On Kirkwood-Salsburg solutions at criticality

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Abstract

In this work we study the Kirkwood-Salsburg equations of equilibrium classical continuous systems. We prove a Laurent expansion for the resolvent, at an eigenvalue of largest modulus of the Kirkwood-Salsburg operator, which is shown to have a pole of order 1. Then we prove that all correlation functions have an asymptotic limit as the activity parameter tends to a smallest zero of the partition function. As corollary, we show that any smallest zero of the partition function is simple. The main consequence is that in case of positive or hardcore potentials we find the spectral radius of the Kirkwood-Salsburg operator and the convergence radius of the solutions.

1 Introduction

The Kirkwood-Salsburg equations, or KS equations for short, is an infinite system of linear inhomogeneous integral equations satisfied by the correlation functions of classical continuous systems in statistical mechanics, see the books Hill [10], Ruelle [15] and Gallavotti [6]. These systems of equations have been developed and studied over the years in statistical mechanics almost always to study phase transitions. In fact, these equations seems to have its origins in the work of Boltzmann about hard sphere fluid, and its modern form is due to Mayer, Kirkwood and their collaborators, see [10]. The KS equations appears in others contexts in statistical mechanics. For instance, in Brascamp [1] is shown that the KS equations for a classical lattice gas are equivalent to the Dobrushin-Lanford-Ruelle equations. Applications of the KS equations on convergence problems of polymer models can be find in Bissacot et al [2] (and references therein) and on point process in Kuna et al [12]. In this work we explore the spectral properties of the parameter activity $z \in \mathbb{C}$ of the Kirkwood-Salsburg equation to study the behavior of the correlation functions near a leading singularity.

The KS operator has non trivial spectral properties which deeply depends on the operator domain as well as the interacting potential. We may see this in Ruelle [16] and [15], Pastur [14], Gorzelaci czyk [8] and Zagrebnov [18] and references therein.

In [15], the Kirkwood-Salsburg equations were introduced as an equation in a Banach space in which is defined the Kirkwood-Salzburg operator. This opened a new way to study classical correlation functions with relevant results. The author in [14] explore the fact that the correlation functions in finite volume are given by the ratio of two entire functions, with the partition function in the denominator, to built a Fredholm Theory for the Kirkwood-Salsburg system in an wider Banach space. The main result in [14] is that the spectrum of the KS operator coincides with the set $z_l^{-1}$, $l = 1, 2, \ldots$, which $z_l$ are zeros of the partition function. In particular, we have only the point spectrum for the KS operator. This same result is find in [8] in a smaller Banach space. In [8] the author also shows a bound on the spectrum of the KS operator in case of Hard Core potentials. Then for such systems it is proved in [8] that all correlation functions are analytic for nonnegative activity $z$ such that $z \leq [(e-1)C]^{-1}$, that is, phase transition does
not occur for these activities values. The positive constant $C$ is defined in the next section in (1.4). From this the system is still in the gas phase if the density $\rho_1$ is such that $\rho_1 < [eC]^{-1}$. This result expanded the previous known region $\rho_1 < [(e + 1)C]^{-1}$, which was proved in [15] for nonnegative potentials. In [18], the author was concern about the spectral properties of the KS operator (defined with nonempty boundary conditions, but the results there follows as well to empty boundary conditions) considering different potentials and domains. He proves, among other things, that the KS operator have only the point spectrum, in which is equal to the inverse of zero’s partition function, if we take the domain of the KS operator as the Banach subspace of symmetric sequences and a interacting potential with repulsive part as a hard core or positive at least in a neighborhood of the origin.

In this work we assume that the interacting potential and the domain of the KS operator, are such that it has the property to have only the point spectrum in which is equal to the set of inverse zeros of the partition function. This a important property to require since, besides we find it in literature on relevant situations (e. g. positive and hard core potentials) as cited above, we can explore the functional analysis. We show, for all volume $\Lambda$, that there exists a Laurent expansion of the resolvent operator, for the KS operator, at an eigenvalue $\lambda_c = z_c^{-1}$ such that its modulus is equal to the spectral radius of the KS operator. Then we prove that $\lambda_c$ is a pole of order 1 and that all $n$–point $\Lambda$ correlation functions are asymptotically to $M_n z \over 1 - z/z_c$ as $z \to z_c$, for some nonzero constants $M_n = M_n(\Lambda)$. From these and the result that inverse of spectral values are zeros of the partition function, mentioned in the preceding paragraph, we have that the smallest zero of the partition function (the closest to the origin) $z_c$ is simple. The main consequence of our tools is the prove, for stable and regular potentials, that the convergence radius of the Cluster expansion is at least $C^{-1}$, with equality if there is a zero of the partition function with modulus equal to $C^{-1}$. For positive or hard core potentials, we then have that the convergence radius of the cluster expansion is equal to $C^{-1}$, with singularity at $z_c = -C^{-1}$. This solves an old problem in classical statistical mechanics of equilibrium of determining the convergence radius of the Cluster expansion of the gas density, as may be seen in the classical works of Groeneveld [9], Penrose [13], Ruelle [16] and [15], as well as in Gorzelańczyk [8] and [7]. We also prove that the convergence radius of the Virial expansion for positive or hard cores potentials is at least $(2C)^{-1}$, which improves the results in Ruelle [15] and Gorzelańczyk [8].

Our method to study correlation functions at criticality are motived from combinatorial problems in which complex analysis play an important role as in analytic combinatorics, see book Flajolet and Sedgewick [5]. In order to find the singularity of the correlation functions, we start looking for an equation involving them (KS equations), following some standard techniques to study singularities of a system of generating functions. The problem is that in case of an infinite system of correlation functions a lot of other questions arise and this approach is not direct. We hope that these ideas can be applied in other problems.

The next Subsection 1.1 contains the definitions. The results and their proofs are in the Section 2. We also include an Appendix, Section 3, and a short discussion of the results in the last Section 4.

### 1.1 Classical continuous systems

Consider a classical continuous system of identical point particles in a bounded volume $\Lambda \subset \mathbb{R}^\nu$. The $n$-point correlation functions in the volume $\Lambda$ with activity $z \in \mathbb{C}$, in the grand canonical ensemble formalism, are defined by
\[ \rho_\Lambda(x)_n = \rho_\Lambda(z; (x)_n) = \Xi_{\Lambda}^{-1}(z) \chi_\Lambda(x)_n \sum_{m=0}^{\infty} \frac{z^{n+m}}{m!} \int_{\Lambda^m} d(y)_m e^{-\beta U((x)_n, (y)_m)}, \] (1.1)

where \((x)_n = (x_1, \ldots, x_n) \in \mathbb{R}^n, \int_{\Lambda^m} d(y)_m = \int_{\Lambda} dy_1 \cdots \int_{\Lambda} dy_m, \)

\[ \chi_\Lambda(x)_n = \prod_{j=1}^{n} \chi_\Lambda(x_j), \quad \chi_\Lambda(x) = \begin{cases} 1, & x \in \Lambda \\ 0, & x \notin \Lambda \end{cases} \]

and \(\Xi_\Lambda(z)\) is the grand-canonical partition function given by

\[ \Xi_\Lambda(z) = 1 + \sum_{m=1}^{\infty} \frac{z^m}{m!} \int_{\Lambda^m} d(y)_m e^{-\beta U(y)_m}. \] (1.2)

