I. INTRODUCTION

In high temperature QCD the self-energies of both quarks and gluons change the particle dispersion relations from their zero-temperature form. The modified dispersion relations indicate that the naturally propagating modes are collective excitations, i.e. quasiparticles. The locations of the poles in the propagators are gauge-fixing invariant. Braaten and Pisarski showed that for a consistent perturbation theory the propagators should not be free but should include part of the self-energies. With this hard-thermal-loop approximation to the quasiparticle effects it is possible to compute gauge invariant rates in which the corrections are controllably small, at least in certain kinematic regimes. A completely different test of the HTL quasiparticle dispersion relations comes from the excellent agreement with lattice data that is obtained by using an ideal gas of quark and gluon quasiparticles. When running coupling constants are incorporated, the agreement holds even very close to the critical temperature of QCD.

Quasiparticles are ubiquitous in finite-temperature calculations, and yet they normally play no role in the operator structure of the theory. This situation contrasts markedly with that of conventional particles at $T=0$. In particular, zero-temperature field theory has the following simple properties:

- The Hamiltonian operator has eigenstates $|\Psi\rangle$ in a linear vector space.
- Poles in the propagator occur at the eigenvalues of the single particle state vectors.
- Branch cuts in the propagator occur at the eigenvalues of the multiparticle state vectors.

This paper will describe a formalism in which these features also hold for finite-temperature quasiparticles. The present discussion will only treat scalar field theory.

Sections II and III summarize the results of a recent paper on branch cuts at finite temperature, which shows that the location of the cuts is determined by the quasiparticle energies. Section IV is based on a new formulation of finite-temperature field theory in which all the Bose-Einstein temperature dependence is contained in the Lagrangian. Green functions are computed as vacuum matrix elements rather than thermal traces. Sections V and VI show how to split the Lagrangian into an unperturbed part that describes free quasiparticles and a remainder that determines the interactions. Several illustrations are worked out. The developments in Sec. IV and subsequently may be read independently of the earlier sections.

II. BREAKDOWN OF PERTURBATION THEORY IN MASSIVE $\phi^3$ FIELD THEORY

Braaten and Pisarski showed that finite-temperature perturbation theory breaks down in field theories with massless particles or those in which the mass is much smaller than the temperature. Resummation of hard thermal loops is then necessary for Green functions evaluated at low energy and momentum.

This section will discuss a more primitive breakdown of perturbation theory that occurs even without soft external momenta and without high temperatures. The discussion will treat scalar field theory having cubic self-interactions,

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - \frac{g}{3!} \phi^3,$$

with counterterms omitted. Perturbation theory is conventionally defined by choosing free thermal propagators that have poles at the zero-temperature mass $m$. This produces two-particle branch points in the self-energy at $k^2 = 4m^2$ and three-particle cuts at $k^2 = 9m^2$. Simple calculations show that at finite temperature perturbation theory fails in the vicinity of these points. The reason is that they are not the true branch points of the full theory.
A. Zero-temperature example

A simple zero-temperature example will illustrate how higher order corrections can shift the location of branch cuts. Even though $m$ is the physical mass for the theory, suppose that one performs perturbative calculations using a free propagator $\Delta(k) = 1/[k^2 - m_0^2]$, where $m_0$ is some different mass. It is important that $m_0$ is finite and not the bare mass. For definiteness choose $m_0 < m$. The one-loop self-energy shown in Fig. 1 has a branch cut for $k^2 \geq 4m_0^2$. The discontinuity across the branch cut is

$$\text{Disc} \, \Pi'(k) = -\frac{i g^2}{16 \pi} (1 - \frac{4m_0^2}{k^2})^{1/2} \theta(k^2 - 4m_0^2).$$  \hfill (2)

Since $m_0$ has no physical significance, this is clearly not a branch cut of the full theory.

