GEPNER TYPE STABILITY CONDITIONS ON GRADED MATRIX FACTORIZATIONS

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ABSTRACT. We introduce the notion of Gepner type Bridgeland stability conditions on triangulated categories, which depends on a choice of an autoequivalence and a complex number. We conjecture the existence of Gepner type stability conditions on the triangulated categories of graded matrix factorizations of weighted homogeneous polynomials. Such a stability condition may give a natural stability condition for Landau-Ginzburg B-branes, and correspond to the Gepner point of the stringy Kähler moduli space of a quintic 3-fold. The main result is to show our conjecture when the variety defined by the weighted homogeneous polynomial is a complete intersection of hyperplanes in a Calabi-Yau manifold with dimension less than or equal to two.

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1. INTRODUCTION

1.1. Motivation. The Donaldson-Thomas (DT) invariants enumerate semistable coherent sheaves on Calabi-Yau 3-folds, which have drawn much attention recently [Tho00]. We are interested in the following two problems in DT theory:

- Find constraints among DT invariants induced by autoequivalences of the derived category of coherent sheaves, e.g. Seidel-Thomas twists [ST01].
- Construct DT type invariants counting B-branes on Landau-Ginzburg (LG) models associated to a superpotential.

As for the former problem, there are several predictions in string theory on generating series of DT invariants, e.g. S-duality conjecture, Ooguri-Strominger-Vafa conjecture [DM], [OSV04]. There seem to be
mysterious constraints among DT invariants behind such predictions, and we hope to reveal their origins via symmetries in the derived category. We believe that a key step toward this problem is to construct a Bridgeland stability condition on the derived category [Bri07] satisfying a certain symmetric property with respect to the given autoequivalence. Indeed a construction of a (weak) stability condition which is preserved under the derived dual, together with wall-crossing arguments [JS], [KS], play crucial roles in the proof of the rationality of the generating series of rank one DT type invariants counting curves [Bri11], [Tod10b], [Tod12].

As for the latter problem, in order to define the DT type invariants, we need to fix a stability condition for B-branes on LG models. A desired stability condition should be natural in some sense, so that it is an analogue of Gieseker stability on coherent sheaves. In a mathematical term, if the superpotential is given by a homogeneous polynomial $W$, the relevant B-brane category is Orlov’s triangulated category of graded matrix factorizations $\text{HMF}^\text{gr}(W)$ [Orl09]. For instance, suppose that $W$ is the defining polynomial of a quintic Calabi-Yau 3-fold $X \subset \mathbb{P}^4$. Then a desired Bridgeland stability condition on $\text{HMF}^\text{gr}(W)$ may correspond to the Gepner point (cf. Figure 1) of the stringy Kähler moduli space of $X$, via mirror symmetry and Orlov equivalence [Orl09] (1)

$$D^b \text{Coh}(X) \sim \text{HMF}^\text{gr}(W).$$

The Gepner point is an orbifold point in the stringy Kähler moduli space of $X$, with the stabilizer group $\mathbb{Z}/5\mathbb{Z}$. Such an orbifold data may be translated into a certain symmetric property of the corresponding Bridgeland stability condition, which we focus and pursue in this paper.

Now we have observed a common keyword regarding the above two problems, that is a Bridgeland stability condition with a symmetric property. The motivating problem of this paper, which is rather ambitious and not able to do at this moment, is to find a natural stability condition on $\text{HMF}^\text{gr}(W)$, and apply its symmetric property to obtain non-trivial constraints among DT invariants on $X$, via Orlov equivalence (1) and wall-crossing arguments [JS], [KS]. (cf. Subsection 1.5 (iii).)

![Figure 1. Stringy Kähler moduli space of a quintic 3-fold](image-url)
1.2. **Gepner type stability conditions.** As we discussed, we are interested in constructing a natural stability condition on $HMF^{gr}(W)$. So far there are few works on this problem, except [KST07], [Tak], [Wal], which will be discussed later. A serious issue is that, since the category $HMF^{gr}(W)$ is not a priori constructed as a derived category of some abelian category, it is not clear what is the meaning of ‘natural’. Our viewpoint is as follows: rather than constructing a stability condition in terms of graded matrix factorizations, we just extract and formulate the symmetric property of a desired stability condition, and try to find a one satisfying such a property. We formulate it as a *Gepner type* property, which depends on a choice of a pair of an autoequivalence and a complex number, given simply as follows:

**Definition 1.1.** A stability condition $\sigma$ on a triangulated category $\mathcal{D}$ is called Gepner type with respect to $(\Phi, \lambda) \in \text{Aut}(\mathcal{D}) \times \mathbb{C}$ if the following condition holds:

$$\Phi_* \sigma = \sigma \cdot (\lambda).$$  

Obviously any stability condition is Gepner type with respect to $([k], k)$ for $k \in \mathbb{Z}$, where $[k]$ is the $k$-times composition of the shift functor $[1]$. On the other hand, there are several interesting examples in which the relation (2) holds with respect to non-trivial pairs $(\Phi, \lambda)$. As the name indicates, if $X \subset \mathbb{P}^4$ is a quintic 3-fold, a Gepner type stability condition on $D^b\text{Coh}(X)$ with respect to the pair

$$((\Phi, \lambda) = \left(ST_{\mathcal{O}_X} \circ \mathcal{O}_X(1), \frac{2}{5}\right)$$

conjecturally corresponds to the Gepner point, where $ST_{\mathcal{O}_X}$ is the Seidel-Thomas twist [ST01] associated to $\mathcal{O}_X$. The fraction $2/5$ appears since the five times composition of the autoequivalence $\Phi$ becomes the twice shift functor. This fact is best understood in terms of $HMF^{gr}(W)$ via the equivalence [ ], as the autoequivalence $\Phi$ corresponds to the grade shift functor on the matrix factorization side.

At this moment, it seems to be a difficult problem to construct a stability condition on a quintic 3-fold corresponding to the Gepner point: we are not even able to construct a Bridgeland stability condition on a quintic 3-fold near the large volume limit point. (cf. [BMT].) However there is a plenty of examples of weighted homogeneous polynomials, which are more amenable than quintic polynomials, but enough interesting to study. The goal of this paper is to construct Gepner type stability conditions on graded matrix factorizations in some of such interesting cases. In the quintic case, an attempt to construct a Gepner point leads to a conjectural stronger version of Bogomolov-Gieseker inequality among Chern characters of stable sheaves on quintic 3-folds. The detail in this case will be discussed in a subsequent paper [Tod].
1.3. Gepner type stability conditions on graded matrix factorizations. Let $W$ be a homogeneous element with degree $d$ in the weighted polynomial ring $W \in A := \mathbb{C}[x_1, x_2, \cdots, x_n], \deg x_i = a_i \in \mathbb{Z}$ such that $(W = 0) \subset \mathbb{C}^n$ has only an isolated singularity at the origin. The triangulated category of graded matrix factorizations of $W$, denoted by $\text{HMF}^{gr}(W)$, is defined to be the homotopy category of the dg category whose objects consist of data

$$P^0 \xrightarrow{p^0} P^1 \xrightarrow{p^1} P^0(d)$$

where $P^i$ are graded free $A$-modules of finite rank, $p^i$ are homomorphisms of graded $A$-modules, $P^i \mapsto P^i(1)$ is the shift of the grading, satisfying the following condition:

$$p^1 \circ p^0 = -W, \quad p^0(d) \circ p^1 = -W.$$

In [Orl09], Orlov proved that the triangulated category $\text{HMF}^{gr}(W)$ is related to the derived category of coherent sheaves on the stack

$$(4) \quad X := (W = 0) \subset \mathbb{P}(a_1, \cdots, a_n)$$

depending on the sign of the Gorenstein index

$$\varepsilon := \sum_{i=1}^{n} a_i - d.$$

Let $\tau$ be the autoequivalence of $\text{HMF}^{gr}(W)$ induced by the grade shift functor $P^\bullet \mapsto P^\bullet(1)$. We propose the following conjecture on the existence of a Gepner type stability condition on $\text{HMF}^{gr}(W)$:

**Conjecture 1.2.** There is a Gepner type stability condition

$$\sigma_G = (Z_G, \{P_G(\phi)\}_{\phi \in \mathbb{R}}) \in \text{Stab}(\text{HMF}^{gr}(W))$$

with respect to $(\tau, 2/d)$, whose central charge $Z_G$ is given by

$$Z_G(P^\bullet) = \text{str}(e^{2\pi \sqrt{-1}/d} : P^\bullet \rightarrow P^\bullet).$$

Here the $e^{2\pi \sqrt{-1}/d}$-action on $P^\bullet$ is induced by the $\mathbb{Z}$-grading on each $P^i$, and ‘str’ is the supertrace which respects the $\mathbb{Z}/2\mathbb{Z}$-grading on $P^\bullet$.

Similarly to the quintic case, the fraction $2/d$ appears since the $d$-times composition of $\tau$ coincides with $[2]$. We propose that, if there is a desired stability condition $\sigma_G$ in Conjecture 1.2, then it may be employed as a ‘natural’ stability condition for graded matrix factorizations. There are at least three reasons for this. Firstly the Gepner type property with respect to $(\tau, 2/d)$ resembles the following property of the classical Gieseker stability on coherent sheaves: for a polarized variety $(X, H)$, a coherent sheaf $E$ on $X$ is $H$-Gieseker semistable if and only if $E \otimes \mathcal{O}_X(H)$ is $H$-Gieseker semistable. In this sense, the desired stability condition $\sigma_G$ seems to be a natural analogue of the
Gieseker stability on Coh(X) for graded matrix factorizations. Secondly, the Gepner type property with respect to (τ, 2/d) turns out to be a very strong constraint for the stability conditions. Indeed, such a property characterizes the central charge $Z_G$ uniquely up to a scalar multiplication. (cf. Subsection 2.8) Also in some cases, we see that $\sigma_G$ is also unique up to shift (cf. Subsection 2.9) and one may expect that this holds in general. Thirdly, if there is such $\sigma_G$, then the $\sigma_G$-semistable objects have a nice compatibility with the Serre functor on $\text{HMF}^{gr}(W)$. As a result, the moduli space of $\sigma_G$-semistable objects should have good local properties, e.g. smoothness, with a perfect obstruction theory, etc, depending on the given data $n, d, \varepsilon$. (cf. Subsection 1.5 (ii).) Therefore, whatever the way $\sigma_G$ is constructed, it seems worth studying moduli spaces of $\sigma_G$-semistable objects, and the enumerative invariants defined by them.

### 1.4. Result.

Before stating our result, we mention the previous beautiful works by Takahashi [Tak] and Kajiura-Saito-Takahashi [KST07]. They study the triangulated category $\text{HMF}^{gr}(W)$ in the following cases: $n = a_1 = 1$ [Tak] and $n = 3, \varepsilon > 0$ [KST07]. In these cases, they show that $\text{HMF}^{gr}(W)$ is equivalent to the derived category of representations of a quiver of ADE type. As a result, $\text{HMF}^{gr}(W)$ has only a finite number of indecomposable objects up to shift, which are completely classified. Using such a classification, they construct a stability condition on $\text{HMF}^{gr}(W)$ by the assignment of a phase for each indecomposable object. Their construction satisfies our Gepner type property, so Conjecture 1.2 is proved in these cases. (cf. Subsection 2.7.)

From our motivation, we are rather interested in a case that there is an infinite number of indecomposable objects up to shift, which are hard to classify, and form non-trivial moduli spaces. For instance if $\varepsilon \leq 0$, then the stack (4) is either Calabi-Yau or general type, and it seems hopeless to construct $\sigma_G$ via classification of indecomposable objects. Our main result, formulated as follows, contains such cases: (cf. Propositions 5.2, 5.4, 5.10, 5.18, 5.26.)

**Theorem 1.3.** Conjecture 1.2 is true if $n - 4 \leq \varepsilon \leq 0$ and the stack $X$ defined by (4) does not contain stacky points.

The assumption that $X$ does not contain stacky points means that $X$ is indeed a smooth projective variety. The inequality $n - 4 \leq \varepsilon \leq 0$ implies that $X$ is contained in a Calabi-Yau manifold of dimension less than or equal to two as a codimension $-\varepsilon$ complete intersection of hyperplanes. These conditions are restrictive, and the possible types $(a_1, \ldots, a_n, d, W, X)$ are completely classified. The classification with $n - \varepsilon = 3, 4$ and $W$ of Fermat type is given in Table 1 below.

Our strategy proving Theorem 1.3 is as follows: by Orlov’s theorem [Orl09], the condition $\varepsilon \leq 0$ allows us to describe $\text{HMF}^{gr}(W)$ as a
Table 1. Possible types in Theorem 1.3

| \( n \) | \( \varepsilon \) | \((a_1, \ldots, a_n)\) | \( d \) | \( W \) | \( X \) |
|--------|--------|----------------|-----|-------|--------|
| 4      | 0      | \((1, 1, 1, 1)\) | 4   | \(x_4^4 + x_2^4 + x_3^4 + x_4^4\) | K3 surface |
| 4      | 0      | \((3, 1, 1, 1)\) | 6   | \(x_4^4 + x_2^6 + x_3^6 + x_4^6\) | K3 surface |
| 3      | −1     | \((1, 1, 1)\)   | 4   | \(x_1^4 + x_2^4 + x_3^4\)      | genus 3 curve |
| 3      | −1     | \((3, 1, 1)\)   | 6   | \(x_1^2 + x_2^5 + x_3^5\)      | genus 2 curve |
| 2      | −2     | \((1, 1)\)      | 4   | \(x_1^4 + x_2^4\)               | 4 points    |
| 2      | −2     | \((3, 1)\)      | 6   | \(x_1^2 + x_2^6\)               | 2 points    |
| 3      | 0      | \((1, 1, 1)\)   | 3   | \(x_1^3 + x_2^3 + x_3^3\)      | elliptic curve |
| 3      | 0      | \((2, 1, 1)\)   | 4   | \(x_1^2 + x_2^4 + x_3^4\)      | elliptic curve |
| 3      | 0      | \((3, 2, 1)\)   | 6   | \(x_1^4 + x_2^6 + x_3^6\)      | elliptic curve |
| 2      | −1     | \((1, 1)\)      | 3   | \(x_1^3 + x_2^3\)               | 3 points    |
| 2      | −1     | \((2, 1)\)      | 4   | \(x_1^2 + x_2^4\)               | 2 points    |
| 2      | −1     | \((3, 2)\)      | 6   | \(x_1^2 + x_2^6\)               | 1 point     |


semiorthogonal decomposition

\[
\text{HMF}^{gr}(W) = \langle \mathbb{C}(-1 - \varepsilon), \ldots, \mathbb{C}(0), \Psi D^b \text{Coh}(X) \rangle
\]

where \(\Psi\) is a fully faithful functor from \(D^b \text{Coh}(X)\) to \(\text{HMF}^{gr}(W)\), and \(\mathbb{C}(i)\) is a certain exceptional object. We show that there is the heart of a bounded t-structure on \(\text{HMF}^{gr}(W)\), given by the extension closure

\[
\mathcal{A}_W := \langle \mathbb{C}(-1 - \varepsilon), \ldots, \mathbb{C}(0), \Psi \text{Coh}(X) \rangle_{\text{ex}}.
\]

We describe the central charge \(Z_G\) in terms of the generators of the heart \(\mathcal{A}_W\), and take a suitable tilting \(\mathcal{A}_G\) of \(\mathcal{A}_W\) using the description of \(Z_G\). We then show that the pair \((Z_G, \mathcal{A}_G)\) determines a stability condition \(\sigma_G\) on \(\text{HMF}^{gr}(W)\). It remains to show that \(\sigma_G\) has a desired Gepner type property, and we reduce it to showing \(\sigma_G\)-stability of certain objects in \(\text{HMF}^{gr}(W)\).

We show that the above construction works in the situation of Theorem 1.3. However unfortunately, several case by case arguments are involved in the proof, and that prevents us to construct \(\sigma_G\) beyond the cases in Theorem 1.3. For instance, they contain the proofs of the inequalities of numerical classes of certain objects in \(\mathcal{A}_W\), which are required in proving the axiom of Bridgeland stability for \((Z_G, \mathcal{A}_G)\). They also contain checking the \(\sigma_G\)-stability of some objects in \(\text{HMF}^{gr}(W)\), which we use to prove the Gepner type property of \(\sigma_G\). The above arguments are in particular hard if the image of \(Z_G\) is not discrete. In the situation of Theorem 1.3, fortunately, the image of \(Z_G\) is always discrete and that makes our situation technically rather amenable. For instance the image of \(Z_G\) is not discrete if \(d = 5\), and the case of \((a_1, a_2, d) = (1, 1, 5)\) is not included in Table 1. Such a case will be discussed in a subsequent paper [Tod], since it involves a subtle argument due to the above non-discrete issue.
Among the list in Table 1, the case of \((n, \varepsilon) = (3, -1)\) seems to be the most interesting case, since we observe a new phenomenon relating graded matrix factorizations and coherent systems on the smooth projective curve \(X\). Recall that a coherent system on \(X\) consists of data

\[ V \otimes \mathcal{O}_X \rightarrow F \]

where \(F\) is a coherent sheaf on \(X\) and \(V\) is a finite dimensional \(\mathbb{C}\)-vector space. We show that the heart \(\mathcal{A}_W\) is equivalent to the abelian category of coherent systems on \(X\). In the construction of \(\sigma_G\), we see that a Clifford type bound on the dimension of \(V\) is involved. Such a Clifford type bound for semistable coherent systems is established by Lange-Newstead \([LN08]\), and we apply their work. If \(X\) has a higher dimension, the conjectural construction of \(\sigma_G\) would predict a higher dimensional analogue of Clifford type bound for semistable coherent systems. In the case of a quintic surface, the detail will be discussed in \([Tod]\).

1.5. **Future directions of the research.** We believe that the work of this paper leads to several interesting directions of the future research. We discuss some of them.

(i) **Descriptions of \(\sigma_G\)-semistable objects in terms of graded matrix factorizations:** As we discussed in the previous subsection, our construction of \(\sigma_G\) relies on Orlov’s theorem, and it is not intrinsic in terms of graded matrix factorizations. It would be an interesting problem to see what kinds of graded matrix factorizations appear as \(\sigma_G\)-semistable objects, and compare them with the \(R\)-stability discussed in string theory \([Wal]\). It may involve deeper understanding of Orlov equivalence, and we are not even able to give a mathematically rigorous candidate of the description of \(\sigma_G\) purely in terms of graded matrix factorizations.

(ii) **Constructing moduli spaces of \(\sigma_G\)-semistable graded matrix factorizations:** It would be an important problem to construct moduli spaces of \(\sigma_G\)-semistable graded matrix factorizations, and study their properties. We expect that, using the argument of \([Tod08]\), there exist Artin stacks of finite type

\[ \mathcal{M}_G^\sigma(\gamma) \subset \mathcal{M}_G^\sigma(\gamma), \quad \gamma \in \text{HH}_0(W) \]

which parameterize \(\sigma_G\)-(semi)stable graded matrix factorizations \(P^*\) with \(\text{ch}(P^*) = \gamma\). Here \(\text{HH}_0(W)\) is the zero-th Hochschild homology group of \(\text{HMF}^\text{gr}(W)\), studied in \([Dyc11]\), \([PV]\), \([PV12]\). On the other hand, we have the following vanishing for \([P^*] \in \mathcal{M}_G^\sigma(\gamma)\):

\[ \text{Hom}^i(P^*, P^*) = 0, \quad i > n - 2 - \frac{2\varepsilon}{d}. \]

The above vanishing, which is proved in Lemma 2.18, is one of the important properties of Gepner type stability conditions. Since the space
Hom$(P^\bullet,P^\bullet)$ is responsible for the local deformation theory of $P^\bullet$, the moduli space $\mathcal{M}^{\text{eff}}_{\mathcal{G}}(\gamma)$ would have good local properties depending on $n,d,\varepsilon$, e.g. it is smooth if $4 > n - 2\varepsilon/d$, has a perfect obstruction theory if $5 > n - 2\varepsilon/d$. Such properties would be important in constructing counting invariants of graded matrix factorizations, even in a non-CY3 situation.

(iii) DT type invariants counting $\sigma_G$-semistable graded matrix factorizations: Suppose that $\text{HMF}^{\text{gr}}(W)$ is a CY3 category. If Conjecture 1.2 is true, then (ii) would imply the existence of the invariants

$$\text{DT}_G(\gamma) \in \mathbb{Q}, \quad \gamma \in \text{HH}_0(W)$$

which count $\sigma_G$-semistable matrix factorizations $P^\bullet$ with $\text{ch}(P^\bullet) = \gamma$. The above invariants may give DT analogue of the Fan-Jarvis-Ruan-Witten theory [FJR] in Gromov-Witten theory. Also, the Gepner type property of $\sigma_G$ should yield an important identity

$$\text{DT}_G(\gamma) = \text{DT}_G(\tau_*\gamma).$$

The above identity, combined with Orlov’s equivalence and wall-crossing argument [JS, KS], may imply non-trivial constraints among the original sheaf counting DT invariants on $X$ induced by the autoequivalence $\text{ST}_{O_X} \circ \otimes O_X(1)$. If the above story works, then it realizes an analogue of Calabi-Yau/Landau-Ginzburg correspondence in FJRW theory [CRI0].

1.6. Plan of the paper. In Section 2, we introduce the notion of Gepner type stability conditions and propose a conjecture on the existence of Gepner type stability conditions on the triangulated category $\text{HMF}^{\text{gr}}(W)$. We also discuss some examples of our conjecture, and their uniqueness. In Section 3 we construct the heart $\mathcal{A}_W$ of a bounded $t$-structure on $\text{HMF}^{\text{gr}}(W)$, and describe it in terms of quiver representations or coherent systems. In Section 4 we explain how to compute the central charge in terms of generators of $\mathcal{A}_W$, and propose a general recipe on a construction of a Gepner type stability condition. In Section 5 we prove Theorem 1.3 by applying the strategy in Section 4.

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2. GEPNER TYPE STABILITY CONDITIONS

In this section, we recall the definition of Bridgeland stability conditions on triangulated categories, group actions, and define the notion
of Gepner type stability conditions. We then recall Orlov’s triangulated categories of graded matrix factorizations, and discuss Gepner type stability conditions on them.

2.1. Definitions. Let \( \mathcal{D} \) be a triangulated category and \( K(\mathcal{D}) \) its Grothendieck group. We first recall Bridgeland’s definition of stability conditions on it.

**Definition 2.1.** ([Bri07]) A stability condition \( \sigma \) on \( \mathcal{D} \) consists of a pair \((Z, \{\mathcal{P}(\phi)\}_{\phi \in \mathbb{R}})\)

\[
Z: K(\mathcal{D}) \to \mathbb{C}, \quad \mathcal{P}(\phi) \subset \mathcal{D}
\]

where \( Z \) is a group homomorphism (called central charge) and \( \mathcal{P}(\phi) \) is a full subcategory (called \( \sigma \)-semistable objects with phase \( \phi \)) satisfying the following conditions:

- For \( 0 \neq E \in \mathcal{P}(\phi) \), we have \( Z(E) \in \mathbb{R}_{>0} \exp(\sqrt{-1} \pi \phi) \).
- For all \( \phi \in \mathbb{R} \), we have \( \mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1] \).
- For \( \phi_1 > \phi_2 \) and \( E_i \in \mathcal{P}(\phi_i) \), we have \( \text{Hom}(E_1, E_2) = 0 \).
- For each \( 0 \neq E \in \mathcal{D} \), there is a collection of distinguished triangles

\[
E_{i-1} \to E_i \to F_i \to E_{i-1}[1], \quad E_N = E, \ E_0 = 0
\]

with \( F_i \in \mathcal{P}(\phi_i) \) and \( \phi_1 > \phi_2 > \cdots > \phi_N \).

The full subcategory \( \mathcal{P}(\phi) \subset \mathcal{D} \) is shown to be an abelian category, and its simple objects are called \( \sigma \)-stable. In [Bri07], Bridgeland shows that there is a natural topology on the set of ‘good’ stability conditions \( \text{Stab}(\mathcal{D}) \), and its each connected component has structure of a complex manifold.

