Optimal factors in Vladimir Markov’s inequality in L2 Norm

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Abstract
In this paper we discuss a problem of computation of constants in Vladimir Markov’s type inequality in $L^2$ norm on the interval $[-1; 1]$.

Key words:
V. Markov’s inequality, L2 norms

1. Vladimir Markov’s inequality.

The famous V. Markov’s inequality is the following bound for a polynomial $P$ in one variable with real coefficients of degree at most $n$

$$\sup_{x \in [-1, 1]} |P^{(k)}(x)| \leq \frac{n^2(n^2 - 1) \cdots (n^2 - (k-1)^2)}{1 \cdot 3 \cdots (2k-1)} \sup_{x \in [-1, 1]} |P(x)|$$

$$= T_n^{(k)}(1) \sup_{x \in [-1, 1]} |P(x)|,$$

where $T_n$ denotes the $n$-th Chebyshev polynomial of the first kind, which is given by the formula $\cos(nx) = T_n(\cos x)$. The case $k = 1$ was firstly considered by Dmitrij Mendeleev (yes, the famous Russian chemist!) and Andrey Markov, and for this reason the above inequality is known as Markov’s inequality.

In the sequel, we shall write $\sup_{x \in [-1, 1]} |P(x)| =: ||P||_\infty$ and for $1 \leq p < \infty$

$$||P||_p = \left(\frac{1}{2} \int_{-1}^{1} |P(x)|^p dx \right)^{1/p}.$$

We can rewrite V. Markov’s inequality in the forms

$$||P^{(k)}||_\infty \leq ||T_n^{(k)}||_\infty ||P||_\infty$$

or

$$||P^{(k)}||_\infty \leq V(n, k)n^{2k}||P||_\infty,$$

where $V(n, k) = \frac{1}{(2k-1)!!} \left(1 - \frac{1}{n^2}\right) \cdots \left(1 - \frac{(k-1)^2}{n^2}\right)$.

The above inequalities imply the following facts

- $V(n + 1, k) \leq V(n, k), \quad n \geq k$.
- $\lim_{n \to \infty} V(n, k + 1)/V(n, k) = \frac{1}{2k-1}$.

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There exists a positive constant $C$ such that for any polynomial $P$ of degree at most $n$, 
\[
||P^{(k)}||_\infty \leq C \frac{1}{k!} n^{2k} ||P||_\infty.
\]

There exists a family of monic polynomials $(\hat{P}_n)$ such that $\hat{P}_n(-x) = (-1)^n \hat{P}_n(x)$ and 
\[
||\hat{P}_n||_\infty = \inf\{||Q||_\infty : Q \text{ is a monic polynomial of degree } n\}
\]
\[
||\hat{P}_n^{(k)}||_\infty / ||\hat{P}_n||_\infty = \sup\{||Q^{(k)}||_\infty / ||Q||_\infty : \deg Q \leq n\}.
\]

In particular,
\[
\left|\frac{d^k}{dx^k} \hat{P}_n\right|_\infty / ||\hat{P}_n||_\infty = \sup\left\{\left|\frac{d^k}{dx^k} \hat{P}^{(a,a)}_n\right|_\infty / ||\hat{P}^{(a,a)}_n||_\infty : a > -1\right\}.
\]

Here $(\hat{P}^{(a,a)}_n)$ is the family of monic ultraspherical polynomials belonging to the larger family of monic Jacobi polynomials $\hat{P}^{(\alpha,\beta)}_n$.

2. Markov’s inequality for the first and the second derivative in $L_p$ norms.

For the first derivative of polynomials, and $p \geq 1$ there exists a positive constant $C_p$ such that 
\[
||P'||_p \leq C_p \cdot (\deg P)^{2} ||P||_p.
\]

This inequality was firstly proved in [18], motivated by Zygmund’s inequality in [36]. In the special case $p = 2$ it was proved by E. Schmidt ([26],[27]) that $C_2 = \sqrt{3}$. He also obtained a remarkable result: in the inequality $||P'||_2 \leq A(\deg P)||P||_2$ we have
\[
\lim_{n \to \infty} A(n)/n^2 = \frac{1}{\pi}.
\]

