Nucleon Solution of the Faddeev Equation
in the Nambu-Jona-Lasinio Model

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Abstract

Given the phenomenological success of the Nambu-Jona-Lasinio model in describing the meson physics in the low energy limit, it is tempting to find the fully relativistically structured nucleon solution in the same model under the similar approximation employed in the mesonic sector. To achieve this goal we need to solve a relativistic Faddeev equation. The factorizability of the two-body T-matrix reduces the three-body Faddeev equation to a tractable two-body Bethe-Salpeter equation. The reduced equation is then solved numerically. Our result indicates that the nucleon consists of three loosely bound constituent quarks.
I. INTRODUCTION

One of the most important features of QCD is the chiral symmetry and its dynamical breaking, which is expected to dictate the low energy hadronic physics. There exist work, such as the QCD sum rule [1], the instanton liquid model [2] and an explicit lattice QCD simulation via cooling technique [3], directly or indirectly confirming this expectation. The Lagrangian introduced by Nambu and Jona-Lasinio [4] long time ago conveniently mimics such an essential aspect of QCD in the low energy limit. Models based on the NJL type of Lagrangians have been demonstrated to be very successful in describing the low energy mesonic physics [5]. On the other hand, due to technical reasons, these models are much less effective in describing low energy physics involving baryons. It is very often that extra assumptions beyond these models have to be used in order to make concrete predictions in the baryonic sector.

While there is very little doubt that the NJL type of models could support bound baryonic states, the direct approach in solving a three-body problem has only been attempted recently [6] with approximations apparently quite different from that of employed in the mesonic sector. An important point is that approximations in the baryonic sector have to be consistent with chiral symmetry, for example, the nucleon solution should approximately satisfy the Goldberger-Treiman relation [7]. Otherwise the very essence of the NJL model, the chiral symmetry, is ruined by the ad hoc approximations. Other indirect attempts in finding the nucleon solution in the NJL-like models, such as the non-topological soliton approach [8], the bosonization approach [9] and undoubtedly others can be found in the literature.

In this paper we undertake the task of finding a nucleon-like solution in the NJL type of models. First we derive the three-body Faddeev equation in the valence constituent quark approximation by ignoring the three-body irreducible graphs. Due to the heaviness of the constituent quark, this approximation is expected to be good at low energies, as shown in the mesonic sector. By observing that the two-body diquark $T$-matrix has a separable
form the Faddeev equation can be reduced to an effective two-body Bethe-Salpeter equation with an energy dependent interaction. Then the reduced problem is solved numerically, without any further approximations. Although we can not explicitly show that our solution respects the exact chiral symmetry, in contrast with the meson solutions in the Hartree-Fock approximation, we believe that our work is a step forward in the right direction. So long as we can find weakly bound nucleon-like state of three constituent quarks, the chiral symmetry should be well protected, since the chiral symmetry is exact at the constituent quark level [5].

This paper is organized as follows. In section 2 we first introduce the model we explicit consider and then briefly review the two-body sector to fix parameters in the model. In section 3 the derivation of the three-body Faddeev equation and its reduction to the effective two-body equation are presented. The numerical technique involved in solving the reduced fully relativistic Bethe-Salpeter equation, based on the work of Rupp and Tjon [10], is recapitulated and then applied to our case in section 4. A summary and some outlook follows in section 5.

II. TWO-BODY SECTOR

The Lagrangian we consider is the two flavored Nambu-Jona-Lasinio model given by

\[ \mathcal{L} = \bar{\psi} i \gamma_\mu \partial^\mu \psi + G_1[(\bar{\psi} \psi)^2 + (\bar{\psi} i \gamma_5 \tau_a \psi)^2] - G_2[\bar{\psi} \gamma_\mu (\lambda_A / 2) \psi]^2, \]  

