A Measure Characterization of Embedding and Extension Domains for Sobolev, Triebel–Lizorkin, and Besov Spaces on Spaces of Homogeneous Type

Ryan Alvarado,* Dachun Yang and Wen Yuan

Abstract. In this article, for an optimal range of the smoothness parameter $s$ that depends (quantitatively) on the geometric makeup of the underlying space, the authors identify purely measure theoretic conditions that fully characterize embedding and extension domains for the scale of Hajłasz–Triebel–Lizorkin spaces $M^s_{p,q}$ and Hajłasz–Besov spaces $N^s_{p,q}$ in general spaces of homogeneous type. Although stated in the context of quasi-metric spaces, these characterizations improve related work even in the metric setting. In particular, as a corollary of the main results in this article, the authors obtain a new characterization for Sobolev embedding and extension domains in the context of general doubling metric measure spaces.

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*Corresponding author, E-mail: rjalvarado@amherst.edu/February 14, 2022/Final version.
1 Introduction

The extendability of a given class of functions satisfying certain regularity properties (for instance, functions belonging to Sobolev, Besov or Triebel–Lizorkin spaces) from a subset of an ambient to the entire space, while retaining the regularity properties in question, is a long-standing topic in analysis that has played a fundamental role in both theoretical and applied branches of mathematics. The literature on this topic is vast; however, we refer the reader to, for instance, [25, 38, 11, 31, 34, 35, 36, 19, 20, 40, 23] and the references therein, for a series of studies on extension problems for function spaces including Sobolev, Besov, and Triebel–Lizorkin spaces. In this article, we investigate the relationship between the extension (as well as embedding) properties of these functions spaces and the regularity of the underlying measure in the most general setting. Functions belonging to Sobolev, Besov or Triebel–Lizorkin spaces) from a subset of an ambient to the entire space, while retaining the regularity properties in question, is a long-standing topic in analysis that has played a fundamental role in both theoretical and applied branches of mathematics. The upper bound 1 for the radii in (1.1) is not essential and can be replaced by any strictly positive finite threshold. From a geometric perspective, the measure density condition implies, among other things, that the category of domains satisfying (1.1) encompasses not only the classes of Lipschitz and (ε, δ) domains, but also fractal sets such as the ‘fat’ Cantor and Sierpiński carpet sets [26, 37].

In the Euclidean setting, that is, X = \( \mathbb{R}^n \) equipped with the Euclidean distance and the Lebesgue measure, Hajłasz et al. [19] proved that if \( \Omega \subset \mathbb{R}^n \) is an extension domain of the classical Sobolev space \( W^{k,p} \) for some positive integer \( k \) and \( p \in [1, \infty) \), that is, if every function in \( W^{k,p}(\Omega) \) can be extended to a function in \( W^{k,p}(\mathbb{R}^n) \), then \( \Omega \) necessarily satisfies the measure density condition (see also [30]). Although there are domains satisfying (1.1) that do not have the extension property (for instance, the ‘slit disk’ example is \( \mathbb{R}^2 \)), it was shown in [19] Theorem 5 that (1.1) together with the demand that the space \( W^{k,p}(\Omega) \) admits a characterization via some sharp maximal functions in spirit of Calderón [7], is enough to characterize all \( W^{k,p} \)-extension domains when \( p \in (1, \infty) \). In this case, there actually exists a bounded and linear operator

\[
\mathcal{E} : W^{k,p}(\Omega) \rightarrow W^{k,p}(\mathbb{R}^n)
\]

satisfying \( \mathcal{E}u|_\Omega = u \) for any \( u \in W^{k,p}(\Omega) \), which is a variant of the Whitney–Jones extension operator; see [36]. Zhou [40] Theorem 1.1] considered the fractional Sobolev space \( W^{s,p} \) with \( s \in (0, 1) \) and \( p \in (0, \infty) \) (which is known to coincide with the classical Besov and Triebel–Lizorkin spaces \( B^s_{p,p} \) and \( F^s_{p,p} \)), and proved that a domain \( \Omega \subset \mathbb{R}^n \) with \( n \geq 2 \) is a \( W^{s,p} \)-extension domain if and only if \( \Omega \) satisfies the measure density condition. Going further, Zhou [40] Theorem 1.2] showed that the measure density condition is also equivalent to the existence of certain Sobolev-type embeddings for the space \( W^{s,p} \) on \( \Omega \). Remarkably, no further conditions
Beyond (1.1) are needed to characterize $W^{s,p}$-embedding and extension domains; however, the extension operator in [40, Theorem 1.1] is a modified Whitney-type operator based on median values that may fail to be linear if $p < 1$. We shall return to this point below.

Characterizations of extension domains for certain classes of function spaces have been further studied in much more general settings than the Euclidean one, namely, in the setting of metric measure spaces $(X, \rho, \mu)$, where the measure $\mu$ satisfies the following doubling condition: there exists a positive constant $C_D$ such that

$$\mu(2B) \leq C_D \mu(B) \quad \text{for any } \rho\text{-ball } B \subset X; \quad (1.2)$$

here and thereafter, for any $\lambda \in (0, \infty)$ and $\rho$-ball $B$ in $X$, $\lambda B$ denotes the ball with the same center as $B$ and $\lambda$-times its radius; see [20, 17, 22, 6]. When $\rho$ is a quasi-metric and $\mu$ is doubling, the triplet $(X, \rho, \mu)$ is called a space of homogeneous type in the sense of Coifman and Weiss [8, 9], and a doubling metric measure space is a space of homogeneous type equipped with a metric.

For doubling metric measure spaces $(X, \rho, \mu)$, Hajłasz et al. [20, Theorem 6] proved that, if a set $\Omega \subset X$ satisfies the measure density condition (1.1) then, for any given $p \in [1, \infty)$, there exists a bounded and linear Whitney-type extension operator from $M^{1,p}(\Omega)$ into $M^{1,p}(X)$, where $M^{1,p}$ denotes the Hajłasz–Sobolev space (introduced in [17]) defined via pointwise Lipschitz-type inequalities. Under the assumption that $s \in (0,1)$, the corresponding Whitney extension theorems for the Hajłasz–Triebel–Lizorkin space $M^s_{p,q}$ and the Hajłasz–Besov space $N^s_{p,q}$ were obtained by Heikkinen et al. [23]. More specifically, they proved in [23, Theorem 1.2] that, if $(X, \rho, \mu)$ is a doubling metric measure space and $\Omega \subset X$ is a $\mu$-measurable set satisfying (1.1) then, for any $s \in (0,1)$, $p \in (0, \infty)$, and $q \in (0, \infty)$, there exists a bounded extension operator from $M^s_{p,q}(\Omega)$ [resp., $N^s_{p,q}(\Omega)$] into $M^s_{p,q}(X)$ [resp., $N^s_{p,q}(X)$]. Similar to [40, Theorem 1.1], the extension operator in [23, Theorem 1.2] may fail to be linear for small values of $p$. The definitions and some basic properties of these function spaces can be found in Section 2.2 below. In particular, it is known that $M^{1,p}$, $M^s_{p,q}$, and $N^s_{p,q}$ coincide with the classical Sobolev, Triebel–Lizorkin, and Besov spaces on $\mathbb{R}^n$, respectively, and that $M^{1,\infty}_{p,\infty} = M^{1,p}$ for any $p \in (0, \infty)$ in general metric spaces; see [32].

The first main result of this work is to provide a uniform approach to extending both [20, Theorem 6] and [23, Theorem 1.2] simultaneously from metric spaces to general quasi-metric spaces, without compromising the quantitative aspects of the theory pertaining to the range of the smoothness parameter $s$. To better elucidate this latter point, we remark here that the upper bound 1 for the range of the smoothness parameter $s$ of the function spaces considered in [23] is related to the fact that one always has nonconstant Lipschitz functions (that is, Hölder continuous functions of order 1) in metric spaces. However, in a general quasi-metric space, there is no guarantee that nonconstant Lipschitz functions exist. From this perspective, it is crucial for the work being undertaken here to clarify precisely what should replace the number 1 in the range for $s$ in the more general setting of quasi-metric spaces. As it turns out, the answer is rather subtle and is, roughly speaking, related to the nature of the ‘best’ quasi-metric on $X$ which is bi-Lipschitz equivalent to $\rho$. More specifically, the upper bound on the range of $s \in (0,1)$ that was considered in [23] for metric spaces should be replaced by the following ‘index’:

$$\text{ind } (X, \rho) := \sup_{\varrho \geq \rho} (\log_2 C_\varrho)^{-1} := \sup_{\varrho \geq \rho} \left\{ \log_2 \left( \sup_{x, y, z \in X \text{ not all equal}} \frac{\varrho(x, y)}{\max(\varrho(x, z), \varrho(z, y))} \right) \right\}^{-1} \in (0, \infty), \quad (1.3)$$
where the first supremum is taken over all the quasi-metrics \( \varrho \) on \( X \) which are bi-Lipschitz equivalent to \( \rho \), and the second supremum is taken over all the points \( x, y, z \) in \( X \) which are not all equal; see \( \text{(2.2)} \) below for a more formal definition of \( C_{\varrho} \in [1, \infty) \). This index was introduced in \( [33] \), and its value encodes information about the geometry of the underlying space, as evidenced by the following examples (see Section 2 for additional examples highlighting this fact):

- \( \text{ind} ((\mathbb{R}^n, |\cdot - \cdot|) = 1 \) and \( \text{ind} ([0, 1]^n, |\cdot - \cdot|) = 1 \), where \( |\cdot - \cdot| \) denotes the Euclidean distance;
- \( \text{ind} ((\mathbb{R}^n, |\cdot - \cdot|^{1/\varepsilon}) = \varepsilon \) and \( \text{ind} ([0, 1]^n, |\cdot - \cdot|^{1/\varepsilon}) = \varepsilon \) for any \( \varepsilon \in (0, \infty) \);
- \( \text{ind} (X, \rho) \geq 1 \) if there exists a genuine distance on \( X \) which is pointwise equivalent to \( \rho \);
- \( (X, \rho) \) cannot be bi-Lipschitzly embedded into some \( \mathbb{R}^n \) with \( n \in \mathbb{N} \), whenever \( \text{ind} (X, \rho) < 1 \);
- \( \text{ind} (X, \rho) = 1 \) if \( (X, \rho) \) is a metric space that is equipped with a doubling measure and supports a weak \( (1, p) \)-Poincaré inequality with \( p > 1 \).
- \( \text{ind} (X, \rho) = \infty \) if there exists an ultrametric\(^1\) on \( X \) which is pointwise equivalent to \( \rho \).

From these examples, we can deduce that the range of \( s \in (0, 1) \) is not optimal, even in the metric setting, as the number 1 fails to fully reflect the geometry of the underlying space.

To facilitate the statement of the main theorems in this work, we make the following notational convention: Given a quasi-metric space \( (X, \rho) \) and fixed numbers \( s \in (0, \infty) \) and \( q \in (0, \infty] \), we will understand by \( s \leq q \text{ ind} (X, \rho) \) that \( s \leq \text{ind} (X, \rho) \) and that the value \( s = \text{ind} (X, \rho) \) is only permissible when \( q = \infty \) and the supremum defining the number \( \text{ind} (X, \rho) \) in \( \text{(1.3)} \) is attained. Also, recall that a measure \( \mu \) is doubling (in the sense of \( \text{(1.2)} \)) if and only if \( \mu \) satisfies the following \( Q \)-doubling property: there exist positive constants \( Q \) and \( \kappa \) such that

\[
\kappa \frac{\mu(B_\rho(x, r))}{\mu(B_\rho(y, R))} \leq \kappa \left( \frac{r}{R} \right)^Q
\]

whenever \( x, y \in X \) and \( 0 < r \leq R < \infty \) satisfy \( B_\rho(x, r) \subset B_\rho(y, R) \). Although \( \text{(1.2)} \) and \( \text{(1.4)} \) are equivalent, the advantage of \( \text{(1.4)} \) over \( \text{(1.2)} \), from the perspective of the work in this article, is that the exponent \( Q \) in \( \text{(1.4)} \) provides some notion of dimension for the space \( X \). With this convention and piece of terminology in hand, the first main result of this article which simultaneously generalizes \([20\text{ Theorem 6]}\) and \([23\text{ Theorem 1.2} \) from metric spaces to general quasi-metric spaces for an optimal range of \( s \) is as follows (see also \([31\text{ Theorem 3.13} \) below).

**Theorem 1.1.** Let \( (X, \rho, \mu) \) be a space of homogeneous type, where \( \mu \) is a Borel regular\(^2\) measure satisfying \( \text{(1.2)} \) for some \( Q \in (0, \infty) \), and fix exponents \( s, p \in (0, \infty) \) and \( q \in (0, \infty] \) such that \( s \leq q \text{ ind} (X, \rho) \). Also, suppose that \( \Omega \subset X \) is a nonempty \( \mu \)-measurable set that satisfies the measure density condition \( \text{(1.1)} \). Then any function belonging to \( M_{p,q}^s (\Omega, \rho, \mu) \) can be extended to the entire space \( X \) with preservation of smoothness while retaining control of the associated ‘norm’. More precisely, there exists a positive constant \( C \) such that,

for any \( u \in M_{p,q}^s (\Omega, \rho, \mu) \), there exists a \( \tilde{u} \in M_{p,q}^s (X, \rho, \mu) \)

satisfying \( u = \tilde{u}|_\Omega \) and \( \|\tilde{u}\|_{M_{p,q}^s (X, \rho, \mu)} \leq C \|u\|_{M_{p,q}^s (\Omega, \rho, \mu)} \).
Consequently,
\[ M^s_{p,q}(\Omega, \rho, \mu) = \{ \bar{u}_{|\Omega} : \bar{u} \in M^s_{p,q}(X, \rho, \mu) \}. \]

Furthermore, if \( p, q > Q/(Q + s) \), then there exists a bounded linear operator
\[ \mathcal{E} : M^s_{p,q}(\Omega, \rho, \mu) \to M^s_{p,q}(X, \rho, \mu) \]
such that \( (\mathcal{E} u)|_{\Omega} = u \) for any \( u \in M^s_{p,q}(\Omega, \rho, \mu) \). In addition, if \( s < \text{ind} (X, \rho) \), then all of the statements above remain valid with \( M^s_{p,q} \) replaced by \( N^s_{p,q} \).

The distinguishing feature of Theorem 1.1 is the range of \( s \) for which the conclusions of this result hold true, because it is the largest range of this type to be identified and it turns out to be in the nature of best possible. More specifically, if the underlying space is \( \mathbb{R}^n \) equipped with the Euclidean distance, then \( \text{ind} (\mathbb{R}^n, | \cdot - \cdot |) = 1 \) and Theorem 1.1 is valid for the Hajłasz–Sobolev space \( M^1_{p,\infty} = M^{1,p}_1(\mathbb{R}^n) \), as well as the Hajłasz–Triebel–Lizorkin space \( M^s_{p,q} \) and the Hajłasz–Besov space \( N^s_{p,q} \) whenever \( s \in (0, 1) \). Therefore, we recover the expected range for \( s \) in the Euclidean setting. Similar considerations also hold whenever the underlying space is a set \( S \subset \mathbb{R}^n \) or, more generally, a metric measure space. Consequently, the trace spaces \( W^{1,p}(\mathbb{R}^n)|_\Omega = M^{1,p}(\mathbb{R}^n)|_\Omega \), \( F^s_{p,q}(\mathbb{R}^n)|_\Omega = M^s_{p,q}(\mathbb{R}^n)|_\Omega \), and \( B^s_{p,q}(\mathbb{R}^n)|_\Omega = N^s_{p,q}(\mathbb{R}^n)|_\Omega \) can be identified with the pointwise spaces \( M^{1,p}(\Omega) \), \( M^s_{p,q}(\Omega) \), and \( N^s_{p,q}(\Omega) \), respectively, for \( s \in (0, 1) \), where \( \Omega \subset \mathbb{R}^n \) is as in Theorem 1.1 and \( M^s_{p,q} \) and \( N^s_{p,q} \) denote the classical Triebel–Lizorkin and Besov spaces in \( \mathbb{R}^n \). Remarkably, there are environments where the range of \( s \) is strictly larger than what it would be in the Euclidean setting. For example, if the underlying space \( (X, \rho) \) is an ultrametric space (like a Cantor-type set), then \( \text{ind} (X, \rho) = \infty \) and, in this case, Theorem 1.1 is valid for the spaces \( M^s_{p,q} \) and \( N^s_{p,q} \) for all \( s \in (0, \infty) \). To provide yet another example, for the metric space \( (\mathbb{R}, | \cdot - \cdot |^{1/2}) \), one has that \( \text{ind} (\mathbb{R}, | \cdot - \cdot |^{1/2}) = 2 \) and hence, Theorem 1.1 is valid for the spaces \( M^s_{p,q} \) and \( N^s_{p,q} \) for any \( s \in (0, 2) \). As these examples illustrate, Theorem 1.1 not only extends to full generality the related work in [23, Theorem 1.2] and [20, Theorem 6], but also sharpens their work even in the metric setting by identifying an optimal range of the smoothness parameter \( s \) for which these extension results hold true. In particular, the existence of an \( M^{1,p} \)-extension operator for \( p < 1 \) as given by Theorem 1.1 is new in the context of doubling metric measure spaces.

Loosely speaking, the extension operator in Theorem 1.1 is of Whitney-type in that it is constructed via gluing various ‘averages’ of a function together with using a partition of unity that is sufficiently smooth (relative to the parameter \( s \)). A key issue in this regard is that functions in \( M^s_{p,q} \) and \( N^s_{p,q} \) may not be locally integrable for small values of \( p \), which, in turn, implies that the usual integral average of such functions will not be well defined. It turns out that one suitable replacement for the integral average in this case is the so-called median value of a function (see, for instance, [13], [40], [23], and also Definition 3.6 below). Although the Whitney extension operator based on median values has the distinct attribute that it extends functions in \( M^s_{p,q}(\Omega) \) and \( N^s_{p,q}(\Omega) \) for any given \( p \in (0, \infty) \), there is no guarantee that such an operator is linear. However, when \( p \) is not too small, functions in \( M^s_{p,q} \) and \( N^s_{p,q} \) are locally integrable and we can consider a linear Whitney-type extension operator based on integral averages.

While our approach for dealing with Theorem 1.1 is related to the work in [20] and [23], where metric spaces have been considered, the geometry of quasi-metric spaces can be significantly more intricate, which brings a number of obstacles. For example, as mentioned above, the extension operators constructed in [20] and [23] rely on the existence of a Lipschitz partition of unity; however,
depending on the quasi-metric space, nonconstant Lipschitz functions may not exist. To cope with this fact, we employ a partition of unity consisting of Hölder continuous functions developed in [3] Theorem 6.3 (see also [2] Theorem 2.5) that exhibit a maximal amount of smoothness (measured on the Hölder scale) that the geometry of the underlying space can permit.

It was also shown in [23, Theorem 1.3] that the measure density condition fully characterizes the $M^s_{p,q}$ and $N^s_{p,q}$ extension domains for any given $s \in (0, 1)$, provided that metric measure space is geodesic (that is, the metric space has the property that any two points in it can be joined by a curve whose length equals the distance between these two points) and the measure satisfies the following $Q$-Ahlfors regularity condition on $X$:

$$\kappa_1 r^Q \leq \mu(B_p(x, r)) \leq \kappa_2 r^Q \quad \text{for any } x \in X \text{ and finite } r \in (0, \text{diam}_p(X)],$$

where $Q, \kappa_1, \kappa_2 \in (0, \infty)$ and $\text{diam}_p(X) := \sup\{\rho(x, y) : x, y \in X\}$. Clearly, every $Q$-Ahlfors regular measure is $Q$-doubling; however, there are non-Ahlfors regular doubling measures. The corresponding result for Hajłasz–Sobolev spaces can be found in [20, Theorem 5]. The geodesic assumption is a very strong connectivity condition that precludes many settings in which the spaces $M^s_{p,q}$ and $N^s_{p,q}$ have a rich theory, such as Cantor-type sets and the fractal-like metric space $(\mathbb{R}, |x-y|^{1/2})$. In fact, one important virtue that the Hajłasz–Sobolev, Hajłasz–Besov, and Hajłasz–Triebel–Lizorkin spaces possess over other definitions of these spaces in the quasi-metric setting, is that there is a fruitful theory for this particular branch of function spaces without needing to assume that the space is geodesic (or even connected) or that the measure is Ahlfors regular.

In this article, we sharpen the results in [23, Theorem 1.3] and [20, Theorem 5] by eliminating the $Q$-Ahlfors regularity and the geodesic assumptions on the underlying spaces, without compromising the main conclusions of these results and, in particular, the quantitative aspects of the theory pertaining to the optimality of the range for the smoothness parameter $s$. Moreover, in the spirit of Zhou [40], we go further and show that the measure density condition also fully characterizes the existence of certain Sobolev-type embeddings for the spaces $M^s_{p,q}$ and $N^s_{p,q}$ on domains, which is a brand new result even in the Euclidean setting. For the clarity of exposition in this introduction, we state in the following theorem a simplified summary of these remaining principal results which, to some degree, represents the central theorem in this article. The reader is directed to Theorems 4.6 and 4.10 below for stronger and more informative formulations of this result.

**Theorem 1.2.** Let $(X, p, \mu)$ be a space of homogeneous type, where $\mu$ is a Borel regular measure satisfying (1.2) for some $Q \in (0, \infty)$, and suppose that $\Omega \subset X$ is a $\mu$-measurable locally uniformly perfect set in the sense of (4.2). Then the following statements are equivalent.

(a) $\Omega$ satisfies the measure density condition (1.1).

(b) $\Omega$ is an $M^s_{p,q}$-extension domain for some (or all) $p \in (0, \infty)$, $q \in (0, \infty]$, and $s \in (0, \infty)$ satisfying $s \leq q \text{ ind}(X, \rho)$.

(c) $\Omega$ is an $N^s_{p,q}$-extension domain for some (or all) $p \in (0, \infty)$, $q \in (0, \infty]$, and $s \in (0, \infty)$ satisfying $s < \text{ ind}(X, \rho)$.

(d) $\Omega$ is a local $M^s_{p,q}$-embedding domain for some (or all) $p \in (0, \infty)$, $q \in (0, \infty]$, and $s \in (0, \infty)$ satisfying $s \leq q \text{ ind}(\Omega, \rho)$. 


(e) $\Omega$ is a local $N_{s_{\rho,q}}^{s}$-embedding domain for some (or all) $p \in (0, \infty)$, $q \in (0, p]$, and $s \in (0, \infty)$ satisfying $s \leq q \text{ind}(\Omega, \rho)$.

If the measure $\mu$ is actually $Q$-Ahlfors regular on $X$, then the following statements are also equivalent to each of (a)-(e).

(f) $\Omega$ is a global $M_{s_{\rho,q}}$-embedding domain for some (or all) $p \in (0, \infty)$, $q \in (0, \infty]$, and $s \in (0, \infty)$ satisfying $s \leq q \text{ind}(\Omega, \rho)$.

(g) $\Omega$ is a global $N_{s_{\rho,q}}^{s}$-embedding domain for some (or all) $p \in (0, \infty)$, $q \in (0, p]$, and $s \in (0, \infty)$ satisfying $s \leq q \text{ind}(\Omega, \rho)$.

The additional demand that $\Omega$ is a locally uniformly perfect set is only used in proving that each of the statements in (b)-(g) imply the measure density condition in (a). Note that this assumption on $\Omega$ is optimal. Indeed, when $\mu$ is $Q$-Ahlfors regular on $X$, the measure density condition in Theorem 1.2(a) implies that $\Omega$ is a locally Ahlfors regular set, and it is known that such sets are necessarily locally uniformly perfect; see, for instance [10, Lemma 4.7].

Regarding how Theorem 1.2 fits into the existing literature, since the metric space $(\mathbb{R}, | \cdot - |^{1/2})$ is not geodesic, one cannot appeal to [20] and [23] in order to conclude that the measure density condition holds true for $M_{1,p}^{1}$, $M_{p,q}^{1}$, and $N_{p,q}^{s}$-extension domains as their results are not applicable in such settings. However, we have $\text{ind}(\mathbb{R}, | \cdot - |^{1/2}) = 2$ and hence, Theorem 1.2 in this work is valid for the spaces $M_{p,q}^{1}$ and $N_{p,q}^{s}$ for any $s \in (0, 2)$ and, in particular, for the space $M_{1,p}^{1}$. In this vein, we also wish to mention that for $q = \infty$, Theorem 1.2 is valid for any $s \in (0, \infty)$ satisfying $s \leq (\log_{2} C_{\rho})^{-1}$, where $C_{\rho} \in [1, \infty)$ is as in (2.2). On the other hand, it follows immediately from (2.2) that if $\rho$ is a genuine metric then $C_{\rho} \leq 2$ and so, $(\log_{2} C_{\rho})^{-1} \geq 1$. Therefore, by combining this observation with the fact that $M_{q,\infty}^{1} = M_{1}^{1}$ (see [3]), we have the following consequence of Theorem 1.2 (or, more specifically, Theorem 4.6), which is a brand new result for Hajłasz–Sobolev spaces in the setting of metric spaces and which generalizes and extends the work in [20] and [27].

**Corollary 1.3.** Let $(X, \rho, \mu)$ be a doubling metric measure space, where $\mu$ is a Borel regular measure satisfying (1.4) for some $Q \in (0, \infty)$, and suppose that $\Omega \subset X$ is a $\mu$-measurable locally uniformly perfect set in the sense of (4.2). Then the following statements are equivalent.

(a) $\Omega$ satisfies the measure density condition (1.1).

(b) $\Omega$ is an $M_{1,p}^{1}$-extension domain for some (or all) $p \in (0, \infty)$ in the sense that there exists a positive constant $C$ such that,

for any $u \in M_{1,p}^{1}(\Omega, \rho, \mu)$, there exists a $\bar{u} \in M_{1,p}^{1}(X, \rho, \mu)$

satisfying $u = \bar{u}|_{\Omega}$ and $\|\bar{u}\|_{M_{1,p}^{1}(X, \rho, \mu)} \leq C\|u\|_{M_{1,p}^{1}(\Omega, \rho, \mu)}$.

(c) For some (or all) $p \in (Q/(Q + 1), \infty)$, there exists a linear and bounded extension operator $\mathcal{E}: M_{1,p}^{1}(\Omega, \rho, \mu) \to M_{1,p}^{1}(X, \rho, \mu)$ such that $(\mathcal{E}u)|_{\Omega} = u$ for any $u \in M_{1,p}^{1}(\Omega, \rho, \mu)$.

(d) $\Omega$ is a local $M_{1,p}^{1}$-embedding domain for some (or all) $p \in (0, \infty)$.

If the measure $\mu$ is actually $Q$-Ahlfors regular on $X$, then the following statement is also equivalent to each of (a)-(d).
(e) $\Omega$ is a global $M^{1,p}$-embedding domain for some (or all) $p \in (0, \infty)$.

The remainder of this article is organized as follows. In Section 2 we review some basic terminology and results pertaining to quasi-metric spaces and the main classes of function spaces considered in this work, including the fractional Hajłasz–Sobolev spaces, the Hajłasz–Triebel–Lizorkin spaces, and the Hajłasz–Besov spaces. In particular, we present some optimal embedding results for these spaces recently established in [5], which extends the work of [11] and [28].

The main aim of Section 3 is to prove Theorem 1.1. This is done in Subsection 3.2, after we collect a number of necessary and key tools in Subsection 3.1, including a Whitney type decomposition of the underlying space (see Theorem 3.1 below) and the related partition of unity (see Theorem 3.2 below), as well as some Poincaré-type inequalities in terms of both the ball integral averages and the median values of functions (see Lemmas 3.4 and 3.8 below). The regularity parameter of the aforementioned partition of unity is closely linked to the geometry of the underlying quasi-metric space, and is important in constructing the desired extension of functions from the spaces $M^s_{p,q}(\Omega, \rho, \mu)$ and $N^s_{p,q}(\Omega, \rho, \mu)$ for an optimal range of $s$. On the other hand, as we consider the extension theorems of $M^s_{p,q}(\Omega, \rho, \mu)$ and $N^s_{p,q}(\Omega, \rho, \mu)$ for full ranges of parameters, whose elements might not be locally integrable when $p$ or $q$ is small, we need to use the median values instead of the usual ball integral averages. Via these tools, in Subsection 3.2 we first successfully construct a local extension for functions in $M^s_{p,q}(\Omega, \rho, \mu)$ [and also $N^s_{p,q}(\Omega, \rho, \mu)$] from $\Omega$ to a neighborhood $V$ of $\Omega$, which, together with the boundedness of bounded Hölder continuous functions with support $V$, operating as pointwise multipliers, on these spaces (see Lemma 3.10 below), further leads to the desired global extension to the entire space $X$.

Finally, in Section 4 we formulate and prove Theorem 1.2 (see Theorems 4.6 and 4.10 below), that is, to show that, on spaces of homogeneous type and, in particular, on Ahlfors regular spaces, the existence of an extension for functions in $M^s_{p,q}(\Omega, \rho, \mu)$ and $N^s_{p,q}(\Omega, \rho, \mu)$, from a $\mu$-measurable locally uniformly perfect set $\Omega \subset X$ to the entire space $X$, is equivalent to the measure density condition. The key tools are the embedding results of these spaces recently obtained in [5] (see also Theorems 2.4 and 2.5 below) and a family of maximally smooth Hölder continuous ‘bump’ functions belonging to $M^s_{p,q}$ and $N^s_{p,q}$ constructed in [5] Lemma 4.6] (see also Lemma 4.3 below). Indeed, as we can see in Theorems 1.2, 4.6, and 4.10, we not only obtain the equivalence between the extension properties for the spaces $M^s_{p,q}(\Omega, \rho, \mu)$ and $N^s_{p,q}(\Omega, \rho, \mu)$, and the measure density condition, but also their equivalence to various Sobolev-type embeddings of these spaces.

In closing, we emphasize again that the main results in this article are not just generalizations of [20, Theorems 5 and 6] and [23, Theorems 1.2 and 1.3] from metric to quasi-metric spaces, but they also sharpen these known results, even in the metric setting, via seeking an optimal range of the smoothness parameter which is essentially determined (quantitatively) by the geometry of the underlying space.

