Ordinal ultrafilters versus P-hierarchy

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Abstract

An earlier paper, entitled “P-hierarchy on $\beta\omega$”, investigated the relations between ordinal ultrafilters and the so-called P-hierarchy. The present paper focuses on the aspects of characterization of classes of ultrafilters of finite index, existence, generic existence and the Rudin-Keisler-order.

1 Introduction

Ultrafilters on $\omega$ may be classified with respect to sequential contours of different ranks, that is, iterations of the Fréchet filter by contour operations. This way an $\omega_1$-sequence $\{P_\alpha\}_{1 \leq \alpha \leq \omega_1}$ of pairwise disjoint classes of ultrafilters - the P-hierarchy - is obtained, where P-points correspond to the class $P_2$, allowing us to look at the P-hierarchy as an extension of P-points. Section 2 recalls all necessary definitions and properties of the P-hierarchy. Section 3 shows some equivalent conditions for an ultrafilter to belong to a class of (fixed) finite index of the P-hierarchy; those conditions appear to be very similar to the behavior of classical P-points. We also obtain another condition for belonging to a class of (fixed) finite index of the P-hierarchy which is literally a part of conditions for being an element of a class of (fixed) finite index of ordinal ultrafilters. Section 4 focuses on the Rudin-Keisler order on P-hierarchy classes. It is shown that RK minimal elements of classes of finite index can exist. Similar results are achieved for ordinal ultrafilters. In section 5 we show evidence for the generic existence of the P-hierarchy being equivalent to $\mathfrak{d} = \mathfrak{c}$, in consequence, being equivalent to the generic existence

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of ordinal ultrafilters. In section 6 we prove that CH implies that each class of the P-hierarchy is not empty, we also presented known results concerning existence of both types of ultrafilters.

We generally use standard terminology, however less popular terms are taken from [6], where key-term ”monotone sequential cascades” has been introduced. All necessary information may also be found in [18]. For additional information regarding sequential cascades and contours a look at [6], [7], [5], [16] is recommended. Below, only the most important definitions and conventions are repeated.

If $u$ is a filter on $A \subset B$, then we identify $u$ with the filter on $B$ for which $u$ is a filter-base.

Let $p$ be a filter on $X$, and let $q$ be a filter on $Y$; we say that $p$ is Rudin-Keisler greater than $q$ (we write $p \geq_{RK} q$) if there is such a map $f : X \to Y$ that $f(p) \supset q$. We say that $p$ is infinite Rudin-Keisler greater than $q$ (we write $p >_{\infty} q$) if there is a map $f : X \to Y$ with $f(p) = q$, but there is no $P \in p$ such that $f |_P$ is finite-to-one. We say that $p$ is greater than $q$ if $q \subset p$.

Recall also that if $p$, $q$ are ultrafilters, $f(p) = q$ and if $p \approx q$ (i.e. $p \geq_{RK} q$ and $q \geq_{RK} p$), then there exists $P \in p$ such that $f |_P$ is one-to-one (see [3, Theorem 9.2]).

The cascade is a well founded tree i.e. a tree $V$ without infinite branches and with a least element $\emptyset_V$. A cascade is sequential if for each non-maximal element of $V$ ($v \in V \setminus \text{max } V$) the set $v^+$ of immediate successors of $v$ (in $V$) is countably infinite. For $v \in V$ we write $v^-$ to denote such an element of $V$ that $v \in (v^-)^+$. For $A \subset V$ we use $A^+ = \bigcup_{v \in A} v^+$, $A^- = \bigcup_{v \in A} v^-$. In symbols $v^+, v^-, A^+, A^-$ we omit the name of cascade (obtaining $v^+$, $v^-$, $A^+$, $A^-$) if it is clear from the context which cascade we have on mind. If $v \in V \setminus \text{max } V$, then the set $v^+$ (if infinite) may be endowed with an order of the type $\omega$, and then by $(v_n)_{n \in \omega}$ we denote the sequence of elements of $v^+$, and by $v_{n+} -$ the $n$-th element of $v^+$.

The rank of $v \in V$ ($r_V(v)$ or $r(v)$) is defined inductively as follows: $r(v) = 0$ if $v \in \text{max } V$, and otherwise $r(v)$ is the least ordinal greater than the ranks of all immediate successors of $v$. The rank $r(V)$ of the cascade $V$ is, by definition, the rank of $\emptyset_V$. If it is possible to order all sets $v^+$ (for $v \in V \setminus \text{max } V$) so that for each $v \in V \setminus \text{max } V$ the sequence $(r(v_n))_{n \in \omega}$ is non-decreasing, then the cascade $V$ is monotone, and we fix such an order on $V$ without indication.

Let $W$ be a cascade, and let $(V_w)_{w \in \text{max } W}$ be a pairwise disjoint sequence of cascades such that $V_w \cap W = \emptyset$ for all $w \in \text{max } W$. Then, the confluence of cascades $V_w$ with respect to the cascade $W$ (we write $W \leftrightarrow V_w$) is defined as a cascade constructed by the identification $w \in \text{max } W$ with $\emptyset_{V_w}$ and according to the following rules: $\emptyset_w = \emptyset_{W \leftrightarrow V_w}$, if $w \in W \setminus \text{max } W$, then
\[ w^+ \rightarrow V_w = w^+ W; \text{ if } w \in V_{w_0} \text{ (for a certain } w_0 \in \max W), \text{ then } w^+ \rightarrow V_w = w^+ V_{w_0}; \text{ in each case we also assume that the order on the set of successors remains unchanged.} \]

If \( P = \{p_s : s \in S\} \) is a family of filters on \( X \) and if \( q \) is a filter on \( S \), then the **contour of \( \{p_s\} \) along \( q \)** is a filter on \( X \) defined by

\[
\int_q P = \int_q p_s = \bigcup_{Q \in q} \bigcap_{s \in Q} p_s.
\]

Such a construction has been used by many authors \([8], [9], [10]\) and is also known as a sum (or as a limit) of filters. On the sequential cascade, we consider the finest topology such that for all but the maximal elements \( v \) of \( V \), the co-finite filter on the set \( v^+ V \) converges to \( v \). For the sequential cascade \( V \) we define the **contour of** \( V \) (we write \( \int V \)) as the trace on \( \max V \) of the neighborhood filter of \( \emptyset V \) (the **trace** of a filter \( u \) on a set \( A \) is the family of intersections of elements of \( u \) with \( A \)). Equivalently we may say that \( \int V \) is a Fréchet filter on \( \max V \) if \( r(V) = 1 \), and \( \int V = \int F \cup \int V_n \) if \( V = (n) \triangleleft \emptyset V \).

Let \( V \) be a monotone sequential cascade and let \( u = \int V \). Then the **rank** \( r(u) \) of \( u \) is, by definition, the rank of \( V \). It was shown in \([7]\) that if \( \int V = \int W \), then \( r(V) = r(W) \).

Let \( S \) be a countable set. A family \( \{u_s\}_{s \in S} \) of filters is referred to as **discrete** if there exists a pairwise disjoint family \( \{U_s\}_{s \in S} \) of sets such that \( U_s \in u_s \) for each \( s \in S \). For \( v \in V \) we denote by \( v^+ \) a subcascade of \( V \) built by \( v \) and all successors of \( v \). If \( U \subset \max V \) and \( U \in \int V \), then by \( U^+ \) we denote the biggest (in the set-theoretical order) \(^1\) monotone sequential subcascade of the cascade \( V \) built of some \( v \in V \) such that \( U \cap \max v^+ \neq \emptyset \). We write \( v^+ \) and \( U^+ \) instead of \( v^+ V \) and \( U^+ V \) if we know in which cascade the subcascade is considered. By \( V_n \) we usually denote \( (\emptyset V)_n \), by \( V_{n,m} \) we understand \( ((\emptyset V)^+_n)^+_m \).

