Are quantum states real?

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Abstract

In this paper we consider theories in which reality is described by some underlying variables, \( \lambda \). Each value these variables can take represents an ontic state (a particular state of reality). The preparation of a quantum state corresponds to a distribution over the ontic states, \( \lambda \). If we make three basic assumptions, we can show that the distributions over ontic states corresponding to distinct pure states are non-overlapping. This means that we can deduce the quantum state from a knowledge of the ontic state. Hence, if these assumptions are correct, we can claim that the quantum state is a real thing (it is written into the underlying variables that describe reality). The key assumption we use in this proof is ontic indifference - that quantum transformations that do not affect a given pure quantum state can be implemented in such a way that they do not affect the ontic states in the support of that state. In fact this assumption is violated in the Spekkens toy model (which captures many aspects of quantum theory and in which different pure states of the model have overlapping distributions over ontic states). This paper proves that ontic indifference must be violated in any model reproducing quantum theory in which the quantum state is not a real thing. The argument presented in this paper is different from that given in a recent paper by Pusey, Barrett, and Rudolph. It uses a different key assumption and it pertains to a single copy of the system in question.

1 Introduction

Ever since the Schroedinger wrote down his wave equation, people have wondered what kind of thing the wave function is. Perhaps the most basic question one can ask is whether it is a real thing. That is, is there something in the underlying reality of the world that corresponds to the wavefunction?

We start by assuming there is still some underlying reality, described by some variables, \( \lambda \). Sometimes these variables are called hidden variables. This is a little inappropriate though since not all the variables have to be “hidden”
Each $\lambda$ describes a possible ontic state - a possible state of the underlying reality. When a quantum state is prepared we imagine that it corresponds to a probability distribution over ontic states. A different quantum state would, in general, correspond to a different distribution over ontic states.

Spekkens has pioneered the idea that we might be able to provide an epistemic interpretation of the quantum state. He has constructed a toy theory \cite{35} (see also \cite{19}, \cite{9} \cite{10}, \cite{3}) which is capable of reproducing a substantial fraction of the predictions of quantum theory. This toy theory has the property that, for different pure states in the theory, the corresponding distribution over ontic states overlap. Thus, there exist ontic states that are consistent with two or more pure states in this toy theory. The toy theory does not reproduce quantum theory however. A natural question to ask is whether there exists a model having this property that can reproduce the predictions of quantum theory.

What if we can prove that pure states, in quantum theory, must correspond to non-overlapping distributions over ontic states? This would imply that, no matter how complicated the variables, $\lambda$, representing the ontic state are, we would be able to deduce from $\lambda$ what the pure quantum state is (since each $\lambda$ would only appear in the distribution of one pure quantum state). In other words, the quantum state would be written into the underlying reality of the world and we could assert that the quantum state is real. Harrigan and Spekkens \cite{21} introduced the following terminology: $\psi$-epistemic models are those in which there exist pairs of pure quantum states for which the distributions over ontic states overlap; $\psi$-ontic models are those in which the distributions over ontic states are non-overlapping for any pair of pure quantum states. In $\psi$-ontic models we can assert that the wavefunction is a real thing. The question is whether all reasonable models reproducing the predictions of quantum theory are $\psi$-ontic.

There has been a lot of interest in this subject recently. In 2008 Montina \cite{26, 27} showed that, under the very natural assumption that the evolution of the $\lambda$ is Markovian, the hidden variables must have at least $2N - 2$ real parameters where $N$ is the dimension of the Hilbert space (this being the same number of real parameters required to describe a pure quantum state). This provides rather strong evidence for the reality of the quantum state. Then, in 2011 Pusey, Barrett, and Rudolph (PBR) \cite{30} obtained the momentous result that, under a certain separability assumption, distinct pure quantum states have non-overlapping probability distributions thus providing the first proof that the quantum state is a real thing in the terms outlined above. The separability assumption employed by PBR is that independently prepared pure quantum states correspond to a product of probability distributions over ontic states.

In this paper we show that, if three assumptions hold, then pure quantum states have non-overlapping ontic support thus implying the reality of the wavefunction. The key assumption employed here, ontic indifference, is quite different from the separability assumption used by PBR. Additionally the proof presented here only requires a single system (whereas PBR required multiple copies of the system of interest). The assumption of ontic indifference is that it is possible to implement any quantum transformation which does not affect
any given pure quantum state, $|\varphi\rangle$, in such a way that the ontic states in the support of that state are not affected.

At the cost of very slightly complicating matters we will also show that can run the proof with an even weaker assumption. This is the assumption of restricted ontic indifference in which it is assumed that there exists at least one pure state, $|0\rangle$, such that quantum transformations leaving this state unchanged can be implemented in a way that does not affect the ontic states in the ontic support of $|0\rangle$. An example is the following. Consider a system consisting of a quantum particle that can be in one of $N+1$ boxes labeled $n = 0$ to $N$ such that we have an $N+1$ dimensional Hilbert space (we do not consider any internal structure in the boxes). Let $|0\rangle$ be the state corresponding to the particle being localized in box 0. Consider further different possible quantum transformations we might perform that leave $|0\rangle$ invariant. Then restricted ontic indifference asserts that we can find an apparatus to implement these quantum transformations in such a way that ontic states in the support of $|0\rangle$ are not affected by which quantum transformation we choose to implement. Transformations that leave $|0\rangle$ invariant can all be implemented without actually touching box 0. Indeed, boxes 1 to $N$ could be at a great distance from box 0. Viewed in this way restricted ontic indifference is a kind of locality assumption. Restricted ontic indifference would follow if (i) all the ontic variables associated with a localized particle are “situated” in the region where that particle is localized, and (ii) these ontic variables are unaffected by transformations that can be implemented without touching this region.