![FIG. 1. One-loop self-energy.](image)

The indication that $k^2 = 4m_0^2$ is not a branch point of the full theory comes from the two-loop contribution. The full propagator is $D'(k) = 1/[k^2 - m^2 - \Pi(k)]$ and by definition $\Pi$ contains the necessary counterterm to vanish at the true mass $k^2 = m^2$. To do perturbation theory with mass $m_0$ the full propagator is written $D'(k) = 1/[k^2 - m_0^2 - \Pi(k)]$ where $\Pi(k) = m^2 - m_0^2 + \Pi(k)$. Of course $\Pi$ does not vanish at $k^2 = m^2$ or at $k^2 = m_0^2$ and this is the source of the problem. A self-energy insertion on the internal lines of Fig. 1 gives the two-loop contribution shown in Fig. 2.

![FIG. 2. Two-loop self-energy due to one insertion.](image)

The corresponding self-energy is

$$\Pi^{(2)}(k) = ig^2 \int \frac{d^4 p}{(2\pi)^4} \left[ \Delta(p) \Pi(p) \Delta(p) \right] \Delta(p - k).$$  \hfill (3)

This has a two-particle and a three-particle discontinuity. The quantity in square brackets has a double pole at $p^2 = m_0^2$ since $\Pi(p)$ does not vanish at $p^2 = m_0^2$. This produces a two-particle discontinuity of the form

$$\text{Disc} \, \Pi^{(2)}(k) = \frac{ig^2}{8\pi} \left( \frac{\delta m^2/k^2}{(1 - 4m_0^2/k^2)^{1/2}} + \ldots \right) \theta(k^2 - 4m_0^2).$$

The correction is in infinite at $k^2 = 4m_0^2$ and large near there. This is a signal that the correct branch point is not at $k^2 = 4m_0^2$. Multiple self-energy insertions produce successively higher powers of the inverse square root:

$$\begin{align*}
\text{Disc} \, \Pi^{(2)}(k) &= \frac{ig^2}{16\pi} \left( (1 - \frac{4m_0^2}{k^2})^{1/2} - \frac{2\delta m^2}{k^2} (1 - \frac{4m_0^2}{k^2})^{-1/2} \\
&\quad - \frac{1}{2} \frac{2\delta m^2}{k^2} (1 - \frac{4m_0^2}{k^2})^{-3/2} + \ldots \right).
\end{align*}$$

This is the beginning of a Taylor series. All the corrections diverge at the false threshold $k^2 = 4m_0^2$. In the range $4m_0^2 < k^2 < 4m^2$ each correction is finite but the Taylor series diverges. Thus perturbation theory fails throughout this region of $k^2$. To obtain a convergent series it is necessary to work in the range $k^2 > 4m^2$. In this region the Taylor series converges and the sum is the full two-particle discontinuity with branch point shifted to the physical mass $m^2 = m_0^2 + \delta m^2$:

$$\text{Disc} \, \Pi(k) = \frac{ig^2}{16\pi} \left( 1 - \frac{4m_0^2}{k^2} \right)^{1/2} \theta(k^2 - 4m^2).$$  \hfill (4)

This has a two-particle threshold is still a square root branch point at $k^2 = 4m^2$. The breakdown of perturbation theory is entirely due to a propagator $\Delta(k) = 1/[k^2 - m_0^2]$ with the the wrong mass $m_0$. The breakdown is easily avoided by using $1/[k^2 - m^2]$ for the free particle propagator.

B. Breakdown of thermal perturbation theory

In the preceding example $m_0$ was the wrong mass. Finite temperature calculations are normally done using free propagators with the wrong mass, i.e. the zero-temperature mass $m$. A typical component of the conventional finite-temperature matrix propagator $\Delta_{ab}(k)$ is

$$\Delta_{++}(k) = \frac{1}{k^2 - m^2 + i\epsilon} - \frac{2\pi i \delta(k^2 - m^2)}{\exp(\beta|k_0|) - 1}.$$ 

The two-loop topology shown in Fig. 2 produces a matrix self-energy

$$\Pi^{(2)}_{aa}(k) = ig^2 \int \frac{d^4 p}{(2\pi)^4} \left[ \Delta_{ab}(p) \Pi^{(1)}_{bc}(p) \Delta_{cd}(p) \right] \Delta_{ad}(k-p).$$

