ALGEBRAIC $K$-THEORY AND CUBICAL DESCENT

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ABSTRACT. In this note we apply Guillén-Navarro descent theorem \cite{GN02}, to define a descent variant of the algebraic $K$-theory of varieties over a field of characteristic zero, $KD(X)$, which coincides with $K(X)$ for smooth varieties. After a result of Haesemeyer, this new theory is equivalent to the homotopy algebraic $K$-theory introduced by Weibel. We also prove that there is a natural weight filtration on the groups $KH_*(X)$.

1. INTRODUCTION

F. Guillén and V. Navarro have proved in \cite{GN02} a general theorem which, in presence of resolution of singularities, permits to extend some contravariant functors defined on the category of smooth schemes to the category of all schemes. In this paper we apply this result to algebraic $K$-theory. More specifically, we consider the algebraic $K$-theory functor which to a smooth algebraic variety over a field of characteristic zero $X$ associates the spectrum of the cofibration category of perfect complexes, $K(X)$. We apply Guillén-Navarro extension criterion to prove that this functor admits an (essentially unique) extension to all algebraic varieties, $KD(X)$, which satisfies a descent property.

It is well known that algebraic $K$-theory of schemes does not satisfies descent. C. Haesmeyer has proved in \cite{H} that the homotopy algebraic $K$-theory $KH$, introduced by Weibel in \cite{W1}, satisfies descent for varieties over a field of characteristic zero. From the uniqueness of our extension $KD$ and Haesmeyer’s result it follows that, for any variety $X$ over a field of characteristic zero, the spectra $KD(X)$ and $KH(X)$ are weakly equivalent.

Following \cite{GN02} we find also an extension of $K$ to a functor with compact support, $K^c$, which once again by uniqueness is weakly equivalent to the algebraic $K$-theory with compact support introduced by Gillet and Soulé in \cite{GS}.

Moreover, by using the extension theorem in analogy of Guillén and Navarro’s paper \cite{GN03}, we are able to prove the existence of some natural filtrations on the $KD$-groups associated to an algebraic variety. In fact, the $KD$-theory of an algebraic variety $X$ is defined by cubical descent and therefore, if $X_\bullet$ is a cubical hyperresolution of $X$ (see \cite{GNPP}), there is a convergent spectral sequence, see proposition 4.4.3

$$E_1^{pq} = \bigoplus_{|\alpha|=p+1} K_q(X_\alpha) \Rightarrow KD_{q-p}(X),$$

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where we have written $KD_*(X) = \pi_*(K\mathcal{D})$. We prove that the associated filtration on $KD_*(X)$ is independent of the chosen hyperresolution $X_\bullet$ of $X$. In the analogous situation for compactly supported algebraic $K$-theory we recover the weight filtration introduced in [GS]. We observe that Cortiñas, Haesemeyer and Weibel have analyzed in [CHW] the fiber of the morphism $K \to KH$ in terms of the negative cyclic homology functor.

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2. The descent theorem of Guillén-Navarro

In this section we recall the main extension theorem proved by Guillén and Navarro and present some corollaries of its proof not explicitly stated in [GN02]. We also fix some notations.

2.1. Descent categories. The descent theorem in [GN02] is stated for functors from the category of smooth varieties to a cohomological descent category. This kind of category is a (higher) variation of the classical triangulated categories. We recall the main features of descent categories and refer to [GN02], (1.5.3) and (1.7), for the precise definitions (see also the proof of proposition 3.2.6).

2.1.1. For any finite set $S$, the associated cubical set $\square_S$ is the ordered set of non-empty subsets of $S$ and the augmented cubical set $\square_S^+$ is the ordered set of subsets of $S$, including the empty set. When $S = \{0, 1, \ldots, n\}$, we simply write $\square_n$ (respectively, $\square_n^+$), which may be identified with the ordered set of $n + 1$-tuples $(i_0, \ldots, i_n)$, where $i_k \in \{0, 1\}$ such that there is a $k$ with $i_k \neq 0$, and including the $(0, \ldots, 0)$ tuple in the augmented case. We will write $|\alpha| = \sum_0^n i_k$.

As usual, we will denote by the same symbol the associated category. Following [GN02], we denote by $\Pi$ the category whose objects are finite products of categories $\square_S$ and whose morphisms are the functors associated to injective maps in each component. The objects of $\Pi$ will be called cubical index categories. $\Pi$ is a symmetric monoidal category.

2.1.2. Let $\mathcal{D}$ be a category. Given a cubical index category $\square$, a $\square$-cubical diagram of $\mathcal{D}$ is a functor $X : \square \to \mathcal{D}$. We denote by $\text{CoDiag}_\Pi \mathcal{D}$ the category of cubical diagrams of $\mathcal{D}$ (according to [GN02] we should call these functors cubical codiagrams, reserving the term diagram for the contravariant functors $X : \square^\text{op} \to \mathcal{D}$): its objects are pairs $(X, \square)$, where $X$ is a $\square$-cubical diagram; a morphism from the diagram $(X, \square)$ to the diagram $(Y, \square')$ is a functor $\delta : \square' \to \square$ together with a natural transformation $\delta^* X = X \circ \delta \Rightarrow Y$.

2.1.3. A descent category is, essentially, a triple $(\mathcal{D}, E, s)$ given by a cartesian category $\mathcal{D}$ with initial object $*$, a saturated class of morphisms $E$ of $\mathcal{D}$, called weak equivalences, and a functor

$$s : \text{CoDiag}_\Pi \mathcal{D} \to \mathcal{D},$$

called the simple functor, which satisfy the following properties:

1. Product: for any object $X$ of $\mathcal{D}$, there is a natural isomorphism $s_{\square_0}(X \times \square_0) \cong X$ and for any $\square \in \text{Ob}\Pi$ and any couple of $\square$-diagrams $(X, Y)$, the morphism

$$s_{\square}(X \times Y) \to s_{\square}X \times s_{\square}Y,$$
is an isomorphism.

2. **Factorisation**: Let $\square, \square' \in \text{Ob}\Pi$. For any $\square \times \square'$-diagram $X = (X_{\alpha\beta})$, there is an isomorphism
   \[ \mu : s_{\alpha\beta}X_{\alpha\beta} \longrightarrow s_{\alpha}s_{\beta}X_{\alpha\beta}, \]
   natural in $X$.

3. **Exactness**: Let $f : X \longrightarrow Y$ be a morphism of $\square$-diagrams, $\square \in \text{Ob}\Pi$. If for all $\alpha \in \square$ the morphism $f_{\alpha} : X_{\alpha} \longrightarrow Y_{\alpha}$ is a weak equivalence (i.e. it is in $E$), then the morphism $s_{\square}f : s_{\square}X \longrightarrow s_{\square}Y$ is a weak equivalence.

4. **Acyclicity criterium**: Let $f : X_1 \longrightarrow X_0$ be a morphism of $\mathcal{D}$. Then, $f$ is a weak equivalence if and only if the simple of the $\square_1$-diagram
   \[ \ast \longrightarrow X_0 \xleftarrow{f} X_1 \]
is acyclic, that is, it is weakly equivalent to the final object of $\mathcal{D}$.

