DEGREES OF SYMMETRIC GROTHENDIECK POLYNOMIALS AND CASTELNUOVO-MUMFORD REGULARITY

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Abstract. We give an explicit formula for the degree of the Grothendieck polynomial of a Grassmannian permutation and a closely related formula for the Castelnuovo-Mumford regularity of the Schubert determinantal ideal of a Grassmannian permutation. We then provide a counterexample to a conjecture of Kummini-Lakshmibai-Sastry-Seshadri on a formula for regularities of standard open patches of particular Grassmannian Schubert varieties and show that our work gives rise to an alternate explicit formula in these cases. We end with a new conjecture on the regularities of standard open patches of arbitrary Grassmannian Schubert varieties.

1. Introduction

Lascoux and Schützenberger [10] introduced Grothendieck polynomials to study the K-theory of flag varieties. Grothendieck polynomials have a recursive definition, using divided difference operators. The symmetric group $S_n$ acts on the polynomial ring $\mathbb{Z}[x_1, x_2, \ldots, x_n]$ by permuting indices. Let $s_i$ be the simple transposition in $S_n$ exchanging $i$ and $i+1$. Then define operators on $\mathbb{Z}[x_1, x_2, \ldots, x_n]$

$\partial_i = \frac{1 - s_i}{x_i - x_{i+1}}$ and $\pi_i = \partial_i(1 - x_{i+1}).$

Write $w_0 = n \ n - 1 \ldots 1$ for the longest permutation in $S_n$ (in one-line notation) and take

$G_{w_0}(x_1, x_2, \ldots, x_n) = x_1^{n-1}x_2^{n-2}\ldots x_{n-1}.$

Let $w_i := w(i)$ for $i \in [n]$. Then if $w_i > w_{i+1}$, we define $G_{s_i w} = \pi_i(G_w)$. We call $\{G_w : w \in S_n\}$ the set of Grothendieck polynomials. Since the $\pi_i$'s satisfy the same braid and commutation relations as the simple transpositions, each $G_w$ is well defined.

Grothendieck polynomials are generally inhomogeneous. The lowest degree of the terms in $G_w$ is given by the Coxeter length of $w$. The degree (i.e. highest degree of the terms) of $G_w$ can be described combinatorially in terms of pipe dreams (see [3, 7]), but this description is not readily computable. We seek an explicit combinatorial formula. In this paper, we give such an expression in the Grassmannian case. Our proof relies on a formula of Lenart [11].

One motivation for wanting easily-computable formulas for degrees of Grothendieck polynomials (for large classes of $w \in S_n$) comes from commutative algebra: formulas for degrees of Grothendieck polynomials give rise to closely related formulas for Castelnuovo-Mumford regularity of associated Schubert determinantal ideals. Recall that Castelnuovo-Mumford
regularity is an invariant of a homogeneous ideal related to its minimal free resolution (see Section 4 for definitions). Formulas for regularities of Schubert determinantal ideals yield formulas for regularities of certain well-known classes of generalized determinantal ideals in commutative algebra. For example, among the Schubert determinantal ideals are ideals of \( r \times r \) minors of an \( n \times m \) matrix of indeterminates and one sided ladder determinantal ideals. Furthermore, many other well-known classes of generalized determinantal ideals can be viewed as defining ideals of Schubert varieties intersected with opposite Schubert cells, so degrees of specializations of double Grothendieck polynomials govern Castelnuovo-Mumford regularities in these cases. Thus, one purpose of this paper is to suggest a purely combinatorial approach to studying regularities of certain classes of generalized determinantal ideals.

2. Background on Permutations

We start by recalling some background on the symmetric group. We follow [12] as a reference. Let \( S_n \) denote the symmetric group on \( n \) letters, i.e. the set of bijections from the set \([n] := \{1, 2, \ldots, n\}\) to itself. We typically represent permutations in one-line notation.

The permutation matrix of \( w \), also denoted by \( w \), is the matrix which has a 1 at \((i, w_i)\) for all \( i \in [n] \), and zeros elsewhere.

The Rothe diagram of \( w \) is the subset of cells in the \( n \times n \) grid
\[
D(w) = \{(i, j) \mid 1 \leq i, j \leq n, \ w_i > j, \ \text{and} \ \ w_j^{-1} > i\}.
\]
Graphically, \( D(w) \) is the set of cells in the grid which remain after plotting the points \((i, w_i)\) for each \( i \in [n] \) and striking out any boxes which appear weakly below or weakly to the right of these points. The essential set of \( w \), denoted \( \text{Ess}(w) \), is the subset of the diagram
\[
\text{Ess}(w) = \{(i, j) \in D(w) \mid (i + 1, j), (i, j + 1) \notin D(w)\}.
\]
Each permutation has an associated rank function defined by
\[
r_w(i, j) = |\{(i', w_{i'}) \mid i' \leq i, w_{i'} \leq j\}|.
\]
We write \( \ell(w) := |D(w)| \) for the Coxeter length of \( w \).