(2.1)

where \( \psi \) is the quark field, \( \tau_a \) (\( a = 1, 2, 3 \)) and \( \lambda_A \) (\( A = 1, 2, \cdots, 8 \)) are the generators of the flavor \( SU_f(2) \) and color \( SU_c(3) \) groups respectively. Small current quark masses are ignored for simplicity. Since the coupling constants \( G_1 \) and \( G_2 \) have negative mass dimension, this model is not renormalizable. An appropriate ultraviolet cutoff procedure has to be specified in order to make the model well defined. In this work we insert a form factor \( g(k) = g(-k) \), whose functional form will be eventually taken to be a four-momentum cutoff \( \Lambda \) in Euclidean space for convenience, at every fermion vertex in the loop integrals.
The justification of using Eq. (2.1) to model the low energy physics of the strong interaction and its phenomenological success in mesonic channels were well studied in the literature. A recent review can be found in reference [5]. Although our primary goal is to find three-body baryonic solutions in this model, it is adequate to recapitulate the essential features of this model in the meson sector, which is used to fix all the parameters but one in the model. Then the two-body $T$-matrix in the scalar-isoscalar diquark channel, which consists of an essential component of the three-body Faddeev equation, will be derived.

### A. Meson Channel

The most important feature that makes the model resembles QCD at low energy domain is that the NJL model and QCD share the same chiral symmetry and its dynamical breaking. The manifestation of this phenomenon in the NJL model is that the massless quarks acquire dynamical masses through the following self-consistent gap equation, when only the fermion bubble chain graphs are included or in the Hartree-Fock approximation,

$$1 = i(a_1G_1 + a_2G_2) \int \frac{d^4k}{(2\pi)^4} \frac{4g(k)}{k^2 - m^2}, \quad (2.2)$$

where $m$ is the constituent quark mass, which is related to the fermion condensate $\langle \bar{\psi}\psi \rangle$ by $m = -(a_1G_1 + a_2G_2)\langle \bar{\psi}\psi \rangle/(N_cN_f)$. For $N_c = 3$ and $N_f = 2$, $a_1 = 13$ and $a_2 = 8/3$.

As a consequence of the chiral symmetry breaking the pion emerges as the massless Goldstone boson, which manifests itself explicitly as the massless pole in the quark-antiquark two-body $T_{\bar{q}q}$-matrix in pseudoscalar channel. If again only the fermion bubble chain graphs are retained or in the RPA approximation, $T_{\bar{q}q}$ given by Fig. [1] can be readily calculated. The residue of $T_{\bar{q}q}$-matrix at this pole, $\Gamma_\pi^a$, has the form

$$\Gamma_\pi^a = g_{\pi\bar{q}q}[\Gamma^C \otimes \tau^a \otimes i\gamma_5], \quad (2.3)$$

where $g_{\pi\bar{q}q}$ is the pion-quark-antiquark coupling constant. The pion decay constant, $f_\pi$, is defined through the axial-vector current matrix element,
Using the chiral Ward identity, or the Goldberger-Treiman relation at the quark level, $f_\pi g_{\pi qq} = m$, one can easily find

$$f_\pi^2 = 4N_c m^2 \int \frac{d^4k}{(2\pi)^4} \frac{i g(k)}{k^2 - m_c^2}.$$  \hspace{1cm} (2.5)

In arriving at the above result the on-shell condition $p^2 = m_\pi^2 = 0$ has been used.

There are three parameters in the model, two couplings $G_1$, $G_2$ and the cutoff $\Lambda$. By equating $f_\pi$ and $m$ or $\langle \bar{\psi} \psi \rangle$ to the phenomenological values through Eq.(2.2) and Eq.(2.5) we can fix two of them, which we pick $\Lambda$ and $G \equiv a_1 G_1 + a_2 G_2$. This more or less fixes the theory in the mesonic sector. The last parameter $\eta \equiv G_1/G_2$ is left free to vary.

