Topological conjugacy of topological Markov shifts and Cuntz–Krieger algebras

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Abstract

For an irreducible non-permutation matrix $A$, the triplet $(\mathcal{O}_A, D_A, \rho^A)$ for the Cuntz-Krieger algebra $\mathcal{O}_A$, its canonical maximal abelian $C^*$-subalgebra $D_A$, and its gauge action $\rho^A$ is called the Cuntz–Krieger triplet. We introduce a notion of strong Morita equivalence in the Cuntz–Krieger triplets, and prove that two Cuntz–Krieger triplets $(\mathcal{O}_A, D_A, \rho^A)$ and $(\mathcal{O}_B, D_B, \rho^B)$ are strong Morita equivalent if and only if $A$ and $B$ are strong shift equivalent. We also show that the generalized gauge actions on the stabilized Cuntz–Krieger algebras are cocycle conjugate if the underlying matrices are strong shift equivalent. By clarifying K-theoretic behavior of the cocycle conjugacy, we investigate a relationship between cocycle conjugacy of the gauge actions on the stabilized Cuntz–Krieger algebras and topological conjugacy of the underlying topological Markov shifts.

1 Introduction and Preliminaries

Let $A = [A(i,j)]_{i,j=1}^N$ be an irreducible matrix with entries in $\{0,1\}$ with $1 < N \in \mathbb{N}$. We assume that $A$ is not any permutation matrix. In [7], J. Cuntz and W. Krieger have introduced a $C^*$-algebra $\mathcal{O}_A$ associated to topological Markov shift $(X_A, \sigma_A)$. The $C^*$-algebra is called the Cuntz–Krieger algebra, which is a universal unique purely infinite simple $C^*$-algebra generated by partial isometries $S_1, \ldots, S_N$ subject to the relations:

$$\sum_{j=1}^N S_j S_j^* = 1, \quad S_i^*S_i = \sum_{j=1}^N A(i,j) S_j S_j^*, \quad i = 1, \ldots, N. \quad (1.1)$$

For $t \in \mathbb{R}/\mathbb{Z} = \mathbb{T}$, the correspondence $S_i \rightarrow e^{2\pi \sqrt{-1} t} S_i, i = 1, \ldots, N$ gives rise to an automorphism of $\mathcal{O}_A$ denoted by $\rho^A_t$. The automorphisms $\rho^A_t, t \in \mathbb{T}$ yield an action of $\mathbb{T}$ on $\mathcal{O}_A$ called the gauge action. Cuntz and Krieger in [7] have shown that the algebra $\mathcal{O}_A$ has close relationships with the underlying dynamical system called topological Markov shift. Let us denote by $X_A$ the shift space

$$X_A = \{(x_n)_{n \in \mathbb{N}} \in \{1, \ldots, N\}^\mathbb{N} \mid A(x_n, x_{n+1}) = 1 \text{ for all } n \in \mathbb{N}\}. \quad (1.2)$$

Define the shift transformation $\sigma_A$ on $X_A$ by $\sigma_A((x_n)_{n \in \mathbb{N}}) = (x_{n+1})_{n \in \mathbb{N}}$, which is a continuous surjection on $X_A$. The topological dynamical system $(X_A, \sigma_A)$ is called the one-sided
topological Markov shift for matrix $A$. The two-sided topological Markov shift $(\bar{X}_A, \bar{\sigma}_A)$ is similarly defined with the shift space

$$\bar{X}_A = \{(x_n)_{n \in \mathbb{Z}} \in \{1, \ldots, N\}^\mathbb{Z} \mid A(x_n, x_{n+1}) = 1 \text{ for all } n \in \mathbb{Z}\}$$

and the shift homeomorphism $\bar{\sigma}_A((x_n)_{n \in \mathbb{Z}}) = (x_{n+1})_{n \in \mathbb{Z}}$ on $\bar{X}_A$.

Let us denote by $\mathcal{D}_A$ the $C^*$-subalgebra of $\mathcal{O}_A$ generated by the projections of the form: $S_{i_1} \cdots S_{i_n} S_{i_n}^* \cdots S_{i_1}^*$, $i_1, \ldots, i_n = 1, \ldots, N$. The subalgebra $\mathcal{D}_A$ is canonically isomorphic to the commutative $C^*$-algebra $C(X_A)$ of the complex valued continuous functions on $X_A$ by identifying the projection $S_{i_1} \cdots S_{i_n} S_{i_n}^* \cdots S_{i_1}^*$ with the characteristic function $\chi_{U_{i_1} \cdots i_n} \in C(X_A)$ of the cylinder set $U_{i_1} \cdots i_n$ for the word $i_1 \cdots i_n$. Let us denote by $\mathcal{K}$ the $C^*$-algebra $\mathcal{K}(\ell^2(\mathbb{N}))$ of compact operators on a separable infinite dimensional Hilbert space $\ell^2(\mathbb{N})$ and by $\mathcal{C}$ its maximal abelian $\mathcal{C}^*$-subalgebra of diagonal operators.

In [24], R. F. Williams proved that the topological Markov shifts $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are topologically conjugate if and only if the matrices $A, B$ are strong shift equivalent. Two nonnegative matrices $A, B$ are said to be elementary equivalent if there exist nonnegative rectangular matrices $C, D$ such that $A = CD, B = DC$. We write it as $A \approx_{C,D} B$. If there exists a finite sequence of nonnegative matrices $A_0, A_1, \ldots, A_n$ such that $A = A_0, B = A_n$ and $A_i$ is elementary equivalent to $A_{i+1}$ for $i = 1, 2, \ldots, n-1$, then $A$ and $B$ are said to be strong shift equivalent. Hence elementary equivalence generates topological conjugacy of two-sided topological Markov shifts.

Let $A$ be an irreducible non-permutation matrix. The triplet $(\mathcal{O}_A, \mathcal{D}_A, \rho^A)$ for the Cuntz-Krieger algebra $\mathcal{O}_A$, its canonical maximal abelian $\mathcal{C}_0$-subalgebra $\mathcal{D}_A$, and its gauge action $\rho^A$ is called the Cuntz-Krieger triplet for the matrix $A$. As pointed out in [10], two elementary equivalence matrices $A = CD, B = DC$ yield $\mathcal{O}_A - \mathcal{O}_B$-imprimitivity bimodule via Cuntz-Krieger algebra $\mathcal{O}_Z$ for the matrix $Z$ defined by $Z = \begin{bmatrix} 0 & C \\ D & 0 \end{bmatrix}$.

In the first part of the paper, We will introduce a notion of strong Morita equivalence in the Cuntz–Krieger triplets, and prove the following theorem.

**Theorem 1.1** (Corollary 2.17). The Cuntz–Krieger triplets $(\mathcal{O}_A, \mathcal{D}_A, \rho^A)$ and $(\mathcal{O}_B, \mathcal{D}_B, \rho^B)$ are strong Morita equivalent if and only if the matrices $A$ and $B$ are strong shift equivalent.

It is well-known that two unital $C^*$-algebras $A$ and $B$ are strong Morita equivalent if and only if their stabilizations $A \otimes \mathcal{K}$ and $B \otimes \mathcal{K}$ are isomorphic by Brown–Green–Rieffel Theorem [3] Theorem 1.2] (cf. [3], [4]). We will next study relationships between stabilized Cuntz–Krieger algebras with their gauge actions and strong shift equivalence matrices. We must emphasize that Cuntz and Krieger in [7] 3.8 Theorem and Cuntz in [6] 2.3 Theorem] have shown that the stabilized Cuntz–Krieger triplet $(\mathcal{O}_A \otimes \mathcal{K}, \mathcal{D}_A \otimes \mathcal{C}, \rho^A \otimes \text{id})$ is invariant under topological conjugacy of the two-sided topological Markov shifts $(\bar{X}_A, \bar{\sigma}_A)$. We will investigate stabilizations of generalized gauge actions from a view point of flow equivalence.

Let us denote by $C(X_A, \mathbb{Z})$ the set of $\mathbb{Z}$-valued continuous functions on $X_A$. For $f \in C(X_A, \mathbb{Z})$, define a one-parameter unitary group $U_t(f), t \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$ in $\mathcal{D}_A$ by

$$U_t(f) = \exp(2\pi \sqrt{-1}tf),$$

and an automorphism $\rho^A_t$ on $\mathcal{O}_A$ for each $t \in \mathbb{T}$ by

$$\rho^A_t(S_i) = U_t(f)S_i, \quad i = 1, \ldots, N.$$
For $f \equiv 1$, the action $\rho^A_1$ is the gauge action denoted by $\rho^A_t$. Suppose that $A = CD$ and $B = DC$ for some nonnegative rectangular matrices $C, D$. Then there exist homomorphisms $\varphi : C(X_A, \mathbb{Z}) \to C(X_B, \mathbb{Z})$ and $\psi : C(X_B, \mathbb{Z}) \to C(X_A, \mathbb{Z})$ such that

$$(\psi \circ \varphi)(f) = f \circ \sigma_A, \quad (\varphi \circ \psi)(g) = g \circ \sigma_B$$

for $f \in C(X_A, \mathbb{Z})$ and $g \in C(X_B, \mathbb{Z})$. Let us denote by $(H^A, H^A_\varphi)$ the ordered cohomology groups for the one-sided topological Markov shift $(X_A, \sigma_A)$ which has been introduced in [17] by setting

$$H^A = C(X_A, \mathbb{Z})/\{\eta - \eta \circ \sigma_A \mid \eta \in C(X_A, \mathbb{Z})\}$$

and its positive cone

$$H^A_+ = \{[\eta] \in H^A \mid \eta(x) \geq 0 \text{ for all } x \in X_A\}.$$

The ordered cohomology group $(H^A, H^A_\varphi)$ for $(\bar{X}_A, \bar{\sigma}_A)$ has been considered by Y. T. Poon in [17]. The latter ordered group $(\hat{H}^A, \hat{H}^A_\varphi)$ has been proved to be a complete invariant of flow equivalence of the two-sided topological Markov shift $(\bar{X}_A, \bar{\sigma}_A)$ by M. Boyle and D. Handelman in [1]. The two ordered groups $(\hat{H}^A, \hat{H}^A_\varphi)$ and $(H^A, H^A_\varphi)$ are actually isomorphic ([15], Lemma 3.1).

In [14], the following result has been proved.

**Theorem 1.2 ([14], Corollary 4.4).** Suppose that $A$ and $B$ are strong shift equivalent. Then there exist an isomorphism $\Phi : \mathcal{O}_A \otimes \mathcal{K} \to \mathcal{O}_B \otimes \mathcal{K}$ satisfying $\Phi(D_A \otimes C) = \mathcal{D}_B \otimes C$ and a homomorphism $\varphi : C(X_A, \mathbb{Z}) \to C(X_B, \mathbb{Z})$ of ordered groups which induces an isomorphism between $(H^A, H^A_\varphi)$ and $(H^B, H^B_\varphi)$ of ordered groups such that for each function $f \in C(X_A, \mathbb{Z})$ there exists a unitary one-cocycle $u^f_t \in \mathcal{U}(M(\mathcal{O}_A \otimes \mathcal{K}))$ relative to $\rho^{A,f} \otimes \text{id}$ satisfying

$$\Phi \circ \text{Ad}(u^f_t) \circ (\rho^{A,f} \otimes \text{id}) = (\rho^{B,f} \otimes \text{id}) \circ \Phi \quad \text{for } t \in \mathbb{T}.$$
In the third part of the paper, we will study the converse of the above theorem for the gauge actions. We will introduce an invariant $K_0^{\text{SSE}}(\mathcal{O}_A)$ which is a non-empty subset of $K_0(\mathcal{O}_A)$. The invariant $K_0^{\text{SSE}}(\mathcal{O}_A)$ is realized as a subset of $\mathbb{Z}^N/(\text{id} - A^t)\mathbb{Z}^N$ consisting of the classes $[v]$ of vectors $v \in \mathbb{Z}^N$ such that $v = D_1^t \cdots D_{n-1}^t D_n^t[1, 1, \ldots, 1]^t$ for some strong shift equivalences $A \approx \cdots \approx D_n C_n$ (Proposition 5.7). We will then prove the following theorem.

**Theorem 1.4** (Theorem [5,8]). Let $A, B$ be irreducible and non-permutation matrices. The following two assertions are equivalent.

(i) Two-sided topological Markov shifts $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are topologically conjugate.

(ii) There exist an isomorphism $\Phi : \mathcal{O}_A \otimes \mathcal{K} \to \mathcal{O}_B \otimes \mathcal{K}$ and a unitary representation $t \in \mathbb{T} \to u_t^A \in M(\mathcal{D}_A \otimes \mathcal{C})$ such that

$$
\Phi(\mathcal{D}_A \otimes \mathcal{C}) = \mathcal{D}_B \otimes \mathcal{C}, \quad \Phi \circ \text{Ad}(u_t^A) \circ (\rho_t^A \otimes \text{id}) = (\rho_t^B \otimes \text{id}) \circ \Phi \text{ for } t \in \mathbb{T},
$$

$$
\Phi_*(K_0^{\text{SSE}}(\mathcal{O}_A)) = K_0^{\text{SSE}}(\mathcal{O}_B).
$$

We say that $A$ has full units if $K_0^{\text{SSE}}(\mathcal{O}_A) = K_0(\mathcal{O}_A)$. The condition $K_0^{\text{SSE}}(\mathcal{O}_A) = K_0(\mathcal{O}_A)$ is able to describe in terms of the matrix $A$ as in Proposition 5.7.

**Corollary 1.5** (Corollary [5,12]). Suppose that matrices $A$ and $B$ have full units. Then two-sided topological Markov shifts $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are topologically conjugate if and only if there exist an isomorphism $\Phi : \mathcal{O}_A \otimes \mathcal{K} \to \mathcal{O}_B \otimes \mathcal{K}$ of $C^*$-algebras and a unitary representation $t \in \mathbb{T} \to u_t^A \in M(\mathcal{D}_A \otimes \mathcal{C})$ such that

$$
\Phi(\mathcal{D}_A \otimes \mathcal{C}) = \mathcal{D}_B \otimes \mathcal{C}, \quad \Phi \circ \text{Ad}(u_t^A) \circ (\rho_t^A \otimes \text{id}) = (\rho_t^B \otimes \text{id}) \circ \Phi.
$$

Throughout the paper, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{Z}_+$ the set of nonnegative integers, respectively. For one-sided topological Markov shift $(X_A, \sigma_A)$, a word $\mu = (\mu_1, \ldots, \mu_k)$ for $\mu_i \in \{1, \ldots, N\}$ is said to be admissible for $X_A$ if $(\mu_1, \ldots, \mu_k) = (x_1, \ldots, x_k)$ for some element $(x_n)_{n \in \mathbb{N}} \in X_A$. The length of $\mu$ is denoted by $|\mu| = k$. We denote by $B_k(X_A)$ the set of all admissible words of length $k$. We similarly denote by $B_k(\bar{X}_A)$ the set of admissible words of length $k$, so that $B_k(\bar{X}_A) = B_k(X_A)$. The cylinder set $\{(x_n)_{n \in \mathbb{N}} \in X_A \mid x_1 = \mu_1, \ldots, x_k = \mu_k\}$ for $\mu = (\mu_1, \ldots, \mu_k) \in B_k(X_A)$ is denoted by $U_\mu$.

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## 2 Strong Morita equivalence for Cuntz–Krieger triplets

There is a standard method to associate a Cuntz–Krieger algebra from a square matrix with entries in nonnegative integers as described in [21] Section 4. Now we suppose that $A = [A(i,j)]_{i,j=1}^N$ is an $N \times N$ matrix with entries in nonnegative integers. Then the associated graph $G_A = (V_A, E_A)$ consists of the vertex set $V_A = \{v_1^A, \ldots, v_N^A\}$ of $N$
vertices and the edge set $E_A = \{a_1, \ldots, a_{N_A}\}$, where there are $A(i, j)$ edges from $v_i$ to $v_j$.

Hence the total number of edges is $\sum_{i,j=1}^{N} A(i, j)$ denoted by $N_A$. For $a_i \in E_A$, denote by $t(a_i), s(a_i)$ the terminal vertex of $a_i$, the source vertex of $a_i$, respectively. The graph $G_A$ has the $N_A \times N_A$ transition matrix $A^G = [A^G(i, j)]_{i,j=1}^{N_A}$ of edges defined by

$$A^G(i, j) = \begin{cases} 1 & \text{if } t(a_i) = s(a_j), \\ 0 & \text{otherwise.} \end{cases} \quad (2.1)$$

The Cuntz–Krieger algebra $\mathcal{O}_A$ for the matrix $A$ with entries in nonnegative integers is defined as the Cuntz–Krieger algebra $\mathcal{O}_{AG}$ for the matrix $A^G$ which is the universal $C^*$-algebra generated by partial isometries $S_{a_i}$ indexed by edges $a_i, i = 1, \ldots, N_A$ subject to the relations:

$$\sum_{j=1}^{N_A} S_{a_j}^* S_{a_j} = 1, \quad S_{a_i}^* S_{a_i} = \sum_{j=1}^{N_A} A^G(i, j) S_{a_j}^* S_{a_j}^* \quad \text{for } i = 1, \ldots, N_A. \quad (2.2)$$

For a word $\mu = (\mu_1, \ldots, \mu_k), \mu_i \in E_Z$, we denote by $S_{\mu}$ the partial isometry $S_{\mu_1} \cdots S_{\mu_k}$.

As in the standard text books [8], [9] of symbolic dynamics, the two-sided topological Markov shift defined by a square matrix with entries in $\{0, 1\}$ is naturally topologically conjugate to a topological Markov shift of the edge shift defined by the underlying directed graph. In what follows, we consider edge shifts and hence square matrices with entries in nonnegative integers (cf. [8], [9], [24], etc.). Such a matrix is simply called a nonnegative square matrix. For a nonnegative square matrix $A$, the two-sided shift space $\hat{X}_A$ is defined by the two-sided shift space $\hat{X}_{AG}$ for the matrix $A^G$ which consists of two-sided bi-infinite sequences of concatenated edges of the directed graph $G_A$.