After Hille, Schegö, Tamarkin and Schmidt, Markov’s inequality in $L_p$ norms (and its generalizations with various weights) was investigated by a number of specialist, especially at the end of twentieth century, when the case $p = 2$ was thoroughly studied, cf. [2], [3],[5], [7],[9], [10], [11], [12], [13], [14], [15], [16], [17], [19], [20], [21], [22], [23], [28], [29], [31], [32], [33], [35]. Sixty years after [18] it was proved in [5] that $\lim_{p \to \infty} C_p = 1$. Thus the classical A. Markov’s inequality is the limit case from [18]. It was conjectured by P. Goetgheluck in [16] that $C_p = (p + 1)^{1/p}$, which agrees with known values for $p = 2, \infty$. Many authors are searching for the best estimates of the type $||P^{(k)}||_p \leq C_p(n, k)||P||_p$, $\deg P \leq n$ or try to describe the asymptotic behavior of $C_p(n, k)$. It was completely solved in the case $p = 2$:
\[ \lim_{n \to \infty} \frac{C_2(n, 1)}{n^2} = \frac{1}{\pi}, \quad \text{E. Schmidt ([26],[27])}; \]
\[ \lim_{n \to \infty} \frac{C_2(n, 2)}{n^4} = \frac{1}{4k_0} = 0.0711, \quad \text{where } k_0 = \inf \{k > 0 : 1 + \cos k \cosh k = 0\}, \quad \text{L. Shampine ([28],[29])}; \]
\[ \lim_{n \to \infty} \frac{C_2(n, k)}{n^{2k}} = ||L_k^*||_{L^2(0,1)}, \quad \text{where } L_k^*f(x) = \frac{1}{2^k(k-1)!} \int_0^x (x-y)^{k-1}f(y)dy, \quad \text{A. Böttcher, P. Dörfler ([9])}. \]

**Theorem 3.1.** Let us observe that \( \lim_{n \to \infty} V(n, k)/n^{2k} = 1/(2k-1)! \) and
\[ \lim_{k \to \infty} (2^k k!/(2k-1)!!)^{1/k} = 1. \]
This suggests a connection between limits \( \lim_{n \to \infty} C_p(n, k) \) and the \( L_p \) norms of Volterra’s operators
\[ L_k^*f(x) = \frac{1}{2^k(k-1)!} \int_0^x (x-y)^{k-1}f(y)dy. \]

By applying Schwarz inequality, it is easy to get the following upper bound
\[ ||L_k^*||_{L^2(0,1)} \leq \frac{1}{2^k(k-1)!} \frac{1}{\sqrt{(2k-1)2k}}. \]
It gives for \( k = 2 \) the upper bound \( \frac{1}{8\sqrt{3}} = 0.0721... \) while the exact value is 0.0711....

Moreover, the lower bound may be found in [9]
\[ ||L_k^*||_{L^2(0,1)} \geq \frac{1}{2^k(k-1)!} \frac{1}{\sqrt{(2k-1)(2k+1)}}. \]
In particular, for \( k = 3 \), we get
\[ 0.010564... \leq ||L_3^*||_{L^2(0,1)} \leq 0.011410... \]

3. **Markov’s inequality for third derivative in \( L_2 \) norm.**

Refining a method from [5], G. Sroka [32] obtained the following non-trivial result.

**Theorem 3.1.** If \( p \geq 1 \), then for an arbitrary polynomial \( P \) of degree \( k \leq \deg P \leq n \) we have inequality
\[ ||P^{(k)}||_p \leq (C(p + 1)k^2)^{1/p}||T_n^{(k)}||_\infty ||P||_p, \]
with \( C = 6\sqrt[4]{e} \) for \( k \geq 3 \).

Later M. Baran and P. Ozorka (see P.Ozorka’s PhD thesis) proved, applying quite different technique, the following theorem.
**Theorem 3.2.** If $1 \leq p \leq 2$, then for an arbitrary polynomial $P$ of degree $k \leq \deg P \leq n$ we have inequality

$$\|P^{(k)}\|_p \leq B_p \max_{k \leq n} \|T_n^{(k)}\|_p n^{\frac{2}{p}} \|P\|_p = B_p \|T_n^{(k)}\|_p n^{\frac{2}{p}} \|P\|_p,$$

with $B_p = (3e/\pi)^{1/p}(p+1)^{1/p}$.

