Difference systems in bond and face variables and non-potential versions of discrete integrable systems

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Abstract

Integrable discrete scalar equations defined on a 2D or 3D lattice can be rewritten as difference systems in bond variables or in face variables, respectively. Both the difference systems in bond variables and the difference systems in face variables can be regarded as vector versions of the original equations. As a result, we link some of the discrete equations by difference substitutions and reveal the non-potential versions of some consistent-around-the-cube equations. We obtain higher-point configurations, including pairs of compatible six point equations on the \(\mathbb{Z}^2\) lattice together with associated seven point equations. Also we obtain a variety of compatible ten point equations together with associated ten and twelve point equations on the \(\mathbb{Z}^3\) lattice. Finally, we present integrable multiquadratic quad relations.

Keywords: discrete integrable systems, difference substitutions, simplex equations

(Some figures may appear in colour only in the online journal)

1. Introduction

Discrete integrable systems (or their ultradiscrete versions) appear in various branches of mathematics and physics such as, algebraic geometry [1], cluster algebras [2–4], group theory [5], difference geometry [6–9], statistical physics [10–17], to name just a few. The aim of this paper is a systematic investigation of links between discrete nonlinear integrable equations. As a discrete nonlinear integrable equation we understand here such a recurrence that possesses a series of interrelated properties. First, inverse scattering and algebro-geometric methods can
be applied to such a recurrence. Second, there exists a Lax pair, so the recurrence can be written as a compatibility condition of a system of linear equations. The so-called Darboux type transformations can be applied in this case. Third, it admits a Bäcklund transformation that allows one to build from a seed solution of the equation a family of new solutions of the equation with at least one essential parameter. A Bianchi type nonlinear superposition principle allows us to superpose these new solutions in a nonlinear way. A particular case of the third property is the popular concept of consistent-around-the-cube equations [18].

Integrable systems can appear in several disguises, this is one of the reasons why the link between integrable systems and the mentioned fields had been undiscovered for years. For instance, Menelaus of Alexandria theorem, the Desargues maps and the star–triangle map were recognized as essential in the theory of the integrable systems quite recently [9, 14, 19].

A given integrable equation can be related to another by a point transformation, difference substitution or by a non-auto Bäcklund transformation [20]. We aim to reveal here some of these links. We focus here on difference substitutions. More precisely, we show that particular difference substitutions, i.e. the ones that come from the mentioned systems in bond variables (or face variables), which we refer to as bond systems (or face systems) or the ones that come from the invariants of the introduced bond systems, see sections 3 and 4, lead to difference systems which either are new or of the form that has not been studied systematically yet.

In particular, we present the following forms of integrable equations:

1. Pairs of compatible six point equations as non-potential versions of known integrable quad equations
2. Seven point equations associated with the pairs of the six point equations
3. A non-potential form of the Hirota–Miwa equation that is referred to as the gauge invariant form of the Hirota–Miwa equation [21]
4. An asymmetric quad equation similar to the ones introduced in [22, 23]
5. Multiquadratic quad relations
6. Pairs of compatible ten point equations
7. Ten point equations and twelve point equations associated with the pairs of the ten point equations

Rewriting a given integrable equation defined on vertices of the \( \mathbb{Z}^2 \) (or \( \mathbb{Z}^3 \)) lattice as a system of difference equations given on edges of the corresponding \( \mathbb{Z}^2 \) lattice (or on faces of the \( \mathbb{Z}^3 \) lattice) [24, 25], opens a new perspective on the theory of discrete integrable systems. As we shall see it serves as a unifying tool in the theory. In fact this type of systems appeared earlier in the integrable systems literature without referring them to as difference systems, in bond or face variables. For instance the first and second potentials of [26] can be regarded as bond and face variables, respectively.

To illustrate the idea (which we will refer here to as vectorization procedure) consider the lattice potential Korteweg de Vries (KdV) equation [27, 28] (which also is referred to as H1)

\[
(x_{m+1,n+1} - x_{m,n})(x_{m+1,n} - x_{m,n+1}) = p_m - q_n,
\]

where \( x \) is the dependent variable and \( p_m \) and \( q_n \) are given functions of the indicated (single) variable. We can vectorize it by introducing the variables \((u, v)\)

\[
u_{m_1,n_1} = x_{m+1,n}, \quad v_{m_2,n_2} = x_{m,n+1},
\]

where we consider \( u \) as a function defined on the set of horizontal edges (i.e. pairs of vertices \( \{(m,n),(m+1,n)\} \) \( m_1, n_1 \in \mathbb{Z}^2 \)) and respectively \( v \) as a function defined on set of vertical edges (i.e. pairs of vertices \( \{(m,n),(m,n+1)\} \) \( m_2, n_2 \in \mathbb{Z}^2 \)), see section 2 for precise definitions. Also \( p_m \) can be regarded as a function given on horizontal edges and \( q_n \) as a
function given on vertical edges i.e. we can write \( p_{m_1} \) and \( q_{n_2} \). We get the following difference system in bond variables

\[
u_{m_2,z_2} = \frac{1 + \frac{p_{m_1} - q_{n_2}}{u_{m_1,z_1}}}{u_{m_1,z_1} - \nu_{m_2,z_2}}; \quad \nu_{m_2,z_1} = \frac{1 + \frac{p_{m_1} - q_{n_2}}{u_{m_1,z_1} - \nu_{m_2,z_2}}}{u_{m_1,z_1} - \nu_{m_2,z_2}}.
\]

We thoroughly studied the path from lattice integrable systems to systems in bond variables and vice versa in series of our papers \([20, 29, 30]\).

The goal of this article is to show that looking from the perspective of difference systems in bond variables we can unify some integrable equations and integrable relations (correspondences) that so far were not connected. We systematically study here two basic procedures.

Namely:

- We rewrite the bond systems in terms of their invariants with separated variables (see section 4) e.g. in the case of the system (2), one of the underlying invariants is \( H_{MN} = \frac{u_{m_1,z_1}}{u_{m_2,z_2}} = \frac{u_{m_2,z_2}}{u_{m_1,z_1}} \), where the subscripts \( M, N \) label anti-diagonals of the \( \mathbb{Z}^2 \)-lattice as we describe it in section 2. The difference system (2) rewritten in terms of \( H_{MN} \), takes the form of a pair of compatible six-point equation (10).
- We rewrite the bond systems in terms of the \( u \) variable only by eliminating the variable \( v \). For example, in case of the system (2) we get the following multiquadratic relation

\[
(u_{m_1+1,n_1}, u_{m_1+1,n_1+1} - u_{m_1,n_1} u_{m_1,z_1+1})^2 + [p_{m_1+1} + u_{m_1+1,n_1} + u_{m_1+1,n_1+1} - \left( p_{m_1} + u_{m_1,n_1} + u_{m_1,z_1+1} \right)] \cdot \left( [p_{m_1+1} + u_{m_1+1,n_1} + u_{m_1+1,n_1+1}] u_{m_1,n_1} u_{m_1,z_1+1} - (p_{m_1} + u_{m_1,n_1} + u_{m_1,n_1+1}) u_{m_1+1,n_1} u_{m_1+1,n_1+1} + q_{n_2} (u_{m_1+1,n_1} u_{m_1+1,n_1+1} - u_{m_1,n_1} u_{m_2,z_1+1}) \right) = 0.
\]

We also apply analogous procedures to integrable equations on the \( \mathbb{Z}^3 \) lattice, revealing the structure of several ten and twelve point schemes. We are starting our paper with the introductory sections 2 and 3 and then we present our main results in sections 4 and 5. We are ending the article with some suggestions for further development in section 6.

