Duality approach to one-dimensional degenerate electronic systems

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We investigate the possible classification of zero-temperature spin-gapped phases of multicomponent electronic systems in one spatial dimension. At the heart of our analysis is the existence of non-perturbative duality symmetries which emerge within a low-energy description. These dualities fall into a finite number of classes that can be listed and depend only on the algebraic properties of the symmetries of the system: its physical symmetry group and the maximal continuous symmetry group of the interaction. We further characterize possible competing orders associated to the dualities and discuss the nature of the quantum phase transitions between them. Finally, as an illustration, the duality approach is applied to the description of the phases of two-leg electronic ladders at incommensurate filling.

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I. INTRODUCTION

Conformal Field Theory (CFT) is a powerful approach which determines the low-energy properties of one-dimensional (1D) quantum systems or two-dimensional classical systems close to criticality [1, 2]. In particular, it allows for a complete classification of 1D gapless phases and quantum
critical points with Lorentz invariance [3]. The low-energy gapless spectrum of the corresponding 1D lattice model with continuous symmetry, like the spin-1/2 Heisenberg chain, is described in terms of representations of certain Kac-Moody (KM) current algebra [4]. This affine symmetry fixes the operator content of the system and the scaling dimensions of the operators are in turn determined by the conformal invariance of the underlying Wess-Zumino-Novikov-Witten (WZNW) model built from the currents of the affine symmetry [2]. The 1D gapless phases are then labelled by the central charge $c$ of the corresponding WZNW CFT. The simplest example of this description is the Luttinger liquid universality class which accounts for the low-energy properties of 1D metals [5–7]. Such a CFT has a U(1) affine KM symmetry with central charge $c = 1$. The critical theory of half-integer Heisenberg spin chains corresponds to the SU(2)$_1$ WZNW whereas the family of integrable spin chains of arbitrary spins $S$ admits for a critical theory the SU(2)$_{2S}$ WZNW model with central charge $c = 3S/(S + 1)$ [8].

In sharp contrast to gapless systems, a general classification of all possible zero temperature 1D gapped phases is still lacking despite numerous works in the past two decades. To the best of our knowledge, there exists no general symmetry based arguments that allow for a quantitative description of 1D quantum problems with a spectral gap. A natural starting approach is to identify the massive phases that occur in the vicinity of quantum critical points or close to gapless phases which are described by some known CFTs. Very special perturbations could be singled out which correspond to massive integrable deformations of the original CFT [9]. The integrability provides, in turn, a non-perturbative basis to construct the massive quasiparticles defining the low-energy spectrum of the fully gapped phase. The stability of these excitations can then be investigated perturbatively with respect to other non-integrable generic perturbations [10] and it will define the extension of the massive phase. A second more phenomenological approach relies directly on orders by considering very special ones and the use of extended symmetries which unify them. A typical example is the SO(5) theory for the competition between $d$-wave superconductivity and antiferromagnetism, where the order parameters of $d$-wave superconductivity and antiferromagnetism are combined to form a unified order parameter quintet [11]. Several different competing orders may be transformed into each other under this extended global symmetry.

In this work, we will combine these two approaches to describe spin-gapped phases for a large class of 1D fermionic models where the gap is exponentially small in the weak coupling regime. More precisely, we will be concerned with weakly interacting $N$-component fermionic models such as $n$-leg electronic spin-1/2 ladders ($N = 2n$) [6, 7] or one-dimensional fermionic cold atoms with hyperfine spin $F$ ($N = 2F + 1$) [12, 13]. In absence of interactions, those systems display a large
degeneracy, with a corresponding large symmetry group mixing microscopic degrees of freedom, to be called \( G \) in the following. For the aforementioned models, the group \( G \) will be \( \text{SU}(N) \) or \( \text{SO}(2N) \), depending on the filling. This degeneracy allows for many possible competing orders. The low-energy properties of the non-interacting system are governed by a CFT built on the group \( G \): the \( G_k \) WZNW CFT. For the applications to coupled fermionic chains, the level of this CFT is \( k = 1 \), but most of our analysis can be carried out in the general \( k \) case. Interactions generically break the symmetry \( G \) down to a subgroup \( H \) which is the “true” physical symmetry of the problem. The degeneracy lifting which arises from the interactions results in general in the opening of a spectral gap, which is accompanied by the stabilization of one of the possible orders.

Questions we will address in the following are: what kind of competing orders can be stabilized? How do they depend on the algebraic properties of the groups involved in the symmetry breaking scheme \( G \sim H \)? Can these orders be related between themselves? And what is the nature of the (zero-temperature) quantum phase transitions?

In this paper, we will investigate these questions by means of a low-energy approach where the symmetry breaking scheme is described by a \( G_k \) WZNW model perturbed by marginal current-current interactions with \( H \) invariance. The main result of this work is the fact that this general class of models possesses hidden non-trivial duality symmetries which are associated with the underlying competing orders. These dualities are emergent, and exact, in the continuum limit. They can be viewed as the naive generalization for continuous symmetry groups of the well-known Kramers-Wannier (KW) duality symmetry of the 1D quantum Ising model. Some duality symmetries are not new and have already been found in specific models such as the two-leg electronic spin-1/2 ladders at half-filling [14, 15] or away from half-filling [16–18], in two-leg spin ladder with four-spin exchange interactions [19, 20], and also in 1D multicomponent cold fermions [12, 13]. Most of them have been revealed within a one-loop renormalization group (RG) approach of the interactions with the emergence of a dynamical symmetry enlargement (DSE) [14]. Such DSE corresponds to a situation where a Hamiltonian is attracted under RG flow to a manifold possessing a higher symmetry than the original bare theory. The main interest of this DSE scenario stems from the fact that the isotropic RG ray is usually described by an integrable field theory. The integrability, in turn, leads to the description of the low-lying excitations and the determination of the physical properties of the corresponding spin-gapped phases [14, 21–23]. However, as it will be shown

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\(^1\) This definition of the group \( G \) as the microscopic degeneracy group of the non-interacting theory, \( G^{\text{micro}} \), is not fully correct. A more rigorous definition of the group \( G \) will be given in section II: it is the maximal symmetry group of the \textit{interacting} sector of the theory, which is of course closely related to \( G^{\text{micro}} \).
in the following, the duality symmetries are not restricted to a two-leg ladder or specific to the one-loop RG calculation but correspond to a general powerful non-perturbative approach in 1D. In particular, they can be classified and depend only on the algebraic properties of \( G \) and of the actual symmetry \( H \) of the lattice model. As it will be shown, the resulting list of dualities turns out to be in correspondence with the classification of \( \mathbb{Z}_2 \) graduations of semi-simple Lie algebra. The interplay between these dualities and the DSE phenomenon will also be clarified in this work. In simple words, the dualities correspond to the different possible DSEs of the problem. The existence of these dualities symmetries enables one to relate different possible competing orders and to shed light on the global structure of the phase diagram of weakly-interacting fermionic models. Of course, our work does not solve the general problem of classification of zero-temperature 1D gapped phases but, at least, the duality approach provides a classification of a subset of gapped phases from symmetry arguments. Moreover, the nature of the quantum phase transitions between these orders can also be determined, within our approach, by investigating the physics along the self-dual manifolds.

The rest of the paper is organized as follows. The general low-energy effective theory, that we will study in this work, is presented in section II. Section III contains our most important results with the definition and characterization of the duality symmetries. These hidden discrete symmetries are introduced from two different complementary approaches. A first definition stems from the covariance of the low-energy effective Hamiltonian. The characterization and classification of these dualities are then deduced in section III A. A second definition is presented in section III B in light of the DSE phenomenon where duality symmetries are viewed as exact isometries of the RG beta-function. The nature of quantum phase transitions between different orders is then investigated. In section IV, the duality approach is applied to weakly-interacting multicomponent fermionic models for incommensurate filling. The special example of a generalized two-leg ladder is then considered in section V. Finally, we conclude in section VI and some technical details on the duality approach are presented in an appendix.

II. GENERAL LOW-ENERGY EFFECTIVE HAMILTONIAN

Let us consider \( n \) chains of weakly-interacting lattice spin-1/2 fermions \( \psi_{i,a} \) \( a = 1, \ldots, N \) where \( N = 2n \) is the total number of internal degrees of freedom of the problem. The generic model, that we have in mind in this paper, is described by a lattice Hamiltonian of the Hubbard type with
contact interactions:

\[ H = -t \sum_i \sum_a \left( c_{i,a}^\dagger c_{i+1,a} + \text{H.c.} \right) + \sum_i \sum_a U_{\{a\}} \left[ c_{i,a_1}^\dagger c_{i,a_2}^\dagger c_{i,a_3} c_{i,a_4} \right], \]

(1)

where \( t \) is the hopping term and the independent couplings (on-site couplings \( U_{\{a\}} \)) depend on the precise model under consideration, that can be specified by fixing its physical symmetry group \( H \). In this respect, many other terms could be added to model (1), without affecting the following discussion which is mostly based on symmetry considerations. For example, more general short-range interaction terms could be considered as long as they respect the symmetry \( H \) of the model. More concrete examples will be discussed in section V.

In the continuum limit, the non-interacting spectrum around the two Fermi points \( \pm k_F \) is linearized and gives rise to left- and right-moving Dirac fermions \( \Psi_{al(r)} \) [6, 7]. The left (right)-moving fermions are holomorphic (antiholomorphic) fields of the complex coordinate \( z = v_F \tau + i x \) (\( \tau \) being the imaginary time and \( v_F \) is the Fermi velocity): \( \Psi_{al(\bar{z})} \), \( \Psi_{ar(z)} \). They are normalized by the following operator product expansion (OPE):

\[
\Psi_{ar}^\dagger (\bar{z}) \Psi_{br} (\bar{\omega}) \sim \frac{\delta_{ab}}{2\pi (\bar{z} - \bar{\omega})},
\]

\[
\Psi_{al}^\dagger (z) \Psi_{bl} (\omega) \sim \frac{\delta_{ab}}{2\pi (z - \omega)}.
\]

(2)

In the weak-coupling regime, the effect of the interactions is described by the following general low-energy effective Hamiltonian density:

\[ \mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{\text{int}} = -iv_F \left( \Psi_{ar}^\dagger \partial_x \Psi_{ar} - \Psi_{al}^\dagger \partial_x \Psi_{al} \right) + g_\alpha J_{\alpha}^A d_{AB}^\alpha J_{\beta}^B , \]

(3)

where velocity renormalization terms and interactions with the same chirality will be neglected in this work and \( g_\alpha \) accounts for the coupling constants of the fermionic models. In the following, repeated indices are summed throughout this paper unless stated otherwise. The continuous symmetry of the non-interacting Hamiltonian of Eq. (1) is \( U(N) = U(1) \times SU(N) \) where \( U(1) \) is the usual global charge symmetry \( (c_{i,a} \rightarrow e^{i\phi} c_{i,a}) \) and the \( SU(N) \) group describes the continuous symmetry of the remaining degrees of freedom, dubbed in the following “spin” sector for simplicity. This basis, which singles out the charge degrees of freedom, is natural for incommensurate filling since a spin-charge separation is expected in the continuum limit so that model (3) decomposes into two commuting pieces: \( \mathcal{H} = \mathcal{H}_c + \mathcal{H}_s \). In this case, the charge degrees of freedom display metallic properties in the Luttinger liquid universality class [5–7]. The Hamiltonian in the spin sector depends only on the currents \( J_{\alpha}^A_{l(r)} = 2\pi \Psi_{al(r)}^\dagger T_{\alpha}^A \Psi_{bl(r)} \), \( T_A^A \ (A = 1, \ldots, N^2 - 1) \) being the generators.
of SU(N) in the fundamental representation (we choose the normalization $\text{Tr}(T^AT^B) = \delta^{AB}$) and reads as follows:

$$H_s = \frac{v_F}{4\pi(N+1)} (\mathcal{J}_R^A \mathcal{J}_R^A + \mathcal{J}_L^A \mathcal{J}_L^A) + g_\alpha \mathcal{J}_R^A d_{AB}^{\alpha} \mathcal{J}_L^B,$$

(4)

where, in the following, normal ordering is always implied for the non-interacting term. Model (4) is nothing but the SU(N)$_1$ WZNW model perturbed by a current-current interaction.

In contrast, there is no spin-charge separation in the half-filled case and the charge degrees of freedom cannot be disentangled from the spin ones due to an umklapp process. A good starting point is then to consider the maximal global continuous symmetry of the non-interacting lattice Hamiltonian (1) i.e. $G = \text{SO}(2N)$. Physically, the appearance of the group SO(2N) can be motivated as follows: due to particle-hole symmetry present at half-filling, the 2N local states \{${c}_{i,a}^{\dagger} |GS\rangle, {c}_{i,a} |GS\rangle$\} can play a symmetric role, where $|GS\rangle$ is a particle-hole symmetric many-body ground state. In the continuum limit, the SO(2N) symmetry can be revealed by introducing 2N real (Majorana) fermions from the N Dirac ones: \(\Psi_{aL(R)} = (\xi_a + i \xi_{a+N})_{aL(R)}/\sqrt{2}\).

The non-interacting Hamiltonian of model (3) then takes a manifestly SO(2N) invariant form:

$$H_0 = -iv_F (\xi_{ar} \partial_x \xi_{ar} - \xi_{al} \partial_x \xi_{al})/2.$$  

Within this representation, the currents $\mathcal{J}^A_{aL(R)}$ in Eq. (3) express as fermionic bilinears: $\mathcal{J}^A_{aL(R)} = i2\pi \xi_{aL(R)} \xi_{bL(R)}$, $A = (a, b), 1 \leq a < b \leq 2N$.

The Hamiltonian (3) contains only marginal interactions – this is the main, though important, restriction to our study – and includes several competing orders which are encoded in the physical symmetry of the problem, the group H. In Eq. (3), these symmetries are taken into account by the set of symmetric matrices $D = \{d^\alpha\}$ that commute with all elements of H: $[d^\alpha, H] = 0$. On top of continuous symmetries, the Hamiltonian (3) has discrete symmetries such as lattice symmetries, chiral invariance, charge conjugation and parity. In particular, those discrete symmetries are responsible for the matrices $d^\alpha$ to be real and symmetric. In addition, model (3) should be stable and renormalizable under the RG approach and we thus require the set of matrices $d^\alpha$ to be closed under anticommutation: $\{D, D\} \subset D$ [24, 25].

In the following, we will generalize the problem and make abstraction of the underlying fermions to consider the G$_k$ WZNW model perturbed by current-current interaction with Hamiltonian density:

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{\text{int}} = \frac{v_F}{4\pi(k + g^\vee)} (\mathcal{J}^A_{aL} \mathcal{J}^A_{aL} + \mathcal{J}^A_{aL} \mathcal{J}^A_{aL}) + g_\alpha \mathcal{J}^A_{aL} d_{AB}^{\alpha} \mathcal{J}^B_{aL},$$

(5)

where $A, B = 1, \ldots, \dim(G)$ and $g^\vee$ is the dual Coxeter number of G (for instance $g^\vee = N$ for G = SU(N) and $g^\vee = 2N - 2$ for G = SO(2N)) [2]. In this paper, G is a regular simple Lie group
and for application to weakly interacting fermionic models (1), one has $G = \text{SO}(2N)$ with the KM level $k = 1$ for model (3) at half-filling and $G = \text{SU}(N)$ and $k = 1$ for the spin degrees of freedom of model (4) away from half-filling. The WZNW description (5) has to be supplemented by the defining $G_k$ KM current algebra:

$$ \mathcal{J}_A^A(z)\mathcal{J}_B^B(w) \sim \frac{k\delta^{AB}}{(z-w)^2} + if^{ABC} \frac{\mathcal{J}_C^C(w)}{z-w},$$

(6)

where a similar relation for right-moving currents holds with the replacement $z, w \rightarrow \bar{z}, \bar{w}$. In Eq. (6), we choose a basis of the Lie algebra $\mathfrak{g}$ of $G$ for which the Killing form is the identity and $f^{ABC}$ denote the structure constants of $\mathfrak{g}$ which are normalized according to: $f^{ABC}f^{ABD} = 2\eta^{\alpha\beta}\delta_{CD}^\alpha$. In absence of interactions, i.e. when $g_\alpha = 0$, the theory (5) is conformally invariant and its critical properties are described by the $G_k$ WZNW CFT with central charge $c(G,k) = k\dim(G)/(k+g^\vee)$. As far as global symmetries are concerned, the non-interacting model displays a $G_L \times G_R$ symmetry, which is generated by the charges $Q^A_{l(r)} = \int dx \mathcal{J}^A_{l(r)}$. One obvious effect of the interactions $g_\alpha \neq 0$, on top of reducing the symmetry down to $H$, is to couple the left and right sectors of the CFT: the continuous part of the physical symmetry group $H$ of model (5) is generated by a subset of the diagonal charges (i.e. that rotate simultaneously the left and right sectors of the theory) $Q^A_L + Q^A_R$, so that more rigorously $H = (H_L \times H_R)_{\text{diag}}$. For generic $g_\alpha \neq 0$, we will investigate situations where the interaction is marginally relevant so that a spectral gap opens and the conformal symmetry is lost. 2 One important point about the interacting part of the Hamiltonian (5) is that it contains a special, G-symmetric ray, i.e. there exist some couplings $g_\alpha^0$ such that $g_\alpha^0d^\alpha = \mathds{1}$ ($\mathds{1}$ being the $\dim(G) \times \dim(G)$ identity matrix). If this were not the case, it would mean that the Hamiltonian could be broken into smaller, commuting pieces, built on subalgebras of $G$, to which our approach should be applied separately. Along the G-symmetric ray, the Hamiltonian (5) simplifies as follows:

$$ \mathcal{H}_G = \frac{v_F}{4\pi(k+g^\vee)} \left( \mathcal{J}_R^A \mathcal{J}_R^A + \mathcal{J}_L^A \mathcal{J}_L^A + g \mathcal{J}_R^A \mathcal{J}_L^A \right), \hspace{1cm} (7)$$

which is the Gross-Neveu (GN) [28] or Thirring model [29] built on the group $G$ at level $k$. On this special ray, the global symmetry of model (7) is given more rigorously by $G = (G_L \times G_R)_{\text{diag}}$, which is generated by the diagonal charges $Q^A_L + Q^A_R$. This last remark finally provides us with

2 An exception corresponds to the case where the marginal current-current of model (5) is described in terms of currents with different KM levels. Even though the perturbation is marginally relevant the conformal invariance is restored at an intermediate infrared fixed point. The so-called chirally stabilized spin-liquids are examples of this class of models [26, 27].
the correct way to introduce rigorously the group $G$, namely the maximal symmetry group of the interacting model (5).

III. EMERGENT DUALITIES

In this section, we study duality transformations on the general model (5). After a brief survey of basic facts about the famous KW duality in the quantum Ising model, we start by introducing duality symmetries that map the set of theories defined by Eq. (5) onto itself, from a rather algebraic point of view. This definition will enable us to obtain a classification of dualities depending on the nature of the group $G$. Then, we study the interplay of such dualities with the phenomenon of DSE. For readers not interested in the technical details, a brief summary of our main results is presented at the end of this section.

