ON THE EMBEDDINGS OF UNIVERSAL TORSORS OVER DEL PEZZO SURFACES IN THE CONES OVER FLAG VARIETIES.

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Abstract. Following Skorobogatov and Serganova we construct the embeddings of universal torsors over del Pezzo surfaces in the cones over flag varieties, considered as the closed orbits in the projectivization of quasiminuscule representations. We give the approach that allows us to construct the embeddings of universal torsors over del Pezzo surfaces of the degree one. This approach uses the Mumford stability and slightly differs from the approach of the authors named above.

Let \( G \) be a simply connected algebraic group over algebraically closed field \( \mathbb{K} \) of characteristics zero, \( Z(G) \) is the center of \( G \), \( T \) is a maximal torus in \( G \), and \( B \) is the Borel subgroup containing this torus, \( U \) is the unipotent radical of \( B \). This define the root system \( \Delta \) and its subsets of positive and negative roots \( \Delta^+ \) and \( \Delta^- \).

Let \( \Delta \) be a root system of one of the following types: \( A_{4}, D_{5}, E_{6}, E_{7}, E_{8} \). Denote by \( \alpha \) the root corresponding to the endpoint of the Dynkin diagram: for the diagram \( D_{5} \) we set \( \alpha = \varepsilon_{4} - \varepsilon_{5} \), for the systems of type \( E_{*} \) and a system \( A_{4} \) \( \alpha = \varepsilon_{1} - \varepsilon_{2} \) (here \( \varepsilon_i \) is the standard basis in the vector space generated by the weights). The root system corresponding to the Dynkin diagram with deleted end vertex we denote by \( \Delta' \). Let us denote by \( \beta \) the root corresponding to the adjacent vertex to \( \alpha \) (cf. fig.1). The fundamental weight dual to \( \alpha \) we denote by \( \pi_{\alpha} \). By \( \pi'_{\beta} \) we denote the fundamental weight, from the set of fundamental weights of the root system \( \Delta' \), dual to \( \beta \) (we have to note that \( (\pi'_{\beta}; \alpha) \neq 0 \)). The systems of simple roots for the root systems \( \Delta \) and \( \Delta' \) we denote by \( \Pi \) and \( \Pi' \) correspondingly.

Let us notice that the representation \( V(\pi_{\alpha}) \) with the highest weight \( \pi_{\alpha} \) is the miniscule representation for all systems but \( E_{8} \), i.e. the Weyl group is acting transitively on the weights of representation. In the case of \( E_{8} \) we the representation in consideration is adjoint and the Weyl group is acting transitively on the weights of representation distinct from zero, and a zero weight has the multiplicity 8.

Let \( P \) be a parabolic subgroup that stabilizes the point \( \langle v_{\pi_{\alpha}} \rangle \in \mathbb{P}(V(\pi_{\alpha})) \), where \( v_{\pi_{\alpha}} \) is the highest weight vector. Let \( L \) be a Levi subgroup of \( P \). The semisimple part of \( L \) we denote by \( G' \), it has a root system \( \Delta' \subset \Delta \). The irreducible representation of \( G' \) with the highest weight \( \omega \) we denote by \( V(\omega) \). The stabilizer in \( G' \) of the point \( \langle v_{\pi'_{\beta}} \rangle \in \mathbb{P}(V'(\pi'_{\beta})) \) is the parabolic subgroup \( P' \) with the Levi subgroup \( L' \), that has the system of simple roots \( \Pi'' := \Pi \setminus \{\alpha, \beta\} \). The semisimple part of \( L' \) is denoted by \( G'' \), and its root system is denoted by \( \Delta'' \).

As it is well known from [7], to a del Pezzo surface one can put in the correspondence the root system \( \Delta \) of one of the types mentioned above. Thus by \( X_{\Delta} \) we denote an arbitrary del Pezzo surface those root system has type \( \Delta \).
Let us remind the reader the definition of the universal torsor. We assume that the action of some torus $T_0$ on the normal variety $\mathcal{T}$ is scheme theoretically free and there exists a geometric quotient $\mathcal{T} \rightarrow X$. Let $\mathcal{O}(\mathcal{T})^*$ be the set of regular invertible functions on $\mathcal{T}$. Then we have the following exact sequence from the work of J-L. Colliot-Thélène and J-J. Sansuc that we denote by (CTS):

$$1 \rightarrow \mathcal{O}(X)^*/\mathbb{K}^* \rightarrow (\mathcal{O}(\mathcal{T})^*/\mathbb{K}^*) \rightarrow \Xi(T_0) \rightarrow \text{Pic}(X) \rightarrow \text{Pic}(\mathcal{T}) \rightarrow 0,$$

where $\Xi(T_0)$ is the character lattice of $T_0$. We note that this sequence is functorial on $\mathcal{T}$.

**Definition 0.1.** The torsor $\mathcal{T}$ is called universal, if the map $\Xi(T_0) \rightarrow \text{Pic}(X)$ is an isomorphism.

For a convenience of the reader we recall the definition of the set of semistable and stable points, introduced by Mumford.

**Definition 0.2.** Let $X$ be an algebraic variety with the action of a reductive group $H$, and $L$ is an invertible ample $H$-linearized sheaf on $X$.

(i) The set of semistable points is equal to

$$X^s_L = \{ x \in X : \exists n > 0, \exists \sigma \in \Gamma(X, L^\otimes n^H), \sigma(x) \neq 0 \}.$$

(ii) The set of stable points is equal to

$$X^s_L = \{ x \in X^s_L : \text{orbit } Hx \text{ closed in } X^s_L \text{ and the stabilizer } H_x \text{ is finite} \}.$$

In our case $H = T$, $X = G/P$, and for the sheaf $L$ we take $i^*\mathcal{O}(1)$, (where $i : G/P \subset \mathbb{P}(V(\pi_\alpha))$), by $(G/P)^{sf}$ we denote the set of points $x \in (G/P)^s$, for which the stabilizer is equal to $T_x = Z(G)$.

Consider the torus $T \times \mathbb{K}^x$. Let us define the action of this torus on the space $V(\pi_\alpha)$ in such way that the component $\mathbb{K}^x$ acts on $V(\pi_\alpha)$ by homotety, and the second component as the subgroup in $G$. Since the representation $V(\pi_\alpha)$ is irreducible, the kernel of the action of $T \times \mathbb{K}^x$ on $V(\pi_\alpha)$ is isomorphic to $Z(G)$. By $\hat{T}$ we denote the quotient of $T \times \mathbb{K}^x$ by this kernel.

Our aim is to prove the theorem in general case that was proved by Skorobogatov and Serganova [2] in the case of root system $\Delta$ distinct from $E_8$.

**0.3. There exists a locally closed $\hat{T}$-equivariant embedding of the universal torsor $\hat{T}$ over del Pezzo surface $X_\Delta$ (in the case $E_8$ we consider a sufficiently general surface $X_\Delta$) in the affine cone in $V(\pi_\alpha)$ over the open subset of points $(G/P)^{sf}$ of the flag variety $G/P \subset \mathbb{P}(V(\pi_\alpha))$ that are stable with respect of the action of maximal torus $T$ and having a stabilizer $Z(G)$. Consider the intersection of $\hat{T}$ with the $\hat{T}$-invariant hyperplane $v_\omega = 0$ (where $\omega$ is the nonzero weight of the representation $V(\pi_\alpha)$). Its image under the quotient by the action of torus $\hat{T}$ is the $(-1)$-curve lying on the surface $X_\Delta$. All $(-1)$-curves on $X_\Delta$ can be obtained in such way.

**Remark 0.4.** The connection of universal torsors over del Pezzo surfaces and flag varieties embedded in the projectivization of miniscule representation was first observed by V.V. Batyrev (see also [3]). The total coordinate rings of del Pezzo surfaces were calculated by V.V. Batyrev and O.N. Popov in [3]. For the detailed history of this question we refer the reader to [2].

Let us give a sketch of the proof of the main theorem. We note that the scheme of the proof was taken from [2], but most steps will be modified, and we shall give some proofs different from [2].

First we prove that the Mumford quotient of the flag variety $G/P$ by a one parameter subgroup $\lambda : \mathbb{K}^x \rightarrow T$ is isomorphic to the blow up of $\mathbb{P}(V(\pi_\beta))$ in the flag variety $G'/P'$, that we denote by $\text{Bl}(\mathbb{P}(V(\pi_\beta)), G'/P')$. On the induction
step we assume that for the $T'$-torsor $\mathcal{T}'$ over $X_{\Delta'}$ we constructed its embedding into $G'/P'$ (it is obtained be the embedding of the universal torsor $\mathcal{T}'$ in the cone over $G'/P'$ and a further projection on $G'/P'$). Let us fix a weight basis in $V'(\pi'_3)$. Then the point $s \in \mathbb{P}(V'(\pi'_3))$ whose all coordinates are nonzero defined the automorphism of $\mathbb{P}(V'(\pi'_3))$. Let us denote the obtained torsor by $\mathbb{P}(V'(\pi'_3))$. Let us introduce an additional notation: $\sigma : X_{\Delta} \longrightarrow X_{\Delta'}$ is a blow up of this point. Consider the point $s$ of the torsor $\mathcal{T}'$, lying in the fiber over the point $e_\Delta$. Let $s \in G'/P'$ be a sufficiently general point. By applying the automorphism defined by $s s^{-1}$ that acts by multiplication of the weight vectors of $V'(\pi'_3)$ we get that the image of the torsor $s s^{-1} \mathcal{T}'$ intersect $G'/P'$ by one $T'$-orbit $Ts'$, lying over the point $e_\Delta$. Then we take a proper transform $\mathcal{T}$ of the torsor $s s^{-1} \mathcal{T}'$ under the blow up $\mathbb{P}(V(\pi'_3))$ in $G'/P'$, and then we take its preimage under the quotient by $\lambda$. Let us denote the obtained torsor by $\mathcal{T}$. We will show that the affine cone over $\mathcal{T}$ will be desired torsor.

In the case of group $G$ of type $E_8$ the quotient $\lambda\gamma(G/P)^{ss}$ is not isomorphic to a blow up. But here we can consider the quotient $\lambda\gamma(G/P)^{ss} \cap D_\alpha$ (for some $T$-invariant divisor $D_\alpha$ defined below), with the projection $p_0$ on $\mathbb{P}(V_1)$. Then there exists a $T$-invariant neighborhood of the point $s' \in G'/P' \subset \mathbb{P}(V_1)$, such that its preimage is isomorphic to a weighted blow up in the subvariety $G'/P'$. The local calculation of the proposition allows us to finish the proof similar to the case of the del Pezzo surfaces of the degree $> 1$.

For the convenience of the reader we supply the commutative diagram illustrating the scheme of the proof.

\[
\begin{array}{ccc}
\mathcal{T}' & \longrightarrow & (G/P)^{sf} \\
\downarrow_{\lambda\gamma} & & \downarrow_{\lambda\gamma} \\
\mathcal{T} & \longrightarrow & \text{Bl}(\mathbb{P}(V(\pi'_3)); G'/P') \\
\downarrow_{\sigma} & & \downarrow_{\sigma} \\
X_{\Delta} & \longrightarrow & \mathbb{P}(V'(\pi'_3)) \\
\downarrow_{\sigma} & & \downarrow_{\sigma} \\
X_{\Delta'} & \longrightarrow & G'/P' \\
\end{array}
\]

**Let us introduce an additional notation:**

Let $W = N_G(T)/T$ be the Weyl group of $G$, $C$ is the dominant Weyl chamber for the root system $\Delta$. We denote by $s_\gamma \in W$ the reflection corresponding to the root $\gamma$. Let $w \in W$, by $n_w$ we denote its representative in $N_G(T)$. Let $H$ be a semisimple subgroup in $G$, normalized by torus $T$. We denote by $W_H$ the Weyl group of $H$, we can assume that $W_H \subset W$. For the Lie algebra $\mathfrak{g}$ let us fix a standard basis that consists of $e_\gamma$, where $\gamma \in \Delta$, and $h_\vartheta$, where $\vartheta \in \Pi$.

Let $V$ be a $T$-module. By $\text{supp}(V)$ we denote the set of weights of the $T$-module $V$.

For the miniscule representation $V(\pi_\alpha)$ let us fix a basis that consists of the weight vectors $v(\omega)$, where $\omega \in \text{supp}(V(\pi_\alpha))$. By $V^*(-\pi_\alpha)$ we denote the dual
module to $V(\pi_\alpha)$ with the lowest weight $-\pi_\alpha$, by $\langle \cdot, \cdot \rangle$ we denote the canonical pairing, and by $\{v^* \langle - \omega \rangle\}$, its basis dual to $\{v(\omega)\}$.

For the birational morphism $p : X \to Y$ and a subvariety $Z \subset Y$ we denote by $p_*^{-1}(Z)$ the proper transform of $Z$ in $X$.

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1. The geometry of embeddings of the flag varieties in the projectivizations of minuscule representations

For describing geometry of the flag varieties we shall need the following lemma from the work of I.N. Berstein, I.M. Gelfand, S.I. Gelfand [4], describing the structure of Schubert varieties in this projective embedding.

Lemma 1.1. [4, 2.12] Let $w \in W$ be an element of the Weyl group, and $B^-w P/P$ is the corresponding Schubert cell. Consider the closed embedding $G/P \hookrightarrow \mathbb{P}(V(\chi))$ (where $\chi$ — is the dominant weight). Let $f \in V(\chi)$ is the vector from the orbit of the highest weight vector. Then $f \in B^-w P/P$ iff $w_\chi \in \text{supp}(f)$ and $f \in \Omega(b^-)v_{w_\chi}$, where $\Omega(b^-)$ is the universal enveloping of the Lie algebra $b^-$, and $v_{w_\chi} = wv_\chi$ is the vector of weight $w_\chi$ (that has multiplicity one in the representation $V(\chi)$).

Let us study the $G'$-submodules in the representation $V(\pi_\alpha)$.

Consider the restriction of the representation $V(\pi_\alpha)$ on the subgroup $G' \subset P$. Let us notice that the pairing $(\pi_\alpha; \cdot)$ defines the grading on $V(\pi_\alpha)$. Indeed, the vector $v = \sum v_\chi \in \text{supp}(V(\pi_\alpha))$ belongs to the component $V_i$ iff $(\pi_\alpha; \chi) = (\pi_\alpha; \pi_\alpha) - i$ for any $\chi$, such that $v_\chi \neq 0$.

There is a decomposition

$$V = \langle v(\pi_\alpha) \rangle \oplus V_1 \oplus V_2 \oplus V_{\geq 3}.$$  

The grading in consideration is $G'$-invariant. To prove this, it is sufficient to check that the action of the element $e_\tau \in \mathfrak{g}'$ of the standard basis $\mathfrak{g}'$, where $\tau \in \Delta'$, maps a weight vector $v_{\chi_0}$ to the weight vector $v_\chi$, where $(\pi_\alpha; \chi_0) = (\pi_\alpha; \chi)$. Let us notice that $e_\tau v_{\chi_0} = v_{\chi_0 + \tau}$, and also $(\pi_\alpha; \tau) = 0$, since $\mathfrak{g}'$ stabilizes the line $\langle v_\pi \rangle$. This gives the required assertion.

Since $\alpha$ is the only simple root with nontrivial pairing with $\pi_\alpha$ it is easy to see that

$$\text{supp}(V_1) = (\pi_\alpha - \alpha - \sum_{\gamma \in \Pi} Q_\gamma \cap \text{supp}(V(\pi_\alpha))),$$

$$\text{supp}(V_2) = (\pi_\alpha - 2\alpha + \sum_{\gamma \in \Delta'} Q \gamma) \cap \text{supp}(V(\pi_\alpha)).$$

The weight $\pi_\alpha - \alpha$ is dominant for the representation $G' : V_1$ (indeed $(\pi_\alpha - \alpha; \beta) = 1$ and $(\pi_\alpha - \alpha; \gamma) = 0$ for $\gamma \in \Pi'$). Thus $\text{supp} V_1 \subset \text{supp} V \subset s_\alpha (\pi_\alpha - \sum_{\gamma \in \Pi} Q_\gamma)$, that means that for $G'$-module $V_1$ we have $\text{supp} V_1 \subset \pi_\alpha - \alpha - \sum_{\gamma \in \Pi} Q_\gamma + \gamma$.

Remark 1.2. The component $V_{\geq 3}$ is nonzero only for the groups of type $E_7$ and $E_8$. In the case of $E_7$ there is an equality $V_{\geq 3} = V_3 = \langle v_{-\pi_\alpha} \rangle$. The equality $V_{\geq 3} = 0$ for the remaining groups follows from the fact that $\langle \pi_\alpha; \pi_\alpha - w_0 \pi_\alpha \rangle < 3$, where $w_0 \pi_\alpha$ is the lowest weight of representation $V(\pi_\alpha)$. 

Lemma 1.3. The representation $G' : V_1$ is irreducible miniscule representation. Let $G' : p^-_u$ be an adjoint representation of the semisimple part of the Levi subgroup $L$ on the unipotent radical $p^-_u$ of the parabolic subalgebra $p^-$. Then we have

- if $\Delta$ is distinct from type $E_8$, then we have isomorphism of the $G'$-modules $V_1 \cong V(\pi_1') \cong p^-_u$,
- in the case when $\Delta$ is the root system of type $E_8$, we have isomorphisms of the $G'$-modules $V_1 \cong V(\pi_2')$ and $V_1 \oplus \langle v_0 \rangle \cong p^-_u$, where $v_0 = e_{-\pi_5} v_{\pi_6}$ is the vector of weight zero (In the case of $E_8$ the weight $\pi_6 = \varepsilon_1 + \varepsilon_9$ is a root).

Proof. First let us prove that the representation $V_1$ is irreducible. Let us recall that all weights of $V_1$ have multiplicity one. (The weights $V_1$ are those weights $\tau$ of the representation $V(\pi)$ that belong to the hyperplane $\langle \pi, \pi - \tau \rangle = 1$. But the nonzero weights of $V(\pi)$ have multiplicity one.) If $V_1$ is reducible, then there are two subrepresentations with the different highest weights $\omega_1$ and $\omega_2$. Since the representation $V(\pi)$ is $G$-irreducible, the difference of these weights is the integral linear combination of simple roots from $\Delta$. In other words there exists a weight $\omega$ such that

$$
\omega = \omega_1 - n_\alpha \alpha - \sum_{\gamma \in I_1 \subset \Pi'} n_\gamma \gamma = \omega_2 - m_\alpha \alpha - \sum_{\gamma \in I_2 \subset \Pi'} m_\gamma \gamma
$$

where all the coefficients $n_\alpha, n_\gamma, m_\alpha, m_\gamma$ are nonnegative and the sets $I_1$ and $I_2$ do not intersect.

Let us notice that $(\pi; \omega_1) = (\pi; \omega_2) = (\pi; \alpha_0) - 1$, taking into account that $(\pi; \gamma) = 0$ for $\gamma \in \Pi'$ we get:

$$(\omega; \pi) = (\omega_1; \pi) - n_\alpha (\alpha; \pi) = (\omega_2; \pi) - m_\alpha (\alpha; \pi) - 1.$$

Thus $n_\alpha = m_\alpha$ and we can assume that $n_\alpha = m_\alpha = 0$.

Let us show that the weight $\omega$ is dominant with respect to $G'$. Let $\gamma_0 \notin I_2 \subset \Pi'$, then

$$(\omega; \gamma) = (\omega_2; \gamma_0) = \sum_{\gamma \in I_2 \subset \Pi'} m_\gamma (\gamma; \gamma_0) \geq 0,$$

since $\omega_2$ is dominant and $\langle \gamma; \gamma_0 \rangle \leq 0$ for $\gamma \neq \gamma_0$. The similar equality holds for $\gamma_0 \notin I_1 \subset \Pi'$. That is required. Since $\omega$ is dominant and can be obtained from $\omega_1$ and $\omega_2$ by subtracting simple roots, this weight belong to the weights of both irreducible modules, that implies that its multiplicity is at least 2. We come to a contradiction since all weights of the representation $V_1$ have multiplicity one.

To show that $V_1 \cong V(\pi_1')$ let us calculate the pairing of $\pi_6 - \alpha$ with the simple roots in $\Delta'$. As we have already seen $(\pi_6 - \alpha; \beta) = 1$ and $(\pi_6 - \alpha; \gamma) = 0$ for $\gamma \in \Pi''$, that is needed.

We notice that $p^+ u v(\pi_6) \subset V_{\geq 2}$ and that $p^- u v(\pi_6)$ is not contained in $V_{\geq 2}$. Indeed $p^- u$ contains the root $-\alpha$, hence $p^- u v(\pi_6)$ contains $e_{-\alpha} v(\pi_6) \in V_1$.

Comparing the dimensions we get $\dim V(\pi_1') = \dim V_1 = \dim (p^-_u)$. Taking into account that there exists a nontrivial projection of $G'$-module $p^-_u$ to $V_1$, we obtain the isomorphism of these modules in the first case.

In the second case when $\Delta$ is of the type $E_8$, the proof is similar. As before all weights of $V_1$ have multiplicity 1 (since the only weight with multiplicity in $V(\pi_6)$ is the zero weight but it lie in $V_2$). We also notice that $v_0 = e_{-\pi_5} v(\pi_6) = h_{-\alpha}$ is $G'$-fixed vector. The irreducibility of $V_1$ and the following equality on dimensions $\dim p^-_u = \dim V_1 + 1$ terminates the proof. \qed
Corollary 1.4. The intervals connecting the weight $\pi_\alpha$ with the weights $\pi_\alpha - \delta$, where $\delta \in \Delta_{P_\alpha}$ (corr. $\delta \in \Delta_{P_\alpha} \setminus \pi_\alpha$ in the case of type $E_8$), are the edges of weight polytop of the representation $V(\pi_\alpha)$.

