Non-radial null geodesics in spherical dust collapse

Filipe C Mena\textsuperscript{\巨大} and Brien C Nolan\textsuperscript{\♭}
\textsuperscript{\巨大} School of Mathematical Sciences, Queen Mary, University of London, London E1 4NS, UK.
\textsuperscript{\♭} School of Mathematical Sciences, Dublin City University, Glasnevin, Dublin 9, Ireland.

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Abstract

The issue of the local visibility of the shell-focussing singularity in marginally bound spherical dust collapse is considered from the point of view of the existence of future-directed null geodesics with angular momentum which emanate from the singularity. The initial data (i.e. the initial density profile) at the onset of collapse is taken to be of class $C^3$. Simple necessary and sufficient conditions for the existence of a naked singularity are derived in terms of the data. It is shown that there exist future-directed non-radial null geodesics emanating from the singularity if and only if there exist future-directed radial null geodesics emanating from the singularity. This result can be interpreted as indicating the robustness of previous results on radial geodesics, with respect to the presence of angular momentum.

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1 Introduction

It has long been known that naked singularities may arise in the gravitational collapse of inhomogeneous dust spheres; see [1], where the existence of naked so-called shell-crossing singularities was first demonstrated. The physical
significance of these singularities has been questioned, primarily on the basis that they are gravitationally weak according to the definition of Tipler [2, 3]. However, it remains to be shown how to obtain a unique evolution to the future of a shell-crossing singularity. A more disturbing type of naked singularity was discovered and christened the shell-focussing singularity in [4]. This singularity was further studied in [3] and [2], where the first results regarding the role of regular initial data in determining the causal nature of the singularity were derived. This question has been studied extensively since then (see [6] and references therein). The main conclusion of these studies is that for a given initial mass distribution, the remaining initial data (namely the velocity distribution of the dust cloud) may be chosen so that the collapse results in either a black hole or a naked singularity [7]. Furthermore, there are open subsets (of the appropriate initial data space) of choices for the velocity distribution which lead to these conclusions. In this sense, both censored and naked singularities are stable [8, 9]. Indications of the instability of the Cauchy horizon associated with the naked singularity have arisen in perturbative studies [10].

The aim of this study is to address the stability of the naked singularity from another point of view. All results to date on the visibility of the singularity relate to the existence of future-pointing radial causal geodesics emanating from the singularity. (In all but one case [11], where radial time-like geodesics are treated, the results relate solely to radial null geodesics.) Our goal is to investigate the occurrence or otherwise of non-radial null geodesics emanating from the singularity. This is of importance as non-radial geodesics provide a better model of physically realistic trajectories than do radial geodesics. On the mathematical side, radial null geodesics passing through a point \( p \) of space-time constitute a set of measure zero in the set of null geodesics through \( p \) and so must be considered to be extremely specialized. Furthermore, investigating non-radial geodesics gives an indication of how angular momentum (at the level of the metric) might influence the system.

The structure of this paper is as follows. In Section 2, we discuss the field equations and occurrence of singularities and give a clear definition of the initial data which are being studied. This is essential for a rigorous study of the possible outcomes. The initial data consist of just one function \( \mu(r) \) of a single variable on an interval \([0, b]\), where \( b \) is the initial radius of the dust cloud. We take \( \mu \in C^3[0, b] \); this is more general than the \( C^\infty \) or analytic data often studied. In Section 3, we derive necessary and sufficient conditions, in terms of the initial data, for the existence of radial
null geodesics emanating from the singularity. Our results agree with those of previous studies [6, 7, 12], but have the two main advantages of (i) overcoming some potential mathematical deficiencies of previous approaches, for example in the case of the proof of necessary conditions for the occurrence of naked singularities; (ii) giving a complete decomposition of the $C^3$ initial data space into regions leading to naked and censored singularities. Also, the analysis used here is quite different from the approach of previous authors and sheds some new light on the nature of the singularity. We study non-radial null geodesics in Section 4. Our main conclusion is this: there exist non-radial null geodesics emanating from the singularity if and only if there exist radial null geodesics emanating from the singularity. Some concluding comments are given in Section 5. All our considerations are restricted to the local visibility of the singularity. A bullet • indicates the end of a proof. We use $8\pi G = c = 1$.

## 2 Field equations and singularities

We study spherical inhomogeneous dust collapse for the marginally bound case. The line element is [13, 14, 15]

$$ds^2 = -dt^2 + (R,r dr)^2 + R^2(r,t)d\Omega^2,$$

(1)

where $d\Omega^2$ is the line element for the unit 2-sphere. We use subscripts to denote partial derivatives. The field equations yield

$$R_t = -\sqrt{\frac{m(r)}{R}},$$

(2)

$$\rho = \frac{m_r}{R^2 R_r},$$

(3)

the latter equation defining the density $\rho(r, t)$ of the dust; $m(r)$ is arbitrary\(^1\). ($m$ will be referred to as the mass, although in fact $m$ equals twice the Misner-Sharp mass. In particular, this locates the apparent horizon at $R = m$.) Solving (2) gives

$$R^3(r, t) = \frac{9}{4}m(t_c(r) - t)^2,$$

(4)

\(^1m(r)$ will, in fact, be taken to satisfy certain differentiability conditions and inequalities defined ahead in the text.
where \( t_c(r) \) is another arbitrary function which, in the process of collapse, corresponds to the time of arrival of each shell \( r \) to the singularity.

**Assumption 1** The collapse proceeds from a regular initial state; i.e. at time \( t = 0 \), (i) there are no singularities (all curvature invariants are finite) and (ii) there are no trapped surfaces:

\[
m(r) < R(r, 0). \tag{5}
\]

We have the freedom of a coordinate rescaling in (1) which we can use to set \( R(r, 0) = r \). This gives

\[
t_c(r) = \frac{2}{3} \sqrt{\frac{r^3}{m}}. \tag{6}
\]

Notice that this leaves just one arbitrary function, \( m(r) \). There is a curvature singularity called the shell focussing singularity along \( t = t_c(r) \), so the ranges of the coordinates \( r, t \) are \( 0 \leq r < \infty \) and \( 0 \leq t < t_c(r) \). Assuming a dust sphere of finite radius, we can restrict the range of \( r \) to \([0, b]\) for some \( b > 0 \), and match it to a Schwarzschild exterior. During the collapse of the dust sphere there can be also a curvature singularity given by \( R_{r,r} = 0 \) along \( t = t_{sc}(r) \), where

\[
t_{sc}(r) = 2 \sqrt{\frac{r m}{m_r}}. \tag{7}
\]

It is known that this so-called shell-crossing singularity is gravitationally weak \([2, 3]\), though what this means in terms of continuability of the geometry is not yet known. Since we are primarily interested in the shell focussing singularity we impose:

**Assumption 2** Along each world-line \( r = \text{constant} \), the shell-crossing singularity does not precede the shell-focussing singularity. That is, \( t_c(r) < t_{sc}(r) \) for all \( r > 0 \).

