Strong Revenue (Non-)Monotonicity of Single-parameter Auctions

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Consider Myerson’s optimal auction with respect to an inaccurate prior, e.g., estimated from data, which is an underestimation of the true value distribution. Can the auctioneer expect getting at least the optimal revenue w.r.t. the inaccurate prior since the true value distribution is larger? This so-called strong revenue monotonicity is known to be true for single-parameter auctions when the feasible allocations form a matroid. We find that strong revenue monotonicity fails to generalize beyond the matroid setting, and further show that auctions in the matroid setting are the only downward-closed auctions that satisfy strong revenue monotonicity. On the flip side, we recover an approximate version of strong revenue monotonicity that holds for all single-parameter auctions, even without downward-closedness. As applications, we get sample complexity upper bounds for single-parameter auctions under matroid constraints, downward-closed constraints, and general constraints. They improve the state-of-the-art upper bounds and are tight up to logarithmic factors.

CCS Concepts: • Theory of computation → Algorithmic mechanism design; Computational pricing and auctions.

Additional Key Words and Phrases: single-parameter auctions, revenue monotonicity, sample complexity

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1 INTRODUCTION

Revenue optimal auction design is a central topic in economics and more recently in algorithmic game theory. For example, consider auctioning an item to some bidders. How shall the auctioneer decide which bidder wins the item and how much the winner pays based on the bids, so that the auctioneer’s revenue is maximized? It is known that we may without loss of generality focus on truthful auctions which incentivize bidders to truthfully report their values. Further, classical auction theory often studied this problem under the Bayesian model, in which the bidders values are drawn from some value distribution known to the auctioneer beforehand. The goal is to maximize the expected revenue over the random realization of bidders’ values.

The simplest case is when there is only one bidder. The problem then becomes choosing a take-it-or-leave-it price $p$ to maximize the product of price $p$ and the probability that the bidder’s value is at least $p$.

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The revenue optimal auction is more involved when there are multiple bidders. Myerson [1981] characterized the revenue optimal auction when the bidders’ values are independently (but not necessarily identically) distributed; the literature often refers to it as Myerson’s optimal auction. In a nutshell, the auctioneer computes for each bidder an ironed virtual value based on the bidder’s value/bid and value distribution. Then, the auctioneer allocates the item to the bidder with the largest nonnegative virtual value, and leaves the item unallocated if all bidders have negative virtual values. Finally, the winner of the item pays the threshold value, i.e., the smallest value at which it would still win.

Myerson’s optimal auction generalizes to all single-parameter auctions, which will be the focus of this paper. For ease of exposition, we consider the following simplified definition of single-parameter auctions in the introduction; Section 2 will give a more general definition. Suppose that the auctioneer has some homogeneous items which it can allocate to certain subsets of bidders. We refer to each subset of bidders that the auctioneer can allocate to as a feasible allocation. Below are some example constraints that may define the set of feasible allocations:

(C1) There are \( k \) copies of the item and therefore at most \( k \) bidders can be allocated an item.
(C2) Two bidders are competitors so the auctioneer can not allocate to both at the same time.
(C3) Two bidders are bundled in the sense that either none or both are allocated an item.

Each bidder has a value for being allocated an item, drawn from a value distribution. In this sense, its value is given by a single parameter and hence the name of the setting. The auctioneer knows the value distribution but not the realized value. Myerson’s optimal auction first asks all bidders to submit bids. Then, treating each bidder’s bid as its value, the auctioneer computes the ironed virtual value of each bidder, chooses an allocation that maximizes the sum of ironed virtual values of the allocated bidders, and finally lets each allocated bidder pays its threshold value.

1.1 Inaccurate Prior and Revenue Monotonicity

To faithfully implement Myerson’s optimal auction, the auctioneer would need to have complete information of each bidder’s value distribution, which is rarely available. Further, it is known that the ironed virtual values are sensitive to smaller changes to the value distributions (e.g., [Guo et al., 2019, Roughgarden and Schrijvers, 2016]). This leads to a natural question: Can the auctioneer use Myerson’s optimal auction with respect to (w.r.t.) an inaccurate prior, e.g., estimated from various data, and still expect good revenue?

Let us first revisit the simplest case of a single bidder. It is folklore that underestimating and overestimating the value distribution and, correspondingly, decreasing and increasing the resulting take-it-or-leave-it prices have contrasting impacts to the revenue. Decreasing the price by 1% would at worst lower the expected revenue by 1%, while increasing the price by 1% might lower the expected revenue to almost zero. The robustness to underestimation could be formalized as an auction-wise revenue monotone property. Consider any value distributions \( D \) and \( \tilde{D} \) such that the former (stochastically) dominates the latter, i.e., the former’s cumulative distribution function (CDF) is point-wise less than or equal to the latter’s CDF. The revenue of any auction (i.e., any take-it-or-leave-it price) w.r.t. \( D \) is greater than or equal to the revenue w.r.t. \( D \). Hence, it is safer to use the optimal take-it-or-leave-it price w.r.t. an inaccurate prior \( \tilde{D} \) that is an underestimation, since the auctioneer can guarantee getting at least the estimated revenue of the chosen auction.

Do similar properties hold when there are multiple bidders? It is easy to construct counter-examples that refute auction-wise revenue monotonicity;\(^1\) but is it still safe to use the optimal auction w.r.t. an underestimation and guarantee getting at least the estimated revenue?

\(^1\)For example, consider two bidders Alice and Bob and the auctioneer can allocate to at most one. Consider an auction that allocates to Alice and charges her $1 if her value is at least $1, and otherwise allocates to Bob and charges him $10 if his
et al. [2016] gave an affirmative answer when the set of feasible allocations form a matroid. They called this property \textit{strong revenue monotonicity}. While we defer the definition of matroids to Section 2, readers may think of the aforementioned constraint (C1) as a running example, which is called a \(k\)-uniform matroid.

Beyond the matroid setting, only a weaker notion of \textit{revenue monotonicity} was known (folklore, c.f., [Devanur et al., 2016]): if value distribution \(D\) dominates \(\tilde{D}\) then the optimal revenue w.r.t. \(D\) is greater than or equal to the optimal revenue w.r.t. \(\tilde{D}\). This weaker notion is insufficient for answering our motivating question, i.e., how much revenue the auctioneer could expect when it uses Myerson’s optimal auction w.r.t. an inaccurate prior. Hence, this paper studies whether strong revenue monotonicity holds for general single-parameter auctions. Before getting to our results, we remark that intriguingly even the weaker notion of revenue monotonicity ceases to hold in the presence of multiple types of items, a.k.a., the multi-parameter setting (c.f., [Hart and Reny, 2015]).

\textbf{Our Contribution.} On the one hand, we prove that strong revenue monotonicity does not hold in general single-parameter auctions (Theorem 3.1), even if the set of feasible allocations is \textit{downward-closed}, i.e., removing a bidder from any feasible allocation would give another feasible allocation, such as the aforementioned constraints (C1) and (C2). In fact, we show that a downward-closed single-parameter auction is strongly revenue monotone if and only if its feasible allocations form a matroid (Theorem 3.3). Further, the decrease in revenue could be as large as a constant fraction of the optimal revenue (Corollary 3.2).

On the other hand, we show that the auctioneer can nonetheless ensure good revenue by using Myerson’s optimal auction w.r.t. an \textit{approximately accurate} underestimation for all single-parameter auctions, including those that are not downward-closed, e.g., with the aforementioned constraint (C3). Concretely, we prove that if \(D\) dominates \(\tilde{D}\) and further \(D\) and \(\tilde{D}\) are \textit{sufficiently close}, then running Myerson’s optimal auction w.r.t. \(\tilde{D}\) when the value distribution is \(D\) gets almost the optimal revenue w.r.t. \(\tilde{D}\) (Theorem 3.5). We call this \textit{approximate strong revenue monotonicity}. Moreover, we supplement this result by examining the revenue bound from the closeness of \(D\) and \(\tilde{D}\) alone, i.e., without stochastic dominance. We call this \textit{strong revenue Lipschitzness} and show that it is strictly weaker than approximate strong revenue monotonicity (Theorems 3.8 and 3.10).

We remark that both matroid and non-matroid downward-closed auctions practically relevant. As a simple example of matroid auctions consider \(k\)-uniform matroid. The corresponding auction sells \(k\) identical goods to \(n\) agents where each agent wants at most 1 good. For an example of non-matroid downward-closed auctions, consider selling \(k\) identical goods to \(n\) agents, but each agent demands a different quantity of the goods: agent \(i\) has value \(v_i\) if she gets \(d_i\) goods, and 0 otherwise. This is downward closed but would be a non-matroid in general if \(d_i\)'s are distinct. For a related practical example, this captures a simplified version of spectrum auctions, where different companies demand different numbers of frequency blocks.

Figure 1 summarizes the revenue monotone properties in different auction settings.