## 2 Ordinal ultrafilters and classes \( P_\alpha \)

In the remainder of this paper each filter is considered to be on \( \omega \), unless otherwise indicated. Let us define \( P_\alpha \) for \( 1 \leq \alpha < \omega_1 \) on \( \beta \omega \) (see \([13]\)) as follows: \( u \in P_\alpha \) if there is no monotone sequential contour \( \mathcal{V}_\alpha \) of rank \( \alpha \) such that \( \mathcal{V}_\alpha \subset u \), and for each \( \beta \) in the range \( 1 \leq \beta < \alpha \) there exists a monotone

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\(^1\) i.e. the cascade \( V \) is greater than the cascade \( W \) if \( V \subset W \), \( \emptyset V = \emptyset W \), \( w^+ W \subset w^+ V \) for all \( w \in W \), \( \max W \subset \max V \).
sequential contour $V_\beta$ of rank $\beta$ such that $V_\beta \subset u$. Moreover, if for each $\alpha < \omega_1$ there exists a monotone sequential contour $V_\alpha$ of rank $\alpha$ such that $V_\alpha \subset u$, then we write $u \in \mathcal{P}_{\omega_1}$.

Let us recall three equivalent definitions of P-points: a point $u \in \beta \omega \setminus \omega$ is a P-point if

A) the intersection of countably many neighborhoods of $u$ is a (not necessarily open) neighborhood of $u$;

B) for each countable set $\{U_n\}_{n<\omega}$ of elements of the ultrafilter $u$ there exists a set $U \in u$ such that $\text{card}(U \setminus U_n)$ is finite for each $n < \omega$;

C) for each function $f : \omega \to \omega$ there exists a set $U \in u$ such that either $f|_U$ is constant or $f|_U$ is finite-to-one.

Remark 2.1. If $u$ is an ultrafilter on $\omega$ then:

1) $u \in \mathcal{P}_1$ if and only if $u$ is a principal ultrafilter;
2) if $u$ is RK-minimal then $u \in \mathcal{P}_2$. $lacksquare$

Let $M$ be a countably infinite set, and let $V$ be a monotone cascade of rank $\alpha < \omega_1$ such that $\max V = M$. Then the set $D = \{D_v = \max v^+: v \in V, r(v) \geq 1\}$ is called an $\alpha$-partition (of $M$).

Thus, the classic “partitions of $\omega$ into infinitely many infinite sets” belong to “2-partitions” in our language. Since a cascade uniquely defines a partition, it is usually identified with its cascade. For an $\alpha$-partition we define by transfinite induction residual sets as follows: a set $A$ is residual for a 1-partition $V$ if $A \cap \max V$ is finite; if residual sets are defined for all $\beta$-partitions for $\beta < \alpha$, then a set $A$ is residual for the $\alpha$-partition $V = (n) \leftrightarrow V_n$ if there exists a finite set $N \subset \omega$ such that for all $n \notin N$ the set $A$ is residual for the partitions $V_n$. For a partition defined by a monotone sequential cascade $V$, equivalently we can say that $U$ is residual if and only if $\omega \setminus U \in \int V$.

Certain properties of the P-hierarchy from [18] are listed below, namely Proposition 2.1 and Theorems 2.3, 2.5, 2.9, 2.8.

Proposition 2.2. An ultrafilter $u$ (on $\omega$) is a P-point if and only if $u \in \mathcal{P}_2$.

Theorem 2.3. Let $u \in \mathcal{P}_\alpha$ and let $f : \omega \to \omega$. Then $f(u) \in \mathcal{P}_\beta$ for a certain $\beta \leq \alpha$.

Let $\alpha$ be an ordinal, by $-1 + \alpha$ we understand $\alpha - 1$ if $\alpha$ is finite, and $\alpha$ if $\alpha$ is infinite.

Theorem 2.4. Let $(\alpha_n)_{n<\omega}$ be a non-decreasing sequence of ordinals less than $\omega_1$, let $\alpha = \lim_{n<\omega}(\alpha_n)$, let $1 < \beta < \omega_1$ and let $(X_n)$ be a partition of $\omega$. If $(p_n)$ is a sequence of ultrafilters such that $X_n \in p_n \in \mathcal{P}_{\alpha_n}$ and $p \in \mathcal{P}_\beta$, then $\int p_n \in \mathcal{P}_{\alpha + (-1 + \beta)}$. 

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Theorem 2.5. Let $\alpha$, $\beta$ be countable ordinals. If $u \in \mathcal{P}_{\alpha+\beta+1}$ then there exists a function $f : \omega \to \omega$ such that $f(u) \in \mathcal{P}_{\beta+1}$.

Theorem 2.6. The following statements are equivalent:
1) $\mathcal{P}$-points exist,
2) the $\mathcal{P}_\alpha$ classes are non-empty for each countable successor $\alpha$,
3) There exists a countable successor $\alpha > 1$ such that the class $\mathcal{P}_\alpha$ is non-empty.

In [1] Baumgartner provides the following definition. Let $I$ be a family of subsets of a set $A$ such that $I$ contains all singletons and is closed under subsets. Given an ultrafilter $u$ on $\omega$, we say that $u$ is an $I$-ultrafilter if for any $f : \omega \to A$ there is $U \in u$ such that $f(U) \in I$. For $\alpha < \omega_1$, let $I_\alpha = \{ B \subset \omega_1 : B$ has an order type $\leq \alpha \}$, $J_\alpha = \{ B \subset \omega_1 : B$ has order type $< \alpha \}$. A proper $I_\alpha$-ultrafilter is one which is not an $I_\beta$-ultrafilter for any $\beta < \alpha$. Denote also proper $J_\alpha$-ultrafilters as $J^*_\alpha$-ultrafilters those being the $J_\alpha$-ultrafilters which are not $J_\beta$-ultrafilters for any $\beta < \alpha$. We can also find in [1] the following statement: If $u$ is a proper $I_\alpha$-ultrafilter, then $\alpha$ must be indecomposable. Recall that

Proposition 2.7. [18, a corollary of Proposition 3.3] If $u \in J^*_\omega$, then $u \in \mathcal{P}_\beta$ for a certain $\beta \leq \alpha$.

3 $\mathcal{P}_\alpha$ classes for finite $\alpha$ and $<\infty$ sequences

Theorem 3.1. If $u \in \beta\omega$, then the following statements are equivalent:
1) There is no monotone sequential contour $C$ of rank $n$ such that $C \subset u$. (i.e., for each $n$-partition there exists a set $U \in u$ residual for this partition)
2) $u \in \bigcup_{i=1}^{n} \mathcal{P}_i$.
3) For each family of functions $\{f_1, \ldots, f_{n-1}\}$, $f_i : \omega \to \omega$ there exists a set $U \in u$ such that
   a) $f_1 \circ \ldots \circ f_{n-1} | U$ is constant or
   b) there exists $i \in \{1, \ldots, n-1\}$ such that $f_i | f_{i+1} \circ \ldots \circ f_{n-1}(U)$ is finite-to-one.
4) For each function $f : \omega \to \omega$ there exists a set $U \in u$ such that
   a) $f^{n-1} | U$ is constant or
   b) there exists $i \in \{1, \ldots, n-1\}$ such that $f | f_{i-1}(U)$ is finite-to-one.