This is equivalent to taking \( R_{r,r} > 0 \) for all \( r > 0 \), and yields

\[
m_{r,r} < \frac{3 m}{r}, \quad r > 0. \tag{8}
\]

We note that a no-shell crossing condition should imply \( t_c'(r) > 0 \) and, in this case, this is in fact equivalent to (8).
In order to obtain rigorous results relating to the initial data, we need to be very clear about what those initial data are. This can be phrased in terms of the class of functions to which \( m(r) \) may belong. The freely specifiable initial data consists of just one function: the initial density

\[
\mu(r) := \rho(r, 0) = \frac{m_r}{r^2}. \tag{9}
\]

Through the next assumptions we ensure the positivity of the matter-density \( \rho \) during the collapse. We impose a certain differentiability level on \( \mu \) which is higher than one would normally like to assume \((C^1)\), but allows for a comprehensive discussion of censorship.

**Assumption 3**

1. The mass function \( m(r) \) is strictly increasing on \([0, b]\).

2. The density function \( \mu(r) \) is such that

\[
\mu \in C^3[0, b], \quad \mu_0 := \mu(0) \neq 0. \tag{10}
\]

We define three functions \( m_i, i = 1, 2, 3 \) as follows. The definitions are meant to apply for whatever values of \( r \) the quantities exist.

\[
m(r) =: r^3\left(\frac{\mu_0}{3} + m_1\right), \quad m_1(0) = 0; \tag{11}
\]

\[
m_r(r) =: r^2(\mu_0 + m_2), \tag{12}
\]

\[
m_{rr}(r) =: r(2\mu_0 + m_3), \quad m_3(0) = 0. \tag{13}
\]

Notice that \( m_2(r) = \mu(r) - \mu_0 \), and so \( m_2 \in C^3[0, b] \) and \( m_2(0) = 0 \).

We now turn to the question of how the no-shell crossing condition restricts the initial data \( m(r) \). In order to do that we prove a number of results which constrain both \( m(r) \) and its derivatives.

**Proposition 1** The no-shell crossing condition (8) is equivalent to

\[
m_2(r) < 3m_1(r), r \in (0, b], \tag{14}
\]

which is equivalent to

\[
m'_1(r) < 0, r \in (0, b]. \tag{15}
\]

**Proof:** This is immediate from the definitions above.
**Corollary 1**  The no-shell crossing condition yields $m_2 < 3m_1 < 0$ on $(0, b]$.

It is important to study in more detail the properties of $m(r)$ at $r = 0$ since they will play an important role in deciding whether a null geodesic can escape from the singularity. We will now demonstrate some conditions on the values of the derivatives of $m(r)$ at $r = 0$. The proof of the next result makes use of elementary calculus to describe some important relationships between the derivatives of the functions $m_i(r), i = 1, 2, 3$ which will be of use in the following section.

**Proposition 2**  $m_1, m_3 \in C^3[0, b]$ and

\[
\begin{align*}
    m_1'(0) &= \frac{1}{4} m_2'(0), & m_3'(0) &= 3m_2'(0); \\
    m_2''(0) &= \frac{1}{5} m_2''(0), & m_3''(0) &= 4m_2''(0); \\
    m_4'''(0) &= \frac{1}{6} m_2'''(0), & m_3'''(0) &= 5m_2'''(0).
\end{align*}
\]

**Proof:**  Clearly, it is only differentiability at the origin which needs to be checked. By the definitions above, we can write

\[
\begin{align*}
    m_1(r) &= \frac{1}{r^3} \int_0^r x^2 m_2(x) \, dx, \quad (19) \\
    m_2(r) &= \frac{1}{r^2} \int_0^r x m_3(x) \, dx. \quad (20)
\end{align*}
\]

Using Taylor’s theorem to first order, we can write

\[
m_2(x) = x m_2'(\hat{x}),
\]

where the mapping $x \mapsto \hat{x}(x) : [0, b] \to [0, x]$ is continuous. Then

\[
\begin{align*}
    m_1(r) &= \frac{1}{r^3} \int_0^r x^3 m_2'(\hat{x}) \, dx \\
    &= \frac{m_2'(\bar{r})}{r^3} \int_0^r x^3 \, dx \\
    &= \frac{r}{4} m_2'(\bar{r}),
\end{align*}
\]
where in the second line we used the generalised mean value theorem for integrals. The mapping \( r \mapsto \bar{r}(r) : [0, b] \rightarrow [0, r] \) is continuous. Then

\[
\lim_{r \to 0^+} \frac{m_1(r) - m_1(0)}{r} = \lim_{r \to 0^+} \frac{m_1(r)}{r} = \lim_{r \to 0^+} \frac{1}{4} m_2'(\bar{r}) = \lim_{r \to 0^+} \frac{1}{4} m_2'(0),
\]

which says that \( m_1'(0) \) exists and equals the quantity asserted. The results on the higher derivatives of \( m_1 \) are obtained in a similar manner. Differentiating (20) and rearranging, we can write

\[
\frac{m_3(r)}{r} = m_2'(r) + 2 \frac{m_2(r)}{r},
\]

then taking the limit and using l’Hôpital’s rule for the second term on the right hand side shows that \( m_3'(0) \) exists and equals the quantity asserted. Results for the higher derivatives of \( m_3 \) are obtained by analysing the derivatives of (22).

As a corollary of the above results one can write useful conditions on the derivatives of the functions \( m_2 \) such that the no-shell crossing condition is satisfied.

**Corollary 2** The no-shell crossing condition yields:

1. \( m_2'(0) \leq 0 \).
2. If \( m_2'(0) = 0 \), then \( m_2''(0) \leq 0 \).
3. If \( m_2'(0) = m_2''(0) = 0 \), then \( m_2'''(0) \leq 0 \).

