Optimal RG-Improvement of Perturbative Calculations in QCD

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Abstract

Using renormalization-group methods, differential equations can be obtained for the all-orders summation of leading and subsequent non-leading logarithmic corrections to QCD perturbative series for a number of processes and correlation functions. For a QCD perturbative series known to four orders, such as the $e^+e^-$ annihilation cross-section, explicit solutions to these equations are obtained for the summation to all orders in $\alpha_s$ of the leading set and the subsequent two non-leading sets of logarithms. Such summations are shown for a number of processes to lead to a substantial reduction in sensitivity to the renormalization scale parameter. Surprisingly, such summations are also shown to lower the infrared singularity within the perturbative expression for the $e^+e^-$ annihilation cross-section to coincide with the Landau pole of the naive one-loop running QCD couplant.
1 Optimal RG Improvement of $\Gamma(B \to X_u\ell^-\bar{\nu}_\ell)$

Optimal renormalization-group (RG) improvement of a perturbative series to a given
order in the expansion couplant is the idea of including within that series all higher-order
contributions that can be extracted by renormalization-group methods [1]. We call such
terms, which involve leading and successive logarithms of the renormalization scale $\mu$,
RG-accessible. Techniques have been developed to obtain closed-form summations of
such RG-accessible contributions to all orders in the perturbative expansion parameter,
and such summations have been shown to lead to “optimally RG-improved” expressions
for perturbative quantities that have significantly diminished dependence on $\mu$ [2, 3].

For example, leading and subleading perturbative QCD corrections to the inclusive
semi-leptonic $B \to X_u\ell^-\bar{\nu}_\ell$ decay rate, which in tree-order is purely a charged-current
weak interaction process, are given by a QCD series [4]

$$S = 1 + \left[ 4.25360 + 5 \log \left( \frac{\mu^2}{m_b^2(\mu)} \right) \left( \frac{\alpha_s(\mu)}{\pi} \right) \right]$$
$$+ \left[ 26.7848 + 36.9902 \log \left( \frac{\mu^2}{m_b^2(\mu)} \right) + \frac{415}{24} \log^2 \left( \frac{\mu^2}{m_b^2(\mu)} \right) \right] \left( \frac{\alpha_s(\mu)}{\pi} \right)^2$$
$$+ \mathcal{O} \left( \left( \frac{\alpha_s(\mu)}{\pi} \right)^3 \right)$$

such that

$$\Gamma(b \to u\ell^-\bar{\nu}_\ell) = \frac{G_F^2 |V_{ub}|^2}{192\pi^3} m_b^5(\mu) S.$$  (2)

If one substitutes eq. (1) into eq. (2), one obtains a decay rate that decreases
monotonically with increasing $\mu$, raising the question as to which value of $\mu$ is most
appropriate for comparing the calculation (2) to experiment. Clearly, such dependence
on the unphysical parameter $\mu$ is an embarrassment; indeed the renormalization group
equation for the series $S$

$$0 = \left[ \mu^2 \frac{\partial}{\partial \mu^2} + \beta(g) \frac{\partial}{\partial g^2} + m_b \gamma_m(g) \frac{\partial}{\partial m_b} + 5 \gamma_m(g) \right] S \left[ \mu^2, g(\mu), m_b(\mu) \right]$$

is nothing more than a chain rule expression for the requirement that the physically
measurable decay rate be impervious to changes in the renormalization scale parameter
$\mu$,

$$0 = \frac{d}{d\mu^2} \Gamma(B \to X_u\ell^-\bar{\nu}_\ell).$$  (4)

The residual $\mu$-dependence of the decay rate obtained from the series (1) is necessarily a
consequence of the truncation of that series, as well as the relatively large value of the
expansion constant $\alpha_s(\mu)/\pi$.

