INSTANTON SYMMETRIES
AND
THE ENTROPY OF COMPACT MANIFOLDS

Marika Taylor-Robinson *
Department of Applied Mathematics and Theoretical Physics,
University of Cambridge, Silver St., Cambridge. CB3 9EW
(March 28, 2022)

Abstract

Many Euclidean Einstein manifolds possess continuous symmetry groups of at least one parameter and we consider here a classification scheme of $d$ dimensional compact manifolds based on the existence of such a one parameter group in terms of the fixed point sets of the isometries. We discuss applications of such a classification scheme, including the geometric interpretation of the entropy; there are intrinsic contributions to the entropy from the volumes of $(d-2)$ dimensional fixed point sets and contributions related to the cohomology structure of the orbit space of the isometry. We consider the relevance of such a decomposition of the entropy in the context of the no boundary proposal and cosmological processes, and generalise the discussion to compact solutions of gravity coupled to scalar and gauge fields.

PACS numbers: 04.50.+h, 04.65.+e, 04.70.Dy

Typeset using REVTeX

*E-mail: mmt14@damtp.cam.ac.uk
I. INTRODUCTION

Euclidean Einstein manifolds arise as instanton solutions of the classical Euclidean field equations of not only $d$ dimensional gravity but also of supergravity theories, with a constant dilaton and all other fields except the graviton vanishing. Many instanton solutions possess continuous symmetry groups of at least one parameter; indeed in many cases we consider dimensional reduction of $d$ dimensional solutions to $(d - 1)$ dimensions along closed orbits of circle isometries. We consider here a classification scheme of $d$ dimensional Euclidean Einstein manifolds based on the existence of such a one parameter group, in terms of the fixed point sets of the isometries, generalised nuts and bolts. This is a generalisation of the four-dimensional case analysed in [1]; the action of fixed point sets of isometries has also been considered in [2], [3] and [4].

Such a classification scheme has various applications, perhaps the most important of which is the geometric interpretation of the entropy. It is well known that black holes have an intrinsic entropy proportional to one quarter of the volume of the horizon. In addition cosmological event horizons have an associated entropy equal to one quarter of their volume; this entropy can be derived using the Euclidean path integral approach. In terms of the nuts and bolt terminology, the $(d - 2)$ dimensional horizon is a fixed point set of the imaginary time Killing vector, and contributes an entropy proportional to its proper volume. Recently the advent of D-brane technology in string theory has permitted a microscopic derivation of black hole entropy for particular classes of near BPS states.

However, for four dimensional compact solutions it is known that there are contributions to the gravitational entropy not only from the areas of the bolts but also from the nut charges of the nuts and bolts. In [1] the entropy of a four dimensional compact Einstein manifold with no boundary admitting at least a circle subgroup was found to be

$$S = \sum_{\text{bolts}} \frac{V_2}{4G_4} + \sum_{\text{nuts,bolts}} \frac{\beta}{16\pi G_4} \Psi^a \int_{M^d_2} F,$$

where the nuts and bolts are zero and two dimensional fixed point sets respectively. $\Psi^a$ is a scalar potential evaluated at the $a$th fixed point set, and $F$ is the Kaluza-Klein two form gauge field obtained upon dimensional reduction along closed orbits of the isometry. Further discussions of the rôle of the nut charge were presented recently in [5].

The main object of our classification scheme is to extend this geometric interpretation of the entropy in terms of fixed point sets to general dimensions. What we find is that $(d - 2)$ dimensional bolts have an intrinsic entropy related to their volume. There are additional contributions to the entropy from all bolts of lower dimension and $(d - 2)$ dimensional bolts which have non-trivial normal bundles; these contributions can be thought of as the generalisations of the four-dimensional nut potential terms.

Although in four dimensions one can represent the nut contributions to the entropy in terms the properties of the fixed point sets only, in higher dimensions the situation is considerably more complicated. Non trivial $(d - 3)$ cohomology of the $(d - 1)$ dimensional orbit space plays a role as does the nut type behaviour of individual fixed point sets. The total contribution to the entropy can best be represented as

$$S = \sum_a \frac{V_a}{4G_d} + \frac{\beta}{16\pi G_d} \int_{\Sigma} F \wedge \bar{G},$$

(2)
where $\bar{G}$ is a $(d - 3)$ form related to the dual of $F$ in the orbit space $\Sigma$, and the integral is taken over this space. In general both $\bar{G}$ and $F$ have non-zero periods and there is no natural way to split the integral into individual contributions from fixed point sets and Dirac string type singularities within the orbit space.

This decomposition of the Euclidean action, and hence the entropy, of compact manifolds in terms of the action of isometries has implications for cosmological processes. In the context of the no boundary proposal, we can express the entropy of the Lorentzian solution in terms of the action of the isometry on the original Euclidean manifold. Thence we are able to demonstrate explicitly that cosmological pair creation of black branes is associated with an entropy dependent on the volumes of the horizons, and that monopole pair creation within an expanding background is associated with an entropy dependent on the monopole charges.

Having discussed the action of isometries on Euclidean Einstein manifolds, it is natural to consider the extensions to Euclidean solutions of gravity coupled to scalar and gauge fields. We find that the same decomposition of the action holds, but if the “electric” part of the gauge field is non vanishing, there is an additional term in the action dependent on this part of the field. In this context, “electric” means that if we consider the action of an isometry $\partial_\tau$ the $\mathcal{H}_{\tau r\cdots}$ components of the gauge field are non zero. If we analytically continue the solution and $\tau$ is interpreted as an imaginary time coordinate, this part of the gauge field will indeed be electric.

In the context of the no boundary proposal, we find that the additional term in the action can be removed by imposing a constraint on the nucleation surface; this constraint has the physical interpretation of fixing the charge on the hypersurface. We can then show that cosmological pair creation of generic charged black dilatonic branes is associated with an entropy dependent on the horizon volumes. The treatment of extreme black holes within this formalism requires a more careful treatment of the boundary terms, and will be considered briefly.

The plan of this paper is as follows. In §II we give a brief discussion of the properties of higher dimensional symmetry groups. In §III we discuss the decomposition of the action of compact solutions in terms of the properties of the fixed point sets of the action of a circle isometry group. In §IV we consider further the contributions to the action from the non-trivial cohomology of $\Sigma$. We discuss the entropy of such solutions, and the cosmological relevance in §V, and consider the generalisation to gauge field theories in §VI. In §VII we discuss the inclusion of boundaries to the compact manifold, and the applications to extreme solutions.

II. PROPERTIES OF SYMMETRIES

We will be considering solutions of the Euclidean action of $d$ dimensional Einstein gravity (omitting cosmological constant and boundary terms for the meantime)

\[
S_E = -\frac{1}{16\pi G_d} \int_M d^d x \sqrt{\tilde{g}} R_d,
\]

(3)
where $\hat{g}$ is the determinant of the $d$ dimensional metric and $R_d$ is the Ricci scalar. $G_d$ is the $d$ dimensional newton constant. The $d$ dimensional oriented manifold $M$ will in general have a $(d - 1)$ dimensional boundary at infinity which we denote as $\partial M$.

Many solutions of interest admit continuous symmetry groups of at least two parameters, and we assume here the existence of at least a one parameter group. A solution admitting a Killing vector $k$ with closed orbits can be written in terms of $(d - 1)$ dimensional fields, which we refer to as the dilaton $\phi$, gauge potential $A_i$ and metric $g_{ij}$, as

$$ds^2 = e^{\frac{4\pi}{d-2}}(dx^d + A_i dx^i)^2 + e^{\frac{4\pi}{(d-3)(d-2)}}g_{ij}dx^i dx^j,$$

where we take the Killing vector to be $\partial_{\nu}d$ of period $\beta = 2\pi \mu$. In the context of Kaluza-Klein theories it is perhaps more conventional to let $A_i \equiv 2A'_i$; we choose the normalisation here for later convenience when we compare the four dimensional limit of our results with those obtained in \[I\]. The action can be expressed in terms of the lower dimensional fields as

$$S_E = -\frac{1}{16\pi G_{d-1}} \int_{\Sigma} d^{d-1}x \sqrt{g} \left[ R - \frac{4}{d-3}(\partial \phi)^2 - \frac{1}{4}e^{-\frac{4\pi}{d-3}\phi} F^2 \right],$$

where $G_d = \beta G_{d-1}$. We refer to the $(d - 1)$ dimensional manifold we obtain after dividing out by the $U(1)$ isometry as $\Sigma$ with $(d - 2)$ dimensional boundary $\partial \Sigma$. This is precisely the dimensional reduction procedure that is used in Kaluza-Klein theories, hence our notation for the $(d - 1)$ dimensional fields.

If the isometry generated by the Killing vector has fixed point sets, then the metric $g_{ij}$ will be singular at these points. Denote by $\mu_\tau : M \rightarrow M$ the action of the group where $\tau$ is the group parameter. At a fixed point, the action of $\mu_\tau$ on the manifold $M$ gives rise to an isometry $\mu_\tau^* : T_p(M) \rightarrow T_p(M)$ where $\mu_\tau^*$ is generated by the antisymmetric tensor $k_{M;N}$. Vectors in the kernel $V$ of $k_{M;N}$ leave directions in the tangent space at a fixed point invariant under the action of the symmetry. The image of the invariant subspace of $T_p(M)$ under the exponential map will not be moved by $\mu_\tau$, and so will constitute a submanifold of fixed points of dimension $p$ where $p$ is the dimension of the kernel of $V$. Since the rank of an antisymmetric matrix must be even, the dimension of the invariant subspace may take the values $0, 2, \ldots, d$ for $d$ even, and $0, \ldots, (d - 1)$ for $d$ odd. If the fixed point set is decomposed into connected components, each connected component is a closed totally geodesic submanifold of even codimension \[I\].

