Analytic geometry

Quot schemes and Ricci semipositivity

Schéma quot et semi-positivité de Ricci

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\section*{A B S T R A C T}

Let $X$ be a compact connected Riemann surface of genus at least two, and let $\mathcal{Q}_X(r,d)$ be the quot scheme that parameterizes all the torsion coherent quotients of $\mathcal{O}_X^{\text{tr}}$ of degree $d$. This $\mathcal{Q}_X(r,d)$ is also a moduli space of vortices on $X$. Its geometric properties have been extensively studied. Here we prove that the anticanonical line bundle of $\mathcal{Q}_X(r,d)$ is not nef. Equivalently, $\mathcal{Q}_X(r,d)$ does not admit any Kähler metric whose Ricci curvature is semi-positive.

\section*{R É S U M É}

Soit $X$ une surface de Riemann compacte et connexe de genre au moins deux, et soit $\mathcal{Q}_X(r,d)$ le schéma quot qui paramètre tous les quotients torsion cohérents de $\mathcal{O}_X^{\text{tr}}$ de degré $d$. L’espace $\mathcal{Q}_X(r,d)$ est aussi un espace de modules de vortex sur $X$. Nous démontrons que le fibré anticanonique de $X$ n’a pas la propriété nef. De façon équivalente, $\mathcal{Q}_X(r,d)$ n’admet aucune métrique kählerienne dont la courbure de Ricci soit semi-positive.

\section*{1. Introduction}

Take a compact connected Riemann surface $X$. The genus of $X$, which will be denoted by $g$, is assumed to be at least two. We will not distinguish between the holomorphic vector bundles on $X$ and the torsion-free coherent analytic sheaves on $X$. For a positive integer $r$, let $\mathcal{O}_X^{\text{tr}}$ be the trivial holomorphic vector bundle on $X$ of rank $r$. Fixing a positive integer $d$, let

$$\mathcal{Q} := \mathcal{Q}_X(r,d)$$

be the quot scheme that parameterizes all (torsion) coherent quotients of $\mathcal{O}_X^{\text{tr}}$ of rank zero and degree $d$ [17]. Equivalently, $\mathcal{Q}$ parametrizes all coherent subsheaves of $\mathcal{O}_X^{\text{tr}}$ of rank $r$ and degree $-d$, because these are precisely the kernels of coherent...
quotients of $O_X^{\text{efr}}$ of rank zero and degree $d$. This $Q$ is a connected smooth complex projective variety of dimension $rd$. See [6,5,4] for the properties of $Q$. It should be mentioned that $Q$ is also a moduli space of vortices on $X$, and it has been extensively studied from this point of view of mathematical physics; see [3,9,12] and references therein.

Bökstedt and Romão proved some interesting differential geometric properties of $Q$ (see [12]). In [10] and [11], we proved that $Q$ does not admit Kähler metrics with semipositive or seminegative holomorphic bisectional curvature. In this note, we continue to study the question of the existence of metrics on $Q$ whose curvature has a sign. Our aim here is to prove the following.

**Theorem 1.1.** The quot scheme $Q$ in (1.1) does not admit any Kähler metric such that the anticanonical line bundle $K_Q^{-1}$ is Hermitian semipositive.

Since semipositive holomorphic bisectional curvature implies semipositive Ricci curvature for a Kähler metric, Theorem 1.1 generalizes the main result of [11].

Recall that a holomorphic line bundle $L$ on a compact complex manifold $M$ is said to be Hermitian semipositive if $L$ admits a smooth Hermitian structure such that the corresponding Hermitian connection has the property that its curvature form is semipositive. The anticanonical line bundle on $M$ will be denoted by $K_M^{-1}$. Note that if $M$ admits a Kähler metric such that the corresponding Ricci curvature is semipositive, then $K_M^{-1}$ is Hermitian semipositive. Indeed, in that case, the Hermitian connection on $K_M^{-1}$ for the Hermitian structure induced by such a Kähler metric has semipositive curvature. The converse statement, that the Hermitian semipositivity of $K_M^{-1}$ implies the existence of Kähler metrics with semipositive Ricci curvature, is also true by Yau's solution to the Calabi conjecture [1,2,19].

The proof of Theorem 1.1 is based on a recent work of Demailly, Campana, and Peternell on the classification of compact Kähler manifolds $M$ with semipositive $K_M^{-1}$ [15,14]. This classification implies that if $K_M^{-1}$ is semipositive, then there is a nontrivial Abelian ideal in the Lie algebra of holomorphic vector fields on $M$, provided $b_1(M) > 0$. On the other hand, for $M = Q$, this Lie algebra is isomorphic to $sl(r, \mathbb{C})$, which does not have any nontrivial Abelian ideal.