**B. Diquark Channel**

If we use the same fermion bubble chain approximation in the quark-quark sector, we can easily calculate the corresponding $T$-matrix. In the color $\bar{3}$ scalar-isoscalar channel the $T$-matrix has the structure, when ignoring the mixing with other channels (for example, the color $\bar{3}$ vector-isoscalar channel)

$$T^{ab,dc}_{qq}(p) = i R(p) [\lambda^{[i]}_A \otimes \tau_2 \otimes C \gamma_5]^{ab} [\lambda^{[j]}_A \otimes \tau_2 \otimes C \gamma_5]^{dc},$$  \hspace{1cm} (2.6)

where $\lambda^{[ij]}_A \equiv (\lambda^{ij}_A - \lambda^{ji}_A)/2$, $C \equiv i \gamma_0 \gamma_2$ is the charge conjugation matrix and $a, b$ label all the color, flavor and Dirac indices. The scalar function $R(p)$ can be obtained straightforwardly by summing the fermion bubble chain, yielding $R(p) = G'/(1 - G' J(p))$ with $G' = (b_1 G_1 + b_2 G_2)/4$ and

$$J(p) = 4i \int \frac{d^4k}{(2\pi)^4} g^2(k) \text{Tr}[C \gamma_5 S_F(k + p/2) C \gamma_5 S_F^T(-k + p/2)],$$

where Tr denotes the trace in Dirac space and $S_F(k)$ and $S_F^T(k)$ are the constituent quark propagator and its transpose (in Dirac space) respectively. Furthermore we have $b_1 = 4$, $b_2 = 8/3$. 

\hspace{1cm}
Whether there exists diquark bound states in this model depends on whether \( R(p) \) develops poles in the time-like region. As shown in [11] it is possible by varying \( \eta \) to find a bound diquark state in this channel. It should be emphasized that the existence of such a diquark bound state is not a necessary condition for the existence of a three-quark bound state, though it might be useful to utilize the diquark concept phenomenologically to explain certain scaling violations in lepton-nucleon experiments. In this paper the diquark state is merely an intermediate device in setting up the three-body Faddeev equation. The phenomenological relevance of the diquark will not be pursued here.

**III. THREE-BODY SECTOR**

Given the fundamental four-fermion vertex by the Lagrangian and the quark-quark two-body \( T_{qq} \)-matrix, and ignoring the three-body irreducible graphs, the three-body \( T \)-matrix can be solved from the Faddeev equation by iterating the fundamental vertex and the two-body \( T_{qq} \)-matrix. Throwing away the three-body irreducible graphs is in some sense equivalent to ignoring the non-valence constituent quark loops in the iteration process. Due to the heaviness of the constituent quark mass (300 \( \sim \) 400MeV) this approximation is justified in the low energy region. Of course, one should realize that we do not invoke more approximation here. Essentially the same kind of approximation was used in the mesonic and diquark cases.

**A. Faddeev Equation**

Since we are only interested at the moment in the three-body bound state, we only need to consider the homogeneous Faddeev equation. If the full three-body amplitude \( \Gamma^{f,d} \) (with \( f \) and \( d \) being the external flavor and Dirac indices) is decomposed as a sum of three partial amplitudes \( \Gamma_i \) \((i = 1, 2, 3)\), with

\[
\Gamma_i^{f,d} = \epsilon_{c_1,c_2,c_3} \gamma_2^{i_2} \gamma_5^{i_3} (\gamma_5)^{d_2} \delta^{f_1} \delta^{f_3} \delta^{f_1} \Gamma^{(1)}_{d_1,d_2,d_3}(p_1,p_2,p_3), \tag{3.1}
\]
and similarly for $\Gamma_2$ and $\Gamma_3$ by cyclically permuting $(1, 2, 3)$, then these partial amplitudes satisfy the following integral equation,

$$
\Gamma^{(3)}(p_1, p_2, p_3) = 2g(p_1 - p_2)iR(p_1 + p_2) \times \\
\left\{ \int \frac{d^4p_1'}{(2\pi)^4} g(p_1' - p_2')[C\gamma_5S_F^T(p_2')C\gamma_5S_F(p_1')]\Gamma^{(1)}(p_1', p_2', p_3) \\
+ \int \frac{d^4p_2'}{(2\pi)^4} g(p_1' - p_2')[C\gamma_5S_F^T(p_1')C\gamma_5S_F(p_2')]\Gamma^{(2)}(p_1', p_2', p_3) \right\}.
$$

