A Non-commutative Version of
the Fundamental Theorem of Asset Pricing*

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Abstract. In this note, a non-commutative analogue of the fundamental theorem of asset
pricing in mathematical finance is proved.

1. Introduction

In retrospect, the field of mathematical finance has undergone a remarkable development
since the seminal papers by F.Black and M.Scholes [2] and R.Merton [15], in which the famous
“Black-Scholes Option Pricing Formula” was derived. The idea of developing a “formula” for
the price of an option actually goes back as far as 1900, when L.Bachelier wrote a thesis with
the title “Théorie de la spéculations” [1]. It was Bachelier who firstly had the innovative idea of
using a stochastic process as a model for the price evolution of a stock. For a stochastic process
\((S_t)_{0 \leq t \leq T}\) he made a natural and far-reaching choice being the first to give a mathematical
definition of Brownian motion, which in the present context is interpreted as follows: \(S_0\) is
today’s (known) price of a stock (say a share of company XYZ to fix ideas) while for the time
\(t > 0\) the price \(S_t\) is a normally distributed random variable.

The basic problem of Bachelier, as well as of modern Mathematical Finance in general,
is that of assigning a price to a contingent claim. Bachelier used the equilibrium argument.
It was the merit of Black and Scholes [2] and Merton [15] to have replaced this argument
by a so-called “no-arbitrage” argument, which is of central importance to the entire theory.
Roughly speaking, an arbitrage is a riskless way of making a profit with zero net investment.
An economically very reasonable assumption on a financial market consists of requiring that
there are no arbitrage opportunities. The remarkable fact is that this simple and primitive

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“principle of no arbitrage” allows already to determine a unique option price in the Black-
Scholes model. This is the theme of the so-called fundamental theorem of asset pricing which
states briefly that a process \( S = (S_t) \) does not allow arbitrage opportunities if and only if
there is an equivalent probability measure under which \( S \) is a martingale.

The history of the fundamental asset pricing theorem goes back to the seminal work
of Harrison, Kreps and Pliska ([11, 12, 14]). After their pioneering work many authors
made contributions to gradually improve the understanding about this fundamental theorem,
e.g., Duffie and Huang [10], Stricker [21], Dalang, Morton, and Willinger [6], and Delbaen
and Schachermayer [7] etc. In [8] this theorem was proved to hold true for very general
(commutative) stochastic processes.

In this note we deal with this issue in the non-commutative (= quantum) setting. After
having formalized the notations of (quantum) arbitrage and quantum trading strategies, we
shall prove a non-commutative analogue of the fundamental theorem of asset pricing. As
shown in [4], there are several reasons why quantizing mathematical finance may be inter-
esting. In particular, classical mathematical finance theory is a well established discipline
of applied mathematics (see [9, 20] and references therein) which has found numerous ap-
lications in financial markets (see for example [13, 16]). Since it is based on probability to
a large extend, there is a fundamental interest in generalizing this theory to the domain of
quantum probabilities. Indeed, recently non-commutative (= quantum) probability theory
has developed considerably. In particular, all sorts of non-commutative analogues of Brown-
nian motion and martingales have been studied. We refer to [17] and references therein.
Moreover, it has recently been shown that the quantum version of financial markets is maybe
much more suited to real-world financial markets rather than the classical one, because the
quantum binomial model ceases to pose the paradox which appears in the classical model of
the binomial market, see [3, 5] for details.

