THE RELATION BETWEEN A GENERALIZED FIBONACCI SEQUENCE AND THE LENGTH OF CUNNINGHAM CHAINS

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ABSTRACT. Let \( p \) be a prime number. A chain \( \{p, 2p + 1, 4p + 3, \cdots, (p + 1)2^{j(p)-1} - 1\} \) is called the Cunningham chain generated by \( p \) if all elements are prime numbers and \( (p + 1)2^{j(p)-1} - 1 \) is composite. Then \( l(p) \) is called the length of the Cunningham chain. It is conjectured by Bateman and Horn in 1962 that the number of prime numbers \( p \leq N \) such that \( l(p) \geq k \) is asymptotically equal to \( B_k N/(\log N)^k \) with a real \( B_k > 0 \) for all natural numbers \( k \). This suggests that \( l(p) = \Omega(\log p / \log \log p) \). However, so far no good estimation is known. It has not even been proven whether \( \limsup_{p \to \infty} l(p) \) is infinite or not. All we know is that \( l(p) = 5 \) if \( p = 2 \) and \( l(p) < p \) for odd \( p \) by Fermat’s little theorem. Let \( \alpha \geq 3 \) be an integer. In this article, a generalized Fibonacci sequence \( F_\alpha = \{F_n\}_{n=0}^\infty \) is defined as \( F_0 = 0, F_1 = 1, F_{n+2} = \alpha F_{n+1} + F_n \) \( n \geq 0 \), and \( F_{\alpha,\sigma}(n) = \sum_{d|n,0 < d \leq F_{\alpha}} d \) is called the divisor function on \( F_\alpha \). Then we obtain an interesting relation between the iteration of \( F_{\alpha,\sigma} \) and the length of Cunningham chains. For two prime numbers \( p \) and \( q \), the fact \( p = 2q + 1 \) or \( 2q - 1 \) is equivalent to \( F_{\alpha,\sigma}(F_p) = F_{\alpha,\sigma}(F_q) \) for some \( \alpha \). By this relation, we get \( l(p) \ll \log p \) under a certain condition. It seems that this sufficient condition is plausible by numerical test. Furthermore, the condition, written in terms of prime numbers, can be replaced by the condition written in terms of natural numbers. This implies that the problem of upper estimation of \( l(p) \) is reduced to that on natural numbers.

1. INTRODUCTION

Notation 1.1. Let \( \mathbb{Z} \) and \( \mathbb{P} \) be the set of integers and prime numbers, respectively.
- Let \( \mathbb{N} \) be \( \{1, 2, 3, \cdots\} \), and its elements are called natural numbers.
- For real \( x \), \( \lfloor x \rfloor \) means to the unique integer such that \( \lfloor x \rfloor \leq x < \lfloor x \rfloor + 1 \); called the integer part of \( x \).
- Unless stated otherwise, the greatest common divisor of two integers \( m \) and \( n \) is denoted by \( (m,n) \).

It is well known that arithmetical progressions are related to prime numbers. For example, the Dirichlet prime number theorem in 1837 [8] and the Green-Tao theorem in 2004 [13]. However, the following conjecture of Dickson has not been well studied.

Conjecture 1.2 (Dickson [6]). Let \( k \) be a natural number. Suppose that \( a_i \) and \( b_i \) are integers with \( b_i \geq 1 \) and \( (a_i,b_i) = 1 \) for all \( i = 1, \cdots, k \). Then, there are infinitely many integers \( n \) such that \( a_1 + b_1 n, \cdots, a_k + b_k n \) are simultaneously primes.

If \( k = 1 \), then this is nothing but the Dirichlet prime number theorem. However, all the other cases are unsolved. In 1962, Bateman and Horn generalized this conjecture to irreducible polynomials.

Conjecture 1.3 (Bateman-Horn [4]). Suppose that \( f_i(x) \) is an irreducible polynomial whose leading coefficient is a positive integer. Let \( N \) be a natural number, and let \( d_i \) be the degree of \( f_i(x) \). Then, the number of \( 0 < n \leq N \) such that \( f_1(n), \cdots, f_k(n) \) are simultaneously prime by given by

\[
\int_2^N \prod_{i=1}^k \frac{1}{\log f_i(x)} dx \cdot \prod_{p \in \mathbb{P}} \frac{1 - w(p)/p}{(1 - p^{-1})^k} \sim \frac{N}{d_1 \cdots d_k \log k} \prod_{p \in \mathbb{P}} \frac{1 - w(p)/p}{(1 - p^{-1})^k}
\]

where \( w(p) \) is the number of \( x \in \{0, 1, \cdots, p-1\} \) which is a solution for \( f_1(x) \times \cdots \times f_k(x) \equiv 0 \) (mod \( p \)).

Caldwell applied Conjecture 1.3 to various progressions in [5]. One of them is the Cunningham chain.

Definition 1.4. Let \( p \) be a prime number. A sequence

\[
\left\{ p, 2p + 1, \cdots, (p + 1)2^{j(p)-1} - 1 \right\} \text{ (resp. } \left\{ p, 2p - 1, \cdots, (p - 1)2^{j(p)-1} + 1 \right\})
\]

are simultaneously primes.
is called the first (resp. second) Cunningham chain generated by \( p \) if all elements are prime numbers and \((p + 1)2^{k_1(p) - 1} - 1\) (resp. \((p - 1)2^{k_2(p) - 1} + 1\)) is composite. As we will see later, \( l_1(p) \) and \( l_2(p) \) are finite. Also when we discuss the first case and the second case simultaneously, we write \( l_1(p) \) and \( l_2(p) \) as \( l(p) \).

If \( p = 2 \), then
\[
2 \times 2 + 1 = 5 \in \mathbb{P}, \quad 5 \times 2 + 1 = 11 \in \mathbb{P}, \quad 11 \times 2 + 1 = 23 \in \mathbb{P}, \quad 23 \times 2 + 1 = 47 \in \mathbb{P}, \quad 47 \times 2 + 1 = 95 \notin \mathbb{P},
\]
\[
2 \times 2 - 1 = 3 \in \mathbb{P}, \quad 3 \times 2 - 1 = 5 \in \mathbb{P}, \quad 5 \times 2 - 1 = 9 \notin \mathbb{P}.
\]
Thus \( l_1(2) = 5, l_2(2) = 3 \). If \( p \neq 2 \), we have \( l(p) \leq \text{ord}(p; 2) \) by Fermat’s little theorem. Here, \( \text{ord}(p; 2) \) is the order of 2 modulo \( p \). In fact, in the case of the first Cunningham chain, we have
\[
(p + 1)2^{\text{ord}(p; 2)} - 1 \equiv 0 \pmod{p},
\]
and hence \((p + 1)2^{\text{ord}(p; 2)} - 1\) is a composite number. The second case is similar. If we assume Artin’s conjecture (cf. \([2]\) pp.viii-x), there exist infinitely many primitive roots of 2. Thus we only have \( l(p) \leq p - 1 \) in this way. However, if we use the theorem of Euler, we obtain the following statement.

**Remark 1.5.** \( l(p) < p/2 \) for all prime \( p \geq 7 \).

**Proof.** First, we consider the case first Cunningham chain. If \( p \equiv 3 \pmod{4} \), then \((p - 1)/2 \) is odd. Let \( k = \varphi((p - 1)/2) \geq 2 \). Here \( \varphi \) denotes the Euler function. We have
\[
(p + 1)2^{k-1} - 1 \equiv 2 \pmod{2}.
\]
Thus \((p + 1)2^{k-1} - 1\) is a composite number for \( p > 3 \), and we get \( l_1(p) < k = \varphi((p - 1)/2) < p/2 \). If \( p \equiv 1 \pmod{4} \),
\[
(p + 1)2^{k-1} - 1 \equiv 2 \pmod{2},
\]
where \( k = \varphi((p - 3)/2) - 1 \geq 2 \), and hence \( l_1(p) < \varphi((p - 3)/2) - 1 < p/2 \). Therefore \( l_1(p) < p/2 \) for all \( p \geq 7 \). Let us next show the second case. If \( p \equiv 3 \pmod{4} \), setting \( k = \varphi((p + 3)/2) - 1 \geq 2 \), we have
\[
(p - 1)2^{k-1} = -2^{k+1} \equiv -1 \pmod{2}.
\]
Thus,
\[
l_1(p) < \varphi \left( \frac{p + 3}{2} \right) - 1 \leq \frac{p + 3}{2} - 2 < \frac{p}{2}.
\]
Similarly, \((p - 1)2^{\varphi((p+1)/2) - 1} + 1\) is composite if \( p \equiv 1 \pmod{4} \). \( \square \)

If \( 2p + 1 \) is also prime, then this prime number is called a safe prime. Its name comes from the fact that it is a prime number useful for Pollard’s \( p - 1 \) algorithm (cf. \([20]\)) that suggests the vulnerability of the RSA cryptosystem. It is useful if \( p - 1 \) is decomposed into the product of small prime factors. Since \((2p+1) - 1\) has 2 and \( p \) as prime factors and \( p \) is big, for two safe primes \( 2p+1, 2q+1 \), it is difficult to factorize \((2p+1)(2q+1)\) by the \( p - 1 \) algorithm. However, the Lenstra elliptic-curve factorization in \([17]\) can overcome this weakness in the probabilistic sense. Thus the meaning of safe primes as the RSA cryptosystem has faded. In 2004, whose first draft is in 2002, Agrawal, Kayal and Saxena found the AKS primality test\([1]\). In the present paper, they show that if in the case \( k = 2, f_1(n) = n, f_2(n) = 2n + 1 \) of \([1]\), the Sophie Germain prime density conjecture is true, the order of the complexity can be reduced to \( \tilde{O}(\log^6 n) \). Here, \( F(n) = \tilde{O}(G(n)) \) is defined by \( F(n) = O(G(n)h(\log G(n))) \) for some polynomial \( h(x) \). But, in 2005, Lenstra and Pomerance found a variant of AKS that runs in \( O(\log^8 n) \) without any conjectures \([15]\). Thus, now it is not necessary to use safe primes on the AKS primality test. Safe primes can be effectively used in the Diffie-Hellman key exchange \([2]\). Since this is the key to the strength of the discrete logarithm problem, we need to consider the question of how long and how well distributed Cunningham chains are.

