Complete embedded minimal $n$-submanifolds in $\mathbb{C}^n$

Claudio Arezzo and Frank Pacard

1 Introduction

A classical problem in the theory of minimal submanifolds of Euclidean spaces is to study the existence of a minimal submanifold with a prescribed behavior at infinity, or to determine from the asymptotes the geometry of the whole submanifold. Beyond the intrinsic interest of these questions, they are also of crucial importance when studying the possible singularities of minimal submanifolds in general Riemannian manifold. When studying minimal surfaces, i.e. two dimensional submanifolds, the standard tool to solve these problems is given by the Weierstrass representation formula which relates the geometry of the minimal surface to complex analytic properties of holomorphic one-forms on Riemann surfaces. Recently, gluing technics have been developed and have provided an abundant number of new examples of minimal hypersurfaces in Euclidean space.

For higher dimensional minimal submanifolds such a link clearly disappears and no complex analysis can be put into play. While, gluing technics have been extensively used in the study of minimal hypersurfaces, they have not been adapted to handle higher codimensional submanifolds. The aim of this paper is to use a gluing technique for minimal submanifolds to make a step towards the understanding of these questions in arbitrary codimension. We will restrict ourselves to the case of real $n$-dimensional submanifolds of $\mathbb{C}^n$. There are two main reasons for doing so. The first one is technical: when trying to desingularize the intersection, for example, of a pair of $n$-planes, which give the desired asymptotic behavior, one needs a model of a minimal submanifold with this behavior at infinity, to rescale and to glue into the pair of planes where a neighborhood of the intersection is removed. This local model needs to be sufficiently simple to allow a very detailed study of the linearized mean curvature operator. In our situation this is provided by a generalization of an area minimizing submanifold found by Lawlor in [6], while in more general cases such an example is not known. The second reason is that among minimal $n$-submanifolds of $\mathbb{C}^n$ there is a special family, namely the special Lagrangian ones, which are of great importance in a variety of geometric and physical problems (see, for example, [3], [5] and [13]).

To better describe our result let us first observe that for minimal surfaces in $\mathbb{C}^2$ graphs of meromorphic (or anti-meromorphic) functions are enough to answer some the above questions.

For example, given $z_1, \ldots, z_k \in \mathbb{C}$ all distinct and $\alpha_1, \ldots, \alpha_k \in \mathbb{C}$, the surface which is the graph of

$$z \in \mathbb{C} \setminus \{z_1, \ldots, z_k\} \mapsto \alpha_1(\bar{z} - \bar{z}_1)^{-1} + \ldots + \alpha_k(\bar{z} - \bar{z}_k)^{-1} \in \mathbb{C}^2$$

(1)

is a complete embedded minimal surface with $k+1$ ends.

When $k = 1$ we get the usual hyperbola

$$z \in \mathbb{C} \setminus \{0\} \mapsto (z, \bar{z}^{-1}) \in \mathbb{C}^2.$$

which can be seen as the two dimensional element of the following family of $n$-submanifolds, $H_I$, of $\mathbb{C}^n$,

$$\left(0, \frac{\pi}{n}\right) \times S^{n-1} \ni (s, \Theta) \mapsto \frac{e^{is}}{(\sin(ns))^{\frac{1}{n}}} \Theta \in \mathbb{C}^n$$
For any $n$, $H_I$ is a complete special Lagrangian (and therefore minimal) submanifold of $\mathbb{C}^n$ with two ends $\beta_3$, $\beta_3$ and $\beta_4$.

Observe that, for any $\mathcal{R} \in O(n)$, we can define $H_\mathcal{R}$ the $n$-submanifold of $\mathbb{C}^n$ which is parameterized by
\[
(0, \frac{\pi}{n}) \times S^{n-1} \ni (s, \Theta) \mapsto \frac{1}{\sin(ns)} (\cos s \Theta + i \sin s \mathcal{R}(\Theta)) \in \mathbb{C}^n.
\]

This is still a minimal submanifold of $\mathbb{C}^n$, since $H_0 \in O(2n)$. However, given the complex structure in $\mathbb{R}^{2n}$ induced by the identification of $\mathbb{R}^{2n}$ with $\mathbb{C}^n$, it fails to be Lagrangian, except when $\mathcal{R}^2 = I$.

In this paper we prove the existence of minimal embedded $n$-submanifolds having $k + 1$ ends which, in some sense are the high dimensional analogues of the surfaces described by (1). Consider the "horizontal" $n$-plane
\[
\Pi_0 := \{ x \in \mathbb{C}^n : x \in \mathbb{R}^n \}
\]
and, for all $j = 1, \ldots, k$, consider the $n$-planes
\[
\Pi_j := \left\{ \left( x_j + \cos \frac{\pi}{n} x + i \sin \frac{\pi}{n} \mathcal{R}_j x \right) \in \mathbb{C}^n : x \in \mathbb{R}^n \right\},
\]
where $x_1, \ldots, x_k \in \mathbb{R}^n$ and $\mathcal{R}_1, \ldots, \mathcal{R}_k \in O(n)$ are fixed. We further assume that all $x_j$ are distinct. One property of this set of planes is that, than all angles between the planes $\Pi_0$ and $\Pi_j$, are equal to $\frac{\pi}{n}$. We refer to [3] for a definition of the angles between pairs of $n$-planes in $\mathbb{C}^n$.

We set
\[
\xi_{j,j'} := \frac{x_j - x_{j'}}{|x_j - x_{j'}|}
\]
for $j \neq j'$ and we define the $k \times k$ matrix $\Gamma := (\gamma_{jj'})_{j,j'}$ by $\gamma_{jj} = 0$ for all $j = 1, \ldots, k$ and
\[
\gamma_{j,j'} := \frac{1}{|x_j - x_{j'}|^n} \int_{S^{n-1}} (\mathcal{R}_j \Theta \cdot \mathcal{R}_{j'} \Theta - n (\Theta \cdot \mathcal{R}_j \xi_{j,j'}) (\Theta \cdot \mathcal{R}_{j'} \xi_{j,j'})) \, d\theta,
\]
for all $j \neq j'$.

Finally, for any $A_0 \in M_n(\mathbb{R})$, we define for all $j = 1, \ldots, k$
\[
\lambda_j := - \int_{S^{n-1}} A_0 \Theta \cdot \mathcal{R}_j \Theta \, d\theta,
\]
and $\Lambda := (\lambda_1, \ldots, \lambda_k)$.

In order for the desingularization to be possible, we will do the following assumptions :

(H1) $\forall j \neq j'$, $\xi_{j,j'} \notin \text{Im} (I - \mathcal{R}_{j'}^{-1} \mathcal{R}_j)$

(H2) The matrix $\Gamma$ is invertible

(H3) $\Gamma^{-1} \Lambda \in (0, +\infty)^k$

These assumptions will be commented in the next section. We can now state the main result of the paper.
Theorem 1 Assume that \( n \geq 3 \), and \((H1), (H2)\) and \((H3)\) are fulfilled. Then, the set of \( n \)-planes \( \Pi_0, \ldots, \Pi_k \) can be desingularized to produce a complete, embedded minimal \( n \)-dimensional submanifolds of \( \mathbb{C}^n \) which has \( k + 1 \) planar ends. This minimal \( n \)-submanifold is of the topological type of a \( n \)-sphere with \( k + 1 \) punctures and has finite total curvature.

More precisely, there exists \( \varepsilon_0 \) and for all \( \varepsilon \in (0, \varepsilon_0) \) there exists \( \Sigma_\varepsilon \) a \( n \)-dimensional submanifold of \( \mathbb{C}^n \) such that

(i) For all \( \varepsilon \in (0, \varepsilon_0) \), \( \Sigma_\varepsilon \) is embedded and minimal.

(ii) For all \( \varepsilon \in (0, \varepsilon_0) \), \( \Sigma_\varepsilon \) has \( k + 1 \) ends, which, up to translations, are given by \( \Pi_j \), for \( j = 1, \ldots, k \) and \( \{ x + i \varepsilon \mathcal{A}_0 x \mid x \in \mathbb{R}^n \} \).

(iii) As \( \varepsilon \) tends to 0, \( \Sigma_\varepsilon \) converges, away from \( \{ x_1, \ldots, x_k \} \), to \( \cup_{j=0}^k \Pi_j \) in \( \mathcal{C}^\infty \) topology.

(iv) For all \( j = 1, \ldots, k \), the rescaled submanifold \( \varepsilon^{-1/n} (\Sigma_\varepsilon - x_j) \) converges to \( \alpha_j H_{\mathcal{R}_j} \), where \( (\alpha_1, \ldots, \alpha_k) = \Gamma^{-1} \Lambda \).

If \((H1) - (H3)\) are not fulfilled it is still possible to desingularize \( \cup_{j=0}^k \Pi_j \). However, this time we only obtain a minimal submanifold which is not embedded.

More generally we should remark that not every configuration of planes and points can be obtained as limit of families of immersed minimal submanifolds. For example, Ross [11] proved that a minimal two sphere with two punctures minimally immersed in \( \mathbb{R}^4 \) with two simple planar ends has to be holomorphic, thus giving a severe restriction on the planes.

A number of questions arise naturally from the above result. In first place it would be interesting to know whether conditions \((H1) - (H3)\) are also necessary for such a desingularization to exist. It is easy to check that \((H1)\) is necessary and we believe that \((H2)\) and \((H3)\) should also be necessary conditions, at least when restricting the topological type of the minimal submanifolds. If this turns out to be the case, since there are special Lagrangian configurations of planes which do not satisfy these conditions, it would mean that they cannot be obtained as limits of special Lagrangian submanifolds, thus helping to understand the possible degenerations of these manifolds.

Not unrelated is the problem of determining whether one can desingularize a configuration of special Lagrangian planes through special Lagrangian submanifolds, possibly leaving free the phase of the calibration to change as in [4]. In the case of two planes Lawlor’s examples answer obviously the question. A. Brown has recently generalized this to the case of the intersection of two special Lagrangian submanifolds with boundary. Nothing is known for more than one point of intersection.

Finally, we would like to mention the recent work of J. Isenberg, R. Mazzeo and D. Pollack [4] where a technical analysis similar to our is developed.

2 Comments and examples

In this section, we give examples of sets of points \( x_1, \ldots, x_k \in \mathbb{R}^n \), orthogonal transformations \( \mathcal{R}_1, \ldots, \mathcal{R}_k \in O(n) \), and matrices \( \mathcal{A}_0 \in M_n(\mathbb{R}) \) for which \((H1), (H2)\) and \((H3)\) hold. To simplify the discussion, let us define

\[
\mathcal{S} := \{(x_1, \ldots, x_k) \in \mathbb{R}^n \times \ldots \times \mathbb{R}^n : x_j \neq x_{j'} \text{ if } j \neq j'\}.
\]

and

\[
\Omega := O(n) \times \ldots \times O(n)
\]
2.1 Comments on the assumptions

Observe that

$$\text{Im} (I - R_j^{-1} R_j') = \text{Im} (I - R_j^{-1} R_j),$$

so that condition (H1) is symmetric in $j$ and $j'$. Now, as may easily be checked, this assumption guarantees that

$$\forall j \neq j', \quad \Pi_j \cap \Pi_{j'} = \emptyset,$$

while $\Pi_0 \cap \Pi_j = \{x_j\}$ for all $j \geq 0$.

Obviously, the set of $((x_j), (R_j), j) \in S \times \Omega$ such that (H1) is fulfilled is an open set which is not equal to $S \times \Omega$, since, for example, if $I - R_j^{-1} R_j$ is invertible for some $j \neq j'$, then (H1) does not hold. Similarly, that the set of $((x_j), (R_j)) \in S \times \Omega$ such that (H2) holds is an open set which is not equal to $S \times \Omega$, since, for example, when all $R_j$ are equal, then $\Gamma \equiv 0$ and hence (H2) does not hold. Finally the set of $((x_j), (R_j), A_0) \in S \times \Omega \times M_n(\mathbb{R})$ for which (H3) is fulfilled is also open.

In $M_n(\mathbb{R})$ we define the relation $A \sim A'$ if and only if there exists $R, R' \in O(n)$ such that

$$A = R A R'.$$

Now if $A_0 \sim A'_0$ and if (H3) is fulfilled for $A_0$ and $(R_j) \in \Omega$ then (H3) is fulfilled for $A'_0$ and $(R R_j R')^{-1} \in \Omega$. Hence the set of $A \in M_n(\mathbb{R})$ such that (H3) is fulfilled for some $((x_j), (R_j), A_0) \in S \times \Omega \times M_n(\mathbb{R})$ only depends on the coset of $A$ in $M_n(\mathbb{R}) / \sim$.

2.2 Examples

When $k = 2$, we give some examples for which (H1), (H2) and (H3) are fulfilled. Without loss of generality, we can assume that $x_1 = -x_2 := e$. Let us assume that both orthogonal transformations $R_j$ leave $e$ unchanged, that is

$$R_1 e = R_2 e = e$$

Therefore

$$e \notin \text{Im}(I - R_1^{-1} R_2)$$

and in particular, (H1) is always fulfilled for such a choice of $R_j$.

We would like to compute

$$\gamma_{12} = \frac{1}{2^n} \int_{S^{n-1}} (R_1 \Theta \cdot R_2 \Theta - n (\Theta \cdot e) (\Theta \cdot e)) \, d\theta,$$

To this aim, decompose $\mathbb{R}^n$ into the direct sum of $E_\pm E_i \oplus (\oplus E_i)$ where $E_\pm$ are the eigenspaces of $R_1^{-1} R_1$ corresponding to the eigenvalues $\pm 1$ and all $E_i$ are two dimensional vector spaces on which the restriction of $R_2^{-1} R_1$ is a rotation of angle $\theta_i \in (0, \pi)$. Since all these spaces are mutually orthogonal, it is easy to compute

$$\gamma_{12} = -\frac{\omega_n}{2^n} \left( \frac{2}{n} \dim E_- + \frac{2}{n} \sum_i (1 - \cos \theta_i) \right).$$

where $\omega_n := |S^{n-1}|$. Hence the matrix $\Gamma$ is invertible except when $R_1 = R_2$. Hence, (H2) holds whenever $R_1 \neq R_2$. Observe that in this case we have $\gamma_{12} < 0$.

Finally it remains to check that it is possible to choose $A_0 \in M_n(\mathbb{R})$ such that

$$\lambda_j := -\int_{S^{n-1}} A_0 \Theta \cdot R_j \Theta \, d\theta < 0,$$
for \( j = 1, 2 \). To this aim, we define
\[
L : \mathcal{A}_0 \in M_n(\mathbb{R}) \rightarrow \Lambda \in \mathbb{R}^2.
\]
This linear map is easily seen to be non zero so, either it is surjective or its image is included in a one dimensional space. Assume that the latter is true, then there exists \( \alpha, \beta \neq 0 \) such that
\[
\alpha \int_{\mathbb{S}^{n-1}} \mathcal{A}_0 \Theta \cdot \mathcal{R}_1 \Theta d\theta = \beta \int_{\mathbb{S}^{n-1}} \mathcal{A}_0 \Theta \cdot \mathcal{R}_2 \Theta d\theta,
\]
for all \( \mathcal{A}_0 \). Using this equality for \( \mathcal{A}_0 = \mathcal{R}_1 \) and \( \mathcal{A}_0 = \mathcal{R}_2 \), we conclude that necessarily \( \alpha = \pm \beta \) and
\[
\int_{\mathbb{S}^{n-1}} \mathcal{R}_1 \Theta \cdot \mathcal{R}_2 \Theta d\theta = \pm \omega_n.
\]
However, direct computation shows that
\[
\int_{\mathbb{S}^{n-1}} \mathcal{R}_1 \Theta \cdot \mathcal{R}_2 \Theta d\theta = \omega_n \left( 1 - \frac{2}{n} \dim E - \frac{2}{n} \sum_i (1 - \cos \theta_i) \right)
\]
Since we already know that this quantity is in absolute value strictly less than \(|\mathbb{S}^{n-1}|\) when \( \mathcal{R}_1 \neq \mathcal{R}_2 \), this implies that \( L \) is surjective and hence \( L^{-1}(0, +\infty)^2 \) is an open nonempty set in \( M_n(\mathbb{R}) \).

To summarize, we have obtained the

**Lemma 1** Assume that \( e \in \mathbb{R}^n \), \( \mathcal{R}_1, \mathcal{R}_2 \in O(n) \) are chosen so that
\[
\mathcal{R}_1 e = \mathcal{R}_2 e = e
\]
then \( (H1) \) holds. If in addition \( \mathcal{R}_1 \neq \mathcal{R}_2 \) then \( (H2) \) holds and the set of \( \mathcal{A}_0 \) such that \( (H3) \) holds
is an open subset of \( M_n(\mathbb{R}) \).

