LOG DEL PEZZO SURFACES WITH LARGE VOLUMES

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ABSTRACT. We classify all of the log del Pezzo surfaces $S$ of index $a$ such that the volume $(−K_S^2)$ is larger than or equal to $2a$.

1. Introduction

The purpose of this paper is to classify all of the log del Pezzo surfaces of fixed index with large anti-canonical volumes. A normal projective variety $S$ is called a log Fano variety if $S$ is log-terminal and the anti-canonical divisor $−K_S$ is ample. We call 2-dimensional log Fano varieties as log del Pezzo surfaces. For a log Fano variety $S$, the index of $S$ is defined as the smallest positive integer $a$ such that $−aK_S$ is Cartier. Log del Pezzo surfaces $S$ with fixed index $a$ have been treated by many authors in various viewpoint.

We recall the viewpoint in terms of the classification problem of log del Pezzo surfaces. If $a = 1$ then all such $S$ are classified by several authors, see [Bre80, Dem80, HW81]; if $a = 2$ then all such $S$ are classified in [AN88, AN89, AN06, Nak07]; and if $a = 3$ then all such $S$ are classified in [FY14]. However, it is not realistic to classify all of the log del Pezzo surfaces of (fixed) index $a \geq 4$ since even for the case $a = 3$ there are 300 types of such log del Pezzo surfaces (see [FY14]).

On the other hand, the problem of the boundedness for log Fano varieties of fixed index $a$ is also important in the area of minimal model program. The problem is equivalent to bound the anti-canonical volume ($−K_S^a$) for such $S$ if the characteristic of the base field is zero. Such problem is called the Batyrev conjecture. See [Bor01, HMX12]. We remark that the authors in [HMX12] showed that the Baryrev conjecture is true (in any dimension, if the characteristic of the base field is zero). Nowadays, a generalized version of the Batyrev conjecture, called the BAB conjecture, is considered by many authors, see [BB92, Ale94, AM04, Lai12, Jia13]. The BAB conjecture is true for...
log del Pezzo surfaces but is open in higher-dimensional case. Note that, as a corollary of \cite{Jia13}, any log del Pezzo surface $S$ of index $a \geq 2$ satisfies that $(-K_S^2) \leq (2a^2 + 4a + 2)/a$ and equality holds if and only if $S$ is isomorphic to the weighted projective plane $\mathbb{P}(1, 1, 2a)$.

In this paper, we classify the log del Pezzo surfaces $S$ of index $a \geq 4$ such that $(-K_S^2) \geq 2a$. The motivation is related to both the classification problem and the boundedness problem. We note that log del Pezzo surfaces $S$ of index $a \leq 3$ are completely classified.

**Theorem 1.1.** Let $S$ be a log del Pezzo surface of index $a \geq 4$. Then $(-K_S^2) \geq 2a$ holds if and only if $S$ is isomorphic to the log del Pezzo surface associated to the $a$-fundamental multiplet of length $b$ with $b = [(a + 1)/2]$ whose type is one of the following:

1. $\langle O \rangle_a$ \quad $((-K_S^2) = (2a^2 + 4a + 2)/a)$,
2. $\langle I \rangle_a$ \quad $((-K_S^2) = (2a^2 + 3a + 2)/a)$,
3. $\langle II \rangle_a$ \quad $((-K_S^2) = (2a^2 + 2a + 2)/a)$,
4. $\langle III \rangle_a$ \quad $((-K_S^2) = (2a^2 + a + 2)/a)$,
5. $\langle A \rangle_5$ \quad $((-K_S^2) = 54/5)$,
6. $\langle IV \rangle_a$ \quad $((-K_S^2) = (2a^2 + 2)/a)$,
7. $\langle B \rangle_4$ \quad if $a = 4$ \quad $((-K_S^2) = 8)$,
8. $\langle C \rangle_4$ \quad if $a = 4$ \quad $((-K_S^2) = 8)$.

We describe the above types in Section 4.1.

By Theorem 1.1 and the arguments in Section 4 we have the following result. We note that we do not use the results in \cite{Jia13} in order to prove Corollary 1.2.

**Corollary 1.2.** Let $S$ be a log del Pezzo surface of index $a \geq 2$. Then $(-K_S^2) > 2(a + 1)$ holds if and only if $S$ is isomorphic to one of the following (We describe the following varieties in Section 4.2):

1. $\mathbb{P}(1, 1, 2a)$ \quad $((-K_S^2) = (2a^2 + 4a + 2)/a)$,
2. $S_{1,a}$ \quad $((-K_S^2) = (2a^2 + 3a + 2)/a)$,
3. $S_{11,a}$ \quad $((-K_S^2) = (2a^2 + 2a + 2)/a)$,
4. $S_{12,a}$ \quad $((-K_S^2) = (2a^2 + 2a + 2)/a)$,
5. $\mathbb{P}(1, 1, 3)$ \quad if $a = 3$ \quad $((-K_S^2) = 25/3)$.

We remark that all of them are toric varieties.

The strategy for the classification is essentially same as the strategy in \cite{FY14} based on the earlier work in \cite{Nak07}. First, we reduce the classification problem of log del Pezzo surfaces of index $a$ to the classification problem of the pair of its minimal resolution and $(-a)$ times the discriminant divisor. We call such pair an $a$-basic pair (see Section 3.1). Next, by contracting $(-1)$-curves very carefully, we get
an \textit{a-fundamental multiplet} from an \textit{a-basic} pair (see Section \S3.2). See also the flowcharts in [FY14] \S1. From the assumption \((-K_S^2) \geq 2a\), the structure of the associated \textit{a-fundamental multiplet} has very special structure. Thus we can get the multiplets in Section 4.1.

\textbf{Acknowledgments.} The author is partially supported by a JSPS Fellowship for Young Scientists.

\textbf{Notation and terminology.} We work in the category of algebraic (separated and of finite type) scheme over a fixed algebraically closed field \(k\) of arbitrary characteristic. A \textit{variety} means a reduced and irreducible algebraic scheme. A \textit{surface} means a two-dimensional variety. For a normal variety \(X\), we say that \(D\) is a \(\mathbb{Q}\)-divisor (resp. divisor or \(\mathbb{Z}\)-divisor) if \(D\) is a finite sum \(D = \sum a_i D_i\) where \(D_i\) are prime divisors and \(a_i \in \mathbb{Q}\) (resp. \(a_i \in \mathbb{Z}\)). For a \(\mathbb{Q}\)-divisor \(D = \sum a_i D_i\), the value \(a_i\) is denoted by \(\text{coeff}_{D_i} D\) and set \(\text{coeff}_{D_i} := \{a_i\}\). For an effective \(\mathbb{Q}\)-divisor or a scheme \(D\) on \(X\), let \(|D|\) be the support of \(D\). A normal variety \(X\) is called \textit{log-terminal} if the canonical divisor \(K_X\) is \(\mathbb{Q}\)-Cartier and the discrepancy \(\text{discrep}(X)\) of \(X\) is bigger than \(-1\) (see [KM98] \S2.3). For a proper birational morphism \(f: Y \to X\) between normal varieties such that both \(K_X\) and \(K_Y\) are \(\mathbb{Q}\)-Cartier, we set

\[ K_{Y/X} := \sum_{E_0 \subset Y \atop f \text{-exceptional}} a(E_0, X) E_0, \]

where \(a(E_0, X)\) is the discrepancy of \(E_0\) with respects to \(X\) (see [KM98] \S2.3). (We note that if \(aK_X\) and \(aK_Y\) are Cartier for \(a \in \mathbb{Z}_{>0}\), then \(aK_{Y/X}\) is a \(\mathbb{Z}\)-divisor.)

For a nonsingular surface \(S\) and a projective curve \(C\) which is a closed subvariety of \(S\), the curve \(C\) is called a \((-n)\)-\textit{curve} if \(C\) is a nonsingular rational curve and \((C^2) = -n\). For a birational map \(M \dashrightarrow S\) between normal surfaces and for a curve \(C \subset S\), the strict transform of \(C\) on \(M\) is denoted by \(C^M\). If the birational map is of the form \(M_i \dashrightarrow M_j\), then the strict transform of \(C \subset M_j\) on \(M_i\) is denoted by \(C^i\).

Let \(S\) be a nonsingular surface and let \(D = \sum a_j D_j\) be an effective divisor on \(S\) \((a_j > 0)\) such that \(|D|\) is simple normal crossing and \(D_i, D_j\) are intersected at most one point. (It is sufficient in our situation.) The \textit{dual graph} of \(D\) is defined as follows. A vertex corresponds to a component \(D_j\). Let \(v_j\) be the vertex corresponds to \(D_j\). The \(v_i\) and \(v_j\) are joined by a (simple) line if and only if \(D_i, D_j\) are intersected. In the dual graphs of divisors, a vertex corresponding to \((-n)\)-curve is expressed as \(\circ\). On the other hand, an arbitrary irreducible curve is expressed by the symbol \(\odot\) when it is not necessary a \((-n)\)-curve.
Let $\mathbb{F}_n \to \mathbb{P}^1$ be a Hirzebruch surface $\mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(n))$ of degree $n$ with the $\mathbb{P}^1$-fibration. A section $\sigma \subset \mathbb{F}_n$ with $(\sigma^2) = -n$ is called a minimal section. If $n > 0$, then such $\sigma$ is unique. We usually denote a fiber of $\mathbb{F}_n \to \mathbb{P}^1$ by $l$.

