Nonholonomic Distributions and
Gauge Models of Einstein Gravity

Sergiu I. Vacaru*

Science Department, University "Al. I. Cuza" Iaşi,
54 Lascar Catargi street, 700107, Iaşi, Romania

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Abstract
For (2+2)–dimensional nonholonomic distributions, the physical
information contained into a spacetime (pseudo) Riemannian metric
can be encoded equivalently into new types of geometric structures
and linear connections constructed as nonholonomic deformations of
the Levi–Civita connection. Such deformations and induced geometric/physical objects are completely determined by a prescribed metric
tensor. Reformulation of the Einstein equations in nonholonomic
variables (tetrads and new connections, for instance, with constant co-
efficient curvatures and/or Yang–Mills like potentials) reveals hidden
geometric and rich quantum structures. It is shown how the Einstein
gravity theory can be re–defined equivalently as certain gauge models
on nonholonomic affine and/or de Sitter frame bundles. We speculate
on possible applications of the geometry of nonholonomic distributions
with associated nonlinear connections in classical and quantum gravity.

Keywords: nonholonomic manifolds, nonlinear connections, Ein-
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*sergiu.vacaru@uaic.ro, Sergiu.Vacaru@gmail.com;
http://www.scribd.com/people/view/1455460-sergiu

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1 Introduction

It is a well known fact that, in a certain sense, the Einstein theory of gravity, i.e. general relativity, can be viewed as a gauge model for the non-semisimple Poincaré/affine group and, in another sense, it can also be considered as a gauge theory of the group of spacetime translations, which are equivalent to arbitrary diffeomorphisms. Details of such and alternative gauge gravity constructions, relevant to this paper, are reviewed in [1, 2, 3, 4, 5, 6].

In the standard approach to general relativity, a (pseudo) Riemannian metric $g$ and the corresponding Levi-Civita connection $\nabla$ are taken as basic variables representing the gravitational field and its interactions. The Einstein equations were originally formulated for geometric objects defined

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1 there were published some tenth of monographs and thousands of articles on "gauge gravity models"; we are not able to discuss here a number of important ideas and results or to provide a comprehensive list of references
2 we use left "up/down" labels in order to emphasize that some geometric/physical objects are defined by the same fundamental geometric structures, see explanations in next section
by such standard \((g, gV)\) variables. Nevertheless, various purposes in classical gravity (for instance, description of gravitational interactions with spinor fields and computation of observable gravitational and light interaction effects) and different attempts to quantize Einstein gravity and formulate generalizations of gravity theory employed the tetrad/spinor fields and different connection forms as (new) fundamental variables, see discussions in Refs. [7] [8] [9].

The dynamics of gravitational fields for a four dimensional spacetime is defined by six independent components (from general ten ones) of a symmetric metric field. Working with different variables in classical and quantum models of Einstein gravity, we have to consider additional, in general, nonholonomic (equivalently, nonintegrable and/or anholonomic) constraints on fundamental field equations and elaborate geometric models in certain forms involving nonholonomic structures. The mathematical formalism for physical theories with constrained dynamics has its origins in the concept of nonholonomic manifold and geometrization of nonholonomic mechanics [10] [11] [10] [13], see a review of results and historical details in [14] [15].

Following methods of Finsler geometry and constructions adapted to nonholonomic distributions and nonlinear connections, it was developed an approach to geometrization of Lagrange and Hamilton mechanics and higher order generalizations [16] [17] [16].

In a series of works, we proved that nonholonomic variables and methods of nonlinear connection geometry (here we note that nonholonomic Lagrange–Finsler variables can be considered even in general relativity) are very effective for developing new geometric methods for constructing exact solutions in gravity theories, analysis of nonlinear solitonic gravitational and matter field interactions, elaborating noncommutative and gauge like generalizations etc, see reviews of results in [18] [19] [20] [21] [22] [19]. Using nonholonomic distributions on (pseudo) Riemannian manifolds (Lagrange–Finsler, or Hamilton–Cartan, ones and their almost Kähler structures), we showed how the Einstein gravity can be quantized [23] [24] [25] [26] [27] [28] following methods of Fedosov (deformation) quantization [29] [30] [31], loop quantum

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3There were elaborated various approaches to geometrization of nonholonomic mechanics and classical and quantum field theories. We cited some monographs related to metric compatible constructions, proposed by E. Cartan, and developed by R. Miron’s school on Lagrange–Hamilton and Finsler–Cartan geometries, and generalizations, following the nonlinear connection formalism.

4We emphasize that in this article we shall not work with more general classes of Lagrange–Finsler geometries elaborated in original form on tangent bundles [16] but apply the nonholonomic manifold geometric formalism in classical gravity and further developments to quantum gravity.
gravity and/or brane A–model approach (for our purposes, such models where considered to enabled with nonholonomic structures). Since the successful standard models in particle physics are related to gauge theories with Yang–Mills type potentials, it is reasonable to expect that there may be constructed a gauge like model of quantum gravity. The perturbation theory of gauge fields is also more familiar for phenomenological physicists. During almost fifty years, this prompted various re–examinations of general relativity from the gauge theoretical points of view and related geometric methods. All such approaches resulted in a general conclusion that if a viable gauge like theory of gravity can be elaborated, it will not be a usual Yang–Mills theory.

The bulk of models on gauge gravity were constructed as certain generalizations, or alternatives, to the Einstein gravity theories. Nevertheless, among thousands of works on gauge gravity, two papers by D. Popov and L. Dikhin play a more special role. They proved that the Yang–Mills equations for the so–called Cartan connection 1–form (in the bundle of affine frames) projected on the base spacetime manifold are just the Einstein equations for general relativity. Nevertheless, that gauge gravity theory (and different similar models, for instance, for the Poincaré gauge group) was considered ‘unphysical’ because the affine structure group is non–semisimple, i.e. with degenerated Killing form, which results in ‘nonvariational’ field equations in the total bundle. Perhaps, that was not a substantial problem because the Yang–Mills equations can be formally derived by ”pure” geometric methods using a formal dualization of 2–forms of curvature and/or by introducing an auxiliary bilinear form on the fiber space. Projecting such gauge gravity equations on the base, it is possible to obtain the same Einstein equations not depending on any auxiliary fiber values. Together with the fact that such constructions had not provided additional tools which could solve the problem of renormalizability of gravitational interactions, that motivated a number of researches to develop different types of generalized models of gauge gravity with nonmetricity and/or dynamical torsion and extensions of the affine structural group to the de Sitter and/or another ones, nonlinear gauge symmetries, Higgs like broken symmetries of gauge group etc. Various sophisticate mechanisms to extract, or to get in certain limits, the classical Einstein gravity were proposed.

Our approach to gauge gravity is different from the former ones. We attempt to construct a gauge like theory in certain bundle spaces enabled with nonholonomic distributions (for simplicity, defining some classes of nonholonomic frames with associated nonlinear connection structure) which will encode equivalently all geometric and physical data for the Einstein equa-
tions on (pseudo) Riemannian spaces. So, we are not going to generalize the Einstein theory but try to reformulate it in certain nonholonomic variables lifted, for instance on affine, or de Sitter bundles. Lifts of geometric/physical objects will be performed in such forms that a new geometric techniques will be possible to be applied in order to develop a formal renormalization scheme of gravitational interactions, see second partner work [33]. We shall use the nonlinear connection formalism and anholonomic frame method which we elaborated for certain gauge models with generalizations of the Einstein gravity to Lagrange–Finsler structures, higher order bundles and noncommutative gravity [34, 35, 36, 20]. Nevertheless, we emphasize that in this work the nonholonomic geometric techniques will be considered only for bundle spaces on (pseudo) Riemannian manifolds, in particular, on Einstein spacetimes.

Our purpose is twofold:

1. To prove that the Einstein theory of gravity can be alternatively formulated in terms of nonholonomic deformations of linear and non-linear connection structures uniquely defined by a fundamental metric structure. The main issue will be related to constructions defining nonholonomic frames and deformations of connections which result in constant coefficient curvatures encoding the information for Einstein spaces.

2. To show that having introduced a corresponding nonholonomic distribution the geometric and physical constructions with the ”standard” Levi–Civita connection split into two categories of classical objects (and quantum objects, see a partner paper [33]): The first ones are in terms of distinguished linear connections, which allows us to encode a part of gravitational data into terms of some constant matrix curvatures, like in [21], or (equivalently) in terms of almost Kähler/Lagrange–Finsler and/or other nonholonomic variables for (pseudo) Riemannian manifolds [19, 23, 25, 27, 28]. The second ones contain contributions of a ’distortion’ tensor which can be encoded into a formal nonholonomic gauge gravity model (we shall analyze two such examples)."
Let us now briefly describe the content of the present work. We outline the geometry of nonholonomic distributions on (pseudo) Riemannian manifolds (Sec. 2); the geometric and physical objects are adapted to an associated nonlinear connection nonholonomic spacetime splitting when various classes of metric compatible connections are defined by a metric structure. It is shown how the Einstein gravity theory can be reformulated equivalently in nonholonomic variables for different linear connection structures (Sec. 3). We construct two models of gauge gravity in nonholonomic affine/ de Sitter frame bundles, with Yang–Mills like equations which can be equivalently re–defined into the standard Einstein equations for the general relativity theory on the base spacetime (Sec. 4). We close with some concluding remarks (Sec. 5). In Appendix, we present some necessary local formulas.

2 Nonholonomic Distributions on (Pseudo) Riemannian Manifolds

In this section we outline some new geometric features of general relativity in order to fix the notations and provide a clear insight of nonholonomic variables. Our goal is to prove that the Levi–Civita connection can be decomposed conventionally into an auxiliary distinguished connection (d–connection) structure and a distorsion tensor. Such decompositions can be performed in a canonical form for some nonholonomic distributions defining $2 + 2$ spacetime decompositions.

2.1 Nonlinear connections and metrics

We consider a four dimensional (in general, nonholonomic) pseudo–Riemannian manifold (spacetime) $V$, endowed with a metric, $g$, for instance, of signature $(-,+,+,+)$, and the corresponding Levi–Civita connection, $g\nabla \equiv \{ i \Gamma^\alpha_{\beta\gamma} \}$, structures such that $g\nabla g = 0$ and torsion $iT = \{ i T^\alpha_{\beta\gamma} \}$ of $g\nabla$ vanishes, $iT = 0$.

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7 The complete system of field equations and constraints for an auxiliary d–connection and distorsion tensor fields will be constructed to be equivalent to the standard Einstein equations.

8 Definitions for different types of connections and torsions are given below, in this section. Here we also note that we follow our system of notations, see details in Refs. [19, 18, 28], when left "up/low" labels show that (for instance) a value $g\nabla = \nabla[ g]$ is completely defined by $g$. Any right indices are usual abstract, or coordinate, tensor ones. In this work, a pair $(V, N)$, where $N$ is a nonintegrable distribution on a (pseudo) Riemannian spacetime $V$ of enough smooth class, is called a nonholonomic manifold.
Local coordinates on $V$ are denoted in the form $u^\alpha = (x^i, y^a)$ (or, in brief, $u = (x, y)$) where indices of type $i, j, ... = 1, 2$ will be considered as formal horizontal/ holonomic ones ($h$–indices), labeling $h$–coordinates, and indices of type $a, b, ... = 3, 4$ will be considered as formal vertical/nonholonomic ones ($v$–indices), labeling $v$–coordinates. We shall also use ‘underlined’ indices $(\alpha = (i, a), \beta = (i, b), ...)$, for local coordinate bases $e_\alpha = \partial_\alpha = (\partial_i, \partial_a)$, equivalently $\partial/\partial u^\alpha = (\partial/\partial x^i, \partial/\partial y^a)$; for dual coordinate bases we shall write $\epsilon^\alpha = du^\alpha = (\epsilon^i = dx^i, \epsilon^a = dy^a)$. There will be considered primed indices $(\alpha' = (i', a'), \beta' = (j', b'), ...)$, with double primes etc, for other local abstract/coordinate bases, for instance, $e_{\alpha'} = (e_{i'}, e_{a'})$, $e^{\alpha'} = (e^{i'}, e^{a'})$ and $e_{\alpha''} = (e_{i''}, e_{a''}), e^{\alpha''} = (e^{i''}, e^{a''})$, where $i', i'' = 1, 2...$ and $a', a'' = 3, 4$.

On a manifold $V$, we can introduce any nonholonomic distribution $N = N_i^a(u)dx^i \otimes \partial_a$ defined by a set of coefficients $N_i^a(u) = N_i^a(x, y)^9$. If such coefficients state also a decomposition of the tangent bundle $TV$ into conventional horizontal ($h$), $hV$, and vertical ($v$), $vV$, subspaces as a Whitney sum

$$TV = hV \oplus vV,$$

we say that $N = \{N_i^a\}$ defines a nonlinear connection ($N$–connection) on $V$ and that such a nonholonomic (equivalently, anholonomic) space is a $N$–anholonomic manifold.