Example 2.1. If \( w = 63284175 \in S_8 \) (in one-line notation) then \( D(w) \) is the following:

Here \( \text{Ess}(w) = \{(1, 5), (2, 2), (4, 5), (4, 7), (5, 1), (7, 5)\} \).

3. Grassmannian Grothendieck Polynomials

A partition is a weakly decreasing sequence of nonnegative integers \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \). We define the length of \( \lambda \) to be \( \ell(\lambda) = |\{h \in [k] \mid \lambda_h \neq 0\}| \) and the size of \( \lambda \), denoted \( |\lambda| \), to be \( \sum_{i=1}^{k} \lambda_i \). Write \( \mathcal{P}_k \) for the set of partitions of length at most \( k \). Here, we conflate partitions with their Young diagrams, i.e. the notation \((i, j) \in \lambda\) indicates choosing the \( j \)th box in the \( i \)th row of the Young diagram of \( \lambda \).
We say $w \in S_n$ has a **descent** at position $k$ if $w_k > w_{k+1}$. A permutation $w \in S_n$ is **Grassmannian** if $w$ has a unique descent. To each Grassmannian permutation $w$, we can uniquely associate a partition $\lambda \in \mathcal{P}_k$:

$$\lambda = (w_k - k, \ldots, w_1 - 1),$$

where $k$ is the position of the descent of $w$.

Let $w_\lambda$ denote the Grassmannian permutation associated to $\lambda$. It is easy to check that

$$|\lambda| = \ell(w_\lambda) = |D(w_\lambda)|.$$

Define $\mathcal{YTab}(\lambda)$ to be the set of fillings of $\lambda$ with entries in $[k]$ so that

- entries weakly increase from left-to-right along rows and
- entries strictly increase from top-to-bottom along columns.

For a partition $\lambda$, the **Schur polynomial** in $k$ variables is

$$s_\lambda(x_1, x_2, \ldots, x_k) = \sum_{T \in \mathcal{YTab}(\lambda)} \prod_{i=1}^k x_i^{\# \text{‘} s \text{ in } T}. $$

**Definition 3.1.** Let $\lambda, \mu \in \mathcal{P}_k$ so that $\lambda \subseteq \mu$. Denote by $\mathcal{Tab}(\mu/\lambda)$ the set of fillings of the skew shape $\mu/\lambda$ with entries in $[k]$ such that

- entries strictly increase left-to-right in each row,
- entries strictly increase top-to-bottom in each column, and
- entries in row $i$ are at most $i - 1$ for each $i \in [k]$.

For ease of notation, let $\mathcal{G}_\lambda := \mathcal{G}_{w_\lambda}$.

**Theorem 3.2.** \cite{11, Theorem 2.2} For a Grassmannian permutation $w_\lambda \in S_n$,

$$\mathcal{G}_\lambda(x_1, x_2, \ldots, x_k) = \sum_{\substack{\mu \in \mathcal{P}_k \\ \lambda \subseteq \mu}} a_{\lambda\mu} s_\mu(x_1, x_2, \ldots, x_k)$$

where $(-1)^{|\mu|-|\lambda|}a_{\lambda\mu} = |\mathcal{Tab}(\mu/\lambda)|$ and $k$ is the unique descent of $w_\lambda$.

**Example 3.3.** The Grassmannian permutation $w = 24813567$ corresponds to $\lambda = (5, 2, 1)$. By Theorem 3.2

$$\mathcal{G}_{(5,2,1)}(x_1, x_2, x_3) = s_{(5,2,1)} - 2s_{(5,2,2)} - s_{(5,3,1)} + 2s_{(5,3,2)} - s_{(5,3,3)}.$$  

This corresponds to the tableaux:

![Tableaux](image)

**Definition 3.4.** We say a partition $\mu$ is **maximal** for $\lambda$ if $\mathcal{Tab}(\mu/\lambda) \neq \emptyset$ and $\mathcal{Tab}(\nu/\lambda) = \emptyset$ whenever $|\nu| > |\mu|$.

The following lemma can be obtained from the proof of \cite{11, Theorem 2.2}, but we include it for completeness.

**Lemma 3.5.** Fix a partition $\lambda \in \mathcal{P}_k$. Define $\mu$ by setting $\mu_1 = \lambda_1$, and $\mu_i = \min\{\mu_{i-1}, \lambda_i + (i - 1)\}$ for each $1 < i \leq k$. Then $\mu$ is the unique partition that is maximal for $\lambda$.  

Lemma 3.5. If \( \mu/\lambda \) has strictly increasing rows, \( \rho/\lambda \) has at most \( i-1 \) boxes in row \( i \) for each \( i \). That is, \( \rho_i \leq \lambda_i + (i-1) \) for each \( i \). It follows that \( \rho_i \leq \mu_i \) for each \( i \). Thus, uniqueness of \( \mu \) will follow once we show that \( \mu \) is maximal for \( \lambda \). It suffices to produce an element \( T \in \text{Tab}(\mu/\lambda) \).