In four dimensions, the possible fixed point sets are 2-dimensional submanifolds (bolts) and fixed points (nuts). In higher dimensions, generalised bolts and nuts are possible. There are at most $[d/2]$ eigenvalues of $k_{M;N}$, $\{n_i\}$, where $[n]$ denotes the integer part of $n$. If the eigenvalues are rationally related, the action of $\mu_\tau$ will be periodic, with the integers relating the eigenvalues determining the number of rotations in distinct orthogonal planes in $T_p(M)$ induced by one orbit of the isometry. If one pair of eigenvalues are not rationally related, the orbits of a vector in $T_p(M)$ under the action of $\mu_\tau^*$ is dense in the torus $C$ consisting of all vectors of the form $\mu_\tau^{n_1} \mu_\tau^{n_2}$, where $\mu_\tau^{n_1}$ has rank $n$, $\mu_\tau^{n_2}$ has rank $(n - 1)$ and $\mu_\tau^* = \mu_\tau^{1*} \mu_\tau^{2*}$. All scalar invariants must be constant over each torus in $M$ of the form $\exp(C(M))$ for each $X \in T_p(M)$, and since scalar invariants characterise the metric it follows that $\mu_\tau^{1*}$ and $\mu_\tau^{2*}$ must actually correspond to independent isometries of the metric on $M$. One can then take
appropriate combinations of the Killing vectors such that the orbits are periodic; we thus assume that the action of the isometry group is periodic.

It is useful to express this action of the group on the fixed point set in the following way. Let $G$ be a finite or closed Lie group acting on the oriented manifold $M$. We consider the action of an element $g \in G$; we denote fixed points of the action of the isometry as $M^g = \{ x : gx = x \}$, and construct the normal bundle $N^g_x$ over $X^g$. Then in the neighbourhood of the fixed point set the action of $g$ on the normal bundle is effective and $N^g$ may be equivariantly identified with a neighbourhood of $M^g$ in $M$:

$$(z_1, \ldots, z_s) \leftrightarrow (z_1, \ldots, z_s, x_1, \ldots, x_{d-2s}), \quad (6)$$

where we choose coordinates on the fixed point set such that:

$$(x_1, \ldots, x_d) = (x_1, \ldots, x_{d-2s}, 0, \ldots 0), \quad (7)$$

and $z_i \equiv \rho_i e^{\psi_i}$ are complex coordinates in a neighbourhood of $M^g$, which are acted on by $g$ as:

$$g(z_1, \ldots, z_s) = (e^{in_1 \theta} z_1, \ldots, e^{in_s \theta} z_s), \quad (8)$$

where $\theta$ is the group parameter and takes values between 0 and $2\pi$. That is, locally we can decompose the normal bundle as a direct sum of complex line bundles. Expressed in terms of the metric, in a small neighbourhood of a fixed point set of dimension $d - 2k$, we can write the metric as

$$ds^2 = \sum_{i=1}^{k} (d\rho_i^2 + \rho_i^2 d\psi_i^2) + ds^2_{d-2k}, \quad (9)$$

where each $\psi_i$ has period $2\pi$, and the Killing vector is $\partial_\theta = \sum_i n_i \partial_\psi_i$. We shall find this form of the metric to be useful in the following sections.

Where the symmetry group is more than one-dimensional, different choices of the one parameter subgroup may lead to different numbers and locations of nuts and bolts. However topological invariants of the manifold - the Euler characteristic and the Hirzebruch signature - are evidently independent of the choice of circle subgroup. By the Lefschetz fixed point theorem, the Euler characteristic for a compact manifold without boundary may be decomposed (see for example [7]) as:

$$\chi(M) = \sum_i \chi(M^g_i), \quad (10)$$

where we sum over the fixed point sets, and the Euler characteristic of a point is one.

In four dimensions, the G-signature theorem takes the particularly simple form

$$\tau = \sum_{nuts} \cot \frac{p \theta}{2} \cot \frac{q \theta}{2} + \sum_{bolts} Y \cosec^2 \frac{\theta}{2}, \quad (11)$$

where $Y$ is the self-intersection number of a bolt, and the integers $p, q$ characterise the normal bundle over the nut fixed point set. Expanding in powers of $\theta$, one then obtains constraints on the nut and bolt parameters given in [1].
However, in general dimensions, the G-signature theorems take a much more complicated form, and we will not use them here. We mention only the most simple case of nut fixed point sets, assuming that $d$ is even. We may then express the signature as:

$$\tau = \sum_{nud/s} \prod_{i=1}^{d/2} (-i \cot \frac{n_i \theta}{2}).$$

(12)

This is imaginary when the dimension is not a multiple of four, whereas the dimensions of the cohomology groups are real, and the signature vanishes. For fixed point sets of general dimension, we would expect the signature theorem to include for example terms involving the signature of the self-intersection manifold of the fixed point set. Schematically the Euler characteristic is a weighted sum over the number of fixed point sets, whilst the signature depends on the structure of the normal bundle over the fixed point set.

In fact, the form of the signature theorem is an indication that the analysis of higher dimensional Einstein manifolds in terms of fixed point sets is much more complicated than in four dimensions. As we shall see, many simplifications of the analysis in §1 arise from the existence of a type of electromagnetic duality in four dimensions.

We briefly mention here examples of complete non-singular Einstein manifolds which are of interest physically. In order for a compact Einstein space $M$ to admit continuous isometries, the Ricci scalar must be non-negative and, if one excludes the case where $M$ contains flat circle factors, $M$ can admit Killing vectors only if the Ricci scalar is strictly positive.

The obvious examples of positive curvature compact Einstein manifolds are homogeneous manifolds, $G/H$, where $G$ is the isometry group and $H$ is the isotropy group, which admit Einstein metrics. The simplest example is the $d$ dimensional sphere, with canonical metric, which may be viewed as the homogeneous manifold $SO(d+1)/O(d)$. The $SO(d+1)$ isometry group is generated by $(d + 1)$-dimensional anti-symmetric matrices of rank $0, 2, ...$. In the case of rank 2, there is a $(d - 1)$-plane through the origin which is not moved by the rotation, and the intersection of this with the $d$-sphere is a $(d - 2)$-dimensional sphere. Higher rank matrices leave smaller spheres invariant. Another interesting example is complex projective space of complex dimension $n$ which may be viewed as the homogeneous manifold $SU(n+1)/S(U(1)U(n))$; we will discuss this further in §IV.

Many of the manifolds discussed have natural extensions to higher dimensions. For example, one may take the metric product of a two sphere with a $(d - 2)$ sphere; this is a (regular) limit of the Schwarzschild de Sitter solution and we shall consider it further in §V. Furthermore, one could consider inhomogeneous Einstein manifolds such as those constructed in §II and §IV, although we shall not do so here.

### III. ACTION OF COMPACT SOLUTIONS

Given an Euclidean Einstein manifold, we are interested in calculating its action, since this is important in describing the thermodynamics of the system, and gives a measure of the probability for a decay into the instanton to occur. In this section we will rewrite the action in terms of the lower dimensional fields and a $(d - 3)$ form which we will define. For
compact manifolds, we can then obtain an expression for the action entirely in terms of characteristics of the orbit space of the isometry.

The action of a circle subgroup of the isometry group of $M$ defines a fibering $\pi : M - C \rightarrow B$, where $C$ is the fixed point set of the isometry and $B$ is a $d$ dimensional space of non-trivial orbits. The metric on $M$ can be expressed in the form \([\mathbb{H}]\), with the gauge field $A_i$ invariant under the gauge transformation $A_i' = A_i - \partial_i a$ and the gauge invariant field strength being $F_{ij} = \partial_i A_j - \partial_j A_i$. The Bianchi identity implies that:

$$D_{[i} F_{jk]} = 0,$$

or in form notation $dF = 0$, where we define the covariant derivative with respect to the metric $g_{ij}$. The equation of motion for $F_{ij}$ derived from the Lagrangian (5) is

$$D_i (e^{\frac{-4\sqrt{d-2}\phi}{d-3}} F^{ij}) = 0,$$

which may be rewritten in the form $D_i F^{ij} = J^i$ where $J^i$ is the conserved current

$$J^i = \frac{4\sqrt{d-2}}{d-3} (\partial_i \phi) F^{ij}.$$

In form notation, we may express this as $dG = *J$, where $G_{i_1 \ldots i_{d-3}}$ is the dual field strength defined by

$$G_{i_1 \ldots i_{d-3}} = \frac{1}{2\sqrt{g}} \epsilon^{i_1 \ldots i_{d-3}} F_{i_{d-2} i_{d-1}},$$

with $\epsilon_1 \ldots (d-1) = 1$. The Bianchi identity for the field $F$ dualises to give the field equation for $G$, $d * G = 0$.

For clarity we mention here that the action may also be dualised by making the transformations

$$\bar{\phi} = -\phi, \quad \bar{F} = e^{\frac{-4\sqrt{d-2}\phi}{d-3}} * F.$$

It is this duality which we commonly use in supergravity theories; it exchanges the field equations and the Bianchi identities. In the absence of the cosmological term, the equations of motion from the resultant action admit solutions in which the metric is unchanged from the corresponding solution in the original theory but “electric” fields are exchanged for “magnetic” fields. In the presence of the cosmological term, solutions of the equations of motion derived from the dualised action are not solutions of the original equations of motion. The “duality” we use here simply re-expresses the original solution in terms of different fields.

Associated with the conserved current \([\mathbb{H}]\), there is a conserved (“electric”) charge:

$$Q_e = \int_{M^{d-2}} J_i d\sigma^i,$$

or in form notation $Q_e = \int_{M^{d-2}} *J = \int_{M^{d-2}} dG$. It is important to note here that in general dimensions there is no such conserved quantity associated with the dual field strength; there is no “magnetic” charge. In four dimensions, one can define a conserved charge by
\( P_m = \int_{M^2} F \). The Bianchi identity implies that the total charge vanishes for compact solutions, although we may define non-zero charges within closed two-dimensional submanifolds. Expressed in the language of [1], the nut charges associated with individual fixed point sets sum to zero for a compact manifold, which has a simple interpretation in terms of the G-signature theorem ([2]).