**Corollary 3.3.** If $1 \leq p \leq 2$ is fixed then there exists a constant $C_p$ independent of $n$ and $k$ such that for all $k \geq 3$ the following Vladimir Markov type inequality holds

$$\|P^{(k)}\|_p \leq C_p \frac{1}{k!} n^{2k} \|P\|_p.$$

**Remark 3.4.** Applying Nikolski inequality (cf. [32] Lemma 3 where a little strong result is given) $\|P\|_\infty \leq (2p + 2)^{1/p}(\deg P)^{2/p} \|P\|_p$, we derive from Theorem 3.1 the following

$$\|P^{(k)}\| \leq (C(p + 1) k^2)^{1/p}(2p + 2)^{1/p} n^{2/p} \|T_n^{(k)}\|_p \|P\|_p, \text{ deg } P \leq n,$$

which is considerably worse than Theorem 3.2.

Now we shall discuss the case $p = 2$ to compare Theorems 3.1 and 3.2 with earlier known results.

Denote

$$B(n, k) = (3e/\pi)^{1/2}\sqrt{3n} \cdot \|T_n^{(k)}\|_2,$$

$$A(n, k) = (6\sqrt[3]{4e^2})^{1/2}\sqrt{3k} \cdot \|T_n^{(k)}\|_\infty$$

and

$$R(n, k) = \frac{A(n, k)}{B(n, k)}.$$

We have

$$R(n, k) = (2e^{3\sqrt{4\pi}})^{1/2}(k/n)\|T_n^{(k)}\|_\infty/\|T_n^{(k)}\|_2 \approx 5.20692 \cdot \rho_n^{(k)},$$

where

$$\rho_n^{(k)} = (k/n) \cdot \|T_n^{(k)}\|_\infty/\|T_n^{(k)}\|_2.$$
Analyzing the above numerical results we see that the bounds in Theorem 3.2 are much better than the bounds in Theorem 3.1. We can also suppose that the factor $k^2/p$ in Theorem 3.1 can be replaced by $k^{1/p}$. Moreover, we can conjecture that

- $(k/n) \cdot ||T_n^{(k)}||_\infty / ||T_n^{(k)}||_2 \leq \sqrt{k}$,
- $(k/n) \cdot ||T_n^{(k)}||_\infty / ||T_n^{(k)}||_2 \to \sqrt{k}$ as $n \to \infty$. 

| $n$ | $||T_n^{(3)}||_2$ | $n \cdot ||T_n^{(3)}||_2$ | $||T_n^{(3)}||_\infty$ | $3 \cdot ||T_n^{(3)}||_\infty$ | $\rho_n^{(3)}$ |
|-----|-----------------|-----------------|-----------------|-----------------|-------|
| 3   | 24              | 72              | 24              | 72              | 1     |
| 4   | 110.851         | 443.405         | 192             | 576             | 1.299 |
| 5   | 349.17          | 1745.85         | 840             | 2520            | 1.443 |
| 6   | 882.842         | 5297.05         | 2688            | 8064            | 1.522 |
| 7   | 1926.46         | 13585.2         | 7056            | 21168           | 1.570 |
| 8   | 3779.46         | 30235.7         | 16128           | 48344           | 1.599 |
| 9   | 6840.29         | 61562.6         | 33264           | 99792           | 1.621 |
| 10  | 11620.5         | 116205          | 63360           | 190080          | 1.636 |
| 11  | 18759.1         | 206350          | 113256          | 339768          | 1.647 |
| 12  | 29036.1         | 348434          | 192192          | 576576          | 1.655 |
| 13  | 43387.6         | 564039          | 312312          | 936936          | 1.661 |
| 14  | 62919.1         | 880867          | 489216          | 1467648         | 1.666 |
| 15  | 88919.9         | 1333800         | 742560          | 2227680         | 1.670 |
| 16  | 121878          | 1966040         | 1096704         | 3290112         | 1.673 |
| 17  | 166491          | 2830360         | 1581408         | 4744224         | 1.676 |
| 18  | 221687          | 3990370         | 2232576         | 6697728         | 1.678 |
| 19  | 290632          | 5522000         | 3093048         | 9279144         | 1.680 |
| 20  | 375745          | 7514910         | 4213440         | 12640320        | 1.682 |
| 30  | 2859070         | 85772200        | 48330240        | 1449900800      | 1.690 |
| 40  | 12056700        | 4822680000      | 272213760       | 8166412800      | 1.693 |
| 50  | 36806500        | 18403250000     | 1039584000      | 31187520000     | 1.695 |
| 100 | 117833300000    | 6663333600000   | 48330240000     | 1449900800000   | 1.697 |

