ON JET BUNDLES AND GENERALIZED VERMA MODULES II

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Abstract. Let $G$ be a semi simple linear algebraic group over a field of characteristic zero and let $V$ be a finite dimensional irreducible $G$-module with highest weight vector $v \in V$. Let $P \subseteq G$ be the parabolic subgroup fixing $v$. Let $\mathfrak{g} = \text{Lie}(G)$. We get a filtration $U^k(\mathfrak{g})v : U^k(\mathfrak{g})v \subseteq V$ of $P$-modules for $1 \leq k \leq N$. The aim of this paper is to use higher direct images of $G$-linearized sheaves, filtrations of generalized Verma modules and annihilator ideals of highest weight vectors to give a natural basis for $U^k(\mathfrak{g})v$ and to compute its dimension. We also relate the filtration $U^k(\mathfrak{g})v$ to $G$-linearized jet bundles on the flag variety $G/P$ for $G = \text{SL}(E)$ where $E$ is a finite dimensional vector space.

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1. Introduction

Let $F$ be a fixed field of characteristic zero and let $G$ be a semi simple linear algebraic group over $F$. Let $V$ be a finite dimensional irreducible $G$-module with highest weight vector $v \in V$ and highest weight $\lambda$. Let $L_v \subseteq V$ be the subspace spanned by $v$. Let $P \subseteq G$ be the parabolic subgroup of elements fixing the subspace $L_v \subseteq V$. It follows the quotient $G/P$ is a smooth projective variety of finite type over $F$. Let $\mathfrak{g} = \text{Lie}(G)$. We get a filtration of $V$ by $P$-modules

\begin{equation}
U^1(\mathfrak{g})v \subseteq U^2(\mathfrak{g})v \subseteq \cdots \subseteq U^N(\mathfrak{g})v = V
\end{equation}

- the canonical filtration. Here $N = N(\lambda)$ is the minimal integer with $U^N(\mathfrak{g})v = V$.

In a previous paper on this subject (see [7]) the filtration (1.0.1) was studied in the case of $V = H^0(G(m, m+n), \mathcal{O}(d))^*$ on the grassmannian $\text{SL}(E)/P = G(m, m+n)$. Here $P \subseteq \text{SL}(E)$ is the parabolic subgroup fixing an $m$-dimensional subspace in $E$. There is an equivalence of categories between the category of $\text{SL}(E)$-linearized locally free sheaves on $\text{SL}(E)/P$ and the category of rational $P$-modules and the

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aim of this paper is to use this equivalence to interpret the filtration \([1.0.1]\) in terms of \(\text{SL}(E)/\text{P}\).

We use higher direct images of \(G\)-linearized sheaves, filtrations of generalized Verma modules and annihilator ideals of highest weight vectors to answer the following questions for any parabolic subgroup \(P \subseteq \text{SL}(E)\) fixing a flag \(E_\bullet\) in \(E\) where \(E\) is a finite dimensional vector space over \(F\):

\[\text{(1.0.2)}\] Construct a basis for \(U^k(\mathfrak{g})v\) as \(F\)-vector space.

\[\text{(1.0.3)}\] Calculate the dimension of \(U^k(\mathfrak{g})v\).

\[\text{(1.0.4)}\] Interpret \(\{U^k(\mathfrak{g})v\}_{k=0}^N\) in terms of geometric objects on \(G/P\).

The strategy of the proof is as follows.

In section two of the paper we consider Question \([1.0.2]\) and Question \([1.0.3]\). Let

\[E_\bullet : 0 \neq E_1 \subseteq E_2 \subseteq \cdots \subseteq E_k \subseteq E_{k+1} = E\]

be a flag in the vector space \(E\) of type \(\underline{d} = (d_1, d_2, \ldots, d_k, d_{k+1})\). This means \(\dim_F(E_i) = d_1 + \cdots + d_i\). Let \(n_i = d_1 + \cdots + d_i\) for \(i = 1, \ldots, k+1\). Let \(P \subseteq G = \text{SL}(E)\) be the parabolic subgroup fixing the flag \(E_\bullet\). The quotient \(G/P = \mathbb{P}(\underline{d}, E)\) is the flag variety of type \(\underline{d}\) parametrizing flags in \(E\). Let \(V\) have highest weight

\[\lambda = \sum_{i=1}^k l_i(L_1 + \cdots + L_{n_i}).\]

Let \(L([\underline{l}]) \in \text{Pic}^G(G/P) = \mathbb{Z}^k\) be the line bundle corresponding to \(\underline{l} = (l_1, \ldots, l_k)\). The \(G\)-module \(H^0(G/P, L([\underline{l}]))^*\) has highest weight \(\lambda\) hence there is an isomorphism of \(G\)-modules \(V \cong H^0(G/P, L([\underline{l}]))^*\) giving a geometric construction of the \(G\)-module \(V\). The highest weight vector \(v\) of \(V\) has a geometric construction. It is the vector defined by \(v : H^0(G/P, L([\underline{l}])) \to F\) and \(v(s) = s(\underline{l})\). There is an inclusion of \(G\)-modules

\[\text{(1.0.5)}\] \[H^0(G/P, L([\underline{l}]))^* \subseteq \text{Sym}^{l_1}(\wedge^{d_1}E) \otimes \cdots \otimes \text{Sym}^{l_k}(\wedge^{d_k}E)\]

and the highest weight vector \(v \in V = H^0(G/P, L([\underline{l}]))^*\) is via the inclusion \([3.9]\) described explicitly as follows:

\[\text{(1.0.6)}\] \[v = \text{Sym}^{l_1}(\wedge^{d_1}E_1) \otimes \cdots \otimes \text{Sym}^{l_k}(\wedge^{d_k}E_k) \in H^0(G/P, L([\underline{l}]))^*.\]

We get an exact sequence of \(G\)-modules

\[0 \to \text{ann}(v, \lambda) \otimes L_v \to U(\mathfrak{g}) \otimes L_v \to H^0(G/P, L([\underline{l}]))^* \to 0\]

and an exact sequence of \(P\)-modules

\[\text{(1.0.7)}\] \[0 \to \text{ann}^k(v, \lambda) \otimes L_v \to U^k(\mathfrak{g}) \otimes L_v \to U^k(\mathfrak{g})v \to 0.\]

Here \(\text{ann}(v, \lambda) \subseteq U(\mathfrak{g})\) is the left annihilator ideal of \(v \in V\). The left \(G\)-module \(U(\mathfrak{g}) \otimes L_v\) is a generalized Verma module and \(U^\bullet(\mathfrak{g}) \otimes L_v \subseteq U(\mathfrak{g}) \otimes L_v\) is the canonical filtration of \(U(\mathfrak{g}) \otimes L_v\). Let \(\mathfrak{p} = \text{Lie}(P)\) be the Lie algebra fixing \(v \in V\). We use the explicit description of \(v \in V = H^0(G/P, L([\underline{l}]))^*\) given in \([1.0.6]\) the exact sequence \([1.0.7]\) and properties of the universal enveloping algebra \(U(\mathfrak{g})\) to prove the following theorem:
**Theorem 1.1.** Let $1 \leq k \leq \min\{l_i + 1\}$. There is an equality of vector spaces
\[
W^k(v, \lambda) = U^k(n_-).
\]
Here $W^k(v, \lambda)$ is a complement of $\operatorname{ann}^k(v, \lambda)$ in $U^k(g)$ and $n_- \subseteq g$ is a sub Lie algebra with $g = n_- \oplus p$.