To summarise the main points of this section, we have proved a number of results which establish conditions on the mass function such that the process of collapse starts from a regular initial state and the shell-crossing of the dust spheres is avoided thus ensuring that the collapse will end in a (central) shell-focussing singularity. In what follows we will consider the question of whether this singularity is naked or covered.
3 Radial null geodesics

We recall that a singularity is called \textit{locally naked} if in its neighbourhood there are future-directed causal geodesics which emanate from the singularity. In this section we study the case of radial null geodesics\footnote{In fact the results of this section apply to arbitrary outgoing radial null curves; an affine parameter is not introduced at any stage.}. Using (1), (4) and (6) one can write the equation governing the outgoing radial null geodesics as

\[ \frac{dt}{dr} = R_r = \frac{1}{2} m^{-\frac{4}{3}} \left( \sqrt{\frac{r^3}{m} - \frac{3}{2} t} \right)^{-\frac{1}{3}} \left( 2\sqrt{rm} - m_r t \right). \] (23)

We know that if a radial null geodesic (RNG) emanates from the shell focussing singularity, then it must immediately move into the regular (untrapped) region. That is, it must precede the apparent horizon, which is given by \( R = m \), i.e. by

\[ t = t_H(r) = \frac{2}{3} \left( \frac{m}{r^3} \right)^{-1/2} - \frac{2}{3} m > 0. \] (24)

Furthermore, the only ‘point’ of the shell focussing singularity which may be visible is \( r = 0 \), whereat \( t_c(0) = \frac{2}{3} \left( \frac{\mu_0}{3} \right)^{-1/2} =: t_0 \). So we are seeking RNG’s that extend back to \( (r = 0, t = t_0) \). Henceforth, we shall refer to this point as ‘the singularity’ which will be called \textit{naked} if it is, at least, locally naked.

We shall now derive sufficient conditions on \( m(r) \) such that the singularity is naked. The main idea of the proof is that in order to find geodesics \( \gamma \) escaping from the singularity one must ensure that they lie below a curve given by \( t = t_H(r) \) (i.e. they precede the time of the apparent horizon) and above another curve \( t = t_s(r) \) which ensures \( R_{rt} > 0 \) as well as the no-shell crossing condition. This idea is represented schematically in Figure 1.

\[ \Omega_{\text{naked}} := \{ (r, t) : 0 < r < a, t_s(r) < t < t_H(r) \} \]

**Proposition 3** If \( m_2'(0) \neq 0 \), or if \( m_2'(0) = 0 \) and \( m_2''(0) \neq 0 \), then the singularity is naked. More precisely, defining

\[ t_s(r) = \left( \frac{m}{r^3} \right)^{-1/2} - \frac{\sqrt{rm}}{m_r} > 0, \] (25)

then there exists \( c > 0 \) such that every outgoing radial null geodesic which passes through the region

...
originates at the singularity.

**Proof:** First, we note that \( t_H > t_* \) if and only if
\[
m_1 > \frac{1}{3} m_2 + \frac{2}{3} r^3 \left( \frac{\mu_0}{3} + m_1 \right)^{3/2} \left( \mu_0 + m_2 \right),
\]
and so subject to the hypotheses of the Proposition and the conditions on the mass, there exists a sufficiently small \( c \) so that the region \( \Omega_{naked} \) is non-empty. Note also that \( t_H(0) = t_*(0) = t_0 \). It is easily established that \( t_H(r), t_*(r) \) are differentiable on \([0, b]\), and we find
\[
t'_H(r) = -\frac{1}{18} m'_2(0) \left( \frac{\mu_0}{3} \right)^{-3/2} - \frac{1}{45} m''_2(0) \left( \frac{\mu_0}{3} \right)^{-3/2} r + O(r^2). \tag{26}
\]
Now let \( p \) be any point of \( \Omega_{naked} \). Then there exists a unique solution of (23) through \( p \), which for \( r > 0 \) is a differentiable curve \( t_{rad}(r) \). Decreasing \( r \) along \( t_{rad} \), the RNG must either cross \( t = t'_H \), \( t = t_* \) or run into the singularity. Using (23), we find that
\[
t'_{rad} \big|_{t=t_H} = t'_H + m \geq t'_H \quad \text{for all } r \in [0, b], \tag{27}
\]
with equality holding only at the origin. Thus in the direction of increasing \( r \), \( t_{rad} \) crosses \( t_H \) from below, and so extending back to \( r = 0 \), we see that \( t_{rad} \) must stay below \( t_H \). We can also calculate that
\[
t'_{rad} \big|_{t=t_*} = 2^{-2/3} \left( \frac{\mu_0}{3} + m_1 \right)^{-1} (\mu_0 + m_2)^{1/3} (3m_1 - m_2)^{2/3}. \tag{28}
\]
It is then a straightforward matter to check that in each of the two cases referred to in the hypotheses, there exists \( d > 0 \) such that, for all \( r \in (0, d) \),
\[
t'_{rad} \big|_{t=t_*} < t'_*(r).
\]
Thus in the direction of increasing \( r \), \( t_{rad} \) crosses \( t_* \) from above, and so extending back to \( r = 0 \), we see that \( t_{rad} \) must stay above \( t_* \).

Hence we see that the geodesic \( t_{rad} \) must extend back to the singularity \((r = 0, t = t_0)\) •

We note that one has \( t > t_*(r) \) if and only if \( R_{rt} > 0 \). This will be of importance when we turn to the study of non-radial null geodesics. We also note that the condition \( t_* > 0 \) gives
\[
m_r > \frac{m}{r}. \tag{29}
\]
In the last proposition we have shown sufficient conditions for nakedness. We will now show necessary conditions on $m(r)$ such that the singularity is censored and which help us to isolate in a final case the remaining possible naked solutions. As in the previous proposition the proof follows by comparing the slopes of the curves depicted in Figure 1.

Figure 1: Schematic diagram of the singularity structure in the radial 2-space. In the region $\Omega = \{(r,t) : r < a, t_* < t < t_H\}$ no outgoing radial null geodesic $\gamma$ can cross either $t_H$ or $t_*$ ($t_q$ in the final case), as $r$ decreases, and so must extend back to the singularity ($r = 0, t = t_0$). If $t_H$ is decreasing in a neighbourhood of the origin, then the singularity must be censored.

**Proposition 4** If $m_2'(0) = m_2''(0) = 0$ and $|m_2'''(0)|$ is sufficiently small, then the singularity is censored.

**Proof:** We know that $dt_{rad}/dr > 0$ for $r > 0$ and that a RNG cannot emanate into the trapped region. Thus a necessary condition for the existence of a naked singularity is that $t_H(r)$ is increasing, i.e. $t'_H(r) > 0$, on some neighbourhood $(0, a)$ of the origin. This is equivalent to $t_H(r) > t_*(r)$ on
(0, a), which in turn is equivalent to
\[ u(r) := m_1 - \frac{1}{3} m_2 - \frac{2}{3} r^3 (\frac{\mu_0}{3} + m_1)^{3/2} (\mu_0 + m_2) > 0, \quad r \in (0, a). \]

Subject to the current hypotheses, we find that
\[ \lim_{r \to 0^+} \frac{u(r)}{r^3} = -2 (\frac{\mu_0}{3})^{5/2} - \frac{1}{36} m_2''', \]
and so the conclusion follows.