We now let \( k \) be a natural number. We consider the number \( 0 < n \leq N \) such that \( n, 2n + 1, \cdots, (n + 1)2^{k-1} - 1 \) are simultaneously primes. Since
\[
M(n) := n(2n + 1) \cdots ((n + 1)2^{k-1} - 1) \equiv n \pmod{2},
\]
\( n = 0 \) is the only solution of \( M(n) \equiv 0 \pmod{2} \) for \( n \in \{0, 1\} \). Thus \( w(2) = 1 \). If \( p > 2 \), then \((n + 1)2^{k-1} - 1 \pmod{p}\) has period ord \( (p, 2) \) with respect to \( k \). Letting \( v(p) = \min\{k, \text{ord} (p; 2)\} \), this implies that \( w(p) \) is equal to the number \( w'(p) \) of solutions of

\[
  n (2n + 1) \cdots (n + 1)2^{(p-1)n - 1} \equiv 0 \pmod{p}.
\]

We observe that each of \( n, 2n + 1, \ldots, (n + 1)2^{(p-1)n - 1} - 1 \) has one solution in \( \{0, \ldots, p - 1\} \). That is, \( w'(p) \geq v(p) \). Assume that there exist \( 1 \leq a \leq b \leq v(p) \) and \( n \in \{0, \ldots, p - 1\} \) satisfy

\[
  (n + 1)2^{a-1} \equiv 1 \pmod{p} \quad \text{and} \quad (n + 1)2^{b-1} \equiv 1 \pmod{p}.
\]

Then, since \( n + 1 \) and \( p \) are relatively prime, \( 2^{a-1} \equiv 2^{b-1} \pmod{p} \) and it implies \( a = b \). We have \( w(p) = w'(p) = v(p) \). The same argument can be applied to the second Cunningham chains. From this argument and Conjecture \[13\] we can expect the following.

**Conjecture 1.6** (Caldwell \[5\]). Fix a natural number \( k \). Then

\[
  \sum_{p \leq N \atop l(p) \geq k} 1 \sim B_k \int_2^N \frac{dx}{(\log x)(\log 2x) \cdots (\log 2^{k-1}x)} \sim B_k \frac{N}{\log^k N}
\]

where the partial sum on the left hand side runs through all prime numbers \( p \leq N \) satisfying \( l(p) \geq k \). Here,

\[
  B_k := \prod_{p \leq 2} \frac{1 - w(p)/p}{(1 - p^{-1})^k} = 2^{k-1} \prod_{p > 2} \frac{1 - \min\{k, \text{ord}\,(p; 2)/p\}}{(1 - p^{-1})^k}.
\]

If we examine the size of \( B_k \), we can get some upper bound of \( l(p) \).

**Proposition 1.7.** Let \( \gamma \) be the Euler constant \( \approx 0.57722 \). Then for \( k \geq 2 \),

\[
  \log \log k + \gamma - 2 + O \left( \frac{1}{\log k} \right) \leq \frac{\log B_k}{k} \leq \log k + \gamma + \log \log 2 - 1 + O \left( \frac{1}{\log k} \right).
\]

**Proof.** The following facts in \[21\] will be used in the proof without mentioning explicitly:

\( (a) \) \( \log \log x + B - \frac{1}{2 \log^2 x} < \sum_{p \leq x} \frac{1}{p} < \log \log x + B + \frac{1}{2 \log^2 x} \),

\( (b) \) \( \prod_{p \leq x} \left( 1 - \frac{1}{p} \right) = O \left( \frac{1}{\log x} \right) \),

\( (c) \) \( \prod_{p \leq x} \left( 1 - \frac{1}{p} \right)^{-1} = e^{\gamma \log x} + O \left( \frac{1}{\log x} \right) \),

\( (d) \) \( \sum_{p \leq x} \log p < \left( 1 + \frac{1}{2 \log x} \right) x \),

where \( x > 2 \) and \( B = \gamma - \sum_{p \leq 2} \sum_{k=2}^{\infty} k^{-1}p^{-k} \). For \( k \geq 2 \) and a sufficiently large real number \( x \), we define

\[
  B_k(x) := 2^{k-1} \prod_{2 < p \leq x} \frac{1 - \min\{k, \text{ord}\,(p; 2)/p\}}{(1 - p^{-1})^k}.
\]

First, we estimate the lower bound of \( B_k(x) \) in \( k \). Write

\[
  B_k(x) = 2^{k-1} \left( \prod_{2 < p \leq x} \left( 1 - \frac{1}{p} \right)^{k-1} \right) \left( \prod_{2 < p \leq x} \prod_{k < p \leq x} \right) \left( 1 - \frac{\min\{k, \text{ord}\,(p; 2)\}}{p} \right) =: 2^{k-1} \Pi_1 \times \Pi_2 \times \Pi_3,
\]

where
say. Then log \( \Pi_1 \), log \( \Pi_2 \) and log \( \Pi_3 \) are evaluated as
\[
\log \Pi_1 = k \log \left( \frac{1}{2} \prod_{p \leq x} \left(1 - \frac{1}{p}\right)^{-1} \right) = k \log \left( \frac{e^\gamma}{2} \log x + O \left( \frac{1}{\log x} \right) \right)
\]
\[
= k \log \frac{e^\gamma}{2} + k \log \log x + O \left( \frac{k}{\log^2 x} \right),
\]
\[
\log \Pi_2 \geq \log \prod_{2 < p \leq k} \left(1 - \frac{p-1}{p}\right) = \log 2 - \sum_{p \leq k} \log p = -k - \frac{k}{\log k} + \log 2,
\]
\[
\log \Pi_3 \geq \prod_{k < p \leq x} \left(1 - \frac{k}{p}\right) = -\sum_{k < p \leq x} \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{k^n}{p^n}\right),
\]
respectively. Since
\[
k \sum_{k < p \leq x} \frac{1}{p} \leq \left( \log \left( \frac{\log x}{\log k} \right) + \frac{1}{\log^2 x} + \frac{1}{2 \log^2 k} \right) k
\]
and
\[
\sum_{k < p \leq x} \sum_{n=2}^{\infty} \frac{1}{n} \left(\frac{k^n}{p^n}\right) = \sum_{n=2}^{\infty} \frac{k^n}{n} \sum_{k < p \leq x} \frac{1}{p^n} < \sum_{n=2}^{\infty} \frac{k^n}{n} \sum_{m=k+1}^{\infty} \frac{1}{m^n}
\]
\[
< \sum_{n=2}^{\infty} \frac{k^n}{n} \left(\frac{1}{(k+1)^n} + \frac{1}{n-1} \right)
\]
\[
< k \sum_{n=2}^{\infty} \frac{1}{n} \left(\frac{1}{n+k} + \frac{1}{n-1}\right)
\]
\[
= k + O \left( \log k \right),
\]
we have
\[
\log \Pi_3 \geq -k \left( \log \left( \frac{\log x}{\log k} \right) + \frac{1}{\log^2 x} + \frac{1}{2 \log^2 k} \right) - k + O \left( \log k \right).
\]
Therefore
\[
\frac{1}{k} \log B_k(x) = \frac{1}{k} \left( (k-1) \log 2 + \log \Pi_1 + \log \Pi_2 + \log \Pi_3 \right)
\]
\[
> \frac{k-1}{k} \log 2 + \log \frac{e^\gamma}{2} + \log \log x + O \left( \frac{1}{\log^2 x} \right) + \frac{2}{k} - \frac{1}{2 \log k}
\]
\[
- \log \left( \frac{\log x}{\log k} \right) - \frac{1}{2 \log^2 k} - \frac{1}{2 \log^2 k} - 1 + O \left( \frac{\log k}{k} \right)
\]
\[
= \log \log k + \gamma - 2 - \frac{1}{2 \log k} - \frac{1}{2 \log^2 k} + O \left( \frac{1}{\log^2 x} \right) + O \left( \frac{\log k}{k} \right)
\]
and we have
\[
\frac{1}{k} \log B_k \geq \log \log k + \gamma - 2 - \frac{1}{2 \log k} - \frac{1}{2 \log^2 k} + O \left( \frac{\log k}{k} \right)
\]
as \( x \to \infty \). We next consider the upper bound of \( B_k(x) \) in \( k \). Write
\[
B_k(x) = 2^{k-1} \left( \prod_{2 < p \leq x} \left(1 - \frac{1}{p}\right)^{-k} \right) \left( \prod_{2 < p \leq k} \prod_{k < p \leq 2k} \prod_{2k < p \leq x} \right) \left(1 - \frac{\min \{k, \text{ord}(p; 2)\}}{p}\right),
\]
\[
=: 2^{k-1} \Pi_1 \times \Pi_2 \times \Pi_3 \times \Pi_4
\]
say. Then \( \log \Pi_1, \log \Pi_2 \) and \( \log \Pi_4 \) are evaluated as

