Sparse Reconstruction for Radar Imaging Based on Quantum Algorithms

Xiaowen Liu®, Chen Dong, Ying Luo®, Le Kang®, Yong Liu, and Qun Zhang®

Abstract—The sparse-driven radar imaging can obtain high-resolution images about a target scene with the down-sampled data. However, the huge computational complexity of the classical sparse recovery method for the particular situation seriously affects the practicality of the sparse imaging technology. In this letter, this is the first time the quantum algorithms are applied to the image recovery for the radar sparse imaging. First, the radar sparse imaging problem is analyzed and the calculation problem to be solved by quantum algorithms is determined. Then, the corresponding quantum circuit and its parameters are designed to ensure extremely low computational complexity, and the quantum-enhanced reconstruction algorithm for sparse imaging is proposed. Finally, the computational complexity of the proposed method is analyzed, and the simulation experiments with the raw radar data are illustrated to verify the validity of the proposed method.

Index Terms—Compressive sensing (CS), quantum algorithm, radar imaging, sparse recovery.

I. INTRODUCTION

MICROWAVE imaging via synthetic aperture radar (SAR) and inverse SAR (ISAR) relying on high-resolution performance is playing a pivotal role in military and civilian applications, such as topographic mapping, marine monitoring, and target identification [1]. However, the amount of radar data needs to be collected due to the wide bandwidth signal and long coherent processing interval for high-resolution images and large-scale scene recovery, which result in a grand challenge to data acquisition and storage [2]. For addressing the issue, compressive sensing (CS) theory which is capable of recovering the sparse signals with a high probability from down-sampled data has been widely used in SAR/ISAR imaging with the sparse aperture measurements [3]. Nevertheless, the sparse-driven radar imaging approaches can improve the imaging performance, but most of them need to transmute 2-D raw data into a vector, which will lead to significant time-consuming, memory cost, and computational complexity, especially for the situation of high-resolution and large-scale scene imaging [4].

To cope with the critical issue, Bi et al. [1], [4] and Fang et al. [5] has constructed the azimuth-range decouple-based sparse SAR imaging methods which replace the observation matrix in the CS-SAR framework with approximated observations derived from the inverse of traditional matched filtering (MF) based procedures. The sparsity ISAR imaging methods with the block-based CS technique and the Kronecker CS were proposed in [6] and [7], respectively, to reduce the computational complexity and the memory cost. In addition, a fast sparsity ISAR imaging method was proposed [8], where the enhanced sparsity constraint shrinks the feasible region of the solution so as to reduce the computational cost. Moreover, the alternating direction method of multipliers (ADMM) based [9], [10] and the smoothed L0 based [11] sparsity recovery methods were proposed, respectively. In [12], the images obtained by Fourier transform, but not the echoes, as the input data are inputted into the deep neural network, and the output is the reconstructed high-resolution image. However, for the sake of low computational complexity and real-time processing, it still needs long-term exertions.

The quantum algorithms profiting from the quantum computer which depends on quantum gates and wires for manipulating the quantum states can bring remarkable exponential speedup or quadratic speedup over some classical algorithms and have been applied in computational chemistry and molecular simulation [13]. Among them the Harrow–Hassidim–Lloyd (HHL) algorithm is proposed in 2009 for solving the linear systems of equations, but there is still no literature about its practical applications and experimental implementations [14]. In order to make the HHL algorithm a useful tool for a specific problem, slightly modifying the quantum circuit design and adding the classical pre-/post-processing are necessary to preserve the exponential speedup [15].

In this letter, we are the first to apply the HHL algorithm to the sparse reconstruction problem for radar imaging. After the sparse reconstruction problem about radar imaging is analyzed, the suitable linear equation for the quantum calculation mechanism is formed. Then the quantum circuit and...
the parameters in the quantum gates are designed, and the quantum-enhanced reconstruction algorithm (QRA) for sparse imaging is proposed. Finally, the computational complexity is analyzed, and the proposed method and the classical recovery algorithms are used to acquire the sparse imaging result, respectively. The proposed method can obtain the similar algorithms are used to acquire the sparse imaging result, respectively. The proposed method and the classical recovery algorithms are categorized as convex relaxations, nonconvex optimization, and greedy algorithms.