As in Eq. (3) the quantity in square brackets has a double pole at $p^2 = m^2$ and is the source of the problem.
The self-energy has branch cuts at two-particle and three-particle thresholds as well as cuts due to absorption of one or more particles from the heat bath. The discontinuities across any of these is given by the cutting rules of Koby and Semenoff [3]. A particular linear combination of the \( \Pi_{\text{out}} \) gives the retarded self-energy. The formula for the discontinuity of the two-loop retarded self-energy is given explicitly in [3] in terms of the one-loop \( \Pi_R^{(1)} \). It is easily evaluated in the limit \( k^2 \to 4m^2 \) with the result

\[
\text{Disc} \Pi_R^{(2)}(k) = -\frac{i g^2}{16\pi} \frac{1 + 2n(k_0/2)}{k^2(1 - 4m^2/k^2)^{1/2}} \text{Re} \Pi_R^{(1)}(k_0/2).
\]

The correction is infinite at \( k^2 = 4m^2 \) indicating the breakdown of finite-temperature perturbation theory in this region. This breakdown is an artifact of using propagators with mass \( m \).

### III. Quasiparticle Resummation for \( \phi^3 \) Field Theory

To cure the breakdown of thermal perturbation theory described above it is necessary to use a free thermal propagator that has the same poles as the exact thermal propagator. The exact thermal propagator matrix may be expressed as a linear combination of the exact retarded and advanced thermal propagators. The exact retarded thermal propagator \( D_R^{(p)}(k) \) has the following properties:

(a) It is analytic for \( \text{Im} \ k_0 > 0 \); (b) It has a pole in the fourth quadrant at \( k_0 = \mathcal{E}(\bar{k}) \); (c) It has a pole in the third quadrant at \( k_0 = -\mathcal{E}^*(\bar{k}) \) because of the relation \( D_R^{(p)}(k_0, \bar{k}) = [D_R^{(p)}(-k_0, \bar{k})]^* \). It will be useful to have an explicit notation for the real and imaginary parts of \( \mathcal{E} \):

\[
\mathcal{E}(\bar{k}) = \mathcal{E}(\bar{k}) - i\Gamma(\bar{k})/2 \quad (E > 0; \Gamma > 0).
\]

The fourth-quadrant pole is thus at \( -\mathcal{E}^* = -E - i\Gamma/2 \). The functions \( E \) and \( \Gamma \) also depend on \( T, g, m \).

It is somewhat misleading to write the retarded propagator as \( D_R^{(p)}(k) = 1/[k^2 - m^2 - \Pi_R(k)] \) since there is no pole at \( k^2 = m^2 \). It is better to express it as

\[
D_R^{(p)}(k) = \frac{1}{(k_0 - \mathcal{E})(k_0 + \mathcal{E}^*) - \Pi_{\text{Rap}}(k)},
\]

where \( \Pi_{\text{Rap}}(k) = \Pi_R(k) + \mathcal{E}^2 + i\Gamma k_0 \). It is natural to treat \( \Pi_{\text{Rap}} \) perturbatively since in vanishes at \( k_0 = \mathcal{E} \) and at \( k_0 = -\mathcal{E}^* \). Consequently the free retarded quasiparticle propagator is

\[
D_R(k) = \frac{1}{(k_0 - \mathcal{E})(k_0 + \mathcal{E}^*)}.
\]

The exact advanced propagator is \( D_A^{(p)}(k) = D_A^{(p)}(-k) \) and has poles in the first quadrant at \( k_0 = \mathcal{E}^* \) and in the second quadrant at \( k_0 = -\mathcal{E} \). Consequently the free advanced quasiparticle propagator is

\[
D_A(k) = \frac{1}{(k_0 + \mathcal{E})(k_0 - \mathcal{E}^*)}.
\]

