Coefficient Bounds For Subclass of $m$-fold Symmetric Bi-Univalent Functions Sense of Yamakawa

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Abstract. In this paper we introduce a new subclass $A_{\alpha}^{\Sigma,m}$ of the bi-univalent function sense of Yamakawa in which both $f$ and $f^{-1}$ are $m$-fold symmetric analytic functions. The coefficient bounds for $|a_{m+1}|$ and $|a_{2m+1}|$ of the subclass $A_{\alpha}^{\Sigma,m}$ also determined.

1. Introduction

Let $A$ denote the class of functions $f$ that are analytic in the open unit disk $U = \{ z \in \mathbb{C} : |z| < 1 \}$ and and normalized by the conditions $f(0) = 0 = f'(0) - 1$ and having the form as follow

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k.$$ (1)

Duren [4] ensures that the image of $U$ under every univalent function $f \in A$ contains the disk of radius $\frac{1}{4}$ as stated in the Koebe one quarter theorem. Thus every univalent function $f$ has an inverse $f^{-1}$ satisfying

$$f^{-1}(f(z)) = z, \quad (z \in U)$$

and

$$f(f^{-1}(w)) = w, \quad (|w| < r_0(f), \ r_0(f) \geq \frac{1}{4})$$

where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 - (5a_2^3 - 5a_2 a_3 + a_4) w^4 + ... .$$

A function $f \in A$ is said to bi-univalent in $U$ if both $f(z)$ and $f^{-1}$ are univalent in $U$. Let $\Sigma$ denote the class of bi-univalent in $U$ given by (1). For a brief history and interesting examples
in the class $\Sigma$, see [10].

Brannan and Taha [1] (see also [13]) introduced certain subclasses of bi-univalent function class $\Sigma$ similar to the familiar subclasses $S^\ast(\alpha)$ and $K(\alpha)$ of starlike and convex functions of order $\alpha$ ($0 \leq \alpha < 1$), respectively (see [2]). Thus, following Brannan and Taha [1] (see also [13]), a function $f \in A$ is in the class $S^\ast_C(\alpha)$ of strongly bi-starlike functions of order $\alpha$ ($0 \leq \alpha < 1$) if each of the following conditions is satisfied:

$$f \in \Sigma \text{ and } \left| \arg \left( \frac{zf'(z)}{f(z)} \right) \right| < \frac{\alpha \pi}{2}, \quad (0 < \alpha \leq 1, \ z \in U)$$

and

$$\left| \arg \left( \frac{g'(w)}{g(w)} \right) \right| < \frac{\alpha \pi}{2}, \quad (0 < \alpha \leq 1, \ w \in U),$$

where $g$ is the extension of $f^{-1}$ to $U$. The classes $S^\ast_C(\alpha)$ and $K^\ast(\alpha)$ of bi-starlike functions of order $\alpha$ and bi-convex functions of order $\alpha$, corresponding (respectively) to the function classes $S^\ast(\alpha)$ and $K(\alpha)$, were also introduced analogously. For each of the function classes $S^\ast_C(\alpha)$ and $K^\ast(\alpha)$, they found non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ (for detail, see [1, 13] and [6]). Some several researchers which as motivator in Bi-univalent are Saibah [8], [9] and [7].

A simple argument shows that $f \in S_m$ is characterized by having a power series of the form below:

$$f(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1}. \quad (z \in U, \ m \in \mathbb{N} = \{1, 2, 3, \ldots \}).$$ (2)

The normalized form of $f$ given by (2), the series expansion for $f^{-1}$ can be obtained as follows

$$g(w) = w - a_{m+1} w^{m+1} + (m + 1)a_{m+1}^2 - a_{2m+1}^2 \ w^{2m+1}$$

$$- \frac{1}{2} (m + 1)(3m + 2)a_{m+1}^3 - (3m + 2)a_{m+1}a_{2m+1} + a_{3m+1}^2 \ w^{3m+1} + \ldots$$ (3)

where $f^{-1} = g$. The class of $m$-fold symmetric bi-univalent functions in $U$ is denoted by $\Sigma_m$.