3 Definition of the connection Laplacian on the tangent
and normal bundles

We define all the operators which will be needed in the subsequent sections. We also recall some well known properties of these operators.

3.1 First order differential operators

To begin with let us define the connections in the tangent bundle and normal bundle of a \( m \)-dimensional submanifold \( M \) of \( \mathbb{R}^n \). Let \( V \) be any vector field on a submanifold \( M \) of \( \mathbb{R}^N \) and \( e \) a tangent vector field on \( M \). We will denote by \( \nabla_e V \) the full derivative of \( V \) along \( e \).

The connection on the tangent bundle \( \nabla^\tau_e \), along \( e \), applied to the tangent vector field \( T \), is defined to be the orthogonal projection of \( \nabla_e T \) on the tangent bundle. We will also write
\[
\nabla^\tau_e T = [\nabla_e T]^\tau_M
\]
where \([\cdot]^\tau_M\) denotes the orthogonal projection over the tangent bundle of \( M \).

Finally, we define the connection on the normal bundle \( \nabla^\nu_e \), along \( e \), applied to the normal vector field \( V \), is defined to be the orthogonal projection of \( \nabla_e N \) on the normal bundle. We will also write
\[
\nabla^\nu_e N = [\nabla_e N]^\nu_M
\]
where \([\cdot]|^\nu_M\) denotes the orthogonal projection over the normal bundle of \(M\).

Let \((e_1, \ldots, e_m)\) be a local orthonormal tangent frame field. For any function \(f\) defined on \(M\), we set
\[
\text{grad}_M f := \sum_{j=1}^m (e_j f) e_j,
\]
and for any tangent vector field \(T\)
\[
\text{div}_M T := \sum_{j=1}^m \nabla^\tau e_j T \cdot e_j.
\]
where \((e_1, \ldots, e_m)\) is a local orthonormal frame.

### 3.2 Second order differential operators

Given \((e_1, \ldots, e_m)\) a local orthonormal tangent frame field, we can define \(\Delta_M\) the Laplace operator on \(M\) acting on the function \(f\) by
\[
\Delta_M f := \sum_{j=1}^m e_j^2 f - \sum_{j=1}^m \left(\nabla^\tau e_j e_j\right) f.
\]
The connection Laplacian on the tangent bundle of \(M\), acting on the tangent vector field \(T\), is defined by
\[
\Delta^\tau_M T := \sum_{j=1}^m \nabla^\nu_{e_j} \nabla^\nu e_j T - \sum_{j=1}^m \nabla^\nu_{\nabla^\tau e_j} e_j T.
\]
This is just the trace of the invariant second derivative defined by
\[
(\nabla^\tau)^2_{V,W} := \nabla^\nu_V \nabla^\nu W - \nabla^\nu_{\nabla^\nu V} W.
\]
Let us recall the main properties of \(\Delta^\tau_M\).

**Proposition 1** \([8]\) *The operator \(\Delta^\tau_M\) is a negative self-adjoint operator and*
\[
\int_M \Delta^\tau_M V \cdot W = -\int_M \nabla^\tau V \cdot \nabla^\tau W.
\]
Here, by definition
\[
\int_M \nabla^\tau V \cdot \nabla^\tau W := \sum_{j=1}^m \int_M \nabla^\nu_{e_j} V \cdot \nabla^\nu e_j W.
\]
Finally, the connection Laplacian operator on the normal bundle of \(M\), acting on the normal vector field \(N\), is defined to be
\[
\Delta^\nu_M N := \sum_{j=1}^m \nabla^\nu_{e_j} \nabla^\nu e_j - \sum_{j=1}^m \nabla^\nu_{\nabla^\tau e_j} e_j N.
\]
This is just the trace of the invariant second derivative defined by
\[
(\nabla^\nu)^2_{V,W} := \nabla^\nu_V \nabla^\nu W - \nabla^\nu_{\nabla^\nu V} W.
\]
Let us recall the main properties of \(\Delta^\nu_M\).
Proposition 2. The operator $\Delta^\nu_M$ is a negative self-adjoint operator and

$$\int_M \Delta^\nu_M V \cdot W = - \int_M \nabla^\nu V \cdot \nabla^\nu W.$$ 

Here, by definition

$$\int_M \nabla^\nu V \cdot \nabla^\nu W := \sum_{j=1}^{n} \int_M \nabla^\nu_{e_j} V \cdot \nabla^\nu_{e_j} W.$$ 

Observe that all these definitions do not depend on the choice of the local orthonormal tangent frame field $(e_1, \ldots, e_m)$.

### 3.3 Operators of order zero

The second fundamental form is a section of the bundle $T^*(M) \otimes T^*(M) \otimes N(M)$ defined by

$$B_{V,W} := \nabla^\nu V \cdot W$$

In other words, at any point $p \in M$, $B_p$ represents a symmetric bilinear map from $T_p(M)$ into $N_p(M)$.

Given any local orthonormal tangent frame field $(e_1, \ldots, e_m)$ we define

$$B_{i,j} = B_{e_i, e_j} = \nabla^\nu_{e_i} e_j$$

This allows to define the linear operator acting on normal vector fields

$$B(N) := \sum_{i,j=1}^{m} (B_{ij} \cdot N) B_{i,j} = \sum_{i,j=1}^{m} (\nabla^\nu_{e_i} e_j \cdot N) \nabla_{e_i} e_j$$

### 4 Differential forms

It will be useful to translate some of the previously defined operator in the language of differential forms. Let $\Omega^p(M)$ denote the space of $p$-forms on $M$. We denote by $d^p$ the exterior derivative

$$d^p : \Omega^p(M) \rightarrow \Omega^{p+1}(M)$$

and define the operator

$$\delta^p : \Omega^p(M) \rightarrow \Omega^{p-1}(M)$$

by

$$\delta^p = (-1)^{d(p+1)+1} \star d \star$$

where $\star$ denotes the usual Hodge operator.

The Hodge Laplacian on $p$ forms is defined by

$$\Delta^p = - (d^{p-1} \delta^p + \delta^{p+1} d^p).$$

Observe that, with this definition, $\Delta^p$ is a negative operator.

The inner product on $\Omega^p(M)$ is defined by

$$<\omega, \tilde{\omega}>_{\Omega^p(M)} := \int_M \omega \wedge \star \tilde{\omega}$$
Recall that, granted the definition of $\delta^p$, for any $p-1$-form $\omega$ and any $p$-form $\tilde{\omega}$, we have
\[
\int_M d^p\omega \wedge \star \tilde{\omega} = \int_M \omega \wedge \star \delta^p \tilde{\omega}
\]
and hence, for any $p$-forms $\omega, \tilde{\omega}$, we have
\[
\int_M \Delta^p \omega \wedge \star \tilde{\omega} = - \int_M (d^p\omega \wedge \star d^p \tilde{\omega} + \delta^p \omega \wedge \star \delta^p \tilde{\omega})
\]

We will use the Hodge decomposition Theorem we recall now.

**Theorem 2** Let $M$ be a compact submanifold of $\mathbb{R}^n$. Then $\Omega^p(M)$ can be uniquely decomposed as
\[
\Omega^p(M) = d^{p-1} \Omega^{p-1}(M) \oplus \delta^{p+1} \Omega^{p+1}(M) \oplus \text{Har}^p(M)
\]
where $\text{Har}^p(M)$ is the set of harmonic $p$-forms.

One can identify functions on $M$ with 0-forms in the obvious way and, using the metric on $M$, one can also identify tangent vector fields with 1-forms in the following way. Assume that we have chosen a local orthonormal tangent frame field $(e_1, \ldots, e_m)$ on $M$ and that the metric is then given by $g = (g_{ij})_{i,j}$ so that
\[
\varepsilon_j = \frac{1}{\sqrt{g_{jj}}} \frac{\partial}{\partial x_j}
\]

Then to any vector field
\[
T = \sum_{j=1}^m T_j e_i
\]
we can associate the 1-form
\[
\omega = \sum_{j=1}^m \sqrt{g_{jj}} T_j dx_j
\]
This identification is coherent with the definition of the inner product on $\Omega^1(M)$ and on $T(M)$, namely, if the vector field $T$ is associated to the 1-form $\omega$, we have
\[
\|T\|_{L^2(M)}^2 := \int_M |T|^2 = \int_M \omega \wedge \star \omega := \|\omega\|^2
\]

Granted this identification of vector fields with 1-forms, we can identify $\Delta^1$, the Hodge Laplacian on 1-forms, with some symmetric second order differential operator acting on tangent vector fields. We will still denote by $\Delta^1$ this operator which is now defined on tangent vector fields. The relation between this operator and the connection Laplacian on the tangent bundle which we have defined in the previous section is given by the following

**Theorem 3** The difference between the Hodge Laplacian on 1-forms and the connection Laplace operator on the tangent bundle is given by
\[
\Delta^1 = \Delta^t_M - \text{Ric},
\]
where $\text{Ric}$ denotes the Ricci tensor.
The hyperbola

The hyperbola $H_I$ in dimension $2n$ is parameterized by

$$X(s,\theta) := \left(\frac{\cos s}{(\sin(ns))^{1/n}} \Theta(\theta), \frac{\sin s}{(\sin(ns))^{1/n}} \Theta(\theta)\right)$$

where $s \in (0, \frac{\pi}{n})$ and where

$$\theta \longrightarrow \Theta(\theta) \in S^{n-1}$$

is a parameterization of the $n - 1$ dimensional sphere. It will be convenient to assume that this parameterization is chosen in such a way that

$$\partial_{\theta_j} \Theta \cdot \partial_{\theta_k} \Theta = 0 \quad \text{if} \quad j \neq k.$$  \hspace{1cm} (2)

**Remark 1** When $n = 2$, we have the equivalent definition of the hyperbola given by

$$z \in \mathbb{C} \setminus \{0\} \longrightarrow \left(z, \frac{1}{\bar{z}}\right) \in \mathbb{C}^2$$

For notational convenient, we will frequently write $S$ instead of $S^{n-1}$ in subscripts or superscripts. It will also be convenient to identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ using

$$\mathbb{R}^n \times \mathbb{R}^n \ni (x, y) \sim x + iy \in \mathbb{C}^n.$$ 

With this identification we will write

$$X(s,\theta) := \frac{e^{is}}{(\sin(ns))^{1/n}} \Theta(\theta).$$

**The tangent bundle.** The tangent space of $H_I$ is spanned by the following set of vectors

$$\partial_s X = -\frac{e^{i(1-n)s}}{(\sin(ns))^{1+1/n}} \Theta$$

and, for all $j = 1, \ldots, n - 1$

$$\partial_{\theta_j} X = \frac{e^{is}}{(\sin(ns))^{1/n}} \Theta_j$$

In order to simplify the notations, we will write for short

$$\Theta_j := \partial_{\theta_j} \Theta.$$ 

and we will define the vectors

$$e_0 := e^{i(1-n)s} \Theta \quad \text{and} \quad e_j := e^{is} \Theta_j$$

so that, thanks to (3), $(e_0, \ldots, e_{n-1})$ is an orthonormal basis of $T(H_I)$ at the point $X$.

Finally, we define, for all $j = 1, \ldots, n - 1$

$$\varepsilon_j := \frac{\Theta_j}{|\Theta_j|}$$

so that, $(\varepsilon_1, \ldots, \varepsilon_{n-1})$ is an orthonormal basis of $T(S^{n-1})$ at the point $\Theta$. 

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The normal bundle. The normal space to $H_I$ is spanned by the following vectors

$$N_0 := i e^{i(1-n)s} \Theta,$$

and, for $j = 1, \ldots, n - 1$

$$N_j := i e^{is} \Theta_j / |\Theta_j|.$$

Hence any normal vector field on $H_I$ can be written as

$$V := i e^{i(1-n)s} f \Theta + i e^{is} T,$$

where $f$ is a real valued function and, for all $s \in (0, \pi/n)$, $T(s, \cdot)$ is a tangent vector field on $S^{n-1}$.

6 Expansion of the lower end of the hyperbola

Recall that the hyperbola is parameterized by

$$X(s, \theta) = \frac{e^{is}}{(\sin(ns))^{1/n}} \Theta.$$

We set

$$r := \cos s \frac{\Theta}{(\sin(ns))^{1/n}}.$$

We can expand $r$ in terms of $s$ as $s$ tends to 0. We get explicitly

$$r = (ns)^{-1/n} (1 + O(s^2))$$

Which in turn yields an expansion of $s$ in terms of $r$, as $s$ tends to 0

$$s = \frac{r^{-n}}{n} (1 + O(r^{-2n}))$$

Finally, we get the expansion as $s$ tends to 0

$$\frac{\sin s}{(\sin(ns))^{1/n}} = \frac{r^{-n}}{n} (1 + O(r^{-2n}))$$

Hence the hyperbola is parameterized by

$$X(r, \theta) = r \Theta + i \frac{r^{1-n}}{n} (1 + O(r^{-2n})) \Theta$$

as $r$ tends to $+\infty$.

We now consider the Hyperbola scaled by a factor $(n \beta \varepsilon)^{1/n}$ for some $\beta > 0$ and $\varepsilon > 0$, namely

$$X_\beta(s, \theta) = (n \beta \varepsilon)^{1/n} \frac{e^{is}}{(\sin(ns))^{1/n}} \Theta.$$

We set $\rho := (\beta \varepsilon)^{1/n} r$ and, as $s$ tends to 0 we can also write

$$X_\beta(\rho, \theta) = \rho \Theta + i \rho^{1-n} (1 + O(\varepsilon^2 \rho^{-2n})) \Theta.$$

(3)
7 The Linearized mean curvature operator about the hyperbola

In order to compute the linearized mean curvature operator, we collect the results of the Appendix 1. We recall from [7] that the linearized mean curvature operator is given by

\[ L_H = \Delta_H + B \]

In the local chart given by \( X \), we have explicitly

**Proposition 3** The linearized mean curvature operator about the hyperbola \( H_I \) reads

\[
(s\sin(ns))^{-\frac{2n}{n}} L_H V = i e^{i(1-n)s} \left[ \left( \sin(ns) \right)^2 - \frac{\sin(ns)}{n} \partial_s \left( \sin(ns) \right) \partial_s f \right] + \Delta_S f - (n - 1)f
+ (n^2 - 1) \sin^2(ns) f - 2 \cos(ns) \text{div}_S T \right] \Theta
+ \left( \sin(ns) \right)^2 \partial_s \left( \sin(ns) \right)^2 \partial_s f
+ \Delta_S f - (n - 1)f
+ 3 \sin^2(ns) T + 2 \cos(ns) \text{grad}_S f
\]

In order to study the spectral properties of \( L_H \), it will be useful to identify tangent vector fields on \( S^{n-1} \) with 1-forms on \( S^{n-1} \). This identification yields a natural identification of normal vector fields on \( H_I \)

\[
V := f i e^{i(1-n)s} \Theta + i e^{is} T \in N(H)
\]

with

\[
U := (f, v) \in C^\infty(\mathbb{R} \times S^{n-1}; \Omega^0 \times \Omega^1)
\]

where \( v(s, \cdot) \) is the 1-form corresponding to \( T(s, \cdot) \).

Granted this identification we can identify \( L_H \) with the following linear operator

\[
(s\sin(ns))^{-\frac{2n}{n}} L_H (f, v) = \left( \sin(ns) \right)^2 \partial_s \left( \sin(ns) \right) \partial_s \left( \sin(ns) \right) \partial_s f, \partial_s v \right) + (\Delta^0 f, \Delta^1 v)
+ 2 \cos(ns) (\delta^1 v, \delta^0 f) + ((1 - n) f, (n - 3) v)
+ \sin^2(ns) ((n^2 - 1)f, 3v)
\]

Also, in order to have a better understanding of the structure of this operator we define the variable \( t \) by

\[
dt := \frac{1}{\sin(ns)} ds,
\]

with \( t(\frac{\pi}{2n}) = 0 \). We obtain explicitly that

\[
e^{-nt} = \frac{\sin(ns)}{1 - \cos(ns)}
\]

which implies that

\[
\sin(ns) = (\cosh(nt))^{-1} \quad \text{and} \quad \cos(ns) = -\tanh(nt).
\]

Then one checks directly that the operator

\[
D := \left( \sin(ns) \right)^2 \partial_s \left( \sin(ns) \right) \partial_s f
\]
\[
\left( \sin(ns) \right)^{\frac{n+2}{n+2}} D \left( \sin(ns) \right)^{\frac{n+2}{n+2}} = \partial_t^2 - \frac{(n-2)^2}{4} - \frac{n^2 - 4}{4} \frac{1}{\cosh^2(nt)}.
\]

Hence the study of \( L_H \) is equivalent to the study of the conjugate operator
\[
L_H(f, v) = (\partial_t^2 f, \partial_t^2 v) + (\Delta^0 f, \Delta^1 v) - 2 \tanh(nt) (\delta^1 v, d^0 f)
\]
\[
- \left( \frac{n^2}{4} f, \frac{(n-4)^2}{4} v \right) + \frac{1}{\cosh^2(nt)} \left( \frac{3n^2}{4} f, \frac{16 - n^2}{4} v \right)
\]

Observe that, when \( t \) tends to \(-\infty\) the operator \( \tilde{L}_H \) is equivalent to the following differential operator
\[
\tilde{L}_0(f, v) = (\partial_t^2 f, \partial_t^2 v) + (\Delta^0 f, \Delta^1 v) + 2 (\delta^1 v, d^0 f) - \left( \frac{n^2}{4} f, \frac{(n-4)^2}{4} v \right)
\]

### 8 Eigendata of \( \Delta^0 \) and \( \Delta^1 \) on \( S^{n-1} \)

The spectrum of \( \Delta^0 \) on \( S^{n-1} \) is well known and given by
\[
\sigma(\Delta^0) = \{ k(n-2+k) | k \geq 0 \}
\]

The spectrum of \( \Delta^1 \) is also well known. In dimension \( n = 2 \) or \( n = 3 \) this spectrum is simply given by
\[
\sigma(\Delta^1) = \{ k(n-2+k) | k \geq 1 \}
\]

and all 1-eigenforms of \( \Delta^1 \) are image of eigenfunctions of \( \Delta^0 \) by the operator \( d^0 \).

