On the Identifiability of the Influence Model for Stochastic Spatiotemporal Spread Processes

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Abstract

The influence model is a discrete-time stochastic model that succinctly captures the interactions of a network of Markov chains. The model produces a reduced-order representation of the stochastic network, and can be used to describe and tractably analyze probabilistic spatiotemporal spread dynamics, and hence has found broad usage in network applications such as social networks, traffic management, and failure cascades in power systems. This paper provides sufficient and necessary conditions for the identifiability of the influence model, and also develops estimators for the model structure through exploiting the model’s special properties. In addition, we analyze conditions for the identifiability of the partially observed influence model (POIM), for which not all of the sites can be measured. We develop an expectation-maximization (EM) algorithm-based estimator to find the identifiable parameters in a POIM.

Index Terms

Stochastic systems, Spatiotemporal spread processes, Networks, Estimation.

I. INTRODUCTION

The influence model was first introduced in [1] to succinctly capture the interaction effects of a network of Markov chains. It is a discrete-time stochastic model composed of a network of sites, the statuses of each evolving according to the local Markov chains and the influences received from neighboring sites. The model produces a reduced-order representation of the aforementioned stochastic network, and exposes the graph structure of network influences [2]. The model simultaneously captures the local- and network- properties of the network, and permits a tractable analysis of network dynamics with computational efficiency, including the evolution of key probabilities and statistics, and the algebraic graph theoretic analysis. Since its development, the influence model has found broad usage in network applications, including social networks [3]–[10], communication networks [11]–[15], power networks [16]–[20], spatiotemporal weather spread [21]–[23], virus propagation [24], and distributed tasks such as graph partitioning [20], [25].

Several studies further the analysis of the influence model. Paper [26] analyzes high-order statistics by placing the influence model in a more general moment-linear stochastic network framework, and also points out the wide

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applicability of the influence model by formulating a general hidden Markov model (HMM) as a two-site influence model. In [27], the reduced-order property of the influence model was further investigated, by interpreting the influence model as a separable Bayesian network. Paper [28] studies the vulnerability of sites in an influence model. Some variants of the influence model have also been developed for various application needs. In [29], a hierarchical mobility framework was developed that integrates the binary influence model and a hierarchical graph-based representations of the mobility area to capture a heterogeneous mobile ad hoc network. Paper [30] generalizes the influence model by relaxing some constraints of the status update behavior to capture more complicated modeling scenarios.

There have also been some studies that investigate the state and parameter estimations of the influence model from measurement data. In [26], estimation methods based on maximum likelyhood (ML) and linear minimum mean square error (LMMSE) are used to estimate the states of influence models, where only some of the sites can be measured. We call such an influence model the partially observed influence model (POIM). In [31], the influence model is placed in a broader framework of dynamic Bayesian networks (DBN), and the inference algorithms developed for DBN were studied. Paper [32] connects POIM with Hidden Markov Models (HMM), and adopts the forward-backward algorithm, Viterbi Algorithm, and Baum-Welch algorithm for the state and parameter estimations of POIM. Moreover, it exploits the graph structure of POIM to expedite the state estimation process. Paper [33] studies a latent structure influence model, and shows that its inference is equivalent to the inference of the corresponding HMM by using a marginalizing operator. In [34], the estimation of influence model parameters for a weather spread application is studied.

None of these existing studies investigate whether the influence model is identifiable or not, i.e., if the model parameters can be uniquely determined from measurement data. Although the influence model can be placed in the HMM and DBN frameworks (see [35] and [36] for examples), the identifiability of these models do not necessarily transfer to the influence model, due to the different mappings from model parameters to measurements. The identifiability problem is important, as it affects the design of the influence model structure and measurement locations in all network applications that adopt the influence model as the modeling framework. The problem is also very challenging, considering the nonlinear mappings between model parameters and measurement locations, and the coupled network influences inherent in the influence model framework. Per our knowledge, this paper is the first to systematically study the identifiability of influence models.

The main contributions of this paper include the following. First, our paper provides an analytical framework to study the mapping properties between the influence model and its corresponding master Markov chain, as a step toward identifiability analysis. Second, our paper provides sufficient and necessary conditions for the influence model to be identifiable. A novel parameter estimator that exploits the identifiability properties of the influence model is developed, which is computationally efficient for small-size networks. Third, we analyze conditions for the identifiability of the partially observed influence model (POIM), for which not all of the sites can be measured. We develop an improved Expectation Maximization (EM) algorithm to find a subset of identifiable parameters in a POIM, which is more efficient compared to that in [32]. Results of this paper provide further insights and guidelines for the design and estimation of influence models from measurement data, and also facilitate their use in various
network applications.

The rest of the paper is structured as follows. In Section II, we review fundamentals of the influence model and formulate the parameter identification problem. In Section III, we study the identifiability of the influence model and develop two estimation algorithms for the identifiable influence models. In Section IV, we study the identifiability of POIM, and present the EM algorithm to estimate identifiable parameters of a POIM. In Section V, we demonstrate the theoretical results using simulation studies. Section VI concludes this paper.

II. Preliminaries and Problem Formulation

The influence model is composed of a network of $N$ sites, each of which has a status that varies stochastically over time. The evolution of statuses work as follows. Each site randomly chooses a neighbor site (including itself) as its determining site, and updates its status based on the status of the determining site. In this section, we describe the influence model, its master Markov chain representation, the POIM, and formulate the parameter identification problem.

A. The Influence Model

The influence model is described at two levels: the network level and the local level (see Figure 1 for an example). At the network level, the interaction among sites is captured by a network influence matrix $D \in \mathbb{R}^{N \times N}$. $D$ is a right stochastic matrix with $d_{ij}$ denoting the probability that site $i$ is influenced by site $j$. At the local level, a Markov chain between each pair of sites $i$ and $j$, with transition matrix $A_{ij} \in \mathbb{R}^{M_i \times M_j}$, is used to describe how one site’s next status is influenced by its neighbor site’s current status, where $M_i$ and $M_j$ are the numbers of statuses in sites $i$ and $j$ respectively. Each entry $a_{mn}$ of $A_{ij}$ denotes the probability for site $i$ to be in status $m$, when site $j$ is in status $n$ at the current time step. In this paper, we focus on the homogeneous influence model, where each site in the network has the same number of statuses $M$, and each pair of sites has the same transition matrix $A \in \mathbb{R}^{M \times M}$. In the following sections, we refer the homogeneous influence model as the influence model when it does not cause confusion.

At time $k$, the status of site $n$ is represented by a row vector of length $M$, $S_n[k]$, where $n \in \{1, 2, ..., N\}$. $S_n[k]$ has a single entry of value ‘1’ at the position corresponding to the status of site $n$ at time $k$, and ‘0’ everywhere
else. Let also a scalar \( s_n[k] \) denote the status index of site \( n \) at time \( k \). For example, \( s_n[k] = 2 \) is equivalent to \( S_n[k] = [0, 1, \ldots, 0] \). The whole influence model’s state matrix \( S[k] \in R^{N \times M} \) at time \( k \) can be captured by cascading \( S_n[k] \) for all sites \( i \),

\[
S[k] = \left[ S^T_1[k], S^T_2[k], \cdots, S^T_N[k] \right]^T.
\]  

(1)

where the superscript \( T \) denotes the transpose operation.  

Likewise, let a length-\( M \) row vector \( p_n[k] \) represent the probability mass function (PMF) for the status of site \( n \) at time \( k \). The influence model’s state probability matrix can then be represented as:

\[
p[k] = \left[ p^T_1[k], p^T_2[k], \cdots, p^T_N[k] \right]^T.
\]  

(2)

According to the influence model’s evolution rules, site \( n \)’s status PMF vector at time \( k+1 \) can be represented by a quasi-linear combination of the statuses of its neighbors and itself at time \( k \),

\[
p_n[k+1] = \sum_{l=1}^{N} d_{nl} S_l[k] A.
\]  

(3)

Therefore, the network’s state probability matrix at time \( k+1 \) can be represented as a matrix multiplication of \( D \), \( A \), and \( S[k] \),

\[
p[k+1] = DS[k]A.
\]  

(4)

The network’s state at time \( k+1 \) is randomly realized according to \( p[k+1] \),

\[
S[k+1] = \text{Realize}(p[k+1]).
\]  

(5)

B. The Master Markov Representation

The influence model captured by (4) and (5) is a reduced-order representation of a stochastic network of influencing Markov chains. Because the influence model’s next state only depends on its current state but nothing else from the past, it also has a Markov representation. In particular, the dynamics of the influence model is equivalent to a master Markov chain of \( M^N \) states, each of which is a combination of \( M \) statuses from the \( N \) sites. The influence model representation is more scalable, as it only contains \( M^N \) number of states, compared to the Master Markov representation which has an exponentially growing number of states at the increase of the number of sites.