To a $N$–connection structure, we can associate a class of vielbein (frame) transforms which are linear on coefficients $N_i^a$.

$$e_\alpha^a = \begin{bmatrix} e_i^i(u) & N_i^a(u)e_b^a(u) \\ 0 & e_a^a(u) \end{bmatrix}, \quad e_a^\alpha = \begin{bmatrix} e_i^i(u) & -N_i^a(u)e_b^k(u) \\ 0 & e_a^a(u) \end{bmatrix},$$

where, in a particular case, $e_i^i = \delta_i^i$ and $e_a^a = \delta_a^a$, with $\delta_i^i$ and $\delta_a^a$ being the Kronecker symbols. Such transform define the so–called $N$–adapted frame and co–frame structures, respectively, $e_\alpha = e_\alpha^a\partial_a$ and $e_\alpha^\beta = e_\alpha^a du^\alpha$, when the $N$–elongated partial derivatives/ frames are

$$e_\alpha \triangleq \delta_\alpha = (\delta_i, \partial_a) \quad \equiv \quad \frac{\delta}{\delta u^\alpha} = \left( e_i = \frac{\delta}{\delta x^i} = \partial_i - N_i^a(u) \partial_a, \frac{\partial}{\partial y^a} \right)$$

\footnote{this is similar to the possibility to consider on a spacetime any frame and/or coordinate systems, or any 2+2 and 3+1 decompositions}
and the N–elongated differentials / co–frames,
\[
e^\beta \doteq \delta^\beta = (dx^i, dy^a)
\]
\[
\equiv \delta u^\alpha = (\delta x^i, e^a = \delta y^a = dy^a + N_i^a(u) dx^i).
\]

There are used both type of denotations \(e^\alpha \doteq \delta^\alpha\) and \(e^\beta \doteq \delta^\beta\) in order to preserve a relation to denotations from Refs. [19, 20, 16].

For (3), there are satisfied the anholonomy relations
\[
[e_\alpha, e_\beta] = e_\alpha e_\beta - e_\beta e_\alpha = w_{\alpha\beta}^\gamma(u) e_\gamma
\]
with nontrivial anholonomy coefficients \(w_{\alpha\beta}^\gamma(u)\) computed as
\[
w_{ji}^a = -w_{ij}^a = \Omega_{ij}^a, \ w_{ia}^b = -w_{ai}^b = \partial_a N_i^b,
\]
where \(\Omega_{ij}^a\) define the N–connection curvature coefficients
\[
\Omega_{ij}^a \doteq e^i (N_i^a) - e^j (N_j^a).
\]

On a N–anholonomic manifold \(V\), we can re–define all geometric objects in a form adapted to a splitting (4), with respect to bases (3) and (4) and theirs tensor products. In such cases, there are used the terms distinguished objects (in brief, d–objects) and, in particular cases, there are considered d–tensors, d–connections, d–metrics etc.

Any metric structure \(g\) on \(V\) can be parametrized in local coordinate form,
\[
g = g_{\alpha\beta} du^\alpha \otimes du^\beta
\]
with
\[
g_{\alpha\beta} = g_{\alpha\beta} = \begin{bmatrix} g_{ij} + N_i^a N_j^b h_{ab} & N_i^e h_{ae} \\ N_j^e h_{be} & h_{ab} \end{bmatrix},
\]
or, equivalently, in N–adapted form, as a d–metric,
\[
g = g_{\alpha\beta}(u) e^\alpha \otimes e^\beta = g_{ij}(u) e^i \otimes e^j + h_{ab}(u) e^a \otimes e^b.
\]

A metric, for instance, parametrized in the form (8) is generic off–diagonal if it can not be diagonalized by any coordinate transforms. The anholonomy coefficients (6) do not vanish for the a generic off–diagonal metric (8) (equivalently, d–metric (9)).

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10We adopt the convention that for the spaces provided with N–connection structure the geometrical objects are denoted by "boldfaced" symbols if it is necessary to distinguish such objects from similar ones for spaces without N–connection.
2.2 The set of metric compatible d–connections

A distinguished connection (in brief, d–connection) on a spacetime \( V \),

\[
\mathbf{D} = (^h D, \ ^v D) = \{ \Gamma^\alpha_{\beta\gamma} = (L^i_{jk}, L^a_{bk}; C^i_{jc}, C^a_{bc}) \}, \tag{10}
\]

is a linear connection preserving under parallel transports the distribution \( (1) \). In explicit form, the coefficients \( \Gamma^\gamma_{\alpha\beta}(u) = (D_{\alpha}\delta_{\beta})] \delta^\gamma \).

The operations of h- and v-covariant derivations, \(^h D_k = \{ L^i_{jk}, L^a_{bk} \} \) and \(^v D_c = \{ C^i_{jk}, C^a_{bc} \} \) from \( (10) \) are introduced as corresponding h- and v–parametrizations of \( (11) \),

\[
L^i_{jk} = (D_k \delta_j)\lceil d_i, L^a_{bk} = (D_k \partial_b)\lceil \delta^a, C^i_{jc} = (D_c \delta_j)\lceil d^i, C^a_{bc} = (D_c \partial_b)\lceil \delta_a, \text{ where } "\lceil" \text{ denotes the interior product defined by a metric } g = \{ g_{\alpha\beta} \} \text{ and its inverse } g^{-1} = \{ g^{\alpha\beta} \}.
\]

We can introduce the d–torsion and d–curvature of a d–connection 1–form \( \Gamma^\gamma_{\alpha\beta} = \Gamma^\gamma_{\alpha\beta} \epsilon^\beta \), respectively, following formulas

\[
\mathcal{T}^\alpha = \mathbf{D} \epsilon^\alpha = \delta \epsilon^\alpha + \Gamma^\gamma_{\beta\gamma} \land \epsilon^\beta \tag{12}
\]

and

\[
\mathcal{R}^\alpha_{\beta} = \mathbf{D} \Gamma^\alpha_{\beta} = \delta \Gamma^\alpha_{\beta} - \Gamma^\gamma_{\beta \gamma} \land \Gamma^\alpha_{\gamma}, \tag{13}
\]

were \( \"\land\" \) is the anti–symmetric product of differential forms. The component formulas for \( \mathcal{T}^\alpha \) and \( \mathcal{R}^\alpha_{\beta} \) are presented in Appendix, see formulas \( (A.1) \) and \( (A.3) \) and details in Refs. \[19, 18, 20, 16\].

By straightforward computations, one proves two important results (see \[37\] and references therein, for N–anholonomic manifolds with symmetric and nonsymmetric metrics, and \[16\], for vector bundles):

1. Kawaguchi’s metrization: To any fixed d–connection \( \circ \mathbf{D} \) we can associate a d–connection \( \mathbf{D} \) being compatible with a metric \( g \), when it is satisfied the condition \( \mathbf{D}_X g = 0 \), for any d–vector \( X = X^i e_i + X^a e_a \in TV \). The coefficients for d–connections are related by formulas

\[
L^i_{jk} = \circ L^i_{jk} + \frac{1}{2} g^{im} \circ D_k g_{mj}, \ L^a_{bk} = \circ L^a_{bk} + \frac{1}{2} g^{ac} \circ D_k g_{cb},
\]

\[
C^i_{jc} = \circ C^i_{jc} + \frac{1}{2} g^{im} \circ D_c g_{mj}, \ C^a_{bc} = \circ C^a_{bc} + \frac{1}{2} g^{ae} \circ D_c g_{eb}.
\]
2. Miron’s procedure: The set of d–connections \( \{D\} \) satisfying the conditions \( D_Xg = 0 \) for a given \( g \) is defined by formulas

\[
L^i_jk = \hat{L}^i_jk + \pm O^{\alpha i}_{\beta j} Y^\alpha_{m e j} \quad L^a_{b k} = \hat{L}^a_{b k} + \pm O^{\alpha a}_{\beta b} Y^d_{c k},
\]

\[
C^i_{j c} = \hat{C}^i_{j c} + \pm O^{\alpha i}_{\beta j} Y^\alpha_{m e c}, \quad C^a_{b c} = \hat{C}^a_{b c} + \pm O^{\alpha a}_{\beta b} Y^d_{e c},
\]

where \( \pm O^{ij} = \frac{1}{2} \left( \delta^i_j \delta^j_i \pm g_{i j} g^{i j} \right) \), \( \pm O^{ca} = \frac{1}{2} \left( \delta^c_a \delta^a_c \pm g_{c a} g^{c a} \right) \).

are the so–called the Obata operators; \( Y^m_{e j}, Y^k_{m c}, Y^d_{c k} \) and \( Y^d_{e c} \) are arbitrary d–tensor fields and \( \hat{\Gamma}^\alpha_{\beta \gamma} = \left( \hat{L}^i_{j k}, \hat{L}^a_{b k}, \hat{C}^i_{j c}, \hat{C}^a_{b c} \right) \), with

\[
\hat{L}^i_{j k} = \frac{1}{2} g^{i r} \left( \mathbf{e}_k g_{j r} + \mathbf{e}_j g_{k r} - \mathbf{e}_r g_{j k} \right),
\]

\[
\hat{L}^a_{b k} = \mathbf{e}_b (N^a_k) + \frac{1}{2} g^{a c} \left( \mathbf{e}_k g_{b c} - g_{b c} \mathbf{e}_b N^a_k - \mathbf{e}_d N^d_k \right),
\]

\[
\hat{C}^i_{j c} = \frac{1}{2} g^{i k} \mathbf{e}_c g_{j k}, \quad \hat{C}^a_{b c} = \frac{1}{2} g^{a d} \left( \mathbf{e}_c g_{b d} + \mathbf{e}_d g_{c d} - \mathbf{e}_b g_{d c} \right)
\]

is the canonical d–connection uniquely defined by the coefficients of d–metric \( g = [g_{i j}, g_{a b}] \) and N–connection \( N = \{N^a_i\} \) in order to satisfy the conditions \( D_Xg = 0 \) and \( \hat{T}^i_{j k} = 0 \) and \( \hat{T}^a_{b c} = 0 \) but with general nonzero values for \( \hat{T}^i_{j a}, \hat{T}^a_{j i} \) and \( \hat{T}^a_{b i} \), see formulas (A.1) computed for the case \( \Gamma^\alpha_{\beta \gamma} = \hat{\Gamma}^\gamma_{\alpha \beta} \).

d–Tensors \( Y^m_{e j}, Y^k_{m c}, Y^d_{c k} \) and \( Y^d_{e c} \) parametrize the set of metric compatible d–connections, with a metric \( g \), on a N–anholonomic manifold \( V \).

Having prescribed any values of such d–tensors (we can follow any geometric, or physical theoretical/experimental, arguments; for instance, we can take some zero, or non–zero, constants), we get a metric compatible d–connection \( ^gD \) (14) completely defined by a (pseudo) Riemannian metric \( g \) (equivalently, d–metric (9)) because formulas (16) and (15) depend only on coefficients of a d–metric and N–connection structures. Here we note that even the N–connection coefficients are contained in the coefficient form of formulas for d–connections the provided constructions hold true for any \( 2 + 2 \) splitting of a spacetime \( V \). Such constructions are coordinate free because there are considered nonholonomic distributions which do not depend on the type of local frame/coordinate systems and their transforms.

As a matter of principle, any (pseudo) Riemannian geometry can be alternatively described in terms of a canonical d–connection \( ^g\hat{D} \) (16) and/or any d–connection \( ^gD = ^g\hat{D} + ^gZ \) if the distortion d–tensor \( ^gZ \) is completely
defined by a metric $g$. In a metric compatible and N–adapted case, $^gZ$ is defined by torsion $^gT^\alpha$ of $^gD$ but this torsion is completely different from the case of the Riemann–Cartan, string or gauge gravity theories with additional field equations for an additional ‘physical’ torsion field. On N–anholonomic (pseudo) Riemannian manifolds, a canonical d–torsion $^\hat{g}T^\alpha$ (in more general cases, any d–torsion $^gT^\alpha$) is generated as a nonholonomic deformation effect, by anholonomic coefficients $^\hat{g}$, including curvature of N–connection $^{\check{g}}$, completely defined by certain generic off–diagonal coefficients with $N^a_i$ in a metric $g$.

Of course, all above presented geometric constructions can be equivalently redefined in terms of (standard, for (pseudo) Riemannian geometry) Levi–Civita connection $^g\nabla$. Nevertheless, for various purposes in deformation quantization, brane A–model quantization and nonholonomic loop quantization of gravity theories and solitonic hierarchies in Einstein and Lagrange–Finsler spaces, it is more convenient to work with nonholonomic variables and generalized connections (we considered, for instance, certain examples of $^gD$ for almost Kähler variables or d–connections with constant curvature d–connections), see details and discussions, respectively, in Refs. [23, 24, 25, 27, 26] and [21, 22].