2 Preliminaries

This section is devoted to presenting some basic assumptions on the underlying quasi-metric measure space, as well as the definitions and some facts of function spaces considered in this work.
2.1 The Setting

Given a nonempty set $X$, a function $\rho : X \times X \to [0, \infty)$ is called a quasi-metric on $X$, provided there exist two positive constants $C_0$ and $C_1$ such that, for any $x, y, z \in X$, one has

$$\rho(x, y) = 0 \iff x = y, \quad \rho(y, x) \leq C_0 \rho(x, y),$$

and

$$\rho(x, y) \leq C_1 \max\{\rho(x, z), \rho(z, y)\}.$$  \hfill (2.1)

A pair $(X, \rho)$ is called a quasi-metric space if $X$ is a nonempty set and $\rho$ a quasi-metric on $X$. Throughout the whole article, we tacitly assume that $X$ is of cardinality $\geq 2$. It follows from this assumption that the constants $C_0$ and $C_1$ appearing in (2.1) are $\geq 1$.

Define $C_\rho$ to be the least constant which can play the role of $C_1$ in (2.1), namely,

$$C_\rho := \sup_{x, y, z \in X, \text{not all equal}} \frac{\rho(x, y)}{\max\{\rho(x, z), \rho(z, y)\}} \in [1, \infty).$$  \hfill (2.2)

Also, let $\overline{C}_\rho$ to be the least constant which can play the role of $C_0$ in (2.1), that is,

$$\overline{C}_\rho := \sup_{x, y \in X, x \neq y} \frac{\rho(y, x)}{\rho(x, y)} \in [1, \infty).$$  \hfill (2.3)

When $C_\rho = \overline{C}_\rho = 1$, $\rho$ is a genuine metric that is commonly called an ultrametric. Note that, if the underlying metric space is $\mathbb{R}^n$, $n \in \mathbb{N}$, equipped with the Euclidean distance $d$, then $C_d = 2$.

Two quasi-metrics $\rho$ and $\varrho$ on $X$ are said to be equivalent, denoted by $\rho \approx \varrho$, if there exists a positive constant $c$ such that

$$c^{-1} \varrho(x, y) \leq \rho(x, y) \leq c \varrho(x, y), \quad \forall x, y \in X.$$  

It follows from [33 (4.289)] that $\rho \approx \varrho$ if and only if the identity $(X, \rho) \to (X, \varrho)$ is bi-Lipschitz (see [33 Definition 4.32]). As such, also refer to $\rho \approx \varrho$ as $\rho$ is $\text{bi-Lipschitz equivalent}$ to $\varrho$.

We now recall the concept of an ‘index’ which was originally introduced in [33 Definition 4.26]. The lower smoothness index of a quasi-metric space $(X, \rho)$ is defined as

$$\text{ind} (X, \rho) := \sup_{\varrho \approx \rho} \left( \log_2 C_\varrho \right)^{-1} \in (0, \infty],$$  \hfill (2.4)

where the supremum is taken over all the quasi-metrics $\varrho$ on $X$ which are bi-Lipschitz equivalent to $\rho$. The index $\text{ind} (X, \rho)$ encodes information about the geometry of the underlying space, as evidenced by the properties listed in the introduction, as well as the following additional ones:

- $\text{ind} (Y, \rho) \geq \text{ind} (X, \rho)$ whenever $Y \subset X$;
- $\text{ind} (X, \| \cdot \|) = 1$ if $(X, \| \cdot \|)$ is a nontrivial normed vector space; hence $\text{ind} (\mathbb{R}^n, \| \cdot \|) = 1$;
- $\text{ind} (Y, \| \cdot \|) = 1$ if $Y$ is a subset of a normed vector space $(X, \| \cdot \|)$ containing an open line segment; hence $\text{ind} ([0, 1]^n, \| \cdot \|) = 1$;
- $\text{ind} (X, \rho) \leq 1$ whenever the interval $[0, 1]$ can be bi-Lipschitzly embedded into $(X, \rho)$;
• \( \text{ind} (X, \rho) \leq Q \) if \((X, \tau_\rho)\) is pathwise connected and \((X, \rho)\) is equipped with a \(Q\)-Ahlfors-regular measure, where \(\tau_\rho\) denotes the topology generated by \(\rho\);
• there are compact, totally disconnected, Ahlfors regular spaces with lower smoothness index \(\infty\); for instance, the four-corner planar Cantor set equipped with \(|· − ·|\);
• \( \text{ind} (X, \rho) = \infty \) whenever the underlying set \(X\) has finite cardinality;

see also \([5, \text{Remark 4.6}]\) and \([33, \text{Section 4.7}]\) or \([2, \text{Section 2.5}]\) for more details.

We conclude this subsection with some notational conventions. Balls in \((X, \rho)\) will be denoted by \(B_\rho(x, r) := \{ y \in X : \rho(x, y) < r \}\) with \(x \in X\) and \(r \in (0, \infty)\), and let

\[ \overline{B}(x, r) := \{ y \in X : \rho(x, y) \leq r \}. \]

As a sign of warning, note that in general \(\overline{B}_\rho(x, r)\) is not necessarily equal to the closure of \(B_\rho(x, r)\). If \(r = 0\), then \(B_\rho(x, r) = \emptyset\), but \(\overline{B}_\rho(x, r) = \{ x \}\). Moreover, since \(\rho\) is not necessarily symmetric, one needs to pay particular attention to the order of \(x\) and \(y\) in the definition of \(B_\rho(x, r)\). The triplet \((X, \rho, \mu)\) is called a quasi-metric measure space if \(X\) is a set of cardinality \(\geq 2\), \(\rho\) is a quasi-metric on \(X\), and \(\mu\) is a Borel measure such that all \(\rho\)-balls are \(\mu\)-measurable and \(\mu(B_\rho(x, r)) \in (0, \infty)\) for any \(x \in X\) and any \(r \in (0, \infty)\). Here and thereafter, the measure \(\mu\) is said to be Borel regular if every \(\mu\)-measurable set is contained in a Borel set of equal measure. See also \([33, 3, 2]\) for more information on this setting.

Let \(\mathbb{Z}\) denote all integers and \(\mathbb{N}\) all (strictly) positive integers. We always denote by \(C\) a positive constant which is independent of the main parameters, but it may vary from line to line. We also use \(C_{(\alpha, \beta, \ldots)}\) to denote a positive constant depending on the indicated parameters \(\alpha, \beta, \ldots\). The symbol \(f \lesssim g\) means that \(f \leq Cg\). If \(f \leq g\) and \(g \leq f\), we then write \(f \approx g\). If \(f \leq Cg\) and \(g = h\) or \(g \leq h\), we then write \(f \lesssim h\) or \(f \lesssim g \lesssim h\), rather than \(f \lesssim g = h\) or \(f \leq g \leq h\). The integral average of a locally \(\mu\)-measurable function \(u\) on a \(\mu\)-measurable set \(E \subset X\) with \(\mu(E) \in (0, \infty)\) is denoted by

\[ u_E := \frac{1}{\mu(E)} \int_E u \, d\mu, \]

whenever the integral is well defined. For sets \(E \subset (X, \rho)\), let \(\text{diam}_\rho(E) := \sup\{\rho(x, y) : x, y \in E\}\) and \(1_E\) be the characteristic function of \(E\). For any \(p \in (0, \infty)\), let \(L^p(X) := L^p(X, \mu)\) denote the Lebesgue space on \((X, \mu)\), that is, the set of all the \(\mu\)-measurable functions \(f\) on \((X, \rho)\) such that

\[ \| f \|_{L^p(X)} := \| f \|_{L^p(X, \mu)} := \begin{cases} \left( \int_X |f(x)|^p \, d\mu(x) \right)^{1/p}, & p \in (0, \infty), \\ \text{ess sup} \{|f(x)| : x \in X\}, & p = \infty, \end{cases} \]

is finite.

### 2.2 Triebel–Lizorkin, Besov, and Sobolev Spaces

Suppose \((X, \rho, \mu)\) is a quasi-metric space equipped with a nonnegative Borel measure, and let \(s \in (0, \infty)\). Following \([32]\), a sequence \(\{g_k\}_{k \in \mathbb{Z}}\) of nonnegative \(\mu\)-measurable functions on \(X\) is
called a fractional $s$-gradient of a $\mu$-measurable function $u$: $X \to \mathbb{R}$ if there exists a set $E \subset X$ with $\mu(E) = 0$ such that
\[
|u(x) - u(y)| \leq [\rho(x, y)]^s [g_k(x) + g_k(y)]
\]
for any $k \in \mathbb{Z}$ and $x, y \in X \setminus E$ satisfying $2^{-k-1} \leq \rho(x, y) < 2^{-k}$. Let $\mathcal{D}_\rho^s(u)$ denote the set of all the fractional $s$-gradients of $u$.

Given $p \in (0, \infty)$, $q \in (0, \infty]$, and a sequence $\vec{g} := \{g_k\}_{k \in \mathbb{Z}}$ of $\mu$-measurable functions on $X$, define
\[
\|\vec{g}\|_{L^p(X, \mu)} := \|\{(g_k)_{k \in \mathbb{Z}}\}_{k \in \mathbb{Z}}\|_{L^p(X, \mu)}
\]
and
\[
\|\vec{g}\|_{\ell^q(L^p(X))} := \left\{\|\{(g_k)_{k \in \mathbb{Z}}\}_{k \in \mathbb{Z}}\|_{L^p(X, \mu)}\right\}_{k \in \mathbb{Z}},
\]
where
\[
\|\{(g_k)_{k \in \mathbb{Z}}\}_{k \in \mathbb{Z}}\|_{\ell^q} := \begin{cases} \left(\sum_{k \in \mathbb{Z}} |g_k|^q\right)^{1/q} & \text{if } q \in (0, \infty), \\ \sup_{k \in \mathbb{Z}} |g_k| & \text{if } q = \infty. \end{cases}
\]

The homogeneous Hajłasz–Triebel–Lizorkin space $M_{p,q}^s(X, \rho, \mu)$ is defined as the collection of all the $\mu$-measurable functions $u$: $X \to \mathbb{R}$ such that
\[
\|u\|_{M_{p,q}^s(X, \rho, \mu)} := \|u\|_{M_{p,q}^s(X, \rho, \mu)} := \inf_{\vec{g} \in \mathcal{D}_\rho^s(u)} \|\vec{g}\|_{L^p(X, \mu)} < \infty.
\]

Here and thereafter, we make the agreement that $\inf \emptyset := \infty$. The corresponding inhomogeneous Hajłasz–Triebel–Lizorkin space is defined as $M_{p,q}^s(X, \rho, \mu) := M_{p,q}^s(X, \rho, \mu) \cap L^p(X)$, and it is equipped with the (quasi-)norm
\[
\|u\|_{M_{p,q}^s(X, \rho, \mu)} := \|u\|_{M_{p,q}^s(X, \rho, \mu)} := \|u\|_{M_{p,q}^s(X, \rho, \mu)} + \|u\|_{L^p(\Omega)}, \quad \forall u \in M_{p,q}^s(X, \rho, \mu).
\]

The homogeneous Hajłasz–Besov space $N_{p,q}^s(X, \rho, \mu)$ is defined as the collection of all the $\mu$-measurable functions $u$: $X \to \mathbb{R}$ such that
\[
\|u\|_{N_{p,q}^s(X, \rho, \mu)} := \|u\|_{N_{p,q}^s(X, \rho, \mu)} := \inf_{\vec{g} \in \mathcal{D}_\rho^s(u)} \|\vec{g}\|_{\ell^q(L^p(X))} < \infty,
\]
and the inhomogeneous Hajłasz–Besov space is defined as $N_{p,q}^s(X, \rho, \mu) := N_{p,q}^s(X, \rho, \mu) \cap L^p(X)$, and it is equipped with the (quasi-)norm
\[
\|u\|_{N_{p,q}^s(X, \rho, \mu)} := \|u\|_{N_{p,q}^s(X, \rho, \mu)} := \|u\|_{N_{p,q}^s(X, \rho, \mu)} + \|u\|_{L^p(\Omega)}, \quad \forall u \in N_{p,q}^s(X, \rho, \mu).
\]

It is known that, when $p \in [1, \infty)$ and $q \in [1, \infty]$, $\|\cdot\|_{M_{p,q}^s(X, \rho, \mu)}$ and $\|\cdot\|_{N_{p,q}^s(X, \rho, \mu)}$ are genuine norms and the corresponding spaces $M_{p,q}^s(X, \rho, \mu)$ and $N_{p,q}^s(X, \rho, \mu)$ are Banach spaces. Otherwise, they are quasi-Banach spaces. We will simply use $M_{p,q}^s(X, \rho, \mu)$, $N_{p,q}^s(X, \rho, \mu)$, $M_{p,q}^s(X, \rho, \mu)$, and $N_{p,q}^s(X, \rho, \mu)$ in place of $M_{p,q}^s(X, \rho, \mu)$, $N_{p,q}^s(X, \rho, \mu)$, $M_{p,q}^s(X, \rho, \mu)$, and $N_{p,q}^s(X, \rho, \mu)$, respectively, whenever the quasimetric and the measure are well-understood from the context. It was shown in [32] that $M_{p,q}^s(\mathbb{R}^n)$ coincides with the classical Triebel–Lizorkin space $F_{p,q}^s(\mathbb{R}^n)$ for any $s \in (0, 1)$, $p \in \left(\frac{n}{n+s}, \infty\right)$, and $q \in (\frac{n+s}{n+1}, \infty]$, and $N_{p,q}^s(\mathbb{R}^n)$ coincides with the classical Besov space $B_{p,q}^s(\mathbb{R}^n)$ for any $s \in (0, 1)$,
\( p \in (\frac{n}{n+1}, \infty) \), and \( q \in (0, \infty] \). We also refer the reader to [14, 23, 24, 27, 28, 1, 21, 4] for more information on Triebel–Lizorkin and Besov spaces on quasi-metric measure spaces.

The following proposition highlights the fact that equivalent quasi-metrics generate equivalent Triebel–Lizorkin and Besov spaces.

**Proposition 2.1.** Let \((X, \rho, \mu)\) be a quasi-metric space equipped with a nonnegative Borel measure, and let \(s, p \in (0, \infty)\) and \(q \in (0, \infty]\). Suppose that \(\varrho\) is a quasi-metric on \(X\) such that \(\varrho \approx \rho\) and all \(\varrho\)-balls are \(\mu\)-measurable. Then \(M^s_{p,q}(X, \rho, \mu) = M^s_{p,q}(X, \varrho, \mu)\) and \(N^s_{p,q}(X, \rho, \mu) = N^s_{p,q}(X, \varrho, \mu)\), as sets, with equivalent (quasi-)norms, and, consequently,

\[
M^s_{p,q}(X, \rho, \mu) = M^s_{p,q}(X, \varrho, \mu) \quad \text{and} \quad N^s_{p,q}(X, \rho, \mu) = N^s_{p,q}(X, \varrho, \mu),
\]

as sets, with equivalent (quasi-)norms.

**Proof.** We only provide the details for the proof of \(M^s_{p,q}(X, \rho, \mu) \subset M^s_{p,q}(X, \varrho, \mu)\) as all other inclusions can be handled similarly. To this end, suppose that \(u \in M^s_{p,q}(X, \rho, \mu)\) and take any \(\tilde{u} := (g_k)_{k \in \mathbb{Z}} \in D^s_p(u)\). Since \(\varrho \approx \rho\), it follows that there exists a constant \(N \in \mathbb{N}\) such that

\[
2^{-N} \varrho(x, y) \leq \rho(x, y) \leq 2^N \varrho(x, y), \quad \forall x, y \in X.
\]

Define

\[
h_k := 2^{sN} \sum_{j=-N}^N g_{k+j}, \quad \forall k \in \mathbb{Z}.
\]

We claim that \(h := \{h_k\}_{k \in \mathbb{Z}} \in D^s_p(u)\). To see this, let \(E \subset X\) be a \(\mu\)-measurable set such that \(\mu(E) = 0\) and

\[
|u(x) - u(y)| \leq [\rho(x, y)]^s \left[ g_k(x) + g_k(y) \right]
\]

for any \(k \in \mathbb{Z}\) and \(x, y \in X \setminus E\) satisfying \(2^{-k-1} \leq \rho(x, y) < 2^{-k}\). Fix a \(k \in \mathbb{Z}\) and take \(x, y \in X \setminus E\) satisfying \(2^{-k-1} \leq \varrho(x, y) < 2^{-k}\). By (2.6), we have \(2^{-k-1-N} \leq \rho(x, y) < 2^{-k+N}\) and hence there exists a unique integer \(j_0 \in [k - N, k + N]\) such that \(2^{-j_0-1} \leq \rho(x, y) < 2^{-j_0}\). From this, (2.7), and (2.6), we deduce that

\[
|u(x) - u(y)| \leq [\rho(x, y)]^s \left[ \sum_{j=-N}^N g_{k+j}(x) + \sum_{j=-N}^N g_{k+j}(y) \right]
\]

\[
\leq [\varrho(x, y)]^s \left[ |h_k(x) + h_k(y)| \right].
\]

Hence, \(h := \{h_k\}_{k \in \mathbb{Z}} \in D^s_p(u)\), as wanted. It is straightforward to prove that \(\|h\|_{L^p(X, \varrho)} \leq \|\tilde{u}\|_{L^p(X, \varrho)}\), where the implicit positive constant only depends on \(s, p, q, \) and \(N\) (which ultimately depends on the proportionality constants in \(\varrho \approx \rho\)). This finishes the proof of Proposition 2.1. \(\Box\)

**Proposition 2.2.** Let \((X, \rho, \mu)\) be a quasi-metric space equipped with a nonnegative Borel measure, and suppose that \(s, p \in (0, \infty)\) and \(q \in (0, \infty]\). For any \(u \in M^s_{p,q}(X, \rho, \mu)\), one has that \(\|u\|_{M^s_{p,q}(X, \rho, \mu)} = 0\) if and only if \(u\) is constant \(\mu\)-almost everywhere in \(X\). The above statement also holds true with \(M^s_{p,q}\) replaced by \(N^s_{p,q}\).
Proof. By similarity, we only consider \( M^s_{p,q}(X, \rho, \mu) \). Fix a \( u \in M^s_{p,q}(X, \rho, \mu) \). If \( u \) is constant \( \mu \)-almost everywhere in \( X \), then it is easy to check that \( \{ g_k \}_{k \in \mathbb{N}} \subseteq D^s_p(u) \), where \( g_k \equiv 0 \) on \( X \) for any \( k \in \mathbb{N} \). Clearly, \( \| u \|_{M^s_{p,q}(X, \rho, \mu)} \leq \| g_k \|_{L^p(X, \rho, \mu)} = 0 \) and so, \( \| u \|_{M^s_{p,q}(X, \rho, \mu)} = 0 \), as desired.

Suppose next that \( \| u \|_{M^s_{p,q}(X, \rho, \mu)} > 0 \). By the definition of \( \| \cdot \|_{M^s_{p,q}(X, \rho, \mu)} \), we can find a collection of sequences, \( \{ \bar{g}_j \}_{j \in \mathbb{N}} \), with the property that, for any \( j \in \mathbb{N} \), \( \bar{g}_j := \{ g_{j,k} \}_{k \in \mathbb{N}} \subseteq D^s_p(u) \) and \( \| g_{j,k} \|_{L^p(X, \rho, \mu)} \leq \| \bar{g}_j \|_{L^p(X, \rho, \mu)} < 2^{-j} \) for any \( k \in \mathbb{Z} \). Then it is straightforward to check that, for any fixed \( k \in \mathbb{Z} \), \( \sum_{j \in \mathbb{N}} g_{j,k} \in L^p(X, \mu) \) and there exists a \( \mu \)-measurable set \( N_k \subset X \) such that \( \mu(N_k) = 0 \) where, for any \( x \in X \setminus N_k \), \( g_{j,k}(x) \to 0 \) as \( j \to \infty \). Going further, by \( (2.5) \), for any \( j \in \mathbb{N} \), there exists a \( \mu \)-measurable set \( E_j \subset X \) such that \( \mu(E_j) = 0 \) and
\[
|u(x) - u(y)| \leq [\rho(x, y)]^s \left[ g_{j,k}(x) + g_{j,k}(y) \right]
\]
for any \( k \in \mathbb{Z} \) and \( x, y \in X \setminus E_j \) satisfying \( 2^{-k-1} \leq \rho(x, y) < 2^{-k} \). Consider the set
\[
E := \left( \bigcup_{k \in \mathbb{Z}} N_k \right) \cup \left( \bigcup_{j \in \mathbb{N}} E_j \right).
\]
Then \( E \) is \( \mu \)-measurable and \( \mu(E) = 0 \). Moreover, if \( x, y \in X \setminus E \), then there exists a unique \( k \in \mathbb{Z} \) such that \( 2^{-k-1} \leq \rho(x, y) < 2^{-k} \) and
\[
|u(x) - u(y)| \leq [\rho(x, y)]^s \left[ g_{j,k}(x) + g_{j,k}(y) \right], \quad \forall \ j \in \mathbb{N}, \tag{2.8}
\]
Passing to the limit in (2.8) as \( j \to \infty \), we find that \( u(x) = u(y) \). By the arbitrariness of \( x, y \in X \setminus E \), we conclude that \( u \) is constant in \( X \setminus E \), as wanted. This finishes the proof of Proposition 2.2. \( \Box \)

Now, we recall the definition of the Hajłasz–Sobolev space. Let \( (X, \rho) \) be a quasi-metric space equipped with a nonnegative Borel measure \( \mu \), and let \( s \in (0, \infty) \). Following [17] [18] [39], a nonnegative \( \mu \)-measurable function \( g \) on \( X \) is called an \( s \)-gradient of a \( \mu \)-measurable function \( u : X \to \mathbb{R} \) if there exists a set \( E \subset X \) with \( \mu(E) = 0 \) such that
\[
|u(x) - u(y)| \leq [\rho(x, y)]^s \left[ g(x) + g(y) \right], \quad \forall \ x, y \in X \setminus E.
\]
The collection of all the \( s \)-gradients of \( u \) is denoted by \( D^s_p(u) \).

Given \( p \in (0, \infty) \), the homogeneous Hajłasz–Sobolev space \( M^{s,p}(X, \rho, \mu) \) is defined as the collection of all the \( \mu \)-measurable functions \( u : X \to \mathbb{R} \) such that
\[
\| u \|_{M^{s,p}(X, \rho, \mu)} := \| u \|_{M^{s,p}(X, \rho, \mu)} := \inf_{g \in D^s_p(u)} \| g \|_{L^p(X, \rho, \mu)} < \infty,
\]
again, with the understanding that \( \inf \emptyset := \infty \). The corresponding inhomogeneous Hajłasz–Sobolev space \( M^{s,p}(X, \rho, \mu) \) is defined by setting \( M^{s,p}(X, \rho, \mu) := M^{s,p}(X, \rho, \mu) \cap L^p(X, \rho, \mu) \), equipped with the (quasi-)norm
\[
\| u \|_{M^{s,p}(X, \rho, \mu)} := \| u \|_{M^{s,p}(X, \rho, \mu)} := \| u \|_{M^{s,p}(X, \rho, \mu)} + \| u \|_{L^p(X, \rho, \mu)} \quad \forall \ u \in M^{s,p}(X, \rho, \mu).
\]
It is known that, for any \( s \in (0, \infty) \) and \( p \in (0, \infty) \),
\[
\dot{M}^{s,p}(X, \rho, \mu) = M^{s,\infty}(X, \rho, \mu) \quad \text{and} \quad M^{s,p}(X, \rho, \mu) = M^{s,\infty}(X, \rho, \mu);
\]
see, for instance, [32] Proposition 2.1] and [5] Proposition 2.4].
2.3 Embedding Theorems for Triebel–Lizorkin and Besov Spaces

In this subsection, we recall a number of embedding results that were recently obtained in [5], beginning with some Sobolev–Poincaré-type inequalities for fractional Hajłasz–Sobolev spaces. To facilitate the formulation of the result, we introduce the following piece of notation: Given \( Q, b \in (0, \infty), \sigma \in [1, \infty), \) and a ball \( B_0 \subset X \) of radius \( R_0 \in (0, \infty) \), the measure \( \mu \) is said to satisfy the \( V(\sigma B_0, Q, b) \) condition provided

\[
\mu(B_\rho(x, r)) \geq br^Q \quad \text{for any } x \in X \text{ and } r \in (0, \sigma R_0] \text{ satisfying } B_\rho(x, r) \subset \sigma B_0.
\]

We remind the reader that the constants \( C_\rho, \tilde{C}_\rho \in [1, \infty) \) were defined in (2.2) and (2.3), respectively. The following conclusion was obtained in [5 Theorem 3.1].

**Theorem 2.3.** Let \((X, \rho, \mu)\) be a quasi-metric measure space. Let \( s, p \in (0, \infty), \sigma \in [C_\rho, \infty) \), and \( B_0 \) be a \( \rho \)-ball of radius \( R_0 \in (0, \infty) \). Assume that the measure \( \mu \) satisfies the \( V(\sigma B_0, Q, b) \) condition for some \( Q, b \in (0, \infty) \). Let \( u \in M^{s/p}(\sigma B_0, \rho, \mu) \) and \( g \in D_0^s(u) \). Then \( u \in L^{p^*}(B_0, \mu) \) and there exists a positive constant \( C \), depending only on \( \rho, s, p, Q, \) and \( \sigma \), such that

\[
\inf_{y \in X} \left( \frac{1}{B_\rho} \int_{B_\rho} |u - y|^{p^*} \, d\mu \right)^{1/p^*} \leq C \left[ \frac{\mu(\sigma B_0)}{b R_0^Q} \right]^{1/p} \left( \frac{1}{\mu(B_\rho(y, R))} \right)^{1/p} R_0^s \left( \frac{1}{\mu(B_\rho(y, R))} \right)^{1/p},
\]

where \( p^* := Qp/(Q - sp) \).

We will also employ the following embeddings of Hajłasz–Triebel–Lizorkin and Hajłasz–Besov spaces on spaces of homogeneous type obtained in [5]. To state these results, let us introduce the following piece of terminology: Given a quasi-metric measure space \((X, \rho, \mu)\), the measure \( \mu \) is said to be \( Q \)-doubling up to scale \( r_\sigma \in (0, \infty] \), provided that there exist positive constants \( \kappa \) and \( Q \) satisfying

\[
\kappa \left( \frac{r}{R} \right)^Q \leq \frac{\mu(B_\rho(x, r))}{\mu(B_\rho(y, R))},
\]

whenever \( x, y \in X \) with \( B_\rho(x, r) \subset B_\rho(y, R) \) and \( 0 < r \leq R \leq r_\sigma \). When \( r_\sigma = \infty, \mu \) is simply \( Q \)-doubling as before [see (1.4)].

The following inequalities were obtained in [5 Theorem 3.7 and Remark 3.8].

**Theorem 2.4.** Let \((X, \rho, \mu)\) be a quasi-metric measure space, where \( \mu \) is \( Q \)-doubling up to scale \( r_\sigma \in (0, \infty] \) for some \( Q \in (0, \infty) \). Let \( s, p \in (0, \infty) \) and \( q \in (0, \infty) \), and set \( \sigma := C_\rho \). Then there exist positive constants \( C, C_1, \) and \( C_2 \), depending only on \( \rho, \kappa, Q, s, \) and \( p \), such that the following statements hold true for any \( \rho \)-ball \( B_0 := B_\rho(x_0, R_0) \), where \( x_0 \in X \) and \( R_0 \in (0, r_\sigma/\sigma] \) is finite:

(a) If \( p \in (0, Q/s) \), then any \( u \in \mathcal{M}^s_{p, q}(\sigma B_0, \rho, \mu) \) belongs to \( L^{p^*}(B_0, \mu) \) with \( p^* := Qp/(Q - sp) \), and satisfies

\[
\|u\|_{L^{p^*}(B_0)} \leq C \frac{1}{[\mu(\sigma B_0)]^{1/Q}} \left[ R_0^s \|u\|_{\mathcal{M}^s_{p, q}(\sigma B_0)} + \|u\|_{L^p(\sigma B_0)} \right]
\]

and

\[
\inf_{y \in X} \|u - y\|_{L^{p^*}(B_0)} \leq C \frac{1}{[\mu(\sigma B_0)]^{1/Q}} R_0^s \|u\|_{\mathcal{M}^s_{p, q}(\sigma B_0)}.
\]
(b) If $p = Q/s$, then, for any $u \in M^s_{p,q}(\sigma B_0, \rho, \mu)$ with $\|u\|_{M^s_{p,q}(\sigma B_0)} > 0$, one has
\begin{equation}
\int_{B_0} \exp \left( C_1 \frac{[\mu(\sigma B_0)]^{1/Q} |u - u_{B_0}|}{R_0^{\ast} \|u\|_{M^s_{p,q}(\sigma B_0)}} \right) \, d\mu \leq C_2 \mu(B_0).
\end{equation}

(c) If $p \in (Q/s, \infty)$, then each function $u \in \dot{M}^s_{p,q}(\sigma B_0, \rho, \mu)$ has a Hölder continuous representative of order $s = Q/p$ on $B_0$, denoted by $u$ again, satisfying
\begin{equation}
|u(x) - u(y)| \leq C[\rho(x,y)]^{s-Q/p} \frac{R_0^{Q/p}}{\mu(\sigma B_0)^{1/p}} \|u\|_{M^s_{p,q}(\sigma B_0)}, \quad \forall \, x, y \in B_0.
\end{equation}

In addition, if $q \leq p$, then the above statements are valid with $\dot{M}^s_{p,q}$ replaced by $N^s_{p,q}$.

The embeddings for the spaces $N^s_{p,q}$ in Theorem 2.4 are restricted to the case when $q \leq p$; however, an upper bound on the exponent $q$ is to be expected; see [5, Remark 4.17]. As the following theorem (see [5, Theorem 3.9]) indicates, one can relax the restriction on $q$ and still obtain Sobolev-type embeddings with the critical exponent $Q/s$ replaced by $Q/\varepsilon$, where $\varepsilon \in (0, s)$ is any fixed number. These embeddings will be very useful in Section 5 when characterizing $M^s_{p,q}$ and $N^s_{p,q}$-extension domains.