**Proof.** See Theorem 2.12.

From Theorem 2.12 we can prove Corollary 2.13 answering Question 1.0.2 and Corollary 2.14 answering Question 1.0.3.

In the final section we consider Question 1.0.4. The vector space $U^k(g)v$ is a $P$-module hence it corresponds to an $SL(E)$-linearized locally free sheaf $J^k$ on $SL(E)/P$. We make the equivalence explicit and give a geometric construction of $J^k$ in terms of $SL(E)$-linearized jet bundles on $SL(E)/P$. We use a vanishing theorem from an earlier paper on the subject (see [7]), Kunne th formulas and general properties of jet bundles on products to construct an exact sequence of $P$-modules
\[
0 \to J^k_0(L(\mathfrak{l})(\mathfrak{f}))^* \to H^0(F, L(\mathfrak{l}))^* \to \phi H^0(F, m^{k+1}L(\mathfrak{l}))^* \to 0
\]
when $1 \leq k \leq \min\{l_i, + 1\}$. We get an injection of $P$-modules
\[
J^k_0(L(\mathfrak{l})(\mathfrak{f}))^* \subseteq V = H^0(F(\mathfrak{d}, E), L(\mathfrak{l}))^*.
\]
Here we use the fact we can give a geometric construction of the $SL(E)$-module $V$. Then we prove the main result of the paper:

**Theorem 1.2.** Let $1 \leq k \leq \min\{l_i + 1\}$. There is an isomorphism of $P$-modules
\[
U^k(g)v \cong J^k_0(L(\mathfrak{l})(\mathfrak{f}))^*.
\]

**Proof.** See Theorem 3.10.

Theorem 3.10 answer Question 1.0.4 giving a geometric interpretation of the filtration $U^*\underline{\mathfrak{g}}v \subseteq V$ in terms of $SL(E)$-linearized jet bundles on $SL(E)/P$.

The motivation for the study of the jet bundle $J^k(L(\mathfrak{l}))$ is partly its relationship with the discriminant $D^k(L(\mathfrak{l}))$ of the line bundle $L(\mathfrak{l})$. Assume $\mathfrak{l} = (l_1, \ldots, l_k) \in \mathbb{Z}^k$ with $l_i \geq 1$ for all $i$. It follows by the results of this paper the $k$th Taylor map
\[
T^k : H^0(G/P, L(\mathfrak{l})) \otimes O_{G/P} \to J^k(L(\mathfrak{l}))
\]
is surjective when $1 \leq k \leq \min\{l_i + 1\}$. We get an exact sequence of locally free sheaves
\[
0 \to Q_{L(\mathfrak{l})} \to H^0(G/P, L(\mathfrak{l})) \otimes O_{G/P} \to J^k(L(\mathfrak{l})) \to 0
\]
of $O_{G/P}$-modules. Dualize this sequence to get an exact sequence
\[
0 \to J^k(L(\mathfrak{l}))^* \to H^0(G/P, L(\mathfrak{l}))^* \otimes O_{G/P} \to Q_{L(\mathfrak{l})} \to 0.
\]
Take relative projective space bundle to get a closed immersion of schemes
\[
P(Q_{L(\mathfrak{l})}) \subseteq P(W^*) \times G/P
\]
where
\[
W = H^0(G/P, L(\mathfrak{l})).
\]
There is a projection map
\[
p : P(W^*) \times G/P \to P(W^*)
\]
and by the results of [3] it follows the direct image scheme \( p(P(\mathcal{O}_\mathcal{L}^*)) \) equals the discriminant \( D^k(\mathcal{L}(\mathcal{U})) \) of the line bundle \( \mathcal{L}(\mathcal{U}) \). There is on \( P(W^*) \) the tautological sequence

\[
0 \to \mathcal{O}(-1) \to W \otimes \mathcal{O}_{P(W^*)}.
\]

pull this and the Taylor map back to \( Y = P(W^*) \times G/P \) to get the composed map

\[
\phi : \mathcal{O}(-1)_Y \to W \otimes \mathcal{O}_{G/P} \to J^k(\mathcal{L}(\mathcal{U}))_Y.
\]

It follows by the results of [3] the scheme theoretic image of the zero scheme \( p(Z(\phi)) \) equals the discriminant \( D^k(\mathcal{L}(\mathcal{U})) \) as subscheme of \( Y \). When the ideal sheaf of \( Z(\phi) \) is locally generated by a regular sequence we get a Koszul complex of locally free sheaves

\[
0 \to \mathcal{O}(-r)_Y \otimes \wedge^r J^k(\mathcal{L}(\mathcal{U}))_Y^* \to \cdots \to \mathcal{O}(-1)_Y \otimes \wedge^1 J^k(\mathcal{L}(\mathcal{U}))_Y^* \to \mathcal{O}_Y \to \mathcal{O}_{Z(\phi)} \to 0
\]

which is a resolution of the ideal sheaf of \( Z(\phi) \). When we push this complex down to \( P(W^*) \) we get a double complex with terms as follows:

\[
\mathcal{R}^i p_* (\mathcal{O}(-j)_Y \otimes \wedge^j J^k(\mathcal{L}(\mathcal{U}))_Y^*) \to \mathcal{O}(-j) \otimes \mathcal{H}^i(G/P, \wedge^j J^k(\mathcal{L}(\mathcal{U}))^*)
\]

By [3], Theorem 5.2 we know knowledge on the \( P \)-module structure of the fiber \( J^k(\mathcal{L}(\mathcal{U}))(e)^* \) will give information on \( \mathcal{H}^i(G/P, \wedge^j J^k(\mathcal{L}(\mathcal{U}))^*) \). Hence we may determine if the double complex

\[
\mathcal{O}(-j) \otimes \mathcal{H}^i(G/P, \wedge^j J^k(\mathcal{L}(\mathcal{U}))^*)
\]

can be used to construct a resolution of the ideal sheaf \( I \) of the discriminant \( D^k(\mathcal{L}(\mathcal{U})) \).

2. Character ideals and annihilator ideals

Let in this section \( E \) be a fixed \( N \)-dimensional vector space over \( F \). Let

\[
E_\bullet : 0 \neq E_1 \subset E_2 \subset \cdots \subset E_k \subset E_{k+1} = E
\]

be a flag in the vector space \( E \) of type \( d = (d_1, d_2, ..., d_k, d_{k+1}) \). This means \( \dim F(E_i) = d_1 + \cdots + d_i \). Let \( n_i = d_1 + \cdots + d_i \) for \( i = 1, ..., k + 1 \). Let \( P \subset \text{SL}(E) \) be the parabolic subgroup fixing the flag \( E_\bullet \). This means \( g(E_i) \subset E_i \) for all \( g \in P \) and \( i = 1, ..., k \). The quotient \( \mathbb{F}(d, E) = \text{SL}(E)/P \) is a smooth projective variety of finite type over \( F \) - the flag variety of type \( d \). It is a geometric quotient in the sense of [12] and it is the parameter space parametrizing flags of type \( d \) in \( E \). This means each point \( x \in \mathbb{F}(d, E) \) with coefficients in \( F \) corresponds to a unique flag

\[
E_1^x \subset E_2^x \subset \cdots \subset E_k^x \subset E
\]

of type \( d \).