The notation of the above equation is depicted in Fig. 3. The factor of 2 in Eq.(3.2) arises from the color sum $\epsilon_{ac_1c_2}\epsilon_{bc_1c_2} = 2\delta_{ab}$. Though formally similar to the non-relativistic Faddeev equation, Eq.(3.2) is exact within the approximation mentioned above. An analogous equation with scalar particles was considered by Rupp and Tjon in a different context [10]. Since we explicitly included the color, flavor and Dirac structures in the definition of the two-body $T_{qq}$-matrix and three-body amplitudes $\Gamma^{f_d}_{1,2,3}$, the recoupling-coefficient matrix has already been automatically taken into account in Eq.(3.2).

B. Reduction to an effective Bethe-Salpeter equation

If the two-body $T_{qq}$-matrix involved has a general form, it would be a formidable task to find the solution for Eq.(3.2). The crucial observation is that $T_{qq}$-matrix has a factorized form and hence we are only dealing with the so-called separable situation. The separability of the two-body interaction leads to a reduction of the three-body problem to an effective Bethe-Salpeter equation. As a matter of fact, this reduction has already been hinted by the explicit form of Eq.(3.2). More concretely, the three-body amplitudes can be written as

$$
\Gamma^{(1)}_{dd'}(p_1, p_2, p_3) = \Psi_{dd'}(p_1)g(p_2 - p_3)R(p_2 + p_3),
$$

and similarly for $\Gamma^{(2,3)}$, with $\Psi$ satisfying

$$
\Psi(p_3) = 4i \int \frac{d^4p_1'}{(2\pi)^4} g(p_1' - p_2')R(p_2' + p_3)g(p_2' - p_3)[C\gamma_5S_F^T(p_2')C\gamma_5S_F(p_1')]\Psi(p_1'),
$$

as a matrix equation in Dirac space. When deriving the above equation the quarks are treated as identical particles.
Diagrammatically, Eq.(3.4) can be represented by Fig. 4, which looks like a boson (with propagator $R$) coupling to a third quark to form a three-body bound state. However, this ought to be distinguished from identifying the “boson” as the diquark bound state. The reduction of the three-body Eq.(3.2) to the effective two-body Eq.(3.4) does not depend on whether the diquark channel has a pole, but rather on the separability of $T_{qq}$-matrix.

Introducing the equal-mass Jacobi momentum variables $q$ and $q'$,

$$p_3 \equiv \frac{P}{3} - q; \quad p'_1 \equiv \frac{P}{3} - q'; \quad (3.5)$$

where the total momentum $P$ is given by

$$P \equiv p_1 + p_2 + p_3 = p'_1 + p'_2 + p_3; \quad (3.6)$$

we find that the reduced Bethe-Salpeter equation can be written as

$$\Psi(P, q) = \frac{i}{4\pi^4} \int d^4q'V(q, q'; P)R\left(\frac{2}{3}P + q'\right)K\Psi(P, q'), \quad (3.7)$$

where we have defined an energy dependent interaction

$$V(q, q'; P) = \frac{g(p'_1 - p'_2)g(p'_2 - p_3)}{(p'_1^2 - m^2)(p'_2^2 - m^2)}. \quad (3.8)$$

The Dirac structure of the kernel is contained in the operator

$$K = [C\gamma_5(\gamma^Tp'_2 + m)C\gamma_5(\gamma p'_1 + m)]. \quad (3.9)$$

Using well-known properties of the charge conjugation operator $C$, this simplifies to

$$K = (\gamma p'_2 + m)(\gamma p'_1 + m). \quad (3.10)$$

In view of Eqs. (3.5-3.6) the various momenta present in Eq. (3.7) can be expressed in terms of the Jacobi variables $q, q'$ and total momentum $P$.