\subsection{Sample Complexity}

Closely related to the analysis of Myerson’s optimal auction obtained from an inaccurate prior, Cole and Roughgarden [2014] introduced a model in which the auctioneer can only access the value distribution through i.i.d. samples. They asked how many samples are sufficient and necessary for learning an auction that is optimal up to an \(\epsilon\) error?\(^2\) Driven by strong revenue monotonicity,
Fig. 1. Revenue monotone properties in different auction settings. A setting satisfies a revenue monotone property if all auctions therein satisfies the property. A setting does not satisfy a revenue monotone property if there is an auction in the setting violating the property.

Guo et al. [2019] proposed to use Myerson’s optimal auction w.r.t. a dominated product empirical distribution derived from samples, which is dominated by the true distribution and is as close to the true distribution as possible. They showed that this approach gives sample complexity upper bounds that are tight up to logarithmic factors for auctions under matroid constraints.

**Our Contribution.** We prove that the approximate strong revenue monotonicity proposed in this paper is good enough for deriving sample complexity upper bounds using Myerson’s optimal auction w.r.t. the dominated product empirical. In fact, our analysis is an improvement over that of Guo et al. [2019] and therefore even in the matroid setting our $O\left(\frac{n^k}{\epsilon^2} \log \frac{n^k}{\epsilon^2} \log \frac{n^k}{\epsilon^2} \delta\right)$ upper bound (Theorem 4.1) is better than theirs by three logarithmic factors. Here $n$ is the number of bidders, $k$ is the maximum number of bidders that can be allocated to in any feasible allocation, a.k.a., the rank, and $\delta$ is the probability that the algorithm fails to obtain an $\epsilon$-additive approximation. For downward-closed auctions, we derive the same bound as in the matroid setting. It improves the best previous bound by Gonczarowski and Nisan [2017] by a multiplicative $\frac{k}{\epsilon}$ factor, and is tight up to a logarithmic factor due to the known lower bound in the more special matroid setting [Guo et al., 2019]. Finally, for arbitrary single-parameter auctions, we obtain an upper bound of $O\left(\frac{n^k}{\epsilon^2} \log \frac{n^k}{\epsilon^2} \log \frac{n^k}{\epsilon^2} \delta\right)$ (Theorem 4.3), which improves the best previous bound by Gonczarowski and Nisan [2017] by a multiplicative $\frac{1}{\epsilon}$ factor. We also prove a lower bound of $\Omega\left(\frac{n^k}{\epsilon^2}\right)$ (Theorem 4.4), matching the upper bound up to logarithmic factors. Our results further demonstrate that general single-parameter auctions are intrinsically harder than downward-closed auctions in terms of sample complexity.

[0,1] and $\epsilon$-additive approximation. Nonetheless, regular distributions and multiplicative approximation and several other settings can be reduced to our setting through appropriate discretizations (see, e.g., [Gonczarowski and Nisan, 2017, Guo et al., 2019]).
Table 1. Summary of sample complexity upper bounds of different single-parameter auctions. Here \( n \) denotes the number of bidders, \( k \) denotes the maximum total allocated amount in any feasible allocation, a.k.a., the rank, \( \epsilon \) denotes the additive approximation factor, and \( \delta \) denotes the algorithm’s failure probability.

| Class                  | Best Previous Bound                              | This Paper                  |
|------------------------|-------------------------------------------------|-----------------------------|
| Single-bidder          | \( O(\frac{1}{\epsilon^2} \log \frac{1}{\epsilon^2}) \) \(^a\) | \( O(\frac{nk}{\epsilon^2} \log \frac{nk}{\epsilon^2}) \) (Thm. 4.1) |
| Matroid                | \( O(\frac{nk}{\epsilon^2} \log^2 \frac{nk}{\epsilon^2}) \) \(^b\) | \( O(\frac{nk}{\epsilon^2} \log \frac{nk}{\epsilon^2}) \) (Thm. 4.1) |
| Downward-closed        | \( O(\frac{nk^2}{\epsilon^2} \log \frac{nk}{\epsilon}) \) \(^c\) | \( O(\frac{nk^2}{\epsilon^2} \log \frac{nk}{\epsilon} \log \frac{nk}{\epsilon^2}) \) (Thm. 4.3) |
| Single-parameter       | \( O(\frac{nk^2}{\epsilon^2} \log \frac{nk}{\epsilon^2}) \) \(^c\) | \( O(\frac{nk^2}{\epsilon^2} \log \frac{nk}{\epsilon} \log \frac{nk}{\epsilon^2}) \) (Thm. 4.3) |

\(^a\) Huang et al. [2018]; \(^b\) Guo et al. [2019]; \(^c\) Gonczarowski and Nisan [2017].

1.3 Related Works

For multi-parameter auctions, Hart and Reny [2015] gave an example of revenue non-monotonicity involving only one bidder and two heterogeneous items. Before this paper, the non-monotonicity of multi-parameter auctions seems to render it impossible to generalize the sample complexity framework of Guo et al. [2021, 2019] beyond single-parameter auctions. This paper shows that approximate strong revenue monotonicity is sufficient for sample complexity analyses. It is an interesting open question whether approximate weak/strong revenue monotonicity and the corresponding sample complexity framework extend to multi-parameter auctions.

Hart and Reny [2015] also showed that two special classes of single-bidder multi-item auctions have monotone payment functions and as a result satisfy strong revenue monotonicity.\(^3\) There is a long line of works proving that simple auctions can guarantee nearly optimal revenue in various multi-parameter auctions, e.g., by bundling all items together and posting a take-it-or-leave-it price, and by selling items separately (e.g., [Babaioff et al., 2014, Cai and Zhao, 2017, Hart and Nisan, 2017, Li and Yao, 2013, Rubinstein and Weinberg, 2018]). Rubinstein and Weinberg [2018] observed that these simple single-bidder auctions satisfy revenue monotonicity, i.e., they yield better revenue on stochastically dominating distributions, and therefore the optimal revenue of the respective multi-parameter auctions is approximately monotone. Yao [2018] extended the approximate revenue monotonicity to multiple bidders with fractionally subadditive valuations.

Following Cole and Roughgarden [2014], there has been a vast literature devoted to the sample complexity of various auctions. Huang et al. [2018] resolved the sample complexity of the single-bidder case up to logarithmic factors. Morgenstern and Roughgarden [2015], Devanur et al. [2016], Gonczarowski and Nisan [2017], and Syrgkanis [2017] built on learning theory to improve the sample complexity upper bound of single-parameter auctions with multiple bidders. Guo et al. [2019] built on strong revenue monotonicity and got sample complexity upper and lower bounds tight up to logarithmic factors for the matroid setting. Although a complete characterization of optimal multi-parameter auctions remains elusive, Gonczarowski and Weinberg [2021] and Guo et al. [2021] showed that polynomially many samples are informationally sufficient for learning a multi-parameter auction optimal up to an \( \epsilon \) error. Last but not least, Guo et al. [2021] extended the notion of strong monotonicity to other Bayesian optimization problems including prophet inequality and Pandora’s problem, and obtained nearly tight sample complexity for them.

This paper is also related to the literature of robust mechanism design, in particular, to the recent research on designing auctions with revenue guarantees for all value distributions that are “close” to a given distribution (e.g., [Brustle et al., 2020, Cai et al., 2021]). Our results may be viewed as

\(^3\)Hart and Reny [2015] only claimed revenue monotonicity but implicitly proved strong revenue monotonicity as well.
improved robustness guarantees for general single-parameter auctions. Qualitatively, our result directly uses the optimal auction w.r.t. a stochastically dominated distribution while previous works on robust mechanism design often need to modify the auction through the “nudge and round” technique (e.g., [Hart and Nisan, 2017]). Quantitatively, the robustness bounds in this paper are tight for general single-parameter auctions.

2 PRELIMINARIES

Notations. Let \( \mathbb{R}_+ \) denote the set of nonnegative real numbers. Let \([n] = \{1, 2, \ldots, n\} \). For any distribution \( D \) on \( \mathbb{R}_+ \), we abuse notation and let \( D \) also denote its CDF, i.e., \( D(v) = \Pr_{u \sim D}[u \leq v] \); thus, its derivative \( D'(v) \) (if exists) is the probability density function (PDF). For distributions \( D \) and \( \tilde{D} \) on \( \mathbb{R}_+ \), we say that \( D \) (first-order) stochastically dominates \( \tilde{D} \), denoted as \( D \succeq \tilde{D} \), if \( D(v) \leq \tilde{D}(v) \) for all \( v \in \mathbb{R}_+ \). For any distribution \( D \) over a domain \( \Omega \) and any function \( f : \Omega \to \mathbb{R} \), we write \( f(D) \) for the expected function value \( \mathbb{E}_{\omega \sim D} f(\omega) \).

