Can Brans-Dicke scalar field account for dark energy and dark matter?

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Abstract. By using a linearized non-vacuum late time solution in Brans-Dicke cosmology we account for the seventy five percent dark energy contribution but not for approximately twenty-three percent dark matter contribution to the present day energy density of the universe.

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Our universe seems, according to the present-day evidence, to be spatially flat and to possess a non vanishing cosmological constant [1, 2]. For a flat matter dominated universe, cosmological measurements [3] imply that the fraction $\Omega_\Lambda$ of the contribution of the cosmological constant $\Lambda$ to present energy density of the universe is $\Omega_\Lambda \sim 0.75$. In standard cosmology $\Omega_\Lambda$ would be induced by a cosmological constant which is a dimensionful parameter with units of $(\text{length})^{-2}$. From the point of view of classical general relativity, there is no preferred choice for what the length scale defined by $\Lambda$ might be. Particle physics, however, gives a different point of view to the issue. The cosmological constant turns out to be a measure of the energy density of the vacuum and although we can not calculate the vacuum energy with any confidence, this allows us to consider the scales of various contributions to the cosmological constant. The energy scale of the constituent(s) of $\Lambda$ which in Planck units is approximated to $10^{-123}$ is problematic since it is lower than the normal energy scale predicted by most particle physics models. To solve this problem, a dynamical $\Lambda$ [4] in the form of scalar field with some self interacting potential [8] can be considered and its slowly varying energy density induces a cosmological constant. This idea called “quintessence” [4] is similar to the inflationary phase of the early universe with the difference that it evolves at a much lower energy density scale. The energy density of this field has to evolve in such a way that it becomes comparable with the mass density fraction $\Omega_M$ now. This type of specific evolution, better known as “cosmic coincidence” [11] problem, needs several constraints and fine tuning of parameters for the potential used to model quintessence with minimally coupled scalar field. To solve the cosmic coincidence problem, a new form of quintessence field called the “tracker field” [11] has been proposed. Such kind of quintessence field is mainly based on an equation of motion with a solution for such that for a wide range of initial conditions the equation of motion converge to the same solution. This type of solution is also called an ‘attractor like’ solution. There are a
number of quintessence models proposed. Most of these involve minimally coupled scalar field with different potentials dominating over the kinetic energy of the field. Purely exponential \([5]-[7]\) and inverse power law \([8]-[11]\) potentials have been extensively studied for quintessence fields to solve the cosmic coincidence problem. However the fact that the energy density is not enough to make up for the missing part of the cosmological constant or that the \(p/\rho \equiv \gamma\) value found for the equation of state of quintessence is not in good agreement with the observed results makes such an explanation unlikely. The investigation of alternative models in which the equation of state parameter \(\gamma\) of the cosmological constant evolves with time has been proposed due to the conceptual difficulties associated with a cosmological constant \([12]-[15]\).

There have been quite a few attempts for treating this problem with non-minimal coupled scalar fields. Studies made by Bartolo et al \([16]\), Bertolami et al \([17]\), Ritis et al \([18]\) have found tracking solutions in scalar tensor theories with different types of power law potential. In another work, Sen et al \([19]\) have found the potential relevant to power law expansion in Brans-Dicke (BD) cosmology and Arık et al \([20]\) have shown that (BD) theory of gravity with the standard mass term potential \(\frac{1}{2}m^2\phi^2\) is a natural model to explain the rapid primordial inflation and the observed slow late-time inflation.

In this paper we show that a linearized non-vacuum solution about the stable cosmological vacuum solution with flat space-like section is capable of explaining how the Hubble parameter evolves with the scale size of the universe \(a(t)\). In this framework, we also show that the standard Friedmann equation changes into a form in which the power of the scale size term with \(\Omega_M\) is corrected by an amount \(1/\omega\)

\[
\left(\frac{H}{H_0}\right)^2 = \Omega_\Lambda + \Omega_M \left(\frac{a_0}{a}\right)^{3+\frac{1}{2}}
\]

where \(\omega\) is the Brans-Dicke parameter with \(\omega \gg 1\) \([21,22]\). Subsequently, under such a linearized solution, we point out that only a very small part of the dark matter can be accommodated into the contribution of the Brans-Dicke scalar field.

