Limit theorems for additive functionals of random walks in random scenery

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Abstract

We study the asymptotic behaviour of additive functionals of random walks in random scenery. We establish bounds for the moments of the local time of the Kesten and Spitzer process. These bounds combined with a previous moment convergence result (and an ergodicity result) imply the convergence in distribution of additive observables (with a normalization in $n^{1/4}$). When the sum of the observable is null, the previous limit vanishes and we prove the convergence in the sense of moments (with a normalization in $n^{1/8}$).

Keywords: Random walk in random scenery; central limit theorem; local limit theorem; local time; Brownian motion; ergodicity; infinite measure; dynamical system.

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1 Introduction

1.1 Description of the model and of some earlier results

We consider two independent sequences $(X_k)_{k \geq 1}$ (the increments of the random walk) and $(\xi_y)_{y \in \mathbb{Z}}$ (the random scenery) of independent identically distributed $\mathbb{Z}$-valued random variables. We assume in this paper that $X_1$ is centered and admits finite moments of all orders, and that its support generates the group $\mathbb{Z}$. We define the random walk $(S_n)_{n \geq 0}$ as follows

$$S_0 := 0 \quad \text{and} \quad S_n := \sum_{i=1}^{n} X_i \quad \text{for all } n \geq 1.$$  

We assume that $\xi_0$ is centered, that its support generates the group $\mathbb{Z}$, and that it admits a finite second moment $\sigma^2_\xi := E[\xi_0^2] > 0$. The random walk in random scenery (RWRS) is the process defined as follows

$$Z_n := \sum_{k=0}^{n-1} \xi_{S_k} = \sum_{y \in \mathbb{Z}} \xi_y N_n(y),$$  

(1)

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where we set $N_n(y) = \#\{k = 0, \ldots, n-1 : S_k = y\}$ for the local time of $S$ at position $y$ before time $n$. This process first studied by Borodin [7] and Kesten and Spitzer [32] describes the evolution of the total amount won until time $n$ by a particle moving with respect to the random walk $S$, starting with a null amount at time 0 and winning the amount $\xi_k$ at each time the particle visits the position $\ell \in \mathbb{Z}$. This process is a natural example of (strongly) stationary process with long time dependence. Due to the first works by Borodin [7] and by Kesten and Spitzer [32], we know that $(n^{-\frac{1}{d}}Z_{\lfloor nt \rfloor})_t$ converges in distribution, as $n$ goes to infinity, to the so-called Kesten and Spitzer process $(\sigma_\ell \Delta_t, t \geq 0)$, where $\Delta$ is defined by

$$\Delta_t := \int_{-\infty}^{+\infty} L_t(x) d\beta_x, \quad (2)$$

with $(\beta_x)_{x \in \mathbb{R}}$ a Brownian motion and $(L_t(x), t \geq 0, x \in \mathbb{R})$ a jointly continuous in $t$ and $x$ version of the local time process of a standard Brownian motion $(B_t)_{t \geq 0}$, where $((B_t)_t, (\beta_x)_{x \in \mathbb{R}})$ is the limit in distribution of $n^{-\frac{1}{d}}((S_{\lfloor nt \rfloor})_t, (\sigma_\ell^{-1} \sum_{k=1}^{\lfloor nt \rfloor} \xi_k)_\ell)$ as $n \to +\infty$. Observe that $\Delta$ is the continuous time analog of the random walk in random scenery.

To be convinced of this fact, one may compare the right hand side of (1) with (2). The values in $\{26\}$ have been established by Campanino and Petritis [11] and which has many generalizations and de Marsily Model [39] of transport in porous media, the transience of which has been extended by Bolthausen [6] (for the case $\alpha = \beta = 2$ for random walks of dimension $d = 2$), by Deligiannidis and Utev [19] (for the case $\alpha = d \in \{1, 2\}$, $\beta = 2$, providing some correction to [6]) and by Castell, Guillotin-Plantard and the author [12] (when $\alpha \leq d$ and $\beta < 2$), completing the picture for the convergence in the sense of distribution and for the functional limit theorem (except in the case $\alpha \leq 1$ and $\beta = 1$ for which the tightness remains an open question). Since the seminal works by Borodin and by Kesten and Spitzer, random walks in random scenery and the Kesten and Spitzer process $\Delta$ have been the object of various studies (let us mention for example $\{33, 50, 29, 3, 27, 25, 28, 2\}$).

Random walks in random scenery are related to other models, such as the Matheron and de Marsily Model [39] of transport in porous media, the transience of which has been established by Campanino and Petritis [11] and which has many generalizations (e.g. [26, 20, 23, 10, 9]), and such as the Lorentz-Lévy process (see [40] for a short presentation of some models linked to random walks in random scenery).

Random walks in random scenery constitute also a model of interest in the context of dynamical systems. They correspond indeed to Birkhoff sums of a transformation called the $T, T^{-1}$ transformation appearing in [49, p. 682, Problem 2] where it was asked whether this Kolmogorov automorphism is Bernoulli or not. In [30], Kalikow answered negatively this question by proving that this transformation is not even loosely Bernoulli.

### 1.2 Main results

Before stating our main results, let us introduce some additional notations. Let $d \in \mathbb{N}$ be the greatest common divisor of the set $\{x \in \mathbb{Z}, P(\xi_0 - \xi_1 = x) > 0\}$ and $\alpha \in \mathbb{Z}$ such that $P(\xi_0 = \alpha) > 0$. This means that the random variables $\xi$ take almost surely their values in $\alpha + d\mathbb{Z}$ and that $d$ is largest positive integer satisfying this property. Since the support of $\xi$ generates the group $\mathbb{Z}$, necessarily $\alpha$ and $d$ are coprime. Recall that the quantity $d$ can be also simply characterized using the common characteristic function $\varphi_\xi$
of the $\xi_k$.\footnote{Indeed $d \geq 1$ is such that $\{u : |\varphi(u)| = 1\} = (2\pi/d)\mathbb{Z}$ and a.s. $e^{2\pi i u/d} = e^{2\pi i u/d}$ which is a primitive $d$-th root of the unity.}

In the present paper we are interested in the asymptotic behaviour of additive functionals of the RWRS $(Z_n)_{n \geq 1}$ that is of quantities of the following form:

$$Z_n := \sum_{k=1}^{n} f(Z_k)$$

where $f : \mathbb{Z} \to \mathbb{R}$ is absolutely summable. This quantity is strongly related to the local time $N_n$ of the RWRS $Z$, which is defined by

$$N_n(a) = \# \{k = 1, \ldots, n : Z_k = a\}.$$ 

Indeed if $f = \mathbb{1}_0$, then $Z_n = N_n(0)$ and if $f = \mathbb{1}_0 - \mathbb{1}_1$, then $Z_n = N_n(0) - N_n(1)$. In the general case, $Z_n$ can be rewritten

$$Z_n := \sum_{a \in \mathbb{Z}} f(a)N_n(a).$$

The asymptotic behaviour of $(N_n(0))_n$ has been studied by Castell, Guillotin-Plantard, Schapira and the author in [14, Corollary 6], in which it has been proved that the moments of $(n^{-\frac{1}{2}}N_n(0))_{n \geq 1}$ converge to those of the local time $L_1(0)$ at position 0 and until time 1 of the process $\Delta$. The proof of this result was based on a multitime local limit theorem [14, Theorem 5] extending a local limit theorem contained in [13] and on the finiteness of the moments of $L_1(0)$ (which was a delicate question). We complete this previous work by establishing in Section 2 the following bounds for the moments of $L_1(0)$.

**Theorem 1** (Bounds for the moments of the local time of the Kesten and Spitzer process). For any $\eta_0 > 0$, there exists $a > 0$ and $C > 0$ such that

$$(Cm)^{\frac{3m}{2n}} \leq \mathbb{E}[(L_1)^m] = \mathcal{O}\left(\frac{a^m (m!)^{\frac{3}{2}+\eta_0}}{\Gamma\left(\frac{m}{2} + 1\right)}\right) \leq \mathcal{O}\left(m^{m\left(\frac{1}{2}+2\eta_0\right)}\right).$$

Even if it uses some ideas that already existed in [14], the proof of Theorem 1 (given in Section 2) is different in many aspects. The proof of Theorem 1 relies on several auxiliary results. We summarize quickly its strategy. We will prove (see (5) coming from [14] and (6)) that

$$\mathbb{E}[(L_1)^m] = \frac{m!}{(2\pi\sigma^2)^{\frac{m}{2}}} \int_{0 < t_1 < \cdots < t_m < 1} \prod_{k=0}^{m-1} (t_{k+1} - t_k)^{-\frac{m}{2}} \mathbb{E}\left[\prod_{k=0}^{m-1} d(L(k+1), W_k)\right]^{-1} dt_1 \cdots dt_m,$$

where we set $W_k := \text{Pct}(L^{(1)}, \ldots, L^{(k)})$ and $L^{(k+1)} := (L_{t_{k+1}} - L_{t_k})/(t_{k+1} - t_k)^{\frac{1}{2}}$ (normalized so that $|L^{(m)}|_{L^2(\mathbb{R})}$ has the same distribution as $|L_1|_{L^2(\mathbb{R})}$). We will prove, in Lemma 7, that

$$\exists c, C > 0, \quad m! \int_{0 < t_1 < \cdots < t_m < 1} \prod_{k=0}^{m-1} (t_{k+1} - t_k)^{-\frac{m}{2}} dt_1 \cdots dt_m \sim c(Cm)^{\frac{3m}{2n}},$$

as $m \to +\infty$ and, in Lemma 6, that

$$\left(\mathbb{E}\left[\frac{L_1}{L^{(1)}(\mathbb{R})}\right]\right)^m \leq \mathbb{E}\left[\prod_{k=0}^{m-1} d(L(k+1), W_k)\right]^{-1} \leq \prod_{k=0}^{m-1} \mathbb{E}\left[\sup_{V \in V_k} (d(L_1, V))^{-1}\right].$$
where $d(\cdot, \cdot)$ is the distance associated with the $L^2$-norm on $L^2(\mathbb{R})$ and where $\mathcal{V}_k$ is the set of linear subspaces of $L^2(\mathbb{R})$ of dimension at most $k$. Theorem 1 will then follow from the next self-interesting estimate on the local time $L_1$ of the Brownian motion $B$ up to time $1$.

**Theorem 2** (An estimate on the distance between the local time of the Brownian motion and a linear subspace).

$$\sup_{V \in \mathcal{V}_k} \mathbb{E}\left[ (d(L_1, V))^2 \right] = k^{4+o(1)}, \quad \text{as } k \to +\infty.$$  

Now we use the following classical argument for positive random variables. The upper bound provided by Theorem 1 allows us to prove that Carleman’s criterion is satisfied for $\mathcal{E}\sqrt{\mathcal{L}_1(0)}$ where $\mathcal{E}$ is a centered Rademacher distribution independent of $\mathcal{L}_1(0)$ and of $Z$, indeed:

$$\sum_{m \geq 1} \mathbb{E}[\mathcal{L}_1(0)^m]^{-\frac{1}{m}} \geq c_1 \sum_{m \geq 1} m^{-\frac{2}{m}} = \infty,$$

for every $\eta_0 \in (0, \frac{4}{5})$. This enables us to deduce from [14, Corollary 6] that $n^{-\frac{1}{2}} \mathcal{E}\sqrt{\mathcal{N}_n(0)}$ converges in distribution to $\mathcal{E}\sigma^{-1}_1 \mathcal{L}_1(0)$ and so that

$$n^{-\frac{1}{2}} \mathcal{N}_n(0) \xrightarrow{\mathcal{L}} \sigma^{-1}_1 \mathcal{L}_1(0), \quad \text{as } n \to +\infty,$$  

where $\xrightarrow{\mathcal{L}}$ means convergence in distribution. This convergence in distribution is extended to more general observables as follows.

**Theorem 3** (Limit theorem for additive functionals of the RWRS $Z$). Let $f : Z \to \mathbb{R}$ be such that $\sum_{a \in Z} |f(a)| < \infty$. Then $n^{-\frac{1}{2}} \sum_{k=0}^{n-1} f(Z_k)$ converges in distribution and in the sense of moments to $(\sum_{a \in Z} f(a)) \sigma^{-1}_1 \mathcal{L}_1(0)$.

The proof of the moments convergence in Theorem 3 is a straightforward adaptation of [14] and is given in Appendix B. Due to Theorem 1 and to the above argument that lead to (3), the convergence in distribution in Theorem 3 is a consequence of the moments convergence. Another strategy to prove the convergence in distribution in Theorem 3 consists in seeing this result as a direct consequence of (3) combined with Proposition 14 stating the ergodicity of the dynamical system $(\tilde{\Omega}, \tilde{T}, \tilde{\mu})$ corresponding to

$$\tilde{\mathcal{L}}^k((X_{m+1})_{m \in \mathbb{Z}}, (\xi_m)_{m \in \mathbb{Z}}, Z_0) = ((X_{k+m+1})_{m \in \mathbb{Z}}, (\xi_{m+n_k})_{m \in \mathbb{Z}}, Z_k).$$

This dynamical system preserves the infinite measure $\mu := \mathbb{P}^{\otimes \mathbb{Z}} \otimes \mathbb{P}^{\otimes \mathbb{Z}} \otimes \lambda_{\mathbb{Z}}$, where $\lambda_{\mathbb{Z}}$ is the counting measure on $\mathbb{Z}$. Actually, thanks to (3) and to the recurrence ergodicity of $(\tilde{\Omega}, \tilde{T}, \tilde{\mu})$, we prove the following stronger version of the convergence in distribution of Theorem 3.

**Theorem 4** (Limit theorem for Birkhoff’s sums of $(\tilde{\Omega}, \tilde{T}, \tilde{\mu})$). For any $\tilde{\mu}$-integrable function

$$f : \tilde{\Omega} \to \mathbb{R},$$

$$n^{-\frac{1}{2}} \sum_{k=0}^{n-1} f \circ \tilde{T}^k \xrightarrow{\mathcal{L}(\tilde{\mu})} \frac{1}{\sigma_1} \mathcal{L}_1(0), \quad \text{as } n \to +\infty,$$  

where $\xrightarrow{\mathcal{L}(\tilde{\mu})}$ means convergence in distribution with respect to any probability measure absolutely continuous with respect to $\tilde{\mu}$.

Theorem 3 can be seen as a weak law of large numbers, with a non constant limit. When $\sum_{a \in Z} f(a) = 0$, the limit given by Theorem 3 vanishes, but then the next result provides a limit theorem for $Z_n = \sum_{k=0}^{n-1} f(Z_k)$ with another normalization. This second...
result corresponds to a central limit theorem for additive functionals of RWRS. Let us indicate that, contrarily to the moments convergence in Theorem 3, the next result is not an easy adaptation of [14], even if its proof (given in Section 4) uses the same initial idea (computation of moments using the local limit theorem) and, at the beginning, some estimates established in [13, 14]. Indeed, important technical difficulties arise from the cancellations coming from the fact that \( \sum_{a \in \mathbb{Z}} f(a) = 0 \).

**Theorem 5** (Convergence of the moments of “centered” additive functionals of the RWRS \( Z \)). Assume moreover that there exists some \( \kappa \in (0, 1] \) such that \( \xi_0 \) admits a moment of order \( 2 + \kappa \). Let \( f : \mathbb{Z} \to \mathbb{R} \) be such that \( \sum_{a \in \mathbb{Z}} (1 + |a|) |f(a)| < \infty \) and that \( \sum_{a \in \mathbb{Z}} f(a) = 0 \). Then

\[
\sum_{k \in \mathbb{Z}} \sum_{a \in \mathbb{Z}} f(a) f(b) \mathbb{P}(Z_{(a+b) mod 2} = a - b) < \infty.
\]

Moreover all the moments of \( \left( n^{-\frac{d}{2}} \sum_{k=0}^{n-1} f(Z_k) \right)_n \) converge to those of \( \sqrt{\frac{\sigma_f^2}{\sigma_\xi}} \mathcal{L}_1(0) \mathcal{N} \), where \( \mathcal{N} \) is a standard Gaussian random variable independent of \( \mathcal{L}_1(0) \)

where

\[
\sigma_f^2 := \sum_{k \in \mathbb{Z}} \sum_{a,b \in \mathbb{Z}^2} f(a) f(b) \mathbb{P}(Z_{|k|} = a - b).
\]

In particular, for any \( a \in \mathbb{Z} \), \( \left( n^{-\frac{d}{2}} (\mathcal{N}_a(a) - \mathcal{N}_a(0)) \right)_n \) converges in the sense of moments to \( \sqrt{\frac{\sigma_{f,a}^2}{\sigma_\xi}} \mathcal{L}_1(0) \mathcal{N} \), with \( \sigma_{f,a}^2 := \sum_{k \in \mathbb{Z}} \left( 2 \mathbb{P}(Z_{|k|} = 0) - \mathbb{P}(Z_{|k|} = a) - \mathbb{P}(Z_{|k|} = -a) \right) \).

Let us point out the similarity between these results and the classical Law of Large Numbers and Central Limit Theorem for sums of square integrable independent and identically distributed random variables. Indeed Theorems 3 and 5 establish convergence results of the respective following forms

\[
\frac{1}{a_n} \sum_{k=1}^n Y_k \to I(Y_1) \mathcal{Y} \quad \text{and} \quad \frac{1}{\sqrt{a_n}} \sum_{k=1}^n (Y_k - Y(Y_k)^0) \to \sqrt{\sigma_f^2} \mathcal{Y} \mathcal{Z}
\]

as \( n \to +\infty \), with \( a_n \to +\infty \), \( I \) an integral (with respect to the counting measure on \( \mathbb{Z} \)) and \( Y_k^0 \) a reference random variable with integral 1 (e.g. \( Y_k^0 = \mathbb{I}_0(Z_k) \)), note that we cannot take \( Y_k^0 = 1 \) since it is not integrable with respect to the counting measure on \( \mathbb{Z} \).

The summation order in the expression (4) of \( \sigma_f^2 \) is important. Indeed recall that \( \mathbb{P}(Z_k = 0) \) has order \( k^{-\frac{d}{2}} \) and so is not summable. The sum \( \sum_{k \in \mathbb{Z}} \) appearing in (4) is a priori non absolutely convergent if \( d \neq 1 \). Indeed, considering for example that \( \xi_0 \) is a centered Rademacher random variable (i.e. \( \mathbb{P}(\xi_0 = 1) = \mathbb{P}(\xi_0 = -1) = \frac{1}{2} \)) and that \( f = \mathbb{I}_0 - \mathbb{I}_1 \), then, for any \( k \geq 0 \),

\[
\sum_{a,b \in \mathbb{Z}^2} f(a) f(b) \mathbb{P}(Z_{|2k|} = a - b) = \mathbb{P}(Z_{|2k|} = 0) + \mathbb{P}(Z_{|2k|} = 1) - 2 \mathbb{P}(Z_{|2k|} = 0)
\]

and

\[
\sum_{a,b \in \mathbb{Z}^2} f(a) f(b) \mathbb{P}(Z_{|2k+1|} = a - b) = -\mathbb{P}(Z_{|2k+1|} = 1 - 0) - \mathbb{P}(Z_{|2k+1|} = 0 - 1) = -\mathbb{P}(Z_{|2k+1|} = 1).
\]

But, \( \sigma_f^2 \) corresponds to the following sum of an absolutely convergent series (in \( k \)):

\[
\sigma_f^2 = \sum_{a,b \in \mathbb{Z}^2} \left( \sum_{k \in \mathbb{Z}} \sum_{a,b \in \mathbb{Z}^2} f(a) f(b) \mathbb{P}(Z_{(a+b) mod 2} = a - b) \right).
\]
Finally, let us point out that $\sigma^2_f$ defined in (4) corresponds to the Green-Kubo formula, well-known to appear in central limit theorems for probability preserving dynamical systems (see Remark 15 at the end of Section 3).

Let us indicate that results similar to Theorem 5 exist for one-dimensional random walks, that is when the RWRS $(S_n)_{n \geq 1}$ is replaced by the RW $(S_n)_{n \geq 1}$, with other normalizations and with an exponential random variable instead of $L_1(0)$. Such results have been obtained by Dobrušin [21], Kesten in [31] and by Csáki and Földes in [17, 18]. The idea used therein was to construct a coupling using the fact that the times between successive return times of $(S_n)_{n \geq 1}$ to 0 are i.i.d., as well as the partial sum of the $f(S_k)$ between these return times to 0 and that these random variables have regularly varying tail distributions. This idea has been adapted to dynamical contexts by Thomine [47, 48].

Still in dynamical contexts, another approach based on moments has been developed in [41, 42] in parallel to the coupling method. This second method based on local limit theorem is well tailored to treat non-markovian situations, such as RWRS. Indeed, recall that the RWRS $(Z_n)_{n \geq 1}$ is (strongly) stationary but far to be markovian (for example it has been proved in [14] that $Z_{n+m} - Z_n$ is more likely to be 0 if we know that $Z_n = 0$ and even more intricate conditionally to the scenery (it has been proved in [25] that the RWRS does not converge knowing the scenery). Luckily local limit theorem type estimates enable to prove moments convergence. But unfortunately Theorem 1 is not enough to conclude the convergence in distribution via Carleman’s criterion.

The paper is organized as follows. In Section 2, we prove Theorem 1 (bounds on moments of the local time of the Kesten Spitzer process) and Theorem 2 (estimate on the distance in $L^2(\mathbb{R})$ between the local time of a Brownian motion and a $k$-dimensional vector space). In Section 3, we establish the recurrence ergodicity of the infinite measure preserving dynamical system $(\tilde{\Omega}, \tilde{T}, \tilde{\mu})$ and obtain the convergence in distribution of Theorem 3 (Law of Large Numbers) as a byproduct of this recurrence ergodicity combined with (3). Section 3 is completed by Appendix B which contains the proof of the moments convergence of Theorem 3. In Section 4 (completed with Appendix A), we prove Theorem 5 (Central Limit Theorem).

2 Upper bound for moments: Proof of Theorem 1

This section is devoted to the study of the behaviour of $E[(L_1(0))^m]$ as $m \to +\infty$. It has been proved in [14] that these quantities are finite, but the estimate established therein was not enough to apply the Carleman criterion. The proof of Theorem 1 requires a much more delicate study, even if it uses some estimates used in [14]. We start by establishing bounds for $E[(L_1(0))^m]$.

