MEROMORPHIC VECTOR FIELDS WITH SINGLE-VALUED SOLUTIONS ON COMPLEX SURFACES

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Abstract. We study ordinary differential equations in the complex domain given by meromorphic vector fields on Kähler compact complex surfaces. We prove that if such an equation has a maximal single valued solution with Zariski-dense image (in particular, if it has an entire one) then, up to a bimeromorphic transformation, either the vector field is holomorphic or it preserves a fibration.

1. Introduction

For algebraic differential equations in the complex domain, the existence of special kinds of solutions often imposes very restrictive conditions on the equations themselves. An instance of this is given by Malmquist’s centennial result [Mal13, Théorème 1]:

Theorem 1 (Malmquist, 1913). Let $R(w, t)$ be a rational function of $w$ and $t$ with complex coefficients. Let $w(t)$ be a transcendental meromorphic solution in $\mathbb{C}$ to the differential equation

$$w' = R(w, t).$$

Then $R(t) = A(t)w^2 + B(t)w + C(t)$, with $A$, $B$ and $C$ rational functions (the equation is a Riccati one).

Through the years, this theorem has known many proofs, from the early one by Yosida using Nevalinna theory [Yos33] to the more recent geometric one of Pan and Sebastiani [PS01]. Generalizations of the theorem range from results valid for more general first-order non-autonomous equations (including some by Malmquist himself [Mal20]), to some valid for higher order non-autonomous equations (although, in general, their conclusions are less categorical than that of Malmquist’s original theorem). They go from fully algebraic to only partially algebraic ones. We refer the reader to [Lai93, Ch. 10] and [Eré82] for a guide through these generalizations.

Our main result is a generalization of Malmquist’s theorem valid for differential equations given by meromorphic vector fields on complex Kähler surfaces (which are, in a
way, autonomous, second-order, algebraic systems). Our result considers not only solutions given by meromorphic functions defined in the whole plane but, more generally, single-valued ones:

**Main Theorem.** Let $S$ be a compact Kähler surface, $X$ a meromorphic vector field on $S$, $\Omega \subset \mathbb{C}$ an open subset and $\Phi : \Omega \rightarrow S$ a maximal solution of $X$ with Zariski-dense image. Up to a birational transformation, either

- $X$ is holomorphic; or
- $S$ fibers over a rational or elliptic curve, with rational or elliptic fiber, and $X$ preserves the fibration (the poles of $X$ are contained in fibers).

A solution $\Phi : \Omega \rightarrow S$ is **maximal** if it has no analytic continuation as a map from $\mathbb{C}$ into $S$ beyond $\Omega$. This notion formalizes that of “single valued solution” and will be made precise in Definition 5. From the classification of holomorphic vector fields on Kähler surfaces (see Proposition 15 for a precise statement), it follows that those that do not preserve a fibration are holomorphic vector fields on Abelian surfaces (without Abelian subgroups). In consequence, in the statement of the Main Theorem, we may replace the possibility “$X$ is holomorphic” by “$X$ is a holomorphic vector field on an Abelian surface”.

One can interpret the fibered case as the existence a partial separation of variables: there is one variable that is integrated independently. For example, for the rational fibrations, we obtain, according to the nature of the vector field on the base, Riccati equations over rational, trigonometric (Clairaut) or elliptic function fields.

This result generalizes Theorem B of Rebelo and the author in [GR12], that states that the same conclusion may be obtained under the hypothesis that there is a maximal solution through every point of $S$ (that vector field is semicomplete). A related result is given by the tandem [Bru04], [BG12]. In the first, Brunella studied complete polynomial vector fields in $\mathbb{C}^2$, giving a complete list of normal forms up to polynomial automorphisms, after proving that such a vector field must have a first integral or preserve a fibration. In the second, Bustinduy and Giraldo, taking further Brunella’s approach, prove that if a polynomial vector field in $\mathbb{C}^2$ has one entire transcendental solution, it is actually complete. These results imply that if a polynomial vector field has one entire transcendental solution, it preserves a fibration.

The structure of the surfaces and vector fields in the Main Theorem imply the existence of a threshold for the number of maximal solutions that a vector field may have:

**Corollary 2.** Let $X$ be a meromorphic vector field in a compact complex Kähler surface having five different Zariski-dense maximal solutions. Then all the solutions are maximal.

(The proof will be given in Section 7) A vector field having exactly four maximal solutions appears in Example 23. The Main Theorem also implies that, in our context, the domains where maximal solutions are defined are very special:
Corollary 3. Let $S$ be a compact Kähler surface, $X$ a meromorphic vector field on $S$. Let $\Omega \subset \mathbb{C}$ and $\phi : \Omega \to S$ a maximal solution. Then $\overline{\Omega} = \mathbb{C}$.

Accordingly, it is not until dimension three that we may witness the existence of natural boundaries for functions satisfying algebraic differential equation (and we indeed do, like in the classical equations of Halphen, Chazy and Ramanujan [Gui07]). These natural boundaries are inevitably accompanied by rich dynamics:

Corollary 4. Let $M$ be a compact complex algebraic threefold, $X$ a meromorphic vector field on $S$, $\phi : \Omega \to S$ a maximal solution such of $X$ that $\overline{\Omega} \neq \mathbb{C}$. Then $\phi(\Omega)$ is Zariski dense.

(See [Gui07] Thm. A], [MS95] for direct proofs for Halphen’s equations). The last two corollaries are the direct analogues of Corollaries C and D from [GR12] to the present setting.

The proof of the Theorem is naturally split into two situations. One of them is when there is a maximal solution $\phi : \Omega \to S$ that cannot be extended to an entire mapping $\hat{\phi} : \mathbb{C} \to S$. In this situation we will adapt (and at some points just refer to) the techniques of Rebelo and the author’s article [GR12]. The idea is to show that such a maximal solution accumulates the locus of poles of $X$ in a complicated way, and that this imposes severe restrictions on the nature of this locus. In the Kähler setting, these restrictions imply the existence of the fibration. In the other situation, where all the maximal solutions are (or can be extended to) entire maps $\hat{\phi} : \mathbb{C} \to S$, we will use McQuillan’s classification of algebraic foliations on surfaces admitting an entire invariant curve (see Theorem 17) to gain information about the structure of the foliation. This information is then used together with the existence of the maximal solutions to conclude. Despite the similarity in the conclusions of McQuillan’s Theorem and of our main one, the conclusion does not follow in a straightforward way (see Example 24). McQuillan’s classification is an important ingredient in Brunella’s Theorem [Bru04] on the classification of complete polynomial vector fields in $\mathbb{C}^2$; it will play an analogue role in this work.

The reader is supposed to be familiar with general (mainly local) facts about foliations on surfaces, like those in [Bru00, Chapter 1]. For definitions related to the single-valuedness of solutions of complex differential equations, we refer the reader to [GR12, Section 2] and the following section.

2. Preliminaries

2.1. Maximal solutions. Let $X$ be a holomorphic vector field on the complex manifold $M$, let $p \in M$. By Cauchy’s theorem on the existence of solutions of ordinary differential equations, for every $p \in M$ there exists an open set $U \subset \mathbb{C}$, $0 \in U$, and a holomorphic map $\phi : (U, 0) \to (M, p)$ that is a solution of $X$ in the sense that for every $t \in U$, $D\phi|_p(\partial/\partial t) = X|_{\phi(t)}$. By considering the maximal domain where the analytic continuation of $\phi$ is defined (as a map from $\mathbb{C}$ into $M$) we obtain a function defined in some domain that spreads over $\mathbb{C}$ but that is not, in general, a subset of $\mathbb{C}$. For example,
the vector field $e^z \partial/\partial z$ in $\mathbb{C}$ has the multivalued solution $t \mapsto -\log(1 - t)$, defined in the Riemann surface of the logarithm. The following formalizes the notion of “single-valued solution”:

**Definition 5.** Let $X$ be a holomorphic vector field in the complex manifold $M$. Let $\Omega \subset \mathbb{C}$. A solution $\phi : \Omega \rightarrow M$ of $X$ is said to be maximal if for every sequence $\{t_i\} \subset \Omega$, converging in $\mathbb{C}$ but not in $\Omega$, the sequence $\{\phi(t_i)\}$ escapes from every compact subset of $M$. If $X$ is a meromorphic vector field in the complex manifold $M$, a maximal solution of $X$ is a maximal solution of the restriction to the open subset where it is holomorphic.

In other words, a solution is maximal if it has no analytic continuation beyond the domain where it is defined. If $\phi : \mathbb{C} \rightarrow M$ is a solution of $X$, it is automatically maximal. For a holomorphic vector field $X$ on a curve $C$, the time form of $X$ is the one-form $\omega$ such that $\omega(X) = 1$. An orbit $L$ of a holomorphic vector field admits a maximal solution if every curve $\gamma : [0, 1] \rightarrow L$ for which $\int_\gamma \omega = 0$ is closed [Reb96, Prop. 2.7]. If all the solutions are maximal the vector field is said to be semicomplete (following Rebelo [Reb96, Def. 2.3]) or univalent (following Palais [Pal57, Def. VI, p. 62]).

**Lemma 6.** If the meromorphic vector field $X$ in the curve $C$ admits a maximal solution, $X$ has no poles (is in fact holomorphic) and has at most two zeros (counted with multiplicity).

**Proof.** The solutions of strictly meromorphic vector fields on curves cannot be single-valued [GR12, Lemma 2]. Up to a change of coordinates, the zeros of a holomorphic vector field admitting maximal solutions are of the form $z^2 \partial/\partial z$ or $\lambda z \partial/\partial z$ [Reb96, Section 3]. In the first case, where the solution is given by $t \mapsto -1/t$, a whole neighborhood of the point at infinity belongs to the domain of definition of the solution. In the second case, the solution is given by $t \mapsto \exp(\lambda t)$ and a set of the form $\{t \mid \Re(\lambda t) < N\}$ belongs to this domain. If the vector field admits a maximal solution, these sets cannot overlap. In consequence, there can be at most two zeros counted with multiplicity (furthermore, if there are two simple ones, the corresponding eigenvalues should have opposite signs). □

**Remark 7.** The same proof shows that if a vector field on a curve has $2n$ zeros (counted with multiplicity), its solutions have at least $n$ determinations. A vector field having a pole of order $n - 1$ has a local solutions with $n$ determinations. However, unlike the case of zeros, this is not cumulative: if $\Sigma$ is a hyperelliptic curve of genus $g$ and $\pi : \Sigma \rightarrow \mathbb{P}^1$ is the quotient by the hyperelliptic involution, the pullback by $\pi$ of a generic vector field with a double zero on $\mathbb{P}^1$ to $\Sigma$ is a meromorphic vector field having two double zeros and $2(g + 1)$ simple poles, but whose solutions have only two determinations.

**Proposition 8.** If a meromorphic vector field on a curve has an essential singularity, its solutions have infinitely many determinations.

**Proof.** Consider the meromorphic vector field $f(z) \partial/\partial z$, defined in a punctured neighborhood $U^*$ of 0 in $\mathbb{C}$. Suppose its solutions have finitely many determinations and, in
particular, that no zeros of $f$ accumulate to 0. The time form $\omega = 1/f(z)dz$ is holomorphic in $U^*$. If the integral of $\omega$ around 0 vanishes, consider the function $h : U \to \mathbb{C}$ given by $\int \omega$. If $h$ has an essential singularity at 0, by Picard’s Great Theorem, it attains most values infinitely many times: the solutions of $f$ (given by the inverse of $h$) have infinitely many determinations. We must suppose that $h$ (and thus $f$) is meromorphic at 0. If the integral of $\omega$ around 0 is $2i\pi$, let $h : U \to \mathbb{C}$, $h(z) = \exp(\int^z \omega)$. Again, if $h$ has an essential singularity at 0, by Picard’s Great Theorem, it attains most values infinitely many times and the solutions of $f$, given by $t \mapsto h^{-1}(e^t)$, have infinitely many determinations. Thus, $h$ is meromorphic and, since $h'/h = 1/f$, $f$ is meromorphic at 0. □

In particular, a holomorphic vector field on $\mathbb{C}$ whose solutions have finitely many determinations extends as a rational vector field to $\mathbb{P}^1$.

2.2. **Affine structures on curves.** An affine structure on a complex curve $C$ is an atlas for its complex structure taking values in $\mathbb{C}$ whose changes of coordinates lie in the affine group $\text{Aff}(\mathbb{C}) = \{z \mapsto az + b\}$. With the affine structure come a developing map $D : \tilde{C} \to \mathbb{C}$ and a monodromy $\text{mon} : \pi_1(C) \to \text{Aff}(\mathbb{C})$ satisfying $D(\alpha \cdot p) = \text{mon}(\alpha)(D(p))$.

If the changes of coordinates of an affine structure lie within the group of translations $\{z \mapsto z + b\}$, the affine structure is said to be a translation structure. On a complex curve, translation structures are in correspondence to holomorphic and nowhere vanishing holomorphic vector fields: the charts of a translation structure are given by the local integrals of the time form of the corresponding vector field.

An affine structure on a curve $C$ is said to be uniformizable if $C$ is affinely equivalent to the quotient of an open subset of $\mathbb{C}$ by a group of affine transformations. In a curve endowed with a vector field, the induced affine structure is uniformizable if and only if the vector field admits a maximal solution.

