QED treatment of linear elastic waves in asymmetric environments

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ABSTRACT
Considering the importance of correctly understanding the dynamics of microstructure materials for their applications in related technologies, by eliminating the shortcomings and some overlooked physical concepts in the existing asymmetric elastic theories, we have presented an asymmetric elastodynamic model based on a \( U(1) \) gauge theory with quantum electrodynamics (QED) structure. Accordingly, we have shown that there is a correspondence between an elastic theory, which can explain the behavior of elastic waves within an asymmetric elastic medium and QED. More specific, we have indicated that the corresponding elastic wave equations are somehow analogous to QED ones. In this regard, by adding vibrational degrees of freedom and introducing a gauge property of the waves of displacement for the waves of rotation, we have generalized and modified the related Cosserat theory (CT) for an elastic environment. Thus on macro scales, the elastic waves can possess the QED treatment. This analogy provides a new paradigm of fermions and bosons. Also, from experimental point of view, we have shown that the behavior of elastic waves in a granular medium is equivalent to behavior of light in dispersive media, which can be explained using QED. Hence, contrary to the Cosserat and discrete models, this amended CT has qualitatively been indicated to be consistent with the corresponding empirical observations.

1. Introduction
There is no doubt that one of the most important and essential discussions in technologies, which are based on polymers, crystals, granular, and cellular materials, are the understanding of internal dynamics among the structures of these materials that have microstructures. As recently, important and significant researches have been assigned on these topics. For example, a lot of articles have been presented on wave propagation in composite materials, see, e.g. [1–6], and on the dynamic properties of nanocomposite materials, see, e.g. [7–13]. In this regard, it is known that discrete techniques can supply a more accurate explanation of discontinuous materials. However, these techniques are complex and necessitate accuracy in modeling of the interface. For instance, inappropriate selection of interface elements in many cases leads to realms of unrealistically high stress gradients and incorrect results.
In addition, in many applications, the definition of the input model becomes impractical as the number of joints gets large [14]. Such issues can be considered as a challenge for discrete models.

In the equivalent continuum techniques, the properties of the interface are mixed with the properties of the intact matrix, and discrete materials are replaced by homogeneous continua [14]. The equivalent continuum techniques that are based on the classical continuum theory have the challenge of not taking into account the effect of the internal length scale, which is an intrinsic characteristic of layered materials. Consequently, these techniques are suitable to cases where no slip occurs at the interface of the layers and/or when the internal length scale of the problem is negligible compared to the dimensions of the engineering structure [14].

Given the important challenges of current discrete and continuous elastic models, for a better description of dynamics of elastic environments, one needs a model without the shortcomings of the current models. Accordingly, from theoretical and experimental points of view, via generalization and modification of some existing elastic models, we have investigated relevance between such an elastic theory (which should explain the behavior of elastic waves within an asymmetric elastic medium) and a $U(1)$ gauge theory (e.g. QED). In particular, we have focused on the equations of motion of elastic waves in asymmetric environments and the corresponding QED ones. In this respect, in addition to the advantages of those continuous elastic models, according to the quantum aspects (e.g. loop corrections), a $U(1)$ gauge theory (e.g. QED), as a continuous model, has other properties that can compensate for the shortcomings of discrete models. Indeed, due to the phenomenon of loop corrections and the existence of cut-off energy in QED, such a relevance can be important when it is employed to describe the effect of internal length scale of discontinuous materials. In this process, we have been looking forward that one would be able to take advantage of continuum techniques without the challenge in describing discontinuous materials.

Previous attempts had been to achieve an elastic model with respect to electromagnetic theory. However, in the scope of classical elastic theory, the similarity between the elastic and electromagnetic waves is incomplete. For example, in [15–17], an analogy between the electromagnetic field and a fluid, which includes a large number of vortex filaments, has been investigated. In these works, the pressure of fluid, the fluid velocity and the density of the turbulence energy respectively are as the electrostatic potential, the magnetic vector potential and the electromotive force, but charge has no independent nature. Also in [18,19], it has been claimed that the electromagnetic equations fit the structure of a linear elastic continuum that is hard to compression though liable to shear deformation. Nevertheless, in these two works, charge still has no independent nature because microstructures have not been considered for the elastic environment. Whereas a physical elastic environment is composed of microstructures.

On the other hand, in the CT [20–23], by generalizing the elastic theory and assuming the presence of microstructures in an elastic environment, the stress tensor has been employed asymmetrically. However, still there are some overlooked physical concepts in these theories. For instance, any microstructure can have a vibrational degree of freedom, which has not been talked about in these theories. Whereas the waves of displacement can play a role in transmission of the waves of rotation as a gauge property, but there is no mention of the gauge property of the waves of displacement for the waves of rotation of the microstructures.
In this regard, to achieve an asymmetric elastodynamic model based on the QED as a $U(1)$ gauge theory with quantum aspects, we have generalized and then have modified the CT. During this process, first by adding vibrational degrees of freedom for microstructures in the CT and using some mathematical methods, we have achieved a covariant form of the CT. Then, while using the gauge property of the waves of displacement and also modeling fermions, by means of the Cosserat elasticity, we have modified the CT in a way that the resulted asymmetric elastodynamic model being fully comparable to QED. Finally, we have made a comparison between the experimental observations and the predictions of the proposed asymmetric elastodynamic model.