Suppose that two nonnegative square matrices $A$ and $B$ are elementary equivalent such that $A = CD$ and $B = DC$. The sizes of the matrices $A$ and $B$ are denoted by $N$ and $M$ respectively, so that $C$ is an $N \times M$ matrix and $D$ is an $M \times N$ matrix, respectively. We set the square matrix $Z = \begin{bmatrix} 0 & C \\ D & 0 \end{bmatrix}$ as a block matrix, and we see

$$Z^2 = \begin{bmatrix} CD & 0 \\ 0 & DC \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}.$$ 

Similarly to the directed graph $G_A = (V_A, E_A)$, let us denote by $G_B = (V_B, E_B), G_C = (V_C, E_C), G_D = (V_D, E_D)$ and $G_Z = (V_Z, E_Z)$ the associated directed graphs to the nonnegative matrices $B, C, D$ and $Z$, respectively. By the equalities $A = CD$ and $B = DC$, we may take bijections $\varphi_{A,CD}$ from $E_A$ to a subset of $E_C \times E_D$ and $\varphi_{B,DC}$ from $E_B$ to a subset of $E_D \times E_C$. Let $S_c, S_d, c \in E_C, d \in E_D$ be the generating partial isometries of the Cuntz–Krieger algebra $\mathcal{O}_Z$ for the matrix $Z$, so that $\sum_{c \in E_C} S_c S_c^* + \sum_{d \in E_D} S_d S_d^* = 1$ and

$$S_c^* S_c = \sum_{d \in E_D} Z(c, d) S_d S_d^*, \quad S_d^* S_d = \sum_{c \in E_C} Z(d, c) S_c S_c^*$$

for $c \in E_C, d \in E_D$. Since $S_c S_d \neq 0$ (resp. $S_d S_c \neq 0$) if and only if $\varphi_{A,CD}(a) = cd$ (resp. $\varphi_{B,DC}(b) = dc$) for some $a \in E_A$ (resp. $b \in E_B$), we may identify $cd$ (resp. $dc$) with $a$ (resp. $b$) through the map $\varphi_{A,CD}$ (resp. $\varphi_{B,DC}$). We may then write $S_{cd} = S_a$ (resp. $S_{dc} = S_b$)
$S_{dc} = S_b$ where $S_{cd}$ denotes $S_cS_d$ (resp. $S_{dc}$ denotes $S_dS_c$). We define two particular projections $P_C$ and $P_D$ in $D_Z$ by $P_C = \sum_{c \in E_C} S_cS_c^*$ and $P_D = \sum_{d \in E_D} S_dS_d^*$ so that $P_C + P_D = 1$. It has been shown in [10] (cf. [14]) that

$$P_C O_Z P_C = O_A, \quad P_D O_Z P_D = O_B, \quad D_Z P_C = D_A, \quad D_Z P_D = D_B. \quad (2.3)$$

As in [10] Lemma 3.1, both projections $P_C$ and $P_D$ are full projections so that $P_C O_Z P_D$ has a natural structure of $O_A - O_B$ imprimitivity bimodule that makes $O_A$ and $O_B$ strong Morita equivalent (cf. [19], [20]).

Let $\rho^Z, \rho^A, \rho^B$ be the gauge actions of $O_Z, O_A, O_B$, respectively. Since $S_cS_d$ (resp. $S_dS_c$) in $O_Z$ is identified with $S_b$ in $O_A$ (resp. $S_b$ in $O_B$) if $\varphi_ACD(a) = cd$ (resp. $\varphi_BDC(b) = dc$, we have

$$\rho_t^Z |_{P_C O_Z P_C} = \rho_{2t}^A \text{ on } O_A, \quad \rho_t^Z |_{P_D O_Z P_D} = \rho_{2t}^B \text{ on } O_B. \quad (2.4)$$

Let $A$ be an irreducible non-permutation matrix. The triplet $(O_A, D_A, \rho^A)$ for the Cuntz-Krieger algebra $O_A$, its canonical maximal abelian $C^*$-subalgebra $D_A$, and its gauge action $\rho^A$ is called the Cuntz–Krieger triplet for the matrix $A$. In this section we will define the notion of strong Morita equivalence in Cuntz–Krieger triplets. We will then prove that the Cuntz–Krieger triplets $(O_A, D_A, \rho^A)$ and $(O_B, D_B, \rho^B)$ are strong Morita equivalent if and only if the matrices $A$ and $B$ are strong shift equivalent. Let $A, B$ be irreducible non-permutation matrices.

**Definition 2.1.** The Cuntz–Krieger triplets $(O_A, D_A, \rho^A)$ and $(O_B, D_B, \rho^B)$ are said to be **strong Morita equivalent in 1-step** if there exist a Cuntz–Krieger triplet $(O_Z, D_Z, \rho^Z)$ for some nonnegative matrix $Z$ and projections $P_A, P_B \in D_Z$ having the following properties:

1. $P_A + P_B = 1$,
2. $P_A O_Z P_A = O_A$ and $P_B O_Z P_B = O_B$,
3. $D_Z P_A = D_A$ and $D_Z P_B = D_B$,
4. $\rho_t^Z |_{P_A O_Z P_A} = \rho_{2t}^A$ on $O_A$ and $\rho_t^Z |_{P_B O_Z P_B} = \rho_{2t}^B$ on $O_B$ for $t \in \mathbb{T}$.

In this case, we say that $(O_A, D_A, \rho^A)$ and $(O_B, D_B, \rho^B)$ are strong Morita equivalent in 1-step via $(O_Z, D_Z, \rho^Z)$. If two Cuntz–Krieger triplets $(O_A, D_A, \rho^A)$ and $(O_B, D_B, \rho^B)$ are connected through $n$-chains of strong Morita equivalences in 1-step, $(O_A, D_A, \rho^A)$ and $(O_B, D_B, \rho^B)$ are said to be strong Morita equivalent in $n$-step, or simply, strong Morita equivalent.

We note that if there exists an isomorphism $\Phi : O_A \to O_B$ satisfying $\Phi(D_A) = D_B$ and $\Phi \circ \rho^A_t = \rho^B_t \circ \Phi$, $t \in \mathbb{T}$, then the one-sided topological Markov shifts $(X_A, \sigma_A)$ and $(X_B, \sigma_B)$ are eventually conjugate ([14] Corollary 3.5]), so that their two-sided topological Markov shifts $(\tilde{X}_A, \tilde{\sigma}_A)$ and $(\tilde{X}_B, \tilde{\sigma}_B)$ are topologically conjugate by [13] Theorem 5.5] (cf. [13] Theorem 6.7], and hence the matrices $A$ and $B$ are strong shift equivalent.

**Proposition 2.2.** If $A$ and $B$ are elementary equivalent, then their Cuntz–Krieger triplets $(O_A, D_A, \rho^A)$ and $(O_B, D_B, \rho^B)$ are strong Morita equivalent in 1-steps,
Proof. Suppose that $A$ and $B$ are elementary equivalent such that $A = CD, B = DC$. Let $Z$ be the square matrix $Z = \begin{bmatrix} 0 & C \\ D & 0 \end{bmatrix}$. By the above discussions, there exist projections $P_C, P_D$ in $D_Z$ satisfying $P_C + P_D = 1$ and (2.3) (2.4). \hfill \Box

The main purpose of this section is to study the converse implication of Proposition 2.2.

We henceforth assume that $(O_A, D_A, \rho^A)$ and $(O_B, D_B, \rho^B)$ are strong Morita equivalent in 1-step via $(O_Z, D_Z, \rho^Z)$ for some matrix $Z$. We may take two projections $P_A, P_B$ in $D_Z$ having the properties (1), (2), (3) and (4) in Definition 2.1. Let us denote by $G_Z = (V_Z, E_Z)$ the directed graph for the matrix $Z$. The Cuntz–Krieger algebra $O_Z$ is then generated by partial isometries $S_{\gamma, \gamma} \in E_Z$ satisfying the relations:

$$\sum_{\eta \in E_Z} S_{\eta} S^*_{\eta} = 1, \quad S^*_{\gamma} S_{\gamma} = \sum_{\eta \in E_Z} Z^G(\gamma, \eta) S_{\eta} S^*_{\eta} \quad \text{for} \, \gamma \in E_Z$$

(2.5)

where $Z^G(\gamma, \eta) = 1$ if $t(\gamma) = s(\eta)$, and 0 otherwise. We have the following lemmas.

Lemma 2.3. Let $S_{\gamma, \gamma} \in E_Z$ be the generating partial isometries of $O_Z$ satisfying (2.5). Then we have

(i) $P_A S_{\gamma} P_A = P_B S_{\gamma} P_B = 0$.

(ii) $S_{\gamma} = P_A S_{\gamma} P_B + P_B S_{\gamma} P_A$.

(iii) $P_A S_{\gamma} = S_{\gamma} P_B$ and $P_B S_{\gamma} = S_{\gamma} P_A$.

Proof. By the equality $P_A + P_B = 1$, we have

$$S_{\gamma} = P_A S_{\gamma} P_A + P_A S_{\gamma} P_B + P_B S_{\gamma} P_A + P_B S_{\gamma} P_B.$$  

Since $P_A S_{\gamma} P_A$ belongs to $P_A O_Z P_A$ which is identified with $O_A$, the condition (4) of Definition 2.1 gives rise to the equality

$$\rho^Z_1 (P_A S_{\gamma} P_A) = \rho^A_2 (P_A S_{\gamma} P_A).$$

(2.6)

As $\rho^Z_1 |_{D_Z} = \text{id}$ and $P_A, P_B \in D_Z$, the left hand side for $t = \frac{1}{2}$ of (2.6) goes to

$$P_A \rho^Z_1 (S_{\gamma}) P_A = -P_A S_{\gamma} P_A.$$  

(2.7)

As $\rho^A_1 = \text{id}$, the right hand side for $t = \frac{1}{2}$ goes to $P_A S_{\gamma} P_A$. Hence we have $P_A S_{\gamma} P_A = 0$ and similarly $P_B S_{\gamma} P_B = 0$. Therefore we know (i), (ii) and (iii). \hfill \Box

Lemma 2.4.

$$\sum_{\gamma \in E_Z} S_{\gamma} P_A S^*_{\gamma} = P_B, \quad \sum_{\gamma \in E_Z} S_{\gamma} P_B S^*_{\gamma} = P_A.$$  

(2.8)

Proof. By Lemma 2.3 we know $S_{\gamma} P_A = P_B S_{\gamma}$ so that

$$\sum_{\gamma \in E_Z} S_{\gamma} P_A S^*_{\gamma} = \sum_{\gamma \in E_Z} P_B S_{\gamma} S^*_{\gamma} = P_B.$$  

(2.9)

Similarly we see that $\sum_{\gamma \in E_Z} S_{\gamma} P_B S^*_{\gamma} = P_A$. \hfill \Box
We notice the following identities which immediately come from Lemma 2.3 (iii).

**Lemma 2.5.** For $\gamma_1, \gamma_2 \in E_Z$, we have the following identities.

(i) $S_{\gamma_1} S_{\gamma_2} P_A = P_A S_{\gamma_1} S_{\gamma_2} \in O_A$ and $S_{\gamma_1} S_{\gamma_2} P_B = P_B S_{\gamma_1} S_{\gamma_2} \in O_B$.

(ii) $S_{\gamma_1} P_B S_{\gamma_2} = P_A S_{\gamma_1} P_B S_{\gamma_2} \in O_A$ and $S_{\gamma_1} P A S_{\gamma_2} = P_B S_{\gamma_1} P A S_{\gamma_2} P_B \in O_B$.

**Lemma 2.6.** Let $\gamma_1, \gamma_2 \in E_Z$. Then $P_A S_{\gamma_1} \neq 0$, $P_B S_{\gamma_2} \neq 0$ and $Z^G(\gamma_1, \gamma_2) = 1$ if and only if $P_A S_{\gamma_1} S_{\gamma_2} \neq 0$.

**Proof.** The if part is obvious. It suffices to show the only if part. Since $P_A S_{\gamma_1} S_{\gamma_2} = P_A S_{\gamma_1} P_B S_{\gamma_2} = S_{\gamma_1} S_{\gamma_2} P_A$, we have

$$(S_{\gamma_1} S_{\gamma_2} P_A)^* S_{\gamma_1} S_{\gamma_2} P_A = P_A S_{\gamma_1} S_{\gamma_2} P_A = \sum_{\gamma_1, \gamma_2} Z^G(\gamma_1, \gamma_2) P_A S_{\gamma_1} S_{\gamma_2} P_A = Z^G(\gamma_1, \gamma_2) P_A S_{\gamma_1} S_{\gamma_2} P_A = Z^G(\gamma_1, \gamma_2)(P_B S_{\gamma_2})^*(P_B S_{\gamma_2}).$$

The above equalities ensure us the only if part. \(\square\)

**Lemma 2.7.** Let $\gamma_1, \gamma_2, \eta_1, \eta_2 \in E_Z$. Then $S_{\gamma_1} S_{\gamma_2} \neq 0, S_{\gamma_2} S_{\eta_1} \neq 0, P_A S_{\eta_1} S_{\eta_2} \neq 0$ if and only if $P_A S_{\gamma_1} S_{\gamma_2} S_{\eta_1} S_{\eta_2} \neq 0$.

**Proof.** Since $P_A S_{\gamma_1} S_{\gamma_2} S_{\eta_1} S_{\eta_2} = S_{\gamma_1} S_{\gamma_2} P_A S_{\eta_1} S_{\eta_2}$, the if part is obvious. It suffices to show the only if part. We have

$$(P_A S_{\gamma_1} S_{\gamma_2} S_{\eta_1} S_{\eta_2})^*(P_A S_{\gamma_1} S_{\gamma_2} S_{\eta_1} S_{\eta_2}) = P_A S_{\gamma_2} S_{\eta_1} S_{\eta_2} P_A = \sum_{\gamma_1, \gamma_2} Z^G(\gamma_1, \gamma_2) P_A S_{\gamma_2} S_{\eta_1} S_{\eta_2} P_A = Z^G(\gamma_1, \gamma_2) P_A S_{\gamma_2} S_{\eta_1} S_{\eta_2} P_A = Z^G(\gamma_1, \gamma_2)(P_B S_{\gamma_2})^*(P_B S_{\gamma_2}).$$

The above equalities ensure us the only if part. \(\square\)

Now we are assuming that the Cuntz–Krieger triplets $(O_A, D_A, \rho^A)$ and $(O_B, D_B, \rho^B)$ are strong Morita equivalent in 1-step via $(O_Z, D_Z, \rho^Z)$. We introduce several directed graphs in this situation. Define edge sets $E_{\tilde{A}}, E_{\tilde{B}}, E_{\tilde{C}}, E_{\tilde{D}}$ by setting

$E_{\tilde{A}} = \{(A, \gamma_1) \in \{A\} \times B_2(X_Z) \mid P_A S_{\gamma_1} S_{\gamma_2} \neq 0\},$

$E_{\tilde{B}} = \{(B, \gamma_1) \in \{B\} \times B_2(X_Z) \mid P_B S_{\gamma_1} S_{\gamma_2} \neq 0\},$

$E_{\tilde{C}} = \{(A, \gamma_1) \in \{A\} \times E_Z \mid P_A S_{\gamma_1} \neq 0\},$

$E_{\tilde{D}} = \{(B, \gamma_1) \in \{B\} \times E_Z \mid P_B S_{\gamma_1} \neq 0\}.$
and vertex sets $V_{A s}, V_{A t}, V_{B s}, V_{B t}, V_{C s}, V_{C t}, V_{D s}, V_{D t}$ by setting

\[
V_{A s} = \{(A, s(\gamma_1)) \in \{A\} \times V \mid (A, \gamma_1 \gamma_2) \in E_A\},
\]

\[
V_{A t} = \{(A, t(\gamma_2)) \in \{A\} \times V \mid (A, \gamma_1 \gamma_2) \in E_A\},
\]

\[
V_{B s} = \{(B, s(\gamma_1)) \in \{B\} \times V \mid (B, \gamma_1 \gamma_2) \in E_B\},
\]

\[
V_{B t} = \{(B, t(\gamma_1)) \in \{B\} \times V \mid (B, \gamma_1 \gamma_2) \in E_B\},
\]

\[
V_{C s} = \{(A, s(\gamma_1)) \in \{A\} \times V \mid (A, \gamma_1) \in E_C\},
\]

\[
V_{C t} = \{(B, t(\gamma_1)) \in \{A\} \times V \mid (A, \gamma_1) \in E_C\},
\]

\[
V_{D s} = \{(B, s(\gamma_1)) \in \{B\} \times V \mid (B, \gamma_1) \in E_D\},
\]

\[
V_{D t} = \{(A, t(\gamma_1)) \in \{A\} \times V \mid (B, \gamma_1) \in E_D\}.
\]

Lemma 2.8. Keep the above notations. We have

(i) $V_{A s} = V_{A t} = V_{C s} = V_{D t}$.

(ii) $V_{B s} = V_{B t} = V_{D s} = V_{C t}$.

Proof. (i) We will first show the equality $V_{A s} = V_{A t}$. Take an arbitrary vertex $(A, s(\gamma_1)) \in V_{A s}$ and $\gamma_2 \in E_Z$ with $P_A S_\gamma_1, S_\gamma_2 \neq 0$, so that $t(\gamma_1) = s(\gamma_2)$. We may find $\eta_1, \eta_2 \in E_Z$ such that $S_{\eta_1} S_{\eta_2} \neq 0$ and $t(\eta_2) = s(\gamma_1)$. By Lemma 2.7, we have $S_{\eta_1} S_{\eta_2} S_{\eta_2} P_A \neq 0$. Since $S_{\gamma_1} S_{\gamma_2} P_A = P_A S_{\gamma_1} S_{\gamma_2} S_{\gamma_2},$ we have $P_A S_{\gamma_1} S_{\gamma_2} \neq 0$ so that $(A, t(\eta_2)) \in V_{A t}$ and hence $(A, s(\gamma_1)) \in V_{A t}$. This shows that the inclusion relation $V_{A s} \subset V_{A t}$ holds. Similarly we know that $V_{A t} \subset V_{A s}$ so that $V_{A s} = V_{A t}$.

We will second show the equality $V_{C s} = V_{D t}$. Take an arbitrary vertex $(A, s(\gamma_1)) \in V_{C s}$. We see that $P_A S_{\gamma_1} \neq 0$ and hence $S_{\gamma_1} P_B \neq 0$. As $\sum_{\gamma' \in E_Z} S_{\gamma' \gamma} S_{\gamma'} \geq 1$, we may find $\gamma_2 \in E_Z$ such that $S_{\gamma_2} S_{\gamma_1} P_B \neq 0$ so that $t(\gamma_2) = s(\gamma_1)$. Since $S_{\gamma_2} S_{\gamma_1} P_B = P_B S_{\gamma_2} S_{\gamma_1},$ we have $P_B S_{\gamma_2} \neq 0$. This implies that $(B, \gamma_2) \in E_D$ and $(A, t(\gamma_2)) \in V_{D t}$. As $t(\gamma_2) = s(\gamma_1)$, we obtain that $(A, s(\gamma_1)) \in V_{D t}$ so that $V_{C s} \subset V_{D t}$. We similarly see that $V_{D t} \subset V_{C s}$ so that $V_{C s} = V_{D t}$.

