THE SIMILARITY TO QUANTUM STATE SPACE, GENERALISED QPLEXES AND 2-DESIGNS

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ABSTRACT: We study the class of quantum measurements with the property that the image of the set of quantum states under the measurement map is similar to this set. This leads to the notion of generalised qplexes, where SIC-POVMs are replaced by a larger class of 2-design POVMs.

1. Introduction

Over the last ten years in a series of papers [1,2,11,14,29] Fuchs, Schack, Appleby and the co–authors have introduced first the idea of QBism (quantum Bayesianism) and then its probabilistic embodiment - the (Hilbert) qplex, both based on the notion of symmetric informationally complete positive operator valued measure (SIC-POVM). In spite of the fact that SIC-POVMs do indeed exhibit some unique properties that distinguish them among other IC-POVMs [8,9,23], we believe that similar (or, in a sense, even richer from purely mathematical point of view) structures, say, generalised qplexes, can be successfully studied and applied, if we replace SIC-POVMs by a broader class of rank-1 measurements, namely, those generated by complex projective 2-designs.

In standard (finite) d-dimensional quantum mechanics the states of a system are identified with the set of positive trace-one operators \( S(\mathbb{C}^d) \) (known also as density operators or density matrices) and the pure states with the extreme boundary of this set, \( P(\mathbb{C}^d) \), i.e. rank-1 orthogonal projections or, equivalently, with the elements of the projective complex vector space \( \mathbb{C}P^{d-1} \). The mixed states can be also described as elements of a \((d^2 - 1)\)-dimensional real Hilbert space \( L_0^d(\mathbb{C}^d) \) of Hermitean traceless operators on \( \mathbb{C}^d \), endowed with the Hilbert-Schmidt product given by \( \langle \sigma | \tau \rangle := \text{tr}(\sigma \tau) \) for \( \sigma, \tau \in L_0^d(\mathbb{C}^d) \). Namely, the map defined by \( S(\mathcal{H}) \ni \rho \rightarrow \rho - I/d \in L_0^d(\mathbb{C}^d) \) gives us an affine embedding of the set of states (resp. pure states) into the (outer) \((d^2 - 1)\)-dimensional ball (resp. sphere) in \( L_0^d(\mathbb{C}^d) \) centered at 0 of radius \( \sqrt{1 - d^{-1}} \). The image of this map is called the Bloch body (resp. the generalized Bloch sphere). Only for \( d = 2 \) the map is onto, and for \( d > 2 \) its image is a convex subset of the ball of full dimension and contains the maximal (inner) ball of radius \( 1/\sqrt{d(d-1)} \).

A measurement with a finite number \( n \) of possible outcomes can be described by positive operator valued measure (POVM), i.e. an ensemble of positive non-zero operators \( \Pi = (\Pi_j)_{j=1,...,n} \) on \( \mathbb{C}^d \) that sum to the identity operator, i.e. \( \sum_{j=1}^n \Pi_j = I \). According to the Born rule, the affine measurement map \( p_\Pi \) sends an input quantum state \( \rho \in S(\mathbb{C}^d) \) into the vector of probabilities of obtaining given outcomes \( (\text{tr}(\rho \Pi_j))_{j=1,...,n} \) that belongs to the standard (probability) simplex \( \Delta_n \). If this map is one-to-one we call \( \Pi \) informationally complete; then, necessarily \( n \geq d^2 \). In this situation the results of the measurement determine uniquely the input state, i.e. quantum tomography, the task of reconstructing quantum state from the outcomes of a measurement can be concluded. In this case \( Q_\Pi := p_\Pi(S(\mathbb{C}^d)) \) (the probability range of \( \Pi \)) is an affine \((d^2 - 1)\)-dimensional image of the Bloch body and, clearly, \( \text{ext} Q_\Pi = p_\Pi(P(\mathbb{C}^d)) \).

The main result of the present paper is Theorem 3 in which we provide a necessary and sufficient condition for \( S(\mathbb{C}^d) \) and \( Q_\Pi \) to be of the same shape (similar), namely the orthogonal projections onto \( L_0^d(\mathbb{C}^d) \) of the effects constituting the POVM in question need to form a tight operator frame in this space. In consequence, we can identify the set of quantum states with a convex subset of \( \Delta_n \). We pay special attention to POVMs, where \( \Pi_j \ (j = 1, \ldots, n) \) are one-dimensional and of equal trace, and so proportional to certain pure states \( \rho_j \ (j = 1, \ldots, n) \). We show that in this case \( p_\Pi \) is a similarity if and only if these states constitute a complex projective 2-design, or, equivalently, their image in the Bloch body is a tight frame in \( L_0^d(\mathbb{C}^d) \). In this case we call \( Q_\Pi \) generalised qplex. Note that the class of 2-designs also contain, next to SIC-POVMs, mutually unbiased bases [22], group designs [18], and many other discrete objects representing quantum measurements [4].
The main difference (between Hilbert) qplexes and their generalisation lies not in their internal geometry, but rather in their location in the underlying real vector space. Both are isomorphic (similar) through the measurement map to the Bloch body. The former, however, is a subset of the standard $(d^2-1)$-dimensional simplex $\Delta_{d^2}$ lying in a $d^2$-dimensional real vector space, whilst the latter is a subset of a $(d^2-1)$-dimensional polytope $\Delta$ being a cross-section of the simplex $\Delta_n$. We show that the cross-section in question has to be central, i.e. it passes through the centre of the simplex $c_n$, and, what's more, medial, i.e. the vertices of the simplex are equidistant from the affine space $A$ generated by $\Delta$. (We borrow the name from [30,39], though the authors considered only hyperplane sections there.) It seems that the latter property is crucial for constructing a generalised qplex. The distance equal $1-d^2/n$ measures the deviation from minimality: for $n = d^2$ we get necessarily a SIC-POVM. Moreover, we consider the polytope $D$ generated by the elements of 2-design transformed by $p n$ that is the image under the homothety (with the centre at $c_n$ and the ratio $1/(d+1)$) of the orthogonal projection of $\Delta_n$ on $A$.

The generalised qplex $Q_\Pi$ (we omit further subscript $\Pi$) lies in-between these two polytopes $D$ and $\Delta$. The situation is somewhat analogous to quantum (Bell) correlation picture, where the quantum convex body lies in-between the local polytope and the non-signaling polytope, see [5,6,10,15,16,28]. In fact, the analogy is even deeper, since also in our situation $D$ represents in a sense classical behaviour and $\Delta$ ‘beyond-quantum’ one. The polytopes $D$ and $\Delta$ are dual in $A$, see [38], namely $D$ and $\iota(\Delta)$, where $\iota$ is the inversion through $c_n$, are polar with respect to the central sphere of radius $m := n^{-1/2}(d+1)^{-1/2}$. Moreover, $Q$ is self-dual in this sense, every pair of elements $p,q \in Q$ fulfills the fundamental inequalities:

$dm^2 \leq \langle p,q \rangle \leq 2dm^2$, and $Q$ lies in-between two central balls: the inscribed ball of $\Delta$ of radius $r := m(d-1)^{-1/2}$ and the circumscribed ball of $D$ of radius $R := m(d-1)^{1/2}$. Note that $m = \sqrt{rR}$. Thus, the overall geometric picture is similar to the (Hilbert) qplex case [1, Fig. 2], but more complicated as our polytopes are not necessarily simplices. These properties, especially mediality of the constitutive cross-section space $A$, can be chosen as the starting point for an abstract definition of generalised qplex (in preparation).