The pairwise interaction potential \(\Phi\) is a function depend on the distance between particles and the energy of \(n\) particles is the function \((x)_n \in \mathbb{R}^n \mapsto U(x)_n = \sum_{i<j} \Phi(|x_i - x_j|) \in \mathbb{R}\) which is symmetric respect with all permutations of their arguments and is stable, that is, for all \(n \geq 1\) and \(x_1, \ldots, x_n \in \Lambda\) there is a constant \(B \geq 0\) such that

\[ U(x)_n \geq -Bn. \] (1.3)

We also assume the regularity of the potential,

\[ C := C(\beta) = \int_{\mathbb{R}^n} |e^{-\beta \Phi(x)} - 1| dx < \infty. \] (1.4)

The correlation functions satisfy an infinite system of integral Kirkwood-Salsburg equations [15]:

\[ \rho_\Lambda(z; x_1) = z\chi_\Lambda(x)_1 (1 + (K_\Lambda \rho_\Lambda)(x)_1), \] (1.5)

\[ \rho_\Lambda(z; (x)_n) = z\chi_\Lambda(x)_n (K_\Lambda \rho_\Lambda)(x)_n, \] (1.6)

where \(K_\Lambda\) is the Kirkwood-Salsburg operator:

\[ (K_\Lambda \rho_\Lambda)(x)_1 = \sum_{m=0}^{\infty} \frac{1}{m!} \int_{\Lambda^m} d(y)_m K(x_1; (y)_m) \rho_\Lambda(y)_m, \] (1.7)

and

\[ (K_\Lambda \rho_\Lambda)(x)_n = e^{-W(x_1; (x)_n)} \sum_{m=0}^{\infty} \frac{1}{m!} \int_{\Lambda^m} d(y)_m K(x_1; (y)_m) \rho_\Lambda((x)_n', (y)_m) \] (1.8)

with \((x)_n' := (x_2, \ldots, x_n), W(x_1; (x)_n') := U(x)_n - U(x)_n'\) and

\[ K(x_1; (y)_m) := \prod_{k=1}^{m} (e^{-\beta \Phi(|x_1 - y_k|)} - 1). \]

The Kirkwood-Salsburg equation is considered in [15] in the Banach space \(E_\xi(\Lambda)\), with respect to the norm \(|| \cdot ||_\xi\).
\[ E_\xi(\Lambda) = \{ \varphi = (\varphi(x))_{n \geq 1} : \| \varphi \| = \sup_{n \geq 1} \xi^{-n} \text{ess sup}_{(x) \in \mathbb{R}^\nu} |\varphi(x)_n| < \infty \}. \] (1.9)

We may see that the correlation functions belongs to \( E_\xi(\Lambda) \) if \(|z| < \xi e^{-\beta B}\).

Note that \( E_{\xi_1}(\Lambda) \subset E_{\xi_1}(\Lambda) \), if \( \xi_1 > \xi_2 \). The KS operator may be defined as \( K_\Lambda : E_\xi(\Lambda) \to E_{\xi e^{-\beta B}(\Lambda)} \), see \([15]\).

In order to get spectral results we will consider the KS operator defined on a Banach subspace \( D_\xi(\Lambda) = \text{domain}(K_\Lambda) \subset E_\xi(\Lambda) \) such that \( K_\Lambda : D_\xi(\Lambda) \to D_\xi(\Lambda) \) has the property that its spectrum consists of only the point spectrum in which is equal to the set of inverse zeros of the partition function. This is not a strong assumption, as we can see below, since we find it in relevant situations.

There are, as we know, two special closed subspace \( D_\xi(\Lambda) \) with the desired property above:

(i) \( D_\xi(\Lambda) = E_{\xi}^s(\Lambda) \), in which \( E_{\xi}^s(\Lambda) \) is the subspace of symmetric sequences.

(ii) \( D_\xi(\Lambda) = D_\xi(\Lambda) \), in which

\[ D_\xi(\Lambda) = \{ \varphi = (\varphi_A(a_m)(x)_{n \geq 1} \in E_\xi(\Lambda) : \varphi_A(a_m)(x)_n = \sum_{m=0}^\infty \frac{a_{n+m}}{m!} \int_\Lambda d(z)e^{-\beta U((x)_{n},(y)_m)} < \infty \forall n \geq 1, \exists (x)_n \in \mathbb{R}^{nu}, \text{ and a complex sequence } \{a_m\}_{m=1}^\infty. \] (1.10)

The subspace in (i) is the natural to carry symmetries, \([16],[18]\) and the subspace in (ii) appears in \([8]\). We give a proof in the Appendix, Section 3, that the subspace in (ii) is a Banach space. This subspace seems to be a natural choice, since besides in \([8]\), the Banach subspace in \([14]\), see Lemma 2 therein, may be thought as some \( D_\xi \).

In the following, we state the Theorems (1.1) and (1.2) below, in which we have the property for the KS operator to have only the point spectrum equal to the set of inverse zeros of the partition function.

**Theorem 1.1.** (Zagrebnov \([18]\), Theorem 3.5) If \( D_\xi(\Lambda) = E_{\xi}^s(\Lambda) \) and the pair potential is regular and its repulsive part has a hard core or is positive in some neighborhood of the origin, then the spectrum of \( K_\Lambda \) on \( D_\xi(\Lambda) \) consists of only the point spectrum, and a complex number \( \lambda \neq 0 \) belongs to the spectrum if and only if \( z = \lambda^{-1} \) is a zero of the partition function.

**Theorem 1.2.** (Gorzelniczyk \([8]\), Proposition 1; Pastur \([14]\), Theorem) If the pair potential is regular and stable, then the spectrum of \( K_\Lambda \) on \( D_\xi(\Lambda) \) consists of only the point spectrum, and a complex number \( \lambda \neq 0 \) belongs to the spectrum of \( K_\Lambda \) on \( D_\xi(\Lambda) \) if and only if \( z = \lambda^{-1} \) is a zero of the partition function. The multiplicity of the eigenvalue \( \lambda \) is equal to the corresponding zero. In \([8]\) we have \( D_\xi(\Lambda) = D_\xi(\Lambda) \) and in \([14]\) we have other subspace \( D_\xi(\Lambda) = D \subseteq \bigcup_{\xi \geq 0} E_{\xi}^s(\Lambda) \).

**Remark 1.3.** Our results are based on the fact that the KS operator \( K_\Lambda \) has the property of have only the point spectrum equal to the set of inverse zeros of the partition function. So, what we need is a Banach subspace \( D_\xi(\Lambda) \) and a suitable potential interaction to have this. Although we work with this property itself, the achievement of our results are conditioning on the more general setting in which we have such property.

