Geometry of Special Galileon

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Abstract

Theory known as Special Galileon has recently attracted considerable interest due to its peculiar properties. It has been shown that it represents an extremal member of the set of effective field theories with enhanced soft limit. This property makes its tree-level $S$-matrix fully on-shell reconstructible and representable by means of the Cachazo-He-Yuan representation. The enhanced soft limit is a consequence of a new hidden symmetry of the Special Galileon action, however, until now, the origin of this peculiar symmetry has remained unclear. In this paper we interpret this symmetry as a special transformation of the coset space $\text{GAL}(D,1)/\text{SO}(1,D-1)$ and show, that there exists a three-parametric family of invariant Galileon actions. The latter family is closed under duality which appears as a natural generalization of the above mentioned symmetry. We also present a geometric construction of the Special Galileon action using $D$-dimensional brane propagating in $2D$-dimensional flat pseudo-riemannian space. Within such framework, the Special Galileon symmetry emerges as an $U(1,D-1)$ symmetry of the target space, which can be treated as a $D$-dimensional Kähler manifold. Such a treatment allows for classification of the higher order invariant Lagrangians needed as counterterms on the quantum level. We also briefly comment on relation between such higher order Lagrangians and the Lagrangians invariant with respect to the polynomial shift symmetry.

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1 Introduction and summary of the results

Galileons are known to be very interesting derivatively coupled real scalar field theories with a rich spectrum of applications. Originally the simplest cubic Galileon emerged as an effective field theory describing the only interacting scalar mode in a decoupling limit of the Dvali-Gabadadze-Porrati model [1, 2]. It also naturally appeared in analogous decoupling limit of theory with massive graviton [3] where it described a zero-helicity mode. It has been soon realized, that generalizations of the cubic Galileon exist, namely $D$ dimensions allow for $D + 1$ independent Galileon Lagrangian terms. Such a general Galileon theory has been then
proposed as a long distance modification of the General relativity [4]. An appealing feature of Galileon in this context is a presence of Vainshtein screening [5] as well as a stability of its basic Lagrangian with respect to the quantum corrections [6, 7, 8]. Other generalizations as a vector or $p$–form Galileons and its covariant form on a curved background has been discussed e.g. in [9, 10].

There exist also close relation between the Galileon theories and theories describing fluctuations of $D$ dimensional brane in $D + 1$ dimensional Minkowski space-time. Within this framework the flat space Galileon emerges as a certain non-relativistic limit of special versions of such theories [11, 7]. This approach allowed for further generalizations changing the Minkowski space for Anti-de Sitter space or for a general curved background and recovering in this way the conformal Galileon and covariant Galileon respectively.

Recently the Galileon theories attracted further attention due to the special distinctive properties of their $S$–matrix. It has been found [12, 13, 14], that the general Galileon is a unique theory (up to the values of the couplings and within a certain class of single scalar theories with specific power-counting) the amplitudes of which have non-trivial one-particle soft limit. More precisely, let us write the interaction Lagrangian of a single scalar effective theory schematically as

$$L_{\text{int}} = \sum_{d,n} \lambda_{d,n} \partial^d \phi^n$$

and let us restrict ourselves to the theories with vertices for which

$$\rho \equiv \frac{d - 2}{n - 2} = 2.$$  \hspace{1cm} (1.2)

Then the general Galileon is the only theory whose scattering amplitudes $A_m(p, \ldots)$ vanish as the second power of momentum,

$$A_m(p, \ldots) \xrightarrow{p \to 0} O(p^2)$$  \hspace{1cm} (1.3)

when one of the particles in the in or out state becomes soft. Moreover, within the class of the general Galileon theories there exists a distinguished one (dubed Special Galileon), which exhibits even more enhanced soft behavior - the corresponding amplitudes vanish as the third power of the soft momentum [12, 15]. The latter feature makes the Special Galileon very peculiar: its $S$–matrix can be fully reconstructed from the lowest nontrivial on-shell amplitude i.e. form the four-point one\textsuperscript{1}[13, 14]. Also, the $S$–matrix of the Special Galileon has the Cachazo-He-Yuan representation [15] (along with other exceptional scalar field theories with enhanced soft limit, namely the $U(N)$ non-linear sigma model and the Dirac-Born-Infeld theory (DBI)). As discussed in [14], all these theories occupy the border line $\rho = \sigma - 1$ which delimits the allowed region for theories with the non-trivial soft limit

$$A_n(p) \xrightarrow{p \to 0} O(p^\sigma)$$  \hspace{1cm} (1.4)

in the $(\sigma, \rho)$ plane and so their soft behavior is extremal. The Special Galileon is in fact “doubly extremal” because is sits in a corner of the allowed region and its soft exponent $\sigma = 3$ is the highest possible one [14].

\textsuperscript{1}Also the $S$–matrix of the general Galileon in $D$ dimensions is on-shell reconstructible [13], however, as an input, it is necessary to know all the on shell amplitudes up to the $D + 1$ point one.
The soft behavior of the scattering amplitudes is closely related to the non-linear symmetries of the underlying theory [14]. For all the above mentioned exceptional scalar field theories there exist a symmetry of the action of the type

$$\delta_{\theta} \phi (x) = \theta_{\alpha_1 \ldots \alpha_n} [x^{\alpha_1} \ldots x^{\alpha_n} + \Delta^{\alpha_1 \ldots \alpha_n} (x)],$$

where $\theta_{\alpha_1 \ldots \alpha_n}$ are infinitesimal parameters and where $\Delta^{\alpha_1 \ldots \alpha_n} (x)$ is a linear combination of local composite operators. The very presence of such a symmetry guarantees, under some regularity assumptions, that the soft exponent satisfies $\sigma \geq n + 1$. For instance the $O(p^2)$ soft limit of the general Galileon is a consequence of the linear shift symmetry

$$\delta_{\theta}^{Gal} \phi (x) = \theta \alpha x^\alpha, \quad (1.5)$$

and the same $\sigma = 2$ soft behavior of the Dirac-Born-Infeld theory is a consequence of the symmetry

$$\delta_{\theta}^{DBI} \phi (x) = \theta [x^\alpha - F^{-D} \phi (x) \partial^\alpha \phi (x)] \quad (1.7)$$

Let us remind, that within the latter theory the field $\phi (x)$ describes a position of the brane in the extra dimension. The symmetry (1.7) of the DBI action has then a nice geometrical interpretation as a non-linearly realized Lorentz transformation of the target $D+1$ dimensional space in which the fluctuating brane propagates. This geometrical picture is very useful because it simplifies considerably the constructions of higher order invariant Lagrangians (i.e. those with more than one derivative per field). The latter are necessary as counterterms when the higher loop corrections are taken into account. The very identification of the DBI symmetry as a group of invariance of the target space metric converts the task of counterterm construction in an almost routine enumeration of the reparameterization invariants built from the induced metric on the brane and its extrinsic curvature. In this sense the DBI theory and its symmetry is fully understood.

This is not the case of the Special Galileon, although a hidden symmetry responsible for its $O(p^3)$ soft behavior has been found shortly after discovery of this peculiar theory [16]. This symmetry can be written in the form

$$\delta_{\theta}^{Gal} \phi (x) = \theta^{\mu \nu} (\Lambda x_\mu x_\nu - \partial_\mu \phi (x) \partial_\nu \phi (x)) \quad (1.8)$$

where $\theta^{\mu \nu}$ is a symmetric traceless tensor and $\Lambda$ is a parameter related to the coupling constant of the Special Galileon. However, the origin and properties of this symmetry have been still unclear and its form does not allow for construction of higher order counterterms in an easy and straightforward way. Even the direct proof that the transformation (1.8) is really a symmetry of the Special Galileon action is rather complicated, because the contributions of different parts of the Lagrangian with different number of fields has to cancel each other in a subtle way.

The aim of this article is to elucidate this issue. We will first show, that the hidden symmetry (1.8) has its origin in a certain finite reparameterization of the coset space $GAL (D, 1) / SO(1, D - 1)$. The latter corresponds to the spontaneous symmetry breaking of the Galileon symmetry

$$\delta_{c,d} \phi (x) = c + d \mu x^\mu, \quad (1.9)$$

according to the pattern\footnote{Here, as above, $D$ is the space-time dimension.} $GAL (D, 1) \rightarrow ISO(1, D - 1)$. As it has been shown in [17], the general Galileon Lagrangian can be recognized as a linear combination of Schwinger terms
related to closed $GAL(D, 1)$-invariant $D + 1$ forms $\omega^{(n)}_{D+1}$ on the above mentioned coset. This coset space can be parameterized by $2D + 1$ coordinates $\phi$, $L_\mu$ and $x^\mu$, which are related to the generators of the general Galileon symmetry (1.9) and space-time translations. The general Galileon Lagrangian in then obtained by means of integrating the forms $\omega^{(n)}_{D+1}$ over $D + 1$ dimensional ball $B^{D+1}$ whose boundary is the compactified space-time and imposing then the inverse Higgs constraint $L_\mu = \partial_\mu \phi$.

We will show that the coset space reparametrization $(\phi, L, x) \rightarrow (\phi, L_\theta, x_\theta)$, where

$$
x^\mu_\theta = [\cosh (2a\theta)]^{\mu}_\nu x^\nu + \frac{1}{a} [\sinh (2a\theta)]^{\mu}_\nu L^\nu
$$

$$
L^\mu_\theta = a [\sinh (2a\theta)]^{\mu}_\nu x^\nu + [\cosh (2a\theta)]^{\mu}_\nu L^\nu
$$

$$
\phi_\theta = \phi - \frac{1}{2} L^\mu x_\mu + \frac{1}{2} L^\mu_\theta \cdot (x_\theta)_\mu
$$

(1.10)
is responsible for the hidden Galileon symmetry (1.8) with $\Lambda = a^2$. Here the matrices $\cosh (2a\theta)$ and $\sinh (2a\theta)$ are corresponding functions of the symmetric tensor $\theta^{\mu\nu}$ which is a free parameter of the transformation. The case $\Lambda < 0$ can be obtained by means of analytic continuation $a \rightarrow ia$. We will also show, that for fixed $a$ there is a two-parametric family of actions, $S(a, c_+, c_-)$ which are invariant with respect to (1.10) for traceless $\theta^{\mu\nu}$. Moreover, in the case when $\theta^{\mu\nu}$ is not traceless, the transformation (1.10) becomes a duality of the family $S(a, c_+, c_-)$, namely such actions transform according to

$$
S(a, c_+, c_-) \rightarrow S(a, c_+ + a\theta^\mu_\mu, c_- - a\theta^\mu_\mu).
$$

(1.11)

The theory known in the literature as a Special Galileon then corresponds (up to an overall normalization) to the case $c_+ = c_-$. We have therefore not only hidden Galileon symmetry, but also a hidden Galileon duality of the whole family of Special Galileons.

Our second result shows, that the above symmetry/duality has a nice geometrical origin. Namely, we will prove that the Galileon field can be interpreted as a scalar degree of freedom which describes position of special $D-$dimensional brane in $2D$ dimensional flat space with pseudo-riemannian metric whose signature is either $(D, D)$ or $(2, 2D - 2)$. The coordinates on such a $2D$ space can be identified with the above mentioned coset space coordinates $x^\mu$ and $L_\mu$ and the metric reads

$$
ds^2 = \eta_{\mu\nu} \left( dX^\mu dX^\nu - \frac{1}{\Lambda} dL^\mu dL^\nu \right).
$$

(1.12)
The inverse Higgs constraint $L_\mu = \partial_\mu \phi$ is implemented demanding the brane to be isotropic with respect to the properly chosen two-form, namely

$$
\omega = \eta_{\mu\nu} dX^\mu \wedge dL^\nu.
$$

(1.13)
The $\omega-$preserving subgroup of isometries of the bulk then acts on the brane configurations, which are described solely by one scalar field $\phi$, and result in nonliner transformations which are exactly the general Galileon symmetry (1.9) (the translations in the $L_\mu$ directions) and the hidden Galileon duality (1.10) (rotations which mix $x^\mu$ and $L_\mu$). As a consequence and

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3In the powers of $\theta$ the indices are contracted with the flat metric tensor.

4In the analytically continued case $a = ia$ and $\alpha$ fixed there is only one-parameter family due to the requirement of reality of the action.
similarly to the construction of the DBI action, the invariant Galileon action can be constructed as brane action which consists of reparameterization invariants build from geometric objects like the induced metric on the brane

\[ g_{\mu\nu} = \eta_{\mu\nu} - \frac{1}{\Lambda} \partial_{\mu} \phi \partial^{\alpha} \partial_{\nu} \phi, \]  

(1.14)

or the brane extrinsic curvature. The higher order Lagrangian terms invariant with respect to (1.10), which are needed as counterterms for the special Galileon, can be then easily classified. We make such a classification up to the terms of the schematic structure \( \partial^{2n+4} \phi^n \) and \( n \) arbitrary.

Our third results concerns the relation between the higher order special Galileon Lagrangians and the Lagrangians invariant with respect to the polynomial shift symmetries. The latter have been studied and classified in [18] and [19]. We will show, that in \( D > 2n \) dimensions the quadratic shift invariants of the general form \( \partial^{2n+4} \phi^{2n+2} \) (i.e. \( (P, N, \Delta) = (2, 2n + 2, 3n + 2) \) when using the notation\(^5\) of [18]) can be easily obtained from our construction. Namely, provided we restrict ourselves to the Lovelock action [20] built form \( n \)–th power of the Riemann tensor corresponding to the metric (1.14), we can obtain the invariant \( (2, 2n + 2, 3n + 2) \) just summing up the all the terms with \( N = 2n + 2 \) legs.

The paper is organized as follows. In the Section 2 we briefly review a coset construction of the general Galileon action and fix our notation. In Section 3 we shortly discuss the relation between transformations of the coset and general Galileon dualities. In Section 4 we introduce a special coset space transformation which generates a duality of certain three-parametric family of Galileon actions and the infinitesimal form of which is identical with the hidden symmetry of Special Galileon for special choice of the parameters. Here we also discuss possible generalizations of the above mentioned duality/symmetry. Sections 5 and 6 are devoted to the geometrical construction of the Special Galileon using the probe brane in \( 2D \)–dimensional space. In Sections 7 and 8 we present a construction of the higher order Lagrangians and their classification up to the terms \( \partial^{2n+4} \phi^n \). Here we also comment on the relation to the actions invariant with respect to the quadratic shift. In Section 9 we briefly discuss the second branch of the Special Galileon symmetry within the probe brane context. Conclusions are summarized in Section 10. Some technical details are postponed to appendices. In Appendix A we prove the invariance of the Inverse Higgs Constraint with respect to the special Galileon duality. The same is done for generalized special Galileon duality in Appendix B. In Appendix C we give an explicit form of some higher order Lagrangian terms. In Appendix D we discuss the properties of the family of invariant Lagrangians under other duality transformations.

2 Coset construction of the Galileon action

In this section we give a brief overview\(^6\) of the geometry of the coset space behind the Galileon Lagrangian and its connection to the more traditional treatment. It appears that such a geometrical language is particularly useful and elegant for investigation of additional symmetries and dualities of the theory while the traditional approach makes these aspects less transparent. In what follows we will also fix our conventions and notation.

\(^5\)Here \( P \) is the order of the polynomial shift, \( N \) is a number of legs and \( 2\Delta \) is number of derivatives.

\(^6\)For more detailed treatment we refer to the original paper [17], as an example of further application see also [21].
It is well known that the Galielon field $\phi(x)$ can be understood as a Goldstone boson of the spontaneously broken Galileon\textsuperscript{7} symmetry,\textsuperscript{8}
\[
\delta_{a,b} \phi(x) = a + b \cdot x,
\]
according to the symmetry breaking pattern
\[
GAL(D, 1) \rightarrow ISO(1, D - 1).
\]
Here $GAL(D, 1)$ is the Galileon group in $D$ space-time dimensions with generators $P_a$, $J_{ab}$ (space-time translations, rotations and boosts) and $A, B_a$ (constant shift and non-uniform linear shift of the field respectively). These generators satisfy the algebra
\[
\begin{align*}
[P_a, P_b] &= [B_a, B_b] = [P_a, A] = [B_a, A] = [J_{ab}, A] = 0 \\
[P_a, B_b] &= i \eta_{ab} A \\
[J_{ab}, P_c] &= i (\eta_{bc} P_a - \eta_{ac} P_b) \\
[J_{ab}, B_c] &= i (\eta_{bc} B_a - \eta_{ac} B_b) \\
[J_{ab}, J_{cd}] &= i (\eta_{bc} J_{ad} + \eta_{ad} J_{bc} - \eta_{ac} J_{bd} - \eta_{bd} J_{ac}).
\end{align*}
\]
In terms of the generators $A$ and $B_a$ we can write the infinitesimal Galileon transformation (2.1) as
\[
\delta_{a,b} = -iaA - ib^a B_a.
\]
The symmetry breaking order parameter is the vacuum expectation value
\[
\langle 0 | \delta_{a,b} \phi | 0 \rangle = a + b \cdot x.
\]
As it has been recognized in [17], the basic (i.e. the lowest order\textsuperscript{9}) Galileon Lagrangian in $D$ dimensional space-time represents a linear combination of $D + 1$ generalized Wess-Zumino-Witten terms [22, 23, 24, 25, 26]. These can be constructed using the standard coset construction of Callan, Coleman, Wess and Zumino [26, 27] which has been adapted for non-uniform symmetries\textsuperscript{10} by Volkov [28] and Ivanov and Ogievetsky [29].