For higher-dimensional solutions, one cannot define a unique two sphere at infinity and there is no “magnetic” charge. So as we stated earlier “electromagnetic” duality is a concept confined to four dimensions. However, as we shall see, there is a straightforward generalisation of the decomposition of the action in terms of the properties of the orbit space of the isometry.

The \( d \) dimensional Euclidean action, including boundary and cosmological constant terms, is

\[
S_E = -\frac{1}{16\pi G_d} \int_M d^d x \sqrt{g} (R_d - m) - \frac{1}{8\pi G_d} \int_{\partial M} d^{d-1} x \sqrt{b} (\mathcal{K} - \mathcal{K}_0),
\]

where \( b \) is the induced metric on the boundary and \( \mathcal{K} \) is the trace of the second fundamental form in the \( d \) dimensional metric defined with respect to a suitable background geometry \( \mathcal{K}_0 \).

We choose the cosmological constant term such the solution is Einstein with \( R_{MN} = \Lambda g_{MN} \), which implies that \( m = (d-2)\Lambda \). After dimensional reduction along a closed orbit of the isometry the volume term in the action, in the Einstein frame, becomes

\[
S_E = -\frac{1}{16\pi G_{d-1}} \int_{\Sigma} d^{d-1} x \sqrt{g} [R - \frac{4}{d-3} (\partial \phi)^2 - m e^{\frac{4\phi}{\sqrt{d-2(d-3)}}} - \frac{1}{4} e^{-\frac{4\sqrt{d-2}}{d-3}} \phi F^2].
\]

We can express the action in terms of the dual field strength \( G = *F \), with the appropriate field equations being:

\[
D_{i1} G^{i1...i_{d-3}} = 0; \quad D_{i[j} (e^{-\frac{4\sqrt{d-2}}{d-3}} G_{i1...i_{d-3}}) = 0,
\]

which are equivalent to those given previously, but expressed in coordinate form. To obtain these field equations from a dualised action, we require that the action is stationary under variations of the fields subject to the constraint that the dual field strength is conserved; we thus define the dualised action a constraint term to the action

\[
\bar{S}_E = -\frac{1}{16\pi G_{d-1}} \int_{\Sigma} d^{d-1} x \sqrt{g} [R - m e^{\frac{4\phi}{\sqrt{d-2(d-3)}}} - \frac{4}{d-3} (\partial \phi)^2 - \frac{1}{2(d-3)!} e^{-\frac{4\sqrt{d-2}}{d-3}} \phi G^2 + \frac{1}{(d-4)!} G^{i1...i_{d-3}} D_{i[j} \Psi_{i2...i_{d-3}}].
\]

Then the field equation for \( \Psi \) gives the constraint equation for the \((d-3)\)-form \( G \), whilst variation of \( G \) gives the defining equation for the potential \( \Psi \):

\[
e^{-\frac{4\sqrt{d-2}}{d-3}} G_{i1...i_{d-3}} = (d-3) D_{i[j} \Psi_{i2...i_{d-3}}.
\]
\[ S_E = -\frac{1}{16\pi G d-1} \int_{\Sigma} d^{d-1}x \sqrt{g} \{ R - me^{\sqrt{d-2} \phi} - \frac{4}{d-3} (\partial \phi)^2 + \frac{(d-3)}{2(d-4)!} e^{\frac{4\sqrt{d-2}}{d-3} \phi} (D_{[i_1} \Psi_{i_2...i_{d-3}]}^2)^2 \}. \]

The field equations may be expressed as:

\[ D_i (e^{\frac{4\sqrt{d-2}}{d-3} \phi} D_{[i_1} \Psi_{i_2...i_{d-3}]}^2) = 0; \quad D_i D_{i_1} \Psi_{i_2...i_{d-3}]} = 0, \]

or in form notation as

\[ G = f \bar{G}, \bar{G} = d \Psi \rightarrow d(*fd \Psi) = 0, dd \Psi = 0, \]

where we have introduced the \((d-3)\) form \(\bar{G}\) which is related to \(G\) by the function

\[ f = \exp(\frac{4\sqrt{d-2}}{(d-3)} \phi). \]

We have so far simply followed the prescription of [1], but we find here an important difference. Although the local existence of the potential \(\Psi\) is ensured by the closure of the \((d-3)\)-form \(\bar{G}\), if the periods of \(\bar{G}\) are non-zero, the potential \(\Psi\) cannot be defined globally. Even if the \((d-3)\) cohomology of \(M\) is trivial we cannot guarantee that \(\bar{G}\) has zero periods, since it is defined within the orbit space \(\Sigma\). Since in four dimensions \(\Psi\) is a scalar, this problem did not arise in the discussions of [1].

Suppose \(\bar{G}\) has non-trivial periods; we can partition the \((d-1)\) dimensional manifold \(\Sigma\) into a finite set of neighbourhoods \(\sigma_m\) with \((d-2)\) dimensional boundaries \(\partial \sigma_m\), such that each point in \(\Sigma\) is covered by a finite number of \(\sigma_m\). Although the original \(d\) dimensional manifold has no boundary by definition the \((d-1)\) dimensional manifold \(\Sigma\) will have boundaries at the fixed points of the circle action; the total boundary \(\partial \Sigma\) consists of a disjoint set of boundaries around each fixed point set. Contributions to the boundaries \(\partial \sigma_m\) thus arise both from the boundary of \(\Sigma\) and from the boundaries dividing the \(\sigma_m\).

Within each of the \(\sigma_m\) we may define a \((d-4)\) potential \(\Psi_m\) such that in the overlap \(\cap_{\sigma_m,n}\) between two neighbourhoods \(\sigma_m\) and \(\sigma_n\) the potentials are related by gauge transformations

\[ \Psi_m - \Psi_n = d\omega_{mn}, \]

where \(\omega_{mn}\) is a \((d-5)\) form. Then we can for example express an integral over the entire \((d-1)\) dimensional manifold in terms of integrals over the boundaries of each neighbourhood

\[ \int_{\Sigma} F \wedge \bar{G} = \sum_m \int_{\sigma_m} F \wedge d\Psi_m = \sum_m \int_{\partial \sigma_m} F \wedge \Psi_m. \]

We will find that this particular integral arises below. For example, in the simplest non-trivial case, where we divide \(\Sigma\) into two regions, each of which contains a single fixed point set, we find that
\[
\int_{\Sigma} F \wedge \bar{G} = \int_{M^{d-2}} F \wedge \Psi_1 + \int_{M^{d-2}} F \wedge \Psi_2 + \int_{\partial \sigma_1 \cap \sigma_{1,2}} F \wedge (\Psi_1 - \Psi_2),
\]

where in the last term we have taken account of the opposite orientations of the boundaries, and the \(M_{i}^{d-2}\) are arbitrary surfaces enclosing the fixed point sets. Thus the total integral can be related to integrals over the two fixed point sets, and to a Dirac string type contribution. Note that although the original integral is manifestly independent of the gauge choices for the potentials, individual contributions to the integral will depend on each gauge choice.

Following the approach of [1], we next look for symmetries of the terms in the Lagrangian depending on the potential and the dilaton under global transformations. Although \(\Psi\) cannot necessarily be defined globally, we will find that looking for symmetries of the action will help to indicate a well-defined way to usefully rewrite the action. There is a manifest symmetry under translations of the form

\[
\bar{G} \rightarrow \bar{G} + dA,
\]

where \(A\) is an arbitrary exact \((d-4)\) form, and all other fields are held constant; the associated \((d-4)\) form translational Noether current is

\[
J_T = \bar{G}.
\]

This is simply a statement that the potentials are only defined modulo exact forms, as we discussed above.

There is also a symmetry under a global dilation of the form

\[
\Psi_m \rightarrow b \Psi_m, \quad e^{\frac{4\sqrt{-1} \phi}{d-3}} \rightarrow b^{-2} e^{\frac{4\sqrt{-1} \phi}{d-3}},
\]

with the associated Noether current within each region being

\[
J_{m(D)} = \frac{2}{\sqrt{d-2}} (\ast d\phi) + \frac{1}{2} F \wedge \Psi_m.
\]

Again, if the periods of \(\bar{G}\) are non-zero the dilation current is defined locally within each submanifold \(\sigma_m\); the dilation currents in the intersections of different regions are related as

\[
J_{m(D)} - J_{n(D)} = \frac{1}{2} F \wedge d\omega_{mn}.
\]

In the absence of the cosmological term, both dilations and translations are symmetries of the effective action, and the Noether currents are conserved, but in the presence of a cosmological term the symmetry under the dilation current is broken. That is, using the field equations, we find that

\[
H_D = dJ_D = \Lambda e^{\frac{4\phi}{\sqrt{d-2(d-3)}}} \eta_{d-1},
\]

\[
= \frac{2}{\sqrt{d-2}} (\ast d\phi) + \frac{1}{2} \int_{\Sigma} F \wedge \bar{G},
\]

where \(\eta_{d-1}\) is the volume form on \(\Sigma\). Note that \(H_D\) is totally independent of potentials and is well defined throughout the \((d-1)\)-dimensional manifold. In fact the existence of \(H_D\)
is implied by the field equation for the dilaton derived from the original action (20), as is easily seen if we rewrite (20) in form notation. For comparison with the work of [1], we have derived the existence of $H_D$ by introducing a dilation of the reduced action, but $H_D$ is defined even when $\bar{G}$ has non-zero periods.