2. The \( \mathbb{Z}^2 \)-lattice and associated union of lattices, notation used in the paper

With the \( \mathbb{Z}^2 \) lattice one can associate (see figure 1):

- the set of vertices \( V = \{(m,n) | m, n \in \mathbb{Z}\} \), which is the original \( \mathbb{Z}^2 \) lattice
- the set of horizontal edges \( E_h = \{(m,n), (m+1,n) | m, n \in \mathbb{Z}\} \)
- the set of vertical edges \( E_v = \{(m,n), (m,n+1) | m, n \in \mathbb{Z}\} \)
- the set of diagonals \( D_{\diagup} = \{(m,n), (m+1,n+1) | m, n \in \mathbb{Z}\} \)
- the set of anti-diagonals \( D_{\diagup} = \{(m+1,n), (m,n+1) | m, n \in \mathbb{Z}\} \)
- the set of faces \( F = \{(m,n), (m+1,n), (m,n+1), (m+1,n+1) | m, n \in \mathbb{Z}\} \)

We label with \((m,n) \in \mathbb{Z}^2, (m_1,n_1) \in \mathbb{Z}^2, (m_2,n_2) \in \mathbb{Z}^2 \) and \((M,N) \in \mathbb{Z}^2 \), the elements of \( V, E_h, E_v \) and \( D_{\diagup}, D_{\diagup} \), respectively. Therefore, each of these sets can be regarded as a \( \mathbb{Z}^2 \) lattice itself.

In this way we also identify the sets \( D_{\diagup} \) and \( D_{\diagup} \) with the set \( F \). On the set \( V \) we define the forward difference operators \( T_1, T_2 \) s.t. \( T_1 : G_{m,n} \mapsto G_{m+1,n} \) and \( T_2 : G_{m,n} \mapsto G_{m,n+1} \). Since the sets \( E_h, E_v \) and \( D_{\diagup}, D_{\diagup} \) are dependent on the elements of the set \( V \), the action of the operators \( T_1 \) and \( T_2 \) is well-defined also on the former sets. On the set \( E_h \), we have \( T_1 : f_{m_1,n_1} \mapsto f_{m+1,n_1} \) and...


\[ T_2 : f_{m_1,n_1} \mapsto f_{m_1,n_1+1}. \]

On the set \( E \), we have
\[ T_1 : g_{m_2,n_2} \mapsto g_{m_2+1,n_2} \text{ and } T_2 : g_{m_2,n_2} \mapsto g_{m_2,n_2+1}. \]

On the set \( D \), we have
\[ T_1 : h_{M,N} \mapsto h_{M+1,N} \text{ and } T_2 : h_{M,N} \mapsto h_{M,N+1}. \]

Moreover, since all these sets can be viewed as congruent (replace the segments by their middle points), we use a concise notation. Namely, we omit the independent variables and we denote the action of the shift operators \( T_1, T_2 \) as

\[ T_1 x_{a,b} = x_{a+1,b} : x_1, \quad T_1 T_1 x_{a,b} = x_{a+2,b} : x_1, \quad T_1 T_2 x_{a,b} = x_{a+1,b+1} : x_2. \]

where \((a, b) \in \{(m, n), (m_1, n_1), (m_2, n_2), (M, N)\}\). Hence we denote the forward shift in \( i \)th direction \((i = 1, 2)\) by the subscript \( i \). For the convenience of the reader we illustrate this concise notation in figure 2.

Generalization of the considerations above to the \( \mathbb{Z}^3 \) lattice is straightforward. We have in addition a shift operator in the third direction \( T_3 : G_{M,N,K} \mapsto G_{M,N,K+1} := G_3 \). We can associate with the \( \mathbb{Z}^3 \) lattice a set of horizontal faces \( F_h \), two sets of vertical faces \( F_v, i = 1, 2 \) and the set of the centers of the cubes:

- the set of horizontal faces \( F_h = \{(m, n, l), (m + 1, n, l), (m, n + 1, l), (m + 1, n + 1, l)\} \quad |m, n, l \in \mathbb{Z}| \)
- two sets of vertical faces \( F_v = \{(m, n, l), (m, n + 1, l), (m, n + 1, l + 1)\} \quad |m, n, l \in \mathbb{Z}| \)

\( F_v = \{(m, n, l), (m + 1, n, l), (m + 1, n + 1, l), (m + 1, n + 1, l + 1)\} \quad |m, n, l \in \mathbb{Z}| \)
- the set of the centers of the cubes

\[ F_c = \{(m, n, l), (m + 1, n, l), (m, n + 1, l), (m + 1, n + 1, l), (m, n + 1, l + 1), (m + 1, n, l + 1), (m + 1, n + 1, l + 1), (m + 1, n + 1, l + 1)\} \quad |m, n, l \in \mathbb{Z}|. \]
3. Difference systems in bond and face variables

We are considering the following systems of difference equations, where \(u_{m_1,n_1}\) and \(v_{m_1,n_1}\) are the dependent variables, while \(p_{m_1}\) and \(q_{m_1}\) are prescribed functional parameters of one variable (for notation see figures 2 and 3).