The following corollary is immediate:

**Corollary 3** If \( m_2'(0) = m_2''(0) = m_2'''(0) = 0 \), then the singularity is censored.

This is a very useful result as it shows that there is only one case left to consider, that where \( m_2'(0) = m_2''(0) = 0 \) but \( m_2'''(0) \neq 0 \). Thus the structure of the mass function is completely decided for the consideration of this case, which we call the final case.

### 3.1 The final case

We assume for the remainder of this section that \( m_2'(0) = m_2''(0) = 0 \) and \( m_2'''(0) < 0 \). The sign of the last term here is required by the no-shell crossing condition. It is convenient to define the number \( \beta > 0 \) by
\[ m_2'''(0) = -12 \beta (\frac{\mu_0}{3})^{5/2}. \]

Then from Proposition 2, we have, for \( r \to 0^+ \),
\[ m_1(r) \sim -\frac{1}{3} \beta (\frac{\mu_0}{3})^{5/2} r^3, \quad m_2(r) \sim -2 \beta (\frac{\mu_0}{3})^{5/2} r^3. \quad (30) \]

Furthermore, from the proof of Proposition 4, we see that we can restrict our attention to \( \beta \geq 6 \); if \( \beta < 6 \), then the singularity is censored. We will now show that with our approach one can also easily recover the known result \[ \text{[7, 16]} \] that, in the final case, sufficient conditions for the existence of naked singularities are obtained through the solutions of an algebraic quartic equation. Again the proof follows by calculating the slopes of the curves in Figure 3.
Proposition 5 Subject to the conditions (30) which define the final case, sufficient conditions for the singularity to be naked are that $\beta > 6$ and that there exists at least two positive roots $q \in (0, \infty)$ of the equation

$$2q\left(\frac{\beta - q}{6}\right)^{1/3} = 3\beta - q. \quad (31)$$

Proof: Let $q > 0$ and define $t_q(r) = t_0 + \frac{q}{9}(\frac{\mu_0}{3})r^3$. Notice that $t_q(r)$ intersects the singularity at $r = 0$. Then the region $\{(t, r) : 0 < r < a, t_q(r) < t < t_H(r)\}$ is non-empty for some $a > 0$ if

$$\beta > q + 6 > 6.$$ 

We calculate that the slope of a radial null geodesic (RNG) crossing $t_q$ is

$$R_{,r}|_{t=t_q(r)} = (\frac{\mu_0}{3})\left(\frac{\beta - q}{6}\right)^{-1/3}(\frac{3\beta - q}{6})^2 + o(r^2),$$

while the slope of $t_q$ is

$$t'_q(r) = \frac{q}{3}(\frac{\mu_0}{3})r^2.$$ 

Thus the former slope is less than the latter, and so the RNG cannot move below $t_q$ as $r$ decreases (hence giving a naked singularity), provided

$$\left(\frac{\beta - q}{6}\right)^{-1/3}(\frac{3\beta - q}{6}) < \frac{q}{3}.$$ 

The result follows $\bullet$

The converse of this result is also true, but is considerably more difficult to prove. We recall that in past works [6, 7, 12] either the emphasis was given on obtaining only sufficient conditions for the occurrence of a naked singularity, or the tangent to a null geodesic emanating from the singularity was assumed to have a finite, positive limit in a particular co-ordinate system. We note that, in principle, the non-existence of such limit might not be a sufficient condition for the non-existence of a geodesic emanating from the singularity. We give an example of this in the last section. We show next a sketch of the rigorous proof that the sufficient conditions given by Proposition 4 for the existence of a naked singularity are in fact also necessary. The details of the proof are omitted. Unfortunately, the usual fixed point theorems do not apply, and so an ad hoc approach is required.
Proposition 6 Subject to the conditions (30) which define the final case, a necessary condition for the singularity to be naked is that there exists a root \( q \in [0, \infty) \) of the equation

\[
2q\left(\frac{\beta - q}{6}\right)^{1/3} = 3\beta - q.
\]

Sketch of proof: In this final case, the governing equation (23) for the geodesics may be written as

\[
\frac{dy}{du} = \left(\frac{\beta}{6}\right)^{-1/3}(1 + f_1(u))(1 - \frac{3y}{2u} + f_2(u))^{-1/3}(1 - \frac{1y}{2u} + f_3(u)),
\]

where \( y = 6\beta^{-1}(\frac{\omega_3}{3})^{-1}(t - t_0) \), \( u = r^3 \) and \( f_i(u), i = 1, 2, 3 \) are \( C^0 \) functions of \( u \) which vanish in the limit \( u \to 0^+ \). We assume that there exists a solution \( y_1(u) \) of (32); i.e. \( y_1 \in C^1(0, u_i] \) for some \( u_i \), \( y_1(0) = 0 \) and \( y_1 \) satisfies (32) on \( (0, u_i] \). Furthermore, the solution must satisfy, for \( u \in (0, u_i] \),

\[
0 < y_1(u) < y_H(u) = \alpha u + f_4(u),
\]

where \( \alpha = (\beta - 6)/\beta \), \( y_H \) corresponds to the apparent horizon, and this defines the continuous \( o(u) \) function \( f_4(u) \).

A key point of the proof, which is used repeatedly, is that if the limit \( \lim_{u \to 0^+} u^{-1}y_1 \) exists, then using l’Hopital’s rule and (32), this limit must satisfy a certain quartic equation which strongly restricts the allowed values of this limit. In fact we do not need to prove that this limit exists as the quartic equation arises elsewhere in the analysis. Furthermore, the existence of this limit is not guaranteed by the proof below, and the limit possibly may not exist.

The proof proceeds by iteratively improving the bounds \( 0 < y_1/u < y_H/u \) by using existing bounds in (32) and integrating. Unfortunately this is not a straightforward process, as the bounding intervals do not automatically contract at each step. However, by studying carefully the subregions of a current bounding interval where contraction does not occur, we can show that the resulting ‘expansion’ leads to a contradiction such as \( y_1/u > \alpha \). The first step is to show that there exists \( u_0 > 0 \) such that

\[
k_0 < \frac{y_1}{u} < l_0
\]
on \((0, u_0]\), where \(k_0, l_0\) are respectively the smaller and greater positive roots of \(g_0(x) = x^4 - x^3 + \lambda_0^3\), where
\[
\lambda_0^3 = \left(\frac{6}{\beta}\right)^3(\beta + 3)^3.
\]