We will denote by \( T(i,j) \) the filling by \( T \) of the box in row \( i \) and column \( j \) of \( \mu \). For each \( i \) and \( j \) with \( \lambda_i < j \leq \mu_i \), set

\[
T(i,j) = i + j - \mu_i - 1.
\]

It is easily seen that \( T \) strictly increases along rows with \( T(i,j) \in [i-1] \) for each \( i \). To see that \( T \in \text{Tab}(\mu/\lambda) \), it remains to note that \( T \) strictly increases down columns. Observe

\[
T(i,j) - T(i-1,j) = \mu_{i-1} - \mu_i + 1 > 0.
\]

\[\square\]

**Example 3.6.** If \( \lambda = (10, 10, 9, 7, 7, 2, 1) \), the unique partition \( \mu \) maximal for \( \lambda \) is \( \mu = (10, 10, 10, 10, 10, 7, 7) \). Below is the tableau \( T \in \text{Tab}(\mu/\lambda) \) constructed in the proof of Lemma 3.5

```
  1 2 3 4 5 6
  2
  1 2 3
  4
```

**Definition 3.7.** Given a partition \( \lambda = (\lambda_1, \ldots, \lambda_k) \), let \( P(\lambda) = (P_1, P_2, \ldots, P_k) \) be the set partition of \([k]\) such that \( i, j \in P_h \) if and only if \( \lambda_i = \lambda_j \), and \( \lambda_i > \lambda_j \) whenever \( i \in P_h \) and \( j \in P_l \) with \( h < l \).

Note that if \( \lambda = (\lambda_1, \ldots, \lambda_k) = (\lambda_1^{p_1}, \ldots, \lambda_k^{p_k}) \) in exponential notation, then \( p_h = |P_h| \) for each \( h \in [r] \). In the following definition, we describe a decomposition of \( \lambda \) into rectangles.

**Definition 3.8.** Let \( \lambda = (\lambda_1, \ldots, \lambda_k) \) be a partition and \( P(\lambda) = (P_1, P_2, \ldots, P_r) \). Set \( m_h = \min P_h \) for each \( h \). Define \( R(\lambda) = (R_1, R_2, \ldots, R_r) \) by setting

\[
R_h := \left\{(i, j) \in \lambda \mid i \in \bigcup_{l=1}^{h} P_l \text{ and } \lambda_{m_{h+1}} < j \leq \lambda_{m_h}\right\},
\]

where we take \( \lambda_{m_{r+1}} := 0 \).

Set \( \lambda^{(h)} \) to be the partition

\[
\lambda^{(h)} = \bigcup_{j=1}^{h} R_j
\]

for \( h \in [r] \). Equivalently, for \( h \in [r - 1] \), \( \lambda^{(h)} = (\lambda_1 - \lambda_i, \lambda_2 - \lambda_i, \ldots, \lambda_{i-1} - \lambda_i) \) where \( i = \min P_{h+1} \), and \( \lambda^{(r)} = \lambda \). Set \( \lambda^{(0)} := \emptyset \).

**Example 3.9.** For \( \lambda \) as in Example 3.6, one has \( P_1 = \{1, 2\} \), \( P_2 = \{3\} \), \( P_3 = \{4, 5\} \), \( P_4 = \{6\} \), and \( P_5 = \{7\} \). The sets in \( R(\lambda) \) are outlined below, with \( R_1 \) the rightmost rectangle and \( R_5 \) the leftmost. Considering \( h = 2 \), \( \lambda^{(h)} = R_1 \cup R_2 = (10 - 7, 10 - 7, 9 - 7) = (3, 3, 2) \).
Definition 3.10. For any \( n \geq 1 \), let \( \delta^n \) denote the staircase shape \( \delta^n = (n, n-1, \ldots, 1) \). Given a partition \( \mu \), let
\[
sv(\mu) = \max \left\{ k \mid \delta^k \subseteq \mu \right\}.
\]
The partition \( \delta^{sv(\mu)} \) is called the Sylvester triangle of \( \mu \).

Proposition 3.11. Suppose \( \mu \) is maximal for \( \lambda \) and \( P(\lambda) = (P_1, \ldots, P_r) \). If \( i \in P_{h+1} \) for some \( 0 \leq h < r \), then
\[
\mu_i = \lambda_i + sv(\lambda^{(h)}).
\]

Proof. By Lemma 3.5, \( \mu_1 = \lambda_1 \) and \( \mu_i = \min\{\mu_{i-1}, \lambda_i + (i-1)\} \) for \( 1 < i \leq k \). Clearly \( P(\lambda) \) refines \( P(\mu) \): if \( \lambda_i = \lambda_j \), then \( \mu_i = \mu_j \). Example 3.6 shows this refinement can be strict. Hence, it suffices to prove the statement when \( i = \min P_{h+1} \). We work by induction on \( h \).