Lemma 6 (Bounds for the moments of the local time $L_1$ of the RWRS $Z$ in terms of the local time $L_1$ of the Brownian motion and of an integral).

\[
\left( E\left[ (L_1)^{-1}_{L^2(\mathbb{R})} \right] \right)^m \frac{m!}{(2\pi \sigma_f^2)^\frac{m}{2}} \int_{0 < t_1 < \ldots < t_m < 1} \prod_{k=0}^{m-1} (t_{k+1} - t_k)^{-\frac{m}{2}} dt_1 \cdots dt_m \leq E[(L_1(0))^m]
\]

and

\[
E[(L_1(0))^m] \leq \prod_{j=0}^{m-1} \left( \sup_{V \in V_k} E\left[ (d(L_1, V))^{-1} \right] \right) \frac{m!}{(2\pi \sigma_f^2)^\frac{m}{2}} \int_{0 < t_1 < \ldots < t_m < 1} \prod_{k=0}^{m-1} (t_{k+1} - t_k)^{-\frac{m}{2}} dt_1 \cdots dt_m,
\]

where $d(f, g) = |f - g|_{L^2(\mathbb{R})}$ and where $V_k$ is the set of linear subspaces of $L^2(\mathbb{R})$ of dimension at most $k$. 
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Proof. Recall that it has been proved in [14, Theorem 3] that

\[ E[(|L_1(0)|)^m] = \frac{m!}{(2\pi \sigma^2)^{\frac{m}{2}}} \int_{0<t_1<\cdots<t_m<1} E[(\det D_{t_1,\ldots,t_m})^{-\frac{1}{2}}] dt_1 \cdots dt_m, \tag{5} \]

with \( D_{t_1,\ldots,t_m} := (\int_{\mathbb{R}} L_{t_1}(x)L_{t_j}(x) dx)_{i,j=1,\ldots,m} \) where \((L_t(x))_{t \geq 0, x \in \mathbb{R}}\) is the local time of the Brownian motion \(B\). Since \(\det D_{t_1,\ldots,t_m}\) is a Gram determinant, we have the iterative relation

\[ \det D_{t_1,\ldots,t_{m+1}}^2 = \det D_{t_1,\ldots,t_m}^2 d(L_{t_{m+1}}, Vect(L_{t_1},\ldots,L_{t_m})) , \]

where \(d(f,g) = \|f-g\|_{L^2(\mathbb{R})}\) and \(Vect(L_{t_1},\ldots,L_{t_m})\) is the sublinear space of \(L^2(\mathbb{R})\) generated by \(L_{t_1}\), \ldots, \(L_{t_m}\). It follows that

\[ \det D_{t_1,\ldots,t_m}^\frac{1}{2} = \prod_{k=0}^{m-1} (d(L_{t_{k+1}}, Vect(L_{t_1},\ldots,L_{t_k}))^{-1}. \tag{6} \]

But, for any \(m \geq 1\) and any \(0 < t_1 < \cdots < t_{m+1} < 1\) and any \(k = 0,\ldots,m-1,\)

\[ E \left[ d \left( L_{t_{k+1}}, Vect(L_{t_1},\ldots,L_{t_k}) \right)^{-1} \right] \]

\[ = E \left[ E \left[ d \left( L_{t_{k+1}} - L_{t_k}, Vect(L_{t_1},\ldots,L_{t_k}) \right)^{-1} \bigg| (B_s)_{s \leq t_k} \right] \right] \]

\[ = E \left[ E \left( d \left( L_{t_{k+1}} - L_{t_k}(B_{t_k} + \cdot), Vect(L_{t_1}(B_{t_1} + \cdot),\ldots,L_{t_k}(B_{t_k} + \cdot)) \right)^{-1} \right| (B_s)_{s \leq t_k} \right] . \]

Therefore

\[ E \left[ L_{t_{k+1}} - L_{t_k} \right] \leq E \left[ d \left( L_{t_{k+1}}, Vect(L_{t_1},\ldots,L_{t_k}) \right)^{-1} \right] \]

and

\[ E \left[ d \left( L_{t_{k+1}}, Vect(L_{t_1},\ldots,L_{t_k}) \right)^{-1} \right] \leq \sup_{V \in \mathcal{V}_k} E \left[ d \left( L_{t_{k+1}} - L_{t_k}(B_{t_k} + \cdot), V \right)^{-1} \right] , \tag{7} \]

where \(\mathcal{V}_k\) is the set of linear subspaces of dimension at most \(k\) of \(L^2(\mathbb{R})\) and where we used the independence of \(L_{t_{k+1}} - L_{t_k}(B_{t_k} + \cdot)\) with respect to \((B_s)_{s \leq t_k}\) and the fact that \(L_{t_1}(B_{t_1} + \cdot),\ldots,L_{t_k}(B_{t_k} + \cdot)\) is measurable with respect to \((B_s)_{s \leq t_k}\). Thus, by induction and using the fact that the increments of \(B\) are (strongly) stationary, it follows from (6) and (7) that

\[ \prod_{k=0}^{m-1} E \left[ d \left( L_{t_{k+1}} - L_{t_k} \right)^{-1} \right] \leq E \left[ \det D_{t_1,\ldots,t_m}^\frac{1}{2} \right] \]

\[ \leq \prod_{k=0}^{m-1} \sup_{V \in \mathcal{V}_k} E \left[ d \left( L_{t_{k+1}} - L_{t_k}(B_{t_k} + \cdot), V \right)^{-1} \right] \]

\[ = \prod_{k=0}^{m-1} \sup_{V \in \mathcal{V}_k} E \left[ d \left( L_{t_{k+1}} - t_k, V \right)^{-1} \right] . \tag{8} \]

with the convention \(t_0 = 0\). Recall that \((L_u(x))_{x \in \mathbb{R}}\) has the same distribution as \((\sqrt{u}L_1(x)/\sqrt{u})_{x \in \mathbb{R}}\) and so \((d(L_u, Vect(g_1,\ldots,g_k)))^2\) has the same distribution as

\[ \min_{a_1,\ldots,a_k} \int_{\mathbb{R}} \left( \sqrt{u}L_1 \left( \frac{x}{\sqrt{u}} \right) - \sum_{i=1}^{k} a_i g_i(x) \right)^2 dx = \min_{a_1,\ldots,a_k} \int_{\mathbb{R}} \left( L_1 \left( \frac{x}{\sqrt{u}} \right) - \sum_{i=1}^{k} a_i g_i(x) \right)^2 dx \]

\[ = u^{\frac{3}{2}} \min_{a_1,\ldots,a_k} \int_{\mathbb{R}} \left( L_1(y) - \sum_{i=1}^{k} a_i g_i(\sqrt{uy}) \right)^2 dy \]

\[ = u^{\frac{3}{2}} (d(L_1, Vect(h_1,\ldots,h_k)))^2 \]
setting \(a' := a/\sqrt{n}\), and making the change of variable \(y = x/\sqrt{n}\), with \(h_i(x) = g_i(\sqrt{n}x)\)
and so (8) becomes

\[
\prod_{k=0}^{m-1} \left((t_{k+1} - t_k)^{-\frac{4}{2}} \mathbb{E}\left[|L_1|_{L^2(\mathbb{R})}^{-1}\right]\right) \leq \mathbb{E}\left[\det D_{t_1, \ldots, t_m}^{-\frac{4}{2}}\right]
\leq \prod_{k=0}^{m-1} (t_{k+1} - t_k)^{-\frac{4}{2}} \sup_{V \in \mathcal{V}_k} \mathbb{E}\left[(d(L_1, V))^{-1}\right],
\]
which ends the proof of the lemma. \(\square\)

We first study the behaviour, as \(m \to +\infty\), of the integral appearing in Lemma 6.

**Lemma 7 (Asymptotic estimate of the integral).**

\[
m! \int_{0 < t_1 < \ldots < t_m < 1} \prod_{k=0}^{m-1} (t_{k+1} - t_k)^{-\frac{2}{2}} dt_1 \cdots dt_m = \frac{m! \Gamma(\frac{1}{2})^m}{\Gamma(\frac{m}{2} + 1)} \sim c(m)^{\frac{m}{4}},
\]
as \(m \to +\infty\).

**Proof.**

\[
a_{m+1} := \int_{0 < t_1 < \ldots < t_{m+1} < 1} \prod_{k=0}^{m} (t_{k+1} - t_k)^{-\frac{2}{2}} dt_1 \cdots dt_{m+1}
= \int_{x_i > 0 : x_1 + \ldots + x_{m+1} < 1} \prod_{k=1}^{m+1} x_k^{-\frac{2}{2}} dx_1 \cdots dx_{m+1}
= \int_{0}^{1} x_{m+1}^{-\frac{2}{2}}(1 - x_{m+1})^{-\frac{2m}{2}} \left(\prod_{k=0}^{m} (x_k/(1 - x_{m+1}))^{-\frac{2}{2}} dx_1 \cdots dx_m\right) dx_{m+1}
= \int_{0}^{1} x_{m+1}^{-\frac{2}{2}}(1 - x_{m+1})^{-\frac{2m}{2}} \left(\prod_{k=0}^{m} u_k^{-\frac{2}{2}} du_1 \cdots du_m\right) dx_{m+1}
= a_m \int_{0}^{1} x_{m+1}^{-\frac{2}{2}}(1 - x_{m+1})^{-\frac{2m}{2}} dx_{m+1} = a_m B \left(1, m, \frac{m}{4} + 1\right),
\]
where \(B(\cdot, \cdot)\) and \(\Gamma(\cdot)\) stand respectively for Euler’s Beta and Gamma functions, and so, by induction, \(a_m = \frac{\Gamma(1/4)^m}{\Gamma((m+1)/4)}\) proving the first point of the lemma. Moreover

\[
m! a_m \sim (\Gamma(1/4))^{m} m^{m+\frac{1}{4}}(m + 4)^{-\frac{m}{2} - \frac{1}{4} + \frac{1}{4} e^{-\frac{3m}{4}} + 1},
\]
where we used the Stirling formulas \(m! = \Gamma(m + 1)\) and \(\Gamma(z) \sim \sqrt{2\pi} z^{z-\frac{1}{2}} e^{-z}\). This ends the proof of the lemma. \(\square\)

Observe that \(\mathbb{E}\left[|L_1|_{L^2(\mathbb{R})}^{-1}\right] > 0\). Thus, the proof of Theorem 1 will be deduced from the two previous lemmas combined with Theorem 2, which can be rewritten as follows

\[
\forall \eta_0 > 0, \exists C > 1, \forall k \in \mathbb{N}^*, \quad C^{-1} k^{\frac{1}{2} - \eta_0} \leq \sup_{V \in \mathcal{V}_k} \mathbb{E}\left[(d(L_1, V))^{-1}\right] \leq C k^{\frac{1}{2} + \eta_0}. \tag{9}
\]

Due to [44, Cor. (1.8) of Chap. VI, Theorem (2.1) of Chap. I], \(L_1\) is almost surely Hölder continuous of order \(\frac{1}{2} - \eta_0\) and its Hölder constant admits moments of any order. The lower bound of theorem 2 follows directly from this fact.
Proof of the lower bound of Theorem 2. We prove the lower bound of (9). Let \( \eta_0 \in (0, 1/2) \).

Let \( C_1 \) be the Hölder constant of order \( 1/2 - \eta_0 \) of \( L_1 \). Let \( V_k \) be the linear subspace of \( L^2(\mathbb{R}) \) generated by the set

\[
\left\{ \mathbf{1}_{[m/k,(m+1)/k]}, m = -\left\lfloor \frac{k}{2} \right\rfloor, \cdots, \left\lfloor \frac{k}{2} \right\rfloor - 1 \right\},
\]

and consider \( \tilde{L}_k \in V_k \) given by

\[
\tilde{L}_k := \sum_{m=-\left\lfloor \frac{k}{2} \right\rfloor}^{\left\lfloor \frac{k}{2} \right\rfloor - 1} L_1 \left( \frac{m}{k} \right) \mathbf{1}_{\left( \frac{m}{k}, \frac{m+1}{k} \right)}.
\]

Thus, \( \tilde{L}_k \) is a basis of \( \mathbb{R}^n \). Let \( \eta > 0 \) be such that \( \left\lfloor \frac{k}{2} \right\rfloor \leq \left\lfloor \frac{k}{2} \right\rfloor - \eta \).

Let \( K_0 > 0 \). We will use the fact that

\[
\mathbb{E} \left[ (d(L_1, V_k))^{-1} \right] \geq \mathbb{E} \left[ (d(L_1, V_k))^{-1} \mathbb{1}_{\{ C_1 \leq K_0, \sup_{[0,1]} |B| \leq \frac{2}{k+1} \}} \right].
\]

Observe that, if \( \sup_{[0,1]} |B| \leq \frac{k-1}{2k} \) and \( C_1 \leq K_0 \), then

\[
d(L_1, V_k)^2 = d(L_1, \tilde{L}_k)^2 = \sum_{m=-\left\lfloor \frac{k}{2} \right\rfloor}^{\left\lfloor \frac{k}{2} \right\rfloor - 1} (L_1(u) - L_1(m/k))^2 du
\]

\[
\leq \left\lfloor \frac{k}{2} \right\rfloor - 1 \left( K_0 k^{-1/2 + \eta_0} \right)^2 \leq \left( K_0 k^{-1/2 + \eta_0} \right)^2.
\]

Thus

\[
\mathbb{E} \left[ (d(L_1, V_k))^{-1} \right] \geq \mathbb{E} \left[ (d(L_1, V_k))^{-1} \mathbb{1}_{\{ C_1 \leq K_0, \sup_{[0,1]} |B| \leq \frac{2}{k+1} \}} \right]
\]

\[
\geq \mathbb{E} \left[ \left( K_0 k^{-1/2 + \eta_0} \right)^{-1} \mathbb{1}_{\{ C_1 \leq K_0, \sup_{[0,1]} |B| \leq \frac{2}{k+1} \}} \right]
\]

\[
\geq K_0^{-1} k^{1/2 - \eta_0} \mathbb{P} \left( C_1 \leq K_0, \sup_{[0,1]} |B| \leq \frac{1}{3} \right).
\]

The rest of this section is devoted to the proof of the upper bound of Theorem 2 (i.e. the upper bound of (9)), which is much more delicate to establish. To this end, we will prove a sequence of estimates. We have chosen to start by listing the different quantities used in this proof, and the relations between them, for two reasons. First, it makes more evident the compatibility between our different conditions. Second, for practical use for the reader who can come back to this page if he or she forget at some point one of these different conditions or relations. We fix \( \eta_0 > 0 \) and \( d' = \frac{1}{2} + \eta_0 > 1/2 \). Choose \( \epsilon_0 \in (0, \frac{1}{10}) \) such that

\[
d' > \frac{1 + \epsilon_0}{2(1 - \epsilon_0)}.
\]

Fix \( a, b, \eta, \gamma \in (0, \frac{1}{10}) \) such that \( 0 < b < \frac{\gamma}{2} < \frac{2}{2} \) and small enough so that

\[
\frac{(1 + \gamma)(1 + \epsilon_0)}{2} + \frac{a}{2} + \frac{b}{8} < 1
\]

and

\[
(2d' - 1 - \epsilon_0 - 1 - \epsilon_0)(1 - 2\eta) - 8\eta > 0.
\]

Let \( \theta > 0 \) such that \((1 - 2\eta)\theta > 1\) and

\[
1 - \frac{b}{4} - \frac{(1 + \gamma)(1 + \epsilon_0)}{2} < \theta(1 - 2\eta) \left( 1 - \frac{(1 + \gamma)(1 + \epsilon_0)}{2} - \frac{a}{2} - \frac{b}{8} \right)
\]
and

\[(1 - \epsilon_0)(1 + 2d') < \theta \left| (2d' (1 - \epsilon_0) - 1 - \epsilon_0)(1 - 2\eta) - 8\eta \right|.\]  

(14)

The existence of such a \(\theta\) is ensured by (11) and (12). Fix then \(K\) such that \(\frac{1}{4 - \gamma} < K\), \(v_0 = \frac{16/b}{\ell}\) and \(\zeta > 0\) such that \(4a - (1 + 4\zeta)b > 0\) and \(K > (4a - (1 + 2\zeta)b)^{-1}\). We further consider the following quantities which will depend on \(k \geq 1\). We set \(M := \lceil k \theta \rceil\) and \(M' := M^{d'}\). For \(x > M'\), we also set:

\[r_0 := (x/M')^{-(1 + \gamma)(1 + \epsilon_0)} M^{-1 - \epsilon_0}, \quad x_0 = (x/M')^a M, \quad x_1 = (x/M')^b.\]

(15)

Let \(V\) be a linear space generated by \(g_1, \ldots, g_k \in L^2(\mathbb{R})\). Observe that

\[
E \left[ (d(L_1, V))^{-1} \right] = \int_0^\infty P \left( (d(L_1, V))^{-1} > x \right) \, dx = O(M') + \int_M^\infty P \left( d(L_1, V) < x^{-1} \right) \, dx.
\]

(16)

**Lemma 8** (An upper bound using a spatial discretization). Uniformly on \(x > M'\):

\[P \left( d(L_1, V) < x^{-1} \right) \leq O \left( (x/M')^{-2} \right) + P \left( \forall \ell = -v_0, \ldots, v_0, D \left( \sum_{n=1}^\infty W_{V}^{(x_1^{\frac{1}{2}})} \right) \right) < 2x^{-1}r_0^{-\frac{1}{2}},\]

where \(W_{V}^{(y_0)} := \text{Span} \left( \left( \int_{y_0 + n/x_0} g_j(y) \, dy \right) \right)_{n=1, \ldots, M}, j = 1, \ldots, k \subset \mathbb{R}^M\) and where \(D\) is the usual euclidean metric in \(\mathbb{R}^M\).

**Proof.** We set

\[C_1 := \sup_{y, z \in \mathbb{R} : y \neq z} \frac{|L_1(y) - L_1(z)|}{|y - z|^u}, \quad \text{with } u := \frac{1}{1 + \epsilon_0} - \frac{1}{2}.\]

Since \(C_1\) admits moments of every order, it follows that

\[P \left( d(L_1, V) < 1/x \right) \leq P \left( d(L_1, V) < 1/x \right) \leq (x/M')^\gamma + O((x/M')^{-2}).\]

Note that, if \(x > M'\), then

\[r_0x_0 = (x/M')^{a-(1+\gamma)(1+\epsilon_0)} M^{-1-\epsilon_0} \leq 1,
\]

since \(a < 1 < (1 + \gamma)(1 + \epsilon_0)\) and since \(M' = M^{d'}\) with \(\frac{1}{2} \leq d'\), and so \(r_0 \leq x_0^{-1}\). Assume moreover that \(d(L_1, V) < 1/x\) and \(C_1 \leq (x/M')^\gamma\). Let \(a_j\) be such that \(d(L_1, \sum_{j=1}^k a_j g_j) < x^{-1}\). Then, for every \(\ell \in \mathbb{Z}\), the following estimate holds true

\[x^{-1} > \left( \sum_{n=1}^M \int_{\ell x_1^{\frac{1}{2}} + \frac{n}{x_0}} \left( L_1(y) - \sum_{j=1}^k a_j g_j(y) \right)^2 \, dy \right)^{\frac{1}{2}}
\]

\[\geq \left( \sum_{n=1}^M \int_{\ell x_1^{\frac{1}{2}} + \frac{n}{x_0}} \left( L_1 \left( \ell x_1^{\frac{1}{2}} + \frac{n}{x_0} \right) - \sum_{j=1}^k a_j g_j(y) \right)^2 \, dy \right)^{\frac{1}{2}} - \left( M r_0 (x/M')^2 r_0^{-\frac{1}{2}} \right)^{\frac{1}{2}}
\]

\[\geq \sqrt{r_0} D \left( \sum_{n=1}^M \int_{\ell x_1^{\frac{1}{2}} + \frac{n}{x_0}} \left( L_1 \left( \ell x_1^{\frac{1}{2}} + \frac{n}{x_0} \right) \right) \, dy \right)^{\frac{1}{2}} - \sqrt{M (x/M')^\gamma r_0^{-\frac{1}{2}}}.
\]
Since \( \frac{1}{2} + u = \frac{1}{1 - r_0} \) and \( r_0 = (x/M')^{-(1+\gamma)(1+\epsilon_0)}M^{-\frac{1+\gamma}{2}}M'^{-1-\epsilon_0} \), we conclude that 
\[
\sqrt{M(x/M')^{\gamma}r_0^{\frac{1}{2} + u}} = x^{-1} \text{ and so}
\]
\[
P( d (L_1, V) < 1/x, C_1 \leq (x/M')^\gamma )
\]
\[
\leq P \left( \forall \ell = -v_0, \ldots, v_0, D \left( \left( L_1 \left( \ell x_1^{-\frac{1}{2}} + \frac{n}{x_0} \right) \right)_{n=1, \ldots, M}, W \left( \tau' x_1^{-\frac{1}{2}} \right) \right) < 2x^{-1}r_0^{-\frac{1}{2}} \right). \quad \square
\]

Recall that \( v_0 = [16/b] \). For every \( \ell = -v_0, \ldots, v_0 \), we set \( t_\ell(x_1) := \inf \{ s > 0 : L_s(\ell x_1^{-\frac{1}{2}}) > x_1^{-\frac{1}{2}} \} \) and \( Y'_s(y) = L_{t_\ell(x_1)}(\ell x_1^{-\frac{1}{2}} + y) \). Due to the second Ray-Knight theorem (see [44, Theorem 2.3, page 456]), \( Y'_s(y) \) has the same distribution as a squared Bessel process \( Y' \) of dimension 0 starting from \( x_1^{-\frac{1}{2}} \) and we set
\[
E_{0,W,\ell,A} := \left\{ D \left( \left( Y'_s \left( \frac{n}{x_0} \right) \right)_{n=1, \ldots, M}, W + A \right) < 2x^{-1}r_0^{-\frac{1}{2}} \right\}
\]
and
\[
E_{0,W} := \left\{ D \left( \left( Y'_s \left( \frac{n}{x_0} \right) \right)_{n=1, \ldots, M}, W \right) < 2x^{-1}r_0^{-\frac{1}{2}} \right\}.
\]
Let \( \tau' := \int_0^\infty Y'(y) \, dy \).

**Lemma 9** (An upper bound involving the square Bessel process \( Y' \) conditionally with respect to \( \tau' \)). The following estimate holds true uniformly on \( x > M' \):
\[
P( d (L_1, V) < x^{-1} ) \leq \mathcal{O} \left( (x/M')^{-2} \right) + (2v_0 + 1)E \left[ \sup_W P \left( E_{0,W} | \tau' \right) \right],
\]
where \( \sup_W \) means the supremum over the set of affine subspaces \( W \) of \( \mathbb{R}^M \) of dimension at most \( k \).

**Proof.** We adapt the proof of [14, Lemma 9]. Setting \( \epsilon' := x_1^{-\frac{1}{2}} \) and \( T_n := \min \{ s > 0 : |B_s| = u \} \) for the first hitting time of \( \{ \pm u \} \) by the Brownian motion \( B \), we observe that there exists \( c_0 > 0 \) such that
\[
P(T_{v_0 \epsilon'} > 1) = P \left( \sup_{s \in [0,1]} |B_s| \leq v_0 \epsilon' \right) = \mathcal{O}(e^{-c_0(v_0 \epsilon')^2}) = \mathcal{O} \left( (x/M')^{-b_{v_0}/8} \right) = \mathcal{O} \left( (x/M')^{-2} \right).
\]

(Using e.g. [43, Proposition 8.4, page 52]). Moreover, due to [44, Exercise 4.12, Chapter VI, p 265], for every \( n = 0, \ldots, v_0 - 1 \),
\[
P \left( L_{T_{n+1}}(B_{T_{n+1}}) - L_{T_{n}}(B_{T_{n}}) \leq (\epsilon')^2 | (B_u)_{u \leq T_{n+1}} \right) \leq P(L_{T_{n}}(0) \leq (\epsilon')^2) \leq \epsilon'
\]
and so, due to the strong Markov property,
\[
P \left( \forall n = 0, \ldots, v_0 - 1, L_{T_{n+1}}(B_{T_{n+1}}) - L_{T_{n}}(B_{T_{n}}) \leq (\epsilon')^2 \right) \leq (\epsilon')^{v_0},
\]
and this, combined with (18), ensures that there exists \( C_0 > 0 \) such that \( P( \forall \ell = -v_0, \ldots, v_0, L_1(\ell \epsilon') \leq (\epsilon')^2) \leq C_0(\epsilon')^{v_0} \) and so
\[
P \left( \forall \ell = -v_0, \ldots, v_0, t_\ell(x_1) > 1 \right) \leq C_0(x/M')^{-b_{v_0}/8} \leq C_0(x/M')^{-2},
\]
recalling that \( t_x(1) := \inf\{s > 0 : L_s(\ell x_1^{-\frac{1}{4}}) > x_1^{-\frac{1}{4}}\} \). As in [14, p. 2430], we write
\[
\tau'_\ell = \int_0^\infty Y'_\ell(y) \, dy
\]
for the time spent by the brownian motion \( B \) above \( \ell x_1^{-\frac{1}{4}} \) before time \( t_x(1) \). For any \( \ell, \ldots, v_0 \), we have
\[
\sup \mathbb{P} \left( t_x(1) < 1, D \left( \left( L_1 \left( \ell x_1^{-\frac{1}{4}} + \frac{n}{x_0} \right) \right)_{n=1,\ldots,M} \right), W'_{1}^{(\ell x_1^{-\frac{1}{4}})} \right) < 2x_1^{-1}r_0^{-\frac{1}{4}}
\]
with \( A^{(\ell)} := \left( L_{t_x(1)} - L_1 \right) \left( \ell x_1^{-\frac{1}{4}} + \frac{n}{x_0} \right) \). To end the proof, we proceed as in [14, p. 2431] and notice that
\[
\mathbb{P} \left( \{t_x(1) < 1\} \cap E_{0,W,A}^{(\ell)} \right) \leq \mathbb{E} \left[ \mathbb{1}_{\{t_x(1) < 1\}} \sup_{A \in \mathbb{R}^M} \mathbb{P} \left( E_{0,W,A}^{(\ell)} \right) \right],
\]
for any \( A^{(\ell)} \) independent of \( Y'_\ell \) conditionally to \( (t_x(1), \tau'_\ell) \). Moreover \( Y'_\ell \) and \( t_x(1) \) are independent conditionally to \( \tau'_\ell \). It follows that
\[
\sup_{A \in \mathbb{R}^M} \mathbb{P} \left( E_{0,W,A}^{(\ell)} \right) \leq \sup_{A \in \mathbb{R}^M} \mathbb{P} \left( E_{0,W,A}^{(\ell)} \right) = \sup_{A \in \mathbb{R}^M} \mathbb{P} \left( E_{0,W,A}^{(\ell)} \right).
\]

The lemma follows from this last identity combined with (19), (20) and (21).

Recall that \( 4a - (1 + 4\zeta)b > 0 \) and that \( K > (4a - (1 + 2\zeta)b)^{-1} \). Set
\[
E_1 := \left\{ \sup_{s \leq M/x_0} \left| Y'(s) - x_1^{-1/4} \right| < \frac{x_1^{(1+\zeta)/4}}{2} \right\} \quad \text{and} \quad E_1' := E_1 \cap \left\{ \tau' \geq 2x_1^{-\frac{2b+4}{4}} \right\}.
\]

**Lemma 10** (Removal of high values of \( \tau' \)). The following estimate holds true uniformly on \( x > M' \):
\[
\mathbb{P} \left( E_1' \right) = 1 \mathcal{O} \left( (x/M')^{-K(4a - (1 + 2\zeta)b)} \right).
\]

**Proof.** As recalled in [14, before (17)], \( \tau' \) has the same distribution as the first hitting time of \( 2x_1^{-\frac{1}{2}} \) by a Brownian motion. Thus there exist two positive real numbers \( c_1 \) and \( c_2 \) such that:
\[
\mathbb{P} \left( \tau' < 2x_1^{-\frac{2b+4}{4}} \right) \leq \mathbb{P} \left( \sup_{s \in [0,2x_1^{-\frac{2b+4}{4}}]} B_s \geq \frac{x_1^{-\frac{1}{2}}}{2} \right)
\]
\[
\leq \mathbb{P} \left( \sqrt{2x_1^{-\frac{2b+4}{4}}} \sup_{s \in [0,1]} B_s > \frac{x_1^{-\frac{1}{2}}}{2} \right) \leq c_1 e^{-c_2 x_1^{-\frac{2b+4}{4}}},
\]
Using the Burkholder-Davis-Gundy inequality, combined with the fact that \( Y' \) is dominated by the square of a Brownian motion starting from \( x_1^{-\frac{1}{2}} \), we observe that
\[
p_x = \mathbb{P} \left( \sup_{s \leq 10M/x_0} \left| Y'(s) - x_1^{-1/4} \right| \geq \frac{x_1^{-(1+\zeta)/4}}{2} \right)
\]
\[
\leq C_K x_1^{2(1+\zeta)K} a^{2K} \mathbb{E} \left[ \left( \int_0^{10M/x_0} Y'(u) \, du \right)^{4K} \right]
\]
\[
\leq C'_K x_1^{2(1+\zeta)K} (10M/x_0)^{4K-1} \int_0^{10M/x_0} \mathbb{E} \left[ Y'(u)^{4K} \right] \, du
\]
with \( C''_K = 2^{8K} C'_K \), and so

\[
p_x \leq C'_K x_1^{2(1+\zeta)K} (M/x_0)^{4K-1} \int_0^{10M/x_0} \mathbb{E} \left[ (x_1^{-1/8} + B_u)^{8K} \right] \, du \\
\leq C'_K x_1^{2(1+\zeta)K} (M/x_0)^{4K} 2^{8K} \left( x_1^{-K} + (M/x_0)^{4K} \right) \\
\leq C''_K x_1^{2K} \left( x_1^K (M/x_0)^{4K} + x_1^2 (M/x_0)^{8K} \right)
\]

with \( C''_K = 2^{8K} C'_K \) and

\[
x_1^K (M/x_0)^{4K} = (x/M')^{-K(4a-b)},
\]

since \( x_0 = (x/M')^a M \) and \( x_1 = (x/M')^b \).