Given two affine structures in a disk $\Delta$ with corresponding coordinate charts $g_i : \Delta \to \mathbb{C}$, $i = 1, 2$, for the map $h = g_2 \circ g_1^{-1}$, the differential

$$\frac{h''(z)}{h'(z)}dz$$

depends only on the affine structures (and not on the coordinate charts). It vanishes if and only if $g_1$ and $g_2$ define the same affine structure. It measures the difference of the affine structures. Reciprocally, given a holomorphic form $\eta$ and an affine structure with on $\Delta$, there is another affine structure such that the difference with the original one is $\eta$.

It will be important to understand the uniformizable affine structures defined in the complement of some points (affine structures with singularities). Given an affine structure in $\Delta^* = \Delta \setminus \{0\}$ and an affine structure on $\Delta$, their difference is a form $\eta$ in $\Delta^*$ having a singularity at 0 if the affine structure does not extend to $\Delta$. The ramification index of the singular affine structure is

$$\text{ind}(\Delta, 0) = \frac{1}{\text{Res}(\eta, 0) + 1}$$
For the affine structure on $\Delta$ whose charts are branches of $\sqrt[n]{z}$, $n \in \mathbb{Z}^*$, the ramification index is $n$; for the one having $\log(z)$, it is $\infty$. In both cases the difference with a regular affine structure on $\Delta$ has a simple pole. All these local affine structures with singularities are uniformizable and are in fact the only ones. Moreover, they are characterized by the invariants of the one-form: an affine structure on $\Delta^*$ with a singularity at 0 is uniformizable if and only if for the one-form $\eta$ measuring its difference with a regular affine structure on $\Delta$, $\eta$ has a simple pole at 0 with residue of the form $1/n - 1/n$, $n \in \mathbb{Z}^* \cup \{\infty\}$.

This local result globalizes on compact curves as follows:

**Proposition 9.** The uniformizable affine structures (with singularities) on compact curves are:

- **Rational orbifolds.** The rational curve $\mathbb{C} \cup \{\infty\}$ (with the tautological affine structure of $\mathbb{C}$) and the quotients of the latter by the cyclic groups of linear transformations of order $n$ fixing 0 and $\infty$, realized by $z \mapsto z^n$. The ramification indices are $n$ at 0 and $-n$ at $\infty$.

- **Parabolic.** The cylinder, $\mathbb{C}/2\pi\mathbb{Z}$ compactified by $\exp: \mathbb{C} \to \mathbb{P}^1$, with two points of ramification index $\infty$ at 0 and $\infty$. The orbifold $(2,2,\infty)$, quotient of the latter by the involution $z \mapsto -z$, compactified by $\cos(z)$, having one point with ramification index 0 and two with ramification index 2.

- **Elliptic.** The curves and orbifolds arising as compact quotients of $\mathbb{C}$ under the action of (crystallographic) subgroups of the affine group. These contain a lattice as a normal subgroup of finite index and are thus quotients of elliptic curves. As curves, they are either elliptic without singularities, or rational curves with singularities of indices $p_i$, such that $\sum (1 - 1/p_i) = 2$. The possibilities are $(2,3,6)$, $(2,4,4)$, $(3,3,3)$ or $(2,2,2,2)$, the last being a one-parameter family parametrized by the cross-ratio.

- **Hyperbolic.** Tori of the form $\mathbb{C}^*/G$, $G \subset \mathbb{C}^*$ a discrete group containing hyperbolic elements (without singularities).

The result is, of course, related to the classification of the crystallographic groups of the plane. We refer to [GR12 Prop. 7] for a proof along the lines of this article (see also [BB55] for an early appearance of this classification among complex differential equations).

### 2.3. Foliations and vector fields on surfaces.

We refer to [Bru00 Ch. 1] for a swift presentation of some general facts about the local theory of foliations and vector fields in surfaces, to [Lor06 Section 5.3] for a more detailed one.

A meromorphic vector field on a surface induces naturally a holomorphic foliation (with singularities). In general, the leaves of this foliation are not closed, but some ends of some leaves may have a particularly mild dynamic behavior. Let $L$ be a leaf of the foliation $\mathcal{F}$. We will say that an end of $L$ (considering the latter as a curve) is an algebraic isolated planar end if there exists a holomorphic map $\gamma: \Delta \to M$ (for $\Delta = \{z; |z| < 1\}$) such that, for $\Delta^* = \Delta \setminus \{0\}$, $\gamma(\Delta^*) \subset L$, $\gamma(0) \notin L$, and such that $\gamma|_{\Delta^*}$ is a biholomorphism between $\Delta^*$ and a neighborhood of the corresponding end of $L$. 

Let $X$ be a meromorphic vector field in the complex manifold $S$. In the open subset where $X$ is holomorphic and nonzero, the leaves of the induced foliation $\mathcal{F}$ are naturally endowed a translation (in particular, affine) structure whose transverse variation is holomorphic. If $C$ is a component of the locus of zeros or poles of $X$ that is invariant by $\mathcal{F}$, the leafwise affine structure induced by $X$ in the leaves other than $C$ extends to $C$ (although the translation structure does not). For example, if $X = f(x, y)y^q \frac{\partial}{\partial x}$ with $f$ holomorphic and nonzero, on the leaf $x = x_0$, $x_0 \neq 0$, the vector fields $X$ and $y^{-q}X$ are proportional and induce the same affine structure. The vector field $y^{-q}X$ induces a honest translation structure on the leaf $x = 0$, whose affine class is well-defined (depends only on $X$). In this way, in the complement of the singularities of $\mathcal{F}$ and of the curves of zeros and poles that are not invariant by $\mathcal{F}$, the foliation $\mathcal{F}$ admits a leafwise affine structure. It extends as an affine structure with singularities to the non-singular points of $\mathcal{F}$ lying at the curve of zeros and poles.

**Definition 10.** Let $X$ be a meromorphic vector field on the complex surface $M$. The uniformizable or univalent locus, denoted by $\mathcal{U}$, is the subset of $M \setminus \text{Sing}(\mathcal{F})$ of leaves carrying a uniformizable affine structure.

The set $\mathcal{U}$, naturally saturated by $\mathcal{F}$, is closed. Let us sketch a proof of this fact. Lack of uniformizability of an affine structure in a curve $C$ is equivalent to the existence of a path $\gamma : [0, 1] \to C$, $\gamma(0) \neq \gamma(1)$ such that the developing map of the affine structure along $\gamma$ is not injective. If $C$ is a leaf of the foliation, we can lift the path $\gamma$ to a path $\gamma'$ in a neighboring leaf $C'$ in such a way that the lift extends to a local isomorphism of the affine structures along neighborhoods of $\gamma$ and $\gamma'$. But this means that the affine structure in $C'$ is not uniformizable (see also [GR12, Cor. 12]).

Within $\mathcal{U}$, the leafwise geometry conditions the holonomy. This is the content of the Fundamental Lemma, for the proof of which we refer the reader to [GR12, Section 4.2]:

**Lemma 11.** Let $X$ be a holomorphic vector field on the manifold $M$. Let $\mathcal{U} \subset M \setminus \text{Sing}(\mathcal{F})$ denote the set given by the leaves where the induced affine structure (with singularities) is uniformizable. Let $p \in \mathcal{U}$, let $\mathcal{L}$ be the leaf of $\mathcal{F}$ passing through $p$ and suppose that the affine structure at $p$ is non-singular. Let $T$ be a transversal of $\mathcal{F}$ through $p$, let $\Sigma = T \cap \mathcal{U}$. The restricted holonomy representation $\text{hol} : \pi_1(\mathcal{L}, p) \to \text{Homeo}(\Sigma, p)$ factors through the monodromy representation $\text{mon} : \pi_1(\mathcal{L}, p) \to \text{Aff}(\mathbb{C})$.

### 2.4. Reduced foliations and vector fields.

A foliation $\mathcal{F}$ in a surface is said to be reduced in Seidenberg’s sense if, in the neighborhood every point $p$, either (i) $\mathcal{F}$ is non-singular at $p$, (ii) $\mathcal{F}$ is tangent to a holomorphic vector field whose linear part has two non-zero eigenvalues whose ratio is not a positive rational or (iii) $\mathcal{F}$ is tangent to a vector field having one zero eigenvalue but a non-nilpotent linear part. Seidenberg’s theorem affirms that every foliation becomes reduced after a locally finite number of blowups.

A meromorphic vector field $X$ in a complex surface inducing the foliation $\mathcal{F}$ is said to be reduced if $\mathcal{F}$ is reduced in Seidenberg’s sense and if, at every point $p$, the curve given by the union of the curve of zeros and poles of $X$ and the union of separatrices of $\mathcal{F}$.

Through $p$ (if $p$ is a singularity of $F$) or the integral curve of $F$ through $p$ (if $F$ is non-singular at $p$), has normal crossings. A meromorphic vector field may be transformed, by a locally finite number of blowups, to a reduced one.

2.5. **Special foliations.** Some special foliations adapted to rational or elliptic fibrations will have a prominent role in our discussion. We refer the reader to [Bru00, Ch. 4] for details.

2.5.1. **Riccati foliations.** A foliation $F$ in a compact complex surface $M$ is a Riccati foliation if there exists a rational fibration $\Pi : M \to S$ that is adapted to $F$ in the sense that $F$ is transverse to the generic fiber of $\Pi$. The non-generic fibers $\{\Pi^{-1}(p_i)\}$ are called special. There is a monodromy representation $\rho : \pi_1(S \setminus \{p_1, \ldots, p_k\}) \to \text{PSL}(2, \mathbb{C})$.

Some natural transformations for Riccati foliations are given by flips: one may blow up a point in a fiber $F_0$, creating an exceptional divisor $D$ of self-intersection $-1$ and making $F_0$ a curve of self-intersection $-1$, which may then be blown down, making $D$ a curve of self-intersection $0$. In coordinates $(z, w)$ in $\Delta \times \mathbb{P}^1$ these flips are given by

$$(2) \quad (z, w) \mapsto (z, zw).$$

It may be useful to override the condition of a locally trivial fibration in the definition of a Riccati foliation, in order to have simpler singularities for $F$. Up to a birational transformation, the foliation can be, in the neighborhood of the special fibers, brought, to one of the following kinds (see [Bru00, Ch. 4, Prop. 2]):

- **Non-singular or transverse:** Those given by $dw$.
- **Non-degenerate, non parabolic:** Given by

  $$\lambda w dz - z dw,$$

  with $\lambda \in \mathbb{C} \setminus \mathbb{Q}$. The monodromy is $w \mapsto e^{2\pi i \lambda} w$. The flip (2) replaces $\lambda$ by $\lambda + 1$.

- **Non-degenerate, parabolic:** Those given by $dz - z dw$. The monodromy is $w \mapsto w + 2\pi i$.

- **Dicritical:** Those of the form (3) with $\lambda \in \mathbb{Q}^* \setminus \mathbb{N}$, $\lambda = p/q$, $(p, q) = 1$, $q > p > 0$. The foliation has the local first integral $w^q/z^p$. The monodromy, $w \mapsto e^{2\pi i p/q} w$, is periodic with period $q$. The integer $q$ is also called the multiplicity of the fiber.

  We may choose to reduce the singularities at the price of complicating the space. Upon reducing the singularity at $(0, 0)$ we will find a chain of rational curves, all of them invariant by the foliation except one, transverse to the foliation, which is not one of the two extremal ones. The two invariant chains can be contracted to produce a finite quotient singularity each. An alternative description of these fibers is the following: given a finite cyclic group acting in the neighborhood of a transverse fiber $F$ preserving the foliation and acting faithfully in the base, the quotient under this action has two cyclic quotient singularities and a rational curve.

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\footnote{Warning!!! we are using the word monodromy for two different concepts. We have the monodromy representation associated to a developing map of an affine structure and the monodromy representation of a Riccati (or turbulent) foliation.}
coming from $F$, transverse to the foliation. The resolution of the two singularities takes us to the previous description.

**Semidegenerate:** There are two saddle-node singularities on $F$ whose strong separatrices are contained in $F$ (in particular, the holonomy of the foliation along $F$ is tangent to the identity). The weak separatrices may or may not converge.

**Nilpotent:** There is only one singular point in $F$, with nilpotent linear part. They can be seen as quotients of semidegenerate models by an involution exchanging the two saddle-nodes. Its resolution has three rational divisors: one central divisor of self-intersection $-1$ having one saddle-node singularity (with strong separatrix contained in the divisor) and two saddles where the other two divisors, of self-intersection $-2$, intersect (there are no further singularities of $F$ in these).

A Riccati foliation having special fibers as above will be said to be in *standard form*.

### 2.5.2. Elliptic fibrations, turbulent foliations.

An **elliptic fibration** on a surface $S$ is a map $\Pi : S \to B$ onto a curve, with connected fibers, such that the generic fiber is an elliptic curve. Kodaira classified the fibers of elliptic fibrations that are not elliptic curves up to a birational transformation. Kodaira’s combinatorial models bear the symbols $I_n$ ($n \geq 0$), $II$, $III$, $IV$, $I_n^*$ ($n \geq 0$), $II^*$, $III^*$, $IV^*$. They are all *divisors of elliptic fiber type*, effective divisors of the form $D = \sum a_iD_i$ such that $D \cdot D_i = 0$ and $K_S \cdot D = 0$. See [BPVdV84] for details. An elliptic fibration is said to be *isotrivial* if the fibration is a holomorphically trivial fibration in the neighborhood of a generic fiber. In an isotrivial fibration the special fibers are restricted to be among Kodaira’s models $II$, $III$, $IV$ or $II^*$, $III^*$, $IV^*$.