Such roots must exist in order that \(y_1\) exists. Next, we define the sequences \(\{\lambda_n\}_{n=0}^\infty, \{k_n\}_{n=0}^\infty\) and \(\{l_n\}_{n=0}^\infty\) as follows. With the \(n = 0\) terms as above, for \(n \geq 0\) we take
\[
\lambda_{n+1} = \frac{3}{2} \left(\frac{6}{\beta}\right)^{1/3} (1 - \frac{l_n}{3}),
\]
and define \(k_n, l_n\) to be the smaller and larger positive roots of \(g_{n+1}(x) = x^4 - x^3 + \lambda_{n+1}^3\). As for the first step, we can force the contraction \(u^{-1}y_1 \in (k_n, l_n)\). Clearly, \(k_{n+1}, l_{n+1}\) are not defined for all \(n\) for particular values of \(\beta\). When this happens, \(u^{-1}y_1\) leaps out of the current interval \((k_n, l_n)\) giving a contradiction and so a solution cannot exist. So we assume that \(\beta\) is such that the three sequences are defined. Then it is readily shown that \(\{\lambda_n\}_{n=0}^\infty, \{k_n\}_{n=0}^\infty\) are strictly increasing, while \(\{l_n\}_{n=0}^\infty\) is strictly decreasing. If \(\{\lambda_n\}_{n=0}^\infty\) is unbounded, we obtain a contradiction. Hence this sequence, and consequently \(\{k_n\}_{n=0}^\infty, \{l_n\}_{n=0}^\infty\) must converge. It is easily demonstrated that the limit \(l := \lim_{n \to \infty} l_n\) must be positive and must satisfy
\[
l(1 - l)^{1/3} = \frac{3}{2} \left(\frac{6}{\beta}\right)^{1/3} (1 - \frac{l}{3}).
\]
Taking \(q = \beta l\) proves the result.

We note that the proof avoids using the assumption that \(\lim_{u \to 0^+} u^{-1}y_1\) exists, nor does it conclude that this is the case (i.e. we do not have \(\lim_{n \to \infty} k_n \neq \lim_{n \to \infty} l_n\)). The necessary and sufficient conditions for the occurrence of naked singularities in the final case can be summarised in the next result which is obtained by carefully studying all the possible roots of \((31)\):

**Corollary 4** In the final case, excluding \(\beta = \beta_\ast := \frac{2}{3}(26 + 15\sqrt{3}) \simeq 77.97\), the singularity is naked if and only if \(\beta > \beta_\ast\).

**Proof**: We will see below that for \(\beta = \beta_\ast\), the equation \((31)\) has exactly one positive root. This does not guarantee that the sufficient condition of Proposition 5 is verified, and so we exclude this case from our considerations. Then
as we have seen, the necessary and sufficient conditions for the singularity to be naked are that $\beta > 6$ and that there exist at least two positive roots of

$$f_\lambda(l) = l^4 - l^3 + \lambda(1 - \frac{l}{3})^3 = 0,$$

where $\lambda = \frac{81}{(4\beta)}$. The values of $\lambda$ for which $f_\lambda$ has roots are determined as follows. First, we look for double roots of $f_\lambda$, i.e. values of $l$ for which $f_\lambda(l) = f'_\lambda(l) = 0$. These occur for two values of $\lambda$ only; $\lambda = \lambda_1 = 27(26 - 15\sqrt{3})/2$ with double root $l_1 = 3(2 - \sqrt{3})$ and $\lambda = \lambda_2 = 27(26 + 15\sqrt{3})/2$ with double root $l_2 = 3(2 + \sqrt{3})$. Notice that since $l_1 < 3$, $f_\lambda(l_1) < 0$ for all $\lambda < \lambda_1$, and since $l_2 > 3$, $f_\lambda(l_2) < 0$ for all $\lambda > \lambda_2$. Thus $f_\lambda$ has positive roots for $\lambda \in (0, \lambda_1) \cup (\lambda_2, \infty)$.

Suppose next that $f_\lambda$ has two positive roots for $\lambda \in (\lambda_1, \lambda_2)$ Then $f'_\lambda$ has a positive root $l_*$ which can be shown to lie in $(l_1, l_2)$. But the conditions $\lambda \in (\lambda_1, \lambda_2)$, $f'(l_*) = 0$, $l_* \in (l_1, l_2)$ yield $f_\lambda(l_*) > 0$, which cannot be true. Hence $f_\lambda$ cannot have two positive roots for $\lambda \in (\lambda_1, \lambda_2)$.

Thus $f_\lambda$ has (at least) two positive roots iff $\lambda \in (0, \lambda_1) \cup (\lambda_2, \infty)$. In terms of $\beta$, this is iff $\beta \in (0, 3(26 - 15\sqrt{3})/2) \cup (3(26 + 15\sqrt{3})/2, \infty)$. The condition $\beta > 6$ restricts us to the latter interval.

For convenience, we summarise the results of this section as follows, phrasing our results in terms of the behaviour of the initial density function $\mu(r)$.

**Theorem 1** Let $b > 0$ and $\mu \in C^3[0, b]$. Let

$$m(r) = \int_0^r x^2 \mu(x)dx$$

defined on $[0, b]$ satisfy the no-shell crossing condition $\Box$. Then the marginally bound collapse of the dust sphere with initial radius $b$ and initial density profile $\mu(r)$ results in a naked singularity if and only if one of the following conditions is satisfied.

1. $\mu'(0) < 0$.
2. $\mu'(0) = 0$ and $\mu''(0) < 0$.
3. $\mu'(0) = \mu''(0) = 0$ and

$$\frac{\mu''(0)}{(\mu(0))^{5/2}} < -\frac{2}{3}(26\sqrt{3} + 45) \simeq -60.0222.$$
We note that the results of this section are in complete agreement with those of previous studies [12, 16, 17]. Our approach, which will be also used in next section, is substantially different and so gives independent verification of these past results on radial null geodesics. Furthermore, our treatment of the final case is more general than it was previously done since we establish rigorously that, from the RNG’s analysis, the sufficient conditions for the occurrence of naked singularities are also necessary. We will comment on this again in the final section.

Having determined fully and rigorously which data give rise to singularities which are radially visible, we may now consider the case of non-radial null geodesics.

4 Non-radial null geodesics

In this section, we determine the circumstances under which a non-radial null geodesic may emanate from the singularity. This is potentially important since, physically, one can expect null geodesics to have some angular momentum. Furthermore, by introducing angular momentum in the geodesics equations we will also be doing a stability study of the naked singularity solutions that appear in the radial case.