In fact, the series (1) may be expressed as a double summation over powers of loga-
risms and the expansion parameter, i.e., in the following form:

$$S[x, L] = \sum_{n=0}^{\infty} \sum_{m=0}^{n} T_{n,m} x^n L^m,$$  (5)
where
\[ x \equiv \alpha_s(\mu)/\pi, \quad L \equiv \log(\mu^2/m_s^2(\mu)). \] (6)

The first few constants of this series, i.e., the set \( \{ T_{0,0}(=1), T_{1,0}, T_{1,1}, T_{2,0}, T_{2,1}, T_{2,2} \} \), are given by eq. (1). However, all higher-order constants of the form \( T_{n,n}, T_{n,n-1} \) and \( T_{n,n-2} \) can be obtained via eq. (3), and, hence, are RG-accessible. In terms of the new variables \( x \) and \( L \), the RG-equation (3) may be expressed as

\[ 0 = \left[ (1 - 2\gamma_m(x)) \frac{\partial}{\partial L} + \beta(x) \frac{\partial}{\partial x} + 5\gamma_m(x) \right] S[x,L]. \] (7)

If we substitute the series (5) into the RG-equation (7), as well as the known QCD series expansions of the RG-functions

\[ \beta(x) = -\sum_{n=0}^{\infty} \beta_n x^{n+2}, \quad \gamma_m(x) = -\sum_{n=0}^{\infty} \gamma_n x^{n+1}, \] (8)

we find for any integer \( p \) that the aggregate coefficients of \( x^p L^{p-1} \), \( x^p L^{p-2} \) and \( x^p L^{p-3} \) on the right hand side of eq. (7) necessarily vanish:

\[ x^p L^{p-1} : \quad 0 = pT_{p,p} - \beta_0 T_{p-1,p-1}(p-1) - 5\gamma_0 T_{p-1,p-1} \] (9)

\[ x^p L^{p-2} : \quad 0 = (p-1)T_{p,p-1} + 2\gamma_0(p-1)T_{p-1,p-1} - \beta_0(p-1)T_{p-1,p-2} - \beta_1(p-2)T_{p-2,p-2} - 5\gamma_0 T_{p-1,p-2} - 5\gamma_1 T_{p-2,p-2} \] (10)

\[ x^p L^{p-3} : \quad 0 = (p-2)T_{p,p-2} + 2\gamma_0(p-2)T_{p-1,p-2} + 2\gamma_1(p-2)T_{p-2,p-2} - \beta_0(p-1)T_{p-1,p-3} - \beta_1(p-2)T_{p-2,p-3} - \beta_2(p-3)T_{p-3,p-3} - 5\gamma_0 T_{p-1,p-3} - 5\gamma_1 T_{p-2,p-3} - 5\gamma_2 T_{p-3,p-3}. \] (11)

Given knowledge of \( T_{0,0}(=1) \), one can calculate any coefficient \( T_{p,p} \) through successive applications of eq. (9). Indeed the eq. (1) values \( T_{1,1} = 5 \) and \( T_{2,2} = 415/24 \) follow from just two successive iterations of (9) using the \( n_f = 5 \) QCD values \( \gamma_0 = 1, \beta_0 = 23/12 \). Similarly, knowledge of all \( T_{p,p} \) coefficients, as obtained via (9), plus knowledge of \( T_{1,0} = 4.25360 \) [eq. (1)] is sufficient via successive applications of (10) to determine all coefficients \( T_{p,p-1} \). Finally, knowledge of all coefficients \( T_{p,p} T_{p,p-1} \) plus the single coefficient \( T_{2,0} = 26.7848 \) is sufficient via successive applications of (11) to determine all coefficients \( T_{p,p-2} \), since the set of \( \{ S \} \) RG-function coefficients \( \beta_0, \beta_1, \beta_2, \gamma_0(=1), \gamma_1 \), and \( \gamma_2 \) have all been calculated [5].