It is straightforward to reorganize the Feynman rules for perturbation theory so as to use the free quasiparticle propagators. The \( 2 \times 2 \) propagator matrix is

\[
D_{ab}(k) = \begin{pmatrix}
D_R(k) & 0 \\
0 & -D_A(k)
\end{pmatrix}
\]

\[
+ \left( D_R(k) - D_A(k) \right) n(k_0) \begin{pmatrix}
e^{\beta k_0} + 1 & -2e^{\sigma k_0} \\
-2e^{-(\beta - \sigma)k_0} & e^{\beta k_0} + 1
\end{pmatrix},
\]

where \( n(k_0) = 1/\left[ \exp(\beta k_0) - 1 \right] \) is the Bose-Einstein function with absolute value bars. This satisfies the KMS condition [4]. Whenever there is a self-energy insertion on a propagator (i.e. \( D \Pi D \)) then a counterterm \( \delta \Pi \) should be inserted where

\[
\delta \Pi_{ab}(k) = \begin{pmatrix}
\bar{k}^2 + m^2 & -\mathcal{E}^*\mathcal{E} \\
0 & 0
\end{pmatrix}
\]

\[
+ ik_0\Gamma n(k_0) \begin{pmatrix}
e^{\beta k_0} + 1 & -2e^{\sigma k_0} \\
-2e^{-(\beta - \sigma)k_0} & e^{\beta k_0} + 1
\end{pmatrix}.
\]

The identity

\[
D_{ab}^{-1}(k) - \delta \Pi_{ab}(k) = \begin{pmatrix}
k^2 - m^2 & 0 \\
0 & -k^2 + m^2
\end{pmatrix}
\]

guarantees that the two approaches are identical when summed to all orders.

#### A. One-loop retarded self-energy

Using the quasiparticle propagator [5] it is straightforward to compute the one-loop self-energy in Fig. 1. The retarded self-energy reduces to

\[
\Pi_R(k) = \frac{g^2}{2} \int \frac{d^3k_1 d^3k_2}{2E_1E_2} \delta^3(\vec{k}_1 + \vec{k}_2 - \vec{k})/(2\pi)^3
\]

\[
\times \left\{ \begin{array}{l}
\frac{1}{k_0 - \mathcal{E}_1 - \mathcal{E}_2} - \frac{1}{k_0 - \mathcal{E}_1 - \mathcal{E}_2^*} \\
\frac{1}{k_0 - \mathcal{E}_1 + \mathcal{E}_2} - \frac{1}{k_0 - \mathcal{E}_1 + \mathcal{E}_2^*} \\
\frac{1}{k_0 - \mathcal{E}_1^* - \mathcal{E}_2 - \mathcal{E}_2^*} - \frac{1}{k_0 - \mathcal{E}_1^* - \mathcal{E}_2 + \mathcal{E}_2^*} \\
\frac{1}{k_0 + \mathcal{E}_1^* + \mathcal{E}_2} - \frac{1}{k_0 + \mathcal{E}_1^* + \mathcal{E}_2 + \mathcal{E}_2^*}
\end{array} \right\}
\]
The statistical factors in the numerators of Eq. (11) describe the emission and absorption of quasiparticles, weighted by $1 + n(E)$ and $n(E)$ respectively, and also the emission and absorption of quasiholes, weighted by $1 + n(E^*)$ and $n(E^*)$. Each of the numerators describes a direct process minus the inverse process. The denominators are singular in the lower half-plane at the four locations

$$k_0 = \lambda(\vec{k}_1) + \lambda(\vec{k}_2)$$

$$\lambda = E - E^*.$$ (12)

All of these lie in the lower half-plane since $E$ and $-E^*$ have the same negative imaginary part. (There are also unphysical cuts related to Matsubara frequencies but these are cancelled by the two-loop contribution.) Eq. (11) can also be written as a dispersion integral over a spectral function as discussed in Eq. (3.124) of [10].

B. Two-loop retarded self-energy

The two-loop self-energy shown in Fig. 2 requires the counterterm matrix. The retarded part of the self-energy has both two-particle and three-particle cuts. The two-particle cuts are not singular (because the integrands contain no double poles) and have the same form as Eq. (11). The two-particle cuts are at

$$k_0 = \lambda(\vec{k}_2) + \lambda(\vec{k}_3) + \lambda(\vec{k}_4).$$ (13)

There are eight possibilities since $\lambda = E$ or $-E^*$, independently.