In the Galileon case the coset space is\textsuperscript{11}
\[
GAL(D, 1) / SO(1, D - 1) = \{gSO(1, D - 1), g \in GAL(D, 1)\}
\]
\footnotetext[7]{The Galileon symmetry is the simplest nontrivial example of the general polynomial shift symmetry discussed in [19] and [18].}
\footnotetext[8]{Here and in what follows the dot means a contraction of the adjacent Lorentz indices, i.e.
$L \cdot x = L^\mu x_\mu = L^\mu \eta_{\mu\nu} x_\nu = L \cdot \eta \cdot x$.
In the same spirit, the double (multiple) dot between symmetric tensors means double (multiple) contraction, e.g.
$\partial \phi: \eta = \partial_\mu \partial_\nu \phi \eta^{\mu\nu}$, $\partial \partial \phi: \partial \partial \phi = \partial_\mu \partial_\nu \partial_\rho \partial_\sigma \phi \partial^{\rho} \partial^{\sigma}$ e.t.c..}
\footnotetext[9]{For the systematic power counting scheme for the Galileon see [21]. According to this counting, the operators are classified using the index $\delta = d - 2(n - 1)$ where $d$ is number of derivatives and $n$ is number of the external lines. The lowest order Galileon Lagrangian corresponds to the linear combination of the operators with $\delta = 0$.}
\footnotetext[10]{That means those symmetries which do not commute with the spacetime translations.}
\footnotetext[11]{Note that the only generators which are realized linearly on the field space are the rotations and boosts $J_{ab}$.}
and can be parametrized by means of the coordinates \( x^\mu \), \( L^\mu \) and \( \phi \) (the latter two are candidates for the Goldstone fields) corresponding to the following choice of the representative \( U \) of each left coset

\[
U = e^{iP \cdot x} e^{iA \phi + iB \cdot L} \in \left\{ e^{iP \cdot x} e^{iA \phi + iB \cdot L} SO(1, D - 1) \right\}.
\]

This choice induces a non-linear realization of the transformation \( g \in GAL(D, 1) \) on the coset according to the prescription

\[
\left\{ e^{iP \cdot x'} e^{iA \phi' + iB \cdot L'} SO(1, D - 1) \right\} = \left\{ g e^{iP \cdot x} e^{iA \phi + iB \cdot L} SO(1, D - 1) \right\}.
\]

The basis of the covariant building blocks (dubbed \( \omega_A \), \( \omega_P^a \) and \( \omega_B^a \) in what follows) which are used for the construction of the \( GAL(D, 1) \) invariant action can be then read off from the Maurer-Cartan form

\[
\frac{1}{i} U^{-1} dU = \omega_P^c P_c + \omega_A A + \omega_B^d B_d.
\]

Explicitly we get

\[
\begin{align*}
\omega_A &= d\phi - L^a \eta_{ab} \cdot dx^b \\
\omega_P^a &= dx^a \\
\omega_B^a &= dL^a.
\end{align*}
\]

(2.10)

Lorentz invariants constructed from the components of the forms \( \omega_A \), \( \omega_P^a \), and \( \omega_B^a \) and the ordinary\(^{12}\) derivative \( \partial_\mu \) can be shown to be automatically invariant with respect to the non-linear realization (2.6) of \( GAL(D, 1) \) on the coset.

Note that the localized form of the Galileon symmetry with space-time dependent parameters \( a(x) \) and \( b(x) \)

\[
\delta_{a(x), b(x)} \phi = a(x) + b(x) \cdot x \equiv \tilde{a}(x) = \delta_{\tilde{a}(x)} \phi
\]

looks like the localization of the constant shift with parameter \( \tilde{a}(x) \). Therefore it is not possible to distinguish between the localization of the constant shift and the linear shift, and as a consequence, there is only one physical Goldstone boson \( \phi(x) \) corresponding to the local fluctuation of the order parameter (2.5) \([30, 31, 32, 33]\). The unphysical Goldstones \( L^\mu(x) \) can be eliminated imposing the Inverse Higgs Constraint (IHC) \([29]\) which in this case reads

\[
\omega_A = 0.
\]

(2.12)

This implies

\[
\omega_B^a = dL^\mu(x) = \partial^\mu \partial_\nu \phi(x) d\nu.
\]

(2.13)

The building blocks for the invariant Lagrangian are therefore the second and higher derivatives of the Galileon field. The most general invariant Lagrangian is then

\[
\mathcal{L}_{\text{inv}} = \mathcal{L}_{\text{inv}}(\partial_\mu \partial_\nu \phi, \partial_\chi \partial_\mu \partial_\nu \phi, \ldots).
\]

(2.14)

It is manifestly invariant with respect to the Galileon symmetry because the number of derivatives acting on each field is sufficient to compensate the linear shift (2.1). However such

\[\text{\textsuperscript{12}}\text{The covariant derivative } \nabla_\mu \text{ used in the general Volkov and Ogievetsky construction is in this case simply the ordinary derivative } \partial_\mu.\]
a Lagrangian is in fact higher order in the power counting with respect to the basic Galileon Lagrangian. The latter has smaller number of derivatives per field, schematically

$$L_{\text{basic}} = \sum_{n=1}^{D+1} \partial^{2n-2} \phi^n. \quad (2.15)$$

This is possible because the basic Lagrangian is invariant only up to a total derivative and therefore the apparently deficient number of derivatives which do not compensate the linear shift of all the fields is in fact sufficient. $L_{\text{basic}}$ thus represents a generalized Wess-Zumino-Witten term [22, 23, 24, 25]. The corresponding action can be written as an integral of closed GAL $(D,1)$-invariant $D + 1$ form $\omega_{D+1}$ (which is however not an exterior derivative of any GAL $(D,1)$-invariant $D$-form). The integration is taken over $D + 1$ dimensional ball $B_{D+1}$ whose boundary is the compactified space-time $S_D = \partial B_{D+1}$

$$S_{\text{WZW}} = \int_{B_{D+1}} \omega_{D+1}. \quad (2.16)$$

The basis\(^{13}\) of such forms was found in [17] and has $D + 1$ elements $\omega_{D+1}^{(n)}$, explicitly

$$\omega_{D+1}^{(n)} = d\beta_D^{(n)} = \varepsilon_{\mu_1,..\mu_D} \omega_A \wedge \omega_B^{\mu_1} \wedge \ldots \wedge \omega_B^{\mu_n} \wedge \ldots \wedge \omega_P^{\mu_D}, \quad (2.17)$$

where the GAL $(D,1)$-non-invariant $D$-forms $\beta_D^{(n)}$ are

$$\beta_D^{(n)} = \varepsilon_{\mu_1,..\mu_D} \phi dL^{\mu_1} \wedge \ldots \wedge dL^{\mu_n} \wedge dx^{\mu_n} \wedge \ldots \wedge dx^{\mu_D} + \frac{n - 1}{2(D - n + 2)} \varepsilon_{\mu_1,..\mu_D} L^2 dL^{\mu_1} \wedge \ldots \wedge dL^{\mu_{n-1}} \wedge dx^{\mu_n} \wedge \ldots \wedge dx^{\mu_D}. \quad (2.18)$$

We can thus write according to the Stokes theorem

$$\int_{B_{D+1}} \omega_{D+1}^{(n)} = \int_{\partial B_{D+1}} \beta_D^{(n)} = \int_{S_D} \beta_D^{(n)} \quad (2.19)$$

As the last step we impose the IHC constraint (2.12) in the last formula. As a result we get

$$\int_{S_D} \beta_D^{(n)} |_{\text{IHC}} = \frac{1}{n} \int_{S_D} d^D x \mathcal{L}_n, \quad (2.20)$$

where

$$\mathcal{L}_n = \phi \varepsilon^{\mu_1,..\mu_D} \varepsilon^{\nu_1,..\nu_D} \prod_{i=1}^{n-1} \partial_{\mu_i} \partial_{\nu_i} \phi \prod_{j=n}^{D} \eta_{\mu_j \nu_j} = (-1)^{D-1} (D - n + 1)! \phi \det \{ \partial^{\mu_i} \partial^{\nu_j} \phi \}_{i,j=1}^{n-1}. \quad (2.21)$$

$\mathcal{L}_n$ is up to a constant one of the traditional forms of the Galileon Lagrangian. Here we use the convention $\eta = \text{diag} (1, -1, \ldots, -1)$ and $\varepsilon^{01..D-1} = 1$. The most general basic Galileon action can be therefore written in the form

$$S_{\text{basic}} = \sum_{n=1}^{D+1} n d_n \int_{B_{D+1}} \omega_{D+1}^{(n)} |_{\text{IHC}} = \int_{S_D} d^D x \sum_{n=1}^{D+1} d_n \mathcal{L}_n. \quad (2.22)$$

where $d_n$ are real constants.

\(^{13}\)I.e. the basis of the cohomology $H^{D+1}(\text{GAL}(D,1)/\text{SO}(1, D - 1), \mathbb{R})$; see [17] for more detail.
3 Coset transformations and Galileon dualities

In the previous section the coset space $GAL(d,1)/SO(1,d-1)$ has been parametrized by means of the coordinates $x^\mu$, $L^\mu$ and $\phi$ according to the choice $U$ of the representative of each coset, where

$$U = e^{iP \cdot x + iA \phi + iB \cdot L},$$

(3.1)

and then the redundant field $L^\mu$ has been fixed by means of imposing the IHC constraint

$$\omega_A = d\phi - L \cdot dx = 0.$$  

(3.2)

Therefore any transformation of the coset space which preserves the IHC constraint defines a consistent transformation of the Galileon field. More formally, suppose that we have a general transformation on the coset space which is expressed in terms of the coordinates $x^\mu$, $L^\mu$ and $\phi$ as

$$
\begin{align*}
x'^\mu &= \xi^\mu (x, L, \phi) \\
L'^\mu &= \Lambda^\mu (x, L, \phi) \\
\phi' &= f (x, L, \phi).
\end{align*}

(3.3)

Provided the IHC is preserved by this transformation, i.e. when the following implication holds

$$\omega_A = 0 \Rightarrow \omega'_A = 0,$$

(3.4)

where

$$\omega'_A \equiv d\phi' - L' \cdot dx',$$

(3.5)

then a well defined transformation of the Galileon field can be obtained as

$$
\begin{align*}
x'^\mu &= \xi^\mu (x, \partial \phi (x), \phi (x)) \\
\phi' (x') &= f (x, \partial \phi (x), \phi (x)).
\end{align*}

(3.6)

As a consequence of $\omega'_A = 0$ we get a consistent relation for the transformation of the derivatives

$$\partial^\mu \phi' (x') = \Lambda^\mu (x, \partial \phi (x), \phi (x)).$$

(3.7)

An important class of such transformations are dualities, i.e. those transformations for which the general basic Galileon action is form-invariant. This means that, provided

$$S_{\text{basic}} [\phi] = \int d^D x \sum_{n=1}^{D+1} d_n L_n [\phi]$$

(3.8)

with some set $\{d_n\}_{n=1}^{D+1}$ of the couplings, then

$$S_{\text{basic}} [\phi'] = \int d^D x \sum_{n=1}^{D+1} d'_n L_n [\phi]$$

(3.9)

with new set $\{d'_n\}_{n=1}^{D+1}$. In the case when $\{d_n\}_{n=1}^{D+1} = \{d'_n\}_{n=1}^{D+1}$ the duality becomes a symmetry of the basic Galileon action.
A large class of such dualities has been classified in [21]. These dualities can be identified with group of matrices $M \in GL(2,\mathbb{R})$ which act on the coset space as follows. The transformation of the coordinated $x^\mu$ and $L^\mu$ are given by the matrix multiplication

$$
\begin{pmatrix}
    x'^\mu \\
    L'^\mu
\end{pmatrix} = M \begin{pmatrix}
    x^\mu \\
    L^\mu
\end{pmatrix}
$$

(3.10)

while the transformation of $\phi$ can be written in a compact form as

$$
\phi' - \frac{1}{2} L' \cdot x' = \left( \phi - \frac{1}{2} L \cdot x \right) \det M.
$$

(3.11)

The new set of couplings is then a linear combination of the original ones

$$
d'_n = \sum_{m=1}^{D+1} A_{nm}(M) d_m
$$

(3.12)

for appropriate matrix $A_{nm}(M)$. Composition of the transformations (3.10), (3.11) is in one-to-one correspondence with the matrix multiplication in $GL(2,\mathbb{R})$ and the theory space (i.e. the $(D+1)$ dimensional vector space with coordinates $d_m$) carries its linear representation (see [21] for more details). Let us note that for special choice of the matrix $M = \alpha_D(\theta)$ where

$$
\alpha_D(\theta) = \begin{pmatrix}
    1 & -2\theta \\
    0 & 1
\end{pmatrix}
$$

(3.13)

we recover a one parametric subgroup of dualities

$$
x' = x - 2\theta \partial \phi (x), \quad \phi' (x') = \phi (x) - \theta \partial \phi (x) \cdot \partial \phi (x)
$$

(3.14)

which has been discussed in [34], [35] and [36]. Let us remind, that the duality (3.14) preserves the on-shell $S-$matrix, i.e. the theories related with this transformation describe the same on-shell physics (see [21] for more detail).

4 Special Galileon duality

In this section we introduce another very special type of the coset space transformations (3.3) satisfying (3.4). We will show that it is possible to find a new set of duality (and even symmetry) transformations outside the class (3.10), (3.11) provided we restrict ourselves to a special two (or three)-parametric families of the basic Galileon actions. Within the new set of dualities we will find a subset corresponding to the symmetry discussed by Hinterbichler and Joyce in [16]. As a particular member of the family of actions mentioned above we will identify a theory known as Special Galileon.

In what follows we will distinguish between two branches of the abovementioned duality transformation. The most natural identification of these two branches is in terms of the appropriately chosen set of complex and real coordinates on the coset space.
4.1 Complex coordinates

For further convenience we introduce first the following complex combinations of the coset coordinates

\[ Z = x + \frac{i}{\alpha} L, \quad \overline{Z} = x - \frac{i}{\alpha} L, \] (4.1)

where \( \alpha \) is a real parameter with canonical dimension \( \text{dim} \alpha = (D+2)/2 \). Note that the building blocks \( \omega_A, \omega_B \) and \( \omega_P \) (see (2.10)) read in these coordinates

\[ \omega_A = d\phi + i \frac{\alpha}{4} (Z - \overline{Z}) \cdot (dZ + d\overline{Z}) \] (4.2)

\[ \omega_P = \frac{1}{2} (dZ + d\overline{Z}) \] (4.3)

\[ \omega_B = -i \frac{\alpha}{2} (dZ - d\overline{Z}). \] (4.4)

Let now \( G^{\mu\nu} = G^{\nu\mu} \) be constant symmetric real tensor and \( \theta \) be a real number which will play a role of the parameter of the transformation (here we assume \( \text{dim} G^{\mu\nu} = 0 \) and \( \text{dim} \theta = -\text{dim} \alpha \)). Let us denote \( U (\theta) \) the following matrix

\[ U (\theta) = \exp (-i \alpha \theta G), \] (4.5)

where in the matrix notation

\[ (G)^\mu_\nu = G^{\mu\alpha} \eta_{\alpha\nu} = G^\mu_\nu, \] (4.6)

and thus

\[ U (\theta)^\mu_\nu = \delta^\mu_\nu - i \alpha \theta G^\mu_\nu - \frac{1}{2} \alpha^2 \theta^2 G^\mu_\alpha G^\alpha_\nu + \ldots \] (4.7)

Let us introduce the following transformation of the coset coordinates\(^\text{14}\)

\[ Z_\theta = U (\theta) \cdot Z \]
\[ \overline{Z}_\theta = U (-\theta) \cdot \overline{Z} \]
\[ \phi_\theta = \phi + i \frac{\alpha}{8} \left( Z^2 - \overline{Z}^2 \right) - i \frac{\alpha}{8} \left( Z_\theta^2 - \overline{Z}_\theta^2 \right) \] (4.8)

As a result of the transformation we get

\[ dZ_\theta = U (\theta) \cdot dZ \]
\[ d\overline{Z}_\theta = U (-\theta) \cdot d\overline{Z}. \] (4.9)

Using the properties of the matrix \( U (\theta) \) it is then an easy exercise (see Appendix A) to show that the form \( \omega_A \) is invariant, i.e.

\[ [\omega_A]_\theta \equiv d\phi_\theta + i \frac{\alpha}{4} (Z_\theta - \overline{Z}_\theta) \cdot (dZ_\theta + d\overline{Z}_\theta) \]
\[ = d\phi + i \frac{\alpha}{4} (Z - \overline{Z}) \cdot (dZ + d\overline{Z}) = \omega_A. \]

Thus the transformation (4.8) respects the IHC constraint in the sense discussed above and therefore induces a well defined transformation of the Galileon field. Let us note that this definition guaranties that the combination

\[ \phi + i \frac{\alpha}{8} \left( Z^2 - \overline{Z}^2 \right) = \phi - \frac{1}{2} L \cdot x \] (4.10)

\(^{14}\)Note that the transformation of \( \overline{Z} \) is consistent with the relation \( \overline{Z}^* = \overline{Z} \).
is invariant with respect to the transformation (4.8) (cf. also (3.11)).

