Quench dynamics of topological Euler class in optical lattices

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The last years have witnessed rapid progress in the topological characterization of out-of-equilibrium systems. We report on robust signatures of a new type of invariant—the Euler class—that recently gained interest in the context of topological crystalline phases, in such a dynamical setting. We find that upon quenching a trivial state in two dimensions with a Hamiltonian having non-trivial Euler class results in the formation of topologically stable monopole-antimonopole pairs. These pairs induce a non-trivial linking of momentum-time trajectories when considered as pre-images of Bloch sphere vectors under the first Hopf map, thereby making the invariant experimentally observable.

Introduction.—Topological insulators (TI) are gapped quantum phases that have a topological nature arising by virtue of protecting symmetries. Following time reversal symmetry (TRS) protected TIs [1, 2], past years have seen remarkable progress in characterizing topological materials taking into account crystal symmetries [3–11]. Using combinatorial arguments that map out topological classes of band structures in momentum space [12], recent schemes further formulated diagnosis criteria upon comparing which of these combinations can be obtained by Fourier transforming trivial atomic configurations, defining topology relative to this subset [13, 14]. These pursuits also revealed novel fragile invariants that can be trivialized by gap closings with trivial bands, rather than involving those having opposite topological charge [15–17], relating to refined partitioning schemes [18]. An archetypal invariant emerging in such studies is Euler class. It acts as the crystalline-protected analogue of archetypal invariant emerging in such studies is Euler class. It acts as the crystalline-protected analogue of Euler class models lead novel charges that have non-Abelian braiding properties [19–21], making an experimental realization even more desirable.

On another note, ultracold atomic gases have proven versatile platforms for exploring topological phenomena [22–33]. In particular, advances in periodic driving and artificial gauge fields [34] have called for expansion of these notions to out-of-equilibrium settings and new classification schemes [35–39]. Development of novel quenching techniques, where a system is driven far from equilibrium by sudden changes in the Hamiltonian—most interestingly between topologically distinct regimes—has not only unearthed new connections between different topological invariants, but also provided a powerful experimental tool to detect topological quantities [40–47]. As versatile as they are, quench protocols have been employed only in the context of Chern insulators and mainly in two-band models. We here consider the dynamics of Euler class that requires a minimum of three bands (although the notion extends to many-band cases [18, 21]), following a quench between topologically distinct regimes. We demonstrate that non-trivial Euler class embodies stable monopole-antimonopole pair production, which provides for distinct experimental signatures in the linking of momentum-time trajectories upon appealing to a Hopf map. Accordingly, we device concrete experimental protocols to investigate this unexplored class of topology in cold-atom systems.

Chern number versus Euler class.—In simplest form, Chern insulators can be described by a two-band Hamiltonian in two dimensions,

$$H_C(k) = d(k) \cdot \sigma + d_0(k)\sigma_0,$$

for momentum $k$ and Pauli matrices $\sigma$. Upon spectral flattening of the Hamiltonian (by replacing $d \rightarrow d/||d||$ and $d_0 \rightarrow 0$ without affecting the topology), the Chern number $C$ coincides with the second homotopy group of the sphere, $\pi_2(S^2) = \mathbb{Z}$. The invariant can readily be calculated via the Pontryagin skyrmion number [48]

$$C = \frac{1}{4\pi} \int_{BZ} d^2k \ d \cdot (\partial_{k_x} d \times \partial_{k_y} d),$$

where the Chern number can be geometrically interpreted as the covering of $S^2$ in terms of $d$.

Similarly, Euler invariant $\xi$ finds a simple incarnation in three-band models. However, rather than a complex variant as in the Chern number, Euler class corresponds to a real characteristic form, which necessitates a protecting symmetry. One of the most rudimental is $C_2T$ as almost all lattice geometries possess two-fold rotational symmetry, and thus will be assumed in the remainder. Consequently, the Hamiltonian can be recast

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into real symmetric matrix [21], having three eigenstates $E = \{|u_1(k)\rangle, |u_2(k)\rangle, |u_3(k)\rangle\}$. The symmetry further dictates that the eigenstates form an orthonormal triad, a dreibein, satisfying $n(k) \equiv |u_3(k)\rangle = |u_1(k)\rangle \times |u_2(k)\rangle$. In this basis, the associated spectral flattened form of the Euler Hamiltonian reads, [21, 49]

$$H(k) = 2 n(k) \cdot n(k)^	op - \mathbb{I}_3.$$  \hfill (3)

The spectrum features two degenerate bands $\{|u_1(k)\rangle, |u_2(k)\rangle\}$ with eigenvalue $-1$ and a 'spectator' third band $n(k)$ of energy $1$. Physically, the Euler invariant $\xi$ is encoded by winding of the degenerate two-band subspace, inducing a number of $2\xi$ band nodes, and traced by the third band due to the special structure of the dreibein (see Appendix A for details). The invariant then takes a particularly easy form [21],

$$\xi = \frac{1}{2\pi} \int_{BZ} d^3 k \cdot n \cdot (\partial_{k_x} n \times \partial_{k_y} n).$$  \hfill (4)

The Euler class, similar to the Chern number, can thus geometrically be represented as a vector tracing the sphere. In fact, $n(k)$ on its own defines a Chern insulator as in Eq. (1), with $d = n(k)$, which we verify for the models studied here (Appendix A).

Nonetheless, care has to be exercised in this sphere analogy. Vectors spanning the dreibein are a priori only defined up to a sign relating to the projective plane $RP^2$, as in a nematic [50–53]. $RP^2$ features $\mathbb{Z}_2$-valued string charges and monopole charges, characterized by the first and second homotopy group (see Appendix E). Assuming absence of the former 'weak invariants', the monopole charges are $\mathbb{Z}$-valued, although the sign is ambiguous, touching upon illustrious Alice dynamics [54, 55]. To construct Euler class, a consistent gauge, or handedness, has to be defined for $E$, making the analogy appropriate.

**Model setting.**—The Hamiltonian can be generally parametrized as $H(k) = R(k)|-1 \oplus 1|R(k)^	op$, where $R(k)$ accomplishes the desired winding via a geometric construction that makes use of a so-called Pliicker embedding (see Appendix A and Ref. [18]). Sampling over a grid set by the target lattice geometry, which we here take to be a square, renders the explicit hopping parameters specifying the Hamiltonian

$$H(k) = \sum_j h_j(k) \lambda_j,$$  \hfill (5)

for the eight Gell-Mann matrices $\lambda_j$. In the following, we consider Euler class $\xi = 2$ and $\xi = 4$, the $h_j(k)$ of which are given in Appendix A and F.