\( \square \)

**Lemma 11** (Removal of the conditioning). There exists \( K > 0 \) such that

\[
\sup_W \mathbb{P} \left( E_0, W \cap E'_1 | \tau' \right) \leq K \sup_W \mathbb{P} \left( E_1 \cap E_0, W \right),
\]

where \( \sup_W \) means the supremum over the set of affine subspaces \( W \) of \( \mathbb{R}^M \) of dimension at most \( k \).

**Proof.** We adapt the proof of [14, Lemma 12]. Let \( W \) be an affine subset of \( \mathbb{R}^M \) of dimension at most \( k \). We decompose \( \tau' \) in \( \tau = \tau + \tau'' \) with

\[
\tau := \int_0^M Y'(s) \, ds \quad \text{and} \quad \tau'' := \int_{M/x_0}^\infty Y'(s) \, ds.
\]

Then, for any bounded measurable function \( \phi : [0, +\infty) \to [0, +\infty) \), the following relations hold true

\[
\mathbb{E} \left[ \phi(\tau') \mathbb{P} \left( E_0, W \cap E'_1 | \tau' \right) \right] = \mathbb{E} \left[ \phi(\tau + \tau'') \mathbb{I}_{E_0, W \cap E'_1} \right] \\
\leq \mathbb{E} \left[ \mathbb{E} \left[ \mathbb{I}_{I_1} (\tau + \tau'') \phi(\tau + \tau'') | (Y_s)_{s \leq M/x_0} \right] \mathbb{I}_{E_0, W \cap E_1} \right], \quad (22)
\]

with \( I_1 = \left( 2x_1^{-2+\zeta} + \infty \right) \). As in the proof of [14, Lemma 12], we use the fact that the probability density functions of \( \tau' \) is \( f_{x_1^{-1/4}} \) with

\[
f_y(t) := \frac{ye^{-\frac{t}{\zeta}}}{\sqrt{\pi(2t)^2}}
\]

and that the probability density functions of \( \tau'' \) conditionally to \( (Y'(s))_{s \leq M/x_0} \) is \( f_{Y'(M/x_0)} \) (due to the strong Markov property). Thus

\[
\mathbb{E} \left[ \mathbb{I}_{I_1} (\tau'') \phi(\tau + \tau'') | (Y_s)_{s \leq M/x_0} \right] = \int_{I_1 - \tau} \phi(\tau + z) f_{Y'(M/x_0)}(z) \, dz \quad (23)
\]

To conclude, we will prove that \( \frac{f_{Y'(M/x_0)}(z)}{x_1^{-1/4}} \) is uniformly bounded on \( E_1 \) and in \( z \in I_1 - \tau \).

We observe that, on \( E_1 \), \( \sup_{s \in [0, M/x_0]} \left| Y'(s) - x_1^{-\frac{1}{4}} \right| < x_1^{-\frac{14a}{2}} \) and so \( \frac{M}{x_0} \frac{1}{4} \leq \tau < \frac{3M}{x_0} \frac{1}{2} \).

Moreover

\[
\frac{M}{x_0} \frac{1}{4} = (x/M')^{-a} \frac{-a-\frac{1}{4}}{4} \leq (x/M')^{-\frac{(1+2b)1}{4}} = x_1^{-\frac{14a}{2}} \leq x_1^{-\frac{2a+4}{8}},
\]

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since $4a - (1 + 4\zeta)b > 0$. Thus, on $E_1$, for any $z \in I_1 - \tau$, we have
\[
\frac{f_{Y'(M/z_0)}(z)}{f_{X_1}^{-1}(\tau + z)} = \frac{Y'(M/z_0)}{x_1^{\frac{1}{4}}} \left(\frac{\tau + z}{z}\right)^{\frac{3}{4}} e^{x_1^{\frac{1}{4}} - (Y'(M/z_0))^2} \leq \frac{3}{2} 4^{\frac{3}{4}x_1^{\frac{1}{4}}} e^{x_1^{\frac{1}{4}} - (Y'(M/z_0))^2},
\]
where we used the fact that, since $z \in I_1 - \tau$, $\tau + z \geq 2x_1^{-\frac{3}{4}}$ and so $z \geq (\tau + z) - \tau \geq (x_1^{-\frac{3}{4}})^{-1}$. This ensures the existence of a constant $K > 0$ such that, on $E_1$ and for all $z \in I_1$, $\frac{f_{Y'(M/z_0)}(z)}{f_{X_1}^{-1}(\tau + z)} \leq K$. This combined with (22) and (23) implies that
\[
E[\phi(\tau')P(E_0, W \cap E_1 | \tau')] \leq KE \left[ \mathbb{I}_{E_0 \cap E_1} \int_{E_1} \phi(y) f_{X_1}^{-1/4}(y) dz \right] \leq KE \left[ \mathbb{I}_{E_0 \cap E_1} E[\phi(\tau')] \right] = K \mathbb{P}(E_0, W \cap E_1) E[\phi(\tau')].
\]

**Lemma 12** (An estimate on the distance between the discretization of the square Bessel process $Y'$ and an affine space). Uniformly on $x > M'$,
\[
\sup_{W} \mathbb{P}(E_0, W \cap E_1) \leq C^\epsilon(x/M') \left[1 - \frac{1}{M(1-2nM)} \right]^{1/2} \left[1 - \frac{1}{M1/2} \right]^{1/2} \leq M^{1/2} \left[1 - \frac{1}{M1/2} \right]^{1/2}.
\]

**Proof of Lemma 12.** Let $W$ be an affine subset of $\mathbb{R}^M$ of dimension at most $k$. Observe that
\[
E_0 \cap E_1 \subset \left\{ (Y'(n/x_0))_{n=1,\ldots,M} \in B_{\infty} \left( x_1^{-1/4}, x_1^{-1/4} \right) \cap W_x \right\},
\]
where $B_{\infty} \left( x_1^{-1/4}, x_1^{-1/4} \right)$ is the ball (for the supremum norm) of radius $x_1^{-1/4}$ and centered on $(x_1^{-1/4}, \ldots, x_1^{-1/4})$, and where $W_x$ is the $\epsilon = 2x_1^{-1/4}$-neighbourhood of $W$ for the metric $D$. Note that
\[
\epsilon = 2(x/M')^{(1-(1+\epsilon)(1+\epsilon))} \left[ 1 + M^{1-\epsilon} - M^{1-\epsilon} \right] \geq 1,
\]
and
\[
R_x := \sqrt{M} x_1^{-1/4} = \frac{1}{2}(x/M')^{1/4} \left[ 1 + (1+\epsilon)(1+\epsilon) \right] M^{1-\epsilon} - M^{1-\epsilon} \geq 1,
\]
uniformly in $x > M'$, since $\frac{1}{4} + (1+\epsilon)(1+\epsilon) < 1$. Observe that $B_{\infty} \left( x_1^{-1/4}, x_1^{-1/4} \right) \cap W_x$ is contained in $B_2 \left( x_1^{-1/4}, \sqrt{M} x_1^{-1/4} \right) \cap W_x$ where $B_2 \left( x_1^{-1/4}, \sqrt{M} x_1^{-1/4} \right)$ is the euclidean ball centered on $(x_1^{-1/4}, \ldots, x_1^{-1/4})$ with radius $\sqrt{M} x_1^{-1/4}$. Let $z_0, z'_0 \in B_2 \left( x_1^{-1/4}, \sqrt{M} x_1^{-1/4} \right) \cap W_x$ and $z_1 \in W \cap B_2(z_0, \epsilon)$, $z'_1 \in W \cap B_2(z'_0, \epsilon)$. Then $z'_1 \in B_2 \left( z_1, 3\sqrt{M} x_1^{-1/4} \right)$. Due to [45, Theorem 3, pages 157], there exists $c > 0$ such that $W \cap B_2 \left( z_1, 3\sqrt{M} x_1^{-1/4} \right)$ is contained in the union of at most $(cR_x)^k$ euclidean balls of radius $\epsilon$ in $W$. Thus $W_x \cap B_2 \left( x_1^{-1/4}, \sqrt{M} x_1^{-1/4} \right) \cap W_x$ is contained in the union of at most $(cR_x)^k$ euclidean balls of radius $2\epsilon$. We conclude that $B_{\infty} \left( x_1^{-1/4}, x_1^{-1/4} \right) \cap W_x$ is contained in
the union of at most \((cR_x)^k\) euclidean balls of radius \(4\varepsilon\) centered at a point contained in 
\(B_\infty \left( x^{-1/4}, z^{-1/4} \frac{1}{2} \right) \cap W_x \). It follows from this combined with (24) that

\[
P(E_0W \cap E_1) \leq (cR_x)^k \sup_{z \in B_\infty} P \left( (Y'(n/x_0))_{n=1, \ldots, M} \in B_2(z, 4\varepsilon) \right). \tag{27}
\]

Note that if \(z = (z_n)_{n=1, \ldots, M} \in B_\infty \left( x^{-1/4}, z^{-1/4} \frac{1}{2} \right) \) and \((Y'(n/x_0))_{n=1, \ldots, M} \in B_2(z, 4\varepsilon)\), then \(\max_{n=0, \ldots, M-1} |z_{n+1} - x^{-\frac{1}{4}}| < x^{-\frac{1}{4}}\) and there exist at most \(\eta M\) indices \(n'\) that \(|Y'(n'/x_0) - z_{n'}| \geq 4\varepsilon / \sqrt{\eta M}\), and so at least \((1 - 2\eta)M\) indices \(n = \{0, \cdots, M - 1\}\) such that

\[
\left| Y' \left( \frac{n}{x_0} \right) - z_n \right|, \left| Y' \left( \frac{n+1}{x_0} \right) - z_{n+1} \right| < 4\varepsilon / \sqrt{\eta M},
\]

with \(z_0 = x^{-\frac{1}{4}}\). Due to [44, after Corollary 1.4, page 441], the distribution of \(Y'(n+1/x_0)\) knowing \(Y'(n/x_0) = y\) is the sum of a Dirac mass at 0 and of a measure with density

\[
z \mapsto q_{x_0}(y, z) := \frac{z_0}{2} \exp \left( -\frac{x_0 (y + z)}{2} \right) I_1 (x_0 \sqrt{y^2}),
\]

where \(I_1\) is the modified Bessel function of index 1 which satisfies \(I_1(z) = O(e^{z^2/\sqrt{z}})\), as \(z \to \infty\), (see [35, (5.10.22) or (5.11.15)].) So

\[
q_{x_0}(y, z) = O \left( \frac{x_0^\frac{1}{2} y^\frac{1}{4}}{2} \exp \left( -\frac{x_0 (\sqrt{y} - \sqrt{z})^2}{2} \right) \right) = O \left( \frac{x_0^\frac{1}{2} z^\frac{1}{4}}{2} \right)
\]

uniformly on \(y, z \in \left[ x_1^{-\frac{1}{4}}, 2x_1^{-\frac{1}{4}} \right]\). We will use the expression \(x_0, x_1\) and \(\varepsilon\) given in (15) and (25). Thus by using the Markov property (and \(\frac{M^\varepsilon}{M(1-2\eta)^2(2\eta M)} \leq M^{2\eta M}\)), we get by induction, that, when \(x > M'\),

\[
\sup_{z \in B_\infty \left( x^{-1/4}, z^{-1/4} \frac{1}{2} \right)} P \left( (Y'(n/x_0))_{n=1, \ldots, M} \in B_2(z, 4\varepsilon) \right)
\]

\[
\leq M^{2\eta M} \left( C'(x/M')^{-1} \left( \frac{1 - 2\eta \gamma + \eta \gamma}{2} \right) \frac{M^\varepsilon}{M(1-2\eta)} M^{1+\eta} M^{-1+\frac{1+\eta}{2}} \right)^{(1-2\eta)M}.
\]

Recalling that \(M = O(k)\), the previous estimate combined with (27) and (26) ensures that

\[
\sup_{W} P (E_0W \cap E_1) \leq C^\varepsilon M^\varepsilon \left( x/M' \right)^{\left[ 1 - \frac{1}{2} - (\varepsilon + \eta)^2 \right]} k^{-M(1-2\eta)} \left[ 1 - (\varepsilon + \eta)^2 \right]^{\left( 1-2\eta \right)M} M^{\left( 1+\eta \right)^2 + 2\eta M} M^{(1-\eta)M} (k(1-2\eta)M), \tag{28}
\]

which ends the proof of the lemma.

\(\square\)

**Proof of the upper bound of Theorem 2.** Formula (9) follows from (16) and Lemmas 9, 10, 11 and 12. We will use the fact that

\[
\forall Q > 1, \quad \int_{M'}^{\infty} (x/M')^{-Q} dx = O(M'). \tag{29}
\]
Thanks to this, the error terms in Lemmas 9 and 10 give directly a term in \(O(M') = O(k^d)\). Let us detail the term coming from Lemma 12. We first observe that the exponent of \((x/M')\) is strictly smaller than \(-1\) for \(k\) large enough. Indeed this exponent is

\[
[1 - \frac{b}{4} - \frac{(1 + \gamma)(1 + \epsilon_0)}{2}] k - M(1 - 2\eta) \left[1 - \frac{(1 + \gamma)(1 + \epsilon_0)}{2} - \frac{a}{2} - \frac{b}{8}\right]
\]

which is smaller than

\[
k \left[1 - \frac{b}{4} - \frac{(1 + \gamma)(1 + \epsilon_0)}{2} - \theta(1 - 2\eta) \left(1 - \frac{(1 + \gamma)(1 + \epsilon_0)}{2} - \frac{a}{2} - \frac{b}{8}\right)\right]
\]

where we used the fact that \(M = \lceil \theta k \rceil \geq \theta k\). The fact that this quantity is strictly smaller than \(-1\) for any \(k\) large enough comes from our conditions (11) and (13). It follows from this combined with (29) and Lemma 12 that

\[
\int_{M'}^\infty \sup_W \mathbb{P}(E_{0,W} \cap E_1) \, dx \leq C_k \left[ \left( \frac{1 - \epsilon_0}{4\theta} \right)^{1+\alpha_0} \left( \frac{1 + \epsilon_0}{4\theta} (1 - 2\eta) M \right)^{1+\gamma} \right] \leq C_k \left[ M^{d+\alpha_0(1+2d')} + (1+\epsilon_0 (1-\epsilon_0)) (1-2\eta) \right],
\]

where we used the fact that \(M' = M^d\) and that \(k \leq \lceil \theta k \rceil / \theta = M / \theta\). Finally, we notice that \(1 + \epsilon_0 - 2d' (1 - \epsilon_0) < 0\) (due to (10)) and that (14) ensures that

\[
(1 - \epsilon_0) (1 + 2d') + (1 + \epsilon_0 - 2d' (1 - \epsilon_0)) (1 - 2\eta) + 2\eta < 0
\]

and conclude that

\[
\int_{M'}^\infty \sup_W \mathbb{P}(E_{0,W} \cap E_1) \, dx = O(1).
\]

Hence we have proved that

\[
\sup_{V \in \mathcal{V}_k} \mathbb{E} \left[ (d(L_1, V))^{-1} \right] = O(M') \, .
\]

\section{Law of large numbers: Proof of Theorem 3}

We complete the sequence \((X_n)_{n \geq 1}\) into a bi-infinite sequence \((X_n)_{n \in \mathbb{Z}}\) of i.i.d. random variables. Theorem 3 could be proved by an adaptation of the proof of [14, Corollary 6] (combined with Theorem 1, see Appendix B). We use here another approach enabling the study of more general additive functionals. Recall that \((\xi_m + S_k)_{m \in \mathbb{Z}}\) is the scenery seen from the particle at time \(k\).

\textbf{Proposition 13} (Limit theorem for Birkhoff’s ratios). Let \(\tilde{f} : \mathbb{Z}^Z \times \mathbb{Z}^Z \times \mathbb{Z} \to \mathbb{R}\) be a measurable function such that

\[
\sum_{\ell \in \mathbb{Z}} |\mathbb{E}[\tilde{f}(X_{n+1})_{n \in \mathbb{Z}}, (\xi_n)_{n \in \mathbb{Z}}, \ell]| < \infty.
\]

Then

\[
\left( \sum_{k=0}^{n-1} \tilde{f}(X_{m+k+1})_{m \in \mathbb{Z}}, (\xi_m + S_k)_{m \in \mathbb{Z}}, Z_{k+n} \right) / \mathcal{N}_n(0)
\]

converges almost surely to \(I(\tilde{f}) := \sum_{\ell \in \mathbb{Z}} \mathbb{E}[\tilde{f}(X_{n+1})_{n \in \mathbb{Z}}, (\xi_n)_{n \in \mathbb{Z}}, \ell]\). In particular, this combined with (3) ensures that
With these notations, 
\[ Z_{λ} \]
where
\[ I = \sum_{n-1}^{n} \sum_{k=0}^{κ} \tilde{f}(X_{m+k+1})_{m∈Z}, (ξ_{m+S_{k}})_{m∈Z}, Z_{k}) \]
converges in distribution to \( I(\tilde{f})σ_{λ}^{-1}L_{1}(0) \).

Our approach to prove Proposition 13 uses an ergodic point of view. Let us consider the probability preserving dynamical system \((Ω, T, µ)\) given by
\[ Ω = Z^{Z} × Z^{Z}, \quad T((x_k)_{k∈Z}, (y_k)_{k∈Z}) = ((x_{k+1})_{k∈Z}, (y_{k+x_0})_{k∈Z}), \quad µ = P_{X_{1}}^{Z} ⊗ P_{ξ_{0}}^{Z}, \]
i.e. \( T(x, y) = (σx, σ^{x_{0}}y) \), where we write \( σ : Z^{Z} → Z^{Z} \) for the usual shift transformation given by \( σ((z_k)_{k∈Z}) = (z_{k+1})_{k∈Z} \).

This system \((Ω, T, µ)\) is known to be ergodic (see [49, 30]). We set \( Φ(x, y) := y_{0} \).

With these notations, \( Z_{k} \) corresponds to the Birkhoff sum \( ∑_{k=0}^{n-1} Φ ⊕ T^{k} \). Consider the \( Z\)-extension \((Ω, T, µ)\) over \((Ω, T, µ)\) with step function \( Φ \). This system is given by
\[ \tilde{Ω} := Ω × Z, \quad \tilde{µ} = µ ⊗ λ_{Z}, \]
where \( λ_{Z} = ∑_{Δ∈Z} δ_{Δ} \) is the counting measure on \( Z \) and with
\[ \tilde{T}(x, y, ℓ) = (T(x, y), ℓ + y_{0}). \]

In particular
\[ \tilde{T}^{k}((x_{m+1})_{m∈Z}, (y_{m})_{m∈Z}, ℓ) = ((x_{m+k+1})_{m∈Z}, (y_{m+x_{0}+⋯+x_{-1}})_{m∈Z}, ℓ + ∑_{j=0}^{k-1} y_{x_{0}+⋯+x_{j}}) \].

Observe that \( N_{1}(0) \) corresponds to the Birkhoff sum \( ∑_{k=0}^{n-1} h_{0} ⊕ \tilde{T}^{k}(x, y, 0) \) with \( h_{0}(x, y, ℓ) = 1_{0}(ℓ) \), and the sum studied in Proposition 13 corresponds to \( ∑_{k=0}^{n-1} \tilde{f} ⊕ \tilde{T}^{k}(x, y, 0) \), while \( I(\tilde{f}) = ∫_{Ω} \tilde{f} d\tilde{µ} \).

**Proposition 14.** The system \((\tilde{Ω}, \tilde{T}, \tilde{µ})\) is recurrent ergodic.

**Proof.** Since \((Ω, T, µ)\) is ergodic and since \( Φ \) is integrable and \( µ \)-centered, we know (by [46, Corollary 3.9] combined with the Birkhoff ergodic theorem) that \( P(Z_{n} = 0 \ i.o.) = 1 \), thus that \((Ω, T, µ)\) is recurrent (i.e. conservative). Now let us prove that this system is also ergodic. Let \( g : \tilde{Ω} → (0, +∞) \) be a positive \( \tilde{µ} \)-integrable function such that \( g(x, y, ℓ) = g_{0}(ℓ) \) does not depend on \((x, y) ∈ Ω \) and with unit integral \( (g \) is a probability density function with respect to \( \tilde{µ}) \). By recurrence of \((Ω, T, µ)\), we know that
\[ \sum_{k≥1} g ⊕ \tilde{T}^{k} = ∞ \quad (30) \]
\( \tilde{µ} \)-almost everywhere. Let \( K ∈ N \). Consider \( f : \tilde{Ω} → R \) a \( \tilde{µ} \)-integrable function constant on the \( K \)-cylinders of the first coordinate, i.e. such that \( f(x, y, ℓ) = f_{0}(x_{m})_{|m|≤K}, y, ℓ \) does not depend on \((x_{k})_{|k|>K} \).

Since \((Ω, T, µ)\) is recurrent, the Hopf-Hurewicz’s theorem (see e.g. [1, p. 56]) ensures that
\[ \lim_{|n|→+∞} \sum_{k=1}^{n} f ⊕ \tilde{T}^{k} = H(f, g) := E_{g} f_{0} \left[ \frac{f}{g} \right]_{\tilde{µ}}. \quad (31) \]
\(\tilde{\mu}\)-almost everywhere, where \(\tilde{T}\) is the \(\sigma\)-algebra of \(\tilde{T}\)-invariant events. Thus, by \(L^1(\tilde{\mu})\)-density, the ergodicity of \((\tilde{\Omega}, \tilde{T}, \tilde{\mu})\) will follow from the fact that \(H_{(f, g)}\) is \(\tilde{\mu}\)-almost everywhere constant for every \(f\) as above \((g\) can be fixed). Observe that, for \(k > K\),

\[
f \circ \tilde{T}^k(x, y, \ell) = f \left( \sigma^k x, \sigma^{x_0 + \cdots + x_{k-1}} y, \ell + \sum_{m=0}^{k-1} y_{x_0 + \cdots + x_m} \right)
\]

does not depend on \((x_k)_{k \leq -1}\). Analogously, for \(k > K\),

\[
f \circ \tilde{T}^{-k}(x, y, \ell) = f \left( \sigma^{-k} x, \sigma^{-x_1 - \cdots - x_{k-1}} y, \ell - \sum_{m=1}^{k} y_{-x_1 - \cdots - x_m} \right)
\]

does not depend on \((x_k)_{k \geq 0}\). Of course \(g \circ \tilde{T}^k\) satisfies the same property. Thus, due to (30) and (31), it follows that \(H_{(f, g)}(x, y, \ell)\) does not depend on \(x\). Thus, \(H_{(f, g)}(x, y, \ell) = H^{(0)}_{(f, g)}(y, \ell)\) for \(\tilde{\mu}\)-almost every \((x, y, \ell) \in \tilde{\Omega}\).

By \(\tilde{T}\)-invariance of \(H_{(f, g)}\), given two distinct points \(x_0, x'_0 \in Z\) such that \(\mathbb{P}(X_1 = x_0) \mathbb{P}(X_1 = x'_0) > 0\), the following equality holds true almost everywhere

\[
H^{(0)}_{(f, g)}(y, \ell) = H^{(0)}_{(f, g)}(\sigma x_0 y, \ell + y_0) = H^{(0)}_{(f, g)}(\sigma x'_0 y, \ell + y_0),
\]

where we write \(\sigma\) for the usual shift on \(\mathbb{Z}^2\) given by \(\sigma((y_k)_{k \in \mathbb{Z}}) = (y_{k+1})_{k \in \mathbb{Z}}\). It follows that, for every \(\ell \in Z\), \(H^{(0)}_{(f, g)}(\cdot, \ell)\) is \(\sigma_{x_0 - x'_0}\)-invariant almost everywhere. By ergodicity of \(\sigma_{x_0 - x'_0}\), we conclude that \(H_{(f, g)}(x, y, \ell) = H^{(1)}_{(f, g)}(\ell)\) depends only on \(\ell\) almost everywhere.

Since it is \(\tilde{T}\)-invariant, for every \(y_0 \in Z\) such that \(\mathbb{P}(\xi_0 = y_0) > 0\), \(H^{(1)}_{(f, g)}(\ell) = H^{(1)}_{(f, g)}(\ell + y_0)\). Since the support of \(y_0\) generates the group \(Z\), we conclude that \(H_{(f, g)}\) is \(\tilde{\mu}\)-almost everywhere equal to a constant.

