Existence of multiple solutions for quasi-linear degenerate elliptic equations

Yawei Wei

School of Mathematical Sciences and LPMC, Nankai University, Tianjin 300071, China
Email: weiyawei@nankai.edu.cn

Received February 12, 2020; accepted July 4, 2020; published online November 23, 2020

Abstract The present paper is concerned with a class of quasi-linear degenerate elliptic equations. The degenerate operator arises from analysis of manifolds with singularities. The variational methods are applied here to verify the existence of infinitely many solutions for the problem.

Keywords quasi-linear, degenerate operator, weighted Sobolev spaces, variational method

MSC(2020) 35J70, 35J20, 58J05

Citation: Wei Y W. Existence of multiple solutions for quasi-linear degenerate elliptic equations. Sci China Math, 2022, 65: 971–992, https://doi.org/10.1007/s11425-020-1732-y

1 Introduction

In this paper, the following quasi-linear degenerate elliptic equation is concerned:

\[
\begin{cases}
-(x_1x_2)^{-p} \text{div}_M(|\nabla_M u|^{p-2} \nabla_M u) = \lambda |u|^{q-2} u & \text{in } \text{int}M, \\
u = 0 & \text{on } \partial M,
\end{cases}
\]

where \(\lambda > 0\), \(2 < p < N\) and \(p \leq q < p^* = \frac{Np}{N-p}\). Here, \(M := (0, \delta) \times (0, \delta) \times X\) with the fixed small positive \(\delta\) and dimension \(N = n + 2\), and \(\partial M := \{0\} \times \{0\} \times X\) denotes the boundary of \(M\), where \(X\) is a bounded open set in the unit sphere of \(\mathbb{R}^{N-2}\) with \(x' := (x'_1, \ldots, x'_n) \in X\), \(\nabla_M := (x_1 \partial_{x_1}, x_1 x_2 \partial_{x_2}, \partial_{x'_1}, \ldots, \partial_{x'_n})\), and \(\text{div}_M := \nabla_M\).

The non-trivial solution \(u \in H^{1, \frac{N-1}{2N-p}}(M)\) (see the definition below) verifies (1.1) in the weak sense, i.e., for any \(\varphi \in C_0^\infty(\text{int}M)\), it holds that

\[
\int_M x_1 |\nabla_M u|^{p-2} \nabla_M u \cdot \nabla_M \varphi \frac{dx_1}{x_1} \frac{dx_2}{x_1 x_2} = \lambda \int_M x_1(x_1x_2)^p |u|^{q-2} u \varphi \frac{dx_1}{x_1} \frac{dx_2}{x_1 x_2}.
\]

In the following calculus, for simplicity, write \(d\sigma := \frac{dx_1}{x_1} \frac{dx_2}{x_1 x_2} dx'\). The weak solutions for (1.1) are the critical points of the energy functional

\[
J(u) = \frac{1}{p} \int_M x_1 |\nabla_M u|^p d\sigma - \frac{\lambda}{q} \int_M x_1(x_1x_2)^p |u|^q d\sigma.
\]
The quasi-linear elliptic equations involving the $p$-Laplacian operator in the smooth domain have been widely studied, where the $p$-Laplacian is one of the typical quasi-linear elliptic operators. García Azorero and Peral Alonso [9] considered the existence and non-uniqueness for the $p$-Laplacian. The work [8] by Drabek concerned the resonance problems for the $p$-Laplacian. Recently, Carl and Motreanu [4] studied the multiple and sign-changing solutions for the multivalued $p$-Laplacian. There are plenty of research works related to the quasi-linear elliptic operators and equations which may not be listed here by the limit of pages.

The present quasi-linear degenerate operator in (1.1) comes from the analysis of manifolds with singularities. This academic field has been studied from various perspectives. In [12], Melrose and Mendoza studied the totally characteristic elliptic operators by the so-called B-calculus. Later the boundary value problems on domains with singularities were investigated by many mathematicians, as Grisvard, Kozlov, Mazya and Rossmann, etc. (see [10, 11] and the references therein). Costabel and Dauge contributed results of analysis and applications on elliptic boundary value problems in corner domains (see [7] and the references therein). Recently, in [15, 16], Roidos and Schrohe concerned the nonstationary problems on manifolds with singularities by applying microlocal tools, semigroup theory, maximal regularity theory and other functional methods. Schulze has been working on this field for decades and gave fruitful results on microlocal analysis and pseudo differential operators on manifolds with singularities (see [18, 19] and the references therein). This paper is based on the framework by Schulze.

Let $X$ be a bounded open subset in the unit sphere of $\mathbb{R}^n$. Define an infinite cone in $\mathbb{R}^{n+1}$ as a quotient space

$$X^\Delta = (\mathbb{R}_+ \times X)/\{\{0\} \times X\},$$

and the stretched cone is defined as

$$X^\wedge = \mathbb{R}_+ \times X.$$

Set $x_1 \in \mathbb{R}_+$, $x' = (x'_1, \ldots, x'_n) \in X$. It is sufficient to consider the case of $x_1$ near to 0, which gives us a finite cone

$$E = ([0, \delta) \times X)/\{\{0\} \times X\}$$

with a small fixed positive $\delta$. The finite stretched cone of $E$ is

$$E = (0, \delta) \times X$$

with the boundary $\partial E = \{0\} \times X$. The typical degenerate differential operator on $E$ is in the form of

$$A = x_1^{-\mu} \sum_{j \leq \mu} a_j(x_1)(x_1 \partial x_1)^j$$

(1.4)

with the coefficients $a_j(x_1) \in C^\infty(\mathbb{R}_+, \text{Diff}^{\mu-j}(X))$. Denote by $\text{Diff}^{\mu}_{\text{deg}}(E)$ the set of cone differential operators as $A$ in (1.4). Schrohe and Seiler [17] discovered the closed extensions of the cone differential operators. Chen with his cooperators in [5] studied the semilinear elliptic equations based on this framework.

An infinite corner can be defined as

$$E^\Delta = (\mathbb{R}_+ \times E)/\{\{0\} \times E\},$$

and the stretched corner is

$$E^\wedge = \mathbb{R}_+ \times E.$$

Let $(x_1, x_2, x') \in E^\wedge$. We focus on the case of $x_2$ small enough, and then the finite corner is

$$M = ([0, \delta) \times E)/\{\{0\} \times E\}$$

and the finite stretched corner is

$$M = (0, \delta) \times E = (0, \delta) \times (0, \delta) \times X.$$
with the boundary $\partial M = \{0\} \times \partial \mathbb{E} = \{0\} \times \{0\} \times X$. The typical differential operator $B$ on $M$ is then

$$B = x_2^{-\nu} \sum_{l \leq \nu} b_l(x_2)(x_2 \partial_{x_2})^l,$$

where the coefficients

$$b_l(x_2) \in C^\infty(\mathbb{R}_+, \text{Diff}_{\text{deg}}^\nu(X))$$

as in (1.4), which is in the form of

$$b_l(x_2) = x_1^{-(\nu-l)} \sum_{j \leq \nu-l} a_{jl}(x_1, x_2)(x_1 \partial_{x_1})^j,$$

with $a_{jl}(x_1, x_2) \in C^\infty(\mathbb{R}_+, \text{Diff}_{\text{deg}}^{\nu-l-j}(X))$. It implies that

$$B = (x_1 x_2)^{-\nu} \sum_{j+l \leq \nu} \tilde{a}_{jl}(x_1, x_2)(x_1 \partial_{x_1})^j(x_1 x_2 \partial_{x_2})^l,$$

where $\tilde{a}_{jl}(x_1, x_2) \in C^\infty(\mathbb{R}_+, \text{Diff}_{\text{deg}}^{\nu-l-j}(X))$. We see the operators are polynomials in vector fields $x_1 \partial_{x_1}, x_1 x_2 \partial_{x_2}, \partial_{x_1}, \ldots, \partial_{x_\nu}$, and here let

$$\nabla_M = (x_1 \partial_{x_1}, x_1 x_2 \partial_{x_2}, \partial_{x_1}, \ldots, \partial_{x_\nu}).$$

The corresponding semilinear elliptic problems have been discussed in [6].

The present paper concerns the quasi-linear problem as in (1.1) and the following results hold.

Theorem 1.1. For $2 < p < N$, $p < q < p^*$, and $\lambda > 0$ the Dirichlet eigenvalue problem (1.1) possesses infinitely many non-trivial weak solutions in the sense of (1.2).

Theorem 1.2. If $(c_m)_{m \in \mathbb{N}}$ is the critical value sequence obtained in Theorem 1.1, then we have $c_m \to \infty$ as $m \to \infty$.

The problem (1.1) with $p < q < p^*$ holding different homogeneity of the right-hand side preserves a sequence of solutions with respect to $\lambda$. In fact, if $u \neq 0$ is a solution of the problem (1.1) with $\lambda = 1$, then for any $\alpha > 0$, $\alpha u$ verifies the problem (1.1) with $\lambda = \alpha^{p-q}$. But for the case of $p = q$, if $(u, \lambda)$ is a solution of (1.1), then for all $\alpha \in \mathbb{R}$, $(\alpha u, \lambda)$ is a solution too. Hence, we need different methods to solve the problem in the two cases. We call the problem (1.1) with $p = q$ the typical Dirichlet eigenvalue problem, which has the following results.

Theorem 1.3. For $2 < p < N$ and $q = p$, the Dirichlet eigenvalue problem (1.1) possesses a sequence of infinitely many non-trivial weak solutions

$$(u_k, \lambda_k) \in H^1_{p, 0}(\mathbb{R}^N, \mathbb{R}) \times \mathbb{R}_+$$

in the sense of (1.2).

Theorem 1.4. The eigenvalues $\lambda_k$ of (1.1) in Theorem 1.3 turn to infinity as $k \to \infty$.

The rest of this paper is organized as follows. In Section 2, the preliminaries are given including the definitions of weighted Sobolev spaces, the necessary definitions, notations and the abstract theory in the variational method. Section 3 is to verify the problem (1.1) in the case of $p < q < p^*$. In Subsection 3.1, to prove Theorem 1.1, we introduce a specific isomorphism to prove the embedding theory, prove the Brezis-Lieb type result in the present weighted Sobolev spaces, and verify the (PS) condition and the conditions in the abstract variational methods, which is the crucial part in this subsection. Theorem 1.2 is proved subsequently in Subsection 3.2. Here, the critical value sequences of the energy functional $J$ turning to infinity have been proved by analysis on the sequence of linear subspaces of the weighted Sobolev spaces. Section 4 is to deal with the case of $p = q$ of the problem (1.1). In Subsection 4.1, the idea of Lusternik-Schnirelman theory is employed to prove the existence of weak solutions which verifies Theorem 1.3. In the following Subsection 4.2, we prove Theorem 1.4, the eigenvalues of (1.1) turning to infinity by constructing contradiction.
2 Preliminaries

In this section, we first review some necessary definitions and notations.

**Definition 2.1.** Let \((x_1, x_2, x') \in \mathbb{R}_+ \times X^\wedge\) with the weighted data \(\gamma_1 \in \mathbb{R}, \gamma_2 \in \mathbb{R}\) and \(1 \leq p < +\infty\). Then \(L_p^{\gamma_1, \gamma_2}(\mathbb{R}_+ \times X^\wedge)\) denotes the space of all \(u(x) \in D'(\mathbb{R}_+ \times X^\wedge)\) such that
\[
\|u\|_{L_p^{\gamma_1, \gamma_2}(\mathbb{R}_+ \times X^\wedge)} = \left( \int_{\mathbb{R}_+ \times X^\wedge} |x_1^{\gamma_1} x_2^{\gamma_2} x' u(x)|^p d\sigma v \right)^{1/p} < +\infty.
\]
(Here and after we write \(d\sigma := \frac{dx_1}{x_1} \frac{dx_2}{x_2} dx'\) for simplicity.) The weighted Sobolev spaces are defined as follows:
\[
\mathcal{H}^{m, (\gamma_1, \gamma_2)}(\mathbb{R}_+ \times X^\wedge) := \{ u \in D'(\mathbb{R}_+ \times X^\wedge) : (x_1 \partial_{x_1})^j (x_1 x_2 \partial_{x_2})^l \partial_{x'}^\beta u \in L_p^{\gamma_1, \gamma_2}(\mathbb{R}_+ \times X^\wedge) \}
\]
for arbitrary \(j, l \in \mathbb{N}, \beta \in \mathbb{N}^{N-2} \) and \(j + l + |\beta| \leq m\).

