On the weighted rearrangement of functions and degenerate nonlinear elliptic equations

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Abstract
Let $\Omega$ be a bounded domain of $\mathbb{R}^n$. We shall deal with boundary value problems of the following form
\[
\begin{cases}
-\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left( |x|^\alpha a_i(x,u,\nabla u) \right) = |x|^\alpha H(x,u) & \text{in } \Omega, \\
u = g & \text{on } \partial \Omega.
\end{cases}
\]
(0.1)

Here $\alpha > 1 - n$, $u$ is the relevant solution, $\nabla u$ is its gradient and $H$ is a given real-valued function. Under proper assumptions a priori estimates of solutions $u$ to the problem (0.1) are established by virtue of weighted rearrangement of functions and weighted isoperimetric inequalities.

1. The weighted rearrangement of functions
For $n \geq 1$ we define two classes of weight functions on $\mathbb{R}^n$.

Definition 1.1. (A weight class $W(\mathbb{R}^n)$)
\[
W(\mathbb{R}^n) = \left\{ f \in L^1_{\text{loc}}(\mathbb{R}^n) : f \text{ is nonnegative and spherically symmetric} \right\}
\]

Let us take and fix any $f \in W(\mathbb{R}^n)$. Then, for any bounded measurable set $A \subset \mathbb{R}^n$ we define a set function $\mu_f$ by
\[
\mu_f(A) = \int_A f(x) \, dx.
\]
(1.1)

Then $\mu_f$ becomes a spherically symmetric nonnegative measure on $\mathbb{R}^n$. Now we introduce a spherically symmetric decreasing rearrangement of function with respect to $\mu_f$ in the following way.
For any $u \in C_0^\infty(\mathbb{R}^n)$ and $t \geq 0$ we set

$$\mu_f^u(t) = \mu_f(\{x \in \mathbb{R}^n : |u(x)| > t\}) = \int_{|u| > t} f(x) \, dx. \quad (1.2)$$

Then $\mu_f^u(t)$ is a right-continuous and non-increasing function of $t$. Finally we define

$$u_f^*(x) = \sup\{t : \mu_f^u(t) > \mu_f(B_{|x|}(0))\}, \quad (1.3)$$

where $B_{|x|}(0)$ is a ball centered at the origin with a radius $|x|$. Here we note that $u_f^*$ can be defined for an integrable function $u$ w.r.t. $\mu_f$ on $\mathbb{R}^n$. Then $u_f^*$ is nonnegative, lower semi-continuous and non-increasing function of $|x|$. We also define

$$u_f^+(s) = \sup\{t : \mu_f^u(t) > s\}. \quad (1.4)$$

Then we see

$$u_f^+(x) = u_f^+(\mu_f(B_{|x|}(0))). \quad (1.5)$$

**Example:** If $f(x) = |x|^\alpha$ with $\alpha > -n$, we have

$$\mu_f(B_{|x|}(0)) = \omega_n|x|^{n+\alpha} \quad \text{for} \quad \omega_n = \frac{n C_n}{n+\alpha}, \quad (1.6)$$

where $C_n$ is the measure of a unit ball and hence $\omega_n$ is the measure of a unit ball with respect to the weighted Lebesgue measure $|x|^\alpha \, dx$. Then we have

$$u_f^*(x) = u_f^+(\omega_n|x|^{n+\alpha})$$

$$= \sup \left\{ t : \mu_f^u(t) \geq \frac{1}{\omega_n^{1/n}} |x| \right\}. \quad (1.7)$$

Since both $u$ and $u_f^*$ have the same distribution function, we have the next lemma.

**Lemma 1.1.** Let $p \geq 1$. Assume that $f \in W(\mathbb{R}^n)$. For any $u \in C_0^\infty(\mathbb{R}^n)$, it holds that

$$\int_{\mathbb{R}^n} |u(x)|^p f(x) \, dx = \int_{\mathbb{R}^n} u_f^+(x)^p f(x) \, dx = \int_0^\infty (u_f^+(s))^p \, ds.$$ 

**Proof:** By the definition we have

$$\int_{\mathbb{R}^n} |u|^p f(x) \, dx = \int_0^\infty pt^{p-1} \int_{|u| > t} f(x) \, dx = \int_0^\infty \mu_f^u(t) \, d(t^p)$$

$$= \int_0^\infty \mu_f^u(t) \, t \, d(t^p) = \int_0^\infty pt^{p-1} \int_{|u_f^*| > t} f(x) \, dx = \int_{\mathbb{R}^n} (u_f^*)^p f(x) \, dx.$$

Since $\mu_f(B_{r}(0)) = \int_{B_r(0)} f(x) \, dx = n C_n \int_0^r f(t) t^{n-1} \, dt$, we also have

$$\int_{\mathbb{R}^n} (u_f^*)^p f(x) \, dx = n C_n \int_0^\infty (u_f^+(r))^p f(r) r^{n-1} \, dr$$

$$= n C_n \int_0^\infty u_f^+(\mu_f(B_r(0)))^p f(r) r^{n-1} \, dr = \int_0^\infty (u_f^+(s))^p \, ds.$$
Theorem 1.1. Assume that \( f \in W(\mathbb{R}^n) \). Then for any \( u, v \in C_0^\infty(\mathbb{R}^n) \) we have
\[
\int_{\mathbb{R}^n} u(x) \cdot v(x) f(x) \, dx \leq \int_{\mathbb{R}^n} u_f^*(x) \cdot v_f^*(x) f(x) \, dx. \tag{1.9}
\]
In particular for any bounded open set \( E \subset \mathbb{R}^n \) and \( u \in C_0^\infty(\mathbb{R}^n) \), we have
\[
\int_E |u(x)| f(x) \, dx \leq \int_0^{\mu_f(E)} u_f^*(s) \, ds = \int_{E_f^*} u_f^*(x) f(x) \, dx \tag{1.10}
\]
Here \( E_f^* \) ia a ball centered at the origin satisfying \( \mu_f(E) = \mu_f(E_f^*) \).

