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BOUNDARY NULL-CONTROLLABILITY OF COUPLED PARABOLIC SYSTEMS WITH
ROBIN CONDITIONS

KUNTAL BHANDARI* AND FRANCK BOYER†

Abstract. The main goal of this paper is to investigate the boundary controllability of a coupled parabolic system in the cascade
form in the case where the boundary conditions are of Robin type. In particular, we prove that the associated controls satisfy suitable
uniform bounds with respect to the Robin parameters, that let us show that they converge towards a Dirichlet control when the Robin
parameters go to infinity. This is a justification of the popular penalisation method for dealing with Dirichlet boundary data in the
framework of the controllability of coupled parabolic systems.

Key words. Control theory, parabolic systems, moments method, spectral estimates.

AMS subject classifications. 35B30 - 35K20 - 93B05

1. Introduction.

1.1. The problem under study. This paper is concerned with the boundary null-controllability problem for
linear coupled parabolic systems with less controls than equations. It is by now well-known that it can be a difficult
problem in various situations and that there is still no complete theory in the literature. We will concentrate here
on a particular case which is in the so-called cascade form and that can be written as follows

\[
\begin{aligned}
\begin{cases} 
\partial_t y_1 - \text{div} (\gamma(x) \nabla y_1) = 0 & \text{in } (0, T) \times \Omega, \\
\partial_t y_2 - \text{div} (\gamma(x) \nabla y_2) + y_1 = 0 & \text{in } (0, T) \times \Omega, \\
y_1(0, \cdot) = y_{0,1} & \text{in } \Omega, \\
y_2(0, \cdot) = y_{0,2} & \text{in } \Omega,
\end{cases}
\end{aligned}
\]

(1.1)

where the control \( v \) will be acting only on the component \( y_1 \) on some part \( \Gamma_0 \) of the boundary \( \Gamma \) of the domain \( \Omega \).

Since we want to control both components of the system and \( v \) has no direct influence in the equation for \( y_2 \), the
role of the coupling term \( y_1 \) in the second equation is fundamental: it acts as an indirect control term. We refer for
instance to the review paper [5] for a general presentation of different results on that topic.

The usually studied case is the one of a Dirichlet control, which means that the above system is supplemented
with the following boundary conditions

\[
\begin{aligned}
\begin{cases} 
y_1 = 1_{\Gamma_0} v & \text{on } (0, T) \times \Gamma, \\
y_2 = 0 & \text{on } (0, T) \times \Gamma.
\end{cases}
\end{aligned}
\]

(1.2)

In the present work we would like to analyse the controllability properties of the same system when one considers,
instead of the Dirichlet boundary conditions, a set of Robin boundary conditions with two non negative parameters
\( \beta_1, \beta_2 \)

\[
\begin{aligned}
\begin{cases} 
\frac{\partial y_1}{\partial \nu_\gamma} + \beta_1 y_1 = 1_{\Gamma_0} v & \text{on } (0, T) \times \Gamma, \\
\frac{\partial y_2}{\partial \nu_\gamma} + \beta_2 y_2 = 0 & \text{on } (0, T) \times \Gamma,
\end{cases}
\end{aligned}
\]

(1.3)

where the conormal derivative operator associated to the diffusion tensor \( \gamma \) is defined by

\[
\frac{\partial}{\partial \nu_\gamma} = \nu \cdot (\gamma \nabla) .
\]

1.2. Motivations and overview of the paper. Our motivation for studying the above problem is two-fold.
The first one comes from the fact that it is an instance of the very popular penalisation approach to deal
with boundary condition that have never been studied, as far as we know, in the framework of the controllability of
coupled parabolic systems. From a numerical point of view, for instance when considering a Galerkin approximation
of an elliptic or parabolic equation, this approach consists in replacing a Dirichlet boundary condition \( y = g \) by

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a Robin boundary condition \( \frac{1}{\gamma} \partial_\nu y + y = g \), with a large penalisation parameter \( \beta \). It generally induces more flexibility and robustness in the computational code. This approach is indeed proposed in many finite element libraries and softwares. Moreover, it is also a suitable way to deal with data \( g \) that have a regularity lower than the one expected to solve the problem in the usual energy spaces (typically if \( g \notin H^{1/2}(\Gamma) \) when solving the Laplace equation). This approach was initially studied in [7] for elliptic problems or in [9] for parabolic problems. In the framework of control theory, this penalisation approach was for instance analyzed in [8, 13] for solving optimal control of elliptic equations. In each case, it is proven that the solution of the penalised problem actually converges to the one of the original problem, with some estimate of the rate of convergence.

Our motivation is thus to investigate the same kind of issues for the coupled parabolic system (1.1) with a single boundary control and in particular to show that, not only the problem (1.1)-(1.3) is null-controllable, but more importantly, that it is possible to prove estimates on the controls that are (in some sense that will be precised later) uniform with respect to the Robin (penalisation) parameter. It will follow that the corresponding controlled solution converges towards a controlled solution of the Dirichlet problem when those parameters go to infinity.

Another motivation for this analysis, related to the discussion above, is that Robin boundary conditions have a regularizing effect on the boundary data. Indeed, as it will be recalled at the beginning of Section 2, the functional analysis adapted to boundary controls in \( L^2 \) for parabolic systems is a little intricate since, with such a low regularity of the data, we cannot expect solutions to exist in the usual energy space \( C^0([0, T], L^2(\Omega)) \cap L^2(0, T, H^1(\Omega)) \) and to satisfy a standard weak formulation. Instead the solutions are known to live in a larger space \( C^0([0, T], H^{-1}(\Omega)) \cap L^2(0, T, L^2(\Omega)) \), the boundary condition being understood in a weak sense. When changing the Dirichlet boundary condition into Robin (or Neumann) boundary conditions, the functional setting is more comfortable and we recover the expected regularity for weak solutions even if the boundary data is only in \( L^2 \).

Finally, we want to recall here that we lack of general mathematical techniques to deal with the controllability issue for those systems; applicability of the few available methods is very dependent on the structure of the underlying operators. Therefore, the analysis of each significantly new system needs to develop more elements (of spectral nature in our case) that are interesting by themselves and possibly useful in other situations. More precisely, there are not that many works regarding the controllability of coupled parabolic systems with less controls than equations, especially for boundary controls. This is mainly due to the fact that the very powerful Carleman estimates approach is essentially inefficient in that context. In particular, we recall that the boundary controllability for such systems is no longer equivalent with distributed controllability as it has been observed for instance in [18], see also [5]. In particular, most of the available controllability results concern the 1D setting since they are based on the moments method (that we will discuss below) which is not straightforward to implement in higher dimension.

Among the few results available, we mention [4] where the authors proved the controllability to trajectories of a 1D system of \( n \) parabolic equations when \( m < n \) number of controls are exerted on a part of the boundary through Dirichlet conditions. They actually proved that a general Kalman condition is a necessary and sufficient controllability condition for this problem. In the multi-dimensional case, we quote [10, 2], where controllability results are obtained in particular cylindrical geometries by exploiting on the one hand a sharp estimate of the control for the associated 1D problem and on the other hand spectral Lebeau-Robbiano inequalities, see also the discussion in Section 2.4.2. We also mention [1] where symmetric parabolic coupled systems are analysed in any dimension, provided that the control region satisfies the Geometric Control Condition.

**Paper organisation.** In Section 2, we first recall the different notions of solutions for (1.1) with boundary conditions (1.2) or (1.3), that we will need in the paper and we give the associated wellposedness and regularity results. In Section 2.4, we give the precise statements of our main results. As mentioned above, those results essentially say that the coupled parabolic system with Robin boundary condition is null-controllable at any time \( T > 0 \) and that we can find uniform bounds on the control that allow to justify the convergence towards a control for the Dirichlet problem when the Robin parameters are large. The proofs are given in Sections 3.4 and 5. They are based on the moments method [17] and on its recent extension called block moments method [11]; they require in particular a careful analysis of spectral properties of the underlying operators, with estimates uniform with respect to the parameters. Some of those spectral estimates are particularly difficult to obtain when the two Robin parameters are different, that is why in that case we restrict our analysis to a constant diffusion coefficient \( \gamma \).

**Notations.** Throughout this paper \( C \) or \( C' \) denotes a generic positive constant (that may vary from line to line) which does not depend on \( T, y_0 \) nor the parameters \( \beta_1, \beta_2 \) but may depend on the diffusion coefficient \( \gamma \). Sometimes, we will make emphasis on the dependence of a constant on some quantities \( \alpha_1, \alpha_2, \ldots, \alpha_n \) (\( n \geq 1 \)) by \( C_{\alpha_1, \alpha_2, \ldots, \alpha_n} \).
Moreover, we shall use the following notation
\[ ((a, b)) := (\min\{a, b\}, \max\{a, b\}), \quad \text{for any } a, b \in \mathbb{R}, \]
which is an open interval in \( \mathbb{R} \).

The euclidean inner product in \( \mathbb{R}^d \), \( d \geq 1 \), will be denoted by \( \xi_1 \cdot \xi_2 \) for any \( \xi_1, \xi_2 \in \mathbb{R}^d \).

2. General setting and main results. In this section, we will first discuss about the well-posedness for our parabolic system with Dirichlet and Robin boundary condition with \( L^2 \) data. We will be particularly interested in estimates on the solutions that are uniform with respect to the Robin parameters. Then, we will give our main results concerning the associated control problems.

Let \( \Omega \subset \mathbb{R}^d \) be a smooth bounded domain and \( \gamma : \Omega \to M_d(\mathbb{R}) \) be a smooth bounded field of symmetric matrices which are uniformly coercive: there is a \( \gamma_{\min} > 0 \) such that
\[ (\gamma(x)\xi) \cdot \xi \geq \gamma_{\min} |\xi|^2, \quad \forall \xi \in \mathbb{R}^d, \forall x \in \Omega. \]

We will first study the scalar problem before discussing the coupled cascade system.

2.1. The scalar problem.

2.1.1. Dirichlet boundary data. We first recall the usual setting adapted to the analysis of the Dirichlet problem
\[
\begin{cases}
\partial_t y - \text{div}(\gamma \nabla y) = f & \text{in } (0, T) \times \Omega, \\
y = g & \text{on } (0, T) \times \Gamma, \\
y(0, \cdot) = y_0 & \text{in } \Omega,
\end{cases}
\]
with non smooth data. In the case where \( g = 0 \), we can easily solve the above problem in a weak sense in \( C^0([0, T], L^2(\Omega)) \cap L^2(0, T, H^1(\Omega)) \) for given \( y_0 \in L^2(\Omega) \). This can be done by using the continuous semigroup in \( L^2(\Omega) \) associated with the operator \( -A_D = \text{div}(\gamma \nabla \cdot) \) with the domain \( D(A_D) = H^2(\Omega) \cap H^1_0(\Omega) \). However, if one considers the case where \( g \) is any data in \( L^2((0, T) \times \Gamma) \) which is the usual framework in control theory, we cannot define as easily a good notion of weak solution because of a lack of regularity of the data. Instead, we have the following well-posedness result in a dual sense, see for instance [14, 25].

**Proposition 2.1.** For any \( y_0 \in L^2(\Omega), f \in L^2((0, T) \times \Omega), g \in L^2((0, T) \times \Gamma) \), there exists a unique \( y \in C^0([0, T], H^{-1}(\Omega)) \cap L^2((0, T) \times \Omega) \) solution of (2.1) in the following sense: for any \( t \in [0, T] \) and \( \zeta \in H^1_0(\Omega) \), we have
\[ \langle y(t), \zeta \rangle_{H^{-1}, H^1_0} = \langle y_0, e^{-tA_D^*} \zeta \rangle_{L^2} + \int_0^t \int_{\Omega} f (e^{-(t-s)A_D^*} \zeta) - \int_0^t \int_{\Gamma} g \frac{\partial}{\partial \nu_\gamma} (e^{-(t-s)A_D^*} \zeta). \]

**Remark 2.2.** The operator \( A_D \) being self-adjoint, we could have replaced \( A_D^* \) by \( A_D \) in the previous statement but we prefer to keep it in order to be consistent with the non-scalar case that we will consider in Section 2.3.

2.1.2. Homogeneous Robin boundary data. For any \( \beta \in [0, +\infty) \), we consider now the following parabolic problem
\[
\begin{cases}
\partial_t y - \text{div}(\gamma \nabla y) = f & \text{in } (0, T) \times \Omega, \\
\frac{\partial y}{\partial \nu_\gamma} + \beta y = 0 & \text{on } (0, T) \times \Gamma, \\
y(0, \cdot) = y_0 & \text{in } \Omega,
\end{cases}
\]
where the regularity of \( y_0 \) and \( f \) will be precised below.

If the data are regular enough, the semigroup theory also gives a solution for this problem. Indeed, if one introduces the (self-adjoint) unbounded operator \( A_\beta = -\text{div}(\gamma \nabla \cdot) \) in \( L^2(\Omega) \) associated with the domain
\[ D(A_\beta) = \left\{ u \in H^2(\Omega) \middle| \frac{\partial y}{\partial \nu_\gamma} + \beta y = 0 \text{ on } \Gamma \right\}, \]
then we can prove that \( -A_\beta \) generates a continuous semigroup in \( L^2(\Omega) \). Hence, the following result holds.
Proposition 2.3. Let \( \beta \in [0, +\infty) \) be given. For any \( y_0 \in D(A_\beta) \) and \( f \in C^1([0, T], L^2(\Omega)) \), there exists a unique strong solution \( y \) to (2.2) in \( C^1([0, T], L^2(\Omega)) \cap C^0([0, T], D(A_\beta)) \), which is given by

\[
y(t) = e^{-tA_\beta}y_0 + \int_0^t e^{-(t-s)A_\beta}f(s) \, ds.
\]

Moreover, this solution satisfies the energy estimates

\[
(2.3) \quad \|y\|_{L^\infty(0, T; L^2(\Omega))} + \|y\|_{L^2(0, T; H^1(\Omega))} + \sqrt{\beta}\|y\|_{L^2((0, T) \times \Gamma)} \leq C_T(\|y_0\|_{L^2(\Omega)} + \|f\|_{L^2((0, T) \times \Omega)}),
\]

and

\[
(2.4) \quad \|y\|_{L^\infty(0, T, H^1(\Omega))} + \|\partial_t y\|_{L^2((0, T) \times \Omega)} + \|y\|_{L^2(0, T, H^2(\Omega))} + \sqrt{\beta}\|y\|_{L^\infty(0, T, L^2(\Gamma))}
\leq C_T(\|\nabla y_0\|_{L^2(\Omega)} + \sqrt{\beta}\|y_0\|_{L^2(\Gamma)} + \|f\|_{L^2((0, T) \times \Omega)}),
\]

where \( C_T > 0 \) does not depend on \( \beta \).

In particular, if \( y_0 \in D(A_\beta) \cap H^1_0(\Omega) \), we have an estimate whose right-hand side does not depend on \( \beta \).

Proof. The existence of a unique strong solution is a standard result from semigroup theory, see for instance [15, Corollary 7.6]. We only sketch the proof of the estimates. The weak estimate (2.3) simply comes by multiplying the equation by \( y \) and using that

\[
(2.6) \quad (A_\beta \zeta, \zeta)_{L^2} = \int_\Omega (\gamma \nabla \zeta \cdot \nabla \zeta + \beta \int_\Gamma |\zeta|^2), \quad \forall \zeta \in D(A_\beta).
\]

To prove the stronger estimate, we first assume that \( y_0 \in D(A_\beta^2) \) and that \( f \in C^1([0, T], D(A_\beta)) \), the final result being deduced by a density argument. With this regularity of the data we can justify that

\[
\frac{1}{2} \frac{d}{dt} (A_\beta y, y)_{L^2} = (A_\beta y, \partial_t y)_{L^2} = -\|\partial_t y\|_{L^2}^2 + (f, \partial_t y)_{L^2}.
\]

Using the Cauchy-Schwarz inequality, and integrating in time, we get

\[
(A_\beta y(t), y(t))_{L^2} + \int_0^t \|\partial_t y\|_{L^2}^2 \leq (A_\beta y_0, y_0)_{L^2} + \int_0^t \|f\|_{L^2}^2.
\]

By (2.6), it follows that

\[
(2.7) \quad \|y\|_{L^\infty(0, T, H^1(\Omega))} + \|\partial_t y\|_{L^2((0, T) \times \Omega)} + \sqrt{\beta}\|y\|_{L^\infty(0, T, L^2(\Gamma))}
\leq C_T(\|\nabla y_0\|_{L^2(\Omega)} + \sqrt{\beta}\|y_0\|_{L^2(\Gamma)} + \|f\|_{L^2((0, T) \times \Omega)}).
\]

It remains to prove the \( L^2(0, T, H^2(\Omega)) \) estimate. To this end, we observe that

\[
\|A_\beta y\|_{L^2((0, T) \times \Omega)} \leq \|f\|_{L^2((0, T) \times \Omega)} + \|\partial_t y\|_{L^2((0, T) \times \Omega)},
\]

and thus the claim is just a consequence of (2.7) and of the following elliptic regularity property: there exists a \( C > 0 \), independent of \( \beta \in [0, +\infty) \), such that

\[
\|\zeta\|_{H^2(\Omega)} \leq C(\|\zeta\|_{L^2(\Omega)} + \|A_\beta \zeta\|_{L^2(\Omega)}), \quad \forall \zeta \in D(A_\beta).
\]

This can be proved, for instance, as in [12, Theorems III.4.2 and III.4.3] and using the fact that \( \beta \geq 0 \) to obtain a constant which is independent of \( \beta \). \( \square \)
2.1.3. Non-homogeneous Robin boundary data. Let us now consider the same problem but with a non-homogeneous boundary data

\[ \begin{cases} \partial_t y - \text{div}(\gamma \nabla y) = f & \text{in } (0, T) \times \Omega, \\ \frac{\partial y}{\partial n} + \beta y = g & \text{on } (0, T) \times \Gamma, \\ y(0, \cdot) = y_0 & \text{in } \Omega. \end{cases} \]

(2.8)

The theory developed in [23] for this problem gives the following result concerning existence and uniqueness of a solution in the natural energy spaces.

**Proposition 2.4.** Let \( \beta \in [0, +\infty) \) be given. For any \( y_0 \in L^2(\Omega) \), \( f \in L^2((0, T) \times \Omega) \), \( g \in L^2((0, T) \times \Gamma) \), there exists a unique weak solution \( y \in C^0([0, T], L^2(\Omega)) \cap L^2(0, T, H^1(\Omega)) \) to (2.8) in the following sense:

- \( y(0) = y_0 \).
- For any test function \( \psi \in H^1(0, T, L^2(\Omega)) \cap L^2(0, T, H^1(\Omega)) \), and any \( t_1, t_2 \in [0, T] \) we have

\[ (2.9) \quad - \int_{t_1}^{t_2} \int_{\Omega} y \psi_t + \int_{t_1}^{t_2} \int_{\Omega} (\gamma \nabla y) \cdot \nabla \psi + \beta \int_{t_1}^{t_2} \int_{\Gamma} y \psi = \int_{\Omega} y(t_1) \psi(t_1) - \int_{\Omega} y(t_2) \psi(t_2) + \int_{t_1}^{t_2} \int_{\Omega} f \psi + \int_{t_1}^{t_2} \int_{\Gamma} g \psi. \]

Moreover, it satisfies the estimate

\[ (2.10) \quad \| y \|_{C^0([0, T], L^2(\Omega))} + \| y \|_{L^2(0, T, H^1(\Omega))} + \| \partial_t y \|_{L^2(0, T, H^{-1}(\Omega))} \leq C_T (\| y_0 \|_{L^2(\Omega)} + \| f \|_{L^2((0, T) \times \Omega)} + \| g \|_{L^2((0, T) \times \Gamma)}), \]

for some \( C_T > 0 \) independent of \( \beta \).