In the master Markov representation, let us use a scalar \( s[k] \) with value ranging from 1 to \( M^N \) to index the state at time \( k \), and \( G \in R^{M^N \times M^N} \) as the state-transition matrix. The mapping from \( s_n[k] \) to \( s[k] \) is \( s[k] = \sum_{n=1}^{N} (s_n[k] - 1) M^{N-n} + 1 \). The state \( s[k] \) of the master Markov representation and the state \( S[k] \) of the influence model

\[
\begin{bmatrix}
1 & 0 & 0 & \cdots & 0 \\
1 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0
\end{bmatrix}
\]

representation have a one-to-one mapping. For example, \( s[k] = 2 \) is corresponding to \( S[k] = \)

\[
\begin{bmatrix}
1 & 0 & 0 & \cdots & 0 \\
1 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0
\end{bmatrix}
\]

which indicates that all the sites are in status 1, except that site \( N \) is in status 2. As such, we denote the mapping
from $s[k]$ to $S[k]$ as follows. For $s[k] = i$, we have $s_n[k] = i_n$ in the influence model, where $i_n$ denotes the corresponding status of site $n$ for $s[k]$ to be $i$. Similarly, the corresponding $S_n[k] = e_{i_n}$, where $e_{i_n}$ is a row vector of zeros, except 1 at the $i_n$th position.

For each entry of $G$, $g_{ij}$ represents the transition probability from state $i$ of the master Markov representation to state $j$. As given $S[k]$, all sites’ statuses at time $k+1$ evolve independently, $g_{ij}$ can be calculated as the product of each site’s corresponding conditional probability,

$$
g_{ij} = P(s[k + 1] = j | s[k] = i)$$

$$= P(s_1[k + 1] = j_1, \ldots, s_N[k + 1] = j_N | s_1[k] = i_1, \ldots, s_N[k] = i_N)$$

$$= P(S_1[k + 1] = e_{j_1}, \ldots, S_N[k + 1] = e_{j_N} | S_1[k] = e_{i_1}, \ldots, S_N[k] = e_{i_N})$$

$$= \prod_{l=1}^{N} P(S_l[k + 1] = e_{j_l} | S_1[k] = e_{i_1}, \ldots, S_N[k] = e_{i_N})$$

$$= \prod_{l=1}^{N} e_{j_l}^T \cdot e_{i_l} \cdot |O|$$

We here denote the function $h : \theta \to G$ as the mapping from $\theta = [A, D]$ ($A \in R^{M \times M}$ and $D \in R^{N \times N}$) to $G \in R^{M \times M \times N}$. Its inverse function is $h^{-1}_G : G \to \theta$.

### C. POIM

POIM is an influence model where not all sites are measured. In other words, the statuses of the unmeasurable sites are unknown. In practical applications, this is often true, due to e.g., limited sensing devices available in the spatial domain.

For a POIM, let $O_P$ denote the set of indices of measured sites, and $|O_P|$ the size of the measured sites, i.e., $O_P = \{l_1, l_2, \ldots, l_{|O_P|}\}$. \overline{O}_P = \{r_1, r_2, \ldots, r_{\overline{|O_P|}}\}$ denotes the set of $\overline{|O_P|}$ indices of sites that are not measured, where $\overline{|O_P|} = N - |O_P|$. Similar to the notations used for the general influence model, the status of the measured site $l_n$ at time $k$ is represented by a row vector of length $M$, $S_{l_n}[k]$. $S_{l_n}[k]$ has a single entry of value ‘1’ at the position corresponding to the status of site $l_n$ at time $k$, and ‘0’ everywhere else. Let also a scalar $s_{l_n}[k]$ denote the status index of site $l_n$ at time $k$. $s_{l_n}[k]$ and $S_{l_n}[k]$ has a one-to-one mapping relationship. The POIM’s measured state matrix $S_P[k] \in R^{(|O_P|) \times M}$ at time $k$ can be captured by cascading $S_{l_n}[k]$ for all sites in $O_P$.

$$S_P[k] = \left[ S_{l_1}^T[k], S_{l_2}^T[k], \ldots, S_{l_{|O_P|}}^T[k] \right]^T.$$  

$S_P[k]$ of the measured state of POIM is a sub-matrix of $S[k]$ of its corresponding influence model.

We construct a reduced-size Markov chain $C \in R^{M^{(|O_P|) \times |O_P|}}$ for the POIM that contains only the statuses of sites in $O_P$. Similar to the master Markov chain of a general influence model, we use a scalar $s_P[k]$ with value ranging from 1 to $M^{(|O_P|)}$ to index the state of the reduced-size Markov chain at time $k$. The state $s_P[k]$ and the measured state matrix of POIM $S_P[k]$ has a one-to-one mapping. We use the function $h_P : \theta_P \to C$ to denote the mapping from $\theta_P = [A, D]$ ($A \in R^{M \times M}$ and $D \in R^{N \times N}$) to $C \in R^{M^{(|O_P|) \times |O_P|}}$. Its inverse function is $h^{-1}_P : C \to \theta_P$. 
We also denote the unmeasured state matrix of POIM as \( S_U[k] = [S_{r_1 T}[k], S_{r_2 T}[k], \ldots, S_{r_{|O_P}| T}[k]]^T \). Then we construct an unmeasured Markov chain \( U \in R^{M|O_P| \times M|O_P|} \) for the POIM that contains only the statuses of sites in \( O_P \). We use a scalar \( s_U[k] \) with value ranging from 1 to \( M|O_P| \) to index the state of the unmeasured Markov chain at time \( k \). The state \( s_U[k] \) and the unmeasured state matrix of POIM \( S_U[k] \) has a one-to-one mapping relationship.

D. Problem Formulation

In this paper, we study the estimation of parameters \( \theta = [A, D] \) for the influence model and POIM.

**Problem 1**: Given \( L \) independent measurement sequences of the influence model, \( Y = [Y^1, Y^2, \ldots, Y^L] \), for each \( i \in \{1, \cdots, L\} \), \( Y^i = [S^i[1], S^i[2], \ldots, S^i[T]] \) is a length-\( T \) sequence starting from a random initial state, find the parameter \( \theta \) that maximizes the likelihood function,

\[
\hat{\theta} = \arg \max_{\theta} P(Y|\theta). \tag{8}
\]

where \( \theta = [A, D] \) are the parameters which completely determine an influence model.

**Definition 1**: An influence model is identifiable if and only if the solution \( \hat{\theta} \) to (8) is unique, with a sufficiently large number of measurement data, i.e., \( L, T \to \infty \).

**Problem 2**: Given \( Y \) with \( L, T \to \infty \), determine the identifiability of the underlying influence model.

**Problem 3**: Given \( L \) independent measurement sequences of the POIM, \( Y_P = [Y^1_P, Y^2_P, \ldots, Y^L_P] \). For each \( i \in \{1, \cdots, L\} \), \( Y^i_P = [S^i_P[1], S^i_P[2], \ldots, S^i_P[T]] \), where \( S^i_P[k] = [S^T_{i_1}[k], S^T_{i_2}[k], \ldots, S^T_{i_{|O_P|}}[k]]^T \). Find the parameter \( \theta_P = [A, D] \) that maximizes the likelihood function,

\[
\hat{\theta}_P = \arg \max_{\theta_P} P(Y_P|\theta_P). \tag{9}
\]

**Definition 2**: A POIM is identifiable if and only if the solution \( \hat{\theta}_P \) to (9) is unique, with a sufficiently large number of measurement data, i.e., \( L, T \to \infty \).

**Problem 4**: Given \( Y_P \) with \( L, T \to \infty \), determine the identifiability of the POIM.

As the solution \( \hat{\theta}_P \) to (9) is always not unique as we will show later, we here instead define the identifiability of POIM for a subset of parameters \( \theta_{P_O} \).

**Definition 3**: The POIM is identifiable for the subset of parameters \( \theta_{P_O} \subset \theta_P \), if and only if the solution of \( \hat{\theta}_{P_O} \) to (9) is unique, with a sufficiently large number of measurement data, i.e., \( L, T \to \infty \).

**Problem 5**: Given \( Y_P \) with \( L, T \to \infty \), determine the identifiability of the POIM for the subset of parameters \( \theta_{P_O} \).