Proof. Let us notice that the cone with the vertex $\pi_\alpha$, spanned by the roots $\delta \in \Delta_{P_\alpha}$ coincides with the cone spanned by the edges of polytop with the end in the vertex $\pi_\alpha$. The proposition follows from the fact that the edges in consideration are the edges of the cone with the vertex $\pi_\alpha$ over the convex polytop formed by the weights $\pi_\alpha - \delta$. One has to use the fact that they the vertices of the weight polytop of the representation $V_1$, that are also the vertices of the weight polytop of the representation $V(\pi_\alpha)$.

We omit the proof for the case of $E_8$ since it is similar. □

In the proposition we obtained that the miniscule representation $V_1$ of the group $G'$ includes in the quasiminiscule representation $V(\pi_\alpha)$. The next lemma relates the flag varieties $G/P$ and $G'/P'$, that lie in the projectivizations of these quasiminiscule representations.

Proposition 1.5. Let $V(\pi_\alpha)$ be a quasiminiscule representation, and $G/P$ be a flag variety that embeds in $\mathbb{P}(V(\pi_\alpha))$ as the projectivization of the orbit of the highest weight vector. Consider the decomposition of $V(\pi_\alpha)$ in the sum of irreducible $G'$-modules:

$$V(\pi_\alpha)|_{G'} = \langle v(\pi_\alpha) \rangle \oplus V_1 \oplus \ldots$$

Then the intersection $G/P \cap \mathbb{P}(V_1)$ is isomorphic to the flag variety $G'/P'$, embedded in $\mathbb{P}(V_1)$ as a $G'$-orbit of the point $\langle e^{-\alpha}v(\pi_\alpha) \rangle$.

Proof. Let $\langle v \rangle \in G/P \cap \mathbb{P}(V_1)$. Since $V_1$ is a miniscule representation, without a loss of generality we may assume that $v_{\pi_\alpha - \alpha} \neq 0$. Indeed we can find a component $v_\omega \neq 0$ for some weight $\omega$. Since the Weyl group of the root system $\Delta'$ is acting transitively on the weights of $V_1$, there exists an element $w$ of this Weyl group, translating the weight $\omega$ in $\pi_\alpha - \alpha$. Let us notice that any representative $n_w \in G'$ of the element $w$ maps $G/P$ and $P(V_1)$ into themselves. Besides the component of the weight $\pi_\alpha - \alpha$ of the vector $n_w v$ is nonzero (since the component $(n_w v)_{\pi_\alpha - \alpha}$ is proportional to $v_\omega$ with nonzero coefficient.) Instead of the point $\langle v \rangle$ we may consider the line $v_v(\langle v \rangle) \in G/P \cap \mathbb{P}(V_1)$.

Let us notice that if $v_{\pi_\alpha - \alpha} \neq 0$, then the component $(s_\alpha v)_{\pi_\alpha} = s_\alpha v_{\pi_\alpha - \alpha}$ of the vector $s_\alpha v$ is nonzero (here we used that $s_\alpha \pi_\alpha = \pi_\alpha - \alpha$). By the Lemma we get that $\langle s_\alpha v \rangle$ belongs to the open cell $P_u^- P/P$. The latter is equivalent to the equality $\langle u(s_\alpha v)_{\pi_\alpha} \rangle$ for $u \in P_u^-$. Applying the exponential presentation of the element $u \in P_u^-$ (i.e. $u = \exp(\sum_{\gamma \in \Delta_{s_\alpha}} c_\gamma e^{-\gamma})$ where $c_\gamma \in \mathbb{K}$ and $e^{-\gamma}$ is an element of the standard basis of the Lie algebra $g$), we can represent $\langle v \rangle$ in the form

$$\langle v \rangle = \langle s_\alpha(\exp(\sum_{\gamma \in \Delta_{s_\alpha}} c_\gamma e^{-\gamma})(s_\alpha v)_{\pi_\alpha}) \rangle = \langle \exp(\sum_{\gamma \in \Delta_{s_\alpha}} c_\gamma s_\alpha e^{-\gamma}) v_{\pi_\alpha - \alpha} \rangle =$$

$$= \langle v_{\pi_\alpha - \alpha} + \sum_{\gamma \in \Delta_{s_\alpha}} c_\gamma s_\alpha e^{-\gamma} v_{\pi_\alpha - \alpha} + \ldots \rangle.$$

Let us show that $\langle v \rangle$ belongs to a $G'$-orbit of the vector $v_{\pi_\alpha - \alpha}$. Assume that it is not so. Then in the exponential representation we can find $\gamma$, such that $s_\alpha \gamma \notin \Delta'$. The weight $\pi_\alpha - \gamma$ is extremal by the Corollary in other words $\gamma$ cannot be represented as a sum of at least two roots from $s_\alpha \Delta_{s_\alpha}$. Let us show that the vector $v$ has nonzero component of weight $s_\alpha(\pi_\alpha - \gamma) = \pi_\alpha - \alpha - s_\alpha \gamma$ that is equal to $c_\gamma e^{-s_\alpha \gamma} v_{\pi_\alpha - \alpha}$. From the exponential representation and extremality of the weight $\pi - \gamma$ we obtain that the component of weight $s_\alpha(\pi - \gamma)$ is equal to $c_\gamma e^{-s_\alpha \gamma} v_{\pi_\alpha - \alpha}$.
Assume that this component is zero. This will imply that the vector \( v_{\pi_\alpha - \alpha} \) is the lowest weight vector of the representation of the three dimensional algebra \( \mathfrak{sl}_2 \), generated by the triple \( \{ e_{-s_\alpha \gamma}, h_{s_\alpha \gamma}, e_{s_\alpha \gamma} \} \). That is impossible by the following chain of inequalities: \( (\pi_\alpha - \alpha; s_\alpha \gamma) = (s_\alpha \pi_\alpha; s_\alpha \gamma) = (\pi_\alpha; \gamma) > 0 \). Where the last inequality is due to the fact that \( \gamma \in p^\alpha_\mu \) and \( P \) is the stabilizer of the line spanned by the highest weight vector with the weight \( \pi_\alpha \).

Let us notice that the component of the vector \( v \) of weight \( \pi_\alpha - \alpha - s_\alpha \gamma \) does not belong to \( V_1 \). We come to the contradiction since in the exponential representation we have \( \gamma \), such that \( s_\alpha \gamma \notin \Delta' \). Thus we obtain that \( \sum_{\gamma \in \Delta s_\alpha} c_\gamma e_{-s_\alpha \gamma} \in g^\prime \). That means that \( \exp(\sum_{\gamma \in \Delta s_\alpha} c_\gamma e_{-s_\alpha \gamma}) \in U \cap G^\prime \).

From the above we have the inclusion \( G/P \cap P(V_1) \subseteq G'/\langle v_{\pi_\alpha - \alpha} \rangle \). And from the \( G' \)-invariance of the intersection we obtain the equality \( G/P \cap P(V_1) = G'/P \). □

**Proposition 1.6.** Let \( G \) be the group of one of the types considered above. And let \( i : G/P \subset P(V(\pi_\alpha)) \) be the embedding of the flag variety in the projectivization of miniscule representation. For the action of torus \( T \) consider the set of stable points \( (G/P)'' \) with respect to the sheaf \( i^*\mathcal{O}(1) \). Then the complement to the set \( (G/P)'' \) in \( G/P \) has the codimension \( > 1 \).

The codimension of the points from \( G/P \) with the stabilizer strictly containing \( Z(G) \) is strictly greater than \( 1 \).

**Proof.** Let us denote by \( W^s_\pi \) the set of elements \( w \in W \), such that \( \langle w \pi_\alpha; \lambda \rangle \leq 0 \) for every \( \lambda \in C \). By the Theorem 1.6 \([10]\) the set of stable points is described by the following formula:

\[
(G/P)^s = \bigcap_{w \in W} \bigcup_{w \in W^s_\pi} \tilde{w}BW/P.
\]

Let us note that in the Schubert decomposition for \( G/P \) there exists only one cell of codimension \( 1 \), i.e. \( BW_0s_\pi P/P \). Using the previous formula we see that for the proof of proposition it is sufficient to show that \( w_0s_\pi \in W^s_\pi \).

Let us set \( w_0 \lambda = -\sum a_\gamma \pi_\gamma \in -C \), where \( a_\gamma \geq 0 \). The following calculation shows that \( w_0s_\pi \in W^s_\pi \):

\[
-\langle w_0s_\pi; \lambda \rangle = (\pi_\alpha - \alpha; -w_0 \lambda) = a_\alpha (\pi_\alpha; \pi_\alpha) - \frac{\alpha; \alpha}{2} + \sum_{\gamma \neq \alpha, \gamma \in \Pi} a_\gamma (\pi_\alpha; \pi_\gamma) > 0,
\]

where we used the fact that in considered cases \( (\pi_\alpha; \pi_\alpha) - \frac{\alpha; \alpha}{2} > 0 \), and that \( (\pi_\alpha; \pi_\gamma) > 0 \) for a simple root system \( \Delta \) (see also \([2]\) Prop. 2.4).

For the last assertion see \([2]\) Prop. 2.4. □

### 2. The Quotient of \( G/P \) by the One-Parameter Subgroup \( \lambda \)

Let us define the one-parameter subgroup \( \lambda \) corresponding to the weight \( \pi_\alpha \), by means of the pairing with the weights \( \chi \in \Xi(T) \) by the following formula:

\[
\chi(\lambda(t)) = t^{\langle \pi_\alpha; \chi \rangle}.
\]

Consider the action \( \lambda \) on \( G/P \) by the left translations. Let us fix a linearization of the action \( \lambda \), defined by a \( G \)-linearized sheaf \( i^*\mathcal{O}(1) \), where \( i : G/P \rightarrow P(V(\pi_\alpha)) \) is the \( G \)-equivariant embedding. We can consider the set of stable points \( (G/P)^s \) with respect to the action \( \lambda \) (from the proof of the next theorem it will be seen that the set of stable points \( (G/P)^s \) in the case of root system distinct from \( E_8 \), coincide with the set \( (G/P)^{ss} \) of semistable points). By \( p_{\pi_\alpha} : (G/P)^s \rightarrow \lambda \backslash (G/P)^s \) we denote a quotient by the action of one-parameter subgroup \( \lambda \).
2.1. Let $G$ be a simple group of one of the considered types except $E_8$. The geometric quotient $\tilde{X} = \lambda\backslash(G/P)^s$ is equal to the blow up of $\mathbb{P}(V_1)$ in the flag variety $G'/P'$.

Proof. We shall need a following lemma.

Lemma 2.2. There exist a projection $p_0$ from the considered quotient $\tilde{X}$ to $\mathbb{P}(V_1)$.

Proof. Let $\langle v \rangle \in (G/P)^{ss}$. To the point $\langle v \rangle$ we assign the point $\langle v_1 \rangle \in \mathbb{P}(V_1)$, where $v_1$ is the projection of $v$ to $V_1$ along the subspace $\mathbb{K}v(\pi_\alpha) \oplus V_{\geq 2}$.

Let us notice that this map factors through $\tilde{X}$. This is a corollary of universal property of quotient $\lambda$ and the fact that action on $\mathbb{P}(V_1)$ is trivial (the action of $\lambda(t)$ multiplies the vectors from $V_1$ by $t^{(\pi_\alpha, \pi_\alpha - \alpha)} = t^{(\pi_\alpha, \pi_\alpha - 1)}$).

It is left to check that $p_0$ is well defined on $(G/P)^{ss}$. To prove this let us notice that for the groups of considered type $G$ (except $E_8$) the linear subspace $\langle v_{\pi_\alpha} \rangle \oplus V_1$ has positive weights with respect to the action of $\lambda$, and the subspaces $V_2 \oplus V_{\geq 3}$ has negative weights (in the case of $E_8$ the subspace $V_2$ has zero weight and $V_{\geq 3}$ has negative). Thus for the stability of $\langle v \rangle$ it is necessary that the component of $v$ in the subspace $\langle v_{\pi_\alpha} \rangle \oplus V_1$ is not equal to zero. Let the map $p_0$ is not defined in $\langle v \rangle$, i.e. $v_1 = 0$. Then since $\langle v \rangle \in (G/P)^{ss}$ we have $v_{\pi_\alpha} \neq 0$. Thus we obtain that the vector $v$ belongs to the open cell $P^{-}_yP/P$. Let us represent $v$ as the exponential map from the element of the Lie algebra $p_{\pi_\alpha}$, applied to the vector $v_{\pi_\alpha}$:

$$v = \exp(\sum_{\gamma \in \Delta_{\pi_\alpha}} c_\gamma e^{-\gamma}v_{\pi_\alpha} = v_{\pi_\alpha} + \sum_{\gamma \in \Delta_{\pi_\alpha}} \frac{c_\gamma e^{-\gamma}}{1!} v_{\pi_\alpha} + \frac{(\sum c_\gamma e^{-\gamma})^2}{2!} v_{\pi_\alpha} + \ldots$$

Since $v_1 = 0$ all the coefficients $c_\gamma$ are equal to zero (the vectors $e^{-\gamma}v_{\pi_\alpha}$ for $\gamma \in \Delta_{\pi_\alpha}$ have different weights and form the basis $p_{\pi_\alpha}v_{\pi_\alpha} \cong V_1$). Thus $\langle v \rangle = \langle v_{\pi_\alpha} \rangle$, but the latter point is unstable.

We split the proof of the theorem into a few steps. First we construct the quotient of stable orbits lying in the open cell, then for the orbits lying in the complement of the open cell (Steps 1, 2). Then using the projection $p_0$ onto the quotient $\tilde{X}$ and the Moishezon contraction theorem [3], we obtain that these sets glue together in a blow up (Step 3).

**Step 1.** Consider the open cell $P^{-}_yP/P$. Let us show that the quotient $\lambda\backslash(P^{-}_yP/P)$ is isomorphic to the projective space $\mathbb{P}(V_1)$ with the deleted flag variety $G'/P$.

As before let us use an exponential representation $v$:

$$v = \exp(\sum_{\gamma \in \Delta_{\pi_\alpha}} c_\gamma e^{-\gamma}v_{\pi_\alpha} = v_{\pi_\alpha} + \sum_{\gamma \in \Delta_{\pi_\alpha}} \frac{c_\gamma e^{-\gamma}}{1!} v_{\pi_\alpha} + \frac{(\sum c_\gamma e^{-\gamma})^2}{2!} v_{\pi_\alpha} + \ldots$$

Let us notice that the set of coefficients $\{c_\gamma\}_{\gamma \in \Delta_{\pi_\alpha}}$ considered up to a multiplication by a constant define the orbit of $\lambda$, thus the quotient $\lambda\backslash((P^{-}_yP/P) \setminus \{v_{\pi_\alpha}\})$, is identified with $\mathbb{P}(V_1)$. To obtain that $\lambda\backslash((P^{-}_yP/P)^{ss})$ we have to delete from $\mathbb{P}(V_1)$ those points $(c_{\gamma_1} : \ldots : c_{\gamma_{\dim V_1}})$ for which the corresponding orbits $\lambda(t)\langle v \rangle$ are unstable.

The orbit of the point $\langle v \rangle$ from the open cell is unstable iff $v$ belongs to $\mathbb{K}v(\pi_\alpha) \oplus V_1$. Let us prove the following lemma:

**Lemma 2.3.** Let $v \in \mathbb{K}v(\pi_\alpha) \oplus V_1$. Then we have an inequality:

$$\lim_{t \to 0} \lambda(t)(G/P \cap \mathbb{P}(\mathbb{K}v(\pi_\alpha) \oplus V_1)) = (G/P) \cap \mathbb{P}(V_1) = G'/P'.$$
Proof. Applying Proposition 1.5 we see that we need to prove the following limit:

$$\lim_{t \to 0} \lambda(t) \langle v \rangle = \lim_{t \to 0} \langle (\pi_{\alpha}; \gamma) v_{\pi_{\alpha}} + \sum_{i \in W} e^{\gamma} e^{-\gamma} e\gamma \rangle =$$

$$= \lim_{t \to 0} (t \pi_{\alpha} + \sum_{i \in W} e^{\gamma} e^{-\gamma} e\gamma) = \langle \sum_{i \in W} e^{\gamma} e^{-\gamma} e\gamma \rangle \in \overline{P}(V_{l}).$$

Here we used that $\langle \pi_{\alpha}; \gamma \rangle = 1$ for all $\gamma \in \Delta_{P^\circ}$. We also notice that $\langle v_1 \rangle = \lim_{t \to 0} \lambda(t) \langle v \rangle$.

By Lemma 2.3 we get that for $\langle v \rangle \notin (P^\circ P/P)^{ss}$ we have the inclusion $\langle v_1 \rangle \in G'/P'$. Let us show that this condition is also sufficient. Consider the vector $v = \exp(e_{-\alpha}) v_{\pi_{\alpha}} = v_{\pi_{\alpha}} + e_{-\alpha} v_{\pi_{\alpha}}$. It is easy to see that $\langle v \rangle \notin (G/P)^{ss}$. Since $G'$ and $\lambda$ commute the orbit $G' \langle v \rangle$ is unstable. Besides the projection is equal to $p_0(G' \langle v \rangle) = G' \langle v_1 \rangle = G'/P' \subset \overline{P}(V_l)$.

From what was said above and from isomorphism $\lambda \langle (P^\circ P/P \setminus \{v_{\pi_{\alpha}}\}) \cong \overline{P}(V_l)$ we get that unstable points project by means of $p_0$ exactly onto $G'/P'$. Thus we proved that $\lambda \langle (P^\circ P/P)^{ss} \cong \overline{P}(V_l) \setminus (G'/P')$.

**Step 2.** The space of stable orbits from the complement to $P^- P/P$ is isomorphic to the projectivization of the conormal bundle in $\overline{P}(V_l)$ to the flag variety $G'/P'$.

From Lemma 1.1 and 2.2 we get that the stable orbits from the complement of to $P^- P/P$ satisfy the following condition:

- The component of vector $v$ of weight $\pi_{\alpha}$ is equal to zero (since $\langle v \rangle \notin P^- P/P$).
- The component of vector $v$ that belong to $V_1$ is nonzero (by the stability of $\langle v \rangle$).

In particular these orbits belong to the closure of the Schubert divisor $B^- \pi_{\alpha} P/P$. Indeed in the Bruhat decomposition of $G/P$ there is a unique cell of codimension one $B^- \pi_{\alpha} P/P$; and it contains in its closure all cells of smaller dimension.

**Lemma 2.4.** Let $\langle v \rangle \in \overline{P}(V_{\pi_{\alpha}})$ be a stable point such that the component of vector $v$ of the weight $\pi_{\alpha}$ is nonzero. Then the component $v_1$ of vector $v$ that belong to $V_1$ is distinct from zero and $\langle v_1 \rangle \in \overline{P}(V_l) \cap G/P$.

Proof. Consider the decomposition $v = \sum_{i \geq 1} v_i$ for $v_i \in V_i$, besides we are given $v_1 \neq 0$. Then we have an equality:

$$\lim_{t \to \infty} (\lambda(t) v) = \lim_{t \to \infty} (v_1 + \sum_{i > 1} t^{-i+1} v_i) = \langle v_1 \rangle.$$

Since $\lim_{t \to \infty} (\lambda(t) v) \in G/P$ this proves the lemma.

Let us notice that the action of $G'$ commutes with $\lambda(t)$, and the intersection $\overline{P}(V_l) \cap G/P = G'/P'$ is a single orbit by Lemma 1.5. By Lemma 2.1 for the component $v \in V_1$ of vector $v$ we have the inclusion $\langle v \rangle \in G'/P'$. Acting by an element from the group $G'$ we may assume that $v_1 = v_{\pi_{\alpha}}$. Let us represent an element $v$ in terms of exponential map (from the element of $u$) applied to the vector $v_{\pi_{\alpha}}$.

$$v = v_{\pi_{\alpha}} + \sum_{i \in W} e^{\gamma} e^{-\gamma} e\gamma v_{\pi_{\alpha}} + \frac{(\sum_{i \in W} e^{\gamma} e^{-\gamma} e\gamma)^2}{2!} v_{\pi_{\alpha}} + \ldots.$$

By the assumptions we have $\pi_{\alpha} - \alpha - \gamma \notin \text{supp}(V_1)$ that implies that the coefficients $c_i$ can be nonzero only for the roots of type $\gamma = m\alpha + k\beta + \sum_{0 \in \Pi^{\circ}} a_0 \theta$ where $a_0 \geq 0$ only when $m, k > 0$.

Since $P'$ stabilizes $\langle v_{\pi_{\alpha}} \rangle$ the vectors with the weights $\pi_{\alpha} - \alpha + \gamma$ form a $P'$-module $N$. (The vector subspace $N$ is isomorphic to the module $\overline{P}_{\alpha} v_{\pi_{\alpha}}$. The
structure of the $P'$-module $p_u^- v_{\pi_\alpha - \alpha}$ is defined as follows. For $p' \in P'$ taking into account the equality $p' v_{\pi_\alpha - \alpha} = v_{\pi_\alpha - \alpha}$ we get $p' p_u^- v_{\pi_\alpha - \alpha} = (\text{Ad}(p')) p_u^- (p' v_{\pi_\alpha - \alpha}) \subset p_u^- v_{\pi_\alpha - \alpha}$, where $\text{Ad}(P') : p_u^-$ is an adjoint representation.)

Let us prove the following lemma on the structure of the $P'$-module $N$.

