AN UPPER BOUND ON THE EXCEPTIONAL CHARACTERISTICS FOR LUSZTIG’S CHARACTER FORMULA

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Abstract. We develop and study a Lefschetz theory in a combinatorial category associated to a root system and derive an upper bound on the exceptional characteristics for Lusztig’s formula for the simple rational characters of a reductive algebraic group. Our bound is huge compared to the Coxeter number. It is, however, given by an explicit formula.

1. Introduction

Lusztig’s conjecture on the irreducible rational characters of a reductive algebraic group over a field $k$ of positive characteristic has in the last 20 years been approached from various directions. It is known that Lusztig’s formula holds for large enough characteristics (with respect to a fixed root system), yet so far it is unknown what “large enough” means in all but low rank cases (conjecturally, the characteristic should be at least the Coxeter number of the root system). So in almost all explicitly given cases the irreducible characters of a reductive group are unknown. This is certainly not a completely satisfying situation.

1.1. Lusztig’s conjecture as a moment graph problem. The most recent approach towards Lusztig’s conjecture, contained in the articles [Fie4] and [Fie5], gives a connection between representations of reductive groups over $k$ (more precisely, representations of their Lie algebras) and the theory of $k$-sheaves on moment graphs. In particular, it is shown in the above papers that Lusztig’s conjecture follows from a similar multiplicity conjecture for the Braden-MacPherson sheaves (with coefficients in the field $k$) on an affine moment graph.

The moment graph theory has the advantage that it can be formulated and studied in a relatively elementary way. In particular, one can determine the $p$-smooth locus on an affine moment graph by quite elementary arguments (cf. [Fie3]). The result is in complete accordance with Lusztig’s conjecture and yields its multiplicity one case for all fields with characteristic at least the Coxeter number. Unfortunately, this cannot be reinterpreted in terms of characters.

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1.2. A Lefschetz theory for sheaves on moment graphs. In the present paper we study a Lefschetz theory for moment graphs and show that the multiplicity conjecture is implied by a Hard Lefschetz conjecture for the Braden-MacPherson sheaves. Then we prove the latter conjecture in the characteristic zero case following the argumentation in \cite{Soc3}, where the equivalent category of special bimodules is considered. The essential ingredient for the proof is the fact that our objects occur as equivariant intersection cohomologies of Schubert varieties. The Hard Lefschetz conjecture for moment graphs then follows from the Hard Lefschetz theorem for projective varieties.

1.3. From $\mathbb{Q}$ to $\mathbb{F}_p$ via $\mathbb{Z}$. In positive characteristic, Soergel's approach does not work any more for several reasons, one of them being that the correspondence between Braden-MacPherson sheaves and intersection cohomology cannot be established so far. Yet one can deduce the conjecture for big enough characteristics from its characteristic zero analogue. For this one first replaces the Braden-MacPherson sheaves by their global sections which naturally are modules over the (commutative, associative) structure algebra $\mathbb{Z}$ associated to the graph. Then one translates the multiplicity conjecture into the language of $\mathbb{Z}$-modules. The advantage is that there is an alternative way to construct those global sections as direct summands of Bott–Samelson modules. These turn out to be defined over the integers, so that we can apply a base change argument. A closer look at the base change procedure opens a way to give an estimate on the exceptional primes.

In the particular case that is connected to Lusztig's conjecture we get the following result. For a fixed root system $R$ we let $\hat{w}_0$ be the element in the affine Weyl group corresponding to the lowest alcove in the anti-fundamental box and we let $l = l(\hat{w}_0)$ be its length. For any reduced expression $s$ of $\hat{w}_0$ we define the numbers $r = r(s), d = d(s)$ and $N = N(s)$ explicitly in terms of the affine Hecke algebra (cf. Section 5.3). Then we set

$$U(\hat{w}_0) := \min_s r!(r - 1)!N^{l(\hat{w}_0) + 2d}r.$$

**Theorem.** Suppose that $\text{char } k > U(\hat{w}_0)$. Then the Hard Lefschetz conjecture for the Braden-MacPherson sheaves, hence Lusztig's conjecture for the algebraic groups with root system $R$, holds.

Admittedly, the above bound is very far from the Coxeter number. It is, however, an explicit number and in the general case the only bound that is known to the author.

After having established the Lefschetz theory, we derive the above estimate using the most basic linear algebra. I am sure that any careful reader can immediately come up with an improved bound. I strongly doubt, however, that the methods outlined in this paper suffice to come anywhere in the vicinity of the Coxeter number.

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2. Sheaves on the affine Bruhat graph

In this section we introduce the Braden–MacPherson sheaves on affine Bruhat graphs. We state the main conjecture on the graded rank of their stalks and quickly review the connection to Lusztig’s conjecture on simple rational characters of reductive groups. As a first step, let us recall the notion of a moment graph.

2.1. Moment graphs. Let $Y \cong \mathbb{Z}^r$ be a lattice.

**Definition 2.1.** A moment graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \alpha, \leq)$ over $Y$ is given by the following data.

1. A graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with set of vertices $\mathcal{V}$ and set of edges $\mathcal{E}$.
2. A map $\alpha: \mathcal{E} \to Y \setminus \{0\}$, called the labelling.
3. A partial order “$\leq$” on $\mathcal{V}$ such that two vertices are comparable if they are connected by an edge.

We use the notation $E: x \longrightarrow y$ for an edge $E$ connecting $x, y \in \mathcal{V}$. The partial order allows us to endow each edge with a direction. We write $E: x \rightarrow y$ if $x < y$. We write $E: x \overset{\alpha}{\longrightarrow} y$ or $E: x \overset{\alpha}{\rightarrow} y$ if we want to specify the label $\alpha = \alpha(E)$ of $E$.

Each subset $\mathcal{I} \subset \mathcal{V}$ defines a (full) sub-moment graph $\mathcal{G}_I$ of $\mathcal{G}$ in the obvious way. For $w \in Y$ we denote by $\mathcal{G}_{\leq w}$ the sub-moment graph associated to the set \{ $x \in \mathcal{V}$ | $x \leq w$ \}. In the following we write $\{ \leq w \}$ instead of $\{ x \in \mathcal{V} | x \leq w \}$, and we write $\{ < w \}, \{ > w \}$, etc. for the similarly defined sets.

We will mostly deal in this article with the moment graphs that are associated to an affine root system. Let us quickly introduce the notions and notations that we need.

2.2. Finite root systems. Let $V$ be a finite dimensional $\mathbb{Q}$-vector space and let $R \subset V$ be an irreducible, reduced, finite root system in the sense of Bourbaki (cf. [Bou]). We denote by $V^* = \text{Hom}(V, \mathbb{Q})$ the dual space, and write $\langle \cdot, \cdot \rangle: V \times V^* \to \mathbb{Q}$ for the natural pairing. For $\alpha \in R$ we denote by $\alpha^\vee \in V^*$ its coroot and we let $R^\vee := \{ \alpha^\vee | \alpha \in R \} \subset V^*$ be the dual root system. Let

$$X := \{ \lambda \in V | \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z} \text{ for all } \alpha \in R \}$$

be the weight lattice and

$$X^\vee := \{ v \in V^* | \langle \alpha, v \rangle \in \mathbb{Z} \text{ for all } \alpha \in R \}$$

the coweight lattice. We have $ZR \subset X$ and $ZR^\vee \subset X^\vee$, and $(ZR, X^\vee)$ and $(X, ZR^\vee)$ are dual lattices under the pairing $\langle \cdot, \cdot \rangle$.

2.3. The affine Weyl group. For $\alpha \in R$ and $n \in \mathbb{Z}$ we define the affine reflection $s_{\alpha, n}: V^* \to V^*$ by

$$s_{\alpha, n}(v) := v - (\langle \alpha, v \rangle - n)\alpha^\vee.$$ 

The affine Weyl group $\widehat{W}$ is the subgroup of affine transformations on $V^*$ that is generated by $\{ s_{\alpha, n} \}_{\alpha \in R, n \in \mathbb{Z}}$. Since $(-\alpha)^\vee = -\alpha^\vee$ we have $s_{\alpha, n} = s_{-\alpha, -n}$.

Adding an additional dimension allows us to linearize the above affine action. If we set $\widehat{V} := V \oplus \mathbb{Q}$, then we have $\widehat{V}^* = V^* \oplus \mathbb{Q}$ where we extend the natural
pairing between $V$ and $V^*$ in the obvious way: $\langle (\lambda, m), (v, n) \rangle = \lambda(v) + mn$. Let $s_{\alpha,n}$ act on $\hat{V}$ by the formula
\[ s_{\alpha,n}(v, m) := (v - \langle \alpha, v \rangle - mn)\alpha^\vee, m. \]
This extends to a linear action of $\hat{W}$ on $\hat{V}$. The level spaces $\hat{V}_m^* := \{(v, m) | v \in V^* \}$ are stabilized by $\hat{W}$, and on the affine space $\hat{V}_1^*$ we recover the affine action that we introduced above.

2.4. **The affine root system.** The hyperplane of fixed points of $s_{\alpha,n}$ in $\hat{V}$ is
\[ \hat{H}_{\alpha,n} := \{(v, m) | \langle \alpha, v \rangle = mn\}. \]
Set $\hat{X} := X \oplus \mathbb{Z}$ and consider $\hat{X}$ as a subset in $\hat{V}$ by the obvious embedding. We denote by $\delta$ the element $(0, -1) \in \hat{X}$. For $\alpha \in R$, $n \in \mathbb{Z}$, the affine root $\alpha + n\delta \in \hat{X}$ is then an equation of $\hat{H}_{\alpha,n}$. The set of (real) affine roots is, by definition,
\[ \hat{R} := \{\alpha + n\delta | \alpha \in R, n \in \mathbb{Z}\} \subset \hat{X}. \]

Now let us choose a system $R^+ \subset R$ of positive roots. Set
\[ \hat{R}^+ := \{\alpha + n\delta | \alpha \in R, n > 0\} \cup \{\alpha + n\delta | \alpha \in R^+, n \geq 0\}. \]
Then $\hat{R} = \hat{R}^+ \cup -\hat{R}^+$. Let us denote by $\Pi \subset R^+$ the subset of simple roots, and by $\gamma \in R^+$ the largest root. The set of simple affine roots is
\[ \hat{\Pi} := \Pi \cup \{-\gamma + \delta\}. \]
Note that each $\beta \in \hat{R}^+$ has a unique expression as a sum of elements in $\hat{\Pi}$. Set
\[ \hat{S} := \{s_{\alpha,0} | \alpha \in \Pi\} \cup \{s_{\gamma,1}\}. \]
Then $\hat{S}$ is a set of generators of $\hat{W}$, and $(\hat{W}, \hat{S})$ is a Coxeter system. This means that we have a corresponding length function $l: \hat{W} \to \mathbb{N}$ and a Bruhat order “$\leq$” on $\hat{W}$. We can now associate the following moment graph to our data.

**Definition 2.2.** The affine moment graph $\hat{\mathcal{G}} = \hat{\mathcal{G}}(\hat{R}, \hat{R}^+)$ is given by the following data:

1. The underlying lattice is $\hat{X}$.
2. The set of vertices is $\hat{V} := \hat{W}$.
3. The vertices $x, y \in \hat{W}$ are connected by a (single) edge $E$ if there are $\alpha \in R$ and $n \in \mathbb{Z}$ with $x = s_{\alpha,n}y$. The edge $E$ is then labelled by the unique positive root in $\{\alpha + n\delta, -\alpha - n\delta\}$.
4. The order “$\leq$” is the affine Bruhat order.

Let us now return to the case of arbitrary moment graphs.
2.5. **Base fields and the GKM-property.** Let $Y$ be a lattice and $\mathcal{G}$ a moment graph over $Y$. Let $K$ be a field of characteristic $\neq 2$ and let

$$Y_K := Y \otimes_\mathbb{Z} K$$

be the $K$-vector space associated to the lattice $Y$. We will often denote by $\lambda$ also the element $\lambda \otimes 1 \in Y_K$.

**Definition 2.3.** (1) Let $\mathcal{I}$ be a subset of $\mathcal{V}$. We say that $(K, \mathcal{I})$ is a \textit{GKM-pair} if the labels on any pair of edges of $\mathcal{G}_\mathcal{I}$ that share a common vertex are linearly independent in $Y_K$, i.e. if for all $x, y, y' \in \mathcal{I}$, $y \neq y'$, and edges $E: x \to y$ and $E': x \to y'$ we have $\alpha(E) \not\in K\alpha(E')$.

(2) If $w \in \mathcal{V}$, then we say that $(K, w)$ is a GKM-pair if $(K, \{ \leq w \})$ is a GKM-pair.

Denote by $S := S_K(Y_K)$ the symmetric algebra of the $K$-vector space $Y_K$. We consider $S$ as a $\mathbb{Z}$-graded algebra. The grading is determined by setting $\deg \lambda := 2$ for all non-zero $\lambda$ in $Y_K$. Almost all $S$-modules in the following will be assumed to be $\mathbb{Z}$-graded.