**Theorem 2.5.** Let $(X, p, \mu)$ be a quasi-metric measure space, where $\mu$ is $Q$-doubling up to scale $r_* = (0, \infty]$ for some $Q \in (0, \infty)$. Let $s$, $p \in (0, \infty)$ and $q \in (0, \infty]$, and set $\sigma := C_{p \ast}$. Then, for any fixed $\varepsilon \in (0, s)$, there exist positive constants $C$, $C_1$, and $C_2$, depending only on $p, \kappa, Q, s, \varepsilon$, and $p$, such that the following statements hold true for any $p$-ball $B_0 := B_p(x_0, R_0)$, where $x_0 \in X$ and $R_0 \in (0, r_\ast/\varepsilon)$ is finite:

(a) If $p \in (0, Q/\varepsilon)$, then any $u \in \dot{N}^s_{p,q}(\sigma B_0, \rho, \mu)$ belongs to $L^{p \ast}(B_0, \mu)$ with $p \ast := (Q p)/(Q - \varepsilon p)$, and satisfies
\[ \|u\|_{L^{p \ast}(B_0)} \leq \frac{C}{[\mu(\sigma B_0)]^{\varepsilon/Q}} \left[ R_0^{\ast} \|u\|_{\dot{N}^s_{p,q}(\sigma B_0)} + \|u\|_{L^{p \ast}(\sigma B_0)} \right], \]
and
\[ \inf_{\gamma \in \mathbb{R}} \|u - \gamma\|_{L^{p \ast}(B_0)} \leq \frac{C}{[\mu(\sigma B_0)]^{\varepsilon/Q}} R_0^{\ast} \|u\|_{\dot{N}^s_{p,q}(\sigma B_0)}. \]

(b) If $p = Q/\varepsilon$, then, for any $u \in \dot{N}^s_{p,q}(\sigma B_0, \rho, \mu)$ with $\|u\|_{\dot{N}^s_{p,q}(\sigma B_0)} > 0$, one has
\begin{equation}
\int_{B_0} \exp \left( C_1 \frac{[\mu(\sigma B_0)]^{1/Q} |u - u_{B_0}|}{R_0^{\ast} \|u\|_{\dot{N}^s_{p,q}(\sigma B_0)}} \right) \, d\mu \leq C_2 \mu(B_0).
\end{equation}

(c) If $p \in (Q/\varepsilon, \infty)$, then each function $u \in \dot{N}^s_{p,q}(\sigma B_0, \rho, \mu)$ has a Hölder continuous representative of order $s = Q/p$ on $B_0$, denoted by $u$ again, satisfying
\[ |u(x) - u(y)| \leq C[\rho(x,y)]^{s-Q/p} \frac{R_0^{Q/p}}{[\mu(\sigma B_0)]^{1/p}} \|u\|_{\dot{N}^s_{p,q}(\sigma B_0)}, \quad \forall \, x, y \in B_0. \]
In the Ahlfors regular setting, we have the following estimates, which follows from [5, Theorem 1.4]. (Note that the uniformly perfect property was not used in proving that (a) implies (b)-(e) in [5, Theorem 1.4].)

**Theorem 2.6.** Let \((X, \rho, \mu)\) be a quasi-metric measure space, where \(\mu\) is a \(Q\)-Ahlfors regular measure on \(X\) for some \(Q \in (0, \infty)\). Let \(s, p \in (0, \infty)\) and \(q \in (0, \infty]\). Then there exists a positive constant \(C\) such that the following statements hold true for any \(u \in \mathcal{M}_{sp,q}^{\mu}(X, \rho, \mu)\):

(a) If \(p \in (0, Q/s)\), then
\[
\|u\|_{L^{p^*}(X)} \leq C\|u\|_{\mathcal{M}_{sp,q}^{\mu}(X)} + \frac{C}{\text{diam}_\rho(X)} \|u\|_{L^p(X)}
\]
and
\[
\inf_{\gamma \in \mathbb{R}} \|u - \gamma\|_{L^{p^*}(X)} \leq C\|u\|_{\mathcal{M}_{sp,q}^{\mu}(X)},
\]
where \(p^* := Qp/(Q - sp)\).

(b) If \(p = Q/s\) and \(\|u\|_{\mathcal{M}_{sp,q}^{\mu}(X)} > 0\), then
\[
\int_B \exp \left( C_1 \frac{|u - u_B|}{\|u\|_{\mathcal{M}_{sp,q}^{\mu}(X)}} \right) \, d\mu \leq C_2 r^Q
\]
for any \(\rho\)-ball \(B \subset X\) having finite radius \(r \in (0, \text{diam}_\rho(X)]\).

(c) If \(p \in (Q/s, \infty)\), then \(u\) has a Hölder continuous representative of order \(s - Q/p\) on \(X\), denoted by \(u\) again, satisfying
\[
|u(x) - u(y)| \leq C[\rho(x,y)]^{s - Q/p} \|u\|_{\mathcal{M}_{sp,q}^{\mu}(X)}, \quad \forall \, x, y \in X.
\]

In addition, if \(q \leq p\), then all of the statements above are valid with \(\mathcal{M}_{sp,q}^{\mu}\) replaced by \(N_{sp,q}^{\mu}\).

### 3 Optimal Extensions for Triebel–Lizorkin Spaces and Besov Spaces on Spaces of Homogeneous Type

This section is devoted to the proof of Theorem 1.1. To this end, we first collect a number of necessary tools and results in Subsection 3.1. The proof of Theorem 1.1 will be presented in Subsection 3.2.

#### 3.1 Main Tools

The existence of \(\mathcal{M}_{sp,q}^{\mu}\) and \(N_{sp,q}^{\mu}\) extension operators on domains satisfying the measure density condition for an *optimal* range of \(s\) relies on two basic ingredients:

(i) the existence of a Whitney decomposition of an open subset of the quasi-metric space under consideration (into the Whitney balls of bounded overlap);
(ii) the existence of a partition of unity subordinate (in an appropriate, quantitative manner) to such a decomposition that exhibits an optimal amount of smoothness (measured on the Hölder scale).

It is the existence of a partition of unity of maximal smoothness order that permits us to construct an extension operator for an optimal range of \( s \). The availability of these ingredients was established in \([3, \text{Theorem 6.3}]\) (see also \([2, \text{Theorems 2.4 and 2.5}]\)). We recall their statements here for the convenience of the reader.

**Theorem 3.1 (Whitney-type decomposition).** Suppose that \((X, \rho, \mu)\) is a quasi-metric measure space, where \( \mu \) is a doubling measure. Then, for any \( \theta \in (1, \infty) \), there exist constants \( \Lambda \in (\theta, \infty) \) and \( M \in \mathbb{N} \), which depend only on both \( \theta \) and the space \((X, \rho, \mu)\), and which have the following significance. For any proper, nonempty, open subset \( O \) of the topological space \((X, \tau_\rho)\), where \( \tau_\rho \) denotes the topology canonically induced by the quasi-metric \( \rho \), there exist a sequence \( \{x_j\}_{j \in \mathbb{N}} \) of points in \( O \) along with a family \( \{r_j\}_{j \in \mathbb{N}} \) of (strictly) positive numbers, for which the following properties are valid:

(i) \( O = \bigcup_{j \in \mathbb{N}} B_\rho(x_j, r_j) \);

(ii) \( \sum_{j \in \mathbb{N}} 1_{B_\rho(x_j, \theta r_j)} \leq M \) pointwise in \( O \). In fact, there exists an \( \varepsilon \in (0, 1) \), depending only on both \( \theta \) and the space \((X, \rho, \mu)\), with the property that, for any \( x_0 \in O \),

\[
\# \left\{ j \in \mathbb{N} : B_\rho(x_0, \varepsilon \text{dist}_\rho(x_0, X \setminus O)) \cap B_\rho(x_j, \theta r_j) \neq \emptyset \right\} \leq M,
\]

where, in general, \( \# \) denotes the cardinality of a set \( E \);

(iii) For any \( j \in \mathbb{N} \), \( B_\rho(x_j, \theta r_j) \subset O \) and \( B_\rho(x_j, \Lambda r_j) \cap [X \setminus O] \neq \emptyset \);

(iv) \( r_i \approx r_j \) uniformly for \( i, j \in \mathbb{N} \) such that \( B_\rho(x_i, \theta r_i) \cap B_\rho(x_j, \theta r_j) \neq \emptyset \).

Given a quasi-metric space \((X, \rho)\) and an exponent \( \alpha \in (0, \infty) \), recall that the homogeneous Hölder space \((\rho^\alpha, X, \rho)\) is the collection of all the functions \( f : X \to \mathbb{R} \) such that

\[
\|f\|_{\rho^\alpha(X, \rho)} := \sup_{x, y \in X, x \neq y} \frac{|f(x) - f(y)|}{[\rho(x, y)]^\alpha} < \infty.
\]

**Theorem 3.2 (Partition of Unity).** Let \((X, \rho, \mu)\) be a quasi-metric measure space, where \( \mu \) is a doubling measure. Suppose that \( \tilde{O} \) is a proper nonempty subset of \( X \). Fix a number \( \theta \in (C_\rho^2, \infty) \), where \( C_\rho \) is as in \([22]\), and consider the decomposition of \( \tilde{O} \) into the family \( \{B_\rho(x_j, r_j)\}_{j \in \mathbb{N}} \) as given by Theorem 3.1 for this choice of \( \theta \). Let \( \theta' \in (C_\rho, \theta/C_\rho) \). Then, for any \( \alpha \in \mathbb{R} \) satisfying

\[
0 < \alpha \leq \left[ \log_2 C_\rho \right]^{-1},
\]

there exists a constant \( C_* \in [1, \infty) \), depending only on \( \rho, \alpha, M, \) and the proportionality constants in Theorem 3.1 iv), along with a family \( \{\psi_j\}_{j \in \mathbb{N}} \) of real-valued functions on \( X \) such that the following statements are valid:
(i) for any \( j \in \mathbb{N} \), one has
\[
\psi_j \in \mathcal{V}^\beta(X, \rho) \quad \text{and} \quad \|\psi_j\|_{\mathcal{V}^\beta(X, \rho)} \leq C_\ast r_j^{-\beta}, \quad \forall \beta \in (0, \alpha];
\]

(ii) for any \( j \in \mathbb{N} \), one has
\[
0 \leq \psi_j \leq 1 \quad \text{on} \quad X, \quad \psi_j \equiv 0 \quad \text{on} \quad X \setminus B_\rho(x_j, \theta r_j), \quad \text{and} \quad \psi_j \geq 1/C_\ast \quad \text{on} \quad B_\rho(x_j, r_j);
\]

(iii) it holds true that
\[
\sum_{j \in \mathbb{N}} \psi_j = 1 \cup_{j \in \mathbb{N}} B_\rho(x_j, r_j) = 1 \cup_{j \in \mathbb{N}} B_\rho(x_j, \theta r_j) = \mu X \quad \text{pointwise on} \quad X.
\]

The range for the parameter \( \alpha \) in (3.1) is directly linked to the geometry of the underlying quasi-metric spaces and it is a result of the following sharp metrization theorem established in [33, Theorem 3.46].

**Theorem 3.3.** Let \((X, \rho)\) be a quasi-metric space and assume that \( C_\rho, \bar{C}_\rho \in [1, \infty)\) are as in (2.2) and (2.3), respectively. Then there exists a symmetric quasi-metric \( \rho_\# \) on \( X \) satisfying \( \rho_\# \approx \rho \) and having the property that for any finite number \( \alpha \in (0, (\log_2 C_\rho)^{-1}] \), the function \((\rho_\#)^\alpha : X \times X \to \mathbb{R}\) is a genuine metric on \( X \), that is,
\[
[\rho_\#(x,y)]^\alpha \leq [\rho_\#(x,z)]^\alpha + [\rho_\#(z,y)]^\alpha, \quad \forall \ x, \ z, \ y \in X.
\]
Moreover, the function \( \rho_\# : X \times X \to [0, \infty) \) is continuous when \( X \times X \) is equipped with the natural product topology \( \tau_\rho \times \tau_\rho \). In particular, all \( \rho_\# \)-balls are open in the topology \( \tau_\rho \).

The following Poincaré-type inequality will play an important role in proving our main extension result.

**Lemma 3.4.** Let \((X, \rho, \mu)\) be a quasi-metric measure space, where \( \mu \) is a \( Q \)-doubling measure for some \( Q \in (0, \infty) \), and suppose that \( \Omega \subset X \) is a nonempty \( \mu \)-measurable set that satisfies the measure density condition (1.1), namely, there exists a positive constant \( C_\mu \) such that
\[
\mu(B_\rho(x, r)) \leq C_\mu \mu(B_\rho(x, r) \cap \Omega), \quad \forall \ x \in \Omega, \ \forall \ r \in (0, 1].
\]

Fix \( t, s, \varepsilon, \varepsilon' \in (0, \infty) \) with \( t < Q / \varepsilon' \) and \( \varepsilon' < \varepsilon < s \), and let \( k_0 \in \mathbb{N} \) be any number such that \( 2^{k_0} \geq C_\rho^2 \bar{C}_\rho \), where \( C_\rho, \bar{C}_\rho \in [1, \infty) \) are as in (2.2) and (2.3), respectively. Then, for any \( r_\ast \in [1, \infty) \), there exists a positive constant \( C, \) depending on \( r_\ast, \mu, Q, t, \varepsilon', \) and \( \varepsilon \), such that, for any \( \mu \)-measurable function \( u : \Omega \to \mathbb{R} \), \( \{g_k\}_{k \in \mathbb{Z}} \subset D^E_{C_\rho}(u) \), \( x \in \Omega \), and \( L \in \mathbb{Z} \) satisfying \( 2^{-L} \leq r_\ast \), one has
\[
\inf_{y \in \mathbb{R}} \left( \int_{B_\rho(x, 2^{-L}y) \cap \Omega} |u - y|^t r_\ast \mu_1 \right)^{1/t} \leq C 2^{-L \varepsilon} \left( \int_{B_\rho(x, C \rho 2^{-L})} \sup_{j \in \mathbb{Z}, L + k_0} \left( 2^{j(s - \varepsilon') L} g_j^\varepsilon \right) \mu_1 \right)^{1/t}, \quad (3.3)
\]
where \( t^* := Q t / (Q - \varepsilon' t) \). Here, for any \( j \in \mathbb{Z} \), the function \( g_j \) is identified with function defined on all of \( X \) by setting \( g_j \equiv 0 \) on \( X \setminus \Omega \).
Remark 3.5. In the context of Lemma 3.4 if the function \( u \) is integrable on domain balls and we take \( t := Q/(Q + \varepsilon') \), then \( t' = 1 \) and we can replace the infimum in (3.3) by the ball average \( u_{B_\varepsilon(x,2^{-L})\cap\Omega} \), and obtain

\[
\int_{B_\varepsilon(x,2^{-L})\cap\Omega} \left| u - u_{B_\varepsilon(x,2^{-L})\cap\Omega} \right| \, d\mu \leq C 2^{-L} \left( \int_{B_\varepsilon(x,2^{-L})\cap\Omega} \sup_{j\geq L-k_0} \left( 2^{-j(s-\varepsilon')} g_j^e \right) \right)^{1/t},
\]

where the positive constant \( C \) is twice the constant appearing in Lemma 3.4. This inequality will be important in the proof of Theorem 3.13 when constructing an extension operator that is linear.

Proof of Lemma 3.4. Fix a \( \mu \)-measurable function \( u : \Omega \to \mathbb{R} \) and suppose that \( r_* \in [1, \infty) \), \( \{g_k\}_{k\in\mathbb{Z}} \subseteq D^\omega(u) \), \( x \in \Omega \), and \( L \in \mathbb{Z} \) satisfies \( 2^{-L} \leq r_* \). Without loss of generality, we may assume that the right-hand side of (3.3) is finite.

If \( \Omega = \{x\} \) then (3.3) trivially holds. Therefore, we can assume that \( \Omega \) as cardinality at least two, in which case, we have that \((\Omega, \rho, \mu)\) is a well-defined quasi-metric measure space where \( \rho \) and \( \mu \) are restricted to \( \Omega \). Our plan is to use (2.10) in Theorem 2.3 for the quasi-metric measure space \((\Omega, \rho, \mu)\) with \( \sigma := C_\rho, R_0 := 2^{-L}, \) and \( B_0 := B_\rho(x,2^{-L}) \). To this end, we first claim that the measure \( \mu \) satisfies the \( V(\sigma B_0 \cap \Omega, Q, b) \) condition [that is, (2.9) with \( \sigma B_0, B_\rho(x, r), \) and \( x \in X \) replaced, respectively, by \( \sigma B_0 \cap \Omega, B_\rho(z, r) \cap \Omega, \) and \( z \in \Omega \)] for some choice of \( b \). To show this, suppose that \( z \in \Omega \) and \( r \in (0, \sigma R_0) \) satisfy \( B_\rho(z, r) \cap \Omega \subseteq \sigma B_0 \cap \Omega \). Then \( (\sigma r_*^{-1})r \leq r \leq \sigma^2 R_0 \) and \( (\sigma r_*^{-1})r \leq r_*^{-1} R_0 = r_*^{-1} 2^{-L} \leq 1 \). Moreover, by \( z \in \sigma B_0, \) we obtain \( B_\rho(z, (\sigma r_*^{-1})r) \subseteq \sigma^2 B_0 \). Indeed, if \( y \in B_\rho(z, (\sigma r_*^{-1})r) \), then

\[
\rho(y, r) \leq C_\rho \max\{\rho(y, z), \rho(z, y)\} < C_\rho \max\{\sigma R_0, \sigma^{-1} r\} = C_\rho \sigma R_0 = \sigma^2 R_0.
\]

Using the \( Q \)-doubling property (1.4) and the measure density condition (3.2), we conclude that

\[
\frac{\mu(B_\rho(z, r) \cap \Omega)}{\mu(\sigma B_0 \cap \Omega)} \geq \frac{\mu(B_\rho(z, (\sigma r_*^{-1})r) \cap \Omega)}{\mu(\sigma B_0)} \geq \frac{C_\mu^{-1} \mu(B_\rho(z, (\sigma r_*^{-1})r))}{\mu(\sigma^2 B_0)} \geq \frac{\kappa C_\mu^{-1} \left( \frac{(\sigma r_*^{-1})r}{\sigma^2 R_0} \right)^Q}{\sigma^2 R_0}.
\]

Hence, \( \mu \) satisfies the \( V(\sigma B_0 \cap \Omega, Q, b) \) condition with \( b := \kappa C_\mu^{-1} \mu(\sigma B_0 \cap \Omega)(\sigma^3 r, R_0)^{-Q} \). This proves the above claim.

Moving on, we claim next that \( u \in M^{\varepsilon', \varepsilon} (\sigma B_0 \cap \Omega) \). To see this, let

\[
g := \sup_{j\geq L-k_0} \left( 2^{-j(s-\varepsilon')} g_j^e \right),
\]

where \( k_0 \in \mathbb{N} \) satisfies \( 2^{k_0} \geq C_\rho^2 \mu \). Then \( g \in D^\omega(u) \). Indeed, observe that, for any points \( y, z \in \sigma B_0 \cap \Omega = B_\rho(x, C_\rho 2^{-L}) \cap \Omega, \) we have

\[
\rho(y, z) \leq C_\rho \max\{\rho(y, x), \rho(x, z)\} < C_\rho^2 \mu \leq 2^{-L+k_0},
\]
which further implies that there exists a unique integer \( j_0 \geq L - k_0 \) satisfying \( 2^{-j_0 - 1} \leq \rho(y, z) < 2^{-j_0} \). Then, by \( \{g_k\}_{k \in \mathbb{Z}} \in D_\rho(u) \), we conclude that there exists an \( E \subset X \) with \( \mu(E) = 0 \) such that, for any \( y, z \in (\sigma B_0 \cap \Omega) \setminus E \),
\[
|u(y) - u(z)| \leq [\rho(y, z)]^s \left[ g_{j_0}(y) + g_{j_0}(z) \right] \\
\quad \quad < [\rho(y, z)]^{t_0} 2^{-j_0(s-t)} \left[ g_{j_0}(y) + g_{j_0}(z) \right] \leq [\rho(y, z)]^{t_0} [g(y) + g(z)],
\]
which implies that \( g \in D_\rho'(u) \), as wanted. Observe that
\[
\|g\|_{L^1(\sigma B_0 \cap \Omega, \mu)} = \left[ \int_{\sigma B_0} \sup_{j \geq L-k_0} \left\{ 2^{-j(s-t)} g_j \right\} \, d\mu \right]^{\frac{1}{t}} \\
\quad \quad \leq \sup_{j \geq L-k_0} 2^{-j(s-t_0)} \left[ \int_{\sigma B_0} \sup_{j \geq L-k_0} \left\{ 2^{-j(s-t)} g_j \right\} \, d\mu \right]^{\frac{1}{t}} \\
\quad \quad \leq 2^{-L-k_0(s-t_0)} \left[ \int_{\sigma B_0} \sup_{j \geq L-k_0} \left\{ 2^{-j(s-t)} g_j \right\} \, d\mu \right]^{\frac{1}{t}} < \infty, \tag{3.4}
\]
where we have used the fact that for any \( j \in \mathbb{Z} \), \( g_j \equiv 0 \) on \( X \setminus \Omega \). It follows that \( u \in M^{t_0, t}(\sigma B_0 \cap \Omega) \).

Since \( t < Q/t_0 \), from (2.10) in Theorem 2.3 (keeping in mind the value of \( b \) above) and (3.4), we deduce that
\[
\inf_{\gamma \in \mathbb{R}} \left[ \int_{B_p(x, 2^{-L} \gamma) \cap \Omega} |u - \gamma|^{t_0} \, d\mu \right]^{1/t_0} \\
\quad \quad \leq 2^{-L} \left[ \int_{B_p(x, C_p 2^{-L}) \cap \Omega} g' \, d\mu \right]^{1/t} \leq 2^{-L} \left[ \int_{B_p(x, C_p 2^{-L}) \cap \Omega} \sup_{j \geq L-k_0} \left\{ 2^{-j(s-t)} g_j' \right\} \, d\mu \right]^{\frac{1}{t}}. \tag{3.5}
\]
Note that the measure density condition (3.2) and the doubling property for \( \mu \) imply (keeping in mind that \( C_p, r_s \geq 1 \) and \( r_s^{-1} 2^{-L} \leq 1 \))
\[
\mu(B_p(x, C_p 2^{-L}) \cap \Omega) \geq \mu(B_p(x, r_s^{-1} 2^{-L}) \cap \Omega) \geq \mu(B_p(x, r_s^{-1} 2^{-L})) \geq \mu(B_p(x, C_p 2^{-L})),
\]
which, together with (3.5), implies the desired inequality (3.3). The proof of Lemma 3.4 is now complete.

The \( M^s_{p,q} \) and \( N^s_{p,q} \) extension operators that we construct in Theorem 3.13 will be a Whitney-type operator which, generally speaking in this context, is based on gluing various ‘averages’ of a function together with using a partition of unity that is sufficiently smooth (related to the parameter \( s \)). Since functions in \( M^s_{p,q} \) and \( N^s_{p,q} \) may not be locally integrable for small values of \( p \), we cannot always use integral averages of these functions when constructing the extension operator. As a substitute, we use the so-called median value of a function in place of its integral average when \( p \) or \( q \) is small (see, for instance, [13], [40], and [23]).
Lemma 3.6. Let \((X, \mu)\) be a measure space, where \(\mu\) is a nonnegative measure. Given a \(\mu\)-measurable set \(E \subset X\) with \(\mu(E) \in (0, \infty)\), the median value \(m_\mu(E)\) of a \(\mu\)-measurable function \(u : X \to \mathbb{R}\) on \(E\) is defined by setting
\[
m_\mu(E) := \max_{\theta \in \mathbb{R}} \left\{ \mu(\{x \in E : u(x) < \theta\}) \leq \frac{\mu(E)}{2} \right\}.
\]

It is straightforward to check that \(m_\mu(E) \in \mathbb{R}\) is a well-defined number using basic properties of the measure \(\mu\). We now take a moment to collect a few key properties of \(m_\mu\) that illustrate how these quantities mimic the usual integral average of a locally integrable function. One drawback of the median value is that there is no guarantee that \(m_\mu(E)\) is linear in \(u\). Consequently, the resulting extension operator constructed using median values may not be linear.

Lemma 3.7. Let \((X, \rho, \mu)\) be a quasi-metric measure space and fix an \(\eta \in (0, \infty)\). Then, for any \(\mu\)-measurable function \(u : X \to \mathbb{R}\), any \(\rho\)-ball \(B \subset X\), and any \(\gamma \in \mathbb{R}\), one has
\[
|m_\mu(B) - \gamma| \leq \left( 2 \int_B |u - \gamma|^\eta \, d\mu \right)^{1/\eta}.
\]

Estimate (3.6) was established in [14] (2.4) under the assumption that \(\rho\) is a genuine metric (see also [13] Lemma 2.2) for a proof in the Euclidean setting). The proof in the setting of quasi-metric spaces is the same and, therefore, we omit the details.

The following lemma is a refinement of [23] Lemmas 3.6 and 3.8], which, among other things, sharpens the estimate [23] (3.9)] in a fashion that will permit us to simultaneously generalize [23] Theorem 1.2] and [20] Theorem 6] in a unified manner; see Theorem 3.13 below.

Lemma 3.8. Let \((X, \rho, \mu)\) be a quasi-metric measure space with \(\mu\) being a doubling measure, and suppose that \(\Omega \subset X\) is a nonempty \(\mu\)-measurable set that satisfies the measure density condition \((1.1)\), namely, there exists a constant \(C_\mu \in (0, \infty)\) satisfying
\[
\mu(B_\rho(x, r)) \leq C_\mu \mu(B_\rho(x, r) \cap \Omega), \quad \forall x \in \Omega, \quad \forall r \in (0, 1].
\]

Then the following statements are valid.

(i) Suppose that \(u : \Omega \to \mathbb{R}\) belongs locally to \(M^{\infty,p}(\Omega, \rho, \mu)\) for some \(\varepsilon, p \in (0, \infty)\), in the sense that \(u \in M^{\infty,p}(B \cap \Omega, \rho, \mu)\) for any fixed \(\rho\)-ball \(B \subset X\) that is centered in \(\Omega\). Then
\[
\lim_{r \to 0^+} m_\mu(B_\rho(x, r) \cap \Omega) = u(x) \quad \text{for } \mu\text{-almost every point } x \in \Omega,
\]
here and thereafter, \(r \to 0^+\) means \(r \in (0, \infty)\) and \(r \to 0\). In particular, (3.8) holds true if \(u \in M^{\infty}_p(\Omega, \rho, \mu)\) or \(u \in N^{\infty}_p(\Omega, \rho, \mu)\) for some \(s, p \in (0, \infty)\) and \(q \in (0, \infty]\).

(ii) Fix \(t, s, \varepsilon \in (0, \infty)\) with \(\varepsilon < s\), and let \(k_0 \in \mathbb{N}\) be any number such that \(2^{k_0} \geq C_\rho^2 C_\rho\), where \(C_\rho, C_\rho \in [1, \infty)\) are as in (2.2) and (2.3), respectively. Then, for any given \(r_0 \in [1, \infty)\), there exists a positive constant \(C\) such that
\[
|m_\mu(B \cap \Omega) - m_\mu(B_\rho(x, 2^{-t} \cap \Omega))|
\]
By [2, Theorem 3.7], we know that the function \( \mathcal{M}_{p_0}(g_k) \) is \( \mu \)-measurable. Also, \( g_k^p \in L^{p/\eta}(X, \mu) \), where \( p/\eta > 1 \). As such, it follows from the boundedness of \( \mathcal{M}_{p_0} \) on \( L^{p/\eta}(X, \mu) \) (see [2, Theorem 3.7]) that \( \mathcal{M}_{p_0}(g_k^\eta) \in L^{\eta/\eta}(X, \mu) \). This, together with the fact that \( g_k \in L^p(B_k \cap \Omega, \mu) \), implies that \( \mu(E_k) = 0 \).

Suppose now that \( x \in (C_{p^*}^{-1}B_k \cap \Omega) \setminus E_k \) for some \( k \in \mathbb{N} \) and that \( r \in (0, 1] \) satisfies \( r \leq C_{p^*}^{-1} \).

Then \( B_r(x, r) \subset B_k \) and, by combining (3.6) with \( \gamma := u(x) \), the measure density condition (3.7), and the fact that \( g_k \) is an \( \varepsilon \)-gradient for \( u \) on \( B_k \cap \Omega \), we conclude that

\[
|m_u(B_r(x, r) \cap \Omega) - u(x)| \leq 2 \left( \frac{1}{2} \sup_{y \in B_r(x, r) \cap \Omega} |u(y) - u(x)|^\eta \, d\mu(y) \right)^{1/\eta}
\]

\[
\leq r^\varepsilon \left( \frac{1}{2} \sup_{y \in B_r(x, r)} |g_k(y)|^\eta \, d\mu(y) \right)^{1/\eta} + r^\varepsilon g_k(x)
\]

\[
\leq r^\varepsilon \left[ \mathcal{M}_{p_0}(g_k^\eta)(x) \right]^{1/\eta} + r^\varepsilon g_k(x).
\]

(3.10)
Note that, in obtaining the third inequality in (3.10), we have relied on the fact that $\rho \approx \rho_y$ implies $M_y(g_k) \approx M_y(g_{k+q})$, granted that $\mu$ is doubling. Given how the sets $E_k$ are defined, looking at the extreme most sides of (3.10) and passing to the limit as $r \to 0^+$ gives (3.8) for any point $x \in (C^{-1}_\rho B_k \cap \Omega) \setminus E_k$. Since we have $\Omega = \bigcup_{k=1}^{\infty} (C^{-1}_\rho B_k \cap \Omega)$ and $\mu(\bigcup_{k=1}^{\infty} E_k) = 0$, the claim in (3.8), as stated, now follows.

The fact that (3.8) holds true whenever $u \in \dot{M}_{p,q}^{s}(\Omega, \rho, \mu)$ or $u \in \dot{N}_{p,q}^{s}(\Omega, \rho, \mu)$ for some exponents $s$, $p \in (0, \infty)$, and $q \in (0, \infty)$, is a consequence of [5] Proposition 2.4, which gives the inclusions

$$\dot{M}_{p,q}^{s}(\Omega, \rho, \mu) \hookrightarrow \dot{M}_{p,\infty}^{s}(\Omega, \rho, \mu) = \dot{M}_{\infty}^{s}(\Omega, \rho, \mu)$$

and

$$\dot{N}_{p,q}^{s}(B \cap \Omega, \rho, \mu) \hookrightarrow \dot{M}_{\infty}^{s}(B \cap \Omega, \rho, \mu),$$

whenever $B$ is a $\rho$-ball centered in $\Omega$ and $\varepsilon \in (0, s)$. This finishes the proof of (i).

