Hard rods: statistics of parking configurations

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Abstract. We compute the correlation function in the equilibrium version of Rényi’s parking problem. The correlation length is found to diverge as $2^{-1} \pi^{-2} (1 - \rho)^{-2}$ when $\rho \nearrow 1$ (maximum density) and as $\pi^{-2} (2\rho - 1)^{-2}$ when $\rho \searrow 1/2$ (minimum density).

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1. Introduction

A parking configuration of hard rods (or cars) on a line or a circle is a configuration where the largest gap between successive rods is less than the length of a rod: the parking is full. The statistical properties of parking configurations obtained by random sequential parking (or adsorption) were first studied by Rényi [1]. The present paper deals with the corresponding equilibrium model.

The correlation function of a gas of hard rods in one dimension was computed by Frenkel [2]. A general one-dimensional fluid with a nearest neighbor interaction, strongly repulsive at short distance and decaying rapidly at large distance, was then solved by Gürsey [3] in a grand canonical ensemble. Salsburg, Zwanzig and Kirkwood [4] derived a similar solution in the canonical ensemble, suitable for testing the then newly invented Kirkwood-Salsburg equations.

We give a solution which is a little simpler, in a canonical isobaric ensemble, and give precise asymptotic forms for the diverging correlation length near maximum density (pressure going to $+\infty$) and near minimum density (pressure going to $-\infty$).

The intervals between hard rods can be mapped onto the gradient variables in a one-dimensional interface model. One could prove mathematically the equivalence of ensembles for the parking model, including uniqueness of correlation functions, using the local central limit theorem like in the proof of the Wulff shape for one-dimensional interfaces [5].

As a by-product, we give the probability for $N$ points distributed uniformly and independently in $(0, L)$ to be in a parking configuration, as a function of $\rho = N/L$, in the thermodynamic limit.
2. The model

A parking configuration of $N$ hard rods (or arcs) of length one on a circle of length $L$ is specified by $N$ positions $X_1, \ldots, X_N \in [0, L) = \mathbb{R}/L\mathbb{Z}$ with

\[
\begin{align*}
|X_i - X_j| & \geq 1 \quad \forall i \neq j \\
\min_j (X_i - X_j)_+ & \leq 2 \quad \forall i \\
\min_j (X_j - X_i)_+ & \leq 2 \quad \forall i
\end{align*}
\] (2.1)

The Lebesgue measure on the set of parking configurations, a Borel subset of $\mathbb{R}^N$, normalised by the total measure of this set, defines a probability measure $P_{N,L}(\cdot)$. This probability measure inherits the rotation invariance of the Haar measure on the circle of length $L$. The canonical partition function $Z_{N,L}$ is defined as

\[
Z_{N,L} = \frac{1}{N!} \int_{\text{Parking}} dx_1 \ldots dx_N
\] (2.2)

where the range of integration “Parking” is defined by (2.1). The free energy is $F(N, L) = -\ln Z_{N,L}$. Temperature plays no role and is omitted.

For $x, y \in [0, L)$ we define

\[
\begin{align*}
\rho_{N,L}^{(1)}(x) &= \sum_{i=1}^{N} \frac{\mathbb{P}_{N,L}(X_i \in (x, x + dx))}{dx} \\
\rho_{N,L}^{(2)}(x, y) &= \sum_{i \neq j=1}^{N} \frac{\mathbb{P}_{N,L}(X_i \in (x, x + dx), X_j \in (y, y + dy))}{dx \, dy}
\end{align*}
\] (2.3)

By rotation invariance $\rho_{N,L}^{(1)}(x)$ is independent of $x$. Since

\[
\int_{0}^{L} dx \, \rho_{N,L}^{(1)}(x) = N,
\]

denoting $N/L = \rho$, we get $\rho_{N,L}^{(1)}(x) = \rho$. Conditions (2.1) imply $1/2 \leq \rho \leq 1$. Similarly $\rho_{N,L}^{(2)}(x, y)$ depends only upon $|x - y|$, and

\[
\int_{0}^{L} dx \int_{0}^{L} dy \, \rho_{N,L}^{(2)}(x, y) = N(N - 1).
\]

