RADIUS OF CLOSE-TO-CONVEXITY OF HARMONIC FUNCTIONS

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Abstract. Let $H$ denote the class of all normalized complex-valued harmonic functions $f = h + g$ in the unit disk $D$, and let $K = H + G$ denote the harmonic Koebe function. Let $a_n, b_n, A_n, B_n$ denote the Maclaurin coefficients of $h, g, H, G$, and

$$F = \{ f = h + g \in H : |a_n| \leq A_n \text{ and } |b_n| \leq B_n \text{ for } n \geq 1 \}.$$ 

We show that the radius of univalence of the family $F$ is 0. We also show that this number is also the radius of the starlikeness of $F$. Analogous results are proved for a subclass of the class of harmonic convex functions in $H$. These results are obtained as a consequence of a new coefficient inequality for certain class of harmonic close-to-convex functions. Surprisingly, the new coefficient condition helps to improve Bloch-Landau constant for bounded harmonic mappings.

1. Introduction and Main Results

Denote by $H$ the class of all complex-valued harmonic functions $f$ in the unit disk $D = \{ z \in \mathbb{C} : |z| < 1 \}$ normalized by $f(0) = 0 = f_z(0) - 1$. Each $f$ can be decomposed as $f = h + g$, where $g$ and $h$ are analytic in $D$ so that

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} b_n z^n.$$ 

Let $S_H$ denote the class of univalent and orientation-preserving functions $f = h + g$ in $H$. Then the Jacobian of $f$ is given by $J_f(z) = |h'(z)|^2 - |g'(z)|^2$. We note that if $f = h + g \in S_H$ and $g(z) \equiv 0$ in $D$, then $f = h \in S$, where $S$ denotes the well-known class of normalized univalent analytic functions in $D$. A necessary and sufficient condition (see [4] or Lewy [10]) for a harmonic function $f$ to be locally univalent in $D$ is that $J_f(z) > 0$ in $D$. The function $\omega(z) = g'(z)/h'(z)$ denotes the complex dilatation of $f$. Thus, for $f = h + g \in S_H$ with $g(0) = b_1$ and $|b_1| < 1$ (because $J_f(0) = 1 - |b_1|^2 > 0$), the function

$$F = \frac{f - \overline{b_1}f}{1 - |b_1|^2}$$

is also in $S_H$. Thus, it is customary to restrict our attention to the subclass

$$S_H^0 = \{ f \in S_H : f_z(0) = 0 \}.$$
The family $S^0_H$ is known to be compact. The uniqueness result of the Riemann mapping theorem does not extend to these classes of harmonic functions, [6, 8]. Several authors have studied the subclass of functions that map $\mathbb{D}$ onto specific domains, e.g., starlike domains, convex and close-to-convex domains. Let $S^0_H (K_H, C_H \text{ resp.})$ consist of all sense-preserving harmonic mappings $f = h + \overline{g} \in \mathcal{H}$ of $\mathbb{D}$ onto starlike (convex, close-to-convex, resp.) domains. Denote by $S^0_H (K^0_H, C^0_H \text{ resp.})$ the class consists of those functions $f$ in $S^0_H (K_H, C_H \text{ resp.})$ for which $f(0) = 0$.

In [6] Lemma 5.15, Clunie and Sheil-Small proved the following result.

**Lemma A.** If $h, g$ are analytic in $\mathbb{D}$ with $|h'(0)| > |g'(0)|$ and $h + \epsilon g$ is close-to-convex for each $\epsilon$, $|\epsilon| = 1$, then $f = h + \overline{g}$ is close-to-convex in $\mathbb{D}$.

This lemma has been used to obtain many important results. In the case of $S^0_H$, we have the harmonic Koebe function $K = H + \overline{G}$ in $S^0_H$, where

$$H(z) = \frac{z - \frac{1}{2}z^2 + \frac{1}{6}z^3}{(1 - z)^3} \quad \text{and} \quad G(z) = \frac{\frac{1}{2}z^2 + \frac{1}{6}z^3}{(1 - z)^3}.$$  

