THE FINITE BASIS PROBLEM
FOR KISELMAN MONOIDS

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Abstract. In an earlier paper, the second-named author has described the identities holding in the so-called Catalan monoids. Here we extend this description to a certain family of Hecke–Kiselman monoids including the Kiselman monoids $K_n$. As a consequence, we conclude that the identities of $K_n$ are nonfinitely based for every $n \geq 4$ and exhibit a finite identity basis for the identities of each of the monoids $K_2$ and $K_3$.

1. Kiselman and Hecke–Kiselman monoids

Kiselman [14] has found all possible compositions of three natural closure operators playing a distinguished role in convex analysis. Namely, let $E$ be a real topological vector space and let $f$ be any function on $E$ with values in the extended real line $\mathbb{R} \cup \{+\infty, -\infty\}$. The first of the three operators associates to $f$ its convex hall or largest convex minorant $c(f)$, defined as the supremum of all its convex minorants. Alternatively, it can be shown that for each $x \in E$, the value of $c(f)$ at the point $x$ can be calculated as follows:

$$c(f)(x) = \inf \left\{ \sum_{i=1}^{N} \lambda_i f(x_i) \mid N \geq 1, \lambda_i > 0, f(x_i) < +\infty, \sum_{i=1}^{N} \lambda_i x_i = x \right\}.$$ 

The second operator is that of taking the largest lower semicontinuous minorant $\ell(f)$ of the function, defined as the supremum of all its lower semicontinuous minorants with respect to the topology on $E$. It can be verified that for each $x \in E$, the value of $\ell(f)$ at $x$ is given by the formula

$$\ell(f)(x) = \lim \inf_{y \to x} f(y).$$

This corresponds to taking the closure of the epigraph of the function $f$ (the set of points lying on or above the graph of $f$) with respect to the Cartesian product of the topology on $E$ and the usual topology on $\mathbb{R}$.

The third operator is the operator $m$, defined as

$$m(f)(x) = \begin{cases} f(x) & \text{if } f(x) > -\infty \text{ for all } x \in E; \\ -\infty & \text{otherwise.} \end{cases}$$

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The operators \( c, \ell, m \) generate a monoid \( G(E) \) with composition as multiplication. Kiselman [14, Theorem 4.1] has shown that the order of \( G(E) \) can be 1, 6, 15, 16, 17, or 18, depending on the dimension of \( E \) and its topology. In particular, for every normed space \( E \) of infinite dimension, \( G(E) \) consists of 18 elements and has the following presentation (as a monoid):

\[
G(E) = \langle c, \ell, m \mid c^2 = c, \, \ell^2 = \ell, \, m^2 = m, \, clc = \ell cl = lce, \, cmc =cmc = mc, \, \ell ml = m \ell m = m \ell \rangle. \tag{1}
\]

Ganyushkin and Mazorchuk (unpublished) have suggested to consider analogous presentations with arbitrarily many generators. Namely, for each \( n \geq 2 \), they have defined the Kiselman monoid \( K_n \) as follows:

\[
K_n = \langle a_1, a_2, \ldots, a_n \mid a_i^2 = a_i, \, i = 1, \ldots, n; \]
\[
a_i a_j a_i = a_j a_i a_j = a_j a_i, \, 1 \leq i < j \leq n \rangle. \tag{2}
\]

Clearly, the monoid defined by the presentation \( \text{(1)} \) is isomorphic to \( K_3 \)—the bijection \( c \mapsto a_1, \ell \mapsto a_2, m \mapsto a_3 \) extends to an isomorphism. Algebraic properties of the family of monoids \( \{K_n\}_{n=2,3,\ldots} \) have been studied in depth by Kudryavtseva and Mazorchuk [16]. Here we only mention that the monoid \( K_n \) is finite for every \( n \) [16, Theorem 3] but its exact order (as a function of \( n \)) is not yet known.

A further generalization motivated by some connections with representation theory has been suggested by Ganyushkin and Mazorchuk in [8]. Fix an integer \( n \geq 2 \) and take an arbitrary anti-reflexive binary relation \( \Theta \) on the set \( \{1, 2, \ldots, n\} \). The Hecke–Kiselman monoid \( \mathcal{H}_K \Theta \) corresponding to \( \Theta \) is the monoid generated by elements \( a_1, a_2, \ldots, a_n \) subject to the relations

\[
a_i^2 = a_i \quad \text{for each } i = 1, \ldots, n;
\]
\[
a_i a_j a_i = a_j a_i a_j = a_j a_i \quad \text{if } (i, j), (j, i) \notin \Theta;
\]
\[
a_i a_j a_i = a_j a_i a_j \quad \text{if } (i, j), (j, i) \in \Theta;
\]
\[
a_i a_j a_i = a_j a_i a_j = a_j a_i \quad \text{if } (i, j) \notin \Theta, (j, i) \in \Theta.
\]

Obviously, the Kiselman monoid \( K_n \) arises as a special case of this construction when the role of \( \Theta \) is played by the strict “descending” order \( \Theta_K = \{(j, i) \mid 1 \leq i < j \leq n\} \). Another important special case is the so-called Catalan monoid, denoted \( \mathcal{C}_{n+1} \), that is generated by elements \( a_1, a_2, \ldots, a_n \) subject to the relations

\[
a_i^2 = a_i \quad \text{for each } i = 1, \ldots, n;
\]
\[
a_i a_j a_i = a_j a_i a_j = a_j a_i \quad \text{if } |i - j| \geq 2, \, i, j = 1, \ldots, n;
\]
\[
a_i a_{i+1} a_i = a_{i+1} a_i a_{i+1} = a_{i+1} a_i \quad \text{for each } i = 1, \ldots, n-1.
\]