The pair distribution function is defined as

\[
g_{N,L}(x) = \frac{\rho_{N,L}^{(2)}(0, x)}{\rho^2}
\] (2.4)
and satisfies
\[ \int_0^L dx \, g_{N,L}(x) = \left(1 - \frac{1}{N}\right) L \]

The degree of correlation or independence between the configuration around \( x \) and the configuration around \( y \) may be estimated by looking at the pair correlation function
\[
\frac{\mathbb{P}_{N,L}( \exists i: X_i \in (x, x + dx), \exists j: X_j \in (y, y + dy) )}{\mathbb{P}_{N,L}( \exists i: X_i \in (x, x + dx) \mathbb{P}_{N,L}( \exists j: X_j \in (y, x + dy) )} - 1 = g_{N,L}(x - y) - 1
\]

3. Spacings

Let \((X_1, X'_2, \ldots, X'_N)\) be obtained by permutation of \((X_1, \ldots, X_N)\) so that
\[ X_1 \leq X'_2 \leq \ldots \leq X'_N \leq X_1 + L \]

and let
\[
S_1 = X'_2 - X_1 \\
S_i = X'_{i+1} - X'_i, \quad i = 2, \ldots, N - 1 \\
S_N = X_1 + L - X'_N
\]

Then \((X_1, S_1, \ldots, S_{N-1})\) is distributed according to the Lebesgue measure on the subset of \(\mathbb{R}^N\) defined by
\[
0 \leq X_1 < L \\
1 \leq S_i \leq 2, \quad i = 1, \ldots, N - 1 \\
1 \leq L - \sum_{i=1}^{N-1} S_i \leq 2
\]

and we may write (2.2) as
\[
Z_{N,L} = \int_0^L dx_1 \int_1^2 ds_1 \ldots \int_1^2 ds_{N-1} 1_{1 \leq L - \sum_{i=1}^{N-1} s_i \leq 2}
= L \int_1^2 ds_1 \ldots \int_1^2 ds_N \delta \left( \sum_{i=1}^N s_i - L \right)
\]

It follows that \((S_1, \ldots, S_N)\) is distributed according to
\[
\mathbb{P}_{N,L}( S_1 \in (s_1, s_1 + ds_1), \ldots, S_N \in (s_N, s_N + ds_N) ) =
= \left( \prod_{i=1}^N 1_{1 \leq s_i \leq 2} ds_i \right) \delta \left( \sum_{i=1}^N s_i - L \right) / \text{normalisation}
\]

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Then, for $x > 0$,

\[
P_{N,L}(\exists i, j : X_i \in (0, dy), X_j \in (x, x + dx)) = N\mathbb{P}_{N,L}(X_1 \in (0, dy), \exists j : X_j \in (x, x + dx))
\]

\[
= N \sum_{m=1}^{N-1} \mathbb{P}_{N,L}(X_1 \in (0, dy), X_1 + S_1 + \ldots + S_m \in (x, x + dx))
\]

\[
= N \frac{dy}{L} \sum_{m=1}^{N-1} \mathbb{P}_{N,L}(S_1 + \ldots + S_m \in (x, x + dx))
\]

so that

\[
g_{N,L}(x) = \sum_{m=1}^{N-1} \frac{\mathbb{P}_{N,L}(S_1 + \ldots + S_m \in (x, x + dx))}{\rho \, dx}, \quad x > 0,
\]

(3.4)

and $g_{N,L}(-x) = g_{N,L}(x)$.

4. Isobaric ensemble

In the thermodynamic limit, $L \to \infty$, $N \to \infty$ with $\rho = N/L$ fixed, asymptotic statistical properties are easier to compute in an ensemble where the pressure $p$ is fixed instead of the system size $L$. This is defined as follows: for $p \in \mathbb{R}$, let $\tilde{S}_1, \ldots, \tilde{S}_N$ be distributed according to

\[
\mathbb{P}_{N,p}(\tilde{S}_1 \in (s_1, s_1 + ds_1), \ldots, \tilde{S}_N \in (s_N, s_N + ds_N)) = \left( \prod_{i=1}^{N} 1_{1 \leq s_i \leq 2} e^{-p s_i} ds_i \right) / \text{norm}.
\]