We will finally show that $V_{A s} = V_{C s}$. Since the condition $P_A S_{\gamma_1}, S_{\gamma_2} \neq 0$ implies $P_A S_{\gamma_1} \neq 0$, we have $V_{A s} \subset V_{C s}$. Conversely, for $(A, s(\gamma_1)) \in V_{C s}$, we have $P_A S_{\gamma_1} \neq 0$ so that $S_{\gamma_1} P_B \neq 0$. Since $P_B = \sum_{\gamma' \in E_Z} S_{\gamma' \gamma} P_A S_{\gamma' \gamma},$ we may find $\gamma_2 \in E_Z$ such that $S_{\gamma_2} S_{\gamma_1} P_B \neq 0$. Hence we see that $P_A S_{\gamma_1} S_{\gamma_2} \neq 0$ so that $(A, s(\gamma_1)) \in V_{A s}$. This shows that $V_{A s} = V_{C s}$. Therefore (i) has been shown. (ii) is similarly shown.

Let us denote by $V_A$ and by $V_B$ the first four vertex sets and the second four vertex sets in Lemma 2.8 respectively. Namely we put

\[
V_A := V_{A s} = V_{A t} = V_{C s} = V_{D t},
\]

\[
V_B := V_{B s} = V_{B t} = V_{D s} = V_{C t}.
\]

For an edge $(A, \gamma_1 \gamma_2) \in E_A$, define its source and terminal vertices by

\[
s(A, \gamma_1 \gamma_2) = (A, s(\gamma_1)) \in V_{A s}, \quad t(A, \gamma_1 \gamma_2) = (A, t(\gamma_2)) \in V_{A t}.
\]

We then have a directed graph $(V_A, E_A)$ denoted by $G_A$. We similarly have a directed graph $G_B = (V_B, E_B)$. From an edge $(A, \gamma_1) \in E_C$, define its source and terminal vertices by

\[
s(A, \gamma_1) = (A, s(\gamma_1)) \in V_{C s}, \quad t(A, \gamma_1) = (A, t(\gamma_1)) \in V_{C t}.
\]
We have a directed graph $G_\tilde{C} = (V_\tilde{A} \xrightarrow{E_\tilde{C}} V_\tilde{B})$ and similarly $G_\tilde{D} = (V_B \xrightarrow{E_\tilde{D}} V_A)$.

Let $\tilde{A}$ be the vertex transition matrix $\tilde{A} : V_\tilde{A} \times V_\tilde{A} \rightarrow \mathbb{Z}_+$ of the directed graph $G_\tilde{A}$ which is defined by

$$\tilde{A}((A,u),(A,v)) = |\{(A, \gamma_1 \gamma_2) \in E_\tilde{A} \mid s(\gamma_1) = u, t(\gamma_2) = v\}|$$

for $(A,u),(A,v) \in V_\tilde{A}$. The edge transition matrix $\tilde{A}^G : E_\tilde{A} \times E_\tilde{A} \rightarrow \{0,1\}$ of $G_\tilde{A}$ is defined by

$$\tilde{A}^G(\gamma_1 \gamma_2, \eta \eta_2) = \begin{cases} 1 & \text{if } t(A, \gamma_1 \gamma_2) = s(A, \eta \eta_2), \\ 0 & \text{otherwise} \end{cases}$$

for $(A, \gamma_1 \gamma_2), (\eta, \eta_2) \in E_\tilde{A}$. We similarly have the vertex transition matrices $\tilde{B}, \tilde{C}, \tilde{D}$ and the edge transition matrices $\tilde{B}^G, \tilde{C}^G, \tilde{D}^G$ of the directed graphs $G_\tilde{B}, G_\tilde{C}, G_\tilde{D}$, respectively.

**Proposition 2.9.** The matrices $\tilde{A}$ and $\tilde{B}$ are elementary equivalent such that

$$\tilde{A} = \tilde{C} \tilde{D} \quad \text{and} \quad \tilde{B} = \tilde{D} \tilde{C}.$$

Hence $\tilde{A}^G = \tilde{C}^G \tilde{D}^G$ and $\tilde{B}^G = \tilde{D}^G \tilde{C}^G$, and the two-sided topological Markov shifts $(X_\tilde{A}, \tilde{\sigma}_A)$ and $(X_\tilde{B}, \tilde{\sigma}_B)$ are topologically conjugate.

**Proof.** For $(A, \gamma_1 \gamma_2)$ with $\gamma_1, \gamma_2 \in E_Z$, Lemma 2.8 ensures us that $(A, \gamma_1) \in E_\tilde{C}, (B, \gamma_2) \in E_\tilde{D}$, $Z^G(\gamma_1 \gamma_2) = 1$ if and only if $(A, \gamma_1 \gamma_2) \in E_\tilde{A}$. Since $t(A, \gamma_1) = s(B, \gamma_2)$ if and only if $Z^G(\gamma_1 \gamma_2) = 1$, we know that $\tilde{A} = \tilde{C} \tilde{D}$, and similarly $\tilde{B} = \tilde{D} \tilde{C}$. The relations $\tilde{A}^G = \tilde{C}^G \tilde{D}^G$ and $\tilde{B}^G = \tilde{D}^G \tilde{C}^G$ automatically come from $\tilde{A} = \tilde{C} \tilde{D}$ and $\tilde{B} = \tilde{D} \tilde{C}$.

Let $E_\tilde{Z} = E_\tilde{C} \cup E_\tilde{D}$ and $V_\tilde{Z} = V_\tilde{A} \cup E_\tilde{B}$. We have a bipartite directed graph $G_\tilde{Z} = (V_\tilde{Z}, E_\tilde{Z})$. Let us denote by $\tilde{Z}$ and $\tilde{Z}^G$ the vertex transition matrix and the edge transition matrix of the directed graph $G_\tilde{Z}$, respectively. Since $G_\tilde{Z}$ is bipartite, by the above proposition, we have

$$\tilde{Z} = \begin{bmatrix} 0 & C \\ \tilde{D} & 0 \end{bmatrix}, \quad \tilde{Z}^2 = \begin{bmatrix} \tilde{A} & 0 \\ 0 & \tilde{B} \end{bmatrix}.$$

We will study the relationship between the two matrices $\tilde{Z}$ and $Z$. For $\gamma \in E_Z$, denote by $S_{(A,\gamma)}, S_{(B,\gamma)}$ the partial isometries $P_A S_\gamma, P_B S_\gamma$, respectively, so that $S_\gamma = S_{(A,\gamma)} + S_{(B,\gamma)}$.

**Lemma 2.10.** Let $\gamma_1, \gamma_2 \in E_Z$ satisfy $Z^G(\gamma_1, \gamma_2) = 1$.

(i) $S_{(B,\gamma_2)} \neq 0$ implies $S_{(A,\gamma_1)} \neq 0$.

(ii) $S_{(A,\gamma_2)} \neq 0$ implies $S_{(B,\gamma_1)} \neq 0$.

**Proof.** (i) Since $S_{(A,\gamma_1)} S_{(B,\gamma_2)} = P_A S_{\gamma_1} P_B S_{\gamma_2} = S_{\gamma_1} S_{\gamma_2} P_A$, we have

$$(S_{(A,\gamma_1)} S_{(B,\gamma_2)})^* (S_{(A,\gamma_1)} S_{(B,\gamma_2)}) = P_A S_{\gamma_2}^* S_{\gamma_1}^* S_{\gamma_1} S_{\gamma_2} P_A$$

$$= \sum_{\eta_1 \in E_Z} Z^G(\gamma_1, \eta_1) P_A S_{\gamma_2}^* S_{\eta_1}^* S_{\eta_1} S_{\gamma_2} P_A$$

$$= Z^G(\gamma_1, \gamma_2) S_{(B,\gamma_2)} S_{(B,\gamma_2)}.$$ 

The above equality ensures us the assertion. (ii) is similarly shown.
Lemma 2.11. Either of the following two situations occurs:

1. Both $S_{(A,\gamma)}$ and $S_{(B,\gamma)}$ are not zero for all $\gamma \in E_Z$. In this case we have $\bar{G}^G = \bar{D}^G = Z^G$ so that $\bar{A} = \bar{B}$ and $\bar{Z} = \begin{bmatrix} 0 & Z \\ Z & 0 \end{bmatrix}$.

2. Either $S_{(A,\gamma)} = 0$ or $S_{(B,\gamma)} = 0$ for all $\gamma \in E_Z$. In this case we have $\bar{Z} = Z$.

Proof. Suppose that there exists $\gamma_0 \in E_Z$ such that both conditions $S_{(A,\gamma_0)} \neq 0$ and $S_{(B,\gamma_0)} \neq 0$ hold. By the preceding lemma, any edge $\eta \in E_Z$ satisfying $Z^G(\eta, \gamma_0) = 1$ forces that $S_{(A,\eta)} \neq 0$ and $S_{(B,\eta)} \neq 0$. Since for any edge $\gamma \in E_Z$, there exists a finite sequence of edges $\gamma_1, \ldots, \gamma_n$ in $E_Z$ such that

$$Z^G(\eta, \gamma_1) = Z^G(\gamma_1, \gamma_2) = \cdots = Z^G(\gamma_n, \gamma_0) = 1$$

so that $S_{(A,\gamma)} \neq 0$ and $S_{(B,\gamma)} \neq 0$. Hence either of the following two cases occurs:

1. Both $S_{(A,\gamma)}$ and $S_{(B,\gamma)}$ are not zero for all $\gamma \in E_Z$.

2. Either $S_{(A,\gamma)} = 0$ or $S_{(B,\gamma)} = 0$ for all $\gamma \in E_Z$.

Case (1): We have the following equalities.

$$S^*_\gamma S_\gamma = (S^*_{(A,\gamma)} + S^*_{(B,\gamma)})(S_{(A,\gamma)} + S_{(B,\gamma)})$$

$$= S^*_{(A,\gamma)} S_{(A,\gamma)} + S^*_{(B,\gamma)} S_{(B,\gamma)}$$

$$= \sum_{(B,\eta) \in E_D} \bar{G}^G((A, \gamma), (B, \eta)) S_{(B,\eta)} S^*_{(B,\eta)} + \sum_{(A,\eta) \in E_C} \bar{D}^G((B, \gamma), (A, \eta)) S_{(A,\eta)} S^*_{(A,\eta)}.$$

On the other hand, we have

$$S^*_\gamma S_\gamma = \sum_{\eta \in E_Z} Z^G(\gamma, \eta) S_\eta S^*_\eta$$

$$= \sum_{\eta \in E_Z} Z^G(\gamma, \eta)(P_B S_\eta S^*_\eta P_B + P_A S_\eta S^*_\eta P_A)$$

$$= \sum_{\eta \in E_Z} Z^G(\gamma, \eta) S_{(B,\eta)} S^*_{(B,\eta)} + \sum_{\eta \in E_Z} Z^G(\gamma, \eta) S_{(A,\eta)} S^*_{(A,\eta)}.$$

Since both $S_{(A,\gamma)} \neq 0$ and $S_{(B,\gamma)} \neq 0$ for all $\gamma \in E_Z$, we have

$$\bar{G}^G((A, \gamma), (B, \eta)) = Z^G(\gamma, \eta), \quad \bar{D}^G((B, \gamma), (A, \eta)) = Z^G(\gamma, \eta)$$

for all $\gamma, \eta \in E_Z$. Hence we have $\bar{G}^G = \bar{D}^G = Z^G$ so that $\bar{A} = \bar{B}$ and hence $\bar{Z} = \bar{Z} = \begin{bmatrix} 0 & Z \\ Z & 0 \end{bmatrix}$.

Case (2): Since either $S_{(A,\gamma)} \neq 0$ or $S_{(B,\gamma)} \neq 0$ for all $\gamma \in E_Z$ occurs, we have a disjoint union $E_Z = E_C \cup E_D$. As $S_{(A,\gamma_1)} S_{(A,\gamma_2)} = 0$, $S_{(B,\gamma_1)} S_{(B,\gamma_2)} = 0$ for all $\gamma_1, \gamma_2 \in E_Z$, we have $Z = \begin{bmatrix} 0 & \bar{C} \\ \bar{D} & 0 \end{bmatrix}$ so that $\bar{Z} = Z$. □

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We will next study the bipartite graph $G_Z$ from the $C^*$-algebraic view point. For $(A, \gamma_1 \gamma_2) \in E_{\hat{A}}$, define the partial isometry

$$S_{(A, \gamma_1 \gamma_2)} = P_A S_{\gamma_1} S_{\gamma_2}.$$

**Lemma 2.12.** The $C^*$-subalgebra $C^*(S_{(A, \gamma_1 \gamma_2)}; (A, \gamma_1 \gamma_2) \in E_{\hat{A}})$ of $O_Z$ is isomorphic to the Cuntz–Krieger algebra $O_{\hat{A}}$ for the matrix $\hat{A}$.

**Proof.** We first notice that

$$\sum_{(A, \gamma_1 \gamma_2) \in E_{\hat{A}}} S_{(A, \gamma_1 \gamma_2)} S^*_{(A, \gamma_1 \gamma_2)} = \sum_{\gamma_1 \gamma_2 \in E_Z} P_A S_{\gamma_1} S^*_{\gamma_1} S_{\gamma_2} P_A = P_A.$$

We also have

$$S^*_{(A, \gamma_1 \gamma_2)} S_{(A, \gamma_1 \gamma_2)} = P_A S^*_{\gamma_2} S^*_{\gamma_1} S_{\gamma_1} S_{\gamma_2} P_A$$

$$= \sum_{\zeta_1 \in E_Z} Z^G(\gamma_1, \zeta_1) P_A S^*_{\gamma_2} S^*_{\gamma_1} S_{\gamma_1} S_{\gamma_2} P_A$$

$$= \sum_{\eta \in E_Z} Z^G(\gamma_1, \gamma_2) Z^G(\gamma_2, \eta_1) P_A S_{\eta_1} S^*_{\eta_1} P_A$$

$$= \sum_{\eta_1, \eta_2 \in E_Z} Z^G(\gamma_1, \gamma_2) Z^G(\gamma_2, \eta_1) Z^G(\eta_1, \eta_2) P_A S_{\eta_1} S_{\eta_2} S^*_{\eta_2} S^*_{\eta_1} P_A$$

For $(A, \gamma_1 \gamma_2), (A, \eta_1 \eta_2) \in E_{\hat{A}}$, the condition $t(A, \gamma_1 \gamma_2) = s(A, \eta_1 \eta_2)$ holds if and only if $Z^G(\gamma_2, \eta_1) = 1$. Hence we know

$$Z^G(\gamma_1, \gamma_2) Z^G(\gamma_2, \eta_1) Z^G(\eta_1, \eta_2) = \hat{A}^G(\gamma_1 \gamma_2, \eta_1 \eta_2).$$

By the above equalities, we have

$$S^*_{(A, \gamma_1 \gamma_2)} S_{(A, \gamma_1 \gamma_2)} = \sum_{(A, \eta_1 \eta_2) \in E_{\hat{A}}} \hat{A}^G(\gamma_1 \gamma_2, \eta_1 \eta_2) S_{(A, \eta_1 \eta_2)} S^*_{(A, \eta_1 \eta_2)}.$$

Hence the $C^*$-subalgebra $C^*(S_{(A, \gamma_1 \gamma_2)}; (A, \gamma_1 \gamma_2) \in E_{\hat{A}})$ of $O_Z$ is isomorphic to the Cuntz–Krieger algebra $O_{\hat{A}}$ for the matrix $\hat{A}$. \(\Box\)

**Lemma 2.13.** The $C^*$-subalgebra $C^*(S_{(A, \gamma_1 \gamma_2)}; (A, \gamma_1 \gamma_2) \in E_{\hat{A}})$ of $O_Z$ is nothing but $P_A O_Z P_A$. Hence the Cuntz–Krieger algebra $O_{\hat{A}}$ is isomorphic to $O_A$.

**Proof.** Since $S_{(A, \gamma_1 \gamma_2)} = P_A S_{\gamma_1} S_{\gamma_2} P_A$ for $(A, \gamma_1 \gamma_2) \in E_{\hat{A}}$, we have $C^*(S_{(A, \gamma_1 \gamma_2)}; (A, \gamma_1 \gamma_2) \in E_{\hat{A}}) \subset P_A O_Z P_A$. We will show the converse inclusion relation. Take an arbitrary fixed $X \in O_Z$ with $P_A X P_A \neq 0$. Let $P_Z$ be the dense $*$-subalgebra of $O_Z$ algebraically generated by $S_\gamma, \gamma \in E_Z$. We may find $X_n \in P_Z$ such that $\|X - X_n\| \to 0$. Since $\|P_A X P_A - P_A X_n P_A\| \leq \|X - X_n\| \to 0$, it suffices to show that $P_A X_n P_A$ belongs to $C^*(S_{(A, \gamma_1 \gamma_2)}; (A, \gamma_1 \gamma_2) \in E_{\hat{A}})$. By [7] 2.2 Lemma, any element of the subalgebra $P_Z$ is a finite linear combination of elements of the form $S_{\mu_1} S_{\mu_2} S^*_{\nu_1} S^*_{\nu_2}$ for some $\mu = (\mu_1, \ldots, \mu_m), \nu = (\nu_1, \ldots, \nu_n) \in B_s(X_Z)$. Assume that $P_A S_{\mu_1} S_{\mu_2} S^*_{\nu_1} S^*_{\nu_2} P_A \neq 0$. Since $P_A S_{\mu} = S_{\mu} P_B$, we have

$$P_A S_{\mu} = P_A S_{\mu_1} \cdots S_{\mu_m} = \begin{cases} S_{\mu_1} \cdots S_{\mu_m} P_A & \text{if } m \text{ is even}, \\ S_{\mu_1} \cdots S_{\mu_m} P_B & \text{if } m \text{ is odd}. \end{cases} \quad (2.9)$$

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The assumption $P_A S_{\mu} S_i S^*_\nu P_A \neq 0$ forces the numbers $m, n$ to be both even, or both odd.