From the theorems above, we can consider the KS operator \( K_\Lambda \) with our property in one of the followings situations:

(a) \( D_\xi(\Lambda) = E_{\xi}^s(\Lambda) \) and the pair potential is regular and its repulsive part has either a hard core or is positive in some neighborhood of the origin.
(b) \( D_\xi(\Lambda) = D_\xi(\Lambda) \) and pair potential is regular and stable.

In front of this, we consider the KS operator \( K_\Lambda : D_\xi(\Lambda) \to D_\xi(\Lambda) \), which is bounded if it is defined following either (a) or (b) above, see Appendix, Proposition 3.3. We will state our theorems under the assumptions in (b) in order to obtain a more general result.

Remark 1.4. The subspaces \( D_\xi(\Lambda) = D_\xi(\Lambda) \) and \( E_\xi(\Lambda) = E_\xi(\Lambda) \) are Banach spaces invariant under the action of \( K_\Lambda \), see [8], [18] or Appendix Proposition 3.2. In fact, the Kirkwood-Salsburg operator \( K_\Lambda \) acting on \( \varphi(x)^n = \varphi_\Lambda \{a_m\}_{m=1}^\infty \) defines an operator (also denoted by \( K_\Lambda \)) in the space of coefficients \( \{a_m\}_{m=1}^\infty \) acting as

\[
K_\Lambda \{a_m\}_{m=1}^\infty = \{a_{m-1}\}_{m=1}^\infty, \quad a_0 = -\sum_{m=1}^\infty \frac{a_m}{m!} \int_{\Lambda^m} d(y) e^{-\beta U((y)_m)}.
\]  

(1.11)

This is already known important fact since from [14], which may be seen by directed calculations. Due this action of the KS operator, which is similar to the right shift, we have for the particular case of the ideal gas, that the spectrum of the KS operator consists of only the residual spectrum. See [18] for a proof and a more involved discussion.

We want to consider the solvability of the Kirkwood-Salsburg (KS) equations on a \( D_\xi(\Lambda) \), such that the KS operator has only the point spectrum in which coincides with the inverse of zeros of the partition function,

\[
(I - z K_\Lambda) \varphi = z \alpha_\Lambda, \tag{1.12}
\]

or

\[
(K_\Lambda - \lambda) \varphi = -\alpha_\Lambda, \tag{1.13}
\]

with \( \lambda = z^{-1} \), \( \varphi \in D_\xi(\Lambda) \), and \( \alpha_\Lambda = (\alpha_\Lambda(z; (x)_n))_{n\geq 1} \), where \( \alpha_\Lambda(z; (x)_n) = 1 \), if \( n = 1 \) and 0 otherwise.

We will consider the resolvent operator \( R(\lambda) := (K_\Lambda - \lambda)^{-1} \) as an operator valued function of one complex variable.

By the symbol \( f \sim g \) we mean that the functions \( f, g : \Delta \to \mathbb{C} \), for \( \Delta \subset \mathbb{C} \setminus \{z_c\}, z_c \in \tilde{\Delta} \), satisfy \( \lim_{z \to z_c} f(z)/g(z) = 1 \). We also use the same notation for sequences of complex numbers \( f_n \sim g_n \) to mean \( \lim_{n \to \infty} f_n/g_n = 1 \).

2 Results: critical behavior

From now on we assume that the Kirkwood-Salsburg operator are defined, for a non ideal gas, as in (b). See Remarks 1.3 and 1.4.

The goal is to explore the spectral properties of the Kirkwood-Salsburg operator in the problem of solving the equation (1.12)-(1.13). It can be seen along this work that our results comes from the fact that the critical behavior of correlation functions are determined by its components in the direction corresponding to a leading simple eigenvalue in a spectral decomposition of the solution of the Kirkwood-Salsburg equation. We shall see that we are able to determine the behavior of the solution of this equation near a smallest zero of the partition function.

The Theorem 2.1 below gives the asymptotic behavior of the correlation functions and its coefficients at a leading singularity. As a Corollary (2.2), we have that a smallest zero of the partition function is simple.

\footnote{We consider the resolvent operator as in [11] instead of \( (\lambda - K_\Lambda)^{-1} \).}
Theorem 2.1. Let be $\lambda_c = z_c^{-1} \neq 0$ in the spectrum of $K_\Lambda$. Then, for all $\Lambda$, in a neighborhood of $\lambda_c$ holds,

$$R(\lambda) = \frac{-P}{\lambda - \lambda_c} + \sum_{n \geq 0} (\lambda - \lambda_c)^n S^{n+1}, \quad (2.14)$$

with

$$P := \frac{-1}{2\pi i} \int_C R(\lambda) \, d\lambda \quad S := \frac{1}{2\pi i} \int_C \frac{R(\lambda)}{\lambda - \lambda_c} \, d\lambda, \quad (2.15)$$

where $C$ is a circle centered at $\lambda_c$ which its interior, except $\lambda_c$, and the own $C$ is contained in the resolvent set of $K_\Lambda$. Then the resolvent operator $R(\lambda) := (K_\Lambda - \lambda)^{-1}$ has a pole of order 1 at $\lambda_c$ and it is an eigenvalue of multiplicity 1.

Consider the complex function $f(z) := v^*_c (P \rho_\Lambda(z))$, defined on a domain $\Delta \subset \mathbb{C} \setminus \{z_c\}$ to be given, where $v^*_c$ is the dual element of the eigenvector $v_c$ associate with $\lambda_c$. Then

$$f(z) = \frac{Mz}{1 - z/z_c}, \quad (2.16)$$

for some nonzero constant $M = v^*_c(\alpha_\Lambda)$. For all $n \geq 1$,

$$\rho_\Lambda(z; (x)_n) \sim \frac{M_n z}{z \to z_c} \frac{M_n z}{1 - z/z_c}, \quad (2.17)$$

for some nonzero constants $M_n = M_n(\Lambda)$, restricting $\rho_\Lambda(z; (x)_n)$ on $\Delta$. Consequently $|\lambda_c| = r(K_\Lambda)$, the spectral radius of $K_\Lambda$.

Corollary 2.2. The smallest zero $z_c = \lambda_c^{-1}$ of the partition function 1.2 is simple.

Corollary 2.3. The $k$-coefficient of the Taylor expansion of $\rho_\Lambda(z; (x)_n)$ around the origin satisfies for all $n \geq 1$,

$$[z^k] \frac{\rho_\Lambda(z; (x)_n)}{z} \sim M_n \lambda_c^k + o(\lambda_c^{-k}), \quad (2.18)$$

as $k \to \infty$.

The last corollary stress that its asymptotic result comes from the fact that we have a pole of order 1. A pole of order $d > 1$, for instance, would imply a subexponential factor of the form $k^{d-1}$, that is, we would have $\sim \lambda_c^k k^{d-1}$.