### 2.3 D–connections with constant coefficient curvatures

Considering nonholonomic frames (vierbeins and, equivalently, tetrads) on $V$ of type $\epsilon^\alpha = \epsilon^{\alpha}_\omega du^\omega$ and $\epsilon^a = \epsilon^a_{\alpha'} e^{\alpha'}$, and their duals defined respectively by matrices $e^{\alpha}_\omega$ and $e^a_{\alpha'}$, we may represent any d–metric $g$ and some related classes of d–connections to be parametrized by constant coefficients with respect to certain N–adapted bases. We should use “boldface” values, for instance, $e^{\alpha}_\omega$ and $e^a_{\alpha'}$, if such transforms deform smoothly, nonholonomically, a N–connection 2+2 splitting into another similar one. For two d–metrics

$$g = g_{\alpha\beta} e^\alpha \otimes e^\beta \quad \text{and} \quad \hat{g} = \hat{g}_{\alpha'\beta'} \hat{e}^{\alpha'} \otimes \hat{e}^{\beta'},$$

we have $g = \hat{g}$ if and only if

$$g_{\alpha\beta} e^\alpha_{\alpha'} e^\beta_{\beta'} = \hat{g}_{\alpha'\beta'},$$

for some ’vierbeins’ $e^{\alpha}_{\alpha'}$, which can be defined algebraically for any given values $g_{\alpha\beta}$ and prescribed constant coefficients $\hat{g}_{\alpha'\beta'}$, with respect to bases

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11We can use any linear connection $^gD = ^gD + ^gZ$, which, in general, may be not adapted to the N–connection splitting $^{\check{g}}$ but subjected to the condition that a distortion tensor $^gZ$ is completely defined by metric structure.
of type \([1]\) elongated, respectively, by some \(N^\alpha_u(u)\) and \(\hat{N}_{i'}^\alpha(u')\) for \(u^{a'} = u^a(u^\alpha)\). In \(N\)–adapted form, we can parametrize \(e^\alpha_{\alpha'} = (e^i_{\alpha'}, e^a_{\alpha'})\) and write \([17]\) and \([18]\) as

\[
\begin{align*}
g &= g_{ij} e^i \otimes e^j + h_{ab} e^a \otimes e^b = \hat{g}_{ij'} e^{i'} \otimes e^{j'} + \hat{h}_{a'b'} \hat{e}^a \otimes \hat{e}^{b'}, \\
\hat{e}^{\alpha'} &= (e^i = dx^i, \hat{e}^{\alpha'} = dy^{a'} + \hat{N}^{a'}_i dx^i), \\
e^\alpha &= (e^i = dx^i, e^a = dy^a + N^a_i dx^i),
\end{align*}
\]

with \(g_{ij} e^i \otimes e^j = \hat{g}_{ij'} e^{i'} \otimes e^{j'}\) and \(h_{ab} e^a \otimes e^b = \hat{h}_{a'b'} \hat{e}^a \otimes \hat{e}^{b'}\) when it is convenient to take \([\hat{e}^{i'}, \hat{e}^{j'}] = 0\).

Introducing the coefficients \(\hat{g}_{\alpha'\beta'}\) and \(\hat{N}^a_{i'}\) (equivalently, \(g_{\alpha\beta}\) and \(N^a_i\)) into formulas \([16]\), we can compute the \(N\)–adapted coefficients of canonical \(d\)–connection \(\hat{\Gamma}^\gamma_{\alpha'\beta'}\) (equivalently, \(\hat{\Gamma}^\gamma_{\alpha\beta}\)). Here we note that having defined the frame transform coefficients \(e^\alpha_{\alpha'}\) from \([18]\), we can consider the corresponding transformation law for any \(d\)–connection 1–form:

\[
\hat{\Gamma}^\alpha_{\alpha'\beta'} \rightarrow \Gamma^\alpha_{\alpha'\beta'} = e^\alpha_{\alpha'} \hat{\Gamma}^\alpha_{\alpha'\beta'} e^{\beta'}_{\beta} + e^\alpha_{\gamma'} \hat{e}^{\gamma'}_{\beta'}. \tag{20}
\]

where \(\hat{e}' = \delta u^{a'} \hat{e}_{a'}\).

By straightforward computations (see details on proof of Proposition 2.1 in Ref. \([21]\)), using formulas \([16]\) for \(\hat{\Gamma}^\gamma_{\alpha'\beta'}\) we find that any (pseudo) Riemannian metric \(g = \hat{g}\) on \(V\) defines a set of metric compatible \(d\)–connections with constant coefficients of type

\[
\hat{\Gamma}^\gamma_{\alpha'\beta'} = \left(\begin{array}{c}
\hat{\tilde{L}}^\gamma_{a'\beta'} = 0, \hat{\bar{L}}^\gamma_{y'k'} = \text{const}, \hat{\tilde{C}}^\gamma_{a'\beta'} = 0, \hat{\bar{C}}^\gamma_{y'k'} = 0
\end{array}\right) \tag{21}
\]

with respect to \(N\)–adapted frames \([3]\) and \([1]\) for any \(\hat{N} = \{\hat{N}_{i'}^\alpha(x', y')\}\) being a nontrivial solution of the system of equations

\[
2 \hat{\tilde{L}}^\gamma_{a'\beta'} = \frac{\partial N^a_{i'}}{\partial y^{b'}} - \hat{h}^{a'\beta'} \hat{h}_{a'b'} \frac{\partial N^{a'}_{i'}}{\partial y^{c'}} \tag{22}
\]

for any nondegenerate constant–coefficients symmetric matrix \(\hat{h}_{a'b'}\) and its inverse \(\hat{h}^{a'\beta'}\).

Putting the coefficients \(\hat{\Gamma}^\gamma_{\alpha'\beta'}\) \([21]\) into formulas \([A.3]\), we obtain constant curvature coefficients if the conditions \([22]\) are satisfied:

\[
\begin{align*}
\hat{R}^\alpha_{\beta'\gamma'\delta'} &= (\hat{R}_{a'\beta'\gamma'\delta'} = 0, \hat{R}_{a'\beta'\gamma'} = \hat{L}^{a'}_{y'k'} \hat{L}^\gamma_{c'k'} - \hat{L}^{a'}_{y'k'} \hat{L}^\gamma_{c'k'} = \text{const}, \hat{\tilde{R}}^\gamma_{a'\beta'\gamma'} = 0, \hat{\bar{R}}_{a'\beta'\gamma'} = 0, \hat{\tilde{S}}_{a'\gamma'\delta'} = 0, \hat{\bar{S}}_{a'\gamma'\delta'} = 0), \tag{23}
\end{align*}
\]
From formula (20) for N–adapted frame transforms of d–connection, one follows that $$\hat{\Gamma}^{\gamma'}_{\alpha'\beta'} \rightarrow \hat{\Gamma}^{\gamma}_{\alpha\beta}(u)$$ and $$\hat{R}^{\alpha'\beta'\gamma'\delta'} \rightarrow \hat{R}^{\alpha\beta\gamma\delta}(u)$$ resulting in non–constant coefficients for such geometric objects defined with respect to bases $$e_\alpha$$ and $$e^\alpha$$.

Nevertheless, even the curvature d–tensor $$\hat{R}^{\alpha\beta\gamma\delta}(u)$$, in general, does not have constant coefficients, the corresponding scalar curvature (A.5) is constant both for $$\hat{\Gamma}^{\gamma'}_{\alpha'\beta'}$$ and $$\hat{\Gamma}^{\gamma}_{\alpha\beta}$$ if the conditions (22) were fixed with respect to a basis $$\hat{e}_{\mu'}$$ and $$\hat{e}^\mu$$. Really, using formulas (20) and (18) and introducing values (21) and (23) into (A.4), we get:

$$\hat{R} = \hat{g}^{\alpha'\beta'} \hat{R}_{\alpha'\beta'} = \hat{g}^{\alpha'\beta'} \hat{R}_{\alpha'\beta'} + \hat{h}^{\alpha'\beta'} \hat{S}_{\alpha'\beta'} = \hat{R} + \hat{S} = const$$

and

$$\overrightarrow{R} = \hat{g}^{\alpha\beta} \hat{R}_{\alpha\beta} = \overrightarrow{R} + \overrightarrow{S} = \hat{g}^{\alpha'\beta'} \hat{R}_{\alpha'\beta'} = \overrightarrow{R} = const.$$  

We conclude that using corresponding nonholonomic frame transform, for any metric $$\mathbf{g} = \hat{\mathbf{g}}$$ on $$\mathbf{V}$$, we can always construct a d–connection with constant scalar curvature and it is also possible to chose such nonholonomic constraints and N–adapted bases when the curvature and Ricci d–tensors have constant coefficients. In Ref. [33], we consider such constraints also for other curvature invariants which is important for elaborating a formal procedure of renormalization by using d–connections of type $$\hat{\Gamma}^{\gamma}_{\alpha'\beta'}$$ (the above considered d–tensors and d–connections are completely defined by a metric $$\mathbf{g}$$ but contain certain freedom in choosing necessary types of nonholonomic distributions/transforms/frames).

Finally, we emphasize that the above mentioned geometric constructions can not be performed for the Levi–Civita connection corresponding to a metric $$\mathbf{g} = \mathbf{g}$$ on $$\mathbf{V}$$, see details in next section.

### 3 Einstein Gravity in N–adapted Variables

The ‘standard’ formulation of classical Einstein theory and first approaches to quantum gravity were elaborated in variables ($$\mathbf{g}, \mathbf{g}^\nabla$$), where $$\mathbf{g}^\nabla = \nabla(\mathbf{g}) = \{ \mathbf{g}^{\Gamma_{\alpha\beta\gamma}} = \Gamma_{\alpha\beta\gamma}[\mathbf{g}] \}$$ is the Levi–Civita connection. This linear connection is completely constructed from the coefficients of a metric $$\mathbf{g} = \{g_{\mu\nu}\}$$, and their first derivatives, on a spacetime manifold $$\mathbf{V}$$ following the condition that $$\mathbf{g}^\nabla \mathbf{g} = 0$$ and $$\mathbf{g}^\nabla T^{\alpha}_{\beta\gamma} = 0$$, where $$\mathbf{g}^\nabla T$$ is the torsion of $$\mathbf{g}^\nabla$$.

In another turn, for different purposes in classical and quantum gravity, it should be noted that this connection is not adapted to the distribution (1) because it does not preserve under parallelism the h- and v–distribution.
gravity, there were introduced various types of tetrac, or spinor, variables
and 3 + 1 spacetime decompositions (for instance, in the so–called
Arnold–Deser–Misner, ADM, formalism, Ashtekar variables and loop quantum
gavity), or nonholonomic 2 + 2 splittings, see reviews of results, discussion
and references in [8] [9] [7] and [27] [28], on nonholonomic variables, and Figure 1.

For a (pseudo) Riemannian metric \( g \), we can construct an infinite num-
ber of linear connections \( gD \) which are metric compatible, \( gD g = 0 \), and
completely defined by coefficients \( g = \{g_{\mu\nu}\} \). Of course, in general, the
torsion \( gT = \varpi D[g] \) of a \( gD \) is not zero. Nevertheless, we can work
equivalently both with \( g\nabla \) and any \( gD \) if the distorsion tensor \( gZ = Z[g] \)
from the corresponding connection deformation,

\[
g\nabla = gD + gZ,
\]

is also completely defined by the metric structure \( g \) (in the metric compatible
cases, \( gZ \) is proportional to \( gT \)). So, any geometric construction/field
equations etc performed in terms of a connection \( gD \) can be rewritten
in terms of \( g\nabla \), and inversely. It is a matter of convenience for some
geometric constructions or physical models to use one of the connections
from an infinite set of metric compatible ones \( \{gD\} \) and \( g\nabla \).

Let us consider two explicit examples: In previous section, we defined
the so–called canonical d–connection \( \hat{\Gamma}_{\alpha\beta} \). The formula (A.6) from
Appendix allows us to compute the coefficients of the Levi–Civita connection
\( g\nabla = \{\hat{\Gamma}_{\alpha\beta}\} \) because the components of both \( \hat{\Gamma}_{\alpha\beta} \) and distorsion d–
tensor \( Z_{\alpha\beta} \) (A.7) depend on coefficients of d–metric and N–connection
(that why we put the left up label \( g \) and use boldface symbols). Similar
splitting formulas hold true for the constant coefficient d–connection \( \hat{\Gamma}_{\alpha'\beta'} \)
but we have to use \( g = \hat{g} \) and \( \hat{N} \), when the d–metric and N–connection
coefficients are recomputed following formulas (19) and (18).

It is possible to compute the distorsion d–tensor \( gZ \) for any metric
compatible d–connection \( gD \) for a splitting of linear connections of type
(A.1), \( g\nabla = gD + gZ \). This follows from formulae (14) and (A.6):

\[
gZ = gZ - OY,
\]

where \( OY \) denotes the terms proportional to Obata operators (15) and
arbitrary d–tensor fields \( Y = \{Y_{e_j}, Y_{m_k}, Y_{e_k}, Y_{e_d}\} \). This emphasizes that in
Riemann geometry a specific ‘gauge freedom’ exists with respect to choosing

\[\text{for a general linear connection, we do not use boldface symbols if such a geometric object is not adapted to a prescribed nonholonomic distribution}\]
a metric linear connection being adapted to a 2+2 splitting. Following Miron’s procedure, such a freedom is parametrized by the set of d–tensor fields $Y_{14}$.

Let us first consider the usual Einstein equations defined in terms of a Levi–Civita connection $\nabla$ on a spacetime $\mathbf{V}$:

$$R_{\alpha\beta} - \frac{1}{2} (\overrightarrow{R} + \Lambda) e_{\alpha\beta} = 8\pi G \overrightarrow{T}_{\alpha\beta},$$  \hspace{1cm} (26)

where $R_{\alpha\beta} = e_{\alpha}\gamma R_{\gamma\beta}$, $\overrightarrow{R}$ is the scalar curvature of $\nabla$, $T_{\alpha\beta}$ is the effective energy–momentum tensor, $\Lambda$ is the cosmological constant, $G$ is the Newton constant in the units when the light velocity $c = 1$, and the coefficients $e_{\alpha\beta}$ of tetradic decomposition $e_{\beta} = e^{\alpha}_{\beta} \partial/\partial u^{\alpha}$ are defined by the N–coefficients of the N–elongated operator of partial derivation, see (3).