**Remark 2.2.** The above ‘good’ conditions are called ‘numerical property’ and ‘support property’ in literatures. Although the above properties are important in considering the space \( \text{Stab}(\mathcal{D}) \), we omit the detail since we focus on the construction of one specific stability condition.

Let \( \text{Aut}(\mathcal{D}) \) be the group of autoequivalences on \( \mathcal{D} \). There is a left \( \text{Aut}(\mathcal{D}) \)-action on the set of stability conditions on \( \mathcal{D} \). For \( \Phi \in \text{Aut}(\mathcal{D}) \), it acts on the pair \((5)\) as follows:

\[
\Phi_*(Z, \{\mathcal{P}(\phi)\}_{\phi \in \mathbb{R}}) = (Z \circ \Phi^{-1}, \{\Phi(\mathcal{P}(\phi))\}_{\phi \in \mathbb{R}}).
\]

There is also a right \( \mathbb{C} \)-action on the set of stability conditions on \( \mathcal{D} \). For \( \lambda \in \mathbb{C} \), its acts on the pair \((5)\) as follows:

\[
(Z, \{\mathcal{P}(\phi)\}_{\phi \in \mathbb{R}}) \cdot (\lambda) = (e^{-\sqrt{-1} \pi \lambda} Z, \{\mathcal{P}(\phi + \text{Re} \lambda)\}_{\phi \in \mathbb{R}}).
\]

The notion of Gepner type stability conditions is defined in terms of the above group actions.
Definition 2.3. A stability condition $\sigma$ on a triangulated category $\mathcal{D}$ is called Gepner type with respect to $(\Phi, \lambda) \in \text{Aut}(\mathcal{D}) \times \mathbb{C}$ if the following condition holds:

$$(6) \quad \Phi_\ast \sigma = \sigma \cdot (\lambda).$$

Here we give some trivial examples:

Example 2.4. (i) For $k \in \mathbb{Z}$, any stability condition $\sigma$ on a triangulated category $\mathcal{D}$ is Gepner type with respect to $([k], k)$.

(ii) Let $A$ be an abelian category and $Z$ a group homomorphism $Z : K(A) \to \mathbb{C}$ such that $Z(A \setminus \{0\})$ is contained in $\mathbb{R}_{>0} e^{\sqrt{-1} \pi \theta}$ for some $\theta \in \mathbb{R}$. We set $P(\phi)$ for $\phi \in [\theta, \theta + 1)$ to be

$$P(\theta) = A, \quad P(\phi) = \{0\} \text{ if } \phi \in (\theta, \theta + 1).$$

Other $P(\phi)$ are determined by the rule $P(\phi + 1) = P(\phi)[1]$. Let $\Phi$ be an autoequivalence of $D^b(A)$ which preserves $A$. Then the pair $(Z, \{P(\phi)\}_{\phi \in \mathbb{R}})$ is a Gepner type stability condition on $D^b(A)$ with respect to $(\Phi, 0)$.

Remark 2.5. A Gepner type stability conditions as in Example (ii) appears at a point in the space of stability conditions on $X = \omega_{\mathbb{P}^2}$ studied by [BM11]. Such a point corresponds to the orbifold point in the stringy Kähler moduli space of $X$, and $A$ is the abelian category of representations of a McKay quiver. The autoequivalence $\Phi$ is given by

$$\Phi = ST_{\mathbb{P}^2} \circ \otimes \mathcal{O}_X(1)$$

which induces the automorphism of the McKay quiver.

As we will see, there will be more interesting examples of Gepner type stability conditions on triangulated categories of graded matrix factorization.

2.2. Triangulated categories of graded matrix factorizations. Here we recall Orlov’s construction of triangulated categories of graded matrix factorizations [Orl09]. Let $A$ be a weighted graded polynomial ring

$$(7) \quad A := \mathbb{C}[x_1, x_2, \cdots, x_n], \quad \deg x_i = a_i$$

and $W \in A$ a homogeneous element of degree $d$. Throughout of this paper, we always assume that $a_1 \geq a_2 \geq \cdots \geq a_n$, and $(W = 0) \subset \mathbb{C}^n$ has an isolated singularity at the origin. For a graded $A$-module $P$, we denote by $P_i$ its degree $i$-part, and $P(k)$ the graded $A$-module whose grade is shifted by $k$, i.e. $P(k)_i = P_{i+k}$. 

Definition 2.6. A graded matrix factorization of $W$ is data
\begin{equation}
\begin{aligned}
P^0 \xrightarrow{p^0} P^1 \xrightarrow{p^1} P^0(d)
\end{aligned}
\end{equation}
where $P^i$ are graded free $A$-modules of finite rank, $p^i$ are homomorphisms of graded $A$-modules, satisfying the following conditions:
\begin{equation}
\begin{aligned}
p^1 \circ p^0 &= \cdot W, & p^0(d) \circ p^1 &= \cdot W.
\end{aligned}
\end{equation}

The category $\text{HMF}_{\text{gr}}(W)$ is defined to be the homotopy category of graded matrix factorizations of $W$. Its objects consist of data (8), and the set of morphisms are given by the commutative diagrams of graded $A$-modules

\begin{equation}
\begin{array}{c}
P^0 \xrightarrow{p^0} P^1 \xrightarrow{p^1} P^0(d) \\
\downarrow{p^0} \downarrow{p^1} \downarrow{p^0(d)} \\
Q^0 \xrightarrow{q^0} Q^1 \xrightarrow{q^1} Q^0(d).
\end{array}
\end{equation}

modulo null-homotopic morphisms: the above diagram is null-homotopic if there are homomorphisms of graded $A$-modules
\begin{equation}
h^0: P^0 \to Q^1(-d), \quad h^1: P^1 \to Q^0
\end{equation}
satisfying
\begin{equation}
\begin{aligned}
f^0 &= q^1(-d) \circ h^0 + h^1 \circ p^0, & f^1 &= q^0 \circ h^1 + h^0(d) \circ p^1.
\end{aligned}
\end{equation}

The category $\text{HMF}_{\text{gr}}(W)$ has a structure of a triangulated category. The shift functor $[1]$ sends data (8) to
\begin{equation}
P^1 \xrightarrow{-p^1} P^0(d) \xrightarrow{-p^0(d)} P^1(d)
\end{equation}
and the distinguished triangles are defined via the usual mapping cone constructions. The grade shift functor $P^\bullet \mapsto P^\bullet(1)$ induces the autoequivalence $\tau$ of $\text{HMF}_{\text{gr}}(W)$, which satisfies the following identity:
\begin{equation}
\tau \times d = [2].
\end{equation}

There is also a Serre functor $S_W$ on $\text{HMF}_{\text{gr}}(W)$, described by $\tau$, $n$ and the Gorenstein index $\varepsilon$. The number $\varepsilon$ is defined as
\begin{equation}
\varepsilon := \sum_{i=1}^{n} a_i - d \in \mathbb{Z}.
\end{equation}

The Serre functor $S_W$ on $\text{HMF}_{\text{gr}}(W)$ is given by (for instance see [KST09, Theorem 3.8])
\begin{equation}
S_W = \tau^{-\varepsilon}[n-2].
\end{equation}

The above description of the Serre functor will be used later in this paper.
2.3. Relation to the triangulated categories of singularities.

The triangulated category $\text{HMF}^{\text{gr}}(W)$ is known to be equivalent to the derived category of singularities of the hypersurface singularity $(W = 0) \subset \mathbb{C}^n$. This equivalence is often useful in doing some computations of matrix factorizations.

Let $R$ be the graded ring $R = A/(W)$ and $\text{gr-R}$ the abelian category of finitely generated graded $R$-modules. We denote by $D^b(\text{grproj-R})$ the subcategory of $D^b(\text{gr-R})$ consisting of perfect complexes of $R$-modules. The triangulated category of singularities is defined to be the quotient category

$$D^\text{gr}_{\text{sg}}(R) := D^b(\text{gr-R})/D^b(\text{grproj-R}).$$

The following result is proved in [Orl09]:

**Theorem 2.7.** ([Orl09, Theorem 3.10]) There is an equivalence of triangulated categories

$$\text{Cok}: \text{HMF}^{\text{gr}}(W) \cong D^\text{gr}_{\text{sg}}(R)$$

sending a matrix factorization (8) to the cokernel of $p^0$.

The cokernel of $p^0$ is easily checked to be annihilated by $W$, so it is $R$-module. Obviously the equivalence (11) commutes with grade shift functors on both sides, so we use the same notation $\tau$ for the grade shift functor on $D^\text{gr}_{\text{sg}}(R)$.

Let

$$m = (x_1, \cdots, x_n) \subset R$$

be the maximal ideal and set the graded $R$-module $\mathbb{C}(j)$ to be $(R/m)(j)$. The graded $R$-module $\mathbb{C}(j)$ determines an object in $D^\text{gr}_{\text{sg}}(R)$. The matrix factorization given by

$$\text{Cok}^{-1}(\mathbb{C}(j)) \in \text{HMF}^{\text{gr}}(W)$$

plays an important role. By [Dyc11] Corollary 2.7, it is given by the matrix factorization of the form

$$\bigoplus_{k \geq 0} \bigwedge^{2k+1} m/m^2 \otimes A(dk + j) \xrightarrow{p^0} \bigoplus_{k \geq 0} \bigwedge^{2k} m/m^2 \otimes A(dk + j)$$

$$\xrightarrow{p^1} \bigoplus_{k \geq 0} \bigwedge^{2k+1} m/m^2 \otimes A(dk + k + j).$$

We omit the descriptions of the above morphisms $p^0$, $p^1$, as we will not use them. By an abuse of notation, we abbreviate $\text{Cok}^{-1}$ and denote by $\mathbb{C}(j)$ the matrix factorization given by (13).
2.4. Conjecture on the existence of Gepner type stability condition. We are interested in constructing a natural stability condition on $\text{HMF}^{gr}(W)$. We require that such a stability condition is preserved under the grade shift functor $\tau$ in some sense. This might be an analogues property for the classical $H$-Gieseker stability on a polarized variety $(X, H)$: $H$-Gieseker semistable sheaves are preserved under $\otimes O_X(H)$. Because of the identity (9), the expected stability condition is not exactly preserved by $\tau$ but the phases of semistable objects should be shifted by $2/d$. This is nothing but the Gepner type property with respect to $(\tau, 2/d)$.

There is a natural construction of a central charge $Z_G$ which might give a desired stability condition, and already appeared in some articles [KST07], [Tak], [Wal]. For a graded matrix factorization (8), its image under $Z_G$ is symbolically defined by

$$\text{str}(e^{2\pi \sqrt{-1}/d}; P^* \to P^*).$$

Here $e^{2\pi \sqrt{-1}/d}$-action is given by the $\mathbb{C}^*$-action on $P^*$ induced by the $\mathbb{Z}$-grading on each $P^i$, and the ‘str’ means the super trace of $e^{2\pi \sqrt{-1}/d}$-action which respects the $\mathbb{Z}/2\mathbb{Z}$-grading of $P^* = P^0 \oplus P^1$. More precisely, since $P^i$ are free $A$-modules of finite rank, they are written as

$$P^i \cong \bigoplus_{j=1}^{m} A(n_{i,j}), \quad n_{i,j} \in \mathbb{Z}.$$  

Then (14) is written as

$$\sum_{j=1}^{m} \left( e^{2n_{0,j}\pi \sqrt{-1}/d} - e^{2n_{1,j}\pi \sqrt{-1}/d} \right).$$

Example 2.8. Let $\mathbb{C}(0)$ be the matrix factorization given by (13) for $j = 0$. By the definition of $Z_G$, we have

$$Z_G(\mathbb{C}(0)) = \sum_{i=1}^{n} e^{-2a_i \pi \sqrt{-1}/d} + \sum_{i_1<i_2<i_3} e^{-2(a_{i_1}+a_{i_2}+a_{i_3})\pi \sqrt{-1}/d} + \ldots - 1 - \sum_{i_1<i_2} e^{-2(a_{i_1}+a_{i_2})\pi \sqrt{-1}/d} - \sum_{i_1<i_2<i_3<i_4} \ldots$$

$$= - \prod_{j=1}^{n} \left( 1 - e^{-2a_{j}\pi \sqrt{-1}/d} \right) \neq 0.$$

It is easy to check that (13) descends to a group homomorphism $Z_G: K(\text{HMF}^{gr}(W)) \to \mathbb{C}$.

Indeed, $Z_G$ is one of the components of the Chern character map of $\text{HMF}^{gr}(W)$ constructed in [PV12]. (cf. Remark 2.16.) As we stated in the introduction, we propose the following conjecture:
Conjecture 2.9. There is a Gepner type stability condition
\[ \sigma_G = (Z_G, \{ P_G(\phi) \}_{\phi \in \mathbb{R}}) \in \text{Stab}(\text{HMF}^{gr}(W)) \]
with respect to \((\tau, 2/d)\), where \(Z_G\) is given by (14).

Note that the central charge \(Z_G\) satisfies the condition (6) with respect to \((\tau, 2/d)\) by the construction. The problem is to construct full subcategories \(P_G(\phi) \subset \text{HMF}^{gr}(W)\) satisfying the desired property.

Remark 2.10. The image of \(Z_G\) is contained in \(Z[e^{2\pi \sqrt{-1}/d}]\), which may or may not be discrete depending on \(d\). For instance, it is discrete if \(d = 3, 4, 6\), but not so if \(d = 5\).

2.5. The case of \(n = 1\). As a toy example of Conjecture 2.9, let us consider the case of \(n = 1\), i.e.
\[ W = x^d \in \mathbb{C}[x], \quad \text{deg } x = a. \]

The case of \(a = 1\) is worked out in [Tak]. In this case, the triangulated category \(\text{HMF}^{gr}(W)\) is equivalent to the derived category of the path algebra of the Dynkin quiver of type \(A_{d-1}\). As a result, we have a complete classification of indecomposable objects in \(\text{HMF}^{gr}(W)\), given by
\[ Q_{j,l} := \left\{ A(j-l) \xrightarrow{x^l} A(j) \xrightarrow{x^{d-l}} A(j-l+d) \right\} \]
for \(1 \leq l \leq d-1\) and \(j \in \mathbb{Z}\). Note that \(Q_{j,1}\) coincides with \(C(j)\) given by (13). We set \(\phi(Q_{j,l}) \in \mathbb{Q}\) to be
\[ \phi(Q_{j,l}) := -\frac{1}{2} - \frac{l}{d} + \frac{2j}{d}. \]

For \(\phi \in \mathbb{R}\), we define \(P_G(\phi) \subset \text{HMF}^{gr}(W)\) to be the subcategory consisting of direct sums of objects \(Q_{j,l}\) with \(\phi(Q_{j,l}) = \phi\). Then the pair \(\sigma_G = (Z_G, \{ P_G(\phi) \}_{\phi \in \mathbb{R}})\) is shown to be a desired stability condition in Conjecture 2.9 by [Tak]. The case of \(a > 1\) follows from the following lemma:

Lemma 2.11. Let \(A\) be the graded ring (7), \(a \in \mathbb{Z}_{\geq 1}\) the greatest common divisor of \((a_1, \cdots, a_n)\), and set \(a'_i = a_i/a\). Let \(A'\) be the graded ring defined by
\[ A' = \mathbb{C}[x'_1, x'_2, \cdots, x'_n], \quad \text{deg } x'_i = a'_i. \]

For a homogeneous element \(W \in A\) of degree \(d\), we regard it as a homogeneous element \(W' \in A'\) of degree \(d' = d/a\) by the identification \(x_i = x'_i\). Then if Conjecture 2.9 holds for \(W' \in A'\), then it also holds for \(W \in A\).

Proof. There is an obvious fully-faithful functor as triangulated categories
\[ i: \text{HMF}^{gr}(W') \to \text{HMF}^{gr}(W) \]
by multiplying \( a \) to each grading of \( A' \)-modules which appear in the LHS. If \( \tau' \) is the grade shift on \( \text{HMF}^{gr}(W') \) and \( Z'_G \) the central charge (14) for \( \text{HMF}^{gr}(W') \), we have
\[
\tau^a \circ i = i \circ \tau', \quad Z_G \circ i = Z'_G.
\]
Also note that if there is a non-zero morphism of graded \( A \)-modules \( A(m) \to A(n) \), then \( m-n \) is divisible by \( a \). This implies that \( \text{HMF}^{gr}(W) \) has the following orthogonal decomposition:
\[
\langle i \text{HMF}^{gr}(W'), \tau_i \text{HMF}^{gr}(W'), \cdots, \tau^{a-1} i \text{HMF}^{gr}(W') \rangle.
\]
(15)
Also note that if there is a non-zero morphism of graded \( A \)-modules \( A(m) \to A(n) \), then \( m - n \) is divisible by \( a \). This implies that \( \text{HMF}^{gr}(W) \) has the following orthogonal decomposition:
\[
\langle i \text{HMF}^{gr}(W'), \tau_i \text{HMF}^{gr}(W'), \cdots, \tau^{a-1} i \text{HMF}^{gr}(W') \rangle.
\]
(16)
Suppose that \((Z'_G, \{P'_G(\phi)\})_{\phi \in \mathbb{R}}\) is a Gepner type stability condition on \( \text{HMF}^{gr}(W') \) with respect to \((\tau', 2/d')\). We set \( P_G(\phi) \) as follows:
\[
P_G(\phi) = \left\{ \bigoplus_{j=0}^{a-1} \tau^j(Q_j) : Q_j \in i'P'_G \left( \phi - \frac{2j}{d} \right) \right\}.
\]
By (15) and (16), it is easy to check that \((Z_G, \{P_G(\phi)\})_{\phi \in \mathbb{R}}\) is a Gepner type stability condition on \( \text{HMF}^{gr}(W) \) with respect to \((\tau, 2/d)\).
\[\square\]

Combined with the argument for \( n = a_1 = 1 \), we obtain the following corollary:

**Corollary 2.12.** Conjecture 2.9 is true if \( n = 1 \).

2.6. Remarks for the case of \( n = 2 \).

We discuss Conjecture 2.9 in some more cases with small \( n \). By Corollary 2.12, the next interesting case may be \( n = 2 \). In this case, the problem is trivial when \( \varepsilon > 0 \). Indeed it is easy to check that
\[
\text{HMF}^{gr}(W) = \{0\}, \quad n = 2, \quad \varepsilon > 0
\]
by using the equivalence (11). On the other hand, the triangulated category \( \text{HMF}^{gr}(W) \) is non-trivial when \( \varepsilon = 0 \). In this case, by applying the coordinate change if necessary, we may assume that
\[
W = x_1x_2 \in \mathbb{C}[x_1, x_2]
\]
with \( a_1 \) and \( a_2 \) coprime by Lemma 2.11. Similarly to the case of \( n = 1 \), it turns out that there is only a finite number of indecomposable objects up to shift in \( \text{HMF}^{gr}(W) \). They consist of the objects
\[
\mathbb{C}(j)[k], \quad 0 \leq j \leq d - 1, \quad k \in \mathbb{Z}
\]
where \( \mathbb{C}(j) \) is given by (13). Furthermore each indecomposable objects are mutually orthogonal. The above fact can be easily checked, for instance using Orlov’s theorem \cite{Orl09}, as given in Example 3.4 below.

A desired stability condition \( \sigma_G \) in Conjecture 2.9 is constructed as follows: we first choose \( \phi_0 \in \mathbb{R} \) so that \( Z_G(\mathbb{C}(0)) \in \mathbb{R}_{>0}e^{2\pi i \phi_0} \) and set
\[
\phi(\mathbb{C}(j)[k]) = \phi_0 + k + \frac{2j}{a_1 + a_2}.
\]
We define $\mathcal{P}_G(\phi)$ to be the subcategory consisting of direct sums of indecomposable objects with $\phi(*) = \phi$. Then by the above argument, $\sigma = (Z_G, \{\mathcal{P}_G(\phi)\}_{\phi \in \mathbb{R}})$ gives a desired stability condition. As a summary, we have the following:

**Proposition 2.13.** Conjecture 2.9 is true if $n = 2$ and $\varepsilon \geq 0$.

The situation drastically changes when $n = 2$ and $\varepsilon < 0$. For instance, let us consider the case $W = x_1^d + x_2^d \in \mathbb{C}[x_1, x_2]$, $\deg x_i = 1$, $d \geq 3$.

Even in such a simple case, Conjecture 2.9 seems to be not obvious, and it requires a deep understanding of the category $\text{HMF}^{gr}(W)$. The above case is treated in this paper when $d \leq 4$. The $d = 5$ case will be studied in [Tod].

2.7. Remarks for the case of $n = 3$. Suppose that $n = 3$ and $\varepsilon > 0$, the case studied by Kajiura-Saito-Takahashi [KST07]. In this case, $W$ is classified into the following ADE types [Sai87]:

$$W(x_1, x_2, x_3) = \begin{cases} 
  x_1x_2 + x_3^{l+1} & A_l \ (l \geq 1) \\
  x_1^2 + x_2^2 + x_3^{l-1} & D_l \ (l \geq 4) \\
  x_1^2 + x_2^3 + x_3^3 & E_6 \\
  x_1^2 + x_2^3 + x_3^3 + x_3^2 & E_7 \\
  x_1^2 + x_3^3 & E_8.
\end{cases}$$

Furthermore, the triangulated category $\text{HMF}^{gr}(W)$ is equivalent to the derived category of quiver representations of a Dynkin quiver of the corresponding ADE type. As a result, the category $\text{HMF}^{gr}(W)$ is shown to have only a finite number of indecomposable objects up to shift, which are completely classified. Similar to the case of $n = 1$ (which is also interpreted as an $A_l$-case in the above $n = 3$, $\varepsilon > 0$ list by Knörrer periodicity [Kno87]) they assign phases to classified indecomposable objects, and prove the following:

**Theorem 2.14.** ([KST07 Theorem 4.2]) Conjecture 2.9 is true if $n = 3$ and $\varepsilon > 0$.

Conjecture 2.9 in the case of $n = 3$ and $\varepsilon \leq 0$ is not obvious, and a part of this case is treated later in this paper.

2.8. Uniqueness of the central charge. It is a natural question whether the Gepner type property uniquely characterize $\sigma_G$ or not in some sense. As for the central charge, this is true: $Z_G$ is characterized by the Gepner type property with respect to $(\tau, 2/d)$ up to a scalar multiplication. Indeed we are only interested in central charges which factors through the Chern character map

$$\text{ch}: \text{HMF}^{gr}(W) \to \text{HH}_0(W).$$
The RHS is the Hochschild homology group of $\text{HMF}^{gr}(W)$, or more precisely of its dg enhancement. A general theory on Hochschild homology groups and Chern character maps on $\text{HMF}^{gr}(W)$ is available in [Dyc11], [PV], [PV12].

Because $K(\text{HMF}^{gr}(W))$ is not finitely generated in general, it would be more natural to define the set of central charges on $\text{HMF}^{gr}(W)$ as the dual space $\text{HH}^{0}(W)^{\vee}$, rather than the original one in Definition 2.1.

On the other hand, the autoequivalence $\tau$ on $\text{HMF}^{gr}(W)$ defines the linear isomorphism $\tau^{\ast}: \text{HH}^{\ast}(W) \cong \rightarrow \text{HH}^{\ast}(W)$.

The above isomorphism induces the isomorphism $\tau^{\vee \ast}$ on the space of central charges $\text{HH}^{0}(W)^{\vee}$. The Gepner type property requires the central charge, regarded as an element in $\text{HH}^{0}(W)^{\vee}$, to be an eigenvector with respect to $\tau^{\vee \ast}$ with eigenvalue $e^{\pm 2\pi \sqrt{-1}/d}$. The lemma below shows that such an eigenspace is one dimensional.

**Lemma 2.15.** The eigenspace of $\tau_{s}$-action on $\text{HH}_{s}(W)$ with eigenvalue $e^{\pm 2\pi \sqrt{-1}/d}$ is one dimensional, and contained in $\text{HH}_{0}(W)$.