The result of this research also motivated by Sümmer Eker, [12], Serap Bulut [3] and Srivastava [11] in their research about coefficient bounds for subclasses of $m$-fold symmetric bi-univalent functions.

The function $f \in A$ is starlike $p$-valent with negative coefficient in unit disc $U$, denote by $T_o(p, n)$, if satisfied

$$\left( \frac{zf'(z)}{f(z)} \right) > 0, \quad z \in U.$$

Lets consider Yamakawas class $T(p, n, \alpha)$.

**Definition 1.1** Yamakawa [14] A function $f \in T(p, n)$ is said to be a member of the class $T(p, n, \alpha)$ if it satisfies the inequality

$$\text{Re} \left\{ \frac{f(z)}{zf'(z)} \right\} > \alpha, \quad (z \in U),$$

where $0 \leq \alpha < 1$.

Since

$$\text{Re} \left\{ \frac{f(z)}{zf'(z)} \right\} > \alpha \Rightarrow \text{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > 0 \quad (z \in U), \quad (0 \leq \alpha < 1).$$

$T(p, n, \alpha)$ is a subclass of $T_o(p, n)$.
2. Preliminary Result

Lemma 2.1 If \( p \in P \) then \( |p_m| \leq 2 \) and \( |q_m| \leq 2 \) for \( m \in N \), where the Caratheodory class \( P \) is the family of all functions \( p \) analytic in \( U \) for which \( \Re p(z) > 0 \),

\[
p(z) = 1 + p_m z^m + p_{2m} z^{2m} + p_{3m} z^{3m} + \ldots
\]

for \( z \in U \).

3. Coefficient bounds for the function class \( A^\alpha_{\Sigma,m} \)

Definition 3.1 A function \( f \) given by (2) is said to be in the class \( A^\alpha_{\Sigma,m} \) \( (0 < \alpha \leq 1, m \in N) \) if the following conditions are satisfied

\[
f \in \Sigma \text{ and } \left| \arg \left( \frac{f(z)}{zf'(z)} \right) \right| < \frac{\alpha \pi}{2}, \quad (0 < \alpha \leq 1, \, z \in U) (4)
\]

and

\[
\left| \arg \left( \frac{g(w)}{wg'(w)} \right) \right| < \frac{\alpha \pi}{2}, \quad (0 < \alpha \leq 1, \, w \in U) (5)
\]

where the function \( g \) is given by

\[
g(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 - (5a_3^2 - 5a_2 a_3 + a_4) w^4 + \ldots
\]

Firstly, we state the following result as below.

Theorem 3.1 Let \( f(z) \) given by (2) be in the class \( A^\alpha_{\Sigma,m} \) \( (0 < \alpha \leq 1, m \in N) \). Then

\[
|a_{m+1}| \leq \frac{2\alpha}{\sqrt{m^2(1-3\alpha)} - 2\alpha m} \quad (7)
\]

and

\[
|a_{2m+1}| \leq \frac{2\alpha^2(m+1)}{m^2} - \frac{\alpha}{m} \quad (8)
\]

Proof. It follows from (4) and (5) that

\[
\frac{f(z)}{zf'(z)} = \left[ p(z) \right]^\alpha
\]

and for its inverse map, \( g = f^{-1} \), we have

\[
\frac{g(w)}{wg'(w)} = \left[ q(w) \right]^\alpha
\]

where \( p(z) \) and \( q(w) \) satisfy the following inequalities:

\[
\Re (p(z)) > 0 \quad (z \in U) \quad \text{and} \quad \Re (q(w)) > 0 \quad (w \in U). \quad (11)
\]

Furthermore, the functions \( p(z) \) and \( q(w) \) have the form

\[
p_m(z) = 1 + p_m z^m + p_{2m} z^{2m} + p_{3m} z^{3m} + \ldots \in P
\]

and

\[
q_m(w) = 1 + q_m w^m + q_{2m} w^{2m} + q_{3m} w^{3m} + \ldots \in P
\]
respectively.