We see that the function $K$ has the dilatation $\omega(z) = z$ and $K$ maps the unit disk $\mathbb{D}$ onto the slit plane $\mathbb{C}\{u + iv : u \leq -1/6, \ v = 0\}$. Moreover,

$$H(z) = \sum_{n=2}^{\infty} A_n z^n \quad \text{and} \quad G(z) = \sum_{n=2}^{\infty} B_n z^n,$$

where

$$A_n = \frac{1}{6}(2n + 1)(n + 1) \quad \text{and} \quad B_n = \frac{1}{6}(2n - 1)(n - 1), \quad n \geq 1.$$  

A well-known coefficient conjecture of Clunie and Sheil-Small [6], is that if $f = h + \overline{g} \in S^0_H$ then the Taylor coefficients of the series of $h$ and $g$ satisfy the inequality

$$|a_n| \leq A_n \quad \text{and} \quad |b_n| \leq B_n \quad \text{for all} \quad n \geq 1.$$  

Although, the coefficients conjecture remains an open problem for the full class $S^0_H$, the same has been verified for certain subclasses, namely, the class $T_H$ (see [8 Section 6.6]) of harmonic univalent typically real functions, the class of harmonic convex functions in one direction, harmonic starlike functions in $S^H$ (see [8 Section 6.7]), and the class of harmonic close-to-convex functions (see [17]).

It is interesting to know to what extent do the conditions (1.4) influence the univalency of the normalized harmonic function $f(z)$ and of all of its partial sums, namely, $f_n(z)$ and $f_{\overline{m}}(z)$, where

$$f_n(z) = h_n(z) + g_m(z) \quad \text{if} \quad n \geq m; \quad f_{\overline{m}}(z) = h_n(z) + \overline{g_m(z)} \quad \text{if} \quad m \geq n.$$  

Here $h_n(z)$ and $g_m(z)$ represent the $n$-th section/partial sums of $h$ and $g$ given by

$$h_n(z) = z + \sum_{k=2}^{n} a_k z^k \quad \text{and} \quad g_m(z) = \sum_{k=1}^{m} b_k z^k,$$

respectively. According to our notation, the degree of the polynomials $f_n(z)$ and $f_{\overline{m}}(z)$ is $n$ if $n = m$. 

Theorem 1.5. Let $h$ and $g$ have the form (1.1) and the coefficients of the series satisfy the conditions (1.4). Then $f = h + \overline{g}$ is close-to-convex (univalent), and starlike in the disk $|z| < r_S$, where

$$r_S = 1 + \frac{\sqrt{2}}{4} - \sqrt{\frac{\sqrt{2}}{2} + \frac{1}{8}} \approx 0.112903$$

is the root of the quadratic equation

$$\sqrt{2}r^2 - (1 + 2\sqrt{2})r + \sqrt{2} - 1 = 0$$

in the interval $(0, 1)$. The result is sharp.

The radii problems for various subclasses of univalent harmonic mappings are open [2, Problem 3.3] (see also [6, 8, 15, 14]). However, Theorem 1.5 quickly yields Corollary 1.6. The radius of close-to-convexity and the radius of starlikeness for mappings in $S_H^0$ (resp. $C_H^0$ and $T_H$) is at least $0.112903$.

Under the hypotheses of Theorem 1.5, all the partial sums of $f$ are close-to-convex (univalent), and starlike in $|z| < r_S$. Similar comments apply to the next two results.

Another well-known result due to Clunie and Sheil-Small [6] states that the coefficients of the series of $h$ and $g$ of every convex function $f = h + \overline{g} \in K_H^0$ satisfy the inequalities

(1.7) $|a_n| \leq \frac{n+1}{2}$ and $|b_n| \leq \frac{n-1}{2}$ for all $n \geq 1$.

Equality occurs for the function $L = M + N \in K_H^0$, where

(1.8) $M(z) = \frac{1}{2} \left( \frac{z}{1-z} + \frac{z}{(1-z)^2} \right)$ and $N(z) = \frac{1}{2} \left( \frac{z}{1-z} - \frac{z}{(1-z)^2} \right)$.

We observe that

$$L(z) = \text{Re} \left( \frac{z}{1-z} \right) + \text{Im} \left( \frac{z}{(1-z)^2} \right) = z + \sum_{n=2}^{\infty} \frac{n+1}{2} z^n - \sum_{n=2}^{\infty} \frac{n-1}{2} z^n.$$

At this place it is worth recalling that the convexity (resp. starlikeness) property is not a hereditary property in the harmonic case, unlike the analytic case. For instance, the convex function $L$ maps the subdisk $|z| < r$ onto a convex domain for $r \leq \sqrt{2} - 1$, but onto a non-convex domain for $\sqrt{2} - 1 < r < 1$.

