ON ESTIMATES OF BIHARMONIC FUNCTIONS
ON LIPSCHITZ AND CONVEX DOMAINS

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Abstract. Using Maz’ya type integral identities with power weights, we obtain new boundary estimates for biharmonic functions on Lipschitz and convex domains in $\mathbb{R}^n$. For $n \geq 8$, combined with a result in [S2], these estimates lead to the solvability of the $L^p$ Dirichlet problem for the biharmonic equation on Lipschitz domains for a new range of $p$. In the case of convex domains, the estimates allow us to show that the $L^p$ Dirichlet problem is uniquely solvable for any $2 - \varepsilon < p < \infty$ and $n \geq 4$.

1. Introduction

Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ with Lipschitz boundary. Let $N$ denote the outward unit normal to $\partial \Omega$. We consider the $L^p$ Dirichlet problem for the biharmonic equation,

\begin{equation}
\begin{cases}
\Delta^2 u = 0 & \text{in } \Omega, \\
u = f \in L^p_1(\partial \Omega), \quad \frac{\partial u}{\partial N} = g \in L^p(\partial \Omega) & \text{on } \partial \Omega, \\
(\nabla u)^* \in L^p(\partial \Omega)
\end{cases}
\end{equation}

where $L^p_1(\partial \Omega)$ denotes the space of functions in $L^p(\partial \Omega)$ whose first order (tangential) derivatives are also in $L^p(\partial \Omega)$. We point out that the boundary values in (1.1) are taken in the sense of non-tangential convergence a.e. with respect to the surface measure on $\partial \Omega$. As such, one requires that the non-tangential maximal function $(\nabla u)^*$ is in $L^p(\partial \Omega)$.

For $n \geq 2$, the Dirichlet problem (1.1) with $p = 2$ was solved by Dahlberg, Kenig and Verchota [DKV], using bilinear estimates for harmonic functions. The result was then extended to the case $2 - \varepsilon < p < 2 + \varepsilon$ by a real variable argument, where $\varepsilon > 0$ depends on $n$ and $\Omega$. They also showed that the restriction $p > 2 - \varepsilon$ is necessary for general Lipschitz domains. In [PV1,PV2], Pipher and Verchota proved that if $n = 3$ (or 2), the $L^p$ Dirichlet problem (1.1) is uniquely solvable for the sharp range $2 - \varepsilon < p \leq \infty$. Moreover, they pointed out that (1.1) is not solvable in general for $p > 6$ if $n = 4$, and for $p > 4$ if $n \geq 5$. Recently in [S1,S2], for $n \geq 4$ and $p$ in a certain range, we established the solvability of the $L^p$ Dirichlet problem for higher order elliptic equations and systems, using a new approach.

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via $L^2$ estimates and weak reverse H"older inequalities. In particular, we were able to solve the $L^p$ Dirichlet problem (1.1) in the following cases,

\begin{equation}
\begin{cases}
2 - \varepsilon < p < 6 + \varepsilon & \text{for } n = 4, \\
2 - \varepsilon < p < 4 + \varepsilon & \text{for } n = 5, 6, 7, \\
2 - \varepsilon < p < 2 + \frac{4}{n-3} + \varepsilon & \text{for } n \geq 8.
\end{cases}
\end{equation}

This gives the sharp ranges of $p$ for $4 \leq n \leq 7$. It should be pointed out that the sharp range $2 - \varepsilon < p < 4 + \varepsilon$ for the case $n = 6, 7$ in (1.2) relies on a classical result of Maz’ya [M1,M2] on the boundary regularity of biharmonic functions in arbitrary domains. The approach we will use in this paper is inspired by the work of Maz’ya [M1,M2,M4] (we shall come back to this point later). We mention that if the domain $\Omega$ is $C^1$, then (1.1) is uniquely solvable for all $n \geq 2$ and $1 < p \leq \infty$ [CG,V1,PV2]. For related work on the $L^p$ Dirichlet problem for the polyharmonic equation and general higher order equations and systems on Lipschitz domains, we refer the reader to [V2,PV3,PV4,K,V3,S1,S2].

The purpose of this paper is twofold. First we study the case $n \geq 8$ for which the question of the sharp ranges of $p$ remains open for Lipschitz domains. Secondly we initiate the study of the $L^p$ Dirichlet problem (1.1) on convex domains. Note that any convex domain is Lipschitz, but may not be $C^1$.

Let $I(Q,r) = B(Q,r) \cap \partial \Omega$ and $T(Q,r) = B(Q,r) \cap \Omega$ where $Q \in \partial \Omega$ and $r > 0$. Our starting point is the following theorem.

**Theorem 1.3.** Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 4$. Suppose that there exist constants $C_0 > 0$, $R_0 > 0$ and $\lambda \in (0, n]$ such that for any $0 < r < R < R_0$ and $Q \in \partial \Omega$,

\begin{equation}
\int_{T(Q,r)} |\nabla v|^2 \, dx \leq C_0 \left( \frac{r}{R} \right)^\lambda \int_{T(Q,R)} |\nabla v|^2 \, dx,
\end{equation}

whenever $v$ satisfies

\begin{equation}
\begin{cases}
\Delta^2 v = 0 & \text{in } \Omega, \\
v = \frac{\partial v}{\partial N} = 0 & \text{on } I(Q,R), \\
(\nabla v)^* \in L^2(\partial \Omega).
\end{cases}
\end{equation}

Then the $L^p$ Dirichlet problem (1.1) is uniquely solvable for

\begin{equation}
2 < p < 2 + \frac{4}{n-\lambda}.
\end{equation}

Moreover, the solution $u$ satisfies

\begin{equation}
\| (\nabla u)^* \|_{L^p(\partial \Omega)} \leq C \{ \| \nabla f \|_{L^p(\partial \Omega)} + \| g \|_{L^p(\partial \Omega)} \},
\end{equation}
where $\nabla_t f$ denotes the tangential derivatives of $f$ on $\partial\Omega$.

Theorem 1.3 is a special case of Theorem 1.10 in [S2] for general higher order homogeneous elliptic equations and systems with constant coefficients. It reduces the study of the $L^p$ Dirichlet problem to that of local $L^2$ estimates near the boundary. The main body of this paper will be devoted to such estimates. In particular, we will prove that if $n \geq 8$, then estimate (1.4) holds for some $\lambda > \lambda_n$, where

(1.8) \[
\lambda_n = \frac{n + 10 + 2\sqrt{2(n^2 - n + 2)}}{7}.
\]

We will also show that if $\Omega$ is convex and $n \geq 4$, then (1.4) holds for any $0 < \lambda < n$. Consequently, by Theorem 1.3, we obtain the following.

Main Theorem. Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^n$.

a) If $n \geq 8$, the $L^p$ Dirichlet problem (1.1) is uniquely solvable for

(1.9) \[
2 - \varepsilon < p < 2 + \frac{4}{n - \lambda_n} + \varepsilon.
\]

b) If $n \geq 4$ and $\Omega$ is convex, the $L^p$ Dirichlet problem (1.1) is uniquely solvable for $2 - \varepsilon < p < \infty$.

We remark that in the case of Laplace’s equation $\Delta u = 0$, the Dirichlet problem in $L^p$ is uniquely solvable on convex domains for all $1 < p \leq \infty$. This follows easily from the $L^\infty$ boundary estimates on the first derivatives of the Green’s functions. Whether a similar result (the $L^\infty$ boundary estimate on the second derivatives) holds for biharmonic functions remains open for $n \geq 3$ (see [KM] for the case $n = 2$). Note that part (b) of the Main Theorem as well as its proof gives the $C^\alpha$ boundary estimate of $u$ for any $0 < \alpha < 1$. This seems to be the first regularity result for biharmonic functions on general convex domains in $\mathbb{R}^n$, $n \geq 4$.