In fact the assumption of ontic indifference (and of restricted ontic indifference) is violated by the Spekkens toy model mentioned above and, even more interestingly, by a particular application of that model due to Elliott Martin and Robert Spekkens (see [33]) which applies to interferometers of the sort that we will consider later. In the model of Martin and Spekkens there are ontic states associated with a path of an interferometer even when the particle goes along the other path (i.e. it violates (i) from the preceding paragraph). This can be viewed as taking a field picture of the ontic variables in which the vacuum has non-trivial structure. The assumption of restricted ontic indifference is not violated in the de Broglie Bohm model for quantum mechanics in which the ontic state is given by $(\psi, x)$ where $x$ is the actual position, in configuration space, of the particles. Thus the result of this paper could be read in the following way. While it is possible to construct $\psi$-ontic models that satisfy the ontic indifference assumption, it is not possible to build $\psi$-epistemic models of this nature. This, then, could be regarded as strong hint on how to construct of $\psi$-epistemic models. In particular, it points to a field (rather than a particle) ontology for $\psi$-epistemic models.

The main proof we will present requires that the Hilbert space associated with the system in question is infinite dimensional. If this dimension, $N + 1$, is finite, then we prove that any pair of pure states, $|\varphi\rangle$ and $|\psi\rangle$, for which $|\langle\varphi|\psi\rangle|^2 \geq \frac{N-1}{N}$ must have non-overlapping distributions over the ontic states (under the given assumptions). We will also discuss the case of a system, $A$, with finite dimensional Hilbert space. We will show how, by introducing an
infinite dimensional ancilla, $B$, in some fixed state, we are able to deduce that different pure states for system $A$ must correspond to different ontic states of $AB$.

We will introduce the assumptions we use in Sec. 2. In Sec. 3 we will illustrate the proof method using a Mach Zehnder interferometer. This works for the $N = 2$ case. Then, in Sec. 4, we will prove the main theorem using the assumption of ontic indifference and assuming that the Hilbert space dimension can be arbitrarily large. In Sec. 5 we will show how to replace the ontic indifference assumption with the restricted ontic indifference assumption. Finally we show how to deal with systems having a finite dimensional Hilbert space by using an ancilla.

2 Assumptions

We make the following assumptions

Realism. Each time a system is prepared there exists an underlying state of reality, $\lambda$, which we will call the ontic state.

Possibilistic completeness. The ontic state, $\lambda$, is sufficient to determine whether any outcome of any measurement has probability equal to zero of occurring or not.

Ontic indifference. Any quantum transformation on a system which leaves unchanged any given pure state, $|\psi\rangle$, can be performed in such a way that it does not affect the underlying ontic states, $\lambda \in \Lambda_{|\psi\rangle}$, in the ontic support of that pure state.

By the ontic support of a given state, $|\psi\rangle$, we mean the set, $\Lambda_{|\psi\rangle}$, of ontic states, $\lambda$ which might be prepared when the given pure state is prepared (i.e. those ontic states that have a non-zero probability of being prepared when the given pure state is prepared).

Rather than assuming possibilistic completeness we might have made the stronger assumption that $\lambda$ determines the probability. However, it is sufficient for our purposes that it only determines whether the probability is zero or non-zero. Of course, in any model having hidden variables, $\lambda$, it is reasonable to expect the hidden variables actually determine the probabilities for measurement outcomes.

We assume that the system has arbitrarily large Hilbert space dimension. We also assume that we can perform arbitrary unitary transformations and arbitrary measurements on the system (or, at least, we assume that we can perform the particular unitary transformations and measurements that we need to run the proof). This is reasonable in view of the fact that arbitrary unitaries can be implemented with beamsplitters and mirrors in the case of interferometry as shown by Reck and Zeilinger [31] and, more generally, we can perform arbitrary unitaries in quantum circuits given access to a universal gate set [12]. An
arbitrary measurement can then be implemented by having the appropriate unitary followed by a measurement in some standard basis.

Similar assumptions to the first two assumptions appear in the work of Pusey, Barrett, and Rudolph and it is difficult to imagine getting any traction on the problem unless we make these assumptions (though, of course, it is still interesting to investigate models that do not satisfy these assumptions). The key assumption here is the third assumption. It turns out that we can weaken this assumption and still run the proof. Thus we can, instead, assume

**Restricted ontic indifference** Any quantum transformation on a system which leaves a particular given pure quantum state, $|0\rangle$, unchanged can be implemented in such a way that it does not affect the underlying ontic states, $\lambda \in \Lambda_{|0\rangle}$, in the ontic support of $|0\rangle$.

This is a considerably weaker assumption and is very well motivated in the case where $|0\rangle$ corresponds to a spatially localized particle as discussed in Sec. I.

We will first run the proof using the assumption of ontic indifference then show how we can, in fact, use the weaker assumption of restricted ontic indifference.

### 3 Interferometric example

In popular accounts of quantum theory an interferometer is often used to argue that something can be in two places at once. For example, consider the Mach Zehnder interferometer in Fig. I. The two beamsplitters, $BS1$ and $BS2$ are 50:50. We assume that the transformation effected at each beamsplitter is given by the unitary matrix

$$
\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}
$$

Such that

$$
|s_1\rangle \rightarrow \frac{1}{\sqrt{2}}(|a_0\rangle + |a_1\rangle)
$$

Figure 1: This apparatus prepares $|\psi\rangle = \frac{1}{\sqrt{2}}(|a_0\rangle + |a_1\rangle)$ which then impinges on $BS2$ followed by two detectors $B1$ and $B2$. A $\phi = \pi$ phase shifter can be placed in path $a_1$. 

We will first run the proof using the assumption of ontic indifference then show how we can, in fact, use the weaker assumption of restricted ontic indifference.
at BS1 and
\[
|a_0⟩ \longrightarrow \frac{1}{\sqrt{2}}(|b_1⟩ + |b_2⟩) \quad |a_1⟩ \longrightarrow \frac{1}{\sqrt{2}}(|b_1⟩ − |b_2⟩)
\] (3)

at BS2. We have chosen the integer subscript labeling here for later convenience.