Moving on to proving (ii), suppose that $u \colon \Omega \to \mathbb{R}$ is a $\mu$-measurable function, $\{g_k\}_{k \in \mathbb{Z}} \in D^s_{\rho}(u)$, $x \in \Omega$, $r_\varepsilon \in [1, \infty)$, and $L \in \mathbb{Z}$ satisfies $2^{-L} \leq r_\varepsilon$. Recall that the doubling property of $\mu$ implies that $u$ is $Q$-doubling with $Q := \log_2 C_D \in (0, \infty)$, where $C_D \in (1, \infty)$ is the doubling constant for $\mu$ [see (1.2)]. Let $\varepsilon' \in (0, \min(\varepsilon, Q/t))$ and suppose that $B \subset B_{\rho}(x, 2^{-L})$ is a $\rho$-ball centered in $\Omega$ such that $\mu(B \cap \Omega) \approx \mu(B_{\rho}(x, 2^{-L}) \cap \Omega)$. Also, let $\gamma_0 \in \mathbb{R}$ be such that

$$\left[ \int_{B_{\rho}(x, 2^{-L}) \cap \Omega} |u - \gamma_0|^{t^*} \, d\mu \right]^{1/t^*} \leq 2 \inf_{\gamma \in \mathbb{R}} \left[ \int_{B_{\rho}(x, 2^{-L}) \cap \Omega} |u - \gamma|^{t^*} \, d\mu \right]^{1/t^*},$$

where $t^* := Qt/(Q - \varepsilon't)$. In concert, (3.6) in Lemma 3.7 [used here with the induced quasi-metric measure space $(\Omega, \rho, \mu)$], the measure density condition (3.7), (3.11), as well as (3.3) in Lemma 3.4 imply

$$\left| m_u(B \cap \Omega) - m_u(B_{\rho}(x, 2^{-L}) \cap \Omega) \right| \leq |m_u(B \cap \Omega) - \gamma_0| + \left| \gamma_0 - m_u(B_{\rho}(x, 2^{-L}) \cap \Omega) \right| \leq 2 \int_{B \cap \Omega} |u - \gamma_0|^{t^*} \, d\mu + 2 \int_{B_{\rho}(x, 2^{-L}) \cap \Omega} |u - \gamma_0|^{t^*} \, d\mu \leq 2^{2L} \left[ \int_{B_{\rho}(x, 2^{-L}) \cap \Omega} 2^{-(s-\varepsilon')\rho J_k} d\mu \right]^{1/s}.$$

This finishes the proof of (ii) and, in turn, the proof of Lemma 3.8

\[ \square \]

\textbf{Lemma 3.10.} Suppose that $(X, \rho, \mu)$ is a quasi-metric measure space and fix an $\alpha \in (0, \infty)$, a $p \in (0, \infty)$, a $q \in (0, \infty]$, and an $s \in (0, \alpha]$, where the value $s = \alpha$ is only permissible when $q = \infty$. Let $f : X \to \mathbb{R}$ be a $\mu$-measurable function with $\dot{h} := \{h_k\}_{k \in \mathbb{Z}} \in D^s_{\rho}(f)$, and assume that
\[ \Psi \in \mathcal{G}^{\alpha}(X, \rho) \text{ is a bounded function that vanishes pointwise outside of a } \mu\text{-measurable set } V \subset X. \]

Then there exists a sequence \( \tilde{g} \in \mathcal{D}_{\rho}^s(\Psi f) \) satisfying
\[
\|\tilde{g}\|_{L^p(X,\rho)} \leq C \left[ \|\Psi\|_{L^\infty(X)} \|\tilde{h}\|_{L^p(V,\rho)} + \|\Psi\|_{\mathcal{G}^{\alpha}(X,\rho)}^{\beta/\alpha} \left( \|\Psi\|_{L^\infty(X)} + 1 \right) \|f\|_{L^p(V,\rho)} \right] \tag{3.12}
\]
and
\[
\|\tilde{g}\|_{L^q(V,\rho)} \leq C \left[ \|\Psi\|_{L^\infty(X)} \|\tilde{h}\|_{L^q(V)} + \|\Psi\|_{\mathcal{G}^{\alpha}(X,\rho)}^{\beta/\alpha} \left( \|\Psi\|_{L^\infty(X)} + 1 \right) \|f\|_{L^q(V)} \right] \tag{3.13}
\]
for some positive constant \( C \) depending only on \( s, p, q, \) and \( \alpha. \)

Consequently, if \( f \in M^{s}_{p,q}(V,\rho,\mu) \), then \( \Psi f \in M^{s}_{p,q}(X,\rho,\mu) \) and
\[
\|\Psi f\|_{M^{s}_{p,q}(X,\rho,\mu)} \leq \|f\|_{M^{s}_{p,q}(V,\rho,\mu)},
\]
where the implicit positive constant depends on \( \Psi \), but is independent of \( f \). This last statement is also valid with \( M^{s}_{p,q} \) replaced by \( N^{s}_{p,q}. \)

**Proof.** If \( \|\Psi\|_{\mathcal{G}^{\alpha}(X,\rho)} = 0 \), then \( \Psi \) is constant. In this case, we take \( g_k := \|\Psi\|_{L^\infty(X)} h_k \) for any \( k \in \mathbb{Z}. \)
Then it is easy to check that \( \tilde{g} := \{g_k\}_{k \in \mathbb{Z}} \in \mathcal{D}_{\rho}^s(\Psi f) \) and satisfies both (3.12) and (3.13).

Suppose next that \( \|\Psi\|_{\mathcal{G}^{\alpha}(X,\rho)} > 0 \) and let \( k_\Psi \in \mathbb{Z} \) be the unique integer such that
\[
2^{k_\Psi - 1} \leq \|\Psi\|_{\mathcal{G}^{\alpha}(X,\rho)}^{1/\alpha} < 2^{k_\Psi}.
\]

For any \( k \in \mathbb{Z} \) and \( x \in X \), define
\[
g_k(x) := \begin{cases} 
\left[ |f(x)| 2^{-k(s-\alpha)} \|\Psi\|_{\mathcal{G}^{\alpha}(X,\rho)} + h_k(x) \|\Psi\|_{L^\infty(X)} \right] 1_V(x) & \text{if } k \geq k_\Psi, \\
2^{k+s+1} |f(x)| + h_k(x) \|\Psi\|_{L^\infty(X)} 1_V(x) & \text{if } k < k_\Psi.
\end{cases}
\]

Note that each \( g_k \) is \( \mu\)-measurable because \( f \) and each \( h_k \) are \( \mu\)-measurable, by assumption. In order to show that this sequence is a fractional \( s\)-gradient of \( \Psi f \) (with respect to \( \rho \)), we fix a \( k \in \mathbb{Z} \) and let \( y, z \in X \) be such that \( 2^{-k-1} \leq \rho(y, z) < 2^{-k}. \) To proceed, we first consider the case when \( k \geq k_\Psi. \) If \( y, z \in X \setminus V \), then \( \Psi(y) = \Psi(z) = 0 \) and there is nothing to show. If, on the other hand, \( y \in V \) and \( z \in X \), then we write
\[
|\Psi(y) f(y) - \Psi(z) f(z)| \leq |f(y)\|\Psi(y) - \Psi(z)| + |\Psi(z)| |f(y) - f(z)|, \tag{3.15}
\]
and use the Hölder continuity of \( \Psi \) and the fact that \( s \leq \alpha \) and \( \text{supp } \Psi \subset V \) in order to estimate
\[
|\Psi(y) f(y) - \Psi(z) f(z)| \\
\leq |f(y)| |\rho(y, z)|^s \|\Psi\|_{\mathcal{G}^{\alpha}(X,\rho)} + |\Psi|_{L^\infty(X)} |\rho(y, z)|^s \left[ h_k(y) + h_k(z) \right] 1_V(z) \\
= |f(y)| |\rho(y, z)|^s \|\Psi\|_{\mathcal{G}^{\alpha}(X,\rho)} + |\Psi|_{L^\infty(X)} |\rho(y, z)|^s \left[ h_k(y) + h_k(z) \right] 1_V(z) \\
\leq |\rho(y, z)|^s \left[ |f(y)| 2^{-(k-s)} \|\Psi\|_{\mathcal{G}^{\alpha}(X,\rho)} + |\Psi|_{L^\infty(X)} \left[ h_k(y) + h_k(z) \right] 1_V(z) \right] \\
\leq |\rho(y, z)|^s \left[ g_k(y) + g_k(z) \right].
\]
The estimate when \( y \in X \) and \( z \in V \) is similar where, in place of (3.15), we use
\[
|\Psi(y)f(y) - \Psi(z)f(z)| \leq |f(z)||\Psi(y) - \Psi(z)| + |\Psi(y)||f(y) - f(z)|.
\]
This finishes the proof of the case when \( k \geq k_\Psi \). Assume next that \( k < k_\Psi \). As in the case \( k \geq k_\Psi \), it suffices to consider the scenario when \( y \in V \) and \( z \in X \). Appealing to (3.15), we conclude that
\[
|\Psi(y)f(y) - \Psi(z)f(z)| \\
\leq 2 |f(y)||\Psi|_{L^\infty(X)} + |\Psi|_{L^\infty(X)} |\rho(y, z)|^s |h_k(y) + h_k(z)| 1_{V}(z) \\
+ |\Psi|_{L^\infty(X)} |\rho(y, z)|^s |h_k(y) + h_k(z)| 1_{V}(z) \\
\leq |\rho(y, z)|^s |\Psi|_{L^\infty(X)} [2^{s(k+1)+1} |f(y)| + [h_k(y) + h_k(z)| 1_{V}(z) \\
\leq |\rho(y, z)|^s [g_k(y) + g_k(z)].
\]
This finishes the proof of the claim that \( \tilde{g} := \{g_k\}_{k \in \mathbb{Z}} \in \mathcal{D}^s_{\rho}(\Psi f) \).

We next show that \( \tilde{g} := \{g_k\}_{k \in \mathbb{Z}} \) satisfies the desired estimates in (3.12) and (3.13). If \( q < \infty \), then [keeping in mind (3.14) and the fact that \( 0 < s < \alpha \) in this case]
\[
\left( \sum_{k \in \mathbb{Z}} g_k^q \right)^{1/q} = \left\{ \sum_{k=-\infty}^{k_\Psi-1} |\Psi|_{L^\infty(X)}^q 2^{(k_\Psi+s+1)q} |f| + h_k \right\}^{1/q} \\
+ \sum_{k=k_\Psi}^{\infty} \left[ \left| f \right| 2^{-k(\alpha-s)} |\Psi|_{L^\infty(X)}^q + h_k \right] |\Psi|_{L^\infty(X)} \right)^{1/q} \\
\leq \left\{ |\Psi|_{L^\infty(X)} \left( \sum_{k \in \mathbb{Z}} h_k^q \right)^{1/q} + |\Psi|_{L^\infty(X)} \left| f \right| \left[ \sum_{k=-\infty}^{k_\Psi-1} 2^{(k_\Psi+s+1)q} \right]^{1/q} \\
+ |\Psi|_{L^\infty(X)} \left| f \right| \sum_{k=k_\Psi}^{\infty} 2^{-k(\alpha-s)q} \right]^{1/q} \\
\leq \left\{ |\Psi|_{L^\infty(X)} \left( \sum_{k \in \mathbb{Z}} h_k^q \right)^{1/q} + |\Psi|_{L^\infty(X)} \left| f \right| 2^{k_\Psi s} + |\Psi|_{L^\infty(X)} \left| f \right| 2^{-k_\Psi(\alpha-s)} \right]^{1/q} \\
\leq \left\{ |\Psi|_{L^\infty(X)} |\Psi|_{L^\infty(X)}^{q/\alpha} |\Psi|_{L^\infty(X)}^{s/\alpha} \left| f \right| + |\Psi|_{L^\infty(X)}^{q/\alpha} \left| f \right|. \right. \tag{3.16}
\]
The estimate in (3.12) (for \( q < \infty \)) now follows from the estimate in (3.16) and the fact that each \( g_k \) is supported in \( V \). To verify (3.13) when \( q < \infty \), observe that
\[
||\tilde{g}||_{L^q(X)} \left( \sum_{k \in \mathbb{Z}} |g_k|^q \right)^{1/q} \\
\leq \sum_{k=-\infty}^{k_\Psi-1} |\Psi|_{L^\infty(X)}^q \left[ 2^{(k_\Psi+s+1)q} |f| + h_k \right]^{1/q} \left| L^q(X) \right|_{L^q(X)} \\
\leq \left\{ |\Psi|_{L^\infty(X)} \left( \sum_{k \in \mathbb{Z}} h_k^q \right)^{1/q} + |\Psi|_{L^\infty(X)} \left| f \right| 2^{k_\Psi s} + |\Psi|_{L^\infty(X)} \left| f \right| 2^{-k_\Psi(\alpha-s)} \right]^{1/q} \\
\leq \left\{ |\Psi|_{L^\infty(X)} |\Psi|_{L^\infty(X)}^{q/\alpha} |\Psi|_{L^\infty(X)}^{s/\alpha} \left| f \right| + |\Psi|_{L^\infty(X)}^{q/\alpha} \left| f \right|. \right. \tag{3.17}
\]
Then there exists a positive constant $C$ such that,

$$q\text{-permissible when } \tag{1.1}$$

measure density condition

Let

$$\in \preceq \sum_{k=0}^{\infty} 2^{-k(a-s)} \|\Psi\|_{\ell^\infty(X,\rho)} + h_k \|\Psi\|_{L^\infty(X)} \left\| \sum_{k=k_q}^{k-1} 2^{(k+1)q} \right\|^{1/q}$$

as wanted. The proof of (3.12) and (3.13) when $q = \infty$ follow along a similar line of reasoning. Note that the choice of $s = \alpha$ is permissible in this case because, in this scenario, the summation $\sum_{k=k_q}^{\infty} 2^{-k(a-s)}$ is replaced by $\sup_{k \geq k_q} 2^{-k(a-s)} = 1$. This finishes the proof of Lemma 3.10.

We end this section by recalling the following inequality established in [23, Lemma 3.1].

**Lemma 3.11.** Let $a \in (1, \infty)$ and $b \in (0, \infty)$. Then there exists a positive constant $C = C(a, b)$ such that, for any sequence $\{c_k\}_{k \in \mathbb{Z}}$ of nonnegative real numbers,

$$\sum_{k \in \mathbb{Z}} \left( \sum_{j \in \mathbb{Z}} a^{-|j-k|} c_j \right)^b \leq C \sum_{j \in \mathbb{Z}} c_j^b.$$  

3.2 Principal Results

At this stage, we are ready to construct a Whitney-type extension operator for the spaces $M_{p,q}^s$ and $N_{p,q}^s$ on domains satisfying the so-called measure density condition for an optimal range of $s$. Before stating this result, we recall the following piece of notational convention.

**Convention 3.12.** Given a quasi-metric space $(X, \rho)$ and fixed numbers $s \in (0, \infty)$ and $q \in (0, \infty)$, we will understand by $s \leq q$ ind $(X, \rho)$ that $s \leq $ ind $(X, \rho)$ and that the value $s = $ ind $(X, \rho)$ is only permissible when $q = \infty$ and the supremum defining ind $(X, \rho)$ in (2.4) is attained.

Here is the extension result alluded to above.

**Theorem 3.13.** Let $(X, \rho, \mu)$ be a quasi-metric measure space, where $\mu$ is a Borel regular $Q$-doubling measure on $X$ for some $Q \in (0, \infty)$, and fix exponents $s, p \in (0, \infty)$ and $a \in (0, \infty)$, where $s \leq q$ ind $(X, \rho)$. Also, suppose that $\Omega \subset X$ is a nonempty $\mu$-measurable set that satisfies the measure density condition (1.1), that is, there exists a positive constant $C_\mu$ such that

$$\mu(B_\rho(x, r)) \leq C_\mu \mu(B_\rho(x, r) \cap \Omega) \quad \text{for any } x \in \Omega \text{ and } r \in (0, 1]. \tag{3.17}$$

Then there exists a positive constant $C$ such that,

for any $u \in M_{p,q}^s(\Omega, \rho, \mu)$, there exists a $\tilde{u} \in M_{p,q}^s(X, \rho, \mu)$

for which $u = \tilde{u}|_{\Omega}$ and $\|\tilde{u}\|_{M_{p,q}^s(X, \rho, \mu)} \leq C \|u\|_{M_{p,q}^s(\Omega, \rho, \mu)}$. \tag{3.18}
Furthermore, if \( p, q > Q/(Q + s) \), then there is a linear and bounded operator
\[
\mathcal{E}: M_{p,q}^s(\Omega, \rho, \mu) \to M_{p,q}^s(X, \rho, \mu)
\]
such that \( (\mathcal{E}u)_{|\Omega} = u \) for any \( u \in M_{p,q}^s(\Omega, \rho, \mu) \). In addition, if \( s < \text{ind} (X, \rho) \), then all of the statements above are also valid with \( M_{p,q}^s \) replaced by \( N_{p,q}^s \). 

**Remark 3.14.** In Theorem 3.13, we only assume that \( \mu \) is Borel regular to ensure that the Lebesgue differentiation theorem holds true. It is instructive to note that one could assume a weaker regularity condition on \( \mu \), namely, the Borel semi-regularity, which turns out to be equivalent to the availability of the Lebesgue differentiation theorem; see [2] Theorem 3.14.

**Remark 3.15.** The linear extension operator constructed in Theorem 3.13 (when \( p \) and \( q \) are large enough) is a Whitney-type operator and it is universal (in the sense that it simultaneously preserves all orders of smoothness).

**Proof of Theorem 3.13** We begin by making a few important observations. First, note that we can assume \( \Omega \) is a closed set. Indeed, the measure density condition (3.17), when put in concert with the Lebesgue differentiation theorem, implies that \( \mu(\partial \Omega) = 0 \) (see, for instance, [37] Lemma 2.1). Hence, \( M_{p,q}^s(\Omega) = M_{p,q}^s(X) \) and \( N_{p,q}^s(\Omega) = N_{p,q}^s(X) \) because the membership to the spaces \( M_{p,q}^s \) and \( N_{p,q}^s \) is defined up to a set of measure zero. Moreover, we can assume that \( \Omega \neq X \) as the claim is trivial when \( \Omega = X \).

Moving on, by \( s \leq \text{ind} (X, \rho) \) and its meaning, there exists a quasi-metric \( \varrho \) on \( X \) such that \( \varrho \approx \rho \) and \( s \leq (\log_2 C_\varrho)^{-1} \), where \( C_\varrho \in [1, \infty) \) is as in (2.2), and the value \( s = (\log_2 C_\varrho)^{-1} \) can only occur when \( q = \infty \) and \( C_\varrho > 1 \). Next, let \( \varrho_\# \) be the regularized quasi-metric given by Theorem 3.3. Then \( \varrho_\# \approx \varrho \) and \( C_{\varrho_\#} \leq C_\varrho \). Thus, \( s \leq (\log_2 C_{\varrho_\#})^{-1} \), where the value \( s = (\log_2 C_{\varrho_\#})^{-1} \) can only occur when \( q = \infty \) and \( C_{\varrho_\#} > 1 \).

Recall that, by Theorem 3.3, all \( \varrho_\# \)-balls are \( \mu \)-measurable. Combining this with the fact that \( \varrho_\# \approx \rho \), we conclude that \( (X, \varrho_\#, \mu) \) is a quasi-metric measure space where \( \mu \) is doubling with respect to \( \varrho_\# \)-balls. As such, we can consider the decomposition of the open set \( O := X \setminus \Omega \) into the family \( \{B_j\}_{j \in \mathbb{N}} \) of \( \varrho_\# \)-balls, where, for any given \( j \in \mathbb{N} \), \( B_j := B_{\varrho_\#}(x_j, r_j) \) with \( x_j \in O \) and \( r_j \in (0, \infty) \), as given by Theorem 3.1 with \( \theta := 2C_{\varrho_\#}^2 \). Also, fix \( \theta' \in (C_{\varrho_\#}, \theta/C_{\varrho_\#}) = (C_{\varrho_\#}, 2C_{\varrho_\#}) \) and choose any finite number \( \alpha \in [s, (\log_2 C_{\varrho_\#})^{-1}] \), where \( \alpha \neq s \) unless \( s = (\log_2 C_{\varrho_\#})^{-1} \). In this context, let \( \{\psi_j\}_{j \in \mathbb{N}} \) be the associated partition of unity of order \( \alpha \) given by Theorem 3.2 (applied here with the space \( (X, \varrho_\#, \mu) \)).

By Theorem 3.1(iii), for any \( j \in \mathbb{N} \), there exists an \( x_j^* \in \Omega \) satisfying \( \varrho_\#(x_j, x_j^*) < \Lambda r_j \), where \( \Lambda \in (\theta, \infty) \) is as in Theorem 3.1. Let
\[
B_j^* := B_{\varrho_\#}(x_j^*, r_j), \quad \forall \ j \in \mathbb{N},
\]
and, for any \( x \in X \setminus \Omega \), define
\[
r(x) := \text{dist}_{\varrho_\#}(x, \Omega)/(4C_{\varrho_\#}) \quad \text{and} \quad B_x := B_{\varrho_\#}(x, \Lambda^2 r(x)),
\]
where \( \text{dist}_{\varrho_\#}(x, \Omega) := \inf_{y \in \Omega} \varrho_\#(x, y) \). We claim that
\[
B_j^* \subset B_x \subset C_{\varrho_\#}^2 \Lambda^3 B_j^* \quad \text{whenever} \ j \in \mathbb{N} \ \text{and} \ x \in \theta B_j.
\]
To show (3.21), we first prove that
\[
\frac{r_j}{2} \leq r(x) \leq \frac{\Lambda r_j}{4} \quad \text{whenever } j \in \mathbb{N} \text{ and } x \in \theta'B_j.
\] (3.21)
Indeed, for any \( j \in \mathbb{N} \) and \( x \in \theta'B_j \), by \( x_j^* \in \Omega \) and \( \Lambda > \theta > C_{\#} \theta' \), we find that (keeping in mind that \( \varrho_\# \) is symmetric)
\[
\text{dist}_{\#}(x, \Omega) \leq \varrho_\#(x, x_j^*) \leq C_{\#} \max \{ \varrho_\#(x, x_j), \varrho_\#(x, x_j^*) \} < C_{\#} \max \{ \theta' r_j, \Lambda r_j \} = C_{\#} \Lambda r_j,
\]
which implies the second inequality in (3.21). To prove the first inequality in (3.21), fix an arbitrary \( z \in \Omega \). By Theorem 3.1(iii), we have \( \theta B_j \subset X \setminus \Omega \). Therefore, \( z \notin \theta B_j \), which, together with \( x \in \theta'B_j \) and \( \theta > C_{\#} \theta' \), implies that
\[
\theta r_j \leq \varrho_\#(x_j, z) \leq C_{\#} \max \{ \varrho_\#(x_j, x), \varrho_\#(x, z) \} < \max \{ C_{\#} \theta' r_j, C_{\#} \varrho_\#(x, z) \} < \max \{ \theta r_j, C_{\#} \varrho_\#(x, z) \} = C_{\#} \varrho_\#(x, z).
\] (3.22)

Given that \( z \in \Omega \) was arbitrary, by looking at the extreme most sides of (3.22) and taking the infimum over all \( z \in \Omega \), we conclude that
\[
2C_{\#}^2 r_j = \theta r_j \leq C_{\#} \text{dist}_{\#}(x, \Omega), \quad \forall x \in \theta'B_j,
\] (3.23)
and the first inequality in (3.21) follows. This finishes the proof of (3.21).

Returning to the proof of (3.20), we fix again a \( j \in \mathbb{N} \) and an \( x \in \theta'B_j \). By (3.21), the choice of \( \theta := 2C_{\#}^2 \), and the fact that \( \Lambda > \theta > \theta' \), we find that, for any \( z \in B_j^* \),
\[
\varrho_\#(x, z) \leq C_{\#}^2 \max \{ \varrho_\#(x, x_j), \varrho_\#(x_j, x_j^*), \varrho_\#(x_j^*, z) \} < C_{\#}^2 \max \{ \theta' r_j, \Lambda r_j, r_j \} = C_{\#}^2 \Lambda r_j \leq 2C_{\#}^2 \Lambda^2 r(x) < \Lambda^2 r(x).
\]
This implies \( B_j^* \subset B_x \).

Next, we show \( B_x \subset C_{\#}^2 \Lambda^3 B_j^* \). Indeed, by (3.21), one has, for any \( z \in B_x \),
\[
\varrho_\#(x_j^*, z) \leq C_{\#}^2 \max \{ \varrho_\#(x_j^*, x_j), \varrho_\#(x_j, x), \varrho_\#(x, z) \} < C_{\#}^2 \max \{ \Lambda r_j, \theta' r_j, \Lambda^2 r(x) \} \leq C_{\#}^2 \max \{ \Lambda r_j, \theta' r_j, \Lambda^3 r_j \} = C_{\#}^2 \Lambda^3 r_j.
\]
Thus, \( B_x \subset C_{\#}^2 \Lambda^3 B_j^* \). This finishes the proof of (3.20).

Moving on, define \( J := \{ j \in \mathbb{N} : r_j \leq 1 \} \). It follows from (3.20), the definition of \( J \), the measure density condition (3.17), and the doubling property of \( \mu \) that
\[
\mu(B_j \cap \Omega) \geq C \mu(B_j^*) \approx \mu(B_x), \quad \text{whenever } j \in J \text{ and } x \in \theta'B_j.
\] (3.24)

Our plan below is to show that each function belonging to \( M_{p,q}^s(\Omega, \varrho_\#, \mu) \) can be extended to the entire space \( X \) with preservation of smoothness and while retaining control of the associated
‘norm’. Then the conclusion of this theorem will follow from Proposition 2.1 which implies $M_{p,q}^s(\Omega, \mathcal{g}_\# , \mu) = M_{p,q}^s(\Omega, \rho, \mu)$ with equivalent ‘norms’. To this end, fix a $u \in M_{p,q}^s(\Omega, \mathcal{g}_\# , \mu)$ and choose a $g := \{ g_k \}_{k \in \mathbb{Z}} \in \mathcal{D}^s_p(u)$ satisfying \[ \| g \|_{L^p(\Omega, \mathcal{g}_\#)} \leq \| u \|_{M_{p,q}^s(\Omega)}. \] Note that such a choice is possible, due to Proposition 2.2. Although, for any $k \in \mathbb{Z}$, $g_k$ and $u$ are defined only in $\Omega$, we will identify them as a function defined on all of $X$ by setting $u \equiv g_k \equiv 0$ on $X \setminus \Omega$.

We first extend the function $u$ to the following neighborhood of $\Omega$:

\[ V := \{ x \in X : \text{dist}_{\mathcal{g}_\#}(x, \Omega) < 2C_{\mathcal{g}_\#} \}. \tag{3.25} \]

To this end, for each $x \in V \setminus \Omega$, let $I_x$ denote the collection of all $j \in \mathbb{N}$ such that $x \in \theta^j B_j$. Since $\theta' < \theta$, by Theorem 3.1(ii), we have $\# I_x \leq M$, where $M$ is a positive integer depending on $\theta$ and the space $(X, \mathcal{g}_\#, \mu)$. Moreover, if $j \in \mathbb{N} \setminus J$, then it follows from (3.23) and the definition of $J$ that

\[ \text{dist}_{\mathcal{g}_\#}(\theta^j B_j, \Omega) \geq 2C_{\mathcal{g}_\#} r_j \geq 2C_{\mathcal{g}_\#}, \]

and hence $\theta^j B_j \cap V = \emptyset$ and $j \notin I_x$. As such, we have $I_x \subset J$. Observe that, if $j \notin I_x$, then, by Theorem 3.2(ii), one has $\psi_j(x) = 0$ which, together with Theorem 3.2(iii), implies that

\[ \sum_{j \in I_x} \psi_j(x) = \sum_{j \in J} \psi_j(x) = 1, \quad \forall \ x \in V \setminus \Omega. \tag{3.26} \]

Define the local extension, $\mathcal{F} u : X \to \mathbb{R}$, of $u$ to $V$ by setting

\[ \mathcal{F} u(x) := \begin{cases} u(x) & \text{if } x \in \Omega, \\ \sum_{j \in \mathbb{N}} \psi_j(x) m_u(B_j^s \cap \Omega) & \text{if } x \in X \setminus \Omega, \end{cases} \tag{3.27} \]

where $m_u$ is as in Definition 3.6. Given the presence of the median value factor $m_u(B_j^s \cap \Omega)$ in (3.27), it is not to be expected that $\mathcal{F}$ is linear. However, if $p, q > Q/(Q + s)$, then we can construct a linear extension as follows: First, observe that $u \in L^1(B_j^s \cap \Omega)$ for any $j \in J$. Indeed, if $\Omega$ contains only one point then this claim is obvious. Otherwise, the measure density condition (3.17) and the doubling property for $\mu$, ensure that $(\Omega, \mathcal{g}_\#, \mu)$ is a quasi-metric measure space, where $\mu$ is $Q$-doubling on domain $\mathcal{g}_\#$-balls up to scale $C_{\mathcal{g}_\#}$. Here, $\mathcal{g}_\#$ is naturally restricted to the set $\Omega$. Let $r := Q/(Q + s) < Q/s$. Then $r < p$ and by $u \in M_{p,q}^s(\Omega, \mathcal{g}_\#, \mu)$ and the Hölder inequality, we have $u \in M_{r,q}^s(\Omega, \mathcal{g}_\#, \mu)$. Hence, $u \in L^r(B_j^s \cap \Omega) = L^1(B_j^s \cap \Omega)$, where $r^* := Qr/(Q - sr)$, by Theorem 2.4(a) [applied here for the space $(\Omega, \mathcal{g}_\#, \mu)$ and with $r_* := \sigma := C_{\mathcal{g}_\#}$]. As such, we can replace the median value $m_u(B_j^s \cap \Omega)$ in (3.27) with the integral average $u_{B_j^s \cap \Omega}$. The resulting operator $\mathcal{F}$ is linear and we can show that it is also a local extension of $u$ by using the estimate in Remark 3.5 with $0 < e' < e < s$ chosen so that $p, q > Q/(Q + e') =: t$ in place of (3.9) in the proof that follows here. We omit the details.

We now prove that $\mathcal{F} u \in M_{p,q}^s(V)$ with $\| \mathcal{F} u \|_{M_{p,q}^s(V, \mu)} \leq \| u \|_{M_{p,q}^s(\Omega, \mu)}$ by establishing the following three lemmas.

**Lemma 3.16.** Let all the notation be as in the above proof of Theorem 3.13. Then $\mathcal{F} u \in L^p(V, \mu)$ and $\| \mathcal{F} u \|_{L^p(V, \mu)} \leq C \| u \|_{L^p(\Omega, \mu)}$, where $C$ is a positive constant independent of $u, V$, and $\Omega$.

\[ \text{Similarly, one can show that restrictions of functions in } N_{p,q}^s(\Omega) \text{ belong to } L^1(B_j^s \cap \Omega) \text{ by choosing an } \varepsilon \in (0, s) \text{ so that } \tilde{r} := Q/(Q + \varepsilon) < p \text{ and then using Theorem 2.3(a) in place of Theorem 2.4(a).} \]
Proof. First observe that $\mathcal{F} u$ is $\mu$-measurable, because $u$ is $\mu$-measurable and, by Theorem 3.1 (ii), and both (i) and (ii) of Theorem 3.2, the sum in (3.27) is a finite linear combination of continuous functions nearby each point in $X \setminus \Omega$.