III. IDENTIFIABILITY AND ESTIMATION OF THE INFLUENCE MODEL

In this section, we study the identifiability of the influence model. We use the master Markov chain as an intermediate step for the identifiability analysis of influence model parameters \( \theta \) from the measurement data \( Y \).
A. Sufficient and Necessary Conditions for the Identifiability of A and D

In the first lemma, we show that the identifiability of the influence model is equivalent to uniqueness of solution for the inverse function \( h^{-1}_{\theta} : G \rightarrow \theta \).

**Lemma 1.** Given \( Y \) with \( L, T \rightarrow \infty \), the influence model is identifiable if and only if the solution of \( h^{-1}_{\theta}(G) \) is unique.

**Proof.** Based on the one-to-one mapping between \( S[k] \) and \( s[k] \), the states of the master Markov chain \( s[1] \cdots s[T] \) can be directly obtained from \( Y \). Given a sufficiently large number of sequences and a sufficiently long sequence length of measurements \( Y \), each entry of the transition matrix of the master Markov chain, \( g_{ij} \), can be uniquely determined according to the law of large numbers by counting the frequencies of state transitions \[37\]. As \( G \) is uniquely identified from \( Y \), the influence model is identifiable if and only if the solution of \( h^{-1}_{\theta}(G) \) is unique according to the Definition 1.

According to the lemma, the influence model identifiability problem is reduced to the uniqueness of mapping \( \theta = h^{-1}_{\theta}(G) \). We first analyze the uniqueness of \( A \) in Theorem 1 and then study the uniqueness of \( D \) based on \( A \) and \( G \) in Theorem 2.

**Theorem 1.** Given a master Markov chain \( G \) constructed from a network influence matrix \( D \) and a local Markov chain \( A \), \( h^{-1}_{A} : G \rightarrow A \) has a unique solution

\[
a_{ij} = \sqrt{g_{n_i,n_j}},
\]

**Proof.** Let \( n_i \) denote the state of the master Markov chain, where all the sites in the corresponding influence model are in the status \( i \). \( g_{n_i,n_j} \) is the corresponding transition probability from the state \( n_i \) to state \( n_j \). According to \[6\], we have

\[
g_{n_i,n_j} = P(s[k+1] = n_j | s[k] = n_i) = \prod_{l=1}^{N} \left( \sum_{r=1}^{N} d_{lr} a_{ij} \right) = a_{ij}^N.
\]

\( a_{ij} \) in \[10\] is then derived naturally.

**Theorem 2.** Given a master Markov chain \( G \) constructed from a local Markov chain \( A \) and a network influence matrix \( D \), the solution of \( h^{-1}_{D} : G \rightarrow D \) is unique if and only if \( \text{Null}(O) \subseteq \text{Null}(V) \), where \( O \in \mathbb{R}^{M \times N} \) with each element

\[
o_m^{MN(i-1)+j} = \prod_{r=1}^{N} a_{i_m,r} a_{j_r,r+1},
\]

with each element
where

\[ 1 \leq i, j \leq M^N, \]
\[ 1 \leq m_1, \ldots, m_N \leq N, \]

and \( V \in \mathbb{R}^{N^2 \times N^N} \) with each element

\[ v_{N(l-1)+n, \sum_{r=1}^{N}(m_r-1)}^{N^N-r+1} = \begin{cases} 1, & \text{if } m_l = n \\ 0, & \text{otherwise} \end{cases}, \]

where

\[ 1 \leq l, n \leq N. \]

**Proof.** The idea of this proof is to express the transition probability of the master Markov chain \( G \) in (6) in a form that further exhibits the relationship between the elements of \( D, A \) and \( G \), and hence allows the interpretation of the uniqueness of \( D \) in terms of the null space of some specially constructed matrices.

From (6), the transition probability of the master Markov chain \( g_{ij} \) can be expressed as a summation of element-wise products of \( D \) and \( A \).

\[
g_{ij} = \prod_{l=1}^{N} P(s_l | k + 1) = j_l | s_1[k] = i_1, \ldots, s_N[k] = i_N \]
\[ = (d_11 a_{i_1,j_1} + d_12 a_{i_2,j_1} + \cdots + d_{1N} a_{i_N,j_1}) \times (d_{21} a_{i_1,j_2} + d_{22} a_{i_2,j_2} + \cdots + d_{2N} a_{i_N,j_2}) \times \cdots \]
\[ \times (d_{N1} a_{i_1,j_N} + d_{N2} a_{i_2,j_N} + \cdots + d_{NN} a_{i_N,j_N}) \]
\[ = \sum_{m_1=1}^{N} \sum_{m_2=1}^{N} \cdots \sum_{m_N=1}^{N} \left( \prod_{r=1}^{N} d_{r,m_r} \right) \left( \prod_{r=1}^{N} a_{i,m_r,j_r} \right). \]

In each summation term, the first parenthesis is a multiplication of one entry selected from each row of \( D \). The summations traverse the combinations of all such \( D \) entry selections. The second parenthesis in each summation term is solely dependent on \( A \), and can be calculated from \( G \) according to Theorem 1.

To facilitate the analysis, we introduce a few notations. We vectorize matrix \( G \) to a column vector \( G_c \in \mathbb{R}^{M^{2N}}, \) where \( G_c = vec(G^T) = \begin{bmatrix} g_{1,1}, g_{1,2}, \ldots, g_{1,N}, g_{2,1}, \ldots, g_{N,N} \end{bmatrix}^T \). Introduce \( f = \begin{bmatrix} f_1, f_2, \ldots, f_{NN} \end{bmatrix}^T \in \mathbb{R}^{N^N} \) with \( f_1, f_2, \ldots, f_{NN} \) as \( d_{1,1} d_{1,1} \cdots d_{N,1}, d_{1,1} d_{2,1} \cdots d_{N,2}, \ldots, d_{1,N} d_{2,1} \cdots d_{N,N} \) respectively, i.e.,

\[
f_{\sum_{r=1}^{N}(m_r-1)N^N-r+1}^{N^N-r+1} = \prod_{r=1}^{N} d_{r,m_r}. \]

Let matrix \( O \) denote a coefficient matrix, where each entry of \( O \in \mathbb{R}^{M^{2N} \times N^N} \) is an element-wise product of \( A \) in (14), as shown in (12). With these new notations, (14) further leads to

\[ O f = G_c. \]

Since \( G \), and hence \( G_c \), is uniquely determined from the measurement sequences \( Y \), and \( O \) can be computed from \( G \) according to Theorem 1 we aim to solve (16) to obtain \( f \).

Based on the fact that row sum of \( D \) equals 1, we have

\[ 1_{NN} f = 1, \]
where \( \mathbf{1}_{N^N} \) denotes an all-one row vector of length \( N^N \). To untangle each element of \( D \) from \( f \), we have

\[
d_{r,m} = \sum_{m_1=1}^{N} \cdots \sum_{m_{r-1}=1}^{N} \sum_{m_{r+1}=1}^{N} \cdots \sum_{m_N=1}^{N} (d_{1,m_1} \cdots d_{r,m_r} \cdots d_{N,m_N})
\]

\[
= \sum_{m_1=1}^{N} \cdots \sum_{m_{r-1}=1}^{N} \sum_{m_{r+1}=1}^{N} \cdots \sum_{m_N=1}^{N} f_{\sum_{i=1}^{r} (m_i-1) + i} \cdot \cdot \cdot f_{\sum_{i=1}^{N-1} (m_i-1) + N-1}.
\]

According to (17) and (18), the mapping from \( f \) to \( D \) can be represented in a matrix form,

\[
D_c = V f,
\]

where \( V \in \mathbb{R}^{N^2 \times N^N} \) has entries shown in (13) and \( D_c \) is a vector form of \( D \), i.e.,

\[
D_c = \text{vec}(D^T) = \begin{bmatrix} d_{11}, d_{12}, \cdots, d_{1N}, d_{21}, \cdots, d_{NN} \end{bmatrix}^T.
\]

We now discuss two cases to analyze the uniqueness of \( D_c \) and hence \( D \).

Case 1: matrix \( O \) has full column rank. In this case, clearly, \( D \) can be uniquely determined, since \( f \) is unique according to (16). In this case, the dimension of \( \text{Null}(O) \) is zero, and \( \text{Null}(O) \subseteq \text{Null}(V) \) holds.

Case 2: matrix \( O \) doesn’t have full column rank. In this case, \( f \) can have infinite many solutions. Let \( f = f_s + f_z \) be the solution of (16), where \( f_s \) denotes a specific solution and \( f_z \) denotes the solution set of the corresponding homogeneous linear equation \( Of_z = 0 \). We have

\[
V f = V f_s + V f_z.
\]

If \( \text{Null}(O) \subseteq \text{Null}(V) \), \( V f_z = 0 \) holds. Therefore \( D_c = V f = V f_s \) is unique, and \( D \) can be uniquely determined as well. Otherwise, \( D \) can not be uniquely determined.