Since we now see that all coefficients \( T_{p,p}, T_{p,p-1} \) and \( T_{p,p-2} \) are RG-accessible, it makes sense to restructure the double-summation series (4) in the form

\[ S[x,L] = \sum_{p=0}^{\infty} T_{p,p} x^p L^p + \sum_{p=1}^{\infty} T_{p,p-1} x^p L^{p-1} + \sum_{p=2}^{\infty} T_{p,p-2} x^p L^{p-2} + \sum_{p=3}^{\infty} T_{p,p-3} x^p L^{p-3} + \ldots, \] (12)
since the first three terms above are completely determined by eqs. (9), (10) and (11). We express (12) in the more compact form
\[ S[x, L] = \sum_{n=0}^{\infty} x^n \left( \sum_{p=n}^{\infty} T_{p,p-n}(xL)^{p-n} \right) \]
\[ \equiv \sum_{n=0}^{\infty} x^n S_n(xL) \] (13)
and note that \( S_0(xL), S_1(xL) \) and \( S_2(xL) \) all correspond to RG-accessible functions, based upon the information given in (1). Indeed, the program of optimal RG-improvement is nothing more than the explicit closed-form evaluation of these functions, and their subsequent incorporation into the calculated decay rate.

To evaluate the summation \( S_0(u) \), as defined by (13), we simply multiply eq. (9) by \( u^{p-1} \) and sum from \( p = 1 \) to infinity:
\[ 0 = \sum_{p=1}^{\infty} pT_{p,p}u^{p-1} - \beta_0 \sum_{p=1}^{\infty} (p-1)T_{p-1,p-1}u^{p-1} - 5\gamma_0 \sum_{p=1}^{\infty} T_{p-1,p-1}u^{p-1} = (1 - \beta_0u)\frac{dS_0}{du} - 5\gamma_0 S_0. \] (14)
We note from the definition (13) of the series \( S_n(xL) \) that
\[ S_n(0) = T_{n,0}. \] (15)
The solution of the differential equation (14) with initial condition \( S_0(0) = T_{0,0} = 1 \) is
\[ S_0(u) = (1 - \beta_0u)^{-5\gamma_0/\beta_0}. \] (16)
A similar procedure is employed to find \( S_1 \), and \( S_2 \). If we multiply eq. (10) by \( u^{p-2} \) and then sum from \( p = 2 \) to \( \infty \), we find after a little algebra that
\[ (1 - \beta_0u)\frac{dS_1}{du} - (\beta_0 + 5\gamma_0)S_1 = 5\gamma_1 S_0 - (2\gamma_0 - \beta_1 u)\frac{dS_0}{du}. \] (17)
Substituting the solution (16) into the right hand side of (17) and noting that \( S_1(0) = T_{1,0} \), we find that
\[ S_1(u) = \frac{5(\gamma_0\beta_1/\beta_0 - \gamma_1)/\beta_0}{(1 - \beta_0u)^{5\gamma_0/\beta_0}} + T_{1,0} - 5(\gamma_0\beta_1/\beta_0 - \gamma_1)/\beta_0 + [5\gamma_0(2\gamma_0 - \beta_1/\beta_0)]/\beta_0 \log(1 - \beta_0u). \] (18)
Similarly, we can multiply eq. (11) by \( u^{p-3} \) and then sum from \( p = 3 \) to \( \infty \) to obtain the differential equation
\[ (1 - \beta_0u)\frac{dS_2}{du} - (2\beta_0 + 5\gamma_0)S_2 \]
\[ = (\beta_1 u - 2\gamma_0)\frac{dS_1}{du} + (\beta_1 + 5\gamma_1)S_1 + (\beta_2 u - 2\gamma_1)\frac{dS_0}{du} + 5\gamma_2S_0 \] (19)
whose solution is given by eq. (2.28) of ref. [2].
2 Order-by-Order Elimination of Renormalization Scale Dependence