As we mentioned earlier, our approach to estimate (1.4) is motivated by the work of Maz’ya [M1,M2,M4]. It is based on certain integral identities for

(1.10) \[
\int_{\Omega} \Delta^2 u \cdot u \frac{dx}{\rho^\alpha} \quad \text{and} \quad \int_{\Omega} \Delta^2 u \cdot \frac{\partial u}{\partial \rho} \frac{dx}{\rho^{\alpha-1}},
\]

where $\rho = |x - Q|$ with $Q \in \partial\Omega$ fixed. See (2.13) and (3.1). These identities with power weights allow us to control the integrals

(1.11) \[
\int_{\Omega} |\nabla u|^2 \frac{dx}{\rho^{\alpha+2}} \quad \text{and} \quad \int_{\Omega} u^2 \frac{dx}{\rho^{\alpha+4}}
\]

for certain values of $\alpha$. We point out that integral identity (2.13) with $\alpha = n - 4$ appeared first in [M1,M2], where it was used to establish a Wiener’s type condition on the boundary.
continuity for the biharmonic equation $\Delta^2 u = f$ on arbitrary domains in $\mathbb{R}^n$ for $n \leq 7$. Since the restriction $n \leq 7$ in [M1,M2] is related to the positivity of a quadratic form (see (1.12) below), the idea to prove part (a) of the Main Theorem is to use the identity (2.13) for certain $\alpha < n - 4$ in the case $n \geq 8$. However, it should be pointed out that the main novelty of this paper is the new identity (3.1), on which the proof of part (b) of the Main Theorem is based. This identity allows us to estimate the integrals in (1.11) on convex domains for any $\alpha < n - 2$. We remark that due to the lack of maximum principles for higher order equations, identities such as (2.13) and (3.1) are valuable tools in the study of boundary regularities in nonsmooth domains.

Finally we mention that the results in [M1,M2] were subsequently extended to the polyharmonic equation [MD,M3] and general higher order elliptic equations [M4]. Also, the related question of the positivity of the quadratic form

$$\int_{\mathbb{R}^n} (-\Delta)^\lambda u \cdot u \frac{dx}{|x|^{n-2\lambda}}$$

for all real function $u \in C_0^\infty(\mathbb{R}^n)$, has been studied systematically by Eilertsen [E1,E2] for all $\lambda \in (0, n/2)$.

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### 2. Boundary Estimates on Lipschitz Domains

The goal of this section is to prove part (a) of the Main Theorem. We begin with a Cacciopoli’s inequality. Recall that for $Q \in \partial \Omega$, $T(Q, R) = B(Q, R) \cap \Omega$ and $I(Q, R) = B(Q, R) \cap \partial \Omega$. We assume that $0 < R < R_0$, where $R_0$ is a constant depending on $\Omega$ so that for any $Q \in \partial \Omega$, $T(Q, 4R_0)$ is given by the intersection of $B(Q, 4R_0)$ and the region above a Lipschitz graph, after a possible rotation.

**Lemma 2.1.** Let $u \in W^{2,2}(T(Q, R))$ for some $Q \in \partial \Omega$ and $0 < R < R_0$. Suppose that $\Delta^2 u = 0$ in $T(Q, R)$ and $u = 0, \nabla u = 0$ on $I(Q, R)$. Then

$$\frac{1}{r^2} \int_{T(Q,r)} |\nabla u|^2 dx + \int_{T(Q,r)} |\nabla^2 u|^2 dx \leq C \frac{r^4}{r^4} \int_{T(Q,2r) \setminus T(Q,r)} |u|^2 dx,$$

where $0 < r < R/4$.

**Proof.** Let $\eta$ be a smooth function on $\mathbb{R}^n$ such that $\eta = 1$ on $B(Q, r)$, supp $\eta \subset B(Q, 2r)$ and $|\nabla^k \eta| \leq C/r^k$ for $0 \leq k \leq 4$. Since $u \in W^{2,2}(T(Q, R))$ and $u = 0, \nabla u = 0$ on $I(Q, R)$, we have $u\eta^2 \in W^{2,2}_0(\Omega)$. We will show that for any $\varepsilon > 0$,

$$\int_\Omega |\nabla^2 (u\eta^2)|^2 dx \leq \varepsilon \int_\Omega |\nabla^2 (u\eta^2)|^2 dx + \varepsilon \int_\Omega |\nabla (u\eta^2)|^2 dx + \frac{C_\varepsilon}{r^4} \int_{T(Q,2r) \setminus T(Q,r)} |u|^2 dx.$$
This, together with the Poincaré inequality

\[(2.4) \int_{T(Q,2r)} |\nabla (u\eta^2)|^2 \, dx \leq C r^2 \int_{T(Q,2r)} |\nabla^2 (u\eta^2)|^2 \, dx,\]
yields the estimate (2.2).

To prove (2.3), we use integration by parts and $\Delta^2 u = 0$ in $T(Q,2r)$ to obtain

\[(2.5) \int_{\Omega} |\nabla^2 (u\eta^2)|^2 \, dx = \int_{\Omega} |\Delta (u\eta^2)|^2 \, dx = \int_{\Omega} \{ |\Delta (u\eta^2)|^2 - \Delta u \cdot \Delta (u\eta^4) \} \, dx.\]

A direct computation shows that

\[(2.6) \Delta (u\eta^2) \cdot \Delta (u\eta^2) - \Delta u \cdot \Delta (u\eta^4) = u \Delta (u\eta^2) \Delta \eta^2 + 4|\nabla u \cdot \nabla \eta^2|^2 + 2u(\nabla u \cdot \nabla \eta^2) \Delta \eta^2 - u \Delta u (2|\nabla \eta^2|^2 + \eta^2 \Delta \eta^2),\]

In view of (2.3), the integral of the first term in the right side of (2.6) can be handled easily by Hölder’s inequality with an $\varepsilon$. The remaining terms may be handled by using integration by parts, together with the following observation. For terms with $u \frac{\partial u}{\partial x_i}$, like the third term in the right side of (2.6), we may write

\[(2.7) u \frac{\partial u}{\partial x_i} \psi = \frac{1}{2} \frac{\partial}{\partial x_i} \left( |u|^2 \psi \right) - \frac{1}{2} |u|^2 \frac{\partial \psi}{\partial x_i}.\]

For terms with $\eta^2 \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j}$, like the second term, we use

\[(2.8) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \eta^2 \psi = \frac{\partial}{\partial x_i} \left( \frac{\partial (u\eta^2)}{\partial x_j} \cdot u \psi \right) - \frac{\partial^2 (u\eta^2)}{\partial x_i \partial x_j} \cdot u \psi - u \frac{\partial u}{\partial x_i} \frac{\partial \psi}{\partial x_j} \eta^2 - u \frac{\partial u}{\partial x_j} \frac{\partial \eta^2}{\partial x_i} \psi - u^2 \frac{\partial \eta^2}{\partial x_i} \frac{\partial \psi}{\partial x_i}.\]

Finally, for the last term which contains $\eta^2 u \Delta u$, we note that

\[(2.9) \eta^2 u \Delta u \cdot \psi = \frac{\partial}{\partial x_i} \left( \eta^2 u \frac{\partial u}{\partial x_i} \psi \right) - u \frac{\partial u}{\partial x_i} \frac{\partial (\eta^2 \psi)}{\partial x_i} - \eta^2 |\nabla u|^2 \psi.\]

The rest of the proof, which we omit, is fairly straightforward.