We require, of course, that
\[
⟨s_m|s_n⟩ = δ_{mn} \quad ⟨a_m|a_n⟩ = δ_{mn} \quad ⟨b_m|b_n⟩ = δ_{mn}
\] (4)

We will assume that the path lengths are such that zero net phase is accumulated as the particle goes along the internal paths of the interferometer. On passing through the first, then the second beamsplitter it follows from the above transformations that the evolution is
\[
|s_1⟩ \longrightarrow \frac{1}{\sqrt{2}}(|a_0⟩ + |a_1⟩) \longrightarrow |b_1⟩
\] (5)

Thus the detector, B1, in path b1 will always fire and there is zero probability that the B2 detector will fire. However, if we insert a phase shifter \(φ = π\) in the a2 path as indicated then the evolution is
\[
|s_1⟩ \longrightarrow \frac{1}{\sqrt{2}}(|a_0⟩ + |a_1⟩) \longrightarrow \frac{1}{\sqrt{2}}(|a_0⟩ − |a_1⟩) \longrightarrow |b_2⟩
\] (6)

With the phase shifter in place the detector B2 will always fire and there is zero probability that the detector B1 will fire.

Given this setup, one version of the popular argument for something going both ways is the following. Let us try to account for this experiment in such a way that the particle either goes along path a0 or a1. If this is the case then, sometimes it must go along path a0. Consider a case in which it does go along path a0. When the particle reaches BS2 it must go along path b1 if there is no phase shifter and it must go along path b2 if the \(φ = π\) phase shifter is in place. Since it did not go along path a1 it does not “know” whether there is a phase shifter in place or not. Hence, it cannot make the correct “decision”. This shows that something must be going along both paths. We will now see how to turn this argument into a formal argument for the reality of the quantum state.

This example can be used to show that the pair of states
\[
|φ⟩ = |a_1⟩ \quad \text{and} \quad |ψ⟩ = \frac{1}{\sqrt{2}}(|a_0⟩ + |a_1⟩)
\] (7)

must have non-overlapping ontic support given three assumptions in Sec. 2 above. We can prepare the state \(|φ⟩ = |a_0⟩\) by the apparatus shown in Fig. 2 and we can prepare the state \(|ψ⟩ = \frac{1}{\sqrt{2}}(|a_0⟩ + |a_1⟩)\) by the apparatus shown in Fig. 1 (in each case we consider the state at a time just before the system passes through the location where the phase shifter may be inserted). Let \(Λ_{|φ⟩}\) be the ontic support of the state \(|φ⟩ = |a_0⟩\). We now allow it to impinge on the measurement apparatus constituted by BS2 and the two detectors B1 and B2.
Figure 2: This apparatus prepares the state $|\varphi\rangle = |a_0\rangle$ which then impinges on BS2 followed by two detectors B1 and B2. A $\phi = \pi$ phase shifter can be inserted in path $a_1$.

Further, we have the phase shifter, $\phi$, which can be inserted in path $a_1$. We have a choice of two setting for the phase shifter. Either $\phi = 0$ (not inserted) or $\phi = \pi$ (inserted). Let $\Lambda_{|\varphi\rangle}^{B_1}[\phi] \subseteq \Lambda_{|\varphi\rangle}$ be the subset of ontic states in the ontic support of $|\varphi\rangle$ that have non-zero probability of giving rise to a click at detector $B_1$ given that the phase shifter in path $a_1$ has setting $\phi$. We are invoking possibilistic completeness in asserting that this set is well defined. Likewise we define $\Lambda_{|\varphi\rangle}^{B_2}[\phi] \subseteq \Lambda_{|\varphi\rangle}$ to be the subset of ontic states in the ontic support of $|\varphi\rangle$ that have non-zero probability of giving rise to a click at detector $B_2$ with phase shifter setting $\phi$. Inserting, or not inserting, the phase shifter has no affect on the quantum state $|\varphi\rangle = |a_0\rangle$. Hence, it follows from the assumption of ontic indifference that inserting, or not inserting, the phase shifter has no affect on the ontic states, $\lambda \in \Lambda_{|\varphi\rangle}$ in the support of $|\varphi\rangle$. It follows that

$$\Lambda_{|\varphi\rangle}^{B_1}[\phi = 0] = \Lambda_{|\varphi\rangle}^{B_1}[\phi = \pi] = \Lambda_{|\varphi\rangle}^{B_1}$$

(8)

$$\Lambda_{|\varphi\rangle}^{B_2}[\phi = 0] = \Lambda_{|\varphi\rangle}^{B_2}[\phi = \pi] = \Lambda_{|\varphi\rangle}^{B_2}$$

(9)

We have

$$\Lambda_{|\varphi\rangle}^{B_1} \cup \Lambda_{|\varphi\rangle}^{B_2} = \Lambda_{|\varphi\rangle}$$

(10)

as either detector $B_1$ or $B_2$ must fire (we are assuming 100% efficient detectors).

Now consider an ontic state, $\lambda$, in the intersection

$$\Lambda_{|\psi\rangle}^{B_2}[\phi = 0] \cap \Lambda_{|\psi\rangle}$$

(11)

where $\Lambda_{|\psi\rangle}$ is the ontic support of $|\psi\rangle = \frac{1}{\sqrt{2}}(|a_0\rangle + |a_1\rangle)$. If $\phi = 0$ and $|\psi\rangle$ has been prepared then we know that detector $B_2$ cannot fire. Hence, using (9),

$$\Lambda_{|\varphi\rangle}^{B_2}[\phi = 0] \cap \Lambda_{|\varphi\rangle} = \Lambda_{|\varphi\rangle}^{B_2} \cap \Lambda_{|\varphi\rangle} = \emptyset$$

(12)

(where $\emptyset$ is the empty set) since $\Lambda_{|\varphi\rangle}^{B_2}[\phi = 0]$ are the ontic states that have non-zero probability of causing $B_2$ to fire. Likewise, if we consider the setting $\phi = \pi$ then we know that detector $B_1$ cannot fire and hence

$$\Lambda_{|\varphi\rangle}^{B_1} \cap \Lambda_{|\psi\rangle} = \emptyset$$

(13)
Using (10) this implies that
\[ \Lambda_{\psi} \cap \Lambda_{\psi} = \emptyset \] (14)

since
\[ \Lambda_{\psi} \cap \Lambda_{\psi} = (\Lambda_{B_1} \cup \Lambda_{B_2}) \cap \Lambda_{\psi} = (\Lambda_{B_1} \cap \Lambda_{\psi}) \cup (\Lambda_{B_2} \cap \Lambda_{\psi}) = \emptyset \] (15)