Lemma 2.5. Let $\gamma \in \Delta_{p_u}$ be a positive root such that $\pi_\alpha - \alpha + \gamma \in \text{supp}(N)$. Consider a decomposition $\gamma = m \alpha + k \beta + \sum_{\theta \in \Pi''} a_\theta \gamma$ of the root $\gamma$ in the sum of simple roots. Then in this decomposition we have $k = 2$, $m = 1$. We have the inclusion $N \subset V_2$. And the equality $(\alpha; \gamma) = 0$ take place.

The module $N$ is a simple $P'$-module with the trivial $P'_u$-action. As $G'' \times P'_u$-module it is isomorphic to the fiber of the conormal bundle to the flag variety $G' / P' \subset \mathbb{P}(V_1)$ in the point $\langle v(\pi_\alpha - \alpha) \rangle$.

Proof. Let us denote by $V'_1$ and $V'_2$ the irreducible $L'$-submodules in $V_1$ that are the graded components of the weights $\langle \pi'_2; \pi'_2 \rangle - 1$ and $\langle \pi'_3; \pi'_3 \rangle - 2$ correspondingly with respect to the pairing with $\pi'_3$.

First let us notice that as $L'$-module the fiber of cotangent bundle to $G' / P'$ in the point $\langle v(\pi_\alpha - \alpha) \rangle$ can be identified with the module $p_u^- v(\pi_\alpha - \alpha)$, that is isomorphic to $V'_1$ by Lemma 1.3. Thus the fiber of conormal bundle in the point $\langle v(\pi_\alpha - \alpha) \rangle$ is identified with factor module $V_1/(\mathbb{K}v(\pi_\alpha - \alpha) \oplus V'_1)$. By the Remark 1.2 we have isomorphism of $L'$-modules $V'_2 \cong V_1/(\mathbb{K}v(\pi_\alpha - \alpha) \oplus V'_1)$. Since the fiber of conormal bundle as $L'$-module is isomorphic to a simple module $V'_2$, $P'_u$ is acting on it trivially and $Z(L')$ is acting by the multiplication with the scalar. Thus for the proof of the last part of the lemma it is sufficient to check the isomorphism of $G''$-modules $N$ and $V'_2$.

Let the coefficient $c_\gamma$ is not equal to zero for the root $\gamma = m \alpha + k \beta + \sum_{\theta \in \Pi''} a_\theta \gamma$ (where $a_\theta > 0$). Since the weight $s_\alpha \pi_\alpha - \gamma$ belongs to the weight polytop, the weight $\pi_\alpha - s_\alpha \gamma$ also lie in this polytop. Thus we obtain that $s_\alpha \gamma \in \Delta_{p_u}$. Using the equality $s_\alpha \beta = \beta + \alpha$ and the fact that $s_\alpha$ fixes $\theta \in \Pi''$ we obtain

$$s_\alpha \gamma = -ma + k(\beta - \alpha) + \sum_{\theta \in \Pi''} a_\theta \gamma = k\beta + (k-m)\alpha + \sum_{\theta \in \Pi''} a_\theta \gamma$$

Since $s_\alpha \gamma \in \Delta_{p_u}$ from Lemma 1.3 it follows that the coefficient of $\alpha$ in the decomposition of $s_\alpha \gamma$ in the sum of simple roots is equal to 1 that implies $k - m = 1$. Since $V_1$ is miniscule representation, for the root systems of types $A_4$, $D_5$ or $E_6$ then the coefficient by $\beta$ in the decomposition of $s_\alpha \gamma$ is not greater than 2 (this follows from the fact that the vector of weight $s_\alpha \gamma - \alpha$ belong to the graded component with the weight not less than $\langle \pi_3; \pi_3 \rangle - 2$ with respect to the pairing $\langle \pi_3; \gamma \rangle$, see Remark 1.2). Thus we get $m = 1$ and $k = 2$. Since the coefficient of $\beta$ in this decomposition is equal to 2 the root $s_\alpha \gamma$ belongs to $V'_2$. In other words we get that the element $s_\alpha$ maps the module $N$ in $V'_2$. Since the element $s_\alpha$ is acting trivially on $G''$ by the conjugations it gives the map of the $G''$-module $N$ into the irreducible $G''$-module $V'_2$. Since $s_\alpha^2 = 1$ this map gives the isomorphism of these modules. That is required.

We are finished by providing the following calculation

$$(\alpha; \gamma) = (\alpha; \alpha + 2\beta + \sum_{\theta \in \Pi''} a_\theta \gamma) = (\alpha; \alpha) + 2(\alpha; \beta) = 0,$$

where we used the equalities $(\alpha; \beta) = -1$, $(\alpha; \theta) = 0$ for $\theta \in \Pi''$.

Since $L'$ differs from $G''$ by the central torus from the isomorphism of $G''$-modules $V'_2$ and $N$ we get the isomorphism of $L'$-varieties $\mathbb{P}(V'_2)$ and $\mathbb{P}(N)$. □
Proposition 2.6. The orbits $\langle \lambda(t) v \rangle$ of the points $\langle v \rangle \in (G/P)^{ss}$ with the condition $v_{\pi_\alpha} = 0$ are parameterized by the variety $G' \ast_{P'} P(\mathbb{N})$. That is isomorphic to the projectivization of conormal bundle to $G'/P'$.

Proof. It is easy to see that the orbits $\langle \lambda(t) v \rangle$ with the conditions $v_{\pi_\alpha} = 0$ and $v_1 = v_{\pi_{\alpha-\alpha}}$ are parameterized by the projective space $P(\mathbb{N})$. Since $N \subset V_2$ the one-parameter subgroup $\lambda(t)$ is acting on $N$ by the multiplication with $t^{-2}$, and on the vector $v_{\pi_{\alpha-\alpha}}$ by the multiplication with $t^{-1}$. Thus the map

$$
\langle v \rangle \rightarrow \left( \langle v, v^*(-\pi_\alpha + \alpha + \gamma_1) \rangle : \ldots : \langle v, v^*(-\pi_\alpha + \alpha + \gamma_{\dim N}) \rangle \right)
$$

identifies the considered space of $\lambda(t)$-orbits and $P(\mathbb{N})$. (Let us notice that $\frac{\langle v, v^*(-\pi_\alpha + \alpha + \gamma) \rangle}{\langle v, v^*(-\pi_\alpha + \alpha) \rangle}$ is the coefficient $c_\gamma$ by the vector $e^{-\gamma v_{\pi_{\alpha-\alpha}}}$ in the exponential representation for $\langle v \rangle$.) By the Lemma 2.5 $P(\mathbb{N})$ is isomorphic to the projectivisation of the fiber of the conormal bundle in the point $\langle v_{\pi_{\alpha-\alpha}} \rangle$.

By Lemma 2.5 the variety of orbits $\langle \lambda(t) v \rangle$ such that $v_{\pi_{\alpha-\alpha}} = 0$ projects surjectively by means of $p_0$ onto the flag variety $G'/P'$. Since the projection $p_0$ is $G'$-equivariant we obtain that $G' \ast_{P'} p_0^{-1}(\langle v_{\pi_{\alpha-\alpha}} \rangle) \cong G' \ast_{P'} P(\mathbb{N})$ (where we used the isomorphism of $P'$-varieties $p_0^{-1}(\langle v_{\pi_{\alpha-\alpha}} \rangle) \cong P(\mathbb{N})$ from the paragraph).

The homogeneous bundle $G' \ast_{P'} P(\mathbb{N})$ is isomorphic to the projectivization of the conormal bundle to $G'/P'$. Indeed by Lemma 2.5 the fibers over $eP'/P'$ of the considered bundles are isomorphic as $P'$-varieties. By the $G'$-equivariance we get the isomorphism of the projective bundles. $\square$

Step 3. Denote by $D_N \subset X$ the divisor corresponding to the subvariety $G' \ast_{P'} P(\mathbb{N})$ in $X$ and by $\mathcal{L}_N$ the line bundle corresponding to this divisor.

To finish the proof of the theorem 2.1 we shall use Moishezon contraction theorem (8).

To apply it we need to proof the following proposition.

Proposition 2.7. For every point $x \in D_N$ the restriction of the line bundle $\mathcal{L}_N$ on the fiber $p_0^{-1}(p_0(x))$ (that is the fiber of the projectivization of the normal bundle $\mathcal{N}$ to the flag variety $G'/P'$) is isomorphic to the line bundle $\mathcal{O}(-1)$.

Proof. First let us notice that we can assume that the line bundle $\mathcal{L}_N$ is $G'$-linearized. The divisor $D_N$ is invariant with respect to the action of $G'$. The projection $p_0$ is $G'$-equivariant and the latter group is acting transitively on $p_0(D_N)$ $\cong G'/P'$. From the above it is sufficient to check the condition of the Moishezon contraction theorem only for one point $x \in D_N$.

Let us describe the line bundle $\mathcal{L}_N$. Let us recall that the preimage of the divisor $D_N$ by the quotient morphism is the divisor that is equal to the intersection of the Schubert divisor $B^- s_{\alpha} P/P$ and $(G/P)^{ss}$. The line bundle corresponding to this divisor can be described as $\mathcal{L} = G \ast_P k_{\pi_\alpha}$ (where $k_{\pi_\alpha}$ is one dimensional module where $P$ is acting by multiplication with the character $\pi_\alpha$). Let us notice that the section of the line bundle $\mathcal{L}_N$ can be considered as the function on $V_{\pi_\alpha}$ (since the linear system corresponding to $\mathcal{L}_N$ defines an embedding $G/P \subset \mathbb{P}(V_{\pi_\alpha})$).

The section $\mathcal{L}_N$ those zero set is equal to $B^- s_{\pi_\alpha} P/P$ is described by the equation $\langle v^*_{\pi_{\alpha-\alpha}} \rangle = 0$, it is semiinvariant with respect to the action of $B^-$ with the weight $-\pi_\alpha$ (cf. for example [10]).

We want to take the decent of the sheaf $\mathcal{L}_N$ to the sheaf $\tilde{\mathcal{L}}_N$ on the quotient of the set $(G/P)^{ss}$ by the one-parameter subgroup $\lambda$ in such way that the Weyl divisor defined by the section $\langle v^*_{\pi_{\alpha-\alpha}} \rangle = 0$ after the quotient morphism $p_{\pi_\alpha}$ maps in the section of the line bundle $\mathcal{L}_N$ (cf. [13]).
We can obtain this by making the section \( (u^*_{\pi - \gamma}) = 0 \) invariant with respect to \( \lambda \). This can be achieved by taking instead of the line bundle \( \mathcal{L}_N \) the line bundle \( \mathcal{L}_N \otimes k_{-\pi} \) isomorphic to the previous one with the action of \( \lambda \) twisted by the character \(-\pi_\alpha\). The image of the divisor \( B^N_{\pi}s_\alpha P/P \) on the quotient \( X \) is defined by the section \( (v^*_{-\pi;}) = 0 \) of the line bundle \( \lambda^\#(\mathcal{L}_N \otimes k_{-\pi}) \).

For the fiber of \( p_0 \) on which we want to restrict the line bundle \( -\mathcal{L}_N \) we chose the fiber over the point \( v_{\pi - \alpha} \). Then the fiber over this point is defined by \( \exp(\sum_{\pi - \alpha - \gamma \in \mathbb{N}} c_{\gamma}e_{-\gamma})s_\alpha P/P \). The restriction of the line bundle \( -\mathcal{L}_N \otimes k_{\pi_\alpha} \) to this fiber is defined by the formula

\[
(-\mathcal{L}_N \otimes k_{\pi_\alpha})|_{D_N} = \exp(\sum_{\pi - \alpha - \gamma \in \mathbb{N}} c_{\gamma}e_{-\gamma})s_\alpha P \ast_P (k_{-\pi} \otimes k_{\pi_\alpha}).
\]

This is the line bundle over the affine space \( \mathbb{N} \) and its quotient by \( \lambda \) is the line bundle over the projective space \( \mathbb{P}(\mathbb{N}) \). To prove that the latter line bundle is isomorphic to \( \mathcal{O}(1) \) one can check that when we multiply the coordinates \( \{c_\gamma\} \) by \( t \) the fiber of the line bundle is multiplied by \( t^{-1} \). This assertion follows from the line of equalities:

\[
\begin{align*}
\lambda(t)^{-1}(-\mathcal{L}_N \otimes k_{\pi_\alpha})|_{D_N} &= \lambda(t)^{-1}(\exp(\sum_{\pi - \alpha - \gamma \in \mathbb{N}} c_{\gamma}e_{-\gamma})s_\alpha P \ast_P k_{-\pi} \otimes \lambda(t)^{-1}k_{\pi_\alpha}) = \\
&= (\exp(\sum_{\pi - \alpha - \gamma \in \mathbb{N}} c_{\gamma}e_{-\gamma})s_\alpha P \ast_P k_{-\pi} \otimes \lambda(t)^{-1}k_{\pi_\alpha}) = \\
&= (\exp(\sum_{\pi - \alpha - \gamma \in \mathbb{N}} c_{\gamma}e_{-\gamma})s_\alpha P \ast_P t^{-1}(\pi - \alpha - \pi_\alpha)k_{-\pi} \otimes t^{-1}(\pi - \alpha - \pi_\alpha)k_{\pi_\alpha}) = \\
&= (\exp(\sum_{\pi - \alpha - \gamma \in \mathbb{N}} c_{\gamma}e_{-\gamma})s_\alpha P \ast_P t^{-1}(\pi - \alpha - \pi_\alpha)k_{-\pi_\alpha},
\end{align*}
\]

where we used the equalities \( \langle \alpha, \pi_\alpha \rangle = 1 \), \( \lambda(t)^{-1}e_{-\gamma}(t) = t^{\langle \pi - \gamma, e_{-\gamma} \rangle} \) and also \( (s_\alpha \lambda)(t) = s_\alpha \lambda(t)s_\alpha = t^{\langle \pi - \alpha - \pi_\alpha \rangle} = t^{\langle \pi - \alpha - \pi_\alpha \rangle} \).

We apply the Moishezon contraction theorem to the morphism \( p_0 : X \rightarrow \mathbb{P}(V_1) \) and exceptional divisor \( D_N \) that contracts onto the flag variety \( G'/P' \). We obtain that \( X \) is the blow up of \( \mathbb{P}(V_1) \) in the flag variety \( G'/P' \). That finishes the proof of the theorem.

**Remark 2.8.** From the proof of the previous theorem we can get the description of \( G'/P' \subset \mathbb{P}(V_1) \) as the set of zeros of the homogeneous system of equations (generating the ideal \( \mathcal{I}_{G'/P'} \)). First let us define the open cell \( P_+^\alpha P/P \) from \( G/P \) as the exponential map of the Lie algebra \( p_+^\alpha \) applied to \( v_{\pi_\alpha} \):

\[
v = \exp(\sum_{\gamma \in \Delta_{P_+}} c_{\gamma}e_{-\gamma})v_{\pi_\alpha} = v_{\pi_\alpha} + \sum_{\gamma \in \Delta_{P_+}} \frac{c_{\gamma}e_{-\gamma}}{1!}v_{\pi_\alpha} + \frac{(\sum_{\gamma \in \Delta_{P_+}} c_{\gamma}e_{-\gamma})^2}{2!}v_{\pi_\alpha} + \ldots.
\]

As we observed the set of nonstable points with respect to \( \lambda(t) \) is defined by the condition \( V_2 = 0 \) that implies that the component in the subspace \( V_{>2} \) also should be zero. These conditions can be rewritten as the system of equations on \( \{c_\gamma\} \):

\[
p_\mu(c) = \sum_{\pi - \gamma - \delta = \mu} c_{\gamma}c_\delta(e_{-\gamma}e_{-\delta}v_{\pi_\alpha}) = 0,
\]

for \( \mu \in \text{Supp}(V_2) \). Let us notice that the weights of all quadratic polynomials are different. I.e. the component \( \mathcal{I}_{G'/P'}(\mu) \) of the weight \( \mu \) in the ideal \( \mathcal{I}_{G'/P'} \) has the dimension 1.
In the Lemma 2.3 it was shown that
\[
\lim_{t \to 0} \lambda(t)(P^-/P \cap \mathbb{P}(Kv(\pi_\alpha) \oplus V_1)) = G/P \cap P(V_1) = G'/P'.
\]
Since the action of \(\lambda(t)\) do not change the set of homogeneous coordinates \(\{c_\gamma\}\) the flag variety \(G'/P'\) in the projective space \(P(V_1)\) is defined by the system of homogeneous equations \(p_\mu(c) = 0\) \((\mu \in \text{Supp}(V_2))\).

Later we shall need the equations corresponding to the vanishing of the components lying in \(V_3\):
\[
q_\nu(c) = \sum_{\pi_\alpha - \mu_1 - \mu_2 - \mu_3 = \nu} c_{\mu_1} c_{\mu_2} c_{\mu_3} (e_{-\mu_1} e_{-\mu_2} e_{-\mu_3} v_{\pi_\alpha}) = 0,
\]
that vanish on the flag variety \(G'/P'\). In the case of \(E_7\) it is a single equation for which \(\nu = -\pi_\alpha\).

3. **The quotient by the subgroup \(\lambda\) of the flag variety \(G/P\) from \(\mathbb{P}(g)\) in the case of type \(E_8\)**

Let us study the case when the root system \(\Delta\) has the type \(E_8\). Instead of miniscule representations \(E_8\) we should consider the case of adjoint representation \(Ad : g\). Our first task is to study a \(P'\)-module \(N = \langle p^- v(\pi_\alpha - \alpha) \rangle\) (cf. fig.2).

The module \(N\) consists of the weights of type \(s_\alpha^p \pi_\alpha - \gamma = s_\alpha \pi_\alpha - m\alpha - k\beta - \sum_{\theta \in \Pi''} a_{\theta} \theta\) where \(m, k > 0\) and \(a_{\theta} \geq 0\). Our aim to define the values of \(m, k\) and to find decomposition of \(N\) in the irreducible \(G''\)-submodules. We shall need the decomposition of \(V_1\) in the sum of simple \(G''\)-modules:
\[
V_1 = \mathbb{K}v(\pi_\alpha - \alpha) \oplus V'_1 \oplus V'_2 \oplus V'_3.
\]

On the figure 2b) we have the orthogonal projection of the weight polytop on the plane generated by \(\alpha\) and \(\pi_\alpha\) (since \(\pi_\beta = \pi_\alpha/2 - \alpha\), supp(\(V'_1\)) projects in the point).

**Proposition 3.1.** Let \(\gamma \in \Delta_{p^-}\) be a positive root such that \(\pi_\alpha - \alpha + \gamma \in \text{supp}(N)\). Consider the decomposition \(\gamma = ma + k\beta + \sum_{\theta \in \Pi'} a_{\theta} \theta\) of the root \(\gamma\) in the sum of simple roots, where \(a_{\theta} \geq 0, m, k > 0\). Then we have the following possibilities for the coefficients \(m, k\):
• The vector with the weights $\pi_\alpha - \alpha + \gamma$ with $m = 1, k = 2$, belong to $V_2$ and form a simple $G''$-module isomorphic to $V'_2$. In this case we have an equality $(\alpha; \gamma) = 0$.

• Let $m = 1, k = 3$. This condition is satisfied by the unique vector $h_{\pi_\alpha - \alpha}$ with zero weight. There is an isomorphism of $G''$-modules $\mathcal{N} \cap V_2 \cong \mathbb{K} h_{\pi_\alpha - \alpha} \oplus V'_2$.

• Let $m = 2, k = 3$. This condition is satisfied by a unique vector $v(-\alpha) \in V_3$.

The projective space $\mathbb{P}(\mathcal{N} \cap V_2)$ is isomorphic as $P'$-variety to the projectivization of the fiber of normal bundle to the flag variety $G'/P' \subset \mathbb{P}(V_1)$ in the point $\langle v(\pi_\alpha - \alpha) \rangle$.

Proof. Let us represent the weights of the module $\mathcal{N}$ in the form $s_\alpha \pi_\alpha - \gamma = s_\alpha \pi - m\alpha - k\beta - \sum \theta \in \Pi'' a_\theta \theta$ for $m, k > 0$. We apply to these weights the reflection $s_\alpha$. The weight $s_\alpha \pi_\alpha$ maps to the $\pi_\alpha$. Let us notice that the weights $\pi - s_\alpha \gamma$ belong to the weight polytop. Since $s_\alpha \gamma$ is a root by Lemma 1.3 we get $s_\alpha \gamma \in \mathfrak{p}_\alpha$, that is equivalent to $s_\alpha \pi_\alpha - \gamma \in V_1 \oplus \mathbb{K} e_{-\pi_\alpha} v(\pi_\alpha)$. In the adjoint representation we have:

$e_{-\pi_\alpha} v(\pi_\alpha) = \langle e_{-\pi_\alpha}; e_{\pi_\alpha} \rangle = h_{\pi_\alpha} \in \mathfrak{h}$.

We have the equality

$s_\alpha \gamma = -m\alpha + k\alpha + k\beta + \sum \theta \in \Pi'' a_\theta \theta$.

Let us notice that $k \leq 3$ since the miniscule representation $G'': V_1$ has the grading not greater than 3 with respect to the pairing with $\pi_\beta$ (the pairing with $\pi_\beta$ is not less than $\langle \pi_\beta; \pi_\beta \rangle - 3$). Let us describe the vectors $\gamma$ and the weights of the module $\mathcal{N}$. Consider the following cases:

Let $\pi_\alpha - s_\alpha \gamma = 0 (k - m = 2)$. From this we get that the weight $(\pi_\alpha - \alpha) - \gamma$ is equal to zero. The corresponding vector is equal to

$e_{-\pi_\alpha} (e_{-\pi_\alpha} v(\pi_\alpha)) = e_{-\pi_\alpha} v(\pi_\alpha - \alpha) = [e_{-\pi_\alpha}; e_{\pi_\alpha} - \alpha] = -h_{\pi_\alpha - \alpha}$.