For a $\mathbb{Z}$-graded space $M$ and $n \in \mathbb{Z}$ we denote by $M_{\{n\}}$ its homogeneous component of degree $n$. In the following, almost all maps $f: M \to N$ of $\mathbb{Z}$-graded spaces will be of degree zero, i.e. they satisfy $f(M_{\{n\}}) \subset N_{\{n\}}$ for all $n \in \mathbb{Z}$. For $l \in \mathbb{Z}$ we denote by $M_{\{l\}}$ the graded space obtained from $M$ by shifting the grading in such a way that $M_{\{l\}_{\{n\}}} = M_{\{l+n\}}$.

2.6. **Sheaves on moment graphs.** Now we come to one of the central definitions for our theory.

**Definition 2.4.** A \textit{$K$-sheaf} $\mathcal{M}$ on the moment graph $\mathcal{G}$ is given by the data $\{(\mathcal{M}^x), \{\mathcal{M}^E\}, \{\rho_{x,E}\}\}$, where

1. $\mathcal{M}^x$ is an $S$-module for any vertex $x$,
2. $\mathcal{M}^E$ is an $S$-module with $\alpha(E).\mathcal{M}^E = 0$ for any edge $E$,
3. $\rho_{x,E}: \mathcal{M}^x \to \mathcal{M}^E$ is a homomorphism of $S$-modules for any vertex $x$ lying on the edge $E$.

The \textit{support} of a sheaf $\mathcal{M}$ is defined as $\text{supp} \mathcal{M} := \{x \in \mathcal{V} \mid \mathcal{M}^x \neq 0\}$. For $x \in \mathcal{V}$ we call the space $\mathcal{M}^x$ the \textit{stalk} of $\mathcal{M}$ at $x$. Let us define $\mathcal{E}^x \subset \mathcal{E}$ as the set of edges that contain the vertex $x$, i.e.

$$\mathcal{E}^x := \{E \in \mathcal{E} \mid E: x \to y \text{ for some } y \in \mathcal{V}\}.$$  

Then we define the \textit{costalk} of $\mathcal{M}$ at $x$ as

$$\mathcal{M}_x := \{m \in \mathcal{M}^x \mid \rho_{x,E}(m) = 0 \text{ for all } E \in \mathcal{E}^x\}.$$  

By definition, $\mathcal{M}_x$ is a sub-$S$-module of $\mathcal{M}^x$. Later we also need the following intermediate module. For $x \in \mathcal{V}$ let $\mathcal{E}^{\delta x} \subset \mathcal{E}^x$ be the set of all directed edges \textit{starting} at $x$, i.e.

$$\mathcal{E}^{\delta x} = \{E \in \mathcal{E} \mid E: x \to y \text{ for some } y \in \mathcal{V}\}.$$  

Recall that the notation $E: x \to y$ implies $x < y$. We set

$$\mathcal{M}_{[x]} := \{m \in \mathcal{M}^x \mid \rho_{x,E}(m) = 0 \text{ for all } E \in \mathcal{E}^{\delta x}\}.$$  

Then we have inclusions $\mathcal{M}_x \subset \mathcal{M}_{[x]} \subset \mathcal{M}^x$.  

2.7. Sections of sheaves. The space of global sections of a sheaf \( \mathcal{M} \) is by definition the space

\[
\Gamma(\mathcal{M}) := \left\{ (m_x) \in \prod_{x \in V} \mathcal{M}^x \mid \rho_{x,E}(m_x) = \rho_{y,E}(m_y) \text{ for all edges } E: x \rightarrow y \right\}.
\]

More generally, for a subset \( \mathcal{I} \) of \( \mathcal{V} \) we define the sections over \( \mathcal{I} \) by

\[
\Gamma(\mathcal{I}, \mathcal{M}) := \left\{ (m_x) \in \prod_{x \in \mathcal{I}} \mathcal{M}^x \mid \rho_{x,E}(m_x) = \rho_{y,E}(m_y) \text{ for } E: x \rightarrow y \text{ with } x, y \in \mathcal{I} \right\}.
\]

Note that for each pair \( \mathcal{I}' \subset \mathcal{I} \) there is an obvious restriction map \( \Gamma(\mathcal{I}, \mathcal{M}) \rightarrow \Gamma(\mathcal{I}', \mathcal{M}) \).

2.8. Braden-MacPherson sheaves. Now we introduce the sheaves \( \mathcal{B}(w) \), for \( w \in \mathcal{V} \), that are most important for our approach. They made their first appearance in the article [BM], where they were called the canonical sheaves.

The construction of \( \mathcal{B}(w) \) is easily motivated by the following problem. Suppose that we want to find a global section \( m = (m_x) \) of a sheaf \( \mathcal{M} \) on \( \mathcal{G} \).

We could try to construct \( m_x \) vertex by vertex following the partial order (note that so far none of our definitions used the partial order), i.e. suppose that we are given an element \( m_y \in \mathcal{M}^y \) for all \( y > x \) such that \( (m_y) \in \Gamma(\{ > x \}, \mathcal{M}) \). Then we want to find an extension to the vertex \( x \), i.e. an element \( m_x \in \mathcal{M}^x \) such that \( (m_y, m_x) \in \Gamma(\{ \geq x \}, \mathcal{M}) \). This means that \( m_x \) should have the following property: For each edge \( E \) starting at \( x \) and ending at some \( y > x \) we have \( \rho_{x,E}(m_x) = \rho_{y,E}(m_y) \). This immediately leads to the following definitions.

Fix \( x \in \mathcal{V} \) and recall that we defined \( \mathcal{E}^{\delta x} \subset \mathcal{E} \) as the set of all edges that start at \( x \). Let

\[
\mathcal{V}^{\delta x} := \{ y \in \mathcal{V} \mid \text{there is } E: x \rightarrow y \in \mathcal{E}^{\delta x} \}
\]

be the set of the corresponding endpoints. For a sheaf \( \mathcal{M} \) on \( \mathcal{G} \) define \( \mathcal{M}^{\delta x} \subset \bigoplus_{E \in \mathcal{E}^{\delta x}} \mathcal{M}^E \) as the image of the composition

\[
\Gamma(\{ \geq x \}, \mathcal{M}) \subset \bigoplus_{y > x} \mathcal{M}^y \xrightarrow{p} \bigoplus_{y \in \mathcal{V}^{\delta x}} \mathcal{M}^y \xrightarrow{\rho} \bigoplus_{E \in \mathcal{E}^{\delta x}} \mathcal{M}^E,
\]

where \( p \) is the projection along the decomposition and \( \rho = \bigoplus_{E: x \rightarrow y} \rho_{y,E} \).

Moreover, we define

\[
\rho_{x,\delta x} := (\rho_{x,E})_{E \in \mathcal{E}^{\delta x}}^T : \mathcal{M}^x \rightarrow \bigoplus_{E \in \mathcal{E}^{\delta x}} \mathcal{M}^E.
\]

The following statement follows directly from the definitions.

**Lemma 2.5.** For each sheaf \( \mathcal{M} \) on \( \mathcal{G} \) the canonical restriction map \( \Gamma(\{ \geq x \}, \mathcal{M}) \rightarrow \Gamma(\{ > x \}, \mathcal{M}) \) is surjective if and only if the set \( \mathcal{M}^{\delta x} \) is contained in the image of the map \( \rho_{x,\delta x} \).

Let us now suppose that for any \( w \in \mathcal{V} \) the set \( \{ \leq w \} \) is finite. The following theorem introduces the Braden–MacPherson sheaves on \( \mathcal{G} \).

**Theorem 2.6.** For each \( w \in \mathcal{V} \) there is an up to isomorphism unique \( K \)-sheaf \( \mathcal{B}(w) \) on \( \mathcal{G} \) with the following properties:
(1) The support of $\mathcal{B}(w)$ is contained in $\{x \in V \mid x \leq w\}$, and $\mathcal{B}(w)^w \cong S$.
(2) If $E: x \xrightarrow{\alpha} y$ is a directed edge, then the map $\rho_{y,E}: \mathcal{B}(w)^y \rightarrow \mathcal{B}(w)^E$ is surjective with kernel $\alpha \mathcal{B}(w)^y$.
(3) For any $x \in V$, the image of $\rho_{x,\delta x}$ is $\mathcal{B}(w)^{\delta x}$, and $\rho_{x,\delta x}: \mathcal{B}(w)^x \rightarrow \mathcal{B}(w)^{\delta x}$ is a projective cover in the category of graded $S$-modules.

By a projective cover of a graded $S$-module $M$ we mean a graded free $S$-module $P$ together with a surjective homomorphism $f: P \rightarrow M$ of graded $S$-modules such that for any homomorphism $g: Q \rightarrow P$ of graded $S$-modules the following holds: If $f \circ g$ is surjective, then $g$ is surjective.

There is, in fact, nothing to prove for the above theorem, as it describes an injective action of $B$. Its multiplication is determined by the formulas

$$B \cdot B = B \cdot B$$

If $m \in \alpha_1 \cdots \alpha_n \beta_1 \cdots \beta_m \mathcal{B}(w)^x$, then $\rho_{x,E}(m) = 0$ for all edges $E$ that contain $x$ as a vertex, since $\mathcal{B}(w)^E$ is annihilated by one of the $\alpha$’s or one of the $\beta$’s. Hence (1). Now suppose that $(K, w)$ is a GKM-pair and let $m \in \alpha_1 \cdots \alpha_n \cdot \mathcal{B}(w)[x]$. If $E$ is an edge that starts at $x$, then $\rho_{x,E}(m) = 0$ by definition of $\mathcal{B}(w)[x]$. If $E$ ends at $x$, then $E$ is labelled by one of the $\alpha_i$, so $\rho_{x,E}(m) = 0$. Hence $\alpha_1 \cdots \alpha_n \cdot \mathcal{B}(w)[x] \subset \mathcal{B}(w)_x$.

Clearly, $\mathcal{B}(w)_x \subset \mathcal{B}(w)[x]$. If $m \in \mathcal{B}(w)_x$, then $\rho_{x,E}(m) = 0$ for each edge $E$ ending at $x$. Since for such $E$ we can find an isomorphism $\mathcal{B}(w)^E \cong \mathcal{B}(w)^x \alpha(E) \mathcal{B}(w)^x$ in such a way that $\rho_{x,E}: \mathcal{B}(w)^x \rightarrow \mathcal{B}(w)^E$ identifies with the canonical quotient map, $m$ must be divisible by $\alpha(E)$ in $\mathcal{B}(w)^x$. Now $\alpha(E)$ acts injectively on $\mathcal{B}(w)^{\delta x}$ (since, by the GKM-property, it acts injectively on $\bigoplus_{E \in \delta x} \mathcal{B}(E)$) and we deduce $\alpha(E)^{-1} m \in \mathcal{B}(w)[x]$. Hence (again by the GKM-property) $\alpha_1^{-1} \cdots \alpha_n^{-1} m \in \mathcal{B}(w)[x]$, and we deduce $\mathcal{B}(w)_x \subset \alpha_1 \cdots \alpha_n \cdot \mathcal{B}(w)[x]$. \hfill $\square$

We now return to the case of affine moment graphs.

2.9. The affine Hecke algebra. Let $\widehat{H} = \bigoplus_{x \in \widehat{W}} \mathbb{Z}[v, v^{-1}] T_x$ be the Hecke algebra associated to the Coxeter system $\langle \widehat{W}, \widehat{S} \rangle$. Its multiplication is determined by the formulas

$$T_x \cdot T_y = T_{xy} \quad \text{if } l(xy) = l(x) + l(y),$$

$$T_x^2 = v^{-2} T_e + (v^{-2} - 1) T_x \quad \text{for } s \in \widehat{S}.$$ 

Then $T_x$ is a unit in $\widehat{H}$ and for any $x \in \widehat{W}$ there exists an inverse of $T_x$ in $\widehat{H}$. For $s \in \widehat{S}$ we have $T_s^{-1} = v^2 T_s + (v^2 - 1)$. There is a duality (i.e. a $\mathbb{Z}$-linear anti-involution) $d: \widehat{H} \rightarrow \widehat{H}$, given by $d(v) = v^{-1}$ and $d(T_x) = T_{x^{-1}}$ for $x \in \widehat{W}$. 

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Set $H_x := v^d(x)T_x$. Recall the following result:

**Theorem 2.8 (KL, Soe1).** For any $w \in \hat{W}$ there exists a unique element $H_w = \sum_{x \in \hat{W}} h_{x,w}(v) H_x \in \hat{H}$ with the following properties:

1. $H_w$ is self-dual, i.e. $d(H_w) = H_w$.
2. $h_{x,w}(v) = 0$ if $x < w$, and $h_{w,w}(v) = 1$,
3. $h_{x,w}(v) \in v\mathbb{Z}[v]$ for $x < w$.

For example, we have $H_e = H_e$ and $H_s = H_s + vH_e$ for each $s \in \hat{S}$. The polynomials $h_{x,w}$ are called the affine Kazhdan–Lusztig polynomials.