**Remark 2.10.** It follows from Lemma 2.1 that for any $0 < r < R/2$ and $\alpha \in \mathbb{R}$,

\[(2.11) \int_{T(Q,r)} \frac{|\nabla u(x)|^2}{|x-Q|^\alpha} \, dx + \int_{T(Q,r)} \frac{|\nabla^2 u(x)|^2}{|x-Q|^\alpha} \, dx \leq C \int_{T(Q,2r)} \frac{|u(x)|^2 \, dx}{|x-Q|^\alpha}.\]

This may be seen by writing $T(Q, r) \cup \bigcup_{j=0}^{\infty} T(Q, 2^{-j}r) \setminus T(Q, 2^{-j-1}r)$.

The key step to establish estimate (1.4) relies on the following extension of an integral identity due to Maz’ya [M1,M2].
Lemma 2.12. Suppose that \( u \in C^2(\overline{\Omega}) \) and \( u = 0, \nabla u = 0 \) on \( \partial \Omega \). Then for any \( \alpha \in \mathbb{R} \),

\[
\int_{\Omega} \Delta u \cdot \Delta \left( \frac{u}{\rho^\alpha} \right) \, dx = \int_{\Omega} |\Delta u|^2 \, \frac{dx}{\rho^\alpha} + 2 \alpha \int_{\Omega} |\nabla u|^2 \, \frac{dx}{\rho^{\alpha+2}} - 2 \alpha(\alpha + 2) \int_{\Omega} \left| \frac{\partial u}{\partial \rho} \right|^2 \, \frac{dx}{\rho^{\alpha+2}} \]

\[
+ \frac{1}{2} \alpha(\alpha + 2)(n - 2 - \alpha)(n - 4 - \alpha) \int_{\Omega} |u|^2 \, \frac{dx}{\rho^{\alpha+4}},
\]

where \( \rho = |x - y| \) and \( \frac{\partial u}{\partial \rho} = <\nabla u(x), (x - y)/\rho> \) with \( y \in \overline{\Omega} \) fixed.

Proof. We will use the summation convention that the repeated indices are summed from 1 to \( n \). First, note that

\[
\int_{\Omega} \Delta u \cdot \Delta \left( \frac{u}{\rho^\alpha} \right) \, dx = \int_{\Omega} |\Delta u|^2 \, \frac{dx}{\rho^\alpha} + 2 \int_{\Omega} \Delta u \cdot \frac{\partial u}{\partial x_j} \left( \frac{1}{\rho^\alpha} \right) \, dx
\]

\[
+ \int_{\Omega} u \Delta u \cdot \Delta \left( \frac{1}{\rho^\alpha} \right) \, dx.
\] (2.14)

Next it follows from integration by parts that

\[
2 \int_{\Omega} \Delta u \cdot \frac{\partial u}{\partial x_j} \frac{\partial}{\partial x_j} \left( \frac{1}{\rho^\alpha} \right) \, dx = \int_{\Omega} |\nabla u|^2 \Delta \left( \frac{1}{\rho^\alpha} \right) \, dx - 2 \int_{\Omega} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \frac{\partial^2}{\partial x_i \partial x_j} \left( \frac{1}{\rho^\alpha} \right) \, dx.
\] (2.15)

Similarly, we have

\[
\int_{\Omega} u \Delta u \cdot \Delta \left( \frac{1}{\rho^\alpha} \right) \, dx = - \int_{\Omega} |\nabla u|^2 \Delta \left( \frac{1}{\rho^\alpha} \right) \, dx + \frac{1}{2} \int_{\Omega} |u|^2 \Delta^2 \left( \frac{1}{\rho^\alpha} \right) \, dx.
\] (2.16)

Substituting (2.15) and (2.16) into (2.14), we obtain

\[
\int_{\Omega} \Delta u \cdot \Delta \left( \frac{u}{\rho^\alpha} \right) \, dx = \int_{\Omega} |\Delta u|^2 \, \frac{dx}{\rho^\alpha} - 2 \int_{\Omega} \frac{\partial u}{\partial x_i} \cdot \frac{\partial u}{\partial x_j} \left( \frac{1}{\rho^\alpha} \right) \, dx
\]

\[
+ \frac{1}{2} \int_{\Omega} |u|^2 \, \Delta^2 \left( \frac{1}{\rho^\alpha} \right) \, dx.
\]

The desired formula (2.13) now follows from the fact that

\[
\frac{\partial^2}{\partial x_i \partial x_j} \left( \frac{1}{\rho^\alpha} \right) = -\alpha \rho^{-\alpha - 2} \delta_{ij} + \alpha(\alpha + 2)(x_i - y_i)(x_j - y_j) \rho^{-\alpha - 4},
\]

\[
\Delta^2 \left( \frac{1}{\rho^\alpha} \right) = \alpha(\alpha + 2)(n - 2 - \alpha)(n - 4 - \alpha) \rho^{-\alpha - 4},
\]

for any \( \rho = |x - y| \neq 0 \). The proof is complete.
Lemma 2.18. Under the same assumption as in Lemma 2.12, we have

\[(2.19) \quad \int_{\Omega} \Delta u \cdot \frac{\partial u}{\partial \rho} \frac{dx}{\rho^{\alpha+1}} = \frac{1}{2} (n - 4 - \alpha) \int_{\Omega} |\nabla u|^2 \frac{dx}{\rho^{\alpha+2}} + (\alpha + 2) \int_{\Omega} \left| \frac{\partial u}{\partial \rho} \right|^2 \frac{dx}{\rho^{\alpha+2}}.\]

Proof. It follows from integration by parts that

\[
\int_{\Omega} \Delta u \cdot \frac{\partial u}{\partial \rho} \frac{dx}{\rho^{\alpha+1}} = \int_{\Omega} \Delta u \cdot \frac{\partial u}{\partial x_i} (x_i - y_i) \frac{dx}{\rho^{\alpha+2}} = \frac{1}{2} \int_{\Omega} |\nabla u|^2 \frac{\partial}{\partial x_i} \left( \frac{x_i - y_i}{\rho^{\alpha+2}} \right) dx - \int_{\Omega} \frac{\partial u}{\partial x_j} \cdot \frac{\partial u}{\partial x_j} \frac{x_i - y_i}{\rho^{\alpha+2}} dx
\]

\[
= \frac{1}{2} (n - 4 - \alpha) \int_{\Omega} |\nabla u|^2 \frac{dx}{\rho^{\alpha+2}} + (\alpha + 2) \int_{\Omega} \left| \frac{\partial u}{\partial \rho} \right|^2 \frac{dx}{\rho^{\alpha+2}}.
\]

Lemma 2.12, together with Lemma 2.18, allows us to estimate

\[
\int_{\Omega} |u|^2 \frac{dx}{\rho^{\alpha+4}} \quad \text{and} \quad \int_{\Omega} |\nabla u|^2 \frac{dx}{\rho^{\alpha+2}} \quad \text{by} \quad \int_{\Omega} \Delta u \cdot \Delta \left( \frac{u}{\rho^{\alpha}} \right) dx
\]

for certain values of \( \alpha \).