\[
\log \Pi_1 = k \log \frac{e^\gamma}{2} + k \log x + O \left( \frac{k}{\log^2 x} \right),
\]

\[
\log \Pi_2 \leq \log \prod_{2 < p \leq k} \left( 1 - \frac{1}{p} \right) = O \left( \log \log k \right),
\]

\[
\log \Pi_4 \leq \log \prod_{2^k < p \leq x} \left( 1 - \frac{k}{p} \right) = - \sum_{2^k < p \leq x} \sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{k}{p} \right)^n < -k \sum_{2^k < p \leq x} \frac{1}{p}
\]

\[
\leq -k \left( \log \left( \frac{\log x}{k \log 2} \right) - \frac{1}{k^2 \log 2} - \frac{1}{2 \log^2 x} \right)
\]

\[
= k \log \left( \frac{k \log 2}{\log x} \right) + \frac{k}{2 \log^2 x} + O \left( \frac{1}{k} \right),
\]

since \( \text{ord} (p; 2) \geq k \) where \( p > 2^k \). If \( p > 2^{j-1} \) for some \( j \in \mathbb{N} \), then \( \text{ord} (p; 2) \geq j \). Thus, putting \( y = \log \frac{k}{\log 2} \), we obtain

\[
\log \Pi_3 \leq \log \prod_{y < j \leq k} \prod_{2^{j-1} < p \leq 2^j} \left( 1 - \frac{j}{p} \right)
\]

\[
< \log \prod_{y < j \leq k} \left( \frac{j-1}{j} \right)^j \exp \left( \frac{1}{2j \log^2 2} + \frac{j}{(j-1)^2 \log^2 2} \right)
\]

\[
= \log \left( \frac{[y]!}{([y]-1)!} \right) \left( \frac{k-1}{kk} \right) + \frac{1}{2 \log^2 2} \sum_{y < j \leq k} \left( \frac{1}{2j} + \frac{j}{(j-1)^2} \right).
\]

Applying \( \log(n^n/n!) = n + O(\log n) \), we find that the right-hand side of the above is

\[
= \log \frac{k!}{k^k} + O(\log k) = -k + O(\log k).
\]

Therefore, we have

\[
\frac{1}{k} \log B_k (x) = \frac{1}{k} \left( (k-1) \log 2 + \log \Pi_1 + \log \Pi_2 + \log \Pi_4 + \log \Pi_4 \right)
\]

\[
\leq \frac{k-1}{k} \log 2 + \log \frac{e^\gamma}{2} + \log \log x + O \left( \frac{1}{\log^2 x} \right) + O \left( \frac{\log \log k}{k} \right)
\]

\[
- 1 + O \left( \frac{\log k}{k} \right) + \log \left( \frac{k \log 2}{\log x} \right) + \frac{1}{2 \log^2 x} + O \left( \frac{1}{k^2} \right)
\]

\[
= \log k + \gamma + \log \log 2 - 1 + O \left( \frac{1}{\log^2 x} \right) + O \left( \frac{\log k}{k} \right).
\]

\[
\square
\]

Let \( k(N) \) be the maximum of the length of the Cunningham chain which is generated by \( p \leq N \), that is,

\[ k(N) = \max \{ l(p) : p \leq N \}. \]

Let \( f(k) = \frac{1}{k} \log B_k \). Then we have

\[
B_{k(N)} \frac{N}{(\log N)^{k(N)}} = \left( \frac{f(k(N))}{\log N} \right)^{k(N)} =: F(N)^{k(N)}.
\]

Suppose that there exists a sequence \( \{ N_n \}_{n=1}^{\infty} \) satisfying

\[
(1 - F(N_n)) k(N_n) \gg 1.
\]
Then, there exists a constant $C > 0$ which satisfies $F(N_n) \leq 1 - C/k(N_n)$. Thus
\[
\limsup_{n \to \infty} B_k(N_n)(\log N_n)^{k(N_n)} \leq \lim_{n \to \infty} \left( 1 - \frac{C}{k(N_n)} \right)^{k(N_n)}
\]
\[
= \begin{cases} 
  e^{-C} & \text{if } \lim_{n \to \infty} k(N_n) = +\infty, \\
  \max_{n \in \mathbb{N}} \left( 1 - \frac{C}{k(N_n)} \right)^{k(N_n)} & \text{if } \lim_{n \to \infty} k(N_n) < +\infty \\
< 1.
\end{cases}
\]

However, unless the order with respect to $k$ of the terms that vanish by approximation of Conjecture 1.6 is small, this result contradicts Conjecture 1.6 and the maximality of $k(N)$. Therefore we may expect that $\lim_{N \to \infty}(1 - F(N))k(N) = 0$. In particular, since $\lim_{N \to \infty} F(N) = 1$, we have $F(N) < 2$ for sufficiently large $N$. Thus
\[
\frac{\log N}{k(N)} + \log f(k(N)) - \log \log N < \log 2.
\]

From Proposition 2.1 and $l(2) \geq 3$, we have
\[
\log f(k(N)) > \log \log k(N) - 2 > \log \log 3 - 2.
\]

Therefore, we obtain
\[
k(N) \geq (1 + o(1)) \frac{\log N}{\log \log N}.
\]

From this, we may conjecture the following:

**Conjecture 1.8.**

\[
l(p) = \Omega \left( \frac{\log p}{\log \log p} \right) \quad \text{on } \mathbb{P}.
\]

In [3] it is reported that, $p_1 := 275983293417386593519$ is the first term of the longest first Cunningham chain in the data up to 2020. Its length is 17. And $p_2 = 42008163485623434922152331$ is the first term of the longest second whose length is 19. Thus
\[
l_1(p_1) = 17, \quad \frac{\log p_1}{\log \log p_1} \simeq 12.661,
\]
\[
l_1(p_2) = 19, \quad \frac{\log p_2}{\log \log p_2} \simeq 14.470.
\]

In this way, a better estimation of $B_k$ implies a better bound of the length of Cunningham chains. However, it is still unknown whether $\limsup_{p \to \infty} l(p)/p = 0$ or not.

In this paper, by using a generalized Fibonacci sequence, we get $l(p) \ll \log p$ under a certain condition (Theorem 4.13). It seems that this sufficient condition is plausible by numerical test. The condition can be extended from prime numbers to natural numbers (Corollary 4.14). This implies that the problem of upper estimation of $l(p)$ is reduced to that on natural numbers. One of the benefit of this is that we can use methods of number theory to solve it.

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2. $\mathcal{F}_n$ NUMBERS

**Definition 2.1.** Let $\alpha \geq 1$ be a natural number. A sequence $\mathcal{F}_\alpha$:
\[
F_0 = 0, \quad F_1 = 1, \quad F_{n+1} = \alpha F_n + F_{n-1} \quad (n \in \mathbb{Z})
\]
is called the generalized Fibonacci sequence, and its elements are called $\mathcal{F}_\alpha$ numbers. For example, enumerating $F_{-5}$ through $F_5$ we have
\[
\cdots, \alpha^4 + 3\alpha^2 + 1, -\alpha^3 - 2\alpha, \alpha^2 + 1, -\alpha, 1, 0, 1, \alpha, \alpha^2 + 1, \alpha^3 + 2\alpha, \alpha^4 + 3\alpha^2 + 1, \cdots.
\]
The following facts on $\mathcal{F}_\alpha$ numbers are well known. For example, in [16], T. Koshy showed the case $\alpha = 1$ of (a) in Section 5.2 pp.88-90, (b) in Section 5 p.82, (c) in Section 5.6 p.103, (d) in Section 20.1 p.397, (e) in Section 10.1 pp.171-173, and (f) in Section 10.1 pp.173-174. We can similarly show the case $\alpha > 1$.

**Fact 2.2.** Let $m, n$ be integers, and put $\varphi_\alpha = (\alpha + \sqrt{\alpha^2 + 4})/2$. We have:

(a) $F_n = \frac{1}{\sqrt{\alpha^2 + 4}} \left( \varphi_\alpha^n - (-\varphi_\alpha)^{-n} \right)$,

(b) $\sum_{i=1}^{n} F_i = \left( 1 + \frac{1}{\alpha} \right) F_n + \frac{F_{n-1} - 1}{\alpha} < \left( 1 + \frac{2}{\alpha} \right) F_n \quad (n \geq 1)$,

(c) $F_{-n} = (-1)^{n+1} F_n$,

(d) $F_{m+n} = F_{m+1} F_n + F_m F_{n-1}$,

(e) $m \mid n \iff F_m \mid F_n$,

(f) $(F_m, F_n) = F_{(m,n)}$.

If $\alpha = 1$, then $m$ should not be 2. Indeed, $F_2 = 1, F_3 = 2$ thus $F_2 \mid F_3$, but $2 \nmid 3$. We next define the divisor function on $\mathcal{F}_\alpha$.

**Definition 2.3.** In this article, a natural number $d$ is called a $\mathcal{F}_\alpha$ divisor of $n$ if $d \in \mathcal{F}_\alpha$ and $d \mid n$. A map $\mathcal{F}_\alpha : \mathbb{N} \to \mathbb{C}$ is defined by

$$\mathcal{F}_\alpha(n) := \sum_{d\mid n, \ 0 < d \in \mathcal{F}_\alpha} d$$

for $n \in \mathbb{N}$, and called the $\mathcal{F}_\alpha$ divisor function. Let $\mathcal{F}_\alpha(1) = \mathcal{F}_\alpha$, and $\mathcal{F}_\alpha(4) = \mathcal{F}_\alpha(2) = 1$.

