The Decay Rate of Patterson–Sullivan Measures with Potential Functions and Critical Exponents

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Abstract Basing upon the recent development of the Patterson–Sullivan measures with a Hölder continuous nonzero potential function, we use tools of both dynamics of geodesic flows and geometric properties of negatively curved manifolds to present a new formula illustrating the relation between the exponential decay rate of Patterson–Sullivan measures with a Hölder continuous potential function and the corresponding critical exponent.

Keywords Geodesic flows, Patterson–Sullivan measures, critical exponent

MR(2010) Subject Classification 37D40, 37B05

1 Introduction

This article is devoted to the study of the properties of Patterson–Sullivan measures with nonzero potential functions on the ideal boundary $X(\infty)$ of a simply connected negatively curved Riemannian manifold $X$.

There are various families of measures on $X(\infty)$ indexed by points of $X$ and the members of each family belong to a same measure class. Among these, three kinds of measures, the Lebesgue measures, the harmonic measures and the Patterson–Sullivan measures, are particularly important.

The topic of this article concerns the Patterson–Sullivan measures, which was first introduced and studied by Patterson in the setting of Fuchsian groups [13]. He constructed a family of absolutely continuous measures supported on the limit set of the ideal boundary of a Fuchsian group. Subsequently Sullivan extended this construction onto general real hyperbolic spaces [19]. Then Yue [21] and Roblin [17, 18] generalized these results to manifolds of negative curvature. Recently, Paulin–Pollicott–Schapira [14] developed a theory of Patterson–Sullivan
measures with a nonzero potential function $F$, and showed that this new kind of measures share many important properties with the classical ones. Pit–Schapira [15] called it the Patterson–Sullivan–Gibbs measure.

The Patterson–Sullivan measures build a connection of the actions of the limit set of a discrete group on the universal covering manifold with the ergodic theory of the geodesic flow on the quotient manifold, hence play a significant role in the study of the dynamics of geodesic flows nowadays ([4–6, 9, 15, 16]). In [4], Kaimanovich showed that there exists a natural 1-to-1 correspondence between the set of finite invariant measures of the geodesic flow on $T^1 M$ and the set of $\Gamma$-invariant Radon measures on $X^2(\infty)$, and explained that how to construct $\Gamma$-invariant measures on $X^2(\infty)$ from the measures on $X(\infty)$ (for example, the Patterson–Sullivan measures and the harmonic measures), where $M = X/\Gamma$ is a compact quotient manifold and $X^2(\infty) = X(\infty) \times X(\infty) \setminus \{(\xi, \xi) | \xi \in X(\infty)\}$. Furthermore he revealed several properties of these measures.

In this article, we will focus on Patterson–Sullivan–Gibbs measures rather than the classical Patterson–Sullivan measures and generalize some results of Kaimanovich’s.

2 Basic Concepts and the Main Result

Let $M$ be a smooth compact negatively curved manifold with pinched sectional curvature $-b^2 \leq K \leq -a^2$ ($b > a > 0$), and $X$ be its Riemannian universal covering manifold. Thus $M = X/\Gamma$ where $\Gamma$ is the fundamental group of $M$. Let $T^1 M$ (resp. $T^1 X$) denote the unit tangent bundle of $M$ (resp. $X$). For any point $p \in M$ or $X$, and for all $v \in T_p M$ or $T_p X$, let $\gamma_v$ be the unique geodesic satisfying the initial conditions $\gamma_v(0) = p$ and $\gamma'_v(0) = v$. In order to simplified the notations, we use $\phi_t(v) = \gamma'_v(t)$ to denote the geodesic flow both on $T^1 M$ and $T^1 X$.

Two geodesics $\gamma_1$ and $\gamma_2$ in $X$ are called positively asymptotic (resp. negatively asymptotic), if there exists $C > 0$ such that

$$d(\gamma_1(t), \gamma_2(t)) \leq C, \quad \forall t \geq 0 \ (\text{resp. } \forall t \leq 0).$$

Here $d$ is the distance function induced by the Riemannian metric. It is easy to see that the positive asymptoticity (resp. negatively asymptoticity) establishes an equivalence relation on the set of all the geodesics on $X$. Given a geodesic $\gamma$, we use $\gamma(+\infty)$ (resp. $\gamma(-\infty)$) to denote the equivalence class of geodesics positively asymptotic (resp. negatively asymptotic) to $\gamma$. Also we can consider this equivalence class as a point at infinity. The set of all points at infinity is usually denoted by $X(\infty)$.