Lemma 2.20. Let \( \Omega \) be a bounded Lipschitz domain in \( \mathbb{R}^n \), \( n \geq 5 \). Suppose that \( u \in C^2(\bar{\Omega}) \) and \( u = 0, \nabla u = 0 \) on \( \partial \Omega \). Then, if \( 0 < \alpha \leq n - 4 \) and \( n^2 + 2n\alpha - 7\alpha^2 - 8\alpha > 0 \), we have

\[(2.21) \quad \int_{\Omega} |\nabla u|^2 \frac{dx}{\rho^{\alpha+2}} \leq C_{n,\alpha} \int_{\Omega} \Delta u \cdot \Delta \left( \frac{u}{\rho^{\alpha}} \right) dx,
\]

where \( C_{n,\alpha} > 0 \) depends only on \( n \) and \( \alpha \).

Proof. We first use (2.19) for \( 0 < \alpha \leq n - 4 \) to obtain

\[
\frac{n + \alpha}{2} \int_{\Omega} \left| \frac{\partial u}{\partial \rho} \right|^2 \frac{dx}{\rho^{\alpha+2}} \leq \int_{\Omega} \Delta u \cdot \frac{\partial u}{\partial \rho} \frac{dx}{\rho^{\alpha+1}} \leq \left\{ \int_{\Omega} |\Delta u|^2 \frac{dx}{\rho^{\alpha}} \right\}^{1/2} \left\{ \int_{\Omega} \left| \frac{\partial u}{\partial \rho} \right|^2 \frac{dx}{\rho^{\alpha+2}} \right\}^{1/2},
\]

where the Cauchy inequality is also used. It follows that

\[(2.22) \quad \frac{1}{4} (n + \alpha)^2 \int_{\Omega} \left| \frac{\partial u}{\partial \rho} \right|^2 \frac{dx}{\rho^{\alpha+2}} \leq \int_{\Omega} |\Delta u|^2 \frac{dx}{\rho^{\alpha}}.
\]

Since \( 0 < \alpha \leq n - 4 \), in view of (2.13) and (2.22), we have

\[
\int_{\Omega} \Delta u \cdot \Delta \left( \frac{u}{\rho^{\alpha}} \right) dx \geq \left\{ \frac{1}{4} (n + \alpha)^2 + 2\alpha - 2\alpha(\alpha + 2) \right\} \int_{\Omega} \left| \frac{\partial u}{\partial \rho} \right|^2 \frac{dx}{\rho^{\alpha+2}}
\]

\[
= \frac{1}{4} (n^2 + 2n\alpha - 7\alpha^2 - 8\alpha) \int_{\Omega} \left| \frac{\partial u}{\partial \rho} \right|^2 \frac{dx}{\rho^{\alpha+2}}.
\]

Thus, if \( n^2 + 2n\alpha - 7\alpha^2 - 8\alpha > 0 \), by (2.13) again,
\[
2\alpha \int_{\Omega} |\nabla u|^2 \frac{dx}{\rho^{\alpha+2}} \leq \int_{\Omega} \Delta u \cdot \Delta \left( \frac{u}{\rho^\alpha} \right) dx + 2\alpha(\alpha + 2) \int_{\Omega} \left| \frac{\partial u}{\partial \rho} \right|^2 \frac{dx}{\rho^{\alpha+2}}
\]
\[
\leq C \int_{\Omega} \Delta u \cdot \Delta \left( \frac{u}{\rho^\alpha} \right) dx.
\]

The proof is finished.

**Remark 2.23.** Let \( \alpha = n - 4 \). Then \( n^2 + 2n\alpha - 7\alpha^2 - 8\alpha = 4(-n^2 + 10n - 20) > 0 \) for \( n = 5, 6, 7 \). It follows that (2.21) holds for \( \alpha = n - 4 \) in the case \( n = 5, 6 \) or 7. This was the result obtained by Maz‘ya in [M1,M2]. If \( n \geq 8 \), then (2.21) holds for \( 0 < \alpha < \alpha_n < n - 4 \), where

\[
(2.24) \quad \alpha_n = \frac{1}{7}(n - 4 + 2\sqrt{2(n^2 - n + 2)})
\]
is the positive root of \( n^2 + 2n\alpha - 7\alpha^2 - 8\alpha = 0 \).

**Remark 2.25.** If \( n \geq 8 \) and \( \alpha = \alpha_n \) given by (2.24), we observe that the first three terms on the right side of (2.13) is nonnegative, by an inspection of the proof of Lemma 2.20. It follows that

\[
(2.26) \quad \int_{\Omega} |u|^2 \frac{dx}{\rho^{\alpha+4}} \leq C_n \int_{\Omega} \Delta u \cdot \Delta \left( \frac{u}{\rho^\alpha} \right) dx.
\]

Since \( C_0^\infty(\Omega) \) is dense in \( W_0^{2,2}(\Omega) \), inequality (2.26) holds for any \( u \in W_0^{2,2}(\Omega) \).

We are now in a position to give the proof of part (a) of the Main Theorem.

**Theorem 2.27.** Let \( \Omega \) be a bounded Lipschitz domain in \( \mathbb{R}^n \), \( n \geq 8 \). Then the \( L^p \) Dirichlet problem (1.1) is uniquely solvable for \( 2 - \epsilon < p < 2 + \frac{4}{n-\lambda_n} + \epsilon \), where \( \lambda_n = \alpha_n + 2 \) is given in (1.8).

**Proof.** By Theorem 1.3, we only need to show that estimate (1.4) holds for some \( \lambda > \lambda_n = \alpha_n + 2 \). To this end, we fix \( Q \in \partial \Omega \) and \( 0 < R < R_0 \), where \( R_0 \) is a constant depending on \( \Omega \). Let \( \eta \) be a function on \( \Omega \) satisfying (1.5). Let \( \eta \) be a smooth function on \( \mathbb{R}^n \) such that \( \eta = 1 \) on \( B(Q,r) \), \( \text{supp} \eta \subset B(Q,2r) \) and \( |\nabla^k \eta| \leq C/r^k \) for \( 0 \leq k \leq 4 \) where \( 0 < r < R/4 \). Since \( \nu = \frac{\partial \nu}{\partial \nu} = 0 \) on \( I(Q,R) \) and \( (\nabla \eta)^\ast \in L^2(\partial \Omega) \), by the regularity estimate \( \|(\nabla \eta)^\ast\|_2 \leq C \|\nabla_t \eta\|_2 \) established in [V2], we know \( \eta \in W_0^{2,2}(\Omega) \). Thus we may apply estimate (2.26) to \( u = \nu \eta \) with \( \alpha = \alpha_n \) and \( \rho = |x - y| \), where \( y \in \Omega^\ast \). We obtain

\[
(2.28) \quad \int_{\Omega} |\eta|^2 \frac{dx}{\rho^{\alpha+4}} \leq C \int_{\Omega} \Delta (\eta \eta) \cdot \Delta (\nu \eta \rho^{-\alpha}) dx.
\]

Using an identity similar to (2.6),
\[
\Delta (\eta \eta) \cdot \Delta (\nu \eta \rho^{-\alpha}) - \Delta \nu \cdot \Delta (\nu \eta \rho^{-\alpha}) = v\Delta (\nu \eta \rho^{-\alpha}) \Delta \eta + 2(\nabla \nu \cdot \nabla \eta) \Delta (\nu \eta \rho^{-\alpha}) - 2(\nabla (\nu \eta \rho^{-\alpha}) \cdot \nabla \eta) \Delta \nu - \nu \eta \rho^{-\alpha} \Delta \nu \cdot \Delta \eta,
\]
and $\Delta^2 v = 0$ in $\Omega$, we get
\begin{equation}
\int_\Omega |v\eta|^2 \frac{dx}{\rho^{\alpha+4}} \leq C \int_\Omega \left\{ v\Delta(v\rho^{-\alpha})\Delta\eta + 2(\nabla v \cdot \nabla\eta)\Delta(v\rho^{-\alpha}) - 2(\nabla(v\rho^{-\alpha}) \cdot \nabla\eta)\Delta v - v\rho^{-\alpha}\Delta v \cdot \Delta\eta \right\} dx.
\tag{2.29}
\end{equation}