In dimension \( n \geq 4 \), things are slightly more involved. The spectrum of \( \Delta^1 \) can be decomposed into two disjoint subsets
\[
\sigma(\Delta^1) = \sigma_{ex}(\Delta^1) \cup \sigma_{coex}(\Delta^1)
\]
where
\[
\sigma_{ex}(\Delta^1) = \{ k(n-2+k) | k \geq 1 \}
\]
corresponds of the eigenvalues associated to exact 1-eigenforms (namely 1-eigenforms which belong to the image of \( \Omega^0(S^{n-1}) \) by \( d^0 \)) and where
\[
\sigma_{coex}(\Delta^1) = \{ (k+1)(n-3+k) | k \geq 1 \}
\]
corresponds of the eigenvalues associated to coexact 1-eigenforms (namely 1-eigenforms which belong to the image of \( \Omega^2(S^{n-1}) \) by \( \delta^2 \)). It may be observed that all exact 1-eigenforms are image of eigenfunctions of \( \Delta^0 \) by \( d^0 \) and all coexact 1-eigenforms are image of exact 2-eigenforms of \( \Delta^2 \) by \( \delta^2 \).

We define
\[
\mathcal{V}^0 := \{ (f, 0) : d^0 f = 0 \}
\]
Next, for all \( k \geq 1 \)
\[
\mathcal{V}_{ex}^k := \{ (f, v) : (\Delta^0 f, \Delta^1 v) = -k(n-2+k)(f, v) \text{ and } d^1 v = 0 \}
\]
the eigenspace corresponding to the eigenvalue \( k(n-2+k) \) and to exact 1-forms, and
\[
\mathcal{V}_{coex}^k := \{ (0, v) : \Delta^1 v = -(k+1)(n-3+k) v \text{ and } \delta^1 v = 0 \}
\]
the eigenspace corresponding to the eigenvalue $k(n - 2 + k)$ and to coexact 1-forms.

This decomposition corresponds to the decomposition for 1-forms on $S^{n-1}$ given by Hodge’s decomposition Theorem
\[ \Omega^1(S^{n-1}) = d^0 \Omega^0(S^{n-1}) \oplus \delta^2 \Omega^2(S^{n-1}), \]
since there is no harmonic 1-form on $S^{n-1}$. The set of exact 1-eigenforms is total in $d^0 \Omega^0(S^{n-1})$ and the set of coexact 1-eigenforms is total in $\delta^1 \Omega^2(S^{n-1})$. In addition the two spaces $d^0 \Omega^0(S^{n-1})$ and $\delta^2 \Omega^2(S^{n-1})$ are orthogonal.

Remark 2 This decomposition of the space of 1-forms induces a natural decomposition of the space of tangent vector fields into vector fields corresponding to $d^0 \Omega^0(S^{n-1})$ and vector fields corresponding to $\delta^2 \Omega^2(S^{n-1})$. Moreover these two sets of vector fields are $L^2$ orthogonal.

9 Indicial roots

To begin with let us compute the indicial roots corresponding to $L_H$. First, observe that, in dimension $n \geq 4$, we can decompose any 1-form $v = v_{\text{ex}} + v_{\text{coex}}$ into the sum of an exact 1-form on $S^{n-1}$ and a coexact 1-form on $S^{n-1}$. In dimension $n = 2$ and $n = 3$, we set $v_{\text{coex}} = 0$. Now, if $(f, v)$ is a solution of the homogeneous equation $L_H(f, v) = 0$ then $v_{\text{coex}}$ satisfies
\[ L_H(0, v_{\text{coex}}) = (0, \partial_t^2 v_{\text{coex}}) + (0, \Delta^1 v_{\text{coex}}) - \left(0, \frac{(n-4)^2}{4} v_{\text{coex}}\right) \]
\[ + \frac{1}{\cosh^2(nt)} \left(0, \frac{16 - n^2}{4} v_{\text{coex}}\right) \]
Recall that, restricted to coexact forms, the spectrum of $\Delta^1$ is given by the set of $(k+1)(n-3+k)$, for $k \geq 1$. In order to find the indicial roots corresponding to this operator, we look for solutions of the homogeneous problem of the form
\[ v_{\text{coex}} = a \psi \] (4)
where $\psi$ is a coexact eigenform of $\Delta^1$ corresponding to the eigenvalue $(k+1)(n-3+k)$ and where $a$ is a scalar function only depending on $t$. It is easy to see that the scalar function $a$ is a solution of the following ordinary differential equation
\[ \ddot{a} - \left(\frac{n-2}{2} + k\right)^2 a + \frac{16 - n^2}{4} \frac{a}{\cosh^2(nt)} = 0 \] (5)
The behavior of the solutions when $t$ tends to either $\pm \infty$ is given by $e^{\gamma_k^\pm} t$ where
\[ \gamma_k^\pm = \pm \left(\frac{n-2}{2} + k\right) \] (6)
These are the indicial roots of $L_H$ when this operator is restricted to coexact 1-forms.

We now turn to the study of indicial roots corresponding to $\omega_{\text{ex}}$. Observe that, this time
\[ L_H(f, v_{\text{ex}}) = (\partial_t^2 f, \partial_t^2 v_{\text{ex}}) + (\Delta^0 f, \Delta^1 v_{\text{ex}}) - 2 \tanh(nt) (\delta^1 v_{\text{ex}}, d^0 f) \]
\[ - \left(\frac{n^2}{4} f, \frac{(n-4)^2}{4} v_{\text{ex}}\right) + \frac{1}{\cosh^2(nt)} \left(\frac{3n^2}{4} f, \frac{16 - n^2}{4} v_{\text{ex}}\right) \] (7)
Assume that $\phi$ is an eigenvalue of $\Delta^0$ associated to the eigenvalue $k(n - 2 + k)$. Then, we look for solutions of (\ref{eq:InnerProduct}) of the form
\[ f = a \phi \quad \text{and} \quad v_{\text{ex}} = b d^0 \phi \]
where $a$ and $b$ are scalar functions only depending on $t$. We obtain the following system of ordinary differential equation
\[
\ddot{a} - \frac{n^2}{4} a + \frac{3n^2}{4} \frac{a}{\cosh^2(nt)} = 0
\]
when $k = 0$ and
\[
\begin{aligned}
\ddot{a} - k(n - 2 + k) a - \frac{n^2}{4} a - 2 \tanh(nt) k(n - 2 + k) b + \frac{3n^2}{4} \frac{a}{\cosh^2(nt)} &= 0 \\
\ddot{b} - k(n - 2 + k) b - \frac{(n - 4)^2}{4} b - 2 \tanh(nt) a + \frac{16 - n^2}{4} \frac{b}{\cosh^2(nt)} &= 0
\end{aligned}
\]
for $k \neq 0$. With little work one finds for $k \neq 0$ the asymptotic behavior of $a$ and $b$ at both $\pm \infty$ is governed by the following sets of indicial roots
\[ \mu_k^\pm = \pm \left( \frac{n}{2} + k \right) \quad \text{and} \quad \nu_k^\pm = \pm \left( \frac{n - 4}{2} + k \right) \]
and for $k = 0$, we find
\[ \mu_0^\pm = \pm \frac{n}{2}. \]
It is worth mentioning that the operators $L_H$ and $L_0$ have the same indicial roots.

10 The Jacobi fields

Some Jacobi fields of $L_H$ are very easy to obtain since they correspond to geometric transformations of the hyperbola. In this paragraph we consider $L_H$ instead of $L_H$ since these Jacobi fields are easier to describe for $L_H$.

**Jacobi fields corresponding to translations** Let $a, b \in \mathbb{R}^n$ be given, the Jacobi field corresponding to the translation of vector $e^{i\alpha} a$, namely
\[ x + i y \in \mathbb{C}^n \mapsto (x + \cos \alpha a) + i( y + \sin \alpha b) \in \mathbb{C}^n \]
is the projection on the normal bundle of the constant vector $e^{i\alpha} a$. We obtain explicitely
\[ \Phi_t(\alpha, a) = i e^{i(1-n)s} \sin((n-1)s + \alpha) (a \cdot \Theta) \Theta + i e^{is} \sin(\alpha - s) (a - (a \cdot \Theta) \Theta) \quad (8) \]

**Jacobi field corresponding to a dilation** This Jacobi field corresponding to the dilation
\[ x + i y \in \mathbb{C}^n \mapsto (1 + \delta) (x + i y) \in \mathbb{C}^n \]
is obtained by projecting over the normal bundle the infinitesimal dilation
\[ \delta \frac{e^{is}}{(\sin(ns))^{1/2}} \Theta \quad (9) \]
We obtain
\[ \Phi_d(\delta) = \delta (\sin(ns))^{1 - \frac{1}{2}} i e^{i(1-n)s} \Theta \quad (10) \]
Jacobi fields corresponding to the action of $SU(n)$ Let $A$ be some $n \times n$ symmetric matrix. The Jacobi fields corresponding to the action of

\[ x + iy \in \mathbb{C}^n \rightarrow e^{iA} (x + iy) \in \mathbb{C}^n \]

are obtained by projecting over the normal bundle the infinitesimal action

\[ i \frac{e^{i s}}{(\sin(ns))^{1/n}} A \Theta \]

We obtain explicitly

\[ \Phi_{SU(n)}(A) = (\sin(ns))^{-\frac{1}{n}} \left( i e^{i(1-n)s} \cos(ns) (A \Theta \cdot \Theta) \Theta + i e^{i s} (A \Theta - (A \Theta \cdot \Theta) \Theta) \right). \] (11)

Jacobi fields corresponding to the action of $O(2n)/SU(n)$ Let $A$ be some $n \times n$ antisymmetric matrix. The Jacobi fields corresponding to the action of

\[ x + iy \in \mathbb{C}^n \rightarrow (e^{-A} x + i e^A y) \in \mathbb{C}^n \]

are obtained by projecting over the normal bundle the infinitesimal action

\[ \frac{e^{-i s}}{(\sin(ns))^{\frac{1}{n}}} A \Theta \]

We obtain explicitly

\[ \Phi_{O(2n)}(A) = (\sin(ns))^{-\frac{1}{n}} \sin(2s) i e^{i s} A \Theta. \] (12)

The Jacobi fields corresponding to the action of

\[ x + iy \in \mathbb{C}^n \rightarrow (\cosh A (x + iy) + \sinh A (y + ix)) \in \mathbb{C}^n \]

are obtained by projecting over the normal bundle the infinitesimal action

\[ i \frac{e^{-i s}}{(\sin(ns))^{\frac{1}{n}}} A \Theta \]

We obtain explicitly

\[ \Psi_{O(2n)}(A) = (\sin(ns))^{-\frac{1}{n}} \cos(2s) i e^{i s} A \Theta. \]

11 The maximum principle

We want to prove the following result

**Proposition 4** Assume that $n \geq 4$ and assume that $U$ is a solution of $\mathcal{L}_H U = 0$ in $(t_1, t_2) \times S^{n-1}$, (with $U = 0$ at the boundary if $t_1 > -\infty$ or $t_2 < +\infty$). Assume that

\[ |U| \leq (\cosh t)^{-\nu}, \]

for some $\nu > \frac{2-n}{2}$. Further assume that, for every $t \in (t_1, t_2)$, $U(t, \cdot)$ is orthogonal to $\mathcal{V}^0$ in the $L^2$-sense on $S^{n-1}$. Then $U \equiv 0$.

**Proof:** We decompose $U = (f, v)$ into $v = v_{ex} + v_{coex}$ where $v_{ex}$ is an exact 1-form on $S^{n-1}$ and where $v_{coex}$ is a coexact 1-form on $S^{n-1}$. 

15
**Step 1.** We multiply the equation $L_H(f, v)$ by $(0, v_{coex})$ and integrate over $(t_1, t_2) \times S^{n-1}$. We obtain

$$0 = \int |\partial_t v_{coex}|^2 + \int |d^0 v_{coex}|^2 + \frac{n^2}{4} \int |v_{coex}|^2 - \frac{n^2 - 16}{4} \int (\cosh(nt))^{-2} |v_{coex}|^2$$

which already implies that $v_{coex} \equiv 0$ since $n \geq 4$. It therefore remains to prove that $(f, v_{ex}) \equiv 0$.

The proof is now quite involved and, in order to simplify the notations, we set

$$A := \int f^2, \quad B := \int (\cosh(nt))^{-2} f^2, \quad C := \int |\partial_t f|^2, \quad D := \int |d^0 f|^2$$

$$A' := \int |v_{ex}|^2, \quad B' := \int (\cosh(nt))^{-2} |v_{ex}|^2, \quad C' := \int |\partial_t v_{ex}|^2, \quad D' := \int |\delta^1 v_{ex}|^2$$

and

$$E := \int \tanh(nt) d^0 f \wedge v_{ex} = - \int \tanh(nt) f \delta^1 v_{ex}$$

**Step 2.** We multiply the equation $L_H(f, v)$ by $(f, v_{ex})$ and integrate. This time, we obtain

$$\int (|\partial_t f|^2 + |\partial_t v_{ex}|^2) + \int (|d^0 f|^2 + |\delta^1 v_{ex}|^2) + \int \left( \frac{n^2}{4} f^2 + \frac{(n-4)^2}{4} |v_{ex}|^2 \right)$$

$$= \int (\cosh(nt))^{-2} \left( \frac{3n^2}{4} f_2 + \frac{16-n^2}{4} |v_{ex}|^2 \right) - 2 \int \tanh(nt) (d^0 f \wedge v_{ex} + f \delta^1 v_{ex})$$

With our notations this can be written as

$$C + C' + D + D' + \frac{n^2}{4} A + \frac{(n-4)^2}{4} A' - \frac{3n^2}{4} B + \frac{n^2 - 16}{4} B' = -4 E \quad (13)$$

Using with Cauchy-Schwarz inequality and integration by parts, we estimate

$$\left| \int \tanh(nt) f \delta^1 v_{ex} \right| \leq \left( \int |\delta^1 v_{ex}|^2 \right)^\frac{1}{2} \left( \int f^2 - \int (\cosh(nt))^{-2} f^2 \right)^\frac{1}{2}.$$ 

This inequality can also be written as

$$E^2 \leq D'(A - B). \quad (14)$$

Now, we use the fact that, for all $t$, $f(t, \cdot)$ is $L^2$-orthogonal to the first eigenfunction on $S^{n-1}$ hence

$$\int_{S^{n-1}} |d^0 f|^2 \geq (n-1) \int_{S^{n-1}} f^2$$

Integrating over $t$ yields

$$\int |d^0 f|^2 \geq (n-1) \int f^2$$

Otherwise stated

$$D \geq (n-1) A. \quad (15)$$

Finally, we use an integration by parts to prove that

$$n \int (\cosh(nt))^{-2} f^2 = -2 \int \tanh(nt) f \partial_t f$$
Using Cauchy-Schwarz inequality, we conclude that
\[ n \int (\cosh(nt))^{-2} f^2 \leq 2 \left( \int |\partial_t f|^2 \right)^{\frac{1}{2}} \left( \int f^2 - \int (\cosh(nt))^{-2} f^2 \right)^{\frac{1}{2}} \]
A similar inequality holds with $f$ replaced by $v_{ex}$. This yields
\[ n^2 B^2 \leq 4 C (A - B)^2 \quad n^2 B'^2 \leq 4 C' (A' - B')^2 \tag{16} \]

**Step 3.** Assume that $A \neq B$ and $A' \neq B'$, we can collect the previous inequalities (14)-(16) to eliminate $C$, $C'$, $D$ and $D'$ in (13). With little work we find
\[ \frac{n^2 (2B - A)^2}{4} - \frac{1}{4} A - \frac{1}{4} A' + (n - 5) A + 4 B + \frac{(E - A + B)^2}{A - B} \leq 0 \]
It is now an easy exercise to check that, when $n \geq 4$, the above inequality implies that $A = A' = 0$. hence $f = 0$ and $v_{ex} = 0$.