The acyclicity criterium has to be verified also for higher cubical diagrams, [GN02]. More specifically, let $X^+$ be a $\square^+_n$-diagram in $\mathcal{D}$ and denote by $X$ the cubical diagram obtained from $X^+$ by restriction to $\square_n$. Then the acyclicity criterium takes the following form (see property (CD8)$_{op}$ of definition (1.5.3) of [GN02]):

4'. **Acyclicity criterium**: The augmentation morphism $\lambda : X_0 \longrightarrow s_{\square}X$ is a weak equivalence if and only if the canonical morphism $\ast \longrightarrow s_{\square^+}X^+$ is a weak equivalence.

We remark that the transformations $\mu$ and $\lambda$ of properties 2 and 4' are, in fact, part of the data of a descent structure.

2.1.4. The categories of complexes give the basic examples of descent categories: if $\mathcal{A}$ is an abelian category, the category of bounded below cochain complexes $C^*(\mathcal{A})$, with the class of quasi-isomorphisms as weak equivalences and the total functor of a multicomplex as simple functor, is a descent category. See [GN02] for other examples.

2.2. **Guillén-Navarro theorem.** Let $k$ be a field of characteristic zero. We denote by $\text{Sch}(k)$ the category of reduced separated schemes of finite type over $k$, simply called *algebraic varieties*, and by $\text{Sm}(k)$ the category of smooth varieties.

2.2.1. Let
   \[
   \begin{array}{ccc}
   \tilde{Y} & \xrightarrow{j} & \tilde{X} \\
   \downarrow{g} & & \downarrow{f} \\
   Y & \xrightarrow{i} & X
   \end{array}
   \]
be a cartesian diagram of schemes, which we may consider as a $\square^+_1$-diagram. We say that it is an *acyclic square* if $i$ is a closed immersion, $f$ is a proper morphism and the induced morphism $\tilde{X} \setminus \tilde{Y} \longrightarrow X \setminus Y$ is an isomorphism.
We say that an acyclic square is an \textit{elementary acyclic square} if all schemes in the diagram are irreducible and smooth, and \( f \) is the blow-up of \( X \) along \( Y \).

\textbf{Theorem 2.2.1.} ([\text{GN02}, (2.1.5)]) Let \( \mathcal{D} \) be a cohomological descent category and
\[ G : \text{Sm}(k) \rightarrow \text{Ho}\mathcal{D} \]
a contravariant \( \Phi \)-rectified functor satisfying the following conditions:

\begin{itemize}
  \item[(F1)] \( G(\emptyset) = 0 \), and the canonical morphism \( G(X \sqcup Y) \rightarrow G(X) \times G(Y) \) is an isomorphism,
  \item[(F2)] if \( X_\bullet \) is an elementary acyclic square in \( \text{Sm}(k) \), then \( sG(X_\bullet) \) is acyclic.
\end{itemize}

Then there is an extension of \( G \) to a \( \Phi \)-rectified functor
\[ GD : \text{Sch}(k) \rightarrow \text{Ho}\mathcal{D} \]
which satisfies the descent condition
\[ (D) \text{ if } X_\bullet \text{ is an acyclic square in } \text{Sch}(k), \text{ then } sGD(X_\bullet) \text{ is acyclic.} \]

Moreover, this extension is essentially unique: if \( G' \) is another extension of \( G \) verifying the descent property (D), then there is a uniquely determined isomorphism of \( \Phi \)-rectified functors \( GD \Rightarrow G' \).

We will say that the functor \( GD \) has been obtained from \( G \) by cubical descent.

The proof of Guillén-Navarro’s theorem gives more than stated above. In fact, if \( X \) is an algebraic variety and \( X_\bullet \rightarrow X \) is any cubical hyperresolution, see [\text{GNPP}], it is proved in [\text{GN02}] that, under the hypothesis of the theorem,
\[ GD(X) = sG(X_\bullet), \]
gives a well defined functor from \( \text{Sch}(k) \) to \( \text{Ho}\mathcal{D} \), independent of the chosen hyperresolution \( X_\bullet \). From this explicit presentation we deduce easily some more properties of the descent extension \( GD \).

\textbf{Proposition 2.2.2.} Suppose that the functor \( G \) in theorem 2.2.1 is already defined for all varieties, that is, we have \( G : \text{Sch}(k) \rightarrow \text{Ho}\mathcal{D} \), and satisfies (F1) and (F2). Then there is a natural transformation of \( \Phi \)-rectified functors \( G \Rightarrow GD \).

\textit{Proof.} Let \( X \) be a variety and \( X_\bullet \) a cubical hyperresolution of \( X \), indexed by a cubical set \( \square \). Taking the simple of the morphism of cubical diagrams \( G(X \times \square) \rightarrow G(X_\bullet) \) we get the morphism \( G(X) = sG(X \times \square) \rightarrow sG(X_\bullet) = GD(X) \).

Looking at the construction and properties of cubical hyperresolutions, it may be proved that the extended functor \( GD \) inherits many properties of the functor \( G \) over the smooth varieties. As an example, and in view of their interest in algebraic \( K \)-theory, let us remark the two properties inclosed in the following proposition.

\textbf{Proposition 2.2.3.} Consider the hypothesis of theorem 2.2.1.
1. Suppose that \( G \) is homotopy invariant, i.e. for any smooth variety \( X \) the projection \( X \times \mathbb{A}^1 \to X \) induces an isomorphism \( G(X) \cong G(X \times \mathbb{A}^1) \). Then \( GD \) is homotopy invariant: for any variety \( X \), there is an isomorphism \( GD(X) \cong GD(X \times \mathbb{A}^1) \).

2. Suppose that \( G \) satisfies the Mayer-Vietoris, i.e. for any smooth variety \( X \) and any open decomposition \( X = U \cup V \) the square, induced by inclusions,

\[
\begin{array}{ccc}
G(X) & \longrightarrow & G(U) \\
\downarrow & & \downarrow \\
G(V) & \longrightarrow & G(U \cap V)
\end{array}
\]

is acyclic in \( D \). Then \( GD \) satisfies Mayer-Vietoris for all varieties.

Proof. Given \( X \) an algebraic variety we fix \( X_{\bullet} \), a cubical hyperresolution of \( X \).

1. By the definition of \( GD \) and the homotopy invariance of \( G \) we have a sequence of weak equivalences

\[
GD(X) \cong sG(X_{\bullet}) \cong sG(X_{\bullet} \times \mathbb{A}^1) \cong GD(X \times \mathbb{A}^1),
\]

so the proof follows.