2. **Proof of Theorem 1.1**

2.1. **Semipositive Ricci curvature**

Let $J^d(X) = \text{Pic}^d(X)$ be the connected component of the Picard group of $X$ that parameterizes the isomorphism classes of holomorphic line bundles on $X$ of degree $d$. Let $S^d(X)$ denote the space of all effective divisors on $X$ of degree $d$, so $S^d(X) = X^d/P_d$ is the symmetric product, with $P_d$ being the group of permutations of $\{1, \cdots, d\}$. Let

$$p : S^d(X) \longrightarrow \text{Pic}^d(X)$$

be the natural morphism that sends a divisor on $X$ to the holomorphic line bundle on $X$ defined by it.

Take any coherent subsheaf $F \subset O_X^{\text{efr}}$ of rank $r$ and degree $-d$. Let

$$s_F : O_X^{\text{efr}} \longrightarrow F^*$$

be the dual of the inclusion of $F$ in $O_X^{\text{efr}}$. Its exterior product

$$\bigwedge^r s_F : O_X = \bigwedge^r O_X^{\text{efr}} \longrightarrow \bigwedge^r F^*$$

is a holomorphic section of the holomorphic line bundle $\bigwedge^r F^*$ of degree $d$. Therefore, the divisor $\text{div}(\bigwedge^r s_F)$ is an element of $S^d(X)$. Consequently, we have a morphism

$$\varphi : Q \longrightarrow S^d(X), \quad F \longmapsto \text{div}(\bigwedge^r s_F),$$

where $Q$ is defined in (1.1). We note that when $r = 1$, then $\varphi$ is an isomorphism.

Assume that $Q$ admits a Kähler metric $\omega$ such that $K_Q^{-1}$ is Hermitian semipositive. Then there is a connected finite étale Galois covering

$$f : \widetilde{Q} \longrightarrow Q$$

such that $(\widetilde{Q}, f^*\omega)$ is holomorphically isometric to a product

$$\gamma : \widetilde{Q} \longrightarrow A \times C \times H \times F,$$

where

- $A$ is an Abelian variety,
- $C$ is a simply connected Calabi–Yau manifold (holonomy is SU($c$), where $c = \dim C$),

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• $H$ is a simply connected hyper-Kähler manifold (holonomy is $\text{Sp}(h)/2$, where $h = \dim H$), and
• $F$ is a rationally connected smooth projective variety such that $K_F^{-1}$ is Hermitian semipositive.

(See [15, Theorem 3.1].) Henceforth, we will identify $\tilde{Q}$ with $A \times C \times H \times F$ using $\gamma$ in (2.4). We note that $F$ is simply connected because it is rationally connected [13, p. 545, Theorem 3.5], [18, p. 362, Proposition 2.3].

2.2. A lower bound of $d$

We know that $b_1(Q) = 2g$, and the induced homomorphism

$$(p \circ \varphi)_* : H_1(Q, \mathbb{Q}) \longrightarrow H_1(\text{Pic}^d(X), \mathbb{Q}),$$

where $p$ and $\varphi$ are constructed in (2.1) and (2.2), respectively, is an isomorphism [5], [6, p. 649, Remark]. Since $f$ in (2.3) is a finite étale covering, the induced homomorphism

$$f_* : H_1(\tilde{Q}, \mathbb{Q}) \longrightarrow H_1(Q, \mathbb{Q})$$

is surjective. Therefore, the homomorphism

$$(p \circ \varphi \circ f)_* : H_1(\tilde{Q}, \mathbb{Q}) \longrightarrow H_1(\text{Pic}^d(X), \mathbb{Q})$$

(2.5)

is surjective.

There is no nonconstant holomorphic map from a compact simply connected Kähler manifold to an Abelian variety. In particular, there are no nonconstant holomorphic maps from $C$, $H$ and $F$ in (2.4) to $\text{Pic}^d(X)$. Hence, the map $p \circ \varphi \circ f$ factors through a map

$$\beta : A \longrightarrow \text{Pic}^d(X).$$

In other words, there is a commutative diagram

$$\begin{array}{ccc}
\tilde{Q} = A \times C \times H \times F & \xrightarrow{(p \circ \varphi \circ f)} & \text{Pic}^d(X) \\
q \downarrow & & \| \downarrow \text{Id} \\
A & \xrightarrow{\beta} & \text{Pic}^d(X)
\end{array}$$

(2.6)

where $q$ is the projection of $A \times C \times H \times F$ to the first factor. Since $H_1(A \times C \times H \times F, \mathbb{Z}) = H_1(A, \mathbb{Z})$ (as $C$, $H$ and $F$ are simply connected), and $(p \circ \varphi \circ f)_*$ in (2.5) is surjective, it follows that the homomorphism

$$\beta_* : H_1(A, \mathbb{Q}) \longrightarrow H_1(\text{Pic}^d(X), \mathbb{Q})$$

induced by $\beta$ is surjective. This immediately implies that the map $\beta$ is surjective. Since $\beta$ is surjective, from the commutativity of (2.6) we know that the map $p$ is surjective. This implies that

$$d = \dim \text{Pic}^d(X) \geq \dim \text{Pic}^d(X) = g \geq 2.$$  

(2.7)