The theorems (2.4) and (2.5) determines respectively, for all $\Lambda$, a bound for the spectral radius $r(K_\Lambda)$ of the Kirkwood-Salsburg operator and the gas phase region for systems with stable and regular pairwise interacting potential.

Theorem 2.4. Let be $r(K_\Lambda)$ the spectral radius in $D_\xi(\Lambda)$ . Then, for all $\Lambda$,

$$r(K_\Lambda) \leq \xi^{-1}. \quad (2.19)$$

Theorem 2.5. Consider a nonzero pair potential stable and regular. If $\Lambda \nearrow \mathbb{R}^n$ is an increasing sequence, then for all $n \geq 1$ the limit

$$\rho_n(z) = \lim_{\Lambda \nearrow \mathbb{R}^n} \rho_\Lambda(z; (x)_n), \quad (2.20)$$

there exists for all $z$ in the disc $|z| < C^{-1}$. All correlation functions $\rho_n(z)$ have convergence radius at least $C^{-1}$. If there is a zero of the partition function with modulus $C^{-1}$, then the convergence radius is $C^{-1}$.
Corollary 2.6. For positive or hard core potentials, the convergence radius of \( \rho_1(z) \) is \( C^{-1} \), with the singularity at \(-C^{-1}\). The convergence radius of the Virial expansion is at least \((2C)^{-1}\).

Before start with the main demonstrations, we give two propositions. The first one is a straightforward consequence of definitions and from the equations (1.5)-(1.6).

Proposition 2.7. The vector \( \rho_\Lambda(z; (x)_n) = \Xi_\Lambda(z)^{-1} \varphi_\Lambda\{z^n\}(x)_n, \ n \geq 1, \)

\[ \varphi_\Lambda\{z^n\}(x)_n = \sum_{m=0}^{\infty} \frac{z^{n+m}}{m!} \int_{\Lambda^m} d(y)_m e^{-\beta U(x)_n(y)_m}, \tag{2.21} \]

satisfies formally the KS equations (1.12)-(1.13).

Remember that the resolvent operator \( R(\lambda) = (K_\Lambda - \lambda)^{-1} \) is unbounded if and only if \( \lambda = z^{-1} \) belongs to the spectrum of \( K_\Lambda \). So, from Theorems 1.1 and 1.2, we have the following proposition.

Proposition 2.8. The complex number \( z_c = \lambda_c^{-1} \neq 0 \) is a zero of the partition function if and only if the operator \( R(\lambda_c) \) is unbounded in \( \mathcal{D}_\xi(\Lambda) \).

From these propositions we have that the correlation functions \( \rho_\Lambda(z) \) are in \( \mathcal{D}_\xi(\Lambda) \) if and only if \( z^{-1} \) is in the resolvent set of \( K_\Lambda \).

2.1 Proof of the Theorem 2.1

Proof. Theorem 2.1.

Let be \( \lambda_c = z_c^{-1} \neq 0 \) a complex number in the spectrum of \( K_\Lambda \).

First, note that the projections operators \( P \) and \( I - P \) are well defined because \( \lambda_c = z_c^{-1} \) is an isolated point of the hole spectrum, since we are assuming that the KS operator has only the point spectrum which is equal to the set of inverse zeros of the partition function. Therefore there exist a Laurent expansion for the resolvent as

\[ R(\lambda) = \frac{-P}{\lambda - \lambda_c} - \sum_{n=1}^{\infty} \frac{D^n}{(\lambda - \lambda_c)^n+1} + \sum_{n \geq 0} (\lambda - \lambda_c)^n S^{n+1}, \tag{2.22} \]

where \( P \) and \( S \) were defined and \( D := \frac{1}{2\pi i} \int_{\mathcal{C}}(\lambda - \lambda_c) R(\lambda) d\lambda \). Both projections \( P \) and \( I - P \) commutate with \( K_\Lambda \), and holds \( PD = DP = D, K_\Lambda S = SK_\Lambda \) and \( SP = PS = 0 \). See book Kato [11] Chapter 3, Section 6 for all these and more properties related with these projections. We will state some others as we need. We can see similar results in the book Dunford and Schwartz [4] Chapter VII.

The series involving the operator \( D \) in (2.22) could be either an infinite series or an expansion with only \( n_c \) finite terms if the subspace \( \mathcal{M}_c = P(\mathcal{D}_\xi(\Lambda)) \) is finite dimensional \( n_c = \dim \mathcal{M}_c < \infty \).

We want to show that \( D = 0 \) and then the series in the expansion (2.22) involving the operator \( D \) vanishes. To see this, first we note that assembling the powers of \( \lambda - \lambda_c \) in \( (K_\Lambda - \lambda) R(\lambda) = I \), we will find for all \( n = 1, 2, \ldots, \)

\[ D^n[K_\Lambda(\lambda_c) - D] = 0, \tag{2.23} \]

with \( K_\Lambda(\lambda_c) := K_\Lambda - \lambda_c. \) Indeed, we have
\[ I = (K_\Lambda - \lambda)R(\lambda) = [K_\Lambda(\lambda_c) - (\lambda - \lambda_c)] \left[ \frac{-P}{(\lambda - \lambda_c)} - \sum_{n=1}^{\infty} \frac{D^n}{(\lambda - \lambda_c)^{n+1}} \right] + (K_\Lambda - \lambda)S \sum_{n \geq 0} (\lambda - \lambda_c)^n S^n. \] (2.24)

With the property that \( K_\Lambda(\lambda_c)P = (K_\Lambda - \lambda_c)P = (K_\Lambda - \lambda_c) = D \) and writing as one series the first term in the sum of r.h.s. of (2.24), we have

\[ I = (K_\Lambda - \lambda)R(\lambda) = P + \sum_{n=1}^{\infty} \frac{D^n[K_\Lambda(\lambda_c) - D]}{(\lambda - \lambda_c)^{n+1}} + (K_\Lambda - \lambda)S \sum_{n \geq 0} (\lambda - \lambda_c)^n S^n. \] (2.25)

Now, since hold the identities \( S = (I - P)S = S(I - P) \), and \( (K_\Lambda - \lambda_c)S = (K_\Lambda - \lambda_c) = I - P \), see [11], we have that

\[ (K_\Lambda - \lambda)S = [K_\Lambda - \lambda_c - (\lambda - \lambda_c)]S = I - P - (\lambda - \lambda_c)S = (I - P)[I - (\lambda - \lambda_c)S]. \] (2.26)

Then, putting the identity (2.26) in the last series of r.h.s. (2.25) we end up with

\[ (K_\Lambda - \lambda)S \sum_{n \geq 0} (\lambda - \lambda_c)^n S^n = I - P, \] (2.27)

which together with (2.25) gives (2.23).

