The Complexity of Optimal Multidimensional Pricing

Xi Chen∗ Ilias Diakonikolas† Dimitris Paparas‡ Xiaorui Sun∗
Mihalis Yannakakis‡

Abstract

We resolve the complexity of revenue-optimal deterministic auctions in the unit-demand single-buyer
Bayesian setting, i.e., the optimal item pricing problem, when the buyer’s values for the items are inde-
pendent. We show that the problem of computing a revenue-optimal pricing can be solved in polynomial
time for distributions of support size 2, and its decision version is NP-complete for distributions of sup-
port size 3. We also show that the problem remains NP-complete for the case of identical distributions.

∗Columbia University. Email: {xichen, xiaoruisun}@cs.columbia.edu. Research supported by NSF grant CCF-
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†University of Edinburgh. Email: ilias.d@ed.ac.uk. Research supported in part by a startup from the University of
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‡Columbia University. Email: {paparas, mihalis}@cs.columbia.edu. Research supported by NSF grant CCF-
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1 Introduction

Consider the following natural pricing scenario: We have a set of $n$ items for sale and a single unit-demand buyer, i.e., a consumer interested in obtaining at most one of the items. The goal of the seller is then to set prices for the items in order to maximize her revenue by exploiting stochastic information about the buyer’s preferences. More specifically, the seller is given access to a distribution $\mathcal{F}$ from which the buyer’s valuations $v = (v_1, \ldots, v_n)$ for the items are drawn, i.e., $v \sim \mathcal{F}$, and wants to assign a price $p_i$ to each item in order to maximize her expected revenue. We assume, as is commonly the case, that the buyer here is quasi-linear, i.e., her utility for item $i \in [n]$ is $v_i - p_i$, and she will select an item with the maximum nonnegative utility or nothing if no such item exists. This is known as the Bayesian Unit-demand Item-Pricing Problem (BUPP) [CHK07], and has received considerable attention in the CS literature during the past few years [GHK+05, CHK07, Bri08, CHMS10, CD11, DDT12b].

Throughout this paper we focus on the well-studied case [CHK07, CHMS10, CD11] that $\mathcal{F} = \times_{i=1}^n \mathcal{F}_i$ is a product distribution, i.e., the valuations of the buyer for the items are mutually independent random variables. We assume that the $n$ (marginal) distributions $\mathcal{F}_i$ are discrete and are known to the seller (i.e., the values of the support and the corresponding probabilities are rational numbers given explicitly in the input). This seemingly simple computational problem appears to exhibit a very rich structure. Prior to our work, even the (very special) case that the distributions $\mathcal{F}_i$ have support 2 was not well understood: First note that the search space is apparently exponential, since the support size of $\mathcal{F}$ is $2^n$. What makes things trickier is that the optimal prices are not necessarily in the support of $\mathcal{F}$ (see [CD11] for a simple example with two items with distributions of support 2). So, a priori, it was not even clear whether the optimal prices can be described with polynomially many bits in the size of the input description.

Revenue-optimal pricing is well-studied by economists (see, e.g., [Wil96] for a survey and [MMW89] for a simple additive case with two items). The pricing problem studied in this work fits in the general framework of optimal multi-dimensional mechanism design, a central question in mathematical economics (see [MV07] and references therein). Finding the optimal deterministic mechanism in our setting is equivalent to finding the optimal item-pricing. A randomized mechanism, on the other hand, would allow the seller to price lotteries over items [BCKW10, CMS10], albeit this may be less natural in this context.

Optimal mechanism design is well-understood in single-parameter settings for which Myerson [Mye81] gives a closed-form characterization for the optimal mechanism. Chawla, Hartline and Kleinberg [CHK07] show that techniques from Myerson’s work can be used to obtain an analogous closed-form characterization (and also an efficient algorithm) for pricing in our setting, albeit with a constant factor loss in the revenue. In particular, they obtain a factor 3 approximation to the optimal expected revenue (subsequently improved to 2 in [CHMS10]). Cai and Daskalakis [CD11] obtain a polynomial-time approximation scheme for distributions with monotone hazard-rate (and a quasi-polynomial time approximation scheme for the broader class of regular distributions). That is, prior to this work, closed-form characterizations (and efficient algorithms) were known for approximately optimal pricing. The question of whether such a characterization exists for the optimal pricing has remained open and was posed as an open problem in these works [CHK07, CD11].

Our Results. In this paper, we take a principled complexity-theoretic look at the BUPP with independent (discrete) distributions. We start by showing (Theorem 1) that the general decision problem is in NP (and as a corollary, the optimal prices can be described with polynomially many bits). We note that the membership proof is non-trivial because the optimal prices may not be in the support. Our proof proceeds by partitioning the space of price-vectors into a set of (exponentially many) cells (defined by the value distributions $\mathcal{F}_i$), so that the optimal revenue within each cell can be found efficiently by a shortest path computation. One
consequence of the analysis is that the optimal pricing problem has the integrality property: if the values in the supports are integer then the optimal prices are also integer (though they may not belong to the support).

We then proceed to show (Theorem 2) that the case in which each marginal distribution has support at most 2 can be solved in polynomial time. Indeed, by exploiting the underlying structure of the problem, we show that it suffices to consider $O(n^2)$ price-vectors to compute the optimal revenue in this case.

Our main result is that the problem is NP-hard, even for distributions of support 3 (Theorem 3) or distributions that are identical but have large support (Theorem 4). This answers an open problem first posed in [CHK07] and also asked in [CD11, DDT12b]. The main difficulty in the reductions stems from the fact that, for a general instance of the pricing problem, the expected revenue is a highly complex nonlinear function of the prices. The challenge is to construct an instance such that the revenue can be well-approximated by a simple function and is also general enough to encode an NP-hard problem.

Previous Work. We have already mentioned the main algorithmic works for the independent distributions case with approximately-optimal revenue guarantees [CHK07, CHMS10, CD11]. On the lower bound side, Guruswami et al. [GHK05] and subsequently Briest [Bri08] studied the complexity of the problem when the buyer’s values for the items are correlated, respectively obtaining APX-hardness and $\Omega(n^\epsilon)$ inapproximability, for some constant $\epsilon > 0$. More recently, Daskalakis, Deckelbaum and Tzamos [DDT12b] showed that the pricing problem with independent distributions is SQRT-SUM-hard when either the support values or the probabilities are irrational. We note that their reduction relies on the fact that, for certain carefully constructed instances, it is SQRT-SUM-hard to compare the revenue of two price-vectors. This has no bearing on the complexity of the problem under the standard discrete model we consider, for which the exact revenue of a price-vector can be computed efficiently.

Related Work. The optimal mechanism design problem (i.e., the problem of finding a revenue-maximizing mechanism in a Bayesian setting) has received considerable attention in the CS community during the past few years. The vast majority of the work so far is algorithmic [CHK07, CHMS10, BGM10, Ala11, DFK11, HN12, CDW12a, CDW12b], providing approximation or exact algorithms for various versions of the problem. Regarding lower bounds, Papadimitriou and Pierrakos [PPTT] show that computing the optimal deterministic single-item auction is APX-hard, even for the case of 3 bidders. We remark that, if randomization is allowed, this problem can be solved exactly in polynomial time via linear programming [DFK11]. In very recent work, Daskalakis, Deckelbaum and Tzamos [DDT12a] show $\#P$-hardness for computing the optimal randomized mechanism for the case of additive buyers. We remark that their result does not have any implication for the unit-demand case due to the very different structures of the two problems.

The rest of the paper is organized as follows. In Section 2 we first define formally the problem, state our main results, and prove some preliminary basic properties. In Section 3 we show that the decision problem is in NP. In Section 4 we give a polynomial-time algorithm for distributions with support size 2. Section 5 shows NP-hardness for the case of support size 3, and Section 6 for the case of identical distributions. We conclude in Section 7.

2 Preliminaries

2.1 Problem Definition and Main Results

In our setting, there are one buyer and one seller with $n$ items, indexed by $[n] = \{1, 2, \ldots, n\}$. The buyer is interested in buying at most one item (unit demand), and her valuation of the items are drawn from $n$ inde-
pendent discrete distributions, one for each item. In particular, we use $V_i = \{v_{i,1}, \ldots, v_{i,|V_i|}\}$, $i \in [n]$, to denote the support of the value distribution of item $i$, where $0 \leq v_{i,1} < \cdots < v_{i,|V_i|}$. We also use $q_{i,j} > 0$, $j \in [|V_i|]$, to denote the probability of item $i$ having value $v_{i,j}$, with $\sum_j q_{i,j} = 1$. Let $V = \times_{i=1}^n V_i$. We use $\Pr[v]$ to denote the probability of the valuation vector being $v = (v_1, \ldots, v_n) \in V$, i.e., the product of $q_{i,j}$’s over $i, j$ such that $i \in [n]$ and $v_i = v_{i,j}$.

In the problem, all the $n$ distributions, i.e., $V_i$ and $q_{i,j}$, are given to the seller explicitly. The seller then assigns a price $p_i \geq 0$ to each item. Once the price vector $p = (p_1, \ldots, p_n) \in \mathbb{R}_+^n$ is fixed, the buyer draws her values $v = (v_1, \ldots, v_n)$ from the $n$ distributions independently, i.e., $v \in V$ with probability $\Pr[v]$. We assume that the buyer is quasi-linear, i.e., her utility for item $i$ equals $v_i - p_i$. Let

$$U(v, p) = \max_{i \in [n]} (v_i - p_i).$$

If $U(v, p) \geq 0$, the buyer selects an item $i \in [n]$ that maximizes her utility $v_i - p_i$, and the revenue of the seller is $p_i$. If $U(v, p) < 0$, the buyer does not select any item, and the revenue of the seller is 0.

Knowing the value distributions as well as the behavior of the buyer described above, the seller’s objective is to compute a price vector $p \in \mathbb{R}_+^n$ that maximizes the expected revenue

$$\mathcal{R}(p) = \sum_{i \in [n]} p_i \cdot \Pr\left[ \text{buyer selects item } i \right].$$

We use ITEM-PRICING to denote the following decision problem: The input consists of $n$ discrete distributions, with $v_{i,j}$ and $q_{i,j}$ all being rational and encoded in binary, and a rational number $t \geq 0$. The problem asks whether the supremum of the expected revenue $\mathcal{R}(p)$ over all price vectors $p \in \mathbb{R}_+^n$ is at least $t$, where we use $\mathbb{R}_+$ to denote the set of nonnegative real numbers.

We note that the aforementioned decision problem is not well-defined without a tie-breaking rule, i.e., a rule that specifies which item the buyer selects when there are multiple items with maximum nonnegative utility. Throughout the paper, we will use the following maximum price tie-breaking rule (which is convenient for our arguments): when there are multiple items with maximum nonnegative utility, the buyer selects the item with the smallest index among items with the highest price. (We note that the critical part is that an item with the highest price is selected. Selecting the item with the smallest index among them is arbitrary — and does not affect the revenue; however we need to make such a choice so that it makes sense to talk about “the” item selected by the buyer in the proofs.) We show in Section 2.2 that our choice of the tie-breaking rule does not affect the supremum of the expected revenue (hence, the complexity of the problem).

We are now ready to state our main results. First, we show in Section 3 that ITEM-PRICING is in NP.

**Theorem 1.** ITEM-PRICING is in NP.

Second, we present in Section 4 a polynomial-time algorithm for ITEM-PRICING when all the distributions have support size at most 2.

**Theorem 2.** ITEM-PRICING is in P when every distribution has support size at most 2.

As our main result, we resolve the computational complexity of the problem. We show that it is NP-hard even when all distributions have support size at most 3 (Section 5), or when they are identical (Section 6).

**Theorem 3.** ITEM-PRICING is NP-hard even when every distribution has support size at most 3.

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1 It may also be called the maximum value tie-breaking rule, since an item with the maximum price among a set of items with the same utility must also have the maximum value.
Theorem 4. Item-Pricing is NP-hard even when the distributions are identical.

2.2 Tie-Breaking Rules

In this section, we show that the supremum of the expected revenue over \( p \in \mathbb{R}_+^n \) is invariant to tie-breaking rules. Formally, a tie-breaking rule is a mapping from the set of pairs \((v, p)\) with \( U(v, p) \geq 0\) to an item \( k \) such that \( v_k - p_k = U(v, p) \).

We will need some notation. Let \( B \) be the maximum price tie-breaking rule described earlier. We will denote by \( R(p) \) the expected revenue of \( p \) under \( B \), and by \( R(v, p) \) the seller’s revenue under \( B \) when the valuation vector is \( v \in V \). Given a price vector \( p \) and a valuation vector \( v \in V \), we also denote by \( T(v, p) \) the set of items with maximum nonnegative utility (so \( T(v, p) = \emptyset \) iff \( U(v, p) < 0 \)).

We show the following:

Lemma 2.1. The supremum of the expected revenue over \( p \in \mathbb{R}_+^n \) is invariant to tie-breaking rules.

Proof. Let \( v_{i,j} \) and \( q_{i,j} \) denote the numbers that specify the distributions. Let \( B' \) be a tie-breaking rule. We will use \( R'(p) \) to denote the expected revenue of \( p \) under \( B' \) and use \( R'(v, p) \) to denote the seller’s revenue under \( B' \) when the valuation vector is \( v \in V \).

It is clear that for any \( p \in \mathbb{R}_+^n \) and \( v \in V \), we have \( R(v, p) \geq R'(v, p) \) since \( B \) picks an item with the highest price among those that maximize the utility. Hence, it follows that \( \sup_p R(p) \geq \sup_p R'(p) \).