A. Dualities and covariance of the Hamiltonian

Dualities are precious tools that help our understanding of strongly correlated systems. Given a set of theories depending on some parameters \( \{g_\alpha\} \), with Hamiltonian \( H_{g_\alpha}[\phi] \) and fluctuating fields \( \{\phi(x)\} \), a duality \( \Omega \) is a symmetry operation that relates different points in this set of theories, \( H_{g_\alpha}[\phi] = H_{\tilde{g}_\alpha}[\tilde{\phi}] \), where parameters as well as fields are acted upon by the duality: \( \Omega(g_\alpha) = \tilde{g}_\alpha \) and \( \Omega(\phi) = \tilde{\phi} \) with \( \Omega^2 = \mathbb{1} \). This ensures that the whole physical content of the theory \( H_{g_\alpha} \) can be deduced from that of the theory \( H_{\tilde{g}_\alpha} \): correlators are invariant, \( \langle \ldots \tilde{\phi}(x) \ldots \rangle_{\tilde{g}_\alpha} = \langle \ldots \phi(x) \ldots \rangle_{g_\alpha} \), where \( \langle \ldots \rangle_{g_\alpha} \) is the quantum average defined by Hamiltonian \( H_{g_\alpha} \).

1. Warming up: Kramers-Wannier duality

The 1D Ising model in a transverse magnetic field provides us with maybe the most famous example of such a duality, the so-called KW duality. It is worth recalling basic facts about this model, since it reveals striking similarities with the more general situation studied later. Its lattice Hamiltonian is:

\[
H = \sum_i \left[ g_1 \sigma_i^z \sigma_{i+1}^z + g_2 \sigma_i^2 \right],
\]

where \( \sigma_i^a \) are Pauli matrices on site \( i \). Defining the operators \( \Omega(\sigma_i^z) = \mu_i^z = \prod_{j<i} \sigma_j^z \) and \( \Omega(\sigma_i^x) = \mu_i^x = \sigma_i^z \sigma_{i+1}^z \) allows a dual description of this model, \( H = \sum_i \left[ g_2 \mu_i^z \mu_{i+1}^z + g_1 \mu_i^x \right] \). The KW duality \( \Omega \) simply exchanges the couplings, \( \Omega(g_{1,2}) = g_{2,1} \). At the self-dual point \( g_1 = g_2 \), the theory is
critical and described by a $c = \frac{1}{2}$ CFT which is the free Majorana fermion theory \cite{2, 6}. Moving away from the self-dual point amounts to adding a mass term to the fermions, yielding the following Hamiltonian density in the continuum limit:

$$H = -\frac{i\nu}{2}(\xi_\alpha \partial_x \xi_\alpha - \xi_\lambda \partial_x \xi_\lambda) + im\xi_\alpha \xi_\lambda,$$  \hspace{1cm} (9)$$

with $m \propto g_1 - g_2$. Then the KW duality exchanges the two massive phases of the Ising model: when $m > 0$, it is in its ordered phase, $\langle \sigma^z \rangle \neq 0$, $\langle \mu^z \rangle = 0$, while the disordered phase is characterized by $\langle \mu^z \rangle \neq 0$ and $\langle \sigma^z \rangle = 0$. On top of Majorana fermions, there are two other fluctuating fields in the theory, $\sigma(x)$ and $\mu(x)$, that are the continuum limits of the Ising spin $\sigma^z$, and of the disorder operator $\mu^z$, respectively. The KW duality has the following possible representation on the continuous fields: $\xi_\lambda \rightarrow -\xi_\lambda$, $\xi_\alpha \rightarrow \xi_\alpha$, $\sigma \leftrightarrow \mu$ which is indeed a symmetry of model (9) with $m \rightarrow -m$. Note also that if the original $\mathbb{Z}_2$ symmetry is broken (e.g. by adding a magnetic field along $\sigma^z$), then the KW duality no longer holds. More precisely, if one then requires the model to be globally invariant under KW (up to a redefinition of the couplings), one is forced to include a new term in the Hamiltonian, namely a term proportional to $\mu^z$ which is not local with respect to the original building blocks $\sigma^z_a$. In other word, preserving the possibility of KW duality requires to enlarge drastically the class of perturbations. This is a general fact: as we will see, for the generic model (5), the set of allowed dualities strongly depends on the symmetry group of the model. Moreover, a quantum critical behavior is also likely to emerge at the self-dual points. In this respect, the KW duality approach has been fruitfully exploited in the past to obtain the massive phases and quantum phase transitions of two-leg spin ladders for weak interchain interactions \cite{30}.

2. Characterization and classification of dualities

Motivated by the example of the Ising model, we now investigate whether the theories defined by (5) admit non-trivial transformations that relates different points in this set of theories. Since the models that are considered are built out of the current fields, it is natural to define such a transformation – call it $\Omega$ – by its action on the currents, $J \rightarrow \tilde{J} = \Omega(J)$. In close parallel to the Ising model, we demand that the model defined in terms of the new variables $\tilde{J}$, corresponds to the original one up to a redefinition of the couplings, $g \rightarrow \tilde{g}$:

$$H(\tilde{g}, \tilde{J}) = H(g, J).$$  \hspace{1cm} (10)$$

We will see shortly that such transformations can exist, and, moreover, under very general assumptions, are involutive; hence it is sensible to coin them dualities.
In this paper, we will restrict ourselves to the simplest case of transformations that act linearly on the currents. Imposing that the KM algebra (6) be invariant prevents left-right mixing of the currents, thus resulting in: 

\[ \tilde{J}^A_{(l)} = (\omega_{(l)}^{AB}) J^B_{(l)}, \quad \omega_{(l)} = \text{dim}(G) \times \text{dim}(G) \text{ matrices representing the action of } \Omega \text{ on the currents.} \]

From the invariance of the OPE (6), \(\omega_{(l)}\) are necessary orthogonal matrices. By writing \(\omega_l = \omega \omega_r\), with \(\omega = \omega_l \omega_l^{-1}\), we recognize that the transformation can be decomposed into the product of a pure diagonal G-rotation \(\omega\) (i.e. that acts simultaneously on the right- and left- chiral sectors) and of a transformation acting only in the left sector, \(\omega\). It can be shown (see Appendix A 1 for details) that \(\omega\) and the diagonal rotation \(\omega_r\) have to leave the Hamiltonian globally invariant (i.e., they separately fulfill Eq. (10)). Therefore, the problem of finding all possible dualities factors out in two independent sub-problems. The diagonal rotation \(\omega_r\) corresponds to a mere change of basis in the representation of the Hamiltonian. If the low-energy theory that one considers is the continuum limit of some lattice model, this diagonal rotation is the continuum representation of a local, unitary transformation of the lattice operators. Here, we will not be interested in these diagonal transformations. We mention that such transformations have been studied on the lattice in the context of the two-leg Hubbard ladder [15] and in the two-leg spin ladder with ring-exchange interactions [19, 20]. One can convince oneself that they appear whenever there is some accidental degeneracy in the decomposition of the Lie algebra of G in irreducible representations of H. 3

In the following, we will investigate the more interesting transformations that affect only one chirality sector, setting therefore \(\omega_r = \mathbb{1}\). These transformations will be called dualities, and generically denoted by \(\Omega\). The matrix \(\omega\) is the representation of \(\Omega\) on the KM current fields; \(\Omega\) twists the current algebra:

\[ \tilde{J}^A_{l} \xrightarrow{\Omega} \omega^{AB} J^B_{l}, \]

\[ \tilde{J}^A_{r} \xrightarrow{\Omega} J^A_{r}. \]  

Contrarily to the diagonal rotations, these transformations that affect differently the left- and right-sector of the theory cannot be viewed as the continuum limit of a local transformation of the underlying lattice fermions. In this sense, they are non-local, like the KW duality in the Ising model which involves non-local transformation of lattice operators. Moreover, in general and contrarily to the KW duality, there is no known, simple lattice realization of the dualities in terms

\[ ^3 \text{An archetypical example of this is the spin-orbital model, in which spin and orbital degrees of freedom are treated on an equal footing, both transforming under SU(2) rotations, with } H=\text{SU(2)}_{\text{orb}} \times \text{SU(2)}_{\text{spin}} \subset G=\text{SU(4)}. \text{ Then, there is a SU(4) automorphism exchanging spin and orbital degrees of freedom. We will come back to this example in section V.} \]
of fractional variables (akin to the Ising spin and disorder operators in the case of KW duality, that are fractional with respect to the fermion) these transformations are thus in general emergent in the continuum limit.\footnote{Note that our study does by no means exclude the possibility of exact, complicated, necessarily non-local lattice realizations of dualities in some specific examples.}

We now proceed to characterize the set $\mathcal{D}$ of possible dualities $\Omega$. It turns out that the structure of $\mathcal{D}$ is remarkably simply extractible from few basis data: the physical symmetry group $H$, and the maximal symmetry group $G$ of the interaction.

First, one observes that $\omega$ should preserve the KM current algebra (6); it results that $\Omega$ is an automorphism of $g$. The covariance of the interaction term imposes further constraints: there should exist couplings $\tilde{g}_\alpha = \Omega(g_\alpha)$, such that

$$g_\alpha d^\alpha = \tilde{g}_\alpha d^\alpha \omega.$$  \hspace{1cm} (12)

Multiplying this relation by constant symmetric matrices and taking a trace, we see that the transformed couplings $\tilde{g}_\alpha$ are linearly related to initial couplings $g_\alpha$. Moreover, condition (12) is strong enough to completely characterize allowed dualities: it is possible to show (see Appendix A.1) that the set of allowed dualities is given by:

$$\mathcal{D} = \mathcal{C}(H)|_{\text{inv}},$$  \hspace{1cm} (13)

i.e., the matrix $\omega$ should be the representation (on the adjoint representation of $G$) of those elements of the center of $H$\footnote{We recall that the center of a group $H$ is defined from the elements which commute with all elements of $H$.}, that are involutive, $\omega^2 = \mathbb{1}$. This characterization of $\mathcal{D}$ has direct, simple consequences. First, it shows that there is a finite number of dualities. $\mathcal{D}$ is indeed isomorphic to the finite group $(\mathbb{Z}_2)^n$, $n$ being some positive integer, and has $2^n$ elements. Furthermore, to each duality, there corresponds a symmetry operation $S$ that belongs to the physical symmetry group $H$, which has the following properties: $S^2 = \mathbb{1}$ ($S$ is a $\mathbb{Z}_2$ symmetry), and $S$ commutes with all other symmetry operations of $H$. Explicitly, a representation of $S$ can be given as: $S = \omega \times \omega$, where the two operands of the tensor product act on the left- and right- sector of the theory, respectively. A duality therefore corresponds to the “square root” of some exact involutive symmetry of the lattice Hamiltonian that commutes with all other symmetries.

Let us denote by $\Omega_a$ the different dualities, $\mathcal{D} = \{\Omega_a\}$. Using the properties $[\omega_a, \omega_b] = 0$ and $\omega^2_a = \mathbb{1}$, we deduce that there exists a basis for the currents that simultaneously diagonalizes the action of dualities on the currents, the eigenvalues being $\pm 1$. In the following we will work in such
This yields a more physical interpretation of dualities. Introducing Noether charge and current densities associated to the G-invariance, $J^A_0 = J^A_k + J^A_l$ and $J^A_1 = J^A_k - J^A_l$ (the subscripts correspond to space-time indices, $x_0 \equiv t$ and $x_1 \equiv x$, and in a G-invariant theory $\partial_t J^A_0 + \partial_x J^A_1 = 0$), they transform as:

$$
\Omega_a (J^A_0) = J^A_0, \quad \Omega_a (J^A_1) = J^A_1, \quad \epsilon^A_a = +1,
$$

$$
\Omega_a (J^A_0) = J^A_1, \quad \Omega_a (J^A_1) = J^A_0, \quad \epsilon^A_a = -1.
$$

(15)

In a good basis, a duality just amounts to the exchange of the role played by Noether charges and currents associated to G-invariance. Thus, it is tempting to view dualities as generalizations of the U(1) duality that exchanges the role played by electric and magnetic charges (this is known as "T-duality" in the context of string theory [31]), to a non-Abelian theory, with the further constraint that they have to be compatible with the original H-invariance of the microscopic theory. This will have important consequences for the labeling of states by quantum numbers. Indeed, if the eigenvalues of the charge operator $Q^A_0 = \int dx J^A_0(x)$ happen to be (approximate) good quantum numbers in a phase $\mathcal{M}$, and if $\epsilon^A_a = -1$, it follows that the (approximate) good quantum numbers in the dual phase $\Omega_a(\mathcal{M})$ will be current quantum numbers: quasiparticles carry generalized currents.

We now make use of a mathematical result on Lie algebras that allows for a complete classification of dualities. Given $\Omega_a \in \mathcal{D}$, using Eq. (14), it is possible to decompose the Lie algebra $\mathfrak{g}$ of G into two orthogonal subspaces: $\mathfrak{g} = \mathfrak{g}^\parallel \oplus \mathfrak{g}^\perp$, $\mathfrak{g}^\parallel$, $\mathfrak{g}^\perp$ (respectively) being generated by those elements of $\mathfrak{g}$ with $\epsilon^A_a = +1$ ($\epsilon^A_a = -1$ respectively). Mathematically, this decomposition can be associated to any involutive automorphism ($\omega$ in our case), which is called a $\mathbb{Z}_2$-grading of $\mathfrak{g}$ [32]. Each $\mathbb{Z}_2$ grading $\Omega$ is characterized by an invariant subspace $\mathfrak{g}^\parallel$, which can be shown to have the structure of a semi-simple Lie algebra, and is defined by $\Omega(X) = X, \forall X \in \mathfrak{g}^\parallel$. One has the orthogonal decomposition (with respect to the Killing form) $\mathfrak{g} = \mathfrak{g}^\parallel \oplus \mathfrak{p}$, with $\Omega(X) = -X, \forall X \in \mathfrak{p}$, and the Lie structure of $\mathfrak{g}$ breaks up as follows under the action of $\Omega$:

$$
[\mathfrak{g}^\parallel, \mathfrak{g}^\parallel] = \mathfrak{g}^\parallel, \quad [\mathfrak{g}^\parallel, \mathfrak{p}] = \mathfrak{p}, \quad [\mathfrak{p}, \mathfrak{p}] = \mathfrak{g}^\parallel.
$$

(16)

Now, there exists a complete classification of $\mathbb{Z}_2$-gradings for simple Lie algebras [33]. This means that once G is given, irrespective of the physical symmetry group H, one knows what are the different possible types of dualities: they are of the form $\omega = U\omega U^{-1}$, where $U$ belongs to G, and
TABLE I: Exhaustive list of $\mathbb{Z}_2$-gradings for simple Lie algebras. For the $\mathbb{Z}_2$-grading of the AIII type, we use a somewhat non-standard notation to designate the algebra of block diagonal matrices, with blocks $A$ and $B$ on the diagonal, where $A$ (respectively $B$) is a $p \times p$ (respectively $q \times q$) Hermitian matrix with the constraint $\text{Tr}(A + B) = 0$. In this case, $g_\parallel$ is isomorphic to $\mathfrak{su}(p) \oplus \mathfrak{su}(q) \oplus \mathfrak{u}(1)$.

where $\bar{\omega}$ is one representative of the possible, classified, $\mathbb{Z}_2$-gradings of $g$. For regular simple Lie groups $G$, the $\mathbb{Z}_2$-gradings are presented in Table I [33].

In section IV, we will investigate in detail the case $G=\text{SU}(N)$ for applications to fermionic models (3). In this picture, the role of the physical symmetry group $H$ is to select what are the realized dualities amongst the set of possible ones. That the set $\mathcal{D}$ of dualities is not reduced to the trivial identity element will be clear later on the example presented in section V.

3. Transformation of fields

So far, one has fully characterized the set of dualities $\mathcal{D}$, that depends only on the maximal symmetry group $G$ and the physical symmetry group $H$. This was done by investigating the action of dualities on particular fields, the left-moving currents of the theory (the action on the right currents being trivial). To investigate the effect of dualities on observables, we need more generally their action on any local field of the $G_k$ WZNW model (the unperturbed theory); this action turns out to be particularly simple.

The whole field content of the WZNW model can be recovered by considering the KM primary operators $\Phi_{\lambda_L,\lambda_R}$ [2]. Since the WZNW model is invariant under independent global chiral $G$-rotations in the right and the left sectors, each primary field is labeled by two $G$-highest weights $\lambda_L$ and $\lambda_R$ that dictate how the primary field transforms under the group $G_L \times G_R$. Each primary field is thus a tensor $(\Phi_{\lambda_L,\lambda_R})_{a_L,a_R} \quad a_L(r) = 1, \ldots, \dim(\lambda_L(r))$. The precise set of representations $\lambda_L(r)$
that occurs in the WZNW model defines the field content that depends on the level \( k \) [2].

Before we give the action of dualities on these primary fields, we need a few basic facts about the group of automorphisms \( \text{Aut}(G) \). It contains the subgroup of *inner* automorphisms \( \text{Aut}_0(G) \), that can be identified with conjugation by elements of \( G \), i.e. those transformations \( A_U \) that act on \( G \) as: \( g \rightarrow A_U(g) = U g U^{-1} \), with \( U \in G \). Thus, inner automorphisms can be viewed as the group of changes of basis in the representations of \( G \) \(^6\). In fact, inner automorphisms do not exhaust \( \text{Aut}(G) \). Indeed, \( \text{Aut}(G)/\text{Aut}_0(G) \) is in general a non-trivial discrete group. Whenever it is not trivial, we will denote its non-trivial elements generically by \( \tau \). This leads us to introduce *outer* automorphisms, that can be written as \( A \circ \tau \) with \( A \in \text{Aut}_0(G) \), and to write \( \text{Aut}(G) = \{ \text{Aut}_0(G), \text{Aut}_0(G) \circ \tau \} \). Contrarily to inner automorphisms, that leave globally invariant any \( G \)-representation, outer automorphisms generically *exchange* representations, i.e. \( \tau(\lambda) \neq \lambda \). Outer automorphisms are in one-to-one correspondence (up to conjugation) with the symmetries of the Dynkin diagram associated to \( G \) [32].

For regular simple Lie algebras, involutive outer automorphisms do exist and will be relevant to our study (they will give rise to “outer” dualities). An exhaustive list of outer automorphisms is: for \( G = SU(N) \), the non-trivial element \( \tau \) corresponds to *charge conjugation*, that maps a representation on its charge conjugated partner – the corresponding outer \( \mathbb{Z}_2 \)-gradings are of type \( \text{A}_I \) and \( \text{A}_{II} \) (see Table I) –; for \( G = SO(2N) \), the non-trivial element \( \tau \) corresponds to “\( SO(2N) \) parity”, that exchanges the two spinorial representations – the corresponding outer \( \mathbb{Z}_2 \)-gradings is of type \( \text{BDI} \) when \( p \) and \( q \) are odd (see Table I) –.