Let \( G_i = G(n_i, N) \) be the grassmannian of \( n_i \)-planes in an \( N \)-dimensional \( F \)-vector space. There is a closed immersion - the generalized Plucker embedding

\[
i : \mathbb{F}(d, E) \to G_1 \times \cdots \times G_k
\]

defined by

\[
i([E_1 \subset E_2 \subset \cdots \subset E_k]) = ([E_1], [E_2], ..., [E_k]).
\]

Let \( \mathcal{L} = (l_1, ..., l_k) \in \mathbb{Z}^k \) be a \( k \)-tuple of integers. We get a line bundle

\[
\mathcal{L}(\mathcal{U}) = i^* \mathcal{O}(l_1) \otimes \cdots \otimes \mathcal{O}(l_k)
\]
and the line bundle $\mathcal{L}(\underline{l})$ has by [12] a unique $\text{SL}(E)$-linearization. We get an isomorphism

$$\text{Pic}^{\text{SL}(E)}(\mathbb{F}(\underline{d}, E)) \cong \mathbb{Z}^k$$

of groups. If $l_i \geq 1$ for $i = 1, \ldots, k$ it follows $\mathcal{L}(\underline{l})$ is very ample. It follows there is a closed immersion

$$j : \mathbb{F}(\underline{d}, E) \rightarrow \mathbb{P}^M$$

with $M >> 0$ and $j^*\mathcal{O}(1) = \mathcal{L}(\underline{l})$.

Let $P \subseteq G$ be the subgroup consisting of matrices $g$ with determinant one of the following type:

$$g = \begin{pmatrix}
A_1 & * & \cdots & * & * \\
0 & A_2 & \cdots & * & * \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & A_k & * \\
0 & 0 & \cdots & 0 & A_{k+1}
\end{pmatrix}$$

where $A_i$ is an $d_i \times d_i$-matrix with coefficients in $F$. It follows $P$ is the group of elements $g \in \text{SL}(E)$ fixing the flag $E_\bullet$. Note: We can define the subgroup $P$ using points with values in $F$ since $F$ has characteristic zero and all group schemes in characteristic zero are smooth. It follows $P$ is a parabolic group and it follows the quotient $G/P$ equals $\mathbb{F}(\underline{d}, E)$. Let $\underline{l} = (l_1, \ldots, l_k) \in \mathbb{Z}^k$ and let $\mathcal{L}(\underline{l}) \in \text{Pic}^G(G/P)$ be the line bundle defined above. There is a unique $P$-stable vector $v$ in $\text{H}^0(G/P, \mathcal{L}(\underline{l}))^*$ defined as follows:

$$v : \text{H}^0(G/P, \mathcal{L}(\underline{l})) \rightarrow F$$

with

$$v(s) = s(\tau)$$

where $\tau \in G/P$ is the class of the identity element.

**Lemma 2.1.** The vector $v$ is a highest weight vector for $\text{H}^0(G/P, \mathcal{L}(\underline{l}))^*$ with highest weight

$$\lambda = \sum_{i=1}^k l_i \omega_{n_i}$$

**Proof.** This is left to the reader as an exercise. \qed

Let $V$ be an arbitrary finite dimensional irreducible $G$-module with highest weight vector $v \in V$ and highest weight $\lambda$. Let $\omega_j = L_1 + \cdots + L_j$ for $j = 1, \ldots, N - 1$ be the fundamental weights for $G = \text{SL}(E)$. It follows $\lambda = \sum_{i=1}^k l_i \omega_{n_i}$ with $l_i \geq 0$ for all $i$. There is the following well known result:

**Theorem 2.2.** There is a parabolic subgroup $P \subseteq G$ and a linebundle $\mathcal{L}(\underline{l}) \in \text{Pic}^G(G/P)$ with an isomorphism $\text{V} \cong \text{H}^0(G/P, \mathcal{L}(\underline{l}))^*$ as $G$-modules.

**Proof.** Let $P \subseteq G$ be the above defined parabolic subgroup and let $\mathcal{L}(\underline{l})$ be the line bundle with $\underline{l} = (l_1, \ldots, l_k) \in \mathbb{Z}^k$ It follows by the Borel-Weil-Bott Theorem (see [11]) that $\text{H}^0(G/P, \mathcal{L}(\underline{l}))^*$ is an irreducible $G$-module. By Lemma 2.1 it follows the $P$-stable vector $v \in \text{H}^0(G/P, \mathcal{L}(\underline{l}))^*$ has highest weight $\lambda$ with

$$\lambda = \sum_{i=1}^k l_i (\text{tr}(A_1) + \cdots + \text{tr}(A_i)) = \sum_{i=1}^k l_i \omega_{n_i}$$
hence there is an isomorphism

\[ V \cong H^0(G/P, L(\mathcal{L}))^* \]

of SL(E)-modules and the Theorem is proved. \(\square\)

Hence Theorem 2.2 gives a geometric construction of all finite dimensional irreducible representations of \(G = \text{SL}(E)\): All irreducible finite dimensional \(G\)-modules may be realized as duals of global sections of \(G\)-linearized line bundles on \(G/P\) for some parabolic subgroup \(P \subseteq G\). Note: The subgroup \(P\) is not unique.

Let \(L_v\) be the subspace spanned by \(v\). The group \(P \subseteq \text{SL}(E)\) is the subgroup of elements \(g \in \text{SL}(E)\) stabilizing the space \(L_v\) defined by the highest weight vector \(v \in V = H^0(G/P, L(\mathcal{L}))^*\). Let \(g = \text{Lie}(\text{SL}(E))\) and let \(U^k(g) \subseteq U(g)\) be the canonical filtration of the universal enveloping algebra of \(g\). The vector space \(U^k(g)\) is a \(G\)-module via the adjoint representation. It follows \(U^k(g)\) is a \(P\)-module. There is a surjective map of \(G\)-modules

\[ \phi : U(g) \otimes L_v \to H^0(G/P, L(\mathcal{L}))^* \]

defined by

\[ \phi(x \otimes v) = x(v). \]

The \(G\)-module \(U(g) \otimes L_v\) is a **generalized Verma module** and the map \(\phi\) realize the \(G\)-module \(H^0(G/P, L(\mathcal{L}))^*\) as a quotient of \(U(g) \otimes L_v\). It is a fact that any finite dimensional irreducible \(G\)-module may be realized as a quotient of a generalized Verma module. The map \(\phi\) induces a surjection of \(P\)-modules

\[ U^k(g) \otimes L_v \to U^k(g)v \]

where \(U^k(g)v\) is the vector space of elements \(x(v)\) with \(x \in U^k(g)\). We get a filtration of \(V\) by \(P\)-modules:

\[ U^*(g)v : U^1(g)v \subseteq \cdots \subseteq U^k(g)v \subseteq V. \]

**Definition 2.3.** Let the filtration \(U^*(g)v \subseteq V\) be the **canonical filtration** of \(V\).

Since the \(P\)-module \(U^k(g)v\) only depends on the vector space \(L_v\) defined by the highest weight vector \(v \in V\) we have defined for an arbitrary irreducible \(\text{SL}(E)\)-module \(V\) a canonical filtration \(U^*(g)v \subseteq V\).