1. The bond system associated to the (non-autonomous) Hirota’s KdV equation [20, 30]

\[
\begin{align*}
-u_2 &= v - p + q_2 + (p_1 - q_1) \frac{v}{u}, \\
v_1 &= u + p_1 - q - (p - q_2) \frac{u}{v}. \\
\end{align*}
\]

(\(H\))

its autonomous case, where \(p\) and \(q\) are regarded as constants

\[
\begin{align*}
-u_2 &= v + (p - q) \left(1 + \frac{v}{u}\right), \\
v_1 &= u + (p - q) \left(1 - \frac{u}{v}\right). \\
\end{align*}
\]

(\(H_a\))

Let us rewrite the \((H_a)\) system in a slightly different form and let us perform the following limiting procedure. Replacing \((p, q)\) with \((\tilde{p}, \tilde{q})\) and making the substitutions \(u = \tilde{u}(\tilde{p} - \tilde{q})(\tilde{p}_1 - \tilde{q}_1), v = \tilde{v}(\tilde{p} - \tilde{q})(\tilde{p} - \tilde{q}_1)\), the bond system \((H)\) reads

\[
\begin{align*}
(\tilde{p}_1 - \tilde{q}_2)\tilde{u}_2 - (\tilde{p} - \tilde{q})(\tilde{p}_1 + \tilde{q}) = (\tilde{p}_1 + \tilde{q}), \\
(\tilde{p}_1 - \tilde{q}_2)\tilde{v}_1 - (\tilde{p} - \tilde{q})(\tilde{p}_1 + \tilde{q}) = (\tilde{p}_1 + \tilde{q}). \\
\end{align*}
\]

(\(\tilde{H}\))

Replacing \(\tilde{u}\) with \(\epsilon \tilde{u} + 1\) and \(\tilde{v}\) with \(\epsilon \tilde{v} + 1\), taking the limit \(\epsilon \to 0\), and removing the tildes from the resulting equations, we get:

\[
\begin{align*}
(p_1 - q_2)u_2 &= (p - q)v + (p_1 + q_2)(v - u), \\
(p_1 - q_2)v_1 &= (p - q)u + (p + q_2)(v - u). \\
\end{align*}
\]

(\(L\))

which in autonomous case reads

\[
\begin{align*}
u_2 &= v + \frac{p + q}{p - q}(v - u), \\
v_1 &= u + \frac{p + q}{p - q}(v - u). \\
\end{align*}
\]

(\(L_a\))

Figure 2. Fields on vertices, edges and diagonals of the \(\mathbb{Z}_2^2\) lattice. Standard notation (left figure) and concise notation used in the paper (right figure).
2. The bond systems associated with some consistent-around-the-cube lattice equations [25]

\[ u_2 = pv \frac{(1-q)u+q-p+(p-1)v}{q(1-p)u+(p-1)v+(p-q)v}, \quad v_1 = qu \frac{(1-q)u+q-p+(p-1)v}{q(1-p)u+(p-1)v+(p-q)v} \] (FI)

\[ u_2 = \frac{v}{p} \frac{pu+q-p-v}{u-v}, \quad v_1 = \frac{v}{p} \frac{pu+q-p-v}{u-v} \] (FII)

\[ u_2 = v \left( 1 + \frac{p-q}{u-v} \right), \quad v_1 = u \left( 1 + \frac{p-q}{u-v} \right) \] (FIII)

\[ u_2 = v + \frac{p-q}{u-v}, \quad v_1 = u + \frac{p-q}{u-v} \] (FIV)

We recall that \( p \) is a given function of the independent variable \( m_1 \) and \( q \) is another given function of the variable \( n_2 \), so the equations above are in general non-autonomous.

3. A system on faces of the \( \mathbb{Z}^3 \) lattice associated with the Hirota–Miwa equation [15, 16]

\[ u_3 = \frac{uv}{u+w}, \quad v_1 = u+w, \quad w_2 = \frac{vw}{u+w} \] (HM)

For all the difference systems presented here except for the (HM), one can associate the maps \( \mathbb{CP}^1 \times \mathbb{CP}^1 \ni (u, v) \mapsto (U, V) = (u_2(u, v), v_1(u, v)) \in \mathbb{CP}^1 \times \mathbb{CP}^1 \). Many of these maps have been studied by various researchers in connection with the Yang–Baxter property. The maps we get from the bond systems \( F_I - F_V \), are the F-list of quadrirational Yang–Baxter maps [31]. From (La) we obtain a linear Yang–Baxter map [32–34]. From the bond system \( (H) \) we obtain the (non-autonomous) Hirota’s KdV map [20, 30], however this map is not Yang–Baxter in the strict sense but it satisfies an entwining Yang–Baxter relation [35]. As a side remark, note that the singularities of the map associated with the bond system \( (H) \), under the limiting procedure to the system \( (L) \) described in \( (i) \), merge into a single singularity [30]. In fact the Yang–Baxter maps associated with \( F_H - F_V \) difference systems, arise under certain limiting procedures that merge the singularities of the map associated with the \( F_I \) system [31]. When the systems are defined on faces of a \( \mathbb{Z}^3 \) lattice, the associated maps reads \( \mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{CP}^1 \ni (u, v, w) \mapsto (U, V, W) = (u_3(u, v, w), v_1(u, v, w), w_2(u, v, w)) \in \mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{CP}^1 \). From the face system (HM), we get a functional tetrahedron map [15, 16].

An important feature of the maps associated with \( (H) \), \( (L) \), \( (F_I - F_V) \), (HM) is their involutivity [36]. Involutive maps admit vast number of invariants and among them one can find
Table 1. Exhaustive list of invariants with separated variables and alternating invariants with separated variables for the maps associated with the difference systems \((F_I - F_V)\), \((H_a)\) and \((L_a)\).

| Map   | Invariants                                                                                                                                       |
|-------|--------------------------------------------------------------------------------------------------------------------------------------------------|
| \(F_I\) | \(\sqrt{\frac{p}{q}}u = \sqrt{\frac{q}{p}}v\) = \(\sqrt{\frac{p-1}{q-1}}u - \frac{1}{\sqrt{q-1}}\) = \(\sqrt{\frac{q-1}{p-1}}v + \frac{1}{\sqrt{p-1}}\) |
| \(F_{II}\) | \(\sqrt{\frac{p}{q}}u = \sqrt{\frac{q}{p}}v\) = \(\sqrt{\frac{p-1}{q-1}}u - \frac{1}{\sqrt{q-1}}\) = \(\sqrt{\frac{q-1}{p-1}}v + \frac{1}{\sqrt{p-1}}\) |
| \(F_{III}\) | \(\sqrt{\frac{p}{q}}u = \sqrt{\frac{q}{p}}v\) = \(pU - \frac{1}{2}p - qV + \frac{1}{2}q = -(pu - \frac{1}{2}p - qv + \frac{1}{2}q)\) |
| \(F_{IV}\) | \(\sqrt{\frac{p}{q}}u = \sqrt{\frac{q}{p}}v\) = \(\sqrt{\frac{p-1}{q-1}}u - \frac{1}{\sqrt{q-1}}\) = \(\sqrt{\frac{q-1}{p-1}}v + \frac{1}{\sqrt{p-1}}\) |
| \(F_V\) | \(U - V = -(u - v)\) \(U^2 + p - V^2 - q = -(u^2 + p - v^2 - q)\) |
| \(H_a\) | \(U = \frac{v}{u}\) \(U = \frac{v}{u}\) |
| \(L_a\) | \(U - V = -(u - v)\) \(U^2 - V^2 = \frac{u^2}{p} - \frac{v^2}{q}\) |

Invariants with separated variables. Following the procedure described in [20, 29, 30], for each of the maps (except for \((HM)\)), we present the exhaustive list of invariants and alternating invariants with separated variables (see table 1).