If one substitutes solutions for \( S_0(xL), S_1(xL) \) and \( S_2(xL) \) into eq. (13), one obtains the following optimally RG-improved version of the series \( S \) [2]:

\[
S \approx S_0(xL) + xS_1(xL) + x^2S_2(xL) \\
= w^{-60/23} + x \left[ -\frac{18655}{3174} w + 10.1310 + \frac{1020}{529} \log w \right] w^{-83/23} \\
+ x^2 \left[ 13.2231\, w^2 - \left( 47.4897 + \frac{3171350}{279841} \log w \right) w \right] \\
+ \left( 61.0515 + 25.5973 \log w + \frac{719610}{279841} \log^2 w \right) w^{-106/23}
\]

(20)

with

\[
w \equiv 1 - \beta_0 xL
\]

and with \( x \) and \( L \) given by eq. (6).

When multiplied by \( m_b^5(\mu) \), this expression has the remarkable property of being almost entirely independent of \( \mu \). Figure 1 of ref. [2] displays a head to head comparison of the \( \mu \) dependence of eq. (20), and the same expression with \( S \) given by the known terms of eq. (1). For the latter case, \([m_b(\mu)]^5S\) is seen to decrease from \( \approx 2500 \) GeV\(^5\) to \( \approx 1500 \) GeV\(^5\) as \( \mu \) decreases from 1.5 GeV to 9.0 GeV. For eq. (20), however, the quantity \([m_b(\mu)]^5S\) is seen to be \( 1816 \pm 6 \) GeV\(^5\) over the same range of \( \mu \), effectively removing all \( \mu \)-dependence from the optimally RG-improved two-loop calculation.

Such elimination of renormalization-scale dependence via optimal RG-improvement is also upheld for a number of perturbative expressions, including QCD corrections to the inclusive semileptonic decay of B-mesons to charmed states \( B \to X_c\ell\bar{\nu}_\ell \), QCD corrections to Higgs boson decays, the perturbative portion of the QCD static potential function, the (Standard-Model) Higgs-mediated \( WW \to ZZ \) cross section at very high energies, and QCD sum-rule scalar- and vector-current correlation functions [2]. This last example is of particular relevance for QCD corrections to the benchmark electromagnetic cross-section ratio \( R(s) = \sigma(e^+e^- \to \text{hadrons})/\sigma(e^+e^- \to \mu^+\mu^-) \). Such QCD corrections are proportional to the imaginary part of the vector-current correlation function series, a series which is fully known to three subleading orders in \( \alpha_s \) [6]. For five active flavours, we have

\[
S \equiv 3R(s)/11 \\
= 1 + x + \left( 1.40924 + \frac{23}{12}L \right) x^2 + \left( -12.8046 + 7.81875L + \frac{529}{144}L^2 \right) x^3 + ...
\]

(22)

where \( x = \alpha_s(\mu)/\pi \), as before, and where \( L \) is now the logarithm

\[
L \equiv \log(\mu^2/s).
\]

(23)
Note that dependence on the physical scale $s$ resides entirely in the logarithm, and that the all-orders series (22) for $S$, a measurable quantity, is necessarily impervious to changes in $\mu$. However, progressive truncations of (22) introduce progressively larger amounts of renormalization scale dependence. For example, if the series $S$ is truncated after all its known terms, as listed in eq. (22), we find for $\sqrt{s} = 15$ GeV that to order $x^3$, $S$ increases modestly from 1.0525 to 1.0540 as $\mu$ increases from 7.5 to 30 GeV. $^1$ Had we truncated the series (22) following its $O(x^2)$ term, we find that such a truncation of $S$ now decreases from 1.056 to 1.053 over the same range of $\mu$, doubling the magnitude of $\mu$-dependence evident over this range. Finally, if we consider only the lowest order correction to unity ($S = 1 + x(\mu)$), we find that $S$ decreases from 1.061 to 1.045 as $\mu$ increases from 7.5 GeV to 30 GeV.