In Section 3, we will investigate the relationship between the iteration of $\mathcal{F}_\alpha$ and Cunningham chains.

**Example 2.4.** Suppose $\alpha = 3$. Since

$$\mathcal{F}_\alpha := \cdots, 0, 1, 3, 10, 33, 109, 360, \cdots,$$

we get

$$\mathcal{F}_\alpha(2) = 1, \mathcal{F}_\alpha(3) = 4, \mathcal{F}_\alpha(4) = 1,$$

$$\mathcal{F}_\alpha(109) = \mathcal{F}_\alpha(2) = 1.$$

We consider the Dirichlet series associated with $\mathcal{F}_\alpha$. Put $\zeta_\alpha(s) = \sum_{n \in \mathcal{F}_\alpha} n^{-s}$ for $\alpha \geq 1$. Then $\zeta_\alpha(s)$ converges for all $s$ with $\Re(s) > 0$. Suppose $f(n)$ is $n$ if $n \in \mathcal{F}_\alpha$, and is 0 otherwise. Then

$$\zeta(s)\zeta_\alpha(s - 1) = \left( \sum_{m=1}^{\infty} \frac{1}{m^{\alpha}} \right) \left( \sum_{0 < n \in \mathcal{F}_\alpha} \frac{1}{n^{s-1}} \right) = \sum_{m,n=1}^{\infty} \frac{f(n)}{mn} = \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{d\mid n, \ 0 < d} f(d) = \sum_{n=1}^{\infty} \frac{\mathcal{F}_\alpha(n)}{n^s}$$

for $\Re(s) > 1$. As we can see from this expression, the research of $\mathcal{F}_\alpha$ will be useful for the study of $\zeta_\alpha$. In particular, $\zeta_1$ is called the Fibonacci zeta function by Egami[9] and Navas[19], and it is a meromorphic function on $\mathbb{C}$. It is the famous unsolved problem whether $\zeta_1(1)$ is transcendental or not.

Hereafter, we suppose that $\alpha \geq 3$ unless explicitly stated otherwise. Let $\text{ind}_\alpha(n)$ be the index of the maximal $\mathcal{F}_\alpha$ number $\leq n$ for $n \in \mathbb{N}$, that is, if we take $k$ satisfying $F_k \leq n < F_{k+1}$, then $\text{ind}_\alpha(n) = k$. Since the sequences $\mathcal{F}_\alpha$ are increasing for arbitrary $\alpha \geq 3$, $\text{ind}_\alpha(n)$ is determined uniquely for all $n$. In addition, $\text{ind}_\alpha(n)$ is not decreasing for $n$.

**Lemma 2.5.** Let $k' := \text{ind}_\alpha(\mathcal{F}_\alpha(n))$. Then $k' \leq \text{ind}_\alpha(n)$ for all $n$, and if $\mathcal{F}_\alpha(n) \neq 1$, then

$$F_{k'} < \mathcal{F}_\alpha(n) < \left( 1 + \frac{2}{\alpha} \right) F_{k'}.$$

In particular, $F_{k'} \nmid \mathcal{F}_\alpha(n)$.
Proof. \( k' = 1 \leq \text{ind}_\alpha(n) \text{ if } F_\alpha \sigma(n) = 1. \) Suppose that \( F_\alpha \sigma(n) > 1. \) Let \( i_0 \) be the maximal \( i \) with \( F_i | n. \) Then \( i_0 > 1 \) and \( n \) has at least two \( F_\alpha \) divisors \( F_{i_0} \) and \( F_1. \) Thus we estimate that

\[
F_{i_0} < F_{i_0} + F_1 \leq F_\alpha \sigma(n) \leq \sum_{i=1}^{i_0} F_i < \left( 1 + \frac{2}{\alpha} \right) F_{i_0} < 2F_{i_0} < F_{i_0+1}
\]

from Fact 2.2 \((b).\) This implies that \( k' = i_0 = \text{ind}_\alpha(F_{i_0}) \leq \text{ind}_\alpha(n). \) \( \square \)

Let \( F_\alpha \sigma(N) \) be the set of all \( F_\alpha \sigma(n) \) for \( n \in \mathbb{N}. \)

**Theorem 2.6.** For every natural number \( n, \) there exists a non-negative integer \( k \leq \log n/ \log \varphi_\alpha + 2 \) such that \( F_\alpha \sigma^k(n) = 1. \)

In Section 4, we will improve this bound to \( k \ll \log \log n. \)

Proof. Let \( a \) be an element of \( F_\alpha \sigma(N), \) and let \( \sigma = \text{ind}_\alpha(a). \) If \( k = 1, \) then \( F_\alpha \sigma(a) = a = 1. \) In addition, since

\[
F_2 + F_1 = \alpha + 1 \neq 0 \pmod{\alpha},
\]

we get \( F_\alpha \sigma(a) = 1 \) if \( k = 2. \) Suppose that \( k \geq 3. \) Since \( F_k \nmid a \) from Lemma 2.5, we have

\[
F_\alpha \sigma(a) \leq \sum_{i=1}^{k-1} F_i < \left( 1 + \frac{2}{\alpha} \right) F_{k-1} < F_k.
\]

Thus \( \text{ind}_\alpha(F_\alpha \sigma(a)) \leq k - 1. \) By iterating this argument, we have \( \text{ind}_\alpha(F_\alpha \sigma^{k-2}(a)) \leq 2. \) Therefore \( F_\alpha \sigma^{k-1}(a) = 1. \) This also holds for \( k = 1, 2. \) Let \( k = \text{ind}_\alpha(F_\alpha \sigma(n)) \) for arbitrary \( n. \) We have

\[
F_\alpha \sigma^{k'}(n) = F_\alpha \sigma^{k'-1}(F_\alpha \sigma(n)) = 1.
\]

Since \( k' \leq \text{ind}_\alpha(n) \) from Lemma 2.5, this leads to \( F_\alpha \sigma^{\text{ind}_\alpha(n)}(n) = 1. \) On the other hand, since

\[
F_\alpha^{\text{ind}_\alpha(n)} = F_{\text{ind}_\alpha(n)} \sqrt{\alpha^2 + 4 + (-\varphi_\alpha)^{-\text{ind}_\alpha(n)}}
\]

from Fact 2.2 \((a)\) and \( F_{\text{ind}_\alpha(n)} \leq n, \) we obtain that

\[
\text{ind}_\alpha(n) \leq \frac{1}{\log \varphi_\alpha} \left( \log n + \log \left( \sqrt{\alpha^2 + 4 + (-\varphi_\alpha)^{-\text{ind}_\alpha(n)}} \right) \right).
\]

Now, \( n \) is a natural number and \( \varphi_\alpha = (\alpha + \sqrt{\alpha^2 + 4})/2 \) is bigger than \( \alpha \geq 3. \) Thus

\[
\frac{1}{\log \varphi_\alpha} \log \left( \sqrt{\alpha^2 + 4 + (-\varphi_\alpha)^{-\text{ind}_\alpha(n)}} \right) < \frac{1}{\log \alpha} \log \left( \sqrt{\alpha^2 + 4 + \frac{1}{\alpha}} \right)
\]

\[
= \frac{1}{\log \alpha} \log \left( \sqrt{1 + \frac{4}{\alpha^2} + \frac{1}{\alpha^2}} \right) + 1
\]

\[
< \frac{1}{\log \alpha} \log \left( 1 + \frac{5}{\alpha^2} \right) + 1 < 2.
\]

This implies \( \text{ind}_\alpha(n) < \log n/ \log \varphi_\alpha + 2. \) \( \square \)

Here, we define

\[
\text{ord}_\alpha(n) = \min \left\{ k \in \mathbb{Z}_{\geq 0} : F_\alpha \sigma^k(n) = 1 \right\}.
\]

In Section 4, we will find that a better estimation of \( \text{ord}_\alpha(n) \) for \( n \) implies a better bound of the length of Cunningham chains.
3. The relation between $\mathcal{F}_\alpha$ numbers and the Cunningham chains

In this section, let $m, n$ be natural numbers. From Fact 2.2 (c) and (d),

$$F_{m+n} + F_{m-n} = F_{m+1}F_n + F_m F_{n+1} + F_{m-1}F_{n+1} + F_{m-1}F_n$$

$$= (1 + (-1)^n) F_m F_{n-1} + (F_{m+1} + (-1)^n F_{m-1}) F_n$$

$$= \begin{cases} F_n (F_{m+1} + F_{m-1}) & (\text{n is odd}) \\ 2F_m F_{n-1} + (F_{m+1} - F_{m-1}) F_n & (\text{n is even}) \end{cases}$$

Since $2F_{m+1}F_n = F_{m+1}F_n + \alpha F_m F_n + F_{m-1}F_n$, we have

$$F_{m+n} + F_{m-n} = \begin{cases} F_n (F_{m+1} + F_{m-1}) & (\text{n is odd}) \\ F_m (F_{n-1} + F_{n+1}) & (\text{n is even}) \end{cases}$$

Note that $F_{m+1} + F_{m-1} = \frac{F_{2m}}{F_m}$ from Fact 2.2 (d).