Proof of Theorem 1.1: Let \( A \) be a bounded measurable subset of \( \mathbb{R}^n \) satisfying \( 0 < |A| < \infty \). By \( A_f^* \) we denote the rearrangement of \( A \) with respect to an admissible weight \( f \), namely, \( A_f^* \) is a ball centered at the origin satisfying
\[
\mu_f(A) = \mu_f(A_f^*). \tag{1.11}
\]
By \( \chi_A(x) \) we denote the characteristic function of \( A \). Then, by the layer cake representation of functions, the desired inequality becomes
\[
\int_0^\infty \int_0^\infty \int_{\mathbb{R}^n} \chi_{\{u > t\}}(x) \chi_{\{v > s\}}(x) f(x) \, dx \, ds \, dt \\
\leq \int_0^\infty \int_0^\infty \int_{\mathbb{R}^n} \chi_{\{u_f^* > t\}}(x) \chi_{\{v_f^* > s\}}(x) f(x) \, dx \, ds \, dt.
\]
Then it suffices to show the next inequality for any bounded Borel sets \( A \) and \( B \).
\[
\int_{\mathbb{R}^n} \chi_A(x) \chi_B(x) f(x) \, dx \leq \int_{\mathbb{R}^n} \chi_{A_f^*}(x) \chi_{B_f^*}(x) f(x) \, dx. \tag{1.12}
\]
We may assume that \( \mu_f(A) \leq \mu_f(B) \). Then we see that \( A_f^* \subset B_f^* \). Therefore we have
\[
\mu_f(A \cap B) \leq \mu_f(A_f^*) = \mu_f(A_f^* \cap B_f^*). \tag{1.13}
\]
This proves the first assertion. Since \( \chi_E \) can be approximated by a sequence of smooth functions, the second assertion follows direct from the previous one. \( \square \)

Now we recall the coarea formula and some related matters.

Lemma 1.2. (the coarea formula) Let \( p \geq 1 \). For any \( u \in C_0^\infty(\mathbb{R}^n) \) and any measurable function \( \Phi \) on \( \mathbb{R}^n \) we have
\[
\int_{\mathbb{R}^n} |\nabla u(x)|^p \Phi(x) \, dx = \int_{-\infty}^\infty ds \int_{\{u=s\}} \left| \nabla u(x) \right|^{p-1} \Phi(x) \, dH^{n-1}. \tag{1.14}
\]
Here \( dH^{n-1} \) is the \((n-1)\)-dimensional Hausdorff measure.

Remark 1.1. This formula is also valid under the assumption that \( u \) is Lipschitz continuous and \( \nabla u \) is integrable. For the proof of this formula, see [Ma; Theorem in §1.2.4] for example.
In this formula we assume that \(u \in C_0^\infty(\mathbb{R}^n)\) is nonnegative and
\[
\Phi = \begin{cases} \chi_{\{u > t\}}(x) \frac{f(x)}{|\nabla u(x)|}, & \text{if } \nabla u(x) \neq 0, \\ 0, & \text{if } \nabla u(x) = 0, \end{cases}
\]
then we have
\[
\mu_f^u(t) = \mu_f(\{u > t\} \cap \{\nabla u = 0\}) + \int_t^\infty ds \int_{\{u = s\}} \frac{f(x)}{|\nabla u(x)|} dH^{n-1}. \tag{1.15}
\]
By Sard’s lemma the set of critical values of \(u \in C_0^\infty(\mathbb{R}^n)\) has a vanishing measure, hence we have
\[
\frac{d}{dt} \mu_f^u(t) = -\int_{\{u = t\}} \frac{f(x)}{|\nabla u(x)|} dH^{n-1}, \quad \text{for almost all } t \in (0, \infty). \tag{1.16}
\]
Now we replace \(u\) by its rearrangement \(u_f^\ast\) in (1.15). We recall that both \(u\) and \(u_f^\ast\) share the same distribution function, and \(u_f^\ast\) is at least Lipschitz continuous as a rearrangement of a smooth \(u\). With somewhat more argument we see that
\[
\frac{d}{dt} \mu_f^u(\{u_f^\ast > t\} \cap \{\nabla u_f^\ast = 0\}) = 0, \quad \text{for almost all } t \in (0, \infty). \tag{1.17}
\]
For the proof of (1.17) see Appendix in [HK] (c.f. [CF;p12, lemma 2.4]). Hence we have the following.

**Lemma 1.3.** Assume that \(f \in W(\mathbb{R}^n)\). Then for any nonnegative \(u \in C_0^\infty(\mathbb{R}^n)\)
\[
\frac{d}{dt} \mu_f^u(\{u_f^\ast > t\} \cap \{\nabla u_f^\ast = 0\}) = 0, \quad \text{for almost all } t \in (0, \infty). \tag{1.18}
\]

### 1.1. The weighted isoperimetric inequalities

In this subsection we shall prepare the weighted isoperimetric inequalities and the CKN-type inequalities from [CKN] and the author’s paper [HK], [Ho1] and [Ho2]. For reader’s convenience we shall review the CKN-type inequalities for all \(p \geq 1\).

**Definition 1.2.** (The noncritical relation (NCR)) For any \(p, q, n, \alpha \) we set
\[
\beta(p, q, \alpha) = n \left( \frac{1}{p} - \frac{1}{q} \right) + \alpha - 1. \tag{1.19}
\]
In the next we define the noncritical (i.e. subcritical or supercritical) relation.