**Quench dynamics.**—We now turn to the quench dynamics. For this purpose, we consider a trivial initial state $\Psi_0(k) = (1, 0, 0)^	op$ and follow its time evolution, $\Psi(k, t) = U(k, t)\Psi_0(k) = e^{-it H(k)}\Psi_0(k)$ resulting from a sudden change of the Hamiltonian to topologically non-trivial Euler form (we set $\hbar = 1$). For simplicity, we spectrally flatten the quench Hamiltonian, which does not affect the topologically robust signatures presented.

The Euler Hamiltonian physically corresponds to a $\pi$-rotation around the vector $n(k)$, as can be easily seen by analysing matrix components of Eq. (3). Since the initial state naturally corresponds to a normalized vector on $S^2$, this gives us the incentive to formulate the quench dynamics as rotations on this sphere. For the flattened Hamiltonian with the energy gap $\Delta = 2$ separating $n(k)$ from the degenerate subspace, the time-evolving state returns to itself, $\Psi(k, t^*) = \Psi_0$ after each $t^* = 2\pi/\Delta$, which characteristically traces a circle $S^1$ in time. Combined with the two-dimensional momentum space ($T^2$), the time evolution forms a three-torus in $(k_x, k_y, t)$-space. Since the weak invariants of this $T^3$ are zero, this corresponds to $S^3$ and gives the possibility of establishing a Hopf map from $S^3$ to $S^2$. We here prove that this is indeed the case. Using that the Hamiltonian squares to unity, the time evolution operator can be written in Rodrigues form, so that $\Psi(k, t) = [\cos (t) - i \sin (t) H(k)]\Psi_0(k)$. This complex time-evolving state can be mapped onto a Bloch sphere as $\hat{\Psi} = \Psi(k, t)\mu \Psi(k, t)$, where

$$\mu_x = \begin{pmatrix} 0 & i & 1 \\ -i & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \mu_y = \begin{pmatrix} 0 & 1 & -i \\ 1 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \mu_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$  \hfill (6)

The above construction can be best motivated upon appealing to the quaternion description of the Hopf map. Quaternions, written as $q = x_0 + x_1 i + x_2 j + x_3 k, x_i \in \mathbb{R}^3$, extend complex numbers, with units satisfying $ij = k, jk = i, ki = j$, and $i^2 = j^2 = k^2 = -1$. Vectors in $\mathbb{R}^3$ can be represented as pure quaternions with real part $x_0 = 0$, using the units $i, j, k$ as basis vectors. Moreover, quaternions having unit norm $\sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2} = 1$ are called versors and implement rotations on 3D vectors. In particular, acting with versor $v = x_0 + v$ on vector $t$ as $R_v : t \mapsto vtv^{-1}$, where $v^{-1}$ is the inverse of $v$, implements a rotation around vector $v$ by an angle $\theta = 2 \arccos (x_0)$. The rotation map can be rewritten in matrix form $R_v t = vtv^{-1}$.
(2π)

θ, with

\[
R_v = \begin{pmatrix}
  x_2^2 + x_3^2 - x_1^2, & 2x_1x_2 - 2x_0x_3, & 2x_0x_2 + 2x_1x_3 \\
  -2x_0x_2 + 2x_1x_3, & x_0^2 + x_1^2 - x_2^2, & 2x_1^2 + x_2^2 - x_3^2 \\
  2x_0^2 + 2x_1^2 + x_3^2, & 2x_0x_1 + 2x_2x_3, & -2x_0^2 - 2x_1^2 + x_2^2 + x_3^2
\end{pmatrix}
\]

This description carries a deeper meaning as a representation of the Hopf map. Since versors have unit norm, they span a three-sphere \( S^3 \subset \mathbb{R}^4 \). In addition, the rotation map preserves the norm and hence, acting on a normalized vector, results in another normalized vector on \( S^2 \). In other words, acting with \( R_v \) on a unit norm vector (from the left or right) induces a map from \( S^3 \) to \( S^2 \), the renowned Hopf map. The inverse image is a circle, constituting the familiar fibre. A key insight is that the elements of \( \Psi(\theta) \) can be related to a quaternion by associating \( \{x_0, x\} = \{\cos t, -\sin t \} H \Psi_0 \}, \) which then connects to \( R_v \) through the \( \mu \)-matrices. We refer to Appendix B for further elaboration on the Hopf parametrization.

Monopole-antimonopole pairs.– We now analyze the physical consequences embodied by the Hopf construction. We identify \( \mathbf{a}(\mathbf{k}) = H(\mathbf{k}) \Psi_0(\mathbf{k}) \) as a \( \pi \)-rotation of the initial state around the vector \( \mathbf{n}(\mathbf{k}) = (n_1, n_2, n_3) \) as illustrated in Fig.1. For \( \Psi_0(\mathbf{k}) = (1, 0, 0)^\top \), this results in \( \mathbf{a}(\mathbf{k}) = (2n_1^2 - 1, 2n_2, 2n_1n_3)^\top \) through Eq. (3). Where \( \mathbf{n}(\mathbf{k}) \) wraps the sphere once [left-purple sphere in Fig.1], the vector \( \mathbf{a}(\mathbf{k}) \) covers the sphere twice within the Brillouin zone. For \( k \)-values where \( \mathbf{n}(\mathbf{k}) \) lies within the same hemisphere with \( \Psi_0 \) (i.e. \( +\hat{x} \)-hemisphere), the orientation of the wrapping of \( \mathbf{a}(\mathbf{k}) \) is same with \( \mathbf{n}(\mathbf{k}) \) –clockwise, encapsulating a monopole [red vector-sphere in Fig.1]. However, when \( \mathbf{n}(\mathbf{k}) \) and \( \Psi_0 \) lie on opposite hemispheres, \( \mathbf{a}(\mathbf{k}) \) covers the sphere anticlockwise [blue sphere], corresponding to an antimonopole.