We claim that the equation (2.23) implies \( D = 0 \). Indeed, either \( D^n = 0 \) for all \( n = 1, 2, \ldots \), or we must have \( 0 = K_\Lambda(\lambda_c) - D = K_\Lambda - \lambda_c - D \), that is, \( D = K_\Lambda - \lambda_c \). Thus, for all \( r \in \mathcal{M}_c \) we have \( Dr = 0 \) and then \( D |_{\mathcal{M}_c} = 0 \). For \( \mathcal{M}_c' = (I - P)(\mathcal{D}_\xi(\Lambda)) \), \( r = (I - P)v \in \mathcal{M}_c' \), \( v \in \mathcal{D}_\xi(\Lambda) \), note that \( Dr = D(I - P)v = (D - DP)v = (D - D)v = 0 \), since holds the identity \( DP = PD = D \). From this we have that \( D |_{\mathcal{M}_c'} = 0 \) and then, we obtain \( D = DP + D(I - P) = D |_{\mathcal{M}_c} + D |_{\mathcal{M}_c'} = 0 \) on \( \mathcal{D}_\xi(\Lambda) = \mathcal{M}_c \oplus \mathcal{M}_c' \). Thus the part involving \( D \) in the expansion (2.22) vanishes, and since \( D^1 = 0 \) we have \( \dim \mathcal{M}_c = 1 \). Then \( \lambda_c \) is an eigenvalue of (algebraic) multiplicity 1, since it belongs to the spectrum of \( K_\Lambda \) restricted to \( \mathcal{M}_c \), which is one dimensional operator, so it is an eigenvalue of the restricted operator and of \( K_\Lambda \) as well. Therefore is showed the expansion (2.14) and \( R(\lambda) \) has a pole of order 1 at the eigenvalue \( \lambda_c \).

Now we will prove the claims (2.16) and (2.17).

In order to get the asymptotic result we need the Laurent expansion of \( \rho_\Lambda \) at \( \lambda_c \). From the expansion for the resolvent (2.14), we get

\[ \rho_\Lambda = -R(\lambda)\alpha_\Lambda = \frac{P\alpha_\Lambda}{\lambda - \lambda_c} - \sum_{n \geq 0} (\lambda - \lambda_c)^n S_{n+1} \alpha_\Lambda. \] (2.28)

The subspace \( \mathcal{M}_c \) is one dimensional and so, let be \( v \) a basis for \( \mathcal{M}_c \). Because of that and by the fact we have an isolated eigenvalue, the subspace \( \mathcal{M}_c^* \), which is the range of \( P^* \), is also one dimensional, see for instance Kato [11] Chapter 3, Section 6 for details. Then there is a well defined basis \( v^*_c \) of \( \mathcal{M}_c^* \), such that \( v^*_c(v_c) = 1 \) and for all \( v \in \mathcal{D}_\xi(\Lambda) \) we may write \( P^*v = v^*_c(v)v_c. \) Since holds the identity \( SP = PS = 0 \), applying \( P \) in the equation (2.28), we have

\[ P\rho_\Lambda = \frac{P\alpha_\Lambda}{\lambda - \lambda_c} = \frac{v^*_c(\alpha_\Lambda)}{\lambda - \lambda_c}v_c. \] (2.29)
Let be $\Delta := \{ z : |z - z_c| < \eta \text{ and } |\arg(z - z_c)| > \theta \} \subset \mathbb{C} \setminus \{z_c\}$, for some $\eta > 0$ and $0 < \theta < \pi/2$ such that it is free of other zeros of the partition function, that is, we want that the image of $\Delta$ by the map $0 \neq z \mapsto z^{-1}$ is contained in the interior of the circle $\mathcal{C}$ except $\lambda_c$. Thus (writing $z = \lambda^{-1}$) we define on $\Delta$

$$f(z) := v_c^*(P \rho_\Lambda(z)) = \frac{v_c^*(\alpha_\Lambda)}{\lambda - \lambda_c} = \frac{z v_c^*(\alpha_\Lambda)}{1 - z/z_c}, \quad (2.30)$$

which proves $(2.16)$ with $M = v_c^*(\alpha_\Lambda)$.

We also have $(I - P)S = S(I - P) = S$, so we rewrite the equation $(2.28)$ as

$$\rho_\Lambda = P \rho_\Lambda + (I - P) \rho_\Lambda, \quad (2.31)$$

and the analytic part of $\rho_\Lambda$ in the expansion $(2.14)$ is

$$(I - P) \rho_\Lambda = - \sum_{n \geq 0} (\lambda - \lambda_c)^n S^{n+1}. \quad (2.32)$$

Then,

$$\left(1 - \frac{z}{z_c}\right)(I - P) \rho_\Lambda(z) = (\lambda - \lambda_c)(I - P) \rho_\Lambda(z) \to 0, \quad (2.33)$$

as $z \to z_c$ in $\Delta$. Therefore, follows from $(2.29)$-$(2.33)$ that

$$\rho_\Lambda(z; (x)_n) = \frac{M_n z}{1 - z/z_c} + o\left(\frac{z}{1 - z/z_c}\right), \quad (2.34)$$

as $z \to z_c$ in $\Delta$, for some constants $M_n = M_n(\Lambda) \neq 0$. This proves $(2.17)$. Finally, from this and since $|z_c| = |\lambda_c|^{-1}$, we have $|\lambda_c| > |\lambda|$ for any $\lambda$ in the spectrum of $K_\Lambda$ and so $|\lambda_c| = r(K_\Lambda)$, the spectral radius of $K_\Lambda$.

Proof. Proof of the Corollary 2.2.

The proof follows from the theorem’s result and Theorem (1.2).

Proof. Proof of the Corollary 2.3.

To prove this corollary we use the Theorem VI.4 in [5] which translate the asymptotic of the correlation function in $(2.17)$ to coefficients. For that, since $\Delta$ is free from other zeros of the partition function, the identity $(2.34)$ implies that $\rho_\Lambda(z; (x)_n)/z$ have an analytic continuation to the region $\Delta$, and

$$\frac{\rho_\Lambda(z; (x)_n)}{z} = \frac{M_n}{1 - z/z_c} + o\left(\frac{1}{1 - z/z_c}\right), \quad (2.35)$$

So applying the theorem cited above for $\rho_\Lambda(z; (x)_n)/z$, in its little-$o$ version, we have the desired result.
2.2 Proofs of the Theorems 2.4 and 2.5

We denote by \( \lambda_c \) the unique eigenvalue of \( K_\Lambda \) such that \( |\lambda_c| = r(K_\Lambda) > 0 \).

We start with the Lemma 2.9 below. The proofs of Theorems (2.4) and 2.5 are possible through this key lemma, in which comes from the Theorem 2.1.

Lemma 2.9. The Kirkwood-Salsburg operator can be written \( D_\xi(\Lambda) \) as

\[
K_\Lambda = \lambda_c P + T,
\]

(2.36)

which \( T \) is an operator that satisfies \( PT = TP = 0 \), \( \lambda_c = r(K_\Lambda) \). As a consequence we have

\[
(\lambda_c^{-1}K_\Lambda)^n \rightarrow P,
\]

(2.37)
as \( n \rightarrow \infty \), in the operator norm.