In [14], Paulin–Pollicott–Schapira has made a comprehensive and systematic study of Patterson–Sullivan measures with nonzero potential function $F$. The following notations and results are cited from §3.1–§3.6 of [14].

Let $F : T^1 M \to \mathbb{R}$ be a Hölder continuous function and $\tilde{F} : T^1 X \to \mathbb{R}$ be the lift function of $F$, thus $\tilde{F}$ is a Hölder continuous, $\Gamma$-invariant function, called a potential.

For any $x, y \in X$, let

$$\int_x^y \tilde{F} = \int_0^{d(x,y)} \tilde{F}(\phi_t(v))dt, $$
where \( v \in T^1 X \) such that \( \pi(\phi_{d(x,y)}(v)) = y \). Here \( \pi : T^1 X \to X \) is the standard projection map and \( \phi_t : T^1 X \to T^1 X \) is the geodesic flow.

Fix \( x, y \in X \), the Poincaré series of \( (\Gamma, F) \) is the map
\[
Q_{\Gamma, F, x, y} : \mathbb{R} \to [0, +\infty],
\]
\[
s \mapsto Q_{\Gamma, F, x, y}(s) = \sum_{\gamma \in \Gamma} e^{\gamma y} \left( \bar{F} - s \right).
\]

We define the critical exponent \( \delta_{\Gamma, F} \) of \( (\Gamma, F) \) by
\[
\delta_{\Gamma, F} = \limsup_{n \to +\infty} \frac{1}{n} \log \sum_{n-1 < d(x, y) \leq n} e^{\gamma x} \in [-\infty, +\infty].
\]

In [14], Paulin–Pollicott–Schapira proved the following result.

**Theorem 2.1** (Paulin–Pollicott–Schapira [14]) If \( \delta_{\Gamma, F} < \infty \), then there exists a family of finite nonzero (positive Borel) measures \( \{\mu^F_x\}_{x \in X} \) on \( X(\infty) \), such that, for any \( \gamma \in \Gamma \), for any \( x, y \in X \), and for each \( \xi \in X(\infty) \), we have
\[
\gamma_* \mu^F_x = \mu^F_{\gamma x},
\]
\[
\frac{d\mu^F_x}{d\mu^F_y}(\xi) = e^{-C_{\Gamma, F, \xi}(x, y)},
\]
where \( C_{\Gamma, F, \xi}(x, y) = \lim_{t \to +\infty} \{f^t_x(F - \delta_{\Gamma, F}) - f^t_y(F - \delta_{\Gamma, F})\} \) is the Gibbs cocycle for the potential function \( F \), here \( t \mapsto \xi_t \) is any geodesic ray ending at \( \xi \in X(\infty) \).

\( \{\mu^F_x\}_{x \in X} \) are called Patterson–Sullivan–(Gibbs) measures of dimension \( \delta_{\Gamma, F} \). When \( F \equiv 0 \), this definition coincides with the classical Patterson–Sullivan measures. By the definition, we know that all \( \mu^F_x \) \((x \in X)\) belong to the same measure class.

Let \( \lambda \) be the Liouville measure on the unit tangent bundle \( T^1 X \). Since the geodesic flow \( \phi_t : T^1 M \to T^1 M \) is ergodic with respect to the Liouville measure on \( T^1 M \) (see [1]), by Birkhoff ergodic theorem, for \( \lambda \)-a.e. \( v \in T^1 X \), the following limit
\[
\lim_{t \to +\infty} \frac{1}{t} \int_0^t \tilde{F}(\gamma_v(s))ds
\]
exists and is independent of \( v \), it is in fact the integral of \( F \) with respect to the Liouville measure on \( T^1 M \). We denote this limit by \( \lambda_F \), i.e., for \( \lambda \)-a.e. \( v \in T^1 X \),
\[
\lambda_F = \lim_{t \to +\infty} \frac{1}{t} \int_0^t \tilde{F}(\gamma_v(s))ds.
\]

The following theorem is the main result of this article. A special case when \( F \equiv 0 \) has been proved by Kaimanovich in [4].