Let us now construct a basic Galileon action with nice transformation properties with respect to the transformation (4.8). As we have discussed above (see (2.22)), any such action is a linear combination of the integrals of the basic forms \( \omega^{(n+1)}_{D+1} \)

\[
\omega^{(n+1)}_{D+1} = \varepsilon_{\mu_1 \mu_2 ... \mu_D} \omega_A \wedge \omega^{\mu_1} \wedge ... \wedge \omega^{\mu_n} \wedge \omega^{\mu_{n+1}} \wedge ... \wedge \omega^{\mu_D}
\]  

(4.11)

over \( D + 1 \) dimensional ball \( B_{D+1} \). Because the basic building blocks \( \omega_P \) and \( \omega_B \) are linear combinations of \( dZ \) and \( d\mathbb{Z} \) (see (2.10)), it is natural to consider the following form

\[
\Omega = \varepsilon_{\mu_1 \mu_2 ... \mu_D} \omega_A \wedge dZ^{\mu_1} \wedge dZ^{\mu_2} \wedge ... \wedge dZ^{\mu_D}
\]

(4.12)

The latter transforms under (4.8) as follows

\[
\Omega_\theta = \varepsilon_{\mu_1 \mu_2 ... \mu_D} [\omega_A]_\theta \wedge dZ^{\mu_1}_\theta \wedge dZ^{\mu_2}_\theta \wedge ... \wedge dZ^{\mu_D}_\theta
\]

(4.13)

and thus

\[
\Omega_\theta = \det U(-\theta) \Omega = e^{-i\alpha \theta \text{tr} \mathcal{G}} \Omega.
\]  

(4.14)

In the same way, the form

\[
\overline{\Omega} = \varepsilon_{\mu_1 \mu_2 ... \mu_D} \omega_A \wedge d\overline{Z}^{\mu_1} \wedge d\overline{Z}^{\mu_2} \wedge ... \wedge d\overline{Z}^{\mu_D}
\]

(4.15)

transforms as

\[
\overline{\Omega}_\theta = \det U(-\theta) \overline{\Omega} = e^{i\alpha \theta \text{tr} \mathcal{G}} \overline{\Omega}.
\]  

(4.16)

This results suggest to construct a two-parametric family of basic Galileon actions\(^\text{15}\)

\[
S(\alpha, \beta) = \frac{1}{2i} \alpha \int_{B_{D+1}} \left( e^{i\beta \Omega} - e^{-i\beta \overline{\Omega}} \right) |_{IHC},
\]

(4.17)

where \( \beta \) is a real parameter. Such actions have (as a consequence of (4.14) and (4.16)) very simple transformation property with respect to (4.8),

\[
S_\theta(\alpha, \beta) = \frac{1}{2i} \alpha \int_{B_{D+1}} \left( e^{i\beta \Omega_\theta} - e^{-i\beta \overline{\Omega}_\theta} \right) |_{IHC}
\]

(4.18)

\[
= \frac{1}{2i} \alpha \int_{B_{D+1}} \left( e^{i\beta \epsilon_{\mu_{n+1}...\mu_D} \omega_{\mu_1}^{\mu_1} \wedge ... \wedge \omega_{\mu_n}^{\mu_n} \wedge \omega_{\mu_{n+1}}^{\mu_{n+1}} \wedge ... \wedge \omega_{\mu_D}^{\mu_D}} \right) |_{IHC},
\]

(4.19)

i.e. the parameter \( \beta \) is shifted according to

\[
S_\theta(\alpha, \beta) = S(\alpha, \beta - \alpha \theta \mathcal{G}_\mu^\mu).
\]

\(^{15}\)The prefactor \( \alpha/2 \) avoids the \( \alpha \)-dependence of the kinetic term.
The transformation \((4.8)\) is therefore a duality transformation and for traceless tensor \(G^{\mu\nu}\) it is a symmetry of the two parameter family of actions \(S(\alpha, \beta)\). In the traditional notation we get using \((2.22)\) and \((4.12)\)

\[
S(\alpha, \beta) = \int d^D x \mathcal{L}(\alpha, \beta),
\]

where

\[
\mathcal{L}(\alpha, \beta) = \sum_{n=1}^{D+1} d_n(\alpha, \beta) \mathcal{L}_n
\]

with \(\mathcal{L}_n\) given by \((2.21)\) and where the couplings \(d_n(\alpha, \beta)\) read explicitly

\[
d_{2n}(\alpha, \beta) = \frac{(-1)^n}{2n} \left( \frac{D}{2n-1} \right) \frac{\cos \beta}{\alpha^{2(n-1)}},
\]

\[
d_{2n+1}(\alpha, \beta) = \frac{(-1)^n}{2n+1} \left( \frac{D}{2n} \right) \frac{\sin \beta}{\alpha^{2n-1}}.
\]

Especially for traceless tensor \(G^{\mu\nu}\) we get invariance of the Lagrangian \(\mathcal{L}(\alpha, \beta)\) under \((4.8)\) (with IHC imposed) up to a total derivative.

The infinitesimal form of the duality transformation reads

\[
Z_\theta^\mu = Z^\mu - i \alpha \theta G^{\mu\nu} Z_\nu
\]

\[
\overline{Z}_\theta^\mu = \overline{Z}^\mu + i \alpha \theta G^{\mu\nu} \overline{Z}_\nu
\]

\[
\phi_\theta = \phi - \frac{\alpha^2}{4} \theta \left( Z^{\mu} G_{\mu\nu} Z^\nu + \overline{Z}^{\mu} G_{\mu\nu} \overline{Z}^\nu \right),
\]

or using the IHC

\[
x_\theta^\mu = x^\mu + \theta G^{\mu\nu} \partial_\nu \phi
\]

\[
\phi_\theta(x_\theta) = \phi(x) - \frac{\alpha^2}{2} \theta G^{\mu\nu} \left( x_\mu x_\nu - \frac{1}{\alpha^2} \partial_\mu \phi(x) \partial_\nu \phi(x) \right).
\]

Equivalently we can write

\[
\phi_\theta(x) = \phi(x) - \frac{\theta}{2} G^{\mu\nu} \left( \alpha^2 x_\mu x_\nu + \partial_\mu \phi(x) \partial_\nu \phi(x) \right).
\]

In the latter formula we can recognize a generalization of the hidden Galileon symmetry of Hinterbichler and Joyce \([16]\). The theory known as a Special Galileon corresponds then (up to an overall normalization) to particular value of parameter \(\beta\), namely for \(\beta = k\pi, k \in \mathbb{Z}\). Therefore the hidden Galileon symmetry is a special case of more general “hidden Galileon duality” \((4.8)\) for traceless \(G^{\mu\nu}\) and there is in fact a two parameter family of Lagrangians invariant with respect to this symmetry.

### 4.2 Real coordinates

We can repeat the above consideration also for imaginary parameter \(\alpha = i a\). Let us introduce real coordinates on the coset space according to

\[
Z^\pm = x \pm \frac{1}{a} L,
\]

\[
\mathcal{L}(\alpha, \beta) = \sum_{n=1}^{D+1} d_n(\alpha, \beta) \mathcal{L}_n
\]

with \(\mathcal{L}_n\) given by \((2.21)\) and where the couplings \(d_n(\alpha, \beta)\) read explicitly

\[
d_{2n}(\alpha, \beta) = \frac{(-1)^n}{2n} \left( \frac{D}{2n-1} \right) \frac{\cos \beta}{\alpha^{2(n-1)}},
\]

\[
d_{2n+1}(\alpha, \beta) = \frac{(-1)^n}{2n+1} \left( \frac{D}{2n} \right) \frac{\sin \beta}{\alpha^{2n-1}}.
\]
and define the transformation matrix \( U(\pm \theta) \) as
\[
U(\pm \theta) = \exp(\pm a\theta G).
\]

Here \( G \) is defined as above (4.6). The transformation of the coset coordinates is then
\[
Z^\pm_\theta = U(\pm \theta) \cdot Z^\pm,
\]
\[
\phi_\theta = \phi - \frac{a}{8}(Z^{+2} - Z^{-2}) + \frac{a}{8}(Z^\theta_{\pm} + Z^{-\theta}_{\pm})^2,
\]
and for the forms \( \omega_A \) and \( dZ^\pm \) we get following transformation rules
\[
[\omega_A]_\theta = \omega_A, \quad dZ^\pm_\theta = U(\pm \theta) \cdot dZ^\pm.
\]

In analogy with (4.12) and (4.15) we can now construct two real forms
\[
\Omega^\pm = \varepsilon_{\mu_1 \mu_2 \ldots \mu_D} \omega_A \wedge dZ^{+\mu_1} \wedge dZ^{+\mu_2} \ldots \wedge dZ^{+\mu_D}
\]
transferring as
\[
\Omega^\pm_\theta = \Omega^\pm \det[\exp(\pm a\theta trG)] = e^{\pm a\theta G_{\mu}^\mu} \Omega^\pm.
\]

Let us therefore assume the following actions
\[
S_\pm(a, c_\pm) = \pm \frac{1}{2}a e^{c_\pm} \int_{B_{D+1}} \Omega^\pm|_{IHC} = \int d^Dx \sum_{n=1}^{D+1} d_n^\pm(a, c_\pm) \mathcal{L}_n,
\]
where
\[
d_n^\pm(a, c_\pm) = 1 \int_{B_{D+1}} \left( e^{c_\pm} \right) \left( \frac{D}{n-1} \right) \left( \frac{1}{a} \right)^n \left( 1 + 2\sigma \right)^{n-1}.
\]

According to (4.32), these actions are related by duality (4.29)
\[
S_\pm(a, c_\pm)_\theta = S_\pm(a, c_\pm \pm a \theta G_{\mu}^\mu),
\]
which becomes a symmetry of \( S_\pm(a, c_\pm) \) for traceless \( G_{\mu}^\mu \).

However, the theories corresponding either to only \( S_+(a, c_+) \) or to only \( S_-(a, c_-) \) are in some sense trivial. As it is shown in Appendix D, under the \( GL(2, \mathbb{R}) \) duality transformation (3.10), (3.11) with matrix \( M = \alpha_D(\sigma) \) (see (3.13)) we transform \( d_n^\pm(a, c_\pm) \) into
\[
d_n^\pm(a, c_\pm)_\sigma = 1 \frac{1}{2n} ae^{c_\pm} \left( \frac{D}{n-1} \right) \left( \frac{1}{a} \right)^{n-1}.
\]

Thus for \( M_\pm = \alpha_D(\pm 1/2\alpha) \), the actions \( S_\pm(a, c_\pm) \) are dual to the theory with action
\[
S_1(a, c_\pm) = \frac{(-1)^{D-1}D!}{2} ae^{c_\pm} \int d^Dx \phi.
\]

To get a nontrivial theory we have therefore to construct a linear combination of both actions \( S_\pm(a, c_\pm) \) with nonzero coefficients,
\[
S(a, c_+, c_-) = \frac{1}{2} \int_{B_{D+1}} \left( e^{c_+} \Omega^+ - e^{c_-} \Omega^- \right)|_{IHC} = \int d^Dx \sum_{n=1}^{D+1} d_n^\pm(a, c_+, c_-) \mathcal{L}_n,
\]
where

\[ d_n^\pm (a, c_+, c_-) = \frac{1}{2n} \left( \begin{array}{c} D \cr n - 1 \end{array} \right) \frac{1}{a^{n-2}} (e^c + (-1)^n e^{-c}). \] (4.39)

Under (4.29) \( S_\pm (a, c_+, c_-) \) transforms according to

\[ S_\pm (a, c_+, c_-) = S_\pm (a, c_+ + a\theta G^\mu_\mu, c_- - a\theta G^\mu_\mu), \] (4.40)

and again, the case of traceless \( G^\mu_\mu = 0 \) yields a symmetry of the whole family \( S (a, c_+, c_-) \).

Note that the special case \( S_\pm (a, b, -b) \) is an analytic continuation of the action \( S (\alpha, \beta) \) (cf. (4.17)) to imaginary values of the parameters, namely \( \alpha = ia \) and \( \beta = -ib \)

\[ S (ia, -ib) = \frac{1}{2} a \int_{B^{D+1}} (e^b \Omega^+ - e^{-b} \Omega^-) \mid_{IHC} = \int a^D x \sum_{n=1}^{D+1} d_n (ia, -ib) L_n, \] (4.41)

where

\[ d_{2n} (ia, -ib) = \frac{1}{2n} \left( \begin{array}{c} D \\
2n - 1 \end{array} \right) \frac{\cosh b}{a^{2(n-1)}} \]

\[ d_{2n+1} (ia, -ib) = \frac{1}{2n + 1} \left( \begin{array}{c} D \\
2n \end{array} \right) \frac{\sinh b}{a^{2n-1}}. \] (4.42)

Under (4.29) this actions transform as

\[ S_\theta (ia, -ib) = S (ia, -ib - i\theta G^\mu_\mu). \] (4.43)

The infinitesimal transformation (4.29) reads in terms of \( x, \phi \) (after using the IHC)

\[ x^\mu_\theta = x^\mu + \theta G^\mu_\nu \partial_\nu \phi \]

\[ \phi_\theta (x_\theta) = \phi (x) + \frac{a^2}{2} \theta G^\mu_\nu \left( x^\mu x_\nu + \frac{1}{a^2} \partial_\mu \phi (x) \partial_\nu \phi (x) \right) \] (4.44)

or finally

\[ \phi_\theta (x) = \phi (x) + \frac{\theta}{2} G^\mu_\nu \left( a^2 x^\mu x_\nu - \partial_\mu \phi (x) \partial_\nu \phi (x) \right) \] (4.45)

This corresponds to the second branch of the hidden Galileon symmetry of Hinterbichler and Joyce [16]. The corresponding Special Galileon is a particular case of \( S (a, c_+, c_-) \) with \( c_+ = c_- \).

### 4.3 Special Galileon duality and \( GL (2, \mathbb{R}) \) duality

Let us add a note on the interrelation of the above Special Galileon duality and the \( GL (2, \mathbb{R}) \) duality (3.10), (3.11) discussed in Section 3. For \( G^\mu_\nu = \eta^\mu_\nu \) the transformations (4.8) and (4.29) correspond to the dualities (3.10), (3.11) with \( SL (2, \mathbb{R}) \) matrices \( M_\alpha (\theta) \) and \( M_\alpha (\theta) \) respectively, namely

\[ M_\alpha (\theta) = \left( \begin{array}{cc} \cos \alpha \theta & \alpha^{-1} \sin \alpha \theta \\
-\alpha \sin \alpha \theta & \cos \alpha \theta \end{array} \right) \] (4.46)

\[ M_\alpha (\theta) = \left( \begin{array}{cc} \cosh \alpha \theta & a^{-1} \sinh a \theta \\
a \sinh a \theta & \cosh a \theta \end{array} \right) \] (4.47)
The $G^{\mu\nu} = \eta^{\mu\nu}$ case is therefore not only duality of the Special Galileon but also a duality of the most general Galileon action. Note that in the limit $\alpha, a \to 0$ we get

$$M_{\alpha, a} (\theta) \xrightarrow{\alpha, a \to 0} \begin{pmatrix} 1 & \theta \\ 0 & 1 \end{pmatrix} = \alpha_D (-\theta/2)$$

(4.48)

and the resulting matrix corresponds to the one parametric subgroup of dualities (3.13), (3.14).

4.4 Possible generalization of Special Galileon duality

In this subsection we will discuss one possible generalization of the Special Galileon duality. Originally this kind of transformation has been discussed by Noller, Sivanesan and von Strauss in [37]. Though this transformation is apparently a nontrivial symmetry of a two parameter family of Galileon actions, it can be shown, that such a family is in some sense trivial being dual (with respect to the duality (3.10), (3.11)) to the free theory with tadpole term. Here we give a coset space formulation of this generalized symmetry and construct the invariant actions as the appropriate manifestly invariant Wess-Zumino terms (2.16).

Let us try to generalize the duality (4.29) and assume the following coordinate transformation on the coset space

$$Z^{+\mu} = Z^{+\mu}$$

$$Z^{-\mu} = Z^{-\mu} + \frac{2\theta}{a} G^{\mu_{1}\mu_{2}...\mu_{N-1}} Z_{\mu_{1}}^{+} Z_{\mu_{2}}^{+} ... Z_{\mu_{N-1}}^{+}$$

$$\phi_{\theta} = \phi + \frac{a}{8} (Z^{+} - Z^{-})^{2} - \frac{a}{8} (Z_{\theta}^{+} - Z_{\theta}^{-})^{2} - \frac{\theta}{N} G^{\mu_{1}\mu_{2}...\mu_{N}} Z_{\mu_{1}}^{+} Z_{\mu_{2}}^{+} ... Z_{\mu_{N}}^{+},$$

(4.49)

where as above

$$Z^{\pm} = x \pm \frac{1}{a} L,$$

(4.50)

and where $G^{\mu_{1}\mu_{2}...\mu_{N-1}}$ is general totally symmetric traceless tensor. Again, it is an easy exercise to show that such a transformation preserves the IHC (see Appendix B, let us note that for the invariance of IHC, $G^{\mu_{1}\mu_{2}...\mu_{N}}$ need not to be traceless), i.e.

$$[\omega_{A}]_{\theta} = \omega.$$  

(4.51)

Therefore (4.49) generates a consistent transformation of the Galileon field. The obvious (though trivial; see Appendix D) candidate for invariant Galileon action can be constructed form the manifestly invariant $D + 1$ form $\Omega^{+}$ defined in the previous section (cf (4.31))

$$\Omega^{+} = \varepsilon_{\mu_{1}\mu_{2}...\mu_{D}} \omega_{A} \wedge dZ^{+\mu_{1}} \wedge dZ^{+\mu_{2}} ... \wedge dZ^{+\mu_{D}}$$

(4.52)

and coincides with $S_{+} (a, c_{+})$ introduced there. However, the analogous form $\Omega^{-}$ is not invariant under (4.49) and we cannot therefore combine it with $\Omega^{+}$ in order to get nontrivial action analogous to $S (ia, -ib)$.