The governing equations for the non-radial null geodesics (NRNG’s) obtained from the Euler-Lagrange equations are

\begin{align}
\dot{t} + \frac{R_{rt}}{R_r} \dot{\theta}^2 + \frac{L^2}{R^2} \left( R_t - \frac{R_{rt}}{R_r} \right) &= 0, \\
R_r \dot{\theta} + R_{rr} \dot{\theta}^2 + 2R_{rt} \dot{t} \dot{\theta} - \frac{L^2}{R^2} &= 0, \\
-\dot{\theta}^2 + (R_r \dot{\theta})^2 + \frac{L^2}{R^2} &= 0.
\end{align}

The over-dot represents differentiation with respect to an affine parameter $s$ along the geodesics, and the constant $L$ is the conserved angular momentum. We are looking for is the existence of a solution of (33-35), which satisfies this

**Existence condition:**

There exists $\epsilon > 0$ such that $\dot{t}$ is a non-negative, integrable function of the affine parameter $s$ and $\dot{\theta}$ is an integrable function of $s$ for $s \in [0, \epsilon)$ and such
that
\[ \lim_{s \to 0^+} r(s) = 0, \quad \lim_{s \to 0^+} t(s) = t_0. \]

Non-negativity of \( \dot{t} \) implies that the geodesic is future-pointing. We immediately see from (35) that \( \dot{t} \) must be infinite in the limit \( s \to 0^+ \), and hence we must have \( \dot{t} \to -\infty \) as \( s \to 0^+ \). In fact this applies to any NRNG which extends back to the center \( R = 0 \) \((r = 0)\). These conditions will be used in the proofs that follow. The main idea of the proofs given in this section is essentially the same as in the case of RNG where in order to have naked singularity solutions, one needed to find geodesics \( \gamma \) which lie in the region between \( t_H \) and \( t_* \) (as shown in Figure 1). The existence of the lower bound \( t_* \) obtained in the next proposition is a simple consequence of the geodesic equations (33-35).

**Proposition 7** A non-radial null geodesic emanating from the singularity must lie in the region \( t_* (r) < t \leq t_H (r) \) for all \( r \in (0, a) \) and some \( a > 0 \).

**Proof:** The second inequality is obvious. For the first, note that using the geodesic and field equations we can write
\[ \ddot{t} = -\frac{R_{rt}}{R_r} (R_r \dot{r})^2 + \frac{L^2}{R^2} \left( \frac{m}{R^3} \right)^{1/2}, \]  
and so
\[ \ddot{t} \geq -\frac{R_{rt}}{R_r} (R_r \dot{r})^2. \]  
Thus for a solution which extends back to the singularity, we must have \( R_{rt} > 0 \), which gives \( t > t_* (r) \) (recall that this is the defining property of \( t_* (r) \)).

We can now consider the following problem: Suppose that for a given mass function \( m(r) \) there are no RNG’s escaping from the singularity. Can we find NRNG’s which escape? Before giving the answer below in Proposition 8 we need to prove the following result:

**Lemma 1** \( \dot{r} \neq 0 \) in a neighbourhood of \( s = 0^+ \) along an NRNG which extends back to the centre \( r = 0 \).

**Proof:** Suppose that \( \dot{r} = 0 \) at some point \( s = s_0 \) on such a geodesic. From (36) this gives \( \ddot{t} \) positive which cannot happen sufficiently close to \( r = 0 \).
Note that this lemma applies to the case where the geodesic extends back to the singularity. We can now give an answer to the previous question by simply using the geodesics equations (33-35) and the results of the last section on RNGs:

**Proposition 8** If the singularity is radially censored, then it is censored. That is, if there are no radial null geodesics emanating from the singularity, then there are no null geodesics emanating from the singularity.

**Proof:** Assume that the singularity is radially censored and suppose that there exists an NRNG which extends back to the centre. By Lemma 1, we have \( \dot{r} \neq 0 \) and since generally \( R_{,r} \neq 0 \), we can rewrite (33) as

\[
\left( \frac{\dot{t}}{R_{,r}} \right)^2 + \left( \frac{L}{RR_{,r}} \right)^2 > 1.
\]

Thus along an NRNG, we have

\[
\frac{dt}{dr} > R_{,r}.
\]

The right hand side here gives the slope of an outgoing radial null geodesic, the left hand side is the slope of the NRNG's. Now let \( p \) be an arbitrary point of \( \Omega = \{(t, r): r > 0, t_s(r) < t < t_H(r)\} \). There is a unique outgoing radial null geodesic through \( p \) which by hypothesis extends back to the regular centre rather than the singularity. The inequality above tells us that any NRNG through \( p \) must cross the radial null geodesic from above, and so must precede this geodesic at all points \( q \) preceding \( p \). Hence no NRNG’s can extend back to the singularity.

Now we consider the converse problem: Suppose that for a given mass function \( m(r) \) there are RNG’s escaping from the singularity. Will there necessarily be NRNGs which also escape? To answer this problem we found it convenient to define new variables:

\[
x = \dot{t}, \quad y = R_{,r} \dot{r}.
\]

Then a crucial point will be with how to control the derivative of \( x/y \) as \( r \to 0 \). We explain why this should be done before we actually prove our result in Proposition 9 ahead.
We assume henceforth that there exist radial null geodesics emanating from the singularity. Choosing data at a point \( p \) with \( x|_p > 0, y|_p > 0 \), which corresponds to a future pointing outgoing null geodesic, we see that \((x, y)\) lies on the positive right-hand branch of the hyperbola \( \mathcal{H}_L \) given by

\[
x^2 - y^2 = \frac{L^2}{R^2}.
\]  

(38)

Clearly \( x/y > 1 \) along this branch. This says that the slope \( dt/dr \) in the \( r-t \) plane exceeds the slope of the unique future-pointing outgoing RNG through \( p \). However, since \( p \) is an interior point of \( \Omega \), we can find \( \delta \) sufficiently small such that the unique solution through \( p \) of \( x/y = 1 + \delta \) will also extend back to the singularity. (Notice that the proofs of existence of naked singularities for the radial case all involved showing that the slope \( dt/dr \) was smaller that the slope of some curve which we know to intersect the singularity.) Now on the hyperbola \( \mathcal{H}_L \), there is a continuum of choices of \((x, y)\) which satisfy \( x/y < 1 + \delta \). Take any such pair and let this give the initial data for \((33-35)\) at some time \( s_0 > 0 \). We follow the NRNG with these data as \( s \) decreases. Letting \( s \to 0^+ \), we must have \( x/y > 1 \) for all \( s > 0 \), which says that \( \gamma \) stays below the RNG through \( p \). By our choice of data,

\[
\frac{x}{y}(s_0) < 1 + \delta;
\]

to prove that \( \gamma \) extends back to the singularity, it would suffice to show that

\[
\frac{x}{y}(s) < 1 + \delta, \quad s \in (0, s_0].
\]

This inequality says that \( \gamma \) stays above the curve with \( x/y = 1 + \delta \), which we know extends back to the singularity. This would prove that the NRNG must also extend back to the singularity. The key to proving that we may proceed in this manner is to identify a subset of \( \Omega \) wherein the rate of change of \( x/y \) may be controlled. This is done by carefully analysing the relative magnitudes of the terms that arise in the equation governing this derivative as will be shown in the next proposition. (Note however that the demonstration of the existence of NRNG’s emanating from the singularity does not quite follow the outline of the paragraph above: this paragraph is intended to motivate our attempt to control the derivative of \( x/y \).)
Proposition 9 Let $m(r)$ be such that there exist radial null geodesics emanating from the singularity. Let

$$G(r, t) = R \frac{R_{rt}}{R_r}.$$ 

For $r_0 > 0$, $n > 0$, $\delta > 0$ define

$$\Omega[n, r_0] = \{(r, t) : 0 < r \leq r_0, G(r, t) > \frac{n-1}{2n} \sqrt{m R}, t < t_H(r)\}$$

and

$$\Omega[\delta, n, r_0] = \{(r, t) : 0 < r \leq r_0, \sqrt{m R} > \frac{2n}{3n-1}(1 + \delta), t < t_H(r)\}.$$ 

Then there exists $r_0 > 0$, $n > 0$ and $\delta > 0$ such that for any $p \in \Omega[\delta, n, r_0]$, an NRNG with initial data satisfying $(\xi_y)_p \leq 1 + \delta$ extends back to the singularity.