On the other hand, given any price vector \( p \in \mathbb{R}_+^n \), we consider

\[
p_\epsilon = (\max(0, p_1 - r_1 \epsilon), \ldots, \max(0, p_n - r_n \epsilon)) \in \mathbb{R}_+^n,
\]

where \( \epsilon > 0 \) and \( r_i \) is the rank of \( p_i \) sorted in increasing order (when there are ties, the item with the smaller index is ranked higher). We claim that

\[
\lim_{\epsilon \to 0^+} R'(p_\epsilon) = R(p).
\]

It then follows from (1) that \( \sup_p R'(p) \geq \sup_p R(p) \), which gives the proof of the lemma.

To prove (1), we show that the following holds for any valuation vector \( v \in V \):

\[
\lim_{\epsilon \to 0^+} R'(v, p_\epsilon) = R(v, p).
\]

Observe that (1) follows from (2) since

\[
R(p) = \sum_{v \in V} R(v, p) \cdot \Pr[v] \quad \text{and} \quad R'(p_\epsilon) = \sum_{v \in V} R'(v, p_\epsilon) \cdot \Pr[v].
\]

To prove (2), we consider two cases. If \( U(v, p) < 0 \), then we have \( U(v, p_\epsilon) < 0 \) when \( \epsilon \) is sufficiently small, and thus, \( R(v, p) = R'(v, p_\epsilon) = 0 \). When \( U(v, p) \geq 0 \), we make the following three observations. First, the utility of an item \( i \in [n] \) under \( p_\epsilon \) is at least as high as that under \( p \). Second, if \( v_i - p_i > v_j - p_j \) for some items \( i, j \in [n] \), then under \( p_\epsilon \) the utility of item \( i \) remains strictly higher than that of item \( j \), for \( \epsilon \) sufficiently small. Third, if \( v_i - p_i = v_j - p_j \) and \( p_i > p_j \) (in particular, \( p_i > 0 \)) for some \( i, j \in [n] \), then under \( p_\epsilon \), the utility of item \( i \) is strictly higher than that of item \( j \) when \( \epsilon \ll p_i \), as \( r_i > r_j \). It follows from these observations that when \( \epsilon \) is sufficiently small, \( B' \) must pick, given \( v \) and \( p_\epsilon \), an item \( k \in [n] \) such that \( p_k = R(v, p) \). (2) then follows from the definition of \( p_\epsilon \).
We will henceforth always adopt the maximum price tie-breaking rule, and use $R(v, p)$ to denote the revenue of the seller with respect to this rule. One of the advantages of this rule is that the supremum of the expected revenue $R(p)$ is always achievable, so it makes sense to talk about whether $p$ is optimal or not. In the following example, we point out that this does not hold for general tie-breaking rules.

**Example:** Suppose item 1 has value 10 with probability 1, item 2 has value 8 with probability 1/2 and value 12 with probability 1/2, and in case of tie the buyer prefers item 1. The supremum in this example is 11: set $p_1 = 10$ for item 1 and $p_2 = 12 - \epsilon$ for item 2. The buyer will buy item 1 with probability 1/2 (if her value for item 2 is 8) and item 2 with probability 1/2 (if her value for item 2 is 12). However, an expected revenue of 11 is not achievable: if we give price 12 to item 2, then the buyer will always buy item 1 and the revenue is 10. Note that the expected revenue for this tie-breaking rule is not a continuous function of the prices.

Before proving that the supremum is indeed always achievable under the maximum price rule, we start by showing that without loss of generality, we may focus the search for an optimal price vector in the set

$$P = \times_{i=1}^n [a_i, b_i], \quad \text{where } a_i = \min_j v_{i,j} \text{ and } b_i = \max_j v_{i,j}$$

denote the minimum and maximum values in the support $V_i$, respectively.

**Lemma 2.2.** For any price vector $p \in \mathbb{R}_+^n$, there exists a $p' \in P$ such that $R(p') \geq R(p)$.

**Proof.** First, it is straightforward that no price $p_i$ should be above $b_i$; if such a price exists, we can simply replace it by $b_i$ and this will not decrease the expected revenue.

The non-trivial part is to argue that it is no loss of generality to assume that no price $p_i$ is below $a_i$. Let $p \in \times_{i=1}^n [0, b_i]$. Suppose that there exists $i \in [n]$ such that $p_i < a_i$, i.e., the set $L(p) = \{ i \in [n] : p_i < a_i \}$ is nonempty; otherwise, there is nothing to prove.

Fix an $i \in L(p)$ arbitrarily and let $S_i = \{ j \in [n] : p_j < a_i \}$. We consider the price vector $\tilde{p}$ defined by $\tilde{p}_j = \min \{ b_j, a_i \}$ for $j \in S_i$ and $\tilde{p}_j = p_j$ otherwise. As $i \in S_i$, it follows that $S_i \neq \emptyset$ and therefore $\tilde{p} \neq p$ (in particular, $\tilde{p}_i = a_i$ now). It is also clear that $\tilde{p} \in \times_{i=1}^n [0, b_i]$. It suffices to show that $R(\tilde{p}) \geq R(p)$.

Indeed, note that $|L(\tilde{p})| < |L(p)|$ so this process will terminate in at most $n$ stages. After the last stage we will obtain a vector $p' \in P$ whose expected revenue is lower bounded by all the previous ones.

To prove that $R(\tilde{p}) \geq R(p)$, we proceed as follows. Given any valuation vector $v \in V$, we compare the revenue $R(v, p)$ to $R(v, \tilde{p})$ and consider the following two cases:

- **Case 1:** On input $(v, p)$, the item selected by the buyer is not from $S_i$. We claim that the same item is selected on input $(v, \tilde{p})$. Indeed, we did not decrease prices of items in $S_i$, hence their utilities did not go up, while the utilities of the remaining items did not change. Therefore, the revenue does not change in this case, i.e., $R(v, \tilde{p}) = R(v, p)$.

- **Case 2:** On input $(v, p)$, the item selected is from $S_i$. Then by the definition of $S_i$, the revenue $R(v, p)$ we get is certainly less than $a_i$. On input $(v, \tilde{p})$, we know that $U(v, \tilde{p}) \geq 0$ (since item $i$ must have nonnegative utility, i.e., $v_i - \tilde{p}_i = v_i - a_i \geq 0$) and thus, $T(v, \tilde{p}) \neq \emptyset$. We claim that $R(v, \tilde{p}) \geq a_i > R(v, p)$. To see this, we consider two sub-cases. If $U(v, \tilde{p}) = 0$, then we must have $i \in T(v, \tilde{p})$ and the claim follows from our choice of the maximum price tie-breaking rule. If $U(v, \tilde{p}) > 0$, then every $j \in T(v, \tilde{p})$ must satisfy $\tilde{p}_j \geq a_i$; otherwise, by definition of $\tilde{p}$ we have $\tilde{p}_j = b_j$ and $v_j - \tilde{p}_j \leq 0$, a contradiction. From $\tilde{p}_j \geq a_i$ and $j \in T(v, \tilde{p})$, we have $R(v, \tilde{p}) \geq a_i$.

The lemma follows by combining the two cases. \qed
Now we show that the supremum can always be achieved under the maximum price rule \( B \).

**Lemma 2.3.** There exists a price vector \( p^* \in P \) such that \( \mathcal{R}(p^*) = \sup_{p} \mathcal{R}(p) \).

**Proof.** By the compactness of \( P \), it suffices to show that if a sequence of vectors \( \{p_i\} \) approaches \( p \), then

\[
\mathcal{R}(p) \geq \lim_{i \to \infty} \mathcal{R}(p_i).
\]

To this end, it suffices to show that, for any valuation vector \( v \in V \),

\[
\mathcal{R}(v, p) \geq \lim_{i \to \infty} \mathcal{R}(v, p_i).
\] (3)

Given any valuation \( v \in V \), it is easy to check that \( \mathcal{T}(v, p_i) \subseteq \mathcal{T}(v, p) \) when \( i \) is sufficiently large. (Again consider two cases: \( \mathcal{U}(v, p) < 0 \) and \( \mathcal{U}(v, p) \geq 0 \) \( \square \) then follows, since \( \mathcal{R}(v, p) \) is the highest price of all items in \( \mathcal{T}(v, p) \) under the maximum price tie-breaking rule.

### 3 Membership in NP

In this section we prove Theorem\( \square \) i.e., ITEM-PRICING is in NP.

**Proof of Theorem.** We start with some notation. Given a price vector \( p \in \mathbb{R}_+^n \) and a valuation \( v \in V \), let \( \mathcal{I}(v, p) \in [n] \cup \{\text{nil}\} \) denote the item picked by the buyer under the maximum price tie-breaking rule, with \( \mathcal{I}(v, p) = \text{nil} \) iff \( \mathcal{U}(v, p) < 0 \). We will partition \( P = \times_{i=1}^n [a_i, b_i] \) into equivalence classes so that two price vectors \( p, p' \) from the same class yield the same outcome for all valuations: \( \mathcal{I}(v, p) = \mathcal{I}(v, p') \) for all \( v \).

Consider the partition of \( P \) induced by the following set of hyperplanes. For each item \( i \in [n] \) and each value \( s_i \in V_i \), we have a hyperplane \( p_i = s_i \). For each pair of items \( i, j \in [n] \) and pair of values \( s_i \in V_i \) and \( t_j \in V_j \), we have a hyperplane \( s_i - p_i = t_j - p_j \), i.e., \( p_i - p_j = s_i - t_j \). These hyperplanes partition our search space \( P \) into polyhedral cells, where the points in each cell lie on the same side of each hyperplane (either on the hyperplane or in one of the two open-halfspaces).

We claim that, for every valuation \( v \in V \), all the vectors in each cell yield the same outcome. Consider any cell \( C \). It is defined by a set of equations and inequalities. Given any price vector \( p \in C \) and any value \( s_i \in V_i \), let \( V(p, s_i) \) be the set of valuation vectors \( v \in V \) such that \( v_i = s_i \) and the buyer ends up buying item \( i \) on \( (v, p) \). We claim that \( V(p, s_i) \) does not depend on \( p \), i.e., it is the same set \( V(s_i) = V(p, s_i) \) over all \( p \in C \). To this end, first, if the points of \( C \) satisfy \( p_i > s_i \) then \( V(p, s_i) = \emptyset \). So suppose that \( C \) satisfies \( p \leq s_i \). Consider any valuation vector \( v \in V \) with \( v_i = s_i \). The valuation \( v \) is in \( V(p, s_i) \) iff for all \( j \neq i \), we have \( s_i - p_i \geq v_j - p_j \), and in case of equality we have \( s_i \geq v_j \) (iff \( p_i \geq p_j \) due to the equality), and in case of further equality \( s_i = v_j \) we have \( i < j \). Because all points of the cell \( C \) lie on the same side of each hyperplane \( s_i - p_i = v_j - p_j \), it follows that \( V(p, s_i) \) does not depend on \( p \). As a result, for any cell \( C \) and any \( v \in V \), all the points \( p \in C \) yield the same outcome \( \mathcal{I}(v, p) \).

Next, we show that it is easy to compute the supremum of the expected revenue \( \mathcal{R}(p) \) over \( p \in C \), for each cell \( C \). To this end, let \( W_i = \cup_{s_i \in V_i} V(s_i) \subseteq V \) denote the set of valuations for which the buyer picks item \( i \) if the prices lie in the cell \( C \), and let \( \gamma_i \) be the probability of \( W_i \): \( \gamma_i = \sum_{v \in W_i} \Pr[v] \). It turns out that \( \gamma_i \) can be computed efficiently, since the probability of \( V(s_i) \) can be computed efficiently as shown below (and \( W_i \) is the disjoint union of \( V(s_i), s_i \in V_i \)).

Given \( s_i \in V_i \), to compute the probability of \( V(s_i) \), we note that \( V(s_i) \) is actually the Cartesian product of subsets of \( V_j, j \in [n] \). For each \( j \neq i \), we can determine efficiently the subset of values \( L_j \subseteq V_j \) such
that the buyer prefers item $i$ to $j$ if $i$ has value $s_i$ and $j$ has value from $L_j$. As a result, we have

$$V(s_i) = L_1 \times \cdots \times L_{i-1} \times \{s_i\} \times L_{i+1} \times \cdots \times L_n,$$

and thus, we multiply the probabilities of these subsets $L_j$, for all $j$, and the probability of $s_i$. Summing up the probabilities of $V(s_i)$ over $s_i \in V_i$ gives us $\gamma_i$, the probability of $W_i$.

Finally, the supremum of the expected revenue $R(p)$ over all $p \in C$ is the maximum of $\sum_{i\in[n]} \gamma_i \cdot p_i$ over all $p$ in the closure of $C$. Let $C'$ denote the closure of $C$; this is the polyhedron obtained by changing all the strict inequalities of $C$ into weak inequalities. The supremum of $\sum_i \gamma_i \cdot p_i$ over all points $p \in C$ can be computed in polynomial time by solving the linear program that maximizes $\sum \gamma_i \cdot p_i$ subject to $p \in C'$. In fact, as we will show below after the proof of Theorem 1, that this LP has a special form: The question of whether a set of equations and inequalities with respect to a set of hyperplanes of the form $p_i = s_i$ and $p_i - p_j = s_i - t_j$ is consistent, i.e., defines a nonempty cell, can be formulated as a negative weight cycle problem, and the optimal solution for a nonempty cell can be computed by solving a single-source shortest path problem. It follows that the specification of a cell $C$ in the partition is an appropriate yes certificate for the decision problem Item-Pricing, and the theorem is proved. □

Next we describe in more detail how to determine whether a set of equations and inequalities defines a nonempty cell, and how to compute the optimal solution over a nonempty cell. The description of a (candidate) cell $C$ consists of equations and inequalities specifying (1) for each item $i$, the relation of $p_i$ to every value $s_i \in V_i$, and (2) for each pair of items $i, j$ and each pair of values $s_i \in V_i$ and $t_j \in V_j$, the relation of $p_i - p_j$ to $s_i - t_j$. Construct a weighted directed graph $G = (N, E)$ over $n + 1$ nodes $N = \{0, 1, \ldots, n\}$ where nodes $1, \ldots, n$ correspond to the $n$ items. For each inequality of the form $p_i < s_i$ or $p_i \leq s_i$, include an edge $(0, i)$ with weight $s_i$, and call the edge strict or weak accordingly as the inequality is strict or weak.