**Definition 1.3.** (The noncritical relation (NCR)) The parameters \(p, q, n, \alpha \) and \(\beta\) are said to satisfy the noncritical relation (NCR) if they satisfy
\[
\begin{align*}
\alpha &\neq 1 - \frac{n}{p}, \\
\beta &= \beta(p, q, \alpha), \\
q &< +\infty,
\end{align*}
\tag{1.20}
\]
Moreover the parameter $\alpha$ is said to be subcritical and supercritical if $\alpha > 1 - \frac{n}{p}$ and $\alpha < 1 - \frac{n}{p}$ holds respectively.

**Remark 1.2.** When $\alpha = 1 - \frac{n}{p}$ is satisfied, one can also define the critical relation and study the corresponding CKN-type inequalities. For the detail see [HK].

Under the condition (NCR), $\beta = \beta(p, q, \alpha)$ satisfies

$$E_{p, q, \alpha}(u) = \frac{\int_{\mathbb{R}^n} |\nabla u|^p |x|^\alpha \, dx}{(\int_{\mathbb{R}^n} |u|^q |x|^\beta \, dx)^{p/q}} \quad (u \in C_0^\infty(\mathbb{R}^n \setminus \{0\}) \setminus \{0\}).$$

**Definition 1.4.** For the noncritical case $\alpha \neq 1 - \frac{n}{p}$ we set for $\beta = \beta(p, q, \alpha)$

$$E^{p, q, \alpha}(u) = \frac{\int_{\mathbb{R}^n} |\nabla u|^p |x|^\alpha \, dx}{(\int_{\mathbb{R}^n} |u|^q |x|^\beta \, dx)^{p/q}} \quad (u \in C_0^\infty(\mathbb{R}^n \setminus \{0\}) \setminus \{0\}).$$

**Remark 1.3.** The functional $E^{p, q, \alpha}(u)$ is clearly invariant under dilations $u_\lambda(x) = u(\lambda x)$.

Using the following notation.

$$C^\infty_0(B_r)_{\text{rad}} = \{ u \in C^\infty_0(B_r) : u \text{ is a spherically symmetric function} \}$$

for $B_r = \{ x : |x| < r \}$ with $0 < r \leq \infty$, we define the best constants of the CKN-type inequalities as follows.

**Definition 1.5.** Under the condition (NCR) we set

$$S(p, q, \alpha) = \inf_{u \in C_0^\infty(\mathbb{R}^n \setminus \{0\}) \setminus \{0\}} E^{p, q, \alpha}(u),$$

$$S_{\text{rad}}(p, q, \alpha) = \inf_{u \in C_0^\infty(\mathbb{R}^n \setminus \{0\})_{\text{rad}} \setminus \{0\}} E^{p, q, \alpha}(u).$$

Here we remark that in $(P_{\text{rad}})$ we need not assume the condition $\beta \leq \alpha$.

**Definition 1.6.** For $\alpha \in \mathbb{R}$, let $L^p(\Omega, |x|^\alpha)$ denote the space of Lebesgue measurable functions, defined on a domain $\Omega$ of $\mathbb{R}^n$, for which

$$||u||_{L^p(\Omega, |x|^\alpha)} = \left( \int_{\Omega} |u(x)|^p |x|^\alpha \, dx \right)^{1/p} < +\infty.$$
Definition 1.7. Let \( p \) and \( \alpha \) satisfy \( 1 \leq p < +\infty \) and \( \alpha \neq 1 - \frac{n}{p} \). Let \( \Omega \) be a domain of \( \mathbb{R}^n \) such that \( 0 \in \Omega \). Then, by \( W_{1,p}^{\alpha,0}(\Omega) \) we denote the completion of \( C_\infty^0(\Omega \setminus \{0\}) \) with respect to a norm defined by

\[
\|u\|_{W_{1,p}^{\alpha,0}(\Omega)} = |||\nabla u|||_{L^p(\Omega, |x|^p \alpha)} + ||u||_{L^p(\Omega, |x|^{p(\alpha-1)})}.
\]

In the next we shall state the imbedding inequalities in the noncritical case which are known in the subcritical case as the classical CKN-type inequalities.

Theorem 1.2. (The imbedding results for \( p = 1 \)) Let \( n \) satisfy \( n \geq 1 \). Assume that the parameters \( q, n, \alpha \) and \( \beta \) satisfy the noncritical relation (NCR) with \( p = 1 \) and \( \alpha - 1 < \beta \).

Then we have \( S(1, q, \alpha) > 0 \) and for any \( u \in W^{1,1}_{\alpha,0}(\mathbb{R}^n) \),

\[
\int_{\mathbb{R}^n} |\nabla u(x)||x|^\alpha \, dx \geq S(1, q, \alpha) \left( \int_{\mathbb{R}^n} |u(x)|^q |x|^{\beta q} \, dx \right)^{1/q}.
\]

If \( \alpha > 1 - n \), then the proof is seen in [CKN]. If \( \alpha < 1 - n \), then the assertion follows the previous case using the argument in [HK].

Corollary 1.1. (Weighted isoperimetric inequality) Let \( n \) satisfy \( n \geq 1 \). Assume that the parameters \( q, n, \alpha \) and \( \beta \) satisfy the noncritical relation (NCR) with \( p = 1 \), \( \alpha - 1 < \beta \) and \( \alpha > 1 - n \) (the subcritical case). Then we have the following:

For any bounded open set \( M \subset \mathbb{R}^n \) with a smooth boundary \( \partial M \), we have

\[
\int_{\partial M} |x|^\alpha \, dH^{n-1} \geq S(1, q, \alpha) \left( \int_M |x|^{\beta q} \, dx \right)^{1/q}.
\]

Here by \( dH^{n-1} \) we denote the \((n-1)\) dimensional Hausdorff measure.