Theorem 1.9. Let $h$ and $g$ have the form (1.1) and the coefficients of the series satisfy the conditions (1.7). Then $f = h + \overline{g}$ is close-to-convex (univalent), and starlike in the disk $|z| < r_S$, where

$$r_S = 1 + \frac{3\sqrt{-18 + \sqrt{330}}}{6^{2/3}} - \frac{1}{\sqrt[3]{6(-18 + \sqrt{330})}} \approx 0.164878$$

is the real root of the cubic equation

$$2r^3 - 6r^2 + 7r - 1 = 0$$

in the interval $(0, 1)$. The result is sharp.
Theorem 1.9 easily gives the following corollary although Theorem 1.9 is much more stronger.

**Corollary 1.10.** The radius of close-to-convexity and the radius of starlikeness for convex mappings in $S^0_H$ is at least 0.164878.

**Theorem 1.11.** Let $h$ and $g$ have the form (1.1) with $|b_1| = |g'(0)| < 1$, and the coefficients of the series satisfy the conditions

$$|a_n| + |b_n| \leq c \quad \text{for all } n \geq 2.$$ 

Then $f = h + g$ is close-to-convex (univalent), and starlike in the disk $|z| < r_S$, where

$$r_S = 1 - \sqrt{\frac{c}{c + 1 - |b_1|}}.$$ 

The result is sharp.

Theorem 1.11 helps to improve the Bloch-Landau’s theorem for bounded harmonic functions. Consider the class $B^M_H$ of a harmonic mapping $f$ of the unit disk $D$ with $f(0) = f_z(0) = f_z(0) - 1 = 0$, and $|f(z)| < M$ for $z \in D$. There are two important constants one is relative to the domain of the function while the other one, namely the Bloch constant, is defined relative to the range. In [3], authors proved that if $f \in B^M_H$ then $f$ is univalent in $|z| < \rho_0$ and $f(|z| < \rho_0)$ contains a disk $|w| < R_0$, where

$$\rho_0 \approx \frac{1}{11.105M} \quad \text{and} \quad R_0 = \frac{\rho_0}{2} \approx \frac{1}{22.21M}.$$ 

Better estimates were given in [7, 9, 11, 12] and later in [5], see Table 1 in which the functions $\phi$ and $\psi$ are explicitly given by

$$\phi(x) = \frac{x}{\sqrt{2(x^2 + x - 1)}} \quad \text{and} \quad \psi(x) = \frac{1}{\sqrt{2}} \left[ 1 + \left( \frac{x^2 - 1}{x} \right) \log \left( \frac{x^2 - 1}{x^2 + x - 1} \right) \right].$$

This result is the best known but not sharp.

The purpose the next theorem is to give a new proof of one of these results. Indeed our method of proof is simple and improves the best known result. In fact our distortion estimate for $f \in B^M_H$ provides the radius of close-to-convexity and the radius starlikeness of $B^M_H$.

**Theorem 1.12.** Let $f \in B^M_H$. Then $f = h + g$ is close-to-convex (univalent) in the disk $|z| < r_0$, where

$$r_S = 1 - \sqrt{\frac{4M}{4M + \pi}}$$

and $f(D_{r_0})$ contains a univalent disk of radius at least

$$R_S = r_S - \frac{4M}{\pi} \frac{r_S^2}{1 - r_S}.$$
Table 1. The left side columns refer to Theorem 4 in [5] and the right side columns refer to Theorem 1.12.

2. Useful Lemmas and their Proofs

We need the following two lemmas to prove our main results.