Optimal RG-improvement of the known terms of the series (22) has been shown by the same methods delineated above to lead to the following expression [2]:

$$S = 1 + x/w + x^2 \left[ 1.49024 - 1.26087 \log w \right] / w^2$$

$$+ \quad x^3 \left[ 0.115003w - 12.9196 - 5.14353 \log w + 1.58979 \log^2 w \right] / w^3 + ... \quad (24)$$

where $w$ is given by the definition (21), but with $L$ now given by eq. (23). Eq. (24) is, of course, really a restructured version of the same infinite series as eq. (21), and similarly must be independent of $\mu$ when taken to all orders. However, for the series (24) such imperviousness to changes in renormalization scale is now evident on an order-by-order basis. Truncation of the series (24) after its first nonleading term (i.e., $S = 1 + x/w$) still provides an expression that exhibits less variation with $\mu$ than all four known terms of the series (22). As $\mu$ increases from 7.5 to 30 GeV, we find for $\sqrt{s} = 15$ GeV that $1 + x/w$ decreases from 1.0524 to 1.0516. Similarly, truncation of the series (24) after its third term leads to a slow decrease from 1.0557 to 1.0553, and retention of all four known terms leads to an almost flat value ($1.05372 \pm 0.00004$) over the same 7.5 GeV - 30 GeV spread in $\mu$.

Consequently, we see that the program of optimal RG-improvement, as described above, is seen to yield order-by-order perturbation-theory predictions which are almost entirely decoupled from the particular choice of renormalization scale.

3 Lowering the Infrared Bound on Perturbative Approximations to $R(s)$

The optimally RG-improved series (24) is term-by-term singular when $w = [1 - \beta_0 x(\mu) \log(\mu^2/s)]$ is zero. Since $s$ is the external momentum scale characterising the physical $e^+e^-$ annihilation process, we see that the use of (24) is possible only if [3]

$$s > \mu^2 \exp \left( -\frac{\pi}{\beta_0 \alpha_s(\mu)} \right). \quad (25)$$

$^1$In all estimates presented here, $\alpha_s(\mu)$ is assumed to evolve via its known 4-loop order $\beta$-function from $\alpha_s(M_Z) = 0.118$ [7].
It is particularly curious that this bound on $s$ corresponds to the infrared bound on the naive one-loop ($1L$) running couplant $(x_{1L} = (\alpha_s(\mu))_{1L}/\pi)$

$$\mu^2 \frac{dx_{1L}}{d\mu^2} = -\beta_0 x_{1L}$$

whose solution

$$\alpha_s(\mu) = \frac{\pi}{\beta_0 \log(\mu^2/\Lambda_{1L}^2)}$$

(27)

can be inverted as follows to express the $1L$ infrared cut-off in terms of some reference value of $\alpha_s(\mu)$:

$$\Lambda_{1L}^2 = \mu^2 \exp \left( -\frac{\pi}{\beta_0 \alpha_s(\mu)} \right).$$

(28)

Consequently, for a given choice of $\mu$ for which $\alpha_s(\mu)$ is known (e.g. the value $\alpha_s(m_\tau)$ extracted from $\tau$-decay experiments), we see that each of the terms in the optimally RG-improved series (24) diverges as $s$ approaches $\Lambda_{1L}^2$ from above.

The idea that QCD perturbative series break down in the infrared is hardly new, but the location of this breakdown is usually identified with an IR-divergence in the all-available-orders evolution of $\alpha_s(\mu)$, not the naive 1L Landau pole of eq. (28). To consider the IR boundary of QCD corrections to $e^+e^-$ annihilation, we find for three active flavours that QCD corrections to the $e^+e^-$ annihilation cross-section are, as before, obtained from the perturbative series within the imaginary part of the QCD vector current correlation function [6]:

$$R(s)/2 = 1 + x + \left(1.63982 + \frac{9}{4}L\right)x^2$$

$$+ \left(-10.2839 + 11.3792L + \frac{81}{16}L^2\right)x^3$$

$$+ ...$$

(29)