2.1 Single-parameter Auctions

In a single-parameter auction with \( n \) bidders, each bidder \( i \) has a private value \( 0 \leq v_i \leq 1 \) drawn independently from a distribution \( D_i \). An auction proceeds as follows. First each bidder \( i \) submits a bid \( b_i \) to the auctioneer. The auctioneer then picks an allocation \( x \) from a set \( \mathcal{X} \subset \mathbb{R}^n_+ \) of feasible allocations, and prices \( p \in \mathbb{R}_+^n \) according to the bids. Note that from now on we consider a more general model that allows the allocation \( x \) to be non-binary. Each bidder \( i \) receives allocation \( x_i \), pays \( p_i \), and gets utility \( v_i x_i - p_i \). Throughout the paper we will focus on truthful auctions in which a bidder can always guarantee a non-negative utility and can maximize its utility by bidding its true value; this is without loss of generality (WLOG) by the revelation principle. Hence, we will assume \( b_i = v_i \) and will no longer talk about bids.

Let \( k = \max_{x \in \mathcal{X}} \|x\|_1 \) be the maximum size of any feasible allocation. Following a terminology from the special case when \( \mathcal{X} \) is the convex hull of a matroid, we refer to \( k \) as the rank. Finally, for ease of exposition we assume WLOG that the problem is unit-demand, i.e., \( x_i \leq 1 \) for any feasible allocation \( x \in \mathcal{X} \) and any bidder \( i \). The general case can be reduced to this unit-demand case, since an \( \epsilon \)-approximation in an auction with \( n \) bidders, rank \( k \), and maximum demand \( d \) is equivalent to an \( \frac{\epsilon}{k} \)-approximation in another auction with \( n \) bidders, rank \( k \), and unit-demand, obtained by scaling all allocations by a factor \( \frac{d}{k} \).

Quantiles and Revenue Curves. For any value distribution \( D \) on \( \mathbb{R}_+ \), the quantile of a value \( v \in \mathbb{R}_+ \) w.r.t. \( D \) is the probability that a sample from \( D \) is greater than \( v \):

\[
q_D(v) = \Pr_{u \sim D}[u > v].
\]

On the other hand, for any quantile \( q \in [0, 1] \) the corresponding value w.r.t. \( D \) is:

\[
v_D(q) = \inf \{ u : q_D(u) \leq q \}.
\]

We remark that \( v_D \) is simply the inverse of \( q_D \) if \( D \) is continuous.

For any value distribution \( D \) and any quantile \( 0 \leq q \leq 1 \), consider \( q^+ = \Pr_{u \sim D}[u \geq v_D(q)] \) and \( q^- = \Pr_{u \sim D}[u < v_D(q)] \) in the next definition. If the distribution is continuous, we would have that \( q = q^+ = q^- \). The revenue curve w.r.t. value distribution \( D \) in the quantile space is:

\[
R_D(q) = \begin{cases} 
q \cdot v_D(q) & \text{if } D \text{ is continuous at } v_D(q); \\
\frac{q - q^-}{q^+ - q^-}v_D(q^+) + \frac{q^+ - q^-}{q^+ - q^-}v_D(q^-) & \text{if } v_D(q) \text{ is a point mass}.
\end{cases}
\]

The ironed revenue curve is its convex hull, i.e.:

\[
\bar{R}_D(q) = \max \{ R_D(P) : P \text{ is a distribution over } [0, 1] \text{ with expectation } q \}.
\]
Fig. 2. Revenue curves. This example corresponds to a value distribution that has a point mass of \( \frac{1}{5} \) at value \( \frac{1}{2} \), and otherwise is uniform over \([0, 1]\).

**Optimal Auction.** Given any truthful auction \( M \), abuse notation and let \( M \) also be a mapping from value profiles to the resulting revenue. That is, for any \( v = (v_1, v_2, \ldots, v_n) \), \( M(v) \) denotes the revenue of \( M \) when bidders bid \( v \); for any product value distribution \( D = D_1 \times D_2 \times \cdots \times D_n \), \( M(D) \) denotes the expected revenue of \( M \). Let \( \text{Orr}(D) \) be the largest expected revenue achievable by truthful auctions when the value distribution is \( D \).

Myerson [1981] introduced the **virtual values** of any bidder \( i \), defined as \( \varphi_i(v_i) = v_i - \frac{1-\text{Orr}(D)}{\partial\text{Orr}(D)/\partial v_i} \) when value distribution \( D_i \) is continuous. For general value distributions, the virtual value is the right derivative of revenue curve \( R_{D_i}(q) \) at \( q_{D_i}(v_i) \). Myerson showed that an auction is truthful if and only if its allocation rule \( x \) is monotone, i.e., if \( x_i \) is nondecreasing in \( v_i \) for all bidders \( i \), and the payment rule \( p \) is determined by a specific formula according to the allocation rule \( x \).\(^4\) Further, the expected revenue equals the expected **virtual welfare**:

\[
\mathbb{E}_{v_i \sim D} \sum_{i=1}^{n} x_i(v_i) \varphi_i(v_i) .
\]

Myerson further defined the ironed virtual values of any bidder \( i \), denoted as \( \bar{\varphi}_i \). It is the right derivative of the ironed revenue curve \( \bar{R}_{D_i}(q) \) at \( q_{D_i}(v_i) \). Finally, Myerson showed that to maximize the expected revenue, the optimal auction always chooses an allocation that maximizes the ironed virtual welfare. Denote Myerson’s optimal auction w.r.t. a value distribution \( D \) as \( M_D \).

A value distribution is **regular** if its revenue curve is concave. For regular distributions, the ironed revenue curve coincides with the revenue curve, and the ironed virtual values coincide with the virtual values.

Figure 2 presents an illustrative example of the revenue curves and virtual values w.r.t. a value distribution that has a point mass of \( \frac{1}{5} \) at value \( \frac{1}{2} \), and is uniform over \([0, 1]\) otherwise. Figure 2a demonstrates the case of value \( v = \frac{1}{5} \) which is not a point mass; it corresponds to quantile \( q = \frac{4}{5} \), and the right derivative of the revenue curve equals the virtual value. Figure 2b shows the case of value \( v = \frac{1}{2} \) which is a point mass; it corresponds to a left-closed-right-open quantile interval \( q \in [\frac{1}{2}, \frac{3}{5}] \), and the right derivative of the revenue curve equals the virtual value. Finally, Figure 2c gives the ironed revenue curve, ironing quantile interval \([\frac{3-\sqrt{3}}{5}, \frac{3}{5}] \). It effectively rounds the values from \( \frac{1}{2} \) to \( \frac{1+\sqrt{3}}{4} \) down to \( \frac{1}{2} \); the ironed revenue curve is the revenue curve of the rounded distribution, with a point mass of \( \sqrt{3}/5 \) at value \( \frac{1}{2} \), and has probability density \( \frac{3}{5} \) in value intervals.

\[\varphi_i(v_i, v_{-i}) = x_i(v_i, v_{-i}) \cdot v_i - \int_{0}^{v_i} x_i(t_i, v_{-i}) dt_i\]
We remark that interpreting ironing as a rounding of values will be a useful viewpoint in an argument in Section 3.2. Hence, we formulate it as a lemma below, whose proof follows directly from the definitions of revenue curve and ironed revenue curve.

**Lemma 2.1.** Consider any value distribution $D$. Define value distribution $\hat{D}$ by rounding values down to the closest value that lies on the convex hull of $D$’s revenue curve, i.e., for any quantile $0 \leq q \leq 1$:

$$v_\hat{D}(q) = \sup \left\{ v_D(q') : q \leq q' \leq 1, R_D(q) = \hat{R}_D(q) \right\}.$$

We have that (1) $\hat{D}$ is regular, and (2) for any value $0 \leq v \leq 1$, $v$’s ironed virtual value w.r.t. $D$ is equal to $v$’s virtual value w.r.t. $\hat{D}$.

Finally, we formally define strong revenue monotonicity below.

**Definition 2.2.** The single-parameter auctions with feasible allocations $X$ satisfy strong revenue monotonicity if for any product value distribution $\tilde{D}$ and any stochastically dominating product value distribution $D \geq \tilde{D}$, Myerson’s optimal auction $M_\tilde{D}$ w.r.t. $\tilde{D}$ gets weakly larger revenue from $D$ than from $\tilde{D}$, i.e.:

$$M_D(D) \geq \text{Opt}(\tilde{D}).$$

### 2.2 Set Systems and Matroids

The simplified definition of single-parameter auctions in the introduction further assumes that the set of feasible allocations consists of some binary allocations, in which every $x_i \in \{0, 1\}$, and their convex combinations through randomized allocations. We can describe the set of feasible allocations of such an auction by a set system. Let $\mathcal{I}$ be the set of subsets of bidders that can be allocated to in any feasible allocation; the notation $\mathcal{I}$ comes from the concept of independent sets in set systems. For any $S \in \mathcal{I}$, let $x^S$ be the corresponding allocation vector, i.e., $x^S_i = 1$ if $i \in S$ and $x^S_i = 0$ otherwise. The set of feasible allocation can then be written as:

$$\left\{ \sum_{S \in \mathcal{I}} \lambda^S x^S : \sum_{S \in \mathcal{I}} \lambda^S = 1 \text{ and } \forall S \in \mathcal{I}, \lambda^S \geq 0 \right\}$$

A set system is a matroid if it satisfies:

(M1) The empty set is feasible, i.e., $\emptyset \in \mathcal{I}$.