Proof: We take a bounded open subset \( M \subset \mathbb{R}^n \) with smooth boundary and construct approximative characteristic functions \( u_\varepsilon \) of \( M \) for sufficiently small \( \varepsilon > 0 \) as follows. Let us set \( M_\varepsilon = \{ x \in M; dist(x, \partial M) / \varepsilon \} \) and

\[
u_\varepsilon(x) = \begin{cases} 1, & x \in M_\varepsilon \\ \frac{dist(x, \partial M)}{\varepsilon}, & x \in M \setminus M_\varepsilon \\ 0, & x \in M^c. \end{cases}
\]

Since \( u_\varepsilon \) for sufficiently small \( \varepsilon > 0 \) belong to the space \( W^{1,1}_{\alpha,0}(\mathbb{R}^n) \) with \( \alpha > 1 - n \), it follows from Theorem 1.2 that

\[
\int_{\mathbb{R}^n} |\nabla u_\varepsilon(x)||x|^\alpha \, dx \geq S(1, q, \alpha) \left( \int_{\mathbb{R}^n} |u_\varepsilon(x)|^q |x|^{\beta q} \, dx \right)^{1/q}.
\]

Then we have, letting \( \varepsilon \to 0 \),

\[
\lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} |u_\varepsilon(x)|^q |x|^{\beta q} \, dx = \int_M |x|^{\beta q} \, dx.
\]
Since $\partial M_\varepsilon$ is a smooth manifold for a sufficiently small $\varepsilon > 0$, we also have
\[
\limsup_{\varepsilon \to 0} \int_{\mathbb{R}^n} |\nabla u_\varepsilon (x)||x|^\alpha \, dx = \limsup_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^\varepsilon dt \int_{\partial M_\varepsilon} |x|^\alpha \, dH^{n-1}
\]
(1.32)
\[
= \int_{\partial M} |x|^\alpha \, dH^{n-1}. \quad (1.33)
\]
Here we used Lemma 1.2 (the coarea formula).

\[ \square \]

**Corollary 1.2.** (The upper bound of the best constant) Let $n$ satisfy $n \geq 1$. Assume that the parameters $q, n, \alpha$ and $\beta$ satisfy the noncritical relation (NCR) with $p = 1$, $\alpha - 1 < \beta$ and $\alpha > 1 - n$ (the subcritical case). Moreover assume that $\alpha = \beta q$. Then we have the following estimate:
\[
S(1, q, \alpha) \leq (n + \alpha)\omega_n^{\frac{q + \alpha}{n}} \quad (\omega_n = \frac{nC_n}{n + \alpha}) \quad (1.34)
\]

**Proof:** It suffices to take $M = B_1(0)$ in the previous Corollary.

Lastly we describe the imbedding results for $p > 1$. For the detailed, see [HK].

**Theorem 1.3.** (The imbedding results for $p > 1$) Let $p$ satisfy $1 < p < +\infty$ and let $n$ satisfy $n \geq 1$. Assume that the parameters $p, q, n, \alpha$ and $\beta$ satisfy the noncritical relation (NCR).

Then we have $S(p, q, \alpha) > 0$ and for any $u \in W^{1, p}_{\alpha, \beta}(\Omega),
\[
\int_{\mathbb{R}^n} |\nabla u|^p |x|^{\alpha p} \, dx \geq S(p, q, \alpha) \left( \int_{\mathbb{R}^n} |u|^q |x|^{\beta p} \, dx \right)^{p/q}. \quad (1.35)
\]

**2. Application**

Let $n$ be a positive integer and let $\Omega$ be a bounded domain of $\mathbb{R}^n$. Throughout this section we adopt $f(x) = |x|^\alpha$ with $\alpha > 1 - n$ as a weight function and employ the results in §1 and 2. For a measurable $E \subset \Omega$ we set $E^* = E^*_{|.|, \alpha}$ for simplicity. In a similar way we put $\mu = \mu_{|.|, \alpha}$, $u^* = u^*_{|.|, \alpha}$ and $u^# = u^#_{|.|, \alpha}$.

We shall deal with boundary value problems of the following form
\[
\begin{cases}
- \sum_{i=1}^n \frac{\partial}{\partial x_i} (|x|^\alpha u_i (x, u, \nabla u)) = |x|^\alpha H(x, u) & \text{in } \Omega, \\
u = g & \text{on } \partial \Omega.
\end{cases} \quad (2.1)
\]

where $\alpha > 1 - n$, $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $u$ is the relevant solution, $\nabla u$ is its gradient and $H$ is a given real-valued function of the specified arguments.

The hypotheses we assume are the following:
1. The functions \(a_i(x,u,\xi)\) are measurable in \(\Omega \times \mathbb{R}_1 \times \mathbb{R}_n\) and a function \(A(r)\) exists such that:

\[
A(r) \in C^2([0,\infty)) \text{ is convex and strictly increasing,}
\]

\[
\lim_{r \to +0} \frac{A(r)}{r} = 0,
\]

\[
\sum_{i=1}^{n} a_i(x,u,\xi)\xi_i \geq A(|\xi|).
\]

2. \(H(x,u)\) is measurable in \(\mathbb{R}^n \times \mathbb{R}_1\) and

\[
(H(x,u) - H(x,0))u \leq 0 \quad \text{for all } (x,u).
\]

3. \(H(\cdot,0)\) is integrable on \(\Omega\).

4. \[
\begin{cases}
\text{either } \lim_{r \to \infty} \frac{A(r)}{r} = \infty \text{ or } \\
\sup_{s>0} s^{-\frac{1}{q}} \sup_{E} \left\{ \int_{E} |H(x,0)||x|^\alpha \, dx : \mu(E) = s \right\} < S(1,q,\alpha) \lim_{r \to \infty} \frac{A(r)}{r}
\end{cases}
\]

Here \(\frac{1}{q} = 1 - \frac{1}{n+\alpha}\), \(\mu = \mu_\cdot|\cdot|_\alpha\) and \(S(1,q,\alpha)\) is the best constant of the CKN-inequality defined by \((P)\).