Proof: Straightforward calculations show that $(r, t) \in \Omega[n, r_0]$ if and only if $t_n(r) < t < t_H(r)$ where

$$t_n = \frac{3n}{3n-1} t_c - \frac{2}{3n-1} \sqrt{r m} \sigma_m,$$

and that $(r, t) \in \Omega[\delta, n, r_0]$ iff $t_\delta < t < t_H$ where

$$t_\delta = t_c - \frac{2}{3} \sigma^{-1} m$$

and $\sigma^{1/3} = 2n(1 + \delta)/(3n - 1)$.

The inequalities $t_\ast(r) < t_n(r) < t_\delta(r) < t_H(r)$ may be shown to be true for sufficiently small $r$. This is easily done except in the final case, i.e. where $m_1(r) \sim -\frac{1}{3} \beta(\frac{m}{3})^{5/2} r^3$. In this case, the relevant inequalities will hold if $n$ is chosen so that

$$\beta > \max\{n_1 = 3(3n - 1)(\frac{3n-1}{2n})^3, n_2 = 6(\frac{4n-1}{n-1})(\frac{3n-1}{2n})^3\}$$

(In fact, the condition that $\beta > n_2$ arises from a later consideration, but we find it convenient to deal with it here). Rather fortuitously, we find that taking $n_1 = n_2$ gives $n_1 = n_2 = \beta_\ast$, and since we must have $\beta > \beta_\ast$, the
inequalities are verified. The inequality $t_\delta < t_H$ requires that $\delta > 0$ be chosen sufficiently small.

Using the geodesic equations (33-35), we find that

$$\frac{d}{ds} \left( \frac{x}{y} \right) = \frac{L^2}{R^2 y} \left( G + \sqrt{m} \frac{x}{R} - \frac{x}{y} \right). \quad (39)$$

Now let $p$ as defined be the initial point at time $s = s_0$ of an NRNG and set the data such that $(\frac{x}{y})|_{s_0} \leq 1 + \delta$. By definitions, while the NRNG remains in $\Omega[\delta, n, r_0]$, we have $\frac{d}{ds} (\frac{x}{y}) > 0$. As $s$ decreases, this geodesic cannot leave $\Omega[\delta, n, r_0]$ through $t = t_H$ (this would correspond to a future-pointing null geodesic exiting the trapped region across its past space-like boundary). Suppose that the geodesic reaches the boundary $t = t_\delta(r)$. In the $r-t$ plane, the geodesic has slope $t' = \frac{\dot{x}}{\dot{y}}R_{,r}$. But for $s \leq s_0$, while the geodesic is in the closure of $\Omega[\delta, n, r_0]$, we have (since $x/y$ decreases as $s$ decreases)

$$t' \leq (1 + \delta)R_{,r}. \quad (39)$$

Also, $(1 + \delta)R_{,r}|_{t=t_\delta(r)} < t'_\delta(r)$ for sufficiently small $r$ and $\delta$. (As above, this is immediate except in the final case, where is necessitates the inequality $\beta > n_2$.) Thus the geodesic cannot exit $\Omega[\delta, n, r_0]$ via this boundary, and therefore must extend back to the singularity $\bullet$

This then proves that if, for a given mass function $m(r)$, RNGs escape from the singularity then NRNGs will also necessarily escape. The following theorem summarises the principal results of this section.

**Theorem 2** Given regular initial data for marginally bound spherical dust collapse subject to a condition which rules out shell-crossing singularities, there are non-radial null geodesics which emanate from the ensuing singularity if and only if there are radial null geodesics which emanate from the singularity.

Finally, it is also of interest to note that there are always NRNG’s which, starting from any point $p$ with $t < t_H(r)$, avoid the singularity in the past:

**Proposition 10** Let $p$ lie in the region $\{(r,t) : r > 0, 0 < t < t_H(r)\}$. If the initial data for an NRNG $\gamma$ through $p$ satisfy $(\frac{x}{y})|_p > \frac{3}{2}$, then $\gamma$ cannot meet the singularity in the past.
Proof: For $t_* < t < t_H$, it is easily shown that

$$G(r, t) < \frac{1}{2} \sqrt{\frac{m}{R}}. \quad (40)$$

Suppose that $\gamma$ meets the singularity. Then from above, it must do so in the region $t_* < t < t_H$ and the condition that $0 > \dot{x} = \ddot{t} \rightarrow -\infty$ in a neighbourhood of the singularity yields, using (33),

$$\left(\frac{x}{y}\right)^2 < \frac{3}{2}.$$

But (38), (40) and the initial condition at $p$ yield $\left.\dot{r}(\frac{x}{y})\right|_p < 0$. Hence the condition $x/y > 3/2$ is maintained along the geodesic moving into the past, and so we obtain a contradiction. Hence $\gamma$ cannot meet the singularity in the past.

Having proved the main results of this paper we will discuss some of their consequences in the next section.

5 Conclusions and comments

We have shown that the existence of radial null geodesics emanating from the shell-focussing singularity in marginally bound spherical dust collapse is a necessary and sufficient condition for the existence of non-radial null geodesics emanating from the singularity. Our interpretation of this result is that it expresses an aspect of stability of the naked singularity: the insertion of angular momentum into the system, at the level of the null geodesics, does not affect the outcome. This result can be interpreted as implying the robustness of the previous results, for radial null geodesics, with respect to the presence of angular momentum. One consequence of this is that all such naked singularities are equally ‘malicious’: previously, it has been noted that those which arise by the conditions (i) or (ii) of Theorem 1 being satisfied are gravitationally weak, while those which arise by condition (iii) of this theorem being satisfied are gravitationally strong [12]. However it is known that a non-radial null geodesic emanating from a central singularity always satisfies the strong singularity condition [13], and so the distinction in terms of singularity strength vanishes when angular momentum is included.