Note that $\rho^{-\alpha}$ and its derivatives are uniformly bounded for $y \in B(Q, r/2) \setminus \overline{\Omega}$ and $x \in \text{supp}(|\nabla\eta|) \subset \{ x \in \mathbb{R}^n : r \leq |x - Q| \leq 2r \}$. It follows by a simple limiting argument that (2.29) holds for $\rho = |x - Q|$. This gives
\begin{equation}
\int_{T(Q,r)} \frac{|v(x)|^2 dx}{|x - Q|^{\alpha+4}} \leq C \int_{T(Q,2r)} \left\{ |v|^2 + r^2|\nabla v|^2 + 4|\nabla^2 v|^2 \right\} dx
\leq \frac{C}{r^{\alpha+4}} \int_{T(Q,4r) \setminus T(Q,2r)} |v|^2 dx
\leq C_1 \int_{T(Q,4r) \setminus T(Q,r)} \frac{|v(x)|^2 dx}{|x - Q|^{\alpha+4}},
\tag{2.30}
\end{equation}

where the second inequality follows from Cacciopoli’s inequality (2.2). By “filling” the hole in (2.30), we obtain
\begin{equation}
\int_{T(Q,r)} \frac{|v(x)|^2 dx}{|x - Q|^{\alpha+4}} \leq \frac{C_1}{C_1 + 1} \int_{T(Q,4r)} \frac{|v(x)|^2 dx}{|x - Q|^{\alpha+4}}.
\end{equation}

This implies that there exists $\delta > 0$ such that
\begin{equation}
\int_{T(Q,r)} \frac{|v(x)|^2 dx}{|x - Q|^{\alpha+4}} \leq C \left( \frac{r}{R} \right)^{\delta} \int_{T(Q,R/4)} \frac{|v(x)|^2 dx}{|x - Q|^{\alpha+4}}
\leq C \left( \frac{r}{R} \right)^{\delta} \frac{1}{R^{\alpha+4}} \int_{T(Q,R)} |v(x)|^2 dx,
\end{equation}

for any $0 < r < R/4$, where the second inequality follows from (2.30). Consequently,
\begin{equation}
\int_{T(Q,r)} |v(x)|^2 dx \leq C \left( \frac{r}{R} \right)^{\alpha+4+\delta} \int_{T(Q,R)} |v(x)|^2 dx.
\end{equation}

This, together with Cacciopoli’s inequality and Poincaré inequality, gives
\begin{equation}
\int_{T(Q,r/2)} |\nabla v|^2 dx \leq C \frac{r^2}{r^2} \int_{T(Q,r)} |v|^2 dx \leq C \left( \frac{r}{R} \right)^{\alpha+4+\delta} \int_{T(Q,R)} |v|^2 dx
\leq C \left( \frac{r}{R} \right)^{\alpha+2+\delta} \int_{T(Q,R)} |\nabla v|^2 dx.
\end{equation}

Thus we have established estimate (1.4) for $\lambda = \alpha_n + 2 + \delta = \lambda_n + \delta$. The proof is finished.
3. Boundary Estimates on Convex Domains

In this section we give the proof of part (b) of the Main Theorem. By Theorem 1.3, it suffices to show that estimate (1.4) holds for any \( \lambda < n \). To do this, the crucial step is to establish the following new integral identity,

\[
(\alpha + 4 - n) \int_{\Omega} \Delta u \cdot \nabla \left( \frac{u}{\rho^\alpha} \right) dx - 2 \int_{\Omega} \Delta^2 u \cdot \frac{\partial u}{\partial \rho} \frac{dx}{\rho^{\alpha-1}}
\]

\[
= \int_{\partial \Omega} |\nabla^2 u|^2 < \frac{x-y}{\rho}, N > \frac{d\sigma}{\rho^{\alpha-1}} + 4\alpha \int_{\Omega} \left| \frac{\partial}{\partial \rho} \left( \frac{\rho^{n-2}}{2} \nabla u \right) \right|^2 \frac{dx}{\rho^{n-2}}
\]

\[
+ 2\alpha(\alpha + 2)(n - \alpha - 2) \int_{\Omega} \left| \frac{\partial}{\partial \rho} \left( \frac{\rho^{n-4}}{4} u \right) \right|^2 \frac{dx}{\rho^{n-2}},
\]

where \( u \in C^4(\overline{\Omega}) \) and \( u = 0, \nabla u = 0 \) on \( \partial \Omega \). Recall that \( N \) denotes the outward unit normal to \( \partial \Omega \). Also in (3.1), as before, \( \rho = \rho(x) = |x-y|, \frac{\partial u}{\partial \rho} = \langle \nabla u, (x-y)/\rho \rangle \) with \( y \in \overline{\Omega} \) fixed. By a limiting argument, it is not hard to see that if \( \alpha < n \), (3.1) holds also for \( y \in \partial \Omega \). We will use (3.1) with \( \alpha = n - 2 \) for convex domain \( \Omega \). The key observation is that if \( \Omega \) is convex, the boundary integral in (3.1) is nonnegative. This is because \( <P-Q,N(P)> \geq 0 \) for any \( P, Q \in \partial \Omega \).

The proof of (3.1), which involves the repeated use of integration by parts, will be given through a series of lemmas.

**Lemma 3.2.** Suppose \( u \in C^2(\overline{\Omega}) \) and \( u = 0, \nabla u = 0 \) on \( \partial \Omega \). Then, for any \( \alpha \in \mathbb{R} \),

\[
\int_{\Omega} \frac{\partial^2 u}{\partial x_i \partial x_j} \cdot \frac{\partial^2}{\partial x_i \partial x_j} \left( \frac{u}{\rho^\alpha} \right) dx = \int_{\Omega} |\nabla^2 u|^2 \frac{dx}{\rho^\alpha} + \alpha(n-\alpha - 1) \int_{\Omega} |\nabla u|^2 \frac{dx}{\rho^\alpha+2}
\]

\[
- \alpha(\alpha + 2) \int_{\Omega} \left| \frac{\partial^2 u}{\partial \rho} \right|^2 \frac{dx}{\rho^\alpha+2} + \frac{1}{2} \alpha(\alpha + 2)(n - \alpha - 2)(n - \alpha - 4) \int_{\Omega} |u|^2 \frac{dx}{\rho^{\alpha+1}},
\]

where the repeated indices are summed from 1 to \( n \).