### 2.10. The main conjecture.

Now let $\hat{G}$ be the moment graph that we associated to the affine root system $\hat{R}$ in Definition 2.2. Let $w \in V = \hat{W}$. By the defining properties of $B(w)$ listed in Theorem 2.6, the $S$-module $B(w)^x$ is graded free for all $x \leq w$. For any graded free $S$-module $M$ we set

$$rk' M := v^{l_1} + \cdots + v^{l_n},$$

where $l_1, \ldots, l_n \in \mathbb{Z}$ are such that $M \cong \bigoplus_{i=1}^n S\{l_i\}$.

**Definition 2.9.** The graded character of $B(w)$ is

$$h(B(w)) := \sum_{x \leq w} rk' B(w)^x v^{l(w)} T_x \in \hat{H}.$$

Here is the main conjecture.

**Conjecture 2.10.** If $(K, w)$ is a GKM-pair, then

$$h(B(w)) = H_w.$$

In the following section we shortly present the main application of the above conjecture.

### 2.11. Lusztig’s conjecture.

Suppose that $k$ is a field of positive non-even characteristic. Let $\bar{k}$ be an algebraic closure of $k$ and let $G^\vee_{\bar{k}}$ be the connected, simply connected simple algebraic group over $\bar{k}$ whose root system is dual to $R$. Let $\hat{w}_0 \in \hat{W}$ be the element that corresponds to the lowest alcove in the anti-fundamental box (cf. [Fie4]). If char $k$ is at least the Coxeter number of our root system $R$, then the pair $(k, w)$ is a GKM-pair for each $w \leq \hat{w}_0$.

The main result in [Fie4] and [Fie5] is that Conjecture 2.10 implies Lusztig’s conjecture on the characters of irreducible rational representations of $G^\vee_{\bar{k}}$. More precisely:

**Theorem 2.11.** Suppose that char $k$ is at least the Coxeter number of $R$. If Conjecture 2.10 holds for all $w \leq \hat{w}_0$, then Lusztig’s conjecture (cf. [Lus]) on the characters of irreducible rational representations of $G^\vee_{\bar{k}}$ holds.

### 3. $\mathcal{Z}$-modules

From now on we fix the root system $R$, a system $R^+ \subset R$ of positive roots and the field $K$. We denote by $\hat{G}$ the moment graph associated to these data. In the following we want to translate Conjecture 2.10 into a similar character conjecture for certain modules over a commutative and associative algebra.
3.1. The structure algebra. The structure algebra of \( \widehat{G} \) over \( K \) is, by definition, the algebra

\[
\mathcal{Z} := \left\{ (z_x) \in \prod_{x \in \widehat{W}} S \mid \begin{array}{ll}
z_x \equiv z_{s_{\alpha,n}x} \pmod{\alpha + n\delta} \\
\text{for all } x \in \widehat{W}, \alpha \in R, n \in \mathbb{Z}
\end{array} \right\}.
\]

For a subset \( \mathcal{I} \) of \( \widehat{W} \) we define the local structure algebra

\[
\mathcal{Z}(\mathcal{I}) := \left\{ (z_x) \in \prod_{x \in \mathcal{I}} S \mid z_x \equiv z_{s_{\alpha,n}x} \pmod{\alpha + n\delta} \right. \]
\[
\left. \text{for all } x \in \mathcal{I}, \alpha \in R, n \in \mathbb{Z} \text{ with } s_{\alpha,n}x \in \mathcal{I} \right\}.
\]

The algebras \( \mathcal{Z} \) and \( \mathcal{Z}(\mathcal{I}) \) are the global and local sections of the structure sheaf \( \mathcal{Z} \) that is defined by \( \mathcal{Z}^x = S, \mathcal{Z}^E = S/\alpha(E)S, \) and \( \rho_{x,E}: S \to S/\alpha(E)S \) the canonical map. There are obvious restriction maps \( \mathcal{Z}(\mathcal{I}) \to \mathcal{Z}(\mathcal{I}') \) whenever \( \mathcal{I}' \subset \mathcal{I} \) are subsets of \( \widehat{W} \). Both \( \mathcal{Z} = \mathcal{Z}(\widehat{W}) \) and \( \mathcal{Z}(\mathcal{I}) \) are commutative, associative, unital \( S \)-algebras with coordinatewise addition and multiplication. More generally, coordinatewise multiplication makes \( \Gamma(\mathcal{I}, \mathcal{M}) \) into a \( \mathcal{Z}(\mathcal{I}) \)-module (hence a \( \mathcal{Z} \)-module) for each sheaf \( \mathcal{M} \) on \( \widehat{G} \).

If \( \mathcal{I} \) is infinite, then \( \mathcal{Z}(\mathcal{I}) \) and \( \prod_{x \in \mathcal{I}} S \) are not graded algebras in any canonical sense. However, we can define the graded component of \( \mathcal{Z}(\mathcal{I}) \) of degree \( n \) as the intersection \( \mathcal{Z}(\mathcal{I}) \cap \prod_{x \in \mathcal{I}} S_{(n)} \). It then makes sense to talk about \( \mathbb{Z} \)-graded \( \mathcal{Z}(\mathcal{I}) \)-modules, i.e. modules that carry a \( \mathbb{Z} \)-grading such that each homogeneous element of \( \mathcal{Z}(\mathcal{I}) \) acts homogeneously.

**Definition 3.1.** For \( w \in \widehat{W} \) we denote by \( \mathcal{B}(w) := \Gamma(\mathcal{B}(w)) \) the \( \mathcal{Z} \)-module of global sections of \( \mathcal{B}(w) \).

Since \( \mathcal{B}(w) \) is supported on \( \{ \leq w \} \), the action of \( \mathcal{Z} \) on \( \mathcal{B}(w) \) factors over an action of \( \mathcal{Z}(\{ \leq w \}) \) on \( \mathcal{B}(w) \).

Our next objective is to translate Conjecture 2.10 into an analogous multiplicity conjecture for the special modules \( \mathcal{B}(w) \). In order to do this we have to recover the stalks \( \mathcal{B}(w)^{\alpha} \) in terms of \( \mathcal{B}(w) \).

3.2. The generic decomposition. Fix a field \( K \) and a finite subset \( \mathcal{I} \) of \( \widehat{W} \) such that \( (K, \mathcal{I}) \) is a GKM-pair. In particular, we have \( \alpha(E) \neq 0 \) in \( S \) for all edges \( E \) of \( \widehat{G} \). Let us denote by \( Q_{\mathcal{I}} \) the ring obtained by adjoining \( \alpha(E)^{-1} \) for all those \( E \) to \( S \) inside the quotient field \( \text{Quot}(S) \). For an \( S \)-module \( M \) we write \( M_{Q_{\mathcal{I}}} := M \otimes_S Q_{\mathcal{I}} \) for the \( Q_{\mathcal{I}} \)-module obtained by extending scalars. In particular, we have a \( Q_{\mathcal{I}} \)-algebra \( \mathcal{Z}(\mathcal{I})_{Q_{\mathcal{I}}} \). The natural inclusion \( \mathcal{Z}(\mathcal{I}) \subset \bigoplus_{x \in \mathcal{I}} S \) induces an inclusion \( \mathcal{Z}(\mathcal{I})_{Q_{\mathcal{I}}} \subset \bigoplus_{x \in \mathcal{I}} Q_{\mathcal{I}} \).

**Lemma 3.2 (Fiebig).** Suppose that \( (K, \mathcal{I}) \) is a GKM-pair and that \( \mathcal{I} \) is finite. Then the following holds:

1. The canonical inclusion \( \mathcal{Z}(\mathcal{I})_{Q_{\mathcal{I}}} \subset \bigoplus_{x \in \mathcal{I}} Q_{\mathcal{I}} \) is a bijection.
2. If \( \mathcal{M} \) is a \( \mathcal{Z}(\mathcal{I}) \)-module, then there is a canonical decomposition \( \mathcal{M}_{Q_{\mathcal{I}}} = \bigoplus_{x \in \mathcal{I}} \mathcal{M}_{Q_{\mathcal{I}}}^{x} \) such that \( (z_x) \in \mathcal{Z}(\mathcal{I})_{Q_{\mathcal{I}}} \) acts on \( \mathcal{M}_{Q_{\mathcal{I}}}^{x} \) as multiplication by

\[
(z_x).
\]
Let $\mathcal{M}$ be a $\mathcal{Z}(\mathcal{I})$-module that is torsion free as an $S$-module. Then we have a canonical inclusion $\mathcal{M} \subset \mathcal{M}_{Q_x}$. For a subset $\mathcal{J}$ of $\mathcal{I}$ let us define

$$\mathcal{M}^\mathcal{J} := \text{im}\left( \mathcal{M} \subset \mathcal{M}_{Q_x} = \bigoplus_{x \in \mathcal{I}} \mathcal{M}_{Q_x}^x \to \bigoplus_{x \in \mathcal{J}} \mathcal{M}_{Q_x}^x \right)$$

and

$$\mathcal{M}_\mathcal{J} := \mathcal{M} \cap \bigoplus_{x \in \mathcal{J}} \mathcal{M}_{Q_x}^x.$$

If $\mathcal{J}'$ is a subset of $\mathcal{J}$, then we have a natural surjection $\mathcal{M}^\mathcal{J} \to \mathcal{M}^{\mathcal{J}'}$ and a natural injection $\mathcal{M}_{\mathcal{J}'} \to \mathcal{M}_{\mathcal{J}}$. We define the stalk of $\mathcal{M}$ at $x$ as $\mathcal{M}^x := \mathcal{M}_{\{x\}}^x$ and the costalk of $\mathcal{M}$ at $x$ as $\mathcal{M}_x := \mathcal{M}_{\{x\}}$. For $x \leq w$ we set $\mathcal{M}^{\geq x} = \mathcal{M}_{\geq x}$ and $\mathcal{M}^{> x} = \mathcal{M}_{> x}$. We let $\mathcal{M}_{[x]}$ be the kernel of the surjection $\mathcal{M}^{\geq x} \to \mathcal{M}^{> x}$. Then we have inclusions

$$\mathcal{M}_x \subset \mathcal{M}_{[x]} \subset \mathcal{M}^x.$$

The following lemma is a consequence of the definitions.

**Lemma 3.3.** The costalk $\mathcal{M}_x$ is the biggest submodule of $\mathcal{M}$ on which $(z_y) \in \mathcal{Z}(\mathcal{I})$ acts as multiplication with $z_x$. Analogously, the stalk $\mathcal{M}^x$ is the biggest quotient of $\mathcal{Z}$ on which $(z_y)$ acts as multiplication with $z_x$.

Now we can compare the stalks and costalks of $\mathcal{B}(w)$ and of $\mathcal{B}(w)$.

**Proposition 3.4.** The canonical map $\mathcal{B}(w) = \Gamma(\mathcal{B}(w)) \to \mathcal{B}(w)^x$ induces an isomorphism

$$\mathcal{B}(w)^x \cong \mathcal{B}(w)^x.$$

This isomorphism restricts to the following isomorphisms on subspaces:

$$\mathcal{B}(w)_x \cong \mathcal{B}(w)_x, \quad \mathcal{B}(w)_{[x]} \cong \mathcal{B}(w)_{[x]}.$$

In particular, the map $\mathcal{B}(w)_x \to \mathcal{B}(w)^x$ identifies with the map $\mathcal{B}(w)_x \to \mathcal{B}(w)^x$.

**Proof.** By the characterization in the previous lemma and the freeness of the stalks $\mathcal{B}(w)^y$ it is clear that we have an embedding $\mathcal{B}(w)^x \subset \mathcal{B}(w)^x$. From the construction of $\mathcal{B}(w)$ it follows that each element in $\mathcal{B}(w)^x$ appears as a component of a global section, hence the above embedding is also a surjection. More generally, since each local section of $\mathcal{B}(w)$ extends to a global section we have induced isomorphisms $\mathcal{B}(w)^{\geq x} \cong \Gamma(\{\geq x\}, \mathcal{B}(w))$ and $\mathcal{B}(w)^{> x} \cong \Gamma(\{> x\}, \mathcal{B}(w))$. From the definition it is clear that $\mathcal{B}(w)_{[x]}$ is the kernel of the restriction map $\Gamma(\{\geq x\}, \mathcal{B}(w)) \to \Gamma(\{> x\}, \mathcal{B}(w))$, hence we get an induced isomorphism $\mathcal{B}(w)_{[x]} \cong \mathcal{B}(w)_{[x]}$. Finally, $\mathcal{B}(w)_x \cong \mathcal{B}(w)_x$ follows directly from the previous lemma. $\square$

Hence, in order to prove Conjecture 2.10 we can calculate the graded character of the stalks $\mathcal{B}(w)^x$ for all pairs $(x, w)$. The $\mathcal{B}(w)$ have the advantage that they are modules over a commutative algebra. Moreover, they can be constructed by an alternative method, namely as direct summands of the Bott–Samelson modules.
3.3. Bott–Samelson modules.Fix a simple affine reflection $s \in \widehat{S}$ and define the sub-$S$-algebra
\[ Z^s := \left\{ (z_x) \in Z \mid z_x = z_{xs} \text{ for all } x \in \widehat{W} \right\}. \]

**Definition 3.5.** The translation functor associated to $s$ is the functor $\vartheta_s$ that maps a $Z$-module $M$ to the $Z$-module $Z \otimes Z_s M$, and a map $f : M \to N$ of $Z$-modules to the map $1 \otimes f : Z \otimes Z_s M \to Z \otimes Z_s N$.