Lemma 2.1. Let \( h \) and \( g \) have the form (1.1) with \( |b_1| < 1 \), \( f = h + \overline{g} \), and satisfy the condition

\[
\sum_{n=2}^{\infty} n|a_n| + \sum_{n=1}^{\infty} n|b_n| \leq 1.
\]

Then \( f \in C^2_H \), where \( C^2_H = \{ f \in S_H : |f(z) - 1| < 1 - |f_{z}(z)| \text{ in } D \} \). The bound in (2.2) is sharp as the harmonic function

\[
f(z) = z + \sum_{n=2}^{\infty} \frac{\epsilon_n}{n} z^n + \sum_{n=1}^{\infty} \frac{\epsilon'_n}{n} \overline{z^n},
\]

for which \( \sum_{n=2}^{\infty} |\epsilon_n| + \sum_{n=1}^{\infty} |\epsilon'_n| = 1 \), shows.

Proof. In [13], it was shown that \( \text{Re } f_z(z) > |f_{\overline{z}}(z)| \) whenever (2.2) holds. The proof of this lemma follows from an easy modification of the proof of the corresponding result from [13]. For the sake of completeness, we include the detail. Note that the coefficient inequality implies that both \( h \) and \( g \) are analytic in \( D \). Thus, \( f = h + \overline{g} \) is harmonic in \( D \). Without loss of generality, we may assume that \( f \) is not affine. Then, as \( f_z = h' \) and \( f_{\overline{z}} = g' \), it follows from the hypotheses that

\[
|h'(z) - 1| \leq \sum_{n=2}^{\infty} n|a_n| \left|\frac{z^n}{n}\right|^{n-1} \leq \sum_{n=2}^{\infty} n|a_n| \leq 1 - \sum_{n=1}^{\infty} n|b_n| \leq 1 - |g'(z)|
\]

implying that \( f \in C^2_H \) (since strict inequality occurs either at the second or fourth inequality). In particular, \( \text{Re } h'(z) > |g'(z)| \) in \( D \) and hence, \( f \) is locally univalent in \( D \).

For example, the functions

\[
f_n(z) = z + \frac{n+1}{2n^2} z^n + \frac{n-1}{2n^2} \overline{z^n} \text{ for } n \geq 2
\]
satisfy the condition (2.2) and hence, belong to the class $C^2_H$. In the following lemma, we show that functions in $C^2_H$ are indeed close-to-convex in $\mathbb{D}$.

**Lemma 2.3.** Let $h$ and $g$ have the form (1.1) with $|b_1| < 1$, $f = h + \overline{g}$. Suppose $f \in C^2_H$. Then, we have the following

(a) $f$ is close-to-convex in $\mathbb{D}$.

(b) $|a_n| - |b_n| \leq 1/n$ for $n \geq 2$ whenever $b_1 = 0$. The equality occurs, for example, for the function

$$f(z) = z + \frac{e^{i\theta}}{n}z^n \quad \text{or} \quad f(z) = z + \frac{e^{i\theta}}{n\sqrt{n}}$$

for $n \geq 2$ and $\theta$ real.

(c) $\sum_{n=2}^{\infty} n^2(|a_n|^2 + |b_n|^2) \leq 1 - |b_1|^2$.

**Proof.** First we prove part (a). Let $f = h + \overline{g} \in C^2_H$ and $F = h + e\overline{g}$, where $|\epsilon| = 1$. Then,

$$|F'(z) - 1| < |h'(z) - 1| + |g'(z)| < 1$$

showing that $F$ is analytic and close-to-convex in $\mathbb{D}$. According to Lemma A, it follows that the harmonic function $f$ is also close-to-convex (and univalent) in $\mathbb{D}$.

Next, set $\omega(z) = F'(z) - 1$. Then, as $b_1 = g'(0) = 0$, we have $\omega(0) = 0$ and $|\omega(z)| < 1$ for $z \in \mathbb{D}$. It is well-known property that the coefficients of such an analytic function $\omega$ satisfy the inequality $|\omega(n)(0)| \leq n!$ for each $n \geq 1$. This gives the estimate

$$|na_n + eb_n| \leq 1 \quad \text{for each} \quad n \geq 2.$$

As $|\epsilon| = 1$, triangle inequality gives the proof for part (b).