Having chosen a metric compatible d–connection $D$, we compute the Ricci d–tensor $R_{\beta\gamma}$ and the scalar curvature $\overrightarrow{R}$, see formulas (A.4) and (A.5) when $\hat{D} \rightarrow D$. For any such d–connections, we can also postulate the (nonholonomic) filed equations

$$R_{\alpha\beta} - \frac{1}{2} (\overrightarrow{R} + \Lambda) e_{\alpha\beta} = 8\pi G \overrightarrow{T}_{\alpha\beta},$$  \hspace{1cm} (27)

but such equations (also defined by the metric structure) are not equivalent to the Einstein equations in general relativity if the tensor $\overrightarrow{T}_{\alpha\beta}$ does not include contributions of distorsion d–tensor $g_{Z}(25)$ in a necessary form. We can use nonholonomic gravitational equations (27) in various types of generalized gravity theories (like noncommutative and/or string gravity [20]). Such equations, with d–connections, and their solutions with additionally constrained integral varieties happen to be also very useful in constructing new classes of exact solutions in general relativity, parametrized by generic off–diagonal matrices [18, 19].

Introducing in (27) the absolute antisymmetric tensor $e_{\alpha\beta\gamma\delta}$ and the effective source 3–form

$$T_{\beta} = \overrightarrow{T}_{\alpha\beta} \epsilon_{\alpha\beta\gamma\delta} du^{\gamma} \wedge du^{\alpha} \wedge du^{\delta},$$

\footnote{As a matter of principle, we can also work with metric noncompatible d–connections and then to use the above mentioned Kawaguchi’s metrization, which results in a similar ‘gauge’ like freedom for d–connections. Nevertheless, to work directly with nonmetricity fields is not only a more difficult technical problem but it is not clear how to provide rigorous motivations following ‘scenarios’ of standard theories in physics, see more discussions in Refs. [15] [19].}
and expressing the curvature tensor $R^\tau_{\gamma\gamma} = R^\tau_{\gamma\alpha\beta} e^\alpha \wedge e^\beta$ of $\Gamma^\alpha_{\beta\gamma} = \Gamma^\alpha_{\beta\gamma} - Z^\alpha_{\beta\gamma}$ as $R^\tau_{\gamma\gamma} = R^\tau_{\gamma\alpha\beta} e^\alpha \wedge e^\beta$ is the curvature 2–form of the Levi–Civita connection $\nabla$ and the distorsion of curvature 2–form $Z^\tau_{\gamma\gamma}$ is defined by $Z^\alpha_{\beta\gamma}$ (25), we derive the equations (27) (varying the action on components of $e_\beta$, see details for $D = \hat{D}$ in Ref. [27]). The gravitational field equations written in terms of an arbitrary metric compatible d–connection $D$ can be represented as 3–form equations,

$$\epsilon_{\alpha\beta\gamma\tau} \left( e^\alpha \wedge R^\beta\gamma + \Lambda e^\alpha \wedge e^\beta \wedge e^\gamma \right) = 8\pi G T^\tau_{\tau}, \quad (28)$$

when

$$mT^\tau_{\tau} = mT^\alpha_{\alpha\beta\gamma} \epsilon_{\alpha\beta\gamma\delta} du^\beta \wedge du^\gamma \wedge du^\delta,$$

$$ZT^\tau_{\tau} = -(8\pi G)^{-1} Z^\alpha_{\alpha\beta\gamma} \epsilon_{\alpha\beta\gamma\delta} du^\beta \wedge du^\gamma \wedge du^\delta,$$

where $mT^\alpha_{\alpha\beta\gamma}$ is the matter tensor field. It should be noted here that the equations (28) are equivalent to the standard Einstein equations (26) for the Levi–Civita connection.

We conclude that all geometric and classical physical information in Einstein gravity formulated for data $g, g \nabla$, can be transformed equivalently into geometric objects and constructions for any nonholonomic variables $g, gD$, from an infinite set of metric compatible d–connections defined by the same metric $g$. A formal scheme for general relativity sketching a "dictionary" between two equivalent geometric "languages" in general relativity (with a Levi–Civita and a metric compatible d–connection ones) is presented in Figure 1.

### 4 N–adapted Gauge Models of Einstein Gravity

There are known gauge gravity models when an equivalent re-definition of the Einstein equations in a Yang–Mills like form implies the non–semisimplicity of the gauge group. Such constructions were performed for nonvariational theories on the total space of the bundle of affine frames (to this class one belong the Poincaré and affine group gauge gravity theories) [1, 2]. For more complete expositions concerning generalizations of the Einstein gravity theory to gauge models for the de Sitter gauge group and nonlinear realizations, with metric–affine spaces, locally anisotropic structures, non–commutative gravity etc, see, e.g., [3, 4, 5, 6, 34, 35, 36] and references therein.

In our works [34, 35, 36], we followed a geometric calculus similar to that developed in Refs. [1, 2, 3] but generalized for different types of (higher
(pseudo) Riemannian metric $g$; tetrads $e^\alpha = e^\alpha_\alpha (u) du^\alpha$.

Arbitrary frames/coordinates, and/or $3+1$ splitting, spinor variables

any $2+2$ nonholonomic splitting, deformations, transform etc

metric compatible $d$–connections defined by $g$

Levi–Civita connection $\mathfrak{g}\nabla$;

$\mathfrak{g}\nabla g = 0$, $\mathfrak{g}\Gamma^\gamma_{\alpha\beta} = 0$,

Levi–Civita variables:

$(\mathfrak{g}, \mathfrak{g}\nabla) = (g_{\mu\nu}, \mathfrak{g}\Gamma^\gamma_{\mu\nu})$

nonholonomic variables:

$(\mathfrak{g}g, \mathfrak{g}D) = (g_{\alpha\beta}, \mathfrak{g}\Gamma^\gamma_{\alpha\beta})$

Classical Einstein spaces

Gauge models

Figure 1: Nonholonomic Variables in Gravity
order) locally anisotropic, supersymmetric, or noncommutative generalizations of gravity. That allowed us to construct certain classes of nonholonomic gauge gravity models, when projections of the generalized Yang–Mills equations, on (pseudo) Riemannian spacetimes, resulted in standard Einstein equations for general relativity and/or field equations for (non) commutative/super Lagrange–Finsler models of gravity.

The goal of this section is to prove that using splitting of connections $\mathfrak{g} \nabla = \mathfrak{g} \mathcal{D} + \mathfrak{g} \mathcal{Z}$, see formulas (24) and (25), the Einstein gravity theory can be encoded equivalently into some nonholonomic gauge gravitational theories when the Yang–Mills equations are defined by gravitational equations (28), but projections on a nonholonomic base result in vacuum gravitational equations (27). Such models provide a new perspective for renormalization perturbative schemes for the Einstein gravity following a two connections formalism (when both connections are defined by the same metric structure with respect to a prescribed nonholonomic distribution) [33].

4.1 Affine gauge distributions induced by Einstein metrics

In our approach, we follow a global geometric calculus elaborated in Ref. [38]. For Yang–Mills gauge fields on bundles spaces and related models of gauge gravity, the formalism was developed in Ref. [1, 2]. In nonholonomic forms, for nonholonomic gauge models of Lagrange and Finsler anisotropic gravity and higher order generalizations, on (higher order) vector/tangent bundles enabled with $N$–connection structure, such computations were performed both in global and component forms in Refs. [34, 35], see also reviews of results, on gauge $d$–groups and $d$–spinors, in [19, 20, 39, 40]. We shall omit details on computations, which on $N$–anholonomic manifolds are very similar to those on nonholonomic vector/tangent bundles, presenting some most important results necessary for our further considerations in quantum gravity [33].

4.1.1 Bundle of $N$–adapted linear frames

Let $e_\alpha = (e_i, e_a)$ be a $N$–adapted frame at a point $u = (x, y) \in V$. We can adapt the frame transform to a 2+2 splitting defined by a $N$–connection $N = \{N_i^a\}$ if there are considered local right distinguished

\[15\text{We note that coordinates are defined for local carts on an atlas covering } V \text{ and bundle spaces on such } N\text{–anholonomic manifolds; for simplicity, we shall omit labels pointing that certain geometric objects are defined on an open region/local cart } U \subset V \text{ and write, for instance, instead of } \mathfrak{u}e_\alpha \text{ and } \mathfrak{u}N, \text{ only } e_\alpha \text{ and } N \text{ etc.}\]
actions \( e_{\alpha'} = e_{\alpha'}^\alpha e_\alpha \) with matrices of type

\[
e_{\alpha'}^\alpha(u) = \begin{pmatrix} e_i^j(u) & 0 \\ 0 & e_{\alpha'}^\alpha(u) \end{pmatrix} \subset GL_{2+2} = GL(2, \mathbb{R}) \oplus GL(2, \mathbb{R}),
\]

when \( e_i^j = e_i^j e_i \) and \( e_{\alpha'} = e_{\alpha'}^\alpha e_\alpha \), for \( e_{\alpha'} = (e_i^j, e_{\alpha'}) \). We denote by \( ^N\mathcal{L}_a(V) = (^N\mathcal{L}_a(V), GL_{2+2}, V) \) the bundle of linear adapted frames on \( V \) defined as the principal bundle with a surjective map \( ^N\pi : ^N\mathcal{L}_a(V) \to V \) transforming any adapted frame \( e_{\alpha'} \) in a point \( u \) on base spacetime. The nonholonomic total space \( ^N\mathcal{L}_a(V) \) (with nonholonomic structure induced by \( N \) on \( V \)) is constructed as the set of adapted frames \( e_{\alpha'} \) in all points of base \( N \)-anholonomic manifold \( V \) enabled with a d–metric structure \([9]\).

The structural (linear distinguished) group \( GL_{2+2} \) (in brief, d–group) acts on a typical fiber space. We denote by \( \theta_{\alpha} = (\theta_i, \theta_\alpha) \) a basis in the Lie algebra \( ^N\mathcal{L}_{2+2} \) of d–group \( GL_{2+2} \) (N–adapting, we get a structural Lie d–algebra) satisfying the commutation relations of type

\[
\left[ \theta_{\alpha}, \theta_{\beta} \right] = f_{\alpha\beta}^\gamma \theta_\gamma, \text{ distinguished as } \left[ \theta_i, \theta_j \right] = f_{ij}^k \theta_k, \left[ \theta_{\alpha}, \theta_\beta \right] = f_{\alpha\beta}^\epsilon \theta_\epsilon,
\]

with structural constants \( f_{\alpha\beta}^\gamma = (f^k_{ij}, f^\epsilon_{\alpha\beta}) \), when conventional h– and v–splitting have to be introduced in order to perform N–adapted geometric constructions.\footnote{It is not obligatory to split the fiber and total space constructions even a base is a N–anholonomic manifold. Nevertheless, to elaborate gauge like models of Einstein gravity, we have to consider pull–backs of geometric objects from the base spacetime manifold which positively results in d–objects with distinguishing of Lie group/algebras indices.}

The Killing form (fiber metric) is defined

\[
K(\theta_{\alpha}, \theta_{\beta}) = K_{\alpha\beta} = f_{\alpha\beta}^\gamma f_{\gamma}^\epsilon f_{\epsilon}^\rho f_{\rho}^\sigma f_{\sigma}^\epsilon f_{\epsilon}^\rho f_{\rho}^\sigma f_{\sigma}^\tau,
\]

\[(29)\]

For semisimple Lie groups, this metric is not degenerate which allows us to define d–metrics on corresponding principal bundles induced by a d–metric \( g \) \([9]\) and Killing forms \([29]\), \( k = g + K \).\footnote{Bundle \( ^N\mathcal{L}_a(V) = (^N\mathcal{L}_a(V), GL_{2+2}, V) \) is associated to \( (P, ^N\pi, V) \).} Using the d–metrics \( g \) on \( V \) and \( k \) on \( ^N\mathcal{L}_a(V) \), we can introduce two operators \( k_* \) and \( k_\delta \), acting on the space of differential \( r \)-forms \( \Lambda^r(V) \) on \( V \). Such operators are corresponding horizontal projections on the base manifold of the total space operators \( k_* \) and \( k_\delta \), acting on the space of differential forms \( \Lambda^r(^N\mathcal{L}_a(V)) \) on \( ^N\mathcal{L}_a(V) \), for \( r = 1, 2, 3 \), if \( \dim V = 4 \).

In a simple form, for instance, the operators acting on a base are defined by their actions on an orthonormalized basis \( e_{\alpha''} = (e_i^r, e_{\alpha''}) \), constructed
by a $N$–adapted transform $e_{\alpha} = \left( e_{\alpha}, e_{\alpha}' \right)$, where $g = \sum_{\mu''=1}^{4} \eta(\mu'') e_{\mu''} \otimes e_{\mu''}$, where $\eta(1) = -1, \eta(2) = 1, \eta(3) = 1, \eta(4) = 1$. For $g_* : \Lambda^r(V) \rightarrow \Lambda^{4-r}(V)$, we have

$$ g_* : \left( e_{\mu''} \wedge \ldots \wedge e_{\mu''} \right) \rightarrow - \text{sign} \left( \begin{array}{cccccc}
1 & 2 & \ldots & r & r+1 & \ldots & 4 \\
\mu''_1 & \mu''_2 & \ldots & \mu''_r & \nu''_1 & \ldots & \nu''_{4-r} 
\end{array} \right) \times e_{\nu''} \wedge \ldots \wedge e_{\nu''_{4-r}} $$

and the adjoint to absolute differential operator $d$ (acting on differential form), associated to the scalar product for forms, specified for $r$–forms, is

$$ g_\delta = (-1)^r \ g_*^{-1} \ cd \circ \ g_* , $$

for $g_*^{-1} = - (1)^{(4-r)} g_*$ and 'o' denoting supperpositions of operators. Introducing corresponding canonical orthonormalized $N$–adapted bases on $N La(V)$, we can define in a similar form the actions of operators $k_*$ and $k_\delta$, see details in [1 34].