Let $M(e)$ be the $Z$-module that is free of rank one over $S$ and on which $(z_x) \in Z$ acts by multiplication with $z_e$.

**Definition 3.6.** Let $s = (s_1, \ldots, s_l)$ be a sequence in $\widehat{S}$. The module $B(s) := \vartheta_{s_l} \cdots \vartheta_{s_1}(M(e))$ is called the **Bott–Samelson module** associated to $s$.

The following theorem shows that it is possible to construct the global sections $B(w)$ of the Braden-MacPherson sheaf $\mathcal{B}(w)$ directly in the category of $Z$-modules, i.e. without refering to sheaves on a moment graph. Its proof is contained in \cite{Fie2} (cf. the proof of Theorem 6.1 and Corollary 6.5 in loc. cit.).

**Theorem 3.7.** Fix $w \in \widehat{W}$ and suppose that $(K, w)$ is a GKM-pair. Let $s = (s_1, \ldots, s_l)$ be a sequence in $\widehat{S}$ such that $w = s_1 \cdots s_l$ is a reduced expression. Then there are $x_1, \ldots, x_n < w$ and $l_1, \ldots, l_n \in \mathbb{Z}$ such that
\[ B(s) \cong B(w) \oplus \bigoplus_{i=1}^{n} B(x_i\{l_i\}). \]

3.4. A duality. For graded $S$-modules $M$ and $N$ let
\[ \text{Hom}_S^n(M, N) := \text{Hom}_S(M, N\{n\}) \]
be the space of degree $n$ homomorphisms between $M$ and $N$, and let
\[ DM := \bigoplus_{n \in \mathbb{Z}} \text{Hom}_S^n(M, S\{n\}) \]
be the dual of $M$. This is a graded $S$-module as well, and we have $DS \cong S$ and $D(M\{l\}) \cong (DM)\{-l\}$ as $S$-modules.

Each $Z$-module is naturally an $S$-module as $Z$ is a unital $S$-algebra. If $M$ is a $Z$-module, then $DM$ is a $Z$-module as well with the action given by
\[ (z.f)(m) := f(z.m) \]
for all $z \in Z$, $f \in DM$ and $m \in M$.

**Theorem 3.8** \cite{Fie2}, Cor. 5.7, Thm. 6.1, Prop. 7.1].

1. For each sequence $s = (s_1, \ldots, s_l)$ of length $l$ the Bott–Samelson module $B(s)$ is self-dual up to a shift by $2l$, i.e. we have
\[ DB(s) \cong B(s)\{2l\}. \]

This isomorphism induces isomorphisms $D(B(s)^x) \cong B(s)_x\{2l\}$ for all $x \in \widehat{W}$. 
(2) For each $w \in \widehat{W}$ the $\mathbb{Z}$-module $B(w)$ is self-dual up to a shift by $2l(w)$, i.e. we have

$$DB(w) \cong B(w)\{2l(w)\}.$$  

This isomorphism induces isomorphisms $D(B(w)^x) \cong B(w)_x\{2l(w)\}$ for all $x \in \widehat{W}$.

Let $s = (s_1, \ldots, s_l)$ be a sequence in $\widehat{S}$, and let $J(s) \subset \widehat{W}$ be the set of all elements in $\widehat{W}$ that are smaller than or equal to some subword of $s_1 \cdots s_l$. Let $K$ be a field.

**Definition 3.9.** We say that $(K, s)$ is a GKM-pair if $(K, J(s))$ is a GKM-pair.

If $(K, w)$ is a GKM-pair, then the Bott–Samelson module $B(s)$ decomposes into a direct sum of shifted copies of various $B(w)$’s with $w \in J(s)$. This follows from Theorem 3.7, Theorem 3.8, Lemma 2.7, Proposition 3.4 and the fact that the number of edges of $\hat{G}_{\leq w}$ that end at $x$ is $l(x)$, yield the following:

**Corollary 3.10.** Suppose that $(K, s)$ is a GKM-pair. Let $x \in \widehat{W}$.

(a) The $S$-modules $B(s)^x$, $B(s)_x$ and $B(s)_{[x]}$ are graded free of finite rank.

(b) Let the numbers $a_1, \ldots, a_r$ be defined by $B(s)^x \cong \bigoplus_{i=1}^{a_r} S\{a_i\}$. Then we have

$$B(s)_x \cong \bigoplus_{i=1}^{a_r} S\{-a_i - 2l\},$$

$$B(s)_{[x]} \cong \bigoplus_{i=1}^{a_r} S\{-a_i - 2l + 2l(x)\}.$$  

(1) Let $w \in \widehat{W}$ and suppose that $(K, w)$ is a GKM-pair. Let $x \in \widehat{W}$.

(a) The $S$-modules $B(w)^x$, $B(w)_x$ and $B(w)_{[x]}$ are graded free of finite rank.

(b) Let the numbers $b_1, \ldots, b_r$ be defined by $B(w)^x \cong \bigoplus_{i=1}^{a_r} S\{b_i\}$. Then we have

$$B(w)_x \cong \bigoplus_{i=1}^{b_r} S\{-b_i - 2l(w)\},$$

$$B(w)_{[x]} \cong \bigoplus_{i=1}^{b_r} S\{-b_i - 2l(w) + 2l(x)\}.$$  

**Remark 3.11.** For a graded free $S$-module $M \cong \bigoplus_{i=1}^{a_r} S\{l_i\}$ we set $\text{rk} M := v^{-l_i} \in \mathbb{Z}[v, v^{-1}]$ (note that in the definition of $\text{rk}'$ in Section 2.10 we used the opposite sign convention). From the corollary above we deduce

$$\text{rk}' B(w)^x = \text{rk} B(w)_{[x]} v^{2l(x) - 2l(w)},$$

and, together with Proposition 3.4 we get

$$h(B(w)) = \sum_{x \leq w} \text{rk} B(w)_{[x]} v^{2l(x) - l(w)} T_x.$$  

We see that our Main Conjecture 2.10 coincides with Conjecture 8.3 of [Fie2] and Vermutung 1.3 in [Soc2] in the cases discussed there. (Note that $T_x = v^{l(x)} T_x$ and that the $B(w)$ in [Fie2] are shifted by $l(w)$ with respect to our
$B(w)$, by Corollary 6.5 in [Fie2]). When we stated Conjecture 2.10 we did not know that $B(w)_x$ is a graded free $S$-module, which is why we used the graded character of the stalks.

Now we can state the following conditions that are equivalent to our main conjecture.

**Proposition 3.12.** Suppose that $(K, w)$ is a GKM-pair. Then Conjecture 2.10 is equivalent to each of the following statements:

1. For each $x < w$ the $S$-module $B(w)^x$ is generated in degrees $< l(w) - l(x)$.
2. For each $x < w$ the $S$-module $B(w)_x$ lives in degrees $> l(w) - l(x)$.
3. For each $x < w$ the $S$-module $B(w)_x$ lives in degrees $> l(w) + l(x)$.

**Proof.** The equivalence between the three statements above is an immediate consequence of Corollary 3.10. That the Conjecture 2.10 and the statement (2) are equivalent is proven in [Fie2, Proposition 8.4]. □

Finally we associate the following character to $B(s)$:

$$h(B(s)) := \sum_{x \in \hat{W}} \text{rk} B(s)^x v^l T_x$$

Up to a factor of $v^{-l}$ on the right hand side this coincides with the definition of $h_{\leq l}$ in Section 4.3 of [Fie4]. The Lemma 4.4. in loc. cit. yields the following result.

**Lemma 3.13.** For a sequence $s = (s_1, \ldots, s_l)$ in $\hat{S}$ we have

$$h(B(s)) = H_{s_1} \cdots H_{s_l}.$$  

In particular, the graded rank of $B(s)^x$ for fixed $x$ is independent of the base field $K$ as long as $(K, s)$ is a GKM-pair.

4. A Lefschetz theory for sheaves on moment graphs

We approach Conjecture 2.10 by studying an analogue of the Hard Lefschetz property of the intersection cohomology of projective varieties. For this we have to replace the symmetric algebra $S$ by a polynomial ring in one variable.

Let $K[t]$ be the polynomial ring in one variable over the field $K$ which we consider as a graded algebra with the grading given by setting $t$ in degree 2. We endow $K[t]$ with the structure of an $S$-algebra via the homomorphism $\tau: S \to K[t]$ of graded algebras that is given by $\tau(\alpha) = t$ for all $\alpha \in \hat{\Pi}$.

Let $s = (s_1, \ldots, s_l)$ be a sequence in $\hat{S}$ and denote by $\hat{R}^+_s \subset \hat{R}^+$ the subset of all positive roots that appear as a label on the graph $\tilde{G}_{J(s)}$ (recall that $J(s) \subset \hat{W}$ is the subset of elements that are smaller than or equal to a subexpression of $s_1 \cdots s_l$). Analogously, for $w \in \hat{W}$ let us define $\hat{R}^+_{\leq w} \subset \hat{R}^+$ as the set of all labels on $\tilde{G}_{\leq w}$. 
We define the height \( \text{ht}(\alpha) \) of a positive affine root \( \alpha \in \hat{R}^+ \) as the number \( n \) such that \( \alpha \) can be written as a sum of \( n \) elements of \( \hat{H} \) and we set

\[
N(s) := \max_{\alpha \in \hat{R}^+_s} \{\text{ht}(\alpha)\},
\]

\[
N(w) := \max_{\alpha \in \hat{R}^+_w} \{\text{ht}(\alpha)\}.
\]

Clearly \( N(w) = N(s) \) if \( w = s_1 \cdots s_l \) is a reduced expression for \( w \). We obviously have:

**Lemma 4.1.** Suppose that \( K \) is a field with \( \text{char} \ K > N(s) \) (char \( K > N(w) \)). Then we have \( \tau(\alpha) \neq 0 \) for all \( \alpha \in \hat{R}^+_s \) (\( \alpha \in \hat{R}^+_w \)).

Note that \( \text{char} \ K > N(s) \) (char \( K > N(w) \)) implies that \((K,s) \) (or \((K,w)\)) is a GKM-pair. For any \( S \)-module \( M \) we let \( \overline{M} := M \otimes_S K[t] \) be the \( K[t] \)-module obtained by extension of scalars. For notational convenience we write \( \overline{B}(w) \) instead of \( \overline{B}(w)_x \) and \( \overline{B}(w)^x \) instead of \( \overline{B}(w)_x^x \), etc. The natural inclusion \( B(w)_x \subset B(w)^x \) yields a homomorphism \( i_x : \overline{B}(w)_x \rightarrow \overline{B}(w)^x \).

**Lemma 4.2.** Let \( w \in \hat{W} \) and suppose that \( \text{char} \ K > N(w) \). Then the homomorphism \( i_x : \overline{B}(w)_x \rightarrow \overline{B}(w)^x \) is an injective map between graded free \( K[t] \)-modules for all \( x \leq w \). It is an isomorphism if we invert the variable \( t \).

**Proof.** We have already seen that \( B(w)^x \) and \( B(w)_x \) are graded free \( S \)-modules. Hence \( \overline{B}(w)_x \) and \( \overline{B}(w)^x \) are graded free \( K[t] \)-modules.

Let \( \alpha_1, \ldots, \alpha_n \) be the labels of all edges of \( \hat{G}_{\leq w} \) that contain \( x \) as a vertex. By Lemma \( 2.7 \) \( \alpha_1 \cdots \alpha_n B(w)^x \subset B(w)_x \). Hence, if we invert the roots \( \alpha_1, \ldots, \alpha_n \), then \( B(w)_x \) and \( B(w)^x \), and hence \( B(w)_x \) and \( B(w)^x \) coincide. Since our assumptions imply that \( \tau(\alpha_i) \neq 0 \) for \( i = 1, \ldots, n \), the map \( i_x : \overline{B}(w)_x \rightarrow \overline{B}(w)^x \) is an isomorphism after inverting the variable \( t \). Since both spaces are free \( K[t] \)-modules, the map \( i_x \) must be injective. \( \square \)

**Proposition 4.3.** Let \( w \in \hat{W} \) and suppose that \( \text{char} \ K > N(w) \). Then the following holds.

1. \( \overline{B}(w)^w/\overline{B}(w)_w \cong K[t]/\langle t^{l(w)} \rangle \).
2. For \( x < w \) let the numbers \( a_i, b_i, i = 1, \ldots, r \) be defined by

\[
\overline{B}(w)^x/\overline{B}(w)_x \cong \bigoplus_{i=1}^r K[t]/\langle t^{a_i} \rangle \{b_i\}.
\]

Then \( r \) is the ungraded rank of \( B(w)^x \) and we have \( a_i > l(x) \) for all \( i = 1, \ldots, r \).