(M2) The feasible sets are downward-closed, i.e., if $S' \subseteq S$ and $S \in \mathcal{I}$ then $S' \in \mathcal{I}$.

(M3) The feasible sets satisfy the exchange property, i.e., for any feasible sets $S, S' \in \mathcal{I}$ such that $|S'| < |S|$, there is a bidder $i \in S \setminus S'$ that can be allocated to on top of $S'$, i.e., $S' \cup \{i\} \in \mathcal{I}$.

Naturally it is a downward-closed set system if it satisfies the first two properties.

### 2.3 Empirical Distributions

Let $D = D_1 \times D_2 \times \cdots \times D_n$ be a product value distribution. Given $N$ i.i.d. samples from $D$, the product empirical distribution $E = E_1 \times E_2 \times \cdots \times E_n$ is defined such that each dimension $E_i$ is the uniform distribution over the samples from $D_i$. The next lemma follows from Bernstein’s inequality and union bound.

**Lemma 2.3** (c.f., Lemma 5 of Guo et al. [2019]). For any product distribution $D$ on $\mathbb{R}_+^n$, any positive integer $N$, and any $0 < \delta < 1$, consider the product empirical distribution $E$ from $N$ i.i.d. samples from $D$. Then, with probability at least $1 - \delta$, for any bidder $1 \leq i \leq n$:

$$\max_{v \in \mathbb{R}_+} \left| D_i(v) - E_i(v) \right| \leq \sqrt{\frac{2D_i(v)(1 - D_i(v)) \ln \frac{2nN}{\delta}}{N}} + \frac{\ln \frac{2nN}{\delta}}{N}.$$
Further define the dominated product empirical distribution $\tilde{E} = \tilde{E}_1 \times \tilde{E}_2 \times \cdots \times \tilde{E}_n$ such that for any bidder $i$ and any value $0 \leq v_i \leq 1$:

$$\tilde{E}_i(v_i) = \min \left\{ 1, E_i(v_i) + \sqrt{\frac{2E_i(v_i)(1 - E_i(v_i)) \ln \frac{2nN}{\delta}}{N}} + \frac{4 \ln \frac{2nN}{\delta}}{N} \right\}.$$ 

The larger constant in the last term upper bounds the difference from having distinct terms inside the square root compared to the previous equation.

**Lemma 2.4 (c.f., Lemmas 6 and 7 of Guo et al. [2019]).** For any product distribution $D$ on $\mathbb{R}_+^n$, any positive integer $N$, and any $0 < \delta < 1$, consider the dominated product empirical distribution $\tilde{E}$ from $N$ i.i.d. samples from $D$. Then, with probability at least $1 - \delta$, for any bidder $1 \leq i \leq n$ and any value $0 \leq v_i \leq 1$:

$$D_i(v_i) \leq \tilde{E}_i(v_i) \leq D_i(v_i) + \frac{8D_i(v_i)(1 - D_i(v_i)) \ln \frac{2nN}{\delta}}{N} + \frac{7 \ln \frac{2nN}{\delta}}{N}.$$ 

Motivated by the above bounds on the differences between the original distribution and empirical distributions and driven by our analysis, we say that two product value distributions $D$ and $E$ are $\epsilon$-close, denoted as $D \approx_\epsilon E$, if for any bidder $i$ and any value $0 \leq v_i \leq 1$ we have:

$$\left| D_i(v_i) - E_i(v_i) \right| \leq \min \left\{ D_i(v_i)(1 - D_i(v_i)), E_i(v_i)(1 - E_i(v_i)) \right\} \cdot \frac{\epsilon^2}{4nk} + \frac{\epsilon^2}{2nk}.$$ 

The value distribution and dominated product empirical distribution are $\epsilon$-close when the number of samples meets the sample complexity bound in this paper. The next lemma, which follows as a corollary of Lemma 2.4, makes this precise.

**Lemma 2.5.** For any product distribution $D$ on $\mathbb{R}_+^n$, any positive integer $N$, and any $0 < \delta < 1$, consider the dominated product empirical distribution $\tilde{E}$ from $N$ i.i.d. samples from $P$. If:

$$N \geq C \cdot \frac{nk}{\epsilon^2} \log \frac{nk}{\epsilon \delta},$$

for a sufficiently large constant $C$, then with probability at least $1 - \delta$ we have both $D \succeq \tilde{E}$ and $D \approx_\epsilon \tilde{E}$.

Some readers may prefer a uniform notion of $\epsilon$-closeness. Two product value distributions $D$ and $E$ are $\epsilon$-close uniformly, denote as $D \approx^{\text{unif}}_\epsilon E$, if for any bidder $i$ and any value $0 \leq v_i \leq 1$:

$$\left| D_i(v_i) - E_i(v_i) \right| \leq \frac{\epsilon}{\sqrt{nk}},$$

where the right-hand-side is scaled by $\frac{1}{\sqrt{nk}}$ for a direct comparison with the nonuniform notion. By definition, $D \approx^{\text{unif}}_\epsilon E$ implies $D \approx^{\text{unif}}_\epsilon E$ but not the other way around.

### 3 STRONG REVENUE (NON-)MONOTONICITY

#### 3.1 Example of Non-monotonicity

Recall that Devanur et al. [2016] proved the strong revenue monotonicity of single-parameter auctions in the matroid setting, i.e., when the set of feasible allocations is the convex hull of the basis of a matroid. This subsection shows that the strong revenue monotonicity fails to generalize to more general single-parameter auctions (for some value distributions) even if we assume downward-closedness of the feasible allocations.
Example (Minimum Non-matroid). Consider a 3-bidder rank-2 auction as follows. We refer to the three bidders as $A$, $B$ and $C$. Let the set of feasible allocations be:

$$\mathcal{X} = \{(x_A, x_B, x_C) \in [0, 1]^3 : x_A + x_B \leq 1 \text{ and } x_A + x_C \leq 1\}.$$  

In other words, it is the convex hull of allocating exclusively to $A$, or $B$ and $C$, or $A$ and $B$, or $A$, $B$, and $C$. Let the set of value distributions be $\hat{\mathcal{D}}$, $\mathcal{D}$.

The above Myerson's optimal auction w.r.t. $\hat{\mathcal{D}}$ is as follows:

$$\varphi_{\hat{\mathcal{D}}_B}(v) = \varphi_{\hat{\mathcal{D}}_C}(v) = \begin{cases} 1 & \text{if } v = 1; \\ 0 & \text{if } 0 \leq v < 1; \\ -\infty & \text{if } 0 \leq v < \epsilon. \end{cases}$$

Therefore, Myerson's optimal auction $M_{\hat{D}}$ w.r.t. $\hat{D}$ is as follows:

- If $v_B = v_C = 1$, allocate to both $B$ and $C$, and let each of them pay $\epsilon$; that is, $x = (0, 1, 1)$ and $p = (0, \epsilon, \epsilon)$.
- If $v_B = 1$, $v_C = \epsilon$, allocate to both $B$ and $C$, and let $B$ pay 1 and let $C$ pay $\epsilon$; that is, $x = (0, 1, 1)$ and $p = (0, \epsilon, 1)$.
- If $v_B = \epsilon$, $v_C = 1$, allocate to both $B$ and $C$, and let $B$ pay $\epsilon$ and let $C$ pay 1; that is, $x = (0, 1, 1)$ and $p = (0, \epsilon, 1)$.
- If $v_B = v_C = \epsilon$, allocate to $A$, let $A$ pay $\frac{1}{2}$; that is, $x = (1, 0, 0)$ and $p = (\frac{1}{2}, 0, 0)$.

The corresponding expected revenue equals:

$$M_{\hat{D}}(\hat{D}) = (1 - \epsilon)^2 \cdot \frac{1}{2} + 2\epsilon(1 - \epsilon) \cdot (1 + \epsilon) + \epsilon^2 \cdot 2\epsilon > \frac{1}{2}.$$  

The above Myerson's optimal auction w.r.t. $\hat{D}$ suffers from non-monotone payments: when only one of $B$ and $C$ has value 1 the total payment is $1 + \epsilon$; when both of them have value 1, however,
the total payment is only $2\varepsilon$. That is why in the bigger value distribution $D$ we let $B$ and $C$’s value distributions be identically a point mass at 1. As a result, the expected revenue is only

$$M_{\hat{D}}(D) = 2\varepsilon.$$  

By our choice of $\varepsilon = \frac{1}{10}$ we have $M_{\hat{D}}(D) < M_{\hat{D}}(\hat{D}) - \Omega(1)$.