III. QRA AND COMPUTATIONAL COMPLEXITY ANALYSIS

A. Quantum-Enhanced Reconstruction Algorithm

For the optimization problem as (4), when the target scene sparsity $K$ is roughly estimated, the optimization problem is recognized as the linear least-squares problem. Thus, solving the optimization problem (4) is equivalent to calculate the linear equation as $\Phi^H\Phi \sigma = \Phi^H Y$, where $(\cdot)^H$ is the conjugate transpose. However, the linear problem cannot be directly solved by the matrix inversion and multiplication due to the low-rank matrix $\Phi^H \Phi$. Thus, the linear equation needs to be transformed, moreover, the new form needs to be eligible for the quantum calculation mechanism to guarantee high recovery precision. The new linear equation can be formed as

$$ (\eta \Phi^H \Phi + \lambda_0 I) \hat{\sigma} = \Phi^H Y $$

where $\mathbf{I}$ is the identity matrix. The factors $\eta$ and $\lambda_0$, which can resize the eigenvalues of the coefficient matrix and ensure the recovery precision, are mainly to control the condition number $\kappa$ and the number of qubits to ensure a low computational complexity. For example, when the maximum eigenvalue of the coefficient matrix is below 8 or 16, the needed number of qubits is 3 or 4, which could be considered as a selection for low complexity. With that, when the recovery error is unacceptable, the factor $\eta$ and $\lambda_0$ should be properly adjusted to obtain a low condition number and a proper ratio of the factors. The sparse processing for $\hat{\sigma}$ with the appropriate $K$ can keep the error between the final recovery result $\hat{\sigma}$ and $\sigma$ very small. Let $\Xi = (\eta \Phi^H \Phi + \lambda_0 \mathbf{I})$ and $\mathbf{y} = \Phi^H \mathbf{Y}$, and thus (5) can be rewritten as $\Xi \hat{\sigma} = \mathbf{y}$. According to (2) and (5), we can draw the conclusion that the coefficient matrix $\Xi$ is Hermitian and has $s$ nonzero entries per row.

As mentioned above, quantum algorithms can achieve exponential speedup or quadratic speedup over some classical algorithms. Among them, the HHL algorithm is a kind of quantum algorithm for linear systems of equations usually expressed as $\mathbf{A} \mathbf{x} = \mathbf{b}$ and requires that the matrix $\mathbf{A}$ is Hermitian, $s$-sparse, and efficiently row computable. Considering the characteristic of the matrix $\Xi$, after normalizing the vector $\mathbf{y}$, the target scene can be obtained by solving the linear equation $\Xi \hat{\sigma} = \hat{\mathbf{y}}$ using the HHL algorithm, where $\hat{\mathbf{y}}$ and $\hat{\sigma}$ are the normalized vector of $\mathbf{y}$ and the product of the vector $\hat{\sigma}$ and the normalized coefficient of $\mathbf{y}$, respectively. The QRA for radar imaging can be summarized as Algorithm 1.

The quantum circuit and parameter selection for the sparse imaging problem is designed as illustrated in Fig. 1. The


Algorithm 1 QRA

**Input:** the measurement matrix $\Phi$ and the downsampled data vector $Y$.

1. Select the parameters $\lambda_0$ and $\eta$ to construct the matrix $\Xi$.
2. Design the quantum circuit for $\Xi \vec{y}$ according to $\Xi$ and $\vec{y}$, and set the parameters of the quantum gates in the circuit.
3. Prepare the initial state $|b\rangle = \sum_{i=0}^{n-1} \hat{\gamma}_{i+1} |i\rangle$ in the Input register $I$.
4. Perform the designed quantum circuit to obtain the outcome $|x\rangle = \hat{\sigma}$.
5. Extract the $K$ largest elements in $\hat{\sigma}$ to form the recovery result $\hat{\sigma}$.

quantum circuit contains five different kinds of registers, i.e., Ancilla register $S$, register $A$, register $B$, register $C$, and Input register $I$, and can be separated into three stages: phase estimation, controlled rotation, and uncomputation. Quite apart from register $I$, the initial quantum state of other registers is $|0\rangle^{\otimes n}$, and $n_r$ which represents the number of qubits in the registers may be different for the different registers. The initial state of register $I$ need to be prepared as the unit vector $\hat{\sigma}$, i.e., the initial state $|b\rangle = \sum_{i=0}^{n-1} \hat{\gamma}_{i+1} |i\rangle$, where $\hat{\gamma}_{i+1}$ is the $(i+1)$th element in $\vec{y}$. The initial state $|i\rangle$ is the basis state of register $I$, and $N_r$ is the dimension of $\vec{y}$. Thus, the number of qubits in the register $I$ is $\lceil \log_2 N_r \rceil$, where $\lceil \cdot \rceil$ is the rounding up operation, so that the theoretical computational complexity of the recovery algorithm for the sparse imaging problem may potentially and obviously be reduced, especially when the dimension of $\vec{y}$ is extremely large.