**Proof.** There are exactly \( l(w) \) edges in \( \hat{G}_{\leq w} \) that contain the vertex \( w \). Let \( \alpha_1, \ldots, \alpha_{l(w)} \) be their labels. Then we have \( B(w)_w = \alpha_1 \cdots \alpha_{l(w)} B(w)_{[w]} \) and, by definition, \( B(w)[w] = B(w)^w \). From this we deduce (1).

Suppose that \( x < w \). From \( 2.7 \) (2) we deduce that the minimal number of generators of \( \overline{B}(w)^x/\overline{B}(w)_x \) is the rank of \( B(w)^x \). Since the number of edges of \( \hat{G}_{\leq w} \) that end at \( x \) is \( l(x) \), we deduce from the same statement that there are \( \alpha_1, \ldots, \alpha_{l(x)} \) such that \( B(w)_x = \alpha_1 \cdots \alpha_{l(x)} B(w)[x] \). So \( a_i \geq l(x) \) in any case.
Now suppose that there are \( i \) with \( a_i = l(x) \). Among those let us choose \( i \) such that \( b_i \) is maximal. Then there is an element \( m \in \mathcal{B}(w)_x \) of degree \(-b_i + l(x)\) that is not contained in the subspace \( \mathcal{B}(w)^{\leq b_i}_x \) of \( \mathcal{B}(w)_x \) that is generated by all homogeneous components of degree \(<-b_i\). By Lemma 2.7, \( m \) is of the form \( \alpha_1 \cdots \alpha_{l(x)} m' \) with \( m' \in \mathcal{B}(w)_{[x]} \). Then \( m' \) is of degree \(-b_i\) and is also not contained in \( \mathcal{B}(w)^{\leq b_i}_x \). Hence there is a homogeneous \( S \)-basis of \( \mathcal{B}(w)^x \) that contains \( m' \). But \( m' \) lies in the kernel of the map \( \mathcal{B}(w)^x \to \mathcal{B}(w)^x/\mathcal{B}(w)_{[x]} \), which, by construction, is a projective cover. But then no basis element can be mapped to zero, hence we have a contradiction. So there is no \( i \) with \( a_i = l(x) \). \( \square \)

### 4.1. The Lefschetz condition.

Suppose that \( A \) and \( B \) are graded free \( K[t] \)-modules of finite rank. Let \( f: A \to B \) be an injective homomorphism of \( K[t] \)-modules that is an isomorphism over the ring \( K[t, t^{-1}] \). So we assume that the cokernel of \( f \) is a torsion \( K[t] \)-module, i.e. \( t^n \text{coker} f = 0 \) for \( N \gg 0 \). Then \( \text{coker} f \) is isomorphic to a direct sum of \( K[t] \)-modules of the form

\[
K[t]/\langle t^a \rangle \{b\}
\]

for some numbers \( a \geq 1 \) and \( b \in \mathbb{Z} \).

**Definition 4.4.** We say that \( f: A \to B \) satisfies the Lefschetz condition with center \( l \in \mathbb{Z} \) if for each \( n \geq 1 \) the multiplication with \( t^n \) induces an isomorphism

\[
(\text{coker} f)_{\{t^{-n}\}} \xrightarrow{\sim} (\text{coker} f)_{\{t^n\}}.
\]

Our definition is motivated by the Hard Lefschetz theorem in complex algebraic geometry. Note that \( f: A \to B \) satisfies the Lefschetz condition with center \( l \) if and only if \( \text{coker} f \) is isomorphic to a direct sum of \( K[t] \)-modules of the form

\[
K[t]/\langle t^a \rangle \{-l + a\}
\]

with \( a \geq 1 \). Here is another conjecture.

**Conjecture 4.5.** Let \( w \in \hat{\mathcal{W}} \) and suppose that \( \text{char} K > N(w) \). Then for all \( x \leq w \) the map

\[
i_x: \mathcal{B}(w)_x \to \mathcal{B}(w)^x
\]

satisfies the Lefschetz condition with center \( l(w) \).

**Theorem 4.6.** For each \( w \in \hat{\mathcal{W}} \) and each field \( K \) with \( \text{char} K > N(w) \) the Conjecture 4.5 implies Conjecture 2.10.

**Proof.** Suppose that Conjecture 4.5 holds for the pair \((K, w)\) and let \( x < w \). Then \( \mathcal{B}(w)^x/\mathcal{B}(w)_x \) is a direct sum of \( K[t] \)-modules of the form \( K[t]/\langle t^a \rangle \{-l(w) + a\} \) for various \( a > 0 \). By Proposition 4.3 only terms with \( a > l(x) \) occur. Now \( K[t]/\langle t^a \rangle \{-l(w) + a\} \) is generated in degree \( l(w) - a \), which is strictly smaller than \( l(w) - l(x) \), and we deduce that \( \mathcal{B}(w)^x \) is generated in degrees \(< l(w) - l(x) \). This statement is equivalent to Conjecture 2.10 by Proposition 3.12. \( \square \)
4.2. The characteristic zero case. In this section we review the topological approach of [Soe3] that yields Conjecture 4.5 in the case that char $K = 0$.

**Theorem 4.7.** Suppose that $K$ is a field of characteristic zero. Then Conjecture 4.5 holds for all $w \in \hat{W}$.

**Sketch of the proof.** The proof is essentially contained in the last section of [Soe3]. However, Soergel works in the category of special $S$-bimodules. But in [Pie2] it is proven that our category of $Z$-modules is equivalent to Soergel’s category of bimodules. Moreover, in [Soe3] the finite dimensional parabolic situation is considered, while we need to work in the infinite dimensional Kac-Moody setting. Here the Schubert varieties are still finite dimensional, so there are no further technical difficulties to overcome. For the Kac-Moody versions of the well-known results about Schubert varieties see [Kum].

Here are Soergel’s arguments: Let $G$ be the complex Kac-Moody group associated to $R$, $B \subset G$ the Borel subgroup associated to $R^+$ and $T \subset B$ the Kac-Moody torus. The first step in the proof is to identify $\mathcal{Z}(\{ \leq w \})$ with the $T$-equivariant cohomology and $\mathcal{B}(w)$ with the $T$-equivariant intersection cohomology (both with coefficients in $K$) of the Schubert variety $\overline{BwB}/B$ associated to $w \in \hat{W}$ (the latter even as a module over the cohomology). The map $i_x: \mathcal{B}(w)_x \to \mathcal{B}(w)^x$ then identifies with the adjunction map between the equivariant local hypercohomologies of the costalk and the stalk of the intersection cohomology complex on $\overline{BwB}/B$. The one-dimensional version $\mathcal{B}(w)_x \to \mathcal{B}(w)^x$ is then obtained by reducing to the action of a one-dimensional subtorus of $T$ that corresponds to the coweight $\rho'$.

Soergel then identifies the cokernel of $\mathcal{B}(w)_x \to \mathcal{B}(w)^x$ with the (ordinary) intersection cohomology of a certain projective variety $\overline{Z}$ in such a way that the multiplication with $t$ becomes a Lefschetz operator. The claim of the theorem hence is translated to complex algebraic geometry and equals the Hard Lefschetz theorem in this specific situation. □

5. From characteristic zero to positive characteristic

The main application of the Lefschetz theory developed in this article is a calculation of an upper bound for the exceptional characteristics for Lusztig’s conjecture. For this we use a base change argument. So we have to develop a version of the Lefschetz theory over the integers. It turns out that the Bott–Samelson modules can indeed be defined over $\mathbb{Z}$ (cf. Section 6). Before we introduce this construction, we associate to each Bott–Samelson module its Lefschetz data and prove Proposition 5.4 which serves us as a bridge from characteristic zero to positive characteristics.

5.1. Lefschetz data. Suppose that $f: A \to B$ is as in Section 4.1.

**Definition 5.1.** Suppose that $(a_1, b_1), \ldots, (a_n, b_n) \in \mathbb{Z}_{>0} \times \mathbb{Z}$ are such that

$$\text{coker } f \cong \bigoplus_{i=1}^n K[t]/(t^{a_i})\{b_i\}.$$ 

We call the multiset $((a_1, b_1), \ldots, (a_n, b_n))$ the Lefschetz datum for $f: A \to B$. 
The following is immediate:

**Lemma 5.2.** The map \( f: A \rightarrow B \) satisfies the Lefschetz condition with center \( l \) if and only if its Lefschetz datum only involves pairs of the form \((a, -l + a)\) for various \( a > 0 \).

### 5.2. The Lefschetz datum of a Bott–Samelson module.

It is from now on convenient to indicate the base field \( K \) in the notation of the relevant objects. So we denote by \( S_K \) the symmetric algebra over the \( K \)-vector space \( \widetilde{X} \otimes_{\mathbb{Z}} K \), by \( Z_K \) the structure algebra over \( K \), by \( B_K(s) \) or \( B_K(w) \) our special modules over \( Z_K \), by \( B_K(s)^x \) or \( B_K(w)^x \) the corresponding stalks and so on.

Let us fix an element \( w \in \widetilde{W} \) and a sequence \( s = (s_1, \ldots, s_j) \) such that \( w = s_1 \cdots s_t \) is a reduced expression. Suppose that the field \( K \) is such that \( \text{char } K > N(w) = N(s) \). In particular, \((K, s)\) and \((K, w)\) are GKM-pairs. By Theorem 3.7, the Bott–Samelson module \( B_K(s) \) decomposes into a direct sum of modules of the form \( B_K(x) \) up to degree shifts. We deduce from Lemma 5.3 that for each \( x \leq w \) the map \( \overline{B}_K(s)_x \rightarrow \overline{B}_K(s)^x \) is injective and that its cokernel is a torsion module. We call the Lefschetz datum of the map \( \overline{B}_K(s)_x \rightarrow \overline{B}_K(s)^x \) the Lefschetz datum of \( B_K(s) \) at \( x \). The following result shows how one can read off the decomposition pattern of \( B_K(s) \) from its Lefschetz data.

**Lemma 5.3.** Suppose that \( B_K(s) = B_K(w) \oplus \bigoplus_{i=1}^r B_K(x_i)\{l_i\} \). Choose \( x < w \) and let \((a_1, b_1), \ldots, (a_n, b_n)\) be the Lefschetz datum of \( B_K(s) \) at \( x \). Then the number of \( i \) such that \( x_i = x \) equals the number of \( j \) such that \( a_j = l(x) \).

**Proof.** By Proposition 4.3 the Lefschetz datum of \( \overline{B}_K(y)_x \rightarrow \overline{B}_K(y)^x \) involves only pairs \((a, b)\) with \( a > l(x) \), unless \( x = y \), in which case the cokernel of \( \overline{B}_K(y)_x \rightarrow \overline{B}_K(y)^x \) is isomorphic to \( K[t]/\langle t^{l(x)} \rangle \). \( \square \)

**Proposition 5.4.** Suppose that the Conjecture 4.3 holds for \( B_K(w') \) for all \( w' < w \) and that the Lefschetz datum of \( B_K(s) \) and \( B_K(s) \) coincide at all \( x \leq w \). Then Conjecture 4.3 holds for \( B_K(w) \).

**Proof.** We know from Theorem 3.7 that Conjecture 4.3 holds for \( B_K(w) \), i.e. that the map \( \overline{B}_K(w)_x \rightarrow \overline{B}_K(w)^x \) satisfies the Lefschetz condition with center \( l(w) \). In view of Lemma 5.3 it suffices for the proof of our claim to show that the Lefschetz data of \( B_K(w) \) and \( B_K(w) \) coincide at all \( x < w \).