Combining the conditions for \( D \) to be unique under both Case 1 and Case 2, we conclude that the network influence matrix \( D \) is unique if and only if \( \text{Null}(O) \subseteq \text{Null}(V) \).

**Corollary 1.** If matrix \( O \) has full column rank, the influence model is identifiable.

**Proof.** As the local Markov chain \( A \) is identifiable according to Theorem 1, this corollary can be naturally derived according to Case 1 in Theorem 2.

**B. Properties of the Matrix \( O \)**

The uniqueness of the network influence matrix \( D \), and hence the identifiability of influence model is solely dependent on matrix \( O \) according to Theorem 2 as matrix \( V \) is a constant matrix for a specific \( N \). Due to the crucial role of matrix \( O \) in determining the identifiability of influence model, we here study properties of matrix \( O \), especially those that are closely related to the column rank of \( O \).

**Theorem 3.** The column sum for each column of matrix \( O \) is \( M^N \).

\[
\mathbf{1}_{M^2 N} O = M^N \mathbf{1}_{N^N}.
\]

**Proof.** Because the row sum of \( G \) is 1, \( G \mathbf{1}_{M^2 N}^T = \mathbf{1}_{M^N}^T \), we have

\[
\mathbf{1}_{M^2 N} G_c = M^N.
\]
where \( \mathbf{1}_{M^2N} \) denotes an all-one row vector of length \( M^{2N} \). Equation (16) further leads to

\[
\mathbf{1}_{M^2N} \mathbf{O} \mathbf{f} = M^N. \tag{23}
\]

As \( f \) is solely dependent on \( D \), and \( \mathbf{O} \) is solely dependent on \( A \), (23) holds for all \( D \) and hence all \( f \). Comparing (17) and (23), (21) holds.

The dimension of matrix \( \mathbf{O} \) grows exponentially with the increase of \( N \), which complicates the computation. Here we decompose \( \mathbf{O} \) into two matrices \( \mathbf{R} \) and \( \mathbf{Q} \), which reduces the computational complexity and also facilitates the analysis.

**Theorem 4.** Matrix \( \mathbf{O} \) can be computed based on two special matrices \( \mathbf{R} \in \mathbb{R}^{MN \times MN} \) and \( \mathbf{Q} \in \mathbb{R}^{M^2N \times N^N} \),

\[
\mathbf{O} = (\mathbf{I}_{MN} \otimes \mathbf{R})\mathbf{Q}, \tag{24}
\]

where \( \mathbf{I}_{MN} \) denotes an identity matrix of dimension \( MN \) and \( \otimes \) is the Kronecker product.

\[
\mathbf{R} = \begin{bmatrix}
a^{N}_{1,1} & a^{N-1}_{1,1} & a_{1,2} & \cdots & a^{N}_{M,1} \\
a^{N-1}_{1,1} & a^{N-1}_{1,1} & a_{1,2} & \cdots & a^{N-1}_{M,1} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
a^{N-1}_{1,M-1} & a^{N-1}_{1,M-1} & a_{1,2} & \cdots & a^{N-1}_{M,M-1} \\
a^{N}_{1,M} & a^{N-1}_{1,M} & a_{2,M-1} & \cdots & a^{N}_{M,M}
\end{bmatrix}, \tag{25}
\]

and

\[
\mathbf{Q} = \left[ \mathbf{Q}_1^T, \mathbf{Q}_2^T, \cdots, \mathbf{Q}_{M^N}^T \right]^T,
\]

\[
\mathbf{Q}_i = \left[ e_{q_1}^T, e_{q_2}^T, \cdots, e_{q_{N^N}}^T \right],
\]

\[
q_{i}^{\sum_{r=1}^{N} (m_r - 1)N^{N-r} + 1} = \sum_{r=1}^{N} (i_{m_r} - 1)M^{N-r} + 1,
\]

where

\[
1 \leq m_1, \cdots, m_N \leq N.
\]

**Proof.** Decompose matrix \( \mathbf{O} \) to \( MN \) blocks

\[
\mathbf{O} = \left[ \mathbf{O}_1^T, \mathbf{O}_2^T, \cdots, \mathbf{O}_{M^N}^T \right]^T. \tag{27}
\]

As the columns of \( \mathbf{R} \) enumerates all the possible columns that may appear in \( \mathbf{O}_i \), \( \mathbf{O}_i \) can be obtained by multiplying a permutation and selection matrix \( \mathbf{Q}_i \) on the right-hand side of \( \mathbf{R} \),

\[
\mathbf{O}_i = \mathbf{RQ}_i. \tag{28}
\]

Each column of \( \mathbf{Q}_i \) serves as an index indicating the column of \( \mathbf{R} \) that is selected. (26) is computed based on (12) and (25).

**Theorem 5.** The column sum for each column of matrix block \( \mathbf{O}_i \) is 1.
Proof. The proof is similar to the proof in Theorem 3 and is omitted here.

The next two theorems and corollary find necessary conditions for matrix $O$ to have full column rank.

**Theorem 6.** $N \leq M^2$ is a necessary condition for matrix $O$ to have full column rank.

**Proof.** As $O$ is of dimension $M^{2N} \times N^N$, it’s straightforward that $N \leq M^2$ is necessary for $O$ to have full column rank.

**Theorem 7.** $N \leq M$ is a necessary condition for matrix $O$ to have full column rank.

**Proof.** According to Theorem 4, $Q_i$ is of dimension $M^N \times N^N$. If $N > M$, every $Q_i$ is column rank deficient. The largest rank of $Q_i$ is $M^N$ and the smallest rank of $Q_i$ is 1. According to (26), we can always find a set of $(M + 1)!$ columns of $Q$ to be linearly dependent. Therefore, $\text{rank}(Q) < N^N$ when $N > M$. Since

$$\text{rank}(O) \leq \min(\text{rank}(I_{MN} \otimes R), \text{rank}(Q)) < N^N,$$

matrix $O$ doesn’t have full column rank when $N > M$. Therefore, $N \leq M$ is a necessary condition for matrix $O$ to have full column rank.

**Corollary 2.** The local Markov chain $A$ has $M_1$ nonidentical rows. $N \leq M_1$ is a necessary condition for matrix $O$ to have full column rank.

**Proof.** We can formulate another matrix $R_{M_1} \in R^{M^N \times M_1^N}$ in the same way as in (25) by using the nonidentical subset of rows in $A$. We have

$$O = (I_{MN} \otimes R_{M_1})Q_{M_1},$$

(30)

Similar to the proof of Theorem 4, the corresponding $Q_{M_1} \in R^{(MM_1)^N \times N^N}$ can be decomposed into $M^N$ blocks, each of dimension $R^{M_1^N \times N^N}$. According to Theorem 7 $N \leq M_1$ is a necessary condition for matrix $O$ to have full column rank.

**C. Indenfinability for Classes of Influence Models**

We here analyze the identifiability of some special classes of influence models, based on the if and only if conditions of identifiability described in Section III-A and the properties of matrix $O$ in Section III-B.

**Theorem 8.** If $N > M$ and matrix $R$ has full rank, the influence model is identifiable.

**Proof.** Represent each entry of matrix $Q$ as follows.

$$q_{M^N(i-1)+j, \sum_{r=1}^{N}(m_r - 1)N^{N-r}+1} = \begin{cases} 1, & \text{if } j = \sum_{r=1}^{N}(i_{m_r} - 1)N^{N-r} + 1 \\ 0, & \text{otherwise} \end{cases}$$

$$1 \leq i, j \leq M^N,$$

$$1 \leq m_1, \cdots, m_N \leq N.$$  

(31)
When \( N > M \), there are \( \binom{N}{M} M! (N - M)^M \) different \( Q_i \) which are of full row rank. The comparison between (31) and (13) shows that matrix \( V \) can be obtained from matrix \( Q \) through elementary row transformations. Therefore, \( \text{Null}(Q) \subseteq \text{Null}(V) \). As \( R \) has full rank, the Kronecker product \( (I_M \otimes R) \) has full rank. According to Theorem 4, matrix \( O \) shares the same null space as matrix \( Q \). Hence, \( \text{Null}(O) \subseteq \text{Null}(V) \). According to Theorem 2, the network influence matrix \( D \) is unique. Due to the fact that the uniqueness of the local Markov chain \( A \) is ensured by Theorem 1, the influence model is identifiable according to Lemma 1.