(4.1)

The normalisation is

\[
Z_{N,p} = \left( \int_{1}^{2} ds e^{-ps} \right)^N = \left( \frac{e^{-p} - e^{-2p}}{p} \right)^N
\]

(4.2)

and the associated potential is the Gibbs potential $G(N, p) = -\ln Z_{N,p}$. The thermodynamic relation

\[
\frac{\partial G}{\partial p} = \mathbb{E}_{N,p} \sum \tilde{S}_i
\]

is the usual one, with $\mathbb{E}_{N,p} \sum \tilde{S}_i = L$, the mean system size. We thus get

\[
L = \frac{\partial G}{\partial p} = \frac{N}{p} + N \frac{e^{-p} - 2e^{-2p}}{e^{-p} - e^{-2p}}
\]

or

\[
\frac{1}{\rho} = \frac{1}{p} + \frac{e^{-p} - 2e^{-2p}}{e^{-p} - e^{-2p}}
\]

(4.3)
Although temperature plays only a dummy role, one may wish to have it in, and also to replace the length one of rods and maximum allowed gap between neighboring rods by a length \( \ell \). Then one gets
\[
\frac{1}{\rho} = \frac{kT}{p} + \frac{e^{-p\ell/kT}}{e^{-p\ell/kT} - e^{-2p\ell/kT}},
\]
which looks more like an equation of state.

Going back to (4.3), for \( p = \pm \infty, \ln 2, 0, -\infty \), we have respectively \( \rho = 1, \ln 2, 2/3, 1/2 \).

In this ensemble we can compute, for \( x > 0 \),
\[
g_{N,p}(x) = \sum_{m=1}^{N-1} \frac{\mathbb{P}_{N,p}(\tilde{S}_1 + \ldots + \tilde{S}_m \in (x, x + dx))}{\rho \, dx}
= \sum_{m=1}^{N-1} \left( \frac{p}{e^p - e^{-2p}} \right)^m \frac{1}{\rho} \int_1^2 e^{-ps_1} \, ds_1 \ldots \int_1^2 e^{-ps_m} \, ds_m \, \delta\left( \sum s_i - x \right)
= \sum_{m=1}^{N-1} \left( \frac{p}{e^p - e^{-2p}} \right)^m \frac{e^{-px}}{\rho} \int_0^1 dt_1 \ldots \int_0^1 dt_m \, \delta\left( \sum t_i - x + m \right)
= \sum_{m=1}^{N-1} \left( \frac{p}{e^p - e^{-2p}} \right)^m \frac{e^{-px}}{\rho} \, u_m(x - m)
\]
where the densities \( u_n(x) \) satisfy the recursion relation
\[
u_{n+1}(x) = \int_0^1 u_n(x - y) \, dy,
\]
which gives [6, p. 27]
\[
u_1(x - 1) = 1_{x \in (1,2)}
\]
\[
u_m(x - m) = \frac{1}{(m-1)!} \sum_{\ell=0}^{m} (-)^\ell \binom{m}{\ell} (x - m - \ell)^{m-1}_+, \quad m \geq 2
\]
The thermodynamic limit \( g(x) = \lim_{N \to \infty} g_{N,p}(x) \) is given by (4.5)(4.6) with the range of \( m \) extended to infinity. Note that for each \( x \) only \( \lfloor x/2 \rfloor \) terms contribute, because \( u_m(x - m) \) vanishes for \( x > 2m \). Plots of \( g(x) \) are given in Figure 1.

\( \mathbb{P}_{N,p}(\cdot) \) also induces a distribution for parking configurations of \( N \) rods on \([0, \infty)\), so that \( g_{N,p}(x) \) may be considered to be defined by (2.3)(2.4) with \( L \) replaced by \( p \) and \( 1/\rho = \mathbb{E}_{N,p}L/N \). Then \( g_{N,p}(-x) = g_{N,p}(x) \) and \( g(-x) = g(x) \).
5. Correlation length

Let us compute for $\lambda > 0$ the Laplace transform,

$$\hat{g}^{\text{Laplace}}(\lambda) = \int_0^{\infty} e^{-\lambda x} g(x) \, dx$$

$$= -\frac{p}{\rho} \frac{e^{-(\lambda+p)} - e^{-2(\lambda+p)}}{-(\lambda + p)(e^{-\lambda} - e^{-2\lambda}) + p(e^{-(\lambda+p)} - e^{-2(\lambda+p)})}$$

Using (4.3), we find $\lambda \hat{g}^{\text{Laplace}}(\lambda) \to 1$ as $\lambda \to 0$, in agreement with $g(x) \to 1$ as $x \to \infty$.