**Remark 2.5** (Strong estimates do not pass to the limit). Note that if the boundary data for the Robin problem (2.8) is chosen in the form \( g = \beta g_D \), with \( g_D \in L^2((0, T) \times \Gamma) \) then the boundary condition reads

\[ \frac{1}{\beta} \frac{\partial y}{\partial n} + y = g_D, \quad \text{for } \beta \in (0, +\infty), \]

and we can formally expect the solution to converge, when \( \beta \to \infty \), towards the one associated with the Dirichlet boundary condition \( y = g_D \), that is to a solution of (2.1).

However, the estimate in the proposition above reads

\[ \| y \|_{C^0([0, T], L^2(\Omega))} + \| y \|_{L^2(0, T, H^1(\Omega))} + \| \partial_t y \|_{L^2(0, T, H^{-1}(\Omega))} \leq C_T (\| y_0 \|_{L^2(\Omega)} + \| f \|_{L^2((0, T) \times \Omega)} + \beta \| g_D \|_{L^2((0, T) \times \Gamma)}), \]

which is not uniform with respect to \( \beta \) and therefore we cannot a priori prove that the associated solution \( y \) is bounded when \( \beta \to \infty \). This is due to the fact that, considering only \( L^2 \) boundary data, we cannot expect a uniform bound in \( L^2(0, T, H^1(\Omega)) \) of the solution that would necessitate at least \( g_D \) to be in \( L^2(0, T, H^{1/2}(\Gamma)) \).

For the reasons above, we need to introduce a weaker formulation of the Robin problem that will allow to analyse the limit towards the Dirichlet problem with \( L^2 \) data in a convenient way.

**Proposition 2.6.** We consider the same assumption as in Proposition 2.4.

1. The weak solution \( y \) to the problem (2.8) is the unique function belonging to \( C^0([0, T], L^2(\Omega)) \) and satisfying, for any \( \zeta \in L^2(\Omega) \) and any \( t \in [0, T] \),

\[ (y(t), \zeta)_{L^2} - (y_0, e^{-\gamma A_0^\alpha} \zeta)_{L^2} = \int_0^t \int_{\Omega} f (e^{-(t-\cdot) A_0^\alpha} \zeta) + \int_0^t \int_{\Gamma} g e^{-(t-\cdot) A_0^\alpha} \zeta \]

in addition with the estimate (2.10).

2. The weak solution \( y \) to the problem (2.8) with \( \beta \in (0, +\infty) \), is also the unique function belonging to \( C^0([0, T], H^{-1}(\Omega)) \) and satisfying, for any \( \zeta \in D(A_0^\alpha) \cap H^1_0(\Omega) \) and any \( t \in [0, T] \),

\[ (y(t), \zeta)_{H^{-1}, H_0^1} - (y_0, e^{-\gamma A_0^\alpha} \zeta)_{L^2} = \int_0^t \int_{\Omega} f (e^{-(t-\cdot) A_0^\alpha} \zeta) - \int_0^t \int_{\Gamma} \frac{g}{\beta} \frac{\partial}{\partial n} (e^{-(t-\cdot) A_0^\alpha} \zeta). \]
Moreover, this weak solution satisfies the estimate

\[
\|y\|_{C^0([0,T],H^{-1}(\Omega))} + \|y\|_{L^2((0,T)\times\Omega)} + \|\partial_t y\|_{L^2(0,T;H^{-2}(\Omega))} \\
\leq C_T \left( \|y_0\|_{L^2(\Omega)} + \|f\|_{L^2((0,T)\times\Omega)} + \left\| \frac{g}{\beta} \right\|_{L^2((0,T)\times\Gamma)} \right),
\]

where \(C_T > 0\) does not depend on \(\beta\).

**Remark 2.7.** As explained in Remark 2.2, we decided to keep the adjoint notation \(A_\beta^*\) instead of \(A_\beta\) in the previous statement, even though it is unnecessary.

**Proof.**

1. Let us first consider any \(\zeta \in D(A_\beta^*) = D(A_\beta)\) and let us choose as a test function in (2.9) the strong solution of the homogeneous backward problem \(t \mapsto \psi(t) = e^{-(T-t)A_\beta^*} \zeta\). The integration by parts are well justified and naturally lead to the expected formula. By density of \(D(A_\beta)\) in \(L^2(\Omega)\) and by the estimate (2.3), we can extend the equality to any \(\zeta \in L^2(\Omega)\).

2. In the case where \(\zeta \in D(A_\beta) \cap H^1_0(\Omega)\), we know that \(\psi(t) \in D(A_\beta)\) for any \(t\), and in particular we have the equality \(\partial_\nu \zeta + \beta \psi = 0\) on \((0, T) \times \Gamma\), which gives the claimed equality. Now, applying the estimates (2.3)

and (2.5) to \(\psi\), we obtain for any \(t \in [0, T]\) and \(\beta \in (0, +\infty)\),

\[
\|\langle y(t), \zeta \rangle_{H^{-1},H^1}\| \leq C_T (\|y_0\|_{L^2(\Omega)} + \|f\|_{L^2((0,T)\times\Omega)}) \|\zeta\|_{L^2(\Omega)} + C_T \left\| \frac{g}{\beta} \right\|_{L^2((0,T)\times\Gamma)} \|\nabla \zeta\|_{L^2(\Omega)}.
\]

Since \(D(A_\beta) \cap H^1_0(\Omega)\) is dense in \(H^1_0(\Omega)\), we get the expected bound on \(\|y\|_{C^0([0,T],H^{-1}(\Omega))}\).

Let us show now the bound in \(L^2((0,T)\times\Omega)\). Consider any \(h \in C_\infty((0,T)\times\Omega)\) and let \(\psi\) be the unique strong solution (as given by Proposition 2.3) to the backward problem

\[
\begin{cases}
-\partial_t \psi - \text{div}(\gamma \nabla \psi) = h & \text{in } (0,T) \times \Omega, \\
\frac{\partial \psi}{\partial \nu} + \beta \psi = 0 & \text{on } (0,T) \times \Gamma, \\
\psi(T,\cdot) = 0 & \text{in } \Omega.
\end{cases}
\]

By (2.5), we have the following estimate, uniformly with respect to the parameter \(\beta\)

\[
\|\psi\|_{L^\infty(0,T,H^1(\Omega))} + \|\psi\|_{L^2(0,T;H^2(\Omega))} + \|\partial_t \psi\|_{L^2(0,T,L^2(\Omega))} \leq C_T \|h\|_{L^2((0,T)\times\Omega)}.
\]

Putting this test function in (2.9) and integrating by parts lead to

\[
\int_0^T \int_\Omega yh = \int_\Omega y_0 \psi(0) + \int_0^T \int_\Omega f \psi - \int_0^T \int_\Gamma \frac{g}{\beta} \frac{\partial \psi}{\partial \nu}, \quad \text{for } \beta \in (0, +\infty),
\]

where we have used the boundary condition satisfied by \(\psi\) at each time \(t\) in the boundary term. Using the Cauchy-Schwarz inequality and (2.13), we finally get

\[
\left| \int_0^T \int_\Omega yh \right| \leq C (\|y_0\|_{L^2(\Omega)} + \|f\|_{L^2((0,T)\times\Omega)}) \|\psi\|_{L^\infty(0,T,L^2(\Omega))} + C \left\| \frac{g}{\beta} \right\|_{L^2((0,T)\times\Gamma)} \|\psi\|_{L^2(0,T;H^2(\Omega))}
\]

\[
\leq C_T \left( \|y_0\|_{L^2(\Omega)} + \|f\|_{L^2((0,T)\times\Omega)} + \left\| \frac{g}{\beta} \right\|_{L^2((0,T)\times\Gamma)} \right) \|h\|_{L^2((0,T)\times\Omega)}.
\]

Since \(C_\infty((0,T)\times\Omega)\) is dense in \(L^2((0,T)\times\Omega)\), we obtain the expected estimate by duality.

Finally, we can easily see that the weak solution \(y\) satisfies, in the distribution sense, the equation \(\partial_t y - \text{div}(\gamma \nabla y) = f\), and the bound of \(\partial_t y\) in \(L^2(0,T;H^{-2}(\Omega))\) immediately follows.

**Remark 2.8** (Weak estimates pass to the limit). Going back to the situation described in Remark 2.5, that is if \(g = \beta y_D\), \(y_D \in L^2((0,T)\times\Gamma)\), we deduce now a bound of the associated solution which is uniform when \(\beta \to +\infty\), yet in weaker norms than above. We shall see in the next section that those estimates allow us to pass to the limit towards the Dirichlet problem.
2.2. Passing to the limit to the Dirichlet problem. With the above existence results and estimates, we can now state and prove a convergence result of the solutions of a suitable Robin problem to the one of a Dirichlet problem.

**Theorem 2.9 (Convergence towards the Dirichlet problem).** Let $y_0 \in L^2(\Omega)$ be a given initial data. For any $\beta > 0$, we consider a source term $f_\beta \in L^2((0, T) \times \Omega)$ and a boundary data $g_\beta \in L^2((0, T) \times \Gamma)$, and we denote by $y_\beta$ the associated weak solution to (2.8).

We assume that, for some $f_D \in L^2((0, T) \times \Omega)$, $g_D \in L^2((0, T) \times \Gamma)$ we have the $L^2$-weak convergences
\begin{equation}
(2.14) \quad f_\beta \xrightarrow{\beta \to +\infty} f_D, \quad g_\beta \xrightarrow{\beta \to +\infty} g_D,
\end{equation}

Then $y_\beta$ converges, when $\beta \to +\infty$, weakly in $L^2((0, T) \times \Omega)$ and strongly in $L^2(0, T, H^{-1}(\Omega))$ towards the unique solution $y_D \in C^0([0, T], H^{-1}(\Omega)) \cap L^2((0, T) \times \Omega)$ to the Dirichlet problem (2.1) associated to the data $f_D$ and $g_D$.

Moreover, for any $t \in (0, T)$, $y_\beta(t) \xrightarrow{\beta \to +\infty} y_D(t)$ weakly in $H^{-1}(\Omega)$.

**Proof.** From the hypothesis, we have a bound on the quantities $\|g_\beta/\beta\|_{L^2((0, T) \times \Gamma)}$ and $\|f_\beta\|_{L^2((0, T) \times \Omega)}$ uniform with respect to $\beta \geq 1$. Hence, from (2.11), we deduce that, for some $C_{T, y_0} > 0$, uniform in $\beta$, we have
\begin{equation}
(2.15) \quad \|y_\beta\|_{C^0([0, T], H^{-1}(\Omega))} + \|y_\beta\|_{L^2((0, T) \times \Omega)} + \|\partial_t y_\beta\|_{L^2(0, T, H^{-2}(\Omega))} \leq C_{T, y_0}.
\end{equation}

We can then find some $y_D \in C^0([0, T], H^{-1}(\Omega)) \cap L^2((0, T) \times \Omega)$ and a subsequence, still denoted by $(y_\beta)_\beta$ such that
\begin{equation}
(2.16) \quad \begin{cases}
y_\beta \xrightarrow{\beta \to +\infty} y_D & \text{ weakly in } L^2((0, T) \times \Omega), \\
\partial_t y_\beta \xrightarrow{\beta \to +\infty} \partial_t y_D & \text{ weakly in } L^2(0, T, H^{-2}(\Omega)), \\
y_\beta \xrightarrow{\beta \to +\infty} y_D & \text{ weakly-* in } C^0([0, T], H^{-1}(\Omega)), \\
y_\beta \xrightarrow{\beta \to +\infty} y_D & \text{ strongly in } L^2(0, T, H^{-1}(\Omega)).
\end{cases}
\end{equation}

The last strong convergence comes from the compactness of the embeddings $L^2(\Omega) \hookrightarrow H^{-1}(\Omega)$ and $H^{-1}(\Omega) \hookrightarrow H^{-2}(\Omega)$ and the Aubin-Lions lemma.

All we need to show is that this limit $y_D$ is indeed the solution to the corresponding Dirichlet problem. By uniqueness of the solution of Dirichlet problem (2.1) with the data $f_D, g_D$, the convergence of the whole family $(y_\beta)_\beta$ will be established.

Let us consider a final data $\zeta \in C^\infty_c(\Omega) \subset D(A^*_\beta) \cap H^1_0(\Omega)$ for the adjoint homogeneous problem. The corresponding strong solution is given by $\psi_\beta(t) = e^{-(T-t)A^*_\beta} \zeta$ and, thanks to (2.5), we have
\begin{equation}
(2.17) \quad \|\psi_\beta\|_{C^0([0, T], H^1(\Omega))} + \|\psi_\beta\|_{L^2(0, T, H^2(\Omega))} + \|\partial_t \psi_\beta\|_{L^2((0, T) \times \Omega)} \leq C_T \|\zeta\|_{H^1_0(\Omega)},
\end{equation}

where $C_T$ is uniform in $\beta$.

We can then extract a subsequence, still denoted by $(\psi_\beta)_\beta$, such that
\begin{equation}
(2.18) \quad \begin{cases}
\psi_\beta \xrightarrow{\beta \to +\infty} \psi_D & \text{ weakly in } L^2(0, T, H^2(\Omega)), \\
\partial_t \psi_\beta \xrightarrow{\beta \to +\infty} \partial_t \psi_D & \text{ weakly in } L^2((0, T) \times \Omega), \\
\psi_\beta \xrightarrow{\beta \to +\infty} \psi_D & \text{ weakly-* in } C^0([0, T], H^1(\Omega)), \\
\psi_\beta \xrightarrow{\beta \to +\infty} \psi_D & \text{ strongly in } L^2(0, T, H^1(\Omega)),
\end{cases}
\end{equation}

for some $\psi_D \in L^2((0, T; H^2(\Omega)) \cap C^0([0, T], H^1(\Omega))$. Here also we have used the Aubin-Lions lemma to obtain the last strong convergence.

Moreover, from the boundary condition satisfied by $\psi_\beta$, we have $\psi_\beta = -\frac{1}{\beta}\frac{\partial \psi_\beta}{\partial \nu}$ on the boundary where the quantity $\|\frac{\partial \psi_\beta}{\partial \nu}\|_{L^2((0, T) \times \Gamma)}$ is bounded for any large $\beta$ (using (2.17)). Hence, it follows that
\begin{equation}
(2.19) \quad \psi_\beta \xrightarrow{\beta \to +\infty} 0 \quad \text{ in } L^2((0, T) \times \Gamma),
\end{equation}

and $\psi_D$ satisfies the same boundary condition.

This completes the statement of Theorem 2.9.

**Note.** This manuscript is for review purposes only.
which actually implies that \( \psi_D = 0 \) on the boundary \((0, T) \times \Gamma \). Moreover, by passing to the limit in the equation in the distribution sense, we finally find that \( \psi_D \) is the unique solution to the backward homogeneous Dirichlet problem, that is \( \psi_D(t) = e^{-(T-t)A_D^\gamma} \).

Now, using the trace theorem, we observe that

\[
\int_0^T \int_\Gamma \left| \frac{\partial \psi_\beta}{\partial \nu_\gamma} - \frac{\partial \psi_D}{\partial \nu_\gamma} \right|^2 \leq C \int_0^T \| \psi_\beta - \psi_D \|_{H^1(\Omega)} \| \psi_\beta - \psi_D \|_{H^2(\Omega)}
\]

By (2.18) we see that the first factor of the very right hand side of the above inequality converges to 0 as \( \beta \to +\infty \), whereas the second factor is bounded. Thus, we have

\[
(2.19) \quad \frac{\partial \psi_\beta}{\partial \nu_\gamma} \xrightarrow{\beta \to +\infty} \frac{\partial \psi_D}{\partial \nu_\gamma} \quad \text{in } L^2((0, T) \times \Gamma).
\]

In particular, from the third point of (2.18) we have

\[
(2.20) \quad \psi_\beta(0, \cdot) \xrightarrow{\beta \to +\infty} \psi_D(0, \cdot) \quad \text{weakly in } H^1(\Omega).
\]

With the convergence results (2.19) and (2.20) above together with (2.14) we get

\[
(y_\beta, \psi_\beta(0))_{L^2} + \int_0^t \int_\Omega f_\beta \psi_\beta - \int_\Omega \int_\Gamma \frac{g_\beta}{\beta} \frac{\partial \psi_\beta}{\partial \nu_\gamma} \xrightarrow{\beta \to +\infty} (y_0, \psi_D(0))_{L^2} + \int_0^t \int_\Omega f_D \psi_D - \int_\Gamma \int_\Gamma g_D \frac{\partial \psi_D}{\partial \nu_\gamma}.
\]

Using the weak formulation (2.9) satisfied by \( y_\beta \) with \( \psi_\beta \) as a test function we see that have actually proved that

\[
(2.21) \quad \psi_D(0, \cdot) \xrightarrow{\beta \to +\infty} \psi_D(0, \cdot)
\]

and in the same time, by (2.16), we have

\[
(y_\beta(t), \zeta)_{L^2} \xrightarrow{\beta \to +\infty} (y_D(t), \zeta)_{H^{-1}, D^{-1}}.
\]

As a conclusion, we have proved that \( y_D \) satisfies

\[
(2.22) \quad \langle y_D(t), \zeta \rangle_{H^{-1}, D^{-1}} = \langle y_0, e^{-tA_D^\gamma} \zeta \rangle_{L^2} + \int_0^t \int_\Omega f_D \left( e^{-(t-s)A_D^\gamma} \zeta \right) - \int_\Omega \int_\Gamma g_D \frac{\partial}{\partial \nu_\gamma} \left( e^{-(t-s)A_D^\gamma} \zeta \right),
\]

which is exactly the definition of the solution of (2.1) with the data \( f_D, g_D \), see Proposition 2.1. \( \square \)

**Remark 2.10** (Convergence towards the Neumann problem). *By similar, and in fact simpler, arguments one can prove that if \((f_\beta)_\beta\) and \((g_\beta)_\beta\) both weakly converge, when \( \beta \to 0 \), towards some \( f_N \) and \( g_N \) in \( L^2((0, T) \times \Omega) \) and \( L^q((0, T) \times \Gamma) \) respectively, then the corresponding solution \( y_\beta \) converges, when \( \beta \to 0 \), to the solution \( y_N \) of the corresponding non-homogeneous Neumann problem.*

### 2.3. The coupled system

We can now move to the cascade coupled parabolic systems we are interested in, namely the one with Dirichlet boundary condition

\[
\begin{cases}
\partial_t y_1 - \text{div}(\gamma \nabla y_1) = f_1 & \text{in } (0, T) \times \Omega, \\
\partial_t y_2 - \text{div}(\gamma \nabla y_2) + y_1 = f_2 & \text{in } (0, T) \times \Omega,
\end{cases}
\]

\[
\begin{align*}
y_1 &= g_1 & & \text{on } (0, T) \times \Gamma, \\
y_2 &= g_2 & & \text{on } (0, T) \times \Gamma, \\
y_1(0, \cdot) &= y_{0, 1} & & \text{in } \Omega, \\
y_2(0, \cdot) &= y_{0, 2} & & \text{in } \Omega,
\end{align*}
\]

(2.21)
and the one with Robin boundary conditions

\[
\begin{aligned}
\partial_t y_1 - \text{div}(\gamma \nabla y_1) &= f_1 \quad \text{in } (0,T) \times \Omega, \\
\partial_t y_2 - \text{div}(\gamma \nabla y_2) + y_1 &= f_2 \quad \text{in } (0,T) \times \Omega, \\
\frac{\partial y_1}{\partial \nu_\gamma} + \beta_1 y_1 &= g_1 \quad \text{on } (0,T) \times \Gamma, \\
\frac{\partial y_2}{\partial \nu_\gamma} + \beta_2 y_2 &= g_2 \quad \text{on } (0,T) \times \Gamma, \\
y_1(0,\cdot) &= y_{0,1} \quad \text{in } \Omega, \\
y_2(0,\cdot) &= y_{0,2} \quad \text{in } \Omega.
\end{aligned}
\]

(2.22)

We can obviously solve those two systems by simply using the results on the scalar case: we first solve the equation for \(y_1\) then we solve the scalar equation for \(y_2\) by considering the coupling term \(y_1\) as an additional \(L^2\) source term.