A foliation $\mathcal{F}$ in a compact complex surface $S$ is a **turbulent foliation** if there exists an elliptic fibration $\Pi : S \to B$ that is adapted to $\mathcal{F}$ in the sense that $\mathcal{F}$ is transverse to the generic fiber of $\Pi$. If adapted to a turbulent foliation, an elliptic fibration is necessarily isotrivial (this limits the special fibers occurring in the fibration: there are no special fibers of type $I_n^*$ and no ones of type $I_n$ for $n \geq 1$). A turbulent foliation comes with a *monodromy* representation $\rho : \pi_1(S \setminus \{p_1, \ldots, p_k\}) \to \text{Aut}(F)$.

In the neighborhood of a fiber, up to a ramified cover of the base and a bimeromorphic transformation, a turbulent foliation is non-singular and adapted to a locally trivial elliptic fibration. In a turbulent foliation of a locally trivial elliptic fibration, a fiber may be everywhere transverse to the foliation or totally invariant by it. This implies that for a general turbulent foliation, we have two coarse kinds of special fibers, *transverse* and *invariant*.

### 3. A dichotomy

We have the following dichotomy:

**Proposition 12.** Let $M$ be a compact complex manifold, $X$ a meromorphic vector field on $M$ and $\phi : \Omega \to M$ a maximal solution of $X$ defined in a domain $\Omega \subset \mathbb{C}$. Either:

- $\Omega = \mathbb{C}$ or $\phi$ extends holomorphically to a function $\overline{\phi} : \mathbb{C} \to M$; or
there exists \( p \in \bar{\phi}(\Omega) \), \( p \) in the locus of poles of \( X \), such that \( \bar{\phi}(\Omega) \) is not contained in a proper analytic subset in a neighborhood of \( p \).

Proof. Let \( L = \phi(\Omega) \). If \( L \) does not accumulate the locus of poles of \( X \), there exists a neighborhood \( U \) of this locus that does not intersect \( L \). By the existence theorem for the solutions of ordinary differential equations in the complex domain and the compactness of \( M \setminus U \), there exists some \( \epsilon > 0 \) such that any solution of \( X \) with initial condition in \( M \setminus U \) is defined for \( \{|t| < \epsilon\} \). In particular, since \( \phi \) is maximal and \( L \cap U = \emptyset \), \( \Omega = \mathbb{C} \).

Hence, if \( \Omega \neq \mathbb{C} \), there exists at least one connected component \( D_0 \) of the locus of poles of \( X \) such that \( L \) intersects any neighborhood of \( D_0 \). Let \( \{t_i\} \subset \Omega \) be a sequence converging to \( t_\infty \in \mathbb{C} \setminus \Omega \) such that \( \{\phi(t_i)\} \) converges to \( p \in M \). The point \( p \) belongs necessarily to the locus of poles of \( X \).

Let \( V \) be an analytic subset in a neighborhood \( U \) of \( p \) containing \( \overline{L \cap U} \) and let \( V_0 \subset V \) be an irreducible component containing infinitely many of the \( \{\phi(t_i)\} \). By Puiseux’s theorem on the parametrization of analytic sets, there exists a holomorphic parametrization \( \gamma : (\Delta, 0) \to (V_0, p) \). It compactifies one algebraic isolated planar end of \( L \). Through \( \gamma \), \( X \) induces a meromorphic vector field \( X_0 \) on \( \Delta^* \) (which, by Lemma \( \mathbb{C} \), extends holomorphically to 0). If \( X_0(0) = 0 \), in \( \Delta \), the solution of \( X_0 \) takes infinite time to reach 0 (as explained in the proof of Lemma \( \mathbb{C} \)). This contradicts the fact that \( \{t_i\} \) converges in \( \mathbb{C} \). We must conclude that \( X_0(0) \neq 0 \). A local parametrization of the solution of \( X_0 \) through \( p \) induces an extension of \( \phi \) to a neighborhood of \( t_\infty \) (which, a posteriori, turns out to be an isolated point of the complement of \( \Omega \)).

In this way, if \( \phi \) cannot be extended to a neighborhood of \( t_\infty \), \( \bar{\phi}(\Omega) \) is not contained in a proper analytic subset of \( M \) in a neighborhood of \( p \). \( \square \)

The proof of the Main Theorem will be made separately for each one of the two situations where Proposition \( \boxed{12} \) leads. Section \( 5 \) will deal with the first alternative and Section \( 6 \) with the second one, after some description of the admissible local models in Section \( 4 \).

4. Local models at the limit set

We will now give local models of reduced holomorphic vector fields in the neighborhood of points that are accumulation points of \( \Omega \) in a non-analytic way. This result is the analogue of Prop. 17 in \[GR12\], which concerns semicomplete vector fields. The local models that we will obtain here are essentially the same as the ones obtained there, although our hypothesis are different. The proof of the present result will follow a path not too far from and at times intersecting that of \[GR12\].

Proposition 13. Let \( S \) be a (not necessarily compact) surface, \( X \) a reduced meromorphic vector field on \( S \) and \( \phi : \Omega \to S \) a maximal solution. Let \( L = \phi(\Omega) \) and let \( p \in \overline{L} \). If \( \overline{L} \) is not algebraic in a neighborhood of \( p \), either

- \( X \) is holomorphic and has at most an isolated singularity at \( p \); or
name & local model & ord & ind & CS \\
--- & --- & --- & --- & --- \\
regular & $y^q \frac{\partial}{\partial x}$ & q & 1 & 0 \\
finite & $x^p y^q \left( m x \frac{\partial}{\partial x} - n y \frac{\partial}{\partial y} \right)$ & $pm - qn = \pm 1$ & & \\
ramification & & & $x = 0$ & \\
 & & & $p^n - q/m = \pm 1$ & \\
infinite & $x^p y^q \left( x^{q/p} + \cdots \frac{\partial}{\partial x} - y^{p/q} + \cdots \frac{\partial}{\partial y} \right)$ & & & \\
ramification & & & $x = 0$ & \\
 & & & $p - q/p = \pm 1$ & \\
saddle node & $y^q \left( x^{q/p} + \cdots \frac{\partial}{\partial x} + y^{k+1} \frac{\partial}{\partial y} \right)$ & q & $\infty$ & 0 \\

Table 1. The (semi) local models for Proposition [13] (up to multiplication by a non-vanishing holomorphic function). In these, $p, q \in \mathbb{Z}$, $m, n > 0$ and $k \geq 0$.

- up to multiplication by a non-vanishing holomorphic function, $X$ is locally of one the forms of Table 1.

We include in Table 1 more information about the local models: the order of the vector field along the curves, the ramification index of the affine structure of vector field and the contribution of each singularity to the self-intersection of a curve according to the Camacho-Sad theorem [CS82].

Let $S$ be a (not necessarily compact) surface, $X$ a meromorphic vector field on $S$ and $\phi : \Omega \rightarrow S$ a maximal solution, $L = \phi(\Omega)$. Let $\hat{L}$ be the curve obtained from $L$ after adding all the algebraic isolated planar ends. By Lemma 6, the restriction of $X$ to $L$ extends holomorphically to $\hat{L}$ and has at most two zeros (counted with multiplicity). Let $\mathcal{L} \subset \hat{L}$ be the curve where the induced vector field is non-vanishing, $L \subset \mathcal{L} \subset \hat{L}$. Upon adding some isolated points to $\Omega$ we can extend $\phi$ to obtain a parametrization $\phi : \Omega \rightarrow \mathcal{L}$. (There is no contradiction with the maximal character of $\phi$: the maximal solution $\phi$ may not be maximal as a function from a subset of $\mathbb{C}$ into $M$). Notice that if $p \in \mathcal{L} \cap \text{Sing}(\mathcal{F})$, $\mathcal{L}$ is a separatrix of $\mathcal{F}$ through $p$.

Let us start with the proof of Proposition [13]. Let $X$ be a reduced meromorphic vector field in a neighborhood of 0 in $\mathbb{C}^2$ and suppose that the curve of zeros and poles of $X$ passes though $p$. Let $\{p_i\}$ be a sequence of points such that $\lim_{i \rightarrow \infty} p_i = 0$ but such that no point belongs to a separatrix of $\mathcal{F}$. We will suppose that the points $p_i$ belong to the same leaf $\mathcal{L}$ of $\mathcal{F}$ in some bigger ambient surface and that this leaf $\mathcal{L}$ corresponds to a maximal solution. The discussion splits naturally into the following cases, according to the local nature of $\mathcal{F}$:
4.1. **Regular point.** Suppose that $\mathcal{F}$ is regular at 0. Under the hypothesis that the vector field is reduced, coordinates can be chosen in a way such that $X = f(x,y)x^p y^q \partial/\partial x$, $p, q \in \mathbb{Z}$ and $f$ a holomorphic function that does not vanish at 0. The leaf $\mathcal{L}$ intersects infinitely many times $\{x = 0\}$. The curve $\hat{\mathcal{L}}$ has infinitely many points where the vector field is $f(x,y)x^p \partial/\partial x$ and thus, by Lemma 6, $p = 0$.

4.2. **Non-degenerate singularity.** If the foliation has a non-degenerate singularity at 0 (if it can be generated by a vector field with an isolated singularity and two non-zero eigenvalues), in suitable coordinates, the vector field is of the form

$$(4) \quad X = x^p y^q \left[ \lambda x f(x,y) \frac{\partial}{\partial x} + \mu y g(x,y) \frac{\partial}{\partial y} \right]$$

with $\mu/\lambda \in \mathbb{C}^* \setminus \mathbb{Q}^+$, $p, q \in \mathbb{Z}$, $p \neq 0$ or $q \neq 0$ and $f$ and $g$ holomorphic functions that do not vanish at the origin and such that $f(0) = g(0)$. The separatrices are given by $\{x = 0\}$ and $\{y = 0\}$.

4.2.1. **In the Siegel domain.** In this case, when $\mu/\lambda \in \mathbb{R}^-$, it is not difficult to see that every closed invariant set invariant by the foliation containing the origin that is not contained in the separatrix contains both separatrices (see [CR15, Lemma 18]). Thus, $\mathcal{L}$ contains both separatrices in its closure, which belongs to $\mathcal{U}$. Let us study the affine structure induced by (4) on the separatrix $\{y = 0\}$. Let us suppose that $\lambda = 1$. We have, from the proof of Proposition 17 in [GR12], that

$$\text{ind}(\{y = 0\}, 0) = -\frac{1}{p + \mu q}, \quad \text{ind}(\{x = 0\}, 0) = -\frac{\mu}{p + \mu q}. \quad \text{If } p + \mu q = 0 \text{ then both ramification indices equal } \infty. \quad \text{We have } \mu = -p/q \in \mathbb{Q} \text{ (in particular, } p \text{ and } q \text{ are have the same sign). This gives the \textit{infinite ramification} local model. If } p + \mu q \neq 0, \text{ ind}(\{y = 0\}, 0) \neq \infty \text{ and thus ind}(\{y = 0\}, 0) \in \mathbb{Z}^* \text{. In particular, } \mu \in \mathbb{Q}^+. \text{ Let } \mu = n/m \text{ with } (m, n) = 1 \text{ (notice that } m \text{ and } n \text{ must have different signs) and let } \Delta = pm + qn \in \mathbb{Z}^* \text{ so that ind}(\{y = 0\}, 0) = -m/\Delta \text{ and ind}(\{x = 0\}, 0) = -n/\Delta. \text{ Since } \Delta \text{ divides } m \text{ and } n, \Delta^2 = 1. \text{ If } q < 0 \text{ and } p \geq 0 \text{ then if } pm + qn = 1, \text{ we necessarily have } n = -1 \text{ and the curve of zeros does not intersect at such a point. The monodromy of the affine structure along the separatrix } \{y = 0\} \text{ is periodic with period } m. \text{ By Lemma 11, the holonomy of } \mathcal{F} \text{ along the separatrix with respect to some transversal has periodic points of period dividing } m \text{ at the intersection of the transversal with } \mathcal{U}. \text{ The existence of periodic points of this period for the holonomy forces the holonomy to be periodic with period } m. \text{ On its turn, this implies that the foliation is linearizable. Thus, the vector field has the sought form. If } p < 0 \text{ or } q < 0, \text{ one can furthermore suppose that } f \equiv 1, \text{ by following the proof of [GR12, Prop. 18].}$$

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2 There is a small fault in the proof of Prop. 18 in [GR12]. At some point, the proof affirms that some period function “must” vanish identically and then uses this vanishing to conclude, although there is not really any reason for this vanishing to take place in full generality. Nevertheless, in any situation that implies that the period function is constant, the conclusion of Prop. 18 in [GR12] holds. One of these
4.2.2. In the Poincaré domain. In this case, $\mu/\lambda \notin \mathbb{R}^-$, $\mu/\lambda \notin \mathbb{Q}^+$. The origin attracts all the orbits that are sufficiently close to it. The foliation is linearizable, so we may suppose that (4) is actually of the form

$$X = f(x, y)x^p y^q \left(\lambda x \frac{\partial}{\partial x} + \mu y \frac{\partial}{\partial y}\right),$$

with $f$ a holomorphic function such that $f(0, 0) = 1$. Up to multiplying $X$ by a constant, we will suppose that $\Re(\lambda) > 0$ and $\Re(\mu) > 0$ (this is possible because we are in the Poincaré domain). Let $V_N = \{ z | \Re(\lambda z) < -N, \Re(\mu z) < -N \}$ and consider the mapping $\rho : V_N \to \mathbb{C}^2; \rho(\zeta) = (x_0 e^{\lambda \zeta}, y_0 e^{\mu \zeta})$, parameterizing injectively an orbit of $X$. For any neighborhood $U$ of the origin of $\mathbb{C}^2$ there exists $N > 0$ such that $V_N$ is in $L \cap U$.