2. The Cosserat theory with Minkowski metric

In the absence of any external force and torque, the corresponding dynamical equations for any homogeneous and isotropic linear asymmetric elastic environment, with the Minkowski metric, can be written as

$$\rho u_{i00}^j + \sigma_{ij} = 0, \quad \mathcal{I} \phi_{i00}^j + \epsilon_{ijk} \sigma_{jk} + m_{ij} = 0,$$

where $\rho$ is constant average mass density of environment, $\mathcal{I}$ is the density of moment of inertia, $\sigma_{ij}$ is the stress tensor, $m_{ij}$ is the couple-stress tensor, $u^j$ and $\phi^j$ respectively are the displacement vector and the rotation pseudovector of any point of environment, and $\epsilon_{ijk}$ is the Levi–Civita alternating symbol. As the displacement and rotation satisfy wave equations and are defined over the entire elastic environment, we also refer to these as waves or fields. The Latin lowercase letters run from one to three, and the unit of speed has been selected to be one.

As Equations (1) and (2) contain four unknown parameters, in order to proceed, we appeal to the definitions of deformation tensors as

$$\gamma_{ij} \equiv u_{ji} - \epsilon_{ijk} \phi^k \quad \text{and} \quad \chi_{ij} \equiv \phi_{ji}, \quad (3)$$

In addition, when an environment behaves linearly, it has been shown that one obtains

$$\sigma_{ij} = (\alpha + \mu) \gamma_{ij} + (\mu - \alpha) \gamma_{ji} + \lambda \delta_{ij} \gamma_{kk}, \quad (4)$$
$$m_{ij} = (\gamma + \epsilon) \chi_{ij} + (\gamma - \epsilon) \chi_{ji} + \beta \delta_{ij} \chi_{kk}, \quad (5)$$

where $\delta_{ij}$ is the Kronecker delta. The six parameters $\alpha, \mu, \gamma, \epsilon, \lambda$ and $\beta$ are the characteristic coefficients of an elastic medium. These coefficients are measured in an adiabatic mode and are constants in a homogeneous environment, in a way that, the first four ones have positive values and the last two ones are required to satisfy the inequalities $3\lambda + 2\mu > 0$ and $3\beta + 2\gamma > 0$ [24]. Accordingly, substituting definitions (4) and (5) with relation (3) into Equations (1) and (2), these equations read

$$\rho u_{i00}^j + (\alpha + \mu) u_{ij}^j + (\mu - \alpha + \lambda) u_{ji}^j + 2\alpha \epsilon_{ijk} \phi_{jk} = 0, \quad (6)$$
$$\mathcal{I} \phi_{i00}^j + (\gamma + \epsilon) \phi_{ij}^j + (\gamma - \epsilon + \beta) \phi_{ji}^j - 4\alpha \phi^j - 2\alpha \epsilon_{ijk} u_{jk} = 0. \quad (7)$$
3. Four-Dimensional representation

To achieve a ‘relativistic’ appearance, we furthermore extend the above obtained equations to four-dimensional spacetimes.

For this purpose, first to introduce a zeroth component for the displacement, we utilize a scalar field in analogy with the issue of longitudinal waves. In there, a scalar field (as a kinetic potential) has been used to explain some parameters\(^7\) including the condensation,\(^8\) which is defined as minus the time derivative of it multiplied by the inverse square of the longitudinal component of the speed of wave, \(v_L\) [25]. Also, in the linear acoustics, it has been shown [26] that the condensation is equal to minus the divergence of the displacement in the Euclidean space. Thus, for such a scalar field/kinetic potential, say \(\phi\), one obtains

\[
\phi,0 - v^2_L u^j,j = 0. \tag{8}
\]

Incidentally, the longitudinal component of the speed of wave is related to the elastic environment characteristic coefficients (the Lamé constants) as \(v^2_L \equiv (\lambda + 2\mu)/\rho\) [27]. With these respects, we rewrite relation (8) as

\[
u_{\hat{\alpha}, \hat{\alpha}} = 0 \tag{9}
\]

with metric \(g_{\hat{\alpha}\hat{\beta}} \equiv (-v^{-2}_L, 1, 1, 1)\) for hat-letters, where the kinetic potential has been assumed as zeroth component of the displacement, \(\phi \equiv u_0\), and the Greek lowercase letters run from 0 to 3.