**Theorem 2.11.** We suppose given \(y_0 := (y_{0,1}, y_{0,2}) \in (L^2(\Omega))^2\), \(f := (f_1, f_2) \in (L^2((0,T) \times \Omega))^2\) and \(g := (g_1, g_2) \in (L^2((0,T) \times \Gamma))^2\).

1. There exists a unique solution \(y = (y_1, y_2) \in (C^0([0,T], H^{-1}(\Omega)) \cap L^2((0,T) \times \Omega))^2\) of (2.21), that is, for any \(i = 1,2\), \(y_i\) satisfies the corresponding scalar problem in the sense of Proposition 2.1.

2. For any \(\beta_1, \beta_2 \in [0, +\infty)\), there exists a unique solution \(y = (y_1, y_2) \in (C^0([0,T], L^2(\Omega)) \cap L^2(0,T, H^1(\Omega)))^2\) of (2.22), that is, for any \(i = 1,2\), \(y_i\) satisfies the corresponding scalar problem in the sense of Proposition 2.4.

3. For any \(\beta = (\beta_1, \beta_2) \in (0, +\infty)^2\), we suppose given \(f_{i,\beta} \in L^2((0,T) \times \Omega)\) and \(g_{i,\beta} \in L^2((0,T) \times \Gamma)\), for \(i = 1,2\) such that

\[
\begin{aligned}
g_{i,\beta} &\xrightarrow{\beta \to +\infty} g_i, \\
f_{i,\beta} &\xrightarrow{\beta \to +\infty} f_i.
\end{aligned}
\]

Then, the solution \(y_{\beta}\) of (2.22) corresponding to the data \(f_{\beta}, g_{\beta}\) converges weakly in \((L^2((0,T) \times \Omega))^2\) and strongly in \((L^2(0,T, H^{-1}(\Omega)))^2\) towards the unique solution of the corresponding Dirichlet problem.

For the analysis of the control problem, it is not convenient to make appear the component \(y_1\) of the solution as a source term in the equation for \(y_2\) since it breaks down the cascade structure of the system which is essential to prove its controllability with only one control. That is the reason why it is necessary to introduce the following unbounded operators in \((L^2(\Omega))^2\): let \(A_{\beta_1,\beta_2}\) and \(A_D\) be defined by the same formal expression

\[
(2.23)\begin{pmatrix}
-\text{div}(\gamma \nabla \cdot) \\
1
\end{pmatrix}
\]

but with the different domains

\[
D(A_{\beta_1,\beta_2}) := \left\{ y \in (H^2(\Omega))^2 \left| \frac{\partial y_1}{\partial \nu_\gamma} + \beta_1 y_1 = 0, \frac{\partial y_2}{\partial \nu_\gamma} + \beta_2 y_2 = 0 \text{ on } \Gamma \right. \right\},
\]

and

\[
D(A_D) := (H^2(\Omega) \cap H^1_0(\Omega))^2,
\]

respectively. Those operators are no more self-adjoint and we define their adjoints by \(D(A_{\beta_1,\beta_2}^*) = D(A_{\beta_1,\beta_2})\) and \(D(A_D^*) = D(A_D)\) and the same formal expression

\[
(2.24)\begin{pmatrix}
-\text{div}(\gamma \nabla \cdot) \\
0
\end{pmatrix}
\]

Standard elliptic theory shows that \(-A_{\beta_1,\beta_2}\) and \(-A_{\beta_1,\beta_2}^*\) as well as \(-A_D\) and \(-A_D^*\) generate continuous semigroups in \((L^2(\Omega))^2\). A similar analysis as in Section 2.1.3 for the scalar case, leads to the following result.

**Proposition 2.12.** We suppose given any \(y_0 \in (L^2(\Omega))^2\), \(f \in (L^2((0,T) \times \Omega))^2\) and \(g \in (L^2((0,T) \times \Gamma))^2\).

1. The solution to (2.21) is the unique element \(y \in (C^0([0,T], H^{-1}(\Omega)))^2\) satisfying, for any \(\zeta \in (H^1_0(\Omega))^2\) and any \(t \in [0,T] \)

\[
\langle y(t), \zeta \rangle_{H^{-1}, H^1_0} = \langle y_0, e^{-tA_D^*} \zeta \rangle_{L^2} + \int_0^t \int_\Omega f \cdot \left( e^{-(t-s)A_D^*} \zeta \right) - \int_0^t \int_\Gamma g \cdot \frac{\partial}{\partial \nu_\gamma} \left( e^{-(t-s)A_D^*} \zeta \right).
\]
2. For any $\beta_1, \beta_2 \in [0, +\infty)$, the solution to (2.22) is the unique element $y \in (C^0([0,T], L^2(\Omega)))^2$ satisfying, for any $\zeta \in (L^2(\Omega))^2$ and any $t \in [0,T]$

\[
(y(t), \zeta)_L^2 = \left(y_0, e^{-tA_{\beta_1, \beta_2}} \right)_L^2 + \int_0^t \int_{\Omega} f \cdot \left(e^{-(t-s)A_{\beta_1, \beta_2}} \zeta\right) - \int_0^t \int_{\Gamma} g \cdot \left(e^{-(t-s)A_{\beta_1, \beta_2}} \zeta\right).
\]

2.4. Main results. Let now $\Gamma_0$ be a part of $\Gamma$. Using the analysis presented in the previous sections we can now formulate the null-control problems we are interested in as follows.

**Proposition 2.13.** Let $y_0 \in (L^2(\Omega))^2$ be given.

1. A function $v \in L^2((0,T) \times \Gamma)$ is a null-control at time $T$ for the Dirichlet problem (1.1)-(1.2), if and only if it satisfies: for any $\zeta \in (H^1_0(\Omega))^2$

\[
(y_0, e^{-tA_D} \zeta)_L^2 = \int_0^T \int_\Gamma \left(1, 0\right) \cdot \frac{\partial}{\partial n} \left(e^{-(T-s)A_D} \zeta\right) \, dt,
\]

2. A function $v \in L^2((0,T) \times \Gamma)$ is a null-control at time $T$ for the Robin problem (1.1)-(1.3), if and only if it satisfies: for any $\zeta \in (L^2(\Omega))^2$

\[
(y_0, e^{-tA_{\beta_1, \beta_2}} \zeta)_L^2 = \int_0^T \int_\Gamma \left(1, 0\right) \cdot \left(e^{-(T-s)A_{\beta_1, \beta_2}} \zeta\right) \, dt.
\]

2.4.1. The 1D case. We start with a discussion of the 1D setting since, as we will see in the next section, we can deduce some null-D results from the 1D analysis.

Hence, we particularize the above control problem to the 1D situation where $\Omega = (0,1)$, $\Gamma_0 = \{0\}$ and the diffusion coefficient is simply a scalar function $\gamma \in C^1((0,1))$ with $\gamma_{\min} = \inf_{[0,1]} \gamma > 0$ and $\gamma_{\max} = \sup_{[0,1]} \gamma < +\infty$.

In that case the control we are looking for is just a scalar function $v \in L^2(0,T)$ and the formulations (2.25), (2.26) just reads

\[
- \left(y_0, e^{-tA_D} \gamma \right)_L^2 = \gamma(0) \int_0^T v(t) \left(1, 0\right) \cdot \frac{\partial}{\partial x} \left|_{x=0} \left(e^{-(T-t)A_D} \gamma\right) \right| dt,
\]

for the Dirichlet problem and

\[
- \left(y_0, e^{-tA_{\beta_1, \beta_2}} \zeta\right)_L^2 = \int_0^T v(t) \left(1, 0\right) \cdot \left(e^{-(T-t)A_{\beta_1, \beta_2}} \zeta\right) \, dt,
\]

for the Robin problem with the same notations for the adjoint of the diffusion operators as in multi-D. It is convenient to introduce the observation operator $B^*$ (that does not depend on the Robin parameters $\beta_1, \beta_2$) defined as follows

\[
B^* : \left(\zeta_1, \zeta_2\right) \in (H^1(0,1))^2 \mapsto \zeta_1(0),
\]

in such a way that (2.28) becomes

\[
(y_0, e^{-tA_{\beta_1, \beta_2}} \zeta)_L^2 = \int_0^T v(t) B^* \left(e^{-(T-t)A_{\beta_1, \beta_2}} \zeta\right) \, dt.
\]

Most of the work in Sections 4 and 5 will consist in solving this problem with suitable estimates of the control with respect to the parameters $\beta_1$ and $\beta_2$. Our main result in that direction is the following.

**Theorem 2.14.** Let $y_0 \in (L^2(0,1))^2$ and $T > 0$ be given.

1. Let $\beta \in (0, +\infty)$ and set $\beta_1 = \beta_2 = \beta$. Then, there exists a null-control $v_\beta \in L^2(0,T)$ for the 1D problem (2.30) that satisfies in addition the estimate

\[
\|v_\beta\|_{L^2(0,T)} \leq C e^{C/T} (1 + \beta) \|y_0\|_{L^2(0,1)},
\]

where $C > 0$ does not depend on $\beta$ and $T$.

2. Assume that $\gamma$ is a positive constant and let $\beta^* > 0$ be given. Let $\beta = (\beta_1, \beta_2) \in (0, +\infty)^2$ be any couple of Robin parameters. Then, there exists a null-control $v_\beta \in L^2(0,T)$ for the 1D problem (2.30) that satisfies in addition the estimate

\[
\|v_\beta\|_{L^2(0,T)} \leq C_{T, \beta^*} (1 + \beta_1) \|y_0\|_{L^2(0,1)},
\]

as soon as either $\beta_1, \beta_2 \in (0, \beta^*)$, or $\beta_1, \beta_2 \in [\beta^*, +\infty)$, where $C_{T, \beta^*} > 0$ does not depend on $\beta$. 

Corollary 2.15 (Convergence towards Dirichlet control). Let $\beta_n = (\beta_{1,n}, \beta_{2,n}) \in (0, +\infty)^2$ be any sequence of Robin parameters such that $\beta_{1,n} \to +\infty$ when $n \to \infty$, for $i = 1, 2$. If the diffusion coefficient $\gamma$ is not a constant, we assume in addition that $\beta_{1,n} = \beta_{2,n}$ for any $n$. For each $n$, let $v_n$ (resp. $y_n$) be the unique null-control of minimal $L^2(0,T)$ norm (resp. the associated trajectory) for the problem (2.30) with Robin parameters $\beta_{1,n}$ and $\beta_{2,n}$.

Then, there exists a subsequence $(n_k)_k$ such that

$$\frac{v_{n_k}}{\beta_{1,n_k}} \to v_D, \text{ in } L^2(0,T),$$

$$y_{n_k} \to y_D, \text{ strongly in } (L^2(0,T,H^{-1}(0,1)))^2 \text{ and weakly in } (L^2((0,T) \times (0,1)))^2,$$

where $v_D$ (resp. $y_D$) is a null-control (resp. the associated trajectory) for the Dirichlet control problem (2.27).

Remark 2.16 (Convergence towards Neumann control). With the same notation as in the previous corollary, if we assume that $\beta_{i,n} \to 0$ when $n \to \infty$, for $i = 1, 2$, then we obtain the convergence, up to a subsequence, of the null-control $v_n$ (resp. of the trajectory $y_n$) towards a null-control $v_N$ (resp. the trajectory $y_N$) corresponding to the Neumann boundary conditions on both components.

Remark 2.17 (The Dirichlet/Neumann case). In point 2 of Theorem 2.14, we needed to assume that either the two Robin parameters are both smaller than some $\beta^*$ or that they are both higher than some $\beta^*$. It is worth noticing that we cannot expect to prove a similar result without those assumptions.

Indeed, if we were able to prove the estimate $\|v_n\| \leq C_T(1 + \beta_1)\|y_0\|_{L^2(0,1)}$, for any couple of parameters $\beta_1$ and $\beta_2$, then by following the same lines as in Corollary 2.15, we would be able to prove the convergence, up to a subsequence, of $v_n/\beta_{1,n}$ when $\beta_{1,n} \to +\infty$ and $\beta_{2,n} \to 0$ to some $v_{DN}$ that would be a null-control for the Dirichlet/Neumann problem (that is system (1.1) in 1D, with a Dirichlet boundary condition for the first component $y_1$ and a Neumann boundary condition for the second component $y_2$). However, we know that this last problem is not even approximately controllable since the underlying operator $A_{\infty,0}^*$ has eigenspaces of dimension higher than 1, which prevents the Fattorini-Hautus criterion (see [16, 24]) from being satisfied.

The same remark holds for the Neumann/Dirichlet problem, that is when $\beta_{1,n} \to 0$ and $\beta_{2,n} \to +\infty$.

2.4.2. A multi-D result. By using the methodology described in [2, 10] it is possible, starting from a suitable null-controllability result for the 1D problem, at least when both Robin parameters are the same, to deduce the corresponding result in any cylinder of $\mathbb{R}^d$ for $d \geq 2$, see Figure 1.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{cylindrical_geometry.png}
\caption{The cylindrical geometry}
\end{figure}

More precisely, we consider a domain $\Omega = (0,1) \times \Omega_2$ in $\mathbb{R}^d$ where $\Omega_2$ is a bounded smooth connected domain in $\mathbb{R}^{d-1}$. The variable in $\Omega$ will be denoted by $(x, \bar{x})$, with $x \in \Omega_1$ and $\bar{x} \in \Omega_2$ and we assume that the diffusion tensor has the following form

$$\gamma(x, \bar{x}) = \begin{pmatrix} \gamma(x) & 0 & \cdots & 0 \\ 0 & \ddots & \vdots & \vdots \\ 0 & \cdots & \gamma(\bar{x}) & 0 \end{pmatrix},$$

with $\gamma : \Omega_1 \to \mathbb{R}$ and $\bar{\gamma} : \Omega_2 \to M_{d-1}(\mathbb{R})$. Let $\omega_2 \subset \Omega_2$ be a non empty open subset of $\Omega_2$. The control region we will consider is $\Gamma_0 = \{0\} \times \omega_2$ so that the control problem is the following

$$\begin{cases}
\partial_t y_1 - \bar{\partial}_x (\gamma(x) \partial_x y_1) - \text{div}_x (\bar{\gamma}(\bar{x}) \nabla_x y_1) = 0 & \text{in } (0,T) \times \Omega, \\
\partial_t y_2 - \bar{\partial}_x (\gamma(x) \partial_x y_2) - \text{div}_x (\bar{\gamma}(\bar{x}) \nabla_x y_2) + y_1 = 0 & \text{in } (0,T) \times \Omega, \\
y_1(0,\cdot) = y_{0,1} & \text{in } \Omega, \\
y_2(0,\cdot) = y_{0,2} & \text{in } \Omega,
\end{cases}$$

(2.31)
associated with either Dirichlet boundary conditions

\[
y_1 = 1_{\{0\} \times \omega_2} v \quad \text{on} \quad (0, T) \times \Gamma,
\]

\[
y_2 = 0 \quad \text{on} \quad (0, T) \times \Gamma,
\]

or Robin boundary conditions with the same parameter

\[
\begin{align*}
\frac{\partial y_1}{\partial \nu_\gamma} + \beta y_1 &= 1_{\{0\} \times \omega_2} v \quad \text{on} \quad (0, T) \times \Gamma, \\
\frac{\partial y_2}{\partial \nu_\gamma} + \beta y_2 &= 0 \quad \text{on} \quad (0, T) \times \Gamma.
\end{align*}
\]

Note that the result below is restricted to the case \( \beta_1 = \beta_2 \) for two reasons. The main one is that when the two parameters are different, the problem has no more a suitable tensor product structure that is crucial in the analysis (see [2]). The second one is that the constant \( C_{T, \beta} \) in point 2 of Theorem 2.14 is not explicit enough with respect to \( T \); we would need an exponential dependence of the constant for the analysis to apply directly.

**Theorem 2.18.** Let \( y_0 \in (L^2(\Omega))^2 \) be given. For any \( T > 0 \) and any \( \beta \in (0, +\infty) \), there exists a null-control \( v_\beta \in L^2((0, T) \times \Gamma) \) for the multi-D problem (2.31)-(2.33) that satisfies in addition the estimate

\[
\|v_\beta\|_{L^2((0, T) \times \Gamma)} \leq Ce^{C/T}(1 + \beta)\|y_0\|_{L^2(\Omega)},
\]

where \( C > 0 \) neither depends on \( \beta \) nor on \( T \).

**Proof.** The proof is mainly based on the strategy developed in [2, 10] which needs the sharp estimate with respect to \( T \) of the 1D control cost given by point 1 of Theorem 2.14 and a Lebeau-Robbiano spectral inequality, uniform in \( \beta \in (0, +\infty) \), relative to our control region \( \omega_2 \), for the eigenfunctions of the diffusion operator \( -\text{div}_x(\gamma(x) \nabla_x) \) with homogeneous Robin boundary condition in \( \Omega_2 \).

The required Lebeau-Robbiano inequality has been proved in [22, Theorem 1.2] for the eigenfunctions of Laplace-Beltrami operator in a multi-dimensional connected compact \( C^1 \)-smooth Riemannian manifold \( M \) with the boundary condition \( \frac{\partial u}{\partial \nu} + lu = 0 \) (\( u \in H^2(M) \)) for \( l \equiv 1 \) and any \( l \in L^\infty(\Gamma) \) with \( l \geq 0 \). Although they did not mention it in the paper, a careful look at their computations ensures us that the Lebeau-Robbiano inequality in this reference is in fact uniform with respect to \( l \geq 0 \). Thus, the required inequality holds true for our operator \( -\text{div}_x(\gamma(x) \nabla_x) \) in \( \Omega_2 \) with homogeneous Robin boundary condition with any parameter \( \beta \in (0, +\infty) \).

**Corollary 2.19** (Convergence towards Dirichlet control). Let \( (\beta_n) \) be any sequence of positive Robin parameters such that \( \beta_n \to +\infty \) when \( n \to +\infty \). For each \( n \) we define \( v_n \) (resp. \( y_n \)) to be the null-control of minimal \( L^2 \) norm (resp. the associated trajectory) for the problem (2.31)-(2.33) with Robin parameter \( \beta_n \).

There exists a subsequence \( (n_k)_k \) such that

\[
v_{n_k, \beta_{n_k}} \quad \text{in} \quad L^2((0, T) \times \Gamma),
\]

\[
y_{n_k} \quad \text{strongly in} \quad (L^2(0, T, H^{-1}(\Omega)))^2
\]

\[
\text{and weakly in} \quad (L^2((0, T) \times \Omega))^2,
\]

where \( v_D \) (resp. \( y_D \)) is a null-control (resp. the associated trajectory) for the Dirichlet control problem (2.31)-(2.32) for the same initial data.

**Remark 2.20** (Convergence towards Neumann control). When \( (\beta_n) \) goes to 0, we obtain the convergence towards a null-control for the Neumann problem as in Remark 2.16.

**2.5. Outline.** The rest of the paper is dedicated to the proof of our main theorem for the 1D case, namely Theorem 2.14. First of all, we establish useful spectral properties for the 1D Robin eigenvalue problem in Section 3. Then, we prove in Section 4 the controllability result in the case of an arbitrary diffusion coefficient but for the same Robin parameter for both components (point 1 of Theorem 2.14). Finally, in Section 5, we investigate the case of a constant diffusion coefficient with two different Robin parameters (point 2 of Theorem 2.14).