For a real number $t$, let $\lfloor t \rfloor$ be the greatest integer not greater than $t$.

2. Elimination of subschemes

In this section, we recall the results in [Nak07, §2] (see also [Fuj14, §2]). Let $X$ be a nonsingular surface and $\Delta$ be a zero-dimensional subscheme of $X$. The defining ideal sheaf of $\Delta$ is denoted by $\mathcal{I}_\Delta$.

**Definition 2.1.** Let $P$ be a point of $\Delta$.

1. Let $\nu_P(\Delta) := \max\{\nu \in \mathbb{Z}_{>0} | \mathcal{I}_\Delta \subset \mathfrak{m}_P^\nu\}$, where $\mathfrak{m}_P$ is the maximal ideal sheaf in $\mathcal{O}_X$ defining $P$. If $\nu_P(\Delta) = 1$ for any $P \in \Delta$, then we say that $\Delta$ satisfies the $(\nu 1)$-condition.

2. The multiplicity $\text{mult}_P \Delta$ of $\Delta$ at $P$ is given by the length of the Artinian local ring $\mathcal{O}_{\Delta, P}$.

3. The degree $\text{deg} \Delta$ of $\Delta$ is given by $\sum_{P \in \Delta} \text{mult}_P \Delta$.

**Definition 2.2.** Assume that $\Delta$ satisfies the $(\nu 1)$-condition. Let $V \to X$ be the blowing up along $\Delta$. The elimination of $\Delta$ is the birational projective morphism $\psi: Y \to X$ which is defined as the composition of the minimal resolution $Y \to V$ of $V$ and the morphism $V \to X$.

For any divisor $E$ on $X$ and for any positive integer $s$, we set $E^{\Delta,s} := \psi^*E - sK_{Y/X}$.

**Proposition 2.3** ([Nak07, Proposition 2.9]).

1. Assume that the subscheme $\Delta$ satisfies the $(\nu 1)$-condition and let $\psi: Y \to X$ be the elimination of $\Delta$. Then the anti-canonical divisor $-K_Y$ is $\psi$-nef. More precisely, for any $P \in \Delta$ with $\text{mult}_P \Delta = k$, the set-theoretic inverse image $\psi^{-1}(P)$ is the straight chain $\sum_{j=1}^k \Gamma_{P,j}$ of nonsingular rational curves and the dual graph of $\psi^{-1}(P)$ is the following:

$$
\begin{array}{cccc}
\Gamma_{P,1} & \Gamma_{P,2} & \Gamma_{P,k-1} & \Gamma_{P,k} \\
2 & 2 & \cdots & 2 & 1
\end{array}
$$

2. Conversely, for a proper birational morphism $\psi: Y \to X$ between nonsingular surfaces such that $-K_Y$ is $\psi$-nef, the morphism $\psi$ is the elimination of $\Delta$ which satisfies the $(\nu 1)$-condition defined by the ideal $\mathcal{I}_\Delta := \psi_*\mathcal{O}_Y(-K_{Y/X})$. 
Definition 2.4. Under the assumption of Proposition 2.3, we always denote the exceptional curves of $\psi$ over $P$ by $\Gamma_{P,1}, \ldots, \Gamma_{P,k}$. The order is determined as Proposition 2.3.

Now we see some examples of the dual graphs of $E^\Delta,s$.

Example 2.5 ([Fuj14, Example 2.5]). Assume that $\Delta$ satisfies the $(\nu_1)$-condition such that $|\Delta| = \{ P \}$. Let $E = eC$ be a divisor on $X$ such that $P \in C$ and $C$ is nonsingular. Let $m := \deg \Delta$ and $k := \text{mult}_P(\Delta \cap C)$. Then we have

$$E^\Delta,s = eC^Y + \sum_{i=1}^{k} i(e - s)\Gamma_{P,i} + \sum_{i=k+1}^{m} (ek - si)\Gamma_{P,i},$$

where $\psi: Y \to X$ is the elimination of $\Delta$. Moreover, the dual graph of $\psi^{-1}(E)$ is the following:

```
2
\Gamma_{P,1} \quad \Gamma_{P,k} \quad \Gamma_{P,m-1} \quad \Gamma_{P,m}
```

```
\bigcirc_{\nabla Y}
2
\bigcirc C_Y
```

Example 2.6. ([Fuj14, Example 2.6]) Assume that $\Delta$ satisfies the $(\nu_1)$-condition such that $|\Delta| = \{ P \}$. Let $E = e_1C_1 + e_2C_2$ be a non-zero effective divisor such that $C_1$ and $C_2$ are nonsingular and intersect transversally at a unique point $P = C_1 \cap C_2$. Let $m := \deg \Delta$ and $k_j := \text{mult}_P(\Delta \cap C_j)$. By [Nak07, Lemma 2.12], we may assume that $k_1 = 1$. Then we have

$$E^\Delta,s = e_1C_1^Y + e_2C_2^Y + \sum_{i=1}^{k_2} (i(e_2 - s) + e_1)\Gamma_{P,i} + \sum_{i=k_2+1}^{m} (e_1 + k_2e_2 - is)\Gamma_{P,i},$$

where $\psi: Y \to X$ is the elimination of $\Delta$. Moreover, the dual graph of $\psi^{-1}(E)$ is the following:

```
\bigcirc_{\bigcirc}
2
C_1^Y \quad \bigcirc \quad \Gamma_{P,1} \quad \Gamma_{P,k_2} \quad \Gamma_{P,m-1} \quad \Gamma_{P,m}
```

```
\bigcirc_{\bigcirc}
2
C_2^Y
```

3. Log del Pezzo surfaces

We define the notion of log del Pezzo surfaces, $a$-basic pairs, and $a$-fundamental multiplets, and we see the correspondence among them.
3.1. Log del Pezzo surfaces and \( a \)-basic pairs.

**Definition 3.1.** (1) A normal projective surface \( S \) is called a log del Pezzo surface if \( S \) is log-terminal and the anti-canonical divisor \( -K_S \) is an ample \( \mathbb{Q} \)-Cartier divisor.

(2) Let \( S \) be a log del Pezzo surface. The index of \( S \) is defined as
\[
\min\{a \in \mathbb{Z}_{>0} \mid -aK_S \text{ is Cartier}\}.
\]

**Remark 3.2.** Any log del Pezzo surface is a rational surface by [Nak07, Proposition 3.6]. In particular, the Picard group \( \text{Pic}(S) \) of \( S \) is a finitely generated and torsion-free Abelian group.

**Definition 3.3** ([FY14, Definition 3.3]). Fix \( a \geq 2 \). A pair \( (M, E_M) \) is called an \( a \)-basic pair if the following conditions are satisfied:

\(\begin{align*}
(C_1) & \ M \text{ is a nonsingular projective rational surface.} \\
(C_2) & \ E_M \text{ is a nonzero effective divisor on } M \text{ such that } \text{coeff } E_M \subseteq \{1, \ldots, a-1\} \text{ and } |E_M| \text{ is simple normal crossing.} \\
(C_3) & \ \text{A divisor } L_M \sim -aK_M - E_M \text{ (called the fundamental divisor of } (M, E_M)) \text{ satisfies that } K_M + L_M \text{ is nef and } (K_M + L_M \cdot L_M) > 0. \\
(C_4) & \ \text{For any irreducible component } C \leq E_M, (L_M \cdot C) = 0 \text{ holds.}
\end{align*}\)

We see the correspondence between log del Pezzo surfaces and \( a \)-basic pairs.

**Proposition 3.4** ([FY14, Proposition 3.4]). Fix \( a \geq 2 \).

(1) Let \( S \) be a non-Gorenstein log del Pezzo surface such that \( -aK_S \) is Cartier. Let \( \alpha : M \to S \) be the minimal resolution of \( S \) and let \( E_M := -aK_{M/S} \). Then \( (M, E_M) \) is an \( a \)-basic pair and the divisor \( \alpha^*(-aK_S) \) is the fundamental divisor of \( (M, E_M) \).

(2) Let \( (M, E_M) \) be an \( a \)-basic pair and \( L_M \) be the fundamental divisor of \( (M, E_M) \). Then there exists a projective and birational morphism \( \alpha : M \to S \) such that \( S \) is a non-Gorenstein log del Pezzo surface with \( -aK_S \) Cartier and \( L_M \sim \alpha^*(-aK_S) \) holds. Moreover, the morphism \( \alpha \) is the minimal resolution of \( S \).

In particular, under the correspondence between \( S \) and \( (M, E_M) \), we have \( a(-K_S^2) = (1/a)(L_M^2) \), where \( L_M \) is the fundamental divisor.

**Definition 3.5.** Let \( a \geq 2 \).

(1) For a log del Pezzo surface \( S \) of index \( a \), the corresponding \( a \)-basic pair \( (M, E_M) \) which is given in Proposition 3.4 (1) is called the associated \( a \)-basic pair of \( S \).

(2) For an \( a \)-basic pair \( (M, E_M) \), the corresponding log del Pezzo surface \( S \) which is given in Proposition 3.4 (2) is called the associated log del Pezzo surface of \( (M, E_M) \). We note that \( a \) is divisible by the index of \( S \).
We discuss that when the log del Pezzo surface $S$ associated to an $a$-basic pair $(M, E_M)$ is of index $a$.