2.3. Albanese for $\tilde{Q}$

The homomorphism of fundamental groups

$$\varphi_* : \pi_1(Q) \longrightarrow \pi_1(\text{Pic}^d(X))$$

induced by $\varphi$ in (2.2) is an isomorphism [8, Proposition 4.1]. Since $d \geq 2$ (see (2.7)), the homomorphism of fundamental groups

$$p_* : \pi_1(\text{Pic}^d(X)) \longrightarrow \pi_1(\text{Pic}^d(X))$$

induced by $p$ in (2.1) is an isomorphism. Indeed, $\pi_1(\text{Pic}^d(X))$ is the Abelianization

$$\pi_1(X)/[\pi_1(X), \pi_1(X)] = H_1(X, \mathbb{Z})$$

of $\pi_1(X)$ [16]. Combining these we conclude that the homomorphism of fundamental groups

$$(p \circ \varphi)_* : \pi_1(Q) \longrightarrow \pi_1(\text{Pic}^d(X))$$

(2.8)

induced by $p \circ \varphi$ is an isomorphism.

Since the homomorphism in (2.8) is an isomorphism, the covering $f$ in (2.3) is induced by a covering of $\text{Pic}^d(X)$. In other words, there is a finite étale Galois covering
\[ \mu : J \longrightarrow \text{Pic}^d(X) \]  
(2.9)
and a morphism \( \lambda : \widetilde{Q} \longrightarrow J \) such that the following diagram is commutative:

\[
\begin{array}{ccc}
\widetilde{Q} & \xrightarrow{f} & Q \\
\downarrow{\lambda} & & \downarrow{p \circ \varphi} \\
J & \xrightarrow{\mu} & \text{Pic}^d(X)
\end{array}
\]  
(2.10)

where \( f \) is the covering map in (2.3). The projection \( q \) in (2.6) is clearly the Albanese morphism for \( \widetilde{Q} \), because \( C, H \) and \( F \) are all simply connected. On the other hand, \( p \circ \varphi \) is the Albanese morphism for \( Q \) [11, Corollary 2.2]. Therefore, its pullback, namely, \( \lambda \), is the Albanese morphism for \( \widetilde{Q} \). Consequently, we have \( A = J \) with \( \lambda \) coinciding with the projection \( q \) in (2.6). Henceforth, we will identify \( A \) and \( q \) with \( J \) and \( \lambda \) respectively.

2.4. Vector fields

The differential \( df \) of \( f \) identifies \( T\widetilde{Q} \) with \( f^*TQ \), because \( f \) is étale. Using the trace homomorphism \( t : f_*\mathcal{O}_{\widetilde{Q}} \longrightarrow \mathcal{O}_Q \), we have

\[ f_*T\widetilde{Q} = f_*f^*TQ \xrightarrow{p_j} (f_*\mathcal{O}_{\widetilde{Q}}) \otimes TQ \xrightarrow{t} \mathcal{O}_Q \otimes TQ = TQ, \]

where \( p_j \) is given by the projection formula. This produces a homomorphism

\[ \Phi : H^0(\widetilde{Q}, T\widetilde{Q}) = H^0(Q, f_*T\widetilde{Q}) \longrightarrow H^0(Q, TQ) \]  
(2.11)

(see (2.12)) is an ideal in this Lie algebra. Since \( A = J \) is a covering of \( \text{Pic}^d(X) \), we have

\[ \dim H^0(A, TA) = \dim \text{Pic}^d(X) = g > 1. \]  
(2.13)

Since \( H^0(A, TA) \) is an ideal in \( H^0(\widetilde{Q}, T\widetilde{Q}) \), it follows immediately that

\[ \Phi(H^0(A, TA)) \subset \Phi(H^0(\widetilde{Q}, T\widetilde{Q})) = H^0(Q, TQ) \]

is an ideal, where \( \Phi \) is constructed in (2.11). Note that \( H^0(A, TA) \) is an Abelian Lie algebra, so the Lie algebra \( \Phi(H^0(A, TA)) \) is also Abelian.

Since \( \mu : J \longrightarrow \text{Pic}^d(X) \) in (2.9) is a covering map between Abelian varieties, the trace map \( H^0(A, TA) \longrightarrow H^0(\text{Pic}^d(X), T\text{Pic}^d(X)) \) is an isomorphism. In view of this, from the commutativity of the diagram in (2.10), it follows that the restriction

\[ \Phi|_{H^0(A, TA)} : H^0(A, TA) \longrightarrow H^0(Q, TQ) \]

is injective (see (2.12) and (2.11)). But \( H^0(Q, TQ) = \mathfrak{sl}(r, \mathbb{C}) \) [7, p. 1446, Theorem 1.1]. Hence the Lie algebra \( H^0(Q, TQ) \) does not contain any nonzero Abelian ideal. This is in contradiction with the earlier result that \( \Phi(H^0(A, TA)) \) is a nonzero Abelian ideal in \( H^0(Q, TQ) \) of dimension \( g \) (see (2.13)). This completes the proof of Theorem 1.1.

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