Entanglement Sharing and Decoherence in the Spin-Bath

Christopher M. Dawson, Andrew P. Hines, Ross H. McKenzie, and G.J. Milburn

1School of Physical Sciences, The University of Queensland, St Lucia, QLD 4072, Australia
2Centre for Quantum Computer Technology, The University of Queensland, St Lucia, QLD 4072, Australia

The monogamous nature of entanglement has been illustrated by the derivation of entanglement sharing inequalities - bounds on the amount of entanglement that can be shared amongst the various parts of a multipartite system. Motivated by recent studies of decoherence, we demonstrate an interesting manifestation of this phenomena that arises in system-environment models where there exists interactions between the modes or subsystems of the environment. We investigate this phenomena in the spin-bath environment, constructing an entanglement sharing inequality bounding the entanglement between a central spin and the environment in terms of the pairwise entanglement between individual bath spins. The relation of this result to decoherence will be illustrated using simplified system-bath models of decoherence.

While entanglement is argued to be the distinguishing feature of quantum computers, responsible for their power [1], it is also the source of one of the major obstacles in their construction. Decoherence, the process by which a quantum superposition state decays into a classical, statistical mixture of states, is caused by entangling interactions between the system and its environment [2]. Somewhat paradoxically, the quantum entanglement between a system and its environment induces classicality in the system. While it is still a contentious topic as to whether quantum computation will be possible in the face of decoherence, Zurek [3] has demonstrated that decoherence is necessary to facilitate the measurement of a quantum system. Understanding decoherence lies at the heart of measurement, quantum information processing and, more fundamentally, the transition from the quantum to the classical world.

The road to studying decoherence by explicitly modeling system-environment interactions has led to simple models of the quantum environment. Environments can be modeled as either baths of harmonic oscillators [4] or spins (with spin-½) argued to represent distinct types of environmental modes [5]. The simplest system-environment models consist of a central spin (or qubit) coupled to the environment - i.e. the spin-boson model [4] - which has applications to the decoherence of qubits for quantum information processing.

Decoherence of a spin-½ particle at low temperatures may be conveniently modeled by the `central spin' model [5], which couples a central spin-½ particle $S$ to a spin-bath $B$ of $N$ spin-½ particles. A typical Hamiltonian for this model may be written in the form

$$H = H_S + H_B + H_{SB},$$

where $H_S$, $H_B$ are the internal Hamiltonians of the central spin and spin-bath respectively, and $H_{SB}$ is the coupling term. Denote the state of the system-environment at time $t$ by $\rho_{SB}(t)$. Initially at $t = 0$ we take the central spin $S$ to be in a pure state, uncorrelated with the bath. That is,

$$\rho_{SB}(0) = |\psi_S\rangle \langle \psi_S| \otimes \rho_B(0)$$

for some initial state of the bath $\rho_B(0)$. Typically $\rho_B(0)$ is taken to be a thermal state of the Hamiltonian $H_B$, or at low temperatures the ground state.

As the system evolves under $H$ the central spin becomes coupled to the bath, and its reduced density matrix $\rho_S(t)$ at later times is no longer pure. The central spin is said to have decohered, and the amount of decoherence is typically quantified by the von Neumann entropy of its reduced density matrix $S(\rho_S(t))$.

More recently interactions between modes within the bath itself have been considered [6, 7, 8], which allow for appreciable correlations, such as entanglement, to arise between the modes of the bath.

In [6], Tessieri and Wilkie introduced coupling terms between spins in the bath Hamiltonian $H_B$ and, taking the initial state of the bath as a thermal state of $H_B$, found that this resulted in a suppression of the decoherence $S(\rho_S(t))$. The amount of suppression increased as the effective energy scale of $H_B$ increased relative to that of $H_{SB}$, ultimately to the point where decoherence was negligible even after long times. This is somewhat surprising, as even small couplings $H_B$ would usually be expected to eventually result in complete decoherence of the central spin. In this article we aim to demonstrate that this suppression effect may be understood to be a consequence of entanglement-sharing, and that it will be common to any central spin whose environment maintains appreciable internal entanglement while involving in time.