**Lemma 3.6.** Let $S$ be a log del Pezzo surface of index $a \geq 2$ and $(M, E_M)$ be the associated $a$-basic pair. Pick any irreducible component $C \leq E_M$ and set $e := \text{coeff}_C E_M$ and $d := -(C^2)$. Then $2 \leq d \leq 2a/(a - e)$. In particular, $d \leq 2a$.

**Proof.** By Proposition 3.4, $d \geq 2$. We know that $-ed \leq (E_M \cdot C) = (-a K_M \cdot C) = a(2 - d)$ since $(L_M \cdot C) = 0$, where $L_M$ is the fundamental divisor. Thus $d \leq 2a/(a - e)$. Since $e \leq a - 1$, we have $d \leq 2a$. □

**Corollary 3.7.** Let $(M, E_M)$ be an $a$-basic pair with $a \geq 2$. Assume that there exists an irreducible component $C \leq E_M$ such that either $(C^2) < -a$, or $\text{coeff}_C E_M$ and $a$ are coprime. Then the associated log del Pezzo surface $S$ is of index $a$.

**Proof.** Let $a'$ be the index of $S$. Then $a' > 1$ and $a$ is divisible by $a'$. Moreover, $\text{coeff}_C E_M$ is divisible by $a/a'$. Thus the assertion follows from Lemma 3.6. □

### 3.2. $a$-fundamental multiplets

In order to determine $a$-basic pairs, we define the notion of $a$-fundamental multiplets given in [FY14, §1].

**Definition 3.8.** Fix $a \geq 2$ and $1 \leq i \leq a - 1$. We define the following notion inductively. A multiplet $(M_i, E_i; \Delta_1, \ldots, \Delta_i)$ is called an $a$-pseudo-fundamental multiplet of length $i$ if the following conditions are satisfied:

1. $(F1)$ $M_i$ is a nonsingular projective rational surface and $E_i$ is a nonzero effective divisor on $M_i$.
2. $(F2)$ A divisor $L_i \sim -a K_{M_i} - E_i$ (called the fundamental divisor) satisfies that $iK_{M_i} + L_i$ is nef and $((i + 1)K_{M_i} + L_i \cdot \gamma) \geq 0$ for any $(-1)$-curve $\gamma \subset M_i$.  
3. $(F3)$ $\Delta_i \subset M_i$ is a zero-dimensional subscheme which satisfies the $(\nu1)$-condition. 
4. $(F4)$ Let $\pi_i : M_{i-1} \to M_i$ be the elimination of $\Delta_i$ and $E_{i-1} := E_i^{A_i, a-i}$. Then the following holds:
   - If $i = 1$, then $(M_0, E_0)$ is an $a$-basic pair.
   - If $i \geq 2$, then $(M_{i-1}, E_{i-1}; \Delta_1, \ldots, \Delta_{i-1})$ is an $a$-pseudo-fundamental multiplet of length $i - 1$.

Moreover, if $(i + 1)K_{M_i} + L_i$ is not nef, then we call the multiplet an $a$-fundamental multiplet of length $i$.

**Definition 3.9.** (1) Let $(M_i, E_i; \Delta_1, \ldots, \Delta_i)$ be an $a$-pseudo-fundamental multiplet. We call the $a$-basic pair $(M_0, E_0)$ (constructed from the multiplet inductively) as the associated $a$-basic pair.
(2) We sometimes call $a$-basic pairs as $a$-pseudo-fundamental multipllets of length zero for convenience.

The following proposition is important.

**Proposition 3.10.** Fix $a \geq 2$ and $0 \leq i \leq a-1$. Let $(M_i, E_i; \Delta_1, \ldots, \Delta_i)$ be an $a$-pseudo-fundamental multiplet of length $i$, $\pi_j: M_{j-1} \to M_j$ be the elimination of $\Delta_j$, $E_{j-1} := E^\Delta_{j, a-j}$ and $L_j$ be the fundamental divisor of $(M_j, E_{j}; \Delta_1, \ldots, \Delta_j)$ for any $0 \leq j \leq i$.

1. Assume that $(i+1)K_M + L_i$ is nef. Then $i \leq a-2$ and $iK_M + L_i$ is nef and big. Moreover, there exists a projective and birational morphism $\pi_{i+1}: M_i \to M_{i+1}$ between nonsingular surfaces such that the following conditions are satisfied:
   - There exists a zero-dimensional subscheme $\Delta_{i+1} \subset M_{i+1}$ which satisfies the $(\nu 1)$-condition such that $\pi_{i+1}$ is the elimination of $\Delta_{i+1}$.
   - Set $E_{i+1} := (\pi_{i+1})_E E_i$ and $L_{i+1} := (\pi_{i+1})_L L_i$. Then $E_i = E^\Delta_{i, a-i} + 1$ and $L_i = L^\Delta_{i+1, i+1}$.
   - $(M_{i+1}, E_{i+1}; \Delta_1, \ldots, \Delta_{i+1})$ is an $a$-pseudo-fundamental multiplet of length $i+1$ and $L_{i+1}$ is the fundamental divisor.

2. Assume that $(i+1)K_M + L_i$ is not nef, that is, the multiplet $(M_i, E_i; \Delta_1, \ldots, \Delta_i)$ is an $a$-fundamental multiplet. Then $M_i$ is isomorphic to either $\mathbb{P}^2$ or $\mathbb{F}_a$, and $((i+1)K_M + L_i \cdot l) < 0$, where $l$ is a line (if $M_i \cong \mathbb{P}^2$; a fiber (if $M_i \cong \mathbb{F}_a$).

3. $L_i$ is nef and big. Moreover, we have the following:
   - $(L_i \cdot E_i) = \sum_{j=1}^i j(a-j) \deg \Delta_j$.
   - $(K_M + L_i \cdot L_i) - (K_{M_0} + L_0 \cdot L_0) = \sum_{j=1}^i j(j-1) \deg \Delta_j$.
   - $(L_i \cdot C) = \sum_{j=1}^i j \deg(\Delta_j \cap C^j)$ for any nonsingular component $C \leq E_i$.
   - $(1/a)(L_0^2) = (K_M \cdot L_i) - \sum_{j=1}^i j \deg \Delta_j$.

**Proof.** We can assume by induction on $i$ that $L_{i-1}$ is nef and big and $L_{i-1} = L^\Delta_{i-1}$ if $i \geq 1$. Thus $L_i$ is nef and big.

1. Assume that $(i+1)K_M + L_i$ is nef. Since $a((i+1)K_M + L_i) \sim -(i+1)E_i + (a - (i + 1))L_i$, we must have $a - (i + 1) > 0$. Moreover, since $(i+1)(iK_M + L_i) = i((i+1)K_M + L_i) + L_i$, $iK_M + L_i$ is nef and big. From now on, we run $((i+2)K_M + L_i)$-minimal model program and let $\pi_{i+1}: M_i \to M_{i+1}$ be the composition of the morphisms in the program. More precisely, we obtain the morphism $\pi_{i+1}$ which is the composition of monoidal transforms such that each exceptional $(-1)$-curve intersects the strict transform of $(i+2)K_M + L_i$ negatively. Furthermore, $((i+2)K_{M+i+1} + L_{i+1} \cdot \gamma) \geq 0$ holds for any $(-1)$-curve


\( \gamma \subset M_{i+1} \). Since \((i+1)K_M + L_1\) is nef, any \((-1)\)-curve in each step of the monoidal transform intersects the strict transform of \((i+1)K_M + L_i\) trivially. Thus \((i+1)K_{M_i} + L_i = \pi_{i+1}(i+1)K_{M_{i+1}} + L_{i+1}\). In particular, 
\(-K_{M_i}\) is \(\pi_{i+1}\)-nef. By Proposition 2.3 there exists a zero-dimensional subscheme \(\Delta_{i+1} \subset M_{i+1}\) which satisfies the \((\nu 1)\)-condition such that \(\pi_{i+1}\) is the elimination of \(\Delta_{i+1}\). Furthermore, we have \(L_i = L^i_{i+1} + 1\) and \(E_i = E^i_{i+1,a-i} - 1\). Since each step of the \(((i+2)K_{M_i} + L_i)\)-minimal model program can be seen as a step of \((-E_i)\)-minimal model program, \(E_i + 1\) is nonzero effective. Thus the assertion \((\dagger)\) follows.

Following from [Mor82 Theorem 2.1].

We know that \((-K_{M_i} \cdot L_i) = (-K_{M_{i-1}} \cdot L_{i-1}) - i(K^2_{M_{i-1}/K_{M_i}}) = (-K_{M_{i-1}} \cdot L_{i-1}) + i \deg \Delta_i\). Thus we have \((1/a)(L_0^2) = (-K_{M_i} \cdot L_i) - \sum_{j=1}^i j \deg \Delta_j\) by induction on \(i\). Other assertions follow similarly. \(\Box\)

As a direct corollary of Proposition 3.11 \((\dagger)\), we have the following.