**Proof.** We consider the $G := \mu_{d}$-action on $\mathbb{C}^{n}$, given by

$$e^{2\pi \sqrt{-1}/d} \cdot (x_{1}, \ldots, x_{n}) = (e^{2\pi \sqrt{-1}/d} x_{1}, \ldots, e^{2\pi \sqrt{-1}/d} x_{n}).$$

By [PV, Theorem 2.6.1 (i)], $\text{HH}_{s}(W)$ is given by

$$\text{HH}_{s}(W) \cong \bigoplus_{\gamma \in G} H(\mathbb{C}_{\gamma}^{n}, W_{\gamma})^{G} \quad (17)$$

Here $H(\mathbb{C}^{n}, W)$ is defined by

$$H(\mathbb{C}^{n}, W) := (\mathbb{C}[x_{1}, \ldots, x_{n}]/(\partial_{x_{1}} W, \ldots, \partial_{x_{n}} W)) \, dx_{1} \wedge \cdots \wedge dx_{n}$$

and the space $H(\mathbb{C}_{\gamma}^{n}, W_{\gamma})$ is given by applying the above construction for

$$\mathbb{C}_{\gamma}^{n} := \{ x \in \mathbb{C}^{n} : \gamma(x) = x \}, \quad W_{\gamma} := W|_{\mathbb{C}_{\gamma}^{n}}.$$

Since $\tau^{\times d} = [2]$ on $\text{HMF}^{gr}(W)$, we have $\tau^{\times d} = \text{id}$ on $\text{HH}_{s}(W)$. This implies that $\tau_{s}$ generates the $\mathbb{Z}/d\mathbb{Z}$-action on $\text{HH}_{s}(W)$. By [PV, Theorem 2.6.1 (ii)], the decomposition (17) coincides with the character decomposition of $\text{HH}_{s}(W)$ with respect to the above $\mathbb{Z}/d\mathbb{Z}$-action. Therefore, noting that $\mathbb{C}_{e^{\pm 2\pi \sqrt{-1}/d}}^{n} = \{0\}$, the desired eigenspace is one dimensional by

$$H(\mathbb{C}_{e^{\pm 2\pi \sqrt{-1}/d}}^{n}, W_{e^{\pm 2\pi \sqrt{-1}/d}}) \cong \mathbb{C}.$$

By the grading of $\text{HH}_{s}(W)$ given in [PV, Theorem 2.6.1], the above eigenspace is contained in $\text{HH}_{0}(W)$. \qed
Remark 2.16. For $E \in \text{HMF}^{gr}(W)$, the $\gamma = e^{2\pi \sqrt{-1}/d}$-component of $\text{ch}(E)$ in the decomposition (17) coincides with the central charge $Z_G$ defined by (14). (cf. [PV12, Theorem 3.3.3].) In particular $Z_G$ is given by an element in $\text{HH}_0(W)^{\vee}$, which gives a basis of the eigenspace of $\tau^\vee$-action with eigenvalue $e^{2\pi \sqrt{-1}/d}$.

Remark 2.17. Obviously the set of Gepner type stability conditions with respect to $(\tau, 2/d)$ is preserved under the natural right action of $\mathbb{C}$ on $\text{Stab}(\text{HMF}^{gr}(W))$. By Bridgeland’s deformation result [Bri07, Theorem 7.1], Lemma 2.15 implies that the set of such stability conditions forms a discrete subset in the quotient space $\text{Stab}(\text{HMF}^{gr}(W))/\mathbb{C}$.

2.9. Uniqueness of $\sigma_G$. There are some cases in which not only $Z_G$ but also $\sigma_G$ is characterized by the Gepner type property. At least this is the case when all the indecomposable objects should become semistable. We first note the following lemma, which is an important property of Gepner type stability conditions:

Lemma 2.18. Suppose that $W \in A$ satisfies Conjecture 2.9 and $\sigma_G = (Z_G, \{P_G(\phi)\}_{\phi \in \mathbb{R}})$ is a Gepner type stability condition with respect to $(\tau, 2/d)$. For $\phi_i \in \mathbb{R}$ with $i = 1, 2$ and $k \in \mathbb{Z}$, suppose that the following inequality holds:

$$\phi_1 > \phi_2 + n - k - 2 + \frac{2\varepsilon}{d}. \tag{18}$$

Then for any $F_i \in P_G(\phi_i)$, we have $\text{Hom}^k(F_2, F_1) = 0$.

Proof. By the Serre functor given by (10), we have the isomorphism

$$\text{Hom}(F_2, F_1[k]) \cong \text{Hom}(F_1, \tau^{-\varepsilon}(F_2)[n - k - 2])^{\vee}. \tag{19}$$

By the Gepner type property with respect to $(\tau, 2/d)$, we have

$$\tau^{-\varepsilon}(F_2)[n - k - 2] \in P_G\left(\phi_2 + n - k - 2 - \frac{2\varepsilon}{d}\right).$$

By the inequality (18), the phase of $F_1$ is bigger than that of the above object, hence the RHS of (19) vanishes. \qed

Using the above lemma, we show the following proposition:

Proposition 2.19. In the same situation of Lemma 2.18, suppose that the following inequality holds:

$$n - 3d \leq 2\varepsilon. \tag{20}$$

Then all the other stability conditions satisfying the conditions in Conjecture 2.9 are obtained as $[2m]_{\sigma_G}$ for $m \in \mathbb{Z}$.

Proof. Let us take $F_i \in P_G(\phi_i)$, $i = 1, 2$ with $\phi_1 > \phi_2$. Then Lemma 2.18 and the assumption (20) show that $\text{Hom}^1(F_2, F_1) = 0$. This implies that any object in $\text{HMF}^{gr}(W)$ whose Harder-Narasimhan factors are $F_1$, $F_2$ decomposes into the direct sum of $F_1$ and $F_2$. Repeating this
argument, any object $E \in \text{HMF}^\text{gr}(W)$ decomposes into the direct sum of $\sigma_G$-semistable objects. In particular any non-zero indecomposable object $E \in \text{HMF}^\text{gr}(W)$ is $\sigma_G$-semistable, whose phase is denoted by $\phi_E$.

Let us fix a non-zero indecomposable object $M \in \text{HMF}^\text{gr}(W)$. By the result of [BFK12, Theorem 5.16], the objects $\tau^i(M)$, $0 \leq i \leq d - 1$ generate the triangulated category $\text{HMF}^\text{gr}(W)$. Therefore for any non-zero indecomposable object $E \in \text{HMF}^\text{gr}(W)$, there is $0 \leq i \leq d - 1$ and $j \in \mathbb{Z}$ such that $\text{Hom}(E, \tau^i(M)[j]) \neq 0$. This implies that

$$\phi_E \leq j + \phi_M + \frac{2i}{d}.$$  

Also by the Serre functor (10), we have $\text{Hom}(\tau^i(M)[j], S_W(E)) \neq 0$, which implies that

$$j + \phi_M - n + 2 + \frac{2i}{d} + \frac{2\varepsilon}{d} \leq \phi_E.$$  

Combined with the assumption (20), we obtain

$$\phi_E \in \left[ j + \phi_M - 1, j + \phi_M + \frac{2i}{d} \right].$$  

(21)

Now suppose that $\sigma'_G$ is another stability condition which satisfies the condition in Conjecture 2.9. Then there is $m \in \mathbb{Z}$ such that the phase of $M$ with respect to $[-2m]_*\sigma'_G$ coincides with $\phi_M$. If $\phi'_E$ is the phase of the indecomposable object $E$ with respect to $[-2m]_*\sigma'_G$, then $\phi'_E$ is also contained in the RHS of (21). Since both of the central charges of $\sigma_G$ and $[-2m]_*\sigma'_G$ are the same $Z_G$, it follows that $\phi'_E = \phi_E$. Therefore $\sigma'_G = [2m]_*\sigma_G$ follows. □

The proof of Proposition 2.19 immediately implies the following:

**Corollary 2.20.** In the same situation of Proposition 2.19, there is a function $\phi(\star)$ from the set of indecomposable objects in $\text{HMF}^\text{gr}(W)$ to real numbers such that $\mathcal{P}_G(\phi)$ consists of direct sums of indecomposable objects $E$ with $\phi(E) = \phi$.

**Remark 2.21.** The inequality (20) is satisfied in the cases of Corollary 2.12, Theorem 2.13 and Theorem 2.14. In the list of Table 1, it is satisfied except $(n, \varepsilon) = (4, 0)$ and $(3, -1)$.

**Remark 2.22.** If we believe Conjecture 2.9, the proof of Proposition 2.19 predicts that $Z_G(E) \neq 0$ for any non-zero indecomposable object $E \in \text{HMF}^\text{gr}(W)$ as long as the inequality (20) is satisfied. This seems to be not an obvious property of graded matrix factorizations.

3. **T-structures on triangulated categories of graded matrix factorizations**

In this section, we construct and study the hearts of bounded t-structures on $\text{HMF}^\text{gr}(W)$, via Orlov’s theorem relating $\text{HMF}^\text{gr}(W)$ with
the derived category of coherent sheaves on \((W = 0)\). In what follows, we use the same notation in the previous section.

3.1. **Orlov’s theorem.** In \[Orl09\], Orlov proves his famous theorem relating the triangulated category \(HMF^{gr}(W)\) with the derived category of coherent sheaves on the Deligne-Mumford stack

\[
X := (W = 0) \subset \mathbb{P}(a_1, \ldots, a_n).
\]

Using the notation in Subsection 2.3, Orlov’s theorem is stated in the following way:

**Theorem 3.1.** (\[Orl09, Theorem 2.5\])

(i) If \(\varepsilon > 0\), there is a fully faithful functor

\[
\Phi_i : HMF^{gr}(W) \hookrightarrow D^b \text{Coh}(X)
\]

such that we have the semiorthogonal decomposition

\[
D^b \text{Coh}(X) = \langle O_X(-i - \varepsilon + 1), \cdots, O_X(-i), \Phi_i HMF^{gr}(W) \rangle.
\]

(ii) If \(\varepsilon \leq 0\), there is a fully faithful functor

\[
\Psi_i : D^b \text{Coh}(X) \hookrightarrow HMF^{gr}(W)
\]

such that we have the semiorthogonal decomposition

\[
HMF^{gr}(W) = \langle \mathbb{C}(-i - \varepsilon), \cdots, \mathbb{C}(-i + 1), \Psi_i D^b \text{Coh}(X) \rangle.
\]

In particular \(\Psi_i\) is an equivalence if \(\varepsilon = 0\).

In this paper we deal with the case of \(\varepsilon \leq 0\), so we only explain the construction of \(\Psi_i\). It is the composition of the following functors:

\[
\Psi_i : D^b \text{Coh}(X) \xrightarrow{\mathcal{R}\omega_i} D^b(\text{gr-}R) \xrightarrow{\pi} D^b_{sg}(R) \xrightarrow{\text{Cok}^{-1}} HMF^{gr}(W)
\]

where \(\text{Cok}^{-1}\) is the inverse of \(\text{(11)}\), \(\pi\) is the natural projection and \(\mathcal{R}\omega_i\) is defined by

\[
\mathcal{R}\omega_i(E) := \bigoplus_{j \geq i} \mathcal{R}\text{Hom}_X(O_X, E(j)).
\]

The functor \(\Psi_i\) is not compatible with grade shift functors. If \(\varepsilon = 0\), i.e. \(X\) is a Calabi-Yau stack, then their difference is described by a Seidel-Thomas twist functor \[ST01\] on \(D^b \text{Coh}(X)\)

\[
\text{ST}_E(*) := \text{Cone}(\mathcal{R}\text{Hom}_X(E, *) \otimes E \to *)
\]

for a spherical object \(E \in D^b \text{Coh}(X)\), e.g. a line bundle. The following result is suggested by Kontsevich and proved in \[BFK12\].
Proposition 3.2. ([BFK12] Proposition 5.8) If \( \varepsilon = 0 \), the following diagram commutes:

\[
\begin{array}{ccc}
D^b \text{Coh}(X) & \xrightarrow{\Psi_i} & \text{HMF}^\text{gr}(W) \\
F_i \downarrow & & \downarrow \tau \\
D^b \text{Coh}(X) & \xrightarrow{\Psi_i} & \text{HMF}^\text{gr}(W).
\end{array}
\]

Here \( F_i := \text{ST}_{O_X(-i+1)} \circ \otimes O_X(1) \).

Remark 3.3. In [BFK12] Proposition 5.8, a comparison result similar to Proposition 3.2 is obtained also for \( \varepsilon \neq 0 \). We only mention the case of \( \varepsilon = 0 \) since we only use the result in this case.

Example 3.4. Let us consider the case of \( n = 2 \) and \( \varepsilon = 0 \). In the same situation as in Subsection 2.6, we have

\[ X \cong \left[ \text{pt}/\mathbb{Z}_{a_1} \right] \coprod \left[ \text{pt}/\mathbb{Z}_{a_2} \right]. \]

Here \( \mathbb{Z}_{a_i} := \mathbb{Z}/a_i \mathbb{Z} \) acts on the smooth one point \( \text{pt} \) trivially, and \( [\ast/\ast] \) means the quotient stack. Therefore we have the orthogonal decomposition

\[ D^b \text{Coh}(X) = \langle V_0^1, \ldots, V^{a_1-1}_1, V_2^0, \ldots, V^{a_2-1}_2 \rangle \]

for one dimensional \( \mathbb{Z}_{a_i} \)-representations \( V^j_i \) with weight \( j \). By Proposition 2.2, the equivalence \( \Psi_1 \) identifies \( \tau \) on \( \text{HMF}^\text{gr}(W) \) with \( F_1 = \text{ST}_{O_X \circ \otimes O_X(1)} \). The equivalence \( F_1 \) transforms \( V^j_i \) in the following way:

\[ V_0^0 \mapsto V_1^1 \mapsto \cdots \mapsto V_1^{a_1-1} \]

\[ \mapsto V_2^0 [1] \mapsto \cdots \mapsto V_2^{a_2-1} [1] \mapsto V_1^0 [2]. \]

In particular \( \text{HMF}^\text{gr}(W) \) has the description stated in Subsection 2.6.

3.2. Construction of t-structures. In this subsection, we construct the hearts of bounded t-structures on \( \text{HMF}^\text{gr}(W) \) when \( \varepsilon \leq 0 \). We introduce the following notation: for a triangulated category \( \mathcal{D} \) and a set of objects \( \mathcal{S} \subset \mathcal{D} \), we denote by \( \langle \mathcal{S} \rangle_\text{ex} \) the extension closure of \( \mathcal{S} \), i.e. the smallest extension-closed subcategory of \( \mathcal{D} \) which contains objects in \( \mathcal{S} \). The constructions of our hearts are based on the semiorthogonal decomposition (23) and the following well-known fact:

Lemma 3.5. Let \( \mathcal{D} \) be a triangulated category and

\[ \mathcal{D} = \langle \mathcal{D}_N, \ldots, \mathcal{D}_2, \mathcal{D}_1 \rangle \]

a semiorthogonal decomposition. Suppose that \( \mathcal{C}_i \subset \mathcal{D}_i \) are hearts of bounded t-structures satisfying \( \text{Hom}^\leq \langle 0 \rangle (\mathcal{C}_j, \mathcal{C}_i) = 0 \) for \( j > i \). Then there is a bounded t-structure on \( \mathcal{D} \) whose heart \( \mathcal{C} \) is given by \( \langle \mathcal{C}_i : 1 \leq i \leq N \rangle_\text{ex} \).
Proof. The result is obviously reduced to the case of $N = 2$, which is proved in [CP10, Lemma 2.1].

Remark 3.6. In the situation of Lemma 3.5, any object $E \in \mathcal{C}$ admits a filtration

$$0 = E_0 \subset E_1 \subset \cdots \subset E_N = E$$

such that $E_i / E_{i-1}$ is an object in $\mathcal{C}_i$.

Remark 3.7. If the abelian category $\mathcal{C}_i$ is generated by an exceptional object $F \in \mathcal{C}_i$, then the heart $\mathcal{C}$ is the extension closure $\langle F_N, \cdots, F_2, F_1 \rangle$. In this case, the sequence $(F_N, \cdots, F_2, F_1)$ is called an ext-exceptional collection.

We have the following proposition:

Proposition 3.8. Suppose that $\varepsilon \leq 0$. For each $i \in \mathbb{Z}$, there is a bounded t-structure on $\text{HMF}^{gr}(W)$ whose heart $\mathcal{A}_i$ is given by

$$\mathcal{A}_i = \langle \mathcal{C}(-i - \varepsilon), \cdots, \mathcal{C}(-i + 1), \Psi_i \text{Coh}(X) \rangle_{ex}.$$

Proof. By following Orlov’s argument in [Orl09, Theorem 2.5], we see the following: there is an admissible subcategory $T_i \subset D^b(\text{gr-R})$ with a semiorthogonal decomposition

$$(26) \quad T_i = \langle \mathcal{C}(-i - \varepsilon), \cdots, \mathcal{C}(-i + 1), \text{R} \omega_i \text{D}^b \text{Coh}(X) \rangle$$

and an equivalence

$$T_i \sim \text{HMF}^{gr}(W)$$

which identifies the semiorthogonal decomposition (26) with the RHS of (23). Here $\text{R} \omega_i$ is the functor defined by (25). Therefore by Lemma 3.5, it is enough to show that

$$(27) \quad \text{Hom}_{\text{gr-R}}^{\leq 0} (\mathcal{C}(j), \mathcal{C}(j')) = 0$$

$$(28) \quad \text{Hom}_{\text{gr-R}}^{\leq 0} (\mathcal{C}(j), \text{R} \omega_i \text{D}^b \text{Coh}(X)) = 0$$

for $j, j' \in [-i + 1, -i - \varepsilon]$ with $j' < j$. The assertion (27) is obvious since $\mathcal{C}(j)$ is a simple object in the heart $\text{gr-R} \subset D^b(\text{gr-R})$. As for the assertion (28), since we have $\text{R} \omega_i (F) \in D^\geq 0(\text{gr-R})$ for $F \in \text{Coh}(X)$, it follows that

$$(29) \quad \text{Hom}_{\text{gr-R}}^{\leq 0} (\mathcal{C}(j), \text{R} \omega_i (F)) \cong \text{Hom}_{\text{gr-R}}^{\leq 0} (\mathcal{C}(j), \omega_i (F)).$$

Here $\omega_i (F) \in \text{gr-R}$ is the zero-th cohomology of $\text{R} \omega_i (F)$. Since $\omega_i (F) \in \text{gr-R}$ is concentrated on degree $\geq i$ parts, and $\mathcal{C}(j)$ is on degree $< i$ parts, the vector space (29) vanishes.

In what follows, we always assume that $\varepsilon \leq 0$. We only focus on the case $i = 1$ in the above proposition:
**Definition 3.9.** Suppose that $\varepsilon \leq 0$. We define $A_W := A_1$ and $\Psi := \Psi_1$, i.e.

\[
A_W = \langle \mathbb{C}(-1-\varepsilon), \cdots, \mathbb{C}(0), \Psi \operatorname{Coh}(X) \rangle_{\operatorname{ex}}.
\]

(30)

3.3. **Description of certain objects in $A_W$.** This subsection is devoted to investigate some objects in $A_W$, which will be used later. By definition, we call a closed point $x \in X$ stacky if the stabilizer group at $x$ is non-trivial. Let us describe $\Psi(O_x)$ for a non-stacky point $x \in X$. Note that $X$ is a closed substack $X := ((\mathbb{C}^n \setminus \{0\})/\mathbb{C}^*)$ where $\mathbb{C}^*$ acts on $\mathbb{C}^n$ via weight $(a_1, \cdots, a_n)$. Hence $x \in X$ is represented by a point $(p_1, \cdots, p_n) \in \mathbb{C}^n$. We define the graded $R$-module $M(x)$ to be

\[
M(x) := \bigoplus_{j \geq 1} \mathbb{C}e_j
\]

where $e_j$ is concentrated on degree $j$, and the action of $x_i$ sends $e_j$ to $p_i e_{j+a_i}$. Obviously if $x \in X$ is non-stacky, then $R\omega_1(O_x)$ is a graded $R$-module and isomorphic to $M(x)$. The object $\Psi(O_x)$ is obtained by applying the inverse of (11) to $M(x)$, after regarding it as an object in $D_{\operatorname{sg}}(R)$. Using the above description, we have the following lemma:

**Lemma 3.10.** For any non-stacky point $x \in X$, we have the exact sequence in $A_W$

\[
0 \to \Psi(O_x) \to \tau \Psi(O_x) \to \mathbb{C}(0) \to 0.
\]

(32)

*Proof.* The result obviously follows from the following exact sequence as graded $R$-modules

\[
0 \to M(x) \to M(x)(1) \to \mathbb{C}(0) \to 0.
\]

\[
\square
\]

Next let us consider the following object

$\mathbb{C}(-\varepsilon) \in \operatorname{HMF}^{\operatorname{gr}}(W)$. The above object is described in terms of the generators of $A_W$. We have the following lemma:

**Lemma 3.11.** We have $\mathbb{C}(-\varepsilon)[-1] \in A_W$. Furthermore there is a filtration in $A_W$

$$0 \subset E_{-1} \subset E_0 \subset \cdots \subset E_{-1-\varepsilon} = \mathbb{C}(-\varepsilon)[-1]$$

such that the following holds for $0 \leq i \leq -1 - \varepsilon$

\[
E_{-1} \cong \Psi(\omega_X), \quad E_i/E_{i-1} \cong \mathbb{C}(i) \otimes R_{-i-\varepsilon}.
\]
Proof. Let \( m \subset R \) be the maximal ideal \( (12) \). We have the exact sequence in \( \text{gr}-R \)
\[
0 \to m(-\varepsilon) \to R(-\varepsilon) \to C(-\varepsilon) \to 0
\]
which implies \( C(-\varepsilon)[-1] \cong m(-\varepsilon) \) in \( D_{\text{sg}}(R) \). Let \( m_{\geq i} \subset m \) be the ideal generated by monomials with degree greater than or equal to \( i \). We have the following filtration in \( \text{gr}-R \)
\[
\begin{align*}
m_{\geq 1-\varepsilon} & \subset m_{\geq -\varepsilon} \subset \cdots \subset m_{\geq 2} \subset m_{\geq 1} = m
\end{align*}
\]
such that the following holds:
\[
(m_{\geq j}/m_{\geq j+1})(-\varepsilon) \cong C(-j - \varepsilon) \otimes R_j, \quad 1 \leq j \leq -\varepsilon.
\]
Therefore it is enough to show that \( m_{\geq 1-\varepsilon}(-\varepsilon) \) is isomorphic to \( R\omega_1(\omega_X) \) in \( \text{gr}-R \). Since \( \omega_X \cong \mathcal{O}_X(-\varepsilon) \) and \( H^k(X, \omega_X(j)) \cong 0 \) for \( k \neq 0 \) and \( j \geq 1 \), we have
\[
R\omega_1(\omega_X) \cong \bigoplus_{j \geq 1} H^0(X, \mathcal{O}_X(-\varepsilon + j)).
\]
Obviously the RHS is isomorphic to \( m_{\geq 1-\varepsilon}(-\varepsilon) \) as a graded \( R \)-module. \( \square \)

The above lemma can be applied to do some computations on the left adjoint of \( \Psi \), denoted by \( \Psi^L : \text{HMF}^{gr}(W) \to D^b\text{Coh}(X) \).