**3. Some spectral properties of the 1D Robin eigenvalue problem.** In this section, we develop some properties of the eigenvalue-eigenfunctions of the 1D scalar operator \( A_\beta \) as introduced in Section 2.1.2. Note that we use same notation as for the general higher dimension case. Those results will be used to draw some spectral properties of our main operator \( A_{\beta_1, \beta_2} \).
3.1. The case of a non-constant diffusion coefficient. We begin with the following scalar eigenvalue problem

\[
\begin{cases}
-\partial_x(\gamma(x)\partial_x\varphi) = \lambda\varphi & \text{in } (0, 1), \\
-\gamma(0)\partial_x\varphi(0) + \beta\varphi(0) = 0, \\
\gamma(1)\partial_x\varphi(1) + \beta\varphi(1) = 0,
\end{cases}
\]

where \( \beta \) is any non-negative parameter and \( \gamma \) is chosen as in Section 2.4.1. Let us denote the eigenvalue-eigenfunction pairs of the Sturm-Liouville problem (3.1) as \( (\lambda_{k,\gamma}^\beta, \varphi_{k,\gamma}^\beta)_{k \geq 0} \). We recall that the eigenvalues are simple and real and can be numbered in such a way that

\[
0 \leq \lambda_{0,\gamma}^\beta < \lambda_{1,\gamma}^\beta < \cdots < \lambda_{k,\gamma}^\beta < \lambda_{k+1,\gamma}^\beta \rightarrow +\infty,
\]

see for instance [6, Theorem 8.4.5]. Also it is well-known that the family \( (\varphi_{k,\gamma}^\beta)_{k \geq 0} \) is a Hilbert basis of \( L^2(0, 1) \), as soon as they are normalized, and indeed each of \( \varphi_{k,\gamma}^\beta \) belongs to the domain of the corresponding differential operator in (3.1).

**Remark 3.1.** Observe that for \( \beta = 0 \), the problem (3.1) reduces to a Neumann eigenvalue problem where we denote the Neumann eigenvalues by \( \lambda_{k,\gamma}^N := \lambda_{k,\gamma}^0 \) for \( k \geq 0 \).

On the other hand, for \( \beta = +\infty \), (3.1) degenerates into a Dirichlet eigenvalue problem and we denote by \( \lambda_{k,\gamma}^D := \lambda_{k,\gamma}^\infty \) for \( k \geq 0 \), the Dirichlet eigenvalues.

For any \( \beta \in [0, +\infty) \) and any \( \varphi \in H^1(0, 1), \varphi \neq 0 \), we define the Rayleigh quotient associated with (3.1),

\[
R_\beta(\varphi) := \frac{\int_0^1 \gamma|\partial_x\varphi|^2 + \beta(|\varphi(0)|^2 + |\varphi(1)|^2)}{\int_0^1 |\varphi|^2}.
\]

For \( \beta = +\infty \), we set

\[
R_\infty(\varphi) := \begin{cases}
\frac{\int_0^1 \gamma|\partial_x\varphi|^2}{\int_0^1 |\varphi|^2}, & \text{if } \varphi \in H_0^1(0, 1), \varphi \neq 0, \\
+\infty, & \text{if } \varphi \notin H_0^1(0, 1).
\end{cases}
\]

Conventionally we set \( R_\beta(0) = 0 \) for any \( \beta \in [0, +\infty] \).

We recall that, for any \( \beta \in [0, +\infty] \) the eigenvalues of our problem can be characterised by the min-max formula

\[
\lambda_{k,\gamma}^\beta = \inf_{X_{k+1} \subset H^1(0, 1)} \sup_{\varphi \in X_{k+1}} R_\beta(\varphi).
\]

**Remark 3.2.** A first consequence of the above formula is that, for any \( 0 < \beta < \beta^* \), we can bound from below the smallest eigenvalue \( \lambda_{0,\gamma}^\beta \) as follows

\[
\lambda_{0,\gamma}^\beta = \inf_{\varphi \in H^1(0, 1), \varphi \neq 0} R_\beta(\varphi) \geq \frac{\beta}{\beta^*} \inf_{\varphi \in H^1(0, 1), \varphi \neq 0} R_{\beta^*}(\varphi) = \frac{\beta}{\beta^*} \lambda_{0,\gamma}^{\beta^*}.
\]

**Lemma 3.3.** For any two parameters \( 0 \leq \alpha < \beta \leq +\infty \), we have the following strict inequality

\[
\lambda_{k,\gamma}^\alpha < \lambda_{k,\gamma}^\beta, \quad \forall k \geq 0
\]

In particular, for any parameter \( 0 < \beta < +\infty \), we have

\[
\lambda_{k,\gamma}^N < \lambda_{k,\gamma}^\beta < \lambda_{k,\gamma}^D, \quad \forall k \geq 0.
\]

**Proof.** From (3.3), we write

\[
\lambda_{k,\gamma}^\alpha = \inf_{X_{k+1} \subset H^1(0, 1)} \sup_{\varphi \in X_{k+1}} R_\alpha(\varphi)
\]

\[
\leq \sup_{\varphi \in \text{span}\{\varphi_{0,\gamma}^\beta, \ldots, \varphi_{k,\gamma}^\beta\}} R_\alpha(\varphi)
\]

\[
\leq \sup_{\varphi \in \text{span}\{\varphi_{0,\gamma}^\beta, \ldots, \varphi_{k,\gamma}^\beta\}} R_\beta(\varphi)
\]

\[
= R_\beta(\varphi_{k,\gamma}^\beta) = \lambda_{k,\gamma}^\beta.
\]
Let us show that the inequality is in fact strict.

Assume first that $\beta < +\infty$ and that there exists some $k \geq 0$ such that $\lambda_{k,\gamma}^\alpha = \lambda_{k,\gamma}^\beta$. This implies that all the inequalities above are, in fact, equalities. Thus, there is some $\tilde{\varphi} = \sum_{j=0}^{k} a_j \varphi_{j,\gamma}^\beta$ with $\sum_{j=0}^{k} |a_j|^2 = 1$, such that

$$\lambda_{k,\gamma}^\alpha = \sup_{\varphi \in \text{span}\{\varphi_{0,\gamma}^\beta, \ldots, \varphi_{k,\gamma}^\beta\}} R_\alpha(\varphi) = R_\alpha(\tilde{\varphi}),$$

which yields that

$$\int_0^1 \gamma |\partial_x \tilde{\varphi}|^2 + \alpha(|\tilde{\varphi}(0)|^2 + |	ilde{\varphi}(1)|^2) = \lambda_{k,\gamma}^\alpha = \lambda_{k,\gamma}^\beta. \tag{3.5}$$

On the other hand, since each $\varphi_{j,\gamma}^\beta$ is a $L^2$-normalized eigenfunction of the operator $-\partial_x (\gamma(x) \partial_x)$ with the Robin boundary condition with parameter $\beta$, corresponding to eigenvalue $\lambda_{j,\gamma}^\beta$, for $0 \leq j \leq k$, we see that $\tilde{\varphi}$ enjoys the following

$$\int_0^1 \gamma |\partial_x \tilde{\varphi}|^2 + \beta(|\tilde{\varphi}(0)|^2 + |	ilde{\varphi}(1)|^2) = \sum_{j=0}^{k} \lambda_{j,\gamma}^\beta |a_j|^2 \leq \lambda_{k,\gamma}^\beta \sum_{j=0}^{k} |a_j|^2 = \lambda_{k,\gamma}^\beta. \tag{3.6}$$

Since $\beta < +\infty$, and $\alpha < \beta$, we can compare (3.5) and (3.6) to deduce that $\tilde{\varphi}(0) = \tilde{\varphi}(1) = 0$ and moreover

$$\sum_{j=0}^{k} \lambda_{j,\gamma}^\beta |a_j|^2 = \lambda_{k,\gamma}^\beta \sum_{j=0}^{k} |a_j|^2.$$

By (3.2), this equality implies that $a_j = 0$ for any $0 \leq j \leq k - 1$ and then that $\tilde{\varphi}$ is proportional to $\varphi_{k,\gamma}^\beta$. However, such an eigenfunction cannot vanish at $x = 0$ (see for instance Theorem 3.5 below) which is a contradiction.

In the case where $\beta = +\infty$, we use the previous results to simply write

$$\lambda_{k,\gamma}^\alpha < \lambda_{k,\gamma}^{\alpha+1} \leq \lambda_{k,\gamma}^\infty,$$

and the proof of the lemma is complete. \hfill \Box

**Remark 3.4.** Let $0 \leq \beta \leq +\infty$. We denote by $\lambda_{k,\gamma}^{\beta_{\min}}$ and $\lambda_{k,\gamma}^{\beta_{\max}}$ for $k \geq 0$, the eigenvalues to the operator $-\gamma_{\min} \partial_x^2$ and $-\gamma_{\max} \partial_x^2$ respectively, with Robin boundary conditions with parameter $\beta$. Then from (3.3), one has the following inequality

$$\lambda_{k,\gamma}^{\beta_{\min}} \leq \lambda_{k,\gamma}^{\beta} \leq \lambda_{k,\gamma}^{\beta_{\max}}, \quad \forall k \geq 0 \quad \text{and} \quad \forall \beta \in [0, +\infty].$$

Let us observe now that, for any non-trivial eigenfunction $\varphi_{k,\gamma}^\beta$ of our problem (3.1), the quantity $\varphi_{k,\gamma}^\beta(0)$ (and hence $(\varphi_{k,\gamma}^\beta)'(0)$) is non-zero for any $k \geq 0$ and $\beta \in (0, +\infty)$. In fact, we prove the following theorem that give bounds from below for those quantities.

**Theorem 3.5.** There exists a constant $C > 0$ depending only on the diffusion coefficient $\gamma$ such that we have

$$|\varphi_{k,\gamma}^\beta(0)|^2 \left(1 + \frac{\beta^2}{\gamma(0) \lambda_{k,\gamma}^\beta}\right) \geq C, \quad \forall k \geq 0, \quad \beta \in (0, +\infty), \tag{3.7}$$

$$|(\varphi_{k,\gamma}^\beta)'(0)|^2 \left(\frac{(\gamma(0))^2}{\beta^2} + \frac{\gamma(0)}{\lambda_{k,\gamma}^\beta}\right) \geq C, \quad \forall k \geq 0, \quad \beta \in (0, +\infty), \tag{3.8}$$

$$\lambda_{k+1,\gamma}^\beta - \lambda_{k,\gamma}^\beta \geq C \sqrt{\lambda_{k+1,\gamma}^\beta}, \quad \forall k \geq 0, \quad \beta \in [0, +\infty]. \tag{3.9}$$

We first state the following lemma which is a straightforward consequence of [3, Lemma 2.2 and Lemma 2.3].
Let $f : [0, 1] \to \mathbb{R}$ be a continuous function and $\lambda > 0$. Suppose that $u : [0, 1] \to \mathbb{R}$ is smooth and satisfies the following second-order differential equation (without any assumptions on the boundary conditions):

\begin{equation}
\partial_x (\gamma(x) \partial_x u)(x) = \lambda u(x) + f(x), \quad \forall x \in (0, 1),
\end{equation}

then there exists $C' > 0$, depending only on $\gamma$, such that we have

\begin{equation}
|u(y)|^2 + \frac{\gamma(y)}{\lambda} |u'(y)|^2 \leq C' \left( |u(x)|^2 + \frac{\gamma(x)}{\lambda} |u'(x)|^2 + \frac{1}{\lambda} \int_0^1 |f(x)|^2 \gamma(x) \right), \quad \forall x, y \in (0, 1).
\end{equation}

Remark 3.7. We recall that in [3, Lemma 2.2], the authors have assumed that $\lambda \geq 1$ but this was related to the fact that they considered the slightly more general second-order differential equation

\begin{equation}
-\partial_x (\gamma(x) \partial_x u)(x) + q(x) u(x) = \lambda u(x) + f(x), \quad \forall x \in (0, 1).
\end{equation}

In our case we have $q(x) \equiv 0$ in (3.10), and so having a careful look at the proof of [3, Lemma 2.3], one can observe that we simply need $\lambda > 0$ for the constant $C'$ in Lemma 3.6 to be uniform with respect to $\lambda$.

Proof of Theorem 3.5.

- We recall the eigenvalue problem (3.1) and apply Lemma 3.6 to $u = \varphi_{k,\gamma}^\beta$, $\lambda = \lambda_{k,\gamma}^\beta$, and $f = 0$ to obtain for each $k \geq 0$ and $\beta \in (0, +\infty)$ that

\begin{equation}
|\varphi_{k,\gamma}^\beta(x)|^2 + \frac{\gamma(x)}{\lambda_{k,\gamma}^\beta} |(\varphi_{k,\gamma}^\beta)'(x)|^2 \geq \frac{1}{C'} \left( |\varphi_{k,\gamma}^\beta(y)|^2 + \frac{\gamma(y)}{\lambda_{k,\gamma}^\beta} |(\varphi_{k,\gamma}^\beta)'(y)|^2 \right), \quad \forall x, y \in (0, 1).
\end{equation}

Putting $x = 0$ above and integrating over $y \in (0, 1)$, we obtain

\begin{equation}
|\varphi_{k,\gamma}^\beta(0)|^2 + \frac{\gamma(0)}{\lambda_{k,\gamma}^\beta} |(\varphi_{k,\gamma}^\beta)'(0)|^2 \geq \frac{1}{C'} \left( ||\varphi_{k,\gamma}^\beta||^2_{L^2(\Omega)} + \int_0^1 \frac{\gamma(y)}{\lambda_{k,\gamma}^\beta} |(\varphi_{k,\gamma}^\beta)'(y)|^2 dy \right).
\end{equation}

Thanks to the normalizing condition $||\varphi_{k,\gamma}^\beta||^2_{L^2(0,1)} = 1$ and due to the positivity of the second integral in the right hand side of the last inequality, we have

\begin{equation}
|\varphi_{k,\gamma}^\beta(0)|^2 + \frac{\gamma(0)}{\lambda_{k,\gamma}^\beta} |(\varphi_{k,\gamma}^\beta)'(0)|^2 \geq \frac{1}{C'}.
\end{equation}

- In one hand, we use the boundary condition of $\varphi_{k,\gamma}^\beta$ at $x = 0$ to express $(\varphi_{k,\gamma}^\beta)'(0)$ as a function of $\varphi_{k,\gamma}^\beta(0)$ and obtain (3.7).
- On the other hand, we use the same boundary condition to express $\varphi_{k,\gamma}^\beta(0)$ as a function of $(\varphi_{k,\gamma}^\beta)'(0)$ and obtain (3.8).

Secondly, for any $k \geq 0$ and $\beta \in [0, +\infty]$, we define

\begin{equation}
u(x) := \varphi_{k+1,\gamma}^\beta(x) \varphi_{k,\gamma}^\beta(0) - \varphi_{k,\gamma}^\beta(x) \varphi_{k+1,\gamma}^\beta(0), \quad \forall x \in (0, 1),
\end{equation}

which satisfies

\begin{equation}
-\partial_x (\gamma(x) \partial_x u)(x) = \lambda_{k+1,\gamma}^\beta u(x) + f(x), \quad \forall x \in (0, 1)
\end{equation}

with

\begin{equation}
f(x) = \left( \lambda_{k+1,\gamma}^\beta - \lambda_{k,\gamma}^\beta \right) \varphi_{k,\gamma}^\beta(x) \varphi_{k+1,\gamma}^\beta(0).
\end{equation}

Moreover, we observe that $u(0) = 0$ and $u'(0) = 0$, from the construction of $u$. So, by taking $x = 0$ in the inequality (3.11), we see

\begin{equation}
|u(y)|^2 + \frac{\gamma(y)}{\lambda_{k+1,\gamma}^\beta} |u'(y)|^2 \leq C' \frac{\left( \lambda_{k+1,\gamma}^\beta - \lambda_{k,\gamma}^\beta \right)^2}{\lambda_{k+1,\gamma}^\beta} |\varphi_{k+1,\gamma}^\beta(0)|^2 \int_0^1 \frac{1}{\gamma(s)} |\varphi_{k,\gamma}^\beta(s)|^2 ds,
\end{equation}
for all \( y \in (0, 1) \). Thanks to the normalizing condition \( \| \varphi_{k, \gamma}^\beta \|_{L^2(0, 1)} = 1 \) and the definition of \( u \) together implies

\[
|\varphi_{k+1, \gamma}^\beta(y)\varphi_{k, \gamma}^\beta(0) - \varphi_{k, \gamma}^\beta(y)\varphi_{k+1, \gamma}^\beta(0)|^2 \leq C' \frac{(\lambda_{k+1, \gamma}^\beta - \lambda_{k, \gamma}^\beta)^2}{\gamma_{\min \lambda_{k+1, \gamma}^\beta}} |\varphi_{k+1, \gamma}^\beta(0)|^2,
\]

for all \( y \in (0, 1) \). Now integrating the left hand side over \( y \in (0, 1) \) and using the \( L^2 \)-orthonormality condition of \( (\varphi_{k, \gamma}^\beta)_{k \geq 0} \) we have

\[
|\varphi_{k, \gamma}^\beta(0)|^2 + |\varphi_{k+1, \gamma}^\beta(0)|^2 \leq C' \frac{(\lambda_{k+1, \gamma}^\beta - \lambda_{k, \gamma}^\beta)^2}{\gamma_{\min \lambda_{k+1, \gamma}^\beta}} |\varphi_{k+1, \gamma}^\beta(0)|^2,
\]

which yields that

\[
\lambda_{k+1, \gamma}^\beta - \lambda_{k, \gamma}^\beta \geq C \sqrt{\lambda_{k+1, \gamma}^\beta}, \quad \forall k \geq 0 \text{ and } \beta \in [0, +\infty].
\]

where the constant \( C \) depends only on \( \gamma \).

The proof is complete. \( \square \)

3.2. The case of a constant diffusion coefficient. In this case, without loss of generality we can assume that \( \gamma \equiv 1 \) on \([0, 1]\) and so we can find a more explicit form of the eigenfunctions to the following problem

\[
\begin{cases}
-\partial_x^2 \varphi = \lambda \varphi & \text{in } (0, 1), \\
-\partial_x \varphi(0) + \beta \varphi(0) = 0, \\
\partial_x \varphi(1) + \beta \varphi(1) = 0.
\end{cases}
\]

(3.12)

Let us first assume that \( \beta \in (0, +\infty) \). Using the boundary condition at \( x = 0 \), and solving explicitly the differential equation, we shall look for \( \varphi_{k}^\beta \) in the following form

\[
\varphi_{k}^\beta(x) = \frac{\sqrt{\lambda_{k}^\beta}}{\beta} \cos \sqrt{\lambda_{k}^\beta} x + \sin \sqrt{\lambda_{k}^\beta} x, \quad \forall x \in (0, 1), \quad \forall \beta > 0 \text{ and } \forall k \geq 0,
\]

(3.13)

where the eigenvalue \( \lambda_{k}^\beta \) will be required to satisfy the following transcendental equation

\[
2\beta \sqrt{\lambda_{k}^\beta} \cos \sqrt{\lambda_{k}^\beta} + (\beta^2 - \lambda_{k}^\beta) \sin \sqrt{\lambda_{k}^\beta} = 0, \quad \forall \beta > 0 \text{ and } \forall k \geq 0.
\]

(3.14)

This equation is obtained from the boundary condition that \( \varphi_{k}^\beta \) should satisfy at \( x = 1 \).

Notice that, in order to simplify the formulas, we do not assume here that \( \varphi_{k}^\beta \) is normalised in \( L^2 \). This will not be a problem in the sequel since we will only use the fact that this family is complete in \( L^2 \).