Another less obvious invariant action can be constructed form the $D + 1$ form

$$\Omega^{L} = \varepsilon_{\mu_{1}\mu_{2}...\mu_{D-1}} \omega_{A} \wedge dZ^{+\mu_{1}} \wedge dZ^{+\mu_{2}} ... \wedge dZ^{+\mu_{D-1}} \wedge dL^{\nu}$$

$$= (-1)^{D-1} \sum_{n=0}^{D-1} \binom{D-1}{n} \frac{1}{a^{n}} \omega_{D+1}^{(n+2)}.$$  

(4.53)
The form $\Omega^L$ transforms under (4.49) as
\[
\left[\Omega^L\right]_\theta = \varepsilon_{\mu_1\mu_2\ldots\mu_{D-1}\nu} [\omega_A]_\theta \wedge dZ^{\mu_1} \wedge dZ^{\mu_2} \ldots \wedge dZ^{\mu_{D-1}} \wedge dL^\nu_	heta
\]
\[
= \varepsilon_{\mu_1\mu_2\ldots\mu_{D-1}\nu} \omega_A \wedge dZ^{\mu_1} \wedge dZ^{\mu_2} \ldots \wedge dZ^{\mu_{D-1}}
\wedge (dL^\nu - \theta (N - 1) G (Z^+)_{\mu} dZ^{\mu})
\]
(4.54)
where we have used
\[
L^\nu_	heta = L^\nu - \theta G^{\mu_1\mu_2\ldots\mu_{N-1}} Z^+_{\mu_1} Z^+_{\mu_2} \ldots Z^+_{\mu_{N-1}},
\]
(4.55)
and where we abbreviated
\[
G (Z^+)_{\mu
\nu} = G^{\mu\alpha_1\alpha_2\ldots\alpha_{N-2}} Z_{\alpha_1}^+ Z_{\alpha_2}^+ \ldots Z_{\alpha_{N-2}}^+.
\]
(4.56)
Note that
\[
\varepsilon_{\mu_1\mu_2\ldots\mu_{D-1}\nu} dZ^{\mu_1} \wedge dZ^{\mu_2} \ldots \wedge dZ^{\mu_{D-1}} \wedge dZ^{\pm \alpha} = \frac{1}{D} \delta^\alpha_{\nu} \varepsilon_{\mu_1\mu_2\ldots\mu_{D}} dZ^{\mu_1} \wedge dZ^{\mu_2} \ldots \wedge dZ^{\pm \mu}
\]
and thus
\[
\left[\Omega^L\right]_\theta = \Omega^L - \frac{\theta (N - 1)}{D} G (Z^+)_{\nu} \varepsilon_{\mu_1\mu_2\ldots\mu_{D}} dZ^{\mu_1} \wedge dZ^{\mu_2} \ldots \wedge dZ^{\mu_{D}}.
\]
(4.57)
Therefore for traceless $G^{\mu_1\mu_2\ldots\mu_{N}}$ the form $\Omega^L$ is invariant with respect to (4.49). Unfortunately, the action
\[
S^L = \int_{B_{D+1}} \Omega^L|_{IHC} = \int d^Dx \sum_{n=2}^{D+1} d_n^L (a) L_n,
\]
(4.59)
where
\[
d_n^L (a) = \left(-1\right)^{D-1} n \left(D - 1 \right) \frac{1}{a^{n-2}},
\]
(4.60)
can be shown to be dual to free theory [37]. Indeed, as we show in Appendix D, under the duality transformation (3.10), (3.11) with matrix $M = \alpha_D (\sigma)$ (see (3.13)), we transform $d_n^L (a)$ into
\[
d_n^L (a)_\sigma = \left(-1\right)^{D-1} \frac{1}{D} n \left(D \right) \left(1 + 2\theta \right)^{-n-2}.
\]
(4.61)
Moreover, this transformation with $\sigma = 1/2a$ (converting $S^L$ to the free theory) transforms at the same time the action $S_+ (a, c_+)$ into the trivial tadpole action $S_1 (a, c_+)$ (see (4.36)). The physical content of any linear combinations of the invariant actions $S_+ (a, c_+)$ and $S^L$ is therefore trivial.

Infinitesimal version of the transformation (4.49) reads after using the the inverse Higgs constraint
\[
x^\mu_\theta = x^\mu + \frac{\theta}{a} G^{\mu_1\mu_2\ldots\mu_{N-1}} (x_{\mu_1} + \frac{1}{a} \partial_{\mu_1} \phi (x)) \ldots (x_{\mu_{N-1}} + \frac{1}{a} \partial_{\mu_{N-1}} \phi (x))
\]
\[
\phi_\theta (x_\theta) = \phi (x) - \frac{\theta}{N} G^{\mu_1\mu_2\ldots\mu_{N}} (x_{\mu_1} + \frac{1}{a} \partial_{\mu_1} \phi (x)) \ldots (x_{\mu_{N}} + \frac{1}{a} \partial_{\mu_{N}} \phi (x))
\]
\[
+ \frac{\theta}{a} \partial_{\mu_1} \phi (x) G^{\mu_1\mu_2\ldots\mu_{N}} (x_{\mu_1} + \frac{1}{a} \partial_{\mu_1} \phi (x)) \ldots (x_{\mu_{N}} + \frac{1}{a} \partial_{\mu_{N}} \phi (x)),
\]
(4.62)
or
\[
\phi_\theta (x) = \phi (x) - \frac{\theta}{N} G^{\mu_1 \mu_2 \ldots \mu_N} \left( x_{\mu_1} + \frac{1}{a} \partial_{\mu_1} \phi (x) \right) \ldots \left( x_{\mu_N} + \frac{1}{a} \partial_{\mu_N} \phi (x) \right). \tag{4.63}
\]

In the latter form we recognize the “extended Galileon symmetry” of Noller, Sivanesan and von Strauss discussed in [37] which (as discussed there and as we have shown here) is in fact dual to the traceless polynomial shift symmetry of the free theory with tadpole.

5 Geometrical origin of the Special Galileon

In this Section we give an alternative geometrical treatment of the symmetries and dualities discussed above. We will show that, using the complex coordinates (4.1) introduced in the previous section, it is possible, at least formally, to construct the Galileon Lagrangian in a way analogous to the DBI one using the probe brane in higher dimensional space. Within this approach we can interpret the Galileon field as a scalar degree of freedom describing fluctuations of appropriately chosen \(D\)-dimensional brane in \(2D\) dimensional space. The latter has a pseudo-riemannian metric with signature \((2, 2D-2)\) and can be formally treated as a complexified Minkowski space. Within such framework, the Galileon linear shift symmetry (2.1) as well as the symmetry (4.8) of the Special Galileon originate from the symmetries of the target space and their non-linear character is a consequence of gauge fixing of the reparametrization freedom describing the embedding of the brane. This construction can be easily repeated for the real coordinates (4.28) and the transformations (4.29) by means of appropriate analytic continuation. Such an approach to Special Galileon, though purely formal, will be useful for construction of the higher order Lagrangians which are necessary as counterterms when the theory is treated on the quantum level.

Let us first describe the target space. Assume a \(D\)-dimensional complex space \(M_D^C = \mathbb{C}^D\) with coordinates
\[
Z = X + \frac{i}{\alpha} L, \quad \overline{Z} = X - \frac{i}{\alpha} L,
\tag{5.1}
\]
where \(X^\mu\) and \(L^\mu\) are real coordinates and \(\alpha\) is fixed real parameter (cf. (4.1)). Let us equip \(M_D^C\) and with a hermitean form \(h\) defined as
\[
h = \eta_{\mu \nu} dZ^\mu \otimes d\overline{Z}^\nu.
\tag{5.2}
\]
The real part of this form defines a metric with signature \((2, 2(D - 1))\) on \(M_D^C\) treated as a real \(2D\) dimensional space
\[
ds^2 = dZ \cdot d\overline{Z} = \eta_{\mu \nu} \left( dX^\mu dX^\nu + \frac{1}{\alpha^2} dL^\mu dL^\nu \right),
\tag{5.3}
\]
while the imaginary part of \(h\) generates a symplectic Kähler form
\[
\omega = \frac{i}{2} \eta_{\mu \nu} dZ^\mu \wedge d\overline{Z}^\nu = \frac{1}{\alpha} \eta_{\mu \nu} dX^\mu \wedge dL^\nu.
\tag{5.4}
\]
All the above forms are invariant with respect to the transformations\(^{16}\)
\[
Z'^\mu = R^\mu_{\nu} Z^\nu + A^\mu,
\tag{5.5}
\]
\(^{16}\)Transformation of \(\overline{Z}\) is defined by means of complex conjugation.
where the rotation matrix $R^\mu_\nu \in U(1, D - 1)$, i.e. it satisfies
\[ R^+ \cdot \eta \cdot R = \eta, \]
and the complex vector $A = c + \frac{i}{\alpha} b \in \mathbb{C}^D$ corresponds to a translation in $M^D_C$. These transformations form a group $\mathbb{C}^D \times U(1, D - 1)$ which can be understood as a complex generalization of the Poincaré group $ISO(1, D - 1) = \mathbb{R}^D \times O(1, D - 1)$, which is its natural real subgroup.

As an analogue of the DBI probe brane, we assume a $D$-dimensional real Minkowski manifold $M^D_R$ embedded into $M^D_C$. The embedding is parametrized by real parameters $\xi^\mu$, $\mu = 0, \ldots, D - 1$
\[ Z^\mu = Z^\mu (\xi) = X^\mu (\xi) + \frac{i}{\alpha} L^\mu (\xi) \]
and we put additional constraint on the functions $Z^\mu (\xi)$, namely we require that the Kähler form vanishes\(^1\) on the brane\(^2\) $M^D_R$
\[ \omega |_{M^D_R} = 0. \] (5.8)
Explicitly we get
\[ \frac{\partial Z}{\partial \xi^\mu} \cdot \frac{\partial \overline{Z}}{\partial \xi^\nu} = \frac{\partial \overline{Z}}{\partial \xi^\mu} \cdot \frac{\partial Z}{\partial \xi^\nu}, \] (5.9)
or using the real coordinates
\[ \frac{\partial X}{\partial \xi^\mu} \cdot \frac{\partial L}{\partial \xi^\nu} = \frac{\partial L}{\partial \xi^\mu} \cdot \frac{\partial X}{\partial \xi^\nu}. \] (5.10)
Let us note, that this constraint is invariant with respect to the transformations (5.5). On the brane $M^D_R$ we get a real induced metric
\[ ds^2 = \eta_{\alpha\beta} \frac{\partial Z^\alpha}{\partial \xi^\mu} \frac{\partial \overline{Z}^\beta}{\partial \xi^\nu} d\xi^\mu d\xi^\nu \equiv g_{\mu\nu} d\xi^\mu d\xi^\nu. \] (5.11)
Because both the target space metric (5.3) and the constraint (5.9) are invariant with respect to (5.5), the geometrical reparametrization invariant actions built from the induced metric on $M^D_R$, the corresponding covariant derivatives and the intrinsic and external curvatures will share the invariance (5.5) too (see [7, 38] for the detailed description the brane construction of effective actions and for discussion of their symmetry properties).

Now we shall identify the Galileon as a scalar degree of freedom which effectively describes the fluctuations of the brane $M^D_C$ in the target space. We shall proceed in the way completely analogous to the construction of the DBI-like actions. Let us first fix the gauge freedom corresponding to the reparametrization invariance and introduce new coordinates $x^\mu$ on the brane according to
\[ x^\mu = X^\mu (\xi). \] (5.12)
In such parametrization the embedding simplifies to
\[ X^\mu (x) = x^\mu, \quad L^\mu = L^\mu (x). \] (5.13)
The fluctuations of the brane are then effectively described by fields $L^\mu (x)$ living on it. However, we have to impose the additional constraint (5.9) which further reduces the number of\(^3\)
\(^1\)Such submanifolds are known as the Lagrangian submanifolds.
\(^2\)Here and in what follows we borrow the terminology from the DBI case.
effective degrees of freedom. Using the parametrization \( (5.12) \), the constraint \( (5.10) \) simplifies to the integrability condition for the field \( L_\mu (x) = \eta_{\mu \alpha} L^\alpha (x) \), namely

\[
\partial_\nu L_\mu = \partial_\mu L_\nu, \tag{5.14}
\]

(here and in what follows, \( \partial_\mu = \partial / \partial x^\mu \)) and as a consequence there exists \( \phi(x) \) such that

\[
L_\mu (x) = \partial_\mu \phi (x). \tag{5.15}
\]

In this gauge we have therefore the embedding of the brane described as

\[
X^\mu (x) = x^\mu, \quad L^\mu (x) = \eta^{\mu \nu} \partial_\nu \phi (x). \tag{5.16}
\]

Similarly to the case of DBI, we are left with one degree of freedom, which is nothing else but the Galileon. To show this, let us discuss the transformation properties of \( \phi(x) \) with respect to the transformations \( (5.5) \). Let us note, that the gauge condition \( (5.13) \) is not invariant under \( (5.5) \). As a consequence, it is necessary to combine the target space transformation with additional gauge transformation in order to ensure the gauge fixing of the form of \( (5.16) \) also for the transformed brane. As a result, the field \( \phi(x) \) will generally transform non-linearly under \( (5.5) \).

Let us first note that there is a residual symmetry

\[
\phi' (x) = \phi (x) + a, \tag{5.17}
\]

which leaves the brane parametrization \( (5.16) \) invariant. The configurations of the brane are therefore parametrized rather by means of the equivalence classes \( [\phi] = \{ \phi + a, a \in \mathbb{R} \} \) modulo this symmetry. This means that the realization of the transformations \( (5.5) \) on the field \( \phi(x) \) described in what follows are unique only up to the constant shift of the field.

Under the complex translation the brane coordinates transform according to

\[
Z'^\mu = Z^\mu + \left( \epsilon^\mu + \frac{i}{\alpha} b^\mu \right), \tag{5.18}
\]

and thus the parameters \( x^\mu \) shift by \( a^\mu \) while the fields \( L^\mu (x) \) shift by \( b^\mu \), explicitly

\[
X'^\mu (x) = x^\mu + \epsilon^\mu, \quad L'^\mu (x) = \eta^{\mu \nu} \partial_\nu \phi (x) + b^\mu = \eta^{\mu \nu} \partial_\nu [\phi (x) + b \cdot x]. \tag{5.19}
\]

Let us define new parameters for the shifted brane

\[
x'^\mu = x^\mu + \epsilon^\mu, \tag{5.20}
\]

and define transformed field \( \phi' (x') \) as

\[
\phi' (x') = \phi (x) + b \cdot x. \tag{5.21}
\]

Using \( \partial'_\mu = \partial_\nu \) we get

\[
L'^\mu (x') \equiv L'^\mu (x(x')) = \eta^{\mu \nu} \partial'_\nu \phi' (x'), \tag{5.22}
\]

and with the new field \( \phi' (x') \) the gauge \( (5.16) \) is preserved. Therefore the complex translation is a combination of the space-time translation and the Galileon transformation corresponding to the linear shift of the Galileon field. The latter corresponds to the purely imaginary shift.
Let us now write the general $U(1, D - 1)$ rotation $R_{\mu}^\nu$ in the form

$$R = \exp (\mathcal{M} + i \mathcal{G}) = \Lambda + i U,$$  \hspace{1cm} (5.23)

where $\Lambda$ and $U$ are real matrices and $\mathcal{M}$ and $\mathcal{G}$ are real generators. The latter satisfy

$$\eta_{\mu \rho} \mathcal{M}^\rho_{\nu} + \eta_{\nu \rho} \mathcal{M}^\rho_{\mu} = 0$$  \hspace{1cm} (5.24)

$$\eta_{\mu \rho} \mathcal{G}^\rho_{\nu} - \eta_{\nu \rho} \mathcal{G}^\rho_{\mu} = 0$$  \hspace{1cm} (5.25)

We get then for the transformed brane

$$X'^\mu (x) = \Lambda^\mu_{\nu} x^\nu - \frac{1}{\alpha} U^\mu_{\nu} L^\nu (x) = \Lambda^\mu_{\nu} x^\nu - \frac{1}{\alpha} U^\mu_{\nu} \eta^{\rho \nu} \partial_\rho \phi (x)$$  \hspace{1cm} (5.26)

$$L'^\mu (x) = \frac{1}{\alpha} \Lambda^\mu_{\nu} L^\nu (x) + U^\mu_{\nu} x^\nu = \frac{1}{\alpha} \Lambda^\mu_{\nu} \eta^{\rho \nu} \partial_\rho \phi (x) + U^\mu_{\nu} x^\nu.$$  \hspace{1cm} (5.27)

For $U = 0$ the matrices $\Lambda = \exp \mathcal{M} \in O(1, D - 1)$ form a subgroup of $U(1, D - 1)$ and the above transformation reduces to the Lorentz transformation with respect to which the field $\phi (x)$ transforms as a scalar. Indeed, after reparametrization

$$x'^\mu = \Lambda^\mu_{\nu} x^\nu,$$  \hspace{1cm} (5.28)

we immediately see that the new field $\phi' (x')$ where

$$\phi' (x') = \phi (x),$$  \hspace{1cm} (5.29)

preserves the gauge condition (5.16). The remaining nontrivial transformations $R = \exp i \mathcal{G}$ are generated by matrices $\mathcal{G}^\rho_{\nu}$ satisfying (5.25), i.e. they are of the form (4.5). According to the results of the previous section we can identify them with the hidden duality transformation discussed there. The compensating gauge transformation corresponds to the reparametrization

$$x'^\mu = \Lambda^\mu_{\nu} x^\nu - \frac{1}{\alpha} U^\mu_{\nu} L^\nu (x) = \Lambda^\mu_{\nu} x^\nu - \frac{1}{\alpha} U^\mu_{\nu} \eta^{\rho \nu} \partial_\rho \phi (x),$$  \hspace{1cm} (5.30)

where $\Lambda = \text{Re} \exp i \mathcal{G}$ and $U = \text{Im} \exp i \mathcal{G}$ and the transformation of the field

$$\phi' (x') = \phi (x) - \frac{1}{2} x \cdot \partial \phi (x)$$

$$+ \frac{1}{\alpha} \eta_{\mu \nu} \left( \Lambda^\mu_{\rho} x^\rho - \frac{1}{\alpha} U^\mu_{\rho} \eta^{\rho \nu} \partial_\nu \phi (x) \right) \left( \frac{1}{\alpha} \Lambda^\nu_{\sigma} \eta^{\sigma \beta} \partial_\beta \phi (x) + U^\nu_{\sigma} x^\sigma \right),$$  \hspace{1cm} (5.31)

are exactly equivalent to (4.8) by construction. Because the latter transformation preserves the IHC, the gauge fixing condition (5.16) is preserved as a consequence.