$\rho_n^{(3)} = (3/n) \cdot ||T_n^{(3)}||_\infty / ||T_n^{(3)}||_2$

| $n$ | $||T_n^{(4)}||_2$ | $n \cdot ||T_n^{(4)}||_2$ | $||T_n^{(4)}||_\infty$ | $4 \cdot ||T_n^{(4)}||_\infty$ | $\rho_n^{(4)}$ |
|-----|-----------------|-----------------|-----------------|-----------------|-------|
| 10  | 178306          | 1783060         | 823680          | 3294270         | 1.848 |
| 20  | 241168.10^4    | 483362.10^4    | 235350720       | 941402880       | 1.948 |
| 30  | 417336.10^4    | 1252008.10^4   | 6151749120      | 2460697965      | 1.965 |
| 40  | 313811.10^4    | 1.25524.10^{11} | 6187029880  | 2.47481.10^{11} | 1.972 |
| 50  | 1.49894.10^{10} | 7.47795.10^{11} | 3699943392.10^{13} | 1.479773560.10^{12} | 1.979 |

$\rho_n^{(4)} = (4/n) \cdot ||T_n^{(4)}||_\infty / ||T_n^{(4)}||_2$.
FURTHER CALCULATIONS.

\[ C(n, 1) = \sup_{\deg P = n} \frac{1}{n^2} \|P'\|_2/\|P\|_2 \]

\[ A(n, 1) = V(n, 1) = \frac{1}{n^2} \sup \left\{ \left\| \frac{dP^{(a,a)}_n}{dx} \right\|_2 : a \geq 0 \right\} = \frac{1}{n^2} \left\| \frac{dP_n^{(\alpha_n, \alpha_n)}}{dx} \right\|_2/\|P_n^{(\alpha_n, \alpha_n)}\|_2. \]

| \( n \) | \( \alpha_n \) | \( A(n, 1) \) | \( C(n, 1) \) |
|---|---|---|---|
| 1 | 0 | 1.732050 | 1.732050 |
| 2 | 0 | 0.968246 | 0.968246 |
| 3 | 0.133222 | 0.724622 | 0.724622 |
| 4 | 0.242328 | 0.60736 | 0.609363 |
| 5 | 0.325474 | 0.53958 | 0.543656 |
| 6 | 0.388334 | 0.49587 | 0.501657 |
| 7 | 0.436555 | 0.465519 | 0.472648 |
| 8 | 0.474328 | 0.443287 | 0.451468 |
| 9 | 0.504555 | 0.426332 | 0.435350 |
| 10 | 0.529225 | 0.41299 | 0.422655 |
| 11 | 0.549714 | 0.402224 | 0.412476 |
| 12 | 0.566993 | 0.393358 | 0.404076 |
| 13 | 0.581761 | 0.385931 | 0.397045 |
| 14 | 0.594533 | 0.37962 | 0.391075 |
| 15 | 0.605691 | 0.374191 | 0.385944 |
| 16 | 0.615528 | 0.369473 | 0.381486 |
| 17 | 0.62427 | 0.365333 | 0.377578 |
| 18 | 0.632094 | 0.361671 | 0.374124 |
| 19 | 0.639143 | 0.358417 | 0.371050 |
| 20 | 0.645528 | 0.355441 | 0.368296 |
| 21 | 0.651342 | 0.353229 | 0.365814 |
| 22 | 0.656662 | 0.350198 | 0.363567 |
| 23 | 0.66155 | 0.348284 | 0.361523 |
| 24 | 0.666058 | 0.346296 | 0.359655 |
| 25 | 0.670232 | 0.344458 | 0.357942 |
| 26 | 0.674108 | 0.342822 | 0.356365 |
| 27 | 0.677718 | 0.341360 | 0.354908 |
| 28 | 0.68109 | 0.335661 | 0.353559 |
| 29 | 0.684249 | 0.339102 | 0.352305 |
| 30 | 0.68724 | 0.339052 | 0.351138 |