In fact, there is a tightest such inequality (i.e., with the smallest value $s_i$) since the cell is in $P$, and it suffices to include the edge for this inequality only. Similarly, for each inequality of the form $p_i > s_i$ or $p_i \geq s_i$ (or only for the tightest such inequality, i.e. the one with the largest value $s_i$) include an edge $(i, 0)$ with weight $-s_i$. For each inequality of the form $p_i - p_j < s_i - t_j$ or $p_i - p_j \leq s_i - t_j$ (or only for the tightest such inequality) include a (strict or weak) edge $(j, i)$ with weight $s_i - t_j$. Similarly, for every inequality of the form $p_i - p_j > s_i - t_j$ or $p_i - p_j \geq s_i - t_j$ (or only for the tightest such inequality) include a (strict or weak) edge $(i, j)$ with weight $t_j - s_i$.

We prove the following connections between $G = (N, E)$ and the cell $C$:

**Lemma 3.1.** 1. A set of equations and inequalities defines a nonempty cell if and only if the corresponding graph $G$ does not contain a negative weight cycle or a zero weight cycle with a strict edge.

2. The supremum of the expected revenue for a nonempty cell is achieved by the price vector $p$ that consists of the distances from node 0 to the other nodes of the graph $G$.

**Proof.** 1. Considering node 0 as having an associated variable $p_0$ with fixed value 0, the given set of equations (i.e., pairs of weak inequalities) and (strict) inequalities can be viewed as a set of difference constraints on the variables $(p_0, p_1, \ldots, p_n)$, and it is well known that the feasibility of such a set of constraints can be formulated as a negative weight cycle problem. If there is a cycle with negative weight $w$, then adding all the inequalities corresponding to the edges of the cycle yields the constraint $0 \leq w$ (which is false); if there is a cycle with zero weight but also a strict edge, then summing the inequalities yields $0 < 0$.

Conversely, suppose that $G$ does not contain a negative weight cycle or a zero weight cycle with a strict edge. For each strict edge $e$, replace its weight $w(e)$ by $w'(e) = w(e) - \epsilon$ for a sufficiently small $\epsilon > 0$ (we
and therefore the buyer should have picked rule. The facts that 

\[ p \in C \]

under any program that maximizes \( \max \{ \gamma_i \} \) subject to \( p' \in C' \), where \( C' \) is the closure of the cell \( C \). Observe that all the coefficients \( \gamma_i \) of the objective function are nonnegative, and clearly \( p \) is in the closure \( C' \). Therefore \( p \) achieves the supremum of the expected revenue over \( C' \).

The NP characterization of \textsc{Item-Pricing} and the corresponding structural characterization of the optimal price vector \( p = p(0) \) of each cell have several easy and useful consequences.

First, we get an alternative proof of Lemma 2.3 regarding the maximum tie-breaking rule:

\textit{Second Proof of Lemma 2.3}. Suppose that the supremum of the expected revenue is achieved in cell \( C \). Let \( G \) be the corresponding graph, and let \( p \) be the price vector of the distances from node 0 to the other nodes. If \( p \in C \) then the conclusion is immediate, so assume \( p \notin C \). From the proof of the above lemma we have that \( p \geq p' \) coordinate-wise for all \( p' \in C \).

We claim that for any valuation \( v \in V \), the revenue \( R(v, p) \) is at least as large as the revenue \( R(v, p') \) under any \( p' \in C \). Suppose that the buyer selects item \( i \) under \( v \) for prices \( p' \). Then \( p'_i \leq v_i \) and thus also \( p_i \leq v_i \) (since \( p \) is in the closure of \( C \)) and thus \( i \) is also eligible for selection under \( p \). If the buyer selects \( i \) under \( p \) then we know that \( p_i \geq p'_i \) and the conclusion follows. Suppose that the buyer selects another item \( j \) under \( p \) and that \( p'_j > p_j \) and hence \( p_i > p_j \). Then we must have \( v_j - p_j > v_i - p_i \) due to the tie-breaking rule. The facts that \( p \) is in the closure of \( C \) and \( v_j - p_j > v_i - p_i \) imply that \( v_j - p'_j > v_i - p'_i \) for all \( p' \in C \), and therefore the buyer should have picked \( j \) instead of \( i \) under prices \( p' \), a contradiction.

We conclude that for any \( v \in V \), \( R(v, p) \geq R(v, p') \) for any \( p' \in C \), and the lemma follows.

Another consequence suggested by the structural characterization of Lemma 3.1 is that the maximum of expected revenue can always be achieved by a price vector \( p \) in which all prices \( p_i \) are sums of a value and differences between pairs of values of items. This implies for example the following useful corollary.

\textbf{Corollary 3.1.} If all the values in \( V_i \), \( i \in [n] \), are integers, then there must exist an optimal price vector \( p \in P \) with integer coordinates.
4 A polynomial-time algorithm for support size 2

In this section, we present a polynomial-time algorithm for the case that each distribution has support size at most 2. In Section 4.1, we give a polynomial-time algorithm under a certain "non-degeneracy" assumption on the values. In Section 4.2 we generalize this algorithm to handle the general case.

4.1 An Interesting Special case.

In this subsection, we assume that every item has support size 2, where \( V_i = \{a_i, b_i\} \) satisfies \( b_i > a_i > 0 \), for all \( i \in [n] \). Let \( a = (a_1, \ldots, a_n) \) and \( b = (b_1, \ldots, b_n) \) satisfy the following "non-degeneracy" assumption:

**Non-degeneracy assumption:** \( b_1 < b_2 < \cdots < b_n, a_i \neq a_j \) and \( t_i \neq t_j \) for all \( i, j \in [n] \).

As we show next in Section 4.2, this special case encapsulates the essential difficulty of the problem.

Let \( \text{OPT} \) denote the set of optimal price vectors in \( P = \times_{i=1}^{n}[a_i, b_i] \) that maximize the expected revenue \( \mathcal{R}(p) \). Next we prove a sequence of lemmas to show that, given \( a \) and \( b \) that satisfy all the conditions above, one can compute efficiently a set \( A \subseteq P \) of price vectors such that \( |A| = O(n^2) \) and \( \text{OPT} \subseteq A \). Hence, by computing \( \mathcal{R}(p) \) for all \( p \in A \), we get both the maximum of expected revenue and an optimal price vector.

We start with the following lemma:

**Lemma 4.1.** If \( p \in P \) satisfies \( p_i > a_i \) for all \( i \in [n] \), then either \( p = b \) or we have \( p \notin \text{OPT} \).

**Proof.** Assume for contradiction that \( p \in P \) satisfies \( p_i > a_i \) for all \( i \in [n] \) but \( p \neq b \). It then follows from the maximum price tie-breaking rule that \( \mathcal{R}(v, b) \geq \mathcal{R}(v, p) \) for all \( v \in V \). Moreover, there is at least one \( v^* \in V \) such that \( \mathcal{R}(v^*, b) > \mathcal{R}(v^*, p) \). If \( p_i < b_i \), then consider \( v^* \) with \( v^*_i = b_i \) and \( v^*_j = a_j \) for all other \( j \). It follows that \( \mathcal{R}(b) > \mathcal{R}(p) \) as we assumed that \( 0 < q_i < 1 \) for all \( i \in [n] \) and thus, \( p \notin \text{OPT} \). \( \square \)

Next we show that there can be at most one \( i \) such that \( p_i = a_i \); otherwise \( p \notin \text{OPT} \). We emphasize that all the conditions on \( V_i \) are assumed in the lemmas below, the non-degeneracy assumption in particular.

**Lemma 4.2.** If \( p \in P \) has more than one \( i \in [n] \) such that \( p_i = a_i \), then we have \( p \notin \text{OPT} \).

**Proof.** Assume for contradiction that \( p \in P \) has more than one \( i \) such that \( p_i = a_i \). We prove the lemma by explicitly constructing a new price vector \( p' \in P \) from \( p \) such that \( \mathcal{R}(v, p') \geq \mathcal{R}(v, p) \) for all \( v \in V \) and \( \mathcal{R}(v^*, p') > \mathcal{R}(v^*, p) \) for at least one \( v^* \in V \). This implies that \( \mathcal{R}(p') > \mathcal{R}(p) \) and thus, \( p \) is not optimal.

We will be using this simple strategy in most of the proofs of this section.

Let \( k \in [n] \) denote the item with the smallest \( a_k \) among all \( i \in [n] \) with \( p_i = a_i \). By the non-degeneracy assumption, \( k \) is unique. Recall that \( t_k = b_k - a_k = b_k - p_k \). We let \( S \) denote the set of \( i \in [n] \) such that \( b_i - p_i = t_k \), so \( k \in S \). By the non-degeneracy assumption again, we have \( p_i > a_i \) for all \( i \in S \setminus \{k\} \). We now construct \( p' \in P \) as follows: For each \( i \in [n] \), set \( p'_i = p_i \) if \( i \notin S \); otherwise set \( p'_i = p_i + \epsilon \) for some sufficiently small \( \epsilon > 0 \). Next we show that \( \mathcal{R}(v, p') \geq \mathcal{R}(v, p) \) for all \( v \in V \). Fix a \( v \in V \). We consider the following three cases:

1. If \( \mathcal{U}(v, p) = t_k \), then \( \mathcal{T}(v, p) \subseteq S \) by the definition of \( S \). When \( \epsilon \) is sufficiently small, we have \( \mathcal{T}(v, p') = \mathcal{T}(v, p) \) and \( \mathcal{R}(v, p') = \mathcal{R}(v, p) + \epsilon > \mathcal{R}(v, p) \).
2. If \( U(v, p) = 0 \) and \( k \in T(v, p) \), then we have \( T(v, p) \cap S = \{k\} \) since \( b_i > p_i > a_i \) for all other \( i \in S \). We claim that \( R(v, p) > p_k \) in this case. To see this, note that there exists an item \( \ell \in [n] \) such that \( p_\ell = a_\ell \) and \( p_\ell > p_k \) by our choice of \( k \). As \( U(v, p) = 0 \), we must have \( v_\ell = a_\ell \) and thus, \( \ell \in T(v, p) \) and \( R(v, p) \geq p_\ell \) is not obtained from selling item \( k \). Therefore, we have

\[
U(v, p') = 0, \quad T(v, p') = T(v, p) - \{k\} \quad \text{and} \quad R(v, p') = R(v, p).
\]

3. Finally, if neither of the cases above happens, then we have \( T(v, p) \cap S = \emptyset \) (note that this includes the case when \( T(v, p) = \emptyset \)). For this case we have \( T(v, p') = T(v, p) \) and \( R(v, p') = R(v, p) \).

The lemma then follows because in the second case above, we indeed showed that the following valuation vector \( v^* \) in \( V \) satisfies \( R(v^*, p') > R(v^*, p) \): \( v_k = b_k \) and \( v_i = a_i \) for all \( i \neq k \). \( \square \)

Lemma 4.2 reduces our search space to \( p \) such that either \( p = b \) or \( p \in P_k \) for some \( k \in [n] \), where we use \( P_k \) to denote the set of price vectors \( p \in P \) such that \( p_k = a_k \) and \( p_i > a_i \) for all other \( i \in [n] \).

The next lemma further restricts our attention to \( p \in P_k \) such that \( p_i \in \{b_i, b_i - t_k\} \) for all \( i \neq k \).

**Lemma 4.3.** If \( p \in P_k \) but \( p_i \notin \{b_i, b_i - t_k\} \) for some \( i \neq k \), then we have \( p \notin \text{OPT} \).

**Proof.** Assume for contradiction that \( p_\ell \notin \{b_\ell, b_\ell - t_k\} \). As \( p \in P_k \), we also have \( p_\ell > a_\ell \). Now we use \( S \) to denote the set of all \( i \in [n] \) such that \( b_i - p_i = b_\ell - p_\ell \). It is clear that \( k \notin S \). We use \( p' \) to denote the following new price vector: \( p'_i = p_i \) for all \( i \notin S \), and \( p'_i = p_i + \epsilon \) for all \( i \in S \), where \( \epsilon > 0 \) is sufficiently small. We use the same proof strategy to show that \( R(p') > R(p) \). Fix any \( v \in V \). We have

1. If \( U(v, p) < 0 \), then clearly \( U(v, p') < 0 \) as well and thus, \( R(v, p') = R(v, p) = 0 \).
2. If \( U(v, p) = b_\ell - p_\ell \), then \( T(v, p) \subseteq S \) by the definition of \( S \). When \( \epsilon \) is sufficiently small,

\[
T(v, p') = T(v, p) \quad \text{and} \quad R(v, p') = R(v, p) + \epsilon > R(v, p).
\]

3. If \( U(v, p) \geq 0 \) but \( U(v, p) \neq b_\ell - p_\ell \), then it is easy to see that \( T(v, p) \cap S = \emptyset \), because \( p_i > a_i \) and \( b_i - p_i = b_\ell - p_\ell \) for all \( i \in S \). It follows that \( T(v, p') = T(v, p) \) and \( R(v, p') = R(v, p) \).

The lemma follows by combining all three cases. \( \square \)

As suggested by Lemma 4.3, for each \( k \in [n] \), we use \( P_k' \) to denote the set of \( p \in P_k \) such that \( p_k = a_k \) and \( p_i \in \{b_i, b_i - t_k\} \) for all other \( i \). In particular, \( p_i \) must be \( b_i \) if \( t_i < t_k \) (\( t_i \neq t_k \), by the non-degeneracy assumption). The next lemma shows that we only need to consider \( p \in P_k' \) such that \( p_i = b_i \) for all \( i < k \).