To conclude, the Galileon field can be formally understood as a scalar degree of freedom describing fluctuations of the $D$ dimensional Lagrangian brane in $2D$ dimension real space $\mathbb{R}^{2,2D-2}$ treated as a Kähler manifold. The symmetries and dualities of the special Galileon can be explained as non-linear realization of the target space symmetry group $\mathbb{C}^D \times U(1, D - 1)$. Its real subgroup $\mathbb{R}^{1, D-1} \times O(1, D - 1)$ corresponds to the Poincaré symmetry $ISO(1, D - 1)$ on the brane. Only $O(1, D - 1)$ subgroup is realized linearly and the linear shift transformations of the Galileon correspond to the pure imaginary translation in the target space.

Note however, that within the target space symmetry group, the generators of real and imaginary translation commute which is not the case of the generators $P_\mu$ and $B_\mu$ of Poincaré
translations and the linear shift transformations of the Galileon field respectively. Let us remind (see (2.3)), that \( [P_\mu, B_\mu] = i \eta_{\mu\nu} A \) where \( A \) is the generator of the residual constant shift symmetry transformation \( \phi \rightarrow \phi + a \) (see (5.17)). However, as explained above, the brane configurations are classes of equivalence modulo this constant shifts and on such classes the generator \( A \) acts trivially, so that \( P_\mu \) and \( B_\mu \) effectively commute.

6 Brane construction of the leading order Lagrangian

Let us now show how to obtain the lowest order special Galileon action within the above framework. It is natural to look for such an action in the form

\[
S_0 = C_0 \int_{M^D_R} \lambda + h.c. \tag{6.1}
\]

where \( C \in \mathbb{C} \) is a complex normalization constant and \( \lambda \) is appropriate \( D \)-form on the target space with as much symmetry as possible with respect to the target space symmetry group. Of course, because the basic Galileon Lagrangian is a Wess-Zumino term, we cannot expect that \( \lambda \) will be completely \( C^D \times U(1, D-1) \) symmetric.

Natural building blocks for construction of such form \( \lambda \) are the holomorphic and antiholomorphic \( D-1 \)-forms \( \frac{1}{(D-1)!} \varepsilon_{\mu_1 \mu_2 \cdots \mu_{D-1}} dZ^\mu_1 \wedge dZ^\mu_2 \cdots \wedge dZ^\mu_{D-1} \), \( \tag{6.2} \)

As we already know, these forms are invariant under \( C^D \times SU(1, D-1) \) and under the general target space symmetry (5.5) they take up a phase \( \det R \) and \( \det R^+ \) respectively. However, because \( d^D Z \) and \( d^D \bar{Z} \) are exact, e.g.

\[
d^D Z = \frac{1}{D!} d(\sigma \cdot Z) \tag{6.4} \]

where

\[
\sigma_\mu = \frac{1}{(D-1)!} \varepsilon_{\mu_1 \mu_2 \cdots \mu_{D-1}} Z^\mu_1 \wedge Z^\mu_2 \cdots \wedge Z^\mu_{D-1} , \tag{6.5} \]

the simplest candidate of the invariant Lagrangian, namely \( \mathcal{L} d^D x = d^D Z |_{M^D_R} \), is a total derivative.

Another possible building block is the Kähler potential

\[
\mathcal{K} = \frac{1}{2} Z \cdot \bar{Z} \tag{6.6} \]

which is invariant with respect to the homogenous subgroup \( U(1, D-1) \) but not with respect to the translations. Nevertheless let us construct the \( SU(1, D-1) \) invariant \( D \)-form \( \lambda \)

\[
\lambda = 2 \mathcal{K} d^D Z = \frac{1}{D!} \varepsilon_{\mu_1 \mu_2 \cdots \mu_D} Z \cdot \bar{Z} Z^\mu_1 \wedge Z^\mu_2 \cdots \wedge Z^\mu_D , \tag{6.7} \]

Writing the action as \( S_0 \) and denoting the coupling constant as \( C_0 \) we anticipate the more systematic notation for the higher order terms explained in the next section.
which transforms under translations $Z'\mu = Z\mu + A\mu$ according to
\[
\lambda' = Z' \cdot \overline{Z} \, d^D Z' = \lambda + Z \cdot \overline{A} d^D Z + A \cdot \overline{Z} d^D Z + A \cdot \overline{A} d^D Z.
\] (6.8)

Thus the induced Lagrangian on the brane apparently breaks the Poincaré translations as well as the linear Galileon shift. However, it can be easily verified\(^{20}\) that the following relation holds
\[
A \cdot \overline{Z} d^D Z + Z \cdot \overline{A} d^D Z + A \cdot \overline{A} d^D Z
\]
\[= d \left[ \frac{1}{D+1} Z \cdot \overline{A} \sigma \cdot Z + \frac{1}{D-1} (A \cdot \overline{Z} \sigma \cdot Z - Z \cdot \overline{Z} \sigma \cdot A) + \frac{1}{D} A \cdot \overline{A} \sigma \cdot Z \right]
\]
\[+ 2i \frac{D-1}{D} \omega \wedge A \cdot \rho \cdot Z,
\] (6.9)

where $\sigma_\mu$ is given by (6.5), $\omega$ is the Kähler for (5.4) and we abbreviated
\[
\rho_{\mu \nu} = \frac{1}{(D-2)!} \varepsilon_{\mu \nu \rho_{D-2}} dZ^{\mu_1} \wedge dZ^{\nu_2} \ldots \wedge dZ^{\mu_{D-2}}.
\] (6.10)

Therefore $\lambda$ is invariant with respect to the complex translation modulo exact form and an additional term proportional to the Kähler form $\omega$. The latter vanishes by definition when restricted to the brane and thus the integral of the form $\lambda$ over the brane $M^D_R$ is invariant with respect to the special Galileon symmetry up to the phase $\text{det } R$. The same is true for the complex conjugated form $\overline{\lambda} = Z \cdot \overline{Z} d^D Z$. Let us therefore assume the following real action defined as
\[
S_0 = \int_{M^D_R} [C_0 \lambda + C_0^* \overline{\lambda}] = \int L_0 d^D x.
\] (6.11)

The individual terms of the corresponding Lagrangian $L_0$ have the right number of derivatives per field (namely $n_\partial - n_x = 2n_{\phi} - 2$)\(^{21}\) as is required for the basic Galileon Lagrangian and we therefore expect that $S_{C,C^*}$ can be identified with the special Galileon action discussed in the previous section. Indeed, using the explicit expressions for the form $d^D Z |_{M^D_R}$ in terms of the Galileon field
\[
d^D Z |_{M^D_R} = \frac{1}{D!} \varepsilon_{\mu_1 \ldots \mu_D} \varepsilon_{\nu_1 \ldots \nu_D} \sum_{n=0}^D \binom{D}{n} \left( \frac{i}{\alpha} \right)^n \prod_{i=1}^n \partial_{\mu_i} \partial_{\nu_i} \phi \prod_{j=n+1}^D \eta_{\mu_j \nu_j} d^D x,
\] (6.12)

performing integration by parts and setting
\[
C_0 = -\frac{i}{4} \alpha^2 (D-1)! e^{i\beta}
\] (6.13)

we reproduce the action (4.22).

\(^{20}\)Here we use formula (6.4) and
\[
dZ^\mu \wedge \sigma_\nu = \delta_\nu^\mu d^D Z.
\]
\[0 = \eta_{\mu_1 \nu_1 \ldots \mu_D} - \eta_{\nu_1 \mu_1 \ldots \mu_D} + \ldots + (-1)^D \eta_{\mu_1 \nu_2 \ldots \mu_D} \varepsilon_{\mu_1 \ldots \mu_{D-1}}
\]
\[= \eta_{\mu_1 \nu_1 \ldots \mu_D} - \eta_{\nu_1 \mu_1 \ldots \mu_D} + \ldots + (-1)^D \eta_{\mu_1 \nu_2 \ldots \mu_D} \varepsilon_{\mu_1 \ldots \mu_{D-1}}
\]

\(^{21}\)Having in mind integration by parts, we count here each explicit $x^\mu$ as "inverse derivative". 23
7 Higher order building blocks

Once we have established the basic (lowest order) action, we can proceed further and try to construct possible higher order Lagrangians. These are necessary as counterterms when the theory is treated on the quantum level. In this section we will restrict ourselves to the case of the Special Galileon $S(\alpha, 0)$ and $S(ia, 0)$ where $\alpha$ and $a$ are real parameters. Apparently only such theories have a well defined quantum version because for $\beta \neq 0$ or $b \neq 0$ the Lagrangians (4.17) and (4.40) contain nonzero tadpole term $L_1$ which makes the perturbation theory ill-defined. In what follows we take $\alpha$ real, the case $\alpha = ia$ can be obtained in a similar way (see Section 9 for details).

The possible counterterms have to share the symmetry of the basic Lagrangian, i.e. the symmetry with respect to the transformations (4.8), (4.29) with traceless $G$. In order to find such symmetric counterterms we need therefore to construct their basic building blocks which are either invariant or have appropriate covariant transformation properties under (4.8) and (4.29). In the previous section we have introduced the geometrical interpretation of the special Galileon, which allows us to use the well established machinery of the probe brane construction. Here this approach gives us the invariants with respect to the special Galileon symmetry.

The basic object of such a construction is the induced effective metric on the brane

$$d{s^2} = dZ \cdot dZ|_{M_D^\mu} \equiv g_{\mu\nu} dx^\mu dx^\nu$$

where explicitly

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{1}{\alpha^2} \partial_\mu \phi \cdot \partial_\nu \phi.$$  \hspace{1cm} (7.1)

As a consequence of the invariance of $d{s^2} = dZ \cdot dZ$ with respect to the target space symmetries, the induced metric on the brane is invariant with respect to transformations (5.20), (5.21) and (5.30), (5.31). Therefore the change of $g_{\mu\nu}$ under these symmetries reduces to a covariant formula

$$g'_{\mu\nu}(x') = \eta_{\alpha\beta}(x) \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu}.$$  \hspace{1cm} (7.2)

Diffeomorphism invariants constructed from $g_{\mu\nu}$ are thus automatically invariant with respect to the transformations (4.8), without any restrictions to $G$ (i.e. $G$ need not to be traceless). Let us now give a list of basic building blocks for construction of such invariants.

The inverse metric $g^{\mu\nu}$ is represented as an infinite series 22

$$g^{\mu\nu} = \eta^{\mu\nu} + \sum_{n=1}^{\infty} \frac{(-1)^n}{\alpha^{2n}} \partial^\mu \phi \cdot (\partial \partial \phi \cdot \partial \partial \phi)^{(n-1)} \cdot \partial \partial \phi.$$  \hspace{1cm} (7.3)

\footnote{In this and in the following formulas, the dot means contraction of the Lorentz indices according to the flat metric $\eta_{\mu\nu}$, i.e. not with $g_{\mu\nu}$. The symbol $(\partial \partial \phi \cdot \partial \partial \phi)^n$ denotes $n$-th matrix power of $\partial^\mu \partial \phi \cdot \partial \partial \phi$. For example, the $n = 2$ term on the RHS of (7.3) reads in detail

$$\frac{1}{\alpha^2} \partial^\mu \partial_{\lambda_1} \phi \partial^\mu_1 \partial_{\lambda_2} \phi \partial^\mu_2 \partial_{\lambda_3} \phi \partial^\mu_3 \partial \partial \phi.$$}

where

$$\partial^\mu \partial \phi = \eta^{\alpha\beta} \partial_\alpha \partial_\beta \phi.$$
It is an easy exercise to calculate the other related objects, namely the Christoffel symbols of the second kind

\[ \Gamma_{\rho\sigma\mu} = \frac{1}{2} (\partial_{\mu} g_{\rho\sigma} + \partial_{\sigma} g_{\rho\mu} - \partial_{\rho} g_{\mu\sigma}) = \frac{1}{\alpha^2} \partial_{\mu} \partial_{\sigma} \partial \phi \cdot \partial \partial_{\rho} \phi, \]  

(7.4)

and the Riemann tensor

\[ R_{\alpha\beta\mu\nu} = \frac{1}{2} (\partial_{\beta} \partial_{\mu} g_{\alpha\nu} + \partial_{\alpha} \partial_{\nu} g_{\beta\mu} - \partial_{\beta} \partial_{\nu} g_{\alpha\mu} - \partial_{\alpha} \partial_{\mu} g_{\beta\nu}) + g^{\rho\sigma} (\Gamma_{\rho\beta\mu} \Gamma_{\sigma\alpha\nu} - \Gamma_{\rho\beta\nu} \Gamma_{\sigma\alpha\mu}) \]  

\[ = \frac{1}{\alpha^2} g^{\rho\sigma} (\partial_{\beta} \partial_{\nu} \partial \phi \partial_{\alpha} \partial \partial_{\mu} \partial \phi - \partial_{\beta} \partial_{\nu} \partial \phi \partial_{\alpha} \partial \partial_{\rho} \partial \phi), \]  

(7.5)

where we have used (7.3).

Other building blocks are the components of the extrinsic curvature \( K_{\alpha\mu} \) and the twist connection \( \beta_{\mu a} \). These structures are defined with help of the basis of tangent vectors \( e_{A\mu} \) and normal vectors \( n_a^A \) to the brane (see [7] for a general construction) as

\[ e^{B}_{\mu} \nabla_B e^{A}_{\nu} = \Gamma^{A}_{\mu\nu} e^{A}_{\nu} - K^{A}_{\mu} n_a^A, \]  

(7.6)

\[ e^{B}_{\mu} \nabla_B n_a^A = \beta^{B}_{\mu a} n_a^A + K^{A}_{\nu} e^{B}_{\nu}. \]  

(7.7)

Here the capital latin letters \( A \equiv (\alpha, \beta), \ldots \) corresponed to the target space indices with respect to the coordinates \( Y^A \equiv (X^\alpha, L^\beta) \), small latin letters \( a, b, \ldots \) denote the \( D \)-bein index of the normal space to the brane, the greek letters \( \mu, \nu, \ldots \) refer to the coordinates \( x^\mu \) on the brane and

\[ \nabla_A = \partial_A. \]  

The tangent vectors \( e_{\mu} \) are given as

\[ e^A_{\mu} = \frac{\partial Y^A}{\partial x^\mu} = (\delta^\alpha_{\mu}, \eta^{\beta\nu} \partial_{\mu} \partial_{\nu} \phi), \]  

(7.8)

while the normal vectors have to satisfy the orthogonality conditions

\[ G_{AB} n^A_a n^B_b = \eta_{ab}, \quad G_{AB} n^A_a e^B_\beta = 0, \]  

(7.9)

where \( G_{AB} = \text{diag} (\eta_{\alpha\beta}, (1/\alpha^2) \eta_{\delta\gamma}) \) is the target space metric in the coordinates \( Y^A \equiv (X^\alpha, L^\beta) \). The \( D \)-bein of the normal vectors \( n_a^A \) can be constructed using the \( D \)-dimensional vectors \( m_a^\alpha \) satisfying

\[ g_{\alpha\beta} m_a^\alpha m_b^\beta = \eta_{ab}, \quad \eta^{ab} m_a^\alpha m_b^\beta = g^{\alpha\beta}. \]  

(7.10)

In terms of these vectors we obtain

\[ n_a^A = \left( -\frac{1}{\alpha} \partial^\alpha \partial \phi \cdot m_a^\beta, \alpha m_a^\beta \right). \]  

(7.11)

Note however, that the constraint (7.10) does not fix the vectors \( m_a \) unambiguously. There is a freedom which allows us to redefine the \( D \)-bein using a local Lorentz transformation acting on the small Latin indices

\[ m'_a = \Lambda_a^b m_b, \quad \Lambda_a^b \eta_{cd} \Lambda_b^d = \eta_{ab}. \]  

(7.12)

\[ ^{23}\text{Note that the target space is flat.} \]

\[ ^{24}\text{In a more detailed notation} \]

\[ e_{\mu} = \frac{\partial}{\partial x^\mu} + \eta^{\nu\rho} \partial_{\mu} \partial \phi \frac{\partial}{\partial L^\rho}. \]
This gives rise to additional gauge invariance of the construction we have to take care of. The twist connection \( \beta^b_{\mu a} \) takes a role of the compensating gauge field corresponding to this gauge transformation.