The existence of the invariants and/or the alternating invariants with separated variables, enable us to introduce a potentialization procedure. This procedure is the ‘opposite’ to the vectorization one that was explained in the introduction. It enables us to introduce a multi-parametric family of potentials, and we recover the lattice equations or the lattice relations (correspondences) out of the bond systems [20, 29, 30]. We illustrate how to apply the potentialization procedure to the bond system \((L_a)\) and to the face system \((HM)\), for the remaining cases we are referring to [20, 29, 30].

The invariants and/or the the alternating invariants with separated variables of the maps, on the level of the difference systems are responsible for the existence of conservation relations. These conservation relations guarantee the existence of a family of potentials. For example, lets consider the difference system \((L_a)\). The associated map has an alternating invariant \(h_1(u, v) = u - v\) and the invariant \(h_2(u, v) = \frac{u^2}{p} - \frac{v^2}{q}\). The following conservation relations hold:

\[
u_2 - v_1 = v - u, \quad \frac{u^2}{p} - \frac{v^2}{q} = \frac{u^2}{p} - \frac{v^2}{q}.
\]

The first of the conservation relations above, guarantees the existence of a potential \(f\) s.t. \(u = f_1 + f, \ v = f_2 + f\). In terms of the potential \(f\) the difference system \((L_a)\) reads:

\[
(p - q)(f_{12} - f) = (p + q)(f_2 - f_1),
\]

which is a linear consistent-around-the-cube equation and serves as a special case of the classification results of [37] (see also [34]).

The second of the conservation relations above, guarantees the existence of a potential \(g\) s.t. \(\frac{g^2}{p} = g_1 - g, \frac{g^2}{q} = g_2 - g\). In terms of the potential \(g\) the difference system \((L_a)\) reads:

\[
[g_{12} - g_2 + (g - g_1)k_1^2]p^2 + 2(g - g_2)[g_{12} - g_2 - (g - g_1)k_1^2](k + 1)^2pq + (g - g_2)^2(k + 1)^4q^2 = 0,
\]

\[
(p - q)(f_{12} - f) = (p + q)(f_2 - f_1),
\]
where $k := \frac{p+q}{p-q}$. The discriminant of this multiquadratic relation with respect to $g_{12}$ reads

$$64(p-q)^4(p+q)^2pq(g-g_1)(g-g_2),$$

compare with [38]. Finally, note that from the defining relations of the potentials $f, g$, by eliminating $u$ and $v$ we obtain:

$$\frac{(f_1+f)^2}{p} = g_1 - g, \quad \frac{(f_2+f)^2}{q} = g_2 - g,$$

which provides a Bäcklund transformation between these two equations. Moreover, by a linear combination of the potentials $f, g$, a two-parameter family of potentials $h$ can be obtained

$$h_1 - h = a(-1)^{m+n}u + b\frac{u^2}{p}, \quad h_2 - h = a(-1)^{m+n}v + b\frac{v^2}{q},$$

In terms of the potential $h$ the difference system ($L_u$) reads

$$2a^2pq(p^2 - q^2)A(h, h_1, h_2, h_{12}) + bB(h, h_1, h_2, h_{12}) = 0,$$

where the expressions $A(h, h_1, h_2, h_{12})$ and $B(h, h_1, h_2, h_{12})$, are the left-hand-sides of the equations (3) and (4) expressed in terms of $h$ instead of $f$ and $g$, respectively.

Now we provide an example of how the potentialization procedure works for the face system ($HM$). This procedure is basically the procedure described in [26] as the introduction of the first and second potentials. By using the first two invariants of table 2, we obtain the following conservation relations:

$$u_3v_1 = uv, \quad v_1w_2 = vw.$$

The first of the conditions above guarantees the existence of a function $f$ s.t. $u = \frac{u}{\tau_1}$, $v = \frac{v}{\tau_2}$, whereas the second condition guarantees the existence of a function $g$ s.t. $v = \frac{v}{\tau_3}$, $w = \frac{w}{\tau_4}$. Following the terminology of the article [26], we refer the functions $f, g$ as first potentials. We observe that $fg = fg_2$ which guarantees the existence of a function $\tau$ (which we refer to as second potential) s.t. $f = \frac{f}{\tau_1}$, $g = \frac{g}{\tau_1}$. Expressing $u, v, w$ in terms of the $\tau$-function, we obtain:

$$u = \frac{\tau_1\tau_2}{\tau\tau_{12}}, \quad v = \frac{\tau_2\tau_3}{\tau\tau_{12}}, \quad w = \frac{\tau_1\tau_3}{\tau\tau_{12}}.$$

Finally, in terms of the $\tau$-function, the ($HM$) difference system reads:

$$\tau\tau_{123} - \tau_2\tau_{13} - \tau_3\tau_{12} = 0,$$

which is the Hirota–Miwa integrable lattice equation in disguise. Indeed by acting on (5) with $T_{-2}T_{-3}$, the later takes a form of the Hirota–Miwa equation.

4. Bond systems in terms of their alternating invariants

In this section we rewrite the bond systems in terms of some of their (alternating) invariants. We present here only the difference schemes that are proper recurrences. By proper
recurrences we mean such recurrences that there exists an initial value problem that assures existence and uniqueness of the solution. We arrive at two cases to well-known quad equations, and in four cases to pairs of compatible six-point equations. As we shall see, the pairs of the six-point equations yield in turn seven-point equations.

4.1. Difference integrable equations on quads and their potential versions, consistent-around-the-cube equations

Before we proceed to the main subject of this section let us remind the Hirota discrete sine-Gordon equation which we are going to refer here to as lattice potential sine-Gordon equation.

Table 3 shows this equation in several disguises. Although all the equations are equivalent up to point transformations they bear different names. We take the stand that it does not make sense to distinguish the particular forms of the equation. Therefore, any of equations from the table we refer to as lattice potential (Hirota’s) sine-Gordon equation.