2. Notational preliminaries and the main result

Throughout this note we shall denote by \((\mathcal{A}, \tau)\) a \(W^*\)-non-commutative probability space,
namely, \(\mathcal{A}\) is a finite von Neumann algebra, and \(\tau\) is a faithful normal tracial state on \(\mathcal{A}\). (See
[18, 23] for details on von Neumann algebras.) We shall denote by \(L^p(\mathcal{A}, \tau)\) or simply \(L^p(\mathcal{A})\)
the non-commutative \(L^p\)-spaces. Note that if \(p = \infty\), \(L^p(\mathcal{A})\) is just \(\mathcal{A}\) itself with the algebra
norm; also recall that the norm in \(L^p(\mathcal{A})\) \((1 \leq p < \infty)\) is defined as
\[
\|a\|_p = \tau[|a|^p]^{\frac{1}{p}},
\]
where \(|a| = (a^*a)^{1/2}\) is the usual absolute value of \(a\). We shall assume that \(\mathcal{A}\) is filtered, so that
there exists a family \((\mathcal{A}_t)_{t \in \mathbb{R}_+}\) of unital weakly closed \(\ast\)-subalgebras of \(\mathcal{A}\), such that \(\mathcal{A}_s \subset \mathcal{A}_t\)
for all \(s, t\) with \(s \leq t\), and \(\mathcal{A}_0 = CI\), \(I\) denoting the unit element in \(\mathcal{A}\). Since the state \(\tau\) is
tracial, for any unital weakly closed \(\ast\)-subalgebra \(\mathcal{B}\) of \(\mathcal{A}\), there exists a unique conditional
expectation onto \(\mathcal{B}\). We shall denote by \(E_\tau[\cdot|\mathcal{B}]\) this conditional expectation. Recall that it
extends to a contraction on all $L^p$-spaces for $1 \leq p \leq \infty$. A map $t \mapsto M_t$ from $[0, +\infty)$ to $L^p(\mathcal{A}, \tau)$ will be called a martingale with respect to the filtration $(\mathcal{A}_t)_{t \in \mathbb{R}_+}$ if for every $s \leq t$ one has that $E_{\tau}|M_t|A_s| = M_s$.

However, even for a state $\sigma$ in a finite dimensional von Neumann algebra $\mathcal{A}$ the conditional expectation operator $E_{\sigma}[.|\mathcal{B}]$ of a $*$-subalgebra $\mathcal{B}$ of $\mathcal{A}$ does not need to exist in general (for details see [22]). Thus we cannot define a martingale under $\sigma$ as in the case of the tracial states or the commutative setting. It seems to us that one needs to generalize the definition of martingales in the non-commutative setting as following:

**Definition 1.** Given any fixed state $\sigma$ on $\mathcal{A}$. A family $\{M_t\}_{t \geq 0}$ in $\mathcal{A}$ is said to be a (non-commutative) martingale with respect to $(\mathcal{A}, (\mathcal{A}_t)_{t \geq 0}, \sigma)$ if it is adapted to $(\mathcal{A}_t)_{t \geq 0}$ and for every $0 \leq s \leq t$,

$$\sigma(aM_t a^*) = \sigma(aM_s a^*),$$

for all $a \in \mathcal{A}_s$.

Clearly, when $\sigma$ is a normal tracial state the above definition coincides to the usual definition of the non-commutative martingales. In the sequel we understand the non-commutative martingales in this sense. We would like to point out that those martingales in the above sense are suitable in the so-called quantum finance, for details see [4].

Together with $(\mathcal{A}, \tau)$ we shall also consider the opposite algebra $\mathcal{A}^{op}$, with the trace $\tau^{op}$, namely $\tau = \tau^{op}$ as a linear map on $\mathcal{A}$, but the notation is meant to stress the algebra structure we are using. The spaces $\mathcal{A}$ and $\mathcal{A} \otimes \mathcal{A}$ have natural $\mathcal{A} - \mathcal{A}$ bimodule structures given by multiplication on the right and on the left, namely $a.u.b = aub$ and $a.(u \otimes v).b = au \otimes vb$, or equivalently they have a left $\mathcal{A} \otimes \mathcal{A}^{op}$-module structure. We shall denote by $\sharp$ these actions, namely one has $(a \otimes b)\sharp u = aub$ and $(a \otimes b)\sharp(u \otimes v) = (au) \otimes (vb)$. The map $\tau \otimes \tau^{op}$ defines a tracial state on the $*$-algebra $\mathcal{A} \otimes \mathcal{A}^{op}$, and we shall denote by $L^p(\tau \otimes \tau^{op})$ the corresponding $L^p$-spaces, thus $L^\infty(\tau \otimes \tau^{op})$ is the von Neumann algebra tensor product of $\mathcal{A}$ and $\mathcal{A}^{op}$.