**Lemma 3.1.** The $\mathcal{F}_\alpha$ divisors of $F_{m+1} + F_{m-1}$ are at most 1 and $\alpha$. Moreover, $\alpha \mid F_{m+1} + F_{m-1}$ if and only if $m \equiv 1 \pmod{2}$, that is,

$$\mathcal{F}_\alpha \sigma (F_{m+1} + F_{m-1}) = \begin{cases} 1 & (m \text{ is even}) \\ \alpha + 1 & (m \text{ is odd}) \end{cases}$$

**Proof.** If $\alpha \mid 2F_m$, then

$$3 \leq \alpha = (\alpha, 2F_m) \leq 2 (\alpha, F_m) = 2F_{(2,m)}$$

and hence $2 \mid n$. Conversely, $\alpha \mid F_m$ if $m$ is even from Fact 2.2 (e). Therefore

$$\alpha \mid 2F_m \iff m \equiv 0 \pmod{2}.$$ 

Also since $F_{m+1} + F_{m-1} = \alpha F_m + 2F_{m-1}$, we find that

$$\alpha \mid F_{m+1} + F_{m-1} \iff \alpha \mid 2F_{m-1} \iff m \equiv 1 \pmod{2}.$$ 

Here, take a natural number $d$ that satisfies $F_d \mid F_{m+1} + F_{m-1}(= \frac{F_{2m}}{F_m})$. Since $F_d \mid F_{2m}$, there exists a natural number $q$ such that $2m = qd$. Suppose that $d$ is odd. Then $q$ is even and $F_d \mid F_{\frac{q}{2}d-1}$ since $F_d \mid F_{\frac{q}{2}d}$ from Fact 2.2 (e). In addition,

$$\left(F_d, \frac{F_{2d}}{F_d-1}\right) = F_{(d, \frac{2}{d}-1)} = F_1 = 1.$$ 

Thus, we get $F_d \mid 2$. This and $\alpha \geq 3$ imply $d = 1$. Let us next suppose that $d$ is even. Then $F_{\frac{q}{2}} \mid F_d$ from Fact 2.2 (e), and this implies $F_{\frac{q}{2}} \mid F_{\frac{q}{2q}-1}$. Similarly, since $F_{\frac{q}{2}} \mid F_{\frac{q}{2}}(\frac{q}{2}) = 1$, we have $F_{\frac{q}{2}} \mid 2$, and hence $d = 2$. From these results, we conclude that there is no $d \geq 3$ which satisfies $F_d \mid F_{m+1} + F_{m-1}$. \qed

**Theorem 3.2.** Let $p$ be a prime number. If $p \neq m < 2p$, then

$$\mathcal{F}_\alpha \sigma (F_{m+p} + F_{m-p}) = \begin{cases} 1 & (m = 1) \\ F_3 + 1 & (m = 3) \\ F_p + 1 & (m \text{ is even}) \\ \alpha + 1 & (m \text{ is odd}) \end{cases} \quad (p = 2),$$

$$\begin{cases} 1 & (m = 1) \\ (\alpha^2 + 2) F_3 & (m = 3) \end{cases} \quad (p \neq 2).$$

**Proof.** Note that $F_{m+p} + F_{m-p} = F_p \cdot F_{2m}/F_m$ from (3). If $p = 2$,

$$F_{m+p} + F_{m-p} = \begin{cases} \alpha^2 + 2 & (m = 1) \\ (\alpha^2 + 2) F_3 & (m = 3) \end{cases}.$$ 

Suppose that $p > 2$. We have

$$\left(\frac{F_p}{F_m}, \frac{F_{2m}}{F_m}\right) \leq (F_p, F_{2m}) = F_{(p,2m)} = F_1 = 1$$

that is $(F_p, F_{2m}/F_m) = 1$. Here, let $D_\alpha(n)$ be the set of all $\mathcal{F}_\alpha$ divisors of $n$. We find that

$$D_\alpha(F_p) \cap D_\alpha\left(\frac{F_{2m}}{F_m}\right) = \{1\}.$$
And in general,

\[ D_\alpha \left( \frac{F_p F_{2m}}{F_m} \right) \supset D_\alpha (F_p) \cup D_\alpha \left( \frac{F_{2m}}{F_m} \right). \]

We will show the inclusion relation of the reverse direction. Take an arbitrary \( F_s \) in \( D_\alpha (F_p \cdot F_{2m}/F_m) \). Then there exists \( a \) and \( b \) satisfying \( a \mid F_p, b \mid F_{2m}/F_m \) and \( ab = F_s \). Since \( (b, F_p) = 1 \),

\[ a = (a, F_p) = (a, F_p) (b, F_p) = (ab, aF_p, bF_p, F_p^2) = (ab, (a, b, F_p) F_p) = (ab, F_p) = F_{(s,p)} = 1 \text{ or } F_p. \]

In the case \( a = 1 \), \( F_s = b \in D_\alpha (F_{2m}/F_m) \) holds. Thus we have \( ab \in \mathcal{F}_\alpha (F_{2m}/F_m) \). Suppose that \( a = F_p \). Then \( k := s/p \) is a natural number from Fact 2.2 (e), and \( F_{kp}/F_p \mid F_{2m}/F_m \) holds. If \( m < p \), then we have \( k = 1 \) since \( F_{kp}/F_p > F_{2m}/F_m \) for \( k \geq 2 \). We consider the case \( p < m < 2p \). Note that

\[ F_{2n}/F_n = F_{n+1} + F_{n-1} \leq F_{n+2} + F_n = \frac{F_{(n+1)}}{F_{n+1}} \]

for all \( n \in \mathbb{N} \). Thus we obtain that

\[ \frac{F_{2m}}{F_m} = \frac{F_{2(m+1)}}{F_{m+1}} \leq \cdots \leq \frac{F_{2(2p-1)}}{F_{2p-1}} = F_{2p} + F_{2p-2} \]

\[ < F_{2p} + F_{2p-2} + F_{p-1} \frac{F_{2p}}{F_p} + F_{2p-3} \]

\[ = (F_p + F_{p-1}) F_{2p} + F_p (F_{2p-2} + F_{2p-3}) \]

\[ < \frac{F_{p+1} F_{2p}}{F_p} + \frac{F_p F_{2p-1}}{F_p} \]

\[ = \frac{F_{3p}}{F_p}. \]

This implies that \( k \leq 2 \). We suppose \( k = 2 \) and let \( r = m - p \). Then \( 1 \leq r < p \) and

\[ \frac{F_{2m}}{F_m} = F_{m+1} + F_{m-1} = F_{p+r+1} + F_{p+r-1} = F_{p+1} F_{r+1} + 2 F_p F_r + F_{p-1} F_{r-1} \]

\[ = (F_{p+1} + F_{p-1}) F_{r+1} + 2 F_p F_r + F_{p-1} F_{r-1} - F_{p-1} F_{r+1} \]

\[ = \frac{F_{2p}}{F_p} F_{r+1} + F_p F_r + (\alpha F_{p-1} + F_{p-2}) F_r + F_{p-1} F_{r-1} - (\alpha F_{p-1} F_r + F_{p-1} F_{r-1}) \]

\[ = \frac{F_{2p}}{F_p} F_{r+1} + 2 F_{(p-1)} F_r. \]

Since \( F_{2p}/F_p \mid F_{2m}/F_m \), we have \( F_{2p}/F_p \mid F_{2(p-1)}/F_{p-1} \cdot F_r \). Thus

\[ \frac{F_{2p}}{F_p} = \left( \frac{F_{2p}}{F_p}, \frac{F_{2(p-1)}}{F_{p-1}} \right) \leq \left( \frac{F_{2p}}{F_p}, \frac{F_{2(p-1)}}{F_{p-1}} \right) = \left( F_{2(p-1)} \right) (F_{2p}, F_r) = F_{2(p-1)} F_{2(p-1)} \leq \frac{F_{2}^2}{2} = \alpha^2. \]

This implies \( F_{2p} \leq \alpha^2 F_p \) whose right-hand side does not exceed \( F_{p+2} \), and hence \( p \leq 2 \). However, this contradicts \( p > 3 \), and we get \( k = 1 \). In other words, \( b = 1 \) in the case \( a = F_p \), and then \( ab \in D_\alpha (F_p) \). From these result, we have

\[ D_\alpha (F_p) \cap D_\alpha \left( \frac{F_{2m}}{F_m} \right) = \{ 1 \} \text{ and } D_\alpha \left( F_p \frac{F_{2m}}{F_m} \right) = D_\alpha (F_p) \cup D_\alpha \left( \frac{F_{2m}}{F_m} \right), \]

and hence

\[ \mathcal{F}_\alpha (F_m + p) = \sum_{d \in D_\alpha (F_m)} d = \left( \sum_{d \in D_\alpha (F_p)} + \sum_{d \in D_\alpha (\frac{F_{2m}}{F_m})} \right) d - 1 = \mathcal{F}_\alpha (F_p) + \mathcal{F}_\alpha \left( \frac{F_{2m}}{F_m} \right) - 1. \]

Apply Lemma 3.1 to complete the proof. □
From this theorem, we find the following corollary. That is a relationship between the iteration of $\varphi_n \sigma$ and Cunningham chains.

**Corollary 3.3.** For every odd prime $p$,

$$\varphi_n \sigma (F_{2p\pm 1} + 1) = F_p + 1.$$  

In particular, if $2p \pm 1$ is also prime, then

$$\varphi_n \sigma (F_{2p\pm 1}) = \varphi_n (F_p).$$

Further, by iterating this argument, we obtain

$$l_{\pm 1}(p) - 1 = \text{ord}_\alpha \left( F_{(p \pm 1)2^\pm 1 (p) - 1} \right) - \text{ord}_\alpha (F_p).$$

**Proof.** It follows $m = p \pm 1$ in Theorem 3.2. Further

$$\text{ord}_\alpha \left( F_{(p \pm 1)2^\pm 1 (p) - 1} \right) = 1 + \text{ord}_\alpha \left( F_{(p \pm 1)2^\pm 1 (p) - 2} \right) = \cdots = l_{\pm 1}(p) - 1 + \text{ord}_\alpha (F_p).$$

\[\square\]

In Section 4, we will show the converse of 4.4.