**Corollary 3.11.** Fix \(a \geq 2\). Let \((M, E_M)\) be an \(a\)-basic pair. Then there exists a positive integer \(1 \leq b \leq a - 1\) and an \(a\)-fundamental multiplet \((M_b, E_b; \Delta_1, \ldots, \Delta_b)\) of length \(b\) such that the associated \(a\)-basic pair is isomorphic to \((M, E_M)\).

From the next proposition, we can replace an \(a\)-fundamental multiplet with another one. The proof is same as that of [Nak07 Proposition 4.4]. See also [Fuj14 Proposition 3.14] and [FY14 Theorem 3.12].

**Proposition 3.12.** Fix \(a \geq 2\) and \(1 \leq b < a\). Let \((M_b, E_b; \Delta_1, \ldots, \Delta_b)\) be an \(a\)-fundamental multiplet of length \(b\) and \(L_b\) be the fundamental divisor. Assume that \(M_b = \mathbb{F}_n\) and \(bK_{M_b} + L_b\) is not big and not trivial. Then there exists an \(a\)-fundamental multiplet \((M'_b, E'_b; \Delta_1, \ldots, \Delta_{b-1}, \Delta'_b)\) such that the associated \(a\)-pseudo-fundamental multiplets of length \(b - 1\) are same, \(M'_b = \mathbb{F}_{n'}\) for some \(n' \geq 0\) and \(\Delta'_b \cap \sigma' = \emptyset\), where \(\sigma' \subset \mathbb{F}_{n'}\) is a section with \((\sigma'^2) = -n'\).

The next proposition is useful to know when a given multiplet is an \(a\)-pseudo-fundamental multiplet.

**Proposition 3.13.** Fix \(1 \leq i < a\). Let \(X\) be a nonsingular projective surface, \(E\) be a divisor on \(X\), \(L \sim -aK_X - E\) be a divisor such that \(iK_X + L\) is nef, \(\Delta \subset X\) be a zero-dimensional subscheme which satisfies the \((\nu 1)\)-condition, \(\psi: Y \to X\) be the elimination of \(\Delta\), \(L_Y := L^i\) and \(E_Y := E^i_{a-i}\). If \(E_Y\) is effective and \((L_Y \cdot C) \geq 0\) for any irreducible component \(C \leq E_Y\), then \(jK_Y + L_Y\) is nef for any \(0 \leq j \leq i\).

**Proof.** Assume that there exists an integer \(0 \leq j \leq i\) such that \(jK_Y + L_Y\) is non-nef. Since \(iK_Y + L_Y = \psi^!(iK_X + L)\), we have \(j < i\). Pick
an irreducible curve $B \subset Y$ such that $(jK_Y + L_Y \cdot B) < 0$. Then
\[
0 > (a - i)(jK_Y + L_Y \cdot B) = (a - j)(iK_Y + L_Y \cdot B) + (i - j)(E_Y \cdot B) \geq (i - j)(E_Y \cdot B).
\]
Thus $B \leq E_Y$. Hence $(L_Y \cdot B) \geq 0$. However,
\[
0 > i(jK_Y + L_Y \cdot B) = j(iK_Y + L_Y \cdot B) + (i - j)(L_Y \cdot B) \geq 0,
\]
which leads to a contradiction. \hfill \Box

### 3.3. Local properties

In this section, we fix the following notation:

Let $1 \leq i < a$, $(M_i, E_i; \Delta_1, \ldots, \Delta_i)$ be an $a$-pseudo-fundamental multiplet of length $i$, $L_i$ be its fundamental divisor, $\pi_j : M_{j-1} \to M_j$ be the elimination of $\Delta_j$, $E_{j-1} := F_{j, a-j}$ and $L_{j-1} := L_{j, a-j}$ for $1 \leq j \leq i$. Moreover, we fix a point $P \in \Delta_i$.

**Lemma 3.14.**

1. The multiplicity of $E_i$ at $P$ is bigger than or equal to $a - i$.
2. Pick any $\pi_{i}$-exceptional curve $\Gamma \subset M_{i-1}$. If $(\Gamma^2) = -1$, then $(L_{i-1} \cdot \Gamma) = i$. If $(\Gamma^2) = -2$, then $(L_{i-1} \cdot \Gamma) = 0$.
3. Assume that $E_i = eC$ around $P$, where $P \in C$ and $C$ is nonsingular and $e \leq a - i$. Then $e = a - i$, $\Delta_i \subset C$ around $P$ and $E_{i-1}$ is the strict transform of $E_i$ around over $P$.
4. Assume that $E_i = (a - 1)C$ around $P$, where $P \in C$ and $C$ is nonsingular. If $2i \leq a + 1$, $\text{mult}_P(\Delta_i \cap C) = 1$ and $\text{mult}_P \Delta_i \geq 2$, then $2i = a + 1$ and $\text{mult}_P \Delta_i = 2$.

**Proof.**

1. The value $\text{coeff}_{\Gamma_{P,i}} E_{i-1}$ must be nonnegative.
2. Follows from the fact $(iK_{M_{i-1}} + L_{i-1} \cdot \Gamma) = 0$.
3. Set $m := \text{mult}_P \Delta_i$ and $k := \text{mult}_P(\Delta_i \cap C)$. By Example 2.5, $\text{coeff}_{\Gamma_{P,i}} E_{i-1} = e - (a - i)$. Thus $e = a - i$. If $k < m$, then $\text{coeff}_{\Gamma_{P,m}} E_{i-1} = (a - i)(k - m) < 0$ by Example 2.5 a contradiction.
4. We know that $\text{coeff}_{\Gamma_{P,2}} E_{i-1} = a - 1 - 2(a - i) \leq 0$ by Example 2.5. Thus $2i = a + 1$ and $\text{mult}_P \Delta_i = 2$. \hfill \Box

**Lemma 3.15.** Assume that $a \geq 4$, $i = 1$, $1 \leq e \leq 2$, $E_1 = (a - 1)C_1 + eC_2$ around $P$, where $C_1, C_2$ are nonsingular and intersect transversally at $P$. Then $e = 2$ and $\text{mult}_P \Delta_1 = \text{mult}_P(\Delta_1 \cap C_2)$. Moreover, $(a, \text{mult}_P \Delta_1) = (5, 2)$ or $(4, 3)$.

**Proof.** Set $m := \text{mult}_P \Delta_1$ and $k : = \text{mult}_P(\Delta_1 \cap C_j)$. By Lemma 3.6, $\text{coeff}_{\Gamma_{P,m}} E_0 = 0$. If $k_1 > 1$, then $k_2 = 1$ by Example 2.6. However, if $m = k_1$, then $\text{coeff}_{\Gamma_{P,m}} E_0 = e$; if $m > k_1$, then $\text{coeff}_{\Gamma_{P,k+1}} E_0 = e - (a - 1) < 0$. This leads to a contradiction. Thus $k_1 = 1$. If $m > k_2$, then $\text{coeff}_{\Gamma_{P,m}} E_0 = a - 1 + ek_2 - m(a - 1) < 0$ by Example.
Lemma 3.16. Assume that \( i \geq 2 \), \( \Delta_j = \emptyset \) for any \( 1 \leq j < i \), \( E_i = (a - i + 1)C \) around \( P \) and \( \text{mult}_P \Delta_i < 2(a - i + 1) \), where \( P \in C \) and \( C \) is nonsingular. Then \( \text{mult}_P \Delta_i = a - i + 1 \) and \( \text{mult}_P(\Delta_i \cap C) = a - i \).

Proof. Set \( m := \text{mult}_P \Delta_i \) and \( k := \text{mult}_P(\Delta_i \cap C) \). By Lemma 3.6, \( E_{i-1} (= E_0) \) does not contain any \((-1)\)-curve. Thus \( 0 = \text{coeff}_{P,m} E_{i-1} = (a - i + 1)k - (a - i)m \) by Example 2.5. Thus the assertion follows. \( \square \)

4. Examples

In this section, we see the \( a \)-fundamental multiplets and the log del Pezzo surfaces which appeared in Section 1.

4.1. Special \( a \)-fundamental multiplets.

Example 4.1. Let \( a \geq 2 \) and \( b := [(a + 1)/2] \). We consider the following \( a \)-fundamental multiplets \((M_b, E_b; \Delta_1, \ldots, \Delta_b)\) of length \( b \):

- \((\langle O \rangle)_a\): \( M_b = \mathbb{F}_{2a}, E_b = (a - 1)\sigma, \Delta_b = 0, \ldots, \Delta_1 = 0 \).
- \((\langle I \rangle)_a\): \( M_b = \mathbb{F}_{2a-1}, E_b = (a - 1)\sigma, \Delta_b = 0, \ldots, \Delta_2 = 0, \Delta_1 \subset \sigma^1 \) with \( \text{deg} \Delta_1 = 1 \).
- \((\langle II \rangle)_a\): \( M_b = \mathbb{F}_{2a-2}, E_b = (a - 1)\sigma, \Delta_b = 0, \ldots, \Delta_2 = 0, \Delta_1 \subset \sigma^1 \) with \( \text{deg} \Delta_1 = 2 \).
- \((\langle III \rangle)_a\): \( M_b = \mathbb{F}_{2a-3}, E_b = (a - 1)\sigma, \Delta_b = 0, \ldots, \Delta_2 = 0, \Delta_1 \subset \sigma^1 \) with \( \text{deg} \Delta_1 = 3 \).
- \((\langle IV \rangle)_a\): \( M_b = \mathbb{F}_{2a-4}, E_b = (a - 1)\sigma, \Delta_b = 0, \ldots, \Delta_2 = 0, \Delta_1 \subset \sigma^1 \) with \( \text{deg} \Delta_1 = 4 \).