When \( h = 0 \), \( i = \min(P_1) = 1 \). Since \( \lambda_1 = \mu_1 \), the result follows. Suppose the claim holds for some \( h - 1 \). We show the claim holds for \( h \). Let \( i = \min P_{h+1} \). Then it suffices to show that
\[
\lambda_i + sv(\lambda^{(h)}) = \min\{\mu_{i-1}, \lambda_i + (i-1)\}.
\]
Since \( i = \min P_{h+1} \), it follows that \( i-1 \in P_h \). By applying the inductive assumption to \( \mu_{i-1} \),
\[
\min\{\mu_{i-1}, \lambda_i + (i-1)\} = \min\{\lambda_{i-1} + sv(\lambda^{(h-1)}), \lambda_i + (i-1)\}.
\]
By Equations (2) and (3), the proof is complete once we show
\[
sv(\lambda^{(h)}) = \min\{(\lambda_{i-1} - \lambda_i) + sv(\lambda^{(h-1)}), i-1\}.
\]
Let \( \omega, \ell \) respectively denote the (horizontal) width and (vertical) length of \( R_h \), and set \( M = sv(\lambda^{(h-1)}) \). Equation (4) is equivalent to proving
\[
sv(\lambda^{(h)}) = \min\{\omega + M, \ell\}.
\]
By definition, \( \lambda^{(h)} = R_h \cup \lambda^{(h-1)} \), so it is straightforward to see that
\[
sv(\lambda^{(h)}) \leq \min\{\omega + M, \ell\}.
\]
Let \( (M, c) \) be the southwest most box in the northwest most embedding of \( \delta^M \subseteq \lambda^{(h-1)} \), with the indexing inherited from \( \lambda \).

Suppose first that \( \ell \geq \omega + M \). Since \( R_h \) is a rectangle, \( (\omega + M, c - \omega) \in \lambda^{(h)} \). Then \( \delta^{\omega+M} \subseteq \lambda^{(h+1)} \) and Equation (4) follows. Otherwise, it must be that \( \ell - M < \omega \). Since \( R_h \) is a rectangle, \( (\ell, c - \ell + M) \in \lambda^{(h)} \). Thus, \( \delta^\ell \subseteq \lambda^{(h+1)} \) and Equation (4) follows.

Theorem 3.12. Suppose \( w_\lambda \in S_n \) is a Grassmannian permutation. Let \( P(\lambda) = (P_1, \ldots, P_r) \). Then
\[
\deg(\mathcal{G}_\lambda) = |\lambda| + \sum_{h \in [r-1]} |P_{h+1}| \cdot sv(\lambda^{(h)}).
\]
Proof. By Theorem 3.2 and Lemma 3.5, the highest nonzero homogeneous component of $G_\lambda$ is $a_\lambda s_\mu$ where $\mu$ is maximal for $\lambda$. Since $\deg(s_\mu) = |\mu|$, Proposition 3.11 implies the theorem, using the fact that $sv(\lambda(0)) = 0$. □

Example 3.13. Returning to $\lambda$ as in Example 3.6, Theorem 3.12 states that $\deg(G_\lambda) = |\lambda| + \sum_{h=1}^{\lambda} |P_{h+1}| \cdot sv(\lambda(h)) = 46 + (1 \cdot 1 + 2 \cdot 3 + 1 \cdot 5 + 1 \cdot 6) = 46 + 18 = 64$.

4. CASTELNUOVO-MUMFORD REGULARITY OF GRASSMANNIAN MATRIX SCHUBERT VARIETIES

In this section, we recall some basics of Castelnuovo-Mumford regularity and then use Theorem 3.12 to produce easily-computable formulas for the regularities of matrix Schubert varieties associated to Grassmannian permutations.

4.1. Commutative algebra preliminaries. Let $S = \mathbb{C}[x_1, \ldots, x_n]$ be a positively $\mathbb{Z}^d$-graded polynomial ring so that the only elements in degree zero are the constants. The multigraded Hilbert series of a finitely generated graded module $M$ over $S$ is

$$H(M; t) = \sum_{a \in \mathbb{Z}^d} \dim_K(M_a) t^a = \frac{K(M; t)}{\prod_{i=1}^n (1 - t^{a_i})}, \quad \deg(x_i) = a_i,$$

where if $a_i = (a_i(1), \ldots, a_i(d))$, then $t^{a_i} = t_1^{a_i(1)} \cdots t_d^{a_i(d)}$. The numerator $K(M; t)$ in the expression above is a Laurent polynomial in the $t_i$’s, called the K-polynomial of $M$. For more detail on $K$-polynomials, see [13, Chapter 8].

We are mostly interested in the case where $S$ is standard graded, that is, $\deg(x_i) = 1$, and the case where $M = S/I$ where $I$ is a homogeneous ideal with respect to the standard grading. Note that, in this case, the $K$-polynomial is a polynomial in a single variable $t$. There is a minimal free resolution

$$0 \to \bigoplus_j S(-j)^{\beta_{i,j}(S/I)} \to \bigoplus_j S(-j)^{\beta_{h-1,j}(S/I)} \to \cdots \to \bigoplus_j S(-j)^{\beta_{0,j}(S/I)} \to S/I \to 0$$

where $l \leq n$ and $S(-j)$ is the free $S$-module obtained by shifting the degrees of $S$ by $j$. The Castelnuovo-Mumford regularity of $S/I$, denoted $\text{reg}(S/I)$, is defined as

$$\text{reg}(S/I) := \max\{j - i | \beta_{i,j}(S/I) \neq 0\}.$$ 

This invariant is measure of complexity of $S/I$ and has multiple homological characterizations. For example, $\text{reg}(S/I)$ is the least integer $m$ for which $\text{Ext}^j(S/I, S)_n = 0$, for all $j$ and all $n \leq -m - j - 1$ (see [2, Proposition 20.16]). We refer the reader to [2, Chapter 20.5] for more information on regularity.