This geometric approach can be supplemented by an algebraic one, namely, for the extreme boundary $\text{ext}Q$ of the qplex we show that it can be described by a set of polynomial equations. For low-dimensional systems ($d = 2,3$) we provide also the description of the whole qplex $Q$ with the help of a set of polynomial equations and inequalities. In particular, the linear equations defining the affine space $A$ are just the famous primal equations (‘Urgleichung’) known from the QBism theory. We complete our paper with a detailed analysis of several examples of 2-designs and the corresponding generalised qplexes: cubical, cuboctahedral and icosahedral POVMs in dimension 2, MUBs, SIC-POVMs in dimension 3, and a two-distance 2-design POVM in dimension 5. Note that in all these cases equations defining $A$ take an unexpectedly simple form.

Finally, we would like to mention that figuring out how the qplex $Q$ is located inside the simplex $\Delta_n$, is also the key to solving the minimization and maximization of the Shannon entropy of POVM problem, i.e. understanding the lower and upper bounds of uncertainty of measurement results [31,33].

2. The probability range of POVM

Let us consider a quantum system represented by $\mathbb{C}^d$ and a general discrete quantum measurement on it described by a POVM (positive operator valued measure) $\Pi = \{\Pi_z\}_{z=1}^n$. In particular, if $\Pi_z$ are orthogonal projections, we call $\Pi$ a PVM (projection valued measure). Let us denote by $Q_\Pi$ the set of ‘allowed’ probabilities, i.e. the set of all possible probability distributions of the measurement outcomes over all quantum states. It is known also as the probability range of $\Pi$ [19]. More formally, if we denote by $\mathcal{L}_s(\mathbb{C}^d)$ the $d^2$-dimensional real Hilbert space of Hermitian operators on $\mathbb{C}^d$, then

$Q_\Pi := pn(\mathcal{L}_s(\mathbb{C}^d))$, where

$pn : \mathcal{L}_s(\mathbb{C}^d) \supset A \mapsto (\text{tr}(A\Pi_1), \ldots, \text{tr}(A\Pi_n)) \in \mathbb{R}^n$. We introduce also $Q^1_\Pi$ for the probability distributions attained for the pure states only, i.e.

$Q^1_\Pi := pn(\mathcal{P}(\mathbb{C}^d))$. As a straightforward consequence, we get the following properties of $Q_\Pi$ and $Q^1_\Pi$:

Fact 1. $Q_\Pi = \text{conv}(Q^1_\Pi)$ and $\text{ext}(Q_\Pi) \subset Q^1_\Pi$.

Let us recall that the joint numerical range of Hermitian matrices $A_1, A_2, \ldots, A_n$ is defined by

$\mathcal{F}(A_1, A_2, \ldots, A_n) = \{\langle z|A_1|z\rangle, \langle z|A_2|z\rangle, \ldots, \langle z|A_n|z\rangle \}^T |z \in \mathbb{C}^d, \|z\| = 1\} \subset \mathbb{R}^n$. 2
Since any pure state \( \rho \) is necessarily the orthogonal projection onto some unit vector \( z \in \mathbb{C}^d \) and then \( \text{tr}(\rho \Pi_j) = \langle z | \Pi_j | z \rangle \), thus \( \mathcal{Q}_\Pi \) can be described also as the joint numerical range of \( \Pi_1, \Pi_2, \ldots, \Pi_n \).

We are particularly interested in the shape of \( \mathcal{Q}_\Pi \). It is easy to see that for \( d = 2 \) we get an (possibly degenerated) ellipsoid, since \( \mathcal{Q}_\Pi \) in that case is a linear image of 3-dimensional ball. Some examples of these ellipsoids are presented below.

**Example 1.** Let us consider a one-parameter family of rank-1 normalized POVMs on \( \mathbb{C}^2 \) consisting of \( n = 4 \) effects:

\[
\Pi_1 := \frac{1}{4} \begin{pmatrix} 1 + \sin \alpha & \cos \alpha \\ \cos \alpha & 1 - \sin \alpha \end{pmatrix}, \quad \Pi_2 := \frac{1}{4} \begin{pmatrix} 1 + \sin \alpha & -\cos \alpha \\ -\cos \alpha & 1 - \sin \alpha \end{pmatrix},
\]

\[
\Pi_3 := \frac{1}{4} \begin{pmatrix} 1 - \sin \alpha & -i \cos \alpha \\ i \cos \alpha & 1 + \sin \alpha \end{pmatrix}, \quad \Pi_4 := \frac{1}{4} \begin{pmatrix} 1 - \sin \alpha & i \cos \alpha \\ -i \cos \alpha & 1 + \sin \alpha \end{pmatrix},
\]

where \( \alpha \in [0, \pi/2] \). The pure states \( \rho_i = 2\Pi_i \) for \( i = 1, \ldots, 4 \) are represented on the Bloch sphere by the vertices of certain tetragonal disphenoid, i.e. a tetrahedron whose four faces are congruent isosceles triangles. Such tetrahedron can be uniquely characterized by the length of the base (or legs) of that triangle. Taking into account possible degenerations, we get five cases. For \( \alpha = \frac{\pi}{2} \) the disphenoid degenerates to a segment (length of the base = 0) and our measurement effectively acts as the projective von Neumann measurement. For \( \alpha \in (0, \pi/2) \) there is no degeneracy and our POVM is informationally complete. In particular, for \( \alpha \in (\arcsin \frac{1}{\sqrt{3}}, \frac{\pi}{2}) \) the length of the base of the face triangle is less than the length of its legs. For \( \alpha = \arcsin \frac{1}{\sqrt{3}} \) we get the tetrahedron (faces being equilateral triangles) representing SIC-POVM, the only case of 2-design here. For \( \alpha \in (0, \arcsin \frac{1}{\sqrt{3}}) \) our disphenoid is such that the length of the base of the face triangle is greater than the length of its legs. Finally, for \( \alpha = 0 \) the disphenoid degenerates to a square (length of the legs = 0) representing a non informationally complete highly symmetric POVM. Its probability range in these cases is presented in Fig. 1 and evolves as follows: for \( \alpha = \frac{\pi}{2} \) we obtain a segment, then for \( \alpha \in (0, \pi/2) \) – an elongated spheroid, for \( \alpha = \arcsin \frac{1}{\sqrt{3}} \) – a flattened spheroid and finally, for \( \alpha = 0 \) – a disk.

**Figure 1.** \( \mathcal{Q}_\Pi \) in the cases: \( \alpha = \frac{\pi}{2} \) – a segment; \( \alpha \in (0, \pi/2) \) – an elongated spheroid; \( \alpha = \arcsin \frac{1}{\sqrt{3}} \) – a sphere; \( \alpha \in (0, \arcsin \frac{1}{\sqrt{3}}) \) – a flattened spheroid; \( \alpha = 0 \) – a disk. All solids are tangent to \( \Delta_4 \).

The shape of \( \mathcal{Q}_\Pi \) in dimensions higher than 2 is much more difficult to describe. However, it seems that there are two cases in some sense ‘nice’. The first one is when \( \mathcal{Q}_\Pi \) is the full simplex. The second one is when it is of the same shape as \( S(\mathbb{C}^d) \). In the following we deal with the sufficient and necessary conditions for these two to appear. We start with the full simplex case, since the following characterization may be categorized as quantum information folklore (see, e.g. [19, p. 4]). The second case will be thoroughly analysed in the next section.