By Proposition 3.13, it is easy to check that these multiplets are exactly \( a \)-fundamental multiplets. Moreover, all of the associated \( a \)-basic pairs \((M_0, E_0)\) satisfy that \( E_0 = (a - 1)\sigma^0 \) and \((\sigma^0)^2) = -2a \). Thus the associated log del Pezzo surfaces \( S \) are of index \( a \) by Corollary 3.7. Furthermore, the value \((-K_S^2)\) is equal to \((2a^2 + 4a + 2)/a \) (if \( \langle O \rangle_a \)), \((2a^2 + 3a + 2)/a \) (if \( \langle I \rangle_a \)), \((2a^2 + 2a + 2)/a \) (if \( \langle II \rangle_a \)), \((2a^2 + a + 2)/a \) (if \( \langle III \rangle_a \)), \((2a^2 + 2)/a \) (if \( \langle IV \rangle_a \)) by Proposition 3.10 (3).

Example 4.2. We consider the case \((a, b) = (5, 3)\). Consider the following 5-fundamental multiplet \((M_3, E_3; \Delta_1, \Delta_2, \Delta_3)\) of length three:

- \((\langle A \rangle)_5\): \( M_3 = \mathbb{F}_8, E_3 = 4\sigma + 2l, \Delta_3 \subset l \setminus \sigma \) with \( \text{deg} \Delta_3 = 2, \Delta_2 = 0 \) and \( \Delta_1 = 0 \).

By Proposition 3.13, it is easy to check that the multiplet is exactly a 5-fundamental multiplet. Moreover, the associated 5-basic pair \((M_0, E_0)\) satisfies that \( E_0 = 4\sigma^0 + 2l^0 \) and the dual graph of \( E_0 \) is the following:
Furthermore, the associated log del Pezzo surface $S$ of index five satisfies that the value $(-K_S^2)$ is equal to 54/5 by Proposition 3.10 (3).

**Example 4.3.** We consider the case $(a, b) = (4, 2)$. Consider the following 4-fundamental multiplets $(M, E; \Delta_1, \Delta_2)$ of length two:

$\langle B \rangle_4$: $M_2 = \mathbb{F}_4$, $E_2 = 3\sigma$, $|\Delta_2| = \{P\}$ with $P \in \sigma$ such that $\deg \Delta_2 = 3$, $\deg(\Delta_2 \cap \sigma) = 2$ and $\Delta_1 = \emptyset$.

$\langle C \rangle_4$: $M_2 = \mathbb{F}_5$, $E_2 = 3\sigma + 2l$, $\deg \Delta_2 = 1$ with $\Delta_2 \subset l \setminus \sigma$, $|\Delta_1| = \{P\}$ such that $P = \sigma_1 \cap l_1$ and $\deg \Delta_1 = \deg(\Delta_1 \cap l_1) = 3$.

By Proposition 3.13 it is easy to check that these multiplets are exactly 4-fundamental multiplets. Moreover, the associated 4-basic pairs $(M_0, E_0)$ satisfy the following:

The case $\langle B \rangle_4$: $E_0 = 3\sigma^0 + 2\Gamma^0_{P,2} + \Gamma^0_{P,1}$ and the dual graph of $E_0$ is the following:

\[
\begin{array}{ccc}
\sigma^0 & \Gamma^0_{P,2} & \Gamma^0_{P,1} \\
6 & 2 & 2 \\
\end{array}
\]

The case $\langle C \rangle_4$: $E_0 = 3\sigma^0 + 2\Gamma_{P,1} + \Gamma_{P,2} + 2l^0$ and the dual graph of $E_0$ is the following:

\[
\begin{array}{ccc}
\sigma^0 & \Gamma_{P,1} & \Gamma_{P,2} & l^0 \\
6 & 2 & 2 & 2 \\
\end{array}
\]

Thus the associated log del Pezzo surfaces $S$ are of index 4 by Corollary 3.7. Furthermore, the value $(-K_S^2)$ is equal to 8 in any case by Proposition 3.10 (3).

**Remark 4.4.** Let $S$ be a log del Pezzo surface of index $a$ such that $(-K_S^2) \geq 2a$.

1. Assume that $a = 2$. Then $S$ is isomorphic to the log del Pezzo surface associated to the fundamental triplet in the sense of [Nak07] whose type is one of the following:
   - $[4; 1, 0]_0$ $((-K_S^2) = 9)$,
   - $[3; 1, 0]_0$ $((-K_S^2) = 8)$,
   - $[2; 1, 0]_0$ $((-K_S^2) = 7)$,
   - $[1; 1, 0]_0$, $[3; 1, 1]_+$ $((-K_S^2) = 6)$,
   - $[0; 1, 0]_0$, $[2; 1, 1]_+(a, b)$ $((-K_S^2) = 5)$,
   - $[1; 1, 1]_0$, $[1; 1, 1]_+(a, b)$ $((-K_S^2) = 4)$. 


We note that the types \([4; 1, 0]_0, [3; 1, 0]_0, [2; 1, 0]_0, [1; 1, 0]_0, [0; 1, 0]_0\) in Nak07 are nothing but the types \((O)_2, (I)_2, (II)_2, (III)_2, (IV)_2\) in Example 4.1, respectively.

(2) Assume that \(a = 3\). Then \(S\) is isomorphic to the log del Pezzo surface associated to the bottom tetrad in the sense of FY14 whose type is one of the following:

- \([6;2,0]\) \(((-K^2_S) = 32/3)\),
- \([5;2,0]\) \(((-K^2_S) = 29/3)\),
- \([4;2,0]\) \(((-K^2_S) = 26/3)\),
- \([3;1,0]\) \(((-K^2_S) = 25/3)\),
- \([5;2,1]_1\) \(((-K^2_S) = 8)\),
- \([3;2,0]\) \(((-K^2_S) = 23/3)\),
- \([2;1,0]\), \([4;2,1]_{1B}\) \(((-K^2_S) = 22/3)\),
- \([4;2,1]_{1A}\) \(((-K^2_S) = 7)\),
- \([4;2,2]_{1F}\) \(((-K^2_S) = 20/3)\),
- \([1;1,0]\), \([2;2,0]\), \([3;2,1]_{1B}\), \([4;2,2]_{1E}\) \(((-K^2_S) = 19/3)\),
- \([3;2,1]_{1A}\), \([4;2,2]_{1C}\), \([4;2,2]_{1D}\), \(((-K^2_S) = 6)\).

We note that the types \([6;2,0]\), \([5;2,0]\), \([4;2,0]\), \([3;2,0]\), \([2;2,0]\) in FY14 are nothing but the types \((O)_3, (I)_3, (II)_3, (III)_3, (IV)_3\) in Example 4.1, respectively. Moreover, the log del Pezzo surface of index three associated to the bottom tetrad of type \([3;1,0]\) is isomorphic to the weighted projective plane \(\mathbb{P}(1, 1, 3)\).

4.2. Special log del Pezzo surfaces. In this section, we determine the log del Pezzo surfaces associated to the \(a\)-fundamental multiplets of types \((O)_a\), \((I)_a\) and \((II)_a\). We freely use the notation of the toric geometry in this section. See Ful93 for example. We fix a lattice \(N := \mathbb{Z}^{\oplus 2}\) and set \(N_R := N \otimes \mathbb{R}(\simeq \mathbb{R}^{\oplus 2})\).

Example 4.5. Fix \(a \geq 2\). It is well-known that the weighted projective plane \(\mathbb{P}(1, 1, 2a)\) is a log del Pezzo surface of index \(a\); the associated \(a\)-basic pair is equal to \((\mathbb{F}_{2a}, (a - 1)\sigma)\). Thus the log del Pezzo surface associated to the \(a\)-fundamental multiplet of type \((O)_a\) is isomorphic to \(\mathbb{P}(1, 1, 2a)\).

Example 4.6. Fix \(a \geq 2\). Let \(\Sigma_{1,a}\) be the complete fan in \(N_R\) such that the set of the generators of one-dimensional cones in \(\Sigma_{1,a}\) is

\[\{(1, 0), (0, 1), (-1, -2a + 1), (-1, -2a)\}\]

Let \(S_{1,a}\) be the projective toric surface associated to the fan \(\Sigma_{1,a}\). Then the minimal resolution \(M_{1,a}\) of \(S_{1,a}\) corresponds to the complete fan \(\Sigma'_{1,a}\) such that the set of the generators of one-dimensional cones in \(\Sigma'_{1,a}\) is

\[\{(1, 0), (0, 1), (-1, -2a + 1), (-1, -2a), (0, -1)\}\].
Moreover, the divisor $-aK_{M_{1,a}/S_{1,a}}$ is equal to $(a - 1)V(\mathbb{R}_{\geq 0}(0, -1))$, where $V(\mathbb{R}_{\geq 0}(0, -1))$ is the torus-invariant prime divisor associated to the cone $\mathbb{R}_{\geq 0}(0, -1) \in \Sigma'_{1,a}$. We note that $M_{1,a}$ is isomorphic to the variety $\mathbb{F}_{2a-1}$ blowing upped along a point on $\sigma$. Moreover, $V(\mathbb{R}_{\geq 0}(0, -1))$ corresponds to the strict transform of $\sigma \subset \mathbb{F}_{2a-1}$. Therefore the log del Pezzo surface associated to the $a$-fundamental multiplet of type $(I)_a$ is isomorphic to $S_{1,a}$.