We will first address the case $f \equiv 1$. In restriction to $\rho(V_N)$, the vector field is, up to a constant factor, $e^{(\lambda p + \mu q)\zeta} \partial / \partial \zeta$ (notice that $\lambda p + \mu q \neq 0$). Integrating the time form, we find that

$$T(\zeta) = \int e^{-(\lambda p + \mu q)\zeta} d\zeta = -\frac{e^{-(\lambda p + \mu q)\zeta}}{\lambda p + \mu q}.$$  

There is no value of $N$ such that the restriction of $T$ to $V_N$ is injective.

Let us now move on to the general case. Let $\alpha \in \mathbb{R}^+$ and consider, for $\alpha \in [0, 1]$, the transformations $h^\alpha(x, y) = (\alpha x, \alpha y)$. Notice that $h$ preserves the foliation while preserving every leaf. Consider the vector field

$$X_\alpha = (\alpha^{\lambda p + \mu q})^{-1} h^\alpha X = f(\alpha x, \alpha y)x^p y^q \left(\lambda x \frac{\partial}{\partial x} + \mu y \frac{\partial}{\partial y}\right).$$

Let $X_0 = \lim_{\alpha \to 0} X_\alpha$. The limit exists since $\Re(\lambda) > 0$ and $\Re(\mu) > 0$. When $\alpha \neq 0$, the vector field $X_\alpha$ is conjugated to $X$, up to a constant multiple. When $\alpha = 0$, it corresponds to $f \equiv 1$ in the definition (5) of $X$. For that vector field, we have shown that the translation geometry in $L$ is not uniformizable. Let $\gamma : [0, 1] \to L$ be a path joining two different points along where (4), the integral of the time form associated to $X_0$, takes the same value. For $\alpha$ sufficiently small, we can slightly deform $\gamma$ into an open path (contained in $L$) along which the integral of the time form of $X_\alpha$ vanishes as well. But this shows that, for $X_\alpha$, time in $L$ is not schlicht.

4.3. Saddle-node. A reduced meromorphic vector field whose induced foliation is a saddle node reduces to the form

$$X = h(x, y)[x - g(y)]^n y^q \left[ f(x, y) \frac{\partial}{\partial x} + y^{k+1} \frac{\partial}{\partial y}\right],$$

with $k \geq 1$, $f$ a holomorphic function such that $f(x, 0) = x$, $g$ a holomorphic function such that $g(0) = 0$, $h$ a non-vanishing holomorphic function and $q, n \in \mathbb{Z}$. The term $(x - g)^n$ accounts for a curve of zeros or poles transverse to the strong separatrix $\{ y = 0 \}$. We claim that $n = 0$ (this would give the sought local model). As remarked in [CR15, Lemma 19], situations is the one given by our setting, the existence some maximal orbit accumulating to the origin. This does not affect in any degree the results in [GRI2].
the description of the saddle-node singularity by sectorial normalizations in [HKM61] implies that with the exception of the central manifold (when it converges), every integral curve accumulates to the strong separatrix of the saddle-node. Hence, \( \mathcal{L} \) accumulates to the strong separatrix, which belongs to \( \mathcal{U} \). For the affine structure along \( \{ y = 0 \} \), the strong separatrix, from the proof of Proposition 17 in [GR12] we have \( \text{ind}(\{ y = 0 \}, 0) = -1/n \) and thus \( n \in \{-1, 0, 1\} \). If \( n = \pm 1 \) then, by Lemma 11, the holonomy has fixed points at the points of \( \mathcal{U} \). However, the holonomy of the strong separatrix is tangent to the identity and has [Lor06, Section 5.3.3] a Leau-Fatou flower dynamics: it cannot have a fixed point. This contradiction proves that \( n = 0 \), giving the sought normal form. In particular, the central manifold, if convergent, does not belong to the curves of zeros and poles.

This finishes the proof of Proposition 13.

5. In the presence of non-entire solutions

We will now prove the Main Theorem in the case where \( \phi : \Omega \to S \) is a maximal solution of a meromorphic vector field in the Kähler compact complex surface \( S \) that does not extend to \( \mathbb{C} \) (one of the two cases where Proposition 12 leads us). Mostly, the proof follows that of Theorem B in [GR12]. We will only sketch some of its lines. The aim is, first, to describe the component the divisor of poles where an orbit accumulates (Theorem 14 below), and then use the topological information (numerical invariants) of the divisor along with with the global geometry of the surface to produce a fibration adapted to the vector field.

In the neighborhood of the divisors. We have the following result, direct analogue of Theorem A in [GR12]:

**Theorem 14.** Let \( S \) be a complex surface, \( X \) a meromorphic vector field on \( S \), \( D \subset S \) a compact connected component of the curve of poles of \( X \), \( \Omega \subset \mathbb{C} \), and \( \phi : \Omega \to S \) a maximal solution of \( X \) that accumulates to \( D \) in a non-algebraic way. Up to a birational transformation, either

- \( D \) can be collapsed to a point where \( X \) is holomorphic;
- \( D \) is a rational curve of vanishing self-intersection; or
- \( D \) is a non-singular elliptic curve of vanishing self-intersection or, more generally, supports a divisor \( Z \) of elliptic fiber type.

The theorem concerns only a neighborhood of \( D \) and, for example, does not require the compactness of \( S \). For the proof, after reducing the vector field (as defined in Section 2.4), the vector field is locally described by the local models of Proposition 13. For each invariant component of the support of the divisor, the induced affine structure is one of those listed in Proposition 9; there is a has a self-intersection number and the vector field has some vanishing order along it. All of these objects come with some numerical invariants that are not independent. The above Theorem follows from investigating the combinatorial relations between all these data while taking into account the fact that we
can sometimes blow down a curve without affecting the reduced nature of the divisor. The proof of Theorem A in [GR12, Section 6] can be used word by word to give a proof the above Theorem. We will not repeat those arguments here. Let us just mention some facts that follow from them.

For the rational curves of vanishing self-intersection in Theorem 14, the curve has a neighborhood \((U,D)\) biholomorphic to a product \(\Delta \times \mathbb{P}^1\) (Savel’ev’s Theorem). In the neighborhood of \(D\), \(\mathcal{F}\) is a Riccati foliation, with \(D\) being a semidegenerate or nilpotent fiber. Locally, \(X\) preserves the rational fibration. For the divisors of elliptic fiber type, the minimal models of the divisors match, from a combinatorial point of view, Kodaira’s singular fibers of elliptic fibrations. In the minimal models of these divisors, the divisor of zeros of \(X\) is of the form \(lZ - Z_{\text{red}}\), \(l \leq 0\), where \(Z\) is the divisor of elliptic fiber type supported in the curve of poles and \(Z_{\text{red}}\) is the associated reduced divisor [GR12, Prop. 24]. This implies that the normal bundle of \(Z\) is of torsion. Let us repeat some of the arguments in [GR12, Section 7] that prove this. The canonical bundle of the surface and the tangent and conormal bundles of the foliation \(\mathcal{F}\) induced by \(X\) satisfy the relation

\[ N^*_{\mathcal{F}} \otimes \mathcal{O}_M(Z_{\text{red}}) = K_M \otimes \mathcal{O}_M(Z) \otimes \mathcal{O}_M([l - 1]Z). \]

For the left-hand side, we have the short exact sequence

\[ 0 \rightarrow N^*_{\mathcal{F}} \rightarrow N^*_X \otimes \mathcal{O}_M(Z_{\text{red}}) \xrightarrow{\text{Res}} \mathcal{O}_Z \rightarrow 0. \]

The sheaf \(N^*_X \otimes \mathcal{O}_M(Z_{\text{red}})\) is that of meromorphic one-forms defining \(\mathcal{F}\) with at most simple poles along \(Z\). The residue map Res is given as follows: if \(f\) is a local equation defining \(Z\) (as a divisor) and \(\eta\) a logarithmic form defining \(\mathcal{F}\) then \(f\eta = hdf + f\eta\) for some holomorphic function \(h\) and some holomorphic one-form \(\eta\). The restriction of \(h\) to \(Z\) is independent of the chosen decomposition. Define \(\text{Res}(\eta)\) as \(h|_Z\). Let us now restrict (7) to \(Z\). The restriction of the left-hand side of (7) to \(Z\) is trivial for \(\text{Res}\) gives the restriction of \(N^*_X \otimes \mathcal{O}_M(Z_{\text{red}})\) to \(\mathcal{O}(Z)\). As for the right-hand side, notice that, by adjunction, \([K_M \otimes \mathcal{O}_M(Z)]|_Z = K_Z\), which is trivial since \(Z\) is of elliptic fiber type. This implies that \(\mathcal{O}_M([l - 1]Z)|_Z\) is trivial.

**Remark 15.** Zeros of semicomplete vector fields have other combinatorics which do not seem to admit a simple classification as in Theorem 14, although they can be described in some particular situations [Gui14].

**Constructing the fibrations.** To prove the Main Theorem out of Theorem 14, one may follow again the proof of Theorem B in [GR12]. Having at hand divisors that are natural candidates to be fibers of the fibration in the Main Theorem, one must now build the fibration out of these divisors: one must show that there exists a (rational or elliptic) fibration on \(S\) having \(D\) as a fiber. The problem is semilocal in nature, for it suffices to prove that some neighborhood of \(D\) has non-constant holomorphic functions. Since the self-intersection of the divisor vanishes, such a function would give, locally, the sought fibration, which could be then globalized to all of \(S\). For rational curves of
vanishing self-intersection, the local fibration always exists (Savel’ev’s Theorem). For
the divisors of elliptic fiber type, Brunella proved that foliations in algebraic surfaces
having invariant elliptic curves (or, more generally, invariant divisors of elliptic fiber

type) are very special ([Bru00, Chapter 9, §3, Corollary 2]). Following Brunella, with
the simplifications that come with our restricted setting, we may proceed as follows in
order to construct holomorphic functions (that will give locally an elliptic fibration) in
a neighborhood of these divisors. Under the hypothesis that $H^1(M, O) = 0$, a result of
Sad [Sad99, Section 2] guarantees the existence of non-constant holomorphic functions
on the neighborhood of a divisor of elliptic fiber type as soon as its normal bundle is,
as in our case, of torsion. In the absence of this hypothesis, the Kähler one implies, by
Hodge decomposition, that there exists a (closed) holomorphic one-form $\eta$ in $M$. If the
meromorphic function $f = X \cdot \eta$ is non-constant, $1/f$ is holomorphic in a neighborhood
of $D$. If $f \equiv c$, $c \in \mathbb{C}$, there is a non-constant holomorphic function $g$ in a neighborhood
of $D$ such that $\eta = dg$, since, for most points in $D$, the local model of $X$ is the regular
one, $f(x, y)y^q \partial/\partial x \ (q < 0)$ with $D$ being $y = 0$, and the unique primitive of $\eta$ vanishing
at the point, of the form $h(y) + cy^{-q} \int_x f(s, y)ds$, vanishes identically along $D$.

We thus establish the Main Theorem in the case where there is a maximal single-valued
solution that cannot be extended to all of $\mathbb{C}$.

Remark 16. Following the arguments in [Gui16] (involving classification results by Koda-
aira and Enoki), one may show that, still in the case where there is a maximal single-
valued solution that cannot be extended to all of $\mathbb{C}$, the result is also valid in the non-
Kähler setting.

6. Entire solutions

Let $M$ be a complex compact Kähler surface, $X$ a meromorphic vector field on $M$
and $\phi : \mathbb{C} \to M$ with Zariski-dense image that is almost everywhere a solution to $X$.
We will further suppose that all the solutions of $X$ in $\mathcal{U}$, the uniformizable locus, can be
extended to entire maps. It is in this setting that we will prove the Main Theorem.

Let $\mathcal{L} = \phi(\mathbb{C})$, $\Lambda$ the module of periods of $\phi$. The latter is not a lattice, for $\mathcal{L}$ is not an
algebraic curve. If $\Lambda$ is isomorphic to $\mathbb{Z}$, $\mathcal{L}$ is isomorphic to $\mathbb{C}^\times$. The curve $\hat{\mathcal{L}}$, obtained
from $\mathcal{L}$ after adding all the isolated planar ends, is either equal to $\mathcal{L}$ or has one extra
point, where the restriction of $X$ has a simple zero (both ends of $\mathcal{L}$ cannot be compactified
in this way, for this would imply that $\mathcal{L}$ is contained in an algebraic curve). If $\Lambda$ is trivial,
$\phi$ is one-to-one and $\hat{\mathcal{L}} = \mathcal{L}$ (for otherwise $\mathcal{L}$ would be contained in an algebraic curve).