For the proof of part (c), we observe that

$$|F'(z) - 1| = \sum_{n=2}^{\infty} n|a_n|z^{n-1} + \epsilon \sum_{n=1}^{\infty} n|b_n|z^{n-1} < 1, \quad z \in \mathbb{D}.$$

Therefore, with $z = re^{i\theta}$ for $r \in (0, 1)$ and $0 \leq \theta \leq 2\pi$, the last inequality gives

$$\sum_{n=2}^{\infty} n^2(|a_n|^2 + |b_n|^2)r^{2(n-1)} + |b_1|^2 = \frac{1}{2\pi} \int_0^{2\pi} |F'(re^{i\theta}) - 1|^2 d\theta \leq 1.$$

Letting $r \to 1^-$, we obtain the inequality

$$\sum_{n=2}^{\infty} n^2(|a_n|^2 + |b_n|^2) \leq 1 - |b_1|^2$$

and the proof is complete. \qed

In [13], under the hypotheses of Lemma 2.1, it was actually shown that $f \in C^1_H$, where

$$C^1_H = \{f \in S_H : \text{Re} \ f_\omega(z) > |f_\omega(z)| \text{ in } \mathbb{D}\}.$$

Clearly, Lemma 2.1 improves this result because of the strict inclusion $C^2_H \subsetneq C^1_H$. Later, in [1], it was also shown that if $b_1 = g'(0) = 0$, then the coefficient condition
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(2.2) ensures that $f \in S_H^0$ (see also [16]). In view of Lemma 2.1, the result of [11, 16] may be stated in an improved form.

**Lemma 2.4.** Let $h$ and $g$ have the form (1.1) with $b_1 = g'(0) = 0$, $f = h + \overline{g}$, and satisfy the condition

\[(2.5) \quad \sum_{n=2}^{\infty} n|a_n| + \sum_{n=2}^{\infty} n|b_n| \leq 1.\]

Then $f \in C_H^2 \cap S_H^0$.

The following generalization of Lemma 2.1 is easy to obtain and so we omit its details.

**Corollary 2.6.** Let $h$ and $g$ have the form (1.1) with $|b_1| < 1 - \beta$ for some $\beta \in [0, 1)$, and $f = h + \overline{g}$. Then we have the following:

(a) If the coefficients of $h$ and $g$ satisfy the condition

\[(2.7) \quad \sum_{n=2}^{\infty} n|a_n| + \sum_{n=1}^{\infty} n|b_n| \leq 1 - \beta,\]

then $f \in C_H^2(\beta)$, where

$$C_H^2(\beta) = \{ f \in S_H : |f_z(z) - 1| < 1 - \beta - |f_z(z)| \text{ in } \mathbb{D} \}.$$  

In particular, $f$ is close-to-convex in $\mathbb{D}$. The bound here is sharp as the harmonic function

\[f(z) = z + \sum_{n=2}^{\infty} \frac{\epsilon_n}{n} z^n + \sum_{n=1}^{\infty} \frac{\epsilon'_n}{n} \overline{z^n},\]

for which $\sum_{n=2}^{\infty} |\epsilon_n| + \sum_{n=1}^{\infty} |\epsilon'_n| = 1 - \beta$, shows.

(b) If $f \in C_H^2(\beta)$, then one has

$$|a_n| - |b_n| \leq (1 - \beta)/n \text{ for } n \geq 2 \text{ whenever } b_1 = 0.$$

The equality occurs, for example, for the function

\[f(z) = z + (1 - \beta)\frac{e^{i\theta}}{n} z^n \text{ or } f(z) = z + (1 - \beta)\frac{e^{i\theta}}{n} \overline{z^n} \text{ for } n \geq 2 \text{ and } \theta \text{ real.}\]

We also have

$$\sum_{n=2}^{\infty} n^2(|a_n|^2 + |b_n|^2) \leq (1 - \beta)^2 - |b_1|^2.$$

It is a matter of routine checking to see that the coefficient condition (2.7) is necessary for $f = h + \overline{g}$ to belong to $C_H^2(\beta)$ whenever the Taylor coefficients $a_n \leq 0$ for all $n \geq 2$, and $b_n \leq 0$ for all $n \geq 1$. 