Note: This notion is well defined for an arbitrary irreducible finite dimensional representation of an arbitrary semi simple Lie algebra.

**Definition 2.4.** Let \(ann(v, \lambda) \subseteq U(g)\) be the left **annihilator ideal** of \(v \in V\) and let \(ann^k(v, \lambda) \subseteq ann(v, \lambda)\) be its canonical filtration.

We get an exact sequence of \(G\)-modules

\[ 0 \to ann(v, \lambda) \otimes L_v \to U(g) \otimes L_v \to H^0(G/P, L(\mathcal{L}))^* \to 0 \]

and an exact sequence of \(P\)-modules

\[ 0 \to ann^k(v, \lambda) \otimes L_v \to U^k(g) \otimes L_v \to U^k(g)v \to 0 \]

where the rightmost map is the obvious action map. The sequence 2.4.2 describe the terms \(U^k(g)v\) in the canonical filtration

\[ U^*(g)v : \{v\} \subseteq U^1(g)v \subseteq U^2(g)v \subseteq \cdots \subseteq U^N(g)v = V \]

of \(V = H^0(G/P, L(\mathcal{L}))^*\) as quotients of the terms in the filtration \(U^*(g) \otimes L_v\) of the generalized Verma module \(U(g) \otimes L_v\). Here \(N = N(\lambda)\) is the minimal integer
with the property that $U^N(g)v = V$. The aim of this section is to answer the questions posed in 1.0.2 and 1.0.3 using the exact sequence 2.4.2 and properties of the universal enveloping algebra.

Let $p = \text{Lie}(P)$ be the Lie algebra of $P$. It follows $p \subseteq g = \mathfrak{sl}(E)$ is the Lie algebra of matrices $x \in \mathfrak{sl}(E)$ of the form

$$x = \begin{pmatrix} A_1 & \cdots & \ast & \ast \\ 0 & A_2 & \cdots & \ast \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_k \\ 0 & 0 & \cdots & 0 & A_{k+1} \end{pmatrix}.$$  

Here $A_i$ is a $d_i \times d_i$-matrix with coefficients in $F$ and $\text{tr}(x) = 0$. We aim to calculate a vector space $W_k(v, \lambda) \subseteq U_k(g)$ with the property that $U_k(g) = W_k(v, \lambda) \oplus \text{ann}_k(v, \lambda)$.

The line $(v)$ defined by $v$ is $p$-stable and we get a character

$$\rho : p \to \text{End}_F(v)$$

defined by

$$\rho(x)(v) = xv.$$ 

Since $\text{End}_F(v) \cong F$ we get a map

$$\rho : p \to F.$$ 

One checks that

$$\rho(x) = \sum_{i=1}^k l_i(\text{tr}(A_1) + \cdots + \text{tr}(A_i)).$$

**Definition 2.5.** Let $\text{char}(\rho) = U(g)\{x - \rho(x) : x \in p\}$ be the left character ideal of $\rho$. Let $\text{char}^k(\rho) = \text{char}(\rho) \cap U_k(g)$ be its canonical filtration.

Let $n_- \subseteq g$ be the complement of $p \subseteq g$. It is a sub Lie algebra.

Let $x \in p$ be the following matrix:

$$x = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & A_k & 0 \\ 0 & 0 & \cdots & 0 & A_{k+1} \end{pmatrix}.$$ 

where $A_k$ is the $d_k \times d_k$ matrix

$$A_k = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$
and $A_{k+1}$ is the $d_{k+1} \times d_{k+1}$-matrix

$$A_{k+1} = \begin{pmatrix}
0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \cdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & -1
\end{pmatrix}.$$

Let $x_{n_j}$ with $n_{i-1} + 1 \leq j \leq n_i$ with $i = 1, \ldots, k$ be the following matrix:

$$x_{n_i} = \begin{pmatrix}
0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \cdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & A_{k+1}
\end{pmatrix}.$$

Where $A_i$ is the matrix with zeros everywhere and 1 on the $j$’th place on the diagonal. The matrix $A_{k+1}$ has zeros everywhere and $-1$ in the lower right corner.

Let $p \subseteq g$ be the stabilizer Lie algebra of $v$ and let $p_v \subseteq p$ be the isotropy Lie algebra of $v \in H^0(G/P, L(l))^*$. Let $p = p_0 \oplus p_D$ where $p_0$ is the subspace of matrices with zeros on the diagonal and $p_D \subseteq p$ is the subspace of diagonal matrices. It follows $p_0 \subseteq p_v$.

**Lemma 2.6.** The set

$$\{x_{n_j} : i = 1, \ldots, k; n_{i-1} + 1 \leq j \leq n_i\}$$

is a basis for the vector space $p_D$.

**Proof.** The proof is left to the reader as an exercise. □

**Proposition 2.7.** Let $v \in H^0(G/P, L(l))^*$ be the unique highest weight vector. It follows $p = p_v \oplus (x)$ where $x$ is the matrix defined above. Furthermore $x(v) = l_kv$.

**Proof.** Let for any matrix $z \in p$

$$z = \begin{pmatrix}
A_1 & * & \cdots & * & * \\
0 & A_2 & * & \cdots & * \\
\vdots & \vdots & \cdots & \vdots & \vdots \\
0 & 0 & \cdots & A_k & * \\
0 & 0 & \cdots & 0 & A_{k+1}
\end{pmatrix},$$

$B_i$ be the associated submatrix

$$B_i = \begin{pmatrix}
A_1 & * & \cdots & * & * \\
0 & A_2 & * & \cdots & * \\
\vdots & \vdots & \cdots & \vdots & \vdots \\
0 & 0 & \cdots & A_{i-1} & * \\
0 & 0 & \cdots & 0 & A_i
\end{pmatrix}.$$

A matrix $z \in p$ is in $p_v$ if and only if

$$\rho(z) = \sum_{i=1}^{k} l_i tr(B_i) = 0.$$
Hence $z \in p - p_v$ if and only if $\rho(z) \neq 0$. Write $p = p_0 \oplus p_D$. In the discussion preceding the Proposition we constructed a basis $x_{n_j}$ for $p_D$ with $n_{i-1} + 1 \leq j \leq n_i$ for $i = 1, \ldots, k$. By definition it follows $x_j \in p_v$ for $n_k + 1 \leq j \leq n_{k+1}$. Hence an element

$$y = \sum_{s=1}^{n_k} a_s x_s$$

is in $p_v$ if and only if

$$\sum_{i=1}^{k} l_i \text{tr}(B_i) = 0.$$ 

This is if and only if there is an equation

$$a_{n_k} = f(a_1, \ldots, a_{n_k-1}).$$

Hence $y \notin p_v$ if and only if

$$a_{n_k} \neq f(a_1, \ldots, a_{n_k-1})$$

and we check that the only element in the above constructed basis for $p_D$ satisfying this condition is the element $x$ defined above, and the first claim of the Proposition follows. One checks the second claim of the Proposition by calculation and the Proposition is proved. \hfill \Box

Lemma 2.8. Let $v \in \text{Sym}^{k+1}(g) \subseteq U^{k+1}(g)$ be an element. We may write $v = v_1 + v_2$ with $v_1 \in \text{Sym}^{k+1}(n_-)$ and $v_2 \in U^k(g)\{y - \rho(y) : y \in p\}$.