Let us now describe the action of dualities on fields. Since a duality \( \Omega \in \mathcal{D} \) affects only the left sector, its action on the primary field \( \Phi_{\lambda_l,\lambda_r} \) is readily found: it is simply given by the action of the automorphism \( \omega \) on the left representation \( \lambda_l \). If \( \Omega \) belongs to the inner class, it just amounts to a change of basis in each representation \( \lambda_l \). Denoting by \( U_{\lambda_l} \) the matrix performing the change of basis, the primary operator becomes:

\[
\Phi_{\lambda_l,\lambda_r} \xrightarrow{\Omega} \tilde{\Phi}_{\lambda_l,\lambda_r} = U_{\lambda_l} \cdot \Phi_{\lambda_l,\lambda_r}, \quad \Omega \text{ inner}.
\]

In this expression, a matrix product is implied; in components it reads as follows: \( \tilde{\Phi}_{\lambda_l,\lambda_r}^{ab} = (U_{\lambda_l})_{a_l,b_l} \Phi_{\lambda_l,\lambda_r}^{b_l,a_r} \). On the other hand, outer dualities map in general the representation \( \lambda_l \) on another representation \( \lambda_l^* = \tau(\lambda_l) \), and also perform a change of basis in \( \lambda_l^* \). Thus, the primary

---

\(^6\) Strictly speaking, \( \text{Aut}_0(G) \) contains more than conjugation by elements of \( G \), since it is the connected component of the identity in \( \text{Aut}(G) \). This subtlety will play no role in our study.
operator transforms under duality according to:

\[ \Phi_{\lambda_l, \lambda_{n}} \xrightarrow{\Omega} \tilde{\Phi}_{\lambda_l, \lambda_{n}} = U_{\lambda_l} \cdot \Phi_{\lambda_l', \lambda_{n}}, \quad \Omega \text{ outer.} \tag{18} \]

Let us now briefly restate the main results of the preceding sections:

- To each family of models (5) defined by (i) an unperturbed critical model, namely a WZNW model built on the symmetry group \( G \), at level \( k \) (for the fermionic models (3), the level is fixed to \( k = 1 \)) and (ii) its bare symmetry group \( H \), there is a finite set of exact dualities \( \mathcal{D} \).

- To each model \( \mathcal{M} \) belonging to this family, defined by its bare couplings \( g_\alpha \), one can associate a finite number of models \( \mathcal{M}_\Omega \) defined by couplings \( \Omega(g_\alpha) \) where \( \Omega \in \mathcal{D} \) is an acceptable duality. These dualities are exact in the continuum limit. The physical properties of \( \mathcal{M}_\Omega \) can in principle entirely be deduced from those of \( \mathcal{M} \). For example, suppose the ground state of model \( \mathcal{M} \) is specified by some order parameter \( O \), that can be expressed in terms of a primary operator:

\[ O = \text{Tr} \left( M \Phi_{\lambda_l, \lambda_{n}} \right), \]

where \( M \) is some matrix encoding the “texture” of the phase of model \( \mathcal{M} \). Then, the ground state for model \( \mathcal{M}_\Omega \) is immediately given, with the help of Eqs. (17,18), by

\[ O_\Omega = \text{Tr} \left( M \tilde{\Phi}_{\lambda_l, \lambda_{n}} \right). \]

It follows that it would be sufficient to understand the physics of one “fundamental chamber” – a subspace of the space of the couplings \( g_\alpha \) – from which physical properties for the whole space of couplings could be deduced by acting with dualities. Nevertheless, understanding the physics of the “fundamental chamber” is still of course a complicated task. In particular, there is no reason for this “fundamental chamber” to display a single type of physical behavior – it may contain many different phases. In section IV, we will give a physical picture of part of the “fundamental chamber” in the SU(\( N \)) case to exhaust its physical properties.

### B. Dualities and DSE

In this section, we present an alternative characterization of the dualities found in the previous section: they are in one-to-one correspondence with the different ways the bare physical symmetry group \( H \) of the model can be promoted to a larger dynamical symmetry group \( G \). This notion of DSE phenomenon, which can be made explicit by means of the RG approach [34], is recalled for completeness and applied to model (5). Then, important consequences for the global phase diagram are drawn, that allow for a general picture thereof.
1. **DSE phenomenon**

Perturbing a 1D critical model with a continuous symmetry offers in general a whole family of possibilities, that one can label by the symmetry breaking pattern and by the type of perturbing operator; in our case, once chosen the general class of perturbations of the current-current type (5), the model is specified by the pattern $G \rightarrow H$. Generically, a marginal relevant perturbation results in the opening of a spectral gap in every sector of the theory. However, in one dimension, a simplification can occur that might allow for a kind of universal description of these fully gapped phases. As we have seen in section II, the family of models (5) contains a special, isotropic ray (with couplings proportional to $g_0^0$), where model (7) displays an extended global $G$ symmetry. This GN model, that we will call $M_0$, has the following Hamiltonian density:

$$\mathcal{H}_G = \mathcal{H}_0 + g \mathcal{J}_r^A \mathcal{J}_l^A.$$  \hspace{1cm} (19)

For $g > 0$, this model is a massive integrable field theory and its spectrum is known for $k = 1$ [35–37] and in the general $k$ case [38–40]. The coupling $g$ flows to strong coupling under RG, and a mass scale is dynamically generated: $m \propto e^{-cst/g}$.

It turns out that physical properties of the generic model (5) can be weakly sensitive to anisotropy, i.e. to departure from the isotropic ray, in a sense we are going to discuss now. To restate it differently, the effective theory describing the bare anisotropic model at low energy, displays approximately a larger symmetry $G$, and the physics is that of the maximally symmetric model $M_0$. This phenomenon has been coined DSE. An indication for DSE can be gained from considering the RG beta-function for generic models of the form (5). As shown in the detailed study of Refs. 34 and 41, the beta-function generically indicates that the more symmetric ray is attractive under RG (writing $g_\alpha = gg_0^\alpha + \delta g_\alpha$ the quantity $\delta g_\alpha/g$ flows to zero at one loop $\lim_{\ell \to 0} (\delta g_\alpha/g) = 0$, with $\ell = \ln(\Lambda a_0/v_f)$ the RG time, $\Lambda$ the running cut-off and $a_0$ the lattice spacing). The model thus flows to more and more symmetric theories at low energies. The RG flow defines a scale $\Lambda_{\text{DSE}}$, akin to Josephson length, where anisotropy becomes negligible: $(\delta g_\alpha/g)(\ln(\Lambda_{\text{DSE}} a_0/v_f)) \ll 1$. Now focusing on the low-energy spectrum (precisely speaking at energies $E < E_{\text{low}} \ll \Lambda_{\text{DSE}}$), we can have a non-trivial regime described by field theory only if one considers the triple limit

$$(\text{strong DSE}) \hspace{1cm} m \ll E_{\text{low}} \ll \Lambda_{\text{DSE}} \ll v_f a_0^{-1},$$ \hspace{1cm} (20)

where $m \propto e^{-cst/g}$ is the typical gap scale of the system (in this asymptotic DSE regime a single coupling $g$ is enough to describe the flow). Regime (20) is universal: in that limit the lattice model
(1) and model (19) share the same quantitative physical low energy \((E < E_{\text{low}})\) properties (this quantitative DSE phenomenon could be checked explicitly in some cases where the anisotropic model itself is integrable, allowing for a complete determination of all physical quantities [34]). It is not very surprising that when dealing with gapped systems, the notion of universality requires far more restrictive conditions – as far as the regime of bare parameters of the underlying lattice model is concerned – than for critical systems (for which a Josephson length \(\Lambda_J\) can also be defined and that only require \(E_{\text{low}} \ll \Lambda_J \ll v_1 a_0^{-1}\)). Note that (20) requires in particular \(m \ll \Lambda_{\text{DSE}},\) which implies that the running coupling constants are small at the DSE scale: \(g_\alpha(\Lambda_{\text{DSE}}) \ll 1.\) This shows that strong DSE is essentially a weak-coupling phenomenon [34, 41], and this justifies the one-loop approximation.

It is also possible to define DSE in a weaker sense, in which one does not require that the physical quantities be quantitatively the same as those of the isotropic point. The meaning of DSE in the weak sense, can maybe be best grasped in light of adiabatic continuity: it simply reflects the fact that the ground state at the isotropic point can be adiabatically connected to the ground state of the (weakly) anisotropic models; moreover, this adiabatic continuity also holds for the quasiparticles that can still be labelled by G quantum numbers: each of the G-multiplets \(\lambda\) of quasiparticles of the isotropic GN model splits under the (small) residual anisotropy breaking, \(m_\lambda \to (m^1_\lambda, m^2_\lambda, \ldots),\) with a splitting \(\delta m_\lambda = \max(|m^i_\lambda - m^j_\lambda|) \ll m_\lambda.\) This gives the hint that if one does not insist on the physical quantities to be asymptotically identical to those on the isotropic ray, then one can allow for finite, small deviations from the limit \(m/\Lambda_{\text{DSE}} \to 0.\) This has been shown in the particular case of the SU(4) Hubbard chain at half-filling [42], where continuity could be checked numerically at finite anisotropy in a non-integrable model. Thus, DSE in the weak sense simply means that the group G is an approximate symmetry of the low-energy sector of the theory, that allows for a description of the low-energy sector – very much in the way approximate symmetries allow for a classification of baryons and mesons in particle physics. As shown in the following section, the splitting of the G-multiplet under the residual anisotropy can be controlled in a perturbative expansion, which means in particular that the quasiparticles of the isotropic point (of model \(\mathcal{M}_0\)) survive anisotropy \(^7\), and establishes the aforementioned adiabatic continuity on a firm ground. This weak-DSE regime, corresponding to:

\[
\text{(weak DSE)} \quad m \lesssim \Lambda_{\text{DSE}}
\]  

\(^7\) Of course, this cannot be the case irrespective of the perturbation type: our statement holds provided the anisotropy is due to current-current terms.
is characterized by adiabatic continuity of the low-energy spectrum (in particular, it is a smooth
crossover and the symmetry of the ground state cannot change). Of course, symmetry enlargement
cannot occur everywhere in the phase diagram: it has been established on the level of a perturbative
expansion around the isotropic ray, and if the bare anisotropy is too strong, one expects it to break
down. In particular, when $\Lambda_{\text{DSE}}/m$ decreases and $\delta g/g$ becomes of order one, level crossings start
to occur in the lowest energy sector of the theory and the labeling in terms of $G$ quantum numbers
loses its meaning. In the following section, we will actually see that the existence of dualities, that
where introduced in section III A purely on group theoretical considerations, implies that is $has$
to break down, and in a rather elegant way, since it will allow to grasp at a glance the (very rough)
structure of the phase diagram.

2. Interplay with dualities

We are now in position to make the connection with the dualities introduced in the previous
section. It is possible to sketch the phenomenon of DSE as follows:

$$G_L \times G_R \xrightarrow{\text{interaction}} H = (H_L \times H_R)_{\text{diag}} \xrightarrow{\text{DSE}} G^{IR},$$

where the global symmetry group is indicated (from left to right, respectively) before perturbation,
in the bare $H$-symmetric model (5), and in the effective low-energy theory in the infrared (IR)
regime after DSE has occurred for instance in the weaker sense with the symmetry group $G^{IR}$.
One can then ask the following question: is diagram (22) unique?

It turns out that the answer is positive up to twists in the current algebra of the form (11), i.e., up
to dualities. Indeed, the final point of diagram (22) is in general of the form $G^{IR} = (G_L \times G_R)_{\text{diag}}$
with generators given by $\int dx \left( \tilde{J}_L^A + J_R^A \right)$, where $\tilde{J}_L^A = \omega^{AB} J_L^B$ and $\omega \in \mathcal{D}$ is a duality (see
Appendix A 2 for a proof). Dualities can thus be viewed as the different possible DSEs compatible
with a given bare symmetry group $H$. To each duality $\Omega \in \mathcal{D}$, there corresponds a generalized
isotropic, $\tilde{G}$-symmetric ray, described by the following Hamiltonian density:

$$H_\Omega = H_0 + g J_R^A \omega^{AB} J_L^B,$$

a model that we will denote $\mathcal{M}_\Omega$. Being related by a duality to the fundamental isotropic model
(19), it is also an integrable field theory.
3. Symmetry enlarged phases

Since dualities are exact isometries of the beta-function, model (23) is also attractive in the RG sense, so that DSE also occurs in the vicinity of the (generalized) isotropic ray. Now, the fact that DSE breaks down if the bare anisotropy is too large acquires a new meaning: it has to be so, since the different DSE schemes corresponding to different dualities are incompatible; in other words, the different symmetry enlarged phases \( \mathcal{M}_\Omega \) compete. We will turn shortly to the study of this competition.

What about the robustness of DSE, i.e. what happens when one moves away, in the space of bare couplings, from the isotropic ray \( g_\Omega^\alpha = \Omega(g_0^\alpha) \) (with \( g_\Omega^\alpha d^\alpha = \omega \), see Eq. (12))? One way to answer this question is to consider the one-loop RG flow, with initial conditions \( g_\alpha(a_0^{-1}) = g(a_0^{-1}) (g_\alpha^\Omega + \delta_\alpha(a_0^{-1})) \), \( \delta_\alpha \ll 1 \). The latter has to be cut-off before the couplings become of order one, i.e. at a scale \( \Lambda > m \), where \( m \) is the mass scale. Departure from the isotropic ray then results in a residual anisotropy, the couplings being \( g_\alpha(\Lambda) = g(\Lambda) (g_\alpha^\Omega + \delta_\alpha(\Lambda)) \). The study of Ref. 34 ensures that \( \delta_\alpha(\Lambda) \) scales to 0 in the scaling limit \( \Lambda a_0 \ll 1 \) when approaching the isotropic ray, resulting in corrections of order \( \left(g(a_0^{-1}) \delta_\alpha(a_0^{-1})\right)^\beta \) in the physical quantities, with \( \beta > 0 \) an exponent that depends on the groups \( G \) and \( H \) under consideration.

An alternative argument, that establishes the adiabatic continuity of the low-energy physics when departing from the isotropic rays, can be given using on the integrability of the isotropic model (23). At scale \( \Lambda \), the model can be described by the following Hamiltonian density:

\[
\mathcal{H} = \mathcal{H}_\Omega + g(\Lambda) \delta_\alpha(\Lambda) J^A_k d^\alpha_{AB} J^B_L,
\]

where the last term is the residual anisotropy. It is possible to investigate, by form factor methods, the effect of this term which constitutes a perturbation on top of an integrable model [10]: if the perturbation is local with respect to the quasiparticles, then the quasiparticles survive the perturbation, i.e. there is adiabatic continuity in the spectrum [10]. Since the perturbation is built on currents, that are the local generators of conserved charges, one concludes to mutual locality between the perturbation and the quasiparticles. It results that the low-energy spectrum of model (24) is described by the quasiparticles of the integrable model (23).

4. Quantum phase transitions

Let us now consider two different symmetry enlarged phases. The low-energy physics of each of them is captured by generalized isotropic models of the form (23), to be called \( \mathcal{M}_{\Omega_1} \) and \( \mathcal{M}_{\Omega_2} \).
These two models are exchanged by the duality $\Omega = \Omega_1 \Omega_2$ (recall that dualities form a group).

In the space of bare couplings, amongst the different trajectories connecting these two isotropic rays, there is a special, maximally symmetric one, that allows to study, in a minimal model, the competition between the two phases $\mathcal{M}_1$ and $\mathcal{M}_2$. Again, we invoke DSE to argue that this maximally symmetric trajectory is attractive under RG.

Recalling that the duality $\Omega$ corresponds to a $\mathbb{Z}_2$ grading of the Lie algebra of $G$ ($g = g\parallel \oplus g\perp$), there exists a basis $\{J^a\parallel , J^b\perp\}$ for the currents with the following properties:

$$\Omega(J^a\parallel_{(r)}) = J^a\parallel_{(r)} \quad \Omega(J^b\perp_{(r)}) = \mp J^b\perp_{(r)}, \quad (25)$$

where $a = 1, \ldots, \dim(g\parallel)$ and $b = 1, \ldots, \dim(g\perp)$. Then, the maximally symmetric model interpolating between the phases $\mathcal{M}_1$ and $\mathcal{M}_2$ has symmetry $G\parallel$ and is given by $^8$:

$$\mathcal{H}_{1-2} = \mathcal{H}_0 + g\parallel J^a\parallel_{(r)} \omega_1 J^a\parallel_{(r)} + g\perp J^b\perp_{(r)} \omega_1 J^b\perp_{(r)}. \quad (26)$$

When choosing $g\parallel = g\perp$, one recovers model $\mathcal{M}_1$, whereas $g\parallel = -g\perp$ yields model $\mathcal{M}_2$. There is a special point where the competition between the orders is at its climax: the self-dual point $\mathcal{M}_{1-2}^*$ corresponding to $g\perp = 0$, or, in term of the original couplings, to $\Omega(g_\alpha) = g_\alpha$.

The model (26) can be further simplified by acting with the duality $\Omega_1$: this amounts to study the competition between the phase $\mathcal{M}_1 = \mathcal{M}_0$ and $\mathcal{M}_1$, and in (26) to replacement $\omega_1 \rightarrow I$, which will be assumed in the following. Then, at the self-dual point, the Hamiltonian density takes a simpler form:

$$\mathcal{H}_\Omega^* = \mathcal{H}_0 + g J^a\parallel_{(r)} J^a\parallel_{(r)}, \quad (27)$$

i.e., the currents $J^a_{\perp}$ do not appear in the interaction term. This means that there is room for criticality at the self-dual point, with the decoupling of some degrees of freedom. The currents $J^a\parallel_{(r)}$ generate an affine algebra $g\parallel$ at level $k^* = \ell k$, where $\ell$, a positive integer, is the Dynkin index for the embedding $g\parallel \subset g$ $^2$. It is then possible to separate the “parallel” and “perpendicular” degrees of freedom at the level of the non-interacting Hamiltonian: $\mathcal{H}_0 = \mathcal{H}_0(g,k) = \mathcal{H}_0(g\parallel, k^*) + \mathcal{H}_A$, where the last piece is the Hamiltonian of model $A$ describing the degrees of freedom that decouple at the self-dual point. Model $A$ (that can be trivial, i.e., with vanishing central charge), is the coset model

$$A = \frac{G_k}{G_{k^*}}, \quad (28)$$

$^8$ In the case $G\parallel$ is not a simple Lie group, there are in general several couplings $g_{\parallel,\alpha}$. This does not affect our analysis.
with central charge \( c_A = c(G, k) - c(G, k^*) \), where we recall that \( c(G, k) = k \dim(G)/(k + g^\vee) \) is the central charge of the \( G_k \) WZNW model. Model \( A \) can be non-trivial, with non-integer central charge. Concrete examples in the context of fermionic models will be given in sections IV and V. Of course, \( c_A \) is only an upper bound to the central charge of the “perpendicular” criticality at the self-dual point: one has to check that no other relevant operator in model \( A \) is allowed by symmetries, that would spoil criticality. One also has to check whether the parallel sector itself can develop criticality.

**FIG. 1**: Sketch of (a 2D-projection of) the phase diagram for the example of a model with four dualities: \( \Omega_0 = 1 \), \( \Omega_1 \), \( \Omega_2 \), and \( \Omega_3 = \Omega_1 \Omega_2 \). Bold lines correspond to symmetry enlarged rays \( \mathcal{M}_\Omega \), dashed ones to self-dual models \( \mathcal{M}^*_{\Omega^*} \) (for clarity some of the self-dual models have been omitted). All lines meet at the \( G_k \) WZNW critical point (i.e. non-interacting point). Gray areas indicate regions of DSE and the dotted lines bounding it marks a cross-over to a regime where DSE looses its meaning (i.e. the splitting of \( G \)-multiplets due to anisotropy becomes of the order of the gap scale \( m \)). Four representative points together with the dualities exchanging them are also depicted. The dashed curve \( \gamma \) symbolizes the path in parameter space which corresponds to the minimal theory (26) interpolating between phases \( \mathcal{M}_1 \) and \( \mathcal{M}_{\Omega_1} \). See figure 2 for a qualitative picture of the low-energy spectrum along this path.