Our results on the relationship between the initial data and the causal nature of the singularity agree with previous results based on the study of
radial null geodesics. There are some differences: we have taken the initial data $\mu(r)$ to be $C^3$, as opposed to analytic or $C^\infty$. The convenience of this assumption is that it is more general and allows a comprehensive and rigorous discussion of all possible cases. Can this level of differentiability be reduced? Some authors have studied this problem (in fact the more general problem where the collapse is not assumed to be marginally bound) using $C^1$ initial data (e.g. [3, 4]; here a differentiability condition is imposed on the mass function $m(r)$ which corresponds to $\mu(r)$ being $C^1$). However in these studies either a complete decomposition of the initial data space into regions giving rise to naked and censored singularities has not been obtained, or further restrictions on the differentiability are imposed. It seems to us that the critical role of the third derivative of $\mu(r)$ indicates that $C^3$ is as low as one can push the differentiability requirement while maintaining a complete description of the possible outcomes.

As mentioned above, the approach used here to prove the existence of null geodesics emanating from the singularity is different from that of previous studies (but is more in line with the earlier studies [3, 4]). One immediate advantage is that we can give independent verification of the previous results on radial null geodesics. We see other two distinct advantages to our (more qualitative) approach. Firstly, we do not need to assume the existence of a limit of $R/r^\alpha$ (for some $\alpha > 0$) in the limit as the singularity is approached. Assuming the existence of such a limit, or equivalently of $y_1(u)/u$ as $u \to 0^+$ above, removes the need for the detailed analysis of Proposition 6: one can simply take the limit of both sides of (32) and use l’Hopital’s rule to produce the quartic equation of Corollary 4. However, while it is almost certain that this limit exists for some of the null geodesics emanating from the singularity, it is by no means certain that this is the case for all such geodesics. Evidence that the limit need not exist arises from the contraction process in the proof of Proposition 6. If the limit did exist, one would expect this to be reflected in the bounding interval $(k_n, l_n)$ having zero length in the limit as $n \to \infty$. In fact this does not happen, so while $y_1(u)/u$ is bounded within $[\lim_{n \to \infty} k_n, \lim_{n \to \infty} l_n]$ in the limit as $u \to 0^+$, there is no evidence that it possesses a finite limit. By way of illustration, we note that the approach relying on the existence of such a limit would not identify the (positive) solution

$$y(x) = x(1 + \epsilon \sin 1/x) \quad (0 < \epsilon < 1)$$

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of the initial value problem
\[ y'(x) = \frac{y}{x} - \frac{\epsilon}{x} \sqrt{1 - \frac{1}{\epsilon^2} \left(1 - \frac{y}{x}\right)^2}, \quad y(0) = 0. \]

Of course it may be that the regularity conditions imposed on the initial data only allow geodesics for which such a limit exists; however we feel that this is something which needs to be checked. Our fundamental point here is that the non-existence of this limit might not be a sufficient condition for the absence of a geodesic emanating from the singularity.

A second advantage which we see is that the qualitative approach applied here lends itself very easily to the analysis of non-radial null geodesics. In future work we hope to show that this is also the case when one considers radial and non-radial time-like geodesics. This would then give a unified treatment of all causal geodesics in marginally bound spherical dust collapse. One clear disadvantage must be mentioned: We have dealt only with marginally bound, and not general, spherical dust collapse. We expect that the analysis used here can be carried over to the general case, but it must be acknowledged that this would not be done without some difficulty. In the general case, one no longer has the explicit solution \[ \text{(1)} \] and consequently, identifying the important surfaces such as \( t = t^*(r) \) becomes problematic.

We comment as follows on the relationship between the initial data and the temporal nature of the singularity. One can argue that \( \mu'(0) \neq 0 \), i.e. \( m_1'(0) \neq 0 \) is unphysical, as it represents a cusp in the initial density distribution. One can also argue that all higher order odd derivatives should vanish at \( r = 0 \), i.e. that one should take \( \mu(r) \) to be an even function of \( r \). The reasoning is that the polar co-ordinates \( (r, \theta, \phi) \) are singular at \( r = 0 \), and so one should move to a Cartesian co-ordinate system \( (x, y, z) \) in a neighbourhood of the origin. But then odd powers of \( r \) lead to non-smooth functions of \( (x, y, z) \). In particular, demanding \( C^2 \) behaviour in the Cartesian co-ordinate system (it is difficult to see where higher derivatives would play a role in the physics, e.g. particle motion, of the space-time on and shortly after the initial slice) would rule out an \( r^3 \) term in \( \mu(r) \) and enforce \( \mu'''(0) = 0 \). This leaves \( \mu''(0) \) as the only feature of regular initial data which has any input into determining whether the collapse results in a naked or censored singularity. Clearly, the generic case is \( \mu''(0) < 0 \) which leads to a naked singularity.

Allowing the rather mild singularities generated by non-vanishing \( \mu'(0) \) or \( \mu'''(0) \), the picture becomes less clear. Obviously naked singularities will
arise as the generic case, but a more interesting question remains open: Do small perturbations of vacuum $\mu \equiv 0$ and homogeneous $\mu \equiv \mu(0)$ initial data generically give rise to naked singularities? Consider, for example, the following situation.

Fix the $C^3$ function $m_1(r)$, satisfying $m_1'(0) = m_1''(0) = 0$. Then we have, as in Section 3, $m_1(r) \sim -\frac{1}{3} \beta (\frac{\mu_0}{3})^{5/2} r^3$. Set initial data $(\mu_0, m_1(r))$ and allow the collapse to proceed. Taking $\beta < \beta_*$, the singularity is censored. Now consider the initial data $(\bar{\mu}_0, \bar{m}_1(r) = m_1(r))$. Then

$$\bar{\beta} = (\frac{\mu_0}{\bar{\mu}_0})^{5/2} \beta.$$

Choosing $\bar{\mu}_0$ sufficiently small, we can arrange that $\bar{\beta}$ exceeds the numerical value $\beta_*$ and so the collapse will result in a naked singularity. The singularity will occur at time

$$\bar{t}_0 = 2 \frac{3}{(\bar{\mu}_0)^{-1/2}} > t_0 = 2 \frac{3}{(\mu_0)^{-1/2}}.$$

Thus there is a tendency for ‘smaller’ initial data to result in naked rather than censored singularities. We note that this pattern reflects what is frequently found in studies of critical phenomenon: for example, in spherical collapse of a massless scalar field, Choptuik space-time, which contains a naked singularity, lies on the boundary between small/weak initial data which lead to dispersal, and large/strong initial data which lead to black hole formation [19]. It would be useful to carefully analyse this issue as it arises in dust using a standard measure of initial data for general relativity [20, 21].

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