Clearly, \( \mathcal{C}_{n+1} \) is nothing but the Hecke–Kiselman monoid corresponding to the covering relation \( \Theta_C = \{(i + 1, i) \mid i = 1, 2, \ldots, n - 1\} \) of the order \( \Theta_K \). Solomon [20] has shown that \( \mathcal{C}_{n+1} \) is isomorphic to the monoid of all order
preserving and decreasing transformations of the chain
\[ 1 < 2 < \cdots < n < n + 1. \]
(Recall that transformation \( \alpha \) of a partially ordered set \( \langle Q, \leq \rangle \) is called order preserving if \( q \leq q' \) implies \( q\alpha \leq q'\alpha \) for all \( q, q' \in Q \), and decreasing if \( q\alpha \leq q \) for every \( q \in Q \).) Hence the monoid \( C_n \) is finite for every \( n \); moreover, it can be shown that the cardinality of \( C_n \) is the \( n \)-th Catalan number \( \frac{1}{n+1} \binom{2n}{n} \); see, e.g., [10, Theorem 3.1].

In [29], the second-named author has investigated the identities holding in each Catalan monoids and solved the finite basis problem for these identities. Here we extend this study to Kiselman monoids and, more generally, to all Hecke–Kiselman monoids \( HK_\Theta \) such that \( \Theta_C \subseteq \Theta \subseteq \Theta_K \).

The paper is basically self-contained and is structured as follows. In Section 2 we briefly discuss the finite basis problem in order to place the present study into a proper perspective. We also summarize few properties of Catalan, Kiselman and Hecke–Kiselman monoids that are essential for the proof of our main result. This result is formulated and proved in Section 3 while Section 4 collects a few related open questions.

2. Preliminaries

A semigroup identity is just a pair of words, i.e., elements of the free semigroup \( X^+ \) over an alphabet \( X \). In this paper identities are written as “bumped” equalities such as \( u \approx v \). A semigroup \( S \) satisfies \( u \approx v \) where \( u, v \in X^+ \) if for every homomorphism \( \varphi : X^+ \to S \), the equality \( u\varphi = v\varphi \) is valid in \( S \); alternatively, we say that \( u \approx v \) holds in \( S \).

Given any system \( \Sigma \) of semigroup identities, we say that an identity \( u \approx v \) follows from \( \Sigma \) if every semigroup satisfying all identities of \( \Sigma \) satisfies the identity \( u \approx v \) as well; alternatively, we say that \( \Sigma \) implies \( u \approx v \). A subset \( \Sigma' \subseteq \Sigma \) is called an identity basis for \( \Sigma \) if each identity in \( \Sigma \) follows from \( \Sigma' \).

Given a semigroup \( S \), its equational theory is the set of all identities holding in \( S \). A semigroup is said to be finitely based if its equational theory admits a finite identity basis; otherwise it is called nonfinitely based.

It was discovered by Perkins [21, 22] that a finite semigroup can be nonfinitely based. In fact, semigroups are the only “classical” algebras for which finite nonfinitely based objects exist: finite groups [20], finite associative and Lie rings [15, 18, 4], finite lattices [19] are all finitely based. This circumstance gave rise to numerous investigations whose final aim was to classify all finite semigroups with respect to the property of having/having no finite identity basis. In spite of many researchers’ efforts, the complete classification has not yet been achieved, and therefore, it appears to be reasonable to look at some restricted versions of the problem where one focuses on certain important classes of finite semigroups. This paper contributes to the study of the finite basis problem within the class of finite \( J \)-trivial monoids.

Recall that a monoid \( M \) is said to be \( J \)-trivial if every principal ideal of \( M \) has a unique generator, that is, \( MaM = MbM \) implies \( a = b \) for all
$a, b \in M$. Finite $J$-trivial monoids attract much attention because of their distinguished role in algebraic language theory \cite{24, 25} and representation theory \cite{7}. The finite basis problem remains extremely hard when restricted to this class of monoids. In fact, one of the very first examples of non-finitely based finite semigroups constructed by Perkins \cite{22} was a $J$-trivial monoid, and further studies of the underlying construction have revealed a very complicated behaviour of $J$-trivial monoids with respect to the finite basis property; see \cite{28, Subsection 4.2} for an overview of related results.

The Catalan monoids $C_n$ defined in Section 1 are known to be $J$-trivial, and moreover, they serve as sort of universal objects for the class of all finite $J$-trivial monoids. Recall that a monoid $M$ is said to divide a monoid $N$ if $M$ is a homomorphic image of a submonoid in $N$. Straubing \cite{27} has shown that a finite monoid is $J$-trivial if and only if it divides some monoid of order preserving and decreasing transformations of a finite partially ordered set. Pin \cite{23, Theorem 4.1.10} has observed that this result can be made more concrete: a finite monoid is $J$-trivial if and only if it divides some Catalan monoid $C_n$. The finite basis problem for the monoids $C_n$ has been solved by the second-named author \cite{29} on the basis of some results by Blanchet-Sadri \cite{5, 6}. The following summarizes this solution.

**Theorem 1.** a) The identities
\[ xyxzx \equiv xyzzx, \quad (xy)^2 \equiv (yx)^2 \] (3)
form an identity basis of the monoid $C_3$.

b) The identities
\[ xyx^2zx \equiv xyzzx, \quad xyx^2tz \equiv xyxx^2tz, \quad zyx^2zt \equiv zyx^2ztx, \]
\[ (xy)^3 \equiv (yx)^3 \] (4)
form an identity basis of the monoid $C_4$.

c) The monoids $C_n$ with $n \geq 5$ are nonfinitely based.