Introducing a dual basis \( e^\mu_A \) and \( n^a_A \)

\[
e^\mu_A e^A_\nu = \delta^\mu_\nu, \quad \eta^{a}_A n^a_B = \delta^a_b, \quad n^a_A e^A_\mu = n^a_A e^A_\lambda = 0, \quad e^A_\mu e^B_\mu + n^A_\mu n^a_B = \delta^A_B, \quad (7.13)
\]

explicitly

\[
e^\mu_A = g^{\mu\sigma} \left( \eta_{\alpha\sigma} \frac{1}{\alpha^2} \partial_\sigma \partial_\beta \phi \right), \quad n^a_A = \frac{1}{\alpha} \eta^{ab} (-m_b \cdot \partial \partial_\alpha \phi, \eta_{\beta\sigma} m^c_{\sigma}), \quad (7.14)
\]

we can express \( K_{\mu\nu}^a \) and \( \beta^b_{\mu a} \) as follows

\[
K_{\mu\nu}^a = -n^a_A e^B_\mu \nabla_B e^A_\nu = -\frac{1}{\alpha} \eta^{ab} m_b \cdot \partial \partial_\nu \partial_\nu \phi \quad (7.15)
\]

\[
\beta^b_{\mu a} = n^b_A e^B_\mu \nabla_B n^a_A = \eta^{bc} \left( m^a_c g_{\alpha\beta} \partial_\mu m^\beta_a + \frac{1}{\alpha^2} m_c \cdot \partial \partial_\phi \cdot \partial \partial_\mu \partial_\phi \cdot m_a \right). \quad (7.16)
\]

and \( K^b_{\mu\nu} = \eta_{ab} g^{\nu\sigma} K_{\mu\sigma}^a \). Under the gauge transformation \((7.12)\) these objects transform as (cf. \((7.10)\))

\[
K^b_{\alpha\nu} = \Lambda^b_a K^a_{\alpha\nu}, \quad (7.17)
\]

\[
\beta^b_{\mu a} = \eta_{ac} \beta^c_{\mu b} = \Lambda^c_a \Lambda^d_b \beta_{\mu cd} + \Lambda^c_a \eta_{cd} \partial_\mu \Lambda^d_b, \quad (7.18)
\]

while under the duality transformation \((4.8)\) they transform covariantly as tensors with corresponding Greek indices.

However, not all these objects are in fact independent. Note that as a consequence of \((7.10)\) we get independently on the choice of \( m_a \)

\[
\eta_{ab} K^a_{\mu\nu} K^b_{\nu\alpha} = \frac{1}{\alpha^2} \partial_\nu \partial_\alpha \phi g^{\mu\sigma} \partial_\sigma \partial_\phi \partial_\nu \partial_\phi, \quad (7.19)
\]

and thus (see \((7.5)\)) we get the Gauss formula

\[
R_{\alpha\beta\mu\nu} = \eta_{ab} \left( K^c_{\mu\alpha} K^b_{\nu\beta} - K^a_{\mu\beta} K^b_{\nu\alpha} \right), \quad (7.20)
\]

which is valid in this form for the flat target space. Also, the curvature of the twist connection

\[
\varphi^a_{\mu\nu} = \partial_\mu \beta^a_{\nu b} - \partial_\nu \beta^a_{\mu b} + \beta^c_{\mu c} \beta^b_{\nu c} - \beta^c_{\nu c} \beta^b_{\mu c} \quad (7.21)
\]

can be expressed in terms of the extrinsic curvature using the flat target space form of the Ricci equation

\[
\varphi^a_{\mu\nu} = g^{\mu\sigma} \eta_{bc} \left( K^a_{\mu\alpha} K^c_{\nu\sigma} - K^a_{\nu\alpha} K^c_{\mu\sigma} \right). \quad (7.22)
\]

Finally, we get also the Codazzi equation in the form

\[
D_\mu K^a_{\nu\alpha} - D_\nu K^a_{\mu\alpha} = 0. \quad (7.23)
\]

Here the covariant derivative \( D_\mu \) acts also on the \( D \)–bein index as

\[
D_\mu K^a_{\nu\alpha} = \partial_\mu K^a_{\nu\alpha} - \Gamma^\sigma_{\mu\nu} K^a_{\sigma\alpha} - \Gamma^\sigma_{\mu\alpha} K^a_{\nu\sigma} + \beta^a_{\mu b} K^b_{\nu\alpha}, \quad (7.24)
\]

Here the covariant derivative \( D_\mu \) acts also on the \( D \)–bein index as
and analogically for other objects carrying the small Latin indices. We can e.g. easily verify, that the vectors $m_a$ are covariantly constat

$$D_\mu m^\alpha_a = \partial_\mu m^\alpha_a + \Gamma^\alpha_{\mu\nu} m^\nu_a - \beta^b_{\mu a} m^\alpha_b = 0.$$  

(7.25)

Note that, using an expansion

$$(1 + x)^{-1/2} = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2} - n\right)_n}{n!} x^n,$$

(7.26)

where $(a)_n = \Gamma(a + n)/\Gamma(a)$ is the Pochammer symbol, a particular solution of the constraints (7.10) can be formally written in terms of an infinite series

$$m^\alpha_a = \eta^{\alpha\beta} \left( \eta_{\alpha\beta} + \sum_{n=1}^{\infty} \frac{\left(\frac{1}{2} - n\right)_n}{n! \alpha^{2n}} [(\partial \partial \phi)^{n}]_{a\beta} \right).$$  

(7.27)

This choice of $m_a$ corresponds to a particular gauge fixing of the local $O(1, D - 1)$ invariance (7.12).

The last building blocks are the invariant measures on the brane. In the previous section we have discussed the forms $d^D Z$ and $d^D \overline{Z}$ which are invariant under (4.8) up to the phase. From the induced metric $g_{\mu\nu}$ we can construct strictly invariant volume element

$$d^D x \sqrt{|\det (g_{\mu\nu})|} = d^D x \left| \det \left( \frac{\partial Z^\alpha}{\partial x^\mu} \right) \right| \equiv \sqrt{d^D Z d^D \overline{Z}}.$$  

(7.28)

After some algebra we get

$$\sqrt{d^D Z d^D \overline{Z}} = d^D x \sum_{M=0}^{\infty} \left( - \frac{1}{\alpha^2} \right)^M \sum_{N=0}^{\infty} \frac{1}{N!} \sum_{n_1 \geq 1}^{\sum_{j=1}^N n_j = M} \prod_{k=1}^N \left( - \frac{1}{2n_k} \eta^{\alpha\beta} \left[ (\partial \partial \phi)^{2n_k} \right]_{\alpha\beta} \right).$$  

(7.29)

This finishes our list of the basic building block of the higher order Lagrangians.

All these object can be visualized and easily manipulated using an efficient graphical representation developed in [18]. Each field $\phi$ is represented with a point and the derivative $\partial_\mu$ acting on $\phi$ is depicted as a line starting at the point representing $\phi$ and carrying a corresponding Lorentz index $\mu$. The flat metric $\eta^{\mu\nu}$ is drawn as line with indices $\mu$, $\nu$ and contraction of the Lorentz indices is then represented as an internal line connecting the points adjacent to the contracted derivatives. Also integration by parts can be visualized within the graphical language: one simply disconnect one end of a chosen line from the corresponding point and the free end of this line attaches successively to all other points in the graph. The sum of resulting graphs is then taken with additional minus sign. The basic graphical rules as well as simple examples of their application are shown in figure 1. The graphical representation of inverse metric and the Riemann tensor are then depicted in figures 2 and 3.

8 Higher order Lagrangians

Having established the basic building blocks, let us proceed to the construction of the higher order Lagrangians. According to the general prescription for the probe brane action [7, 38],
\[ \mu \nu \equiv \eta^{\mu \nu} \]

\[ \mu_1 \mu_2 \ldots \mu_n \equiv \partial_{\mu_1} \partial_{\mu_2} \ldots \partial_{\mu_n} \phi \]

\[ \equiv \partial_\mu \partial_\nu \phi \partial^\mu \partial^\nu \phi \]

\[ \equiv \partial_\mu \phi \]

\( (a) \)

\( (b) \)

\( (c) \)

Figure 1: The rules for graphical description of the invariant Lagrangians. The basic graphical building block corresponds to (a) while simple examples are depicted as (b) and (c).

\[ g^{\mu \nu} \equiv \begin{array}{c} \mu \\ \nu \end{array} = \begin{array}{c} \mu \\ \nu \end{array} + \sum_{n=1}^\infty \begin{array}{c} n \\ \nu \end{array} \]

\[ \equiv \frac{(-1)^n}{\alpha^{2n}} \]

\[ (n-1) \text{ times} \]

Figure 2: The graphical representation of the inverse effective metric \( g^{\mu \nu} \).

The most general case can be written in the form

\[ S_{\text{inv}} = \int_{M^D} d^DZ L_Z + \int_{M^D} d^DZ L_{Z^*} + \int_{M^D} \sqrt{-g} \ud^DZ \ud^DZ L_{ZZ^*}. \] 

(8.1)

Here the functions \( L_Z \), \( L_{Z^*} \equiv L_{Z^*}^* \) and \( L_{ZZ^*} \equiv L_{ZZ^*}^* \) are diffeomorphism invariants and invariants with respect to the local gauge transformation (7.12) constructed from the building blocks listed in the previous section and their covariant derivatives. The action \( S_{\text{inv}} \) is then invariant with respect to the transformations (4.8) with traceless \( G^{\mu \nu} \) and form-invariant (i.e. invariant up to a change of couplings) for general \( G^{\mu \nu} \).

Thanks to the Gauss-Codazzi formulae (7.20) and (7.22), it is sufficient to use only the extrinsic curvature \( K^{\mu \nu}_{\rho \sigma} \), its covariant derivatives and the \( D \)-bein \( m^a_\alpha \). The invariants are then obtained contracting the Greek indices of such building blocks with respect to the induced metric \( g_{\mu \nu} \) and its inverse \( g^{\mu \nu} \), while the small Latin indices have to be contracted using the flat metric \( \eta_{ab} \). Because \( m_a \) is the \( D \)-bein with respect to the induced metric \( g_{\mu \nu} \), we can convert freely the Greek indices into small Latin indices and vice versa. For instance, instead of \( K^a_{\mu \nu} \) we can use solely the following rank-three tensor \( K_{\alpha \mu \nu} \)

\[ K_{\alpha \mu \nu} = g_{\alpha \beta} m^a_\beta K^a_{\mu \nu} = -\frac{1}{\alpha} \partial_\alpha \partial_\mu \partial_\nu \phi, \] 

(8.2)

and its covariant derivatives as the basic building blocks. In terms of \( K_{\alpha \mu \nu} \) the Gauss formula reads (see figures 4 and 3)

\[ R_{\alpha \beta \mu \nu} = g^{\rho \sigma} (K_{\rho \mu \alpha} K_{\sigma \nu \beta} - K_{\rho \mu \beta} K_{\sigma \nu \alpha}). \] 

(8.3)

8.1 Hierarchy of the counterterms

Before giving some explicit examples of the higher order actions, let us briefly comment on the hierarchy of the counterterms. As explained in [21], in the Galileon theories the higher
order Lagrangians with $n_\phi$ derivatives and $n_\phi$ fields can be classified according to the index $\delta$ where
\[
\delta = n_\phi - 2n_\phi + 2.
\] (8.4)

In what follows we will denote therefore the higher order actions and Lagrangians as $S_\delta$ and $\mathcal{L}_\delta$ respectively in order to indicate the index of their vertices. While all the terms of the basic Lagrangian have $\delta = 0$, the indices of the counterterms which are necessary to cancel UV divergences of the graph with $L$ loops and vertices $V_i$ with indices $\delta_i$ in $D$ spacetime dimensions are [21]
\[
\delta_{CT} = (D + 2)L + \sum_i \delta_i.
\] (8.5)

Therefore the first Lagrangians which are renormalized by loop corrections has index $\delta = D + 2$.

The assignment of the index $\delta$ to a concrete term in the Lagrangian is easy. Note that $\sqrt{d^DZd^D\bar{Z}}$, $d^DZ$ and $d^D\bar{Z}$ depend only on second derivatives of $\phi$ and therefore they do not contribute to $\delta$ at all. The same is true for $g^{\mu\nu}$, $g_{\mu\nu}$, while each $K_{\alpha\mu\nu}$ as well as each covariant derivative $D_\mu$ increases $\delta$ by one. As a result
\[
\delta = n_K + n_D + 2
\] (8.6)

where $n_K$ is number of $K_{\alpha\mu\nu}$ and $n_D$ is number of covariant derivatives. In what follows we will discuss several lowest $\delta$ actions in more detail.

### 8.2 Lagrangians with $\delta = 2$

Let us start with the action $S_2$. Because, unlike $d^DZ$ and $d^D\bar{Z}$, the invariant volume element $\sqrt{d^DZd^D\bar{Z}}$ is not trivial when integrated over the brane, the next to lowest action with $\delta = 2$, i.e. with $n_\phi = 2n_\phi$ derivatives, can be constructed as
\[
S_2 = B_2 \int_{M^D_R} \sqrt{d^DZd^D\bar{Z}}.
\] (8.7)
\[
d^DZ = d^Dx[1 + \frac{i}{\alpha} \bigcirc - \frac{1}{2\alpha^2} \bigcirc \bigcirc - \bigtriangleup] - \frac{i}{6\alpha^3} \bigcirc \bigcirc \bigcirc + 2 \bigtriangleup - 3 \bigcirc \bigcirc \bigcirc + O(\phi^4)]
\]

Figure 5: The measure \(d^DZ\) in terms of the Galileon field.

\[
\sqrt{d^DZ d^\bar{D}Z} = d^Dx[1 + \frac{1}{2\alpha^2} \bigcirc - \frac{1}{2\alpha^4} \square + \frac{1}{8\alpha^4} \bigcirc \bigcirc \bigcirc + O(\phi^6)]
\]

Figure 6: The measure \(\sqrt{d^DZ d^\bar{D}Z}\) in terms of the Galileon field.

Such a term is unique up to a real constant \(B_2\) and corresponds to the cosmological constant term for the induced metric \(g_{\mu\nu}\). Using (7.29) we get explicitly

\[
\mathcal{L}_2 = B_2 \sqrt{\det (g_{\mu\nu})} = B_2 \left[ 1 + \frac{1}{2\alpha^2} \langle \partial \partial \phi \cdot \partial \partial \phi \rangle - \frac{1}{2\alpha^4} \langle \partial \partial \phi \cdot \partial \partial \phi \cdot \partial \partial \phi \cdot \partial \partial \phi \rangle + \frac{1}{8\alpha^4} \langle \partial \partial \phi \cdot \partial \partial \phi \rangle^2 + O(\phi^6) \right]. \tag{8.8}
\]

Here and in what follows we abbreviated by \(\langle \cdot \rangle\) a trace of rank two Minkowski tensor with respect to the flat metric \(\eta_{\mu\nu}\). The action \(S_2\) introduces a higher order kinetic term as well as an infinite tower of related interaction terms. The constant \(B_2\) is not renormalized by loop corrections in any dimensions.