In the context of (BD) theory \([23]\) with self interacting potential and matter field, the action in the canonical form is given by

\[
S = \int d^4x \sqrt{g} \left[ -\frac{1}{8\omega} \phi^2 R + \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} m^2 \phi^2 + L_M \right].
\]

In particular we may expect that \(\phi\) is spatially uniform, but varies slowly with time. The nonminimal coupling term \(\phi^2 R\) where \(R\) is the Ricci scalar, replaces with the Einstein-Hilbert term \(\frac{1}{G_N} R\) in such a way that \(G_{eff}^{-1} = \frac{2\pi}{\omega}\phi^2\) where \(G_{eff}\) is the effective gravitational constant as long as the dynamical scalar field \(\phi\) varies slowly. In units where \(c = \hbar = 1\), we define Planck-length, \(L_p\), in such a way that \(L_p^2 \phi_0^2 = \omega/2\pi\) where \(\phi_0\) is the present value of the scalar field \(\phi\). Thus, the dimension of the scalar field is chosen to be \(L_p^{-1}\) so that \(G_{eff}^{-1}\) has a dimension \(L_p^2\). The signs of the non-minimal coupling term and the kinetic energy term are properly adopted to \((+---)\) metric signature. The Lagrangian of the scalar field, in addition to non-minimal coupling term and the kinetic term, is composed of a potential which consists of only a standard mass
term. $L_M$, on the other hand, is the matter part of the Lagrangian which in accordance with the weak equivalence principle is decoupled from $\phi$ as has been assumed in the original (BD) theory. Excluding $\phi$, as the matter field, we consider a classical perfect fluid with the energy-momentum tensor $T_\mu^\nu = \text{diag} (\rho, -p, -p, -p)$ where $\rho$ is the energy density term. The gravitational field equations derived from the variation of the action (2) with respect to Robertson- Walker metric is

\[
\frac{3}{4\omega} \phi^2 \left( \frac{\ddot{a}}{a^2} + \frac{k}{a^2} \right) - \frac{1}{2} \ddot{\phi}^2 - \frac{1}{2} m^2 \dot{\phi}^2 + \frac{3}{2\omega} \frac{\dot{a}}{a} \phi = \rho_M \tag{3}
\]

\[
\frac{-1}{4\omega} \phi^2 \left( 2 \frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} \right) - \frac{1}{\omega} \frac{\dot{a}}{a} \phi - \frac{1}{2\omega} \ddot{\phi} - \frac{1}{2\omega} \phi \ddot{\phi} = \rho_M \tag{4}
\]

\[
\ddot{\phi} + 3 \frac{\ddot{a}}{a} \phi + \left[ m^2 - \frac{3}{2\omega} \left( \frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} \right) \right] \phi = 0 \tag{5}
\]

where $k$ is the curvature parameter with $k = -1, 0, 1$ corresponding to open, flat, closed universes respectively and $a(t)$ is the scale factor of the universe (dot denotes $\frac{d}{dt}$). Since in the standard theory of gravitation, the total energy density $\rho$ is assumed to be composed of $\rho = \rho_\Lambda + \rho_M$ where $\rho_\Lambda$ is the energy density of the universe due to the cosmological constant which in modern terminology is called as “dark energy”, the right hand sides of (3-4) are adopted to the matter energy density term $\rho_M$ instead of $\rho$ and $p_M$ instead of $p$ where $M$ denotes everything except the $\phi$ field. The main reason behind doing such an organization is that whether if the $\phi$ terms on the left-hand side of (3) can accommodate a contribution to due to what is called dark matter. In addition, the right hand side of the $\phi$ equation (5) is set to be zero according to the assumption imposed on the matter Lagrangian $L_M$ being independent of the scalar field $\phi$. By defining the fractional rate of change of $\phi$ as $F(a) = \dot{\phi}/\phi$ and the Hubble parameter as $H(a) = \dot{a}/a$, we rewrite the left hand-side of the field equations (3-5) in terms of $H(a), F(a)$ and their derivatives with respect to the scale size of an universe $a$ (prime denotes $\frac{d}{da}$)

\[
H^2 - \frac{2\omega}{3} F^2 + 2HF + \frac{1}{a^2} - \frac{2\omega}{3} m^2 = \left( \frac{4\omega}{3} \right) \frac{\rho_M}{\phi^2} \tag{6}
\]

\[
H^2 + \left( \frac{2\omega}{3} + \frac{4}{3} \right) F^2 + \frac{4}{3} HF + \frac{2a}{3} \left( H \dot{H} + H \dot{F} \right) + \frac{1}{3 a^2} - \frac{2\omega}{3} m^2 = \left( \frac{-4\omega}{3} \right) \frac{p_M}{\phi^2} \tag{7}
\]

\[
H^2 - \frac{\omega}{3} F^2 - \omega H F + a \left( \frac{H \dot{H}}{2} - \frac{\omega}{3} H \dot{F} \right) + \frac{1}{2a^2} - \frac{\omega}{3} m^2 = 0. \tag{8}
\]