Proof. The proof is left to the reader as an exercise. \hfill \Box

The element $x$ from Proposition 2.7 depends on the decomposition $p = p_v \oplus (x)$ but this fact will not be important in what follows.

Proposition 2.9. There is for all $k \geq 1$ an equality

$$U^k(g) = U^k(n_-) \oplus \text{char}^k(\rho)$$

of vector spaces.

Proof. One checks that there is an equality of vector spaces

$$\text{char}^1(\rho) = \{x - l_k 1, y : y \in p_v\}.$$ 

Using the Poincare-Birkhoff-Witt Theorem one checks there is an equality of vector spaces

$$\text{char}^k(\rho) = U^{k-1}(g)\{y - \rho(y) : y \in p\}.$$ 

We prove the claim in the Proposition using induction on $k$. We first check it for $k = 1$. We get

$$U^1(g) = 1 \oplus g = 1 \oplus n_- \oplus (x) \oplus p_v = \\
1 \oplus n_- \oplus (x - l_k 1) \oplus p_v = U^1(n_-) \oplus \{x - l_k 1, y : y \in p\} = \\
U^1(n_-) \oplus \{y - \rho(y) : y \in p\} = U^1(n_-) \oplus \text{char}^1(\rho),$$

and the claim of the Proposition is proved for $k = 1$. Assume the claim is true for $k$. We get

$$U^k(g) = U^k(n_-) \oplus \text{char}^k(\rho).$$

Using the symmetrization map we may identify

$$U^k(g) = \oplus_{i=0}^{k} \text{Sym}(g)$$
where \( \text{Sym}(g) \) is the \( i \)th symmetric power of \( g \) with the adjoint representation. It follows there is an isomorphism

\[
U^{k+1}(g) \cong U^k(g) \oplus \text{Sym}^{k+1}(g)
\]

of \( g \)-modules. Since the symmetrization map is an isomorphism of vector spaces we may identify \( \text{Sym}^{k+1}(g) \) with its image in \( U(g) \). All calculations in what follows are done inside \( U(g) \) via the symmetrization map. Clearly there is an inclusion

\[
U^{k+1}(n_-) \oplus U^k(g) \{ y - \rho(y) : y \in p \} \subseteq U^{k+1}(g).
\]

We prove the reverse inclusion. Write

\[
U^{k+1}(g) = U^k(g) \oplus \text{Sym}^{k+1}(g) = U^k(n_-) \oplus U^{k-1}(g) \{ y - \rho(y) : y \in p \} \oplus \text{Sym}^{k+1}(g).
\]

Let \( v \in \text{Sym}^{k+1}(g) \). From Lemma \([2,8]\) one may write \( v = v_1 + v_2 \) with \( v_1 \in \text{Sym}^{k+1}(n_-) \) and \( v_2 \in U^k(g) \{ y - \rho(y) : y \in p \} \). It follows

\[
U^{k+1}(g) = U^{k+1}(n_-) \oplus U^{k-1}(g) \{ y - \rho(y) : y \in p \} = U^{k+1}(n_-) \oplus char^{k+1}(\rho)
\]

and the claim of the Proposition is proved. \( \square \)

Consider the Plucker embedding

\[
i : \mathbb{F}(d, E) \to \mathbb{G}_1 \times \cdots \times \mathbb{G}_k \subseteq \mathbb{P}^M
\]

defined in the beginning of this section. The flag variety is projectively normal hence there is an injection of vector spaces

\[
H^0(\mathbb{F}(d, E), L(l))^* \subseteq \text{Sym}^{i_1}(\wedge^{n_1} E) \otimes \cdots \otimes \text{Sym}^{i_k}(\wedge^{n_k} E).
\]

There is a \( P \)-stable line

\[
\text{Sym}^{i_1}(\wedge^{n_1} E_1) \otimes \cdots \otimes \text{Sym}^{i_k}(\wedge^{n_k} E_k) \subseteq \text{Sym}^{i_1}(\wedge^{n_1} E) \otimes \cdots \otimes \text{Sym}^{i_k}(\wedge^{n_k} E).
\]

**Lemma 2.10.** Let \( v \in H^0(\mathbb{F}(d, E), L(l))^* \) be the highest weight vector. There is an equality

\[
v = \text{Sym}^{i_1}(\wedge^{n_1} E_1) \otimes \cdots \otimes \text{Sym}^{i_k}(\wedge^{n_k} E_k).
\]

**Proof.** This is left to the reader as an exercise. \( \square \)

Let \( v_i = \wedge^{n_i} E_i \) for \( i = 1, \ldots, k \). We write \( v = v_1^k \otimes \cdots \otimes v_k^k \).

We use the notation of \([2]\) Chapter 7. Let \( P \) be the dominant weights of \( g = \mathfrak{sl}(E) \) and let \( B \) be a basis for the roots of \( g \). Let \( \rho \) be the character associated to \( v \). It follows for \( x \in p \) we have

\[
\rho(x) = \sum_{i=1}^k l_i(\text{tr}(A_1) + \cdots + \text{tr}(A_i)).
\]

Let \( g = g_- \oplus h \oplus g_+ \) be a Cartan decomposition of \( g \). Let \( n_+ = g_+ \oplus h \). Let \( I(v) \) be the left ideal in \( U(g) \) defined by

\[
I(v) = U(g)n_+ + \sum_{x \in h} U(g)(x - \lambda(x)).
\]

By \([2]\) Proposition 7.2.7 it follows

\[
ann(v, \lambda) = I(v) + \sum_{\beta \in B} U(n_-)X^{m_{\beta}}_{-\beta}.
\]
Let \( I^k(v) = I(v) \cap U^k(g) \) be the canonical filtration of \( I(v) \). Let \( B = \{L_i - L_{i+1}\}_{i=1, \ldots, N} \). Let \( \beta_i = L_i - L_{i+1} \) and let \( g^{\beta_i} = F(E_{i,i+1}) \). It follows \( g^{-\beta_i} = F(E_{i+1,i}) \). We have by definition \( X_{-\beta_i} = E_{i,i+1} \) and since \([E_{i,j}, E_{j,i}] = E_{i,i} - E_{j,j}\) and \( 0 \neq H_{\beta_i} \in [g^{\beta_i}, g^{-\beta_i}] \) it follows
\[
H_{\beta_i} = E_{i,i} - E_{i+1,i+1}.
\]

By definition we have
\[
m_{\beta_i} = \lambda(H_{\beta_i}) + 1.
\]

**Lemma 2.11.**

\[
(2.11.1) \quad m_{\beta_i} = l_j + 1 \text{ if } i = n_j
\]

\[
(2.11.2) \quad m_{\beta_i} = 1 \text{ if } i \neq n_j
\]