We see here the crucial role the assumption of ontic indifference is playing since if we had to keep the \( \phi \) argument on \( \Lambda_{B_1} \) and \( \Lambda_{B_2} \) we could not use the two different contexts in the same equation. It follows from (14) that the states \( |\psi\rangle \) and \( |\psi\rangle \) have non-overlapping ontic support. This particular proof applies only to a pair of states having \( |\langle \psi | \psi \rangle|^2 = \frac{1}{2} \). We would like to prove that any pair of distinct pure states has non-overlapping ontic support.

Before we prove this we notice that the above proof can be modified to apply to a pair of states having \( |\langle \psi | \psi \rangle|^2 \leq \frac{1}{2} \). Consider the experiment in Fig. 3. Now assume that the first beamsplitter, \( BS_1 \), is uneven such that it prepares a state
\[ |\psi\rangle = \alpha|a_0\rangle + \beta|a_1\rangle \] where \( \alpha \) and \( \beta \) are real and positive and \( \alpha \leq \beta \). After the phase shifter the state will impinge on the beamsplitter \( BS_3 \) in path \( a_1 \). We set the transmissivity, \( T \), of this beamsplitter such that
\[ |\psi\rangle = \alpha|a_0\rangle + \beta|a_1\rangle \longrightarrow \alpha|a_0\rangle + \alpha|a_1'\rangle + \gamma|b_0\rangle \] (16)

This is achieved if \( \sqrt{T} \beta = \alpha \). Note that if the phase shifter had been inserted there would be a minus sign in front of the second term on the right. Rather than sending in the state \( |\psi\rangle \), we can send in the state \( |\phi\rangle = |a_0\rangle \) as shown in Fig. 4. Since the amplitudes in front of the \( |a_0\rangle \) and \( |a_1'\rangle \) terms are now equal we have probability zero of getting a detection in the \( B_2 \) detector if \( \phi = 0 \) (phase shifter not inserted) and probability zero of getting a detection in the \( B_1 \) detector if \( \phi = \pi \) (phase shifter inserted). We define \( \Lambda_{\phi}^{B_1} \) as being the subsets
of $\Lambda_{|\varphi\rangle}$ which have non-zero probability of giving rise to a detector click in $B_n$ (where $n = 0, 1, 2$) when the phase shifter is set to $\phi$. By ontic indifference, we can put

$$\Lambda_{B_n |\varphi\rangle} = \Lambda_{B_n |\psi\rangle}$$  \hspace{1cm} (17)$$

We have that

$$\Lambda_{B_0 |\varphi\rangle} \cup \Lambda_{B_1 |\varphi\rangle} \cup \Lambda_{B_2 |\varphi\rangle} = \Lambda_{|\varphi\rangle}$$  \hspace{1cm} (18)$$

Now, we know that the probability of a click at detector $B_0$ given that the state was $|\varphi\rangle$ is zero. Hence,

$$\Lambda_{B_0 |\varphi\rangle} = \emptyset$$  \hspace{1cm} (19)$$

Also, by the same reasoning as before, we have that

$$\Lambda_{B_1 |\varphi\rangle} \cap \Lambda_{|\psi\rangle} = \emptyset \quad \Lambda_{B_2 |\varphi\rangle} \cap \Lambda_{|\psi\rangle} = \emptyset$$  \hspace{1cm} (20)$$

It follows from these equations that

$$\Lambda_{|\varphi\rangle} \cap \Lambda_{|\psi\rangle} = (\Lambda_{B_0 |\varphi\rangle} \cup \Lambda_{B_1 |\varphi\rangle} \cup \Lambda_{B_2 |\varphi\rangle} \cap \Lambda_{|\psi\rangle})$$

$$= (\Lambda_{B_0 |\varphi\rangle} \cap \Lambda_{|\psi\rangle}) \cup (\Lambda_{B_1 |\varphi\rangle} \cap \Lambda_{|\psi\rangle}) \cup (\Lambda_{B_2 |\varphi\rangle} \cap \Lambda_{|\psi\rangle}) = \emptyset$$  \hspace{1cm} (21)$$

Since we can choose transmissivity, $T$, for $BS3$ such that $\sqrt{T} \beta = \alpha$ whenever $\beta \geq \frac{1}{\sqrt{2}}$ it follows that we can argue that the ontic support of the states $|\varphi\rangle$ and $|\psi\rangle$ are non-overlapping whenever $|\langle \varphi |\psi\rangle|^2 \leq \frac{1}{2}$.

4 General proof

The above examples were illustrated with interferometers. Now, for the sake of generality, we will simply consider abstract Hilbert spaces without specifying a
particular physical instantiation so that the argument is fully general. Consider a \( N + 1 \) dimensional Hilbert space. We will need to allow \( N \) to be arbitrarily large to run the proof (we will discuss ways around this later). We wish to show that any pair of distinct pure states, \(|\varphi\rangle\) and \(|\psi\rangle\), have non-overlapping ontic support. We can write

\[
|\varphi\rangle = |a_0\rangle \quad |\psi\rangle = \alpha|a_0\rangle + \beta|a_1\rangle
\]  

(22)

for some orthonormal basis \(|a_n\rangle : n = 0 \text{ to } N\) (having \( \langle a_m|a_n\rangle = \delta_{mn} \)). We take \( \alpha \) and \( \beta \) to be real and positive (we can always do this since we can absorb any phase into the definition of \(|a_0\rangle\) and \(|a_1\rangle\)).