Remark 3.8. 1. We know that the family of eigenvalue-eigenfunctions of the operator \(-\partial_x^2 \) with Dirichlet and Neumann boundary conditions are

\[
\varphi_{k}^D(x) = \sin((k + 1)\pi x), \quad x \in [0, 1] \quad \text{with} \quad \lambda_{k}^D = (k + 1)^2 \pi^2, \quad \forall k \geq 0, \text{ and}
\]

\[
\varphi_{k}^N(x) = \cos(k\pi x), \quad x \in [0, 1] \quad \text{with} \quad \lambda_{k}^N = k^2 \pi^2, \quad \forall k \geq 0.
\]

From above, our first obvious observation is \( \lambda_{k}^D = \lambda_{k+1}^N, \forall k \geq 0 \).

2. Secondly, one has \( \lambda_{k}^\beta \in (k^2\pi^2, (k+1)^2\pi^2), \forall k \geq 0 \text{ and } \beta \in (0, +\infty) \), thanks to Lemma 3.3. To be more precise, \( \lambda_{k}^\beta \) is the unique solution of (3.14) in the interval \((k^2\pi^2, (k+1)^2\pi^2)\) for each \( k \geq 0 \).
Remark 3.10. One can obtain from the transcendental equation (3.14) that 
\[
\sqrt{\lambda_k^\beta} = k\pi + \frac{2\beta}{k\pi} + O_{\beta}\left(\frac{1}{k^3}\right), \quad \forall k \geq 1.
\]

See, for instance [19, Problem 1b].

We have seen in Remark 3.8 that the sequence of eigenvalues for the Dirichlet boundary condition and the one for the Neumann boundary condition almost coincide. The following lemma shows that, for other pairs of Robin parameters the sequences of eigenvalues are in fact disjoint.

Lemma 3.11. Consider two parameters \(\beta_1, \beta_2 \in [0, +\infty)\), such that \(\beta_1 < \beta_2\). If for some \(k, l\) we have
\[
\lambda_k^{\beta_2} = \lambda_l^{\beta_1},
\]
then we necessarily have
\[
\beta_1 = 0, \quad \beta_2 = +\infty, \quad \text{and} \quad l = k + 1.
\]

Proof. If \(\beta_1 > 0\), then \(\lambda_k^{\beta_1} \in (l^2\pi^2, (l + 1)^2\pi^2)\) and thus \(\sqrt{\lambda_k^{\beta_2}} \notin \pi\mathbb{N}^*\) and thus \(\beta_2 < +\infty\). Similarly, if we assume \(\beta_2 < +\infty\) then we necessarily have \(\beta_1 > 0\).

Therefore, there are now two cases:

- First case \((\beta_1, \beta_2) = (0, +\infty)\): the result follows from Remark 3.8.
- Second case \(0 < \beta_1 < \beta_2 < +\infty\): the common value \(\lambda\) of \(\lambda_k^{\beta_1}\) and \(\lambda_k^{\beta_2}\) simultaneously belong to \((l^2\pi^2, (l + 1)^2\pi^2)\) and \((k^2\pi^2, (k + 1)^2\pi^2)\), which implies that \(k = l\) and thus we have a contradiction with Lemma 3.3.\(\square\)

4. Boundary controllability of the 1D problem with single Robin parameter. This section is devoted to establish the one-dimensional boundary null-controllability of our cascade system with same non negative Robin parameter on both components and for any diffusion coefficient \(\gamma\) as defined in section 2.4.1. In that case, the system (1.1)-(1.3) simply reads as

\[
\begin{cases}
\partial_t y_1 - \partial_x (\gamma(x)\partial_x y_1) &= 0 \quad \text{in} \quad (0, T) \times (0, 1), \\
\partial_t y_2 - \partial_x (\gamma(x)\partial_x y_2) + y_1 &= 0 \quad \text{in} \quad (0, T) \times (0, 1), \\
\gamma(x) \frac{\partial y_1}{\partial v}(t, x) + \beta y_1(t, x) &= \mathbf{1}_{\{x = 0\}} v(t) \quad \text{on} \quad (0, T) \times \{0, 1\}, \\
\gamma(x) \frac{\partial y_2}{\partial v}(t, x) + \beta y_2(t, x) &= 0 \quad \text{on} \quad (0, T) \times \{0, 1\}, \\
y_1(0, \cdot) &= y_{0,1} \quad \text{in} \quad (0, 1), \\
y_2(0, \cdot) &= y_{0,2} \quad \text{in} \quad (0, 1),
\end{cases}
\]

(4.1)

which is associated with the operator \(A_{\beta, \beta}\) as introduced in Section 2.3, but specialized here to the one-dimensional setting, that is for \(\Omega = (0, 1)\). To simplify the notation, we will simply denote this operator by \(A_\beta\), since the two Robin parameters are equal.

4.1. Spectrum of \(A_\beta^*\). We consider the eigenvalue problem

\[
A_\beta^* u = \lambda u, \quad \lambda \in \mathbb{C},
\]

for a complex-valued function \(u\), that is

\[
\begin{cases}
-\partial_x (\gamma(x)\partial_x u_1) + u_2 &= \lambda u_1 \quad \text{in} \quad (0, 1), \\
-\partial_x (\gamma(x)\partial_x u_2) &= \lambda u_2 \quad \text{in} \quad (0, 1), \\
\gamma(x) \frac{\partial u_1}{\partial v}(x) + \beta u_1(x) &= 0 \quad \text{for} \ x \in \{0, 1\}, \\
\gamma(x) \frac{\partial u_2}{\partial v}(x) + \beta u_2(x) &= 0 \quad \text{for} \ x \in \{0, 1\}.
\end{cases}
\]

(4.2)
Assume first that $u_2 \neq 0$. Multiplying the second equation by $\pi_2$ and integrating by parts, we obtain that $\lambda = \lambda_{k,\gamma}^\beta$ for some $k$ and that we can assume that $u_2 = \varphi_{k,\gamma}^\beta$. Moreover, taking the real or imaginary part, we can assume that $u_1$ is real-valued, then multiplying the first equation by $u_2$ and integrating by parts, we obtain that $\int_0^1 |u_2|^2 = 0$ which is a contradiction.

We have proved that, necessarily, $u_2 = 0$. From the first equation we deduce that $\lambda = \lambda_{k,\gamma}^\beta$ for some $k$ and that, up to a multiplicative constant, we have $u_1 = \varphi_{k,\gamma}^\beta$.

Hence, the eigenfunctions of $A_{\beta}^*$ are

$$\Phi_{k,\gamma}^\beta := \begin{pmatrix} \varphi_{k,\gamma}^\beta \\ 0 \end{pmatrix}$$

corresponding to the eigenvalues $\lambda_{k,\gamma}^\beta$, $\forall k \geq 0$.

We observe that the set $\{\Phi_{k,\gamma}^\beta \}_{k \geq 0}$ is not sufficient generate the whole space $(L^2(0,1))^2$ because the second component of $\Phi_{k,\gamma}^\beta$ is 0 for each $k \geq 0$. Hence we need to look for the generalized eigenfunctions by solving the following problem

$$A_{\beta}^* u = \lambda_{k,\gamma}^\beta u + \Phi_{k,\gamma}^\beta, \ \forall k \geq 0,$$

that is

$$\begin{cases}
-\partial_t (\gamma(x) \partial_x u_1) + u_2 = \lambda_{k,\gamma}^\beta u_1 + \varphi_{k,\gamma}^\beta & \text{in } (0,1), \\
-\partial_t (\gamma(x) \partial_x u_2) = \lambda_{k,\gamma}^\beta u_2 & \text{in } (0,1), \\
\gamma(x) \frac{\partial u_1}{\partial x}(x) + \beta u_1(x) = 0 & \text{for } x \in \{0,1\}, \\
\gamma(x) \frac{\partial u_2}{\partial x}(x) + \beta u_2(x) = 0 & \text{for } x \in \{0,1\}.
\end{cases}$$

The second equation shows that $u_2 = a \varphi_{k,\gamma}^\beta$ for some $a \in \mathbb{R}$. But multiplying the first equation by $(u_2 - \varphi_{k,\gamma}^\beta)$, i.e., $(a - 1) \varphi_{k,\gamma}^\beta$ and performing an integration by parts yields us that the only admissible value for $a$ is 1.

Now its enough to take $u_1 = 0$, which is by default an admissible solution of the system (4.4) and hence the generalized eigenfunctions can be interpreted as

$$\Psi_{k,\gamma}^\beta = \begin{pmatrix} 0 \\ \varphi_{k,\gamma}^\beta \end{pmatrix}, \ \forall k \geq 0.$$

We observe now that, the family $\{\Phi_{k,\gamma}^\beta, \Psi_{k,\gamma}^\beta \}_{k \geq 0}$ is a Riesz basis of $(L^2(0,1))^2$, made of eigenfunctions and generalized eigenfunctions of the operator $A_{\beta}^*$. By construction we simply have

$$\begin{cases}
e^{-t \lambda_{k,\gamma}^\beta} \Phi_{k,\gamma}^\beta = e^{-t \lambda_{k,\gamma}^\beta} \Phi_{k,\gamma}^\beta, & \forall t \in [0,T], \\
e^{-t \lambda_{k,\gamma}^\beta} \Psi_{k,\gamma}^\beta = e^{-t \lambda_{k,\gamma}^\beta} (\Psi_{k,\gamma}^\beta - t \Phi_{k,\gamma}^\beta), & \forall t \in [0,T].
\end{cases}$$

Remark 4.1 (Approximate controllability). We observe that the eigenfunctions of $A_{\beta}^*$ are observable, in the sense that

$$B^* \Phi_{k,\gamma}^\beta = \varphi_{k,\gamma}^\beta(0) \neq 0, \ \forall k \geq 0,$$

where $B^*$ is given by (2.29).

By using the Fattorini-Hautus test (the hypothesis of which are fulfilled in our case, see for instance [16, 24]), we deduce that the control system (4.1), with any $\beta > 0$, is approximately controllable at any time $T > 0$.

4.2. Null-controllability. We are now in position to prove the null-controllability of our system, with a precise bound of the control with respect to $\beta$, that is the first point of Theorem 2.14.

4.2.1. The moments problem. We recall that $\{\Phi_{k,\gamma}^\beta, \Psi_{k,\gamma}^\beta \}_{k \geq 0}$ (defined by (4.3)-(4.5)) forms a complete family in $(L^2(0,1))^2$, so it is enough to check the controllability equation (2.30) (with the operator $A_{\beta}^*$ here) for $\Phi_{k,\gamma}^\beta$ and $\Psi_{k,\gamma}^\beta$ for each $k \geq 0$. This indeed tells us, for any $y_0 \in (L^2(0,1))^2$, that the input $v \in L^2(0,T)$ is a null control for (4.1) if and only if we have

$$\begin{cases}
e^{-T \lambda_{k,\gamma}^\beta} (y_0, \Phi_{k,\gamma}^\beta)_{L^2(0,1)} = \int_0^T v(t) e^{-(T-t) \lambda_{k,\gamma}^\beta} B^* \Phi_{k,\gamma}^\beta \ dt, & \forall k \geq 0, \\
e^{-T \lambda_{k,\gamma}^\beta} (y_0, \Psi_{k,\gamma}^\beta - T \Phi_{k,\gamma}^\beta)_{L^2(0,1)} = \int_0^T v(t) e^{-(T-t) \lambda_{k,\gamma}^\beta} B^*(\Psi_{k,\gamma}^\beta - (T-t) \Phi_{k,\gamma}^\beta) \ dt, & \forall k \geq 0,
\end{cases}$$
using the formulas given by (4.6).

Now since $B^*\Phi_k^\beta = \varphi_k^\beta(0) \neq 0$ and $B^*\Psi_k^\beta = 0$ for each $k \geq 0$, we can simplify the above set of equations as

$$
\begin{align}
&\begin{cases}
\frac{e^{-T\lambda_k^\beta}}{\varphi_k^\beta(0)}(y_{0,1}, \varphi_k^\beta)_{L^2(0,1)} = \int_0^T v(t)e^{-\lambda_k^\beta(t-t)}\,dt, \\
\frac{e^{-T\lambda_k^\beta}}{\varphi_k^\beta(0)}(T(y_{0,1}, \varphi_k^\beta)_{L^2(0,1)} - (y_{0,0}, \varphi_k^\beta)_{L^2(0,1)}) = \int_0^T v(t)(T-t)e^{-\lambda_k^\beta(T-t)}\,dt,
\end{cases} \\
\forall k \geq 0,
\end{align}
$$

The above set of equations is the moments problem that we shall solve in our case.

4.2.2. Existence of a bi-orthogonal family to real exponentials. To construct our control $v$ by solving the moments problem above, the existence of a suitable bi-orthogonal family to time-dependent exponential functions is one the most important ingredient. In this context, it is worth mentioning [10, Theorem 1.5] where the authors proved the existence of bi-orthogonal families to $(t^j e^{-\lambda t})_{k\geq 0, 0 < j < \eta}$ for a complex sequence $(\lambda_k)_{k\geq 0}$ with non-decreasing modulus. This proof is based on a proper gap condition of $|\lambda_k - \lambda_n|$ for all $k \neq n$ and some property of the counting function associated with $(\lambda_k)_{k\geq 0}$ which has been introduced by point 5 and 6 of their proof. In fact, concerning this hypothesis on the counting function, a slightly more general version has been introduced in [2, Remark 4.3] and we indeed make use of this fact in the present study.

We deal with the real sequence $(\lambda_k^\beta)_{k\geq 0}$ and we show that this sequence satisfies all the assumptions of [10, Theorem 1.5] uniformly with respect to the parameter $\beta$.

1. The gap condition:

Without loss of generality we assume that $k > n$ and therefore, $k = n + m$ for some $m \in \mathbb{N}$. We recall (3.9) and Lemma 3.3 to observe that

$$
\lambda_{k+1,\gamma}^\beta - \lambda_k^\beta \geq C\sqrt{\alpha_{k+1,\gamma}^\beta} \geq C\sqrt{\alpha_k^{N,\gamma}} \quad \forall k \geq 0, \quad \forall \beta \geq 0.
$$

Also, by Remark 3.4, we have $\lambda_k^{N,\gamma} = \gamma_{\min}^N k^2\pi^2$ for each $k \geq 0$, so the inequality (4.8) is simplified as

$$
\lambda_{k+1,\gamma}^\beta - \lambda_k^\beta \geq C\gamma_{\min}^N \pi(k + 1), \quad \forall k \geq 0, \quad \forall \beta \geq 0,
$$

which gives us

$$
\lambda_{n+m,\gamma}^\beta - \lambda_n^\beta \geq C\gamma_{\min}^N \pi \sum_{j=n+1}^{n+m} j = C\gamma_{\min}^N \left[ mn + \frac{m(m+1)}{2} \right] = \frac{C}{2} \gamma_{\min}^N \left[ (m+n)^2 - n^2 + m \right].
$$

Thus for any $k, n$ with $k \geq n + 1$, and for any $\beta \geq 0$, we have

$$
\lambda_{k,\gamma}^\beta - \lambda_n^\beta \geq \rho(k^2 - n^2), \quad \forall k, n : k - n \geq 1,
$$

with $\rho := \frac{C}{2} \gamma_{\min}^N$.

2. The counting function: Let $N$ be the counting function associated with the sequence $(\lambda_k^\beta)_{k\geq 0}$, defined by

$$
N(r) = \# \{k : \lambda_k^\beta \leq r \}, \quad \forall r > 0.
$$

We observe that, the function $N$ is piecewise constant and non-decreasing in the interval $[0, +\infty)$. Also for every $r \in [0, +\infty)$ we have $N(r) < +\infty$ and $\lim_{r \to +\infty} N(r) = +\infty$. Moreover,

$$
N(r) = k \iff \lambda_k^\beta \leq r < \lambda_{k+1,\gamma}^\beta,
$$

so that, in particular, if $N(r) = k$, we have

$$
\sqrt{\lambda_{k,\gamma}^\beta} \leq \sqrt{r} < \sqrt{\lambda_{k+1,\gamma}^\beta},
$$

which yields, by Lemma 3.3 and Remark 3.4, that

$$
\sqrt{\lambda_k^{N,\gamma}} \leq \sqrt{\lambda_k^\beta} \leq \sqrt{r} < \sqrt{\lambda_{k+1,\gamma}^\beta} \leq \sqrt{\lambda_{k+1,\gamma}^{D}} \leq \sqrt{\lambda_{k+1,\gamma}^{D,\max}}.
$$
But we have $\lambda_{k,\gamma_{\text{min}}}^N = \gamma_{\text{min}} k^2 \pi^2$ and $\lambda_{k+1,\gamma_{\text{max}}}^D = \gamma_{\text{max}} (k+2)^2 \pi^2$, hence
\[
\sqrt{\gamma_{\text{min}}} k \pi \leq \sqrt{r} < \sqrt{\gamma_{\text{max}} (k+2) \pi}.
\]
Replacing $k$ by $N(r)$, we determine that
\[
\frac{1}{\sqrt{\gamma_{\text{min}}} \pi} \sqrt{r} - 2 < N(r) \leq \frac{1}{\sqrt{\gamma_{\text{min}}} \pi} \sqrt{r} < \frac{1}{\sqrt{\gamma_{\text{min}}} \pi} \sqrt{r} + 2,
\]
which is the point 6 given just before [2, Remark 4.3] with $\alpha = 2$, $p_{\text{min}} = \frac{1}{\sqrt{\gamma_{\text{min}}} \pi}$ and $p_{\text{max}} = \frac{1}{\sqrt{\gamma_{\text{min}}} \pi}$.

From the discussion above, and any given $T > 0$, we can ensure that the existence of a bi-orthogonal family in $L^2(0, T)$, denoted by $(q_{k,j}^\beta)_{k \geq 0, j \leq 1}$, to the family of exponential functions $((T - \cdot)^i e^{-\lambda_{k,\gamma}^D (T - \cdot)})_{k \geq 0, 0 \leq i \leq 1}$. Moreover, this family satisfies the following estimates
\[
(4.11) \quad \|q_{k,j}^\beta\|_{L^2(0, T)} \leq C e^{C \sqrt{k,\gamma + T}}, \quad \forall k \geq 0, \ j = 0, 1,
\]
where the constant $C > 0$ is independent on $T > 0$ and uniform with respect to $k \geq 0$ and to the parameter $\beta \geq 0$ since all the quantities $\rho, \alpha, p_{\text{min}}$, and $p_{\text{max}}$ introduced above do not depend on the Robin parameter $\beta$.

4.2.3. The controllability result. We can now proceed to the proof of the null-controllability result in that case.

Proof of Point 1 of Theorem 2.14. Consider
\[
(4.12) \quad v_\beta(t) = \sum_{k \geq 0} v_k^\beta(t), \quad \forall t \in (0, T),
\]
where
\[
v_k^\beta(t) = \frac{e^{-T \lambda_{k,\gamma}^D}}{\varphi_{k,\gamma}^\beta(0)} (y_{0,1}, \varphi_{k,\gamma}^\beta)_{L^2(0,1)} q_{k,0}^\beta(t) - \frac{e^{-T \lambda_{k,\gamma}^D}}{\varphi_{k,\gamma}^\beta(0)} (T (y_{0,1}, \varphi_{k,\gamma}^\beta)_{L^2(0,1)} - (y_{0,2}, \varphi_{k,\gamma}^\beta)_{L^2(0,1)}) q_{k,1}^\beta(t).
\]

With this choice of $v = v_\beta$, one can observe that the set of moment equations in (4.7) are formally satisfied. Now, all we have to check is the convergence of the series (4.12) in $L^2(0, T)$, with explicit bounds with respect to $\beta \in (0, +\infty)$. To this end, for each $k \geq 0$, we compute
\[
\|v_k^\beta\|_{L^2(0, T)} \leq \frac{\|y_{0,1}\|_{L^2(0,1)} e^{-T \lambda_{k,\gamma}^D}}{\|\varphi_{k,\gamma}^\beta(0)\|} \|q_{k,0}^\beta\|_{L^2(0, T)} + \frac{T \|y_{0,1}\|_{L^2(0,1)} + \|y_{0,2}\|_{L^2(0,1)} e^{-T \lambda_{k,\gamma}^D}}{\|\varphi_{k,\gamma}^\beta(0)\|} \|q_{k,1}^\beta\|_{L^2(0, T)},
\]
thanks to the normalizing condition $\|\varphi_{k,\gamma}^\beta\|_{L^2(0,1)} = 1$. Moreover, the result (3.7) gives us
\[
(4.13) \quad \frac{1}{\|\varphi_{k,\gamma}^\beta(0)\|} \leq C \left(1 + \frac{\beta}{\sqrt{k,\gamma}}\right), \quad \forall k \geq 0, \ \beta \in (0, +\infty),
\]
where $C$ depends only on $\gamma$.