We will begin with the case where $M$ is algebraic. In a series of works, Mendes,
Brunella and McQuillan established a classification of foliations in algebraic surfaces,
in the spirit of Kodaira’s classification of algebraic surfaces. Given a foliation $\mathcal{F}$ in an
algebraic surface $S$, the Kodaira dimension of $\mathcal{F}$, $\kod(\mathcal{F}) \in \{-\infty, 0, 1, 2\}$ is defined, and
the classification concerns those foliations whose Kodaira dimension is less than two. We
refer the reader to [Bru00], [Bru03] for a detailed presentation. For our study, we will
use a remarkable outcome of this classification: the description of foliations on algebraic surfaces admitting an entire curve tangent to them \[ \text{McQ01}, \text{McQ08}, \text{Bru00}. \]

**Theorem 17** (McQuillan). Let \( S \) be an algebraic surface and \( \mathcal{F} \) a holomorphic foliation in \( M \). Let \( \phi : \mathbb{C} \to S \) be a map tangent to the foliation having Zariski-dense image. Then the foliation is either Riccati or turbulent, or, up to a birational transformation, \( \mathcal{F} \) is induced by a holomorphic vector field in a ramified cover of \( S \).

Our proof of the theorem will follow some case-by-case considerations.

6.1. **Covered by a vector field.** The last possibility in Theorem 17 is that, up to a birational map on \( S \), there is a finite ramified covering \( \Pi : M \to S \) of a compact surface \( M \) where the pull-back of \( \mathcal{F} \) is generated by a holomorphic (hence complete) vector field \( V \) with isolated singularities. Up to birational maps, the ramified covering is Galois, and acts by preserving the foliation induced by \( V \) (but not necessarily \( V \) itself) \[ \text{McQ08, Fact IV.3.3} \]. Holomorphic vector field on algebraic surfaces are classified (see \[ \text{Bru00, Ch. 6, Props. 5 and 6} \]):

**Proposition 18.** The holomorphic vector fields in projective surfaces that do not have first integrals are:

- constant vector fields on Abelian surfaces,
- product vector fields in \( \mathbb{P}^1 \times \mathbb{P}^1 \), or
- vector fields in ruled surfaces over elliptic curves that project into a holomorphic vector field in the base.

A classification of the birational automorphisms of the associated foliations, from which we will borrow some arguments, appears in \[ \text{Per05} \].

The study of foliations in the third case will be postponed until Section 6.2, where they will be considered among other Riccati foliations (the automorphisms of such foliated surfaces map fibers to fibers so the quotients are still Riccati foliations). We will now prove the Main Theorem for meromorphic vector fields covered by foliations associated to the first two cases. For them, we will denote by \( R \) the meromorphic function in \( M \) such that \( \Pi^{-1}_*X = RV \). The Galois group of the ramified covering acts by preserving \( RV \).

6.1.1. **Vector fields on Abelian surfaces.** We will need to understand how do the leaves of these foliations intersect algebraic curves in the surface.

**Lemma 19.** Let \( M \) be an Abelian surface, \( \mathcal{F} \) a foliation given by a constant vector field \( V \), \( E \subset M \) an algebraic curve which is not invariant by \( \mathcal{F} \), \( L \) a leaf of \( \mathcal{F} \). Either \( L \) is closed or it intersects \( E \) infinitely many times.

**Proof.** Let \( \eta \) be a holomorphic one-form in \( M \) such that the restriction of \( \eta \) to \( V \) vanishes. The integral of \( \eta \) in \( M \) measures distances to leaves of \( \mathcal{F} \): if \( \rho : [0, 1] \to M \) is such that \( \int_0^1 \eta = 0 \), \( \rho(0) \) and \( \rho(1) \) belong to the same leaf of \( \mathcal{F} \). Let \( \gamma : [0, 1] \to M \) be a path joining a point \( p \in E \) to a point in \( L \). The (ramified) translation structure induced
by \( \eta \) on \( E \) is geodesically complete. Thus, there is a curve \( \hat{\gamma} : [0,1] \to E, \hat{\gamma}(0) = p \), such that \( \gamma^* \eta = \hat{\gamma}^* \eta \). Hence, \( \hat{\gamma}(1) \in E \cap L \). The curve \( L \) is dense in its closure \( \overline{T} \). Thus, if \( L \neq \overline{T} \), \( L \) will intersect \( E \) infinitely many times (at infinitely many different points).

Let thus \( M \) be an Abelian surface and \( V \) a constant vector field in \( M \). If \( V \) has one closed orbit then, by translating it by \( M \), we find that all orbits of \( V \) are closed. This implies the existence of a first integral. Suppose that no orbit of \( V \) is closed. Let \( E \) an irreducible component of the curve of zeros of \( R \). Let \( L \) be the orbit of \( V \) that projects onto \( L \). Since \( V \) has no closed orbit, \( E \) is generically transverse to \( F \) and \( L \) is not closed. By Lemma 19, \( L \) intersects \( E \) infinitely many times. But this means that the solution of \( \Pi^{-1}X \) taking values in \( L \) has infinitely many determinations (Proposition 8) and so must the solution of \( X \) in \( L \). We conclude that \( \Pi^{-1}X \) is a constant multiple of \( V \) (in particular, it is holomorphic), and thus \( X \) is holomorphic as well.

6.1.2. Product vector fields. Product vector fields in \( P_1 \times P_1 \) that do not admit a first integral are, in suitable coordinates and up to a constant factor, of the form \( x \partial/\partial x \) or \( x \partial/\partial x + \mu y \partial/\partial y, \mu \notin \mathbb{Q} \). Multiplying these by rational functions often produces vector fields whose solutions have infinitely many determinations:

**Lemma 20.** If a meromorphic vector field in \( P_1 \times P_1 \) admits a transcendental solution with finitely many determinations and is of the form

- \( R(z,w)[z\partial/\partial z + \partial/\partial w], R \) is a function of \( w \);
- \( R(z,w)[z\partial/\partial z + \mu w \partial/\partial w], \mu \notin \mathbb{Q}, R \) is constant.

**Proof.** In the first case, consider a transcendental leaf parametrized by \( s \mapsto (z_0 e^s, s) \). The induced vector field is \( R(z_0 e^s, s) \partial/\partial s \). If its solutions have finitely many determinations, from Proposition 8 this vector field should be rational and thus \( R \) is a rational function of \( w \) (and does not depend upon \( z \)). In the second, parametrize an orbit of by \( s \mapsto (x_0 e^s, y_0 e^{\mu s}), s \in \mathbb{C} \). The induced vector field is \( R(x_0 e^s, y_0 e^{\mu s}) \partial/\partial s \). By Proposition 8 if this solution has finitely many determinations, \( R(x_0 e^s, y_0 e^{\mu s}) \) is a rational function of \( s \). But since \( \mu \notin \mathbb{Q}, R \) must be constant.

To prove the Main Theorem in this situation, consider first the case \( V = z \partial/\partial z + \partial/\partial w \). By Lemma 20, \( \Pi^{-1}(X) = R(w)[z\partial/\partial z + \partial/\partial w] \). In order to understand the group of birational automorphisms preserving \( RV \), let us begin by describing the larger group of birational transformations preserving the foliation induced by \( V \). We claim that the latter is isomorphic to \( \mathbb{Z}/2\mathbb{Z} \ltimes (C \times C^*) \), where \( C \) acts via the flow of \( \partial/\partial w \), \( C^* \) via that of \( z \partial/\partial z \) and \( \mathbb{Z}/2\mathbb{Z} \) by

\[
(z,w) \mapsto (1/z,-w).
\]

Since the transcendental leaves are Zariski-dense, an automorphism is determined by its action upon such a leaf. The group \( C \times C^* \) acts transitively upon the set of pointed leaves. If an automorphism acts upon the leaf parametrized by \( s \mapsto (e^s, s), s \in C \), fixing the point \( s = 0 \), it is of the form \( (e^{as}, as), a \in C \). But if such an automorphism extends
to \(\mathbb{C}^2\), it extends as \((z, w) \mapsto (z^a, aw)\), and we must have \(a^2 = 1\), proving our claim. The projection onto \(w\) is equivariant with respect to the action of this group, invariant under the action of \(\mathbb{C}^*\). The group acting on \(w\) is \(\mathbb{Z}/2\mathbb{Z} \rtimes \mathbb{C}\). All finite subgroups of the latter are either trivial or have order two and are generated by a map of the form \(w \mapsto -w + c\) (in this case we will suppose that \(c = 0\)). If \(R(w)[z\partial/\partial z + \partial/\partial w]\) is invariant under the action of some group acting trivially on \(w\), the projection onto \(w\) maps the vector field onto \(R(w)\partial/\partial w\). If it is invariant under the action of some group acting on \(w\) by \(w \mapsto -w\), \(R(-w) = -R(w)\) and thus \(R(w) = wR_0(w^2)\). The image of the vector field under the quotient map \((z, w) \mapsto w^2\) is, for \(\xi = w^2\), \(2\xi R_0(\xi)\partial/\partial \xi\). The poles of the vector field are contained in the fibers of the map and the vector field preserves this fibration.

Consider now the case where \(V = x\partial/\partial x + \mu y\partial/\partial y\), \(\mu \not\in \mathbb{Q}\). By Lemma 20, \(\Pi^{-1}(X)\) is a constant multiple of \(V\) and is thus holomorphic. The Galois group of \(\Pi\) acts by preserving this vector field and thus \(X\) is holomorphic as well. This proves the Main Theorem in this case.

6.2. Riccati. Let \(\Pi : S \to B\) be a rational fibration and \(X\) a meromorphic vector field on \(S\) inducing a Riccati foliation \(\mathcal{F}\) adapted to \(\Pi\). Let \(F_1, \ldots, F_n\) be the special fibers of \(\Pi\). We will suppose that the Riccati foliation is in standard form, as discussed in Section 2.3. Let \(\phi : \mathbb{C} \to S\) be a map with Zariski-dense image that is almost everywhere a solution to \(X\).

The vector field may preserve the rational fibration \(\Pi\). For this to happen, it is sufficient for \(X\) to be holomorphic in the neighborhood of a generic fiber. In this situation, the local flow will map fibers to fibers (by the maximum principle) and will induce a nowhere zero holomorphic vector field \(Y\) in the complement of a finite set of points in \(B\). By Lemma 6, \(Y\) will extend holomorphically to \(B\), proving the Main Theorem for this case. However, this is not the only possibility, and there will be cases where the vector field will not preserve the rational fibration. In this case, we will prove the Main Theorem by showing the existence of another fibration that is invariant under the flow. Example 24 gives an example of such a situation: a vector field with a transcendental maximal solution such that the foliation is a Riccati one with respect to some rational fibration, but where this fibration is not preserved by the flow (in this example, there is an elliptic fibration that is preserved).

In general, the divisor of zeros and poles of \(X\) is of the form \(\sum_{j=1}^{n+k} a_j F_j + \sum_i b_i E_i\), where \(F_1, \ldots, F_n\) are special fibers and \(F_{n+1}, \ldots, F_{n+k}\) are transverse ones; the divisors \(E_i\) are irreducible and not supported in fibers (they may or may not be transverse to \(\mathcal{F}\)). For a generic fiber \(F\),

\[
F \cdot \sum_i b_i E_i = 0
\]

(see [Bru00, Ch. 4, Prop. 6]). Let \(S' = S \setminus \{F_1, \ldots, F_n\}\), \(B' = \Pi(S')\). Let \(\rho : \pi_1(S') \to \text{PSL}(2, \mathbb{C})\) be the monodromy representation, \(\Gamma\) its image. The existence of the entire
map $\phi$ implies that $B$ is either rational or elliptic. Let $F$ be a generic fiber, $u = U \cap F$. The solution $L$ gives an orbit of $\Gamma$ within $u$, which we will denote by $\ell$.

If $u$ is big enough, the coexistence of some types of special fibers with some curves of zeros/poles transverse to $\Pi$ may be ruled out:

**Proposition 21.** Let $X$ be a vector field in the surface $S$ inducing a Riccati foliation adapted to the fibration $\Pi : S \to B$. Let $F$ be a generic fiber. Let $L$ be a maximal solution and $\ell = L \cap F$. If $\ell$ has at least three points, either

- $X$ preserves the fibration $\Pi$ or
- all the special fibers are non-degenerate with parabolic monodromy or dicritical.

**Proof.** Consider a special fiber $F_0$. If the vector field is holomorphic in the neighborhood of $F_0$ or if its poles are contained in $F_0$, it preserves the fibration. If not, it has some component of the locus of poles transverse to $\Pi$ and, by (9) some components of the curve of zeros that do so as well. Let $E$ be one such component of the curve of zeros.

If $F_0$ is non-degenerate with hyperbolic monodromy, the foliation is given by the form (9). Once a branch of $L$ that is not one of the two separatrices (there is at least one, since $\ell$ has at least three points) is close enough to $F_0$, it gets trapped by the holonomy of $\mathcal{F}$ along its leaf $F_0$, which is hyperbolic, and the leaf accumulates the singularities of $\mathcal{F}$ in $F_0$, which are in the Poincaré domain. By Proposition 13 no component of the curve of zeros and poles may be invariant by the foliation ($E$ is not invariant by $\mathcal{F}$). Consider a Puiseux parametrization of $E$ around its intersection with the special fiber $F_0$. Up to a ramified cover of the base, suppose that, in the neighborhood of its intersection with $F_0$, $E$ is a section of $\Pi$. The parametrization of $E$ has the form $(t, t^q v(t))$, $q \in \mathbb{Z}$, $v(0) \neq 0$, and is transverse to $F_0$. Up to the birational transformation $(\overline{z}, \overline{w}) = (z, w/z^q)$, where the form giving the foliation reads $(\lambda - q) \overline{w} d\overline{z} - \overline{z} d\overline{w}$, $E$ intersects $F_0$ transversely at a non-singular point and thus intersects every leaf that is not a separatrix infinitely many times. In particular, since $\ell$ has more than two points, it is not contained in the set of separatrices of $F_0$ and must intersect $E$ infinitely many times, contradicting its maximality.