Let $x \in X \setminus \Omega$ and fix an $\eta \in (0, \min\{1, p\})$. Combining (3.26), (3.6) in Lemma 3.7 with $\gamma = 0$, (3.20), (3.24), the fact that $#I_x \leq M$, and this choice of $\eta$, we conclude that

$$|\mathcal{F} u(x)| \leq \sum_{j \in I_x} \psi_j(x) |m_u(B_j' \cap \Omega)| \leq \sum_{j \in I_x} \left( \int_{B_j' \cap \Omega} |u|^\alpha \, d\mu \right)^{1/\alpha},$$

which, together with the definition of $\mathcal{F} u$ and the boundedness of the Hardy-Littlewood maximal operator on $L^{p/\eta}(X)$, implies the desired estimate $\|\mathcal{F} u\|_{L^p(\Omega, \mu)} \leq \|u\|_{L^{p/\eta}(\Omega, \mu)}$. This finishes the proof of Lemma 3.16.

Lemma 3.17. Let all the notation be as in the above proof of Theorem 3.13. Fix an $\varepsilon \in (0, s)$ and choose any number $\delta \in (0, \min\{\alpha - s, s - \varepsilon\})$ if $\alpha \neq s$, and set $\delta = 0$ if $\alpha = s$. Suppose that $t \in (0, \min\{p, q\})$ and let $k_0 \in \mathbb{Z}$ be such that $2^{k_0 - 1} \leq 16C^{4s}_\delta A^{2} < 2^{k_0}$. In this context, define a sequence $\hat{h} := \{h_k\}_{k \in \mathbb{Z}}$ by setting, for any $k \in \mathbb{Z}$ and $x \in X$,

$$h_k(x) := \begin{cases} M_{\mathcal{F} u} \left( \sup_{j \in \mathbb{Z}} \left\{ 2^{-j-k} \hat{g}^j \right\} \right)^{1/\eta} & \text{if } k \geq k_0, \\ 2^{(k+1)s} |\mathcal{F} u(x)| & \text{if } k < k_0. \end{cases}$$

Then there exists a positive constant $C$, independent of $u$, $\hat{g}$, $\hat{h}$, and $\Omega$, such that $\{Ch_k\}_{k \in \mathbb{Z}}$ is an $s$-fractional gradient of $\mathcal{F} u$ on $V$ with respect to $\mathcal{Q}_u$.

Remark 3.18. The sequence in (3.28) is different from than one considered in [23] (5.3), and it is this new sequence defined in (3.28) that allows us to generalize [23] Theorem 1.2 and [20] Theorem 6] simultaneously and in a unified manner.

Proof of Lemma 3.17. We first prove that, for any $k \in \mathbb{Z}$, $h_k$ is a well-defined function. Indeed, by the choice of $\hat{g}$, one has $\|\hat{g}(\cdot)\|_{\mathcal{E}^t} \leq L^{p}(X, \mu)$. Then, when $q < \infty$, we have $\delta > 0$ and, by the Hölder inequality (used with the exponent $r := q/t > 1$),

$$\sup_{j \in \mathbb{Z}} \left\{ 2^{-j-k} \hat{g}^j \right\} \leq \sum_{j \in \mathbb{Z}} 2^{-j-k} \hat{g}^j \leq \left( \sum_{j \in \mathbb{Z}} \left( \sum_{j \in \mathbb{Z}} \hat{g}^j \right)^{q} \right)^{1/q} \in L^{p/\eta}(X, \mu);$$

while, when $q = \infty$, one also has

$$\sup_{j \in \mathbb{Z}} \left\{ 2^{-j-k} \hat{g}^j \right\} \leq \left( \sup_{j \in \mathbb{Z}} g^j \right)^t \in L^{p/t}(X, \mu).$$
 Altogether, we have proved that \( \sup_{j \in \mathbb{Z}} \{ 2^{-k-j} \delta^j g_j \} \) \( \in L^{p/t}(X, \mu) \). Consequently, by \( p/t > 1 \), we conclude that \( \sup_{j \in \mathbb{Z}} \{ 2^{-k-j} \delta^j g_j \} \) is locally integrable on \( X \). From this and Lemma 3.16 we further deduce that \( h_k \) is a well-defined \( \mu \)-measurable function on \( X \).

Moving on, to prove Lemma 3.17 we need to show that there exists a positive constant \( C \) such that, for any fixed \( k \in \mathbb{Z} \), the following inequality holds true, for \( \mu \)-almost every \( x, y \) satisfying \( 2^{-k-1} \leq \varrho_\#(x, y) < 2^{-k} \):

\[
|\mathcal{F} u(x) - \mathcal{F} u(y)| \leq [\varrho_\#(x, y)]^t [Ch_k(x) + Ch_k(y)].
\] (3.29)

To this end, fix \( x, y \in V \) and suppose that \( 2^{-k-1} \leq \varrho_\#(x, y) < 2^{-k} \) for some \( k \in \mathbb{Z} \). If \( k < k_0 \), then

\[
|\mathcal{F} u(x) - \mathcal{F} u(y)| \leq |\mathcal{F} u(x)| + |\mathcal{F} u(y)|
\leq 2^{(k+1)t} [\varrho_\#(x, y)]^t [|\mathcal{F} u(x)| + |\mathcal{F} u(y)|]
= [\varrho_\#(x, y)]^t [h_k(x) + h_k(y)],
\]

which is a desired estimate in this case.

Suppose next that \( k \geq k_0 \). By the choice of \( k_0 \), we actually have \( k \geq k_0 \geq 5 \). Moreover, since \( \varrho_\# \) is symmetric (that is, \( \overline{C}_{\varrho_\#} = 1 \)), our choice of \( k_0 \) ensures that \( 2^{k_0} > 2^{C_{\varrho_\#}} = C_{\varrho_\#} \varrho_\# \). We proceed by considering four cases based on the locations of both \( x \) and \( y \).

**CASE 1:** \( x, y \in \Omega \).

In this case, without loss of generality, we may assume that \( x \) and \( y \) satisfy the inequality (2.5) with \( \rho \) therein replaced by \( \varrho_\# \), as (2.5) holds true \( \mu \)-almost everywhere in \( \Omega \). Recall that we have already shown that \( \sup_{j \in \mathbb{Z}} \{ 2^{-k-j} \delta^j g_j \} \) is locally integrable. As such, in light of the Lebesgue differentiation theorem ([2, Theorem 3.14]), we can further assume that both \( x \) and \( y \) are Lebesgue points of \( \sup_{j \in \mathbb{Z}} \{ 2^{-k-j} \delta^j g_j \} \). For any \( k \in \mathbb{Z} \), since \( g_k \leq \sup_{j \in \mathbb{Z}} \{ 2^{-k-j} \delta^j g_j \} \), by appealing once again to the Lebesgue differentiation theorem, we have \( g_k \leq h_k \) pointwise \( \mu \)-almost everywhere on \( X \) which, in turn, allows us to conclude that

\[
|\mathcal{F} u(x) - \mathcal{F} u(y)| = |u(x) - u(y)| \leq [\varrho_\#(x, y)]^t [h_k(x) + h_k(y)].
\]

This is a desired estimate in Case 1.

**CASE 2:** \( x \in V \setminus \Omega \) and \( y \in \Omega \).

In this case, we have

\[
r(x) := \text{dist}_{\varrho_\#}(x, \Omega) / (4C_{\varrho_\#}) \leq \varrho_\#(x, y) / 4 < 2^{-k-2}.
\] (3.30)

Recall that \( \Lambda^2 > \theta^2 \geq 4C_{\varrho_\#} \), where \( \theta := 2C_{\varrho_\#}^2 \) and \( \Lambda \in (\theta, \infty) \) as in Theorem 3.1. Then it follows from the definition of \( B_\# := B_{\varrho_\#}(x, \Lambda^2 r(x)) \) that we can find a point \( x^* \in B_\# \cap \Omega \). Let \( m \in \mathbb{Z} \) be such that \( 2^{-m-1} \leq C_{\varrho_\#} \Lambda^2 r(x) < 2^{-m} \) and set \( B_{x^*} := B_{\varrho_\#}(x^*, 2^{-m}) \). The existence of such an \( m \in \mathbb{Z} \) is guaranteed by the fact that \( x \) belongs to the open set \( X \setminus \Omega \), which implies that \( r(x) > 0 \). In order to obtain (3.29), we write

\[
|\mathcal{F} u(x) - \mathcal{F} u(y)| \leq [\mathcal{F} u(x) - m_\#(B_{x^*} \cap \Omega)] + |m_\#(B_{x^*} \cap \Omega) - u(y)|.
\] (3.31)

\footnote{Note that the consideration of the Hardy-Littlewood maximal operator with respect to the regularized quasi-metric, \( \varrho_\# \), ensures that the function \( M_\#(\sup_{j \in \mathbb{Z}} \{ 2^{-k-j} \delta^j g_j \}) \) is \( \mu \)-measurable (see [2, Theorem 3.7]).}
We now separately estimate the two terms appearing in the right-hand side of (3.31). For the first term, using the fact that \( \sum_{i \in I_x} \psi_i(x) = 1 \) in (3.26) and the definition of \( F \) in (3.27), we have

\[
|F(u(x) - m_u(B_x \cap \Omega)| \leq \sum_{i \in I_x} \psi_i(x) \left| m_u(B_i^* \cap \Omega) - m_u(B_x \cap \Omega) \right|, \tag{3.32}
\]

where \( B_i^* \) is as in (3.19). From (3.30), the choices of both \( m_0 \) and \( k_0 \), and \( k \geq k_0 \), it follows that

\[
2^{-m} \leq 2C_{e_u} \lambda^2 r(x) < 2^{-k-1} C_{e_u} \lambda^2 < 1 \tag{3.33}
\]

and, moreover,

\[
B_x \subset B_{x^*} \quad \text{and} \quad C_e B_{x^*} \subset 2C_{e_u} B_x. \tag{3.34}
\]

Combining this with (3.20), the definition of \( I_x \), and the doubling and the measure density conditions of \( \mu \), we have, for any \( i \in I_x \),

\[
\mu(B_i^* \cap \Omega) \approx \mu(B_{x^*} \cap \Omega) \quad \text{and} \quad \mu(C_e B_{x^*}) \approx \mu(2C_{e_u} B_x).
\]

As such, using both (3.20) and (3.34), and employing (3.9) in Lemma 3.8 (with \( 2^{-L} = 2^{-m} < 1 = r_x \)) allow us to further bound the right-hand side of (3.32) as follows

\[
\sum_{i \in I_x} \psi_i(x) \left| m_u(B_i^* \cap \Omega) - m_u(B_{x^*} \cap \Omega) \right| \leq 2^{-me} \left[ \sup_{B_{x^*} \subset C_e B_x} \left\{ 2^{-j(s-e)} g_j^l \right\} \right]^{1/t} \sup_{j \geq m-k_0} \left\{ 2^{-j(s-e)} g_j^l \right\} \left| \mu \right| \tag{3.35}
\]

where we have also used the fact that \( \#I_x \leq M \) and \( \psi_i(x) \leq 1 \) for any \( i \in I_x \). Moreover, by (3.33) and the choice of \( k_0 \), we have \( 2^{-m} < 2^{-k-1} C_{e_u} \lambda^2 < 2^{-k+k_0} \), so \( m > k - k_0 \) and by (3.32) and (3.35), we can estimate

\[
|F(u(x) - m_u(B_x \cap \Omega)| \leq 2^{-me} \left[ \sup_{B_{x^*} \subset C_e B_x} \left\{ 2^{-j(s-e)} g_j^l \right\} \right]^{1/t} \sup_{j \geq m-k_0} \left\{ 2^{j-k(s-e)} g_j^l \right\} \left| \mu \right| \tag{3.36}
\]

Observe next that, by \( \delta \in [0, s-e) \), if \( k - 2k_0 - j \leq 0 \), then

\[
2^{k-j(s-e)} < 2^{2k_0(s-e)} 2^{(k-2k_0-j)\delta} = 2^{k_0(s-e)2^{-k-2k_0-j}\delta} \leq 2^{k_0(s-e)} 2^{-k-j\delta}, \tag{3.37}
\]
where, in obtaining the second inequality in (3.37), we have simply used the fact that \(|k - 2k_0 - j| \leq -|k - j| + 2k_0\). This, in conjunction with (3.36) and the definition of \(h_k\), now gives

\[
|\mathcal{F} u(x) - m_u(B_{2^k} \cap \Omega)| \leq [g_\#(x,y)]^t \left[ \mathcal{M}_{g_\#} \left( \sup_{j \geq k - 2k_0} \{2^{-|k-j|} g_j^t (y)\} \right) \right]^{1/t} \\
\leq [g_\#(x,y)]^t h_k(x),
\]

which is a desired estimate for the first term in the right-hand side of (3.31).

We now turn our attention to estimating the second term in the right-hand side of (3.31). Let \(n \in \mathbb{Z}\) be the least integer satisfying \(B_{2^n}(y, 2^{-k}) \subset 2^n B_{2^k}\). We claim that

\[
2^{-k-1} C_{2^n}^{-1} \leq 2^{n-m} < C_{2^n}^2 \Lambda^2 2^{-k+1} < 2^{k_0-k} \leq 1.
\]

Given that we are assuming \(k \geq k_0\), our choice of \(k_0\) ensures that \(C_{2^n}^2 \Lambda^2 2^{-k+1} < 2^{k_0-k} \leq 1\). To prove the second inequality in (3.39), note that, by the choice of \(n\), we necessarily have \(B_{2^n}(y, 2^{-k}) \not\subset 2^n B_{2^k}\). Therefore, we can find a point \(z_0 \in X\) satisfying \(g_\#(y, z_0) < 2^{-k}\) and \(g_\#(x^*, z_0) \geq 2^{n-1-m}\), which, together with (3.30), implies that

\[
2^{n-1-m} \leq g_\#(x^*, z_0) \leq C_{2^n}^2 \max\{g_\#(x^*, x), g_\#(x, y), g_\#(y, z_0)\} \\
< C_{2^n}^2 \max\{\Lambda^2 \delta(x), 2^{-k}, 2^{-k}\} < C_{2^n}^2 \Lambda^2 2^{-k}.
\]

The second inequality in (3.39) now follows. Furthermore, by \(x, y \in B_{2^n}(y, 2^{-k}) \subset 2^n B_{2^k}\), we have

\[
2^{-k-1} \leq g_\#(x,y) \leq C_{2^n} \max\{g_\#(x, x^*), g_\#(x^*, y)\} \\
< C_{2^n} \max\{2^{n-m}, 2^{n-m}\} = C_{2^n} 2^{n-m}.
\]

Hence, \(2^{-k-1} C_{2^n}^{-1} \leq 2^{n-m}\), that is, the first inequality in (3.39) holds true. This proves the above claim (3.39).

Moving on, we write

\[
|m_u(B_{2^k} \cap \Omega) - u(y)| \\
= |u(y) - m_u(B_{2^k} \cap \Omega)| \\
\leq |u(y) - m_u(B_{2^n}(y, 2^{-k}) \cap \Omega)| + |m_u(B_{2^n}(y, 2^{-k}) \cap \Omega) - m_u(2^n B_{2^k} \cap \Omega)| \\
+ |m_u(2^n B_{2^k} \cap \Omega) - m_u(B_{2^k} \cap \Omega)| \\
=: I + II + III.
\]

Since (3.8) holds true \(\mu\)-almost everywhere in \(\Omega\), without loss of generality, we may assume that (3.8) holds true for \(y\). Since \(k \in \mathbb{N}\) implies that \(2^{-i-k} < 1\) for any \(i \in \mathbb{N}_0\), appealing to (3.9) in Lemma 3.8 with \(2^{-i} = 2^{-i-k} < 1 = r_s\) and (3.37), we conclude that

\[
I \leq \sum_{i=0}^{\infty} \left| m_u(B_{2^n}(y, 2^{-(i+1)-k}) \cap \Omega) - m_u(B_{2^n}(y, 2^{-i-k}) \cap \Omega) \right| \\
\leq \sum_{i=0}^{\infty} 2^{-(i+k)\epsilon} \left[ \sup_{B_{2^n}(y, 2^{-(i+k)})} \left\{ \int_{j \geq k - 2k_0} \{2^{-j(s-k)\epsilon} g_j^t \} \right\} d\mu \right]^{1/t}
\]

Since (3.8) holds true \(\mu\)-almost everywhere in \(\Omega\), without loss of generality, we may assume that (3.8) holds true for \(y\). Since \(k \in \mathbb{N}\) implies that \(2^{-i-k} < 1\) for any \(i \in \mathbb{N}_0\), appealing to (3.9) in Lemma 3.8 (with \(2^{-i} = 2^{-i-k} < 1 = r_s\)) and (3.37), we conclude that

\[
I \leq \sum_{i=0}^{\infty} \left| m_u(B_{2^n}(y, 2^{-(i+1)-k}) \cap \Omega) - m_u(B_{2^n}(y, 2^{-i-k}) \cap \Omega) \right| \\
\leq \sum_{i=0}^{\infty} 2^{-(i+k)\epsilon} \left[ \sup_{B_{2^n}(y, 2^{-(i+k)})} \left\{ \int_{j \geq k - 2k_0} \{2^{-j(s-k)\epsilon} g_j^t \} \right\} d\mu \right]^{1/t}
\]
As concerns III, note that, when \( n = k_0 \), and \( 2^{n-m} < 2^{k_0-k} \leq 1 \). On the other hand, appealing once again to \((3.39)\), we have

\[
B_{\hat{g}}(y, 2^{-k}) \subset 2^{n} B_x \subset B_{\hat{g}}(y, \Lambda^2 C_{\hat{g}} 2^{-k+1}),
\]

which, in conjunction with the doubling and the measure density properties of \( \mu \), gives

\[
\mu(B_{\hat{g}}(y, 2^{-k}) \cap \Omega) \approx \mu(2^n B_x \cap \Omega).
\]

Putting these all together, we can then use \((3.9)\) in Lemma \(3.8\) (with \( 2^{-L} = 2^{n-m} < 1 = r_s \)) and \((3.37)\) to estimate

\[
\begin{align*}
\text{III} &= \left| m_n(B \cap \Omega) - m_n(2^n B_x \cap \Omega) \right| \\
&\leq \sum_{i=0}^{n-1} \left| m_n(2^i B_x \cap \Omega) - m_n(2^{i+1} B_x \cap \Omega) \right|.
\end{align*}
\]

For any \( i \in \{0, \ldots, n-1\} \), by \((3.39)\), we have \( 2^{i+1-m} \leq 2^{n-m} < 1 \), which, in conjunction with the doubling and the measure density properties of \( \mu \), gives \( \mu(2^i B_x \cap \Omega) \approx \mu(2^{i+1} B_x \cap \Omega) \). As such, we use \((3.9)\) in Lemma \(3.8\) (with \( 2^{-L} = 2^{i+1-m} < 1 = r_s \)) again to obtain

\[
\sum_{i=0}^{n-1} \left| m_n(2^i B_x \cap \Omega) - m_n(2^{i+1} B_x \cap \Omega) \right| \\
\leq \sum_{i=0}^{n-1} 2^{-(m-i-1)\epsilon} \left[ \int_{C_{\hat{g}} 2^{i+1} B_x} \sup_{j \geq m-i-1-k_0} \left| 2^{-j(s-\epsilon)} g_j' \right| d\mu \right]^{1/t}.
\]
Moreover, since \( B_x \subset B_x^r \subset 2C_{\tilde{\varrho}}^2 B_x \) [see (3.34)], we deduce that
\[
C_{\tilde{\varrho}} 2^{j+1} B_x \subset C_{\tilde{\varrho}} 2^{j+1} B_x^r \subset 2^{j+2} C_{\tilde{\varrho}}^3 B_x.
\]
Hence, by (3.37), the doubling property for \( \mu \), and the fact that \( m - n > k - k_0 \) and \( 2^{m-n} < 2^{k_0-k} \), we further estimate the last term in (3.43) as follows
\[
\sum_{i=0}^{n-1} 2^{-(m-i-1)e} \int_{C_{\tilde{\varrho}} 2^{j+1} B_x} \sup_{j \geq m - n - k_0} \left\{ 2^{-j(s-\varepsilon)} \right\}^{\frac{1}{j}} d\mu \leq \sum_{i=0}^{n-1} 2^{-(m-i-1)e} \left\{ \mathcal{M}_{\tilde{\varrho}} \left( \sup_{j \geq m - n - k_0} \left\{ 2^{-j(s-\varepsilon)} \right\} \right) \right\} \left( \frac{1}{2^{n-m}} \right) \left\{ \mathcal{M}_{\tilde{\varrho}} \left( \sup_{j \geq k - 2k_0} \left\{ 2^{j(s-\varepsilon)} \right\} \right) \right\} \left( \frac{1}{2^{n-m}} \right) \leq 2^{-ks} \left\{ \mathcal{M}_{\tilde{\varrho}} \left( \sup_{j \geq k - 2k_0} \left\{ 2^{j(s-\varepsilon)} \right\} \right) \right\} \left( \frac{1}{2^{n-m}} \right) \leq [\varrho(x,y)]^s h_k(x). \tag{3.44}
\]
Combining (3.42), (3.43), and (3.44), we conclude that
\[
\text{III} \leq [\varrho(x,y)]^s h_k(x) \quad \text{whenever } n \geq 1.
\]
If \( n \leq 0 \), then \( 2^n B_x^r \subset B_x^r \subset B_{\tilde{\varrho}}(y, \varrho_{\tilde{\varrho}}^2 \Lambda^{2-2k+1}) \), where the second inclusion follows from the fact that, for any \( z \in B_x^r \), one has (keeping in mind that \( y \in 2^n B_x^r \) and \( 2^{m-n} < C_{\tilde{\varrho}} \varrho_{\tilde{\varrho}}^2 \Lambda^{2-k+1} \))
\[
\varrho(y,z) \leq C_{\tilde{\varrho}} \max\{\varrho(y,x^r), \varrho(x^r,z)\} \leq C_{\tilde{\varrho}} \max\{2^{n-m}, 2^{-m}\} = C_{\tilde{\varrho}} 2^{-m} < C_{\tilde{\varrho}}^2 \varrho_{\tilde{\varrho}}^2 \Lambda^{2-k+1}.
\]
From this, (3.41), and the doubling and the measure density properties of \( \mu \), it follows that
\[
\mu(B_x^r \cap \Omega) \leq \mu(B_{\tilde{\varrho}}(y, \varrho_{\tilde{\varrho}}^2 \Lambda^{2-2k+1})) \leq \mu(B_{\tilde{\varrho}}(y, 2^{-k})) \leq \mu(2^n B_x) \leq \mu(2^n B_x^r \cap \Omega),
\]
where we have used the fact that \( 2^{n-m} < 1 \) [see (3.39)] when employing the measure density condition. Thus, we actually have \( \mu(2^n B_x^r \cap \Omega) \approx \mu(B_x^r \cap \Omega) \). Recall that \( 2^{-m} < 1 \) by (3.37). Thus, appealing again to (3.9) in Lemma 3.8 (with \( 2^{-k} = 2^{-m} < 1 = r_\varepsilon \)) and (3.37), as well as recycling some of the estimates in (3.45)-(3.48), we find that
\[
\text{III} = \left| m_{\mu}(2^n B_x^r \cap \Omega) - m_{\mu}(B_x^r \cap \Omega) \right| \leq 2^{-me} \int_{C_{\tilde{\varrho}} B_x^r} \sup_{j \geq m - n - k_0} \left\{ 2^{-j(s-\varepsilon)} \right\}^{1/j} d\mu \leq [\varrho(x,y)]^s h_k(x). \tag{3.45}
\]
Combining the estimates for I, II, and III, we have proved that the second term in the right-hand side of (3.31) can be bounded by \([q_u(x, y)]^t [h_k(x) + h_k(y)]\) multiplied a positive constant. This gives the desired estimate in Case 2.

**CASE 3:** Let \(x, y \in V \setminus \Omega\) with \(q_u(x, y) \geq \min\{\text{dist}_{y_u}(x, \Omega), \text{dist}_{y_v}(y, \Omega)\} \).

In this case, as in the beginning of the proof for Case 2, we start by choosing points \(x^* \in B_x \cap \Omega\) and \(y^* \in B_y \cap \Omega\), where \(B_x := B_{q_u}(x, \Lambda^2r(x))\) and \(B_y := B_{q_v}(y, \Lambda^2r(y))\), with \(r(x)\) and \(r(y)\) defined as in (3.30). Let \(m_x, m_y \in \mathbb{Z}\) satisfy \(2^{-m_x} - 1 \leq C_{q_u} \Lambda^2 r(x) < 2^{-m_x}\) and \(2^{-m_y} - 1 \leq C_{q_v} \Lambda^2 r(y) < 2^{-m_y}\), respectively, and set \(B_{x^*} := B_{q_u}(x^*, 2^{-m_x})\) and \(B_{y^*} := B_{q_v}(y^*, 2^{-m_y})\).

In order to establish (3.29) in this case, we write

\[
|\mathcal{F} u(x) - \mathcal{F} u(y)| \leq |\mathcal{F} u(x) - m_u(B_{x^*} \cap \Omega)| + |m_u(B_{x^*} \cap \Omega) - m_u(B_{y^*} \cap \Omega)| + |m_u(B_{y^*} \cap \Omega) - \mathcal{F} u(y)|,
\]

and estimate each term on the right-hand side of (3.46) separately. To this end, without loss of generality, we may assume that \(\text{dist}_{y_u}(x, \Omega) \leq \text{dist}_{y_v}(y, \Omega)\). Then, by the assumption of Case 3, one has

\[
r(x) := \frac{\text{dist}_{y_u}(x, \Omega)}{4C_{q_u}} \leq \frac{1}{4} q_u(x, y) < 2^{-k-2},
\]

where we have also used the fact that \(C_{q_u} \geq 1\). Moreover,

\[
r(y) := \frac{\text{dist}_{y_v}(y, \Omega)}{4C_{q_v}} \leq \frac{C_{q_v} \max\{q_u(x, y), \text{dist}_{y_v}(x, \Omega)\}}{4C_{q_v}} \leq \frac{1}{4} q_u(x, y) < 2^{-k-2}.
\]

In view of these, the first and the third terms in the right-hand side of (3.46) can be estimated, in a fashion similar to the arguments for (3.32–3.38), in order to obtain

\[
|\mathcal{F} u(x) - m_u(B_{x^*} \cap \Omega)| + |m_u(B_{x^*} \cap \Omega) - \mathcal{F} u(y)| \lesssim [q_u(x, y)]^t \left\{ \left[ \mathcal{M}_{y_u} \sup_{j \geq k-2k_0} \left\{ 2^{-jk(j+\delta)} g_j \right\} (x) \right]^{1/t} + \left[ \mathcal{M}_{y_v} \sup_{j \geq k-2k_0} \left\{ 2^{-jk(j+\delta)} g_j \right\} (y) \right]^{1/t} \right\} \lesssim [q_u(x, y)]^t [h_k(x) + h_k(y)],
\]

as wanted.

We still need to estimate the second term in the right-hand side of (3.46). To this end, Let \(n_x, n_y \in \mathbb{Z}\) be the least integers satisfying

\[
B_{q_u}(y, 2^{-k}) \subset 2^{n_x} B_{x^*} \quad \text{and} \quad 2^{n_y} B_{y^*} \subset 2^{n_x} B_{x^*},
\]

and write

\[
|m_u(B_{x^*} \cap \Omega) - m_u(B_{y^*} \cap \Omega)| \leq |m_u(B_{x^*} \cap \Omega) - m_u(2^{n_y} B_{x^*} \cap \Omega)| + |m_u(2^{n_x} B_{x^*} \cap \Omega) - m_u(2^{n_x} B_{y^*} \cap \Omega)|
\]
\[ + \left| m_u(2^n B_{y^*} \cap \Omega) - m_u(B_{y^*} \cap \Omega) \right| =: \Pi + \Pi'. \]  

(3.51)

In order to estimate both \( \Pi \) and \( \Pi' \), we first show that \( 2^{n_y - m_y} \approx 2^{-k} \approx 2^{n_i - m_i} \). Indeed, by the choice of \( n_y \), there exists a point \( z_0 \in X \) satisfying \( \varrho(y, z_0) < 2^{-k} \) and \( \varrho\tilde{y}(y^*, z_0) \geq 2^{n_y - m_y - 1} \), which, together with (3.48), allows us to estimate

\[ 2^{n_y - m_y - 1} \leq \varrho(y^*, z_0) \leq C_{\varrho} \max\{ \varrho(y^*, y), \varrho(y, z_0) \} \]

\[ < C_{\varrho} \max\{ \Lambda^2 r(y), 2^{-k} \} \]

\[ < C_{\varrho} \max\{ \Lambda^2 2^{-k-2}, 2^{-k} \} < C_{\varrho} \Lambda^2 2^{-k}. \]

Hence, \( 2^{n_y - m_y} < C_{\varrho} \Lambda^2 2^{-k+1} \), which further implies \( m_y - n_y > k - k_0 \), because \( C_{\varrho} \Lambda^2 2^{-k+1} < 2^{k_0-k} \) [see (3.59)]. On the other hand, since \( x, y \in B_{y^*}(y, 2^{-k}) \subset 2^n B_{y^*} \), one has

\[ 2^{-k-1} \leq \varrho(y, x) \leq C_{\varrho} \max\{ \varrho(y, x^*), \varrho(y^*, y) \} < C_{\varrho} 2^{n_i - m_i}, \]

which implies \( 2^{-k-1} C_{\varrho}^{-1} < 2^{n_i - m_i} \). Thus, by (3.59), we obtain

\[ 2^{-k-1} C_{\varrho}^{-1} < 2^{n_i - m_i} < C_{\varrho} \Lambda^2 2^{-k+1} < 1. \]  

(3.52)

Similarly, by the choice of \( n_x \), we know that there exists a point \( z_0 \in X \) satisfying \( \varrho(y^*, z_0) < 2^{n_y - m_y} \) and \( \varrho\tilde{y}(x, z_0) \geq 2^{n_x - m_x - 1} \). Making use of the second inequality in (3.52), as well as (3.47) and (3.48), we find that

\[ 2^{n_x - m_x - 1} \leq \varrho(x^*, z_0) \leq C_{\varrho}^3 \max\{ \varrho(x^*, x), \varrho(x, y), \varrho(y, y^*), \varrho(y^*, z_0) \} \]

\[ < C_{\varrho}^3 \max\{ \Lambda^2 r(x), 2^{-k}, \Lambda^2 r(y), 2^{n_y - m_y} \} \]

\[ < C_{\varrho}^3 \max\{ \Lambda^2 2^{-k-2}, 2^{-k}, \Lambda^2 2^{-k-2}, C_{\varrho} \Lambda^2 2^{-k+1} \} = C_{\varrho}^4 \Lambda^2 2^{-k+1}. \]

From this and the definition of \( k_0 \), we deduce that \( 2^{n_x - m_x} < C_{\varrho}^4 \Lambda^2 2^{-k+2} \leq 2^{k_0-k} \), which further implies that

\[ m_x - n_x > k - k_0. \]  

(3.53)

On the other hand, since \( x, y \in 2^n B_{y^*} \subset 2^n B_{y^*} \), we have

\[ 2^{-k-1} \leq \varrho(y, x) \leq C_{\varrho} \max\{ \varrho(y, x^*), \varrho(y^*, y) \} < C_{\varrho} 2^{n_i - m_i}. \]

Thus,

\[ 2^{-k-1} C_{\varrho}^{-1} < 2^{n_i - m_i} < C_{\varrho}^4 \Lambda^2 2^{-k+2} \leq 2^{k_0-k} \leq 1, \]  

(3.54)

and so \( 2^{n_i - m_i} \approx 2^{-k} \approx 2^{n_i - m_i} \), as desired.