Analytically, this can be most easily seen by writing \( \mathbf{n} = (\cos \alpha, \sin \alpha \cos \beta, \sin \alpha \sin \beta) \) in spherical coordinates, where the polar angle \( \alpha \) is defined with respect to \( +\hat{x} \)-axis, and the azimuthal angle \( \beta \) from \( +\hat{y} \)-axis for simplicity. Correspondingly, the vector \( \mathbf{a} = (\cos 2\alpha, \sin 2\alpha \cos \beta, \sin 2\alpha \sin \beta) \) then features doubling of the polar angle as compared to \( \mathbf{n} \), hence covering the sphere twice. For \( \alpha \in [0, \pi/2] \) this is clockwise, and anticlockwise for \( \alpha \in [\pi/2, \pi] \) as \( \sin 2\alpha \) changes sign. Alternatively, considering the stereo-graphic representation of \( \mathbf{a}(\mathbf{k}) \) results in the same conclusion. We emphasize that this monopole–antimonopole pair cannot annihilate each other through recombination and are topologically stable, since the patches of the Brillouin zone (BZ) hosting them belong to the clockwise and anticlockwise cover and thus are naturally separated, akin to the double cover of the BZ for Chern insulators having invariant 2.

Following the quench, the time-evolving state inherits this monopole–antimonopole structure through

\[
\Psi(\mathbf{k}, t) = \cos(t)\Psi_0(\mathbf{k}) - i\sin(t)\mathbf{a}(\mathbf{k}), \tag{7}
\]

which will be also imprinted on the linking number. Under the Hopf construction, each point in the \((k_x, k_y, t)\)-space maps to a vector \( \mathbf{p} \) on the Bloch sphere via Eqs. (7) and (6). When the Hopf map is non-trivial, inverse images of two such vectors on \( S^2 \) manifest linking in \( T^3 \) \([56]\), the number of which signifies the underlying invariant.

To demonstrate this, we numerically evolve the initial state following a quench with a Hamiltonian having Euler invariant \( \xi \). In Fig.2, we present the inverse images of two vectors \( \mathbf{p}_1 = (1, 0, 0) \) and \( \mathbf{p}_2 = (-1, 0, 0) \), i.e. we mark each \((k_x, k_y, t)\)-triple at which the state points along \( \mathbf{p}_1 \) or \( \mathbf{p}_2 \). For the case of \( \xi = 2 \) in Fig.2a, the inverse images link twice with opposite sign in separate patches of the BZ, conforming the monopole–antimonopole pair. Similarly, when the Euler invariant is \( \xi = 4 \) as in Fig.2b, there are four different regions attributed to two linkings of positive, and two of negative sign.

We quantify our findings upon appealing to the Hopf invariant \( \mathcal{H} \). Here it is convenient to use a complex two-vector basis, as it reinforces the quaternion analogy. A quaternion can be rewritten as \( q = z_1 + iz_2 \) by using two complex numbers \( z_1 = x_0 + ix_3 \) and \( z_2 = -x_2 + ix_3 \). Accordingly, a two-vector description of \( \Psi(\mathbf{k}, t) \) may be established by identifying its first component (the one harboring the real part \( x_0 \) as \( z_1(\mathbf{k}, t) \), and combining the second and third components into a single function as \( z_2(\mathbf{k}, t) = i\Psi_2(\mathbf{k}, t) + \Psi_3(\mathbf{k}, t) \). We thus define \( \zeta(\mathbf{k}, t) = (z_1(\mathbf{k}, t), z_2(\mathbf{k}, t))^\top \). The Bloch vector is then reproduced via Pauli matrices amouting to the standard Hopf parametrization, \( \mathbf{p} = \zeta(\mathbf{k}, t) \sigma \zeta(\mathbf{k}, t)^\dagger \). This was notably employed to reveal the Chern number \([40]\). After inserting the two-vector \( \zeta(\mathbf{k}, t) \) in the Hopf invariant,

\[
\mathcal{H} = -\frac{1}{4\pi^2} \int d^2k \ det \ e^{ik \cdot \mathbf{p}} \partial_{\mathbf{k}}^\dagger \partial_{\mathbf{k}} \zeta, \tag{8}
\]

the time integral decouples as in the Chern case \([40, 57, 58]\), and we find that the Hopf invariant \( \mathcal{H} \) reduces to the winding of \( \mathbf{a} \),

\[
\mathcal{H} = \frac{1}{4\pi} \int_{BZ} d^2k \mathbf{a} \cdot (\partial_{k_x} \mathbf{a} \times \partial_{k_y} \mathbf{a}). \tag{9}
\]

Due to monopole–antimonopole pairs, \( \mathcal{H} \) evidently attains zero value, which we also confirm numerically.
implemented in optical lattices for the measurement of Chern number in two-band models created by encoding the pseudospin as hyperfine degrees of freedom \[42, 43\] and as sublattice degrees of freedom \[25, 41\]. While the former can be in principle extended to an arbitrary number of bands, the latter relies on adapting state tomography techniques \[59, 60\] which have been restricted to two bands so far for detecting the linking structure.

When the pseudospin structure is induced by using different hyperfine states \((|A\rangle, |B\rangle, |C\rangle)\), the quench from a spin-polarized trivial initial state \((\Psi_0 = |A\rangle)\) to a nontrivial Euler form initializes Raman-induced oscillations in spin polarization that can be measured locally for each \(k\) \[42, 43\]. The \(\hat{z}\)-component of the Bloch vector defined via Eq. (6) is then given by \(P_z(k, t) = (N_A - N_B - N_C)/(N_A + N_B + N_C)\), where \(N_C\) is the number of particles detected in spin state \(\gamma = A, B, C\).

However, as it does not correspond to the Bloch sphere, \(m(k, t)\) on its own does not capture the winding of the Euler class on \(S^2\). By analyzing the two-vector \(\zeta(k, t)\) description of the Hopf map, one can see that the north and south poles of the sphere can be associated to \(|A\rangle\) and \(|D\rangle\), \(
\), so that \(\zeta(k, t) = \sin(\theta/2)|A\rangle + \cos(\theta/2)e^{i\phi}|D\rangle\) in spherical coordinates. To access the Bloch sphere in TOF, we apply a \(\pi/2\)-pulse to the state with respect to the sublattice \(B\).

Moreover, we observe \(\Psi(k, t)\) with the flat-band Hamiltonian \(H_{\text{pulse}} = (\nu/2)\text{diag}(1, -1, 1)\), for a time \(\pi/2\nu\) given by the energy difference \(\nu\), so that the second component of the state acquires a phase \(e^{i\nu t}\); i.e. \(\Psi(k, t) = \Psi_1(k, t) + i\Psi_2(k, t)\), \(\Psi_3(k, t)\) \textsuperscript{1}, directly corresponding to \(\zeta(k, t)\). Upon further projecting on flat bands for tomography, \(H_{\text{tom}} = (\omega/2)\mu_z\), the state \(\Psi'(k, t)\) starts precessing around the \(\hat{z}\)-axis with frequency \(\omega\), adding a phase \(e^{-i\omega t_{\text{tom}}/2}\) to \(\Psi(t_{\text{tom}})\). As such, the momentum distribution after TOF reveals the Bloch vector \(m(k, t_{\text{tom}}) = \tilde{w}(k)\), making a quench from topologically non-trivial to trivial regime possible and detectable by linking structure.

