ON THE TOPOLOGY OF MANIFOLDS WITH
POSITIVE ISOTROPIC CURVATURE

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Abstract. We show that a closed orientable Riemannian \(n\)-manifold, \(n \geq 5\), with positive isotropic curvature and free fundamental group is homeomorphic to the connected sum of copies of \(S^{n-1} \times S^1\).

1. Introduction

Let \((M, g)\) be a closed, orientable, Riemannian manifold with positive isotropic curvature. By [9], if \(M\) is simply-connected then \(M\) is homeomorphic to a sphere of the same dimension. We shall generalise this to the case when the fundamental group of \(M\) is a free group.

Theorem 1.1. Let \(M\) be a closed, orientable Riemannian \(n\)-manifold with positive isotropic curvature. Suppose that \(\pi_1(M)\) is a free group on \(k\) generators. Then, if \(n \neq 4\) or \(k = 1\) (i.e. \(\pi_1(M) = \mathbb{Z}\)), \(M\) is homeomorphic to the connected sum of \(k\) copies of \(S^{n-1} \times S^1\).

We note that a conjecture of M. Gromov ([4] Section 3 (b)) and A. Fraser [2], based on the work of Micallef-Wang [8], states that any compact manifold with positive isotropic curvature has a finite cover satisfying our hypothesis.

Conjecture 1 (M. Gromov-A. Fraser). \(\pi_1(M)\) is virtually free, i.e., it is a finite extension of a free group.

It is known by the work of A. Fraser [2] and A. Fraser and J. Wolfson [3] that \(\pi_1(M)\) does not contain any subgroup isomorphic to the fundamental group of a closed surface of genus at least one.

Our starting point is the following fundamental result of M. Micallef and J. Moore [9].

Theorem 1.2 (M. Micallef-J. Moore). Suppose \(M\) is a closed manifold with positive isotropic curvature. Then \(\pi_i(M) = 0\) for \(2 \leq i \leq \frac{n}{2}\).

It is clear that the following purely topological result, together with the Micallef-Moore theorem, implies Theorem 1.1.

Theorem 1.3. Let \(M\) be a smooth, orientable, closed \(n\)-manifold such that \(\pi_1(M)\) is a free group on \(k\) generators and \(\pi_i(M) = 0\) for \(2 \leq i \leq \frac{n}{2}\). If \(n \neq 4\) or \(k = 1\), then \(M\) is homeomorphic to the connected sum of \(k\) copies of \(S^{n-1} \times S^1\).

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Henceforth let $M$ be a smooth, orientable, closed $n$-manifold such that $\pi_1(M)$ is a free group on $k$ generators and $\pi_i(M) = 0$ for $2 \leq i \leq \frac{n}{2}$. We assume throughout that all manifolds we consider are orientable.

Let $\tilde{M}$ be the universal cover of $M$. Hence $\pi_1(\tilde{M})$ is trivial and so is $\pi_i(\tilde{M}) = \pi_i(M)$ for $2 \leq i \leq \frac{n}{2}$. We shall show that the homology of $\tilde{M}$ is isomorphic as $\pi_1(M)$-modules to that of the connected sum of $k$ copies of $S^{n-1} \times S^1$. We then show that $\tilde{M}$ is homotopy equivalent to the connected sum of $k$ copies of $S^{n-1} \times S^1$ using Theorems of Whitehead. Finally, recent results of Kreck and Lück allow us to conclude the result.

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2. The homology of $\tilde{M}$

Let $X$ denote the wedge $\bigvee_{i=1}^k S^1$ of $k$ circles and let $x$ denote the common point on the circles. Choose and fix an isomorphism $\varphi$ from $\pi_1(M, p)$ to $\pi_1(X, x)$ for some basepoint $p \in M$. We shall use this identification throughout. Denote $\pi_1(M, p) = \pi_1(X, x)$ by $\pi$.

As $X$ is an Eilenberg-MacLane space, there is a map $f : (M, p) \to (X, z)$ inducing $\varphi$ on fundamental groups and a map $s : (X, z) \to (M, p)$ so that $f \circ s : X \to X$ is homotopic to the identity.

We deduce the homology of $\tilde{M}$ using the Hurewicz Theorem and Poincaré duality.