At this stage, we can then argue as in (3.42)-(3.45) to obtain

\[ \Pi + \Pi' \leq [\varrho(y, x, y) + h_x(y)] \cdot [h_x(x) + h_y(y)]. \]  

(3.55)

So it remains to estimate \( \Pi' \). With the goal of using (3.39) in Lemma 3.8, we make a few observations. First, \( 2^n B_{y^*} \subset 2^n B_{y^*} \) by design and, from (3.54), we deduce that \( 2^{n_i - m_i} < 1 \). Moreover, if
\( z \in 2^{n_s} B_{x'} \), then, by (3.52) and the fact that \( 2^{n_s - m_s} < C_{\tilde{v}_y} \Lambda^2 2^{-k+2} \) and \( \varrho_\#(y^*, x^*) < C_{\tilde{v}_y} \Lambda^2 2^{-k} \), one has

\[
\varrho_\#(y^*, z) \leq C_{\tilde{v}_y} \max\{\varrho_\#(y^*, x^*), \varrho_\#(x^*, z)\} \\
< C_{\tilde{v}_y} \max\{C_{\tilde{v}_y}^2 \Lambda^2 2^{-k}, 2^{n_s - m_s}\} \\
< C_{\tilde{v}_y} \max\{2C_{\tilde{v}_y}^3 \Lambda^2 2^{n_s - m_s}, 8C_{\tilde{v}_y}^5 \Lambda^2 2^{n_s - m_s}\} < 8C_{\tilde{v}_y}^6 \Lambda^2 2^{n_s - m_s},
\]

which implies that \( 2^{n_s} B_{x'} \subset 8C_{\tilde{v}_y}^6 \Lambda^2 2^{n_s} B_{y^*} \). From this, \( 2^{n_s - m_s} < 1 \), and the doubling and the measure density conditions of \( \mu \), it follows that

\[
\mu(2^{n_s} B_{x'} \cap \Omega) \leq \mu(2^{n_s} B_{x'}) \leq \mu(8C_{\tilde{v}_y}^6 \Lambda^2 2^{n_s} B_{y^*}) \leq \mu(2^{n_s} B_{y^*}) \leq \mu(2^{n_s} B_{x'} \cap \Omega).
\]

Hence,

\[
\mu(2^{n_s} B_{x'} \cap \Omega) \approx \mu(2^{n_s} B_{x'} \cap \Omega) \approx \mu(B_{x'}),
\]

where in obtaining the last inequality we have used the measure density condition (3.17) again and the fact that \( 2^{n_s - m_s} < 1 \). Going further, observe that, if \( z \in C_{\tilde{v}_y} 2^{n_s} B_{x'} \), then, by \( y \in 2^{n_s} B_{x'} \) and (3.54), it holds true that

\[
\varrho_\#(y, z) \leq C_{\tilde{v}_y} \max\{\varrho_\#(y, x^*), \varrho_\#(x^*, z)\} < C_{\tilde{v}_y} \max\{2^{n_s - m_s}, C_{\tilde{v}_y} 2^{n_s - m_s}\} \\
= C_{\tilde{v}_y}^2 2^{n_s - m_s} < C_{\tilde{v}_y}^6 \Lambda^2 2^{-k+2},
\]

which implies

\[
C_{\tilde{v}_y} 2^{n_s} B_{x'} \subset B_{\tilde{v}_y}(y, C_{\tilde{v}_y}^6 \Lambda^2 2^{-k+2}).
\]

By this, (3.50), (3.54), (3.9) in Lemma 3.8 (with \( 2^{-L} = 2^{n_s - m_s} < 1 = r_\varepsilon \)), (3.53), and the doubling property for \( \mu \), we conclude that

\[
\Pi = |m_\mu(2^{n_s} B_{y^*} \cap \Omega) - m_\mu(2^{n_s} B_{x'} \cap \Omega)| \\
\leq 2^{-(m_s - n_s)k} \left| \int_{C_{\tilde{v}_y} 2^{n_s} B_{x'}} \sup_{j \geq m_s - n_s - k_0} \left[ 2^{-j(s-\varepsilon)} g_j^f \right] \frac{d\mu}{s_j} \right|^{1/t} \\
\leq 2^{-ke} \left| \int_{B_{\tilde{v}_y}(y, C_{\tilde{v}_y}^6 \Lambda^2 2^{-k+2})} \sup_{j \geq k-2k_0} \left[ 2^{-j(s-\varepsilon)} g_j^f \right] \frac{d\mu}{s_j} \right|^{1/t} \\
\leq 2^{-k_0} \left| \mathcal{M}_{\tilde{v}_y} \left[ \sup_{j \geq k-2k_0} \left[ 2^{(k-j)(s-\varepsilon)} g_j^f \right] \right](y) \right|^{1/t} \leq [\varrho_\#(y, y)]^t h_k(y).
\]

This, together with (3.55), (3.51), (3.49), and (3.46), then finishes the proof of the desired estimate (3.29) in Case 3.

**CASE 4:** \( x, y \in V \setminus \Omega \) with \( \varrho_\#(x, y) < \min\{\text{dist}_{\tilde{v}_y}(x, \Omega), \text{dist}_{\tilde{v}_y}(y, \Omega)\} \).
Again, in this case, without loss of generality, we may assume dist$_{\bar{\Omega}}(x, \Omega) \leq$ dist$_{\bar{\Omega}}(y, \Omega)$. Clearly, this assumption implies $r(x) \leq r(y)$, where $r(x)$ and $r(y)$ are defined as in (3.30). Let $\ell_0 \in \mathbb{Z}$ be the least integer satisfying $C_\varepsilon^2 \leq 2^{\ell_0}$ and note that $\ell_0$ is non-negative. Then

$$
\begin{align*}
\frac{r(y)}{4C_\varepsilon^2} &= \frac{\text{dist}_{\bar{\Omega}}(y, \Omega)}{4C_\varepsilon^2} \leq \frac{C_\varepsilon^2 \max\{\varrho\#(y, x, \Omega), \text{dist}_{\bar{\Omega}}(x, \Omega)\}}{4C_\varepsilon^2} \\
&= \frac{1}{4} \text{dist}_{\bar{\Omega}}(x, \Omega) = C_\varepsilon^2 r(x) \leq C_\varepsilon^2 r(x) \leq 2^{\ell_0} r(x),
\end{align*}
$$
and so $r(x) \approx r(y)$. Consequently, since $r_i \approx r(x)$ for any $i \in I_x$, and $r_i \approx r(y)$ for any $i \in I_y$ [see (3.21)], we have $r_i \approx r(x)$ for any $i \in I_x \cup I_y$; moreover, by (3.26), we have

$$
\sum_{i \in I_x \cup I_y} [\psi_i(x) - \psi_i(y)] = \sum_{i \in I_x \cup I_y} \psi_i(x) - \sum_{i \in I_x \cup I_y} \psi_i(y) = 1 - 1 = 0.
$$

From these and Theorem 3.2 (i) with $\beta = \alpha$, we deduce that

$$
|\mathcal{F} u(x) - \mathcal{F} u(y)| = \sum_{i \in I_x \cup I_y} [\psi_i(x) - \psi_i(y)] \cdot |m_u(B_i^* \cap \Omega) - m_u(2^0 B_{x^*} \cap \Omega)|
\leq \frac{[\varrho\#(x, y)]^\alpha}{|r(x)|^\alpha} \sum_{i \in I_x \cup I_y} |m_u(B_i^* \cap \Omega) - m_u(2^0 B_{x^*} \cap \Omega)|,
$$

(3.57)

where $B_x$ and $B_{x^*}$ are defined as in the proof for Case 3.

In order to bound the sum in (3.57), we will once again appeal to Lemma 3.3. With this idea in mind, fix an $i \in I_x \cup I_y$. Observe that, if $i \in I_x$, then, by (3.20), (3.34), and $\ell_0 \geq 0$, we have $B_i^* \subset B_x \subset C_{\bar{\Omega}} \subset 2^0 B_{x^*}$; if $i \in I_y$, then, with $B_y$ and $B_{x^*}$ maintaining their significance from Case 3, we have $B_i^* \subset B_y \subset 2^0 B_{x^*}$. Indeed, the first inclusion follows from the definition of $I_x$ and the analogous version of (3.20) for $B_y$. To see the second inclusion, observe that, if $z \in B_y$, then we can use (3.56) to obtain

$$
\varrho\#(x^*, z) \leq C_{\varepsilon x}^2 \max\{\varrho\#(x^*, x), \varrho\#(x, y), \varrho\#(y, z)\}
\leq \varrho\#(x^*, x) \leq \varrho\#(x, y) \leq \varrho\#(y, z),
$$

where the second equality follows from the estimate $\lambda^2 > \theta^2 = 4C_{\varepsilon y}^4 \geq 4C_{\varepsilon x}^4$, and the last inequality is obtained from the choices of $\ell_0$ and $m_x$. Hence, $B_y \subset 2^0 B_{x^*}$. Altogether, we proved that $B_i^* \subset 2^0 B_{x^*}$ whenever $i \in I_x \cup I_y$.

Next, we show that $\mu(B_i^* \cap \Omega) \approx \mu(2^0 B_{x^*} \cap \Omega)$ for any $i \in I_x \cup I_y$. Fix $i \in I_x \cup I_y$. It follows from what we have just shown that $\mu(B_i^* \cap \Omega) \leq \mu(2^0 B_{x^*} \cap \Omega)$. To see $\mu(2^0 B_{x^*} \cap \Omega) \leq \mu(B_i^* \cap \Omega)$, observe that, if $i \in I_x$, then the inclusions $B_{x^*} \subset 2^0 B_x$ in (3.34) and $B_x \subset C_{\bar{\Omega}}^2 \Lambda^3 B_i^*$ in (3.20), in conjunction with the doubling and the measure density properties for $\mu$, give

$$
\mu(2^0 B_{x^*} \cap \Omega) \leq \mu(2^0 B_{x^*}) \leq \mu(B_x) \leq \mu(C_{\bar{\Omega}}^2 \Lambda^3 B_i^*) \leq \mu(B_i^*) \leq \mu(B_i^* \cap \Omega).
$$

On the other hand, if $i \in I_y$, then similar to the proof of (3.20), we find that $B_x \subset C_{\bar{\Omega}}^2 \Lambda^3 B_i^*$. Moreover, we have $B_{x^*} \subset 2^0 B_y$ since, for any $z \in B_{x^*}$, there holds

$$
\varrho\#(y, z) \leq C_{\varepsilon y}^2 \max\{\varrho\#(y, x), \varrho\#(x, x^*), \varrho\#(x^*, z)\}
$$

and

$$
\varrho\#(x^*, z) \leq C_{\varepsilon x}^2 \max\{\varrho\#(x^*, x), \varrho\#(x, y), \varrho\#(y, z)\}
\leq \varrho\#(x^*, x) \leq \varrho\#(x, y) \leq \varrho\#(y, z),
$$

(3.57)
\[
< C^2 \max \{ \text{dist}_{\mathcal{U}_4}(x, \Omega), \Lambda^2 r(x), 2^{-m_x} \} \\
< C^2 \max \{ 4C_{\mathcal{U}_4} r(x), \Lambda^2 r(x), 2C_{\mathcal{U}_4} \Lambda^2 r(x) \} = 2C_{\mathcal{U}_4} \Lambda^2 r(x) \leq 2C_{\mathcal{U}_4} \Lambda^2 r(y).
\]

These inclusions, together with the measure density and doubling conditions imply that
\[
\mu(2^{\ell_0} B_{x'} \cap \Omega) \leq \mu(2^{\ell_0} B_{x'}) \leq \mu(B_y) \leq \mu(C_{\mathcal{U}_4} \Lambda^2 B^*_y) \leq \mu(B^*_y) \leq \mu(B^*_y \cap \Omega).
\]

This finishes the proof of the fact that \( \mu(B^*_y \cap \Omega) \approx \mu(2^{\ell_0} B_{x'} \cap \Omega) \).

To invoke Lemma 3.8 we finally need to uniformly bound the radius of the ball \( 2^{\ell_0} B_{x'} \), namely, \( 2^{-(m_x-\ell_0)} \). Indeed, the choices of both \( \ell_0 \) and \( m_x \), the definition of \( r(x) \) [see (3.47)], and the membership of \( x \in V \) [see (3.25)] imply that
\[
2^{-(m_x-\ell_0)} \leq 4C_{\mathcal{U}_4} \Lambda^2 r(x) = C_{\mathcal{U}_4} \Lambda^2 \text{dist}_{\mathcal{U}_4}(x, \Omega) \leq 2C_{\mathcal{U}_4} \Lambda^2 < 2^{\ell_0}.
\]

From this and (3.9) in Lemma 3.8 with \( 2^{-L} = 2^{-(m_x-\ell_0)} < 2^{\ell_0} = r_x \), we deduce for any \( i \in I_x \cup I_y \) that (keeping in mind \( B_x \subset C_{\mathcal{U}_4} 2^\ell B_{x'} \subset 2^\ell + 1 C_{\mathcal{U}_4} B_{x} \))
\[
|m_u(B^*_y \cap \Omega) - m_u(2^{\ell_0} B_{x'} \cap \Omega)|
\]
\[
\leq 2^{-(m_x-\ell_0)e} \left[ \int_{C_{\mathcal{U}_4} 2^{\ell_0} B_{x'}} \sup_{j \geq m_x - \ell_0 - k_0} \left\{ 2^{-j(s+\delta)} \mathcal{L}_j^x \right\} \, d\mu \right]^{1/2}
\]
\[
\leq 2^{-m_x e} \left[ \int_{2^{\ell_0} C_{\mathcal{U}_4} B_{x}} \sup_{j \geq m_x - \ell_0 - k_0} \left\{ 2^{-j(s+\delta)} \mathcal{L}_j^x \right\} \, d\mu \right]^{1/2}
\]
\[
\leq 2^{-m_x e} \left[ \mathcal{M}_{\mathcal{U}_4} \left( \sup_{j \geq m_x - \ell_0 - k_0} \left\{ 2^{-j(s+\delta)} \mathcal{L}_j^x \right\} \right) (x) \right]^{1/2},
\]

which, together with (3.57) and the fact that \( \#(I_x \cup I_y) \leq 2M \) (see Theorem 3.1), further implies that
\[
|\mathcal{F} u(x) - \mathcal{F} u(y)| \leq \frac{[\mathcal{Q}_\#(x, y)]^\alpha \sum_{i \in I_x \cup I_y} |m_u(B^*_y \cap \Omega) - m_u(2^{\ell_0} B_{x'} \cap \Omega)|}{[r(x)]^\alpha}
\]
\[
\leq [\mathcal{Q}(x, y)]^\alpha [r(x)]^{-\alpha} |2^{-m_x e} \left[ \mathcal{M}_{\mathcal{U}_4} \left( \sup_{j \geq m_x - \ell_0 - k_0} \left\{ 2^{-j(s+\delta)} \mathcal{L}_j^x \right\} \right) (x) \right]^{1/2}|. \tag{3.58}
\]

Since \( 2^{-m_x} \leq 2C_{\mathcal{U}_4} \Lambda^2 r(x) \), \( \mathcal{Q}(x, y) < \min(2^{-k}, 4C_{\mathcal{U}_4} r(x)) \), by the ranges of \( s \) and \( \alpha \), and \( \delta \geq 0 \), we conclude that
\[
[\mathcal{Q}(x, y)]^\alpha [r(x)]^{-\alpha} |2^{-m_x e} \left[ \mathcal{M}_{\mathcal{U}_4} \left( \sup_{j \geq m_x - \ell_0 - k_0} \left\{ 2^{-j(s+\delta)} \mathcal{L}_j^x \right\} \right) (x) \right]^{1/2}|
\]
\[
\leq [\mathcal{Q}(x, y)]^\alpha [r(x)]^{s+\delta-\alpha} [r(x)]^{-s-\delta} |2^{-m_x e} \left[ \mathcal{M}_{\mathcal{U}_4} \left( \sup_{j \geq m_x - \ell_0 - k_0} \left\{ 2^{-j(s+\delta)} \mathcal{L}_j^x \right\} \right) (x) \right]^{1/2}|
\]
\[
\leq [\mathcal{Q}(x, y)]^{s+\delta} 2^{m_x (s-\delta)} \leq [\mathcal{Q}(x, y)]^{s+\delta} 2^{(m_x-k)\delta + m_x (s-\delta)},
\]
which, together with (3.58), allows us to write

$$|\mathcal{F} u(x) - \mathcal{F} u(y)| \leq (\mathcal{D}^k(x,y))^t \left[ M_{\mathcal{D}^k} \left( \sup_{j \geq 0, t \geq 0} \left\{ 2^{j(k-\delta)t + (m_j - k)^{1} g_j^t} \right\} \right)(x) \right]^{1/t}. \tag{3.59}$$

To bound the supremum in (3.59), observe that, by the choice of $m_x$, the definition of $r(x)$, the fact that $\Lambda > \theta = 2C^2_{\mathcal{D}^k}$ and the assumption of Case 4, one has

$$2^{-m_x} > C_{\mathcal{D}^k} \Lambda^2 r(x) > C^4_{\mathcal{D}^k} \text{dist}_{\mathcal{D}^k}(x, \Omega) > \mathcal{D}^k(x,y) \geq 2^{-k-1}.$$ Hence, $m_x < k + 1$. As $\ell_0$ and $k_0$ are non-negative, we further have $m_x - \ell_0 - k_0 < m_x \leq k$. Using this and the fact that $m_x \leq j + \ell_0 + k_0$, we can bound the supremum in (3.59) as

$$\left[ M_{\mathcal{D}^k} \left( \sup_{j \geq 0, t \geq 0} \left\{ 2^{j(k-\delta)t + (m_j - k)^{1} g_j^t} \right\} \right)(x) \right]^{1/t} \leq \left[ M_{\mathcal{D}^k} \left( \sup_{j \geq -k-1} \left\{ 2^{(j-k)^{1} g_j^t} \right\} \right)(x) \right]^{1/t} + \left[ M_{\mathcal{D}^k} \left( \sup_{j \geq k} \left\{ 2^{(k-j)^{1} g_j^t} \right\} \right)(x) \right]^{1/t}.$$ For the first supremum in the last line above, since $j < k$, we easily have

$$\left[ M_{\mathcal{D}^k} \left( \sup_{j \geq -k-1} \left\{ 2^{(j-k)^{1} g_j^t} \right\} \right)(x) \right]^{1/t} \leq h_k(x),$$

while, for the second one, we use the fact that $k - j \leq 0$ and $\delta < s - \varepsilon$ to obtain

$$\left[ M_{\mathcal{D}^k} \left( \sup_{j \geq k} \left\{ 2^{(k-j)^{1} g_j^t} \right\} \right)(x) \right]^{1/t} \leq \left[ M_{\mathcal{D}^k} \left( \sup_{j \geq k} \left\{ 2^{(k-j)^{1} g_j^t} \right\} \right)(x) \right]^{1/t} \leq h_k(x).$$

From these estimates and (3.59), we deduce the desired estimate (3.29) in Case 4.

Combining all obtained results in Cases 1-4, we complete the proof of Lemma 3.17. \hfill \Box

Recall that $\tilde{h} := \{h_k\}_{k \in \mathbb{Z}}$ [originally defined in (3.28)] has the following formula: for any $x \in X$,

$$h_k(x) := \begin{cases} M_{\mathcal{D}^k} \left( \sup_{j \in \mathbb{Z}} \left\{ 2^{(k-j)^{1} g_j^t} \right\} \right)(x) \left( \mathcal{F} u(x) \right)^{1/t} & \text{if } k \geq k_0, \tag{3.60} \\ 2^{(k+1)^{1} \mathcal{F} u(x)} & \text{if } k < k_0. \end{cases}$$

We then have the following result.
Lemma 3.19. Let all the notation be as in the above proof of Theorem 3.17. The sequence $\vec{h}$ defined in (3.60) satisfies
\[
\left\| \vec{h} \right\|_{L^p(\mathbb{R}^d)} \lesssim \left\| \vec{g} \right\|_{L^p(\Omega, \mathbb{C}^d)} + \left\| u \right\|_{L^p(\Omega, \mathbb{C}^d)},
\]
where the implicit positive constant is independent of $u$, $\vec{g}$, and $\vec{h}$.

Proof. Suppose first that $q < \infty$ and recall that $\delta > 0$, by design, in this case. Dominating the supremum by the corresponding summation in the definition of $h_k$ in (3.60) and using the sublinearity of $\mathcal{M}_{\psi_n}$, we find that
\[
\left[ \mathcal{M}_{\psi_n} \left( \sup_{j \in \mathbb{Z}} 2^{-|k-j|\delta t} \| g'_j \| \right) \right]^{1/t} \leq \left[ \sum_{j \in \mathbb{Z}} 2^{-|k-j|\delta t} \mathcal{M}_{\psi_n} \left( g'_j \right) \right]^{1/t},
\]
(3.61)
Using this and $s > 0$, and applying Lemma 3.11 with $a = 2^{\delta t}$, $b = q/t$, and $c_j = \mathcal{M}_{\psi_n}(g'_j)$, one has
\[
\left( \sum_{k \in \mathbb{Z}} |h_k|^q \right)^{1/q} = \left( \sum_{k = k_0}^{k_{1}} \left[ \sum_{j \in \mathbb{Z}} \left\{ 2^{-|k-j|\delta t} \mathcal{M}_{\psi_n} \left( g'_j \right) \right\}^{q/t} + |\mathcal{F} u|^{q/t} \sum_{k = -\infty}^{k_0-1} 2^{(k+1)q} \right]^{1/q} \right) \lesssim \left\{ \sum_{j \in \mathbb{Z}} \left[ \mathcal{M}_{\psi_n} \left( g'_j \right) \right]^{q/t} + 2^{k_{j_{0}} s} |\mathcal{F} u|^{q} \right\}^{1/q} \lesssim \left\{ \sum_{j \in \mathbb{Z}} \left[ \mathcal{M}_{\psi_n} \left( g'_j \right) \right]^{q/t} \right\}^{1/q} + |\mathcal{F} u|,
\]
which, together with the Fefferman–Stein inequality (see [16], p. 4, Theorem 1.2)) as well as Lemma 3.16 further implies that
\[
\left\| \vec{h} \right\|_{L^p(\mathbb{R}^d)} = \left\{ \sum_{k \in \mathbb{Z}} |h_k|^q \right\}^{1/q} \lesssim \left\{ \sum_{j \in \mathbb{Z}} \left[ \mathcal{M}_{\psi_n} \left( g'_j \right) \right]^{q/t} \right\}^{1/q} \leq \left\| \mathcal{M}_{\psi_n} \left( g'_j \right) \right\|_{L^p(V, \mu)}^{1/q} + \left\| \mathcal{F} u \right\|_{L^p(V, \mu)}^{1/q} \lesssim \left\{ \sum_{j \in \mathbb{Z}} \left[ \mathcal{M}_{\psi_n} \left( g'_j \right) \right]^{q/t} \right\}^{1/q} + \left\| u \right\|_{L^p(\Omega, \mathbb{C}^d)} \lesssim \left\| \vec{g} \right\|_{L^p(\Omega, \mathbb{C}^d)} + \left\| u \right\|_{L^p(\Omega, \mathbb{C}^d)} + \left\| u \right\|_{L^p(\Omega, \mathbb{C}^d)},
\]
(3.62)
where we have used the fact that for any $j \in \mathbb{Z}$, $g_j \equiv 0$ on $X \setminus \Omega$ in obtaining the last inequality of (3.62). Also, note that we have used the assumption $\min\{q/t, p/t\} > 1$ in applying the Fefferman–Stein inequality.

Suppose next that $q = \infty$. Then $\delta \geq 0$ and we use the fact that $2^{-|k-j|\delta t} \leq 1$ to estimate
\[
\sup_{k \in \mathbb{Z}} |h_k| \leq \left[ \mathcal{M}_{\psi_n} \left( \sup_{j \in \mathbb{Z}} g'_j \right) \right]^{1/t} + 2^{k_{j_{0}} s} |\mathcal{F} u|.
\]
Note that, by the definition of \( M^s_{p,q}(X,\mu) \), we have \( \sup_{j \in \mathbb{Z}} g'_j \in L^{p/t}(X,\mu) \), where \( p/t > 1 \). From these observations, Lemma 3.16 and the boundedness of \( M^s_{p,q} \) on \( L^{p/t}(X,\mu) \), it follows that

\[
\| \tilde{h} \|_{L^p(V;\ell^{s})} = \left\| \sup_{k \in \mathbb{Z}} |h_k| \right\|_{L^p(V;\mu)} \lesssim M^s_{p,q} \left( \sup_{j \in \mathbb{Z}} g'_j \right)_{L^{p/t}(X,\mu)}^{1/t} + \| \mathcal{F} u \|_{L^p(V;\mu)}
\]

\[
\lesssim \left\| \sup_{j \in \mathbb{Z}} g'_j \right\|_{L^{p/t}(X,\mu)}^{1/t} + \| \mathcal{F} u \|_{L^p(V;\mu)}
\]

\[
\lesssim \left\| \sup_{j \in \mathbb{Z}} g_j \right\|_{L^p(X;\mu)} + \| u \|_{L^p(\Omega;\mu)} \lesssim \| \tilde{g} \|_{L^p(\Omega;\ell^{s})} + \| u \|_{L^p(\Omega;\mu)}.
\]

This finishes the proof of Lemma 3.19. \( \square \)

Combining Lemmas 3.16, 3.17 and 3.19 we conclude that

\[
\mathcal{F} u \in M^s_{p,q}(V) \quad \text{and} \quad \| \mathcal{F} u \|_{M^s_{p,q}(V)} \lesssim \| u \|_{M^s_{p,q}(\Omega)},
\]  

(3.63)

that is, \( \mathcal{F} \) serves as a bounded extension operator from \( \Omega \) to \( V \) for functions in Hajłasz–Triebel–Lizorkin spaces.

We now define the final extension of \( u \) to the entire space \( X \). Let \( \Psi : X \to [0, 1] \) be any Hölder continuous function of order \( \alpha \) on \( X \) such that \( \Psi|_{\Omega} \equiv 1 \) and \( \Psi|_{X \setminus \Omega} \equiv 0 \), where here and thereafter, for any set \( E \subset X \), the symbol \( \Psi|_E \) means the restriction of \( \Psi \) on \( E \). The existence of such a function having this order with these properties is guaranteed by [3, Theorem 4.1] (see also [2, Theorem 2.6]). Define \( \tilde{u} : X \to \mathbb{R} \) by setting

\[
\tilde{u} := \Psi \mathcal{F} u.
\]

Then \( \tilde{u}|_{\bar{\Omega}} = u \) and it follows from Lemma 3.10 and (3.63) that

\[
\| \tilde{u} \|_{M^s_{p,q}(X)} \lesssim \| \mathcal{F} u \|_{M^s_{p,q}(V)} \lesssim \| u \|_{M^s_{p,q}(\Omega)}.
\]

This finishes the proof of (3.18) in the statement of Theorem 3.13 and, in turn, the proof of this theorem for \( M^s_{p,q} \) spaces.

Finally, let us turn our attention to extending Hajłasz–Besov functions when \( s < \text{ind}(X,\rho) \). To construct an \( N^s_{p,q} \) extension operator, we will proceed just as we did in the Triebel–Lizorkin case with a few key differences. More specifically, since \( s < \text{ind}(X,\rho) \), we can choose a number \( \alpha \in \mathbb{R} \) and a quasi-metric \( g \) on \( X \) such that \( \rho \approx g \) and \( s < \alpha \leq (\log_2 C_g)^{-1} \). For any \( u \in N^s_{p,q}(\Omega,\mathcal{G},\mu) \), we define the local extension \( \mathcal{F} u \) as in (3.27). Then Lemma 3.17 still gives an \( s \)-fractional gradient of \( \mathcal{F} u \) that is given (up to a multiplicative constant) by (3.28), where \( \tilde{g} \in D^s_{\mathcal{G}}(u) \) is such that \( \| \tilde{g} \|_{\mathcal{G}}^{(s)}(\Omega) \lesssim \| u \|_{N^s_{p,q}(\Omega)} \). By Proposition 2.2 such a choice of \( \tilde{g} \) is possible. Moreover, we have that \( \delta > 0 \) by its definition, because \( \alpha \neq s \). In place of the norm estimate in Lemma 3.19 we use the following lemma.