### 8.3 Lagrangians with \(\delta = 4\)

Next invariant action correspond to \(\delta = 4\), i.e. \(n_\phi = 2n_\phi + 2\). There are three independent invariants \(I_{4}^{(j)}\), \(j = 1, 2, 3\) with \(n_K + n_D = 2\) available for the construction of \(\mathcal{L}_Z, \mathcal{L}_{\bar{Z}}\) and \(\mathcal{L}_{ZZ}\), namely (see Fig.7 for graphical representation)

\[
I_{4}^{(1)} = g^{\alpha\beta} g^{\mu\nu} g^{\rho\sigma} K_{\alpha\mu\nu} K_{\beta\rho\sigma} = \frac{1}{\alpha^2} g^{\alpha\beta} g^{\mu\nu} g^{\rho\sigma} \partial_\alpha \partial_\beta \partial_\mu \partial_\nu \partial_\rho \partial_\sigma \phi \tag{8.9}
\]

\[
I_{4}^{(2)} = g^{\alpha\beta} g^{\mu\rho} g^{\nu\sigma} K_{\alpha\mu\nu} K_{\beta\rho\sigma} = \frac{1}{\alpha^2} g^{\alpha\beta} g^{\mu\rho} g^{\nu\sigma} \partial_\alpha \partial_\beta \partial_\mu \partial_\nu \partial_\rho \partial_\sigma \phi \tag{8.10}
\]

\[
I_{4}^{(3)} = g^{\alpha\beta} g^{\mu\nu} D_\alpha K_{\beta\mu\nu} = \frac{1}{\sqrt{|g|}} \partial_\alpha \left( |g| g^{\alpha\beta} K_{\beta\mu\nu} g^{\mu\nu} \right) = -\frac{1}{\alpha} g^{\alpha\beta} g^{\mu\nu} \partial_\alpha \partial_\beta \partial_\mu \partial_\nu \phi + \frac{1}{\alpha^3} g^{\rho\sigma} \left( g^{\alpha\beta} g^{\mu\nu} + 2 g^{\alpha\mu} g^{\beta\nu} \right) \partial_\alpha \partial_\beta \partial_\mu \partial_\nu \partial_\rho \partial_\sigma \phi, \tag{8.11}
\]

30
The three independent invariants for construction of the general \( \delta = 4 \) Lagrangian.

where \( g = \text{det} (g_{\mu\nu}) \). From these building blocks we get five independent Lagrangian terms, namely

\[
S_4 = \int_{M^D_\mathbb{R}} \sqrt{d^DZ} d^DZ \sum_{i=1}^{2} B_4(i) I_4^{(i)} + \frac{(-1)^{D-1}}{2} \int_{M^D_\mathbb{R}} d^DZ \sum_{i=1}^{3} C_4(i) T_4^{(i)} + \frac{(-1)^{D-1}}{2} \int_{M^D_\mathbb{R}} d^DZ \sum_{i=1}^{3} C_4(i) T_4^{(i)}. \tag{8.12}
\]

Here the coupling constants \( B_4(i) \) are real while \( C_4(i) \) are generally complex. These couplings are not renormalized for \( D > 2 \) (cf. \( (8.5) \)). The first two terms are invariant under the transformation \( (4.8) \), while the remaining two are only form-invariant. The couplings \( C_4(i) \) transform under \( (4.8) \) as

\[
C_4(i) \rightarrow C_4(i) \det (\theta). \tag{8.13}
\]

Using \( (7.3) \), \( (8.9) \), \( (8.10) \) and with help of \( (6.12) \) and \( (7.29) \) we obtain for the corresponding Lagrangian \( \mathcal{L}_4 \) to the third order in \( \phi \) (modulo integration by parts)

\[
\mathcal{L}_4 = \frac{1}{\alpha^2} \left( B_4^{(1)} + B_4^{(2)} + \text{Re}C_4^{(1)} + \text{Re}C_4^{(2)} - \text{Im}C_4^{(3)} \right) \Box \phi \cdot \Box \phi \tag{8.14}
+ \frac{1}{\alpha^3} \left[ \left( \text{Im}C_4^{(1)} - \text{Re}C_4^{(3)} \right) \Box \phi \cdot \Box \phi + \text{Im}C_4^{(2)} (\Box \phi : \Box \phi : \Box \phi) \right] \phi + O \left( \phi^4 \right).
\]

We generate again a higher order kinetic term and an infinite tower of related interaction terms starting with cubic one. The kinetic term vanishes for

\[
B_4^{(1)} + B_4^{(2)} + \text{Re}C_4^{(1)} + \text{Re}C_4^{(2)} - \text{Im}C_4^{(3)} = 0. \tag{8.15}
\]

\[\text{Note that } \sqrt{|g|}Z^{(3)} \text{ is total derivative so that this term is absent in } S_4. \text{ The factors } (-1)^{D-1}/2 \text{ are inserted for further convenience.}\]
\[ \mathcal{L}_4^{\phi^4} = \chi \alpha^4 \left( \begin{array}{c} \ \end{array} \right) + \left( \begin{array}{c} \ \end{array} \right) + \left( \begin{array}{c} \ \end{array} \right) - \frac{\rho}{2\alpha^4} \left( \begin{array}{c} \ \end{array} \right) - \frac{\kappa}{2\alpha^4} \left( \begin{array}{c} \ \end{array} \right) \]

Figure 8: The Lagrangian \( \mathcal{L}_4^{\phi^4} \) (in the case when the higher order kinetic term vanishes) after integration by parts.

In such a case, the quartic term \( 26 \) of the Lagrangian \( \mathcal{L}_4 \) reads up to a total derivative

\[
\mathcal{L}_4^{\phi^4} = \chi \alpha^4 \left( \begin{array}{c} \ \end{array} \right) - 2 \Box \partial \phi \cdot \partial \phi \cdot (\Box \partial \phi) - \rho \frac{1}{2} \alpha^2 G^{\mu\nu} x_\mu x_\nu, \quad (G^{\mu}_\mu = 0). \tag{8.16}
\]

where \( \chi = B_4^{(1)} + \text{Re} C_4^{(1)} - \text{Im} C_4^{(3)} \), \( \rho = \text{Re} C_4^{(1)} - \text{Im} C_4^{(3)} \) and \( \kappa = \text{Re} C_4^{(2)} \).

Let us note, that the quartic term \( \mathcal{L}_4^{\phi^4} \) is very special. Namely, it is invariant under the quadratic shift \( \phi \rightarrow \phi + \delta^{\text{shift}}_{\phi} \), where

\[
\delta^{\text{shift}}_{\phi}(x) = -\frac{\theta}{2} \alpha^2 G^{\mu\nu} x_\mu x_\nu, \quad (G^{\mu}_\mu = 0). \tag{8.17}
\]

The reason is as follows. Note that \( \delta^{\text{shift}}_{\phi}(x) \) is a truncated version of the infinitesimal form \( 4.27 \) of the symmetry \( 4.8 \) of the action \( S_4 \), namely

\[
\delta_{\theta}(x) = -\frac{\theta}{2} \alpha^2 G^{\mu\nu} (x_\mu x_\nu + \partial_\mu \phi(x) \partial_\nu \phi(x)), \quad (G^{\mu}_\mu = 0). \tag{8.18}
\]

Such a transformation converts a general Lagrangian term with \( n_\phi \) fields into terms with \( n_\phi - 1 \) and \( n_\phi + 1 \) fields respectively. Symmetry of the complete action \( S_4 \) under \( \phi \rightarrow \phi + \delta^{\text{shift}}_{\phi} \) therefore requires cancelations between such terms. For the the first two terms in the action with lowest \( n_\phi = n_{\phi}^{\text{min}} \), \( n_{\phi}^{\text{min}} + 1 \) there are no terms available for the cancelation of the \( n_\phi - 1 \) part of their transforms. Because the latter appear as a result of the truncated transformation \( \phi \rightarrow \phi + \delta^{\text{shift}}_{\phi} \), the first two terms, with lowest \( n_\phi \) have to be invariant also with respect to the quadratic shift symmetry with traces tensor parameter \( G^{\mu\nu} \). For the general action \( 8.14 \) this applies to the quadratic and cubic terms, for the case when \( 8.15 \) holds and the quadratic term is absent, also the quartic term \( \mathcal{L}_4^{\phi^4} \) is invariant under quadratic shift \( \delta^{\text{shift}}_{\phi} \).

Using integration by parts it can be easily shown that \( \mathcal{L}_4^{\phi^4} \) can be rewritten as

\[
\mathcal{L}_4^{\phi^4} = \frac{\chi}{\alpha^4} (L_1 + L_2 + L_3 + L_4)
\]

\( 26 \) The general quartic interaction term is rather lengthy and is given in the appendix.
\[-\frac{P}{2\alpha^4} \langle \Box \phi \rangle^2 \Box \phi \cdot \Box \phi - \frac{\kappa}{2\alpha^4} \langle \Box \phi \Box \phi \phi \rangle \langle \Box \phi \Box \phi \phi \rangle , \quad (8.19)\]

where \( L_i \) have been introduced in [18] and are depicted in the figure 8. In such a form \( L_4^\phi \) fits into the classification of the polynomial shift symmetric Lagrangians performed in [19] and [18]. The first term corresponds to the Lagrangian denoted in [18] as \((P, N, \Delta) = (2, 4, 5)\), where \( P \) is the order of the polynomial shift symmetry, \( N = n_\phi \) and \( \Delta = n_\phi / 2 \) in terms of our notation. It is invariant under \( \delta_\phi \) even for \( G_\mu^\phi \neq 0 \). This behavior can be easily understood within our construction. Note that such a term is present even for \( C_4^{(i)} = 0 \), when the condition (8.15) means \( B_4^{(2)} = -B_4^{(1)} \). In such a case, the action \( S_4 \) reduces as a consequence of the Gauss formula (8.3) to the Einstein-Hilbert action for the induced metric \( g_{\mu\nu} \)

\[ S_4^R = B_4^{(1)} \sqrt{\det (g_{\mu\nu})} R. \quad (8.20)\]

As discussed above, it is invariant under (4.8) without any restriction on the symmetric tensor \( G_{\mu\nu} \).

The remaining two terms of the Lagrangian \( L_4^\phi \) are both of the general form \( (\Box \phi)^n \phi \Box \phi^{2m} \). Such terms have been constructed in [19] as the basic interaction terms invariant with respect to quadratic shift \( \delta_\phi \) with traceless \( G_{\mu\nu} \).

### 8.4 Lagrangians with \( \delta \geq 6 \) and a role of Lovelock invariants

The first Lagrangian \( L_5 \) which is renormalized by loop corrections in \( D = 4 \) dimensions (and which is not renormalized for \( D > 4 \)) corresponds to \( \delta = 6 \), i.e. \( n_\phi = 2n_\phi + 4 \). According to (8.6) the invariants \( L_Z, L_\bar{Z} \) and \( L_{Z\bar{Z}} \) are linear combinations of the terms with \( n_K + n_D = 4 \), schematically

\[ K^4, \quad DK^3, \quad D^2 K^2, \quad D^3 K. \quad (8.21)\]

There is a large (but finite) number of combinations how to contract the indices of the above structures with the inverse metric \( g^{\mu\nu} \); we will not list them explicitly.

As discussed above, for each resulting independent Lagrangian, the first two terms with with lowest \( n_\phi = n_\phi^\text{min} \), \( n_\phi^\text{min} + 1 \) have to be invariant with respect to the quadratic shift (8.17). Such terms (provided they are invariant also for \( G_\mu^\phi \neq 0 \), i.e. if they originate in \( L_{Z\bar{Z}} \)) correspond to Lagrangians \((P, N, \Delta) = (2, n_\phi + n_\phi + 2) \) (using the nomenclature of [18]). Here the most interesting are those with maximal possible \( n_\phi^\text{min} \), i.e. those with minimal number of derivatives per field (especially \( n_\phi < 3n_\phi \)). Naively \( n_\phi^\text{min} = n_K \) and therefore we apparently cannot go beyond the uninteresting \((P, N, \Delta) = (2, 4, 6) \) terms with three derivatives per field, which originate from the \( K^4 = O (\phi^4) \) part of the general action. However, in analogy with the \( \delta = 4 \) case, for \( D > 4 \) there exist a special combination of the \( K^4 \) invariants for which \( n_\phi^\text{min} = 6 \). Not accidentally such a combination of \( K^4 \) terms is the Lovelock invariant [20]

\[ L_{ZZ}^{R^2} = R^2 - 4R_{\alpha\beta}R^{\alpha\beta} + R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} \]

\[ = g^{\rho\sigma}g^{\nu\lambda} \left( K_{\rho\mu\alpha}K_{\sigma\nu\beta} - K_{\rho\mu\beta}K_{\sigma\nu\alpha} \right) \left( K_{\kappa\tau\rho}K_{\lambda\nu\delta} - K_{\kappa\tau\delta}K_{\lambda\nu\rho} \right) \]

\[ \times \left[ g^{\rho\mu}g^{\nu\omega}g^{\phi\tau}g^{\delta\gamma} - 4g^{\rho\mu}g^{\nu\omega}g^{\phi\delta}g^{\tau\gamma} + g^{\rho\nu}g^{\phi\delta}g^{\mu\gamma}g^{\tau\omega} \right]. \quad (8.22)\]

The \( n_\phi = 6 \) term of the invariant action

\[ S_6^{R^2} \equiv B_6^{R^2} \int_{M_6^4} \sqrt{d^D Z d^D \bar{Z} L_{ZZ}^{R^2}} \quad (8.23)\]
built from the Lovelock invariant corresponds then to quadratic shift symmetric term \((P, N, \Delta) = (2, 6, 8)\).

For \(\delta = 2n + 2 > 6\) we meet an analogous situation, namely the terms \(K^{2n}\) can be combined for \(D > \delta - 2 = 2n\) into the Lovelock invariant \(L^{R^n}_{ZZ}\) where

\[
L^{R^n}_{ZZ} = \frac{1}{2^n} \delta^{\mu_1 \nu_1} \cdots \delta^{\alpha_n \beta_n} R_{\mu_1 \nu_1} \alpha_1 \beta_1 \cdots R_{\mu_n \nu_n} \alpha_n \beta_n ,
\]

where the generalized Kronecker delta is

\[
\delta^{\mu_1 \cdots \mu_n}_{\alpha_1 \cdots \alpha_n} = \frac{1}{n!} \delta^{\mu_1}_{\alpha_1} \delta^{\mu_2}_{\alpha_2} \cdots \delta^{\mu_n}_{\alpha_n}.
\]

Then the corresponding invariant action

\[
S^{R^n}_{\delta = 2n+2} = B^{R^n}_\delta \int_{\mathbb{M}^D_{\mathbb{R}}} \sqrt{d^DZ d\bar{D}Z} L^{R^n}_{ZZ}
\]

starts with a term with \(n_\phi = \delta = 2n + 2\). Indeed, using\(^{27}\)

\[
R_{\mu \nu}^{\alpha \beta} = \frac{1}{\alpha^2} \left( \partial^\beta \partial_\nu \partial_\phi \partial_\mu - \partial^\beta \partial_\mu \partial_\nu \partial_\phi - \partial^\alpha \partial_\mu \partial_\nu \partial^\beta \partial_\phi \right) + O(\phi^4)
\]

and expressing \(S^{R^n}_{\delta = 2n+2}\) in terms of the Galileon field we get

\[
\sqrt{\det (g_{\mu \nu})} L^{R^n}_{ZZ} = \frac{1}{\alpha^{2n}} \delta^{\mu_1 \nu_1}_{\alpha_1 \beta_1} \cdots \delta^{\mu_n \nu_n}_{\alpha_n \beta_n} \prod_{i=1}^{n} \partial^{\beta_i} \partial_\nu_i \partial_\phi - \partial^{\alpha_i} \partial_\mu_i \partial_\phi + O(\phi^{2n+2})
\]

\[
= \frac{1}{\alpha^{2n}} \delta^{\mu_1 \nu_1}_{\alpha_1 \beta_1} \cdots \delta^{\mu_n \nu_n}_{\alpha_n \beta_n} \partial_\mu_i \partial^{\alpha_i} \left[ \partial^{\beta_i} \partial_\nu_i \partial_\phi \cdot \partial_\phi \prod_{i=2}^{n} \partial^{\beta_i} \partial_\nu_i \partial_\phi \cdot \partial_\mu_i \partial^{\alpha_i} \right] + O(\phi^{2n+2})
\]

(8.28)

where we have used the antisymmetry of \(\delta^{\mu_1 \nu_1}_{\alpha_1 \beta_1} \cdots \delta^{\mu_n \nu_n}_{\alpha_n \beta_n}\). The first term is a total derivative and thus \(S^{R^n}_{\delta = 2n+2} = O(\phi^{2n+2})\).

For general \(\delta = 2n + 2\), for which \(n_\phi = \delta + 2n_\phi - 2 = 2n + 2n_\phi\) and for \(D > \delta - 2 = 2n\) we can therefore generate the quadratic shift invariants \((P, N, \Delta) = (2, 2n + 2, 3n + 2)\) as the sum of the lowest \(n_\phi = 2n + 2\) terms of the Lovelock action \(S^{R^n}_{\delta = 2n+2}\). Of course, the complete Lovelock action is symmetric with respect to the complete transformation (4.8).

9 Note on the analytic continuation \(\alpha \to i\alpha\)

Let comment briefly on the case of imaginary \(\alpha\), i.e. when the coset space transformation is formulated in terms of the real coordinates \(Z^\pm, \mu\) (see (4.29)). In such a case we can repeat the above brane constructions almost literally, however the target space in not a Kähler manifold any more. Instead, we start with a flat space with the metric with signature \((2D, 2D)\)

\[
ds^2 = dZ^+ \cdot dZ^- = \eta_{\mu \nu} \left( dX^\mu dX^\nu - \frac{1}{a^2} dL^\mu dL^\nu \right)
\]

(9.1)

\(^{27}\)On the right hand side the indices are raised by the flat metric \(\eta^{\mu \nu}\).
and the form $\omega$ we replace with

$$\omega = -\frac{1}{2} \eta_{\mu\nu} dZ^+ \wedge dZ^- = \frac{1}{a} \eta_{\mu\nu} dX^\mu \wedge dL^\nu.$$  \hfill (9.2)

The group of simultaneous symmetry of both these structures is now $\mathbb{R}^{2D} \times O(D, D) \cap Sp(2D)$. The induced metric on the brane reads in this case

$$g_{\mu\nu} = \eta_{\mu\nu} - \frac{1}{a^2} \partial_\mu \partial_\phi \cdot \partial_\nu \varphi.$$  \hfill (9.3)

and the other invariant building blocks derived from $g_{\mu\nu}$ can be easily obtained from the previous case by means of analytic continuation $\alpha \rightarrow i\alpha$; e.g., the Christoffel symbols of the second kind are now

$$\Gamma_{\rho\sigma\mu} = -\frac{1}{a} \eta_{\alpha\beta} m_b \cdot \partial_\alpha \varphi \cdot \partial_\beta \varphi \cdot \partial_\mu \varphi.$$  \hfill (9.4)

The normal vectors $n^A_a$ are again constructed in terms of $D$-bein $m_a$ with respect to the induced metric (9.3), (cf. also (7.10)) as

$$n^A_a = \left( \frac{1}{a} \eta^{\alpha\beta} m_b \cdot \partial_\alpha \varphi \cdot \partial_\beta \varphi, m_a \right),$$  \hfill (9.5)

and are normalized according to

$$G_{AB} n^A_a n^B_b = -\eta_{ab}, \quad G_{AB} n^A_a n^B_b = 0.$$  \hfill (9.6)

The dual basis reads then

$$e^\mu = g^{\mu\sigma} \left( \eta_{\sigma\beta}, -\frac{1}{a^2} \partial_\sigma \partial_\beta \varphi \right), \quad n^a_A = \frac{1}{a} \eta^{ab} \left( m_b \cdot \partial_\alpha \varphi, \eta_{\alpha\beta} m_b^\alpha \right),$$  \hfill (9.7)

and thus the extrinsic curvature $K^a_{\mu\nu}$ and twist connection are now

$$K^a_{\mu\nu} = -\frac{1}{a} \eta^{ab} m_b \cdot \partial_\mu \partial_\nu \varphi,$$$$

\beta^b_{\mu\nu} = \eta^{bc} \left( m_c g_{\alpha\beta} \partial_\mu m_b^\alpha - \frac{1}{a^2} m_c \cdot \partial_\phi \cdot \partial_\mu \partial_\nu \varphi \cdot \partial_\alpha \varphi \cdot \partial_\beta \varphi \cdot \partial_\alpha \varphi \cdot \partial_\beta \varphi \right).$$  \hfill (9.8)

Again, as a consequence of the Gauss-Codazzi relations the basic building blocks for the higher order lagrangians can be taken to be the induce metric $g_{\mu\nu}$, its inverse $g^{\mu\nu}$ and the covariant derivatives of the tensor

$$K_{\alpha\beta\mu\nu} = g_{\alpha\beta} m_b^\alpha K_{\mu\nu} = -\frac{1}{a} \partial_\alpha \partial_\beta \partial_\mu \partial_\nu \varphi$$

together with measures $d^DZ^+$, $d^DZ^-$ and $d^Dx \sqrt{|\det (g_{\mu\nu})|} \equiv \sqrt{d^DZ^+ d^DZ^-}$. The invariant action has the general form

$$S_{inv} = \int_{M^D_R} d^DZ L_{Z^+} + \int_{M^D_R} d^DZ L_{Z^-} + \int_{M^D_R} \sqrt{d^DZ^+ d^DZ^-} L_{Z^+ Z^-}.$$  \hfill (9.10)

where in contrast to the case of real $\alpha$ the invariants $L_{Z^+}$ and $L_{Z^-}$ are not correlated.