The two-loop retarded self-energy shown in Fig. 3 has a variety of branch cuts. It has two-particle cuts at

$$k_0 = \lambda(\vec{k}_1) + \lambda(\vec{k}_2)$$

and three-particle cuts at

$$k_0 = \lambda(\vec{k}_1) + \lambda(\vec{k}_3) + \lambda(\vec{k}_4).$$ (14)

and

$$k_0 = \lambda(\vec{k}_2) + \lambda(\vec{k}_3) + \lambda(\vec{k}_4).$$ (15)

A complete account of the two-loop calculations is given in [4].

IV. FIELD THEORY OF QUASIPARTICLES

Reorganizing the Feynman rules so as to use the free quasiparticle propagator $\tilde{L}$ accomplishes two things. First, the perturbative expansion does not break down near the branch points. Second, the branch points of individual diagrams are the correct branch points of the full theory. The original motivations given in Sec. I can now be restated more specifically:

- Do the quasiparticle Feynman rules come from an operator theory with a Hamiltonian that acts in a linear vector space?
- Are the poles in the retarded propagator at $k_0 = E(\vec{k})$ and at $k_0 = -E^*(\vec{k})^*$ eigenvalues of a Hamiltonian?
- Are the energies of two-particle branch cuts at $k_0 = \lambda(\vec{k}_1) + \lambda(\vec{k}_2)$, where $\lambda(\vec{k}) = E(\vec{k})$ or $-E^*(\vec{k})$, eigenvalues of states containing two quasiparticles?

The answer to all these questions is “yes”. However since the energies $\lambda(\vec{k})$ are temperature-dependent and complex the relevant Hamiltonian cannot be the usual one. Previous treatments have not used the complex energy pole to define quasiparticles, but instead have used a continuous mass spectrum [4, 11]. These approaches are not closely related to methods for calculation as it is unclear how to incorporate interactions.

The treatment of quasiparticles described below is based on a Lagrangian $\tilde{L}$ that enjoys two properties. First, it generates the conventional finite-temperature perturbation series with free propagator poles at $k^2 = m^2$. Second, by adding and subtracting the same terms the Lagrangian is written as $\tilde{L} = L_0 + L_I$ in which $L_0$ describes free quasiparticles with the propagator $\tilde{L}$ and $L_I$ generates the counterterm $\delta II$ and the self-interactions.

A. Lagrangian for free quasiparticles

The conjecture for the free quasiparticle Lagrangian is based on the fact that the matrix propagator $\tilde{L}$ has poles at four distinct locations: $E, -E^*, -E, E^*$. A field equation that is second order in the time derivative can only contain two of these energies. If the fields are self-adjoint, the only possible self-adjoint equations are

$$\bigg(i \frac{\partial}{\partial t} + E\bigg)(-i \frac{\partial}{\partial t} + E^*)\phi_R(t, \vec{k}) = 0$$

$$\bigg(i \frac{\partial}{\partial t} + E^*\bigg)(-i \frac{\partial}{\partial t} + E)\phi_A(t, \vec{k}) = 0.$$ (16)

The Lagrangian that produces these equations is

$$\tilde{L}_{00} = \int \frac{d^3k}{(2\pi)^3} \bigg[(-i \frac{\partial}{\partial t} + E^*)\phi_R(t, -\vec{k}) \bigg [(i \frac{\partial}{\partial t} - E)\phi_A(t, \vec{k}) \bigg].$$

Fig 3: Two-loop self-energy due to vertex correction.
The free quasiparticle action can be expressed as
\[
\Gamma(\vec{x} - \vec{x}') = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k} \cdot (\vec{x} - \vec{x}')} \Gamma(\vec{k})
\]
and similarly for \(\mathcal{E}^* \mathcal{E}(\vec{k})\). The Lagrangian becomes
\[
L_{00}(t) = \int d^3x \dot{\phi}_A(x)\dot{\phi}_R(x) + \int d^3x d^3x' \left[ -\phi_A(x)\mathcal{E}^* \mathcal{E}(\vec{x} - \vec{x}')\phi_R(x') \right. \quad \text{(18)}
\]
\[
\left. -\frac{1}{2}\phi_A(x)\Gamma(\vec{x} - \vec{x}')\phi_R(x') + \frac{1}{2}\phi_A(x)\Gamma(\vec{x} - \vec{x}')\phi_R(x') \right].
\]
The fields are all evaluated at time \(t = x_0 = x_0'\).