**Example 4.8.** Fix $a \geq 2$. Let $\Sigma_{II_1,a}$ be the complete fan in $N_{\mathbb{R}}$ such that the set of the generators of one-dimensional cones in $\Sigma_{II_1,a}$ is

$$\{(1, 0), (0, 1), (-1, -2a + 2), (-1, -2a), (0, -1)\}.$$  

Let $S_{II_1,a}$ be the projective toric surface associated to the fan $\Sigma_{II_1,a}$. Then the minimal resolution $M_{II_1,a}$ of $S_{II_1,a}$ corresponds to the complete fan $\Sigma'_{II_1,a}$ such that the set of the generators of one-dimensional cones in $\Sigma'_{II_1,a}$ is

$$\{(1, 0), (0, 1), (-1, -2a + 2), (-1, -2a + 1), (-1, -2a), (0, -1)\}.$$  

Moreover, the divisor $-aK_{M_{II_1,a}/S_{II_1,a}}$ is equal to $(a - 1)V(\mathbb{R}_{\geq 0}(0, -1))$. We note that the pair $(M_{II_1,a}, (a - 1)V(\mathbb{R}_{\geq 0}(0, -1)))$ is isomorphic to the $a$-basic pair associated to the $a$-fundamental multiplet of type $(II)_a$ such that $\#|\Delta_1| = 1$. Therefore the log del Pezzo surface associated to the $a$-fundamental multiplet of type $(II)_a$ such that $\#|\Delta_1| = 1$ is isomorphic to $S_{II_1,a}$.

**Example 4.8.** Fix $a \geq 2$. Let $\Sigma_{II_2,a}$ be the complete fan in $N_{\mathbb{R}}$ such that the set of the generators of one-dimensional cones in $\Sigma_{II_2,a}$ is

$$\{(1, 0), (0, 1), (-1, -2a + 2), (-1, -2a + 1), (1, -1)\}.$$  

Let $S_{II_2,a}$ be the projective toric surface associated to the fan $\Sigma_{II_2,a}$. Then the minimal resolution $M_{II_2,a}$ of $S_{II_2,a}$ corresponds to the complete fan $\Sigma'_{II_2,a}$ such that the set of the generators of one-dimensional cones in $\Sigma'_{II_2,a}$ is

$$\{(1, 0), (0, 1), (-1, -2a + 2), (-1, -2a + 1), (0, -1), (1, -1)\}.$$  

Moreover, the divisor $-aK_{M_{II_2,a}/S_{II_2,a}}$ is equal to $(a - 1)V(\mathbb{R}_{\geq 0}(0, -1))$. We note that the pair $(M_{II_2,a}, (a - 1)V(\mathbb{R}_{\geq 0}(0, -1)))$ is isomorphic to the $a$-basic pair associated to the $a$-fundamental multiplet of type $(II)_a$ such that $\#|\Delta_1| = 2$. Therefore the log del Pezzo surface associated to the $a$-fundamental multiplet of type $(II)_a$ such that $\#|\Delta_1| = 2$ is isomorphic to $S_{II_2,a}$.

From the arguments in this section, Corollary 1.2 is deduced from Theorem 1.1 immediately.
5. Proof of Theorem 1.1

In this section, we prove Theorem 1.1. From now on, let $a \geq 4$, $S$ be a log del Pezzo surface of index $a$ with $(-K_S^2) \geq 2a$, $(M_b, E_0)$ be the associated $a$-basic pair, $(M_b, E_0; \Delta_1, \ldots, \Delta_b)$ be an $a$-fundamental multiplet of length $b$ with $1 \leq b \leq a-1$ such that the associated $a$-basic pair is equal to $(M_0, E_0)$, and $L_b$ be the fundamental divisor of the multiplet. We remark that the existence of such multiplet is proven in Corollary 3.11. By Proposition 3.12, if $b$ is a multiplet of length $\sum b_i \geq 2$ such that $\Delta_0$ in Corollary 3.11. By Proposition 3.12, if $b$ is a multiplet of length $\sum b_i \geq 2$ such that $\Delta_0$ is isomorphic to $F$. Therefore, we can assume that $\Delta_b \cap \Delta = \emptyset$. We know that $a(-K^2_b) = (-K_b \cdot L_b) - \sum_{i=1}^b i \deg \Delta_i$ by Proposition 3.10 (2).

5.1. Structure of $M_b$. In this section, we prove the following lemma.

Lemma 5.1. $M_b$ is isomorphic to $F_n$ for some $n \geq 0$.

**Proof.** Assume not. By Proposition 3.10 (2), we can assume that $M_b = \mathbb{P}^2$. Set $h \in \mathbb{Z}_{>0}$ such that $L_b \sim hl$, where $l$ is a line. By Proposition 3.10, $h \leq 3b+2$ and $2a^2 \leq a(-K^2_b) = 3h - \sum_{i=1}^b i \deg \Delta_i \leq 3(3b+2) \leq 3(3a-1)$. Thus $a = 4$, $b = a-1 = 3$, $h = 3b+2 = 11$ and $\sum_{i=1}^b i \deg \Delta_i \leq 1$. However, $11 = (L_b \cdot E_b) = \sum_{i=1}^3 i(4-i) \deg \Delta_i \leq 4 \sum_{i=1}^3 i \deg \Delta_i \leq 4$. This leads to a contradiction. □

Therefore, we can assume that $M_b = \mathbb{P}^2$. Set $h_0, h \in \mathbb{Z}$ such that $L_b \sim h_0 \sigma + hl$. Then $E_b \sim (2a-h_0)\sigma + ((n+2)a-h)l$ and $a(-K^2_b) = -nh_0 + 2h_0 + 2h - \sum_{i=1}^b i \deg \Delta_i$. We note that $b = \lfloor h_0/2 \rfloor$ by Proposition 3.10 (2). Moreover, $h \geq nh_0$ and $h_0 \geq 1$ since $L_b$ is nef and big. In particular, $0 < h_0 < 2a$.

5.2. Determine the value $h_0$. In this section, we prove that $h_0 = a+1$ and $b = \lfloor (a+1)/2 \rfloor$. To begin with, we see the following claim.

Claim 5.2. We have $h_0 \geq a + 1$.

**Proof.** Assume that $h_0 \leq a$. Since $\text{coeff}_a E_b \leq a - 1$, we have $h \leq 2a + n(h_0 - 1)$. Since $h \geq nh_0$, this implies that $n \leq 2a$. If $h_0 \leq a-1$, then $a(-K^2_b) \leq -nh_0 + 2h_0 + 4a + 2n(h_0 - 1) \leq (n+2)(a-1) + 4a - 2n \leq 2a^2 - 2$, which leads to a contradiction. Thus $h_0 = a$ (under the assumption $h_0 \leq a$). If $n \leq 2a - 5$, then $a(-K^2_b) \leq -na + 2a + 4a + 2n(a-1) = (a-2)n + 6a \leq 2a^2 - 3a + 10$, which leads to a contradiction. Thus $2a-4 \leq n \leq 2a$. In particular, $n \geq a$. Moreover, $h \geq 2a + (a-2)n$ since $h \geq an \geq an + 2(a-n)$. If $h = 2a + (a-2)n$, then $a = n = 4$ and $h = 16$. However, in this case, $a(-K^2_b) \leq -4 \cdot 4 + 2 \cdot 4 + 2 \cdot 16 < 32 = 2 \cdot 4^2$, which leads to a contradiction. Thus we have $h > 2a + (a-2)n$. Since
Assume that \( h \) to contradiction. Thus \( h \geq 2a^2 + a(n - 2) \). However, we know that \( h \leq (n + 2)a - n \), which leads to a contradiction. □

By Claim 5.2, \( h_0 \geq a + 1 \). Since \( E_b \) is effective, we have \( h \leq (n + 2)a \). Since \( h \geq nh_0 \), we have \( n \leq 2a/(h_0 - a) \). From now on, we assume that \( h_0 \geq a + 2 \). Then

\[
a(-K_S^2) \leq n(2a - h_0) + 2h_0 + 4a - \sum_{i=1}^{b} i \deg \Delta_i
\]

\[
\leq \frac{2h_0^2}{h_0 - a} - \sum_{i=1}^{b} i \deg \Delta_i \leq \frac{2h_0^2}{h_0 - a} \leq (a + 2)^2.
\]

Thus \( a = 4 \). In this case, \( h_0 = 6 \) or 7. Assume that \( n \leq 2 \). Then \( a(-K_S^2) \leq n(2a - h_0) + 2h_0 + 4a - \sum_{i=1}^{b} i \deg \Delta_i \leq 2(8 - h_0) + 2h_0 + 16 = 32 \). Thus \( n = 2 \), \( h = (n + 2)a = 16 \) and \( \sum_{i=1}^{b} i \deg \Delta_i = 0 \). However, in this case, \( (L_b \cdot E_b) = 2(8 - h_0)^2 \neq 0 \), which leads to a contradiction. Hence \( n \geq 3 \). Thus \( h_0 = 6, b = 3 \), and \( n = 3 \) or 4 since \( n \leq 2a/(h_0 - a) \). Assume that \( h < a(n + 2) = 4(n + 2) \). Since \( a(-K_S^2) = 6(2 - n) + 2h - \sum_{i=1}^{b} i \deg \Delta_i \geq 32 \), we have \( h \geq 4(n + 2) - 2 \) and \( \sum_{i=1}^{b} i \deg \Delta_i \leq 2 \). We note that there exists a fiber \( l \leq E_b \). By Proposition 3.10 (3),

\[6 = h_0 = (L_b \cdot l) = \sum_{i=1}^{b} i \deg(\Delta_i \cap l) \leq \sum_{i=1}^{b} i \deg \Delta_i \leq 2, \text{ which leads to contradiction. Thus } h \text{ must be equal to } 4(n + 2). \]

In particular, \( L_3 \sim 6\sigma + 4(n + 2)l \), \( E_3 = 2\sigma, \sum_{i=1}^{3} i(4 - i) \deg \Delta_i = 16 - 4n \) and \( \sum_{i=1}^{3} i \deg(\Delta_i \cap \sigma) = 8 - 2n \).