Case 1: $m, n$ are both even. We have

$$P_A S_{\mu} S_i S^*_\nu P_A = P_A S_{\mu_1} S_{\mu_2} P_A S_{\mu_3} S_{\mu_4} P_A \cdots P_A S_{\mu_{m-1}} S_{\mu_m} P_A S_i S^*_\nu P_A \cdots S^*_\nu P_A S_{\nu_{m-1}} P_A \cdots S^*_{\nu_1} P_A S_{\nu_1} S^*_\nu P_A = S_{(A,\mu_1 \mu_2)} S_{(A,\mu_2 \mu_3)} \cdots S_{(A,\mu_{m-1} \mu_m)} P_A S_i S^*_\nu P_A S^*_{(A,\nu_{m-1} \nu_m)} \cdots S^*_{(A,\nu_1 \nu_2)} S^*_{(A,\nu_1 \nu_2)}.$$  

Now we have

$$P_A S_i S^*_\nu P_A = \sum_{j \in E_Z} P_A S_j S^*_j S^*_i P_A = \sum_{j \in E_Z} S_{(A,ij)} S^*_{(A,ij)}$$

so that $P_A S_{\mu} S_i S^*_\nu P_A$ is a finite linear combination of products of the elements $S_{(A,\gamma_1 \gamma_2)}, S^*_{(A,\gamma_1 \gamma_2)}$ for $(A, \gamma_1 \gamma_2) \in E_{\tilde{A}}$ and hence it belongs to $C^*(S_{(A,\gamma_1 \gamma_2)}; (A, \gamma_1 \gamma_2) \in E_{\tilde{A}}).$

Case 2: $m, n$ are both odd. Similarly to Case 1, we have

$$P_A S_{\mu} S_i S^*_\nu P_A = S_{(A,\mu_1 \mu_2)} \cdots S_{(A,\mu_{m-2} \mu_{m-1})} S_{(A,\mu_{m-1} \mu_m)} S^*_{(A,\nu_{m-2} \nu_{m-1})} \cdots S^*_{(A,\nu_1 \nu_2)}$$

so that $P_A S_{\mu} S_i S^*_\nu P_A$ belongs to $C^*(S_{(A,\gamma_1 \gamma_2)}; (A, \gamma_1 \gamma_2) \in E_{\tilde{A}}).$

\[ \square \]

**Proposition 2.14.** The Cuntz–Krieger triplet $(O_{\tilde{A}}, \mathcal{D}_{\tilde{A}}, \rho^{\tilde{A}})$ for the matrix $\tilde{A}$ is isomorphic to $(O_A, \mathcal{D}_A, \rho^A).$

**Proof.** By Lemma 2.12 and Lemma 2.13 we know that

$$O_{\tilde{A}} = C^*(S_{(A,\gamma_1 \gamma_2)}; (A, \gamma_1 \gamma_2) \in E_{\tilde{A}}) = P_A O_Z P_A = O_A. \tag{2.10}$$

Under the identification between $C^*(S_{(A,\gamma_1 \gamma_2)}; (A, \gamma_1 \gamma_2) \in E_{\tilde{A}})$ and $P_A O_Z P_A$ in Lemma 2.13 the $C^*$-subalgebra

$$C^*(S_{(A,\gamma_1 \gamma_2)} \cdots S_{(A,\gamma_{n-1} \gamma_n)} S^*_{(A,\gamma_{n-1} \gamma_n)} \cdots S^*_{(A,\gamma_1 \gamma_2)}; (A, \gamma_1 \gamma_2), \ldots, (A, \gamma_{n-1} \gamma_n) \in E_{\tilde{A}})$$

of $C^*(S_{(A,\gamma_1 \gamma_2)}; (A, \gamma_1 \gamma_2) \in E_{\tilde{A}})$ generated by the projections

$$S_{(A,\gamma_1 \gamma_2)} \cdots S_{(A,\gamma_{n-1} \gamma_n)} S^*_{(A,\gamma_{n-1} \gamma_n)} \cdots S^*_{(A,\gamma_1 \gamma_2)}$$

for $(A, \gamma_1 \gamma_2), \ldots, (A, \gamma_{n-1} \gamma_n) \in E_{\tilde{A}}$ is naturally identified with the $C^*$-subalgebra $P_A O_Z P_A$ of $\mathcal{D}_Z$. Hence we know that $\mathcal{D}_{\tilde{A}} = \mathcal{D}_A$. By regarding the generating partial isometry $S_{(A,\gamma_1 \gamma_2)}$ for $(A, \gamma_1 \gamma_2) \in E_{\tilde{A}}$ as an element of $P_A O_Z P_A = O_A$, we have

$$\rho^{\tilde{A}}_t(S_{(A,\gamma_1 \gamma_2)}) = e^{2\pi \sqrt{-1} t} S_{(A,\gamma_1 \gamma_2)}$$

and

$$P_A e^{2\pi \sqrt{-1} t} S_{\gamma_1} e^{2\pi \sqrt{-1} t} S_{\gamma_2}$$

and

$$P_A \rho^Z_t(S_{\gamma_1}) \rho^Z_t(S_{\gamma_2})$$

and

$$\rho^Z_t(P_A S_{\gamma_1} S_{\gamma_2})$$

Since $P_A S_{\gamma_1} S_{\gamma_2} \in P_A O_Z P_A = O_A$ and $\rho^Z_t |_{P_A O_Z P_A} = \rho^A_t$ on $O_A$, we have

$$\rho^Z_t(P_A S_{\gamma_1} S_{\gamma_2}) = \rho^A_t(P_A S_{\gamma_1} S_{\gamma_2}) = \rho^A_t(S_{(A,\gamma_1 \gamma_2)})$$

so that $\rho^{\tilde{A}}_t = \rho^A_t$ for all $t \in \mathbb{T}$ and hence $\rho^{\tilde{A}} = \rho^A.$  \[ \square \]
Proposition 2.15. Suppose that the Cuntz–Krieger triplets \((\mathcal{O}_A, \mathcal{D}_A, \rho^A)\) and \((\mathcal{O}_B, \mathcal{D}_B, \rho^B)\) are strongly Morita equivalent in 1-step. Then the two-sided topological Markov shifts \((\tilde{X}_A, \tilde{\sigma}_A)\) and \((\tilde{X}_B, \tilde{\sigma}_B)\) are topologically conjugate.

Proof. Assume that the Cuntz–Krieger triplets \((\mathcal{O}_A, \mathcal{D}_A, \rho^A)\) and \((\mathcal{O}_B, \mathcal{D}_B, \rho^B)\) are strongly Morita equivalent in 1-step. By Proposition 2.9 the matrices \(\tilde{A}, \tilde{B}\) are elementary equivalent so that their two-sided topological Markov shifts \((\tilde{X}_A, \tilde{\sigma}_A)\) and \((\tilde{X}_B, \tilde{\sigma}_B)\) are topologically conjugate. Proposition 2.14 with [14, Corollary 3.5] ensures us that the onesided topological Markov shifts \((X_A, \sigma_A)\) and \((X_A, \sigma_A)\) are eventually conjugate and hence strongly continuous orbit equivalent in the sense of [14]. Since the latter property yields topological conjugacy of their two-sided topological Markov shifts, the two-sided topological Markov shifts \((\tilde{X}_A, \tilde{\sigma}_A)\) and \((\tilde{X}_B, \tilde{\sigma}_B)\) are topologically conjugate. Similarly we know that the two-sided topological Markov shifts \((\tilde{X}_B, \tilde{\sigma}_B)\) and \((\tilde{X}_B, \tilde{\sigma}_B)\) are topologically conjugate. Therefore we get the assertion.

Now we reach one of the main results of the paper.

Theorem 2.16. Let \(A, B\) be irreducible non-permutation matrices. The Cuntz–Krieger triplets \((\mathcal{O}_A, \mathcal{D}_A, \rho^A)\) and \((\mathcal{O}_B, \mathcal{D}_B, \rho^B)\) are strongly Morita equivalent if and only if their two-sided topological Markov shifts \((\tilde{X}_A, \tilde{\sigma}_A)\) and \((\tilde{X}_B, \tilde{\sigma}_B)\) are topologically conjugate.

Proof. If part comes from Proposition 2.14 The only if part follows from Proposition 2.15

As a corollary we have

Corollary 2.17. Let \(A, B\) be irreducible non-permutation matrices. The Cuntz–Krieger triplets \((\mathcal{O}_A, \mathcal{D}_A, \rho^A)\) and \((\mathcal{O}_B, \mathcal{D}_B, \rho^B)\) are strongly Morita equivalent if and only if the matrices \(A\) and \(B\) are strongly shift equivalent.

3 Strong shift equivalence and circle actions on \(\mathcal{O}_A\)

It is well-known that two unital \(C^\ast\)-algebras \(A\) and \(B\) are strongly Morita equivalent if and only if their stabilizations \(A \otimes \mathcal{K}\) and \(B \otimes \mathcal{K}\) are isomorphic by Brown–Green–Rieffel Theorem [3] Theorem 1.2 (cf. [3], [4]). We will next study relationships between stabilized Cuntz–Krieger algebras with their gauge actions and strong shift equivalence matrices. We will investigate stabilizations of generalized gauge actions from a viewpoint of flow equivalence.

Recall that for a function \(f \in C(X_A, \mathcal{Z})\) and \(t \in \mathbb{T}\), an automorphism \(\rho^A_t \in \text{Aut}(\mathcal{O}_A)\) is defined by \(\rho^A_t(S_i) = U_t(f)S_i, i = 1, \ldots, N, t \in \mathbb{T}\) for the unitary \(U_t(f) = \exp(2\pi \sqrt{-1}tf) \in \mathcal{D}_A\) as in [15]. It is easy to see that the automorphisms \(\rho^A_t, t \in \mathbb{T}\) yield an action of \(\mathbb{T}\) to \(\mathcal{O}_A\) such that \(\rho^A_t(a) = a\) for all \(a \in \mathcal{D}_A\). For \(f \in C(X_A, \mathcal{Z})\) and \(n \in \mathbb{Z}_+\), let us denote by \(f^n\) the function \(f^n(x) = \sum_{i=0}^{n-1} f(\sigma_A^i(x)), x \in X_A\). We know that the following identity holds (cf. [14], Lemma 3.1)

\[
\rho^A_t(S_\mu) = U_t(f^n)S_\mu, \quad f \in C(X_A, \mathcal{Z}), \quad \mu = (\mu_1, \ldots, \mu_n) \in B_n(X_A), \quad t \in \mathbb{T}.
\]  

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For a $C^*$-algebra $A$ without unit, let $M(A)$ stand for its multiplier $C^*$-algebra defined by
\[ M(A) = \{ a \in \mathcal{A}^* | aA \subset A, Aa \subset A \} \]
where $\mathcal{A}^*$ denotes the second dual ($\mathcal{A}^*$)$^*$ of the $C^*$-algebra $\mathcal{A}$. An action $\alpha$ of $\mathbb{T}$ to $\mathcal{A}$ extends to $M(\mathcal{A})$ and is still denoted by $\alpha$. For an action $\alpha$ of $\mathbb{T}$ to $\mathcal{A}$, a unitary one-co-cycle $u_t, t \in \mathbb{T}$ relative to $\alpha$ is a continuous map $t \in \mathbb{T} \rightarrow u_t \in U(M(\mathcal{A}))$ to the unitary group $U(M(\mathcal{A}))$ satisfying $u_{t+s} = u_s \alpha_s(u_t), s, t \in \mathbb{T}$. The following proposition has been proved in \[14\].

**Proposition 3.1** (\[14\] Proposition 4.3). Suppose that $A = CD$ and $B = DC$. Then there exists an isomorphism $\Phi : \mathcal{O}_A \otimes K \rightarrow \mathcal{O}_B \otimes K$ satisfying $\Phi(D_A \otimes C) = D_B \otimes C$ and a homomorphism $\varphi : C(X_A, \mathbb{Z}) \rightarrow C(X_B, \mathbb{Z})$ of ordered groups such that for each function $f \in C(X_A, \mathbb{Z})$ there exists a unitary one-co-cycle $u_t^f \in U(M(\mathcal{O}_A \otimes K))$ relative to $\rho^{A,f} \otimes \text{id}$ such that
\[ \Phi \circ \text{Ad}(u_t^f) \circ (\rho^{A,f} \otimes \text{id}) = (\rho^{B,\varphi(f)} \otimes \text{id}) \circ \Phi \quad \text{for } t \in \mathbb{T}. \] (3.2)

In this section, we will first review the proof in \[14\] of the above proposition to investigate the K-theoretic behavior of the above isomorphism $\Phi : \mathcal{O}_A \otimes K \rightarrow \mathcal{O}_B \otimes K$. The proof of the above proposition is based on the the proof of \[10\], in which Morita equivalence of $C^*$-algebras has been used (cf. \[2, 3, 4, 5, 11, 16, 22\]).

Suppose that two nonnegative square matrices $A$ and $B$ are elementary equivalent such that $A = CD$ and $B = DC$. As in the previous section, we may take and fix bijections $\varphi_{A,CD}$ from $E_A$ to a subset of $E_C \times E_D$ and $\varphi_{B,DC}$ from $E_B$ to a subset of $E_D \times E_C$. We set the square matrix $Z = \begin{bmatrix} 0 & C \\ D & 0 \end{bmatrix}$ as block matrix, and use the same notation as in the previous sections.

For an arbitrary fixed function $f \in C(X_A, \mathbb{Z})$, we may regard it as an element of $D_A$ and hence of $D_Z$ by identifying it with $f \odot 0$ in $D_A \oplus D_B = D_Z$. As
\[ \exp(2\pi \sqrt{-1}tf(f \odot 0)) = \exp(2\pi \sqrt{-1}tf) \oplus P_D \in U(D_Z), \]
the automorphism $\rho^{Z,f \odot 0}_t$ of $O_Z$ for $t \in \mathbb{T}$ defined by (1.5) satisfies
\[ \rho^{Z,f \odot 0}_t(S_c) = \exp(2\pi \sqrt{-1}tf)S_c \quad \text{for } c \in E_C, \quad \rho^{Z,f \odot 0}_t(S_d) = S_d \quad \text{for } d \in E_D. \] (3.3)

Take $a \in E_A, b \in E_B$ satisfying $\varphi_{A,CD}(a) = cd, \varphi_{B,DC}(b) = dc$, The equalities (3.3) imply
\[ \rho^{Z,f \odot 0}_t(S_cS_d) = \exp(2\pi \sqrt{-1}tf)S_cS_d = \rho^{A,f}_t(S_a), \]
\[ \rho^{Z,f \odot 0}_t(S_dS_c) = S_d\exp(2\pi \sqrt{-1}tf)S_c = S_d\exp(2\pi \sqrt{-1}tf)S_d^*S_b. \]

We set $\varphi(f) = \sum_{d \in E_D} S_d f S_d^* \in D_Z$. As $P_D \varphi(f)P_D = \varphi(f)$, we see that $\varphi(f) \in D_B$ and hence $\varphi(f) \in C(X_B, \mathbb{Z})$ which satisfies
\[ \sum_{d \in E_D} S_d\exp(2\pi \sqrt{-1}tf)S_d^* = \exp(2\pi \sqrt{-1}t\varphi(f)) \in U(D_B). \]

We similarly set $\psi(g) = \sum_{c \in E_C} S_c g S_c^* \in C(X_A, \mathbb{Z})$ for $g \in C(X_B, \mathbb{Z})$. We thus see the following lemma.
Lemma 3.2 ([13] Lemma 4.1]). For \( f \in C(X_A, \mathbb{Z}) \), \( g \in C(X_B, \mathbb{Z}) \) and \( t \in \mathbb{T} \), we have

\[
\begin{align*}
\rho_t(Zf^{\equiv_0}(ScDc)) &= \rho_t^{A,f}(ScDc), \\
\rho_t(Zf^{\equiv_0}(ScSd)) &= \rho_t^{B,f}(ScSd), \\
\rho_t(Zg^{\equiv_0}(ScDc)) &= \rho_t^{A,g}(ScDc), \\
\rho_t(Zg^{\equiv_0}(ScSd)) &= \rho_t^{B,g}(ScSd),
\end{align*}
\] (3.4)

where \( a \in E_A, b \in E_B \) and \( c \in E_C, d \in E_D \) are satisfying \( \varphi_{A,CD}(a) = cd \) and \( \varphi_{B,DC}(b) = dc \), respectively.

We note that the homomorphisms \( \varphi : C(X_A, \mathbb{Z}) \to C(X_B, \mathbb{Z}) \) and \( \psi : C(X_B, \mathbb{Z}) \to C(X_A, \mathbb{Z}) \) satisfy the equalities

\[
(\psi \circ \varphi)(f) = f \circ \sigma_A, \quad (\varphi \circ \psi)(g) = g \circ \sigma_B
\] (3.6)

for \( f \in C(X_A, \mathbb{Z}) \) and \( g \in C(X_B, \mathbb{Z}) \) ([13] Lemma 4.2]).

By [10] Proposition 4.1, one may find partial isometries \( v_A, v_B \in M(O_Z \otimes K) \) such that

\[
v_A^*v_A = v_B^*v_B = 1 \otimes 1, \quad v_Av_A^* = P_C \otimes 1, \quad v_Bv_B^* = P_D \otimes 1.
\] (3.7)

Since

\[
\text{Ad}(v_A^*): O_A \otimes K \to O_Z \otimes K \quad \text{and} \quad \text{Ad}(v_B^*): O_B \otimes K \to O_Z \otimes K
\] (3.8)

are isomorphisms satisfying

\[
\text{Ad}(v_A^*)(D_A \otimes C) = D_Z \otimes C \quad \text{and} \quad \text{Ad}(v_B^*)(D_B \otimes C) = D_Z \otimes C.
\]

By putting

\[
w = v_Bv_A^* \in M(O_Z \otimes K), \quad \Phi = \text{Ad}(w) : O_A \otimes K \to O_B \otimes K,
\] (3.9)

\[
u_t^{A,f} = w^*(\rho_t(Zf^{\equiv_0} \otimes \text{id})(w)) \quad \text{for} \quad f \in C(X_A, \mathbb{Z}),
\] (3.10)

\[
u_t^{B,g} = w(\rho_t(Zg^{\equiv_0} \otimes \text{id})(w^*)) \quad \text{for} \quad g \in C(X_B, \mathbb{Z}),
\] (3.11)

they satisfy \( \Phi(D_A \otimes C) = D_B \otimes C \) and the equalities

\[
\Phi \circ \text{Ad}(u_t^{A,f}) \circ (\rho_t^{A,f} \otimes \text{id}) = (\rho_t^{A,f} \otimes \text{id}) \circ \Phi \quad \text{for} \quad f \in C(X_A, \mathbb{Z}),
\]

\[
\Phi \circ (\rho_t^{A,g} \otimes \text{id}) = \text{Ad}(u_t^{B,g}) \circ (\rho_t^{B,g} \otimes \text{id}) \circ \Phi \quad \text{for} \quad g \in C(X_B, \mathbb{Z}).
\]

The above discussion is a sketch of the proof of Proposition 3.1 given in [14].