**Theorem 2.2** Let \( M, X, \tilde{F}, F \) and \( \Gamma \) are the ones as mentioned above. Let \( \{\mu^F_x\}_{x \in X} \) be the Patterson–Sullivan measures constructed in Theorem 2.1, then for any \( x \in X \) and \( \lambda \)-a.e. \( v \in T^1 X \),
\[
\lim_{t \to +\infty} \frac{1}{t} \log \mu^F_x(B_{x, t}(\gamma_v(\infty))) = -\delta_{\Gamma, F} + \lambda_F.
\]

Here \( B_{x, t}(\gamma_v(\infty)) \subseteq X(\infty) \) is a neighborhood of \( \gamma_v(\infty) \) in \( X(\infty) \) and its detailed definition will be given in the next section.
Obviously the limit value $-\delta_{\Gamma,F} + \lambda_F \leq 0$ since the measure $\mu_x^F$ is finite. A straightforward corollary follows from this observation.

**Corollary 2.3** Under the same condition in Theorem 2.2, we have the inequality

$$\lambda_F \leq \delta_{\Gamma,F}.$$ 

We must point out that this corollary presents the relation between the critical exponent of $(\Gamma,F)$ and the average of the Hölder continuous potential $F$. In the case $F \equiv 0$, this is straightforward. However, when $F$ is a non-zero potential function, this relation is highly nontrivial in general.

Although it is in the early stage of development, we are optimistic about broad application prospects of this theorem on both dynamics and geometry, and outline our future research regarding related topics. Our plan for the next paper will be on computing the Hausdorff dimension of $\mu_x^F$ (see for example [4]).

In the next section we will give the proof of Theorem 2.2. Unlike the traditional method based on the theory of hyperbolic groups like in [4], we introduce the dynamical properties of negatively curved manifolds to build this formula.

### 3 Proof of Theorem 2.2

Let $\text{Isom}(X)$ be the group of all isometry transformations of $X$. We call an isometry $\alpha \in \text{Isom}(X)$ an *axial element* if there exists a geodesic $\gamma$ in $X$ and a $T > 0$ such that for any $t \in \mathbb{R}$, $\alpha(\gamma(t)) = \gamma(t+T)$. Correspondingly $\gamma$ is called the *axis* of $\alpha$. In fact when $M = X/\Gamma$ is compact, every element of $\Gamma$ is an axial element. The following result is useful to us.

**Proposition 3.1** (Ballmann [2]) Let $X$ be a simply connected manifold with pinched negative curvature and $X/\Gamma$ is compact, if $\gamma$ is an axis of $\alpha \in \Gamma \subset \text{Isom}(X)$, then for any neighborhoods $U \subset X(\infty)$ of $\gamma(-\infty)$ and $V \subset X(\infty)$ of $\gamma(+\infty)$, there exists $N \in \mathbb{Z}^+$ such that

$$\alpha^n(X(\infty) - U) \subset V, \quad \alpha^{-n}(X(\infty) - V) \subset U, \quad \forall n \geq N.$$ 

In fact, Ballmann proved this result for a broader class of manifolds known as rank 1 manifolds of non-positive curvature. Later Watkins ([20]) and Liu–Wang–Wu ([9]) extended his result onto the rank 1 manifolds without focal points. Refer to [7–12] for more information about recent works of the geometries and dynamics of geodesic flows on manifolds without focal/conjugate points.

In general, $X(\infty)$ can be identified with the $(\dim X - 1)$-sphere $T_x^1X$ at any point $x \in X$. For any geodesic $\gamma$ in $X$ and for any point $x \in X$, we can find a unique geodesic $\beta$ in $X$ starting from $x$ and is positive asymptotic to $\gamma$, i.e. $\beta(0) = x$ and $\beta(+\infty) = \gamma(+\infty)$. For more details, see [3]. Thus for any $x \in X(\infty)$ and $x \in X$, we use $\gamma_{x,\xi}$ to denote the geodesic connecting $x$ and $\xi$. That is to say $\gamma_{x,\xi}(0) = x$ and $\gamma_{x,\xi}(+\infty) = \xi$. Furthermore, for each $x \in X$ and $\xi, \eta \in X(\infty)$, we can define the angle between $\xi$ and $\eta$ seen from $x$ by