Now, using (4.13) and the bounds on bi-orthogonal functions in (4.11), we deduce for each $k \geq 0$ and for any finite $T > 0$ that
\[
(4.14) \quad \|v_k^\beta\|_{L^2(0, T)} \leq C(T + 2) \left(1 + \frac{\beta}{\sqrt{k,\gamma}}\right) e^{-T \lambda_{k,\gamma}^D} e^{C \sqrt{k,\gamma} + T} \|y_{0}\|_{L^2(0,1)},
\]
since $\gamma(0) > \gamma_{\text{min}} > 0$. Now, Young’s inequality gives us
\[
(4.15) \quad C \sqrt{\lambda_{k,\gamma}^D} \leq \frac{T}{2} \lambda_{k,\gamma}^D + \frac{C^2}{2T}, \quad \forall k \geq 0.
\]
Thus, we see
\[
(4.16) \quad e^{-T \lambda_{k,\gamma}^D + C \sqrt{k,\gamma} + T} \leq e^{-\frac{T}{2} \lambda_{k,\gamma}^D + \frac{C}{2}}, \quad \forall k \geq 0.
\]
But we have, $\lambda_{k,\gamma}^\beta \geq \lambda_{k,\gamma}^N \geq \lambda_{k,\gamma_{min}}^N = \gamma_{min} k^2 \pi^2$ for $k \geq 0$, and so, in one hand, we have

$$\sum_{k \geq 0} e^{-\frac{T}{2} \lambda_{k,\gamma}^\beta} \leq \sum_{k \geq 0} e^{-C k^2 T} \leq \frac{1}{1 - e^{-C T/2}} \leq C \left(1 + \frac{1}{T}\right).$$

On the other hand, we see

$$\sum_{k \geq 0} \frac{\beta}{\sqrt{\lambda_{0,\gamma}^\beta}} e^{-\frac{T}{2} \lambda_{k,\gamma}^\beta} \leq \frac{\beta}{\sqrt{\lambda_{0,\gamma}^\beta}} e^{-\frac{T}{2} \lambda_{0,\gamma}^\beta} + \frac{\beta}{\pi \sqrt{\gamma_{min}}} \sum_{k \geq 1} \frac{1}{k} e^{-\pi^2 \gamma_{min} k^2 T} \leq \frac{\beta}{\sqrt{\lambda_{0,\gamma}^\beta}} + C \beta \sum_{k \geq 1} e^{-C k^2 T}. \leq \frac{\beta}{\sqrt{\lambda_{0,\gamma}^\beta}} + C \beta \sum_{k \geq 1} e^{-C k^2 T}.$$\leq \frac{\beta}{\sqrt{\lambda_{0,\gamma}^\beta}} + C \beta \sum_{k \geq 1} e^{-C k^2 T}.$$

The second quantity in the right hand side of (4.18) has the bound $C \beta \left(1 + \frac{1}{T}\right)$ where the bound of first quantity is not so obvious because we see that $\lambda_{0,\gamma}^\beta$ is getting smaller with respect to smaller $\beta > 0$. But, we have

$$\frac{\beta}{\sqrt{\lambda_{0,\gamma}^\beta}} \leq \frac{\sqrt{\pi}}{\sqrt{\lambda_{0,\gamma}^\beta}} \leq C (1 + \beta), \text{ for } 0 < \beta < 1, \text{ by Remark 3.2, and}$$

$$\frac{\beta}{\sqrt{\lambda_{0,\gamma}^\beta}} \leq C \beta, \text{ for } \beta \geq 1, \text{ which is easy to observe.}$$

Now, using this in (4.18), one can deduce that

$$\sum_{k \geq 0} \frac{\beta}{\sqrt{\lambda_{k,\gamma}^\beta}} e^{-\frac{T}{2} \lambda_{k,\gamma}^\beta} \leq C (1 + \beta) \left(1 + \frac{1}{T}\right).$$

Now, we take summation over $k \geq 0$ in (4.14), and using the estimates (4.16), (4.17) and (4.19), we get for any $\beta \in (0, +\infty)$ and finite $T > 0$ that

$$\sum_{k \geq 0} \|v^\beta_k\|_{L^2(0,T)} \leq C (1 + \beta) e^{C T} \|y_0\|_{L^2(0,1)}.$$\leq C (1 + \beta) e^{C T} \|y_0\|_{L^2(0,1)}.$$

This completes the proof. \qed

5. Boundary controllability result of the 1D problem with different Robin parameters. In this section, we discuss about the boundary controllability of the system (1.1)-(1.3) in 1D with two different parameters $\beta_1 \neq \beta_2$ with $\beta_1, \beta_2 \in (0, +\infty)$ for the two components of the system and as mentioned in the introduction of this paper, we assume now that $\gamma$ is a positive constant that we arbitrarily choose to be equal to 1. We rewrite the control system (1.1)-(1.3) in that setting below

$$\begin{aligned}
\frac{\partial_t y_1 - \partial_x^2 y_1}{\partial_t y_1 - \partial_x^2 y_2 + y_1} &= 0 \quad \text{in } (0, T) \times (0, 1), \\
\frac{\partial_t y_2 - \partial_x^2 y_2}{\partial_t y_2 - \partial_x^2 y_2 + y_1} &= 0 \quad \text{in } (0, T) \times (0, 1), \\
\frac{\partial y_1}{\partial \nu}(t, x) + \beta_1 y_1(t, x) &= 1_{\{x=0\}} v(t) \quad \text{on } (0, T) \times \{0, 1\}, \\
\frac{\partial y_2}{\partial \nu}(t, x) + \beta_2 y_2(t, x) &= 0 \quad \text{on } (0, T) \times \{0, 1\}, \\
y_1(0, \cdot) &= y_{0,1} \quad \text{in } (0, 1), \\
y_2(0, \cdot) &= y_{0,2} \quad \text{in } (0, 1).
\end{aligned}$$

In this case, we recall that the associated operator is $A_{\beta_1, \beta_2}$, as defined in Section 2.3, specified here for $\Omega = (0, 1)$ and for $\gamma \equiv 1$.\leq A_{\beta_1, \beta_2}$, as defined in Section 2.3, specified here for $\Omega = (0, 1)$ and for $\gamma \equiv 1$.\leq \begin{aligned}
\end{aligned}$

The main difference between the present section and Section 4 concerns the spectral properties of the adjoint operators. Unlike the previous case, we will have here a possible condensation of eigenvalues with two different sets of eigenfunctions that form a complete family of the state space, instead of having well-separated eigenvalues and associated generalized eigenfunctions.
5.1. Spectrum of \( A_{\beta_1, \beta_2}^* \). In the present situation, the eigenvalue problem associated with \( A_{\beta_1, \beta_2}^* \) is explicitly given by

\[
\begin{aligned}
-\partial_x^2 u_1 + u_2 &= \lambda u_1 \quad \text{in } (0, 1), \\
-\partial_x^2 u_2 &= \lambda u_2 \quad \text{in } (0, 1), \\
\frac{\partial u_1}{\partial \nu} + \beta_1 u_1 &= 0 \quad \text{on } \{0, 1\}, \\
\frac{\partial u_2}{\partial \nu} + \beta_2 u_2 &= 0 \quad \text{on } \{0, 1\}.
\end{aligned}
\]

(5.2)

First case: Assume that \( u_2 = 0 \), then our system (5.2) reduces to the Robin eigenvalue problem (3.12) with positive parameter \( \beta_1 \) and this gives us the solution \( u_1 = \varphi_{k}^{\beta_1} \) corresponding \( \lambda = \lambda_k^{\beta_1} \) which is real for any \( k \geq 0 \) (recall that, \( \varphi_k^{\beta_1} \) has already been given by (3.13) for all \( k \geq 0 \) and \( \beta \in (0, +\infty) \)). This gives us the following set of eigenfunctions (without normalizing) of \( A_{\beta_1, \beta_2}^* \)

\[
\Phi_{k,1} := \begin{pmatrix} \varphi_{k}^{\beta_1} \\ 0 \end{pmatrix}
\]

(5.3)

Second case: Assume now that \( u_2 \neq 0 \), then we first solve the second set of equations of (5.2), i.e.,

\[
\begin{aligned}
-\partial_x^2 u_2 &= \lambda u_2 \quad \text{in } (0, 1), \\
\frac{\partial u_2}{\partial \nu} + \beta_2 u_2 &= 0 \quad \text{on } \{0, 1\},
\end{aligned}
\]

(5.4)

which gives \( u_2 = \varphi_{k}^{\beta_2} \), up to a multiplicative constant (which we can take 1), corresponding to \( \lambda = \lambda_k^{\beta_2} \) for all \( k \geq 0 \).

Now by implementing \( u_2 = \varphi_{k}^{\beta_2} \) for each \( k \geq 0 \) to the first equation of (5.2) address us the following problem

\[
\begin{aligned}
-\partial_x^2 u_1 + \varphi_{k}^{\beta_2} &= \lambda_k^{\beta_2} u_1 \quad \text{in } (0, 1), \\
-\partial_x u_1(0) + \beta_1 u_1(0) &= 0, \\
\partial_x u_1(1) + \beta_1 u_1(1) &= 0.
\end{aligned}
\]

(5.5)

The existence and uniqueness of the solution to (5.4) follows from the Fredholm alternative theorem and to the fact that \( \lambda_k^{\beta_2} \notin (\lambda_k^{\beta_1})_{k \geq 0} \) for any \( k \geq 0 \) and \( \beta_1 \neq \beta_2 \) (by Lemma 3.11). Let us denote the unique solution \( u_1 \) of (5.4) by \( \psi_{k}^{\beta_1, \beta_2} \), for \( k \geq 0 \) and hence the second set of eigenfunctions (without normalizing) of \( A_{\beta_1, \beta_2}^* \) is given by

\[
\Phi_{k,2} := \begin{pmatrix} \psi_{k}^{\beta_1, \beta_2} \\ \varphi_{k}^{\beta_2} \end{pmatrix}
\]

(5.6)

The family \( \{\Phi_{k,1}, \Phi_{k,2}\}_{k \geq 0} \) is complete in \( (L^2(0,1))^2 \), and we observe that

5.1.1. More on spectral properties and approximate controllability. This section is devoted to show some properties of the first component \( \psi_{k}^{\beta_1, \beta_2} \) of the eigenfunction \( \Phi_{k,2} \) and how the spectral gap \( |\lambda_k^{\beta_1} - \lambda_k^{\beta_2}| \) depends on the parameters \( \beta_1, \beta_2 \) (for any \( k \geq 0 \)). We need all these to find a proper bound of our null-control.

Proving the estimates of this section for any non constant diffusion coefficient \( \gamma \) is still an open problem, that is why we restrict here our attention to the constant coefficient case.

**Lemma 5.1.** Let \( \beta_1 \neq \beta_2 \) be any two real parameters with \( \beta_1, \beta_2 \in (0, +\infty) \) and \( (\psi_{k}^{\beta_1, \beta_2})_{k \geq 0} \) be the set of solutions to (5.4) as introduced before. Then, we have

\[
|\psi_{k}^{\beta_1, \beta_2}(0)| \geq \frac{\lambda_k^{\beta_2} + \beta_2}{4 \beta_2 \sqrt{\lambda_k^{\beta_2} |\beta_1 - \beta_2|}}, \quad \forall k \geq 0.
\]
Proof. 1. Observe that $\psi^{\beta_1,\beta_2}_k$ satisfies the second order ordinary differential equation (5.4), i.e.,

$$
\frac{d^2 \psi^{\beta_1,\beta_2}_k}{dx^2} + \lambda^2_k \psi^{\beta_1,\beta_2}_k = \varphi^{\beta_2}_k, \quad \text{for each } k \geq 0.
$$

To solve this, recall the explicit form of $\varphi^{\beta_2}_k$ from (3.14) (with $\beta = \beta_2$), and hence for each $k \geq 0$, we are looking for $\psi^{\beta_1,\beta_2}_k$ in the following form

$$
\psi^{\beta_1,\beta_2}_k(x) = (Ax + B) \cos(\sqrt{\lambda^2_k}x) + (Cx + D) \sin(\sqrt{\lambda^2_k}x), \quad \forall x \in [0, 1].
$$

Substituting (5.8) into the equation (5.7), we get

$$
A = -\frac{1}{2\sqrt{\lambda^2_k}} \quad \text{and} \quad C = \frac{1}{2\beta_2}.
$$

Then, from the boundary conditions at $x = 0$ and 1 satisfying by $\psi^{\beta_1,\beta_2}_k$, one can obtain

$$
D \sqrt{\lambda^2_k} = \beta_1 B - A, \quad \text{and}
$$

$$(5.10) \quad B \left( 2\beta_1 \sqrt{\lambda^2_k} \cos \sqrt{\lambda^2_k} + (\beta_2^2 - \lambda^2_k) \sin \sqrt{\lambda^2_k} \right)$$

$$
= \left( \frac{\beta_1 \beta_2 - \lambda^2_k}{2\beta_2} \right) \cos \sqrt{\lambda^2_k} - \left( \frac{(1 + \beta_1 + \beta_2)\lambda^2_k + \beta_1 \beta_2}{2\beta_2 \sqrt{\lambda^2_k}} \right) \sin \sqrt{\lambda^2_k}
$$

respectively. We must mention here that the coefficient of $B$ in the left hand side of (5.10) never vanishes due to Remark 3.9.

2. By (3.14), it is known that the eigenvalue $\lambda^2_k$ is the unique solution to

$$
(5.11) \quad 2\beta_2 \sqrt{\lambda^2_k} \cos \sqrt{\lambda^2_k} + (\beta_2^2 - \lambda^2_k) \sin \sqrt{\lambda^2_k} = 0, \quad \text{for each } k \geq 0.
$$

Now by substituting the expression of $\cos \sqrt{\lambda^2_k}$ from (5.11) into (5.10) and replacing $B$ by $\psi^{\beta_1,\beta_2}_k$, we get

$$
\frac{1}{\beta_2} \psi^{\beta_1,\beta_2}_k(0) \left[ \beta_1(\lambda^2_k - \beta_2^2) + \beta_2(\beta_2^2 - \lambda^2_k) \right] = \frac{(\beta_1 \beta_2 - \lambda^2_k)(\lambda^2_k - \beta_2^2)}{4\beta_2^2 \sqrt{\lambda^2_k}} - \frac{(1 + \beta_1 + \beta_2)\lambda^2_k + \beta_1 \beta_2}{2\beta_2 \sqrt{\lambda^2_k}},
$$

where we omitted $\sin \sqrt{\lambda^2_k}$ from both sides since $\sin \sqrt{\lambda^2_k} \neq 0$ for all $k \geq 0$ and $\beta_2 \in (0, +\infty)$. Now by simplifying the above equality provides us

$$
(5.12) \quad \psi^{\beta_1,\beta_2}_k(0) = \frac{-2(\lambda^2_k)^2 - \beta_1 \beta_2 \lambda^2_k - \beta_2^2 \lambda^2_k - 2\beta_2 \lambda^2_k - 2\beta_1 \beta_2^2 - 2\beta_1 \beta_2}{4\beta_2 \sqrt{\lambda^2_k}} \left[ \beta_1(\lambda^2_k - \beta_2^2) + \beta_2(\beta_2^2 - \lambda^2_k) \right] =: \frac{J_k}{J_k}.
$$

Here one can rewrite the quantity $J_k$ as

$$
(5.13) \quad J_k = 4\beta_2 \sqrt{\lambda^2_k} \left( \beta_1 - \beta_2 \right)(\lambda^2_k + \beta_1 \beta_2), \quad \forall k \geq 0,
$$

whereas, $I_k$ enjoys the following

$$
(5.14) \quad |I_k| \geq (\lambda^2_k)^2 + \beta_1 \beta_2 \lambda^2_k + \beta_2^2 \lambda^2_k + \beta_1 \beta_2^2 = (\lambda^2_k + \beta_2^2)(\lambda^2_k + \beta_1 \beta_2), \quad \forall k \geq 0.
$$

To this end, we use (5.13) and (5.14) in the expression (5.12) to deduce that

$$
|\psi^{\beta_1,\beta_2}_k(0)| \geq \frac{(\lambda^2_k + \beta_2^2)}{4\beta_2 \sqrt{\lambda^2_k} |\beta_1 - \beta_2|}, \quad \forall k \geq 0,
$$

and this concludes the lemma.
Remark 5.2 (Approximate controllability). The control problem (5.1) is approximately controllable for any finite time $T > 0$.

To prove this, we will again use Fattorini-Hautus test as mentioned in Remark 4.1, that is to show $B^* \Phi_{k,1} \neq 0$ and $B^* \Phi_{k,2} \neq 0$ for each $k \geq 0$ (where $B^*$ has been defined in (2.29)). But (3.13) and Lemma 5.1 respectively ensure us

$B^* \Phi_{k,1} = \varphi_k^{\beta_1} (0) \neq 0$ and $B^* \Phi_{k,2} = \psi_k^{\beta_1, \beta_2} (0) \neq 0, \forall k \geq 0$,

which gives the claim.

Lemma 5.3. Let $\beta_1 \neq \beta_2$ be any two parameters such that $\beta_1, \beta_2 \in (0, +\infty)$ and $\lambda_k^{\beta_1}, \lambda_k^{\beta_2}, k \geq 0$, be defined as before. Let $\beta^* > 0$ be a fixed finite number. Then

1. for $0 < \beta_1, \beta_2 < \beta^*$, we have

$$|\beta_1 - \beta_2| \leq C_{\beta^*} |\lambda_k^{\beta_1} - \lambda_k^{\beta_2}|, \quad \forall k \geq 0,$$

2. for $\beta_1, \beta_2 \geq \beta^*$, we have

$$\frac{1}{\beta_1} - \frac{1}{\beta_2} \leq C_{\beta^*} |\lambda_k^{\beta_1} - \lambda_k^{\beta_2}|, \quad \forall k \geq 0.$$

Proof. We begin with the fact that any $\beta \in (0, +\infty)$ can be represented by

$$\beta = \sqrt{\lambda_k^2} \tan \frac{\sqrt{\lambda_k^2}}{2}, \quad \text{for } k \text{ even and }$$

$$= -\sqrt{\lambda_k^2} \cot \frac{\sqrt{\lambda_k^2}}{2}, \quad \text{for } k \text{ odd (by Remark 3.9).}$$

Also, since we have $\sqrt{\lambda_k^2} \in (k\pi, (k + 1)\pi)$ by point 2 of Remark 3.8, so one can write

$$\sqrt{\lambda_k^2} = k\pi + \delta_k^{\beta_j} \text{ for } j = 1, 2, \text{ and } \sqrt{\lambda_k^{\beta^*}} = k\pi + \delta_k^{\beta^*}, \quad \forall k \geq 0,$$

where $\delta_k^{\beta_j}, \delta_k^{\beta^*} \in (0, \pi)$.