**Lemma 4.4.** If \( p \in P_k' \) satisfies \( p_\ell = b_\ell - t_k > a_\ell \) for some \( \ell < k \), then we have \( p \notin \text{OPT} \).

**Proof.** We construct \( p' \) from \( p \) as follows. Let \( S \) denote the set of all \( i < k \) such that \( p_i = b_i - t_k > a_i \). By our assumption, \( S \) is nonempty. Then set \( p'_i = p_i \) for all \( i \notin S \) and \( p'_i = p_i + \epsilon \) for all \( i \in S \), where \( \epsilon > 0 \) is sufficiently small. Similarly we show that \( R(p') > R(p) \) by considering the following cases:

1. If \( U(v, p) = t_k \) and \( T(v, p) \cap S \neq \emptyset \), we consider the following cases. If \( T(v, p) \subseteq S \), then

\[
T(v, p') = T(v, p) \quad \text{and} \quad R(v, p') = R(v, p) + \epsilon > R(v, p).
\]
Otherwise, there exists a \( j \geq k \) such that \( j \in T(v, p) \). This implies that \( R(v, p) \geq b_j - t_k \) is not obtained from any item in \( S \). As a result, \( T(v, p') = T(v, p) - S \) and \( R(v, p') = R(v, p) \).

2. If the case above does not happen, then we must have \( T(v, p) \cap S = \emptyset \) (this includes the case when \( T(v, p) = \emptyset \)). As a result, we have \( T(v, p') = T(v, p) \) and \( R(v, p') = R(v, p) \).

The lemma follows by combining the two cases. \( \square \)

Finally, we use \( P_k^* \) for each \( k \in [n] \) to denote the set of \( p \in P \) such that \( p_k = a_k; p_i = b_i \) for all \( i < k; p_i = b_i \), for all \( i > k \) such that \( t_i < t_k \); and \( p_i \in \{b_i, b_i - t_k\} \), for all other \( i > k \). However, \( P_k^* \) may still be exponentially large in general. Let \( T_k \) denote the set of \( i > k \) such that \( t_i > t_k \). Given \( p \in P_k^* \), our last lemma below implies that, if \( i \) is the smallest index in \( T_k \) such that \( p_i = b_i - t_k \), then \( p_j = b_j - t_k \) for all \( j \in T_k \) larger than \( i \); otherwise \( p \) is not optimal. In other words, \( p \) has to be monotone in setting \( p_j, j \in T_k \), to be \( b_j - t_k \); otherwise \( p \) is not optimal. As a result, there are only \( O(n^2) \) many price vectors that we need to check, and the best one among them is optimal. We use \( A \subseteq \cup_k P_k^* \) to denote this set of price vectors.

**Lemma 4.5.** Given \( k \in [n] \) and \( p \in P_k^* \), if there exist two indices \( c, d \) in \( T_k \) such that \( c < d \), \( p_c = b_c - t_k \) but \( p_d = b_d \), then we must have \( p \not\in OPT \).

**Proof.** We use \( t \) to denote \( t_k \) for convenience. Also we may assume, without loss of generality, that there is no index between \( c \) and \( d \) in \( T_k \); otherwise we can use it to replace either \( c \) or \( d \), depending on its price.

We define two vectors from \( p \). First, let \( p' \) denote the vector obtained from \( p \) by replacing \( p_c = b_c - t_k \) by \( p_d = b_d \). Let \( p^* \) denote the vector obtained from \( p \) by replacing \( p_c = b_c - t \) by \( p_c^* = b_c \). In other words, the \( c \)th and \( d \)th entries of \( p, p', p^* \) are \((b_c - t, b_d), (b_c - t, b_d - t), (b_c, b_d)\), respectively, while all other \( n - 2 \) entries are the same. Our plan is to show that if \( R(p) \geq R(p') \), then \( R(p^*) > R(p) \). This implies that \( p \) cannot be optimal and the lemma follows.

We need some notation. Let \( V' \) denote the projection of \( V \) onto all but the \( c \)th and \( d \)th coordinates:

\[
V' = \times_{i \in [n] - \{c, d\}} V_i.
\]

We use \([n] - \{c, d\}\) to index entries of vectors \( u \in V' \). Let \( U \subseteq V' \) denote the set of vectors \( u \in V' \) such that \( u_i - p_i < t \) for all \( i > d \). (This just means that for each \( i \in T_k \), if \( i > d \) and \( p_i = b_i - t \), then \( u_i = a_i \).) Given \( u \in V' \), \( v_c \in \{a_c, b_c\} \) and \( v_d \in \{a_d, b_d\} \), we use \((u, v_c, v_d)\) to denote a \( n \)-dimensional price vector in \( V \). Now we compare the expected revenue \( R(p), R(p') \) and \( R(p^*) \).

First, we claim that, if \( v = (u, v_c, v_d) \in V \) but \( u \not\in U \), then we have \( R(v, p) = R(v, p') = R(v, p^*) \). This is simply because there exists an item \( i > d \) such that \( v_i - p_i = t \), so it always dominates both items \( c \) and \( d \). As a result, the difference among \( p, p' \) and \( p^* \) no longer matters. Second, it is easy to show that for any \( v = (u, a_c, a_d) \in V \), then \( R(v, p) = R(v, p') = R(v, p^*) \) as the utility from \( c \) and \( d \) are negative.

Now we consider a vector \( v = (u, v_c, v_d) \in V \) such that \( u \in U \) and \((v_c, v_d)\) is either \((a_c, b_d), (b_c, a_d)\), or \((b_c, b_d)\). For convenience, for each \( u \in U \) we use \( u_2^+ \) to denote \((u, a_c, b_d); u_2^- \) to denote \((u, b_c, a_d); \) and \( u_3^+ \) to denote \((u, b_c, b_d)\). By the definition of \( U \), we have the following simple cases:

1. For \( p \), we have \( R(u_2^+, p) = b_c - t \) and \( R(u_3^+, p) = b_c - t \);
2. For \( p' \), we have \( R(u_1^+, p') = b_d - t \), \( R(u_2^+, p') = b_c - t \) and \( R(u_3^+, p') = b_d - t \).

We need the following equation:

\[
R(u_1^+, p) = R(u_1^+, p^*) = R(u_3^+, p^*) \tag{4}
\]
as well as the following two inequalities:

\[ R(u_i^+, p^*) - (b_d - b_c) \leq R(u_2^+, p^*) \leq R(u_1^+, p^*) \]  \hspace{1cm} (5)

Given a \( v \in V \), recall that \( \Pr[v] \) denotes the probability of the valuation vector being \( v \). Given a \( u \in U \), we also use \( \Pr[u] \) to denote the probability of the \( n - 2 \) items, except items \( c \) and \( d \), taking values \( u \). Let

\[ h_1 = (1 - q_c)q_d, \quad h_2 = q_c(1 - q_d) \quad \text{and} \quad h_3 = q_cq_d. \]

Clearly we have \( h_1, h_2, h_3 > 0 \) and \( \Pr[u_i^+] = \Pr[u] \cdot h_i \), for all \( u \in U \) and \( i \in [3] \).

In order to compare \( R(p) \), \( R(p^*) \) and \( R(p^*) \), we only need to compare the following three sums:

\[
\sum_{i \in [3]} \sum_{u \in U} \Pr[u_i^+] \cdot R(u_i^+, p), \quad \sum_{i \in [3]} \sum_{u \in U} \Pr[u_i^+] \cdot R(u_i^+, p') \quad \text{and} \quad \sum_{i \in [3]} \sum_{u \in U} \Pr[u_i^+] \cdot R(u_i^+, p^*).
\]

For the first sum, we can rewrite it as (here all sums are over \( u \in U \)):

\[
h_1 \cdot \sum_u \Pr[u] \cdot R(u_1^+, p) + h_2 \cdot \sum_u \Pr[u] \cdot (b_c - t) + h_3 \cdot \sum_u \Pr[u] \cdot (b_c - t), \]

while the sum for \( R(p') \) is the following:

\[
h_1 \cdot \sum_u \Pr[u] \cdot (b_d - t) + h_2 \cdot \sum_u \Pr[u] \cdot (b_c - t) + h_3 \cdot \sum_u \Pr[u] \cdot (b_d - t). \]  \hspace{1cm} (7)

Since \( c < d \) and \( b_c < b_d \), \( R(p) \geq R(p') \) would imply that

\[
\sum_u \Pr[u] \cdot R(u_1^+, p) > \sum_u \Pr[u] \cdot (b_d - t). \]  \hspace{1cm} (8)

On the other hand, we can also rewrite the sum for \( R(p^*) \) as

\[
h_1 \cdot \sum_u \Pr[u] \cdot R(u_1^+, p^*) + h_2 \cdot \sum_u \Pr[u] \cdot R(u_2^+, p^*) + h_3 \cdot \sum_u \Pr[u] \cdot R(u_3^+, p^*). \]  \hspace{1cm} (9)

The first sum in (9) is the same as that of (5). For the second sum, from (5), (4) and (8) we have

\[
\sum_u \Pr[u] \cdot R(u_2^+, p^*) \geq \sum_u \Pr[u] \cdot \left( R(u_1^+, p) - (b_d - b_c) \right) > \sum_u \Pr[u] \cdot \left( b_c - t - (b_d - b_c) \right) = \sum_u \Pr[u] \cdot (b_c - t).
\]

The third sum in (9) is also strictly larger than that of (6) as \( R(u_3^+, p^*) = R(u_1^+, p^*) \geq R(u_2^+, p^*) \) while the second and third sums in (6) are the same, ignoring \( h_2 \) and \( h_3 \). Thus, \( R(p^*) > R(p) \).

\[ \square \]

### 4.2 General Case

Now we deal with the general case. Let \( I \) denote an input instance with \( n \) items, in which \( \left| V_i \right| \leq 2 \) for all \( i \).

For each \( i \in [n] \), either \( V_i = \{a_i, b_i\} \) where \( b_i > a_i \geq 0 \), or \( V_i = \{b_i\} \), where \( b_i \geq 0 \). We let \( D \subseteq [n] \) denote the set of \( i \in [n] \) such that \( \left| V_i \right| = 2 \). For each item \( i \in D \), we use \( q_i : 0 < q_i < 1 \) to denote the probability of
its value being $b_i$. Each item $i \notin D$ has value $b_i$ with probability 1. As permuting the items does not affect the maximum expected revenue, we may assume without loss of generality that $b_1 \leq b_2 \leq \cdots \leq b_n$.

The idea is to perturb $I$ (symbolically), so that the new instances satisfy all conditions described at the beginning of the section, which we know how to solve efficiently. For this purpose, we define a new $n$-item instance $I_\epsilon$ from $I$ for any $\epsilon > 0$: For each $i \in D$, the support of item $i$ is $V_{i,\epsilon} = \{a_i + \epsilon i, b_i + 2\epsilon i\}$, and for each $i \notin D$, the support of item $i$ is $V_{i,\epsilon} = \{b_i + \epsilon i, b_i + 2\epsilon i\}$. For each $i \in D$, the probability of the value being $b_i + 2\epsilon i$ is still set to be $q_i$, while for each $i \notin D$, the probability of the value being $b_i + 2\epsilon i$ is set to be $1/2$.

In the rest of the section, we use $R(p)$ and $R(v, p)$ to denote the revenue with respect to $I$, and use $R_\epsilon(p)$ and $R_\epsilon(v, p)$ to denote the revenue with respect to $I_\epsilon$. Let $V_\epsilon = \times_{i=1}^n V_{i,\epsilon}$. Let $\rho$ denote the following map from $V_\epsilon$ to $V$: $\rho$ maps $u \in V_\epsilon$ to $v \in V$, where

1) $v_i = b_i$ when $i \notin D$;
2) $v_i = a_i$ if $u_i = a_i + \epsilon i$ and $v_i = b_i$ if $u_i = b_i + 2\epsilon i$ when $i \in D$.

It is easy to verify that, when $\epsilon > 0$ is sufficiently small, the new instance $I_\epsilon$ satisfies all conditions given at the beginning of the section, including the non-degeneracy assumption. Moreover, we show that

**Lemma 4.6.** The limit of $\max_p R_\epsilon(p)$ exists as $\epsilon \to 0$, and can be computed in polynomial time.

**Proof.** Since $I_\epsilon$ satisfies all the conditions, we know there is a set of $O(n^2)$ price vectors, denote by $A_\epsilon$ for $I_\epsilon$, such that the best vector in $A_\epsilon$ is optimal for $I_\epsilon$ and achieves $\max_p R_\epsilon(p)$.

Furthermore, from the construction of $A_\epsilon$, we know that every vector $p_\epsilon$ in $A_\epsilon$ has an explicit expression in $\epsilon$: each entry of $p_\epsilon$ is indeed an affine linear function of $\epsilon$. As a result, the limit of $R_\epsilon(p_\epsilon)$ as $\epsilon$ approaches 0 exists and can be computed efficiently. Since $\lim_{\epsilon \to 0} (\max_p R_\epsilon(p))$ is just the maximum of these $O(n^2)$ limits, it also exists and can be computed in polynomial time in the input size of $I$.

Finally, the next two lemmas show that this limit is exactly the maximum expected revenue of $I$.