**Definition 1.** We say that a POVM \( \Pi \) has the norm-1-property if \( \| \Pi_i \| = 1 \) for \( i = 1, \ldots, n \) (here \( \| \cdot \| = \max_{\|x\|_2 = 1} \| A x \|_2 \)).

**Theorem 1.** \( \mathcal{Q}_\Pi = \Delta_n \) if and only if \( \Pi \) has the norm-1-property. Moreover, in such case, \( \mathcal{Q}_\Pi^1 = \Delta_n \), \( n \leq d \) and \( n = d \) if and only if \( \Pi \) is a rank-1 PVM.

3. THE PROBABILITY RANGE SIMILAR TO QUANTUM STATE SPACE

In order to provide a characterization of the second case we need to make some preparations. From now on for \( A, B \in \mathcal{L}_n(\mathbb{C}^d) \) we will denote their Hilbert-Schmidt inner product by \( (A|B) \) and the induced
norm of $A$ by $\|A\|_{HS}$. We also consider $\mathcal{L}_s^0(\mathbb{C}^d) := \{A \in \mathcal{L}_s(\mathbb{C}^d) | \text{tr}A = 0\}$, i.e. the $(d^2 - 1)$-dimensional subspace of traceless Hermitian operators. It is easy to see that

$$\pi_0 : \mathcal{L}_s(\mathbb{C}^d) \ni A \mapsto A - (\text{tr}A/d)I \in \mathcal{L}_s(\mathbb{C}^d)$$

is the orthogonal projection onto $\mathcal{L}_s^0(\mathbb{C}^d)$. The following definition gives a precise description of what we mean by ‘the same shape’ in terms of similarity:

**Definition 2.** We say that $Q_\Pi$ is similar to quantum state space $\mathcal{S}(\mathbb{C}^d)$ if there exists $\alpha > 0$ such that

$$\|p_\Pi(\rho) - p_\Pi(\sigma)\|^2 = \alpha \|\rho - \sigma\|^2_{HS} \quad \text{for all } \rho, \sigma \in \mathcal{S}(\mathbb{C}^d).$$

**Remark 1.** The above definition can be expressed equivalently as

$$\langle p_\Pi(\rho) - c_\Pi, p_\Pi(\sigma) - c_\Pi \rangle = \alpha (\rho - I/d|\sigma - I/d)$$

for all $\rho, \sigma \in \mathcal{S}(\mathbb{C}^d)$, where $c_\Pi := p_\Pi(I/d) = (\text{tr}\Pi_1/d, \ldots, \text{tr}\Pi_k/d)$. (Note that $\rho - I/d = \pi_0(\rho)$ and $p_\Pi(\rho) - c_\Pi = p_\Pi(\pi_0(\rho))$.) In particular, $\|p_\Pi|_{\mathcal{L}_s^0(\mathbb{C}^d)}$ is a similarity with similarity ratio $s = \alpha^{1/2}$.

Let us recall some basic facts about finite tight frames. Let $V$ be a finite-dimensional Hilbert space with inner product $\langle \cdot | \cdot \rangle$ and let $F := \{f_1, \ldots, f_m\} \subset V$.

**Definition 3.** $F$ is a frame if there exist $\alpha, \beta > 0$ such that

$$\alpha \|v\|^2 \leq \sum_{i=1}^{m} |\langle v | f_i \rangle|^2 \leq \beta \|v\|^2 \quad \text{for all } v \in V.$$

$F$ is called a tight frame if $\alpha = \beta$. In such situation $\alpha$ is referred to as the frame bound. The operator $S := \sum_{i=1}^{m} |f_i\rangle\langle f_i| \in \mathcal{L}(V)$ is called the frame operator.

**Theorem 2.** The following conditions are equivalent:

i. $F$ is a tight frame with frame constant $\alpha$;

ii. $\sum_{i=1}^{m} \langle u | f_i \rangle \langle f_i | v \rangle = \alpha \langle u | v \rangle$ for all $u, v \in V$;

iii. $S = \alpha I$.

**Proof.** See, e.g. [37]. \qed

Now we can proceed to the characterization of POVMs for which $Q_\Pi$ is similar to the quantum state space.

**Theorem 3.** Let $\Pi$ be a POVM. Then $Q_\Pi$ is similar to quantum state space if and only if $\pi_0(\Pi)$ is a tight operator frame in $\mathcal{L}_s^0(\mathbb{C}^d)$. Moreover, the frame bound and the square of similarity ratio coincide and are equal to

$$\alpha = \frac{1}{d^2 - 1} \left( \sum_{i=1}^{n} \text{tr}\Pi_i^2 - \frac{1}{d} \sum_{i=1}^{n} (\text{tr}\Pi_i)^2 \right).$$

**Proof.** Since $\text{span}\{\rho - I/d : \rho \in \mathcal{S}(\mathbb{C}^d)\} = \mathcal{L}_s^0(\mathbb{C}^d)$, then (2) holds for all quantum states if and only if it holds for all $\rho, \sigma \in \mathcal{L}_s(\mathbb{C}^d)$ such that $\text{tr} \rho = \text{tr} \sigma = 1$. Thus, by the linearity, we can extend this condition to arbitrary Hermitian operators by equivalently writing it as

$$\langle p_\Pi(A) - \text{tr}\Pi_1B, p_\Pi(B) - \text{tr}\Pi_1C \rangle = \alpha(\text{tr}(A - (\text{tr}A/d)I)B - (\text{tr}B/d)I) \quad \text{for all } A, B \in \mathcal{L}_s(\mathbb{C}^d).$$

Transforming the lhs we get

$$\langle p_\Pi(A) - \text{tr}\Pi_1B, p_\Pi(B) - \text{tr}\Pi_1C \rangle = \sum_{i=1}^{n} (\text{tr}(\Pi_i) - \text{tr}\Pi_i/d)(\text{tr}(\Pi_i) - (\text{tr}\Pi_i/d))$$

$$= \sum_{i=1}^{n} \text{tr}((A - (\text{tr}A/d)I)\Pi_i - (\text{tr}\Pi_i/d)I)(B - (\text{tr}B/d)I)I_i - (\text{tr}\Pi_i/d)I)$$

$$= \sum_{i=1}^{n} (A - (\text{tr}A/d)I)(\pi_0(\Pi_i))(\pi_0(\Pi_i))B - (\text{tr}B/d)I)$$

$$= (A - (\text{tr}A/d)I)S |B - (\text{tr}B/d)I),$$

where $S := \sum_{i=1}^{n} |\pi_0(\Pi_i)(\pi_0(\Pi_i))|$. Thus (3) is equivalent to

$$\langle A|S|B \rangle - \alpha(A|B), \quad \forall A, B \in \mathcal{L}_s^0(\mathbb{C}^d),$$

for all $A, B \in \mathcal{L}_s(\mathbb{C}^d)$. \qed
which is equivalent to $\tau_0(\Pi)$ being a tight operator frame in $\mathcal{L}_d^0(\mathbb{C}^d)$ with the frame bound $\alpha$. In order to calculate $\alpha$, let us take the trace on both sides of the equation

\[(5) \quad \sum_{i=1}^{n} |\tau_0(\Pi_i))\rangle\langle\tau_0(\Pi_i)| = |\tau_0(I)|,\]

where $I$ denotes the identity superoperator on $\mathcal{L}_d^0(\mathbb{C}^d)$. On the lhs we get $\alpha(d^2-1)$, since dim $\mathcal{L}_d^0(\mathbb{C}^d) = d^2 - 1$. On the rhs we get

\[
\sum_{i=1}^{n} (\tau_0(\Pi_i))\tau_0(\Pi_i)) = \sum_{i=1}^{n} \left( |\Pi_i\rangle\langle\Pi_i| - \frac{tr\Pi_i}{d} |\Pi_i\rangle\langle I| - \frac{tr\Pi_i}{d} |I\rangle\langle\Pi_i| + \frac{(tr\Pi_i)^2}{d^2} |I\rangle\langle I| \right)
\]

\[= \sum_{i=1}^{n} tr\Pi_i^2 - \frac{1}{d} \sum_{i=1}^{n} (tr\Pi_i)^2,
\]

which completes the proof.