4. The upper bound of $\text{ord}_\alpha (n)$

In this section, our aim is to prove Theorem 4.6 and Theorem 4.13. Those are important results that suggest the relationship between $\varphi_n \sigma$ and Cunningham chains. In the process of the proof, we will use the following theorem which is called the generalized Zeckendorf's theorem by Hoggatt [14] and Keller [15].

**Definition 4.1 (Zeckendorf-Hoggatt-Keller).** Every natural number $n$ has the unique representation:

$$n = \sum_{i=1}^{r} a_i F_{c_i}$$

where $r$ is a natural number and sequences $\{a_i\}_{i=1}^{r}$, $\{c_i\}_{i=1}^{r} \subset \mathbb{N}$ satisfy the following conditions.

(i) $0 < a_i \leq \alpha \ (i = 1, \ldots, r)$, $a_1 < \alpha$,

(ii) $1 \leq c_1 < \cdots < c_r$,

(iii) $c_{i+1} = c_i + 1 \Rightarrow a_{i+1} < \alpha \ (i = 1, \ldots, r - 1)$.

Such a sum is called the Zeckendorf representation of $n$.

**Remark 4.2.** By definition, $\sum_{0<d\in F_n} d$ is the Zeckendorf representation of $\varphi_n \sigma (n)$.

**Lemma 4.3.** For every $k > i \geq 0$,

$$F_{k+i} \equiv (-1)^{i+1} F_{k-i} \pmod{F_k},$$

$$F_{2k+i} \equiv (-1)^k F_i \pmod{F_k},$$

$$F_{3k+i} \equiv (-1)^{k+i+1} F_{k-i} \pmod{F_k},$$

$$F_{4k+i} \equiv F_i \pmod{F_k}.$$  

More generally, for every two non-negative integers $a, b$ with $a \equiv b \pmod{4}$,

$$F_{ak+i} \equiv F_{bk+i} \pmod{F_k}.$$  

**Proof.** Fix a natural number $k$. First, we prove 6. From Fact 2.2 (d),

$$F_{k+i} = F_k F_{i+1} + F_{k-1} F_i \equiv F_{k-i+1} F_i \pmod{F_k}.$$  

Thus 6 holds for $i = 0, 1$. Suppose that 6 holds for all natural numbers less than $i \geq 2$. The right hand side is

$$(-1)^{i+1} F_{k-i} = (-1)^{i+1} \left( F_{k-i+2} - \alpha F_{k-i+1} \right) = (-1)^{i-1} F_{k-i+2} + \alpha (-1)^i F_{k-i+1}.$$  

Using the assumptions of induction, we have

$$(-1)^{i-1} F_{k-i+2} + \alpha (-1)^i F_{k-i+1} \equiv F_{k+i+2} + \alpha F_{k+i} = F_{k+i} \pmod{F_k}.$$  

\[\square\]
The proof of (6) runs as
\[ F_{2k+i} = F_{2k}F_{i+1} + F_{2k-1}F_i \equiv F_{2k-1}F_i \equiv (-1)^k F_i \pmod{F_k} \]
from the case of \( i = k - 1 \) of [5]. Similarly, [7] and [8] is obtained from

\[ F_{3k+i} = F_{3k}F_{i+1} + F_{3k-1}F_i \equiv F_{2k+(k-1)}F_i \equiv (-1)^k F_{k-1}F_i \]
\[ \equiv (-1)^k (F_kF_{i+1} + F_{k-1}F_i) \equiv (-1)^k F_k+i \equiv (-1)^{k+i+1} F_{k-i} \pmod{F_k}, \]
\[ F_{4k+i} \equiv F_{4k-1}F_i \equiv (-1)^{k+(k-1)+1} F_1F_i = F_i \pmod{F_k}. \]

Next, we observe that [9] holds for every non-negative \( a, b \) with \( a \equiv b \pmod{4} \) if and only if for every non-negative \( a, \)

\[ F_{2k+i} \equiv F_{3k+i} \equiv F_{4k+i} \pmod{F_k} \]
holds where \( a \equiv \delta \pmod{4} \) with \( \delta \in \{0, 1, 2, 3\} \). We prove (10) by using induction with respect to \( a \). The case \( a = 0, 1, 2, \) and 3 are trivial since \( a = \delta \). And [8] is nothing but the case of \( a = 4 \). Suppose that (10) holds also all natural numbers less than \( a \geq 5 \), and we take \( \delta' \in \{1, 2, 3, 4\} \) satisfying \( \delta' \equiv a \pmod{4} \). Then

\[ F_{2k+i} = F_{2k}F_{i+1} + F_{2k-1}F_i \equiv F_{2k-1}F_i = F_{(a-1)k+(k-1)}F_i. \]

Using the assumption of induction, we have

\[ F_{(\delta'-1)k+(k-1)}F_i = F_{\delta'-1}F_i \equiv F_{\delta'k}F_{i+1} + F_{\delta'k-1}F_i = F_{\delta'k+i} \pmod{F_k}. \]

\[ \Box \]

**Theorem 4.4.** Suppose \( a \in \mathcal{F}_\sigma(N) \), and put \( k = \text{ind}_\alpha(a) \). If \( F_i \mid a \), then \( i \leq (k+1)/2 \), that is

\[ \text{ind}_\alpha(\mathcal{F}_\sigma(a)) \leq \frac{k+1}{2}. \]

**Proof.** Let \( i_0 \) be the maximal \( i \) satisfying \( F_i \mid a \). The case \( k = 1 \) is \( i_0 = 1 \) since \( a = 1 \), and hence the claim holds. Assume that \( i_0 > (k+1)/2 \) for \( k \geq 2 \). In particular, \( i_0 \geq 2 \). Then \( a > 1 \). Thus, there exist natural numbers \( c_1, \ldots, c_n \) such that

\[ a = F_{i_0+c_1} + F_{i_0+c_2} + \cdots + F_{i_0+c_m} + F_{c_{m+1}} + \cdots + F_{c_n}, \]
\[ k = i_0 + c_1 > i_0 + c_2 > \cdots > i_0 + c_m > c_{m+1} > \cdots > c_n = 1. \]

Note that \( c_1 \leq i_0 - 2 \). Since \( F_{i_0} \mid a \),

\[ a = \sum_{j=1}^{m} F_{i_0+c_j} + \sum_{j=m+1}^{n} F_{c_j} \equiv 0 \pmod{F_k}. \]

From Lemma 4.3 (5),

\[ \sum_{j=1}^{m} F_{i_0+c_j} \equiv \sum_{j=1}^{m} (-1)^{c_j+1} F_{i_0-c_j} \pmod{F_k}. \]

It is enough to consider only the fractional part, and hence we suppose \( c_{m+1} < i_0 \) without loss of generality. Since \( \alpha \geq 3 \) and \( i_0 \geq 2 \), we estimate

\[ \left| \sum_{j=1}^{m} (-1)^{c_j+1} F_{i_0-c_j} + \sum_{j=m+1}^{n} F_{c_j} \right| \leq 2 \sum_{j=1}^{i_0-1} F_j < F_{i_0}. \]

Therefore, we have

\[ \sum_{j=m+1}^{n} F_{c_j} = \sum_{j=1}^{m} (-1)^{c_j} F_{i_0-c_j}. \]
Assume that there exist some \( j \in \{1, \ldots, m\} \) such that \( c_j \) is odd, and denote all of them by \( c_{d_1}, \ldots, c_{d_h} \). Then
\[
\sum_{j=m+1}^{n} F_{c_j} + \sum_{i=1}^{h} F_{i_0 - c_{d_i}} = \sum_{j=1}^{m} \sum_{j \neq c_{d_i}, (i=1, \ldots, h)} F_{i_0 - c_j}.
\]
In both sides, each coefficient of \( F_n \) numbers are \( \leq 2 \). Thus, they are Zeckendorf representations of a natural number. However, the left hand side has \( F_{i_0 - c_{d_h}} \) and the right-hand side does not. This is a contradiction to the uniqueness of the Zeckendorf representation. Therefore \( c_j \) is even for all \( j = 1, \ldots, m \), and
\[
\sum_{j=m+1}^{n} F_{c_j} = \sum_{j=1}^{m} F_{i_0 - c_j}
\]
holds. This both sides are Zeckendorf representations. But the left hand side has \( F_1 \) and the right-hand side does not have it since \( c_1 \leq i_0 - 2 \). This is also a contradiction to the uniqueness, and hence we obtain that \( i_0 \leq (k+1)/2 \).

**Remark 4.5.** The estimate given by Theorem 4.4 is best-possible. Let \( a = F_{2p-1} + 1 \) with \( p, 2p-1 \in \mathbb{P} \). For example, \( p = 3 \). From Theorem 4.4, \( i \leq p \) if \( F_p \mid a \). On the other hand, we get \( F_p \mid a \) if \( p \) is odd from Corollary 3.3 [4].

We can obtain that the converse of Corollary 3.3 from this theorem.