This relationship between the cosmological constant and $H_D$ can then be used to rewrite the on-shell action for a compact manifold without boundary as

$$S_E = \frac{1}{8\pi G_d} \int_M \eta \Lambda;$$

$$= -\frac{\beta}{8\pi G_d} \int_{\Sigma} \eta_{d-1} (\Lambda e^{\frac{4\phi}{d-2(d-3)}});$$

$$= -\frac{\beta}{8\pi G_d} \int_{\Sigma} H_D,$$

where in the first equality we express the volume form of $M$ as $\eta_d$. So using the explicit form for $H_D$ we find that

$$S_E = -\frac{\beta}{8\pi G_d} \left\{ \frac{2}{\sqrt{d-2}} \int_{\Sigma} d(\ast d\phi) + \frac{1}{2} \int_{\Sigma} F \wedge G \right\}. \quad (38)$$

We defer discussion of the second term to the following section; the first term can be related to the $(d-2)$ volumes of the fixed point sets as follows. Since this term is globally exact it can be converted into an integral over the boundaries of the manifold $\Sigma$, that is, to an integral over the fixed point set boundaries. Thus we may introduce invariant quantities, the dilation charges, such that at the $a$th fixed point set

$$M_a = \int_{M_a} d^{d-2} x \sqrt{c} (\tilde{n} \cdot \partial \phi), \quad (39)$$

where we integrate over any $(d-2)$ dimensional boundary surrounding the fixed point set. All physical quantities are of course independent of the particular choice of $(d-2)$ dimensional manifold around each fixed point set; all surfaces surrounding the nut or bolt that can be continuously transformed into one another will give the same action.

It is useful at this point to rewrite the dilation charges in terms of a conformally rescaled metric; that is, we decompose the $d$ dimensional metric as

$$ds^2 = e^{-\frac{4\phi}{\sqrt{d-2}}} (dx^d + A_i dx^i)^2 + \tilde{g}_{ij} dx^i dx^j, \quad (40)$$

where $\tilde{g}$ is conformally related to the metric $g$ given in (4). $M_a$ can then be expressed as

$$M_a = \int_{M_a}^{d-2} d^{d-2} x \sqrt{c} (\tilde{n} \cdot \sqrt{g_{dd}}), \quad (41)$$

where $\tilde{c}$ is the induced (conformally rescaled) metric on the boundary and $\tilde{n}$ is the normal to the boundary.

Expressed in this form, it is evident that this term vanishes except when the fixed point set is of dimension $(d-2)$. The integral must be independent of the choice of boundary around the fixed point set. So we can take an arbitrary boundary and then take the limit
that it is the boundary of the fixed point set itself, which necessarily has a vanishing \((d-2)\) dimensional volume element. As we take this limit, the normal derivative is finite, since by definition \(\hat{g}_{dd}\) vanishes on the fixed point set, but is non-zero on any boundary surrounding the fixed point set. Hence the integral must vanish, unless the \((d-2)\) dimensional volume of the fixed point set is non-zero.

It is straightforward to evaluate the term for a \((d-2)\) dimensional fixed point set. In the neighbourhood of the bolt, we can express the metric in the form

\[
ds^2 = d\rho^2 + \rho^2 \left( d\left(\frac{x^d}{\mu}\right) + A_\alpha dx^\alpha \right)^2 + g_{\alpha\beta} dx^\alpha dx^\beta,
\]

where the periodicity of \(x^d\) is as usual \(2\pi\mu\). The \((d-2)\) dimensional metric \(g_{\alpha\beta}\) is independent of \(\rho\), and \(A\) is pure gauge if the normal bundle over the bolt is trivial. The boundary of the fixed point set is at the origin \(\rho = 0\), and we choose the boundary in (41) to be the fixed point set itself. Then

\[
M_a = \frac{1}{\mu} \int_{M_a^{d-2}} d^{d-2}x \sqrt{g} = \frac{V_a}{\mu},
\]

where \(V_a\) is the \((d-2)\) volume of the fixed point set, evaluated in the original metric. This gives the intrinsic contribution to the action from \((d-2)\) dimensional bolts. So the total action becomes:

\[
S_E = -\sum_a \frac{V_a}{4G_d} - \frac{\beta}{16\pi G_d} \int_{\Sigma} F \wedge \bar{G},
\]

(44)

where only \((d-2)\) dimensional fixed point sets contribute to the first term. We hence find that there are intrinsic contributions to the action from the \((d-2)\) volumes of the fixed point sets. Another way of stating this is to say that there is a contribution to the action from the volume of the boundary of the orbit space. There are additional contributions arising from the nut behaviour of the fixed point sets and from non-trivial \((d-3)\) cohomology of \(\Sigma\) which we shall now discuss.

IV. COHOMOLOGY CONTRIBUTIONS TO THE ACTION

In four dimensions, our expression for the action reduces to (1) in agreement with (1). Since we can write \(\bar{G} = d\Psi\) globally, where \(\Psi\) is a scalar function, we can express the integral over \(\Sigma\) as an integral over only boundaries of fixed point sets and thus

\[
S_E = -\sum_a \frac{V_a}{4G_4} - \frac{\beta}{16\pi G_4} \sum_a \int_{M_a^2} F \wedge \Psi.
\]

(45)

The potential is a scalar function and so

\[
\int_{M_a^2} F \wedge \Psi = \Psi_a \int_{M_a^2} F,
\]

(46)
i.e. the integral over the potential terms in the action reduces to an integral of the 2-form over a surface surrounding the fixed point set. This integral is related to the first Chern
number of the $U(1)$ bundle over the space of non-trivial orbits, and, as is discussed in \[1\], one can show that the nut charge is given by $\beta/8\pi pq$ for a nut of type $(p, q)$, and by $\mathcal{Y}/8\pi$ for a bolt of self-intersection number $\mathcal{Y}$.

In higher dimensions we cannot in general reduce the integral over $\Sigma$ to integrals over only fixed point sets; there can also be contributions related to the non-trivial $(d - 3)$ cohomology of $\Sigma$. We will postpone the discussion of the general case, and assume that $\bar{G}$ has zero periods so that we can introduce a global potential $\Psi$. Then the integral over $\Sigma$ becomes

$$-\frac{\beta}{16\pi G_d} \int_{\Sigma} F \wedge \bar{G} = -\sum_a \frac{\beta}{16\pi G_d} \int_{M_{d-2}^a} F \wedge \Psi,$$

(47)

where we take the integrals over $(d - 2)$ dimensional manifolds surrounding each fixed point set. Thus we can associate contributions to the action from the nut behaviour of each fixed point set. As in four dimensions, the total contribution is gauge invariant, but individual contributions do depend on the choice of gauge.

The form of (47) will be particularly simple when the gauge field is independent of the coordinates of the fixed point set. That is, if the fixed point set can be surrounded by a $(d - 2)$ manifold which is the product of the the $(d - 2k)$ dimensional fixed point set, and a $(2k - 2)$ dimensional hypersurface of small characteristic size $\epsilon$, the form of the integral simplifies because the metric is a product metric. In physical terms, the requirement is that there are no Dirac string type singularities associated with the fixed point set in the $d$ dimensional manifold. So such a decomposition will always be possible when the second cohomology class of the $d$ dimensional manifold is trivial.

We can show this as follows; if in the neighbourhood of the fixed point set the metric can be expressed as a product of a $(d - 2k)$ dimensional metric $g(x^i)$ and a $(2k - 2)$ dimensional metric $g(\theta^i)$, then $\bar{G}$ is also a product

$$\bar{G} = \bar{G}_1(\theta^i) \wedge \bar{G}_2(x^i).$$

(48)

That is, the dual field strength can be expressed as the exterior product of a $(d - 2k)$ form and a $(2k - 3)$ form, whose only non vanishing components are the $\theta^i$ and $x^i$ components respectively. The potential can also be expressed as the exterior derivative $\Psi = \Psi_1(\theta^i) \wedge \Psi_2(x^i)$ where $\Psi_1$ is a $(2k - 4)$ form and $\Psi_2$ is a $(d - 2k)$ form. Then,

$$d\Psi_1 \wedge \Psi_2 + \Psi_1 \wedge d\Psi_2 = \bar{G}_1 \wedge \bar{G}_2.$$  

(49)

Now it is easy to see that $\bar{G}_2 = \Psi_2$ is closed and $\bar{G}_1 = d\Psi_1$ and so

$$\int_{M_{d-2}} F \wedge \Psi = \int_{M_{2k-2}} F \wedge \Psi_1 \int_{M_{d-2k}} \bar{G}_2.$$  

(50)

Furthermore, $\bar{G}_2$ is the volume form of the fixed point set, and hence,

$$\int_{M_{d-2}} F \wedge \Psi = V_{d-2k} \int_{M_{2k-2}} F \wedge \Psi_1,$$

(51)

where $V_{d-2k}$ is the $(d - 2k)$ volume of the fixed point set in the metric $g(x^i)$.
For a \((d-4)\) dimensional fixed point set, the integral reduces further to

\[
\int_{M^{d-2}} F \wedge \Psi = \Psi f V \int_{M^2} F,
\]

(52)

where \(\Psi f\) is a scalar function, evaluated at the fixed point set. The integral is as usual given by \(\beta/pq\) where the normal bundle over the fixed point set is characterised by the two integers \((p,q)\), and hence

\[
-\frac{\beta}{16 \pi G_d} \int_{M^{d-2}} F \wedge \Psi = -\frac{\beta^2}{16 \pi pq G_d} \Psi f V_{d-4},
\]

(53)

which gives the required answer in four dimensions.

We can also obtain this answer by working with an explicit form of the metric. In a small neighbourhood of the fixed point set, we can always express the metric in the form

\[
ds^2 = (d\rho_1^2 + \rho_1^2 d\psi_1^2) + (d\rho_2^2 + \rho_2^2 d\psi_2^2) + d\bar{s}_{d-4}^2,
\]

(54)

since the normal bundle can always be locally be decomposed into a sum of complex line bundles. If the second cohomology class is trivial such a decomposition is valid throughout the neighbourhood of the fixed point set. In general, although locally we can bring the metric into this form, there will be non-trivial mappings between different neighbourhoods of the fixed point set. This will always be so if the second cohomology of the original \(d\)-dimensional manifold is non-trivial.