4. Multi-metric instead of multi-speed

For simplicity of the appearances and also for clarity of the resulted equations, to raise the covariant components of any tensor (i.e. to get any covariant components from the corresponding contravariant ones), we manifestly define five different diagonal metrics specified with normal-letters, hat-letters, bar-letters, check-letters, and tilde-letters through the work. We already have defined the metric with hat-letters, the metric with the normal-letters is \(g_{\alpha\beta} = (-1, 1, 1, 1)\), and the remaining ones will be defined later. Indeed, the need for these five metrics is due to the fact that there are six parameters characterizing the elastic continuum. These mentioned metrics are defined in a way that only differ with each other in their ‘timelike’ 00 components. These components are related to different speeds (like \(g_{\hat{0}\hat{0}}\)), and actually, we prefer to somehow specify those rather than assuming all of those to be one (as in the usual natural units and/or in the case with the normal-letters). Of course, we assume that, for all of these different kinds of letters, the corresponding covariant components of any tensor (obviously, except the corresponding metrics themselves) being the same, say for example \(T_{\alpha} = T_{\hat{\alpha}} = T_{\tilde{\alpha}} = T_{\check{\alpha}} = T_{\tilde{\alpha}}\). Hence, as desired, this restriction makes only the contravariant zeroth components being different, e.g. \(T^{\hat{0}} = v^{-2}_L T^{\hat{0}}\). Nevertheless, for symmetry of appearances of relations, wherein the summation rule is used, we specify the notation of any covariant component with the same notation of its corresponding contravariant one, e.g. as done in relation (9).

With known definitions

\[
\varphi^{ij} \equiv \epsilon^{ijk} \varphi_k \quad \text{and} \quad F^{\alpha\beta} \equiv u^{\alpha,\beta} - u^{\beta,\alpha}, \tag{10}
\]
while using relation (9), we can rewrite Equation (6) as
\[(\alpha + \mu)F^i_{\cdot \alpha} - 2\alpha\varphi^i_{\cdot j} = 0\]  
(11)
with metric $g^{\tilde{\alpha}\tilde{\beta}} \equiv (-v_Q^{-2}, 1,1,1)$ for bar-letters, where $v_Q^2 \equiv (\alpha + \mu)/\rho$ is the transverse component of the speed of wave for a divergence-free wave of displacement.

5. Generalization of CT via new degrees of freedom

We secondly need to introduce another three components for the existing antisymmetric tensor of the wave of rotation $\varphi^{ij}$, which we indicate those as $\varphi^{0ij}$'s components. For this purpose, as the moment of inertia density is proportional to the mass density of environment, thus varying it will cause that to change. The simplest way to explain variation of moment of inertia density is to assume the microstructures of medium as two same balls, each with mass $m$, connected by a massless spring, i.e. a three-dimensional isotropic harmonic oscillator where its symmetry group is isomorphic to $SO(4)$. To show this, let us write the energy of these mentioned two balls as the energy of another two balls, each also with mass $m$ in a four-dimensional Euclidean space, connected by a massless rod of length $2R$, which are rotating around their center of mass. That is, a four-dimensional linear rigid rotor with total energy $E$ and angular momentum $l_{(4)}$ of each ball, where $l^2_{(4)} = \mu^{AB}l_{AB}/2$ and the Latin uppercase letters run from 1 to 4. Then, one can easily show that
\[\frac{l^2_{(4)}}{2mR^2} = \frac{m^2}{2} + \frac{E^2}{R^2} + \frac{l^2}{2mr^2},\]  
(12)
where $l$ and $r$ respectively are the three-dimensional angular momentum of each ball (with $l^2 = \mu_{ij}/2$) and the distance between two balls in 3-dimensions. To prove relation (12), note that $x^A x_A = R^2$ and $x^A \dot{x}_A = 2E/m$ and $l^{AB} = mx^A x^B - mx^B x^A$. This relation indicates that the symmetry group of the isotropic harmonic oscillator in 3-dimensions is isomorphic to $SO(4)$.