1. Now, we assume that $0 < \beta_1, \beta_2 < \beta^*$ which implies $0 < \delta_k^{\beta_1}, \delta_k^{\beta_2} < \delta_k^{\beta^*} < \pi, \forall k \geq 0$. We denote

$$0 < \delta_\beta : = \sup_{k \geq 0} \delta_k^{\beta^*} < \pi,$$

where $\delta_{\beta^*} < \pi$ since the quantity $\delta_k^{\beta^*}$ is getting smaller as $k \geq 1$ getting larger due to the asymptotic behavior of $\sqrt{\lambda_k^2}$ given by Remark 3.10.

We discuss the proof for $k$ even, for odd $k$ the steps will be similar. We have

$$(5.18) \quad \beta_1 - \beta_2 = (k\pi + \delta_k^{\beta_1}) \tan \left( \frac{k\pi}{2} + \frac{\delta_k^{\beta_1}}{2} \right) - (k\pi + \delta_k^{\beta_2}) \tan \left( \frac{k\pi}{2} + \frac{\delta_k^{\beta_2}}{2} \right)$$

$$= k\pi \left( \tan \frac{\delta_k^{\beta_1}}{2} - \tan \frac{\delta_k^{\beta_2}}{2} \right) + \delta_k^{\beta_1} \tan \frac{\delta_k^{\beta_1}}{2} - \delta_k^{\beta_2} \tan \frac{\delta_k^{\beta_2}}{2}.$$

Applying Mean value theorem to the functions $\tan \frac{\mu}{2}$ and $\mu \tan \frac{\mu}{2}$ on $\mu \in ([\delta_k^{\beta_1}, \delta_k^{\beta_2}])$, we have for some $\delta_k^{\beta_1}$ and $\delta_k^{\beta_2}$ in $([\delta_k^{\beta_1}, \delta_k^{\beta_2}])$ that

$$\left| \tan \frac{\delta_k^{\beta_1}}{2} - \tan \frac{\delta_k^{\beta_2}}{2} \right| \leq |\delta_k^{\beta_1} - \delta_k^{\beta_2}| \sec^2 \frac{\delta_k^{\beta_1}}{2} \leq C_{\beta^*} |\delta_k^{\beta_1} - \delta_k^{\beta_2}|,$$

and

$$\left| \delta_k^{\beta_1} \tan \frac{\delta_k^{\beta_1}}{2} - \delta_k^{\beta_2} \tan \frac{\delta_k^{\beta_2}}{2} \right| \leq |\delta_k^{\beta_1} - \delta_k^{\beta_2}| \left| \frac{\delta_k^{\beta_1}}{2} \sec^2 \frac{\delta_k^{\beta_1}}{2} + \tan \frac{\delta_k^{\beta_1}}{2} \right|$$

$$\leq |\delta_k^{\beta_1} - \delta_k^{\beta_2}| \left| \frac{\delta_k^{\beta_1}}{2} \sec^2 \frac{\delta_k^{\beta_1}}{2} + \frac{\sin(\delta_k^{\beta_1}/2)}{\delta_k^{\beta_1}/2} \sec \frac{\delta_k^{\beta_1}}{2} \right|$$

$$\leq C_{\beta^*} (\delta_k^{\beta_1} + \delta_k^{\beta_2}) |\delta_k^{\beta_1} - \delta_k^{\beta_2}|,$$
where we make use of the fact that the quantities \(\sec \frac{\beta_k}{2}\) and \(\sec \frac{\beta_k}{4}\) can be bounded by \(\sec \frac{\beta_\infty}{2}\) which is some constant \(C_{\beta_\infty}\). Now, we turn back to (5.18) to deduce that

\[
|\beta_1 - \beta_2| \leq C_{\beta_\infty}(k\pi + \delta^{\beta_1}_k + \delta^{\beta_2}_k)|\delta^{\beta_1}_k - \delta^{\beta_2}_k|
\]

\[
= C_{\beta_\infty} \left( \sqrt{\lambda_k^{\beta_1}} + \sqrt{\lambda_k^{\beta_2}} \right) \left| \sqrt{\lambda_k^{\beta_1}} - \sqrt{\lambda_k^{\beta_2}} \right|
\]

\[
\leq C_{\beta_\infty}|\lambda_k^{\beta_1} - \lambda_k^{\beta_2}|.
\]

2. Here also, we demonstrate the result for \(k\) even. We recall (5.15) to observe that

\[
\frac{1}{\beta_1} - \frac{1}{\beta_2} = \frac{1}{k\pi + \delta_k^{\beta_1}} \cot \left( \frac{k\pi}{2} + \frac{\delta_k^{\beta_1}}{2} \right) - \frac{1}{k\pi + \delta_k^{\beta_2}} \cot \left( \frac{k\pi}{2} + \frac{\delta_k^{\beta_2}}{2} \right)
\]

\[
= \frac{1}{k\pi + \delta_k^{\beta_1}} \cot \frac{\delta_k^{\beta_1}}{2} - \frac{1}{k\pi + \delta_k^{\beta_2}} \cot \frac{\delta_k^{\beta_2}}{2},
\]

Let us define the function

\[
g(\mu) = \frac{1}{k\pi + \mu} \cot \frac{\mu}{2}, \quad \text{for } \mu \in ((\delta_k^{\beta_1}, \delta_k^{\beta_2})) \subset (0, \pi).
\]

Consequently,

\[
g'(\mu) = -\frac{1}{2(k\pi + \mu)} \csc^2 \frac{\mu}{2} - \frac{1}{(k\pi + \mu)^2} \cot \frac{\mu}{2},
\]

and \(|g'|\) is monotonically decreasing function in \((0, \pi)\). Now, applying Mean value theorem on \(g(\mu)\), we have from (5.19)

\[
\left| \frac{1}{\beta_1} - \frac{1}{\beta_2} \right| \leq |\delta_k^{\beta_1} - \delta_k^{\beta_2}| |g'(\tilde{\delta}_k)|, \quad \text{for some } \tilde{\delta}_k \in ((\delta_k^{\beta_1}, \delta_k^{\beta_2})).
\]

But we have \(\delta_k^{\beta_1}, \tilde{\delta}_k \geq \delta_k^{\beta_*}\) (\(j = 1, 2\)), since \(\beta_1, \beta_2 \geq \beta_*\), and hence

\[
|g'(\tilde{\delta}_k)| \leq \frac{1}{k\pi + \delta_k} \left[ \frac{1}{2\sin^2 \left( \frac{\tilde{\delta}_k}{2} \right)} + \frac{1}{k\pi + \delta_k} \cot \frac{\tilde{\delta}_k}{2} \right]
\]

\[
\leq \frac{1}{k\pi + \delta_k^{\beta_*}} \left[ \frac{1}{2\sin^2 \left( \frac{\beta_\infty}{2} \right)} + \frac{1}{k\pi + \delta_k^{\beta_*}} \sin \left( \frac{\beta_\infty}{2} \right) \right].
\]

Let us recall the asymptotic formula of \(\sqrt{\lambda_k^{\beta_*}}\) from Remark 3.10 to observe

\[
\delta_k^{\beta_*} = \frac{2\beta_*}{k\pi} + O_{\beta_*}\left( \frac{1}{k^3} \right) \geq \frac{\overline{\beta}}{k\pi}, \quad \forall k \geq 1,
\]

for some \(\overline{\beta} > 0\) depending only on \(\beta_*\). As a consequence, \(\sin \frac{\beta_*}{2k\pi} \geq \sin \frac{\overline{\beta}}{k\pi}\), since \(\sin x\) is monotonically increasing on \((0, \pi/2)\). Also, since \(\sin x \geq \frac{2}{\pi}x\) for all \(x \in (0, \pi/2)\), eventually we have

\[
\sin \frac{\overline{\beta}}{2k\pi} \geq \frac{\overline{\beta}}{k\pi}, \quad \forall k \geq 1.
\]

Now we come back to (5.21), we obtain for any \(k \neq 0\) even, that

\[
|g'(\tilde{\delta}_k)| \leq \frac{1}{\sqrt{\lambda_k^{\beta_*}}} \left[ \frac{k^2\pi^4}{2\overline{\beta}^2} + \frac{k\pi^2}{\overline{\beta} \sqrt{\lambda_k^{\beta_*}}} \right]
\]

\[
\leq C_{\beta_*} \sqrt{\lambda_k^{\beta_*}} \leq C_{\beta_*} \left( \sqrt{\lambda_k^{\beta_1}} + \sqrt{\lambda_k^{\beta_2}} \right).
\]
Implementing this estimate into (5.20), we obtain that

\[
(5.22) \quad \left| \frac{1}{\beta_1} - \frac{1}{\beta_2} \right| \leq C_{\beta^*} \sqrt{\lambda_{k_1}^{\beta_1} - \lambda_{k_2}^{\beta_2}} \left( \sqrt{\lambda_{k_1}^{\beta_1}} + \sqrt{\lambda_{k_2}^{\beta_2}} \right)
\]

\[
= C_{\beta^*} |\lambda_{k_1}^{\beta_1} - \lambda_{k_2}^{\beta_2}|, \quad \forall k \neq 0 \text{ even.}
\]

For \( k = 0 \), one can observe from (5.21) that

\[
|g'(\delta_0)| \leq \frac{1}{\delta_0} \left[ \frac{1}{2 \sin^2 \left( \frac{\delta_0}{2} \right)} \right] \leq \frac{1}{\delta_0} \left[ \frac{1}{2 \sin^2 \left( \frac{\delta_0^*}{2} \right)} \right] \leq C_{\beta^*},
\]

and so (5.20) now provides us the same inequality (5.22) for \( k = 0 \). The proof is complete. \( \square \)

Lemma 5.3 now helps us to prove the following proposition which is the key point to obtain a uniform \( L^2(0,T) \)-bound of a control that we construct in next section, with respect to the parameters \( \beta_1, \beta_2 \).

**Proposition 5.4.** Let \( \beta_1 \neq \beta_2 \) be any two parameters with \( \beta_1, \beta_2 \in (0, +\infty) \) and \( (\lambda_{k_1}^{\beta_1}, \Phi_{k_1}) \), \( (\lambda_{k_2}^{\beta_2}, \Phi_{k_2}) \) be the eigenvalue-eigenfunction pairs of the operator \( A_{\beta_1,\beta_2}^* \) for each \( k \geq 0 \). Also, assume that \( \beta^* \) be any positive real number. Then, for any \( k \geq 0 \), we have

\[
(5.23) \quad \frac{1}{|\lambda_{k_1}^{\beta_1} - \lambda_{k_2}^{\beta_2}|} \left\| \Phi_{k_1} - \Phi_{k_2} \right\|_{L^2(0,1)} \leq \begin{cases} C_{\beta^*}, & \text{if } 0 < \beta_1, \beta_2 < \beta^*, \\ C_{\beta^*} \sqrt{\lambda_{k}^{\beta_1}} & \text{if } \beta_1, \beta_2 \geq \beta^*, \end{cases}
\]

where \( B^* \) is defined in (2.29).

Proof. Since \( B^* = I_{\{x=0\}} (1 \ 0) \) (introduced in (2.29)), and using the definitions (5.3) and (5.5), the quantity we want to estimate can be denoted by

\[
\Theta_k := \left( \begin{array}{c} \Theta_k^1 \\ \Theta_k^2 \end{array} \right),
\]

where the two components are

\[
\Theta_k^1(x) := \frac{1}{(\lambda_{k_1}^{\beta_1} - \lambda_{k_2}^{\beta_2})} \left( \frac{\varphi_{k_1}^{\beta_1}(x)}{\varphi_{k_1}^{\beta_1}(0)} - \frac{\psi_{k_1,\beta_2}^{\beta_1}(x)}{\psi_{k_1,\beta_2}^{\beta_1}(0)} \right), \quad x \in (0,1),
\]

\[
\Theta_k^2(x) := \frac{1}{(\lambda_{k_1}^{\beta_1} - \lambda_{k_2}^{\beta_2})} \frac{\varphi_{k_1}^{\beta_2}(x)}{\psi_{k_1,\beta_2}^{\beta_1}(0)}, \quad x \in (0,1).
\]

In order to ease the computations, we will denote by \( \mu_i = \sqrt{\lambda_{k_1}^{\beta_i}} \) for \( i = 1,2 \), the dependence in \( k \) being now implicit.

1. We first assume that \( 0 < \beta_1, \beta_2 < \beta^* \). Using this assumption, and the fact that \( \lambda_{0}^{\beta_i} \geq \frac{\beta_i}{\beta^*} \lambda_{0}^{\beta_i} \) (by Remark 3.2) and simply \( \lambda_{k_1}^{\beta_1} \geq k^2 \pi^2, \forall k \geq 1 \) (by point 2 of Remark 3.8), we obtain uniformly in \( k \geq 0 \), that

\[
(5.24) \quad \mu_i \geq C_{\beta^*} \sqrt{\beta_i}, \quad i = 1,2.
\]

- Estimate of the first component \( \Theta_k^1 \):
  Recall the expression \( \varphi_{k_1}^{\beta_1} \) (with \( \beta = \beta_1 \)) and \( \psi_{k_1,\beta_2}^{\beta_1} \) from (3.13) and (5.8) respectively and following some steps of computations we obtain

\[
(5.25) \quad \left( \frac{\varphi_{k_1}^{\beta_1}(x)}{\varphi_{k_1}^{\beta_1}(0)} - \frac{\psi_{k_1,\beta_2}^{\beta_1}(x)}{\psi_{k_1,\beta_2}^{\beta_1}(0)} \right) = \cos(\mu_1 x) - \cos(\mu_2 x) + \beta_1 \left[ \frac{\sin \mu_1 x}{\mu_1} - \frac{\sin \mu_2 x}{\mu_2} \right] - \frac{A x}{B} \cos(\mu_2 x) - \frac{C}{B} \sin(\mu_2 x) + \frac{A}{B \mu_2} \sin(\mu_2 x).
\]

Let us bound the contribution of each term in the \( L^2 \) norm of \( \Theta_k^1 \), recalling that \( (\lambda_{k_1}^{\beta_1} - \lambda_{k_2}^{\beta_2}) = \mu_i^2 - \mu_j^2 \).
– First, it is easy to deduce that for any \( \beta_1 \neq \beta_2 \) and any \( x \in (0,1] \), we have

\[
\frac{\cos(\mu_1 x) - \cos(\mu_2 x)}{\mu_1^2 - \mu_2^2} = \frac{x^2}{2} \left| \frac{\sin(\mu_1 - \mu_2 x)}{\mu_1 - \mu_2} \frac{\sin(\mu_1 + \mu_2 x)}{\mu_1 + \mu_2} \right| \leq \frac{1}{2}.
\]

– For \( x \in (0,1] \) fixed, let us define the function \( \mu \mapsto f(\mu) := \frac{\sin(\mu x)}{\mu} \), whose derivative is \( f'(\mu) = \frac{1}{\mu} x \cos(\mu x) - \frac{1}{\mu^2} \sin(\mu x) \). Applying Mean value theorem, we have

\[
|f(\mu_1) - f(\mu_2)| \leq |\mu_1 - \mu_2| |f'(\bar{\mu})|, \quad \text{for some } \bar{\mu} \in (\mu_1, \mu_2).
\]

Now, if \( 0 < \bar{\mu} < 1 \) (consequently \( 0 < \bar{\mu} x < 1 \) for \( x \in (0,1] \)), then we have

\[
|f'(\bar{\mu})| = \left| \frac{1}{\mu} x \cos(\bar{\mu} x) - \frac{1}{\mu^2} \sin(\bar{\mu} x) \right| = \frac{\bar{\mu} x^3}{(\bar{\mu} x)^2} \left| \frac{\cos(\bar{\mu} x) - 1}{\bar{\mu} x} - \frac{\sin(\bar{\mu} x) - \bar{\mu} x}{(\bar{\mu} x)^3} \right| \\
\leq \bar{\mu} x^3 \leq 1, \quad \forall x \in (0,1],
\]

since \( \frac{\cos(\bar{\mu} x) - 1}{(\bar{\mu} x)^2} \leq C \) and \( \frac{\sin(\bar{\mu} x) - \bar{\mu} x}{(\bar{\mu} x)^3} \leq C \), for \( 0 < \bar{\mu} x < 1 \).

On the other hand, for \( \bar{\mu} \geq 1 \), it is quite obvious to see that \( |f'(\bar{\mu})| \leq C \), and so finally we have uniformly in \( x \),

\[
|f(\mu_1) - f(\mu_2)| \leq C |\mu_1 - \mu_2|, \quad \forall \beta_1 \neq \beta_2 \text{ positive}.
\]

This implies that for \( 0 < \beta_1 < \beta^* \) one has

\[
\frac{\beta_1}{|\mu_1^2 - \mu_2^2|} \left| \frac{\sin(\mu_1 x)}{\mu_1} - \frac{\sin(\mu_2 x)}{\mu_2} \right| \leq \frac{C \beta_1}{\mu_1 + \mu_2} \leq C \beta^*,
\]

where the last inequality follows from (5.24).

– For estimating the remaining three terms, we use the values of \( A \) and \( C \) from (5.9) and Lemma 5.1, that gives the bound from below

\[
|B| \geq \frac{\mu_2}{4 \beta_2 |\beta_1 - \beta_2|}.
\]

With the inequality proved in the first point of Lemma 5.3 and (5.24) we see that

\[
\frac{1}{|\mu_1^2 - \mu_2^2|} \left| \frac{Ax}{B} \cos(\mu_2 x) \right| \leq \frac{2 \beta_2 |\beta_1 - \beta_2|}{\mu_2^2 |\mu_1^2 - \mu_2^2|} \leq \frac{C \beta_2 \beta_1}{\mu_2^2} \leq C \beta^*.
\]

Similarly, we have

\[
\frac{1}{|\mu_1^2 - \mu_2^2|} \left| \frac{Cx}{B} \sin(\mu_2 x) \right| \leq \frac{\beta_1 - \beta_2}{|\mu_1^2 - \mu_2^2|} \left| \frac{x^2 \sin(\mu_2 x)}{\mu_2 x} \right| \leq C \beta^*,
\]

and

\[
\frac{1}{|\mu_1^2 - \mu_2^2|} \left| \frac{Ax}{B \mu_2} \sin(\mu_2 x) \right| \leq \frac{1}{|\mu_1^2 - \mu_2^2|} \left| \frac{A x}{B} \right| \leq \frac{2 \beta_2 |\beta_1 - \beta_2|}{\mu_2^2 |\mu_1^2 - \mu_2^2|} \leq \frac{C \beta_2 \beta_1}{\mu_2^2} \leq C \beta^*.
\]

Hence, gathering all the estimates from (5.26)-(5.30), one can deduce that \( \|\Theta_k^1\|_{L^2(0,1)} \leq C \beta^* \) for any \( k \geq 0 \).

• Estimate of the second component \( \Theta_k^2 \).

By using the expression of \( \phi_k^{\beta_2} \) from (3.13), we have for each \( k \geq 0 \) that

\[
\frac{|\phi_k^{\beta_2}(x)|}{|\psi_k^{\beta_1,\beta_2}(0)|} \leq \frac{\mu_2}{\beta_2} \frac{|\cos(\mu_2 x)|}{|\psi_k^{\beta_1,\beta_2}(0)|} + \frac{|\sin(\mu_2 x)|}{|\psi_k^{\beta_1,\beta_2}(0)|}
\]

\[
\leq 4|\beta_1 - \beta_2| + 4 \beta_2 |\beta_1 - \beta_2| \left| \frac{x \sin(\mu_2 x)}{\mu_2 x} \right|
\]

\[
\leq 4(1 + \beta_2) |\beta_1 - \beta_2| \leq C \beta^*(1 + \beta_2) |\mu_1^2 - \mu_2^2|,
\]

where we make use the facts that \( |\psi_k^{\beta_1,\beta_2}(0)| \geq \frac{\mu_2}{4 \beta_2 |\beta_1 - \beta_2|} \) and the estimate in the first point of Lemma 5.3. Consequently, we deduce that \( \|\Theta_k^2\|_{L^2(0,1)} \leq C \beta^* \) for any \( k \geq 0 \).
This completes the proof of the uniform estimate of $\Theta_k$ in $L^2$ for $0 < \beta_1, \beta_2 < \beta^*$, $\beta_1 \neq \beta_2$.