We consider unitary transformations which leave \(|\varphi\rangle\) unchanged (so that we can use the ontic indifference assumption). Such transformations take the form

\[
U = \begin{pmatrix}
1 & 0 & 0 & \cdots \\
0 & 0 & & \\
\vdots & & \ddots & \\
& & & V
\end{pmatrix}
\]  

(23)

in the \(|a_n\rangle\) basis where \( V \) is any unitary matrix acting in the subspace spanned by \(|a_1\rangle\) to \(|a_N\rangle\). In particular, we will consider the set of such transformations

\[
U[m] = \begin{pmatrix}
1 & 0 & 0 & \cdots \\
0 & 0 & & \\
\vdots & & \ddots & \\
& & & V[m]
\end{pmatrix}
\]  

(24)

for some choice of \( V[m] \). All of these leave the state \(|\varphi\rangle\) unchanged and hence can be implemented in such a way that they leave ontic states \( \lambda \in \Lambda|\varphi\rangle \) (in the ontic support of \(|\varphi\rangle\)) unchanged by the ontic indifference assumption (we will always assume that such transformations are implemented in such a way).

After the transformation \( U[m] \) we have

\[
|\psi\rangle = \alpha|a_0\rangle + \beta|a_1\rangle \longrightarrow \alpha|a_0\rangle + \beta|b[m]\rangle
\]  

(25)

where \(|b[m]\rangle = V[m]|a_1\rangle\) is normalized and orthogonal to \(|a_0\rangle\). By appropriate choice of \( V[m] \), the \(|b[m]\rangle\) can be equal to any normalized vectors in the subspace orthogonal to \(|a_0\rangle\). The \(|b[m]\rangle\) vectors do not have to be orthogonal to one another. We put

\[
|c[m]\rangle = \alpha|a_0\rangle + \beta|b[m]\rangle
\]  

(26)

The states \(|c[m]\rangle\) are normalized but not necessarily orthogonal to each other.

Following the transformation \( U[m] \) we subject the system to a maximal measurement (i.e., a measurement corresponding to a non-degenerate observable) onto a basis \(|d_n\rangle : n = 0 \text{ to } N\) (we denote by \( D_n \) the measurement outcome associated with the basis vector \(|d_n\rangle\)). We wish to choose this measurement and the \( V[n] \) such that:

For all \( n \) EITHER \( \langle d_n|a_0\rangle = 0 \) OR \( \langle d_n|c[n]\rangle = 0 \)  

(27)
(the OR is not exclusive - it is ok if both conditions are satisfied). First we will show that this condition can be satisfied for any situation where \( \beta > 0 \) (so \( |\varphi\rangle \) and \( |\psi\rangle \) are distinct) and then we will show that it implies that disjoint pure states have non-overlapping ontic support.

We will prove that we can actually satisfy the condition (27). We will work in the \( \{|d_n\rangle : n = 1 to N\} \) basis. In this basis we put

\[
|a_0\rangle = \frac{1}{\sqrt{M}} (0,1,1,\ldots,1,0,0,\ldots,0)
\]

where \( M \) is the smallest integer such that

\[
M \geq \frac{1}{\beta^2}
\]

(we require \( M \leq N \). Since \( \alpha^2 + \beta^2 = 1 \) this is equivalent to

\[
\frac{\alpha^2}{\beta^2} \leq M - 1
\]

We are free to choose the \( |b[m]\rangle \) states to be any normalized states orthogonal to \( |a_0\rangle \). We choose

\[
|b[m]\rangle = \frac{1}{\sqrt{M-1}} \alpha |b[m]\rangle + \delta |d_0\rangle
\]

where \( |b[m]\rangle \) is orthogonal to \( |d_0\rangle \). It follows from (28) that the first coefficient on the right hand side is less than one. We choose the coefficient \( \delta \) to guarantee the state is normalized. It follows that

\[
|c[m]\rangle = \alpha |a_0\rangle + \frac{1}{\sqrt{M-1}} |b[m]\rangle + \gamma |d_0\rangle
\]

where \( \gamma = \beta \delta \). We wish to choose these states so that (27) is satisfied. One way of doing this is if we can arrange things so that, in the \( \{|d_n\rangle : n = 1 to N\} \) basis,

\[
|c[m]\rangle = \frac{1}{\sqrt{M-1}} (0,1,1,\ldots,1,0,1,\ldots,1,0,\ldots,0)
\]

for \( m = 1 to M \). Then \( \langle d_n|c[m]\rangle = 0 \) for \( n = 1 to M \). For \( m = 0 \) and \( m = M + 1 \) to \( N \) we put \( U[m] = U[1] \). It follows that \( \langle d_n|a_0\rangle = 0 \) for \( n = 0 \) and \( n = M + 1 \) to \( N \). Hence, if we can choose \( |c[m]\rangle \) as in (33) then condition (27) is satisfied. To see that we can choose \( |c[m]\rangle \) in this way we note that it follows from (28), (32), and (33) after a little algebra that

\[
\langle a_0|b[m]\rangle = 0 \quad \text{for} \quad m = 0 to N
\]

(indeed, the choice in (31) was made to make this work). This is the only condition that has to be satisfied - it ensures that we can find \( U[m] \) of the form given in (24) such that \( |c[m]\rangle = U[m]|\psi\rangle \).
Now we will show that it follows from (27) that $|\varphi\rangle$ and $|\psi\rangle$ have non-overlapping ontic support (as long as they are distinct). Let $\Lambda_{|\varphi\rangle}^{D_n}[m]$ be the subset of the ontic support of $|\varphi\rangle$ whose members have a non-zero probability of giving rise to measurement outcome $D_n$ when we precede this measurement by the unitary $U[m]$. By ontic indifference,

$$\Lambda_{|\varphi\rangle}^{D_n}[m] = \Lambda_{|\varphi\rangle}^{D_n}[m'] = \Lambda_{|\varphi\rangle}^{D_n}$$  \hspace{1cm} (35)

since the state $|\varphi\rangle$ is unaffected by $U[m]$. We also have

$$\Lambda_{|\varphi\rangle} = \bigcup_{n=0}^{N} \Lambda_{|\varphi\rangle}^{D_n}$$  \hspace{1cm} (36)

since we must see some outcome for the measurement. Assume that we perform the unitary $U[n]$ and then measure in the $\{|d_n\rangle\}$ basis. Then

$$\Lambda_{|\varphi\rangle}^{D_n} \cup \Lambda_{|\psi\rangle} = \emptyset$$  \hspace{1cm} (37)