**Definition 2.2.** Let \(W^{m,p}_{\text{locc}}(\text{int}\mathbb{M})\) denote the classical local Sobolev space (here, \(\text{int}\mathbb{M}\) is the interior of \(\mathbb{M}\)). For \(1 \leq p < \infty, m \in \mathbb{N}\) and the weighted data \(\gamma_1 \in \mathbb{R}, \gamma_2 \in \mathbb{R}\) then \(\mathcal{H}^{m, (\gamma_1, \gamma_2)}(\mathbb{M})\) denotes the subspace of all \(u \in W^{m,p}_{\text{locc}}(\text{int}\mathbb{M})\), such that
\[
\mathcal{H}^{m, (\gamma_1, \gamma_2)}(\mathbb{M}) := \{ u \in W^{m,p}_{\text{locc}}(\text{int}\mathbb{M}) : (\omega \sigma) u \in \mathcal{H}^{m, (\gamma_1, \gamma_2)}(\mathbb{R}_+ \times X^\wedge) \}
\]
for any cut-off functions \(\omega = \omega(x_1, x')\) and \(\sigma = \sigma(x_2, x')\), supported by collar neighborhoods of \((0, 1) \times \partial\mathbb{M}\) and \((0, 1) \times \partial\mathbb{M}\), respectively. Moreover, define \(L_p^{\gamma_1, \gamma_2}(\mathbb{M}) := \mathcal{H}_p^{0,(\gamma_1, \gamma_2)}(\mathbb{M})\).

**Remark 2.3.** Although the definitions of weighted Sobolev spaces on manifolds with this kind of singularity are complex (see more in [19]), Definitions 2.1 and 2.2 fit the present problem (1.1). Here, since the paper concentrates on \(\mathbb{M} = (0, \delta) \times (0, \delta) \times X\) with small enough positive \(\delta\), it is sufficient to consider the case in the support of \(\omega\) and \(\sigma\) in Definition 2.2. Moreover, let \(\mathcal{H}_p^{1,0,(\gamma_1, \gamma_2)}(\mathbb{M})\) denote the closure of \(C_0^{\infty} \subset \mathcal{H}_p^{1,0,(\gamma_1, \gamma_2)}(\mathbb{M})\).

**Proposition 2.4.** Let \(1 \leq p < \infty\) and \(\gamma_1, \gamma_2 \in \mathbb{R}\). If \(u(x) \in \mathcal{H}_p^{1,0,(\gamma_1, \gamma_2)}(\mathbb{M})\), then
\[
\|u(x)\|_{L_p^{\gamma_1, \gamma_2}(\mathbb{M})} \leq c \|\nabla_M u(t, x)\|_{L_p^{\gamma_1, \gamma_2}(\mathbb{M})},
\]
where the constant \(c\) depends only on \(\mathbb{M}\) and \(p\).

**Proof.** Follow the same process of [6, Proposition 3.2]. \(\square\)

**Remark 2.5.** Proposition 2.4 implies that the norm \(\|u\|_{\mathcal{H}_p^{1,0,(\gamma_1, \gamma_2)}(\mathbb{M})}\) is equivalent to the norm \(\|\nabla_M u\|_{L_p^{\gamma_1, \gamma_2}(\mathbb{M})}\).

Next, we introduce some concepts in variational methods in the following. Let \(E\) be a Banach space.

**Definition 2.6.** The functional \(I\) satisfies the \((PS)_c\) condition, if for any sequence \(\{u_k\} \subset E\) with the properties
\[
I(u_k) \rightarrow c \quad \text{and} \quad \|I'(u_k)\|_{E'} \rightarrow 0,
\]
there exists a subsequence which is convergent, where \(I'(\cdot)\) is the Fréchet derivative of \(I\) and \(E'\) is the dual space of \(E\). If it holds for any \(c \in \mathbb{R}\), we say that \(I\) satisfies the \((PS)\) condition.

**Definition 2.7.** In the Banach space \(E\), define the class
\[
\Sigma(E) = \{ A \subset E \mid A \text{ is closed, and } A = -A \}.
\]
For \(A \in \Sigma(E)\), define the genus of \(A\), denoted by \(\gamma(A)\), as
\[
\gamma(A) = \begin{cases} 0, & \text{if } A = \emptyset, \\ \infty, & \text{if } \{m \in \mathbb{N}_+ ; \exists h \in C(A, \mathbb{R}^m \setminus \{0\}) , h(-x) = -h(x) \} = \emptyset, \\ \inf\{m \in \mathbb{N}_+ ; \exists h \in C(A, \mathbb{R}^m \setminus \{0\}) , h(-x) = -h(x) \}. 
\end{cases}
\]
Proposition 2.8 (See [13, Section 3]). Let \( A, B \in \Sigma(E) \). The genus \( \gamma \) possesses the following properties:

1. if \( \psi \in C(A, B) \) is odd, then \( \gamma(A) \leq \gamma(B) \);
2. if \( \psi \in C(A, B) \) is an odd homeomorphism, then \( \gamma(A) = \gamma(B) = \gamma(\psi(A)) \);
3. if \( A \subset B \), then \( \gamma(A) \leq \gamma(B) \);
4. if \( \gamma(B) < \infty \), \( \gamma(A - B) \geq \gamma(A) - \gamma(B) \);
5. \( \gamma(A \cup B) \leq \gamma(A) + \gamma(B) \);
6. if \( S^{n-1} \) is the sphere in \( \mathbb{R}^n \), then \( \gamma(S^{n-1}) = n \);
7. if \( A \) is compact, then \( \gamma(A) < \infty \);
8. if \( A \) is compact, there exists \( \delta > 0 \) such that for \( N_{\delta}(A) = \{ x \in X : d(x, A) < \delta \} \) we have \( \gamma(A) = \gamma(N_{\delta}(A)) \).

The abstract theory in [2] will be employed to investigate the existence of solutions for the Dirichlet problem (1.1). We recall it in the following. Let the functional \( I \in C^1(E, \mathbb{R}) \) and

\[ B_r = \{ u \in E \mid \| u \|_E \leq r \}. \]

Assume \( I \) satisfies \( I(0) = 0 \) and the following five properties:

1. the functional \( I \) satisfies that \( I(u) = I(-u) \) for all \( u \in E \);
2. the functional \( I \) verifies the Palais-Smale condition;
3. there exists a \( \rho > 0 \) such that \( I > 0 \) in \( B_\rho \setminus \{0\} \) and \( I \geq \alpha > 0 \) on \( \partial B_\rho \);
4. there exists a \( v \in E \) such that \( \| v \|_E > \rho \) and \( I(v) < \alpha \);
5. for any finite-dimensional subspaces \( E_m \subset E \), it holds that \( E_m \cap A_0 \) is bounded, where \( A_0 = \{ u \in E \mid 0 \leq I(u) < +\infty \} \).

Define

\[ \Gamma := \{ h \in C(E, E) \mid h(0) = 0; h \text{ is an odd homeomorphism; } h(B) \subset A_0 \}, \]
\[ \Gamma_m := \{ K \subset E \mid K \text{ is compact; } K = -K; \gamma(K \cap h(\partial B)) \geq m, \forall h \in \Gamma \}. \]

Lemma 2.9 (See [2, Theorem 2.8]). Suppose \( I \) satisfies (I1)–(I5). For each \( m \in \mathbb{N} \), let

\[ b_m = \inf_{K \in \Gamma_m} \max_{u \in K} I(u) \tag{2.2} \]

Then \( 0 < \alpha \leq b_m \leq b_{m+1} \) and \( b_m \) is a critical value of \( I \). Moreover, if \( b_{m+1} = \cdots = b_{m+r} = b \), then \( \gamma(K_b) \geq r \), where \( K_b = \{ u \in E \mid I(u) = 0, I(u) = b \} \).

Let \( \{ E_m \}_{m \in \mathbb{N}} \) be a sequence of subspaces of \( E \), such that \( \dim(E_m) = m \) and \( E_m \subset E_{m+1} \); \( L(\bigcup_{m \in \mathbb{N}} E_m) \) denotes the linear manifold generated by \( \bigcup_{m \in \mathbb{N}} E_m \) which is dense in \( E \). By \( E_m^c \) we denote the algebraical and topological complement of \( E_m \).

Lemma 2.10 (See [2, Theorem 2.13]). Let \( I \) satisfy (I1)–(I5). For each \( m \in \mathbb{N} \), let

\[ c_m = \sup_{h \in \Gamma} \inf_{u \in \partial B_{1} \cap E_{m-1}} I(h(u)) \tag{2.3} \]

Then \( 0 < \alpha \leq c_m \leq b_m \leq \infty \), \( c_m \leq c_{m+1} \), and \( c_m \) is a critical value of \( I \).

3 The case of \( p < q < p^\ast \)

3.1 The proof of Theorem 1.1

The idea of the proof here is to verify the conditions (I1)–(I5) in Lemmas 2.9 and 2.10. The following lemmas will be applied in the proof.

Lemma 3.1. For \( 1 < p < N \) and \( 1 \leq q < p^\ast = \frac{Np}{N-p} \), the embedding

\[ H^1_{p,0}(\gamma_1,\gamma_2)(M) \hookrightarrow L^q_{\gamma_1,\gamma_2}(M) \]

is compact, if \( \frac{N}{q} - \gamma_1 > \frac{N}{p} - \gamma_1 \) and \( \frac{N}{q} - \gamma_2 > \frac{N}{p} - \gamma_2 \).
Proof. Since the embedding $H^{0,(\gamma_1,\gamma_2)}_{p,0}(M) \hookrightarrow L^q_{\gamma_1,\gamma_2}(M)$ is continuous, it is sufficient to prove

$$[\omega][\sigma]H^{0,(\gamma_1,\gamma_2)}_{p,0}(R_+ \times R_+ \times X) \hookrightarrow [\omega][\sigma]H^{0,(\gamma_1,\gamma_2)}_{q,0}(R_+ \times R_+ \times X)$$

is compact.

Set $1 \leq l < \infty$; for any $v(x) \in H^m_{l,0}(R_+ \times R_+ \times X)$, define

$$(\tilde{S}_{l,\gamma_2})v(x_1, y, x') = e^{-y(1-\gamma_2)}v(x_1, e^{-y}, x') =: w(x_1, y, x').$$

(3.1)

Then $\tilde{S}_{l,\gamma_2}$ induces an isomorphism

$$\tilde{S}_{l,\gamma_2} : [\omega][\sigma]H^{m,(\gamma_1,\gamma_2)}_{l,0}(R_+ \times R_+ \times X) \rightarrow [\omega][\sigma]H^{m,\gamma_1}_{l,0}(R_+ \times R_+ \times X),$$

(3.2)

where $\hat{\sigma}(y) = \sigma(e^{-y})$ and $H^{m,\gamma}_{l,0}(R_+ \times R_+ \times X)$ (see more in [18, 19]) denotes the space of all $w(x_1, y, x') \in D'(R_+ \times R_+ \times X)$ such that, for $k, j \in \mathbb{N}$ and $\alpha \in \mathbb{N}^{N-2}$,

$$\|w\|_{H^{m,\gamma}_{l,0}(R_+ \times R_+ \times X)} = \sum_{k+j+|\alpha| \leq m} \int_{R_+ \times R_+ \times X} |x_1^{N-1}(x_1 \partial_{x_1})^k(x_1 \partial_{y})^j\partial_{\alpha}w(x)| \, dx_1 \, dy \, dx' < \infty.$$  

(3.3)

In fact, we have

$$\|(\tilde{S}_{l,\gamma_2})v(x_1, y, x')\|_{H^{m,\gamma_1}_{l,0}(R_+ \times R_+ \times X)} = \|w\|_{H^{m,\gamma_1}_{l,0}(R_+ \times R_+ \times X)}$$

$$= \sum_{k+j+|\alpha| \leq m} \int_{R_+ \times R_+ \times X} \left| \frac{x_1^{N-1}}{x_1^{N}}(x_1 \partial_{x_1})^k(x_1 \partial_{y})^j\partial_{\alpha}w(x_1, e^{-y}) \right| \, \frac{dx_1 \, dy}{x_1} \, dx'$$

$$= \sum_{k+j+|\alpha| \leq m} \int_{R_+ \times R_+ \times X} \left| \frac{x_1^{N-1}}{x_1^{N}}(x_1 \partial_{x_1})^k(x_1 \partial_{\gamma})^j\partial_{\alpha}w(x_1, y, x') \right| \, \frac{dx_1 \, dy}{x_1} \, dx'$$

$$= \|v(x)\|_{H^{m,\gamma_1}_{l,0}(R_+ \times R_+ \times X)} < \infty.$$  

It proves the isomorphism of $\tilde{S}_{l,\gamma_2}$ in (3.2). Moreover, we need the following map to deal with the singularity caused by $x_1$:

$$(\tilde{S}_{l,\gamma_1})w(\rho, \xi, x') = e^{-\rho(\frac{N}{2}-\gamma_1)}w(e^{-\rho}, e^{-\rho} \xi, x') =: a(\rho, \xi, x'),$$

