MONGE-AMPÈRE MEASURE AT THE BOUNDARY OF SOME DOMAINS WITH CORNERS

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Abstract. Let $\mu^z$ be the measure obtained by sweeping out the Monge-Ampère measure of the pluricomplex Green function with pole at $z$. We prove that $\mu^z$ vanish on Levi flat parts of the boundary for 1) every relatively compact analytic polyhedron in complex space, 2) product domains of hyperconvex sets in Stein manifolds.

1. Introduction

Let us denote the plurisubharmonic (henceforth abbreviated as psh) functions on a complex manifold $X$ by $\mathcal{PSH}(X)$, strictly psh functions by $\mathcal{SPSH}(X)$ and non-positive psh functions by $\mathcal{PSH}^-(X)$.

For notations of the complex Monge-Ampère measure at the boundary of an hyperconvex manifold we refer to [Dem85, Dem87]. Let us just recall some basic definitions.

Suppose $\varphi$ is a psh weight on a hyperconvex manifold $\Omega$, i.e. $\varphi \in \mathcal{PSH}^-(\Omega)$, the pseudo balls $B_r = \{ z : \varphi(z) < r \}$ are relatively compact in $\Omega$ for some $r \in \mathbb{R}$, and $e^{\varphi}$ is a continuous function. We denote the pseudo spheres by $S_r = \{ z : \varphi(z) = r \}$.

Denote the characteristic function of a set $A$ by $1_A$. Suppose $X$ is Stein, for any psh weight $u$ on $X$ Demailly defined a “swept out” measure of $(dd^c u)^n$ on $S_r$ by

$$\mu_{u,r} = 1_X \setminus B_r (dd^c u)^n - (dd^c (\max(u,r)))^n.$$  \hfill (1)

For smooth plurisubharmonic $u$ the formula $\mu_{u,r} = (dd^c u)^n - (dd^c u)|_{S_r}$ holds (cf. [Dem85]). Using continuity under monotonic sequences the measure $\mu_u$ for a psh weight $u$ on a hyperconvex domain $\Omega$ is defined by

$$\mu_u = \lim_{r \to 0} \mu_{u,r}.$$  \hfill (2)

In $\mathbb{C}^n$ we may define the Lelong number of a psh function $u$ at the point $w$ as

$$\nu(u, w) = \lim_{r \to 0} \frac{M(u, r, w)}{\log r},$$

where $M(u, r, w) = \sup\{ u(z) : |z - w| < r \}$. Since the Lelong number of a psh function is well known to be independent of a biholomorphic change of variables the definition readily carry over to any complex manifold.

The pluricomplex Green function with pole at $w$ was introduced by Klimek [Kli83] and Zahariuta [Zah84]. For any connected relatively open subset $\Omega$ of $X$ it is defined as

$$g_{\Omega}(z, w) = \sup\{ u(z) : u \in \mathcal{PSH}^-(\Omega), \nu(u, w) \geq 1 \}.$$  \hfill (3)

Finally, fix a pole $w$ in $\Omega$, and define a measure on the boundary by

$$\mu^w = \lim_{r \to 0} \mu_{g_{\Omega}(z, w), r}.$$  \hfill (4)
In the original paper by Demailly following geometric classification theorem was obtained.

**Theorem 1.** (Demailly [Dem87]). For any domain $\Omega$ with defining psh function $\rho \in C^2(\overline{\Omega})$ the measure $\mu^w$ is supported on the strict pseudoconvex points of the boundary, for every $w \in \Omega$.

In this note we give two examples that this theorem may be generalized to some domains with corners.

2. Hyperconvex domains with corners

Let us start with a smoothing Lemma, similar to Lemma 3.2 in [Gua02].

**Lemma 1.** Let $U$ be an open domain in a complex space $X$. Suppose $u_i \in \mathcal{PSH} \cap C^k(U), i = 1, 2,$ for some $k \in \mathbb{Z}_{\geq 0}$ and let $\chi_r : \mathbb{R} \rightarrow \mathbb{R}$ be a smooth convex function such that $\chi_r(x) = |x| > r$. Define $M_r(u_1, u_2) = (\chi_r(u_1 - u_2) + u_1 + u_2)/2$. Then $M_r(u_1, u_2) \in \mathcal{PSH}(U)$ and $M_r(u_1, u_2) \geq \max(u_1, u_2)$ with $M_r(u_1, u_2) = \max(u_1, u_2)$ except on a small neighbourhood of the arête $\{|u_1| = |u_2|\}$. Furthermore $M_r \in C^k$.