**Lemma 4.7.** $\max_p R(p) \leq \lim_{\epsilon \to 0} (\max_p R_\epsilon(p))$.

**Proof.** Let $p^*$ denote an optimal price vector of $I$. It suffices to show that, when $\epsilon$ is sufficiently small,

$$\max_p R_\epsilon(p) \geq \mathcal{R}(p^*) - 4n^2\epsilon. \quad (10)$$

The proof is similar to that of Lemma 2.1. Let $p'$ denote the vector in which $p'_i = \max(0, p_i^* - 4r_i\epsilon)$, where $r_i$ is the rank of $p_i^*$ among $\{p_1^*, \ldots, p_n^*\}$ sorted in the increasing order (when there are ties, items with lower index are ranked higher). We claim that, when $\epsilon > 0$ is sufficiently small,

$$\mathcal{R}_\epsilon(u, p') \geq \mathcal{R}(\rho(u), p^*) - 4n^2\epsilon, \quad \text{for any } u \in V_\epsilon, \quad (11)$$

from which we get $R_\epsilon(p') \geq R(p^*) - 4n^2\epsilon$ and (10) follows.

To prove (11) we fix a $u \in V_\epsilon$ and let $v = \rho(u) \in V$. (11) holds trivially if $\mathcal{R}(v, p^*) = 0$. Assume that $\mathcal{R}(v, p^*) > 0$, and let $k$ denote the item selected in $I$ on $(v, p^*)$. (11) also holds trivially if $p_k^* < 4n^2\epsilon$, so without loss of generality, we assume that $p_k \geq 4n^2\epsilon$. For any other item $j \in [n]$, we compare the utilities of items $k$ and $j$ in $I_\epsilon$ on $(u, p')$. We claim that

$$u_k - p'_k > u_j - p'_j \quad (12)$$

because 1) if $v_k - p_k^* > v_j - p_j^*$, then (12) holds when $\epsilon$ is sufficiently small; 2) if $v_k - p_k^* = v_j - p_j^*$ and $p_k^* > p_j^*$, then (12) holds because $p_k^* - p_k' - (p_j^* - p_j') \geq 4n\epsilon > (v_k - u_k) + (u_j - v_j)$; 3) finally, the case
when \( v_k - p_k^* = v_j - p_j^* = p_k = p_j \) and \( k < j \) follows similarly from \( r_k > r_j \). Therefore, \( k \) remains to be the item being selected in \( I_t \) on \((u, p')\). (11) then follows from the fact that \( p_k^* \geq p_k^* - 4n^2 \epsilon \) by definition. 

\[ \text{Lemma 4.8.} \quad \max_p R(p) \geq \lim_{\epsilon \to 0} \left( \max_p R_\epsilon(p) \right). \]

**Proof.** From the proof of Lemma 4.6 there is a price vector \( p_\epsilon \in A_\epsilon \) in which every entry is an affine linear function of \( \epsilon \), such that (as the cardinality of \(|A_\epsilon|\) is bounded from above by \( O(n^2) \))

\[
\lim_{\epsilon \to 0} \left( \max_p R_\epsilon(p) \right) = \lim_{\epsilon \to 0} R_\epsilon(p_\epsilon).
\]

Let \( \tilde{p} \in \mathbb{R}_{+}^n \) denote the limit of \( p_\epsilon \), by simply removing all the \( \epsilon \)’s in the affine linear functions. Moreover, we note that \(|\tilde{p}_i - p_{\epsilon,i}| = O(n\epsilon)\) by the construction of \( A_\epsilon \), where we use \( p_{\epsilon,i} \) to denote the \( i \)th entry of \( p_\epsilon \).

Next, let \( q_\epsilon \) denote the vector in which the \( i \)th entry \( q_{\epsilon,i} = \max(0, \tilde{p}_i - r_i n^2 \epsilon) \) for all \( i \in [n] \), where \( r_i \) is the rank of \( \tilde{p}_i \) among entries of \( \tilde{p} \) sorted in increasing order (again, when there are ties, items with lower index are ranked higher). To prove the lemma, it suffices to show that, when \( \epsilon \) is sufficiently small,

\[ R(q_\epsilon) \geq R_\epsilon(p_\epsilon) - O(n^3 \epsilon). \]

To this end, we show that for any vector \( u \in V_\epsilon \) with \( v = \rho(u) \),

\[ R(v, q_\epsilon) \geq R_\epsilon(u, p_\epsilon) - O(n^3 \epsilon). \tag{13} \]

Finally we prove (13). First, we note that if \( U(v, \tilde{p}) < 0 \), then \( R(v, q_\epsilon) = R_\epsilon(u, p_\epsilon) = 0 \) when \( \epsilon > 0 \) is sufficiently small (as \( u \) approaches \( v \) and \( p_\epsilon, q_\epsilon \) approach \( \tilde{p} \)). Otherwise, we have \( U(v, q_\epsilon) > U(v, \tilde{p}) \geq 0 \) and we use \( k \) to denote the item selected in \( I_\epsilon \) on \((v, q_\epsilon)\). To violate (13), the item selected in \( I_\epsilon \) on \((u, p_\epsilon)\) must be an item \( \ell \) different from \( k \) satisfying \( \tilde{p}_\ell > \tilde{p}_k \). Below we show that this cannot happen. Consider all the cases: 1) if \( v_k - \tilde{p}_k < v_\ell - \tilde{p}_\ell \), we get a contradiction since item \( k \) is dominated by \( \ell \) in \( I_\epsilon \) on \((v, q_\epsilon)\) when \( \epsilon \) is sufficiently small; 2) if \( v_k - \tilde{p}_k > v_\ell - \tilde{p}_\ell \), we get a contradiction with \( \ell \) being selected in \( I_\epsilon \) on \((u, p_\epsilon)\) when \( \epsilon \) is sufficiently small; 3) if \( v_k - \tilde{p}_k = v_\ell - \tilde{p}_\ell \) and \( \tilde{p}_\ell > \tilde{p}_k \), we conclude that \( v_k - q_{\epsilon,k} < v_\ell - q_{\epsilon,\ell} \), contradicting again with \( k \) being selected in \( I_\epsilon \) on \((v, q_\epsilon)\). (13) follows by combining all these cases. \( \square \)

## 5 NP-hardness for support size 3

In this section, we give a polynomial-time reduction from PARTITION to ITEM-PRICING for distributions with support (at most) 3. Recall that in the PARTITION problem [GJ79] we are given a set \( C = \{c_1, \ldots, c_n\} \) of \( n \) positive integers and wish to determine whether it is possible to partition \( C \) into two subsets with equal sum. We may assume without loss of generality that \( c_1 = \max(c_1, \ldots, c_n) \).

Given an instance of PARTITION, we construct an instance of ITEM-PRICING as follows. We have \( n \) items. Each item \( i \in [n] \) can take \( 3 \) possible integer values \( 0, a, b \), where \( b > a > 0 \), i.e., \( V_i = \{0, a, b\} \) for all \( i \in [n] \). Let \( q_i = \Pr[v_i = b] \) and \( r_i = \Pr[v_i = a] \). We set \( q_i = c_i/M \) where \( M = 2^n c_1^3 \) and

\[ r_i = \frac{b - a}{a(1 - t_i)} \cdot q_i, \quad \text{where} \quad t_i = \frac{b}{2a} \cdot \sum_{j \neq i, j \in [n]} q_j. \]

The two parameters \( a \) and \( b \) should be thought of as universal constants (independent of the given instance of PARTITION) throughout the proof. We will eventually set these constants to be \( a = 1, b = 3 \) (this choice
is not necessary, there is flexibility in our proof and indeed any values with \( b > 2a \) will work). However, for the sake of the presentation, we will keep \( a, b \) as generic parameters for most of the calculations till the end.

Note that the definition of \( r_i \) implies that

\[
 bq_i = a(q_i + r_i) - ar_it_i. \tag{14}
\]

Let \( N = 2^n c_n^2 \). Then we have \( q_i, r_i = O(1/N) \) and \( t_i = O(n/N) \) for all \( i \). Thus, each distribution assigns most of its probability mass to the point 0. This is a crucial property which allows us to get a handle on the optimal revenue. For an arbitrary general instance of the pricing problem, the expected revenue is a highly complex nonlinear function. The fact that most of the probability mass in our construction is concentrated at 0 implies that valuation vectors with many nonzero entries contribute very little to the expected revenue. As we will argue, the revenue is approximated well by its 1st and 2nd order terms with respect to \( 0 \) implies that valuation vectors with many nonzero entries contribute very little to the expected revenue. Thus, each distribution assigns most of its probability mass to the point 0. This is a crucial property which allows us to get a handle on the optimal revenue. For an arbitrary general instance of the pricing problem, the expected revenue is a highly complex nonlinear function. The fact that most of the probability mass in our construction is concentrated at 0 implies that valuation vectors with many nonzero entries contribute very little to the expected revenue. As we will argue, the revenue is approximated well by its 1st and 2nd order terms with respect to \( 0 \).

Our main claim is that, for an appropriate value \( t^* \), there exists a price vector with expected revenue at least \( t^* \) if and only if there exists a solution to the original instance of the Partition problem.

Before we proceed with the proof, we will need some notation. For \( \epsilon > 0 \), we write \( T_1 = T_2 = \epsilon \) to denote that \( |T_1 - T_2| \leq \epsilon \).

Note that, as both the \( q_i \)'s and the \( t_i \)'s are very small positive quantities, we have that \( r_i \approx (b - a)q_i/a \). Formally, with the above notation we can write

\[
 r_i = \frac{b - a}{a(1 - t_i)} \cdot q_i = \frac{b - a}{a} \cdot q_i + \frac{b - a}{a} \cdot q_i = \frac{b - a}{a} \cdot O(n/N^2). \tag{15}
\]

Lemma 2.2 and Corollary 3.1 imply that a revenue maximizing price vector can be assumed to have nonnegative integer coefficients of magnitude at most \( b \). The following lemma establishes the stronger statement that, for our particular instance, an optimal price vector \( p \) can be assumed to have each \( p_i \) in the set \( \{a, b\} \).

**Lemma 5.1.** There is an optimal price vector \( p \in \{a, b\}^n \).

**Proof.** By Lemma 2.2 and Corollary 3.1, there is an optimal price vector with integer coordinates in \([0 : b]\). Let \( p \) be any (integer) vector in \([0 : b]^n\) that has at least one coordinate \( p_j \not\in \{a, b\} \). We will show below that \( \mathcal{R}(p) < \mathcal{R}(b) \), where \( b \) denotes the all-\( b \) vector, and hence \( p \) is not optimal.

Consider an index \( i \in [n] \) with \( p_i > 0 \). The probability the buyer selects item \( i \) is bounded from above by \( \Pr[v_i \geq p_i] \), the probability that item \( i \) has value at least \( p_i \), and is bounded from below by

\[
 \Pr[v_i \geq p_i] \cdot \prod_{j \neq i, j \in [n]} (1 - q_j - r_j) \geq \Pr[v_i \geq p_i] \cdot (1 - O(n/N)).
\]

Note that the second term in the LHS above is the probability that all items other than \( i \) have value 0 and the inequality uses the fact that \( q_i, r_i = O(1/N) \). Applying these two bounds on \( p \) and \( b \) we obtain

\[
 \mathcal{R}(b) \geq \sum_{i \in [n]} q_i (1 - O(n/N)) \cdot b \quad \text{and} \quad \mathcal{R}(p) \leq \sum_{i : p_i > 0} \Pr[v_i \geq p_i] \cdot p_i.
\]

So \( \mathcal{R}(b) \geq (\sum_{i \in [n]} q_i) \cdot b - O(n^2/N^2) \). Regarding \( \mathcal{R}(p) \), we consider the following three cases. For \( i \in [n] \)
with $p_i = b$, the probability that $v_i \geq p_i$ is $q_i$ and the contribution of such an item to the second sum is $q_i b$.
Similarly, for $i \in [n]$ with $p_i = a$, the probability that $v_i \geq p_i$ is $q_i + r_i$ and the contribution to the sum is

$$(q_i + r_i)a \leq q_i b + O(n/N^2),$$

where the inequality follows from (15). Finally, we consider an item $i \in [n]$ with $p_i \notin \{a, b\}$. If $a < p_i < b$ then the contribution is $q_i p_i$, which is at most $q_i(b - 1) = q_i b - q_i$, since $p_i$ is integer. If $p_i < a$, then the contribution is $(q_i + r_i)p_i$, which is at most $(q_i + r_i)(a - 1) = q_i b + ar_i t_i - q_i - r_i = q_i b - q_i - r_i(1 - at_i)$. In both cases, the contribution to the sum is at most

$$q_i b - q_i \leq q_i b - (1/M).$$

Note that the definition of $M$ and $N$ implies that $1/M \gg n^2/N^2$. Because there exists at least one $j$ with $p_j \notin \{a, b\}$, it follows that $R(p) < R(b)$ which completes the proof of the lemma.

As a result, to maximize the expected revenue it suffices to consider price vectors in $\{a, b\}^n$. Given any price-vector $p \in \{a, b\}^n$, we let $S = S(p) = \{i \in [n] : p_i = a\}$ and $T = T(p) = \{i \in [n] : p_i = b\}$. The main idea of the proof is to establish an appropriate quadratic form approximation to the expected revenue $R(p)$ that is sufficiently accurate for the purposes of our reduction.

**Approximating the Revenue.** We appropriately partition the valuation space $V$ into three events that yield positive revenue. We then approximate the probability of each and its contribution to the expected revenue up to, and including, 2nd order terms, i.e., terms of order $O(\text{poly}(n)/N^2)$, and we ignore 3rd order terms, i.e., terms of order $O(\epsilon)$ where $\epsilon = n^3/N^3$.