(3.4)

which induces an isomorphism

$$\tilde{S}_{l,\gamma_1} : [\omega][\sigma]H^{m,(\gamma_1,\gamma_2)}_{l,0}(R_+ \times R_+ \times X) \rightarrow [\omega][\sigma]W^{m,\gamma_1}_{0,0}(R_+ \times R_+ \times X),$$

(3.5)

where $\tilde{\omega}(\rho) = \omega(e^{-\rho})$, and $\tilde{\sigma}(\rho)$ is a cut-off function in $\xi$ for $\xi = \frac{\rho}{x_1}$ with $y \in \text{supp}\tilde{\sigma}(y)$ and $x_1 \in \text{supp} \omega(x_1)$, and $W^{m,\gamma_1}(\cdot)$ denotes the classical Sobolev spaces. In fact, the rule of changing variables implies that

$$\|\tilde{S}_{l,\gamma_1}w\|_{W^{m,\gamma_1}_{0,0}(R_+ \times R_+ \times X)} = \sum_{k+j+|\alpha| \leq m} \int_{R_+ \times R_+ \times X} \left| \frac{\partial_{\rho}^k \partial_{\gamma}^j \partial_{\alpha}w(\rho, \xi, x')}{x_1^{N}} \right| \, d\rho \, dx_1 \, dx'$$

$$= \sum_{k+j+|\alpha| \leq m} \int_{R_+ \times R_+ \times X} \left| \frac{\partial_{\rho}^k \partial_{\gamma}^j \partial_{\alpha}w(\rho, e^{-\rho} \xi, x')}{x_1^{N}} \right| \, d\rho \, dx_1 \, dx'$$

$$= \sum_{k+j+|\alpha| \leq m} \int_{R_+ \times R_+ \times X} \left| \frac{\partial_{\rho}^k \partial_{\gamma}^j \partial_{\alpha}w(x_1, y, x')}{x_1^{N}} \right| \, \frac{dx_1 \, dy}{x_1} \, dx'$$

$$= \|w(x_1, y, x')\|_{H^{m,\gamma_1}_{l,0}(R_+ \times R_+ \times X)} < \infty.$$  

This induces the isomorphism of $\tilde{S}_{l,\gamma_1}$ in (3.5). Then setting $S_{l,\gamma_1,\gamma_2} = \tilde{S}_{l,\gamma_1} \circ \tilde{S}_{l,\gamma_2}$ for $v(x) \in H^{m,(\gamma_1,\gamma_2)}_{l,0}(R_+ \times R_+ \times X)$, we have

$$(S_{l,\gamma_1,\gamma_2}v)(\rho, \xi, x') = e^{-\rho(\frac{N}{2}-\gamma_1)}e^{-\xi e^{-\rho}(\frac{N}{2}-\gamma_2)}v(e^{-\rho}, e^{-\xi e^{-\rho}}, x').$$
which induces the following isomorphism:

\[ S_{1}(\gamma_1, \gamma_2) = \tilde{S}_{1, \gamma_1} \circ \tilde{S}_{1, \gamma_2} : [\omega][\sigma]H^{m, (\gamma_1, \gamma_2)}_{0, 0}(\mathbb{R}^+ \times \mathbb{R}^+ \times X) \to [\omega][\tilde{\sigma}]W^m_0(\mathbb{R} \times \mathbb{R} \times X). \]  

(3.6)

Now in the present case, for \( u_q \in H^{q, (\gamma_1, \gamma_2)}_{0, 0}(\mathbb{R}^+ \times \mathbb{R}^+ \times X) \), we have

\[ (S_{q, (\gamma_1, \gamma_2)}[\omega][\sigma]u_q)(\rho, \xi, x') = [\omega][\tilde{\sigma}]e^{-\rho(\frac{N}{p} - \gamma_1)}e^{-\xi \cdot \omega - \rho(\frac{N}{p} - \gamma_2)}u_q(e^{-\rho}, e^{-\xi \cdot \omega}, x'), \]

which gives the following isomorphism:

\[ S_{q, (\gamma_1, \gamma_2)} : [\omega][\sigma]H^{q, (\gamma_1, \gamma_2)}_{0, 0}(\mathbb{R}^+ \times \mathbb{R}^+ \times X) \to [\omega][\tilde{\sigma}]W^0_0(\mathbb{R} \times \mathbb{R} \times X). \]

On the other hand, the map \( S_{q, (\gamma_1, \gamma_2)} \) induces another isomorphism for \( u_p \in H^{1, (\gamma_1, \gamma_2)}_{p, 0}(\mathbb{R}^+ \times \mathbb{R}^+ \times X) \) as follows. Setting \( \delta_1 := (\frac{N}{q} - \gamma_1) - (\frac{N}{p} - \gamma_1) \) and \( \delta_2 := (\frac{N}{q} - \gamma_2) - (\frac{N}{p} - \gamma_2) \), we have

\[ (S_{q, (\gamma_1, \gamma_2)}[\omega][\sigma]u_p)(\rho, \xi, x') = [\omega][\tilde{\sigma}]e^{-\rho \delta_1}e^{-\xi \cdot \omega - \rho \delta_2}e^{-\rho(\frac{N}{p} - \gamma_1)}u_p(e^{-\rho}, e^{-\xi \cdot \omega}, x'), \]

which gives the isomorphism

\[ S_{q, (\gamma_1, \gamma_2)} : [\omega][\sigma]H^{1, (\gamma_1, \gamma_2)}_{p, 0}(\mathbb{R}^+ \times \mathbb{R}^+ \times X) \to [\omega][\tilde{\sigma}]W^1_0(\mathbb{R} \times \mathbb{R} \times X). \]

For \( 1 < q < p^* \) and \( \delta_1 > 0, \delta_2 > 0 \), the following embedding is compact:

\[ [\omega][\tilde{\sigma}]e^{-\rho \delta_1}e^{-\xi \cdot \omega - \rho \delta_2}W^1_0(\mathbb{R} \times \mathbb{R} \times X) \hookrightarrow [\omega][\tilde{\sigma}]W^0_0(\mathbb{R} \times \mathbb{R} \times X) \]

since the functions \( e^{-\rho \delta_1} \) and \( e^{-\xi \cdot \omega - \rho \delta_2} \) vanish rapidly as \( \rho \to \infty \) and \( \xi \to \infty \), and then the function \( \varphi(\rho, \xi) = e^{-\rho \delta_1}e^{-\xi \cdot \omega - \rho \delta_2} \) and all the derivatives in \( \rho \) and \( \xi \) are uniformly bounded on \( \text{supp} \omega \) and \( \text{supp} \tilde{\sigma} \) for every \( s_1, s_2 \in \mathbb{R} \).

**Remark 3.2.** By applying the same idea as in Lemma 3.1, for \( 1 < p < N \) and \( 1 \leq q < p^* \) the embedding

\[ H^{1, (\gamma_1, \gamma_2)}_{p, 0}(\mathbb{M}) \hookrightarrow L^{\gamma_1 + \gamma_2}_q(\mathbb{M}) \]

is continuous, if \( \frac{N}{q} - \gamma_1 \leq \frac{N}{p} - \gamma_1 \) and \( \frac{N}{q} - \gamma_2 \geq \frac{N}{p} - \gamma_2 \). The embedding

\[ H^{m, (\gamma_1, \gamma_2)}_{p, 0}(\mathbb{M}) \hookrightarrow H^{m, (\gamma_1, \gamma_2)}_{p, 0}(\mathbb{M}) \]

is continuous if \( m' \geq m \), \( \gamma_1 \geq \gamma_1 \) and \( \gamma_2 \geq \gamma_2 \).

**Lemma 3.3 (Breizis-Lieb type result).** Let \( 1 \leq p < \infty \) and \( \{u_k\} \subset L^{p, \gamma_2}_p(\mathbb{M}) \). If the following conditions are satisfied:

(i) \( \{u_k\} \) is bounded in \( L^{p, \gamma_2}_p(\mathbb{M}) \),

(ii) \( u_k \rightharpoonup u \text{ a.e. in } \text{int} \mathbb{M}, \text{ as } k \to \infty \),

then

\[ \lim_{k \to \infty} (\|u_k\|^{p, \gamma_2}_L(\mathbb{M}) - \|u_k - u\|^{p, \gamma_2}_L(\mathbb{M})) = \|u\|^{p, \gamma_2}_L(\mathbb{M}). \]  

(3.7)

**Proof.** Due to the Fatou lemma, it holds that

\[ \|u\|^{p, \gamma_2}_L(\mathbb{M}) = \int_{\mathbb{M}} |x_1^{\frac{N}{p} - \gamma_1} x_2^{\frac{N}{p} - \gamma_2} u|^p \, d\sigma \]

\[ \leq \liminf_{k \to \infty} \int_{\mathbb{M}} |x_1^{\frac{N}{p} - \gamma_1} x_2^{\frac{N}{p} - \gamma_2} u_k|^p \, d\sigma = \liminf_{k \to \infty} \|u_k\|^{p, \gamma_2}_L(\mathbb{M}) < \infty. \]
For simplicity, we set here $\tilde{u}_k = x_1^{N-\gamma_1} x_2^{N-\gamma_2} u_k$ and $\tilde{u} = x_1^{N-\gamma_1} x_2^{N-\gamma_2} u$. Since $p > 1$, $j(t) = t^p$ is convex. For any fixed $\varepsilon > 0$, there exists a constant $c_\varepsilon$, such that

$$||\tilde{u}_k - \tilde{u} + \tilde{u}|^p + |\tilde{u}_k - \tilde{u}|^p| \leq \varepsilon ||\tilde{u}_k - \tilde{u}|^p + c_\varepsilon ||\tilde{u}|^p,$$

and then

$$||\tilde{u}_k - \tilde{u} + \tilde{u}|^p - |\tilde{u}_k - \tilde{u}|^p| \leq \varepsilon ||\tilde{u}_k - \tilde{u}|^p + (1 + c_\varepsilon) ||\tilde{u}|^p.$$

Therefore, we obtain

$$f_k^\varepsilon := (||\tilde{u}_k| - |\tilde{u}_k - \tilde{u}| - |\tilde{u}| - \varepsilon ||\tilde{u}_k - \tilde{u}|^p)^+ \leq (1 + c_\varepsilon) ||\tilde{u}|^p.$$

Then Lebesgue’s dominated convergence theorem induces

$$\lim_{k \to \infty} \int_M f_k^\varepsilon(x) d\sigma = \lim_{k \to \infty} \int_M f_k(x) d\sigma = 0.$$

Since

$$||x_1^{N-\gamma_1} x_2^{N-\gamma_2} u_k|^p - |x_1^{N-\gamma_1} x_2^{N-\gamma_2} u_k - x_1^{N-\gamma_1} x_2^{N-\gamma_2} u|^p| \leq f_k^\varepsilon + \varepsilon ||x_1^{N-\gamma_1} x_2^{N-\gamma_2} u_k - x_1^{N-\gamma_1} x_2^{N-\gamma_2} u|^p,$$

for any arbitrary small $\varepsilon$, it follows that

$$\limsup_{k \to \infty} \int_M ||x_1^{N-\gamma_1} x_2^{N-\gamma_2} u_k|^p - |x_1^{N-\gamma_1} x_2^{N-\gamma_2} u_k - u|^p| |x_1^{N-\gamma_1} x_2^{N-\gamma_2} (u_k - u)|^p |d\sigma \leq c \cdot \varepsilon,$$

where $c := \sup_M |x_1^{N-\gamma_1} x_2^{N-\gamma_2} (u_k - u)|^p |d\sigma$. It verifies the result.

By a direct calculation, one can derive that the energy functional

$$J(u) = \frac{1}{p} \int_M x_1 |\nabla_M u|^p d\sigma - \frac{\lambda}{q} \int_M x_1 (x_1 x_2)^p |u|^q d\sigma \in C^1 (H_{p,0}^{1,\frac{N-\gamma_1}{p}} \cap H_{p,0}^{1,\frac{N-\gamma_2}{p}} (M), \mathbb{R})$$

satisfies $J(0) = 0$ and $J(u) = J(-u)$ for any $u \in H_{p,0}^{1,\frac{N-\gamma_1}{p}} \cap H_{p,0}^{1,\frac{N-\gamma_2}{p}} (M)$.

**Lemma 3.4.** Let $p < q < p^*$. Then the functional

$$J(u) = \frac{1}{p} \int_M x_1 |\nabla_M u|^p d\sigma - \frac{\lambda}{q} \int_M x_1 (x_1 x_2)^p |u|^q d\sigma$$

verifies the (PS) condition in Definition 2.6.