In the stage of well-known quantum phase estimation, the first quantum gate, i.e., Hadamard gate, takes the state $|0\rangle^{\otimes n_r}$ to the superposition state $\sum_{i=0}^{2^{n_r}-1} |i\rangle/(2^{n_r})^{1/2}$, where $n_r$ affecting the precision of the eigenvalues of $\Xi$ is the number of qubits in the register $C$. Applying the unitary operator $e^{2\pi i} 2^{n_r}$ to the state $|b\rangle$ is implemented using the second quantum operation achieved by Hamiltonian simulation. $n_j$ is the bit number of the binary integer value regarding the maximum eigenvalue of $\Xi$. After the former two steps, the initial state is transformed into $\sum_{i=0}^{2^{n_j}-1} \sum_{k=0}^{2^{n_r}-1} \beta_j e^{2\pi i} 2^{n_j} |k\rangle |u_j\rangle/(2^{n_r})^{1/2}$, where the superscript in the state, such as $|i\rangle^C$, indicates the register storing the state. $\lambda_j$ represents an eigenvalue of $\Xi$, and $n_j$ is the number of the eigenvalues. Meanwhile, the initial state $|b\rangle^I$ is decomposed in the eigenvector $|u_j\rangle^I$ of $\Xi$ so that the state $|b\rangle^I$ becomes the weighted sums of the eigenvectors, i.e., $|b\rangle^I = \sum_{j=0}^{n_j} \beta_j |u_j\rangle^I$. The third quantum operation in the stage is inverse quantum Fourier transform achieved by a series of controlled-$R_z$ gates can move the phase information $\lambda_j$ from the probability amplitude to the quantum bases. Therefore, the quantum state in the register $C$ and register $I$ after quantum phase estimation implements the evolution as follows:

$$|0\rangle^{\otimes n_r} |b\rangle^I \mapsto \sum_{j=1}^{n_j} \beta_j |\tilde{\lambda}_j\rangle^C |u_j\rangle^I$$

where $\tilde{\lambda}_j = 2^{(n_r-n_j)} \lambda_j$ in binary format.

In the stage of the controlled rotation, the value of the exponent $i_a$ is decided by the control bit in the register $A$ and $C$, and the value of the exponent $i_b$ is decided by the control bit in the register $A$, as shown in Fig. 1(b). The first quantum operation implemented by a series of controlled-$R_z$ gates maps the state $\sum_{p=0}^{1} \sum_{l=0}^{2^{n_r}-1} |l\rangle^A |p\rangle^B / (2^{n_r})^{1/2}$ in the register A and B to $\sum_{p=0}^{1} \sum_{l=0}^{2^{n_r}-1} e^{2\pi p} |l\rangle^A |p\rangle^B / (2^{n_a})^{1/2}$. $n_a$ is the number of qubits in the register A and its size is related to the lowest common multiple $N_{sa}$ of the scaling eigenvalues $|\tilde{\lambda}_j\rangle$, while the number of qubits in the register B is 1. The second quantum operation is to apply the controlled-$R_z$ gates to the state in the register $B$ controlled by the qubits in the register $A$ and $C$. After this step, for a certain eigenvalue, the state in the register $A$, $B$, and $C$ becomes as

$$\sum_{p=0}^{1} \sum_{j=0}^{2^{n_j}-1} e^{2\pi i} (N_{sa} - \tilde{\lambda}_j) |l\rangle^A |p\rangle^B |\tilde{\lambda}_j\rangle^C / \sqrt{2^{n_a}+1}$$

with the constraint $N_{sa} - \tilde{\lambda}_j = 0$ to make the state vanish, thus the value of state $|l\rangle^A$ is $l = N_{sa} / \tilde{\lambda}_j$. Finally, a series of $R_y$ gates controlled by the state $|l\rangle^A$ are acted on the ancilla register $S$ so that the ancilla state becomes as $\cos(l/N_{sa}) |0\rangle + \sin(l/N_{sa}) |1\rangle$. Due to the arbitrary integer $N_{sa}$, the state can be written as

$$\sum_{j=1}^{n_j} \sum_{p=0}^{2^{n_a}+1} \frac{\beta_j}{\sqrt{2^{n_a}+1}} \left( 1 - \left( \frac{N_{sa}}{\lambda_j N_{sa}} \right)^2 \right)^{1/2} |p\rangle |\tilde{\lambda}_j\rangle |u_j\rangle$$