Proof. 1 $\iff$ 2 is trivial.
2 $\Rightarrow$ 3: Let $u \in \mathcal{P}_i$ for some $i \leq n$ and let us take any functions $f_1, \ldots, f_{n-1} : \omega \to \omega$. 5
Let $A_k^\infty = \{ m < \omega : \text{card} (f_k^{-1}(m)) = \omega \}$ and $A_k^{fin} = \{ m < \omega : \text{card} (f_k^{-1}(m)) < \omega \}$ for $k \in \{1, \ldots, n-1\}$. Since $u$ is an ultrafilter, and $A_k^\infty \cup A_k^{fin} = \omega$, for each $k$ one of those sets belongs to $f_{k+1} \circ \ldots \circ f_{n-1}(u)$. If for some $k$ it is $A_k^{fin}$, then case 3b) holds, so we can assume that for each $k$, each function $f_k$ is infinite-to-one on elements of $f_{k+1} \circ \ldots \circ f_{n-1}(u)$. Since our research is restricted to elements of images of $u$, without loss of generality we may assume that card $(f_k^{-1}(m)) = \omega$ for each $k \in \{1, \ldots, n-1\}$ and for each $m \in \omega$.

Note the following obvious claim: Let $u$ be an ultrafilter and let $f$ be a function such that $f^{-1}(n)$ is infinite for all $n < \omega$. Then for each monotone sequential cascade $V$ of rank $\alpha$ such that $\int V \subset f(u)$, there is $\int f^{-1}(V) \subset u$, and $r(f^{-1}(u)) = 1 + \alpha$, where $f^{-1}(V) = V \cap f^{-1}(u)$.

If $f_1 \circ \ldots \circ f_{n-1}(u)$ is not a principal ultrafilter, then $f_k \circ \ldots \circ f_{n-1}(u)$ contain a contour of rank $k$, and thus $u$ contain a contour of rank $n$ - a contradiction.

$3 \Rightarrow 4$ is trivial.

$4 \Rightarrow 1$: Let us assume that there exists a monotone sequential contour $C_n$ of rank $n$ such that $C_n \subset u$. There exists a monotone sequential cascade $V$ such that $\int V = C_n$. Naturally, $r(V) = n$. Without loss of generality we may assume that max $V = \omega$ and the cascade $V$ is complete, i.e. each branch has the same length $n$. We identify elements of max $V$ with $n$-sequences of natural numbers which label that elements i.e. $\emptyset_V = \emptyset$, $(i_1, \ldots, i_k)^+_{k+1} = (i_1, \ldots, i_{k+1})$. We define the function $f : \text{max} V \rightarrow \text{max} V$ as follows:

$f((i_1, 1, 1, \ldots, 1, 1)) = (1, 1, \ldots, 1, 1)$ for each $i < \omega$;

if $v = (k_1, \ldots, k_n)$ and if there exists $l \in \{2, \ldots, n\}$ such that $k_l \neq 1$ then let $m(v) = \min \{ t \in \{1, \ldots, n\} : \forall \ell > t, \ell \leq n : k_\ell = 1 \}$ and let $f(v) = (k_1, \ldots, k_{m(v)}-2, k_{m(v)}-1 + 1, 1, \ldots, 1, 1)$.

It may be noticed without difficulty that $f^i(\text{max} V) = \{ v \in \text{max} V : m(v) < n - i \}$ for $i \in \{0, \ldots, n-1\}$. Let $V(i) = (f^i(\text{max} V))^{iV}$. Take any $G \subset f^i(\text{max} V)$. If $f$ is finite-to-one on $G$, then $G \cap \text{max} v^{iV(i)}$ is finite for each $v \in V(i)$ such that $v$ is a sequence of length $n - i$. Thus, $G$ is residual for $V(i)$ and so does not belong to $f^i(u)$.

Thus, the ultrafilter $u$ does not have the property described in point 4 of Theorem 3.1. 

It is worth comparing the definitions of P-points from page 3 with the conditions of Theorem 3.1 in order to see that the behavior of P-points is, in a very natural way, extended onto the behavior of elements of $\bigcup_{i=1}^n \mathcal{P}_i$. Condition 1 of Theorem 3.1 is the extension of the equivalent definition of P-point from Theorem 2.3, Condition 2 can be expressed as “$u$ is no more than $\mathcal{P}_n$-point”,

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and Conditions 3 and 4 of Theorem 3.1 extend the definition “C” of P-point from page 3.

**Proposition 3.2.** Let \( u \in \mathcal{P}_n \), \( n \in \omega \), and \( f : \omega \to \omega \). If \( f(u) \in \mathcal{P}_n \) then there exists a set \( U \in u \) such that \( f \upharpoonright U \) is finite-to-one.

**Proof.** Proof basis on the same idea as a proof of \( 2 \Rightarrow 3 \) in the previous Theorem 3.1. Suppose on the contrary, that there is no \( U \in u \) that \( f \upharpoonright U \) is finite-to-one, thus \( \{ i < \omega : \text{card}(f^{-1}(i)) = \omega \} \in f(u) \), and so without loss of generality we may assume that \( \{ i < \omega : \text{card}(f^{-1}(i)) = \omega \} = \omega \). If \( f(u) \in \mathcal{P}_n \), then there exists a monotone sequential contour \( \mathcal{V} \subset f(u) \) of rank \( n - 1 \). Consider a monotone sequential cascade \( \mathcal{V} \) such that \( \int \mathcal{V} = \mathcal{V} \) and \( W = \mathcal{V} \leftrightarrow f^{-1}(v) \), where \( v \in \max \mathcal{V} \). Since \( W \) is a cascade of rank \( n \) and \( u \in \mathcal{P}_n \), there exists a set \( U \in u \) residual for \( W \). Consider sets \( W^i \) for \( i \in \{ 0, \ldots, n - 1 \} \) defined by \( W_0 = U \) and \( w \in W^{i+1} \Leftrightarrow \text{card} (w^{i+1}W \cap W^i) = \omega \) (sets \( W^i \) are subsets of \( W \) and of \( V \) as well, for \( i > 0 \)). Split \( U \) into \( n \) pieces: \( U_{n-1} = (W^{n-1})^W \cap U, U^{i-1} = ((W^{i-1})^W \cap U) \setminus \bigcup_{j=i}^n U_k \). Notice that \( U_i \not\subset u \) for \( i > 0 \). Indeed, \( f(U_i) \subset (W^i)^V \cap \max V \) and \( (W^i)^V \cap \max V \) is residual for \( V \). Thus \( U_0 \in u \), clearly \( f \upharpoonright U_0 \) is finite-to-one. \( \blacksquare \)

By Proposition 3.2 and Theorems 2.3 and 2.5 we obtain the following

**Corollary 3.3.** If \( u \) is an ultrafilter, then \( u \in \mathcal{P}_n \) if and only if there exists an \( n \)-element \( \prec_{\omega} \)-decreasing sequence below it that contain ”\( u \)” and a principal ultrafilter, and there is no such chain of length \( n + 1 \).

**Proof.** Non existence of such \( n + 1 \) chain follows from Proposition 3.2. Existence of \( n \) chain follows inductively from a following fact: If \( \int \mathcal{V} \subset u \) for monotone sequential cascade \( \mathcal{V} \), then \( \int f(\mathcal{V}) \subset f(u) \) where \( f[v^+] = v_1 \) for all \( v \in \mathcal{V} : r(v) = 1 \), and \( f \) is an identity on the rest of \( \mathcal{V} \) - such defined \( f \) is not finite-to-one on each \( U \in u \) (for details see proof of Proposition 3.5). Note that if \( r(\mathcal{V}) \) is finite then \( r(f(\mathcal{V})) = r(\mathcal{V}) - 1 \). \( \blacksquare \)

In [14] Laflamme shows:

**Proposition 3.4.** [14] *Reformulation of Proposition 2.3*

Let \( k \in \omega \) and \( u \) an ultrafilter such that

\((\ast)\) (\( \forall h \in \omega_{\omega} \)) \( \exists X \in u \) the order type of \( h(X) \) is strictly less than \( \omega^\omega \)

Then \( u \) is an \( J_{\omega_k}^* \)-ultrafilter precisely if it has a \( \prec_{\omega} \)-chain of length \( k \) below it that contain \( u \) and a principal ultrafilter but no such a chain of length \( k + 1 \).