In particular, we consider the following disjoint events:

- **First Event:** $E_1 = \{v \in V \mid \exists i \in S : v_i = b\}$.
  Note that for any $v \in E_1$ we have $R(v, p) = a$. The probability of this event is
  \[\Pr[E_1] = 1 - \prod_{i \in S} (1 - q_i) = \sum_{i \in S} q_i - \sum_{i \neq j \in S} q_i q_j \pm O(\epsilon).\]

- **Second Event:** $E_2 = E_1 \cap \{v \in V \mid \exists i \in S : v_i = a \text{ and } \forall i \in T : v_i \in \{0, a\}\}$.
  Note that for any $v \in E_2$ we have $R(v, p) = a$. The probability of this event is
  \[\Pr[E_2] = \prod_{j \in T} (1 - q_j) \left[\prod_{i \in S} (1 - q_i) - \prod_{i \in S} (1 - q_i - r_i)\right]\]

Using the elementary identities

\[
\prod_{j \in T} (1 - q_j) = 1 - \sum_{j \in T} q_j + \sum_{i \neq j \in T} q_i q_j \pm O(\epsilon)
\]
\[
\prod_{i \in S} (1 - q_i) = 1 - \sum_{i \in S} q_i + \sum_{i \neq j \in S} q_i q_j \pm O(\epsilon)
\]
\[
\prod_{i \in S} (1 - q_i - r_i) = 1 - \sum_{i \in S} (q_i + r_i) + \sum_{i \neq j \in S} (q_i + r_i)(q_j + r_j) \pm O(\epsilon),
\]

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we can write

\[
\Pr[E_2] = \left[1 - \sum_{j \in T} q_j + \sum_{i \neq j \in T} q_i q_j \pm O(\varepsilon)\right] \cdot \left[\sum_{i \in S} r_i + \sum_{i \neq j \in S} q_i q_j - \sum_{i \neq j \in S} (q_i + r_i)(q_j + r_j) \pm O(\varepsilon)\right]
\]

\[
= \sum_{i \in S} r_i - \sum_{i \in S} r_i \sum_{j \in T} q_j + \sum_{i \neq j \in S} q_i q_j - \sum_{i \neq j \in S} (q_i + r_i)(q_j + r_j) \pm O(\varepsilon).
\]

- Third Event: \(E_3 = E_1 \cap \{v \in V \mid \exists i \in T : v_i = b\}\).

Note that for any \(v \in E_3\) we have \(\mathcal{R}(v, p) = b\). The probability of this event is

\[
\Pr[E_3] = \prod_{i \in S} (1 - q_i) \left[1 - \prod_{j \in T} (1 - q_j)\right]
\]

\[
= \left(1 - \sum_{i \in S} q_i + \sum_{i \neq j \in T} q_i q_j \pm O(\varepsilon)\right) \left(\sum_{j \in T} q_j - \sum_{i \neq j \in T} q_i q_j \pm O(\varepsilon)\right)
\]

\[
= \sum_{j \in T} q_j - \sum_{i \neq j \in T} q_i q_j - \sum_{i \in S} q_i \sum_{j \in T} q_j \pm O(\varepsilon).
\]

Therefore, for the expected revenue \(\mathcal{R}(p)\) we have:

\[
\mathcal{R}(p) = \left(\Pr[E_1] + \Pr[E_2]\right) \cdot a + \Pr[E_3] \cdot b
\]

\[
= a \cdot \left(\sum_{i \in S} (q_i + r_i) - \sum_{i \neq j \in S} (q_i + r_i)(q_j + r_j) - \sum_{i \in S} r_i \sum_{j \in T} q_j\right)
\]

\[
+ b \cdot \left(\sum_{j \in T} q_j - \sum_{i \neq j \in T} q_i q_j - \sum_{i \in S} q_i \sum_{j \in T} q_j\right) \pm O(\varepsilon).
\]

Using (14) it follows that the first order term of the revenue is

\[
b \sum_{j \in T} q_j + a \sum_{i \in S} (q_i + r_i) = b \sum_{j \in [n]} q_j + a (\sum_{i \in [n]} (a(q_i + r_i) - bq_i) = b \sum_{j \in [n]} q_j + \sum_{i \in S} (ar_i t_i).
\]

Observe that the first term \(b \sum_{j \in [n]} q_j\) in the above expression is a constant \(L_1\), independent of the pricing (i.e., the partition of the items into \(S\) and \(T\)).
In the second order term, we can rewrite the expression \( a \sum_{i \neq j \in S} (q_i + r_i)(q_j + r_j) \) as

\[
\frac{1}{2} \cdot \sum_{i \in S} (q_i + r_i) \sum_{j \in S, j \neq i} a(q_j + r_j)
\]

\[
= \frac{1}{2} \cdot \sum_{i \in S} (q_i + r_i) \sum_{j \in S, j \neq i} (bq_j + ar_j t_j)
\]

\[
= \frac{b}{2} \cdot \sum_{i \in S} q_i \sum_{j \in S, j \neq i} q_j + \frac{b}{2} \sum_{i \in S} r_i \sum_{j \in S, j \neq i} q_j + \frac{1}{2} \cdot \sum_{i \in S} (q_i + r_i) \sum_{j \in S, j \neq i} ar_j t_j
\]

\[
= b \sum_{i \neq j \in S} q_i q_j + \frac{b}{2} \sum_{i \in S} r_i \sum_{j \in S, j \neq i} q_j \pm O(\epsilon)
\]

where in the first expression above, the double summation is multiplied by \(1/2\) because each unordered pair \(i \neq j \in S\) is included twice. Thus, the second order term of the expected revenue \( \mathcal{R}(p) \) is

\[
- a \sum_{i \neq j \in S} (q_i + r_i)(q_j + r_j) - a \sum_{i \in S} r_i \sum_{j \in T} q_j - b \sum_{i \in S} q_i q_j - b \sum_{i \in S} q_i \sum_{j \in T} q_j
\]

\[
= -b \sum_{i \neq j \in S} q_i q_j - b \sum_{i \in S} r_i \sum_{j \in S, j \neq i} q_j - a \sum_{i \in S} r_i \sum_{j \in T} q_j - b \sum_{i \in S} q_i q_j - b \sum_{i \in S} q_i \sum_{j \in T} q_j \pm O(\epsilon)
\]

\[
= -b \sum_{i \neq j [n]} q_i q_j - b \sum_{i \in S} r_i \sum_{j \in S, j \neq i} q_j - a \sum_{i \in S} r_i \sum_{j \in T} q_j \pm O(\epsilon)
\]

The first term in the last expression is a constant \(L_2\) independent of the pricing. As a result, we can rewrite the second order term as follows:

\[
L_2 - \frac{b}{2} \sum_{i \in S} r_i \sum_{j \in S, j \neq i} q_j - a \sum_{i \in S} r_i \sum_{j \in T} q_j \pm O(\epsilon) = L_2 - \sum_{i \in S} r_i \left( b \sum_{j \in S, j \neq i} q_j + a \sum_{j \in T} q_j \right) \pm O(\epsilon).
\]

Summing with the first order term and letting \(L = L_1 + L_2\), we have:

\[
\mathcal{R}(p) = L + \sum_{i \in S} r_i \left( at_i - \frac{b}{2} \sum_{j \in S, j \neq i} q_j - a \sum_{j \in T} q_j \right) \pm O(\epsilon)
\]

\[
= L + \sum_{i \in S} r_i \left( \frac{b}{2} \sum_{j \neq i} q_j - \frac{b}{2} \sum_{j \in S, j \neq i} q_j - a \sum_{j \in T} q_j \right) \pm O(\epsilon)
\]

\[
= L + \sum_{i \in S} r_i \left( \frac{b}{2} - a \right) \sum_{j \in T} q_j \pm O(\epsilon)
\]

\[
= L + \frac{b - a}{a} \cdot \left( \frac{b}{2} - a \right) \cdot \frac{1}{M^2} \sum_{i \in S} c_i \sum_{j \in T} c_j \pm O(\epsilon).
\]
Now setting \( a = 1, b = 3 \) in the previous expression, we have that for any \( p \in \{a, b\}^n \),

\[
R(p) = L + \frac{1}{M^2} \left( \sum_{i \in S} c_i \right) \cdot \left( \sum_{j \in T} c_j \right) \pm O(\epsilon).
\] (16)

At this point, we observe that the sum of the two factors \( \sum_{i \in S} c_i, \sum_{j \in T} c_j \) in (16) is a constant (independent of the partition). Thus, their product is maximized when they are equal. Because \( \epsilon = o(1/M^2) \), it follows that the revenue is maximized when the product of the two factors is maximized. In particular, if there exists a partition of the set \( C = \{c_1, \ldots, c_n\} \) into two sets with equal sums \( H = (\sum_{i \in [n]} c_i)/2 \), then the corresponding partition of the indices into the sets \( S \) and \( T \) yields revenue \( L + \frac{1}{M^2} \cdot H^2 \pm O(\epsilon) \). On the other hand, if there is no such equipartition of the set \( C \), then for any partition of the indices, the revenue will be at most \( L + \frac{1}{M^2} (H + 1)(H - 1) \pm O(\epsilon) = L + \frac{1}{M^2} (H^2 - 1) \pm O(\epsilon) \). Since \( \epsilon = o(1/M^2) \) it follows that there exists a partition of the set \( C = \{c_1, \ldots, c_n\} \) into two sets with equal sums if and only if there exists a price vector \( p \in \{a, b\}^n \) with \( R(p) \geq t^* = L + \frac{1}{M^2} (H^2 - \frac{1}{2}) \). This completes the proof.

**Remark.** In the above construction, the support \( \{0, a, b\} \) of the distributions includes the value 0 (which in fact has most of the probability mass). It is easy to modify the construction, if desired, so that the support contains only positive values: shift all the values of the distributions up by 1 (thus, the supports now become \( V_i = \{1, 2, 4\} \)) and add an additional \((n+1)\)-th item which has value 1 with probability 1. This transformation increases the expected revenue by 1. It is easy to see that an optimal price vector \( p' \) for the new instance will give price \( p'_{n+1} = 1 \) to the \((n+1)\)-th item and price \( p'_i = p_i + 1 \) to each other item \( i \in [n] \), where \( p \) is an optimal vector for the original instance.

### 6 NP-hardness for identical distributions

In this section we show that ITEM-PRICING is NP-hard even for identical distributions. For this purpose we reduce from the following (still NP-complete) version of Integer Knapsack.

**Definition 6.1 (Integer Knapsack with repetitions).**

**Input:** \( n + 1 \) positive integers \( a_1 < \cdots < a_n \) and \( L \).

**Problem:** Do there exist nonnegative integers \( x_1, \ldots, x_n \) such that \( \sum_{i \in [n]} x_i = n \) and \( \sum_{i \in [n]} x_i a_i = L \)?

The NP-hardness of this version of Integer Knapsack is likely known in the literature, but for completeness we include below a quick proof via a reduction from Subset-Sum.

**Lemma 6.1.** Integer Knapsack with repetitions is NP-hard.

**Proof.** Let \( b_1 < \cdots < b_n \) and \( T \) denote an instance of Subset-Sum, where \( b_i \) and \( T \) are all positive integers. Without loss of generality, we assume that \( T > b_n \). Let \( K = n^2 T \). For each \( i \in [n] \), set \( a_i = K^i + b_i \) and \( c_i = K^i \). Then one can show that \( \{K^{n+1}, a_i, c_i : i \in [n]\} \), a set of \( 2n + 1 \) positive integers, together with

\[
L = T + K + K^2 + \cdots + K^n + (n + 1)K^{n+1}
\]

form a yes-instance of the special Integer Knapsack problem iff a subset of \( \{b_1, \ldots, b_n\} \) sums to \( T \). \( \square \)
6.1 Reduction

Let \( a_1 < \cdots < a_n \) and \( L \) denote an instance of INTEGER KNAPSACK WITH REPETITIONS. Without loss of generality, we assume that \( L \leq na_n \); otherwise the problem is trivial. Our goal is to construct a distribution \( Q \) over nonnegative integers, and reduce the Integer Knapsack problem to the problem ITEM-PRICING with \( n \) items, each of which has its value drawn from \( Q \). We require \( Q \) to be such that the optimization of the expected revenue amounts to a quadratic optimization problem that mimics the Integer Knapsack problem with repetitions.

Let \( m = \max(n^5, a_n) \), and let \( N = m^{n^2} \) denote a large integer. For each \( i \in \{1, \ldots, n\} \), let \( v_i = m^{n+i} \). For each \( i \in \{1, \ldots, n-1\} \), let

\[
\gamma_i = \frac{1}{N} \left( \frac{1}{m^{n+i}} - \frac{1}{m^{n+i+1}} \right) = \frac{m-1}{Nm^{n+i+1}}.
\]

Let \( \gamma_n = 1/(Nm^{2n}) \). For convenience, we also let \( \Gamma_i = \sum_{j=1}^{n} \gamma_j = 1/(Nm^{n+i}) \) for each \( i \in \{n\} \).

We record a property that follows directly from our choices of \( v_i \) and \( \gamma_i \).

**Property 6.1.** For each \( i \in \{n\} \), we have \( v_i \Gamma_i = 1/N \).

Let now \( q_1, \ldots, q_n \) denote \( n \) probability distributions. They are closely related to the instance of Integer Knapsack and will be specified later in this section. The support of each \( q_i \) is a subset of \([2n^3] \) and for each \( j \in [2n^3] \), we use \( q_i(j) \) to denote the probability of \( j \) in \( q_i \). Finally, let \( t_1, \ldots, t_n \) denote a sequence of (not necessarily positive) numbers, also to be specified later, with \( |t_i| = O(1/N^2) \) for all \( i \in \{n\} \).

We are ready to define \( Q \) using \( v_i, \gamma_i, t_i \) and \( q_i \). First, the support of \( Q \) is

\[
\left\{ 0, v_i, v_i + j : i \in \{n\} \text{ and } j \in [2n^3] \right\}.
\]

Note that all values in the support are bounded by \( O(n^2) \), and the size of the support is \( O(n^4) \).