After the final stage, i.e., uncomputation, the state in the register $A$, $B$, and $C$ is set back to the initial state $|0\rangle^{\otimes n_r}$, thus
the final state of the quantum circuit takes the form as
\[ \sum_{j=1}^{n_j} \beta_j \left( \sqrt{1 - \frac{Na_0}{\lambda_j N_{\text{sa}}}^2} |0\rangle + \frac{Na_0}{\lambda_j N_{\text{sa}}} |1\rangle \right) |0\rangle^A |0\rangle^B |\tilde{\sigma}_j\rangle. \] (9)

For (9), when we measure the ancilla state and obtain $|1\rangle$, the state in the register $I$ is $|\chi\rangle = \sum C\beta_j |u_j\rangle/\lambda_j$ which is proportional to the solution of $\mathbf{X}\tilde{\sigma} = \mathbf{y}$, where the constant $C = N_a/N_{\text{sa}}$ is the proportional constant.

B. Computational Complexity Analysis

When the range profile is obtained and the range cell migration is corrected, the measurement matrix $\Phi$ is designed based on the product of the sparse sampling matrix and inverse discrete Fourier transform (DFT) matrix, and the high-resolution image is reconstructed by sparse recovery in each range cell. Suppose the dimensionality of $\Phi$ is $M_r \times M_{\text{all}}$ and the target scene sparsity in each range cell is $K_t$, the computational complexity of the QRA for the sparse imaging problem is $O(\kappa L_t \log(M_{\text{all}})/\varepsilon)$, while that of the RD algorithm, the OMP algorithm, and the FISTA are $O(L_t M_t \log(L_t M_t))$, $O(K_t L_t M_{\text{all}} M_t)$, and $O(n_\eta L_t M_{\text{all}} M_t)$, respectively. $n_\eta$ is the number of iterations. The factor $\kappa$ is the condition number of $\mathbf{X}$, and $\varepsilon$ is the reconstruction error in the output state $|\chi\rangle$. It is obviously that the computational complexity brought by the data in the cross-range direction is reduced exponentially, thus the quantum-enhanced method is beneficial to the imaging problem with the long coherent processing interval. If the image is reconstructed by processing the data in the whole range cell, the computational complexity of the quantum method is $O(\kappa \log(L_t^2 M_{\text{all}} M_t)/\varepsilon)$. When the appropriate matrix $\mathbf{X}$ and number of qubits are selected to guarantee $\kappa$ and $\varepsilon$ to be $\text{poly} \log(M_{\text{all}})$ or $\text{poly} \log(L_t^2 M_{\text{all}} M_t)$, the QRA method can roughly achieve an exponential speedup.

IV. EXPERIMENTAL RESULTS AND ANALYSIS

In this section, the well-known raw radar data of the F-16 model with 1:8 scaling factor in microwave anechoic chamber and the Yak-42 airplane, which have been widely used in the existing literature, are processed by the proposed method to verify its reconstruction performance.

In the all-metal F-16 scaling model experiment, the central azimuth, synthetic aperture angle, and sampling interval are 180°, 5°, and 0.04°, respectively. The frequency range and the number of frequency sampling points are selected as 34.2857–37.9428 GHz and 401, respectively. To obtain the ideal complexity and precision, the factors $\eta$ and $\lambda_0$ are set as 23 and 1 to bring the appropriate values about the eigenvalues $\{\lambda_j\}$ of $\mathbf{X}$. In the designed quantum circuit, the register $A$, $C$, and $I$ contain 2, 3, and 7 qubits, respectively. The sparse imaging results of the RD algorithm, the OMP algorithm, the FISTA and the QRA are obtained, as shown in Fig. 2.

In the Yak-42 experiment, the center frequency, bandwidth, and pulse-repetition frequency (PRF) are 5.52 GHz, 400 MHz, and 400 Hz. The factors $\eta$ and $\lambda_0$ are set as 33 and 1 and the register $A$, $C$, and $I$ contain 2, 3, and 8 qubits, respectively.

The sparse imaging results are acquired by the classical and quantum-enhanced method, as shown in Fig. 3. From Figs. 2 and 3, the sparse imaging results obtained by the quantum-enhanced method are similar to that of the classical method.

The entropy of the image (ENT), the peak signal-to-noise ratio (PSNR), the target-to-clutter ratio (TCR), and the computational complexity are calculated to evaluate the quantitative performance of the four algorithms are given, as shown in Table I. Among them, by comparing with the reconstructed image without noise, the PSNR of the reconstructed image under the 15 dB SNR is obtained. The TCR is based on the comparison of the target region and clutter region in the
reconstructed image, where the target region and clutter region are determined by performing a binarization processing on the RD image [12]. It is observed that the proposed QRA can dramatically reduce the computational complexity and has a similar or even slightly better image reconstruction performance compared with the classical recovery algorithm.