**Proof.** First we note that

\[
\int_{\Omega} \frac{\partial^2 u}{\partial x_i \partial x_j} \cdot \frac{\partial^2}{\partial x_i \partial x_j} \left( \frac{u}{\rho^\alpha} \right) dx
\]

\[
= \int_{\Omega} \frac{\partial^2 u}{\partial x_i \partial x_j} \cdot \left\{ \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{1}{\rho^\alpha} + 2 \frac{\partial u}{\partial x_i} \cdot \frac{\partial \rho^{-\alpha}}{\partial x_j} + u \frac{\partial^2 \rho^{-\alpha}}{\partial x_i \partial x_j} \right\} dx.
\]

Next it follows from integration by parts and \( u = 0, \nabla u = 0 \) on \( \partial \Omega \) that

\[
2 \int_{\Omega} \frac{\partial^2 u}{\partial x_i \partial x_j} \cdot \frac{\partial u}{\partial x_i} \cdot \frac{\partial \rho^{-\alpha}}{\partial x_j} dx = - \int_{\Omega} |\nabla u|^2 \Delta(\rho^{-\alpha}) dx.
\]

and

\[
\int_{\Omega} \frac{\partial^2 u}{\partial x_i \partial x_j} \cdot u \cdot \frac{\partial^2 \rho^{-\alpha}}{\partial x_i \partial x_j} dx
\]

\[
= - \int_{\Omega} \frac{\partial u}{\partial x_i} \cdot \frac{\partial u}{\partial x_i} \cdot \frac{\partial^2 \rho^{-\alpha}}{\partial x_i \partial x_j} dx + \frac{1}{2} \int_{\Omega} |u|^2 \Delta^2(\rho^{-\alpha}) dx.
\]
Substituting (3.5) and (3.6) into (3.4), we obtain
\[
\int_{\Omega} \frac{\partial^2 u}{\partial x_i \partial x_j} \cdot \frac{\partial^2}{\partial x_i \partial x_j} \left( \frac{u}{\rho^\alpha} \right) \, dx = \int_{\Omega} |\nabla^2 u|^2 \frac{dx}{\rho^\alpha} - \int_{\Omega} |\nabla u|^2 \Delta (\rho^{-\alpha}) \, dx
\]
\[
- \int_{\Omega} \frac{\partial u}{\partial x_i} \cdot \frac{\partial u}{\partial x_i} \cdot \frac{\partial^2 \rho^{-\alpha}}{\partial x_i \partial x_j} \, dx + \frac{1}{2} \int_{\Omega} |u|^2 \Delta^2 (\rho^{-\alpha}) \, dx.
\]

The desired formula now follows from this and (2.17).

**Lemma 3.7.** Suppose \( u \in C^4(\overline{\Omega}) \) and \( u = 0 \), \( \nabla u = 0 \) on \( \partial \Omega \). Then, for any \( \alpha \in \mathbb{R} \),
\[
\int_{\Omega} \Delta^2 u \cdot \frac{\partial u}{\partial \rho} \frac{dx}{\rho^{\alpha-1}} = -\frac{1}{2} \int_{\partial \Omega} |\nabla^2 u|^2 < \frac{x-y}{\rho}, N > \frac{d\sigma}{\rho^{\alpha-1}}
\]
\[
+ \frac{1}{2} (\alpha + 4 - n) \int_{\Omega} |\nabla^2 u|^2 \frac{dx}{\rho^\alpha} - 2\alpha \int_{\Omega} |\partial_{\rho} \nabla u|^2 \frac{dx}{\rho^\alpha}
\]
\[
+ \frac{1}{2} \alpha (n - \alpha) \int_{\Omega} |\nabla u|^2 \frac{dx}{\rho^{\alpha+2}} - \frac{1}{2} \alpha (\alpha + 2) (n - \alpha) \int_{\Omega} |\partial_{\rho} u|^2 \frac{dx}{\rho^{\alpha+2}}
\]
where \( \rho = |x - y| \) with \( y \in \overline{\Omega} \) fixed.

**Proof.** By translation we may assume that \( y = 0 \). Using integration by parts, we obtain
\[
\int_{\Omega} \Delta^2 u \cdot \frac{\partial u}{\partial \rho} \frac{dx}{\rho^{\alpha-1}} = \int_{\Omega} \partial^4 u \cdot \frac{\partial u}{\partial x_k} \frac{x_k}{\rho^\alpha} \, dx
\]
\[
= - \int_{\Omega} \frac{\partial^3 u}{\partial x_i \partial x_j^2} \cdot \frac{\partial^2 u}{\partial x_i \partial x_k} \frac{x_k}{\rho^\alpha} \, dx - \frac{1}{2} \int_{\partial \Omega} |\nabla^2 u|^2 < \frac{x}{\rho}, N > \frac{d\sigma}{\rho^{\alpha-1}}
\]
\[
- \frac{1}{2} \int_{\Omega} |\nabla^2 u|^2 \frac{\partial}{\partial x_k} \frac{x_k}{\rho^\alpha} \, dx + \int_{\Omega} \frac{\partial^2 u}{\partial x_i \partial x_j} \cdot \frac{\partial^2 u}{\partial x_i \partial x_k} \frac{\partial}{\partial x_j} \frac{x_k}{\rho^\alpha} \, dx,
\]
where we also used the observation that \( \nabla u = 0 \) on \( \partial \Omega \) implies
\[
\frac{\partial}{\partial N} \left( \frac{\partial u}{\partial x_i} \right) \cdot \frac{\partial}{\partial \rho} \left( \frac{\partial u}{\partial x_i} \right) = |\nabla^2 u|^2 < \frac{x}{\rho}, N >.
\]
For the second term on the right side of (3.9), again from integration by parts, we have
\[
- \int_{\Omega} \frac{\partial^3 u}{\partial x_i \partial x_j} \cdot \frac{\partial^2 u}{\partial x_i \partial x_k} \frac{x_k}{\rho^\alpha} \, dx = -\frac{1}{2} \int_{\partial \Omega} |\nabla^2 u|^2 < \frac{x}{\rho}, N > \frac{d\sigma}{\rho^{\alpha-1}}
\]
\[
- \frac{1}{2} \int_{\Omega} |\nabla^2 u|^2 \frac{\partial}{\partial x_k} \frac{x_k}{\rho^\alpha} \, dx + \int_{\Omega} \frac{\partial^2 u}{\partial x_i \partial x_j} \cdot \frac{\partial^2 u}{\partial x_i \partial x_k} \frac{\partial}{\partial x_j} \frac{x_k}{\rho^\alpha} \, dx,
\]
where we also used the observation that \( \nabla u = 0 \) on \( \partial \Omega \) implies
\[
\frac{\partial}{\partial N} \left( \frac{\partial u}{\partial x_i} \right) \cdot \frac{\partial}{\partial \rho} \left( \frac{\partial u}{\partial x_i} \right) = |\nabla^2 u|^2 < \frac{x}{\rho}, N >.
\]
For the second term on the right side of (3.9), we have
\[
- \int_{\Omega} \frac{\partial^3 u}{\partial x_i \partial x_j} \cdot \frac{\partial u}{\partial x_i} \frac{\partial}{\partial x_i} \frac{x_k}{\rho^\alpha} \, dx
\]
\[
= \int_{\Omega} \frac{\partial^2 u}{\partial x_i \partial x_j} \cdot \frac{\partial^2 u}{\partial x_i \partial x_j} \cdot \frac{\partial}{\partial x_i} \frac{x_k}{\rho^\alpha} \, dx + \int_{\Omega} \frac{\partial^2 u}{\partial x_i \partial x_j} \cdot \frac{\partial u}{\partial x_i} \cdot \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{x_k}{\rho} \, dx.
\]
Substituting (3.10) and (3.12) into (3.9) and using
\[ \frac{\partial}{\partial x_i} \left( \frac{x_j}{\rho^\alpha} \right) = \frac{\delta_{ij}}{\rho^\alpha} - \alpha \frac{x_i x_j}{\rho^{\alpha+2}}, \]
\[ \frac{\partial^2}{\partial x_i \partial x_j} \left( \frac{x_k}{\rho^\alpha} \right) = -\alpha (\delta_{ik} x_j + \delta_{ij} x_k + \delta_{jk} x_i) \rho^{-\alpha-2} + \alpha (\alpha + 2) \frac{x_i x_j x_k}{\rho^{\alpha+4}}, \]
we obtain
\[ \int_\Omega \Delta^2 u \cdot \frac{\partial u}{\partial \rho} \frac{dx}{\rho^{\alpha-1}} = -\frac{1}{2} \int_{\partial \Omega} |\nabla^2 u|^2 \frac{dx}{\rho^\alpha} + \frac{1}{2} (\alpha + 4 - n) \int_\Omega |\nabla^2 u|^2 \frac{dx}{\rho^\alpha} \]
\[ - 2\alpha \int_\Omega \left| \frac{\partial u}{\partial \rho} \nabla u \right|^2 \frac{dx}{\rho^\alpha} - 2\alpha \int_\Omega \frac{\partial^2 u}{\partial x_i \partial x_j} \cdot \frac{\partial u}{\partial x_j} \frac{x_i}{\rho^{\alpha+2}} dx \]
\[ - \alpha \int_\Omega \Delta u \cdot \frac{\partial u}{\partial \rho} \frac{dx}{\rho^{\alpha+1}} + \alpha (\alpha + 2) \int_\Omega \frac{\partial^2 u}{\partial x_i \partial x_j} \cdot \frac{\partial u}{\partial x_k} \frac{x_i x_j x_k}{\rho^{\alpha+4}} dx. \]