We thus arrive to the following general picture of the phase diagram for model (5) – see figure 1:

- There is a finite set of generalized isotropic rays where the bare model displays an *exact* enlarged \( \tilde{G} \)-invariance. Each ray \( \mathcal{M}_\Omega \) is labelled by a duality \( \Omega \in D \), and the global invariance
group $\tilde{G}$ is obtained from $G = (G_l \times G_r)_{\text{diag}}$ by acting with $\Omega$ on the currents (see Eq. (11)). The Hamiltonian on those rays is given by Eq. (23).

- Each of those isotropic rays is attractive in the RG sense, and to it there corresponds a pocket with finite extension in the space of bare couplings. In this pocket, the low-energy sector of the theory is adiabatically connected to that of the isotropic model. The low-energy spectrum can be described by making use of the approximate $\tilde{G}$-invariance. In particular, the integrability of the resulting model gives a description of the massive quasiparticles which are organized in $\tilde{G}$-multiplets.

- When the bare anisotropy is too large, DSE cannot hold any longer, i.e. the energy splitting of the $\tilde{G}$-multiplets due to anisotropy becomes of the order of the gap scale $m$: the low-energy physics cannot any longer be described in terms of a single energy scale $m$, and the situation is in general complex.

- In between the symmetry enlarged pockets lie self-dual manifolds, which are the loci where anisotropy is maximal. A simplification occurs, with in general a decoupling of degrees of freedom, accompanied with criticality. The quantum phase transition that results – which in the simplest cases describes the transition between the DSE phases – is captured by the minimal model with Hamiltonian (26). Figure 2 presents a qualitative picture of the low-energy spectrum of this minimal model, along a path joining two distinct symmetry enlarged phases. For clarity, one chooses a path in parameter space along which the mass spectrum scales in a convenient way: apart from possible phase transition points, the typical mass scale is held constant.

IV. FERMIONIC MODELS

In this section, we come back to the fermionic models (3) defined in terms of $N$ chiral Dirac fermions $\Psi_{a l,(k)}$. We have shown, in the previous section, that, irrespective of the bare symmetry group $H$ of the model, there exists a finite number of possible dualities that can be classified. To each of them there corresponds a symmetry enlarged phase, which we now characterize. There will result an exhaustive list of DSE phases possibly supported by models (3). For each of those phases, the duality symmetries together with the integrability yield precious informations on the low-energy physics. As explained above, the maximal symmetry group supported by those models
FIG. 2: Sketch of the low-energy spectrum of the minimal model (26) interpolating between two symmetry enlarged phases (see the path $\gamma$ in figure 1).

is $G = SU(N)$ or $G = SO(2N)$, depending on the filling. Here we analyze the general $G = SU(N)$ case which is relevant to weakly-interacting fermions away from half-filling. The special $N = 4$ case, i.e. two-leg electronic ladders, will be described in detail in section V. The general half-filled case with $G = SO(2N)$ is much more complicated to analyze and will be investigated elsewhere.

A. Different classes away from half-filling

For incommensurate filling, there is no umklapp process coupling spin and charge degrees of freedom: $H = H_c + H_s$. The charge fluctuations are decoupled and described by a massless bosonic field $\Phi_c$ and its dual $\Theta_c$ field, with Hamiltonian density:

$$H_c = \frac{v_c}{2} \left( \frac{1}{K_c} (\partial_x \Phi_c)^2 + K_c (\partial_x \Theta_c)^2 \right), \quad (29)$$

which belongs to the Luttinger liquid universality class [5–7]. In Eq. (29), $v_c$ and $K_c$ are the charge velocity and Luttinger parameter, respectively, that depend on the microscopic details of the Hamiltonian and should be regarded as phenomenological parameters. For a commensurate filling, like one-electron per site, additional umklapp processes may appear depending only on the charge degrees of freedom and the low-energy charge Hamiltonian (29) becomes a sine-Gordon model. A charge gap might open for a sufficiently strong value of the interaction which results in a Mott transition. The non-trivial physics of model (3) corresponds to the remaining “spin” degrees of freedom which are governed by model (4) with $G = SU(N)$.

According to the analysis of section III, we know that in the spin sector, there can only be a finite number of symmetry enlarged phases, that are classified by the $SU(N) \mathbb{Z}_2$ gradings listed in
Table I. Due to DSE, each of these phases has a finite extension in the space of bare couplings, and its low-energy sector is described by the representative Hamiltonian (23), which is labelled by an involutive automorphism $\Omega$ of $\text{SU}(N)$. This results in four classes of symmetry enlarged phases $\mathcal{M}_\Omega$, corresponding to the trivial automorphism $\Omega = 1$, and to the three automorphisms belonging to the classes $\text{AI}$, $\text{AII}$, $\text{AIII}$ of Table I. Recall that those are only the possible dualities: for them to be realized in a given model defined by the symmetry breaking pattern $\text{SU}(N) \sim H$, $H$ must contain in its center an involutive element that gives rise to the duality. As we will see, physical properties show a strong distinction between symmetry enlarged phases $\mathcal{M}_\Omega$ according to whether they correspond to inner or outer automorphisms $\Omega$ (here $\text{AI}$ and $\text{AII}$ correspond to outer automorphisms, and $\text{AIII}$ is inner). While for inner dualities, the order is of the density-wave type, phases associated to outer dualities display off-diagonal order, that comes along with superfluidity in $\text{SU}(N)$ spin space. In addition, we will see that outer dualities conjugate, i.e. maps fermions onto holes. A representative for these three classes for the $\text{SU}(N)$ group, in a suitable basis for the fermions $\Psi_{al}$ ($\Psi_{ar}$ being invariant in the definition of the dualities (11)), is given by:

- **$\text{AI}$ class**: $\Psi_{al} \rightarrow \Omega(\Psi_{al}) = \Psi_{al}^\dagger$. The invariant subspace $g_\parallel$ is $\mathfrak{so}(N)$, with a basis defined by the subset of $\text{SU}(N)$ generators $\Psi_{al}^\dagger T_{ab} \Psi_{bl}$, $T^i$ being real and antisymmetric. Obviously, this automorphism conjugates representations, i.e. it maps the representation $\lambda$ onto its charge conjugated partner $\lambda^*$. The lattice symmetry (the element of $\mathcal{C}(H)_{\text{inv}}$ giving rise to $\Omega$ is charge conjugation: $c_{j,a} \rightarrow c_{j,a}^\dagger$.

- **$\text{AII}$ class**: $\Psi_{al} \rightarrow \Omega(\Psi_{al}) = J_{ab} \Psi_{bl}^\dagger$. This duality exists only for even $N$ and the matrix $J$ ($J$ being the $\text{Sp}(N)$ metric) is given by $-i\sigma^2 \times I_{N/2}$. The invariant subspace $g_\parallel$ is $\mathfrak{sp}(N)$, with a basis defined by the subset of $\text{SU}(N)$ generators $\Psi_{al}^\dagger T_{ab} \Psi_{bl}$ such that $JT^i J = t^i T^i$. This automorphism also conjugates representations. The $N = 2$ is very special since the duality corresponds to the identity for the currents: $\text{Sp}(2) \sim \text{SU}(2)$. The lattice symmetry giving rise to $\Omega$ is an “antisymmetric charge conjugation”: $c_{j,a} \rightarrow \sum_b J_{ab} c_{j,b}^\dagger$.

- **$\text{AIII}$ class**: $\Psi_{al} \rightarrow \Omega(\Psi_{al}) = (I_{p,q})_{ab} \Psi_{bl}$, where $I_{p,q}$ ($0 < p \leq q < N$) is the diagonal matrix with $p$ entries 1 and $q$ entries -1. The automorphism $\Omega$ belongs to $\text{Aut}_0(G)$, i.e. it is not an outer automorphism. One has $g_\parallel = \mathfrak{su}(p) \oplus \mathfrak{u}(q)$ as an invariant subspace. The lattice symmetry that gives rise to this duality is: $c_{j,a} \rightarrow -c_{j,a}$ for $a > p$. Dualities in the $\text{AIII}$ class are thus associated to a symmetry breaking pattern $G \sim H$ that splits the $N$ fermionic chains into two bunches of $p$ and $q = N - p$ chains.
B. Physical properties of the fundamental phase $M_1$

The physical properties of the symmetry enlarged phases can be deduced from those of the GN model (19), the representative Hamiltonian corresponding to the phase $M_1$ labelled by the trivial duality $\Omega = \mathbb{1}$.

- **Ground state and spectrum:** Its exact spectrum is known from integrability [35] and consists in $N-1$ branches of quasiparticles transforming in the Young tableau $\lambda_r$ with $r$ boxes and one column ($r = 1 \ldots N-1$), with masses $m_r = m \sin(\pi r/N)$. The quasiparticles with label $r \neq 1, N-1$ can be seen as bound states of the “fundamental” quasiparticles with $r = 1$ and $r = N-1$. The states are labelled by SU$(N)$ quantum numbers, i.e. by the eigenvalues of the $N-1$ diagonal Cartan generators $Q^a_l + Q^a_r$, $\alpha = 1, \ldots, N-1$.

The order parameter of this spin-gapped phase expresses simply in terms of the SU$(N)_1$ WZNW matrix field $g$ which is defined by [4]:

$$g_{ab} = e^{-i \sqrt{\frac{4\pi}{N}} \Phi_c} \Psi^+_a \Psi_b,$$

(30)

which is a pure spin field since it commutes with all operators of the charge sector. This matrix field identifies with the “fundamental” primary operator, $g = \Phi_{\lambda_{N-1}, \lambda_1}$. The order parameter reads then as follows:

$$\langle O_{\Phi} \rangle = \langle \text{Tr}(g) \rangle \neq 0.$$  

(31)

An order parameter on the lattice is readily found by considering the $2k_f$ oscillating part of the total density operator $n_i = \sum_{a=1}^N c^+_i a c_{i,a}$, whose continuum limit is related to $g$: $n_{2k_f} \sim e^{i \sqrt{\frac{4\pi}{N}} \Phi_c} \text{Tr}(g)$. This operator develops quasi-long-range order with a power-law behavior for the equal-times two-point functions:

$$\left\langle n^{\dagger}_{2k_f}(x) n_{2k_f}(0) \right\rangle \sim x^{-\frac{2k_f}{N}}.$$  

(32)

If the charge sector develops a gap by some process (for instance a pure charge umklapp term at commensurate filling $1/N$ (one electron per site) [43]), the resulting phase is a Mott insulator of the charge-density wave (CDW) type with $\langle n_{2k_f} + \text{H.c.} \rangle \neq 0$. This order may coexist with a $2k_f$ bond-ordering or spin-Peierls phase which is defined as: $O_{SP} = \sum_{j,a} e^{-2ik_f j a_0} c_{j,a}^\dagger c_{j+1,a} + \text{H.c.}$

A lattice order parameter involving only the “spin” degrees of freedom can also be found by considering a generalized dimerization operator in SU$(N)$ space: $O_D = \sum_{j,A} S_j^A S_{j+1}^A e^{-2ik_f j a_0}$, where $S_j^A = c_{j,a}^\dagger T_{a}^A c_{j,b}$ is the SU$(N)$ spin lattice operator on site $j$. The latter operator has a
due to the spin gap, the susceptibility at zero temperature $$N$$ electron per site, which is formed from local SU($$N$$) of the harmonics in all directions of spin space. The rigidity is defined as $$\rho$$ rotations. When computed, the continuum limit of $$\phi$$ in the boundary conditions, results in the following boundary conditions for the bosonic fields:

$$\hat{A}_c$$ conditions ($$t$$)ability, namely the rigidity that measures the response of the system to twists in the boundary

$$V$$ of products vertex operators $$\delta$$ develop a non-zero average value:

We thus conclude, from the spin sector point view, that phase $$\mathcal{M}_0$$ corresponds to a generalized dimerization phase, with an $$N$$-fold ground-state degeneracy for a commensurate filling of one electron per site, which is formed from local SU($$N$$) spin-singlet condensation.

- Susceptibilities and rigidities: Due to the spin gap, the susceptibility at zero temperature vanishes in all directions in spin space. Indeed, coupling the model to some field $$h^A$$, one has

$$\chi_{AB} \equiv \frac{\partial^2 E[h]}{\partial h^A \partial h^B} = 0,$$

where $$E[h]$$ is the energy of the system with the field $$h$$. In contrast, the quantity dual to susceptibility, namely the rigidity that measures the response of the system to twists in the boundary conditions ($$c_{j+L/a_0,a} = U_{ab} c_{j,b}$$ with $$U[\gamma] = e^{i\gamma A T^A}$$) for a system on a ring of size $$L$$, is non-vanishing in all directions of spin space. The rigidity is defined as $$\rho_{AB} \equiv \frac{\partial^2 E[\gamma]}{\partial \gamma^A \partial \gamma^B}$$, and one has $$\rho_{AA} \neq 0$$ for all $$A = 1, \ldots, N^2 - 1$$. This is easily seen by bosonizing model (19): first one chooses a Cartan basis ($$H^\alpha$$), $$\alpha = 1, \ldots, N - 1$$ with the first element $$H^1$$ along the direction $$A$$, and one introduces $$N - 1$$ bosonic fields $$\varphi_{\alpha \ell(k)}$$, $$\alpha = 1, \ldots, N - 1$$, whose gradients represent the currents in the Cartan directions: $$\mathcal{J}_{\ell(k)}^\alpha = \sqrt{4\pi} \partial_x \varphi_{\alpha \ell(k)}$$. The non-interacting part of the Hamiltonian (19) is that of $$N - 1$$ free bosons $$\mathcal{H}_{0\ell} = u \sum_\alpha \left( (\partial_x \varphi^2) + (\partial_x \theta^2) \right) / 2$$, with $$\varphi = \varphi_{\alpha \ell} + \varphi_{\alpha R}$$ the total boson field and $$\theta = \varphi_{\alpha L} - \varphi_{\alpha R}$$ its dual field, while the interacting part of model (19), $$\mathcal{H}_{\text{int,s}}$$, is built from a sum of products vertex operators $$\mathcal{V}_{\delta_{\alpha} \beta_{\alpha}} = e^{i\beta_{\alpha} \varphi_{\alpha L} + i\delta_{\alpha} \varphi_{\alpha R}}$$. The twist $$U = e^{i\gamma H^1}$$, along the direction $$A$$ in the boundary conditions, results in the following boundary conditions for the bosonic fields: $$\phi_{\alpha}(x + L) = \phi_{\alpha}(x)$$ and $$\theta_{\alpha}(x + L) = \theta_{\alpha}(x) + (\gamma / \sqrt{\pi}) \delta_{\alpha,1}$$. Now, due to global (diagonal) SU($$N$$)
invariance, and in particular to the invariance under the transformations generated by the Cartan $H^\alpha$ that are represented by a shift on the bosonic fields, $\varphi_{\alpha L(n)} \to \varphi_{\alpha L(n)} \pm C_\alpha$, $\mathcal{H}_{\text{int},s}$ can involve only vertex operators with $\beta_\alpha = \bar{\beta}_\alpha$, i.e. it does not depend on the dual field $\theta_\alpha$. It results that the twist in the boundary condition can be absorbed via the following canonical transformation affecting the boson 1 only: $\tilde{\theta}_1(x) = \theta_1(x) - \frac{\gamma}{L \sqrt{\pi}} x$, $\tilde{\phi}_1(x) = \phi_1(x)$, yielding

$$\rho_{AA} = \frac{1}{u\pi} \neq 0,$$

(36)

which signals the emergence of a finite rigidity.

- **Spin excitations:** Another quantity of interest characterizing the phase is the dynamical spin structure factor $S(q, \omega) = \int dx dt e^{-iqx-i\omega t} \langle S^A_{j+x/a_0}(t)S^A_j(0) \rangle$. Using the continuum description of the spin operators (33), the Lehmann representation of the zero-temperature spin structure factor involves the form factors

$$\mathcal{F}_r(\xi) = \langle 0|\mathcal{N}_r^A|\xi \rangle,$$

(37)

where $|\xi\rangle$ is an eigenstate of the GN model (19). At low frequency, the behavior of (37) is fixed by one-particle states $|\xi_{r'}\rangle$ transforming in the representation $\lambda_{r'}$. Since the SU($N$) singlet representation does not appear in the tensor product of the adjoint by $\lambda_{r'}$, we deduce a strong selection rule: $\mathcal{F}_r(\xi_{r'}) = 0 \forall r, r'$. The consequence is that the spin structure factor displays no sharp peak structure at the quasiparticle poles $q \approx 2rk_f$ and $\omega \approx m_r$. This situation is sometimes referred to as incoherence of the spin excitations.

C. Effect of dualities

Acting with a duality $\Omega$ on the GN model (19) yields the generalized GN model (23), which, in spite of its simple connection to the original GN model, displays qualitatively different physical features. We now deduce the non-trivial consequences of the duality for the SU($N$) group on the low-energy physics of the symmetry enlarged phases.

- **Ground state:** For inner dualities, using Eq. (17), we observe that the WZNW field $g = \Phi_{\lambda_{N-1}, \lambda_1}$ is mapped onto itself $g \xrightarrow{\Omega} U_{\lambda_{N-1}} g$. As a result, the order parameter of this phase is of CDW type, with a texture depending on the precise form of the duality symmetry encoded in the change of basis matrix $U_{\lambda_{N-1}}$:

$$\langle O_\Omega \rangle = \langle \text{Tr}(U_{\lambda_{N-1}} g) \rangle \neq 0.$$

(38)
In a metallic phase, the $2k_F$ component of the corresponding lattice operator $n_j^\Omega = \sum_{ab} c_{j,a} \dagger (U_{\lambda_{N-1}})_{ba} c_{j,b}$ develops quasi-long-range correlations:

$$\langle n_j^\Omega(x) n_j^\Omega(0) \rangle \sim x^{-2Kc/N}. \quad (39)$$

If some umklapp process opens a gap in the charge sector, the ground state displays a CDW ordering characterized by $\langle n_j^\Omega \rangle \neq 0$. The lattice generalized SU($N$) dimerization operator is now $O_D^\Omega = \sum_{j,A} \epsilon^A S_j^A S_{j+1}^A e^{-2ik_F j a_0}$ with $\epsilon^A = \pm 1$ according to whether the direction $A$ belongs to $g_\parallel$ or not. Using the result (34), we now conclude that $O_D^\Omega \sim \text{Tr}(U_{\lambda_{N-1}} g)$, i.e. it develops a non-zero expectation value indicating a bond ordering.

In contrast, for outer dualities, using Eq. (18), we see that the WZNW field is now changed into its dual, $g \xrightarrow{\Omega} U_{\lambda_1} \bar{g}$ where:

$$\bar{g}_{ab} = e^{i\sqrt{\frac{4\pi}{N}}} \Theta \Psi_{al} \Psi_{br}. \quad (40)$$

As a result, the order parameter of this phase becomes:

$$\langle O_\Omega \rangle = \langle \text{Tr}(U_{\lambda_1} \bar{g}) \rangle \neq 0. \quad (41)$$

This translates into non-diagonal ordering for the lattice fermions and the order parameter becomes of a pairing type. The uniform component of the pairing density $\rho_j^\Omega = \sum_{ab} c_{j,a} \dagger (U_{\lambda_1})_{ba} c_{j,b}$ develops critical correlations

$$\langle \rho_j^\Omega(x) \rho_j^\Omega(0) \rangle \sim x^{-2/NK_c}. \quad (42)$$

In fact, phases associated to outer dualities spontaneously break a discrete symmetry of the original lattice Hamiltonian (1): the center $\mathbb{Z}_N$ of SU($N$), that consists of a discrete gauge redefinition of the phase of the underlying fermions $c_{j,a} \rightarrow e^{2i\pi/N} c_{j,a}$. In the continuum limit, this transformation corresponds to special phase factors on the Dirac fermions:

$$\mathbb{Z}_N : \Psi_{al(r)} \rightarrow e^{2i\pi/N} \Psi_{al(r)}. \quad (43)$$

For $N > 2$, $\rho_j^\Omega$ and $O_\Omega$ are clearly not invariant under this discrete symmetry. The spontaneous breaking of the center group symmetry is thus responsible for the formation of the spin gap of this phase. Note that it is likewise sensible to interpret the formation of a gap in phases associated to inner dualities as the spontaneous breaking of a dual group $\bar{\mathbb{Z}}_N$:

$$\Psi_{al(r)} \rightarrow e^{\pm 2i\pi/N} \Psi_{al(r)}, \quad (44)$$
and this $\mathbb{Z}_N$ symmetry has no local representation on the original lattice fermions.