We have the following lemma:

**Lemma 3.12.** The object \( \Psi^L(\mathbb{C}(0)) \) is isomorphic to \( \mathcal{O}_X[1] \).

*Proof.* Let \( \Psi^R \) is the right adjoint of \( \Psi \). Note that \( \Psi^L \) and \( \Psi^R \) are related by
\[
\Psi^L = S^{-1}_X \circ \Psi^R \circ S_W
\]
where \( S_X = \otimes \omega_X[n - 2] \) is the Serre functor of \( D^b\text{Coh}(X) \). By \((10)\), we have
\[
\Psi^L(\mathbb{C}(0)) \cong S^{-1}_X \circ \Psi^R(\mathbb{C}(-\varepsilon))[n - 2] \cong S^{-1}_X(\omega_X[1])[n - 2] \cong \mathcal{O}_X[1].
\]
Here the second isomorphism follows from Lemma 3.11. \( \square \)

**Remark 3.13.** By the above lemma, it follows that
\[
\text{Hom}_{\text{HMF}^{gr}(W)}^1(\mathbb{C}(0), \Psi(\mathcal{O}_x)) \cong \text{Hom}_X(\mathcal{O}_x, \mathcal{O}_x)
\]
which is one dimensional for \( x \in X \). Since \( \tau \Psi(\mathcal{O}_x) \) is indecomposable, the exact sequence \((32)\) is a unique non-trivial extension.
Remark 3.14. For $F \in \text{Coh}(X)$, suppose that $R\omega_1(F)$ is a graded $R$-module. Then Lemma 3.12 and the argument in Proposition 3.8 imply
\[
\text{Ext}^1_{\text{gr-}R}(\mathcal{C}(0), R\omega_1(F)) \cong H^0(X, F).
\]
For $u \in H^0(X, F)$, we have the corresponding extension in $\text{gr-R}$
\[
0 \to R\omega_1(F) \to M_u \to \mathcal{C}(0) \to 0.
\]
The graded $R$-module $M_u$ is described in the following way: as a graded $\mathbb{C}$-vector space, it is the direct sum $\mathbb{C}(0) \oplus R\omega_1(F)$, and the action of $x_i \in R$ sends $1 \in \mathbb{C}(0)$ to $u \cdot x_i \in H^0(X, F(1))$.

3.4. Description of $A_W$ via quiver representations. In this subsection, we assume that $n = 2$ and describe the heart $A_W$ in terms of certain quiver representations. In this case, $X$ is a smooth zero dimensional Deligne-Mumford stack, and $\text{Coh}(X)$ is generated by mutually orthogonal exceptional objects. Therefore by (30), the abelian category $A_W$ is the extension closure of an ext-exceptional collection. We first compute other Hom groups between these exceptional objects, and then describe $A_W$ via the ext-quivers with relations.

For some technical reason, we assume that $X$ does not contain stacky points, so it consists of finite number of smooth points. Then, after applying the coordinate change if necessary, $W$ is written as
\[
W = x_1W_1 + x_2W_2
\]
for some homogeneous elements $W_i \in A$ such that $x_1$ (resp. $x_2$) does not divide $W_2$ (resp. $W_1$). The heart $A_W$ is described in the following way:
\[
A_W = \langle \mathbb{C}(d - a_1 - a_2 - 1), \cdots, \mathbb{C}(0), \Psi(O_x) : x \in X \rangle_{\text{ex}}.
\]
Below we calculate the Hom groups between the above generators.

Lemma 3.15. For $0 < j < d - a_1 - a_2$, we have the following:
\[
\text{Hom}_{\text{HMF}^0(W)}^i(\mathbb{C}(j), \mathbb{C}(0)) \cong \begin{cases} R'_j, & (i, j) = (1, a_1), (1, a_2) \\ R'_0, & (i, j) = (2, a_1 + a_2) \\ 0, & \text{otherwise}. \end{cases}
\]
Moreover the natural map
\[
\bigoplus_{j, j' \in \{a_1, a_2\}} \text{Hom}^1(\mathbb{C}(j), \mathbb{C}(0))^\vee \otimes \text{Hom}^1(\mathbb{C}(j + j'), \mathbb{C}(j'))^\vee \otimes \text{Hom}^1(\mathbb{C}(j), \mathbb{C}(0))^\vee 
\]
sends $1 \in R_0$ to $x_1 \otimes x_2 - x_2 \otimes x_1$ under the isomorphism (34).

Proof. By the same argument of Proposition 3.8, we have
\[
\text{Hom}_{\text{HMF}^0(W)}^i(\mathbb{C}(j), \mathbb{C}(0)) \cong \text{Ext}^i_{\text{gr-}R}(\mathbb{C}(j), \mathbb{C}(0)).
\]
Then (34) is easily obtained by computing the RHS of (36) using the resolution:

\[
(37) \quad \cdots \to R(-a_1 - d) \oplus R(-a_2 - d) \xrightarrow{h'} R(-d) \oplus R(-a_1 - a_2) \xrightarrow{h} R(-a_1) \oplus R(-a_2) \xrightarrow{(x_1, x_2)} R \to \mathbb{C}(0) \to 0.
\]

Here \( h \) and \( h' \) given by matrices

\[
(38) \quad h = \begin{pmatrix} W_1 & -x_2 \\ W_2 & x_1 \end{pmatrix}, \quad h' = \begin{pmatrix} x_1 & x_2 \\ -W_2 & W_1 \end{pmatrix}.
\]

Next we consider the map (35). We write \( W_1, W_2 \) as

\[
W_1 = x_1W_{11} + x_2W_{12}, \quad W_2 = x_1W_{21} + x_2W_{22}
\]

for homogeneous elements \( W_{k,l} \in A \). Let \( x_k^\vee \in R_{a_k}^\vee \) be the dual basis of \( x_k \in R_{a_k} \), and we regard them as elements of the RHS of (36) for \((i, j) = (1, a_k)\). Then \( x_1^\vee \) is represented by the morphism of complexes

\[
\begin{array}{c}
R(a_1 - d) \oplus R(-a_2) \xrightarrow{h(a_1)} R(0) \oplus R(a_1 - a_2) \xrightarrow{\pi_1} R(a_1) \\
\downarrow g_1 \\
R(-a_1) \oplus R(-a_2) \xrightarrow{(x_1, x_2)} R(0) \xrightarrow{\pi_2} 0.
\end{array}
\]

Similarly \( x_2^\vee \) is represented by

\[
\begin{array}{c}
R(a_2 - d) \oplus R(-a_1) \xrightarrow{h(a_2)} R(a_2 - a_1) \oplus R(0) \xrightarrow{\pi_2} R(a_2) \\
\downarrow g_2 \\
R(-a_1) \oplus R(-a_2) \xrightarrow{(x_1, x_2)} R(0) \xrightarrow{\pi_2} 0.
\end{array}
\]

Here \( \pi_i \) are projections onto the \( i \)-th factor, and \( g_i \) are given by matrices

\[
g_1 = \begin{pmatrix} W_{11} & 0 \\ W_{12} & -1 \end{pmatrix}, \quad g_2 = \begin{pmatrix} W_{21} & 1 \\ W_{22} & 0 \end{pmatrix}.
\]

The image of \( x_2^\vee \otimes x_1^\vee \) by the dual of (35) is computed by composing the above morphisms of complexes. By restricting the map \( \pi_2 \circ g_1(a_2) \) to the second component of \( R(a_1 + a_2 - d) \oplus R(0) \), we see that \( x_2^\vee \otimes x_1^\vee \) is mapped to \(-1^\vee\), where \( 1^\vee \) is the dual basis of \( 1 \in R_0 \). Similarly \( x_1^\vee \otimes x_2^\vee \) is mapped to \( 1^\vee \). By dualizing, we obtain the result. \( \square \)

**Remark 3.16.** By (34) and (36), an element \( u \in R_{a_i}^\vee \) determines the extension in \( \text{gr}-R \)

\[
0 \to \mathbb{C}(0) \to M_u \to \mathbb{C}(a_i) \to 0.
\]

The graded \( R \)-module \( M_u \) is isomorphic to \( \mathbb{C}(0) \oplus \mathbb{C}(a_i) \) as a graded \( \mathbb{C} \)-vector space, and the action of \( x_i \) is given by sending \( 1 \in \mathbb{C}(0) \) to \( u(x_i) \in \mathbb{C}(a_i) \).
Next we compute the Hom groups between $\mathbb{C}(j)$ and $\Psi(O_x)$ for closed points $x \in X$. Let $M(x)$ be the graded $R$-module defined by (31). We have the following lemma:

**Lemma 3.17.** Suppose that $0 \leq j < d - a_1 - a_2$, and $x \in X$ is represented by $(p_1, p_2) \in \mathbb{C}^2$. Then we have

$$\text{Hom}^i_{\text{HMF}^W}(\mathbb{C}(j), \Psi(O_x)) \cong \begin{cases} \mathbb{C}u_j, & i = 1, j \in [0, a_2) \\ \mathbb{C}v_j, & i = 2, j \in [a_1, a_1 + a_2) \\ 0, & \text{otherwise.} \end{cases}$$

Here $u_j$ and $v_j$ are regarded as elements

$$u_j = p_1 e_{-j+a_1} + p_2 e_{-j+a_2} \in M(x)_{-j+a_1} \oplus M(x)_{-j+a_2}$$

$$v_j = ve_{-j+d} \oplus e_{-j+a_1+a_2} \in M(x)_{-j+d} \oplus M(x)_{-j+a_1+a_2}$$

where $v := W_2(p_1, p_2)/p_1 = -W_1(p_1, p_2)/p_2$. If $j \in [a_1, a_1 + a_2)$, the natural map

$$\text{Hom}^2(\mathbb{C}(j), \Psi(O_x))^\vee \to \bigoplus_{j' \in \{j-a_1, j-a_2\}} \text{Hom}^1(\mathbb{C}(j), \mathbb{C}(j'))^\vee \otimes \text{Hom}(\mathbb{C}(j'), \Psi(O_x))^\vee$$

sends the dual basis $v_j^{\vee}$ to $p_2 x_1 \otimes u_{j-a_1}^{\vee} - p_1 x_2 \otimes u_{j-a_2}^{\vee}$ under the isomorphisms (32), (39). (Here we set $u_{j-a_1}^{\vee} = 0$ if $j-a_1 \geq a_1$.)

**Proof.** Similarly to the proof of Lemma 3.15, we have the isomorphism

$$\text{Hom}^i_{\text{HMF}^W}(\mathbb{C}(j), \Psi(O_x)) \cong \text{Ext}^i_{\text{gr-R}}(\mathbb{C}(j), M(x)).$$

Applying $\text{Hom}_{\text{gr-R}}(\ast, M(x)(-j))$ to the exact sequence (57), the RHS of (51) is computed by the $i$-th cohomology group of the following complex

$$0 \to M(x)_{-j} \xrightarrow{(p_1, p_2)} M(x)_{-j+a_1} \oplus M(x)_{-j+a_2}$$

$$\xrightarrow{\cdot h(p_1, p_2)} M(x)_{-j+d} \oplus M(x)_{-j+a_1+a_2}$$

$$\xrightarrow{\cdot h'(p_1, p_2)} M(x)_{-j+d+a_1} \oplus M(x)_{-j+d+a_2} \to \cdots.$$  

Here $h(p_1, p_2)$, $h'(p_1, p_2)$ are the substitution of $(x_1, x_2) = (p_1, p_2)$ to the matrices (35). Then (39) easily follows by noting that every non-zero maps in the complex (42) are rank one. The image of $v_j^{\vee}$ by the map (40) is computed similarly to (35), so we omit the detail. \qed

The above computations enable us to describe $A_W$ in terms of quiver representations with relations. Here we only discuss the case of $a_1 = a_2 = 1$. Recall that, given a set of objects $(F_N, \cdots, F_2, F_1)$, the ext-quiver $Q(F_\bullet)$ is defined as follows: the set of vertices is

$$\{1, 2, \cdots, N\}$$
and the number of edges from \( j \) to \( j' \) is the dimension of \( \text{Ext}^1(F_j, F_{j'}) \), which we identify with a basis of \( \text{Ext}^1(F_j, F_{j'})^\vee \). The following lemma may be well-known, but we include the proof later in Subsection 3.6 because of a lack of a reference.

**Lemma 3.18.** Let \( \mathcal{D} \) be a triangulated category with finite dimensional \( \text{Hom} \) spaces, generated by an ext-exceptional collection \((F_N, \cdots, F_2, F_1)\). Let \( \mathcal{A} \) be the heart of a bounded t-structure on \( \mathcal{D} \) given by the extension closure of all \( F_i \) for \( 1 \leq i \leq N \). Suppose that there is a partition 
\[
\{1, \ldots, N\} = P_1 \sqcup \cdots \sqcup P_l, \quad j' > j \text{ if } j \in P_k, j' \in P_{k'}, k > k'.
\]
such that, by setting \( \hat{F}_k := \oplus_{j \in P_k} F_j \), the following condition holds:
\[
\text{Ext}^i(\hat{F}_{k'}, \hat{F}_k) = 0 \quad \text{unless } (i, k' - k) = (1, 1), (2, 2).
\]
Then \( \mathcal{A} \) is equivalent to the abelian category of \( \mathbb{Q}(F_\cdot) \)-representations with relations generated by the images of the following natural maps for all \( 1 \leq k \leq l \):
\[
\text{Ext}^2(\hat{F}_{k+2}, \hat{F}_k) \to \text{Ext}^1(\hat{F}_{k+2}, \hat{F}_{k+1}) \otimes \text{Ext}^1(\hat{F}_{k+1}, \hat{F}_k).
\]

The following corollary directly follows from Lemma 3.15, Lemma 3.17 and Lemma 3.18.

**Corollary 3.19.** Suppose that \( a_1 = a_2 = 1 \) and we write
\[
X = \{p^{(i)} = (p_1^{(i)}, p_2^{(i)}) \in \mathbb{P}^1 : 1 \leq i \leq d\}.
\]
Then \( \mathcal{A}_W \) is equivalent to the category of representations of the quiver of the form
\[
\begin{array}{c}
\bullet & \xrightarrow{\pi^{(d)}} & \bullet \\
X_1^{(d-3)} & & X_2^{(d-3)} & \cdots & X_1^{(1)} & \xrightarrow{\pi^{(1)}} & \bullet \\
X_2^{(d-3)} & & & & X_2^{(1)} & \\
\end{array}
\]
with relations given by
\[
X_2^{(i-1)} X_1^{(i)} = X_1^{(i-1)} X_2^{(i)}, \quad p_2^{(j)} \pi^{(j)} X_1^{(1)} = p_1^{(j)} \pi^{(j)} X_2^{(1)}
\]
for all \( 2 \leq i \leq d - 3 \) and \( 1 \leq j \leq d \). The vertex \( v^{(i)} \) corresponds to \( \mathbb{C}(i) \) and \( w^{(j)} \) corresponds to \( \Psi(\mathcal{O}_{p^{(j)}}) \).

By investigating the filtration (33), we are able to describe \( \mathbb{C}(-\varepsilon)[-1] \) in terms of a representation of a quiver (45). The following corollary is a straightforward adaptation of Corollary 3.19, Remark 3.14 and Remark 3.16.
Corollary 3.20. In the situation of Corollary 3.19, the object $\mathbb{C}(-\varepsilon)[-1]$ in $\mathcal{A}_W$ is the representation of the quiver (45) given as follows:

\[
\begin{array}{cccccccc}
R_1 & \xrightarrow{x_1} & R_2 & \xrightarrow{x_2} & \cdots & \xrightarrow{x_1} & R_{d-3} & \xrightarrow{x_2} & R_{d-2} \\
\cdot & & \cdot & & \cdots & & \cdot & & \cdot \\
\end{array}
\]

Here $\pi^{(j)} : R_{d-2} \to \mathbb{C}$ is the evaluation at $p^{(j)} = (p^{(j)}_1, p^{(j)}_2)$.

3.5. Description of $\mathcal{A}_W$ via coherent systems. In this subsection, we assume that $\varepsilon = -1$ and describe the heart $\mathcal{A}_W$ in terms of coherent systems on $X$. Let us recall the definition of coherent systems.

Definition 3.21. A coherent system on a Deligne-Mumford stack $X$ is data

\[ V \otimes \mathcal{O}_X \xrightarrow{s} F \]

where $V$ is a finite dimensional $\mathbb{C}$-vector space, $F \in \text{Coh}(X)$ and $s$ is a morphism in $\text{Coh}(X)$.

The category of coherent systems on $X$ is denoted by $\text{Syst}(X)$. The set of morphisms is given by the commutative diagrams in $\text{Coh}(X)$

\[
\begin{array}{ccc}
V \otimes \mathcal{O}_X & \xrightarrow{s} & F \\
\downarrow & & \downarrow \\
V' \otimes \mathcal{O}_X & \xrightarrow{s'} & F'.
\end{array}
\]

Obviously $\text{Syst}(X)$ is an abelian category. We have the following proposition:

Proposition 3.22. Suppose that $\varepsilon = -1$. Then we have an equivalence of abelian categories

\[ \Theta : \text{Syst}(X) \xrightarrow{\sim} \mathcal{A}_W. \]

Proof. Let us take a coherent system $(V \otimes \mathcal{O}_X \xrightarrow{s} F)$ on $X$. By Lemma 3.12 the morphism $s$ is regarded as an element

\[ s' \in \text{Hom}_{\text{HMF}^\psi(W)}(V \otimes \mathbb{C}(0), \Psi(F)[1]). \]

The cone of $s'$ determines an object in $\mathcal{A}_W$. The correspondence

\[ \Theta : (F, s) \mapsto \text{Cone}(s') \]

is a functor from $\text{Syst}(X)$ to $\mathcal{A}_W$ because, as in the proof of Proposition 3.8 we have the vanishing $\text{Hom}(\mathbb{C}(0), \Psi\text{Coh}(X)) = 0$. 

Conversely, let us take an object \( E \in \mathcal{A}_W \). There is an exact sequence
\[
0 \to \Psi(F) \to E \to V \otimes \mathbb{C}(0) \to 0 \tag{47}
\]
for a finite dimensional vector space \( V \) and \( F \in \text{Coh}(X) \). By Lemma \( \text{3.12} \) the extension class \( \xi \) of \( \mathcal{E} \) is regarded as an element \( \xi' \in \text{Hom}(V \otimes \mathcal{O}_X, F) \).

The pair \( (F, \xi') \) determines an object in \( \text{Syst}(X) \). The correspondence
\[
\Theta': E \mapsto (F, \xi')
\]
is a functor from \( \mathcal{A}_W \) to \( \text{Syst}(X) \) since \( \langle \mathbb{C}(0), \Psi \mathcal{D}^b \text{Coh}(X) \rangle \) is a semiorthogonal decomposition of \( \text{HMF}^{gr}(W) \).

Obviously we have
\[
\Theta' \circ \Theta \cong \text{id}_{\text{Syst}(X)}, \quad \Theta \circ \Theta' \cong \text{id}_{\mathcal{A}_W}
\]
hence \( \Theta \) is an equivalence. \( \square \)

Combined with Lemma \( \text{3.11} \) and Remark \( \text{3.14} \) we immediately obtain the following corollary:

**Corollary 3.23.** Suppose that \( \varepsilon = -1 \). Then the object \( \mathbb{C}(1)[-1] \in \mathcal{A}_W \) is given by
\[
\mathbb{C}(1)[-1] \cong \Theta \left( H^0(X, \mathcal{O}_X(1)) \otimes \mathcal{O}_X \xrightarrow{\delta} \mathcal{O}_X(1) \right).
\]
Here \( s \) is the canonical evaluation morphism.

### 3.6. Proof of Lemma \( \text{3.18} \)

Finally in this section, we give a proof of Lemma \( \text{3.18} \). The proof is straightforward, and probably well-known. We recommend the readers to skip this subsection at the first reading.

**Proof.** We denote by \( I \) the set of relations generated by the images of \( \mathcal{I} \). Let \( \text{Rep}(\mathcal{Q}(F_*), I) \) be the category of \( \mathcal{Q}(F_*) \)-representations with relation \( I \). We divide the proof into three steps.

**Step 1.**

We construct the functor
\[
\Phi: \mathcal{A} \to \text{Rep}(\mathcal{Q}(F_*), I)
\]
in the following way: for an object \( E \in \mathcal{A} \), it admits a filtration
\[
0 = E_0 \subset E_1 \subset \cdots \subset E_l = E
\]
such that \( E_k/E_{k-1} \) is written as
\[
E_k/E_{k-1} \cong \bigoplus_{j \in \mathcal{I}_k} F_j \otimes V_j
\]
for finite dimensional vector spaces \( V_j \). By the exact sequence
\[
0 \to E_k/E_{k-1} \to E_{k+1}/E_{k-1} \to E_{k+1}/E_k \to 0 \tag{48}
\]
we obtain the linear maps

\[(49) \quad \phi_{j'j} : V_{j'} \otimes \text{Ext}^1(F_{j'}, F_j)^\vee \to V_j\]

for \(j' \in P_{k+1}, j \in P_k\), which defines the \(Q(F_\bullet)\)-representation \(\Phi(E)\). In order to show that the representation \(\Phi(E)\) satisfies the relation \(I\), consider the composition of extension classes of \((48)\):

\[(50) \quad \bigoplus_{j'' \in P_{k+2}} F_{j''} \otimes V_{j''} \to \bigoplus_{j' \in P_{k+1}} F_{j'} \otimes V_{j'}[1] \to \bigoplus_{j \in P_k} F_j \otimes V_j[2].\]

The above composition must vanish since it coincides with the composition

\[E_{k+2}/E_{k+1} \to E_{k+1}/E_{k-1}[1] \to E_{k+1}/E_k[1] \to E_k/E_{k-1}[2]\]

where the left morphism is the extension class of

\[0 \to E_{k+1}/E_{k-1} \to E_{k+2}/E_{k-1} \to E_{k+2}/E_{k+1} \to 0.\]

By applying \(\text{Hom}(F_{j''}, \ast)\) for \(j'' \in P_{k+2}\) to \((50)\), taking the adjunction and \((j'', j)\)-component for \(j \in P_k\), we see that the map

\[V_{j''} \otimes \bigoplus_{j' \in P_{k+1}} (\text{Ext}^1(F_{j''}, F_{j'})^\vee \otimes \text{Ext}^1(F_{j'j}, F_j)^\vee) \to V_j\]

given by the sum of the composition

\[(51) \quad \sum_{j' \in P_{k+1}} \phi_{j'j} \circ \phi_{j''j'}\]

is zero on \(V_{j''} \otimes I_{j'', j}\), where \(I_{j'', j}\) is the image of \((44)\) restricted to \(\text{Ext}^2(F_{j''}, F_{j'})^\vee\)-component. This implies that \(\Phi(E)\) satisfies the relation \(I\), hence it is an object in \(\text{Rep}(Q(F_\bullet), I)\).

**Step 2.**

The correspondence \(E \mapsto \Phi(E)\) obviously determines a fully faithful functor from \(\mathcal{A}\) to \(\text{Rep}(Q(F_\bullet), I)\), since \((F_N, \cdots, F_2, F_1)\) is an exceptional collection. It remains to show that \(\Phi\) is essentially surjective. Let us take an object

\[W \in \text{Rep}(Q(F_\bullet), I).\]

It consists of finite dimensional vector spaces \(V_j\) for \(1 \leq j \leq N\) and linear maps \((49)\) whose composition \((51)\) is zero on \(V_{j''} \otimes I_{j'', j}\) for \((j'', j) \in P_{k+2} \times P_k\). We need to show the existence of \(E \in \mathcal{A}\) so that \(\Phi(E) \cong W\).

By the induction on \(l\), we may assume that the assertion holds for \(l-1\). We set full subcategories \(\mathcal{A}_k \subset \mathcal{D}\) as follows:

\[\mathcal{A}_k := \langle F_j : j \in P^{k'}, 1 \leq k' \leq k\rangle_{\text{ex}}.\]
Let $Q(F'_*)$ be the ext-quiver for $A_{l-1}$ and define the relation $I'$ by restricting $I$ to $Q(F'_*)$. The category $\text{Rep}(Q(F'_*), I')$ is naturally considered as a subcategory of $\text{Rep}(Q(F_*), I)$, and there is an exact sequence

\begin{equation}
0 \to W' \to W \to W_l \to 0
\end{equation}

where $W' \in \text{Rep}(Q(F'_*), I')$ and $W_l$ is written as

$$W_l \cong \bigoplus_{j \in P_l} e_j \otimes V_j.$$ 

Here $e_j$ is the simple object in $\text{Rep}(Q(F_*), I)$ corresponding to the vertex $j$.