3. Proofs of Main Theorems

Proof of Theorem 1.5. Let \( h \) and \( g \) have the form (1.1) satisfying the coefficient conditions (1.4). First we observe that \( b_1 = g'(0) = 0 \). The conditions (1.4) implies that the series (1.1) are convergent in the unit disk \( |z| < 1 \), and hence, the sum \( h \) and \( g \) are analytic in \( \mathbb{D} \). Thus, \( f = h + g \) is harmonic in \( \mathbb{D} \). Let \( 0 < r < 1 \), we let \( f_r(z) := r^{-1}f(rz) = r^{-1}h(rz) + r^{-1}g(rz) \) so that \( f_r(z) = h_r(z) + g_r(z) \) and

\[
f_r(z) = z + \sum_{n=2}^\infty a_n r^{n-1} z^n + \sum_{n=2}^\infty b_n r^{n-1} z^n, \quad z \in \mathbb{D}.
\]

By hypotheses, \( |a_n| \leq A_n \) and \( |b_n| \leq B_n \) for \( n \geq 2 \), where \( A_n \) and \( B_n \) are given by (1.3). Using these coefficient estimates, we obtain

\[
S = \sum_{n=2}^\infty n|a_n|r^{n-1} + \sum_{n=2}^\infty n|b_n|r^{n-1} \\
\leq \sum_{n=2}^\infty nA_n r^{n-1} + \sum_{n=2}^\infty nB_n r^{n-1}.
\]

We show that \( f_r \in C^2_H \cap S^0_H \). According to Lemma 2.4, it suffices to show that \( S \leq 1 \). By the last inequality, \( S \leq 1 \) if \( r \) satisfies the inequality

\[
\sum_{n=2}^\infty nA_n r^{n-1} \leq 1 - \sum_{n=2}^\infty nB_n r^{n-1},
\]

or equivalently (as \( A_n + B_n = (2n^2 + 1)/3 \)),

\[
2 \sum_{n=2}^\infty n^3 r^{n-1} + \sum_{n=2}^\infty nr^{n-1} \leq 3.
\]

As

\[
\frac{r}{(1-r)^2} = \sum_{n=1}^\infty nr^n \quad \text{and} \quad \frac{r(1+r)}{(1-r)^3} = \sum_{n=1}^\infty n^2 r^n,
\]

it follows that

\[
\frac{(1-r)(1+2r) + 3r(1+r)}{(1-r)^4} = \sum_{n=1}^\infty n^3 r^{n-1}
\]

and (3.1) reduces to the inequality,

\[
\frac{2(r^2 + 4r + 1)}{(1-r)^4} + \frac{1}{(1-r)^2} \leq 6, \quad \text{i.e.} \quad 2(1-r)^4 - (1+r)^2 \geq 0.
\]

This gives

\[
\sqrt{2}(1-r)^2 - (1+r) = \sqrt{2}r^2 - (1 + 2\sqrt{2})r + \sqrt{2} - 1 \geq 0.
\]
Thus, from Lemma 2.4, \( f_r \) is close-to-convex (univalent) in \( D \) and starlike in \( D \) for all \( 0 < r \leq r_S \), where \( r_S \) is the root of the quadratic equation
\[
\sqrt{2} r^2 - (1 + 2\sqrt{2}) r + \sqrt{2} - 1 = 0
\]
in the interval \((0, 1)\). In particular, \( f \) is close-to-convex (univalent) and starlike in \( |z| < r_S \).

Next, to prove the sharpness part of the statement of the theorem, we consider the function
\[
F_0(z) = H_0(z) + \overline{G_0(z)}
\]
with
\[
H_0(z) = 2z - H(z) \quad \text{and} \quad G_0(z) = -\overline{G(z)}.
\]
Here \( H \) and \( G \) are defined by (1.2). We note that
\[
F_0(z) = z - \sum_{n=2}^{\infty} A_n z^n - \sum_{n=2}^{\infty} B_n z^n.
\]
As \( F_0 \) has real coefficients we obtain.
\[
J_{F_0}(r) = (H'_0(r) + G'_0(r))(H'_0(r) - G'_0(r))
\]
\[
= \left( 1 - \sum_{n=2}^{\infty} nA_n r^{n-1} - \sum_{n=2}^{\infty} nB_n r^{n-1} \right) \left( 1 - \sum_{n=2}^{\infty} n(A_n - B_n) r^{n-1} \right)
\]
\[
= \left( 1 - \sum_{n=2}^{\infty} \frac{n(2n^2 + 1)}{3} r^{n-1} \right) \left( 1 - \sum_{n=2}^{\infty} n^2 r^{n-1} \right)
\]
\[
= \left( 1 - \frac{-4r^2 + 3r^3 - r^4}{(-1 + r)^3 r} \right) \left( 1 + \frac{-6r^2 + 5r^3 - 4r^4 + r^5}{(-1 + r)^4 r} \right)
\]
\[
= \frac{(-1 + 7r - 6r^2 + 2r^3)(1 - 10r + 11r^2 - 8r^3 + 2r^4)}{(-1 + r)^7}.
\]
Thus \( J_{F_0}(r) = 0 \), \( 0 < r < 1 \) if and only if
\[
r = r_S = \frac{1}{4} \left( 4 + \sqrt{2} - \sqrt{2 + 16\sqrt{2}} \right) \approx 0.112903
\]
or
\[
r = r'_S = 1 + \left( -18 + \sqrt{330} \right)^{1/3} 6^{-2/3} - \left( 6 \left( -18 + \sqrt{330} \right) \right)^{-1/3} \approx 0.164878.
\]
Moreover for \( r_S < r < r'_S \) we have \( J_{F_0}(r) < 0 \). The graph of the function \( J_{F_0}(r) \) for \( r \in (0, 0.25) \) is shown in Figure 1.