2. Let us assume now that $\beta_1, \beta_2 \geq \beta^* > 0$. Using this assumption, and the fact that $\lambda_0^\beta_i \geq \lambda_0^\beta^*$ and $\lambda_k^\beta \geq k^2 \pi^2, \forall k \geq 1$, we obtain uniformly in $k \geq 0$, that

$$\mu_i \geq C_{\beta^*}, \quad i = 1, 2. \quad (5.31)$$

We need to prove that $\frac{1}{\beta_1} \Theta_k$ is bounded uniformly in $k, \beta_1$ and $\beta_2$.

- **Estimate of the first component $\Theta^1_k$:**
  - We still start from (5.25) and we estimate each term as follows.
    - Analogous to (5.26) and (5.27), we respectively have
      
      $$\left| \frac{\cos(\mu_1 x) - \cos(\mu_2 x)}{\mu_1^2 - \mu_2^2} \right| \leq \frac{1}{2}, \quad (5.32)$$
      
      and
      
      $$\frac{\beta_1}{\mu_1^2 - \mu_2^2} \left[ \frac{\sin(\mu_1 x) - \sin(\mu_2 x)}{\mu_1} - \frac{\sin(\mu_2 x)}{\mu_2} \right] \leq \frac{\beta_1}{\mu_1} \leq C_{\beta^*} \beta_1, \quad (5.33)$$
      
      by (5.31).
    - Now using the fact $|B| \geq \frac{\mu_1^2 + \beta_1^2}{3 \beta_1 \mu_2^2 |\beta_1 - \beta_2|}$ from Lemma 5.1, and the second point of Lemma 5.3, we can bound the remaining three terms as follows
      
      $$\frac{1}{|\mu_1^2 - \mu_2^2|} \left| \frac{Ax}{B} \cos(\mu_2 x) \right| \leq \frac{2}{|\beta_1 - \beta_2|} \frac{\beta_1^2}{\beta_2 |\mu_1^2 - \mu_2^2| (\mu_2^2 + \beta_2^2)} \leq \frac{2 \beta_1}{\beta_2} \frac{1}{|\beta_1 - \beta_2|} = C_{\beta^*} \beta_1, \quad (5.34)$$
      
      Similarly, we have
      
      $$\frac{1}{|\mu_1^2 - \mu_2^2|} \left| \frac{Cx}{B} \sin(\mu_2 x) \right| \leq \frac{2 \beta_2 \mu_2}{\beta_1 |\mu_1^2 - \mu_2^2| (\mu_2^2 + \beta_2^2)} \leq C_{\beta^*} \beta_1, \quad (5.35)$$
      
      and finally
      
      $$\frac{1}{|\mu_1^2 - \mu_2^2|} \left| \frac{Ax}{B \mu_2} \sin(\mu_2 x) \right| \leq \frac{1}{|\mu_1^2 - \mu_2^2|} \left| \frac{A}{B} \right| \leq C_{\beta^*} \beta_1, \quad (5.36)$$
      
      which is obtained by a similar type of computations as in (5.34).

Gathering all the estimates from (5.32)-(5.36), we get that $\|\Theta^1_k\|_{L^2(0,1)} \leq C_{\beta^*} \beta_1$ for any $k \geq 0$.

- **Estimate of the second component $\Theta^2_k$:**
  - Using the same ingredients as before, we compute
    
    $$\left| \frac{\varphi^\beta_k (x)}{\psi^\beta_k (0)} \right| \leq \frac{\mu_1^2 |\cos(\mu_2 x)| + |\sin(\mu_2 x)|}{|B|}$$
    
    $$\leq 2 \mu_2 \frac{\beta_1 \beta_2 |\mu_1^2 - \mu_2^2|}{(\mu_2^2 + \beta_2^2)} \frac{1}{|\beta_1|} \frac{1}{|\beta_2|} + 4 \mu_2 \frac{\beta_1 \beta_2^2}{(\mu_2^2 + \beta_2^2)} \frac{1}{|\beta_1|} \frac{1}{|\beta_2|}$$
    
    $$\leq C_{\beta^*} \beta_1 \mu_2 \frac{\mu_1^2 - \mu_2^2}{|\beta_2|} \frac{1}{|\beta_2|}, \quad (5.37)$$
    
    by the second point of Lemma 5.3. This implies
    
    $$\frac{1}{|\mu_1^2 - \mu_2^2|} \left| \frac{\varphi^\beta_k (x)}{\psi^\beta_k (0)} \right| \leq C_{\beta^*} \beta_1 \mu_2, \quad (5.38)$$
    
    and thus the expected bound $\|\Theta^2_k\|_{L^2(0,1)} \leq C_{\beta^*} \beta_1 \mu_2$ for any $k \geq 0$. \qed

**5.2. Null-controllability.** We can now prove the null-controllability of our system, with a precise bound of the control with respect to $\beta_1$ and $\beta_2$, that is the second point of Theorem 2.14.
5.2.1. The moments problem. In the present context, we recall that the family \( \{\Phi_{k,1}, \Phi_{k,2}\}_{k \geq 0} \) (defined by (5.3)-(5.5)) is complete in \((L^2(0,1))^2\) and so, by checking the equation (2.28) for \(\Phi_{k,1} \) and \(\Phi_{k,2} \) for each \(k \geq 0\), indeed tells us that for any \(y_0 \in (L^2(0,1))^2 \) the input \(v \in L^2(0,T) \) is a null control for (5.1) if and only if one has

\[
\begin{align*}
-e^{-\tau \lambda_k^1} (y_0, \Phi_{k,1})_{L^2(0,1)} &= \int_0^T v(t) e^{-\tau \lambda_k^1(t)} dt, \quad \forall k \geq 0, \\
-e^{-\tau \lambda_k^2} (y_0, \Phi_{k,2})_{L^2(0,1)} &= \int_0^T v(t) e^{-\tau \lambda_k^2(t)} dt, \quad \forall k \geq 0,
\end{align*}
\]

(5.37)

where we used the formulas given in (5.6).

5.2.2. The block moment method. It is known that for any \(k \geq 0\) the eigenvalue \(\lambda_k^\beta\) is continuous with respect to the parameter \(\beta \in [0, +\infty]\), see for instance [21, Theorem 3.1] and as a consequence, it may occur that the two eigenvalues \(\lambda_k^{\beta_1}\) and \(\lambda_k^{\beta_2}\) are arbitrarily close if \(\beta_1\) and \(\beta_2\) are close. This phenomenon is called spectral condensation and may, in general, prevent us from obtaining uniform bounds on the controls when \(\beta_1\) and \(\beta_2\) are getting closer (see for instance a discussion on the influence of the condensation index on controllability properties of parabolic systems in [20]).

Indeed, the classic way to solve the moments problem, as we did in Section 4 is inadequate. More precisely, it is not true anymore that any bi-orthogonal family to \((e^{-\lambda_k^\beta(t)} \cdot (T-t))_{k \geq 0, j=0,1}\) will satisfy uniform \(L^2(0,T)\)-bound with respect to the parameters \(\beta_1\), \(\beta_2\) since the gap \(\inf_k |\lambda_k^{\beta_1} - \lambda_k^{\beta_2}|\) may be arbitrary small when \(|\beta_1 - \beta_2|\) is small (see Lemma 5.3).

To overcome this situation, and still prove uniform controllability result, we will use the block moment approach developed in [11] to solve problems like (5.37) when a weak gap condition holds, instead of a usual uniform gap condition. This method let us take benefit of the condensation of eigenfunctions that actually compensate the condensation of the eigenvalues. Let us go into the details.

We first define \(\Lambda^{\beta_i} := \{\lambda_k^{\beta_i}, k \geq 0\}\) for \(i = 1, 2\), the two families of eigenvalues we are concerned with and we set \(\Lambda^{\beta_1,\beta_2} = \Lambda^{\beta_1} \cup \Lambda^{\beta_2}\).

As we have seen in (4.9), each of the two families satisfies a uniform spectral gap property

\[
\inf_k |\lambda_{k+1}^{\beta_i} - \lambda_k^{\beta_i}| \geq C\pi, \quad i = 1, 2,
\]

(5.38)

and their reciprocal values are uniformly summable in the sense that, there exists a function \(N : (0, +\infty) \rightarrow (0, +\infty)\) that does not depend on \(\beta_1\) and \(\beta_2\), such that

\[
\sum_{\lambda \in \Lambda^{\beta_i}, \lambda > N(\varepsilon)} \frac{1}{\lambda} \leq \varepsilon,
\]

(5.39)

for any \(\varepsilon > 0\) and any \(i = 1, 2\).

Therefore, by [11, Lemma 2.1], we know that the union family \(\Lambda^{\beta_1,\beta_2}\) satisfies a weak-gap property : for any \(\rho > 0\) (independent of \(\beta_1\) and \(\beta_2\)) such that

\[
\rho < C\pi,
\]

(5.40)

we have that \(\Lambda^{\beta_1,\beta_2} \cap [\mu, \mu + \rho]\), contains at most 2 elements for any \(\mu > 0\). Moreover, the reciprocal values of \(\Lambda^{\beta_1,\beta_2}\) are also uniformly summable as in (5.39) but with a possibly different function \(N\).

By [11, Proposition 7.1] we know that, for each value of \(\beta_1\) and \(\beta_2\), we can find a family of disjoint non empty groups \((G_n)_n\) each of them having a cardinal less or equal than 2 and such that

\[
\Lambda^{\beta_1,\beta_2} = \bigcup_n G_n,
\]

(5.41)

\[
(\min G_{n+1}) - (\max G_n) \geq \rho/2,
\]

\[
diam(G_n) < \rho.
\]

Let us prove now that, for \(\rho\) small enough, the structure of those groups is actually simple.
Lemma 5.5. Let $\beta^* > 0$ be fixed and for $\beta_1 \neq \beta_2$ assume that either $\beta_1, \beta_2 < \beta^*$ or $\beta_1, \beta_2 \geq \beta^*$.

There exists $\rho^*$ depending only on $\beta^*$ such that, if we assume that $\rho < \rho^*$ in the above construction in addition to (5.40), then for any group $G_n$ of cardinal 1, there exists an integer $k$ such that

$$G_n = \{\lambda^\beta_1, \lambda^\beta_2\}.$$ 

Proof. Without loss of generality, we assume that $\beta_1 < \beta_2$. Since the diameter of $G_n$ is less than $\rho$ and using (5.38) and (5.40) we know that $G_n$ contains exactly one element from $\Lambda^{\beta_1}$ and one element from $\Lambda^{\beta_2}$, that is

$$G_n = \{\lambda^\beta_1, \lambda^\beta_2\},$$

for some integers $k$ and $j$. We want to show that $j = k$.

By Lemma 3.3 and Lemma 3.11 we know that

$$\lambda_k^\beta_1 < \lambda_k^\beta_2 < \lambda_{k+1}^\beta_1,$$

thus the only possibilities are $j = k$ or $j = k + 1$.

- First, we treat the case when $0 < \beta_1 < \beta_2 < \beta^*$. We have

$$\sqrt{\lambda_k^\beta_1} - \sqrt{\lambda_k^\beta_2} \geq \sqrt{\lambda_k^\beta_1} - \sqrt{\lambda_k^\beta_2} = \pi - \delta_{\beta^*} > 0,$$

since $\lambda_k^\beta_1 = (k + 1)^2 \pi^2$ by point 1 of Remark 3.8; $\lambda_k^\beta^*$ and $\delta_{\beta^*}$ have been introduced in (5.16) and (5.17) respectively. Hence, for all $k \geq 0$, we have

$$\lambda_k^\beta_1 - \lambda_k^\beta_2 = (\sqrt{\lambda_k^\beta_1} + \sqrt{\lambda_k^\beta_2}) (\sqrt{\lambda_k^\beta_1} - \sqrt{\lambda_k^\beta_2})$$

$$\geq \pi (\pi - \delta_{\beta^*}) > 0.$$ 

We choose $\rho^* = \pi (\pi - \delta_{\beta^*})$. From the computation above we see that if $\rho < \rho^*$, then $\lambda_k^\beta_1$ and $\lambda_k^\beta_2$ cannot belong to the same group $G_n$ and thus we necessarily have $j = k$ and the claim is proved.

- Assume now that $\beta_2 > \beta_1 \geq \beta^* > 0$. Then, from the asymptotic formula given by Remark 3.10, we see

$$\sqrt{\lambda_k^\beta_1} - \sqrt{\lambda_k^\beta_2} \geq \lambda_k^\beta_1 - \lambda_k^\beta_2 = \frac{2\beta^*}{(k + 1)\pi} + O_{\beta^*} \left(\frac{1}{k^2}\right), \quad \forall k \geq 1,$$

since $\lambda_k^\beta_1 = (k + 1)^2 \pi^2$ by point 1 of Remark 3.8. Now, it is obvious that $\sqrt{\lambda_k^\beta_1} + \sqrt{\lambda_k^\beta_2} \geq (k + 1)\pi$, for all $k \geq 0$ and so, there exists a $k_{\beta^*} \geq 1$, depending only on $\beta^*$ such that

$$\lambda_{k+1}^\beta_1 - \lambda_k^\beta_2 \geq \beta^* > 0, \quad \forall k \geq k_{\beta^*}.$$ 

It remains to deal with the other values of $k = 0, 1, \ldots, k_{\beta^*} - 1$. We simply use the fact that

$$\lambda_k^\beta_1 - \lambda_k^\beta_2 \geq \lambda_{k+1}^\beta_1 - \lambda_k^\beta_2,$$

to define

$$\rho^* := \min_{0 \leq k < k_{\beta^*}} (\lambda_{k+1}^\beta_1 - \lambda_k^\beta_2) > 0.$$ 

Here also we conclude that if $\rho < \rho^*$, $\lambda_k^\beta_1$ and $\lambda_k^\beta_2$ cannot be in the same group $G_n$ and thus $j = k$, and the proof is complete. \qed

Proof of Point 2 of Theorem 2.14. We proved above that the sequence of eigenvalues $\Lambda^{\beta_1, \beta_2}$ satisfy the good weak-gap and summability conditions required by the block moment method. More precisely, we can apply [11, Theorem 2.1] to find a solution to the set of equations (5.37) as an infinite sum of terms, each of them corresponding to the resolution of the contribution of the group $G_n$. In our case, we can observe that, by Lemma 5.5, the set \{ $\lambda_k^\beta_1, \lambda_k^\beta_2$ \} for any $k \geq 0$, is either exactly one of the groups $G_n$ or the union $G_n \cup G_{n+1}$ of two distinct groups of cardinal 1.
It follows that, the result of [11, Theorem 2.1] can be reformulated as follows: there exist functions $v_k^{\beta_1, \beta_2} \in L^2(0, T)$ for each $k \geq 0$, which satisfy the following

$$
\begin{align*}
\int_0^T v_k^{\beta_1, \beta_2}(t) e^{-\lambda_k^1 (T-t)} \, dt &= -e^{-T\lambda_k^1} \frac{(y_0, \Phi_k, 1)_{L^2(0,1)}}{B^* \Phi_k, 1}, \\
\int_0^T v_k^{\beta_1, \beta_2}(t) e^{-\lambda_k^2 (T-t)} \, dt &= -e^{-T\lambda_k^2} \frac{(y_0, \Phi_k, 2)_{L^2(0,1)}}{B^* \Phi_k, 2}, \\
\int_0^T v_k^{\beta_1, \beta_2}(t) e^{-\lambda_k^i (T-t)} \, dt &= 0, \quad \forall l \neq k, \, \forall i = 1, 2.
\end{align*}
$$

(5.42)

and satisfying the following bound, for any $\varepsilon > 0$,

$$
\|v_k^{\beta_1, \beta_2}\|_{L^2(0, T)} \leq C_{T, \varepsilon, N, \rho^*} e^{(\varepsilon - T)\lambda_k^{\beta_1}} \max \left\{ \left\| \Phi_k, 1 \right\|_{L^2(0, 1)}, \left\| \Phi_k, 2 \right\|_{L^2(0, 1)} - \frac{\Phi_k, 2}{\lambda_k^{\beta_2} - \lambda_k^{\beta_1}} \right\} \|y_0\|_{L^2(0, 1)}.
$$

(5.43)

Note that in [11] it is assumed that all the eigenvalues in the system are greater than 1, whereas in our case we only know that they are non-negative (we recall that $\lambda_0^\beta$ goes to 0 when $\beta \to 0$). However, one can check that this does not change significantly the result since it simply amounts to add a factor $e^T$ in front of the constant $C_{T, \varepsilon, N, \rho^*}$ in the estimate.

We now define $v_\beta$ as

$$
v_\beta(t) := \sum_{k \geq 0} v_k^{\beta_1, \beta_2}(t), \quad \forall t \in [0, T],
$$

(5.44)

so that $v_\beta$ formally satisfies the set of moments problem (5.37), it remains to show that the series converges and to obtain the expected bound on $v_\beta$.

- In the case when $\beta_1 = \beta_2$, the result is just a particular case of point 1 of Theorem 2.14.

- Assume that $\beta_1 \neq \beta_2$. We observe that $\left\| \frac{\Phi_k, 1}{\Phi_k, 1} \right\|_{L^2(0, 1)} = \left\| \frac{\Phi_k, 1}{\Phi_k, 1}(0) \right\|_{L^2(0, 1)}$ which can be bounded by $C(1 + \beta_1)$ for any $\beta_1 \in (0, +\infty)$ (recall the expression of $v_k^{\beta_1}$ from (3.13)). We can then choose $\varepsilon = T/2$ and apply Proposition 5.4 to obtain that, for $0 < \beta_1, \beta_2 < \beta^*$,

$$
\|v_k^{\beta_1, \beta_2}\|_{L^2(0, T)} \leq C_{T, \beta^*} (1 + \beta_1) e^{-T\lambda_k^{\beta_1}} \|y_0\|_{L^2(0, 1)},
$$

(5.45)

and for $\beta_1, \beta_2 \geq \beta^*$,

$$
\|v_k^{\beta_1, \beta_2}\|_{L^2(0, T)} \leq C_{T, \beta^*} (1 + \beta_1) \sqrt{\lambda_k^{\beta_2}} e^{-T\lambda_k^{\beta_1}} \|y_0\|_{L^2(0, 1)},
$$

(5.46)

From (5.44), it follows that

$$
\|v_\beta\|_{L^2(0, T)} \leq \sum_{k \geq 0} \|v_k^{\beta_1, \beta_2}\|_{L^2(0, T)},
$$

in which we can plug (5.45) or (5.46) to finally obtain

$$
\|v_\beta\|_{L^2(0, T)} \leq C_{T, \beta^*} (1 + \beta_1) \|y_0\|_{L^2(0, 1)},
$$

where $C_{T, \beta^*} > 0$ does not depend explicitly on the parameters $\beta_1, \beta_2$ because the two series $\sum_{k \geq 0} e^{-T\lambda_k^{\beta_1}}$ and $\sum_{k \geq 0} \sqrt{\lambda_k^{\beta_2}} e^{-T\lambda_k^{\beta_1}}$ converges uniformly with respect to the parameters due to the fact $\lambda_k^N < \lambda_k^{\beta_1} < \lambda_k^D$, $\forall k \geq 0, \, i = 1, 2$, by Lemma 3.3.

The proof of the theorem is complete. 

ponsors

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