**Remark.** For compactness, when presenting the identity bases in Theorem 1, we have taken the liberty to use bases in the sense of monoid identities. For instance, we have not included in (3) the identity $x^3 \equiv x^2$ (which does hold in $C_3$) because in the monoid setting it can be deduced from the identity $xyxzx \equiv xyzx$ by substituting 1 for $y$ and $z$. However, as observed in \cite{28, p.173}, the property of a monoid to be finitely based or nonfinitely based does not depend on using the semigroup or the monoid deduction rules.

We also need a characterization of equational theories of Catalan monoids. Let $X^*$ stand for the free monoid over an alphabet $X$. A word $u = x_1 \cdots x_k$ with $x_1, \ldots, x_k \in X$ is a **scattered subword** of $v \in X^*$ whenever there exist words $v_0, v_1, \ldots, v_{k-1}, v_k \in X^*$ such that $v = v_0 x_1 v_1 \cdots v_{k-1} x_k v_k$; in other terms, one can extract $u$ treated as a sequence of letters from the sequence $v$. We denote by $J_n$ the set of all identities $w \equiv w'$ such that the words $w$ and $w'$ have the same set of scattered subwords of length at most $n$. The following is a combination of \cite{29, Proposition 4} and \cite{29, Corollary 2}.
Proposition 2. The equational theory of the monoid $C_{n+1}$ is equal to $J_n$.

Now we turn to Kiselman monoids. We represent elements of the Kiselman monoid $K_n$ by words over the alphabet $A_n = \{a_1, a_2, \ldots, a_n\}$. For a word $w$, its content $c(w)$ is the set of all letters that occur in $w$ (in particular, the content of the empty word 1 is the empty set). The relations in $\{2\}$ are such that the same letters occur in both sides of each relation whenever $w, w' \in A_n^*$ represent the same element of $K_n$. Therefore one can speak of the content of an element from $K_n$. The following reduction rules underlie all combinatorics of Kiselman monoids.

Lemma 3 ([16, Lemma 1]). a) Let $s \in K_n$ and $c(s) \subseteq \{a_{i+1}, \ldots, a_n\}$ for some $i$. Then $a_isa_i = sa_i$.

b) Let $t \in K_n$ and $c(t) \subseteq \{a_1, \ldots, a_{i-1}\}$ for some $i$. Then $a_iota_i = a_it$.

For applications in the present paper we modify Lemma 3 as follows.

Lemma 4. a) Let $s \in K_n$ and $c(s) \subseteq \{a_i, a_{i+1}, \ldots, a_n\}$ for some $i$. Then $a_isa_i = s'a_i$, where $s'$ is obtained from $s$ by removing all occurrences of $a_i$ if there were some.

b) Let $t \in K_n$ and $c(t) \subseteq \{a_1, \ldots, a_{i-1}, a_i\}$ for some $i$. Then $a_ita_i = a_it'$, where $t'$ is obtained from $t$ by removing all occurrences of $a_i$ if there were some.

Proof. We verify only claim a); the argument for claim b) is symmetric.

Thus, we have to show that if $c(s) \subseteq \{a_i, a_{i+1} \ldots, a_n\}$, then we can remove all occurrences of $a_i$ in $a_isa_i$ except the rightmost one without changing the value of this expression in $K_n$. We induct on the number of occurrences of $a_i$ in $s$. If $a_i$ does not occur in $s$, Lemma 3a applies immediately. Suppose that $a_i$ occurs in $s$ and represent $s$ as $s = s_1a_is_2$ where $s_2$ contains no occurrences of $a_i$. Then $a_isa_i = a_is_1a_is_2a_i = a_is_1s_2a_i$ by Lemma 3b, and the induction hypothesis applies to $a_is_1s_2a_i$ since $s_1s_2$ contains fewer occurrences of $a_i$. \[\Box\]

Finally, we recall the following property of Hecke–Kiselman monoids.

Proposition 5 ([8, Proposition 14]). Let $\Theta$ and $\Phi$ be anti-reflexive binary relations on the set $\{1, 2, \ldots, n\}$ and $\Phi \subseteq \Theta$. The map $a_i \mapsto a_i$ uniquely extends to a homomorphism from the Hecke–Kiselman monoid $HK_{\Theta}$ onto the monoid $HK_{\Phi}$.

3. Equational theories of Hecke–Kiselman monoids

For each positive integer $n$ and each alphabet $X$, consider the following relation $\sim_n$ on the free monoid $X^*$:

$w \sim_n w'$ iff $w$ and $w'$ have the same scattered subwords of length $\leq n$.

It is convenient for us to introduce also the relation $\sim_0$ by which we merely mean the universal relation on $X^*$. We need a few facts from combinatorics on words dealing with the relation $\sim_n$. They all come from Simon’s seminal paper [25] and are collected in the following lemma.
Lemma 6. Let \( n \geq 1 \) and \( u, v \in X^* \).

a) If \( u \sim_n v \), then there exists a word \( w \in X^* \) that has each of the words \( u \) and \( v \) as a scattered subword and such that \( u \sim_n w \sim_n v \).

b) The relation \( u \sim_n uv \) holds if and only if there exist \( u_1, \ldots, u_n \in X^* \) such that \( u = u_n \cdots u_1 \) and \( c(u_n) \supseteq \cdots \supseteq c(u_1) \supseteq c(w) \).

c) The relation \( v \sim_n vw \) holds if and only if there exist \( v_1, \ldots, v_n \in X^* \) such that \( v = v_1 \cdots v_n \) and \( c(v) \subseteq c(v_1) \subseteq \cdots \subseteq c(v_n) \).

d) Let \( x \in X \). The relation \( uxv \sim_n uv \) holds if and only if there exist \( k, \ell \geq 0 \) with \( k + \ell \geq n \) and such that \( u \sim_k ux \) and \( xv \sim_\ell v \).