If $A = B$, we first use (16) to conclude that $B = 0$, hence we conclude that $f = 0$. In this case, it readily follows from (13) that $A' = 0$ which implies that $v_{ex} = 0$.

If $A' = B'$, we first use (16) to conclude that $B' = 0$, so $v_{ex} = 0$. In this case, (13) reduces to
\[ C + D + \frac{n^2}{4} A - \frac{3n^2}{4} B = 0 \]
And, using (14)-(16) we conclude that $A = 0$ which implies that $f = 0$.

The proof is therefore complete. \qed

When $n = 3$ some modifications are needed in the corresponding statement. Indeed, we shall prove, and this will be sufficient for our purposes that the maximum principle as stated in Proposition 6 holds on any interval containing a large interval centered at the origin. This additional assumption is probably not needed but we have not been able to get ride of it. Let us emphasize that the result needed below is just what we need for all the remaining analysis to hold.

**Proposition 5** Assume that $n = 3$. Then there exists $\hat{t}_0 > 0$ such that, if $U$ is a solution of $L_H U = 0$ in $(t_1, t_2) \times S^2$, for some $t_1 \leq -\hat{t}_0$ and $t_2 \geq \hat{t}_0$, , (with $U = 0$ at the boundary if $t_1 > -\infty$ or $t_2 < +\infty$) satisfying
\[ |U| \leq (\cosh t)^{-\nu}, \]
for some $\nu > -\frac{1}{2}$, and with the property that, for every $t \in (t_1, t_2)$, $U(t, \cdot)$ is orthogonal to $\mathcal{V}^0$ in the $L^2$-sense on $S^2$, then $U \equiv 0$.

**Proof:** First we assume that $U(t, \cdot)$ is not only orthogonal to $\mathcal{V}^0$ in the $L^2$-sense on $S^2$ but also orthogonal to $\mathcal{V}^1$. Then (19) can be improved into
\[ D \geq 6 A. \]
where 6 corresponds to the next eigenvalue. And we can proceed as in Step 2 and Step 3 of the proof of the previous result, to show that $f = 0$ and $v = 0$. Observe that, at this point, there is no additional restriction needed.

Therefore is just remains to prove the result when $U(t, \cdot)$ belongs to $\mathcal{V}^1$. In this case we are reduced to study some coupled system of ordinary differential equation. Indeed, we can
decompose \( f \) and \( v \) over eigenfunctions \( \Delta^0 \) and \( \Delta^1 \) associated to the eigenvalue 2. Hence we can reduce to the case where

\[
f = a \phi \quad \text{and} \quad v_{ex} = b d^0 \phi
\]

where \( a \) and \( b \) are scalar functions only depending on \( t \) and \( \phi \) is an eigenfunction of \( \Delta^0 \) corresponding to the eigenvalue 2. We obtain the following system of ordinary differential equation

\[
\begin{align*}
\ddot{a} - 2a - \frac{9}{4} a - 4 \tanh(3t) b + \frac{27}{4} \frac{a}{\cosh^2(3t)} &= 0 \\
\ddot{b} - 2b - \frac{1}{4} b - 2 \tanh(3t) a + \frac{7}{4} \frac{b}{\cosh^2(3t)} &= 0.
\end{align*}
\]

(17)

For the time being let us assume that \( t_1 = -\infty \) and \( t_2 = +\infty \) and show that any solution of (17) which is bounded by \( \cosh s\nu \) vanishes. As already mentioned, the asymptotic behavior of \( a \) and \( b \) at both \( \pm \infty \) is governed by the following sets of indicial roots

\[
\mu_1^\pm = \pm \frac{5}{2} \quad \text{and} \quad \nu_k^\pm = \pm \frac{1}{2}
\]

Observe that we know explicitly some solutions of (17), namely the solutions corresponding to the Jacobi field \( \Phi_1(\alpha, a) \). In particular, when \( \alpha = 0 \), we obtain the solution

\[
\begin{align*}
 a_1 &= (\sin(3s))^{-\frac{1}{2}} (\sin(3s) \cos s - \sin s \cos(3s)) \\
 b_1 &= - (\sin(3s))^{-\frac{1}{2}} \sin s
\end{align*}
\]

Recall that \( s \) is a function of \( t \). Since \( \sin(3s) = (\cosh(3t))^{-1} \) and \( \cos(3s) = -\tanh(3t) \), we can easily obtain the asymptotic behavior of this explicit solution. Near \( -\infty \) it is given by

\[
\begin{align*}
 a_1 &\sim (\cosh t)^{-\frac{3}{2}} \\
 b_1 &\sim -\frac{1}{3} (\cosh t)^{-\frac{3}{2}}
\end{align*}
\]

and near \( +\infty \) it is given by

\[
\begin{align*}
 a_1 &\sim \sin \frac{3}{2} (\cosh t)^{-\frac{3}{2}} \\
 b_1 &\sim -\sin \frac{3}{2} (\cosh t)^{-\frac{3}{2}}
\end{align*}
\]

Now, assume that we have a solution of (17) which is bounded by \( (\cosh t)^\nu \) for some \( \nu < -\frac{1}{2} \). The inspection of the indicial roots shows that this solution is bounded by a constant times \( (\cosh t)^{-\frac{3}{2}} \). However all solutions of (17) which are bounded by \( (\cosh t)^{-\frac{3}{2}} \) have to be a multiple of the solution \((a_1, b_1)\) described above. Clearly these are not bounded by \( (\cosh t)^{-\frac{3}{2}} \) at \( +\infty \) unless it is identically 0.

In order to complete the proof of the Proposition, we argue by contradiction and assume that the result is not true. There would exist sequences \((t'_i)_i \in [-\infty, 0] \) and \((t''_i)_i \in [0, +\infty] \) tending to \( -\infty \) and \( +\infty \) respectively, and for each \( i \) a solution \((a_i, b_i)\) of (17) defined in \((t'_i, t''_i)\) and bounded by a \( (\cosh s)^\nu \). These solutions have 0 boundary data whenever \( t'_i \) or \( t''_i \) are finite.

Our problem being linear we can assume that the solution \((a_i, b_i)\) is normalized in such a way that

\[
\sup_{(t'_i, t''_i)} (\cosh t)^{-\nu} (|a_i| + b_i) = 1
\]

Let \( t_i \in (t'_i, t''_i) \) a point where the above maximum is achieved. To begin with, observe that the sequence \( t_i - t'_i \) remains bounded away from 0. This is obvious if \( t'_i = -\infty \) and if not, this follows from the fact that since \((a_i,b_i)\) and therefore \((\bar{a}_i, \bar{b}_i)\) are bounded by a constant (independent of \( i \))
times \((\cosh t_i)''\) in \([t_i', t_i' + 1]\) and since \(a_i = b_i = 0\) at \(t_i'\), standard ordinary differential arguments show that \((\tilde{a}_i, \tilde{b}_i)\) is also bounded by a constant (independent of \(i\)) times \((\cosh t_i')''\) in \([t_i', t_i' + \frac{1}{2}]\). As a consequence the above supremum cannot be achieved at a point which is too close to \(t_i'\).

Similarly one proves that the sequence \(t_i'' - t_i\) also remains bounded away from 0.

We now define the sequence of rescaled functions

\[(\tilde{a}_i, \tilde{b}_i)(t) := (\cosh t_i)''(a_i, b_i)(t + t_i).\]

**Case 1**: Assume that the sequence \(t_i\) converges to \(t_\infty \in \mathbb{R}\). After the extraction of some subsequences, if this is necessary, we may assume that the sequence \((\tilde{a}_i, \tilde{b}_i)(· - t_\infty)\) converges to some nontrivial solution of \((17)\). Furthermore this solution is bounded by a constant times \((\cosh t)''\). However, we have just proved that this is not possible.

**Case 2**: Assume that the sequence \(t_i\) converges to \(-\infty\). After the extraction of some subsequences, if this is necessary, we may assume that the sequence \((\tilde{a}_i, \tilde{b}_i)\) converges to \((a_\infty, b_\infty)\) some nontrivial solution of

\[
\begin{align*}
\dot{a} - \frac{17}{4} a + 4 b &= 0 \\
\dot{b} - \frac{9}{4} b + 2 a &= 0
\end{align*}
\]

in some interval \((t_*, +\infty)\), with boundary condition \(a_\infty = b_\infty = 0\), if \(t_* := \lim_{t \to -\infty} t_i - t_i\) is finite. Furthermore this solution is bounded by a constant times \((\cosh t)''\). It is a simple exercise to show that \((18)\) has no such solutions.

**Case 3**: Finally, we assume that the sequence \(t_i\) converges to \(+\infty\). This case being similar to Case 2, we shall omit it.

Since we have ruled out every possible case, the proof of the result is complete. \(\square\)

We will also need the following simpler result for the differential operator which appears in \(\mathcal{L}_H\) when \(t\) tends to \(-\infty\). Here no restriction are needed.

**Proposition 6** Assume that \(U\) is a solution of \(\mathcal{L}_0 U = 0\) in \((t_1, t_2) \times S^{n-1}\), \((U = 0\) at the boundary if \(t_1 > -\infty\) or \(t_2 < +\infty\)). Assume that

\[|U| \leq (\cosh t)^{-\nu},\]

for some \(\nu > \frac{2-n}{2}\). Then \(U \equiv 0\).

**Proof**: This time the proof can be obtain be first decomposing \(U = (f, v)\) over eigenspaces of \(\Delta^0\) and \(\Delta^1\) respectively and then compute explicitly the solutions of the ordinary differential equation and finally show that \(U = 0\).

However, one can also proceed as in the former proof. To begin with, let us assume that \(n \geq 4\). We decompose \(v = v_{\text{ex}} + v_{\text{coex}}\) where \(v_{\text{ex}}\) is an exact 1-form on \(S^{n-1}\) and where \(v_{\text{coex}}\) is a coexact 1-form on \(S^{n-1}\).

We multiply the equation \(\mathcal{L}_H U\) by \((0, v_{\text{coex}})\) and integrate over \((t_1, t_2) \times S^{n-1}\). We obtain

\[0 = \int |\partial_t v_{\text{coex}}|^2 + \int |d^1 v_{\text{coex}}|^2 + \frac{n^2}{4} \int |v_{\text{coex}}|^2\]

which already implies that \(v_{\text{coex}} \equiv 0\). It therefore remains to prove that \((f, v_{\text{ex}}) \equiv 0\).

The proof is now quite involved and, in order to simplify the notations, we set

\[A := \int f^2, \quad B := \int |\partial_t f|^2, \quad C := \int |\Delta^0 f|^2\]

\[A' := \int |v_{\text{ex}}|^2, \quad B' := \int |\partial_t v_{\text{ex}}|^2, \quad C' := \int |\delta^1 v_{\text{ex}}|^2.\]
We multiply the equation $L_H U$ by $(f, v_{ex})$ and integrate. This time, we obtain

$$\int (|\partial_t f|^2 + |\partial_t v_{ex}|^2) + \int (|\partial^0 f|^2 + |\delta^1 v_{ex}|^2) + \int \left( \frac{n^2}{4} f^2 + \frac{(n-4)^2}{4} |v_{ex}|^2 \right) = -2 \int (d^0 f \wedge v_{ex} + f \delta^1 v_{ex})$$

Using with Cauchy-Schwarz inequality, we estimate

$$\left| \int d^0 f \wedge v_{ex} \right| \leq \left( \int |d^0 f|^2 \right)^{\frac{1}{2}} \left( \int |v_{ex}|^2 \right)^{\frac{1}{2}}$$

$$\left| \int f \delta^1 v_{ex} \right| \leq \left( \int |\delta^1 v_{ex}|^2 \right)^{\frac{1}{2}} \left( \int f^2 \right)^{\frac{1}{2}}$$

(19)

To estimate the first RHS, we will use $2a b \leq a^2 + b^2$ and in order to estimate the second one, we use $2a b \leq \frac{1}{2} a^2 + 2b^2$.

Collecting these, together with (19), we conclude that

$$C + \frac{n^2}{4} A + C' + \frac{(n-4)^2}{4} A' \leq \frac{1}{2} C' + 2A + C + A'$$

To finish, we use the fact that, as in (15) we have

$$\int |\delta^1 v_{ex}|^2 \geq (n-1) \int |v_{ex}|^2$$

Otherwise stated that $C' \geq (n-1) A'$ to conclude that

$$\frac{n^2 - 8}{4} A + \frac{(n-4)^2 + 2n - 6}{4} A' \leq 0$$

which proves the desired claim.

\[\square\]

12 Mapping properties of $L_H$ on a half hyperbola

As in [1], the analysis of the mapping properties of $L_H$ is easy to do in some weighted Hölder spaces we are now going to define.

**Definition 1** For all $\delta \in \mathbb{R}$ and for all $t_0 \in \mathbb{R}$, the space $C^k_{\delta, \alpha}([t_0, +\infty) \times S^{n-1}; \Omega^0 \times \Omega^1)$ is defined to be the space of $U \in C^k_{\delta, \alpha}([t_0, +\infty) \times S^{n-1}; \Omega^0 \times \Omega^1)$ for which the following norm is finite

$$\|U\|_{k, \alpha, \delta} := \sup_{t \geq t_0} |e^{-\delta t} U|_{k, \alpha, [t, t+1] \times S^{n-1}}.$$  

Here $|U|_{k, \alpha, [t, t+1] \times S^{n-1}}$ denotes the usual Hölder norm in $[t, t+1] \times S^{n-1}$.

Observe that $U = (f, v)$ where $v$ is a 1-form, hence, in the above defined norm it is the coefficients of $v$ and the function $f$ which are estimated.

To begin with, we investigate the mapping properties of $L_H$ when defined between the above weighted spaces. These mapping properties crucially depend on the choice of $\delta$. We prove the

**Proposition 7** Assume that $\delta \in \left( \frac{\sqrt{3} - \sqrt{2}}{2}, \frac{\sqrt{2} - 2}{2} \right)$ and $\alpha \in (0, 1)$ are fixed. There exists some constant $c > 0$ and, for all $t_0 \in \mathbb{R}$ (when $n = 3$, $t_0$ has to be chosen larger than $t_0$ defined in Proposition 3), there exists an operator

$$G_{t_0} : C^0_{\delta, \alpha}([t_0, +\infty) \times S^{n-1}; \Omega^0 \times \Omega^1) \rightarrow C^2_{\delta, \alpha}([t_0, +\infty) \times S^{n-1}; \Omega^0 \times \Omega^1),$$
such that, for all $V \in C^0_{\alpha}([t_0, +\infty) \times S^{n-1}; \Omega^0 \times \Omega^1)$, if for all $t > t_0$, $V(t, \cdot)$ is orthogonal to $\mathcal{V}^0$ in the $L^2$ sense on $S^{n-1}$, then $U = \mathcal{G}_{t_0}V$ is the unique solution of

$$\begin{cases} \mathcal{L}_H U = V & \text{in } [t_0, +\infty) \times S^{n-1} \\ V \in \mathcal{V}^0 & \text{on } \{t_0\} \times S^{n-1}. \end{cases}$$

Furthermore, $\|U\|_{2, \alpha, \delta} \leq c\|V\|_{0, \alpha, \delta}$.

**Proof**: Uniqueness of $\mathcal{G}_{t_0}$ follows Proposition 4. We therefore concentrate our attention on the existence of $\mathcal{G}_{t_0}$ and the derivation of the uniform estimate for the inverse.

Our problem being linear, we can assume that

$$\sup_{(t', +\infty) \times S^{n-1}} |e^{-\delta t} V| \leq 1.$$

Now, it follows from Proposition 4 that, when restricted to the space of $U$ which are orthogonal to $\mathcal{V}^0$ in the $L^2$-sense on $S^{n-1}$, the operator $\mathcal{L}_H$ is injective on $(t', t'') \times S^{n-1}$. As a consequence, for all $t'' > t' + 1$ we are able to solve $\mathcal{L}_H U = V$, in $(t', t'') \times S^{n-1}$, with $U = 0$ on $\{t', t''\} \times S^{n-1}$.