In this way, we assume the microstructures of a medium being as four-dimensional linear rigid rotors with four-dimensional angular rotation $\varphi^{\alpha\beta}$, which $\varphi^{0ij}$'s (as vibration and relative velocity of parts of each microstructure) are related to $\varphi^{4ij}$'s (as rotation in fourth dimension) via the Wick rotation. On the other hand, by selecting an appropriate gauge, we choose a gauge fixing in such a way that the displacement and rotation wave equations of motion to be covariant, and in general, the corresponding velocities of waves, masses and coupling (to the displacement field) of $\varphi^{0i}$ being different from those of $\varphi^{ij}$. In this respect, we assume the gauge
\[\varphi^{\tilde{\alpha}\tilde{\beta}}_{\cdot \gamma \cdot \tilde{\gamma}} \equiv 0\]  
(13)
with metric $g^{\tilde{\alpha}\tilde{\beta}} \equiv (-v_N^{-2}, 1, 1, 1)$ for check-letters, where $v_N^2 \equiv (\gamma + \varepsilon)/\Im$ is the speed of wave for $\varphi^{0i}$ waves of vibration. Then, we plausibly generalize Equation (7) as
\[(2\gamma + \beta)\left(\varphi^{\tilde{\alpha}\tilde{\beta}}_{\cdot \gamma \cdot \tilde{\gamma}} + \varphi^{\tilde{\alpha}\tilde{\beta}}_{\cdot \gamma \cdot \tilde{\gamma}} - \varphi^{\tilde{\alpha}\tilde{\beta}}_{\cdot \gamma \cdot \tilde{\gamma}}\right) - 4\alpha\varphi^{\tilde{\alpha}\tilde{\beta}} - 2\alpha F^{\tilde{\alpha}\tilde{\beta}} = 0\]  
(14)
with metric $g^{\tilde{\alpha}\tilde{\beta}} \equiv (-v_R^{-2}, 1, 1, 1)$ for tilde-letters, where $v_R^2 \equiv (2\gamma + \beta)/\Im$ is the speed of wave for $\varphi^{ij}$ waves of rotation. In Equation (14), the part
\[(2\gamma + \beta)\left(\varphi^{ij}_{\cdot \gamma \cdot l} - \varphi^{ij}_{\cdot \gamma \cdot l}\right) - 4\alpha\varphi^{ij} - 2\alpha F^{ij} = 0,\]  
(15)
using gauge (13), is just Equation (7), and the extra part
\[
(2\gamma + \beta) \left( \varphi^{\hat{o}i j} + \varphi^{\hat{\jmath} j} - \varphi^{\hat{j} j} \right) - 4\alpha \varphi^{\hat{o}i j} - 2\alpha F^{\hat{o}i j} = 0,
\]
again due to gauge (13), is the wave equation of vibration \(\varphi^{O_i}\), i.e.
\[
(\gamma + \varepsilon) \varphi^{\hat{o}i j} \varphi^{\hat{\jmath} j} - 4\alpha \varphi^{\hat{o}i j} - 2\alpha F^{\hat{o}i j} = 0.
\]
Equation (14) is invariant under a gauge transformation with respect to an arbitrary gauge field, say \(\zeta_{\alpha}\), as
\[
\varphi^{\hat{o}i j} \to \varphi^{\hat{o}i j} - \left( \zeta^{\hat{o}i j} - \zeta^{\hat{\jmath} j} \right),
\]
\[
F^{\hat{ij}} \to F^{\hat{ij}} + 2\left( \xi^{\hat{ij}} - \xi^{\hat{j} j} \right),
\]
\[
F^{\hat{0i}} \to F^{\hat{0i}} + 2\frac{v_T^2}{v_R^2} \left( \varphi^{\hat{0i} j} - \varphi^{\hat{j} j} \right),
\]
provided that we generalize Equation (11) in a more general form
\[
(\alpha + \mu) F^{\hat{0i} \hat{\alpha} \hat{\beta}, \hat{\alpha}} - 2\alpha \varphi^{\hat{0i} \hat{\alpha}, \hat{\alpha}} = 0,
\]
however with \(\varphi^{O_i} = 0\) in order to reproduce Equation (11). The generalization of adding such a degree of freedom (i.e. \(\varphi^{O_i}\)) is physically meaningful. The reason is that, \(F^{O_i}\) is composed of the velocity of microstructures and gradient of the kinetic potential (that, in turn, is related to change of the density of microstructures). On the other hand, the vibration \(\varphi^{O_i}\) has also two features, namely the relative velocity of parts of each microstructure and alteration in the volume (and hence the density) of each one. Thus, \(F^{O_i}\) and \(\varphi^{O_i}\) have a similar role, which is missing in Equation (11).

Now, performing the divergence of Equation (20) yields
\[
(\alpha + \mu) F^{\hat{0i} \hat{\alpha} \hat{\beta}, \hat{\alpha}} - 2\alpha \varphi^{\hat{0i} \hat{\alpha}, \hat{\alpha}} = 0.
\]
Integrating Equation (21) over time while considering the zeroth component of the displacement as a wave, it gives
\[
(\alpha + \mu) F^{\hat{0i} \hat{\alpha} \hat{\beta}, \hat{\alpha}} - 2\alpha \varphi^{\hat{0i} \hat{\alpha}, \hat{\alpha}} = 0.
\]
Combining Equations (20) and (22), we finally end up with a generalized form of Equation (11) as
\[
(\alpha + \mu) F^{\hat{0i} \hat{\alpha} \hat{\beta}, \hat{\alpha}} - 2\alpha \varphi^{\hat{0i} \hat{\alpha}, \hat{\alpha}} = 0.
\]
Now that we have obtained the necessary equations in the covariant form, it would be instructive to use the exterior algebra notations, in order to concisely rewrite those.
Indeed, it gives us a better and simpler sight to investigate the correspondence between the QED equations and the elastic wave ones. In this regard, we respectively rewrite gauge conditions (9) and (13), and Equations (14) and (23) as

\[
\delta \hat{u} = 0 \quad \text{and} \quad \delta \hat{\phi} = 0, \\
(2\gamma + \beta) \delta \hat{d}\hat{\phi} - 4\alpha\hat{\phi} - 2\alpha\hat{d}\hat{u} = 0, \\
(\alpha + \mu)\delta \hat{d}\hat{u} - 2\alpha\delta \hat{\phi} = 0,
\]

where \( u \) and \( \phi \) are one-form and two-form fields, respectively.