The condition of $\pi_0(\Pi)$ being a tight operator frame in $\mathcal{L}_d^0(\mathbb{C}^d)$ brings to mind the definition of tight IC-POVM given by Scott in [30]. Let us introduce the following notation: $P_i := \Pi_i/\text{tr}\Pi_i$, $P := \{P_i\}_{i=1}^{n}$ and $F := \sum_{i=1}^{n} \text{tr}\Pi_i |P_i\rangle\langle P_i|/\sum_{i=1}^{n} 1/\text{tr}\Pi_i |\Pi_i\rangle\langle\Pi_i|.$

**Definition 4.** $\Pi$ is a tight IC-POVM if $\tau_0(\Pi)$ is a tight operator frame with respect to the trace measure $\tau (\tau(i) = \text{tr}\Pi_i, i = 1, \ldots, n)$ in $\mathcal{L}_d^0(\mathbb{C}^d)$, i.e. if there exists $\beta > 0$ such that

\[
\sum_{i=1}^{n} \text{tr}\Pi_i |\tau_0(\Pi_i))\rangle\langle\tau_0(\Pi_i)| = \beta I.
\]

**Remark 2.** The definition of a tight IC-POVM can be expressed equivalently in the following way:

\[F = \beta I + \frac{1-\beta}{d} |I\rangle\langle I|,
\]

where $I$ stands for the identity superoperator on $\mathcal{L}_d(\mathbb{C}^d)$. Note that $\beta$ can be calculated by taking the trace on both sides of one of the above equations:

\[\beta = \frac{1}{d^2-1} \left( \sum_{i=1}^{n} \text{tr}\Pi_i^2 - \frac{1}{d} \right).
\]

The following statement is now a simple observation.

**Corollary 4.** Let $\Pi$ be a POVM consisting of effects of equal trace. Then $\Delta_\Pi$ is similar to quantum state space if and only if $\Pi$ is a tight IC-POVM. The square of similarity ratio $\alpha$ is then equal to $\frac{1}{d^2-1} \left( \sum_{j=1}^{n} \text{tr}\Pi_j^2 - \frac{d}{n} \right)$.

4. The probability range of 2-design POVMs

**Definition 5.** We say that $\{\rho_j\}_{j=1}^{n} \subset \mathcal{P}(\mathbb{C}^d)$ is a $t$-design if

\[\frac{1}{n^2} \sum_{j,k=1}^{n} f(\text{tr}(\rho_j \rho_k)) = \int \int_{(\mathbb{P}(\mathbb{C}^d))^2} f(\text{tr}(\rho \sigma))d\mu(\rho)d\mu(\sigma)
\]

holds for every $f : \mathbb{R} \to \mathbb{R}$ polynomial of degree $t$ or less, where by $\mu$ we mean the unique unitarily invariant measure on $\mathcal{P}(\mathbb{C}^d)$.

Obviously any $t$-design is also an $s$-design for $s < t$. Note that $\{\rho_j\}_{j=1}^{n} \subset \mathcal{P}(\mathbb{C}^d)$ is 1-design if and only if $\{\frac{1}{n}\rho_j\}_{j=1}^{n}$ is a POVM. Moreover, such ensemble is a 2-design if and only if $\{\frac{d}{n}\rho_j\}_{j=1}^{n}$ is a tight IC-POVM [30] Prop. 13. We will refer to such rank-1 POVM as 2-design POVM. That property allows us to rephrase the definition of 2-design in a more comprehensible way, namely $\{\rho_j\}_{j=1}^{n} \subset \mathcal{P}(\mathbb{C}^d)$ is a 2-design if and only if

\[(6) \quad \tau = (d+1) \sum_{j=1}^{n} \frac{d}{n} \text{tr}(\tau \rho_j)\rho_j - I
\]

for every $\tau$ such that $\tau^* = \tau$ and $\text{tr}\tau = 1$.  

5
Corollary 5. Let \( \Pi \) be a rank-1 POVM consisting of effects of equal trace. Then \( \mathcal{Q}_\Pi \) is similar to quantum state space if and only if \( \Pi \) is a 2-design POVM. The square of similarity ratio \( \alpha \) is then equal to \( \frac{d}{nd+1} \).

From now on we assume that \( \Pi = \{ \frac{d}{n} \rho_j \}_{j=1}^n \) is a 2-design POVM. For \( \tau \in \mathcal{S}(\mathbb{C}^d) \), \( p_j(\tau) := (\rho_{\Pi_j} \tau)_{j} = \frac{d}{n} \text{tr}(\tau \rho_j) \) is the probability of obtaining the \( j \)-th outcome. The following equalities are directly derived from (8):

\[
\sum_{j=1}^{n} p_j(\tau) = 1,
\]

\[
\sum_{j=1}^{n} (p_j(\tau))^2 = \frac{d(\text{tr}(\tau^2) + 1)}{n(d+1)},
\]

\[
\sum_{j,k,l=1}^{n} p_j(\tau)p_k(\tau)p_l(\tau)\text{tr}(\rho_j\rho_k\rho_l) = \frac{\text{tr}(\tau + I)^3}{(d+1)^3}.
\]

In particular, for pure states we get

\[
\sum_{j=1}^{n} (p_j(\tau))^2 = \frac{2d}{n(d+1)},
\]

\[
\sum_{j,k,l=1}^{n} p_j(\tau)p_k(\tau)p_l(\tau)\text{tr}(\rho_j\rho_k\rho_l) = \frac{d + 7}{(d+1)^3}.
\]

Remark 3. Note that the rhs of the last equation does not depend on \( n \).

The proof of the following fact can be found in [21].

Fact 2. A self-adjoint operator \( \tau \) is a pure quantum state if and only if \( \text{tr}\tau = 1, \text{tr}\tau^2 = 1 \) and \( \text{tr}\tau^3 = 1 \).

This fact will be helpful for proving the theorem characterising the probability distributions belonging to \( \mathcal{Q}_\Pi^1 \). Note that the statement is more general than [17, Cor. 2.3.4].

Theorem 6. Let \( \Pi = \{ \frac{d}{n} \rho_j \}_{j=1}^n \) be a 2-design POVM. Then \( (p_1, \ldots, p_n) \in \mathcal{Q}_\Pi^1 \) if and only if

i. \( \frac{d}{n} p_l = (d+1) \sum_{j=1}^{n} p_j \text{tr}(\rho_j \rho_l) - 1 \) for \( l = 1, \ldots, n \);

ii. \( \sum_{j=1}^{n} p_j^2 = \frac{2d}{n(d+1)} \); 

iii. \( \sum_{j,k,l=1}^{n} p_j p_k p_l \text{tr}(\rho_j \rho_k \rho_l) = \frac{d+7}{(d+1)^3} \).