- **Spectrum:** The energy spectrum of model (23) is still that of the GN model (19), but now the charges labeling states, associated with the global symmetry $G^{IR} = (\tilde{G}_L \times G_r)_{\text{diag}}$, are the $N-1$ Cartan generators $H^\alpha = Q^\alpha_r + \tilde{Q}^\alpha_L$, $\alpha = 1, \ldots, N-1$, with $\tilde{Q}_L^\alpha = \epsilon^\alpha Q^\alpha_L$, $\epsilon^\alpha = \pm 1$ according to whether the direction $\alpha$ belongs to $g_{\parallel}$ or not. It is possible to show that for inner dualities, one can always find a Cartan basis untouched by $\Omega$, i.e. with $\epsilon^\alpha = 1 \ \forall \alpha$, and we will work in such a basis. It results that states can be labelled by Noether charges $J^\alpha_0$, see Eq. (15). For outer dualities though, it is not the case (for otherwise any representation $\lambda$ would be mapped to itself). Instead, there is at least one direction $\alpha$ in Cartan space such that $H^\alpha = J^\alpha_1$, i.e., the Noether current. We conclude that in the diagram (22), there is some sort of transmutation from Noether charges to Noether currents when going from UV to IR.

- **Susceptibilities and rigidities:** Following again the same line of reasoning that lead to the results (35,36), one sees that dualities, in spite of the presence of the mass gap in every sector of the spectrum, results in the existence of directions in $su(N)$ with *non-vanishing* susceptibilities $\chi_{AA}$, and vanishing rigidity $\rho_{AA}$, namely those “transmuted” directions for which $\epsilon^A = -1$. This is obviously consistent with the fact the associated quantum number along the direction $A$ is a current and not a charge. If the duality is outer, *any* Cartan basis (that is, any choice of the quantization axis for the quantum numbers labeling the states) will be touched by the duality, i.e. it will contain at least one direction with $\epsilon^A = -1$, and vanishing rigidity in that direction. It turns out that this is connected to an observable physical effect, that allows to distinguish between inner and outer dual phases: as shown in Ref.12, the breaking of the $\mathbb{Z}_N$ symmetry (43) in outer phases leads to a ”confinement” of the current, elementary gapless excitations $^9$ carrying currents quantized in units of $Ne$. For a finite system put on a rotating ring, this results in a modified periodicity of the groundstate energy as a function of tangential velocity $[12, 45]$.

- **Spin excitations:** For inner dualities, the conclusions drawn in the case of the trivial duality $\Omega = \mathbb{1}$ are not affected: spin excitations are still incoherent. However, for outer dualities, the spin density $N^A_r$ at wave vector $q = 2\pi k_f$ in Eq. (33) is mapped onto $\Omega(N^A_r) = \text{Tr}(T^A_r U_{\lambda_r} \Phi_{\lambda_r,\lambda_r})$, and owing to the tensor product $\lambda_r \otimes \lambda_r = \lambda_{2r} \oplus \ldots$ (indices are understood modulo $N$), the form factor of Eq. (37) $F_r(\xi_{N-2r})$ is non-vanishing. Hence, the dynamical spin structure factor displays a sharp peak at $q = 2\pi k_f$ and $\omega = m_r$: spin excitations in a charge-gapped system are now *coherent*

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9 This effect appears at incommensurate filling, i.e. with no gap in the charge sector.
Outer dualities thus have a simple spectroscopic signature.

| Duality Ω & phase | order parameter | inn/out | lattice symmetry giving rise to Ω |
|-------------------|-----------------|---------|---------------------------------|
| Trivial SU(N) CDW-SP | \( c_{j,a}^\dagger c_{j,a} e^{-2ik_{j}ja_{0}} \) | inner | identity |
| | \( c_{j,1,a}^\dagger c_{j+1,1,a} e^{-2ik_{j}ja_{0}} \) | | |
| A I | \( c_{j,a} c_{j+1,a} \) | outer | \( c_{j,a} \rightarrow c_{j,a}^\dagger \) |
| A II | \( c_{j,a} J_{ab} c_{j,b} \) | outer | \( c_{j,a} \rightarrow J_{ab} c_{j,b}^\dagger \) (\( J = -i\sigma_{2} \otimes I_{N/2} \)) |
| A III | \( \sum_{a \leq p} - \sum_{a > p} \) \( c_{j,a}^\dagger c_{j,a} e^{-2ik_{j}ja_{0}} \) | inner | \( c_{j,a} \rightarrow -c_{j,a} \) (\( a > p \)) |

TABLE II: The different classes of possible symmetry enlarged phases supported by the \( N \)-component degenerate fermions in the absence of spin-charge coupling (incommensurate case). For each duality Ω, we give the corresponding lattice order parameter, the nature (inner or outer) of the duality, and the lattice symmetry giving rise to Ω.

In Table II, we present a summary of the different possible symmetry enlarged phases for the \( N \)-component degenerate fermions close to the SU(\( N \))_1 critical point, considered away from half-filling.

**D. Cross-overs and quantum phase transitions**

In this section we investigate self-dual points and their vicinity. We have seen in section III B 4 that those are points where the competition between two different symmetry enlarged phases is maximal, and can possibly lead to a quantum phase transition. For each pair of dualities \( \Omega_1 \) and \( \Omega_2 \), the region lying between phases \( \mathcal{M}_{\Omega_1} \) and \( \mathcal{M}_{\Omega_2} \) associated to them is best described in terms of the simplest model (26). By acting with the duality \( \Omega_1 \), one can reduce the family of those interpolating models that need to be investigated: it is enough to consider the models interpolating between \( \mathcal{M}_{\mathbb{I}} \) and \( \mathcal{M}_{\Omega} \), where \( \Omega \neq \mathbb{I} \) is any non-trivial duality:

\[
\mathcal{H}_{\mathbb{I}-\Omega} = \mathcal{H}_0 + g_\parallel \mathcal{J}_\parallel^a \mathcal{J}_\parallel^a + g_\perp \mathcal{J}_\perp^b \mathcal{J}_\perp^b. \tag{45}
\]

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10 This conclusion only holds in a charge gapped phase. If charge excitations are gapless, then the fluctuations in the charge sector (because of the presence of the operators \( \hat{\alpha}_\epsilon \) in the spin operator, see Eq.(33)) will wash out the coherence in the spin-spin correlation function \([22, 46]\).
At the self-dual point ($g_\perp = 0$), the interaction in the parallel sector gives rise to a gap for $g_\parallel > 0$ (this is the case we will consider in the following $^{11}$): the spectrum in this sector is this of the $G_\parallel$ GN model, where $t$ is the embedding index of the embedding $G_\parallel \subset G$ associated to $\Omega$. There are, however, other degrees of freedom in general, that are not affected by the interaction: this decoupled sector is described by the coset model $A = \frac{\text{SU}(N)_1}{G_\parallel}$ (see Eq. (28)). If model $A$ is non-trivial, i.e. with central charge $c_A \neq 0$, this is an indication that there is room for a continuous phase transition, in the universality class of the CFT $A$. If it is trivial, a phase transition might occur as we will see. The inner or outer nature of the duality will have different striking effects.

- **$A$/$A$ transition**: The Dynkin index for the embedding $\text{SO}(N) \subset \text{SU}(N)$ is $t = 2$, so that model $A$ is trivial: $c_A = 0$. The only degrees of freedom at the self-dual point are thus those of the parallel sector, which are governed by the following Hamiltonian:

$$
H^s_\Omega = H_0 + \lambda_J^{a}_{\parallel} J^a_{\parallel} (46)
$$

where $J^a_{\parallel}$ are $\text{SO}(N)_2$ currents. $^{12}$ This is the $k = 2 \text{SO}(N)$ GN model which is fully gapped for $\lambda > 0$. $^{13}$ In particular, it has been shown in Ref. 47 that this model is related to the Andrei-Destri integrable model which is exactly solvable by means of the Bethe ansatz approach [49, 50]. Therefore, no criticality emerges at the self-dual point. However, recalling that $\Omega$ is outer and conjugates representations, we can show that the order parameters on both sides cannot coexist: while $\text{Tr}(g)$ is invariant under the center of $\text{SU}(N)$, that is under the $\mathbb{Z}_N$ transformations (43), it is not the case for the off-diagonal order parameter $\text{Tr}(U_{\lambda_1} \bar{g})$ of Eq. (41) for $N > 2$. The $\mathbb{Z}_N$ symmetry is thus spontaneously broken in phase $\mathcal{M}_\Omega$. The opposite is true for the dual symmetry $\mathbb{Z}_N$ (44), which leaves invariant $\bar{g}$ but not $g$, so that it is spontaneously broken in phase $\mathcal{M}_\parallel$. We conclude to the occurrence of a first-order transition.

$^{11}$ The case $g_\parallel < 0$, for which the interaction at the self-dual point is marginally irrelevant, will not be analyzed here. It corresponds to situations where additional criticality is present at the self-dual point: criticality of the multicritical point extends in parameter space. We will present in detail an example of this in section V.

$^{12}$ We consider here the generic case $N > 2$ since the $N = 2$ case is very special. As discussed in Ref. 47, the resulting transition for $N = 2$ is described by a $\text{SU}(2)$ self-dual sine-Gordon model which displays an $U(1)$ quantum criticality [48].

$^{13}$ A subtlety arises here from the fact that the conformal embedding $\text{SO}(N)_2 \subset \text{SU}(N)_1$, yields a non-diagonal invariant for the partition function in the $\text{SO}(N)$ variables. Namely, some $\text{SO}(N)_2$ states have to be projected out. Specifically, when $N = 2n$ is even, the $\text{SO}(2n)_2$ primary operators constituting the non-interacting spectrum (that are necessary to reconstruct the $\text{SU}(2n)_1$ Hilbert space), together with their degeneracy, are given by: $1 \times \mathbb{I}, \{2 \times \Phi_{\lambda_1, \lambda_1} \}, 1 \times \Phi_{\lambda_1, \lambda_1} + 1 \times \Phi_{\lambda_1, \lambda_1} + 1 \times \Phi_{\lambda_1, \lambda_1} + 1 \times \Phi_{\lambda_1, \lambda_1}$. Here the integer $j$ runs from 1 to $n - 2$, and $\lambda_1$ is the highest weight of the representations of $\text{SO}(N)$, while the spinorial representations (which do not appear in the spectrum) have highest weight $\lambda_{I}^{\parallel}$. When $N = 2n + 1$ is odd, the spectrum is given by: $1 \times \mathbb{I}, \{2 \times \Phi_{\lambda_1, \lambda_1} \}, 1 \times \Phi_{\lambda_1, \lambda_1} + 1 \times \Phi_{\lambda_1, \lambda_1}$, the integer $j$ running from 1 to $n - 1$. We make the reasonable hypothesis that this modified $\text{SO}(N)$ GN model at level 2 is still fully gapped. The study of its spectrum goes beyond the scope of this work.
\( \bullet A_{II} \text{ transition:} \) The embedding index is one in this case and one has the conformal embedding [51] \( SU(2n)_1 \sim Sp(2n)_1 \times \mathbb{Z}_n \), (recall that \( N = 2n \) is even in this case) where the last piece, corresponding to \( A \), stands for the parafermionic minimal model with \( \mathbb{Z}_n \) symmetry \( (n \geq 2) \) and central charge \( c_A = 2(n - 1)/(n + 2) \) which is a generalization of the Ising \( \mathbb{Z}_2 \) critical theory [52]. This transition was studied at length in Ref. 12, 13. In particular, it was shown that for \( N < 4 \), this criticality is stable, i.e. no relevant operator in the \( \mathbb{Z}_n \) sector is allowed by symmetries that would spoil criticality. Once again, the quantum phase transition is associated to the spontaneous breaking of the \( \mathbb{Z}_n \) symmetry – with respect to the \( AI \) case, a \( \mathbb{Z}_2 \) subgroup of the center of \( SU(2n) \) survives that corresponds to a redefinition of the (lattice) fermion sign \( (c_{i,a} \rightarrow -c_{i,a}) \).

\( \bullet A_{III} \text{ transition:} \) The embedding index is one, and the coset model \( A \) is trivial, with \( c_A = 0 \). Recall that the automorphism \( \Omega \) is defined by splitting the set of \( N \) fermionic chains into two clusters of size \( p \) and \( q \) with \( p + q = N \). The affine symmetry \( g_{\|} \) decomposes in three simple factors: \( su(p)_1 \oplus su(q)_1 \oplus u(1) \) (see Table I). The first two factors are generated by the affine currents \( J_{L(n)}^{(p)} \) and \( J_{L(n)}^{(q)} \), that are respectively the uniform components of the lattice spin operators \( \sum_{a,b \leq p} c_{j,a}^{\dagger} T_{a,b}^{(p)} c_{j,b} \) and \( \sum_{p < a,b} c_{j,a}^{\dagger} T_{a,b}^{(q)} c_{j,b} \) where \( T_{a,b}^{(p)} \) (respectively \( T_{a,b}^{(q)} \)) are traceless Hermitian matrices with non-zero entries only in the upper-left \( p \times p \) block (lower-right \( q \times q \) block respectively).

The affine current \( J_{0} \), generating the \( U(1) \) piece, is the uniform component of the relative charge of the two chain clusters \( Q_f = \frac{1}{\sqrt{N pq}} \left( q \sum_{j,a \leq p} c_{j,a}^{\dagger} c_{j,a} - p \sum_{j,a > p} c_{j,a}^{\dagger} c_{j,a} \right) \). This current can be represented by chiral bosonic fields \( \phi_{f_{L(n)}} \) according to \( J_{L(n)}^{00} = 4\pi \partial_x \phi_{f_{L(n)}} \). The order parameter in phase \( \mathcal{M}_3 \) can then be expressed in terms of the WZNW matrix fields \( g_{(p)}^{(p)}, g_{(q)}^{(q)} \) of \( SU(p)_1 \) and \( SU(q)_1 \):

\[
\text{Tr}(g) = e^{i \sqrt{4\pi \phi_f} \text{Tr}(g_{(p)})} + e^{-i \sqrt{4\pi \phi_f} \text{Tr}(g_{(q)})},
\]

with \( \phi_f = \phi_{f_{L}} + \phi_{f_{R}} \). A very similar expression holds for the order parameter in phase \( \mathcal{M}_1 \):

\[
\text{Tr}(\Omega(g)) = e^{i \sqrt{4\pi \phi_f} \text{Tr}(g_{(p)})} - e^{-i \sqrt{4\pi \phi_f} \text{Tr}(g_{(q)})}.
\]

Generically, the two order parameters have the same symmetry and one cannot exclude coexistence. The self-dual model is described by the following Hamiltonian density:

\[
\mathcal{H}_1 = \mathcal{H}_f + \mathcal{H}_{(p)} + \mathcal{H}_{(q)},
\]

where \( \mathcal{H}_{(p)} \) is the \( SU(p) \) GN model at level one (similarly for \( \mathcal{H}_{(q)} \)), so that the corresponding parallel degrees of freedom are fully gapped. The boson \( \phi_f \) decouples and is governed by:

\[
\mathcal{H}_f = \frac{v_f}{2} \left( (\partial_x \phi_f)^2 + (\partial_x \theta_f)^2 \right) + g_f J_{L(n)}^{00} J_{L(n)}^{00} = \frac{uf}{2} \left( \frac{1}{K_f} (\partial_x \phi_f)^2 + K_f (\partial_x \theta_f)^2 \right),
\]

\[
K_f = \frac{1}{4\pi} \text{Tr}(g_{(p)}) + \frac{1}{4\pi} \text{Tr}(g_{(q)}).
\]
which is nothing but the Luttinger Hamiltonian, with velocity \( u_f = \sqrt{v_f^2 - 4\pi^2 g_f^2} \) and Luttinger parameter \( K_f = \frac{v_f - 2\pi g_f}{\sqrt{v_f + 2\pi g_f}} \). This approach predicts a central charge \( c = 1 \) on the self-dual manifold. Of course, naively irrelevant operators (i.e. operators that are irrelevant with respect to the fully unperturbed SU(\( N \)) WZNW model) could be present that would spoil this criticality in the \( \phi_f \) sector. The situation is complex in general, and we will only investigate here the incommensurate case, thus ignoring umklapp oscillating terms. Of course, since we start with a Hamiltonian local in terms of the lattice fermions, the perturbing operators can be built out of the fermions \( \Psi_{l,k} \). The only \( \text{SU}(p) \times \text{SU}(q) \times U(1) \) candidates are of the form \( \mathcal{O}_k = \left( \prod_{j=1}^k \Psi_{a_j L}^\dagger \Psi_{a_j R} \right) \left( \prod_{j=1}^k \Psi_{b_j R}^\dagger \Psi_{b_j L} \right) + \text{H.c.} \), where antisymmetrization is implied in the indices \( a_j \leq p \) and \( b_j > p \), and thus necessarily \( k \leq p, q \). Moreover, the self-dual symmetry forces \( k \) to be an even integer. These operators can be expressed in terms of the \( \text{SU}(p)_1 \) and \( \text{SU}(q)_1 \) primary operators: \( \mathcal{O}_k \sim \text{Tr}(\Phi_{\lambda_k, \lambda_k}^{(p)}) \text{Tr}(\Phi_{\lambda_k, \lambda_k}^{(q)}) e^{i\sqrt{\frac{2\pi N}{pq}} \phi_f} + \text{H.c.} \), where in obvious notations \( \Phi_{\lambda_k, \lambda_k}^{(p)} \) denotes the \( \text{SU}(p)_1 \) primary operator in the “\( p \)” sector. On the self-dual manifold, it is legitimate to average out the gapped degrees of freedom, thus yielding the following effective perturbing operator in the \( \phi_f \) sector:

\[
\mathcal{O}_k \sim g \cos \left( \sqrt{\frac{4\pi N k^2}{pq}} \phi_f \right), 
\]

where \( g \propto \left\langle \text{Tr}(\Phi_{\lambda_k, \lambda_k}^{(p)}) \right\rangle_{\text{GN}} \left\langle \text{Tr}(\Phi_{\lambda_k, \lambda_k}^{(q)}) \right\rangle_{\text{GN}} \) (the average is taken in the \( \text{SU}(p) \) and \( \text{SU}(q) \) GN model) is a non-universal prefactor. The dominant perturbing operator is thus \( \mathcal{O}_2 \) with a scaling dimension (with respect to the Luttinger liquid fixed point) \( \Delta_2 = 4NK_f/pq \). We thus see that depending on the value of \( K_f \) the model exhibits different behaviors on the self-dual manifold: when \( K_f < pq/2N \), the operator \( \mathcal{O}_2 \) is relevant, and spoils criticality: in this case, model (48) describes a smooth cross-over between phases \( \mathcal{M}_\parallel \) and \( \mathcal{M}_\Omega \). On the other hand, if \( K_f > pq/2N \), there is a true continuous phase transition, with a \( c = 1 \) criticality emerging for the degrees of freedom describing the relative charge on the two clusters of fermionic chains. In the latter case, it is instructive to consider the Hamiltonian describing the vicinity of the self-dual manifold. Writing the currents \( J_\perp \) in terms of the \( \text{SU}(p)_1 \times \text{SU}(q)_1 \times U(1) \) variables yields:

\[
\mathcal{H}_\Omega = \mathcal{H}_{\Omega}^* + g_\perp \text{Tr} \left( g^{(p)} \right) \text{Tr} \left( g^{(q)\dagger} \right) e^{i\sqrt{\frac{4\pi N}{pq}} \phi_f} + \text{H.c.} 
\]

The perturbing operator is nothing but \( \mathcal{O}_1 \). Averaging out the gapped degrees of freedom (this is legitimate in the limit of small \( g_\perp \)), one concludes that the low-energy degrees of freedom close to the self-dual manifold, described by the bosonic field \( \phi_f \), are again governed by a sine-Gordon model. The scaling dimension of its cosine term \( \Delta_1 = NK_f/pq \) so that a gap opens if \( K_f < 2pq/N \).
Table III: Different classes of quantum phases transitions between the symmetry enlarged phases, away from half-filling.

| Duality Type | Criticality on the Self-Dual Manifold | Critical Degrees of Freedom |
|--------------|--------------------------------------|-----------------------------|
| A\(I\) (symmetric pairing) | No criticality (first order transition) | - |
| A\(II\) (antisymmetric pairing) | \(N = 2n \leq 8\): \(Z_n\) parafermions | Associated to the symmetry \(c_{i,a} \rightarrow e^{i\pi/N}c_{i,a}\) |
| | \(N = 2n > 8\): first order transition | - |
| A\(III\) (\(p\) and \(q\) bunches) | For \(K_f < \frac{pq}{2N}\): smooth crossover | - |
| | \(\frac{pq}{2N} < K_f < \frac{2pq}{N}\): \(c = 1\) transition | Relative charge of the two bunches of chains |
| | \(K_f > \frac{2pq}{N}\): \(c = 1\) pocket | Relative charge of the two bunches of chains |

As soon as \(g_{\perp} \neq 0\). If \(K_f > \frac{2pq}{N}\), the perturbation is irrelevant and one can only conclude that the \(c = 1\) criticality extends on both sides of the self-dual manifold. Though this criticality is doomed to disappear when one departs too far from the self-dual manifold, our approach does not allow to put a bound to this criticality in parameter space.