Let $\pi_\alpha - s_\alpha \gamma \in V_1$ (in this case $m - k = 1$). Then we can decompose the representation of the group $G'$ on $V_1$ in the graded components with the grading defined by the pairing with $\pi_\beta$:

$V_1 = \mathbb{K} v(\pi_\alpha - \alpha) \oplus V'_1 \oplus V'_2 \oplus V'_3$.

Let $k = 3, m = 2$. Then the weight space in consideration belong to $V'_3$, that is one dimensional and spanned by the lowest weight vector $v_\tau$ of the representation $G'' : V_1$. Since $G' : V_1$ is self dual (the root system $E_7$ does not have automorphisms that are not inner) $\pi_\alpha - \alpha \equiv -\tau \mod \mathbb{Q} \pi$ thus we have $\tau = \alpha$. The corresponding weight vector is $e_{-\pi_\alpha} v(\pi_\alpha - \alpha) = v(-\alpha)$.

If $k = 2, m = 1$ then $\pi_\alpha - s_\alpha \gamma \in V'_2$. In this case the element $s_\alpha$ gives an isomorphism of the simple $G''$-module $V'_2$ and a submodule in $\mathcal{N}$. In this case we have an equality $(\gamma; \alpha) = 0$. That follows from $(\gamma; \alpha) = (\alpha + 2\beta + \sum \theta \in \Pi'' a_\theta \theta; \alpha) = (\alpha; \alpha) + 2(\beta; \alpha) = 0$.

Let us prove the following proposition.

**Proposition 3.2.** The projectivizations of factor module $V_1/(V'_1 \oplus \mathbb{K} v(\pi_\alpha - \alpha))$ and module $\mathcal{N} \cap V_2$ are isomorphic as $P'$-varieties. The first module is identified with the fiber of normal bundle to the flag variety $G'/P' \subset \mathbb{P}(V_1)$ in the point $\langle v(\pi_\alpha - \alpha) \rangle$.

**Remark 3.3.** The modules $\mathcal{N} \cap V_2$ and $V_1/(V'_1 \oplus \mathbb{K} v(\pi_\alpha - \alpha))$ are isomorphic as $G'' \times P'_u$-modules but not isomorphic as $P'$-modules.
Proof. First let us notice that the module $V_1/(V_1' \oplus \mathbb{K}v(\pi_\alpha - \alpha))$ can be identified with the following $L'$ module $\tilde{V} = V_3' \oplus V_2'$ with the following action of $e_\beta$ ($P'$ is generated by $\exp(e_\beta)$ and $L'$):

- On the component $V_2'$ the action of $e_\beta$ is trivial.
- On $V_3'$ the action of $e_\beta$ comes from the action on $V_1$.

Remark 3.4. To prove that $P'$ is generated by $\exp(e_\beta)$ and $L'$ it is sufficient to notice that the linear span of the elements $L'e_\beta$ generates the Lie algebra $\mathfrak{p}_u'$. The last assertion is due to the fact that on the unipotent radical $\mathfrak{p}_u'$ the action of $L'$ is irreducible.

From the above we have a decomposition $N \cap V_2 := \mathbb{K}h_{\pi_\alpha - \alpha} \oplus N_2$ as the sum of irreducible $G'$-modules; where the module $N_2$ is generated by the vectors $e_\gamma v(\pi_\alpha - \alpha)$ for the roots $\gamma \in \mathfrak{p}_u$ such that $e_{\gamma, \gamma} v(\pi_\alpha) \in V_2$.

Our aim is to show that the element $e_{-\alpha}$ maps $\tilde{V}$ in the module $\mathbb{K}(h_{\pi_\alpha - \alpha} + v_0) \oplus N_2$, where $v_0$ is some vector of weight zero. We shall show that $e_\beta v_0 = 0$ that will imply that $\mathbb{G}' \ltimes P'_u$-module $\mathbb{K}(h_{\pi_\alpha - \alpha} + v_0) \oplus N_2$ is isomorphic to $N \cap V_2$.

Let us prove that $e_{-\alpha}$ maps the $G''$-module $V_2'$ in $N_2$. Indeed the weights of $V_2'$ are of the form $\pi_\alpha - (\alpha + 2\beta + \sum_{\theta \in \mathbb{P}''} a_\theta \theta)$ and the map into the set of weights of $N_2$ by the action of $s_\alpha$. The weights from the latter module can be written in the form $\pi_\alpha - (2\beta + \sum_{\theta \in \mathbb{P}''} a_\theta \theta)$. This implies that they are obtained from the weights of $V_2'$ by subtracting $\alpha$. Let us notice that the action $e_{-\alpha}$ commutes with $G''$ (the root $\alpha$ is orthogonal to the roots from $\Delta''$ corresponding to the group $G''$). Since $G''$-module $V_2'$ is simple, for the proof of the isomorphism it is sufficient to check that $e_{-\alpha} V_2' \neq 0$. Let us choose a weight vector $v_\chi \in V_2'$. The weights $\chi$ and $\chi - \alpha$ are the ends of the edge of a weight polytop (see Corollary 3.4), that implies that the vector $e_{-\alpha} v_\chi$ is nonzero.

Next let us notice that $e_{-\alpha}$ maps the weight vector from $V_2'$ to the vector $e_{-\alpha} v_\alpha = h_\alpha$. We have $[e_\beta; h_\alpha] = (\beta; \alpha)e_\beta = - (\beta; \pi_\alpha - \alpha)e_\beta = -[e_\beta; h_{\pi_\alpha - \alpha}]$ since $(\beta; \pi_\alpha) = 0$. Thus we obtain that $e_{-\alpha} v_\alpha = h_\alpha = -h_{\pi_\alpha - \alpha} + v_0$ for some vector $v_0$, annihilated by $e_\beta$.

At last we are left to show that the morphism $e_\alpha : \tilde{V} \to \mathbb{C} h_{\pi_\alpha - \alpha} \oplus N_2$ is equivariant with respect to the action of $e_\beta$. Since $[e_\beta; e_{-\alpha}] \in \mathfrak{g}_{-\alpha} = 0$ (this holds since the difference of the simple roots is not a root) we get a chain of equalities:

$$e_\beta e_{-\alpha} V_2' = e_{-\alpha} e_\beta V_2' + [e_\beta; e_{-\alpha}] V_2' = e_{-\alpha} e_\beta V_2',$$

that is required.

Let us notice that the element $h_{\pi_\alpha}$ and the algebra $\mathfrak{g}'' + \mathfrak{p}_u'$ generate $\mathfrak{p}'$. Thus for the proof it is sufficient to check that $h_{\pi_\alpha}$ act in the same manner on $\mathbb{P}(N \cap V_2)$ and $\mathbb{P}(V_1/(V_1' \oplus \mathbb{K}v(\pi_\alpha - \alpha)))$. But this is a corollary of the equality

$$\text{supp}(N \cap V_2) = \text{supp}(V_1/(V_1' \oplus \mathbb{K}v(\pi_\alpha - \alpha))) + \alpha.$$

This proof of the lemma finishes the proof of the proposition.

Let us introduce some additional notation.

Consider the set of stable points $(G/P)_c^*$ for the action of $\lambda$ (corr. $T$) With respect to the very ample bundle $\mathcal{M} = \mathcal{O}(2)|_{G/P} \otimes k_{-\pi_\alpha}$. As a sheaf it is isomorphic to $\mathcal{O}(2)|_{G/P}$ but the action of $\lambda$ (corr. $T$) is twisted by a character $-\pi_\alpha$. After such
linearization we can assume that $\lambda$ is acting on the component $V_1$ with the weight $t^{k-2}$; and the weights of the subspace $\langle v_{\pi} \rangle \oplus V_1$ are positive and the weights of $V_2$ are negative.

**Weighted blow up and weighted projective space.**

Here we recall the definition of the weighted blow up adapted to our situation. Let $\mathbb{P}(V) \subset \mathbb{P}(\mathbb{K}w \oplus V)$ be a natural embedding of a projective space as a subspace $w = 0$. Let us define on $\mathbb{P}(\mathbb{K}w \oplus V)$ the involution $\iota$ acting by the formula $\iota(w) = -w$ and $\iota(v) = v$ for $v \in V$. Consider the quotient $\mathbb{P}(\mathbb{K}w \oplus V)/\iota$ of the subspace $\mathbb{P}(\mathbb{K}w \oplus V)$ by the action of involution $\iota$. This quotient is the weighted projective space with the weights $(2, 1, \ldots, 1)$ (cf. [5]). The quotient map

$$p_r : \mathbb{P}(\mathbb{K}w \oplus V) \to \mathbb{P}_{wt}(\mathbb{K}w \oplus V)$$

can be written down in homogenous coordinates as

$$p_r : (z_0 : z_1 : \ldots : z_n) \mapsto (z_0^2 : z_1 : \ldots : z_n).$$

On the variety $\mathbb{P}(\mathbb{K}w \oplus V)$ the ramification locus of $p$ is equal to the set of fixed points with respect to the involution $\iota$ consists of the divisor $\mathbb{P}(V) = (0 : 1 : \ldots : z_n)$ and of the isolated ramification point $(1 : 0 : \ldots : 0)$. The points of the divisor $p((0 : z_1 : \ldots : z_n)) \subset \mathbb{P}_{wt}(\mathbb{K}w \oplus V)$ are smooth points of $\mathbb{P}_{wt}(\mathbb{K}w \oplus V)$, and a point $p((1 : 0 : \ldots : 0))$ is an isolated singular point. Let us note that the weighted projective space $\mathbb{P}(\mathbb{K}w \oplus V)/\iota$ is the cone over quadratic Veronese map of the projective space $\mathbb{P}(V)$ (cf. [5]). This map can be written in the weighted homogeneous coordinates as

$$\mathbb{P}(\mathbb{K}w \oplus V) \to \mathbb{P}_{wt}(\mathbb{K}w \oplus V) \to \mathbb{P}(\mathbb{K}w \oplus V^{\otimes 2})$$

$$(z_0 : z_1 : \ldots : z_n) \mapsto (z_0^2 : z_1 : \ldots : z_n) \mapsto (z_0^2 : z_1^2 : \ldots : z_j z_i : \ldots : 0 < j \leq i).$$

Consider $V = V_0 \oplus V_1$ where $(z_1, \ldots, z_k)$ are the coordinates in $V_0$ and $(z_{k+1}, \ldots, z_n) V_1$. Then we can consider a weighted blow up of $\mathbb{K}w \oplus V$ in the subspace $V_1$ with the weight 2 on the subspace $\mathbb{K}w$ and weight 1 on the subspace $V_0$. It can be described as a subvariety of $V \times \mathbb{P}_{wt}(\mathbb{K}v \oplus V_1)$ (where $\mathbb{P}_{wt}(\mathbb{K}v \oplus V_0)$ is the weighted projective space with the weights $(2, 1, \ldots, 1)$ and the coordinates $(\xi_0 : \xi_1 : \ldots : \xi_k)$) by the following system of equations:

\[
\begin{align*}
\xi_0 \xi_j &= z_j \xi_i \\
\xi_0^2 \xi_j &= z_j^2 \xi_0
\end{align*}
\]

On the invariant language it is a weighted $\text{Proj}_{wt}(\bigoplus J^n)$ where $J = (z_0, z_1, \ldots, z_k)$ is the ideal defining $V_1$. Here we assume that the coordinate $z_0$ has weight 2, and the latter coordinates have weight 1.

Our main task is to study the quotient $\lambda \backslash (G/P)_e^{\lambda}$ for the case of the simple group of type $E_8$ The latter set probably has more complicated description than the corresponding set from the Proposition 2.1 but for construction of the embedding of $\mathcal{F}$ it will be sufficient to apply the results about this set obtained below.

The arguments repeat in general the proof of Theorem 2.1. We shall give their modifications.

**Proposition 3.5.** Let $\mathbb{P}_{wt} := \mathbb{P}_{wt}(\langle e_{-\pi_\pi^0} v_{\pi_\pi} \rangle \oplus V_1)$ be a weighted projective space with the weights $(2, 1, \ldots, 1)$ obtained as a quotient of the vector space $\langle e_{-\pi_\pi^0} v_{\pi_\pi} \rangle \oplus V_1$ by the action of one-parameter subgroup $\lambda$. The flag variety $G'/P'$ embeds in $\mathbb{P}_{wt}$ as a composition of natural incusions $G'/P' \subset \mathbb{P}(V_1) \subset \mathbb{P}_{wt}$. 
The quotient $\lambda\|(P_u^- P/P)^*_{\mathfrak{z}}$ is isomorphic to $\mathbb{P}_{\text{wt}} \setminus (G' / P')$ the subset of $\mathbb{P}_{\text{wt}}$ with deleted flag variety $G'/P'$.

**Proof.** Consider the orbit $\langle \lambda(t)v \rangle \subset P_u^- P/P$. Let us represent the vector $v$ as the exponential map of the Lie algebra $p_u^-$ applied to the vector $v_{\pi o}$:

$$v = \exp\left( \sum_{\gamma \in \Delta_{P_u}} c_{\gamma} e^{-\gamma} \right) v_{\pi o} = v_{\pi o} + \sum_{\gamma \in \Delta_{P_u} \setminus \{\pi o\}} \frac{c_{\gamma} e^{-\gamma}}{1!} v_{\pi o} + \sum_{\gamma \in \Delta_{P_u} \setminus \{\pi o\}} \frac{(\sum_{\gamma \in \Delta_{P_u} \setminus \{\pi o\}} c_{\gamma} e^{-\gamma})^2}{2!} v_{\pi o} + \ldots .$$

(In other words we consider the embedding of the open cell $P_u^- P/P$ in the space $\mathbb{P}(V(\pi o))$ as an orbit of the point $\langle v_{\pi o} \rangle$.)

Consider the expression for the component $v_0$ of the weight zero for the vector $v$:

$$v_0 = c_{\pi o} e^{-\pi o} v_{\pi o} + \frac{1}{2} \sum_{\gamma \in \Delta_{P_u} \setminus \{\pi o\}} c_{\gamma} c_{\pi o - \gamma}(e^{-\gamma} e^{-\pi o + \gamma} + e^{-\pi o + \gamma} e^{-\gamma}) v_{\pi o}.$$

Using the inclusion $\mathfrak{sl}_3 \subset \mathfrak{g}$ defined by the positive roots $\{\gamma, \pi o - \gamma, \pi o\}$ we can show by the explicit calculations for $\mathfrak{sl}_3$ that the expression $(e^{-\gamma} e^{-\pi o + \gamma} + e^{-\pi o + \gamma} e^{-\gamma}) v_{\pi o}$ is proportional to $(h_{-\gamma - h_{-\pi o + \gamma}}) = h_{\pi o - 2\gamma}$. Since $(\pi o - 2\gamma; \pi o) = 0$ we get that when $c_{\pi o} = 0$ the vector $v_0$ satisfies the equality $\langle v_0; h_\gamma \rangle = 0$. That implies that orthogonal (with respect to Cartan-Killing form) projection of $v$ to the component $h_{\pi o} \in V_2$ is equal $c_{\pi o} e_{-\pi o} v_{\pi o}$.

Let us notice that

$$\langle \lambda(t)v \rangle = \langle v_{\pi o} + \sum_{\gamma \in \Delta_{P_u} \setminus \{\pi o\}} tc_{\gamma} e^{-\gamma} v_{\pi o} + t^2 c_{\pi o} e^{-\pi o} v_{\pi o} + \frac{1}{2!} (\sum_{\gamma \in \Delta_{P_u} \setminus \{\pi o\}} tc_{\gamma} e^{-\gamma})^2 v_{\pi o} + \ldots \rangle .$$

It is easy to see that the set of coefficients $(t^2 c_{\pi o}, tc, \ldots, tc_{\text{dim} V_1})$, where $\gamma_i \in \Delta_{P_u} \setminus \{\pi o\}$ defines the orbit of $\lambda(t)$ from the open cell. Considering $\lambda$-equivariant map $(P_u^+ P/P) \setminus \langle v(\pi o) \rangle \rightarrow \mathbb{P}_{\text{wt}}$:

$$\langle v \rangle \rightarrow \left( \begin{array}{c} (v, h_{\pi o}) \varepsilon (v, v(-\pi o + \gamma_i)) \varepsilon \ldots \varepsilon (v, v(-\pi o + \gamma_{\text{dim} V_1})) \\ (v, v(-\pi o)) \end{array} \right) .$$

we get that the isomorphism of the considered space of orbits to the weighted projective space $\mathbb{P}_{\text{wt}}(\langle h_{\pi o} \rangle \oplus V_1)$.

It remains to settle which points of $\mathbb{P}_{\text{wt}}$ correspond to the unstable orbits and should be excluded. The orbit $\lambda(v)$ is unstable only in the case when the component of vector $v$ belonging to $V_2$ is equal to zero. In particular the components $c_{\pi o} e_{-\pi o} v_{\pi o} \left( \sum_{\gamma \in \Delta_{P_u} \setminus \{\pi o\}} c_{\gamma} e^{-\gamma} \right) v_{\pi o}$ that are orthogonal with respect to the Cartan-Killing form should be zero as well. Since $v \in \mathbb{K} v_{\pi o} \oplus V_1$, By Lemma 23 we get $\lim_{t \rightarrow 0} \lambda(t) \langle v \rangle = \langle v_1 \rangle \in \mathbb{P}(V_1) \cap G'/P = G'/P'$, (where $v_1$ is the projection of $v$ on $V_1$).

Thus for the inclusion $\langle v \rangle \notin (G/P)^{ss}$ it is necessary that $c_{\pi o} = 0$ and $\langle v_1 \rangle \in G'/P'$. This condition is also sufficient. Let $v = \exp(e_{-\pi o}) v_{\pi o} = v_{\pi o} + e_{-\pi o} v_{\pi o}$ it is easy to see that $\langle v \rangle \notin (G/P)^{ss}$. In the same time since $G'$ and $\lambda$ commutes, the orbit $G' \langle v \rangle$ is unstable. In this case the projection of $G' \langle v \rangle$ on $\mathbb{P}(V_1)$ is equal to $G' \langle v_1 \rangle \cong G'/P'$. Since the map $\lambda\|(P_u^+ P/P) \setminus \langle v_{\pi o} \rangle \rightarrow \mathbb{P}_{\text{wt}}$ is an isomorphism we
obtain that the unstable orbits project exactly to $G' / P' \subset P_{wt}$. i.e. \( \lambda \mathcal{H}(P_u - P)/P^{ss} = P_{wt} \setminus (G' / P') \).

Let us describe the set of $\lambda$-orbits that do not belong to the open cell. Applying Lemma 2.4 and 3.3 we obtain that for the component $v_1 \in V_1$ and vector $v$ we have the inclusion $\langle v_1 \rangle \in G' / P'$. Acting by $G'$ we can assume that $v_1 = v_{\pi_\alpha}$ (since the action of $G'$ commute with $\lambda$). We also obtain that the image of orthogonal projection on $P(V_1)$ of the set of $\lambda$-orbits that do not belong to the open cell is equal to $G' / P'$.

**Lemma 3.6.** Consider the space of $\lambda$-orbits of the points $\langle v \rangle$ such that the projection of $v$ on the subspace \( \langle v_{\pi_\alpha} \rangle \oplus V_1 \) is proportional to $v_{\pi_\alpha} - \alpha$. This orbit space as a $G'$-variety is identified with the quotient of the module \( \langle e_{-\pi_\alpha} v(\pi_\alpha - \alpha) \rangle \oplus N \cap V_2 \) by the one-parameter subgroup $\lambda$ where $\lambda$ is acting on the first component with the weight 2 and on $N \cap V_2$ with the weight 1.

**Proof.** Let us represent $v$ as the exponential map from the element of the Lie algebra $u$, applied to $v_{\pi_\alpha} - \alpha$. We get that

\[
v = v_{\pi_\alpha} - \alpha + \sum c_{\gamma} e^{-\gamma} v_{\pi_\alpha} - \alpha + c_{\pi_\alpha} e^{-\pi_\alpha} v_{\pi_\alpha} - \alpha + \frac{1}{2!} (\sum c_{\gamma} e^{-\gamma})^2 v_{\pi_\alpha} - \alpha + \ldots,
\]

where both sums are taken over the roots $\gamma$ such that $\pi_\alpha - \alpha + \gamma \in N \cap V_2$.

Let us show that the component $v_{\pi_\alpha} - \alpha$ of the vector $v$ is equal to $c_{\pi_\alpha} e^{-\pi_\alpha} v_{\pi_\alpha} - \alpha$.

Let us write down its value:

\[
c_{\pi_\alpha} e^{-\pi_\alpha} v_{\pi_\alpha} - \alpha + \frac{1}{2!} \sum_{\gamma + \delta = \pi_\alpha} c_{\gamma} c_{\delta} e^{-\gamma} e^{-\delta} v_{\pi_\alpha} - \alpha,
\]

where $\gamma, \delta \in \Delta_{P_\alpha}$ are such that $\pi_\alpha - \gamma, \pi_\alpha - \delta \in \text{supp}(V_2')$. But from the last inclusion we obtain that the equality $\gamma + \delta = \pi_\alpha$ is not possible. Indeed the last equality contradicts the equalities $\gamma = \pi_\alpha; \delta = \pi_\beta = 0$ and $\delta = \pi_\alpha; \pi_\beta = 0$.