Connecting to observables, quench protocols have been implemented in optical lattices for the measurement of Chern number in two-band models created by encoding the pseudospin as hyperfine degrees of freedom \[42, 43\] and as sublattice degrees of freedom \[25, 41\]. While the former can be in principle extended to an arbitrary number of bands, the latter relies on adapting state tomography techniques \[59, 60\] which have been restricted to two bands so far for detecting the linking structure.

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show that the nested Hopf tori and the non-trivial linking in $(k_x, k_y, t)$-space can be detected in spin-resolved measurements in momentum space, or via a modified state tomography technique which we expand to the three-band Euler model. Our work opens up avenues for the exploration of new crystalline and exotic fragile topologies, that have attracted much interest in the last couple of years, in the versatile setting of ultracold quantum gases of three-band systems and beyond.

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Appendix A: Hamiltonians for three band models having non-trivial Euler class

We here briefly comment on the explicit models of the main text that were derived using the techniques of Ref. [18]. These include systems having Euler class $\xi = 2, 4$ and different spectral orderings of the bands. Finally, we also illustrate the derived Chern models induced by the non-degenerate subspace.

Model Hamiltonians.— Chern insulator models find their simplest incarnation in two-band systems of the form

$$H_C = \mathbf{d}(k) \cdot \mathbf{\sigma} + d_0(k)\sigma_0,$$  

(A1)

on the bases of Pauli matrices $\mathbf{\sigma}$. Hamiltonians exhibiting Euler class have a similar tractable form in terms of real symmetric three-band models, that have two degenerate bands featuring a number of $2\xi$ of isolated band nodes, and a third separated band. A priori, one may expect that separating the system into a two-band \{\{u_1(k)\}, \{u_2(k)\}\} and one-band subspace \{\{u_3(k)\}\} can give non-trivial behavior as the stable homotopy classes of Wilson flows, characterized by the first homotopy class of $\pi_1$, cover $S^2$ [16, 49]. The two subblocks then accordingly relate to $\pi_1(SO(2)) = \mathbb{Z}$, conveying the gapless charges within the gap between these two bands, whereas $\pi_1(SO(1)) = 0$ relates to the third band. However, these guiding intuitions should be taken with caution. First of all, the orthonormal frame spanned by $E = \{\{u_1(k)\}, \{u_2(k)\}, \{u_3(k)\}\}$ is invariant under reversing the sign of each of the vectors. Hence, the space of Hamiltonians is actually given by the projective plane $S^2/\mathbb{Z}_2 = RP^2$, coinciding with bi-axial nematic descriptions [51, 52]. Fortunately, as for the Euler classes, we are only interested in the orientable case of the vector bundle which can actually be related to the sphere [21]. Secondly, the facts that all bands must sum up to a trivial charge and $\pi_1(SO(3)) = \mathbb{Z}_2$ also suggest that the integer winding of the two band subspace has to be even, which is consistent with earlier findings that an odd Euler class would require a four-band model [61]. Finally, $|u_3(k)\rangle$ is related to the other two eigenstates as $|u_3(k)\rangle = |u_1(k)\rangle \times |u_2(k)\rangle \equiv n(k)$. Although these hints are merely a motivation, the spectrally flattened form for such real three-band systems can indeed be shown to be [18, 21, 49],

$$H(k) = 2\mathbf{n}(k) \cdot \mathbf{n}(k)^\top - \mathbb{I}_3.$$  

(A2)

With these insights it is possible to construct Euler Hamiltonians systematically using a geometric construction on arbitrary lattice geometries [18]. In a nutshell, the idea is that the winding can be encoded via a so-called Plücker embedding. That is, via a pullback map to coordinates parameterizing the sphere, the winding can be formulated in terms of rotation matrix $R(k)$. Concretely, this means that we can obtain Hamiltonians with Euler class $\xi$ as

$$H(k) = R(k)[-1 \oplus 1]R(k)^T,$$  

(A3)

where $R(k)$ implements a $\xi$-times winding of the sphere and we take the flattened energies of degenerate/third band subspace to be $-/+ 1$.

From $H(k)$ we can then get an explicit tight-binding model upon sampling over a grid set by the lattice, e.g a square lattice. Applying an inverse Fourier transform then results in the hopping parameters $t_{\alpha\beta}(R_j - \mathbb{0}) = \mathcal{F}^T \{ H_{\alpha\beta}(k_m) \}_{m \in \Lambda^{s}} \{ (R_j - \mathbb{0}) \}$ around a specific site. Formally this has to be executed over the whole lattice, but due to fast decay of the Fourier envelope we can get simple models by truncating this process, meaning that we consider a specific number of neighbors $N$ for tunneling.

For the $\xi = 2$ case, we restrict tunneling to $N = 2$ neighbors. The three-band Hamiltonian then takes the form,

$$H(k) = h_j(k)\lambda_j,$$  

(A4)

in terms of the eight Gell-Mann matrices $\lambda_j$, where the sum over $j$ is implied. The specific form of the $h_j(k)$ then read

$$h_j(k) = \sum_{\alpha,\beta} t_{j}(\alpha, \beta) e^{i\alpha k_x + \beta k_y},$$  

(A5)

where $\alpha, \beta$ are integers running from $-N$ to $N$ and $t_j(\alpha, \beta)$ defines a matrix element of hopping parameters obtained by the truncation procedure defined above. We have specified these eight $5 \times 5$-matrices for completeness in Appendix F.

Similarly, it is straightforward to obtain models having Euler class $\xi = 4$, where we truncate to $N = 3$ neighbors to accommodate the higher winding. The hopping matrices of this higher winding are also specified in Appendix F. Finally, the construction is also flexible to shift the third band to the bottom of the spectrum, corresponding to the flattened form,

$$H(k) = \mathbb{I}_3 - 2\mathbf{n}(k) \cdot \mathbf{n}(k)^\top,$$  

(A6)

by acting with $R$ as $H(k) = R(k)[-1 \oplus 1]R(k)^T$. We refer to these systems as “inverted”.