We induce a $d$–connection 1–form $\omega$ on $N La(V)$ by a pull–back of a metric compatible $d$–connection $D$ [10] on $V$,

$$ \omega^{\tilde{\alpha}} = \left\{ \omega^{\alpha'' \beta''} \right\} \Gamma^{\alpha'' \beta''}_{\gamma} = e^{\alpha''}_{\alpha} \Gamma^{\beta}_{\beta \gamma} e^{\beta}_{\beta} + e^{\alpha''}_{\gamma} \left( e_{\beta''} \right), $$

with a formal splitting of Lie $d$–algebra indices as $\tilde{\alpha} \rightarrow (\alpha'' \beta'')$. If the coefficients $\Gamma^{\alpha}_{\beta \gamma}$ are related to a nonholonomic deformation of a Levi–Civita connection $\Gamma^{\alpha}_{\beta \gamma}$, following formula [24], we get a nonholonomic lift of $g \nabla$ on $V$ into a $\omega D = \{ \omega^{\tilde{\alpha}} \}$ on $N La(V)$. The curvature of $d$–connection $\omega$ is computed

$$ \omega R = d \omega + \omega \wedge \omega = g_{\alpha'' \beta''} \omega^{\alpha'' \beta''} \wedge e_{\mu''} \wedge e_{\nu''}, $$

with the coefficients $\omega R^{\alpha'' \beta''}_{\mu \nu} = R^{\alpha''}_{\beta'' \mu \nu} = e^{\alpha''}_{\alpha} \Gamma^{\beta}_{\beta \nu} e^{\beta}_{\beta} \wedge e_{\gamma} \wedge e_{\gamma'}$ are computed following formulas [33] and $g_{\alpha'' \beta''} = \left( \begin{array}{cc}
\rho_{\nu'' \nu''} & 0 \\
0 & \rho_{\nu'' \nu''} \nu'' 
\end{array} \right)$ being the standard basis for the Lie $d$–algebra of matrices $gl_{2+2}$, when $(\rho_{\nu'' \nu''})_{k''q''} = \delta_{\nu'' k''} \delta_{q'' q''}$ and $(\rho_{\nu'' \nu''})_{c'' q''} = \delta_{c'' c''} \delta_{q'' q''}$. For the 1–form [30], we can also compute another 1–form on $N La(V)$,

$$ \triangle \ \omega R = k_\delta \omega R + k_*^{-1} \left[ \omega, k_* \omega R \right]. $$
where \([\omega, k \ast \omega] = \omega \wedge k \ast \omega - k \ast \omega \wedge \omega\), and verify that \(d \omega \mathcal{R} + [\omega, \omega \mathcal{R}] = 0\).

One defines the operator \(\triangle = \hat{H} \circ k \hat{\delta}\), where \(\hat{H}\) is the operator of horizontal projection on base.\(^{18}\) Locally, the 1–form \(\triangle \omega \mathcal{R}\) is computed
\[
\triangle \omega \mathcal{R} = \kappa_\alpha'' \tau'' \dot{\tau}'' \otimes g^{\mu \lambda} (D_\lambda \omega \mathcal{R}^{\alpha'' \tau'' \mu \nu} + f^{\alpha'' \tau'' \beta'' \delta'' \gamma'' \varepsilon'' \omega^\beta'' \delta'' \gamma'' \varepsilon'' \omega^\mu \nu}).
\]

We get that the 1–form \(\triangle \omega \mathcal{R}\) vanishes for a standard curvature of a gauge \(GL\)–field. The last two equations can be written in abstract form
\[
\begin{align*}
\omega D \omega \mathcal{R} &= 0, \\
\triangle \omega \mathcal{R} &= 0,
\end{align*}
\]
where the structure equations \(33\) are just the Biachi identities and the field equations \(34\) are just the Yang–Mills equations for the structure (gauge) d–group \(GL_{2+2}\).\(^{19}\)

If \(\omega \mathcal{R}\) is defined by \(g^{\alpha \beta \mu \nu}\) determined by a (pseudo) Riemannian metric on \(\mathcal{V}\), we get a 1–form \(\triangle \omega \mathcal{R}\) on \(NLa(\mathcal{V})\) induced nonholonomically by the d–metric structure on the base N–anholonomic manifold. Nevertheless, by lifts and nonholonomic deformations on the bundle of linear N–adapted frames \(NLa(\mathcal{V})\), we are still not able to generate certain types of Yang–Mills equations which would be equivalent to the Einstein equations on \(\mathcal{V}\). We have to extend the constructions to the bundle of affine N–adapted frames.

### 4.1.2 Bundle of N–adapted affine frames

There is another natural (in our case, nonholonomic) bundle on a space-time \(\mathcal{V}\), extending \(NLa(\mathcal{V})\), called the bundle of N–adapted affine frames \(N\mathcal{Aa}(\mathcal{V}) = (N\mathcal{Aa}(\mathcal{V}), Af_{2+2}, \mathcal{V})\) with the structure affine d–group \(Af_{2+2} = GL_{2+2} \otimes \mathbb{R}^4\) and the total space \(N\mathcal{Aa}(\mathcal{V})\) consisting from the set of affine frames on base spacetime \(\mathcal{V}\) constructed such a way that \(N\mathcal{Aa}(\mathcal{V})\) is a principal bundle for \(Af_{2+2}\). The corresponding Lie d–algebra of \(Af_{2+2}\) is \(af_{2+2} = gl_{2+2} \oplus \mathbb{R}^4\). So, any form \(\Theta\) on \(N\mathcal{Aa}(\mathcal{V})\) can be expressed as \(\Theta = (\Theta, \Theta^\omega)\), where \(\Theta\) is the \(gl_{2+2}\)–component and \(\Theta^\omega\) is the \(\mathbb{R}^4\)–component.

Any d–connection 1–form \(\omega\) \(^{30}\) in \(NLa(\mathcal{V})\) induces a (canonical) Cartan connection \(\varpi\) in \(N\mathcal{Aa}(\mathcal{V})\) which can be represented as \(i^* \varpi = (\omega, e_\omega)\),

\(^{18}\) A form \(\varpi\) on a principal bundle \(P\) with values in a Lie algebra \(\mathcal{G}\) is called horizontal if \(\hat{H} \varpi = \varpi\).

\(^{19}\) We derived the Yang–Mills equations following “pure” geometric methods; for semisimple structure groups, an equivalent variational method also holds true, see Refs. [1][44].
where \( i : \mathcal{N} Aa \to \mathcal{N} La \) is the trivial reduction of bundles and the shifting form \( e_i \chi = e_\hat{\alpha} \otimes \chi^\hat{\alpha}_{\mu''} e^{\mu''} \), with \( e_\hat{\alpha} \) being the standard basis in \( \mathbb{R}^4 \) and \( \chi^\hat{\alpha}_{\mu''} \) defining a frame decomposition of d–metric \( \mathfrak{g} \), when \( \mathfrak{g}_{\alpha''\beta''} = \chi^\hat{\alpha}_{\alpha''} \chi^\hat{\beta}_{\beta''} \eta^\hat{\alpha\beta} \), for \( \eta^\hat{\alpha\beta} = diag[-1, 1, 1, 1] \). For a d–connection \( \mathfrak{d} \), we can define the curvature d–tensor in \( \mathcal{N} Aa \),

\[
\omega^R = d\omega + \omega \wedge \omega,
\]

for which the \( gl_{2+2} \), \( \mathbb{R}^4 \)–components are written in \( \omega^R = (\omega^R, \omega^T) \), where \( \omega^R \) is just the curvature (31) and the torsion

\[
\omega^T = d e_i \chi + \omega \wedge e_i \chi - e_i \otimes T^\hat{\alpha}_{\beta\gamma} e^\beta \wedge e^\gamma
\]

with \( T^\hat{\alpha}_{\beta\gamma} = \chi^\hat{\alpha}_{\mu''} e^{\mu''} T^\mu_{\beta\gamma} \), when the coefficients \( T^\mu_{\beta\gamma} \) are computed following formulas (A.1). If \( \omega \) is induced by a metric compatible d–connection \( D \) (10) on \( V \) related to a nonholonomic deformation of a Levi–Civita connection \( \Gamma^\alpha_{\beta\gamma} \), following formula (24), the values \( \mathfrak{d} \) and resulting \( \omega^R \) and \( \omega^T \) are induced by a d–metric \( \mathfrak{g} \).

By straightforward computations, using the total space formula (32) but for \( \mathfrak{d} \) in \( \mathcal{N} Aa(\mathbf{V}) \), see similar details in Refs. [1, 34, 35] (those constructions are for usual affine frame bundles and respective generalizations on higher order Lagrange–Finsler spaces), we obtain

\[
\triangle \omega^R = (\triangle \omega^R, \mathcal{R}_T + \mathcal{R}_i),
\]

with the standard \( gl_{2+2} \)–component \( \triangle \omega^R \) and the \( \mathbb{R}^4 \)–component defined by the sum of

\[
\mathcal{R}_T = \mathbb{g}_\delta \omega^T + \mathbb{g}_*^{-1} [\omega, \mathbb{g}_* \omega^T]
\]

and

\[
\mathcal{R}_i = \mathbb{g}_*^{-1} [e_i \chi, \mathbb{g}_* \omega^R] = -e_\hat{\alpha} \otimes \chi^\hat{\alpha}_{\alpha\beta\gamma} R_{\beta\nu} e^{\nu},
\]

where the components of the Ricci d–tensor \( R_{\beta\nu} \) are computed following formulas (A.4).

The nonholonomic field equations (27) for variables \( (\mathfrak{g}, \mathbf{D}) \), where a d–connection \( \mathbf{D} \) (10) is metric compatible, induce an equivalent system of nonholonomic Yang–Mills equations for \( N \)–adapted Cartan 1–form \( \mathfrak{d} \) in \( \mathcal{N} Aa(\mathbf{V}) \),

\[
\triangle \omega^R = \mathfrak{J},
\]

with the source \( \mathfrak{J} = (\mathfrak{J}, \mathfrak{e} \mathfrak{J}) \). Taking \( \mathfrak{e} \mathfrak{J} = \mathfrak{J}_T + \mathfrak{J}_i \), for

\[
\mathfrak{J}_i = -e_\hat{\alpha} \otimes \chi^\hat{\alpha}_{\alpha\beta\gamma} (\mathbf{E}_\beta_{\nu} - \frac{1}{2} \mathbf{g}_{\beta\nu} \mathbf{E}) e^\nu,
\]

22
where \( \overrightarrow{E} = E^\nu_\mu \), for \( \overrightarrow{E}_{\beta\nu} = 8\pi G \overrightarrow{T}^\alpha_\beta + \frac{\Lambda}{2} g_{\beta\nu} \), with a spin source \( S^\alpha_\beta \mu \) for an arbitrary torsion \( T \) in \( R_\tau \) and \( \triangle \overrightarrow{\mathcal{R}} = \mathcal{J} = 0 \), we generate nonholonomic gauge gravitational equations induced by a d–connection \( D \). In local form, the equations (36) are usual gauge field equations for the 1–form

\[
\overrightarrow{\omega} = \mathcal{g}\mathbf{\Gamma} = \begin{pmatrix} \mathcal{g}\mathbf{\Gamma}^{\alpha''}_\beta \mu & l_0^{-1} \mathbf{e}^\alpha''_\mu \\ 0 & 0 \end{pmatrix}
\]

and \( \omega = \{ \mathcal{g}\mathbf{\Gamma}^{\alpha''}_\beta \mu \} \), for an arbitrary dimension constant \( l_0^{-1} \), when instead of the Killing form (29), which is degenerated for the affine structural groups, \( K = 0 \); we can consider any auxiliary bilinear form \( a_{\alpha''_\beta} \) in order to have a well defined d–metric \( k = g + a \) in the total space of \( N_{\mathcal{A}}(V) \).

Such a dependence on an auxiliary bilinear form consists a particular case of constructions related to Miron’s procedure on N–adapted affine frame bundles when the pull–backs of d–tensors \( Y^m_{ej}, Y^k_{mc}, Y^d_{ck} \) and \( Y^d_{ec} \), see (14), to the total space are such way taken that the metric compatibility \( D_X g = 0 \) on \( V \) is generalized to \( \omega D_X k = 0 \) for ant \( X \in T N_{\mathcal{A}}(V) \). This type of constructions do not affect the physical/geometrical objects on physical spacetime \( V \) because projections on a base spacetime do not depend on components of \( a \).