From which we get $v_{\alpha} = c_{\pi_\alpha} e^{-\pi_\alpha} v_{\pi_\alpha} - \alpha$.

The set of coefficients $(t_1, c_{\gamma_1}, \ldots, t_{\gamma_k})$, where $\gamma_i \in \Delta_{P_\alpha} \setminus \{\pi_\alpha\}$ define the orbit uniquely. Thus the set of orbits can be identified with the weighted projective space $P_{wt}(Kc_{-\pi_\alpha} v(\pi_\alpha - \alpha) \oplus N \cap V_2)$.

**Remark 3.7.** For the points $\langle v \rangle \in (G / P)^{ss}$ different from $\langle v_{\pi_\alpha} + Kh_{\pi_\alpha} \rangle$, the orthogonal projection on the subspace $V_1$ is not equal to zero. This defines a $\lambda$-equivariant projection on $P(V_1)$ as well as the map $\sigma_0 : \lambda \mathcal{H}((G / P)^{ss} \setminus \langle v_{\pi_\alpha} + Kh_{\pi_\alpha} \rangle) \rightarrow P(V_1)$.

By $D_{h_{\pi_\alpha}}$ we denote a $G'$-invariant hyperplane section in $P(V)$ defined by the vanishing of the coordinate by the vector $h_{\pi_\alpha}$. From the Remark 3.7 it follows that we have well defined $\lambda$-equivariant projection on $P(V_1)$ of the set $(G / P)^{ss} \cap D_{h_{\pi_\alpha}}$ that defines a map $\sigma_0 : \lambda \mathcal{H}((G / P)^{ss} \cap D_{h_{\pi_\alpha}}) \rightarrow P(V_1)$. Our next aim is to study the quotient $\lambda \mathcal{H}((G / P)^{ss} \cap D_{h_{\pi_\alpha}})$ in the neighbourhood $\sigma_0^{-1}(z)$ where $z \in G' / P' \subset P(V_1)$ is the sufficiently general point.

Let us choose in $P(V)$ the affine chart $\mathcal{U}_0 = \{ \langle v \rangle \in P(V) | v_{\pi_\alpha} - \alpha \neq 0 \}$. We recall that conormal bundle to $G' / P' \subset P(V_1)$ is identified with $G' / P' \subset \mathcal{U}_0$. On $P'_u / P' / P'$ we have a transitive action of the group $P'^{-}$, and the stabilizer of the point $eP' / P'$ is equal to $L'$. This implies

\[
\mathcal{N} \cong V_1 / \langle v_{\pi_\alpha} - \alpha \rangle \oplus V_1'
\]
that the restriction of the conormal bundle is isomorphic to $P^*-\ast L^r\mathcal{N}|_{L'}$, where $\mathcal{N}|_{L'} \cong V_2' \oplus \langle v_\alpha \rangle$ is a $P'$-module $\mathcal{N}$ considered as an $L'$-module. In particular the last isomorphism claims the restriction of the conormal bundle on the open cell $P^*_{u'}P^*/P'$ is isomorphic to the direct sum of subbundles $P^*_{\ast L^r}V_2'$ and $P^*_{\ast L^r}\langle v_\alpha \rangle$.

**Proposition 3.8.** The quotient $\lambda_1((G/P)_c^\ast \cap D_{h\pi_2} \cap \Omega_u)$ is isomorphic to the weighted blow up of the variety $\mathbb{P}(V_1) \cap \Omega_u$ in the subvariety $P^*_{u'}P^*/P'$. The preimage of $P^*_{u'}P^*/P'$ is isomorphic to the projectivization of the conormal bundle to $P^*_{u'}P^*/P'$ in $\mathbb{P}(V_1) \cap \Omega_u$, and the fibers of subbundle $P^*_{\ast L^r}\langle v_\alpha \rangle$ have the weight 2, and the fibers $P^*_{\ast L^r}V_2'$ have the weight 1.

**Remark 3.9.** During the proof it will be stated in a precise way how we define the weights for the weighted blow up and we shall give the explicit equations defining it.

**Proof.** Let $(v) \in (G/P)_c^\ast \cap \Omega_u$. Applying Lemma [11] let us represent the vector $v$ as the exponential map from the element of the Lie algebra $\tilde{u} \in \text{Ad}(s_\alpha)p_u^-$ applied to the vector $v_{\pi_\alpha-a}$:

$$v = \exp(\tilde{u})v_{\pi_\alpha-a}$$

The map $\text{Ad}(s_\alpha)p_u^- \rightarrow \text{Ad}(s_\alpha)p_u^-v_{\pi_\alpha-a} \subset V$ defines an $L'$-equivariant embedding of the Lie algebra in the $G$-module $V$. We have the following isomorphism of the $L'$-modules:

$$\text{Ad}(s_\alpha)p_u^-v_{\pi_\alpha-a} \cong \langle v_{\pi_\alpha} \rangle \oplus V_1' \oplus s_\alpha V_2' \oplus \langle v_{-a} \rangle$$

The last decomposition allows us to represent the vector $v$ in the form

$$v = \exp(b_\alpha e_\alpha + \sum_{\pi_\alpha-a-\delta \in \text{supp } V_1'} b_\gamma e_\gamma + \sum_{\pi_\alpha-a-\gamma \in \text{supp } V_2'} c_\beta e_{-\beta} + c_\alpha e_{-\pi_\alpha-a})v_{\pi_\alpha-a},$$

for $\gamma, \delta \in s_\alpha \Delta_{\pi_\alpha-}$. Also we should note that the pairing with the weights $\pi_\alpha$ and $\pi_\alpha'$ define $L'$-invariant gradings on $V(\pi_\alpha)$. Their values define uniquely the components of the decomposition of the module $\text{Ad}(s_\alpha)p_u^-v_{\pi_\alpha-a}$ into irreducible $L'$-modules given above. The latter means that using this gradings we can define in modules the lie the corresponding monomials in the exponential representation of vector $v$. This implies that the component of vector $v$ with the weight $\pi_\alpha-a-\delta \in \text{supp } V_1'$ is equal to $b_\delta e_{-\delta}v_{\pi_\alpha-a}$, the component of weight $\pi_\alpha-a-\gamma \in \text{supp } V_2'$ is equal to $c_\gamma e_{-\gamma}v_{\pi_\alpha-a}$, and a component of weight $\pi_\alpha-a-\delta \in \text{supp } V_2'$ is equal to

$$v_{\pi_\alpha-a} = \frac{1}{2!} \sum_{\gamma=\delta+\alpha} b_\delta b_\gamma (e_{-\gamma}e_{-\gamma} + e_{-\delta}e_{-\delta})v_{\pi_\alpha-a} + \frac{1}{2!} b_\alpha c_\gamma (e_\alpha e_{-\gamma} + e_{-\gamma}e_\alpha)v_{\pi_\alpha-a} =$$

$$= \sum_{\gamma=\delta+\alpha} b_\delta b_\gamma e_{-\gamma}v_{\pi_\alpha-a} + b_\alpha c_\gamma e_{-\gamma}v_{\pi_\alpha-a},$$

where $\pi_\alpha-a-\delta \in \text{supp } V_1'$, and an equality holds since $[e_{-\delta}; e_{-\gamma}] = 0$, $[e_\alpha; e_{-\gamma}] = 0$.

Now let us find a coefficient of $h_{\pi_\alpha}$. Let us begin with writing down the component of the weight zero for the vector $v$:

$$\frac{1}{2!} \sum_{\delta+\gamma=\pi_\alpha-a} b_\delta c_\gamma (e_{-\delta}e_{-\gamma} + e_{-\gamma}e_{-\delta})v_{\pi_\alpha-a} + c_{\pi_\alpha-a} e_{-\pi_\alpha-a} v_{\pi_\alpha-a} = \frac{1}{2!} h_{\pi_\alpha} h_{\pi_\alpha+a} =$$

$$= \frac{1}{2} \sum_{\delta+\gamma=\pi_\alpha-a} b_\delta c_\gamma h_{-\pi_\alpha+\alpha+2\delta} + c_{\pi_\alpha-a} h_{-\pi_\alpha-a} - \frac{1}{2} b_\alpha e_{\pi_\alpha} h_{\pi_\alpha+a},$$

where we used the equality $e_{-\delta}e_{-\gamma} + e_{-\gamma}e_{-\delta} = h_{\pi_\alpha-a-2\delta}$ (obtained by the calculation in $\mathfrak{sl}_2$-subalgebra generated by $e_{-\delta}$ and $e_{-\gamma}$), and the equality $e_{-\pi_\alpha+a} v_{\pi_\alpha-a} = h_{\pi_\alpha-a}$ (that is rewritten equality $[e_{-\pi_\alpha+a}, e_{\pi_\alpha-a}] = -h_{\pi_\alpha-a}$).
Taking into account that \( \langle \delta, \pi_\alpha \rangle = 0 \), and that the lengths of the projections \( h_{\pi_\alpha - \alpha} \) and \( h_{\pi_\alpha + \alpha} \) on the component \( h_{\pi_\alpha} \) are equal to \((\pi_\alpha - \alpha, \pi_\alpha)/(\pi_\alpha, \pi_\alpha) = 1/2 \) and \((\pi_\alpha + \alpha, \pi_\alpha)/(\pi_\alpha, \pi_\alpha) = 3/2 \) respectively we obtain that the projection of \( v \) on the component \( h_{\pi_\alpha} \) is equal to:

\[
x_{h_{\pi_\alpha}} := -\frac{1}{2}(e_{\pi_\alpha - \alpha} + \frac{1}{2} \sum_{\delta + \gamma = \pi_\alpha - \alpha} b_\delta e_\gamma) = \frac{3}{2} e_{-\alpha} e_{\pi_\alpha}.
\]

Let us write down the component of weight \( \alpha \) for the vector \( v \):

\[
v_\alpha = \sum_{\delta_1 + \delta_2 + \delta_3 = \pi_\alpha - 2\alpha} b_{\delta_1} b_{\delta_2} b_{\delta_3} e_{-\delta_1} e_{-\delta_2} e_{-\delta_3} v_{\pi_\alpha - \alpha} + \frac{1}{6} e_{-\pi_\alpha} b^2 \left( e^2 e_{-\pi_\alpha} + e_{\alpha} e_{-\pi_\alpha} e_\alpha \right) v_{\pi_\alpha - \alpha} + \frac{1}{2} b_\alpha e_{\pi_\alpha - \alpha} (e_\alpha e_{-\pi_\alpha + \alpha} + e_{-\pi_\alpha + \alpha} e_\alpha) v_{\pi_\alpha - \alpha} + \frac{1}{6} b_\alpha \sum_{\delta + \gamma = \pi_\alpha - \alpha} c_\delta b_\gamma (e_{-\delta} e_\alpha e_{-\gamma} + e_{-\delta} e_{-\gamma} e_\alpha + e_\alpha e_{-\delta} e_{-\gamma} + e_\alpha e_{-\gamma} e_{-\delta}) v_{\pi_\alpha - \alpha}.
\]

Let us notice that in the last summand we omitted the zero monomials \( e_{-\gamma} e_\alpha e_{-\delta} v_{\pi_\alpha - \alpha} \) and \( e_{-\gamma} e_{-\delta} e_{-\alpha} v_{\pi_\alpha - \alpha} \). They are trivial since the weights of the vectors \( e_\alpha e_{-\delta} v_{\pi_\alpha - \alpha} = e_{-\delta} e_{-\alpha} v_{\pi_\alpha - \alpha} \) are equal to \( \pi_\alpha - \delta \) and these weight do not belong to the weight polytop of the representation (that follows from the equality \( (\pi_\alpha; \delta) = 0 \)). Also we have \( e_\alpha e_{-\gamma} e_{-\delta} v_{\pi_\alpha - \alpha} = [e_{-\gamma} [e_{-\delta}, e_{\pi_\alpha - \alpha}]] = N_\delta \pi_\alpha - \alpha [e_{-\gamma}, e_\gamma] = -N_\delta \pi_\alpha - \alpha h_\gamma \). But we have \( [e_\gamma; h_\gamma] = 0 \), since \( (\gamma, \alpha) = 0 \); that proves the formula.

Let us simplify expression for \( v_\alpha \) using commutation relations \( [e_{-\pi_\alpha + \alpha}, e_\alpha] = 0 \), \( [e_{-\delta}, e_\alpha] = 0 \) and \( [e_\gamma; e_\alpha] = 0 \):

\[
v_\alpha = \sum_{\delta_1 + \delta_2 + \delta_3 = \pi_\alpha - 2\alpha} b_{\delta_1} b_{\delta_2} b_{\delta_3} e_{-\delta_1} e_{-\delta_2} e_{-\delta_3} v_{\pi_\alpha - \alpha} - \frac{1}{2} e_{-\pi_\alpha} b^2 \alpha e_\alpha + \frac{1}{2} b_\alpha e_{\pi_\alpha - \alpha} (e_\alpha e_{-\pi_\alpha + \alpha} + e_{-\pi_\alpha + \alpha} e_\alpha) v_{\pi_\alpha - \alpha} + \frac{1}{6} b_\alpha \sum_{\delta + \gamma = \pi_\alpha - \alpha} c_\delta b_\gamma (e_{-\delta} e_\alpha e_{-\gamma} + e_{-\delta} e_{-\gamma} e_\alpha + e_\alpha e_{-\delta} e_{-\gamma} + e_\alpha e_{-\gamma} e_{-\delta}) v_{\pi_\alpha - \alpha}.
\]

Denoting the first summand by \( q_\alpha \) we obtain that:

\[
v_\alpha = q_\alpha + Ae_{-\pi_\alpha} b^2 \alpha (e_\alpha^2 e_{-\pi_\alpha}) v_{\pi_\alpha - \alpha} + x_{h_{\pi_\alpha}} b_\alpha e_{\pi_\alpha} e_{-\pi_\alpha + \alpha} v_{\pi_\alpha - \alpha},
\]

where \( A \) is nonzero constant whose value is not important for us.

To find the intersection of \( U_\alpha \) with the divisor \( D_{h_{\pi_\alpha}} \) we must put the coordinate \( x_{h_{\pi_\alpha}} \), equal to zero, that implies \( (v) \in G/P \cap U_\alpha \cap D_{h_{\pi_\alpha}} \):

\[
v_\alpha = q_\alpha + Ae_{-\pi_\alpha} b^2 \alpha (e_\alpha^2 e_{-\pi_\alpha}) v_{\pi_\alpha - \alpha}.
\]

Denote by \( x_\omega \) the coordinate of the weight vector \( v(\omega) \). The ideal \( \mathfrak{j} \) of the subvariety \( P^{-}_u P' / P' \subset \mathbb{P}(V) \cap U_\alpha \) is generated by the following elements:

\[
Q_\alpha v(\alpha) := x_\alpha v(\alpha) - \sum_{\delta_1 + \delta_2 + \delta_3 = \pi_\alpha - 2\alpha} x_{\delta_1} x_{\delta_2} x_{\delta_3} e_{-\delta_1} e_{-\delta_2} e_{-\delta_3} v_{\pi_\alpha - \alpha},
\]

\[
P_{\pi_\alpha - \gamma} v(\pi_\alpha - \gamma) := x_{\pi_\alpha - \gamma} v(\pi_\alpha - \gamma) - \sum_{\mu = \delta_1 + \delta_2 + \alpha} x_{\mu} x_{\delta_2} e_{-\delta_1} e_{-\delta_2} v_{\pi_\alpha - \alpha},
\]

where \( \pi_\alpha - \gamma \in \text{supp}(V'_2) \). The last assertion can be obtained from the expression of the element of \( P^{-}_u P' / P' \) as the exponential map \( \exp(\sum_{\pi_\alpha - \alpha - \delta \in \text{supp}(V')} b_\delta e_{-\delta}) v_{\pi_\alpha - \alpha} \). The conormal bundle to \( P^{-}_u P' / P' \) in \( \mathbb{P}(V) \) is identified with \( \mathfrak{j} / \mathfrak{j}^2 \). It is generated by the elements described above. Indeed the number of these elements is equal to the dimension of the fiber of conormal bundle.
and the linear parts of these equations are equal to the coordinates $x_\alpha$ and $x_{\pi_\alpha - \gamma}$ for $\pi_\alpha - \gamma \in \text{supp}(V_2^\gamma)$, that implies the linear independence of these equations modulo $J^2$ (since the elements from this ideal do not have linear parts).

The relations for the coordinates of the vector $(v) \in ((G/P)_L^\ast \cap D_h^{\pi_\alpha} \cap \cup_{\alpha})$ can be written in the form:

$$
\begin{align*}
\{ Q_\alpha &= c_{-\pi_\alpha} b^2_{\pi_\alpha} N_\alpha \\
\pi_\alpha - \gamma &= b_0 c_{\pi_\alpha - \gamma} N_\pi
g(N_\alpha = A[(e_2 \epsilon_{-\pi_\alpha})v_{\pi_\alpha - \gamma}]/|v(\alpha)|, N_{\pi_\alpha - \gamma} = |e_2 \epsilon_{\pi_\alpha} v_{\pi_\alpha - \gamma}|/|v(\pi_\alpha - \gamma)| \}
\end{align*}
$$

where $N_\alpha = A[(e_2 \epsilon_{-\pi_\alpha})v_{\pi_\alpha - \gamma}]/|v(\alpha)|$, $N_{\pi_\alpha - \gamma} = |e_2 \epsilon_{\pi_\alpha} v_{\pi_\alpha - \gamma}|/|v(\pi_\alpha - \gamma)|$ are the normalizing constants.

Let us notice that since we have a decomposition $J/J^2$ in the direct sum of two vector bundles $P^* - \ast L \langle v_\alpha \rangle$ and $P^* - \ast L V_2^\gamma$, we can consider a weighted projectivisation $P^* - \ast L \mathbb{P}(\mathbb{K} \nu_\alpha \oplus V_2^\gamma)$ with the weight 2 on the first subbundle and with the weight 1 on the second. We also have a correctly defined weighted blow up $\mathbb{B}(\nu_\alpha \cap D_h^{\pi_\alpha} \cap \cup_{\alpha})$ in $P^*_u P'/P'$. In other words we can consider the weighted Proj$_{\nu_\alpha} \bigoplus \mathcal{J}_n$, where the generator $Q_\alpha$ of the module $J$ has weight 2, and other generaters $P_{\pi_\alpha - \gamma}$ have weight 1.

We shall give the argument that provide us the explicit equations of the quotient morphism that maps $V_\pi \cap D_h^{\pi_\alpha} \cap \cup_{\alpha}$ to the weighted projective space $P(\nu_\alpha \cap D_h^{\pi_\alpha} \cap \cup_{\alpha})$. The quotient map is the restriction of the map.

**Remark 3.10.** For some purposes it is useful to have the following interpretation of the quotient morphism $\tau : \mathbb{K}^l \to \mathbb{K}^l$ by the action of the group $\mu$, defined on the $i$-th coordinate as $z_i \to z_i^\mu$. We also consider the quotient morphism $\tau^* \mu$ of the projective space $\mathbb{P}(\mathbb{K}^l)$ by the action of $\mu$, that maps $\mathbb{P}(\mathbb{K}^l)$ to the weighted projective space $\mathbb{P}(a_1, \ldots, a_l)$. In the homogeneous coordinates we have:

$$(z_1 : \ldots : z_l) \mapsto (z_1^a : \ldots : z_l^a)$$

The action of $\mu$ can be defined on the blow up $W \otimes \mathbb{K}^l$ in $W$ (defined as $Bl := \{(v, z, \xi) \in (G/P)_L^\ast \times \mathbb{P}(\mathbb{K}^l)|z_i = z_i \xi_i\}$). The generator $z_i \in \mu_{\alpha_1}$ acts as $z_i \mapsto z_i^{\alpha_i}$ on the coordinates of $\mathbb{K}^l$ and as $z_i \mapsto z_i^{\alpha_i}$ on the homogeneous coordinates of $\mathbb{P}(\mathbb{K}^l)$. The quotient $\mathbb{P}(\mathbb{K}^l)$ by the considered action of $\mu$ is identified with the weighted blow up $Bl_{\cup_{\alpha} \cap D_h^{\pi_\alpha} \cap \cup_{\alpha}} = \{(v, z, \xi) \in (G/P)_L^\ast \times \mathbb{P}(a_1, \ldots, a_l)|z_i^{\alpha_i} \xi_i^{a_i} = z_i \xi_i\}$. The quotient map is the restriction of the map

$$(w \otimes (z_1, \ldots, z_l)) \times (\xi_1 : \ldots : \xi_l) \mapsto (w \otimes (z_1^{a_i}, \ldots, z_l^{a_i})) \times (\xi_1^{a_i} : \ldots : \xi_l^{a_i})$$
Indeed making a substitute \( z_i \to z_i^{a_i}, \xi_i \to \xi_i^{a_i} \) in the equation \( z_j^{a_j} \xi_j^{a_j} = z_j^{a_j} \xi_j^{a_j} \), we get \( (z_j \xi_j)^{a_j} = (z_j \xi_j)^{a_j} \), that is the system of equations defining the weighted blow up \( Bl \).

Summarizing what was said above we obtain the following commutative diagram:

\[
\begin{array}{ccc}
\mathcal{K}_{l_0+t}^l & \xrightarrow{\tau} & \mathcal{K}_{l_0+t}^l \times \mathbb{P}(\mathcal{K}_l^l) \\
\tau \downarrow & & \tau \times \tau \\
\mathcal{K}_{l_0+t}^l & \xrightarrow{Bl_{\text{lift}}} & \mathcal{K}_{l_0+t}^l \times \mathbb{P}(a_1, \ldots, a_1)
\end{array}
\]

4. The Embedding of the Torsor over \( X_\Delta \) in the Affine Cone over \( G/P \)

Let us formulate the main theorem from the paper of Skorobogatov and Sernagorova [2]. We shall follow their main scheme of the proof but we shall modify some details that will allow us to get the analogous results in the case of the root system \( E_6 \).