We are ready now to present the only two examples when the systems of $F$-list rewritten in terms of their invariants are equations on a quad\(^3\). Namely, expressing the difference system $(F_{V})$ in terms of its invariant $H = u - v$ we obtain the lattice Hirota’s KdV equation

$$H_2(H_1H_{12} + q - p_1) = H_1(HH_2 + q_2 - p).$$

Whereas by expressing the difference system $F_{III}$ in terms of its invariant $H = \sqrt{\frac{u}{q}}$, we obtain the lattice sine-Gordon equation

$$H_1(\sqrt{p}H_2 - \sqrt{q_2})(\sqrt{q}H_1 - \sqrt{p_1}) = H(\sqrt{p}H_2 - \sqrt{q})(\sqrt{q_2}H_2 - \sqrt{p}).$$

Both cases have been widely discussed in the literature [41, 43–46]. These equations are related to equations that are consistent-around-the-cube. The first equation, by the substitution $H = u - v = x_1 - x_2$, is related to lattice potential KdV equation ($H1$). The second equation, by the substitution $H = \sqrt{\frac{u}{q}} = \frac{\lambda}{\lambda_2}$, is related to lattice potential sine-Gordon equation (and hence $H3^{(6)}$) see table 4. Traditionally the adjective potential is added because

\[^3\text{Note that the linear bond system (L$_n$) expressed in terms of its invariant } H = u - v \text{ leads to the following linear quad equation}

$$H_{12} - H = \frac{p + q_2}{p - q_2}H_2 - \frac{p_1 + q}{p_1 - q}H_1.$$
Table 4. Lattice Korteweg de Vries and sine-Gordon equations.

| Equation                                           | Description                                                                 |
|----------------------------------------------------|-----------------------------------------------------------------------------|
| $H_{12} - H = (T_1 - T_2)\frac{\partial^2}{\partial \tau_1^2}$ lattice Hirota’s KdV equation [47] | $H_{12} - H = (T_1 - T_2)\frac{\partial^2}{\partial \tau_1^2}$ lattice Hirota’s KdV equation [47] |
| $H_{12} = T_1 \left( \frac{\frac{\partial H}{\partial \tau_1}}{\sqrt{\frac{\partial H}{\partial \tau_1}}} \right)$ lattice sine-Gordon equation [43, 44] or lattice modified KdV equation | $H_{12} = T_1 \left( \frac{\frac{\partial H}{\partial \tau_1}}{\sqrt{\frac{\partial H}{\partial \tau_1}}} \right)$ lattice sine-Gordon equation [43, 44] or lattice modified KdV equation |

of the connections of the type $H = x_1 - x_2$, or $H = \frac{\partial}{\partial \tau}$. Moreover, since potential versions of equations are consistent-around-the-cube, and therefore more popular (more in use), the remaining equations are sometimes referred to as non-potential versions of the underlying consistent-around-the-cube equations. The point is that one can find three parameter families of non-potential versions of given consistent-around-the-cube equations. Only a few of them are proper recurrences. Two examples of these proper recurrences are the lattice Hirota’s KdV and the lattice sine-Gordon equations and we present the remaining ones in the next subsection.

4.2. Pairs of six point equations as non-potential versions of consistent-around-the-cube equations and seven point schemes associated with them

As for the remaining potential relations $H = f(x_1, x_2)$ that arise from table 1 we arrive at the systems of compatible six point equations (see figure 4)

$$Q_E(H, H_1, H_2, H_{12}, H_{11}, H_{112}) = 0, \quad Q_N(H, H_1, H_2, H_{12}, H_{22}, H_{122}) = 0.$$  

To assure that these equations are proper recurrences, we confine ourselves to the equations which have the following property:

- $Q_E$ is multilinear in variables $(H, H_1, H_2, H_{11}, H_{112})$ and $Q_N$ is multilinear in variables $(H, H_1, H_{22}, H_{122})$. We refer to this property as multilinearity in the corners.

Indeed, prescribing the initial conditions on the two lines $\{(m, n) \in \mathbb{Z}^2 | m \in \mathbb{Z}, n = 0\}$, $\{(m, n) \in \mathbb{Z}^2 | m = 0, n \in \mathbb{Z}\}$ and at the point $(m, n) = (1, 1)$ one can uniquely find the solution on the whole $\mathbb{Z}^2$ lattice due to multilinearity in the corners and due to compatibility of $Q_E^0$ and $Q_N^0$ equations (see figure 4).

The prototypical example of such equations is the system introduced by Nijhoff et al [46]

$$(p + q)\tau_1 \tau_2 + (p - q)\tau_1 \tau_{12} = 2p\tau_1 \tau_{12}, \quad (p + q)\tau_1 \tau_{12} - (p - q)\tau_{12} = 2q\tau_2 \tau_{12},$$

which is a reformulation of the lattice Hirota’s KdV equation in terms of the $\tau$ function.

We arrive at four examples and we present them with the associated seven point equations i.e. first, the equation $Q_E^N(H, H_1, H_2, H_{12}, H_{11}, H_{112}, H_{122}) = 0$ that arise from elimination of $H_{22}$ from the equations $T_3 Q_E(H, H_1, H_2, H_{12}, H_{11}, H_{112}, H_{122}) = 0$ and $Q_N(H, H_1, H_2, H_{12}, H_{22}, H_{122}) = 0$, second, the equation $Q_E^N(H_1, H_2, H_{11}, H_{112}, H_{122}, H_{122}, H_{112}) = 0$ that arise from elimination of $H$ from the $Q_E(H, H_1, H_2, H_{12}, H_{11}, H_{112}) = 0$ and $Q_N(H, H_1, H_2, H_{12}, H_{22}, H_{122}) = 0$ see figures 5 and 6. Now we list these systems.
In case of the system \( J \). Phys. A: Math. Theor. the six-point equation

We observe that and a seven-point equation of type 

where we introduced auxiliary variables

Thanks to that one of the invariants is \( H = \frac{u}{v} \). Rewriting the system in terms of \( H \) we arrive at a pair of six-point equations, the first equation of type \( Q_E \) reads

and the second one of type \( Q_N \)

where we introduced auxiliary variables

Out of these two six-point equations we get a seven-point equation of type \( Q_{ES} \)

and a seven-point equation of type \( Q_{EN} \)

We observe that \( X = 0 \) solves each of these four equations \((6)–(9)\). Writing \( X = 0 \) explicitly, namely

we recognize the lattice sine-Gordon equation. So every solution of lattice sine-Gordon equation is a solution of each of the four equations presented above.

\* \( F_H \) bond system in terms of the invariant \( H = \frac{u}{v} \) (pair of six-point equations) and associated seven-point equations.

In case of the system \( F_H \) we slightly change it by introducing the variables \( u' = \sqrt{pu}, v' = \sqrt{qv} \). Omitting the primes the new form of the system reads

Thanks to that one of the invariants is \( H = \frac{u}{v} \). Rewriting the system in terms of \( H \) we arrive at a pair of six-point equations, the first equation of type \( Q_E \) reads

and the second one of type \( Q_N \)

where we introduced auxiliary variables

Out of these two six-point equations we get a seven-point equation of type \( Q_{ES} \)

and a seven-point equation of type \( Q_{EN} \)

We observe that \( X = 0 \) solves each of these four equations \((6)–(9)\). Writing \( X = 0 \) explicitly, namely

we recognize the lattice sine-Gordon equation. So every solution of lattice sine-Gordon equation is a solution of each of the four equations presented above.