**Proof.** The proof is left to the reader as an exercise. \(\square\)

Let \( K^k(v) \) be the following vector space:
\[
K^k(v) = \left( \sum_{\beta \in B} U(n_-)X^{m_{\beta}} \right) \cap U^k(g).
\]

It follows
\[
K^k(v) = \sum_{i \neq n_j} U^{k-1}(n_-)X_{-\beta_i} + \sum_{i = n_j} U^{k-l_j-1}(n_-)X^{l_j+1}_{-\beta_{n_j}}.
\]

If \( 1 \leq k \leq \min\{l_i + 1\} \) it follows
\[
K^k(v) = \sum_{i \neq n_j} U^{k-1}(n_-)X_{-\beta_i}.
\]

We have
\[
\text{ann}^k(v, \lambda) = I^k(v) + K^k(v).
\]

**Theorem 2.12.** Let \( 1 \leq k \leq \min\{l_i + 1\} \) be an integer. The following holds:
\[
W^k(v, \lambda) = U^k(n_-).
\]

**Proof.** By definition it follows \( \text{char}^k(\rho) \subseteq \text{ann}^k(v, \lambda) \) for all \( k \geq 1 \). We want to prove the reverse inclusion
\[
\text{ann}^k(v, \lambda) \subseteq \text{char}^k(\rho)
\]
in the case when \( 1 \leq k \leq \min\{l_i + 1\} \).

There is an inclusion \( I^k(v) \subseteq \text{char}^k(\rho) \) for all \( k \geq 1 \). When \( 1 \leq k \leq \min\{l_i + 1\} \) we get \( K^k(v) \subseteq \text{char}^k(\rho) \) and it follows
\[
\text{ann}^k(v, \lambda) \subseteq \text{char}^k(\rho).
\]

From this we deduce an equality
\[
\text{ann}^k(v, \lambda) = \text{char}^k(\rho)
\]
when \( 1 \leq k \leq \min\{l_i + 1\} \). By Proposition 2.13, the following holds:
\[
U^k(g) = U^k(n_-) \oplus \text{char}^k(\rho)
\]
It follows \( W^k(v, \lambda) = U^k(n_-) \) and the claim of the Theorem is proved. \(\square\)

**Corollary 2.13.** Let \( 1 \leq k \leq \min\{l_i + 1\} \). There is an equality of vector spaces
\[
U^k(g) = U^k(n_-) \oplus \text{ann}^k(v, \lambda).
\]
Proof. This follows from Theorem 2.12.

Corollary 2.14. Let $v_1, ..., v_D$ be a basis for $\mathfrak{n}_- \subseteq \mathfrak{g}$ and let $1 \leq k \leq \min\{l_i + 1\}$. It follows the set
\[
\{v_1^{a_1} \cdots v_D^{a_D}(v) : 0 \leq \sum_i a_i \leq k\}
\]
is a basis for $U^k(\mathfrak{g})v$. 

Proof. There is by Corollary 2.13 an equality
\[
U^k(\mathfrak{g}) = U^k(\mathfrak{n}_-) \oplus \text{ann}^k(v, \lambda).
\]
It follows from this there is an isomorphism of vector spaces
\[
U^k(\mathfrak{n}_-) \otimes L_v \to U^k(\mathfrak{g})v.
\]
From this isomorphism and the Poincare-Birkhoff-Witt Theorem the claim of the Corollary follows.

Let $D = \sum_{1 \leq i < j \leq k+1} d_i d_j$ It follows $\dim_F(\mathfrak{n}_-) = D$. 

Corollary 2.15. Let $1 \leq k \leq \min\{l_i + 1\}$. The following holds:
\[
\dim_F(U^k(\mathfrak{g})v) = \binom{D+k}{D}.
\]

Proof. There is by Theorem 2.12 an isomorphism of vector spaces $U^k(\mathfrak{n}_-) \cong U^k(\mathfrak{g})v$. It follows
\[
\dim_F(U^k(\mathfrak{g})v) = \dim_F(U^k(\mathfrak{n}_-)) = \dim_F(\text{Sym}^k(\mathfrak{n}_- \oplus 1)) = \binom{D+k}{D}
\]
and the claim of the Corollary is proved.

Corollaries 2.14 and 2.15 answer the questions 1.0.2 and 1.0.3 posed in the introduction of the paper.

3. Filtrations of SL(E)-modules and jet bundles

In this section we relate filtration $U^k(\mathfrak{g})v \subseteq V$ studied in the previous section to the jet bundle $J^k(L(\underline{l}))$ of the line bundle $L(\underline{l}) \in \text{Pic}^G(G/P)$ with $H^0(G/P, L(\underline{l}))^* = V$. Recall: The vector space $U^k(\mathfrak{g})v$ is a $P$-module hence it corresponds to an $\text{SL}(E)$-linearized locally free sheaf $J^k$ on $\text{SL}(E)/P$. In this section we make the equivalence explicit and give a geometric construction of $J^k$ in terms of $\text{SL}(E)$-linearized jet bundles on $F(d, E) = \text{SL}(E)/P$.

Let $F(d, E) = \text{SL}(E)/P$ be the flag variety parametrizing flags of type $d$ in an $N$-dimensional vector space $E$. Recall the Plucker embedding
\[
i : F(d, E) \to G = G_1 \times \cdots \times G_k
\]
defined by
\[
i([E_1 \subseteq \cdots \subseteq E_k]) = [E_1] \times \cdots \times [E_k].
\]
Let $q_i : G \to G_i$ be the projection morphism. Let $\underline{l} = (l_1, ..., l_k) \in \mathbb{Z}^k$ and let $O(\underline{l}) = q_1^*O(l_1) \otimes \cdots \otimes q_k^*O(l_k)$ be the associated line bundle on $G$. We get a linebundle $L(\underline{l}) = i^*O(\underline{l})$ on $F = F(d, E)$. Let $p, q : F \times F \to F$ be the projection morphisms and let $I \subseteq F \times F$ be the ideal of the diagonal.
Definition 3.1. Let $J^k_F(L(1)) = p_*(\mathcal{O}_{\mathbb{P}^k / \mathbb{P}^{k+1}} \otimes q^* L(1))$ be the $k$'th order jet bundle of $L(1)$.

We first prove some general facts on jet bundles on arbitrary products of schemes.

Let $A, B$ be arbitrary commutative $F$-algebras and let $P^k_A = A \otimes_F A/I_{k+1}$ where $I$ is the ideal of the diagonal. If $X = \text{Spec}(A)$ it follows $\tilde{P}^k_A = J^k_X$. There are natural maps of rings $p : A \rightarrow A \otimes B$ and $q : B \rightarrow A \otimes B$.

Lemma 3.2. There is for every $k \geq 1$ a surjection of $A \otimes B$-modules $P^k_A \otimes P^k_B \rightarrow P^k_{A \otimes B}$.