We take the Killing vector to be \(k = \partial_{\psi_1} + \partial_{\psi_2}\) which has a zero at \(\rho_1 = \rho_2 = 0\) and introduce the coordinate \(\bar{\psi}_2 = \psi_2 - \psi_1\) such that \(\partial_{\bar{\psi}_2}\) is invariant along orbits of \(\partial_{\psi_1}\). We then find that

\[
ds^2 = (d\rho_1^2 + \rho_1^2 d\psi_1^2) + (d\rho_2^2 + \rho_2^2 (d\bar{\psi}_2 + d\psi_1)^2) + d\bar{s}_{d-4}^2.
\]

(55)

We introduce new coordinates

\[
\rho_1 = \epsilon \cos \theta, \quad \rho_2 = \epsilon \sin \theta,
\]

(56)

where the range of the angular coordinate is from 0 to \(\pi/2\). Dimensionally reducing, we find that the \((d-1)\) dimensional fields are

\[
ds^2 = d\epsilon^2 + \epsilon^2 d\theta^2 + \epsilon^2 \sin^2 \theta \cos^2 \theta d\bar{\psi}_2^2 + d\bar{s}_{d-4}^2;
\]

\[
A_{\bar{\psi}_2} = \sin^2 \theta;
\]

\[
g_{\psi_1\psi_1} = \epsilon^2 = \exp(-4\phi/\sqrt{d-2}),
\]

(57)

where we give the conformally rescaled metric defined in (40) for notational simplicity. We can then show that

\[
F_{\theta\bar{\psi}_2} = 2 \sin \theta \cos \theta,
\]

(58)

is the only independent component of the gauge field strength. Dualising the two form field, we find that the only independent component of the \((d-3)\) form field is
where \( g_{d-4} \) is the determinant of the metric on the fixed point set in the conformally rescaled metric. Using the relation between this field and the potential, we can extract the form of the potential in the vicinity of the fixed point sets as

\[
\Psi = (\Psi_f + O(\epsilon^2))\eta,
\]

where \( \Psi_f \) is a constant, and \( \eta \) is the volume form of the fixed point set. Now the potential term in the action can be written as in (57), where \( M^{d-2} \) is any \((d-2)\) manifold surrounding the fixed point set; thus choosing it to be the product manifold of the \((d-4)\) dimensional fixed point set and a surface of constant \( \epsilon \to 0 \), we find

\[
S_E^{(2)} = -\frac{\beta^2}{16\pi G_d} \Psi_f V_{d-4}.
\]

where in the limit \( \epsilon \to 0 \) this is the only contributing term. So we have explicitly shown that the integral may be decomposed as in (53), where we set \( p, q = 1 \); it is straightforward to extend the proof to general integers by taking a Killing vector \( k = p\partial_\psi_1 + q\partial_\psi_2 \).

Returning to the general case, when \( \bar{G} \) has non-zero periods, following (29) we can express the Dirac string terms in the action as

\[
-\frac{\beta}{16\pi G_d} \int_{\Sigma} F \wedge \bar{G} = -\frac{\beta}{16\pi G_d} \left\{ \sum_a \int_{M_a^{d-2}} F \wedge \Psi_a + \sum_{m<n} \int_{\partial \sigma_m \cap \sigma_{m,n}} F \wedge d\omega_{mn} \right\}.
\]

That is, we decompose the integral over the entire manifold into contributions from each fixed point set, and from the Dirac string type behaviour of the \((d-1)\)-dimensional manifold. We have implicitly assumed here that we can introduce a single potential within the neighbourhood of each fixed point set but it is straightforward to relax this condition. Note that even if one can further decompose the integral of \( F \wedge \Psi \) at individual fixed point sets, one still has to allow for the non-trivial \((d-3)\) cohomology of the \((d-1)\)-dimensional manifold, and the integrals do not take simple forms.

Since under gauge transformations of the potentials both the fixed point set and Dirac string terms change it is more useful to evaluate the total contribution to the action from the nut behaviour and \((d-3)\) cohomology; one cannot associate a gauge invariant contribution to the action from the nut behaviour of any one fixed point set.

In decomposing the integral over \( \Sigma \) we have used the fact that \( F \wedge \bar{G} \) is closed to introduce a local potential \( F \wedge \Psi \). We could of course introduce a local potential \( A \wedge \bar{G} \) instead. By definition, \( F \) is not globally exact; if it were we could gauge transform our original circle coordinate \( \tau \) and remove all gauge field contributions. So on introducing a local potential of this form we would still have to partition the orbit space and define the transformations of the potentials between regions. This illustrates further that there is little meaning in identifying contributions to the action from particular parts of \( \Sigma \) particularly when \( \bar{G} \) is not globally exact.

It is interesting to consider a class of compact Einstein manifolds admitting no non-trivial fixed point sets. One can regard the \((2n + 1)\) sphere as a \(U(1)\) bundle over \( CP^n \) with the
action of the $U(1)$ being trivial in the sense of having no fixed point sets. The Kaluza Klein two form is the unique self dual two form in $CP^n$ and the action of the dilaton is trivial. The dual field is defined by $G = \ast F = F^{n-1}$, which necessarily has non trivial periods, and thence the potential $\Psi$ is not well defined globally.

Since there are no fixed point sets, the boundary of $\Sigma$, the set of boundaries of the fixed point sets, vanishes. We can integrate the potential term over $\Sigma$ to find

$$S_E = -\frac{\beta}{8\pi G_d} \int_{\Sigma} (F \wedge \ast F), \quad (63)$$

and the integral reduces to the volume of the base manifold. Evaluation of this integral by the division of $\Sigma$ into regions and the introduction of potentials would give the same answer. Now this compares to an action which we can explicitly evaluate to be

$$S_E = -\frac{\Lambda}{8\pi G_d} V_{2n+1}(S^{2n+1}). \quad (64)$$

These two expressions appear different. However, if we chose the standard $SU(n + 1)$ invariant metric on the sphere, the metric will not be Einstein. We thus choose on the sphere the canonical metric with a constant curvature of one, which is $SU(n + 1)$ invariant, so that the fibration has totally geodesic fibres onto the symmetric metric on the base manifold [3]. The fibre is a great circle of the sphere, with length $2\pi$, and one can then see that the two expressions for the action are equivalent.

Let us consider two further examples for which $G$ is globally exact. We discuss first the $d$-dimensional sphere, endowed with canonical metric, of radius $(d - 1)^{1/2} \Lambda^{1/2}$, which satisfies $R_{ij} = \Lambda g_{ij}$. Evaluating the action directly from (37) we find that

$$S_E = -\frac{\Lambda}{8\pi G_d} V_d(\sqrt{\frac{d - 1}{\Lambda}}), \quad (65)$$

where $V_d(a) = 2\pi^{(d+1)/2}a^d/\Gamma(\frac{d}{2}+1)$ is the volume of a $d$-dimensional sphere of radius $a$.

The action of a rank 2 generator of the $SO(d+1)$ isometry group will leave fixed a single $(d - 2)$ dimensional spherical bolt. There is no contribution to the action from the potential term, since the second cohomology class is trivial, and hence the field in (42) is pure gauge. So we can obtain the action as:

$$S_E = -\frac{V_{d-2}(\sqrt{\frac{d-1}{\Lambda}})}{4G_d}, \quad (66)$$

which is equivalent to the previous expression.

The fixed points of the action of a rank 4 or higher generator of the $SO(d+1)$ isometry group can be regarded as the intersection of the $(d - 2)$ dimensional fixed point sets of independent generators of the Lie algebra; that is, we can decompose the circle action as $q = \sum n_i \partial \psi_i$. The action evaluated using the fixed points of such a circle subgroup will be a potential term specified entirely by the periodicity of the action, these integers and the volume of the fixed point set. The Lefschetz fixed point theorem tells us that the action of a rank $2k$ generator leaves fixed a single sphere of dimension $(d - 2k)$, or two points if $d = 2k$. 

16
If we consider radially extended $U(1)$ bundles over compact homogeneous manifolds, such as those constructed in [3], we can progress further in the evaluation of the potential term. The simplest example is a complex projective space $CP^n$ of real dimension $d = 2n$; although such instantons seem to have little physical relevance, since there exists no Lorentzian continuation, they illustrate several important points. Suppose that

$$g = \{e^{i\xi_0}, ..., e^{i\xi_n}\},$$

(67)

is an element of the torus group $T^{n+1}$ acting on the complex coordinates $z_i$ of $CP^n$ as:

$$(e^{i\xi_0}, ..., e^{i\xi_n}) \cdot (z_0, ..., z_n) = (e^{i\xi_0}z_0, ..., e^{i\xi_n}z_n),$$

(68)

where the definition of complex projective space is that $(z_0, ..., z_n) \in \{C^{n+1} - \{0\}\}/C = CP^n$. The action of $g$ leaves a point fixed if

$$(e^{i\xi_0}z_0, ..., e^{i\xi_n}z_n) = (z_0, ..., z_n).$$

(69)

This requires that

$$e^{i\xi_k}z_k = e^{i\xi}z_k \quad k = 0, ..., s$$

(70)

for some $\xi$ which is determined uniquely since at least one of the $z_k$ is non-zero. In fact, $\xi$ must equal one of the $\xi_k$. Then we can express the manifold $X^g$ which is fixed under the action of $g$ as $X^g = \bigcup X(\xi)$, with

$$X(\xi) = \{(z_0, ..., z_n) \in CP^n : \xi_k \neq \xi \rightarrow z_k = 0, \}$$

(71)

Thus the action of the isometry leaves fixed a set of complex projective spaces of various dimensions.

Now, for a complex projective space $CP^n$, the odd Betti numbers vanish and the even Betti numbers are all equal to one; so the Euler characteristic is given by $(n+1)$. From the Lefschetz fixed point theorem (11), we can then constrain the fixed point sets of the group action. For example, if an element of the isometry group leaves invariant a submanifold isomorphic to $CP^{n-1}$, it must also leave invariant a single point. The natural interpretation of the action of this element is that it leaves fixed the origin $(z_0, ..., 0)$ and the $CP^{n-1}$ submanifold at “infinity” $(0, z_1, ..., z_n)$, where we use quotation marks because the manifold is of course compact. Evidently we may similarly constrain the action of other elements of the isometry group. In particular, there must exist a generator $g$ contained in the isometry group which has $(n+1)$ nut fixed point sets, at the origin $(z_0 : ... : 0)$ and at the “poles at infinity” $(0 : ... : z_i : ... : 0)$.