By the assumption of the induction, there is an object $E'_* \in A_{l-1}$ such that $\Phi(E') \cong W'$. By the exact sequence (52), it is enough to show that the map

\begin{equation}
\alpha : \text{Ext}^1_A(F_j, E') \to \text{Ext}^1_{\text{Rep}}(e_j, W')
\end{equation}

induced by $\Phi$ is an isomorphism for all $j \in P_l$. Here we have written $\text{Rep}(Q(F_*), I)$ just as $\text{Rep}$ for simplicity.

**Step 3.**

We show that the morphism (53) is an isomorphism. Let us consider the exact sequence in $A$

\begin{equation}
0 \to E'' \to E' \to E_{l-1} \to 0
\end{equation}

with $E'' \in A_{l-2}$ and $E_{l-1}$ is written as

$$E_{l-1} \cong \bigoplus_{j' \in P_{l-1}} F_{j'} \otimes V_{j'}.$$ 

By the condition (43), for $j \in P_l$, we see that $\text{Ext}^1_A(F_j, E'') = 0$ and there is a natural isomorphism

$$\text{Ext}^2_D(F_j, E'') \xrightarrow{\cong} \bigoplus_{j'' \in P_{l-2}} \text{Ext}^2_D(F_j, F_{j''}) \otimes V_{j''}.$$ 

Therefore applying $\text{Hom}(F_j, *)$ to (54), we obtain the exact sequence

\begin{equation}
0 \to \text{Ext}^1_A(F_j, E') \to \bigoplus_{j' \in P_{l-1}} \text{Ext}^1_A(F_j, F_{j'}) \otimes V_{j'}
\end{equation}

$$\xrightarrow{\beta} \bigoplus_{j'' \in P_{l-2}} \text{Ext}^2_D(F_j, F_{j''}) \otimes V_{j''}.$$ 

On the other hand, there is an exact sequence in $\text{Rep}(Q(F_*), I)$

$$0 \to W'' \to W' \to \bigoplus_{j' \in P_{l-1}} e_{j'} \otimes V_{j'} \to 0$$
such that $W'' \cong \Phi(E'')$. Since $\text{Ext}^1_{\text{Rep}}(e_a, e_a') = 0$ unless $a \in P_k$, $a' \in P_{k'}$ with $k - k' = 1$, we have $\text{Ext}^1_{\text{Rep}}(e_j, W'') = 0$ for any $j \in P_l$. Therefore we obtain the commutative diagram

$$
\begin{array}{ccc}
0 & \longrightarrow & \text{Ext}^1_A(F_j, E') \\
\downarrow & & \downarrow \\
0 & \longrightarrow & \bigoplus_{j' \in P_{l-1}} \text{Ext}^1_A(F_j, F_{j'}) \otimes V_{j'}
\end{array}
$$

Here $\gamma$ is an isomorphism induced by $\Phi$. By the above diagram, it follows that $\alpha$ is injective. On the other hand, the composition

$$
\beta \circ \gamma^{-1} \circ \delta : \text{Ext}^1_{\text{Rep}}(e_j, W') \rightarrow \bigoplus_{j'' \in P_{l-2}} \text{Ext}^2_D(F_j, F_{j''}) \otimes V_{j''}
$$

vanishes, since any object given by an extension class in the LHS satisfies the relation $I$. Therefore $\alpha$ is surjective, hence an isomorphism. □

4. Construction of Gepner type stability conditions

In this section, we propose a general recipe on a construction of a desired Gepner type stability condition. We first compute the central charge $Z_G$ in terms of generators of $A_W$, and try to construct $\sigma_G$ via tilting of $A_W$. In what follows we assume that the stack $X$ in (22) is a smooth projective variety, i.e. $X$ does not contain stacky points. As in the previous section, we denote by $\Psi := \Psi_1$ the Orlov’s fully faithful functor from $D^b\text{Coh}(X)$ to $\text{HMF}^{gr}(W)$, which is an equivalence if $\varepsilon = 0$.

4.1. Computation of the central charge ($\varepsilon = 0$ case). In this subsection, we explain how to compute $Z_G$ in the case of $\varepsilon = 0$. If $\varepsilon = 0$, $X$ is a Calabi-Yau manifold of dimension $n - 2$, and $A_W$ is equivalent to $\text{Coh}(X)$ via $\Psi$. Let us consider the group homomorphism given by

$$
Z_G \circ \Psi : K(X) \rightarrow \mathbb{C}.
$$

The above group homomorphism is described in terms of Chern characters on $K(X)$. Indeed, a fundamental theory on Hochschild homology groups implies that $\Psi$ induces the isomorphism $\Psi_* : \text{HH}_0(X) \xrightarrow{\cong} \text{HH}_0(W)$ such that the following diagram commutes: (cf. [PV12, Section 1])

$$
\begin{array}{ccc}
D^b\text{Coh}(X) & \xrightarrow{\Psi} & \text{HMF}^{gr}(W) \\
\downarrow & & \downarrow \\
\text{HH}_0(X) & \xrightarrow{\Psi_*} & \text{HH}_0(W).
\end{array}
$$
We also have the Hochschild-Kostant-Rosenberg isomorphism
\[ \text{HH}_0(X) \cong H\Omega_0(X) := \bigoplus_{j=0}^{n-2} H^j(X, \Omega^j_X) \]
such that its composition with \( \text{ch} : D^b \text{Coh}(X) \to \text{HH}_0(X) \) coincides with the classical Chern character map \([\text{C˘ al05}, \text{Theorem 4.5}]\). Since our central charge \( Z_G \) factors through the Chern character map on \( H^\text{MF}_\text{gr}(W) \) (cf. Remark 2.16) and the Poincaré pairing on \( H\Omega_0(X) \) is perfect, the group homomorphism (56) is written as
\[ E \mapsto \sum_{j=0}^{n-2} \int_X \alpha_j \cdot \text{ch}_j(E) \]
for some \( \alpha_j \in H^{n-2-j,n-2-j}(X) \). Here by an abuse of notation, we also denote by \( \text{ch}(E) \in H\Omega_0(X) \) the classical Chern character of \( E \in K(X) \), and by \( \text{ch}_j(E) \) its \( H^{j,j}(X) \)-component.

On the other hand, let us consider the autoequivalence \( F \) of \( D^b \text{Coh}(X) \) defined by
\[ F := \text{ST}_{O_X} \circ \otimes O_X(1). \]
By Proposition 3.2, the above autoequivalence corresponds to the grade shift functor \( \tau \) on \( H^\text{MF}_\text{gr}(W) \) via the equivalence \( \Psi \). By the Riemann-Roch theorem, the autoequivalence \( F \) acts on \( \text{ch}(E) \) for \( E \in D^b \text{Coh}(X) \) in the following way
\[ F_* : \text{ch}(E) \mapsto e^H \text{ch}(E) - \left( \int_X e^H \text{ch}(E) \text{td}_X \right) \cdot 1. \]
Here \( H \) is the first Chern class of \( O_X(1) \). The above action naturally extends to the linear isomorphism on \( H\Omega_0(X) \), given by the composition of the matrices
\[ M := \begin{pmatrix} 1 - t_{n-2} & -t_{n-3} & \cdots & -t_0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ H & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ \frac{H^{n-2}}{(n-2)!} & \frac{H^{n-3}}{(n-3)!} & \cdots & 1 \end{pmatrix}. \]
Here \( t_j \) is the \( H^{j,j}(X) \)-component of \( \text{td}_X \), and we regard an element in \( H\Omega_0(X) \) as a column vector. The Gepner type property of the central charge \( Z_G \) is translated into the following linear equation on \( \alpha_i \):
\[ (I) \quad (\alpha_0, \cdots, \alpha_{n-2}) \cdot M = e^{2\pi \sqrt{-1}/d} \cdot (\alpha_0, \cdots, \alpha_{n-2}). \]
By Lemma 2.15, the solution space (57) must be one dimensional, so it determines the group homomorphism (56) uniquely up to a scalar multiplication.
In practice, it is more convenient to work with a smaller subspace in $H\Omega_0(X)$. Let $\mathbb{C}\langle H \rangle$ be the subspace in $H\Omega_0(X)$ defined by

$$\mathbb{C}\langle H \rangle := \bigoplus_{j=0}^{n-2} CH^j.$$ 

We have the following lemma:

**Lemma 4.1.** The solution space of (57) is contained in $\mathbb{C}\langle H \rangle$.

**Proof.** Let $\mathbb{C}\langle H \rangle^\perp$ be the orthogonal complement of $\mathbb{C}\langle H \rangle$ in $H\Omega_0(X)$ with respect to the Poincaré paring. We have the direct sum decomposition

$$H\Omega_0(X) = \mathbb{C}\langle H \rangle \oplus \mathbb{C}\langle H \rangle^\perp$$

and $F_*$ preserves the above direct summands. Since $\text{td}_X \in \mathbb{C}\langle H \rangle$, $F_*$ acts on $\mathbb{C}\langle H \rangle^\perp$ via multiplication by $e^H$, which is unipotent. Hence all the eigenvectors of the action $F_*$ on $\mathbb{C}\langle H \rangle^\perp$ have eigenvalue 1, which implies that the solution space of (57) is contained in $\mathbb{C}\langle H \rangle$. 

By the above lemma, it is enough to solve the equation (57) for $\alpha_j \in \mathbb{CH}^{n-2-j}$. The ambiguity of the scalar multiplication is fixed by the following lemma:

**Lemma 4.2.** We have the equality

$$\int_X \alpha_0 = \prod_{j=1}^{n} \left(1 - e^{-2a_j \pi \sqrt{-1/d}}\right).$$

**Proof.** By Lemma 3.12 the object $\Psi^L(\mathbb{C}(0))$ is isomorphic to $O_X[1]$. Since $\varepsilon = 0$, the functor $\Psi$ is an equivalence, hence $\Psi^L = \Psi^{-1}$. It follows that $\Psi(O_X)$ is isomorphic to $\mathbb{C}(0)[-1]$. Then the equality (58) follows by applying the homomorphism (56) to $O_X$, and using the computation in Example 2.8.

Now the $\alpha_j \in \mathbb{CH}^{n-2-j}$ are uniquely determined by the equation (57) and the normalization (58). However for our purpose, it is more convenient to consider a different normalization of $Z_G \circ \Psi$. Namely we write $Z_G \circ \Psi$ as a multiple of some non-zero complex number and a central charge on $\text{HMF}_{\text{gr}}(W)$ whose image of $\Psi(O_x)$ is $-1$. This is possible by the following lemma:

**Lemma 4.3.** For any $x \in X$, we have $Z_G(\Psi(O_x)) = -C_W$ where $C_W$ is given by

$$C_W := -(1 - e^{2\pi \sqrt{-1/d}})^{-1} \prod_{j=1}^{n} \left(1 - e^{-2a_j \pi \sqrt{-1/d}}\right).$$
which satisfies
\begin{equation}
C_W \in \mathbb{R}_{>0} e^{\sqrt{-1} \pi \theta_W}, \quad \theta_W = \frac{1}{2} (n-1) - \frac{1}{d} \left( \sum_{j=1}^{n} a_j + 1 \right).
\end{equation}

Proof. The equality (60) follows from Example 2.8 and Lemma 3.10. The property (60) follows from
\begin{equation*}
1 - e^{-2\pi \sqrt{-1} \theta} = 2 \sin \pi \theta \cdot e^{(\frac{1}{2} - \theta) \pi \sqrt{-1}}.
\end{equation*}

□

We summarize the result in this subsection as follows:

**Proposition 4.4.** Suppose that \( \varepsilon = 0 \). Then for \( E \in D^b \operatorname{Coh}(X) \), the central charge \( Z_G(\Psi(E)) \) is written as
\begin{equation}
Z_G(\Psi(E)) = C_W \sum_{j=0}^{n-2} \int_X \alpha_j^1 \cdot \operatorname{ch}_j(E)
\end{equation}
where \((\alpha_0^1, \ldots, \alpha_{n-2}^1)\) satisfies \( \alpha_j^1 \in \mathbb{C} H^{n-2-j} \), and it is the unique solution of the linear equation
\begin{equation}
(\alpha_0^1, \ldots, \alpha_{n-2}^1) \cdot M = e^{2\pi \sqrt{-1} / d} \cdot (\alpha_0^1, \ldots, \alpha_{n-2}^1), \quad \alpha_{n-2}^1 = -1.
\end{equation}

4.2. **Computation of the central charge** (\( \varepsilon < 0 \) case). The purpose of this subsection is to reduce the computation of \( Z_G \) in the case \( \varepsilon < 0 \) to that of the case \( \varepsilon = 0 \). The strategy is to embed \( X \) into \( n-2-\varepsilon \)-dimensional Calabi-Yau manifold \( \hat{X} \) and relate \( Z_G \) with the central charge on \( \hat{X} \). We set
\begin{equation*}
\hat{A} := \mathbb{C}[x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n-\varepsilon}]
\end{equation*}
and consider the element \( \hat{W} \in \hat{A} \) defined by
\begin{equation*}
\hat{W} := W + x_{n+1}^d + \cdots + x_{n-\varepsilon}^d.
\end{equation*}
Since we assume that \( X \) does not contain stacky points, the stack
\begin{equation*}
\hat{X} := (\hat{W} = 0) \subset \mathbb{P}(a_1, \ldots, a_n, 1, \ldots, 1)
\end{equation*}
also does not contain stacky points. The variety \( \hat{X} \) is a projective Calabi-Yau manifold with dimension \( n-2-\varepsilon \), which contains \( X \) as a zero locus \( x_{n+1} = \cdots = x_{n-\varepsilon} = 0 \).

Let \( \hat{R} \) be the graded ring \( \hat{A}/(\hat{W}) \). There is a natural push-forward functor
\begin{equation*}
i_* : D^{gr}_{sg}(R) \to D^{gr}_{sg}(\hat{R})
\end{equation*}
by regarding a graded \( R \) module as a graded \( \hat{R} \)-module via the surjection \( \hat{R} \twoheadrightarrow R \). (cf. [Ued].) Combined with the equivalence (11) and the
functor $\Psi = \Psi_1$ in (24), we obtain the diagram

\[
\begin{array}{c}
\text{HMF}^{\text{gr}}(W) & \xrightarrow{i_*} & \text{HMF}^{\text{gr}}(\hat{W}) \\
\uparrow \Psi & & \uparrow \hat{\Psi} \\
D^b\text{Coh}(X) & \xrightarrow{i_*} & D^b\text{Coh}(\hat{X}).
\end{array}
\]

Here $i: X \to \hat{X}$ is the inclusion and $\hat{\Psi}$ is the equivalence, obtained by applying the same construction of $\Psi$ to $\hat{W}$ and $\hat{X}$. We have the following lemma:

**Lemma 4.5.** The diagram (63) is commutative.

**Proof.** The result follows from the definitions of $\Psi$, $\hat{\Psi}$ and the adjunction for $E \in D^b\text{Coh}(X)$

\[
\bigoplus_{j \geq 1} R\text{Hom}_{\hat{X}}(O_{\hat{X}}(j), i_*E) \cong \bigoplus_{j \geq 1} R\text{Hom}_{X}(O_X(j), E).
\]

\[
\square
\]

The top arrow $i_*$ of the diagram (63) obviously commutes with grade shift functors on both sides. Also by the functoriality of the Hochschild homologies, we have the push-forward functor $i_*: \text{HH}_*(W) \to \text{HH}_*(\hat{W})$ which preserves the one dimensional eigenspaces in Lemma 2.15 on both sides. By Remark 2.16, the composition

\[
\hat{Z}_G \circ i_*: \text{K}(\text{HMF}^{\text{gr}}(W)) \to \text{K}(\text{HMF}^{\text{gr}}(\hat{W})) \to \mathbb{C}
\]

differs from $Z_G$ by a scalar constant, where $\hat{Z}_G$ is the central charge (12) on $\text{HMF}^{\text{gr}}(\hat{W})$ applied for $\hat{W}$. Since $i_*\mathbb{C}(0) = \mathbb{C}(0)$, it follows that

\[
Z_G(P) = (1 - e^{2\pi \sqrt{-1}/d})^\varepsilon \hat{Z}_G(i_*P)
\]

for any $P \in \text{HMF}^{\text{gr}}(W)$ by comparing $Z_G(\mathbb{C}(0))$ and $\hat{Z}_G(\mathbb{C}(0))$ given in Example 2.8. In particular, using the diagram (63), it follows that

\[
Z_G(\Psi(O_x)) = (1 - e^{2\pi \sqrt{-1}/d})^{-1} \prod_{j=1}^{n} \left(1 - e^{2\pi \sqrt{-1}/d}\right)
\]

\[
=: -C_W
\]

where $C_W$ coincides with the one defined in (59). As a summary, we have the following:

**Proposition 4.6.** Suppose that $\varepsilon < 0$. For $E \in D^b\text{Coh}(X)$, the central charge $Z_G(\Psi(E))$ is written as

\[
Z_G(\Psi(E)) = C_W \sum_{j=0}^{n-2-\varepsilon} \int_{\hat{X}} \hat{\alpha}_j^1 \cdot \text{ch}_j(i_*E)
\]
where \((\hat{a}^1_0, \cdots, \hat{a}^1_{n-2-\varepsilon})\) satisfies
\[
\hat{a}^1_j \in \mathbb{C}\hat{H}^{n-2-\varepsilon-j}, \quad \hat{H} := c_1(\mathcal{O}_{\hat{X}}(1))
\]
and it is the unique solution of the equation \((62)\) for \(\hat{X}\).

Note that \(\hat{a}^1_j\) is computed by the argument in the previous subsection, since \(\hat{X}\) is Calabi-Yau. Later we will use the following data:
\[
Z_{\mathcal{G}}(\mathcal{O}_x) = -C_w \\
\in \mathbb{R}_{>0}e^{\sqrt{-1\pi}(\theta_w+1)} \\
Z_{\mathcal{G}}(\mathcal{C}(j)) = C_w e^{2\pi j\sqrt{-1/d}} \left(1 - e^{2\pi\sqrt{-1/d}}\right) \\
\in \mathbb{R}_{>0}e^{\sqrt{-1\pi}(\theta_w+\frac{1}{2}+\frac{1}{d}+\frac{1}{d})}.
\]

Here \(x \in X\) and \(\theta_w \in \mathbb{Q}\) is defined by \((60)\). The relation \((65)\) is a consequence of the above arguments and the computation in Example \(2.8\).

4.3. A recipe constructing Gepner type stability conditions. In this subsection, we explain how desired Gepner type stability conditions are constructed. We divide the construction into 3-steps: construction of a slope stability on \(\mathcal{A}_W\), construction of \(\sigma_G\) via tilting of \(\mathcal{A}_W\), and checking the Gepner type property of \(\sigma_G\). In the next section, we will apply the recipe here in the case of \(n-4 \leq \varepsilon \leq 0\).

Step 1.

Our first step is to construct an analogue of a slope stability on \(\mathcal{A}_W\). This is a map
\[
\mu : \mathcal{A}_W \to \mathbb{R} \cup \{\pm \infty\}
\]
satisfying the weak seesaw property: for any exact sequence \(0 \to F \to E \to G \to 0\) in \(\mathcal{A}_W\), we have either
\[
\mu(F) \leq \mu(E) \leq \mu(G) \quad \text{or} \quad \mu(F) \geq \mu(E) \geq \mu(G).
\]
The above slope function defines the \(\mu\)-stability in \(\mathcal{A}_W\):

**Definition 4.7.** An object \(E \in \mathcal{A}_W\) is \(\mu\)-(semi)stable if for any exact sequence \(0 \to F \to E \to G \to 0\) in \(\mathcal{A}_W\), we have the inequality
\[
\mu(F) < (\leq) \mu(G).
\]

We require that \(\mu\)-stability satisfies the Harder-Narasimhan property, i.e. for any \(E \in \mathcal{A}_W\), there is a filtration in \(\mathcal{A}_W\)
\[
0 = E_0 \subset E_1 \subset \cdots \subset E_N = E
\]
such that each subquotient \(F_i = E_i / E_{i-1}\) is \(\mu\)-semistable with \(\mu(F_i) > \mu(F_{i+1})\) for all \(i\).

Step 2.
Suppose that there is a slope function $\mu$ as above. We define a pair of full subcategories $(T_\mu, F_\mu)$ by
\[
T_\mu := \{ E \in A_W : E \text{ is } \mu\text{-semistable with } \mu(E) > 0 \}_{\text{ex}}
\]
\[
F_\mu := \{ E \in A_W : E \text{ is } \mu\text{-semistable with } \mu(E) \leq 0 \}_{\text{ex}}.
\]
The existence of Harder-Narasimhan filtrations in $\mu$-stability implies that $(T_\mu, F_\mu)$ is a torsion pair on $A_W$. (cf. [HRS96].) We define $\mathcal{A}_G$ to be the associated tilting:
\[
\mathcal{A}_G := \langle F_\mu, T_\mu[-1] \rangle_{\text{ex}} \subset \text{HMF}^\text{gr}(W).
\]
The category $\mathcal{A}_G$ is the heart of a bounded t-structure on $\text{HMF}^\text{gr}(W)$. We try to construct a desired stability condition from the heart $\mathcal{A}_G$, by the following:

**Definition 4.8.** We say that a triple
\[
(\mathcal{Z}_G, \mathcal{A}_G, \theta), \quad \theta \in \mathbb{R}
\]
determines a stability condition if the following condition holds:

- For any $0 \neq E \in \mathcal{A}_G$, we have
\[
\mathcal{Z}_G(E) \in \{ re^{-\sqrt{r^2-\pi\phi}} : r > 0, \phi \in (\theta, \theta + 1) \}.
\]

- Any object in $A_W$ admits a Harder-Narasimhan filtration with respect to $\mathcal{Z}_G$-stability.

Here $\mathcal{Z}_G$-stability and its Harder-Narasimhan filtrations are defined by the same way as in the $\mu$-stability, by replacing $\mu$ by $\arg \mathcal{Z}_G(\ast) \in (\theta, \theta+1]$. If the triple (69) determines a stability condition, it associates a pair
\[
\sigma_G = (\mathcal{Z}_G, \{ \mathcal{P}_G(\phi) \}_{\phi \in \mathbb{R}}), \quad \mathcal{P}_G(\phi) \subset \text{HMF}^\text{gr}(W)
\]
in the following way: we define $\mathcal{P}_G(\phi)$ for $\phi \in (\theta, \theta + 1]$ to be
\[
\mathcal{P}_G(\phi) = \{ E \in A_W : \mathcal{Z}_G\text{-semistable with } \mathcal{Z}_G(E) \in \mathbb{R}_{>0} e^{-\sqrt{r^2-\pi\phi}} \} \cup \{0\}
\]
and other $\mathcal{P}_G(\phi)$ are determined by the rule
\[
\mathcal{P}_G(\phi + 1) = \mathcal{P}_G(\phi)[1].
\]
If $\theta = 0$, the above construction is nothing but the one given in [Bri07, Proposition 5.3], and the same argument applies to show that (71) is a stability condition. Below for an interval $I \subset \mathbb{R}$, we set
\[
\mathcal{P}_G(I) := \langle \mathcal{P}_G(\phi) : \phi \in I \rangle_{\text{ex}}.
\]
Note that $\mathcal{P}_G((\theta, \theta + 1])$ coincides with $\mathcal{A}_G$ by our construction. We require the local finiteness of our stability condition, i.e. for any $\phi \in \mathbb{R}$, the quasi-abelian category $\mathcal{P}_G((\phi - \delta, \phi + \delta))$ is noetherian and artinian for $0 < \delta \ll 1$. (cf. [Bri07, Definition 5.7].) It in particular implies that any object $E \in \mathcal{P}_G(\phi)$ admits a Jordan-Hölder filtration.
Remark 4.9. The local finiteness condition holds if the image of $Z_G$ is discrete. By Remark 2.10, this is always satisfied in the cases studied in the next section, so we will not take care of the local finiteness.

Step 3.