A simple biprocess is a piecewise constant map $t \mapsto H_t$ from $\mathbb{R}_+$ into the algebraic tensor product $\mathcal{A} \otimes \mathcal{A}^{op}$, such that $H_t = 0$ for $t$ large enough. It is called to be adapted if one has $H_t \in \mathcal{A}_t \otimes \mathcal{A}_t$ for all $t \geq 0$. In this case, it is clear that one can choose a decomposition

$$H_t = \sum_{j=1}^n A_{j,t} \otimes B_{j,t} \quad (1)$$

such that there exist times $0 = t_0 \leq t_1 \leq ... \leq t_m$ with $A_{j,t} = A_{j,t_k}$, $B_{j,t} = B_{j,t_k} \in \mathcal{A}_{t_k}$ for $t \in [t_k, t_{k+1})$, $A_{j,t} = B_{j,t} = 0$ for all $t \geq t_m$ (in the sequel we shall always assume that the decompositions we choose satisfy such properties).

In the sequel we always assume that $X = (X_t)_{t \geq 0}$ is a self-adjoint stochastic process adapted to the filtered space $(\mathcal{A}, (\mathcal{A}_t)_{t \geq 0}),$ i.e., for every $t \geq 0, X_t \in \mathcal{A}_t$ and $X_t^* = X_t$.

**Definition 2.** Let $H$ be a simple adapted biprocess with a decomposition as above, then
the stochastic integral of $H$ with respect to $X = (X_t)_{t \geq 0}$ is
\[
\int_0^\infty H_s \mathbb{d}X_s = \sum_{k=0}^{m-1} H_{tk} (X_{tk+1} - X_{tk}) = \sum_{k=0}^{m-1} \sum_{j=1}^{n_k} A_{j,tk} (X_{tk+1} - X_{tk}) B_{j,tk}, \tag{2}
\]
This is clearly independent of the decomposition chosen.

For a simple adapted biprocess $H$, and $s < t$, we shall denote $H_{r}^{(s,t)}$ the stopped simple adapted biprocess given by $H_{r}^{(s,t)} = H_r$ for $s \leq r < t$ and $H_{r}^{(s,t)} = 0$ for $r < s$ or $r \geq t$. Then we define
\[
\int_s^t H_r \mathbb{d}X_r = \int_0^\infty H_{r}^{(s,t)} \mathbb{d}X_r.
\]
We shall write $(H_t X)_t = \int_0^t H_r \mathbb{d}X_r$.

**Remark 1.** The space of adapted simple biprocesses has an antilinear involution, coming from the antilinear involution on $\mathcal{A} \otimes \mathcal{A}$
\[
(\sum A_j \otimes B_j)^* = \sum B_j^* \otimes A_j^*.
\]
The adjoint of the stochastic integral is again a stochastic integral, namely with the adjoint of a biprocess as above, one has that
\[
(\int_0^\infty H_t \mathbb{d}X_t)^* = \int_0^\infty H_t^* \mathbb{d}X_t.
\]

**Definition 3.** $\mathcal{H}$ denotes the set of simple quantum trading strategies for $X = (X_t)_{t \geq 0}$. An element $H = (H_t)_{t \geq 0} \in \mathcal{H}$ is a simple biprocess of the form
\[
H_t = \sum \alpha_j a_j \otimes a_j^*,
\]
with $a_j \in \mathcal{A}_t$, where $\alpha_j$ are all real numbers.