Proof. The natural map $A \otimes A \otimes B \otimes B \rightarrow A \otimes B \otimes A \otimes B$ defined by $\phi(a \otimes b \otimes x \otimes y) = a \otimes x \otimes b \otimes y$ induce a well defined map as claimed. It is surjective and the Lemma is proved. □

Corollary 3.3. There is for every $k \geq 1$ a surjective map of $A_1 \otimes \cdots \otimes A_s$-modules $P^k_{A_1} \otimes \cdots \otimes P^k_{A_s} \rightarrow P^k_{A_1 \otimes \cdots \otimes A_s}$.

Proof. The proof follows from Lemma 3.2 and an induction. □

Let $E_i$ be an $A_i$-module for $i = 1, \ldots, s$.

Corollary 3.4. There is for every $k \geq 1$ a surjection $P^k_{A_1}(E_1) \otimes \cdots \otimes P^k_{A_s}(E_s) \rightarrow P^k_{A_1 \otimes \cdots \otimes A_s}(E_1 \otimes \cdots \otimes E_s)$ of $A_1 \otimes \cdots \otimes A_s$-modules.

Proof. This follows directly from Corollary 3.3. □

Let $X_1, \ldots, X_s$ be arbitrary schemes and let $\mathcal{E}_i$ be a quasi coherent $\mathcal{O}_{X_i}$-module for $i = 1, \ldots, s$. Let $X = X_1 \times \cdots \times X_s$. Let $p_i : X \rightarrow X_i$ be the $i$'th projection and let $\mathcal{F} = p_1^* \mathcal{E}_1 \otimes \cdots \otimes p_s^* \mathcal{E}_s$.

Corollary 3.5. There is for every $k \geq 1$ a surjection $p_1^* J^k_{X_1}(\mathcal{E}_1) \otimes \cdots \otimes p_s^* J^k_{X_s}(\mathcal{E}_s) \rightarrow J^k_X(\mathcal{F})$ of $\mathcal{O}_X$-modules.

Proof. The Corollary is a global version of Corollary 3.4. □

Let $q_i : G \rightarrow \mathbb{G}_i$ be the projection morphism and let $\mathcal{O}(l) = q_1^* \mathcal{O}(l_1) \otimes \cdots \otimes q_k^* \mathcal{O}(l_k)$ be the line bundle on $\mathbb{G}$ defined above.

Proposition 3.6. Let $1 \leq k \leq \min\{l_i + 1\}$. The $k$'th Taylor morphism $T^k : H^0(G, \mathcal{O}(l)) \rightarrow J^k_G(\mathcal{O}(l))(\mathcal{F})$ is surjective.
Proof. There is by the Künneth formula an isomorphism
\[ H^0(G, \mathcal{O}(\mathfrak{l})) \cong H^0(G_1, \mathcal{O}(l_1)) \otimes \cdots \otimes H^0(G_k, \mathcal{O}(l_k)) \]
of vector spaces. The Taylor map \( T_k^i \) is by [7] a surjective map
\[ T_k^i : H^0(G_i, \mathcal{O}(l_i)) \to J_{G_i}^k(\mathcal{O}(l_i))(\mathfrak{p}) \]
for \( i = 1, \ldots, k \). We get a surjective map
\[ \tilde{T}_k : H^0(G, \mathcal{O}(\mathfrak{l})) = \otimes_{i=1}^k \mathcal{O}(l_i) \to \otimes_{i=1}^k J_{G_i}^k(\mathcal{O}(l_i))(\mathfrak{p}) \]
of vector spaces. By Corollary 3.5 we get a surjection
\[ H^k(\mathcal{O}(\mathfrak{l}))(\mathfrak{p}) \to \mathcal{O}(\mathfrak{l}))(\mathfrak{p}) \]
of vector spaces. This induce the surjection \( T_k \)
\[ T_k : H^0(G, \mathcal{O}(\mathfrak{l})) \to J_{G}^k(\mathcal{O}(\mathfrak{l}))(\mathfrak{p}) \]
and the Proposition is proved. \( \square \)

Theorem 3.7. Let \( L(\mathfrak{l}) \in \text{Pic}^{SL(E)}(\mathbb{F}(d, E)) \) be a line bundle with \( l_i \geq 1 \) for all \( i \). Let \( 1 \leq k \leq \min\{l_i + 1\} \). The Taylor map
\[ T_k : H^0(\mathbb{F}(d, E), L(\mathfrak{l})) \to J_{\mathbb{F}}^k(L(\mathfrak{l}))(\mathfrak{p}) \]
is a surjective map of vector spaces.

Proof. Since \( L(\mathfrak{l}) = i^* \mathcal{O}(\mathfrak{l}) \) where \( i : \mathbb{F} \to G \) is the Plücker embedding, and the Taylor map is surjective on \( G \) the Theorem follows from [7] Theorem 4.4. \( \square \)

Corollary 3.8. There is for \( 1 \leq k \leq \min\{l_i + 1\} \) an exact sequence
\[ 0 \to H^0(\mathbb{F}(d, E), m^{k+1}L(\mathfrak{l})) \to H^0(\mathbb{F}(d, E), L(\mathfrak{l})) \to J_{\mathbb{F}}^k(L(\mathfrak{l}))(\mathfrak{p}) \to 0 \]
of \( P \)-modules.

Proof. Let \( p, q : \mathbb{F} \times \mathbb{F} \to \mathbb{F} \) be the projection morphisms and let \( I \subseteq \mathcal{O}_{\mathbb{F} \times \mathbb{F}} \) be the ideal of the diagonal. Using higher direct images and the functor \( p_*(- \otimes q^*L) \) we get a long exact sequence of \( SL(E) \)-linearized locally free sheaves
\[ 0 \to p_*(I^{k+1} \otimes q^*L) \to p_*q^*L(\mathfrak{l}) \to J_{\mathbb{F}}^k(L(\mathfrak{l})) \to \]
\[ R^1 p_*(I^{k+1} \otimes q^*L) \to R^1 p_*q^*L(\mathfrak{l}) \to \ldots. \]
Recall there is an equivalence of categories between the category of \( SL(E) \) linearized vector bundles on \( \mathbb{F}(d, E) \) and the category of rational \( P \)-modules. We take the fiber a \( \mathfrak{p} \in SL(E)/P \) to get an exact sequence of \( P \)-modules
\[ 0 \to H^0(\mathbb{F}, m^{k+1}L(\mathfrak{l})) \to H^0(\mathbb{F}, L(\mathfrak{l})) \to T_k \]
\[ H^1(\mathbb{F}, m^{k+1}L(\mathfrak{l})) \to H^1(\mathbb{F}, L(\mathfrak{l})) \to \ldots \]
and since \( H^1(\mathbb{F}, L(\mathfrak{l})) = 0 \) and \( T_k \) is surjective, the Corollary follows. \( \square \)

Dualize the exact sequence from Corollary 3.8 to get an exact sequence of \( SL(E) \)-modules
\[ 0 \to J_{\mathbb{F}}^k(L(\mathfrak{l}))(\mathfrak{p})^* \to H^0(\mathbb{F}, L(\mathfrak{l}))^* \to \phi H^0(\mathbb{F}, m^{k+1}L(\mathfrak{l}))^* \to 0. \]
The highest weight vector \( v \in H^0(\mathbb{F}, L(\mathfrak{l}))^* \) induce a \( P \)-module
\[ U^k(\mathfrak{g}) \subset H^0(\mathbb{F}, L(\mathfrak{l}))^*. \]
Lemma 3.9. There is an inclusion of $P$-modules

$$U^k(g)v \subseteq \mathcal{J}_k^b(\mathcal{L}(\overline{\lambda}))^*.$$

Proof. One checks that $\phi(U^k(g)v) = 0$ and the Lemma follows. \qed

We can now prove the main theorem of the paper:

Theorem 3.10. Let $1 \leq k \leq \min\{l_i + 1\}$. There is an isomorphism of $P$-modules

$$U^k(g)v \cong \mathcal{J}_k^b(\mathcal{L}(\overline{\lambda}))^*.$$

Proof. There is by Lemma 3.9 an inclusion of $P$-modules

$$U^k(g)v \subseteq \mathcal{J}_k^b(\mathcal{L}(\overline{\lambda}))^*.$$

By Corollary 2.15 this inclusion is an isomorphism and the Theorem is proved. \qed

We have for any finite dimensional irreducible $\text{SL}(E)$-module $V$ with weight $\lambda$ constructed a line bundle $\mathcal{L}(\overline{\lambda})$ on $\text{SL}(E)/P$ where $P \subseteq \text{SL}(E)$ is a parabolic subgroup with the following property: There is an isomorphism $V \cong H^0(\text{SL}(E)/P, \mathcal{L}(\overline{\lambda}))^*$ of $\text{SL}(E)$-modules. Moreover the canonical filtration of $P$-modules

(3.10.1) \hspace{1cm} U^1(g)v \subseteq \cdots \subseteq U^k(g)v \subseteq V

equals the filtration

(3.10.2) \hspace{1cm} \mathcal{J}_k^b(\mathcal{L}(\overline{\lambda}))^* \subseteq \cdots \subseteq \mathcal{J}_k^b(\mathcal{L}(\overline{\lambda}))^* \subseteq V = H^0(\text{SL}(E)/P, \mathcal{L}(\overline{\lambda}))^*$

given by the jet bundle $\mathcal{J}^i$. Here $1 \leq k \leq \min\{l_i + 1\}$. It follows Question 1.0.4 from the introduction is settled.

Assume $l_i^j = (l_i^1, \ldots, l_i^d) \in \mathbb{Z}^d$ with $l_i^j \geq 1$ for all $i, j$. Let $E = \bigoplus_{i=1}^d \mathcal{L}(\overline{l_i^j})$. Let $v_1 \in H^0(G/P, \mathcal{L}(\overline{l_i^j}))$ be the unique highest weight vector. Let $W \subseteq H^0(G/P, E)^*$ be the subspace generated by $v_1, \ldots, v_d$. Let

$$U^l(g)W \subseteq H^0(G/P, E)^*$$

be the $P$-module generated by $W$ and $U^l(g)$.

Corollary 3.11. There is an isomorphism

$$\mathcal{J}^l(E)(v_1)^* \cong \bigoplus_{i=1}^d U(g)v_i \cong U^l(g)W$$

of $P$-modules for all $1 \leq l \leq \min\{l_i^j + 1\}$.

Proof. We get by Theorem 3.10 an isomorphism

$$\mathcal{J}^l(E)(v_1)^* \cong \bigoplus_{i=1}^d \mathcal{J}^l(\mathcal{L}(\overline{l_i^j}))^* \cong \bigoplus_{i=1}^d U^l(g)v_i$$

of $P$-modules, and the claim of the Corollary follows. \qed

Problem 3.12. Canonical filtrations for semi simple algebraic groups.

Let $G$ be any semi simple linear algebraic group over $F$ and let $V$ be any finite dimensional irreducible $G$-module with highest weight vector $v \in V$. Let $L_v \subseteq V$ be the line spanned by $v$. Let $P \subseteq G$ be the subgroup fixing the line $L_v$. It follows $P$ is a parabolic subgroup and the quotient $G/P$ is canonically a smooth projective variety of finite type over $F$. Let $\lambda$ be the weight of $v$ and let $\mathfrak{g} = \text{Lie}(G)$. Let $\text{ann}(v, \lambda) \subseteq U(\mathfrak{g})$ be the left annihilator ideal of $v$ and let $\text{ann}^k(v, \lambda) = \text{ann}^k(v, \lambda) \cap U^k(\mathfrak{g})$ be its canonical filtration. We get an exact sequence

$$0 \rightarrow \text{ann}(v, \lambda) \otimes L_v \rightarrow U(\mathfrak{g}) \otimes L_v \rightarrow V \rightarrow 0$$
of $G$-modules and an exact sequence

$$0 \to \text{ann}^k(v, \lambda) \otimes L_v \to U^k(\mathfrak{g}) \otimes L_v \to U^k(\mathfrak{g})L_v \to 0$$

of $P$-modules.

**Definition 3.13.** Let $U^*(\mathfrak{g})L_v \subseteq V$ be the *canonical filtration* of $V$.

There is work in progress giving a geometric interpretation of the canonical filtration $U^*(\mathfrak{g})L_v$ in terms of $G$-linearized $\mathcal{O}_{G/P}$-modules $J^k$ (see [10]).

**Example 3.14.** *Morphisms of generalized Verma modules*.

Let $G$ be an arbitrary semi simple linear algebraic group and let $P \subseteq G$ be a parabolic subgroup. Let $\mathfrak{g} = \text{Lie}(G)$ and $\mathfrak{p} = \text{Lie}(P)$. Assume $U$ is a $G$-module and let $W, V \subseteq U$ be sub $P$-modules. Assume

$$f : U(\mathfrak{g}) \otimes W \to U(\mathfrak{g}) \otimes V$$

is a map of $G$-modules with

(3.14.1) \[ f(U^l(\mathfrak{g}) \otimes W) \subseteq U^l(\mathfrak{g}) \otimes V \]

(3.14.2) \[ f(\text{ann}(W) \otimes W) \subseteq \text{ann}(V) \otimes V. \]

The modules $U(\mathfrak{g}) \otimes W$ and $U(\mathfrak{g}) \otimes V$ are *generalized Verma modules*. It follows $f$ induce a map

$$f^l : U^l(\mathfrak{g})W \to U^l(\mathfrak{g})V$$

of $P$-modules. Here $U^l(\mathfrak{g})W$ and $U^l(\mathfrak{g})V$ are the sub-$P$-modules generated by $U^l(\mathfrak{g})$, $W$ and $V$ as sub modules of the $G$-module $U$. By the result of Corollary [3.11] we describe $U^l(\mathfrak{g})W$ in terms of $J^l(E)(\overline{\tau})^*$ for a locally free $\mathcal{O}_{G/P}$-module $E$ when $W$ is the $P$-submodule generated by the highest weight vectors $v_i \in V_{\lambda_i}$ for $i = 1, \ldots, d$.

One seek to give a geometric construction of the morphism $f^l$ in terms of $G$-linearized locally free $\mathcal{O}_{G/P}$-modules: We seek a morphism

$$\phi : E \to F$$

of $G$-linearized $\mathcal{O}_{G/P}$-modules $E, F$ with $\phi(\overline{\tau}) = f^l$. This problem will be considered in later paper on this subject (see [11] for results on morphisms between generalized Verma modules).

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