This is because (using (27)) EITHER $\langle d_n | \psi \rangle = \langle d_n | a_0 \rangle = 0$ in which we must have case $\Lambda_{|\varphi\rangle}^{D_n} = \emptyset$ from which (37) follows, OR $\langle d_n | U[n] | \psi \rangle = \langle d_n | c[n] \rangle = 0$ in which case (37) follows since there can be no ontic states which both have non-zero probability of giving rise to outcome $D_n$ (since they belong to $\Lambda_{|\varphi\rangle}^{D_n}$) and have zero probability of giving rise to this same outcome (as they belong to $\Lambda_{|\psi\rangle}$). It then follows that

$$\Lambda_{|\varphi\rangle} \cap \Lambda_{|\psi\rangle} = \left( \bigcup_{n=0}^{N} \Lambda_{|\varphi\rangle}^{D_n} \right) \cap \Lambda_{|\psi\rangle} = \bigcup_{n=0}^{N} \left( \Lambda_{|\varphi\rangle}^{D_n} \cap \Lambda_{|\psi\rangle} \right) = \emptyset$$  \hspace{1cm} (38)

Hence the ontic supports of distinct states are non-overlapping. As long as the Hilbert space dimension, $N + 1$, of the system in question is large enough then however small $\beta$ is, we can always find a $M \leq N$ such that the condition in (29) is satisfied. Hence, in the limit of $N \to \infty$, any pair of pure states must have non-overlapping ontic support and therefore the quantum state is a real thing. In the case that we are restricted to finite dimensional Hilbert spaces it follows from the present argument using (29) that any pair of states for which

$$\langle \varphi | \psi \rangle \geq \frac{N - 1}{N}$$  \hspace{1cm} (39)

have non-overlapping ontic support.

5 Proof with restricted ontic indifference

As before we take the distinct pure states $|\varphi\rangle$ and $|\psi\rangle$. Previously we subjected these states to a transformation $U[m]$ then performed a measurement in the $\{|d_n\rangle\}$ basis. Now we will complicate the experiment a little. First we subject
these states to a unitary transformation $W$ chosen such that $W|\varphi\rangle = |0\rangle$. Next, at this intermediate time, we perform the transformation $\tilde{U}[m] = WU[m]W^\dagger$. Then we perform the transformation $W^\dagger$. Finally we perform a measurement in the $\{|d_n\rangle\}$ basis. In this new configuration we can consider the set $\Lambda_{|\varphi\rangle}^{|0\rangle}[m] \subseteq \Lambda_{|\varphi\rangle}$ of ontic states in the support of $|\varphi\rangle$ at the initial time that have non-zero probability of giving rise to measurement output $D_n$ later given that we are to implement the transformation $\tilde{U}[m]$ at the intermediate time. We can also consider the set of ontic states, $\Lambda_{|\varphi\rangle}^{|0\rangle}[m] \subseteq \Lambda_{|0\rangle}$ at the intermediate time which are in the ontic support of $|0\rangle$ and have a non-zero probability of giving rise to measurement outcome $D_n$ given that we implement transformation $\tilde{U}[m]$ at the intermediate time. Now it must be the case that

$$\Lambda_{|\varphi\rangle} \rightarrow \Lambda_{|0\rangle}$$

under the evolution $W$. Hence, it must be true that

$$\Lambda_{|\varphi\rangle}^{|0\rangle}[m] \rightarrow \Lambda_{|0\rangle}^{|0\rangle}[m]$$

as the set on the right is just the time evolved version of the set on the left. By restricted ontic indifference we have

$$\Lambda_{|\varphi\rangle}^{|0\rangle}[m] = \Lambda_{|\varphi\rangle}^{|0\rangle}[m'] = \Lambda_{|0\rangle}^{|0\rangle}$$

Hence

$$\Lambda_{|\varphi\rangle}^{|0\rangle}[m] = \Lambda_{|\varphi\rangle}^{|0\rangle}[m'] = \Lambda_{|\varphi\rangle}^{|0\rangle}$$

Thus we have the necessary property to run the proof as in the previous section.

6 Systems with finite Hilbert space dimension

Imagine that the system, $A$, under consideration necessarily has a finite Hilbert space dimension, $L$. Can we still argue that its pure states are real? One way to do this is to allow the use of an ancillary system, $B$, having arbitrarily large $N$ prepared in some initial state $|1\rangle$. We can ask whether $|\varphi\rangle_A$ and $|\psi\rangle_A$ for system $A$ must have non-overlapping ontic support. Then the states of the joint system under consideration are $|\varphi\rangle_A|1\rangle_B$ and $|\psi\rangle_A|1\rangle_B$. We can now argue as before that these states must have non-overlapping ontic support. That is

$$\Lambda_{|\varphi\rangle_A|1\rangle_B} \cap \Lambda_{|\psi\rangle_A|1\rangle_B} = \emptyset$$

for any pair of states $|\varphi\rangle_A$ and $|\psi\rangle_A$. Assume that, in the case for independently prepared pure quantum states, we have

$$\Lambda_{|\psi\rangle_A|\varphi\rangle_B} = \Lambda_{|\psi\rangle_A} \times \Lambda_{|\varphi\rangle_B}$$

for any states $|\psi\rangle_A$ and $|\varphi\rangle_B$ for systems $A$ and $B$ respectively. Here the $\times$ represents the cartesian product of the sets. We will call this assumption ontic product separability. It follows from (44) and (45) that

$$\Lambda_{|\varphi\rangle_A} \cap \Lambda_{|\psi\rangle_A} = \emptyset$$
This implies that any pair of pure states for the finite dimensional Hilbert space must have non-overlapping ontic support. The ontic product separability assumption in (45) is very natural if we admit the possibility of considering subsystems of the universe (such as \(A\)) in isolation from the rest of the universe because then we would want to give the system its own ontic states. The assumption in (45) is part of the PBR separability assumption [30]. If we do not make the ontic product separability assumption then we can still make a strong assertion based on (44). Assuming that system \(B\) is simply the rest of the universe it follows from this equation that the universal ontic situation in which we prepare \(|\varphi\rangle_A\) for subsystem \(A\) is distinct from the universal ontic situation in which we prepare \(B\). In the case where we do not assume ontic product separability this is as strong a result as we can expect.