$$\angle_x(\xi, \eta) := \angle_x(\gamma_{x,\xi}'(0), \gamma_{x,\eta}'(0)).$$

Now for each $x \in X$ and $\xi \in X(\infty)$, $\forall a > 0$, define

$$A_{x,a}(\xi) = \{\eta \in X(\infty) | \angle_x(\xi, \eta) < a\}$$
to be the cone neighborhood of $\xi$ in $X(\infty)$. For each $x \in X$, define a distance function $d_x$ on $X(\infty) = \{(\xi, \eta) \in X(\infty) \times X(\infty)|\xi \neq \eta\}$ by
\[
d_x(\xi, \eta) = t \iff d(\gamma_{x,\xi}(t), \gamma_{x,\eta}(t)) = 1.
\]
Under this distance $d_x$, $\forall t \geq \frac{1}{2}$, we define a neighborhood of $\xi \in X(\infty)$ by
\[
B_{x,t}(\xi) = \{\eta \in X(\infty)|d_x(\xi, \eta) > t\}.
\]

**Lemma 3.2** There exists $C > 0$ such that for any $x \in X$ and for any $\xi \in X(\infty)$, we have $\mu^F_x(A_{x,\xi}(\xi)) > C$.

**Proof** We know the Patterson–Sullivan measure $\{\mu^F_x\}_{x \in X}$ and the Gibbs cocycle $C_{F-\delta_{F,\xi}}(\cdot,\cdot)$ are $\Gamma$-invariant, and $C_{F-\delta_{F,\xi}}(\cdot,\cdot)$ is continuous [14]. In fact, given $M = X/\Gamma$ is compact, we only need to prove Lemma 3.2 for one fixed point $x_0 \in X$.

As we have mentioned in Section 2, $X(\infty)$ is homeomorphic to a $(\dim X - 1)$-sphere, thus compact. Therefore there exist finite many points $\{\xi_i\}_{i \in I} \in X(\infty)$ and an angle $\theta > 0$ such that $\forall \xi \in X(\infty)$, there exists at least one $\xi_i$ satisfying
\[
A_{x_0,\theta}(\xi_i) \subseteq A_{x_0,\xi}(\xi).
\] (3.1)

Since the axes are dense in the space of geodesics in $X$ (see [2, 5, 6]), by changing $\theta$ smaller, we can choose each point in $\{\xi_i\}_{i \in I}$ to be an endpoint of some axis of an axial element $\alpha_i \in \text{Isom}(X)$.

The compactness of $M = X/\Gamma$ also implies that the total mass of the Patterson–Sullivan measure $\{\mu^F_x\}_{x \in X}$ is uniformly bounded both from above and below. Since for any $x \in X$, $\text{Supp} \mu_x = X(\infty)$ (see [5, 9]), we can choose two separated open subsets $S_1$ and $S_2$ of $X(\infty)$ such that
\[
\mu^F_x(S_i) > 0, \quad i = 1, 2.
\]

For each $\xi_i (i \in I)$, as $S_1$ and $S_2$ are separated, we know either $\xi_i \notin S_1$ or $\xi_i \notin S_2$. Thus by Proposition 3.1, there exists $n_i \in \mathbb{Z}$ such that either $\alpha_i^{n_i}S_1$ or $\alpha_i^{n_i}S_2$ is contained in $A_{x_0,\theta}(\xi_i)$.

Combining (3.1), we can conclude at least one of the following relations holds
\[
\alpha_i^{n_i}S_1 \subseteq A_{x_0,\theta}(\xi_i) \subseteq A_{x_0,\xi}(\xi),
\]
\[
\alpha_i^{n_i}S_2 \subseteq A_{x_0,\theta}(\xi_i) \subseteq A_{x_0,\xi}(\xi).
\]

Thus in order to prove this lemma, by the finiteness of the set $\{\xi_i\}_{i \in I} \subseteq X(\infty)$, we only need to show that
\[
\mu^F_{x_0}(\alpha_i^{n_i}S_1) > 0, \quad \mu^F_{x_0}(\alpha_i^{n_i}S_2) > 0, \quad i \in I.
\]

Without loss of generality, we will show $\mu^F_{x_0}(\alpha_i^{n_i}S_1) > 0$.

