BIAŁYNICKI-BIRULA DECOMPOSITION FOR REDUCTIVE GROUPS IN POSITIVE CHARACTERISTIC

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ABSTRACT. We prove the existence of Białynicki-Birula decomposition for Kempf monoids, which is a large class that contains for example monoids with reductive unit group in all characteristics. This extends the existence statements from [JS19, AHR20].

1. INTRODUCTION

Let $k$ be a field. By a linear group $G$ we mean a smooth affine group scheme of finite type over $k$. When talking about a $G$-action we always mean a left $G$-action. Let $G$ be a linear group and $G \subset \overline{G}$ be a geometrically integral affine algebraic monoid with zero. For a fixed $G$-scheme $X$ over $k$, its Białynicki-Birula decomposition is a set-valued functor $\text{Map}(G, X)^G$. Explicitly, it is given by

$$X^+(S) = \{ \varphi: \overline{G} \times S \to X \mid \varphi \text{ is } G\text{-equivariant} \}.$$

Intuitively, the functor $X^+$ parameterizes $G$-orbits in $X$ which compactify to $\overline{G}$-orbits. Indeed, its $k$-points are $G$-equivariant morphisms $f: \overline{G} \to X$. The restrictions $f \mapsto f(1) \in X$ and $f \mapsto f(0) \in X^G$ give maps $X^+(k) \to X(k)$ and $X^+(k) \to X^G(k)$. The map $f$ is a partial compactification of the $G$-orbit of $f(1)$ and $f(0)$ is “limit” or the most degenerate point of this compactification. The evaluations at zero and one extend to maps of functors:

- the map $i_X: X^+ \to X$ that sends $\varphi$ to $\varphi|_{1 \times S}: S \to X$.
- the limit map $\pi_X: X^+ \to X^G$ that sends $\varphi$ to $\varphi|_{0 \times S}: S \to X^G$. The limit map has a section $s_X$ that sends $\varphi_0: S \to X^G$ to the constant family $\varphi = \varphi_0 \circ \text{pr}_2: \overline{G} \times S \to X$.

Altogether, we obtain the diagram $X^G \leftarrow X^+ \xrightarrow{i_X} X$.

Figure 1 – The natural maps associated to $X^+$. The blue curves in the middle denote the $G$-orbits, while the transversal black curve is $S$ and the thick blue curve is the limit of $S$.

To obtain the classical positive (resp. negative) Białynicki-Birula decomposition [BB73] we take a smooth proper $X$ and pairs $(G, \overline{G}) = (\mathbb{G}_m, \mathbb{A}^1 = \mathbb{G}_m \cup \{0\})$ and $(G, \overline{G}) = (\mathbb{G}_m, \mathbb{A}^1 = \mathbb{G}_m \cup \{\infty\})$ respectively. For a connected linearly reductive group $G$, the functor $X^+$ is represented by a scheme, as proven in [JS19] and $X^+$ is smooth for smooth $X$. The proof of representability proceeds in three steps

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(I) introduce a formal version \( \bar{X} \) of the functor \( X^+ \) and prove its representability. The stage for this part is the formal neighbourhood of \( X^G \), hence the question becomes essentially affine and the representation theory of \( G \) plays a central role,

(II) prove that the formalization map \( X^+ \to \bar{X} \) is an isomorphism for affine schemes,

(III) for a \( G \)-scheme \( X \) find an affine \( G \)-equivariant étale cover of fixed points of \( X \) and use an easy descent argument to show that the natural map \( X^+ \to \bar{X} \) is an isomorphism. The existence of such a cover is proven in [AHR20], which crucially depends on the linear reductivity of \( G \), see [AHR20, Prop 3.1].

In positive characteristic the only connected linearly reductive groups are tori. Thus it is a natural question whether one could extend those existence results to reductive groups in positive characteristic. This seems also interesting from the point of view of geometric representation theory, similarly to how the \( \mathbb{G}_m \)-case is used in [DG14].

The aim of the present article is to prove that \( X^+ \) is representable for a large class of algebraic monoids \( \mathcal{T} \) with zero, so called Kempf monoids. A monoid \( \mathcal{T} \) with zero is a Kempf monoid if there exists a central one-parameter subgroup \( \mathbb{G}_m \rightarrow Z(G) \subset G \) such that the induced map \( \mathbb{G}_m \times \mathcal{T} \rightarrow \mathcal{T} \) extends to a map \( \mathbb{A}_1^+ \rightarrow \mathcal{T} \) that sends 0 to \( 0_{\mathcal{T}} \). We prove that every monoid \( \mathcal{T} \) with zero and with reductive unit group is a Kempf monoid. We stress that there are no assumptions on the characteristic here, reductive means as usual that the unipotent radical of \( G \) is trivial. There are plenty of Kempf monoids with non-reductive unit group as well, such as the monoid of upper-triangular matrices.

The main result of this paper is the following representability result.

**Theorem 1.1.** Let \( \mathcal{T} \) be a normal Kempf monoid with zero and with unit group \( G \) and \( X \) be a Noetherian \( G \)-scheme over \( k \). Then the functor \( X^+ \) is representable and affine of finite type over \( X^G \).

This directly extends the previous results for the one-dimensional torus [Dri13, AHR20] and linearly reductive groups [JS19, AHR20]. This extension is particularly far-reaching in positive characteristic, where tori are the only geometrically connected linear groups. It also clarifies the situation in general in that the representation theory for \( G \) at the end of the day poses no obstructions to representability.

In its ideas, the proof proceeds along the steps (I)-(III) above. However, there are two fundamental problems along the way:

- the representation theory for \( G \) is complicated and thus step (I) requires much more care. We introduce finitely generated Serre subcategories of representations and heavily employ the Kempf torus \( (\mathbb{G}_m)_{\mathcal{T}} \rightarrow Z(G) \subset G \) and the corresponding \( \mathbb{A}_1^+ \) inside \( \mathcal{T} \).
- the analogue of (III) is not known and it is clear that the current ideas are insufficient (the problems arising are similar to the ones with smoothness discussed below). We overcome this by employing Tannakian formalism of Hall-Rydh [HR19] to get a map \( i_X: \bar{X} \rightarrow X \) mimicking the unit map \( i_k \). In this way, \( \bar{X} \) becomes a \( \mathcal{T} \)-scheme with a map to \( X \). This induces a section of the formalization map \( X^+ \rightarrow \bar{X} \) and implies that it is an isomorphism.

While the language of stacks would be most appropriate for (III), we avoid it in order to make the paper more accessible. To make it self contained, we provide an elementary proof of Tannaka duality in the required generality in the appendix. More precisely, we prove that global quotient stacks \( [X/G] \) are tensorial, see Remark A.3. The proof is perfectly down-to-earth and much in the spirit of Brandenburg’s works [Bra14, BC14].

In the smooth case, we can say a little more about the morphism \( \pi_X \).

**Proposition 1.2.** Let \( x \in X^G \) be such that \( \pi_X: X^+ \rightarrow X^G \) is smooth at \( s_X(x) \in X^+ \). Then locally near \( x \) the map \( \pi_X \) is an affine space fiber bundle with an action of \( \mathcal{T} \) fiberwise.

It would be desirable to have Proposition 1.2 for every smooth \( X \), without any assumptions on \( X^+ \). However, this is still open. The main problem is that the linear map \( m_x \rightarrow m_x/m_x^2 \) may not have a \( G \)-equivariant splitting, so the regularity of \( x \in X \) does not immediately imply the regularity of \( x \in X^+ \), see Example 2.16. This is the same issue which implies that \( X^G \) may not be smooth for smooth \( G \)-schemes \( X \). Indeed, for example \( SL_p \) acting on itself by conjugation has
fixed points \( \mu_p \), which is non-reduced. In general [FN77] shows that smoothness of fixed points for actions on smooth varieties characterizes linearly reductive groups. Very curiously, it seems that very few examples of non-smooth fixed points are known and all known examples seem to have smooth underlying reduced schemes. Also, by Lemma 2.1 below and affineness of \( \pi_X \) we have \( X^G = (X^+) G = (X^+) \mathbb{G}^{m \times} \), so we cannot hope for \( X^G \) to be smooth in general, as \( G_m \)-fixed points would be smooth while \( X^G \) can be singular.

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2. **MONOIDS AND AFFINE SCHEMES**

Throughout, we fix a base field \( k \). We do not impose any characteristic, algebraically closed, perfect or other assumptions on \( k \). An algebraic monoid is a geometrically integral affine variety \( \overline{G} \) together with an associative multiplication \( \mu : \overline{G} \times \overline{G} \to \overline{G} \) that has an identity element \( 1 \in \overline{G}(k) \).

The unit group of \( \overline{G} \) is the open subset consisting of all invertible elements. We denote it by \( \overline{G}^\times \), or simply by \( G \). The unit group \( G \) is open in \( \overline{G} \), so it is dense and connected. The monoid is reductive if \( G \) is reductive, i.e., \( G T \) contains no nonzero normal unipotent subgroups. Every \( G \)-representation is assumed to be rational, that is, a union of finite dimensional subrepresentations on which \( G \) acts regularly. A canonical reference for reductive groups in all characteristics is [SGA3], for a summary see [Dem65]. Great introductions to algebraic monoids are for example [Bri14, Ren05]. We remark that the affineness assumption on \( \overline{G} \) is redundant for reductive \( G \): it follows by descent from [Ri98, Lemma 2] that every geometrically integral variety over \( k \) with an associative multiplication and with dense, reductive unit group is affine.

2.1. **Kempf monoids.** Let \( \overline{G} \) be a Kempf monoid with unit group \( G \) and \( G_{m, \overline{G}} \to G \) be the Kempf torus. The Kempf line is the corresponding map \( A^1_k \to \overline{G} \). Let \( T \) be the image of \( G_{m, \overline{G}} \) in \( G \). This is a one-dimensional algebraic group and it is geometrically connected, since it is connected and has rational point \( 1_T \), see [sta17, Tag 04KV]. Therefore, \( T \) is a torus; in particular it is linearly reductive.

**Lemma 2.1.** Let \( X = \text{Spec}(A) \) be a \( \overline{G}_T \)-scheme such that the action of the Kempf torus on \( X \) is trivial. Then the \( \overline{G}_T \)-action on \( X \) is trivial.

**Proof.** Since \( 1_G, 0_{\overline{G}} \) are in the closure of Kempf’s torus, the assumption implies in particular that the actions of \( 1_{\overline{G}} \) and \( 0_{\overline{G}} \) are equal. For every \( \overline{G} \)-scheme \( S \) and \( S \)-points \( x \in X(S) \) and \( g \in \overline{G}(S) \) we have \( g x = (g \cdot 1)x = g \cdot (1 \cdot x) = g \cdot (0 \cdot x) = (g \cdot 0) \cdot x = 0 \cdot x = 1 \cdot x \), whence the action is trivial. \( \square \)

**Corollary 2.2.** Let \( X = \text{Spec}(A) \) be \( \overline{G} \)-scheme. Consider the \( \mathbb{N} \)-grading on \( A_T \) associated to the Kempf torus. Then \( (A_T)_{>0} \) is the ideal of \( X^G \) in \( X_T \).

**Proof.** Since \( \overline{G} \) acts trivially on \( X^G \), also the Kempf line acts trivially, so \( I(X^G) \supseteq (A_T)_{>0} \). Conversely, the ideal \( (A_T)_{>0} \) is \( G_T \)-stable since the Kempf torus is central, so it is \( G_T \)-stable and so \( Z = V((A_T)_{>0}) \subseteq X_T \) is a \( \overline{G}_T \)-scheme with a trivial action of the Kempf torus. By Lemma 2.1 also the \( \overline{G}_T \)-action on \( Z \) is trivial, so \( Z \subseteq X^G \) and so equality holds. \( \square \)
2.2. Reductive monoids are Kempf.

Lemma 2.3. Let $G$ be a semisimple group and $\mathfrak{G}$ be a monoid with unit group $G$. Then $G$ is closed in $\mathfrak{G}$. 