**Proof.** Let $\{u_k(x)\} \in H_{p,0}^{1,\frac{N-\gamma_1}{p}} \cap H_{p,0}^{1,\frac{N-\gamma_2}{p}} (M)$ be a (PS) sequence. Then

$$J(u_k) = \frac{1}{q} \langle J'(u_k), u_k \rangle = \left( \frac{1}{p} - \frac{1}{q} \right) \int_M x_1 |\nabla_M u|^p d\sigma < \infty,$$

which implies that $\{\|u_k\|_{H_{p,0}^{1,\frac{N-\gamma_1}{p}} \cap H_{p,0}^{1,\frac{N-\gamma_2}{p}} (M)}\}$ is bounded. Hence

$$u_k \to u \quad \text{in} \quad H_{p,0}^{1,\frac{N-\gamma_1}{p}} \cap H_{p,0}^{1,\frac{N-\gamma_2}{p}} (M), \quad \text{as} \ k \to \infty,$$

and from it together with Lemma 3.1, it follows that

$$u_k \to u \quad \text{in} \quad L_{q_1}^{1,\gamma_2} (M), \quad \text{as} \ k \to \infty,$$

for $1 < q < p^*$ and $\frac{1}{p} < \frac{N}{q} - \gamma_1 < p + 1, 0 < \frac{N}{q} - \gamma_2 < p$. Let us calculate that

$$o(1) = \langle J'(u_k) - J'(u), u_k - u \rangle$$
Due to Hölder’s inequality, we derive that $I_2 \leq \lambda T_1 \cdot T_2$, with

$$T_1 := \left( \int_M \left| x_1^{\frac{\alpha}{\alpha - \gamma_1}} x_2^{\frac{\alpha}{\alpha - \gamma_2}} (u_k - u)^\gamma \right| d\sigma \right)^{\frac{1}{\gamma}},$$

$$T_2 := \left( \int_M \left| x_1^{p + 1 - \left( \frac{\alpha}{\alpha - \gamma_1} \right)} x_2^{p - \left( \frac{\alpha}{\alpha - \gamma_2} \right)} \left| |u_k|^{q - 2} u_k - |u|^{q - 2} u \right|^p \right| d\sigma \right)^{\frac{1}{p}}.$$

Since $\{u_k\}$ is bounded in $\mathcal{H}^{1,\frac{N-1}{p - \frac{2}{p}}}(M)$ and $u_k \to u$ in $L^\gamma_0(M)$, we derive that $T_1 \to 0$ and $T_2$ is bounded, which implies $I_2 \to 0$, as $k \to \infty$. Then we obtain that

$$I_1 = \int_M P_k(x) d\sigma \to 0,$$

(3.8)

where $P_k(x) = x_1(|\nabla_M u_k|^{p - 2} \nabla_M u_k - |\nabla_M u|^{p - 2} \nabla_M u_k)(\nabla_M u_k - \nabla_M u)(x)$. Here, denote the $i$-th component of $\nabla_M u$ by $(\nabla_M u)_i$. It is easy to verify that $P_k(x) > 0$; $P_k(x) > 0$, if $\nabla_M u_k \neq \nabla_M u$. In the following, we show that

$$(\nabla_M u_k)_i \to (\nabla_M u)_i, \quad \text{for } 1 \leq i \leq N, \quad \text{as } k \to \infty$$

(3.9)

a.e. in $\text{int} M$, which can be deduced by contradiction. Assume that there exist a point $x_p \in \text{int} M$ and its neighborhood $U_{x_p}$ such that for any $x_0 \in U_{x_p}$,

$$\lim_{k \to \infty} \nabla_M u_k(x_0) \neq \nabla_M u(x_0).$$

Since $x_1(|\nabla_M u_k|^{p - 2} \nabla_M u_k - |\nabla_M u|^{p - 2} \nabla_M u_k)(x_0)(\nabla_M u_k - \nabla_M u)_i(x_0) \leq c$, it follows that

$$x_1(|\nabla_M u_k|^{p - 2} \nabla_M u_k)_i(x_0)(\nabla_M u_k)_i(x_0) \leq c + x_1(|\nabla_M u_k|^{p - 2} + |\nabla_M u|^{p - 2})(x_0)(\nabla_M u_k)_i(x_0)(\nabla_M u)_i(x_0),$$

which indicates that $\{x_1|\nabla_M u(x_0)|^p\}$ is bounded. There exists a subsequence, here still denoted by $\{u_k\}$ such that

$$(\nabla_M u_k)_i(x_0) \to \xi_i', \neq \xi_i = \nabla_M u(x_0), \quad \text{as } k \to \infty.$$

This induces that

$$P_k(x_0) = x_1(|\nabla_M u_k|^{p - 2} \nabla_M u_k - |\nabla_M u|^{p - 2} \nabla_M u)(\nabla_M u_k - \nabla_M u)(x_0) \to c_0 > 0$$

for any $x_0 \in U_{x_p}$, as $k \to \infty$. It follows that

$$I_1 = \int_M P_k(x)d\sigma \to c \neq 0, \quad \text{as } k \to \infty,$$

which contradicts (3.8), and then (3.9) is obtained.

Applying Lemma 3.3 to $(\nabla_M u_k)_i$, for $1 \leq i \leq N$, we have

$$\lim_{k \to \infty} \left( \frac{\|\nabla_M u_k\|_{L^p(M)}^{\frac{\alpha}{\alpha - \gamma_1}}}{\|\nabla_M u\|_{L^p(M)}^{\frac{\alpha}{\alpha - \gamma_1}}} - \|\nabla_M u_k - \nabla_M u\|_{L^p(M)}^{\frac{\alpha}{\alpha - \gamma_1}} \right) = \|\nabla_M u\|_{L^p(M)}^{\frac{\alpha}{\alpha - \gamma_1}}.$$  

(10.10)

To the end, what is left is to show that

$$\int_M x_1|\nabla_M u_k|^p d\sigma \to \int_M x_1|\nabla_M u|^p d\sigma, \quad \text{as } k \to \infty.$$  

(10.11)
Due to the Egorov theorem, we obtain that for any $\delta > 0$, there exists a subset $E \subset \text{int} \mathcal{M}$ with the measure $m(E) < \delta$, such that

$$(\nabla M u_k)_i \to (\nabla M u)_i \quad \text{for} \quad 1 \leq i \leq N, \quad \text{as} \quad k \to \infty,$$

uniformly on $\text{int} \mathcal{M} \setminus E$. It follows that

$$\int_{\mathcal{M} \setminus E} x_1 |\nabla M u_k|^p d\sigma \to \int_{\mathcal{M} \setminus E} x_1 |\nabla M u|^p d\sigma, \quad \text{as} \quad k \to \infty. \quad (3.12)$$

Now we claim that for any $\varepsilon > 0$, there are $\delta(\varepsilon) > 0$, and a subset $E \subset \mathcal{M}$ with the measure $m(E) < \delta(\varepsilon)$, such that

$$\int_{E} x_1 |\nabla M u_k|^p d\sigma < \varepsilon. \quad (3.13)$$

In fact,

$$o(1) = I_1 = \int_{\mathcal{M}} x_1 |(\nabla M u_k)^{p-2}\nabla M u_k - |\nabla M u|^{p-2}\nabla M u)(\nabla M u_k - \nabla M u) d\sigma,$$

which implies that for any $E \subset \mathcal{M}$, we have

$$\int_{E} x_1 |\nabla M u_k|^p d\sigma \leq \int_{E} x_1 |\nabla M u_k|^{p-1}|\nabla M u| + x_1 |\nabla M u|^{p-1}|\nabla M u_k| + x_1 |\nabla M u_k|^p d\sigma + o(1). \quad (3.14)$$

By applying Hölder’s inequality to (3.14), we verify (3.13). Hence, for any $\varepsilon > 0$, there exist $\delta(\varepsilon) > 0$ and a subset $E \subset \text{int} \mathcal{B}$, such that both (3.12) and (3.13) hold. This gives (3.11). \hfill \Box

The following two propositions verify that the functional $J(u)$ satisfies the conditions (I_3)–(I_5) in Lemmas 2.9 and 2.10.

**Proposition 3.5.** If $p < q < p^*$, then there exists $r > 0$ such that

(i) $J(u) > 0$ if $0 < \|u\|_{H^{1,(\frac{N-1}{p^*}, \frac{N}{p})}_{p,0}} < r$, and $J(u) \geq \alpha > 0$ if $\|u\|_{H^{1,(\frac{N-1}{p^*}, \frac{N}{p})}_{p,0}} = r$;

(ii) there exists $v \in H^{1,(\frac{N-1}{p}, \frac{N}{p})}_{p,0}$ such that $\|v\|_{H^{1,(\frac{N-1}{p}, \frac{N}{p})}_{p,0}} > r$ and $J(v) < \alpha$.

**Proof.** According to both Lemma 3.1 and the condition $q < p^* < p(p+1)$, it holds that

$$J(u) \geq \frac{1}{p} \|u\|_{H^{1,(\frac{N-1}{p^*}, \frac{N}{p})}_{p,0}}^p - \frac{c\lambda}{q} \|u\|_{H^{1,(\frac{N-1}{p^*}, \frac{N}{p})}_{p,0}}^q = \|u\|_{H^{1,(\frac{N-1}{p^*}, \frac{N}{p})}_{p,0}}^p \left( \frac{1}{p} - \frac{c\lambda}{q} \|u\|_{H^{1,(\frac{N-1}{p^*}, \frac{N}{p})}_{p,0}}^{q-p} \right).$$

Let $r = \left( \frac{q}{2p\lambda} \right)^{\frac{1}{q-p}} > 0$. If $\|u\|_{H^{1,(\frac{N-1}{p^*}, \frac{N}{p})}_{p,0}} = r$, then

$$J(u) \geq \alpha = \frac{1}{2p} r^p > 0,$$

and if $0 < \|u\|_{H^{1,(\frac{N-1}{p^*}, \frac{N}{p})}_{p,0}} < r$, then

$$J(u) > \alpha > 0.$$

Then the condition (i) is proved. By setting $\|v\|_{H^{1,(\frac{N-1}{p^*}, \frac{N}{p})}_{p,0}} = r$ and $\theta > 0$, we know that

$$J(\theta v) \to -\infty \quad \text{as} \quad \theta \to \infty.$$

Therefore, by choosing a large enough positive constant $\theta_1$ such that $v = \theta_1 u$ and

$$\|v\|_{H^{1,(\frac{N-1}{p^*}, \frac{N}{p})}_{p,0}} > r,$$

one has that $J(v) < 0 < \alpha$, which implies the condition (ii). \hfill \Box
Recalling the notations in Section 2, we see that \( \{ E_m \}_{m \in \mathbb{N}} \) is a sequence of subspaces of \( \mathcal{H}^{1, \left( \frac{\alpha-1}{\rho}, \frac{\beta}{\rho} \right)}(\mathcal{M}) \), such that \( \dim(E_m) = m \) and \( E_m \subset E_{m+1} \); \( \mathcal{L}(\bigcup_{m \in \mathbb{N}} E_m) \) denotes the linear manifold generated by \( \bigcup_{m \in \mathbb{N}} E_m \) which is dense in \( \mathcal{H}^{1, \left( \frac{\alpha-1}{\rho}, \frac{\beta}{\rho} \right)}(\mathcal{M}) \). By \( E_m^c \) we denote the algebraical and topological complement of \( E_m \).

**Proposition 3.6.** Let \( E_m \subset \mathcal{H}^{1, \left( \frac{\alpha-1}{\rho}, \frac{\beta}{\rho} \right)}(\mathcal{M}) \) be defined as above. We have

\[
P_m = E_m \cap \{ u \in \mathcal{H}^{1, \left( \frac{\alpha-1}{\rho}, \frac{\beta}{\rho} \right)}(\mathcal{M}) | 0 \leq J(u) < +\infty \}
\]

is a bounded set.

**Proof.** For any \( u \in P_m \), the energy functional

\[
J(u) = \frac{1}{p} \int_{\mathcal{M}} x_1 |\nabla u|^p d\sigma - \frac{\lambda}{q} \int_{\mathcal{M}} x_1^{p+1} x_2^p |u|^q d\sigma
\]

\[
= \frac{1}{p} \| u \|_{\mathcal{H}^{1, \left( \frac{\alpha-1}{\rho}, \frac{\beta}{\rho} \right)}(\mathcal{M})}^p - \frac{\lambda}{q} \| u \|_{L_q}^q \left( \frac{\alpha-1}{p} + \frac{\beta}{q} \right).
\]

Set \( u = \| u \|_{\mathcal{H}^{1, \left( \frac{\alpha-1}{\rho}, \frac{\beta}{\rho} \right)}(\mathcal{M})} v =: \rho v \). Then we have

\[
+\infty > J(u) = J(\rho v) = \frac{1}{p} \rho^p - \frac{\lambda}{q} \rho^q \| v \|_{q, p}^q \left( \frac{\alpha-1}{p} + \frac{\beta}{q} \right).
\]

\[
= \rho^p \left( \frac{1}{p} - \frac{\lambda}{q} \| v \|_{q, p}^q \left( \frac{\alpha-1}{p} + \frac{\beta}{q} \right) \rho^{q-p} \right) \geq 0.
\]

Therefore, we have \( \| u \|_{\mathcal{H}^{1, \left( \frac{\alpha-1}{\rho}, \frac{\beta}{\rho} \right)}(\mathcal{M})} \) is bounded, and it follows that \( P_m \) is bounded.