### C. Decomposition of the Reduced Amplitudes

To see the Dirac structure more clearly let us reduce the operator $K$ into the Pauli form. Using the $\rho$-spin notation of Ref. [12] for the upper and lower components of four-spinors, we get for the matrix elements $K(\rho, \rho')$ (with $\rho, \rho' = \pm$)
\begin{align*}
K(\mp) &= (p_\mp + m)(p_\mp' + m) - \vec{\sigma} \cdot \vec{p}_\mp \vec{p}_\mp' \\
K(\mp) &= -(p_\mp + m)\vec{\sigma} \cdot \vec{p}_\mp' - (m - p_\mp')\vec{\sigma} \cdot \vec{p}_\mp' \\
K(\mp) &= (p_\mp' + m)\vec{\sigma} \cdot \vec{p}_\mp' - (m - p_\mp')\vec{\sigma} \cdot \vec{p}_\mp' \\
K(\mp) &= (-p_\mp' + m)(-p_\mp + m) - \vec{\sigma} \cdot \vec{p}_\mp' \vec{p}_\mp \\
K(\mp) &= -(p_\mp' + m)(p_\mp + m) - \vec{\sigma} \cdot \vec{p}_\mp' \vec{p}_\mp'.
\end{align*}

The simplest approximation which can be made is to neglect the lower components, i.e. the kernel is replaced by \(K(\mp, \pm)\). The resulting eigenvalue equation becomes in this case

\[
\frac{i}{4\pi^4} \int d^4q' V(q, q'; P) R\left(\frac{2}{3} P + q'\right)[(p_\mp + m)(p_\mp' + m) - \vec{\sigma} \cdot \vec{p}_\mp \vec{p}_\mp'] \chi(q') = \lambda \chi(q),
\]

where a physical bound state solution corresponds to the eigenvalue \(\lambda = 1\). Assuming we are in the overall three-quark c.m. system \(P = (\sqrt{s}, \vec{0})\), we see that there are two classes of solutions to Eq. (3.12)

\[
\chi_1 = \Phi_1(q_0, |\vec{q}|) \\
\chi_2 = \vec{\sigma} \cdot \vec{q} \Phi_2(q_0, |\vec{q}|)
\]

These classes are not coupled to each other in the integral equations. This is due to parity and angular momentum conservation. \(\chi_1\) is a s-wave solution with \((l = 0, s = 1/2, j = 1/2)\) and \(\chi_2\) is a p-wave \((l = 1, s = 1/2, j = 1/2)\). In view of the simple form of the \(\chi\)'s the angular integration in Eq. (3.12) can explicitly be carried out. As a result, we obtain a two-dimensional integral equation of the form

\[
\chi_n(q) = \frac{i}{2\pi^3} \int_{-\infty}^{\infty} dq_0' \int_{0}^{\infty} q'^2 dq' V_n(q, q'; P) R\left(\frac{2}{3} P + q'\right) \chi_n(q')
\]

where \(q \equiv |\vec{q}|\) and

\[
V_n(q, q'; P) = \int_{-1}^{1} dx \frac{g(p_1' - p_2') g(p_2' - p_3)}{(p_2'^2 - m^2)(p_1'^2 - m^2)} \text{Tr}_2 \left[O_n K(\pm, \pm)\right]
\]

with \(x = \cos(\theta_{qq'})\) and \(\text{Tr}_2\) is the trace to be taken in Pauli space. Furthermore, the operator \(O_n\) is given by \(1/2\) and \((\vec{\sigma} \cdot \vec{q})/(2|\vec{q}|^2)\) for \(n=1\) and \(2\) respectively.
This analysis can be extended to the full equation. From Eq. (3.11) we see that the Pauli spin dependence in \( \Psi \) can be either the unit operator or \( \vec{\sigma}.\vec{q} \). In view of parity conservation there are also two classes of solutions, which are given by four spinors of the form

\[
\Psi_1 = \begin{pmatrix}
\phi_1(q_0, q) \\
\vec{\sigma}.\vec{q} \phi_2(q_0, q)
\end{pmatrix}
\]

(3.16)

and

\[
\Psi_2 = \begin{pmatrix}
\vec{\sigma}.\vec{q} \phi_3(q_0, q) \\
\phi_4(q_0, q)
\end{pmatrix}
\]