Next, \( Q \) has probability \( (\gamma_i/m) + t_i \) at \( v_i \) for each \( i \in \{n\} \); probability \( q_i(j) \cdot \gamma_i (m-1)/m \) at \( v_i + j \) for each \( i \in \{n\} \) and \( j \in [2n^3] \); and probability \( 1 - (\sum_{i=1}^{n} \gamma_i + t_i) \) at 0. It is easy to verify that \( Q \) is a probability distribution since the probabilities sum to 1.

For convenience, we also let \( T_i = \sum_{j=i}^{n} t_j \), and \( r_i = \sum_{j=i}^{n} (\gamma_j + t_j) = \Gamma_i + T_i \), for each \( i \in \{n\} \). The latter quantity, \( r_i \), is the probability that the value is at least \( v_i \).

Even though \( t_i \) and \( q_i \) have not been specified yet, we still can prove the following useful lemma about optimal price vectors, as long as \( |t_i| = O(1/N^2) \) for each \( i \in \{n\} \):

**Lemma 6.2.** There is an optimal price vector \( p \in \{v_1, \ldots, v_n\}^n \).

**Proof.** By Lemma 2.2 and Corollary 3.1, there must be an (integral) optimal price vector in \([0 : v_n + 2n^3]^n\).

Let \( p = (p_1, \ldots, p_n) \in [0 : v_n + 2n^3]^n \) be a price vector with \( p \notin \{v_1, \ldots, v_n\}^n \). We will prove below that \( R(p) < R(b) \), where \( b \) is the vector in which all entries are \( v_n \). The lemma then follows.

For convenience, we use \( F(s) \) to denote the probability of a random variable drawn from the distribution \( Q \) being at least \( s \). For each index \( i \in \{n\} \) such that \( p_i > 0 \), the probability that the buyer picks item \( i \) can be
bounded from above by $F(p_i)$, and can be bounded from below by

$$F(p_i) \cdot (1 - r_1)^{n-1} \geq F(p_i) \cdot \left(1 - \frac{1}{m^{n+1}N} - O\left(\frac{n}{N^2}\right)\right) \geq F(p_i) - O\left(\frac{n}{m^{2n+2}N^2}\right),$$

where we used $r_1 = \Gamma_1 + T_1$, $\Gamma_1 = 1/(m^{n+1}N)$, $T_1 = O(n/N^2)$ and $F(p_i) \leq r_1 = O(1/(m^{n+1}N))$ if $p_i > 0$. Applying the upper bound on $\mathcal{R}(p)$ and the lower bound on $\mathcal{R}(b)$, we have

$$\mathcal{R}(p) \leq \sum_{i: p_i > 0} F(p_i) \cdot p_i \quad \text{and} \quad \mathcal{R}(b) \geq n v_n \left(F(v_n) - O\left(\frac{n}{m^{2n+2}N^2}\right)\right) \geq n v_n F(v_n) - O\left(\frac{n^2}{m^2 N^2}\right).$$

We now examine $p_i F(p_i)$ and $v_n F(v_n)$. We have three cases on $sF(s)$:

**Case 1:** $s = v_i$ for some $i \in [n]$. Then we have

$$sF(s) = v_i(\Gamma_i + T_i) = \frac{1}{N} + O\left(\frac{n m^{2n}}{N^2}\right).$$

**Case 2:** $s = v_i + j$ for some $i \in [n]$ and $j \in [2n^3]$. We then have $F(s) \leq r_i - (\gamma_i/m) - t_i$ and

$$sF(s) \leq (v_i + 2n^3) \left(r_i - \frac{\gamma_i}{m} - t_i\right) = \frac{1}{N} \cdot \frac{m^2 - m + 1}{m^2} + O\left(\frac{n^3}{m^{n+1}N}\right) = \frac{1}{N} - \Omega\left(\frac{1}{mN}\right)$$

when $i < n$, and similarly when $i = n$,

$$sF(s) \leq (v_n + 2n^3) \cdot \frac{\gamma_n (m - 1)}{m} = \frac{m - 1}{m} \cdot \frac{1}{N} + O\left(\frac{n^3}{m^{2n}N}\right) = \frac{1}{N} - \Omega\left(\frac{1}{mN}\right).$$

**Case 3:** Otherwise, let $i \in [n]$ denote the smallest index such that $s < v_i$. Then we have

$$sF(s) \leq (v_i - 1)r_i = v_i(\Gamma_i + T_i) - r_i = \frac{1}{N} - \Omega\left(\frac{1}{m^{2n}N}\right).$$

From Case 1, we have

$$\mathcal{R}(b) \geq \frac{n}{N} - O\left(\frac{n^2 m^{2n}}{N^2}\right).$$

Regarding $\mathcal{R}(p)$, combining all three cases, we have that

$$\mathcal{R}(p) \leq \frac{n}{N} - \Omega\left(\frac{1}{m^{2n}N}\right)$$

because there is at least one index $i \in [n]$ such that $p_i \notin \{v_1, \ldots, v_n\}$ by the assumption. As $N \gg n^2 m^{4n}$, we conclude that $\mathcal{R}(p) < \mathcal{R}(b)$. The lemma then follows.

### 6.2 Analysis of the Expected Revenue

Given a price vector $p \in \{v_1, \ldots, v_n\}^n$, we let $x_i$ denote the number of items priced at $v_i$. Then $\sum_i x_i = n$. We will only consider the contribution of two types of valuation vectors to the expected revenue $\mathcal{R}(p)$: those
with exactly one positive entry and those with exactly two positive entries. The following lemma shows that the total contribution from all other valuation vectors is of third order with respect to (roughly) $1/N$.

**Lemma 6.3.** The revenue from valuation vectors with at least three positive entries is $O(n^3/(m^{n+3}N^3))$.

**Proof.** The probability that a valuation vector has at least three positive entries can be bounded by

$$O(n^3r^3_1) = O\left(\frac{n^3}{m^{3n+3}N^3}\right).$$

Thus, the total contribution is at most $O(m^{2n}) \cdot O(n^3r^3_1)$, and the lemma follows.

Let $\epsilon = n^3/(m^{n+3}N^3)$ in the rest of the section.

Next we examine valuation vectors with exactly one positive entry. Their total contribution is

$$\sum_{i \in [n]} x_i v_i r_i (1 - r_1)^{n-1}.$$ 

Since $r_1 = O(1/(m^{n+1}N))$ is of first order, approximating the sum up to second order yields

$$\sum_{i \in [n]} x_i v_i r_i (1 - r_1)^{n-1} = \sum_{i \in [n]} x_i v_i r_i \left(1 - (n - 1)r_1 \pm O(n^2r^2_1)\right)$$

$$= \sum_{i \in [n]} x_i v_i r_i - (n - 1) \sum_{i \in [n]} x_i v_i r_i r_1 \pm O(\epsilon). \quad (17)$$

The contribution of valuation vectors with two positive entries is more involved. First, from those whose two positive entries are over items of the same price, the total contribution to $R(p)$ is

$$\sum_{i \in [n]} \frac{x_i(x_i - 1)}{2} \cdot v_i \left(r^2_i - (r_1 - r_i)^2\right)(1 - r_1)^{n-2}. \quad (18)$$

For each pair $i < j \in [n]$, we use $p(i, j) \in [0, 1]$ to denote the probability of $\alpha - v_i > \beta - v_j$, where $\alpha$ and $\beta$ are drawn independently from $Q$ conditioning on $\alpha \geq v_i$ and $\beta \geq v_j$. Using the $p(i, j)$'s, the contribution from value vectors whose two positive entries are over items of different prices is

$$\sum_{i < j \in [n]} x_i x_j \left(v_i r_i (r_1 - r_j) + v_j r_j (r_1 - r_i) + r_i r_j \left(v_i p(i, j) + v_j (1 - p(i, j))\right)\right)(1 - r_1)^{n-2}. \quad (19)$$

Approximating to the second order, (18) can be simplified to

$$\sum_{i \in [n]} \frac{x_i(x_i - 1)}{2} \cdot v_i (2r_i r_1 - r^2_i) (1 \pm O(nr_1)) = \sum_{i \in [n]} x_i(x_i - 1)v_i r_i r_1 - \sum_{i \in [n]} \frac{x_i(x_i - 1)}{2} \cdot v_i r^2_i \pm O(\epsilon) \quad (20)$$

and (19) can be simplified similarly to

$$\sum_{i < j \in [n]} x_i x_j \left(v_i r_i (r_1 - r_j) + v_j r_j (r_1 - r_i) + r_i r_j \left(v_i p(i, j) + v_j (1 - p(i, j))\right)\right) \pm O(\epsilon). \quad (21)$$

Next we show that, for each $i \in [n]$, all terms of $v_i r_i r_1$ in (17), (20) and (21) cancel each other. This is
because the overall coefficient of $v_i r_i r_j$ is
\[-(n-1)x_i + x_i(x_i-1) + \sum_{j:j \neq i} x_i x_j = -(n-1)x_i + nx_i - x_i = 0,\]
where the first equality uses the fact that $\sum_{j \in [n]} x_j = n$. This allows us to further simplify the sum of (17), (20) and (21), with an error of $O(\epsilon)$, to
\[
\sum_{i \in [n]} x_i v_i r_i - \sum_{i \in [n]} \frac{x_i(x_i - 1)}{2} \cdot v_i r_i^2 - \sum_{i < j \in [n]} x_i x_j r_i r_j (v_i + v_j) + \sum_{i < j \in [n]} x_i x_j r_i r_j (v_i p(i,j) + v_j (1-p(i,j)))
\]
\[
= \sum_{i \in [n]} x_i v_i T_i - \sum_{i \in [n]} \frac{x_i r_i (x_i - 1)}{2N} - \sum_{i < j \in [n]} \frac{x_i x_j (r_i + r_j)}{N} + \sum_{i < j \in [n]} \frac{x_i x_j}{N} \cdot (r_j p(i,j) + r_i (1-p(i,j))) + O(\epsilon).
\]
Note that, by Lemma 6.3, this is also an approximation of $\mathcal{R}(p)$, with an error of $O(\epsilon)$.

Let $\epsilon' = n^3 n^{n-1} / N^3$. By plugging in $v_i r_i = v_i (\Gamma_i + T_i) = (1/N) + v_i T_i$ (note that $T_i = O(n/N^2)$ is of second order), (22) can be further simplified to the following:
\[
\frac{n}{N} + \sum_{i \in [n]} x_i v_i T_i - \sum_{i \in [n]} \frac{x_i r_i (x_i - 1)}{2N} - \sum_{i < j \in [n]} \frac{x_i x_j (r_i + r_j)}{2N} + \sum_{i < j \in [n]} \frac{x_i x_j}{N} \cdot (1/2 - p(i,j)) (r_i - r_j) + O(\epsilon').
\]
Extracting $x_i x_j (r_i + r_j)/(2N)$ from the last sum above, we get
\[
\frac{n}{N} + \sum_{i \in [n]} x_i v_i T_i - \sum_{i \in [n]} \frac{x_i r_i (x_i - 1)}{2N} - \sum_{i < j \in [n]} \frac{x_i x_j (r_i + r_j)}{2N} + \sum_{i < j \in [n]} \frac{x_i x_j}{N} \cdot (1/2 - p(i,j)) (r_i - r_j) + O(\epsilon').
\]
Also note that the second and third sums above can be combined into a linear form of the $x_i$’s:
\[
\sum_{i \in [n]} x_i r_i (x_i - 1) + \sum_{i < j \in [n]} x_i x_j (r_i + r_j) = - \sum_{i \in [n]} x_i r_i + \left( \sum_{i \in [n]} x_i \right) \left( \sum_{i \in [n]} x_i r_i \right) = (n-1) \sum_{i \in [n]} x_i r_i.
\]
As a result, we get the following approximation of the expected revenue $\mathcal{R}(p)$, with an error of $O(\epsilon')$:
\[
\frac{n}{N} + \sum_{i \in [n]} x_i v_i T_i - \frac{n-1}{2N} \sum_{i \in [n]} x_i r_i + \sum_{i < j \in [n]} \frac{x_i x_j}{N} \cdot (1/2 - p(i,j)) (\Gamma_i - \Gamma_j).
\]
(23)
Note that in (23), we also replaced $r_i - r_j$ at the end with $\Gamma_i - \Gamma_j$ since the error introduced is $O(n^3 / N^3)$.

### 6.3 Reverse Engineering of $t_i$ and $p(i,j)$

Our ultimate goal is to set $t_i$’s and $q_i$’s carefully so that (23) by the end has the following form:
\[
\frac{n}{N} + \frac{L^2}{N^2 m^{3n}} - \frac{1}{N^2 m^{3n}} \cdot \left( \sum_{i \in [n]} x_i a_i - L \right)^2.
\]
Recall that $L$ is the target integer in the Knapsack instance. If this is the case, then we obtain a polynomial-time reduction from the special Knapsack problem to ITEM-PRICING, since the difference between (24) and
\( \mathcal{R}(p) \) is at most \( O(\epsilon') \) and thus (24) is at least

\[
\frac{n}{N} + \frac{L^2}{N^2m^{3n}} - \frac{1}{2N^2m^{3n}}
\]

if and only if \( a_1, \ldots, a_n \) and \( L \) is a yes-instance of the special Knapsack problem.