To restate our conclusions for the A\(III\) class, we find that the behavior close to the self-dual manifold depends on the value of the Luttinger parameter \(K_f\), that can vary continuously. If \(K_f < \frac{pq}{2N}\), one has a smooth crossover from phase \(\mathcal{M}_1\) to phase \(\mathcal{M}_\Omega\). If \(\frac{pq}{2N} < K_f < \frac{2pq}{N}\), there is a quantum phase transition with \(c = 1\), the self-dual model being a Luttinger liquid for the relative charge between the two clusters of chains. If \(K_f > \frac{2pq}{N}\), this criticality acquires a finite extension around the self-dual manifold.

Table III presents a summary of the quantum phase transitions that occur between the different classes for general fermionic models away from half-filling.

V. DUALITIES AT WORK: THE EXAMPLE OF THE TWO-LEG ELECTRONIC LADDER

In this section we will apply the duality approach, presented in the preceding sections, to the study of a specific example. The novelty of this approach stems from the fact the different possible symmetry enlarged phases can be exhausted by means of the algebraic properties of the groups \(G\) and \(H\), instead of solving numerically the one-loop RG flow and scanning the different phases. In addition, once a phase is identified and characterized, there is a systematic way to obtain the physics of the other duality-related phases. Finally, the duality approach also offers the possibility
to study directly phase transitions between the different symmetry enlarged phases, by investigating
the self-dual manifolds. We now turn to a careful examination of a particular example, namely a
generalized two-leg electronic ladder.

A. Lattice model

Here we apply the ideas exposed in the preceding sections to the model of two-leg Hubbard
ladder with Hamiltonian:

$$\mathcal{H} = -\sum_i \sum_{\ell \ell' \sigma \sigma'} t_{\ell \sigma, \ell' \sigma'} (c_{i, \ell \sigma}^{\dagger} c_{i+1, \ell' \sigma'} + \text{H.c.}) + \sum_i \sum_{\ell \sigma_j} U_{\ell \sigma_j} c_{i, \ell \sigma_1}^\dagger c_{i, \ell' \sigma_2}^\dagger c_{i, \ell' \sigma_3} c_{i, \ell' \sigma_4},$$

where $c_{i, \ell \sigma}^{\dagger}$ creates an electron with spin $\sigma = \uparrow, \downarrow$ on chain $\ell = 1, 2$, at site $i$. The explicit form of the hopping parameters $t_{\ell \sigma, \ell' \sigma'}$ and on-site couplings $U_{\ell \sigma_j}$ depends on the precise model under consideration, that can be specified by fixing the bare physical symmetry group $H$.

For incommensurate filling, there is a spin-charge separation and, as already discussed in section
II, the relevant maximal symmetry group $G$ supported by the two-leg ladder (52) is $G = SU(4)$ that
mixes the four different local states $\{c_{i, \ell \sigma}^{\dagger} | 0 \}_{\ell \sigma = 1, 2}$ (here $| 0 \rangle$ is the vacuum state). We now need to specify the bare symmetry group $H$. Here, we have chosen to present the example of a model that retains three basic symmetries: first, a natural $SU(2)$ symmetry acting in the spin sector; second, a $Z_2$ symmetry that exchanges the two chains of the ladder; and finally, a U(1) orbital symmetry generated by $N_1 - N_2$, where $N_\ell = \sum_{i \sigma} c_{i, \ell \sigma}^\dagger c_{i, \ell \sigma}$ is the relative charge on the two
chains. This leads to the following symmetry breaking pattern ignoring the U(1) charge symmetry:

$$G \twoheadrightarrow H = SU(2)^{\text{spin}} \times U(1)_{\text{orb}} \times Z_2 \times H_{\text{discrete}},$$

where $H_{\text{discrete}}$ accounts for the remaining discrete symmetries of model (52). As discussed later,
without affecting our conclusions regarding the number and labeling of phases, the constraint of the
U(1) orbital symmetry can also be relaxed: in this case, and in the generic case (i.e. with a
non-vanishing interchain hopping that is permitted in this case), we recover in the continuum limit
an effective $\tilde{U}(1)$ symmetry group, so that the symmetry breaking pattern (53) still holds. As a
consequence, the structure of the phase diagram, that relies on the pattern (53), will be the same.

Given the symmetry $H$, we can construct the Hubbard-like Hamiltonian exhibiting this symmetry
(for simplicity, we consider only on-site couplings) [53]. The hopping amplitudes are constrained
to take the value $t_{\ell \ell' \sigma \sigma'} = t \delta_{\ell \ell'} \delta_{\sigma \sigma'}$. As for the terms quartic in fermions, they are obtained by
considering the general two-fermion states at site $i$, $c_{i, \ell \sigma}^\dagger c_{i, \ell' \sigma'}^\dagger | 0 \rangle$. These states decompose under
the group \( H \) as \((1; 0) \oplus (0; 0) \oplus (0; \pm 1)\), where \((S; L)\) denotes the multiplet of spin \( S \) that carries orbital U(1) charge \( L \). Denoting the 6 local two-particle states by \(| S, S^z; L \rangle\), they read explicitly as follows:

\[
egin{align*}
|1, 1; 0\rangle &= c_{i1\uparrow}^\dagger c_{21\uparrow}^\dagger |0\rangle \\
|1, -1; 0\rangle &= c_{i1\uparrow}^\dagger c_{21\downarrow}^\dagger |0\rangle \\
|1, 0; 0\rangle &= \frac{1}{\sqrt{2}} \left( c_{i1\uparrow}^\dagger c_{21\downarrow}^\dagger + c_{i1\downarrow}^\dagger c_{21\uparrow}^\dagger \right) |0\rangle \\
|0; 1\rangle &= c_{i1\uparrow}^\dagger c_{i1\uparrow}^\dagger |0\rangle \\
|0; -1\rangle &= c_{21\uparrow}^\dagger c_{21\downarrow}^\dagger |0\rangle \\
|0; 0\rangle &= \frac{1}{\sqrt{2}} \left( c_{i1\uparrow}^\dagger c_{21\downarrow}^\dagger - c_{i1\downarrow}^\dagger c_{21\uparrow}^\dagger \right) |0\rangle,
\end{align*}
\]

and they allow for three different invariants under the group \( H \), namely \( \sum_{S^z=\pm 1} |1, S^z; 0\rangle \langle 1, S^z; 0|, |0, 0; 1\rangle \langle 0, 0; 1| + |0, 0; -1\rangle \langle 0, 0; -1|, \) and \(|0, 0; 0\rangle \langle 0, 0; 0|\). In this respect, we choose to parametrize the interacting-part of the Hamiltonian (52) as:

\[
H_{\text{int}} = \sum_i \left[ \frac{U}{2} \sum_{(\ell \sigma) \neq (\ell' \sigma')} n_{i,\ell \sigma} n_{i,\ell' \sigma'} + J_H \vec{S}_{i,1} \cdot \vec{S}_{i,2} + J_t (T^z_i)^2 \right],
\]

where \( n_{i,\ell \sigma} = c_{i,\ell \sigma}^\dagger c_{i,\ell \sigma} \) is the local electronic density for species \((\ell \sigma)\), \( \vec{S}_{i,\ell} = \frac{1}{2} \sum_{\sigma'} c_{i,\ell \sigma}^\dagger \vec{\sigma}_{\sigma'} c_{i,\ell \sigma} \) is the spin operator on chain \( \ell \), and \( T^z_i = \frac{1}{2} \sum_{\sigma} \left( n_{i,1 \sigma} - n_{i,2 \sigma} \right) \) is the difference of electronic densities on the two chains, that generates the U(1) orbital symmetry.

The resulting Hamiltonian depends on three microscopic couplings: an (overall) Coulombic interaction \( U \), a Hund coupling \( J_H \), and an “orbital crystal field anisotropy” \( J_t \). Playing with these three parameters allows to recover several limiting cases that have already been studied: when \( J_H = J_t = 0 \), one recovers the SU(4) Hubbard model, that has been extensively analyzed in recent years [42, 43, 54–56]. The case where only Hund coupling and Coulomb interaction are present \((J_t = 0)\) has been studied in Refs. 17 and 57, where it has been shown that the Hund coupling can stabilize a \( d \)-wave superconducting instability. When \( J_t = 3J_H/4 \), the symmetry is promoted to SU(2)_{spin} \times SU(2)_{orb}, and one recovers the so-called spin-orbital model [58–62]. When \( J_t = J_H/4 \), the symmetry is enlarged to SO(5) \times \mathbb{Z}_2, and this model has been studied in the context of cold-atoms physics [12, 13, 56, 63–65]. Finally, as it will be discussed later, we note that two-leg electronic ladders with a \( t_\perp \) hopping process in the low-energy limit have the symmetry group (53) [14] so that our duality approach is also relevant to this case.
B. Low-energy Hamiltonian

Let us now derive the continuous description of model (55). The procedure is standard and makes use of the continuum limit of the lattice fermionic operators $c_{i,l\sigma}$ in terms of right-and left-moving Dirac fermions $\Psi_{l\sigma R,L}$ [6, 7]:

\[
\frac{c_{i,l\sigma}}{\sqrt{a_0}} \simeq \Psi_{l\sigma R}(x)e^{-ik_fx} + \Psi_{l\sigma L}(x)e^{ik_fx},
\]

with $x = ia_0$. We now introduce the SU(4) currents which are defined from chiral fermion bilinears:

\[
J^A_{R,L} = 2\pi \Psi^\dagger_{l\sigma R,L} T^A_{l\sigma\ell\sigma',\ell'} \Psi_{l\sigma' R,L}, T^A \text{ being SU(4) generators in the fundamental representation which are normalized according to } \text{Tr}(T^A T^B) = \delta^{AB}.
\]

The set of these 15 generators $T^A$ can be organized as 3 SU(2) spin generators $T^a_s$, $a = x, y, z$, that mix spin indices $\sigma$, 3 SU(2) orb generators $T^a_t$, $a = x, y, z$, that mix orbital indices $\ell$, and 9 mixed spin-orbital generators that act simultaneously in spin and orbital spaces, $T^a_{st}$, $a, b = x, y, z$.

\[
\begin{align*}
T^a_s &= \frac{1}{2} \sigma^a \otimes \mathbb{1}, & T^a_t &= \frac{1}{2} \mathbb{1} \otimes \sigma^a, & T^a_{st} &= \frac{1}{2} \sigma^a \otimes \sigma^b, \\
\end{align*}
\]

where $\sigma^a$ are Pauli matrices and the first (respectively second) matrix in the tensor product acts on spin (orbital respectively) indices. Correspondingly, we will introduce the following decomposition for the 15 components of the SU(4) current:

\[
\begin{align*}
J^a_{SR(L)} &= 2\pi \Psi^\dagger_{l\sigma R(L)} (T^a_s)_{l\sigma\ell\sigma',\ell'} \Psi_{l\sigma' R(L)}, & J^a_{TR(L)} &= 2\pi \Psi^\dagger_{l\sigma R(L)} (T^a_t)_{l\sigma\ell\sigma',\ell'} \Psi_{l\sigma' R(L)} \\
J^a_{ST(R)} &= 2\pi \Psi^\dagger_{l\sigma R(L)} (T^a_{st})_{l\sigma\ell\sigma',\ell'} \Psi_{l\sigma' R(L)}. \\
\end{align*}
\]

For incommensurate filling, the low-energy Hamiltonian separates into two commuting pieces: $\mathcal{H} = \mathcal{H}_c + \mathcal{H}_s$ ($[\mathcal{H}_c, \mathcal{H}_s] = 0$) where the charge degrees of freedom are described by a Luttinger Hamiltonian:

\[
\mathcal{H}_c = \frac{v_c}{2} \left[ \frac{1}{K_c} (\partial_x \Phi_c)^2 + K_c (\partial_x \Theta_c)^2 \right],
\]

with the following Luttinger parameters for model (55):

\[
\begin{align*}
v_c &= v_F \left( 1 + a_0 \frac{6U - J_t}{2\pi v_F} \right)^{1/2} \\
K_c &= \left( 1 + a_0 \frac{6U - J_t}{2\pi v_F} \right)^{-1/2}.
\end{align*}
\]

For generic fillings, no umklapp terms appear and the charge sector displays metallic properties in the Luttinger liquid universality class. All non-trivial physics corresponding to the spin degeneracy
is described by the spin part, i.e., $\mathcal{H}_s$. Neglecting spin-velocity renormalization terms, the latter Hamiltonian corresponds to an anisotropic SU(4) model with marginal current-current interactions:

$$
\mathcal{H}_s = \frac{v v}{20\pi} (J_{Ja}^A J_{Ja}^A + J_{La}^A J_{La}^A) + \sum_{\alpha=1}^{5} g_{\alpha} J_{Jr}^A d_{\alpha}^A J_{Ja}^B,
$$

(61)

where the five matrices $d_{\alpha}$ encode the symmetry group (53) of the low-energy effective Hamiltonian. Those matrices are diagonal and they read explicitly (we choose the following ordering for the 15 SU(4) currents: $T_{s}^{a}, T_{t}^{a}, T_{st}^{x}, T_{st}^{y}, T_{st}^{z}$):

$\begin{align*}
\begin{pmatrix} 1 & 1 \\ 0 & 12 \\ \end{pmatrix},
\begin{pmatrix} 0 & 6 \\ 6 & 0 \\ \end{pmatrix},
\begin{pmatrix} 0 & 12 \\ 12 & 0 \\ \end{pmatrix},
\begin{pmatrix} 0 & 3 \\ 3 & 0 \\ \end{pmatrix},
\begin{pmatrix} 0 & 5 \\ 5 & 0 \\ \end{pmatrix},
\begin{pmatrix} 0 & 9 \\ 9 & 0 \\ \end{pmatrix},
\end{align*}$

(62)

In the continuum limit, and at first order in the lattice couplings ($U, J_t, J_H$), the coupling constants in Eq. (61) are given by:

$$
\begin{align*}
g_{1} &= -a_{0} \frac{2}{8\pi^{2}} (2U + J_t - J_H), \\
g_{2} &= a_{0} \frac{2}{16\pi^{2}} (-4U + 2J_t + J_H), \\
g_{3} &= -a_{0} \frac{2}{8\pi^{2}} (2U + J_t + J_H), \\
g_{4} &= -a_{0} \frac{16}{16\pi^{2}} (4U + 2J_t + 3J_H), \\
g_{5} &= -a_{0} \frac{2}{8\pi^{2}} (2U - 3J_t). \\
\end{align*}
$$

(63)

C. Duality approach

We are now in position to apply the general duality approach to the specific model (61) to fully determine the nature of its spin-gapped phases. The general phase diagram of the lattice model (55) will then be deduced.

1. Duality symmetries

The Hamiltonian in the spin sector (61) takes the form of an SU(4) anisotropic current-current model so that we can apply the general result of section IV for the special $N = 4$ case. In this respect, we find three non-trivial duality symmetries which exhaust the possible classes for incommensurate filling of the general classification listed in Table I.

A first one, to be called $\Omega_1$, belongs to the $AI$ class of symmetric pairing with $g_{\|} = so(4)$ (see Table I). It is associated to the following involutive element of the center of the symmetry group...
H: \( c_{\sigma} \rightarrow R_{ls,l's'} c_{l's'}^\dagger \), where the symmetric matrix \( R \) reads: 
\[ R = -\sigma^y \otimes \sigma^y \] (in a good basis \( \tilde{c}_a \) for the fermions, this corresponds to \( \tilde{c}_a \rightarrow \tilde{c}_a^\dagger \)). At the level of the SU(4)\(_1\) currents, \( \Omega_1 \) affects the following components:

\[ \Omega_1: \mathcal{J}^{ab}_{stl} \rightarrow -\mathcal{J}^{ab}_{stl}, \ a,b = x,y,z. \] (64)

This transformation is indeed a symmetry of Eq. (61) provided that the couplings are changed as follows: \( g_2 \rightarrow -g_2 \) and \( g_3 \rightarrow -g_3 \).

A second one, to be called \( \Omega_2 \), belongs to type AII of antisymmetric pairing with \( g_\parallel = \mathfrak{sp}(4) \). It is associated to the following antisymmetric charge conjugation symmetry: 
\[ c_{l\sigma} \rightarrow J_{l\sigma,l's'} c_{l's'}^\dagger, \] where the antisymmetric matrix \( J \) (the Sp(4) metric in the \( c_{l\sigma} \) basis) reads: 
\[ J = i\sigma^y \otimes \sigma^x. \] The SU(4)\(_1\) currents affected by \( \Omega_2 \) are:

\[ \Omega_2: \left\{ \begin{array}{l}
\mathcal{J}^x_{tl} \rightarrow -\mathcal{J}^x_{tl} \\
\mathcal{J}^y_{tl} \rightarrow -\mathcal{J}^y_{tl} \\
\mathcal{J}^{az}_{stl} \rightarrow -\mathcal{J}^{az}_{stl}, \ a = x,y,z.
\end{array} \right. \] (65)

It has the following representation on the couplings: \( g_3 \rightarrow -g_3 \) and \( g_4 \rightarrow -g_4 \).