We claim that, there exists some constant $c > 0$ independent of $t'' > t' + 1$ and $t_0$ and of $V$ such that

$$\sup_{(t', t'') \times S^{n-1}} |e^{-\delta t} U| \leq c.$$

We argue by contradiction and assume that the result is not true. In this case, there would exist sequences $t''_i > t'_i + 1$, a sequence of functions $V_i$ satisfying

$$\sup_{(t'_i, t''_i) \times S^{n-1}} |e^{-\delta t} V_i| = 1,$$

and a sequence $U_i$ of solutions of $\mathcal{L}_H (f_i, v_i) = (g_i, w_i)$, in $(t'_i, t''_i) \times S^{n-1}$, with $(f_i, v_i) = 0$ on $\{t'_i, t''_i\} \times S^{n-1}$ such that

$$A_i := \sup_{(t'_i, t''_i) \times S^{n-1}} |e^{-\delta t} U_i| \longrightarrow +\infty.$$

Furthermore, $U_i(t, \cdot)$ and $V_i(t, \cdot)$ are orthogonal in the $L^2$ sense to $\mathcal{V}^0$ on $S^{n-1}$. Let us denote by $(t_i, \theta_i) \in (t'_i, t''_i) \times S^{n-1}$, a point where the above supremum is achieved. We now distinguish a few cases according to the behavior of the sequence $t_i$ (which, up to a subsequence can always be assumed to converge in $[\pm \infty, +\infty]$). Up to some subsequence, we may also assume that the sequences $t''_i - t_i$ (resp. $t_i - t'_i$) converges to $\tau \in (0, +\infty]$ (resp. to $\tau \in (-\infty, 0)$).

Observe that the sequence $t_i - t'_i$ remains bounded away from 0. Indeed, since $U_i$ and $\mathcal{L}_H U_i$ are bounded by a constant (independent of i) times $e^{\delta t_i} A_i$ in $[t'_i, t''_i + 1] \times S^{n-1}$ and since $U_i = 0$ on $\{t''_i\} \times S^{n-1}$, standard elliptic estimates allow us to conclude that the partial derivative of $U_i$ with respect to $t$ is also uniformly bounded by a constant times $e^{\delta t'_i} A_i$ in $[t'_i, t''_i + \frac{\tau}{2}] \times S^{n-1}$. As a consequence the above supremum cannot be achieved at a point which is too close to $t'_i$. Similarly one proves that the sequence $t''_i - t_i$ also remains bounded away from 0.

We now define the sequence of rescaled functions

$$\hat{U}_i(t, \theta) := \frac{e^{-\delta t_i}}{A_i} U_i(t + t_i, \theta).$$

**Case 1**: Assume that the sequence $t_i$ converges to $t_\infty \in \mathbb{R}$. After the extraction of some subsequences, if this is necessary, we may assume that the sequence $\hat{U}_i$ converges to some nontrivial solution of

$$\mathcal{L}_H U_\infty = 0,$$

21
in \((t_*, t^*) \times S^{n-1}\), with boundary condition \(U_\infty = 0\), if either \(t_*\) or \(t^*\) is finite. Furthermore
\[
\sup_{(t_*, t^*) \times S^{n-1}} |e^{-\beta(t-t_\infty)} U_\infty| = 1. 
\] (20)
and \(U_\infty(t, \cdot)\) orthogonal in the \(L^2\) sense to \(V^0\) on \(S^{n-1}\). If \(t_* = -\infty\), the inspection of the indicial roots shows that, any solution of the homogeneous equation which is bounded by \(e^{\delta t}\) for some \(\delta \in (\frac{2-n}{2}, \frac{n-2}{2})\) is bounded by a constant times \(e^{\frac{\alpha - \beta}{2} t}\) at \(-\infty\). Similarly, if \(t^* = +\infty\) we find that any solution of the homogeneous equation which is bounded by \(e^{\delta t}\) for some \(\delta \in (\frac{2-n}{2}, \frac{n-2}{2})\) is bounded by a constant times \(e^{\frac{\alpha - \beta}{2} t}\) at \(-\infty\). But, applying Proposition \(8\), this implies that \((f, v) = 0\), contradicting (20).

Case 2: Assume that the sequence \(t_i\) converges to \(-\infty\). After the extraction of some subsequences, if this is necessary, we may assume that the sequence \(U_i\) converges to some nontrivial solution of \(\mathcal{L}_0 U_\infty = 0\) (21)
in \((t_*, t^*) \times S^{n-1}\), with boundary condition \(U_\infty = 0\), if either \(t_*\) or \(t^*\) is finite. Furthermore
\[
\sup_{(t_*, t^*) \times S^{n-1}} |e^{-\beta t} U_\infty| = 1. 
\] (22)
As already mentioned, the indicial roots corresponding to (21) are the same as the indicial roots corresponding to \(\mathcal{L}_H\). Again, if \(t_* = -\infty\), the inspection of the indicial roots shows that, any solution of the homogeneous equation which is bounded by \(e^{\delta t}\) for some \(\delta \in (\frac{2-n}{2}, \frac{n-2}{2})\) is bounded by a constant times \(e^{\frac{\alpha - \beta}{2} t}\) at \(-\infty\). Similarly, if \(t^* = +\infty\) we find that any solution of the homogeneous equation which is bounded by \(e^{\delta t}\) for some \(\delta \in (\frac{2-n}{2}, \frac{n-2}{2})\) is bounded by a constant times \(e^{\frac{\alpha - \beta}{2} t}\) at \(-\infty\). But, applying Proposition \(8\), this implies that \(U_\infty = 0\), contradicting (22).

Case 3: Assume that the sequence \(t_i\) converges to \(+\infty\). This case being similar to Case 2, we shall omit it.

Now that the proof of the claim is finished, we may pass to the limit \(t'' \to +\infty\) and obtain a solution of \(\mathcal{L}_H U = V\), in \((t', +\infty) \times S^{n-1}\), with \(U = 0\) on \([t'] \times S^{n-1}\), which satisfies
\[
\sup_{(t', +\infty) \times S^{n-1}} |e^{-\beta t} U| \leq c, 
\]
for some constant \(c > 0\) independent of \(S\). To complete the proof of the Proposition, it suffices to apply Schauder’s estimates in order to get the relevant estimates for all the derivatives. \(\square\)

We now extend the right inverse \(\mathcal{G}_{t_0}\) to the set of \(V := (g, 0)\) where \(g\) only depends on \(t\). For the sake of simplicity in the notations, we keep the same notation for the right inverse since they are defined on orthogonal spaces. This is the content of the following

**Proposition 8** Assume that \(\delta' < -\frac{n}{2}\) and \(\alpha \in (0, 1)\) are fixed. There exists some constant \(c > 0\) and, for all \(t_0 \in \mathbb{R}\), there exists an operator
\[ \mathcal{G}_{t_0} : C^{0,\alpha}_{\delta'}([t_0, +\infty) \times S^{n-1}; \Omega^0 \times \Omega^1) \rightarrow C^{2,\alpha}_{\delta'}([t_0, +\infty) \times S^{n-1}; \Omega^0 \times \Omega^1), \]
such that, for all \((g, 0) \in C^{0,\alpha}_{\delta'}([t_0, +\infty) \times S^{n-1}; \Omega^0 \times \Omega^1)\), if for all \(t > t_0\), \(g(t, \cdot)\) is constant on \(S^{n-1}\), then \((f, 0) = \mathcal{G}_{t_0}(g, 0)\) is the unique solution of
\[
\mathcal{L}_H(f, 0) = (g, 0) \quad (23)
\]
in \([t_0, +\infty) \times S^{n-1}\) which belongs to the space \(C^{2,\alpha}_{\delta'}([t_0, +\infty) \times S^{n-1}; \Omega^0 \times \Omega^1)\). Furthermore,
\[ \|(f, 0)\|_{2,\alpha,\delta'} \leq c \|(g, 0)\|_{0,\alpha,\delta'}. \]
**Proof:** The existence is easy. We already know an explicite solution of the homogeneous problem $L_H(f_0, 0) = 0$. This solution is given by

$$f_0(t) := (\cosh(nt))^{-\frac{1}{2}}.$$  

We define the solution of (23) by

$$f(t) := f_0(t) \int_{t}^{+\infty} (f_0(\zeta))^{-2} \int_{\zeta}^{+\infty} f_0(\zeta) g(\zeta) \, d\zeta \, d\zeta.$$  

It is a simple exercise to show that this is a solution which is well defined and that the estimate is satisfied, since we have chosen $\delta' < -\frac{c}{2}$. 

We will also need the

**Proposition 9** There exists $c > 0$ such that, for all $t_0 \in \mathbb{R}$ and all $W \in C^2,\alpha([t_0, +\infty) \times S^{n-1}; \Omega^0 \times \Omega^1)$, there exists a unique $U_0 \in C^2,\alpha([t_0, +\infty) \times S^{n-1}; \Omega^0 \times \Omega^1)$ solution of

$$\begin{cases} L_0 U_0 &= 0 \quad \text{in } (t_0, +\infty) \times S^{n-1}, \\ U_0 &= W \quad \text{on } \{t_0\} \times S^{n-1}. \end{cases} \tag{24}$$

Furthermore, we have

$$||U_0||_{2,\alpha, S^{n-1}} \leq c e^{-\frac{c}{2} t_0} ||W||_{2,\alpha},$$

**Proof:** Uniqueness of the solution follows Proposition 8. We therefore concentrate our attention on the existence of the solution and the derivation of the uniform estimate for the inverse.

Our problem being linear, we can assume that

$$\sup_{S^{n-1}} |e^{-\delta t} W| \leq 1.$$  

**Step 1** Assume that the boundary data $W$ is orthogonal, in the $L^2$ sense, to $V_{ex}^1$; the eigenspace of $(\Delta^0, \Delta^1)$ corresponding to the eigenvalue $1 - n$. We apply Proposition 8 which implies that the operator $L_0$ is injective over $(t', t'') \times S^{n-1}$. As a consequence, for all $t'' > t' + 1$ we are able to solve $L_0 U = 0$, in $(t', t'') \times S^{n-1}$, with $U = W$ on $\{t'\} \times S^{n-1}$ and $U = 0$ on $\{t''\} \times S^{n-1}$.

We claim that, for any $\delta \in (-\frac{n}{2}, 0)$, there exists some constant $c > 0$ independent of $t'' > t' + 1$ and $t_0$ and of $W$ such that

$$\sup_{(t', t'') \times S^{n-1}} |e^{-\delta t} U| \leq c e^{-\delta t'}.$$  

We argue by contradiction and assume that the result is not true. In this case, there would exist sequences $t''_i > t'_i + 1$, a sequence of functions $W_i$ satisfying

$$\sup_{S^{n-1}} |e^{-\delta t} W_i| = 1,$$

and a sequence $U_i$ of solutions of $L_H U_i = 0$, in $(t'_i, t''_i) \times S^{n-1}$, with $U_i = W_i$ on $\{t'_i\} \times S^{n-1}$ and $U_i = 0$ on $\{t''_i\} \times S^{n-1}$ such that

$$A_i := \sup_{(t'_i, t''_i) \times S^{n-1}} |e^{\delta (t' - t)} U_i| \rightarrow +\infty.$$  

Furthermore, $U_i(t, \cdot)$ are orthogonal in the $L^2$ sense to $V_{ex}^1$ on $S^{n-1}$. Let us denote by $(t_i, \theta_i) \in (t'_i, t''_i) \times S^{n-1}$, a point where the above supremum is achieved. We now distinguish a few cases...
according to the behavior of the sequence \( t_i \) (which, up to a subsequence can always be assumed to converge in \([-\infty, +\infty]\)). Up to some subsequence, we may also assume that the sequences \( t''_i - t_i \) (resp. \( t_i - t'_i \)) converges to \( t^* \in (0, +\infty] \) (resp. to \( t_* \in [-\infty, 0) \)).

As in the proof of Proposition [3], observe that the sequences \( t_i - t'_i \) and \( t''_i - t_i \) remain bounded away from 0.

We now define the sequence of rescaled functions

\[
\hat{U}_i(t, \theta) := \frac{e^{\delta(t'_i - t_i)}}{A_i} U_i(t + t_i, \theta).
\]

Up to a subsequence, we can assume that the sequence \( t''_i - t_i \) converges to \( t^* \in (0, +\infty] \) and that \( t_i - t'_i \) converges to \( t_* \in [-\infty, 0) \). After the extraction of some subsequences, if this is necessary, we may assume that the sequence \( \hat{U}_i \) converges to some nontrivial solution of

\[
L_0 U_\infty = 0,
\]

in \((t_*, t^*) \times S^{n-1}\), with boundary condition \( U_\infty = 0 \), if either \( t_* \) or \( t^* \) is finite. Furthermore

\[
\sup_{(t_*, t^*) \times S^{n-1}} |e^{-\delta t} U_\infty| = 1.
\]

and \( U_\infty(t, \cdot) \) orthogonal in the \( L^2 \) sense to \( V_1 \) on \( S^{n-1} \). If \( t_* = -\infty \), the inspection of the indicial roots shows that, any solution of the homogeneous equation which is bounded by \( e^{\delta t} \) for some \( \delta \in (-\frac{n}{2}, \frac{2-n}{2}) \) is bounded by a constant times \( e^{\frac{\delta}{2} t} \) at \( -\infty \). Applying Proposition [3], this implies that \( U_\infty = 0 \), contradicting [20].

Now that the proof of the claim is finished, we may pass to the limit \( t'' \to +\infty \) and obtain a solution of \( L_0 U = 0 \), in \((t', +\infty) \times S^{n-1}\), with \( U = W \) on \( \{t'\} \times S^{n-1} \), which satisfies

\[
\sup_{(t', +\infty) \times S^{n-1}} |e^{-\delta t} U| \leq c,
\]

for some constant \( c > 0 \) independent of \( t' \). Finally, Schauder's estimates yield the relevant estimates for all the derivatives.

**Step 2** Assume that the boundary data \( W \) belongs to \( V_1 \) the eigenspace of \((\Delta^0, \Delta^1)\) corresponding to the eigenvalue \( 1 - n \). In this case, the partial differential equation \( L_0 U = 0 \) reduces to a finite number of coupled ordinary differential equations of the form

\[
\begin{align*}
\tilde{a} - (n - 1) \tilde{a} - \frac{n^2}{4} \tilde{a} + 2(n - 1) \tilde{b} &= 0 \\
\tilde{b} - (n - 1) \tilde{b} - \frac{(n-1)^2}{4} \tilde{b} + 2 \tilde{a} &= 0
\end{align*}
\]

with boundary data \( a(t') = a_0 \) and \( b(t') = b_0 \). Then, provided \( \Delta^0 \phi = -(n-1) \phi, U = (a \phi, b d^0 \phi) \) will be a solution of \( L_0 U = 0 \) in \((t', +\infty) \times S^{n-1}\) and \( U(t', \cdot) = (a_0 \phi, b_0 d^0 \phi) \) on \( \{t'\} \times S^{n-1} \).

But the above system can be solved explicitly

\[
\begin{align*}
a &= (n - 1) A e^{-\frac{n^2}{4} t(t')} + B e^{\frac{2-n}{2} t(t')}
\end{align*}
\]

and \( (n-2)A = a_0 + b_0, (n-2)B = (n-1)b_0 - a_0 \). It is easy to check that \( \|U\|_{2, \alpha, \frac{2-n}{2}} \leq c \). This completes the proof of the result. □
13 Structure of the mean curvature operator about the Hyperbola

In order to understand the structure of the mean curvature operator for any surface close enough to the hyperbola, we go back to the variational definition of the minimal surface. Let $V$ be a normal perturbation of the hyperbola parameterized by $X$. Recall that, with our notations,

$$X = \frac{e^{is}}{(\sin(ns))^{1/n}}\Theta$$

and that the normal vector field can be taken to be

$$V = ie^{i(1-n)s} f \Theta + i e^{is} T$$

We will denote by

$$Y(t, \theta) := X(t, \theta) + V(t, \theta)$$

First, let us compute the vectors which span the tangent space of the surface parameterized by $Y$.

$$\partial_t Y = -(\cosh(nt))^{\frac{1}{n}} e^{i(1-n)s} \Theta + e^{i(1-n)s} \left( \frac{(n-1)}{\cosh(nt)} f + i \partial_t f \right) \Theta - e^{is} \left( \frac{1}{\cosh(nt)} T - i \partial_t T \right)$$

and, for all $j = 1, \ldots, n - 1$

$$\partial_j Y = (\cosh(nt))^{\frac{1}{n}} e^{i\theta} \Theta_j + i e^{i(1-n)s} \partial_j f \Theta - i e^{is} \left( \tanh(ns) + \frac{i}{\cosh(ns)} \right) \Theta_j + i e^{is} \partial_j T$$

The coefficients of the first fundamental form associated to $Y$ are given by

$$|\partial_t Y|^2 = (\cosh(nt))^\frac{2}{n} - 2 (n - 1) (\cosh(nt))^\frac{1}{n} f + |\partial_t f|^2 + |\partial_t T|^2$$

and, for all $j = 1, \ldots, n - 1$

$$|\partial_j Y|^2 = \left( ((\cosh(nt))^{\frac{2}{n}} + (\epsilon_j f)^2 + |f|^2 + \nabla_{\epsilon_j} T|^2 + 2 (\cosh(nt))^\frac{1}{n} f \right.$$

$$\left. - 2 \tanh(nt) (\epsilon_j f) T \cdot \epsilon_j - 2 \tanh(nt) f (\nabla_{\epsilon_j} T \cdot \epsilon_j) \right) |\Theta_j|^2$$

Finally, if $j \neq k$ we have

$$|\partial_j Y \cdot \partial_k Y|^2 = \left( ((\cosh(nt))^{\frac{2}{n}} + (\epsilon_j f)^2 + |f|^2 + \nabla_{\epsilon_j} T|^2 + 2 (\cosh(nt))^\frac{1}{n} f \right.$$

$$\left. - 2 \tanh(nt) (\epsilon_j f) T \cdot \epsilon_j - 2 \tanh(nt) f (\nabla_{\epsilon_j} T \cdot \epsilon_j) \right) |\Theta_j| |\Theta_k|$$

The exact value of the coefficients of the first fundamental form is not needed. We just observe that the first fundamental form of the surface parameterized by $Y$ is given by

$$I_Y = I_X + (\cosh(nt))^\frac{1}{n} L(f, T) + Q(f, T)$$
where $I_X$ is the first fundamental form of the hyperbola in the variables $(t, \theta)$, namely

$$I_X = (\cosh(nt))^\frac{2}{n} (dt^2 + (\cosh(nt))^\frac{2}{n} d\theta^2)$$

And where $L$ is linear and $Q$ is quadratic in the variables $f, T, \partial_t f, \partial_t T$ and $\varepsilon f_j, \nabla_\tau \varepsilon$. Both $L$ and $Q$ have coefficients depending on $t$ but they are all bounded and have bounded derivatives with respect to $t$.