Finally we note that
\[ 2 \int_\Omega \frac{\partial^2 u}{\partial x_i \partial x_j} \cdot \frac{\partial u}{\partial x_j} \frac{x_i}{\rho^{\alpha+2}} dx = -\int_\Omega |\nabla u|^2 \frac{\partial}{\partial x_i} \left( \frac{x_i}{\rho^{\alpha+2}} \right) dx \]
\[ = (\alpha + 2 - n) \int_\Omega |\nabla u|^2 \frac{dx}{\rho^{\alpha+2}}, \]
and
\[ \int_\Omega \frac{\partial^2 u}{\partial x_i \partial x_j} \cdot \frac{\partial u}{\partial x_k} \frac{x_i x_j x_k}{\rho^{\alpha+4}} dx = -\frac{1}{2} \int_\Omega \frac{\partial u}{\partial x_j} \frac{\partial}{\partial x_k} \left( \frac{x_i x_j x_k}{\rho^{\alpha+4}} \right) dx \]
\[ = \frac{1}{2} (\alpha + 2 - n) \int_\Omega |\nabla u|^2 \frac{dx}{\rho^{\alpha+2}}. \]

The desired formula (3.8) follows by substituting (3.15), (3.16) as well as (2.19) into (3.14). The proof is complete.

Lemma 3.17. Suppose that \( u \in C^1(\overline{\Omega}) \) and \( u = 0 \) on \( \partial \Omega \). Then, for any \( \alpha \in \mathbb{R} \),
\[ \int_\Omega \left| \frac{\partial}{\partial \rho} \left( u \rho^{n-\alpha} \right) \right|^2 \frac{dx}{\rho^{n-2}} = \int_\Omega \left| \frac{\partial u}{\partial \rho} \right|^2 \frac{dx}{\rho^{n-2}} - \frac{1}{4} (n - \alpha)^2 \int_\Omega |u|^2 \frac{dx}{\rho^\alpha}, \]
where \( \rho = |x - y| \) with \( y \in \overline{\Omega}^c \) fixed.

Proof. To see (3.18), we note that
\[ \left| \frac{\partial}{\partial \rho} \left( u \rho^{n-\alpha} \right) \right|^2 \]
\[ = \left| \frac{\partial u}{\partial \rho} \right|^2 \rho^{-n} + (n - \alpha) u \frac{\partial u}{\partial \rho} \rho^{n-\alpha-1} + \frac{1}{4} (n - \alpha)^2 |u|^2 \rho^{n-\alpha-2}. \]
Also, using integration by parts and $u = 0$ on $\partial \Omega$, we have
\begin{equation}
(n - \alpha) \int_\Omega u \frac{\partial u}{\partial \rho} \rho^{n-\alpha-1} \frac{dx}{\rho^{n-2}} = -\frac{1}{2} (n - \alpha) \int_\Omega |u|^2 \frac{\partial}{\partial x_i} \left( \frac{x_i}{\rho^\alpha} \right) \frac{dx}{\rho^{n-2}} = -\frac{1}{2} (n - \alpha)^2 \int_\Omega |u|^2 \frac{dx}{\rho^\alpha}.
\end{equation}

In view of (3.19), this gives (3.18).

We are now ready to prove the integral identity (3.1).

**Lemma 3.21.** Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$. Suppose that $u \in C^4(\Omega)$ and $u = 0$, $\nabla u = 0$ on $\partial \Omega$. Then (3.1) holds for any $\alpha < n$ and any $y \in \partial \Omega$.

**Proof.** By the Lebesgue Dominated Convergence Theorem, it suffices to establish (3.1) for $y \in \Omega^c$. To this end, we note that
\begin{equation}
\int_\Omega \Delta u \cdot \Delta \left( \frac{u}{\rho^{\alpha}} \right) \frac{dx}{\rho^{\alpha}} = \int_\Omega \frac{\partial^2 u}{\partial x_i \partial x_j} \cdot \frac{\partial^2 u}{\partial x_i \partial x_j} \left( \frac{u}{\rho^{\alpha}} \right) \frac{dx}{\rho^{\alpha}},
\end{equation}
from integration by parts. Thus, by (3.3) and (3.8), we have
\begin{align*}
(\alpha + 4 - n) \int_\Omega \Delta u \cdot \Delta \left( \frac{u}{\rho^{\alpha}} \right) \frac{dx}{\rho^{\alpha}} - 2 \int_\Omega \Delta^2 u \cdot \frac{\partial u}{\partial \rho} \frac{dx}{\rho^{\alpha-1}}
&= \int_{\partial \Omega} |\nabla^2 u|^2 < \frac{x - y}{\rho}, N > \frac{d\sigma}{\rho^{\alpha-1}}
+ 4\alpha \int_\Omega \left| \frac{\partial}{\partial \rho} \nabla u \right|^2 \frac{dx}{\rho^{\alpha}} - \alpha(n - \alpha - 2)^2 \int_\Omega |\nabla u|^2 \frac{dx}{\rho^{\alpha+2}}
+ 2\alpha(\alpha + 2)(n - \alpha - 2) \int_\Omega \left| \frac{\partial u}{\partial \rho} \right|^2 \frac{dx}{\rho^{\alpha+2}}
- \frac{1}{2} \alpha(\alpha + 2)(n - \alpha - 2)(n - \alpha - 4)^2 \int_\Omega |u|^2 \frac{dx}{\rho^{\alpha+4}}.
\end{align*}

In view of (3.18), this gives the integral identity (3.1).

Next we will use (3.1) to derive estimate (1.4) on convex domains with smooth boundaries for any $\lambda < n$.

**Lemma 3.23.** Let $\Omega$ be a convex domain in $\mathbb{R}^n$, $n \geq 4$ with smooth boundary. Let $0 < \lambda < n$. Then there exist constants $C_0 > 0$ and $R_0 > 0$ depending only on $n$, $\lambda$ and the Lipschitz character of $\Omega$ such that estimate (1.4) holds for any $v$ satisfying (1.5).