Derived Chern model.— As stated in the main text, the class of Euler Hamiltonians that are studied here directly induce Chern insulator models upon promoting the normalized $\mathbf{n}(k) = |u_3(k)\rangle$-eigenstate to $d(k)$ in Eq. (A1). Since $\mathbf{n}(k)$ is always normalized and real (due to the symmetries), it can be treated as a Bloch vector that corresponds to a flattened Hamiltonian as $H'(k) = \mathbf{n}(k) \cdot \mathbf{\sigma}$. We consider the Euler Hamiltonian
with $\xi = 2$ and display the complete coverage of two-sphere by $n(k)$ in Fig. 4a. Accordingly, when we quench a trivial initial state ($\Psi_0(k) = \frac{1}{\sqrt{2}}(1, 2)^\dagger$) to suddenly evolve with $H'(k)$, we obtain linking number (Hopf invariant) 1 in Fig. 4b.

**Appendix B: Quaternions, Rotations and Hopf Fibration**

We find that the Hopf map description of quench dynamics can be most conveniently described in terms of quaternions, that directly relate to usual characterizations of rotations in terms of $SU(2)$ and $SO(3)$ matrices, thereby also exposing the intimate relation between the two. Note that here we deal with the first Hopf map, where higher Hopf maps can also be represented by similar constructions, for example using octonions.

**Quaternions.**—Recall that quaternions in essence constitute a generalization of complex numbers in an analogous manner in which the latter extends the real numbers. Specifically, a quaternion $q$ is written as

$$q = x_0 + x_1i + x_2j + x_3k,$$

where the quaternion units satisfy $ij = k, jk = i, ki = j$ and $i^2 = j^2 = k^2 = -1$. We hence see that the real coefficients identify $q$ with $\mathbb{R}^4$ similar to how $\mathbb{C}$ relates to $\mathbb{R}^2$. Accordingly, we refer to $x_0$ and $\{x_1, x_2, x_3\}$ as the real part and pure quaternion part of $q$, respectively. Using the relations between the units and defining the conjugate of $q$ as reversing the sign of the units, that is $q^* = -\frac{1}{2}(q + iq_1 + jq_2 + kq_3)$, the norm of $q$ is readily found to be $|q| = \sqrt{qq^*} = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}$.

**Rotations and versors.**—In the subsequent, we will be particularly interested in quaternions $v$ having unit norm, or so-called versors. Due to the constraint $|v| = 1$, it is evident that the subset of versors span the hypersphere $S^3 \subset \mathbb{R}^4$. Versors are of particular use in describing rotations in three spatial dimensions. Note that we can represent a vector $t \in \mathbb{R}^3$ as a pure quaternion, having a zero real part and with the quaternion units corresponding to the unit vectors parametrizing $\mathbb{R}^3$. It is easy to see that upon multiplying a pure quaternion with an arbitrary quaternion results in another pure quaternion. Rotations on vectors can then be implemented using versors, that, due to the unit norm, indeed generate an isometry. In particular, it is well known that acting with a versor $v = x_0 + v$ on a vector $t$ as

$$R_v : t \mapsto vtv^{-1},$$

where $v^{-1}$ refers to the inverse of $v$, implements a rotation around the vector $v$ by an angle $\theta = 2\arccos(x_0) = 2\arcsin(1 - x_0)$.

**Hopf map.**—The formulation of rotations in terms of versors provides for a direct parametrization of the Hopf map. Indeed, we can reinterpret Eq. (B2) as a map relating the versor $v$ with a vector $t' = vtv^{-1}$. Moreover, as this entails an isometry we can already expect that when $t$ is a unit vector, and thus an element corresponding to the two-sphere $S^2$, the map returns another unit vector $t' \in S^2$. In other words, the rotation isometry acts on $S^2$ transversely. As the set of versors span $S^3$, this directly induces a map of $S^3$ to $S^2$, the famous Hopf fibration.

As a specific example one may consider the versor $v = x_0 + x_1i + x_2j + x_3k$ and the unit vector in the $\hat{x}$-direction that we thus represent as $i$ in the quaternion language, $t = i$. Simple algebra then shows that applying the rotation isometry, Eq. (B2), results in

$$t' = \begin{pmatrix} x_0^2 + x_1^2 - x_2^2 - x_3^2 \\ 2x_1x_2 + 2x_0x_3 \\ x_0^2 - x_1^2 + x_2^2 - x_3^2 \\ -2x_0x_2 + 2x_1x_3 \end{pmatrix},$$

which evidently has unit norm, thereby inducing the Hopf map $H : S^3 \to S^2$. Indeed, it is easy to verify that the inverse image is a circle, which constitutes the familiar fibre under this map.

We can make the above even more insightful upon representing the action of the versor $v$ in terms of a rotation matrix acting on a vector. The columns represent the action on the unit vectors $i, j, k$ and a similar calculation to the one above then gives us the general rotation matrix

$$R_v = \begin{pmatrix} x_0^2 + x_1^2 - x_2^2 - x_3^2 & 2x_1x_2 - 2x_0x_3 & 2x_0x_2 + 2x_1x_3 \\ 2x_1x_2 + 2x_0x_3 & x_0^2 - x_1^2 + x_2^2 - x_3^2 & -2x_0x_1 + 2x_2x_3 \\ -2x_0x_2 + 2x_1x_3 & 2x_0x_1 + 2x_2x_3 & x_0^2 - x_1^2 - x_2^2 + x_3^2 \end{pmatrix}.$$
rotation matrix in 3D. From the viewpoint of the Hopf map, we note that Eq. (B4) is nothing but an explicit parameterization. Indeed, any row, column or linear linear combination generated by acting on $R$ with a unit norm vector from the left or right implements the first Hopf map.

Quenches in two-band systems.— We can directly put the above notions in use to reproduce the dynamics of two band models quenched between trivial and non-trivial Chern numbers [40, 41, 43, 57]. We consider the model given in Eq. (A1) for $d_0 = 0$, where the relation of $d$ to the Chern number is given in the main text Eq. (2). An essential role is then played by the time evolution operator $U_C$, which takes the particularly easy form

$$U_C = e^{-itH_C} = \cos (t) - i \sin (t) \sigma \cdot d(k).$$

Writing this in matrix form one obtains

$$U_C = \begin{pmatrix} x_0 + ix_3 & ix_1 + x_2 \\ ix_1 - x_2 & x_0 - ix_3 \end{pmatrix},$$

where we already suggestively identified the elements $\{x_0, x_1, x_2, x_3\} = \{\cos t, -\sin (t) d_1, -\sin (t) d_2, -\sin (t) d_3\}$ in terms of the components $d_i(k)$ of $d(k)$. Indeed, assuming that upon spectral flattening $d(k)$ is a unit vector, we can directly see that, by relating the above to a versor with components $x_a$, the action of the time evolution operator on the initial state $\Psi_0(k)$ is simply to induce a rotation around the vector $d(k)$ with a period that is set by $|d|$. In fact, the above identification is the standard one to relate a quaternion of unit norm to a $SU(2)$ matrix representation.