Let $K(S/I; t)$ denote the $K$-polynomial of $S/I$ with respect to the standard grading. Assume that $S/I$ is Cohen-Macaulay and let $ht_sI$ denote the height of the ideal $I$. Then,

$$\text{reg}(S/I) = \deg K(S/I; t) - ht_sI.$$

See, for example, [11, Lemma 2.5] and surrounding explanation. In this paper, we will use this characterization of regularity.
4.2. Regularity of Grassmannian matrix Schubert varieties. Let $X$ be the space of $n \times n$ matrices with entries in $\mathbb{C}$, let $\bar{X} = (x_{ij})$ denote an $n \times n$ generic matrix of variables, and let $S = \mathbb{C}[x_{ij}]$. Given an $n \times n$ matrix $M$, let $M_{[i,j]}$ denote the submatrix of $M$ consisting of the top $i$ rows and left $j$ columns of $M$. Given a permutation matrix $w \in S_n$ we have the matrix Schubert variety

$$X_w := \{ M \in X \mid \text{rank } M_{[i,j]} \leq \text{rank } w_{[i,j]} \},$$

which is an affine subvariety of $X$ with defining ideal

$$I_w := \langle (r_w(i,j) + 1) - \text{size minors of } \bar{X}_{[i,j]} \mid (i,j) \in \mathcal{E}ss(w) \rangle \subseteq S.$$

The ideal $I_w$, called a Schubert determinantal ideal, is prime [4] and is homogeneous with respect to the standard grading of $S$.

By [6, Theorem A], we have $K(S/I_w; t) = \mathfrak{G}_w(1-t, \ldots, 1-t)$, which has the same degree as $\mathfrak{G}_w(x_1, \ldots, x_n)$, since the coefficients in homogeneous components of single Grothendieck polynomials have the same sign (see, for example, [6]). Thus,

$$\text{reg}(S/I_w) = \deg \mathfrak{G}_w(x_1, \ldots, x_n) - h_t \mathfrak{I}_w = \deg \mathfrak{G}_w(x_1, \ldots, x_n) - |D(w)|,$$

where the second equality follows because

$$h_t \mathfrak{I}_w = \text{codim}_X X_w = |D(w)|$$

by [4]. We now turn our attention to the case where $w$ is a Grassmannian permutation and retain the notation from the previous section.

**Corollary 4.1.** Suppose $w_\lambda \in S_n$ is a Grassmannian permutation. Let $P(\lambda) = (P_1, \ldots, P_r)$. Then

$$\text{reg}(S/I_{w_\lambda}) = \sum_{h \in [r-1]} |P_{h+1}| \cdot \text{sv}(\lambda^{(h)}).$$

**Proof.** This is immediate from Theorem 3.12 Equation (6), and Equation (11).

**Example 4.2.** Continuing Example 3.13 Corollary 4.1 states that $\text{reg}(S/I_{w_\lambda}) = 18$.

**Example 4.3.** The ideal of $(r+1) \times (r+1)$ minors of a generic $n \times m$ matrix is the Schubert determinantal ideal of a Grassmannian permutation $w \in S_{n+m}$. Indeed, $w$ is the permutation of minimal length in $S_{n+m}$ such that rank $w_{[n,m]} = r$.

The corresponding partition is $\lambda = (m-r)^{(n-r)}0^r$. We have $\lambda^{(1)} = (m-r)^{(m-r)}$ and so $\text{sv}(\lambda^{(1)}) = \min\{m-r, n-r\}$. Furthermore, $|P_2| = r$. Therefore,

$$\text{reg}(S/I_w) = r \cdot \min\{m-r, n-r\} = r \cdot (\min\{m,n\} - r).$$

We claim no originality for the formula in Example 4.3; minimal free resolutions of ideals of $r \times r$ minors of a generic $n \times m$ matrix are well-understood (see [9] or [14, Chapter 6]).