Binary influence model (or called the copy influence model) is a special class of influence models that finds broad applications in e.g., power network [18], mobile ad hoc network [29], and distributed networks [20]. It captures the dynamics of a stochastic network, in which each site has only two statuses, ‘0’ or ‘1’. Each site \( i \) picks a neighboring site \( j \) as its determining site with probability \( d_{ij} \), and copies the current status of site \( j \) as its next status. All sites in a binary influence model converge to the statuses of all-ones or all-zeros. Corollary 3 shows that all binary influence models are identifiable.

**Corollary 3.** Any binary influence model is identifiable.

**Proof.** For any binary influence model, \( M = 2 \) and its local Markov chain \( A \) is an identity matrix of dimension \( 2 \times 2 \). Therefore, the corresponding matrix \( R \) is a \( 4 \times 4 \) identity matrix. Clearly it has full rank.

Case 1: \( N > 2 \). According to Theorem 8, the binary influence model with \( N > 2 \) is identifiable.

Case 2: \( N = 2 \). In this case, matrix \( O \) has full column rank. The network influence matrix \( D \) is unique according to Theorem 2. Hence the binary influence model with \( N = 2 \) is identifiable according to Corollary 1.

**Theorem 9.** The influence model is not identifiable if all the rows of the local Markov chain \( A \) are identical.

**Proof.** When all the rows of \( A \) are identical, we have \( \text{rank}(A) = 1 \). According to Theorem 4, \( \text{rank}(R) = 1 \) and hence \( \text{rank}(O) = 1 \). \( \text{Null}(O) \supset \text{Null}(V) \) holds for all \( N \). According to Theorem 2, the network influence matrix \( D \) is not unique, and hence the influence model is not identifiable.

It is worthwhile to interpret this from another point of view. Let \( A_i \) denotes the \( i \)-th row of \( A \), we have \( A_1 = A_2 = \cdots = A_M \). According to (14),

\[
g_{ij} = \prod_{l=1}^{N} A_{l}^{T} e_{j_l}^{T}.
\]

Clearly, the information of the network influence matrix \( D \) is completely hidden in \( g_{ij} \). Therefore, \( D \) is not identifiable given measurements \( Y \).

In practical applications, some prior information of the network influence matrix \( D \) can be available. For instance, in spatiotemporal spread models, it is natural to assume that the influence becomes smaller with the increase of distances among the agents. Denote the spatial distance among agents \( i \) and \( j \) as \( r_{ij} \), we can capture this spatial relationship as

\[
d_{ij} = \frac{e^{-\mu r_{ij}}}{\sum_{j=1}^{N} e^{-\mu r_{ij}}}.
\]
where \( \overline{r} \) denotes the average distance of all the distances between any two agents, and \( \mu \geq 0 \) is a weighting factor. The next theorem studies the identifiability of such influence model formulation.

**Theorem 10.** The network influence matrix \( D \) is given in (33). If there exists at least one row of the local Markov chain \( A \) that is not identical with other rows, the influence model is identifiable.

**Proof.** When not all the rows of \( A \) are identical, we can always find a transition probability, where \( g_{ij} = \prod_{l=1}^{N} \left( \sum_{r=1}^{N} d_{lr} e_{ir} A \right) e_{ji}^{T} \) that contains information of \( D \). As there is only one unknown \( \mu \) in \( D \), \( \mu \) can be solved immediately. Therefore, with the prior information stated in (33), the matrix \( D \) can be uniquely determined. According to Lemma II this class of influence models is identifiable.

### D. Estimation Algorithms

In this subsection, we provide estimation algorithms for identifiable influence models. We introduce two estimators, the first of which is based on maximum likelihood estimator (MLE), and the second is based on the properties of influence model studied in this paper.

1) **Maximum likelihood estimation approach:** As (8) shows in Section II, our goal is to find the underlying parameter \( \hat{\theta} \) which maximizes the likelihood function given the measurements \( Y \).

Consider the mapping from the influence model to the master Markov chain \( G \). \( P(Y|\theta) \) can be expressed as:

\[
P(Y|\theta) = \prod_{t=1}^{L} P(Y^t|\theta) = \prod_{l=1}^{L} P(S^l[1], S^l[2], \ldots, S^l[T]|\theta) = \prod_{l=1}^{L} P(s^l[1], s^l[2], \ldots, s^l[T]|\theta) = \prod_{l=1}^{L} \prod_{k=1}^{T-1} P(s^l[k+1]|s^l[k], \theta).
\]

(34)

\( P(Y|\theta) \) is a function of unknown parameters \( \theta \). To facilitate computation, we take the log of (34).

\[
\log P(Y|\theta) = \sum_{l=1}^{L} \sum_{k=1}^{T-1} \log P(s^l[k+1]|s^l[k], \theta),
\]

(35)

\[
\hat{\theta} = \arg \max_{\theta} \log P(Y|\theta).
\]

(36)

By applying the Lagrange multiplier to (35), \( \hat{\theta} \) is obtained.

2) **Linear algebra based estimation approach:** As studied in Section III, the local Markov chain \( A \) and the network influence matrix \( D \) can be computed through the master Markov chain \( G \) using linear algebra. According to Lemma I \( G \) can be uniquely determined from the measurements \( Y \). Next, \( A \) and \( D \) can be obtained separately.
We develop the following linear algebra based estimator which is based on the specific properties of the influence model studied in this paper, thus avoiding the calculation of the likelihood functions.

**Algorithm 1: Linear Algebra Based Influence Model Estimator**

**Input:**
- A sequence of measurements $Y$.

**Output:**
- Matrices $A$ and $D$.

1. Count state transition frequencies according to measurements $Y$, then calculate matrix $G$.
2. Compute the local Markov chain $A$ from $G$ according to (10).
3. Compute matrix $O$ according to Theorem 4.
4. if matrix $O$ has full column rank then
   5. Apply the least squares regression to (16) to obtain $f = (O^T O)^{-1} O^T G_c$.
   6. else
   7. Find a specific solution of $f$ by applying the Gaussian elimination.
   8. end if
9. Compute network influence matrix $D$ from $f$ according to (19).

When $N$ is small, the linear algebra based influence model estimator is more computationally efficient than MLE, since it avoids using the Newton-Raphson iteration or other gradient descent methods. With the increase of $N$, the matrix operations become more complicated, and affect the performance of this linear algebra-based estimation approach.

IV. IDENTIFIABILITY AND ESTIMATION OF POIM

In this section, we first study the identifiability of POIM using algebraic geometry, and then develop an EM-based estimation algorithm that exploits the properties of POIM.

A. Identifiability of POIM

Clearly, a POIM’s parameters $\theta_P$ can not be identified, if the corresponding influence model where all sites can be measured is not identifiable. As such, we here only study the identifiability of the POIM when its corresponding influence model is identifiable. We first show that all POIMs are not identifiable in the sense that the parameters $D$ and $A$ cannot be uniquely determined. After that, for a special case of POIM, we show conditions such that a portion of the parameters can be determined uniquely.

In the following lemma, we show that the identifiability of POIM is equivalent to the uniqueness of the solution for the inverse function $h_p^{-1} : C \rightarrow \theta_P$.

**Lemma 2.** Given $Y_P$ with $L, T \rightarrow \infty$, the POIM is identifiable if and only if the solution of $h_p^{-1}(C)$ is unique.

**Proof.** Based on the one-to-one mapping between $S_P[k]$ and $s_P[k]$, the states of this reduced-size POIM Markov chain can be directly obtained from $Y_P$. Given a sufficiently large number of sequences and a sufficiently long
sequence length of $Y_p$, each entry of the transition matrix of the reduced-size POIM Markov chain $c_{i,j}$ can be uniquely determined according to the law of large numbers by counting the frequencies of state transitions. As $C$ is uniquely identified from $Y_p$, the POIM is identifiable if and only if the solution of $h_p^{-1}(C)$ is unique according to Definition 2.

According to Lemma 2, the POIM identifiability problem is reduced to the uniqueness of mapping $\theta_p = h_p^{-1}(C)$. To simplify the presentation of the rest analysis, we relabel the measured and unmeasured sites as follows, $\mathcal{O}_p = \{1, 2, \cdots, |\mathcal{O}_p|\}$ and $\overline{\mathcal{O}}_p = \{|\mathcal{O}_p| + 1, |\mathcal{O}_p| + 2, \cdots, N\}$.