Starting from a trivial $\Psi_0(k)$, we quench the system suddenly with a non-trivial Hamiltonian so that the state evolves as $\Psi(k,t) = U_C \Psi_0(k)$. One can then map the evolving state back to the Bloch sphere upon considering $\hat{p} = (\Psi(k,t) \sigma_x \Psi(k,t)^\dagger, \Psi(k,t) \sigma_y \Psi(k,t)^\dagger, \Psi(k,t) \sigma_z \Psi(k,t)^\dagger)^\dagger$, thereby establishing a Hopf map from $\{k_x, k_y, t\}$, that is identified with $S^2$, to the Bloch sphere constituting the $S^2$ [40]. This can be directly checked from the above formulæ by taking, e.g. $\Psi_0 = (1,0)^\dagger$, which gives

$$\hat{p} = \begin{pmatrix} 2x_1 x_3 - 2x_0 x_2 \\ 2x_0 x_1 + 2x_2 x_3 \\ x_0^2 + x_3^2 - x_1^2 - x_2^2 \end{pmatrix},$$

thus parameterizing the Hopf map with the last row (or column upon taking a minus sign in the definition of $x_a$) of Eq. (B4). Similarly, one can verify that taking $\Psi_0(k) = (0,1)^\dagger$ results in a similar expression. This, therefore, shows that starting from an arbitrary normalized initial state, the first Hopf map is realized with the construction of Eq. (B4). Most interestingly, the Hopf map directly exposes the non-trivial Chern number of the quench Hamiltonian as it implies a non-trivial Hopf invariant $\mathcal{H}$ [40, 57], which is also manifested in non-trivial linking of the trajectories under the inverse image of $\mathcal{H}$ that can be measured in experiments [41–43]. We note that, the quaternion description also naturally captures quenches from topologically non-trivial to trivial Hamiltonians.

Appendix C: Quench dynamics in three-band models having non-trivial Euler class

We now turn our attention to further detailing the main subject, topological aspects and dynamics of quenches involving non-trivial Euler class Hamiltonian. We thus assume $C_2T$-symmetry in all instances, which in fact is rather rudimentary and hence does not impose a serious limitation with regard to experimental implementation.

Quenching with non-trivial Chern Hamiltonian.— We first consider quenching an initial state with a Hamiltonian having a non-trivial Euler class in which the third band $|n(k)\rangle$ has the highest energy, topping the other two degenerate bands by an energy gap. We then return to the inverted situation, with $|n(k)\rangle$ being the bottom band, at the end of this section.

As shown in the main text, upon spectral flattening, the Hamiltonian takes the form of Eq. (A2). A non-trivial Euler class, in analogy to the Chern number, is then manifested by the geometrical interpretation that $n(k)$ traces the unit sphere, in fact by a factor 2 as compared to the Chern number case. Moreover, we see that Hamiltonian (A2) physically represents a rotation by angle $\pi$ around the vector $n(k)$. Indeed, the quaternion description in this case reads $v = \cos(\pi/2) + \sin(\pi/2)\{n_1 i + n_2 j + n_3 k\}$ in terms of the components of $n(k) = (n_1, n_2, n_3)^\dagger$. As a result, the time evolution operator $U$ still assumes the simple form

$$U = e^{-itH} = \cos (t) - i \sin (t) H(k),$$

reminiscent of the two-band case.

Starting with a trivial normalized state $\Psi_0(k)$ we then want to appeal to the above Hopf construction. In this regard we first focus on the case $\Psi_0(k) = (1,0,0)^\dagger$. Applying $U$ on $\Psi_0(k)$ we obtain

$$\Psi(k,t) = \begin{pmatrix} \cos (t) - i \sin (t)(2n_1^2 - 1) \\ -i \sin (t) 2n_1 n_2 \\ -i \sin (t) 2n_1 n_3 \end{pmatrix}.$$  

A priori, it seems hard to relate to the aforementioned Hopf construction. However, recall that we are free to reparametrize the Hopf map upon applying rotations to the initial vector $t$ as in Eq. (B4). Additionally, $H$ is nothing but a rotation of $\Psi_0(k)$ around $n(k)$. We therefore relabel $(2n_1^2 - 1, 2n_1 n_2, 2n_1 n_3)$ as $\{a_1, a_2, a_3\}$, which evidently still amounts to a unit vector. Consequently, we thus find the following form of the time-evolved state

$$\Psi(k,t) = \begin{pmatrix} \cos (t) - i \sin (t)a_1 \\ -i \sin (t) a_2 \\ -i \sin (t) a_3 \end{pmatrix}.$$
These matrices then project $\Psi(t)$ back to a ‘Bloch vector’ $\hat{\rho} \in S^2$ in the desired manner, upon contracting

$$\hat{\rho} = (\Psi^\dagger(t)\mu_x \Psi(t), \Psi^\dagger(t)\mu_y \Psi(t), \Psi^\dagger(t)\mu_z \Psi(t))^\top. \quad (C5)$$

Indeed, identifying $\{x_0, x_1, x_2, x_3\} = \{\cos t, -\sin(t)a_1, -\sin(t)a_2, -\sin(t)a_3\}$, we observe that this parametrizes the first Hopf map by the first column of Eq. (B4).

A few remarks on the above are in place. First, we note that multiplying the above matrices with rotations (by an angle $\pi/2$) that interchange the bottom two components merely interchanges the components of the Hopf parameterization, thus preserving the map. Secondly, we observe a close analogy to the two-band case. A quaternion can alternatively be rewritten as two complex numbers. That is, we can define $a,b,c \mapsto \frac{a+bi}{\sqrt{2}}$, $d \mapsto \frac{c-idi}{\sqrt{2}}$, which the results are given in Fig. 5.

With the above Eq. C3 in hand, we subsequently define

$$H(k) = -2 n(k) \cdot n(k)^\top + \mathbb{I}_3. \quad (C9)$$

Repeating the procedure above, we notice that the effect of this change in Hamiltonian is to induce an extra minus sign in the $x_1, x_2$ and $x_3$ components. Hence, we observe that this merely changes the parametrization of the Hopf map by interchanging the respective row with a column in Eq. (B4).