Notice that Proposition 3.4 for ordinal ultrafilters is very similar to Corollary 3.3, the only difference being the extra assumption \((\ast)\).

As opposed to Proposition 3.2, for infinite \( \alpha \)'s we have the following
Proposition 3.5. If $\alpha$ is a countably infinite successor ordinal, then for each $u \in P_\alpha$ there exists a function $f : \omega \to \omega$ such that: $f(u) \in P_\alpha$ and $f \upharpoonright U$ is not finite-to-one for each $U \in u$.

Proof. Let $u \in P_\alpha$, let $\alpha$ be as in the assumptions. Let us take a monotone sequential contour $V$ of rank $\alpha - 1$ such that $V \subset u$; consider a monotone sequential cascade $V$ such that $\int V = V$; without loss of generality we may assume that $\max V = \omega$. For each $v \in V$ such that $r(v) = 1$ choose $\tilde{v} \in v^+$ and define $f : \max V \to \omega$ as follows: if $v \in w^+$ for $v \in V$, $r(w) = 1$ then $f(v) = \tilde{w}$.

We will prove that the function $f$ fulfills the claim. Clearly, the function $f$ is not constant on any $U \in u$. Consider $T = \{v \in V : r(v) = \omega\}$. It is sufficient to prove that $r(f(v^\uparrow)) = \omega$ for each $v \in T$. Let $v^\uparrow = (n) \uparrow v_n$ for $v \in T$. We have $r(f(v_n)^\uparrow) = r(f(\{f(v_n)\}^\uparrow)) + 1$, so $\lim_{n<\omega} r(f(\{f(v_n)\}^\uparrow)) = \omega = \lim_{n<\omega} r(f(v_n)^\uparrow)$, and so $r(f(\int V)) = r(\int V) = \alpha - 1$.

Suppose that $f \upharpoonright U$ is finite-to-one for some $U \in u$. Then $\omega \setminus U \in \int V$, contradiction with $\int V \subset u$.

On the other hand by Theorem 2.3, $f(u) \in P_\gamma$ for a certain $\gamma \leq \alpha$.

Theorem 3.6. If $\alpha$ is a countably infinite successor ordinal and $u \in P_\alpha$, then there exists a function $f : \omega \to \omega$ such that:

1) $f^n \upharpoonright U$ is not finite-to-one for any $n \in \omega$ and any $U \in f^{n-1}(u)$ ($f^0(u) = u$),

2) the sequence $(f^n)_{n<\omega}$ is (pointwise) convergent;

3) $f^n(u) \in P_\alpha$ for each $n < \omega$, and $(\lim_{n<\omega} f^n)(u) \in P_\alpha$.

Proof. Let $V$ be a monotone sequential cascade of rank $\alpha - 1$ such that $\int V \subset u$.

Let $T = \{t \in V : r(v) = \omega\}$. Without loss of generality we may assume that for each $v \in T$, for all $n < \omega$ each branch of $v_n^\uparrow$ has length $r(v_n)$. For each $v \in V$ take a non decreasing sequence $(a_n^v)_{n<\omega}$ of natural numbers, such that $a_n^v \leq r(v_n)$, $\lim_{n \to \infty} a_n^v = \omega$, $\lim_{n \to \infty} (r(v_n) - a_n^v) = \omega$.

For each pair $(v, n)$ where $V \in T$, $n < \omega$ take a set $T_{v,n} = \{t \in v_n^\uparrow : r(t) = a_n^v\}$. For each $t \in T_{v,n}$ take a function $f_{v,n} : \max t^\uparrow \to \max t^\uparrow$ defined like in the proof of case 4 ⇒ 1 in Theorem 3.1, and glue all this functions in a function $f : \max V \to \max V$ which satisfies a claim.

4 Relatively RK-$\alpha$-minimal points.

Recall that a free ultrafilter $u \in \beta \omega$ is RK-minimal \footnote{Also known as Ramsey ultrafilters or selective ultrafilters.} if for each $f : \omega \to \omega$ either $f(u)$ is a principal filter or $f(u) \approx u$. The existence of RK-minimal
there exists a set $H$. Proof. Let $\int_\omega$ Thus, Theorem 4.4. \]

The remainder of this paper a function $F$ finite sequence) of filters such that $\omega \to \omega$ there exists a set $U \in u$ such that either $f \mid_U$ is constant or $f \mid_U$ is one-to-one;

3) $u$ is a $P$-point and for each finite-to-one function $f : \omega \to \omega$ there exists a set $U \in u$ such that $f \mid_U$ is one-to-one;

4) For each partition $d = \{d_n; n < \omega\}$ either there exists a set $U \in u$ such that card $(U \cap d_n) \leq 1$ for each $n < \omega$, or there exists $n_0$ such that $d_{n_0} \in u$.

An ultrafilter $u \in \mathcal{P}_\alpha$ is referred to as relatively RK-$\alpha$-minimal if for each $f : \omega \to \omega$ there is either $u \approx f(u)$ or $f(u) \in \mathcal{P}_\beta$ for a certain $\beta < \alpha$; relatively $<\infty$-$\alpha$-minimal ultrafilters are those $u \in \mathcal{P}_\alpha$ which each not finite-to-one (on each set $U \in u$) image is not in $\mathcal{P}_\alpha$.

The following two propositions, admitting straightforward proofs, are useful in investigations of images of contours.

**Proposition 4.2.** If $(p_n)$ is a sequence of filters, $p$ is a filter and $f : \omega \to \omega$ is a function, then $f(\int_p p_n) = \int_{F(p)} o_m$, where $(o_n)$ is a sequence (possibly a finite sequence) of filters such that $o_i \neq o_j$ for $i \neq j$ and $\{o_j : j < \omega\} = \{f(p_n) : n < \omega\}$, $F(n) = i$ iff $f(p_n) = o_i$.

Notice that $F$ depends on the order on the set $f(\{p_n : n < \omega\})$, so in the remainder of this paper a function $F$ for $f$ is an arbitrary (but fixed) function among such functions.

**Proposition 4.3.** Let $(p_n)$, $p$, $(o_n)$ and $F$ be as in Proposition 4.2. Suppose that there exists a set $P \in F(p)$ such that the sequence $(o_i)_{i \in P}$ is discrete and there exists a set $H \in p$ such that $F \mid_H$ is one-to-one and $p_n \approx o_{F(n)}$ for each $n \in H$. Then $\int_p p_n \approx \int_{F(p)} o_i = f(\int_p p_n)$.

**Theorem 4.4.** Let $m < \omega$. If $(p_n)$ is a discrete sequence of relatively RK-$m$-minimal free ultrafilters on $\omega$ and $p$ is an RK-minimal free ultrafilter, then $\int_p p_n$ is relatively RK-$m + 1$-minimal.

**Proof.** Let $p$ and $(p_n)$ be as in the assumptions. Let $f$ be a function $f : \omega \to \omega$. By Theorem 2.4 $\int_p p_n \in \mathcal{P}_{m+1}$. Take $(o_i)$ and $F$ as in Proposition 4.2. Thus, $f(\int_p p_n) = \int_{F(p)} o_i$. Without loss of generality we may assume that $\int_{F(p)} o_i \in \mathcal{P}_{m+1}$. We want to prove that $\int_{F(p)} o_i \not\in \mathcal{P}_{m+1}$ or $\int_{F(p)} o_i \approx \int_p p_n$. For this and consider two cases:
Case 1. $F(p)$ is a principal filter. In this case there exists $i < \omega$ such that $\{i\} \in F(p)$ and thus $o_i = \int_{F(p)} o_i$. Since $o_i = f(p_j)$ for some $j < \omega$, and by Theorem 2.3 $o_i \in \mathcal{P}_\beta$ for some $\beta \leq m$, we have $f(\int_p p_n) \notin \mathcal{P}_{m+1}$.