Note that the system in infinite measure \((\tilde{\Omega}, \tilde{T}, \tilde{\mu})\) describes the evolution in time \(m\) of \(((X_{m+k+1})_{k \in \mathbb{Z}}, \xi_{S_{m+k+1}})\). In comparison, the system corresponding to \(((X_{m+k+1})_{k \in \mathbb{Z}}, S_m)\) is also recurrent ergodic, but the analogous system corresponding to \(((X_{m+k+1})_{k \in \mathbb{Z}}, \xi_{S_{m+k}})\) is recurrent (since \(\mathbb{P}(S_n = 0) = 1\) not ergodic (since the sets of the form \(\{(x, y, \ell) : (y_{n-\ell})_n \in A_0\}\) are invariant).

**Proof of Proposition 13.** Since \((\tilde{\Omega}, \tilde{T}, \tilde{\mu})\) is recurrent ergodic, the Hopf ergodic theorem ensures that, for any \(\tilde{f} \in L^1(\tilde{\mu})\), the sequence \(\left(\sum_{k=0}^{n-1} f \circ \tilde{T}^k \mathbb{E}[T^n h_{\tilde{\mu}}] \right)_{n \geq 0}\) converges \(\tilde{\mu}\)-almost everywhere to \(\int_{\tilde{\Omega}} \tilde{f} d\tilde{\mu} = I(\tilde{f})\). Thus

\[
\left(\sum_{k=0}^{n-1} \tilde{f}((X_{m+k+1})_{m \in \mathbb{Z}}, \xi_{m+S_{m+k+1}})_{m \in \mathbb{Z}}, Z_{k+m}) \right)_{n \geq 0} \mathbb{N}_n(0) \left(\sum_{k=0}^{n-1} \tilde{f} \circ \tilde{T}^k \mathbb{E}[T^n h_{\tilde{\mu}} \cdot g \tilde{T}^m ((X_m)_{m \in \mathbb{Z}}, (\xi_m)_{m \in \mathbb{Z}}, 0)] \right)_{n \geq 0}
\]

converges almost surely to \(I(\tilde{f})\), and we have proved the first part of the proposition. The second part comes from the first part combined with (3) and the Slutsky theorem.  

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We will then decompose in blocks the product \( Z \) with the indexes are chosen so the \( m \) These quantities will be studied in Proposition 16 below. It will be proved therein that \( \mu \) is the distribution, with respect to \( \mu \otimes \delta_0 \ll \tilde{\mu} \), to \( \int_{\Omega} \tilde{f} \, d\tilde{\mu} \). Thus, Theorem 4 follows from Proposition 13 combined with [51, Theorem 1].

We end this section with an interpretation of \( \sigma_f^2 \) in terms of the famous Green-Kubo formula.

**Remark 15.** Assume the assumptions of Theorem 5. Consider the function \( \tilde{f} : \Omega \rightarrow \mathbb{Z} \) given by \( f(x, y, \ell) := f(\ell) \). Then \( \sigma_f^2 \) can be rewritten

\[
\sigma_f^2 = \sum_{k \in \mathbb{Z}} \int_{\Omega} \tilde{f} \circ \tilde{T}^{|k|} \, d\tilde{\mu}.
\]

## 4 Proof of the central limit theorem: proof of Theorem 5

We start by presenting the strategy of the proof of Theorem 5. We will write the moment of order \( M \) of \( Z_n = \sum_{k=1}^n f(Z_k) \) as follows

\[
E \left[ \left( \sum_{k=1}^n f(Z_k) \right)^M \right] = \sum_{1 \leq m_1 \leq \cdots \leq m_M \leq n} c_m E \left[ \prod_{j=1}^M f(Z_{m_j}) \right],
\]

where, for \( m = (m_1, \cdots, m_M) \), \( c_m \) is the number of \( (r_1, \cdots, r_M) \in \{1, \cdots, n\}^M \) such that \( r_1, \cdots, r_M \) and \( m_1, \cdots, m_M \) contain the same values with same multiplicities.

We will then decompose in blocks the product \( \prod_{j=1}^M f(Z_{m_j}) \) appearing in the right hand side of (32) by gathering the \( Z_{m_j} \)'s that are close one to the others. We will then write

\[
E \left[ \prod_{j=1}^M f(Z_{m_j}) \right] = E \left[ \prod_{j=1}^{m} \left( f(Z_{k_j}) \prod_{s=1}^{s_j} f(Z_{k_j+\ell_j,s}) \right) \right]
\]

with the indexes are chosen so the \( Z_{k_j} \) are far away one from the others and such that the \( Z_{k_j+\ell_j,s} \) are close to \( Z_{k_j} \) (i.e. the \( \ell_j,s \) are small). Recalling that \( \sum_{a \in \mathbb{Z}} f(a) = 0 \), a more convenient form for this expression is the following one:

\[
E \left[ \prod_{j=1}^M f(Z_{m_j}) \right] = \sum_{a_j, b_j \in \mathbb{Z}} \left( \prod_{j=1}^m \left( f(a_j) \prod_{s=1}^{s_j} f(b_j) \right) \right) \mathbb{P} \left( \forall j, s, Z_{k_j} = a_j, Z_{k_j+\ell_j,s} = b_j,s \right).
\]

These quantities will be studied in Proposition 16 below. It will be proved therein that the dominating terms (33) of (32) are the terms made of pairs, that is corresponding to the case \( m = M/2 \) and \( s_1 = \cdots = s_m = 1 \) and that these terms behave as follows

\[
E \left[ \prod_{j=1}^m \left( f(Z_{k_j}) f(Z_{k_j+\ell_j}) \right) \right] = \sum_{a_1, \cdots, a_m, b_1, \cdots, b_m} \left( \prod_{j=1}^m \left( f(a_j) f(b_j) \right) \right) \mathbb{P} \left( \forall j, Z_{k_j} = a_j, Z_{k_j+\ell_j} = b_j - a_j \right)
\]

\[
\sim \mathbb{P} \left( Z_{k_1} = \cdots = Z_{k_m} = 0 \right) \sum_{a_j, b_j \in \mathbb{Z}} f(a_j) f(b_j) \mathbb{P}(Z_{\ell_j} = b_j - a_j).
\]

In this formula, two different behaviours occur depending on the scale: at large scale there is a strong dependence between the \( Z_{k_j} \), but, at small scale, the random variable
Z_{k_j + \ell} \) depends strongly on the closest \( Z_{k_j} \), but (asymptotically) not on the other \( Z_{k_i} \)’s (that are far away). In other words, asymptotically, the long-time dependence is fully supported by the \( Z_{k_i} \). Let us now state the key intermediate results. We recall that \( d \) and \( \alpha \) have been introduced in the beginning of Section 1.2.

**Proposition 16** (Asymptotic behaviour of expectations appearing in the computation of the moments of additive functional of RWRS). Assume the assumptions of Theorem 5. Let \( M, m \in \mathbb{N}^* \) and \( m \) non negative integers \( s_1, \ldots, s_m \geq 0 \) be such that \( M = \sum_{j=1}^{m} (s_j + 1) \).

We set \( \mathcal{J} := \{ j = 1, \ldots, m : s_j = 0 \} \) and \( k' = 0 \) if \( j \notin \mathcal{J} \). Let \( \eta > 0 \). There exists \( L \in (0, 1) \) such that for every \( \theta \in (0, 1) \) the following holds true, as \( n \) varies, with the notations \( n_j := k_j - k_{j-1} \), with the convention \( k_0 = 0 \).

First,

\[
\sum_{k' = 0, \ldots, d-1, \forall j \in \mathcal{J}} \mathbb{E} \left[ \prod_{j=1}^{m} \left( f(Z_{k_j + k'}) \prod_{s=1}^{s_j} f(Z_{k_j + \ell_s}) \right) \right] = \mathcal{O} \left( \prod_{i=1}^{m} n_i^{-\frac{1}{2}} \mathcal{E}_k \right),
\]

uniformly over the \( k = (k_1, \ldots, k_m) \) and \( \ell = (\ell_{j,s})_{j=1,\ldots,m,s=1,\ldots,s_j} \) such that \( n > k_j > k_{j-1} + n^\theta \) (with convention \( k_0 = 0 \)) and \( \ell_{j,s} \in \{0, \ldots, [n^\theta] \} \) with

\[
\mathcal{E}_k = \mathcal{O} \left( \sum_{J' \subset \{1, \ldots, m\} : \# J' \geq \# J / 2} \left( \prod_{j \in J'} n_j^{-\frac{1}{2} + \eta} \right) \right)
\]

Second, if \( s_j = 1 \) for all \( j \), then

\[
\mathbb{E} \left[ \prod_{j=1}^{m} (f(Z_{k_j}) f(Z_{k_j + \ell})) \right] = \frac{d^m E_k}{(2 \pi \sigma_{j/\theta})^2} \prod_{j=1}^{m} A_{k_j, \ell_j} + \mathcal{O} \left( n^{-L(M+1)\theta} \right),
\]

uniformly on \( k, \ell \) as above, with \( E_k \) depending on \( k \) but not on \( \ell \) and such that \( E_k = \mathcal{O} \left( \prod_{j=1}^{m} n_j^{-\frac{1}{2}} \right) \) uniformly on \( k \) as above, and \( E_k \sim n^{-\frac{3m}{4}} \mathbb{E} \left[ \det \mathcal{D}_{t_1, \ldots, t_m} \right] \) (with \( t_1 < \cdots < t_m \)) as \( k_j / n \to t_j \) and \( n \to +\infty \), with \( \mathcal{D}_{t_1, \ldots, t_m} = \left( \int_{\mathbb{R}} L_{t_s}(x) L_{t_s}(x) \, dx \right)_{i,j=1,\ldots,m} \) where \( L \) is the local time of the brownian motion \( B \), limit of \((S_{nt}/\sqrt{n})\), as \( n \) goes to infinity, and where

\[
A_{k, \ell} := \sum_{a \in \mathcal{K} \cap \mathbb{Z}} \left( f(a) \prod_{s=1}^{m} f(b) \right) \mathbb{P}(Z_{k_\ell} = b - a)
\]

Third, also with \( s_j = 1 \) for all \( j \),

\[
\sum_{k'_1, \ldots, k'_m = 0}^{d-1, \eta} \sum_{\ell_1, \ldots, \ell_m = 0}^{n^\theta m \choose (10M)} \mathcal{O} \left( \prod_{j=1}^{m} A_{k'_j, \ell_j} \right) = \sigma_{2m}^2 + o(1),
\]

as \((k_1 / n, \ldots, k_m / n) \to (t_1, \ldots, t_m) \) and \( n \to +\infty \).

**Proof.** The proof of Proposition 16 is based on several technical lemmas. For reader’s convenience, the most technical points are proved in Appendix A. Let \( M \geq 1, \theta \in (0, 1) \) and \( \eta \in (0, \frac{1}{100}) \). Choose \( \mathcal{L} = \frac{\mathcal{L}}{100M} \). Assume \( n^\theta < n_j < n \) and let \( \ell_{j,1}, \ldots, \ell_{j,s_j} = 0, \ldots, [n^\theta] \) with \( \sum_{j=1}^{m} (1 + s_j) = M \). We set \( N_j(y) := \# \{ s = 0, \ldots, n_j - 1 : S_{k_j+s} = y \} \), \( N_j := \sup_y N_j \) and \( R_j := \# \{ y \in \mathbb{Z} : N_j(y) > 0 \} \). Analogously, we set \( N_j(y) := \# \{ m = 0, \ldots, \ell_{j,s} - 1 : S_{k_j+m} = y \} \). The terms appearing in left hand side of (34) can be expressed thanks to the following quantity

\[
B_{k, \ell} := \mathbb{E} \left[ \prod_{j=1}^{m} \left( f(Z_{k_j}) \prod_{s=1}^{s_j} f(Z_{k_j + \ell_s}) \right) \right] = \sum_{a,b} \left( \prod_{j=1}^{m} \left( f(a_j) \prod_{s=1}^{s_j} f(b_j) \right) \right) p_{k, \ell}(a, b),
\]

(35)
where $\sum_{a,b}$ means the sum over $(a,b) \in \mathbb{Z}^M$ with $a = (a_1, \cdots, a_m)$ and $b = (b_j,s)_{j=1,\ldots,m}$, with the convention $a_0 = 0$ and

$$p_{k,l}(a,b) = P(\forall j = 1, \cdots, m, Z_{k_j} = a_j, \forall s = 1, \cdots, s_j, Z_{k_j+1} = b_j,s).$$

Recall that $Z_n = \sum_{y \in \mathbb{Z}} \xi_y N_n(y)$, with $(\xi_y)_{y \in \mathbb{Z}}$ a sequence of independent identically distributed random variables, with common characteristic function $\varphi_\xi$, and that the sequence $(\xi_y)_{y \in \mathbb{Z}}$ is independent of the random walk $(S_n)_{n \geq 0}$ and thus of $(N'_j(y), N'_{j,s}(y))_{j,s}$. A classical computation (detailed in Appendix A) ensures the following.

**Lemma 17** (Finite dimensional distributions of the RWRS $Z$ expressed in terms of integrals of characteristic functions).

$$p_{k,l}(a,b) = \mathbb{I}_{\{y_1, a_i \in k, \alpha + dZ\}} \frac{d^n}{(2\pi)^M} \times \int_{[-\pi, \pi]^M} e^{-i \sum_{j=1}^m [(a_j - a_{j-1})\theta_j + \sum_{s=1}^s (b_{j,s} - a_j)\theta_j', s]} \varphi_{k,l}(\theta, \theta') \, d\theta \, d\theta',$$

with $\theta = (\theta_j)_{j=1,\cdots,m}$ and $\theta' = (\theta'_j,s)_{j=1,\cdots,m; s=1,\cdots,s_j}$ and

$$\varphi_{k,l}(\theta, \theta') = E \left[ \prod_{y \in \mathbb{Z}} \varphi_\xi \left( \sum_{j=1}^m \left( \theta_j N'_j(y) + \sum_{s=1}^s \theta'_j,s N'_{j,s}(y) \right) \right) \right].$$

For any event $E$ and any $I \subset [-\pi, \pi]^m \times [-\pi, \pi]^M$ we also set

$$\varphi_{k,l}(\theta, \theta', E) = \mathbb{I}_{\{y \in \mathbb{Z}\}} \prod_{I \in E} \varphi_\xi \left( \sum_{j=1}^m \left( \theta_j N'_j(y) + \sum_{s=1}^s \theta'_j,s N'_{j,s}(y) \right) \right),$$

$$p_{k,l}(a,b, I, E) = \mathbb{I}_{\{y_1, a_i \in k, \alpha + dZ\}} \frac{d^n}{(2\pi)^M} \times \int_I e^{-i \sum_{j=1}^m [(a_j - a_{j-1})\theta_j + \sum_{s=1}^s (b_{j,s} - a_j)\theta_j', s]} \varphi_{k,l}(\theta, \theta', E) \, d\theta \, d\theta',$$

and

$$B_{k,l,I,E} = \sum_{a,b} \prod_{j=1}^m \left( f(a_j) \prod_{s=1}^s f(b_{j,s}) \right) p_{k,l}(a,b, I, E).$$

Let $\gamma < \min(L\theta, \frac{M}{2\pi})$. Let $\theta' \in (0, \frac{2\pi}{M})$ such that $\theta' \leq \frac{\theta}{2} - 2ML\theta$. We consider the set

$$\Omega_k := \left\{ \det D_k \geq n^{-\gamma} \prod_{i=1}^m n_i^{\gamma} \right\} \cap \bigcap_{j=1}^m \Omega_{k,j}^{(j)},$$

where

$$\Omega_{k}^{(j)} := \left\{ \sup_{r=0, \cdots, n_j} |S_{r+j-1} - S_{j-1}| \leq \frac{n_j^{\frac{1}{2} + \gamma}}{3}, \sup_{y \neq z} \left| \frac{N'_j(y) - N'_j(z)}{|y - z|^{\frac{3}{2}}} \right| \leq n_j^{\frac{1}{2} + \gamma} \right\},$$

and with $D_k = \left( \sum_{y \in \mathbb{Z}} N'_j(y) N'_{j,s}(y) \right)_{i,j}$. The following lemma follows from [14] (see appendix A for details).\(^2\)

\(^2\)The set $\Omega_{k}^{(j)}$ in [14, Lemma 16] coming from [13, Lemma 6] is expressed in terms of the range but is controlled with the infinite norm since the $X'_{j}$’s admit moment of any order.
Lemma 18 (Reduction to a nice event). For any \( p > 1 \), \( P(\Omega_k) = 1 - o(n^{-p}) \), and so \( B_{k,\ell}[-\pi,\pi]^M,\Omega_k = o(n^{-p}) \).

Note that, on \( \Omega_k \),
\[
R_j' \leq n_j^{\frac{1}{2} + \gamma},
\]
(36)
\[
N_j' := \sup_{y \in \mathbb{Z}} N_j'(y) \leq n_j^{\frac{1}{2} + \frac{2}{3}} ((n_j)^{\frac{1}{2} + \gamma})^{\frac{1}{2}} \ll n_j^{\frac{1}{2} + \frac{2}{3}},
\]
(37)
\[
V_j := \sum_{z \in \mathbb{Z}} (N_j'(z))^2 \geq \frac{\left( \sum_{z \in \mathbb{Z}} N_j'(z) \right)^2}{R_j'} \geq \frac{n_j^2}{R_j'} \geq n_j^{\frac{3}{2} - \frac{2}{3}},
\]
(38)
\[
V_j \leq R_j'(N_j')^2 \leq n_j^{\frac{3}{2} + \frac{1}{3} + \frac{1}{4}}.
\]
(39)

It will be useful to notice that
\[
|\phi_{k,\ell}(\theta, \theta', E)| \leq E \left[ I_E \prod_{y \in F} \phi_{\xi} \left( \sum_{j=1}^m \theta_j N_j'(y) \right) \right]
\]
(40)
with
\[
F := \{ y \in \mathbb{Z} : \forall (j,s), \ N_j'(y) = 0 \},
\]
and that
\[
\#(\mathbb{Z} \setminus F) \leq \sum_{j=1}^m \sum_{s=1}^{s_j} \ell_{j,s} \leq Mn^{L_\theta} = o(n^{\frac{3}{2}}).
\]
(41)

Using a straightforward adaptation of the proof of [13, Proposition 10], we prove (see Appendix A) that

Lemma 19 (Reduction to a smaller domain of integration).
\[
B_{k,\ell,1_k^{(1)},\Omega_k} = o \left( e^{-n^\nu} \right),
\]
uniformly on \( k, \ell \) as in Proposition 16, where \( 1_k^{(1)} \) is the set of \( (\theta, \theta') \in [-\pi, \pi]^m \times [-\pi, \pi]^{M-m} \) such that there exists \( j = 1, \ldots, m \) so that \( n_j^{-\frac{1}{2}+\gamma} < |\theta_j| \).

Lemma 20 (Reduction to an even smaller domain of integration).
\[
B_{k,\ell,1_k^{(2)},\Omega_k} = O \left( \prod_{j=1}^m n_j^{-\frac{1}{2}+\gamma} \right),
\]
uniformly on \( k, \ell \) as in Proposition 16, where \( 1_k^{(2)} \) is the set of \( (\theta, \theta') \in [-\pi, \pi]^m \times [-\pi, \pi]^{M-m} \) such that for all \( j = 1, \ldots, m, |\theta_j| < n_j^{-\frac{1}{2}+\gamma} \) and there exists \( j' = 1, \ldots, M \) such that \( n_j^{-\frac{1}{2}+\gamma} < |\theta_j'| \).

It remains to estimate the integral over \( 1_k^{(3)} \), the set of \( (\theta, \theta') \in [-\pi, \pi]^m \times [-\pi, \pi]^{M-m} \) such that for all \( j = 1, \ldots, m, |\theta_j| < n_j^{-\frac{1}{2}-\eta} \).

We set \( J := \{ j = 1, \ldots, m : s_j = 0 \} = \{ j(1), \ldots, j(J) \} \).

Lemma 21 (Study of the integral with the above restrictions). Assume the assumptions of Theorem 5. Let \( J' \subset J \), then
\[
\sum_{k'_{j'}=0, \ldots, d-1, \forall j' \in J'} B_{k+k',\ell,1_k^{(3)},\Omega_k}
\]
\[
= O \left( \prod_{j=1}^m n_j^{-\frac{1}{2}} \right) \sum_{J'' \subset J' \cup (J'+1)} \sum_{J'' \geq J/J/2} \left( \prod_{j \in J''} n_j^{-\frac{1}{2}+\eta} \right),
\]
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uniformly on \( k, \ell \) as in Proposition 16, and where we set \( k' = (k'_1, \ldots, k'_m) \) with \( k'_j = 0 \) if \( j \not\in J' \).

Moreover, if \( s_j = 1 \) for all \( j \) (and \( J' = \emptyset \)), then,

\[
B_{k, \ell} f^{(\alpha)}_n = \left( \frac{d}{2\pi \sigma} \right)^m \sum_{a_1, \ldots, a_m \in \mathbb{Z}} \mathbb{1}_{\{a_i, a_i + a + dz \} \in \mathcal{I}} \mathbb{E} \left[ (\det D_k)^{\alpha} \right] f(a_j) \mathbb{E} \left[ f(a_j + Z_{\ell_j}) \right] = \mathcal{O} \left( n^{-(M+1) \theta} \prod_{j=1}^m n^{-\frac{j}{2}} \right),
\]

uniformly on \( k, \ell \) as above, with

\[
\mathbb{E} \left[ (\det D_k)^{\alpha} \right] = \mathcal{O} \left( \prod_{j=1}^m n^{-\frac{j}{2}} \right),
\]

uniformly on \( k \) as above, and

\[
\mathbb{E} \left[ (\det D_k)^{-\frac{1}{3}} \right] \sim n^{-2/3} \mathbb{E} \left[ (\det D_{\ell_1, \ldots, \ell_m})^{-\frac{1}{3}} \right].
\]

as \( k_j/n \to t_j \) and \( n \to +\infty \).

We can now complete the proof of Proposition 16. The two first points of Proposition 16 come from the upper bounds provided by Lemmas 17, 18, 19, 20 and 21, with \( E_k := \mathbb{E} \left[ (\det D_k)^{-\frac{1}{3}} \right] \). It remains to prove the last point of Proposition 16. We assume that \( s_j = 1 \) for all \( j \) and that \( k_j/n \to t_j \) and \( n \to +\infty \). Observe that, since \( d \) and \( \alpha \) are coprime, for every \( a_j \in \mathbb{Z} \) there is a unique \( k'_j \in \{0, \ldots, d - 1\} \) such that \( a_j \in (k_j + k'_j) \alpha + dz \). Thus

\[
\sum_{k'_1, \ldots, k'_m = 0}^{d} \sum_{\ell_1, \ldots, \ell_m = 0}^{d} 2^{#(j: \ell_j > 0)} \prod_{j=1}^m A_{k_j + k'_j, \ell_j} = \sum_{\ell_1, \ldots, \ell_m = 0}^{d} 2^{#(j: \ell_j > 0)} \prod_{a_j, b_j \in \mathbb{Z}} f(a_j) f(b_j) P(Z_{\ell_j} = b_j - a_j).
\]

Finally, due to the last point of Lemma 21 and to the next lemma, this quantity converges to

\[
\sum_{\ell_1, \ldots, \ell_m \geq 0} 2^{#(j: \ell_j > 0)} \prod_{a_j, b_j \in \mathbb{Z}} f(a_j) f(b_j) P(Z_{\ell_j} = b_j - a_j),
\]

as \( k_j/n \to t_j \) and \( n \to +\infty \) and the last point of Lemma 21 ensures that \( E_k \sim n^{-2/3} \mathbb{E} \left[ (\det D_{\ell_1, \ldots, \ell_m})^{-\frac{1}{3}} \right] \) (with \( t_1 < \cdots < t_m \)) as \( k_j/n \to t_j \) and \( n \to +\infty \). \( \square \)

**Lemma 22** (Summability). Under the assumptions\(^3\) of Theorem 5,

\[
\sum_{\ell \geq 1} \sum_{a, b \in \mathbb{Z}} \left| \sum_{\ell' = 0}^{\ell - 1} f(a) f(b) P(Z_{\ell'} + \ell d = b - a) \right| < \infty.
\]

\(^3\)Our proof is valid in a more general context. The assumptions on \( f \) and \( S \) can be relaxed in \( \sum_{a \in \mathbb{Z}} |a f(a)| < \infty \), \( \sum_{a \in \mathbb{Z}} f(a) = 0 \), and \( \|S_n\|_{L^2} = O(\sqrt{n}) \).
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Proof. The proof of this lemma only uses estimates established in [13]. Since $\sum_{a,b} |f(a)f(b)| < \infty$ and using Lemma 17, we observe that

$$\left| \sum_{\ell' = 0}^{d-1} \sum_{a,b \in \mathbb{Z}} f(a)f(b) P(Z_{\ell' + \ell d} = b - a) \right|$$

$$= \left| \sum_{\ell' = 0}^{d-1} \sum_{a \in \mathbb{Z}, b \in a + (\ell d + \ell') \alpha + \mathbb{Z}} f(a)f(b) \frac{d}{2\pi} \int_{[-\frac{\pi}{2}, \frac{\pi}{2}]} e^{-i(t(b-a))} E \left[ \prod_{y \in \mathbb{Z}} \varphi_\xi(tN_{\ell'd + \ell'}(y)) \right] dt \right|, \quad (42)$$

with $(N_n(y))_{n,y}$ the local time of $(S_n)$ and using the fact that the random variables $\xi_k$'s take their values in $a + d\mathbb{Z}$. Moreover, due to [13, Propositions 8, 9, 10], with the notations therein, there exists an event $\Omega_k$ (depending on $k$) such that $P(\Omega_k) = 1 - o(k^{-1-\eta_0})$ ([14, Lemma 16]) and such that $|\varphi_\xi(tN_k(y))| \leq e^{-\frac{a^2(tN_k(y))^2}{4}}$ on $\Omega_k$ when $|t| \leq k^{-\frac{1}{2}+\eta}$. It comes that

$$\int_{[-\frac{\pi}{2}, \frac{\pi}{2}]} e^{-i(t(b-a))} E \left[ \prod_{y \in \mathbb{Z}} \varphi_\xi(tN_{\ell'd + \ell'}(y)) \right]$$

$$= \int_{|t| \leq k^{-\frac{1}{2}+\eta}} e^{-i(t(b-a))} E \left[ \prod_{y \in \mathbb{Z}} \varphi_\xi(tN_{\ell'd + \ell'}(y)) I_{\Omega_{\ell'd + \ell'}} \right] dt + O(\ell^{1-\eta_0})$$

$$= \int_{|t| \leq k^{-\frac{1}{2}+\eta}} e^{-i(t(b-a))} E \left[ \prod_{y \in \mathbb{Z}} \varphi_\xi(tN_{\ell'd}(y)) I_{\Omega_{\ell'd}} \right] dt + O(\ell^{1-\eta_0}), \quad (43)$$

using also the fact that $\# \{ y \in \mathbb{Z} : N_{\ell d}(y) \neq N_{\ell d + \ell'}(y) \} \leq d$. Since $\alpha$ and $d$ are coprime, $\ell' + d\mathbb{Z} \mapsto \ell' \alpha + d\mathbb{Z}$ defines a bijection on $\mathbb{Z}/d\mathbb{Z}$. Thus, it follows from (42) and from (43) that

$$\left| \sum_{\ell' = 0}^{d-1} \sum_{a,b \in \mathbb{Z}} f(a)f(b) P(Z_{\ell' + \ell d} = b - a) \right|$$

$$= \left| \frac{d}{2\pi} \int_{|t| \leq k^{-\frac{1}{2}+\eta}} \sum_{a,b \in \mathbb{Z}} f(a)f(b) \left( e^{-i(t(b-a))} - 1 \right) E \left[ \prod_{y \in \mathbb{Z}} \varphi_\xi(tN_{\ell'd}(y)) I_{\Omega_{\ell'd}} \right] dt \right| + O(\ell^{1-\eta_0})$$

$$\leq \frac{d}{2\pi} \int_{|t| \leq k^{-\frac{1}{2}+\eta}} \sum_{a,b} |f(a)f(b)(b-a)| E \left[ e^{-\frac{a^2(tN_{\ell'd}(y))^2}{4d}} I_{\Omega_{\ell'd}} \right] dt + O(\ell^{1-\eta_0})$$

$$\leq C E \left[ V_{\ell'd}^{-1} I_{\Omega_{\ell'd}} \right] + o(\ell^{1-\eta_0}),$$

with $V_{\ell'd} := \sum_{y \in \mathbb{Z}} (N_{\ell'd}(y))^2$. Since $\sum_{a,b \in \mathbb{Z}} f(a)f(b) = 0$, $\sum_{a \in \mathbb{Z}} |af(a)| < \infty$ and using the change of variable $v = tV_{\ell'd}^{-1}$. Now, due to (38), $V_{\ell'd}^{-1} I_{\Omega_{\ell'd}} \leq \ell^{-\frac{1}{2}+\gamma_0} = O(\ell^{1-\eta_0})$ up to take $\eta_0$ small enough, which ends the proof of the lemma. \qed

Theorem 5 follows directly from the following corollary of Proposition 16 and Lemma 22, since $E[\Lambda^{2N}] = \frac{(2N)!}{N!2^{N}}$ and $E[\ell_i(0)^N] = \int_{[0,1]^N} \frac{\det D_{i_1 \cdots i_N}^{1 \cdots N}}{(2\pi)^{\frac{N}{2}}} \, dt_1 \cdots dt_N$ (due to [14]).