A simple example of such a system is a single spin in a bath of spins with antiferromagnetic interactions between them. In the absence of the spin the ground state of the $N$ bath spins would be something like a spin singlet which is highly entangled. If the single spin interacts antiferromagnetically with the bath spins all it can do is flip individual spins in the bath. The total spin has to be conserved and hence will have a value of order $1/2$. 

*Electronic address: dawson@physics.uq.edu.au
†Electronic address: hines@physics.uq.edu.au
If the bath is initialized in such a state, it will remain highly entangled throughout its interaction with system spin.

Entanglement sharing refers to a striking difference between classical and quantum correlations — the latter may not be shared arbitrarily amongst several observables. The connection with decoherence is readily seen in a system three-spin $\frac{1}{2}$ particles, labeled $S,B_1,B_2$ respectively. It has been shown that entanglement between $B_1$ and $B_2$ limits the individual and collective entanglement they may have with $S$. If a state of the system $\rho(t)$ is evolving under a Hamiltonian such as $H_B$, and moreover if the ‘bath’ $B_1B_2$ maintains appreciable entanglement, then it follows there is a restriction on the entanglement between the ‘central spin’ $S$ and $B_1B_2$. For pure states this equivalent to a restriction on the amount that $S$ may decohere. For mixed states we must also bound the classical correlations between $S$ and $B_1B_2$ which may be done using a recent result of Koashi et al. Entanglement between $B_1$ and $B_2$ thus suppresses all correlations between the central spin and the bath.

The situation becomes far more complicated for spin-baths of $N$ particles. The main difficulty is the plethora of different types of entanglement which exist in these baths, and the absence of good entanglement measures for them. To overcome this difficulty we will assume there is some symmetry in the Hamiltonians $H_S$ and $H_{SB}$. If the initial bath state $\rho_B(t)$ is taken to be a thermal or eigenstate of $H_B$ then the reduced state of the bath $\rho_B(t)$ at later times will also obey this symmetry. For example, the simplest case is that considered by Tessieri and Wilkie where $H_{SB}$ and $H_B$ are completely symmetric. Here the pairwise entanglement between any two bath spins is the same, allowing us to quantify the bath entanglement by a single parameter.

In this paper we will obtain an entanglement-sharing inequality relating the entanglement between a central spin and a completely symmetric spin-bath to the pairwise entanglement in the bath. This inequality is applicable to both pure and mixed states, and is sufficient to restrict decoherence where $\rho_{SB}(t)$ is pure. We will then illustrate this damping effect in a simple model of decoherence originally proposed by Zurek and the Tessieri and Wilkie model. To conclude we will discuss possible extensions of this result to the bounding of classical correlations between the central spin and the bath.

To begin, let $S$ be a central spin-1/2 particle and $B = B_1B_2...B_N$ a completely symmetric spin-bath. As indicated above, the symmetry implies that the entanglement between any pair of bath spins $B_i,B_j$ is the same, allowing us to use a single parameter as a measure of bath entanglement. This entanglement will be called the intra-bath entanglement, while the entanglement between the central spin and the bath will be called the system-bath entanglement. To quantify these we will make use of a measure known as the tangle, whose definition we now briefly recall. For the reduced density matrix $\rho_{B_iB_j}$ of a pair of bath spins $B_i,B_j$ define the spin-flipped density matrix

$$\hat{\rho}_{B_iB_j} = (\sigma_y \otimes \sigma_y) \rho^*_{B_iB_j} (\sigma_y \otimes \sigma_y).$$

The asterisk denotes complex conjugation in the standard basis and $\sigma_y$ is the Pauli Y matrix. The matrix $\rho_{B_iB_j}\hat{\rho}_{B_iB_j}$ can be shown to have real non-negative eigenvalues, and we write their square roots in decreasing order as $\lambda_1, \lambda_2, \lambda_3, \lambda_4$. The tangle between $B_i$ and $B_j$ is then defined as

$$\tau_{B_i|B_j} = (\max \{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\})^2.$$ (4)

This expression is for two spin-1/2 particles, however the tangle between the central spin $S$ and the bath $B$ is also well-defined for pure states of the combined system. The key point is that, because $S$ is a spin-1/2 particle, only two dimensions of the bath state-space are required to expand the pure state in its Schmidt decomposition. The bath may therefore be imagined as a single spin-1/2 particle, with the tangle defined as before. Eq. (4) can be further simplified for pure states so that the system-bath tangle is

$$\tau_{S|B} = 4 \det \rho_S.$$ (5)

For further properties of the tangle, in particular its validity as an entanglement measure, we refer the reader to.