If $F_0$ is a semi-degenerate fiber, the holonomy of the foliation along $F_0$ is tangent to the identity (but not the identity) and has a Leau-Fatou flower dynamics. The orbit $L$ accumulates to the two saddle-node singularities of $F_0$. By Proposition 13 $E$ cannot be a separatrix of $\mathcal{F}$ (which are central manifolds of the saddle-nodes). Up to a ramified cover of the base, suppose that, in the neighborhood of its intersection with $F_0$, $E$ is a section of $\Pi$ (and thus transverse to $F_0$). Up to some flips in $F_0$, we can suppose that $E$ intersects $F_0$ away from its two singular points. After this, $E$ will be transverse to $F_0$ at a regular point of $\mathcal{F}$, and must intersect $\ell$ infinitely many times. As before, this contradicts the maximality of $L$. Essentially the same arguments deal with the case where $F_0$ is a nilpotent fiber. \hfill \Box

For the proof of the Main Theorem we now begin a series of cases that will eventually cover all the possibilities.
6.2.1. When $\ell$ is finite. We begin with the cases where $B = P^1$. If $\phi$ does not have periods, let $\psi : C \to P^1$ be given by $\psi = \Pi \circ \phi$. Our assumption is that, for a generic $t_0$, $\{ t \in C | \psi(t) = \psi(t_0) \}$ is finite. This implies that $\psi$ is a rational function of $t$ (for otherwise it would have an essential singularity at $t = \infty$ and, by Picard’s Great Theorem, our finiteness assumption would be violated). Under $t \mapsto \psi(t)$, $X$ induces a rational multivalued vector field $V$ in $P^1$. The set of points in $S$ such that $\Pi$ maps $X$ onto one of the determinations of $V$ is a closed analytic subset of $S$ which contains the Zariski-dense curve $\mathcal{L}$. We must conclude that this set is all of $S$. Along a fiber, the projection of the vector field onto $B$ is locally constant and thus constant. We conclude that $V$ is a honest rational vector field. Since it has the single-valued solution $\psi$, it is a holomorphic vector field and, since $\psi$ is rational, this vector field is $\partial/\partial z$ in a suitable coordinate.

The case where $\ell$ is finite and $B$ is elliptic does not arrive. Suppose that $B$ is an elliptic curve and that $\phi$ has no periods and consider the function $\psi : C \to B$ as before. Let $\theta : B \to P^1$ be an auxiliary rational map. Since $\psi$ has only finitely many points in the preimage of any given one, so does $\theta \circ \psi$. By Picard’s Great Theorem, $\theta \circ \psi$ must extend holomorphically to $\infty$, and so must $\psi$. This yields a function $\overline{\psi} : P^1 \to B$, which is impossible. Likewise, if $\phi(t) = \rho(e^t)$, a function $\psi : C^* \to B$ as above extends holomorphically to $0$ and $\infty$. Again, this is impossible.

We may now proceed to study the cases where $\ell$ is infinite. The cases of interest are, according to Proposition 21, the ones where the special fibers are non-degenerate with parabolic monodromy or dicritical. If the vector field has zeros or poles along a non-singular fiber $F$ then, since $\ell$ is infinite, $\hat{\mathcal{L}}$ will have, at its intersections with $F$, infinitely many points where the vector field has zeros or poles, and we must conclude that $X$ is holomorphic and non-zero at the generic point of $F$. For the same reasons, for the dicritical fibers, the vector field is holomorphic and non-zero at the generic point of the dicritical component of the fiber $F$.

6.2.2. Discrete, elementary monodromy. If the monodromy $\Gamma$ is discrete, elementary and infinite, it either [For51, Sections 59–62]

(1) contains a group of translations with finite index, or
(2) is contained in the group that permutes $\{0, \infty\}$, contains a subgroup of $C^*$ with index at most two.

In the first case there is one fixed point for $\Gamma$; in the second, two fixed points or an orbit of size two. In both cases, all orbits accumulate to these finite ones, which thus belong to $u$. We will suppose that there is an invariant algebraic curve $E$ giving such a finite
We begin with Case 1, where $\Gamma$ contains a group of translations with finite index. Each fiber intersects only one point of $E$, $\Pi$ identifies $E$ and $B$. Let $\gamma \in \pi_1(E)$ be a path along which the monodromy of the affine structure is trivial. By the Fundamental Lemma (Proposition 11), the holonomy of $F$ along $\gamma$ will fix pointwise the points of $u$ sufficiently close to $F \cap E$ and, since this set is infinite, the monodromy of the Riccati equation along $\gamma$ is trivial. This implies that the fibers of $\Pi$ corresponding to points of $E$ where the affine structure is non-singular are transverse fibers, for, according to Proposition 21, all other special fibers have non-trivial monodromy. Lemma 11 takes here a strong global form: there is a well-defined (abstract) map from the monodromy of the affine structure on $E$ to the monodromy of the Riccati equation. The problem is now to represent the infinite subgroups of the affine group that can be (abstract) images of the monodromies of the uniformizable affine structures on curves.

We first address the cases where $E$ is a parabolic orbifold. If the affine structure in $E$ is that of a parabolic cylinder (two singularities with infinite ramification indices), $B$ is $\mathbb{P}^1$ and there are two singularities. The monodromy is of infinite order and is thus parabolic. By placing the singularities of $E$ above 0 and $\infty$, the foliation is birationally equivalent to the foliation in $\mathbb{P}^1 \times \mathbb{P}^1$ given by $zdw - dz$, with $E$ being the curve $\{w = \infty\}$. By Lemma 20, $X = R(w)[\partial/\partial z + \partial/\partial w]$ for some rational function $R$. The projection $(z, w) \mapsto w$ maps $X$ onto $R(w)\partial/\partial w$. If the affine structure makes $E$ an orbifold $(2, 2, \infty)$ then, $S$ and $X$ are quotients of the above vector field under $(\mathbb{R}, 2)$. According to the results in Section 6.1.2, the function in the quotient induced by $(z, w) \mapsto w^2$ maps $X$ into a vector field in $\mathbb{P}^1$, with the poles are contained in the fibers.

We now deal with the cases where $E$ is an elliptic orbifold, the quotient of $\mathbb{C}$ under the action of a crystallographic group $G \subset \text{Aff}(\mathbb{C})$. By taking the cover of $B$ according to this quotient, we get a ramified cover $\mathbb{C} \times \mathbb{P}^1 \to S$, such that, in the coordinates $(z, w)$ in $\mathbb{C} \times \mathbb{P}^1$, we may suppose that the pull-back of the foliation is given by $dw = 0$ and the curve $E$ is the image of $w = \infty$. The action of $G$ on this space is given by the monodromy of the affine structure on $E$ in the first factor, via the monodromy $\rho : G \to \text{Aff}(\mathbb{C})$ of the Riccati foliation in the second. The vector field $X$ is the quotient of a meromorphic vector field in $\mathbb{C} \times \mathbb{P}^1$ of the form $P(z, w)\partial/\partial z$. Let $\Lambda \subset \mathbb{C}$ be the lattice that gives the smallest étale covering of $E$. The monodromy of the Riccati foliation can be restricted to $\Lambda$, inducing the morphism of abelian groups $\rho : \Lambda \to \mathbb{C}$, for which

\begin{equation}
(10) \quad P(z + \lambda, w + \rho(\lambda)) = P(z, w) \quad \text{for every } \lambda \in \Lambda.
\end{equation}

For each $z_0$ fixed, $P(z_0, w)$ has finitely many zeros and poles and thus

\begin{equation}
(11) \quad P(z, w) = \mu(z) \frac{w^n + a_{n-1}(z)w^{n-1} + \cdots + a_0(z)}{w^m + b_{m-1}(z)w^{m-1} + \cdots + b_0(z)}
\end{equation}
with $\mu$, $a_i$ and $b_i$ meromorphic functions. The curve $E$ does not intersect the curve of zeros/poles of $P$ for, in that case, all the orbits accumulating to $E$ would gain infinitely many zeros/poles. Thus, $\mu$, $a_i$ and $b_i$ are holomorphic, $\mu$ without zeros. The invariant property of $P$ imposes conditions on these. From

$$
\mu(z)^{w_n + \cdots + a_0(z)} = \mu(z + \lambda) \left[ \frac{w + \rho(\lambda)}{w + \rho(\lambda)} \right]^{n+\cdots+a_0(z + \lambda)} = \mu(z + \lambda) \frac{w_n + \cdots}{w^m + \cdots},
$$

we have that $\mu$ is elliptic and thus constant. We will suppose $\mu \equiv 1$. Both the numerator and the denominator of $P$ must satisfy (10). For the numerator,

$$
w^n + a_{n-1}(z)w^{n-1} + \cdots = \left[ w + \rho(\lambda) \right]^{n+\cdots+a_{n-1}(z + \lambda)}[w + \rho(\lambda)]^{n-1} + \cdots
$$

$$
= w^n + [n\rho(\lambda) + a_{n-1}(z + \lambda)]w^{n-1} + \cdots.
$$

Hence, $a_{n-1}(z) = a_{n-1}(z + \lambda) = n\rho(\lambda)$. Derivating, $a'_{n-1}(z) - a'_{n-1}(z + \lambda) = 0$ and thus $a'_{n-1}$ is elliptic, thus constant, $a_{n-1}(z) = \alpha z + \beta$, for some $\alpha, \beta \in \mathbb{C}$, and $\rho(\lambda) = -(\alpha/n)\lambda$. In particular, $\rho$ (which was a priori only a morphism of abelian groups) is actually induced by a $\mathbb{C}$-linear map. Up to a linear change of coordinates in $w$, we will suppose that $\rho(\lambda) = \lambda$ (this is, $\alpha = -n$, $a_{n-1}(z) = -nz + \beta$). The function $w - z$ belongs to the ring of functions satisfying (10). Since

$$(w^n + a_{n-1}w^{n-1} + \cdots + a_0) - (w - z)^n = \beta w^{n-1} + \cdots,$$

belongs also to this ring, we find, recursively, that the numerator of $P$ is a polynomial of $z - w$. We conclude that $P$ is a rational function of $\xi = z - w$.

Let us come back to the full monodromy group. Let $g \in G$ be given by $z \mapsto \omega z$ and let $\rho(g)$ be given by $w \mapsto \nu w + c$. Up to a translation, we may suppose that $c = 0$ (this does not alter the fact that $\rho(\lambda) = \lambda$). Let $\tau_\lambda \in \text{Aff}^1(\mathbb{C})$ be the translation by $\lambda$. Then $\tau_\omega = \rho(\tau_\lambda g^{-1}) = \rho(\tau_\lambda)\rho(g)^{-1} = \tau_{\nu \lambda}$. We conclude that $\omega = \nu$ and thus that $\rho : G \to \text{Aff}^1(\mathbb{C})$ is the identity. The condition that $P$ must satisfy in order to induce a vector field in $S$ is that $P(\omega \xi) = \omega P(\xi)$ for every $\omega \in \mathbb{C}^*$ in the rotation group of $G$. By Luroth’s theorem, $P(\xi) = \xi R(\xi^k)$, where $R$ is a rational function and $k$ is the smallest common order of the rotations in $G$. The projection onto $\mathbb{P}^1$ given by $\zeta = \xi^k$ maps $X$ to the vector field $k\zeta R(\zeta)\partial/\partial \zeta$. We conclude that there is an elliptic fibration from $S$ onto $\mathbb{P}^1$ preserved by $X$ (the curves of zeros and poles are contained in the fibers), proving the Main Theorem in this case.

We address now Case 2, where $\Gamma$ is in the group that permutes $\{0, \infty\}$. This group is the normalizer $N^1(\mathbb{C}^*)$ of $\mathbb{C}^*$ in $\text{PSL}(2, \mathbb{C})$. It is the group that, in a convenient coordinate, is generated by $\mathbb{C}^*$ and by the involution $z \mapsto 1/z$. All the elements in $\sigma \mathbb{C}^*$ have order two. The center of $N^1(\mathbb{C}^*)$ is generated by the involution $z \mapsto -z$. The couples in $N(\mathbb{C}^*)$ that commute are either couples in $\mathbb{C}^*$, couples in $\sigma \mathbb{C}^*$ differing by $\kappa$ or couples where $\kappa$ is one of their members.

Let us begin with the cases where $\Gamma$ is actually a subgroup of $\mathbb{C}^*$. The only uniformizable affine structures on curves admitting infinite representations in the abelian group $\mathbb{C}^*$ are the parabolic cylinder and those supported on elliptic curves.
If one of the curves corresponding to one of the fixed points of \( \Gamma \) is a parabolic cylinder, the curve has two singularities for its affine structure and the monodromy of the affine structure is cyclic. The monodromy of the Riccati foliation is cyclic, generated by the local monodromy around one of the special fibers. Since this monodromy must be infinite, the special fibers cannot be dicritical. Since the monodromy may not be parabolic, the special fibers cannot be non-degenerate parabolic. According to Proposition 21, this case does not occur.