We then consider the Laplace transform of the pair correlation function $h(x) = g(x) - 1$,

$$\hat{h}^{\text{Laplace}}(\lambda) = \hat{g}^{\text{Laplace}}(\lambda) - \frac{1}{\lambda}$$

and look for its poles, which can only be in the complex half-plane $Re(\lambda) \leq 0$. We must solve

$$\frac{e^{-(\lambda+p)} - e^{-2(\lambda+p)}}{\lambda + p} = \frac{e^{-p} - e^{-2p}}{p}$$

(5.2)
Setting $\lambda = a + ib$ and $\tilde{p} = p + a$, the modulus and the phase in (5.2) give

$$
\frac{(e^{-\tilde{p}} - e^{-2\tilde{p}})^2 + 2e^{-3\tilde{p}}(1 - \cos b)}{\tilde{p}^2 + b^2} = \frac{(e^{-p} - e^{-2p})^2}{p^2}
$$

(5.3)

$$
\frac{\sin b - e^{-\tilde{p}}\sin 2b}{\cos b - e^{-\tilde{p}}\cos 2b} = -\frac{b}{\tilde{p}}
$$

When $p \to +\infty$, looking only in the strip $-1 \leq a \leq 0$, we get $b$ from the second equation and then $a$ from the first:

$$
b = 2n\pi \left(1 - \frac{1}{p} + \mathcal{O}\left(\frac{1}{p^2}\right)\right)
$$

(5.4)

$$
a = -\frac{2n^2\pi^2}{p^2} \left(1 + \mathcal{O}\left(\frac{1}{p}\right)\right)
$$

(5.5)

where $n \in \mathbb{Z} \setminus \{0\}$. When $p \to -\infty$, we get similarly

$$
b = n\pi \left(1 - \frac{1}{2p} + \mathcal{O}\left(\frac{1}{p^2}\right)\right)
$$

$$
a = -\frac{n^2\pi^2}{4p^2} \left(1 + \mathcal{O}\left(\frac{1}{p}\right)\right)
$$

(5.6)

The pair correlation function $h(x)$ is conveniently recovered by Fourier transform:

$$
\hat{h}(k) = \int_{-\infty}^{\infty} e^{ikx} g(x) dx = \hat{h}^{\text{Laplace}}(ik) + \hat{h}^{\text{Laplace}}(-ik)
$$

Using (5.1)(5.4)(5.5), the contour of integration can be shifted past the nearest poles, whose residues are the dominant part of $h(x)$ when $x \to \infty$, giving the inverse correlation length

$$
\frac{1}{\xi} = \lim_{x \to +\infty} \frac{1}{x} \ln |h(x)| = \begin{cases} 
\frac{2\pi^2}{p^2} \left(1 + \mathcal{O}\left(\frac{1}{p}\right)\right) & \text{as } p \to +\infty \\
\frac{\pi^2}{4p^2} \left(1 + \mathcal{O}\left(\frac{1}{p}\right)\right) & \text{as } p \to -\infty
\end{cases}
$$

(5.7)

or, using (4.3),

$$
\xi \sim \begin{cases} 
\frac{1}{2\pi^2} (1 - \rho)^{-2} & \text{as } \rho \nearrow 1 \\
\frac{1}{\pi^2} (2\rho - 1)^{-2} & \text{as } \rho \searrow 1/2
\end{cases}
$$

(5.8)