Assume that \( n = 3 \). Then \( \Delta_1 = \emptyset, \Delta_3 = \emptyset, \text{ deg } \Delta_2 = 1 \) and \( \Delta_2 \subset \sigma^2 \).

Then the associated log del Pezzo surface is isomorphic to \( S_{1,2} \) having described in Example 4.6. However, the index of \( S_{1,2} \) is equal to two, a contradiction.

Assume that \( n = 4 \). Then \( \Delta_1 = \emptyset, \Delta_2 = \emptyset \) and \( \Delta_3 = \emptyset \). then the associated log del Pezzo surface is isomorphic to \( \mathbb{P}(1, 1, 4) \). However, the index of \( \mathbb{P}(1, 1, 4) \) is equal to two (see Example 4.5), a contradiction.

Consequently, we have \( h_0 = a + 1 \) and \( b = \lfloor h_0/2 \rfloor = \lfloor (a + 1)/2 \rfloor \).
5.3. **Determine the values** $h$ **and** $n$. In this section, we prove that $h \geq 2a^2 - 2a - 2$ and $2a - 4 \leq n \leq 2a$. To begin with, we see the following claim.

**Claim 5.3.** We have $\sigma \leq E_b$.

**Proof.** If $\sigma \not\subset E_b$, then $(n+2)a-h \geq (a-1)n$ since $E_b \sim (a-1)\sigma + ((n+2)a-h)l$. However, in this case, $a(-K_S^2) \leq (-n+2)(a+1)+2(2a+n) = n(1-a) + 2a \leq 6a + 2 < 2a^2$, which leads to a contradiction. $\square$

By Proposition 3.10 (3), we have

$$-n(a+1) + h = (L_b \cdot \sigma) = \sum_{i=1}^{b} i \deg(\Delta_i \cap \sigma^i) \leq \sum_{i=1}^{b} i \deg \Delta_i$$

$$\leq (-n+2)(a+1) + 2h - a(-K_S^2)$$

Thus $h \geq 2a^2 - 2a - 2$. Since $(a+1)n \leq h \leq (n+2)a$, we have $2a - 4 \leq n \leq 2a$.

5.4. **The case** $h = (n+2)a$. We consider the case $h = (n+2)a$. In this case, $L_b \sim (a+1)\sigma + (n+2)al$, $E_b = (a-1)\sigma$, $2a-n = (L_b \cdot \sigma) = \sum_{i=1}^{b} i \deg(\Delta_i \cap \sigma^i)$, $(a-1)(2a-n) = (L_b \cdot E_b) = \sum_{i=1}^{b} i(a-i) \deg \Delta_i$ and $a(-K_S^2) = (n-(2a-4))(a-1) + 2a^2 + 6 - \sum_{i=1}^{b} i \deg \Delta_i$. In particular, $\sum_{i=1}^{b} i \deg \Delta_i \leq 6 + (n-(2a-4))(a-1)$.

If $\Delta_2 = \emptyset, \ldots, \Delta_b = \emptyset$, then $\deg \Delta_1 = 2a - n$ and $\Delta_1 \subset \sigma^1$. For the case $n = 2a$ (resp. $n = 2a-1$, $n = 2a-2$, $n = 2a-3$, $n = 2a-4$), the $a$-fundamental multiplet $(M_b, E_b; \Delta_1, \ldots, \Delta_b)$ is of type $\langle O \rangle$ (resp. $\langle I \rangle$, $\langle II \rangle$, $\langle III \rangle$, $\langle IV \rangle$).

From now on, we assume that $\Delta_{i_0} \neq \emptyset$ for some $2 \leq i_0 \leq b$. We can assume that $\Delta_j = \emptyset$ for any $j \geq i_0 + 1$. Then $\Delta_{i_0} \cap \sigma^{i_0} \neq \emptyset$. Thus $i_0 \leq 2a - n \leq 4$.

Assume that $i_0 = 4$. Then $2a - n = 4$, $\deg(\Delta_4 \cap \sigma^4) = 1$ and $\sum_{i=1}^{b} i \deg \Delta_i \leq 6$. Let $\Gamma \subset M_3$ be the $\pi_4$-exceptional $(-1)$-curve. Then $\Delta \leq E_3$ and $\sum_{i=1}^{3} i \deg(\Delta_i \cap \Gamma^i) = 4$ by Lemma 3.14. This leads to a contradiction.

Assume that $i_0 = 3$. Since $b \geq 3$, we have $a \geq 5$. Moreover, $\deg(\Delta_3 \cap \sigma^3) = 1$, $\deg(\Delta_2 \cap \sigma^2) = 0$ and $\deg(\Delta_1 \cap \sigma^1) = 2a - n - 3 \leq 1$. If $\deg \Delta_3 \geq 2$ (and $i_0 = 3$), then $a = 5$ and $\deg \Delta_3 = 2$ by Lemma 3.14 (4). However, this contradicts to the assumption $\Delta_b \cap \sigma = \emptyset$. Assume that $\deg \Delta_3 = 1$ (and $i_0 = 3$). Let $\Gamma \subset M_2$ be the $\pi_3$-exceptional $(-1)$-curve. Since $\text{coeff}_\Gamma E_2 = 2$, we have $\Delta_2 = \emptyset$ and $|\Delta_1| = \{P\}$, where
P = σ^1 ∩ Γ^1. Since ∑_{i=1}^2 i \deg(Δ_i ∩ Γ^i) = 3, we have \mult_P(Δ_1 ∩ Γ^1) = 3. This contradicts to Lemma 3.15 and the fact a ≥ 5.

Assume that \( i_0 = 2 \). If \deg(Δ_2 ∩ σ^2) = 1, then \deg Δ_2 = 1 by Lemma 3.13 (1). In this case, coeff 1 = 1 and \deg(Δ_1 ∩ Γ) = 2, where Γ ⊂ M_1 is the π_2-exceptional (-1)-curve. For \( P := σ^1 ∩ Γ \), \mult_P(Δ_1 ∩ Γ) = 2 and \mult_P(Δ_1 ∩ σ^1) = 1. This contradicts to Lemma 3.15. Thus \deg(Δ_2 ∩ σ^2) = 2. In particular, \( n = 2a - 4 \) and \( Δ_1 ∩ σ^1 = \emptyset \). Thus \( a = 4 \), \( |Δ_2| = \{ P \} \) such that \mult_P Δ_2 = 3, \mult_P(Δ_2 ∩ σ^2) = 2 and \( Δ_1 = \emptyset \) by Lemma 3.16 and the fact ∑_{i=1}^b i \deg Δ_i ≤ 6. Then the 4-fundamental multiplet \( (M_2, E_2; Δ_1, Δ_2) \) is of type \( (B) \).

5.5. The case \( h = (n+2)a - 1 \). We consider the case \( h = (n+2)a - 1 \). In this case, \( n ≤ 2a - 1 \), \( L_b \sim (a+1)σ + ((n+2)a - 1)l \), \( E_b = (a-1)σ + l \), \( 2a - n - 1 = (L_b, σ) = ∑_{i=1}^b i \deg(Δ_i ∩ σ^i) \), \( a+1 = (L_b, l) = ∑_{i=1}^b i \deg(Δ_i ∩ l^i) \), \( 4a - 2 - (a-1)(n-2a-4) = (L_b, E_b) = ∑_{i=1}^b i(a-i) \deg Δ_i \) and \( a(-K_3^b) = (n-2a-4)(a-1) + 2a^2 + 4 - ∑_{i=1}^b i \deg Δ_i \). In particular, \( ∑_{i=1}^b i \deg Δ_i ≤ 4 + (n-2a-4)(a-1) \). Hence \( n ≥ 2a - 3 \).