Suppose now that one of the fixed points of \( \Gamma \) corresponds to an elliptic curve \( E = C/\Lambda \), (having no singularities for its affine structure). The vector field \( X \) is a meromorphic vector field inducing a Riccati foliation over the elliptic curve \( E \) with holonomy in \( \mathbb{C}^* \). There is a vector field \( V = P(z, w)\partial/\partial z \) on \( C \times P^1 \) such that, for some representation \( \rho : \Lambda \to C^* \), \( V \) is invariant under the transformations

\[
(z, \lambda) \mapsto (z + \lambda, \rho(\lambda)w),
\]

and such that \( (S, \Pi, B, X) \) is bimeromorphic to the quotient of \( (C \times P^1, \pi, C, V) \), \( \pi : C \times \mathbb{P}^1 \to C \) denoting the projection onto the first factor and \( E \) being \( \{w = 0\} \) or \( \{w = \infty\} \).

Since \( \Gamma \) discrete and infinite, it contains some hyperbolic elements. The vector field is necessarily of the form (11), with \( \mu \) holomorphic without zeros, \( a_i \) and \( b_i \) holomorphic without zeros or identically zero. In order for \( V \) to be invariant under (12), we must have

\[
P(z + \lambda, \rho(\lambda)w) = P(z, w)
\]

and thus

\[
\mu(z + \lambda) = \mu(z)\frac{\rho^n(\lambda)w^n + \rho^{n-1}(\lambda)a_{n-1}(z + \lambda)w^{n-1} + \cdots + a_0(z)\lambda^n}{\rho^m(\lambda)w^m + \rho^{m-1}(\lambda)b_{m-1}(z + \lambda)w^{m-1} + \cdots + b_0(z)\lambda^m} = \mu(z)\frac{w^n + a_{n-1}(z)w^{n-1} + \cdots + a_0(z)}{w^m + b_{m-1}(z)w^{m-1} + \cdots + b_0(z)}.
\]

Hence, \( \mu(z + \lambda) = \rho(\lambda)^{m-n}\mu(z) \). This implies that \( \mu' / \mu \) is elliptic (and thus constant). We must conclude that \( \mu(z) = e^{\delta z} \) for some \( \delta \in \mathbb{C}^* \), and that \( \rho^{m-n}(\lambda) = e^{\delta \lambda} \). From (13), \( \rho^m(\lambda)\alpha_i(z + \lambda) = \alpha_i(z) \). This implies that either \( \alpha_i \equiv 0 \) or that \( \alpha_i' / \alpha_i \) is elliptic, thus constant, \( \alpha_i(z) = a_i e^{s_i z} \), \( \alpha_i \neq 0 \), \( \rho^{m-n}(\lambda) = e^{s_i \lambda} \). Likewise, \( b_i = \beta_i e^{t_i z} \), \( \rho^{m-n}(\lambda) = e^{t_i \lambda} \).

Let \( d \) be the greatest common divisor of

\[
\{n - m\} \cup \{n - i | a_i \neq 0\} \cup \{m - i | b_i \neq 0\}.
\]

since

\[
\mu(z)\frac{w^n + \cdots + a_0(z)}{w^m + \cdots + b_0(z)} = \mu(z)\frac{w^n(1 + \sum_j a_{n-jd}w^{-jd})}{w^m(1 + \sum_j b_{m-jd}w^{-jd})} = \mu(z)\frac{w^{dk}(1 + \sum_j a_{n-jd}w^{-jd})}{(1 + \sum_j b_{m-jd}w^{-jd})},
\]

we have that \( P \) is, as a function of \( w \), a function of \( w^d \). There exists \( \sigma \in \mathbb{C} \) such that \( \rho^{d}(\lambda) = e^{s \lambda} \). Thus, \( s_i = (n - i)\sigma \), \( t_i = (m - i)\sigma \), \( \delta = (m - n)\sigma \), and

\[
P(z, w) = e^{-s \sigma z}w^n + a_{n-1}e^{-(n-1)s \sigma}w^{n-1} + \cdots + a_0,
\]

\[
e^{-m \sigma z}w^m + b_{m-1}e^{-(m-1)s \sigma}w^{m-1} + \cdots + b_0.
\]
this is, \( P(z, w) \) is a rational function of \( \xi^d, \xi = we^{-\sigma z}, P = R(\xi) \). The projection onto \( \mathbb{P}^1 \) given by \( (z, w) \mapsto \xi \) is invariant under \((12)\) and the image of the vector field is

\[
-\sigma \xi R(\xi) \frac{\partial}{\partial \xi}
\]

(14)

The poles are contained in the fibers.

We will address now the case where \( \Gamma \) contains a subgroup of \( \mathbb{C}^* \) with index two. Let \( E \) be the algebraic curve in \( S \) associated to the finite orbit of \( \Gamma \). Since the group does not contain parabolic elements, the special fibers of the Riccati foliation are, as established previously, dicritical ones. There are two kinds of them, which we will now describe.

We begin with the case where the two fixed points of the local monodromy around a special fiber \( F_0 \) are the separatrices corresponding to \( E \). Let \( n \) be the order of \( X \) along \( E \). Suppose that the vector field in the neighborhood of \( F_0 \) is the quotient of the vector field \( w^n f(z, w) \partial/\partial z \) under \( (z, w) \mapsto (\omega z, \omega^k w) \), where \( \omega^p = 1 \), \( f \) is some meromorphic function, with \( E \) corresponding to \( w \in \{0, \infty\} \). Notice that \( f \) is holomorphic and non-zero at \((0, 0)\) since any orbit accumulating to \( E \) (all of them do) would otherwise gain infinitely many zeros/poles at the zeros/poles of \( f \). A necessary condition for invariance is that \( p|(kn - 1) \). The same reasoning applies in the coordinates \((z, W) = (z, 1/w)\): the vector field has the form \( W^n g(z,W) \partial/\partial z \) and the cyclic group acts now by \((z, W) \mapsto (\omega z, \omega^{-k} W)\). Thus, \( p|(-kn - 1) \). We must conclude that \( p = 2 \). The local monodromy around \( F_0 \) is, \( w \mapsto -w \), and is \( \kappa \), the non-trivial element of the center of the global monodromy of the Riccati equation. The affine structure induced by \( w^n f(z, w) \partial/\partial z \) on \( w = 0 \) is non-singular in the neighborhood of \( z = 0 \). Thus, at each one of the two points where \( E \) intersects \( F_0 \), the affine structure in \( E \) will have a ramification index that is a multiple of 2 (Proposition 11).

If the two fixed points of the local monodromy are not the ones coming from \( E \) then the holonomy is one of the elements in \( \Gamma \) of order two that exchanges them. In this case the restriction of \( \Pi \) to \( E \) is locally a ramified cover of order two.

The ramified cover \( \Pi|_E : E \to B \) of order two has \( r \) ramification points corresponding to the special fibers of the second type described. By the Riemann-Hurwitz formula, \( \chi(E) = 2\chi(B) - r \). Since these curves are both either rational or elliptic, there are three possibilities:

- \( E \) and \( B \) are both elliptic, \( r = 0 \);
- \( E \) and \( B \) are both rational, \( r = 2 \);
- \( E \), is elliptic, \( B \) is rational and \( r = 4 \).

In the first case, there are no special fibers, for \( E \) has no singularities for its affine structure. The monodromy of \( \mathcal{F} \) is a representation in \( N(\mathbb{C}^*) \) of the fundamental group of \( E \) intersecting the class of \( \sigma \). But such abelian subgroups of \( N(\mathbb{C}^*) \) are finite.

The second case cannot arrive. The monodromy would be a representation of the fundamental group of \( \mathbb{C} \setminus \{0, p_1, \ldots, p_k\} \), where the image of small loops around 0 and \( \infty \).
are in the class of $\sigma$ and that of a small one around $p_i$ (assuming $p_i$ give points of the first type, and 0 and $\infty$ of the second one). Such groups are finite as well.

In the last case, since the affine structure in the elliptic curve has no singularities, the local holonomy of all of the special fibers permutes the two local branches of $E$ (is of the second type). Thus, there are four special fibers, the two-point orbit of the monodromy is an elliptic curve $E$ ramifying above the four points of $B$. Up to twofold ramified cover of $B$ along these points, we have a meromorphic vector field inducing a Riccati foliation over the elliptic curve $E$ with holonomy in $\mathbb{C}^*$. By our previous discussion, it will be given by the quotient of the vector field $R(w e^{-\sigma z})\partial/\partial z$ in $\mathbb{C} \times \mathbb{P}^1$. In order to be well defined after the involution, we must have that the vector field is also invariant under $(z, w) \mapsto (-z, 1/w)$, which acts upon $\xi$ by $\xi \mapsto 1/\xi$. Since $R(\xi)\partial/\partial z$ is invariant under this transformation, we have that $R(1/\xi) = -R(\xi)$. The vector field (14) is invariant under the involution, and well-defined in the quotient. Again, there is an elliptic fibration from $S$ onto $\mathbb{P}^1$ preserved by $X$ (the curves of zeros and poles are contained in the fibers).

6.2.3. *In the presence of good orbifold coverings.* For this part, we borrow some arguments from [Bru04]. Let $B^* \subset B$ be the union of $B'$ plus the dicritical fibers $F_i$ (where we have seen that the vector field is necessarily regular), $S^* = \Pi^{-1}(B^*)$. Let $m_i$ be the corresponding multiplicity. We will consider $B^*$ as an orbifold by affecting $p_i$ with the angle $2\pi/m_i$. The monodromy $\Gamma$ is, naturally, a representation of the orbifold fundamental group of $B^*$. Affect the leaves of $\mathcal{F}$ in $S^*$ with an orbifold structure, by declaring that, at a dicritical fiber of multiplicity $m$, each one of the two special curves (the separatrices corresponding to the fixed points of the local monodromy), the angle is $2\pi/m$. (The angle being $2\pi$ for all other points). With respect to these structures, if $L$ is a leaf of $\mathcal{F}|^*_S$, $\Pi|_L$ is tautologically an orbifold covering map.

We will now suppose that

\begin{equation}
L \subset S^*, \ L \text{ has no orbifold points.}
\end{equation}

In other words, that if $L$ intersects a special fiber at finite time, this special fiber is dicritical, and the point of intersection is not one of the two special points of the dicritical component. Recall that we have denoted by $\hat{\mathcal{L}}$ the curve obtained from $\mathcal{L}$ after compactifying all of its algebraic isolated planar ends. Since $\hat{\mathcal{L}}$ is not algebraic, either $\hat{\mathcal{L}} = \mathcal{L}$ or $\hat{\mathcal{L}} \setminus \mathcal{L}$ is a single point, where the vector field has a simple zero. Two cases arise:

When $\hat{\mathcal{L}} \setminus \mathcal{L} \notin S^*$. When $\hat{\mathcal{L}} \setminus \mathcal{L}$ is either empty or consists of a single point that is not in $S^*$. The mapping $\Pi|_{\hat{\mathcal{L}}} : \mathcal{L} \rightarrow B^*$ is a (orbifold) covering of $B^*$ and thus $\mathcal{L}$ is covered by the (orbifold) universal covering of $B^*$. Since $\mathcal{L}$ is an entire curve, its universal covering is biholomorphic to $\mathbb{C}$. Since the group of biholomorphisms of $\mathbb{C}$ preserves (tautologically) an affine structure, this induces an affine structure in $B^*$. We may find coordinates $(z, w) \in \Delta \times \mathbb{P}^1$ around a regular fiber where the vector field has the form $f(z, w)\partial/\partial z$, $f$ meromorphic in $z$ and algebraic in $w$, and where the coordinate $z$ in the base is an affine one (with respect to the affine structure we just defined). Every
leaf $L$ is naturally endowed with two affine structures (with singularities), one induced by $X$ and one by $\Pi|_L$. We may consider, as in (1), the difference of these two affine structures. It is $-f_z/f \, dz$. It vanishes for all the values of $w$ belonging to $L$, and thus vanishes identically: the fibration establishes an isomorphism of affine structures between the base and the one induced by the vector field on every leaf. Up to a finite cover $\rho : \hat{B} \to B$ of the base by a manifold $\hat{B}$, the affine structure on $B$ is induced by a vector field $V$. Let $\hat{\Pi} : \hat{S} \to \hat{B}$ the ruled surface obtained by pulling back $\Pi$ along $\rho$. It has a vector field $\hat{X}$ coming from the natural projection $\hat{S} \to S$. There is a meromorphic function $h$ on $\hat{S}$ such that $\Pi^*(h \hat{X}) = V$. Since the projection is an isomorphism of affine structures in restriction to a Zariski-dense leaf, $h$ is constant on a Zariski-dense leaf, and thus globally constant. The vector field $\hat{X}$ projects to $V$ via $\hat{\Pi}$. But this implies that $\hat{X}$ is holomorphic above a generic fiber of $\hat{\Pi}$. On its turn, this implies that $X$ is holomorphic above a generic fiber of $\Pi$, establishing the result in this case.