This observation together with Lewy’s theorem gives that (as the Jacobian changes sign), the function \( F_0(z) \) is not univalent in \( |z| < r \) if \( r > r_S \), and thus, \( r_S \) cannot be replaced by a larger number.

**Proof of Theorem 1.9.** Following the notation and the method of the proof of Theorem 1.5, it suffices to show that \( f_r \in \mathcal{C}_H^2 \cap \mathcal{S}_H^0 \). According to Lemma 2.4
Figure 1. The graph of the Jacobian $J_{F_0}(r)$ for $r \in (0, 0.25)$.

$f_r \in C_H^2 \cap \mathcal{S}_H^{s_0}$ whenever $S \leq 1$, where

$$S = \sum_{n=2}^{\infty} n|a_n|r^{n-1} + \sum_{n=2}^{\infty} n|b_n|r^{n-1}$$

when $a_n$ and $b_n$ satisfy the coefficient inequalities given by (1.7). Finally, using (1.7), we see that $S \leq 1$ if $r$ satisfies the inequality

$$\sum_{n=2}^{\infty} \frac{n(n+1)}{2}r^{n-1} \leq 1 - \sum_{n=2}^{\infty} \frac{n(n-1)}{2}r^{n-1}.$$ 

The last inequality is easily seen to be equivalent to

$$\frac{1}{2} \left[ \frac{1}{(1-r)^2} + \frac{1+r}{(1-r)^3} - 1 \right] \leq 1 + \frac{1}{2} \left[ \frac{1}{(1-r)^2} - \frac{1+r}{(1-r)^3} - 1 \right]$$

which upon simplification reduces to

$$2(1-r)^3 - 1 - r = -(2r^3 - 6r^2 + 7r - 1) \geq 0.$$ 

The first part of the conclusion easily follows as in the proof of Theorem 1.5.

The sharpness part of the statement of Theorem 1.9 follows if we consider the function

$$L_0(z) = 2z - M(z) - N(z),$$

where $M$ and $N$ are defined by (1.8). We note that

$$L_0(z) = z - \sum_{n=2}^{\infty} \frac{n+1}{2}z^n + \sum_{n=2}^{\infty} \frac{n-1}{2}z^n.$$
Figure 2. The graph of the Jacobian \( J_{L_0}(r) \) for \( r \in (0, 0.35) \).

Again, as \( L_0 \) has real coefficients, we can easily obtain that for \( r \in (0, 1) \)
\[
J_{L_0}(r) = (2 - M'(r))^2 - (N'(r))^2 \\
= (2 - M'(r) + N'(r))((2 - M'(r) - N'(r)) \\
= \left(2 - \frac{1 + r}{(1 - r)^3}\right)\left(2 - \frac{1}{(1 - r)^2}\right) \\
= \frac{2}{(1 - r)^3} (2 - (1 + r)) \left(r - 1 - \frac{\sqrt{2}}{2}\right) \left(r - 1 + \frac{\sqrt{2}}{2}\right).
\]

We see that \( J_{L_0}(r_S) = 0, 0 < r < 1 \) if and only if
\[
r = r_s \approx 0.16487
\]
or
\[
r = r'_s = \frac{2 - \sqrt{2}}{2} \approx 0.292893.
\]

Moreover for \( r_S < r < r'_S \) we have \( J_{L_0}(r) < 0 \). The graph of the function \( J_{L_0}(r) \) for \( r \in (0, 0.35) \) is shown in Figure 2.