Before examining how to quantize \(L_{00}\), it is helpful to present an alternative argument that provides the relation between the fields \(\phi_{R/A}\) and the original fields. The free quasiparticle action may be written as
\[
\int dt L_{00} = \int \frac{d^4k}{(2\pi)^4} \phi_R(-k)(k_0 + \mathcal{E}^*)(k_0 - \mathcal{E})\phi_A(k).
\]
This form is closely related to the results of Aurenche and Becherrawy [12] and of van Eijck, Kobes, and van Weert [13,14] on transforming the matrix propagator \(\tilde{\mathcal{D}}\) to a skew-diagonal form. In the notation of [14] the propagator may be ‘diagonalized’ as
\[
D_{ab}(k) = V_{\gamma}^T(-k_0) \begin{pmatrix} 0 & D_A(k) \\ D_R(k) & 0 \end{pmatrix}_{\beta\gamma} V_{\gamma}(k_0). \quad \text{(19)}
\]
Latin letters \(a, b, c, \ldots\) denote the original \(\pm\) components; Greek letters \(\alpha, \beta, \gamma, \ldots\) denote \(R/A\). The matrix \(D_{\beta\gamma}\) has diagonal entries \(D_{RR} = D_{AA} = 0\) and off-diagonal entries \(D_{RA} \equiv D_A, D_{AR} \equiv D_R\). The matrix \(V_{\gamma}(k_0)\) has rows labelled by \(R/A\) and columns labelled by \(\pm\). The inverse of the center matrix \(D_{\beta\gamma}(k)\) in Eq. (19) is
\[
D^{-1}_{\alpha\beta}(k) = \begin{bmatrix} 0 \\ (k_0 - \mathcal{E}^*)(k_0 + \mathcal{E}) \end{bmatrix} \begin{bmatrix} 0 \\ (k_0 - \mathcal{E})(k_0 + \mathcal{E}^*) \end{bmatrix}.
\]
This is a Hermitian matrix. Using the doublet notation
\[
\phi_\alpha = \begin{pmatrix} \phi_R \\ \phi_A \end{pmatrix},
\]
the free quasiparticle action can be expressed as
\[
\int dt L_{00} = \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \phi_\alpha(-k)D^{-1}_{\alpha\beta}(k)\phi_\beta(k). \quad \text{(20)}
\]
This formulation of the action suggests that the usual fields \(\phi_b\) \((b = \pm)\) be related to the new fields \(\phi\gamma\) \((\gamma = R/A)\) by the matrix transformation
\[
\phi_b(k) \sim \phi_{\gamma b}(k)\mathcal{V}(k_0).
\]

The transformation is non-local in time,
\[
\phi_b(t, \vec{x}) \sim \int_{-\infty}^{\infty} dt' \phi_{\gamma}(t + t', \vec{x})\mathcal{V}_{\gamma b}(t'). \quad \text{(22)}
\]
and is certainly not a canonical transformation. The symbol \(\sim\) does not imply mathematical equality. However it will suggest the full Lagrangian (26) below.