Proof. Since $\mathfrak{G}$ is affine of finite type, it has a faithful finite-dimensional representation. We fix such a representation $V$ and the associated embedding $i: \mathfrak{G} \subset \text{End}_k(V)$. The group $i(G)$ is an image of semisimple group, hence it has no non-zero characters so in particular $	ext{det}_{i(G)}$ is trivial and so we have $i(G) \subset \text{SL}(V)$. The map $i: G \to \text{SL}(V)$ is a group homomorphism, so its image is closed. Also $\text{SL}(V) = (\det = 1)$ is closed in $\text{End}(V)$. Summing up, the group $i(G)$ is closed in $\text{End}(V)$ so also in $\mathfrak{G}$. □

Corollary 2.4. In the setting of Lemma 2.3 assume moreover that $\mathfrak{G}$ has a zero and that its unit group $G \subset \mathfrak{G}$ is dense. Then $\mathfrak{G}$ is the zero monoid.

Proof. Lemma 2.3 shows that $G$ is closed in $\mathfrak{G}$. As it is dense, we have $\mathfrak{G} = G$. But then $0_{\mathfrak{G}} \in G$ is an invertible element and this happens only if $G = \{0\}$. □

Proposition 2.5. Let $\mathfrak{G}$ be a reductive monoid with zero. Let $G$ be its unit group and let $Z$ be the connected component of identity in the center $Z(G)$ of its reduced structure. Then for every point $g \in \mathfrak{G}(k)$, the point $\{g\} \subset \mathfrak{G}$ lies in the closure of $Z : g$.

Proof. Since $G$ is reductive, $Z$ is an algebraic torus [SGA3, XII.4.11]. Consider the quotient variety

$$Q := \mathfrak{G} / Z = \text{Spec} \left( \text{H}^0(\mathfrak{G}, \mathcal{O}_\mathfrak{G})^Z \right)$$

and the quotient map $\pi: \mathfrak{G} \to Q$. Since $Z$ is a torus, this is a categorical quotient, so the map $\pi \circ \mu: \mathfrak{G} \times \mathfrak{G} \to \mathfrak{G} \to Q$ descends to a map $\beta: Q \times Q \to Q$ which makes $Q$ an algebraic monoid with zero. The map $\pi$ and the inclusion $G \to \mathfrak{G}$ are both dominant, hence so is their composition, which implies that the subgroup $\pi(G) \subset Q^\times$ is dense. It is closed as well, as an algebraic subgroup, so $\pi(G) = Q^\times$. The group $G/Z$ is semisimple [SGA3, XXII,4.3.5], so also its image $Q^\times$ is semisimple. Now the monoid $Q$ satisfies assumptions of Corollary 2.4, so $Q$ is a point.

The points of $Q = \mathfrak{G} / Z$ correspond to closed $Z$-orbits in $\mathfrak{G}$, so there is only one closed orbit and it is equal to $\{0\}$. By general theory, the closure of every $Z$-orbit in $\mathfrak{G}$ contains this closed orbit. This concludes the proof. □

Corollary 2.6 (Reductive monoids are Kempf). Let $\mathfrak{G}$ be a reductive monoid with zero. Then $\mathfrak{G}$ is a Kempf monoid.

Proof. Let $Z := Z(G)$ be the connected component of identity in the center of the unit group $G$ of $\mathfrak{G}$. Since $Z$ is a torus, $G$ is an algebraic torus. Let $\mathfrak{G}$ be the closure of $Z$ in $\mathfrak{G}$. By Proposition 2.5 it contains the point $0_{\mathfrak{G}}$, which is a fixed point of the torus $Z$. Then the cone corresponding to the normalization of $Z_{\mathfrak{G}}$ is pointed, so any general one-parameter subgroup $G \to \mathfrak{G}$ extends to $\mathbb{A}^1 \to \mathfrak{G}$ that sends 0 to $0_{\mathfrak{G}}$. □

Remark 2.7. Upon completion of this paper, we learned that over an algebraically closed field the existence of Kempf torus is proven in [Rit98, Proposition 3]. We thank Michel Brion for the reference.

2.3. Representability for affine schemes: general setup. In this section we make no reductivity assumptions on $G$. Fix an algebraic monoid $\mathfrak{G}$ with unit group $G$. Let $V$ be a $G$-representation. We say that $V$ is a $\mathfrak{G}$-representation if the $G$-action extends to a map $\mathfrak{G} \times V \to V$. By the coaction map, $V$ is a $\mathfrak{G}$-representation if and only if there exists an embedding of $V$ into $H^0(\mathfrak{G}, \mathcal{O}_\mathfrak{G})^{\oplus \dim V}$. A $G$-representation that is a subrepresentation or quotient of $\mathfrak{G}$-representation is a $\mathfrak{G}$-representation itself. Also the tensor product of $\mathfrak{G}$-representations is a $\mathfrak{G}$-representation, but the dual of a $\mathfrak{G}$-representation is not necessarily a $\mathfrak{G}$-representation.
Lemma 2.8. Let $V$ be a rational $G$-representation. Then there exists a quotient $X \twoheadrightarrow W$ of rational $G$-representations such that

1. $W$ is a $\overline{G}$-representation,
2. for every other $G$-equivariant map $V \twoheadrightarrow W'$ with $W'$ a rational $\overline{G}$-representation there exists a unique factorization $V \twoheadrightarrow W \twoheadrightarrow W'$ where $W \twoheadrightarrow W'$ is a $\overline{G}$-equivariant map.

In other words, the quotient $V \twoheadrightarrow W$ represents the functor $\text{Rep}_G \to \text{Set}$ given by $\text{Hom}_G(V, -)$.

Proof. Consider first the case when $V$ is finite-dimensional and view it as $V = \text{Spec} \text{Sym}(V^\vee)$. The coaction map becomes $\sigma_V : V^\vee \to H^0(G, \mathcal{O}_G) \otimes V^\vee$. Let $U_{-1} := V^\vee$ and let $U_0$ be the pullback defined by the diagram

$$
\begin{array}{ccc}
U_0 & \xrightarrow{\iota} & H^0(\overline{G}, \mathcal{O}_{\overline{G}}) \otimes_k V^\vee \\
\downarrow & & \downarrow \\
V^\vee & \xrightarrow{\sigma_V} & H^0(G, \mathcal{O}_G) \otimes_k V^\vee
\end{array}
$$

Let $U_n$ for $n = 1, 2, \ldots$ be constructed inductively as the pullback

$$
\begin{array}{ccc}
U_n & \xrightarrow{\iota} & H^0(\overline{G}, \mathcal{O}_{\overline{G}}) \otimes_k U_{n-1} \\
\downarrow & & \downarrow \\
U_{n-1} & \xrightarrow{\iota} & H^0(\overline{G}, \mathcal{O}_{\overline{G}}) \otimes_k U_{n-2}
\end{array}
$$

Finally let $U := \bigcap_n U_n$. Diagram (2.9) implies that the coaction map on $U$ factors as $U \twoheadrightarrow H^0(\overline{G}, \mathcal{O}_{\overline{G}}) \otimes_k U$.

Therefore the $G$-representation $W := \text{Spec} \text{Sym}(U)$ is in fact a $\overline{G}$-representation. The inclusion $U \twoheadrightarrow V^\vee$ induces a surjective map $V \twoheadrightarrow W$. Consider any other map $V \twoheadrightarrow W'$ to a rational $\overline{G}$-representation; then replacing $W'$ by the image of $V$ we may assume $W'$ is finite-dimensional, so $W' = \text{Spec} \text{Sym}(U')$ for some $i: U' \twoheadrightarrow V^\vee$ and we obtain a commutative diagram of coactions

$$
\begin{array}{ccc}
U' & \xrightarrow{\iota_{U'}} & H^0(\overline{G}, \mathcal{O}_{\overline{G}}) \otimes_k U' \\
\downarrow & & \downarrow{\text{id} \otimes i} \\
V^\vee & \xrightarrow{\sigma_V} & H^0(G, \mathcal{O}_G) \otimes V^\vee
\end{array}
$$

It follows from the construction of $\{U_n\}$ that $i$ factors as $U' \twoheadrightarrow U \twoheadrightarrow V^\vee$. Consider now the case of a general rational representation $V$. For every finite-dimensional subrepresentation $V'$ consider the corresponding quotient $V' \twoheadrightarrow W(V')$ and its kernel $K(V')$. The universal property of $W(-)$ gives a map $W(V') \twoheadrightarrow W(V'')$ whence an inclusion $K(V') \hookrightarrow K(V'')$. The union $K = \bigcup_{\nu'} K(V')$ is a $G$-subrepresentation of $V$ and $V/K$ has the required universal property. \hfill $\square$

Proposition 2.10. Let $X = \text{Spec}(A)$ be an affine $G$-scheme. Then $X^+$ is represented by a closed subscheme.

Proof. Let $A \twoheadrightarrow A/\mathcal{G}$ be the universal quotient $\overline{G}$-representation as in Lemma 2.8. Let $I \subset A$ be the ideal generated by $\mathcal{G}$. The closed scheme $\text{Spec}(A/I)$ is $G$-stable and in fact its $G$-actions extends to $\overline{G}$. We claim that $X^+ = V(I) \simeq \text{Spec}(A/I)$. Pick a $\overline{G}$-scheme $Y$ with a $G$-equivariant map $\varphi : Y \to X$. The map $\varphi$ factors through the $\overline{G}$-scheme $\text{Spec} H^0(Y, \mathcal{O}_Y)$, so we may assume that $Y$ is affine. The pullback $A \twoheadrightarrow H^0(Y, \mathcal{O}_Y)$ maps $A$ to a $G$-representation, hence kills $\mathcal{G}$ so also $I$ and thus factors through $X^+$. \hfill $\square$

Now we slightly generalize Proposition 2.10 taking into account open immersions. We say that a $G$-scheme $X$ is locally $G$-linear if it is covered by $G$-stable open affine subschemes. We say that $X$ is locally $\overline{G}$-linear if it is covered by $\overline{G}$-stable open affine subschemes.
Lemma 2.11 (open immersions, [JS19, Prop 5.2]). Let \( U \hookrightarrow X \) be an open immersion of \( G \)-schemes. Then the diagram

\[
\begin{array}{ccc}
U^+ & \to & U^G \\
\downarrow & & \downarrow \\
X^+ & \to & X^G
\end{array}
\]

is cartesian.

Sketch of proof. For every scheme \( S \), the locus \( 0 \times S \subset \overline{G} \times S \) has no \( G \)-stable open neighbourhoods except the whole \( \overline{G} \times S \). Hence \( \varphi : \overline{G} \times S \to X \) factors through \( U \) if and only if \( \varphi \circ 0 \times S : S \to X^G \) factors through \( U^G \).

Proposition 2.12. Let \( X \) be a locally \( G \)-linear scheme. Then \( \pi_X : X^+ \to X^G \) is affine, so \( X^+ \) is represented by a locally linear \( \overline{G} \)-scheme.

Proof. Let \( \{ U_i \} \) be a \( G \)-stable open affine cover of \( X \). By Lemma 2.11 the pullback of \( \pi_X \) via \( U_i^G \to X^G \) is the map \( \pi|_U : U^+ \to U^G \). This map is affine by Proposition 2.10. It follows that the map \( \pi_X \) becomes affine after the pullback to the cover \( \bigsqcup_i U_i \to X \), so \( \pi_X \) is affine.

Remark 2.13. Proposition 2.12 can be stated with weaker assumptions: we do not really need \( \{ U_i \} \) to cover \( X \), just the inclusion \( \bigsqcup_i U_i \supset X^G \).

2.4. Representability for affine schemes: reductive case. Proposition 2.10, while satisfactory for our general purposes, says little about the resulting scheme. In this section we prove that one can say more in the reductive case.

Fix a reductive monoid \( \overline{G} \) with unit group \( G \). We additionally assume that the variety \( \overline{G} \) is normal. All maximal tori of \( T \) are conjugate. We fix one such torus \( T \) and its closure \( \overline{T} \subset \overline{G} \). We say that a \( T \)-representation is a \( \overline{T} \)-representation if the action of \( T \) extends to an action of \( \overline{T} \). A simple \( T \)-representation is an outsider representation if it is not a \( \overline{T} \)-representation.

Lemma 2.14 (Extension principle). Let \( V \) be a \( G \)-representation. Then the following are equivalent

1. the representation \( V \) is a \( \overline{G} \)-representation,
2. the representation \( V \) is a \( \overline{T} \)-representation.