Solving the field equations (3-5) for the closed universe stable-vacuum solution, we get $\phi = \phi_p e^{F_p t}$ and $a(t) = a_* \approx 1/\sqrt{\omega m}$ where $F_p \approx 0.7m$ is the primordial fractional rate of change of $\phi$, $a_*$ is the constant size of this static universe, $\phi_p$ is the constant value of the field $\phi$ as $t \to 0$ and $m$ is the mass of the scalar field $\phi$. The point to note here is that the closed universe ($k = 1$) vacuum solution becomes important for the primordial universe since homogeneity of the universe only makes sense if a
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closed universe undergoes big-bang. Under these closed universe vacuum solutions, we have shown [20] how the presence of radiation changes the behavior of the universe compared to these stable solutions. Putting the scalar field vacuum solution $\phi \sim e^{F_p t}$ in (5) and imposing initial conditions as the big-bang time and the corresponding time dependence of the scale size of the universe, we have found the solution for the scale size of the universe in the form of

$$a^2(t) = a^2_\ast \left[1 - (1 + c)e^{-2F_p t} + ce^{2H_p t}\right]$$

where $c$ is an integration constant and $H_p$ is the primordial Hubble parameter with $H_p = \omega F_p \approx 0.7\omega m$ for $\omega \gg 1$. This solution provides importance for the following reasons:

(1) It is a natural solution. Namely, it does not need any “special” equation of state for the matter. It is just deduced from the theory by putting the stable-empty universe solution $\phi \sim e^{F_p t}$ into the equation (5) [20].

(2) If one examines this inflationary solution concerning as $t \to 0$ and as $t \gg 0$, it is seen that (9) is both consistent with $a(t) \sim \sqrt{t}$ as $t \to 0$ and also with primordial rapid inflation described by $a(t) \sim e^{H_p t}$ for $\omega \gg 1$.

(3) We have also checked that for $\omega \gg 1$ and in the limit as $t \to 0$, if one substitutes $\phi \sim e^{F_p t}$ and $a \sim \sqrt{t}$ into (3-5) then the equation of state $p = 1/3\rho$ is satisfied automatically as expected in the radiation dominated epoch of the standard Einstein cosmology.

In the light of this encouraging result obtained by using the instability caused by the nonvacuum in the closed stable vacuum solution in explaining the rapid primordial inflation, we will show that a linearized non-vacuum solution about the flat stable vacuum solution can also be powerful in explaining the slow late time expansion. Since the universe becomes (approximately) flat in late times, we ignore the curvature parameter $k/a^2$ as $a(t)$ increases with the expansion of the universe. Under these considerations, in analogy with the assumption we use in explaining rapid primordial inflation, we first propose $a = e^{H_\infty t}$ and $\phi = e^{F_\infty t}$ and put into (3-5) and search for a zeroth order stable vacuum (empty except the $\phi$ field) solution. $H_\infty, F_\infty$ are the constants to be determined named as the late time Hubble parameter and the fractional rate of change of $\phi$ in the late time regime respectively. We have the following coupled equations for $H_\infty$ and $F_\infty$:

$$H^2_\infty - \frac{2}{3}\omega F^2_\infty + 2H_\infty F_\infty - \frac{2\omega}{3}m^2 = 0$$

(10)

$$H^2_\infty + \left(\frac{2}{3}\omega + \frac{4}{3}\right)F^2_\infty + \frac{4}{3}H_\infty F_\infty - \frac{2\omega}{3}m^2 = 0$$

(11)

$$H^2_\infty - \frac{\omega}{3}F^2_\infty - \omega H_\infty F_\infty - \frac{\omega}{3}m^2 = 0$$

(12)

and get the solution;

$$H_\infty = 2(\omega + 1)\left(\frac{\omega}{6\omega^2 + 17\omega + 12}\right)^{1/2}m \approx 0.8\sqrt{\omega m}$$

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\[ F_\infty = \left( \frac{\omega}{6\omega^2 + 17\omega + 12} \right)^{1/2} m \approx \frac{0.4}{\sqrt{\omega}} m \]  \quad (14)

where the approximations are again for \( \omega \gg 1 \). Thus, in our one hand, we have had an exact zeroth order stable-vacuum solution as,

\[ a = e^{H_\infty t} \quad (15) \]

\[ \phi = \phi_\infty e^{F_\infty t} \quad (16) \]

where \( H_\infty \approx 0.8\sqrt{\omega} m \), \( F_\infty \approx \frac{0.4}{\sqrt{\omega}} m \), \( \phi_\infty \) is a constant.