Roughly speaking, as solutions for this problem we adopt weak solutions \(u\), which belong to the so-called Orlicz-Sobolev class. Namely we take weak solutions \(u\) whose distributional derivatives in the domain \(\Omega\) are functions from the convex Orlicz class

\[
\left\{ \varphi \text{ is measurable} : \int_{\Omega} A(|\varphi(x)|)|x|^\alpha \, dx < +\infty \right\},
\]

hence

\[
\int_{\Omega} A(|\nabla u(x)|)|x|^\alpha \, dx < +\infty.
\]

Now let us define the Orlicz-Sobolev spaces.

**Definition 2.1.** \((W^{1,A}_0(\Omega), W^{1,A}(\Omega))\) By \(L^A(\Omega)\) we denote the Orlicz space defined by the closure of all test functions \(\varphi \in C_0^\infty(\Omega)\) with respect to the Orlicz norm

\[
||\varphi||_{L^A(\Omega)} = \inf \left\{ \lambda > 0 : \int_{\Omega} A\left(\frac{|\varphi(x)|}{\lambda}\right) |x|^\alpha \, dx \leq 1 \right\}.
\]

The Orlicz-Sobolev space \(W^{1,A}(\Omega)\) is defined as the collection of those members of \(L^A(\Omega)\) whose distributional first derivatives are in \(L^A(\Omega)\) as well. Then the closure of test functions \(C_0^\infty(\Omega)\) with respect to the Orlicz-Sobolev space \(W^{1,A}(\Omega)\) is called \(W^{1,A}_0(\Omega)\).
Remark 2.1. By a usual approximation argument the twice differentiability and
the strictly increasingness of $A(r)$ are not really needed, but for the sake of simplicity we impose them on $A(r)$.

Under this definition, we further assume that
\[
g \in L^\infty(\Omega) \cap W^{1,A}(\Omega). \tag{2.10}
\]

Then we define a weak solution to (2.1) with boundary value $g$ satisfying (2.10).

Definition 2.2. A function $u$ in the Orlicz-Sobolev class is called a weak solution to (2.1) with boundary value $g$ satisfying (2.10) if $u - g \in W^{1,A}_0(\Omega)$ and for any $\varphi \in C_0^\infty(\Omega)$ we have
\[
\int \sum_{i=1}^n a_i(x,u(x),\nabla u(x)) \varphi_{x_i}(x)|x|^\alpha \, dx = \int \Omega H(x,u(x)) \varphi(x)|x|^\alpha \, dx. \tag{2.11}
\]

Now we state our main result which is a generalization of the one in [Ta].

Theorem 2.1. Let $\Omega$ be a bounded domain of $\mathbb{R}^n$. Assume that the hypotheses 1, 2, 3, 4 and (2.10). Further we assume that $u$ is a weak solution to (2.1) from Orlicz-Sobolev class (2.8). Let us set $\Omega^* = \{ x \in \mathbb{R}^n : \omega_1^n \frac{|x|}{|\Omega|^{1/n}} < |\Omega|^{1/n} \}$ be the ball with the same measure as $\Omega$ with respect to the weighed measure $\mu = \mu_1|.|^\alpha$ and $v$ is the function defined in $\Omega^*$ by
\[
v(x) = \sup \Omega |g| + \int_{\omega_1^n|x|^{n+\alpha}}^{\Omega(x)} B^{-1} \left( \frac{\int_0^r (H(x,0))^\#(s) \, ds}{S(1,q,\alpha)r^{\frac{1}{q}}} \right) \frac{r^{-\frac{1}{q}} \, dr}{S(1,q,\alpha)}, \tag{2.12}
\]

where $B(r) = \frac{A(r)}{C_n}$ and $C_n$ is the measure of the $n$ dimensional unit ball. $(H(x,0))^\#(s)$ is a rearrangement of $H(x,0)$ as a function of a variable $x$ w.r.t. the weighted measure $|x|^\alpha \, dx$. Then we have
1. $u^*(x) \leq v(x)$ for all $x \in \Omega^*$.
2. $\int \Omega^* M(|\nabla u(x)||x|^\alpha \, dx \leq \int \Omega M(|\nabla v(x)||x|^\alpha \, dx,$

where $\Omega^* = \Omega^*_{|.|^\alpha}$ and $M$ is any twice continuously differentiable function such that
\[
\begin{aligned}
M'(r) &> 0 \quad \text{and} \quad \frac{M''(r)}{M'(r)} \leq \frac{A''(r)}{A'(r)} \quad \text{if } 0 < r < \infty, \\
M(+0) &\equiv 0.
\end{aligned} \tag{2.13}
\]

For the proof of this we begin with preparing a lemma on the distribution function of $A(|\nabla u||x|^\alpha$ with respect to $u$. 

Lemma 2.1. Under the same assumptions of Theorem 2.1, the function
\[ \nu(t) = \int_{\{ |u| > t \}} A(|\nabla u(x)|)|x|^\alpha \, dx \tag{2.14} \]
is Lipschitz continuous on the interval \((\sup_{x \in \Omega} |g(x)|, +\infty)\), and the inequality
\[ 0 \leq -\frac{d}{dt} \nu(t) \leq \int_0^\nu(t) (H(\cdot, 0))^\#(s) \, ds \tag{2.15} \]
holds for almost all \( t > \sup_{x \in \Omega} |g(x)| \).

Here \((H(\cdot, 0))^\#(s)\) is the value at \( s \) of the decreasing rearrangement of \( H(\cdot, 0) \) into \([0, \infty)\) defined by \((H(\cdot, 0))^\#(s) = \sup\{ t : \int_{H(x,0) > t} |x|^\alpha \, dx > s \}\).