Proof. The implication \( 2 \implies 1 \) is trivial, and the converse is [Ren05, Theorem 5.2]. (The referenced theorem requires \( \overline{G} \) to be normal and this is the main point where we use normality of \( \overline{G} \)).

We now prove the representability of \( X^+ \) for \( X \) affine.

Proposition 2.15. Let \( X = \text{Spec} \ A \) be an affine \( G \)-scheme. Then \( X^+ \) is represented by a closed sub-scheme whose ideal is the smallest \( G \)-ideal containing all outsider representations of \( T \) in \( A \).

Proof. Let us discuss the easiest case: \( X \) a \( \overline{G} \)-scheme. Let \( \varphi : \overline{G} \times X \to X \) denote the action. Then every family \( \varphi_1 : S \to X \) extends uniquely to a \( G \)-equivariant family \( \varphi : \overline{G} \circ (\text{id} \times \varphi_1) : \overline{G} \times S \to X \), hence \( i_X : X^+ \to X \) is an isomorphism.

Let us return to the general case. Consider \( X \) as a \( T \)-variety and let \( \overline{X} \) be the Białynicki-Birula decomposition for \( T \). As \( T \) is linearly reductive, so by [JS19, Proposition 4.5] the scheme \( \overline{X} \) is a closed subscheme of \( X \). Let \( I := I(\overline{X}) \subset A \) be its ideal and let \( I \subset A \) be the ideal generated by \( G \cdot J \). By [JS19, Proposition 4.5] the ideal \( J \) is generated by all irreducible subrepresentations of \( A \) that are not \( \overline{T} \)-representations. From linear reductivity of \( T \) we deduce that \( A/I \) is a \( \overline{T} \)-representation.

Let \( X' = \text{Spec} \ A/I \). By construction, \( X' \) is the largest \( G \)-scheme contained in \( \overline{X} \). The coordinate ring \( A/I \) of \( X' \) is a rational \( G \)-representation that is also a \( \overline{T} \)-representation, hence by Extension Principle 2.14 it is a \( \overline{G} \)-representation, so \( X' \) is a \( \overline{G} \)-scheme. Every \( G \)-equivariant family \( \varphi : \overline{G} \times \text{Spec} \ C \to X \) induces a \( G \)-equivariant pullback map

\[
\varphi^\#: A \to H^0(\overline{G}, O_{\overline{G}}) \otimes_k C.
\]
The right-hand-side $G$-representation is a $\mathcal{G}$-representation, so a $T$-representation, so $\ker(\varphi^g)$ contains $I$. The pullback is $G$-equivariant, so $\ker(\varphi^g)$ contains $I$ as well. This shows that $\varphi$ factors through $X' \hookrightarrow X$ and consequently that $X' = (X')^+$. But $X'$ is an affine $\mathcal{G}$-scheme, hence the "easiest case" from the beginning of the present proof applies and gives $X' = X'$. □

**Example 2.16** (Issues with regularity). Let $x \in X = \text{Spec} (A)$ be a $G$-fixed $k$-point which we identify with $s_X(x) \in X^+$. Applying Proposition 2.15 we get that for every $n \in \mathbb{N}$ the truncated local ring $A^{n,+} := \mathcal{O}_{X,x}/m_x^n$ is the quotient of $A^n := \mathcal{O}_{X,x}/m_x^{n+1}$ by the smallest $G$-ideal containing all outsider representations of $A^n$ with respect to $T$.

If the group $G$ was linearly reductive, we could take a $G$-equivariant section $s$ of $m_x \to m_x/m_x^2$ and conclude that $A^{n,+}$ is the quotient of $A^n$ by the image under $s$ of the $G$-representation generated by the $T$-outsider representations of $m_x/m_x^2$. Consequently, the complete local ring $\hat{\mathcal{O}}_{X,x}$ would be the quotient of $\hat{\mathcal{O}}_{X,x}$ by a sequence of elements whose images in the cotangent space are linearly independent. When $x \in X$ is regular, we would conclude that $x \in X^+$ is regular. This would be essentially the classical Iversen's [Ive72] argument on the smoothness of fixed points of $G$ for $G$ acting on smooth $X$. However, for groups $G$ that are not linearly reductive Iversen's argument fails and so we cannot hope for the existence of section $s$ in our case.

### 3. Formal $G$-schemes

In this section we prove the main results for the formal Białynicki-Birula functor $\hat{X}$ that we will introduce in the next section. The main advantage of the formal functor over $X^+$ is that it is defined on the affine level; correspondingly in this section we speak the language of algebra rather than geometry.

#### 3.1. Setup

We begin with basic definitions. A *formal abelian group* is a sequence $(\mathcal{A}_n)_{n \in \mathbb{N}}$ of abelian groups together with surjections $\pi_n: \mathcal{A}_{n+1} \to \mathcal{A}_n$. A morphism of formal abelian groups $(\mathcal{A}_n) \to (\mathcal{A}_n')$ is a family of morphisms $f_n: \mathcal{A}_n \to \mathcal{A}_n'$ such that the diagram

\[
\begin{array}{ccc}
\mathcal{A}_{n+1} & \xrightarrow{\pi_n} & \mathcal{A}_n \\
\downarrow{f_{n+1}} & & \downarrow{f_n} \\
\mathcal{A}_{n+1}' & \xrightarrow{\pi_n} & \mathcal{A}_n'
\end{array}
\]

commutes for every $n$. We obtain an abelian category of formal abelian groups. Using abstract-nonsense for this category, we can define formal algebras (as algebra objects), $G$-actions etc. Below we gather some formal cases. We say that a formal abelian group is a $B$-algebra if all $\mathcal{A}_n$'s are $B$-algebras and $\pi_n$ are maps of $B$-algebras. A module over a formal $B$-algebra $(\mathcal{A}_n)$ is an abelian group $(\mathcal{M}_n)$ where additionally each $\mathcal{M}_n$ is an $\mathcal{A}_n$-module and $\pi_n$ is a homomorphism of $\mathcal{A}_{n+1}$-modules. We say that a formal $B$-algebra $(\mathcal{A}_n)$ is a $G$-algebra if all algebras $\mathcal{A}_n$ are $G$-algebras, the maps $\pi_n$ are $G$-equivariant homomorphisms of algebras and moreover the structure maps $B \to \mathcal{A}_n$ are $G$-equivariant for the trivial $G$-action on $B$. Similarly we define $\mathcal{G}$-algebras and $G$- or $\mathcal{G}$-modules.

**Definition 3.1** (Formalization). Let $A$ be an abelian group and $(I_n \subseteq A)_{n \in \mathbb{N}}$ an increasing sequence of subgroups. The *associated formal abelian group* is $(A/I_n)_n$ with the natural surjections. When $A$ is a ring and $I_n := I^{n+1}$ for an ideal $I \subseteq A$ then the associated formal abelian group is a ring. When $M$ is an $A$-module, then $(M/I^{n+1}M)_n$ is an $(A/I^{n+1})$-module. We say that $M$ is *separated* if $\bigcap_{n \in \mathbb{N}} I^{n+1}M = 0$.

Conversely, for a formal algebra $(\mathcal{A}_n)$ its *algebraization* is an algebra $A$ with an ideal $I$ such that the algebras $(A/I^n)$ and $(\mathcal{A}_n)$ are isomorphic.

#### 3.2. Serre subcategories of $G$-linearized sheaves

We will be interested in constructing algebraizations of a given formal algebra in an $G$-equivariant way (see Section 3.3). If $G$ was linearly
reductive, this would mean that we just look separately at each $\lambda$-isotypic component for an simple $G$-representation $\lambda$. But the category of $G$-representations is far from semisimple, so working with irreducible representations makes little sense.

We need a generalization of them. While in the linearly reductive case every finite dimensional representation is a direct sum of simple representations, here every representation has a filtration with simple subquotients.

**Lemma 3.2** (Jordan-Hölder). Let $V$ be a finite dimensional $G$-representation. Then there exists a filtration $0 = V_0 \subset V_1 \subset \ldots \subset V_{r+1} = V$ with simple subquotients $E_i = V_{i+1} / V_i$. The set $\{E_0, \ldots, E_r\}$ does not depend on the filtration chosen. The representations $E_0, \ldots, E_r$ are called the composition factors of $V$.

*Proof.* This follows from the Jordan-Hölder theorem. $\square$

In the linearly reductive case, for a simple representation $\lambda$ and a finite dimensional representation $V$, one has the isotypic component $V[\lambda] \subset V$ associated to $\lambda$: the largest subrepresentation that is a direct sum of $\lambda$'s. In our situation, for a set of simple $G$-representations $\lambda = \{\lambda_1, \ldots, \lambda_r\}$ and a representation $V$ we define $V[\lambda] \subset V$ to be the largest subrepresentation whose composition factors belong to $\{\lambda_1, \ldots, \lambda_r\}$. To put this idea on a solid footing we use Serre subcategories.

**Definition 3.3.** Let $C$ be an abelian category (such as $B - \text{Mod}_G$). A Serre subcategory $S$ is a full subcategory closed under direct sums and such that for any short exact sequence

$$0 \to V_1 \to V_2 \to V_3 \to 0$$

we have $V_2 \in S$ if and only if $V_1, V_3 \in S$.

**Lemma 3.4.** Let $S \subset C$ be a Serre subcategory and $V \in C$ be an object. Then there exists a unique largest subobject of $V$ that lies in $S$. We denote it by $V[S]$. For a morphism $f : V_1 \to V_2$ in $C$ the image $f(V_1[S])$ lies in $V_2[S]$, so we get an induced morphism $f[S] : V[S] \to W[S]$. For an exact sequence

$$0 \to V_1 \to V_2 \to V_3$$

we get an exact sequence $0 \to V_1[S] \to V_2[S] \to V_3[S]$, so the functor $(-)[S]$ is left exact.

*Proof.* Consider the family $\{W_i\}$ of all the subobjects of $V$ that lie in $S$. Then $\bigoplus W_i$ lies in $S$, hence its quotient $\sum W_i \subset V$ lies in $S$. Take $V[S] := \sum V_i$. For $f : V_1 \to V_2$ the object $f(V_1[S])$ is a quotient of $V_2[S]$, hence $f(V_1[S]) \subset S$, so $f(V_1[S]) \subset V_2[S]$ by definition of $V_2[S]$. Finally, $\ker(V_2[S] \to V_3[S])$ is in $S$ as a subobject of $V_2[S]$ and is in $V_1$ since the sequence is exact. Thus $\ker(V_2[S] \to V_3[S]) \subset V_1[S]$ which proves exactness. $\square$

It is in general not true, even in our setup, that $(-)[S]$ is right exact.

In the following, we will use the two main examples of Serre subcategories.

**Example 3.5.** Fix a set $\lambda = \{\lambda_1, \ldots, \lambda_r\}$ of simple $G$-representations and consider the full subcategory of $\text{Rep}_G$ consisting of representations $V$ that are unions of finite dimensional subrepresentations with composition factors in $\lambda$. This is a Serre category, which we denote by $\text{Rep}_G[\lambda]$ and call the Serre category generated by $\lambda$. We denote by $(-)[\lambda]$ the associated functor.

For a $G$-representation $V$ the inclusion of sets induces a partial order on $\lambda$ which makes the set $\{V[\lambda]\}_\lambda$ a direct system and $V$ its colimit.

**Remark 3.6.** For a rational representation $\lambda$, the subrepresentation $V[\lambda]$ is obviously rational. But also a stronger “global” rationality condition is satisfied: there exists a finite dimensional $k$-linear subspace $W \subset H^0(G, O_G)$ such that the coaction map for any $V[\lambda]$ has image in $W \otimes V[\lambda] \subset H^0(G, O_G) \otimes_k V[\lambda]$. Indeed, let $W$ be the minimal subspace such that the coactions of all (finitely many!) simple representations make sense. Then $W$ always satisfies the assumptions.

**Example 3.7.** Let $T \subset Z(G)$ be a central torus (which means that $T \cong G_m^w$ for some $m$, in particular $T$ is linearly reductive). Let $\mathcal{X}$ be a finite set of simple $T$-representations and let consider the full subcategory of $\text{Rep}_G$ consisting of $G$-representations $V$ which as a $T$-representation are
3.21 The functor \( \lambda \)-algebraizations. The functor \((-)\langle \lambda \rangle \) is exact.