Assume that \( Δ_1 ∩ σ^i = \emptyset \) for all \( i ≥ 2 \). Then \( Δ_1 = \emptyset \) for all \( i ≥ 2 \). However, by Lemma 3.15 \( P ∉ Δ_1 \), where \( P = σ^1 ∩ l^1 \). Thus \( Δ_1 ∩ l^1 = \emptyset \), which leads to a contradiction. Thus \( n = 2a - 3 \), \deg(Δ_2 ∩ σ^2) = 1 and \deg(Δ_2 ∩ σ^i) = 0 for all \( i ≠ 2 \). Then \| Δ_2 \| = \{ P \} \) with \( P = σ^2 ∩ l^2 \) by Lemma 3.15. Moreover, since \( E_1 \) is effective, by Example 2.6 \deg Δ_2 = \deg(Δ_2 ∩ l^2) \) and \deg Δ_2 ≤ 3 (if \( a = 4 \)); \deg Δ_2 ≤ 2 (if \( a = 5 \)); \deg Δ_2 = 1 (if \( a ≥ 6 \)), unless \( (a, \deg Δ_2, \deg(Δ_2 ∩ l^2)) = (4, 2, 1) \). If \( (a, \deg Δ_2, \deg(Δ_2 ∩ l^2)) = (4, 2, 1) \), then \( Δ_1 ∩ l^1 = \emptyset \), a contradiction. Thus \deg(Δ_2 ∩ l^2) = \deg Δ_2. If \deg Δ_2 = 1, then \deg Δ_1 = (a+3)/(a-1) \) and \deg(Δ_1 ∩ l^1) = a-1, a contradiction. If \deg Δ_2 = 3, then \deg(Δ_1 ∩ l^1) = a-5 = -1, a contradiction. Thus \deg Δ_2 = 2. In this case, \deg Δ_1 = (-a+7)/(a-1) \) and \deg(Δ_1 ∩ l^1) = a-3, \( a = 4 \), \deg Δ_1 = \deg(Δ_1 ∩ l^1) = 1, a contradiction. However, in this case, \( E_1 = 3σ^1 + 2Γ_{P,1} + Γ_{P,2} + l^1 \). Thus \Δ_1 ∩ l^1 = 0, which leads to a contradiction. Therefore \( h ≠ (n+2)a - 1 \).

5.6. The case \( h ≤ (n+2)a - 2 \). We consider the case \( h ≤ (n+2)a - 2 \). In this case, \( n ≤ 2a - 2 \). Assume that \( h ≤ (n+2)a - 3 \). Then \( n = 2a - 3 \) and \( h = 2a^2 - a - 3 \) since \( h ≥ 2a^2 - 2a - 2 \). In this case, \( E_b \sim (a-1)σ + 3l \) and \( a(-K_3^b) = 2a^2 + a - 1 - ∑_{i=1}^b i \deg Δ_i \). Hence there exists a fiber \( l ≤ E_b \). Thus \( a+1 = (L_b, l) = ∑_{i=1}^b i \deg(Δ_i ∩ l^i) ≤ ∑_{i=1}^b i \deg Δ_i ≤ a - 1 \), which leads to a contradiction. Thus \( h \) must be equal to \( (n+2)a - 2 \). In this case, \( L_b \sim (a+1)σ + ((n+2)a - 2)l \) \( E_b \sim (a-1)σ + 2l \), \( 2a - n - 2 = (L_b, σ) = ∑_{i=1}^b i \deg(Δ_i ∩ σ^i) \).
Indeed, if $a\equiv b\sum l$ for such $(3)$. In particular, $(b-\deg\Delta_i)\leq 2+(n-(2a-4))(a-1)$. We note that there exists a fiber $l\leq E_b$ and $a+1=(L_b\cdot l)=\sum_{i=1}^{b} i\deg(\Delta_i\cap l)\leq \sum_{i=1}^{b} i\deg\Delta_i\leq 2a$ for such $l$. Thus $E_b=(a-1)\sigma+2l$. Moreover, we have $n\geq 2a-3$. Indeed, if $n=2a-4$, then $a+1\leq \sum_{i=1}^{b} i\deg\Delta_i\leq 2$, a contradiction.

If $n=2a-2$, then $0=(L_b\cdot \sigma)=\sum_{i=1}^{b} i\deg(\Delta_i\cap \sigma^i)$. Since $a-b\geq 2$, we have $a-b=2$, $\Delta\subset l^b \setminus \sigma^b$, $\Delta_1=\emptyset, \ldots, \Delta_{b-1}=\emptyset$ by Lemma 3.14 (3). In particular, $b+3=a+1=b\deg\Delta_b$. Thus $a=5$, $b=3$ and $\deg\Delta_3=2$. Then the 5-fundamental multiplet $(M_3;E_3;\Delta_1,\Delta_2,\Delta_3)$ is of type $\langle A \rangle_5$.

The remaining case is that $n=2a-3$. Since $a+1=\sum_{i=1}^{b} i\deg(\Delta_i\cap l^i)\leq \sum_{i=1}^{b} i\deg\Delta_i\leq a+1$, we have $\sum_{i=1}^{b} i\deg\Delta_i=a+1$ and $\Delta_i\subset l^i$ for any $i$. Since $\sum_{i=1}^{b} i\deg(\Delta_i\cap \sigma^i)\leq (a,\mult{\Delta_1})=(4,3)$ or $(5,2)$ holds. If $(a,\mult{\Delta_1})=(5,2)$, then $\deg\Delta_2=2$ and $\Delta_3=\emptyset$ since $6=\sum_{i=1}^{3} i\deg\Delta_i$. However, $\Delta_2=\emptyset$ by Lemma 3.14 (3), a contradiction. Thus $(a,\mult{\Delta_1})=(4,3)$. In this case, $\deg\Delta_2=1$ since $\sum_{i=1}^{2} i\deg\Delta_i=5$. Then the 4-fundamental multiplet $(M_2;E_2;\Delta_1,\Delta_2)$ is of type $\langle C \rangle_4$.

As a consequence, we have complete the proof of Theorem 1.1.

References

[Ale94] V. A. Alexeev, Boundedness and $K^2$ for log surfaces, Internat. J. Math. 5 (1994), no. 6, 779–810.

[AM04] V. A. Alexeev and S. Mori, Bounding singular surfaces of general type, Algebra, arithmetic and geometry with applications (West Lafayette, IN, 2000), 143–174, Springer, Berlin, 2004.

[AN88] V. A. Alexeev and V. V. Nikulin, Classification of del Pezzo surfaces with log-terminal singularities of index $\leq 2$, involutions on K3 surfaces, and reflection groups in Lobachevskiǐ spaces (Russian), Lectures in mathematics and its applications, Vol. 2, No. 2 (Russian), 51–150, Ross. Akad. Nauk, Inst. Mat. im. Steklova, Moscow, 1988.

[AN89] V. A. Alexeev and V. V. Nikulin, Classification of del Pezzo surfaces with log-terminal singularities of index $\leq 2$ and involutions on K3 surfaces (Russian), Dokl. Akad. Nauk. SSSR 306 (1989), no. 3, 525–528; translation in Soviet Math. Dokl. 39 (1989), no. 3, 507–511.

[AN06] V. A. Alexeev and V. V. Nikulin, Del Pezzo and K3 surfaces, MSJ Memoirs, 15. Math. Soc. of Japan, Tokyo, 2006.

[BB92] A. A. Borisov and L. A. Borisov, Singular toric Fano three-folds, Math. Sb. 183 (1992), no. 2, 134–141.
A. A. Borisov, *Boundedness of Fano threefolds with log-terminal singularities of given index*, J. Math. Sci. Univ. Tokyo 8 (2001), no. 2, 329–342.

L. Brenton, *On singular complex surfaces with negative canonical bundle, with applications to singular compactifications of \( \mathbb{C}^2 \) and to 3-dimensional rational singularities*, Math. Ann. 248 (1980), no. 2, 117–124.

M. Demazure, *Surfaces de del Pezzo II–V*, in Séminaire sur les Singularités des Surfaces (eds. M. Demazure, H. Pinkham and B. Teissier), Lecture Notes in Math., 777 (1980), Springer, Berlin, pp. 35–68.

K. Fujita, *Log del Pezzo surfaces with not small fractional indices*, arXiv:1401.0988.

W. Fulton, *Introduction to toric varieties*, Annals of Mathematics Studies, 131. The William H. Roever Lectures in Geometry. Princeton University Press, Princeton, NJ, 1993.

K. Fujita and K. Yasutake, *Classification of log del Pezzo surfaces of index three*, arXiv:1401.1283.

C. D. Hacon, J. Mckernan and C. Xu, *ACC for log canonical thresholds*, arXiv:1208.4150.

F. Hidaka and K-i. Watanabe, *Normal Gorenstein surfaces with ample anti-canonical divisor*, Tokyo J. Math. 4 (1981), no. 2, 319–330.

C. Jiang, *Bounding the volumes of singular weak log del Pezzo surfaces*, arXiv:1305.6435.

J. Kollár and S. Mori, *Birational geometry of algebraic varieties*, With the collaboration of C. H. Clemens and A. Corti. Cambridge Tracts in Math., 134, Cambridge University Press, Cambridge, 1998.

C-J. Lai, *Bounding the volumes of singular Fano threefolds*, arXiv:1305.6435.

S. Mori, *Threefolds whose canonical bundles are not numerically effective*, Ann. of Math. 116 (1982), no. 1, 133–176.

N. Nakayama, *Classification of log del Pezzo surfaces of index two*, J. Math. Sci. Univ. Tokyo 14 (2007), no. 3, 293–498.

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