The last non-trivial duality, \( \Omega_3 \), belongs to the AIII class with \( g_\parallel = \mathfrak{su}(2) \times \mathfrak{su}(2) = \mathfrak{u}(1) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2) \), and is associated with the orbital rotation of angle \( \pi \): 
\[ c_{l\sigma} \rightarrow -i(-)^c c_{l\sigma} \] which is indeed an involutive symmetry of the bare Hamiltonian commuting with all other symmetries. It affects the SU(4)\(_1\) currents as follows:

\[ \Omega_3: \left\{ \begin{array}{l}
\mathcal{J}^x_{tl} \rightarrow -\mathcal{J}^x_{tl} \\
\mathcal{J}^y_{tl} \rightarrow -\mathcal{J}^y_{tl} \\
\mathcal{J}^{ax}_{stl} \rightarrow -\mathcal{J}^{ax}_{stl}, \ a = x,y,z \\
\mathcal{J}^{ay}_{stl} \rightarrow -\mathcal{J}^{ay}_{stl}, \ a = x,y,z.
\end{array} \right. \] (66)

This duality has the following action on the couplings: \( g_2 \rightarrow -g_2 \) and \( g_4 \rightarrow -g_4 \).

Amongst those non-trivial dualities, two of them, namely \( \Omega_1 \) and \( \Omega_2 \), are outer dualities. It is coherent to christen the fourth duality, namely the trivial (identity) operation, \( \Omega_0 \). Note that the set of the 4 dualities \( \{\Omega_a\}_{a=0,1,2,3} \) has the structure of the Klein four-group, or \( \mathbb{Z}_2 \times \mathbb{Z}_2 \), with the following multiplication table for non-trivial elements: 
\[ \Omega_1\Omega_2 = \Omega_3, \ \Omega_1\Omega_3 = \Omega_2, \ \Omega_2\Omega_3 = \Omega_1. \]

Before we move on to the description of the phase diagram, it is useful to give an alternative form for the low-energy effective Hamiltonian. Indeed, there is a possible representation of the unperturbed SU(4)\(_1\) model in terms of 6 free Majorana fermions, which has been widely used in the literature on the 2-leg ladders [16–18, 20, 22, 61, 66, 67]. This Majorana description of
the SU(4)$_1$ WZNW model is the quantum translation of the classical equivalence between the Lie algebras of SU(4) and SO(6). As we will see, this equivalence also allows for a very convenient representation of the possible dualities, that are realized as SO(6) dualities.

To this end, one introduces six free massless Majorana fermions $\xi^a_{l(r)}$ that allow to represent the non-interacting part of model (61) as

$$\mathcal{H}_{s0} = -\frac{i e}{2} \sum_{a=1}^{6} \left( \xi^a_R \partial_x \xi^a_R - \xi^a_L \partial_x \xi^a_L \right). \quad (67)$$

Under SU(2)$_{\text{spin}}$ rotations, the three Majorana $\xi^a$ ($a = 1, 2, 3$) transform as a spin one, while the three remaining Majorana $\xi^a$ ($a = 4, 5, 6$) transform as a spin one under orbital rotations of SU(2)$_{\text{orb}}$. The SU(4)$_1$ currents can then be written as fermionic bilinears. Explicitly, the triplet of spin currents reads $J^a_{sl(r)} = -i \pi \epsilon^{abc} \xi^b_{l(r)} \xi^c_{l(r)}$, the triplet of orbital currents reads $J^a_{sl(r)} = -i \pi \epsilon^{abc} \xi^b_{l(r)} \xi^{c+3}_{l(r)}$, while the remaining 9 mixed spin-orbital SU(4)$_1$ currents read $J^{ab}_{s l(r)} = -2i \pi \xi^a_{l(r)} \xi^{b+3}_{l(r)}$ ($a, b = 1, 2, 3$). The full interacting Hamiltonian (61) can then be conveniently written as:

$$\mathcal{H}_s = \mathcal{H}_{s0} + 4\pi^2 \left[ g_1 \sum_{1 \leq a < b \leq 3} \kappa^a \kappa^b + g_2 \sum_{1 \leq a \leq 3} \kappa^a (\kappa^4 + \kappa^5) + g_3 \sum_{1 \leq a \leq 3} \kappa^a \kappa^6 + g_4 (\kappa^4 + \kappa^5) \kappa^6 + g_5 \kappa^4 \kappa^5 \right]. \quad (68)$$

where one has introduced the energy density (or “thermal operator”) $i \kappa^a = i \xi^a_{l(r)} \xi^a_{l(r)}$ of the Ising model associated to each of the 6 free Majorana fermion theories.

In terms of these fermionic variables, the three non-trivial dualities (64,65,66) read:

$$\Omega_1 : \xi^a_l \rightarrow -\xi^a_l \quad a = 4, 5, 6$$
$$\Omega_2 : \xi^6_l \rightarrow -\xi^6_l$$
$$\Omega_3 : \xi^a_l \rightarrow -\xi^a_l \quad a = 4, 5. \quad (69)$$

In other words, the non-trivial dualities can be constructed from “elementary” KW dualities on the Ising models attached to each Majorana fermion theory. This is very peculiar to the $N = 4$ case, i.e. two-leg ladders, and this result does not generalize for $N > 4$. Of course, restrictions on the way they can be combined come from demanding that they be compatible with H-invariance. Note that all of these dualities, once expressed in the fermionic language, become SO(6) dualities of type BDI (see Table I). By using the well-known Lie algebra isomorphisms $u(1) = \mathfrak{so}(2)$, $su(2) = \mathfrak{so}(3)$, $\mathfrak{sp}(4) = \mathfrak{so}(5)$ and $\mathfrak{so}(4) = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$, one can easily check that for each duality $\Omega_1, \Omega_2, \Omega_3$ respectively, the fermionic representation (69) yields the correct invariant subspace $\mathfrak{g}_\parallel$, namely $\mathfrak{g}_\parallel = \mathfrak{so}(3) \oplus \mathfrak{so}(3), \mathfrak{so}(5), \mathfrak{so}(2) \oplus \mathfrak{so}(4)$ respectively.
2. Phase diagram

According to the analysis of section IV, the knowledge of the different allowed dualities $\Omega_a$ gives direct access to the different symmetry enlarged phases $\mathcal{M}_a$. Explicitly, the lattice order parameters, that will develop quasi-long-range order in phase $\mathcal{M}_a$ at incommensurate filling are obtained from the “fundamental” CDW order parameter ($O_0^{(CDW)}$) by acting on one of the two involved lattice fermions with the involutive lattice symmetry associated to the duality $\Omega_a$ (see Table II). Using the dualities (69) that we have found, we can therefore readily conclude that model (55) with bare symmetry $H$ (53) can support the following symmetry enlarged phases:

- Two phases $\mathcal{M}_0$ and $\mathcal{M}_3$ with coexisting SP and CDW instabilities at wave vector $2k_F$. Both CDW-SP phases are distinguished by their properties under the $\mathbb{Z}_2$ exchange of the two chains: $\mathcal{M}_0$ is even while $\mathcal{M}_3$ is odd. The explicit lattice order parameters of $\mathcal{M}_{0,3}$ phases and their continuum descriptions read as follows:

$$\begin{align*}
\mathcal{M}_0 : & \quad O_0^{(CDW)} = O_0 = \sum_{\ell,\sigma} \Psi_{\ell L}^\dagger \Psi_{\ell R}, & & e^{-i2k_Fa_0} \sum_{\ell,\sigma} c_{j,\ell,\sigma} c_{j,\ell,\sigma} \\
& \quad O_0^{(SP)} = e^{ik_Fa_0} \sum_{\ell,\sigma} \Psi_{\ell L}^\dagger \Psi_{\ell R}, & & e^{-i2k_Fa_0} \sum_{\ell,\sigma} c_{j,\ell,\sigma} c_{j+1,\ell,\sigma} \\
\mathcal{M}_3 : & \quad O_3^{(CDW)} = O_3 = \sum_{\ell,\sigma} (-)^{\ell+1} \Psi_{\ell L}^\dagger \Psi_{\ell R}, & & e^{-i2k_Fa_0} \sum_{\ell,\sigma} (-)^{\ell+1} c_{j,\ell,\sigma} c_{j,\ell,\sigma} \\
& \quad O_3^{(SP)} = e^{ik_Fa_0} \sum_{\ell,\sigma} (-)^{\ell+1} \Psi_{\ell L}^\dagger \Psi_{\ell R}, & & e^{-i2k_Fa_0} \sum_{\ell,\sigma} (-)^{\ell+1} c_{j,\ell,\sigma} c_{j+1,\ell,\sigma}
\end{align*}$$

- Two phases $\mathcal{M}_1$ and $\mathcal{M}_2$ characterized by a superconducting pairing $O_1$ ($O_2$ respectively) instability at wave vector $k = 0$, are of BCS type with s-wave symmetry ($d$-wave symmetry respectively). The two BCS phases, are distinguished by the symmetry of the pairing operators: $O_1$ (BCS$s$) is odd under chain exchange, while $O_2$ (BCS$d$) is even.

$$\begin{align*}
\text{BCS}s = \mathcal{M}_1 : & \quad O_s^{(SC)} = O_1 = \Psi_{aL} R_{ab} \Psi_{bR} \leftrightarrow c_{j,a} R_{ab} c_{j+1,b} = c_{j+1,\uparrow,\downarrow} c_{j,\downarrow,\uparrow} - c_{j,\uparrow,\downarrow} c_{j+1,\downarrow,\uparrow} - (1 \leftrightarrow 2) \\
\text{BCS}d = \mathcal{M}_2 : & \quad O_d^{(SC)} = O_2 = \Psi_{aL} J_{ab} \Psi_{bR} \leftrightarrow c_{j,a} J_{ab} c_{j,b} = c_{j,\uparrow,\downarrow} c_{j,\downarrow,\uparrow} - c_{j,\uparrow,\downarrow} c_{j,\downarrow,\uparrow},
\end{align*}$$

with $a = (\ell, \sigma)$ and $b = (\ell', \sigma')$. The BCS instability is accompanied with a generalized spin superfluidity, that would reveal itself for example in a rotating system put on a ring, with a quantization of the current in a mixed spin-orbital direction. As it has been described in section IV C, this result stems from the fact that $\Omega_1$ and $\Omega_2$ are outer dualities. Moreover, if the charge sector is gapped by some process (for example, by considering a suitable commensurate filling away from half filling), one can also characterize the BCS phases by the appearance of coherence of the spin excitations ($\delta$-peak in the zero temperature spin structure factor), contrarily to the CDW-SP phases which correspond to inner dualities.
The general analysis of section IV also allows for a determination of the quantum phase transitions between the SU(4)-symmetry enlarged phases: there are as many different transitions as there are non-trivial dualities. Close to a quantum phase transition, the physics of the two-leg ladder spin sector is captured by a minimal theory with special self-dual symmetry (which is simply the invariant subspace of the $\mathbb{Z}_2$ grading $\Omega_a$), which is maximal in some sense: quantum phase transitions require in general that the bare anisotropy be large enough so that DSE cannot fully, but only partially, develop. What happens is sketched in the following diagram:

\[
\begin{array}{ccc}
\text{SO(3)×SO(3)} & \text{SO(4)×SO(2)} & \text{SU(4)} \\
\text{SO(3)×U(1)×Z}_2 & \text{SO(5)×Z}_2
\end{array}
\]

with the bare theory symmetry indicated on the left, and the RG flows towards low energy when moving to the right. In generic situations (i.e. far away from multicritical points), there is a hierarchy of couplings and DSE develops in several steps (in our case, two steps) as long as the energy cut-off is reduced. Close to a quantum phase transition this process is stopped at an intermediate stage, and the low-energy theory has an intermediate symmetry that eventually labels the quantum phase transition. We list in Table IV the three different quantum phase transitions occurring in the model.

| Symmetry                  | Order | Critical degrees of freedom |
|---------------------------|-------|-----------------------------|
| $\mathcal{M}_0 \leftrightarrow \mathcal{M}_1$ | 1     | $g_1, g_4 = g_5 > 0$       |
| and $\mathcal{M}_2 \leftrightarrow \mathcal{M}_3$ | 2     | $\text{SO(3)}_1 \ (c = \frac{\pi}{2})$ $g_1g_4 = g_1g_5 < 0$ |
| $\text{SO(5)×Z}_2$       | 2     |                               |
| and $\mathcal{M}_0 \leftrightarrow \mathcal{M}_2$ | 2     | $\text{SO(6)}_1 \ (c = 3)$ $g_1, g_4 = g_5 < 0$ |
| and $\mathcal{M}_3 \leftrightarrow \mathcal{M}_1$ | 2     | $\text{Ising} \ (c = \frac{1}{2})$ $g_1 = g_2 = g_5 > 0$ |
| $\text{SO(4)×SO(2)}$    | 2     |                               |
| and $\mathcal{M}_0 \leftrightarrow \mathcal{M}_3$ | 2     | $\text{L.L.} \ (c = 1)$ $g_1 = g_3 > 0$ |
| and $\mathcal{M}_2 \leftrightarrow \mathcal{M}_1$ | 2     | $\text{SO(6)}_1 \ (c = 3)$ $g_1 = g_3 < 0$ |

TABLE IV: Summary of the different possible quantum phase transitions for the generalized two-leg electronic ladder (61) for incommensurate filling.

The next question is whether those four phases are actually realized in the lattice model (55), and what are the values of the microscopic couplings that will yield each of those phases. To answer
this question, one needs to perform, in a weak-coupling approach, a RG analysis of the continuous effective theory (61). The two-loop $\beta$-function can be obtained for example in a point splitting regularization [24] to yield:

\[
\begin{align*}
\dot{g}_1 &= (1 - g_1)(g_1^2 + 2g_2^2 + g_3^2) \\
\dot{g}_2 &= g_2(2g_1 + g_3) + g_3g_4 - \frac{g_2}{2}(2g_1^2 + 3g_2^2 + g_3^2 + g_4^2) \\
\dot{g}_3 &= 2g_1g_3 + 2g_2g_4 - g_3(g_1^2 + g_2^2 + g_3^2 + g_4^2) \\
\dot{g}_4 &= g_4g_5 + 3g_2g_3 - \frac{g_4}{2}(3g_2^2 + 3g_3^2 + g_4^2 + g_5^2) \\
\dot{g}_5 &= (1 - g_5)(3g_2^2 + g_4^2),
\end{align*}
\]

where $\dot{g} = \frac{dg}{d \ln(a/a_0)}$, with $a_0^{-1}$ the UV cutoff (which is of the order of the inverse of the lattice spacing) and $a^{-1}$ a running RG scale, and the couplings in Eqs. (72) have been rescaled according to $g_a \rightarrow 2\pi g_a/v_F$.

One observes that the RG equations are invariant under the duality action on the couplings, as it should be (in fact, since dualities are exact symmetries of the continuous model (61), the all-order $\beta$-function is invariant). Numerical integration of the RG equations yield the phase diagram presented in figure 3. The four possible symmetry enlarged phases $M_0$ to $M_3$ are reached within this model for repulsive Coulombic interaction $U$ (see figure 3).

DSE ensures that all bare theories sitting between the transition lines (bold lines) has a low-energy physics described by an adiabatic deformation of the corresponding representative ($\Omega_a$-twisted) SU(4) GN model. Closer to transition lines, a cross-over regime occurs (gray areas). Even closer to them, one finds again a universal regime described by the quantum phase transition. Note that the curvature of the phase transition lines visible in figure 3, is not a two-loop effect, but rather stems from the non-linearity of the flow equation that are already present at one-loop order. One notices a quite complex topography of the phase diagram, with for example an $s$-wave superconducting pocket for ferromagnetic Hund coupling.

D. Allowing inter-leg hopping

In this section we show how the previous discussion can also be applied to the case of a two-leg ladder with legs connected by transversal hopping $t_\perp$, with a SU(2) spin symmetry, and a $Z_2$ symmetry that exchanges the two legs, but no U(1) orbital symmetry. This model has been examined in several works, with the conclusion that there were four phases at incommensurate fillings [14, 18, 66, 68, 69]. We will see that this model is in fact in closely related to the SU(2)×U(1)×Z_2...
FIG. 3: Phase diagram of model (55). The Coulombic repulsion $U = 0.01t$ is fixed, while Hund coupling $J_H$ and orbital anisotropy $J_t$ are varied. Bold lines indicate the location of the transitions between the different symmetry enlarged phases $\mathcal{M}_a$ whose nature is discussed in the text. The small pocket lying between points $O$ and $D$ is of type $\mathcal{M}_3$. The dashed line extending between points $O$ and $A$ is fully critical, with central charge $c = 3$. Thin rays correspond to fine-tuned bare theories with a larger, exact, bare symmetry $H'$ with $H=SU(2)\times U(1)\times \mathbb{Z}_2 \subset H' \subset SU(4)=G$. On those lines, the effective theory has symmetry $H'$ at any stage of the RG flow, but flows at low energies to a symmetry enlarged phase with symmetry $SU(4)$ in the sense of section III B. On point $O$, the bare theory is maximally symmetric, with a $SU(4)$ invariance. On point $A = (-8U, -2U)$, the continuous theory enjoys an enlarged twisted $SU(4)$ symmetry $\Omega_2(SU(4))$. Self-dual manifolds intersect the $(J_H, J_T)$ plane at the discrete points $B = (-\frac{8}{5}U, -\frac{2}{5}U)$, $C = (-8U, 6U)$ and $D = (0, 2U)$, with $\Omega_1(C) = C$, $\Omega_2(B) = B$ and $\Omega_3(D) = D$.

model studied in the preceding subsections: it is simply obtained therefrom by acting with a duality.

Introducing the bonding and anti-bonding modes that diagonalize the kinetic term, and expanding those modes around the corresponding two Fermi points $k_{\uparrow 1}$ and $k_{\uparrow 2}$, one obtains fermionic fields $\tilde{\Psi}_{\ell\sigma p}$, $\sigma = \uparrow, \downarrow$, $\ell = 1, 2$, $p = L, R$. The Fermi wave vectors satisfy $k_{\uparrow 1} \equiv k_{\uparrow 1} + k_{\uparrow 2} = \frac{\pi n}{a_0}$, with $n$ the electronic density, while the difference $k_{\uparrow 2} \equiv k_{\uparrow 1} - k_{\uparrow 2}$ is a function of $t_\perp$ and $n$. At fillings and $t_\perp$ such that $k_{\uparrow 2}$ is incommensurate, it is easy to see that for an arbitrary $2N$-fermion term $\prod_{i=1}^N \tilde{\Psi}_{\ell_1\sigma_1 p_1} \tilde{\Psi}_{\ell_2\sigma_2 p_2}'$ to conserve lattice momentum, it has to conserve separately the combinations $\rho_+ = N_{1R} + N_{2L}$ and $\rho_- = N_{2R} + N_{1L}$, where $N_{\ell\sigma} = \sum_\sigma \tilde{\Psi}_{\ell\sigma p}^\dagger \tilde{\Psi}_{\ell\sigma p}$ is the total number of chiral
fermions in mode $\ell$. Therefore, on top of conserving the SU(2) spin and the total number of electron $\rho_+ + \rho_-$, the Hamiltonian has to conserve the combination $I_\ell = \frac{1}{2} (\rho_+ - \rho_-) = J_{\ell L}^x - J_{\ell L}^z$. This operator is nothing but the “orbital current”, i.e. the space-like component of the Noether current associated with U(1) orbital symmetry. It results that the model generically enjoys in fact a larger symmetry SU(2) × $\tilde{\text{U}}(1) \times \mathbb{Z}_2$, where the twisted orbital symmetry $\tilde{\text{U}}(1)$ is generated by $I_{\text{t}}$.