Thus, according to Lewy’s theorem, \( L_0(z) \) is not univalent in \( |z| < r \) if \( r > r_S \) and this observation shows that \( r_S \) cannot be replaced by a larger number.

Proof of Theorem 1.11. This time we apply Lemma 2.1 and show that \( f_r \) defined by \( f_r(z) := r^{-1}f(rz) = r^{-1}h(rz) + r^{-1}g(rz) \) belongs to \( C^2_H \).

As in the proof of previous two theorems, it suffices to show the corresponding coefficient inequality (2.2), namely,
\[
S = \sum_{n=2}^{\infty} n(|a_n| + |b_n|)r^{n-1} + |b_1| \leq 1.
\]
By the hypothesis, \(|a_n| + |b_n| \leq c\) for all \(n \geq 2\) and so, the last inequality \(S \leq 1\) clearly holds if \(r\) satisfies the inequality

\[
c \left( \frac{1}{(1-r)^2} - 1 \right) \leq 1 - |b_1|, \quad \text{i.e.} \quad r \leq r_S = 1 - \sqrt{\frac{c}{c+1-|b_1|}}.
\]

Thus, by Lemma 2.1

\[
|h'_r(z) - 1| < 1 - |g'_r(z)|
\]

holds for all \(z \in \mathbb{D}\) whenever \(r \leq r_S\). Thus, \(f \in C^2_H\).

The function \(f_0(z) = h_0(z) + g_0(z)\), where

\[
h_0(z) = z - \frac{c}{2} \left( \frac{z^2}{1-z} \right) \quad \text{and} \quad g_0(z) = -|b_1|z - \frac{c}{2} \left( \frac{z^2}{1-z} \right),
\]

shows that the result is sharp. Indeed, it is easy to compute that

\[
J_{f_0}(r) = |h'_r(r)|^2 - |g'_r(r)|^2 = (1 + |b_1|) \left( 1 + c - |b_1| - \frac{c}{(1-r)^2} \right)
\]

which shows that \(J_{f_0}(r_S) = 0\) and \(J_{f_0}(r) < 0\) for \(r > r_S\). The proof of the theorem is complete. □

**Proof of Theorem 1.12.** Let \(f = h + \overline{g}\) be a harmonic mapping defined on the unit disk \(\mathbb{D}\) with \(f(0) = f_\overline{z}(0) = f_z(0) - 1 = 0\), and \(|f(z)| < M\) for \(z \in \mathbb{D}\), where \(h\) and \(g\) have the form (1.1) with \(b_1 = 0\). According to [4, Lemma 1] (see also [5]), we obtain the sharp estimates

(3.2) \[|a_n| + |b_n| \leq \frac{4M}{\pi} \quad \text{for any} \quad n \geq 1.\]

As \(b_1 = 0\) and \(a_1 = 1\), it follows that \(M \geq \pi/4 \approx 0.785398\). By Theorem 1.11 with \(c = 4M/\pi\), we conclude that \(f\) is close-to-convex and starlike (because \(b_1 = 0\)) for \(|z| < 1 - \sqrt{c/(c+1)} = r_S\).

In particular, \(f\) is univalent for \(|z| < r_S\) and furthermore, we have for \(|z| = r_S\),

\[
|f(z)| = \left| z + \sum_{n=2}^{\infty} (a_n z^n + \overline{b_n} z^n) \right| \\
\geq \quad |z| - \sum_{n=2}^{\infty} (|a_n| + |b_n|) r_S^n \\
\geq \quad r_S - \sum_{n=2}^{\infty} (|a_n| + |b_n|) r_S^n \\
\geq \quad r_S - \frac{4M}{\pi} \sum_{n=2}^{\infty} r_S^n \\
\geq \quad r_S - \frac{4M}{\pi} \frac{r_S^2}{1-r_S} = R_S
\]

and the proof is complete. □
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