2. By the definition of cubical hyperresolutions (see \([\text{GNPP}]\)), the restrictions of \( X_{\bullet} \) to \( U, V \) and \( U \cap V \) give hyperresolutions of these varieties. Let’s denote by \( U_{\bullet}, V_{\bullet} \) and \( (U \cap V)_{\bullet} \), respectively, these restrictions. By construction, for any index \( \alpha \) we have an open decomposition \( X_{\alpha} = U_{\alpha} \cup V_{\alpha} \) with \( U_{\alpha} \cap V_{\alpha} = (U \cap V)_{\alpha} \), so from the Mayer-Vietoris property for \( G \) on the category of smooth schemes, we deduce that the morphisms

\[
G(X_{\alpha}) \to s \begin{pmatrix}
G(U_{\alpha}) \\
\downarrow \\
G(V_{\alpha}) \to G((U \cap V)_{\alpha})
\end{pmatrix}
\]

are weak equivalences for any \( \alpha \). By the exactness property of descent categories, we have that

\[
sG(X_{\bullet}) \to s_{\alpha}s \begin{pmatrix}
G(U_{\alpha}) \\
\downarrow \\
G(V_{\alpha}) \to G((U \cap V)_{\alpha})
\end{pmatrix}
\]

is also a weak equivalence. But, by the factorization axiom of descent categories, the simple on the right is weak equivalent to

\[
s \begin{pmatrix}
sG(U_{\bullet}) \\
\downarrow \\
sG(V_{\bullet}) \to sG((U \cap V)_{\bullet})
\end{pmatrix}
\]

So, taking into account the definition of \( GD \) we finally deduce that the morphism \( GD(X) \to s(GD(U) \leftarrow GD(U \cap V) \to GD(V)) \) is a weak equivalence, hence the Mayer-Vietoris property for open sets follows. \( \square \)
2.3. Extension with compact support. In [GN02], the authors present some variations on the main theorem. In particular, they prove in [GN02], (2.2.2), that with the same hypothesis of theorem 2.2.1 there is an extension $G^c$ of $G$ with compact support: if $\text{Sch}_c(k)$ denotes the category of varieties and proper morphisms, there is an extension of $G$ to a $\Phi$-rectified functor

$$G^c : \text{Sch}_c(k) \longrightarrow \text{HoD}$$

which satisfies the descent property (D) and, moreover,

$$(D_c) \text{ if } Y \text{ is a subvariety of } X, \text{ then there is a natural isomorphism } G^c(X - Y) \cong s_{|c^0}|(G^c(X) \longrightarrow G^c(Y)).$$

3. The descent category of Spectra

In this section we prove that the category of $\Omega$-spectra, with the homotopy limit as a simple functor, is a (cohomological) descent category in the sense of [GN02].

3.1. Fibrant spectra. We will work in the category of fibrant spectra of simplicial sets. Our main references will be the paper by Bousfield-Friedlander [BF] and section 5 of Thomason’s [T80].

Recall that a prespectrum is a sequence of pointed simplicial sets $X_n, n \geq 0$, together with structure maps $\Sigma X_n \longrightarrow X_{n+1}$, where for a pointed simplicial set $K$, $\Sigma K = S^1 \wedge K$. A prespectrum $X$ is a fibrant spectrum, also called $\Omega$-spectrum, if each $X_n$ is a fibrant simplicial set and the maps $X_n \longrightarrow \Omega X_{n+1}$, obtained by adjunction of the structure maps, are weak equivalences. Morphisms between preespectra and between fibrant spectra are defined as maps in each degree that commute with the structure maps. We denote by $\text{PreSp}$ and $\text{Sp}$ the categories of prespectra and fibrant spectra, respectively.

The homotopy groups of a prespectrum $X$ are defined by the direct limit

$$\pi_k(X) = \lim_{\longrightarrow} \pi_{k+n}(X_n), \quad k \in \mathbb{Z},$$

so that if $X$ is a fibrant spectrum, $\pi_k(X) = \pi_{k+n}(X_n)$ for $k + n \geq 0$, and, more specifically, for $k \geq 0, \pi_k(X) = \pi_k(X_0)$. A map $f : X \longrightarrow Y$ of prespectra is a weak equivalence if it induces an isomorphism on homotopy groups. In this way, a map of fibrant spectra is a weak equivalence if and only if it induces weak equivalences in each degree.

3.2. Homotopy limit. Let $X$ be a functor from an index category $I$ to $\text{Sp}$. The homotopy limit spaces $\text{holim} X_n, n \geq 0$, in the sense of Bousfield-Kan, [BK], chapter XI, define a fibrant spectrum, $\text{holim} X$, see [T80], 5.6. In fact, one can see that $\text{PreSp}$ has a structure of simplicial closed model category, see [S], so that we can apply the general theory of homotopy limits for theses categories, [H].

The main properties we need of homotopy limits between fibrant spectra are:

(i) Functoriality and exactness on fibrant spectra: Let $f : X \longrightarrow Y$ be a morphism of $I$-diagrams spectra. Then, there is a natural morphism $\text{holim} f : \text{holim} X \longrightarrow \text{holim} Y$. 
If for each $\alpha \in I$ the morphism $f_\alpha : X_\alpha \rightarrow Y_\alpha$ is a weak equivalence, then $\text{holim} f$ is a weak equivalence.

(ii) Functoriality on the index category and cofinality theorem: Given a functor $\delta : I \rightarrow J$ and a diagram $X : J \rightarrow \text{Sp}$, there is a natural map $\text{holim}_J X \rightarrow \text{holim}_I \delta^* X,$ where $\delta^* X = X \circ \delta$. If $\delta$ is left cofinal, this morphism is a weak equivalence.

(iii) For any diagram $X : I \rightarrow \text{Sp}$, there is a natural map $\text{lim} X \rightarrow \text{holim} X$.

3.2.1. For a cubical diagram of spectra $X : \Box \rightarrow \text{Sp}$ we define the simple spectrum of $X$ as the homotopy limit

$$s_{\Box}(X) = \text{holim}_{\Box} X.$$ 

For a fixed cubical category $\Box$, $s_{\Box}$ defines a functor $s_{\Box} : \text{CoDiag}_{\Box}\text{Sp} \rightarrow \text{Sp}$, and by the functoriality of the homotopy limit with respect to the index category $\Box$, we obtain a functor

$$s : \text{CoDiag}_{\Box}\text{Sp} \rightarrow \text{Sp}.$$

3.2.2. Following [GN02] (1.4.3), we extend the functor $s$ to augmented cubical diagrams by using the cone construction. For instance, if $f : X \rightarrow Y$ is a $\Box_0^+$-diagram of spectra, that is to say, a morphism, it follows from loc. cit. that

$$s_{\Box_0^+}(f) = s_{\Box_1}(X \xrightarrow{f} Y \leftarrow \ast),$$

which is weakly equivalent to the homotopy fiber of $f$.

Take an isomorphism $\Box_0^+ \cong \Box_0^+ \times \Box_{n-1}^+$. As the cone construction respects this product structure, we find

$$s_{\Box_0^+} X^+ = s_{\Box_{n-1}^+} (s_{\Box_0^+} X^+),$$

that is, by viewing $X^+$ as a morphism of two $\Box_{n-1}^+$-diagrams, $f : X_0^+ \rightarrow X_1^+$, the simple spectrum associated to $X^+$ is obtained as the simple of the $\Box_{n-1}^+$-cubical diagram which in each degree $\alpha$ has the homotopy fiber of $f_\alpha$. As a consequence, the simple spectrum $s_{\Box_n^+} X^+$ is isomorphic to the total fiber space of $X^+$ as defined by Goodwillie in [G].

3.2.3. If $X^+$ is a $\Box_0^+$-diagram and $X$ denotes its restriction to $\Box_n$, it follows from the general properties of homotopy limits outlined above that there is a natural map $X_0 \rightarrow \text{holim} X$. As a consequence of [G], 1.1.b (compare also with [P], proposition (3.3), for a similar situation), we obtain:

**Proposition 3.2.1.** Let $X^+ : \Box_n^+ \rightarrow \text{Sp}$ be an augmented cubical diagram of spectra and $X$ the cubical diagram obtained by restriction to $\Box_n$. The simple $s_{\Box_n^+} X^+$ is isomorphic to the homotopy fiber of the morphism $X_0 \rightarrow s_{\Box_n} X = \text{holim} X$.