**Remark 3.1.** In the above construction, $\varepsilon$ can be arbitrarily small. Therefore, the multiplicative gap between the optimal revenue of a value distribution $D$ and the revenue of running Myerson’s optimal auction w.r.t. $D$ on a stochastically dominating distribution $\tilde{D}$ can be arbitrarily large.

By making $\lfloor \frac{k}{2} \rfloor$ copies of the minimum non-matroid and the value distributions in Theorem 3.1, we get that the decrease in revenue could be as large as $\Omega(k)$, which is at least a constant factor of the optimal revenue since the latter cannot exceed $k$.

**Corollary 3.2.** For any positive integer $k$, there is a rank-$k$ downward-closed single-parameter auction, and two value distributions $D \geq \tilde{D}$ such that:

$$M_{\tilde{D}}(D) < M_{\tilde{D}}(\tilde{D}) - \Omega(k).$$

Finally, we build on the above example to show that the matroid setting is the only case satisfying strong revenue monotonicity among all downward-closed set systems.\(^5\)

**Theorem 3.3.** If the set of feasible allocations of a single-parameter auction is a downward-closed set system, then it satisfies strong revenue monotonicity (for all value distributions) if and only if the set system is a matroid.

**Proof.** Devanur et al. [2016] proved the direction from matroid to strong revenue monotonicity. It remains to prove that any downward-closed non-matroid auction does not satisfy strong revenue monotonicity. Consider any such auction and its set $\tilde{I}$ of feasible subsets of bidders who could be allocated to. Since it is downward-closed and is not a matroid, it must violate the exchange property (M3). We will build on this fact to find an embedded structure that resembles the aforementioned minimum non-matroid example.

By the violation of exchange property, there are two feasible subsets of bidders $S, S' \in \tilde{I}$ such that $|S'| < |S|$, yet for any bidder $i \in S \setminus S'$, it is infeasible to allocate to $i$ on top of $S'$, i.e.:

$$\forall i \in S \setminus S': \quad S' \cup \{i\} \notin \tilde{I}. $$

Note that the above implies $S' \not\subset S$. By removing bidders from $S$, we may assume WLOG that $|S| = |S'| + 1$. Among all such pairs of $S$ and $S'$, consider the one with the maximum $|S \cap S'|$. Define $S_\cap = S \cap S', S_A = S' \setminus S$, and $S_{BC} = S \setminus S'$. Further let $m = |S_\cap|$ and $\ell = |S_{BC}|$. Then, we have $|S| = m + \ell$ and $|S'| = m + \ell - 1$. The above greedy choice of $S$ and $S'$ gives a useful property.

**Lemma 3.4.** Suppose that a feasible subset of bidders $T$ satisfies that (1) $S_\cap \subseteq T$, (2) $T \cap S_A \neq \emptyset$, and (3) $T \cap S_{BC} \neq \emptyset$. We have $|T| \leq m + \ell - 2$.

It follows from a proof by contradiction. Suppose on the contrary that there exists such a $T$, and consider the one with the most elements. If $|T| = m + \ell - 1$, we may replace $S'$ by $T$ and increase $|S \cap S'|$. If $|T| \geq m + \ell$, we may replace $S$ by $T$ and increase $|S \cap S'|$.

We next describe the value distributions $D$ and $\tilde{D}$. Let the values of all bidders in $S_\cap$ be 1 deterministically in both distributions, and thus their virtual values are also deterministically 1. Recall that $n$ is the number of bidders. The values of other bidders will be at most $\frac{1}{n}$ so that their

\(^5\)We need the assumption of downward-closedness because otherwise it could be a non-matroid but also effectively equivalent to a matroid, e.g., by adding a dummy bidder $i$ who must be allocated to, i.e., $x_i = 1$, to a matroid setting.
total contribution to virtual welfare is less than that of a single bidder in \( S_\gamma \). Hence, Myerson’s optimal auction always allocates to bidders in \( S_\gamma \) and gets \( m = |S_\gamma| \) total virtual values from them.

Further let the values of all bidders not in \( S \cup S' \) be 0 deterministically so that they may be ignored in our discussion.

Next, let \( A \) be an arbitrary bidder in \( S_A \). Let \( B \) and \( C \) be two arbitrary bidders in \( S_{BC} \). Let the values of all bidders in \( S_A \setminus \{A\} \) and \( S_{BC} \setminus \{B,C\} \) be \( \frac{1}{n} \) deterministically in both distributions. Let the value distributions of \( A, B, \) and \( C \) be the same as the construction in Theorem 3.1, scaled by a \( \frac{1}{n} \) factor so that they are at most \( \frac{1}{n} \) as claimed earlier.

Consider a feasible allocation that allocates to all bidders in \( S_\gamma \). If it is \( S_\gamma \cup S_A \), the virtual welfare equals:

\[
m + \frac{\ell - 2}{n} + \varphi_A(v_A).
\]

If it is \( S_\gamma \cup S_{BC} \), the virtual welfare equals:

\[
m + \frac{\ell - 2}{n} + \varphi_B(v_B) + \varphi_C(v_C).
\]

If it involves bidders from both \( S_A \) and \( S_{BC} \), by Lemma 3.4 the virtual welfare is at most:

\[
m + \frac{\ell - 2}{n}.
\]

Therefore, Myerson’s optimal auction allocates to either \( S_\gamma \cup S_A \) or \( S_\gamma \cup S_{BC} \) at all time, and the contributions to the virtual welfare from bidders other than \( A, B, \) and \( C \) are constant. The rest reduces to the aforementioned example and Theorem 3.1. \( \square \)

### 3.2 Approximate Strong Revenue Monotonicity

In the above example, distribution \( D \) not only stochastically dominates \( \tilde{D} \) but is also much bigger. We next show that if \( D \) is instead \( \epsilon \)-close to \( \tilde{D} \) then we recover an approximate version of strong revenue monotonicity.

**Theorem 3.5.** For any \( 0 < \epsilon \leq 1 \), if product value distributions \( D \) and \( \tilde{D} \) satisfy \( D \succeq \tilde{D} \) and \( D \approx_\epsilon \tilde{D} \), then we have:

\[ M_D(D) \geq M_{\tilde{D}}(\tilde{D}) - \epsilon. \]

If instead \( D \approx_\epsilon \tilde{D} \), then we have:

\[ M_D(D) \geq M_{\tilde{D}}(\tilde{D}) - \sqrt{\frac{n}{k}} \cdot \epsilon. \]

To prove Theorem 3.5 we need to analyze \( M_{\tilde{D}}(D) \). Recall that Myerson’s optimal auction \( M_D \) w.r.t. \( \tilde{D} \) chooses an allocation based on the virtual values w.r.t. \( \tilde{D} \). When it allocates to a bidder \( i \) who has quantile \( q_i \) and thus value \( v_{\tilde{D}_i}(q_i) \), it expects a contribution of \( \varphi_{\tilde{D}_i}(v_{\tilde{D}_i}(q_i)) \) to the expected revenue. The actual contribution, however, depends on the virtual values w.r.t. \( D \), and therefore is \( \varphi_{D_i}(v_{D_i}(q_i)) \). The next lemma upper bounds how much auction \( M_D \) might have overestimated a bidder \( i \)'s contribution to the expected revenue because of the aforementioned mismatch. Here recall that if a bidder \( i \) gets allocation 1 at some quantile \( \theta_i \) then it also gets this allocation for any quantile less than \( \theta_i \), and that \( \int_0^{\theta_i} \varphi_{D_i}(v_{D_i}(q_i)) dq_i = R_{D_i}(\theta_i) \).
LEMMA 3.6. Suppose that product value distributions $D$ and $\hat{D}$ satisfy (1) $D \succeq \hat{D}$, (2) $D \approx \varepsilon \hat{D}$ or $D \approx_{\text{unif.}} \hat{D}$, and (3) $\hat{D}$ is regular. For any bidder $1 \leq i \leq n$ and any threshold quantile $0 \leq \theta_i \leq 1$ such that $\varphi_{\hat{D}_i}(v_{\hat{D}_i}(\theta_i)) \geq 0$, we have:

$$\int_0^{\theta_i} \varphi_{\hat{D}_i}(v_{\hat{D}_i}(q)) dq \leq R_{\hat{D}_i}(\theta_i) + \alpha(\theta_i),$$

where:

$$\alpha(\theta_i) = \begin{cases} \sqrt{\frac{\theta_i \varepsilon^2}{4nk}} + \frac{\varepsilon^2}{2nk} & \text{if } D \approx \varepsilon \hat{D}; \\ \frac{\varepsilon}{\sqrt{n}k} & \text{if } D \approx_{\text{unif.}} \hat{D}. \end{cases}$$

PROOF. We will use the following lemma to compare the integrals of a nondecreasing function w.r.t. two different distributions. Its proof is deferred to the full version [Chen et al., 2022].