Case 2. $F(p)$ is a free filter. Then $F(p)$ is a free ultrafilter, and thus, $F(p) \approx p$, since $p$ is RK-minimal. Define sets $D_i = \{n < \omega : o_n \in \mathcal{P}_i\}$, for $i \in \{1, ..., m\}$. Since $o_n \in \bigcup_{i=1}^{m} \mathcal{P}_i$, there exists exactly one $i_0 \in \{1, ..., m\}$ such that $D_{i_0} \in F(p)$.

Subcase 2.1. $i_0 < m$. Let us take a discrete sequence $(q_i)$ of ultrafilters such that $q_i \approx o_i$ in this aim consider a partition $(A_i)_{i<\omega}$ of $\omega$ into infinite sets, and a sequence $(f_n)$ of bijections $f_i : \omega \rightarrow A_i$ and put $q_i = f_i(o_i)$. By Theorem 2.4 $\int_{F(p)} q_i \in \mathcal{P}_{m+1}$ there is $\int_{F(p)} o_i \leq \int_{F(p)} q_i$ and so $\int_{F(p)} o_i \notin \mathcal{P}_{m+1}$.

Subcase 2.2. $i_0 = m$. For each $i \in D_m$ and for each $n$ with $F(n) = i$ we have $p_n \approx f(p_n) = o_{F(n)} = o_i$ by RK-minimality of $p_n$.

Let $i_1 = \min D_m$. There exists a set $A_1$ such that $A_1 \in o_{i_1}$ and $A_1 \notin \int_{F(p)} o_i$ (because we are not in case 1). If numbers $i_r$ and sets $A_r$ for $r < t$ are already defined, we define $i_t = \min \{i \in D_m : (\bigcup_{r=1}^{t-1} A_r)^c \in o_i\}$, and let $A_t$ be such a set that $A_t \subset (\bigcup_{r=1}^{t-1} A_r)^c$, $A_t \in o_{i_t}$, $(A_t)^c \in \int_{F(p)} o_t$ (such a set exists because we are not in case 1, and $(\bigcup_{r=1}^{t-1} A_r)^c \in o_{i_t}$).

In this way we obtain a sequence $(A_r)_{r<\omega}$ of pairwise disjoint sets such that $(A_r)^c \in \int_{F(p)} o_i$ for each $r < \omega$, and for each $i < \omega$ there exists a number $r < \omega$ such that $A_r \in o_i$. Thus, the sequence $(A_r)_{r<\omega}$ defines a partition $s = (S_n)_{n<\omega}$ of $D_m$ into non-empty sets by letting $i \in S_n$ if and only if $A_n \in o_i$. There is no $n$ such that $S_n \in F(p)$ and $F(p)$ is RK-minimal, so by Theorem 4.1 there exists a set $P \in F(p)$ with $P \subset D_m$ such that $\text{card}(P \cap S_n) \leq 1$ for each $n \in \omega$ (the sequence $(o_i)_{i \in P}$ is discrete). The same Theorem 4.1 shows that there exists a set $H \in p$ such that $F|_H$ is one-to-one. Without loss of generality $F(H) \subset D_m$ and since $P_i \approx o_{F(i)}$ for all $i \in H$ we are in the assumption of Proposition 4.3 so we conclude $\int_{F(p)} o_i \approx \int_p p_n$.

By induction, by Theorem 4.4 one can easily prove the following Corollary 4.5:

**Corollary 4.5.** If there exist RK-minimal ultrafilters in $\beta \omega \setminus \omega$, then for each $n < \omega$ there exist relatively RK-\(n\)-minimal ultrafilters.

In contrast to the above Corollary 4.5, for infinite $\alpha$’s we obtain the following from Theorem 3.6:

**Corollary 4.6.** There are no $< \omega$ (and so no RK) relatively minimal ultrafilters in classes of infinite successor index of the $P$-hierarchy.

**Problem 1.** Do relatively RK-$\alpha$-minimal ultrafilters exist for limit ordinals $\alpha \leq \omega_1$?
We may also consider RK-minimal elements in classes of ordinal ultrafilters. An ultrafilter \( u \in J^*_\omega \) is referred to as a relatively ordinal RK-\( \alpha \)-minimal if for each \( f : \omega \to \omega \) either \( u \approx f(u) \) or \( f(u) \in J^*_\omega \beta \) for a certain \( \beta < \alpha \).

One can get a very similar result to Theorem 4.4 for ordinal ultrafilters:

**Theorem 4.7.** Let \( m < \omega \). If \( (p_n) \) is a discrete sequence of relatively ordinal RK-\( m \)-minimal free ultrafilters on \( \omega \) and \( p \) is a RK-minimal free ultrafilter, then \( \int_p p_n \) is relatively ordinal RK-\( m+1 \)-minimal.

The proof is very similar to the proof of Theorem 4.4 and uses the following reformulation of a theorem of Baumgartner [1, Theorem 4.2]:

**Theorem 4.8.** Let \( (\alpha_n)_{n<\omega} \) be a non-decreasing sequence of ordinals less than \( \omega_1 \), let \( \alpha = \lim_{n<\omega} (\alpha_n) \) and let \( (X_n) \) be a partition of \( \omega \). If \( (p_n) \) is a sequence of ultrafilters such that \( X_n \in p_n \in J^*_\omega \alpha_n \) and \( p \) is a P-point, then \( \int_p p_n \in J^*_\omega \alpha+1 \).

Notice also that, by Proposition 3.2, each element of a class of finite index of the P-hierarchy is relatively \( <_\infty \)-minimal.

In [13, Theorem 3.3] Laflamme built (under MA for \( \sigma \)-centered partial orderings) a special ultrafilter \( u_0 \in J^*_\omega \omega+1 \) the only RK-predecessor of which is a Ramsey ultrafilter. In the proof of [13, Theorem 3.13] it is shown that \( u_0 = P_3 \) and that \( U_0 \) is not in the form of a contour, note also that Laflamme’s ultrafilter \( u_0 \) is different then ultrafilter build in Theorem 4.4 which is a contour. Therefore, we have:

**Theorem 4.9.** It is consistent with ZFC that there exists a relatively-RK-3-minimal ultrafilter that is not in the form of a contour.

## 5 Generic existence

In this section, the P-hierarchy is understood as \( \bigcup_{1<\alpha<\omega} P_\alpha \).

Let \( V \) be a cascade. Denote \( r_\alpha(V) = \{ v \in V : r(v) = \alpha \} \). Let \( h \) be a function with the domain \( V \) such that \( h(v) \in \omega \) for \( v \in \max V \), and \( h(v) \) is a filter on the set \( \omega \), otherwise. We define \( f^h V \) inductively as follows: \( f^h v^+ \) is a principal ultrafilter generated by \( h(v) \), for \( v \in \max V \). If \( f^h w^+ \) is defined for all \( v_n \in v^+ \) then \( f^h v^+ = \int_{h(v)} f v_n^+ \).