where $x = \alpha_s(\mu)$ and where $L = \log(\mu^2/s)$. The standard phenomenological approach to this series is to first recognize its all-orders invariance under changes in $\mu$, and then to assume such invariance applies to truncation of the series after its four known terms. This (seldom stated) assumption [8] motivates the choice $\mu^2 = s$ (i.e. $L = 0$) leading to the usual $n_f = 3$ expression [7],

$$R(s) = 2 \left[1 + x(\sqrt{s}) + 1.63982 x^2(\sqrt{2}) - 10.2839 x^3(\sqrt{s}) + ... \right].$$

(30)

Such an expansion necessarily falls apart in the infrared when $x(\sqrt{s})$ becomes large. Indeed, the large coefficient of $x^3(\sqrt{s})$ in (30) manifests itself in a sharp drop in $R(s)$ at $\sqrt{s} \approx 650$ MeV [3].

It is interesting to compare the known terms in eq. (30) to the optimally RG-improved version of the known terms in eq. (29) [2, 3]:

$$R(s) = s \left[1 + x(\mu)/w(\mu, s)\right]$$

(31)
\[ + x^2(\mu) \left( 1.63982 - \frac{16}{9} \log(w(\mu, s)) \right) / w^2(\mu, s) \\
+ x^3(\mu) \left( -1.31057 w(\mu, s) - 8.97333 \\
- 8.99096 \log(w(\mu, s)) + 3.16049 \log^2(w(\mu, s)) \right) / w^3(\mu, s) \]  

(31)

where

\[ w(\mu, s) = 1 - \frac{9}{4} x(\mu) \log(\mu^2/s). \]  

(32)

We first note that \( x(\mu) \) and \( x(\sqrt{s}) \) occurring in eqs. (29), (30), (31) and (32) are evolved through all known orders of the \( \beta \)-function (8). Consequently, we are free to assign to eq. (31) an \( n_f = 3 \) empirical value for \( \mu (\mu = m_r \text{ or, alternatively, } \mu = 1 \text{ GeV}) \) safely outside the infrared region. To facilitate comparison of eqs. (30) and (31), we will assume that \( x(\sqrt{s}) \) devolves via the full \( \beta \)-function from this same initial choice of \( \mu \) until, for a sufficiently small value of \( s \), \( x(\sqrt{s}) \) becomes infinite. The point here is that for \( n_f = 3 \), the first four known terms of the \( \beta \)-function are all same-sign: \( (\beta_0 x^2 + \beta_1 x^3 + \beta_2 x^4 + \beta_3 x^5) > \beta_0 x^2 \). Thus for a given value of \( x \), the full \( \beta \)-function is more negative than the one-loop \( \beta \)-function of eq. (26). Since the evolution of both equations is referenced to the same initial value \( \mu \), the all-orders couplant \( x(\sqrt{s}) \) will diverge at a value of \( s \) that is larger than \( \Lambda_{1L}^2 \), the Landau pole of eq. (26). Hence the series (31) will probe more deeply into the infrared than the series (30) for \( R(s) \).

As an example, consider the running couplant \( x(\sqrt{s}) \) obtained via a two-loop \( \beta \)-function \( \beta(x) = -\beta_0 x^2 - \beta_1 x^3 \). Solution of the differential equation \( s dx/ds = \beta(x) \) with the initial value \( x(\mu) \) yields the exact constraint

\[ \beta_0 \log\left( \frac{\mu^2}{s} \right) = \frac{1}{x(\mu)} - \frac{1}{x(\sqrt{s})} + \frac{\beta_1}{\beta_0} \log\left( \frac{x(\mu) x(\sqrt{s}) + \beta_0/\beta_1}{x(\sqrt{s}) x(\mu) + \beta_0/\beta_1} \right). \]  

(33)