To compare (24) and (23), we use \( \sum_{i \in [n]} x_i = n \) in (24) and it becomes

\[
\frac{n}{N} - \frac{1}{N^2m^{3n}} \cdot \left( \sum_{i \in [n]} a_i^2 x_i^2 + 2 \sum_{i < j \in [n]} x_i x_j a_i a_j - 2 \sum_{i \in [n]} a_i L x_i \right)
\]

\[
= \frac{n}{N} - \frac{1}{N^2m^{3n}} \cdot \left( \sum_{i \in [n]} a_i^2 x_i \left( n - \sum_{j \neq i} x_j \right) + 2 \sum_{i < j \in [n]} x_i x_j a_i a_j - 2 \sum_{i \in [n]} a_i L x_i \right)
\]

\[
= \frac{n}{N} - \frac{1}{N^2m^{3n}} \cdot \left( \sum_{i \in [n]} (na_i^2 - 2a_i L)x_i - \sum_{i < j \in [n]} x_i x_j (a_i - a_j)^2 \right)
\]

(25)

By comparing (25) with (23), our goal is achieved if the following two conditions hold: First,

\[
T_i = \frac{1}{v_i} \cdot \left( \frac{(n - 1)r_i}{2N} - \frac{1}{N^2m^{3n}} \cdot (na_i^2 - 2a_i L) \right),
\]

(26)

for all \( i \in [n] \) (note that the absolute value of the right side of (26) is \( O(n/(m^{2n+2}N^2)) \)); Second,

\[
\frac{((1/2) - p(i, j))(\Gamma_i - \Gamma_j)}{N} = \frac{(a_i - a_j)^2}{N^2m^{3n}}, \quad \text{for all pairs } i < j \in [n].
\]

(27)

For the first condition, we note that the equations (26) for all \( i \in [n] \) actually form a triangular system of \( n \) equations in the \( n \) variables \( t_1, \ldots, t_n \), and thus there exists a unique sequence \( t_1, \ldots, t_n \) such that (26) holds for all \( i \in [n] \). Moreover, as the absolute value of the right side of (26) is \( O(n/(m^{2n+2}N^2)) \), the \( t_i \)'s are \( O(1/N^2) \) as we promised earlier. To see this, we let \( s \) denote the maximum of the absolute value of the right side of (26), over all \( i \in [n] \). Then one can show by induction on \( i \) that \( |t_i| \leq 2^{n-1}s \) for all \( i \) from \( n \) to \( 1 \). The claim now follows using \( 2^n \ll m^n \).

The second condition is more difficult to satisfy. From (27), we know that the condition is met if

\[
\frac{1}{2} - p(i, j) = \frac{(a_i - a_j)^2}{Nm^{3n}(\Gamma_i - \Gamma_j)}, \quad \text{for all } i < j \in [n].
\]

(28)

We will define below the \( n \) distributions \( q_i, i \in [n] \), so that their induced values for the probabilities \( p(i, j) \) satisfy (28). An important property that we will need for the construction of the \( q_i \)'s is that all the desired probabilities \( p(i, j) \) are very close to \( 1/2 \). Specifically, using \( \Gamma_i - \Gamma_j \geq \gamma_i \geq \gamma_n = 1/(m^{2n}N) \), we have

\[
0 < \frac{1}{2} - p(i, j) \leq \frac{(a_i - a_j)^2 \cdot Nm^{2n}}{Nm^{3n}} = o \left( \frac{1}{m} \right),
\]

(29)

since \( m = \max(n^5, a_n) \) and \( a_n = \max_{i \in [n]} a_i \).
6.4 Connecting \( p(i, j) \) with \( q_i \) and \( q_j \)

Fixing a pair \( i < j \in [n] \), we examine \( p(i, j) \) closer. Recall that \( p(i, j) \) is the probability of \( \alpha - v_i > \beta - v_j \) when \( \alpha \) and \( \beta \) are drawn independently from \( Q \), conditioning on \( \alpha \geq v_i \) and \( \beta \geq v_j \).

For convenience, we use block \( k \) to denote the subset \( \{v_k, v_k + 1, \ldots, v_k + 2n^3\} \) of the support of \( Q \). Note that due to the exponential structure of the support of \( Q \) (and the assumption of \( i < j \)), if \( \alpha \) is in block \( k \geq i \) and \( \beta \) is in block \( \ell \geq j \) with \( \ell \geq k \) then \( \beta - v_j > \alpha - v_i \). Therefore, for \( \alpha - v_i > \beta - v_j \) to happen, we only need to consider the following three cases:

**Case 1:** \( \alpha \) is from block \( k \) and \( \beta \) is from block \( \ell \), where \( k, \ell \in [n] \) satisfy \( k \geq \ell > j \). Then the total contribution of this case to probability \( p(i, j) \) is:

\[
\frac{1}{r_i r_j} \cdot \sum_{k \geq \ell > j} (\gamma_k + t_k)(\gamma_\ell + t_\ell).
\]

**Case 2:** \( \alpha \) is from block \( k \) and \( \beta \) is from block \( j \), where \( k > i \). Then the total contribution is

\[
\frac{1}{r_i r_j} \cdot \sum_{k > i} (\gamma_k + t_k)(\gamma_j + t_j).
\]

**Case 3:** Finally, \( \alpha \) is from block \( i \) and \( \beta \) is from block \( j \), with \( \alpha - v_i > \beta - v_j \). Let \( q(i, j) \) denote the probability of \( \alpha > \beta \), when \( \alpha \) is drawn from \( q_i \) and \( \beta \) is drawn from \( q_j \) independently. Using \( q(i, j) \), the total contribution of this case to \( p(i, j) \) is

\[
\frac{1}{r_i r_j} \cdot \left( \left( \frac{\gamma_i}{m} + t_j \right) \frac{(m - 1)\gamma_i}{m} + q(i, j) \cdot \frac{(m - 1)\gamma_i}{m} \cdot \frac{(m - 1)\gamma_j}{m} \right).
\]

The probability \( p(i, j) \) is equal to the sum of the above three quantities for the three cases. Hence, \( q(i, j) \) is uniquely determined by the \( p(i, j) \) we aim for, i.e., the unique \( p(i, j) \) that satisfies (28), because all other parameters have been well defined by now, including \( t_1, \ldots, t_n \).

We show below that, if \( |p(i, j) - 1/2| = o(1/m) \), then the \( q(i, j) \) it uniquely determines must satisfy \( |q(i, j) - 1/2| = O(1/m) \).

To see this, note first that since \( i < j \leq n \) and \( r_i = T_i / \gamma_i = 1/(N m^{n+i}) = O(n/N^2) \), we have that

\[
\gamma_i = \frac{m - 1}{m^{n+i+1} N} = \frac{m - 1}{m} \cdot r_i \pm O \left( \frac{n}{N^2} \right).
\]

Thus, \( \sum_{k > i} (\gamma_k + t_k) = r_i - \gamma_i - t_i = r_i/m \pm O(n/N^2) \).

Using this fact in the above expressions for the three cases, it is easy to show that, other than

\[
\frac{1}{r_i r_j} \cdot q(i, j) \cdot \frac{(m - 1)\gamma_i}{m} \cdot \frac{(m - 1)\gamma_j}{m},
\]

the contribution of other terms is bounded from above by \( O(1/m) \) (note that \( k \geq \ell > j \) implies \( k > i \)). Since \( |p(i, j) - 1/2| = o(1/m) \), it follows that the term in (30) is between \( 1/2 - O(1/m) \) and \( 1/2 + O(1/m) \). Note that \( \gamma_i = (m - 1)r_i/m \pm O(n/N^2) \) (since \( i < n \)), and \( \gamma_j \) is either \( (m - 1)r_j/m \pm O(n/N^2) \) if \( j < n \) or \( r_j \pm O(n/N^2) \) if \( j = n \). Therefore, the coefficient of \( q(i, j) \) in (30) is \( 1 - O(1/m) \). Since the expression in (30) is \( 1/2 \pm O(1/m) \), it follows that \( |q(i, j) - 1/2| = O(1/m) \).
6.5 Reverse Engineering of $q$

Given $q(i, j)$ for each pair $i < j \in [n]$, our final technical step of the reduction is to construct a sequence of probability distributions $q_1, \ldots, q_n$ over $[2n^3]$ such that, for each pair $i < j \in [n]$, the probability of $\alpha > \beta$, where $\alpha$ is drawn from $q_i$ and $\beta$ is drawn from $q_j$ independently, is exactly $q(i, j)$.

In general, such a sequence of distributions may not exist, e.g., consider $n = 3$, $q(1, 2) = 1, q(2, 3) = 1$ and $q(1, 3) = 0$. But here we are guaranteed that the $q(i, j)$’s are close to $1/2$: $|q(i, j) - 1/2| = O(1/m)$. We shall show that in this case the desired distributions exist, and we can construct them.

To construct $q_1, \ldots, q_n$, we define $\binom{n}{2}$ subsets of $[2n^3]$, called sections. Each section consists of $2n + 3$ consecutive integers. The first section is $\{1, \ldots, 2n + 3\}$, the second section is $\{2n + 4, \ldots, 4n + 6\}$, and so on and so forth. (Note that $2n^3$ is clearly large enough for $\binom{n}{2}$ sections.) Each section is labeled, arbitrarily, by a distinct pair $(i, j)$ with $i < j \in [n]$. We let $t_{i,j,k}$ denote the $k$th smallest integer in section (labeled) $(i, j)$, where $k \in [2n + 3]$. Now we define $q_{i,j} \in [n]$. For each section $(i, j)$, $i < j \in [n]$, we have:

**Case 1:** If $\ell \neq i$ and $\ell \neq j$, then we set

$$q_{\ell}(t_{i,j,\ell}) = q_{\ell}(t_{i,j,2n+4-\ell}) = \frac{1}{2\binom{n}{2}}$$

and $q_{\ell}(t_{i,j,k}) = 0$ for all other $k \in [2n + 3]$.

**Case 2:** If $\ell = j$, then we set $q_{\ell}(t_{i,j,n+2}) = 1/\binom{n}{2}$ and $q_{\ell}(t_{i,j,k}) = 0$ for all other $k \in [2n + 3]$.

**Case 3:** If $\ell = i$, then we set

$$q_{\ell}(t_{i,j,n+1}) = \frac{1}{2\binom{n}{2}} - \binom{n}{2}(q(i, j) - 1/2) \quad \text{and} \quad q_{\ell}(t_{i,j,n+3}) = \frac{1}{2\binom{n}{2}} + \binom{n}{2}(q(i, j) - 1/2),$$

and $q_{\ell}(t_{i,j,k}) = 0$ for all other $k \in [2n + 3]$.

This finishes the construction of $q_1, \ldots, q_n$. Using $|q(i, j) - 1/2| = O(1/m)$ and $m \geq n^5$, we know that $q_1, \ldots, q_n$ are probability distributions: all entries are nonnegative and sum to 1.

It is also not hard to verify that the distributions satisfy the desired property, i.e., for each pair $i < j \in [n]$ the probability of $\alpha > \beta$, where $\alpha$ is drawn from $q_i$ and $\beta$ is drawn from $q_j$ independently, is exactly $q(i, j)$. First observe that every section of each distribution $q_i$ has the same probability $1/\binom{n}{2}$. If $\alpha$ and $\beta$ belong to different sections then the order between $\alpha$ and $\beta$ is determined by the order of the sections, and both orders have obviously the same probability.

So suppose that $\alpha, \beta$ belong to the same section labeled $(g, h)$, where $g, h \in [2n + 3]$. If $g \neq i$ or $h \neq j$, then it is easy to check that both orders between $\alpha$ and $\beta$ have the same probability. To see this, suppose first that $i \notin \{g, h\}$. Then $\alpha = t_{g,h,i}$ or $t_{g,h,2n+4-i}$ with equal probability. If $\alpha = t_{g,h,i}$, then $\alpha < \beta$ because $\beta$ is either $t_{g,h,j}$ or $t_{g,h,2n+4-j}$ (if $j \notin \{g, h\}$), or $\beta = t_{g,h,n+2}$ (if $h = j$) or $\beta = t_{g,h,n+1}$ or $t_{g,h,n+3}$ (if $g = j$); similarly, if $\alpha = t_{g,h,2n+4-i}$ then $\alpha > \beta$. Therefore, if $i \notin \{g, h\}$, then there is equal probability that $\alpha < \beta$ and $\alpha > \beta$. Similarly, the same is true if $j \notin \{g, h\}$.

Suppose that $i \in \{g, h\}$ and $j \in \{g, h\}$. Since $i < j$ and $g < h$, we must have $i = g$ and $j = h$. In this case, $\beta = t_{i,j,n+2}$, and $\alpha = t_{i,j,n+1}$ or $\alpha = t_{i,j,n+3}$, hence $\alpha > \beta$ iff $\alpha = t_{i,j,n+3}$.
The probability that \( \alpha > \beta \) and \( \alpha, \beta \) are not both in section \((i, j)\) is
\[
\frac{1}{2} \cdot \left( 1 - \frac{1}{\binom{n}{2}} \right).
\]
The probability that \( \alpha > \beta \) and \( \alpha, \beta \) are both in section \((i, j)\) is
\[
\frac{1}{\binom{n}{2}} \cdot \left( \frac{1}{2} \binom{n}{2} + \left( \frac{n}{2} \right) \left( q(i, j) - \frac{1}{2} \right) \right) = \frac{1}{2\binom{n}{2}} + \left( q(i, j) - \frac{1}{2} \right).
\]
Thus, the total probability that \( \alpha > \beta \) is exactly \( q(i, j) \) as desired.
This concludes the construction and the proof of the theorem.

7 Conclusions

In this paper, we studied the complexity of the Bayesian Unit-Demand Item-Pricing problem with independent distributions. We showed that the decision problem is NP-complete even when the distributions are of support size 3 or when they are identical. We also presented a polynomial-time algorithm for distributions of support size 2.

Several interesting open questions remain. Is there a PTAS for general distributions? Note that our NP-hardness results do not preclude the existence of an FPTAS. Actually, by adapting techniques from \[\text{CDTT}\] we can give an FPTAS for the case when the supports of the distributions are integers in a bounded interval. Moreover, we conjecture that the IID case can be solved in polynomial time when the size of the support is constant.

A related question concerns the complexity of the randomized case (i.e., lottery pricing). We conjecture that this problem is intractable, but new ideas are needed to prove this.

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