Proof. From the proof of the Theorem 2.1, applying \( K_\Lambda \) in the equation (2.31) we have (2.36) with \( T = K_\Lambda(I-P) \). Remember that \( K_\Lambda \) also commutes with \( P \). Since \( P^2 = P \), we have \( PT = TP = 0 \) and so for all \( n \in \mathbb{N} \)

\[
(\lambda_c^{-1}K_\Lambda)^n - P = (\lambda_c^{-1}T)^n, \quad \forall n \in \mathbb{N}.
\]

(2.38)
The right hand side (2.38) above tends to zero in the operator norm. In fact, the operator \( T \) is the KS operator restricted to \( (I-P)D_\xi(\Lambda) \) and \( \lambda_c \) is the unique eigenvalue of \( K_\Lambda \) of largest modulus. Thus we have that the spectral radius of \( \lambda_c^{-1}T \) is at most \( |\lambda_c^{-1}\lambda_2| < 1 \), where \( |\lambda_2| < |\lambda_c| \) is the second largest eigenvalue of the KS operator in modulus. Then, the function \( f_n(z) = z^n \) converges uniformly to 0 on \( |z| \leq |\lambda_c^{-1}\lambda_2| \) and therefore, by Lemma 13 in [4] Chapter VII, we have \( \| (\lambda_c^{-1}K_\Lambda)^n - P \|_\xi = \| (\lambda_c^{-1}T)^n \|_\xi = \| f_n(\lambda_c^{-1}T) \|_\xi \rightarrow 0 \) as \( n \rightarrow \infty \). \( \square \)

Proof. Theorem 2.4.

We want to show \( |\lambda_c| = r(K_\Lambda) \leq \xi^{-1} \). So, suppose the strict inequality \( |\lambda_c| > \xi^{-1} \) in order to obtain a contradiction. Thus, let be \( \lambda_* \) any positive number such that \( |\lambda_c| > \lambda_* > \xi^{-1} \).

Now we claim the following.

Claim: We have,

\[
\lim_{n \rightarrow \infty} \| (\lambda_*^{-1}K_\Lambda)^n - P \|_\xi < \infty.
\]

(2.39)

Proof of the Claim:

By definition of the projection \( P \) in Theorem 2.1, the interior of the curve \( \mathcal{C} \), except at \( \lambda_c \), and the own \( \mathcal{C} \) are in the resolvent set of the KS operator. So, of course, \( P \) is bounded and there exists a constant \( J > 0 \) such that \( \| P \|_\xi < J \). Thus

\[
\| (\lambda_*^{-1}K_\Lambda)^n - P \|_\xi \leq \| (\lambda_*^{-1}K_\Lambda)^n \|_\xi + J.
\]

(2.40)

Let be \( \varphi = (\varphi[z^m]_{m \geq 1}(x))_{m \geq 1} \in D_\xi(\Lambda) \subset \hat{\mathcal{E}}_\xi(\Lambda) \). Then, since \( U \geq -2B \) and \( |z| < \xi \), and because the action of \( K_\Lambda \), see Remark (1.4), we have,

\[
|K_\Lambda^n \varphi[z^m]_{m \geq 1}(x)| \leq \sum_{m \geq 0} \frac{|z|^{m+l-n}}{m!} \left| \int_{\Lambda^n} d(y)m e^{-B_U((x)_n(y)_m)} \right| < \xi^l n e^{2B} e^{\xi \Lambda}.
\]

(2.41)
Thus, for all \( \varphi \in \mathcal{D}_\xi(\Lambda) \),

\[
\| (\lambda^*_L K_\Lambda)^n \varphi \|_\xi = \sup_{i \geq 1} (\xi^{-1})^i \lambda^*_L e^{\epsilon} \sup_{l \geq 1} |K_\Lambda^m \varphi \{z^m\}_{n \geq 1} (x)\| \leq \sup_{l \geq 1} (\xi^{-1})^i \lambda^*_L \xi^{l-n} e^{\epsilon} = (\lambda^*_L \xi^{-1})^n e^{\epsilon} e^2B. \tag{2.42}
\]

Therefore (2.40), (2.42) and \( |\lambda_c| > \lambda_* > \xi \) imply

\[
\lim_{n \to \infty} \| (\lambda^*_L K_\Lambda)^n - P \|_\xi \leq J. \tag{2.43}
\]

The Claim is proved.

We see by Lemma (2.9) and the Claim (2.39) that for all \( \epsilon > 0 \) there is a \( n_0 \in \mathbb{N} \) such that for all \( n > n_0 \),

\[
\| (\lambda^*_L K_\Lambda)^n - (\lambda^*_c K_\Lambda)^n \|_\xi \leq \| (\lambda^*_L K_\Lambda)^n - P \|_\xi + \| P - (\lambda^*_c K_\Lambda)^n \|_\xi = J + 2\epsilon. \tag{2.44}
\]

So, since \( r(K_\Lambda)^n = \lambda_c^n \leq K_\Lambda^n \), we have from (2.44) above that for all \( n > n_0 \),

\[
|\lambda^*_L - \lambda^*_c|^n \leq (J + 2\epsilon) K_\Lambda^n \|_\xi \leq (J + 2\epsilon) \lambda^*_c^n, \tag{2.45}
\]

Finally, this implies that the sequence \( \{(\lambda_c L)^n - 1\} \) is bounded, an absurd if \( |\lambda_c| > \lambda_* > \xi \). Therefore we have \( |\lambda_c| \leq \xi^{-1} \).

\[ \square \]

**Proof. Theorem 2.5.**

Assume \( \xi^{-1} = C \). From the Theorem 2.4 we have that the spectral radius of \( K_\Lambda \), for any \( \Lambda \), is at most \( C \), that is, \( |\lambda_c|^{-1} = |\lambda_*| = r(K_\Lambda) \leq C \). Then for all \( n \geq 1 \) the thermodynamic limits, \( \rho_n(z) \), are analytic functions of \( z \), if \( |z| < C^{-1} \). Indeed, for any measurable compact set \( \Lambda \in \mathbb{R}^n \), if \( |\lambda| = |z|^{-1} > C \geq r(K_\Lambda) \), then

\[
\| \chi_\Lambda[\rho_{A_i}(z, (x)\| - \rho_{A_j}(z, (x)\|) \|_\xi \leq \sum_{n \geq 0} \| K^n_{\Lambda_i} \alpha_{A_i} - K^n_{\Lambda_j} \alpha_{A_j} \|_\xi |\lambda|^{-n-1} < \infty. \tag{2.46}
\]