Proof. We already know that the probability distribution from \( \mathcal{Q}_\Pi^1 \) satisfies all equalities above. Thus to complete the proof it suffices to show that any vector \( (p_1, \ldots, p_n) \in \mathbb{R}^n \) fulfilling equalities (i)-(iii) belongs to \( \mathcal{Q}_\Pi^1 \). Let us define \( \tau := (d+1) \sum_{j=1}^{n} p_j \rho_j - I \). Then

\[
\text{tr}(\tau p_l) = (d+1) \sum_{j=1}^{n} p_j \text{tr}(\rho_j p_l) - \text{tr}(p_l) = \frac{n}{d} p_l
\]

and thus \( p_l = \frac{d}{n} \text{tr}(\tau p_l) \). It is now enough to show that \( \tau \in \mathcal{P}(\mathbb{C}^d) \). Obviously \( \tau^* = \tau \). Using (i), we show that the numbers \( p_1, p_2, \ldots, p_n \) sum up to 1:

\[
\sum_{l=1}^{n} p_l = \frac{d}{n} \left( (d+1) \sum_{j=1}^{n} \text{tr}(\rho_j \rho_l) - n \right) = \frac{d}{n} \left( (d+1) \sum_{j=1}^{n} p_j - \frac{d}{n} n = (d+1) \sum_{j=1}^{n} p_j - d, \right)
\]

which gives us \( \sum_{l=1}^{n} p_l = 1 \). In consequence, \( \text{tr}\tau = (d+1) \sum_{j=1}^{n} \rho_j \text{tr}(\rho_j) - d = 1 \). Since (8) and (9) hold true for any Hermitian operator with trace equal to 1 we can compare them with our assumptions concerning sum of squares and triple products respectively to obtain that \( \text{tr}\tau^2 = \text{tr}\tau^3 = 1 \) which completes the proof. 

Let us take a closer look at equations (i)-(iii) from Theorem 6. As we already know from Corollary 5 they need to describe a \((2d - 2)\)-dimensional submanifold of a \((d^2 - 2)\)-dimensional sphere. From the proof above we can see that the first \( n \) linear equations (i) define a \((d^2 - 1)\)-dimensional affine subspace \( \mathcal{A} \) in \( \mathbb{R}^n \). It is easy to see that \( \mathcal{A} \) is the affine span of \( \mathcal{Q}_\Pi^1 \). In particular, \( \mathcal{A} \subset \{ x \in \mathbb{R}^n : x_1 + \ldots + x_n = 1 \} \). The number of equations can be reduced to \( n - (d^2 - 1) \) since the rest of them need to be linearly
dependent. In particular, if \( n = d^2 \), this system of equations reduces to a single equation \( \sum_{j=1}^{d^2} p_j^2 = 1 \). Note that the only 2-design POVMs with \( d^2 \) elements are SIC-POVMs, i.e. the ones with \( \text{tr}(\rho_j \rho_k) = \frac{d+1}{d} \) for \( j \neq k \). The quadratic form \([3]\) is obviously the equation of the sphere in \( \mathbb{R}^d \) centred in 0 and of radius \( \sqrt{\frac{2d}{n(d+1)}} \). The intersection of this sphere with the affine subspace \( \mathcal{A} \) gives us the \( (d^2 - 2) \)-dimensional sphere we were looking for. Finally, the cubic form \([4]\) is responsible for cutting the \( (2d-2) \)-dimensional submanifold from the sphere in question. We can now state the following:

### Corollary 7

The set \( \mathcal{Q}_\Pi \) is fully described by the equations \([3]-[5]\) if and only if \( d = 2 \).

The linear equations \([4]\) can be reduced to a single equation \( \sum_{j=1}^{d^2} p_j^2 = 1 \) if and only if \( \Pi \) is a SIC-POVM.

A natural question arises, whether we can provide a similar characterization of \( \mathcal{Q}_\Pi \) as we did it for \( \mathcal{Q}_1 \) in Theorem 6. As we can see, the proof is based on the fact that the pure states are fully described by \( \text{tr} \tau = \text{tr} \tau^2 = \text{tr} \tau^3 = 1 \). The mixed states obviously satisfy \( \text{tr} \tau = 1 \) and \( \text{tr} \tau^2 \leq 1 \) and it is tempting to write down the third inequality in the similar manner. However, it turns out that in order to provide a complete characterization of \( \tau \in \mathcal{S}(\mathbb{C}^d) \) in terms of \( \tau \tau^k \), we need additional \( d - 2 \) inequalities. Put together they take the form \( S_{k,j} \geq 0 \) \( (k \in \{2, \ldots, d\}) \), where \( S_k \) can be defined recursively by \( S_k = \frac{1}{2} \sum_{j=1}^{d} (-1)^{j-1} \text{tr}(\tau) S_{k-1,j} \) and \( S_0 = 1 \). For example, the condition on \( \text{tr} \tau^3 \) is as follows:

\[ 3 \text{tr} \tau^2 - 2 \text{tr} \tau^3 \leq 1. \]

Thus such characterization is possible but it becomes more and more complicated with the growth of \( d \). However, for \( d = 2, 3 \) we can characterise \( \mathcal{Q}_\Pi \) in a relatively simple way.

### Corollary 8

Let \( \Pi = \{ \frac{1}{d} \rho_j \}_{j=1}^{d} \) be a 2-design POVM. If \( (p_1, \ldots, p_n) \in \mathcal{Q}_\Pi \), then

\[ \sum_{i=1}^{n} p_i = (d+1) \sum_{j=1}^{n} p_j \text{tr}(\rho_j p_1) - 1 \text{ for } l = 1, \ldots, n; \]

\[ \sum_{l=1}^{d} p_j^2 \leq \frac{2d}{n(d+1)}; \]

\[ \sum_{l=1}^{d} p_j^2 \leq \sum_{l=1}^{d} p_j^4 + \frac{4}{(d+1)^2}. \]

Moreover, the conditions (i)-(iii) are sufficient for \( d = 3 \) and (i)-(ii) are sufficient for \( d = 2 \).

### 5. Examples

In the following examples we provide some explicit formulae for the \( (d^2 - 1) \)-dimensional affine subspaces \( \mathcal{A} \) defined by the system of linear equations \([6]\).

#### Example 2 (Cubical POVM)

Let us consider a cubical POVM \( \Pi = \{ \frac{1}{d} \rho_j \}_{j=1}^{d} \), i.e. the states \( \rho_1, \ldots, \rho_d \) are represented on the Bloch sphere by the vertices of the cube, e.g. \( \frac{1}{\sqrt{6}} (\pm1, \pm1, \pm1) \). Let us label these vertices in the following way: the vertices of the bottom face are labelled in sequence with 1-4 and if \( i \) and \( j \) are labels of the opposite vertices then \( j = i + 4 \) (mod 8). The eight linear equations \([6]\) from Theorem 7 reduce to following 8 - \( (2^2 - 1) = 5 \) linearly independent ones:

\[
\begin{align*}
p_1 + p_5 & = \frac{1}{2} \\
p_2 + p_6 & = \frac{1}{2} \\
p_3 + p_7 & = \frac{1}{2} \\
p_4 + p_8 & = \frac{1}{2} \\
p_1 + p_3 + p_6 + p_8 & = \frac{1}{2}.
\end{align*}
\]

It is easy to see that the first four of them guarantee that \( p_1 + \ldots + p_8 = 1 \). Obviously the 3-dimensional affine subspace of \( \mathbb{R}^8 \) defined by these equations is label-dependent. The number of possible labellings is \( 8! = 40320 \). However, some of them produce the same set of equations, namely all that come as a result of any isometric operation on the cube. The isometries of the cube are described by the elements of the octahedral group \( O_8 \). Thus the number of different subspaces is \( \frac{8!}{|O_8|} = \frac{40320}{48} = 840 \).