Set

\[
A_0 = \{ u \in \mathcal{H}^{1, \left( \frac{\alpha-1}{\rho}, \frac{\beta}{\rho} \right)}(\mathcal{M}) | 0 \leq J(u) < +\infty \},
\]

\[
B = \{ u \in \mathcal{H}^{1, \left( \frac{\alpha-1}{\rho}, \frac{\beta}{\rho} \right)}(\mathcal{M}) | \| u \|_{\mathcal{H}^{1, \left( \frac{\alpha-1}{\rho}, \frac{\beta}{\rho} \right)}(\mathcal{M})} \leq 1 \},
\]

\[
\Gamma := \{ h \in \mathcal{C}(\mathcal{H}^{1, \left( \frac{\alpha-1}{\rho}, \frac{\beta}{\rho} \right)}(\mathcal{M}), \mathcal{H}^{1, \left( \frac{\alpha-1}{\rho}, \frac{\beta}{\rho} \right)}(\mathcal{M})) | h(0) = 0 ; h \text{ is an odd homeomorphism}; h(B) \subset A_0 \},
\]

\[
\Gamma_m := \{ K \subset \mathcal{H}^{1, \left( \frac{\alpha-1}{\rho}, \frac{\beta}{\rho} \right)}(\mathcal{M}) | K \text{ is compact}; K = -K; \gamma(K \cap h(\partial B)) \geq m, \forall h \in \Gamma \}.
\]

Combining Lemmas 2.9 and 2.10, we complete the proof of Theorem 1.1.

### 3.2 The proof of Theorem 1.2

In this proof, the following definition and lemma will be employed.

**Definition 3.7.** Define the manifold \( M \) as follows:

\[
M = \left\{ u \in \mathcal{H}^{1, \left( \frac{\alpha-1}{\rho}, \frac{\beta}{\rho} \right)}(\mathcal{M}) \setminus \{ 0 \} \mid \| u \|_{\mathcal{H}^{1, \left( \frac{\alpha-1}{\rho}, \frac{\beta}{\rho} \right)}(\mathcal{M})}^p = \lambda \int_{\mathcal{M}} x_1^{p+1} x_2^p |u|^q d\sigma \right\}.
\]

**Lemma 3.8.** For any \( u \in \mathcal{H}^{1, \left( \frac{\alpha-1}{\rho}, \frac{\beta}{\rho} \right)}(\mathcal{M}) \setminus \{ 0 \} \), there exists a unique

\[
\beta := \beta(u) \geq 0 \text{ such that } \beta u \in M.
\]

The maximum of \( J(\beta u) \) for \( \beta \geq 0 \) is achieved at \( \beta = \beta(u) > 0 \). The function \( u \mapsto \beta = \beta(u) \) is continuous.
Proof. Let \( u \in \mathcal{H}^{1,(\frac{N-1}{p}, \frac{N}{p})}_{p,0}(M) \setminus \{0\} \) be fixed. Define \( g(\beta) := J(\beta u) \) on \([0, \infty)\). Then it follows that

\[
g'(\beta) = 0 \Leftrightarrow \beta u \in M \Leftrightarrow \|u\|^p_{\mathcal{H}^{1,(\frac{N-1}{p}, \frac{N}{p})}_{p,0}(M)} = \frac{1}{\beta^p} \lambda \int_M x_1^{p+1} x_2^p |\beta u|^q d\sigma. \tag{3.15}
\]

It is obvious that \( g(0) = 0 \), and

\[
g(\beta) > 0 \quad \text{for } \beta > 0 \text{ small enough,}
\]

\[
g(\beta) < 0 \quad \text{for } \beta > 0 \text{ large.}
\]

Therefore, \( \max_{[0, \infty)} g(\beta) \) is achieved at a unique \( \beta = \beta(u) \) such that

\[
g'(\beta) = 0 \quad \text{and} \quad \beta u \in M.
\]

To prove the continuity of \( \beta(u) \), let us assume that

\[
u_n \to u \quad \text{in} \quad \mathcal{H}^{1,(\frac{N-1}{p}, \frac{N}{p})}_{p,0}(M) \setminus \{0\}.
\]

Then \( \{\beta(u_n)\} \) is bounded. If a subsequence of \( \{\beta(u_n)\} \) converges to \( \beta_0 \), then it follows from the right-hand side of (3.15) that \( \beta_0 = \beta(u) \).

By Definition 3.7, there exists \( r > 0 \), such that

\[
\int_M x_1^{p+1} x_2^p |u|^q d\sigma > r \quad \text{for any} \quad u \in M. \tag{3.16}
\]

Indeed, if \( u \in M \), then by Lemma 3.1, it follows

\[
\|u\|^p_{\mathcal{H}^{1,(\frac{N-1}{p}, \frac{N}{p})}_{p,0}(M)} = \lambda \|u\|^q_{L_q(\frac{N-1}{p}, \frac{N}{p})} \leq c\lambda \|u\|^q_{\mathcal{H}^{1,(\frac{N-1}{p}, \frac{N}{p})}_{p,0}(M)}. \tag{3.17}
\]

For \( q > p \), then it follows

\[
\|u\|^p_{\mathcal{H}^{1,(\frac{N-1}{p}, \frac{N}{p})}_{p,0}(M)} \geq \left( \frac{1}{c\lambda} \right)^{\frac{q}{p}}.
\]

By setting \( r = \frac{1}{2\lambda} (\frac{\lambda}{c})^{\frac{q}{p}}, \) we find (3.16) holds.

Let

\[
d_m = \inf_{u \in M \cap E_m^c} \{\|u\|_{\mathcal{H}^{1,(\frac{N-1}{p}, \frac{N}{p})}_{p,0}(M)} \mid u \in M \cap E_m^c\}.
\]

Then we claim that

\[
d_m \to \infty \quad \text{as} \quad m \to \infty. \tag{3.18}
\]

In fact, if there exist \( d > 0 \) and \( u_m \in M \cap E_m^c \), such that \( \|u_m\|_{\mathcal{H}^{1,(\frac{N-1}{p}, \frac{N}{p})}_{p,0}(M)} \leq d \) for all \( m \in \mathbb{N}_+ \), then there exists \( u \in \mathcal{H}^{1,(\frac{N-1}{p}, \frac{N}{p})}_{p,0}(M) \), such that \( u_m \rightharpoonup u \) in \( \mathcal{H}^{1,(\frac{N-1}{p}, \frac{N}{p})}_{p,0}(M) \). Since \( u_m \in E_m^c \) and \( L(\bigcup_{m \in \mathbb{N}} E_m) \) is dense in \( \mathcal{H}^{1,(\frac{N-1}{p}, \frac{N}{p})}_{p,0}(M) \), we have \( u = 0 \). According to Lemma 3.1, it follows that

\[
u_m \to 0 \quad \text{in} \quad L_q(\frac{N-1}{p}, \frac{N}{p})(M).
\]

This is a contradiction to (3.16). That means \( d_m \) will be unbounded as \( m \to \infty \), which proves the claim (3.18).

Next, for some \( R > 1 \), we define a homeomorphism

\[
h_m = R^{-1}d_m u : E_m^c \to E_m^c. \tag{3.19}
\]
By Lemma 3.8, let $\beta := \beta(u)$ such that $\beta u \in M$. Set

$$
B = \{ u \in \mathcal{H}_{p,0} \cap \mathcal{M} | \| u \|_{\mathcal{H}_{p,0}^{(1,\frac{n-1}{p})}} \leq 1 \}.
$$

For $u_1 \in E_0^c \cap B$, $u_1 \neq 0$ and $R > 1$, we have

$$
R^{-1}d_m < d_m = \inf \{ \| u \|_{\mathcal{H}_{p,0}^{(1,\frac{n-1}{p})}} | u \in M \cap E_0^c \} \leq \| \beta u_1 \|_{\mathcal{H}_{p,0}^{(1,\frac{n-1}{p})}} \leq \beta := \beta(u_1).
$$

(3.20)

It follows that

$$
h_m(E_0^c \cap B) \subset A_0 := \{ u \in \mathcal{H}_{p,0}^{(1,\frac{n-1}{p})} | 0 \leq J(u) < +\infty \}.
$$

(3.21)

In fact, if $u \in E_0^c \cap B$, with $\beta$ chosen as above, such that $\beta u \in M$ and $d_m \leq \beta$, then

$$
J(h_m(u)) = \frac{1}{p} (R^{-1}d_m)^p \| u \|_{\mathcal{H}_{p,0}^{(1,\frac{n-1}{p})}}^p - \lambda \frac{R^{-1}d_m}{\beta} \| \beta u \|_{\mathcal{H}_{p,0}^{(1,\frac{n-1}{p})}}
$$

$$
= \frac{1}{p} (R^{-1}d_m)^p \| u \|_{\mathcal{H}_{p,0}^{(1,\frac{n-1}{p})}}^p - \frac{1}{q} \frac{(R^{-1}d_m)^q}{\beta} \| \beta u \|_{\mathcal{H}_{p,0}^{(1,\frac{n-1}{p})}}^q.
$$

(3.22)

Then let $R$ be large enough, and it gives $J(h_m(u)) \geq 0$, which proves (3.21).

Therefore, we can define

$$
\tilde{h}_m(u) = \begin{cases} h_m(u), & \text{if } u \in E_0^c, \\ \varepsilon_j, & \text{for } j = 1, 2, \ldots, m \text{ and } \{ \varepsilon_j \}_{j=1}^m \text{ is a basis of } E_0^c, \end{cases}
$$

if $u \in E_0^c$, and

$$
\tilde{h}_m(u) = \begin{cases} \varepsilon_j, & \text{if } u \in E_0^c, \\ u, & \text{if } u \in E_0^c, \end{cases}
$$

for $\varepsilon$ small enough. In this way, it is shown that for $R$ large enough, the mapping $h_m$ in (3.19) defined on $E_0^c$ admits an extension $\tilde{h}_m \in \Gamma$ for each $m$. Finally, we take $u \in \partial B \cap E_0^c$. Then

$$
J(\tilde{h}_m(u)) = (R^{-1}d_m)^p \left( \frac{1}{p} - \frac{1}{q} \frac{(R^{-1}d_m)^q}{\beta} \right) \| u \|_{\mathcal{H}_{p,0}^{(1,\frac{n-1}{p})}}^p.
$$

(3.23)

where the calculus in (3.23) is the same as that in (3.22). Since $d_m \leq \beta := \beta(u)$ proved in (3.20), we choose $R$ large enough to deduce that

$$
J(\tilde{h}_m(u)) \geq \frac{1}{2p} (R^{-1}d_m)^p \to \infty \text{ as } m \to \infty.
$$

Since $c_m$ is a critical value sequence of the functional $J$ which is given in Lemma 2.10, we have $c_m \to \infty$ as $m \to \infty$. Theorem 1.2 is proved.

4 The case of $p = q$

4.1 The proof of Theorem 1.3

The idea of Lusternik-Schnirelman theory in [1] is adapted here for the proof. Consider the following two operators:

$$
B(u) = \frac{1}{p} \int_M x_1^{p+1} x_2^{p-1} |u|^p d\sigma : \mathcal{H}_{p,0}^{(1,\frac{n-1}{p})}(\mathcal{M}) \to \mathbb{R},
$$

(4.1)

$$
b(u) = x_1^{p+1} x_2^{p-2} u : \mathcal{H}_{p,0}^{(1,\frac{n-1}{p})}(\mathcal{M}) \to \mathcal{H}_{p}^{-1,(-\frac{n-1}{p} - \frac{1}{p})}(\mathcal{M}),
$$

(4.2)

where $\mathcal{H}_p^{-1,(-\frac{n-1}{p} - \frac{1}{p})}(\mathcal{M})$ is the dual space of $\mathcal{H}_{p,0}^{(1,\frac{n-1}{p})}(\mathcal{M})$ with the norm as follows:

$$
\| g \|_{\mathcal{H}_p^{-1,(-\frac{n-1}{p} - \frac{1}{p})}} = \sup_{\varphi} \frac{|(g, \varphi)|}{\| \varphi \|_{\mathcal{H}_{p,0}^{(1,\frac{n-1}{p})}}},
$$

where $(\cdot, \cdot)$ is the duality pairing.
Lemma 4.1. We have the following properties of the above two operators.

(i) The operator $b$ defined in (4.2) is odd, compact and uniformly continuous in bounded sets.

(ii) The functional $B$ defined in (4.1) is even and compact.