**Theorem 4.6.** For two odd prime numbers \( p, q \), the following are equivalent:

(i) \( p = 2q + 1 \) or \( 2q - 1 \),
(ii) \( x_\alpha \sigma^2 (F_p) = x_\alpha \sigma (F_q) \) for some \( \alpha \geq 3 \),
(iii) \( x_\alpha \sigma^2 (F_p) = x_\alpha \sigma (F_q) \) for all \( \alpha \geq 3 \).

**Proof.** From Corollary 3.3 (i) implies (iii). Also it is clear that (ii) follows from (iii). Suppose that (ii) is true, and let \( p = 2q' + 1 \). Since
\[
F_p + 1 = F_{(q'+1)+q'} + F_{(q'+1)-q'} = \begin{cases} F_{q'} (F_{q'+2} + F_{q'}) & (q' \text{ is odd}), \\ F_{q'+1} (F_{q'+1} + F_{q'-1}) & (q' \text{ is even}) \end{cases}
\]
from (3),
\[
x_\alpha \sigma^2 (F_p) = x_\alpha \sigma (F_p + 1) \geq \begin{cases} x_\alpha \sigma (F_{q'}) & (q' \text{ is odd}), \\ x_\alpha \sigma (F_{q'+1}) & (q' \text{ is even}) \end{cases}.
\]
Thus
\[
q = \text{ind}_\alpha (F_q + 1) = \text{ind}_\alpha (x_\alpha \sigma (F_q)) = \text{ind}_\alpha (x_\alpha \sigma^2 (F_p)) \geq q' + \frac{1 + (-1)^{q'}}{2}.
\]
On the other hand,
\[
\text{ind}_\alpha (x_\alpha \sigma^2 (F_p)) = \text{ind}_\alpha (x_\alpha \sigma (F_{p+1})) \leq \frac{p+1}{2} = q' + 1
\]
from Theorem 4.4. If \( q' \) is even, then we have \( q = q' + 1 \), and hence \( p = 2q - 1 \). If \( q' \) is odd, then we get \( q' = q \) since \( q' \leq q \leq q' + 1 \) and \( q \) is odd. Consequently, we obtain that \( p = 2q + 1 \).

**Theorem 4.7.** For every natural number \( n \),
\[
\text{ord}_\alpha (n) \leq \frac{1}{\log 2} \log \left( \text{ind}_\alpha (n) \right) + 2.
\]

**Proof.** Let \( a \) be an arbitrary element in \( x_\alpha \sigma (\mathbb{N}) \), and take a natural number \( k \) such that \( 2^k \leq \text{ind}_\alpha (a) < 2^{k+1} \). From Theorem 4.4 we estimate that
\[
\text{ind}_\alpha (x_\alpha \sigma^k (a)) \leq \frac{1}{2} \text{ind}_\alpha (x_\alpha \sigma^{k-1} (a)) + \frac{1}{2} \leq \cdots \leq \left( \frac{1}{2} \right)^k \text{ind}_\alpha (a) + \sum_{m=1}^{k-1} \frac{1}{2^m} < 3.
\]
That is ind_\(\alpha\) (\(x_\alpha \sigma^k(a)\)) \(\leq 2\). For \(b \in x_\alpha \sigma(\mathbb{N})\), if ind_\(\alpha\) (\(b\)) = 2, then \(b\) is \(F_2 + F_1\). Since \(\alpha\) does not divide them, we get \(x_\alpha \sigma(b) = 1\). Therefore ind_\(\alpha\) (\(x_\alpha \sigma^{k+1}(a)\)) = 1. Since \(k \leq \log(\text{ind}_\alpha (a))/\log 2\), we find that
\[
\text{ord}_\alpha (a) \leq \frac{1}{\log 2} \log (\text{ind}_\alpha (a)) + 1.
\]
From this, for every \(n\),
\[
\text{ord}_\alpha (n) = \text{ord}_\alpha (x_\alpha \sigma (n)) + 1 \leq \frac{1}{\log 2} \log (\text{ind}_\alpha (x_\alpha \sigma (n))) + 2 \leq \frac{1}{\log 2} \log (\text{ind}_\alpha (n)) + 2.
\]
\(\square\)

Let us remember that ind_\(\alpha\) (\(n\)) is the maximal index of \(F_\alpha\) numbers \(\leq n\). Since \(F_\alpha\) is defined by the linear recurrence sequence whose coefficients are \(\geq 1\), it diverges exponentially. Therefore, we expect that ord_\(\alpha\) (\(n\)) \(\ll \log \log n\). In fact, we obtain the following more explicit inequalities.

**Theorem 4.8.** Let \(\alpha \geq 3\) be an integer. Then we have
\[
\text{ord}_\alpha (n) \begin{cases} 
= 0 & (n = 1), \\
< \frac{1}{\log 2} \log \log n + 3 & (n \geq 2).
\end{cases}
\]
In particular, if \(\alpha \geq 55\) and
\[
n > \exp \left( \frac{4}{\log \varphi_\alpha} \log \left( \sqrt{\alpha^2 + 4 + \frac{1}{\varphi_\alpha}} \right) \right),
\]
then we get
\[
(11) \quad \text{ord}_\alpha (n) < \frac{1}{\log 2} \log \log n.
\]
Moreover,
\[
\text{ord}_\alpha (n) \begin{cases} 
= 0 & (n = 1), \\
= 1 & (2 \leq n \leq 7), \\
< \frac{1}{\log 2} \log \log n & (n \geq 8)
\end{cases}
\]
holds at least in the case \(\alpha \geq 2981\).

**Proof.** We see that ord_\(\alpha\) (1) = 0 by definition. Hereafter let \(n \geq 2\) and \(k = \text{ind}_\alpha (n) \geq 1\). Since \(F_k \leq n\), we have
\[
k \leq \frac{1}{\log \varphi_\alpha} \log \left( n \sqrt{\alpha^2 + 4 + (-\varphi_\alpha)^{-k}} \right)
\]
by the argument in the proof of Theorem 2.6. From Theorem 4.7, we estimate that
\[
\text{ord}_\alpha (n) \leq \frac{1}{\log 2} \left( \log \log \left( n \sqrt{\alpha^2 + 4 + (-\varphi_\alpha)^{-k}} \right) - \log \log \varphi_\alpha \right) + 2
\]
\[
< \frac{1}{\log 2} \log \log n + \frac{1}{\log 2} \left( \log \left( 1 + \frac{1}{\log n} \log \left( \sqrt{\alpha^2 + 4 + \frac{1}{\varphi_\alpha}} \right) \right) - \log \log \varphi_\alpha \right) + 2.
\]
(12)

Here, we define
\[
f_\alpha (x) := \log \left( 1 + \frac{1}{\log x} \log \left( \sqrt{\alpha^2 + 4 + \frac{1}{\varphi_\alpha}} \right) \right) - \log \log \varphi_\alpha,
\]
\[
g_\alpha (x) := \exp (f_\alpha (x)) = \frac{1}{\log \varphi_\alpha} \left( 1 + \frac{1}{\log x} \log \left( \sqrt{\alpha^2 + 4 + \frac{1}{\varphi_\alpha}} \right) \right)
\]
with \(x \geq 2\). For real \(y > (\log \varphi_\alpha)^{-1}\), we have
\[
x > \exp \left( \frac{1}{y \log \varphi_\alpha} - \frac{1}{\log \left( \sqrt{\alpha^2 + 4 + \frac{1}{\varphi_\alpha}} \right)} \right).
\]
Denote by $A_\alpha(y)$ the right-hand side of this inequality. Then $A_\alpha(y)$ is decreasing with respect to $\alpha$. (We will prove this in Remark 4.9). Since $\varphi_3 = (3 + \sqrt{13})/2 > 3.3$, we have

$$\log A_\alpha(2) \leq \log A_3(2) = \frac{1}{2 \log \varphi_3 - 1} \log \left( \sqrt{13} + \varphi_3^{-1} \right) < 1$$

with $\alpha \geq 3$, that is $A_\alpha(2) < 3$. Therefore, we have $g_\alpha(x) < 2$ for $\alpha \geq 3$ and $x \geq 3$. Thus for every $\alpha \geq 3$,

$$\text{ord}_\alpha(n) < \frac{1}{\log 2} \log \log n + \frac{1}{\log 2} \log g_\alpha(n) + 2 < \frac{1}{\log 2} \log \log n + 3$$

with $n \geq 3$. In addition, this also holds the case $n = 2$ since $\text{ord}_\alpha(2) = 1$ and $\log \log 2/\log 2 + 3 \simeq 2.47$.