Once we have obtained the structure of the first fundamental form, it is a simple exercise to obtain the structure of the volume functional and then the structure of the Euler-Lagrange equation. Hence, we conclude that the surface parameterized by $Y$ is minimal if and only if $V$ is a solution of the following partial differential equation

$$L_1 V + (\cosh t)^{-n} \tilde{Q}_2((\cosh t)^{-1} V) + (\cosh t)^{-1} \tilde{Q}_3((\cosh t)^{-1} V) = 0$$

where $L_1$ is the linearized mean curvature operator, $\tilde{Q}_2$ is homogeneous of degree 2 and where $\tilde{Q}_3$ collects all the higher order terms. Observe that the Taylor’s coefficients of $\tilde{Q}_i$ are bounded functions of $t$ and so are the derivatives of any order of these functions.

When we conjugate this operator, as we have done with $L_H$, we obtain the

**Proposition 10** The surface parameterized by

$$Y(t, \theta) = X(t, \theta) + (\cosh(nt))^\frac{2}{n} V(t, \theta)$$

is minimal if and only if

$$\mathcal{L}_H V + (\cosh t)^{-\frac{2}{n}} Q_2((\cosh t)^{-\frac{2}{n}} V) + (\cosh t)^{-\frac{2}{n}} Q_3((\cosh t)^{-\frac{2}{n}} V) = 0$$

(27)

where $Q_2$ is homogeneous of degree 2 and where $Q_3$ collects all the higher order terms. Observe that the Taylor’s coefficients of $Q_i$ are bounded functions of $t$ and so are the derivatives of any order of these functions.

14 Minimal $n$-submanifolds close to the truncated hyperbola $H_I$

For all $\varepsilon \in (0, 1)$, $\rho_* > 0$, $\kappa > 1$ and $\beta \in [\frac{1}{\kappa}, \kappa]$, we define $s_* \in (0, \frac{\pi}{n})$ and $t_* \in \mathbb{R}$ by the identities

$$\rho_* = (n \beta \varepsilon)^\frac{1}{\kappa} \frac{\cos s_*}{(\sin(ns_*))^{1/\kappa}},$$

and

$$e^{-nt_*} = \frac{\sin(n s_*)}{1 - \cos(n s_*)}.$$ 

Hence, $t_* < 0$ for $\varepsilon$ small enough.

Consider any normal perturbation of the rescaled hyperbola

$$Y(t, \theta) = (n \beta \varepsilon)^\frac{1}{\kappa} \left( X(t, \theta) + (\cosh(nt))^\frac{2}{n} V(t, \theta) \right).$$

As usual, we identify the normal vector field $V$ with $U = (f, v)$. We have seen that $Y$ describes a minimal $n$-submanifold if and only if $U$ is a solution of

$$\mathcal{L}_H U = (\cosh t)^{-\frac{2}{n}} Q_2((\cosh t)^{-\frac{2}{n}} U) + (\cosh t)^{-\frac{2}{n}} Q_3((\cosh t)^{-\frac{2}{n}} U).$$
We modify the normal bundle when \((t, \theta) \in [t_*, t_* + 2] \times S^{n-1}\) so that it is now given by the "vertical plane" \(\{x : x \in \mathbb{R}^n\}\) for all \((t, \theta) \in [t_*, t_* + 1] \times S^{n-1}\). More precisely, instead of considering the normal bundle spanned by

\[ N_0 = i e^{i(1-n)s} \Theta \quad \text{and} \quad \forall j = 1, \ldots, n \quad N_j = i e^{i \frac{\Theta_j}{|\Theta_j|}}, \]

we want to choose the bundle spanned by the vectors

\[ \tilde{N}_0 := i \Theta \quad \text{and} \quad \forall j = 1, \ldots, n \quad \tilde{N}_j := i \frac{\Theta_j}{|\Theta_j|} \]

in \((t_*, t_* + \frac{1}{2}) \times S^{n-1}\). As explained in 8, this modifies slightly the equation we have to solve into

\[ \mathcal{L}_H U = (\cosh t)^{-2n} L U + (\cosh t)^{-\frac{n}{2}} \tilde{Q}_2((\cosh(t)^{-\frac{n}{2}} U) + (\cosh t)^{\frac{n}{2}} \tilde{Q}_3((\cosh(t)^{-\frac{n}{2}} U), \quad (28) \]

where the linear operator \(L\) has coefficients which are bounded and supported in \([t_*, t_* + 2] \times S^{n-1}\) and where \(\tilde{Q}_2\) and \(\tilde{Q}_3\) enjoy the properties of \(Q_2\) and \(Q_3\).

**Definition 2** We define \(P_0\) to be the \(L^2\) projection over the space orthogonal to \(\nu^0\) in the \(L^2\) sense on \(S^{n-1}\).

We want to find \(U\) solution of (28) in \((t_*, +\infty) \times S^{n-1}\), boundary data \(P_0 U = W\) on \(\{t_*\} \times S^{n-1}\) where

\[ ||W||_{L^2} \leq \kappa (\varepsilon \rho^*)^{\frac{1}{2}}, \]

for some fixed constant \(\kappa > 0\). To begin with, let us solve

\[
\left\{\begin{array}{ll}
\mathcal{L}_0 U_0 &= 0 & \text{in} & (t_*, +\infty) \times S^{n-1} \\
U_0 &= W & \text{on} & \{t_*\} \times S^{n-1}
\end{array}\right. \quad (29)
\]

We already know from Proposition 9 that there exists some constant \(c > 0\) such that

\[ ||U_0||_{L^2} \leq c e^{\frac{\rho^*}{\kappa^2} t_*} ||W||_{L^2}. \]

It remains to solve

\[
\mathcal{L}_H U = (\cosh t)^{-2n} L(U + U_0) - \mathcal{L}_0 U_0 + (\cosh t)^{-\frac{n}{2}} \tilde{Q}_2((\cosh(t)^{-\frac{n}{2}} U + U_0)) \\
+ (\cosh t)^{\frac{n}{2}} \tilde{Q}_3((\cosh(t)^{-\frac{n}{2}} U + U_0)
\]

in \((t_*, +\infty) \times S^{n-1}\) with \(P_0 U = 0\) on \(\{t_*\} \times S^{n-1}\). We may then rewrite the previous equation as

\[ U = G_{t_*} N(U), \]

where by definition the nonlinear operator \(N\) is given by

\[ N(U) := ((\cosh t)^{-2n} L(U + U_0) - \mathcal{L}_0 U_0 + (\cosh t)^{-\frac{n}{2}} Q_2((\cosh(t)^{-\frac{n}{2}} U + U_0)) \\
+ (\cosh t)^{\frac{n}{2}} Q_3((\cosh(t)^{-\frac{n}{2}} U + U_0))
\]

and where \(G_{t_*}\) is the right inverse constructed in Proposition 9. In order to solve this equation, we use a fixed point theorem for contraction mapping. This is the content of the following.
Proposition 11 Let $\delta \in (\frac{2-\kappa}{2}, \frac{3}{2})$ be fixed. There exists $c_0 > 0$ and for all $\kappa > 0$, there exists $\varepsilon_0 > 0$ such that, for all $W$ which are orthogonal to $V^0$ in the $L^2$ sense on $S^{n-1}$ and which satisfy
\[ \|W\|_{2,\alpha} \leq \kappa (\varepsilon \rho^\alpha)^{\frac{1}{2}}, \]
we have
\[ \|G_t, N(0)\|_{2,\alpha,\delta} \leq \frac{c_0}{2} (\cosh t_s)^{\frac{2-n}{2}} \|W\|_{2,\alpha} \]
and
\[ \|G_t, (N(U_2) - N(U_1))\|_{2,\alpha,\delta} \leq \frac{1}{2} \|U_2 - U_1\|_{2,\alpha,\delta} \]
for all $U_1, U_2 \in C^3_\delta([t_*, +\infty) \times S^{n-1})$ such that $\|U_t\|_{2,\alpha,\delta} \leq c_0 (\cosh t_s)^{\frac{2-n}{2}} \|W\|_{2,\alpha}$.

In particular $G_t, N$ has a unique fixed point in the ball of radius $c_0 (\cosh t_s)^{\frac{2-n}{2}} \|W\|_{2,\alpha}$ in $C^3_\delta([t_*, +\infty) \times S^{n-1}; \Omega^0 \times \Omega^1)$.

Proof: To begin with let us estimate
\[ \|((\cosh t_s)^{-2n} L U_0)\|_{2,\alpha,\delta} \leq (\cosh t_s)^{\delta-2n} \|W\|_{2,\alpha} \]
Now
\[ \mathcal{L}_H U_0 = (\mathcal{L}_H - \mathcal{L}_0) U_0 \]
hence
\[ \|\mathcal{L}_H U_0\|_{2,\alpha} \leq c \left( (\cosh t_s)^{\delta-n} + (\cosh t_s)^{\frac{2-n}{2}} \right) \|W\|_{2,\alpha}. \]
Now, using the properties of $\tilde{Q}_2$ and $\tilde{Q}_3$, we estimate for $\varepsilon$ small enough
\[ \|((\cosh t_s)^{-\frac{3}{2}} \tilde{Q}_2((\cosh t_s)^{-\frac{3}{2}} U_0))\|_{2,\alpha,\delta} \leq c \left( (\cosh t_s)^{\delta-\frac{3}{2}n} + (\cosh t_s)^{2-n} \right) \|W\|_{2,\alpha} \]
and
\[ \|((\cosh t_s)^{\frac{3}{2}} \tilde{Q}_3((\cosh t_s)^{-\frac{3}{2}} U_0))\|_{2,\alpha,\delta} \leq c \left( (\cosh t_s)^{\delta-n} + (\cosh t_s)^{\frac{3(2-n)}{2}} \right) \|W\|_{2,\alpha} \]
We can now use the result of Proposition 7 to conclude that
\[ \|G_t, P_0 N(0)\|_{2,\alpha,\delta} \leq c (\cosh t_s)^{\frac{2-n}{2}} \|W\|_{2,\alpha} \]
where we recall that $P_0$ to be the $L^2$ projection over the space orthogonal to $V^0$ in the $L^2$ sense on $S^{n-1}$.

Using the explicit representation of $G_t$, when acting on functions only depending on $t$, we obtain
\[ \|G_t, (I - P_0) ((\cosh t_s)^{-2n} L U_0)\|_{2,\alpha,\delta} \leq c (\cosh t_s)^{\delta-\frac{3}{2}} \|W\|_{2,\alpha} \]
Now
\[ (I - P_0) \mathcal{L}_H U_0 = 0 \]
Finally, we estimate for all $\varepsilon$ small enough
\[ \|G_t, (I - P_0) ((\cosh t_s)^{-\frac{3}{2}} \tilde{Q}_2((\cosh t_s)^{-\frac{3}{2}} U_0))\|_{2,\alpha,\delta} \leq c \left( (\cosh t_s)^{\delta+\frac{3}{2}n} + (\cosh t_s)^{2-n} \right) \|W\|_{2,\alpha} \]
and
\[ \|G_t, (I - P_0) ((\cosh t_s)^{\frac{3}{2}} \tilde{Q}_3((\cosh t_s)^{-\frac{3}{2}} U_0))\|_{2,\alpha,\delta} \leq c \left( (\cosh t_s)^{\delta+3-n} + (\cosh t_s)^{\frac{3(2-n)}{2}} \right) \|W\|_{2,\alpha} \]
Collecting these estimates together with (30), we obtain
\[ \|N(0)\|_{2,\alpha,\delta} \leq c (\cosh t_s)^{\frac{2-n}{2}} \|W\|_{2,\alpha} \]
provided $\varepsilon$ is chosen small enough. We fix $c_0 > 0$ to be equal to twice the constant which appears on the right hand side of this estimate. The second estimate requires that $\delta \in (\frac{2-\kappa}{2}, \frac{3}{2})$. \(\Box\)

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15 Family of half hyperbola parameterized by their boundary

We summarize what we have obtained so far. Given $\kappa > 1$, and $\rho_* > 0$, there exists $\varepsilon_0 > 0$ such that for all $\beta \in [\frac{\varepsilon_0}{n}, \kappa]$, for all $\varepsilon \in (0, \varepsilon_0)$ and all $\Phi \in C^{2,\alpha}(S^{n-1}; \mathbb{R}^n)$ which is orthogonal to $\Theta$ in the $L^2$ sense and which satisfies

$$\|\Phi\|_{2,\alpha} \leq \kappa \rho_*^2 \varepsilon,$$

we have obtained a minimal $n$-submanifold which is $C^{2,\alpha}$ close to the truncated hyperbola, more precisely this submanifold is parameterized by

$$Y : (t, \theta) \in [t_*, +\infty) \times S^{n-1} \rightarrow (n \beta \varepsilon)^{\frac{1}{n}} \left( X(t, \theta) + (\cosh(n \rho_* t))^\frac{2-n}{n} V(t, \theta) \right)$$

where $V = U + U_0 \in C^{2,\alpha}_0([t_*, +\infty) \times S^{n-1})$. Here $U$ is the solution of (29) and $U_0$ is the solution of (29) obtained in the previous Proposition with $W$ given by

$$W := (n \beta \varepsilon)^{\frac{1}{n}} (\cosh(nt_*)^\frac{2-n}{n} \Phi).$$

Now, we would like to describe the boundary of this $n$-submanifold. To this aim, let us define

$$r := (n \beta \varepsilon)^{\frac{1}{n}} \frac{\cos s}{(\sin(ns))^{\frac{1}{n}}}.$$

Thanks to (3), we see that the above minimal $n$-submanifold can be parameterized, in some neighborhood of its boundary as

$$(r, \theta) \rightarrow r \Theta + i \left( \varepsilon \beta r^{1-n} \Theta + F_0 + F \right) (1 + O(\varepsilon^2 r^{-2n}))$$

Where $F_0$ is the harmonic extension of the boundary data $\Phi$ in $B_{\rho_*}$, and where

$$\|F(\rho_*, \cdot)\|_{C^{2,\alpha}(S^{n-1})} + \|\rho_* \partial_r F(\rho_*, \cdot)\|_{C^{1,\alpha}(S^{n-1})} \leq c (\cosh t_*)^{\frac{2-n}{n}} \|\Phi\|_{2,\alpha}.$$

In other words we can also write this $n$-submanifold near its boundary as the graph of

$$(r, \theta) \rightarrow \varepsilon \beta r^{1-n} \Theta + F_0 + \tilde{F},$$

where $\tilde{F}$ depends smoothly on $\beta$ and $\Phi$ and satisfies

$$\|\tilde{F}(\rho_*, \cdot)\|_{C^{2,\alpha}(S^{n-1})} + \|\rho_* \partial_r \tilde{F}(\rho_*, \cdot)\|_{C^{1,\alpha}(S^{n-1})} \leq c_\kappa \left( \varepsilon^3 \rho_*^{1-3n} + \varepsilon \rho_* (\varepsilon^{-\frac{1}{n}} \rho_*)^{\frac{2-n}{n}} \right),$$

where the constant $c_\kappa > 0$ only depends on $\kappa$.