Proof. It is obvious that $B$ is even and $b$ is odd. First, we verify the uniform continuity of $b$ in the bounded set. Let $u_1$ and $u_0$ be in the bounded set in $H^{1, (\frac{N-1}{p}, \frac{N}{p})}_p (\mathbb{M})$, and set $\delta := u_1 - u_0 \in H^{1, (\frac{N-1}{p}, \frac{N}{p})}_p (\mathbb{M})$. Then for any $\varphi \in H^{1, (\frac{N-1}{p}, \frac{N}{p})}_p (\mathbb{M})$ we have

$$|\langle b(u_1) - b(u_0), \varphi \rangle| = \left| \int_{\mathbb{M}} x_1^{p+1} x_2^p (|u_0| + \delta)^{p-2}(u_0 + \delta) - |u_0|^{p-2} u_0 \varphi d\sigma \right|,$$

where the binomial theorem implies that

$$|u_0 + \delta|^{p-2}(u_0 + \delta) - |u_0|^{p-2} u_0$$

$$\leq \sum_{l=1}^{\frac{p-1}{2}} C_p - 2 u_0^{p-1} \delta^l + u_0^{p-2} \delta^l (u_0 + \delta) - |u_0|^{p-2} u_0$$

$$\leq \sum_{l=1}^{\frac{p-1}{2}} C_p - 2 u_0^{p-1} \delta^l + \left| \int_{\mathbb{M}} x_1^{p-1} x_2^{p-2} \varphi d\sigma \right| \leq C \sum_{l=1}^{\frac{p-1}{2}} |u_0^{p-1} \delta^l|.$$

Then by applying Hölder’s inequality and Lemma 3.1, we see that

$$|\langle b(u_1) - b(u_0), \varphi \rangle| \leq C \sum_{l=1}^{\frac{p-1}{2}} \left( \int_{\mathbb{M}} x_1^{p-1} x_2^{p-2} |u_0|^{p-1} \delta^l \varphi d\sigma \right) \leq C \sum_{l=1}^{\frac{p-1}{2}} \left( \int_{\mathbb{M}} x_1^{p-1} x_2^{p-2} \varphi d\sigma \right) \leq C \sum_{l=1}^{\frac{p-1}{2}} \left( \int_{\mathbb{M}} x_1^{p-1} x_2^{p-2} \varphi d\sigma \right).$$

Due to the assumption that $u_1$ and $u_2$ are in the bounded set and $\delta = u_1 - u_0$, we have

$$\|b(u_1) - b(u_2)\|_{H^{1, (\frac{N-1}{p}, \frac{N}{p})}_p} := \sup_{\varphi} \frac{|\langle b(u_1) - b(u_0), \varphi \rangle|}{\|\varphi\|_{H^{1, (\frac{N-1}{p}, \frac{N}{p})}_p}} \leq C \sum_{l=1}^{\frac{p-1}{2}} \|u_1 - u_0\|_{H^{1, (\frac{N-1}{p}, \frac{N}{p})}_p},$$

which verifies the uniform continuity of $b$ in the bounded set.

Now we show that $b$ is a compact operator. If $\{u_k\}$ is bounded in $H^{1, (\frac{N-1}{p}, \frac{N}{p})}_p (\mathbb{M})$, then there exists a subsequence of $\{u_k\}$ such that

$$u_k \rightharpoonup u \quad \text{in} \quad H^{1, (\frac{N-1}{p}, \frac{N}{p})}_p (\mathbb{M}), \quad \text{as} \quad k \to \infty.$$

By proper choices of $\gamma_1$ and $\gamma_2$, Lemma 3.1 implies that

$$u_k \to u \quad \text{in} \quad L^{\gamma_1, \gamma_2}(\mathbb{M}) \quad \text{as} \quad k \to \infty.$$

Then we claim that there exists a subsequence satisfying that

$$x_1^{\frac{N}{p} - \gamma_1} x_2^{\frac{N}{p} - \gamma_2} u_k \to x_1^{\frac{N}{p} - \gamma_1} x_2^{\frac{N}{p} - \gamma_2} u \quad \text{a.e. in} \quad \text{int} \mathbb{M}.$$
In fact, there is a subsequence \( \{u_{k_j}\} \) such that
\[
\|u_{k_{j+1}} - u_{k_j}\|_{L^{p_1}_{\gamma_1}} \leq \frac{1}{2^j} \quad \text{for } j = 1, 2, \ldots
\]
Let
\[
x_1^{\frac{N}{p} - \gamma_1} x_2^{\frac{N}{p} - \gamma_2} v_k = \sum_{j=1}^{k} \left( x_1^{\frac{N}{p} - \gamma_1} x_2^{\frac{N}{p} - \gamma_2} u_{k_{j+1}} - x_1^{\frac{N}{p} - \gamma_1} x_2^{\frac{N}{p} - \gamma_2} u_{k_j} \right).
\]
Then Minkowski’s inequality gives
\[
\|tv_k\|_{L^{p_1}_{\gamma_1}} \leq \sum_{j=1}^{k} \|u_{k_{j+1}} - u_{k_j}\|_{L^{p_1}_{\gamma_1}} \leq 1.
\]
We set
\[
x_1^{\frac{N}{p} - \gamma_1} x_2^{\frac{N}{p} - \gamma_2} v(x) = \lim_{k \to \infty} x_1^{\frac{N}{p} - \gamma_1} x_2^{\frac{N}{p} - \gamma_2} v_k(x).
\]
By the Fatou lemma, it follows that
\[
\int_M |x_1^{\frac{N}{p} - \gamma_1} x_2^{\frac{N}{p} - \gamma_2} v(x)|^p d\sigma \leq \liminf_{k \to \infty} \int_M |x_1^{\frac{N}{p} - \gamma_1} x_2^{\frac{N}{p} - \gamma_2} v_k(x)|^p d\sigma \leq 1.
\]
The absolute convergence implies that
\[
x_1^{\frac{N}{p} - \gamma_1} x_2^{\frac{N}{p} - \gamma_2} u_{k_1} + \sum_{j=1}^{k} \left( x_1^{\frac{N}{p} - \gamma_1} x_2^{\frac{N}{p} - \gamma_2} u_{k_{j+1}} - x_1^{\frac{N}{p} - \gamma_1} x_2^{\frac{N}{p} - \gamma_2} u_{k_j} \right)
\]
\[
\to x_1^{\frac{N}{p} - \gamma_1} x_2^{\frac{N}{p} - \gamma_2} u(x) \quad \text{a.e. in } M,
\]
which verifies the claim (4.6).

Then for any \( v \in H^{1,\frac{N-1}{p-1}}_{p,0}(M) \), we choose the same \( \gamma_1 \) and \( \gamma_2 \) as in (4.6), such that Lemma 3.1 can be applied, and it follows that
\[
|\langle b(u_k) - b(u), v \rangle| \leq \left( \int_M |x_1^{\frac{N}{p} - \gamma_1} x_2^{\frac{N}{p} - \gamma_2} v|^p d\sigma \right)^{\frac{1}{p}}
\]
\[
\times \left( \int_M |x_1^{p+1-(\frac{N}{p} - \gamma_1)} x_2^{p-(\frac{N}{p} - \gamma_2)} (|u_k|^{p-2} u_k - |u|^{p-2} u)| \right)^{\frac{p-1}{p}} d\sigma
\]
\[
\leq C\|v\|_{H^{1,\frac{N-1}{p-1}}_{p,0}(\frac{N}{p}, \frac{N}{p})}
\]
\[
\times \left( \int_M |x_1^{p+1-(\frac{N}{p} - \gamma_1)} x_2^{p-(\frac{N}{p} - \gamma_2)} (|u_k|^{p-2} u_k - |u|^{p-2} u)| \right)^{\frac{p-1}{p}} d\sigma.
\]
Due to the claim (4.6), we have that
\[
x_1^{\frac{N}{p} - \gamma_1} x_2^{\frac{N}{p} - \gamma_2} u_{k_1} \to x_1^{\frac{N}{p} - \gamma_1} x_2^{\frac{N}{p} - \gamma_2} u \quad \text{a.e. in } \text{int} M, \quad \text{as } k \to \infty.
\]
Then we apply Lebesgue’s dominated convergence theorem to (4.7), and then get the compactness of the operator \( b \).

For the compactness of the operator \( B \), we take a bounded sequence \( \{u_k\} \) in \( H^{1,\frac{N-1}{p-1}}_{p,0}(\frac{N}{p}, \frac{N}{p}) (M) \), and then as before, up to a subsequence we have
\[
u_k \to u \quad \text{in } L^{p_{\frac{N-1}{p-1}}}_{p_{\frac{N-1}{p-1}}}(M), \quad \text{as } k \to \infty.
\]
Then
\[
B(u_k) = \frac{1}{p} \|u_k\|_{L^{p_{\frac{N-1}{p-1}}}_{p_{\frac{N-1}{p-1}}}} \to \frac{1}{p} \|u\|_{L^{p_{\frac{N-1}{p-1}}}_{p_{\frac{N-1}{p-1}}}} = B(u).
\]
This completes the proof.
The main idea of the proof is to obtain the critical points of \( B(u) \) on the manifold

\[
M = \left\{ u \in \mathcal{H}_{p,0}^{1,\frac{N-1}{p},\frac{p}{2}}(\mathcal{M}) \mid \frac{1}{p} \int_{\mathcal{M}} x_1 |\nabla_M u|^p d\sigma = \alpha \right\},
\]

(4.9)

where \( \alpha > 0 \) is fixed. For each \( u \in \mathcal{H}_{p,0}^{1,\frac{N-1}{p},\frac{p}{2}}(\mathcal{M}) \setminus \{0\} \), we can find \( \lambda(u) > 0 \) such that \( \lambda(u)u \in M \) in the following way:

\[
\lambda(u) = \left( \frac{p \alpha}{\int_{\mathcal{M}} x_1 |\nabla_M u|^p d\sigma} \right)^{\frac{1}{p}}.
\]

(4.10)

Hence

\[
\lambda : \mathcal{H}_{p,0}^{1,\frac{N-1}{p},\frac{p}{2}}(\mathcal{M}) \setminus \{0\} \to (0, +\infty).
\]

It is obvious that \( \lambda(u) \) is uniformly continuous on manifold \( M \). By a direct computation, the derivative of \( \lambda \) is as follows:

\[
\langle \lambda'(u), \varphi \rangle = -(p\alpha)^{\frac{1}{p}} \left( \int_{\mathcal{M}} x_1 |\nabla_M u|^p d\sigma \right)^{\frac{p-1}{p}} \int_{\mathcal{M}} x_1 |\nabla_M u|^{p-2} \nabla_M u \cdot \nabla_M \varphi d\sigma
\]

(4.11)

for any \( \varphi \in \mathcal{H}_{p,0}^{1,\frac{N-1}{p},\frac{p}{2}}(\mathcal{M}) \). Therefore,

\[
\int_{\mathcal{M}} x_1 |\nabla_M u|^{p-2} \nabla_M u \cdot \nabla_M \varphi d\sigma = 0
\]

implies \( \langle \lambda(u), \varphi \rangle = 0 \).

**Lemma 4.2.** The functional \( \lambda(\cdot) \) is uniformly continuous on \( M \).

**Proof.** Let \( u_1 \) and \( u_0 \) be in \( M \) defined in (4.9), and set

\[
u_1 - u_0 =: \delta \in \mathcal{H}_{p,0}^{1,\frac{N-1}{p},\frac{p}{2}}(\mathcal{M}).
\]

For any \( \varphi \in \mathcal{H}_{p,0}^{1,\frac{N-1}{p},\frac{p}{2}}(\mathcal{M}) \), we apply the binomial theorem and Lemma 3.1 as we did in (4.3) and (4.4), and it follows that

\[
|\langle \lambda'(u_1) - \lambda'(u_0), \varphi \rangle| \leq C \sum_{l=1}^{p} \|\delta\|^l_{\mathcal{H}_{p,0}^{1,\frac{N-1}{p},\frac{p}{2}}} \|\nabla_M u_0\|^p_{\mathcal{H}_{p,0}^{1,\frac{N-1}{p},\frac{p}{2}}} \|\varphi\|^l_{\mathcal{H}_{p,0}^{1,\frac{N-1}{p},\frac{p}{2}}}
\]

which, as (4.5), leads to the uniform continuity of \( \lambda(\cdot) \) in \( M \).