Let us next prove (11). It is enough to consider the domain of $\alpha \geq 3$ and $\alpha \geq 2$ which satisfies $f_\alpha(x)/\log 2 + 2 < 0$. Since this condition can be transformed into $g_\alpha(s) < 1/4$ it is enough to assume the condition $x > A_\alpha(1/4)$ for all sufficient large $\alpha$. Then $\log \varphi_\alpha > 4$, that is $\alpha \geq 55$. Thus for every $\alpha \geq 55$, we have

$$\text{ord}_\alpha(n) < \frac{1}{\log 2} \log \log n$$

with $x > A_\alpha(1/4)$. Since $A_\alpha(y)$ is decreasing in $\alpha$, $\alpha > A_\alpha(1/4)$ holds for all sufficiently large $\alpha \geq 55$. The lower bound is $\alpha \geq 2981$ from computer calculations, and hence for every $\alpha \geq 2981$ we get

$$\text{ord}_\alpha(n) < \frac{1}{\log 2} \log \log n$$

with $n \geq \alpha$. $\text{ord}_\alpha(n) = 1$ if $2 \leq n < \alpha$, and $\log \log n/\log 2 > 1$ for $n \geq 8$. This implies

$$\text{ord}_\alpha(n) \begin{cases} = 1 & (2 \leq n \leq 7), \\ < \frac{1}{\log 2} \log \log n & (n \geq 8). \end{cases}$$

\[ \square \]

**Remark 4.9.** We show that $A_\alpha(y)$ is decreasing monotonically in $\alpha$. By definition,

$$\log A_\alpha(y) = \frac{\log \sqrt{\alpha^2 + 4}}{y \log \varphi_\alpha - 1} + \frac{1}{y \log \varphi_\alpha - 1} \log \left( 1 + \frac{1}{\varphi_\alpha \sqrt{\alpha^2 + 4}} \right) =: B(\alpha) + C(\alpha)$$

say, with $y \log \varphi_\alpha > 1$. It is clear that $C(\alpha)$ is decreasing, and hence it is sufficient to discuss on $B(\alpha)$. For real $\alpha$ with $y \log \varphi_\alpha > 1$,

$$\frac{d}{d\alpha} B(\alpha) = \frac{1}{(y \log \varphi_\alpha - 1)^2} \left( \frac{2y\alpha}{\alpha^2 + 4} \log \varphi_\alpha - \frac{2\alpha}{\alpha^2 + 4} - \frac{y\varphi_\alpha'}{\varphi_\alpha} \log (\alpha^2 + 4) \right).$$

Since $\varphi_\alpha < \sqrt{\alpha^2 + 4}$, we estimate that the right-hand side of the above is

$$< \frac{1}{(y \log \varphi_\alpha - 1)^2} \left( \frac{2y\alpha}{\alpha^2 + 4} \log \sqrt{\alpha^2 + 4} - \frac{y\varphi_\alpha'}{\varphi_\alpha} \log (\alpha^2 + 4) \right)$$

$$= \frac{y \log (\alpha^2 + 4)}{(y \log \varphi_\alpha - 1)^2} \left( \frac{\alpha}{\alpha^2 + 4} - \frac{\varphi_\alpha'}{\varphi_\alpha} \right)$$

$$= \frac{y \log (\alpha^2 + 4)}{(y \log \varphi_\alpha - 1)^2} \left( \alpha (\alpha + \sqrt{\alpha^2 + 4}) - \left( 1 + \frac{\alpha}{\sqrt{\alpha^2 + 4}} \right) (\alpha^2 + 4) \right)$$

$$= -\frac{4y \log (\alpha^2 + 4)}{(y \log \varphi_\alpha - 1)^2} (\alpha + \sqrt{\alpha^2 + 4}) (\alpha + \sqrt{\alpha^2 + 4}) < 0.$$

Let us consider the case $\alpha = 55$.

**Remark 4.10.** Since $\log A_{55}(1/4) \simeq 2091.79$,

$$\text{ord}_{55}(n) < \frac{1}{\log 2} \log \log n$$

holds at least for $n > e^{2092}$.

In addition, we obtain the following corollary from Theorem 4.8.
Corollary 4.11. For every $\alpha \geq 3$,
\[
\limsup_{n \to \infty} \frac{\text{ord}_\alpha (n)}{\log \log n} \leq \frac{1}{\log 2}.
\]

Corollary 4.12.
\[
\lim_{\alpha \to \infty} \liminf_{n \to \infty} \left( \frac{1}{\log 2} \log \log n \right) - \text{ord}_\alpha (n) = +\infty.
\]

Proof. From (12), we have
\[
\liminf_{n \to \infty} \left( \frac{1}{\log 2} \log \log n - \text{ord}_\alpha (n) \right) \geq \frac{1}{\log 2} \log \varphi - 2.
\]

From Theorem 4.7, we have $\text{ord}_\alpha (F_p) \leq \log p/\log 2 + 2$ for prime $p$. In fact, there is quite a big difference between them, which can be observed by numerical tests (FIGURE 1, FIGURE 2). Thus, the author believes that the following theorem will be useful in the future.

**Figure 1.** $\text{ord}_3 (F_n)$ and $\log n/\log 2 + 2$ ($n \leq 80000$)

**Figure 2.** $\text{ord}_3 (F_n)/\log(n + 1)$ and $1/\log 2$ ($n \leq 80000$)
Theorem 4.13. Suppose \( \alpha \geq 3 \), and put \( C_\alpha := \limsup_{p \to \infty} \frac{\text{ord}_\alpha(F_p)}{\log p} \). If \( C_\alpha < 1/\log 2 \) for some \( \alpha \), then

\[
\limsup_{p \to \infty} \frac{l(p)}{\log p} \leq \frac{C_\alpha}{1 - C_\alpha \log 2}.
\]

Proof. Suppose that \( C_\alpha < 1/\log 2 \). For every \( 0 < \varepsilon < 1/\log 2 - C_\alpha \),

\[
\text{ord}_\alpha(F_p) < (C_\alpha + \varepsilon) \log p
\]

with sufficiently large \( p \). And

\[
\log \left( (p \pm 1) 2^{l_{\pm 1}(p) - 1} \right) = (l_{\pm 1}(p) - 1) \log 2 + \log p + O\left( \frac{1}{p} \right).
\]

We replace \( p \) by \( (p \pm 1) 2^{l_{\pm 1}(p) - 1} \) in (13). Then

\[
l(p) - 1 + \text{ord}_\alpha(F_p) < (C_\alpha + \varepsilon) \left( (l_{\pm 1}(p) - 1) \log 2 + \log p + O\left( \frac{1}{p} \right) \right)
\]

from Corollary 3.3 and hence we have

\[
l(p) < 1 + \frac{C_\alpha + \varepsilon}{1 - (C_\alpha + \varepsilon) \log 2} \log p + O\left( \frac{1}{p} \right).
\]

Divide the both sides by \( \log p \), and take limit superior with respect to \( p \). Then we find that

\[
\limsup_{p \to \infty} \frac{l(p)}{\log p} \leq \frac{C_\alpha + \varepsilon}{1 - (C_\alpha + \varepsilon) \log 2}.
\]

□

The sufficient condition of Theorem 4.13, written in terms of prime numbers, can be replaced by the condition written in terms of natural numbers.

Corollary 4.14. Suppose that \( \alpha \geq 3 \), and put \( D_\alpha := \limsup_{n \to \infty} \frac{\text{ord}_\alpha(n)}{\log \log n} \). If \( D_\alpha < 1/\log 2 \) for some \( \alpha \), then

\[
\limsup_{p \to \infty} \frac{l(p)}{\log p} \leq \frac{D_\alpha}{1 - D_\alpha \log 2}.
\]

Proof. For all natural \( n \),

\[
\log F_n = n \log \varphi + \log \left( \frac{1 - (-\varphi^2)^{-n}}{\sqrt{\alpha^2 + 4}} \right)
\]

from Fact 2.2 (a). In particular, \( \log \log F_n \sim \log n \). Then, for every \( \varepsilon > 0 \), we have \( \log p/\log \log F_p > 1 - \varepsilon \) with any sufficiently large \( p \). Thus we estimate that

\[
D_\alpha \geq \limsup_{p \to \infty} \frac{\text{ord}_\alpha(F_p)}{\log \log F_p} = \limsup_{p \to \infty} \frac{\text{ord}_\alpha(F_p)}{\log p} \cdot \frac{\log p}{\log \log F_p} \geq (1 - \varepsilon) \limsup_{p \to \infty} \frac{\text{ord}_\alpha(F_p)}{\log p}.
\]

Now, since \( \varepsilon > 0 \) is arbitrary, we get

\[
\limsup_{p \to \infty} \frac{\text{ord}_\alpha(F_p)}{\log p} \leq D_\alpha.
\]

Here we suppose \( D_\alpha < 1/\log 2 \) and apply Theorem 4.13 then the proof is complete. □

The advantage of this corollary is that the problem of upper estimation of \( l(p) \) is reduced to the situation that we can use number theoretic methods which cannot be applied to prime numbers.
5. Remaining Problems

In this paper, an experimentally reliable sufficient condition for \( l(p) \ll \log p \) was obtained using elementary methods that do not involve differentiation and integration. If we could successfully use analytical methods, perhaps we would obtain better estimation. For example, it is describable to find some analogy of \( \sigma \) with respect to \( F, \sigma^2, F, \sigma^3, \cdots \), or some non-trivial order of \( \sum \text{ord}_a(n) \). However, the difficulty lies in the fact that \( \text{ord}_a(n) \) is defined by the iterations of \( F, \sigma \). The iterations of the divisor function \( \sigma(n) \) and the Euler function \( \varphi(n) \) have been considered in \([10],[11],[12]\) and so on; however those researches seem to be possible because \( \sigma, \varphi \) are number theoretically easier to treat. Even though \( F, \sigma \) is not multiplicative, \( F_\alpha \) numbers has some nice properties related to multiplication, such as Fact \( \[2.2\](e),(f) \). In the future, it is expected that such properties will be used well to obtain further results on the function \( \text{ord}_a \).

Finally, we list the problems to be solved.

**Problem 5.1.**

\[
\limsup_{n \to \infty} \frac{\text{ord}_a(n)}{\log \log n} < \frac{1}{\log 2}
\]

It is shown in Theorem \[4.14\] that \( l(p) \ll \log p \) holds if this inequality is true. Here we recall Conjecture \[1.8\] which is not so far from the above inequality.

**Problem 5.2. Is there a sequence that is different from \( F_\alpha \) with “similar properties”?**

The term “similar properties” means those that are related to a chain of prime numbers.

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