Guan proved this Lemma for twice differentiable functions by direct computation of the Levi form for $M_r$. In [Ceg01] a much simplified proof of this Lemma was given, where the Lemma was stated for all psh functions $u_i$ regardless of continuity; and it is not clear from Cegrell’s proof whether $M_r$ is upper semicontinuous. For the convenience of the reader we repeat the argument here.

**Proof.** Since $2\max(u_1, u_2) = |u_1 - u_2| + u_1 + u_2$ the latter part is clear. For the first part note that

$$\chi_r(x) = \sup\{kx + l, |k| \leq 1, \text{ and } kt + l \leq \chi_r(t), t \in \mathbb{R}\},$$

thus

$$\chi_r(u_1 - u_2) + u_1 + u_2 = \sup\{(1 + k)u_1 + (1 - k)u_2 + l\},$$

and $(M_r)^* \text{ is psh, but since } u_i \text{ are continuous we have that } (M_r)^* = M_r.$

**Theorem 2.** Suppose $P$ is a relatively compact analytic polyhedron, and let $w \in P$, then $\mu^w$ is supported in a subset of the strictly pseudoconvex points of $\partial P$.

**Proof.** Let $P$ be an analytic polyhedra defined by

$$P = P(f_1, \ldots, f_N) = \{z \in W : |f_i| < 1, i = 1, \ldots, N\}$$

where $W \Subset G \subset \mathbb{C}^n$, and $f_i \in \mathcal{O}(G, \mathbb{C})$ is not identically zero. $P$ is clearly hyperconvex.

Define $\tilde{\varphi} := \max_k \{|f_i|\}$, then $\tilde{\varphi}$ is a psh weight on $P$, furthermore $\exp(\tilde{\varphi}) \in \mathcal{C}(P)$.

From the defining Equation (1) for the swept out Monge-Ampère measure it is clear that $g_P(z, w)$ and $\max\{g_P(z, w), -1\}$ have the same boundary measure on the pseudo spheres $\{z : g_P(z, w) = t\}$ for $t > -1$, thus they have the same boundary measure on $\partial P$.

By the continuity and maximality of $g_P$ there is a positive constant $c$ such that $\max\{g_P(z, w), -1\} \geq c\tilde{\varphi}(z), \forall z \in \overline{P}$. Let $\varphi = c\tilde{\varphi}$. Thus by the comparison principle for the boundary measures (cf. Theorem 3.4 [Dem87]) we have

$$(3) \quad \mu^w \leq \mu_\varphi.$$
Now a straightforward calculation gives that

\( (dd^c \log |f_i|)^{n-1} \wedge d^c \log |f_i| = \frac{1}{(2|f_i|^2)^n} (dd^c |f_i|^2)^{n-1} \wedge d^c |f_i|^2 = \frac{1}{(2|f_i|^2)^n} (dd^c |f_i|^2 - 1)^{n-1} \wedge d^c (|f_i|^2). \)

Take a point \( x \) in a Levi flat part of the boundary of \( P \) then \( |f_i|^2 - 1 \) is a local defining function for \( P \) around \( x \), thus the last expression in Equation (4) vanish.

Take any point \( x \in \partial P \), away from the corners of \( P \). Take \( R > 0 \) fixed and set \( u_R = M_R(\log |f_1|, \ldots, \log |f_N|) \), where we have extended \( M_R \) to \( N \) variables in the natural way.