Then, after finding such a zeroth order exact stable solution, the question that stimulates us, similar to the primordial regime analysis, is that how the presence of matter affects this flat stable vacuum solution \( 15, 16 \). To understand such a perturbation phenomenon, we impose the following linearized first order non-vacuum solution for \( H \equiv \dot{a}/a \) and \( F \equiv \dot{\phi}/\phi \) which includes first order perturbation functions of \( h(a) \) and \( f(a) \) in addition to the constant terms \( H_\infty \) and \( F_\infty \) which appear in the flat stable vacuum solution \( 15, 16 \) respectively.

\[ H = H_\infty + h(a) \]

\[ F = F_\infty + f(a). \]  \quad (17, 18)

Since solving the field equations \( 3, 5 \) exactly for \( a(t) \) and \( \phi(t) \) under the condition \( p = 0 \) is hard enough, we put our imposed solution \( 17, 18 \) into the modified field equations \( 7, 8 \) for \( p = 0 \) and neglect higher terms in \( h(a), f(a) \) then we get \( h(a), f(a) \) for all \( \omega \) in the form of,

\[ h(a) = C_1 H_0 \left( \frac{a_0}{a} \right)^{\frac{3\omega+4}{2\omega+1}} - \left( \frac{1}{H_\infty a_0^2} \right) (\omega + 1)(\omega + 3) \left( \frac{a_0}{a} \right)^2 \]

\[ f(a) = C_2 H_0 \left( \frac{a_0}{a} \right)^{\frac{3\omega+4}{2\omega+1}} + \left( \frac{3}{2H_\infty a_0^2} \right) (\omega + 1) (\omega + 2)(\omega + 3) \left( \frac{a_0}{a} \right)^2 \]

where \( a_0 \) is present size of the universe and \( H_0 \) is the present Hubble parameter. \( C_1 \) and \( C_2 \) are, on the other hand, dimensionless integration constants. Since letting \( \omega \to \infty \) has a special meaning in the sense that the Brans-Dicke scalar tensor theory matches with standard Einstein theory under such limit, we display the linearized solution \( 17, 18 \) in the following form as \( \omega \to \infty \),

\[ H = H_\infty + C_1 H_0 \left( \frac{a_0}{a} \right)^{\frac{3\omega+4}{2\omega+1}} - \frac{1}{2} \left( \frac{1}{H_\infty a_0^2} \right) \left( \frac{a_0}{a} \right)^2 \]

\[ F = F_\infty + C_2 H_0 \left( \frac{a_0}{a} \right)^{\frac{3\omega+4}{2\omega+1}}. \]  \quad (21, 22)

Hence, putting the solution \( 21 \) in the standard Friedmann equation,

\[ \left( \frac{H}{H_0} \right)^2 = \Omega_\Lambda + \Omega_R \left( \frac{a_0}{a} \right)^2 + \Omega_M \left( \frac{a_0}{a} \right)^3 \]  \quad (23)
which is used for fitting Hubble parameter to the measured density parameters of
universe in such a way that $\Omega_\Lambda + \Omega_R + \Omega_M = 1$ and using the present observational
results on density parameters $\Omega_\Lambda \simeq 0.75$, $\Omega_M \simeq 0.25$, $\Omega_R \simeq 0$ [3], we get $C_1 \simeq 0.15$
and $H_\infty = 0.86 H_0$ so that $F_\infty \approx H_\infty / 2 \omega \approx (0.43 / \omega) H_0$ which provides $|C_2| \ll 0.43 / \omega$. 
Namely, the first term in a linearized solution (22) is much greater than the second term.
The curvature density parameter $\Omega_R$, on the other hand, is found to be in accordance
with the recent measurements since the term $(1 / H_0 a_0)^2 \approx \Omega_R \simeq 0$ [3].