**Lemma 2.1.** For $1 \leq i \leq n/2$, $H_i(\tilde{M}, \mathbb{Z}) = 0$

**Proof.** As $\tilde{M}$ is simply-connected and $\pi_i(\tilde{M}) = \pi_i(M) = 0$ for $1 < i \leq n/2$ (by hypothesis), by Hurewicz theorem $H_i(\tilde{M}, \mathbb{Z}) = 0$ for $1 \leq i \leq n/2$. $\square$

We deduce the homology in dimensions above $n/2$ using Poincaré duality for $M$ with coefficients in the module $\mathbb{Z}[\pi]$, namely

$$H_{n-i}(M, \mathbb{Z}[\pi]) = H^i(M, \mathbb{Z}[\pi])$$

Recall that $H_k(M, \mathbb{Z}[\pi]) = H_k(\tilde{M}, Z)$ and the group $H^i(M, \mathbb{Z}[\pi])$ is the cohomology with compact support $H^i_c(M, \mathbb{Z})$. Hence Poincaré duality with coefficients in $\mathbb{Z}[\pi]$ is the same as Poincaré duality for a non-compact manifold relating homology to cohomology with compact support.

To apply Poincaré duality, we need the following lemma.

**Lemma 2.2.** For $1 \leq i \leq n/2$, the map $s : (X, z) \to (M, p)$ induces isomorphisms of modules with $s_* : H^i(M; \mathbb{Z}[\pi]) \to H^i(X; \mathbb{Z}[\pi])$.

**Proof.** As the map $s$ induces an isomorphism on homotopy groups in dimensions at most $n/2$, it induces isomorphisms on the cohomology groups with twisted coefficients. Specifically, we can add cells of dimensions $k \geq n/2 + 2$ to $M$ to obtain an Eilenberg-MacLane space $\tilde{M}$ for the group $\pi$, which is thus homotopy equivalent to $X$. For $i \leq n/2$ and any $\mathbb{Z}[\pi]$-module $A$, it follows that

$$H_i(M, A) = H_i(\tilde{M}, A) = H_i(X, A)$$

where the first equality follows as the cells added to $M$ to obtain $\tilde{M}$ are of dimension at least $n/2 + 2$ and the second as the spaces are homotopy equivalent. $\square$

By applying Poincaré duality, we obtain the following result.
Lemma 2.3. Let $M$ be a smooth, orientable, closed $n$-manifold such that $\pi_1(M)$ is a free group on $k$ generators and $\pi_i(M) = 0$ for $2 \leq i \leq \frac{n}{2}$. Then, for the universal cover $\tilde{M}$ of $M$,

1. $H_i(\tilde{M}, \mathbb{Z}) = 0$ for $1 \leq i < n - 1$
2. We have an isomorphism $H_n(\tilde{M}, \mathbb{Z}) = H_n(\tilde{X}, \mathbb{Z})$, where $\tilde{X}$ is the universal cover of $X$, determined by the isomorphisms $s_* : \pi_n(X, z) \to \pi_n(M, p)$ on fundamental groups.

Proof. The statements follow from Lemmas 2.1 and 2.2 by using $H_*(\tilde{M}, \mathbb{Z}) = H_*(M, \mathbb{Z}[[\pi]])$. \hfill $\square$

3. Homotopy type

We now show that $M$ is homotopy equivalent to the connected sum $Y$ of $k$ copies of $S^{n-1} \times S^1$. Our first step is to construct a map $g : Y \to M$. We shall then show that it is a homotopy equivalence.

Note that $Y$ has the structure of a CW-complex obtained as follows. The 1-skeleton of $Y$ is the wedge $X$ of $k$ circles. Let $\alpha_i$ denote the $i$th circle with a fixed orientation.

We attach $k$ $(n-1)$-cells $D_j$, with the $j$th attaching map mapping $\partial D^{n-1}$ to the midpoint $x_j$ of the $j$th circle. Finally, we attach a single $n$-cell $\Delta$.

We associate to $D_j$ an element $A_j \in \pi_{n-1}(Y, x)$. Namely, as the attaching map is constant, the $j$th $(n-1)$-cell gives an element $B_j \in \pi_{n-1}(Y, x_j)$. We consider the subarc $\beta_j$ of $\alpha_j$ joining $z_j$ to $x$ in the negative direction and let $A_j$ be obtained from $B_j$ by the change of basepoint isomorphism using $\beta_j$.