Denote by $\ast$ the initial object of $\text{Sp}$. The following corollary relates the simple of a cubical diagram with the simple of an augmented diagram.

**Corollary 3.2.2.** Let $X : \Box_n \rightarrow \text{Sp}$ be a cubical diagram of spectra and let $\tilde{X}$ the augmented cubical diagram obtained from $X$ by adding $X_0 = \ast$. Then,

$$s_{\Box_n} \tilde{X} = \Omega s_{\Box_n} X.$$
We also deduce the following result, which will be used later:

**Corollary 3.2.3.** Let $X_\bullet$ be a $\square_n$-diagram of spectra. Then, there is a convergent spectral sequence

$$E_1^{pq} = \bigoplus_{|\alpha|=p+1} \pi_q(X_\alpha) \implies \pi_{q-p}(s_{\square_n}X_\bullet).$$

*Proof.* Consider the cubical diagrams $F^pX_\bullet$ defined by

$$(F^pX_\bullet)_\alpha = \begin{cases} X_\alpha, & \text{if } |\alpha| \leq p + 1, \\ \ast, & \text{if } |\alpha| > p + 1. \end{cases}$$

Observe that $F^{-1}X_\bullet$ is the constant diagram defined by $\ast$ and that $F^nX_\bullet = X_\bullet$. We obtain a sequence of cubical diagrams

$$F^nX_\bullet \rightarrow F^{n-1}X_\bullet \rightarrow \ldots \rightarrow F^0X_\bullet \rightarrow \ast$$

which is a degreewise sequence of fibrations of spectra. Hence, taking homotopy limits there is a sequence of fibrations

$$s_{\square}(F^nX_\bullet) \rightarrow s_{\square}(F^{n-1}X_\bullet) \rightarrow \ldots \rightarrow s_{\square}(F^0X_\bullet) \rightarrow \ast.$$ 

The Bousfield-Kan spectral sequence associated to the tower of fibrations obtained by adjoining identities from the left converges to the homotopy of $s_{\square}(F^nX_\bullet) = s_{\square}X_\bullet$. The $E_1$ terms are

$$E_1^{pq} = \pi_{q-p}(sGr^pX_\bullet),$$

where $Gr^pX_\bullet$ is the $\square$-diagram obtained degree wise as the fibers of the morphism $s_{\square}(F^pX_\bullet) \rightarrow s_{\square}(F^{p-1}X_\bullet)$. But, reasoning as in the proof of proposition (3.3) of [P], for these diagrams we have

$$s_{\square_n}Gr^pX_\bullet = \prod_{|\alpha|=p+1} \Omega^pX_\alpha,$$

hence it follows that

$$E_1^{pq} = \pi_{q-p}(\prod_{|\alpha|=p+1} \Omega^pX_\alpha) = \bigoplus_{|\alpha|=p+1} \pi_q(X_\alpha).$$

Convergence is a consequence of lemma 5.48 of [T80].

3.2.4. We say that an augmented cubical diagram of spectra $X^+$ is *acyclic* if the canonical morphism $\ast \rightarrow s_{\square^+}X^+$ is a weak equivalence. The acyclic diagrams are also called *homotopy cartesian* diagrams, see [G] and [W1]. From proposition 3.2.1 it follows immediately (see also [W1], proposition 1.1):

**Corollary 3.2.4.** Let $X^+: \square_n^+ \rightarrow \text{Sp}$ be an augmented cubical diagram of spectra and $X$ the cubical diagram obtained by restriction to $\square_n$. Then $X^+$ is acyclic if and only if the natural morphism $X_0 \rightarrow \text{holim}X$ is a weak equivalence. 

**Remark 3.2.5.** Observe that for $n = 2$ this result reduces to the well known fact that a square of fibrant spectra

$$\begin{array}{ccc}
X & \longrightarrow & Y \\
\downarrow & & \downarrow \\
X' & \longrightarrow & Y'
\end{array}$$
is acyclic (or homotopy cartesian) if and only if the natural map from \( X \) to the homotopy limit of \( X' \xrightarrow{k} Y' \xleftarrow{h} Y \) is a weak equivalence.

3.2.5. After the remarks above, on \( \text{Sp} \) we have a class of weak equivalences and a simple functor \( s : \text{CoDiag}_\text{Sp} \rightarrow \text{Sp} \). According to [GN02], définition (1.7.1), to have a (cohomological) descent category on \( \text{Sp} \) we also need the following data:

(i) a natural transformation \( \mu : s\Box o s\Box \Rightarrow s\Box \times s\Box \),

(ii) a natural transformation \( \lambda : \text{id}_\text{Sp} \Rightarrow s\Box o i\Box \),

in such a way that \( (s, \mu, \lambda) : \Pi \rightarrow \text{CoReal}_\Pi \text{Sp} \) defines a comonoidal quasi-strict functor, see loc.cit.

As the homotopy limit is the end of a functor, by Fubini theorem (see [M]) there is a natural transformation \( \mu : s\Box o s\Box \Rightarrow s\Box \times s\Box \),

such that for any diagram \( X \), \( \mu_X \) is an isomorphism. As for \( \lambda \), recall that \( s\Box (X \times \Box) \) is the function space from the classifying space of the index category \( \Box \) to \( X \), so one defines

\[ \lambda_{\Box}(X) : X \rightarrow s\Box (X \times \Box), \]

by constant functions.

**Proposition 3.2.6.** The category of fibrant spectra \( \text{Sp} \) with weak homotopy equivalences as weak equivalences and the homotopy limit \( \text{holim} \) as simple functor for cubic diagrams, and the natural transformations \( \mu, \lambda \) defined above, is a cohomological descent category.

**Proof.** The actual definition of cohomological descent category consist of 8 axioms, which are dual to the axioms (CD1)-(CD8) of [GN02], définition (1.5.3), (see also their (1.7)). Much of them are immediate from the definitions and the properties of homotopy limits, so we comment the four axioms summarized in section 2.1.3 (see also [R] for an extension of this result to stable simplicial model categories).

It is clear from the definitions that \( \text{Sp} \) is a cartesian category with initial object \( * \).

1. **Product:** since the homotopy limit is an end, it is compatible with products, so for any \( \Box \)-diagrams \( X, Y \) of \( \text{Sp} \) there is a natural isomorphism

\[ s\Box (X \times Y) \cong s\Box (X) \times s\Box (Y). \]

2. **Factorisation:** also because of the Fubini theorem for ends, if \( X \) is a \( \Box \times \Box' \)-diagram, there are natural isomorphism

\[ s\Box s\Box X_{\alpha \beta} \cong s\Box \times s\Box X_{\alpha \beta} \cong s\Box s\Box X_{\alpha \beta}, \]

see [T80], lemma 5.7.

3. **Exactness:** If \( f : X \rightarrow Y \) is a morphism of \( \Box \)-diagrams in \( \text{Sp} \) such that for any \( \alpha \in \Box \) the morphism \( f_\alpha \) is a weak equivalence, then \( s\Box f : s\Box X \rightarrow s\Box Y \) is a weak equivalence, since the homotopy limit preserves weak equivalences between fibrant spectra, see [T80], 5.5. Observe that this property is not true for prespectra.