Now, let us state the conditions for the source \( \mathcal{J} = (\mathcal{J}, e^J) \) in (36) when the nonholonomic Yang–Mills equations are equivalent to the Einstein equations in nonholonomic variables (28). By a straightforward computation using 1–form (36), we get a source (37), when

\[
\overrightarrow{\omega} = \mathcal{g}\mathbf{\Gamma} = \begin{pmatrix} \mathcal{g}\mathbf{\Gamma}^{\alpha''}_\beta \mu & l_0^{-1} \mathbf{e}^\alpha''_\mu \\ 0 & 0 \end{pmatrix}
\]

and \( \mathcal{R}_\tau \) is induced by d–torsion field \( \mathcal{g}\mathbf{T}^h_\beta \gamma \) of a \( \mathcal{g}D = \mathcal{g}\nabla - \mathcal{g}Z \), subjected to conditions (24) and (25). Such gauge equations,

\[
\triangle \overrightarrow{\mathcal{R}} = \mathcal{g}\mathcal{J},
\]

with \( \mathcal{g}\mathcal{J} = (0, e^J = e^J_\tau + e^J_i) \), where \( e^J_i \) is defined by \( e^J_{\tau''} = e^J_{\tau'} \), \( e_J^\tau = e^J_{\tau''} \), \( e^J_i = e^J_{\tau''} \), \( e^J_{\tau'} = e_J^\tau \) and \( e^J_{\tau''} \), for \( e^J_{\tau'} = e^J_{\tau''} \), are also equivalent to the standard Einstein equations (26). For trivial N–anholonomic structures, with vanishing d–torsion (A.1), nonholonomy coefficients (6) and N–connection curvature (7), the equations (40) transform into similar ones derived by Popov and Dikhin [2], for the Levi–Civita connection \( \mathcal{g}\mathbf{\Gamma}^{\alpha''}_\beta \mu \), when \( \mathcal{g}\mathbf{\Gamma}^{\alpha''}_\beta \mu \rightarrow \mathcal{g}\mathbf{\Gamma}^{\alpha''}_\beta \mu \) in (36). The Einstein equations written in such a gauge like form are

\[
\triangle \overrightarrow{\mathcal{R}} = \mathcal{g}\mathcal{J},
\]
where \( \mathcal{J} \) is constructed following formulas (39) for distorsion \( Z^{\alpha''}_{\mu} = 0 \) and \( \mathcal{J}_\tau = 0 \) because \( \mathcal{T}^{\alpha}_{\beta\gamma} = 0 \) and \( \mathcal{R}_\tau = 0 \).

The gravitational Yang–Mills equations (41) in the bundle of affine frames \( \mathcal{Aa}(V) = (\mathcal{Aa}(V), Af_4, V) \) on a (pseudo) Riemannian spacetime \( V \) not enabled with a nonholonomic distribution structure, can be derived using "pure" geometric methods acting with operators "\( d \)" and \( \mathcal{g}\delta \) on the Cartan 1–form defined by the Levi–Civita connection on the base \( V \). On total spaces, such equations are not variational because of degenerate Killing form for the affine group. Nevertheless, they can be written in a formal variational form by introducing an auxiliary bilinear symmetric form \( a_{\alpha''\beta''} \) like we discussed above; projecting on a base manifold one transforms the gauge equations into standard Einstein equations.

In the language of nonholonomic distributions on (pseudo) Riemannian manifolds and principal bundles (with various types of structure groups, semisimple and non–semisimple ones) on such manifolds, geometrically derived nonholonomic Yang–Mills equations in total spaces are subjected to certain types of non–integrable constraints. It is not surprising that violations of former prescribed gauge symmetries may result in some formal nonvariational gauge theories (as a matter of principle, such theories also can be quantized following almost standard methods, see Ref. [11]). Nevertheless, because the equations (41) are equivalent to (41), and both types of nonholonomic and holonomic gravitational Yang–Mills equations are equivalent to the standard Einstein equations (26), quantization of such gauge gravity models contains the same renormalizability problems as for the standard perturbative quantum approach to general relativity.

### 4.2 De Sitter nonholonomic Einstein distributions

We can elaborate another type of variational gauge gravity model with an extension of the structural group. Geometrically, such constructions are quite similar to the nonholonomic 'nonvariational' constructions on \( \mathcal{N}\mathcal{Aa}(V) = (\mathcal{N}\mathcal{Aa}(V), Af_{2+2}, V) \), when the data \((\mathcal{g}, \mathcal{g}\nabla)\) and/or \((\mathcal{g}, \mathcal{g}\mathcal{D})\) for the Einstein equations induces equivalently the data \( \mathcal{g} = \mathcal{g}\Gamma \) solve a Yang–Mills equation. In general, the so–called affine and de Sitter gauge gravities (see, for instance, reviews [3, 4, 5, 6, 36]) are not equivalent to the Einstein gravity. One has to consider various types of mechanisms with nonlinear realizations, to break symmetries with a Higgs like inducing of gravitational

\(^{20}\)see Ref. [33] with a proposal how to perform a formal renormalization scheme of Einstein gravity using nonholonomic distributions and bi–connections defined by the same metric structures
fields etc which do not provide a complete and generally accepted reduction to the general relativity theory. Our approach is in some lines "inverse" to the former ones: we construct such nonholonomic distributions on a de Sitter (principal) bundle \( N^S a(V) = (N a(V), S f_{2+2+1}, V) \), for which the Einstein equations in nonholonomic variables on a spacetime \( V \) induce equivalently some nonholonomic gravitational gauge equations in the total space \( N^S a(V) \). This type of gravitational field equations are variational, subjected to a class of nonholonomic constraints, and can be redefined, after projection on a base spacetime into the system of field equations (28) (such base manifold equations are nonholonomic deformations of the standard Einstein equations).

Let us consider a de Sitter space \( \Sigma \) defined as a hypersurface \( \eta_{\alpha\beta}u^\alpha u^\beta = -1 \) in a four–dimensional flat space endowed with a diagonal metric \( \eta_{\alpha\beta} = \text{diag}[\pm 1, \ldots, \pm 1] \), where \{\( u^\alpha \)\} are global Cartesian coordinates in \( \mathbb{R}^5 \), indices \( A, B, C \ldots \) run values 1, 2, ..., 5 and \( l > 0 \) is the constant curvature of de Sitter space. The isometry group of \( 5 \Sigma–\)space is the de Sitter group \( \eta^S = SO(5) \).

There are 6 operators \( M_{AB} \) of Lie algebra \( so(5) \) satisfying the commutation relations

\[
[M_{AB}, M_{CD}] = \eta_{AC} M_{BD} - \eta_{BC} M_{AD} - \eta_{AD} M_{BC} + \eta_{BD} M_{AC}.
\]

A canonical \( 4 + 1 \) splitting is parametrized by \( A = (\alpha, 5), B = (\beta, 5), \ldots; \eta_{AB} = (\eta_{\alpha\beta}; \eta_{55}) \) and \( P_\alpha = l^{-1} M_{5\alpha} \), for \( \alpha, \beta, \ldots = 1, 2, 3, 4 \) when the commutation relations are written

\[
[M_{\alpha\beta}, M_{\gamma\delta}] = \eta_{\alpha\gamma} M_{\beta\delta} - \eta_{\beta\gamma} M_{\alpha\delta} + \eta_{\beta\delta} M_{\alpha\gamma} - \eta_{\alpha\delta} M_{\beta\gamma},
\]

\[
[P_\alpha, P_\beta] = -l^{-2} M_{\alpha\beta}, [P_\alpha, M_{\beta\gamma}] = \eta_{\alpha\beta} P_\gamma - \eta_{\alpha\gamma} P_\beta.
\]

The commutators (43) are for a direct sum \( so(5) = so(4) \oplus 4V \), where \( 4V \) is the four dimensional vector space stretched on vectors \( P_\alpha \). We remark that \( 4\Sigma = \eta S/ \eta L \), where \( \eta L = SO(4) \). Choosing signature \( \eta_{AC} = \text{diag}[-1, 1, 1, 1, 1] \) and \( \eta S = SO(1, 4) \), we get the group of Lorentz rotations \( \eta L = SO(1, 3) \). Here we also note that the commutation relations (42) define a non–Abelian Lie algebra \( A_I \) for a gauge group with generators \( I_\perp, \ldots, I_\Sigma \) and structure constants \( f_{I J}^{\ L} \) in \([I_\Sigma, I_\Sigma] = if_{I J}^{\ L} I_\Sigma \) parametrized in a form to obtain the formulas (43).

\[^{21}\text{In order to preserve similarity with usual gauge field theories in particle physics, it is convenient to use the complex unity } "i" \text{ in Lie algebras commutators even our nonholonomic gauge models are for the Einstein gravity on real base spacetime manifolds.}\]
At the next step, we consider \(4 \times 4\) matrix parametrizations of actions of the group \(\eta S\) adapted to splitting \(so(5) = so(4) \oplus 4V\) and distinguishing the subgroup \(\eta L\). We write

\[
Q = q^L Q, \tag{44}
\]

where \(LQ = \begin{pmatrix} L & 0 \\ 0 & 1 \end{pmatrix}\), \(q = \begin{pmatrix} \delta^\alpha_\beta + \frac{\tau^\alpha_\beta}{1 + \tau^5} \tau^\alpha_5 \\ \tau^\alpha_\beta \end{pmatrix}\) and \(L \in \eta L\) is the de Sitter bust matrix transforming the vector \((0, 0, ..., \rho) \in \mathbb{R}^5\) into a point \((V^1, V^2, ..., V^5) \in 5\Sigma_\rho \subset 5\mathcal{R}\), where \(5\mathcal{R}\) is a 'sphere' with curvature \(\rho\), \((V_AV^A = -\rho^2, V^A = \tau^A\rho)\).

A de Sitter gauge field is given by a \(so(5)\)–valued connection 1–form

\[
\tilde{\Omega} = \begin{pmatrix} \omega^\alpha_\beta + \tilde{\theta}^\alpha_\beta \\ \tilde{\theta}_\beta \end{pmatrix}, \tag{45}
\]

where \(\omega^\alpha_\beta \in so(4), \tilde{\theta}^\alpha_\beta \in \mathcal{R}^4, \tilde{\theta}_\beta \in \eta_\beta \tilde{\theta}^\alpha_\beta\).

The introduced parametrization is invariant on action of \(SO(4)\) group. So, it is not possible to say that \(\omega^\alpha_\beta\) and \(\tilde{\theta}^\alpha_\beta\) are induced directly by a d–connection \(\Gamma^\alpha_\beta_\gamma = \Gamma^\alpha_\beta_\gamma - Z^\alpha_\beta_\gamma \tag{24}\) and a 1–form \(e^\alpha \tag{4}\) because the actions of \(\eta S\) mix the components of the matrix \(\omega^\alpha_\beta\) and \(\tilde{\theta}^\alpha_\beta\) fields in \((45)\). This problem can be solved on de Sitter affine bundles with N–connections and nonholonomic frames induced from the base nonholonomic manifold. We also have to consider nonlinear gauge realizations of the de Sitter group \(\eta S\) when the nonlinear gauge field is

\[
\Gamma = q^{-1} \tilde{\Omega} q + q^{-1} dq = \begin{pmatrix} \Gamma^\alpha_\beta' & \theta^\alpha_\beta' \\ \theta^\alpha_\beta' & 0 \end{pmatrix}, \tag{46}
\]

for \(\Gamma^\alpha_\beta' = \omega^\alpha_\beta + \left(\tau^\alpha_\beta D\tau_\beta - \tau_\beta D\tau^\alpha\right) / (1 + \tau^5), \theta^\alpha_\beta' = \tau^5 \tilde{\theta}_\beta + D\tau^\alpha - \tau^\alpha \left(\tau^5 + \tilde{\theta}_\beta \tau^\gamma\right) / (1 + \tau^5),\)

\[
D\tau^\alpha' = d\tau^\alpha + \omega^\alpha_\beta \tau^\beta'.
\]

The action of the group \(\eta S\) is nonlinear and nonholonomically deformed,

\[
\Gamma' = L\Gamma \left(L'\right)^{-1} + L'd \left(L'\right)^{-1}, \theta' = L\theta,
\]

where the nonlinear matrix–valued function \(L' = L' \left(\tau^\alpha, q, Q\right)\) is defined from \(Q_q = q' L' Q\) (see the parametrization \((44)\)). The de Sitter 'nonlinear'
algebra is defined by generators (43) and nonlinear gauge transforms of type (46). In the language of nonholonomic distributions, such nonlinear gauge transforms can be considered as certain nonholonomic deformations of some linear gauge transforms.

In our approach, the de Sitter nonlinear gauge gravitational theory is to be constructed from the coefficients of a $d$–metric $g$ and $N$–connection $N$ in a form when the Einstein equations on the base nonholonomic spacetime are equivalent to the Yang–Mills equations in the total space. For this, we state that the $d$–connection (46) is defined with respect to $N$–adapted frames (3) and a $d$–connection $g\Gamma_{\beta\gamma}^\alpha$, where

$$g\Gamma_{\beta\gamma}^\alpha = g\Gamma_{\alpha' \beta' \mu} \delta u^\mu;$$

for

$$g\Gamma_{\beta\gamma}^\alpha = e_{\alpha'}^{\alpha'} e_{\beta'}^{\beta'}, g\Gamma_{\alpha' \beta' \mu} \delta u^\mu + e_{\alpha'}^{\alpha'} \delta \mu e_{\beta'}^{\beta'}, e_{\alpha'}^{\alpha'} = e_{\mu}^{\alpha'} \delta u^\mu,$$

with $l_0$ being a dimensional constant. The formulas (49) are similar to (20) but, in this section, the indices $\alpha', \beta'$ take values in the typical fiber/de Sitter space.