Recall that \( J_n \) stands for the set of all identities \( w \equiv w' \) such that \( w \) and \( w' \) have the same scattered subwords of length \( \leq n \); that is,

\[
J_n = \{ w \equiv w' \mid w \sim_n w' \}.
\]

Lemma 6 implies that the set

\[
J_n' = \{ w \equiv w' \mid w \sim_n w' \text{ and } w \text{ is a scattered subword of } w' \}
\]

forms an identity basis for \( J_n \). Now let \( w \equiv w' \) be an identity in \( J_n' \). Since \( w \) is a scattered subword of \( w' \), we have \( w = x_1 \cdots x_m \) and

\[
w' = v_0x_1v_1 \cdots v_{m-1}x_mv_m
\]

for some letters \( x_1, \ldots, x_m \) and some \( v_0, v_1, \ldots, v_{m-1}, v_m \in X^* \). If we represent the word \( v_0v_1 \cdots v_{m-1}v_m \) as a product of letters,

\[
v_0v_1 \cdots v_{m-1}v_m = y_1y_2 \cdots y_p \text{ with } y_1, y_2, \ldots, y_p \in X,
\]

we can think of \( w' \) as of the result of \( p \) successive insertions of the letters \( y_1, \ldots, y_p \) in the word \( w \). If we let \( w_0 = w \) and denote by \( w_i, i = 1, \ldots, p \), the word obtained after the \( i \)-th insertion, then \( w_p = w' \), and since \( w \sim_n w' \), we conclude that \( w_{i-1} \sim_n w_i \) for each \( i \). Therefore the identities \( w_0 \equiv w_1, w_1 \equiv w_2, \ldots, w_{i-1} \equiv w_p \) all belong to \( J_n' \) and, clearly, together they imply the identity \( w \equiv w' \). Thus, we can substitute the set \( J_n' \) by the following smaller identity basis for \( J_n \):

\[
J_n'' = \{ w \equiv w' \mid w \sim_n w' \text{ and } w = uv, w' = uvx \text{ for some } u, v \in X^*, x \in X \}.
\]

Combining this observation with Lemma 6-d, we get the following.

Corollary 7. For each \( n \geq 1 \), the collection of all identities of the form

\[
u_k \cdots u_1v_1 \cdots v_\ell \equiv u_k \cdots u_1xv_1 \cdots v_\ell, \tag{5}
\]

where \( k, \ell \geq 0 \), \( k + \ell \geq n \), \( x \in X \), and

\[
c(u_k) \supseteq \cdots \supseteq c(u_1) \supseteq \{ x \} \subseteq c(v_1) \subseteq \cdots \subseteq c(v_\ell) \tag{6}
\]

forms an identity basis for the set \( J_n \).

Let us comment on the meaning of the formulas (5) and (6) in the extreme situations \( k = 0 \) or \( \ell = 0 \). If \( k = 0 \), the identity (5) becomes \( v_1 \cdots v_\ell \equiv xv_1 \cdots v_\ell \), while the inclusions (6) reduce to \( \{ x \} \subseteq c(v_1) \subseteq \cdots \subseteq c(v_\ell) \);
dually, if \( \ell = 0 \), then (5) and (6) become respectively \( u_k \cdots u_1 \equiv u_k \cdots u_1 x \) and \( c(u_k) \supseteq \cdots \supseteq c(u_1) \supseteq \{ x \} \).

Fix an integer \( n \geq 2 \). Recall that
\[
\Theta_K = \{(j, i) \mid 1 \leq i < j \leq n\},
\]
\[
\Theta_C = \{(i + 1, i) \mid i = 1, 2, \ldots, n - 1\}.
\]

Our main result describes the identities holding in the Hecke–Kiselman monoid \( \mathcal{H}K_\Theta \) for every relation \( \Theta \) situated between \( \Theta_K \) and \( \Theta_C \).

**Theorem 8.** Let \( n \geq 2 \). Then for every relation \( \Theta \) on the set \( \{1, 2, \ldots, n\} \) such that \( \Theta_C \subseteq \Theta \subseteq \Theta_K \), the set of identities of the Hecke–Kiselman monoid \( \mathcal{H}K_\Theta \) coincides with \( J_n = \{ w \equiv w' \mid w \sim_n w' \} \).

**Proof.** Let \( \Sigma_\Theta \) stand for the set of identities holding in the monoid \( \mathcal{H}K_\Theta \). By Proposition 5 the Catalan monoid \( C_{n+1} = \mathcal{H}K_{\Theta'} \) is a homomorphic image of \( \mathcal{H}K_\Theta \) whence every identity of \( \Sigma_\Theta \) holds in \( C_{n+1} \). By Proposition 2 this means that \( \Sigma_\Theta \subseteq J_n \). On the other hand, by Proposition 5 the monoid \( \mathcal{H}K_\Theta \) is a homomorphic image of the Kiselman monoid \( K_n = \mathcal{H}K_{\Theta''} \) whence \( \mathcal{H}K_\Theta \) satisfies all identities of \( K_n \). Therefore, in order to prove that \( J_n \subseteq \Sigma_\Theta \), it suffices to check that every identity in the set \( J_n \) holds in \( K_n \). Corollary 7 reduces the latter task to verifying that \( K_n \) satisfies each identity of the form (5) obeying the conditions (6).