**Lemma 3.20.** The sequence \( \tilde{h} := \{h_k\}_{k \in \mathbb{Z}} \) defined in (3.28) satisfies

\[
\| \tilde{h} \|_{\mathcal{G}(L^p(V))} \lesssim \| \tilde{g} \|_{\mathcal{G}(L^p(\Omega))} + \| u \|_{L^p(\Omega;\mu)},
\]

where the implicit positive constant is independent of \( u, \tilde{g}, \) and \( \tilde{h} \).
Proof. By the definition of $N^s_{p,q}(X,\mu)$, we have $g'_j \in L^{p/t}(X,\mu)$ for any $j \in \mathbb{Z}$. Since $p/t > 1$, using (3.61), the Minkowski inequality, and the boundedness of $\mathcal{M}_{\varphi_n}$ on $L^{p/t}(X,\mu)$, one has, for any $k \in \mathbb{Z}$,

$$
\left\| \mathcal{M}_{\varphi_n} \left( \sup_{j \in \mathbb{Z}} \left( 2^{-|k-j|\delta t} g'_j \right) \right) \right\|_{L^p(X,\mu)}^{1/t} \leq \left\| \sum_{j \in \mathbb{Z}} 2^{-|k-j|\delta t} \mathcal{M}_{\varphi_n} \left( g'_j \right) \right\|_{L^p(X,\mu)}^{1/t} \leq \left[ \sum_{j \in \mathbb{Z}} 2^{-|k-j|\delta t} \left\| \mathcal{M}_{\varphi_n} \left( g'_j \right) \right\|_{L^p(X,\mu)} \right]^{1/t} \leq \left[ \sum_{j \in \mathbb{Z}} 2^{-|k-j|\delta t} \left\| g_j \right\|_{L^p(X,\mu)}^{p/t} \right]^{1/t} \cdot (3.64)
$$

Therefore, if $q < \infty$, then Lemma 3.11 (used here with $a = 2^{\delta t}$, $b = q/t$, and $c_j = \|g_j\|_{L^p(X,\mu)}^p$) and Lemma 3.16 give

$$
\left\| \tilde{g} \right\|_{L^q(L^p(V))} = \left( \sum_{k \geq k_0} \left\| \mathcal{M}_{\varphi_n} \left( \sup_{j \in \mathbb{Z}} \left( 2^{-|k-j|\delta t} g'_j \right) \right) \right\|_{L^p(X,\mu)}^q + \left\| \mathcal{F} u \right\|_{L^p(V,\mu)}^q \right)^{1/q} \leq \left( \sum_{k \geq k_0} \left\| g_j \right\|_{L^p(X,\mu)}^q + \left\| \mathcal{F} u \right\|_{L^p(V,\mu)}^q 2^{k_0 q} \right)^{1/q} \leq \left\| \tilde{g} \right\|_{L^q(L^p(\Omega))} + \left\| u \right\|_{L^q(L^p(\Omega))},
$$

where we have used the fact that, for any $j \in \mathbb{Z}$, $g_j \equiv 0$ on $X \setminus \Omega$. If $q = \infty$, then, by (3.64), we find that (keeping in mind that $\delta > 0$)

$$
\left\| \tilde{g} \right\|_{L^\infty(L^p(V))} \leq \sup_{k \geq k_0} \left\| \mathcal{M}_{\varphi_n} \left( \sup_{j \in \mathbb{Z}} \left( 2^{-|k-j|\delta t} g'_j \right) \right) \right\|_{L^p(X,\mu)}^{1/t} + \left\| \mathcal{F} u \right\|_{L^p(V,\mu)} \sup_{k \leq k_0} 2^{(k+1)s} \leq \left\| \tilde{g} \right\|_{L^\infty(L^p(\Omega))} + \left\| u \right\|_{L^\infty(L^p(\Omega))}.
$$

This finishes the proof of Lemma 3.20.

The remainder of the proof of Theorem 3.13 for $N^s_{p,q}$ spaces, including the definition for the final extension of $\mu$ to the entire space $X$, is carried out as it was in the Triebel–Lizorkin case. This finishes the proof of Theorem 3.13.
4 Equivalency of The Extension and Embedding Properties of The Domain and The Measure Density Condition

In this section, we fully characterize the $M^s_{p,q}$ and the $N^s_{p,q}$ extension domains which are locally uniformly perfect via the so-called measure density condition in general spaces of homogeneous type, for an optimal range of $s$; see Theorem 4.6 below.

4.1 Main Tools

Before formulating the main theorems in this section, we first collect a few technical results that were established in [1] and [5]. Given the important role that these results play in the proofs of the main theorems in this article, we include their statements for the convenience of the reader.

We begin with a few definitions. Throughout this section, let $(X, \rho, \mu)$ be a quasi-metric measure space and suppose that $\Omega \subset X$ is a $\mu$-measurable set with at least two points and has the property that $\mu(B(x, r) \cap \Omega) > 0$ for any $x \in \Omega$ and $r \in (0, 1]$. Then $(\Omega, \rho, \mu)$ can be naturally viewed as a quasi-metric measure space, where the quasi-metric $\rho$ and the measure $\mu$ are restricted to the set $\Omega$. In this context, for any $x \in X$ and $r \in (0, \infty)$, we let

$$\varphi^x_{\Omega, \rho}(r) := \sup \{ s \in [0, r] : \mu(B_p(x, s) \cap \Omega) \leq \frac{1}{2} \mu(B_p(x, r) \cap \Omega) \}. \quad (4.1)$$

Note that, when $s = 0$, $B_p(x, s) \cap \Omega = \emptyset$ and so $\varphi^x_{\Omega, \rho}(r) \geq 0$. When $\Omega = X$, we abbreviate $\varphi^x_{\rho} := \varphi^x_{X, \rho}$.

In the Euclidean setting $(\mathbb{R}^n, |·|, L^n)$, where $L^n$ denotes the $n$-dimensional Lebesgue measure, for any $x \in \Omega$ and $r \in (0, \infty)$, it is always possible to find a radius $\tilde{r} < r$ with the property that $L^n(B(x, \tilde{r}) \cap \Omega) = \frac{1}{2} L^n(B(x, r) \cap \Omega)$, whenever $\Omega \subset \mathbb{R}^n$ is an open connected set. In a general quasi-metric measure space there is no guarantee that there exists such a radius $\tilde{r}$; however, $\varphi^x_{\Omega, \rho}(r)$ will be a suitable replacement of $\tilde{r}$ in this more general setting.

In the context above, the quasi-metric space $(\Omega, \rho)$ (or simply the set $\Omega$) is said to be \textit{locally uniformly perfect} if there exists a constant $\lambda \in (0, 1)$ with the property that, for any $x \in \Omega$ and $r \in (0, 1]$,

$$(B_p(x, r) \cap \Omega) \setminus B_p(x, \lambda r) \neq \emptyset \quad \text{whenever} \quad \Omega \setminus B_p(x, r) \neq \emptyset. \quad (4.2)$$

If (4.2) holds true for any $x \in \Omega$ and $r \in (0, \infty)$, then $\Omega$ is simply referred as \textit{uniformly perfect}. It is easy to see that every connected quasi-metric space is uniformly perfect; however, there are very disconnected Cantor-type sets that are also uniformly perfect. Moreover, every uniformly perfect set is locally uniformly perfect, but the set $\Omega := \bigcup_{n=0}^{\infty} (2^{2^n}, 2^{2^n} + 1)$, when equipped with the restriction of the standard Euclidean distance on $\mathbb{R}$, is a locally uniformly perfect set which is not uniformly perfect (see [15] for additional details). Finally, observe that it follows immediately from (4.2) that, whenever (4.2) holds true for some $\lambda \in (0, 1)$, then it holds true for any $\lambda' \in (0, \lambda]$.

The first main result of this subsection is a technical lemma which is needed in proving that $M^s_{p,q}$ and $N^s_{p,q}$ extension domains necessarily satisfy the measure density condition.

\textbf{Lemma 4.1.} Let $(X, \rho, \mu)$ be a quasi-metric measure space, where $\mu$ is a doubling measure on $X$, and suppose that $\Omega \subset X$ is a $\mu$-measurable locally uniformly perfect set. Let $\lambda \in (0, C_{p,\lambda}(C_{p,\lambda})^{-2})$

\textsuperscript{5}The upper bound 1 on the radius can be arbitrarily replaced by any other threshold $\delta \in (0, \infty)$.
\textsuperscript{6}The upper bound 1 can be replaced by any positive and finite number.
be as in (4.2), where \( \rho_\Omega \) denotes the restriction of \( \rho \) to \( \Omega \), and assume that there exists a positive constant \( C \) such that
\[
\mu(B_\rho(x, r)) \leq C \mu(B_\rho(x, r) \cap \Omega),
\]
whenever both \( x \in \Omega \) and \( r \in (0, 1] \) satisfy \( r \leq C_{\rho, \Omega} \varphi^\lambda_\rho(r)/\lambda^2 \). Then there exists a positive constant \( \bar{C} \), depending only on \( C, \rho, \lambda \), and the doubling constant for \( \mu \), such that
\[
\mu(B_\rho(x, r)) \leq \bar{C} \mu(B_\rho(x, r) \cap \Omega)
\]
holds true for any \( x \in X \) and \( r \in (0, 1] \).

We will employ the following result from [5] Lemma 4.2 in the proof of Lemma 4.1. Strictly speaking, the result in [5] Lemma 4.2 was proven for uniformly perfect spaces; however, the same proof is valid for the class of locally uniformly perfect spaces.

**Lemma 4.2.** Let \((X, \rho, \mu)\) be a locally uniformly perfect quasi-metric measure space and suppose that \( \lambda \in (0, (C_p \tilde{C}_\rho)^{-3}) \) is the constant appearing in (4.2), where \( C_p, \tilde{C}_\rho \in [1, \infty) \) are as in (2.2) and (2.3), respectively. If \( x \in X \) and \( r \in (0, \text{diam}_p(X)] \) satisfy \( C_p \varphi^\lambda_\rho(r)/\lambda^2 < r \leq 1 \), then there exist a point \( \tilde{x} \in X \) and a radius \( \tilde{r} \in (0, 1] \) such that \( \lambda r \leq \tilde{r} \leq \min\{r, C_p \varphi^\lambda_\rho(\tilde{r})/\lambda^2\} \) and \( B_\rho(\tilde{x}, \tilde{r}) \subset B_\rho(x, r) \).

**Proof of Lemma 4.2.** We first remark that, by the definition of a locally uniformly perfect set, we conclude that \((\Omega, \rho_\Omega, \mu_\Omega)\) is a locally uniformly perfect quasi-metric-measure space, where the quasi-metric \( \rho \) and the measure \( \mu \) are restricted to the set \( \Omega \). As such, we can invoke Lemma 4.2 for the quasi-metric measure space \((\Omega, \rho_\Omega, \mu_\Omega)\).

With this in mind, fix a point \( x \in \Omega \) and a radius \( r \in (0, 1] \). If \( r \leq C_{\rho, \Omega, \mu} (\varphi^\lambda_\rho(r)/\lambda^2) \), then we immediately have \( \mu(B_\rho(x, r) \cap \Omega) \geq \mu(B_\rho(x, r)) \) by the assumption of Lemma 4.1. Thus, it suffices to assume that \( r > C_{\rho, \Omega, \mu} (\varphi^\lambda_\rho(r)/\lambda^2) \). Observe that (4.3) implies that \( C \geq 1 \). Therefore, if \( r > \text{diam}_p(\Omega) \), then \( B_\rho(x, r) = \Omega \) and (4.4) holds true trivially. Thus, we may further assume that \( r \leq \text{diam}_p(\Omega) \). By Lemma 4.2, there exist a point \( \tilde{x} \in \Omega \) and a radius \( \tilde{r} \in (0, 1] \) such that \( \lambda r \leq \tilde{r} \leq \min\{r, C_p \varphi^\lambda_\rho(\tilde{r})/\lambda^2\} \) and \( B_\rho(\tilde{x}, \tilde{r}) \cap \Omega \subset B_\rho(x, r) \cap \Omega \). Since \( \tilde{x} \in B_\rho(x, r) \), we infer that \( B_\rho(x, r) \subset B_\rho(\tilde{x}, C_p \tilde{C}_\rho r) \), which, together with (4.3) and the doubling property for \( \mu \), implies that
\[
C \mu(B_\rho(x, r) \cap \Omega) \geq C \mu(B_\rho(\tilde{x}, \tilde{r}) \cap \Omega) \geq \mu(B_\rho(\tilde{x}, \tilde{r})) \\
\geq \mu(B_\rho(\tilde{x}, \lambda r)) \geq \mu(B_\rho(\tilde{x}, C_p \tilde{C}_\rho r)) \geq \mu(B_\rho(x, r)).
\]
This finishes the proof of Lemma 4.1 \( \square \)

The following result was recently established in [5] Lemma 4.7] and it concerns the ability to construct certain families of maximally smooth Hölder continuous ‘bump’ functions belonging to \( M^s_{p, q} \) and \( N^s_{p, q} \).

**Lemma 4.3.** Suppose \((X, \rho, \mu)\) is a quasi-metric measure space and \( \varphi \) any quasi-metric on \( X \) such that \( \varphi \approx \rho \). Let \( C_\varphi \in [1, \infty) \) be as in (2.2) and fix exponents \( p \in (0, \infty) \) and \( q \in (0, \infty) \), along with a finite number \( s \in (0, (\log_2 C_\varphi)^{-1}) \), where the value \( s = (\log_2 C_\varphi)^{-1} \) is only permissible when \( q = \infty \). Also suppose that \( \varphi_{\delta, q} \) is the regularized quasi-metric given by Theorem 3.3. Then there exists a number \( \delta \in (0, 1) \), depending only on \( s, \varphi \), and the proportionality constants in \( \varphi \approx \rho \), such that, for any \( x \in X \) and any finite \( r \in (0, \text{diam}_p(X)] \), there exist a sequence \( \{r_j\}_{j \in \mathbb{N}} \) of radii and a collection \( \{u_j\}_{j \in \mathbb{N}} \) of functions such that, for any \( j \in \mathbb{N} \),
(a) \(B_{\varphi}(x, r_j) \subset B_p(x, r)\) and \(\delta r < r_{j+1} < r_j < r\);

(b) \(0 \leq u_j \leq 1\) pointwise on \(X\);

(c) \(u_j \equiv 0\) on \(X \setminus B_{\varphi}(x, r_j)\);

(d) \(u_j \equiv 1\) on \(\overline{B_{\varphi}(x, r_{j+1})}\);

(e) \(u_j\) belongs to both \(M_{p, q}^s(X, \rho, \mu)\) and \(N_{p, q}^s(X, \rho, \mu)\), and

\[
0 < \|u_j\|_{M_{p, q}^s(X, \rho, \mu)} \leq C2^j r^{-s} \left[\mu(B_{\varphi}(x, r_j))\right]^{1/p}
\]

and

\[
0 < \|u_j\|_{N_{p, q}^s(X, \rho, \mu)} \leq C2^j r^{-s} \left[\mu(B_{\varphi}(x, r_j))\right]^{1/p}
\]

for some positive constant \(C\) which is independent of \(x\), \(r\), and \(j\).

Furthermore, if there exists a constant \(c_0 \in (1, \infty)\) such that \(r \leq c_0 \varphi^s_t(r)\), where \(\varphi^s_t(r)\) is as in (4.1), then the radii \(\{r_j\}_{j \in \mathbb{N}}\) and the functions \(\{u_j\}_{j \in \mathbb{N}}\) can be chosen to have the following property in addition to (a)-(e) above:

(f) for any fixed \(j \in \mathbb{N}\) and \(\gamma \in (0, \infty)\), there exists a \(\mu\)-measurable set \(E^\gamma_j \subset B_p(x, r)\) such that \(\mu(E^\gamma_j) \geq \mu(B_{\varphi}(x, r_{j+1}))\) and \(|u_j - \gamma| \geq \frac{1}{2}\) pointwise on \(E^\gamma_j\).

In this case, the number \(\delta\) depends also on \(c_0\).

Remark 4.4. In the context of Lemma 4.3 we say that both the sequence \(\{r_j\}_{j \in \mathbb{N}}\) of radii and the collection \(\{u_j\}_{j \in \mathbb{N}}\) of functions are associated to the ball \(B_p(x, r)\).

The last result in this subsection is an abstract iteration scheme that will be applied many times in the proofs of our main results (see also [5 Lemma 4.9]). A version of this lemma in the setting of metric-measure spaces was proven in [1 Lemma 16], which itself is an abstract version of an argument used in [29].

Lemma 4.5. Let \((X, \rho, \mu)\) be a quasi-metric measure space. Suppose that \(a, b, p, t, \theta \in (0, \infty)\) satisfy \(a < b\) and \(p < t\). Let \(x \in X\) and \(\{r_j\}_{j \in \mathbb{N}}\) be a sequence of radii such that

\[
a \leq r_j \leq b \quad \text{and} \quad \left[\mu(B_p(x, r_{j+1}))\right]^{1/p} \leq \theta 2^j \left[\mu(B_p(x, r_j))\right]^{1/p}, \quad \forall j \in \mathbb{N}.
\]

Then

\[
\mu(B_p(x, r_1)) \geq \theta^{-pt/(t-p)} 2^{-pt^2/(t-p)^2}.
\]

4.2 Principal Results

The stage is now set to prove our next main result which, among other things, includes a converse to Theorem 3.13. The reader is first reminded of the following pieces of terminology:
Given a quasi-metric measure space \((X, \rho, \mu)\) and an exponent \(Q \in (0, \infty)\), a measure \(\mu\) is said to be \(Q\)-doubling (on \(X\)) provided that there exists a positive constant \(\kappa\) such that

\[
\kappa \left( \frac{r}{R} \right)^Q \leq \frac{\mu(B_p(x, r))}{\mu(B_p(y, R))},
\]

whenever \(x, y \in X\) and \(0 < r \leq R < \infty\) satisfy \(B_p(x, r) \subset B_p(y, R)\), and \(\mu\) is said to be \(Q\)-Ahlfors regular (on \(X\)) if there exists a positive constant \(\kappa\) such that

\[
\kappa^{-1} r^Q \leq \mu(B_p(x, r)) \leq \kappa r^Q
\]

for any \(x \in X\) and any finite \(r \in (0, \text{diam}_p(X))\).

**Theorem 4.6.** Let \((X, \rho, \mu)\) be a quasi-metric measure space where \(\mu\) is a Borel regular \(Q\)-doubling measure for some \(Q \in (0, \infty)\), and suppose that \(\Omega \subset X\) is a \(\mu\)-measurable locally uniformly perfect set. Then the following statements are equivalent.

(a) The measure \(\mu\) satisfies the following measure density condition on \(\Omega\): there exists a positive constant \(C_\mu\) such that

\[
\mu(B_p(x, r)) \leq C_\mu \mu(B_p(x, r) \cap \Omega) \quad \text{for any } x \in \Omega \text{ and } r \in (0, 1].
\]

(b) The set \(\Omega\) is an \(M_{p,q}^s\)-extension domain for some \(p \in (0, \infty), q \in (0, \infty]\), and \(s \in (0, \infty)\) satisfying \(s \leq q \text{ ind}(X, \rho)\), in the sense that there exists a positive constant \(C\) such that,

\[
\text{for any } u \in M_{p,q}^s(\Omega, \rho, \mu), \quad \text{there exists a } \overline{u} \in M_{p,q}^s(X, \rho, \mu)
\]

\[
\text{for which } u = \overline{u}|_\Omega \quad \text{and} \quad \|u\|_{M_{p,q}^s(X, \rho, \mu)} \leq C \|u\|_{M_{p,q}^s(\Omega, \rho, \mu)}.
\]

(c) There exist \(s, p \in (0, \infty)\) and \(a, q \in (0, \infty]\), satisfying \(s \leq q \text{ ind}(X, \rho)\) and \(p, q > Q/(Q + s)\), and a linear and bounded operator \(\mathcal{E}: M_{p,q}^s(\Omega, \rho, \mu) \to M_{p,q}^s(X, \rho, \mu)\) with the property that

\[
(\mathcal{E}u)|_\Omega = u \text{ for any } u \in M_{p,q}^s(\Omega, \rho, \mu).
\]

(d) There exist an \(s \in (0, \infty)\), \(a p \in (0, Q/s)\), \(a q \in (0, \infty]\), and a \(C_S \in (0, \infty)\) satisfying \(s \leq q \text{ ind}(\Omega, \rho)\) such that, for any ball \(B_0 := B_p(x_0, R_0)\) with \(x_0 \in \Omega\) and \(R_0 \in (0, 1]\),

\[
\|u\|_{L^{p^*}(B_0 \cap \Omega)} \leq \frac{C_S}{[\mu(B_0)]^{1/Q}} \left[ R_0^s \|u\|_{M_{p,q}^s(\Omega, \rho, \mu)} + \|u\|_{L^p(\Omega)} \right],
\]

whenever \(u \in M_{p,q}^s(\Omega, \rho, \mu)\). Here, \(p^* := Qp/(Q - sp)\).

(e) There exist an \(s \in (0, \infty)\), \(a p \in (0, Q/s)\), \(a q \in (0, \infty]\), and a \(C_P \in (0, \infty)\) satisfying \(s \leq q \text{ ind}(\Omega, \rho)\) such that, for any ball \(B_0 := B_p(x_0, R_0)\) with \(x_0 \in \Omega\) and \(R_0 \in (0, 1]\),

\[
\inf_{\gamma \in \mathbb{R}} \|u - \gamma\|_{L^{p^*}(B_0 \cap \Omega)} \leq \frac{C_P R_0^s}{[\mu(B_0)]^{1/Q}} \|u\|_{M_{p,q}^s(\Omega, \rho, \mu)},
\]

whenever \(u \in M_{p,q}^s(\Omega, \rho, \mu)\). Here, \(p^* := Qp/(Q - sp)\).
There exist constants $C_1$, $C_2$, $\omega \in (0, \infty)$, $q \in (0, \infty)$, and $s \in (0, \infty)$ satisfying $s \lesssim_q \text{ind}(\Omega, \rho)$ such that

$$\int_{B_0 \cap \Omega} \exp \left( C_1 \frac{[\mu(B_0)]^{\omega} |u - u_{B_0 \cap \Omega}|}{R_0^{\omega} \|u\|_{M^s_{\Omega, \rho}}(\Omega, \rho, \mu)} \right) \, d\mu \leq C_2 \mu(B_0),$$

whenever $B_0$ is a $\rho$-ball centered in $\Omega$ having radius $R_0 \in (0, 1]$, and $u \in M^s_{p, q}(\Omega, \rho, \mu)$ with $\|u\|_{M^s_{p, q}(\Omega, \rho, \mu)} > 0$.

There exist an $s \in (0, \infty)$, a $p \in (Q/s, \infty)$, a $q \in (0, \infty)$, and a $C_H \in (0, \infty)$ satisfying $s \lesssim_q \text{ind}(\Omega, \rho)$ such that, for any ball $B_0 := B_0(x_0, R_0)$ with $x_0 \in \Omega$ and $R_0 \in (0, 1]$, every function $u \in M^s_{p, q}(\Omega, \rho, \mu)$ has a Hölder continuous representative of order $s - Q/p$ on $B_0 \cap \Omega$, denoted by $u$ again, satisfying

$$|u(x) - u(y)| \leq C_H [\rho(x, y)]^{s - Q/p} \frac{R_0^{Q/p}}{\|u\|_{M^s_{p, q}(\Omega, \rho, \mu)}} \|u\|_{M^s_{p, q}(\Omega, \rho, \mu)}, \quad \forall x, y \in B_0 \cap \Omega. \quad (4.10)$$

If the measure $\mu$ is actually $Q$-Ahlfors-regular on $X$ for some $Q \in (0, \infty)$, then the following statements are also equivalent to each of (a)-(g).

There exist an $s \in (0, \infty)$, a $p \in (0, Q/s)$, a $q \in (0, \infty)$, and a $c_S \in (0, \infty)$ satisfying $s \lesssim_q \text{ind}(\Omega, \rho)$ such that

$$\|u\|_{L^{p, s}(\Omega)} \leq c_S \|u\|_{M^s_{p, q}(\Omega, \rho, \mu)},$$

whenever $u \in M^s_{p, q}(\Omega, \rho, \mu)$. Here, $p^* := Qp/(Q - sp)$.

There exist an $s \in (0, \infty)$, a $p \in (0, Q/s)$, a $q \in (0, \infty)$, and a $c_P \in (0, \infty)$ satisfying $s \lesssim_q \text{ind}(\Omega, \rho)$ such that

$$\inf_{\gamma \in \mathbb{R}} \|u - \gamma\|_{L^{p, s}(\Omega)} \leq c_P \|u\|_{M^s_{p, q}(\Omega, \rho, \mu)},$$

whenever $u \in M^s_{p, q}(\Omega, \rho, \mu)$. Here, $p^* := Qp/(Q - sp)$.

There exist constants $c_1$, $c_2$, $\omega \in (0, \infty)$, $q \in (0, \infty)$, and $s \in (0, \infty)$ satisfying $s \lesssim_q \text{ind}(\Omega, \rho)$ such that

$$\int_{B_0 \cap \Omega} \exp \left( c_1 \frac{|u - u_{B_0 \cap \Omega}|}{\|u\|_{M^s_{\Omega, \rho}}(\Omega, \rho, \mu)} \right)^{\omega} \, d\mu \leq c_2 R_0^Q$$

for any $\rho$-ball $B_0$ centered in $\Omega$ with finite radius $R_0 \in (0, \text{diam}_p(X))$, and any nonzero function $u \in M^s_{p, q}(\Omega, \rho, \mu)$.

There exist an $s \in (0, \infty)$, a $p \in (Q/s, \infty)$, a $q \in (0, \infty)$, and a $c_H \in (0, \infty)$ satisfying $s \lesssim_q \text{ind}(\Omega, \rho)$ such that every function $u \in M^s_{p, q}(\Omega, \rho, \mu)$ has a Hölder continuous representative of order $s - Q/p$ on $\Omega$, denoted by $u$ again, satisfying

$$|u(x) - u(y)| \leq c_H [\rho(x, y)]^{s - Q/p} \|u\|_{M^s_{p, q}(\Omega, \rho, \mu)}, \quad \forall x, y \in \Omega.$$

**Remark 4.7.** Theorem 4.6 asserts that in particular, if just one of the statements in (b) - (k) holds for some $p, q, s$ (in their respective ranges), then all of the statements (b) - (k) hold for all $p, q, s$ (in their respective ranges).
Remark 4.8. In the context of Theorem 4.6 if we include the additional demand that $q \leq p$ in statements (d)-(k), and require that $s < \text{ind}(X, \rho)$ in statements (b) and (c), then all of the statements in Theorem 4.6 remain equivalent with $M^p_{\rho,q}$ and $N^p_{\rho,q}$ replaced by $\hat{N}^p_{\rho,q}$ and $\hat{N}^s_{\rho,q}$, respectively. The case for the full range of $q \in (0, \infty)$ is addressed in Theorem 4.10 below. Note that an upper bound on the exponent $q$ for the Besov spaces is to be expected; see [5] Remark 4.17.

Remark 4.9. Recall that we assume that $\Omega$ satisfies $\mu(B_{\rho}(x, r) \cap \Omega) > 0$ for any $x \in \Omega$ and $r \in (0, 1]$, and so $(\Omega, \rho, \mu)$ in Theorem 4.6 is a well-defined locally uniformly perfect quasi-metric-measure space. This is the desired inequality in (4.8) and hence, (d) holds true. A similar line of reasoning will show that (a) also implies each of (e)-(g). This finishes Step 1.

Proof of Theorem 4.6 We prove the present theorem in seven steps.

Step 1: Proof that (a) implies (d)-(g): Assume that the measure density condition in (4.6) holds true. Our plan is to use the embeddings in Theorem 2.4 with $r_* := C_{\rho}$ for the induced quasi-metric-measure space $(\Omega, \rho, \mu)$. To justify our use of Theorem 2.4 with this choice of $r_*$, we need to show that $\mu$ is $Q$-doubling on $\Omega$ up to scale $r_*$. That is, we need to show that

$$\left( \frac{r}{R} \right)^Q \leq \frac{\mu(B_{\rho}(x, r) \cap \Omega)}{\mu(B_{\rho}(y, R) \cap \Omega)}$$

(4.11)

for any balls $B_{\rho}(x, r) \cap \Omega \subset B_{\rho}(y, R) \cap \Omega$ with $x, y \in \Omega$ and $0 < r \leq R \leq C_{\rho}$. Indeed, by $C_{\rho} \geq 1$, we have $C_{\rho}^{-1}r \leq r \leq C_{\rho}R$ and $C_{\rho}^{-1}r \leq C_{\rho}^{-1}R \leq 1$. Moreover, since $x \in B_{\rho}(y, R)$, we deduce that $B_{\rho}(x, C_{\rho}^{-1}r) \subset B_{\rho}(y, C_{\rho}R)$. Then the $Q$-doubling property (4.5) and the measure density condition (4.6) imply

$$\frac{\mu(B_{\rho}(x, r) \cap \Omega)}{\mu(B_{\rho}(y, R) \cap \Omega)} \geq \frac{\mu(B_{\rho}(x, C_{\rho}^{-1}r) \cap \Omega)}{\mu(B_{\rho}(y, R))} \geq \frac{C_{\mu}^{-1} \mu(B_{\rho}(x, C_{\rho}^{-1}r))}{\mu(B_{\rho}(y, C_{\rho}R))} \geq \kappa C_{\mu}^{-1} \left( \frac{C_{\rho}^{-1}}{C_{\rho}R} \right)^Q,$$

and so (4.11) holds true. With (4.11) in hand, applying Theorem 2.4 with $r_* := C_{\rho}$ implies that the embeddings in (2.11), (2.12), (2.13), and (2.14) hold true for any domain ball $B_{\rho}(x_0, R_0) \cap \Omega$, where $x_0 \in \Omega$ and $R_0 \in (0, 1]$.

Moving on, fix exponents $s, p \in (0, \infty)$ and $q \in (0, \infty]$. Consider a $\rho$-ball $B_{\rho} := B_{\rho}(x_0, R_0)$, where $x_0 \in \Omega$ and $R_0 \in (0, 1]$, and suppose that $u \in M^1_{\rho,q}(\Omega, \rho, \mu)$. Note that the pointwise restriction of $u$ to $\sigma B_0 \cap \Omega$, also denoted by $u$, belongs to $M^1_{\rho,q}(\sigma B_0 \cap \Omega, \rho, \mu)$. As such, Theorem 2.4(a) implies that, if $p \in (0, Q/s)$, then there exists a positive constant $C$ such that

$$\|u\|_{L^p((\sigma B_0 \cap \Omega))} \leq \frac{C}{[\mu(\sigma B_0 \cap \Omega)]^{Q/s}} \left[ R_0^s \|u\|_{M^p_{\rho,q}(\sigma B_0 \cap \Omega)} + \|u\|_{L^p(\sigma B_0 \cap \Omega)} \right].$$

(4.12)

where, in obtaining the second inequality in (4.12), we have used the measure density condition (4.6) and the fact that $\sigma \geq 1$ to write

$$\mu(\sigma B_0 \cap \Omega) \geq \mu(B_0 \cap \Omega) \geq C_{\mu}^{-1} \mu(B_0).$$

This is the desired inequality in (4.8) and hence, (d) holds true. A similar line of reasoning will show that (a) also implies each of (e)-(g). This finishes Step 1.
Step 2: Proof that each of (d)-(g) implies (a): As a preamble, we make a few comments to clarify for the reader how we will employ Lemma 4.3 for the space $(\Omega, \rho, \mu)$ in the sequel. First, observe that the assumptions on $\Omega$ ensure that $(\Omega, \rho)$ is a well-defined locally uniformly perfect quasi-metric measure space. Moreover, if $s \leq q \leq \text{ind}(\Omega, \rho)$, then we can always choose a quasi-metric $\varrho$ on $\Omega$ such that $\varrho \approx \rho|_{\Omega}$ and $s \leq (\log C_\varrho)^{-1}$, where $C_\varrho \in [1, \infty)$ is as in (2.2), and the value $s = (\log C_\varrho)^{-1}$ can only occur when $q = \infty$. Throughout this step, we let $\varrho$ denote such a quasi-metric whenever $s \leq q \leq \text{ind}(\Omega, \rho)$, and let $\varrho^\#$ be the regularized quasi-metric given by Theorem 3.3. Also, $\lambda \in (0, 1)$ will stand for the local uniformly perfect constant as in (4.2) for the quasi-metric space $(\Omega, \rho, \mu)$. Recall that there is no loss in generality by assuming that $\lambda < (C_{\rho|\Omega} C_{\rho|\Omega})^{-2}$.