**Remark 2.** Evidently, $\int_0^\infty H_t \mathbb{d}X_t$ is self-adjoint provided $H \in \mathcal{H}$.

We define $K^s$ the set of all self-adjoint elements of form $(H \circ X)_\infty$, where $H \in \mathcal{H}$, and $C^s$ the convex cone of self-adjoint elements $a$ in $\mathcal{A}$ with the property that $a \leq b$ for some $b \in K^s$. We denote by $\tilde{C}^*$ the closure of $C^s$ with respect to the weak-star topology $\sigma(\mathcal{A}, \mathcal{A}_*)$ of $\mathcal{A}$, where $\mathcal{A}_*$ is the predual space of $\mathcal{A}$. It is well known that
\[
\mathcal{A}_* = L^1(\mathcal{A}, \tau)
\]
via the correspondence that
\[
b \mapsto \tau[ab], \quad b \in L^1(\mathcal{A}, \tau),
\]
for each $a \in \mathcal{A}$.
Definition 4 (e.g., [14]). We say that \( X = (X_t)_{t \geq 0} \) satisfies the condition of no free lunch (NFL) if
\[
\bar{C}^* \cap \mathcal{A}_+ = \{0\}.
\] (3)

Definition 5. A normal state \( \sigma \) on \( \mathcal{A} \) is called a martingale state of \( X = (X_t)_{t \geq 0} \), if \( X = (X_t)_{t \geq 0} \) is a martingale on \( (\mathcal{A}, (\mathcal{A})_{t \geq 0}, \sigma) \).

We denote by \( M_f(X) \) the family of all such faithful normal states, and say that \( X = (X_t)_{t \geq 0} \) satisfies the condition of the existence of a faithful martingale state (EMS) if \( M_f(X) \neq \emptyset \).

As following is a non-commutative analogue of the fundamental theorem of asset pricing in mathematical finance:

**Theorem.** A non-commutative self-adjoint stochastic process \( X = (X_t)_{t \geq 0} \) satisfies the condition of no free lunch (NFL) if and only if the condition (EMS) of the existence of a faithful martingale state is satisfied.

**Remark 3.** In [4] the author has proved a special case of the above theorem on finite dimensional von Neumann algebras, whose proof is different from that presented here. By using this theorem we present a quantum version of the classical asset pricing theory of multi-period financial markets based on finite dimensional quantum probability spaces.

3. Proofs

**Lemma 1.** Let \( H \) be in \( \mathcal{H} \) and let \( \sigma \) be a state on \( \mathcal{A} \). If \( X = (X_t)_{t \geq 0} \) is a martingale under \( \sigma \), then \( t \to (H^* X)_t \) is also a martingale under \( \sigma \).

**Proof.** Let \( H_t = a \otimes a^* 1_{(t_1, t_2)}(t) \) where \( a \in \mathcal{A}_{t_1} \). Let \( s \leq t \) and \( y \in \mathcal{A}_s \). We have to prove that
\[
\sigma[y \int_s^t H_r^* dX_r y^*] = 0.
\]
One has that
\[
\int_s^t H_r^* dX_r = a(X_{\max(t_1, t_2)} - X_{\max(s, t_2), t_1}) a^*.
\]
Since \( X = (X_t)_{t \geq 0} \) is a martingale, we get the result. The general case follows since linear combinations of martingales are martingales.

**Lemma 2.** Let \( \sigma \) be a state on \( \mathcal{A} \). Then, \( X = (X_t)_{t \geq 0} \) is a martingale under \( \sigma \) if and only if
\[
\sigma[(H^* X)_\infty] = 0,
\]
for every \( H \in \mathcal{H} \).
Proof. Suppose that $X = (X_t)_{t \geq 0}$ is a martingale. By Lemma 1 one concludes that
\[ \sigma[(H_\sharp X)_\infty] = \sigma[(H_\sharp X)_0] = 0, \]
for every $H \in \mathcal{H}$.