To compare (6), with standard Friedmann-Lamaitre cosmology, we put the linearized solution (21, 22) into this equation and transfer all terms except for
$H^2 = (\dot{a} / a)^2$ to the right hand side. Neglecting the $1 / a^2$ term

$$H^2 = \frac{4\omega}{3\phi^2} (\rho_\Lambda + \rho_M + \rho_D).$$

Noting that, in the late time regime, $\phi \sim a^{1 / 2 \omega}$ is approximately constant as $a$ changes
we identify the terms which do not explicitly depend on $a$ with $\rho_\Lambda$ and terms which
depend on $a$ as $a^{-3}$ with the dark matter energy density $\rho_D$ so that

$$\rho_D = (C_2 F_\infty H_0 \phi^2 - \frac{3C_2}{2\omega} H_\infty H_0 \phi^2 - \frac{3C_1}{2\omega} H_0 F_\infty \phi^2) (\frac{a_0}{a})^3$$

$$\rho_\Lambda = \frac{1}{2} F_\infty^2 \phi^2 - \frac{3}{2\omega} H_\infty F_\infty \phi^2 + \frac{1}{2} m^2 \phi^2.$$  \(25\)

Using the recent observational results on density parameters of the universe ($\Omega_\Lambda$ \equiv $\rho_\Lambda / \rho_0 \simeq 0.75$, $\Omega_D \equiv \rho_D / \rho_0 \simeq 0.23$) where $\rho_0$ is the present measured energy density
of the universe and the relations $F_\infty \approx H_\infty / 2 \omega \approx (0.43 / \omega) H_0$ and $H_\infty \approx 0.8\sqrt{\omega m}$ as
$\omega \rightarrow \infty$, we fit (25, 26) to the ratio $\Omega_\Lambda / \Omega_D \simeq 75 / 23$ and determine the $|C_2|$ integration
constant to be $|C_2| \approx 0.20$ which is inconsistent with the requirement $|C_2| \ll 0.43 / \omega$
imposed by the theory.

In conclusion, the first remarkable feature of this work that a linearized non-vacuum
solution (21) about the stable cosmological vacuum solution (13) with flat ($k = 0$) space-
like section is capable of explaining how the Hubble parameter $H \equiv \dot{a} / a$ evolves with
the scale size of the universe $a(t)$.

The second remarkable feature of this theory is that by fitting the linearized solutions (21, 22)
of the theory to the recent observations [3], the late-time Hubble parameter $H_\infty = 0.86 H_0$ and the fractional rate of change of $\phi$ in the late time regime
$F_\infty = (0.43 / \omega) H_0$ are successfully predicted in terms of today’s observational measured
value of Hubble parameter $H_0$.

Another important prediction we note from this theory is that for a fixed $H_0$, since
$F_\infty \approx (0.43 / \omega) H_0$, $F_\infty$ may not attain a large value because of its inverse dependence on
$\omega$ which is measured to be, according to the recent observational data, as $\omega > 10^4 \gg 1$
[21, 22]. This is the reason why $F_\infty$ can not let the scalar field $\phi = \phi_\infty e^{F_\infty t}$ to blow up rapidly so that $\rho_\Lambda$ (26), the energy density of the universe due to the cosmological
constant, can grow slowly and reasonably. Hence we strictly agree on that this theory is
successful in explaining the dark energy though the Brans-Dicke scalar field $\phi$ can not
account for dark matter.
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The last remarkable feature of this theory is that it enables us to estimate some dimensionful parameters displayed in the theory. Using the relation 
\[ H_\infty \approx 0.86 H_0 \approx 0.8 \sqrt{\omega} m \]
and the restriction on \( \omega \), we may estimate \( m \) for a fixed \( H_0 \) as
\[ m \lesssim 10^{-2} H_0 \]  
(27)
where the present value of Hubble constant \( H_0 = 720 \pm 8 \) km s\(^{-1}\) Mpc\(^{-1}\)\cite{24}. Using appropriate conversion relation in relativistic units, present Hubble parameter is found to be \( H_0 \approx 10^{-26} \) m\(^{-1}\) \approx 2 \times 10^{-42} \) GeV.

By using the relation \( G_{eff}^{-1} = (2\pi/\omega)\phi^2 \), the time variation in Newtonian gravitation constant can be written as \( |\dot{G}/G| = 2F = 2 \left( \dot{\phi}/\phi \right) \) so that we can estimate the time variation in Newtonian constant \( G, F_P, F_\infty \) for the primordial and the late time epochs considered in this theory as
\[ \left| \frac{\dot{G}}{G} \right|_P = 2F_P \approx (1.5/\sqrt{\omega}) H_0 \]  
(28)
\[ \left| \frac{\dot{G}}{G} \right|_\infty = 2F_\infty \approx \frac{0.86}{\omega} H_0 \]  
(29)
\[ F_P \lesssim 7 \times 10^{-3} H_0 \]  
(30)
\[ F_\infty < 43 \times 10^{-6} H_0 . \]  
(31)
The Hubble parameters, on the other hand, are given by
\[ H_P \approx 0.7m\omega > 70 H_0 \]  
(32)
\[ H_\infty \approx 0.8\sqrt{\omega} m \approx 0.86 H_0 . \]  
(33)

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