\* \( F_V \) bond system in terms of the invariant \( H = \frac{u}{v} \) (pair of six-point equations) and associated seven-point equations.

If we rewrite the system \( F_V \) in terms of its invariant \( H = \frac{u}{v} \) we get the six-point equation \( Q_E \)

the six-point equation \( Q_N \)
\[ X_2(k_1 G_2 - k_2 G_1 H H_2) + X H_2 T_2 (k_2 G_1 H_2 H_{12} - k_1 G_2) = 0, \]  

(11) the seven-point equation \( Q_{ES} \)

\[ X_1 H_2 H_{12} T_2 (k_2 G_1 H_2 H_{12} - k_1 G_2) = X_2 T_1 (k_2 G_1 H_2 H_{12} - k_1 G_2), \]  

(12) the seven-point equation \( Q_{EN} \)

\[ k_{12} X_1 X + X_1 G_{12} (k_1 G_2 - k_2 G_1 H H_2) + X G_{12} T_1 T_2 (k_2 G_1 H_2 H_{12} - k_1 G_2) = 0, \]  

(13) where this time we used the following auxiliary variables 

\[ k := p - q, \quad G := H - 1, \quad X = G_1 G_2 (H_{12} - H). \]

The equations above can be obtained from equations (6)–(9) by the coalescence \( A = B =: G \), in addition \( H = f_{m-n} \) is a solution of the equations above.

- \( F_{IV} \) bond system in terms of the invariant \( H = u - v + p/2 - q/2 \) (pair of six-point equations) and associated seven-point equations.

This set of six-points and seven-point equations arises from system \( F_{IV} \) rewritten in invariant \( H = u - v + p/2 - q/2 \). Namely, denoting 

\[ k := p - q, \quad A := 2H + p - q, \quad B := 2H - p + q, \quad k_1 A_2 - k_2 A_1 = k_1 B_2 - k_2 B_1 := X \]

we get the six-point equation \( Q_{E} \)

\[ X B_{12} T_1 (B_1 B_{12} - AA_1) + X_1 A_1 (B_2 B_{12} - AA_2) = 0, \]  

(14) the six-point equation \( Q_{N} \)

\[ X B_{12} T_2 (B_2 B_{12} - AA_2) + X_2 A_2 (B_1 B_{12} - AA_1) = 0, \]  

(15) the seven-point equation \( Q_{ES} \)

\[ X X_1 T_2 (B_2 B_{12} - AA_2) - XX_2 T_1 (B_1 B_{12} - AA_1) = X_1 X_2 (A_1 B_2 - A_2 B_1), \]  

(16) the seven-point equation \( Q_{EN} \)

\[ X B_{12} B_{12} T_1 T_2 (B_1 B_{12} - AA_1) = X_{12} A_2 A_{12} (B_1 B_{12} - AA_1). \]  

(17)
Also in this case \( X = 0 \) yields a set of particular solutions of the equations, this time the solutions are \( H = (p - q)/m + r \).

- Non-autonomous Hirota’s bond system \((H)\) in terms of the invariant \( H = u/v \) (pair of six-point equations) and associated seven-point equations.

The last set of six-point and seven-point equations arises from system \((H)\) rewritten in its invariant \( H = u/v \). Namely, by denoting

\[
k := p - q, \quad G := k_2H - k_1, \quad X := H - H_12
\]

we get the six-point equation \( Q_E \)

\[
XH_1T_1(G_1 - HH_1G_2) - X_1H_1H_12(G_1 - H_1H_12G_2) + XX_1H_1H_12T_2(G_1 + H_1G) = 0, \quad (18)
\]

the six-point equation \( Q_N \)

\[
XH_1H_2T_2(G_1 - HH_1G_2) - X_2H_12(G_1 - H_1H_12G_2) + XX_2H_1H_12T_2(G + HG_2) = 0. \quad (19)
\]

The seven-point equation \( Q_{ES} \)

\[
X_1H_1H_1T_2 (G_1 - HH_1G_2) - X_2T_1(G_1 - HH_1G_2) + X_1X_2H_1T_2(G_1 - HH_1G_2) = 0 \quad (20)
\]

and the seven-point equation \( Q_{EN} \)

\[
XH_1H_12T_1T_2(G_1 - HH_1G_2) - X_12H_12H_112(G_1 - H_1H_12G_2) + XX_12H_1H_12T_2(G_1 + H_1G_12 + H_1G) = 0. \quad (21)
\]

Also in this case \( X = 0 \) yields to a special solution for all the equations presented above. This solution reads: \( H = f_{m-n} \).

### 4.3. A non-potential form of the Hirota–Miwa equation

We denote the three invariants (presented in table 2) of the \((HM)\) face system as:

\[
I = uv, \quad J = vw, \quad H = \frac{u}{w} \quad (22)
\]

Expressing the \((HM)\) system in terms of \( I, J \) we get:

\[
\frac{I_{13}J_3}{J_{1J}} = \frac{(I + J)_{3}}{(I + J)_{1}}, \quad \frac{I_{23}J_3}{J_{2J}} = \frac{J_{23}J_2}{J_{3J}(I + J)_{2}}, \quad \frac{J_{12}J_2}{J_{1J}} = \frac{J_{12}J_2}{J_{1J}(I + J)_{1}}. \quad (23)
\]

These equations can be rewritten in terms of the invariant \( H = II/J \) only, namely

\[
H_{123}H(1 + H_1)(1 + H_12) = H_{12}H_3(1 + H_13)(1 + H_2). \quad (24)
\]

Equation (24) can be regarded as the non-potential version of the Hirota–Miwa equation. However, in the literature it is referred to as the gauge invariant form of the Hirota–Miwa equation [21, 48, 49].

### 5. Elimination of variables

In this section we rewrite each of the bond systems presented in the introduction, in terms of the \( u \) variable only by the elimination of the dependent variable \( v \) and its shift \( v_1 \). From the bond systems \((F_1 - F_7)\) we obtain multi-quadratic quad correspondences. Whereas for the
bond system ($L$), we obtain a linear quad equation. For the bond system associated to the Hirota map we obtain a multilinear quad equation. Finally, for the difference system ($HM$) we obtain a system of compatible ten-point equations as well as ten-point and 12-point equations.