Since \( u_R \) is smooth on \( P \) we have in \( U_x \cap P \), where \( U_x \) is some neighbourhood of \( x \),

\[
\mu_{u_R}|S_i = (dd^c u)^{n-1} \wedge d^c u = \frac{1}{(2|f_i|^2)^n} (dd^c |f_i|^2 - 1)^{n-1} \wedge d^c (|f_i|^2) = 0.
\]

By continuity under decreasing sequences we have that \( \lim_{R \to 0} \mu_{u_R}|S_i = \mu_u|S_i = 0 \) and then letting \( t \nearrow 0 \) we may conclude that \( \mu_v = 0 \) on Levi flat parts of the boundary of \( P \). By Equation (3) \( \mu^w \leq \mu_v = 0 \) on Levi flat part of \( \partial P \). \( \square \)

**Theorem 3.** Suppose \( \Omega_i \) is a hyperconvex domain in a Stein manifold \( X_i \), of dimension \( n_i \), for \( i = 1, 2 \), and let \( X = X_1 \times X_2 \), and \( \Omega = \Omega_1 \times \Omega_2 \subset X \). Take \( w \in \Omega \), then \( \mu^w \) vanish except on the corner \( \partial \Omega_1 \times \partial \Omega_2 \).

**Proof.** Fix the pole \( w = (w_1, w_2) \in \Omega_1 \times \Omega_2 \), and let \( u_1 \) and \( u_2 \) be the pluricomplex Green function of \( \Omega_1 \) and \( \Omega_2 \) with poles at \( w_1 \) and \( w_2 \) respectively.

Let \( u = \max\{u_1, u_2\} \). Then \( u \) is a psh weight by the continuity result in [Dem87], and therefore \( \Omega \) is a hyperconvex set in \( X \). Clearly \( u = 0 \) on \( \partial \Omega \). Directly from Equation (2) we have \( \nu(u, w) = \min\{\nu(u_1, w_1), \nu(u_2, w_2)\} = 1 \). Since \( (dd^c u_i)^{n_i} = 0 \) away from \( w_i \), for \( i = 1, 2 \), we have by Proposition 3.4 in [Zer90] that \( (dd^c u)^{n} = 0 \) outside the pole \( w \). By [Dem87] \( \mu^{u} \) is in fact the pluricomplex Green function for \( \Omega \), with pole at \( w \).

Now, for any twice differentiable function \( v \) depending only on \( n < n_1 + n_2 = N \) variables we have either that \( n < N - 1 \) which implies that \( (dd^c v)^{N-1} = 0 \), or we have \( n = N - 1 \). If \( n = N - 1 \) we get, in local coordinates, \( (dd^c v)^{N-1} = f(z) dz_1 \wedge dz_1 \wedge \ldots \wedge dz_{N-1} \wedge d\bar{z}_{N-1} \), for some function \( f \). Thus

\[
(dd^c v)^{N-1} \wedge d^c v = f(z) dz_1 \wedge d\bar{z}_1 \ldots \wedge dz_{N-1} \wedge d\bar{z}_{N-1} \wedge \left( \sum_{k=1}^{N-1} \frac{\partial v}{\partial z_k} dz_k - \frac{\partial v}{\partial \bar{z}_k} d\bar{z}_k \right) = 0.
\]

Fix \( r < 0 \). Take \( \rho_i \in C_0^\infty(X_i) \), \( i = 1, 2 \) then there is a decreasing sequence of smooth psh functions \( \{u_i^j\}_{j=1}^\infty \) on \( \Omega_i \) such that \( \rho_i/(j+1) + u_i < u_i^j < \rho_i/j + u_i \), since the \( u_i^j \)'s are at least continuous. Thus on a neighbourhood of \( S_r \) we have a decreasing sequence of psh functions \( u^r = \max\{u_1^r, u_2^r\} \), smooth away from a neighbourhood of the arête \( \{|u_1| = |u_2|\} \). But by Equation (5) we have \( (dd^c u^r)^{N-1} \wedge d^c u^r \equiv 0 \), away from the arête.

By continuity of the currents under decreasing limit we have \( \mu_{u^r} = 0 \). Thus, away from the arête \( \{z \in \Omega : |u_1(z)| = |u_2(z)|\} \) we have \( \mu_w|S_r = 0 \). Letting \( r \to 0 \) gives the result. \( \square \)
Let us just add the remark that the proof of Theorem 3 could be made a lot cleaner by using the definition of the boundary measure from an unpublished manuscript of Cegrell [Ceg05]. Alas, the boundary measure is only defined on hyper-convex set in $\mathbb{C}^n$ in that paper.

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