(3.17)

The vertex functions \( \Psi_1 \) and \( \Psi_2 \) are again not coupled to each other. With this form for \( \Psi \) a partial wave decomposed set of coupled integral equations can be derived. Inserting Eq. (3.16) in Eq. (3.7) we get

\[
\phi_n(q) = \frac{i}{2\pi^2} \sum_{m=1}^{2} \int_{-\infty}^{\infty} dq_0' \int_{0}^{\infty} q'^2 dq' V_{nm}(q, q'; P) R(\frac{2}{3}P + q') \phi_m(q')
\]

(3.18)

with \( n = 1, 2 \) and

\[
V_{nm}(q, q'; P) = \int_{-1}^{1} dx \frac{g(p_1' - p_2')g(p_2' - p_3)}{(p_2'^2 - m^2)(p_1'^2 - m^2)} K_{nm}
\]

(3.19)

The explicit expression for the matrix \( K_{nm} \) can be determined by noting that

\[
\Psi_1 = \left[ \frac{1 + \gamma_0}{2} \phi_1 - \vec{\gamma}.\vec{q} \frac{1 + \gamma_0}{2} \phi_2 \right] w,
\]

(3.20)

where \( w \) is a four-spinor with every component equal 1. The operator \((1 + \gamma_0)/2\) is in the three-quark c.m. system nothing else as the projection operator \( \Omega = (M_N + \gamma P)/(2M_N) \).

With this we can now calculate \( K_{nm} \) by projecting out the Dirac form on \( \Omega \) and \( \vec{\gamma}.\vec{q} \). In so doing we get for the Dirac part of the kernel

\[
\begin{align*}
K_{1m} &= \frac{1}{2} \text{Tr}[K\kappa_m(q')] \\
K_{2m} &= \frac{\text{Tr}[\vec{\gamma}.\vec{q}K\kappa_m(q')]}{\text{Tr}[\vec{\gamma}.\vec{q}\kappa_2(q)]}
\end{align*}
\]

(3.21)

where \( \kappa_1 = \Omega, \kappa_2 = \vec{\gamma}.\vec{q}\Omega \). Eq. (3.21) can be evaluated in a straightforward way. We find
\[ K_{11} = -q_0 q_0' + q_0 (m + \frac{M_N}{3}) - q^2 - q.q' + (m + \frac{M_N}{3})^2 \]  
\[ K_{12} = -(q_0 - 2 \frac{M_N}{3}) \vec{q}^{\prime 2} + (q_0' + m - \frac{M_N}{3})(\vec{q} \cdot \vec{q}') \]  
\[ K_{21} = q_0' - m - \frac{M_N}{3} - (q_0 + 2 \frac{M_N}{3}) \frac{\vec{q} \cdot \vec{q}'}{\vec{q}^2} \]  
\[ K_{22} = \vec{q}^{\prime 2} + \left[ -q_0(q_0' - m + \frac{M_N}{3}) - q^2 + (m - \frac{M_N}{3})^2 \right] \frac{\vec{q} \cdot \vec{q}'}{\vec{q}^2} \]  

In a similar way the coupled set of equations can be derived for \( \Psi_2 \). It should be noted that possible solutions of this type correspond to p-wave like states and as a result are expected not to be the ground state of the three quark system due to the centrifugal term. Since we are interested in this paper in the nucleon, it is natural to confine ourself to the solutions of the s-wave type, given by \( \Psi_1 \).  