5. On the regularity of coordinate rings of Grassmannian Schubert varieties intersected with the opposite big cell

In this section, we discuss a conjecture of Kummini-Lakshmibai-Sastry-Seshadri [8] on Castelnuovo-Mumford regularity of coordinate rings of certain open patches of Grassmannian Schubert varieties. We provide a counterexample to the conjecture, and then we state and prove an alternate explicit formula for these regularities. We end with a conjecture on regularities of coordinate rings of standard open patches of arbitrary Schubert varieties in Grassmannians.
5.1. Grassmannian Schubert varieties in the opposite big cell. Fix \( k \in [n] \) and let \( Y \) denote the space of \( n \times n \) matrices of the form

\[
\begin{bmatrix}
M & I_k \\
I_{n-k} & 0
\end{bmatrix},
\]

where \( M \) is a \( (k \times (n-k)) \) matrix with entries in \( \mathbb{C} \) and \( I_k \) is a \( k \times k \) identity matrix. Let \( P \subseteq GL_n(\mathbb{C}) \) denote the maximal parabolic of block lower triangular matrices with block rows of size \( k, (n-k) \) (listed from top to bottom). Then the Grassmannian of \( k \)-planes in \( n \)-space, \( \text{Gr}(k, n) \), is isomorphic to \( P \setminus GL_n(\mathbb{C}) \). Further, the map \( \pi : GL_n(\mathbb{C}) \to \text{Gr}(k, n) \) given by taking a matrix to its coset mod \( P \) induces an isomorphism from \( Y \) onto an affine open subvariety \( U \) of \( \text{Gr}(k, n) \) (often called the opposite big cell).

Let \( B \subseteq GL_n(\mathbb{C}) \) be the Borel subgroup of upper triangular matrices. Schubert varieties \( X_w \) in \( P \setminus GL_n(\mathbb{C}) \) are closures of orbits \( P \setminus PwB \), where \( w \in S_n \) is a Grassmannian permutation with descent at position \( k \). Let \( Y_w \) denote the affine subvariety of \( Y \) defined to be \( \pi^{-1}_Y(X_w \cap U) \).

Let \( \tilde{Y} \) denote the matrix that has the form given in (7) with variables \( m_{ij} \) as the entries of \( M \). Then, the coordinate ring of \( Y \) is \( \mathbb{C}[Y] = \mathbb{C}[m_{ij} \mid i \in [k], j \in [n-k]] \), and the prime defining ideal \( J_w \) of \( Y_w \) is generated by the essential minors of \( \tilde{Y} \). That is,

\[
J_w = \langle (r_w(i, j) + 1) - \text{size minors of } \tilde{Y}_{[i,j]} \mid (i, j) \in \text{Ess}(w) \rangle.
\]

5.2. A conjecture, counterexample, and correction. We now consider a conjecture of Kummini-Lakshmibai-Sastry-Seshadri from [8] on regularities of coordinate rings of standard open patches of certain Schubert varieties in Grassmannians. We show that this conjecture is false by providing a counterexample, and then state and prove an alternate explicit combinatorial formula for these regularities. This latter result follows immediately from our Corollary 4.1.

To state the conjecture from [8], we first translate the conventions from their paper to ours. Indeed, we use the same notation as the previous section and assume that \( w \in S_n \) is a Grassmannian permutation with unique descent at position \( k \). Suppose that \( w = w_1 w_2 \cdots w_n \) in one-line notation. Observe that \( w \) is uniquely determined from \( n \) and \( (w_1, \ldots, w_k) \). Suppose further that for some \( r \in [k-1] \),

\[
w_{k-r+i} = n - k + i \quad \text{for all } i \in [r]
\]

and \( w_1 = 1 \). Let \( \tilde{w} \) be defined by \( (\tilde{w}_1, \ldots, \tilde{w}_k) = (n-w_k+1, \ldots, n-w_1+1) \). Then we have

\[
(\tilde{w}_1, \ldots, \tilde{w}_k) = (k-r+1, k-r+2, \ldots, k, a_{r+1}, \ldots, a_{n-1}, n)
\]

for some \( k < a_{r+1} < \cdots < a_{n-1} < n \). Let \( a_r = k \) and \( a_k = n \). For \( r \leq i \leq k-1 \), define \( m_i = a_{i+1} - a_i \).

Conjecture 5.1 ([8 Conjecture 7.5]).

\[
\text{reg}(\mathbb{C}[Y]/J_w) = \sum_{i=r}^{k-1} (m_i - 1)i.
\]

Example 5.2. We consider [8 Example 6.1]. Let \( J \) be the ideal generated by 3 \( \times \) 3 minors of a \( 4 \times 3 \) matrix of indeterminates. Then \( J = J_w \) for \( w = 1245367 \in S_7 \), where \( k = 4 \) and \( n = 7 \). Then \( \tilde{w} = (3, 4, 6, 7) \). Here we see that Equation (10) yields a regularity of 2. This matches the regularity we computed in Example 4.3.
We now show that Conjecture 5.1 is not always true.

**Example 5.3.** Let \( k = 4, n = 10, w = 145723689(10) \) so that \( \tilde{w} = (4, 6, 7, 10) \). Then \( \tilde{w} \) has the desired form. Furthermore, we have that \( m_1 = 2, m_2 = 1, m_3 = 3 \). Thus, by Conjecture 5.1 the regularity should be \( (2 - 1)1 + (1 - 1)2 + (3 - 1)3 = 1 + 6 = 7 \). However, a check in Macaulay2 [3] yields a regularity of 5. In fact, \( J_w \), once induced to a larger polynomial ring, is a Schubert determinantal ideal for \( w \), so we can use our formula from Corollary 4.4. Notice \( w \) has associated partition \( \lambda = (3, 2, 2, 0) \). Then \( \lambda^{(1)} = (1) \) and \( \lambda^{(2)} = (3, 2, 2) \), giving \( \text{reg}(\mathbb{C}[Y]/J_w) = 2 \cdot \text{sv}(\lambda^{(1)}) + 1 \cdot \text{sv}(\lambda^{(2)}) = 2 \cdot 1 + 1 \cdot 3 = 5 \).