In what follows, we will reconstruct partial isometries \( v_A, v_B \) satisfying (3.7) to investigate the \( K \)-theoretical behavior of the map \( \Phi : O_A \otimes K \to O_B \otimes K \) in the following section.

The idea of the reconstruction is due to the proof of [2] Lemma 2.5 [cf. [10] Proposition 4.1]].

We are assuming that \( A = CD, B = DC \). Keep the notations as in the preceding section. Put \( E_C = \{e_1, \ldots, e_{N_C}\} \) and \( E_D = \{d_1, \ldots, d_{N_D}\} \) for the matrices \( C \) and \( D \) respectively. For \( k = 1, \ldots, N_D \), take \( c(k) \in E_C \) such that \( c(k)d_k \in B_2(X_Z) \) so that we have

\[
S_{c(k)}^*S_{c(k)} \geq S_{d_k}^*S_{d_k}.
\] (3.13)
Similarly for \( l = 1, \ldots, N_C \), take \( d(l) \in E_D \) such that \( d(l)c_l \in B_2(X_Z) \) so that we have
\[
S^*_{d(l)}S_{d(l)} \geq S^*_{c_l}S^*_{c_l}.
\]

Put
\[
U_0 = P_C, \quad U_k = S_{c(k)}S_{d_k}S_{d_k}^* \quad \text{for } k = 1, \ldots, N_D, \quad T_0 = P_D, \quad T_l = S_{d(l)}S^*_{c_l}S^*_{c_l} \quad \text{for } l = 1, \ldots, N_C.
\]

We then have
\[
\sum_{k=1}^{N_D} U_k^*U_k = \sum_{k=1}^{N_D} S_{d_k}S_{d_k}^*S_{c(k)}S_{c(k)}S_{d_k}S_{d_k}^* = \sum_{k=1}^{N_D} S_{d_k}S_{d_k}^* = P_D,
\]
\[
\sum_{k=1}^{N_C} T_l^*T_l = \sum_{l=1}^{N_C} S^*_{c_l}S^*_{d(l)}S_{d(l)}S_{c_l}S_{c_l}^* = \sum_{l=1}^{N_C} S^*_{c_l}S^*_{c_l} = P_C.
\]

We decompose the set \( \mathbb{N} \) of natural numbers into disjoint infinite subsets \( \mathbb{N} = \bigcup_{j=1}^{\infty} N_j \), and decompose \( N_j \) for each \( j \) once again into disjoint infinite sets \( N_j = \bigcup_{k=0}^{\infty} N_{j,k} \). Let \( \{e_{i,j}\}_{i,j\in\mathbb{N}} \) be a set of matrix units which generate the algebra \( \mathcal{K} = \mathcal{K}(\ell^2(\mathbb{N})) \). Put the projections \( f_j = \sum_{i\in N_j} e_{i,i} \) and \( f_{j,k} = \sum_{i\in N_{j,k}} e_{i,i} \). Take a partial isometry \( s_{j_k,j} \) such that \( s^*_{j_k,j}s_{j_k,j} = f_j \), \( s_{j_k,j}s^*_{j_k,j} = f_{j,k} \) and put \( s_{j,k} = s^*_{j_k,j} \). We set for \( n = 1, 2, \ldots, \)
\[
    u_n = \sum_{k=1}^{N_D} U_k \otimes s_{n_k,n}, \quad w_n = P_C \otimes s_{n_0,n} + u_n, \\
    t_n = \sum_{l=1}^{N_C} T_l \otimes s_{n_1,n}, \quad z_n = P_D \otimes s_{n_0,n} + t_n.
\]

Then we have

**Lemma 3.3.** *Keep the above notations.*

(i) \( w_n^*w_n = 1 \otimes f_n \) and \( w_nw_n^* \leq P_C \otimes f_n \).

(ii) \( z_n^*z_n = 1 \otimes f_n \) and \( z_nz_n^* \leq P_D \otimes f_n \).

**Proof.** (i) Since \( u_n^*u_n = P_D \otimes f_n \), we have
\[
w_n^*w_n = P_C \otimes f_n + u_n^*u_n = P_C \otimes f_n + P_D \otimes f_n = 1 \otimes f_n.
\]

On the other hand, we know that \( u_n(P_C \otimes s_{n,n_0}) = (P_C \otimes s_{n,n_0})u_n^* = 0 \) so that we have
\[
w_nw_n^* = P_C \otimes f_{n_0} + u_nu_n^* = P_C \otimes f_{n_0} + \sum_{k=1}^{N_D} S_{c(k)}S_{d_k}S_{d_k}^*S_{c(k)}^* \otimes f_{n_k}.
\]

As \( f_{n_0}, f_{n_k} \leq f_n \), we have
\[
w_nw_n^* \leq P_C \otimes f_n.
\]

(ii) is similarly shown to (i). \( \square \)
We will reconstruct and study the unitary $v_A$ in (3.7). Let $f_{n,m}$ be a partial isometry satisfying $f_{n,m}^* f_{n,m} = f_m$, $f_{n,m} f_{n,m}^* = f_n$. We put
\[
\begin{aligned}
v_1 &= w_1 = P_C \otimes s_{1,0,1} + u_1, \\
v_{2n} &= (P_C \otimes f_n - v_{2n-1}^* v_{2n-1}) (P_C \otimes f_{n,n+1}) \quad \text{for } 1 \leq n \in \mathbb{N}, \\
v_{2n-1} &= w_n (1 \otimes f_n - v_{2n-2}^* v_{2n-2}) \quad \text{for } 2 \leq n \in \mathbb{N}.
\end{aligned}
\]

**Lemma 3.4.** Keep the above notations.

(i) $v_{2n-2}^* v_{2n-2} + v_{2n-1}^* v_{2n-1} = 1 \otimes f_n$.

(ii) $v_{2n-1}^* v_{2n-1} + v_{2n} v_{2n}^* = P_C \otimes f_n$.

**Proof.** (i) As $w_n^* w_n = 1 \otimes f_n$, we have
\[
\begin{aligned}
v_{2n-2}^* v_{2n-2} + v_{2n-1}^* v_{2n-1} &= v_{2n-2}^* v_{2n-2} + (1 \otimes f_n - v_{2n-2}^* v_{2n-2}) w_n^* w_n (1 \otimes f_n - v_{2n-2}^* v_{2n-2}) \\
&= v_{2n-2}^* v_{2n-2} + 1 \otimes f_n - v_{2n-2}^* v_{2n-2} \\
&= 1 \otimes f_n.
\end{aligned}
\]

(ii) We have
\[
\begin{aligned}
v_{2n-1}^* v_{2n-1} + v_{2n} v_{2n}^* &= v_{2n-1}^* v_{2n-1} + (P_C \otimes f_n - v_{2n-1}^* v_{2n-1}) (P_C \otimes f_n - v_{2n-1}^* v_{2n-1}) \\
&= v_{2n-1}^* v_{2n-1} + P_C \otimes f_n - v_{2n-1}^* v_{2n-1} \\
&= P_C \otimes f_n.
\end{aligned}
\]

By the above lemma, one may see that the summations $\sum_{n=1}^{\infty} v_{2n-2}$ and $\sum_{n=1}^{\infty} v_{2n-1}$ converge in $M(O_Z \otimes K)$ to certain partial isometries written $v_{ev}$ and $v_{od}$ respectively in the strict topology of the multiplier algebra of $O_Z \otimes K$. Similarly we obtain a partial isometry $v_A = \sum_{n=1}^{\infty} v_n$ in $M(O_Z \otimes K)$ in the strict topology. Therefore we have the next lemma.

**Lemma 3.5.** The partial isometries $v_{ev}, v_{od}$ and $v_A$ defined above satisfy the following relations:

(i) $v_A = v_{od} + v_{ev}$.

(ii) $v_{od}^* v_{od} + v_{ev}^* v_{ev} = 1 \otimes 1$.

(iii) $v_{od} v_{od}^* + v_{ev} v_{ev}^* = P_C \otimes 1$.

(iv) $v_A^* v_A = 1 \otimes 1$ and $v_A v_A^* = P_C \otimes 1$.

We put
\[
\begin{aligned}
q_{od}^C &= \sum_{n=1}^{\infty} v_{2n-1} (P_C \otimes 1) v_{2n-1}^* \\
q_{od}^D &= \sum_{n=1}^{\infty} v_{2n-1} (P_D \otimes 1) v_{2n-1}^*
\end{aligned}
\]
so that
\[
q_{od}^C + q_{od}^D = v_{od} v_{od}^* \quad \text{and hence} \quad q_{od}^C + q_{od}^D + v_{ev} v_{ev}^* = P_C \otimes 1.
\]

We will show the following lemma.
Lemma 3.6. \( v_A(\rho^n_z \otimes 0 \otimes 1)(v_A^*) = q_{od}^0 + (U_t(-f) \otimes 1)q_{ov}^D + v_{ev}v_{ev}^* \).

**Proof.** We notice that \( \rho^n_z(S_c) = U_t(f)S_c \) for \( c \in E_C \) and \( \rho^n_z(S_d) = S_d \) for \( d \in E_D \). As \( v_{2n-1}v^*_{2n-1} \in DZ \otimes C \) so that \( (\rho^n_z \otimes \text{id})(v_{2n-1}v^*_{2n-1}) = v_{2n-1}v^*_{2n-1} \) and hence \( (\rho^n_z \otimes \text{id})(v_{ev}) = v_{ev} \). We then have

\[
v_A(\rho^n_z \otimes \text{id})(v_A^*) = v_{od}(\rho^n_z \otimes \text{id})(v_{od}^*) + v_{ev}(\rho^n_z \otimes \text{id})(v_{ev}^*) = \sum_{n=1}^{\infty} v_{2n-1}(\rho^n_z \otimes \text{id})(v_{2n-1}^*) + v_{ev}v_{ev}^*.
\]

Since

\[
v_1(P_C \otimes 1) = P_C \otimes s_{10,1} \quad \text{and} \quad v_1(P_D \otimes 1) = \sum_{k=1}^{N_D} S_{c(k)}S_{d_k}S_{d_k}^* \otimes s_{1k,1},
\]

we have

\[
(\rho^n_z \otimes \text{id})(v_1^*) = (P_C \otimes 1)v_1^* + (P_D \otimes 1)v_1^* = (P_C \otimes 1)v_1^* = \sum_{k=1}^{N_D} S_{d_k}S_{d_k}^* \rho^n_z(S_{c(k)}) \otimes s_{1k,1}^*
\]

so that

\[
v_1(\rho^n_z \otimes \text{id})(v_1^*) = v_1(P_C \otimes 1)v_1^* + v_1(P_D \otimes 1)v_1^* = v_1(P_C \otimes 1)v_1^* + (U_t(-f) \otimes 1)v_1(P_D \otimes 1)v_1^*.
\]

For \( 2 \leq n \in \mathbb{N} \), we have

\[
v_{2n-1}(P_C \otimes 1) = (P_C \otimes s_{n_0,n})(1 \otimes f_n - v^*_{2n-2}v_{2n-2})
\]

\[
v_{2n-1}(P_D \otimes 1) = \sum_{k=1}^{N_D} (S_{c(k)}S_{d_k}S_{d_k}^* \otimes s_{n_k,n})(1 \otimes f_n - v^*_{2n-2}v_{2n-2}),
\]

and hence

\[
(\rho^n_z \otimes \text{id})(P_D \otimes 1)v_{2n-1}^* = (1 \otimes f_n - v^*_{2n-2}v_{2n-2}) \sum_{k=1}^{N_D} S_{d_k}S_{d_k}^* \rho^n_z(S_{c(k)}^*) \otimes s_{n_k,n}^*
\]

\[
= (1 \otimes f_n - v^*_{2n-2}v_{2n-2}) \sum_{k=1}^{N_n} S_{d_k}S_{d_k}^* S_{c(k)}^* U_t(-f) \otimes s_{n_k,n}^*
\]

\[
= (P_D \otimes 1)v_{2n-1}^*(U_t(-f) \otimes 1)
\]

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so that
\[ v_{2n-1}(\rho_t^{Z,f\oplus 0} \otimes \text{id})(v_{2n-1}^*) = v_{2n-1}(P_C \otimes 1)v_{2n-1}^* + v_{2n-1}(P_D \otimes 1)v_{2n-1}^*(U_t(-f) \otimes 1) \]
\[ = v_{2n-1}(P_C \otimes 1)v_{2n-1}^* + (U_t(-f) \otimes 1)v_{2n-1}(P_D \otimes 1)v_{2n-1}^*. \]

Therefore we have
\[ v_{od}(\rho_t^{Z,f\oplus 0} \otimes \text{id})(v_{od}^*) = q_{od}^C + (U_t(-f) \otimes 1)q_{od}^D \]
and hence
\[ v_A(\rho_t^{Z,f\oplus 0} \otimes \text{id})(v_A^*) = q_{od}^C + (U_t(-f) \otimes 1)q_{od}^D + v_{ev}v_{ev}^*. \]

By using \( t_n, z_n \) instead of \( u_n, w_n \) respectively, we similarly obtain a partial isometry \( v_B \) in \( M(O_Z \otimes K) \) in the strict topology. We then see the following lemmas.

**Lemma 3.7.**

(i) The partial isometry \( v_A(\rho_t^{Z,f\oplus 0} \otimes \text{id})(v_A^*) \) for \( f \in C(X_A, Z), t \in \mathbb{T} \) belongs to \( M(D_A \otimes C) \) and satisfies
\[ v_A(\rho_t^{Z,(f_1+f_2)\oplus 0} \otimes \text{id})(v_A^*) = v_A(\rho_t^{Z,f_1\oplus 0} \otimes \text{id})(v_A^*)v_A(\rho_t^{Z,f_2\oplus 0} \otimes \text{id})(v_A^*) \] (3.19)
for \( f_1, f_2 \in C(X_A, Z), t \in \mathbb{T}. \)

(ii) The partial isometry \( v_B(\rho_t^{Z,0\oplus g} \otimes \text{id})(v_B^*) \) for \( g \in C(X_B, Z), t \in \mathbb{T} \) belongs to \( M(D_B \otimes C) \) and satisfies
\[ v_B(\rho_t^{Z,0\oplus (g_1+g_2)} \otimes \text{id})(v_B^*) = v_B(\rho_t^{Z,0\oplus g_1} \otimes \text{id})(v_B^*)v_B(\rho_t^{Z,0\oplus g_2} \otimes \text{id})(v_B^*) \] (3.20)
for \( g_1, g_2 \in C(X_B, Z), t \in \mathbb{T}. \)

**Proof.** (i) Since the projections \( q_{od}^C, q_{od}^D, v_{ev}, v_{ev}^* \) are all belong to the multiplier algebra \( M(D_A \otimes C) \) of \( D_A \otimes C \), the preceding lemma ensures us that the partial isometry \( v_A(\rho_t^{Z,f\oplus 0} \otimes \text{id})(v_A^*) \) belongs to \( M(D_A \otimes C) \). As \( U_t(f_1 + f_2) = U_t(f_1)U_t(f_2) \), the equality (3.19) follows.

(ii) is similarly shown to (i). \( \square \)

**Lemma 3.8.**

(i) \( (\rho_t^{Z,0\oplus g} \otimes \text{id})(v_A) = v_A \) for \( g \in C(X_B, Z), t \in \mathbb{T}. \)

(ii) \( (\rho_t^{Z,f\oplus 0} \otimes \text{id})(v_B) = v_B \) for \( f \in C(X_A, Z), t \in \mathbb{T}. \)

**Proof.** (i) Since \( \rho_t^{Z,0\oplus g}(S_c) = S_c, \rho_t^{Z,0\oplus g}(S_d) = e^{2\pi \sqrt{-1}t(g)}S_d, \) we have
\[ \rho_t^{Z,0\oplus g}(U_k) = \rho_t^{Z,0\oplus g}(S_{c(k)}S_{d(k)}S_{d(k)}^*) = S_{c(k)}e^{2\pi \sqrt{-1}t(g)}S_{d(k)}S_{d(k)}^* = S_{c(k)}S_{d(k)}S_{d(k)}^* = U_k. \]
Hence \( (\rho_t^{Z,0\oplus g} \otimes \text{id})(u_n) = u_n \) so that \( (\rho_t^{Z,0\oplus g} \otimes \text{id})(w_n) = w_n. \) We then have
\[ (\rho_t^{Z,0\oplus g} \otimes \text{id})(v_1) = (\rho_t^{Z,0\oplus g} \otimes \text{id})(P_C \otimes s_{10,1} + u_1) = P_C \otimes s_{10,1} + u_1 = v_1. \]
Since \( v_{2n-1}v_{2n-1}' \), \( v_{2n-2}^*v_{2n-2} \in D_Z \otimes C \) and the restriction of \( \rho_t^{Z,0\otimes g} \otimes \text{id} \) to \( D_Z \otimes C \) is the identity, we easily know that
\[
(\rho_t^{Z,0\otimes g} \otimes \text{id})(v_{2n}) = v_{2n}, \quad (\rho_t^{Z,0\otimes g} \otimes \text{id})(v_{2n-1}) = v_{2n-1} \quad \text{for } n \in \mathbb{N}.
\]
We thus have \( (\rho_t^{Z,0\otimes g} \otimes \text{id})(v_n) = v_n \) for all \( n \in \mathbb{N} \) and hence \( (\rho_t^{Z,0\otimes g} \otimes \text{id})(v_A) = v_A \).
\( \Box \)

We put
\[
w = v_B v_A^* \in M(O_Z \otimes K),
\]
\[
\tag{3.21}
u_t^{A,f} = w^*(\rho^Z_t f^{\otimes 0} \otimes \text{id})(w) \quad \text{for } f \in C(X_A, Z),
\]
\[
\tag{3.22}
u_t^{B,g} = w(\rho^{Z,0\otimes g}_t \otimes \text{id})(w^*) \quad \text{for } g \in C(X_B, Z).
\]

By Lemma 3.8, we have
\[
u_t^{A,f} = v_A v_B^* (\rho^Z_t f^{\otimes 0} \otimes \text{id}) (v_B^*) (\rho^{Z,0\otimes g}_t \otimes \text{id})(v_A^*) = v_A (\rho^Z_t f^{\otimes 0} \otimes \text{id})(v_A^*)
\]
and similarly \( u_t^{B,g} = v_B (\rho^{Z,0\otimes g}_t \otimes \text{id})(v_B^*) \).