The next step is to construct a flow on \( M \) (defined in (4.9)) related to the functional \( B(u) \) and the corresponding deformation result allows us to apply the min-max theory (see [14]). Let \( D(u) \) denote the derivative of \( B(\lambda(u)u) \) for \( u \in \mathcal{H}_{p,0}^{1,\frac{N-1}{p},\frac{p}{2}}(\mathcal{M}) \setminus \{0\} \). Then we have for any \( v \in \mathcal{H}_{p,0}^{1,\frac{N-1}{p},\frac{p}{2}}(\mathcal{M}) \),

\[
\langle D(u), v \rangle = \frac{p \alpha}{\int_{\mathcal{M}} x_1 |\nabla_M u|^p d\sigma} \langle b(u), v \rangle - \frac{\langle b(u), u \rangle}{\int_{\mathcal{M}} x_1 |\nabla_M u|^p d\sigma} \int_{\mathcal{M}} x_1 |\nabla_M u|^{p-2} \nabla_M u \cdot \nabla_M v d\sigma,
\]

where \( D(u) \in \mathcal{H}_{p,0}^{1,\frac{N-1}{p},\frac{p}{2}}(\mathcal{M}) \). If \( u \in M \), then

\[
\langle D(u), v \rangle = \langle b(u), v \rangle - \frac{\langle b(u), u \rangle}{\int_{\mathcal{M}} x_1 |\nabla_M u|^p d\sigma} \int_{\mathcal{M}} x_1 |\nabla_M u|^{p-2} \nabla_M u \cdot \nabla_M v d\sigma.
\]

We claim that \( D(u) \) is uniformly continuous in \( M \). As proved in Lemmas 4.1 and 4.2, \( b(u) \) and

\[
\int_{\mathcal{M}} x_1 |\nabla_M u|^{p-2} \nabla_M u \cdot \nabla_M v d\sigma
\]
are uniformly continuous on $M$. Then it is sufficient to verify that $\langle b(u), u \rangle$ satisfies this property on $M$. In fact, let $u_1, u_0 \in M$, and set
\[ \delta := u_1 - u_0 \in H^{1, \frac{N-1}{p}, \frac{N}{p}}(M). \]
Applying the binomial theorem, Hölder’s inequality and Lemma 3.1 as in (4.3) and (4.4), we obtain that
\[ |\langle b(u_1), u_1 \rangle - \langle b(u_0), u_0 \rangle| \leq C \sum_{i=1}^{p} \|u_0\|_{H^{1, \frac{N-1}{p}, \frac{N}{p}}(M)}^\ell \|u_1 - u_0\|_{H^{1, \frac{N-1}{p}, \frac{N}{p}}(M)}^i, \]
which implies the uniform continuity of $\langle b(u), u \rangle$ and $D(u)$ on $M$. Recall the definition of the duality map.

**Definition 4.3** (See [3, Chapter 1]). Let $E$ be the normed vector space, and $E^*$ be the dual space of $E$. We set, for every $x_0 \in E$,
\[ \mathcal{J}(x_0) = \{ f_0 \in E^* \mid \|f_0\|_{E^*} = \|x_0\|_E \text{ and } \langle f_0, x_0 \rangle = \|x_0\|^2 \}. \]
The map $x_0 \mapsto \mathcal{J}(x_0)$ is called the duality map from $E$ into $E^*$.

Here define the duality map
\[ \mathcal{J} : H^{1, \frac{N-1}{p}, \frac{N}{p}}(-\delta, \delta)(M) \to H^{1, \frac{N-1}{p}, \frac{N}{p}}(M) \] (4.12)
for all $f \in H^{1, \frac{N-1}{p}, \frac{N}{p}}(-\delta, \delta)(M)$, such that $\mathcal{J}$ verifies
\begin{enumerate}
  \item $\|\mathcal{J}(f)\|_{H^{1, \frac{N-1}{p}, \frac{N}{p}}(M)} = \|f\|_{H^{1, \frac{N-1}{p}, \frac{N}{p}}(M)}$,
  \item $\langle f, \mathcal{J}(f) \rangle = \|f\|^2_{H^{1, \frac{N-1}{p}, \frac{N}{p}}(M)}$,
  \item $\mathcal{J}(\cdot)$ is uniformly continuous in bounded sets.
\end{enumerate}
For each $u \in M$, we define the tangent component as follows:
\[ T(u) = \mathcal{J}(D(u)) - \frac{\int_M x_1 |\nabla_M u|^{p-2} \nabla_M u \cdot \nabla_M (\mathcal{J}(D(u)))d\sigma}{\int_M x_1 |\nabla_M u|^p d\sigma} u \] (4.13)
such that $T : M \to H^{1, \frac{N-1}{p}, \frac{N}{p}}(M)$ and
\[ \int_M x_1 |\nabla_M u|^{p-2} \nabla_M u \cdot \nabla_M (T(u))d\sigma = 0, \]
which implies that if $u \in M$ then
\[ \langle \lambda'(u), T(u) \rangle = 0. \] (4.14)

**Lemma 4.4.** The tangent component $T(u)$ holds the following properties:
\begin{enumerate}
  \item $T(u)$ is odd,
  \item $T(u)$ is uniformly continuous on $M$,
  \item $T(u)$ is bounded on $M$.
\end{enumerate}

**Proof.** According to the definition of the duality map and the fact that $D(u)$ is odd, we have that $T(u)$ is odd. Since both $D(\cdot)$ and $\mathcal{J}(\cdot)$ are uniformly continuous in bounded sets, one can deduce that $T(u)$ is uniformly continuous on $M$ by applying a very similar procedure as in (4.3) and (4.4).

On the manifold $M$, by (4.13), we have
\[ \|T(u)\|_{H^{1, \frac{N-1}{p}, \frac{N}{p}}(M)} \leq I_1 + I_2 \]
with
\[ \begin{align*}
I_1 &= \|\mathcal{J}(D(u))\|_{H^{1, \frac{N-1}{p}, \frac{N}{p}}(M)}, \\
I_2 &= \frac{\int_M x_1 |\nabla_M u|^{p-2} \nabla_M u \cdot \nabla_M (\mathcal{J}(D(u)))d\sigma}{\int_M x_1 |\nabla_M u|^p d\sigma} \|u\|_{H^{1, \frac{N-1}{p}, \frac{N}{p}}(M)}. 
\end{align*} \]
By applying Hölder’s inequality and Lemma 3.1, we obtain that
\[
I_1 = \|J(D(u))\|_{\mathcal{H}^p,0}^{\frac{1}{2},\frac{1}{2}} = \|D(u)\|_{\mathcal{H}^p,0}^{\frac{1}{2},\frac{1}{2}} \leq C\|u\|_{H^{\frac{N-1}{p}-\frac{N}{p}}}^{\frac{1}{2},\frac{1}{2}}.
\]
\[
I_2 \leq \frac{\|u\|^{p-1}_{H^{\frac{N-1}{p}-\frac{N}{p}},0}}{H^{\frac{N-1}{p}-\frac{N}{p}}} \|J(D(u))\|_{\mathcal{H}^p,0}^{\frac{1}{2},\frac{1}{2}} = \|D(u)\|_{H^{\frac{N-1}{p}-\frac{N}{p}},0}^{\frac{1}{2},\frac{1}{2}} \leq C\|u\|_{H^{\frac{N-1}{p}-\frac{N}{p}}}^{p-1}.
\]

Then we have that \(T(u)\) is bounded on \(M\).

For all \(u \in M\), there exist \(\gamma_0 > 0\) and \(t_0 > 0\) such that for all \((u,t) \in M \times [-t_0,t_0]\) it holds that
\[
\|u + tT(u)\|_{H^{\frac{N-1}{p}}_{p,0}}^{\frac{1}{2},\frac{1}{2}} \geq \gamma_0 > 0.
\]

As a consequence we define the flow
\[
\sigma(u,t) := \lambda(u + tT(u)) (u + tT(u)) : M \times [-t_0,t_0] \rightarrow M.
\]

Then \(\sigma(u,t)\) verifies the following properties:

(i) \(\sigma(u,t)\) is odd with respect to \(u\) for fixed \(t\);
(ii) \(\sigma(u,t)\) is uniformly continuous with respect to \(u\) on \(M\);
(iii) \(\sigma(u,0) = u\) for \(u \in M\).

Indeed, it is obvious that the properties (i) and (iii) of \(\sigma(u,t)\) hold. The uniform continuity of \(\sigma(u,t)\) can be induced from the uniform continuity of both \(\lambda(\cdot)\) and \(T(\cdot)\).

In order to obtain the deformation result, we first discover the relation between the functional \(B(u)\) and the flow \(\sigma(u,t)\) on \(M\).

**Lemma 4.5.** Let \(\sigma(u,t)\) be defined in (4.15). Then there exists
\[
r : M \times [-t_0,t_0] \rightarrow \mathbb{R}
\]
such that \(\lim_{\tau \to 0} r(u,\tau) = 0\) uniformly on \(M\) and
\[
B(\sigma(u,t)) - B(u) = \int_0^t (\|D(u)\|_{H^{\frac{N-1}{p}}_{p,0}}^{\frac{1}{2},\frac{1}{2}} + r(u,s))ds
\]
for all \(u \in M\), where \(t \in [-t_0,t_0]\).

**Proof.** Since \(\sigma(u,0) = u\), we have \(B(u) = B(\sigma(u,0))\). By the definitions of the functional \(B\) in (4.1) and the operator \(b\) in (4.2), for any \(v \in H^{\frac{N-1}{p}}_{p,0}(\mathbb{B})\), we have
\[
\langle B'(u),v \rangle = \langle b(u),v \rangle.
\]

Hence,
\[
B(\sigma(u,t)) - B(u) = \int_0^t \langle b(\sigma(u,s)),\partial_s \sigma(u,s) \rangle ds.
\]
Due to the fact that \(\langle \lambda'(u),T(u) \rangle = 0\) in (4.14) and \(\lambda(u) = 1\) on \(M\) by (4.9), one can derive
\[
\partial_s \sigma(u,s) = \partial_s (\lambda(u + sT(u))(u + sT(u)))
\]
\[
= \langle \lambda'(u + sT(u)), T(u) \rangle (u + sT(u)) + \lambda(u + sT(u))T(u)
\]
\[
= \langle \lambda'(u + sT(u)) - \lambda'(u), T(u) \rangle (u + sT(u)) + (\lambda(u + sT(u)) - \lambda(u))T(u) + T(u)
\]
\[
=: R(u,s) + T(u),
\]
where

\[ R(u, s) = \langle \lambda'(u + sT(u)) - \lambda'(u), T(u) \rangle (u + sT(u)) + (\lambda(u + sT(u)) - \lambda(u))T(u). \]

Because \( T \) is bounded on \( M \), and both \( \lambda(u) \) and \( \lambda'(u) \) are uniformly continuous, we have \( \lim_{s \to 0} R(u, s) = 0 \) uniformly on \( M \). Therefore,

\[
B(\sigma(u, t)) - B(u) = \int_0^t \langle b(\sigma(u, s)), R(u, s) + T(u) \rangle ds
\]

\[
= \int_0^t \langle b(u), T(u) \rangle + r(u, s) ds,
\]

where \( r(u, s) = \langle b(\sigma(u, s)) - b(u), R(u, s) + T(u) \rangle + \langle b(u), R(u, s) \rangle \).

Since \( b \) is uniformly continuous as proved in Lemma 4.1, the properties that \( \lim_{s \to 0} \sigma(u, s) = u \) and \( \lim_{s \to 0} R(u, s) = 0 \) lead to

\[
\lim_{s \to 0} r(u, s) = 0
\]

uniformly on \( M \). Moreover, a direct computation implies that

\[
\langle b(u), T(u) \rangle = \langle b(u), J(D(u)) \rangle - \frac{\langle b(u), \int_M x_1|\nabla_M u|^{p-2}\nabla_M u \cdot \nabla_M (J(D(u)))ds \rangle}{\int_M x_1|\nabla_M u|^p ds}
\]

\[
= \langle D(u), J(D(u)) \rangle = \|D(u)\|_{H_{p}^{-1, \left(\frac{n-1}{p} - \frac{1}{p} \right)}}^2,
\]

which verifies (4.16).

For \( \beta > 0 \), consider the level set

\[
\Phi_{\beta} = \{u \in M \mid B(u) \geq \beta\}. \tag{4.17}
\]

Then we have the following deformation result.

**Lemma 4.6.** Let \( \beta > 0 \) be fixed. Assume that there exists an open set \( U \subset M \) such that for some constants \( \delta > 0, 0 < \rho < \beta \), it holds that

\[
\|D(u)\|_{H_{p}^{-1, \left(\frac{n-1}{p} - \frac{1}{p} \right)}} \geq \delta \quad \text{if} \quad u \in V_\rho = \{u \in M \mid u / \notin U, |B(u) - \beta| \leq \rho\}. \tag{4.18}
\]

Then there exist \( \varepsilon > 0 \) and an operator \( \eta_\varepsilon \) such that

(i) \( \eta_\varepsilon \) is odd and continuous;

(ii) \( \eta_\varepsilon(\Phi_{\beta-\varepsilon} - U) \subset \Phi_{\beta+\varepsilon} \).