We can express the metric on $CP^n$ in the following way. Constructing a $U(1)$ bundle over $CP^{n-1}$, with its standard Fubini Study Einstein-Kähler metric, we obtain

$$ds^2 = 2(n+1)\Lambda^{-1}\{d\theta^2 + \sin^2\theta \cos^2\theta(d\tau - A)^2 + \sin^2\theta ds^2_{2(n-1)}\},$$

(72)

with endpoints at $\theta = 0$ and $\theta = \pi/2$. We choose the normalisation of the metric on $CP^{n-1}$ such that $R_{ij} = 2ng_{ij}$, and $dA$ can be chosen as the Kähler form on $CP^{n-1}$. Then the
resulting metric is isometric to the standard Fubini Study metric on $CP^n$. In particular, the Killing vector $\partial_\tau$ has a nut at the “origin” $\theta = 0$ and a $CP^{n-1}$ bolt at “infinity” $\theta = \pi/2$. The period of this circle action is $2\pi$.

This form of the metric is particularly useful in the evaluation of the action; we find that

$$S_E = -\frac{\Lambda}{8\pi G_d} V_d(M), \quad (73)$$

where $V_d(M)$ denotes the volume of the $CP^n$ in the metric (72), or explicitly,

$$S_E = -\frac{d}{8n G_d} \left( \frac{d}{\Lambda} \right)^{n-1} V_{2(n-1)}, \quad (74)$$

where $V_{2(n-1)}$ is the volume of the base manifold. It can be verified that the bolt at infinity contributes a volume term to the action

$$S_E^{(1)} = -\frac{1}{4 G_d} \left( \frac{d}{\Lambda} \right)^{n-1} V_{2(n-1)}, \quad (75)$$

whilst the cohomology of $\Sigma$ contributes a term

$$S_E^{(2)} = -\frac{1}{4n G_d} \left( \frac{d}{\Lambda} \right)^{n-1} V_{2(n-1)}, \quad (76)$$

and thence the two contributions do sum to (74) as required. This provides a verification that our decomposition of the action holds; the reason for choosing projective spaces as an example derives from the simplification in the integral over $\Sigma$ because of the form of the cohomology structure for such spaces. Part of the integral reduces to an integral of $F \wedge *F$ over $CP^{n-1}$, where we take the dual in the metric on the base space. Since there is only one independent closed but non exact $(n-1)$ form, the integral is then proportional to the volume of the base space.

Since the $(d-3)$ form $\bar{G}$ has trivial cohomology, one can find a potential globally of the form

$$\Psi = \gamma(\theta, \Lambda)(*F), \quad (77)$$

where we take the dual in the metric on $CP^{n-1}$; the function $\gamma$ can of course be explicitly determined. One can add an arbitrary constant to this function to ensure that it vanishes either at the origin or at “infinity”, with the latter being the more natural choice. Thus we can convert the volume integral over $\Sigma$ to integrals over the boundaries of the fixed point sets; for instance, in the vicinity of the bolt, the form of the reduced metric is

$$ds^2 \propto (d\epsilon^2 + ds_{2n}^2), \quad (78)$$

and we can take the surrounding surface $M^{2n}$ to be surface of arbitrary small $\epsilon$. The two integrals over the fixed point sets will sum to (76) but the individual contribution from each will depend on the gauge choice for $\Psi$. 

18
V. ENTROPY AND THE COSMOLOGICAL INTERPRETATION

For vacuum gravity, our results have an interpretation in terms of the entropy associated with the fixed point sets and Dirac string behaviour. The derivation of the gravitational entropy of compact solutions is perhaps less familiar than that of black hole solutions; following [1] we introduce a partition function \( Z \) for the canonical ensemble

\[
Z = \sum_m \langle g_m | g_m \rangle ,
\]

(79)

where \( |g_m\rangle \) is an orthonormal basis of states for the gravitational field with a given value of \( \Lambda \). It is important to realise that for compact solutions there is no externally imposed temperature or potential; the probability of each state is then \( p_m = Z^{-1} \) and the entropy is given by

\[
S = \sum_m p_m \ln p_m = \ln Z.
\]

(80)

As usual, by the stationary phase approximation, one would expect the dominant contribution to the partition function \( Z \) which is defined as:

\[
Z = \int d[\hat{g}] \ e^{-S_E[\hat{g}]},
\]

(81)

to come from metrics near a solution \( \hat{g}_0 \) of the classical field equations and thus the value of \( \ln Z \) to be approximately \( S_E[\hat{g}_0] \). Then the entropy will be given by:

\[
S = \sum_a \frac{V_a}{4G_d} + \frac{\beta}{16\pi G_d} \int_\Sigma F \wedge \bar{G}.
\]

(82)

Hence not only do the \((d-2)\) bolts have an entropy equal to a quarter of their \((d-2)\) volume, but there is also a contribution to the entropy from the nut behaviour of the fixed point sets, and from the \((d-3)\) cohomology of \( \Sigma \). This gives the extension to general dimensions of the result given in [1].

The cosmological relevance of this result is as follows; according to the no boundary proposal, the quantum state of the universe is defined by path integrals over Euclidean metrics on compact manifolds \( M \). One usually considers this proposal in a four dimensional context, but the same ideas follow for higher dimensional theories. We give only a brief summary of the procedure here; further discussion may be found in [10]. The subsequent Lorentzian evolution is described by initial data on a zero momentum hypersurface \( \Sigma_i \) of dimension \((d-1)\). If \( M \) is simply connected, then the hypersurface divides the manifold into two parts, \( M_\pm \), which are usually assumed to have equal action.

One then defines a path integral over all metrics on \( M_\pm \) that agree with the induced metric on the hypersurface \( \Sigma_i \); this gives the wavefunction of the universe \( \Psi(b_{ij}) \)

\[
\Psi(b_{ij}) = \int d[g] \exp(-S_E(g)),
\]

(83)

where \( b \) is the induced metric on the boundary. The Euclidean action is given by
\[ S_E = -\frac{1}{16\pi G_d} \int_{M_+} d^d x \sqrt{g} [R - m] - \frac{1}{8\pi G_d} \int_{\Sigma_i} d^{d-1} x \sqrt{b} K_{\Sigma_i}, \] (84)

where \( K_{\Sigma_i} \) is the trace of the second fundamental form on the boundary \( \Sigma_i \); since we regard the hypersurface as initial data for the subsequent Lorentzian evolution, the hypersurface must have zero momentum, and this geometry term vanishes.

The absence of any externally imposed temperature allows us to define a partition function, \( Z = |\Psi(b_{ij})|^2 \), which we interpret as a probability of the process occurring, and an entropy which is given by

\[ S = -2S_E. \] (85)

Now one can decompose the volume term in the original action in terms of the fixed point sets of a Killing vector; we are assuming that the manifold can be divided symmetrically, so that the action of \( M_+ \) is equal to half the action of \( M \). Thence the entropy will be given by the sum over the fixed point sets in the original compact manifold; in the subsequent Lorentzian evolution, these fixed point sets will have the interpretation of, for example, black hole horizons.

We give two examples here; creation of a \( d \) dimensional universe with positive cosmological constant is described by a \( d \) sphere. For definiteness, we consider the five dimensional sphere with its standard round metric

\[ ds_5^2 = d\chi^2 + \cos^2 \chi [d\rho^2 + \frac{1}{4} \sin^2 \rho((d\theta^2 + \sin^2 \theta d\psi^2)) + (dx^5 + \cos \theta d\psi)^2]], \] (86)

where \(-\pi/2 \leq \chi \leq \pi/2, 0 \leq \rho \leq \pi, 0 \leq \theta \leq \pi, 0 \leq \psi \leq 2\pi\) and \(0 \leq x^5 \leq 4\pi\). Then we can calculate the action directly to be

\[ S_E = -\frac{\pi^2}{2G_5}. \] (87)

One can also calculate the action by looking at the fixed point sets of the Killing vector \( \partial_5 \), which generates the Hopf fibration on the three sphere \( \rho, \chi = \) constant; there are fixed point sets at \( \rho = 0, \pi \). We can choose the potential so that each fixed point set contributes equally to the action, that is, so that the contribution from each is \(-\pi^2/4G_5\).

One can consider the creation of a five dimensional universe by taking a tunnelling geometry \(-\pi/2 \leq \chi \leq 0\), with the Lorentzian section described by \( \chi = it \) with \( t \) positive. This gives a five dimensional expanding de Sitter universe. We now only include half of each fixed point set, and thus the action for the compact manifold is half of (87), as required. The total entropy is given by minus twice the action, and we can associate it with contributions from each of the fixed point sets.

If we treat the Killing direction as compact, these fixed point sets correspond to monopoles; the effective four-dimensional Lorentzian solution describes pair creation of monopoles within an expanding background, where there is an entropy associated with each monopole. This interpretation is discussed in [11]; note that the example is illustrative but not physically realistic, since the compact direction is also expanding.
As a second example, we could consider the Euclidean Schwarzschild de Sitter solution in general dimensions, with suitable choice of parameters to ensure regularity. The latter choice is somewhat subtle [12], and we shall not discuss it here; one would however expect to be able to choose an imaginary time Killing vector with fixed point sets such that the action for the compact solution is one quarter of the volumes of the black hole and cosmological horizons.

Upon choosing an appropriate initial value hypersurface, the subsequent Lorentzian evolution will describe pair creation of Schwarzschild black holes within a cosmological background. The entropy for the process will be given by minus the original action, that is, one quarter of the horizon volumes; the exponential of the entropy gives a measure of the probability of the pair creation. We could take these ideas further; for example, the Page solution [13] would be an instanton for cosmological pair creation of Taub-Bolts.