4' **Acyclicitiy criterium:** this is exactly the result of corollary 3.2.4
4. Descent algebraic $K$-theory

4.1. Let $X$ be a noetherian separated scheme, we denote by $\mathcal{K}(X)$ the $K$-spectrum associated to the category of perfect complexes on $X$, see [TT], definition 3.1. It defines a contravariant functor from the category of noetherian separated schemes to the category of spectra $\text{Sp}$ ([TT], 3.14). Moreover, it is a covariant functor for perfect projective maps and for proper flat morphisms, ([TT], 3.16).

**Theorem 4.1.1.** Let $k$ be a field of characteristic zero. The functor

$$K : \text{Sm}(k) \rightarrow \text{HoSp},$$

admits a unique extension, up to unique isomorphism of $\Phi$-rectified functors, to a functor

$$KD : \text{Sch}(k) \rightarrow \text{HoSp}$$

such that satisfies the descent property (D):

(D) if $X_*$ is an acyclic square in $\text{Sch}(k)$, $sKD(X_*)$ is acyclic.

**Proof.** By proposition 3.2.6 we know that $\text{Sp}$ is a descent category. So, in order to apply Guillén-Navarro descent theorem 2.2.1 we have to verify properties $(F1), (F2)$. The first one is immediate, while $(F2)$ follows from Thomason’s calculation in [T93] of the algebraic $K$-theory of a blow up along a regularly immersed subscheme, as has been observed by many authors (see, for example, in [H], [GS] and [CHSW]).

In the context of cubical spectra we propose the following presentation of property $(F2)$. Consider an elementary acyclic square as in 2.2.1 and the square of spectra obtained by application of the algebraic $K$ functor

$$\begin{array}{ccc}
\mathcal{K}(X) & \xrightarrow{i^*} & \mathcal{K}(Y) \\
\downarrow f^* & & \downarrow g^* \\
\mathcal{K}(\tilde{X}) & \xrightarrow{j^*} & \mathcal{K}(\tilde{Y})
\end{array}$$

We have to prove that this square is an acyclic square of spectra. If $N$ is the conormal bundle of $Y$ in $X$, then $\tilde{Y} = \mathbb{P}(N)$, so the morphism

$$\Psi : \prod_{i=0}^{d-1} \mathcal{K}(Y) \rightarrow \mathcal{K}(\tilde{Y}),$$

induced by the functor which is defined on a sequence of perfect complexes by

$$(E_0, \ldots, E_{d-1}) \mapsto \bigoplus_{i=0}^{d-1} O_{\mathbb{P}(N)}(-i) \otimes Lg^*E_i,$$

is a weak equivalence, see [TT], theorem 4.1, and also [T91].

For the blown up variety $\tilde{X}$, it has been proved by Thomason, see [T93], théorème 2.1, that the morphism

$$\Phi : \mathcal{K}(X) \times \prod_{i=0}^{d-1} \mathcal{K}(Y) \rightarrow \mathcal{K}(\tilde{X})$$
which is induced by the functor on perfect complexes given by

\[(F, E_1, \ldots, E_{d-1}) \mapsto f^* F \oplus \bigoplus_{i=1}^{d-1} j_i^* (\mathcal{O}_{\mathcal{F}(N)}(-i) \otimes Lg^* E_i),\]

is also a weak equivalence.

Define \( j' : \mathcal{K}(X) \times \prod^{d-1} \mathcal{K}(Y) \to \prod^d \mathcal{K}(Y) \) componentwise by \( g^* i^* \) on the first component and the morphism given by multiplication by \( \lambda_{-1}(N) \) in the \( Y \)-components. After the self-intersection formula, \([T93], (3.1.4)\), the diagram

\[
\begin{array}{ccc}
\mathcal{K}(X) \times \prod^{d-1} \mathcal{K}(Y) & \xrightarrow{\Phi} & \mathcal{K}(\tilde{X}) \\
j' \downarrow & & \downarrow j^* \\
\prod^d \mathcal{K}(Y) & \xrightarrow{\Psi} & \mathcal{K}(\tilde{Y})
\end{array}
\]

is commutative. Since \( \Phi, \Psi \) are weak equivalences, it is an acyclic diagram.

Consider now the augmented commutative cubical diagram

\[
\begin{array}{ccc}
\mathcal{K}(X) & \xrightarrow{f^*} & \mathcal{K}(\tilde{X}) \\
j' \downarrow & & \downarrow j^* \\
\mathcal{K}(\tilde{Y}) & \xrightarrow{g^*} & \mathcal{K}(\tilde{Y})
\end{array}
\]

where the horizontal back arrows are the inclusion on the first factor.

As the right and left side squares are acyclic, it follows from the definition in \([3.2.2]\) that it is an acyclic cubical diagram. But the back square is acyclic because the two horizontal morphisms have the same cofiber, so the front square must be acyclic, which is what has to be proved. \( \square \)

For a \( k \)-variety \( X \), we will denote by \( KD_s(X) \) the homotopy groups of \( \mathcal{KD}(X) \),

\[KD_s(X) := \pi_s(\mathcal{KD}(X)).\]

The descent property (D) gives rise to exact sequences:

**Corollary 4.1.2.** Let \( X_\bullet \) an acyclic square in \( \textbf{Sch}(k) \). Then there is an exact sequence

\[
\ldots \to KD_n(X) \xrightarrow{f^* - i^*} KD_n(\tilde{X}) \oplus KD_n(Y) \xrightarrow{j^* + g^*} KD_n(\tilde{Y}) \xrightarrow{\delta} KD_{n-1}(X) \to \ldots
\]

More generally, if \( X \) is a \( k \)-variety, then \( \mathcal{KD}(X) \) is defined as the simple of the cubical diagram of spectra \( \mathcal{K}(X_\bullet) \), where \( X_\bullet \) is a cubical hyperresolution, so from proposition \([3.2.3]\) we deduce:
Proposition 4.1.3. Let \( k \) be a field of characteristic zero and \( X \) be an algebraic \( k \)-variety. Let \( X_{\bullet} \) be a cubical hyperresolution of \( X \). Then, there is a convergent spectral sequence

\[
E_1^{pq} = \bigoplus_{|\alpha|=p+1} K_q(X_{\alpha}) \implies KD_{q-p}(X).
\]

If \( X \) is of dimension \( d \), we can take cubical hyperresolutions of size \( d \) (see [GNPP], I.2.15), so it follows:

Corollary 4.1.4. Let \( k \) be a field of characteristic zero and \( X \) be an algebraic \( k \)-variety of dimension \( d \). Then,

\[
KD_n(X) = 0, \quad n < -d.
\]

4.2. Some properties of \( KD \). As explained in section 1, \( KD \) inherits many properties of the algebraic \( K \)-theory of smooth schemes. For example, from proposition 2.2.3 and the properties of homotopy invariance and Mayer-Vietoris for the \( K \)-theory of smooth schemes, (see [Q]), we deduce immediately:

Proposition 4.2.1. The descent \( KD \)-theory satisfies:

1. \( KD \) is homotopy invariant, that is, for any variety \( X \) the projection \( X \times \mathbb{A}^1 \longrightarrow X \) induces a weak equivalence \( KD(X) \cong KD(X \times \mathbb{A}^1) \).
2. \( KD \) has the Mayer-Vietoris property, that is, if \( X = U \cup V \), with \( U, V \) open sets, then the square

\[
\begin{array}{ccc}
KD(X) & \longrightarrow & KD(U) \\
\downarrow & & \downarrow \\
KD(V) & \longrightarrow & KD(U \cap V)
\end{array}
\]

is homotopy cartesian. \( \square \)

One may prove in a similar manner that \( KD \) satisfies the fundamental Bass theorem. Also following [T80], [W1] one can prove the existence of a Brown-Gersten type spectral sequence:

\[
E_2^{pq} = H^p(X, \widetilde{KD}_{-q}) \Rightarrow KD_{-p-q}(X).
\]

where \( \widetilde{KD}_* \) stands for the sheaf in the Zariski topology associated to the presheaf \( KD_* \).