LEMMA 3.7. Let $f : [0,1] \rightarrow [0,1]$ be a nondecreasing function with $f(0) = 0$. Let $P$ and $\hat{P}$ be distributions over $[0,1]$ such that $P \succeq \hat{P}$ and $P(v) - \hat{P}(v) \leq \alpha(1 - \hat{P}(v))$, where $\alpha : [0,1] \rightarrow \mathbb{R}$ is nondecreasing. For any $0 \leq \theta \leq 1$ we have:

$$\int_0^{\theta} f(v_\hat{P}(q)) dq \leq \int_0^{\theta} f(v_P(q)) dq + \alpha(\theta).$$

If follows from the definitions of $\varepsilon$-closeness and $\varepsilon$-uniform closeness that for any $0 \leq \theta_i \leq 1$:

$$\hat{D}_i(v_i) - D_i(v_i) \leq \alpha(1 - \hat{D}_i(v_i)).$$

By Lemma 3.7 with $f(q) = \max \left\{ \varphi_{\hat{D}_i}(v_{\hat{D}_i}(q)), 0 \right\}$, $P = D_i$, $\hat{P} = \hat{D}_i$, $\theta = \theta_i$:

$$\int_0^{\theta_i} \varphi_{\hat{D}_i}(v_{\hat{D}_i}(q)) dq \leq \int_0^{\theta_i} \max \left\{ \varphi_{\hat{D}_i}(v_{\hat{D}_i}(q)), 0 \right\} dq + \alpha(\theta_i).$$

It remains to prove that:

$$\int_0^{\theta_i} \max \left\{ \varphi_{\hat{D}_i}(v_{\hat{D}_i}(q)), 0 \right\} dq \leq R_{\hat{D}_i}(\theta_i).$$

If we have $\varphi_{\hat{D}_i}(v_{\hat{D}_i}(\theta_i)) \geq 0$, the left-hand-side is simply $R_{\hat{D}_i}(\theta_i)$, which is at most $R_{D_i}(\theta_i)$ because $D \succeq \hat{D}$. Otherwise, let $\nu^*_i$ be the monopoly price w.r.t. $\hat{D}_i$, i.e., $\nu^*_i \in \operatorname{arg \max}_p \mathbb{P} \cdot \Pr_{v \sim \hat{D}_i} [v \geq p]$. Further let $q^*_i = \Pr_{\nu \sim D_i} [\nu \geq \nu^*_i]$. Recall that $\hat{D}_i$ is regular. The left-hand-side above equals $\nu^*_i q^*_i$ because it integrates the derivative of a concave revenue curve $R_{\hat{D}_i}(q)$ past its peak, rounding negative derivatives up to 0. On the one hand, by $\varphi_{\hat{D}_i}(v_{\hat{D}_i}(\theta_i)) < 0$ we have that $q^*_i \leq \theta_i$. On the other hand, by the lemma assumption that $\varphi_{\hat{D}_i}(v_{\hat{D}_i}(\theta_i)) \geq 0$, we get that $\nu^*_i \leq v_{\hat{D}_i}(\theta_i)$. Putting together, $q^*_i \nu^*_i$ is at most $\theta_i \nu_{\hat{D}_i}(\theta_i) \leq R_{\hat{D}_i}(\theta_i)$ (it holds with equality if $v_{\hat{D}_i}(\theta_i)$ is not a point mass). \qed

Theorem 3.5 now follows by applying Lemma 3.6 to all bidders and by using the Cauchy-Schwarz Inequality to bound the square-roots of the threshold quantiles $\theta_i$’s, as we shall explain next.

PROOF OF THEOREM 3.5. Let $\hat{x}$ denote the allocation rule of $M_{\hat{D}}$, Myerson’s optimal auction w.r.t. $\hat{D}$. By definition $\hat{x}$ allocates according to the bidders’ ironed virtual values. Further recall that we may interpret ironing as rounding each bidder $i$’s value down to the closest value on the convex hull of revenue curve $R_{\hat{D}_i}$, as explained in the Section 2. Hence, $\hat{x}$ effectively allocates according to the virtual values w.r.t. the distributions of the rounded values. By redefining both $D$ and $\hat{D}$ w.r.t. the rounded values, we may assume WLOG that $\hat{D}$ is regular and the corresponding virtual values $\varphi_{\hat{D}_i}$’s are nondecreasing and coincide with the ironed virtual values.
For example, suppose that for some bidder $i$, $\tilde{D}_i$ has a point mass of $1/5$ at value $1/2$ and is otherwise uniform over $[0, 1]$, as in the example from Section 2, and $D_i$ is a uniform distribution over $[1/2, 1]$. Then, we may instead consider a regular value distribution $\tilde{D}_i'$ with a point mass of $\sqrt{3}/5$ at value $1/2$ and has probability density $4/5$ in value intervals $[0, 1/2)$ and $(1+\sqrt{3})/4, 1]$. Myerson’s optimal auctions w.r.t. $\tilde{D}_i$ and $\tilde{D}_i'$ are the same. Further, its allocations and payments when applied to $D_i$ and $D_i'$ are identical for any given quantile of bidder $i$.

Next consider allocation $\tilde{x}$ when bidders’ values are drawn from $D$. For any bidder $i$, any quantile profile $q_{-i}$ of the other bidders, and any allocation level $0 \leq y \leq 1$, define $\theta_i(y, q_{-i})$ as the largest quantile below which bidder $i$ gets allocation at least $y$ when the other bidders’ values are $v_{D_{-i}}(q_{-i})$. Formally:

$$\theta_i(y, q_{-i}) = \sup \left\{ 0 \leq q_i \leq 1 : \tilde{x}_i(v_{D_i}(q_i), v_{D_{-i}}(q_{-i})) \geq y \right\}.$$  

On the one hand, we write the expected revenue on the left-hand-side of the theorem as:

$$M_{\tilde{D}}(D) = \sum_{i=1}^{n} \int_{[0,1]^n} \tilde{x}_i(v_D(q)) \varphi_{D_i}(v_{D_{-i}}(q_{-i})) dq$$

$$= \sum_{i=1}^{n} \int_{[0,1]^{n-1}} \int_{0}^{1} \theta_i(y, q_{-i}) \varphi_{D_i}(v_{D_{-i}}(q_{-i})) dq_{1} dy dq_{-i}$$

$$= \sum_{i=1}^{n} \int_{[0,1]^{n-1}} \int_{0}^{1} R_{D_{-i}}(\theta_i(y, q_{-i})) dy dq_{-i}. \quad (1)$$

Here, Eqn. (1) holds because the previous step may be interpreted as integrating $\varphi_{D_i}(v_{D_{-i}}(q_{-i}))$ over the area below the curve of $\tilde{x}_i(x_{D_{-i}}(q_{-i}), x_{D_{-i}}(q_{-i}))$ over the quantile space of $q_{-i}$, while its right-hand-side may be viewed as integrating over the area on the left of curve $\theta_i(y, q_{-i})$ which is the inverse of $\tilde{x}_i$. See Figure 4 for an illustration.
On the other hand, we bound the expected revenue on right-hand-side by:

\[ M_{\tilde{D}}(\tilde{D}) = \int_{[0,1]^n} \sum_{i=1}^{n} \xi_i(v_{\tilde{D}}(q)) \varphi_{\tilde{D}_i}(v_{\tilde{D}_i}(q_i)) dq \]

\[ \leq \int_{[0,1]^n} \sum_{i=1}^{n} \xi_i(v_{D}(q)) \varphi_{D_i}(v_{D_i}(q_i)) dq \]

\[ \leq \int_{[0,1]^n} \sum_{i=1}^{n} \xi_i(v_{D}(q)) \varphi_{D_i}(v_{D_i}(q_i)) dq \quad (D \geq \tilde{D}) \]

\[ = \sum_{i=1}^{n} \int_{[0,1]^{n-1}} \int_0^1 \int_0^1 \varphi_{D_i}(v_{D_i}(q_i)) dq_i dy d\tilde{q}_i . \quad (\text{Same as Eqn. (1)}) \]

By Lemma 3.6:

\[ \int_0^1 \varphi_{D_i}(v_{D_i}(q_i)) dq_i - R_{D_i}(\theta_i(y, q_{-i})) \leq \begin{cases} \frac{\sqrt{\theta_i(y, q_{-i})}}{4nk} + \frac{\varepsilon^2}{2nk} & \text{if } D \approx \varepsilon \tilde{D} ; \\ \frac{\varepsilon}{\sqrt{n}k} & \text{if } D \approx \varepsilon \tilde{D}. \end{cases} \]