**Proof of Lemma 2.1:** From (2.8) and the remark 2.1 we may assume that \( u \) belongs to Orlicz-Sobolev space \( W^{1,A}(\Omega) \) and that (2.11) holds for any \( \varphi \in W^{1,A}_0(\Omega) \). Hence we set
\[ \varphi(x) = \begin{cases} u(x) - t, & \text{for } u(x) > t, \\ u(x) + t, & \text{for } u(x) < -t, \\ 0, & \text{otherwise.} \end{cases} \tag{2.16} \]
and put it into (2.11). Then we see that the function \( \Phi \) defined by
\[ \Phi(t) = \int_{\{ |u| > t \}} \sum_{i=1}^n a_i(x, u(x), \nabla u(x)) u_{x_i}(x)|x|^\alpha \, dx \tag{2.17} \]
satisfies
\[ \Phi(t) = \int_{\{ |u| > t \}} H(x, u(x))(|u(x)| - t) \text{sgn } u(x) |x|^\alpha \, dx \tag{2.18} \]
for all \( t > \sup_{x \in \Omega} |g(x)| \). Then for any \( h > 0 \) and \( t > \sup_{x \in \Omega} |g(x)| \) we have
\[ \Phi(t) - \Phi(t + h) = \int_{\{ |u| > t \}} H(x, u(x)) \text{sgn } u(x) |x|^\alpha \, dx \tag{2.19} \]
\[ - h \int_{\{ |u| \leq t + h \}} H(x, u(x)) \left( 1 - \frac{|u(x)| - t}{h} \right) \text{sgn } u(x) |x|^\alpha \, dx. \]
Then we have
\[ \frac{\Phi(t) - \Phi(t + h)}{h} = \int_{\{ |u| > t \}} H(x, u(x)) \text{sgn } u(x) |x|^\alpha \, dx \tag{2.20} \]
\[ - \int_{\{ |u| \leq t + h \}} H(x, u(x)) \left( 1 - \frac{|u(x)| - t}{h} \right) \text{sgn } u(x) |x|^\alpha \, dx \]
\[ \leq \int_{\{ |u| > t \}} |H(x, u(x))| |x|^\alpha \, dx + \int_{\{ |u| \leq t + h \}} |H(x, 0)| |x|^\alpha \, dx \]
\[ \leq 2 \int_\Omega |H(x, 0)| |x|^\alpha \, dx. \tag{2.21} \]
Therefore we have
\[
\frac{\Phi(t) - \Phi(t + h)}{h} \leq 2 \int_\Omega |H(x, 0)||x|^\alpha \, dx \tag{2.22}
\]
and for almost all \(t > \sup_{x \in \Omega} |g(x)|\)
\[
- \frac{d}{dt} \Phi(t) \leq \int_{|u| > t} |H(x, 0)||x|^\alpha \, dx. \tag{2.23}
\]

Then it follows from Theorem 1.1 with \(v\) being the characteristic function of \(\{x; |u(x)| > t\}\) that the desired inequality (2.15) holds. \(\square\)

**Proof of Theorem 2.1:** Let \(u\) be a function in Orlicz-Sobolev space \(W^{1,A}(\Omega)\) such that \(\int_\Omega A(|\nabla u(x)||x|^\alpha \, dx < +\infty\) and \(u\) agrees on the boundary of \(\Omega\) with a bounded function \(g \in L^\infty(\Omega) \cap W^{1,A}(\Omega)\) in the sense that \(u - g \in W^{1,A}_0(\Omega)\). Then we have

\[
\begin{align*}
\mu^u(t) &= \mu^{u|\alpha}_{|\cdot|}(t) = \int_{|u| > t} |x|^\alpha \, dx, \\
\frac{d}{dt} \mu^u(t) &= \int_{\{u = t\}} \frac{|x|^\alpha}{|\nabla u(x)|} \, dH^{n-1}.
\end{align*}
\tag{2.24}
\]

We define auxiliary functions \(B(r)\) and \(C(r)\) as follows:

\[
B(r) = \frac{A(r)}{r}, \tag{2.25}
\]
\[
C(s) = \frac{1}{B^{-1}(s)} \quad \text{if } 0 < s < \lim_{r \to +\infty} \frac{A(r)}{r}, = 0 \quad \text{otherwise}. \tag{2.26}
\]

Then we see that \(B\) is increasing with \(B(0) = 0\). For \(s = B(r)\), we see that \(C'(s) = -\frac{1}{r B'(r)}\) and \(C''(s) = \frac{A''(r)}{(r B'(r))^2} > 0\), and hence \(C(r)\) is a convex function. Then we see that

\[
C \left( \frac{\int_{t < |u| \leq t + h} A(|\nabla u(x)||x|^\alpha \, dx}{\int_{t < |u| \leq t + h} |\nabla u(x)||x|^\alpha \, dx} \right)
\leq C \left( \frac{\frac{1}{n} \int_{t < |u| \leq t + h} B(|\nabla u||x|^\alpha \, dx}{\frac{1}{n} \int_{t < |u| \leq t + h} |\nabla u||x|^\alpha \, dx} \right)
\leq \frac{1}{n} \int_{t < |u| \leq t + h} C \left( \frac{B(|\nabla u(x)||x|^\alpha \, dx}{|\nabla u(x)||x|^\alpha \, dx} \right)
\leq \frac{1}{n} \int_{t < |u| \leq t + h} \left( -\mu^u(t + h) + \mu^u(t) \right)
\leq \frac{1}{n} \int_{t < |u| \leq t + h} |\nabla u(x)||x|^\alpha \, dx.
\tag{2.27}
\]