6. Modification of generalized CT via gauge property of waves of displacement

As the motion of an environment can make the motion of a wave within it ineffective, it is noteworthy to mention that when a wave of displacement moves a microstructure of medium, then the microstructure movement can make the movement of waves of rotation (the gradient or the momentum of waves of rotation) ineffective. This matter means that the displacement field is capable to play the role of the momentum of waves of rotation, similar to the photon field that acts as the role of the momentum or a gauge for the fermion field. However, there is no trace of such a feature in Equations (25) and (26). Hence, we still need to modify the generalized CT further in order to present such a gauge characteristic. For this purpose, let us first check whether the existed generalized CT without the interacting terms between the waves of displacement and waves of rotation (i.e. without the last terms in Equations (25) and (26)), is related to the theory of Yang-Mills or not. If the result being positive, then we will need only to modify the interaction terms.

The non-interactive part of Equation (26) with its gauge condition is well known to be in accord with boson fields. For the other equation, it is also known [29] that a spinor field, say \( \psi \), can be defined by a two-form field as

\[
\psi \equiv \varphi_{\alpha\beta}\sigma^{\alpha\beta}\vartheta,
\]

where \( \sigma^{\alpha\beta} = \frac{i}{2} [\gamma^{\alpha}, \gamma^{\beta}] \), \( \gamma^{\alpha} \)'s are the Dirac matrices and \( \vartheta \) is an arbitrary constant nonzero fiducial spinor for which \( \gamma^{5}\vartheta \neq i\vartheta \). Multiply the transpose conjugate of \( \vartheta \) to a Klein–Gordon equation [31] for \( \bar{\psi} \equiv \varphi_{\alpha\beta}\sigma^{\alpha\beta}\vartheta \), i.e.

\[
\vartheta^\dagger \left( i\tilde{\partial} + m \right) \left( i\tilde{\partial} - m \right) \bar{\psi} = 0,
\]

where \( \tilde{\partial} \equiv \gamma^{\alpha}\partial_{\alpha} \). Then, substituting for \( \bar{\psi} \) while using relation [29]

\[
8\vartheta^\dagger \vartheta = 2\vartheta^\dagger \gamma^{\alpha}\vartheta \gamma^{\alpha} + \omega_{\alpha\beta}\sigma^{\alpha\beta},
\]

in the exterior algebra, it yields

\[
\bar{\omega} \wedge \ast \left( \delta \tilde{d}\tilde{\phi} - m^2\tilde{\phi} \right) = 0,
\]

where \( \wedge \) is the wedge product, \( \ast \) is the Hodge operator [28], and \( \omega \) is an arbitrary two-form defined as \( \omega^{\alpha\beta} \equiv \vartheta^\dagger \sigma^{\alpha\beta}\vartheta \). Hence, in turn, while assuming \( m \equiv \sqrt{4\alpha/(2\gamma + \beta)} \), it gives the non-interactive part of Equation (25).
On the other hand, if the Dirac equation for a massive fermion [31], i.e.
\[
\left( i \tilde{\partial} - m \right) \tilde{\psi} = 0,
\]
(31)
is satisfied, then its corresponding Klein–Gordon equation, and in turn Equation (30), will be held. To check the corresponding gauge condition (i.e. the second one in gauges (24)), let us employ the trivial condition
\[
\vartheta^\dagger \partial_{\tilde{\alpha}} \partial_{\tilde{\beta}} \sigma^{\tilde{\alpha} \tilde{\beta}} \tilde{\psi} = 0
\]
(32)
that, in turn by relation (29), gives
\[
\hat{\omega} \wedge * \left( \tilde{d} \delta \phi \right) = 0.
\]
(33)
Now, as \( \omega \) is arbitrary, it yields the desired result. In addition, the rotation field has two states of right and left, and analogously, the spin 1/2 of fermions with its gyromagnetic ratio causes the effective amount of spin, in reaction to a magnetic field, being \( \pm 1 \). Therefore, the non-interacting QED equations result the non-interacting generalized CT.