**Proof.** Take \( t_0 \) and \( r(u, s) \) as in Lemma 4.5. Consider \( t_1 \in [0, t_0] \), such that for \( s \in [-t_1, t_1] \), and all \( u \in M \), we have \( |r(u, s)| \leq \frac{1}{2} \delta^2 \). Then for \( u \in V_\rho \) (defined in (4.18)) and \( t \in [0, t_1] \), we have

\[
B(\sigma(u, t)) - B(u) = \int_0^t \left( \|D(u)\|_{H_{p}^{-1, \left(\frac{n-1}{p} - \frac{1}{p} \right)}} + r(u, s) \right) ds
\]

\[
\geq \int_0^t \left( \delta^2 - \frac{1}{2} \delta^2 \right) ds = \frac{1}{2} \delta^2 t. \tag{4.19}
\]

Choose \( \varepsilon = \min\{\rho, \frac{1}{4} \delta^2 t_1\} \). If \( u \in V_\rho \cap \Phi_{\beta-\varepsilon} \), then \( |B(u) - \beta| \leq \rho \), and from (4.19) we have

\[
B(\sigma(u, t_1)) \geq B(u) + \frac{1}{2} \delta^2 t_1 \geq \beta + \varepsilon. \tag{4.20}
\]

By Lemma 4.5 and by fixing \( u \in V_\rho \), we know that the functional \( B(\sigma(u, \cdot)) \) is increasing in some interval \([0, s_0) \subset [0, t_1] \). Then for

\[
u \epsilon = \{u \in M \mid u / \notin U, |B(u) - \beta| \leq \varepsilon\},
\]

the functional

\[
t_\varepsilon(u) = \min\{t \geq 0 \mid B(\sigma(u, t)) = \beta + \varepsilon\} \tag{4.21}
\]
is well defined. The inequality (4.20) implies $0 < t_{\varepsilon}(u) \leq t_1$. The continuity of $\sigma(\cdot, s)$ and the continuity of $B(\cdot)$ induce that $t_{\varepsilon}(u)$ is continuous in $V_{\varepsilon}$.

Define

$$\eta_{\varepsilon}(u) = \begin{cases} \sigma(u, t_{\varepsilon}(u)), & \text{if } u \in V_{\varepsilon}, \\ u, & \text{if } u \in \Phi_{\beta_{\varepsilon}} - (U \cup V_{\varepsilon}) \end{cases}$$

such that

$$\eta_{\varepsilon}: \Phi_{\beta_{\varepsilon}} - U \to \Phi_{\beta_{\varepsilon} + \varepsilon}.$$ 

Since $\sigma(u, t)$ is odd and uniformly continuous with respect to $u$, we have $\eta_{\varepsilon}(u)$ is odd and continuous.

We now prove the existence of a sequence of critical values and critical points by applying a min-max argument. For each $k \in \mathbb{N}$, consider the class

$$A_k = \{ A \subset M \mid A \text{ is closed}, A = -A, \gamma(A) \geq k \},$$

where $\gamma$ is the genus as in Definition 2.7.

**Lemma 4.7.** Let $A_k$ be defined in (4.23). Define $\beta_k$ as follows:

$$\beta_k = \sup_{A \in A_k} \min_{u \in A} B(u).$$

Then for each $k$, $\beta_k > 0$, and there exists a sequence $\{u_k\} \subset M$ such that as $j \to \infty$ it holds that

(i) $B(u_{kj}) \to \beta_k$,

(ii) $D(u_{kj}) \to 0$.

**Proof.** By Definition 2.7, for the manifold $M$ as in (4.9), $\gamma(M) = +\infty$. Hence it holds that $A_k \neq \emptyset$ for all $k > 0$. For each $k$, given $A \in A_k$, we have $\min_{u \in A} B(u) > 0$, which implies that $\beta_k > 0$ for all $k$.

Assume that there is no sequence in $M$ verifying the conditions (4.25). Then there must exist constants $\delta > 0$ and $\rho > 0$ such that

$$\|D(u)\|_{H^{-1,\varepsilon}} \geq \delta \quad \text{if } u \in \{ u \in M \mid |B(u) - \beta_k| \leq \rho \}.$$ 

Without loss of generality, assume $\delta < \beta_k$. By applying Lemma 4.6 with $U = \emptyset$, we find that there exist $\varepsilon > 0$ and the odd continuous mapping $\eta_{\varepsilon}$ defined in (4.22) such that

$$\eta_{\varepsilon}(\Phi_{\beta_k - \varepsilon}) \subset \Phi_{\beta_k + \varepsilon}.$$ 

By the definition of $\beta_k$ in (4.24), there exists a set $A_{\varepsilon} \in A_k$ such that

$$B(u) \geq \beta_k - \varepsilon \quad \text{in } A_{\varepsilon},$$

namely, $A_{\varepsilon} \subset \Phi_{\beta_k - \varepsilon}$. Then $B(u) \geq \beta_k + \varepsilon$ in $\eta_{\varepsilon}(A_{\varepsilon})$. Since $A_{\varepsilon} \in A_k$, we have $\gamma(A_{\varepsilon}) \geq k$. By Proposition 2.8, and the fact that $\eta_{\varepsilon}$ is odd and continuous, we get

$$\gamma(\eta_{\varepsilon}(A_{\varepsilon})) \geq k,$$

which implies

$$\eta_{\varepsilon}(A_{\varepsilon}) \in A_k.$$ 

This is a contradiction to the definition of $\beta_k$ in (4.24). In this way, for each $k$, we obtain the sequence $\{u_{kj}\} \subset M$ verifying the conditions (4.25).

To the end, we need the following local (PS) condition.
Lemma 4.8. Let \( \{u_j\} \subset M \) and \( \beta > 0 \) such that as \( j \to \infty \),

\[
\begin{align*}
(i) & \quad B(u_j) \to \beta, \\
(ii) & \quad D(u_j) \to 0 \text{ in } \mathcal{H}_p^{1,-\frac{\lambda-1}{p},\frac{\lambda}{p}}(\mathbb{B}).
\end{align*}
\] (4.26)

Then there exists a convergent subsequence of \( \{u_j\} \) in \( M \).

Proof. Apply the similar process to that in the proof of Lemma 3.4.

Combining Lemmas 4.7 and 4.8, then for each \( k \), we have a sequence \( u_{kj} \subset M \) such that \( u_{kj} \to u_k \) in \( M \) which gives that \( u_k \in M \) with

\[
B(u_k) = \beta_k \quad \text{and} \quad D(u_k) = 0.
\]

This induces that for any \( \varphi \in \mathcal{H}_p^{1,-\frac{\lambda-1}{p},\frac{\lambda}{p}}(\mathbb{M}) \), and for each \( k \in \mathbb{N} \),

\[
\int_{\mathbb{M}} x_1|\nabla M u_k|^{p-2} \nabla M u_k \cdot \nabla M \varphi \, d\sigma = \lambda_k \int_{\mathbb{M}} x_1^{p+1} x_2^p |u_k|^{p-2} u_k \varphi \, d\sigma
\]

by the setting \( \lambda_k = \frac{\beta_k}{\beta} \). This completes the proof of Theorem 1.3.

4.2 The proof of Theorem 1.4

Consider \( \{E_k\} \) to be a sequence of linear subspaces of \( \mathcal{H}_p^{1,-\frac{\lambda-1}{p},\frac{\lambda}{p}}(\mathbb{M}) \), such that \( E_k \subset E_{k+1} \), \( \mathcal{L}(\bigcup_k E_k) = \mathcal{H}_p^{1,-\frac{\lambda-1}{p},\frac{\lambda}{p}}(\mathbb{M}) \) and \( \dim E_k = k \). Define

\[
\breve{\beta}_k = \sup_{A \in A_k} \inf_{u \in A \cap E_k} B(u),
\]

(4.27)

where \( E_k^c \) is the linear and topological complement of \( E_k \). It is obvious that \( \breve{\beta}_k \geq \beta_k > 0 \).

Hence it is sufficient to show that

\[
\lim_{k \to \infty} \breve{\beta}_k = 0,
\]

which will be verified by contradiction as follows. Assume for some positive constant \( \gamma > 0 \), we have \( \breve{\beta}_k > \gamma > 0 \) for all \( k \in \mathbb{N} \). Then for each \( k \in \mathbb{N} \), there exists \( A_k \in A_k \) such that

\[
\breve{\beta}_k \geq \inf_{u \in A_k \cap E_k^c} B(u) > \gamma.
\]

Then there exists \( u \in A_k \cap E_k^c \) such that \( \breve{\beta}_k \geq B(u_k) > \gamma \).

In this way, we have formed a sequence \( \{u_k\} \subset M \) such that \( B(u_k) > \gamma \) for all \( k \in \mathbb{N} \). Since \( \{u_k\} \subset M \), as before we know that \( \{u_k\} \) is bounded in \( \mathcal{H}_p^{1,\frac{\lambda}{p}}(\mathbb{B}) \), which implies that there exists \( v \in \mathcal{H}_p^{1,\frac{\lambda}{p}}(\mathbb{M}) \) such that

\[
u_k \to v \quad \text{in } \mathcal{H}_p^{1,\frac{\lambda}{p}}(\mathbb{M})
\]

and

\[
u_k \to v \quad \text{in } L_p^{\frac{N-\lambda-1}{p} - \frac{\lambda}{p}}.
\]

Hence we have

\[
B(v) = \frac{1}{p} \|v\|_p^{\frac{N-\lambda-1}{p} - \frac{\lambda}{p}} > \gamma.
\]

(4.28)

But the fact that \( u_k \in E_k^c \) implies \( v = 0 \) which induces the contradiction to (4.28), and then we finish the proof of Theorem 1.4.

Acknowledgements This work was supported by National Natural Science Foundation of China (Grant Nos. 11771218, 11371282 and 11631011) and the Fundamental Research Funds for the Central Universities. The author is grateful to the referees for their careful reading and helpful comments.
References

1. Amann H. Lusternik-Schnirelman theory and non-linear eigenvalue problems. Math Ann, 1972, 199: 55–72
2. Ambrosetti A, Rabinowitz P. Dual variational methods in critical point theory and applications. J Funct Anal, 1973, 14: 349–381
3. Brezis H. Functional Analysis, Sobolev spaces and Partial Differential Equations. New York: Springer, 2011
4. Carl S, Motreanu D. Multiple and sign-changing solutions for the multivalued $p$-Laplacian equation. Math Nachr, 2010, 283: 965–981
5. Chen H, Liu X, Wei Y. Multiple solutions for semilinear totally characteristic elliptic equations with subcritical or critical cone Sobolev exponents. J Differential Equations, 2012, 252: 4200–4228
6. Chen H, Liu X, Wei Y. Multiple solutions for semi-linear corner degenerate elliptic equations. J Funct Anal, 2014, 266: 3815–3839
7. Dauge M. Elliptic Boundary Value Problems in Corner Domains: Smoothness and Asymptotics of Solutions. Lecture Notes in Mathematics, vol. 1341. Berlin: Springer-Verlag, 1988
8. Drabek P. Resonance Problems for the $p$-Laplacian. J Funct Anal, 1999, 169: 189–200
9. García Azorero J P, Peral Alonso I. Existence and nonuniqueness for the $p$-Laplacian: Nonlinear eigenvalues. Comm Partial Differential Equations, 1987, 12: 1389–1430
10. Grisvard P. Boundary Value Problems in Non-Smooth Domains. London: Pitman, 1985
11. Kozlov A, Mazya V, Rossman J. Elliptic Boundary Value Problems in Domains with Point Singularities. Mathematical Surveys and Monographs, vol. 52. Providence: Amer Math Soc, 1997
12. Melrose R, Mendoza G. Elliptic operators of totally characteristic type. In: Mathematical Sciences Research Institute Publications. Cambridge: Cambridge University Press, 1983, 47–83
13. Rabinowitz P. Some aspects of nonlinear eigenvalue problems. Rocky Mountain J Math, 1973, 3: 161–202
14. Rabinowitz P. Minimax Methods in Critical Point Theory with Applications to Differential Equations. CBMS Regional Conference Series in Mathematics, vol. 65. Providence: Amer Math Soc, 1986
15. Roidos N, Schrohe E. The Cahn-Hilliard equation and the Allen-Cahn equation on manifolds with conical singularities. Comm Partial Differential Equations, 2014, 38: 925–943
16. Roidos N, Schrohe E. Existence and maximal $L^p$-regularity of solutions for the porous medium equation on manifolds with conical singularities. Comm Partial Differential Equations, 2016, 41: 1441–1471
17. Schrohe E, Seiler J. The resolvent of closed extensions of cone differential operators. Canad J Math, 2005, 57: 771–811
18. Schulze B-W. Boundary Value Problems and Singular Pseudo-Differential Operators. Chichester: Wiley, 1998
19. Schulze B-W, Wei Y. The Mellin-edge quantisation for corner operators. Complex Anal Oper Theory, 2014, 8: 803–841