As illustrated in Example 5.3, our formula for the regularity of a Grassmannian matrix Schubert variety given in Corollary 4.4 corrects Conjecture 5.1 whenever the ideal \( J_w \) is equal (up to inducing the ideal to a larger ring) to the Schubert determinantal ideal \( I_w \). In fact, each Grassmannian permutation considered in [5] Conjecture 7.5 is of this form. This follows because all the essential set of such \( w \) is contained in \( w_{[k,n-k]} \) by Equation (9).

**Corollary 5.4.** Let \( w_\lambda \in S_n \) be a Grassmannian permutation with descent at position \( k \) such that \( w_1 = 1 \) and for some \( r \in [k-1], w_{k-r+i} = n-k+i \) for \( i \in [r] \). Let \( P(\lambda) = (P_1, \ldots, P_r) \). Then
\[
\text{reg}(\mathbb{C}[Y]/J_{w_\lambda}) = \sum_{h \in [r-1]} |P_{h+1}| \cdot \text{sv}(\lambda^{(h)}).
\]

### 5.3. A conjecture for the general case.

We end the paper with a conjecture for the regularity of \( \mathbb{C}[Y]/J_w \) where \( w \) is an arbitrary Grassmannian permutation with descent at position \( k \). We begin with some preliminaries.

First note that \( \mathbb{C}[Y]/J_w \) is a standard graded ring. Indeed, the torus \( T \subseteq GL_n(\mathbb{C}) \) of diagonal matrices acts on \( \mathbb{C}[Y] \) and on \( X_w \cap U \) by right multiplication. This action induces a \( \mathbb{Z}^n \)-grading on \( \mathbb{C}[Y] \) such that \( m_{i,j} \) has degree \( \bar{e}_i - \bar{e}_j \) and \( J_w \) is homogeneous. This \( \mathbb{Z}^n \)-grading can be coarsened to the standard \( \mathbb{Z} \)-grading because the \( T \)-action contains the dilation action\(^1\) embed \( \mathbb{C}^* \to T \) by sending \( z \in \mathbb{C}^* \) to the diagonal matrix that has its \((i,i)\)-entry equal to \( 1 \) when \( 1 \leq i \leq n-k \) and equal to \( z \) when \( n-k+1 \leq i \leq n \).

The codimension of \( Y_w \) in \( \mathbb{C}^n \) is equal to the number of boxes in the diagram \( D(w) \). So, to compute the regularity \( \text{reg}(\mathbb{C}[Y]/J_w) \), it remains to find the degree of the \( K \)-polynomial of \( \mathbb{C}[Y]/J_w \). By [15] Theorem 4.5, this \( K \)-polynomial can be expressed in terms of a double Grothendieck polynomial, \( \mathfrak{G}_w(x; y) \), which is defined as follows:
\[
\mathfrak{G}_w(x; y) = \prod_{i+j \leq n} (x_i + y_j - x_i y_j).
\]

The rest are defined recursively, using the same operator \( \pi_i \) and recurrence defined in Section 4. Note that if \( G_w(x; y) \) denotes the double Grothendieck polynomials in [6], we have \( G_w(x; y) = \mathfrak{G}_w(1 - x; 1 - \frac{1}{y}) \).

Let \( c = ((1-t), (1-t), \ldots, (1-t), 0, 0, \ldots, 0) \) be the list consisting of \( k \) copies of \( 1-t \) followed by \( n-k \) copies of \( 0 \), and let \( \tilde{c} = (0, 0, \ldots, 0, 1-\frac{4}{t}, 1-\frac{4}{t}, \ldots, 1-\frac{4}{t}) \) be the list consisting of \( n-k \) copies of \( 0 \) followed by \( k \) copies of \( 1 - \frac{1}{t} \). By [15] Theorem 4.5, the \( K \)-polynomial

---

\(^1\)More generally, coordinate rings of Kazhdan-Lusztig varieties \( X_w \cap X_v \subseteq B^- \backslash GL_n(\mathbb{C}) \) are standard graded when \( v \), the permutation defining the opposite Schubert cell \( X_v^o = B^- \backslash B_- v B_- \), is 321-avoiding. See [?; pg. 25] or [15] Section 4.1 for further explanation.
of \(S/J_w\) is the specialized double Grothendieck polynomial \(G_w(c; \bar{c})\). Consequently, we are reduced to computing the degree of this polynomial.