Thus, we fix an arbitrary identity of the form (5) satisfying (6) and an arbitrary homomorphism \( \varphi: X^+ \rightarrow K_n \). Given a word \( w \in X^+ \), we write \( \bar{w} \) instead of \( w\varphi \) for the image \( w \) under \( \varphi \), just to lighten notation. Let
\[
b = \bar{u}_k \cdots \bar{u}_1 \bar{v}_1 \cdots \bar{v}_\ell, \quad d = \bar{u}_k \cdots \bar{u}_1 \bar{x} \bar{v}_1 \cdots \bar{v}_\ell.
\]

We aim to prove that \( b \equiv d \).

Recall that we represent elements of \( K_n \) as words over the alphabet \( A_n = \{ a_1, a_2, \ldots, a_n \} \), and it makes sense to speak about occurrences of a letter in an element \( s \in K_n \) since all words representing \( s \) have the same content. If \( p, q \in \{1, \ldots, n\} \), we denote by \( s^{[p, q]} \) the element of \( K_n \) obtained from \( s \) by removing all occurrences of the letters \( a_i \) such that either \( i < p \) or \( q < i \) if there were some. If no letter \( a_i \) with either \( i < p \) or \( q < i \) occurs in \( s \), we let \( s^{[p, q]} = s \). Observe that, by this definition, \( s^{[p, q]} = 1 \) whenever \( p > q \).

Clearly, the inclusions (6) imply that
\[
c(\bar{u}_k) \supseteq \cdots \supseteq c(\bar{u}_1) \supseteq c(\bar{x}) \subseteq c(\bar{v}_1) \subseteq \cdots \subseteq c(\bar{v}_\ell).
\]
\( (7) \)

Suppose that \( \ell > 0 \) and \( a_1 \in c(\bar{v}_\ell) \). Then we can apply Lemma 4a to remove all occurrences of \( a_1 \) from each of the factors \( \bar{v}_1, \ldots, \bar{v}_{\ell-1} \) of \( b \) and from each of the factors \( \bar{x}, \bar{v}_1, \ldots, \bar{v}_{\ell-1} \) of \( d \) without changing \( b \) nor \( d \). Using the notation introduced above, we get the equalities
\[
b = \bar{u}_k \cdots \bar{u}_1 \bar{v}_1^{[2, n]} \cdots \bar{v}_{\ell-1}^{[2, n]} \bar{v}_\ell, \quad (8)
\]
\[
d = \bar{u}_k \cdots \bar{u}_1 \bar{x} \bar{v}_1^{[2, n]} \cdots \bar{v}_{\ell-1}^{[2, n]} \bar{v}_\ell. \quad (9)
\]
However, the inclusions (7) ensure that the equalities (8) and (9) hold true also in the case when \( a_1 \notin c(\bar{v}_\ell) \): if the letter \( a_1 \) does not occur in \( \bar{v}_\ell \), it occurs in none of the factors \( \bar{x}, \bar{v}_1, \ldots, \bar{v}_{\ell-1} \), whence \( \bar{x} = \bar{x}^{[2,n]} \), \( \bar{v}_1 = \bar{v}_1^{[2,n]} \), \( \ldots, \bar{v}_{\ell-1} = \bar{v}_{\ell-1}^{[2,n]} \). Thus, the equalities (8) and (9) hold whenever \( \ell > 0 \).

Suppose that \( \ell > 1 \) and \( a_2 \in c(\bar{v}_{\ell-1}) \). Then we can apply Lemma 4a to remove all occurrences of \( a_2 \) from each of the factors \( \bar{v}_1^{[2,n]}, \ldots, \bar{v}_{\ell-1}^{[2,n]} \) in the representation (8) of \( b \) and from each of the factors \( \bar{x}^{[2,n]}, \bar{v}_1^{[2,n]}, \ldots, \bar{v}_{\ell-1}^{[2,n]} \) in the representation (9) of \( d \) so that neither \( b \) nor \( d \) will change. Thus, we get

\[
\begin{align*}
\ell,n\] & \quad b = \bar{u}_k \cdots \bar{u}_1 \bar{v}_1^{[3,n]} \cdots \bar{v}_{\ell-2}^{[2,n]} \bar{v}_{\ell-1} \bar{v}_\ell, \\
\ell,n\] & \quad d = \bar{u}_k \cdots \bar{u}_1 \bar{x}^{[3,n]} \bar{v}_1^{[2,n]} \cdots \bar{v}_{\ell-2}^{[2,n]} \bar{v}_{\ell-1} \bar{v}_\ell.
\end{align*}
\]

(10)

(11)

Again, the inclusions (7) imply that the equalities (10) and (11) hold also in the case when \( a_2 \notin c(\bar{v}_{\ell-1}) \).