Lemma 5.3 together with our second assumption implies that the decomposition of \( B_K(s) \) into a direct sum of shifted copies of \( B_K(y) \) for various \( y \leq w \) parallels the analogous decomposition of \( B_K(s) \). By our first assumption and Lemma 5.2 the Lefschetz data of \( B_K(w') \) at all \( x < w' \) equals the corresponding Lefschetz data of \( B_K(w') \) for all \( w' < w \). Since \( B_K(w) \) occurs as a direct summand in \( B_K(s) \) with multiplicity one and else only shifted \( B_K(w') \) with \( w' < w \) occur, the Lefschetz data of \( B_K(w) \) and \( B_K(w) \) coincide, which is what we wanted to show. \( \square \)

### 5.3. The main theorem.

We will now formulate the main result of this article, which gives an upper bound on the exceptional characteristics for the Conjecture 2.10. In order to state our result, we need to introduce the following numbers.
Let $s = (s_1, \ldots, s_l)$ be a sequence in $\hat{S}$. Let us define the corresponding Bott–Samelson element $H(s)$ in the affine Hecke algebra by

$$H(s) := H_{s_1} \cdots H_{s_l}.$$ 

Then there are $x_1, \ldots, x_n \in \hat{W}$ and $l_1, \ldots, l_n \in \mathbb{Z}$ with

$$H(s) = \sum_{i=1}^{n} u_i^l H_{x_i}.$$ 

Let the polynomials $a_x \in \mathbb{Z}[v]$ be defined by

$$H(s) = \sum_{x \in \hat{W}} a_x H_x.$$ 

Set $r_x = a_x(1)$ and set

$$r = r(s) := \max_x \{r_x\}.$$ 

Let $d_x = \left( \frac{d}{dv} a_x \right)(1)$ be the sum of the exponents of $a_x$, and set

$$d = d(s) := \max_x \{d_x\}.$$ 

Now we associate to $s$ the number $U(s) := r! (r! (r-1)! N(s)^{l+2d})^r$ ($N(s)$ was defined in Section 4). We set

$$U(w) := \min_s U(s),$$

where the minimum is taken over all reduced expressions for $w$. Note that $U(w') \leq U(w)$ for $w' \leq w$. Here is our result:

**Theorem 5.5.** Let $w \in \hat{W}$ and suppose that $\text{char } K > U(w)$. Then the Conjectures 2.10 and 4.5 hold for $B_K(w)$.

By Proposition 5.4, Theorem 5.5 is a consequence of the following result.

**Theorem 5.6.** Let $s$ be a sequence in $\hat{S}$ and suppose that $\text{char } K > U(s)$. Then the Lefschetz data of $B_K(s)$ and $B_Q(s)$ coincide at all $x \in \hat{W}$.

We prove the latter theorem by studying a base change. For this we have to give an alternative construction of the Bott–Samelson modules that also makes sense over $\mathbb{Z}$.

6. **ANOTHER CONSTRUCTION OF THE BOTT–SAMELSNON MODULAS**

For a sequence $s = (s_1, \ldots, s_l)$ in $\hat{S}$ we let $I(s)$ be the set of all subwords of the product $s_1 \cdots s_l$, i.e.

$$I(s) := \{ (i_1, \ldots, i_n) \mid 0 \leq n \leq l, 1 \leq i_1 < \cdots < i_n \leq l \}.$$ 

Then there is an evaluation map

$$\text{ev} : I(s) \rightarrow \hat{W},$$

$$(i_1, \ldots, i_n) \mapsto s_{i_1} \cdots s_{i_n}.$$ 

If $l \geq 1$ then we denote by $s' = (s_1, \ldots, s_{l-1})$ the truncated sequence. We consider $I(s')$ as a subset of $I(s)$ in the obvious way. If we define for $\sigma = (i_1, \ldots, i_n) \in I(s')$ the element $\sigma s_l \in I(s)$ by appending the reflection $s_l$ on
the right, i.e. $\sigma s_l := (i_1, \ldots, i_n, l)$, then $I(s)$ is the disjoint union of $I(s')$ and $I(s')s_l$.

In the following we will denote by $T$ a commutative unital domain in which 2 is not a zero divisor and which has furthermore the property that the elements $\alpha \otimes 1 \in \w X \otimes \mathbb{Z} T$ with $\alpha \in \w R$ are non-zero. The examples that we need are $T = \mathbb{Z}$ and $T = K$, where $K$ is a field of characteristic $\neq 2$. We denote by $S_T := S_T(\w X \otimes \mathbb{Z} T)$ the symmetric algebra over the free $T$-module associated to the affine weight lattice $\w X$. Again we consider $S_T$ as a graded algebra with $\deg \lambda = 2$ for each non-zero $\lambda \in \w X \otimes \mathbb{Z} T$. For $\lambda \in \w X$ we denote by $\lambda$ also the image of $\lambda \otimes 1$ in $S_T$. We denote by $Q_T$ the $S_T$-algebra that is obtained from $S_T$ by localization at the multiplicatively closed set generated by 2 and the set $\hat{R}$, i.e.

$$Q_T := S_T[2^{-1}][\alpha^{-1}, \alpha \in \hat{R}]$$

We define the structure algebra $Z_T$ in a way analogous to the original definition in Section 3.11 i.e.

$$Z_T := \left\{(z_x) \in \prod_{x \in \w W} S_T \mid \begin{array}{l}
\text{(1)} \quad z_x \equiv z_{k_x,n_x} \text{ mod } \alpha + n\delta \\
\text{for all } x \in \w W, \alpha \in R, n \in \mathbb{Z}
\end{array}\right\}.$$ 

There is a natural action of $Z_T$ on $\bigoplus_{\sigma \in I(s)} S_T$, given as follows: on the $\sigma$-component the element $(z_x) \in Z_T$ acts as multiplication with $z_{ev(\sigma)}$.

6.1. Generically graded $S_T$-modules. We are now going to construct, for each $T$ as above and each sequence $s = (s_1, \ldots, s_l)$ in $\hat{S}$ a sub-$S_T$-module $X_T(s)$ of $\bigoplus_{\sigma \in I(s)} S_T$. We call such a datum a generically graded $S_T$-module. We show that $X_T(s)$ is stable under the action of $Z_T$ on $\bigoplus_{\sigma \in I(s)} S_T$ that we defined above. In case $T = K$ is a field of characteristic $\neq 2$ we show that we have an isomorphism $X_K(s) \cong B_K(s)$ of $Z_K$-modules.

The advantage of this new construction is two-fold. Firstly, it provides a distinguished $S_T$-basis of $X_T(s)$. Secondly, it allows us to view each element of $X_T(s)$ as an $I(s)$-tuple of elements in $S_T$. We use this to construct a basis of the the stalk $B_K(s)^x$ for any $x$, which is then used to gain arithmetic information on a matrix describing the homomorphism $B_K(s)_x \to B_K(s)^x$.

Before we come to the definition of the generically graded modules we need some preparations. For each sequence $s$ of length $l \geq 1$ let us denote by

$$\Delta: \bigoplus_{\gamma \in I(s')} S_T \to \bigoplus_{\sigma \in I(s)} S_T$$

the diagonal embedding, i.e. the map that sends $(z_{\gamma})$ to $(z'_{\sigma})$ with $z'_{\sigma} = z_{\gamma}$ if $\sigma \in \{\gamma, \gamma s_l\}$. We extend $\Delta$ to the diagonal embedding $\bigoplus_{\gamma \in I(s')} Q_T \to \bigoplus_{\sigma \in I(s)} Q_T$.

Let $\iota: I(s) \to I(s')$ be the fixed point free involution that interchanges $\gamma$ and $\gamma s_l$ for all $\gamma \in I(s')$. We denote by the same letter the algebra isomorphism $\iota: \bigoplus_{\sigma \in I(s)} S_T \to \bigoplus_{\sigma \in I(s)} S_T$ that sends $(z_{\sigma})$ to $(z'_{\sigma})$ with $z'_{\sigma} = z_{\iota(\sigma)}$. Then the image of $\Delta$ coincides with the set of $\iota$-invariant elements in $\bigoplus_{\sigma \in I(s)} S_T$.
If 2 is invertible in $T$ then we have a canonical decompositon
\[
\bigoplus_{\sigma \in I(s)} S_T = \left( \bigoplus_{\sigma \in I(s)} S_T \right)^{\iota} \oplus \left( \bigoplus_{\sigma \in I(s)} S_T \right)^{-\iota},
\]
where by $(\cdot)^{\iota}$ we denote the $\iota$-invariant elements and by $(\cdot)^{-\iota}$ the $\iota$-anti-invariant elements. We have already identified the first direct summand with $\bigoplus_{\gamma \in I(s')} S_T$ via $\Delta$. We also identify the second summand with $\bigoplus_{\gamma \in I(s')} S_T$ via the map $\Delta^-$ which send $(z'_{\gamma})$ to $(z_{\gamma})$ with $z_{\gamma} = z'_{\gamma} = -z_{\gamma s_i}$ (recall that we consider $I(s')$ as a subset of $I(s)$). For an endomorphism $f$ of $\bigoplus_{\gamma \in I(s')} S_T$ we denote by $\Delta f$ the endomorphism on $\bigoplus_{\sigma \in I(s)} S_T$ that is diagonal with respect to the direct sum decomposition above and is given by $f$ on both direct summands under the identifications that we just constructed. Explicitly, for $(x, y) \in (\bigoplus_{\sigma \in I(s)} S_T)^{\iota} \oplus (\bigoplus_{\sigma \in I(s)} S_T)^{-\iota}$ we have $x = \Delta(x')$ and $y = \Delta^-(y')$ for some $x', y' \in \bigoplus_{\gamma \in I(s')} S_T$ and we set
\[
\Delta f (x, y) = (\Delta(f(x')), \Delta^-(f(y'))).
\]
For $\lambda \in \widehat{X}$ define $c^\lambda = (c^\lambda_x) \in \bigoplus_{x \in \widehat{W}} S_T$ by $c^\lambda_x = x(\lambda) \otimes 1 \in \widehat{X} \otimes_z T \subset S_T$. For each $\alpha \in R$ and $n \in \mathbb{Z}$ the linear form $x(\lambda) - s_{\alpha n} x(\lambda)$ vanishes on $\widehat{H}_{\alpha n}$, hence $x(\lambda) - s_{\alpha n} x(\lambda)$ is proportional to $\alpha + n \delta$, hence $c^\lambda \in \mathcal{H}_T$. We get a linear map $\lambda \mapsto c^\lambda$ from $\widehat{X}$ to $\mathcal{H}_T$. Using the $\mathcal{H}_T$-action defined before we consider $c^\lambda$ as an $S_T$-linear endomorphism on $\bigoplus_{\sigma \in I(s)} S_T$ for all sequences $s$.

Now we can define the generically graded $S_T$-modules.

**Definition 6.1.** We define for all sequences $s$ in $\widehat{S}$ the sub-$S_T$-module $\mathcal{X}_T(s) \subset \bigoplus_{\sigma \in I(s)} S_T$ by the following algorithm:

1. $\mathcal{X}_T(\emptyset) \subset S_T$ is given by the identity $S_T = S_T$.
2. For $l \geq 1$ and $s = (s_1, \ldots, s_l)$ let $s' = (s_1, \ldots, s_{l-1})$ be the truncated sequence. Then

$$
\mathcal{X}_T(s) := \Delta(\mathcal{X}_T(s')) + c^{\alpha_l} (\Delta (\mathcal{X}_T(s'))) \subset \bigoplus_{\sigma \in I(s)} S_T.
$$

Here $\alpha_l \in \widehat{1}$ is the simple affine root associated to $s_l$.

We list some easy to deduce properties.

**Lemma 6.2.** The sum in the inductive definition of $\mathcal{X}_T(s)$ is direct, i.e.

$$
\mathcal{X}_T(s) = \Delta(\mathcal{X}_T(s')) \oplus c^{\alpha_l} (\Delta (\mathcal{X}_T(s'))) \subset \bigoplus_{\sigma \in I(s)} S_T.
$$

In particular, $\mathcal{X}_T(s)$ is a graded free $S_T$-module of rank $(1 + v^2)^l$.

**Proof.** By construction the image of $\Delta$ consists of $\iota$-invariant elements. Since $c^{\alpha_l}_{x_{s}} = -c^{\alpha_l}_{x_{s_i}}$, the image of $c^{\alpha_l} \circ \Delta$ consists of $\iota$-anti-invariant elements. The claim now follows from our general assumption that 2 is not a zero divisor in $T$. 

The following lemma justifies the name “generically graded module”.

\[\text{\Box}\]
Lemma 6.3. The inclusion $\mathcal{X}_T(s) \subset \bigoplus_{\sigma \in I(s)} S_T$ is an isomorphism after application of the functor $\cdot \otimes_{S_T} Q_T$.

Proof. This is immediate for $s = \emptyset$. It then follows inductively from the equation $\mathcal{X}_T(s) = \Delta(\mathcal{X}_T(s')) \oplus c^{o_1} \Delta(\mathcal{X}_T(s'))$ since after tensoring with $Q_T$ the first summand yields all invariant elements, while the second yields all anti-invariant elements in $\bigoplus_{\sigma \in I(s)} Q_T$; by the induction hypothesis (recall that $2$ is invertible in $Q_T$). □

6.2. Base change. Let $T \to T'$ be a homomorphism of unital rings which satisfy our general assumptions at the beginning of Section 6.

Lemma 6.4. The canonical isomorphism $\left( \bigoplus_{\sigma \in I(s)} S_T \right) \otimes_T T' = \bigoplus_{\sigma \in I(s)} S_T$ identifies $\mathcal{X}_T(s) \otimes_T T'$ with $\mathcal{X}_{T'}(s)$.