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28 The minus sign on the right hand side of the first normalization condition reflects the different signature of the target space metric $G_{AB}$ in comparison with the real $\alpha$ case.
10 Conclusion

The aim of this work was to discuss in detail the symmetry properties of the Special Galileon in flat space. We have found that the hidden symmetry which dictates the form of its Lagrangian and which is responsible for its peculiar $O(p^3)$ soft limit behavior can be understood as a special case of more general family of duality transformations. Similarly, the Special Galileon itself turns out to be a special member of a wider family of theories which are interrelated by the duality transformations mentioned above. These originate in special transformation of the coset space $\text{GAL} (D, 1) / \text{SO}(1, D − 1)$. There are two branches of such transformations which can be parameterized either by the matrix $U = \exp (i \mathcal{G})$ or by the matrix $U = \exp (\mathcal{G})$ where the real rank two tensor $G^\mu_\nu$ satisfies $\eta^\mu_\rho G^\rho_\nu = \eta^\nu_\rho G^\rho_\mu$. The hidden Galileon symmetry found previously in [16] is then identified with an infinitesimal form of this symmetries for $\text{Tr} \mathcal{G} = 0$ after imposing the Inverse Higgs Constraint.

As we have shown, the Special Galileon in $D$ dimensions can be also interpreted within a different geometrical language. Within such a framework, the Galileon field $\phi$ is a scalar degree of freedom describing (in particular gauge) the position of the $D$–dimensional probe brane embedded in a $2D$–dimensional target space. The Inverse Higgs Constraint can be formulated geometrically as a constraint on the brane embedding, namely forcing an appropriate target space two-form $\omega$ to vanish when restricted to the brane. The ordinary Galileon symmetry and the symmetry/duality of Special Galileon can be then identified with a subgroup of isometries of the target space which are at the same time symmetries of the form $\omega$. For the two branches of the Special Galileon, the metric of the target space is flat with signature either $(2, 2D − 2)$ or $(2D, 2D)$. In the first case, the target space has a natural structure of a Kähler manifold and the group of Special Galileon symmetry/duality consists of symmetries of the corresponding hermitian form, namely $\mathbb{R}^{2D} \rtimes O(2, D − 2) \cap \text{Sp}(2D) \approx \mathbb{C}^D \rtimes U(1, D − 1)$. The complex translations represent spatial translation and the Galileon linear shift symmetries, while the $U(1, D − 1)$ transformations can be interpreted as Lorentz transformations and hidden symmetry/duality of the Special Galileon. In the second case, the group is $\mathbb{R}^{2D} \rtimes O(D, D) \cap \text{Sp}(2D)$ and the target space has not any compatible complex structure.

The Lagrangians which are symmetric/dual with respect to these symmetries can be then constructed according the general probe brane prescription. The building blocks can be reduced, as a consequence of the Gauss-Codazzi formulae, to the induced metric on the brane, the extrinsic curvature tensor and its covariant derivatives as well as invariant/covariant measures on the brane. This allows to reconstruct the Special Galileon action, and more importantly simplifies considerably the classification of the higher order counterterms. We have performed such classification up to the invariants of the schematic form $\partial^{2n+2} \phi^n$ ($n$ arbitrary).

As a byproduct we have established a close relations between such higher order invariants and the Lagrangians invariant with respect to the quadratic shift symmetry. We have found, that for each higher order invariant, which in general consist of infinite tower of terms, the sum of vertices with minimal (and next to minimal) number of fields is automatically quadratic shift invariant. Moreover, the invariants constructed as the Lovelock terms, schematically $R^n$ in $D > 2n$ dimensions, where $R$ stays for intrinsic curvature tensor on the brane, allow us to easily obtain the $(P, N, \Delta) = (2, 2n + 2, 3n + 2)$ polynomial shift invariant Lagrangians.

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A Invariance of IHC under special Galileon duality

Here we give a proof of the invariance of the IHC constraint under the transformation (4.8). Let us calculate the transformed form \([\omega_A]_\theta\) defined as

\[
[\omega_A]_\theta = d\phi_\theta + i\frac{\alpha}{4}(Z_\theta - \bar{Z}_\theta)^T \cdot \eta \cdot (dZ_\theta + d\bar{Z}_\theta)
\]

where

\[
Z_\theta = U(\theta) \cdot Z, \quad \bar{Z}_\theta = U(-\theta) \cdot \bar{Z}
\]

\[
\phi_\theta = \phi + i\frac{\alpha}{8}\left(Z^2 - \bar{Z}^2\right) - i\frac{\alpha}{8}\left(Z^2_{\theta} - \bar{Z}^2_{\theta}\right)
\]

(A.1)

Inserting this into \([\omega_A]_\theta\) we get

\[
[\omega_A]_\theta = d\phi + i\frac{\alpha}{4}\left(Z^T \cdot \eta \cdot dZ - \bar{Z}^T \cdot \eta \cdot d\bar{Z}\right) - i\frac{\alpha}{4}\left(Z^T_{\theta} \cdot \eta \cdot dZ_{\theta} - \bar{Z}^T_{\theta} \cdot \eta \cdot d\bar{Z}_{\theta}\right) + i\frac{\alpha}{4}\left(Z^T_{\theta} \cdot \eta \cdot dZ_{\theta} - \bar{Z}^T_{\theta} \cdot \eta \cdot d\bar{Z}_{\theta}\right)
\]

Because

\[
dZ_\theta = U(\theta) \cdot dZ, \quad d\bar{Z}_\theta = U(-\theta) \cdot d\bar{Z}
\]

and

\[
U(\theta)^T \cdot \eta \cdot U(-\theta) = U(-\theta)^T \eta \cdot U(\theta) = \eta
\]

we get

\[
Z^T_{\theta} \cdot \eta \cdot d\bar{Z}_{\theta} = Z^T \cdot \eta \cdot d\bar{Z}, \quad \bar{Z}^T_{\theta} \cdot \eta \cdot dZ_{\theta} = \bar{Z}^T \cdot \eta \cdot dZ
\]

and therefore

\[
[\omega_A]_\theta = d\phi + i\frac{\alpha}{4}(Z - \bar{Z})^T \cdot \eta \cdot (dZ + d\bar{Z}) = \omega_A.
\]

B Invariance of IHC under generalized special Galileon duality

In this appendix we demonstrate invariance of the IHC constraint under (4.49). For this purpose we rewrite (4.49) in the form

\[
Z^+_{\theta} = Z^+ = x + \frac{1}{a} L
\]

(B.1)

\[
L - L_{\theta} = \theta G\left(Z^+\right) \cdot Z^+
\]

(B.2)

\[
\phi_\theta = \phi + \frac{1}{2a} L^2 - \frac{1}{2a} L^2_{\theta} - \frac{\theta}{N} Z^+ \cdot G\left(Z^+\right) \cdot Z^+
\]

(B.3)

where we abreviated

\[
G\left(Z^+\right)^{\mu\nu} = G^{\mu\nu_1\alpha_2\ldots\alpha_{N-2}} Z^+_{\alpha_1} Z^+_{\alpha_2} \ldots Z^+_{\alpha_{N-2}}.
\]
Inserting (B.3) into

\[ [\omega_A]_\theta = d\phi_\theta - L_\theta \cdot dx_\theta \]

we get

\[ [\omega_A]_\theta = d\phi + \frac{1}{a} L \cdot dL - \frac{1}{a} L_\theta \cdot dL_\theta - \theta Z^+ \cdot G \left( Z^+ \right) \cdot dZ^+ - L_\theta \cdot dx_\theta \]

\[ = d\phi + \frac{1}{a} L \cdot dL - L_\theta \cdot dL_\theta - (L - L_\theta) \cdot dZ^+ \]

\[ = d\phi + \frac{1}{a} (L - L_\theta) \cdot dL - L_\theta \cdot dx - (L - L_\theta) \cdot dZ^+ \]

\[ = d\phi - (L - L_\theta) \cdot dx - L_\theta \cdot dx = d\phi - L \cdot dx = \omega_A \]

where we have used (B.1) and (B.2) to get the second line, (B.1) to get the third line and then twice (B.1).

C  Explicit form of the quartic term of \( \delta = 4 \) Lagrangian

In this appendix we give explicit form of the quartic term of the most general \( \delta = 4 \) Lagrangian

\[ L^4 = \frac{\chi}{2\alpha^4} \left[ 6 \langle \partial \partial \partial \partial \phi : \partial \partial \partial \partial \phi \cdot \partial \partial \partial \partial \phi \cdot \partial \partial \partial \partial \phi \rangle - 4 \langle \square \partial \phi \cdot \partial \partial \partial \partial \phi \cdot \partial \partial \partial \partial \phi \cdot \partial \partial \partial \partial \phi \rangle \right] 

-2\langle \square \partial \phi \cdot \partial \partial \partial \phi \cdot \partial \partial \partial \phi \cdot \partial \partial \partial \phi \rangle (\partial \partial \partial \partial \phi \cdot \partial \partial \partial \partial \phi \rangle]

- \frac{\rho}{2\alpha^4} (\phi)^2 \langle \partial \partial \partial \partial \phi : \partial \partial \partial \partial \phi \rangle - \frac{\kappa}{2\alpha^4} (\phi)^2 \langle \partial \partial \partial \partial \phi : \partial \partial \partial \partial \phi \rangle

+ \frac{\xi}{2\alpha^4} \left[ \langle \partial \partial \partial \partial \phi : \partial \partial \partial \partial \phi \rangle \langle \partial \partial \partial \partial \phi : \partial \partial \partial \partial \phi \rangle - 6 \langle \partial \partial \partial \partial \phi : \partial \partial \partial \partial \phi \cdot \partial \partial \partial \partial \phi \cdot \partial \partial \partial \partial \phi \rangle \right], \tag{C.1} \]

where we abbreviated

\[ \xi = B_4^{(1)} + B_4^{(2)} + \text{Re}C_4^{(1)} + \text{Re}C_4^{(2)} - \text{Im}C_4^{(3)} \]

\[ \chi = B_4^{(1)} + \text{Re}C_4^{(1)} - \text{Im}C_4^{(3)} \]

\[ \rho = \text{Re}C_4^{(1)} - \text{Im}C_4^{(3)}, \quad \kappa = \text{Re}C_4^{(2)} \tag{C.2} \]

D  Transformation of the actions with respect to \( GL(2, \mathbb{R}) \) duality

As we have discussed in the main text, not all the Galileon actions have nontrivial physical content. As we have mentioned, some of them might be transformed into free theory or to the tadpole term by means of the \( GL(2, \mathbb{R}) \) duality (3.14) mentioned in section 3. In this appendix we give a proof of such relations in more detail.

The Lagrangians under consideration can be written as

\[ \mathcal{L} = \mathcal{N} \sum_n d_n \mathcal{L}_n \tag{D.1} \]
where $N$ is a real overall constant (which may be needed e.g. in order to get a canonical normalization of the kinetic term). Under the duality transformation (3.14)

$$x_\theta = x - 2\theta \partial \phi(x), \quad \phi_\theta(x_\theta) = \phi(x) - \theta \partial \phi(x) \cdot \partial \phi(x)$$

the constants $d_n$ change according to the relation (see [21] for details)

$$d_n(\theta) = \frac{1}{n} \sum_{m=1}^{n} m \left( \frac{D - m + 1}{n - m} \right) (-2\theta)^{n-m} d_m$$

(D.3)

Paricularly we get

$$d_1(\theta) = d_1.$$  

(D.4)

For the case of the action $S(\alpha, \beta)$ (see (4.17)) we have

$$d_n(\alpha, \beta) = \frac{1}{2in} \alpha e^{i\beta} \left( \frac{D}{n - 1} \right) \left( \frac{i}{\alpha} \right)^{n-1} + h.c.$$  

Therefore inserting this in D.3() for $n > 1$ we get

$$d_n(\theta) = \frac{1}{2in} \alpha e^{i\beta} \left( \frac{D}{n - 1} \right) (-2\theta)^{n-1} \sum_{m=0}^{n-1} \left( \frac{n - 1}{m} \right) \left( \frac{-i}{2\theta \alpha} \right)^m + h.c.$$  

As a result we obtain

$$d_n(\theta) = \frac{1}{2in} \alpha e^{i\beta} \left( \frac{D}{n - 1} \right) \left( \frac{i}{\alpha - 2\theta} \right)^{n-1} + h.c.$$  

(D.5)

$$d_n(\theta) = \frac{1}{2in} \alpha e^{i(\beta-(n-1)\beta_\theta)} \left( \frac{D}{n - 1} \right) \left( \frac{i}{\alpha} \right)^{n-1} + h.c.,$$  

(D.6)

where we expressed

$$\frac{i}{\alpha - 2\theta} = \frac{i}{\alpha} e^{i\beta_\theta}$$  

(D.7)

in terms of real $\alpha_\theta > 0$ and $-3\pi/2 < \beta_\theta < \pi/2$ for finite $\theta \neq 0$. For $n$ odd we can force $d_n(\theta)$ to vanish for

$$\sin(\beta - (n - 1) \beta_\theta) = 0$$  

(D.8)

For $n$ even we can do the same provided

$$\cos(\beta - (n - 1) \beta_\theta) = 0.$$  

(D.9)

For the action $S_\pm(a, c_\pm)$ (see (4.33)) we have

$$d_n^\pm(a, c_\pm) = \frac{1}{n} e^{c_\pm} \left( \frac{D}{n - 1} \right) \left( \frac{(-1)^n}{a^{n-2}} \right)$$  

(D.10)

and repeating the above calculation with imaginary $\alpha = \pm i a$ and $\beta = c_\pm/i$ we get finally for $n > 1$

$$d_n^\pm(\theta) = \frac{1}{2n} ac_\pm \left( \frac{D}{n - 1} \right) \left( \pm \frac{1}{a} - 2\theta \right)^{n-1}.$$  

(D.11)
Therefore for $\theta = \pm 1/2a$ we can make all the coefficients with $n > 1$ vanish. The theory described with action $S_+(a, c_+)$ or $S_-(a, c_-)$ is therefore dual to the action

$$S_\pm(a, c_\pm) \stackrel{\theta=\pm 1/2a}{\rightarrow} \frac{(-1)^{D-1} D!}{2} a e^{c_\pm} \int d^D x \phi$$

with only the tadpole term.

Finally, let us assume the action $S^L$ (see (4.59)) with

$$d^L_1(a) = 0, \quad d^L_n(a) = \frac{(-1)^{D-1}}{n} \left( \frac{D}{n-2} \right) \frac{1}{a^{n-2}}. \quad (D.12)$$

Note that we can write

$$d^L_n(a) = -\frac{(-1)^{D-1}}{D} a^2 \frac{\partial}{\partial a} \left[ \frac{1}{a^n} \left( \frac{D}{n-1} \right) \frac{1}{a^{n-2}} \right]$$

and therefore using (D.11) we get for $n = 1, 2$

$$d^L_1(\theta) = 0, \quad d^L_2(\theta) = \left( \frac{-1}{2} \right)^{D-1}$$

while for $n > 2$

$$d^L_n(\theta) = -\frac{(-1)^{D-1}}{D} a^2 \frac{\partial}{\partial a} \left[ \frac{1}{a^n} \left( \frac{D}{n-1} \right) \left( \frac{1}{a} - 2\theta \right)^{n-1} \right]$$

$$= \frac{(-1)^{D-1}}{D} \frac{1}{n} \left( \frac{D}{n-1} \right) \left( \frac{1}{a} - 2\theta \right)^{n-2}.$$

The choice $\theta = 1/2a$ transforms $S^L$ to the free action with only the kinetic term. Therefore any linear combination of the actions $S_+$ and $S^L$ can be simultaneously transformed to the free theory with tadpole term.

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