**Lemma 3.9.**

(i) For each \( f \in C(X_A, Z) \), the partial isometries \( u_t^{A,f}, t \in T \) give rise to a unitary representation of \( T \) in \( M(D_A \otimes C) \) and satisfies \( u_t^{A,f_1+f_2} = u_t^{A,f_1} u_t^{A,f_2} \) for \( f_1, f_2 \in C(X_A, Z) \).

(ii) For each \( g \in C(X_B, Z) \), the partial isometries \( u_t^{B,g}, t \in T \) give rise to a unitary representation of \( T \) in \( M(D_B \otimes C) \) and satisfies \( u_t^{B,g_1+g_2} = u_t^{B,g_1} u_t^{B,g_2} \) for \( g_1, g_2 \in C(X_B, Z) \).

**Proof.** (i) By Lemma 3.6 and (3.21), we have
\[
u_t^{A,f} = v_A (\rho^Z_t f^{\otimes 0} \otimes \text{id}) (v_A^*) = (q^D + (U_t(-f) \otimes 1)q^D_{od} + ev_{ev}^*) (q^D + (U_t(-f) \otimes 1)q^D_{od} + ev_{ev}^*) = v_A (\rho^Z_t f^{\otimes 0} \otimes \text{id})(v_A^*).
\]
The equality \( u_t^{A,f_1+f_2} = u_t^{A,f_1} u_t^{A,f_2} \) immediately follows from Lemma 3.7. (ii) is similarly shown to (i).
\( \Box \)

We thus have

**Proposition 3.10.** Let \( A, B \) be nonnegative irreducible and non-permutation matrices. Suppose that \( A = CD, B = DC \) for some nonnegative rectangular matrices \( C, D \). Then there exist an isomorphism \( \Phi : O_A \otimes K \to O_B \otimes K \) satisfying \( \Phi(D_A \otimes C) = D_B \otimes C \), and unitary representations \( t \in T \to u_t^{A,f} \in M(D_A \otimes C) \) for each \( f \in C(X_A, Z) \) and \( t \in T \to u_t^{B,g} \in M(D_B \otimes C) \) for each \( g \in C(X_B, Z) \) such that
\[
\Phi \circ \text{Ad}(u_t^{A,f}) \circ (\rho^Z_t \otimes \text{id}) = (\rho^{B,\psi(f)}_t \otimes \text{id}) \circ \Phi \quad \text{for } f \in C(X_A, Z),
\]
\[
\tag{3.25}
\Phi \circ (\rho^A_t \otimes \text{id}) = \text{Ad}(u_t^{B,g}) \circ (\rho^{B,g}_t \otimes \text{id}) \circ \Phi \quad \text{for } g \in C(X_B, Z).
\]
\[
\tag{3.26}
\]
Proposition 3.13. Suppose that two stabilized Cuntz–Krieger triplets \((\mathcal{O}_A \otimes \mathbb{K}), \mathcal{D}_A \otimes \mathbb{K}, \rho^A \otimes \text{id}\) and \((\mathcal{O}_B \otimes \mathbb{K}), \mathcal{D}_B \otimes \mathbb{K}, \rho^B \otimes \text{id}\) are elementary equivalent such that \(\Phi_A : \mathcal{D}_A \otimes \mathbb{K} \to \mathcal{D}_B \otimes \mathbb{K}\) and \(\Phi_B : \mathcal{D}_B \otimes \mathbb{K} \to \mathcal{D}_A \otimes \mathbb{K}\) satisfy
\[
\Phi_A(D_A \otimes \mathbb{K}) = D_B \otimes \mathbb{K}, \quad \Phi_B(D_B \otimes \mathbb{K}) = D_A \otimes \mathbb{K},
\]
\[
\rho^Z \otimes \text{id} = (\Phi_B^{-1} \circ \rho^B \otimes \text{id} \circ \Phi_B) \circ (\Phi_A^{-1} \circ \rho^A \otimes \text{id} \circ \Phi_A)
\]
\[
= (\Phi_A^{-1} \circ \rho^A \otimes \text{id} \circ \Phi_A) \circ (\Phi_B^{-1} \circ \rho^B \otimes \text{id} \circ \Phi_B).
\]
If two stabilized Cuntz–Krieger triplets \((\mathcal{O}_A \otimes \mathbb{K}, \mathcal{D}_A \otimes \mathbb{K}, \rho^A \otimes \text{id})\) and \((\mathcal{O}_B \otimes \mathbb{K}, \mathcal{D}_B \otimes \mathbb{K}, \rho^B \otimes \text{id})\) are connected by \(n\)-chains of strong Morita equivalences in 1-step, they are said to be strong Morita equivalent in \(n\)-step, or simply strong Morita equivalent.

Proposition 3.13. Suppose that \(A, B\) are elementary equivalent such that \(A = CD, B = DC\). Then the stabilized Cuntz–Krieger triplets \((\mathcal{O}_A \otimes \mathbb{K}, \mathcal{D}_A \otimes \mathbb{K}, \rho^A \otimes \text{id})\) and \((\mathcal{O}_B \otimes \mathbb{K}, \mathcal{D}_B \otimes \mathbb{K}, \rho^B \otimes \text{id})\) are strong Morita equivalent in 1-step.
Proof. Let \( Z = \begin{bmatrix} 0 & C \\ D & 0 \end{bmatrix} \). Take partial isometries \( v_A, v_B \in M(O_Z \otimes K) \) satisfying (3.7). By Lemma 3.8 the following identities hold
\[
(\rho_t^{Z,0\oplus 1} \otimes \text{id})(v_A) = v_A, \quad (\rho_t^{Z,1\oplus 0} \otimes \text{id})(v_B) = v_B.
\]
Define \( \Phi_A = \text{Ad}(v_A), \Phi_B = \text{Ad}(v_B) \). As in (3.8), they give rise to isomorphisms
\[
\Phi_A : O_Z \otimes K \rightarrow O_A \otimes K, \quad \Phi_B : O_Z \otimes K \rightarrow O_B \otimes K
\]
satisfying
\[
\Phi_A(D_Z \otimes C) = D_A \otimes C, \quad \Phi_B(D_Z \otimes C) = D_B \otimes C.
\]
Since we see
\[
\rho_t^{Z,0\oplus 1}(S_c) = S_c, \quad \rho_t^{Z,0\oplus 1}(S_d) = e^{2\pi \sqrt{-1} t} S_d,
\]
\[
\rho_t^{Z,1\oplus 0}(S_c) = e^{2\pi \sqrt{-1} t} S_c, \quad \rho_t^{Z,1\oplus 0}(S_d) = S_d
\]
for \( c \in C, d \in D \), we have for \( x \otimes K \in O_Z \otimes K \)
\[
((\rho_t^A \otimes \text{id}) \circ \Phi_A)(x \otimes K) = (\rho_t^{Z,0\oplus 1} \otimes \text{id})(v_A(x \otimes K)v_A^*)
\]
\[
= v_A(\rho_t^{Z,0\oplus 1} \otimes \text{id})(x \otimes K)v_A^*
\]
\[
= \Phi_A \circ (\rho_t^{Z,0\oplus 1} \otimes \text{id})(x \otimes K).
\]
Hence we have \( (\rho_t^A \otimes \text{id}) \circ \Phi_A = \Phi_A \circ (\rho_t^{Z,0\oplus 1} \otimes \text{id}) \) and similarly \( (\rho_t^B \otimes \text{id}) \circ \Phi_B = \Phi_B \circ (\rho_t^{Z,1\oplus 0} \otimes \text{id}) \). Since \( \rho_t^Z \otimes \text{id} = (\rho_t^{Z,1\oplus 0} \otimes \text{id}) \circ (\rho_t^{Z,0\oplus 1} \otimes \text{id}) = (\rho_t^{Z,0\oplus 1} \otimes \text{id}) \circ (\rho_t^{Z,1\oplus 0} \otimes \text{id}) \), we know the assertion. \( \square \)

Therefore we have the following corollary.

**Corollary 3.14.** If \( A, B \) are strong shift equivalent, then the stabilized Cuntz–Krieger triplets \((O_A \otimes K, D_A \otimes C, \rho^A \otimes \text{id})\) and \((O_B \otimes K, D_B \otimes C, \rho^B \otimes \text{id})\) are strong Morita equivalent.

## 4 Behavior on K-theory

In this section we will study the behavior of the isomorphism \( \Phi : O_A \otimes K \rightarrow O_B \otimes K \) in Proposition 3.10 on their K-groups \( \Phi_* : K_0(O_A) \rightarrow K_0(O_B) \) under the condition \( A = CD, B = DC \).

Recall that \( A = [A(i,j)]_{i,j=1}^N \) is an \( N \times N \) matrix with entries in nonnegative integers. Then the associated graph \( G_A = (V_A, E_A) \) consists of the vertex set \( V_A = \{v^A_1, \ldots, v^A_N\} \) of \( N \) vertices and edge set \( E_A = \{a_1, \ldots, a_N\} \), where there are \( A(i,j) \) edges from \( v^A_i \) to \( v^A_j \). Denote by \( t(a_i), s(a_i) \) the terminal vertex of \( a_i \), the source vertex of \( a_i \), respectively. The graph \( G_A \) has the \( N_A \times N_A \) transition matrix \( A^G = [A^G(i,j)]_{i,j=1}^{N_A} \) of edges defined by (2.1).

The Cuntz–Krieger algebra \( O_A \) is defined as the Cuntz–Krieger algebra \( O_{AC} \) for the matrix \( A^G \) which is the universal \( C^* \)-algebra generated by partial isometries \( S_{a_i}, i = 1, \ldots, N_A \) subject to the relations (2.2). We similarly consider the \( N_B \times N_B \) matrix \( B^G \) with entries
in \(\{0,1\}\) for the graph \(G_B = (V_B, E_B)\) of the matrix \(B\) with vertex set \(V_B = \{v_1^B, \ldots, v_M^B\}\) and edge set \(E_B = \{b_1, \ldots, b_{NB}\}\), so that we have the other Cuntz-Krieger algebra \(O_{BG}\) for the matrix \(B^G\) which is denoted by \(O_B\).

Now we are assuming that \(A = CD\) and \(B = DC\) for some nonnegative rectangular matrices \(C\) and \(D\). Both \(A\) and \(B\) are also assumed to be irreducible and not any permutations. Since \(A = CD\), the edge set \(E_A\) is regarded as a subset of the product \(E_C \times E_D\) of those of \(E_C\) and \(E_D\). As in Section 2, we may take a bijection \(\varphi_{A,CD}\) from \(E_A\) to a subset of \(E_C \times E_D\). For any \(a_i \in E_A\), there uniquely exist \(c(a_i) \in E_C\) and \(d(a_i) \in E_D\) such that \(\varphi_{A,CD}(a_i) = c(a_i)d(a_i)\). We write it simply as \(a_i = c(a_i)d(a_i)\). Similarly, for any edge \(b_l \in E_B\), there uniquely exist \(d(b_l) \in E_D\) and \(c(b_l) \in E_C\) such that \(\varphi_{B,DC}(b_l) = d(b_l)c(b_l)\), simply written \(b_l = d(b_l)c(b_l)\). We define \(A \times N_B\) matrix \(\hat{D}(i, l) = [\hat{D}(i, l)]_{i = 1}^{N_A} \in \mathbb{N}\) by

\[
\hat{D}(i, l) = \begin{cases} 
1 & \text{if } d(a_i) = d(b_l), \\
0 & \text{otherwise.}
\end{cases}
\]

### Lemma 4.1

The matrix \(\hat{D}^t : \mathbb{Z}^{N_A} \rightarrow \mathbb{Z}^{N_B}\) induces a homomorphism from \(\mathbb{Z}^{N_A}/(\text{id} - (A^G)^t)\mathbb{Z}^{N_A}\) to \(\mathbb{Z}^{N_B}/(\text{id} - (B^G)^t)\mathbb{Z}^{N_B}\) as abelian groups.

**Proof.** For \(i = 1, \ldots, N_A\) and \(l = 1, \ldots, N_B\), we know that both

\[
[A^G \hat{D}](i, l) = \sum_{j=1}^{N_A} A^G(i, j) \hat{D}(j, l) \quad \text{and} \quad [\hat{D} B^G](i, l) = \sum_{k=1}^{N_B} \hat{D}(i, k) B^G(k, l)
\]

are the cardinal number of the set \(\{c \in E_C \mid d(a_i)c(b_l) \in B_3(X_Z)\}\). Hence we have \(A^G \hat{D} = \hat{D} B^G\). We then have that \(\hat{D}^t(\text{id} - (A^G)^t)\mathbb{Z}^{N_A} \subset (\text{id} - (B^G)^t)\mathbb{Z}^{N_B}\) so that \(\hat{D}^t\) induces a desired homomorphism.

The above homomorphism from \(\mathbb{Z}^{N_A}/(\text{id} - (A^G)^t)\mathbb{Z}^{N_A}\) to \(\mathbb{Z}^{N_B}/(\text{id} - (B^G)^t)\mathbb{Z}^{N_B}\) induced by \(\hat{D}^t\) is denoted by \(\Phi_{D^t}\).

Let us denote by \([e_i^{NA}]\) the class of the vector \(e_i^{NA} = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}^{N_A}\) in \(\mathbb{Z}^{N_A}/(\text{id} - (A^G)^t)\mathbb{Z}^{N_A}\). It was shown in [6] that the correspondence \(\epsilon_{AG} : K_0(O_{AG}) \rightarrow \mathbb{Z}^{N_A}/(\text{id} - (A^G)^t)\mathbb{Z}^{N_A}\) defined by \(\epsilon_{AG}([S_a, S_a^*]) = [e_i^{NA}]\) yields an isomorphism of abelian groups. We then have

### Proposition 4.2

Suppose that \(A = CD\), \(B = DC\). Let \(\Phi : O_A \otimes \mathcal{K} \rightarrow O_B \otimes \mathcal{K}\) be the isomorphism in Proposition 3.10. Then the diagram

\[
\begin{array}{ccc}
K_0(O_{AG}) & \xrightarrow{\Phi_*} & K_0(O_{BG}) \\
\epsilon_{AG} \downarrow & & \downarrow \epsilon_{BG} \\
\mathbb{Z}^{N_A}/(\text{id} - (A^G)^t)\mathbb{Z}^{N_A} & \xrightarrow{\Phi_{D^t}} & \mathbb{Z}^{N_B}/(\text{id} - (B^G)^t)\mathbb{Z}^{N_B}. \\
\end{array}
\]

is commutative.

**Proof.** We note that \(\mathcal{K} = \mathcal{K}(l^2(\mathbb{N}))\) has a countable basis and \(\mathbb{N}\) is decomposed such as \(\mathbb{N} = \bigcup_{j=1}^{\infty} N_j\) where \(N_j\) is also disjoint infinite set such as \(N_j = \bigcup_{k=0}^{\infty} N_{jk}\) with disjoint

\[24\]
Infinite sets $\mathbb{N}_j$ for every $k = 0, 1, 2, \ldots$ We write $\mathbb{N}_j$ as $\mathbb{N}_j = \{j_k(0), j_k(1), j_k(2), \ldots\}$. In particular for $j = 1, k = 0$, we denote by $\bar{n} = 1_n(n)$ for $n = 0, 1, 2, \ldots$ so that $\mathbb{N}_{l_0} = \{0, 1, 2, \ldots\}$. Let $p_n, n = 0, 1, 2, \ldots$ be the sequence of projections of rank one in $K$ such that $\sum_{n=0}^{\infty} p_n = f_{l_0}$. By [3], the group $K_0(O_{AC})$ is generated by the projections of the form

$$S_{a_i}S_{a_i}^* \otimes p_0, \quad i = 1, \ldots, N_A.$$ Denote by $1_A$ the unit of $O_{AC}$ so that $[1_A] = \sum_{i=1}^{N_A} [S_{a_i}S_{a_i}^* \otimes p_0]$ in $K_0(O_{AC})$. Let $\Phi = \text{Ad}(w) : O_{AC} \otimes K \to O_{BC} \otimes K$ be the isomorphism in Proposition [3.10]. Hence $\Phi_* : K_0(O_{AC}) \to K_0(O_{BC})$ satisfies $\Phi_*([S_{a_i}S_{a_i}^* \otimes p_0]) = [w(S_{a_i}S_{a_i}^* \otimes p_0)^* w^*]$. To complete the proof of the proposition, we provide the following two lemmas.

