coordinates.

The uniqueness of residuals in any quantale implies the following quite obvious fact.

**Proposition 2.** Let $A$ be a relational quantale over a base set $X$, then for each $a, b \in A$

1. $a \backslash b = \{ (x, y) \mid \forall z \in X((z, x) \in a \Rightarrow (z, y) \in b) \}$,
2. $a/b = \{ (x, y) \mid \forall z \in X((y, z) \in b \Rightarrow (x, z) \in b) \}$.
The following representation theorem for quantale has been proved by Brown and Gurr [3].

**Theorem 3.** Every quantale \( Q = \langle Q, ;, \vee \rangle \) is isomorphic to the relational quantale on \( Q \) as a base set.

**Proof.** Let us describe a proof sketch. Let \( Q \) be a quantale and \( \mathcal{G}(Q) \) a set of its generators. We define:

\[
\hat{a} = \{ \langle g, q \rangle | g \in \mathcal{G}(Q), q \in Q, g \leq a \cdot q \} \quad \hat{Q} = \{ \hat{a} | a \in Q \}
\]

The rest of the sketch consists of the following claims.

**Claim 1.** \( a \leq b \iff \hat{a} \subseteq \hat{b} \).

**Claim 2.** \( \bigvee A = \bigvee \hat{A}, \hat{a} \hat{b} = \hat{a} \cdot \hat{b} \), and \( (\hat{Q}, \subseteq, \bigvee) \) is a complete semilattice.

**Claim 3.** \( (\hat{Q}, \subseteq, ;) \) is a relational quantale.

**Claim 4.** \( Q \) is isomorphic to \( (\hat{Q}, \subseteq, ;) \).

By Proposition 2 and Theorem 3, and the proposition above imply the following statement.

**Corollary 1.** Every residuated semigroup is isomorphic to the subalgebra of some relational quantale.

### 3.2 The main result

Given a finite algebra \( A \in \mathbb{R}(; , \setminus , / , \leq) \), we show that \( A \) is isomorphic to some semigroup over a finite base. As we have already said, theories of residuated semigroups and representable ones coincide, so \( A \in \mathbb{R}(; , \setminus , / , \leq) \) if \( A \) is a residuated semigroup.

**Lemma 1.** Let \( A \) be a residuated semigroup, then an interpretation \( R : A \rightarrow \hat{Q}_A \) such that \( R : a \mapsto a^R = \downarrow a \) is a \( \tau \)-representation, where \( \tau = \{ ; , \setminus , / , \leq \} \).

**Theorem 3** and the proposition above imply the following statement.

**Corollary 1.** Every residuated semigroup is isomorphic to the subalgebra of some relational quantale.
A representation of $\mathcal{A}$ by means of Theorem 2 belongs to $\mathcal{R}(; \setminus, /, \leq)$. Such a representation of $\mathcal{A}$ has the form $\langle \{ \hat{a} \} \mid a \in \mathcal{A} \setminus \setminus, /, \leq \rangle$.

Using the similar construction, one may also show that any residuated semigroup is representable and the theory of representable residuated semigroups is finitely axiomatisable. Such a construction might simplify significantly the argument provided by Andréka and Mikulás for the same facts [2].

The following observation is quite obvious as well:

**Corollary 2.** Let $\mathcal{A}$ be a finite residuated semigroup, then its representation with the corresponding relational quantale has the finite base.

**Proof.** The base of the quantale $\hat{\mathcal{Q}}_\mathcal{A}$ is the set of Galois stable subsets of $\mathcal{A}$, the cardinality of which is clearly finite.

Theorem 2, Theorem 3, Lemma 2, and Corollary 2 provide the solution to [15, Problem 19.17].

**Theorem 4.** $\mathcal{R}(; \setminus, /, \leq)$ has the finite representation property.

Clearly that the desired $W$ is the domain of Galois stable subsets of $\mathcal{A}$.

### 3.3 Corollaries

The main corollary of Theorem 4 is that the Lambek calculus has the finite model property. Thus, we have a semantical proof of decidability of the Lambek calculus. Before that, there were several algebraic proofs that the Lambek calculus has the FMP [7], but the authors considered arbitrary algebras, not representable ones. Alternatively, one may show that the Lambek calculus is decidable syntactically, that is, via cut elimination and the subformula property [22].

**Corollary 3.** The Lambek calculus is complete w.r.t finite relational models (has the fmp).

Moreover, finite axiomatisability and having the finite representation property for $\text{RRS}$ imply the following corollaries:

**Corollary 4.** Let $\tau = \{ ; \setminus, /, \leq \}$, then

1. The membership of $\mathcal{R}(\tau)$ is decidable, that is, there exists an algorithm determining whether a finite $\tau$-structure belongs to $\text{RRS}$,
2. The class $\mathcal{R}(\tau)_{\text{fin}} = \{ \mathcal{A} \in \mathcal{R}(\tau) \mid |\mathcal{A}| < \omega \}$ is recursive.

As we said, the universal theory of $\text{RRS}$ is NP-complete. Corollary 3 with NP-completeness also imply NP-completeness of finite representable residuated semigroups.
4 Further work

We already know that the class of representable residuated monoid is finitely axiomatisable [2]. The construction we described does not work properly for residuated monoids. The interpretation of the identity has the form $1^R = \downarrow 1$ that does not have to be the identity relation. Thus, the finite representation property for residuated monoids has to be shown differently. The author’s conjecture claims that one may modify the argument Andrěka and Mikulás [2] using games on networks and extract the finite representation from that. The similar idea has been applied by McLean and Mikulás to show that structures of the signature containing meet, join, domain, and range has the finite representation property [26].

The other subsignature of the relation algebras signature is of interest as well. For instance, Hirsch formulated the conjecture according to which $R(\tau)$ does not have the finite representation property iff $\tau$ contains composition and either meet or negation [11]. In particular, we do not know whether $R(\cdot, -)$ has the finite representation property. The negative solution of this problem would imply that the right-to-left implication of Hirsch’s conjecture is valid since the $R(\cdot, \cdot)$ case has been already considered by Maddux [25].

5 Acknowledgements

The author is sincerely grateful to Robin Hirsch, Ian Hodkinson, and Stepan Kuznetsov for valuable conversations and advice.

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