When $\hat{L} \setminus L \in S^*$. If $\hat{L} \setminus L = \{p\}, p \in S^*$. The curve $\hat{L}$ is entire and there is a global coordinate $\zeta$ for $L$, centered at $p$, where, up to multiplication by a constant, $X = \zeta \partial/\partial \zeta$. In this coordinate, the affine defect of the affine structure induced by $X$ is $\zeta^{-1} \, d\zeta$. Pairing this form with $X$ gives the constant function $1$. In the coordinate $\zeta$, $\hat{L}$ (but not $L$!) is a covering of the orbifold $S^*$ and gives it an affine structure which, in turn, can be lifted to any leaf of $F$. In a leaf, the difference of this affine structure with the one induced by $X$ is a one-form, which may be contracted with $X$. This produces a function whose restriction to $\hat{L}$ equals $1$, and that is thus the constant function $1$. In open subsets of $C$, a vector field $f(\zeta) \partial/\partial \zeta$ induces an affine structure. The difference of this affine structure and the one induced by $\zeta$ is $-f'/f \, d\zeta$, and pairing this form with the vector field gives the function $f'$. If $f' = 1$ then $f(\zeta) = \zeta + c$ for some $c \in C$. Thus, for every leaf where it makes sense, in the affine coordinate inherited from $B$, the vector field $X$ is linear. In particular,

- in the neighborhood of $p$ there is a component $E_0$ of the curve of zeros that intersects transversely every leaf; and
- no curve intersects the curve of poles (which is thus invariant by $F$).

From the first fact, since $L$ intersects only once the curve of zeros, it cannot self-accumulate and, since it intersects infinitely many times each fiber, the monodromy of the equation is discrete. From the second one, since there must be at least one component of the curve of poles by (9) and since this component is invariant, the monodromy group of the equation has a finite orbit. We thus conclude that $\Gamma$ is discrete and elementary, a case for which we have already established our result.

If $u$ is uncountable, condition (15) is automatically satisfied, for only finitely many orbits do not satisfy (15), and thus only countably many elements of $u$ may fail to satisfy condition (15). The situations where we have proved the Main Theorem are

- when $\Gamma$ is finite, for $\ell$ is finite;
- when $\Gamma$ is discrete, infinite and non-elementary;
• when $\Gamma$ is discrete and non-elementary, for $u$, being closed and invariant, contains the limit set of the action of $\Gamma$ on $\mathbb{P}^1$ and is thus uncountable;
• when $\Gamma$ is not discrete, for $u$, being the union of orbits of $\Gamma$, is either finite (implying the finiteness of $\ell$) or uncountable.

We have thus proved the Main Theorem in all cases where the induced foliation is a Riccati one.

6.3. **Turbulent.** The proof will follow the lines of that of the Riccati case. Let $\Pi : S \to B$ be an elliptic fibration, $X$ a meromorphic vector field on $S$ inducing a turbulent foliation adapted to $\Pi$. Let $E$ be a component of the curve of zeros that is not contained in a fiber (it intersects all the fibers).

If $E$ is invariant, the intersection of $E$ with a generic fiber $F_0$ gives a finite orbit for the monodromy of the foliation on $F_0$. Since a group of automorphisms of an elliptic curve with one finite orbit is finite, the monodromy must be finite. Further, all special fibers must be dicritical, for non-dicritical fibers do not have separatrices. In this case there is a first integral for $F$.

If $E$ is not invariant, in the neighborhood of the non-dicritical special fibers, every leaf will intersect $E$ infinitely many times. Hence, all the special fibers are dicritical. If the monodromy is finite, there is a first integral for $F$. We will suppose that the monodromy is infinite and, in particular, that all orbits self-accumulate (in particular, the closure of any orbit contains uncountably many others). If $p \in \hat{\mathcal{L}} \setminus \mathcal{L}$, $p$ lies at the intersection of $\hat{\mathcal{L}}$ with the curve of zeros. But, by the self-accumulation of $\hat{\mathcal{L}}$, this would imply that the orbit intersects infinitely many times the curve of zeros. Thus, $\mathcal{L} = \hat{\mathcal{L}}$. As previously, consider $B$ with its orbifold structure. Let us go again through the arguments of the Riccati case in Section [6.2.3]. Up to replacing $\mathcal{L}$ by another orbit, the projection $\Pi|_{\mathcal{L}}$ is a covering of $B$ (in the orbifold sense). Thus, $B$ carries an affine structure making the restriction of $\Pi$ to every leaf is an affine map. From the cover $\rho : \hat{B} \to B$ where the affine structure of $B$ is induced by a vector field, we may construct a cover $\hat{S} \to S$ where the induced vector field is holomorphic above every fiber. But this implies that $X$ is holomorphic along the generic fiber of $\Pi$, and this establishes the result.

We thus finish the proof of our Main Theorem in the case where the surface is algebraic and the solution is entire.

6.4. **The non-algebraic cases.** The remaining case is the one where $S$ is Kähler but not algebraic. The classification of foliations in such surfaces is due to Brunella [Bru03, Section 10]. Let $a(S)$ denote the algebraic dimension of $S$.

**Theorem 22** (Brunella). Let $S$ be a non-projective Kähler complex surface and $\mathcal{F}$ a holomorphic foliation in $M$. Either:

1. $a(S) = 1$: $S$ is an elliptic fibration and $\mathcal{F}$ is turbulent with respect to it.
2. $S$ is an Abelian surface, $a(S) = 0$ and $\mathcal{F}$ is induced by a constant vector field.
(3) $S$ is a $K3$ surface with $a(S) = 0$, there is an Abelian surface $A$ and a covering $\Pi : A \to S$ such that $\mathcal{F}$ is induced through $\Pi$ by a constant vector field in $A$.

In the first case all the algebraic curves are tangent to the fibration and so are the curves in support of $D$ (one can also repeat the arguments for turbulent foliations in the algebraic case). In the second case, the ratio between $X$ and a constant vector field inducing $\mathcal{F}$ is a meromorphic function on $S$, which must be constant. The same argument applies in the third case to the ratio between the pull-back of $X$ and of the constant vector field: $X$ is a holomorphic vector field on $S$.

This finishes the proof of the Main Theorem.

7. Examples and Comments

For Riccati equations, having a first integral implies that the monodromy group is finite. The reciprocal is not true. The following example is attributed to Wittich [Hil97, Section 4.1]. The Riccati differential equation

$$y' = y^2 + \frac{t^3 + 2}{t(t^3 - 1)}y + \frac{(t^3 - 1)^2}{t^4}$$

has the transcendental solutions

$$y(t) = \left(t - \frac{1}{t^2}\right) \tan \left(\frac{t^2}{2} + \frac{1}{t} + c\right).$$

Since all the solutions are single-valued, the monodromy is trivial.

The relation between the maximal solutions and the fixed points of the monodromy representation in Riccati and turbulent foliations is behind Corollary 2.

Proof of Corollary 2. If the vector field $X$ is birationally equivalent to a holomorphic vector field, all its solutions are maximal. Let us address the case where there is a rational fibration $\Pi : S \to C$ preserved by $Z$, let $C_0 \subset \mathcal{C}$ be the image of the non-special fibers. Let $\rho : \pi_1(C_0) \to \text{PSL}(2, \mathbb{C})$ be the monodromy of the Riccati foliation. Let $Y = \Pi_* Z$ and let $\mu : \pi_1(C_0) \to \mathcal{C}$ be the monodromy of the translation structure induced by $Y|_{C_0}$. The maximal solutions of $X$ correspond to the global fixed points of the restriction of $\rho$ to $\ker(\mu) \subset \pi_1(C_0)$. If this representation has three fixed points, it is trivial, and all of its points are fixed. We proceed similarly for an elliptic fibration: the maximal solutions are the fixed points of some representation. The corollary follows from the fact that an automorphism of an elliptic curve having five fixed points is the identity.

The elliptic involution of an elliptic curve has four fixed points. We have the following example:

Example 23. Consider, in $\mathbb{C} \times \mathbb{C}$, the meromorphic vector field

$$(16) \quad X = \frac{1}{2x} \frac{\partial}{\partial x} - \frac{1}{x^3} \frac{\partial}{\partial y}.$$
It is invariant by translations in $y$ and by the order two-mapping $\sigma$

$$\tag{17} (x, y) \xrightarrow{\sigma} (-x, -y).$$

It has the first integral $y - \frac{2}{x}$. Each one of its level curves is may be parametrized by $x$ and is biholomorphic to $\mathbb{C}^*$. In this global coordinate the restriction of the vector field is $(2x)^{-1} \partial / \partial x$. Let $\Lambda \subset \mathbb{C}$ be a lattice. Since the vector field (10) is invariant by translations in $y$, it induces a vector field in $S = \mathbb{C} \times (\mathbb{C} / \Lambda)$, which is an elliptic fibration with respect to the projection onto the first factor. The foliation $\mathcal{F}$ induced by $X$ is a turbulent one. This elliptic fibration is equivariant with respect to the involution induced by (17) on $S$ and by $x \mapsto -x$ on $\mathbb{C}$. The quotient $E$ of $S$ under $\sigma$ is, after resolution of the singularities produced by the four fixed points, an elliptic fibration $\Pi : E \to \mathbb{C}$. It is still endowed with a vector field, projecting onto $\partial / \partial z$ by the projection $(x, y) \mapsto x^2$. For $c \in \frac{1}{2} \Lambda$, the leaf $y = c + 2/x$ of (10) is fixed under the involution and the projection $\Pi$ in restriction to this leaf is injective (and corresponds thus to a maximal solution). In the other leaves the solution is multivalued and has two determinations. This produces an elliptic fibration $\Pi : E \to \mathbb{C}$ endowed with a meromorphic vector field preserving it (inducing a turbulent foliation) having exactly four single-valued solutions.

Some classical examples of algebraic differential equations on surfaces corresponding to the first possibility of the Main Theorem (giving holomorphic vector fields on surfaces) are given, for example, by the Chazy VI equation $\phi''' = \phi\phi'' + 5(\phi')^2 - \phi^2\phi'$ and the Chazy IX equation $\phi''' = 18(\phi' + \phi^2)(\phi' + 3\phi^2) - 6(\phi')^2$. They both have algebraic first integrals and their restriction to a generic level surface integrals respectively, as a linear vector field in $\mathbb{P}^2$ or as a constant vector field in an Abelian surface (see [Gui12] for details). Let us now describe some examples related to the second possibility of the Main Theorem.

**Example 24.** Consider the vector field $X$ in $\mathbb{C} \times \mathbb{P}^1$ given in $\mathbb{C} \times \mathbb{C}$ by $(z - w)\partial / \partial z$. It is invariant under the diagonal action of $\text{Aff}(\mathbb{C})$ on $\mathbb{C} \times \mathbb{C}$. Let $\Gamma \subset \text{Aff}(\mathbb{C})$ be a crystallographic group. Let $\nu$ be the smallest natural such that $\gamma^\nu$ is a translation for every $\gamma \in \Gamma$. Let $B$ be the quotient of $\mathbb{C}$ under $\Gamma$. The quotient of $\mathbb{C} \times \mathbb{P}^1$ under the diagonal action of $\Gamma$ on produces a surface $S$ endowed with a rational fibration $\Pi : S \to B$ with respect to which the foliation induced by $X$ is a Riccati one. Notice that this fibration is not preserved by the flow of $X$ on $S$. However, there is an elliptic fibration $\Xi : S \to \mathbb{P}^1$ induced by $(z, w) \mapsto (z - w)^\nu$ that maps $X$ to the vector field $\nu \xi \partial / \partial \xi$. This second fibration is preserved by the vector field. There are two special fibers: one dicritical and one invariant.

The Main Theorem is of a birational nature and applies more generally to vector fields on analytic surfaces having a Kähler desingularization, like meromorphic vector fields in algebraic surfaces. Consider the following example, analogue to Example 30 in [GR12]:

...
Example 25. Consider, in $\mathbb{C}^3$, the $E_8$ singularity $x^2 + y^3 + z^5 = 0$, together with the vector field given by the restriction of

$$(15x + 3y^2)\frac{\partial}{\partial x} + (10y - 2x)\frac{\partial}{\partial y} + 6z\frac{\partial}{\partial z}.$$ 

It has the maximal solution

$$\left(\frac{1}{2}e^{15t}\varphi\left(\frac{5}{e^{5t}}\right), -e^{10t}\varphi\left(\frac{5}{e^{5t}}\right), e^{6t}\right),$$

for the Weierstrass elliptic function $\varphi$ such that $(\varphi')^2 = 4\varphi^3 - 4$. Under the mapping $(x, y, z) \mapsto z$, the vector field maps to $6z\frac{\partial}{\partial z}$.

Example 26 (Briot-Bouquet [BB55], Ghys-Rebelo [GR97]). Consider the vector fields

$$(18) \eta \frac{\partial}{\partial \zeta} - Q(\zeta)\eta^2 \frac{\partial}{\partial \eta},$$

where $Q$ is a rational function. It corresponds to the equation $\zeta'' = -(\varphi')^2Q(\varphi)$. Briot and Bouquet classified the rational functions giving univalent equations, which correspond to the uniformizable affine structures with singularities on rational curves. For example, if $Q = (1 - n)/\xi$ the solution is $\xi = t^n$. For $S(\zeta) = 4\zeta^3 - g_2\zeta - g_3$ and $Q(\zeta) = -S'(\zeta)/S(\zeta)$, a solution is the Weierstrass function $\wp(t)$ satisfying $(\wp')^2 = S(\wp)$. The image of the homogeneous polynomial vector field of degree $d$ in $\mathbb{C}^2 A(x, y)\frac{\partial}{\partial x} + B(x, y)\frac{\partial}{\partial y}$ under $(x, y) \mapsto (x/y, [yA - xB]/y^2)$ is (18) for

$$Q(\zeta) = \frac{(2 - d)B - y(A_x - \zeta B_x)}{A - \zeta B}.$$ 

The classification of semicomplete homogeneous quadratic vector fields in $\mathbb{C}^2$ may be obtained as a consequence of Briot and Bouquet’s classification.

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