The matrix components of the curvature of the connection (47),

$$\Gamma_{\mathcal{R}} = d \, g\Gamma - g\Gamma \wedge g\Gamma,$$

can be parametrized in an invariant 4+1 form

$$\Gamma_{\mathcal{R}} = \left( \begin{array}{cc} \mathcal{R}_{\alpha'}^{\beta'} + l_0^{-1} \pi_{\beta'}^{\alpha'} & l_0^{-1} \mathcal{T}_{\alpha'}^{\beta'} \\ l_0^{-1} \mathcal{T}_{\beta'}^{\alpha'} & 0 \end{array} \right),$$

where

$$\pi_{\beta'}^{\alpha'} = e_{\alpha'}^{\alpha'} \wedge e_{\beta'}^{\beta'}, \mathcal{T}_{\beta'}^{\alpha'} = \frac{1}{2} g\mathcal{T}_{\mu\nu}^{\beta'} \delta u^\mu \wedge \delta u^\nu,$$

$$\mathcal{R}_{\alpha'}^{\beta'} = \frac{1}{2} \mathcal{R}_{\mu\nu}^{\alpha'} \delta u^\mu \wedge \delta u^\nu, \mathcal{R}_{\beta'}^{\alpha'} \mu_{\nu} = e_{\beta'}^{\beta'} e_{\alpha'}^{\alpha'} g\mathcal{R}_{\beta'}^{\alpha'} \mu_{\nu},$$

when the coefficients $g\mathcal{T}_{\mu\nu}^{\beta'}$ are of type (A.1) and $g\mathcal{R}_{\beta'}^{\alpha'} \mu_{\nu}$ are of type (A.3).

The de Sitter group is semisimple which allows us to construct a variational gauge gravitational theory with the Lagrangian

$$L = gL + m L$$

(51)
where the gauge gravitational Lagrangian is

\[ g_L = \frac{1}{4\pi} T_r (\Gamma \wedge \ast \Gamma \mathcal{R}) = g_L |g|^{1/2} \delta^4 u, \]

\[ g_L = \frac{1}{2l^2} g_{T\alpha'\mu\nu} g_{T\alpha'\mu\nu} + \frac{1}{8\lambda} \mathcal{R}_{\alpha'\beta\mu\nu} \mathcal{R}^{\beta'}_{\alpha'\mu\nu} - \frac{1}{l^2} (\mathcal{R} - 2\lambda_1), \]

for \( \delta^4 u \) being the volume element. The Hodge operator \( \ast \) in \( g_L \) is constructed from the metric in the total space, determinant \( |g| \) is computed from the coefficients of a d–metric \( g \) stated with respect to N–elongated frames, the curvature scalar \( \mathcal{R} \) is computed as in (A.5), \( g_{T\alpha'\mu\nu} = e^{\alpha'\alpha} g_{T\alpha\mu\nu} \) (the constant \( l^2 \) satisfies the relations \( l^2 = 2l_0^2\lambda, \lambda_1 = -3/l_0 \)), \( Tr \) denotes the trace on \( \alpha', \beta' \) indices.

Let us discuss the meaning of constants (physical or non–physical ones) introduced in the formulas of the above formulated nonholonomic gauge models of gravity. We emphasize that such constants have different physical interpretations in generalized gauge gravity theories and in the case of nonholonomically gauge like structures induced by solutions of the Einstein equations on the base spacetime. In the nonholonomic approach, the value \( 2\lambda_1/l^2 \) can be considered as a usual cosmological constant \( \Lambda/2 \) in (26). The constant \( l^2 \) characterizes the intensity of torsion field interactions in a nonlinear de Sitter gauge theory but it can be interpreted as a formal (nonphysical) constant defining the nonholonomic constraints imposed on nonholonomic structure in order to redefine the standard data \( (g, \, \mathcal{g}) \), or \( (g, \, D) \), for the Einstein gravity theory into a nonholonomic de Sitter model with variables \( \mathcal{g} \) (47). The constants \( l^2, l_0^2 \) and \( \lambda \) can be included as particular cases of constant d–tensor \( \mathcal{Y} \) fields used in the Miron’s procedure, see formula (14), generalized for nonlinear de Sitter bundles. Such constants and fields characterize the type of nonholonomic deformations of certain ”usual Einstein data” into certain, another type, but equivalent, ”nonholonomic data”. It is like we can chose any equivalent, but convenient, frame or coordinate transform on a spacetime, which for nonholonomic lifts on certain bundle spaces characterize the properties of such lifts keeping the proprieties of ”Einstein data” even they are nonholonomically deformed under such lifts.

In a more general context, such constructions reflect the fact that nonholonomic distributions provide more rich geometric and physical structures, and a number of more sophisticate geometric methods and constructions, than in the case of unconstrained dynamics of gravitational fields. Such constructions provide more possibilities in developing quantum models, but also for applications of classical gravity in cosmology and astrophysics because of a
multi–connection character of theories with nonholonomic distributions.\footnote{All connections under consideration being defined by the same metric but with possible additional interaction constants, nonholonomic constraints and prescribed symmetries and parametrizations; from formal point of view, we can redefine the geometric constructions for the Levi–Civita connection but this results in a more sophisticated "mixture" of nonholonomic deformations and constraints, nonlinear interactions and broken symmetries and associated constants and/or chosen parametrizations of fields}

The matter field Lagrangian from (51) is

\[ mL = -\frac{1}{2} Tr (g\Gamma \wedge *g\Gamma) = mL|g|^{1/2} \delta^4 u, \]

when the Hodge operator \(*g\) is defined by \(|g|\), where

\[ mL = \frac{1}{2} g_{\alpha' \beta' \mu} S^{\beta' \alpha' \mu} - \tau_{\mu}^{\alpha'} e^{\alpha' \mu}. \]

The matter field source \(J\) is constructed as a variational derivation of \(mL\) on \(g\Gamma\) and is parametrized in the form

\[ J = \begin{pmatrix} S^{\alpha' \beta} & -l_0 \tau^{\alpha'} \\ -l_0 \tau_{\beta'} & 0 \end{pmatrix}, \]

with \(\tau^{\alpha'} = \tau^{\alpha' \mu} \delta u^\mu\) and \(S^{\alpha' \beta'} = S_{\alpha' \beta' \mu} \delta u^\mu\) being respectively the canonical tensors of energy–momentum and spin density.

Varying the action \(S = \int \delta^4 u (gL + mL)\) on the \(\Gamma\)–variables (47), we obtain the gauge–gravitational field equations:

\[ d(*\Gamma R) + g\Gamma \wedge (*\Gamma R) - (*\Gamma R) \wedge g\Gamma = -\lambda (*J). \tag{52} \]

This equations can be alternatively derived in geometric form by applying the absolute derivation and dual operators.

Distinguishing the variations on \(g\Gamma\) and \(e\)–variables, we rewrite (52)

\[ \bar{D} (*\Gamma R) + \frac{2\lambda}{l^2} \left( \bar{D} (*\pi) + e \wedge *T^f - *T \wedge e^f \right) = -\lambda (*S), \tag{53} \]

\[ \bar{D} (*T) - (*\Gamma R) \wedge e - \frac{2\lambda}{l^2} *\pi \wedge e = \frac{l^2}{2} \left( *\tau + \frac{1}{\lambda} *\varsigma \right), \]

\(e^f\) being the transposition of \(e\), where

\[ T^f = \{ T_\alpha = \eta_{\alpha' \beta'} T^{\beta'} \}, \quad e^T = \{ e_\alpha = \eta_{\alpha' \beta'} e^{\beta'}, \quad e^{\beta'} = e^{\beta' \mu} \delta u^\mu \}. \]
for $\overline{\mathcal{D}} = \delta + \bar{\Gamma}$; the operators $\bar{\Gamma}$ acts as $\mathcal{g}^{\alpha' \beta' \mu}$ on indices $\gamma', \delta', \ldots$ and as $\mathcal{g}^{\alpha \beta \mu}$ on indices $\gamma, \delta, \ldots$. For this model, we can construct an energy–momentum tensor for the gauge gravitational field $\bar{\Gamma}$,

$$\mathcal{g}_{\mu \nu} (\bar{\Gamma}) = \frac{1}{2} \mathcal{T}_\tau \left( \mathcal{g}^{\alpha \beta \mu} \mathcal{R}_\alpha^\gamma \mathcal{R}_\beta^\delta - \frac{1}{4} \mathcal{g}^{\gamma \delta} \mathcal{R}_\alpha^\beta \mathcal{R}_\beta^\gamma \mathcal{g}_\mu \nu \right). \quad (54)$$

Equations (52) make up the complete system of variational field equations for nonlinear de Sitter gauge gravity. We note that we can obtain a nonvariational Poincaré gauge gravitational theory if we consider the contraction of the gauge potential (47) to the 1–form (38), i.e.

$$\mathcal{g}^{\Gamma} = \left( \begin{array}{c} \mathcal{g}^{\alpha' \beta' \mu} \\ l_0^{-1} e_{\beta'} \\ 0 \end{array} \right) \rightarrow \omega = \mathcal{g}^{\Gamma}.$$

For $\mathcal{g}^{\Gamma}$, or $\mathcal{g}^{\bar{\Gamma}}$, projected on the base nonholonomic spacetime, our nonholonomic de Sitter gauge equations are completely equivalent to the Einstein equations (28), and (reintroducing the Levi–Civita connection) to (26). We have to use deformations of type $\mathcal{\Gamma}_\beta^\gamma = \mathcal{\Gamma}_\beta^\gamma - \mathcal{Z}_\beta^\gamma$ and $\mathcal{R}_\gamma^\tau = \mathcal{R}_\gamma^\tau - \mathcal{Z}_\gamma^\tau$, where $\mathcal{R}_\gamma^\tau = \mathcal{R}_\gamma^\tau e^\alpha \wedge e^\beta$ is the curvature 2–form of the Levi–Civita connection $\nabla$ and $\mathcal{R}_\gamma^\tau = \mathcal{R}_\gamma^\tau e^\alpha \wedge e^\beta$ the curvature tensor of a $D$, related distorsion of curvature 2–form $\mathcal{Z}_\gamma$ defined by $\mathcal{Z}_\gamma^\tau$ (25). We have to include such terms in an induced spin source $S^{\alpha' \beta' \mu}$, similarly as we induced the distorsion $Z^\tau_{\alpha' \mu}$ of matter energy–momentum tensor for (39).

It is a well known fact that in general relativity it is not possible to define a usual energy–momentum tensor for the gravitational field. Nevertheless, for its nonholonomic deformations induced on some bundle spaces such tensors can be defined at least formally, see (51). For nonholonomic distributions with $3 + 1$ and $2 + 2$ spacetime decompositions (for a de Sitter extension, we have to consider a $3 + 2$ splitting) certain constructions with effective gravitational energy and momenta, like in [7, 8, 9], for usual $3 + 1$ variables in classical and quantum gravity, can be performed.

### 4.3 Gravitational Yang–Mills equations for distorsion $d$–tensors

The $d$–connection and $d$–tensor formalism allows us to adapt the geometric constructions to a prescribed nonholonomic distribution (N–connection) structure. N–anholonomic distributions can be defined in arbitrary form on a spacetime manifold and bundle spaces on such a manifold like we are free
to chose any frame or coordinate structure on a curved spacetime. But we can also chose a nonholonomic distribution, with corresponding constraints on dynamics of nonlinerly interacting gravitational and matter fields, when certain configurations of fields and nonintegrable constraints became stable or subjected to generate certain interactions and evolutions of fields and geometries with prescribed symmetry and/or nonholonomic deformations\[42,43\].

The fact that from a given metric tensor on a nonholonomic manifold one can be constructed different classes of linear connections, all defined by the same metric structure, allows us to develop us new models of gauge gravitational interactions which are equivalents of the Einstein gravity theory but with a more rich nonholonomic geometric structure.