### B. Full Lagrangian

The full Lagrangian will determine the evolution of the fields \(\phi_{R/A}\). It is helpful to examine the contour Lagrangian used in the conventional path integral formulation of real-time, finite-temperature field theory:
\[
\mathcal{L}^c = \frac{1}{2}(\partial_{\mu}\phi_+)(\partial^\mu\phi_+) - \frac{1}{2}m^2\phi_+^2 - g \frac{3}{3!}\phi_+^3 - \frac{1}{2}(\partial_{\mu}\phi_-)(\partial^\mu\phi_-) + \frac{1}{2}m^2\phi_-^2 + g \frac{3}{3!}\phi_-^3.
\]
Under the transformation (22) the quadratic terms of \(\mathcal{L}^c\) become
\[
(\partial_{\mu}\phi_R)(\partial^\mu\phi_A) - m^2\phi_R\phi_A.
\]
However under (22) the cubic powers \((\phi_\pm)^3\) contain products of the fields \(\phi_{R/A}\) at different times. To avoid this it is necessary to make a truncated Taylor series expansion
\[
\phi_\pm(x) = \sum_{t_0}^{N-1} \frac{\partial^t \phi_\gamma(x)}{\partial x_0^t} C_{\gamma t}^t,
\]
where the coefficients are temporal moments:
\[
C_{\gamma t}^t = \int_{-\infty}^{\infty} dt' \frac{t'^t}{t!} \mathcal{V}_{\gamma b}(t'). \quad \text{(24)}
\]
(The parameter \(a\) guarantees convergence later. It has positive real part. At the end of any calculation \(a \rightarrow 0\).) With the definition (23), cubic powers of \(\phi_\pm\) will only involve products of fields evaluated at the same time. Calculations must be done at arbitrary \(N\) and subsequently the limit \(N \rightarrow \infty\) is performed. In this limit the moments have the the property
\[
\lim_{a \rightarrow 0} \sum_{t=0}^{\infty} (ik_0)^{t} C_{t b}^t = \mathcal{V}_{\gamma b}(k_0), \quad \text{(25)}
\]
which will be useful later.

When the definition (23) is substituted into \(\mathcal{L}^c\) the Lagrangian will contain time derivatives up to order \(N-1\). Consequently there will be \(N-1\) canonical momenta. To construct the Hamiltonian requires inverting the relation between the highest time derivative of the field and the highest momentum. This inversion is not tractable if the highest time derivative is buried in the series definition of \(\phi_\pm\). The solution is to add a quadratic term containing
a time derivative of order \( N \). The full Lagrangian density is then
\[
\mathcal{L} = (\partial_{\mu} \phi_R)(\partial^{\mu} \phi_A) - m^2 \phi_R \phi_A - h \frac{\partial^N \phi_R}{\partial x_0^N} \frac{\partial^N \phi_A}{\partial x_0^N} - \frac{g}{3!}(\phi^+)^3 + \frac{g}{3!}(\phi_-)^3.
\] (26)
Because of the auxiliary term it is trivial to invert the relation between the highest time derivative and the \( N \)th canonical momentum. In order that \( h \) has no physical effects it will have to approach zero as \( N \to \infty \). The specific choice is
\[
h = h_0 N^{-N}.
\] (27)

C. Comments on quantization of \( \dot{L} \)

The Lagrangian density (26) has all time derivatives up through order \( N \). The construction of a Hamiltonian for such systems was developed by the mathematician Ostrogradski in 1850 and is discussed in the mechanics book by E.T. Whittaker [15]. For the full Lagrangian density the \( 2N \) canonical coordinates are the fields \( \phi_R/A \) and their time derivatives up through order \( N-1 \). There are \( 2N \) canonical momenta. The classical Poisson bracket and their time derivatives up through order \( N-1 \). There are \( 2N \) canonical momenta. The classical Poisson bracket is replaced by the quantum mechanical commutator in order to quantize.

To compute in perturbation theory it is necessary to choose the unperturbed Lagrangian. It must include the term proportional to \( h \) in order for all the canonical momenta to appear in the unperturbed theory. One choice for the unperturbed Lagrangian density would be the three quadratic terms on the first line of Eq. (26). This choice ultimately leads, in the limit \( N \to \infty \), to conventional Feynman rules with propagators containing poles at \( k^2 = m^2 \) as shown in [15].