Let $l(i)$ be the number $l = 1, \ldots, N_C$ satisfying $c_i = c(a_i)$ so that $d(l(i)) \in E_D$ satisfies $T_{l(i)} = S_{d(l(i))}S_{c(a_i)}S_{c(a_i)}^*$ in [3.16]. We put $s_{1(l(i)), 1_0} = s_{1(l(i)), 1_0} = s_{1(l(i)), 1_0}$.\n
**Lemma 4.3. Keep the above notation.**

(i) $w(S_{a_i}S_{a_i}^* \otimes p_0)w^* = v_B(S_{a_i}S_{a_i}^* \otimes s_{1_0}p_0s_{1_0}l_0) v_B^*$.\n
(ii) $v_B(S_{a_i}S_{a_i}^* \otimes s_{1_0}p_0s_{1_0}l_0) v_B = S_{d(l(i))}S_{c(a_i)}S_{c(a_i)}S_{c(a_i)}^* S_{d(l(i))}^* \otimes s_{1_0}p_0s_{1_0}l_0$.\n
**Proof.** (i) The unitary $w$ is given by $w = v_Bv_A$. We know $v_A = \sum_{n=1}^{\infty} v_n$ and $v_1 = P_C \otimes s_{1_0} + \sum_{k=1}^{N_D} U_k \otimes s_{1_0}$. As $d_0s_{1_0}l_0 = 0$ for $k = 1, \ldots, N_D$, we have\n
$$v_A(S_{a_i}S_{a_i}^* \otimes p_0)v_A = v_1(S_{a_i}S_{a_i}^* \otimes p_0)v_1 = (P_C \otimes s_{1_0})^*(S_{a_i}S_{a_i}^* \otimes p_0)(P_C \otimes s_{1_0}) = S_{a_i}S_{a_i}^* \otimes s_{1_0}p_0s_{1_0}l_0.$$\n
(ii) For $c_i \in E_C = \{c_1, \ldots, c_{N_C}\}$ and $a_i \in E_A$, we note that $S_{c_i}S_{a_i} = S_{c_i}S_{c(a_i)}S_{d(a_i)}$ if $c_i = c(a_i)$, otherwise zero. Hence we have\n
$$v_B(S_{a_i}S_{a_i}^* \otimes s_{1_0}p_0s_{1_0}l_0) v_B = \sum_{l=1}^{N_C} T_{l} \otimes s_{1_0} \sum_{l=1}^{N_C} T_{l} \otimes s_{1_0} = \sum_{l=1}^{N_C} S_{d(l)}S_{c(a_i)}S_{c(a_i)}S_{d(l)}^* \otimes s_{1_0}.$$\n
**Lemma 4.4.** $S_{d(a_i)}S_{d(a_i)}^* = \sum_{l=1}^{N_B} \tilde{D}(i, l) S_{b_l}S_{b_l}^*$.\n
**Proof.** In the algebra $O_{BC}$, we have $\sum_{l=1}^{N_B} S_{b_l}S_{b_l}^* = 1$. As $b_l = d(b_l)c(b_l)$, it implies that $\sum_{l=1}^{N_B} S_{d(b_l)}S_{c(b_l)}S_{c(b_l)}^* S_{d(b_l)}^* = P_D$ in $O_{C}$. By multiplying $S_{d(a_i)}S_{d(a_i)}^*$ to the equality we have\n
$$\sum_{l=1}^{N_B} S_{d(a_i)}S_{d(a_i)}^* S_{d(b_l)}S_{c(b_l)}S_{c(b_l)}^* S_{d(b_l)}^* = S_{d(a_i)}S_{d(a_i)}^*.$$
Since
\[ S_{d(a_i)}S_{d(a_j)}^* S_{d(b_i)} = \hat{D}(i, l) S_{d(b_i)}, \]
we have
\[ \sum_{i=1}^{N_B} \hat{D}(i, l) S_{d(b_i)} S_{c(b_i)}^* S_{d(b_i)} = S_{d(a_i)} S_{d(a_j)}^*. \]
As \( S_{b_i} = S_{d(b_i)} S_{c(b_i)} \), we get the desired equality. \( \Box \)

**Proof of Proposition 4.2**

By using Lemma 4.3 we have the equalities in \( K_0(\mathcal{O}_B^C) \):
\[ \Phi_s([S_{a_i}, S_{a_i}^* \otimes p_0]) = [S_{d(l(i))} S_{c(a_i)} S_{d(a_i)}^* S_{d(l(i))}^* \otimes s_{1_{(i)}}, 1_{(i)} p_0 s_{1_{(i)}}, 1_{(i)}]. \]
Since
\[ [S_{d(l(i))} S_{c(a_i)} S_{d(a_i)}^* S_{d(l(i))}^* \otimes s_{1_{(i)}}, 1_{(i)} p_0 s_{1_{(i)}}, 1_{(i)}] = [S_{d(a_i)} S_{d(a_i)}^* \otimes p_0] \]
and \( f_1 p_0 f_1 \geq p_0 \), we have
\[ \Phi_s([S_{a_i}, S_{a_i}^* \otimes p_0]) = [S_{d(a_i)} S_{d(a_i)}^* \otimes p_0]. \]
As \( \epsilon_A^C([S_{a_i}, S_{a_i}^* \otimes p_0]) = [e_i^N] \) and \( \epsilon_C^B([S_{b_i}, S_{b_i}^* \otimes p_0]) = [e_i^N] \), By using Lemma 4.4 we complete the proof of Proposition 4.2. \( \Box \)

Let \( S_A \) and \( R_A \) be the \( N_A \times N \) matrix and \( N \times N_A \) matrix defined by
\[ S_A(i, j) = \begin{cases} 1 & \text{if } t(a_i) = v^A_j, \\ 0 & \text{otherwise,} \end{cases} \quad R_A(j, i) = \begin{cases} 1 & \text{if } v^B_j = s(a_i), \\ 0 & \text{otherwise,} \end{cases} \]
for \( i = 1, \ldots, N_A \) and \( j = 1, \ldots, N \), respectively. We then have \( A = R_A S_A \) and \( A^G = S_A R_A \). We similarly have the matrices \( S_B, R_B \) for the other matrix \( B \) such that \( B = R_B S_B \) and \( B^G = S_B R_B \). The matrix \( S_A^t : \mathbb{Z}^{N_A} \rightarrow \mathbb{Z}^N \) induces a homomorphism \( \mathbb{Z}^{N_A} / (\text{id} - (A^G)^t) \mathbb{Z}^{N_A} \rightarrow \mathbb{Z}^N / (\text{id} - A^t) \mathbb{Z}^N \) of abelian groups which is actually an isomorphism since its inverse is given by a homomorphism induced by \( R_A^t \). The above isomorphism is denoted by \( \Phi_{S_A^t} \). We have an isomorphism \( \Phi_{S_B^t} : \mathbb{Z}^{N_B} / (\text{id} - (B^G)^t) \mathbb{Z}^{N_B} \rightarrow \mathbb{Z}^{M} / (\text{id} - B^t) \mathbb{Z}^{M} \) in a similar way.

Now we are assuming that \( A = CD, B = DC \) so that \( AC = CB \) and hence \( A^t = B^t C^t \). The matrix \( C^t : \mathbb{Z}^N \rightarrow \mathbb{Z}^M \) induces a homomorphism from \( \mathbb{Z}^N / (\text{id} - A^t) \mathbb{Z}^N \) to \( \mathbb{Z}^M / (\text{id} - B^t) \mathbb{Z}^M \) as abelian groups, which is denoted by \( \Phi_{C^t} \). It is actually an isomorphism with \( \Phi_{D^t} \) as its inverse. We notice the following lemma. The second assertion (ii) is pointed out by Hiroki Matui. The author thanks him for his advice.

**Lemma 4.5.** (i) The diagram
\[
\begin{array}{c}
\mathbb{Z}^{N_A} / (\text{id} - (A^G)^t) \mathbb{Z}^{N_A} \xrightarrow{\Phi_{S_A^t}} \mathbb{Z}^{N_B} / (\text{id} - (B^G)^t) \mathbb{Z}^{N_B} \\
\Phi_{S_B^t} \downarrow \quad \Phi_{S_B^t} \\
\mathbb{Z}^N / (\text{id} - A^t) \mathbb{Z}^N \xrightarrow{\Phi_{C^t}} \mathbb{Z}^M / (\text{id} - B^t) \mathbb{Z}^M
\end{array}
\]
is commutative.
Proof. (i) Since $\Phi^*_D$ is induced by the matrix $D^t$, it suffices to prove the equality $\hat{D}S_B = S_AC$. Let $(i, j)$ be $i = 1, \ldots, N_A$ and $j = 1, \ldots, M$ so that $a_i \in E_A$ and $v^B_j \in V_B$. Let $k$ be such that $t(a_i) = v^A_k$. Hence we have

$$[S_AC](i, j) = \sum_{n=1}^{N} S_A(i, n)C(n, j) = C(k, j)$$

which is the number of edges of $E_C$ leaving $v^A_k$ and terminating at $v^B_j$. On the other hand,

$$[\hat{D}S_B](i, j) = \sum_{l=1}^{M_B} \hat{D}(i, l)S_B(l, j).$$

It is easy to see that the above number is also $C(k, j)$.

(ii) Since $A = R_AS_A$, for each $k = 1, \ldots, N_A$ with $a_k \in E_A$ there exists a unique $i = 1, \ldots, N_A$ such that $s(a_k) = v^A_i$. Hence $\sum_{i=1}^{N_A} R_A(i, k) = 1$ so that we have for each $j = 1, \ldots, N_A$

$$\sum_{i=1}^{N_A} A^t(j, i) = \sum_{i=1}^{N_A} \sum_{k=1}^{N_A} R_A(i, k)S_A(k, j) = \sum_{k=1}^{N_A} (\sum_{i=1}^{N_A} R_A(i, k))S_A(k, j) = \sum_{k=1}^{N_A} S_A^t(j, k).$$

We then see

$$\Phi^*_{S_A}([[(1, 1, \ldots, 1)]] = [(\sum_{k=1}^{N_A} S_A(k, 1), \sum_{k=1}^{N_A} S_A(k, 2), \ldots, \sum_{k=1}^{N_A} S_A(k, N))]$$

$$= [(\sum_{i=1}^{N} A^t(1, i), \sum_{i=1}^{N} A^t(2, i), \ldots, \sum_{i=1}^{N} A^t(N, i))]$$

$$= [(1, 1, \ldots, 1)] \text{ in } \mathbb{Z}^N/(id - A^t)\mathbb{Z}^N.$$

$\square$

Put $\epsilon_A = \Phi^*_{S_A} \circ \epsilon_{AC} : K_0(O_A) \to \mathbb{Z}^N/(id - A^t)\mathbb{Z}^N$, which is an isomorphism of groups such that $\epsilon_A([1_A]) = [(1, 1, \ldots, 1)]$. We thus reach the following theorem:

**Theorem 4.6.** Suppose that two nonnegative irreducible matrices $A, B$ satisfy $A = CD, B = DC$ for some nonnegative rectangular matrices $C, D$. Then the diagram

$$
\begin{array}{ccc}
K_0(O_A) & \xrightarrow{\Phi^*} & K_0(O_B) \\
\epsilon_A \downarrow & & \downarrow \epsilon_B \\
\mathbb{Z}^N/(id - A^t)\mathbb{Z}^N & \xrightarrow{\Phi_{C^t}} & \mathbb{Z}^M/(id - B^t)\mathbb{Z}^M
\end{array}
$$

is commutative, where the two vertical arrows and the two horizontal arrows are all isomorphisms of abelian groups.
We write $A \approx B$ if $A = CD$, $B = DC$. Recall that $A, B$ are said to be strong shift equivalent in $n$-step if there exist a finite sequence of square matrices $A_1, \ldots, A_{n-1}$ and two finite sequences of rectangular matrices $C_1, \ldots, C_n$ and $D_1, \ldots, D_n$ such that

$$A = A_0 \approx_{C_1, D_1} A_1, \quad A_1 \approx_{C_2, D_2} A_2, \ldots, \quad A_{n-1} \approx_{C_n, D_n} A_n = B.$$ 

This situation is written

$$A \approx_{C_1, D_1 \cdots C_n, D_n} B.$$ 

R. F. Williams proved that two-sided topological Markov shifts $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are topologically conjugate if and only if $A$ and $B$ are strong shift equivalent in $n$-step for some $n$ ([24]). Hence we have the following corollary.

**Corollary 4.7.** Suppose that two matrices $A, B$ are strong shift equivalent in $n$-step for some two sequences of rectangular matrices $C_1, \ldots, C_n$ and $D_1, \ldots, D_n$ as in (4.2). Then there exist an isomorphism $\Phi : O_A \otimes K \to O_B \otimes K$ of $C^*$-algebras and a unitary representation $t \in \mathbb{T} \to u^A_t \in M(D_A \otimes C)$ such that

$$\Phi(D_A \otimes C) = D_B \otimes C, \quad \Phi \circ \text{Ad}(u^A_t) \circ (\rho^A_t \otimes \text{id}) = (\rho^B_t \otimes \text{id}) \circ \Phi,$$

and the following diagram is commutative

$$\begin{array}{ccc}
K_0(O_A) & \xrightarrow{\Phi_*} & K_0(O_B) \\
\epsilon_A \downarrow & & \downarrow \epsilon_B \\
\mathbb{Z}^N/(\text{id} - A^t)\mathbb{Z}^N & \xrightarrow{\Phi((C_1 C_2 \cdots C_n)^t)} & \mathbb{Z}^M/(\text{id} - B^t)\mathbb{Z}^M.
\end{array}$$

We note that the inverse of $\Phi((C_1 C_2 \cdots C_n)^t) : \mathbb{Z}^N/(\text{id} - A^t)\mathbb{Z}^N \to \mathbb{Z}^M/(\text{id} - B^t)\mathbb{Z}^M$ is given by $\Phi_{(D_n \cdots D_2 D_1)} : \mathbb{Z}^M/(\text{id} - B^t)\mathbb{Z}^M \to \mathbb{Z}^N/(\text{id} - A^t)\mathbb{Z}^N$.

## 5 Converse and Invariant

In this section, we will study the converse of Corollary 3.11 by using Corollary 4.7. We fix a projection $p_1$ of rank one in $K$.

**Proposition 5.1.** The following assertions are equivalent.

(i) There exist an isomorphism $\Phi : O_A \otimes K \to O_B \otimes K$ of $C^*$-algebras and a unitary one-cocycle $u_t \in M(O_B \otimes K), t \in \mathbb{T}$ relative to $\rho_t^B \otimes \text{id}$ such that

$$\Phi(D_A \otimes C) = D_B \otimes C, \quad \Phi \circ (\rho_t^A \otimes \text{id}) = \text{Ad}(u_t) \circ (\rho_t^B \otimes \text{id}) \circ \Phi,$$

$$\Phi_*([1_A \otimes p_1]) = [1_B \otimes p_1] \text{ in } K_0(O_B).$$

(ii) There exist an isomorphism $\varphi : O_A \to O_B$ and a unitary one-cocycle $v_t \in U(O_B), t \in \mathbb{T}$ relative to $\rho_t^B$ on $O_B$ such that

$$\varphi(D_A) = D_B \quad \text{and} \quad \varphi \circ \rho_t^A = \text{Ad}(v_t) \circ \rho_t^B \circ \varphi, \quad t \in \mathbb{T}.$$
Proof. The implication (ii) $\implies$ (i) is obvious by putting $\Phi = \varphi \otimes \text{id}$ and $u_t = v_t \otimes 1$. We will show the implication (i) $\implies$ (ii) in the following way. By [12, Proposition 3.13], the condition $\Phi_*([1_A \otimes p_1]) = [1_B \otimes p_1]$ in $K_0(\mathcal{O}_B)$ ensures us that there exists a partial isometry $V \in \mathcal{O}_B \otimes \mathcal{K}$ satisfying the following conditions:

$$V(\mathcal{D}_B \otimes \mathcal{C})V^* \subset \mathcal{D}_B \otimes \mathcal{C}, \quad V^*(\mathcal{D}_B \otimes \mathcal{C})V \subset \mathcal{D}_B \otimes \mathcal{C},$$

$$VV^* = 1_B \otimes p_1, \quad V^*V = \Phi(1_A \otimes p_1).$$

Put $\Psi = \text{Ad}(V) \circ \Phi : \mathcal{O}_A \otimes \mathcal{K} \to \mathcal{O}_B \otimes \mathcal{K}$. It is straightforward to see that

$$\Psi(\mathcal{O}_A \otimes \mathcal{C}_p) = \mathcal{O}_B \otimes \mathcal{C}_p, \quad \Psi(\mathcal{D}_A \otimes \mathcal{C}_p) = \mathcal{D}_B \otimes \mathcal{C}_p, \quad \Psi(1_A \otimes p_1) = 1_B \otimes p_1.$$

It is clear that $\Psi_* = \Phi_* : K_0(\mathcal{O}_A) \to K_0(\mathcal{O}_B)$. We identify $\mathcal{O}_B \otimes \mathcal{C}_p$ with $\mathcal{O}_B$. Put the partial isometry $v_t = Vu_t(\rho_t^B \otimes \text{id})(V^*) \in \mathcal{O}_B \otimes \mathcal{K}$. Since $v_t = (1_B \otimes p_1)u_t(1_B \otimes p_1)$, by this identification, $v_t$ belongs to $\mathcal{O}_B$. Define $\varphi : \mathcal{O}_A \to \mathcal{O}_B$ by setting $\varphi(a) = \Psi(a \otimes p_1)$ for $a \in \mathcal{O}_A$. It then follows that

$$\varphi(\rho_t^A(a)) \otimes p_1 = V\Phi(\rho_t^A(a) \otimes p_1)V^*$$

$$= V(\text{Ad}(u_t) \circ (\rho_t^B \otimes \text{id}) \circ \Phi)(a \otimes p_1)V^*$$

$$= Vu_t(\rho_t^B \otimes \text{id})(V^*)(\rho_t^B \otimes \text{id})\Phi(V(a \otimes p_1)V^*)(\rho_t^B \otimes \text{id})(V)u_t^*V^*$$

$$= v_t((\rho_t^B \otimes \text{id}) \circ \Phi)(a \otimes p_1)v_t^*$$

$$= (\text{Ad}(v_t) \circ (\rho_t^B \circ \varphi)(a)) \otimes p_1$$

so that we have $\varphi(\rho_t^A(a)) = (\text{Ad}(v_t) \circ \rho_t^B \circ \varphi)(a)$. Since we have

$$(\rho_t^B \otimes \text{id})(\Phi(1_A \otimes p_1)) = (\text{Ad}(u_t^*) \circ \Phi \circ (\rho_t^A \otimes \text{id}))(1_A \otimes p_1) = u_t^*\Phi(1_A \otimes p_1)u_t = u_t^*V^*Vu_t,$$

we have

$$v_t\rho_t^B(v_s) = Vu_t(\rho_t^B \otimes \text{id})(V^*)(\rho_t^B \otimes \text{id})(Vu_s(\rho_s^B \otimes \text{id})(V^*))$$

$$= Vu_t(\rho_t^B \otimes \text{id})(V^*V)(\rho_t^B \otimes \text{id})(u_s)(\rho_t^B \otimes \rho_s^B \otimes \text{id})(V^*)$$

$$= Vu_tu_s^*V^*Vu_t(\rho_t^B \otimes \text{id})(u_s)(\rho_t^B \otimes \rho_s^B \otimes \text{id})(V^*)$$

$$= Vu_tu_s^*V^*Vu_t(\rho_t^B \otimes \text{id})(u_s)(\rho_{t+s}^B \otimes \text{id})(V^*)$$

$$= Vu_tu_s^*V^*Vu_t(\rho_{t+s}^B \otimes \text{id})(V^*)$$

$$= vu_{t+s}.$$

Hence $v_t, t \in \mathbb{T}$ is a unitary one-cocycle relative to $\rho^B \otimes \text{id}$. \hfill \Box

**Remark 5.2.** Let $v_t$ in $\mathcal{O}_B$ be a unitary one-cocycle relative to $\rho_t^B$ satisfying (5.3). For $a \in \mathcal{D}_A$, we see that $\varphi(\rho_t^A(a)) = \text{Ad}(v_t)(\rho_t^B(\varphi(a)))$. As $\rho_t^A(a) = a$ and $\varphi(a)$ belongs to $\mathcal{D}_B$ so that we have $\varphi(a) = \text{Ad}(v_t)(\varphi(a))$. Hence $v_t$ commutes with any element of $\mathcal{D}_B$. This implies that $v_t$ belongs to $\mathcal{D}_B$ and hence it is fixed by the action $\rho^B$. Therefore a unitary one-cocycle $v_t$ in $\mathcal{O}_B$ relative to $\rho_t^B$ satisfying (5.3) automatically belongs to $\mathcal{D}_B$ and yields a unitary representation $t \in \mathbb{T} \to v_t \in \mathcal{D}_B$. Since the unitary $u_t$ in (5.1) is given by $u_t = v_t \otimes 1$ from the unitary $v_t$ satisfying (5.3), the unitary one-cocycle $u_t$ in the statement (i) of the above proposition can be taken as a unitary representation $t \in \mathbb{T} \to u_t \in M(\mathcal{D}_B \otimes \mathcal{C})$ which is fixed by the action $\rho_t^B \otimes \text{id}$.
Corollary 5.3. If there exist an isomorphism \( \Phi : \mathcal{O}_A \otimes \mathcal{K} \to \mathcal{O}_B \otimes \mathcal{K} \) of \( C^* \)-algebras and a unitary one-cocycle \( u_t \) in \( M(\mathcal{O}_B \otimes \mathcal{K}) \) relative to \( \rho_t^B \otimes \text{id} \) such that
\[
\Phi(D_A \otimes C) = D_B \otimes C, \quad \Phi \circ (\rho_t^A \otimes \text{id}) = \text{Ad}(u_t) \circ (\rho_t^B \otimes \text{id}) \circ \Phi,
\]
\[
\Phi_*([1_A \otimes p_1]) = [1_B \otimes p_1] \text{ in } K_0(\mathcal{O}_B),
\]
then two-sided topological Markov shifts \((\bar{X}_A, \bar{\sigma}_A)\) and \((\bar{X}_B, \bar{\sigma}_B)\) are topologically conjugate.