Conversely, let $s \leq t$ and $y \in \mathcal{A}_s$. Set $H_r = y \otimes y^* 1_{[s,t)}(r)$. Then
\[ (H_\sharp X)_\infty = y(X_t - X_s)y^*, \]
and hence $\sigma[y(X_t - X_s)y^*] = 0$. The proof is complete.

Proof of the Theorem. (EMS) $\implies$ (NFL): By Lemma 2 we have that $\sigma(c) \leq 0$ for each $\sigma \in M_f(X)$ and $c \in C^*$, and this inequality also extends to the weak-star closure $\bar{C}^*$. However, if (EMS) would hold and (NFL) were violated, there would exist a $\sigma \in M_f(X)$ and $c \in \bar{C}^*, c > 0$, whence $\sigma(c) > 0$ since $\sigma$ is faithful, a contradiction.

(NFL) $\implies$ (EMS): We claim that, for fixed $a_0 \in \mathcal{A}, a_0 > 0$, there is $b \in L^1(\mathcal{A})$ which defines a positive linear functional $\tau_b$ on $\mathcal{A}$ via
\[ \tau_b(a) = \tau(ab), \quad a \in \mathcal{A}, \]
such that $\tau_b$ is less or equal to zero on $\bar{C}^*$, and $\tau_b(a_0) > 0$. To see this, apply the separation theorem (e.g., [19, Theorem II.9.2]) to the $\sigma(\mathcal{A}, \mathcal{A}_c)$-closed convex set $\bar{C}^*$ and the compact set $\{a_0\}$ to find a $b \in L^1(\mathcal{A})$ and $\alpha < \beta$ such that $\tau_b[c] \leq \alpha$ for all $c \in \bar{C}^*$ and $\tau_b(a_0) > \beta$. Since $0 \in C^*$ we concludes that $\alpha \geq 0$. As $\bar{C}^*$ is a cone, we have that $\tau_b$ is zero or negative on $\bar{C}^*$ and, in particular, nonnegative on $\mathcal{A}_+$. Noting that $\beta > 0$ we have proved the claim.

Denote by $\mathcal{B}$ the set of all $b \in L^1(\mathcal{A})$ so that $\tau_b$ is a positive linear functional on $\mathcal{A}$ which is less or equal to zero on $\bar{C}^*$. Clearly $0 \in \mathcal{B}$ and hence $\mathcal{B}$ is nonempty.

Let $\mathcal{S}$ be the set of all supports $s(\tau_b)$ of $\tau_b, b \in \mathcal{B}$. Note that $\mathcal{S}$ is a $\sigma$-lattice in the usual order, as for a sequence $b_n \in \mathcal{B}$, we may find strictly positive scalars $\alpha_n$ such that $\sum_n \alpha_n b_n \in \mathcal{B}$.

Hence there is $b_0 \in \mathcal{B}$ such that
\[ s(\tau_{b_0}) = \sup\{s(\tau_b) : b \in \mathcal{B}\}. \]

We now claim that $s(\tau_{b_0}) = 1$, which readily shows that $\tau_{b_0}$ is faithful. Indeed, if $s(\tau_{b_0}) < 1$, then we could apply the above claim to $1 - s(\tau_{b_0})$ to find $b_1 \in \mathcal{B}$ with
\[ \tau[b_1(1 - s(\tau_{b_0}))] > 0. \]
Hence, $b_0 + b_1$ would be an element of $\mathcal{B}$ whose support is bigger than $s(\tau_{b_0})$, a contradiction.

Normalize $\tau_{b_0}$ so that $\tau_{b_0}[1] = 1$, we concludes from Lemma 2 that $\sigma = \tau_{b_0}$ is a martingale state for $X$ and thus, $M_f(X) \neq \emptyset$. The proof is complete.

Remark 4. The exhaustion argument in the above proof goes back to Yan [24].
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