Since all the pairwise intra-bath tangles are the same we write $\tau_B = \tau_{B_i|B_j}$ for all $i,j$. Our aim is to show how this $\tau_B$ constrains the system-bath tangle $\tau_{S|B}$. We will first consider the simplest case of pure states for an $N = 2$ bath, since much is known about states of three spin-1/2 particles. Intuition built in this case will enable us to derive a related inequality for pure states of arbitrary sized baths.

For the two-spin bath, it was shown in and that there are two distinct types of entanglement between $S$ and $B_1B_2$. $S$ can be entangled with the spins $B_1$ and $B_2$ individually, or with the bath $B_1B_2$ as a whole. The latter type is quantified by the three-tangle which we denote by $\tau_{S|B_1|B_2}$. The total entanglement between $S$ and $B$ can now be written as

$$\tau_{S|B} = \tau_{S|B_1} + \tau_{S|B_2} + \tau_{S|B_1|B_2}.$$ (6)

The three-tangle is invariant under permutations of the three spins, and may be written alternatively as

$$\tau_{S|B_1|B_2} = \tau_{S|B_1B_2} - \tau_{S|B_1} - \tau_{S|B_2}$$ (7)

$$\tau_{S|B_1|B_2} = \tau_{B_1|S|B_2} - \tau_B - \tau_{B_1|S}.$$ (8)

A simple consequence of this, together with the fact that the tangle is a positive quantity less than or equal to one, is

$$\tau_B + \tau_{S|B_1|B_2} \leq 1.$$ (9)

This inequality says that the intra-bath entanglement plus the three-tangle part of the system-bath entanglement is always less than 1. On the other hand, the sum
of \( \tau_{SB} + \tau_{S|B_1} + \tau_{S|B_2} \) can be greater than 1 — it can take any value up to and including 4/3. This suggests that intra-bath entanglement has a stronger damping effect on the three-tangle component of \( \tau_{SB} \) than it does on the pairwise tangle component. We will therefore assume that, for a fixed intra-bath tangle, a maximum system-bath entanglement is obtained when \( \tau_{S|B_1} | B_2 = 0 \). That is, when it is composed entirely of the pairwise components in Eq. (8).

States of the \( SB_1B_2 \) system with \( \tau_{S|B_1} | B_2 = 0 \) are equivalent under local unitary operations to so called W-class states of the form

\[
|\psi\rangle = a|\uparrow\rangle_S|\downarrow\rangle_B + b|\uparrow\rangle_S|\downarrow\rangle_B
+ c|\downarrow\rangle_S|\uparrow\rangle_B + d|\uparrow\rangle_S|\uparrow\rangle_B
\]

where \( a, b, c, d \) are real and non-negative and \( a^2 + b^2 + c^2 + d^2 = 1 \). The tensor factors in each term refer to the state of the central spin and of the two bath spins respectively. It is a simple matter to calculate the relevant tangles from Eqs. (4,5)

\[
\tau_B = 4a^2b^2
\]

\[
\tau_{S|B} = 4(a^2 + b^2)c^2.
\]

We will solve the equivalent, and as it turns out slightly easier, problem of maximizing \( \tau_B \) for fixed \( \tau_{S|B} = T' \). That is, we must maximize

\[
g(a, b, c, d) = 4a^2b^2
\]

subject to the constraints

\[
F_1(a, b, c, d) = 4(a^2 + b^2)c^2 - T = 0
\]

\[
F_2(a, b, c, d) = a^2 + b^2 + c^2 + d^2 - 1 = 0.
\]

This can be solved by the method of Lagrange multipliers, and we find the maximum \( \tau_B \) is given by

\[
\tau_B = \frac{1}{4}(1 + \sqrt{1 - \tau_{S|B}})^2.
\]

The corresponding entanglement-sharing inequality for the system-bath and intra-bath tangles is then

\[
\tau_{S|B} \leq \left\{ \begin{array}{ll}
\frac{1}{4} & \tau_B \leq \frac{1}{4} \\
\frac{1}{4}(\sqrt{1-\tau_B} - \tau_B) & \tau_B \geq \frac{1}{4}.
\end{array} \right.
\]

For values of the intra-bath tangle less than 1/4 the system and the bath may be maximally entangled. As \( \tau_B \) increases however, we find that \( \tau_{S|B} \) falls in an approximately linear fashion, and is 0 when the intra-bath tangle is at a maximum. This confirms our expectation that strong quantum correlations in the environment limit decoherence effects, at least for pure states of the combined system.