By the definition, $\mu^F_{x_0}(\alpha_i^{n_i}S_1) = (\alpha_i^{-n_i} \mu^F_{x_0})(S_1)$. Theorem 2.1 implies that
\[
\mu^F_{\alpha_i^{-n_i}, x_0}(S_1) = \mu^F_{x_0}(\alpha_i^{n_i}S_1),
\]
and
\[
\frac{d\mu^F_{\alpha_i^{-n_i}, x_0}}{d\mu^F_{x_0}}(\xi) = e^{-C_{F-\delta_{F,\xi}}(\alpha_i^{-n_i}x_0, x_0)}.
\]

By the facts that $C_{F-\delta_{F,\xi}}(\alpha_i^{-n_i}x_0, x_0)$ is continuous with respect to $\xi \in X(\infty)$ and $X(\infty)$ is compact, we know the function
\[
C_{F-\delta_{F,\xi}}(\alpha_i^{-n_i}x_0, x_0) : X(\infty) \to \mathbb{R}
\]
is uniformly bounded from both above and below, thus $\mu_{x_0}(S_1) > 0$ implies that

$$\mu_{\alpha_i^{-n_i}x_0}^F(S_1) > 0,$$

therefore

$$\mu_{x_0}(\alpha_i^{-n_i}S_1) = \mu_{\alpha_i^{-n_i}x_0}^F(S_1) > 0.$$

We complete the proof of this lemma.

\begin{flushright}
$\Box$
\end{flushright}

**Lemma 3.3** For any $x \in X$ and $\xi \in X(\infty)$, let $\gamma := \gamma_{x,\xi}$ denote the geodesic ray connecting $x$ and $\xi$. There exist positive constants $N$ and $K$ depending only on the curvature bounds, satisfying

$$A_{\gamma(t+K),\frac{\pi}{2}}(\xi) \subseteq B_{x,t}(\xi), \quad \forall t \geq N.$$

**Proof** We will prove this lemma by contradiction. Assume this Lemma fails, then there exists $x \in X$, $\xi \in X(\infty)$, and sequences $\{t_i\}_{i=1}^{+\infty} \subseteq \mathbb{R}$, $\{s_{i,j}\}_{i,j=1}^{+\infty}$ with $\lim_{i \to +\infty} t_i = +\infty$, $\lim_{j \to +\infty} s_{i,j} = +\infty$ such that for any $i, j$

$$A_{\gamma(t_i+s_{i,j}),\frac{\pi}{2}}(\xi) \not\subseteq B_{x,t_i}(\xi). \quad (3.2)$$

We will need the following facts.

**Facts** Let $\beta$ be a geodesic in $X$. $\forall t \in \mathbb{R}$, let $D_t$ be the hyper-surface that orthogonal to $\beta$ at $\beta(t)$. We denote the points at infinity of this hyper-surface by $D_t(\infty)$, then we have

$$D_{t_1}(\infty) \cap D_{t_2}(\infty) = \emptyset, \quad D_{t_1} \cap D_{t_2} = \emptyset, \quad \forall t_1 \neq t_2, \quad (3.3)$$

$$\bigcup_{t \in \mathbb{R}} D_t(\infty) = X(\infty) - \{\beta(-\infty), \beta(+\infty)\}, \quad (3.4)$$

$$A_{\beta(t),\frac{\pi}{2}}(\xi) - \{\xi\} = \bigcup_{s \geq t} D_s(\infty). \quad (3.5)$$

We will only prove (3.3) here. (3.4) and (3.5) are straightforward corollaries of (3.3).

Suppose $D_{t_1}(\infty) \cap D_{t_2}(\infty) \neq \emptyset$, take $\eta \in D_{t_1}(\infty) \cap D_{t_2}(\infty)$, then by the definitions of $D_t$ and $D_t(\infty)$, the sum of interior angles of the geodesic triangle $\triangle_{\beta(t_1)\beta(t_2)\eta}$ is greater than $\frac{\pi}{2} + \frac{\pi}{2} = \pi$, which is not possible because $X$ is a negatively curved manifold with the curvature $K \leq -a^2 < 0$. Thus $D_{t_1}(\infty) \cap D_{t_2}(\infty) = \emptyset$. Similarly we can show that $D_{t_1} \cap D_{t_2} = \emptyset$.