Applying this argument \( \ell - 1 \) times to \( b \) and \( \ell \) times to \( d \), we eventually arrive at the equalities

\[
\begin{align*}
\ell,n\] & \quad b = \bar{u}_k \cdots \bar{u}_1 \bar{x}^{[\ell,n]} \cdots \bar{v}_{\ell-2}^{[3,n]} \bar{v}_{\ell-1} \bar{v}_\ell, \\
\ell,n\] & \quad d = \bar{u}_k \cdots \bar{u}_1 \bar{x}^{[\ell,n]} \cdots \bar{v}_{\ell-2}^{[3,n]} \bar{v}_{\ell-1} \bar{v}_\ell.
\end{align*}
\]

(12)

(13)

Now we can apply the symmetric argument on the left. If \( k > 0 \) and \( a_n \in c(\bar{u}_k) \), we use Lemma 4b to remove all occurrences of \( a_n \) from each of the factors \( \bar{u}_k, \ldots, \bar{u}_1 \) in the representation (12) of \( b \) and from each of the factors \( \bar{u}_k, \ldots, \bar{u}_1, \bar{x}^{[\ell+1,n]} \) in the representation (13) of \( d \) with no effect on the value of \( b \) or \( d \). This leads to the equalities

\[
\begin{align*}
\ell,n\] & \quad b = \bar{u}_k \bar{u}_k^{[1,n-1]} \cdots \bar{u}_1^{[1,n-1]} \bar{x}^{[\ell,n]} \cdots \bar{v}_{\ell-2}^{[3,n]} \bar{v}_{\ell-1} \bar{v}_\ell, \\
\ell,n\] & \quad d = \bar{u}_k \bar{u}_k^{[1,n-1]} \cdots \bar{u}_1^{[1,n-1]} \bar{x}^{[\ell+1,n-1]} \bar{v}_1^{[\ell,n]} \cdots \bar{v}_{\ell-2}^{[3,n]} \bar{v}_{\ell-1} \bar{v}_\ell,
\end{align*}
\]

that hold also in the case when \( a_n \notin c(\bar{u}_k) \). Applying the “left” argument \( k - 1 \) times to \( b \) and \( k \) times to \( d \), we finally arrive at the following equalities:

\[
\begin{align*}
\ell,n\] & \quad b = \bar{u}_k \bar{u}_k^{[1,n-1]} \cdots \bar{u}_1^{[1,n-2]} \bar{x}^{[\ell,n]} \cdots \bar{v}_{\ell-2}^{[3,n]} \bar{v}_{\ell-1} \bar{v}_\ell, \\
\ell,n\] & \quad d = \bar{u}_k \bar{u}_k^{[1,n-1]} \cdots \bar{u}_1^{[1,n-2]} \bar{x}^{[\ell+1,n-k]} \bar{v}_1^{[\ell,n]} \cdots \bar{v}_{\ell-2}^{[3,n]} \bar{v}_{\ell-1} \bar{v}_\ell.
\end{align*}
\]

(14)

(15)

Since \( k + \ell \geq n \), we have \( \ell + 1 > n - k \) whence \( \bar{x}^{[\ell+1,n-k]} = 1 \). Thus, from the representations (14) and (15) we conclude that \( b = d \), as required.

Combining Theorem 8, Proposition 2, and Theorem 1, we immediately obtain a solution to the finite basis problem for the Hecke–Kiselman monoids \( \mathcal{H}K_\Theta \) with \( \Theta \) satisfying \( \Theta_C \subseteq \Theta \subseteq \Theta_K \).

**Corollary 9.** Let \( n \geq 2 \) and let \( \Theta \) be a binary relation on the set \( \{1, 2, \ldots, n\} \) satisfying \( \Theta_C \subseteq \Theta \subseteq \Theta_K \).

a) If \( n = 2 \) or \( n = 3 \), then the monoid \( \mathcal{H}K_\Theta \) is finitely based and has the identities (3) or respectively (4) as an identity basis.

b) If \( n \geq 4 \), then the monoid \( \mathcal{H}K_\Theta \) is nonfinitely based.
Specializing Corollary 9 for $\Theta = \Theta_K$, we get a complete solution to the finite basis problem for the Kiselman monoids $K_n$.

**Corollary 10.** The monoids $K_2$ and $K_3$ are finitely based while the monoids $K_n$ with $n \geq 4$ are nonfinitely based.

**Remark.** Even though, for the sake of completeness, we have included the case $n = 2$ in the formulations of Theorem 8 and Corollaries 9 and 10, none of these results are new for the special case. Indeed, on the set $\{1, 2\}$, the relations $\Theta_C$ and $\Theta_K$ coincide, whence the Kiselman monoid $K_2$ is nothing but the Catalan monoid $C_3$ and so are all the Hecke–Kiselman monoids $H(K_\Theta)$ such that $\Theta_C \subseteq \Theta \subseteq \Theta_K$.

Finally, we present an example demonstrating that the above description of the equational theory of the Kiselman monoids may also be used to obtain information about their structural properties. The following result was announced in [9] while its first proof appeared in [16, Theorem 22]; this proof uses a specific representation of $K_n$ by integer $n \times n$-matrices and involves a clever estimation of an ad hoc numerical parameter of such matrices. In contrast, our proof is rather straightforward and works for all Hecke–Kiselman monoids $H(K_\Theta)$ with $\Theta \subseteq \Theta_K$.

**Corollary 11.** The Kiselman monoids $K_n$ are $J$-trivial.

**Proof.** Lemma 6b,c implies that $(xy)^n \sim_n (xy)^nx$ and $x(yx)^n \sim_n (yx)^n$. By Theorem 8 the identities $(xy)^n \Rightarrow (xy)^nx$ and $x(yx)^n \Rightarrow (yx)^n$ hold in the monoid $K_n$. It is well known that every monoid $M$ satisfying these two identities is $J$-trivial but, for completeness, we reproduce an elementary proof of this fact.