4.3. Equivalence with homotopy algebraic \( K \)-theory. In [H], theorem 3.5, Haesemeyer has proved that the homotopy algebraic \( K \)-theory \( KH \) of an algebraic variety \( X \) defined by Weibel in [W1] (see also [TT]), satisfies the descent axiom (D). As the \( KH \)-theory coincides with \( K \)-theory for smooth varieties, we can apply the uniqueness property of the extension theorem 2.2.1 to obtain:

Corollary 4.3.1. Let \( X \) be an algebraic variety over a field of characteristic zero. There is a natural morphism \( KD(X) \longrightarrow KH(X) \), in \( \text{HoSp} \), which is a weak equivalence. \( \square \)

This may also be stated as a uniqueness result for \( KH \)-theory:
Corollary 4.3.2. Let $k$ be a field of characteristic zero. The homotopy algebraic $K$-theory $KH$ is the unique ($\Phi$-rectifiable) functor $\text{Sch}(k) \to \text{HoSp}$, up to equivalence, which satisfies the descent property $(D)$ and is equivalent to the algebraic $K$-functor $K$ over smooth algebraic varieties. □

4.4. Algebraic $K$-theory with compact support. We can apply the same arguments of the proof of theorem 4.1.1 jointly with the compact support extension theorem in [GN02] to extend the algebraic $K$-theory of smooth projective varieties over a field of characteristic zero to a theory with compact support:

Theorem 4.4.1. Let $k$ be a field of characteristic zero and $V(k)$ be the category of smooth projective $k$-varieties. The rectified contravariant functor

$$K : V(k) \to \text{HoSp}$$

admits a unique extension, up to unique isomorphism of $\Phi$-rectified functors, to a functor

$$K^c : \text{Sch}_c(k) \to \text{HoSp}$$

such that satisfies the descent property $(D)$ and the compact support descent property:

$$(D_c)$$ if $Y$ is a subvariety of $X$, then there is a natural isomorphism

$$K^c(X \setminus Y) \cong \text{holim}(K^c(X) \to K^c(Y) \leftarrow *) .$$

In other words, property $(D_c)$ says that the sequence

$$K^c(X \setminus Y) \to K^c(X) \to K^c(Y),$$

is a fibration sequence in $\text{HoSp}$, so that taking homotopy groups it gives rise to a long exact sequence

$$\ldots \to K^c_n(X \setminus Y) \to K^c_n(X) \to K^c_n(Y) \to K^c_{n-1}(X \setminus Y) \to \ldots$$

In [GS], theorem 7, Gillet-Soulé defined a $K$-theory with compact support satisfying $(D_c)$, so by the uniqueness of the compact support extension we find:

Corollary 4.4.2. Let $X$ be an algebraic variety over a field of characteristic zero. Then $K^c(X)$ is naturally isomorphic in $\text{HoSp}$ to the algebraic $K$-theory with compact support introduced by Gillet and Soulé in [GS], theorem 7. □

We will write $K^c_*(X) = \pi_*(K^c(X))$.

5. Weight filtration

In this section we prove that there are well defined filtrations on the groups $KD_*(X)$, or equivalently on $KH_*(X)$, and on the groups $K^c_*(X)$, which are trivial for $X$ smooth. In the compact support case we recover the weight filtration obtained by Gillet-Soulé, [GS].

We fix a field $k$ of characteristic zero.
5.1. Let $X$ be an algebraic variety. The spectral sequence \[4.1.3\] associated to a cubical hyper-resolution $X_\bullet$ of $X$ induces a filtration on the groups $KD_n(X)$. Our next goal is to prove that this filtration on $KD_n(X)$ is independent of the cubical hyper-resolution $X_\bullet$. We will follow section 3 of [GN03] closely, where the authors analyze the weight filtration in an abelian setting.

5.2. Towers of fibrant spectra. First, we introduce a cohomological descent structure on the category of towers of fibrations $\text{tow}(\text{Sp})$.

5.2.1. A tower of fibrations $X(-)$ is a sequence of fibrations of spectra

$$\ldots \to X(n) \to X(n-1) \to \ldots \to X(1) \to X(0) \to *$$

A morphism of towers of fibrations is a morphism of diagrams. We denote by $\text{tow}(\text{Sp})$ the category of towers of fibrations.

Defining weak equivalences of towers of fibrations and simple functors for cubical diagrams degree wise, it is immediate to prove the following result:

**Proposition 5.2.1.** The category of towers of fibrations $\text{tow}(\text{Sp})$ together with weak equivalences and simple functors for cubical diagrams defined degree wise is a descent category. \[\square\]

5.2.2. We now introduce a second descent structure on $\text{tow}(\text{Sp})$. Recall that if $X(-)$ is a tower of fibrations, there is a functorial spectral sequence

$$E^{pq}_1 = \pi_{q-p}(F(p)) \Rightarrow \pi_{q-p}(X),$$

where $F(p)$ is the fiber of the morphism $X(p) \to X(p-1)$ and $X = \lim X(p)$, see [T80], 5.43 (where convergence is understood in the sense of Bousfield-Kan).

**Definition 5.2.2.** We say that a morphism of towers $f : X(-) \to Y(-)$ is an $E_2$-weak equivalence if the morphism $E_2^{**}(f)$ induced on the $E_2$-terms of the corresponding spectral sequences is an isomorphism.

Observe that if $f_p : X(p) \to Y(p)$ is a weak equivalence, for all $p \geq 0$, then $f$ induces an isomorphism in the $E_1$ terms of the spectral sequence and hence it is also an $E_2$-weak equivalence.

5.2.3. Now we define a simple construction, $s_2 : (\square, \text{tow}(\text{Sp})) \to \text{tow}(\text{Sp})$, compatible with the $E_2$-weak equivalences: given a tower of fibrations $X(-)$ and a positive integer $n \geq 0$, we denote by $X[n](-)$ the tower of fibrations defined by

$$X[n](p) := \begin{cases} *; & 0 \leq p < n, \\ X(p-n); & p \geq n, \end{cases}$$

with the evident morphisms, so that the new tower is obtained by translating $n$ places to the left the tower $X(-)$.