**Theorem 11.** The reduced-size POIM Markov chain $C$ and the master Markov chain $G$ of its corresponding influence model has the following relationship.

$$C = \frac{1}{M^{[\mathcal{O}_p]}} F G F^T,$$

(37)

where $F = I_{M^{[\mathcal{O}_p]}} \otimes 1_{M^{[\overline{\mathcal{O}}_p]}} \in R^{M^{[\mathcal{O}_p]} \times M^{[\overline{\mathcal{O}}_p]}}$, and each entry of $C$

$$c_{i,j} = \frac{1}{M^{[\mathcal{O}_p]}} \sum_{m_1=1}^{M^{[\mathcal{O}_p]}} \sum_{m_2=1}^{M^{[\overline{\mathcal{O}}_p]}} g_{(i-1)M^{[\mathcal{O}_p]}+m_1,(j-1)M^{[\overline{\mathcal{O}}_p]}+m_2},$$

(38)

$$1 \leq i, j \leq M^{[\mathcal{O}_p]}.$$

**Proof.** We partition the master Markov chain $G$ into $M^{[\mathcal{O}_p]} \times M^{[\overline{\mathcal{O}}_p]}$ blocks, each of which of dimension $M^{[\mathcal{O}_p]} \times M^{[\overline{\mathcal{O}}_p]}$. Each entry of the reduced-size POIM Markov chain $C$ is a scaled sum of all entries in the corresponding block of $G$. Clearly, (38) holds. Therefore, (37) can be obtained by multiplying a specific matrix $F$ and its transpose on both sides of $G$. Consider the property of $G$ and $F$, we have

$$FGF^T1_{M^{[\mathcal{O}_p]}\otimes 1_{M^{[\overline{\mathcal{O}}_p]}}} = (I_{M^{[\mathcal{O}_p]}} \otimes 1_{M^{[\overline{\mathcal{O}}_p]}})G(I_{M^{[\mathcal{O}_p]}} \otimes 1_{M^{[\overline{\mathcal{O}}_p]}})1_{M^{[\mathcal{O}_p]}} = (M^{[\mathcal{O}_p]}1_{M^{[\overline{\mathcal{O}}_p]}})^T.$$

The row sum of each row of $FGF^T$ is $M^{[\mathcal{O}_p]}$. To obtain a right stochastic matrix $C$ for the reduced-size POIM Markov chain, $FGF^T$ is divided by $M^{[\mathcal{O}_p]}$.

**Theorem 12.** Given a reduced-size POIM Markov chain $C$ constructed from a network influence matrix $D$ and a local Markov chain $A$, $h_p^{-1}: C \to D$ doesn’t have a unique solution $\forall A \in R^{M \times M}, \forall D \in R^{N \times N}$, and $\forall C \in R^{M^{[\mathcal{O}_p]} \times M^{[\overline{\mathcal{O}}_p]}}$.

**Proof.** According to (3), (38) can be calculated as follows:

$$c_{i,j} = \frac{1}{M^{[\mathcal{O}_p]}} \sum_{m_1=1}^{M^{[\mathcal{O}_p]}} \sum_{m_2=1}^{M^{[\overline{\mathcal{O}}_p]}} g_{(i-1)M^{[\mathcal{O}_p]}+m_1,(j-1)M^{[\overline{\mathcal{O}}_p]}+m_2}$$

$$= \frac{1}{M^{[\mathcal{O}_p]}} \sum_{m_1=1}^{M^{[\mathcal{O}_p]}} \sum_{m_2=1}^{M^{[\overline{\mathcal{O}}_p]}} \prod_{l=1}^{N} \left( \sum_{r=1}^{d_{l\tau}} A_{c_{i,m_1},l} A_{c_{j,m_2},l} \right)^T$$

$$= \frac{1}{M^{[\mathcal{O}_p]}} \sum_{m_1=1}^{M^{[\mathcal{O}_p]}} \sum_{m_2=1}^{M^{[\overline{\mathcal{O}}_p]}} \prod_{l=1}^{N} \left( \sum_{r=1}^{d_{l\tau}} A_{c_{i,m_1},l} A_{c_{j,m_2},l} \right)^T.$$
where \( c_{(i,m)} \) denotes a row vector of zeros, except ’1’ at the \((i,m)\)th position which represents the status of site \( r \) corresponding to the master Markov state \((i-1)M(\Omega) + m_1\).

The rows related to the sites in \( \Omega_P \) of the network influence matrix \( D \) are totally hidden in \( c_{ij} \). Therefore, the network interaction information of the unmeasured sites cannot be obtained from measurements. The mapping \( h^{-1}_{D'_0} : C \rightarrow D \) doesn’t have a unique solution for all matrices \( D, A, \) and \( C \).

Theorem 13 shows that all POIMs are non-identifiable, as the the mapping \( h^{-1}_{D'_0} : C \rightarrow D \) is not unique. The statuses of sites in \( \Omega_P \) are all hidden. Hence, we instead focus on the identifiability for a subset of parameters \( \theta_{P_0} = [A,D_O] \subset \theta_P \) which determine the dynamics of the sites that are measured, where \( D = \begin{bmatrix} D_O \\ D_C \end{bmatrix} \) with \( D_C \in \mathbb{R}^{|\Omega_P| \times N} \). In the following theorem, we study the identifiability of \( \theta_{P_0} \) for a POIM of two sites.

**Theorem 13.** Consider a POIM of 2 sites, each of which has 2 statuses. Site 1 is measured and site 2 is not measured. \( \theta_{P_0} \) is identifiable if and only if both of the following two conditions hold: 1) the two rows of \( A \) are not the same, and 2) \( D_C \), i.e., the first row of \( D \), does not equal to \( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \).

**Proof.** Let

\[
A_1 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, D_1 = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix}, A_2 = \begin{bmatrix} a_{11} + \delta_1 & a_{12} - \delta_1 \\ a_{21} + \delta_2 & a_{22} - \delta_2 \end{bmatrix}, D_2 = \begin{bmatrix} d_{11} + \epsilon_1 & d_{12} - \epsilon_1 \\ d_{21} + \epsilon_2 & d_{22} - \epsilon_2 \end{bmatrix},
\]

where \(-a_{11} \leq \delta_1 \leq a_{12}, -a_{21} \leq \delta_2 \leq a_{22}, -d_{11} \leq \epsilon_1 \leq d_{12}, \) and \(-d_{21} \leq \epsilon_2 \leq d_{22} \). According to (38), the reduced-size Markov chains of these two POIMs are

\[
C_1 = \begin{bmatrix} a_{11} + d_{11}a_{11} + d_{12}a_{21} & a_{12} + d_{11}a_{12} + d_{12}a_{22} \\ a_{21} + d_{12}a_{11} + d_{11}a_{21} & a_{22} + d_{12}a_{12} + d_{11}a_{22} \end{bmatrix},
\]

\[
C_2 = \begin{bmatrix} a_{11} + d_{11}a_{11} + d_{12}a_{21} + \xi_{11} & a_{12} + d_{11}a_{12} + d_{12}a_{22} + \xi_{12} \\ a_{21} + d_{12}a_{11} + d_{11}a_{21} + \xi_{21} & a_{22} + d_{12}a_{12} + d_{11}a_{22} + \xi_{22} \end{bmatrix},
\]

where \( \xi_{11} = \delta_1 + \delta_1d_{11} + \epsilon_1d_{11} + \epsilon_1\delta_1 + \delta_2d_{12} - \epsilon_1a_{21} - \epsilon_1\delta_2, \xi_{12} = -\xi_{11}, \xi_{21} = \delta_2 + \delta_2d_{11} - \epsilon_1a_{11} - \epsilon_1\delta_1 + \delta_1d_{12} + \epsilon_1a_{21} + \epsilon_1\delta_2, \) and \( \xi_{22} = -\xi_{21} \). Simple algebra leads to the conclusion that \( C_1 = C_2 \) if and only if \( \delta_2 = -\delta_1, a_{11} = a_{21}, \) and \( \epsilon_1 = -d_{11} \).

Theorem 13 naturally follows.

The first condition in term of matrix \( A \) in Theorem 13 matches with the conclusion obtained in Theorem 9 as a POIM is not identifiable if the corresponding fully measured influence model is not identifiable. In this case, the network influence matrix \( D \) has infinite many solutions. The second condition indicates that the local Markov chain \( A \) of the POIM can not be uniquely determined if site 1 is solely influenced by the unmeasured site 2 with probability 1. Since the status of site 2 is unknown, \( A \) has infinite many solutions.

**B. EM Algorithm**

In this subsection, we provide an estimation algorithm based on EM for the identifiable \( \theta_{P_0} \) of POIM.