Taking a limit as \(h \to 0\) we have for almost all \(t\)
\[ C \left( \frac{\frac{d}{dt} \int_{\{|u| > t\}} A(|\nabla u(x)|)|x|^\alpha \, dx}{\frac{d}{dt} \int_{\{|u| > t\}} |\nabla u(x)||x|^\alpha \, dx} \right) \leq \frac{\frac{d}{dt} \mu^n(t)}{\frac{d}{dt} \int_{\{|u| > t\}} |\nabla u(x)||x|^\alpha \, dx}. \] (2.28)

Note that
\[ -\frac{d}{dt} \int_{\{|u| > t\}} |\nabla u(x)||x|^\alpha \, dx = \int_{\{|u| = t\}} |x|^\alpha \, dH^{n-1}. \] (2.29)

Now we recall the weighted isoperimetric inequality for \( q = \frac{n+\alpha}{n+1} > 1 \) and \( \beta = \frac{\alpha}{q} \)
\[ \int_{\Omega^M} |x|^\alpha \, dH^{n-1} \geq S(1, q, \alpha) \left( \int_M |x|^\alpha \, dx \right)^{1/q} \] (2.30)
and set \( M = \{ x : |u| > t \} \). Then we obtain for almost all \( t \)
\[ \int_{\{|u| = t\}} |x|^\alpha \, dH^{n-1} \geq S(1, q, \alpha) \left( \int_{\{|u| > t\}} |x|^\alpha \, dx \right)^{1/q}. \] (2.31)

As a result we have
\[ -\frac{d}{dt} \int_{\{|u| > t\}} |\nabla u(x)||x|^\alpha \, dx \geq S(1, q, \alpha) \mu^n(t)^{1/q}. \] (2.32)

By using Lemma 2.1 we have
\[ 1 \leq \frac{-\frac{d}{dt} \mu^n(t)}{\frac{d}{dt} \int_{\{|u| > t\}} |\nabla u(x)||x|^\alpha \, dx} C \left( \frac{\frac{d}{dt} \int_{\{|u| > t\}} A(|\nabla u(x)|)|x|^\alpha \, dx}{\frac{d}{dt} \int_{\{|u| > t\}} |\nabla u(x)||x|^\alpha \, dx} \right) \] (2.33)
\[ \leq \frac{-\frac{d}{dt} \mu^n(t)}{S(1, q, \alpha) \mu^n(t)^{1/q}} B^{-1} \left( \int_{E^*} (H(\cdot, 0))^\alpha(x)|x|^\alpha \, dx \right) \frac{1}{S(1, q, \alpha) \mu^n(t)^{1/q}}, \]
\[ \leq \frac{-\frac{d}{dt} \mu^n(t)}{S(1, q, \alpha) \mu^n(t)^{1/q}} B^{-1} \left( \int_0^t (H(\cdot, 0))^\#(s) \, ds \right) \frac{1}{S(1, q, \alpha) \mu^n(t)^{1/q}}, \]
where \( E^* = \{ x : |u(x)| > t \}^* \). By integrating w.r.t. \( t \) from \( \sup_{x \in \Omega} |g(x)| \) to \( t \), we have
\[ t \leq \sup_{x \in \Omega} |g(x)| + \int_{\sup_{x \in \Omega} |g(x)|}^t \left( \int_0^\tau (H(\cdot, 0))^\#(s) \, ds \right) \frac{1}{S(1, q, \alpha) \mu^n(t)^{1/q}} \, d\tau \] (2.34)
\[ \leq \sup_{x \in \Omega} |g(x)| + \int_{\mu^n(t)}^{\mu^n(\Omega)} \left( \int_0^\tau (H(\cdot, 0))^\#(s) \, ds \right) \frac{1}{S(1, q, \alpha) \mu^n(t)^{1/q}} \, d\tau \leq \frac{\mu^n(\Omega)}{S(1, q, \alpha) \mu^n(t)^{1/q}}. \]
where we used a change of variables by \( r = \mu(\tau) \). Then by the definition of \( u^\#(t) \) we have

\[
u^\#(s) \leq \sup_{x \in \Omega} |q(x)| + \int_s^{\mu(\Omega)} B^{-1} \left( \frac{\int_0^s (H(\cdot, 0))^\#(t) \, dt}{S(1, q, \alpha) r^{\frac{n}{q}}} \right) \frac{dr}{S(1, q, \alpha) r^{1/q}}. \tag{2.35}\]

This proves the assertion 1.

Now we proceed to the proof of the assertion 2. First we assume that \( g = 0 \) and \( A = M \). By the assumption (2.4) and (2.11) we see that

\[
\mathcal{J} \leq \int_{\Omega} H(x, u(x)) |x|^\alpha \, dx \\
\leq \int_{\Omega} H(x, 0) |x|^\alpha \, dx \\
\leq \int_{\Omega} H(x, 0) u(x) |x|^\alpha \, dx \\
\leq \int_0^{\mu(\Omega)} (H(\cdot, 0))^\#(s) u^\#(s) \, ds \\
\leq \int_0^{\mu(\Omega)} (H(\cdot, 0))^\#(s) \left( \int_s^{\mu(\Omega)} B^{-1} \left( \frac{\int_0^s (H(\cdot, 0))^\#(t) \, dt}{S(1, q, \alpha) r^{\frac{n}{q}}} \right) \frac{dr}{S(1, q, \alpha) r^{1/q}} \right) \, ds \\
= \int_0^{\mu(\Omega)} \int_0^s (H(\cdot, 0))^\#(t) \, dt \frac{B^{-1}}{S(1, q, \alpha) s^{\frac{n}{q}}} \left( \frac{\int_0^s (H(\cdot, 0))^\#(t) \, dt}{S(1, q, \alpha) s^{\frac{n}{q}}} \right) \, ds.
\]