### Appendix D: Tomography in 3-band Euler model

As explained in the main text, a state tomography can be employed to measure the linking in $(k_x, k_y, t)$-space. Here we present the linking structure acquired through the proposed tomography scheme, for another initial state $\Psi_0 = (0, 0, 1)^\top$ to simultaneously illustrate the effect of different initial states. Our tomography scheme involves first applying a $\pi$-pulse with respect to sublattice $B$ to access the two-vector (depending on the initial state, here $z_1$ is identified with the last component of $\Psi(k, t)$) as it involves the real $x_0$-term, $\zeta = (\Psi_3, \Psi_1 + i\Psi_2)^\top$. Secondly, we quench with the tomography Hamiltonian having flat bands with respect to sublattice $C$, since we start with an initial state completely localized in $C$. Namely, $\mu_\zeta = \nu \mu_\zeta \nu = \text{diag}(-1, -1, 1)$ and $H_{\text{tom}} = (\omega/2)\mu_\zeta$. After evolving with $H_{\text{tom}}$ for a time $t_{\text{tom}}$, we obtain the momentum distribution (which can be measured in TOF) as $m(k, t_{\text{tom}}) \propto (1 + \sin \phi_k \cos(\phi_k + \omega t_{\text{tom}}))$. The azimuthal angle $\phi_k$ and the amplitude sin $\phi_k$ is calculated by fitting $m(k, t_{\text{tom}})$ with a cosine at each $k$ and quench time $t$, for which the results are given in Fig. 5.

Note that, for the initial state $(0, 0, 1)$ the monopole-antimonopole pair now resides in patches aligned at the
FIG. 5. Linking of the inverse images of $\pm \hat{x}$ for initial state $\Psi_0 = (0, 0, 1)^T$, where the Bloch vector $\hat{p}$ is reconstructed from the momentum distribution via tomography.

center and outer circle of the BZ as can be seen in Fig. 5, as opposed to left/right division visible in Fig. 2 and Fig. 3 given for $(1, 0, 0)$. This can also be seen from the stereographic projection of the components of $H \Psi_0(k)$. We emphasize that despite the different alignments of the BZ patches, the clear separation can be easily seen in the azimuthal angle profile given in Fig. 5b, and the monopole–antimopole pair is topologically stable.

Appendix E: String and monopole charges on the real projective plain

We recall some aspects of the real projective plane. Turning first to the Euler Hamiltonian, we note that the models have a gauge symmetry, relating each eigenstate spanning the dreibein $E = \{|u_1(k)\}, |u_2(k)\}, |u_3(k)\rangle$ with its negative partner. This therefore relates to the real protective plane $RP^2$, as in a biaxial nematic [50–53, 62]. The string charges corresponding to first homotopy group $\pi_1$ are thus characterized by $\pi_1(RP^2) = Z_2$, whereas the monopole charges are given by the second homotopy group $\pi_2(RP^2) = Z$. The non-triviality of the string charges dictates caution in considering maps from the torus by using intuition from the sphere. Their presence ensures the existence of weak invariants. In each of the two directions the weak index can take either a trivial or non-trivial value, giving four possibilities. This effects the monopole charge possibilities. Namely, when there are nonzero weak invariants in either direction or both, an even multiple of monopole charges can adiabatically be split in two equal pieces and transferred around the non-trivial direction. The action of $\pi_1$ then ensures that the charge gets opposite values and thus can annihilate the other half, rendering a $Z_2$ classification. We therefore restrict attention to systems having trivial string charges. In fact the models presented are designed to meet this criterion. However even in this case, the direction is not defined and hence skyrmions and anti-skyrmions can not be discriminated in an absolute sense, showing that these charges are characterized by the absolute value of the winding number. This also relates to Alice dynamics, as in certain scenarios the monopole can be adiabatically deformed into an Alice string, which changes the sign of the charge charge upon passing through [54, 55]. Details of these features are beyond the scope of this paper and will be reported elsewhere. To construct Euler class, an orientation must be fixed, which physically amounts to specifying a handiness. Once this has been taken into account the sphere analogy becomes appropriate.

Appendix F: Specific matrix parametrization of the models

We here give the explicit forms of the tunneling matrix elements given in Eq. (A5).

The Euler Hamiltonian with $\xi = 2$ can be constructed by restricting $N = 2$-neighbor tunneling [18]. The eight $t_{ij}(\alpha, \beta)$ multiplying the Gell-Mann matrices are $5 \times 5$ matrices given as:

$$
t_1 = \begin{pmatrix}
0.0089 - 0.0151i & -0.0761 + 0.0309i & -0.0025 - 0.0076i & 0.0811 - 0.0158i & -0.0139 + 0.0000i \\
-0.0761 + 0.0309i & -0.1205 - 0.0467i & 0.0025 + 0.0233i & 0.1155 + 0.0000i & 0.0811 + 0.0158i \\
-0.0025 - 0.0076i & 0.0025 + 0.0233i & -0.0025 + 0.0000i & 0.0025 - 0.0233i & -0.1205 + 0.0467i & -0.0761 - 0.0309i \\
0.0811 - 0.0158i & 0.1155 + 0.0000i & 0.0025 - 0.0233i & -0.1205 + 0.0467i & -0.0761 - 0.0309i & 0.0811 + 0.0158i \\
-0.0139 + 0.0000i & 0.0811 + 0.0158i & -0.0025 + 0.0076i & -0.0761 - 0.0309i & -0.0025 + 0.0076i & 0.0811 + 0.0158i
\end{pmatrix}
$$

$$
t_3 = \begin{pmatrix}
-0.0025 - 0.0883i & -0.1727 - 0.0883i & -0.0025 - 0.0025i \\
0.0833 - 0.0025i & 0.0375 + 0.0025i & 0.0833 \\
0.1677 - 0.0425i & -0.0025 + 0.0425i & 0.1677 \\
0.0833 - 0.0025i & 0.0375 + 0.0025i & 0.0833 \\
-0.0025 - 0.0883i & -0.1727 - 0.0883i & -0.0025 - 0.0025i
\end{pmatrix}
$$

$$
t_4 = \begin{pmatrix}
0.0278i & 0.0636i & 0 & -0.0636i & -0.0278i \\
0.2275i & 0.1142i & 0 & -0.1142i & -0.2275i \\
0.4917i & -0.2810i & 0 & 0.2810i & -0.4917i \\
0.2275i & 0.1142i & 0 & -0.1142i & -0.2275i \\
0.0278i & 0.0636i & 0 & -0.0636i & -0.0278i
\end{pmatrix}
$$

$$
t_6 = \begin{pmatrix}
0.0278i & 0.0636i & 0 & -0.0636i & -0.0278i \\
0.2275i & 0.1142i & 0 & -0.1142i & -0.2275i \\
0.4917i & -0.2810i & 0 & 0.2810i & -0.4917i \\
0.2275i & 0.1142i & 0 & -0.1142i & -0.2275i \\
0.0278i & 0.0636i & 0 & -0.0636i & -0.0278i
\end{pmatrix}
$$

$$
t_8 = \begin{pmatrix}
0 & -0.1879 & -0.4330 & -0.1879 & 0 \\
-0.1879 & 0 & 0.3083 & 0 & -0.1879 \\
-0.4330 & 0.3083 & 0 & 0.3083 & -0.4330 \\
-0.1879 & 0 & 0.3083 & 0 & -0.1879 \\
0 & -0.1879 & -0.4330 & -0.1879 & 0
\end{pmatrix}
$$