**Example 5.5.** Let \(w = 132\) and \(k = 2\). Then
\[
G_w(x; y) = (x_2 + y_1 - x_2y_1) + (x_1 + y_2 - x_1y_2) - (x_1 + y_2 - x_1y_2)(x_2 + y_1 - x_2y_1).
\]
Letting \(c = (1-t, 1-t, 0)\) and \(\bar{c} = (0, 1-\frac{1}{t}, 1-\frac{1}{t})\), one checks that \(G_w(c; \bar{c}) = (1-t)\) which is the \(K\)-polynomial of \(S/J_w\) with respect to the standard grading.

For the reader familiar with pipe dreams (see, e.g. [3] and [7]), we note that the degree of \(G_w(c; \bar{c})\) is the maximum number of plus tiles in a (possibly non-reduced) pipe dream for \(w\) with all of its plus tiles supported within the northwest justified \(k \times (n - k)\) subgrid of the \(n \times n\) grid. This follows from [15]. However, this is not a very explicit formula for degree.

We now turn to our conjecture. It asserts that the degree of the \(K\)-polynomial of \(\mathbb{C}[Y]/J_w\) for a Grassmannian permutation \(w \in S_n\) with descent at position \(k\) can be computed in terms of the degree of a Grothendieck polynomial of an associated vexillary permutation. This will be a much more easily computable answer than a pipe dream formula because the first, third, and fifth authors will give an explicit formula for degrees of vexillary Grothendieck polynomials in the sequel.

A permutation \(w \in S_n\) is **vexillary** if it contains no 2143-pattern, i.e. there are no \(i < j < k < l\) such that \(w_j < w_i < w_l < w_k\). For example, \(w = 325164\) is not vexillary since it contains the underlined 2143 pattern.

Suppose \(w_\lambda \in S_n\) is Grassmannian with descent \(k\). Define \(\lambda' = (\lambda_1, \ldots, \lambda_{\ell(\lambda)})\) and \(\phi(\lambda) = (\phi_1, \ldots, \phi_{\ell(\lambda)})\) as follows. For \(i \in [\ell(\lambda)]\),
\[
\phi_i = \begin{cases} 
  i + \min\{(n - k) - \lambda_i, k - i\} & \text{if } \lambda_i > \lambda_{i+1} \text{ or } i = \ell(\lambda), \\
  \phi_{i+1} & \text{otherwise.}
\end{cases}
\]
A vexillary permutation \(v\) is determined by the statistics of a partition and a flag, computed using \(D(v)\) (see [12] Proposition 2.2.10). Thus, the partition \(\lambda'\) and flag \(\phi\) defined above from \(w_\lambda\) define at most one vexillary permutation.

**Conjecture 5.6.** Fix \(w_\lambda \in S_n\) Grassmannian with descent \(k\). Then \(\lambda', \phi(\lambda)\) define a vexillary permutation \(v\), and \(\deg(G_{w_\lambda}(c; \bar{c})) = \deg(G_v(x))\). In particular, \(\text{reg}(\mathbb{C}[Y]/J_{w_\lambda}) = \deg(G_v(x)) - |\lambda|\).

While we state this as a conjecture here, the first, third, and fifth authors will prove this in the sequel and furthermore give an explicit combinatorial formula for \(\deg(G_v(x))\), as mentioned above.

**Example 5.7.** Let \(k = 5\), \(n = 10\) and \(w_\lambda = 1489(10)23567\). Then \(\lambda' = (5, 5, 5, 5, 2)\) and \(\phi(\lambda) = (3, 3, 3, 5)\), which corresponds to the vexillary permutation \(v = 678142359(10)\). Thus Conjecture 5.6 states that \(\deg(G_{w_\lambda}(c; \bar{c})) = \deg(G_v(x)) = 18\), so \(\text{reg}(\mathbb{C}[Y]/J_{w_\lambda}) = 18 - 17 = 1\).

To compute this regularity directly, take \(R = \mathbb{C}[Y] = \mathbb{C}[m_{ij} \mid 1 \leq i, j \leq 5]\). Let \(G\) denote the set of \(2 \times 2\) minors of \(\begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \end{bmatrix}\), and let \(H\) be the set of entries in the bottom

\[\text{The conventions used in [15] differ from ours, so the given formula is a translation of their formula to our conventions.}\]
three rows of the matrix of variables $M = (m_{ij})_{1 \leq i,j \leq 5}$. Then $G \cup H$ is a minimal generating set of $J_{w_{\lambda}}$. The Eagon-Northcott complex is a minimal free resolution of $R/\langle G \rangle$:

$$0 \to R(-3)^2 \to R(-2)^3 \to R \to R/\langle G \rangle \to 0.$$  

From this, one directly observes that the regularity of the $R$-module $R/\langle G \rangle$ is 1. Modding out $R/\langle G \rangle$ by the linear forms in $H$ does not change the regularity (see, e.g. [2, Proposition 20.20]), and hence the regularity of $R/J_{w_{\lambda}}$ is also 1.

ACKNOWLEDGEMENTS

The authors would like to thank Daniel Erman, Reuven Hodges, Patricia Klein, Claudiu Raicu, Alexander Yong, and the anonymous referee for their helpful comments and conversations.

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