4.1. Consider the embedding of the variety \( G/P \) in the projectivization of minuscule representation \( \mathbb{P}(V(\pi_\alpha)) \). For the left action of maximal torus \( T \) on \( G/P \) we denote by \( (G/P)^s \) the set of stable points with respect to \( G \)-linearized sheaf \( \mathcal{O}(1)_{G/P} \). Let \( \tau : (G/P)^s \to T \backslash (G/P)^s \) be a quotient morphism. There exists an embedding \( \iota \) of the del Pezzo surface \( X_\Delta \) with the degree \( 9 - \text{rk} \Delta \) into the quotient \( T \backslash (G/P)^s \) such that for the torsor \( \mathcal{T} = \tau^{-1}(s(X_\Delta)) \) the following conditions hold

- Consider the hypersurface \( H_\Delta = \{ \langle v \rangle \in \mathbb{P}(V(\pi_\alpha)) \mid v_\omega = 0 \} \). The divisor \( E_\omega = \tau(H_\omega \cap \mathcal{T}) \subset X_\Delta \) is equal to a \((-1)\)-curve on \( X_\Delta \).
- The \((-1)\) curves \( E_{\omega_1} \) and \( E_{\omega_2} \) do not intersect if the weights \( \omega_1 \) and \( \omega_2 \) are not adjacent in the weight polytop.

We construct the required embedding by the induction. The base of the induction follows from the isomorphism \( X_A \cong T \backslash (G/P) \) (for \( G = SL(5) \)), that was proved by Skorobogatov in [2].

Let us assume that embedding of the torsor \( \mathcal{T}' \subset G'/P' \subset \mathbb{P}(V_1) \) is already constructed. Let us denote by \( \tau \) the quotient by the left action of the torus \( T' \). The quotient \( \tau'(\mathcal{T}) = T' \mathcal{T}' \) is a del Pezzo surface \( X_{\Delta'} \) of the degree \( 9 - \text{rk} \Delta' \). This is a plane with the blown up \( \text{rk} \Delta \) points \( e_i \) in a general position. (Cf. [7]. By the general position it is assumed that no line contain three points and no conic contain 6 points from this set)

The divisors corresponding to the exceptional curves lying over \( e_i \), we denote by \( E_i \). We want to construct torsor \( \mathcal{T} \subset G/P \subset \mathbb{P}(V(\pi_\alpha)) \) over del Pezzo surface \( X_\Delta \), that is obtained from \( X_{\Delta'} \) by the blow up of the point \( e_{\text{rk} \Delta} \in X_{\Delta'} \) that is in general position with the points \( e_i \) where \( i \leq \text{rk} \Delta' \). (The image of the point \( e_{\text{rk} \Delta} \) under the contraction \( X_{\Delta'} \to \mathbb{P}^2 \) is also denoted by \( e_{\text{rk} \Delta} \).

Let \( s \in G'/P' \) be a point for which \( p_{\pi_\alpha}(s) = e_{\text{rk} \Delta} \), and \( s' \in G'/P' \) is another point on the flag variety (later we assume that it is sufficiently general). Let \( \{s_\omega\} \) be the set of homogenous coordinates of the point \( s \in \mathbb{P}(V_1) \) with respect to the weight basis of the subspace \( V_1 \) (it is well defined since \( V_1 \) — is minuscule representation).

By the action of \( s \) on the point of projective space \( x = \langle v \rangle \) we mean the action \( sx = (\sum s_\omega v_\omega) \).

Let us prove the following theorem (cf. [2 6.3]).

4.2. Consider the embedding \( G'/P' \subset \mathbb{P}(V_1) \). Let \( s \in G'/P' \) be a point of a flag variety such that \( p_{\pi_\alpha}(s) = e_{\text{rk} \Delta} \). Consider the equations \( p_{\mu}(x) = 0 \) and \( q_{\mu}(x) = 0 \) from the Remark 23. For the general point \( s' \in G'/P' \) we have the following statements
1) The restrictions of \( p_\nu(x) = 0 \) and \( q_\nu(x) = 0 \) to \( s^{-1}T' \) nontrivial. The image of the set of zeros of these equations define nontrivial divisors on \( X' \), those proper transforms with respect to the blow up \( \sigma : X' \rightarrow X' \) are \((-1)\)-curves.

2) The varieties \( s^{-1}T' \) and \( G'/P' \) intersect in a single \( T'-\text{orbit} \) i.e. \( s^{-1}T' \cap G'/P' = T' \).

**Remark 4.3.** We give an alternative proof that differs from [2, 6.3] and can be written uniformly for \( p_\nu(x) \) and \( q_\nu(x) \) that generalizes to the case of \( E_8 \).

**Remark 4.4.** For the points \( x \in T' \) by the phrase: “the equation \( x_{w} = 0 \) (or \( p_\nu(x) = 0, q_\nu(x) = 0 \)) defines a divisor (curve) on the surface \( X' \),” for the brevity we mean the following: the equality \( x_{w} = 0 \) (or \( p_\nu(x) = 0, q_\nu(x) = 0 \)) defines a divisor of zeros of the section of the line bundle \( O(1) \) (corr. \( O(2), O(3) \)) on \( \mathbb{P}(V_i) \). We can intersect it with the torsor \( T' \). In the case when this intersection is nontrivial since the sections are seminvariant with respect to the action \( T' \) it follows that \( x_{w}|_{T'} = 0 \) (or \( p_\nu(x)|_{T'} = 0, q_\nu(x)|_{T'} = 0 \)) define also the divisors on the quotient \( X' = T'/\mathbb{P}(T') \).

**Proof.** The first part of the theorem is equivalent to the statement that the restrictions on \( T' \) of the equations \( p_\nu(s^{-1}x) = 0 \) and \( q_\nu(s^{-1}x) = 0 \) are nontrivial. Indeed let us notice that these polynomials are zero in the point \( s \), since \( p_\nu(x) \) and \( q_\nu(x) \) are equal to zero on the flag variety \( G'/P' \). By the induction assumption the equations \( x_{w} = 0 \) define the \((-1)\)-curves \( E_w \) on \( X' \), and also on \( T'/\mathbb{P}(s^{-1}T') \).

Consider the curves \( C \) of the degree 2 on \( X' \) with respect to the pairing with the canonical class (i.e. \( (K_{X'}, C) = 2 \)) passing through \( e_{rk} \) \( \Delta \) (it will become a \((-1)\)-curve after the blow up of the point \( e_{rk} \)). Let us describe these curves by the equations. From [2] it is easy to see that after the contraction of \( X_\Delta \) to \( \mathbb{P}^2 \) such curve maps to the line passing through \( e_{rk} \) and \( e_i \) for some \( i \) (for example when \( i = 1 \) we obtain \( C = L_0 - E_1 - e_{rk} \), or to the conic passing through any four points and the point \( e_{rk} \) \( (C = L_0 - E_1 - E_2 - E_3 - E_4 - e_{rk}) \). Or in the cubic (for example \( C = L_0 - 2E_1 - E_2 - E_3 - E_4 - E_5 - E_6 - E_7 - e_{rk} \)) that has the ordinary double point in some point \( e_i \) and passes through all points (this case occurs only when \( \Delta \) is of type \( E_7 \)). Let us notice that the dimensions of the linear systems defined by these curves is equal to 0. Let us show that \( p_\nu(s^{-1}x) = 0 \) defines a curve of the considered type (in the case when \( p_\nu(s^{-1}x) \) is not equal to zero on \( T' \)).

In the first case we have the following presentations of the class of curve \( C \) as the sum of two classes of \((-1)\)-curves (cf. fig.3): \( C = (L_0 - E_1 - E_i) + E_i \), where \( i \neq 1 \). In the second (cf. fig.3) and in the third case we have: \( C = (2L_0 - E_1 - E_2 - E_3 - E_4) + (L_0 - E_1 - E_2) = (L_0 - E_1 - E_2) + (L_0 - E_3 - E_4) = (L_0 - E_1 - E_2 - E_3 - E_4 - E_5) + E_5 = (2L_0 - E_1 - E_2 - E_3 - E_4 - E_5) + E_0 \).
Thus we get that \( p \) defines some element of this linear system (or a zero element) passing through the considered decompositions correspond to the reducible elements of this linear system. There linear combination \( \pi \) that \( \in \) the following decompositions correspond to presentation of some weight \( p \) in the form \( T \). The classes of \( (\mu G) \) curves. Let us take a point in its preimage in the torsor \( T \). Let us recall that equations \( x^0 \) defined by the equation \( x_{\gamma_i} = 0 \), and \( \delta_i \) is the weight corresponding to \( E_i \) with the equation \( x_{\delta_i} = 0 \). The linear system \( |L_0 - E_1| \) has the dimension 1 and the considered decompositions correspond to the reducible elements of this linear system. There linear combination

\[
p_{\mu}(s^{'-1}x) = \sum_{\pi - \gamma - \delta = \mu} x_{\gamma x_\delta} s_{\gamma} s_{\delta} (e_{-\gamma e - s v_{\pi}}) = 0,
\]

defines some element of this linear system (or a zero element) passing through \( e_{r \Delta} \).

Thus we get that \( p_{\mu}(s^{'-1}x) = 0 \) defines a curve \( C \).

Let us rewrite \( p_{\mu}(s^{'-1}x) = 0 \) as a polynomial on \( s' \)

\[
\tilde{p}_{\mu}(s') = p_{\mu}(s^{'-1}x) = \sum_{\pi - \gamma - \delta = \mu} p_{\gamma \delta} \frac{x_{\gamma x_\delta}}{s_{\gamma} s_{\delta}} s_{\gamma} s_{\delta} = 0,
\]

Now let us prove nontriviality of the restriction \( p_{\mu}(s^{'-1}x) = 0 \) on the torsor \( T' \) for a general point \( s' \in G'/P' \). Assume the contrary, then the restriction \( p_{\mu}(s^{'-1}x)|_{T'} \) is trivial for all \( s' \in G'/P' \), in other words \( \tilde{p}_{\mu}(s') \in \delta_{G'/P'} \) for all \( x \in T' \). It is easy to see that for \( \tilde{p}_{\mu}(s') \in \delta_{G'/P'}(\mu) \) all monomials from \( \tilde{p}_{\mu}(s') \) have the form \( s's' \) for \( \pi - \gamma - \delta = \mu \) (where the weight \( \mu \) is considered as the \( T' \)). Since \( \delta_{G'/P'}(\mu) \) is generated by the equation \( p_{\mu}(s') = 0 \), we get that the set of coefficients \( \{p_{\gamma \delta} \frac{x_{\gamma x_\delta}}{s_{\gamma} s_{\delta}}\} \) is proportional to the set \( \{p_{\gamma \delta}\} \). Thus for any pair \( \gamma_1, \delta_1 \) and \( \gamma_2, \delta_2 \), such that \( \pi - \gamma_1 - \delta_1 = \mu \) \( \pi - \gamma_2 - \delta_2 = \mu \) we have:

\[
\frac{x_{\gamma_1 x_{\delta_1}}}{s_{\gamma_1} s_{\delta_1}} = \frac{x_{\gamma_2 x_{\delta_2}}}{s_{\gamma_2} s_{\delta_2}}.
\]

Let us recall that equations \( x_{\omega} = 0 \) define \((-1)\)-curves on \( X_{\Delta'} \). Let us choose some point on the \((-1)\)-curve defined by \( x_{\gamma_1} = 0 \) and not lying on the other \((-1)\)-curves. Let us take a point in its preimage in the torsor \( T' \) with the coordinates \( \{x_{\omega}^0\} \). Then \( x_{\gamma_1}^0 = 0 \), but also \( x_{\gamma_2}^0 \neq 0 \) and \( x_{\delta_2}^0 \neq 0 \) that contradicts with the equality \((*)\). That proves the assertion.

Let us give a second proof of this fact and its generalization in the case of the cubic curve \( q_p(x) \).
Assume that claim 1) is not true. Then the restrictions \( \tilde{p}_\mu(s') = p_\mu(s's^{-1}x) \) (corr. \( \tilde{q}_\mu(s') = q_\mu(s's^{-1}x) \)) for \( x \in T' \) are identically equal to zero on \( G'/P' \). Consider the restriction \( \tilde{p}_\mu(s') (\tilde{q}_\mu(s')) \) on the open cell \( P_u' / P' = \exp(\sum c_\gamma e^{-\gamma}v_{\gamma\alpha - \alpha}) \) Then \( \tilde{p}_\mu(s') \) (corr. \( \tilde{q}_\mu(s') \)) are the polynomials (on the open cell identified with the linear space) from the coordinates \( c_\gamma \) where \( \gamma \in \Delta(p_u') \). We shall show that for some point \( x \in T' \) for the polynomial \( \tilde{p}_\mu(s') (\tilde{q}_\mu(s')) \) the coefficient by some monomial on the coordinates \( c_\gamma \) is not equal to zero. That implies nontriviality of \( \tilde{p}_\mu(s') \) (corr. \( \tilde{q}_\mu(s') \)) as a polynomial on \( s' \).

Let us fix the weight bases \( \{v_\omega\} \) and \( \{v'_\omega\} \) of the subspace \( V(\pi_\alpha) \) and \( V_1 \). Let us notice that we do not assume that these bases are concordant. The coefficients of the polynomials \( \tilde{p}_\mu(s') \) (corr. \( \tilde{q}_\mu(s') \)) we choose with respect to the basis \( \{v_\omega\} \). In other words

\[
\begin{align*}
v_{\pi\alpha - \gamma} &= e^{-\gamma}v_{\pi\alpha}, \quad \gamma \in \Delta(p_u), \\
e^{-\gamma}e^{-\delta}v_{\pi\alpha} &= p_\delta v_\mu, \quad \pi - \gamma - \delta = \mu, \\
e^{-\mu_1}e^{-\mu_2}e^{-\mu_3}v_{\pi\alpha} &= q_{\mu_1, \mu_2, \mu_3} v_\alpha, \quad \pi_\alpha - \mu_1 - \mu_2 - \mu_3 = \nu.
\end{align*}
\]

Correspondingly for the subspace \( V_1 \) we have:

\[
\begin{align*}
v'_{(\pi\alpha - \gamma)'} &= e^{-\gamma'}(e^{-\alpha}v_{\pi\alpha}) = n_{\gamma'}v_{\pi\alpha - \gamma}, \quad \gamma' \in \Delta(p_u'), \quad \gamma = \alpha + \gamma', \\
e^{-\gamma}e^{-\delta'}(e^{-\alpha}v_{\pi\alpha}) &= p_{\gamma'} v'_{\gamma'}, \quad (\pi_\alpha - \alpha) - \gamma' - \delta' = \mu'.
\end{align*}
\]

**Lemma 4.5.** We have the equality \( [e^{-\alpha}, e^{-\gamma}] = -n_{\gamma}e^{-\gamma}, \quad \gamma' \in \Delta(p_u'), \quad \gamma = \alpha + \gamma' \).

**Proof.** Since \( \gamma' \in \Delta(p_u') \) we have \( \{\alpha, \gamma\} = 0 \), that implies \( e_{\gamma'} v_{\pi\alpha} = 0 \). From the chain of equalities

\[
\begin{align*}
\gamma e^{-\gamma}v_{\pi\alpha} &= [e^{-\alpha}, e^{-\gamma}]v_{\pi\alpha} = (e^{-\alpha}e^{-\gamma} - e^{-\gamma}e^{-\alpha})v_{\pi\alpha} = -e^{-\gamma}e^{-\alpha}v_{\pi\alpha} = -n_{\gamma}e^{-\gamma}v_{\pi\alpha}.
\end{align*}
\]

we get that \( c = -n_{\gamma} \).

Without the loss of generality we can assume that the \((-1)\)-curve that correspond to \( v_{\pi\alpha - \gamma} = 0 \) is \( E_{\rk} \Delta' \), and one of the monomials \( p_\mu \) is equal to \( s_\alpha s_\omega \) where \( \omega = L - E_1 - \Omega_{\rk} \Delta' \). Let us recall that we consider the points \( s' = \{v\} \) that belong to the open cell i.e.

\[
v = v_{\pi\alpha - \gamma} + \sum_{\gamma \in \Delta(p_u)} \frac{c_{\gamma} e^{-\gamma}}{1!} v_{\pi\alpha - \gamma} + \frac{(\sum_{\gamma \in \Delta(p_u)} c_{\gamma} e^{-\gamma})^2}{2!} v_{\pi\alpha - \gamma} + \ldots.
\]

Consider the curves \( s' = 0 \) and \( s'_\delta = 0 \) where \( \pi_\alpha - \gamma = L - E_1 - E_2 \) and \( \pi_\alpha - \delta = E_2 \) (the weights \( \pi_\alpha - \alpha - \gamma \) and \( \pi_\alpha - \alpha - \delta \) as the weights of module \( V_1 \) correspond to the curves \( L - E_1 - E_2 \) and \( E_2 \). But we consider them as the curves on the del Pezzo surface where the curve \( E_{\rk} \Delta' \) is contracted). It is easy to check (by a direct calculation with weights or by considering all the curves in the linear systems to which belong the curves \( L - E_1 - E_2 \) and \( E_2 \) that only \( \gamma \) are independent). for \( \omega = \gamma + \delta - \alpha \), contain the monomial \( c_{\gamma} c_{\delta} \).

We obtain that \( v_{\pi\alpha - \omega} = p_{\gamma'} s'_\gamma c_{\gamma'} c_{\delta'} + \ldots \), where we omitted the monomials that do not containing \( c_{\gamma'} \) and \( c_{\delta'} \) (here we used that \( e_{\gamma'} \) for \( \gamma_i \in \Delta(p_u) \) commutes pairwise). Then the coefficient by \( c_{\gamma} c_{\delta} \) in the polynomial \( \tilde{p}_\mu(s') \) is equal to

\[
P_{\gamma'\delta}\frac{\sum_{\gamma, \delta} c_{\gamma} c_{\delta} x_{\gamma} x_{\delta}}{s_{\gamma} s_{\delta}} = n_{\gamma} s_{\gamma} p_{\gamma} s_{\delta} x_{\gamma} x_{\delta}
\]

\[
(*)
\]

Let us note that \( x_{\gamma} x_{\omega} \) and \( x_{\gamma} x_{\delta} \) are linearly independent in the linear system of conics on the del Pezzo surface \( X_{\Delta'} \). By contracting some \((-1)\)-curves we can obtain
that these monomials define a pair of lines that are the elements of the linear system of conics passing through 4 fixed points \( e_1, e_2, e_3, e_4 \) thus the equation (*) is not identically zero on the del Pezzo surface.

Let us prove the nontriviality of \( \tilde{q}_v(s') \). In the case of \( E_8 \) without loss of generality we can assume that the cubic is of the form \( C = L_0 - E_1 - E_2 - E_3 - E_4 - E_5 - E_7 - 2e_{rk} \Delta \) (this can be obtained by the contraction \( X_{rk} \Delta' \rightarrow \mathbb{P}^2 \) since the Weyl group \( W' \) is acting transitively on the weights of \( V_3 \) corresponding to the considered cubics). In the case of \( E_7 \) we consider the surface obtained by blowing up points \( e = i \) for \( i = 1 \ldots 7 \), \( i \neq 6 \). In this case such cubic is unique. Let us fix the (−1)-curves \( H_\alpha = E_7, H_\mu = E_2, H_\nu = L_0 - E_1 - E_2, H_\delta = E_1, H_\gamma = 2L_0 - E_1 - E_2 - E_3 - E_4 - E_5 \). The required cubic can be represented as

\[
C = L_0 - E_1 - E_2 - E_3 - E_4 - E_5 - E_7 - 2e_{rk} \Delta = E_7 + (E_2 + (L_0 - E_1 - E_2)) + (E_1 + (2L_0 - E_1 - E_2 - E_3 - E_4 - E_5)).
\]

Consider the expressions \( s_{\theta_1}'s_{\theta_2}'s_{\theta_3}' \) from the variables \( \{c_\theta\}_{\theta \in P_u'} \) and let us find those that contain the monomials \( c_\mu c_\nu c_\delta c_3 \). Let us notice that \( s_{\theta_1}' \) define some linear system; more precisely when \( s_{\theta_1}' = n_\zeta c_\zeta (\zeta \in P_u) \) it is a curve from \( X_{\Delta'} \) corresponding to the weight \( \pi - \alpha - \zeta \) (the same curve can be considered on \( X_{\Delta''} \), but then it will correspond to the weight \( \pi_{\beta} - \zeta \). If \( s_{\theta_1}' = \sum_{\zeta_1 + \zeta_2 = \gamma + \delta - 2\alpha} p_{\zeta_1} c_{\zeta_1} c_{\zeta_2} \) (for \( \zeta_1, \zeta_2 \in P_u \)) then the curve \( s_{\theta_1}' = 0 \) considered as the curve on \( X_{\Delta''} \) lie in the linear system generated by the pairs of (−1)-curves \( c_{\zeta_1} c_{\zeta_2} = 0 \) for all \( \zeta_1, \zeta_2 \in P_u \) (the argument is analogous in the case when \( s_{\theta_1}' \) is the cubic from the variables \( c_\zeta \)).