Therefore, summing over \( i \) and integrating over \( q_{-i} \) and \( y \) we have:

\[ M_{\tilde{D}}(\tilde{D}) - M_{\tilde{D}}(D) \leq \begin{cases} \sum_{i=1}^{n} \int_{[0,1]^{n-1}} \int_0^1 \theta_i(y, q_{-i}) dy d\tilde{q}_{-i} & \text{if } D \approx \varepsilon \tilde{D} ; \\ \sqrt{n}k & \text{if } D \approx \varepsilon \tilde{D}. \end{cases} \]

It remains to finish proving the \( D \approx \varepsilon \tilde{D} \) case. By Cauchy-Schwarz:

\[ \sum_{i=1}^{n} \int_{[0,1]^{n-1}} \int_0^1 \theta_i(y, q_{-i}) dy d\tilde{q}_{-i} \leq \left( \sum_{i=1}^{n} \int_{[0,1]^{n-1}} \int_0^1 \theta_i(y, q_{-i}) dy d\tilde{q}_{-i} \right)^{\frac{1}{2}} \]

\[ = \left( \int_{[0,1]^n} \xi_i(v_{D}(q)) dq \right)^{\frac{1}{2}} \quad (\text{Same as Eqn. (1)}) \]

\[ \leq \sqrt{nk} . \quad (\text{Rank } k) \]

Further:

\[ \sum_{i=1}^{n} \int_{[0,1]^{n-1}} \int_0^1 \frac{\varepsilon^2}{2nk} dy d\tilde{q}_{-i} = \frac{\varepsilon^2}{2k} \leq \frac{\varepsilon}{2} . \]

Combining the bounds proves the theorem. \( \square \)

### 3.3 Strong Revenue Lipschitzness

We remark that it is possible to remove the stochastic dominance condition from Theorem 3.5 but we would need the two distributions to be closer than in that theorem. We refer to this property as **strong revenue Lipschitzness** of single-parameter auctions; the next two theorems make it precise.

**Theorem 3.8.** For some sufficiently small constant \( c > 0 \), and for any \( 0 < \varepsilon \leq 1 \) if product value distributions \( D \) and \( \tilde{D} \) satisfy \( D \approx \varepsilon \tilde{D} \) for:

\[ \varepsilon' = c \cdot \frac{\varepsilon}{\sqrt{k \log nk / \varepsilon}}, \]

then we have:

\[ M_{\tilde{D}}(D) \geq M_{\til{D}}(\til{D}) - \varepsilon \].
Proof. Define an auxiliary product value distribution \( \hat{D} \) such that for any bidder \( i \) and any value \( 0 \leq v_i \leq 1 \):

\[
\hat{D}_i(v_i) = D_i(v_i) - \sqrt{\frac{D_i(v_i)(1 - D_i(v_i))(\epsilon')^2}{4nk} - \frac{2(\epsilon')^2}{nk}} - \frac{2c^2e^2}{nk^2 \log \frac{nk}{\epsilon}}
\]

By this construction we get that \( \hat{D} \approx O(\epsilon') D \). Combining this with \( D \approx \epsilon' \tilde{D} \), we get \( \hat{D} \approx O(\epsilon') \tilde{D} \) which for a sufficiently small \( c \) implies:

\[ \hat{D} \approx \epsilon^2 \tilde{D} \]

By definition of \( \epsilon \)-close, \( D \approx \epsilon \tilde{D} \) implies:

\[
\hat{D}_i(v_i) \geq D_i(v_i) - \sqrt{\frac{D_i(v_i)(1 - D_i(v_i))(\epsilon')^2}{4nk} - \frac{2(\epsilon')^2}{nk}} = \hat{D}(v_i),
\]

which means \( \hat{D} \geq \tilde{D} \).

Therefore by Theorem 3.5 we have:

\[ M_{\hat{D}}(\hat{D}) \geq M_{\tilde{D}}(\tilde{D}) - \frac{\epsilon}{2} \]

Lemma 3.9 (c.f., Lemmas 1 and 11 of Guo et al. [2021]). For \( D \) and \( \hat{D} \) constructed above, and for any function \( f : [0, 1]^n \rightarrow [0, 1] \) we have:

\[ \left| f(D) - f(\hat{D}) \right| \leq \frac{\epsilon}{2k} . \]

Letting \( f = \frac{1}{k} M_{\tilde{D}} \) gives \( M_{\tilde{D}}(D) \geq M_{\tilde{D}}(\hat{D}) - \frac{\epsilon}{2} \). Combining it with the previous inequality proves the theorem. \(\square\)

Theorem 3.10. For any \( n \geq 1 \), any \( k \leq n \), and any \( 0 \leq \epsilon \leq 1 \), there is a single-parameter auction with \( n \) bidders and rank \( k \), and two product value distribution \( D \approx \epsilon \tilde{D} \) such that:

\[ \text{Opt}(D) \leq \text{Opt}(\tilde{D}) - \frac{\epsilon \sqrt{k}}{8} , \]

and as a corollary:

\[ M_{\tilde{D}}(D) \leq M_{\tilde{D}}(\tilde{D}) - \frac{\epsilon \sqrt{k}}{8} . \]

Theorem 3.10 implies two conceptual messages when we compare its bound with those in Theorems 3.5 and 3.8. First, the bound of strong revenue Lipschitzness given by Theorem 3.8 is tight up to a logarithmic factor. Second and more relevant to the theme of this paper, the closeness of value distributions on its own (strong revenue Lipschitzness) is strictly weaker than its combination with stochastic dominance (strong revenue monotonicity). Therefore, if one has to use an inaccurate prior to design the auction, it is more robust to employ an underestimation.

Proof. Consider an \( n \)-bidder rank-\( k \) auction as follows. Let the set of feasible allocations be:

\[ X = \left\{ (0, 0, \ldots, 0), \left( \frac{k}{n}, \frac{k}{n}, \ldots, \frac{k}{n} \right) \right\} . \]

In other words, we allocate either to none of the bidders, or to all of them each by a \( \frac{k}{n} \) amount. We next define the product value distributions \( D \) and \( \hat{D} \). The bidders’ values are binary, either 0 or 1, and are independently and identically distributed. In the first distribution \( D \), the value of
each bidder is 0 with probability $\frac{1}{2n}$ and 1 with probability $1 - \frac{1}{2n}$. In second distribution $\tilde{D}$, the value of each bidder is 0 with a slightly smaller probability $\frac{1}{2n} - \frac{\epsilon}{(4n\sqrt{k})}$, and is 1 otherwise. By our construction it is easy to verify that $D \approx \epsilon \tilde{D}$.

For both distributions a bidder’s virtual value is 1 when its value is 1, and is smaller than $-(n-1)$ when its value is 0. Therefore, Myerson’s optimal auctions w.r.t. these two distributions are the same: If all bidders have value 1, allocate to all bidders each by a $\frac{k}{n}$ amount, and each bidder pays $\frac{k}{n}$. Otherwise, allocate to none of them, and the bidders pay nothing.

Therefore, the optimal revenue w.r.t. the two distributions are:

$$\text{Opt}(D) = \left(\frac{k}{n}\right) \cdot n \cdot \left(1 - \frac{1}{2n}\right)^n = k\left(1 - \frac{1}{2n}\right)^n,$$

payment per bidder number of bidders probability that all values are 1

and similarly:

$$\text{Opt}(\tilde{D}) = \frac{k}{n} \cdot n \cdot \left(1 - \frac{1}{2n} + \frac{\epsilon}{4n\sqrt{k}}\right)^n = k\left(1 - \frac{1}{2n}\right)^n\left(1 + \frac{\epsilon}{2(2n-1)\sqrt{k}}\right)^n.$$

The difference is therefore:

$$k\left(1 - \frac{1}{2n}\right)^n\left(\left(1 + \frac{\epsilon}{2(2n-1)\sqrt{k}}\right)^n - 1\right) \geq k \cdot \frac{1}{2} \cdot \frac{\epsilon n}{2(2n-1)\sqrt{k}} \cdot (1 + x)^n \geq 1 + nx \text{ for } x > -1$$

$$\geq \frac{\epsilon \sqrt{k}}{8}. \quad \left(\frac{n}{2(2n-1)} \geq \frac{1}{2}\right)$$

\[\square\]

4 APPLICATIONS IN SAMPLE COMPLEXITY

In practice auctioneers do not have accurate knowledge of the bidders’ value distributions. Instead they often have various kinds of data such as the bidders’ bids in previous auctions. Cole and Roughgarden [2014] introduced the sample complexity model of single-parameter auction design, in which the auctioneer can only access the value distributions $D$ through i.i.d. samples from it. They asked how many samples are sufficient for learning an auction that is an (additive) $\epsilon$-approximation. Guo et al. [2019] used the strong revenue monotonicity of single-parameter auctions in the matroid setting to obtain sample complexity bounds that are tight up to logarithmic factors. This section shows that the approximate version of strong revenue monotonicity in the previous section is also sufficient for getting the same bound (with better logarithmic factors), but more generally in the downward-closed setting. For arbitrary single-parameter auctions, we resort to the strong revenue Lipschitzness and get a sample complexity upper bound with an additional $\tilde{O}(k)$ factor.