VI. GENERALISATION TO SUPERGRAVITY THEORIES

It is interesting to consider whether one can extend the ideas of entropy associated with the fixed point sets of isometries to compact solutions of theories involving not only the graviton, but also other fields. We will consider here a generic action of the form

$$S_E = -\frac{1}{16\pi G_d} \int_M d^d x \sqrt{\hat{g}} [R - e^{-b\Phi} m - (\partial \Phi)^2 - e^{-a\Phi} H_{p+1}^2],$$

(88)

where $\Phi$ is the dilaton, and $H_{p+1}$ is a $(p+1)$ form. Depending on the values of $a$, $b$ and $p$, this will give the appropriate action for Einstein-Maxwell theories coupled to a dilaton, and for particular limits of supergravity theories. Using the field equations we can rewrite the action as

$$S_E = -\frac{1}{16\pi G_d} \int_M d^d x \sqrt{\hat{g}} [2e^{-b\Phi} \Lambda - \frac{2p}{(d-2)} e^{-a\Phi} H_{p+1}^2].$$

(89)

Isometries of solutions must map not just the graviton, but also the other fields, into themselves. If we assume the existence of a one parameter isometry group, we can dimensionally reduce along closed orbits of the Killing vector and re-express the action in terms of the $(d-1)$ dimensional fields.

From the $d$ dimensional gauge field, we will obtain a $(d-1)$ dimensional $(p+1)$ form $H_m$ and a $(d-1)$ dimensional $p$ form $H_e$. We will call the former the “magnetic” part of the field, and the latter the “electric” part of the field. The reason for this terminology is that we will later analytically continue the solution, and interpret the Killing direction as the imaginary time. With this interpretation, the $(d-1)$ dimensional gauge field arising from the metric must vanish if a Lorentzian evolution is to exist, since otherwise the Lorentzian and Euclidean metrics could not both be real. This then implies that the imaginary time Killing vector has only $(d-2)$ dimensional fixed point sets, which we will interpret as horizons in the Lorentzian continuation.

It is perhaps unnecessary to assume that the Euclidean metric is real. Since we allow electric gauge fields which are pure imaginary on the Euclidean section, we should also permit the Euclidean metric to be complex provided that the metric is real in the Lorentzian
continuation. However, few useful complex metrics of this type are known, and we shall not consider them here.

Let us take the \((p+1)\) form to be pure magnetic. Since \(F\) is pure gauge, the potential \(\Psi\) vanishes and the dilation current is well-defined globally. We can regard the cosmological constant and magnetic field terms in the \((d-1)\) dimensional action as breaking the symmetry under dilations so that

\[
dJ_D = \eta_{d-1} \left\{ e^{-b\Phi} \Lambda e^{\frac{4\phi}{\sqrt{\Lambda} - 2(d-3)}} - \frac{p}{(d-2)} e^{-a\Phi} \mathcal{H}_{p+1}^2 e^{-\frac{4\phi}{\sqrt{\Lambda} - 2(d-3)}} \right\},
\]

where the dilation current is defined as before. Relating this to the on shell action given above, we find that

\[
S_E = -\frac{\beta}{8\pi G_d} \int \Sigma dJ_D = -\sum_a \frac{V_a}{4G_d},
\]

where we need only sum over \((d-2)\) dimensional fixed point sets, since we are assuming that the \((d-1)\) dimensional gauge field is trivial. Note that the gradient of the dilaton field \(\Phi\) does not contribute to the divergence of the dilation current, and its contribution to the action vanishes on shell. So we can see that the entropy, which is still given by minus the action in the saddle-point approximation, since there is no externally imposed temperature, is again given entirely by the contributions from the fixed point sets.

If we assume instead that the field is pure electric, the action can be decomposed in terms of the fixed point sets, and a volume integral of the field

\[
S_E = -\sum_a \frac{V_a}{4G_d} + \frac{1}{8\pi G_d} \int_M d^d x \sqrt{\tilde{g}} e^{-a\Phi} \mathcal{H}_{p+1}^2.
\]

That is, the action for the solution depends not only on the fixed point sets, but also on a volume integral of the gauge field.

These results again have an interesting interpretation in the context of cosmological pair creation. Let us take the Killing vector to be \(\partial_\tau\), and interpret \(\tau\) as the imaginary time; we then divide the compact manifold along an appropriate zero momentum hypersurface. For a pure magnetic two form, there are no boundary contributions to the action, and the total entropy is given by

\[
S = \sum_a \frac{V_2}{4G_4}
\]

where the summation runs over all fixed point sets of the Killing vector \(\partial_\tau\) in \(M\) and we restrict to four dimensions since one cannot define magnetic charge in higher dimensions.

For a pure electric field, one should however include a boundary term in the action to obtain the correct equations of motion when the electric charge is held fixed [14]. That is, we want to use an action whose variations give the Euclidean equations of motion when the variation fixes the boundary data on the initial value hypersurface \(\Sigma_i\). The appropriate action is then
\[ S_E^{\text{tot}} = \frac{1}{2} S_E - \frac{1}{4\pi G_d} \int_{\Sigma_i} d^{d-1}x \sqrt{g} e^{-a\Phi} \bar{F}^{\mu\nu} n_\mu \bar{A}_\nu, \]  
(94)

where \( S_E \) is the action for the total manifold given above and \( n \) is the normal to the boundary. Note that our notation for the gauge field is intended to differentiate between the \( d \) dimensional fields, and the \((d-1)\) dimensional gauge field we obtain upon dimensional reduction. One can convert the remaining Maxwell field term in the volume part of the action to a surface term
\[ S_E^{(2)} = \frac{1}{8\pi G_d} \int_{M^+} d^d x \sqrt{g} e^{-a\Phi} \bar{F}^{\mu\nu} D_\mu \bar{A}_\nu; \]
\[ = \frac{1}{4\pi G_d} \int_{\Sigma_i} d^{d-1}x \sqrt{g} e^{-a\Phi} \bar{F}^{\mu\nu} n_\mu \bar{A}_\nu, \]  
(95)

where we have used the equation of motion for the gauge field
\[ D_\mu (e^{-a\Phi} \bar{F}^{\mu\nu}) = 0. \]  
(96)

Thus, the two gauge field terms in the action cancel out, and the entropy is again given by the summation over the fixed point sets in \( M \) (93). Such a result was found explicitly in [15] for cosmological production of black hole pairs in four dimensions; the entropy is generally given by the one quarter of the area of the black hole horizon, plus one quarter of the area of the cosmological horizon. Note that our analysis breaks down for production of extreme black holes, since we would find that there was an inner boundary of which we would have to take account. We should also mention that we are implicitly assuming that we can find an appropriate non-singular choice of gauge.

We give as an example a particular limit of the four dimensional Reissner-Nordström de Sitter instanton
\[ ds^2 = \frac{1}{A} (d\chi^2 + \sin^2 \chi d\psi^2) + \frac{1}{B} (d\theta^2 + \sin^2 \theta d\phi^2), \]  
(97)

where \( \chi \) and \( \theta \) both run from 0 to \( \pi \), and the other coordinates have period \( 2\pi \). The dilaton field is trivial; this is a solution of Einstein-Maxwell theory. The interpretation of this solution is of pair creation of charged black holes within a cosmological background. This solution is obtained from the Reissner-Nordström de Sitter solution in the limit that the black hole and cosmological horizons are at the same radius.

The cosmological constant is given by \( \Lambda = (A + B)/2 \), and the magnetic and electric gauge fields are
\[ \bar{F}_{\text{magn}} = q \sin \theta d\theta \wedge d\phi; \]
\[ \bar{F}_{\text{elec}} = -iq \frac{B}{A} \sin \chi d\chi \wedge d\psi, \]  
(98)

where the magnetic/electric charge is defined by \( q^2 = (B - A)/2B^2 \).

One can compute the action for the magnetic instanton by looking at the fixed point sets of \( \partial_\psi \); there are two spherical fixed point sets at \( \chi = 0, \pi \), and thus the action is given by
\[ S_E = -\frac{1}{4G_4} (2 \times \frac{4\pi}{B}) = -\frac{2\pi}{BG_4}. \] (99)

One can compute the action for the electric instanton by looking at the fixed point sets of \( \partial \phi \), and treating the field as “magnetic” with respect to this isometry; we then find that the action for the electric instanton is \(-2\pi/AG_4\).

This gives the correct actions for the compact solutions; to describe the Lorentzian evolution, we choose the boundary which subdivides the instanton to be the hypersurface \( \psi = 0, \; \psi = \pi \), where the coordinate \( \psi \) parametrises the imaginary time. Since the division of the manifold \( M \) must divide the action symmetrically, and both fixed point sets of \( \partial \phi \) in the original manifold contribute equally, one must include half of each fixed point set in each of \( M_+ \) and \( M_- \).

For the magnetic solution, the action is half of (99), and thus the total entropy for the process is given by \( 2\pi/B \). For the electric solution, we must add a boundary Maxwell field term, and the entropy is given by the same expression. The pair creation rates, which are given by the exponentiation of the entropy, are identical in the two cases. Note that our actions and entropies are in agreement with those calculated in [14], [15].

In discussing higher dimensional generalisations we run into the problem that few compact solutions of such theories have been constructed. We therefore discuss here the simplest generalisation to a limit of the five dimensional Reissner-Nordström de Sitter solution which is given by

\[ ds^2 = \frac{1}{A} (d\chi^2 + \sin^2 \chi d\psi^2) + \frac{2}{B} d\Omega_3^2, \] (100)

with the latter term being the standard metric on a unit three sphere. Again we work within Einstein-Maxwell theory, since the dilaton field is unimportant here.

It is straightforward to determine the cosmological constant as \( \Lambda = (A + 2B)/3 \) where the electric field is given by

\[ F = -iq \left( \frac{B^{3/2}}{2\sqrt{2A}} \right) \sin \chi d\chi \wedge d\psi, \] (101)

with the charge being given by

\[ q^2 = \frac{(B - A)\sqrt{2}}{AB^{3/2}}. \] (102)

Evidently there is no corresponding magnetic solution. As in the four dimensional solution, the coordinate \( \chi \) runs from 0 to \( \pi \), and \( \psi \) will parametrise the imaginary time. One can then verify explicitly that the action for the compact manifold is given by the fixed point set expression, where we take the Killing vector to be \( \partial \psi \) which has fixed three spheres at the poles.