Now return to the proof of the lemma. If (3.2) holds, we can fix $i \in \mathbb{Z}$ and take

$$\xi_{i,j} \in A_{\beta(t_i+s_{i,j}),\frac{\pi}{2}}(\xi) - B_{x,t_i}(\xi),$$

By (3.5), there exists $t_{i,j} > t_i + s_{i,j}$, such that

$$\xi_{i,j} \in D_{t_{i,j}}(\infty).$$

We can choose $\{t_{i,j}\}_{j=1}^{+\infty}$ to be an increasing sequence with respect to the index $j$. (3.3), (3.4) and (3.5) imply that $\lim_{j \to +\infty} t_{i,j} = +\infty$. Thus for the fixed $i \in \mathbb{Z}$, we obtain a sequence of points at infinity

$$\{\xi_{i,j}\}_{j=1}^{+\infty} \subseteq X(\infty) - B_{x,t_i}(\xi). \quad (3.6)$$

By the compactness of $X(\infty)$, we can assume that

$$\overline{\xi}_i = \lim_{j \to +\infty} \xi_{i,j}.$$
The fact $\xi \in B_{x,t}(\xi)$ and (3.6) imply that $\overline{\xi}_i \neq \xi$, then (3.5) shows that there exists $t > t_i$, such that
$$\overline{\xi}_i \in D_t(\infty).$$
(3.7)
By (3.5), for a fixed $i \in \mathbb{Z}$, the sets $\{A_{(t_i+s_n,j)} \}$ are nested, thus (3.7) is not possible. Therefore (3.2) is not true. We complete the proof of this lemma. $\square$

Finally we will prove Theorem 2.2.

Proof. By Birkhoff ergodic theorem, for $\lambda$-a.e. $v \in T^1X$, the limit
$$\lambda_F = \lim_{t \to +\infty} \frac{1}{t} \int_0^t \tilde{F}(\gamma_v(s))ds$$
exists and is independent of $v$. For any $v \in T^1X$, let $y = \gamma_v(t)$, $t > 0$. Given that $X/\Gamma$ is compact and the Gibbs cocycle $C_{\Gamma,F}(x,y)$ is continuous and $\Gamma$-invariant (see [14]), we know there exists a constant $D$ depending only on the curvature bounds and the function $F$, such that
$$|C_{\Gamma,F}(\gamma_v(+\infty))(x,y) - C_{\Gamma,F}(\xi,x,y)| \leq D, \quad \forall \xi \in B_{x,t}(\gamma_v(+\infty)).$$

By the definition,
$$C_{\Gamma,F}(\xi,x,y) = -\int_0^t \tilde{F}(\gamma_v(s))ds + t \cdot \delta_{\Gamma,F}.$$
thus
$$\int_{B_{x,t}(\gamma_v(+\infty))} e^{-D} d\mu^\xi_F(\xi) \leq e^{t \cdot \delta_{\Gamma,F} - \int_0^t \tilde{F}(\gamma_v(s))ds} \int_{B_{x,t}(\gamma_v(+\infty))} e^{-C_{\Gamma,F}(\xi,x,y)} d\mu^\xi_F(\xi)$$
$$\leq \int_{B_{x,t}(\gamma_v(+\infty))} e^D d\mu^\xi_F(\xi),$$
i.e.,
$$e^{-D} \cdot \mu^\xi_F(B_{x,t}(\gamma_v(+\infty))) \leq e^{t \cdot \delta_{\Gamma,F} - \int_0^t \tilde{F}(\gamma_v(s))ds} \cdot \mu^x_F(B_{x,t}(\gamma_v(+\infty)))$$
$$\leq e^D \cdot \mu^x_F(B_{x,t}(\gamma_v(+\infty))).$$
Therefore
$$\left| t \cdot \delta_{\Gamma,F} - \int_0^t \tilde{F}(\gamma_v(s))ds + \ln \mu^x_F(B_{x,t}(\gamma_v(+\infty))) - \ln \mu^x_F(B_{x,t}(\gamma_v(+\infty))) \right| \leq D.$$
(3.8)
By Lemma 3.3, we know there exist positive constants $N$ and $K$ depending only on the curvature bounds such that
$$A_{\gamma_v(t+K),\mathbb{S}}(\gamma_v(+\infty)) \subseteq B_{x,t}(\gamma_v(+\infty)), \quad \forall t \geq N.$$
(3.9)
Lemma 3.2 implies that there exists a constant $C$ which is independent of $\gamma_v(t+K)$, such that
$$\mu^x_{\gamma_v(t+K)}(A_{\gamma_v(t+K),\mathbb{S}}(\gamma_v(+\infty))) \geq C.$$  
(3.10)
(3.8), (3.9) and (3.10) imply that
$$\lim_{t \to +\infty} \frac{1}{t} \ln \mu^x_F(B_{x,t}(\gamma_v(+\infty))) = -\delta_{\Gamma,F} + \lambda_F.$$  
This completes the proof of Theorem 2.2. $\square$