According to Problem 3 and Definition 3, the goal of the estimator is to find the parameter \( \theta_{P_0} \) that maximizes the likelihood function in (9). For the ease of presentation, we assume \( L = 1 \) and drop the superscript of \( Y^i \). We
reformulate the log likelihood function as follows using the law of total probability, by separating the measured and unmeasured sites.

\[
\log P(Y_P | \theta_P) = \log \sum_Z P(Y_P, Z | \theta_P),
\]

where \( Z = [S_U[1], S_U[2], \ldots, S_U[T]] \) denotes the sequence of the unmeasured state matrix from time 1 to \( T \).

The calculation is intractable due to the exponentially growing statuses with the sequence length \( T \). We therefore apply Jensen’s inequality to (39) to transform the log of summation to the summation of log functions. The EM algorithm is summarized as follows.

**E-step:** Calculate the expected value of the log likelihood function, with respect to the conditional distribution of \( Z \) given \( Y_P \) under the current estimate of the parameter \( \theta_P \).

\[
\phi(\theta_P | \theta_P^l) = E_{(Z|Y_P, \theta_P^l)}[\log P(Y_P, Z | \theta_P)],
\]

where \( \theta_P^l \) is the parameter estimated from previous iteration \( l \).

\[
P(Z | Y_P, \theta_P^l) = \frac{P(Y_P, Z | \theta_P^l)}{P(Y_P | \theta_P^l)}.
\]

Therefore, \( \phi(\theta_P | \theta_P^l) \) can be described as,

\[
\phi(\theta_P | \theta_P^l) = \sum_Z \frac{P(Y_P, Z | \theta_P^l)}{P(Y_P | \theta_P^l)} \log P(Y_P, Z | \theta_P).
\]

\( P(Y_P | \theta_P) \) is independent of the unmeasured sequence \( Z \).

According to the Markov property of the master Markov chain, \( P(Y_P, Z | \theta_P) \) can be computed as:

\[
P(Y_P, Z | \theta_P) = \pi(s_U[1]) \prod_{k=1}^{T-1} P(s[k+1] | s[k]),
\]

where \( \pi(s_U[1]) \) denotes the initial state distribution of the unmeasured Markov chain \( U \) of POIM, and \( s[k] \) denotes the corresponding network state of influence model at time \( k \), which includes the statuses of the measured sites in \( O_P \) and the unmeasured sites in \( \overline{O_P} \). (42) can thus be expressed as,

\[
\phi(\theta_P | \theta_P^l) = \sum_{m_U=1}^{M(\overline{O_P})} \frac{P(s[1] = m_P, Y_P | \theta_P^l)}{P(Y_P | \theta_P^l)} \log \pi(s_U[1] = m_U) + \sum_{k=1}^{T-1} \sum_{i=1}^{M} \sum_{j=1}^{M} \frac{P(s[k] = i, s[k+1] = j, Y_P | \theta_P^l)}{P(Y_P | \theta_P^l)} \log g_{ij}
\]

\[
= \sum_{m_U=1}^{M(\overline{O_P})} P(s[1] = m_P, Y_P | \theta_P^l) \log \pi(s_U[1] = m_U) + \sum_{k=1}^{T-1} \sum_{i=1}^{M} \sum_{j=1}^{M} P(s[k] = i, s[k+1] = j | Y_P, \theta_P^l) \log g_{ij},
\]

where \( s_U[1] = m_U \) with \( m_U \) ranging from 1 to \( M(\overline{O_P}) \) denotes the state of the unmeasured Markov chain \( U \) at time 1, and \( s[1] = m_P \) denotes the corresponding network state at time 1, which is consistent with the measured state \( s_P[1] \) and the unmeasured state \( s_U[1] \).
**M-step:** Find the parameter that maximizes $\phi(\theta_P|\theta_P^l)$ with constraints that the row sum of each row of the master Markov chain $G$ is one.

$$\theta_P^{l+1} = \arg\max_{\theta_P} \phi(\theta_P|\theta_P^l),$$

s.t. \[\sum_{j=1}^{M^N} g_{ij} = 1, \quad \sum_{m_U=1}^{M|OP|} \pi(s_U[1] = m_U) = 1, \quad \forall 1 \leq i \leq M^N.\] (45)

By applying the Lagrange multiplier to (45), the estimated $g_{ij}$ at iteration $l + 1$ can be expressed as,

$$\hat{\pi}(s_U[1] = m_U) = \gamma_i[1],$$

$$\hat{g}_{ij}^{l+1} = \frac{\sum_{k=1}^{T-1} \xi_{ij}[k]}{\sum_{k=1}^{T-1} \gamma_i[k]},$$ (46)

where $\xi_{ij}[k] = P(s[k] = i, s[k+1] = j|Y_P, \theta_P)$, $\gamma_i[k] = P(s[k] = i|Y_P, \theta_P)$.

To obtain $\xi_{ij}[k]$ and $\gamma_i[k]$, a forward-backward algorithm is introduced as follows. Let

$$\alpha_{iu}[k] = P(s_U[k] = i_U, S_P[1] : S_P[k]|\theta_P),$$

$$\beta_{iu}[k] = P(S_P[k+1] : S_P[T]|s_U[k] = i_U, S_P[k], \theta_P),$$ (47)

where $S_P[1] : S_P[k]$ denotes the measured state matrix of POIM from time 1 to time $k$, and $i_U$ denotes the state of the unmeasured Markov chain which is consistent with the network state $i$. $\alpha_{iu}[k]$ and $\beta_{iu}[k]$ are the forward and backward variables, respectively.

The recursive calculation of $\alpha_{ju}[k]$ is expressed as:

$$\alpha_{ju}[1] = \pi(s_U[1] = j_U),$$

$$\alpha_{ju}[k+1] = \sum_{i_U=1}^{M^{[OP]}} \alpha_{iu}[k]g_{ij},$$ (48)

$$1 \leq k \leq T - 1,$$

$$1 \leq j_U \leq M^{[OP]},$$

where $g_{ij} = P(s_U[k+1] = j_U, S_P[k+1]|s_U[k] = i_U, S_P[k], \theta_P)$.

The recursive calculation of $\beta_{ju}[k]$ is expressed as:

$$\beta_{ju}[T] = 1,$$

$$\beta_{ju}[k] = \sum_{i_U=1}^{M^{[OP]}} \beta_{iu}[k+1]g_{ji},$$ (49)

$$1 \leq k \leq (T - 1),$$

$$1 \leq j_U \leq M^{[OP]},$$

Similarly, $g_{ji} = P(s_U[k+1] = i_U, S_P[k+1]|s_U[k] = j_U, S_P[k], \theta_P).$
Since
\[ \alpha_{i_U}[k] \beta_{i_U}[k] = P(s_U[k] = i_U, Y_P|\theta_P), \]  
which can be expressed as below.
\[ P(Y_P|\theta_P) = \sum_{i_U=1}^{M^{(CP)}} \alpha_{i_U}[k] \beta_{i_U}[k], \]  
(51)

\[ \xi_{ij}[k] \] and \( \gamma_{i}[k] \) can be expressed as below.
\[ \xi_{ij}[k] = \frac{P(s[k] = i, s[k + 1] = j|Y_P, \theta_P)}{P(Y_P|\theta_P)} = \frac{\alpha_{i_U}[k] g_{ij}[k]}{\sum_{i_U=1}^{M^{(CP)}} \alpha_{i_U}[k] \beta_{i_U}[k]}, \]  
(52)

\[ \gamma_{i}[k] = \frac{P(s[k] = i|Y_P, \theta_P)}{P(Y_P|\theta_P)} = \frac{\alpha_{i_U}[k] \beta_{i_U}[k]}{\sum_{i_U=1}^{M^{(CP)}} \alpha_{i_U}[k] \beta_{i_U}[k]}, \]  
(53)

The entry \( \hat{g}_{i,j} \) of the master Markov chain \( G \) is thus estimated. We then recover the network influence matrix \( D \) and the local Markov chain \( A \) from \( \hat{g}_{i,j} \). The following algorithm summarizes the full estimation process.

\textbf{Algorithm 2: EM algorithm for POIM}

\textbf{Input:}
A sequence of measurements \( Y_P \) of POIM.

\textbf{Output:}
Matrices \( A \) and \( D \).