We now prove that (d) implies (a). Fix a ball $B := B_\rho(x, r)$ with $x \in \Omega$ and $r \in (0, 1]$. By assumption, there exist a $p \in (0, Q/s)$, a $q \in (0, \infty]$, an $s \in (0, \infty)$ satisfying $s \leq q \leq \text{ind}(\Omega, \rho)$, and a $C_S \in (0, \infty)$ such that (4.8) holds true whenever $u \in M_{p,q}^s(\Omega, \rho, \mu)$. Note that, if $r > \text{diam}_\rho(\Omega)$, then $B = \Omega$ and (4.6) is trivially satisfied with any $C_\rho \in [1, \infty)$. As such, we assume that $r \leq \text{diam}_\rho(\Omega)$. Let $\{r_j\}_{j \in \mathbb{N}}$ and $\{u_j\}_{j \in \mathbb{N}}$ be as in Lemma 4.3 associated to the domain ball $B \cap \Omega$ in the induced quasi-metric measure space $(\Omega, \rho, \mu)$. Then, for any $j \in \mathbb{N}$, we have $u_j \in M_{p,q}^s(\Omega, \rho, \mu)$ which, in turn, implies that $u_j$ satisfies (4.8) with $B_0$ replaced by $B$. Moreover, $B_{\varrho^\#}(x, r_j) \cap \Omega \subset B \cap \Omega$. As such, it follows from the properties listed in (b)-(e) of Lemma 4.3 that, for any $j \in \mathbb{N}$,

$$
\|u_j\|_{M_{p,q}^s(\Omega)} \leq r^{-s} 2^j \left[\mu(B_{\varrho^\#}(x, r_j) \cap \Omega)\right]^{1/p}
$$

and

$$
\|u_j\|_{L^p(\Omega)} \leq \left[\mu(B_{\varrho^\#}(x, r_j) \cap \Omega)\right]^{1/p}.
$$

Therefore, we have

$$
\rho^s \|u_j\|_{M_{p,q}^s(\Omega)} + \|u_j\|_{L^p(\Omega)} \leq (2^j + 1) \left[\mu(B_{\varrho^\#}(x, r_j) \cap \Omega)\right]^{1/p} 
\leq 2^{j+1} \left[\mu(B_{\varrho^\#}(x, r_j) \cap \Omega)\right]^{1/p}.
$$

Moreover, since $u_j \equiv 1$ on the set $B_{\varrho^\#}(x, r_{j+1}) \cap \Omega$, one deduces that

$$
\|u_j\|_{L^{p^\#}(B \cap \Omega)} \geq \|u_j\|_{L^{p^\#}(B_{\varrho^\#}(x, r_{j+1}) \cap \Omega)} = \left[\mu(B_{\varrho^\#}(x, r_{j+1}) \cap \Omega)\right]^{1/p^\#}.
$$

Combining (4.8), (4.15), and (4.16) gives

$$
\left[\mu(B_{\varrho^\#}(x, r_{j+1}) \cap \Omega)\right]^{1/p^\#} \leq C [\mu(B)]^{-s/Q} 2^j \left[\mu(B_{\varrho^\#}(x, r_j) \cap \Omega)\right]^{1/p}, \quad \forall j \in \mathbb{N},
$$

where $C \in (0, \infty)$ is independent of both $x$ and $r$. Since $0 < \delta r < r < r < \infty$ for any $j \in \mathbb{N}$, where $\delta \in (0, 1)$ is as in Lemma 4.3, we invoke Lemma 4.3 with the quasi-metric $\varrho^\#$, and

$$
p := p, \quad t := p^\#, \quad \text{and} \quad \theta := C [\mu(B)]^{-s/Q},
$$

to conclude that

$$
\mu(B \cap \Omega) \geq \mu(B_{\varrho^\#}(x, r_1) \cap \Omega) \geq 2^{-\frac{\theta^s}{\delta^s}} \left[\mu(B)^{-s/Q}\right]^{-Q/s},
$$

which implies the measure density condition (4.6). This finishes the proof of the statement that (d) implies (a).
We next prove that (e) implies (a). Fix a ball \( B := B_{r}(x, r) \) with \( x \in \Omega \) and \( r \in (0, 1] \) and recall that we can assume that \( r \leq \text{diam}_{\rho}(\Omega) \). By the statement in (e), there exist \( s \in (0, \infty) \) satisfying \( s \leq \text{ind}(\Omega, \rho) \), a \( p \in (0, Q/s) \), a \( q \in (0, \infty) \), and a \( C_{p} \in (0, \infty) \) such that (4.9) holds true whenever \( u \in M^{s}_{q, \rho}(\Omega, \rho, \mu) \), which further leads to the following weaker version of (4.9) (stated here with \( B_{0} = B \) for the arbitrarily fixed ball \( B \))

\[
\inf_{y \in \mathbb{R}} \|u - \gamma\|_{L^{p}(B \cap \Omega)} \leq \frac{C_{pr}^{1/\mu}}{r^{1/\mu} \mu_{B}(\Omega, \rho, \mu)} \|u\|_{M^{s}_{q, \rho}(\Omega, \rho, \mu)}. \tag{4.19}
\]

We next show that (4.19) implies (4.6). In light of Lemma 4.1, we may assume that

\[
r \leq C_{\rho, \Omega, \mu}^{r} \|u\|_{\mu_{B}(\Omega, \rho, \mu)}/r^{2}.
\]

Then, by Lemma 4.3, we find that there exist \( \{r_{j}\}_{j \in \mathbb{N}} \subset (0, \infty) \) and a collection \( \{u_{j}\}_{j \in \mathbb{N}} \) of functions that are associated to \( B \cap \Omega \) and have the properties (a)-(f) listed in Lemma 4.3 with the choice \( c_{0} := C_{\rho, \Omega, \mu}^{r} \|u\|_{\mu_{B}(\Omega, \rho, \mu)}/r^{2} \in (1, \infty) \). In particular, for any \( j \in \mathbb{N} \), \( u_{j} \in M^{s}_{q, \rho}(\Omega, \rho, \mu) \). Hence, for any \( j \in \mathbb{N} \), \( u_{j} \) satisfies (4.19) and the estimates in both (4.13) and (4.14). In particular, we obtain

\[
\|u_{j}\|_{M^{s}_{q, \rho}(\Omega)} \leq (r^{-s}2^{j} + 1) \left[ \mu(B_{O}(x, r_{j}) \cap \Omega) \right]^{1/p} \leq r^{-s}2^{j} \left[ \mu(B_{O}(x, r_{j}) \cap \Omega) \right]^{1/p}, \tag{4.20}
\]

where the last inequality follows from the fact that \( 1 \leq r^{-s}2^{j} \). On the other hand, Lemma 4.3(f) gives

\[
\inf_{y \in \mathbb{R}} \|u_{j} - \gamma\|_{L^{p}(B \cap \Omega)} \geq \inf_{y \in \mathbb{R}} \frac{1}{2} \left[ \mu(B_{O}(x, r_{j+1}) \cap \Omega) \right]^{1/p}, \quad \forall j \in \mathbb{N}. \tag{4.21}
\]

In concert, (4.19), (4.21), and (4.20) give

\[
\left[ \mu(B_{O}(x, r_{j+1}) \cap \Omega) \right]^{1/p} \leq \frac{1}{\|u\|_{\mu_{B}(\Omega, \rho, \mu)}} 2^{j} \left[ \mu(B_{O}(x, r_{j}) \cap \Omega) \right]^{1/p}, \quad \forall j \in \mathbb{N}.
\]

At this stage, we have arrived at the inequality (4.17). Therefore, arguing as in the proofs of both (4.17) and (4.18) leads to (4.6). This finishes the proof of the statement that (e) implies (a).

Next, we turn our attention to showing that (f) implies (a) by proving that the measure density condition (4.6) can be deduced from the following weaker version of (f): There exist constants \( C_{1}, C_{2}, \omega \in (0, \infty) \), \( q \in (0, \infty) \), and \( s \in (0, \infty) \) satisfying \( s \leq \text{ind}(\Omega, \rho) \) such that, for any \( \rho \)-ball \( B := B_{\rho}(x, r) \) with \( x \in \Omega \) and \( r \in (0, 1] \), one has

\[
\inf_{y \in \mathbb{R}} \int_{B \cap \Omega} \exp \left( C_{1} \frac{\|u\|_{M^{s}_{q, \rho}(\Omega, \rho, \mu)}}{r^{1/\mu} \mu_{B}(\Omega, \rho, \mu)} \right)^{\omega} d\mu \leq C_{2} \mu(B), \tag{4.22}
\]

whenever \( u \in M^{s}_{q, \rho}(\Omega, \rho, \mu) \) is nonconstant. To this end, fix a ball \( B := B_{\rho}(x, r) \) with \( x \in \Omega \) and \( r \in (0, 1] \). As before, it suffices to consider the case when \( r \leq \text{diam}_{\rho}(\Omega) \) and \( r \leq C_{\rho, \Omega, \mu}^{r} \|u\|_{\mu_{B}(\Omega, \rho, \mu)}/r^{2} \). Now, let \( \{r_{j}\}_{j \in \mathbb{N}} \) and \( \{u_{j}\}_{j \in \mathbb{N}} \) be as in Lemma 4.3 associated to the domain ball \( B \cap \Omega \). Then, by Lemma 4.3(e), for any \( j \in \mathbb{N} \), \( u_{j} \in M^{s}_{q, \rho}(\Omega, \rho, \mu) \) is nonconstant in \( \Omega \) and hence satisfies (4.22) with \( B \) and the estimate in (4.20). Therefore, for any \( \gamma \in \mathbb{R} \) and any \( j \in \mathbb{N} \) we have

\[
\frac{\|u\|_{M^{s}_{q, \rho}(\Omega, \rho, \mu)}}{r^{1/\mu} \mu_{B}(\Omega, \rho, \mu)} \leq \frac{C_{1} \|u\|_{M^{s}_{q, \rho}(\Omega, \rho, \mu)}}{2^{j} \left[ \mu(B_{O}(x, r_{j}) \cap \Omega) \right]^{1/p}}.
\]
for some positive constant \(C\) independent of \(x, r, \gamma,\) and \(j\). Combining this with Lemma 4.3(f) and (4.22), we conclude that

\[
\mu(B_{u\gamma}(x, r_{j+1}) \cap \Omega) \exp \left( \frac{C[\mu(B)]^{s/J}}{2^{j+1}[\mu(B_{u\gamma}(x, r_j) \cap \Omega)]^{s/J}} \right) ^{\omega} \leq \inf_{y \in \mathbb{R}} \int_{E_y} \exp \left( C_1 \frac{[\mu(B)]^{s/J}[\mu_j - \gamma]}{r^s\|u_j\|_{M^J_{\Omega/\omega}(\Omega, \rho, \mu)}} \right) \, d\mu \leq C_2 \mu(B). \tag{4.23}
\]

By increasing the positive constant \(C_2\), we may assume that \(C_2 > 1\). As such, using the elementary estimate \(\log(z) \leq 2Q(s\omega)^{-1} z^{s/(2Q)}\) for any \(z \in (0, \infty)\), a rewriting of (4.23) implies that

\[
\frac{C[\mu(B)]^{s/J}}{2^{j+1}[\mu(B_{u\gamma}(x, r_j) \cap \Omega)]^{s/J}} \leq \left[ \log \left( \frac{C_2 \mu(B)}{\mu(B_{u\gamma}(x, r_{j+1}) \cap \Omega)} \right) \right]^{1/\omega} \\
\leq \left[ 2Q(s\omega)^{-1} \right]^{1/\omega} C_2^{s/(2Q)} \left[ \frac{\mu(B)}{\mu(B_{u\gamma}(x, r_{j+1}) \cap \Omega)} \right]^{s/(2Q)}.
\]

Therefore,

\[
[\mu(B_{u\gamma}(x, r_{j+1}) \cap \Omega)]^{s/(2Q)} \leq \frac{C}{[\mu(B_{0})]^{s/(2Q)}} 2^j [\mu(B_{u\gamma}(x, r_j) \cap \Omega)]^{s/J}.
\]

Now, applying Lemma 4.5 with the quasi-metric \(\omega\),

\[
p := Q/s, \quad t := 2Q/s, \quad \text{and} \quad \theta := \frac{C}{[\mu(B)]^{s/(2Q)}},
\]

we obtain

\[
\mu(B \cap \Omega) \geq \mu(B_{u\gamma}(x, r_1) \cap \Omega) \geq \left( \frac{C}{[\mu(B)]^{s/(2Q)}} \right)^{-\theta} 2^{-\frac{4Q}{s^2}}, \tag{4.24}
\]

which proves (4.6). The proof of the statement that (e) implies (a) is now complete.

As concerns the implication \((g) \implies (a)\), note that, by \((g)\), we find that there exist an \(s \in (0, \infty)\) satisfying \(s \leq \bar{q} \text{ ind}(\Omega, \rho)\), a \(p \in (Q/s, \infty)\), a \(q \in (0, \infty)\), and a \(C_H \in (0, \infty)\) such that the following weaker version of (4.10) holds true, whenever \(B := B_{\rho}(x, r)\) with \(x \in \Omega\) and \(r \in (0, 1]\) is a \(\rho\)-ball and \(u \in \dot{M}_{\rho, \dot{p}}(\Omega, \rho, \mu)\),

\[
|u(x) - u(y)| \leq C_H [\rho(x, y)]^{s/Q} \frac{r^Q/p}{[\mu(B)]^{1/p}} \|u\|_{M^J_{\rho, p}(\Omega, \rho, \mu)}, \quad \forall x, y \in B \cap \Omega. \tag{4.25}
\]

We now show that (a) follows from (4.25). To this end, fix an \(x \in \Omega\) and an \(r \in (0, 1]\) again. If \(B_{\rho}(x, r) = \Omega\), then (4.6) easily follows with any choice of \(C_H \in (1, \infty)\). Thus, in what follows, we assume that \(\Omega \setminus B_{\rho}(x, r) \neq \emptyset\). Granted this, using the local uniform perfectness of \((\Omega, \rho)\), we can choose a point \(x_0 \in [B_{\rho}(x, r) \cap \Omega] \setminus B_{\rho}(x, \lambda r)\). By Lemma 4.3, we know that there exist a sequence \(\{r_j\}_{j \in \mathbb{N}}\) of radii and a family \(\{u_j\}_{j \in \mathbb{N}}\) of functions that are associated to the domain ball \(B_{\rho}(x, \lambda r) \cap \Omega\). Consider the function \(u_1\) from this collection. According to Lemma 4.3, we have...
we know that there exists a constant $\gamma$ such that $\mu(B_{\mathcal{S}}(x,r)) \leq \gamma r^s \mu(B_{\mathcal{S}}(x,\Lambda r))$. Thus, we have

$$\gamma \leq \frac{1}{\gamma}.$$  

(4.25)

Note that we have simply used the inclusion $B_{\mathcal{S}}(x,r) \subseteq B_{\mathcal{S}}(x,\Lambda r)$ and the fact that $\Lambda < 1$ in obtaining the last inequality in (4.25). Now, the choice of the point $x_0$ ensures that $x_0 \notin B_{\mathcal{S}}(x,r_1)$ and so $u_1(x_0) = 0$. Thus, it follows from (4.25) (used with $u = u_1$) and (4.26) that

$$1 = |u_1(x) - u_1(x_0)| \leq \gamma r^s \mu(B_{\mathcal{S}}(x,r))^{1/p} \mu(B_{\mathcal{S}}(x,\Lambda r))^{1/p} \mu(B_{\mathcal{S}}(x,\Lambda^s r))^{1/p}.$$

The desired estimate in (4.6) now follows. This finishes the proof that $(g) \implies (a)$ and concludes Step 2.

**Step 3: Proof that (a) implies (c):** This implication follows immediately from Theorem 5.3.

**Step 4: Proof that (c) implies (b):** This trivially holds true.

**Step 5: Proof that (b) implies (a):** Assume that the statement in (b) holds true. That is, $\Omega$ is an $M_{\rho,\mu}^s$-extension domain in the sense of (4.7) for some exponents $p \in (0, \infty)$, $q \in (0, \infty)$, and $s \in (0, \infty)$ satisfying $s \leq \min \{X, \rho\}$. Since $C_{\rho,\mu}^s \leq C_{\rho}$ [due to their definitions in (2.2)], it follows from (2.4) that $\min \{X, \rho\} \leq \min \{\Omega, \rho\}$. Hence, we have $s \leq \min \{\Omega, \rho\}$. To ease the notation in what follows, we will omit the notation ‘$|\Omega|$’ on the quasi-metric. In what follows, we let $\sigma := C_{\rho}$. We now proceed by considering three cases depending on the size of $p$.

**CASE 1:** $p \in (0, Q/s)$. In this case, we first show that inequality (4.19) holds true for any ball $B := B_r(x, r)$ with $x \in \Omega$ and $r \in (0, 1]$. Fix such a ball $B$ and a function $u \in M_{\rho,\mu}^s(\Omega, \rho, \mu)$. By (4.7), we know that there exists a $\tilde{u} \in M_{\rho,\mu}^s(X, \rho, \mu)$ such that $u = \tilde{u}|_\Omega$ and $\|\tilde{u}\|_{M_{\rho,\mu}^s(X, \rho, \mu)} \leq \|u\|_{M_{\rho,\mu}^s(\Omega, \rho, \mu)}$. Since $\mu$ is $Q$-doubling on $X$ and $u \in M_{\rho,\mu}^s(\sigma B)$, (2.12) in Theorem 2.4 (applied with $B_0$ therein replaced by $B$) gives

$$\|\tilde{u}\|_{M_{\rho,\mu}^s(\sigma B, \rho, \mu)} \geq \inf_{\gamma \in \mathbb{R}} \|u - \gamma\|_{L^p(B)} \geq \frac{\mu(\sigma B)^{s/p}}{r^s} \inf_{\gamma \in \mathbb{R}} \|u - \gamma\|_{L^p(B \setminus \Omega)}.$$

where $p^* := Qp/(Q - sp)$. As such, we have

$$\inf_{\gamma \in \mathbb{R}} \|u - \gamma\|_{L^p(B \setminus \Omega)} \leq \frac{r^s}{\mu(B)^{s/p}} \|\tilde{u}\|_{M_{\rho,\mu}^s(\sigma B, \rho, \mu)} \leq \frac{r^s}{\mu(B)^{s/p}} \|u\|_{M_{\rho,\mu}^s(\Omega, \rho, \mu)}.$$
Hence, (4.19) holds true whenever \( u \in M^s_{p,q}(\Omega, \rho, \mu) \) which, in turn, implies that the measure density condition (4.6), as we have proved in Step 2.

**CASE 2:** \( p = \frac{Q}{s} \). In this case, we claim that (4.22) holds true. To see this, let \( B := B_r(x, r) \) with \( x \in \Omega \) and \( r \in (0, 1] \), and fix a nonconstant function \( u \in M_{p,q}^{s}(\Omega, \rho, \mu) \). By (4.7), we know that there exists a \( \tilde{u} \in M_{p,q}^{s}(X, \rho, \mu) \) such that \( u = \tilde{u}|_\Omega \) and \( \|\tilde{u}\|_{M_{p,q}^{s}(X, \rho, \mu)} \leq C\|u\|_{M_{p,q}^{s}(\Omega, \rho, \mu)} \) for some positive constant \( C \) independent of \( u \). If \( \|\tilde{u}\|_{M_{p,q}^{s}(\sigma B)} = 0 \), then, by Proposition 2.2, we know that \( \tilde{u} \) is constant \( \mu \)-almost everywhere in \( \sigma B \). Hence, \( u = \tilde{u} \) is constant \( \mu \)-almost everywhere in \( B \cap \Omega \) and (4.22) is trivial. If \( \|\tilde{u}\|_{M_{p,q}^{s}(\sigma B)} > 0 \), since \( \mu \) is \( Q \)-doubling on \( X \), applying Theorem 2.4(b) with \( B_0 = B \), we find that

\[
\begin{align*}
\mu(B) & \geq \int_B \exp \left( C_1 \frac{\mu(\sigma B)^{s/Q} |\tilde{u} - \tilde{u}|}{r^{s}} \right)^\omega \ d\mu \\
& \geq \inf_{y \in \mathbb{R}} \int_{B^c \cap \Omega} \exp \left( C_1 \frac{\mu(B)^{s/Q} |u - \gamma|}{r^{s}} \right)^\omega \ d\mu \\
& \geq \inf_{y \in \mathbb{R}} \int_{B^c \cap \Omega} \exp \left( C_1 \frac{\mu(B)^{s/Q} |u - \gamma|}{C r^{s}} \right)^\omega \ d\mu.
\end{align*}
\]

Thus, (4.22) holds true, and the measure density condition (4.6) now follows from the argument presented in (4.22)-(4.24).

**CASE 3:** \( p \in (0, \frac{Q}{s}) \). In this case, fix a ball \( B := B_r(x, r) \) with \( x \in \Omega \) and \( r \in (0, 1] \), and let \( u \in M_{p,q}^{s}(\Omega, \rho, \mu) \). Again, by (4.7), we know that there exists a \( \tilde{u} \in M_{p,q}^{s}(X, \rho, \mu) \) such that \( u = \tilde{u}|_\Omega \) and \( \|\tilde{u}\|_{M_{p,q}^{s}(X, \rho, \mu)} \leq \|u\|_{M_{p,q}^{s}(\Omega, \rho, \mu)} \). Since \( \mu \) is \( Q \)-doubling on \( X \), applying Theorem 2.4(c) with \( B_0 = B \), we conclude that, if \( x, y \in B \cap \Omega \), then

\[
|u(x) - u(y)| = |\tilde{u}(x) - \tilde{u}(y)| \leq [p(x, y)]^{-Q/p} \frac{r^{Q/p}}{\mu(\sigma B)^{1/p}} \|\tilde{u}\|_{M_{p,q}^{s}(\sigma B, \rho, \mu)}.
\]

This implies that (4.25) holds true, which further implies the measure density condition (4.6), as proved in Step 2. This finishes the proof of the statement that \( (b) \implies (a) \), and hence of Step 5.

There remains to show that (a)-(k) are equivalent under the assumption that \( \mu \) is \( Q \)-Ahlfors regular on \( X \). Since a \( Q \)-Ahlfors regular measure is also a \( Q \)-doubling measure (with the same value of \( Q \)), we immediately have that (a)-(g) are equivalent in light of Steps 1-5. As such, there are only two more steps left in the proof of the present theorem.

**Step 6: Proof that each of \((h)-(k)\) implies \((a)\):** Note that the \( Q \)-Ahlfors regularity condition ensures that \( \mu(B(x,r)) \approx r^Q \) for any \( x \in X \) and \( r \in (0,1] \). With this observation in hand, it is straightforward to check that the statements (h), (i), (j), and (k) of Theorem 4.6 imply the statements (d), (e), (f), and (g), respectively. Hence, (a) follows as a consequence of (i)-(k), given what we have already established in Step 2. This finishes the proof of Step 6.
Step 7: Proof that (b) implies each of (h)-(k): This step proceeds along a similar line of reasoning as the one in Step 5 where, in place of the local embeddings for \( Q \)-doubling measures in Theorem 2.4, we use the global embeddings for \( Q \)-Ahlfors regular measures in Theorem 2.6. This finishes the proof of Step 7 and, in turn, the proof of Theorem 4.6. □

The following theorem is a version of Theorem 4.6 adapted to \( N^s_{p,q} \) spaces which, to our knowledge, is a brand new result even in the Euclidean setting.

**Theorem 4.10.** Let \((X,\rho,\mu)\) be a quasi-metric measure space where \( \mu \) is a Borel regular \( Q \)-doubling measure for some \( Q \in (0,\infty) \), and suppose that \( \Omega \subset X \) is a \( \mu \)-measurable locally uniformly perfect set. Then the following statements are equivalent.

(a) The measure \( \mu \) satisfies the following measure density condition on \( \Omega \): there exists some constant \( C_\mu \in (0,\infty) \) with the property that

\[
\mu(B_\rho(x,r)) \leq C_\mu \mu(B_\rho(x,r) \cap \Omega) \quad \text{for any } x \in \Omega \text{ and } r \in (0,1].
\]

(b) The set \( \Omega \) is an \( N^s_{p,q} \)-extension domain for some \( p \in (0,\infty), q \in (0,\infty), \) and \( s \in (0,\text{ind}(X,\rho)) \) in the sense that there exists a positive constant \( C \) such that,

for any \( u \in N^s_{p,q}(\Omega,\rho,\mu) \), there exists a \( \tilde{u} \in N^s_{p,q}(X,\rho,\mu) \)

for which \( u = \tilde{u}_\Omega \) and \( \| \tilde{u} \|_{N^s_{p,q}(X,\rho,\mu)} \leq C \| u \|_{N^s_{p,q}(\Omega,\rho,\mu)} \).

(c) There exist an \( s \in (0,\text{ind}(X,\rho)), a p \in (0,\infty), a q \in (0,\infty), \) and \( \mu \)-measurable locally uniformly perfect set \( \Omega \subset X \) such that, for any ball \( B_0 := B_\rho(x_0,R_0) \) with \( x_0 \in \Omega \) and \( R_0 \in (0,1] \), one has

\[
\| u \|_{L^{p^*}(B_\rho(\Omega \cap \Omega))} \leq \frac{C_\mu}{\mu(B_0)} \left[ \frac{R_0^s}{\mu(B_0)} \right]^{1/Q} \| u \|_{N^s_{p,q}(\Omega,\rho,\mu)} + \| u \|_{L^p(B_\rho(\Omega \cap \Omega))},
\]

whenever \( u \in N^s_{p,q}(\Omega,\rho,\mu). \) Here, \( p^* := Qp/(Q-ep) \).

(d) There exist \( s, \epsilon, \omega \in (0,\infty) \) satisfying \( \epsilon < s \leq \text{ind}(\Omega,\rho) \), a \( p \in (0,Q/\epsilon), a q \in (0,\infty), \) and \( C_\omega \in (0,\infty) \) such that, for any ball \( B_0 := B_\rho(x_0,R_0) \) with \( x_0 \in \Omega \) and \( R_0 \in (0,1] \), one has

\[
\inf_{\gamma \in \mathbb{R}} \| u - \gamma \|_{L^{p^*}(B_\rho(\Omega \cap \Omega))} \leq \frac{C_\omega R_0^s}{\mu(B_0)} \left[ \frac{R_0^s}{\mu(B_0)} \right]^{1/Q} \| u \|_{N^s_{p,q}(\Omega,\rho,\mu)},
\]

whenever \( u \in N^s_{p,q}(\Omega,\rho,\mu). \) Here, \( p^* := Qp/(Q-ep) \).

(e) There exist \( c_1, c_2, \omega \in (0,\infty), a q \in (0,\infty), \) and \( \epsilon, s \in (0,\infty) \) satisfying \( \epsilon < s \leq \text{ind}(\Omega,\rho) \) such that

\[
\int_{B_0 \cap \Omega} \exp\left( c_1 \frac{[\mu(B_0)]^{1/Q}[\mu - u_{B_0 \cap \Omega}]}{R_0^s \| u \|_{N^s_{p,q}(\Omega,\rho,\mu)}} \right)^\omega \ d\mu \leq c_2 \mu(B_0),
\]

whenever \( B_0 \) is a \( \rho \)-ball centered in \( \Omega \) having radius \( R_0 \in (0,1], \) and \( u \in N^s_{Q/\epsilon,q}(\Omega,\rho,\mu) \) with \( \| u \|_{N^s_{Q/\epsilon,q}(\Omega,\rho,\mu)} > 0. \)
(g) There exist $\varepsilon, s \in (0, \infty)$, $p \in (Q, \infty)$, $q \in (0, \infty)$, and $C_H \in (0, \infty)$ satisfying $\varepsilon < s \leq q \text{ind}(\Omega, \rho)$ such that, for any ball $B_0 := B_{p}(x_0, R_0)$ with $x_0 \in \Omega$ and $R_0 \in (0, 1]$, every function $u \in N_{p,q}^{s}(\Omega, \rho, \mu)$ has a Hölder continuous representative of order $s - Q/p$ on $B_0 \cap \Omega$, denoted by $u$ again, satisfying

$$|u(x) - u(y)| \leq C_H [\rho(x, y)]^{s - Q/p} \frac{R_0^{Q/p}}{[\mu(B_0)]^{1/p}} \|u\|_{N_{p,q}^{s}(\Omega, \rho, \mu)}, \quad \forall \, x, y \in B_0 \cap \Omega.$$ 

The proof of this theorem follows along a similar line of reasoning as in the proof of Theorem 4.6 where, in place of the $\dot{M}_{p,q}^{s}$-embeddings from Theorem 2.4, we use the $N_{p,q}^{s}$-embeddings from Theorem 2.5. We omit the details.

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Ryan Alvarado (Corresponding author)
Department of Mathematics and Statistics, Amherst College, Amherst, MA, USA
E-mail: rjalvarado@amherst.edu

Dachun Yang and Wen Yuan
Laboratory of Mathematics and Complex Systems (Ministry of Education of China), School of Mathematical Sciences, Beijing Normal University, Beijing 100875, People’s Republic of China
E-mails: dcyang@bnu.edu.cn (D. Yang)
wenyuan@bnu.edu.cn (W. Yuan)