We saw above that the three-tangle component of the system-bath entanglement was more strongly limited by the intra-bath entanglement than the pairwise components \( \tau_{S|B_1}, \tau_{S|B_2} \). In the case of an \( N \)-spin bath it seems reasonable that we should expect the same, this time potentially for three-party and other higher order quantum correlations between \( S \) and the bath. We will therefore assume that analogues of the W-class states are able to achieve maximum system-bath entanglement for a given intra-bath entanglement. An inequality similar to Eq. (17) follows from this assumption and has been confirmed numerically for small values of \( N \).

An analogue of a W-class state should ideally be one where the system is only entangled with each of the bath spins individually. We will use a generalization of the states (10) given by

\[
|W\rangle = a_1|\uparrow\rangle_S|\uparrow\rangle_B + a_2|\uparrow\rangle_S|\uparrow\rangle_B
+ \cdots
+ a_N|\uparrow\rangle_S|\uparrow\rangle_B + c|\downarrow\rangle_S|\uparrow\rangle_B
+ \cdots
+ d|\uparrow\rangle_S|\uparrow\rangle_B
\]

for real \( a_i, c, d \) where \( \sum_{i=1}^{N} a_i^2 + c^2 + d^2 = 1 \). \( a_i \) is the coefficient of the state where the \( i \)-th bath spin is down. From Eqs. (11,5) we find that the tangle between any pair of bath-spins is given by

\[
\tau_{B_i | B_j} = 4a_i^2a_j^2,
\]

and the tangle between the central spin and the bath is given by

\[
\tau_{S|B} = 4c^2 \sum_{i=1}^{N} a_i^2.
\]

The symmetry constraint implies that \( a_i = a_j = a \) for all \( i, j \leq N \), and it follows that

\[
\tau_B = \tau_{B_i | B_j} = 4a^4
\]

\[
\tau_{S|B} = 4Na^2c^2.
\]

Fixing \( \tau_{S|B} = D \) we can maximize \( \tau_B \) as we did for the \( N = 2 \) case, and subsequently obtain a maximum \( \tau_B \) at

\[
\tau_B = \frac{1}{N^2}(1 + \sqrt{1 - \tau_{S|B}})^2
\]

with the corresponding entanglement-sharing inequality

\[
\tau_{S|B} \leq \left\{ \begin{array}{ll}
N & \tau_B \leq \frac{1}{4} \\
N(2\sqrt{\tau_B} - N\tau_B) & \tau_B \geq \frac{1}{4}.
\end{array} \right.
\]

This inequality is identical to Eq. (17) up to a dimensional scaling. Note that the maximum possible pairwise tangle for a symmetric bath of \( N \) spins has been shown to be \( 4/N^2 \), and that the system-bath tangle falls to 0 for this value of \( \tau_B \).

Of course, we have only demonstrated this inequality for the W-class states Eq. (13). To verify the inequality numerically for small values of \( N \) we calculated \( \tau_{S|B} \) and \( \tau_B \) for random states having the appropriate bath symmetry. A sample size of \( 1 \times 10^7 \) was used, and to reduce the sample space we used the generalized Schmidt
decomposition \cite{footnote}. No violations of Eq. \eqref{eq:violation} were found for \( N \leq 5 \).

The extension of Eq. \eqref{eq:violation} to mixed states \( \rho \), where the formula \eqref{eq:noschmidt} no longer valid is straightforward. Given a pure state decomposition \( \rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| \) we may define the average system-bath tangle by

\[
\bar{\tau}_{SB}(\rho) = \sum_i p_i \tau_{SB}(|\psi_i\rangle).
\] (25)

The minimum \( \tau_{SB}(\rho) \) over all pure-state decompositions \( \{p_i, |\psi_i\rangle\} \) of \( \rho \) can then be used to quantify the quantum correlations between the system and the bath.