With respect to the gauge characteristic mentioned before, regarding the necessity of modifying the generalized CT, if we choose Equation (31) to be modified as
\[
\left( i \tilde{\partial} - e \tilde{u} - m \right) \tilde{\psi} = 0,
\]
(34)
then consequently, Equation (25) will be amended as
\[
(2 \gamma + \beta) \tilde{\Delta} \tilde{D} \tilde{\phi} - 4 \alpha \tilde{\phi} = 0,
\]
(35)
where \( e \) is a constant parameter depended on the characteristics of the elastic wave of environment, \( D \equiv d + ie \mathbf{u} \) and \( \Delta \equiv *D* \). Equation (34) is invariant under \( U(1) \) gauge transformation, which indicates that the waves of displacement play the role of momentum for the waves of rotation. Moreover, we need to modify Equation (23) in such a way that also remains invariant under \( U(1) \). Accordingly, we choose
\[
2(\alpha + \mu) F_{\tilde{\alpha} \tilde{\beta}} \tilde{\phi} + e \bar{\psi} \gamma^\dagger \tilde{\psi} = 0.
\]
(36)
With the above choices, Equations (34) and (36) can be gained from the variation of Lagrangian
\[
L = \bar{\psi} \left( i \tilde{\partial} - e \tilde{u} - m \right) \tilde{\psi} - \frac{1}{2} (\alpha + \mu) F_{\tilde{\alpha} \tilde{\beta}} F^{\tilde{\alpha} \tilde{\beta}},
\]
(37)
with respect to \( \bar{\psi} \) and \( u_{\tilde{\alpha}} \), where \( \bar{\psi} \) is the adjoint of \( \tilde{\psi} \).

7. QED treatment

Now, by employing the exchange of variables as
\[
\sqrt{2(\alpha + \mu)} u^\alpha \longrightarrow u^\alpha \quad \text{and} \quad e/\sqrt{2(\alpha + \mu)} \longrightarrow e,
\]
(38)
Lagrangian (37) can be rewritten as

\[ L = \bar{\psi} \left( i \partial_t - e \mathbf{A} - m \right) \psi - \frac{1}{4} F^{\bar{\alpha} \bar{\beta}} F_{\bar{\alpha} \bar{\beta}}. \]  

(39)

This Lagrangian is equivalent to the QED Lagrangian for photon with zero rest-mass and fermion with mass \( m \) and negative charge \( e \) [37], except that the metrics of different parts of it are different. However, for a specific elastic environment via the thermodynamic characteristics, namely the one with properties

\[ \alpha = \mu + \lambda, \quad \varepsilon = \gamma + \beta \quad \text{and} \quad \frac{\alpha + \mu}{\rho} = \frac{\gamma + \varepsilon}{3}, \]  

(40)

this equivalence is exact. In Lagrangian (39), the displacement vector \( u^\alpha \) is analogous with the electromagnetic potential four-vector in electrodynamics, and the tensor \( \varphi^{\alpha \beta} \) is comparable to the fermion spin tensor in the theory of QED. In particular, components \( \varphi^{0i} \) are analogous with the electric dipole ones and components \( \varphi^{ij} \) are comparable to the magnetic dipole ones.

8. Empirical evaluation

The behavior of light in dispersive media can be explained using QED [38]. On the other hand, the behavior of elastic waves in a granular medium can be described by an elastic wave theory in an asymmetric environments [24,39]. In this situation, if the behavior of light in dispersive media being similar to the behavior of elastic waves in granular environments, it is plausible to conclude that such an elastic wave theory in an asymmetric medium would also correspond to the theory of QED.

Having this in mind, let us qualitatively justify our modifications. Accordingly, from experimental point of view, a report has been presented [40] on the empirical evaluation of the CT. Therein, a comparison has been made between their experimental results with the predictions of the CT and also a discrete lattice model [39]. At last, there has been shown that the CT is at least inadequate and the other model is better than the CT. For our purpose, we have qualitatively compared the experimental results of [40] with the presented amended CT. In this regard, we will mainly indicate that the elastic waves behavior in granular environments is similar to the behavior of light in dispersive media.

In Figure 3 of [40] (repeated here as Figure 1 with some subregions marked on it), two regions have been specified. The first region is the low-frequency region (0−80 kHz) or the L-TR region. In this region, in two subregions of less than 20 kHz and 20−50 kHz (the subregions 1 and 2 in Figure 1, respectively), the intensity of wave is maximal, and the arrival-time is almost constant with respect to the variations of frequency (as shown by the arrows in the subregions 1 and 2). Thus the environment for the waves of displacement is transparent and the waves of displacement have constant speed and are massless. However, according to their figure, this environment consists of the subregion of less than 20 kHz (that is related to the longitudinal wave) and the subregion 20−50 kHz (that corresponds to the shear wave which, due to the intensity of wave in these two subregions, is linearly coupled to the longitudinal wave). Now, let us compare this result with the situation considered in [41–43], wherein the motion of light in a dispersive medium has been considered. Their results, particularly Figure 9 of [43], in the region of low-energies and long-wavelengths, illustrate that
Figure 1. (colored online) This figure (the Figure 3 of [40] with some marks on it) shows the arrival-time of received signals from a shear transducer (left) and a longitudinal transducer (right) in terms of frequency and intensity after transmission through a crystal, that we have identified thereon the variations of the wave velocity based on the frequency variation.

the refractive index of the medium (and therefore the speed of light) is almost constant and does not change with respect to changes in the light frequency.