Note that if we instead chose the arc joining $z_j$ to $x$ in the positive direction, then the resulting element is $-\alpha_j \cdot A_j$. By the construction of $Y$, it follows that the attaching map of the $(n-1)$-cell represents the element

$$\partial \Delta = \Sigma_j(A_j - \alpha_j \cdot A_j)$$

in $\pi_{n-1}(Y)$ regarded as a module over $\pi_1(Y)$. This can be seen for instance by using Poincaré duality.

We now construct the map $g : Y \to M$. Recall that we have a map $s : (X, z) \to (M, p)$ inducing the isomorphism $\varphi^{-1}$ on fundamental groups. We define $g$ on the 1-skeleton $X$ of $Y$ by $g|_X = s$. We henceforth identify the fundamental groups of $Y$ and $M$ using the isomorphism $\varphi$, i.e., $\pi_1(Y, z)$ is identified with $\pi$.

We next extend $g$ to the $n$-cell of $Y$ as follows. By Hurewicz theorem and Lemma 2.3 we have isomorphisms of $\pi$-modules $\pi_{n-1}(M, p) = H_{n-1}(\tilde{M}, \mathbb{Z})$ and $\pi_{n-1}(Y, z) = H_{n-1}(\tilde{X}, \mathbb{Z})$. By Lemma 2.3 each of these modules is isomorphic to $H_n^c(\tilde{X}, \mathbb{Z})$ with the isomorphisms determined by the identifications of the fundamental groups.

Under the above isomorphisms the elements $A_j$ correspond to elements $A_j'$ in $\pi_{n-1}(M, p)$. Consider the element $B_j'$ of $\pi_{n-1}(M, g(z_j))$ obtained from $A_j'$ by the basechange map using the arc $f(\beta_j)$. We define the map $g$ on $D_j$ extending the constant map on its boundary to be a representative of $B_j'$.

As the $\pi$-modules $\pi_{n-1}(M, p)$ and $\pi_{n-1}(Y, z)$ are isomorphic, the image $g$ of $\partial \Delta$ is homotopically trivial. Hence we can extend the map $g$ across the cell $\Delta$.

Lemma 3.1. The map $g : Y \to M$ is a homotopy equivalence.
Proof. Let $G: \tilde{Y} \to \tilde{M}$ be the induced map on the universal covers. By Lemma 2.3 applied to $M$ and $Y$, we see that $H_p(\tilde{Y}) = H_p(\tilde{M}) = 0$ for $0 < p \neq n - 1$ and $G$ induces an isomorphism on $H_{n-1}$. Thus the map $G$ is a homology equivalence. By a theorem of Whitehead [10], a homology equivalence between simply-connected CW-complexes is a homotopy equivalence.

It follows that $G$ induces isomorphisms $G_* : \pi_k(\tilde{Y}) \to \pi_k(\tilde{M})$ for $k > 1$. As covering maps induce isomorphisms on higher homotopy groups, and $g$ induces an isomorphism on $\pi_1$, it follows that $g$ is a weak homotopy equivalence, hence a homotopy equivalence(see [5]).

4. Proof of Theorem 1.3

The rest of the proof of Theorem 1.3 is based on results of Kreck-Lück [7]. In [7], the authors define a manifold $N$ to be a Borel manifold if any manifold homotopy equivalent to $N$ is homeomorphic to $N$. We have shown that a manifold $M$ satisfying the hypothesis of Theorem 1.3 is homotopy equivalent to the connected sum $Y$ of $k$ copies of $S^{n-1} \times S^1$. Hence it suffices to observe that $Y$ is Borel.

By Theorem 0.13(b) of [7], the manifold $S^{n-1} \times S^1$ is Borel for $n \geq 4$. This completes the proof in the case when $\pi_1(M) = \mathbb{Z}$. Further, if $n \geq 5$, then Theorem 0.9 of [7] says that the connected sum of Borel manifolds is Borel, hence $Y$ is Borel. This concludes the proof for $\pi_1(M)$ a free group and $n \geq 5$.

Finally, in the case when $n = 3$ by the Kneser conjecture (proved by Stallings) the manifold $M$ is a connected sum of manifolds whose fundamental group is $\mathbb{Z}$. As $M$ is orientable, it follows that if $M$ is expressed as a connected sum of prime manifolds (such a decomposition exists and is unique by the Kneser-Milnor theorem), then each prime component is either $S^2 \times S^1$ or a homotopy sphere. By the Poincaré conjecture (Perelman’s theorem), every homotopy 3-sphere is homeomorphic to a sphere. It follows that $M$ is the connected sum of $k$ copies of $S^2 \times S^1$. □

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