---

4One may observe that $V_{\ell'd}$ corresponds to the unique entry of the matrix $D_{\ell(d)}$ with the notation $D_k$ introduced before Lemma 18.
Corollary 23 (A rewriting of Theorem 5). Under the assumptions of Theorem 5,

\[ \mathbb{E} \left[ \left( \sum_{k=1}^{n} f(Z_k) \right)^{2N+1} \right] = o \left( n^{2N+1} \right), \]

and

\[ \mathbb{E} \left[ \left( \sum_{k=1}^{n} f(Z_k) \right)^{2N} \right] = \frac{(2N)!}{N!2^N} \frac{\sigma_2^2}{(2\pi\sigma_2^2)^\frac{N}{2}} \int_{[0,1]^N} \mathbb{E} \left[ \det \mathcal{D}_{t_1,\ldots,t_N} \right] dt_1 \cdots dt_N + o(\frac{2^N}{n}). \]

Proof. Since \( f \) is bounded, it is enough to prove the result for \( n = n'd \). We start by writing

\[ \mathbb{E} \left[ \left( \sum_{k=1}^{n} f(Z_k) \right)^{M} \right] = \sum_{1 \leq m_1 \leq \cdots \leq m_M \leq n} c_m \mathbb{E} \left[ \prod_{j=1}^{M} f(Z_{m_j}) \right], \]

where \( c_m \) is the number of \((r_1, \ldots, r_M) \in \{1, \ldots, n\}^M\) such that \( r_1, \ldots, r_M \) and \( m_1, \ldots, m_M \) contain the same values with the same multiplicities.

Let \( \theta_0 \in \left( 0, \frac{1}{M+1} \right) \). Given a sequence \( 1 \leq m_1 \leq \cdots \leq m_M \leq n \) with convention \( m_0 = 0 \), we consider \( p \in \{0, \ldots, M\} \) such that no \( m_j - m_{j-1} \) (for \( j = 1, \ldots, M \)) is in \((n^{Lp+\theta_0}, n^{L\theta_0}]\).

Set \( \theta = L^p\theta_0 \). We write \( k_1 = m_1 \) and, inductively, if \( k_j = m_{u(j)} \), we set \( k_{j+1} = m_{u(j+1)} \) for the smallest integer \( m_r \) such that \( m_r > k_j + n^\theta \), \( s_j = u(j+1) - u(j) - 1 \) and then \( \ell_j = s_{u(j)} \).

Thus each \( m = (m_1, \ldots, m_M) \) with \( 1 \leq m_1 \leq \cdots \leq m_M \leq n \) can be represented by at least one

\[ (k, \ell) \in \bigcup_{p=0}^{M} \bigcup_{m=1}^{M} \bigcup_{s_j=0}^{M} F_{n,L^p\theta_0,m_1,\ldots,m_M}, \]

with \( F_{n,\theta,m_1,\ldots,m_m} \) the set of \( M \)-uple \((k, \ell)\) of nonnegative integers with \( k = (k_j)_{j=1,\ldots,m} \), \( \ell = (\ell_{j,s})_{j=1,\ldots,m; s_j=1,\ldots,s_j} \) such that, for all \( j = 1, \ldots, m \), \( k_j \geq k_{j-1} + n^\theta \) (with convention \( k_0 = 0 \)) and, for all \( j = 1, \ldots, m \) and all \( s = 1, \ldots, s_j \), \( 0 \leq s_j \leq n^L \theta \) and, with this representation,

\[ \mathbb{E} \left[ \prod_{j=1}^{M} f(Z_{m_j}) \right] = \mathbb{E} \left[ \prod_{j=1}^{m} f(Z_{k_j}) \prod_{s=1}^{s_j} f(Z_{k_j+\ell_{j,s}}) \right]. \]

We first study separately the following sums

\[ \sum_{(m,s) \in G_M} c_{(k,\ell)} \mathbb{E} \left[ \prod_{j=1}^{m} f(Z_{k_j}) \prod_{s=1}^{s_j} f(Z_{k_j+\ell_{j,s}}) \right], \]

with \( G_M \) the set of \((m,s)\) with \( m \in \{1, \ldots, M\} \) and \( s = (s_1, \ldots, s_m) \) with \( s_j \geq 0 \) for all \( j = 1, \ldots, m \) and such that \( M = \sum_{j=1}^{m} (s_j+1) \), and with \( c_{(k,\ell)} = c_{(k_1,\ldots,k_m,\ell_{j,s})} \).

Let us fix for the moment \((m,s) \in G_M \). With the notation (35), we wish to study

\[ \sum_{(k,\ell) \in F_{n,\theta,m,s_1,\ldots,s_m}} \mathbb{E} \left[ \prod_{j=1}^{m} f(Z_{k_j}) \prod_{s=1}^{s_j} f(Z_{k_j+\ell_{j,s}}) \right] = \sum_{(k,\ell) \in F_{n,\theta,m,s_1,\ldots,s_m}} B_{k,\ell}. \]

Recall \( J := \{ j = 1, \ldots, m : s_j = 0 \} \). We say that \((k, \ell)\) and \((k', \ell')\) belong to the same block if

\[ \forall r \not\in J, k_r = k'_r, \quad \forall j \in J, \quad |k_j| = |k'_j|, \quad \ell = \ell'. \]
A block is an equivalence class for this equivalence relation. We write $F'_{n, \theta, m, s_1, \ldots, s_m}$ for the set of $(k, \ell)$ such that their block is contained in $F_{n, \theta, m, s_1, \ldots, s_m}$. We will see that the contribution of the sum over $F_{n, \theta, m, s_1, \ldots, s_m} \setminus F'_{n, \theta, m, s_1, \ldots, s_m}$ is negligible in (46). Indeed, observe that if $(k, \ell) \in F_{n, \theta, m, s_1, \ldots, s_m} \setminus F'_{n, \theta, m, s_1, \ldots, s_m}$, then at least one of the following conditions holds true:

(a) $|k_j/d|d - k_j - 1 < n^0 \leq ((|k_j/d|+1)d-1 - k_j - 1)$ if $j \neq 1 \notin \mathcal{J}$ (or $|k_j/d|d - (|k_j-1/d|+1)d - d < n^0 \leq ((|k_j/d|+1)d-1 - [k_j-1/d]d)$ if $j = 1 \notin \mathcal{J}$)

(b) $m \in \mathcal{J}$ and $d[k_m/d] + \max_s \ell_m,s < n \leq d([k_j/d]+1) + \max_s \ell_m,s$

Let us fix $\mathcal{J'} \subset \mathcal{J}$. Due to the first point of Lemma 21, the contribution to (46) of blocks having a type (a) or (b) problem at indices $\mathcal{J'}$ is in

$$\sum_{(k, \ell) \in F_{n, \theta, m, s_1, \ldots, s_m}} B_{(k, \ell)} = O \left( n^{LM_{\theta}} \sum_{n_1, \ldots, n_m = n^0} \left( \prod_{i=1}^m n_i^{-2} \right) \left( \sum_{\mathcal{J'} \subset \{1, \ldots, m\} : \# \mathcal{J'} \geq (\# \mathcal{J})/2} \prod_{j \in \mathcal{J'}} n_j^{-\frac{1}{2} + \eta} \right) \right).$$

Analogously (up to taking $\mathcal{J'} = \emptyset$), it follows from (34) that

$$\sum_{(k, \ell) \in F_{n, \theta, m, s_1, \ldots, s_m}} B_{(k, \ell)} = O \left( n^{LM_{\theta}} \sum_{n_1, \ldots, n_m = n^0} \left( \prod_{i=1}^m n_i^{-2} \right) \left( \sum_{\mathcal{J'} \subset \{1, \ldots, m\} : \# \mathcal{J'} \geq (\# \mathcal{J})/2} \prod_{j \in \mathcal{J'}} n_j^{-\frac{1}{2} + \eta} \right) \right).$$

The above quantity is in

$$O \left( n^{LM_{\theta}} \sum_{\mathcal{J'} : \# \mathcal{J'} \geq (\# \mathcal{J})/2} \sum_{n_1, \ldots, n_m = n^0} \left( \prod_{i=1}^m n_i^{-2} \right) \prod_{r \in \mathcal{J'}} n_r^{-\frac{1}{2} - \eta} \right)$$

$$= O \left( n^{LM_{\theta} + \frac{1}{2} \left( m - \frac{\#(\mathcal{J})}{2} \right) - \left( \frac{1}{4} - \eta \right) \theta \left( \frac{\#(\mathcal{J})}{2} \right)} \right).$$

where we used the fact that $\sum_{r=1}^m r^{-\frac{1}{2}} = O \left( n^{\frac{1}{2}} \right)$ and that $\sum_{r \geq n^0} r^{-\frac{1}{2}} = O \left( n^{-\frac{1}{2}} \right)$. Observe moreover that $M = \sum_{j=1}^m (s_j + 1) \geq 2(m - \# \mathcal{J}) + \# \mathcal{J} = 2m - \# \mathcal{J}$, with equality if and only if $s_j \in \{0, 1\}$ for all $j = 1, \ldots, m$. It follows that

$$\sum_{(k, \ell) \in F_{n, \theta, m, s_1, \ldots, s_m}} \left| \mathbb{E} \left[ \prod_{j=1}^m \left( f(Z_{k_j}) \prod_{s=1}^{s_j} f(Z_{k_j+s_j}) \right) \right] \right|$$

$$= O \left( n^{LM_{\theta} + \frac{1}{2} \left( M - \frac{2m - \# \mathcal{J}}{2} \right) + \theta \left( \frac{\#(\mathcal{J})}{2} - \# \mathcal{J} \right)} \right).$$

In particular this is in $o(n^\frac{1}{2})$ as soon as $M > 2m - \# \mathcal{J}$ or $\mathcal{J} \neq \emptyset$.

This ends the proof of the first point of Corollary 23 (since, when $M$ is odd, we cannot
have \( M = 2m - \# \mathcal{J} \) and \( \mathcal{J} = \emptyset \) and ensures that, for \( M \) even,
\[
n^{-\frac{m}{2}} E \left[ \sum_{k=1}^{n} f(Z_k) \right]^{M} = n^{-\frac{m}{2}} \sum_{(k, \ell) \in \bigcup_{j=1}^{M} F_{n, L_p, \theta_0, M/2, 1, \ldots, 1}} c_{(k, \ell)} E \left[ \prod_{j=1}^{m} f(Z_{k_j}) f(Z_{k_j + \ell_j + 1}) \right].
\]

Assume from now on that \( \theta = \theta_0 \) and that \( M = 0 \) and \( M = 2m \), which means that \( s_j = 1 \) for every \( j = 1, \ldots, m \) and let us estimate the following quantity
\[
\mathcal{E}_{n,M,\theta} = \sum_{(k, \ell) \in F_n \theta, M/2, 1, \ldots, 1} c_{(k, \ell)} E \left[ \prod_{j=1}^{m} f(Z_{k_j}) f(Z_{k_j + \ell_j + 1}) \right].
\]

Note that, when \( (k, \ell) \in F_{n, \theta, M/2, 1, \ldots, 1} \), then \( c_{(k, \ell)} = \frac{(2m)!}{2^n (d_{k_j})!}. \) Using this and applying Proposition 16 combined with the dominated convergence theorem, we obtain that
\[
n^{-\frac{m}{2}} \mathcal{E}_{n,M,\theta} = \frac{(2m)!}{2^m n - \frac{m}{2}} \sum_{0 \leq k_1 < \cdots < k_m \leq n \# k_j > n^p \ell_1, \ldots, \ell_m = 0} 2^{\# (j: \ell_j > 0)} E \left[ \prod_{j=1}^{m} f(Z_{k_j}) f(Z_{k_j + \ell_j}) \right] \\
= \frac{(2m)!}{2^m n - m} \sum_{0 \leq k_1 < \cdots < k_m \leq n /d k_j, n^p k_j > n^p} \sum_{\ell_1, \ldots, \ell_m = 0} 2^{\# (j: \ell_j > 0)} E \left[ \prod_{j=1}^{m} f(Z_{d k_j + \ell_j}) f(Z_{d k_j + \ell_j}) \right] + o(1)
\]
\[
= \frac{(2m)!}{2^m} \int_{0 \leq \ell_1 < \cdots < \ell_m \leq 1/d} d^m \sigma_2^m E \frac{\det D_{\ell_1, \ldots, \ell_m}}{(2\pi \sigma_\ell^2)^{\frac{m}{2}}} \, d\ell_1 \cdots d\ell_m + o(1).
\]

Indeed, we transform the first sum in an integral by making the change of variable \( k_j = [nt_j] \) and, for the domination, it follows from the second point of Proposition 16 that there exists \( \tilde{C} \) such that, for every \( 0 = t_0 < t_1 < \cdots < t_m \leq 1/d \) and every positive integer \( n \) such that
\[
\mathbb{I}_{\{\forall j=1, \ldots, m, \, [nt_{j+1} - [nt_j]] > n^p\}} E \left[ \prod_{j=1}^{m} f(Z_{d[t_{j}]} + k_j) f(Z_{d[t_{j}]} + k_j + \ell_j) \right] \leq \tilde{C} \prod_{i=1}^{m} (d_{j+1} - t_j)^{-\frac{1}{2}}.
\]

Therefore we have proved that
\[
\lim_{n \to +\infty} n^{-\frac{m}{2}} \mathcal{E}_{n,M,\theta} = \frac{(2m)!}{2^m} \int_{0 \leq s_1 < \cdots < s_m \leq 1} \sigma_2^m E \frac{\det D_{s_1, \ldots, s_m}}{(2\pi \sigma_\ell^2)^{\frac{m}{2}}} \, ds_1 \cdots ds_m \\
= \frac{(2m)!}{m!^2} \sigma_2^m \int_{[0,1]^m} \det D_{s_1, \ldots, s_m} \, ds_1 \cdots ds_m.
\]

It remains now to prove that we can neglect the contribution of the \( (k, \ell) \in \bigcup_{p=1}^{M} F_{n, L_p, \theta_0, M/2, 1, \ldots, 1} \setminus F_n \theta_0, M/2, 1, \ldots, 1 \). Fix some \( p = 1, \ldots, M \). It follows from (34)
that
\[
n^{-\frac{3}{4}} \sum_{(k,\ell)\in F_{n,L^p \theta_0,M/2,1,\ldots,1} \setminus F_{n,\theta_0,M/2,1,\ldots,1}} c_{(k,\ell)} \mathbb{E} \left[ \prod_{j=1}^{m} (f(Z_{k_j}) f(Z_{k_j+\ell_j,1})) \right] = O \left( n^{-\frac{3}{4}} \frac{n^m}{n^m L^{p+1} \theta_0} \right) = O \left( n^{-\frac{3}{4}} \frac{n^m}{n^m L^p \theta_0} \right) = o(1).
\]

The last part of Theorem 5 corresponds to the particular case \( f = \delta_0 - \delta_n \). In this case
\[
\sigma_f^2 = \sigma_{\tilde{f}}^2 = \sum_{k \in \mathbb{Z}} \left[ 2 \mathbb{P}(Z[k] = 0) - \mathbb{P}(Z[k] = a) - \mathbb{P}(Z[k] = -a) \right].
\]

## A Proofs of technical lemmas for Theorem 5

Recall the context. Let \( M \geq 1, \theta \in (0,1), \eta \in (0, \frac{1}{100}), L = \frac{n_\eta}{100 \eta} \). Recall that \( n_j = k_j - k_{j-1} \) (with convention \( k_0 = 0 \)). Assume \( n^2 < n_j < n \) and let \( \ell_{j,1}, \ldots, \ell_{j,s_j} = 0, \ldots, [n \theta] \) with \( \sum_{j=1}^{s_j} (1 + s_j) = M \).

### Proof of Lemma 17

We start by writing
\[
p_{k,\ell}(a, b) = \frac{1}{(2\pi)^M} \int_{[-\pi,\pi]^M} e^{-i \sum_{j=1}^{m} \left( (a_{j} - a_{j-1}) \theta_j + \sum_{i=1}^{s_j} (b_{j,i} - a_{j}) \theta_{j,i} \right)} \varphi_{k,\ell}(\theta, \theta') \, d(\theta, \theta').
\]

But, due to the definition of \( d \), for any \( u, v \in \mathbb{Z} \), \( \varphi_{k}(u + \frac{2\pi}{d}v) = (\varphi_{k}(\frac{2\pi}{d}))^v \varphi_{k}(u) \) and so, for any \( u \in \mathbb{R}^M \) and \( v \in \mathbb{Z}^M \),
\[
\varphi_{k,\ell}(u + \frac{2\pi}{d} v) = \mathbb{E} \left[ \prod_{y \in \mathbb{Z}} \varphi_{\ell} \left( \sum_{j=1}^{m} \left( u_j + \frac{2\pi v_j i}{d} \right) N_j(y) + \sum_{s=1}^{s_j} \left( u_j + \frac{2\pi v_j s}{d} \right) N_j^s(y) \right) \right] = \mathbb{E} \left[ \prod_{y \in \mathbb{Z}} \left( \varphi_{\ell} \left( \frac{2\pi}{d} \right) \right)^{\sum_{j=1}^{m} v_j N_j(y) + \sum_{s=1}^{s_j} \sum_{i=1}^{s_j} \ell_{j,i} v_{j,i}} \varphi_{k,\ell}(u) \right].
\]

and so
\[
p_{k,\ell}(a, b) = \frac{1}{(2\pi)^M} \int_{[-\pi,\pi]^M} e^{-i \sum_{j=1}^{m} \left( (a_{j} - a_{j-1}) \theta_j + \sum_{i=1}^{s_j} (b_{j,i} - a_{j}) \theta_{j,i} \right)} \varphi_{k,\ell}(\theta) \, d(\theta, \theta').
\]

Moreover, for any \( a \in \mathbb{Z} \), then \( \sum_{r=0}^{d-1} e^{-\frac{2\pi a r}{d}} \left( \varphi_{\ell} \left( \frac{2\pi}{d} \right) \right)^r = 0 \) except if \( e^{-\frac{2\pi a}{d}} \left( \varphi_{\ell} \left( \frac{2\pi}{d} \right) \right)^v = 1 \) (i.e. if \( a - v = d \mathbb{Z} \)) and then this sum is equal to \( d \). This ends the proof of Lemma 17. \( \Box \)

### Proof of Lemma 18

Due to [14, Lemma 16], \( \mathbb{P}(\Omega_k^{(j)}) = 1 - o(n_j^{-p}) \) for any \( p > 1 \) and so, since \( n_j > n^\theta \), it follows that for all \( p > 1 \), \( \mathbb{P}(\Omega_k^{(j)}) = 1 - o(n^{-p}) \). Moreover, since \( \theta' \in (0, \frac{1}{4}) \), due to [14, Lemma 21],
\[
\forall p > 1, \quad \mathbb{P} \left( \det D_k < n^{-\theta} \prod_{i=1}^{m} n_i \right) = o(n^{-p}),
\]
uniformly on \( k \) as above. \( \Box \)
Proof of Lemma 19. Recall that $F = \{ y \in \mathbb{Z} : \forall (j, s), N'_{j,s}(y) = 0 \}$. Due to (40), Lemma 19 follows from the following estimate

$$
\exists c > 0, \quad \int_{\{y \in F : \frac{1}{2} + q < |\theta| \}} \mathbb{E} \left[ \prod_{y \in F} |\varphi_\xi \left( \sum_{j=1}^m \theta_j N'_j(y) \right) | I_{\Omega_k} \right] d\theta = o\left( e^{-cn} \right),
$$

(47)

uniformly on $k, \ell$ as in Proposition 16. To this end, we follow and slightly adapt the proof of [13, Proposition 10] as explained below. Observe that, up to conditioning with respect to $(S_{k+1} - S_k)_{k \in \{k_1, \ldots, k_{j-1}\}}$, this will be a consequence of

$$
\forall j = 1, \ldots, m, \ \forall u \in \mathbb{R}, \quad \int_{n_j^{-\frac{1}{2}} + u < |\theta| < n_j^{\frac{1}{2}} - u} \mathbb{E} \left[ \prod_{y \in F} |\varphi_\xi (u + \theta N'_j(y)) | I_{\Omega_k} \right] d\theta = o\left( e^{-cn} \right),
$$

(48)

uniformly on $k_j, \ell_j$ as above. Recall that $\#(\mathbb{Z} \setminus F) \leq \sum_{j=1}^m \sum_{s=1}^{n_j} \ell_{j,s} \leq M_1 n^L$. As in [13, after Lemma 16], we observe that, for $n$ large enough,

$$
\prod_{y \in F} |\varphi_\xi (u + \theta N'_j(y)) | \leq \exp \left( -\frac{\sigma^2_1}{4} n^{-\frac{1}{2} + 4\gamma} \# \left\{ y : d \left( u + \theta N'_j(y), \frac{2\pi}{d} Z \right) \geq n^{-\frac{1}{4} + 2\gamma} \right\} \right),
$$

(49)

and that

$$
d \left( u + \theta N'_j(y), \frac{2\pi Z}{d} \right) \geq n^{-\frac{1}{4} + 2\gamma} \iff \frac{u}{\theta} + N'_j(y) \in I := \bigcup_{k \in \mathbb{Z}} I_k,
$$

(50)

where, for all $k \in \mathbb{Z}$,

$$
I_k := \left[ \frac{2k\pi}{d\theta} + n^{-\frac{1}{4} + 2\gamma}, \frac{2(k+1)\pi}{d\theta} - n^{-\frac{1}{4} + 2\gamma} \right].
$$

In particular $\mathbb{R} \setminus I = \bigcup_{k \in \mathbb{Z}} J_k$, where for all $k \in \mathbb{Z}$,

$$
J_k := \left( \frac{2k\pi}{d\theta} - n^{-\frac{1}{4} + 2\gamma}, \frac{2k\pi}{d\theta} + n^{-\frac{1}{4} + 2\gamma} \right).
$$