V. CONCLUSION

In this letter, a quantum-enhanced sparse reconstruction method for radar imaging is proposed. After analyzing the sparse imaging problem and determining the mathematical problem to be solved, the algorithm flow of QRA and the quantum circuit with the appropriate parameters is presented to ensure a low system complexity. The radar data collected from microwave anechoic chamber and the real airplane echo data are processed to verify the performance of the proposed method. Comparing with the classical algorithm, the similar or even slightly better image reconstruction performance are acquired by the proposed method. By theoretically analyzing on the computational complexity, the conclusion that the quantum-enhanced sparse reconstruction method achieves an approximate exponential speedup is verified.

REFERENCES

[1] H. Bi, G. Bi, B. Zhang, and W. Hong, “Complex-image-based sparse SAR imaging and its equivalence,” *IEEE Trans. Geosci. Remote Sens.*, vol. 56, no. 9, pp. 5006–5014, Sep. 2018.
[2] G. Xu, L. Yang, G. Bi, and M. Xing, “Enhanced ISAR imaging and motion estimation with parametric and dynamic sparse Bayesian learning,” *IEEE Trans. Comput. Imag.*, vol. 3, no. 8, pp. 940–952, Dec. 2017.
[3] M. Çetin et al., “Sparsity-driven synthetic aperture radar imaging: Reconstruction, autofocus, moving targets, and compressed sensing,” *IEEE Signal Process. Mag.*, vol. 31, no. 4, pp. 27–40, Jul. 2014.
[4] H. Bi, G. Bi, B. Zhang, W. Hong, and Y. Wu, “From theory to application: Real-time sparse SAR imaging,” *IEEE Trans. Geosci. Remote Sens.*, vol. 58, no. 4, pp. 2928–2936, Apr. 2020.
[5] J. Fang, Z. B. Xu, B. C. Zhang, W. Hong, and Y. R. Wu, “Fast compressed sensing SAR imaging based on approximated observation,” *IEEE J. Sel. Topics Appl. Earth Observ. Remote Sens.*, vol. 7, no. 1, pp. 354–363, Jan. 2014.
[6] J. Fang, L. Zhang, and H. Li, “Two-dimensional pattern-coupled sparse Bayesian learning via generalized approximate message passing,” *IEEE Trans. Image Process.*, vol. 25, no. 6, pp. 2920–2930, Jun. 2016.
[7] W. Qiu, J. Zhou, and Q. Fu, “Tensor representation for three-dimensional radar target imaging with sparsely sampled data,” *IEEE Trans. Comput. Imag.*, vol. 6, pp. 263–275, Jan. 2020.
[8] G. Zhao, F. Shen, J. Lin, G. Shi, and Y. Niu, “Fast ISAR imaging based on enhanced sparse representation model,” *IEEE Trans. Antennas Propag.*, vol. 65, no. 10, pp. 5453–5461, Oct. 2017.
[9] S. Zhang, Y. Liu, and X. Li, “Fast sparse aperture ISAR autofocus and imaging via ADMM based sparse Bayesian learning,” *IEEE Trans. Image Process.*, vol. 29, pp. 3213–3226, 2020.
[10] C. Hu, Z. Li, L. Wang, J. Guo, and O. Loffeld, “Inverse synthetic aperture radar imaging using a deep ADMM network,” in *Proc. 20th Int. Radar Symp. (IRS)*, Jun. 2019, pp. 1–5.
[11] E. Giusti, D. Cataldo, A. Bacci, S. Tomei, and M. Martorella, “ISAR image resolution enhancement: Compressive sensing versus state-of-the-art super-resolution techniques,” *IEEE Trans. Aerosp. Electron. Syst.*, vol. 54, no. 4, pp. 1983–1997, Aug. 2018.
[12] C. Hu, L. Wang, Z. Li, and D. Zhu, “Inverse synthetic aperture radar imaging using a fully convolutional neural network,” *IEEE Geosci. Remote Sens. Lett.*, vol. 17, no. 7, pp. 1203–1207, Jul. 2020.
[13] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information: 10th Anniversary Edition*, Cambridge, U.K.: Cambridge Univ. Press and Tsinghua Univ. Press, 2015.
[14] A. W. Harrow, A. Hassidim, and S. Lloyd, “Quantum algorithm for linear systems of equations,” *Phys. Rev. Lett.*, vol. 103, no. 15, Oct. 2009, Art. no. 150502.
[15] S. Aaronson, “Read the fine print,” *Nature Phys.*, vol. 11, no. 4, pp. 291–293, 2015.