Proof. By [13, Theorem 6.7], the equality (5.3) implies strongly continuous orbit equivalence between the one-sided topological Markov shifts \((X_A, \sigma_A)\) and \((X_B, \sigma_B)\). It also implies topological conjugacy of their two-sided topological Markov shifts \((\bar{X}_A, \bar{\sigma}_A)\) and \((\bar{X}_B, \bar{\sigma}_B)\) by [13, Theorem 5.5]. \(\Box\)

Definition 5.4. We say that an isomorphism \( \xi : \mathcal{O}_B \otimes \mathcal{K} \to \mathcal{O}_A \otimes \mathcal{K} \) of \( C^* \)-algebras is induced from strong shift equivalence \( A \approx_{C_1, D_1} \cdots \approx_{C_n, D_n} B \) if there exists a unitary one-cocycle \( u_t \) in \( M(\mathcal{O}_A \otimes \mathcal{K}) \) relative to \( \rho_t^A \otimes \text{id} \) such that
\[
\xi(D_B \otimes C) = D_A \otimes C, \quad \xi \circ (\rho_t^B \otimes \text{id}) = \text{Ad}(u_t) \circ (\rho_t^A \otimes \text{id}) \circ \xi,
\]
\[
\xi_* = \epsilon_A^{-1} \circ \Phi(D_{D_1} \cdots D_{D_n}) \circ \epsilon_B : K_0(\mathcal{O}_B) \to K_0(\mathcal{O}_A).
\]

We will define the strong shift equivalence invariant subset of \( K_0(\mathcal{O}_A) \) as follows.

Definition 5.5.
\[
K_0^{\text{SSE}}(\mathcal{O}_A) = \{ [p] \in K_0(\mathcal{O}_A) \mid \exists B \text{ a square matrix and } \exists \xi : \mathcal{O}_B \otimes \mathcal{K} \to \mathcal{O}_A \otimes \mathcal{K}
\]
an isomorphism induced from strong shift equivalence;
\[
A \approx_{C_1, D_1} \cdots \approx_{C_n, D_n} B \text{ and } \xi_*([1_B]) = [p] \text{ in } K_0(\mathcal{O}_A)\}.
\]

We note that the class \([1_A]\) in \( K_0(\mathcal{O}_A) \) of the unit \( 1_A \) of \( \mathcal{O}_A \) always belongs to the set \( K_0^{\text{SSE}}(\mathcal{O}_A) \), because we may take \( B = A \) and \( \xi = \text{id} \).

Proposition 5.6. Suppose that there exists a topological conjugacy between \((\bar{X}_A, \bar{\sigma}_A)\) and \((\bar{X}_B, \bar{\sigma}_B)\). Then there exists an isomorphism \( \eta : K_0(\mathcal{O}_A) \to K_0(\mathcal{O}_B) \) satisfying \( \eta(K_0^{\text{SSE}}(\mathcal{O}_A)) = K_0^{\text{SSE}}(\mathcal{O}_B) \). Hence the pair \((K_0(\mathcal{O}_A), K_0^{\text{SSE}}(\mathcal{O}_A))\) is an invariant under topological conjugacy of two-sided topological Markov shifts.

Proof. Suppose that \((\bar{X}_A, \bar{\sigma}_A)\) and \((\bar{X}_B, \bar{\sigma}_B)\) are topologically conjugate so that \( A \approx_{C_1, D_1} \cdots \approx_{C_n, D_n} B \) for some nonnegative rectangular matrices \( C_1, D_1, \ldots, C_n, D_n \). The strong shift equivalence induces that there exist an isomorphism \( \xi_{BA} : \mathcal{O}_A \otimes \mathcal{K} \to \mathcal{O}_B \otimes \mathcal{K} \) and a unitary one-cocycle \( u_t \) in \( M(\mathcal{O}_B \otimes \mathcal{K}) \) relative to \( \rho_t^B \otimes \text{id} \) such that
\[
\xi_{BA}(D_A \otimes C) = D_B \otimes C, \quad \xi_{BA} \circ (\rho_t^A \otimes \text{id}) = \text{Ad}(u_t) \circ (\rho_t^B \otimes \text{id}) \circ \xi_{BA},
\]
\[
\xi_{BA^*} = \Phi(C_1 \cdots C_n) : K_0(\mathcal{O}_A) \to K_0(\mathcal{O}_B).
\]

Put \( \eta = \xi_{BA^*} : K_0(\mathcal{O}_A) \to K_0(\mathcal{O}_B) \). Take an element \([p] \in K_0^{\text{SSE}}(\mathcal{O}_A)\). There exists a square nonnegative matrix \( A' \) and an isomorphism \( \xi_{AA'} : \mathcal{O}_A' \otimes \mathcal{K} \to \mathcal{O}_A \otimes \mathcal{K} \) of \( C^* \)-algebras
induced from strong shift equivalence $A' \approx \cdots \approx A$ such that $\xi_{AA'}([1_A]) = [p]$ in $K_0(O_A)$. Then the isomorphism $\xi_{BA} \circ \xi_{AA'} : O_A \otimes K \to O_B \otimes K$ is induced from strong shift equivalence

$$A' \approx \cdots \approx A \approx \cdots \approx B,$$

such that $\eta([p]) = (\xi_{BA} \circ \xi_{AA'})([1_A])$ in $K_0(O_B)$ so that $\eta([p]) \in K_0^{\text{SSE}}(O_B)$. \hfill \square

Suppose that two matrices $A, B$ are strong shift equivalent in $n$-step such as (4.2). The matrix $B$ in (4.2) is given by $B = D_n C_n$ so that (4.2) is written as

$$A \approx \cdots \approx D_n C_n.$$

We set the following sequence $\text{SSE}_n(A), n = 1, 2, \ldots$ of subsets of the group $\mathbb{Z}^N$

$$\text{SSE}_n(A) = \{v \in \mathbb{Z}^N \mid v = D_1^t \cdots D_{n-1}^t D_n [1, 1, \ldots, 1]^t, A \approx \cdots \approx D_n C_n\},$$

where $[1, 1, \ldots, 1]^t$ denotes the (the row size of $D_n$) $\times 1$ matrix whose entries are all 1’s. We define the sequence $K_{\text{alg}, n}(A), n = 1, 2, \ldots$ of subsets of the group $\mathbb{Z}^N/(\text{id} - A^t)\mathbb{Z}^N$ by

$$K_{\text{alg}, n}(A) = \{[v] \in \mathbb{Z}^N/(\text{id} - A^t)\mathbb{Z}^N \mid v \in \text{SSE}_n(A)\}, \quad n = 1, 2, \ldots.$$

Then we define the subset $K_{\text{alg}}^{\text{SSE}}(A)$ of $\mathbb{Z}^N/(\text{id} - A^t)\mathbb{Z}^N$ by

$$K_{\text{alg}}^{\text{SSE}}(A) = \cup_{n=1}^{\infty} K_{\text{alg}, n}(A).$$

By Corollary 4.7 we have the following proposition

**Proposition 5.7.** Let $\epsilon_A : K_0(O_A) \to \mathbb{Z}^N/(\text{id} - A^t)\mathbb{Z}^N$ be the isomorphism in Corollary 4.7. Then we have

$$\epsilon_A(K_0^{\text{SSE}}(O_A)) = K_{\text{alg}}^{\text{SSE}}(A).$$

**Proof.** For $[p] \in K_0^{\text{SSE}}(O_A)$, there exist a nonnegative square matrix $B$ with a strong shift equivalence $A \approx \cdots \approx B$ and an isomorphism $\xi : O_B \otimes K \to O_A \otimes K$ of $C^*$-algebras and a unitary one-cocycle $u_t, t \in \mathbb{T}$ relative to $\rho^A \otimes \text{id}$ such that

$$\xi(D_B \otimes C) = D_A \otimes C, \quad \xi \circ (\rho_t^{A} \otimes \text{id}) = \text{Ad}(u_t) \circ (\rho_t^A \otimes \text{id}) \circ \xi,$$

$$\xi_* = \epsilon_A^{-1} \circ \Phi_{(D_n \cdots D_2 D_1)^t} \circ \epsilon_B : K_0(O_B) \to K_0(O_A) \quad \text{and} \quad \xi_*([1_B]) = [p].$$

(5.5)

Since $\epsilon_B([1_B]) = [[1, 1, \ldots, 1]^t]$ in $\mathbb{Z}^M/(\text{id} - B^t)\mathbb{Z}^M$, we have

$$\epsilon_A([p]) = \epsilon_A \circ \xi_*([1_B]) = \Phi_{(D_n \cdots D_2 D_1)^t} \circ \epsilon_B([1_B]) = \Phi_{(D_n \cdots D_2 D_1)^t}([[1, 1, \ldots, 1]^t])$$

(5.7)

so that $\epsilon_A([p]) \in K_{\text{alg}}^{\text{SSE}}(A)$ and hence $\epsilon_A(K_0^{\text{SSE}}(O_A)) \subset K_{\text{alg}}^{\text{SSE}}(A)$.

Conversely, take an arbitrary element $[v] \in K_{\text{alg}}^{\text{SSE}}(A)$. We may find a strong shift equivalence $A \approx \cdots \approx D_n C_n$ such that $v = (D_n \cdots D_2 D_1)^t [1, 1, \ldots, 1]^t$. Put $B = D_n C_n$. By Corollary 4.7 there exists an isomorphism $\xi : O_B \otimes K \to O_A \otimes K$ of $C^*$-algebras and a unitary one-cocycle $u_t, t \in \mathbb{T}$ relative to $\rho^A \otimes \text{id}$ satisfying (5.5) and $\xi_* = \epsilon_A^{-1} \circ \Phi_{(D_n \cdots D_2 D_1)^t} \circ \epsilon_B : K_0(O_B) \to K_0(O_A)$. Put $[p] = \xi_*([1_B])$ which belongs to $K_0^{\text{SSE}}(O_A)$. By the same equalities as (5.7), we get $\epsilon_A([p]) = \Phi_{(D_n \cdots D_2 D_1)^t}([[1, 1, \ldots, 1]^t])$ which is the class of $[v]$. This shows that $\epsilon_A(K_0^{\text{SSE}}(O_A)) \supset K_{\text{alg}}^{\text{SSE}}(A)$. \hfill \square
Theorem 5.8. Let $A, B$ be nonnegative irreducible and non-permutation matrices. The following two assertions are equivalent.

(i) Two-sided topological Markov shifts $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are topologically conjugate.

(ii) There exist an isomorphism $\Phi : \mathcal{O}_A \otimes \mathcal{K} \to \mathcal{O}_B \otimes \mathcal{K}$ of $C^*$-algebras and a unitary one-cocycle $u_t$ in $M(\mathcal{O}_B \otimes \mathcal{K})$ relative to $\rho_B^t \otimes \text{id}$ such that

$$\Phi(D_A \otimes \mathcal{C}) = D_B \otimes \mathcal{C}, \quad \Phi \circ (\rho_A^t \otimes \text{id}) = \text{Ad}(u_t) \circ (\rho_B^t \otimes \text{id}) \circ \Phi,$$

$$\Phi_*([1_{\mathcal{K}}]) = [1_{\mathcal{K}}] \quad \text{in} \ K_0(\mathcal{O}_B).$$

Proof. (i) $\implies$ (ii) comes from Corollary 3.11 and Proposition 5.6.

(ii) $\implies$ (i): Suppose that there exist an isomorphism $\Phi : \mathcal{O}_A \otimes \mathcal{K} \to \mathcal{O}_B \otimes \mathcal{K}$ of $C^*$-algebras and a unitary one-cocycle $u_t$ in $M(\mathcal{O}_B \otimes \mathcal{K})$ relative to $\rho_B^t \otimes \text{id}$ satisfying the conditions of (ii). Put the projection $p = \Phi(1_A \otimes 1) \in \mathcal{O}_B \otimes \mathcal{K}$. As $[1_A] \in K_0^\text{SSE}(\mathcal{O}_A)$ and $\Phi_*([1_{\mathcal{K}}]) = [1_{\mathcal{K}}]$, $p \in K_0(\mathcal{O}_B)$ belongs to $K_0^\text{SSE}(\mathcal{O}_B)$. One may take a nonnegative square matrix $B'$ and an isomorphism $\gamma : \mathcal{O}_B \otimes \mathcal{K} \to \mathcal{O}_B' \otimes \mathcal{K}$ with a unitary one-cocycle $u'_t$ in $M(\mathcal{O}_B' \otimes \mathcal{K})$ relative to $\rho_B^{t'} \otimes \text{id}$ induced from strong shift equivalence $B \approx_{c_1, d_1} \cdots \approx_{c_n, d_n} B'$ satisfying

$$\gamma(D_B \otimes \mathcal{C}) = D_{B'} \otimes \mathcal{C}, \quad \gamma \circ (\rho_B^t \otimes \text{id}) = \text{Ad}(u'_t) \circ (\rho_B^{t'} \otimes \text{id}) \circ \gamma,$$

$$\gamma_*([1_{\mathcal{K}}]) = [1_{\mathcal{K}}] \quad \text{in} \ K_0(\mathcal{O}_B').$$

Then the isomorphism $\gamma \circ \Phi : \mathcal{O}_A \otimes \mathcal{K} \to \mathcal{O}_B' \otimes \mathcal{K}$ satisfies the conditions

$$(\gamma \circ \Phi)(D_A \otimes \mathcal{C}) = D_{B'} \otimes \mathcal{C}, \quad (\gamma \circ \Phi) \circ (\rho_A^t \otimes \text{id}) = \text{Ad}(\gamma(u_t)u'_t) \circ (\rho_B^{t'} \otimes \text{id}) \circ (\gamma \circ \Phi),$$

$$(\gamma \circ \Phi)_*([1_A]) = [1_{\mathcal{K}}] \quad \text{in} \ K_0(\mathcal{O}_B').$$

By Corollary 5.3, the two-sided topological Markov shifts $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_{B'}, \bar{\sigma}_{B'})$ are topologically conjugate. Since $(\bar{X}_B, \bar{\sigma}_B)$ and $(\bar{X}_{B'}, \bar{\sigma}_{B'})$ are topologically conjugate, so are $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_{B'}, \bar{\sigma}_{B'})$. \hfill $\Box$

Remark 5.9. The unitary one-cocycle $u_t$ in $M(\mathcal{O}_B \otimes \mathcal{K})$ in (ii) of the above theorem can be taken as a unitary representation $t \in T \to u_t \in M(\mathcal{O}_B \otimes \mathcal{K})$ by Corollary 3.11.

Definition 5.10. A nonnegative square matrix $A = [A(i, j)]_{i,j=1}^N$ is said to have full strong shift equivalent units in $K_0$-group if $K_0^\text{SSE}(A) = \mathbb{Z}^N / (1 - A^t)\mathbb{Z}^N$. We simply call it that $A$ has full units.

By Proposition 5.7, $A$ has full units if and only if $K_0^\text{SSE}(A) = K_0(\mathcal{O}_A)$. Since the subset $K_0^\text{SSE}(\mathcal{O}_A) \subset K_0(\mathcal{O}_A)$ is invariant under topological conjugacy of two-sided topological Markov shifts by Proposition 5.6, we have

Proposition 5.11. Suppose that two-sided topological Markov shifts $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are topologically conjugate. Then $A$ has full units if and only if $B$ has full units.

As a corollary of Theorem 5.8, we have the following corollary.
Corollary 5.12. Suppose that both $A$ and $B$ have full units. Then the following two assertions are equivalent.

(i) Two-sided topological Markov shifts $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are topologically conjugate.

(ii) There exist an isomorphism $\Phi : \mathcal{O}_A \otimes K \to \mathcal{O}_B \otimes K$ of $C^*$-algebras and a unitary one-cocycle $u_t$ in $M(\mathcal{O}_B \otimes K)$ relative to $\rho^B_t \otimes \text{id}$ such that

$$
\Phi(D_A \otimes C) = D_B \otimes C, \quad \Phi \circ (\rho_t^A \otimes \text{id}) = \text{Ad}(u_t) \circ (\rho^B_t \otimes \text{id}) \circ \Phi.
$$

Example 5.13.

1. If $K_0(\mathcal{O}_A) = 0$, then $A$ has full units.

2. If $A = [N]$ for some $1 < N \in \mathbb{N}$, then the matrix $A$ has full units. For any $0 \leq k \leq N - 1$, let $C$ be the $1 \times (k+1)$ matrix $[1, \ldots, 1, N-k]$ and $D$ the $(k+1) \times 1$ matrix $(1, 1, \ldots, 1)^t$. Then $A = CD$ and $D^t[1, \ldots, 1]^t = k + 1$. Hence $[k+1] \in \mathbb{Z}/(1-N)\mathbb{Z}$ so that $K_{\text{alg}}^S(A) = \mathbb{Z}/(1-N)\mathbb{Z} = K_0(\mathcal{O}_A)$.

There is no known example of irreducible, non permutation matrix $A$ such that $A$ does not have full units.

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