The two-loop Landau pole \( s_{2L} \) occurs when \( x(\sqrt{s_{2L}}) \to \infty \), i.e., when

\[ s_{2L} = \mu^2 \exp\left( -\frac{1}{\beta_0 x(\mu)} \right) \left[ 1 + \frac{\beta_0}{\beta_1 x(\mu)} \right]^{\beta_1/\beta_0^2} = \Lambda_{1L}^2 \left[ 1 + \frac{\beta_0}{\beta_1 x(\mu)} \right]^{\beta_1/\beta_0^2}. \]  

(34)

Since \( \beta_0, \beta_1 \) and \( x(\mu) \) are all positive, \( s_{2L} > \Lambda_{1L}^2 \). Note that \( \beta_2 \) and \( \beta_3 \) [9] persist in being positive. Consequently, for a given initial value \( x(\mu) \), the singularity in eq. (31) that occurs at \( w(\mu, s) = 0 \) (i.e., at \( s = \Lambda_{1L}^2 \)), continues to precede the Landau singularity of \( x(\sqrt{s}) \) characterizing eq. (30). Thus, the optimally RG-improved eq. (31) extends the applicability of perturbative QCD to lower values of \( s \) than in the conventional eq. (30) approach to \( R(s) \), as is explicitly shown in Fig. 2 of ref. [3].

Finally, we note that one must distinguish between the infrared limitations on the domain of perturbative approximations to \( R(s) \), and any such limitations on \( R(s) \) itself. For example, each term within the toy series \( \sum_{n=0}^{\infty} (-x/(s - \Lambda^2))^n \) diverges at \( s = \Lambda^2 \), but the function \( (s - \Lambda^2)/(s - \Lambda^2 + x) \) from which this series is extracted, is clearly finite.
at $s = \Lambda^2$. Similarly, the all-orders function $R(s)$, as opposed to truncations of its series representations, may indeed proceed smoothly to a finite limit as $s \to 0$ [10]. If such is the case, the best one can hope for in a perturbative series representation of $R(s)$ is the deepest possible penetration of that series into the low-$s$ region.

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References

[1] C. J. Maxwell, Nucl. Phys. B (Proc. Suppl.) 86, 74 (2000).
[2] M. R. Ahmady et al., Phys. Rev. D66, 014010 (2002).
[3] M. R. Ahmady et al., hep-ph/0208025.
[4] T. van Ritbergen, Phys. Lett. B454, 353 (1999).
[5] D. J. Gross and F. Wilczek, Phys. Rev. Lett. 30, 1343 (1973); H. D. Politzer, Phys. Rev. Lett. 30, 1346 (1973); W. E. Caswell, Phys. Rev. Lett. 33, 244 (1974); D. R. T. Jones, Nucl. Phys. B75, 531 (1974); E. S. Egorian and O. V. Tarasov, Theor. Mat. Fiz. 41, 26 (1979); O. V. Tarasov, A. A. Vladimirov and A. Yu. Zharkov, Phys. Lett. B93, 429 (1980); S. A. Larin and J. A. M. Vermaseren, Phys. Lett. B303, 334 (1993).
[6] S. G. Gorishny, A. L. Kataev and S. A. Larin, Phys. Lett. B259, 144 (1991); L. R. Surguladze and M. A. Samuel, Phys. Rev. Lett. 66, 560 (1991) and 2416 (E); K. G. Chetyrkin, Phys. Lett. B391, 402 (1997); F. Chishtie, V. Elias and T. G. Steele, Phys. Rev. D59 105013 (1999).
[7] Particle Data Group, D. E. Groom et al., Eur. Phys. J. C15, 1 (2000).
[8] F. J. Yndurain, Quantum Chromodynamics (Springer, New York, 1983) 58-62.
[9] T. van Ritbergen, J. A. M. Vermaseren and S. A. Larin, Phys. Lett. B400, 379 (1997).
[10] D. M. Howe and C. J. Maxwell, hep-ph/0204036.