Recall that the family \( F \) of functions \( \omega \to \omega \) is a dominating family if for each function \( g : \omega \to \omega \) there exists \( f \in F \) such that \( f(n) \geq g(n) \) for almost all \( n < \omega \), i.e. there is \( n_0 \) such that \( f(n) \geq g(n) \) for all \( n > n_0 \). The dominating number \( \mathfrak{d} \) is the minimum of cardinalities of dominating families, and \( \mathfrak{c} \) is the cardinality of the continuum.
We say that filters belonging to the family $\mathbb{F}$ exist generically if each filterbase of size less than $c$ can be extended to a filter belonging to $\mathbb{F}$.

In [2] Brendle showed that:

**Theorem 5.1.** [2, part of Theorem E] The following are equivalent:

(a) $\mathfrak{d} = c$;

(b) ordinal ultrafilters exist generically.

We obtain the same result for the P-hierarchy. For this we need to prove the following theorem.

**Theorem 5.2.** For each ordinal $1 < \alpha < \omega_1$ and for each monotone sequential contour $\mathcal{V}$ of rank $\alpha$, the minimum of cardinalities of filterbases of $\mathcal{V}$ is $\mathfrak{d}$.

**Proof.** First, we will show that there exists a base of cardinality $\mathfrak{d}$. Let $\mathcal{V} = \int V$ for a monotone sequential cascade $V$. Let $\mathbb{D} = \{d_\beta : \beta < \mathfrak{d}\}$ be a dominating family for functions $V \to \omega$ (there exists such a family since $V$ is countable). Define family $\mathbb{F} = \{f_{d,n} : d \in \mathbb{D}, n < \omega\}$ as follows: $f_{d,n}(v) = d(v)$ for $v \neq \emptyset_V$ and $f_{d,n}(\emptyset_V) = n$ for $v = \emptyset_V$. For each $v \in \max V$ let $(\emptyset_V = v_{0,v} \subseteq v_{1,v} \subseteq \cdots \subseteq v_{n,v,v} = v)$ be a branch of $V$ with maximal node $v$. For each function $f : V \to \omega$ define sets $V(f) \subset V$ by the condition: $v \in V(f)$ if and only if for each $i$ there exists $k$ such that $v^{i+1,v} = v_k^{i,v}$ and $k \geq f(v^{i,v})$. A typical base of the cascade $V$ is as follows: $\{V(g) : g : V \to \omega\}$. Take any $g : V \to \omega$. Since $\mathbb{D}$ is a dominating family, there exists $d_{\beta_0} \in \mathbb{D}$ such that $g \leq^* d_{\beta_0}$. Thus, the set $A = \{v \in V : g(v) > d_{\beta_0}(v)\}$ is finite, so we can define $n_0 = \max\{n : A \cap (\emptyset_V)^{n+1} \neq \emptyset\} + 1$. Therefore, $V(f_{d_{\beta_0},n_0}) \subset V(g)$, thus $\{V(f) : F \in \mathbb{F}\}$ is a base.

Now, let us assume that there exists a base $\mathbb{B} = \{B_\beta : \beta < \gamma\}$ of $\mathcal{V}$ with $\gamma < \mathfrak{d}$. Since $\{V(f) : f : V \to \omega\}$ constitutes the base for $\mathcal{V}$, for each $\beta < \gamma$ there exists $f_\beta$ such that $f_\beta : V \to \omega$ and $W(f_\beta) \subset B_\beta$. Let $\mathbb{G} = \{f_\beta : \beta < \gamma\}$. Since card $(\mathbb{G}) < \mathfrak{d}$, for each $v \in \mathcal{V}$ such that $r(v) = 2$ the family $\{f_\beta \upharpoonright v^+ : \beta < \gamma\}$ is not a dominating family on the set $v^+$. For each $v$ such that $r(v) = 2$, take a function $g_v : v^+ \to \omega$ such that $g_v \not\leq^* f_\beta \upharpoonright v^+$ for each $\beta < \gamma$. Now, let $g : V \to \omega$ be a function that $g(v) = g_v(v)$ if $r(v) = 1$ and $v \in \check{v}^+$; otherwise, $g(v) = 1$. We have $V(g) \in \mathcal{V}$ and $V(g) \cap V(f_\beta) \neq V(f_\beta)$ for each $\beta < \gamma$. ■

The supercontour is a filter of type $\bigcup_{\alpha<\omega_1} \mathcal{V}_\alpha$, where $(\mathcal{V}_\alpha)_{\alpha<\omega_1}$ is an increasing sequence of monotone sequential contours such that $r(\mathcal{V}_\alpha) = \alpha$.

**Corollary 5.3.** Generic existence of the P-hierarchy is equivalent to $\mathfrak{d} = c$. 

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Proof. By the Ketonen’s theorem [13, Theorem 1.1] generic existence of P-points is equivalent to $d = c$. By Proposition 2.2 P-points belong to $P_\omega$ class; thus, if $d = c$, then the P-hierarchy generically exists.

Let $d < c$, and take an increasing $\omega_1$-sequence $(V_\alpha)$ of monotone sequential contours such that $r(V_\alpha) = \alpha$. Let $B_\alpha$ be a base of cardinality $d$ of $V_\alpha$ (there exist by Theorem 5.2). Let $B = \bigcup_{\alpha \in \omega_1} B_\alpha$. Obviously, $\text{card}(B) = d$ and $B$ is the base for a supercontour, so it cannot be extended to any element of the P-hierarchy. ■

By Ketonen’s Theorem [13] each ultrafilterbase of cardinality less than $d$ is the base of a P-point. In order to obtain a similar result for other classes, we need the extra assumptions that such a base can be extended to infinitely many ultrafilters. To prove this we need to quote the following two results:

**Theorem 5.4.** [1, Theorem 4.1] The $J^{*}_{\omega_2}$-ultrafilters are the P-point ultrafilters.

We say that families $u$ and $o$ mesh (and we write $u \# o$) whenever $U \cap O \neq \emptyset$ for every $U \in u$ and $O \in o$.

**Proposition 5.5.** The following statements are equivalent:

a) For each successor ordinal $1 < \alpha < \omega_1$ each filterbase of cardinality less than $c$ which can be extended to infinitely many ultrafilters, can also be extended to some elements of $P_\alpha$;

b) For each successor ordinal $1 < \alpha < \omega_1$ each filterbase of cardinality less than $c$ which can be extended to infinitely many ultrafilters, can also be extended to some elements of $J^*_{\omega_1}$;

c) $d = c$.

**Proof.** For $d < c$ a proof is analogical to the second part of the proof of Theorem 5.3, with an additional use of Proposition 2.8 for the case of ordinal ultrafilters.

Now let $d = c$, and let $B$ be a proper filterbase of cardinality $< c$. By the Ketonen’s Theorem [13] $B$ can be extended to a P-point, and so we can assume that $\alpha > 2$. by the assumption, there exists a family $\{F_n\}_{n < \omega}$ of pairwise disjoint sets such that $F_n \# B$ for each $n < \omega$. Let $(p_n)$ be a sequence of P-points such that $B \cup \{F_n\} \subset p_n$. Take a monotone sequential cascade $V$ of rank $\alpha$. Put $R = \{v \in V : r(v) = 1\}$ and without loss of generality assume that for each $v \notin R$ a cascade $v^+$ has no branches of length 1. Let $g$ be an arbitrary bijections $g : R \to \omega$ and let $f_v : v^+ \to F_n$ be an arbitrary bijection for each $v \in R$. Let $h$ be a function which domain is $V$, defined as follows:

$h(v') = f_v(v')$ for $v' \in v^+$, $v \in R$

$h(v) = p_{g(v)}$ for $v \in R$
\[ h(v) = p_1 \text{ for other } v \in V. \]

Consider \( J^h \) and note following facts:
1) \( J^h \# B \), since \( p_n \# B \) for all \( n \);
2) \( J^h \in P_\alpha \), inductively by Theorem 2.5;
3) \( J^h \in J^*_{\alpha+1} \) by Corollary 5.4 and inductively by Theorem 4.8. \( \blacksquare \)

6 Existence

We say that a cascade \( V \) is built by destruction of nodes of rank 1 in a cascade \( W \) of rank \( r(W) \geq 2 \) iff for a set \( R = \{ w \in W : w(w) = 1, r(w^-) = 2 \} \) there is: \( V = W \setminus R \) and if \( v \in R^{-W} \) then \( v^+V = (v^+W \setminus R) \cup (v^+W \setminus R^+) \), i.e. order on the cascade is unchanged.