Therefore, in the continuum limit one recovers exactly the model studied in the preceding subsections, provided one performs a duality $\Omega$ that changes the sign of the following component of the currents $\bar{\Omega}$:

$\bar{\Omega} : \begin{cases}
J_{\ell L}^x & \rightarrow -J_{\ell L}^x \\
J_{\ell L}^z & \rightarrow -J_{\ell L}^z \\
J_{\text{stL}}^a & \rightarrow -J_{\text{stL}}^a, \quad a = x, y, z.
\end{cases}$ \hspace{1cm} (73)

One thus immediately deduces that the maximal number of symmetry enlarged phases is four. Since (73) corresponds to an outer automorphism of $\mathfrak{su}(4)$, one readily knows that pairing phases are mapped onto density-wave phases, and vice versa. Note that in terms of the six Majorana fermions $\xi^a$, this duality bears the very simple form: $\xi^5_L \rightarrow -\xi^5_L$. Using the representation of $\Omega$ on the original bonding and anti-bonding fermions:

$\tilde{\Psi}_{\ell\uparrow} \rightarrow +\tilde{\Psi}_{\ell\uparrow}, \quad \tilde{\Psi}_{\ell\downarrow} \rightarrow -\tilde{\Psi}_{\ell\downarrow}$ \hspace{1cm} (74)

one can readily identify the four phases $\mathcal{M}_a = \bar{\Omega}(\mathcal{M}_a)$ of the $t_\perp$-ladder, with order parameters $\bar{O}_a = \bar{\Omega}(O_a)$ that read explicitly:

$\bar{O}_0 = \sum_\ell \left( \tilde{\Psi}_{\ell\uparrow L} \tilde{\Psi}_{\ell\uparrow R} - \tilde{\Psi}_{\ell\downarrow L} \tilde{\Psi}_{\ell\downarrow R} \right)$

$\bar{O}_1 = \sum_\sigma \left( \tilde{\Psi}_{1\sigma L} \tilde{\Psi}_{2\sigma R} - \tilde{\Psi}_{2\sigma L} \tilde{\Psi}_{1\sigma R} \right)$

$\bar{O}_2 = \sum_\sigma \left( \tilde{\Psi}_{1\sigma L} \tilde{\Psi}_{1\sigma R} + \tilde{\Psi}_{2\sigma L} \tilde{\Psi}_{1\sigma R} \right)$

$\bar{O}_3 = \sum_\ell (-)^{\ell+1} \left( \tilde{\Psi}_{\ell\uparrow L} \tilde{\Psi}_{\ell\uparrow R} - \tilde{\Psi}_{\ell\downarrow L} \tilde{\Psi}_{\ell\downarrow R} \right)$.

These phases correspond to the four phases found in generalized two-leg ladders with interchain hopping for incommensurate filling: coexistence of CDW and SP phases, d-wave and s-wave superconducting phases (DSC and SSC respectively) and coexistence of time-reversal breaking phases.

---

\footnote{Our choice is dictated by the following constraints: $\bar{\Omega}$ has to preserve the SU(4) Lie algebra (it is an automorphism), it has to leave $J_{\text{t}}^z$ invariant, and has to change the sign of $J_{\text{t}}^x$. There is in fact a whole family of solutions, that are labelled by an angle $\alpha$. Of course, our conclusions do not depend on this choice.}
like d-density wave (DDW) and diagonal current (DC) phases (see for instance Ref. [69]). The explicit connection, together with lattice order parameter in terms of the original fermions $c_1$ and $c_2$ on the two legs of the ladder, is given by:

$$SSC = \tilde{\mathcal{M}}_0: \Delta_s = -\tilde{\mathcal{O}}_0$$

$$DSC = \tilde{\mathcal{M}}_3: \Delta_d = -\tilde{\mathcal{O}}_3$$

$$CDW+SP = \tilde{\mathcal{M}}_2: O^{CDW} = 2\Re(e^{ik_r x} \tilde{\mathcal{O}}_2)$$

$$O^{SP} = 4\cos(k_{r+1/2})\Re(e^{i k_{r+1/2} x} \tilde{\mathcal{O}}_2)$$

$$DDW+DC = \tilde{\mathcal{M}}_1: O^{DDW} = -2\Im e^{ik_r x} \tilde{\mathcal{O}}_1$$

$$O^{DC} = 4\sin(k_{r+1/2})\Re(e^{ik_{r+1/2} x} \tilde{\mathcal{O}}_1)$$

We thus see that the duality approach allows to capture the four phases of the generalized two-leg ladder with interchain hopping at incommensurate filling. In addition, the nature of the different quantum phase transitions between these phases is still determined by Table IV. Finally, our approach leads us to conclude that more exotic phases can only be found by (i) considering theories with large anisotropic bare couplings, i.e. theories that flow, in the IR limit, close to the quantum phase transitions described previously, or by (ii) breaking the remaining bare SU(2)\text{spin} \times \mathbb{Z}_2 symmetry.

### VI. CONCLUSIONS

In this paper, we have developed a general non-perturbative approach to describe spin-gapped phases of weakly interacting one-dimensional degenerate fermions. In the continuum limit, the low-energy properties of these systems are described by a WZNW CFT perturbed by marginal relevant current-current interaction with H invariance. At the heart of the analysis is the existence, in this general class of model, of emergent duality symmetries which enable one to relate different competing orders between themselves and to shed light on the nature of the zero-temperature quantum phase transitions. In particular, we have shown that these dualities can be classified and depend on the algebraic properties of the problem: the physical symmetry group H of the system and the maximal continuous symmetry group of the interaction G. For $N$-component degenerate fermions, this maximal symmetry group is $G = \text{SU}(N)$ or $\text{SO}(2N)$ away from half-filling and at half-filling respectively. The duality symmetries can be identified as the involutions which belong to the center group $C(H)_{\text{inv}}$ of the symmetry group H of the problem so that for each duality, there is an involutive $\mathbb{Z}_2$ lattice symmetry that gives rise to it. Alternatively, based on the DSE phenomenon, the duality symmetries are also in one-to-one correspondence with the
different symmetry enlarged phases that can be supported by the generic fermionic model with marginal current-current interactions. Those symmetry enlarged phases are fully gapped phases, and all meet at the multicritical point, that is obtained by fine-tuning all interactions to zero. Moreover, and this should not be a surprise since we are dealing with one-dimensional models, those massive phases do not break spontaneously any continuous symmetries. Hence, our approach allows to draw a picture of a quite large class of possible generalized spin-liquid phases supported by coupled fermionic chains in one dimension, at least those that develop close to the multicritical (non-interacting) point.

In the course of our study, there naturally appears a strong distinction between two kinds of gapped spin-liquid phases for incommensurate filling according to whether the duality they are associated to is an *inner* and *outer* automorphism. This quite mathematical distinction turns out to have important physical signatures. “Inner” phases, appear to display “conventional” properties, in the following sense: the ordering is of charge-density type, with a lattice order parameter of the form $c_{j,a}^\dagger M_{ab} c_{j,b}$. On the other hand, “outer” phases display off-diagonal order, with the development of quasi-long-range correlations for pairing operators $c_{j,a}^\dagger M_{ab} c_{j,b}^\dagger$. This superconducting instability is accompanied by spin superfluidity: the low-energy collective modes carry spin currents. In spite of the gap, the system has non-vanishing susceptibility in some spin directions, that results for example in a quantization of the spin current for a rotating system put on a ring. All these “unconventional” properties can be ultimately connected to the spontaneous breaking of a discrete symmetry of the bare theory: the discrete phase redefinition of the fermions (43), which it is tempting to connect to the usual breaking of U(1) gauge invariance in a superconducting state in higher dimensions.

We can also address the issue of quantum phase transitions between the different phases, by means of “minimal theories” interpolating between them. The nature of the phase transition between the two symmetry enlarged phases $\mathcal{M}_{\Omega_1}$ and $\mathcal{M}_{\Omega_2}$ (associated to dualities $\Omega_1$ and $\Omega_2$) depends only on the duality $\Omega = \Omega_1 \Omega_2$, so that the same set of dualities can be used to label those minimal models. The self-dual manifold of these minimal models exhibit in general criticality, that captures the nature of the quantum phase transition between the phases. We have shown in the specific example of two-leg electronic ladders at incommensurate filling how this approach allows for an immediate determination of the possible symmetry enlarged phases on the basis of the analysis of the bare symmetry group (of the continuous theory) and the determination of the different quantum phase transitions.

As perspective, it would be interesting to study the interplay of symmetry enlargement with
doping – i.e. how commensurability generically affect the general picture of the phase diagram. In this respect, the half-filled case is very special since there is no spin-charge separation and the charge degrees of freedom cannot be disentangled from the spin ones due to an umklapp process. As it has been discussed in this paper, the relevant maximal symmetry group for $N$ fermionic species at half-filling is $G = \text{SO}(2N)$. The detail of the analysis of the dualities for this case turns out to be more complicate than for the incommensurate case, i.e. with $G = \text{SU}(N)$, and will be investigated elsewhere. Dualities of the kind studied in this work can have further applications in other contexts. For example, in degenerate quantum impurity problems where a localized spin is coupled to electronic spin currents, they can be used to relate different IR boundary fixed points. More generally, the duality symmetries can be viewed as automorphisms of the fusion algebra of a CFT. We focused here on WZNW models perturbed marginally by current-current interactions. Generalized dualities could be used to shed light on the phase diagram of other CFT’s perturbed by relevant operators. In this respect, this case will be useful to investigate the possible classification of 1D spin-liquid phases where the spin gap is opened by a strongly relevant perturbation. We hope to come back to these issues elsewhere.

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APPENDIX A: CHARACTERIZATION OF DUALITIES

In this Appendix, we first characterize the set of linear transformations of the currents that leave the Hamiltonian (5) globally invariant, by (i) showing that these transformations can be decomposed in the product of diagonal and chiral transformations and by (ii) showing that the chiral transformations, that correspond to dualities, are given by Eq. (13), i.e. they have to belong to the involutive part of the center of the bare symmetry group $H$. Then we show that for a given model, 15 An exception occurs for $N = 4$, the 2-leg electronic ladder, for which the group identity $\text{SU}(4)\sim\text{SO}(6)$ allows to factorize the $\text{SO}(8)$ dualities into a charge part and a spin part. One can then show that with respect to the incommensurate case, a doubling of the number of allowed dualities occurs, yielding 8 different phases for the example studied in section V. It is important to realize that the triality – a remarkable, non-involutive automorphism of $\text{SO}(8)$ – that was invoked in Ref.14 in the study of the half-filled 2-leg electronic ladder has nothing to do with the dualities studied here. Rather, it is connected to the identity $\text{SU}(4)\sim\text{SO}(6)$, and helps understanding why $N = 4$ is so particular. We shall come back to this in a forthcoming publication.
the set of dualities coincides with the different possible inequivalent symmetry enlargements. We recall that $\mathcal{D}$ denotes the set of allowed dualities, $D$ is the vector space spanned by the matrices $d^\alpha$. In particular, $D$ is the set of real (from hermiticity) symmetric (from parity) matrices that commute with the action of $H$. From renormalizability, $D$ is closed under anticommutation: $\{D, D\} \subset D$.

1. Dualities are the involutive part of the center of $H$

We investigate here the general case of a transformation $\Omega$, that acts both on the currents and the coupling constants, such as to leave Hamiltonian (5) globally invariant. $\Omega$ acts linearly on the currents, and one can write its action as $\omega_l \times \omega_r$ as in section III:

$$\tilde{J}^A_{L(R)} = (\omega_l(R))^AB \tilde{J}^B_L(R),$$

(A1)

while it changes the coupling constants as $g_\alpha \rightarrow \tilde{g}_\alpha = \Omega(g_\alpha)$. From the invariance of the current OPE’s (6), it follows that both $\omega_l$ and $\omega_r$ must be automorphisms of $G$. One useful remark is that $G$-automorphisms automatically belong to $O(\dim(G))$, i.e. the matrices $\omega_{l(r)}$ are orthogonal. The covariance of the interacting part of the Hamiltonian (5) leads to an identity that generalizes Eq. (12):

$$g_\alpha d^\alpha = \tilde{g}_\alpha t^\alpha \omega_r d^\alpha \omega_l,$$

(A2)

where $t^\omega_r$ denotes the transpose of the matrix $\omega_r$. One starts by decomposing $\omega_l \times \omega_r = (\omega_r \times \omega_l) \circ (\omega \times 1)$, with $\omega = \omega_l \omega_r^{-1}$, as the product of a diagonal and a chiral part. Considering the special isotropic ray $g_\alpha = g_\alpha^0$, with $g_\alpha^0 d^\alpha = 1$, and multiplying (A2) by $\omega_r$ on the left, and by $t^\omega_l$ on the right, one deduces that the matrix $\omega^{-1}$ belongs to $D$: $\omega^{-1} = f_\alpha d^\alpha$, with $f_\alpha = \Omega(g_\alpha^0)$. It results that $\omega^{-1}$ is a symmetric matrix; but $\omega$ being also orthogonal, one has $\omega^{-1} = t^\omega = \omega$. We deduce thus

$$\omega \in D, \text{ and } \omega^2 = 1.$$  

(A3)

Now, we want to show that the diagonal rotation $\omega_r \times \omega_r$ alone leaves the Hamiltonian (5) globally invariant. This is equivalent to showing that $t^\omega_r d^\alpha \omega_r \in D$ for all $\alpha$. Multiplying relation (A2) by $\omega_r$ on the left and by $\omega_r^{-1}$ on the right, one gets the following relation, valid for any couplings $g_\alpha$:

$$g_\alpha \omega_r d^\alpha t^\omega_r = \tilde{g}_\alpha d^\alpha \omega.$$  

(A4)
Transposing this relation, and recalling that \( \omega \) and \( d^\alpha \) are symmetric, we deduce that \( \omega \) commutes with \( d^\alpha \) for all \( \alpha \) (for this, one chooses couplings \( g_\beta = \Omega^{-1}(\delta_{\alpha\beta}) \), so that \( \tilde{g}_\beta = \delta_{\alpha\beta} \)). It results that 
\[
\omega d^\alpha = \frac{1}{2}\{d^\alpha, \omega\} \in D \quad \text{since} \quad \{D, D\} \subset D.
\]
We deduce then from Eq. (A4) the announced result:
\[
t^\omega_r d^\alpha \omega_r \in D, \tag{A5}
\]
i.e. the diagonal rotation \( \omega_r \times \omega_r \) leaves the Hamiltonian (5) globally invariant. This proves our claim that the general transformation \( \Omega \) given by Eq. (A1) can be decomposed into the product of a diagonal rotation, that corresponds to a global change of basis, and of a transformation \( \omega \) affecting only one chirality sector, both of these transformations separately leaving the Hamiltonian (5) globally invariant.

We now proceed to characterize the set of chiral transformations \( \omega \), that it is legitimate to term dualities in view of their involutive character. To study the dualities, we therefore set \( \omega_r = \mathbb{1} \) in the following, with no loss of generality, so that \( \Omega \) is a transformation of the left currents and of the couplings, that satisfies:
\[
g_\alpha d^\alpha = \tilde{g}_\alpha d^\alpha \omega. \tag{A6}
\]
We already know that \( \omega \) belongs to \( D \). Moreover, from Eq. (A6), fixing \( \alpha \) and choosing \( g_\beta = \Omega(\delta_{\alpha\beta}) \) (i.e. \( \tilde{g}_\beta = \delta_{\alpha\beta} \)), one gets \( d^\alpha \omega = \Omega(\delta_{\alpha\beta})d^\beta \in D \). Transposing this relation, and recalling that \( d^\beta \) and \( \omega \) are symmetric, we deduce that \( [\omega, d^\alpha] = 0 \), or \( \omega d^\alpha \omega = d^\alpha, \forall \alpha \). It results that \( \omega \) is an element of the physical symmetry group \( H \). Since \( \omega \in D \) and \( [H, D] = 0 \), \( \omega \) is in the center of \( H \). Thus, one has proved that \( \mathcal{D} \subset C(H)|_{\text{inv.}} \).

Reciprocally, let us take some \( \omega \in C(H)|_{\text{inv.}} \). Being involutive and orthogonal, \( \omega \) has to be a real symmetric matrix. Furthermore, it commutes with \( H \). This is enough to ensure that \( \omega \in D \). To show that \( \omega \) is a duality, it suffices to prove that Eq. (A6) holds. This is easily done: \( d^\alpha \omega = \frac{1}{2}\{\omega, d^\alpha\} \in D \).

This ends the proof of Eq. (13).

### 2. Dualities correspond to the different possible DSE’s

It is not difficult to realize that each duality \( \Omega \in \mathcal{D} \) defines a different DSE. Considering the G-isotropic model \( \mathcal{M}_0 \) with interacting part of the Hamiltonian \( \mathcal{H}^{0\text{int}}_{\text{int}} = g \mathcal{J}^A_r \mathcal{J}^A_l \) and acting on it with a duality \( \Omega \in \mathcal{D} \), one gets the model \( \mathcal{M}_\Omega \) with interacting part \( \mathcal{H}^{\Omega\text{int}}_{\text{int}} = g \mathcal{J}^A_r \Omega(\mathcal{J}^A) = g \mathcal{J}^A_r \tilde{\mathcal{J}}^A_l \).

Since \( \tilde{\mathcal{J}}^A_l = \Omega(\mathcal{J}^A) \) are the generators of
the (twisted) group \( \tilde{G}_l \). \( \mathcal{M}_\Omega \) is thus obviously invariant under the group \(( \tilde{G}_l \times G_r )_{\text{diag}}\) generated by \( J^A_l + \tilde{J}^A_l \).

Reciprocally, let us suppose there exists some global invariance group \( \tilde{G} \), isomorphic to \( G \), that leaves invariant a model of the form (5), with (global) generators \( \tilde{Q}^A \) linearly related to the original ones in each chirality sector, \( \tilde{Q}^A = \int dx (\tilde{J}^A_l + \tilde{J}^A_r) \), with \( \tilde{J}^A_{l(r)} = \omega^{AB} J^B_{l(r)} \). Then, considering the non-interacting part of the Hamiltonian, it follows that \( \omega_{l(r)} \) must belong to \( \text{Aut}(G) \); considering the interacting part, which must be of the form \( g \tilde{J}^A_l \tilde{J}^A_r \) from \( \tilde{G} \)-invariance, leads to the conclusion that \( \omega \equiv \omega^{-1} \omega_l \in \text{D} \). Since \( \omega \) belongs to \( \text{D} \cap \text{Aut}(G) \), following the same line of reasoning as in A 1, we conclude that \( \omega \in \mathcal{C}(\text{II})|_{\text{inv.}} = \mathcal{D} \). Thus, up to a global rotation \( \omega_r \times \omega_l \) (affecting identically both chirality sectors), the generators \( \tilde{Q}^A \) are nothing but the dual generators \( \int dx (J^A_r + \omega^{AB} J^B_l) \).

This completes the proof.

Hence, it results that the set of dualities identifies with the different possible DSE patterns, i.e. they correspond to the different possibilities to glue together the two chiral invariance groups \( G_l \) and \( G_r \) in a way consistent with \( \text{H-inv} \).

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