In this step, we assume that the triple (69) determines a stability condition $\sigma_G$. We expect that $\sigma_G$ is a Gepner type stability condition with respect to $(\tau, 2/d)$. To show this, we consider the following stability condition

$$\tau^{-1} \sigma_G \left( \frac{2}{d} \right) = (Z_G, \{ P'_G(\phi) \}_{\phi \in \mathbb{R}})$$

where $P'_G(\phi)$ is given by

$$P'_G(\phi) = \tau^{-1} P_G \left( \phi + \frac{2}{d} \right).$$

It is enough to show that (72) coincides with $\sigma_G$. This is equivalent to that $P_G((\theta, \theta + 1]) = P_G((\theta, \theta + 1])$, or equivalently

$$\tau(A_G) = P_G \left( \left( \theta + \frac{2}{d}, \theta + \frac{2}{d} + 1 \right) \right).$$

We show the equality (73) by investigating $\sigma_G$-stability of simple objects in $A_W$. When $n = 2$ we have the following lemma:

Lemma 4.10. Suppose that $n = 2$ and the following inequality holds:

$$\theta_W - \frac{1}{d} - \frac{2\varepsilon}{d} - \frac{1}{2} \leq \theta < \theta_W + 1.$$  

If $\tau(\mathcal{O}_x)$, $\mathbb{C}(1)$, $\cdots$, $\mathbb{C}(\varepsilon)$ are $\sigma_G$-semistable for all $x \in X$, then the equality (73) holds.

Proof. Let $\phi_x$ be the phase of $\tau(\mathcal{O}_x)$ for $x \in X$ and $\phi_j$ the phase of $\mathbb{C}(j)$ for $1 \leq j \leq -\varepsilon$. Since $\tau(\mathcal{O}_x)$ and $\mathbb{C}(j)$ for $1 \leq j \leq -1 - \varepsilon$ are objects in $A_W$, and $A_G$ is obtained as a tilting of $A_W$, the phases $\phi_x$, $\phi_j$ are contained in $(\theta, \theta + 2]$ for $1 \leq j \leq -1 - \varepsilon$. On the other hand, the condition (74) implies that

$$\theta < \theta_W + 1 < \theta_W + 1 + \frac{2}{d} < \theta_W + \frac{1}{d} + \frac{3}{2} < \cdots$$

$$\cdots < \theta_W + \frac{1}{d} + \frac{2(-1 - \varepsilon)}{d} + \frac{3}{2} \leq \theta + 2.$$

By comparing (75) with (65), we obtain

$$\phi_x = \theta_W + 1 + \frac{2}{d}, \quad \phi_j = \theta_W + \frac{1}{d} + \frac{2j}{d} + \frac{3}{2}.$$
for $1 \leq j \leq -1 - \varepsilon$. We show that (76) also holds for $j = -\varepsilon$. By Lemma 3.11, we have $\mathbb{C}(-\varepsilon)[-1] \in \mathcal{A}_W$. Therefore if $\mathbb{C}(-\varepsilon)$ is $\sigma_G$-stable, then we have either

(77) \[ \mathbb{C}(-\varepsilon) \in \mathcal{A}_G[1], \quad \phi_{-\varepsilon} \in (\theta + 1, \theta + 2] \] or

(78) \[ \mathbb{C}(-\varepsilon) \in \mathcal{A}_G[2], \quad \phi_{-\varepsilon} \in (\theta + 2, \theta + 3]. \]

In the case of (77), the equality (76) also holds for $j = -\varepsilon$ by the inequalities (75). In the case of (78), we need to exclude the case of

\[ \phi_{-\varepsilon} = \phi_{-1-\varepsilon} + 2 + \frac{2}{d}. \]

If this happens, then $\phi_{-1-\varepsilon} < \theta + 1$, and Lemma 3.11 and (75) imply that

\[ Z_G(\mathbb{C}(-\varepsilon)[-1]) \in \{ \mathbb{R}_{>0}e^{-1}\pi\phi : \phi \in (\theta, \theta + 1) \}. \]

This contradicts to that $\phi_{-\varepsilon} \in (\theta + 2, \theta + 3]$, hence (76) also holds for $j = -\varepsilon$.

Note that $\tau(\mathcal{A}_W)$ is generated by $\tau\Psi(\mathcal{O}_x)$ for all $x \in X$ and $\mathbb{C}(j)$ for $1 \leq j \leq -\varepsilon$, whose phases are given by (76). By the inequality (74), we have

\[ \theta + \frac{2}{d} \geq \theta_W + \frac{1}{d} - \frac{2\varepsilon}{d} + \frac{3}{2} - j = \phi_{-\varepsilon} - j \]

for $j \geq 2$. Noting that there is no non-trivial homomorphism from $\mathcal{P}(\phi)$ to $\mathcal{P}(\phi')$ if $\phi > \phi'$, it follows that

(79) \[ \text{Hom}^{<-1} \left( \mathcal{P}_G \left( \left( \theta + \frac{2}{d}, \theta + \frac{2}{d} + 1 \right) \right), \tau(E) \right) = 0 \]

for any $E \in \mathcal{A}_W$. Similarly the inequality (74) implies

\[ \phi_x = \theta_W + 1 + \frac{2}{d} > \theta + \frac{2}{d} + 1 - j \]

for $j \geq 1$. It follows that

(80) \[ \text{Hom}^{<-1} \left( \tau(E), \mathcal{P}_G \left( \left( \theta + \frac{2}{d}, \theta + \frac{2}{d} + 1 \right) \right) \right) = 0 \]

for any $E \in \mathcal{A}_W$. The above vanishing (79), (80) imply that the RHS of (73) is obtained as a tilting of $\tau(\mathcal{A}_W)$. Hence the result follows from Lemma 4.11 below:

\[ \square \]

We have used the following lemma:

Lemma 4.11. Let $\mathcal{D}$ be a triangulated category, $\mathcal{A} \subset \mathcal{D}$ the heart of a bounded t-structure on $\mathcal{D}$, and $Z: K(\mathcal{D}) \to \mathbb{C}$ a group homomorphism. Suppose that there are torsion pairs $(\mathcal{T}_k, \mathcal{F}_k)$, $k = 1, 2$ on $\mathcal{A}$ such that, for $\mathcal{B}_k = \langle \mathcal{F}_k, \mathcal{T}_k[-1] \rangle_{ex}$ the associated tilting, both of the triples

(81) \[ (Z, \mathcal{B}_1, \theta), \quad (Z, \mathcal{B}_2, \theta) \]

determine stability conditions. Then $\mathcal{B}_1 = \mathcal{B}_2$. 

Proof. We may assume that $\theta = 0$. Let us take an object $E \in \mathcal{T}_1$. Since $(\mathcal{T}_2, \mathcal{F}_2)$ is a torsion pair, there is an exact sequence in $\mathcal{A}$

$$0 \to F \to E \to G \to 0$$

such that $F \in \mathcal{T}_2$ and $G \in \mathcal{F}_2$. Also since $(\mathcal{T}_1, \mathcal{F}_1)$ is a torsion pair, objects in $\mathcal{T}_1$ are closed under quotients, hence $G \in \mathcal{T}_1$. Suppose that $G \neq 0$. Then since (81) determine stability conditions for $\theta = 0$, it follows that $\text{Im} \ Z(G) = 0$. Hence the condition $G \in \mathcal{T}_2$ implies $\text{Re} \ Z(G) \in \mathbb{R}_{<0}$ but the condition $G \in \mathcal{T}_1$ implies $\text{Re} \ Z(G) \in \mathbb{R}_{>0}$, which is a contradiction. Therefore $G = 0$, and $\mathcal{T}_1 \subset \mathcal{T}_2$ follows. Similarly $\mathcal{T}_2 \subset \mathcal{T}_1$ also holds, hence $\mathcal{T}_1 = \mathcal{T}_2$ holds. Because $\mathcal{F}_k$ is an orthogonal complement of $\mathcal{T}_k$ in $\mathcal{A}$, we have $\mathcal{F}_1 = \mathcal{F}_2$, hence $\mathcal{B}_1 = \mathcal{B}_2$ holds. \hfill $\square$

Next we discuss the case of $n = 3$. In this case, we need to add an extra check of $\sigma_G$-stability.

**Lemma 4.12.** Suppose that $n = 3$ and the following inequality holds:

$$\theta_W - \frac{1}{d} - \frac{2\varepsilon}{d} - \frac{1}{2} \leq \theta \leq \theta_W. \tag{82}$$

Suppose furthermore that $\tau \Psi(\mathcal{O}_x)$ is $\sigma_G$-stable, $\mathcal{C}(1), \cdots, \mathcal{C}(-\varepsilon)$ are $\sigma_G$-semistable and the following holds for all $x \in X$:

$$\tau^{1-\varepsilon} \Psi(\mathcal{O}_x) \in \mathcal{P}_G \left( 1 + \theta_W + \frac{2(1-\varepsilon)}{d} \right). \tag{83}$$

Then the equality (73) holds.

**Proof.** By following the proof of Lemma 4.10, we have the vanishing (79), (80) for $E = \Psi(\mathcal{O}_x)$, $\mathcal{C}(0), \cdots, \mathcal{C}(-1-\varepsilon)$ with $x \in X$. Below we show the above vanishing also holds for $E = \Psi(F)$ for any $F \in \text{Coh}(X)$. Then the RHS of (73) is shown to be a tilting of $\tau(\mathcal{A}_W)$, hence Lemma 4.11 is applied to give the result.

Let us take an object

$$A \in \tau^{-1} \mathcal{P}_G \left( \left( \theta + \frac{2}{d}, \theta + \frac{2}{d} + 1 \right) \right) \tag{84}$$

and let

$$\Psi^R \colon \text{HMF}^\mathbb{R}(W) \to D^b \text{Coh}(X)$$

be the right adjoint functor of $\Psi$. We claim that $\Psi^R(A) \in D^b \text{Coh}(X)$ satisfies

$$\mathcal{H}^j(\Psi^R(A)) = 0, \quad \text{for } j \neq 0, 1. \tag{85}$$

The above property will be proved in Sublemma 4.13 below, and we continue the proof assuming this fact. Let us take the distinguished triangle in $\text{HMF}^\mathbb{R}(W)$

$$\Psi \Psi^R(A) \to A \to B \tag{86}$$
where $B$ satisfies

$$B \in (C(-1 - \varepsilon), \cdots , C(0)).$$

For a coherent sheaf $F$ on $X$, we apply $\text{Hom}(\Psi(F), *)$ to the distinguished triangle (86). Since $B$ is right orthogonal to $\Psi D^b \text{Coh}(X)$, and $\Psi^R(A)$ satisfies the condition (85), we have

$$\text{Hom}^{<0}(\tau \Psi(F), \tau(A)) \cong \text{Hom}^{<0}(F, \Psi^R(A))$$

$$\cong 0,$$

which proves the vanishing (80) for $E = \Psi(F)$.

Similarly applying $\text{Hom}(\ast, \Psi(F))$ to the triangle (86), and noting that

$$\text{Hom}^{-1}(\Psi^R(A), \Psi(F)) \cong \text{Hom}^{-1}(\Psi^R(A), F)$$

$$\cong 0,$$

by the property (85), we see that the vanishing (79) for $E = \Psi(F)$ is equivalent to

(87)

$$\text{Hom}^{-1}(B, \Psi(F)) \cong 0.$$

To show (87), note that the vanishing (79) for $E = \mathbb{C}(j), 0 \leq j \leq -1 - \varepsilon$ and the triangle (86) imply

$$\text{Hom}^{-1}(B, \mathbb{C}(j)) \cong 0, \quad j = 0, \cdots , -1 - \varepsilon.$$

Therefore, if we denote by $H^i_A \mathcal{W}(B) \in \mathcal{A}_W$ the $i$-th cohomology with respect to the t-structure on $\text{HMF}^{\mathcal{W}}(W)$ with heart $\mathcal{A}_W$, then we have $H^i_A \mathcal{W}(B) = 0$ for $i > 1$. Since $\Psi(F) \in \mathcal{A}_W$, this implies that (87) holds. Therefore the vanishing (79) for $E = \Psi(F)$ holds.

We have used the following sublemma:

**Sublemma 4.13.** The condition (85) holds.

**Proof.** We investigate the vanishing

(88)

$$\text{Hom}^j(\Psi^R(A), \mathcal{O}_x) = 0$$

for $x \in X$ and $j \in \mathbb{Z}$. By the Serre duality on $X$, adjunction and applying $\tau$, the vanishing (88) is equivalent to

(89)

$$\text{Hom}(\tau \Psi(\mathcal{O}_x)[j - 1], \tau(A)) = 0.$$

We first show that (89) holds for any $j < -1$ and $x \in X$. Applying the Serre functor $S_W = \tau^{-\varepsilon}[1]$ on $\text{HMF}^{\mathcal{W}}(W)$, the vanishing (88) is equivalent to

(90)

$$\text{Hom}(\tau(A)[-j], \tau^{1-\varepsilon}\Psi(\mathcal{O}_x)) = 0.$$

On the other hand, by the assumption (82), we have the inequality

$$\theta + \frac{2}{d} - j \geq 1 + \theta_W + \frac{2(1 - \varepsilon)}{d}.$$
for \( j < -1 \). Therefore by our assumptions (83) and (84), the vanishing (90) holds for \( j < -1 \).

Next, we have the inequality
\[
\theta_W + 1 + \frac{2}{d} + j - 1 > \theta + \frac{2}{d} + 1
\]
for \( j \geq 2 \) by (82), hence the vanishing (89) holds for \( j \geq 2 \). Moreover the above inequality, hence the vanishing (89), also holds for \( j = 1 \) unless \( \theta = \theta_W \) holds. Suppose that \( \theta = \theta_W \) holds, and let \( P \) be the \( \sigma_G \)-Harder-Narasimhan factor of \( \tau(A) \) with the maximum phase. Note that \( \tau \Psi(O_x) \) is \( \sigma_G \)-stable with phase \( \theta_W + 1 + 2/d \), which is bigger than or equal to the phase of \( P \). Therefore the vanishing of (89) for \( j = 1 \) does not hold only if we have
\[
P \in \mathcal{P}_G \left( \theta_W + 1 + \frac{2}{d} \right)
\]
and \( \tau \Psi(O_x) \) is one of the Jordan-Hölder factors of \( P \). It follows that, by taking the Jordan-Hölder filtration of \( P \), there is a distinguished triangle
\[
(91) \quad \tau \Psi(Q) \to \tau(A) \to \tau(A')
\]
where \( Q \) is a zero dimensional coherent sheaf on \( X \), \( A' \) is an object in the RHS of (84), such that the vanishing (89) holds for \( j \geq 1 \) after replacing \( A \) by \( A' \).

Applying \( \Psi^R \circ \tau^{-1} \) to (91), we have the distinguished triangle
\[
Q \to \Psi^R(A) \to \Psi^R(A').
\]

The above argument shows that, after replacing \( A \) by \( A' \), the vanishing (88) holds unless \( j = -1, 0 \). It follows that \( \Psi^R(A') \) is a two term complex of vector bundles on \( X \), whose cohomologies are concentrated on degrees 0 and 1. Since \( Q \in \text{Coh}(X) \), we conclude that \( H^j(\Psi^R(A)) = 0 \) for \( j \neq 0, 1 \).

By the above results, the problem is reduced to showing \( \sigma_G \)-stability of some objects in \( \text{HMF}^G(W) \). The following lemma is useful in checking the \( \sigma_G \)-stability of these objects. The proof is obvious, and we omit

**Lemma 4.14.** Let \( \mathcal{D} \) be a triangulated category, \( Z: K(\mathcal{D}) \to \mathbb{C} \) a group homomorphism and \( \mathcal{A} \subset \mathcal{D} \) the heart of a bounded t-structure on \( \mathcal{D} \). Suppose that the triple \( (Z, \mathcal{A}, \theta = 0) \) determines a stability condition on \( \mathcal{D} \), and the following condition holds:
\[
c_{\min} := \inf \{ \text{Im} Z(E) > 0 : E \in \mathcal{A} \} > 0.
\]
Then an object \( E \in \mathcal{A} \) with \( \text{Im} Z(E) = c_{\min} \) is \( \sigma_G \)-stable if and only if \( \text{Hom}(P, E) = 0 \) for any \( P \in \mathcal{A} \) with \( \text{Im} Z(P) = 0 \).
5. Proof of Theorem 1.3

In this section, we apply the strategy in the previous section and prove Theorem 1.3. Below we use the same notation in the previous section. In particular the constants $C_W \in \mathbb{C}^*$ and $\theta_W \in \mathbb{Q}$ are defined as in Lemma 4.3 both in $\varepsilon = 0$ and $\varepsilon < 0$ case. The goal is to prove Conjecture 2.9 when $n - 4 \leq \varepsilon \leq 0$ and $X$ does not contain stacky points. Since we already discussed the case with $n = 1$ and $n = 2$, $\varepsilon = 0$, the following five possibilities are left:

$$(n, \varepsilon) = (3, 0), (2, -1), (4, 0), (3, -1), (2, -2).$$

We divide the proof into the five subsections, so that each subsection corresponds to one of the above types. We repeat similar arguments in these subsections, so we recommend the readers to follow only one or two cases, e.g. $(n, \varepsilon) = (4, 0)$ or $(3, -1)$, at the first reading of this paper.

5.1. The case of $n = 3$, $\varepsilon = 0$. In this subsection, we study the case of $n = 3$ and $\varepsilon = 0$. In this case, $X$ is a smooth elliptic curve, and the heart $A_W$ is given by

$$A_W = \Psi \text{Coh}(X).$$

Furthermore possible data $(a_1, a_2, a_3, d, \int_X H)$ are classified into the three types [Sai87, Table 2]

$$(92) \quad \left( a_1, a_2, a_3, d, \int_X H \right) = \begin{cases} (1, 1, 1, 3, 3) \\ (2, 1, 1, 4, 2) \\ (3, 2, 1, 6, 1). \end{cases}$$

The central charge $Z_G$ is described as follows:

**Lemma 5.1.** Suppose that $n = 3$ and $\varepsilon = 0$. For any $E \in D^b \text{Coh}(X)$, we have

$$(93) \quad Z_G(\Psi(E)) = C_W \left\{ -d(E) + r(E) \left( \cos \frac{2\pi}{d} - 1 \right) + r(E) \sin \frac{2\pi}{d} \sqrt{-1} \right\}. $$

Here we set $(r(E), d(E)) = (\text{rank}(E), \text{deg}(E))$.

**Proof.** The equation (62) becomes

$$(\alpha_0^\dagger, -1) \left( 1 - e^{2\pi \sqrt{-1}/d} \int_X H - \int_X H \frac{-1}{1 - e^{2\pi \sqrt{-1}/d}} \right) = 0,$$

giving $\int_X \alpha_0^\dagger = e^{2\pi \sqrt{-1}/d} - 1$. \qed
We set the slope function $\mu$ in (66) to be the constant function $\mu = -1$, so that the resulting heart $A_G$ in (68) coincides with $A_W$. We have the following result:

**Proposition 5.2.** Suppose that $n = 3$ and $\varepsilon = 0$. Then the triple

$$ (Z_G, A_W, \theta = \theta_W) $$

determines a Gepner type stability condition $\sigma_G$ on $\text{HMF}^R(W)$ with respect to $(\tau, 2/d)$.

**Proof.** Note that, since $X$ is an elliptic curve, the space of stability conditions on $D^b\text{Coh}(X)$ is completely described in [Bri07, Section 9]. Since $A_W = \Psi \text{Coh}(X)$, and the central charge $Z_G$ is given by (93), it follows that the triple (94) satisfies the condition (70). Then the same argument of [Bri07, Example 5.4] shows that the triple (94) determines a stability condition $\sigma_G$ on $\text{HMF}^R(W)$. Now we are going to apply Lemma 4.12. Note that the inequality (82) is satisfied in this case. For a point $x \in X$, we have

$$ \tau \Psi(O_x) \cong \Psi(O_X(-x)[1]) $$

by Proposition 3.2. Since the above object is $\sigma_G$-stable with phase $1 + \theta_W + 2/d$, Lemma 4.12 implies that $\sigma_G$ is a Gepner type stability condition with respect to $(\tau, 2/d)$. \qed

### 5.2. The case of $n = 2$, $\varepsilon = -1$.

In this subsection, we study the case of $n = 2$, $\varepsilon = -1$. In this case, $X$ is a finite number of smooth points, and the heart $A_W$ is given by

$$ A_W = \langle \mathbb{C}(0), \Psi(O_x) : x \in X \rangle_{\text{ex}}. $$

The following lemma immediately follows from (65):

**Lemma 5.3.** If we write the K-theory class of an object $E \in A_W$ as

$$ [E] = v_0[\mathbb{C}(0)] + \sum_{x \in X} w_x[\Psi(O_x)] $$

and set $w := \sum_{x \in X} w_x$, then $Z_G(E)$ is given by

$$ Z_G(E) = C_W \left\{ -w + v_0 \left( 1 - \cos \frac{2\pi}{d} \right) - v_0 \sin \frac{2\pi}{d} \sqrt{-1} \right\}. $$

In the notation used in Subsection 4.2, the polynomial $\hat{W}$ is of type $(a_1, a_2, 1, d)$, which must belong to one of the classifications in (92). Furthermore, applying Lemma 3.17 and Lemma 3.18 we see that $A_W$ is equivalent to the abelian category of representations of a certain quiver $Q$. By the above arguments, the possible types of $(a_1, a_2, d, \sharp X, Q)$ are
classified as follows:

$$\begin{align*}
(a_1, a_2, d, \#X) = (1, 1, 3, 3), & \quad Q = \bullet \\
(a_1, a_2, d, \#X) = (2, 1, 4, 2), & \quad Q = \\
(a_1, a_2, d, \#X) = (3, 2, 6, 1), & \quad Q = \bullet
\end{align*}$$

Here the left vertex of the quiver $Q$ corresponds to $C(0)$, and the right vertices correspond to $\Psi(O_x)$ for $x \in X$. In all the above cases, we set $\mu = -1$ so that the heart (68) coincides with $A_W$. We have the following result:

**Proposition 5.4.** Suppose that $n = 2$ and $\varepsilon = -1$. Then the triple

$$Z_G, A_W, \theta$$

(95)

$$\frac{5}{6} + \theta_W \leq \theta < 1 + \theta_W$$

determines a Gepner type stability condition $\sigma_G$ on $\text{HMF}^{\text{gr}}(W)$ with respect to $(\tau, 2/d)$.

**Proof.** By Lemma 5.3, it follows that the triple (95) satisfies the condition (70). Since $A_W$ is noetherian and artinian, the triple (95) satisfies the Harder-Narasimhan property (cf. [Bri07, Proposition 2.4]) hence it determines a stability condition $\sigma_G$ on $\text{HMF}^{\text{gr}}(W)$. In order to show the Gepner type property of $\sigma_G$, it is enough to check the assumption of Lemma 4.10. Since the inequality (74) is satisfied, it remains to show the $\sigma_G$-stability of $\tau \Psi(O_x)$ and $C(1)$ for all $x \in X$, which we prove in the next lemma.

**Lemma 5.5.** The objects $\tau \Psi(O_x)$ and $C(1)$ are $\sigma_G$-stable.

**Proof.** As for $\tau \Psi(O_x)$, it fits into a unique non-trivial extension in $A_W$ by Lemma 3.10 and Remark 3.13

$$0 \rightarrow \Psi(O_x) \rightarrow \tau \Psi(O_x) \rightarrow C(0) \rightarrow 0.$$ 

Therefore $\Psi(O_x)$ is the only non-trivial subobject of $\tau \Psi(O_x)$ in $A_W$. Comparing the argument of $Z_G(*)$, we see that $\tau \Psi(O_x)$ is $Z_G$-stable.