On the right hand side above, for all \( n \geq 1 \), the term \( \| K^n_{\Lambda_i} \alpha_{A_i} - K^n_{\Lambda_j} \alpha_{A_j} \|_\xi \) tends to zero as \( l, j \to \infty \). So the left side of (2.46) tends to zero as \( l, j \to \infty \), on \( |z| < C^{-1} \). Then the sequence of correlation functions tends to the thermodynamical limit \( \rho_n(z) \) uniformly on compact subsets of \( |z| < C^{-1} \). Then, for each \( n \in \mathbb{N} \), \( \rho_n(z) \) is analytic on \( |z| < C^{-1} \) and therefore the radius of convergence of \( \rho_n(z) \) is at least \( C^{-1} \). Note as well that if there is some zero such that its modulus is \( C^{-1} = \xi \), then the inequality for \( r(K_\Lambda) \) in the Theorem 2.4 becomes equality, and we have the convergence radius equal to \( C^{-1} \).

\[ \square \]

**Proof. Corollary 2.6.**

Let be \( \mathcal{R} \) be the convergence radius of \( \rho_1(z) \). For these particular cases of potentials we have, from [15] Chapter 4, Section 5 Theorem 4.5.3 (see inequalities 5.17, 5.18 and 5.19), that holds:

\[ \mathcal{R} \leq C^{-1}, \]
and for all $s > 0$,

$$\rho_1 \geq \frac{s}{1 + Cs}.$$ 

We also have that $\rho_1(s)$ is an increasing function for $s > 0$.

Then, the proof of the Corollary (2.6) follows from the Theorem 2.5 and the inequalities above. Indeed, the first one together theorems result, give us that $\mathcal{R} = C^{-1}$ and the fact that the singularity of the cluster expansion is at $-\mathcal{R}$ comes from the alternating sign property of its coefficients. Note that this implies, in fact, that $-\mathcal{C}$ belongs to the spectrum. Another way to prove that $\mathcal{R} = C^{-1}$ is the following: the vector $\varphi_{\Lambda}(z) = (\varphi_{\Lambda}(z^m)(x)_n)_{n \geq 1}$,

$$\varphi_{\Lambda}(z^m)(x)_n = \sum_{m=0}^{\infty} \frac{z^{n+m}}{m!} \int_{\Lambda^m} d(y)_m e^{-\beta U((x)_n,(y)_m)}, \quad (2.47)$$

for $\Phi > 0$, belongs to $\mathcal{D}_\xi$, in both cases $\mathcal{D}_\xi(\Lambda) = \mathcal{E}_\xi(\Lambda)$ and $\mathcal{D}_\xi(\Lambda) = \mathcal{D}_\xi(\Lambda)$ (see Proposition 3.2 in the Appendix), if and only if $|z| \leq \xi$. Then the same holds for $\rho_{\Lambda}(z) = \Xi_{\Lambda}^{-1}(z)\varphi_{\Lambda}(z)$. Then, because Proposition 2.8, we may not have a bound on the spectral radius less than $\xi^{-1}$. Therefore we must have $|\lambda_c| = \xi^{-1} = C$.

To see the bound on the convergence radius of the Virial expansion, note that if $0 < s < C^{-1}$ then $\rho_1(s) \geq s(1 + Cs)^{-1} \geq s2^{-1}$ and so we have that radius of convergence of $z$ in powers of $\rho_1$ is at least $(2C)^{-1}$. This implies that the same holds for the Virial expansion. Therefore there does not exist phase transition if $0 < \rho_1 < (2C)^{-1}$.

\[\square\]

3 Appendix

Proposition 3.1. The subspace

$$D_\xi(\Lambda) = \{ \varphi = (\varphi_{\Lambda}\{a_m\}(x)_n)_{n \geq 1} \in \mathcal{E}_\xi(\Lambda) : \varphi_{\Lambda}\{a_m\}(x)_n = \sum_{m=0}^{\infty} \frac{a_{n+m}}{m!} \int_{\Lambda^m} d(y)_m e^{-\beta U((x)_n,(y)_m)} < \infty \} \quad \forall n \geq 1, \forall (x)_n \in \mathbb{R}^{nu}, \text{ and a complex sequence } \{a_m\}_{m=1}^{\infty}. \quad (3.48)$$

is a Banach space.

Proof. Let $\{\varphi^q\}_{q \geq 1} = \{\varphi_{\Lambda}\{a_m^q\}\}_{q \geq 1}$ be a Cauchy sequence in $D_\xi(\Lambda)$, in which,

$$\varphi_{\Lambda}\{a_m^q\} = (\varphi_{\Lambda}\{a_m^q\}(x)_1, \varphi_{\Lambda}\{a_m^q\}(x)_2, \ldots), \quad (3.49)$$

and

$$\varphi_{\Lambda}\{a_m^q\}(x)_n = \sum_{m=0}^{\infty} \frac{a_{n+m}^{q}}{m!} \int_{\Lambda^m} d(y)_m e^{-\beta U((x)_n,(y)_m)} < \infty, \quad \forall q, n \geq 1, \forall (x)_n \in \mathbb{R}^{nu}. \quad (3.50)$$

Let be $\varepsilon > 0$ and $n \in \mathbb{N}$. Because $\{\varphi^q\}_{q \geq 1}$ is a Cauchy sequence and $\varepsilon/\xi^n > 0$, there is an integer $q_0$ such that if $q, l \geq q_0$, then $\|\varphi^q - \varphi^l\|_{\xi^n} < \varepsilon/\xi^n$. So, in particular, for all $n \geq 1$ and almost every $(x)_n$, if $q, l \geq q_0$, then

$$|\varphi_{\Lambda}\{a_m^q\}(x)_n - \varphi_{\Lambda}\{a_m^l\}(x)_n| < \xi^n \|\varphi^q - \varphi^l\|_{\xi^n} < \varepsilon, \quad (3.51)$$
that is, the sequence \( \{ \varphi_A(a_m^q)(x_n) \}_{q \geq 1} \) is a Cauchy sequence in \( C \), for all \( n \geq 1 \) and almost every \( (x)_n \). Define for all \( n \geq 1 \) and almost every \( (x)_n \),

\[
f(x)_n = \lim_{q \to \infty} \varphi_A(a_m^q)(x)_n.
\]

Now, taking the limit \( q \to \infty \) in (3.51) we have that \( f = (f(x)_1, f(x)_2, \ldots) \) is such that \( \| f - \varphi^l \|_{\xi} < \varepsilon \), if \( l \geq q_0 \), and (3.51) also implies \( f \in D_\xi(\Lambda) \). \( \square \)

Next we give a characterization of the subspace \( D_\xi \) of the former Proposition 3.1. It is similar to Lemma 2 in [14] for the subspace considered there, which the next proposition is already implicit.