**Definition 5.2.3.** Let $\square$ be a cubical category and $X_\bullet(-)$ be a $\square$-diagram of towers of fibrations. Denote by $dX(-)$ the $\square$-diagram of towers of fibrations given by

$$(dX)_a(-) = X_a[|\alpha| - 1](-),$$
with morphisms induced by $X_\bullet$. We define the $s_2$ simple of $X_\bullet(-)$ as the tower of fibrations obtained by applying homotopy limits in each cubical degree of $dX_\bullet(-)$, that is,

$$s_2(X_\bullet)(p) := s(dX_\bullet(p)) = \text{holim}_\alpha X_\alpha(p - |\alpha| + 1).$$

For example, given a $\square_1$-diagram $X_\bullet(-)$ of towers of fibrations

$$\begin{array}{ccccccc}
... & \longrightarrow & X(1) & \longrightarrow & X(0) & \longrightarrow & * \\
\downarrow & & \downarrow & & \downarrow & & \\
... & \longrightarrow & Y(1) & \longrightarrow & Y(0) & \longrightarrow & * \\
\uparrow & & \uparrow & & \uparrow & & \\
... & \longrightarrow & Z(1) & \longrightarrow & Z(0) & \longrightarrow & *
\end{array}$$

the new diagram $dX_\bullet(-)$ is the diagram

$$\begin{array}{ccccccc}
... & \longrightarrow & X(1) & \longrightarrow & X(0) & \longrightarrow & * \\
\downarrow & & \downarrow & & \downarrow & & \\
... & \longrightarrow & Y(0) & \longrightarrow & * & \longrightarrow & * \\
\uparrow & & \uparrow & & \uparrow & & \\
... & \longrightarrow & Z(1) & \longrightarrow & Z(0) & \longrightarrow & *
\end{array}$$

and it follows that its $s_2$ simple in degree $p$ corresponds to the spectrum

$$\text{holim}(X(p) \longrightarrow Y(p - 1) \leftarrow Z(p)).$$

**Lemma 5.2.4.** For any cubical diagram of towers of fibrations $X_\bullet(-)$ there is a canonical isomorphism of complexes of abelian groups

$$E_{1}^{aq}(s_2X_\bullet(-)) \longrightarrow s(\alpha \mapsto E_{1}^{aq}(X_\alpha(-))).$$

**Proof.** The notation $s(\alpha \mapsto E_{1}^{aq}(X_\alpha(-)))$ refers to the ordinary simple functor for complexes of abelian groups, also called the total complex associated to a cubical complex. The group in degree $p$ of this complex is

$$s(\alpha \mapsto E_{1}^{aq}(X_\alpha(-)))^p = s(\alpha \mapsto \pi_{q-r}F_\alpha(r)) = \bigoplus_{|\alpha|+r=p+1} \pi_{q-r}F_\alpha(r),$$

while the differential is induced by the differentials of the Bousfield-Kan spectral sequence of the tower $X_\alpha(-)$.

On the other hand, by definition, for each $p$, $s_2X_\bullet(p)$ is the ordinary simple of the cubical diagram of spectra $dX_\bullet(p)$, so the complex $E_{1}^{aq}(s_2X_\bullet(-))$ is the $E_1$-term of the Bousfield-Kan spectral sequence associated to the tower of fibrations

$$\begin{array}{ccccccc}
... & \longrightarrow & sdX_\bullet(p) & \longrightarrow & sdX_\bullet(1) & \longrightarrow & sdX_\bullet(0) & \longrightarrow & *
\end{array}$$
Denote by $F_\alpha(p)$ the fiber of the fibration $X_\alpha(p) \to X_\alpha(p-1)$. Since homotopy limits commute, the fiber of the fibration $sdX_\alpha(p) \to sdX_\alpha(p-1)$ is isomorphic to the simple spectrum associated to the cubical diagram $dF_\bullet(p)$. But, in this diagram all morphisms are constant, so $sdF_\bullet(p) = \prod_{|\alpha|+r=p+1} \Omega |\alpha| F_\alpha(p-|\alpha|+1) = \prod_{|\alpha|+r=p+1} \Omega p^{-r} F_\alpha(r)$, hence its homotopy groups are given by

$$E^{pq}_1 = \pi_{q-p}(sdF_\bullet(p)) = \bigoplus_{|\alpha|+r=p+1} \pi_{q-r}(F_\alpha(r)).$$

The differential is also induced by the differentials of the Bousfield-Kan spectral sequence of the tower $X_\alpha(-)$.

**Proposition 5.2.5.** The simple $s_2$ and the $E_2$-weak equivalences define a cohomological descent category structure on $\text{tow}(\text{Sp})$.

**Proof.** Observe that a morphism between towers of fibrations $f$ is an $E_2$-weak equivalence if and only if the morphism $E_1(f)$ of the corresponding spectral sequence is a quasi-isomorphism of complexes. If $GrC_\bullet(\mathbb{Z})$ denotes the category of graded complexes of abelian groups, the functor

$$E_1 : \text{tow}(\text{Sp}) \to GrC_\bullet(\mathbb{Z})$$

$$X(-) \mapsto E_1^*$$

commutes with direct sums and, by the previous result, it commutes with the simple $s_2$ functor, so the result follows from [GN02], (1.5.12). \qed

5.3. **An extension criterion for towers.** In the next result we write $\text{Ho}_2(\text{tow}(\text{Sp}))$ for the homotopy category obtained from $\text{tow}(\text{Sp})$ by inverting $E_2$-weak equivalences.

The following result, remarked by Navarro several years ago in the abelian context, is the key point in order to extend some functors on $\text{Sm}(k)$ with values in the category of spectra to functors defined for all varieties and taking values in $\text{Ho}_2(\text{tow}(\text{Sp}))$.

**Proposition 5.3.1.** [Compare with [GN03], proposition (3.10)]. Let $G : \text{Sm}(k) \to \text{Ho}\text{Sp}$ be a $\Phi$-rectifiable functor and denote also by

$$G : \text{Sm}(k) \to \text{Ho}_2(\text{tow}(\text{Sp}))$$

the associated constant functor. Then, $G$ satisfies property $(F2)$ if and only if for every elementary acyclic square the sequence

$$0 \to \pi_n G(X) \xrightarrow{f^*-g^*} \pi_n G(\bar{X}) \oplus \pi_n G(Y) \xrightarrow{f^*+g^*} \pi_n G(\bar{Y}) \to 0,$$

is exact.

**Proof.** The $(F2)$ property for the extended functor $G$ says that the morphism

$$E_1^{*q}(G(X)) \to E_1^{*q}s_2 G(X_\bullet)$$

is exact. \qed
is a quasi-isomorphism. Observe that we have
\[
E_1^{*,q}(G(X), \tau) = \begin{cases} \pi_q(G(X)), & p = 0, \\ 0, & p \neq 0. \end{cases}
\]
By the other hand, the $E_1$-page of the spectral sequence of $s_2^*G(X_\ast)$ reduces to the exact sequence
\[
\pi_n(G(\tilde{X}) \oplus \pi_n(G(Y)) \xrightarrow{j^* + g^*} \pi_n(G(\tilde{Y})),
\]
so the $(F2)$ property is equivalent to the the fact that the morphism of complexes of abelian groups
\[
\pi_n(G(X) \xrightarrow{f^* - i^*} \pi_n(G(\tilde{X}) \oplus \pi_n(G(Y)) \xrightarrow{j^* + g^*} \pi_n(G(\tilde{Y}))),
\]
is a quasi-isomorphism, which is precisely the condition stated in the proposition.

5.3.1. We return now to the applications to algebraic $K$-theory. The following proposition has also been proved by Gillet-Soulé directly from Thomason’s calculations, see [GS], theorem 5:

**Proposition 5.3.2.** For any elementary acyclic square of $\text{Sm}(k)$ and any $n \geq 0$, the sequence
\[
0 \longrightarrow K_n(X) \xrightarrow{f^* - i^*} K_n(\tilde{X}) \oplus K_n(Y) \xrightarrow{j^* + g^*} K_n(\tilde{Y}) \longrightarrow 0,
\]
is exact.