Let $a, b \in M$ and $MaM = MbM$. Then $a = qbr$ and $b = sat$ for some $q, r, s, t \in M$. Substituting the second equality into the first one, we get $a = qsat$ whence $a = (qs)^n a(tr)^n$. Since $M$ satisfies $(xy)^n \Rightarrow (xy)^nx$ and $x(yx)^n \Rightarrow (yx)^n$, we have

$$a = (qs)^n a(tr)^n = s(qs)^n a(tr)^n t = sat = b.$$  

Similar syntactic arguments can be used to reprove some other structural results in [16].

4. **Open questions**

4.1. **The finite basis problem for monoids $G(E)$**. Corollary 10 shows, in particular, that the “classic” Kiselman monoid $K_3$ originated in [14] is finitely based. Recall that $K_3$ has 18 elements and arises as the monoid of the form $G(E)$ where the underlying space $E$ is a normed space of infinite dimension. If $E$ is finite-dimensional, then the order of the monoid $G(E)$ is always less than 18 and the value of the order varies, depending on the two parameters: the dimension of $E$ and the dimension of $\{0\}$, the closure of the origin ($E$ is not assumed to be Hausdorff so that singletons need not be closed). Moreover, it follows from [14, Theorem 4.1] that each monoid of
the form \( G(E) \) for finite-dimensional \( E \) is isomorphic to the Rees quotient of \( K_3 \) over a certain non-singleton ideal. We present a classification of the monoids in Table 1, labelling them as in [14, Theorem 4.1].

### Table 1. The monoids \( G(E) \) for finite-dimensional \( E \)

| Label | Relation between \( n = \dim E \) and \( k = \dim \{0\} \) | Order of \( G(E) \) | The ideal \( I \) with \( K_3/I \cong G(E) \) |
|-------|--------------------------------------------------|------------------|----------------------------------|
| \( A_1 \) | \( n = 0 \) | 1 | \( K_3 \) |
| \( A_{15} \) | \( n = 1, k = 0 \) | 15 | \( \{a_2a_3a_1, a_2a_3a_1a_2, a_3a_1a_2, a_3a_2a_1\} \) |
| \( A_{16} \) | \( n \geq 2, k = 0 \) | 16 | \( \{a_2a_3a_1, a_2a_3a_1a_2, a_3a_2a_1\} \) |
| \( B_6 \) | \( n = k > 0 \) | 6 | \( K_3a_2K_3 \) |
| \( B_{16} \) | \( n - 1 = k > 0 \) | 16 | \( \{a_2a_3a_1a_2, a_3a_1a_2, a_3a_2a_1\} \) |
| \( B_{17} \) | \( n - 2 \geq k > 0 \) | 17 | \( \{a_2a_3a_1a_2, a_3a_2a_1\} \) |

It may be worth commenting on the topology of \( E \) in each of the non-trivial cases in Table 1. In case \( A_{15} \) the space \( E \) is nothing but the real line \( \mathbb{R} \) with the usual topology while case \( A_{16} \) corresponds to \( E = \mathbb{R}^n, n \geq 2 \), again with the usual topology. Case \( B_6 \) arises when \( E \) is nonzero and equipped with the so-called chaotic topology, i.e., the topology such that the only neighborhood of the origin is the whole space. In the remaining two cases (\( B_{16} \) and \( B_{17} \)), one has a mixture between the usual and chaotic topologies: the space \( E \) is isomorphic to the product of a \( k \)-dimensional chaotic space and the space \( \mathbb{R}^{n-k} \) with the usual topology.

It can be readily verified that each of the monoids in Table 1 satisfies the identity \( x^2 \simeq x^3 \) that fails in \( K_3 \). Therefore their equational theories differ from that of \( K_3 \), and one cannot solve the finite basis problem for these monoids by the arguments applied in Section 3. Of course, being homomorphic images of \( K_3 \), the monoids \( G(E) \) satisfy the identities (4) but it is known that these identities are not strong enough to ensure the finite basis property in every finite semigroup satisfying them. Indeed, Lee (unpublished) has observed that the identities (4) hold in the 6-element semigroup \( \mathcal{L} \) given by the presentation

\[
\mathcal{L} = \langle e, f \mid e^2 = e, f^2 = f, efe = 0 \rangle
\]

in the class of semigroups with 0, while Zhang and Luo [30] have proved that this semigroup is nonfinitely based. Furthermore, the fact that all monoids \( G(E) \) are Rees quotients of a finitely based \( J \)-trivial monoid cannot help too since Jackson [11] has constructed numerous examples of finitely based \( J \)-trivial monoids possessing nonfinitely based Rees quotients. Thus, we
see that the finite basis problem for each monoid of the form $G(E)$ with finite-dimensional $E$ should be treated separately.

Clearly, the 1-element monoid in case $A_1$ is finitely based. Also, the 6-element monoid $K_3/a_2K_3$ in case $B_6$ is finitely based: it can be shown that this monoid has the same equational theory as the 5-element monoid $K_2 = C_3$, and therefore, has the system (3) as its identity basis.

As for the monoids in the four remaining rows of Table 1, the finite basis problem for each of them remains open. In fact, only three cases are to be analyzed here since one can observe that the 16-element monoids in cases $A_{16}$ and $B_{16}$ are anti-isomorphic: the anti-isomorphism is induced by the anti-automorphism of the monoid $K_3$ that swaps the generators $a_1$ and $a_3$ and fixes the generator $a_2$. Thus, we formulate our first question.

**Question 1.** Consider the following ideals of the Kiselman monoid $K_3$:

- $I_1 = \{a_2a_3a_1, a_2a_3a_1a_2, a_3a_1a_2, a_3a_2a_1\}$,
- $I_2 = \{a_2a_3a_1, a_2a_3a_1a_2, a_3a_2a_1\}$,
- $I_3 = \{a_2a_3a_1a_2, a_3a_2a_1\}$.