Now we recall (2.35) and \( q = \frac{n+\alpha}{n+\alpha-1} \) to have

\[
\int_{\Omega} A(\nabla u(x)) |x|^\alpha \, dx \quad (sB^{-1}(s) = A(B^{-1}(s))) \tag{2.37}
\]

\[
\leq \int_0^{\mu(\Omega)} A \left( B^{-1} \left( \frac{\int_0^s (H(\cdot, 0))^\#(t) \, dt}{S(1, q, \alpha) s^{\frac{n}{q}}} \right) \right) \, ds \\
= \int_0^{\mu(\Omega)} A \left( B^{-1} \left( \frac{\int_0^s (H(\cdot, 0))^\#(t) \, dt}{S(1, q, \alpha) s^{\frac{n}{q}}} \right) \right) nC_n r^{n+\alpha-1} \, dr \\
= \int_{\Omega^*} A \left( \frac{\int_0^\omega |x|^{\alpha+\alpha}(H(\cdot, 0))^\#(t) \, dt}{S(1, q, \alpha) \omega^{\frac{n}{q}} |x|^{\alpha+\alpha-1}} \right) |x|^\alpha \, dx \\
= \int_{\Omega^*} A \left( \frac{\int_0^\omega \omega^{n+\alpha}(H(\cdot, 0))^\#(t) \, dt}{S(1, q, \alpha) \omega^{\frac{n}{q}} |x|^{\alpha+\alpha-1}} \right) |x|^\alpha \, dx \\
\leq \int_{\Omega^*} A(\nabla v(x)) |x|^\alpha \, dx.
\]

Here we remark

\[
|\nabla v(x)| = K(n, \alpha) B^{-1} \left( \frac{\int_0^\omega |x|^{\alpha+\alpha}(H(\cdot, 0))^\#(t) \, dt}{S(1, q, \alpha) \omega^{\frac{n}{q}} |x|^{\alpha+\alpha-1}} \right), \tag{2.38}
\]
Hence we have

\[ K(s) = A(M^{-1}(s)). \]  

(2.39)

Since \( K \) is a convex function, we see that for \( h > 0 \)

\[ K \left( \frac{\int_{|t| \in [t+\varepsilon)} M(|\nabla u(x)||x|^\varepsilon \, dx}{-\mu^n(t + h) + \mu^n(t)} \right) \leq \frac{\int_{|t| \in [t+h]} A(|\nabla u(x)||x|^\varepsilon \, dx}{-\mu^n(t + h) + \mu^n(t)}. \]  

(2.40)

Hence we have

\[
M^{-1} \left( \frac{1}{\frac{d}{dt} \mu^n(t)} \int_{|u| > t} M(|\nabla u(x)||x|^\alpha \, dx \right) \leq A^{-1} \left( \frac{-\frac{d}{dt} \int_{|u| > t} A(|\nabla u(x)||x|^\alpha \, dx}{S(1, q, \alpha) \mu^n(t) \frac{1}{2}} \right) \left( \int_{S(1, q, \alpha) \mu^n(t) \frac{1}{2}} A^{-1} \left( \frac{s}{C(s)} \right) \right) \right) \]  

\[
\leq \left( \frac{\int_0^{\omega(u)} (H\{0\}, \#(s) \, ds}{S(1, q, \alpha) \mu^n(t) \frac{1}{2}} \right). \]  

\[
\int_{|u| > \sup|g|} M(|\nabla u(x)||x|^\alpha \, dx \right) \]  

\[
= \int_{\sup|g|} \int_{|u| > t} M(|\nabla u(x)||x|^\alpha \, dx \right) \]  

\[
\leq \int_{0}^{M} \left( B^{-1} \left( \int_0^{\omega(u)} (H\{0\}, \#(s) \, ds}{S(1, q, \alpha) \mu^n(t) \frac{1}{2}} \right) \left( \frac{-d}{dt} \mu^n(t) \right) \right) \]  

\[
\leq \int_{0}^{M} \left( B^{-1} \left( \int_0^{\omega(u)} (H\{0\}, \#(s) \, ds}{S(1, q, \alpha) \mu^n(t) \frac{1}{2}} \right) \right) \]  

\[
= \int \left( \frac{1}{\omega(u)} \right) \frac{1}{S(1, q, \alpha) \mu^n(t) \frac{1}{2}} \) \right) \) \]  

\[
= \int_{\Omega} \left( B^{-1} \left( \int_0^{\omega(u)} (H\{0\}, \#(s) \, ds}{S(1, q, \alpha) \mu^n(t) \frac{1}{2}} \right) \right) \) \]  

\[
|\nabla u(x)||x|^\alpha \, dx \]  

\[
= \int \left( B^{-1} \left( \int_0^{\omega(u)} (H\{0\}, \#(s) \, ds}{S(1, q, \alpha) \mu^n(t) \frac{1}{2}} \right) \right) \) \]  

\[
|\nabla u(x)||x|^\alpha \, dx \]  

\[
= \int \left( B^{-1} \left( \int_0^{\omega(u)} (H\{0\}, \#(s) \, ds}{S(1, q, \alpha) \mu^n(t) \frac{1}{2}} \right) \right) \) \]  

\[
|\nabla u(x)||x|^\alpha \, dx \]  

\[
= \int \left( B^{-1} \left( \int_0^{\omega(u)} (H\{0\}, \#(s) \, ds}{S(1, q, \alpha) \mu^n(t) \frac{1}{2}} \right) \right) \) \]  

\[
|\nabla u(x)||x|^\alpha \, dx \]
On the weighted rearrangement of functions and degenerate nonlinear elliptic equations

\[
\int_{\Omega} M \left( \frac{|\nabla v(x)|}{K(n, \alpha)} \right) |x|^\alpha \, dx \\
\leq \int_{\Omega} M (|\nabla v(x)|) |x|^\alpha \, dx. \quad (K(n, \alpha) > 1)
\]

Thus we have the desired estimate. \( \square \)

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