As for $C(1)$, note that the object $C(1)[-1]$ is contained in $A_W$ by Lemma 3.11. Let $0 \neq F \in A_W$ be a proper subobject of $C(1)[-1]$ in
As in Corollary 3.20, the inclusion \( F \subset \mathbb{C}(1)[-1] \) is represented by the following inclusion of quiver representations:

\[
F = V^{(0)} \subset \mathbb{C}(1)[-1] = W^{(1)} \]

Here \( R_1 \) is the space of degree one elements in \( R = \mathbb{C}[x_1, x_2] \), whose dimension is \( \sharp X - 1 \). The morphisms \( \pi^{(i)} : R_1 \to \mathbb{C} \) are evaluations at the points in \( X \), and \( V^{(0)} , W^{(i)} \) are finite dimensional vector spaces with \( V^{(0)} \subset R_1 \), \( \dim W^{(i)} \leq 1 \). Let \( I \) be the subset of \( i \in \{1, \cdots, \sharp X\} \) satisfying \( W^{(i)} = 0 \). Then we have

\[
V^{(0)} \subset \bigcap_{i \in I} \ker(\pi^{(i)})
\]

hence \( \dim V^{(0)} \leq \sharp X - 1 - |I| \). Since we have

\[
Z_G(\mathbb{C}(1)[-1]) = C_W \left\{ -\sharp X + (\sharp X - 1) \cdot (1 - e^{2\pi \sqrt{-1}/d}) \right\}
\]

\[
Z_G(F) = C_W \left\{ |I| - \sharp X + \dim V^{(0)} \cdot (1 - e^{2\pi \sqrt{-1}/d}) \right\}
\]

it follows that the argument of \( Z_G(F) \) is less than that of \( Z_G(\mathbb{C}(1)[-1]) \) in \( (\pi\theta, \pi\theta + \pi) \). Therefore \( \mathbb{C}(1)[-1] \) is \( \sigma_G \)-stable.

5.3. The case of \( n = 4, \varepsilon = 0 \). In this subsection, we study the case of \( (n, \varepsilon) = (4, 0) \). In this case, \( X \) is a smooth projective K3 surface, and the heart \( \mathcal{A}_W \) is given by

\[
\mathcal{A}_W = \Psi \text{Coh}(X).
\]

By Theorem 3.1, the triangulated category \( \text{HMF}^{gr}(W) \) is equivalent to \( D^b \text{Coh}(X) \) via the equivalence \( \Psi \). On the other hand, the spaces of stability conditions on K3 surfaces are studied by Bridgeland [Bri08]. Combining the techniques in [Bri08] with the arguments in Subsection 4.3, we construct a desired Gepner type stability condition on \( \text{HMF}^{gr}(W) \).

Let us first describe the matrix \( M \) which appears in the equation (57). By writing \( \int_X H^2 = 2m \) for \( m \in \mathbb{Z}_{\geq 1} \), the matrix \( M \) is given by

\[
M = \begin{pmatrix}
-1 - m & -H & -1 \\
H & 1 & 0 \\
m & H & 1
\end{pmatrix}.
\]

By using the above description, we give a classification of possible types.
Lemma 5.6. If \( n = 4 \), the possible data \((a_1, a_2, a_3, a_4, d, \int_X H^2)\) is classified into the two types:

\[
(97) \quad (a_1, a_2, a_3, a_4, d, \int_X H^2) = \begin{cases} 
(1, 1, 1, 1, 4, 4) \\
(3, 1, 1, 1, 6, 2).
\end{cases}
\]

Proof. By (96), we have

\[
\det(M - \lambda \cdot \text{id}) = -(\lambda + 1)\{\lambda^2 + (m - 2)\lambda + 1\}.
\]

Since \( \det(M - \lambda \cdot \text{id}) = 0 \) for \( \lambda = e^{\pm 2\pi \sqrt{-1}/d} \), we have

\[
2 \cos(2\pi/d) = m - 2.
\]

This is only possible for \((m, d) = (2, 4), (1, 6)\). In the latter case, the possibility \((a_1, a_2, a_3, a_4) = (2, 2, 1, 1)\) is excluded since the stack \(X\) always contains a stacky point. Therefore we obtain the classification (97).

\[\square\]

Remark 5.7. In the type \((1, 1, 1, 1, 4, 4)\) case, \(X\) is a quartic K3 surface. In the type \((3, 1, 1, 1, 6, 2)\) case, \(X\) is a double cover of \(\mathbb{P}^2\).

In order to describe the central charge \(Z_G\), it is convenient to use the twisted Mukai vector. For \(B \in H^2(X, \mathbb{R})\), we set

\[
(98) \quad v^B(E) := e^{-B \cdot \text{ch}(E)} \sqrt{\text{td}_X}.
\]

We denote by \(v^B_i(E)\) the \(H^{i,i}(X)\)-component of \(v^B(E)\). The central charge \(Z_G\) is computed in the following way:

Lemma 5.8. By setting \(B = -H/2\), the following holds for \(E \in D^b \text{Coh}(X)\):

\[
(99) \quad Z_G(\Psi(E)) = C_W \left(-v^B_2(E) + \frac{d}{8}v^B_0(E) + \frac{1}{2}\sqrt{\frac{d}{H^2}}H \cdot v^B_1(E)\sqrt{-1} \right).
\]

Proof. By (96), the equation (62) becomes

\[
(\alpha_0^\dagger, \alpha_1^\dagger, -1) \begin{pmatrix} -1 - m - e^{2\pi \sqrt{-1}/d} & -H & -1 \\
H & 1 - e^{2\pi \sqrt{-1}/d} & 0 \\
m & H & 1 - e^{2\pi \sqrt{-1}/d} \end{pmatrix} = 0
\]

which gives

\[
\int_X \alpha_0^\dagger = e^{2\pi \sqrt{-1}/d} - 1, \quad \alpha_1^\dagger = \frac{e^{2\pi \sqrt{-1}/d}}{1 - e^{2\pi \sqrt{-1}/d}}H.
\]

Applying the classification in Lemma 5.6 a straightforward computation shows the result. \[\square\]
Remark 5.9. The equality (97) is written as the integral

\[ Z_G(\Psi(E)) = -C_W \int_X e^{-\sqrt{-1}\omega} v^{-H/2}(E) \]

where \( \omega \in \mathbb{R}_{>0} H \) satisfies \( \int_X \omega^2 = d/4 \).

By setting \( B = -H/2 \), we consider the slope function \( \mu \) on \( A_W = \text{Coh}(X) \), defined by

\[ \mu(\Psi(E)) = \frac{v^B(E) \cdot H}{v^B_0(E)}. \]

Here \( \mu(\Psi(E)) \) is defined to be \( \infty \) if \( E \) is a torsion sheaf. The above \( \mu \)-stability coincides with the classical slope stability condition on \( \text{Coh}(X) \) via \( \Psi \). As we discussed in Subsection 4.3, the slope function \( \mu \) defines a torsion pair (67) on \( A_W \), and the associated tilting \( A_G \) is given by (68). We have the following result:

Proposition 5.10. Suppose that \( n = 4 \) and \( \varepsilon = 0 \). Then the triple

\[ (Z_G, A_G, \theta = \theta_W - 1) \]  

(101)

determines a Gepner type stability condition \( \sigma_G \) on \( \text{HMF}^W(W) \) with respect to \( (\tau, 2/d) \).

Proof. Instead of the triple (101), we consider the triple

\[ (Z^*_G := Z_G/C_W, A_G[1], \theta = 0). \]  

(102)

Obviously, the triple (101) determines a stability condition if and only if the triple (102) determines a stability condition. If the latter holds, then by Remark 5.9, the resulting stability condition is \( \Psi \), one of stability conditions on \( D^b \text{Coh}(X) \) constructed in [Bri08, Section 6]. Therefore, applying [Bri08] Lemma 6.2, it is enough to show the following: for any spherical torsion free sheaf \( E \) on \( X \), one has \( Z_G^*(\Psi(E)) \notin \mathbb{R}_{<0} \). The proof of this fact requires some more arguments, so we leave the proof to Lemma 5.11 below. We continue the proof assuming this fact.

By the above argument and Lemma 5.11, the triple (101) determines a stability condition \( \sigma_G \) on \( \text{HMF}^W(W) \). It remains to show that \( \sigma_G \) is Gepner type with respect to \( (\tau, 2/d) \). To show this, let us consider the objects \( \tau \Psi(O_x[-1]) \) for \( x \in X \). By Proposition 3.2, we have

\[ \tau \Psi(O_x[-1]) \cong \Psi(I_x) \in A_G[1] \]

where \( I_x \subset O_X \) is the ideal sheaf which defines \( x \). In Lemma 5.12 below, we see that \( \Psi(I_x) \) is \( \sigma_G \)-stable, hence it lies in \( P_G(\theta_W + 2/d) \). If we set \( \sigma'_G \) to be the stability condition defined as in (72), this implies that

\[ \Psi(O_x[-1]) \in \tau^{-1}P'_G \left( \theta_W + \frac{2}{d} \right) = P'_G(\theta_W). \]
for any $x \in X$, and it is $\sigma'_G$-stable. Since $X$ is a K3 surface, this implies that $\mathcal{P}_G(\theta_W - 1, \theta_W)$ is obtained as a tilting of $\mathcal{A}_W = \Psi(\text{Coh}(X))$, by the argument of [Bri08, Lemma 10.1]. Therefore $\mathcal{P}_G(\theta_W - 1, \theta_W)$ coincides with $\mathcal{A}_G$ by Lemma 5.11, which shows that $\sigma_G$ is Gepner type with respect to $(\tau, 2/d)$. □

Lemma 5.11. For any spherical torsion free sheaf $E$ on $X$, one has $Z^G(\Psi(E)) \notin \mathbb{R}_{\leq 0}$.

Proof. This is equivalent to saying that, for any spherical torsion free sheaf $E$ on $X$ with $v_B^1(E) \cdot H = 0$, one has

$$-v_2^B(E) + \frac{d}{8} v_0^B(E) > 0. \tag{103}$$

Since $E$ is a spherical sheaf, we have $v_0^B(E)^2 = -2$, where the square is defined in the Mukai lattice of $X$. (cf. [Bri08, Lemma 5.1].) Combined with $v_1^B(E) \cdot H = 0$ and the Hodge index theorem, we have

$$0 \geq v_1^B(E)^2 = 2v_0^B(E)v_2^B(E) - 2.$$

Noting that $v_0^B(E) > 0$, we obtain $v_0^B(E) \leq 1/v_0^B(E)$, hence

$$-v_2^B(E) + \frac{d}{8} v_0^B(E) \geq \frac{1}{v_0^B(E)} \left\{ \frac{d}{8} v_0^B(E)^2 - 1 \right\}. \tag{104}$$

The RHS of (104) is positive if $v_0^B(E) \geq 2$. Therefore we may assume that $v_0^B(E) = 1$, i.e. $E$ is rank one. Since $E$ is a spherical sheaf, this implies that $E$ is a line bundle.

Let $l$ be the first Chern class of $E$. Since $v_1^B(E) \cdot H = 0$, we have $l \cdot H = -H^2/2$, and $v_2^B(E)$ is written as

$$v_2^B(E) = \frac{l^2}{2} - \frac{H^2}{8} + 1.$$

Suppose by a contradiction that (103) does not hold for a line bundle $E$. Then, combining with the Hodge index theorem, we obtain

$$\frac{H^2}{4} - 2 + \frac{d}{4} \leq l^2 \leq \frac{H^2}{4}.$$

Since we have only two cases, $(d, H^2) = (4, 4), (6, 2)$, and $l^2$ is an even integer, it follows that $l^2 = 0$. Also since $-l \cdot H$ is either 2 (when $X$ is a quartic K3 surface) or 1 (when $X$ is a double cover of $\mathbb{P}^2$) it follows that the linear system $| - l|$ defines an elliptic fibration $X \to \mathbb{P}^1$. In particular, its general fiber is a smooth elliptic curve $C \in | - l|$ in $X$. However this is a contradiction: when $X$ is a quartic K3 surface, we have the Castelnuovo inequality

$$g(C) \leq \frac{1}{2}(H \cdot C - 1)(H \cdot C - 2) = 0$$

which contradicts to $g(C) = 1$. When $X$ is a double cover of $\mathbb{P}^2$, if $\pi: X \to \mathbb{P}^2$ is the double cover, then $H \cdot C = 1$ implies that $\pi|_C$ is an
isomorphism between $C$ and a line in $\mathbb{P}^2$, which contradicts to that $C$ is an elliptic curve. Therefore (103) also holds for a line bundle $E$. □

**Lemma 5.12.** The object $\Psi(I_x) \in \mathcal{A}_G[1]$ is $\sigma_G$-stable.

**Proof.** It is enough to show the $Z_G^\dagger$-stability of $\Psi(I_x)$ with respect to the triple (102). By Lemma 5.8, we have the following:

\[
\text{Im} Z_G^\dagger(\mathcal{A}_G[1]) \subset \left\{ \frac{1}{2} \sqrt{\frac{d}{H^2}} \times \mathbb{Z}_{\geq 0} \right\}.
\]

We first consider the case $(d, H^2) = (6, 2)$. In this case, the imaginary part of $Z_G^\dagger(\Psi(I_x))$ is $\sqrt{3}/2$, that is the smallest positive value of the RHS of (105). Therefore by Lemma 4.14, the $Z_G^\dagger$-stability of $\Psi(I_x)$ follows from

\[
\text{Hom}(P, \Psi(I_x)) = 0
\]

for any $P \in \mathcal{A}_G[1]$ with $\text{Im} Z_G^\dagger(P) = 0$. The above vanishing holds since $\Psi^{-1}(P)$ is given by an iterated extensions of zero dimensional sheaves and objects $U[1]$ for torsion free $\mu$-stable sheaves with $\mu(U) = 0$.

Next we consider the case $(d, H^2) = (4, 4)$. In this case, $Z_G^\dagger(\Psi(I_x)) = \sqrt{-1}$, whose imaginary part is the twice of the smallest positive value $1/2$ of the RHS of (105). Therefore, besides the vanishing (106), we need to show the following: if there is an exact sequence in $\mathcal{A}_G[1]$,

\[
0 \to P_1 \to \Psi(I_x) \to P_2 \to 0
\]

with $\text{Im} Z_G^\dagger(P_1) = 1/2$, then we have the inequality in $(0, \pi]$

\[
\arg Z_G^\dagger(P_1) < \arg Z_G^\dagger(\Psi(I_x)) = \frac{\pi}{2}.
\]

In order to show this, we first observe that $\Psi(\mathcal{O}_X) \in \mathcal{A}_G[1]$ is $Z_G^\dagger$-stable by [BMT, Proposition 7.4.1]. Since $\Psi(I_x)$ is a subobject of $\Psi(\mathcal{O}_X)$ in $\mathcal{A}_G[1]$, and $Z_G^\dagger(\Psi(\mathcal{O}_X)) = -1 + \sqrt{-1}$, we have

\[
\arg Z_G^\dagger(P_1) < \arg Z_G^\dagger(\Psi(\mathcal{O}_X)) = \frac{3}{4} \pi.
\]

Since the image of $Z_G^\dagger$ is contained in $\frac{1}{2} \mathbb{Z} + \sqrt{-1} \mathbb{Z}$, and $\text{Im} Z_G^\dagger(P_1) = 1/2$, the above inequality implies that $\arg Z_G^\dagger(P_1) \leq \pi/2$. It remains to exclude the following case:

\[
Z_G^\dagger(P_1) = Z_G^\dagger(P_2) = \frac{\sqrt{-1}}{2}.
\]

Indeed, we see that there is no object $P \in \mathcal{A}_G[1]$ such that $Z_G^\dagger(P) = \sqrt{-1}/2$. Suppose that such an object $P$ exists, and let

\[
\text{ch}(\Psi^{-1}P) \sqrt{\text{td}_X} = (r, l, s) \in H^0(X) \oplus H^2(X) \oplus H^4(X)
\]
be the (untwisted) Mukai vector of $\psi^{-1} P$. Then the condition $Z_G^t(P) = \sqrt{-1}/2$ is equivalent to

$$l \cdot H + 2r = 1, \quad l \cdot H + 2s = 0.$$ 

This is a contradiction since both of $r$ and $s$ are integers.

5.4. The case of $n = 3$, $\epsilon = -1$. In this subsection, we study the case of $(n, \epsilon) = (3, -1)$. This case seems to be the most interesting case studied in this paper, as a construction of $\sigma_G$ has to do with the study of coherent systems on the smooth projective curve $X$. Indeed in Proposition 3.22 we constructed an equivalence $\Theta$ between the category of coherent systems on $X$ and the heart $\mathcal{A}_W$, given by

$$\mathcal{A}_W = \langle \mathbb{C}(0), \psi \text{Coh}(X) \rangle_{ex}.$$ 

Below we abbreviate $\Theta$ and regard any coherent system $(\mathcal{O}_X \oplus \mathcal{R}_X \to \mathcal{F})$ on $X$ as an object in $\mathcal{A}_W$.

In the notation of Subsection 4.2, the polynomial $\hat{W}$ is of type $(a_1, a_2, a_3, 1, d)$, which must belong to one of the classifications (97). There are two possibilities:

$$(a_1, a_2, a_3, d) = \begin{cases}(1, 1, 1, 4), & X \text{ is a genus 3 curve} \\ (3, 1, 1, 6), & X \text{ is a genus 2 curve.}\end{cases}$$

The Calabi-Yau manifold $\hat{X}$ is a K3 surface which is either a quartic surface or a double cover of $\mathbb{P}^2$, and it contains $X$ as an element in the linear system $X \in |\hat{H}|$. The central charge $Z_G$ is described in terms of coherent systems as follows:

**Lemma 5.13.** For any coherent system $(\mathcal{O}_X^{\oplus R} \to \mathcal{F})$ on $X$, we have

$$Z_G(\mathcal{O}_X^{\oplus R} \to \mathcal{F}) = C_W \left\{ -d(F) + R \left( 1 - \cos \frac{2\pi}{d} \right) + \frac{\sqrt{H^2d}}{2} \left( r(F) - \frac{R}{2} \right) \sqrt{-1} \right\}.$$ 

Here we set $(r(F), d(F)) = (\text{rank}(F), \text{deg}(F))$.

**Proof.** The result follows from (64), (65), Lemma 5.8 and an easy computation. \qed

By setting $Z_G^t := Z_G/C_W$, we define the slope function $\mu$ on $\mathcal{A}_W$ by

$$\mu(\mathcal{O}_X^{\oplus R} \to \mathcal{F}) := -\text{Im} \frac{Z_G^t(\mathcal{O}_X^{\oplus R} \to \mathcal{F})}{R}$$

$$= \frac{\sqrt{H^2d}}{2} \left( \frac{1}{2} - \frac{r(F)}{R} \right).$$

Here we set $\mu(*) = -\infty$ if $R = 0$. 

\[\text{(107)}\]
Lemma 5.14. Any object in $\mathcal{A}_W$ admits a Harder-Narasimhan filtration with respect to $\mu$-stability.

Proof. Although $\mathcal{A}_W$ is a noetherian abelian category, we have to take a little care since the condition in [Tod10a, Proposition 2.12] is not satisfied in this case. Instead, we apply the argument used in [Tod09, Theorem 2.29]. Let $\mathcal{C} \subset \mathcal{A}_W$ be the subcategory consisting of objects $(O \oplus R \xrightarrow{X} s \rightarrow F)$ such that $s$ is surjective. Note that the right orthogonal complement $\mathcal{C}^\perp$ consists of objects of the form $(0 \rightarrow F')$. Any object $(O \oplus R \xrightarrow{X} s \rightarrow F) \in \mathcal{A}_W$ fits into the exact sequence

$$0 \rightarrow (O \oplus R \xrightarrow{X} \text{Im } s) \rightarrow (O \oplus R \xrightarrow{X} F) \rightarrow (0 \rightarrow \text{Cok}(s)) \rightarrow 0$$

showing that $(\mathcal{C}, \mathcal{C}^\perp)$ is a torsion pair on $\mathcal{A}_W$. Furthermore, since $\mu(*) = -\infty$ on $\mathcal{C}^\perp$, we can easily see the following: an object $E \in \mathcal{C}$ is $\mu$-semistable if and only if for any exact sequence $0 \rightarrow E_1 \rightarrow E \rightarrow E_2 \rightarrow 0$ in $\mathcal{C}$, we have $\mu(E_1) \leq \mu(E_2)$. (cf. [Tod09, Lemma 2.27].) Since $\mathcal{C}$ is a noetherian and artinian quasi-abelian category, an argument similar to [Tod09, Theorem 2.29] shows that any object in $\mathcal{C}$ admits a $\mu$-Harder-Narasimhan filtration. Combined with (108), any object in $\mathcal{A}_W$ admits a $\mu$-Harder-Narasimhan filtration. □

Remark 5.15. By the proof of the above lemma, we see the following: if an object $(O \oplus R \xrightarrow{X} s \rightarrow F)$ is $\mu$-semistable, then either $s$ is surjective or $R = 0$.

Before discussing the construction of $\sigma_G$, we review Clifford type theorem for coherent systems established by Newstead-Lange [LN08]. For a smooth projective curve $C$ and $\alpha \in \mathbb{R}_{>0}$, recall that the $\alpha$-stability on $\text{Syst}(C)$ is defined by the slope function

$$\mu_\alpha(O \oplus R \xrightarrow{X} F) \mapsto \frac{d(F) + \alpha \cdot R}{r(F)}.$$  

(109)

Here the above slope function is set to be $\infty$ if $r(F) = 0$.

Theorem 5.16. ([LN08, Theorem 2.1, Remark 2.3]) Let $C$ be a smooth projective curve of genus $g(C) \geq 2$, and $\alpha \in \mathbb{R}_{>0}$. Then for any $\alpha$-stable coherent system $(O \oplus R \xrightarrow{X} F)$ with $0 \leq d(F) < 2g(C) \cdot r(F)$, we have

$$R \leq \frac{d(F)}{2} + r(F).$$

(110)

Moreover if $C$ is non-hyperelliptic, then the equality holds in (110) only if $(O \oplus R \xrightarrow{X} F)$ is isomorphic to either

$$H^0(O_C) \otimes O_C \rightarrow O_C \quad \text{or} \quad H^0(\omega_C) \otimes O_C \rightarrow \omega_C.$$  

(111)

Using the above result, we have the following lemma, which plays a crucial role in constructing a Gepner type stability condition on $\text{HMF}^{\alpha}(W)$:
Lemma 5.17. Let \((\mathcal{O}_X^R \xrightarrow{s} F)\) be a \(\mu\)-stable object in \(\mathcal{A}_W\) such that \(R = 2r(F) > 0\). Then we have
\[
d(F) > R \left(1 - \cos \frac{2\pi}{d}\right).
\]

Proof. By the \(\mu\)-stability, the morphism \(s\) is surjective (cf. Remark 5.15) hence \(d(F) \geq 0\). Also the RHS of (112) is \(R\) if \(d = 4\) and \(R/2\) if \(d = 6\). In both cases, they are smaller than \(2g(X) \cdot r(F) = g(X) \cdot R\), so we may assume that \(d(F) < 2g(X) \cdot r(F)\). Comparing the \(\mu\)-stability in (107) and the \(\alpha\)-stability in (109), we see that the \(\mu\)-stable object \((\mathcal{O}_X^R \to F)\) is \(\alpha\)-stable for \(\alpha \gg 0\), since the set of quotient sheaves of \(F\) with bounded above degrees is bounded. Therefore, applying Theorem 5.16, we obtain the inequality \(d(F) \geq R\). It remains to check that the equality \(d(F) = R\) does not hold if \(d = 4\). In this case, \(X\) is a quartic curve, so it is not hyperelliptic. Also \(H^0(\mathcal{O}_C)\) is one dimensional, \(H^0(\mathcal{O}_C^\vee)\) is three dimensional, hence the objects (111) do not satisfy our assumption \(R = 2r(F)\). Therefore the case \(d(F) = R\) is excluded. \(\Box\)

As we discussed in Subsection 4.3, the slope function \(\mu\) defines a torsion pair (67) on \(\mathcal{A}_W\), and the associated tilting \(\mathcal{A}_G\) is given by (68). We have the following result:

Proposition 5.18. Suppose that \(n = 3\) and \(\varepsilon = -1\). Then the triple
\[
(Z_G, \mathcal{A}_G, \theta = \theta_W)
\]
determines a Gepner type stability condition \(\sigma_G\) on \(\text{HMF}^{\text{gr}}(W)\) with respect to \((\tau, 2/d)\).