Let $N_{\pm}$ be two positive integers such that $\mathbb{P}(X_1 = N_+) \mathbb{P}(X_1 = -N_-) > 0$. Let $C_{\pm} = (C_{k,\pm})_{k=1, \ldots, r} \in \mathbb{Z}^r$ with $C_{\pm} = N_+ + N_-$ and $C_{k,\pm} = N_+$ for $k \leq N_-$ and $C_{k,\pm} = -N_-$ otherwise, and symmetrically $C_{k,\pm} = -N_-$ for $k \leq N_+$ and $C_{k,\pm} = N_+$ otherwise. It has been proved in [13] (see Lemma 15 therein combined with the estimate $\mathbb{P}(D_n) = 1 - o(e^{-cn})$ in Section 2.8 therein) that, for $n$ large enough,

$$
\mathbb{P}(\Omega_k \setminus \mathcal{E}_j) = o(e^{-cn_j}),
$$

(51)

with

$$
\mathcal{E}_j = \left\{ \# \{ y \in \mathbb{Z} : C_j(y) \geq n_j^{\frac{1}{4} - 2\gamma} \} \geq 3N_+N_- n_j^{\frac{1}{4} - 2\gamma} \right\}
$$

and, where, for any $y \in \mathbb{Z}$,

$$
C_j(y) := \# \left\{ k = 0, \ldots, \left\lfloor \frac{n_j}{T} \right\rfloor - 1 : S_{k-1+kT} - S_{k-1} = y \right\}.
$$

(52)

Now, on $\mathcal{E}_j$, we define $Y_i$ for $i = 1, \ldots, \left\lfloor n_j^{\frac{1}{4} - 2\gamma} \right\rfloor$, by

$$
Y_i := \min \left\{ y \in \mathbb{Z} : C_j(y) \geq n_j^{\frac{1}{4} - 2\gamma} \right\},
$$

(53)

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We define the events with probability at least 

Thus, conditionally to \( n \), let \( t'_j = m_n(j) + T \) be the \( j \)-th time of the form \( mT \) when a peak of the form \( C^+ \) is based on the site \( Y_i \). We also define \( N_j^0(Y_i + N+ N_\cdot) \) as the number of visits of \( \{ S_{k_{j-1} + k} - S_{k_{j-1}} \}_{k \geq 0} \) before time \( n_j \) to \( Y_i + N_+ N_\cdot \), which do not occur during the time intervals \([t'_j, t'_j + T]\), for \( u \leq n_j^{1/2-2\gamma} \). We proved in [13, Lemma 16] that, for any \( H \geq 0 \),

\[
P \left( u + N'_j(Y_i + N_+ N_\cdot) \in I | E_n, N_j^0(Y_i + N_+ N_\cdot) = H \right) = P \left( H + u + b_j \in I \right),
\]

where \( b_j \) is a random variable with binomial distribution \( B \left( \left[ n_j^{1/2-2\gamma} \right]; \frac{1}{2} \right) \) and finally we proved in [13, Lemmas 17 and 18] (see in particular the last formula in the proof of Lemma 17) that

\[
\forall H' \in \mathbb{R}, \quad P (H' + b_n \in I) \geq \frac{1}{3}.
\]

Thus, conditionally to \( \{ S_{k+1} - S_k \}_{k \geq \{ k_j - 1 \} \in \mathbb{Z}, \}, \mathcal{E}_j \) and \( \{ N_j^0(Y_i + N+ N_\cdot), i \geq 1 \} \), the events \( \{ u + N_j^0(Y_i + N_+ N_\cdot) \in I \}, i \geq 1 \), are independent of each other, and all happen with probability at least 1/3. We conclude that

\[
P \left( \mathcal{E}_j \cap \left\{ \# \left\{ i : u + N'_j(Y_i + N_+ N_\cdot) \in I \right\} \leq n_j^{1/2-2\gamma} \right\} \right) \leq P \left( B_j \leq n_j^{1/2-2\gamma} \right) = O(e^{-c''n_j}), \tag{52}
\]

where \( B_j \) has binomial distribution \( B \left( \left[ n_j^{1/2-2\gamma} \right]; \frac{1}{2} \right) \).

But if \( \# \{ y \in \mathbb{Z} : N'_j(y) \in I \} \geq n_j^{1/2-2\gamma}/4 \), then, by (49) and (50) there exists a constant \( c'' \) such that, for any \( n \) large enough,

\[
\prod_{y \in \mathcal{F}} |\varphi_\xi (u + \theta N'_j(y))| \leq \exp \left( -c''n_j^{1/2-2\gamma} n_j^{-1/4+\gamma} \right),
\]

since \( \#(\mathbb{Z} \setminus \mathcal{F}) \ll n_j^{1/2-2\gamma}/4 \). This, combined with (51) and (52), ends the proof of (48) and so of Lemma 19.

**Proof of Lemma 20.** We have to estimate \( B_k, k \in \mathbb{N}, \ell \) uniformly on \( k, \ell \) as in Proposition 16, where \( \ell_k \) is \( V_k \times [-\pi, \pi]^{M-m} \) and where \( V_k \) is the set of \( \theta \in \mathbb{R}^m \) such that for all \( j = 1, \cdots, m \), \( |\theta_j| < n_j^{-1/2+m} \) and such that there exists some \( j_0 = 1, \cdots, m \) satisfying

\[
n_j^{-1/2+\gamma} < |\theta_{j_0}|. \quad \text{Let } \varepsilon_0 > 0 \text{ be such that}
\]

\[
\forall u \in [-\varepsilon_0, \varepsilon_0], \quad |\varphi_\xi (u)| \leq e^{-\frac{\varepsilon_0^2 u^2}{2}}. \tag{53}
\]

We define the events \( H_k = \Omega_k \cap \{ \forall y \in \mathbb{Z}, | \sum_{j=1}^m \theta_j N'_j(y) | \leq \varepsilon_0/2 \} \) and

\[
H_k := \left\{ \# \left\{ y \in \mathbb{Z} : \left| \sum_{j=1}^m \theta_j N'_j(y) \right| \leq \left[ \frac{\varepsilon_0}{4}, \frac{\varepsilon_0}{2} \right] \right\} > n_j^{1/2} \right\}.
\]

Due to [14, Lemma 21 and last formula of p. 2446],

\[
\exists c' > 0, \quad P ( \Omega_k \ \setminus \ (H_k \cup H_k') ) = O \left( \prod_{j=1}^m n_j^{-\frac{1}{4}} \right),
\]
uniformly on $k$ as above and uniformly on $\theta \in V_k$. Thus,

$$B_{k,t,E^{(2)}_{u_k}\Omega_k\setminus(H_k\cup H_k')} = O\left(\prod_{j=1}^{m} n_j^{\frac{3}{2} + \eta}\right),$$  

(54)

where we used the fact that $\int_{V_k} d\theta \leq \prod_{j=1}^{m} n_j^{\frac{3}{2} + \eta}$. Moreover, for $n$ large enough, it follows from the definition of $H_k'$, from (41) and (53) that

$$B_{k,t,E^{(2)}_{u_k}\Omega_k\cap H_k'} = O\left(\int_{V_k} E \left| \prod_{y \in \mathcal{F}} \left| \sum_{j=1}^{m} \theta_j N_j'(y) \right| \mathbb{I}_{\Omega_k\cap H_k'} \right) d\theta \right) \leq e^{-\frac{n^{3/2} + \eta}{\nu}}. \tag{55}$$

Finally, it remains to estimate $B_{k,t,E^{(2)}_{u_k}\Omega_k\cap H_k'}$. To this end we write

$$\int_{V_k} E \left[ \prod_{y \in \mathcal{F}} \left| \sum_{j=1}^{m} \theta_j N_j'(y) \right| \mathbb{I}_{\Omega_k\cap H_k'} \right] d\theta$$

$$\leq \int_{V_k} E \left[ e^{-\frac{\sigma^2}{2} \sum_{y \in \mathcal{F}} \left( \sum_{j=1}^{m} \theta_j n_j^{-\frac{3}{2}} N_j'(y) \right)^2} \mathbb{I}_{\Omega_k} \right] d\theta$$

$$\leq \int_{V_k'} \left( \prod_{j=1}^{m} n_j^{-\frac{3}{2}} \right) E \left[ e^{-\frac{\sigma^2}{2} \sum_{y \in \mathcal{F}} \left( \sum_{j=1}^{m} \theta_j n_j^{-\frac{3}{2}} N_j'(y) \right)^2} \mathbb{I}_{\Omega_k} \right] d\theta'$$

$$\leq \left( \prod_{j=1}^{m} n_j^{-\frac{3}{2}} \right) E \left[ \int_{(\tilde{D}_k')^{\frac{3}{2}}} \left( \det \tilde{D}_k' \right)^{-\frac{1}{2}} e^{-\frac{\sigma^2|v|^2}{2}} \mathbb{I}_{\Omega_k} dv \right], \tag{56}$$

with the successive changes of variable $\theta'_j = n_j^{\frac{3}{2}} \theta_j$ and $v = (\tilde{D}_k')^{\frac{3}{2}} \theta''$, with

$$\tilde{D}_k' = \left( (n_i n_j)^{-\frac{3}{2}} \sum_{y \in \mathcal{F}} N_j'(y) N_j'(y) \right)_{i,j} \quad \text{and} \quad V_k'' = \text{Diag}(n_j^{\frac{3}{2}}) V_k.$$

Note that $V_k''$ is the set of $(\theta'_1, \cdots, \theta'_m)$ such that $|\theta'_{j_0}| \leq n_j^{\frac{3}{2} + \eta}$ and such that there exists $j_0 = 1, \cdots, m$ such that $|\theta'_{j_0}| \geq n_j^{\frac{3}{2} - \eta}$.

Let us prove that, in the above formula, we can approximate the determinant of $\tilde{D}_k'$ by the one of $\hat{D}_k := \left( (n_i n_j)^{-\frac{3}{2}} \sum_{y \in Z} N_j'(y) N_j'(y) \right)_{i,j}$. To this end, writing $\Sigma_m$ for the set of permutations of the set $\{1, \cdots, m\}$ and $\nu(\sigma)$ for the signature of $\sigma \in \Sigma_m$, we observe that, on $\Omega_k$,

$$\left| \det \tilde{D}_k' - \det \hat{D}_k \right|$$

$$= \left( \prod_{j=1}^{m} n_j^{-\frac{3}{2}} \right) \sum_{\sigma \in \Sigma_m} (-1)^{\nu(\sigma)} \prod_{j=1}^{m} \left( \sum_{y \in \mathcal{F}} N_j'(y) N_{\sigma(j)}'(y) \right)$$

$$- \sum_{\sigma \in \Sigma_m} (-1)^{\nu(\sigma)} \prod_{j=1}^{m} \left( \sum_{y \in \mathcal{F}} N_j'(y) N_{\sigma(j)}'(y) \right)$$

$$\leq \left( \prod_{j=1}^{m} n_j^{-\frac{3}{2}} \right) \sum_{\sigma \in \Sigma_m} \sum_{j=1}^{m} \sum_{z \in \mathcal{F} \setminus \mathcal{F}} \sum_{j' \neq j} N_j'(z) N_{\sigma(j)}'(z) \prod_{j' \neq j} \left( \sum_{y \in \mathcal{F}} N_j'(y) N_{\sigma(j')}'y \right)$$

$$\leq \left( \prod_{j=1}^{m} n_j^{-\frac{3}{2}} \right) \sum_{\sigma \in \Sigma_m} \sum_{j=1}^{m} \frac{1}{2} n_j^{1+2\eta} n_{\sigma(j)}^{1+2\eta} \prod_{j' \neq j} \sqrt{V_j V_{\sigma(j')}},$$
where we used the Cauchy-Schwarz inequality together with the notations and estimates given after Lemma 18. Using (39) and (41), it follows that, on $\Omega_k$,

$$\left| \det \tilde{D}_k - \det \tilde{D}_k \right| \leq \left( \sum_{j=1}^{m} n_j^{-2} \right)^{\frac{1}{2}} \leq \left( \sum_{j=1}^{m} n_j^{-2} \right)^{\frac{1}{2}} \leq \frac{1}{2} n^{-\gamma \theta + \frac{d}{2} + \frac{d}{2} \theta} \leq \frac{1}{2} n^{-\gamma \theta - (M-1) \theta} \leq \frac{n^{-(M-1) \theta}}{2} \det \tilde{D}_k,$$

since $\theta' \leq \frac{\theta}{2} - 2ML\theta < \frac{\theta}{2} - m\gamma - ML\theta$ and where we used the fact that $\det \tilde{D}_k = \det D_k \prod_{j=1}^{m} n_j^{-2}$ together with the definition of $\Omega_k$. Therefore, on $\Omega_k$, $\det \tilde{D}_k \geq \frac{1}{2} \det \tilde{D}_k$. Thus, due to (56),

$$\int_{V_k} E \left[ \prod_{j \in \mathcal{F}} |\varphi_j \sum_{j=1}^{m} \theta_j N_j'(y) - 1 \mathbb{1}_{\Omega_k \cap H_k} | \right] d\theta \leq O \left( \left( \prod_{j=1}^{m} n_j^{-2} \right) E \left[ (\det \tilde{D}_k)^{-\frac{1}{2}} \mathbb{1}_{\Omega_k} e^{-\frac{\sigma_2^2 |w|^2}{2}} dv \right] \right) = O \left( \left( \prod_{j=1}^{m} n_j^{-2} \right) E \left[ (\tilde{D}_k)^{-\frac{1}{2}} \mathbb{1}_{\Omega_k} e^{-\frac{\sigma_2^2 |w|^2}{2}} dv \right] \right).$$

By definition of $V''_k$, for any $v \in (\tilde{D}_k)^{-\frac{1}{2}} V''_k$, $|v|_2 \geq (\tilde{\lambda}_k)^{1/2} n^{\left(\frac{3}{2} - \eta\theta\right)}$, where $\tilde{\lambda}_k$ is the smallest eigenvalue of $\tilde{D}_k$. Since all the eigenvalues of $\tilde{D}_k$ are nonnegative ($\tilde{D}_k$ being symmetric and nonnegative), it follows that all the eigenvalues of $\tilde{D}_k'$ are smaller than $\text{trace}(\tilde{D}_k') \leq \sum_{j=1}^{m} V_j' \leq mn^{\frac{\eta\theta}{2}}$ (on $\Omega_k$). Thus, on $\Omega_k$,

$$\tilde{\lambda}_k \geq \frac{\text{det} (\tilde{D}_k)^{1/2} n^{\left(\frac{3}{2} - \eta\theta\right)}}{(mn^{\frac{\eta\theta}{2}})^{m-1}} \geq \frac{n^{\left(\frac{3}{2} - \eta\theta\right) - \frac{\theta}{2} - \frac{3(\gamma(m-1))}{2}}}{2m^{\frac{m-1}{2}}} \gg n^\eta.$$

since $\eta\theta$, $\frac{\theta}{2}$, and $\frac{3\gamma(m-1)}{2}$ are all strictly smaller than $\frac{\theta}{16}$. Hence

$$E \left[ (\tilde{D}_k)^{-\frac{1}{2}} \mathbb{1}_{\Omega_k} e^{-\frac{\sigma_2^2 |w|^2}{2}} dv \right] \leq E \left[ (\tilde{D}_k)^{-\frac{1}{2}} \mathbb{1}_{\Omega_k} e^{-\frac{\sigma_2^2 |w|^2}{2}} dv \right] = O (n^{-p}),$$

for any $p > 0$. This combined with (54), (55) and (57) ends the proof of the lemma. It will be worthwhile to note that the previous estimate also holds true when $\tilde{\lambda}_k$ is replaced by the smallest eigenvalue $\tilde{\lambda}_k'$ of $\tilde{D}_k'$.

Before proving Lemma 21, we state a useful coupling lemma allowing us to replace $\det D_k$ by a copy independent of $(N'_j)_{j,s}$. Up to enlarging the probability space if necessary, we consider $X' = (X'_t)_k \geq 0$ an independent copy of the increments $X = (X_t)_k \geq 0$ of the random walk $S$. We then define the random walk $S''$ as follows: $S''_t = \sum_{k=1}^{m} X''_k$ with $X''_k = X_k$ if $j_1 + \ell_1 - 1 \leq k < j_2$ and $X''_k = X'_k$ if $j_2 \leq k < j_2 + \ell_2$, with $\ell_2 = \max_{s=1, \ldots, s} \ell_j$. We define $\Omega''_k$, $N''_k$ and $D''_k$ for the space as we have defined $\Omega_k$, $N''_k$ and $D''_k$ up to replacing $S$ by $S''$.

**Lemma 24** (Replacing a part of the RW by an independent copy). There exists $\Omega'_k \subset \Omega_k \cap \Omega''_k$ such that

$$\forall p > 0, \quad P (\Omega_k \cap \Omega''_k \setminus \Omega'_k) = O (n^{-p})$$

(59)
and such that, on $\Omega'_k$,
\[
\left| (\det D_k)^{-\frac{1}{4}} - (\det D_k')^{-\frac{1}{4}} \right| \leq n^{-\frac{3}{2} - L_\theta} (\det D_k^{-\frac{3}{2}} + (\det D_k')^{-\frac{3}{2}}).
\]
Moreover
\[
\mathbb{E} \left[ \left| (\det D_k)^{-\frac{1}{4}} - (\det D_k')^{-\frac{1}{4}} \right| 1_{\Omega_k} \right] \leq n^{-\frac{3}{2} - L_\theta} \prod_{j=1}^m n_j^{-\frac{1}{4}}.
\] (60)

**Proof of Lemma 24.** Observe that
\[
h_j := S_{k_j} - S_{k_j} = \sum_{j' < j} \left(S_{k_{j'}} + e_{j'} - (S_{k_{j'}} + e_{j'} - S_{k_{j'}})\right)
\]
and, on $\Omega_k \cap \Omega'_k$,
\[
|N_j'(z) - N_j''(z)| = |N_j'(z) - N_j'(z + h_j)| \leq n_j^\frac{1}{2} + \frac{2}{3} |h_j|^{\frac{3}{2}},
\]
for all $z \in \mathbb{Z} \setminus \bigcup_{m=k_j-1}^{k_j} \{ S_m, S_m' \}$.

We will prove that $\det D_k$ is close enough to $\det D_k'' = \det \left( \left( \sum_{y \in \mathbb{Z}} N''_i(y)N''_j(y) \right)_{i,j} \right)$. Due to the Markov inequality,
\[
\forall p > 0, \quad \mathbb{P} \left( |S_{\ell_j}| > h \right) \leq \mathcal{O} \left( \frac{\ell_j^{\frac{p}{2}}}{h^p} \right) = \mathcal{O} \left( n^{-r_j} \right),
\]
where we set $h = n^{\gamma' + \frac{a_n}{a_m}} \geq n^{\gamma' \ell_j^2}$. Thus we set
\[
\Omega_k' := \Omega_k \cap \Omega'_k \cap \{ \forall j = 1, \ldots, m, \ |h_j| \leq h \}
\]
and we observe that $\mathbb{P}((\Omega_k \cap \Omega'_k) \setminus \Omega_k') = \mathcal{O}(n^{-p})$ for all $p > 0$. Moreover, on $\Omega_k'$,
\[
|N_j'(z) - N_j''(z)| \leq 2\ell_j + n_j^{\frac{1}{2} + \frac{2}{3}} h_j^{\frac{1}{2}} \leq 3n_j^{\frac{1}{2} + \frac{2}{3}} n^{\frac{2}{3} + \frac{a_n}{a_m}}.
\]
Moreover
\[
V_j'' := \sum_{y \in \mathbb{Z}} (N''_i(y))^2 \leq \sum_{y \in \mathbb{Z}} (N'_i(y))^2 + 2\ell_j^3 \leq n_j^{2 + 3\gamma}.
\]
This allows us to observe that, on $\Omega_k'$,
\[
|\det D_k - \det D_k''| \leq 3n_j^{\frac{1}{2} + \frac{a_n}{a_m}} m \prod_{j'=1}^m \max \left( \sum_{y \in \mathbb{Z}} N'_j(y)N'_j(y'), \sum_{y \in \mathbb{Z}} N''_j(y)N''_j(y') \right) \prod_{j' \neq j} \left( V''_{j'} - V''_{j'} \right) \left( V''_{j'} V''_{j'} \right)^{\frac{1}{a}} \leq 3n_j^{\frac{1}{2} + \frac{a_n}{a_m}} m \prod_{j'=1}^m n_j^{2 + 3\gamma} \prod_{j=1}^m n_j^{-\frac{1}{2} - 2} \leq \prod_{j'=1}^m n_j^{\frac{1}{2} - \frac{1}{2} n_j^{-L_\theta}},
\]

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We conclude thanks to [14, Lemma 21] which ensures that
\( (\det D_k)^{-\frac{1}{2}} - (\det D_k')^{-\frac{1}{2}} \leq n^{-\frac{3}{2}} - L \delta (\det D_k)^{-\frac{3}{2}} + (\det D_k')^{-\frac{3}{2}} \).

We conclude thanks to [14, Lemma 21] which ensures that
\( \mathbb{E} \left[ (\det D_k)^{-\frac{1}{2}} \mathbb{I}_{\Omega_k} \right] = O \left( \prod_{j=1}^m n_j^{-\frac{1}{2}} \right) \).

The proof of Lemma 21 will also use the following result. Recall that we set \( \mathcal{J} = \{ j = 1, \ldots, m : s_j = 0 \} \) and that \( \mathcal{J}' \) is a subset contained in \( \mathcal{J} \).

**Lemma 25** (An estimate using the “centering” assumption). Under the assumptions of Lemma 21,
\[
\sum_{k_j' = 0, \ldots, d-1, \forall j \in \mathcal{J}'} B_{k+k' \ell} \mathbb{I}_{\Omega_{k}} = \frac{d^m}{(2\pi)^{2d}} \int_{\mathbb{R}^m} \mathbb{E} \left[ \mathbb{I}_{\Omega_{k}} F(\theta, \theta') G(\theta, \theta') \right] d(\theta, \theta'),
\]
with \( k' \in \mathbb{Z}^m \) such that \( k'_j = 0 \) for all \( j \not\in \mathcal{J}' \), with
\[
G(\theta, \theta') := \prod_{j \not\in \mathcal{J}'} \left( \sum_{a_j \in \alpha_{k_j} + d\mathbb{Z}} \sum_{j \not\in \mathcal{J}'} f(a_j) \left( \prod_{j=1}^{s_j} f(b_{j,s}) \right) e^{i(a_j \theta_{j+1} - \theta_j - 1) \sum_{s=1}^{s_j} \theta_{j,s}} \right) \times \prod_{y \in \mathbb{Z}^m} \varphi_{\xi} \left( \sum_{j=1}^{s_j} \theta_{j} N'_{j}(y) + \sum_{s=1}^{s_j} \theta'_{j,s} N'_{j,s}(y) \right),
\]
with \( \mathcal{J}' = \bigcup_{j \in \mathcal{J}} \mathcal{S}'_j, \mathcal{S}'_j := \{ S_{k_j} \} \), so that \( \{ S_{k_j} \} \) and with \( F \) satisfying
\[
F(\theta, \theta') = O \left( \sum_{\mathcal{J}' \subseteq \mathcal{J}'' \subseteq \mathcal{J} \setminus \mathcal{J}'} (|\theta_{\mathcal{J}'}| + |\theta_{\mathcal{J}''}|) \mathbb{I}_{\Omega_{k}} \right),
\]
uniformly on \( \mathcal{J} \), and on \( \Omega_{k} \), with
\[
B_j = \{ S'_{j} \cap \bigcup_{j' \not\in \mathcal{J}''} \mathcal{S}'_{j'} \neq \emptyset \}.
\]
If \( \sum_{a \in \mathbb{Z}} f(b + ad) = 0 \) for all \( b \in \mathbb{Z} \) (true if \( d = 1 \)), then
\[
F(\theta, \theta') = O \left( \prod_{j \in \mathcal{J}''} (|\theta_{\mathcal{J}''}| + |\theta_{\mathcal{J}''}|) \right)
\]
(with convention \( \theta_{m+1} = 0 \)).