Furthermore, in the subregion 50−80 kHz (the subregion 3 in Figure 1), in addition to the decrement of the wave intensity, the wave arrival-time increases with the increment of the wave frequency (as shown by the arrows in the subregion 3) and thus, the wave speed decreases. Again, let us compare this new result with the mentioned situation in [41–43]. Their results, particularly Figure 9 of [43], in the region of high-energies and short-wavelengths, indicate that the refractive index of the medium (due to the induction of electric and magnetic dipoles and in turn, increase of its permittivity and permeability) increases. Hence, the speed of light in the dispersive medium decreases with respect to changes in the light frequency. In analogous, this situation means that in this subregion, components $\phi_{0i}$ and $\phi_{ij}$ get excited and the wave $u^\alpha$ gradually decays into the wave $\phi^{\alpha\beta}$. Such a decay is visible in Figure 2b of [40] at the end of this region. Also, this decay is similar to decay of high-energy cosmic gamma-rays in collision with low-energy background lights and their pair production. In Figure 9 of [44], the spectral energy distribution of the Crab Nebula from soft to very high-energy gamma rays has been depicted, which is similar to Figure 2b of [40] that shows transmittance measured in the granular crystal. This point indicates similarity between gamma-rays treatment and the presented amended CT.

The second region in Figure 3 of [40] is the high-frequency region (80−150 kHz, the subregion 4 in Figure 1) or the RT-region. In this region, waves show different behaviors compared to the first region. That is, while the intensity of waves is very low, the wave arrival-time decreases with the increment of frequency (as shown by the arrows in the left-hand side of Figure 1) and thus the wave speed increases. This issue means waves in this region are massive. Hence, the hypothesis of the decay of massless waves $u^\alpha$ into massive waves $\phi^{\alpha\beta}$ is confirmed, although due to the high mass of the vibrational waves $\phi^0$, there is no trace of this wave in the subregion 4 in the right-hand side of Figure 1. This result can also be compared to a situation where light with low-energies and long-wavelengths
is moving in a dispersive medium. In this case, a new phenomenon occurs after the environmental refractive index increases sufficiently with increment of the energy of light. This phenomenon is known as the photovoltaic effect, which is the generation of voltage and electric current in a material upon exposure to light [45,46]. For instance, in Figure 3 of [46], it has been illustrated that, in the collision of light with a wavelength in the range of 300 to 400 nm, the photoelectric effect produces electric current in TiO2 nanotubes, however, at shorter and longer wavelengths, almost no current is produced.

Also, the comparison of difference in the intensity of wave, in the subregions 1 and 4 in Figure 1, indicates that the wave intensity of $\phi^{\alpha\beta}$ is approximately the square of the wave intensity of $u^{\alpha}$. This result displays that, contrary to the Cosserat and discrete models, $\phi^{\alpha\beta}$ wave is non-linearly coupled to $u^{\alpha}$ wave, which is consistent with our modification of interaction parts of the generalized CT.

Another point, which is not a new one, is how to apply a continuous theory to a discrete environment. In this respect, when one quantizes a theory, one needs an energy scale to get a renormalized theory. Such an energy scale should be proportional to the size of grains in a granular system. If the frequency of wave in a granular medium is close to such an energy scale, it will be equivalent to when the wavelength approaches to the size of system grains. In this situation, loop-corrections should be considered and the coupling constant of U(1) gauge theory running and being a function of energy of system [37]. Such a situation is equivalent with an effective higher-gradient theory with increasing energy levels as it had been suggested in [40] that the CT should been combined with higher-gradient theories like those in [47], however therein, such terms have been manually added.

9. Conclusions

By considering the importance of correctly understanding the dynamics of microstructure materials, particularly when applied in the related technologies, while eliminating the shortcomings of current elastic models, we have presented an asymmetric elastodynamic model based on the QED as a U(1) gauge theory with quantum aspects. During this process, first we have noticed some overlooked physical concepts in the Cosserat [20] and corresponding discrete models [39], such as disregarding the possibility of vibrational degrees of freedom for microstructures and also the gauge property of $u^{\alpha}$ wave for $\phi^{\alpha\beta}$ wave. Accordingly, we have studied a particular version of linear Cosserat elasticity, i.e. elasticity theory which allows for microrotations of material points.

Then, we have generalized the CT in a four-dimensional form by adding vibrational degrees of freedom for microstructures. For this purpose, we first have shown an equivalency between, a microstructure with rotational and vibrational degrees of freedom in 3-dimensions, and the same microstructure with a rotational degree of freedom in 4-dimensions. Also, by using some mathematical methods and substituting the notion of multi-metrics instead of applying several velocities for different waves (such as longitudinal, transverse, and rotational waves), we have expressed the wave equations in a covariant form.