**IV. CALCULATIONS**

Following Ref. [10] the resulting integral equations can be studied by performing a Wick rotation of the \( q_0 \) and \( q_0' \) variables to the complex plane. Assuming that the diquark system supports a boundstate at \( M_{qq} \), we find that at the threshold point of quark-diquark scattering a pinching singularity can occur in the kernel of Eq. (3.18) at \( q_0 = \hat{q}_0 = \frac{1}{3}(2m - M_{qq}) \). It can readily be verified that in the triquark boundstate region, corresponding to \( \sqrt{s} < m_q + M_{qq} \), the \( q_0 \) and \( q_0' \) variables can be rotated to a path going through the point \( \hat{q}_0 \) and parallel to the imaginary axis without encountering any singularities in the kernel. In so doing we implicitly assume that eventual singularities in the form factors \( g(q) \) do not cross the imaginary \( q_0 \) axis. Furthermore the arguments of the form factors are approximated by neglecting the \( \hat{q}_0 \) dependence. The resulting Euclidean form of the integral equation is regular in the boundstate region and as a result it can in principle be solved by standard discretization procedures. Because of the actual size of the resulting matrix equations we have adopted the method described in Ref. [10]. The perturbation series is determined by iterating the equations, while the occurring two-dimensional integrals are evaluated using standard Gaussian quadratures. From this series the energy of the boundstate is determined.
using the ratio method of Malfliet and Tjon \cite{13}. It should noted that as a byproduct also the corresponding wavefunction can be found in this way.

There are several parameters in the model: the cutoff \( \Lambda \), the coupling constants \( G_1 \) and \( G_2 \). The overall mass scale can be set by the choice of the the cutoff mass \( \Lambda \). There are two constraints we would like to satisfy. From the pion decay constant \( f_\pi = 93\,\text{MeV} \), we can determine according to Eq. (2.5) the constituent quark mass \( m \). Taking a value of \( \Lambda = 750\,\text{MeV} \) we find \( m = 375\,\text{MeV} \). Decreasing \( \Lambda \) for instance to 739 MeV, the quark mass increases to \( m = 400 \,\text{MeV} \). Secondly, from the self-consistent mass gap equation, the value of \( G \equiv 13G_1 + \frac{8}{3}G_2 \) is fixed. As a consequence the only free parameter is the ratio \( G_1/G_2 \), which can be used as the parameter to determine the diquark mass. In Fig. 5 the diquark mass dependence on this ratio is shown.

Once the parameters of the model have been fixed we may study the three quark bound state. In Table I we list the diquark masses needed to get a nucleon solution at \( M_N = 939 \,\text{MeV} \) in three approximations, non-relativistic (or static limit) \( K_{11} \to 4m^2 \); one channel defined by Eq.(3.12) and two-channel defined by Eq.(3.18). As one can see, a stable nucleon solution always requires the scalar-isoscalar diquark state lie below two-quark threshold. The binding of the three quark system clearly depends on the choice of the diquark energy. In Fig. 6 are shown for two cases of \( \Lambda \) the results of the calculated mass of the three quark ground state as a function of the diquark mass (solid line). Also are plotted the results when we only keep the s-wave components of the three quark wave function (dot-dash line) and the static limit of \( K_{11} \to 4m^2 \) (dash line). From this we see that at lower diquark masses the static limit predicts a substantially deeper binding than the full 2-channel result and hence it can be an unreliable approximation.

To have a feeling on the quality of the static and one channel approximations we list in Table II the nucleon masses with diquark mass fixed at the value where the two-channel calculation would yield \( M_N = 939 \,\text{MeV} \). It is clear from Table I that the relativistic forms give rise to less attraction, leading to a slightly higher lying ground state, though there is no qualitative difference from the static approximation \cite{6}. From these results we may conclude
that over a wide range of $M_{qq}$ a stable nucleon solution indeed exists in the considered NJL model. Increasing the diquark mass leads to a weakening of the quark-quark interaction and as result the nucleon mass increases.

In the range of diquark mass we considered, the existence of the nucleon solution near its experimental value required about 150 to 300 MeV binding in the scalar-isoscalar diquark. This kind of diquark clustering is also observed in a recent instanton model calculation by Shuryak et al \[14\] in the nucleon channel and qualitatively confirmed by the lattice simulation through cooling \[15\]. Since the NJL type of models are practically effective theories for these instanton models, the similar diquark clustering in our case may not be a mere coincidence.