Comparing the corresponding coefficients of (9) and (10) yields

\[ -m a_{m+1} = \alpha p_m \]  
\[ (-4m) a_{2m+1} = 2\alpha p_{2m} + \alpha(\alpha - 1)p_m^2 + 2\alpha p_m(m + 1)a_{m+1} \]  
\[ m a_{m+1} = \alpha q_m \]  
\[ 4m \left[ -(m + 1)a_{m+1}^2 + a_{2m+1} \right] = 2\alpha q_{2m} + \alpha(\alpha - 1)q_m^2 - 2\alpha q_m(m + 1)a_{m+1} \]  
(17)

From (14) and (16), we get the equation below

\[ p_m = -q_m \]  
(18)

and

\[ 2m^2 a_{m+1}^2 = \alpha^2(p_m^2 + q_m^2). \]  
(19)

Also, from (15) and (17), we will obtain

\[ -4m(m + 1)a_{m+1}^2 - 2\alpha p_{2m} - \alpha(\alpha - 1)p_m^2 = 2\alpha q_{2m} + \alpha(\alpha - 1)q_m^2 \]  
(20)

A rearrangement together with the second identity in (19) and (20), we get as equation below,

\[ -4m(m + 1)a_{m+1}^2 = 2\alpha(p_{2m} + q_{2m}) + \alpha(\alpha - 1)(p_m^2 + q_m^2). \]  
(21)

And by the short calculation (19) and (21), we obtain

\[ a_{m+1} = \frac{\alpha^2(p_{2m} + q_{2m})}{m^2(1 - 3\alpha) - 2\alpha m}. \]  
(22)

According to Lemma 2.1, \(|p_m| \leq 2\) and \(|q_m| \leq 2\) for \(m \in N\). Now, taking the absolute value of (22) and applying the Lemma 2.1 for coefficients \(p_{2m}\) and \(q_{2m}\) we obtain

\[ |a_{m+1}| \leq \frac{2\alpha}{\sqrt{m^2(1 - 3\alpha) - 2\alpha m}}. \]

This gives the desired estimate for \(|a_{m+1}|\) as asserted (7). Next, in order to determine the bound on \(|a_{m+1}|\), by subtracting (17) from (15), we can get

\[ 2(-4m)a_{2m+1} - (-4m)(m + 1)a_{m+1} = 2\alpha(p_{2m} - q_{2m}) + \alpha(\alpha - 1)(p_m^2 - q_m^2) \]

By substituting the value of \(a_{m+1}^2\) from (19) and observing that \(p_m^2 = q_m^2\) it follows that

\[ 2(-4m)a_{2m+1} - (-4m)(m + 1)a_{m+1} \frac{\alpha^2(p_m^2 + q_m^2)}{2m^2} = 2\alpha(p_{2m} - q_{2m}) \]

\[ 2(-4m)a_{2m+1} - \frac{(-4m)(m + 1)\alpha^2(p_m^2)}{m^2} = 2\alpha(p_{2m} - q_{2m}) \]

\[ 2m^2(-4m)a_{2m+1} = 2m^2\alpha(p_{2m} - q_{2m}) + (-4m)(m + 1)\alpha^2(p_m^2) \]

\[ a_{2m+1} = \frac{\alpha(p_{2m} - q_{2m})}{-4m} + \frac{\alpha^2(m + 1)p_m^2}{2m^2} \]

Apply the lemma 2.1 for coefficients \(p_m, p_{2m}\) and \(q_{2m}\), we will have

\[ |a_{2m+1}| \leq \frac{2\alpha^2(m + 1)}{m^2} - \frac{\alpha}{m}. \]

This complete the proof of Theorem 3.1. If \(m = 1\) that means the result is first-fold, as follow.
Corollary 3.1 Let $f$ given by (2) be in the class $A_{\alpha,1}^\alpha$ ($0 < \alpha \leq 1$). Then

$$|a_2| \leq \frac{2\alpha}{\sqrt{(1-5\alpha)}}$$

and

$$|a_3| \leq \alpha(1-4\alpha).$$

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