Let us write the gauge like equivalent of Einstein equations (41), in the bundle of affine frames $A\alpha(V) = (A\alpha(V), AF, V)$ on a (pseudo) Riemannian spacetime $V$, in the form

$$d(\star \omega^{\mathcal{R}}) + \mathcal{T}^{\alpha} \wedge (\star \omega^{\mathcal{R}}) - (\star \omega^{\mathcal{R}}) \wedge \mathcal{T} = - \mathcal{J},$$

for a gravitational connection $\omega = \mathcal{T}$\[41\] defined by the Levi–Civita connection and $\mathcal{J}$ induced by the energy–momentum tensor of matter fields in general relativity\[24\]. Any nonholonomic splitting on $V$ of type $\Gamma^{\alpha\beta\gamma} = \Gamma^{\alpha\beta\gamma} + \mathcal{Z}^{\alpha\beta\gamma}$, see formulas (24) and (25), results in $\mathcal{R}^{\tau}_{\gamma} = \mathcal{R}^{\tau}_{\gamma} + \mathcal{Z}^{\tau}_{\gamma}, \text{i.e.} \mathcal{T}^{\tau} = \mathcal{T}^{\tau} + \mathcal{Z}^{\tau}$ generates $\omega^{\mathcal{R}} = \mathcal{T}^{\tau} + \mathcal{Z}^{\tau}$. Introducing such distorsion relations into (55), for a curvature $\mathcal{T}^{\alpha}$ solving the equations

$$d(\star \mathcal{T}^{\alpha}) + (\star \mathcal{T}^{\alpha}) \wedge (\star \mathcal{T}^{\alpha}) - (\star \mathcal{T}^{\alpha}) \wedge \mathcal{T} = 0,$$

(they are equivalent to equations (27) for $\Lambda = 0$ and $\mathcal{T}^{\alpha\beta\gamma} = 0$), we get

$$d(\star \mathcal{Z}^{\alpha}) + (\star \mathcal{Z}^{\alpha}) \wedge (\star \mathcal{Z}^{\alpha}) - (\star \mathcal{Z}^{\alpha}) \wedge \mathcal{T} = - \mathcal{J},$$

where the nonholonomically deformed source is

$$\mathcal{J}^{\alpha} = \mathcal{J}^{\alpha} + \mathcal{T}^{\alpha} \wedge (\star \mathcal{Z}^{\alpha}) - (\star \mathcal{Z}^{\alpha}) \wedge \mathcal{T} + \mathcal{T}^{\alpha} \wedge (\star \mathcal{R}^{\alpha}) - (\star \mathcal{R}^{\alpha}) \wedge \mathcal{Z}^{\alpha}.$$

Source $\mathcal{J}^{\alpha}$ is the total space analog of base source $T^{\tau} = m T^{\tau} + Z T^{\tau}$ for the Einstein equations in nonholonomic variables (28). The induced from base

\[24\] we emphasize that similar constructions, but for different nonholonomic de Sitter distributions, can be performed if we begin with the gauge gravity equations (52); for simplicity, in this section, we shall consider only nonholonomic affine frame bundles and connections on such spaces.
N–anholonomic distribution in $Aa(V)$ states a nonlinear relation between the distortion of curvature $\mathring{g}\bar{Z}$ and distortion of the Cartan d–connection $\mathring{\gamma}$, which is different from that in standard Yang–Mills theory, of type (35).

The effective Yang–Mills gravitational equations and their source can be substantially simplified for nonholonomic distributions when $g = \hat{g}$ on base spacetime $V$ induces a d–connection $\hat{\Gamma}_{\alpha'\beta'}^\gamma = (0, \hat{L}_{\alpha'\beta'}, 0, 0)$ (21) with constant curvature coefficients $\hat{\mathcal{R}}_{\alpha'\beta'\gamma\delta'} = (0, \hat{L}_{\alpha'\beta'} = \hat{L}_{\alpha'\beta'} - \hat{\mathcal{R}}_{\alpha'\beta'} = (0, \hat{L}_{\alpha'\beta'}, 0, 0, 0, 0)$ (23), with respect to a class of N–adapted frames. The corresponding distortion of the Levi–Civita connection with respect to $\hat{\Gamma}_{\alpha'\beta'}^\gamma$ is written in the form $\hat{\gamma} = \hat{\Gamma}_{\alpha'\beta'}^\gamma + \hat{Z}_{\alpha'\beta'}$. The related distortions in the total space are $\mathring{\mathcal{R}} = \mathring{\Gamma} + \mathring{Z}$ and $\mathring{\gamma} = \mathring{\mathcal{R}} + \mathring{Z}$.

For a four dimensional nonholonomic (pseudo) Riemannian base $V$, one could be maximum eight nontrivial components $\hat{L}_{\alpha'\beta'}$. We can prescribe such a nonholonomic distribution with some nontrivial values $\hat{L}_{\alpha'\beta'}$ when $\hat{R}_{\alpha'\beta'} = \hat{L}_{\alpha'\beta'} - \hat{\mathcal{R}}_{\alpha'\beta'} = 0$ Choosing $\mathring{\mathcal{R}} = 0$, we write the gauge like gravitational equations (56) in a simplified form,

$$d\left(\mathring{\gamma}\right) + \mathring{\mathcal{R}} = \mathring{\Gamma} + \mathring{Z} = \mathring{\gamma} = \mathring{\mathcal{R}} = 0,$$ (57)

where the nonholonomically deformed source is

$$\mathring{\gamma} = \mathring{\mathcal{R}} + \mathring{\Gamma} + \mathring{Z} = \mathring{\gamma} = \mathring{\mathcal{R}} + \mathring{\Gamma}.$$

Induced gauge like gravitational equations of type (56), or (57), with constraints of type $\hat{\mathcal{R}}_{\beta'\gamma\delta'} = \text{const}$, or $= 0$, are important for elaborating a formal scheme for perturbative quantization and renormalization of general relativity using the so–called ”nonholonomic two–connection formalism” in [33].

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25 It should be emphasized here that the geometric properties of curvature d–tensor for a d–connection are very different from those of usual tensors and linear connections. Even the coefficients of a d–tensor may vanish with respect to a particular class of nonholonomic distributions, the real spacetime may be a general (pseudo) Riemannian one with nontrivial curvature of the Levi–Civita connection and nonzero associated/induced nonholonomically d–torsions, nonholonomy coefficients and curvature of N–connection.
5 Concluding Remarks

In this work the Einstein gravity theory was reformulated in some different, but equivalent, forms on manifolds and affine/de Sitter frame bundles enabled with nonholonomic distributions. There were considered distributions with 2+2 splitting of base spacetime and associated nonholonomic frame structures. Using the formalism of nonlinear connections, the geometric constructions were adapted to formal decompositions into holonomic and nonholonomic variables.

We followed two guiding geometric principles: 1) for a (pseudo) Riemannian metric, it is possible to construct an infinite number of metric compatible linear connections, all determined by a metric structure in unique forms following certain well defined geometric/physical conditions; 2) imposing certain types of nonintegrable/nonholonomic constraints, we can select some classes of connections and nonholonomic deformations of geometric objects which are most convenient for various purposes in modern gravity (for instance, to define some invariants and conservation laws, to elaborate new quantization schemes for gravity theory etc); all such geometric constructions can be equivalently re-defined in terms of a Levi–Civita connection.

The gauge gravitational models presented here are related to a number of attempts to find a Yang–Mills formulation/ generalization for Einstein gravity. The bulk of former constructions were oriented to elaborate various gauge gravity theories which in certain limits model classical and possible quantum effects for general relativity. In our approach, we considered some inverse constructions. We defined such nonholonomic distributions on a base spacetime, and their lifts on bundle spaces, when the geometric data for a solution of Einstein equations can be encoded equivalently into some classes of nonholonomic frames and linear and nonlinear connections stating the geometric structure of such bundles.

The formulated equivalents of the Einstein equations on nonholonomic manifolds and (affine/de Sitter) frame bundle spaces contain non–trivial torsion contributions. For various generalizations of Einstein gravity to other gauge/ string / Riemann–Cartan gravity theories, the torsion fields are different from the metric one and subjected to certain additional field/algebraic equations. In our approach, the torsion structure is induced nonholonomically by certain off–diagonal metric (equivalently, nonholonomy) coefficients. So, we do not generalize the Einstein gravity theory, but represent it equivalently in terms of some new types of nonholonomic variables, which is more convenient for certain constructions in quantum gravity (preserving an analogy with the usual Yang–Mills fields) and, for instance, to develop
new geometric methods for constructing exact solutions in gravity and Ricci flow theories (see new results and reviews in our works \[18, 19, 20, 21, 23, 26, 27, 28, 42\]).

Finally, we note that this paper provides a nonholonomic geometric formalism for a second partner works on a model of perturbative quantum gauge gravity \[33\].

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A Coefficients of d–Connections and d–Tensors

For convenience, in this Appendix, we outline some important coefficient formulas for fundamental geometric objects on N–anholonomic manifolds. Details and proofs are presented in Refs. \[19, 20, 21\].

Locally it is characterized by (N–adapted) d–torsion coefficients

\[
T^i_{jk} = L^i_{jk} - L^i_{kj}, \quad T^i_{ja} = -T^i_{aj} = C^i_{ja}, \quad T^a_{ji} = \Omega^a_{ji}, \quad (A.1)
\]

The curvature of a d–connection \(D\),

\[
\mathcal{R}^\alpha_{\beta\gamma\delta} \equiv D\Gamma^\alpha_{\beta\gamma} = d\Gamma^\alpha_{\beta\gamma} - \Gamma^\alpha_{\gamma\delta} \wedge \Gamma^\gamma_{\beta\gamma}, \quad (A.2)
\]
splits into six types of N–adapted components with respect to \(3\) and \(4\),

\[
\mathcal{R}^\alpha_{\beta\gamma\delta} = (R^i_{hjk}, R^a_{bjk}, P^i_{hja}, P^c_{bjc}, S^i_{jbc}, S^a_{bcd}),
\]

\[
R^i_{hjk} = e_k L^i_{hj} - e_j L^i_{hk} + L^m_{hj} L^i_{mk} - L^m_{hk} L^i_{mj} - C^i_{hmk} \Omega^a_{kj}, \quad (A.3)
\]

\[
R^a_{bjk} = e_k L^a_{bj} - e_j L^a_{bk} + L^c_{bj} L^a_{ck} - L^c_{bk} L^a_{cj} - C^a_{bc} \Omega^c_{kj},
\]

\[
P^i_{jka} = e_a L^i_{jk} - D_h C^i_{ja} + C^i_{jb} T^b_{ka},
\]

\[
P^c_{bka} = e_a L^c_{bk} - D_h C^c_{ba} + C^c_{bd} T^d_{ka},
\]

\[
S^i_{jbc} = e_c C^i_{jb} - e_b C^i_{jc} + C^h_{jb} C^i_{hc} - C^h_{jc} C^i_{hb},
\]

\[
S^a_{bcd} = e_d C^a_{bc} - e_c C^a_{bd} + C^e_{be} C^a_{ed} - C^e_{bd} C^a_{ec}.
\]

Contracting respectively the components, \(R_\alpha^\beta \equiv R^r_{\alpha\beta\tau} r\), one computes the h–v–components of the Ricci d–tensor

\[
R_{ij} \equiv R^k_{ijk}, \quad R_{ia} \equiv -P^k_{ika}, \quad R_{ai} \equiv P^b_{aib}, \quad S_{ab} \equiv S^c_{abc}. \quad (A.4)
\]
The scalar curvature is defined by contracting the Ricci d–tensor with the inverse metric $g^{\alpha\beta}$,

$$\overrightarrow{R} \doteq g^{\alpha\beta} R_{\alpha\beta} = g^{ij} R_{ij} + h^{ab} S_{ab} = \overrightarrow{R} + \overleftarrow{S}. \tag{A.5}$$

Finally, we emphasize that formulas presented in this section hold true both for any d–connection and/or not N–adapted linear connection (in the last case, one should not use ‘boldface’ symbols) with respect to arbitrary (non) holonomic frames. Working with metric compatible linear connections, we get coefficient formulas which are very similar to those for the standard (in (pseudo) Riemannian geometry) Levi–Civita connection. Nevertheless, nonholonomic configurations may change certain symmetry properties of geometric objects and physical equations/conservation laws. For instance, the Ricci d–tensor [A.4], in general, is not symmetric, i.e. $R_{\alpha\beta} \neq R_{\beta\alpha}$ even for symmetric metrics. Working with d–connections/linear connections completely defined by a metric structure, we can always redefine the constructions in an equivalent ‘Levi–Civita form’ when the usual symmetries and Bianchi identities are present.

Any geometric construction for the canonical d–connection $\hat{D}$ [16] can be re–defined for the Levi–Civita connection $\nabla = \{ \hat{\Gamma}^\gamma_{\alpha\beta} \}$ by using the formula

$$\Gamma^\gamma_{\alpha\beta} = \hat{\Gamma}^\gamma_{\alpha\beta} + Z^\gamma_{\alpha\beta}, \tag{A.6}$$

where the both connections $\Gamma^\gamma_{\alpha\beta}$ and $\hat{\Gamma}^\gamma_{\alpha\beta}$ and the distorsion tensor $Z^\gamma_{\alpha\beta}$ with N–adapted coefficients where

$$Z^a_{jk} = -\hat{C}^i_{jk} g^i_{ab} - \frac{1}{2} \Omega^a_{jk}, \ Z^i_{kk} = \frac{1}{2} \Omega^i_{jk} h_{cb} g^{ji} - \Xi^{ih}_{jk} \hat{C}^j_h, \ Z^i_{kj} = 0, \ Z^i_{jk} = 0, \ Z^i_{ij} = 0, \ Z^i_{ab} = 0, \ Z^i_{ij} = 0,$$

$$Z^i_{ab} = \frac{1}{2} [\hat{L}_c^i h_{cb} + \hat{L}_b^c h_{ca}].$$

For $\Xi^{ij}_{jk} = \frac{1}{2} (\delta^i_j \delta^h_k - g_{jk} g^{ih})$, $\Xi^{ab}_{cd} = \frac{1}{2} (\delta^a_d \delta^b_c + h_{cd} h^{ab})$ and $\hat{L}_a^c = \hat{L}_{a}^c - e_a (N^c)$. If we work with nonholonomic constraints on the dynamics/geometry of gravity fields, it is more convenient to use a N–adapted approach. For other purposes, it is preferred to use only the Levi–Civita connection.

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