Proof. We prove this by induction, the case $s = \emptyset$ being clear. If the length of $s$ is $\geq 1$, then we have, by definition and Lemma 6.2

$$
\mathcal{X}_T(s) \otimes_T T' = \Delta(\mathcal{X}_T(s')) \otimes_T T' + c^{o_1} \Delta(\mathcal{X}_T(s')) \otimes_T T'
$$

$$= \Delta(\mathcal{X}_T(s') \otimes T') + c^{o_1} (\Delta(\mathcal{X}_T(s') \otimes T'))
$$

$$= \Delta(\mathcal{X}_{T'}(s')) + c^{o_1} (\Delta(\mathcal{X}_{T'}(s'))),
$$

which is the statement we wanted to prove. □

6.3. A duality. For an $S_T$-module $M$ we denote by $DM = \text{Hom}_{S_T}^*(M, S_T)$ its graded dual. For a sequence $s$ in $\widehat{S}$ let us consider the standard pairing

$$
\bigoplus_{\sigma \in I(s)} Q_T \times \bigoplus_{\sigma \in I(s)} Q_T \to Q_T,
$$

$$
((x_\sigma), (y_\sigma)) \mapsto \sum_{\sigma} x_\sigma y_\sigma.
$$

Lemma 6.5. Using the above pairing we can canonically identify

$$
\text{DX}_T(s) = \left\{ (z_\sigma) \in \bigoplus_{\sigma \in I(s)} Q_T \biggm| \sum_{\sigma} z_\sigma m_\sigma \in S_T \right. \text{ for all } (m_\sigma) \in \mathcal{X}_T(s) \right\}.
$$

Proof. Each $S_T$-linear homomorphism $\phi : \mathcal{X}_T(s) \to S_T$ induces, after application of the functor $\cdot \otimes_{S_T} Q_T$, a homomorphism $\phi \otimes 1 : \bigoplus_{\sigma \in I(s)} Q_T \to Q_T$, by Lemma 6.3. After we identified $\text{Hom}_{Q_T}(\bigoplus_{\sigma \in I(s)} Q_T, Q_T)$ with $\bigoplus_{\sigma \in I(s)} Q_T$ in the obvious way, we have in fact associated to $\phi$ an element in the set on the right hand side of the lemma’s claim. A moment’s thought shows that this yields a bijection. □

Now we come to a definition that is very similar to the definition of the $\mathcal{X}_T(s)$’s.

Definition 6.6. We define for all sequences $s$ in $\widehat{S}$ the sub-$S_T$-module $\mathcal{Y}_T(s) \subset \bigoplus_{\sigma \in I(s)} Q_T$ by the following algorithm:

1. $\mathcal{Y}_T(\emptyset) \subset Q_T$ is given by the inclusion $S_T \subset Q_T$. 
(2) For \( l \geq 1 \) and \( s = (s_1, \ldots, s_l) \) let \( s' = (s_1, \ldots, s_{l-1}) \) be the truncated sequence. Then
\[
\mathcal{Y}_T(s) := \Delta(\mathcal{Y}_T(s')) + (c^{\alpha_l})^{-1}(\Delta(\mathcal{Y}_T(s'))) \subset \bigoplus_{\sigma \in I(s)} Q_T.
\]

Here \( \alpha_l \in \widehat{\Pi} \) is the simple affine root associated to \( s_l \).

Note that \( \text{ev}(\sigma)(\alpha_l) \in \widehat{X} \) is an affine root, so its image in \( Q_T \) is invertible. Hence \( c^{\alpha_l} \) is an automorphism of \( \bigoplus_{\sigma \in I(s)} Q_T \), so is invertible. As in Lemma 6.2, we show that the sum in part (2) of the definition is direct, i.e. we have

**Lemma 6.7.** For each sequence \( s \) of length \( \geq 1 \) we have \( \mathcal{Y}_T(s) = \Delta(\mathcal{Y}_T(s')) \oplus (c^{\alpha_l})^{-1}(\Delta(\mathcal{Y}_T(s'))) \subset \bigoplus_{\sigma \in I(s)} Q_T \).

Now we identify \( \mathcal{Y}_T(s) \) with the dual of \( \mathcal{X}_T(s) \).

**Lemma 6.8.** Suppose that 2 is invertible in \( T \). For each sequence \( s \) we then have
\[
\mathcal{Y}_T(s) = \left\{ (z_\sigma)_{\sigma \in I(s)} \in \bigoplus_{\sigma \in I(s)} Q_T \mid \sum_{\sigma \in I(s)} z_\sigma m_\sigma \in S_T \text{ for all } (m_\sigma) \in \mathcal{X}_T(s) \right\}.
\]
Hence, using Lemma 6.3 we get an identification \( D\mathcal{X}_T(s) = \mathcal{Y}_T(s) \).

**Proof.** For \( s = \emptyset \) the claim follows immediately from the definitions. So suppose that the length of \( s \) is at least one and that the statement holds for the truncated sequence \( s' \). We have \( \mathcal{X}_T(s) = \Delta(\mathcal{X}_T(s')) \oplus c^{\alpha_l} \Delta(\mathcal{X}_T(s')) \) and \( \mathcal{Y}_T(s) = \Delta(\mathcal{Y}_T(s')) \oplus (c^{\alpha_l})^{-1}\Delta(\mathcal{Y}_T(s')) \). For \( x \in \mathcal{X}_T(s') \) and \( y \in \mathcal{Y}_T(s') \) we have
\[
(\Delta(x), \Delta(y)) = 2(x, y),
\]
\[
(c^{\alpha_l}\Delta(x), (c^{\alpha_l})^{-1}\Delta(y)) = 2(x, y).
\]
Since 2 is supposed to be invertible in \( T \) we deduce from the induction hypothesis that \( (\cdot, \cdot) \) pairs either \( \Delta(\mathcal{X}_T(s')) \) and \( \Delta(\mathcal{Y}_T(s')) \) or \( c^{\alpha_l}\Delta(\mathcal{X}_T(s')) \) and \( (c^{\alpha_l})^{-1}\Delta(\mathcal{X}_T(s')) \) perfectly.

On the other hand, \( c^{\alpha_l} \) acts by a diagonal multiplication with a \( \iota \)-anti-invariant element, and we see that
\[
(\Delta(\mathcal{X}_T(s')), (c^{\alpha_l})^{-1}\Delta(\mathcal{Y}_T(s'))) = 0,
\]
\[
(c^{\alpha_l}\Delta(\mathcal{X}_T(s')), \Delta(\mathcal{Y}_T(s'))) = 0.
\]

\( \square \)

### 6.4. Stalks and costalks.

The inclusion \( \mathcal{X}_T(s) \subset \bigoplus_{\sigma \in I(s)} S_T \) allows us to define stalks and costalks of \( \mathcal{X}_T(s) \) in a way analogous to the case of \( \mathbb{Z} \)-modules (cf. Section 3). Let \( I \) be a subset of \( \widehat{\mathbb{W}} \) and define \( I(s)_{I} \subset I(s) \) as the set of elements whose evaluation is contained in \( I \), i.e. \( I(s)_{I} = \text{ev}^{-1}(I) \). We view \( \bigoplus_{\gamma \in I(s)_{I}} S_T \) as a direct summand in \( \bigoplus_{\gamma \in I(s)} S_T \). For a subspace \( M \) of
\[ \bigoplus_{\gamma \in I(s)} S_T \] we define
\[ M_T := M \cap \bigoplus_{\gamma \in I(s)_{x}} S_T \]
\[ M^x := \text{im} \left( M \subset \bigoplus_{\gamma \in I(s)} S_T \to \bigoplus_{\gamma \in I(s)_{x}} S_T \right). \]

In particular, we define the stalk of \( M \) at \( x \) as \( M^x := M(x) \) and the costalk of \( M \) at \( x \) as \( M_x := M(\{x\}) \).

**Lemma 6.9.** Let \( s \) be a sequence in \( \hat{S} \) and \( x \in \hat{W} \). The identification in Lemma 6.8 induces an identification
\[
D(\mathcal{X}_T(s^x)) = \left\{ (z_\sigma) \in \bigoplus_{\sigma \in I(s)_x} Q_T \ \bigg| \ \sum_\sigma z_\sigma m_\sigma \in S_T \right. \\
\left. \text{for all } (m_\sigma) \in \mathcal{X}_T(s)^x \right\} \\
= \mathcal{Y}_T(s)_x.
\]

**Proof.** Note that \( D(\mathcal{X}_T(s^x)) \subset D\mathcal{X}_T(s) \) is the subspace of all linear forms \( \phi: \mathcal{X}_T(s) \to S_T \) that vanish on the kernel of the map \( \mathcal{X}_T(s) \to \mathcal{X}_T(s)^x \). Using Lemma 6.3 we see that this is identified with the set of \( (z_\sigma) \in \bigoplus_{\sigma \in I(s)} Q_T \) with \( z_\sigma = 0 \) if \( \sigma \notin I(s)_x \). From this we get the first identity. The second identity follows from Lemma 6.8. \( \square \)

### 6.5. Distinguished bases.

By Lemma 6.2 and Lemma 6.7, the \( S_T \)-modules \( \mathcal{X}_T(s) \) and \( \mathcal{Y}_T(s) \) are free of (ungraded) rank \( 2^l \).

**Definition 6.10.** Let \( E_T(s) \subset \mathcal{X}_T(s) \) and \( F_T(s) \subset \mathcal{Y}_T(s) \) be the homogeneous \( S_T \)-bases that are defined inductively by the following algorithms:

1. \( E_T(\emptyset) = \{1\} \subset \mathcal{X}_T(\emptyset) = S_T \) and \( F_T(\emptyset) = \{1\} \subset \mathcal{Y}_T(\emptyset) = S_T. \)
2. If \( s \) is a sequence of length \( l \geq 1 \), then
\[ E_T(s) := \Delta(E_T(s')) \cup e^{\alpha_l} \Delta(E_T(s')), \]
\[ F_T(s) := \Delta(F_T(s')) \cup (e^{\alpha_l})^{-1} \Delta(F_T(s')). \]

Note that for a homomorphism \( T \to T' \) of unital domains the identification \( \mathcal{X}_T(s) \otimes_T T' = \mathcal{X}_T(s) \) of Lemma 6.4 identifies \( E_T(s) \otimes 1 \) with \( E_{T'}(s) \).

Using the inclusion \( \mathcal{X}_T(s) \subset \bigoplus_{\sigma \in I(s)} S_T \) we can view each element of \( E_T(s) \) as an \( I(s) \)-tuple of elements in \( S_T \). The following statement is clear from the definition.

**Lemma 6.11.** Each coordinate of each element \( e \) of \( E_T(s) \) is a product of \( 1/2 \) degree roots.

We need the following endomorphism \( P(s) \) on \( \bigoplus_{\sigma \in I(s)} Q_T \).

**Definition 6.12.** Let \( P(s) \) be defined for any sequence \( s \) in \( \hat{S} \) inductively by the following algorithm:

1. We let \( P(\emptyset) \) be the identity on \( Q_T \).
(2) If \( P(s') \) is already defined then we set
\[
P(s) := c^{\alpha_l} \circ \Delta^{P(s')}.\]

By definition, \( P(s) \) acts diagonally and on each direct summand it is given by multiplication with a product of \( l \) roots, where \( l \) is the length of \( s \).

**Lemma 6.13.** We have \( E_T(s) = P(s)(F_T(s)) \). In particular, we have \( X_T(s) = P(s)(\mathcal{Y}_T(s)) \).

**Proof.** Again we use induction on the length of \( s \). For \( s = \emptyset \) we have \( E_T(\emptyset) = F_T(\emptyset) = \{1\} \) and \( P(\emptyset) = \text{id} \). Suppose that \( s \) is of length \( l \geq 1 \) and that the statement is true for the truncated sequence \( s' \). Then
\[
P(s)(\Delta(F_T(s'))) = c^{\alpha_l} \Delta^{P(s')}(\Delta(F_T(s')))
= c^{\alpha_l}(\Delta(P(s')(F_T(s'))))
= c^{\alpha_l}(\Delta(E_T(s))).
\]

Analogously,
\[
P(s)((c^{\alpha_l})^{-1}\Delta(F_T(s'))) = c^{\alpha_l} \Delta^{P(s')}( (c^{\alpha_l})^{-1}\Delta(F_T(s')))
= \Delta(P(s')(F_T(s')))
= \Delta(E_T(s))
\]
and the claim follows. \( \square \)

### 6.6. The relation to Bott-Samelson modules.

Now we show that we indeed have given an alternative construction of the Bott–Samelson modules.

**Proposition 6.14.**
(1) Suppose that 2 is invertible in \( T \). Then for all sequences \( s \), the subset \( X_T(s) \) of \( \bigoplus_{s \in I(s)} S_T \) is stable under the action of \( Z_T \) defined above.

(2) If \( T = K \) is a field of characteristic \( \neq 2 \), then \( X_K(s) \) identifies with the Bott-Samelson module \( B_K(s) \) as a \( Z_K \)-module.