It seems that the sequence \( A(n, 1) \) is decreasing. If it is true, the sequence \( A(n, 1) \) is convergent, but it is not clear that the limit is \( 1/\pi = 0.31830... \).
\[ V(n, k) = \frac{1}{n^{2k}} \sup \left\{ \left\| \frac{d^k P_n^{(a,a)}}{dx^k} \right\|_2 : a \geq 0 \right\} \]

\[ = \frac{1}{n^{2k}} \left\| \frac{d^k P_n^{(\alpha_n^{(k)}, \alpha_n^{(k)})}}{dx^k} \right\|_2 / \left\| P_n^{(\alpha_n^{(k)}, \alpha_n^{(k)})} \right\|_2, \]

\[ \| U_{n-1}^{(k)} \|_2^* = \| U_{n-1}^{(k)} \|_2 / n^{2k}. \]

| n  | \(a_n^{(2)}\) | \(V(n, 2)\) | \(a_n^{(3)}\) | \(V(n, 3)\) | \(\| U_{n-1}^{(k)} \|_2^*\) | \(\| U_{n-1}^{(k)} \|_2^*\) |
|----|----------------|-------------|-------------|-------------|----------------|----------------|
| 1  | 0              | 0           | 0           |             |                |                |
| 2  | 0              | 0           |             |             |                |                |
| 3  | 0.0626545     | 0.21803     | 0           | 0.044401    | 0.5123        | 0.108253       |
| 4  | 0.126775      | 0.181934    | 0.0354246   | 0.0373206   | 0.510283      | 0.111735       |
| 5  | 0.182076      | 0.159283    | 0.075835    | 0.0325326   | 0.508472      | 0.113534       |
| 6  | 0.227988      | 0.143887    | 0.113921    | 0.0290943   | 0.507044      | 0.114622       |
| 7  | 0.266016      | 0.132788    | 0.148038    | 0.0266326   | 0.505933      | 0.11534        |
| 8  | 0.297795      | 0.124424    | 0.17816     | 0.0246069   | 0.506062      | 0.115841       |
| 9  | 0.32467       | 0.117902    | 0.204704    | 0.0230634   | 0.504371      | 0.11205        |
| 10 | 0.347674      | 0.112677    | 0.228163    | 0.0218158   | 0.503813      | 0.116479       |
| 11 | 0.36759       | 0.108422    | 0.248998    | 0.0207856   | 0.503358      | 0.11669        |
| 12 | 0.385008      | 0.104829    | 0.267603    | 0.0199217   | 0.502982      | 0.116856       |
| 13 | 0.400382      | 0.101808    | 0.28431     | 0.0191873   | 0.502668      | 0.116988       |
| 14 | 0.414062      | 0.0992172   | 0.299393    | 0.0185554   | 0.502402      | 0.117096       |
| 15 | 0.426323      | 0.0969702   | 0.31308     | 0.0180062   | 0.502175      | 0.117185       |
| 16 | 0.437383      | 0.0950031   | 0.325558    | 0.0175245   | 0.501979      | 0.117259       |
| 17 | 0.447419      | 0.0932664   | 0.336984    | 0.0170678   | 0.50181       | 0.117322       |
| 18 | 0.456573      | 0.0917213   | 0.34749     | 0.0167193   | 0.501663      | 0.117375       |
| 19 | 0.464961      | 0.0903384   | 0.357186    | 0.016380    | 0.501533      | 0.11742        |
| 20 | 0.47268       | 0.0890925   | 0.366166    | 0.0160731   | 0.501418      | 0.11746        |
| 21 | 0.479812      | 0.0879644   | 0.374509    | 0.0157956   | 0.501316      | 0.117494       |
| 22 | 0.486425      | 0.0869379   | 0.382284    | 0.0155430   | 0.501226      | 0.117524       |
| 23 | 0.492576      | 0.085999    | 0.38955     | 0.0153065   | 0.501144      | 0.11755        |
| 24 | 0.498316      | 0.0851355   | 0.396358    | 0.0151038   | 0.501071      | 0.117583       |
| 25 | 0.503685      | 0.0843456   | 0.402751    | 0.0148929   | 0.501005      | 0.117594       |
| 26 | 0.508722      | 0.0835003   | 0.408769    | 0.0147388   | 0.500945      | 0.117612       |
| 27 | 0.513457      | 0.0825563   | 0.414445    | 0.0144975   | 0.500896      | 0.117629       |
| 28 | 0.51792       | 0.0826320   | 0.41981     | 0.0144838   | 0.50084       | 0.117644       |
| 29 | 0.522134      | 0.0806537   | 0.424889    | 0.0142467   | 0.500795      | 0.117657       |

From the above numerical data one can conjecture that

\[ \lim_{n \to \infty} V(n, k) = \lim_{n \to \infty} C_2(n, k). \]
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