, $t_8 = \begin{pmatrix}
0 & -0.1879 & -0.4330 & -0.1879 & 0 \\
-0.1879 & 0 & 0.3083 & 0 & -0.1879 \\
-0.4330 & 0.3083 & 0 & 0.3083 & -0.4330 \\
-0.1879 & 0 & 0.3083 & 0 & -0.1879 \\
0 & -0.1879 & -0.4330 & -0.1879 & 0
\end{pmatrix}$,
whereas \( t_2 = t_5 = t_7 = 0 \) as they correspond to the complex Gell-Mann matrices.

Similarly, the Euler Hamiltonian with \( \xi = 4 \) can be constructed by restricting \( N = 3 \)-neighbor tunneling. Accordingly, \( t_{ij}(\alpha, \beta) \) are 7 \( \times \) 7 matrices given by:

\[
t_1 = \begin{pmatrix}
0 & 0.0626 - 0.0037i & 0.1137 - 0.0240i & -0.0001i & -0.1137 + 0.0242i & -0.0626 + 0.0036i & 0.0001i \\
-0.0626 + 0.0037i & 0 & 0.0421 + 0.0278i & -0.0037i & -0.0421 - 0.0204i & -0.0073i & 0.0626 + 0.0036i \\
-0.1137 + 0.0240i & -0.0421 - 0.0278i & 0 & -0.0241i & 0.0482i & 0.0421 - 0.0204i & 0.1137 + 0.0242i \\
0.0001i & 0.0037i & 0.0241i & 0 & -0.0241i & -0.0073i & -0.0001i \\
0.1137 - 0.0242i & 0.0421 + 0.0204i & -0.0482i & 0.0241i & 0 & -0.0241 + 0.0278i & -0.1137 - 0.0240i \\
0.0626 - 0.0036i & 0.0073i & -0.0421 + 0.0204i & 0.0037i & 0.0421 - 0.0278i & 0 & -0.0626 - 0.0037i \\
0.0001i & -0.0626 - 0.0036i & -0.1137 - 0.0242i & 0 & 0.1137 + 0.0240i & 0.0626 + 0.0037i & 0
\end{pmatrix}
\]

\[
t_3 = \begin{pmatrix}
-0.0298 & -0.0125 & 0.0432 & 0.0432 & 0.0432 & -0.0125 & -0.0298 \\
-0.0125 & -0.1652 & -0.0550 & 0.0753 & -0.0550 & -0.1652 & -0.0125 \\
0.0432 & -0.0550 & 0.1086 & -0.0601 & 0.1086 & -0.0550 & 0.0432 \\
0.0432 & 0.0753 & -0.0601 & -0.0073 & -0.0601 & 0.0753 & 0.0432 \\
0.0432 & -0.0550 & 0.1086 & -0.0601 & 0.1086 & -0.0550 & 0.0432 \\
-0.0125 & -0.1652 & -0.0550 & 0.0753 & -0.0550 & -0.1652 & -0.0125 \\
-0.0298 & -0.0125 & 0.0432 & 0.0432 & 0.0432 & -0.0125 & -0.0298
\end{pmatrix}
\]

\[
t_4 = \begin{pmatrix}
0 & -0.0531 & -0.1480 & -0.1995 & -0.1480 & -0.0531 & 0 \\
0.0531 & 0 & -0.0655 & -0.0840 & -0.0655 & 0 & 0.0531 \\
0.1480 & 0.0655 & 0 & 0.3123 & 0 & 0.0655 & 0.1480 \\
0.1995 & 0.0840 & -0.3123 & 0 & -0.3123 & 0.0840 & 0.1995 \\
0.1480 & 0.0655 & 0 & 0.3123 & 0 & 0.0655 & 0.1480 \\
0.0531 & 0 & -0.0655 & -0.0840 & -0.0655 & 0 & 0.0531 \\
0 & -0.0531 & -0.1480 & -0.1995 & -0.1480 & -0.0531 & 0
\end{pmatrix}
\]

\[
t_5 = \begin{pmatrix}
-0.0508 & -0.0735 & 0.0490 & 0 & 0.0490 & 0.0735 & 0.0508 \\
-0.0735 & -0.1579 & -0.2956 & 0 & 0.2956 & 0.1579 & 0.0735 \\
0.0490 & -0.2956 & 0.2493 & 0 & 0.2493 & 0.2956 & -0.0490 \\
0.0490 & 2.9393 & 0.2493 & 0 & 0.2493 & 0.2956 & -0.0490 \\
0.0490 & 0.2493 & 0.0490 & 0 & 0.0490 & 0.2493 & -0.2956 \\
0.0735 & 0.1579 & 0.2956 & 0 & 0.2956 & 0.1579 & -0.0735 \\
0.0508 & 0.0735 & -0.0490 & 0 & 0.0490 & 0.0735 & -0.0508
\end{pmatrix}
\]

\[
t_6 = \begin{pmatrix}
0 & -0.0461 & 0 & 0.1148 & 0 & -0.0461 & 0 \\
-0.0461 & 0 & -0.1879 & -0.4330 & -0.1879 & 0 & -0.0461 \\
0 & -0.1879 & 0 & 0.3083 & 0 & -0.1879 & 0 \\
0.1148 & -0.4330 & 0.3083 & 0 & 0.3083 & -0.4330 & 0.1148 \\
0 & -0.1879 & 0 & 0.3083 & 0 & -0.1879 & 0 \\
-0.0461 & 0 & -0.1879 & -0.4330 & -0.1879 & 0 & -0.0461 \\
0 & -0.0461 & 0 & 0.1148 & 0 & -0.0461 & 0
\end{pmatrix}
\]

(F2)