**Proof.** We start by writing
\[
\sum_{k_j' = 0, \ldots, d-1, \forall j \in \mathcal{J}'} B_{k+k' \ell} \mathbb{I}_{\Omega_{k}} = \frac{d^m}{(2\pi)^{2d}} \int_{\mathbb{R}^m} \mathbb{E} \left[ \mathbb{I}_{\Omega_{k}} F(\theta, \theta') G(\theta, \theta') \right] d(\theta, \theta'),
\]
where we set
\[
F(\theta, \theta') := \sum_{k_j' = 0, \ldots, d-1, \forall j \in \mathcal{J}'} \prod_{j \in \mathcal{J}'} \left( \sum_{a_j \in (k_j + k'_j) + \mathbb{Z}} \left( f(a_j) e^{-i(a_j \theta_{j+1})} \right) \right) \times \prod_{y \in \mathbb{Z}^m} \varphi_{\xi} \left( \sum_{r=1}^{s_r} \theta_{r,s} N'_{r,s}(y) + \sum_{s=1}^{s_j} \theta'_{j,s} N'_{j,s}(y) \right),
\]
with
\[
\tilde{N}'_{r,k}(y) = \# \{ u = k_{r-1} + k'_{r-1}, \ldots, k_r + k'_r - 1 : S_u = y \}.
\]
If \( \sum_{a \in u + d\mathbb{Z}} f(a) = 0 \) for all \( u \in \mathbb{Z} \), the proof of Lemma 25 ends by noticing that
\[
\sum_{a_j \in (k_j + k'_j) + \mathbb{Z}} \left( f(a_j) e^{-i(a_j \theta_{j+1})} \right) = \sum_{a_j \in (k_j + k'_j) + \mathbb{Z}} \left( f(a_j) e^{-i(a_j \theta_{j+1})} - 1 \right),
\]
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which is in \( \mathcal{O}(|\theta_j| + |\theta_{j+1}|) \) since \( \sum_{a \in \mathbb{Z}} |af(a)| < \infty \). Since we just assume here that \( \sum_{a \in \mathbb{Z}} f(a) = 0 \), we need a more delicate approach. We rewrite \( F \) as follows

\[
F(\theta, \theta') := \sum_{k_j' = 0} \left( \prod_{j \in J'} H_{j, k_j'}(\theta_j - \theta_{j+1}) \right) \Psi(k')
\]

with

\[
H_{j, k_j'}(\theta) := \sum_{a_j \in (k_j + k_j')\alpha + dZ} (f(a_j)e^{-ia_j\theta}),
\]

\[
\Psi(k') = \prod_{y \in S'} \varphi_y \left( \sum_{r=1}^{s_r} \left( \theta \tilde{N}_{r,k}(y) + \sum_{s=1}^{s_r} \theta_{r,s} \tilde{N}_{r,s}(y) \right) \right),
\]

recalling that \( \tilde{N}_{r,k}(y) = \# \{|u = k_{r-1} + k_{r-1}' + \cdots + k_r + k_r' - 1 : S_n = y\} \). Note that \( \tilde{N}_{r,k}(y) = N_r(y) \) except maybe if \( r \in J' \) and \( y \in S_r' \) or if \( r - 1 \in J' \) and \( y \in S_{r-1}' \). We order the elements of \( J' \) as follows: \( j_1' < \cdots < j_{j_f}' \), and write

\[
F(\theta, \theta') = F_0(\theta, \theta') + F_1(\theta, \theta')
\]

with

\[
F_1(\theta, \theta') = \sum_{k_j = 0}^{d-1} H_{j_1', k_j}(0) \sum_{k_{j_2}, \ldots, k_{j_f}}^{d-1} \left( \prod_{j \in J' \setminus \{j_1\}} H_{j, k_j}(\theta_j - \theta_{j+1}) \right) \Psi(k')
\]

and

\[
F_0(\theta, \theta') = \sum_{k_j = 0}^{d-1} \left( H_{j_1', k_j}(\theta_{j_1} - \theta_{j_1+1}) - H_{j_1', k_j}(0) \right) \sum_{k_{j_2}, \ldots, k_{j_f} = 0}^{d-1} \left( \prod_{j \in J' \setminus \{j_1\}} H_{j, k_j}(\theta_j - \theta_{j+1}) \right) \Psi(k').
\]

Note that \( H_{j_1', k_j}(\theta_{j_1} - \theta_{j_1+1}) - H_{j_1', k_j}(0) \) is in \( \mathcal{O}(|\theta_{j_1}| + |\theta_{j_1+1}|) \). Since \( \sum_{a \in \mathbb{Z}} f(a) = 0 \), \( F_1 \) satisfies

\[
F_1(\theta, \theta') = \sum_{k_j = 0}^{d-1} H_{j_1', k_j}(0) \sum_{k_{j_2}, \ldots, k_{j_f} = 0}^{d-1} \left( \prod_{j \in J' \setminus \{j_1\}} H_{j, k_j}(\theta_j - \theta_{j+1}) \right) \Delta_{j_1'} \Psi(k')
\]

with \( \Delta_{j'}(\theta') = \phi(\theta') - \phi(\theta'_j) \), where \( k'_j \in \mathbb{N}^m \) is such that \( (k'_j)_i = k'_i \) for \( i \neq j \), and \( (k'_j)_j = 0 \). Indeed

\[
\sum_{k_j = 0}^{d-1} \left( \Psi(k') - \Delta_{j_1'} \Psi(k') \right) H_{j_1', k_j}(0) = \Psi(k'_j) \sum_{k_j = 0}^{d-1} H_{j_1', k_j}(0) = \Psi(k'_j) \sum_{a_j \in (k'_j + k_j')\alpha + dZ} f(a_j) = 0.
\]

Proceding iteratively on \( J' \), we obtain

\[
F(\theta, \theta') = \sum_{\epsilon_1, \ldots, \epsilon_{j_f} \in \{0,1\}} F_{\epsilon_1, \ldots, \epsilon_{j_f}}(\theta, \theta'),
\]

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with
\[ F_{\epsilon, \ldots, \epsilon'}(\theta, \theta') \]
\[ = \left( \prod_{j' \in J'} \left( H_{j', k'_j} \theta_{j'} - \theta_{j'+1} \right) - H_{j', k'_j} \left( 0 \right) \right) \left( \prod_{j \in J''} H_{j, k_j} \left( 0 \right) \right) \Delta_{j'} \Delta_{j} \Psi(k'), \]
with convention \( \Delta^0_j = \text{Id} \). The first part will be easily dominated by \( O\left( \prod_{j' \in J'} \left( |\theta_{j'}| + |\theta_{j'+1}| \right) \right) \). Let us study the second part of the formula exploiting the fact that \( \sum_{a \in \mathbb{Z}} f(a) = 0 \). The difficulty here is that \( k' \) appears both in \( \prod_{j' \in J'} H_{j', k'_j} \left( 0 \right) \) and \( \Delta \cdot \Psi(k') \). The value of \((\epsilon_1, \ldots, \epsilon_{J'})\) being fixed, we consider the set \( \mathcal{J}'' \) of the \( j' \in J' \) such that \( \epsilon_{j'} = 1 \).

Observe that, if \( S''_j \cap S'_j = \emptyset \), then
\[ \Delta_{j'} \Delta_j \Psi(k') = \left( \Delta_{j'} \Psi_{S''_j}(k'_j) \right) \left( \Delta_j \Psi_{S'_j}(k'_j) \right) \]
with
\[ \Psi_{S''_j}(k'_j) = \prod_{y' \in S''_j} \left( \sum_{r=1}^{m} \left( \theta_r \tilde{N}_{r, k'_j}(y) + \sum_{s=1}^{k_r} \theta_r \tilde{N}_{r, s}(y) \right) \right), \]
and where we set \( \tilde{k}'_j \) for the vector of \( \mathbb{Z}^m \) with \( j \)-th coordinate equal to \( k'_j \), all the other coordinates being null. Let \( \mathcal{J}'_0'' \) be the set of \( j \in \mathcal{J}'' \) such that \( S''_j \cap \bigcup_{j' \in \mathcal{J}'' \setminus (j)} S'_{j'} = \emptyset \). Then
\[ \sum_{j' \in \mathcal{J}'_0''} \left( \prod_{j \in \mathcal{J}''} H_{j, k_j} \left( 0 \right) \right) \Delta_{j'} \Delta_j \Psi_{S''_j}(k'_j) = \prod_{j \in \mathcal{J}''} \left( \sum_{k_j' = 0}^{d-1} H_{j, k_j} \left( 0 \right) \Delta_j \Psi_{S''_j}(k'_j) \right) \Delta_{\mathcal{J}'' \setminus \mathcal{J}'_0''} \Psi(k'_0), \]
with \( k'_0' \in \mathbb{N}^m \) such that \( (k'_j)_i = k'_i \) for \( i \not\in \mathcal{J}'_0'' \), the other coordinates being null, the notation \( \Delta, \Delta_{\mathcal{J}'' \setminus \mathcal{J}'_0''} \) standing for the composition of all the operators \( \Delta_j \) for \( j \in \mathcal{J}'' \setminus \mathcal{J}'_0'' \). We conclude by using (61) and by noticing that
\[ \left( \prod_{j' \in \mathcal{J}'' \setminus \mathcal{J}'_0''} \left( H_{j', k'_j} \theta_{j'} - \theta_{j'+1} \right) - H_{j', k'_j} \left( 0 \right) \right) = O \left( \prod_{j' \in \mathcal{J}'' \setminus \mathcal{J}'_0''} \left( |\theta_{j'}| + |\theta_{j'+1}| \right) \right), \]
\[ \prod_{j' \in \mathcal{J}'' \setminus \mathcal{J}'_0''} \left( \sum_{k_j' = 0}^{d-1} H_{j', k_j} \left( 0 \right) \Delta_j \Psi_{S''_j}(k'_j) \right) = O \left( \prod_{j' \in \mathcal{J}'' \setminus \mathcal{J}'_0''} \left( |\theta_{j'}| + |\theta_{j'+1}| \right) \right) \]
and that
\[ j \in \mathcal{J}'' \setminus \mathcal{J}'_0'' \implies S_j'' \cap \bigcup_{j' \in \mathcal{J}'' \setminus (j)} S'_{j'} \neq \emptyset. \]

The following lemma will be useful to estimate the term \( F \) appearing in Lemma 25. It is not needed when \( \sum_{a \in \mathbb{Z}} f(b + ad) = 0 \) for all \( b \in \mathbb{Z} \).

**Lemma 26.** For any \( \mathcal{J}' \subset \mathcal{J} \),
\[ \mathbb{P} \left( \Omega_k \cap \bigcap_{j \in \mathcal{J}'} B_j \right) = O \left( \prod_{j \in \mathcal{J}'' \setminus \mathcal{J}' \setminus \mathcal{J}_k''} \sum_{j' \in \mathcal{J}''} n^{\gamma} \prod_{j \in \mathcal{J}''} \left( k_j - k_{j'} \right)^{-\frac{3}{2}} \right), \]
where \( k_j^- = \max\{k_s \leq k_j, s \in \mathcal{J}' \} \) and with the notation \( B_j \) introduced in Lemma 25.
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Proof. It is enough to study

$$\Pr \left( \Omega_k \cap \bigcap_{j \in J'} \left\{ S_{k_j + r_j} = S_{k_{m(j)} + s_j} \right\} \right)$$

for any $m(j) \in J' \setminus \{ j \}$, $r_j, s_j \in \{ 0, \ldots, d-1 \}$. This probability is dominated by

$$\Pr \left( \Omega_k \cap \left\{ \forall j \in J', \ |S_{k_j} - S_{k_{m(j)}}| \leq n^p \right\} \right) + \mathcal{O}(n^{-p}),$$

for all $p, v > 0$. We partition the set $J'$ by the equivalence relation generated by the relation $j \sim m(j)$. We write $R(j)$ for the class of $j$ and $R$ for the set of these equivalence classes. Observe that the number of equivalent classes is at most \#\([J'/2]\). We order the set $J'$ in $j'_1 < \cdots < j'_p$. We wish to estimate

$$\sum_{A_r \in R} \Pr \left( \Omega_k, \forall i = 1, \ldots, J' - 1, S_{k_{j'_i+1} - k_{j'_i}} = A_{R(j'_i)} - A_{R(j'_i)} + \mathcal{O}(n^v) \right),$$

where the sum is over $(A_r)_{r \in R} \in \mathbb{Z}^R$ such that $A_{R(1)} = 0, A_{R(j'_i+1)} - A_{R(j'_i)} = \mathcal{O}((k_{j'_i+1} - k_{j'_i})^{1+\frac{1}{2}})$. Due to the local limit theorem and the independence of the increments of $S$, the above probability is in

$$\sum_{A_r \in R} \prod_{i=1}^{J'-1} n^v \left( (k_{j'_i+1} - k_{j'_i})^{1-\frac{1}{2}} \right).$$

Now let us control the cardinal of the admissible $(A_r, r \in R)$. To this end, consider the set $J''$ of the smallest representants of $R$. Then the above quantity is smaller than

$$n^{J'(1+\frac{1}{2})} \prod_{j \in J' \setminus J''} (k_j - k_{j'1})^{-\frac{1}{2}}. \tag{62}$$

Proof of Lemma 21. All the estimates below are uniformly in $k$. For the first estimate, we have to estimate the following integral

$$\int_{\|\theta_j\| < \eta \frac{1}{2}} \prod_{j \in J'} \left( \sum_{g_j \in G} f(a_j) e^i a_j \theta_j \prod_{s=1}^{s_j} \sum_{b_j,s \in \mathbb{Z}} f(b_j,s) e^{-i(b_j,s-a_j)\theta_j} \right) \times \mathbb{E} \left[ \mathbf{1}_{\Omega_k} F(\theta, \theta') \prod_{y \in \mathbb{Z} \setminus S'} \mathcal{A}_y \right] d\theta,$$

where we set

$$\mathcal{A}_y := \varphi_\xi \left( \sum_{j=1}^m \left( \theta_j N_j(y) + \sum_{s=1}^{s_j} \theta_j N_j^s(y) \right) \right).$$

Let us study

$$E_{k,\varepsilon}(\theta, \theta') := \prod_{y \in \mathbb{Z} \setminus S'} \mathcal{A}_y - \prod_{y \in \mathbb{Z} \setminus S'} \mathcal{B}_y,$$

with

$$\mathcal{B}_y := \exp \left( \frac{\sigma_\xi^2}{2} \left( \sum_{j=1}^m \theta_j N_j(y) \right)^2 \right) \varphi_\xi \left( \sum_{j=1}^m \sum_{s=1}^{s_j} \theta_j N_j^s(y) \right).$$
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But, on $\Omega_k$, we have $|\theta_j| \leq n_j^{-\frac{1-\eta}{2}}$ for all $j = 1, \ldots, m$, and so

$$\forall y \in \mathbb{Z}, \quad \sum_{j=1}^{m} \theta_j N_j'(y) \leq \sum_{j=1}^{m} |\theta_j| N_j' \leq \sum_{j=1}^{m} n_j^{-\frac{\eta}{2}} \leq \frac{m n^{-\frac{\eta}{2}}}{2} < \varepsilon_0,$$

as soon as $n$ is large enough (uniformly on $n_j \in [n^0, n]$). Thus $|E_{k,\ell}(\theta, \theta')|$ is dominated by

$$\sum_{y \in \mathbb{Z}} |\mathcal{A}_y - \mathcal{B}_y| e^{-\frac{\sigma_x^2}{2} \sum_{z \in \mathcal{F}(\mathcal{S} \cup \{y\})} (\sum_{j=1}^{m} \theta_j N_j'(z))^2}$$

for $n$ large enough. Now, on $\Omega_k$, according to (41),

$$\forall y \in \mathbb{Z}, \quad \sum_{z \in \mathcal{F}(\mathcal{S} \cup \{y\})} \left( \sum_{j=1}^{m} \theta_j N_j'(z') \right)^2 \geq \sum_{z' \in \mathbb{Z}} \left( \sum_{j=1}^{m} \theta_j N_j'(z') \right)^2 - M(d + n \frac{n^{2\theta}}{m^2}) n^{-\theta_0}.$$ 

It follows that

$$|E_{k,\ell}(\theta, \theta')| \leq (A + B) \exp \left( -\frac{\sigma_x^2}{4} \sum_{z' \in \mathbb{Z}} \left( \sum_{j=1}^{m} \theta_j N_j'(z') \right)^2 - O(n^{-\frac{2\theta}{m^2}}) \right), \quad (63)$$

with

$$A := \sum_{y \in \mathbb{Z} \setminus \mathcal{S}} \left| \varphi_{\xi} \left( \sum_{j=1}^{m} \theta_j N_j'(y) \right) - e^{-\frac{\sigma_x^2}{2} (\sum_{j=1}^{m} \theta_j N_j'(y))^2} \right| \leq \sum_{y \in \mathbb{Z}} \sum_{j=1}^{m} \theta_j N_j'(y) C' n^{-\frac{2\theta}{m^2}} \quad (64)$$

where we used the fact that

$$\left| \varphi_{\xi}(u) - \exp \left( -\frac{\sigma_x^2 |u|^2}{2} \right) \right| \leq |u|^{2+\kappa} \quad \text{for all } u \in \mathbb{R},$$

since $\xi$ admits a moment of order $2 + \kappa$ and there exists $C_0 > 0$ such that

$$B := \sum_{y \in \mathbb{Z} \setminus \mathcal{S}} \left| \varphi_{\xi} \left( \sum_{j=1}^{m} \left( \theta_j N_j'(y) + \sum_{s=1}^{s_j} \theta_{j,s} N_{j,s}'(y) \right) \right) \right| e^{-\frac{\sigma_x^2}{2} (\sum_{j=1}^{m} \theta_j N_j'(y))^2} \varphi_{\xi} \left( \sum_{j=1}^{m} \sum_{s=1}^{s_j} \theta_{j,s} N_{j,s}'(y) \right) \leq C_0 \sum_{y \in \mathbb{Z} \setminus \mathcal{S}} \sum_{j=1}^{m} \theta_j N_j'(y) \leq C_0 m \sum_{j=1}^{m} \sum_{s=1}^{s_j} \ell_{j,s} n^{-\frac{\eta}{2}} = O \left( n^{-\frac{2\eta}{m^2}} \right) = O \left( n^{-\frac{\eta}{2}} \right), \quad (65)$$

since $\varphi_{\xi}$ and $u \mapsto e^{-\frac{\sigma_x^2}{2} u^2}$ are Lipschitz continuous. Recall that it has been proved in [14, Lemma 21] that

$$E \left[ \left| \det D_k \right|^{-\frac{1}{2}} \mathbb{I}_{\Omega_k} \right] = O \left( \prod_{j=1}^{m} n_j^{-\frac{1}{2}} \right), \quad (66)$$

uniformly on $k$. 

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Combining Lemmas 25 and 26 with (63), (64), (65), (66) and using the change of variable $\nu = (D_k)^{\frac{1}{2}} \theta$ with $D_k = \left( \sum_{y \in \mathbb{Z}} N_y(y) N'_y(y) \right)_{i,j}$, it follows that there exists $C_1 > 0$ such that

$$
\int_{\nu, |\theta| \leq n_j^{-\frac{1}{2}-\eta}} E \left[ |F(\theta, \theta') E_{k, \xi_i}(\theta, \theta')| \right] \Omega_k \, d\theta, d\theta',
$$

$$
\leq C_1 \int_{\mathbb{R}^2} \left( n^{-\frac{n}{2}} |w|^{2} + O(n^{-\frac{n}{2}}) \right) e^{-\frac{\sigma_0^2 |w|^2}{2}} \, dv
$$

$$
\sum_{J_{\nu} \subset J^{\prime}} \prod_{j \in J \setminus J_{\nu}} \left( n_j^{\frac{1}{2}-\eta} + n_{j+1}^{\frac{1}{2}-\eta} \right) E \left[ |\det D_k|^{-\frac{1}{2}} \chi_{\Omega_k \cap \bigcap_{j} S_j} \right]
$$

$$
= O \left( n^{-\frac{n}{2}} \prod_{j=1}^{m} n_j^{-\frac{1}{2}} \mathcal{E}_k(J^{\prime}) \right),
$$

with

$$
\mathcal{E}_k(J^{\prime}) = \sum_{J'' \subset J^{\prime}} \prod_{j \in J^{\prime} \setminus J''} \left( n_j^{\frac{1}{2}-\eta} + n_{j+1}^{\frac{1}{2}-\eta} \right) \sum_{J_0 \subset J^{\prime} \setminus \min J''} \eta_j^{\frac{1}{2}+\eta} \left( \prod_{j \in J_0} (k_j - k_j^-)^{-\frac{1}{2}} \right)
$$

$$
= O \left( \sum_{J'' \subset J^{\prime} \cup \{ J^{\prime} + 1 \} : \#J'' \geq \#J^{\prime}/2} \prod_{j \in J''} n_j^{-\frac{1}{2}+\eta} \right).
$$

where $k_j^- = \max \{ k_s \leq k_j, s \in J'' \}$. Combining this last estimate with (62) and Lemmas 25 and 26,

$$
\sum_{k' = 0, \ldots, d-1, \forall j \in J^{\prime}} B_{k + k', k, t_k^{(1)}, \Omega_k}
$$

$$
= \frac{d^m}{(2\pi)^M} \sum_{(a)_{j} \in J^{\prime}, (a_{i,j})_{i,s}} \mathbb{1}_{\{ a \in k, a + d \in \mathbb{Z} \}} \int_{[-\pi, \pi]^{M-1}} E \left[ I_1(a) I_2(a, b) \Omega_k \right] \, d\theta'
$$

$$
= O \left( n^{-\frac{n}{2}} \prod_{j=1}^{m} n_j^{-\frac{1}{2}} \mathcal{E}_k(J^{\prime}) \right),
$$

with

$$
I_1(a) := \int_{\nu, |\theta| \leq n_j^{-\frac{1}{2}-\eta}} \left( \prod_{j \in J^{\prime}} e^{-i \sum_{j=1}^{m} (a_{j} - a_{j-1}) \theta_j} \right) F(\theta, \theta') e^{-\frac{\sigma_0^2}{2} \sum_{y \in \mathbb{Z}, s} \left( \sum_{j=1}^{m} \theta_j N'(y) \right)^2} \, d\theta
$$

$$
= O \left( \int_{\nu, |\theta| \leq n_j^{-\frac{1}{2}-\eta}} F(\theta, \theta') e^{-\frac{\sigma_0^2}{2} \sum_{y \in \mathbb{Z}, s} \left( \sum_{j=1}^{m} \theta_j N'(y) \right)^2} \, d\theta \right)
$$

$$
= O \left( \det D_k^{-\frac{1}{2}} \sup_{\theta \in \mathbb{V}_a} F(\theta, \theta') \int_{\mathbb{R}^2} e^{-\frac{\sigma_0^2 |w|^2}{2}} \, dv \right)
$$

(67)

with the change of variable $\nu = D_k^{\frac{1}{2}} \theta$ and

$$
I_2(a, b) := \left( \prod_{j \in J^{\prime}} \left( f(a_{j}) \prod_{s=1}^{s_j} f(b_{j,s}) e^{-i \sum_{j,s} (b_{j,s} - a_{j,s}) \theta_j} \right) \right) \prod_{y \in \mathbb{Z} \setminus S'} \varphi \xi \left( \sum_{j,s} (\theta'_{j,s} N'_{j,s}(y)) \right)
$$

$$
= O \left( \prod_{j \in J^{\prime}} \left( f(a_{j}) \prod_{s=1}^{s_j} f(b_{j,s}) \right) \right).
$$

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Since \( \sum_{a \in \mathbb{Z}} |I(a)| < \infty \), it follows from (66), (67), (68) and (69) that

\[
\sum_{k'=0, \ldots, d-1, \forall j \in \mathcal{J}'} B_{k+k', \ell, f_k, \Omega_k} = \mathcal{O} \left( \mathcal{E}_k(\mathcal{J}') \left( \prod_{j=1}^m n_j^{-2} \right) \right).
\]

This ends the proof of the first point of Lemma 21.