Acknowledgements. Part of this work was accomplished during the visit of Southern University of Science and Technology (SUSTech), Shenzhen. We would like to express our gratitude to Prof. Zhihong Xia for his accommodation and help.
References

[1] Anosov, D. V.: Geodesic flows on closed Riemann manifolds with negative curvature, Proceedings of the Steklov Institute of Mathematics, 90 (1967). Translated from the Russian by S. Feder American Mathematical Society, Providence, R. I. 1969

[2] Ballmann, W.: Axial isometries of manifolds of nonpositive curvature. Math. Annalen, 259, 131–144 (1982)

[3] Eberlein, P., O’Neill, B.: Visibility manifolds. Pacific J. Math., 46, 45–109 (1973)

[4] Kaimanovich, V. A.: Invariant measures of the geodesic flows and measures at infinity on negatively curved manifolds. Annales de l’I.H.P., section A, 53, 361–393 (1990)

[5] Knieper, G.: The uniqueness of the measure of maximal entropy for geodesic flows on rank 1 manifolds. Ann. of Math. (2), 148, 291–314 (1998)

[6] Knieper, G.: Closed geodesics and the uniqueness of the maximal measure for rank 1 geodesic flows, Smooth Ergodic Theory and its Applications (Seattle, WA, 1999), 573–590, Proc. Sympos. Pure Math., 69, Amer. Math. Soc., Providence, RI, 2001

[7] Liu, F., Liu, X., Wang, F.: On the mixing and Bernoulli properties for geodesic flows on rank 1 manifolds without focal points, arXiv:1812.00377, 19 pages

[8] Liu, F., Wang, F.: Entropy-expansiveness of geodesic flows on closed manifolds without conjugate points. Acta Math. Sin., Engl. Ser., 32, 507–520 (2016)

[9] Liu, F., Wang, F., Wu, W.: On the Patterson–Sullivan measure for geodesic flows on rank 1 manifolds without focal points, arXiv:1812.04398, 49 pages

[10] Liu, F., Wang, F., Wu, W.: The topological entropy for autonomous Lagrangian systems on compact manifolds whose fundamental groups have exponential growth. To appear on Sci. China Math.

[11] Liu, F., Wang, Z., Wang, F.: Hamiltonian systems with positive topological entropy and conjugate points. J. Appl. Anal. Comput., 5, 527–533 (2015)

[12] Liu, F., Zhu, X.: The transitivity of geodesic flows on rank 1 manifolds without focal points. Diff. Geom. Appl., 60, 49–53 (2018)

[13] Patterson, S.: The limit set of a Fuchsian group. Acta Math., 136, 241–273 (1976)

[14] Paulin, F., Pollicott, M., Schapira, B.: Equilibrium states in negative curvature. Asterisque, 373, viii+281 (2015)

[15] Pit, V., Schapira, B.: Finiteness of Gibbs measures on non-compact manifolds with pinched negative curvature. Ann. Inst. Fourier, 68, 457–510 (2018)

[16] Ricks, R.: Flat strips, Bowen–Margulis measures, and mixing of the geodesic flow for rank one CAT(0) spaces. Ergodic Theory Dyn. Syst., 37, 939–970 (2017)

[17] Roblin, T.: Sur l’ergodicité rationelle et les propriétés ergodiques du flot géodésique dans les variétés hyperboliques. Ergodic Theory Dyn. Syst., 20, 1785–1819 (2000)

[18] Roblin, T.: Ergodicité et équidistribution en courbure négative. Mém. Soc. Math. Fr., 95, vi+96 (2003)

[19] Sullivan, D.: The density at infinity of a discrete group of hyperbolic motions. Publ. Math. IHES, 50, 171–202 (1979)

[20] Watkins, J.: The higher rank rigidity theorem for manifolds with no focal points. Geom. Dedicata, 164, 319–349 (2013)

[21] Yue, C.: The ergodic theory of discrete isometry groups of manifolds of variable negative curvature. Trans. Amer. Math. Soc., 348, 4965–5005 (1996)