The constructions \((-)[\lambda]\) and \((-)[\lambda']\) are connected as follows. For \( \lambda \), let \( \chi(\lambda) \) be the set of all \( T \)-weights that appear in \( \bigoplus_{\lambda \in \lambda'} \lambda \). For a rational \( G \)-representation \( V \), we have \( V[\lambda] \subset V[\chi(\lambda)] \). On the other hand, if \( w \in V[\chi(\lambda)] \), then we may form \( \mu(w) = \{ \mu_1, \ldots, \mu_n \} \) where \( \mu_i \) are composition factors of the representation generated by \( w \). Since \( G \cdot w \subset V[\chi(\lambda)] \), we have \( \chi(\mu(w)) \subset \chi(\lambda) \). By definition \( w \) lies in \( V[\mu(w)] \), therefore we obtain

\[
V[\chi] = \bigcup_{\mu: \chi(\mu) \subset \chi} V[\mu].
\]

3.3. \( G \)-equivariant algebraizations. In this subsection we fix a torus \( T \subset Z(G) \), not necessarily split. We assume that the Kempf torus \( (G_m)_T \to G \) factors through \( T \).

**Definition 3.10.** Let \((M_n)\) be a formal abelian group with a \( G \)-action. Its \( \lambda \)-algebraization is

\[
\lim_G(M_n) := \operatorname{colim}_\lambda \lim_n M_n[\lambda],
\]

By Remark 3.6, \( \lim_G(M_n) \) is a rational \( G \)-representation. It is also the limit of the diagram \( \ldots \to M_{n+1} \to M_n \to \ldots \to M_0 \) in the category of abelian groups with \( G \)-action; hence the notation. If \( \mathcal{A} = (A_n)_n \) is a formal algebra, then \( \lim_G(\mathcal{A}) \) is an algebra as well. Indeed for \( \lambda \) and \( \mu \) let \( \lambda \otimes \mu \) denote the Serre subcategory generated by composition factors of all \( \{ A_i \otimes \mu_j \} \). Then the multiplication on an \( A_n \) restricts to \( A_n[\lambda] \otimes A_n[\mu] \to A_n[\lambda \otimes \mu] \) and induces multiplication \( \lim_n(A_n[\lambda]) \otimes \lim_n(A_n[\mu]) \to \lim_n A_n[\lambda \otimes \mu] \) and thus a multiplication on \( \lim_G(\mathcal{A}) \).

**Remark 3.11.** It is not yet clear what is the connection between algebraizations from Definition 3.1 and \( \lambda \)-algebraizations. We will see in Theorem 3.21 that in favorable conditions a \( \lambda \)-algebraization is an algebraization.

**Definition 3.12.** Let \((M_n)\) be a formal abelian group with \( T \)-action. Its \( \chi \)-algebraization is

\[
\lim_T(M_n) := \operatorname{colim}_\chi \lim_n M_n[\chi],
\]

where the colimit is taken over all finitely generated Serre subcategories of \( \operatorname{Rep}_T \).

**Lemma 3.13.** Let \( 0 \to \mathcal{M} \to \mathcal{N} \to \mathcal{P} \) be an exact sequence of formal abelian groups with \( G \)-action (resp. \( T \)-action). Then the induced sequence \( 0 \to \lim_G(\mathcal{M}) \to \lim_G(\mathcal{N}) \to \lim_G(\mathcal{P}) \) is exact (resp., the induced sequence of \( \operatorname{lim}_T(-) \) is exact).

**Proof.** This follows from left exactness of \((-)[\chi]\) and \((-)[\lambda]\) and the fact that colimits in both algebraizations are taken over filtered sets. \( \square \)

Since \( \operatorname{Rep}_T \) is semisimple we have a canonical isomorphism

\[
\lim_T(M_n) = \bigoplus_{\chi} \lim_n M_n[\chi],
\]

where \( \chi \) runs through all simple \( T \)-representations. It follows that

\[
(\lim_T(M_n))[\chi] = \lim_n M_n[\chi]
\]

for every set of simple \( T \)-representation \( \chi \) and we see that \((-)[\chi]\) is right-exact. The \( T \)-module \( \operatorname{lim}_T(M_n) \) is the limit of \( \ldots \to M_{n+1} \to M_n \to \ldots \to M_0 \) in the category of abelian groups with \( T \)-action. For a formal abelian group \((M_n)\) with \( G \)-action, the restriction \( \operatorname{Rep}_G \hookrightarrow \operatorname{Rep}_T \) induces a injective \( \lambda \)-\( \chi \)-comparison map:

\[
i: \lim_G(M_n) \hookrightarrow \lim_T(M_n),
\]
that embeds $M_n[\lambda]$ into $M_n[\chi(\lambda)]$ and so $\lim_n M_n[\lambda] \to \lim_n M_n[\chi(\lambda)]$. The reason to introduce both algebraizations is that $\lambda$-algebraization is more canonical but also more challenging to work with, mostly because $[-](\lambda)$ is not right exact. We will prove that the $\lambda$-$\chi$-comparison map is an isomorphism in situations of interest, see Corollary 3.20. For this we employ various stabilization results that we now prove.

3.4. Stabilization. We say that a formal abelian group $(M_\lambda)$ with $G$-action stabilizes $G$-equivariantly if for every $\lambda$ the map $M_{n+1}[\lambda] \to M_n[\lambda]$ is an isomorphism for all $n$ large enough. Similarly, we say that $(M_\lambda)$ stabilizes $T$-equivariantly if for every $\chi$ the map $M_{n+1}[^X\chi] \to M_n[^X\chi]$ is an isomorphism for all $n$ large enough. If $(M_\lambda)$ stabilizes $T$-equivariantly then it also stabilizes $G$-equivariantly. Indeed, for every $\lambda$ the map $M_{n+1}[\lambda] \to M_n[\chi(\lambda)]$ is an isomorphism for $n \gg 0$ and so using (3.9) we deduce that also the map

$$M_{n+1}[\lambda] = M_{n+1}[\chi(\lambda)][\lambda] \to M_n[\chi(\lambda)][\lambda] = M_n[\lambda]$$

is an isomorphism. We now give an instance where those stabilizations hold. We say that a formal algebra $A$ is adic if for every $m \geq n$ the map $A_m \to A_n$ is surjective and

$$\ker(A_m \to A_n) = \ker(A_m \to A_0)^{n+1}.$$

This implies in particular that $\ker(A_m \to A_0)^{m+1} = 0$ for every $m$.

**Definition 3.16 (standard formal algebra).** We say that an formal algebra $A$ is a standard adic formal $\mathcal{G}$-algebra if all of the following hold

1. $A$ is an adic algebra,
2. $A$ is an $A_0$-algebra (which means that every $A_n$ is an $A_0$-algebra and $A_n \to A_{n-1}$ are surjections of $A_0$-algebras),
3. $\text{Spec}(A_0)^G \to \text{Spec}(A_0)$ is an isomorphism for every $n$.

Every formal algebra obtained using formalization is adic and every formal algebra obtained from a $G$-scheme $X = \text{Spec}(A)$ with $I = I(X^0)$ is standard adic, as we will see in Section 4.2. In the same section we will prove that under good conditions standard adic algebras come from $G$-schemes.

Let $M = (M_n)_n$ be module over an adic algebra $A = (A_n)_n$. Then $M$ is adic if the maps $M_n \to M_{n-1}$ are surjective and for every $m \geq n$ we have $\ker(M_m \to M_n) = \ker(M_m \to A_n)M_m$ so that the maps induce isomorphisms $M_m \otimes_{A_n} A_n \to M_n$. If $A$ is standard adic and the module $M$ is equipped with a $T$-action then we say that $M$ is grounded if there exists a finite sum of $T$-characters $\chi$ such that $M_0 = M_0[^X\chi]$. Since the $G$-action on $A_0$ is trivial, the module $M$ is grounded whenever $M_0$ is a finitely generated $A_0$-module.

**Example 3.17.** Let $T$ be a $G$-group and let $A = (k[t]/t^n)_n$ be equipped with a $T$-action coming from the standard grading. The $A$-modules $M^1 = (k[t]/t^n \oplus (k[t]/t^n)t^{-1})_n$ and $M^2 = (\bigoplus_{i \in \mathbb{N}}(k[t]/t^n)t^{-i})_n$ are grounded, while $M^3 = (\bigoplus_{i \in \mathbb{N}}(k[t]/t^n)t^{-i})_n$ is not grounded. The difference is that $M^3$ has generators of arbitrarily negative weights, while the generators $M^2$ are infinite, but their weights are bounded from below.

**Definition 3.18 (stabilization indices).** For a simple $T$-representation $\chi$, we define $n_\chi \in \mathbb{Z}$ as the maximal weight of the Kempf torus in $\chi_T$. We set $n_\chi = \max(n_\lambda : \chi \in \chi')$ and $n_\lambda := n_\chi(\lambda)$.

For example, $n_{\chi}$ for $\lambda$ is a trivial $T$-representation.

**Lemma 3.19 ($T$-equivariant stabilization lemma).** Let $A = (A_n)$ be a standard adic formal $\mathcal{G}$-algebra. Let $M = (M_n)$ be an adic $A$-module with a $T$-action. Assume that $M_m$ is grounded. (For example, this is the case when $M_0$ is a finitely generated $A_0$-module.)

Fix $\chi$. Then there exists a natural number $n_{\chi,M}$ such that for every $n > n_{\chi,M}$ the surjection

$$M_n[\chi] \to M_{n-1}[\chi]$$

is an isomorphism. If $M = A$ then we may take $n_{\chi,M} = n_{\chi}$.

**Proof.** Fix $\chi'$ so that $M_0 = M_0[^X\chi']$. The claim is invariant under field extensions, so we pass to the algebraic closure of $k$. Let $m$ be the minimal weight of the Kempf torus on $M_0$; such a weight
exists since the set of weights is finite by $M_0 = M_0[\chi]$. We claim that $n_{\chi,M} = n_{\chi} + |m|$ satisfies the assumptions.

By Corollary 2.2 for every $n$ the ideal $\ker(A_n \to A_0)$ consists entirely of positive weights. Since $\mathcal{M}$ is adic, we have $M_0 = M_0/(\ker(A_n \to A_0)M_n)$. By the graded Nakayama lemma, the minimal weight of the Kempf torus on $M_n$ is equal to $m$.

We have a short exact sequence

$$0 \to \ker(M_n \to M_{n-1})[\chi] \to M_n[\chi] \to M_{n-1}[\chi] \to 0$$

so it is enough to prove that $\ker(M_n \to M_{n-1})[\chi] = 0$ for all $n > n_{\chi,M}$. Since $\mathcal{A}$ and $\mathcal{M}$ are adic, we have $\ker(M_n \to M_{n-1}) = \ker(A_n \to A_0)^n M_n$. If $n > n_{\chi} + |m|$ then the minimal weight of the Kempf torus in $\ker(A_n \to A_0)^n M_n$ is greater than $n_{\chi} + |m| + m$ which is at least $n_{\chi}$. The isotypic component $(\ker(A_n \to A_0)^n M_n)[\chi]$ is thus equal to zero, as claimed. If $\mathcal{M} = \mathcal{A}$ then $m = 0$ and so $n_{\chi,M} = n_{\chi}$.

As discussed above, for a $\mathcal{G}$-module $\mathcal{M}$ its $T$-equivariant stabilization implies $\mathcal{G}$-equivariant stabilization.

**Corollary 3.20.** Let $\mathcal{A}$ be a standard adic formal $\mathcal{G}$-algebra and let $\mathcal{M}$ be an adic $\mathcal{G}$-module. Then the $\lambda$-$\lambda'$-comparison map (3.15) for $\mathcal{M}$ is an isomorphism. Consequently, the map $\lim_{\mathcal{G}}(M_n) \to M_n$ is surjective for every $n$.