**Proposition 3.2.** The vector \( \varphi_A = \left( \varphi_A(a_m)(x)_n \right)_{n \geq 1}, \) with

\[
\varphi_A(a_m)(x)_n = \sum_{m=0}^{\infty} \frac{a_{n+m}}{m!} \int_{\Lambda^m} d(y)_m e^{-\beta U((x)_n,(y)_m)},
\]

belongs to \( D_\xi \), defined in (3.48), if and only if the sequence \( \{a_m\}_{n=1}^{\infty} \) satisfies,

\[
\sup_{m \geq 1} \frac{|a_m|}{\xi^m} \sup_{(x)_{m} \in \mathbb{R}^\nu} \exp^{-\beta U((x)_m)} < \infty.
\]

**Proof.** Consider the vector \( \varphi_A' = \left( a_n e^{-\beta U((x)_n)} \right)_{n \geq 1} \). Observe that \( \varphi_A' \in \mathcal{E}_\xi(\Lambda) \) if and only if

\[
\| \varphi_A' \|_{\xi} = \sup_{m \geq 1} \frac{|a_m|}{\xi^m} \sup_{(x)_{m} \in \mathbb{R}^\nu} \exp^{-\beta U((x)_m)} < \infty.
\]

From the inequality \( |a_n e^{-\beta U((x)_n)}| \leq \xi^n \| \varphi_A' \|_{\xi} < \infty \), we have

\[
a_n e^{-\beta U((x)_n)} = \varphi_A'(b_m)(x)_n = \sum_{m=0}^{\infty} \frac{b_{n+m}}{m!} \int_{\Lambda^m} d(y)_m e^{-\beta U((x)_n,(y)_m)} < \infty,
\]

with \( b_{n+m} = a_n \) or 0, if \( m = 0 \) or \( m \geq 1 \), respectively. So, if \( \| \varphi_A' \|_{\xi} < \infty \), then \( \varphi_A' \in D_\xi \).

Now, consider the following bounded operators in \( D_\xi \), \( A_{\Lambda} \) and \( B_{\Lambda} \) defined in [14],

\[
A_{\Lambda} \varphi = \left( A_{\Lambda}(\varphi(x))_n \right)_{n \geq 1}, \quad B_{\Lambda} \varphi = \left( B_{\Lambda}(\varphi(x))_n \right)_{n \geq 1},
\]

where,

\[
A_{\Lambda} \varphi(x)_n = \sum_{m=0}^{\infty} \frac{1}{m!} \int_{\Lambda^m} \varphi((x)_n,(y)_m)d(y)_m, \quad B_{\Lambda} \varphi(x)_n = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \int_{\Lambda^m} \varphi((x)_n,(y)_m)d(y)_m.
\]

We showed in the preceding paragraph that if \( \| \varphi_A' \|_{\xi} < \infty \), then \( \varphi_A' \in D_\xi \). Because it holds \( A_{\Lambda} \varphi_A' = \varphi_A \), we have that if \( \| \varphi_A' \|_{\xi} < \infty \) then \( \varphi_A \in D_\xi \), by the Closed Graph Theorem (CGT). Reciprocally, by the CGT, if \( \varphi_A \in D_\xi \), then \( \varphi_A' = B_{\Lambda}\varphi_A \) belongs as well and so \( \| \varphi_A' \|_{\xi} < \infty \). \( \square \)

**Proposition 3.3.** The Kirkwood-Salsburg operator \( K_\Lambda : D_\xi(\Lambda) \to D_\xi(\Lambda) \) is bounded, if we consider it on the setting (a) or (b) in Remark 1.3.
Proof. If we consider the KS operator defined under the conditions in (a), that is, \( D_\xi(\Lambda) = E_\xi^s \) with positive/hard core potentials, then we can see straightforward from definitions that \( K_\Lambda \) has finite norm \( \| \cdot \|_\xi \) in \( E_\xi \supset E_\xi^s \) and that \( K_\Lambda E_\xi^s \subset E_\xi^s \).

Consider now the KS operator defined under the conditions in (b), that is, \( D_\xi(\Lambda) = D_\xi(\Lambda) \) with regular and stable potentials. For a vector \( \varphi_\Lambda \in D_\xi \) defined by

\[
\varphi_\Lambda \{ a_m \}(x)_n = \sum_{m=0}^{\infty} \frac{a_{n+m}}{m!} \int_{\Lambda^m} d(y)_m e^{-\beta U((x)_n,(y)_m)},
\]

we have by definition of the KS operator that the vector \( K_\Lambda \varphi \) is given by

\[
K_\Lambda \varphi \{ a_m \}(x)_{n+1} = \sum_{m=0}^{\infty} \frac{a_{n+m}}{m!} \int_{\Lambda^m} d(y)_m e^{-\beta U((x)_{n+1},(y)_m)},
\]

in which for \( n = 0 \), the constant \( a_0 \) is given in the Remark 1.4. Therefore, if \( \varphi_\Lambda \in D_\xi(\Lambda) \) then \( K_\Lambda \varphi_\Lambda \in D_\xi(\Lambda) \), and this also proves that \( K_\Lambda \) has finite norm \( \| \cdot \|_\xi \) by definition of \( D_\xi(\Lambda) \).

4 Discussion

The study of the spectral properties of the Kirkwood-Salsburg operator have shown to be challenge due the influence in the spectrum of the interaction potential as well as the operator domain. Fortunately, for representative situations like positive, or more general, stable and regular potentials, and a suitable domain operator, we have the good property to have a point spectrum equal to the set of inverse zeros of the partition function. This allow us to use different techniques to obtain precise bounds on the convergence radius, instead, for instance, to deal with the hard problem of find the exact form of the series coefficients.

The spectrum of the Kirkwood-Salsburg operator formed by eigenvalues equal to inverse zeros of the partition function, give us a lot of similarities with compact operators. This fact was already discussed in Zagrebnov [18], which the author shows, for instance, that for hard core potentials \( K_\Lambda^n \) is compact for some \( n \) big enough. Our Theorem 2.1 in which we prove the Laurent expansion of the resolvent has indeed some similarities with compact operators which is known to have a similar expansion, see Steinberg [17]. From Theorem 2.1, we obtain the key Lemma 2.9. The result of the lemma in (2.37), confirms and expands for more general potentials, the results of [18], see Remark 3.5 therein, that the KS operator is quasipotentially compact in the sense that \( K_\Lambda^n \) can be arbitrarily close to a compact operator for some \( n \) big enough, here the finite dimensional rank spectral projection \( P \).

The asymptotic result (2.17) in Theorem 2.1 is new for classical continuous models and it shows how correlation functions behave close the singularity. Other interesting property proved is the simplicity of the closest zero of the partition function. The prove of this fact seems to be new as far as we know. A result related with this appears in Csikvári [3] where the author proves, among other things, the simplicity of the smallest root of the Independence polynomial in which is related with the partition function of the Hard Core lattice model.

The main consequence of this work is, indeed, the lower bound of at least \( C^{-1} \) of the convergence radius of the Cluster expansion of the density \( \rho_1 \) for stable and regular potentials, Theorem 2.5. The equality for the convergence radius holds if there is a zero of the partition function with modulus \( C^{-1} \). In case of positive or hard core potentials, Corollary 2.6, we have the convergence radius equal to \( C^{-1} \). From the Corollary 2.6 we also have that the convergence
radius of the Virial expansion for positive or hard core potentials is at least $0.5/C$, which improves the bounds $1/(1 + e)C = 0.27/C$ in Ruelle [15] and $1/eC = 0.37/C$ in Gorzeláńczyk [8].

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