To do this we have to write down all linear systems corresponding to \( s_{\theta_1}', s_{\theta_2}', s_{\theta_3}' \), such that each linear system corresponding to \( s_{\theta_1}' \) contain the monomial that is a submonomial in \( c_\mu c_\nu c_\zeta c_3 \), and the product of such monomials is equal to \( c_\mu c_\nu c_\zeta c_3 \).

All possible generators \( s_{\theta_1}, s_{\theta_2}, s_{\theta_3} \) are given on fig. 5. (For the brevity on fig. 5, by the sum \( \sum c_{\zeta_1} c_{\delta} \), we mean a sum of type \( \sum_{\zeta_1 + \zeta_2 = \gamma + \delta - 2\alpha} p_{\zeta_1} c_{\zeta_1} c_{\zeta_2} \) in which we can find monomial \( c_\mu c_\nu c_\zeta c_3 \)).

Now it is not difficult to calculate the coefficient by the monomial \( c_\mu c_\nu c_\zeta c_3 \):

\[
q_{\alpha(\mu)\nu(\gamma)p_{\delta}}' x_\alpha x_{(\mu)}x_{(\gamma)}x_{(\delta)} + q_{\alpha(\mu)\nu(\gamma)p_{\delta}'} x_\alpha x_{(\mu)} x_{(\gamma)} x_{(\delta)} + q_{\alpha(\mu)\nu(\gamma)p_{\delta}''} x_\alpha x_{(\mu)} x_{(\gamma)} x_{(\delta)} + q_{\alpha(\mu)\nu(\gamma)p_{\delta}'''} x_\alpha x_{(\mu)} x_{(\gamma)} x_{(\delta)}.
\]
Then the restriction of the expression $e$ to the curve $l$ is identically equal to zero.

Lemma 4.6.

Let us do one of the calculations that are equivalent to each other:

$$q_{\mu\gamma(\nu\delta)}P_{\nu'\gamma'}^q R_{\gamma'}^{\mu'}n_{\mu'}^{n_{\gamma'}} \frac{x_{\mu}x_{\nu}(x_{\gamma})}{s_{\mu}s_{\nu}s_{(\nu\delta)}} + q_{\mu\nu(\delta\gamma)}P_{\gamma'\delta'}^p R_{\delta'}^{\nu'}n_{\nu'}^{n_{\gamma'}} \frac{x_{\mu}x_{\nu}(x_{\gamma})}{s_{\mu}s_{\nu}s_{(\nu\delta)}} \quad (**)
$$

(*** for brevity by $(\zeta_1, \zeta_2)$ we mean the weight $\zeta_1 + \zeta_2 - \alpha$)

The coefficients of the fractions are equal to $\pm 1$). To prove this let us notice that $e_{-\vartheta}$ for $\vartheta \in p_\mu$ commutes pairwise ($[e_{-\vartheta_1}, e_{-\vartheta_2}] \neq 0$ only in the case of $E_5$, but in this case they do not commute only when $\vartheta_1 + \vartheta_2 = \pi_\alpha$, that is not the case since such curves correspond to the cubic curves passing through the points $e_1 \ldots e_7$). Any two monomials in the expression (***) have a common variable $x_\vartheta$. Let us do one of the calculations that are equivalent to each other:

$$q_{\mu\gamma(\nu\delta)}v_{(\mu\gamma\nu\delta)} = e_{-(\nu\delta)}e_{\mu}e_{-\gamma}v_{\pi\alpha} = n_{\gamma}^{-1}n_{\mu}^{-1}e_{-(\nu\delta)}[e_{-\alpha}, e_{-\gamma}][e_{-\alpha}, e_{-\gamma}]v_{\pi\alpha} =$$

$$n_{\gamma}^{-1}n_{\mu}^{-1}e_{-(\nu\delta)}(e_{-\alpha}e_{-\mu'} - e_{-\mu}e_{-\alpha})(e_{-\alpha}e_{-\gamma'} - e_{-\gamma}e_{-\alpha})v_{\pi\alpha} =$$

$$= -n_{\gamma}^{-1}n_{\mu}^{-1}e_{-(\nu\delta)}e_{-\alpha}e_{-\mu'}e_{-\gamma'}e_{-\alpha}v_{\pi\alpha} = -n_{\gamma}^{-1}n_{\mu}^{-1}P_{\mu'}^{\gamma'}e_{-\alpha}e_{-(\nu\delta)}e_{-\alpha}v_{\pi\alpha} =$$

where we used that $e_{-\gamma}v_{\pi} = 0$, $e_{-\alpha}e_{-\gamma}v_{\pi} = 0$ and applied Lemma 4.5.

From the following elementary lemma it follows that the expression (**) cannot be identically equal to zero.

Lemma 4.6. Consider the points $e_i$ that are in the general position. Let $l_{ij}$ be a line passing through the points $e_i, e_j$ and $q_{ij}$ is a conic passing through the points $e_i, e_j, e_3, e_4, e_5$. $E_i$ as before denote the exceptional curves that are the preimages of $e_i$. Let us normalize the equations in such way that $l_{ij}(e_{\pi\Delta}) = q_{ij}(e_{\pi\Delta}) = 1$. Then the restriction of the expression

$$l_{12}q_{26}E_2 - l_{12}q_{16}E_1 + l_{16}q_{12}E_1 - l_{16}q_{26}E_6 + l_{26}q_{16}E_6 - l_{26}q_{12}E_2$$

on the curve $l_{\pi} = L - E_1 - e_{\pi\Delta}$ is not equal to zero (let us note that the line $l_{\Delta}$ considered as the curve on $X_{\Delta}$ has the degree 2 with respect to $K_{\Delta}$).

![fig.5](image-url)
Proof. Let us notice that the curve \( l_{\Delta} \) intersect each curve \( E_1, q_{16}, l_{26}, q_{12} \) in a single point that we denote by \( a, b, c \) and \( d \) correspondingly, and with the curve \( q_{26} \) in two points (see fig.7).

Since the lines \( l_{12} \) and \( l_{16} \) do not intersect \( l_{\Delta} \) the curves \( l_{16}q_{26}E_6 \) and \( l_{12}q_{26}E_2 \) define the same section on the line bundle \( O(2) \) on the curve \( l_{\Delta} \), that is also equal to \( q_{26}|_{l_{\Delta}} \). Indeed all three curves intersect \( l_{\Delta} \) in two fixed points and all the three sections are equal to 1 in the point \( e_{rk \Delta} \). From this we obtain that \( (l_{16}q_{26}E_6 - l_{12}q_{26}E_2)|_{l_{\Delta}} = 0 \). Without the loss of generality we assume that all points lie in the affine chart \( x = 0 \). Then from the normalizing condition in the point 0 (i.e. \( l_{ij}(e_{rk \Delta}) = q_{ij}(e_{rk \Delta}) = 1 \)) our expression can be written as

\[
- l_{12}q_{16}E_1 + l_{26}q_{16}E_6 - l_{26}q_{12}E_2 + l_{16}q_{12}E_1 =
\]

\[
- \frac{1}{ab}(x - a)(x - b) + \frac{1}{bc}(x - b)(x - c) - \frac{1}{cd}(x - c)(x - d) + \frac{1}{da}(x - d)(x - a) =
\]

\[
= \frac{(b - d)(c - a)}{abcd} x^2,
\]

thus we get that the last expression is equal to zero iff \( a = c \) or \( b = d \). But the first means that the points \( e_1, e_2, e_3 \) lie on the same line and the second states that the conics \( q_{12} \) and \( q_{16} \) intersect in 5 points that implies that 6 belong to the same conic that contradicts the generality of position.

The assertion 2) from the Proposition 4.2 follows easily from the fact that \( p_\mu(x) = 0 \) for \( \mu \in \text{supp}(V_1) \) defines \( G'/P' \), and from the fact that on the quotient \( X_{\Delta'} = T \setminus s's^{-1}T' \) these equations define conics intersecting transversally in a single point \( e_{rk \Delta} \).

\[\square\]

Let us prove the following proposition

**Proposition 4.7.** Let \( \mathcal{T} = p_{s^{-1}e^{-1}} \sigma_{0^{-1}} (s's^{-1}T') \) be a composition of proper transform of \( s's^{-1}T' \) with respect to the blow up \( \sigma_0 \) and preimage of the quotient morphism \( p_{s^{-1}e^{-1}} \). Then the quotient \( T \setminus \mathcal{T} \) is the del Pezzo surface \( X_{\Delta} \) obtained by the blow up of del Pezzo surface \( X_{\Delta'} \) in the point \( e_{rk \Delta} = p_{T'}(s) \).

Proof. Consider the quotient \( \mathcal{T} = \lambda \setminus \mathcal{T} \). It is easy to see that the proper transform of \( s's^{-1}T' \) with respect to the blow up \( Bl(\mathbb{P}(V_1), G'/P') \) of the space \( \mathbb{P}(V_1) \) in the flag subvariety \( G'/P' \) (contraction morphism we denote by \( \lambda_0 \)).

\[\square\]
Consider the following commutative diagram

\[
\begin{array}{ccc}
\hat{X} & \xrightarrow{p} & \hat{T} \\
\sigma & \downarrow & \downarrow \\
X_{\Delta'} & \xrightarrow{p_{T'}} & s's^{-1}T' \\
\end{array}
\]

Where the surface $\hat{X}$ is defined as the quotient $T'\backslash \hat{T}$. We also notice that since $s's^{-1}T' \cap G'/P' = T's'$ we have an isomorphism:

$$
\hat{T} \setminus \sigma_0^{-1}(T's') = s's^{-1}T' \setminus T's'.
$$

Passing to quotients we get the isomorphism

$$
\hat{X} \setminus \sigma^{-1}(e_{rk}\Delta) = X_{\Delta'} \setminus (e_{rk}\Delta).
$$

This implies that $\sigma$ is the composition of the blow ups with the centers outside $X_{\Delta'} \setminus (e_{rk}\Delta)$ or in other words $\hat{X}$ is obtained by the blow up of $X_{\Delta'}$ in the subscheme with the support in $e_{rk}\Delta$. Our aim is to show that $\sigma = X_{\Delta}$ or in other words that the ideal $\mathcal{I}_{e_{rk}\Delta}$ is maximal. Applying the universal property of the blow up (cf. [9, Prop. 7.14, Corr. 7.15]) to the right hand of the diagram we get that the ideal defining the blow up is equal to the restriction of the ideal $\mathcal{I}_{\mathcal{G}/P'}$ on $s's^{-1}T'$ in other words $\mathcal{I}_0 = \mathcal{I}_{\mathcal{G}/P'}|_{s's^{-1}T'}$.

Let us notice that we can check our proposition locally. Consider the $T'$-invariant affine chart $U = \text{Spec}(A)$ on $s's^{-1}T'$ containing the orbit $T's'$. The morphism $p_{T'}$ maps it to the affine chart $T'\backslash U = \text{Spec}(A^{T'})$ containing the point $e_{rk}\Delta$. Shrinking the map we assume that the divisors defined by $p_\mu(x)$ are principal and defined by the regular functions $f_\mu \in A^{T'} = \mathcal{O}(U)$. The inverse images of these functions are invariant functions on $U$ with the same divisors of zeros as $p_\mu(x)$ (Since the zeros of $p_\mu(x)$ are $T'$-invariant), thus they also define $\mathcal{I}_0|_U$.

Let us apply the universal property of the blow up to the left hand of the diagram. Reformulating [9, Prop. 7.14, Corr. 7.15], we get that $\mathcal{I}_0$ is equal to the ideal of $\mathcal{A}$, generated by the ideal $\mathcal{I}_{e_{rk}\Delta} \subset \mathcal{A}^{T'}$. Thus $\mathcal{I}_0' = \mathcal{I}_{e_{rk}\Delta}$ (cf. [9]). As we have seen the ideal $\mathcal{I}_{e_{rk}\Delta} = \mathcal{I}_0'$ contain the functions $f_\mu$, but the zeros of these functions coincide with the curves $p_\mu(x) = 0$ on the surface $X_{\Delta'}$. Any two curves from this set intersect transversally in the point $e_{rk}\Delta$ (though it is sufficient to know the intersection of the curves $L - E_1 - e_{rk}\Delta$ or $L - E_2 - e_{rk}\Delta$). This implies that the ideal $\mathcal{I}_{e_{rk}\Delta}$ coincide with the maximal ideal of the point $e_{rk}\Delta$.

For the case of $E_8$ let us prove an analog of Proposition 4.7

**Proposition 4.8.** Let $G$ be a group of type $E_8$, $p_{\sigma_0}$ is the quotient morphism with respect to the action of $\lambda$, and $\sigma_0$ is the morphism $\lambda\backslash((G/P)^{ss} \cap D_{h_{ss}}) \rightarrow \mathbb{P}(V_1)$, defined by the projection $p_0$ on $\mathbb{P}(V_1)$. Assume we have an embedding of the universal torsor $T'$ over del Pezzo surface of the degree 2 in $G'/P' \subset \mathbb{P}(V_1)$, that is not contained in any $T'$-invariant hyperplane. Let us fix sufficiently general points $s, s' \in G'/P'$ and denote by $T = p_{s^{-1}\sigma_0^{-1}}(s's^{-1}T')$ the composition of the proper transform of $s's^{-1}T'$ with respect to the weighted blow up $\sigma_0$ and its inverse image of the quotient morphism by $\lambda$. Then the quotient $T\backslash T$ is the surface $X_{\Delta}$ obtained by blowing up the del Pezzo surface $X_{\Delta'}$ in $e_{rk}\Delta = p_{T'}(s)$.

**Proof.** For the proof of the proposition let us consider the $T'$-invariant affine chart $(G/P)^{ss} \cap D_{h_{ss}} \cap \U_{ss}$. According to Proposition 4.8 $\lambda\backslash((G/P)^{ss} \cap D_{h_{ss}} \cap \U_{ss}) \rightarrow \mathbb{P}(V_1) \cap \U_{ss}$ is isomorphic to the weighted blow up $\sigma_0$ of the variety $\mathbb{P}(V_1) \cap \U_{ss}$ in...
The preimage $\sigma_0^{-1}(P_u^r - P'/P')$ is identified with $P^r - s_L: \mathbb{P}_{\text{wt}}(Kv_\alpha \oplus V_2')$. Since the stabilizer of the action of $T'$ in the point $s'$ is trivial the variety $\sigma_0^{-1}(T^{s'})$ is identified with $T^{s'} \times \mathbb{P}_{\text{wt}}(Kv_\alpha \oplus V_2')$. In particular $T^{s'} \setminus \sigma_0^{-1}(T^{s'}) \cong \mathbb{P}_{\text{wt}}(Kv_\alpha \oplus V_2')$.

By the induction assumption we have the embedding $\mathcal{T}' \subset G'/P'$. As it was proved $s^{-1}\mathcal{T}'$ intersects $G'/P'$ transversally in the orbit $T^{s'}$. The quotient $T^{s'} \setminus T^{s^{-1}\mathcal{T}'}$ a smooth del Pezzo surface of degree 2. This implies that in the neighbourhood of the point $p_{G'}(s')$ the germ of this surface is the complete intersection and is defined by the system of equations $(l = \dim V_1 - \dim T' - 4)$. Since $T'$-variety can be covered by $T'$-invariant affine maps we can assume that $s' \in U_{\gamma'}$ for some $T'$-invariant affine chart and $f_i \in K[T^{s'}]/U_{\gamma'} = K[U_{\gamma'}][T']$. From the above we see that the germ of the torus $s^{-1}\mathcal{T}'$ is also defined by the considered regular sequence $(f_0, \ldots, f_l)$.

Let $\mathcal{F} = \lambda\mathbb{A}_{s'}$ be the proper transform $s^{-1}\mathcal{T}'$ with respect of the weighted blow up $\sigma_0$. Our aim is to show that the quotient $\mathcal{F} \setminus \sigma_0^{-1}(s') \cong \mathcal{T} \setminus \mathbb{P}_{\text{wt}}(Kv_\alpha \oplus V_2')$ by the action of torus $T'$, is isomorphic to $\mathbb{P}^1$.

On the considered $T'$-invariant chart $U_{\gamma'}$ we can assume that the variety $P^r_u - P'/P'$ is defined by the vanishing of the equations $p_{\pi_\alpha-\gamma_i}, q_\alpha$. It is not difficult to check that these equations after restriction to the affine chart $v_\pi_\alpha \neq 0$ define equations corresponding to vanishing of the components with the weights $s_{\pi_\alpha}(\pi_\alpha-\gamma_i)$ and $\gamma_i$ correspondingly for the flag variety $G'/P$. In particular if we restrict on the quotient $T^{s'} \setminus s^{-1}\mathcal{T}'$, the equations $p_{\pi_\alpha-\gamma_i} = 0$ define on the del Pezzo surface of degree 2 the conics passing through the point $e_\gamma$, and the equation $q_\alpha = 0$ defines a cubic with the double point in $e_\gamma$.

Let us recall that the weighted blow up $\sigma_0 : \mathbb{A}_{s'}((G/P)_{ss} \times D_{h_{\pi_\alpha}}) \rightarrow \mathbb{P}(V_1)$ is defined as the subvariety in $\mathbb{P}^{\dim V_1-1} \times \mathbb{P}_{\text{wt}}(Kv_\alpha \oplus V_2')$ by the system of equations:

\[
\begin{align*}
p_{\pi_\alpha-\gamma_i}c_{\gamma_i} &= p_{\pi_\alpha-\gamma_i}c_{\gamma_i} \\
q_\alpha c_{\gamma_i} &= q_\alpha c_{\gamma_i}
\end{align*}
\]

Let us represent the equations $f_i$ defining $s^{-1}\mathcal{T}'$ as the sum of linear part from the variables $p_{\pi_\alpha-\gamma_i}$ (that we assume to be nonzero) and a remainder that consists of the monomials of the bigger weight with respect to the weighted grading in consideration:

\[
f_i = \sum a_{ij}p_{\pi_\alpha-\gamma_j} + f_i^{>1},
\]

where the coefficients $a_{ij}$ depend only on variables $b_i$. Moreover we assume that $a_{ij}$ are written as the polynomials from $(b_i - (b_i)_\alpha \cdot v)$ with the constant terms $\alpha_{ij}$, where $(b_i)_\alpha$ are the coordinates of the point $s'$ in $P^r_u - P'/P'$. Our aim is to investigate which variety is the defined by the equation $f_i = 0$ in the fiber over the point $s'$, where all the coordinates $p_{\pi_\alpha-\gamma_i}, q_\alpha$ vanish. In the chart $c_{\gamma_i} \neq 0$ we get:

\[
f_i = p_{\pi_\alpha-\gamma_i} \sum a_{ij}c_{\gamma_i} + p_{\pi_\alpha-\gamma_i}^{>1},
\]

where $f_i^{>1}$ is the polynomial from $p_{\pi_\alpha-\gamma_i}$. Dividing the expression for $f_i$ by $p_{\pi_\alpha-\gamma_i}$, setting $p_{\pi_\alpha-\gamma_i} = 0$ and considering the homogenous equation with the variables $c_{\pi_\alpha-\gamma_i}$, we obtain that the intersection of the proper transform of the variety $f_i = 0$ and a fiber $\sigma_0^{-1}(s') \cong \mathbb{P}_{\text{wt}}(Kv_\alpha \oplus V_2')$ is defined in the fiber $\mathbb{P}_{\text{wt}}(Kv_\alpha \oplus V_2')$ by the equation

\[
f_\gamma = \sum a_{ij}c_{\gamma_j}.
\]

In a similar way we get that the equation

\[
f = q_\alpha + \sum d_{ij}p_{\pi_\alpha-\gamma_j}p_{\pi_\alpha-\gamma_j} + f_i^{>2}
\]
in the fiber over the point \( s' \) isomorphic to \( \mathbb{P}_{\text{wt}}(\mathbb{K}v_o \oplus V_2') \) defines a hyperplane \( f = c_\alpha + \sum \overline{d}_{ij} c_{\gamma_i} c_{\gamma_j}, \) where \( \overline{d}_{ij} \) is defined similar to \( \pi_{ij}. \)

Let us show that the matrix that consists of the homogenous components \( \mathbf{T}_{ji} \) of the degree 1 (of the polynomials \( f_j \)) has a rank equal to \( \dim \mathbb{P}_{\text{wt}}(\mathbb{K}v_o \oplus V_2') - 2. \) Let us notice that on the surface \( \mathbb{T}' \setminus \mathbb{s}' \times \mathbb{T}' \) the equation \( q_\alpha = 0 \) defines the cubic with a double point in \( e_s. \) In particular this implies that the differential \( dq_\alpha = 0 \) after the restriction on the tangent space to the torsor in the point \( s'. \) Thus we have the linear dependence between \( dq_\alpha \) and \( df_0, \ldots, df_\ell. \) In other words we can assume that \( df_1, \ldots, df_\ell \) and \( dq_\alpha \) are linear dependent, and a linear part of the polynomial \( df_0 \) is proportional to \( q_\alpha. \) Let us notice that the restrictions of \( df_i \) to the subspace \( u s' \subset T_{s'}(G'/P') \) are zero. In the space generated by the forms \( db_\gamma, \) consider the space of forms that are zero on the subspace \( u s'. \) Let us chose a basis \( \overline{d}_b \overline{k} \) in this space, where \( k = \dim G'/P' - \dim T'. \) Let us write down \( df_i \) in the basis \( dp_{\pi_{\gamma_i} - \gamma_j}, dq_\alpha, \overline{d}_b \overline{k} \) for \( i \geq 1. \) We get a matrix in which in the first \( \dim (V_1) - \dim G'/P' - 2 \) rows we have elements \( \pi_{ij}. \) Since the matrix formed by \( df_1, \ldots, df_\ell \) has the rank \( \dim V_1 - \dim T' - 3, \) than the matrix of the coefficients \( \pi_{ij} \) has the rank not less than \( \dim \mathbb{P}(V_1) - \dim T' - 3 = (\dim G'/P' - \dim T') = \dim V_2' - 2. \) Let us notice that on the surface \( \mathbb{T}' \setminus \mathbb{s}' \times \mathbb{T}' \) the equations \( p_{\pi_{\gamma_i} - \gamma_j} = 0 \) define the conics passing through \( e_s \) and intersecting transversally there. This implies that the differentials \( dp_{\pi_{\gamma_i} - \gamma_j}, dp_{\pi_{\gamma_i} - \gamma_j} \) generate the tangent space to the surface in \( e_s, \) that gives the linear independence of \( dp_{\pi_{\gamma_i} - \gamma_j}, dp_{\pi_{\gamma_i} - \gamma_j}, df_0, \ldots, df_\ell. \) Thus the rank of the matrix \( \pi_{ij} \) is not bigger than \( \dim V_2' - 2 \) and should be equal to this number.