#### Example 3 (Cuboctahedral and icosahedral POVMs)

The polyhedra defining these POVMs (cuboctahedron and icosahedron) have the same number of vertices: \( n = 12 \), which can be represented on the Bloch sphere by \( \frac{1}{\sqrt{2(s+1)}} (\pm s, \pm 1, 0), \frac{1}{\sqrt{2(s+1)}} (0, \pm s, \pm 1) \) and \( \frac{1}{\sqrt{2(s+1)}} (\pm 1, 0, \pm s) \), where \( s = 1 \) for cuboctahedron and \( s = (1 + \sqrt{5})/2 \) (golden ratio, usually denoted by \( \varphi \)) for icosahedron. If we label these vertices as follows: \( v_1 = (+, +, 0), v_3 = (+, -, 0), v_5 = (0, +, +), v_7 = (0, +, -), v_9 = (+, 0, +), \)
$v_{11} = (-0, +)$ and $v_{2j} = -v_{2j-1}$ for $j \in \{1, \ldots, 6\}$, then the different values of the parameter $s$ result in slightly different affine subspaces defined by the following systems of 9 linear equations:

\[
\begin{align*}
\text{cuboctahedron:} & \quad \begin{cases}
p_1 + p_2 = \frac{1}{6} \\
p_3 + p_4 = \frac{1}{6} \\
p_5 + p_6 = \frac{1}{6} \\
p_7 + p_8 = \frac{1}{6} \\
p_9 + p_{10} = \frac{1}{6} \\
p_{11} + p_{12} = \frac{1}{6} \\
p_1 - p_3 + p_6 + p_8 = \frac{1}{6} \\
p_5 - p_7 + p_{10} + p_{12} = \frac{1}{6} \\
p_9 - p_{11} + p_2 + p_4 = \frac{1}{6}
\end{cases} & \quad \text{icosaedron:} \quad \begin{cases}
p_1 + p_2 = \frac{1}{6} \\
p_3 + p_4 = \frac{1}{6} \\
p_5 + p_6 = \frac{1}{6} \\
p_7 + p_8 = \frac{1}{6} \\
p_9 + p_{10} = \frac{1}{6} \\
p_{11} + p_{12} = \frac{1}{6} \\
\varphi(p_1 - p_3) + p_6 + p_8 = \frac{1}{6} \\
\varphi(p_5 - p_7) + p_{10} + p_{12} = \frac{1}{6} \\
\varphi(p_9 - p_{11}) + p_2 + p_4 = \frac{1}{6}
\end{cases}
\end{align*}
\]

As in the previous case, the definitions of the subspaces are label-dependent. The number of possible labellings is $12!$, but taking into account the symmetry groups of cuboctahedron and icosahedron, i.e. the octahedral group $O_h$ and the icosahedral group $I_h$, we get that the numbers of different subspaces are $\frac{12!}{24} = 12499200$ and $\frac{12!}{120} = 3991680$.

**Example 4** (MUBs). Another example of a 2-design POVM is a complete set of mutually unbiased bases (MUB) in $\mathbb{C}^d$, where $d$ is a prime power (the question whether complete sets of MUBs exist in other dimensions remains open). Such POVM $\Pi$ consists of $d(d + 1)$ effects of the form $\frac{1}{d} \rho_j^2$, where $j \in \{1, 2, \ldots, d + 1\}$ can be uniquely written as $kd + l$ for some $k \in \{0, \ldots, d\}$ and $l \in \{1, \ldots, d\}$. The states $\rho_j$ fulfill the following conditions:

$$\text{tr}(\rho_{kd+l}\rho_{k'd+l'}) = \begin{cases} 1, & \text{for } k = k', l = l' \\ 0, & \text{for } k = k', l \neq l' \\ \frac{1}{d}, & \text{for } k \neq k' \end{cases}$$

The system of linear equations takes now a particularly nice form since instead initial $n = d(d + 1)$ equations it suffices to consider the following $d+1$ equations representing simple fact that the probabilities over any basis need to sum up to $\frac{1}{d+1}$:

$$\sum_{l=1}^{d} p_{(k-1)d+l} = \frac{1}{d+1}, \quad k \in \{1, \ldots, d + 1\}.$$  

Complete sets of MUBs are example of two-distance 2-designs, i.e. the 2-designs with the inner products between different states taking exactly two values. In case of MUBs these values are $0$ and $\frac{1}{d}$.

In the following we consider a two-distance 2-design which is not of this form.

**Example 5** (Two-distance 2-design in dimension 5). This example comes from [20] Ex.18. The 2-design consists of 45 projections onto vectors $(1, 0, 0, 0, 0)$ and $\frac{1}{\sqrt{2}}(0, 1, \pm \eta, \pm \eta, \pm 1)$ under all cyclic permutations of their coordinates, where $\eta = e^{2\pi i/3}$. The 2-design POVM is then formed by rescaling these projections by $\frac{5}{\sqrt{2}} = \frac{1}{5}$. The inner products between different states of this 2-design take values $0$ and $\frac{1}{5}$, which slightly resembles the MUB case. As a two-distance 2-design it carries a strongly regular graph $\text{sr}(45, 12, 3, 3)$, constructed as follows: the vertices represent vectors and two vertices are adjacent if the vectors are orthogonal [20] [24] [25]. It turns out that among 78 nonisomorphic strongly regular graphs with such parameters, our graph is the one provided by the point graph of the generalized quadrangle $GQ(4, 2)$, meaning that the 45 vectors constituting the 2-design can be arranged into 27 orthonormal bases in such a way that every vector belongs to 3 bases, every 2 vectors belong to at most 1 basis and for every vector $v$ and a base $B$ such that $v \notin B$ there exist a unique vector $v'$ and a basis $B'$ such that $v' \in B'$, $v \in B'$ and $v' \subseteq B$. Moreover, the vectors from different bases have the same squared overlaps, which makes the similarity with MUB even stronger. Obviously, sum of the probabilities over any basis is equal to $\frac{1}{5}$. But there are 27 bases and we need just $45 - (25 - 1) = 21$ linear equations to describe $\mathcal{A}$. An example how to get rid of 6 such equations in order to obtain a system of 21 linearly independent equations is presented in Fig. [2].