**Proof.** The comparison map is always injective, so it is enough to prove surjectivity. Let $m \in \lim_{\mathcal{G}}(M_n)$ and choose $\mathcal{X}$ such that $m \in \lim_n M_n[\mathcal{X}]$. Let $n' := n_{\lambda,M}$. For $n > n'$ the map $M_n[\mathcal{X}] \to M_{n-1}[\mathcal{X}]$ is an isomorphism, so $\lim_n M_n[\mathcal{X}] \to M_{n'}[\mathcal{X}]$ is an isomorphism. Since $T$ is central in $\mathcal{G}$, the subgroup $M_{n'}[\mathcal{X}]$ is $\mathcal{G}$-stable and so there exists a finite set of $\mathcal{G}$-representations $\mu$ such that $m \in M_{n'}[\mu]$. But then $\mathcal{X}(\mu) \subset \mathcal{X}$ and by the discussion above, also $(\lim_n M_n)[\mu] \to M_{n'}[\mu]$ is an isomorphism, so $m$ lifts to an element of $(\lim_n M_n[\mu])$ which in turn lies in $\lim_{\mathcal{G}}(M_n)$ and so $m$ is in the image of the $\lambda$-$\lambda'$-comparison map. The final claim follows because $\lim_{\mathcal{G}}(M_n) \to M_n$ is surjective by right-exactness of $(-)[\mathcal{X}]$.

**Theorem 3.21 (G-equivariant algebraization exists).** Let $\mathcal{A}$ and $\mathcal{M}$ be as in Lemma 3.19. Let $M = \lim_{\mathcal{G}}(M)$. Then $\ker(M \to M_n) = \ker(A \to A_0)^{n+1}M$ for all $n$. Therefore $\mathcal{M}$ is a formalization of the $A$-module $(M, \ker(A \to A_0))$. In particular, $\mathcal{A}$ is a formalization of $(A, \ker(A \to A_0))$.

**Proof.** Let $\pi_n: A \to A_n$ be the canonical maps and let $I_n = \ker(\pi_n)$. Let $\pi^M_n: M \to M_n$ be the canonical maps. The maps $\pi^M_n$ and $\pi_n$ are surjective by exactness of $(-)[\mathcal{X}]$. This implies that $\pi^M_n(I_0^{n+1}M) = \pi_n(I_0^{n+1})M_n = \ker(A_n \to A_0)^{n+1}M_n = 0$ because $\mathcal{A}$ and $\mathcal{M}$ are adic. We have proven $\ker(\pi^M_n) \supset I_0^{n+1}M$. It remains to prove the other containment. Take an element $m \in \ker(\pi^M_n)$ and fix $\mathcal{X}$ such that $m \in \lim_n M_n[\mathcal{X}]$. By $T$-stabilization for the module $\mathcal{M}$, for any $n' > n_{\lambda,M}$ the map $\lim_n M_n[\mathcal{X}] \to M_{n'}[\mathcal{X}]$ is an isomorphism. This isomorphism maps the element $m$ to an element of $\ker(M_{n'} \to M_n) = \ker(A_{n'} \to A_0)^{n+1}M_{n'}$. Since $\pi_{n'}$ and $\pi^M_{n'}$ are surjective, we have

$$\pi^M_{n'}(I_0^{n+1}M) = \pi_{n'}(I_0^{n+1})\pi^M_{n'}(M) = \ker(A_{n'} \to A_0)^{n+1}M_{n'}.$$

Since $(-)[\mathcal{X}]$ is right-exact, we see that $\pi^M_{n'}((I_0^{n+1}M)[\mathcal{X}])$ contains $m$. So we pick an element $j \in (I_0^{n+1}M)[\mathcal{X}]$ such that $\pi^M_{n'}(j) = \pi^M_{n'}(i)$. Then $\pi^M_{n'}(i - j) = 0$ and by (3.14) we have $i - j \in \lim_n M_n[\mathcal{X}]$. But $\pi^M_{n'}$ restricted to $\lim_n M_n[\mathcal{X}]$ is an isomorphism, so $i - j = 0$.

**Proposition 3.22 (generating sets for grounded modules).** Let $\mathcal{A}$ and $\mathcal{M}$ be as in Lemma 3.19 and let $M = \lim_{\mathcal{G}}(M)$. Let $F$ be a $A$-module with a $T$-equivariant map $p: F \to \lim_{\mathcal{G}}(M)$ such that the associated map $p: F_0 \to M_0$ is surjective. Then $p$ is surjective.

**Proof.** Consider the map $p_n: F_n \to M_n$ which is just $p_n = p \otimes_A (A/I_0^{n+1})$. Let $G_n = p_n(F_n)$. Since $p_0$ is surjective and $\mathcal{M}$ is adic, we have $M_n = G_n + I^n M_n$. But $I^{n+1}M_n = 0$ so

$$M_n = G_n + I M_n = G_n + I(G_n + I M_n) = G_n + I^2 M_n = \ldots = G_n + I^{n+1}M_n = G_n.$$
Let $G = p(F) \subset M$. It is enough to prove that $G[\mathcal{X}] = M[\mathcal{X}]$ for all $\mathcal{X}$. By exactness, we have $M[\mathcal{X}] = \lim_n M_n[\mathcal{X}]$ for all $\mathcal{X}$. By Stabilization Lemma 3.19, the map $M[\mathcal{X}] \rightarrow M_n[\mathcal{X}]$ is an isomorphism for $n$ large enough and we obtain the following diagram

$$
\begin{array}{ccc}
G[\mathcal{X}] & \rightarrow & M[\mathcal{X}] \\
\downarrow & & \downarrow \simeq \\
G_n[\mathcal{X}] & \rightarrow & M_n[\mathcal{X}]
\end{array}
$$

so $G[\mathcal{X}] \rightarrow M[\mathcal{X}]$ is surjective as well as injective. $\square$

Corollary 3.23 (finitely generated modules). Let $A$ be a standard adic formal $\mathcal{G}$-algebra with algebraization $A$ and let $\mathcal{M} = (M_n)_n$ be an adic module over $A$ with a $T$-action. Then, the following are equivalent

1. the $A_0$-module $M_0$ is finitely generated,
2. every $A_n$-module $M_n$ is finitely generated,
3. $\lim_T(\mathcal{M})$ is a finitely generated $A$-module.

If these hold, we say that $\mathcal{M}$ is a finitely generated $A$-module.

Proof. 3. $\Rightarrow$ 2. Since $[-](\mathcal{X})$ is right-exact for every $\mathcal{X}$, the map $\lim_T(\mathcal{M}) \rightarrow M_n$ is surjective so $M_n$ is a finitely generated $A$-module. But $M$ is adic, so $I^{n+1}$ annihilates $M_n$ and so $M_n$ is a finitely generated $A_n$-module.

The implication 2. $\Rightarrow$ 1. is trivial and 1. $\Rightarrow$ 3. follows from Proposition 3.22 since $T$ is linearly reductive, so we can find a $T$-equivariant surjection from a linearized free $A$-module $F$ onto $M_0$ and its $T$-equivariant lift to $\lim_T(\mathcal{M})$. $\square$

We are now ready to prove the main equivalence statement for grounded modules. Consider a standard adic formal $\mathcal{T}$-algebra $A = (A_n)_n$ with algebraization $A$ and $I = \ker(A \rightarrow A_0)$. By Theorem 3.21, the algebra $A$ is a formalization of $A$. Consider the two functors

$$
\text{Alg} := \lim_G(-) : \text{Mod}_{A}^{\text{adic},G} \rightarrow \text{Mod}_{A}^{G} \quad \text{and} \quad \text{Formalize} : \text{Mod}_{A}^{G} \rightarrow \text{Mod}_{A}^{\text{adic},G},
$$

defined by respectively $\text{Formalize}(M) = (M/I^{n+1}M)_n \in \mathbb{N}$ and $\text{Alg}(\mathcal{M}) = \lim_G(\mathcal{M})$, where $\text{Mod}_{A}^{G}$ denotes the category of $A$-modules with a $G$-action and $\text{Mod}_{A}^{\text{adic},G}$ of adic $G$-modules over $A$.

Theorem 3.24 (Algebraization for grounded $G$-modules). The functors $\text{Alg}$ and $\text{Formalize}$ restrict to an equivalence between the category of finitely generated $A$-modules with a $G$-action and finitely generated $\text{adic} A$-modules (in the sense of Corollary 3.23) with $G$-action.

Proof. By Corollary 3.20 we can speak about $\mathcal{A}$- and $\mathcal{X}$-algebraizations interchangeably. By Corollary 3.23 the functors $\text{Alg}$ and $\text{Formalize}$ map finitely generated modules to finitely generated ones. It remains to prove that they give an equivalence.

By Theorem 3.21, the composition $\text{Formalize} \circ \text{Alg}$ is isomorphic to identity. Consider $\text{Alg} \circ \text{Formalize}$. Choose an $A$-module $M$ with $G$-action, let $\mathcal{M} = \text{Formalize}(M)$ and $M' = \text{Alg}(\mathcal{M})$ and consider the natural map $M \rightarrow M'$. By Stabilization Lemma 3.19, the map $M[\mathcal{X}] \rightarrow M'[\mathcal{X}]$ is surjective for every $\mathcal{X}$. Since every element of $M'$ sits inside some $M'[\mathcal{X}]$, we get that $M \rightarrow M'$ is surjective. By the same argument as in Stabilization Lemma, for every $\mathcal{X}$ the map $M[\mathcal{X}] \rightarrow (M/I^{n+1}M)[\mathcal{X}]$ is an isomorphism for $n$ large enough, so $M \rightarrow M'$ is injective, so finally it is an isomorphism. $\square$

4. Formal Bialynicki-Birula functors

In this section we introduce the formal version of Bialynicki-Birula functors. The notation and general outline are consistent with [JS19, Section 6]. For an $n \in \mathbb{Z}_{\geq 0}$ let $\mathcal{T}_n = V(m_0^{n+1}) \subset \mathcal{T}$ be
the \( n \)-th thickening of the \( k \)-point \( 0 \in \mathcal{Z} \). Consider the set-valued functor
\[
\hat{X}(S) = \left\{ (\varphi_n)_{n \in \mathbb{Z}_{\geq 0}} \mid \varphi_n : \mathcal{T}_n \times S \to X, \varphi_n is \mathcal{G}\text{-equivariant} \text{ and } (\varphi_{n+1})_{\mathcal{T}_n \times S} = \varphi_n \text{ for all } n \right\}.
\]
We have a natural formalization map \( X^+ \to \hat{X} \) given by \( \varphi \mapsto (\varphi_{|\mathcal{T}_n \times S})_n \). Eventually, we will prove that it is an isomorphism. The crucial technical advantage of \( \hat{X} \) over \( X^+ \) is that it avoids the topological issues altogether: the set-theoretic image of each \( \varphi_n \) lies in \( X^G \).

4.1. Formal \( \mathcal{T} \)-schemes and \( \mathcal{G} \)-linearized sheaves. We introduce some notation for the formal neighbourhoods of fixed points in \( X \). It is convenient to make this in a very general way.

**Definition 4.1.** A formal \( \mathcal{G} \)-scheme consists of a sequence of \( \mathcal{G} \)-schemes \( Z = (Z_n)_{n \in \mathbb{Z}_{\geq 0}} \) and equivariant closed immersions
\[
Z_0 \xrightarrow{\varphi} Z_1 \xrightarrow{\varphi} \ldots \xrightarrow{\varphi} Z_n \xrightarrow{\varphi} Z_{n+1} \xrightarrow{\varphi} \ldots
\]
such that
1. the \( \mathcal{G} \)-action on \( Z_0 \) is trivial and \( Z_0 \xrightarrow{\varphi} Z_0^\mathcal{G} \) is an isomorphism,
2. if \( \mathcal{I} \subset O_{Z_n} \) is the ideal sheaf defining \( Z_0 \subset Z_n \), then for all \( m \leq n \) the ideal sheaf \( \mathcal{I}^m \subset Z_n \).

A morphism of formal \( \mathcal{G} \)-schemes \( f : (W_n) \to (Z_n) \) is a family of \( \mathcal{G} \)-equivariant morphisms \( (f_n : W_n \to Z_n)_n \) compatible with closed immersions \( \mathcal{I}_W \subset W_{n+1} \) and \( Z_n \to Z_{n+1} \). If every \( Z_n \) is a \( \mathcal{T} \)-scheme and the closed immersions \( Z_n \to Z_{n+1} \) are \( \mathcal{G} \)-equivariant then we say that \( Z \) is a formal \( \mathcal{T} \)-scheme.