**Proof.** We prove the proposition by induction on the length of \( s \). For \( s = \emptyset \) both statements immediately follow from the definitions. So suppose that \( s = (s_1, \ldots, s_l) \) with \( l \geq 1 \) and that we have proven the claims for \( s' \). Now note that \( \Delta: \bigoplus_{\gamma \in \mathcal{I}(s') S_T \to \bigoplus_{s \in I(s)} S_T \) is actually a homomorphism of \( Z_T^{s_l} \)-modules. From our induction assumption we hence get that \( \Delta(X_T(s')) \) and \( c^{\alpha_l}(\Delta(X_T(s'))) \) are stable under the action of \( Z_T^{s_l} \).

By Lemma 5.1 in [Fie2] we have \( Z_T = Z_T^{s_l} \oplus c^{\alpha_l}Z_T^{s_l} \) under the assumption that 2 is invertible in \( T \). Note that \( (c^{\alpha_l})^2 \in Z_T^{s_l} \). Hence giving a \( Z_T \)-module structure on an abelian group \( M \) is the same as giving a \( Z_T^{s_l} \)-module structure on \( M \) together with a \( Z_T^{s_l} \)-linear endomorphism \( f \) on \( M \) such that \( f^2 \) coincides with the action of \( (c^{\alpha_l})^2 \). From this, statement (1) for \( X_T(s) = \Delta(X_T(s') \oplus c^{\alpha_l}\Delta(X_T(s')) \) easily follows by defining the map \( f \) on the components as \( \Delta(X_T(s')) \to c^{\alpha_l}\Delta(X_T(s')) \), \( x \mapsto c^{\alpha_l}x \) and \( c^{\alpha_l}\Delta(X_T(s')) \to \Delta(X_T(s')) \), \( c^{\alpha_l}x \mapsto (c^{\alpha_l})^2x \). Similarly, statement (2) follows after we considered the identities
\[
B_K(s) = Z \otimes_{Z^{s_l}} B_K(s') = (Z_K^{s_l} \oplus c^{\alpha_l}Z_K^{s_l}) \otimes_{Z^{s_l}} B_K(s')
= B_K(s') \oplus c^{\alpha_l}B_K(s').
\]
Finally, we compare the stalks of $\mathcal{B}_K(s)$ and the stalks of $\mathcal{X}_K(s)$.

**Lemma 6.15.** Let $s$ be a sequence in $\hat{S}$ and $x \in \hat{W}$. If $T = K$ is a field of characteristic $\neq 2$, then the isomorphism $\mathcal{X}_K(s) \cong \mathcal{B}_K(s)$ of $\mathcal{Z}_K$-modules induces isomorphisms $\mathcal{X}_K(s)_x \cong \mathcal{B}_K(s)_x$ and $\mathcal{X}_K(s)^x \cong \mathcal{B}_K(s)^x$.

**Proof.** The statement follows immediately from the definition of the action of $\mathcal{Z}_K$ on $\mathcal{X}_K(s)$ and Lemma 3.3. 

### 6.7. Integral matrices

Now we want to prove Theorem 5.6 so we fix a sequence $s = (s_1, \ldots, s_l)$ in $\hat{S}$ and a field $k$ of positive characteristic such that $\text{char } k > U(s)$. Note that this implies that $(k, s)$ is a GKM-pair. Let us also fix an element $x$ of $\hat{W}$ that is contained in $\text{ev}(I(s))$. We want to prove that the Lefschetz data of $\mathcal{B}_k(s)$ and $\mathcal{B}_Q(s)$ coincide at $x$. For this we construct a matrix $X$ with entries in $S_Z$ that describes both the inclusions $\mathcal{B}_k(s)_x \rightarrow \mathcal{B}_k(s)^x$ and $\mathcal{B}_Q(s)_x \rightarrow \mathcal{B}_Q(s)^x$ and then give an estimate on the minors of $X$.

In the following $K$ denotes either the field $k$ or the rational numbers $\mathbb{Q}$. Recall that we have a distinguished basis $E_Z(s)$ of $\mathcal{X}_Z(s)$. We next want to choose a subset $E(x)$ of $E_Z(s)$ that has the property that the images of its elements in both $\mathcal{X}_k(s)^x$ and $\mathcal{X}_Q(s)^x$ form a basis. (We consider the natural maps $\mathcal{X}_Z(s) \rightarrow \mathcal{X}_Z(s) \otimes_Z K = \mathcal{X}_k(s) \rightarrow \mathcal{X}_k(s)^x$ for $K = k$ and $K = \mathbb{Q}$).

**Lemma 6.16.** There is a subset $E(x)$ of $E_Z(s)$ with the property that the images of its elements in $\mathcal{X}_k(s)^x$ form an $S_K$-basis for both $K = k$ and $K = \mathbb{Q}$.

**Proof.** The images of the elements of $E_Z(s)$ generate the $S_k$-module $\mathcal{X}_k(s)^x$. Since this is a graded free $S_k$-module we can find a subset $E(x)$ of $E(s)$ such that its image forms a basis. Then $E(x)$ is $S_Z$-linearly independent in $\mathcal{X}_Z(s)^x$ since we have a commutative diagram

$$
\begin{align*}
\mathcal{X}_Z(s) & \longrightarrow \mathcal{X}_Z(s)^x \\
\mathcal{X}_k(s) & \longrightarrow \mathcal{X}_k(s)^x.
\end{align*}
$$

Hence $E(x)$ is $S_\mathbb{Q}$-linearly independent in $\mathcal{X}_\mathbb{Q}(s)^x$. Since the graded ranks of $\mathcal{X}_k(s)^x$ and $\mathcal{X}_\mathbb{Q}(s)^x$ coincide, by Lemma 6.15 and Lemma 3.13, the image of $E(x)$ forms a basis of $\mathcal{X}_\mathbb{Q}(s)^x$ as well.  

We now fix such a subset $E(x)$ of $E_Z(s)$. From Lemma 6.3 we deduce that the ungraded rank of $\mathcal{X}_k(s)^x$ equals the order of $I(s)_x$ which coincides with the number $r_x$ defined in Section 5.3. Let us identify $I(s)_x$ with $\{1, \ldots, r_x\}$ and let us choose an enumeration $\{v_1, \ldots, v_{r_x}\}$ of the images of the elements of $E(x)$ in $\mathcal{X}_Z(s)^x$. Let $E \subset S_k^{x \times r_x}$ be the matrix $(v_1, \ldots, v_{r_x})^T$, i.e. the matrix with row vectors $v_1, \ldots, v_{r_x}$.

We apply the base change $S_Z \rightarrow S_K$ and consider the matrix $E$ now as an element in $S_K^{x \times r_x}$. Its row vectors form a basis of $\mathcal{X}_K(s)^x$ by construction. By Lemma 6.3 the matrix $E$ is invertible over $Q_K$. The row vectors of the matrix
\[(E^{-1})^T \in Q^{r_x \times r_x}_K\] now form a basis of the space
\[
\left\{(z_\sigma) \in \bigoplus_{\sigma \in I(s)_x} Q_K \left| \sum_{\sigma} z_\sigma m_\sigma \in S_K \right. \text{ for all } (m_\sigma) \in \mathcal{X}_K(s) \right\}
\]
which we have identified, as a subspace of \(\bigoplus_{\sigma \in I(s)_x} Q_K\), with \(\mathcal{Y}_K(s)_x\) in Lemma 6.9.

Recall the endomorphism \(P(s)\) on \(\bigoplus_{\sigma \in I(s)} Q_Z\). It acts diagonally and so we can consider its restriction \(P(s)^x\) to the direct summand \(\bigoplus_{\sigma \in I(s)_x} Q_Z\). We consider the latter endomorphism as a diagonal \(r_x \times r_x\)-matrix which we denote by \(P\) for simplicity. Now Lemma 6.9 and Lemma 6.13 show that the row vectors \(w_1, \ldots, w_{r_x}\) of the matrix \((E^{-1})^T P\) form a basis of the \(S_K\)-module \(\mathcal{X}_K(s)_x\).

Hence we have constructed bases \((w_i)\) and \((v_i)\) of \(\mathcal{X}_K(s)_x\) and \(\mathcal{X}_K(s)^x\). Now consider the matrix
\[
X = (x_{ij}) := (E^{-1})^T P E^{-1}.
\]
Then we have
\[
w_i = \sum_{j=1}^{r_x} x_{ij}v_j.
\]

The matrix \(X\) already appears in \(\text{H"{a}rterich}\)'s paper [Ha] in which he studies the push-forward of the equivariant constant sheaf on a Bott-Samelson variety to a Schubert variety, cf. the remark following Proposition 6.8 in loc. cit.

Recall that we defined \(\tau : S_Z \rightarrow \mathbb{Z}[t]\) by the property \(\tau(\alpha) = t\) for all simple affine roots \(\alpha\). We denote by \(\tau\) also the homomorphism \(S_K \rightarrow K[t]\) obtained by base change. Recall that we assume \(\text{char } k > N(s)\). In order to determine the Lefschetz data we have to study the inclusion \(\overline{\mathcal{X}}_K(s)_x \subset \overline{\mathcal{X}}_K(s)^x\) which is given by the image \(\overline{X}\) of the matrix \(X\) in \(K[t]^{r_x \times r_x}\) under the map \(\tau\). In order to prove Theorem 5.6 it is enough to prove the following claim:

Under the assumption \(\text{char } k > U(s)\), a minor of the matrix \(X\) vanishes in \(k[t]\) if and only if it vanishes in \(\mathbb{Q}[t]\).

For convenience we study, instead of \(X\), the matrix
\[
X' = (x'_{ij}) := \det(E)^2 X' = (\det E)(E^{-1})^T P (\det E) E^{-1}.
\]
It has the advantage that its entries are elements in \(S_Z\), and it is certainly enough to prove the above claim for \(X'\) instead of \(X\).

Let us denote by \(d_i\) the half degree of \(v_i\). Set \(d = \sum_{i=1}^{r_x} d_i\). Then we have:

**Lemma 6.17.** The element \(x'_{ij} \in S_Z\) can be written as a sum of \(r_x! (r_x - 1)!\) products of \(2d - d_i - d_j + l\) roots.

**Proof.** Recall that the \(ij\)-entry \(v_{ij}\) of the matrix \(E\) is a product of \(d_i\) roots. The \(ij\)-entry of \((\det E)^{-1}\) is the determinant of the matrix obtained from \(E\) by deleting the \(j\)-th row and the \(i\)-th column. Hence it can be written as a sum of \((r_x - 1)!\) products of \(d - d_j\) roots. It follows that the \(ij\)-entry of \(P(\det E)E^{-1}\) can be written as a sum of \((r_x - 1)!\) products of \(d - d_j + l\) roots. Finally, the \(ij\)-entry \(x'_{ij}\) of \(X'\) can be written as a sum of \(r_x (r_x - 1)!^2 = r_x! (r_x - 1)!\) products of \(2d - d_i - d_j + l\) roots. \(\square\)
Let $J$ and $J'$ be subsets of $\{1, \ldots, r_x\}$ of the same size $s$ and denote by $\xi_{J,J'} \in S_Z$ the corresponding minor of the matrix $X'$. Set $d_J = \sum_{i \in J} d_i$ and $d_{J'} = \sum_{i \in J'} d_i$. Then we have:

**Lemma 6.18.** The element $\xi_{J,J'}$ can be written as a sum of $s!(r_x!(r_x - 1)!^s$ products of $s(2d + l) - d_J - d_{J'}$ roots.

**Proof.** Each entry of the submatrix of $X'$ corresponding to $J$ and $J'$ is a sum of $r_x!(r_x - 1)!$ products of roots by Lemma 6.17. Hence $\xi_{J,J'}$ can be written as a sum of $s!(r_x!(r_x - 1)!^s$ products of roots. Moreover, the $ij$-entry of $X'$ is a sum of products of $2d + l - d_i - d_j$ roots. So each summand in $\xi_{J,J'}$ is a product of $s(2d + l) - d_J - d_{J'}$ roots. □

One should keep in mind that the above estimate is very rough. For example, up to a power of 2, the determinant of the matrix $X'$ is a product of roots by Lemma 6.13.

The map $\tau: S_Z \to \mathbb{Z}[t]$ maps a positive root $\alpha$ to $\text{ht}(\alpha)t$. By definition of $N(s)$ we have $\text{ht}(\alpha) \leq N(s)$ for all roots that occur in the situation of Lemma 6.18. If we denote by $\xi_{J,J'}$ the image of $\xi_{J,J'}$ in $\mathbb{Z}[t]$, then we can deduce from Lemma 6.18 that $\xi_{J,J'} = a_{J,J'} t^{s(2d+l)-d_J-d_{J'}}$ with

$$|a_{J,J'}| \leq s!(r_x!(r_x - 1)!^s N(s)^{s(2d+l)-d_J-d_{J'}}.$$

Since $r_x \leq r$, $s \leq r$ and $d_J, d_{J'} \geq 0$ we have

$$|a_{J,J'}| \leq r!(r!(r-1)! N(s)^{2d+l})^r = U(s).$$

Hence, for $\text{char } k > U(s)$ the minor $\xi_{J,J'}$ vanishes in $k[t]$ if and only if it vanishes in $\mathbb{Q}[t]$, which is the statement we wanted to show in order to prove Theorem 5.6.

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