7 Violating ontic indifference

In Spekkens’s toy model a toy bit is modeled by a system having four underlying ontic states. States in the theory correspond to having equal probability of two of these four ontic states. We can imagine a state in the theory which is a equal distribution over the first two ontic states (i.e. with equal probabilities). This can be regarded as being analogous to the \(|0\rangle\) state in quantum theory. Another state would be an equal distribution over the second and third ontic state which would be analogous to a superposition \(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\). An equal distribution over the first and fourth ontic state would be analogous to \(\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\) in quantum theory. Measurements course grain over pairs of ontic states (and remix the underlying states so it is impossible to prepare a given ontic state with probability one). Transformations correspond to permutations of the underlying ontic states. Thus, we can perform a transformation that interchanges the first and second, and the third and fourth ontic states. This leaves the state analogous to \(|0\rangle\) unchanged but transforms the state analogous to \(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\) to the state analogous to \(\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\). It does so by interchanging the ontic states in the support of the state analogous to \(|0\rangle\). Thus, ontic indifference is violated in the Spekkens toy model.

An application of this model, due to Martin and Spekkens [34], pertains to interferometers of the sort illustrated in Fig. 1 and Fig. 2. In this case there are ontic variables associated with the occupation number of the path (take this to be 0 or 1) and a phase associated with the path (take this to be 0 or \(\pi\)). Even if the occupation number is 0 there is still the phase variable which will be affected by a phase shifter. Thus a path with no particle in it still has non-trivial degrees of freedom associated with it. This allows the model to violate ontic indifference in a local way.
8 Previous work and discussion

In this paper we have been interested in theories that attempt to provide a description of the underlying reality of the world. This is sometimes referred to as providing an ontology and sometimes referred to as providing a hidden variable interpretation. It is only in the context of such theories it is natural to ask whether wave functions are real things. If we deny that there is an underlying reality (assuming this is a consistent thing to do) then it does not make sense to ask whether the wave function is a real thing.

The pilot wave model of de Broglie (see [2]) and Bohm [7] is an interesting example of a theory that attempts to provide a description of the underlying reality. In the usual interpretation of this model the ontic state is normally taken to be given by $\lambda = (\psi, \mathbf{X})$ where $\psi$ is the wavefunction and $\mathbf{X}$ is the position, in configuration space, of the particles (the particles are taken to actually exist). Thus, distinct pure states correspond to distinct ontic states and so the pilot wave model is $\psi$-ontic. The influence of the pilot wave model of de Broglie and Bohm on the study of quantum foundations has been significant. Bell was motivated by the pilot wave model to ask whether any hidden variable model must be nonlocal. Valentini was motivated by the pilot wave model to ask whether any deterministic hidden variable model leads to signaling for non-equilibrium distributions of the hidden variables [37]. Likewise, the question of whether any hidden variable model must be $\psi$-ontic is also motivated by the pilot wave model. The pilot wave model satisfies the restricted ontic indifference assumption (it is a matter of calculation as to whether it satisfies ontic indifference).

It is worth mentioning, as an aside, that the pilot wave model acts as a counterexample to the often encountered claim that quantum theory does not admit a hidden variable (or ontic) model. It does. The pilot wave model is one example. Further, there exist versions of the pilot wave model capable of reproducing enough quantum (field) theory to account for the standard model (see, for example, [36]) so such models can account for all of present day quantum phenomena.

Other examples of $\psi$-ontic models are the many worlds interpretation [14], collapse models in the standard ontology where the wavefunction is taken to be real [29, 17] (though the status of the flash ontology of Bell [5] is not so clear), and the Beltrametti Bugajski model [6].

There have been other takes on the question of whether the wavefunction is real. Aharonov and Vaidman [1] showed that it is possible to directly measure the wave function with weak measurements using a protective measurement scheme given only a single copy of the system. For this to work, however, it is necessary that the wavefunction is an eigenstate of the acting Hamiltonian and so a measurement onto the energy basis would enable us to deduce the wavefunction. Nevertheless, the fact that the shape of the wavefunction can be directly measured is quite remarkable. Fuchs regards the quantum state as a degree of belief by taking a Bayesian interpretation of probabilities [15]. In particular, he and collaborators have shown that there can be situations in...
which it is consistent for different agents to have different beliefs as to what
the state of a given system is and yet for them all to believe the system is in a
pure state \[8\]. There is clearly a tension between that result and any proof that
the wavefunction is a real thing which would be interesting to explore further.
Indeed, one can think of a quantum state as being represented by a list of
probabilities (corresponding to a complete set of fiducial measurements). Then
the only difference between two different pure quantum states is that they have
different probabilities for these fiducial measurements. If we adopt the point of
view that probabilities are nothing more than degrees of belief then it is clearly
possible for two agents to ascribe different pure states to the same situation. It
follows that there is a conflict between this view of probabilities and the three
assumptions used in this paper. Indeed, in the point of view developed by Fuchs
\[15\], the first assumption (that there is a hidden variable \(\lambda\) associated with a
system) is not true. Although this assumption was dubbed “realism” we should
admit that it is particular way of implementing the notion of realism. We may
have models do not deny realism in the broader sense of the word but which are
inconsistent with this assumption.