As an illustrative example we show how the elimination procedure works in the case of the $F_{III}$ bond system. Namely, the $F_{III}$ bond system reads:

$$u_2 = \frac{v pu - qv}{p - u - v}, \quad v_1 = \frac{u pu - qv}{q - u - v}. \tag{25}$$

Shifting the first equation of (25) in the first direction we obtain

$$u_{12} = \frac{v_1 p u_1 - qv_1}{p_1 - u_1 - v_1}. \tag{26}$$

From equations (25) and (26) we eliminate the fields $v, v_1$ and we obtain the equation $dQ^{30*}$ (see also table 5)

$$(y_1 - y)(z_1y - zy_1) - q(z_1 - z)^2 = 0, \quad \text{where} \quad z = puu_2, \quad y = p(u + u_2).$$

5.1. A quad equation associated to the bond system of Hirota’s map

Applying the elimination procedure to the non-autonomous Hirota’s difference system ($H$)

$$u_2 = v - k_2 + k_1 \frac{v}{u}, \quad v_1 = u + k_1 - k_2 \frac{u}{v}, \quad \text{where} \quad k := p - q$$

we obtain

$$\frac{(u_{12} + k_{12})(u_2 + k_2)}{(u_1 + k_{11})(u + k_1)} = \frac{u_2}{u_1}. \tag{27}$$

For the autonomous case, where $p, q$ are considered constant, the system ($H$) reduces to ($H_a$) and from the equation above we obtain

$$\frac{u_1 u_{12} - k^2}{u u_2 - k^2} = \frac{u_1 + k}{u_2 + k}. \tag{28}$$

Equations (27) and (28) are examples of asymmetric multilinear quad equations. Similar equations have been presented in [22, 23]. We have not been able to identify yet the lattice equations (27) and (28) with known ones.

Finally, when we eliminate the fields $v, v_1$ from the system ($L$) we get the linear equation

$$k(k_1 + k_{11})(k + k_1 - k_2)u - (k + k_1)k_{11}k_{12}u_1 + (k_1 + k_{11})k_{12}k_{12}u_2 - (k + k_1)k_{12}k_{112}u_{12} = 0.$$

5.2. Multi-quadratic quad correspondences associated to the bond systems of quadrirational Yang–Baxter maps

In [38] a list of integrable multi-quadratic quad equations were introduced. Their construction is based on the hypothesis that discriminants of the defining polynomials factorize in a particular way that allows one to reformulate the equation as a single-valued system [50]. More precisely, a list of eight multi-quadratic quad equations $Q^1, Q^2, Q^3, Q^4, A^1, A^2, H^2, H^3$, were derived with the property that the discriminants of these polynomials wrt. any of its arguments is proportional to the factor of two bi-quadratic polynomials. These polynomials
are the ones used by Adler et al [51] in their list of integrable quad equations. There exists a
generalization of the construction presented in [38]. Namely, in [52] it was introduced a multi-
quadratic polynomial respecting the symmetry of the Fano plane, such that its discriminants
wrt. any of its arguments is proportional to the factor of three bi-quadratic polynomials.

Here we show that from the bond systems $F_I - F_V$ one can obtain multi-quadratic quad
equations which are degenerate cases of the $Q^4$. $Q^3$, $Q^2$, $Q^1$, and $Q^{0*}$ (where the super-
script ‘0’ stands for $Q^3$ with $\delta = 0$) equations. Namely, by applying the elimination proce-
dure at the bond systems $F_I - F_V$, we obtain five multi-quadratic quad equations which are
presented in table 5. We denote these degenerate cases as $Q^4$, $Q^3$, $Q^2$, $Q^1$, and $Q^{0*}$
respectively. The degeneration lies in the fact that the discriminants of the polynomials wrt.
any of its arguments is proportional to one bi-quadratic polynomial. The proportionality fac-
tor is a perfect square of an expression so, in principle, it can be removed by dividing the
multiquadratic equation with this expression. In table 6 we present the discriminants of the
obtained multiquadratic equations wrt. $u_{12}$ which are bi-quadratic polynomials of $u$ and $u_2$
(we omit the perfect square factors since they can be removed by appropriate divisions of the
original equations).}

Table 5. Multi-quadratic quad equations obtained from bond systems $F_I - F_V$.

| Name | Equation |
|------|----------|
| dQ4* | $(y_1 - y)(z_1 - z)$ $\frac{(z_1 y - z_1)(y_1 - y + z_1 y - z_1)}{p_1}$ $q(z_1 - z) = 0,$ $z := \frac{u_1 - u_2}{u_1 + u_2}$, $y := \frac{u_1 - u_2}{u_1 + u_2}$ |
| dQ3* | $(y_1 - y)(z_1 y - z_1) - q(z_1 - z)^2 + q(z_1 - z)(y_1 - y) = 0,$ $z := \frac{u_1 - u_2}{u_1 + u_2}$, $y := \frac{u_1 - u_2}{u_1 + u_2}$ |
| dQ2* | $(y_1 - y)(y_1 z - y_1) + (z_1 - z)^2 + q(z_1 - z)(y_1 - y) = 0,$ $z := \frac{u_1 - u_2}{u_1 + u_2}$, $y := \frac{u_1 - u_2}{u_1 + u_2}$ |
| dQ1* | $(y_1 - y)(y_1 z - y_1) - (z_1 - z)^2 - q(y_1 - y)^2 = 0,$ $z := \frac{u_1 - u_2}{u_1 + u_2}$, $y := \frac{u_1 - u_2}{u_1 + u_2}$ |

5.3. Ten-point compatible equations on the $\mathbb{Z}^3$ lattice associated to the Hirota–Miwa
difference system

A difference system associated to the Hirota–Miwa equation reads:

$$u_3 = \frac{uv}{u + w},$$ (29)

$$v_1 = u + w,$$ (30)

$$w_2 = \frac{vw}{u + w}.$$ (31)

In what follows, we apply the elimination procedure to the face system (29)--(31).

5.3.1. Elimination of the fields $v$, $w$. Shifting (29) by $T_1$ and by using (30) we eliminate $v_1$ to obtain:

$$w_1 = -u_1 \left(1 - \frac{u + w}{u_{13}}\right).$$ (32)
Note that by acting on (35) with $T_1$ and by using (30) to eliminate $u_{13}$, we eliminate $w_{12}$, and from the resulting equation by the use of (32) and (31) we obtain $w_{12} = (u + w)[u_{123}(u + u_{12} - u_{13} + w) - u_2 u_{12}]$.

Now we are solving the system of equations (29) and (33) for $v$ and $w$ and we get

$$v = u_3 u_{123}(u_{12} - u_{13}) + u_{12}(u_3 - u_2), \quad w = u u_{123}(u - u_3) + u_{12}(u_{123} - u_2).$$

Finally, we use (34) to eliminate the fields $v$, $w$ and their shifts from (29) to (31). Then, equation (29) is identically satisfied, whereas equations (30) and (31) lead to the following system of homogeneous quintic ten-point equations which are compatible on the 3D lattice.

$$X u_{13} T_1 (u_{123}(u_{12} - u_{13}) + u_{12}(u_3 - u_2)) = X_1 u (u_{123}(u_{12} - u_{13}) + u_{12}(u_3 - u_2)), \quad (35)$$

$$X u_2 T_2 (u_{123}(u - u_3) + u_{12}(u_{123} - u_2)) = X_2 u_3 (u_{123}(u - u_3) + u_{12}(u_{123} - u_2)), \quad (36)$$

where

$$X := u_{123} - u_{13} u_2.$$  

Here compatibility means that the equations provide unique values at points $u_{1123}$ and $u_{1223}$ and the two different ways to evaluate $u_{1123}$ leads to the same result (see figure 7). A consequence of this compatibility is that by imposing a correct initial value problem f.i. the initial data provided on the three perpendicular coordinate planes, the solution exists on the whole $\mathbb{Z}^3$ lattice.