**Proof.** As we have recalled in the proof of theorem 4.1.1, an elementary acyclic square gives rise to a homotopy cartesian square of algebraic $K$-theory spectra, so we have an exact sequence
\[
\ldots \longrightarrow K_n(X) \xrightarrow{f^* - i^*} K_n(\tilde{X}) \oplus K_n(Y) \xrightarrow{j^* + g^*} K_n(\tilde{Y}) \xrightarrow{\delta} K_{n-1}(X) \longrightarrow \ldots
\]
But, by Thomason calculation of the algebraic $K$-theory of a blow up ([T93]), there are isomorphisms
\[
\varphi : K_n(X) \bigoplus_{i=1}^{d-1} K_n(Y) \longrightarrow K_n(\tilde{X}),
\]
\[
\psi : \bigoplus_{i=0}^{d-1} K_n(Y) \longrightarrow K_n(\tilde{Y}),
\]
given, respectively, by
\[
\varphi(x, y_1, \ldots, y_{d-1}) = f^*(x) + \bigoplus_{i=1}^{d-1} j_*(\ell^{-i} \cup g^*(y_i)),
\]
\[
\psi(y_0, y_1, \ldots, y_{d-1}) = \bigoplus_{i=0}^{d-1} j_*(\ell^{-i} \cup g^*(y_i)).
\]
With this identifications the morphism $f^*$ corresponds to the inclusion of $K_n(X)$ on the first factor of $K_n(\tilde{X})$, and so the morphism $f^* - i^*$ is injective. This splits the exact sequence above into the required short exact sequences. □
Now, by propositions 5.3.1 and 5.3.2 we can apply the extension criterion of theorem 2.2.1, so we find:

**Corollary 5.3.3.** Let \( k \) be a field of characteristic zero. The constant algebraic \( K \)-theory functor \( K : \text{Sm}(k) \to \text{Ho}_2(\text{tow}(\text{Sp})) \) admits an essentially unique extension \( KD(-) : \text{Sch}(k) \to \text{Ho}_2(\text{tow}(\text{Sp})) \) which satisfies the descent property \( (D) \). Moreover, for any variety \( X \), the tower of fibrations \( KD(-)(X) \) satisfies \( KD(n)(X) = KD(X) \) for \( n \gg 0 \).

**Proof.** We have only to justify the last sentence. Take an algebraic variety \( X \) and an hyperresolution \( X_\bullet \), whose type \( \square \) is of length \( \ell \). By the definition of the descent functor \( KD(-) \), the tower \( KD(-)(X) \) is the \( s_2 \)-simple tower associated to the diagram of constant towers \( K(X_\bullet) \), that is, it is the tower whose spectra are the homotopy limits of the diagram \( dK(X_\bullet)(n) \) for each \( n \) (see definition 5.2.3). Observe that this diagram is constant for \( n \geq \ell \) and, moreover, it is precisely the cubical diagram \( X_\bullet \), so the result follows.

Since the spectral sequence of a tower of fibrations is functorial in the category \( \text{Ho}_2(\text{tow}(\text{Sp})) \) from the \( E_2 \)-term on, we deduce from the corollary above:

**Corollary 5.3.4.** There is a well defined and functorial finite increasing filtration \( F^p \) on \( KD_n(X) \) which is trivial for smooth varieties.

**Remark 5.3.5.** Equivalently, by 4.3.1 for any variety \( X \) the last corollary defines a functorial finite filtration on the homotopy algebraic \( K \)-theory groups \( KH_n(X) \).

5.4. Finally, we observe that the same procedure may be applied to the algebraic \( K \)-theory with compact support. In this case, from the uniqueness property of descent extensions and [GS], theorem 7, we deduce:

**Corollary 5.4.1.** There is a well defined and functorial finite increasing filtration \( W^pK^c_n(X) \) which is trivial for complete smooth varieties. This filtration coincides with the weight filtration defined by Gillet-Soulé in [GS].

**References**

[BF] A. Bousfield, E. Friedlander, *Homotopy theory of \( \Gamma \)-spaces spectra and bisimplicial sets*. In Geometric Applications of Homotopy Theory. Springer LNM 658, (1978), 80–130.

[BK] A. Bousfield, D. Kan, *Homotopy limits, completions, and localizations*. Springer LNM 304, 1972.

[CHSW] G. Cortiñas, C. Haesemeyer, M. Schlichting, C. Weibel, *Cyclic homology, cdh-cohomology and negative \( K \)-theory*. ArXiv preprint, math.KT/0502255.

[CHW] G. Cortiñas, C. Haesemeyer, C. Weibel, *K-regularity, cdh-fibrant hochschild homology, and a conjecture of Vorst*. ArXiv preprint, math.KT/0605367.

[GN02] F. Guillén, V. Navarro Aznar, *Un critère d’extension des foncteurs définis sur les schémas lisses*. Publ. Math. I.H.E.S. 95, (2002), 1–91.

[GN03] F. Guillén, V. Navarro Aznar, *Cohomological descent and weight filtration*. Communication at the Joint Meeting of the AMS and the Spanish Math. Soc. Sevilla, 2003.

[GNPP] F. Guillén, V. Navarro, P. Pascual, F. Puerta, *Hyperrésolutions cubiques et descente cohomologique*. Springer LNM 1335, 1988.

[GS] H. Gillet, C. Soulé, *Descent, motives and \( K \)-theory*. Crelle J. Reine Angew. Math. 478, (1996), 127–176.

[G] T. Goodwillie, *Calculus II: Analytic functors. \( K \)-theory 5*, (1992), 295–332.

[H] C. Haesemeyer, *Descent properties for homotopy \( K \)-theory*. Duke Math. J. 125, 2004, 589–620.

[M] S. MacLane, *Categories for the working mathematician*. Springer GTM, 1970.
[P] P. Pascual, On the simple object associated to a diagram in a closed model category. Math. Proc. Cambridge Phil. Soc. 100, (1986), 459–474.

[Q] D. Quillen, Higher algebraic K-theory I. In Higher K-theories. Springer LNM 341 (1973), 85–147.

[R] Ll. Rubió Pons, Stable homotopy categories and cohomological descent. Preprint, 2006.

[S] S. Schwede, Spectra in model categories and applications to the algebraic cotangent complex. JPAA 120, (1997), 77–104.

[T80] R. Thomason, Algebraic K-theory and étale cohomology. Annals Sci. École Normale Sup. 13, (1980), 437–552.

[T91] R. Thomason, Les K-groupes d’un fibré projectif. In J. Jardine et alt. (eds), “Algebraic topology and algebraic K-theory” Lake Louise 1991. Kluwer.

[T93] R. Thomason, Les K-groupes d’un schéma éclaté et une formule d’intersection excédentaire. Invent. math. 112, (1993), 195–215.

[TT] R. Thomason, T. Trobaugh, Higher Algebraic K-theory of schemes and of derived categories. The Grothendieck Festschrift, vol III, Progress in Math. 88, 247–436. Birkhäuser V., 1990.

[W1] C. Weibel, Homotopy algebraic K-theory. In Algebraic K-theory and Number Theory, Contemp. Math. 83, 461-488. American Math. Soc., 1989.

[W2] C. Weibel, A Quillen-type spectral sequence for the K-theory of varieties with isolated singularities. K-Theory 3, (1989), 261–270.

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