If we restrict ourselves to the case of a spin half particle then there are two
interesting models in the literature. Bell \[4\] constructed a simple hidden variable
model in which distinct pure spin half states correspond to hidden variables
represented by distinct unit vectors and the probability of seeing certain outcome
is given by the overlap between this vector and the vector representing the
measurement direction. This is a \(\psi\)-ontic model. On the other hand, Kochen
and Specker \[22\] gave an example in which a pure spin state corresponds to a
distribution of unit vector hidden variables. This is a \(\psi\)-epistemic model. Both
these models are restricted to standard spin measurements on spin half particles
and do not work for Hilbert space dimension greater than 2.

An intermediate question we can ask is what the size of the ontic space must
be to reproduce quantum theory. The toy bit in the Spekkens toy theory has
only four underlying ontic states. However, although it behaves in a similar
way to a qubit, it does not reproduce all predictions of quantum theory. A
preliminary result in this direction was provided by Galvão and the present
author \[16\]. It was shown that a single qubit can substitute for an infinite
number of classical bits in a certain information processing task suggesting that
we need an infinite number of ontic states to model a single qubit. Then it was
proven by the present author \[20\] that the number of ontic states required to
reproduce quantum predictions for even a single qubit must, indeed, be infinite.
This proof depends critically on the fact that there are an infinite number of
distinct pure quantum states for a qubit (there exist an infinite number of states
\(|\psi\rangle = \alpha|0\rangle + \beta|1\rangle\) since \(\alpha\) and \(\beta\) vary continuously).

The proof in \[20\] only showed that there must be an infinite number of ontic
states in models reproducing quantum theory. As discussed in the introduction,
a big step forward was made by Montina \[26\] when he showed, by making some
very reasonable assumptions (most importantly he assumed that the model was
Markovian), that the hidden variables for a system with Hilbert space dimension
\(N\) must contain at least \(2N - 2\) real parameters - this being the same as the
number of real parameters required to describe a state $|\psi\rangle = \sum c_n |n\rangle$ after normalization and an overall phase has been set. Further in [27], Montina showed that in models for which the ontic space is specified by the minimum number of real parameters, the ontic state is isomorphic to a vector in an $N$ dimensional Hilbert space and, further, this vector evolves according to the Schroedinger equation. These results of Montina provide very strong evidence under rather weak assumptions that the quantum state is real. Even if the quantum state is not real, Montina’s results show that, in Markovian models, there is something at least as complicated that is real.

Montina’s work falls just short of actually proving that quantum theory is $\psi$-ontic. This was first proven by Pusey, Barrett, and Rudolph. The result of PBR represents a significant breakthrough in quantum foundations and has already been the subject of much study [18, 11, 25, 13, 32] (see in particular Leifers illuminating discussion [23]). The key assumption that PBR make to get their result is that the distribution over ontic states for independently prepared pure quantum states should factorize.

It is worth remarking, in view of the general acclaim for the extraordinary PBR result that, in some respects, the results of Montina are stronger than those of PBR and, indeed, the present paper. By assuming less, Montina proves less. However, his result is a bigger obstacle to finding worthwhile $\psi$-epistemic models - namely those in which the ontic space is simpler than the pure quantum state space. One of the biggest motivations to construct a $\psi$-epistemic model must be to achieve a simpler description of reality than is afforded by the quantum state. It is possible that, by violating the assumptions PBR and the present paper, we can build $\psi$-epistemic models. Further, by violating Montina’s Markovian assumption, we may even be able to build $\psi$-epistemic models that have fewer real variables than required to describe the quantum state. Montina has shown that, for Hilbert space dimension 2, it is possible to build a non-Markovian model having only one (rather than two) real parameters in the ontic state. This theme has been continued by Montina in [28]. Lewis, Jennings, Barrett, and Rudolph [24], by violating the separability assumption of PBR, have shown how to build a $\psi$-epistemic model reproducing quantum theory for any dimension of the Hilbert space (this model, however, requires as many real parameters as there are independent real parameters in the quantum state and it also violates the ontic indifference assumption of the present paper).

The restricted ontic indifference assumption is a much stronger assumption than either Montina’s Markovian assumption or PBR’s separability assumption. The real result of this paper is, then, that any $\psi$-epistemic model must violate restricted ontic indifference (and therefore also violate ontic indifference). The fact that the Spekkens’s toy model violates ontic indifference suggests that it is a step in the right direction. As demonstrated by the pilot wave model for non-relativistic particles, $\psi$-ontic models need not violate restricted ontic indifference. This property seems, then, to be a key difference between the two approaches (though, of course, $\psi$-ontic approaches do not have to satisfy ontic indifference).

One attitude one could take to the results of this paper and the work of
Montina and PBR is that we should give up on trying to find interpretations of quantum theory in which the wave function is not a real thing. However, the benefits of finding an interpretation in which the wave function is not a real thing may far outweigh the problems caused by having to violate the various assumptions used in the afore mentioned work. In fact it is quite easy to violate the assumption of ontic indifference in hidden variable models. Further, one can question how well motivated the PBR separability assumption (for independently prepared systems) is in view of the fact that, generically, we do not expect hidden variable interpretations of quantum theory to be separable when the systems are not independently prepared (see Spekkens discussion of the PBR result in [34]). Further, as just mentioned, Montina has already embarked on a project looking for models in which the number of real parameters required for the ontic state is fewer than required to specify the quantum state (such models are necessarily \( \psi \)-epistemic). Further, the work of Spekkens and collaborators [35, 34] points to the advantages of a \( \psi \)-epistemic approach. An alternative, and even more radical, approach is that of Fuchs [15] and collaborators. In this approach the quantum state is not a real thing since different agents can attribute different states for the same physical situation.

The area of realistic interpretations of quantum theory in which the quantum state is not a real thing remains relatively unexplored. It represents a very different take on the problem of interpretation to the old school approaches of de Broglie and Bohm (the pilot wave model), of Everett (the many worlds interpretation), and of Pearl, Ghirardi, Rimini, and Weber (the collapse models) from the last century. The search for reasonable \( \psi \)-epistemic models (violating the assumptions implicit in the various no-go theorems) is likely to be come an increasingly exciting area of research in quantum foundations.

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assumption is possible in a field ontology.

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