**Proof.** Let $R_0 > 0$ be a constant so that for any $Q \in \partial \Omega$, $T(Q, 4R_0)$ is given by the intersection of $B(Q, 4R_0)$ and the region above a Lipschitz graph, after a possible rotation. Fix $Q \in \partial \Omega$ and $0 < R < R_0$. Let $v$ be a biharmonic function in $\Omega$ such that $v = \frac{\partial v}{\partial N} = 0$.
on \(I(Q, R)\) and \((\nabla v)^* \in L^2(\partial \Omega)\). Since \(\Omega\) has smooth boundary, by the classical regularity theory for elliptic equations, \(v \in C^4(T(Q, R/2))\).

Let \(\eta\) be a smooth function on \(\mathbb{R}^n\) such that \(\eta = 1\) on \(B(Q, R/8)\), \(\text{supp} \eta \subset B(Q, R/4)\) and \(|\nabla^k \eta| \leq C/R^k\) for \(0 \leq k \leq 4\). Note that \(u = v \eta \in C^4(\bar{\Omega})\) and \(u = 0, \nabla u = 0\) on \(\partial \Omega\). Thus we may apply integral identity (3.1) to \(u\) with \(\alpha = n - 2\) and \(y = Q\). This gives

\[
(3.24) \quad \int_\Omega \left| \frac{\partial}{\partial \rho} (\nabla u) \right|^2 \frac{dx}{\rho^{n-2}} \leq C \left\{ \int_\Omega \Delta^4 u \cdot \frac{\nabla u}{\rho^{n-3}} dx + \left| \int_\Omega \Delta^4 u \cdot \frac{u}{\rho^{n-2}} dx \right| \right\}.
\]

Since \(\Delta^2 v = 0\) in \(\Omega\), we have

\[
(3.25) \quad \Delta^2 u = 2 < \nabla (\Delta v), \nabla \eta > + \Delta v \cdot \Delta \eta + \Delta \{2 < \nabla v, \nabla \eta > + \eta \Delta \eta \}.
\]

Substituting (3.25) into (3.24) and using integration by parts as well as Cauchy inequality, we obtain

\[
(3.26) \quad \int_\Omega \left| \frac{\partial}{\partial \rho} (\nabla u) \right|^2 \frac{dx}{\rho^{n-2}} \leq \frac{C}{R^{n-2}} \int_{T(Q, R/4)} \left\{ |\nabla^2 v|^2 + |\nabla v|^2 + \frac{|v|^2}{R^2} \right\} dx \\
\leq \frac{C}{R^n} \int_{T(Q, R/2)} |\nabla v|^2 dx,
\]

where we also used the Caccioppoli’s inequality (2.2) and Poincaré inequality in the second inequality.

Since \(\text{supp}(u) \subset B(Q, R)\), for any \(\delta \in (0, n - 2)\), we have

\[
\int_\Omega \left| \frac{\partial}{\partial \rho} (\nabla u) \right|^2 \frac{dx}{\rho^{n-2}} \geq \frac{1}{R^{n-2}} \int_\Omega \left| \frac{\partial}{\partial \rho} (\nabla u) \right|^2 \frac{dx}{\rho^{n-2-\delta}} \geq \frac{\delta^2}{4R^\delta} \int_\Omega |\nabla u|^2 \frac{dx}{\rho^{n-\delta}},
\]

where the second inequality follows from (3.18) with \(\alpha = n + 2 - \delta\), which also holds for \(y \in \partial \Omega\) if \(\alpha < n + 2\). In view of (3.26), this gives

\[
\int_{T(Q, r)} |\nabla v|^2 dx \leq r^{n-\delta} \int_{T(Q, r)} |\nabla u|^2 \frac{dx}{\rho^{n-\delta}} \leq C_\delta \left( \frac{r}{R} \right)^{n-\delta} \int_{T(Q, R)} |\nabla v|^2 dx,
\]

for any \(0 < r < R/8\). Estimate (1.4) is thus proved for \(\lambda = n - \delta\).

Lemma 3.23, together with a well known approximation argument, gives part (b) of the Main Theorem.

**Theorem 3.27.** Let \(\Omega\) be a bounded convex domain in \(\mathbb{R}^n\), \(n \geq 4\). Then the \(L^p\) Dirichlet problem (1.1) is uniquely solvable for any \(2 - \varepsilon < p < \infty\).

**Proof.** Let \(p > 2\) and \(f \in L_1^p(\partial \Omega), g \in L^p(\partial \Omega)\). We need to show that the unique solution \(u\) to the \(L^2\) Dirichlet problem (1.1) satisfies estimate (1.7). To this end, we first note that
by an approximation argument (e.g. see [JK] for Laplace’s equation), we may assume that 
\( f, g \in C^\infty_0(\mathbb{R}^n) \).

Next we approximate \( \Omega \) from outside by a sequence of convex domains \( \{ \Omega_j \} \) with smooth boundaries, \( \Omega_1 \supset \Omega_2 \supset \cdots \supset \Omega \). Let \( u_j \) be the solution to the \( L^2 \) Dirichlet problem (1.1) on \( \Omega_j \) with boundary data \((u_j, \frac{\partial u_j}{\partial N}) = (f|_{\partial \Omega_j}, g|_{\partial \Omega_j}) \) on \( \partial \Omega_j \). By Lemma 3.23 and Theorem 1.3, we have

\[
(3.28) \quad \| (\nabla u_j)^* \|_{L^p(\partial \Omega)} \leq C \| (\nabla u_j)^* \|_{L^p(\partial \Omega_j)} \leq C \left\{ \| \nabla t f \|_{L^2(\partial \Omega_j)} + \| g \|_{L^p(\partial \Omega_j)} \right\},
\]

where \((\nabla u_j)^* \) denotes the non-tangential maximal function of \( \nabla u_j \) with respect to \( \Omega_j \), and \( C \) is a constant independent of \( j \). Estimate (3.28) implies that the sequence \( \{ \nabla u_j \} \) is uniformly bounded on any compact subset of \( \Omega \). It follows that there exist a subsequence, which we still denoted by \( \{ \nabla u_j \} \), and a function \( u \) on \( \Omega \) such that \( u_j \) converges to \( u \) uniformly on any compact subset of \( \Omega \). It is easy to show that \( u \) is biharmonic in \( \Omega \). Also by (3.28) and Fatou’s Lemma,

\[
(3.29) \quad \| (\nabla u)^* \|_{L^p(\partial \Omega)} \leq C \left\{ \| \nabla t f \|_{L^p(\partial \Omega)} + \| g \|_{L^p(\partial \Omega)} \right\},
\]

where \( K \) is a compact subset of \( \Omega \), and \((\nabla u)^* (Q) = \sup\{|\nabla u(x)| : \ x \in K \text{ and } |x - Q| < 2 \text{dist}(x, \partial \Omega)\}\). By the monotone convergence theorem, this gives the estimate (1.7) on \( \Omega \).

Finally one may use \( L^2 \) estimates on \( \| (\nabla u_i - \nabla u_j)^* \|_{L^2(\partial \Omega_i)} \) for \( i \geq j \) as well as \( L^2 \) regularity estimate, \( \| (\nabla^2 u_j)^* \|_{L^2(\partial \Omega_j)} \leq C \left\{ \| \nabla^2 f \|_{L^2(\partial \Omega_j)} + \| \nabla g \|_{L^2(\partial \Omega_j)} \right\} \) (see [V2]) to show that \( u = f \) and \( \frac{\partial^2 u}{\partial N^2} = g \) on \( \partial \Omega \) in the sense of non-tangential convergence. We leave the details to the reader.

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