The preceding arguments show that the equations \( \mathbf{T}_{ji} \) in the space \( \mathbb{P}_{\text{wt}}(\mathbb{K}v_o \oplus V_2') \) the weighted projective subspace \( \mathbb{P}(2, 1, 1). \) The equation \( \mathbf{T}_0 = c_\alpha + \sum \pi_{ij} c_{\gamma_i} c_{\gamma_j} = 0 \) with respect to the quotient by the action of the involution \( \mathbb{P}^2 \to \mathbb{P}(2, 1, 1) \) that reverses sign of the first coordinate. Thus we get that the fiber over the point \( s' \) consists of the smooth rational curve (A simple calculation in the local coordinates shows that the obtained surface is smooth, see next remarks).

That we get that \( \mathbb{T}' \setminus \mathbf{T} \) is the surface \( X_{\Delta'} \) obtained by the blow up of del Pezzo surface \( X_{\Delta} \) in the point \( e_{ki} \Delta = pr'(s). \)

**Remark 4.9.** To define the intersection of the fiber \( \sigma_0^{-1} \) and a proper transform of the hypersurface \( f = 0 \) we can argue in the following way. Let us use the realization of the blow up from the Remark [3.10]. Let \( \mathbf{T} \) be a component of \( f \) that consists of the monomials with the minimal grading \( d \) in the variables \( z_i: \)

\[
\mathbf{T} = \sum_{a_{ij} = d} p_{i_1, \ldots, i_j} (w) z_i^{a_{i_1}} \cdots z_i^{a_{i_j}}
\]

Then we have \( \tau^* f = \sum_{a_{ij} = d} p_{i_1, \ldots, i_j} (w) z_i^{a_{i_1}} \cdots z_i^{a_{i_j}} + f > d, \) where the component \( f > d \) has degree strictly bigger than \( d. \) The intersection of \( \sigma_0^{-1} \) and of the proper transform \( f = 0 \) for the ordinary blow up \( Bl \) is defined by the homogeneous component of minimal degree \( \tau^* \mathbf{T} = \sum_{a_{ij} = d} p_{i_1, \ldots, i_j} (w) z_i^{a_{i_1}} \cdots z_i^{a_{i_j}} \) \( ( \text{In terms of } \text{Proj} \) this is the projection on \( \mathbb{g}^d / \mathbb{g}^{d+1}, \) where \( \mathbb{g} = (z_1, \ldots, z_l) \)). To get the equation of the intersection of \( \sigma_0^{-1} \) and a proper transform of the hypersurface \( f = 0 \) for the weighted blow up \( Bl, \) consider the quotient \( (\tau^* \mathbf{T} = 0) \) by \( \tau_\mathbb{g}. \) It is easy to see that it is defined by the equation \( \sum_{a_{ij} = d} p_{i_1, \ldots, i_j} (w) z_i^{a_{i_1}} \cdots z_i^{a_{i_j}} = 0. \) In more invariant terms the latter equation is the projection on the component of the degree \( d \) in the algebra \( \text{Proj}_{\mathbb{P}_{\text{wt}}(\mathbb{K}v_o \oplus V_2')} \mathbb{g} / \mathbb{g}^{d+1}, \) where we consider the weighted grading where the generators \( \mathbb{g} = (z_1, \ldots, z_l) \) carry the weights \( (a_1, \ldots, a_l). \)
Remark 4.10. The preceding remark provides us an illustration of the situation described in Proposition 4.3. Consider the blow up in zero point of the cone \( z_0^2 = z_1z_2 \), lying in the affine space \( \mathbb{A}^3 \). This blow up give a resolution of the singularity. And over the single point we have a conic \( \xi_0^2 = \xi_1\xi_2 \), that is a \((-2)\)-curve. Now consider the involution on \( \mathbb{A}^3 \times \mathbb{P}^3 \) acting on the coordinates \( z_0 \) and \( \xi_0 \) by multiplication with \(-1\). The quotient of the blow up \( \mathbb{A}^3 \) in zero by the action of this involution is the weighted blow up defined in \( \mathbb{A}^3 \times \mathbb{P}(2,1,1) \) by the equations
\[
\begin{cases}
z_0\xi_0^2 = z_1^2\xi_0 \\
z_1\xi_2 = z_2\xi_1
\end{cases}
\]
The quotient of the considered blown up cone is the proper transform with respect to the weighted blow up of the surface defined by the equation \( z_0 = z_1z_2 \). Let us notice that after taking the quotient by this involution the conic \( \xi_0^2 = \xi_1\xi_2 \), that is exceptional curve over zero become a two sheeted covering of the rational curve in \( \mathbb{P}(2,1,1) \) defined by the equation \( \xi_0 = \xi_1\xi_2 \). By the projection formula from the intersection theory the latter curve is the \((-1)\)-curve. Besides it is easy to see that the image of the blown up cone is the smooth surface that is the ordinary blow up of affine plane.

From this proposition it is easy to get the second assertion of the theorem. Indeed let \( \omega \) be the neighbour weight to \( p_\alpha \), then \( \omega \in V_1 \). The curve \( E_\pi,\alpha \) is the blow up of the point \( e_\pi,\alpha \) on the surface \( X_\Delta' \), the curves \( E_\omega \) are \((-1)\)-curves of the surface \( X_\Delta' \), that do not intersect with \( E_\pi,\alpha \), since they do not pass through \( e_\pi,\alpha \).

The curves \( E_\mu \) and \( E_\nu \) correspond to nonadjacent weights are defined by the equations \( p_\mu = 0 \) and \( q_\nu = 0 \) where \( \mu \in V_2 \), \( \nu \in V_3 \) are the curves of the degree 2 and with respect to \(-K_{X_\Delta'}\). The intersection indexes with the curve \( E_\pi,\alpha \) are equal to 1 and 2, since the first curve passes through the point \( e_\pi,\alpha \), and the second has a double point in \( e_\pi,\alpha \).

Now we are able to prove the main theorem (in this step we give the argument of [2]).

We have constructed embedding \( \mathcal{I} \subset (G/P)^* \). Consider the weight hyperplanes \( H_\omega = \{(v) \in \mathbb{P}(V(\pi,\alpha)) \mid v_\omega = 0\} \). By construction the divisor \( E_\omega = \tau(H_\omega \cap \mathcal{I}) \subset X_\Delta \) is the \((-1)\)-curve on \( X_\Delta \), besides all \((-1)\)-curve can be obtained in such way. It is well known [2] that \( E_\omega \) generate \( \text{Pic}(X_\Delta) \).

For \( \tilde{T} \) let us take the preimage in \( V(\pi,\alpha) \) of the torsor \( \mathcal{I} \) with respect to the projectivization \( V(\pi,\alpha) \setminus \{0\} \rightarrow \mathbb{P}(V(\pi,\alpha)) \).

Let \( (G/P)^*_T \) be the set of stable points for the sheaf \( i^*\mathcal{O}(1) \) and the action of \( T \) (where \( i : G/P \subset \mathbb{P}(V(\pi,\alpha)) \)). Consider the subset \( (G/P)^s_T \subset (G/P)^s \) of the points with the stabilizer \( Z(G) \). As it will be showed the set \( (G/P)^s_T \) has the codimension \( > 1 \) in \( G/P \), thus we have \( \text{Pic}((G/P)^s_T) = \text{Pic}(G/P) = \Xi(P) = \Xi(\pi,\alpha) \) (cf. [14]). Let us denote by \( G/P \) the preimage of \( (G/P)^s_T \) with respect to the projectivization \( V(\pi,\alpha) \setminus \{0\} \rightarrow \mathbb{P}(V(\pi,\alpha)) \).

Thus it is easy to see that \( \text{Pic}(G/P) = 0 \). Since there are no regular \( \tilde{T} \)-semivariant invertible functions on \( G/P \), then from the exact sequence we get that the torsor \( (G/P)^s_T \) is universal. The images of the hyperplanes \( \tilde{H}_\omega = \{(v) \in V(\pi,\alpha) \mid v_\omega = 0\} \) by the quotient morphism generate \( \text{Pic}(\tilde{T}\setminus(G/P)) \cong \text{Pic}(G/P) \).

Let us show that the torsor \( \tilde{T} \) is also universal. Using the exact sequence (CTS) for the torsors \( \tilde{T} \) and \( G/P \) we obtain a commutative diagram where the right vertical arrow is induced by the embedding \( \tilde{T} \hookrightarrow G/P \):
But we have seen that the quotients $\overset{\sim}{\Gamma} \backslash \overset{\sim}{H}_\omega$ of the hypersurfaces $\overset{\sim}{H}_\omega$ generate $\text{Pic}(X_\Delta)$. Thus we get that the vertical map is surjective that implies that all maps in commutative diagram are isomorphisms. That implies the universality of the torsor $\overset{\sim}{\Gamma}$.

In the case when the root system of type $E_8$ the main theorem is formulated by the following way.

\textbf{4.11.} Let $\Delta$ be the root system of type $E_8$. Consider the embedding of $G/P$ in the projectivization of adjoint representation $\mathbb{P}(V(\pi_{\alpha})) = \mathbb{P}(g)$. For the left action of maximal torus $T$ on $G/P$ let us denote by $(G/P)_s^\omega$ the set of stable points with respect to the $G$-linearized line bundle $O(2)|_{G/P} \otimes k_{-\pi_{\alpha}}$. Let $\tau : (G/P)^s_\omega \rightarrow T \backslash (G/P)^s_\omega$ be a quotient morphism. Consider a sufficiently general del Pezzo $X_\Delta$ of degree $1$. There exists an embedding $\iota$ of the surface $X_\Delta$ in the quotient $T \backslash (G/P)^s_\omega$, such that for the torsor $T = \tau^{-1}(\iota(X_\Delta))$ the following conditions hold:

- Consider a hyperplane $H_\omega = \{\langle v \rangle \in \mathbb{P}(V(\pi_{\alpha})) \mid v_\omega = 0\}$. The divisor $E_\omega = \tau(H_\omega \cap T) \subset X_\Delta$ is the $(-1)$-curve on $X_\Delta$.
- The $(-1)$-curves $E_{\omega_1}$ and $E_{\omega_2}$ do not intersect iff the weights $\omega_1$ and $\omega_2$ are not adjacent on the weight polytop.

\textbf{Proof.} Assume we constructed embedding of the torsor $\overset{\sim}{T}'$ over $X_\Delta'$. in the flag variety $G'/P' \subset \mathbb{P}(V_1)$. Let $e_8 \in X_\Delta$ be a point that we blow up to obtain $X_\Delta$ from $X_\Delta'$. Let us choose a point $s \in G'/P'$, such that $\tau(s) = e_8$. By theorem \textbf{4.2} for a sufficiently general $s' \in G'/P'$ we have $s' s^{-1} T' \cap G'/P' = T s'$. Let us notice by Proposition \textbf{4.8} the preimage of $s' s^{-1} T'$ for the weighted blow up $\text{Bl}_{\text{wt}}(\mathbb{P}(V(\pi_{\beta}')), G'/P') \rightarrow \mathbb{P}(V(\pi_{\beta}'))$ is isomorphic to the $T'$-torsor over $X_\Delta$, that we denote by $\overset{\sim}{T}'$.

By the Theorem \textbf{3.5} the quotient $\lambda \backslash (G/P)^s_\omega \cap D_{h_\omega} \cap U_{\alpha}$ is isomorphic to $\text{Bl}_{\text{wt}}(\mathbb{P}(V(\pi_{\beta}')), G'/P')$. Let us define a required torsor taking the preimage of $\overset{\sim}{T}'$ with respect to the quotient by $\lambda$, and then taking affine cone over it in $V(\pi_{\alpha})$ over it.

The arguments are completely analogues to the proof of the main Theorem \textbf{4.1}. Thus we give only the missing steps. Let us prove the first assertion of the theorem.
Our aim is to show that for a sufficiently general point \( e_8 \) the homogeneous polynomial \( q_4(s' s^{-1} x) \) of weight 4 is not identically zero when we restrict it to the torsor for a sufficiently general point \( s' \) (for the equation of lower degrees we proved it in the Theorem 4.2).

Consider the exponential representation for the point \( \langle v \rangle \in P_u/P'P \):

\[
\langle v \rangle = \langle \exp \left( \sum_{\gamma \in \Delta P_u} x_\gamma e_{-\gamma} v_{\pi_\alpha} \right) \rangle.
\]

Let us recall that the equation \( q_4(x) \) of the degree 4 from \( \{ x_\gamma \}_{\gamma \in \Delta P_u} \) is obtained by requiring that the coordinate by the vector \( v(-\pi_\alpha) \) is zero. This polynomial define on the surface \( X_{\Delta'} \) a curve of the degree four with respect to the pairing with \(-K_{X_{\Delta'}}\).

Let us argue as in the proof of proposition 4.2. We assume that the point \( s' \) belong to the open cell of \( G'/P' \). Thus we can represent it as the exponential map

\[
s' = \langle \exp \left( \sum_{\gamma \in \Delta P_u} c_\gamma e_{-\gamma} v(\pi_\alpha - \alpha) \right) \rangle.
\]

Substitute \( s' \) by this expression in \( q_4(s' s^{-1} x) \).

And consider the coefficient \( R \) by the monomial \( c_{\delta_1}^2 c_{\delta_2}^2 c_{\delta_3}^2 \) where \( \delta_1 = L_1 = L - E_1 - E_2, \delta_2 = L_2 = L - E_3 - E_4, \delta_3 = L_3 = L - E_5 - E_6 \) (see fig.8). Our aim is to prove the non-triviality of this coefficient.

The submonomials of this monomial occur in expressions of the linear systems generating the following curves:

\[
C = 3L - E_1 - E_2 - E_3 - E_4 - E_5 - E_6 - 2E_7 \quad \text{corr. to the weight } \delta_1 + \delta_2 + \delta_3 - 2\alpha,
\]

\[
Q_1 = 2L - E_3 - E_4 - E_5 - E_6 - E_7, \quad \text{corr. to the weight } \delta_2 + \delta_3 - \alpha,
\]

\[
Q_2 = 2L - E_1 - E_2 - E_5 - E_6 - E_7, \quad \text{corr. to the weight } \delta_1 + \delta_3 - \alpha,
\]

\[
Q_3 = 2L - E_1 - E_2 - E_3 - E_4 - E_7, \quad \text{corr. to the weight } \delta_1 + \delta_2 - \alpha.
\]

Using the notation of Proposition 4.2 by \( (\delta_1, \delta_2, \delta_3) \) we denote the weight \( \delta_1 + \delta_2 + \delta_3 - 2\alpha \), corresponding to the cubic \( C \). In a similar way denote by \( (\delta_i, \delta_j) \) the weight \( \delta_i + \delta_j - \alpha \) corresponding to the conic \( Q_k \) \( (k \neq i, j) \).

The coefficient by the monomial \( c_{\delta_1}^2 c_{\delta_2}^2 c_{\delta_3}^2 \) we denote by \( R \). Let us notice that \( x_2^2 \delta_1, \delta_2, \delta_3 \) (the last monomial defining \( C'^2 \)) is contained only in one monomial \( \sum x_2^2 \delta_1, \delta_2, \delta_3 \delta_4 \delta_5 \delta_6 \delta_7 \), from \( R \).
Let us choose the coordinate \( L \) in such way that the point \( e \) is strictly bigger than zero for sufficiently small \( \varepsilon \). Let us denote \( r \) a bounded by a constant \( r_{max} \).

The coefficient by \( x^6 \) in the expression for \( R \) is bigger than \( \frac{1}{e^6(1+c^2)} - C_1 \frac{r_{max}r_{max}}{c} - C_2 \frac{r_{max}r_{max}}{c} \) where \( C_1 \) is the number of monomials in \( R \) containing \( x_{(s_1,s_2,s_3)} \), and \( C_2 \) is the number of monomials in \( R \) not containing \( x_{(s_1,s_2,s_3)} \), and the last expression is strictly bigger than zero for sufficiently small \( \varepsilon \). Thus \( R \) as a function of \( x \) is not identically zero for a sufficiently general point \( e_8 \). That proves the theorem.

5. Appendix

The Proposition \[ \] is a particular case of the following proposition:

**Proposition 5.1.** Let \( H \subset G \) be some semisimple subgroup normalized by the torus \( T \). Let \( V(\omega) \) be an irreducible \( G \)-module with the highest weight \( \omega \). Consider the weight vector \( v_\chi \in V(\omega) \) for which the weight \( \chi \) is the vertex of the weight polytop of the representation \( V(\omega) \), in other words \( \chi = w\omega \) for some \( w \in W \). Consider the module \( V_H = \langle H \rangle \) generated by the vector \( v_\chi \). Then we have an equality

\[
G(\langle v_\omega \rangle) \cap \mathbb{P}(V_H) = H(\langle v_\chi \rangle).
\]

**Proof.** Without the loss of generality we can assume that \( \chi = \omega \). Let us denote by \( P_\omega \) the stabilizer in \( G \) of the line \( \langle v_\omega \rangle \). Since \( v_\omega \) is the highest weight vector for the group \( G \), it is highest weight vector for \( H \). Thus \( V_H \) is the irreducible \( H \)-module. Let us notice that we can identify \( \text{supp}_H(V_H) \) with the subset in \( \text{supp}(V) \cap (\omega - \sum_{\gamma \in \Delta_H} \mathbb{Z}\gamma) \). From the inclusion \( \omega \in \text{supp}(V_H) \) for a dominant weight \( \omega \), it follows that the vertices of the weight polytop of \( V \), that belong to \( \text{supp}_H(V_H) \) are the vertices of the weight polytop \( V_H \).
Let \( \langle v \rangle \in G/P_w \cap \mathbb{P}(V_{Hv}) \). Let us prove that the vector \( v \) has a nonzero component of weight \( w_\omega \) for \( w \in W_H \) (we recall that \( w_\omega \) being a vertex of the weight polytop has multiplicity one). Indeed \( \langle v \rangle \in \bigcup_{w \in W} B^w P_w \cap \mathbb{P}(V_{Hv}) \), in particular \( \langle v \rangle \in B^\omega \overline{w} P_w \cap \mathbb{P}(V_{Hv}) \) for some \( \overline{w} \in W \). By Lemma 1.1 we get that the component \( v_{\overline{w}w_\omega} \) of vector \( v \) is not equal to zero. Since \( \langle v_{\overline{w}w_\omega} \rangle \in \mathbb{P}(V_{Hv}) \) we can assume that the element \( \overline{w} \in W \) belong to \( W_H \). Applying the element \( n_\omega^{-1} \) to the vector \( v \) we can assume that its component \( v_\omega \) is not equal to zero. Then from Lemma 1.1 we get that \( \langle v \rangle \) belong to the open cell \( P^-_w P/P \) in \( G/P \). Thus \( \langle v \rangle = \langle w_\omega \rangle \) for some \( u \in P^- \). Using the presentation of \( u \) as the exponential map we get:

\[
\langle v \rangle = \langle \exp( \sum_{\gamma \in \Delta^+_{\mu}} c_\gamma e^{-\gamma}) v_\omega \rangle.
\]

Assume that there exists \( \gamma \notin \Delta^+_H \) such that \( c_\gamma \neq 0 \). From such roots let us choose the root \( \gamma_0 \) that cannot be obtained as the positive integer linear combination of the roots \( \mu \in \Delta_{P_u} \) for which \( c_\mu \neq 0 \). Then the component of vector \( v \) of weight \( \omega - \gamma_0 \), is equal to \( v_{\omega - \gamma_0} = c_{\gamma_0} e^{-\gamma_0} v_\omega \). Let us show that this component is not equal to zero. The weight \( \omega \) is dominant and \( \langle \omega; \gamma_0 \rangle > 0 \) (the last equality follows from the fact that \( P \) is the stabilizer of the line spanned by the highest weight vector with the weight \( \omega \) and the fact that \( \gamma_0 \in \Delta_{P_u} \)). This implies that it cannot be the lowest weight of the representation of the three dimensional subalgebra generated by the triple \( \{ e_{-\gamma_0}, h_{\gamma_0}, e_{\gamma_0} \} \) thus we get \( e_{-\gamma_0} v_\omega \neq 0 \).

On the other hand the component \( v_{\omega - \gamma_0} \) cannot be nonzero since \( v \in V_H \) and \( \text{supp}(V_H) \subset \omega - \sum_{\gamma \in \Delta^+_{H}} \mathbb{Z}_+ \gamma \). Thus we come to the contradiction with the existence of \( \gamma_0 \notin \Delta^+_H \) such that \( c_{\gamma_0} \neq 0 \); this implies the inclusion \( u \in P^-_w \cap H \). The proof finishes similar to the proof of Proposition 1.11.\( \square \)

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