Similarly, a morphism of formal \( \mathcal{T} \)-schemes is a morphism of formal \( \mathcal{T} \)-schemes such that every \( f_n \) is \( \mathcal{T} \)-equivariant.

**Definition 4.2 (Quasi-coherent sheaves on \( Z \)).** Let \( Z = (Z_n)_{n \in \mathbb{Z}_{\geq 0}} \) be a formal \( \mathcal{G} \)-scheme. The closed inclusions \( Z_n \hookrightarrow Z_{n+1} \) induce a sequence of restriction maps
\[
\ldots \xrightarrow{\varphi} \text{Qcoh}_G(Z_{n+1}) \xrightarrow{\varphi} \text{Qcoh}_G(Z_n) \xrightarrow{\varphi} \ldots \xrightarrow{\varphi} \text{Qcoh}_G(Z_2) \xrightarrow{\varphi} \text{Qcoh}_G(Z_1) \xrightarrow{\varphi} \text{Qcoh}_G(Z_0)
\]
A quasi-coherent \( \mathcal{G} \)-linearized sheaf on \( Z \) is a sequence \( \left( F_n \in \text{Qcoh}_G(Z_n) \right)_n \) together with isomorphisms \( i_n : F_{n+1}|_{Z_n} \to F_n \). A morphism of quasi-coherent \( \mathcal{G} \)-linearized sheaves \( \varphi : (F_*, i_*) \to (G_*, j_*) \) is a sequence of morphisms \( \varphi_n : F_n \to G_n \) such that for every \( n \) the following diagram commutes
\[
\begin{array}{c}
F_{n+1}|_{Z_n} = F_{n+1} \otimes O_{Z_{n+1}} \xrightarrow{i_n} F_n \\
\downarrow \varphi_{n+1} \otimes \text{id} \downarrow \varphi_n \\
G_{n+1}|_{Z_n} = G_{n+1} \otimes O_{Z_{n+1}} \xrightarrow{j_n} G_n
\end{array}
\]
These objects and morphisms form the category \( \text{Qcoh}_G(Z) \) of \( \mathcal{G} \)-linearized quasi-coherent sheaves on \( Z \) which is a symmetric monoidal category, that is a 2-categorical limit of the diagram (4.3).

We say that a \( \mathcal{G} \)-linearized sheaf \( F \) of \( Z \) is of finite type if there exists an affine open cover of \( Z_0 \) such that the restrictions of \( F \) to each member of this open cover are finitely generated modules in the sense of Corollary 3.23. We denote the by \( \text{Qcoh}_G^H(Z) \) the full subcategory of \( \text{Qcoh}_G(Z) \) consisting of finite type sheaves.

We have a natural formalization and — much subtler — algebraization in the \( \mathcal{G} \)-equivariant setting, that are the geometric analogues of the ones from Section 3. Eventually we will use them to show that the formalization map \( X^+ \to \hat{X} \) is an isomorphism.

**Definition 4.4 (Formalization).** Let \( Z \) be a \( \mathcal{G} \)-scheme and \( \mathcal{I} \subset O_Z \) be the ideal defining \( Z^G \).

The formal \( \mathcal{G} \)-scheme \( \hat{Z} \) associated to \( Z \) is the sequence \( (Z_n) \) where \( Z_n = V(\mathcal{I}^{n+1}) \) for all \( n \). The inclusions \( Z_n \to Z \) induce restrictions \( \text{Qcoh}_G(Z) \to \text{Qcoh}_G(Z_n) \) which together give a natural comparison functor \( \text{Qcoh}_G(Z) \to \text{Qcoh}_G(Z_n) \) that is cocontinuous and tensor (see appendix for definitions) and preserves finite type sheaves.
**Definition 4.5** (Algebraization). Let \( Z \) be a formal \( \mathcal{G} \)-scheme. An algebraization of \( Z \) is a \( \mathcal{G} \)-scheme \( Z \) such that \( Z \) is isomorphic to the formal \( \mathcal{G} \)-scheme associated to \( Z \) and the associated restriction functor \( \text{Qcoh}_{\mathcal{G}}^0(Z) \to \text{Qcoh}_{\mathcal{G}}^0(\hat{Z}) \) is an equivalence.

4.2. Algebraization, geometric counterpart. To a formal \( \mathcal{G} \)-scheme \( Z = (Z_n) \) we may associate the family of maps \( \pi_n : Z_n \to Z_0 \) which are multiplications by \( 0 \in \mathcal{G} \). In particular, \( \pi_n|_{Z_n} = \pi_n \) for all \( n \). Every \( Z_n \) is defined in \( Z_{n+1} \) by the \( n \)-power of the ideal \( I(Z_0) \), so \( (Z_0)_{\text{red}} \to (Z_n)_{\text{red}} \to (Z_0)_{\text{red}} \) are isomorphisms. In particular, \( Z_n \to Z_0 \) is affine. The maps \( \pi_n \) give a sequence of sheaves of quasi-coherent \( \mathcal{G} \)-algebras on \( Z_0 \):

\[
A_n := (\pi_n)_* \mathcal{O}_{Z_n}
\]

and surjections \( A_{n+1} \to A_n \). We call these the sheaves of \( \mathcal{G} \)-algebras associated to \( Z \). The category \( \text{Rep}_\mathcal{G} [\lambda] \) behaves well when we pass from \( \text{Rep}_\mathcal{G} \) to \( \text{Rep}_\mathcal{G} \) and then to \( \mathcal{G} \)-equivariant sheaves on some \( \mathcal{G} \)-scheme \( Z_0 \) with trivial \( \mathcal{G} \)-action. Indeed, fix a quasi-coherent sheaf \( \mathcal{A} \) with \( \mathcal{G} \)-action and define \( \mathcal{A}[\lambda] \subset \mathcal{A} \) by setting \( H^0(U, \mathcal{A}[\lambda]) := H^0(U, \mathcal{A}[\lambda]). \) Since \( (-)[\lambda] \) is left exact, we obtain a subsheaf. For each open \( U \subset Z_0 \) the algebra \( H^0(U, \mathcal{O}_{Z_0}) \) is a trivial \( \mathcal{G} \)-representation, so the structural map \( H^0(U, \mathcal{O}_{Z_0}) \otimes_k H^0(U, \mathcal{A}) \to H^0(U, \mathcal{A}) \) descends to

\[
H^0(U, \mathcal{O}_{Z_0}) \otimes_k H^0(U, \mathcal{A})[\lambda] \to H^0(U, \mathcal{A}[\lambda]).
\]

and so \( \mathcal{A}[\lambda] \subset \mathcal{A} \) is a subsheaf of \( \mathcal{G} \)-modules and \( \mathcal{O}_{Z_0} \)-modules; since the action of \( \mathcal{G} \) on \( Z_0 \) is trivial those structures commute. We can thus form \( \lambda \)-algebraizations of sheaves. For a sheaf of algebras \( \mathcal{A} \) its \( \lambda \)-algebraization is a sheaf of algebras as well. If we fix a central torus \( T \) the same holds for \( (-)[\lambda] \) instead of \( (-)[\lambda] \).

**Theorem 4.6** (algebraization of a formal \( \mathcal{G} \)-scheme). Let \( Z = (Z_n) \) be a formal Noetherian formal \( \mathcal{G} \)-scheme with associated sheaf of algebras \( \mathcal{A}_n \). The scheme \( Z = \text{Spec} \mathcal{O}_{Z_0}(\lim_n \mathcal{A}_n) \) is the colimit of \( Z_n \) in the category of locally linear \( \mathcal{G} \)-schemes and \( Z \) is an algebraization of \( Z \). Moreover, the scheme \( Z \) is locally linear and \( \pi : Z \to Z^G \) is affine of finite type, so \( Z \) is locally Noetherian.

**Proof.** Let \( \mathcal{A} = \lim_n \mathcal{A}_n \). We also fix central Kempf torus \( \mathcal{G}_{m,T} \to \text{Z} \) and its image \( T \subset \text{Z} \). This is a one-dimensional torus, possibly non-split, see Subsection 2.1. By Theorem 3.21 applied to sections of \( \mathcal{A} \), the map \( \mathcal{A} \to \mathcal{A}_n \) is surjective and its kernel is equal to \( \ker(\mathcal{A} \to \mathcal{A}_0)^{n+1} \) section-wise. Therefore indeed the formalization of \( Z \) is \( Z \).

Now we prove that \( Z = \text{colim} Z_n \) in the category of locally linear \( \mathcal{G} \)-schemes. Fix a locally linear \( \mathcal{G} \)-scheme \( W \) with coherent maps \( f_n : Z_n \to W \). In particular, we get a map \( f_0 : Z_0 \to W^G \).

By Proposition 2.12 the multiplication by \( \mathcal{O}_{\mathcal{G}} \) map \( \pi_W : W \to W^G \) is affine. Let \( W' = W \times_{W^G} Z_0 \). The map \( W' \to Z_0 \) is affine and \( \mathcal{G} \)-equivariant. We have induced maps \( f_n' = f_n \times \pi_n : Z_n \to W' \) over \( Z_0 \). To sum up, we have the following situation and we want to find the dashed arrow

\[
\begin{array}{ccc}
Z_n & \xrightarrow{f_n} & W' \\
\downarrow \pi_n & & \downarrow \text{aff} \\
Z & \xrightarrow{f_0} & W \\
\end{array}
\]

All schemes \( Z_n, Z \) and \( W' \) are affine over \( Z_0 \). Let \( B \) be the sheaf of \( \mathcal{G} \)-algebras on \( Z_0 \) such that \( W' = \text{Spec} \mathcal{O}_{Z_0}(B) \). Note that \( B = \bigcup B[\lambda] \). The maps \( f_n' \) induce compatible \( \mathcal{G} \)-equivariant morphisms \( f_n : B \to \mathcal{A}_n \). By \( \mathcal{G} \)-equivariance, for every \( \lambda \), they restrict to \( f_n' : B[\lambda] \to \mathcal{A}_n[\lambda] \) and give morphisms

\[
F[\lambda] : B[\lambda] \to \mathcal{A}[\lambda].
\]
Composing those with $A[\lambda] \to A$, we obtain $F[\lambda]: B[\lambda] \to A$ which in turn glue to $F: B \to A$. (Compatibility of any two $F[\lambda_1]$ and $F[\lambda_2]$ follows from compatibility of $f^{\#}_{\mu}$, because of stabilization.) This is a $G$-equivariant homomorphism of $O_{Z_0}$-algebras and it gives the required map $Z \to W'$ and in turn $Z \to W$.

By assumption the scheme $Z$ is locally Noetherian; we will use it to prove that $Z \to Z_0$ is of finite type. The argument of [JS19, Theorem 6.8] generalizes verbatim; we repeat it for completeness. By assumption the scheme $Z_0$ is locally Noetherian. For every $n$, in the $O_{Z_0}$-module $A_n$ we have a coherent ideal sheaf $K_n := \ker(A_n \to A_0)$ and by definition of formal schemes we have $K_n^m = \ker(A_n \to A_m) = 0$, so $0 = K_n^m \subset K_n^{m-1} \subset \ldots \subset K \subset A_n$ is a finite filtration whose quotients are coherent $O_{Z_0}$-modules that are in fact $O_{Z_0}$-modules. Thus the $O_{Z_0}$-module $A_n$ is coherent as well. The geometric analogue of Stabilization Lemma 3.19 implies that for every $\lambda$ the $O_{Z_0}$-module $A[\lambda]$ is coherent. The $A$-module $J_0/J_0^2 \subset A_2$ is coherent, so we can fix $\lambda$ and a map $A[\lambda]^{gr} \to J_0$ that restricts to a surjection $A[\lambda]^{gr} \to J_0/J_0^2$. We claim that $A$ is generated as an $O_{Z_0}$-algebra by the image of $A[\lambda]^{gr}$. To prove this, we pass to $\tilde{\mathfrak{k}}$ and use the grading induced by the Kempf torus. By Stabilization Lemma for the trivial representation, the degree zero part of $A$ is indeed $A_0 = O_{Z_0}$. Therefore $J_0$ is positively graded and the result follows from graded Nakayama lemma.