V. SUMMARY AND OUTLOOK

We have been able to demonstrate that the NJL type of models can easily accommodate the nucleon-like state under similar type approximations employed in the mesonic sector. Although we were not able to explicitly show, from the derived Faddeev equation, that the solution of the nucleon state satisfies the Goldberger-Treiman relation, we indeed found that the nucleon be a loosely bound state of the constituent quarks. In order to have the nucleon solution as a true bound state a bound diquark in the scalar-isoscalar channel is necessary in our model.

There are clearly some interesting questions which can be addressed in such a model. Using the wavefunction corresponding to the three-quark boundstate, the properties of the various form factors for the nucleon can in principle be studied. It is also of interest to examine possible Delta isobar states in the same model we have considered. The mass splitting between the baryon decuplet and octet constitutes a non-trivial test of the NJL type of models, while the mass splittings within the same baryon multiplets are less stringent due to the fact that the latter splittings mainly come from quark masses. Since the dominant diquark configuration in the Delta should be vector-isovector, the resulting three-body bound state could be a resonance rather than a bound state. The numerical method we
used in the nucleon case needs to be modified if the Delta lies in the continuum. A more
careful examination of the compatibility of the Faddeev equation and the chiral symmetry
could provide useful insight on how the Goldberger-Treiman relation at the nucleon level
is realized. Finally, also pion-nucleon and nucleon-nucleon low energy scattering processes
can in principle be studied. It is easy to anticipate that the meson-exchange potential could
merge between nucleons, if only the valence quark lines are included at each instance in
the Feynman graphs. Furthermore, since the nucleon in this model is a loosely bound state
of three constituent quarks, it is very likely that there are anomalous singularities in the
scattering processes, which could potentially modify the one-meson exchange nuclear force
picture in the low energy limit. The role of the anomalous singularity in the form factor for
loosely bound states in similar models was studied recently [16].

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FIGURES

FIG. 1. Feynman graphs for $T_{qq}$ in pseudoscalar channel.

FIG. 2. Feynman graphs for $T_{qq}$ in scalar-isoscalar diquark channel.

FIG. 3. Faddeev equation for the nucleon.

FIG. 4. Reduced effective Bethe-Salpeter equation for the nucleon.

FIG. 5. Scalar-isoscalar diquark mass as a function of $G_1/G_2$ for two values of constituent quark masses. The horizontal lines indicate the quark-quark thresholds.

FIG. 6. Nucleon mass as a function of the diquark mass, with the dash line corresponding to static limit $K_{11} \rightarrow 4m^2$, dot-dash line corresponding to one channel defined by Eq. (3.12), solid line corresponding to full two-channel defined by Eq. (3.18) solutions respectively. The dotted line indicates the quark-diquark scattering threshold.
TABLES

TABLE I. Diquark masses needed to get a nucleon solution at $M_N = 939$ MeV in various approximations.

| $m$ (MeV) | $\Lambda$ (MeV) | $M_{qq}^a$ (MeV) | $M_{qq}^b$ (MeV) | $M_{qq}^c$ (MeV) |
|-----------|-----------------|-----------------|-----------------|-----------------|
| 375       | 750             | 579.0           | 570.9           | 576.8           |
| 400       | 739             | 572.8           | 554.7           | 564.7           |
| 450       | 728             | 577.0           | 531.5           | 547.8           |

$^a$static limit.

$^b$one channel defined by Eq. (3.12).

$^c$two-channel defined by Eq. (3.18).

TABLE II. Comparison of predictions of the nucleon mass in various approximations. The value of $M_{qq}$ is fixed so that $M_N = 939$ MeV in the full two-channel calculation.

| $m$ (MeV) | $\Lambda$ (MeV) | $M_{qq}$ (MeV) | $M_N^a$ (MeV) | $M_N^b$ (MeV) |
|-----------|-----------------|----------------|----------------|----------------|
| 375       | 750             | 576.8          | 936.4          | 945.3          |
| 400       | 739             | 564.7          | 928.2          | 950.0          |
| 450       | 728             | 547.3          | 892.7          | 957.9          |

$^a$static limit.

$^b$one channel defined by Eq. (3.12).