1: \textbf{for} \( |	heta_P^l - \theta_P^{l-1}| \geq \delta_P \), where \( \delta_P \) is a small threshold, \textbf{do}
2: \hspace{1em} Apply E-step in (40) and M-step in (45) to \( Y_P \) to obtain \( \hat{g}_{i,j} \) of the estimated master Markov chain \( G \).
3: \hspace{1em} Compute the estimated local Markov chain \( A \) from the estimated \( G \) according to (10).
4: \hspace{1em} Compute matrix \( O \) according to Theorem 4.
5: \hspace{1em} if matrix \( O \) has full column rank \textbf{then}
6: \hspace{2em} Apply the least squares regression to (16) to obtain \( f = (O^T O)^{-1} O^T G_v \).
7: \hspace{1em} else
8: \hspace{2em} Find a specific solution of \( f \) by applying the Gaussian elimination.
9: \hspace{1em} end if
10: \hspace{1em} Compute the estimated network influence matrix \( D \) from \( f \) according to (19).
11: \textbf{end for}

V. SIMULATION STUDIES

In this section, we conduct several simulation studies to demonstrate the theorems and algorithms.
A. Case 1: Matrix $O$ Has Full Column Rank

We consider an influence model of 3 sites, each of which has 3 statuses. The network influence matrix $D \in R^{3 \times 3}$ and the local Markov chain $A \in R^{3 \times 3}$. Generate a measurement sequence of length 40000 and apply the MLE and linear algebra based approach, respectively. Matrices $A$ and $D$ are as follows.

$$
A = \begin{bmatrix}
0.7 & 0.2 & 0.1 \\
0.1 & 0.85 & 0.05 \\
0.2 & 0.15 & 0.65 \\
\end{bmatrix}, \quad D = \begin{bmatrix}
0.1 & 0.1 & 0.8 \\
0.5 & 0.2 & 0.3 \\
0.05 & 0.6 & 0.35 \\
\end{bmatrix}.
$$

According to Theorem 1, $A$ can be uniquely determined. Matrix $O$ can be computed according to (24) and is proven to have full column rank. Therefore, $D$ can be uniquely determined according to Corollary 1 and hence the influence model is identifiable.

Apply MLE to the measurements to estimate $A$ and $D$, we have

$$
\hat{A}_{\text{MLE}} = \begin{bmatrix}
0.7017 & 0.2014 & 0.0969 \\
0.1023 & 0.8475 & 0.0502 \\
0.1938 & 0.1509 & 0.6553 \\
\end{bmatrix}, \quad \hat{D}_{\text{MLE}} = \begin{bmatrix}
0.1034 & 0.0916 & 0.8050 \\
0.4944 & 0.2011 & 0.3045 \\
0.0501 & 0.5969 & 0.3531 \\
\end{bmatrix},
$$

where the subscript MLE denotes the MLE method. The mean squared error of MLE is $1.5 \times 10^{-5}$ and the execution time is 385.31 seconds.

Next, we apply the linear algebra approach. We first identify matrix $A$ and then identify matrix $D$ according to Algorithm 1 and find

$$
\hat{A}_{\text{LAE}} = \begin{bmatrix}
0.7029 & 0.1968 & 0.1003 \\
0.0991 & 0.8525 & 0.0484 \\
0.2042 & 0.1492 & 0.6466 \\
\end{bmatrix}, \quad \hat{D}_{\text{LAE}} = \begin{bmatrix}
0.1144 & 0.0967 & 0.7889 \\
0.4941 & 0.2090 & 0.2969 \\
0.0512 & 0.6058 & 0.3430 \\
\end{bmatrix},
$$

where the subscript LAE denotes the linear algebra based influence model estimator. The mean squared error of the linear algebra based approach is $3.4 \times 10^{-5}$ and execution time is 0.98 seconds.

The results show that with a sufficiently long sequence of measurements, both the MLE and the linear algebra based estimation approaches achieve accurate estimate results. However, the MLE takes much longer execution time compared to the linear algebra based influence model estimator.

B. Case 2: A Large-Size Binary Influence Model to Capture Weather Spread

We use a binary influence model to describe the spatiotemporal spread of a cold front from the west to the east in a space of $20 \times 20$ grids (see Figure 2).

To capture such spread, we construct a binary influence model of $N = 400$ sites to represent the 400 grids, and each site has two statuses $M = 2$. The local Markov chain is $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. The network influence matrix $D$ is designed in a way that each site (grid) is influenced by its immediate neighboring sites (grid). Since the cold front comes from the west, the neighboring site on the left exerts an larger influence. Figure 3 shows the influence map, where the left site exerts an influence of 0.6 to site $i$, and the upper, below, right sites and site $i$ itself each exerts
an influence of 0.1. One spatiotemporal cold front spread scenario generated from the above influence model is shown in Figure 2. According to Corollary 3, the binary influence model is identifiable. Here we apply MLE to estimate the scenario considering the large number of sites. The estimated matrices $A$ and $D$ converge to the real $A$ and $D$ with a mean squared error 0.0056.

C. Case 3: A Non-identifiable Influence Model

In this example, all the rows of the local Markov chain $A$ are identical. Here we consider an influence model with 3 sites, and each of which has 3 statuses. The network influence matrix $D \in \mathbb{R}^{3 \times 3}$ and the local Markov
chain $A \in \mathbb{R}^{3 \times 3}$.

$$A = \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.7 & 0.2 & 0.1 \\ 0.7 & 0.2 & 0.1 \end{bmatrix}, \quad D = \begin{bmatrix} 0.1 & 0.1 & 0.8 \\ 0.5 & 0.2 & 0.3 \\ 0.05 & 0.6 & 0.35 \end{bmatrix}. $$

Generate 100 scenarios with each one of length 40000.

According to Theorem 9 the network influence matrix $D$ cannot be identified. We apply MLE to this problem 100 times. $A$ converges to the real $A$. However, matrix $D$ is different in each of these 100 scenarios, verifying the non-identifiability of $D$.

D. Case 4: A POIM

We consider a POIM with 2 sites, each of which has 2 statuses. Three POIM models with different parameters are shown below. According to Theorem 13 the first one is identifiable for the subset of parameters $\theta_{PO}$, while the second and third ones are not identifiable, as they violate the two conditions stated in Theorem 13.

$$A_1 = \begin{bmatrix} 0.8 & 0.2 \\ 0.1 & 0.9 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0.4 & 0.6 \\ 0.7 & 0.3 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0.8 & 0.2 \\ 0.8 & 0.2 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 0.4 & 0.6 \\ 0.7 & 0.3 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 0.8 & 0.2 \\ 0.1 & 0.9 \end{bmatrix}, \quad D_3 = \begin{bmatrix} 0 & 1 \\ 0.7 & 0.3 \end{bmatrix}. $$

Generate 3 length-1000 scenarios of POIM for each model and apply the EM algorithm to estimate the parameters, we obtained the estimated parameters as follows.

$$A_{1EM} = \begin{bmatrix} 0.8036 & 0.1964 \\ 0.1042 & 0.8958 \end{bmatrix}, \quad D_{1EM} = \begin{bmatrix} 0.3732 & 0.6268 \\ 0.6092 & 0.3908 \end{bmatrix},$$

$$A_{2EM} = \begin{bmatrix} 0.7927 & 0.2073 \\ 0.7804 & 0.2196 \end{bmatrix}, \quad D_{2EM} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix},$$

$$A_{3EM} = \begin{bmatrix} 0.0628 & 0.9372 \\ 0.4999 & 0.5001 \end{bmatrix}, \quad D_{3EM} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. $$

The first model is identifiable except for the second row of the network interaction matrix $D_{1EM}$ corresponding to the unmeasured site 2. The matrix $D$ of the second model is not identifiable. The matrix $A$ and the second row of matrix $D$ are not identifiable for the third model. The three estimation results verify Theorem 13.

VI. Conclusion

In this paper we investigated the identifiability of the influence model, which has broad applications in capturing stochastic spatiotemporal spread processes. Using a master Markov chain as an intermediate step, we showed the sufficient and necessary conditions for the local Markov chain $A$ and the network influence matrix $D$ to be identifiable from measurements $Y$. Matrix $A$ is always identifiable, while the properties of matrix $O$ play crucial
roles in determining the identifiability of $D$, which were further investigated. We also provided conditions for the identifiability of some special classes of influence model. One interesting finding is that all binary influence models are identifiable. We then developed two estimators for identifiable influence models, one based on MLE, and the other based on linear algebra that exploit properties of the influence model developed in this paper. In addition, we analyzed the identifiability of POIM. We showed that for any POIM, the network interaction of the unmeasured sites is not identifiable. We therefore focused on the estimation of POIM for a subset of parameters associated with the dynamics of the measured sites, and provided conditions for the identifiability of a special case. An estimator based on the EM algorithm was developed for POIM. In the end, simulation studies were conducted to demonstrate the theoretical results.

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