The concavity of Eq. \eqref{eq:concavity} allows us to write

\[
\frac{1}{N^2} \left( 1 + \sqrt{1 - \tau_{SB}^{\min}(\rho)} \right)^2 \geq \sum_i p_i \tau_B(|\psi_i\rangle).
\] (26)

On the other hand the tangle is convex so we have

\[
\sum_i p_i \tau_B(|\psi_i\rangle) \geq \tau_B(\rho),
\]

and thus obtain the following inequality

\[
\frac{1}{N^2} \left( 1 + \sqrt{1 - \tau_{SB}^{\min}(\rho)} \right)^2 \geq \tau_B(\rho)
\] (27)

which we can invert to obtain the entanglement-sharing inequality for mixed states.

One simple model of decoherence where the inequality \eqref{eq:violation} is immediately applicable is an exactly solvable model introduced by Zurek \cite{zurek} and recently used to investigate the structure of the decoherence induced by spin environments \cite{footnote}. The system is always in a pure state, so there are no classical correlations and a bound on the system-bath entanglement is a bound on the decoherence.

The Hamiltonian of this model, after applying the complete symmetry constraint, is written

\[
H_{SB} = \frac{1}{2} g \sum_{k=1}^{N} \sigma_z^{(a)} \sigma_z^{(B_k)}.
\] (28)

It is possible to analytically solve this model to give a good illustration of how the decoherence of the central spin - as quantified by the decay of the off-diagonal elements of the reduced density operator of the system \cite{footnote} - is suppressed by the presence of entanglement between the bath spins. Starting with a separable system-bath (SB) state

\[
|\Psi_{SB}\rangle = (|\downarrow\rangle_s \chi |\uparrow\rangle_B + \gamma |\downarrow\rangle_B |\uparrow\rangle_s) \otimes |B(0)\rangle,
\] (29)

the state of \( SB \) at an arbitrary time \( t \) is

\[
|\Psi_{SB}(t)\rangle = \chi |\downarrow\rangle_s |B_1(t)\rangle + \gamma |\uparrow\rangle_s |B_1(t)\rangle
\] (30)

where

\[
|B_1(t)\rangle = e^{igt} \sum_{k=1}^{N} \sigma_z^{(B_k)} |B(0)\rangle.
\] (31)

The state of the system is then described by the reduced density operator,

\[
\rho_S = |\chi|^2 |\downarrow\rangle_s \langle \downarrow | + \chi \gamma r(t) |\downarrow\rangle_s |\uparrow\rangle_s + \gamma \chi r^*(t) |\uparrow\rangle_s \langle \downarrow | + |\gamma|^2 |\uparrow\rangle_s \langle \uparrow | + \chi \gamma^* r^*(t) |\downarrow\rangle_s \langle \uparrow | + \gamma \chi^* r(t) |\uparrow\rangle_s \langle \downarrow |
\] (32)

where the \textit{decoherence factor} \cite{footnote}, \( r(t) = \langle B_1(t)|B_1(t)\rangle \) can be easily calculated. The absolute value of this factor is bounded by \( 0 \leq |r(t)|^2 \leq 1 \), corresponding to complete decoherence to a statistical mixture (0) and no loss of coherence (1), respectively. The \textit{SB} tangle, \( \tau_{SB}(t) \), can be written in terms of this factor by

\[
\tau_{SB}(t) = 4|\chi|^2 |\gamma|^2 (1 - |r(t)|^2)
\] (33)

We first consider an initial bath state of the form

\[
|B(0)\rangle = \bigotimes_{k=1}^{N} (|\downarrow\rangle_B + |\uparrow\rangle_B)
\] (34)

which is completely separable, with each individual bath spin in an identical state (preserving the symmetry). It is a relatively simple exercise to calculate the decoherence factor,

\[
|r(t)|^2 = [|\alpha|^4 + |\beta|^4 + 2|\alpha|^2 |\beta|^2 \cos(2gt)]^N.
\] (35)

As argued in Zurek \textit{et. al.} \cite{footnote}, as \( N \to \infty \), the average value, \( \langle |r(t)|^2 \rangle \to 0 \), implying complete decoherence of the initial state. This is the average over time, since for large \( N \), \( |r(t)|^2 \) is predominantly zero (over time) but will revive to one periodically. However as \( N \to \infty \), these revival approach delta functions in time. With no intra-bath entanglement (\( \tau_B = 0 \)), there is no bound on \( \tau_{SB} \), resulting in maximal possible entanglement between system and bath. Unentangled baths of this form were the topic of Ref. \cite{footnote}.