Observe that if \( W \) is a monotone sequential cascade then \( V \) is also a monotone sequential cascade and if \( r(W) \) is finite then \( r(V) = r(W) - 1 \), if \( r(W) \) is infinite, then \( r(V) = r(W) \).

Assume that we are given a cascade of rank \( \alpha \) and an ordinal \( 1 < \beta \leq \alpha \). We shall describe an operation of decreasing the rank of a cascade \( W \). The construction is inductive:

For finite \( \alpha \), we can decrease rank of \( W \) from \( \alpha \) to \( \beta \) by applying \( \alpha - \beta \) times an operation of destroying nodes of rank 1 (i.e. if \( \alpha = \beta \) then the cascade is unchanged).

For infinite \( \alpha \). Suppose that for each pair \((\delta, \gamma)\) where \( 1 < \delta \leq \gamma < \alpha \), and for each cascade \( W \) of rank \( \gamma \) the operation of decreasing the rank of \( W \) from \( \gamma \) to \( \delta \) is defined. Let \( W \) be a monotone sequential cascade of rank \( \alpha \), let \((\beta_n)\) be a nondecreasing sequence of ordinals such that: \( \beta_n = 0 \) if and only if \( r(W_n) = 0 \), \( \beta_n \leq r(W_n) \) and \( \lim_{n \to \infty} (\beta_n + 1) = \delta \). Let, for each \( n < \omega \), \( V_n \) be the cascade obtained by decreasing of rank of \( W_n \) to \( \beta_n \). Finally let \( V = (n \mapsto V_n) \).

Clearly for infinite \( \alpha \) the operation of decreasing of rank is not defined uniquely. Observe also that the above described decreasing of rank of a cascade \( W \) does not change \( \text{max}W \). If a cascade \( V \) is obtained from \( W \) by decreasing of rank, then we write \( V \triangleleft W \). Trivially \( V \triangleleft W \) and inductively \( \int V \subset \int W \).

\textbf{Theorem 6.1.} [5] If \((V_n)_{n<\omega}\) is a sequence of monotone sequential contours of rank less than \( \alpha \) and \( \bigcup_{n<\omega} V_n \) has the finite intersection property, then there is no monotone sequential contour \( W \) of rank \( \alpha + 1 \) such that \( W \subset \langle \bigcup_{n<\omega} V_n \rangle. \)

Before we prove the main Lemma we shall prove a technical claim;
Lemma 6.2. Let $V$ be the cascade of rank $\alpha$, $W$ be cascade obtained from $V$ by decreasing the rank of $V$ to $\beta < \alpha$ and let $\beta < \gamma < \alpha$. Then there is a cascade $T$ of rank $\gamma$ such that $W \triangleleft T \triangleleft V$.

Proof. If $\beta = 1$ then it suffice to take any monotone sequential cascade $T$ obtained by decreasing of the rank of $W$ to $\gamma$.

If $\beta > 1$ then take $(\beta_n)$ - a nondecreasing sequence of ordinals such that: $\beta_n = 0$ if and only if $r(W_n) = 0$, $r(V_n) \leq \beta_n \leq r(W_n)$ and $\lim_{n \to \infty}(\beta_n + 1) = \gamma$. By inductive assumption one can find $(T_n)$ a sequence of monotone sequential contours such that $V_n \triangleleft T_n \triangleleft W_n$. Put $T = (n) \leftrightarrow T_n$. ■

We write $V \blacktriangleright_1 W$ if $\max W \in \int V$ and $V \triangleleft W$. We write $V \blacktriangleright_2 W$ if $\max V \in \int W$ and $V \triangleleft W$. Trivially Lemma 6.2 is true also for $\blacktriangleright_1$, $\blacktriangleright_2$ instead of $\triangleleft$.

Lemma 6.3. Let $\alpha < \omega_1$ be a limit ordinal and let $(\mathcal{V}^n : n < \omega)$ be a sequence of monotone sequential contours such that $r(\mathcal{V}^n) < r(\mathcal{V}^{n+1}) < \alpha$ for every $n$ and such that $\bigcup_{n<\omega} \mathcal{V}^n$ has the finite intersection property. Then there is no monotone sequential contour $W$ of rank $\alpha$ such that $W \subset \langle \bigcup_{n<\omega} \mathcal{V}^n \rangle$.

Proof. Put $\alpha_n = r(\mathcal{V}^n)$, without loss of generality we may assume that $\alpha_1 \geq 3$. Assume that there exists a monotone sequential contour $W$ of rank $\alpha$ such that $W \subset \langle \bigcup_{n<\omega} \mathcal{V}^n \rangle$. We build a cascade $W$ and a sequence of cascades $(W^n)_{n<\omega}$ such that:

- $\int W = W$;
- $W^n \blacktriangleright_1 W^{n+1}$ for all $n$;
- $W^n \blacktriangleright_2 W$ for all $n$;
- $r(W^n) = \alpha_n + 3$ for all $n$;
- $r(W^n_i) = \alpha_n + 2$ for all $n$ and all $i$;
- $r(W^n_{i,j}) = \alpha_n + 1$ for all $n$, $i$ and $j$.

Fix any monotone sequential cascade $\bar{W}$ such that $\int \bar{W} = W$. Let $\bar{W}^m$ be the cascade obtained from $\bar{W}$ by cutting every subcascade $\bar{W}_i$ of rank smaller than $\alpha_m + 2$ and every subcascade $\bar{W}_{i,j}$ of rank smaller than $\alpha_m + 1$. Observe that we cut only finitely many subcascades $\bar{W}_i$ and for the other $\bar{W}_i$ only finitely many subcascades $\bar{W}_{i,j}$. Thus $\int \bar{W}^m = \int \bar{W} = W$ for every $m$.

Let $W = \bar{W}^1$ and $W$ be a cascade obtained from $\bar{W}^1$ by decreasing ranks of $W_{i,j}$ to $\alpha_1 + 1$. Thus $W \blacktriangleright_2 W$.  

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Since cascades $\bar{W}^n$ and $\bar{W}^n$ are subcascades of $W$ thus for nodes (and so subcascades) of $\bar{W}^n$ and $\bar{W}^n$ we may keep the indexation from $W$, to avoid the collision of notation we put those indexes in parenthesis.

Assume that $W^1 \blacktriangleright W^2 \blacktriangleright \ldots \blacktriangleright W^m$ have been defined. We apply Lemma 6.2 to cascades $W^m_{(i,j)}$ and $\bar{W}^m_{(i,j)}$ to define $W^{m+1}_{(i,j)}$ of rank $\alpha_{m+1} + 1$ for those $(i, j)$ that $w_{i,j} \in W^m \cap \bar{W}^{m+1}$.

Let $K^{m+1}$ be a subcascade of $W$ with $K^{m+1} = \{\emptyset W\} \cup (\emptyset W)^{m+1} \cup ((\emptyset W)^{m+1} + W^{m+1}) + W^{m+1}$. Put $W^{m+1} = K^{m+1} \mapsto W^{m+1}_{(i,j)}$.