For further use it will be convenient to define

**Definition 3** For all $\Phi \in C^{2,\alpha}(S^{n-1}; \mathbb{R}^n)$, we define $\mathcal{P}_{\mathrm{int}}(\Phi)$ to be equal to $\partial_r F$ where $F$ is the harmonic extension of $\Phi$ in the unit ball of $\mathbb{R}^n$.

Finally, given any $\mathcal{R} \in \mathcal{O}(n)$ we consider the image of the above defined family of minimal $n$-submanifolds by

$$x + iy \in \mathbb{C}^n \rightarrow x + i \mathcal{R} y \in \mathbb{C}^n.$$

We denote by $H_{\varepsilon, \mathcal{R}, \rho_*}(\beta, \Phi)$ this submanifold.
16 Mean curvature operator for normal graphs over $\mathbb{R}^n$

Let a normal graph over the $x$ space in $\mathbb{C}^n$, namely $\Omega \subset \mathbb{R}^n \ni x \rightarrow x + i F(x) \in \mathbb{C}^n$. The first fundamental form is given by

$$\mathcal{I} = \sum_{j} dx_j dx_j + 2 \sum_{j < j'} \partial_{x_j} F \cdot \partial_{x_{j'}} F dx_j dx_{j'}$$

It is an easy exercise to derive the equation which ensures that the graph of $F$ will be minimal. The exact expression of this equation will not be needed but we only need its structure

$$\Delta F + \text{div} (Q_3(\nabla F)) = 0$$

where $Q_3$ is analytical and has coefficients which do not depend on $x$. Furthermore, $Q_3, \nabla Q_3$ and $\nabla^2 Q_3$ all vanish at 0.

17 Mapping properties of the Laplace operator in $\mathbb{R}^n$

Choose $\rho_0 > 0$ such that, for all $j \neq j'$, we have $B(x_j, \rho_0) \cap B(x_{j'}, \rho_0) = \emptyset$ and also $B(x_j, \rho_0) \subset B(0, \rho_0^{-1})$.

The weighted space we will be working with is defined by

**Definition 4** Given $\nu, \mu \in \mathbb{R}$ and $\alpha \in (0, 1)$. The space $C^{k, \alpha}_{\nu, \mu}(\mathbb{R}^n \setminus \{x_1, \ldots, x_N\}; \mathbb{R}^n)$ is defined to be the space of functions $F \in C^{k, \alpha}_{\text{cal}}(\mathbb{R}^n \setminus \{x_1, \ldots, x_N\}; \mathbb{R}^n)$ for which, the following norm is finite

$$\|F\|_{k, \alpha, \nu, \mu} := \sum_{r=0}^{\infty} \sup_{r \in (0, \rho_0)} r^{-\nu} |F|_{k, \alpha, B_r \setminus B_{r/2}(x_j)} + \sup_{r \geq 1/\rho_0} r^{-\mu} |F|_{k, \alpha, B_{r/2} \setminus B_r(x_j)}$$

where

$$[F]_{k, \alpha, B_r \setminus B_{r/2}} := \sum_{j=0}^{k} \sup_{B_l \setminus B_{l/2}} |\nabla^j F| + \sup_{x, y \in B_r \setminus B_{r/2}} \frac{|\nabla^k F(x) - \nabla^k F(y)|}{|x - y|^\alpha}.$$

and where $|F|_{k, \alpha, \Omega}$ is the usual Hölder norm in $\Omega$.

**Proposition 12** Assume that $2 - n < \nu < 0$ and also that $1 - n < \mu < 2 - n$. Then

$$\Delta : C^{2, \alpha}_{\nu, \mu}(\mathbb{R}^n \setminus \{x_1, \ldots, x_N\}; \mathbb{R}^n) \oplus (1 + |x|^2)^{\frac{\mu - 2}{2}} \times \mathbb{R}^n \rightarrow C^{0, \alpha}_{\nu - 2, \mu - 2}(\mathbb{R}^n \setminus \{x_1, \ldots, x_N\}; \mathbb{R}^n),$$

is an isomorphism.

**Proof**: For all $H \in C^{0, \alpha}_{\nu - 2, \mu - 2}(\mathbb{R}^n \setminus \{x_1, \ldots, x_N\}; \mathbb{R}^n)$, $H \in L^1(\mathbb{R}^n)$. Hence there exists $F$ weak solution of $\Delta F = H$ in $\mathbb{R}^n$. Now, the function $x \rightarrow |x - x_j|^\nu$ can be used as a barrier function to prove that $F$ is bounded by a constant times $\|H\|_{0, \alpha, \nu - 2, \mu - 2}$ times $|x - x_j|^\nu$ in each $B(x_j, \rho_0)$.

Moreover, it is classical to prove that $F$ is bounded by a constant times $\|H\|_{0, \alpha, \nu - 2, \mu - 2}$ times $|x - x_j|^{2-n}$ outside $\cup_j B(x_j, \rho_0)$. The estimates for the derivatives follow from rescaled Schauder’s estimates.

As above, we assume that $\rho_0 > 0$ is chosen such that, for all $j \neq j'$, we have $B(x_j, \rho_0) \cap B(x_{j'}, \rho_0) = \emptyset$. Using the maximum principle and the previous result, it is a simple exercise to show that
**Proposition 13** Assume that $2 - n < \nu < 0$ and also that $1 - n < \mu < 2 - n$. Then, for all $\rho \in (0, \rho_0)$

$$\Delta : \mathcal{C}^{2,\alpha}_{\nu,\mu,\mathcal{D}}(\mathbb{R}^n \setminus \bigcup_{j=1}^{N} B(x_j, \rho); \mathbb{R}^n) \oplus (1 + \|x\|^2)^{\frac{2-n}{2}} \times \mathbb{R}^n \rightarrow \mathcal{C}^{0,\alpha}_{\nu-2,\mu-2}(\mathbb{R}^n \setminus \bigcup_{j=1}^{N} B(x_j, \rho); \mathbb{R}^n),$$

is an isomorphism. Moreover, there exists $c > 0$ such that, for all $\rho \in (0, \rho_0)$ the norm of its inverse $\mathcal{G}_\rho$ is bounded by $c$.

Here the subscript $\mathcal{D}$ refers to the fact that all functions have 0 boundary data on each $\partial B(x_j, \rho)$.

### 18 Expansion of Green’s function

Assume that $x_1, \ldots, x_N \in \mathbb{R}^n$, $\alpha_1, \ldots, \alpha_N \in \mathbb{R}$, $A_0 \in M_n(\mathbb{R})$ and $\mathcal{R}_1, \ldots, \mathcal{R}_N \in O(n)$ are given, we set

$$G(x) = \sum_{j=1}^{N} \alpha_j \mathcal{R}_j \left( \frac{x - x_j}{|x - x_j|^n} \right) + A_0 x.$$

It is an easy exercise to perform an expansion of $G$ near one the point $x_{j_0}$. We get

$$G(x) = \alpha_{j_0} \mathcal{R}_{j_0} \left( \frac{x - x_{j_0}}{|x - x_{j_0}|^n} \right) - \left( \sum_{j \neq j_0} \alpha_j \mathcal{R}_j \left( \frac{x_j}{|x_j|^n} \right) - A_0 x_{j_0} \right) + A_0 (x - x_{j_0})$$

$$+ \sum_{j \neq j_0} \frac{\alpha_j}{|x_j - x_{j_0}|^n} \mathcal{R}_j \left( x - x_{j_0} - n \left( (x - x_{j_0}) \cdot (x_j - x_{j_0}) \right) \frac{x_j - x_{j_0}}{|x_j - x_{j_0}|^2} \right)$$

$$+ \mathcal{O}\left( |x - x_{j_0}|^2 \right).$$

We set

$$\rho \Theta := x - x_{j_0} \quad \text{and, for all } j \neq j' \quad \xi_{jj'} := \frac{x_j - x_{j'}}{|x_j - x_{j'}|},$$

and we define $\gamma_{jj} = 0$ for all $j$ and

$$\gamma_{jj'} := \frac{1}{|x_j - x_{j'}|^n} \left( \int_{S_{n-1}} \mathcal{R}_j \Theta \cdot \mathcal{R}_{j'} \Theta d\theta - n \int_{S_{n-1}} (\Theta \cdot \mathcal{R}_j \xi_{jj'}) (\Theta \cdot \mathcal{R}_{j'} \xi_{jj'}) d\theta \right), \quad (31)$$

for all $j \neq j'$, and

$$\lambda_j = - \int_{S_{n-1}} A_0 \Theta \cdot \mathcal{R}_j \Theta d\theta$$

so that we can write

$$G(x) = \left( \alpha_{j_0} \rho^{1-n} + \frac{1}{\omega_n} \left( \sum_{j \neq j_0} \alpha_j a_{jj_0} - \lambda_{j_0} \right) \rho \right) \mathcal{R}_{j_0} \Theta$$

$$- \left( \sum_{j \neq j_0} \alpha_j \mathcal{R}_j \left( \frac{x_j}{|x_j|^n} \right) - A_0 x_{j_0} \right) + \mathcal{O}(\rho^2) + \mathcal{O}_\perp(\rho), \quad (32)$$

where $\mathcal{O}_\perp(\rho)$ is orthogonal to $\mathcal{R}_{j_0} \Theta$ in the $L^2$ sense and where $\omega_n := |S_{n-1}|$.

### 19 Minimal graphs

Assume that $x_1, \ldots, x_N \in \mathbb{R}^n$, $\alpha_1, \ldots, \alpha_N \in \mathbb{R}$, $A_0 \in M_n(\mathbb{R})$ and $\mathcal{R}_1, \ldots, \mathcal{R}_N \in O(n)$ are given, we have defined

$$G(x) := \sum_{j=1}^{N} \alpha_j \mathcal{R}_j \left( \frac{x - x_j}{|x - x_j|^n} \right) + A_0 x$$

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We would like to solve
\[
\begin{cases}
\Delta F + \text{div} (Q_3(\nabla F)) = 0 & \text{in } \mathbb{R}^n \setminus \bigcup_j B(x_j, \rho_s) \\
F = \varepsilon G + \mathcal{R}_j \Phi_j & \text{on } \partial B(x_j, \rho_s)
\end{cases}
\]
where \( \Phi_j \) is small.

To begin with, let us define \( F_j \) to be the solution of
\[
\begin{cases}
\Delta F_j = 0 & \text{in } \mathbb{R}^n \setminus \bigcup_j B(x_j, \rho_s) \\
F_j = \mathcal{R}_j \Phi_j & \text{on } \partial B(x_j, \rho_s)
\end{cases}
\]
Using barrier arguments, we obtain the

**Lemma 2** There exists a constant \( c > 0 \) such that
\[
\| F_j \|_{2, \alpha, 2-n, 2-n} \leq c \rho_s^{2-n} \| \Phi_j \|_{2, \alpha}.
\]

**Proof:** Observe that
\[
x \mapsto \left( \frac{|x - x_j|}{\rho^*_s} \right)^{2-n} \| \Phi_j \|_{2, \alpha}
\]
is a supersolution for our problem, hence we already have
\[
|F_j| \leq \left( \frac{|x - x_j|}{\rho^*_s} \right)^{2-n} \| \Phi_j \|_{2, \alpha}.
\]
The estimates for the other derivatives follow from Schauder’s estimates. \( \square \)

As above, we choose \( \rho_0 > 0 \) such that, for all \( j \neq j' \), we have \( B(x_j, \rho_0) \cap B(x_{j'}, \rho_0) = \emptyset \) and also \( B(x_j, \rho_0) \subset B(0, \rho_0^{-1}) \). Using cutoff functions, we can restrict the definition of \( F_j \) to \( B(x_j, \rho_0) \) and sum all these functions to obtain a mapping
\[
\tilde{F}_0 := \sum_j \chi_{\rho_0}(x - x_j) F_j(x)
\]
where \( \chi_{\rho_0} \) is equal to 1 in \( B(0, \rho_0/2) \) and equal to 0 outside \( B(0, \rho_0) \).

It is an easy task to evaluate the mean curvature of the graph of \( F = \varepsilon G + \tilde{F}_0 \). This is the contain of the following

**Lemma 3** Assume that \( \nu \in (2-n, 0) \) and that \( \mu \in (1-n, 2-n) \). There exists a constant \( c > 0 \) such that
\[
\| \Delta F + \text{div} (Q_3(\nabla F)) \|_{0, \alpha, \nu-2, \mu-2} \leq c \left( \varepsilon^3 \rho_s^{1-3n-\nu} + \sup_j \| \Phi_j \|_{2, \alpha} \rho_s^{n-2} \right)
\]

**Proof:** In order to evaluate the mean curvature of the graph of \( F \), it suffices to evaluate \( \Delta F + \text{div} (Q_3(\nabla F)) \). We use the fact that \( \Delta F = 0 \) away from each \( B(x_j, \rho_0) \) and also in \( B(x_j, \frac{\rho_0}{2}) \setminus B(x_j, \rho_s) \) while \( \Delta F \) and all its derivatives are bounded by a constant times \( \rho_s^{n-2} \| \Phi_j \|_{2, \alpha} \) in each \( B(x_j, \rho_0) \setminus B(x_j, \frac{\rho_0}{2}) \). Hence
\[
\| \Delta F \|_{2, \alpha, \nu-2, \mu-2} \leq c \rho_s^{n-2} \sup_j \| \Phi_j \|_{2, \alpha}
\]
In order to estimate the norm of $\text{div}(Q_3(\nabla F))$ we first observe that the mean curvature of the graph of the function $F_0 : x \to \varepsilon A_0 x$ is 0 hence $\text{div}(Q_3(\nabla F)) = \text{div}(Q_3(\nabla F_0))$. We can thus evaluate
\[
\|\text{div}(Q_3(\nabla F))\|_{2,\alpha,\nu-2,\mu-2} \leq c \varepsilon^3 \rho_*^{1-3n-\nu} + \sup_j \|\Phi_j\|_{2,\alpha} \rho_*^{n-2}
\]
This ends the proof of the Lemma. \hfill \Box

The graph of $F = \varepsilon G + F_0$ over the horizontal plane $\Pi_0$ can also be viewed as a normal graph over the submanifold $\Pi_\varepsilon$ which is the graph of $x \to x + i (1 - \chi_\rho) \varepsilon A_0 x$, where $\chi_\rho$ is equal to 0 in $B(0, \frac{1}{\rho_0})$ and equal to 0 outside $B(0, \frac{2}{\rho_0})$. By construction, this submanifold $\Pi_\varepsilon$ is equal to $\Pi_0$ in $B(0, \frac{1}{\rho_0})$ and is equal to the graph of the mapping $F_0$ away from $B(0, \frac{2}{\rho_0})$. Hence we can say that the vertical graph of $F = \varepsilon G + F_0$ over $\Pi_0$ is a normal graph of some mapping $F$ over $\Pi_\varepsilon$.

Now we want to perturb this normal graph in order to obtain a minimal submanifold. The equation we have to solve now reads
\[
\begin{align*}
\Delta(F + \tilde{F}) + \varepsilon L(F + \tilde{F}) + \text{div}\left(\tilde{Q}_3(\nabla(F + \tilde{F}))\right) &= 0 \quad \text{in} \quad \mathbb{R}^n \setminus \cup_j B(x_j, \rho_*) \\
F &= 0 \quad \text{on} \quad \partial B(x_j, \rho_*).
\end{align*}
\]
Where $L$ is a second order linear differential operator which takes into account the fact that $\tilde{\Pi}_\varepsilon$ has some region where it is not planar. The coefficients of $L$ are bounded independently of $\Pi_0$ and can be assumed to be supported in $B(0, \frac{2}{\rho_0}) \setminus B(0, \frac{1}{\rho_0})$.

Making use of Proposition 13, we can rewrite this equation as
\[
F = -\mathcal{G}_{\rho_*} \left(\Delta \tilde{F} + \varepsilon L(F + \tilde{F}) + \text{div}\left(\tilde{Q}_3(\nabla(F + \tilde{F}))\right)\right).
\]
For the sake of simplicity, let us call $\mathcal{M}(F)$ this operator which is defined on the space
\[
\mathcal{E}_{\rho_*} := \left[C_{\epsilon,\mu}^{2,\alpha}(\mathbb{R}^n \setminus \cup_j B(x_j, \rho_*); \mathbb{R}^n) \oplus (1 + |x|^2)^{\frac{2-n}{2}} \times \mathbb{R}^n\right]_D,
\]
which is naturally endowed with the product norm, with values into $C_{\nu-2,\mu-2}^{2,\alpha}(\mathbb{R}^n \cup \cup_j B(x_j, \rho_*); \mathbb{R}^n)$.