We now consider an initial entangled environment state. Following from the previous construction of the entanglement sharing constraint, we choose an initial state of the form

\[
|B(0)\rangle = \frac{a}{\sqrt{N}} \left( \bigotimes_{k=1}^{N} (|\downarrow\rangle_B + |\uparrow\rangle_B) + \sum_{d} |\downarrow\rangle_B \bigotimes_{k=1}^{N} |\downarrow\rangle_B + d |\downarrow\rangle_B \bigotimes_{k=1}^{N} |\downarrow\rangle_B \right)
\] (36)

where \( a^2 + d^2 = 1 \), such that the entanglement between any two bath spins is \( \tau_B = 4a^2 \). Since the system-bath interaction does not flip spins, for such initial states the intra-bath entanglement is invariant over the evolution. In other words, the bath spins maintain their entanglement. From this initial bath state, the decoherence factor is

\[
|r(t)|^2 = |a|^4 + |d|^4 + 2|a|^2 |d|^2 \cos(2gt),
\] (37)

which, firstly, does not average to zero in the limit of large \( N \) and, in fact, will not be zero at anytime for given values of \( a \) and \( d \) (see Figure \ref{fig:decoherence}). This can be interpreted as a suppression of decoherence, since at no time will the system ever be a complete statistical mixture of states.

The inequality only places a nontrivial upper bound on the system-bath entanglement when \( \tau_B \geq 1/N^2 \). For the states considered here, this corresponds to the parameter range \( 1/\sqrt{2} \leq a \leq 1 \), to which we will now restrict ourselves. The system-bath tangle is given by

\[
\tau_{SB} = 2|a|^2 (1 - |a|^2)(1 - \cos(2gt)).
\] (38)
Following Ref. [6], $\beta$ and the interaction, however, we set $\omega$ critical. The bath starts in the thermal state, a reservoir of (identical) qubits has considered the process of homogenization [16], of which thermalization is a special case [17]. The system qubit is initially in some special case [17]. The system qubit is initially in some...
state \( \rho \), with the all bath spins each in the identical state, \( \xi \). The aim of the process is to output all qubits in some arbitrarily small neighborhood of \( \xi \). Thermalization is the case were \( \xi \) corresponds to the thermal state. This thermalization process is equivalent to the decoherence of the system qubit to a thermal state.

In this discrete time process, the system qubit interacts with a only single bath qubit at each time step, and never the same qubit twice. It is shown that the partial swap operation uniquely determines a universal quantum homogenizer [16]. While there is no explicit interaction between bath qubits, their mutual interaction with the system qubit generates entanglement not only between the system and reservoir, but also intra-bath entanglement. This entanglement is studied in [16] and the results agree with the entanglement sharing arguments we have made here. Specifically, in the example considered, the entanglement between system and bath decreases in the long term, as more bath qubits become entangled with each other. Interestingly, it is shown that all entanglements are pairwise, with no multi-party entanglement present [18]. It would be interesting to extend the work in these articles by considering thermalization in the presence of a self-interacting bath. Of course, different methods would have to be employed, since the state of the bath qubits would change after each interaction.

Decoherence is the major stumbling block on the road to quantum computing. Here we have introduced a novel way of constraining the decoherence effects from a spin-bath environment. Such environmental models are of particular importance for predicting decoherence effects in solid-state qubits in the low temperature regime [19, 20].

We have used two simplified models as examples of how entanglement in the environmental bath may suppress decoherence. While we have only discussed spin-baths, one could also envision similar effects for oscillator baths, where entangled spins may be replaced by multi-mode squeezed states. As well we have focussed on two-party entanglement in the bath. The effects of \( m \)-party entangled states may be quite different.

The types of entangled states of the bath that may be created and maintained will depend explicitly upon the physical system in question. To discover if entanglement-sharing can suppress decoherence in realistic situations requires calculations for specific quantum computer architectures. Only then will it be apparent if this unique property of entanglement can be used to our advantage in overcoming decoherence.

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