It remains to prove that $Qcoh_G^d(Z) \to Qcoh_G^d(Z)$ is an equivalence, but this follows from Theorem 3.24.

4.3. Putting it all together. In this short section we apply the results of the previous one to prove Theorem 1.1. We begin by constructing a formal $G$-scheme and its algebraization. Let $X$ be a $G$-scheme and $X_n = V(\mathcal{I}^{n+1})$ where $\mathcal{I} \subset O_X$ is the sheaf defining $X^G$. Then $X_0 = X^G$ and moreover $X_0 \hookrightarrow X_n$ induces a homeomorphism of topological spaces, so topologically the $G$-action is trivial. In particular each $X_n$ is a locally linear $G$-scheme because every open subset is $G$-stable.

Let $Z_n = X_n^+ \subset X_n$ be the closed subscheme of $X_n$ which is its Białynicki-Birula decomposition, Proposition 2.12. Let $Z = (Z_n)$ and let $\hat{X}$ be the algebraization of $Z$. The space $\hat{X}$ has a natural map $\hat{X} \to X^G$, but a priori lacks the “inclusion of cells” map $\hat{X} \to X$. In fact, the construction of this inclusion map is subtle topologically, as $\hat{X}$ is obtained from a formal neighbourhood of $X^G$ and does not see directly the topology of $X$. We will construct the map using the formalism of Tannaka duality, see Appendix. The main input is that, by Theorem 4.6, we have $Qcoh_G^d(\hat{X}) = \lim_n Qcoh_G^d(Z_n)$.

The pullback via closed inclusions $Z_n \hookrightarrow X$ gives a functor $F: Qcoh_G^d(X) \to \lim_n Qcoh_G^d(Z_n)$. The composition $F: Qcoh_G^d(X) \to Qcoh_G^d(\hat{X})$ is a cocontinuous tensor functor. Since $X$ is Noetherian, each $G$-linearized quasi-coherent sheaf on $X$ is a filtered limit of coherent $G$-linearized sheaves, see for example [sta17, Tag 07TU]. The category of quasi-coherent sheaves of $\hat{X}$ with $G$-linearization admits all limits, so the functor $F$ extends to a functor $\text{Qcoh}_G(X) \to \text{Qcoh}_G(\hat{X})$ that we also denote by $F$. Let $p_X: X \to \text{Spec } k$ and $p_{\hat{X}}: \hat{X} \to \text{Spec } k$ be the structure maps. Then $F \circ p_X^* = p_{\hat{X}}^*$ and $p_X^*$ are isomorphic as this holds coherently for all $\lambda$. By Theorem A.2, we obtain a $G$-equivariant map

$$I: \hat{X} \to X$$

such that $I|_{Z_n}$ is the inclusion of $Z_n$, for all $n$.

Proof of Theorem 1.1. The $G$-scheme $\hat{X}$ is a scheme over $X$ via the equivariant map $I$, so we have a family

$$\Phi: G \times \hat{X} \to X.$$
universal family Φ, hence φ is equal to this pullback. This proves that (X, Φ) represents
the functor X∗.

Proposition 1.2 now follows from the construction of X very similarly as the proof in the
linearly reductive case [JS19, §7]. The key point is the mere existence and affineness of πX : X+ → XG.

Proof of Proposition 1.2. Since πX is affine and smooth at x by assumption, we can write it locally
on XG as π : Spec (B) → Spec (A), where B is a G-algebra and π is G-invariant. Since π has a
section, the map πB : A → B is injective. Suppose first k = F. Then a Kempf line sits inside G and
induces an N-grading on B such that A ⊂ B0. By Corollary 2.2, we in fact have A = B0. In this
setup, the claim follows from [JS19, Lemma 7.2].

For general k, we may assume x is closed. Now the morphism X+ → (XG)♯ is an affine
space fibration near x by the previous case, so in fact near x it a trivial affine space fibration, i.e., a
projection from a trivial vector bundle, so X+ → XG is a GL-torsor. But being a GL-torsor bundle is
Zariski-local, so we get that also πX is such locally near x and that concludes the proof. □

Appendix A. Tannakian Formalism

Let G be an affine algebraic group over k and X be a G-scheme. In this appendix we recover
G-equivariant morphisms from pullback-like maps. This is used crucially in the proof of Theorem
1.1 to obtain the “embedding of cells” map X → X.

We need some preliminary notions. A cocontinuous functor F between tensor categories is a
functor that preserves all small colimits; in particular it is right-exact. A tensor functor is a strong
symmetric monoidal functor, which means that F(M ⊗ N) ≃ F(M) ⊗ F(N) and F(1) ≃ 1 and
these isomorphisms are subject to some compatibility conditions [SR72, I.4.1.1, 4.2.4]. For a G-
scheme Z the equivariant map p2 : Z → Spec (k) induces a pullback p2♯ : RepG → QCohG(Z)
which maps a rational G-representation V to V ⊗ k OZ with the natural linearization. Our main
results are as follows.

Proposition A.1. Let X be a quasi-compact quasi-separated G-scheme. Let Y be a scheme and F : QCohG(X) → QCoh(Y) be a cocontinuous tensor functor. Then there exists a principal G-bundle π : P → Y and an
G-equivariant map γ : P → X such that F is isomorphic to π♯γ♯.

Theorem A.2. Let X be a quasi-compact quasi-separated G-scheme. Let Y be a G-scheme and F : QCohG(X) → QCohG(Y) be a cocontinuous tensor functor. Assume moreover that the functors
F ◦ pX♯, pY♯ : RepG → QCohG(Y)
are isomorphic. Then there exists a unique G-equivariant morphism f : Y → X such that F ∼ f♯.

Remark A.3 (Stacky point of view). In this appendix we do not assume any familiarity with
stacks. However, the stacky language is natural when speaking about equivariant geometry and
so we summarize here the stacky perspective. For an introductions to stacks, see [Ols16].

The main results above tell that for a G-scheme X the stack [X/G] is tensorial in the sense of
[Bra14, HR19]. Indeed, QCohG(Z) ≃ QCoh([Z/G]). Let ∗ = Spec (k). Theorem A.2 and Proposition
A.1 read: every cocontinuous tensor functor F : QCoh([Y/G]) → QCoh([X/G]) is isomorphic to a pullback map f♯∗ for a morphism f♯ : [X/G] → [Y/G]. If moreover F ◦ pX♯ and pY♯ are isomorphic then we have the following diagram with upper row obtained from the lower
one by pullback by the quotient map ∗ → ∗/G and with f♯ = f♯/G:
Let us begin the proof. In the special case $X = \text{Spec}(k)$, Proposition A.1 yields a bundle $P \to Y$. In this case $F$ is trivially faithfully flat, hence is $F$ is in fact exact. We prove this first.

**Lemma A.4.** Let $F : \text{Rep}_G \to \text{Qcoh}(Y)$ be a cocontinuous tensor functor. Let $0 \to V_1 \to V_2 \to V_3 \to 0$ be an exact sequence of representations and $M$ be a quasi-coherent sheaf. Then the sequence $0 \to M \otimes F(V_1) \to M \otimes F(V_2) \to M \otimes F(V_3) \to 0$ is exact. In particular, $F$ is exact.

**Proof.** The key observation is that finite dimensional representations admit duals (see [Bra14, §4.7] for a categorical abstraction of this notion). Every rational representation is a filtered union of finite dimensional representations and $M \otimes (-)$ commutes with filtered limits, so it is enough to show that $M \otimes F(-)$ maps a short exact sequence $0 \to V_1 \to V_2 \to V_3 \to 0$ of finite dimensional representations to a short exact sequence. By right-exactness, the sequence $M \otimes F(V_1) \to M \otimes F(V_2) \to M \otimes F(V_3) \to 0$ is exact and it remains to prove that the kernel of $M \otimes F(V_1) \to M \otimes F(V_2)$ is zero. Denote this kernel by $K$.

From the dual of the above sequence, we get a short exact sequence

$$F(V_3') \otimes K \to F(V_2') \otimes K \to F(V_1') \otimes K \to 0.$$  

After application of $\text{Hom}(-, M)$ we obtain an inclusion

(A.5) \quad $\text{Hom}(F(V_1') \otimes K, M) \to \text{Hom}(F(V_2') \otimes K, M)$.

For a finite dimensional representation $V_1$ the functor $V_1 \otimes (-)$ is right adjoint to $V_1' \otimes (-)$. The functor $F$ preserves adjointness, so $F(V_1)$ is right adjoint to $F(V_1')$ and the inclusion (A.5) completes to a diagram

$$\text{Hom}(F(V_1') \otimes K, M) \quad \text{Hom}(F(V_2') \otimes K, M)$$

$$\downarrow \cong \quad \downarrow \cong$$

$$\text{Hom}(K, F(V_1) \otimes M) \quad \text{Hom}(K, F(V_2) \otimes M)$$

where the bottom row arrow comes from the map $F(V_1) \to F(V_2)$. The inclusion $K \to F(V_1) \otimes M$ is an element of $\text{Hom}(K, F(V_1) \otimes M)$ which maps to zero, hence is zero, so $K = 0$. 

**Lemma A.6.** Let $F : \text{Rep}_G \to \text{Qcoh}(Y)$ be a cocontinuous tensor functor and $V$ be a rational $G$-representation. Then $F(V)$ is a flat sheaf. If $V$ is finite dimensional, then $F(V)$ is locally free sheaf of finite rank.

**Proof.** If $V$ is finite dimensional, then $F(V)$ is a dualizable object in $\text{Qcoh}(Y)$, so it is locally free of finite presentation [Bra14, Proposition 4.7.5]. If $V$ is rational, then it is a filtering limit of finite subrepresentations, so $F(V)$ is a filtering limit of locally free sheaves, so it is flat [Eis95, Theorem A6.6].

Let $(C, \otimes_C, \mathcal{O})$ be a tensor category (we will be interested in the case of quasi-coherent sheaves with linearization). The notions of algebra and $G$-action are naturally defined for $C$. Namely, an associative and commutative algebra in $C$ is an object $R \in C$ together with morphisms $R \otimes_C R \to R$ and $\mathcal{O} \to R$ satisfying the axioms of associative commutative multiplication and unity. Similarly, a $G$-action on an object $R \in C$ is a morphism

$$R \to R \otimes_C \mathcal{O} \otimes \dim_k H^0(G, \mathcal{O}_C) \simeq R \otimes \dim_k H^0(G, \mathcal{O}_C),$$

which satisfies the axioms of a $G$-coaction if written informally as $R \to R \otimes_k H^0(G, \mathcal{O}_G)$. Finally, a $G$-algebra is an algebra with a $G$-action which is a homomorphism of algebras. Every tensor functor preserves algebras, $G$-actions and $G$-algebras.

We will work with $G$-algebras in the category of $G$-representations, so our objects are equipped with two $G$-actions. To minimize notational collision, we let $G_1, G_2$ be algebraic groups equal to $G$. We will work in the category of $\text{Rep}_{G_1}$ and consider $G_2$-algebras in this category.
Let $A$ be equal to $G$ which we view as a $(G_1 \times G_2)$-variety by $\mu: G_1 \times A \times G_2 \to A$ defined by $\mu(g_1, a, g_2) = g_1 a g_2^{-1}$. Let $A := H^0(A, O_A)$, viewed as a $k$-vector space. The action $\mu$ gives a coaction

$$\Delta = \mu^\#: \overline{A} \to H^0(G_1, O_{G_1}) \otimes \overline{A} \otimes H^0(G_2, O_{G_2}).$$

Evaluating on $1 \in G_2$ gives a coaction $\Delta_1: \overline{A} \to H^0(G_1, O_{G_1}) \otimes \overline{A}$ so we obtain a rational $G_1$-representation $\Lambda := (\overline{A}, \Delta_1)$. Since $A$ is a variety, the vector space $\overline{A} = H^0(G, O_G)$ is an algebra. Its multiplication and unity are $G_1$-equivariant morphisms, hence they induce multiplication $\Lambda \otimes \Lambda \to \Lambda$ and unity $k \to \Lambda$ which make $\Lambda \in \text{Rep}_{G_1}$ an algebra object. The $G_2$-action on the algebra $\overline{A}$ commutes with the $G_1$-action, so $G_2$ acts on the algebra $\Lambda$. Summarizing, $\Lambda$ is an algebra in $\text{Rep}_{G_1}$ with an action of $G_2$.