The existence of a fixed point for this operator is the content of the following

**Proposition 14** Let $\nu \in (2-n, 0)$, $\mu \in (1-n, 2-n)$ and $\kappa > 0$ be fixed. There exists $c_0 > 0$ and $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ and for all $\Phi_j \in C_{\nu-2,\mu-2}^{2,\alpha}(S^{n-1}; \mathbb{R}^n)$ which satisfy $\|\Phi_j\|_{2,\alpha} \leq \kappa \varepsilon \rho_*^2$, we have
\[
\|\mathcal{M}(0)\|_\varepsilon \leq \frac{c_0}{2} (\varepsilon^3 \rho_*^{1-3n-\nu} + \rho_*^{2-n} \sup_j \|\Phi_j\|_{2,\alpha})
\]
and
\[
\|\mathcal{M}(F_2) - \mathcal{N}(F_1)\|_\varepsilon \leq \frac{1}{2} \|F_2 - F_1\|_\varepsilon
\]
for all $F_1, F_2 \in \mathcal{E}_{\rho_*}$ such that $\|F_1\|_\varepsilon \leq c_0 (\varepsilon^3 \rho_*^{1-3n-\nu} + \rho_*^{2-n} \sup_j \|\Phi_j\|_{2,\alpha})$.

In particular $\mathcal{M}$ has a unique fixed point in this ball.

**Proof:** The estimate for $\mathcal{M}(0)$ follows from the result of Lemma 1. The other estimate is left to the reader. \hfill \Box
20 Minimal $n$-submanifolds close to the graph of Green’s function, parameterized by their boundaries

Again, we give a summary of what we have obtained in the last sections. Assume that $x_1, \ldots, x_N \in \mathbb{R}^n$, $\alpha_1, \ldots, \alpha_N \in \mathbb{R}$, $A_0 \in M_n(\mathbb{R})$ and $R_1, \ldots, R_N \in O(n)$ are fixed. Given $\kappa > 0$, $\rho > 0$, there exists $\varepsilon_0 > 0$ such that for all $\alpha_j \in \left[1, \kappa, \kappa \right]$, for all $\varepsilon \in (0, \varepsilon_0)$ and all $\Phi_j \in C^{2,\alpha}(S^{n-1}; \mathbb{R}^n)$ which is orthogonal to $R_j \Theta$ in the $L^2$ sense and which satisfies

$$\|\Phi_j\|_{2,\alpha} \leq \kappa \rho^2 \varepsilon,$$

we have obtained a minimal $n$-submanifold which is $C^{2,\alpha}_{\nu,\mu}$ close to the $n$-plane

$$x \longrightarrow x + i \varepsilon A_0 x,$$

and which, up to some translation, can be parameterized, in some neighborhood of each of its boundary as the graph of

$$(r, \theta) \longrightarrow R_{j_0} \left( \varepsilon \beta_{j_0} r^{1-n} + \frac{\varepsilon}{\omega_n} \left( \sum_{j \neq j_0} \gamma_{j_0} \alpha_j - \lambda_{j_0} \right) r \Theta + J_{j_0} + J + J^\perp \right),$$

where we have set

$$r \Theta := x - x_{j_0}$$

and where $J_{j_0}$ is the unique harmonic extension of the boundary data $\Phi_j$ outside $B_{\rho_*}$ which tends to $0$ at $\infty$ and where $J_{j_0}$ satisfies

$$\|J_{j_0}(\rho_* \cdot)\|_{2,\alpha} + \|\rho_* \partial_r J_{j_0}(\rho_* \cdot)\|_{1,\alpha} \leq c \varepsilon \rho_*^2 + c_\varepsilon \rho_*^{3-n-\nu}.$$

and

$$\|J^\perp_{j_0}(\rho_* \cdot)\|_{2,\alpha} + \|\rho_* \partial_r J^\perp_{j_0}(\rho_* \cdot)\|_{1,\alpha} \leq c \varepsilon \rho_*.$$

In addition $J^\perp_{j_0}(r, \cdot)$ is orthogonal to $\Theta$ in the $L^2$ sense on $S^{n-1}$ while $J_{j_0}(r, \cdot)$ is collinear to $\Theta$. Observe, and this is important, that the constants $c > 0$ do not depend on $\kappa$ while the constants $c_\varepsilon > 0$ does.

We denote by $\mathcal{P}_E((\alpha_j)_j, (\Phi_j)_j)$ this submanifold. Though this depends on $(R_j)_j$, on $(x_j)_j$ and on $\rho_*$, we do not write this dependence in the notation.

For further use it will be convenient to define

**Definition 5** For all $\Phi \in C^{2,\alpha}(S^{n-1}; \mathbb{R}^n)$, we define $\mathcal{P}_\text{ext}(\Phi)$ to be equal to $\partial_r F$ where $F$ is the unique harmonic extension of $\Phi$ outside the unit ball of $\mathbb{R}^n$, which tends to $0$ at $\infty$.

21 Gluing procedure

Assume that $x_1, \ldots, x_N \in \mathbb{R}^n$, $A_0 \in M_n(\mathbb{R})$ and $R_1, \ldots, R_N \in O(n)$ are fixed in such a way that (H1), (H2) and (H3) hold. By assumption, we can choose $(\alpha_1^*, \ldots, \alpha_N^*) \in \mathbb{R}_+^N$ such that

$$\sum_j \gamma_{j,j'} \alpha_j^* = \lambda_j.$$

We set

$$\beta_j^* = \alpha_j^*$$

Finally we fix $\rho > 0$ small enough and $\kappa > 0$ large enough. By the previous analysis there exists $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$ and for all $\alpha_j, \beta_j \in \left[\frac{1}{\kappa}, \kappa \right]$, for all $\Phi_j, \hat{\Phi}_j \in C^{2,\alpha}(S^{n-1}; \mathbb{R}^n)$
where \( \tilde{x} \) vector and \( \Theta \) are C functions over the space of functions spanned by \( \Theta \).

In the \( L^2 \) sense on \( S^{n-1} \), we can define \( \Pi_{\varepsilon,(x_j),\rho_s}((\alpha_j)_j,(\Phi_j)_j) \) this submanifold. This submanifold is, near each of its boundaries, the graph of

\[
(r, \theta) \mapsto \mathcal{R}_{j_0} \left( \varepsilon \beta_{j_0} r^{1-\alpha} + \frac{\varepsilon}{\omega_n} \sum_{j \neq j_0} \gamma_{j_0,j} (\alpha_j - \alpha_j^*) r \Theta + J_{j_0} + J + J^\perp \right)
\]

where we have set

\[ r \Theta := x - x_{j_0} \]

and where \( J_{j_0} \) is the unique harmonic extension of the boundary data \( \Phi_j \) outside \( B_{\rho_s} \) which tends to 0 at \( \infty \) and where both \( J \) and \( J^\perp \) satisfies

\[
\| J_{j_0}(\rho_s, \cdot) \|_{2,\alpha} + \| \rho_s \partial_r J_{j_0}(\rho_s) \|_{1,\alpha} \leq c \varepsilon \rho_s^2 + c_\alpha \rho_s^{3-\alpha}.
\]

And

\[
\| J^\perp_{j_0}(\rho_s, \cdot) \|_{2,\alpha} + \| \rho_s \partial_r J^\perp_{j_0}(\rho_s) \|_{1,\alpha} \leq c \varepsilon \rho_s.
\]

In addition \( J^\perp_{j_0}(r, \cdot) \) is orthogonal to \( \Theta \) in the \( L^2 \) sense on \( S^{n-1} \) while \( J_{j_0}(r, \cdot) \) is collinear to \( \Theta \).

We can also define a minimal \( n \)-submanifold \( H_\varepsilon_{\mathbb{R}}(\beta_j, \tilde{\Phi}_j) \) which, once translated by the vector \( x_j \) will be denoted by \( H_\varepsilon(\beta_j, \tilde{\Phi}_j) \). Its boundary is parameterized by

\[
(r, \theta) \mapsto \varepsilon \beta r^{1-n} \Theta + F_j + \tilde{F}_j,
\]

where \( \tilde{F}_j \) depends smoothly on \( \beta_j \) and \( \Phi_j \) and satisfies

\[
\| \tilde{F}(\rho_s, \cdot) \|_{C^{2,\alpha}(S^{n-1})} + \| \rho_s \partial_r \tilde{F}(\rho_s, \cdot) \|_{C^{1,\alpha}(S^{n-1})} \leq c_\alpha \left( \varepsilon^3 \rho_s^{1-3\alpha} + \varepsilon \rho_s (\varepsilon^{-\frac{1}{2}} \rho_s) \right),
\]

and where \( F_j \) is the harmonic extension of \( \tilde{F}_j \) in \( B_{\rho_s} \).

Our aim will now be to find \( \Phi_j, \tilde{\Phi}_j, \alpha_j \) and \( \beta_j \) in such a way that

\[
M_\varepsilon := \Pi_{\varepsilon}((\alpha_j)_j,(\Phi_j)_j) \cup_{j=1}^N H_\varepsilon(\beta_j, \Phi_j),
\]

is a \( C^1 \) hypersurface.

Writing that the boundary of these submanifolds coincide yields the following system of equations

\[
\Phi_j - \tilde{\Phi}_j = (P_0 F_j - J^\perp_j)(\rho_s, \cdot)
\]

\[
\varepsilon (\beta_{j_0} - \alpha_{j_0}) \rho_s^{1-\alpha} \Theta + \frac{\varepsilon}{\omega_n} \sum_{j \neq j_0} \gamma_{j_0,j} (\alpha_j - \alpha_j^*) \rho_s \Theta = ((I - P_0) F_j - J_{j_0})(\rho_s, \cdot)
\]

where the first equation corresponds to the projection over the space of functions orthogonal to \( \Theta \) in the \( L^2 \) sense on \( S^{n-1} \) and where the second equation corresponds to the orthogonal projection over the space of functions spanned by \( \Theta \).

Writing that the conormals at the boundaries coincides yields the following system of equations

\[
\mathcal{P}_{ext} \Phi_j - \mathcal{P}_{int} \tilde{\Phi}_j = \rho_s (P_0 \partial_r F_j - \partial_r J^\perp_j)(\rho_s, \cdot)
\]

\[
(1-n) \varepsilon (\beta_{j_0} - \alpha_{j_0}) \rho_s^{1-\alpha} + \frac{\varepsilon}{\omega_n} \sum_{j \neq j_0} \gamma_{j_0,j} (\alpha_j - \alpha_j^*) \rho_s \Theta = \rho_s ((I - P_0) \partial_r F_j - \partial_r J_{j_0})(\rho_s, \cdot)
\]

where the first equation corresponds to the projection over the space of functions orthogonal to \( \Theta \) in the \( L^2 \) sense on \( S^{n-1} \) and where the second equation corresponds to the orthogonal projection over the space of functions spanned by \( \Theta \).

We will now use the well known result
Lemma 4 The mapping \( \mathcal{P}_{\text{ext}} - \mathcal{P}_{\text{int}} \) is an isomorphism from \( C^{2,\alpha}(S^{n-1}; \mathbb{R}^n) \) into \( C^{1,\alpha}(S^{n-1}; \mathbb{R}^n) \).

Proof: Observe that both \( \mathcal{P}_{\text{int}} \) and \( \mathcal{P}_{\text{ext}} \) are self-adjoint first order pseudodifferential operators which are elliptic, with principal symbols \( |\xi| \) and \( -|\xi| \), respectively, hence the difference is also elliptic and semibounded. This means that \( \mathcal{P}_{\text{ext}} - \mathcal{P}_{\text{int}} \) has discrete spectrum, and thus we need only prove that it is injective. The invertibility in Hölder spaces then follows by standard regularity theory.

To prove this, we argue by contradiction. Assume that \( \mathcal{P}_{\text{ext}} - \mathcal{P}_{\text{int}} \) is not injective. Then, there would exist some function \( \Phi \in C^{2,\alpha}(S^{n-1}; \mathbb{R}^n) \) for which \( (\mathcal{P}_{\text{ext}} - \mathcal{P}_{\text{int}})\Phi = 0 \). We may extend the Dirichlet data \( \Phi \) to a harmonic mapping \( F \) on the \( B_1 \) and also on \( \mathbb{R}^n \setminus B_1 \). In addition \( F \) tends to 0 at \( \infty \). Since \( (\mathcal{P}_{\text{ext}} - \mathcal{P}_{\text{int}})\Phi = 0 \), \( F \) is \( C^1 \) and hence is \( C^\infty \) and tends to 0 at \( \infty \). Thus \( F \equiv 0 \). This completes the proof of the injectivity of \( \mathcal{P}_{\text{ext}} - \mathcal{P}_{\text{int}} \).

We set
\[
\mathcal{F}^\alpha := [C^{2,\alpha}(S^{n-1}; \mathbb{R}^n) \times \mathbb{R}]^{2N}
\]
endowed with the product norm. Using this result, the previous system of equations \( (36) \) and \( (37) \) reduces to
\[
(\Phi_j)_j, (\tilde{\Phi}_j)_j, (\varepsilon (\alpha_j - \alpha_j^*))_j) = C \left( (\Phi_j)_j, (\tilde{\Phi}_j)_j, (\varepsilon (\alpha_j - \alpha_j^*))_j, (\varepsilon (\beta_j - \beta_j^*))_j \right),
\]
where the nonlinear mapping \( C \) satisfies
\[
\| C \left( (\Phi_j)_j, (\tilde{\Phi}_j)_j, (\varepsilon (\alpha_j)_j (\varepsilon (\beta_j)_j) \right) \|_{\mathcal{F}} \leq c \rho \varepsilon
\]
for some constant which does not depend on \( \kappa \), provided \( \varepsilon \) is chosen small enough, say \( \varepsilon \in (0, \varepsilon_0) \). This last claim is a simple consequence of \( (33)-(35) \).

We denote by \( B_\kappa \) the ball of radius \( \kappa \rho \varepsilon \) in \( \mathcal{F}^\alpha \). It follows from our previous analysis that, for fixed \( \kappa > 0 \) large enough, the mapping \( C \) is well defined in \( B_\kappa \) provided the parameter \( \varepsilon \) is small enough.

This zero of \( C \) produces a \( C^{1,\alpha} \) minimal \( n \)-submanifold \( M_\varepsilon \). It is then a simple exercise to see, thanks to regularity theory, that \( M_\varepsilon \) is in fact a \( C^\infty \) minimal hypersurface with \( N + 1 \) ends.

To conclude, we want to use Schauder’s fixed point theorem which will ensure the existence of at least one fixed point of \( C \) in \( B_\kappa \). However, since \( C \) is not compact it is not possible to apply directly Schauder’s Theorem. This is the reason we introduce a family of smoothing operators \( D^q \), for all \( q > 1 \), which satisfy for fixed \( 0 < \alpha' < \alpha < 1 \)
\[
\| D^q \Phi \|_{C^{2,\alpha}(S^{n-1})} \leq c_0 \| \Phi \|_{C^{2,\alpha'}(S^{n-1})} \quad \quad \| D^q \Phi \|_{C^{1,\alpha}(S^{n-1})} \leq c_0 q^{\alpha' - \alpha} \| \Phi \|_{C^{2,\alpha'}(S^{n-1})},
\]
and
\[
\| \Phi - D^q \Phi \|_{C^{2,\alpha}(S^{n-1})} \leq c_0 q^{\alpha' - \alpha} \| \Phi \|_{C^{2,\alpha'}(S^{n-1})},
\]
for some constant \( c_0 > 0 \) which does not depend on \( q > 1 \). The existence of such smoothing operators is available in [1], Proposition 1.6, page 97. To keep the notation short, we use the same notation for the smoothing operator defined on \( \mathcal{F}^\alpha \) and acting on all function spaces.

Now we fix \( \kappa > 0 \) large enough. For all \( q > 1 \), we may apply Schauder’s fixed point theorem to \( D^q C \) to obtain the existence of \( P_q \) fixed point of \( D^q C \) in \( B_\kappa \), provided \( \varepsilon \) is chosen small enough, say \( \varepsilon \in (0, \varepsilon_0) \).

Since \( P_q \) has norm bounded uniformly in \( q \), we may extract a sequence \( q_j \to +\infty \) such that \( P_{q_j} \) converges in \( \mathcal{F}^{\alpha'} \) for some fixed \( \alpha' < \alpha \). Thanks to the continuity of \( C \) (with respect to the \( C^{2,\alpha'} \) and \( C^{1,\alpha'} \) topology) and also to \( (38) \), the limit of this sequence is a fixed point of the mapping \( C \) and hence, produces a zero of \( C \), for all \( \varepsilon \in (0, \varepsilon_0) \). This completes our proof of the theorem.
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Claudio Arezzo
Dipartimento di Matematica
Università di Parma
Via M. D’Azeglio 85
43100, Parma
Italy
e-mail: claudio.arezzo@unipr.it

Frank Pacard
Centre de Mathématiques - Faculté de Sciences et Technologie
Université Paris XII - Val de Marne
61, Avenue du Général de Gaulle
94 010 Creteil Cedex
France
e-mail: pacard@univ-paris12.fr