Are the monoids $K_3/I_1$, $K_3/I_2$, $K_3/I_3$ finitely based?

### 4.2. The finite basis problem for general Hecke–Kiselman monoids.

Theorem 8 describes the equational theory of the Hecke–Kiselman monoids $HK_\Theta$ such that the anti-reflexive binary relation $\Theta$ satisfies $\Theta_C \subseteq \Theta \subseteq \Theta_K$, and Corollary 9 solves the finite basis problem problem for these monoids. However, questions of the same sort are of interest for an arbitrary $\Theta$. Here we discuss two open problems related to the equational theory of Hecke–Kiselman monoids.

Let $\Theta$ be an anti-reflexive binary relation on the set $V_n = \{1, 2, \ldots, n\}$. It is known [8, Theorem 16] that the Hecke–Kiselman monoid $HK_\Theta$ determines the graph $(V_n, \Theta)$ up to isomorphism, that is, for an arbitrary anti-reflexive binary relation $\Phi$ on the set $V_m = \{1, 2, \ldots, m\}$, the monoids $HK_\Theta$ and $HK_\Phi$ are isomorphic if and only if the graphs $(V_n, \Theta)$ and $(V_m, \Phi)$ are isomorphic.

On the other hand, Theorem 8 reveals that Hecke–Kiselman monoids with non-isomorphic underlying graphs can be *equationally equivalent*, i.e., can have the same equational theory. For instance, if $n \geq 3$, then the Catalan monoid $C_{n+1} = HK_{\Theta_C}$ and the Kiselman monoid $K_n = HK_{\Theta_K}$ are non-isomorphic but these monoids are equationally equivalent by Theorem 8.

This observation gives rise to the following question.

**Question 2.** Let $\Theta$ and $\Phi$ be anti-reflexive binary relations on the sets $V_n = \{1, 2, \ldots, n\}$ and respectively $V_m = \{1, 2, \ldots, m\}$. Under which necessary and sufficient conditions on the graphs $(V_n, \Theta)$ and $(V_m, \Phi)$ are the Hecke–Kiselman monoids $HK_\Theta$ and $HK_\Phi$ equationally equivalent?

Similarly, with respect to the finite basis problem, the next question appears to be quite natural.
Question 3. Let \( \Theta \) be an anti-reflexive binary relation on the set \( V_n = \{1, 2, \ldots, n\} \). Under which necessary and sufficient conditions on the graph \((V_n, \Theta)\) is the Hecke–Kiselman monoid \( HK_\Theta \) finitely based?

Questions 2 and 3 relate the study of Hecke–Kiselman monoids to the promising area of investigations whose aim is to interpret graph-theoretical properties within equational properties of semigroups. Such an interpretation may shed new light on complexity-theoretical aspects of the theory of semigroup identities as a whole and of the finite basis problem in particular; see [12] for an impressive instance of this approach. It is to expect, however, that Questions 2 and 3 may be rather hard—for comparison, recall that in general it is still unknown for which graphs \((V_n, \Theta)\) the Hecke–Kiselman monoid \( HK_\Theta \) is finite, and even for \( n = 4 \), the classification of graphs with finite Hecke–Kiselman monoids has turned out to be non-trivial, see [1].

### 4.3. The finite basis problem for involuted Kiselman monoids.

A semigroup \( S \) is called an involuted semigroup if it admits a unary operation \( s \mapsto s^* \) (called involution) such that \((st)^* = t^*s^* \) and \((s^*)^* = s \) for all \( s, t \in S \). Whenever \( S \) is equipped with a natural involution, it appears to be reasonable to investigate the identities of \( S \) as an algebra of type \((2,1)\); see [2, 3] for numerous examples of recent studies along this line. Observe that adding an involution may radically change the equational properties of a semigroup: a finitely based semigroup may become a nonfinitely based involuted semigroup and vice versa. Infinite examples of this sort have been known since 1970s (see [28] Section 2 for references and a discussion); more recently, Jackson and the second-named author [13] have constructed a finitely based finite semigroup that becomes a nonfinitely based involuted semigroup after adding a natural involution, while Lee [17] has shown that the 6-element nonfinitely based semigroup \( L \) defined by [16] admits an involution under which it becomes a finitely based involuted semigroup.

Since the relations in (2) are preserved by the map \( a_i \mapsto a_{n-i+1} \), this map uniquely extends to an involution of the Kiselman monoid \( K_n \); in fact, this is the only anti-automorphism of \( K_n \); see [16] Proposition 20b]. In the same way, this map extends to an involution of the Catalan monoid \( C_{n+1} \). To the best of our knowledge, the equational properties of Kiselman and Catalan monoids treated as involuted semigroups have not been considered so far, and the examples mentioned in the preceding paragraph indicate that these properties need not necessarily follow the patterns revealed by the results of Section 3. Thus, we conclude with a pair of interrelated questions.

**Question 4.** Are the Kiselman monoid \( K_n \) and the Catalan monoid \( C_{n+1} \) equationally equivalent as involuted semigroups?

It is easy to see that the epimorphism \( K_n \rightarrow C_{n+1} \) constructed according to Proposition 5 is in fact a homomorphism of involuted semigroups so that \( C_{n+1} \) satisfies all involuted semigroup identities that hold in \( K_n \). The converse, however, is very far from being clear.
**Question 5.** a) For which $n$ is the Kiselman monoid $K_n$ finitely based as an involuted semigroup?

b) For which $n$ is the Catalan monoid $C_n$ finitely based as an involuted semigroup?

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