Let $\Lambda' \in \text{Rep}_{G_1}$ be the $G_2$-algebra equal to the algebra $\Lambda$ but equipped with the trivial $G_2$-action. Let $H^0(G_2, O_{G_2}) \simeq k^{\oplus \dim H^0(G_2, O_{G_2})} \subseteq \text{Rep}_{G_1}$ be the trivial $G_1$-representation equipped with the usual algebra structure and $G_2$-action, so $H^0(G_2, O_{G_2}) \subseteq \text{Rep}_{G_1}$ is a $G_2$-algebra. We summarize the objects in the table below. In this table, usual $G_1$-action means coming from the action $G_1 \times G \ni (g_1, g) \mapsto g_1 g \in G$ and usual $G_2$-action means coming from the action $G_2 \times G \ni (g_2, g) \mapsto g g_2^{-1} \in G$.

| object          | algebra structure | $G_1$-representation | $G_2$-action |
|-----------------|-------------------|----------------------|--------------|
| $H^0(G_1, O_G)$ | usual             | not specified        | not specified|
| $\overline{A}$  | usual             | usual                | not specified|
| $\Lambda$       | usual             | usual                | usual        |
| $\Lambda'$      | usual             | usual                | trivial      |
| $H^0(G_2, O_{G_2})$ | usual          | trivial              | usual        |

Remark A.7. In the language of stacks, the map $\ast \to [\ast / G_1]$ is affine and in fact $\ast \simeq \text{Spec}_{[\ast / G_1]}(\Lambda)$.

Definition A.8. For a $G_1$-representation $V$, the coaction map is a morphism $V \to |V| \otimes_k \overline{A}$, where $|V|$ is the trivial representation on the underlying vector space. Using the coaction on $\overline{A}$, we obtain a left-exact sequence of $G_1$-representations, called the copresentation of $V$

$$0 \to V \to |V| \otimes_k \overline{A} \to |V| \otimes_k \overline{A} \otimes \overline{A} = |V| \otimes H^0(G, O_G) \otimes \overline{A}.$$  

Lemma A.9. The $G_2$-algebras $\Lambda \otimes \Lambda'$ and $H^0(G_2, O_{G_2}) \otimes \Lambda'$ are isomorphic.

Proof. Consider the isomorphism $\varphi: G \times G \to G \times G$ given by $\varphi(g, h) = (h^{-1} g h)$. On the level of functions it induces the isomorphism

$$\varphi^\#: H^0(G, O_G) \otimes H^0(G, O_G) \to H^0(G, O_G) \otimes H^0(G, O_G).$$

For every $g_1 \in G_1$ we have $\varphi(g_1, g_1) = (h^{-1} g_1 h)$, so the pullback $\varphi^#$ is an isomorphism of algebras in $\text{Rep}_{G_1}$:

$$\varphi^\#: \overline{A} \otimes \overline{A} \to H^0(G, O_G) \otimes \overline{A}.$$  

For every $g_2 \in G_2$ we have $\varphi(g_2 g_2, h) = (h^{-1} g g_2, h)$, so the isomorphism of $\varphi^#$ descends to an isomorphism of $G_2$-algebras in $\text{Rep}_{G_1}$:

$$\varphi^\#: \Lambda \otimes \Lambda' \to H^0(G_2, O_{G_2}) \otimes \Lambda'.$$

Let $X$ be a $G_1$-scheme over $k$ and denote by $X'$ the same scheme but equipped with the trivial $G_1$-action. Then the projection $\text{pr}_X$ and $G_1$-action $\sigma_X$ map below are equivariant

$$
\begin{array}{ccc}
G_1 \times X' & \xrightarrow{\sigma_X} & X \\
\downarrow \text{pr}_X & & \\
X' & & \\
\end{array}
$$
Let $\overline{\Lambda}^X := \overline{\Lambda} \otimes_k \mathcal{O}_X$ be the pullback of $\overline{\Lambda}$ via the structural morphism $\iota: X \to \text{Spec}(k)$, so that $\overline{\Lambda}^X$ is a quasi-coherent sheaf of $\mathcal{O}_X$-algebras with a $G_1$-linearization. Let $\text{Mod}_{\overline{\Lambda}^X} \subset \text{Qcoh}_{\mathcal{G}_1}(X)$ denote the full category of modules over this algebra. We have a natural map

$$\Psi: \text{Qcoh}_X \to \text{Mod}_{\overline{\Lambda}^X}$$

(A.10) defined by $\Psi(M) = (\sigma_X)_* \text{pr}_{X_Y}(M)$. When we forget about the $G_1$-action, this map is just tensoring with the linear space $H^0(G, \mathcal{O}_G)$. We also have natural maps in the opposite direction. Let $\Phi_1: \text{Mod}_{\overline{\Lambda}^X} \to \text{Mod}_{\overline{\Lambda}'X}$ be the map that forgets the $G_1$-linearization and $\Phi_2: \text{Mod}_{\overline{\Lambda}'X} \otimes_{H^0(G, \mathcal{O}_G)} \text{Mod}_{\overline{\Lambda}X} = \text{Qcoh}_X$ be the map associated to $H^0(G, \mathcal{O}_G) \to k$ which is evaluation at identity. Let

$$\Phi = \Phi_2 \circ \Phi_1: \text{Mod}_{\overline{\Lambda}^X} \to \text{Qcoh}_X$$

Lemma A.12. The maps (A.10)-(A.11) give an equivalence of tensor categories.

**Proof.** This is well-known, see for example [Ols16, Example 9.1.19].

**Proof of Proposition A.1.** We keep the notation $G_2 = G$ to emphasize that $G_1$-action on $\Lambda$ is absorbed by $F$ while the $G_2$-action survives and gives rise to the $G$-action.

**Step 1, $X = k$.** Let $A = F(\Lambda)$. Since $F$ is a tensor functor, the sheaf $A \in \text{Qcoh}(Y)$ is an algebra with a $G_2$-action. Then the scheme $P = \text{Spec}_Y(A)$ has a natural $G$-action and $\pi: P \to Y$ is $G_2$-equivariant with respect to the trivial action on $Y$. Let $P' \to Y$ denote the scheme $P$ but equipped with the trivial $G_2$-action.

The sheaf $A$ is flat by Lemma A.6. For a sheaf $M$ the inclusion $k = \Lambda^{G_1} \to \Lambda$ of representations induces an inclusion $M \to M \otimes F(\Lambda) = M \otimes A$ by Lemma A.4, so $A$ and $P, P'$ are all faithfully flat over $Y$. Lemma A.9 implies that $P \times_Y P' \simeq G_2 \times P'$ as $G_2$-schemes over $P'$. Hence $\pi: P \to Y$ becomes a $G_2$-bundle after pullback by itself. By faithfully flat descent [Ols16, §4.5.3] this means that $\pi: P \to Y$ is a $G_2$-bundle. The unique map $\gamma: P \to k$ is obviously $G_2$-equivariant and faithfully flat. It remains to prove that $F$ and $\pi_* \gamma^*$ are isomorphic. Both are exact functors that map $\Lambda$ to $F(\Lambda)$. By exactness, both functors preserve kernels, so using copresentations (Definition A.8) we get the desired isomorphism $F \simeq \pi_* \gamma^*$. Recalling that $G_2 = G$, we get the desired statement.

**Step 2, $X$ general.** The unique map $\iota: X \to \text{Spec}(k)$ is equivariant, so induces a pullback $\iota^*: \text{Qcoh}_{G_1}(\text{Spec}(k)) \to \text{Qcoh}_{G_1}(X)$. The composition $H := F \circ \iota^*$ is a cocontinuous tensor functor $\text{Qcoh}_{G_1}(\text{Spec}(k)) \to \text{Qcoh}_X$. Step 1 yields the corresponding $G_2$-bundle

$$\pi: P = \text{Spec}_Y(A) \to Y,$$

where $A = H(\Lambda)$.

From $H = F \circ \iota^*$, we get $A = F(\Lambda^X)$, so the functor $F$ restricts to a cocontinuous tensor functor $\overline{F}: \text{Mod}_{\overline{\Lambda}^X} \to \text{Mod}_A$. The pushforward $\pi_*$ gives an equivalence of $\text{Qcoh}_P$ with $\text{Mod}_A$. Lemma A.12 gives an equivalence of $\text{Mod}_{\overline{\Lambda}^X}$ with $\text{Qcoh}_X$. Combining those, we see that the functor $\overline{F}$ induces a cocontinuous tensor functor $\text{Qcoh}_X \to \text{Qcoh}_P$:

$$\text{Qcoh}_X \xrightarrow{\sim} \text{Mod}_{\overline{\Lambda}^X} \xrightarrow{\overline{F}} \text{Mod}_{\overline{F}(\Lambda^X)} \xrightarrow{\sim} \text{Qcoh}_P$$

(A.13)

For further reference, we note that the map $\text{Qcoh}_G(X) \to \text{Qcoh}_X$ is just forgetting about the $G$-linearization and the map $\text{Qcoh}_Y \to \text{Qcoh}_P$ is the pullback via $\pi$.

By the Tannaka duality for schemes [BC14], taking into account that $X$ is qcs, such a functor $\text{Qcoh}_X \to \text{Qcoh}_P$ is isomorphic to a pullback via a uniquely determined map $\gamma: P \to X$. Moreover, the functor $\overline{F}$ is equivariant on $\Lambda^X$ in the sense that it induces a commutative coaction
diagram where horizontal maps are $G_2$-coactions
\[
\begin{array}{c}
\Lambda^X \\
\downarrow F
\end{array}
\begin{array}{c}
H^0(G_2, \mathcal{O}_{G_2} \otimes_k \Lambda^X) \\
\downarrow F
\end{array}
\begin{array}{c}
F(\Lambda^X) \\
\downarrow \overline{F}
\end{array}
\begin{array}{c}
F(H^0(G_2, \mathcal{O}_{G_2} \otimes_k \Lambda^X)) = H^0(G_2, \mathcal{O}_{G_2} \otimes_k F(\Lambda^X)).
\end{array}
\]

Therefore also the diagram
\[
\begin{array}{c}
\text{Qcoh}_X \\
\downarrow \overline{F}
\end{array}
\begin{array}{c}
\text{Mod}_{\Lambda^X} \\
\downarrow \overline{F}
\end{array}
\begin{array}{c}
\text{Mod}_{F(\Lambda^X)} \\
\downarrow \overline{F}
\end{array}
\begin{array}{c}
\text{Qcoh}_{G_2 \times X} \\
\text{Qcoh}_{G_2 \times P}
\end{array}
\]

is commutative. By Tannaka duality for schemes again, this induces a commutative diagram below with vertical maps being the usual actions.
\[
\begin{array}{c}
X \\
\gamma \\
\overline{\gamma}
\end{array}
\begin{array}{c}
G_2 \times X' \\
\text{id} \times \gamma \\
\overline{\gamma}
\end{array}
\begin{array}{c}
P \\
G_2 \times P'
\end{array}
\]

This exactly means that $\gamma$ is equivariant. It remains to prove that $F$ is isomorphic to $\pi_1, \gamma$. By construction, $\gamma$ is isomorphic to the composition of the forgetful map $j: \text{Qcoh}_G(X) \to \text{Qcoh}_X$ and $\overline{F}: \text{Qcoh}_X \to \text{Qcoh}_P$. Therefore, $\pi_1 \circ F = \overline{F} \circ j = \gamma$. By adjunction $F \simeq \pi_2 \gamma$. This completes the proof.

\textbf{Proof of Theorem A.2.} Let $i: \text{Qcoh}_G(Y) \to \text{Qcoh}_Y$ be the forgetful functor. By Proposition A.1 the functor $i \circ F$ corresponds to a bundle $p: P \to Y$ together with an equivariant map to $X$. By the same construction, the functor $i \circ F \circ p^X_*$ corresponds to $p: P \to Y$ (without the map to $X$). Using this proposition for the third time, we have that $i \circ p^Y_*$ corresponds to the trivial bundle $G \times Y$. The isomorphism $F \circ p^X_* \simeq p^Y_*$ implies that $P$ is trivial, hence has a section and thus induces an equivariant morphism $f: Y \to X$. \hfill $\square$

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