Subsequently, via a gauge property of the waves of displacement relative to waves of rotation, we have amended the existing interaction terms of the current elastic models instead of somehow modifying those manually. Thus we have established a fruitful analogy
between different subfields of physics and have shown that in asymmetric elastic environments, the elastic wave equations have the QED structure as a $U(1)$ gauge theory. Indeed, we have shown that the presented asymmetric elastodynamic model, based on this gauge theory, produces solutions which exhibit similarities with bosons and fermions in the theory of QED. That is, on macro scales, elastic waves can possess the QED treatment.\textsuperscript{12} This analogy provides a new paradigm of fermions and bosons. On the other hand, due to the phenomenon of loop corrections and the existence of cut-off energy in $U(1)$ gauge theories (such as QED), the internal length scale of discontinuous materials can be comparable to the inverse of cut-off energy. Hence in this process, in addition of being able to take the advantage of continuum techniques, the challenge in describing discontinuous materials can be remedied.

Moreover, from the experimental point of view, due to the analogy between wave intensity of $\varphi^{ij\alpha\beta}$ and $u^\alpha$, we have qualitatively indicated that, contrary to the Cosserat and the corresponding discrete models, our amended CT, as an asymmetric elastodynamic model based on a $U(1)$ gauge theory, are consistent with the corresponding empirical observations. In this point of view, an elastic environment resembles an ether\textsuperscript{13} or a dispersive medium, and the elastic waves resemble the electromagnetic waves in that ether or dispersive medium.\textsuperscript{14} Therefore, we in fact have shown that the behavior of elastic waves in granular media is equivalent to behavior of light in dispersive media, which can be explained using QED, i.e. a kind of asymmetric-elastodynamics/QED correspondence.

The importance of this correspondence, as an attempt to arrive at an acceptable elastodynamic model, becomes more apparent when it is applied in the engineering of modern composite materials, e.g. the functionally graded materials (FGMs) that have had significant impact on design and construction technology. In this regard, discussion on the behavioral analysis of the wave propagation in the FGMs has recently gained a wide range of research interests, see e.g. [1–13]. However, the point of recent studies is that, basically, the Coserat degree of freedom is not usually considered [3,52,53]. Whereas, as mentioned in the previous section, the effect of such a degree of freedom can be observed at high-frequencies and plays an important role in the wave behavior. Thus, if one wants to study the behavior of high-frequency waves in these materials, one will need a genuine elastodynamic model that can accurately explain the wave behavior in these materials, wherein the presented correspondence with the QED might also be fruitful.

Notes

1. In the classical elastic theory, the stress tensor has normally been considered to be symmetric.
2. The displacement waves play the role of a gauge field for the rotation waves and can, in a way, change the transmission speed or momentum of the rotation waves. This behavior is similar to the behavior of a photon relative to an electron in QED.
3. In principle, the idea of modeling fermions by means of the Cosserat elasticity is a challenging task, though there are not that many publications on the subject. Somehow this approach has never really taken off.
4. In Ref. [24], these equations have been written in the Euclidean three-dimensional space, however, we have rewritten those based on the Minkowski metric with signature $+2$. Also, we indicate the partial derivatives with commas and use the Einstein summation convention.
5. $\Upsilon_{ij}$ is the deformation gradient tensor.
6. In [24], the term $4\varphi^{ij}$ in Equation (7) is missing, and it has been stated that total initial conditions have been marked with a symbol therein. However, an initial condition is obviously something
different from a term in field equations, besides the mentioned term is a field, which its existence changes the nature of waves of rotation to a massive wave (as it will be described below). Also, field equations have to be the same independently of any initial conditions, hence for simplicity, we consider initial conditions to be zero.

7. In linear acoustics, a real potential has been presented in a way that the turbulence of pressure and mass density, and speed can be provided in terms of it [25].

8. The ratio of difference between the instantaneous and the equilibrium mass densities to the equilibrium mass density of a medium, at a point, is called condensation [26].

9. Equation (22), without the second interaction term while using relation (9), explains the kinetic potential as a wave [25].

10. See, e.g. [28].

11. The comparison between the waves of fermions and rotation has also been considered in [30,32,33]. Besides, the anticommutation nature of the spinor field (27) has been described in [34], which guarantees the characteristics of the Fermi–Dirac statistics of the waves of rotation. However, by such a comparison, it does not mean that the classical behavior of waves of rotation is quantized. This is just a comparison between the only two left- and right-turning rotations of the classical waves of rotation and the only two up- and down-spin of the waves of fermions. In this respect, the waves of rotation of microstructures can be visualized as the classic states of the waves of fermions. Incidentally, in [34–36], the curl of the waves of displacement has been considered equivalent to the waves of fermions.

12. Also, in [48], we have shown a kind of symmetry that statistically implies uniformity of physics in large and small scales.

13. For a review on the ether, see, e.g. [49] and references therein.

14. Indeed, in another work [50], we have proposed an ethereal model based on a ‘third kind’ of quantization approach [51], in which the electromagnetic waves analogize the elastic waves in an elastic environment.

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No potential conflict of interest was reported by the author(s).

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