**Theorem 4.1 (Downward-closed Feasibility, Unit-demand).** Consider an arbitrary downward-closed single-parameter auction with $n$ bidders and rank $k$. Suppose that we have $N$ i.i.d. samples where $N$ is at least:

$$C \cdot \frac{nk}{\epsilon^2} \log \frac{nk}{\epsilon \delta}$$

for a sufficiently large constant $C$. Then Myerson’s optimal auction w.r.t. the dominated product empirical distribution $\tilde{E}$ is an $\epsilon$-approximation with probability at least $1 - \delta$.\(^6\)

---

\(^6\) Readers may wonder if Myerson’s optimal auction w.r.t. the product empirical distribution $E$, a.k.a., the Empirical Myerson Auction, also yields nearly optimal sample complexity. For $[0, 1]$-bounded values, Guo et al. [2021] gave an affirmative answer for the single-item setting, and we remark that our result for general (non-downward-closed) setting in Theorem 4.3 also holds for the Empirical Myerson Auction. For regular value distributions, however, the revenue of Empirical Myerson Auction also yields nearly optimal sample complexity. For $[0, 1]$-bounded values, Guo et al. [2021] gave an affirmative answer for the single-item setting.
We first establish that the optimal revenue satisfies a Lipschitzness-style property.

**Lemma 4.2 (Downward-closed, Lipschitzness of Optimal Revenue).** Consider an arbitrary downward-closed single-parameter auction. If we have both \( D \succeq \tilde{D} \) and \( D \approx_\varepsilon \tilde{D} \), then:

\[
\text{Opt}(\tilde{D}) \geq \text{Opt}(D) - \varepsilon.
\]

A similar property was shown by Guo et al. [2019] for the more restrictive matroid setting using an indirect information theoretic argument and revenue monotonicity. By comparison, our argument directly constructs an auction for \( \tilde{D} \), which essentially runs the optimal auction for \( D \) in the quantile space. The lower bound of this auction’s revenue then follows from a calculation similar to the proof of approximate revenue monotonicity in the previous section. This more direct argument reduces the logarithmic factors and applies more generally to the downward-closed setting. The detailed proof is deferred to the full version [Chen et al., 2022].

Readers may also notice that a weaker version of the lemma follows as a corollary of Theorem 3.8 and holds more generally for arbitrary single-parameter auctions, although we would need the two distributions to be closer than the stated lemma assumption.

Next we explain how our sample complexity bound (Theorem 4.1) follows from a simple combination of the Lipschitzness of optimal revenue (Lemma 4.2) and approximate strong revenue monotonicity (Theorem 3.5).

**Proof of Theorem 4.1.** With a sufficiently large \( N \) as assumed in the theorem, by Lemma 2.3 with probability at least \( 1 - \delta \) we have that \( D \succeq \tilde{E} \) and \( D \approx_\varepsilon \tilde{E} \). Therefore, the theorem follows by:

\[
M_{\tilde{E}}(D) \geq M_{\tilde{E}}(\tilde{E}) - \frac{\varepsilon}{2} \tag{Theorem 3.5}
\]

\[
= \text{Opt}(\tilde{E}) - \frac{\varepsilon}{2} \geq \text{Opt}(D) - \varepsilon. \tag{Lemma 4.2}
\]

\[\square\]

**Theorem 4.3 (General Feasibility, Unit-demand).** Consider an arbitrary single-parameter auction with \( n \) bidders and rank \( k \). Suppose that we have \( N \) i.i.d. samples where \( N \) is at least:

\[
C \cdot \frac{nk^2}{\varepsilon^2} \log \frac{nk}{\epsilon} \log \frac{nk}{\epsilon \delta}
\]

for a sufficiently large constant \( C \). Then Myerson’s optimal auction w.r.t. the dominated product empirical distribution \( \tilde{E} \) is an \( \varepsilon \)-approximation with probability at least \( 1 - \delta \).

**Proof.** We will prove a \( 2\varepsilon \)-approximation with the understanding that having the approximation factor does not change the sample complexity asymptotically. With the stated lower bound of \( N \), we get that \( D \approx_\varepsilon \tilde{E} \) for:

\[
e' = c \cdot \frac{\varepsilon}{\sqrt{k \log \frac{nk}{\epsilon}}}.
\]

Therefore we have:

\[
M_{\tilde{E}}(D) \geq M_{\tilde{E}}(\tilde{E}) - \varepsilon \tag{D \approx_\varepsilon \tilde{E} and Theorem 3.8}
\]

\[
\geq M_D(\tilde{E}) - \varepsilon \tag{optimality of M_{\tilde{E}}}
\]

\[
\geq M_D(D) - 2\varepsilon. \tag{D \approx_\varepsilon \tilde{E} and Theorem 3.8}
\]

Auction does not even converge to the optimal in general when the number of samples tends to infinity (c.f., Dhangwatnotai et al. [2015] and Guo et al. [2019]). It is an interesting open question whether the Empirical Myerson Auction achieves the sample complexity in Theorem 4.1 for matroid and downward-closed settings.
We next complement Theorem 4.3 with an almost matching lower bound, demonstrating that the sample complexity bound therein is tight up to logarithmic factors. This lower bound also shows a separation between the sample complexity of downward-closed and non-downward-closed single-parameter auctions.

**Theorem 4.4.** There is a constant $c$ such that for any number of bidders $n$, any rank $k \leq n$, and any $0 < \epsilon \leq \frac{1}{100}$, there is a set of feasible allocations for which any algorithm needs at least:

$$c \cdot \frac{nk^2}{\epsilon^2}$$

samples to learn an auction with an $\epsilon$-approximation in expectation, over the randomness of the algorithm, the samples, and the bidders’ valuations in the auction.

The proof of this theorem is deferred to the full version [Chen et al., 2022]. Below we sketch the main ideas. The lower bound uses the same set of feasible allocations as in Theorem 3.10: $X = \{(0, 0, \ldots, 0), (\frac{k}{n}, \frac{k}{n}, \ldots, \frac{k}{n})\}$.

Then, we design two possible distributions $D^+$ and $D^-$ with binary support such that (1) the Hellinger distance between them is sufficiently small so that one cannot distinguish them using only $c \cdot \frac{nk^2}{\epsilon^2}$ samples, (2) with at least a constant probability there is exactly one bidder with value 0, and (3) when this happens, the optimal auction’s allocation depends on whether this bidder’s value distribution is $D^+$ or $D^-$, and choosing the wrong allocation leads to a revenue loss of $\Omega(\epsilon)$. Putting together these properties yields the desired lower bound.

**Beyond Unit Demand.** Recall the reduction that we mentioned in Section 2: an $\epsilon$-approximation in an auction with $n$ bidders, rank $k$, and maximum demand $d$ is equivalent to an $\frac{\epsilon}{d}$-approximation in another auction with $n$ bidders, rank $\frac{k}{d}$, and unit-demand, obtained by scaling all allocations by a factor $\frac{1}{d}$. Using this reduction we get the following corollaries when bidders’ maximum demand $d$ can be larger than 1. Note that $d \leq k$ and thus $d$ is subsumed by $k$ inside the logarithmic factors.

**Corollary 4.5 (Downward-closed Feasibility, General Demand).** Consider an arbitrary downward-closed single-parameter auction with $n$ bidders, rank $k$, and maximum demand $d$ of any single bidder. Suppose that we have $N$ i.i.d. samples where $N$ is at least:

$$C \cdot \frac{nk}{\epsilon^2} \log \frac{nk}{\epsilon \delta}$$

for a sufficiently large constant $C$. Then Myerson’s optimal auction w.r.t. the dominated product empirical distribution $\tilde{E}$ is an $\epsilon$-approximation with probability at least $1 - \delta$.

**Corollary 4.6 (General Feasibility, General Demand).** Consider an arbitrary single-parameter auction with $n$ bidders, rank $k$, and maximum demand $d$ of any single bidder. Suppose that we have $N$ i.i.d. samples where $N$ is at least:

$$C \cdot \frac{nk^2}{\epsilon^2} \log \frac{nk}{\epsilon} \log \frac{nk}{\epsilon \delta}$$

for a sufficiently large constant $C$. Then Myerson’s optimal auction w.r.t. the dominated product empirical distribution $\tilde{E}$ is an $\epsilon$-approximation with probability at least $1 - \delta$.

Readers may notice that Corollary 4.6 gives the same sample complexity bound as Theorem 4.3, independent of the maximum demand $d$. This is not a typo but instead follows from the fact that $k$ and $\epsilon$ are homogeneous in the bound of Theorem 4.3. The reduction in Section 2 scales both $k$ and $\epsilon$ by a factor $\frac{1}{d}$ and therefore the two effects cancel out.
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