Proof. Obviously the triple (113) determines a stability condition if and only if the triple
\[
(Z_G^\dagger = Z_G/C_W, \mathcal{A}_G, \theta = 0)
\]
determines a stability condition. By the construction of \(\mathcal{A}_G\), any non-zero object \(E \in \mathcal{A}_G\) satisfies \(\text{Im} Z_G^\dagger(E) \geq 0\). Moreover Lemma 5.13 and Lemma 5.17 imply that if \(\text{Im} Z_G^\dagger(E) = 0\), then \(\text{Re} Z_G^\dagger(E) < 0\) holds. Therefore the triple (114) satisfies the condition (70). Also by Lemma 5.13, the image of \(Z_G^\dagger\) is discrete, which enables us to apply the same argument of [Bri08, Proposition 7.1] to prove the Harder-Narasimhan property of the triple (114). Therefore the triple (114), hence (113), determines a stability condition. Let
\[
\sigma_G = (Z_G, \{\mathcal{P}_G(\phi)\}_{\phi \in \mathbb{R}})
\]
be the stability condition on \(\text{HMF}^{\text{gr}}(W)\) determined by the triple (113). We need to show that \(\sigma_G\) is Gepner type with respect to \((\tau, 2/d)\). Note that, in our situation, the triple (113) satisfies the inequality in (82). Therefore, by Lemma 4.12 it is enough to check the \(\sigma_G\)-stability of
\[ \tau \Psi(\mathcal{O}_x), \mathbb{C}(1) \text{ and } \tau^2 \Psi(\mathcal{O}_x). \] The stabilities of these objects are proved in Lemma 5.19, Lemma 5.21 and Lemma 5.23 below.

Below we check the \( \sigma_G \)-stability of the objects \( \tau \Psi(\mathcal{O}_x), \mathbb{C}(1) \) and \( \tau^2 \Psi(\mathcal{O}_x). \) Let \( \sigma^\dagger_G \) be the stability condition on \( \text{HMF}^W(W) \) determined by the triple (114). It differs from \( \sigma_G \) by an action of \( \mathbb{C} \), so it is enough to check the \( \sigma^\dagger_G \)-stability of these objects.

**Lemma 5.19.** For any \( x \in X \), the object \( \tau \Psi(\mathcal{O}_x) \) is \( \sigma^\dagger_G \)-stable.

**Proof.** By Lemma 3.10, the object \( \tau \Psi(\mathcal{O}_x) \) is an object in \( \mathcal{A}_W \), given by the coherent system \( (\mathcal{O}_X \to \mathcal{O}_x) \). It is \( \mu \)-stable with \( \mu(\ast) > 0 \), hence \( \tau \Psi(\mathcal{O}_x)[-1] \in \mathcal{A}_G \). On the other hand, we have

\[
\text{Im } Z^\dagger_G(\mathcal{A}_G) \subset \left\{ \frac{\sqrt{H^2d}}{4} \times \mathbb{Z}_{\geq 0} \right\}.
\]

The imaginary part of \( Z^\dagger_G(\tau \Psi(\mathcal{O}_x)[-1]) \) is \( \sqrt{H^2d}/4 \), which is the smallest positive number in the RHS of (115). By Lemma 4.14, it is enough to check that there is no non-zero morphism from any object \( P \in \mathcal{A}_G \) with \( \text{Im } Z^\dagger_G(P) = 0 \) to \( \Psi(\mathcal{O}_x)[-1] \). Since \( \tau \Psi(\mathcal{O}_x)[-1] \in \mathcal{A}_W[-1] \), and \( P \in \mathcal{A}_W \) by Sublemma 5.20 below, there is no non-zero morphism from \( P \) to \( \tau \Psi(\mathcal{O}_x)[-1] \).

We have used the following sublemma. The proof is obvious from the construction of \( \mathcal{A}_G \), and we omit it.

**Sublemma 5.20.** A non-zero object \( P \in \mathcal{A}_G \) satisfies \( \text{Im } Z^\dagger_G(P) = 0 \) if and only if \( P \in \mathcal{A}_W \), and it is given by an iterated extensions of \( \mu \)-stable coherent systems \( (\mathcal{O}^\oplus_R \to F) \) with \( \mu(\mathcal{O}^\oplus_R \to F) = 0 \), and coherent systems of the form \( (0 \to \mathcal{O}_y) \) for \( y \in X \).

Next we check the stability of \( \mathbb{C}(1) \).

**Lemma 5.21.** The object \( \mathbb{C}(1) \) is \( \sigma^\dagger_G \)-stable.

**Proof.** By Corollary 3.23, we have the isomorphism (again we abbreviate \( \Theta \))

\[ \mathbb{C}(1)[-1] \cong (\mathcal{O}_X \otimes R_1 \overset{s}{\to} \mathcal{O}_X(1)) \]

such that \( s \) is the natural evaluation map. In particular \( H^0(s) \) is an isomorphism, and \( s \) is surjective since \( \mathcal{O}_X(1) \) is globally generated. Then we apply Sublemma 5.22 below to show that \( \mathbb{C}(1)[-1] \) is \( \mu \)-stable. The slope \( \mu(\mathbb{C}(1)[-1]) \) equals to 1/6 if \( d = 4 \) and 0 if \( d = 6 \). In the former case, \( \mathbb{C}(1)[-2] \in \mathcal{A}_G \), and the imaginary part of \( Z^\dagger_G(\mathbb{C}(1)[-2]) \) is 1, which is the smallest positive value of the RHS of (115). Hence \( \sigma^\dagger_G \)-stability of \( \mathbb{C}(1)[-2] \) follows from the same argument of Lemma 5.19. In the latter case, \( \mathbb{C}(1)[-1] \in \mathcal{A}_G \) and the imaginary part of \( Z^\dagger_G(\mathbb{C}(1)[-1]) \) is 0. Hence the \( \sigma^\dagger_G \)-stability of \( \mathbb{C}(1)[-1] \) follows from its \( \mu \)-stability and Sublemma 5.20.

\[ \square \]
Sublemma 5.22. For an object \((\mathcal{O}_X^R \to \mathcal{L}) \in \mathcal{A}_W\) with \(R > 0\) and \(\mathcal{L} \in \text{Pic}(X)\), it is \(\mu\)-stable if \(s\) is surjective and \(H^0(s)\) is injective.

**Proof.** Suppose that \(s\) is surjective and \(H^0(s)\) is injective. Let

\[
0 \to (\mathcal{O}_X^R \to \mathcal{L}_1) \to (\mathcal{O}_X^R \to \mathcal{L}) \to (\mathcal{O}_X^R \to \mathcal{L}_2) \to 0
\]

be an exact sequence of coherent systems. It is enough to show the inequality

\[
\mu(\mathcal{O}_X^R \to \mathcal{L}) < \mu(\mathcal{O}_X^R \to \mathcal{L}_2).
\]

(116)

By our assumption, \(\mathcal{L}_1 \neq 0\) and \(\mathcal{L}_2 \neq 0\), hence \(\mathcal{L}_2\) is a zero dimensional sheaf. Therefore the RHS of (116) equals to \(1/2\), while the LHS of (116) is less than \(1/2\). Hence (116) holds. \(\square\)

It remains to prove the following lemma:

**Lemma 5.23.** The object \(\tau^2 \Psi(\mathcal{O}_x)\) is an object in \(\mathcal{P}_G(\theta_W + 1 + 4/d)\).

**Proof.** Applying \(\tau\) to the exact sequence (32), we obtain the distinguished triangle in \(\text{HMF}^{gr}(W)\)

\[
\tau \Psi(\mathcal{O}_x) \to \tau^2 \Psi(\mathcal{O}_x) \to \mathbb{C}(1).
\]

Combined with Lemma 3.10 and Corollary 3.23, the object \(\tau^2 \Psi(\mathcal{O}_x)\) is obtained as a cone of the morphism of coherent systems:

\[
\begin{array}{ccc}
\mathcal{O}_X \otimes R_1 & \xrightarrow{s} & \mathcal{O}_X(1) \\
\downarrow \gamma_1 & & \downarrow \gamma_2 \\
\mathcal{O}_X & \xrightarrow{\gamma} & \mathcal{O}_x.
\end{array}
\]

Here \(s\) is the evaluation map and \((\gamma_1, \gamma_2)\) is the morphism of coherent systems. Since \((\gamma_1, \gamma_2)\) is non-zero, both of \(\gamma_1, \gamma_2\) are non-zero, hence they are surjective. Therefore the object \(\tau^2 \Psi(\mathcal{O}_x)[-1]\) is given by the coherent system

\[
(\mathcal{O}_X \otimes R_{1,x} \xrightarrow{s_x} \mathcal{O}_X(1) \otimes I_x).
\]

Here \(R_{1,x}\) is the subspace of \(R_1\) which vanishes at \(x\), and \(I_x\) is the ideal sheaf which defines \(x\).

First suppose that \(d = 4\). In this case, \(R_{1,x}\) is two dimensional, and \(s_x\) is surjective since any of two lines in \(\mathbb{P}^2\) determined by generic two elements in \(R_{1,x}\) intersect only at \(x\) transversally. Also \(H^0(s_x)\) is injective since \(H^0(s)\) is an isomorphism. Therefore the coherent system (117) is \(\mu\)-stable by Sublemma 5.22. Since \(\mu = 0\) for the coherent system (117), we have \(\tau^2 \Psi(\mathcal{O}_x)[-1] \in \mathcal{A}_G\), and it is \(\sigma_G\)-stable. In particular \(\tau^2 \Psi(\mathcal{O}_x)\) is an object in \(\mathcal{P}_G(\theta_W + 2)\).

Next suppose that \(d = 6\). In this case, \(R_{1,x}\) is one dimensional, and \(s_x\) is not surjective at \(x\). In particular the coherent system (117) is not \(\mu\)-semistable. However we can show the \(Z_G^t\)-stability, hence \(\sigma_G^t\)-stability, of (117) in the following way: the sheaf \(\mathcal{O}_X(1) \otimes I_x\) is isomorphic to
$\mathcal{O}_X(x')$ for another point $x' \in X$, and (117) is isomorphic to the coherent system

$$(\mathcal{O}_X \xrightarrow{s'} \mathcal{O}_X(x'))$$

where $s'$ is a natural inclusion. There is an exact sequence of coherent systems

$$(118) \quad 0 \to (\mathcal{O}_X \xrightarrow{id} \mathcal{O}_X) \to (\mathcal{O}_X \to \mathcal{O}_X(x')) \to (0 \to \mathcal{O}_{x'}) \to 0.$$ 

Both of the objects $(\mathcal{O}_X \xrightarrow{id} \mathcal{O}_X)$ and $(0 \to \mathcal{O}_{x'})$ are $\mu$-stable with negative slopes, hence the object (117) is an object in $\mathcal{A}_G$. Also the imaginary part of $Z^G_1(\mathcal{O}_X \to \mathcal{O}_X(x'))$ is $\sqrt{3}/2$, which is the smallest positive value of the RHS of (115). Therefore by Lemma 4.14 it is enough to check that there is no non-zero morphism from $P \in \mathcal{A}_G$ with $\text{Im} Z^G_1(P) = 0$ to the object $(\mathcal{O}_X \to \mathcal{O}_X(x'))$. This follows from Sublemma 5.20, since both of $(\mathcal{O}_X \xrightarrow{id} \mathcal{O}_X)$ and $(0 \to \mathcal{O}_{x'})$ are $\mu$-stable with negative slopes, and the exact sequence (118) does not split. □

5.5. The case of $n = 2$, $\varepsilon = -2$. Finally in this subsection, we study the case of $(n, \varepsilon) = (2, -2)$. In this case, $X$ is a finite number of smooth points, represented by points $p^{(j)} = (p_1^{(j)}, p_2^{(j)}) \in \mathbb{C}^2$ for $1 \leq j \leq \#X$.

The heart $\mathcal{A}_W$ is given by

$$\mathcal{A}_W = \langle \mathbb{C}(1), \mathbb{C}(0), \Psi(\mathcal{O}_x) : x \in X \rangle_{\text{ex}}.$$ 

By Lemma 3.17 and Lemma 3.18 the heart $\mathcal{A}_W$ is equivalent to the abelian category of representations of a certain quiver $Q$ with relations. As in the previous subsection, we have the following possibilities:

$$(a_1, a_2, d, \#X) = (1, 1, 4, 4), \quad Q = \begin{array}{c}
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$$\pi(4) \quad \pi(3) \quad \pi(2) \quad \pi(1)$$

$$(a_1, a_2, d, \#X) = (3, 1, 6, 2), \quad Q = \begin{array}{c}
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Also by Corollary 3.19 in the $d = 4$ case, we put the following relation:

$$(119) \quad p_2^{(j)} \pi^{(j)} X_1 = p_1^{(j)} \pi^{(j)} X_2, \quad 1 \leq j \leq 4.$$ 

There is no relation in the $d = 6$ case. By (65) and the above classification, the central charge $Z_G$ is given as follows:
Lemma 5.24. If we write the K-theory class of \( E \in A_W \) as
\[
[E] = v_1[C(1)] + v_0[C(0)] + \sum_{j=1}^{\sharp X} w_j[\Psi(O_{\mu(j)})]
\]
for \( v_j, w_j \in \mathbb{Z}_{\geq 0} \), then we have
\[
Z_G(E) = CW\left\{ -w + \left( 1 - \cos \frac{2\pi}{d} \right) v_0 + v_1 \right. \\
+ \left. \left\{ -\sin \frac{2\pi}{d} v_0 + \left( \sin \frac{2\pi}{d} - \sin \frac{4\pi}{d} \right) v_1 \right\} \sqrt{-1} \right\}.
\]
Here we set \( w := \sum_{j=1}^{\sharp X} w_j \).

By setting \( Z_G^\dagger := Z_G/C_W \), we define the slope function \( \mu \) on \( A_W \) by
\[
\mu(E) := \frac{\text{Re} Z_G^\dagger(E)}{w} = \frac{1}{w} \left( v_1 + \left( 1 - \cos \frac{2\pi}{d} \right) v_0 \right) - 1.
\]
Here we set \( \mu(E) = \infty \) if \( w = 0 \). The above slope function defines the \( \mu \)-stability on \( A_W \). Since \( A_W \) is noetherian and artinian, any object in \( A_W \) admits a \( \mu \)-Harder-Narasimhan filtration. (cf. [Tod10a, Proposition 2.12].) We prepare the following lemma:

Lemma 5.25. For any non-zero \( \mu \)-stable object \( E \in A_W \) with \( \mu(E) = 0 \), we have \( \text{Im} Z_G^\dagger(E) < 0 \).

Proof. In the case of \( d = 6 \), we have \( \text{Im} Z_G^\dagger(E) = -\sqrt{3} v_0/2 \), which is non-positive. If \( v_0 = 0 \), the condition \( \mu(E) = 0 \) implies \( v_1 = w \neq 0 \). Therefore \( E \) decomposes into direct sums, which contradicts to the \( \mu \)-stability of \( E \).

In the case of \( d = 4 \), the claim is equivalent to that if \( E \) is \( \mu \)-stable with \( v_0 + v_1 = w \neq 0 \), then \( v_1 < v_0 \). Let us represent \( E \) as a representation of \( Q \):
\[
\begin{align*}
V^{(1)} & \xrightarrow{x_1} \begin{array}{ccc} & \pi^{(2)} & \\
X_1 & V^{(0)} & W^{(2)} \end{array} \\
\pi^{(3)} & \pi^{(4)} & W^{(3)} \end{align*}
\]
where \( V^{(i)} \) are \( v_i \)-dimensional and \( W^{(i)} \) are \( w_i \)-dimensional. Suppose by a contradiction that \( v_1 \geq v_0 \). By the relation (119), we have the linear maps for \( 1 \leq i \leq 4 \)
\[
p_1^{(i)} X_1 - p_2^{(i)} X_1: V^{(1)} \to \text{Ker}(\pi^{(i)}).
\]
Because of the condition $w \neq 0$, there is $1 \leq i \leq 4$ such that $w^{(i)} \neq 0$. Also the $\mu$-stability of $E$ implies that $\pi^{(i)}$ is surjective. Therefore the assumption $v_1 \geq v_0$ implies that $\dim V^{(1)} > \dim \text{Ker}(\pi^{(i)})$, and there is a non-zero element $v \in V^{(1)}$ which is mapped to zero by (122). Let us consider the sub quiver representation of (121) generated by $v$. By taking account the relation (119) into consideration, the corresponding subobject $F \subset E$ in $A_W$ has the K-theory class either $[\mathcal{C}(1)]$ or $[\mathcal{C}(1)] + [\mathcal{C}(0)]$ or $[\mathcal{C}(1)] + [\mathcal{C}(0)] + [\Psi(O_{\mu(o)})]$. Hence we have $\mu(F) > 0$, which contradicts to the $\mu$-stability of $E$ with $\mu(E) = 0$. □

As before, the slope function $\mu$ defines a torsion pair (67) on $A_W$, and the associated tilting $A_G$ given by (68). We have the following result:

**Proposition 5.26.** Suppose that $n = 2$ and $\varepsilon = -2$. Then the triple 
\[
(123) \quad \left( Z_G, A_G, \theta = \theta_W + \frac{1}{2} \right)
\]
determines a Gepner type stability condition $\sigma_G$ on $\text{HMF}^{\text{st}}(W)$ with respect to $(\tau, 2/d)$.

**Proof.** The triple (123) determines a stability condition if and only if the triple 
\[
(124) \quad \left( Z_G^\dagger = Z_G/C_W, A_G, \theta = \frac{1}{2} \right)
\]
determines a stability condition. By Lemma 5.24, Lemma 5.25 and the construction of $A_G$, the triple (124) satisfies the condition (70). Also since the image of $Z_G^\dagger$ is discrete, the same argument of [Bri08, Proposition 7.1] shows the Harder-Narasimhan property of (124). Therefore the triples (123), (124) determine stability conditions $\sigma_G$, $\sigma_G^\dagger$ respectively. Since they only differ by a $\mathbb{C}$-action, by Lemma 4.10, it is enough to check the $\sigma_G^\dagger$-stability of $\tau \Psi(O_x)$, $\mathcal{C}(1)$ and $\mathcal{C}(2)$. These are checked in Lemma 5.27, Lemma 5.28 and Lemma 5.29 below. □

**Lemma 5.27.** For any $x \in X$, the object $\tau \Psi(O_x)$ is $\sigma_G^\dagger$-stable.

**Proof.** By Lemma 3.10 we have $\tau \Psi(O_x) \in A_W$. By the exact sequence (32), the object $\tau \Psi(O_x)$ is $\mu$-stable with non-positive slope. Therefore $\tau \Psi(O_x)$ is an object in $A_G$. In the $d = 4$ case, we have $Z_G^\dagger(\tau \Psi(O_x)) = -\sqrt{-1}$. Since the image of $Z^\dagger_G$ is $\mathbb{Z} + \mathbb{Z}\sqrt{-1}$, this immediately implies that $\tau \Psi(O_x)$ is $\sigma_G^\dagger$-stable. In the $d = 6$ case, note that $\text{Re} Z_G^\dagger(\tau \Psi(O_x))$ on $A_G$ is contained in $\mathbb{Z}_{\leq 0} \times 1/2$. Since $\text{Re} Z_G^\dagger(\tau \Psi(O_x)) = -1/2$, by Lemma 4.14, it is enough to check that $\text{Hom}(P, \tau \Psi(O_x)) = 0$ for any $P \in A_G$ with $\text{Re} Z_G^\dagger(P) = 0$. By our construction of $A_G$, we have
Lemma 5.28. The object $C(1)$ is $\sigma_G^1$-stable.

Proof. Since $C(1)$ is $\mu$-stable with $\mu(C(1)) = \infty$, we have $C(1)[-1] \in A_G$. In the $d = 4$ case, note that $Re Z_G^1(\ast)$ is contained in $\mathbb{Z}_{\leq 0}$. Since $Re Z_G^1(C(1)[-1]) = -1$, by Lemma 4.14, it is enough to check that $Hom(P, C(1)[-1]) = 0$ for any $P \in A_G$ with $Re Z_G^1(P) = 0$, which follows since $P$ is an object in $A_W$. In the $d = 6$ case, let $E$ be a non-trivial quotient of $C(1)[-1]$ in $A_G$. Since $Z_G^1(C(1)[-1]) = -1$, it is enough to check that $Im Z_G^1(E) < 0$. To check this, note that we have $E \in A_W$ since $C(1)$ is a simple object in $A_W$. Let us write the K-theory class of $E$ as in (120). Since $Im Z_G^1(E) = -\sqrt{3}v_0/2$ is non-positive, it is enough to exclude the case of $v_0 = 0$. If $v_0 = 0$, then $E$ is a direct sum of objects $C(1)$ and $\Psi(O_x)$ for $x \in X$. Then there is a non-zero morphism from $C(1)[-1]$ to $\Psi(O_x)$, which contradicts to Lemma 3.17.

Lemma 5.29. The object $C(2)$ is $\sigma_G^1$-stable.

Proof. We first discuss the case of $d = 4$. In this case, by Corollary 3.20, the object $C(2)[-1]$ is an object in $A_W$ represented by the following quiver representation

\[
\begin{array}{c}
\bullet & \overset{x_1}{\longrightarrow} & \bullet \\
R_1 & & R_2 \\
\end{array}
\]

(125)

Here $R_1$ is two dimensional and $R_2$ is three dimensional. We first show that $C(2)[-1]$ is $\mu$-stable. Let $E$ be a subobject of $C(2)[-1]$ in $A_W$ given by a representation \([121]\), with $v^{(j)} := \dim V^{(j)}$ and $w := \sum_{j=1}^{4} \dim W^{(j)}$. Let $I$ be the subset of $i \in \{1, 2, 3, 4\}$ such that $W^{(i)} = 0$. Similarly to the proof of Lemma 5.3, we have the inequality $v^{(0)} \leq 3 - \frac{1}{\ell}I$. Also we may assume that $v^{(1)} \leq 1$, since otherwise $F$ coincides with $C(2)[-1]$. Therefore we obtain the inequality

\[v^{(0)} + v^{(1)} \leq 4 - |I| = w\]

which implies that $\mu(E) \leq 0 < \mu(C(2)[-1]) = 1/4$. Therefore $C(2)[-1]$ is $\mu$-stable with positive slope, hence we have $C(2)[-2] \in A_G$. Since $Z_G^1(C(2)[-2]) = -1 + \sqrt{-1}$, and the image of $Z_G^1$ is $\mathbb{Z} + \mathbb{Z}\sqrt{-1}$, by Lemma 4.14, the $\sigma_G^1$-stability of $C(2)[-2]$ follows if we check that $Hom(P, C(2)[-2]) = 0$ for any $P \in A_G$ with $Re Z_G^1(P) = 0$. This follows since $P \in A_W$ and $C(2)[-2] \in A_W[-1]$. 

$P \in A_W$, and it is $\mu$-semistable with $\mu(P) = 0$. Since $\tau \Psi(O_x)$ is $\mu$-stable with negative slope, we have $Hom(P, \tau \Psi(O_x)) = 0$. \qed
Next we discuss the case of $d = 6$. By Lemma 3.11, the object $C(2)[-1] \in \mathcal{A}_W$ is represented by the following quiver representation

\[
\begin{array}{ccc}
0 & \rightarrow & C \\
\text{id} & & \text{id}
\end{array}
\]

The above object is obviously $\mu$-stable. Since $\mu(C(2)[-1]) = -3/4$, it follows that $C(2)[-1] \in \mathcal{A}_G$. Also we have $Z_G^1(C(2)[-1]) = -(1/2 + 3\sqrt{-3}/2)$, and $\text{Re} Z_G^1(*)$ is contained in $\mathbb{Z}_{\leq 0} \times 1/2$ on $\mathcal{A}_G$. Applying Lemma 4.14, the $\sigma^1_{\mathcal{G}}$-stability of $C(2)[-1]$ follows if we check that $\text{Hom}(P, C(2)[-1]) = 0$ for any $P \in \mathcal{A}_G$ with $\text{Re} Z_G^1(P) = 0$. Since such $P$ is a $\mu$-semistable object in $\mathcal{A}_W$ with $\mu(P) = 0$, and $C(2)[-1]$ is $\mu$-stable with negative slope, it follows that we have $\text{Hom}(P, C(2)[-1]) = 0$. \hfill \Box

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