Starting from dimension $d = 3$, the equation of third degree plays crucial role in defining $Q^d_{11}$. In the following example we take a closer look at the 1-parameter family of 2-design POVMs for which the equations of first and second degree coincide, but not these of third degree. But first let us observe that
the lhs of (iii) in Theorem 6 can be written in a slightly different way if we use the assumption that both $\eta$ and $\xi$ from the same theorem hold. We start with:

$$\sum_{j,k,l=1}^{n} p_j p_k p_l \text{tr}(\rho_j \rho_k \rho_l) = \sum_{j=1}^{n} p_j^3 \text{tr}(\rho_j) + 3 \sum_{j \neq k} p_j^2 p_k \text{tr}(\rho_j^2 \rho_k) + \sum_{j \neq k \neq l} p_j p_k p_l \text{tr}(\rho_j \rho_k \rho_l).$$

But $p_j^2 = \rho_j$ and $\text{tr}(\rho_j^3) = 1$. Moreover,

$$\sum_{j \neq k} p_j^2 p_k \text{tr}(\rho_j \rho_k) = \sum_{j=1}^{n} p_j^2 \left( \sum_{k=1}^{n} p_k \text{tr}(\rho_j \rho_k) - p_j \text{tr}(\rho_j^2) \right) = \sum_{j=1}^{n} p_j^2 \left( \frac{1}{d+1} \left( \frac{n}{d} \eta_j + 1 \right) - p_j \right)$$

$$= \left( \frac{n}{d(d+1) - 1} \right) \sum_{j=1}^{n} p_j^3 + \frac{1}{d+1} \sum_{j=1}^{n} p_j^2 = \left( \frac{n}{d(d+1) - 1} \right) \sum_{j=1}^{n} p_j^3 + \frac{2d}{n(d+1)^2}.$$

Thus, the third degree equation now takes the form:

$$\left( \frac{3n}{d(d+1) - 2} \right) \sum_{j=1}^{n} p_j^3 + \sum_{j \neq k \neq l \neq j} p_j p_k p_l \text{tr}(\rho_j \rho_k \rho_l) = \frac{n(d + 7) - 6d(d+1)}{n(d+1)^3}$$

**Example 6** (SIC-POVMs in dimension 3). Let

$$v_{0,j} = \frac{1}{\sqrt{2}}(-e^{it}\eta^j, 0, 1), \quad v_{t,j} = \frac{1}{\sqrt{2}}(1, -e^{it}\eta^j, 0), \quad v_{\infty,j} = \frac{1}{\sqrt{2}}(0, 1, -e^{it}\eta^j),$$

where $\eta = e^{2\pi i/3}$, $j \in \{0, 1, 2\}$ and $t \in [0, \pi/3)$. Then $\Pi^t := \{ \frac{1}{2} |v_{m,j}^t\rangle \langle v_{m,j}^t| : m, j = 0 \}$ is a SIC-POVM and for $t \neq t'$ sets $\Pi^t$ and $\Pi^{t'}$ are not unitarily equivalent. It is easy to see that the equations of first and second degree are the same for every $t$, but that is not the case for the third degree equation. Let us introduce the following notations:

$$J := \{(\alpha, \beta, \gamma) \in (Z_3 \times Z_3)^3 | \alpha \neq \beta \neq \gamma \neq \alpha \}$$

$$J_k := \{(\alpha, \beta, \gamma) \in (Z_3 \times Z_3)^3 | \alpha \neq \beta \neq \gamma \neq \alpha, \alpha_2 + \beta_2 + \gamma_2 = k \pmod{3} \}, \quad k = 0, 1, 2$$

$$J_3 := \{(\alpha, \beta, \gamma) \in (Z_3 \times Z_3)^3 | \alpha_2 = \beta_2 = \gamma_2 \neq \alpha_2 \}$$

$$J' := J \setminus (J_0 \cup J_1 \cup J_2 \cup J_3).$$

**Figure 2.** Two-distance 2-design in $\mathbb{C}^5$ as GQ(4, 2). Vertices represent vectors and elliptic curves (lines, circles or arcs of ellipses) represent bases. Sums of probabilities over each of blue bases can be omitted in the definition of the affine subspace $A$. 
Now, using (12), we get (for the sake of clarity we omit the label $t$)

$$
\frac{1}{32} = \frac{1}{4} \sum_{\alpha \in Z_3} p_\alpha^3 + \sum_{(\alpha,\beta,\gamma) \in J} p_\alpha p_\beta p_\gamma \tr(p_\alpha p_\beta p_\gamma)
$$

$$
= \frac{1}{4} \sum_{\alpha \in Z_3} p_\alpha^3 - \frac{1}{8} \cos(3t) \sum_{(\alpha,\beta,\gamma) \in J_0} p_\alpha p_\beta p_\gamma + \frac{1}{16} \left( \cos(3t) + \sqrt{3} \sin(3t) \right) \sum_{(\alpha,\beta,\gamma) \in J_1} p_\alpha p_\beta p_\gamma
$$

$$
+ \frac{1}{16} \left( \cos(3t) - \sqrt{3} \sin(3t) \right) \sum_{(\alpha,\beta,\gamma) \in J_2} p_\alpha p_\beta p_\gamma - \frac{1}{8} \sum_{(\alpha,\beta,\gamma) \in J_3} p_\alpha p_\beta p_\gamma + \frac{1}{16} \sum_{(\alpha,\beta,\gamma) \in J_4} p_\alpha p_\beta p_\gamma
$$

For $t = 0$ we get the Hesse configuration and the above equation takes a particularly nice form $35$:

$$
\sum_{\alpha \in Z_3} p_\alpha^3 - \frac{1}{2} \sum_{(\alpha,\beta,\gamma) \in J_0} p_\alpha p_\beta p_\gamma = 0.
$$

6. GEOMETRIC PROPERTIES OF GENERALISED QPLEX

To understand a geometric picture hidden behind the algebraic equations presented in the previous sections let us consider a 2-design POVM $\Pi = \{\rho_j\}_{j=1}^n$, where $\{\rho_j\}_{j=1}^n \subset \mathcal{P}(\mathbb{C}^d)$. We show that the geometry of generalised qplex generated by this design is quite similar to that of (Hilbert) qplex, in spite of the fact that now this object is not located between two simplices, but between two dual polytopes lying in a medial cross-section of the probability simplex by an affine space. Also our approach is quite similar to presented in $1$ although some alterations were necessary.

Let us recall that from Corollary $5$ it follows that the measurement map $p_\Pi : S(\mathbb{C}^d) \to Q_\Pi$ is a similarity of ratio $d^{-1/2}m$, where $m := n^{-1/2}(d + 1)^{-1/2}$. Clearly, the uniform distribution $c_n := (1/n,\ldots,1/n) = p_\Pi(I/d) \in Q_\Pi$. We know that the maximal ball centred at $I/d$ and contained in $S(\mathbb{C}^d)$ has radius $(d-1)^{-1/2}$. Hence the maximal ball centred at $c_n$ and contained in $Q_\Pi$ has radius $r := m(d-1)^{-1/2}$. On the other hand, $S(\mathbb{C}^d)$ is contained in the ball with centre $I/d$ and radius $((d-1)/d)^{1/2}$ with the pure states contained in the corresponding sphere. Thus $Q_\Pi$ is contained in the ball centred at $c_n$ with radius $R := m(d-1)^{-1/2}$ and $Q^1_\Pi$ is contained in the corresponding sphere. In short,

$$
B_{d^A}(c_n,r) \subset Q_\Pi \subset B_{d^A}(c_n,R) \quad \text{and} \quad Q^1_\Pi \subset \partial B_{d^A}(c_n,R).
$$

For $j = 1,\ldots,n$ introduce the basis distributions given by $f_j := p_\Pi(\rho_j)$ generating the basis polytope $D := \text{conv} \{f_j : j = 1,\ldots,n\}$.

Note that the uniform distribution $c_n := (1/n,\ldots,1/n) = p_\Pi(I/d) = p_\Pi \left( \frac{I}{d} + n^{-1} \sum_{j=1}^n \rho_j \right) \in D$, and $(f_j)_k = (f_k)_j$ for $j,k = 1,\ldots,n$.