The Lorentzian evolution is again described by taking the boundary surface \( \psi = 0, \pi \), and choosing half of each fixed point set to lie in each half of the manifold. The entropy of the solution will then be given by one quarter of the three volume of the fixed point set at \( \chi = 0 \), which we interpret as a cosmological horizon, and one quarter of that of the three
sphere \( \chi = \pi \), which we interpret as a black hole horizon. Thus our results demonstrate not only the well-known result that cosmological and black hole horizons have an intrinsic entropy equal to one quarter of their areas, but also extend the proof to gauge field theories and higher dimensions.

Our treatment of \((p+1)\) form gauge fields also demonstrates that pair creation of \(p\)-branes is associated with an entropy equal to the volumes of the horizons. For electric fields we will again need to include a boundary term to ensure that we take variations over solutions with constant charges. The boundary term that we require is

\[
S_{\text{boundary}} = -\frac{(p+1)}{8\pi G_d} \int_{\Sigma_i} d^{d-1}x e^{-\alpha\Phi} H^{\mu_1 \ldots \mu_{p+1}} n_{\mu_1} B_{\mu_2 \ldots \mu_{p+1}},
\]

(103)

where \(n\) is the normal to the boundary, and \(B_p\) is a \(p\) form potential such that \(H = dB_p\). It is straightforward to verify that such a choice of boundary term gives the required variational behaviour. As before this boundary term precisely cancels out the gauge field volume term, and thus the entropy for the process is given by the sum over fixed point sets.

**VII. EXTREME SOLUTIONS**

We now consider the treatment of extreme solutions, which in this context means compact Euclidean solutions with internal boundaries to the manifold. The action for a solution of the generic theory discussed in the previous section will then have an additional term deriving from the extrinsic curvature of the boundary \(\partial M\) of the \(d\) dimensional manifold \(M\)

\[
S_{\text{boun}} = -\frac{1}{8\pi G_d} \int_{\partial M} d^{d-1}x \kappa \sqrt{b}.
\]

(104)

As before one can decompose the volume term in terms of the dilation current; for a magnetic solution the total action for the Euclidean solution then becomes

\[
S_E = -\frac{\beta}{8\pi G_d} \int_{\Sigma} dJ_D + S_{\text{boun}}.
\]

(105)

The boundary of the \((d-1)\) dimensional hypersurface now includes contributions not only from the fixed point sets of the imaginary time Killing vector, but also contributions from the dimensional reduction of the original boundary \(\partial M\). For a pure electric field we again obtain the additional term from a volume integral of the field

\[
\frac{1}{8\pi G_d} \int_{M} d^d x \sqrt{g} e^{-\alpha\Phi} E^2.
\]

(106)

In the context of the no boundary proposal this term cancels with that on the initial value hypersurface, and the total action for both electric and magnetic pair creation processes is given by one half of (105), where only \((d-2)\) dimensional fixed point sets are possible.

Now one cannot relate the boundary geometry term to the gauge fields without further assumptions about the topology. We shall discuss two generic types of solution with internal boundaries that are physically interesting, both of which in some sense represent pair creation
of extreme black holes \[1\]. The first, usually referred to as “cold” cosmological pair creation, is a solution of topology \(R^2 \times S^{d-2}\), which has a boundary of topology \(S^1 \times S^{d-2}\) and a fixed point set of the imaginary time Killing vector within the manifold. The form of the metric is

\[
d s^2 = dR^2 + R^2 d\psi^2 + \alpha d\Omega^2_{d-2},
\]

where \(\psi\) is the imaginary time, \(\alpha\) is related to the cosmological constant and the coordinate \(R\) runs from the origin \(R = 0\) to \(R_{\infty}\) which we can take to infinity at the end of the calculation. One can choose the other fields so that the solution satisfies the field equations. The metric is globally a product, and the extrinsic curvature of the boundary is non trivial, so that the boundary term \((104)\) is given by minus one quarter of the volume of the \((d - 2)\) sphere. The volume term in \((105)\) can be decomposed as

\[
S_E = \frac{V_{d-2}(R_{\infty})}{4G_d} - \frac{V_{d-2}(R = 0)}{4G_d},
\]

where we have taken account of the direction of the normal to the boundary at infinity. Note that the term at infinity is obtained by integrating the dilation current over the boundary; the form of the metric then implies that it reduces to the volume. Thence the boundary extrinsic curvature term cancels with the term from the dilation current, and the total entropy for the pair creation process is given by

\[
S = \frac{V_{d-2}(R = 0)}{4G_d},
\]

that is, by the volume term for the horizon contained within the Euclidean solution.

The second type of solution, referred to as “ultracold” cosmological pair creation, is a solution of topology \(R^2 \times S^{d-2}\), which has a boundary of topology \(R^1 \times S^{d-2}\) and no fixed point sets of the imaginary time Killing vector within the manifold. The form of the metric is

\[
d s^2 = dx^2 + dy^2 + \alpha_1 d\Omega^2_{d-2},
\]

where \(y\) is the imaginary time, \(\alpha_1\) is related to the cosmological constant and the coordinate \(x\) runs from \(-x_{\infty}\) to \(x_{\infty}\), where we can take \(x_{\infty}\) to infinity at the end of the calculation. The extrinsic curvatures of the boundaries at \(\pm x_{\infty}\) vanish, and hence there is no boundary term in the action. The volume term in \((105)\) can be decomposed in terms of the integral of the dilation current over only the \((d - 2)\) volumes of the boundaries, since there are no fixed point sets of the Killing vector, where we take account correctly of the directions of the normals. Since the dilation current vanishes throughout the manifold, the total action for the solution vanishes, as does the entropy of the pair creation process, as we would expect, since there are no horizons contained within the manifold. Our results are in agreement with those given in \[15\] for four dimensional solutions.

\[1\]For discussions of the physical interpretations of all these pair creation solutions see \[15\].
VIII. CONCLUSIONS

In this paper, we have discussed the action of a circle isometry group on compact Euclidean Einstein manifolds, and on compact Euclidean solutions of supergravity theories. For the former, we can decompose the action of the solution in terms of characteristic properties of the action of any Killing vector with closed orbits. Although in four dimensions the characteristic properties of the fixed point sets, that is the volume and nut behaviour, alone determine the action, in higher dimensions one has to take account also of non-trivial \((d - 3)\) cohomology of the orbit space. It is natural to characterise the contribution from the cohomology as an integral over the entire orbit space rather than from individual fixed point sets. Introducing a microcanonical ensemble, the entropy is equal to minus the action, and thence we obtain an expression for the entropy in terms of the volume of \((d - 2)\) dimensional fixed point sets, and the cohomology of the orbit space.

In the context of the no boundary proposal, our decomposition of the action allows us to calculate the entropy of a particular solution, and thence the probability of a process occurring, in terms of these fixed point sets. In particular, we can demonstrate explicitly that pair creation of (neutral) black \(p\)-branes in a cosmological background is associated with an entropy equal to one quarter of the volumes of the horizons.

It is natural to extend the analysis to compact Euclidean solutions of Einstein gravity, coupled to scalar and gauge fields; with suitable couplings between the fields this includes solutions of supergravity theories. We find that the action can again be decomposed in terms of characteristics of the action of an isometry but that for electric fields there is an additional term left over. In the context of the no boundary proposal, this additional term can be regarded as a constraint on the initial value hypersurface that we fix the electric charge (per unit area). Thus we are able to demonstrate that generic black \(p\)-branes pair created in an appropriate background have an entropy equal to one quarter of the volumes of the event and cosmological horizons. We also considered briefly the extremal limits of pair creation solution, for which we must consider internal boundaries to the manifold.

The analysis of non-compact Euclidean Einstein manifolds is more subtle. Firstly, we will require the existence of a suitable background with respect to which all thermodynamic quantities can be defined. Secondly, we have to identify an appropriate temperature before we can define the entropy; we must work within a canonical or grand canonical ensemble. When we attempt to decompose the action in terms of the action of an isometry in general there will be surface terms on the boundary at infinity which are left over. If we decompose the action in terms of the action of a Killing vector which has null fixed point sets in the Lorentzian continuation, these surface terms will be related to the energy and angular momentum, and the sum over the fixed point sets can again be identified as the entropy. A discussion of the action of isometries on non compact manifolds is contained in \[16\].
REFERENCES

[1] G. Gibbons and S. Hawking, Commun. Math. Phys. 66, 291 (1979).
[2] G. Gibbons, D. Page and C. Pope, Commun. Math. Phys. 127, 529 (1990).
[3] F. Dowker, J. Gauntlett, G. Gibbons and G. Horowitz, Phys. Rev. D 53, 7115 (1996).
[4] P. Ramond, in Lattice Gauge Theories, Supersymmetry, and Grand Unification: Proceedings of the 6th Workshop on Current Problems in High-Energy Particle Theory, John Hopkins University (1982).
[5] C. Hunter, gr-qc/9807010.
[6] A. Besse, Einstein manifolds (Springer Verlag, 1987).
[7] T. Eguchi, P. Gilkey and A. Hanson, Phys. Rep. 66 (1980).
[8] F. Hirzebruch and D. Zagier, The Atiyah-Singer Theorem and Elementary Number Theory (Publish or Perish, 1974).
[9] D. Page and C. Pope, Class. Quantum Grav., 4, 213 (1987).
[10] Euclidean Quantum Gravity, edited by G. Gibbons and S. Hawking (Cambridge University Press, 1992).
[11] R. Caldwell, A. Chamblin and G. Gibbons, Phys. Rev. D 53, 7103 (1996).
[12] R. Bousso and S. Hawking, Phys. Rev. D 54, 6312 (1996).
[13] D. Page, Phys. Let. B 79, 235 (1979).
[14] S. Hawking, G. Horowitz and S. Ross, Phys. Rev. D 51, 4302 (1995); S. Hawking and S. Ross, ibid. 52, 5865 (1995).
[15] R. Mann and S. Ross, Phys. Rev. D 52, 2254 (1995).
[16] M. Taylor-Robinson, to appear.