Assume now that \( s_j = 1 \) for all \( j = 1, \ldots, m \) (in particular \( \mathcal{J} = \emptyset \)). Then

\[
I_1(a) = \int_{\mathbb{R}_+ \times \{0\}} e^{-i \sum_{j=1}^m (a_j - a_{j-1}) \theta_j} e^{-\frac{\sigma^2}{2} \sum_{y \in \mathbb{Z}} \sigma_j N_j(y)^2} d\theta
\]

\[
= \left( \prod_{j=1}^m n_j^{\frac{1}{2}} \right) \int_{\mathbb{R}_+ \times \{0\}} e^{-i \sum_{j=1}^m n_j^{-\frac{1}{2}} (a_j - a_{j-1}) \theta_j} e^{-\frac{\sigma^2}{2} \sum_{y \in \mathbb{Z}} \sigma_j n_j^{-\frac{1}{2}} N_j(y)^2} d\theta
\]

\[
= \left( \prod_{j=1}^m n_j^{\frac{1}{2}} \right) \int_{\mathbb{R}_+ \times \{0\}} e^{-i \sum_{j=1}^m n_j^{-\frac{1}{2}} (a_j - a_{j-1}) \theta_j} e^{-\frac{\sigma^2}{2} \sum_{y \in \mathbb{Z}} \sigma_j n_j^{-\frac{1}{2}} N_j(y)^2} d\theta
\]

where \( U_k \) is the set of \( \theta'' = (\theta_1'', \ldots, \theta_m'') \) such that \( |\theta_j''| \leq n_j^{\frac{1}{2} - \eta} \) for all \( j = 1, \ldots, m \) and with \( \bar{D}_k = \left( (n_j, n_j)^{-\frac{1}{2}} \sum_{y \in \mathbb{Z}} N'_j(y) N_j(y) \right) \). Moreover

\[
I_2(a, b)
\]

\[
= (2\pi)^{m+1} \left( \prod_{j=1}^m \left( f(a_j) f(b_j, 1) \right) \right) P \left( \forall j, \sum_{y \in \mathbb{Z}} N'_j(y) \xi_y = b_{j, 1} - a_j \left( N'_j, 1 \right) \right)
\]

\[
= (2\pi)^{M-m} \left( \prod_{j=1}^m f(a_j) \right) E \left[ f \left( a_j + \sum_{y \in \mathbb{Z}} N'_j(y) \xi_y \right) I \left( \sum_{y \in \mathbb{Z}} N'_j(y) \xi_y = b_{j, 1} \right) \left( N'_j, 1 \right) \right]
\]

Thus, it follows that, uniformly in \( k \) and on \( \Omega_k \),

\[
\frac{d^n}{(2\pi)^M} \sum_{b_1, \ldots, b_m, a \in \mathbb{Z}} I_1(a) I_2(a, b) = \left( \frac{d}{2\pi} \right)^n \left( \prod_{j=1}^m f(a_j) \right) \left( \det D_k \right)^{-\frac{1}{2}} \int_{\mathbb{R}_+} e^{-i \sum_{j=1}^m \frac{\sigma^2}{2} (a_j - a_{j-1})} e^{-\frac{\sigma^2}{2} n_j^{\frac{1}{2} - \eta}} d\theta + \mathcal{O}(n^{-p})
\]

\[
E \left[ f \left( a_j + \sum_{y \in \mathbb{Z}} N'_j(y) \xi_y \right) \left( N'_j, 1 \right) \right]
\]

for all \( p > 0 \), as seen at the end of the proof of Lemma 20 (applied with \( \bar{D}_k \)) and so

\[
\frac{d^n}{(2\pi)^M} \sum_{b_1, \ldots, b_m, a \in \mathbb{Z}} I_1(a) I_2(a, b) = \left( \frac{d}{2\pi} \right)^n \left( \prod_{j=1}^m f(a_j) \right) \left( \det D_k \right)^{-\frac{1}{2}}
\]

\[
\times \left( \int_{\mathbb{R}_+} 1 + \mathcal{O} \left( \left( \left( \sum_{j=1}^m n_j^{-\frac{1}{2}} (a_j - a_{j-1}) \right)^2 \right) \right) \right) e^{-\frac{\sigma^2}{2} n_j^{\frac{1}{2} - \eta}} d\theta + \mathcal{O}(n^{-p})
\]

\[
\times E \left[ f \left( a_j + \sum_{y \in \mathbb{Z}} N'_j(y) \xi_y \right) \left( N'_j, 1 \right) \right]
\]
for all $p$. Due to (67), we obtain that

$$B_{k,t,k}^{(3)},\Omega_k = \left(\frac{d}{2\pi}\right)^m \sum_{a_1,\cdots,a_m \in Z} 1_{\{\forall i, a_i = k, \alpha + dZ\}} \times \mathbb{E} \left[ (\det D_k)^{-\frac{1}{2}} \mathbb{1}_{\Omega_k} \prod_{j=1}^m f(a_j) f \left( a_j + \sum_{y \in Z} N'_j, (y) \xi_y \right) \right] \left( \frac{\sqrt{2\pi}}{\sigma_\xi} \right)^m + O \left( \prod_{j=1}^m n_j^{-\frac{1}{2}} \mathbb{E} \left[ (\det D_k)^{-\frac{1}{2}} \mathbb{1}_{\Omega_k} \right] \right),$$

(70)

where $\tilde{\lambda}_k$ is the smallest eigenvalue of $\tilde{D}_k$. For the last term, we use (58) (applied for $\tilde{D}_k$), which ensures that on $\Omega_k$,

$$\tilde{\lambda}_k \geq \frac{\det \tilde{D}_k}{(mn^3)^{m-1}}$$

and so

$$(\min_j n_j)^{-\frac{1}{2}} \mathbb{E} \left[ (\det D_k)^{-\frac{1}{2}} \mathbb{1}_{\Omega_k} \right] \leq (mn^3)^{m-1} (\min_j n_j)^{-\frac{1}{2}} \left( \prod_{j=1}^m n_j^{-\frac{1}{2}} \right) \mathbb{E} \left[ (\det \tilde{D}_k)^{-\frac{1}{2}} \mathbb{1}_{\Omega_k} \right] = O \left( \prod_{j=1}^m n_j^{-\frac{1}{2}} \right)^{n-\frac{3}{2}+3(m-1)\gamma},$$

(73)

where we used [14, Lemma 21] which ensures that $\mathbb{E} \left[ (\det \tilde{D}_k)^{-\frac{1}{2}} \mathbb{1}_{\Omega_k} \right] = O(1)$ uniformly in $k$. This combined with (72) implies that

$$B_{k,t,k}^{(3)},\Omega_k = O \left( \prod_{j=1}^m n_j^{-\frac{1}{2}} \right)^{n-(M+1)\theta} + \left( \frac{d}{\sqrt{2\pi}\sigma_\xi n^2} \right)^m \sum_{a_1,\cdots,a_m \in Z} 1_{\{\forall i, a_i = k, \alpha + dZ\}} \times \mathbb{E} \left[ (\det D_k)^{-\frac{1}{2}} \mathbb{1}_{\Omega_k} \prod_{j=1}^m f(a_j) f \left( a_j + \sum_{y \in Z} N'_j, (y) \xi_y \right) \right],$$

(74)

since $L < \min \left( \frac{3n}{4M}, \frac{m}{n^2} \right)$ and since $L(M+1)\theta < \frac{3\theta}{2} - 3(m-1)\gamma$. The last step of the proof of the lemma consists in studying the following quantity

$$G_k := \mathbb{E} \left[ (\det D_k)^{-\frac{1}{2}} \mathbb{1}_{\Omega_k} \prod_{j=1}^m f(a_j) f \left( a_j + \sum_{y \in Z} N'_j, (y) \xi_y \right) \right].$$

Due to Lemma 24,

$$G_k = \mathbb{E} \left[ (\det D_k)^{-\frac{1}{2}} \mathbb{1}_{\Omega_k} \prod_{j=1}^m f(a_j) f \left( a_j + \sum_{y \in Z} N'_j, (y) \xi_y \right) \right] + O \left( n^{-\frac{\theta}{4} - L\theta} \prod_{j=1}^m n_j^{-\frac{1}{2}} \right)$$

$$= \mathbb{E} \left[ (\det D_k)^{-\frac{1}{2}} \mathbb{1}_{\Omega_k} \prod_{j=1}^m f(a_j) \mathbb{E} \left[ f \left( a_j + \sum_{y \in Z} N'_j, (y) \xi_y \right) \right] \right] + O \left( n^{-\frac{\theta}{4} - L\theta} \prod_{j=1}^m n_j^{-\frac{1}{2}} \right),$$

(71)
where we used the fact that $D_k'$ has the same distribution as $D_k$ and is independent of $N_{j,i}$. This combined with (74), (73), (59) and (60) ensures that

$$B_{k,l,f_k^{(i)},\Omega_k} = \left(\frac{d}{\sqrt{2\pi \sigma_k}}\right)^m \sum_{a_1,\ldots,a_m \in \mathbb{Z}} 1_{\{\forall i, a_i = k_i + dZ_i\}} E \left[ (\det D_k)^{-\frac{1}{2}} \mathbb{I}_{\Omega_k} \right] \prod_{j=1}^m f(a_j) E \left[ f(a_j + Z_{t_j}) \right] + O \left( n^{-L(M+1)\theta} \prod_{j=1}^m n_j^{-\frac{1}{2}} \right).$$

Moreover [14, Lemmas 21 and 23] ensure that

$$E \left[ (\det D_k)^{-\frac{1}{2}} \mathbb{I}_{\Omega_k} \right] = O \left( \prod_{j=1}^m n_j^{-\frac{1}{2}} \right),$$

and that

$$E \left[ (\det D_k)^{-\frac{1}{2}} \mathbb{I}_{\Omega_k} \right] \sim n^{-\frac{L}{2}} E \left[ \det D_{i_1,\ldots,i_m} \right]$$

as $k_j/n \to t_j$ and $n \to +\infty$. This ends the proof of the lemma. 

\[\square\]

**B Moment convergence in Theorem 3**

Let $f : \mathbb{Z} \to \mathbb{R}$ be such that $\sum_{a \in \mathbb{Z}} |f(a)| < \infty$. In this appendix we prove that all the moments of $n^{-\frac{1}{2}} \sum_{a=0}^{n-1} f(Z_k)$ converge to those of $\sum_{a=0}^{\infty} f(a)\sigma_k^{-1} L_1(0)$, as $n \to +\infty$.

Due to Theorem 1, it is enough to prove the convergence of every moment. The key result is the following proposition.

**Proposition 27** (Multi-time local limit theorem for the RWRS $Z$). For all $a_1, \ldots, a_k \in \mathbb{Z}$,

$$P \left( Z_{n_1} = a_1, \ldots, Z_{n_k} = a_k \right) \sim E \left[ 1_{\{\forall i, a_i = n_i + dZ_i\}} \left( \frac{d}{\sqrt{2\pi \sigma_k}} \right)^k \right] E[\det D_{i_1,\ldots,i_k}^{-\frac{1}{2}}] n^{-3k/4},$$

as $n \to +\infty$ and $n_i/n \to t_i$, where $D_{i_1,\ldots,i_k} = (\int_{\mathbb{R}} L_{i_j}(x) L_{i_j}(x) dx)_{i,j=1,\ldots,k}$ where $L$ is the local time of the brownian motion $B$, limit of $(S_{nt}/\sqrt{n})$, as $n$ goes to infinity.

Moreover, for every $k \geq 1$ and every $\theta \in (0, 1)$, there exists $C = C(k, \theta) > 0$, such that

$$P \left[ Z_{n_1} = a_1, \ldots, Z_{n_1+\ldots+n_k} = a_k \right] \leq C \prod_{j=1}^k n_j^{-3/4},$$

for all $n \geq 1$, all $a_1, \ldots, a_k \in \mathbb{Z}$ and all $n_1, \ldots, n_k \in [n^\theta, n]$.

**Proof.** The lemma has been proved for $a_i \equiv 0$ in [14, Theorem 5]. The proof in the general case is the straightforward adaptation of [14, Section 5]. For completeness, we explain the required adaptations. The proof of the present result follows by line by line the same proof with the adjonction of a term $e^{-i \sum_{j=0}^k (a_j - a_{j-1})\theta_j}$ (with convention $a_0 = 0$) in the integrals appearing in [14, Lemma 15] (see Lemma 17 with $M = m = k$ and $s_j \equiv 0$). Lemma 16 (definition of the good set) and Propositions 18 and 19 (estimates of the integral of the absolute values) of [14] are unchanged. The only difference in the proof concerns [14, Proposition 17] and more specifically [14, Lemma 23] for which there is a multiplication by $e^{-i \sum_{j=1}^k (a_j - a_{j-1})\theta_j}$ in the integral. The only difference in the proof of [14, Lemma 23] is that the quantity $I_{n_1,\ldots,n_k}$ considered therein ($n_i$ corresponding to $[nT_i] - [nT_{i-1}]$) is slightly modified with the multiplication in the integral by a quantity converging in probability to 1 (with the notations of the proof...
Let $\alpha$ be such that $\alpha \alpha_0 \in 1 + dZ$. Then $a_i = q_i \alpha + dZ$ is equivalent to $q_i \in a_i \alpha_0 + dZ$. Thus

$$
E \left[ \left( \sum_{q=0}^{n-1} f(Z_q) \right)^k \right] = \sum_{a_1, \ldots, a_k \in \mathbb{Z}} f(a_1) \cdots f(a_k) \sum_{q_1, \ldots, q_k=0}^{n-1} P(Z_{q_1} = a_1, \ldots, Z_{q_k} = a_k) 
$$

$$
= O(n^{k-1}) + \sum_{r_1, \ldots, r_d=0}^{d-1} \sum_{a_1, \ldots, a_k \in \mathbb{Z}} f(a_1) \cdots f(a_k) \sum_{q_1, \ldots, q_k=0}^{n-1} P(Z_{r_1+a_1+q_1 d} = a_1, \ldots, Z_{r_k+a_k+q_k d} = a_k)
$$

$$
= O(n^{k-1}) + \sum_{a_1, \ldots, a_k \in \mathbb{Z}} f(a_1) \cdots f(a_k) \sum_{q_1, \ldots, q_k=0}^{n-1} P(Z_{a_1+q_1 d} = a_1, \ldots, Z_{a_k+q_k d} = a_k),
$$

with $\pi$ the representant of $x + dZ$ belonging to $\{0, \ldots, d-1\}$. It follows that

$$
E \left[ \left( \sum_{q=0}^{n-1} f(Z_q) \right)^k \right] = o(n^k) + \sum_{a_1, \ldots, a_k \in \mathbb{Z}} f(a_1) \cdots f(a_k) n^k H_k
$$

$$
= o(n^k) + \sum_{a_1, \ldots, a_k \in \mathbb{Z}} f(a_1) \cdots f(a_k) n^k H_k',
$$

with

$$
H_k := \int_{[0,1/d]^k} P\left( Z_{a_1+\lfloor t_1 n \rfloor d} = a_1, \ldots, Z_{a_k+\lfloor t_k n \rfloor d} = a_k \right) dt_1 \cdots dt_k
$$

$$
H_k' = \int_{[0,1/d]^k} n^{\frac{k}{2}} P\left( Z_{a_1+\lfloor t_1 n \rfloor d} = a_1, \ldots, Z_{a_k+\lfloor t_k n \rfloor d} = a_k \right)
\times \mathbb{1}_{\min_{i,j} \lfloor |t_i n| - |t_j n| \rfloor > 2n^a} dt_1 \cdots dt_k.
$$

Proof of the convergence of moments in Theorem 3. Take $\vartheta < \frac{1}{2}$. Note that the last point of the lemma ensures that

$$
P \left[ Z_{n_1} = a_1, \ldots, Z_{n_1 + \cdots + n_k} = a_k \right] \leq C \left( \prod_{i : n_i > n^a} n_i \right)^{-3/4}.
$$

Let $\alpha_0$ be such that $\alpha \alpha_0 \in 1 + dZ$. Then $a_i = q_i \alpha + dZ$ is equivalent to $q_i \in a_i \alpha_0 + dZ$. Thus

$$
E \left[ \left( \sum_{q=0}^{n-1} f(Z_q) \right)^k \right] = \sum_{a_1, \ldots, a_k \in \mathbb{Z}} f(a_1) \cdots f(a_k) \sum_{q_1, \ldots, q_k=0}^{n-1} P(Z_{q_1} = a_1, \ldots, Z_{q_k} = a_k)
$$

$$
= O(n^{k-1}) + \sum_{r_1, \ldots, r_d=0}^{d-1} \sum_{a_1, \ldots, a_k \in \mathbb{Z}} f(a_1) \cdots f(a_k) \sum_{q_1, \ldots, q_k=0}^{n-1} P(Z_{r_1+q_1 d} = a_1, \ldots, Z_{r_k+q_k d} = a_k)
$$

$$
= O(n^{k-1}) + \sum_{a_1, \ldots, a_k \in \mathbb{Z}} f(a_1) \cdots f(a_k) \sum_{q_1, \ldots, q_k=0}^{n-1} P(Z_{a_1+q_1 d} = a_1, \ldots, Z_{a_k+q_k d} = a_k),
$$

with $\pi$ the representant of $x + dZ$ belonging to $\{0, \ldots, d-1\}$. It follows that

$$
E \left[ \left( \sum_{q=0}^{n-1} f(Z_q) \right)^k \right] = o(n^k) + \sum_{a_1, \ldots, a_k \in \mathbb{Z}} f(a_1) \cdots f(a_k) n^k H_k
$$

$$
= o(n^k) + \sum_{a_1, \ldots, a_k \in \mathbb{Z}} f(a_1) \cdots f(a_k) n^k H_k',
$$

with

$$
H_k := \int_{[0,1/d]^k} P\left( Z_{a_1+\lfloor t_1 n \rfloor d} = a_1, \ldots, Z_{a_k+\lfloor t_k n \rfloor d} = a_k \right) dt_1 \cdots dt_k
$$

$$
H_k' = \int_{[0,1/d]^k} n^{\frac{k}{2}} P\left( Z_{a_1+\lfloor t_1 n \rfloor d} = a_1, \ldots, Z_{a_k+\lfloor t_k n \rfloor d} = a_k \right)
\times \mathbb{1}_{\min_{i,j} \lfloor |t_i n| - |t_j n| \rfloor > 2n^a} dt_1 \cdots dt_k.
$$
Due to the dominated convergence theorem, we conclude that

\[
E \left[ (n-1) \sum_{q=0}^{n-1} f(Z_q) \right] = o(n^{\frac{k}{2}}) + n^{\frac{k}{2}} \sum_{a_1, \ldots, a_k \in \mathbb{Z}} f(a_1) \cdots f(a_k) \int_{[0,1]^k} \left( \frac{d}{\sqrt{2\pi \sigma^2}} \int_{[0,1]^k} \text{det} \begin{pmatrix} \frac{2}{\sqrt{2\pi \sigma^2}} \end{pmatrix} \right)^{-k} \text{det} \begin{pmatrix} D_{1d, \ldots, t_k d} \end{pmatrix} dt_1 \cdots dt_k
\]

due to \([14, \text{Theorem 3}].\)

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\begin{thebibliography}{99}

[1] Aaronson, J.: An introduction to infinite ergodic theory. Mathematical Surveys and Monographs, AMS 50, Providence, RI, 1997. xii+284 pp. MR1450400

[2] Aurzada, F., Guillotin-Plantard, F. and Pène, F.: Persistence probabilities for stationary increment processes. Stochastic Process. Appl. 128 (2018), no. 5, 1750–1771. MR3780696

[3] Berger, N. and Peres, Y.: Detecting the trail of a random walker in a random scenery. Electron. J. Probab. 18, (2013), no. 87, 18 pp. MR3119085

[4] Billingsley, P.: Probability and measure, third edition. Wiley Series in Probability and Mathematical Statistics. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1995. xiv+593 pp. MR1324786

[5] Blachère, S., den Hollander, F. and Steif, J. E.: A crossover for the bad configurations of random walk in random scenery. Ann. Probab. 39, (2011), no. 5, 2018–2041. MR2884880

[6] Bolthausen, E.: A central limit theorem for two-dimensional random walks in random sceneries. Ann. Probab. 17, (1989), no. 1, 108–115. MR0972774

[7] Borodin, A. N.: A limit theorem for sums of independent random variables defined on a recurrent random walk. (Russian) Dokl. Akad. Nauk SSSR 246, (1979), no. 4, 786–787. MR0543530

[8] Borodin, A. N.: On the character of convergence to Brownian local time. II. Probab. Theory Relat. Fields 72, (1986), 251–277. MR0836277

[9] Brémond J.: Planar random walk in a stratified quasi-periodic environment. Markov Processes and Related Fields. To appear. 29 pp. MR2502473

[10] Brémond J.: On planar random walks in environments invariant by horizontal translations. Markov Processes and Related Fields 22, (2016), no 2, 267–310. MR3561139

[11] Campanino M. and Petritis D.: Random walks on randomly oriented lattices. Mark. Proc. Relat. Fields 9, (2003), 391–412. MR2028220

[12] Castell, F., Guillotin-Plantard, N. and Pène, F.: Limit theorems for one and two-dimensional random walks in random scenery. Ann I.H.P. (B) Probabilités et Statistiques 49, (2013), no 2, 506–528. MR3088379

[13] Castell, F., Guillotin-Plantard, N., Pène, F. and Schapira, Br.: A local limit theorem for random walks in random scenery and on randomly oriented lattices. Ann. Probab. 39, (2011), 2079–2118. MR2932665

[14] Castell, F., Guillotin-Plantard, N., Pène, F. and Schapira, Br.: On the local time of random processes in random scenery. Ann. Probab. 42, (2014), no 6, 2417–2453. MR3265171

EJP 26 (2021), paper 128.

https://www.imstat.org/ejp

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Limit theorems for additive functionals of random walks in random scenery

[15] Chen, X.: Random Walk Intersections: Large Deviations and Related Topics. *Mathematical Surveys and Monographs, AMS, 157*, Providence, RI (2009). MR2584458

[16] Chen, X., Li, W. V., Rosiński, J. and Shao, Q.-M.: Large deviations for local times and intersection local times of fractional Brownian motions and Riemann-Liouville processes. *Ann. Probab. 39*, (2011), 729–778. MR2789511

[17] Csáki, E. and Földes, A.: On asymptotic independence and partial sums. *Asymptotic methods in probability and statistics, A volume in honour of Miklós Csörgő*, Elsevier, (2000), 373–381. MR1661493

[18] Csáki, E. and Földes, A.: Asymptotic independence and additive functionals. *Journal of Theoretical Probability, 13*, (2000), 1123–1144. MR1820506

[19] Deligiannidis, G. and Utev, S.: Asymptotic variance of the self-intersections of stable random walks using Darboux-Wiener theory, *Sib Math J. 52*, 639 (2011), 14 p. MR2883216

[20] Devulder, A. and Pène, F.: Random walk in random environment in a two-dimensional stratified medium with orientations. *Electron. J. Probab. 18*, (2013), no. 88, 1–23 MR3035746

[21] Dobrušin, R. L. Two limit theorems for the simplest random walk on a line. (Russian) *Uspehi Mat. Nauk (N.S.) 10*, (1955), no. 3(65), 139–146. MR0071662

[22] Dombry, C. and Guillotin-Plantard, N.: Discrete approximation of a stable self-similar stationary increments process. *Bernoulli 15*, (2009), no. 1, 195–222. MR2546804

[23] Gantert, N., Kochler, M. and Pène, F.: On the recurrence of some random walks in random environment. *ALEA 11*, (2014), 483–502. MR3274642

[24] Geman, D. and Horowitz, J.: Occupation densities. *Ann. Probab. 8*, (1980), 1–67. MR0556414

[25] Guillotin-Plantard, N., Hu, Y. and Schapira, Br.: The quenched limiting distributions of a one-dimensional random walk in random scenery. *Electron. Commun. Probab. 18*, (2013), no. 85, 7 pp. MR3141794

[26] Guillotin-Plantard, N. and Le Ny, A.: Transient random walks on 2d-oriented lattices. *Theory of Probability and Its Applications 52*, (2007), no 4, 815–826. MR2742878

[27] Guillotin-Plantard and N., Pène, F.: Renewal theorems for random walks in random scenery. *Electron. J. Probab. 17*, (2012), no. 78, 1–22. MR2981903

[28] Guillotin-Plantard, P., Pène and Wendler, M.: Empirical processes for recurrent and transient random walks in random scenery. *ESAIM Probab. Stat. 24*, (2020), 127–137. MR4071316

[29] Guillotin-Plantard, N. and Poisat, J.: Quenched central limit theorems for random walks in random scenery. *Stochastic Process. Appl. 123*, (2013), no. 4, 1348–1367. MR3016226

[30] Kalikow S. A.: T, T-1 Transformation is Not Loosely Bernoulli. *Ann. Math. 115*, (1982), no 2, 393–409. MR0647812

[31] Kesten, H.: Occupation times for Markov and semi-Markov chains. *Trans. Amer. Math. Soc. 103*, (1962), 82–112. MR0138122

[32] Kesten, H. and Spitzer, F.: A limit theorem related to a new class of self-similar processes. *Z. Wahrsch. Verw. Gebiete 50*, (1979), 5–25. MR0550121

[33] Khoshnevisan, D.: The codimension of the zeros of a stable process in random scenery. *Séminaire de Probabilités XXXVII*, 236–245, Lecture Notes in Math. 1832, Springer, Berlin, (2003). MR2053048

[34] Khoshnevisan, D. and Lewis, T.M.: Iterated Brownian motion and its intrinsic skeletal structure. Seminar on Stochastic Analysis, Random Fields and Applications (Ascona, 1996), In: *Progr. Probab. 45*, Birkhäuser, Basel, (1999), 201–210. MR1712242

[35] Lebedev, N. N.: Special functions and their applications, Revised edition, translated from the Russian and edited by Richard A. Silverman. Unabridged and corrected republication. *Dover Publications, Inc.*, New York, (1972). xii+308 pp. MR0350075

[36] Le Doussal, P.: Diffusion in layered random flows, polymers, electrons in random potentials, and spin depolarization in random fields. *J. Statist. Phys. 69*, (1992), no. 5-6, 917–954. MR1192029

[37] Le Gall, J.-F.: Mouvement brownien, processus de branchement et superprocessus, Master course, available on http://www.math.u-psud.fr/ jflegall.
Limit theorems for additive functionals of random walks in random scenery

[38] Marcus, M. B. and Rosen, J.: Markov processes, Gaussian processes, and local times. Cambridge Studies in Advanced Mathematics 100, Cambridge University Press, Cambridge, 2006. x+620 pp. MR2250510

[39] Matheron G. and de Marsily G.: Is transport in porous media always diffusive? A counterexample. Water Resources Res. 16, (1980) 901–907. doi:10.1029/WR016i005p00901

[40] Pène F.: Random walks in random sceneries and related models. ESAIM: Proceedings and surveys 68, (2020), 35–51. MR4142497

[41] Pène F. and Thomine D.: Potential kernel, hitting probabilities and distributional asymptotic. Ergodic Theory and Dynamical Systems, 1–74 (online:2019). doi:10.1017/etds.2018.136. MR4108909

[42] Pène, F. and Thomine, D.: Central limit theorems for the $Z_2$-periodic Lorentz gas. Israel Journal of Mathematics, to appear. MR4242538

[43] Port, S. C. and Stone, C. J.: Brownian motion and classical potential theory. Probability and Mathematical Statistics, Academic Press, Harcourt Brace Jovanovich, Publishers, New York-London, 1978. xii+236 pp. MR0492329

[44] Revuz, D. and Yor, M.: Continuous martingales and Brownian motion. Third edition. Grundlehren der Mathematischen Wissenschaften 293, Springer-Verlag, Berlin, 1999. xiv+602 pp. MR1725357

[45] Rogers, C. A.: Covering a sphere with spheres. Mathematika 10, (1963), 157-164. MR0166687

[46] Schmidt, K.: On recurrence. Z. Wahrsch. Verw. Gebiete 68, (1984) 75–95. MR0767446

[47] Thomine, D.: Théorèmes limites pour les sommes de Birkhoff de fonctions d’intégrale nulle en théorie ergodique en mesure infinie. PhD Thesis, Université de Rennes 1, 2013 version (in French). MR0959369

[48] Thomine, D.: A generalized central limit theorem in infinite ergodic theory. Probab. Theory Related Fields 158, (2014), no. 3-4, 597–636. MR3176360

[49] Weiss B.: The isomorphism problem in ergodic theory. Bull. AMS 78, (1972), 668–684. MR0304616

[50] Xiao, Y.: The Hausdorff dimension of the level sets of stable processes in random scenery. Acta Sci. Math. (Szeged) 65, (1999), 385–395. MR1702175

[51] Zweimüller, R.: Mixing limit theorems for ergodic transformations. Journal of Theoretical Probability 20, (2007), 1059–1071. MR2359068