The imaginary part of the poles gives the pseudo-period of oscillation, $\Delta x = 1$ when $\rho \nearrow 1$ and $\Delta x = 2$ when $\rho \searrow 1/2$, in accordance with Figure 1 and with the Dirac train limits

$$
\lim_{p \to \pm\infty} \hat{g}^{\text{Laplace}}(\lambda) = \begin{cases} 
\frac{e^{-\lambda}}{1 - e^{-\lambda}} & \text{as } p \to +\infty \\
\frac{2e^{-2\lambda}}{1 - e^{-2\lambda}} & \text{as } p \to -\infty
\end{cases}
$$

(5.9)
\[ \lim_{p \to \pm \infty} g(x) = \begin{cases} 
\sum_{n \in \mathbb{Z}} \delta(x - n) & \text{as } p \to +\infty \\
2 \sum_{n \in \mathbb{Z}} \delta(x - 2n) & \text{as } p \to -\infty \end{cases} \tag{5.10} \]

6. Equivalence of ensembles

For any \( p \in \mathbb{R} \) we may write (3.2) as

\[
\frac{Z_{N,L}}{L} = \int_1^2 ds_1 \ldots \int_1^2 ds_N \delta\left(\sum s_i - L\right) \\
= \int_1^2 ds_1 \ldots \int_1^2 ds_N e^{-p\left(\sum s_i - L\right)} \delta\left(\sum s_i - L\right) \\
= e^{pL} Z_{N,p} \mathbb{E}_{N,p} \delta\left(\sum \tilde{S}_i - L\right) \tag{6.1} \]

where \( \mathbb{E}_{N,p}(\cdot) \) is defined by (4.1)(4.2). Let us choose \( p \) so that

\[ \mathbb{E}_{N,p} \sum \tilde{S}_i = L \]

where \( L \) is the fixed value in the fixed \( L \) ensemble. Then the Central Limit Theorem implies

\[ \mathbb{E}_{N,p} \delta\left(\sum \tilde{S}_i - L\right) = \mathcal{O}(L^{-1/2}) \]

and we get

\[ \frac{Z_{N,L}}{L} = e^{pL} Z_{N,p} \mathcal{O}(L^{-1/2}) \]

The free energy per particle at density \( \rho \),

\[ f(\rho) = \lim_{N \to \infty} \frac{F(N, N/\rho)}{N} \]

and the Gibbs potential per particle at the corresponding pressure \( p \),

\[ g(p) = \lim_{N \to \infty} \frac{G(N, p)}{N} = -\ln \frac{e^{-p} - e^{-2p}}{p} \tag{6.2} \]

are therefore related by

\[ f(\rho) = -\frac{p}{\rho} + g(p) \tag{6.3} \]

The local version of the central limit theorem can then be used to prove

\[ \lim_{N \to \infty} g_{N,N/\rho}(x) = \lim_{N \to \infty} g_{N,p}(x) = g(x) \]
along the lines of the proof of the Wulff shape for one-dimensional interfaces in [5]. Equivalence of ensembles is of course very standard, but its justification or derivation is generally more involved than for the present model as outlined above.

The explicit form of the free energy $f(\rho)$ allows to answer an old question in statistics [7,8]: let $X_1, \ldots, X_N$ be $N$ independent random variables each distributed uniformly over the interval $(0, L)$. Denote $\mathbb{P}^\text{free}_{N,L}$ the corresponding probability distribution (the ideal gas), and call Parking the event that the smallest and largest spacings are respectively larger than one and smaller than two. We have

$$
\mathbb{P}^\text{free}_{N,L}(\text{Parking}) = \frac{\int_1^2 ds_1 \cdots \int_1^2 ds_N \delta\left(\sum s_i - L\right)}{\int_0^L ds_1 \cdots \int_0^L ds_N \delta\left(\sum s_i - L\right)} = \frac{Z_{N,L}/L}{L^{N-1}/(N-1)!}
$$

Let $1/2 < \rho < 1$. Then (6.1)(6.2)(6.3)(6.4) and Stirling’s formula give

$$
\lim_{N \to \infty} -\frac{1}{N} \ln \mathbb{P}^\text{free}_{N,N/\rho}(\text{Parking}) = 1 - \frac{p}{\rho} - \ln \rho - \ln \frac{e^{-p} - e^{-2p}}{p}
$$

where $p$ is related to $\rho$ by (4.3).

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