Next we build a decreasing sequence $(U_n)_{n<\omega}$ satisfying conditions $U_A$-$U_D$:

1. $U_A(n)$: $U_n \in \int W^n$;
2. $U_B(n)$: $U_n \notin \{\cup_{i\leq n} V^i\}$;
3. $U_C(n)$: $U_n \cap (\omega \setminus \max W^{n+1}) = U_{n+1} \cap (\omega \setminus \bar{W}^{n+1})$;
4. $U_D(n)$: $U_n \cap \max W_i \in \int W_i$ for all $n$ and all $i$.

In this aim first we built an additional sequence $(\tilde{W}^n_n)$ of cascades by $\tilde{W}^n_n = W^n \setminus \emptyset W^n$ such that $\emptyset W^n = \bigcup \{w^+: w \in \emptyset W^n\}$, and that the rest of cascades we leave unchanged (we may say that $\tilde{W}^n$ is obtained from $W^n$ by destroying all nodes of rank $\alpha_m + 2$). Notice that $\tilde{W}^n$ is a monotone sequential cascade of rank $\alpha_m + 2$, and that if a set $U_n$ fulfills conditions $U_B(n), U_C(n)$ and belongs to $\tilde{W}^n_n$ then the same set $U_n$ fulfills all conditions $U_A(n) - U_D(n)$.

Put $U_0 = \omega$. Assume that $U_0, U_1, \ldots, U_{n-1}$ was defined, but it is impossible to define $U_n$. This means that every set $U \in \int \tilde{W}_n$ is contained in $\{\cup_{i\leq n} V_i\}$. On the other side $\max \tilde{W}_n \in \mathcal{W}$ and so the family $\{U \cap \max \tilde{W}_n : U \in \bigcup_{i\leq n} V_i\}$ has the finite intersection property. By the theorem of Dolecki $\{\max \tilde{W}_n : U \in \bigcup_{i\leq n} V_i\}$ do not contain any monotone sequential contour of rank $\alpha_n + 2$ and so do not contain $\int \tilde{W}_n$. A contradiction. On each step of induction we can put $\bigcap_{i\leq n} U_i$ instead of $U_n$ and assume that the sequence $(U_n)_{n<\omega}$ is decreasing.

Notice that $\bigcup_{n<\omega} (\max W_{n+1})^c = \max W$, let $U = \bigcap_{n<\omega} U_n$. Conditions (1)-(4) guarantee that
1. $U \in \int W$ and
2. $U \notin \{\cup_{n<\omega} V_n\}$.
To see 1) fix any $t < \omega$, note that $\max W^m \in \int W_t$ only for finite number of $m$. So the sequence $(U_n \cup R)_{n<\omega}$ is (decreasing and) almost constant on some $R \in \int W_t$. Therefore $\bigcap_{n<\omega} U_n \cap R$ is indeed a finite intersection of $R$ and $U_n$ all of which by condition $U_D$ belongs to $\int W_t$. So $\bigcap_{n<\omega} U_n \in \int W_t$ for all $t$, and so $U \in \int W$.

To see 2), assume that $U \notin (\bigcup_{n<\omega} V_n)$, then there is a finite $M < \omega$ such that $U \notin (\bigcup_{n<M} V_n)$. But $U_M \notin (\bigcup_{n\leq M} V_n)$ and $U \subset U_M$. Thus $U \notin (\bigcup_{n<\omega} V_n)$. A contradiction. ■

**Proposition 6.4.** ([18, part of corollary 2.6] (ZFC) Classes $\mathcal{P}_1$ and $\mathcal{P}_{\omega_1}$ are nonempty.

**Theorem 6.5.** (CH) Each class of the $P$-hierarchy is nonempty.

*Proof.* For successor $\alpha$’s and for 1 for $\omega_1$ we deal in Proposition 6.4. By well known result of W. Rudin CH implies existing of P-points so for successor $\alpha$ we are done by Theorem 6.5. Let $\alpha < \omega_1$ be limit ordinal. Let $(V_n)$ be an increasing sequence of monotone sequential contours such that $r(V_n)$ is an increasing sequence with $\lim_{n<\omega} r(V_n) = \alpha$. By CH we can order all $\alpha$-partitions in an $\omega_1$ sequence $(P_\beta)$.

We will build a sequence $(Q_\beta)_{\beta<\omega_1}$ of subsets of $\omega$ such that $Q_\beta$ is residual for the partition $P_\beta$ and a family $\{Q_\beta : \beta < \omega_1\} \cup \bigcup_{\beta<\omega} V_n$ has the finite intersection property. Since $\bigcup_{\beta<\omega} V_n$ is a filter and, by the Lemma 6.3 above, does not contain any monotone sequential contour of rank $\alpha$, thus there exists a set $Q_1$ residual for the partition $P_1$ such that the family $\{Q_1\} \cup \bigcup_{\beta<\omega} V_n$ has the finite intersection property. Suppose now that the sequence $(Q_\beta)_{\beta<\gamma}$ is already built. If $\gamma < \omega$ then consider the sequence $(V_n |_{\beta<\gamma}, Q_\beta)_{\beta<\omega}$, this is an increasing sequence of monotone sequential contours with $r(V_n) = r(V_n |_{\beta<\gamma}, Q_\beta)$ thus by the Lemma 6.3 there exist a set $Q_\gamma$ residual for the partition $P_\gamma$ and such that a family $\{Q_\gamma\} \cup \bigcup_{\beta<\omega} (V_n |_{\beta<\gamma}, Q_\beta)$ has the finite intersection property and thus also a family $\{Q_\beta : \beta \leq \gamma\} \cup \bigcup_{\beta<\omega} V_n$ has the finite intersection property. If $\gamma \geq \omega$ then we enumerate the sequence $(Q_\beta)_{\beta<\gamma}$ by natural numbers and obtain the sequence $(Q_{\gamma}^m)_{m<\omega}$. Consider the sequence $(V_n |_{m<\gamma}, Q_{\gamma}^m)_{m<\omega}$, this is an increasing sequence of monotone sequential contours with $r(V_n) = r(V_n |_{m<\gamma}, Q_{\gamma}^m)$. Thus by the Lemma 6.3 there exist a set $Q_\gamma$ residual for the partition $P_\gamma$ and such that a family $\{Q_\gamma\} \cup \bigcup_{m<\omega} (V_n |_{m<\gamma}, Q_{\gamma}^m)$ has the finite intersection property and thus also a family $\{Q_\beta : \beta \leq \gamma\} \cup \bigcup_{\beta<\omega} V_n$ has the finite intersection property. Thus a sequence $(Q_\beta)_{\beta<\omega_1}$ with described properties exists.

Now it is sufficient to take any ultrafilter $u$ that contains $\{Q_\beta : \beta < \omega_1\} \cup \bigcup_{\beta<\omega} V_n$. Since $\bigcup_{\beta<\omega} V_n \subset u$ then $u$ contains a monotone sequential contour of each rank less then $\alpha$. Since $u$ contains $\{Q_\beta : \beta < \omega_1\}$ thus $u$ contains
residual set for each \(\alpha\)-partition, and thus \(u\) do not contain any monotone sequential contour of rank \(\alpha\). \(\blacksquare\)

Notice that it was also shown

**Theorem 6.6.** [18, reformulation of Theorem 3.12]

\[ MA_{\sigma - \text{center}} \] implies \(P_{\alpha + \omega} \neq \emptyset\).

It is worth to compare the above results with [1, Theorem 4.2], where Baumgartner proved that if P-points exist then for each successor \(\alpha < \omega_1\) the class of \(J^*_{\omega}\) ultrafilters is nonempty, and with our theorem from [17] where we proved (in ZFC) that a class of \(J^*_{\omega}\) ultrafilters is empty.

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