Fact 3. $B_{d^A}(c_n,R)$ is the circumscribed ball of $D$.

Immediately from Theorem $6$ we get that $\mathcal{A}$ is the affine span of $D$ in $\mathbb{R}^n$. Now, the following statement is the result of simple calculation.

Fact 4. Let $p \in \mathbb{R}^n$. The following conditions are equivalent:

1. $p \in \mathcal{A}$;
2. $p = (d+1) \sum_{j=1}^n p_j f_j - dc_n$;
3. $p - c_n = (d+1) \sum_{j=1}^n p_j (f_j - c_n)$;
4. $\frac{1}{\sqrt{d}} \langle p, e_k \rangle = (p, f_k) - dm^2$ for every $k = 1,\ldots,n$;
5. $p - c_n \perp e_k - (d+1) (f_k - c_n)$ for every $k = 1,\ldots,n$.

Note that (iii) is just the famous primal equation (‘Urgleichung’) introduced by the founders of quantum Bayesianism.

Let us consider now the homothety in $\mathcal{A}$ with the centre at $c_n$ and the ratio equal $1/(d+1)$, i.e. the map $h : \mathcal{A} \to \mathcal{A}$ given by

$$
h(p) := c_n + \frac{p - c_n}{d+1} = \frac{1}{d+1}p + \frac{d}{d+1}c_n
$$

for $p \in \mathcal{A}$. Let $P_{d^A} : \mathbb{R}^n \to \mathcal{A}$ denote the orthogonal projection on $\mathcal{A}$.

Fact 5. In the above situation $h(P_{d^A}e_k) = f_k$ for every $k = 1,\ldots,n$. In particular, $h(P_{d^A}D_n) = D$.
Thus it follows from Theorem 6 that

Theorem 9. The above cross-section is medial, i.e. the vertices of \( \Delta_n \) are equidistant from \( A \). More precisely,

\[
\operatorname{dist}(e_k, A) = 1 - \frac{d^2}{n}
\]

for every \( k = 1, \ldots, n \).

Proof. Let \( k = 1, \ldots, n \). Then it follows from condition (v) in Fact 6 that

\[
P_A e_k = c_n + (d + 1)(f_k - c_n).
\]

Hence \( h (P_A e_k) = f_k \), as desired. \( \square \)

Define the primal polytope \( \Delta \) as the cross-section of \( \Delta_n \) by \( A \), i.e.

\[
\Delta := A \cap \Delta_n.
\]

It follows from Theorem 6 that

\[
D \subset Q_\Pi \subset \Delta.
\]

Thus

\[
D \cup B_A(c_n, r) \subset Q_\Pi \subset \Delta \cap B_A(c_n, R).
\]

Theorem 10. The polytopes \( D \) and \( \Delta \) are dual in \( A \) with respect to the central sphere of radius \( m = \sqrt{n} \), as well as dual are balls \( B_A(c_n, r) \) and \( B_A(c_n, R) \).

Now we show that the elements of \( Q_\Pi \) fulfill the following fundamental inequalities

\[
\langle p, q \rangle = \frac{1}{n} - s^2
\]

for every \( q \in C \) and \( \langle q, p \rangle \leq \langle q, c_n \rangle - \frac{1}{n} \) for \( p \in A \). Now we can characterise \( A \) more precisely. The following fact is the result of direct calculations.

Fact 6. Let \( p \in A \). The following conditions are equivalent:

i. \( p \in \Delta \);
ii. \( p_k \geq 0 \) for every \( k = 1, \ldots, n \);
iii. \( \langle p, f_k \rangle \geq dm^2 \) for every \( k = 1, \ldots, n \);
iv. \( \langle p, q \rangle \geq dm^2 \) for every \( q \in D \);
v. \( m^2 \geq \langle p - c_n, q - c_n \rangle \) for every \( q \in \iota(D) \).

Fact 7. \( B_A(c_n, r) \) is the inscribed (maximal central) ball contained in \( \Delta \).

Proof. Taking \( \rho'_k := (1 - p_k)/(d - 1) \) and \( f'_k := p_\Pi(\rho'_k) \), from the similarity of \( S(C^d) \) and \( Q_\Pi \) we get

\[
\|f'_k - c_n\|^2 = dm^2\|p_k - I/d\|^2_{HS} = \frac{m^2}{n} = r^2 \quad \text{and} \quad \langle f'_k, f_k \rangle = dm^2 (\operatorname{tr}(\rho'_k p_k) + 1) = dm^2.
\]

Thus from Fact 6 it follows that \( f'_k \in \partial_A \Delta \cap B(c_n, r) \). \( \square \)

Now we discuss the polarity of basic and primal polytopes.

Definition 6. Let \( C \) be a convex subset of \( A \), \( s > 0 \). Define polar and dual of \( C \) in \( A \) with respect to the central sphere (with centre at \( c_n \)) of radius \( s \) by

\[
C^o := \{ p \in A : s^2 \geq \langle p - c_n, q - c_n \rangle \text{ for every } q \in C \} \quad \text{and} \quad C^* := \iota(C^o).
\]

Note that \( C^* = \{ p \in A : \langle p, q \rangle \geq s^2 \text{ for every } q \in C \} \).

From the convex analysis we know that

Fact 8. In the above situation

i. \( C^o = C \);
ii. \( C^* = (\iota(C))^\circ \);
iii. \( C^{**} = C \).

Note that the last statement is a direct consequence of the preceding two. For two convex sets \( C \) and \( G \) we call them polar (resp. dual) in \( A \) with respect to the central sphere of radius \( s \) if \( C^o = G \) (resp. \( C^* = G \)). From condition (v) in Fact 6 we deduce

Theorem 10. The polytopes \( D \) and \( \Delta \) are dual in \( A \) with respect to the central sphere of radius \( m = \sqrt{n} \), as well as dual are balls \( B_A(c_n, r) \) and \( B_A(c_n, R) \).
Theorem 11. For $p, q \in Q_\Pi$ we have

$$dm^2 \leq (p, q) \leq 2dm^2.$$ 

Proof. Take $\rho, \sigma \in S(C^d)$ such that $p_{\Pi}(\rho) = p$ and $p_{\Pi}(\sigma) = q$. From the similarity of $S(C^d)$ and $Q_\Pi$ it follows that $(p, q) = dm^2(\text{tr}(\rho\sigma) + 1)$. Now, it is enough to apply the well-known inequality $0 \leq \text{tr}(\rho\sigma) \leq 1$. □

Finally, let us set $s = m$ in the definition of dual set. Then

Theorem 12. The probability range $Q_\Pi$ is self-dual, i.e.

$$Q_\Pi^* = Q_\Pi.$$

Proof. It follows from the first inequality in Theorem 11 that $Q_\Pi \subset Q_\Pi^*$. Let $p \in Q_\Pi$ and $\tau \in L_s(C^d)$ be such that $\text{tr}\tau = 1$ and $p = p_{\Pi}(\tau)$. Since the equation defining similarity (2) holds for all $\rho, \sigma \in L_s(C^d)$ such that $\text{tr}\rho = \text{tr}\sigma = 1$, then from the definition of duality we get $\text{tr}(\rho\sigma) \geq 0$ for every $\rho \in S(C^d)$. Hence $\tau \geq 0$ and, in consequence, $\tau \in S(C^d)$ and $p \in Q_\Pi$. □

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