Table 6. Discriminants of multiquadratic quad equations of the table 5 (with omitted perfect square factors).

| Discriminant |
|---------------|
| $\Delta(d Q^4, u_{12}) \propto (y - q z - q + 1)^2 - 4 q (q - 1) z$ |
| $\Delta(d Q^3, u_{12}) \propto (y + q)^2 - 4 q z$ |
| $\Delta(d Q^{8'}, u_{12}) \propto v^2 - 4 q z$ |
| $\Delta(d Q^2, u_{12}) \propto (y - q)^2 - 4 z$ |
| $\Delta(d Q^1, u_{12}) \propto y^2 + 4(z - q)$ |

Shifting (31) by $T_1$ and by using (30) we eliminate $v_1$, also we shift (32) by $T_2$ and we obtain the equations

$$w_{21} = \frac{(u + w) w_1}{u_1 + u + w}, \quad w_{12} = -u_{12} \left(1 - \frac{u_2 + w_2}{u_{132}}\right).$$

Finally, we use (34) to eliminate the fields $v$, $w$ and their shifts from (29) to (31). Then, equation (29) is identically satisfied, whereas equations (30) and (31) lead to the following system of homogeneous quintic ten-point equations which are compatible on the 3D lattice.

$$X u_{13} T_1 (u_{123}(u_{12} - u_{13}) + u_{12}(u_3 - u_2)) = X_1 u (u_{123}(u_{12} - u_{13}) + u_{12}(u_3 - u_2)), \quad (35)$$

$$X u_2 T_2 (u_{123}(u - u_3) + u_{12}(u_{123} - u_2)) = X_2 u_3 (u_{123}(u - u_3) + u_{12}(u_{123} - u_2)), \quad (36)$$

where

$$X := u_{123} - u_{13} u_2.$$  

Note that by acting on (35) with $T_{23} T_{-3}$ followed by the change of the independent variables $(m, n) \mapsto (-n, -m)$, the latter takes exactly the form of a ten-point equation first presented in [53], which in this paper is referred to as the non-potential version of the Hirota–Miwa equation.
Whereas, from (35) and (36) by eliminating the variable \( u \), we obtain the following 12-point homogeneous sextic equation (see figure 8)

\[
X_1 T_2 (u_{123} (u_{13} - u_{12}) - u_{12} (u_{3} - u_{2})) = 0.
\]

(37)

Note that \( X = 0 \) serves as a special solution of (37) and of (38), as well as for the system (35) and (36). Explicitly this solution reads

\[
u_{i,m,n} = f_{i-m,n} g_{n,m},
\]

where \( f, g \) are arbitrary functions of 2 independent variables.

5.3.2. Elimination of the fields \( u, w \). Working similarly, we can express \( u, w \) in terms of \( v \) and its shifts to obtain:

\[
u = \frac{v_1 v_{13} v_{12} (v_{23} - v_3) + v_2 (v - v_{12})}{v_2 v_{13} - v_3 v_{12}}, \quad w = \frac{v_1 v_{12} v_{13} (v_2 - v_{123}) + v_3 (v_{13} - v)}{v_2 v_{13} - v_3 v_{12}}.
\]

(39)

The compatible system of ten-point equations now reads:

\[
X v_{13} T_3 (v_{12} (v_{123} - v_3) + v_2 (v - v_{12})) = X v_3 (v_{12} (v_{123} - v_3) + v_2 (v - v_{12})),
\]

(40)

\[
X v_{12} T_2 (v_{13} (v_{123} - v_2) + v_3 (v - v_{13})) = X v_2 (v_{13} (v_{123} - v_2) + v_3 (v - v_{13})),
\]

(41)

where we have introduced the variable

\[
X := v_2 v_{13} - v_3 v_{12}.
\]

From (40) and (41) by eliminating \( v \), we obtain the following ten-point homogeneous quartic equation

\[
X_1 T_3 (v_{13} (v_{123} - v_3) + v_3 (v - v_{13})) + X_3 T_2 (v_3 (v_{123} - v_3) + v_2 (v_{12} - v)) = 0.
\]

(42)
On the other hand, if we shift the equation (40) by $T_2$ and we eliminate $v_{22}$ by the use of (41), we obtain:

$$X_{23}v_3 v_2 v_2 (v_{12} (v_{123} - v_2) + v_2 (v - v_{13})) = X T_2 T_3 (v_1 v_{12} (v_{13} (v_{123} - v_2) + v_3 (v - v_{13}))),$$  (43)

that is a 12-point homogeneous sextic equation. Here it holds that $X = 0$ is a special solution for the system (40) and (41), as well as for the equations (42) and (43). Explicitly this solution reads

$$v_{l,m,n} = F_{l,m+n} G_{m,n},$$

where $F, G$ are arbitrary functions of two independent variables.

6. Conclusions

The theory of 2D discrete integrable systems has focused so far mainly on equations defined on quads. Equations defined on higher point stencils of a 2D lattice appeared occasionally, for instance the discrete analogue of the Toda-lattice that is defined on a five-point stencil [54], an equation defined on a hexagonal configuration [55], the discrete analogue of the Boussinesq equation [56] that is defined on a nine-point stencil, a nine-point stencil discretization of the Liouville equation that was considered in [57] and various six-point and seven-point schemes [46, 58–61]. In this article we presented various equations defined on six- and seven-point stencils, which to the best of our knowledge are novel. The novelty can be inferred from the degrees on which the variables appear in the equations. Moreover these equations serve as non-potential versions of some known integrable equations defined on quads [42]. Note that through our procedure we can also obtain multiquadratic, or even higher degree six-point equations. We have omitted the presentation of these exotic systems here, however, they require further investigation. In addition, a challenging problem is to classify integrable six-point equations and seven-point equations.

Finally, we have presented multiquadratic quad relations, as well as compatible equations of homogeneous degree defined on ten-point stencils of the $\mathbb{Z}^3$-lattice, as well as various equations defined on ten- and twelve-point stencils. Equations defined on ten-point stencils of the $\mathbb{Z}^3$-lattice have appeared rarely in the literature so far. The only example that we can point out is in [53] where such an equation was introduced but without its compatible partner.
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