For the unperturber operator theory to describe quasiparticles, a different choice of unperturbed Lagrangian is necessary:
\[
\dot{L}_0 = \dot{L}_{00} - h \frac{\partial^N \phi_R}{\partial x_0^N} \frac{\partial^N \phi_A}{\partial x_0^N}.
\] (28)
This also has \( 2N \) canonical coordinates and \( 2N \) canonical momenta. In the limit \( N \to \infty \) the free propagators will be \( D_R \) and \( D_A \). The full Lagrangian (26) is separated into \( \dot{L} = \dot{L}_0 + \dot{L}_I \), where the interaction term is
\[
\dot{L}_I = \int d^3x [- \nabla \phi_A \cdot \nabla \phi_R - m^2 \phi_A \phi_R - \frac{g}{3!}(\phi_-)^3 + \frac{g}{3!}(\phi_+)^3]
+ \int d^3x d^3x' \left[ \phi_A(x) E^* E(\vec{x} - \vec{x}') \phi_R(x') \right.
+ \frac{1}{2} \phi_A(x) \Gamma(\vec{x} - \vec{x}') \phi_R(x') - \frac{1}{2} \phi_A(x) \Gamma(\vec{x} - \vec{x}') \phi_R(x') \left. \right].
\] (29)
The quadratic terms in \( \dot{L}_I \) will produce the counter term matrix (9) and the cubic terms produce the interaction vertices.

V. QUANTIZATION OF \( L_{00} \)

Although the unperturbed Lagrangian \( \dot{L}_0 \) produces field equations with time derivatives of order \( 2N \), the vanishing of \( h \) according to Eq. (27) is chosen so that as \( N \to \infty \) the extra modes become infinitely heavy and decouple [16]. Consequently the essential physics of the free quasiparticles is contained in \( L_{00} \). It it therefore quite instructive to quantize \( L_{00} \) first. This will lead to a self adjoint Hamiltonian \( \dot{H}_{00} \) whose eigenvalues are the locations of the poles and cuts in the exact thermal propagators.

A. Field operators

The field equations that follow from \( \dot{L}_{00} \) are given in Eq. (16). The solutions can be expanded in terms of spatial Fourier transforms in a box of volume \( V \) with periodic boundary conditions:
\[
\phi_R(t, \vec{x}) = \sum_k \frac{1}{\sqrt{2EV}} \left( a_R^k e^{i \vec{k} \cdot \vec{x} - i\epsilon t} + a_R^{k \dagger} e^{-i \vec{k} \cdot \vec{x} + i\epsilon t} \right)
\] (30)
\[
\phi_A(t, \vec{x}) = \sum_k \frac{1}{\sqrt{2EV}} \left( a_A^k e^{i \vec{k} \cdot \vec{x} - i\epsilon t} + a_A^{k \dagger} e^{-i \vec{k} \cdot \vec{x} + i\epsilon t} \right).
\]
Note that \( \phi_R(t, \vec{k}) \sim \exp(+\Gamma t/2) \) rises exponentially with time and \( \phi_A(t, \vec{k}) \sim \exp(-\Gamma t/2) \) falls exponentially. The Hamiltonian will contain products of the two fields and will be time-independent. For a fixed wave vector \( \vec{k} \), one may interpret \( \phi_A(t, \vec{k}) \) and \( \phi_R(t, \vec{k}) \) as the coordinates of a damped harmonic oscillator an anti-damped harmonic oscillator, respectively. The oscillator problem, devoid of the field concept, has a long history. Bateman [16] in 1931 invented a Lagrangian of the form (18) to treat the classical problem of one damped and one anti-damped oscillator. Morse and Feshbach [15] also use this same oscillator Lagrangian. The quantum mechanics of the system is discussed in [16].

B. Canonical momenta and commutation relations

Since there are only first order time derivatives, the usual canonical formalism applies directly. The conjugate momenta are \( \pi_\alpha = \delta L_{00}/\delta \dot{\phi}_\alpha \):
\[
\pi_R(x) = \dot{\phi}_R(x) + \frac{1}{2} \int d^3x' \Gamma(\vec{x} - \vec{x}') \dot{\phi}_R(x')
\]
\[
\pi_A(x) = \dot{\phi}_A(x) - \frac{1}{2} \int d^3x' \Gamma(\vec{x} - \vec{x}') \dot{\phi}_R(x').
\]
The canonical commutation relations are
\[
[\pi_\alpha(x), \phi_\beta(y)]_{x_0 = y_0} = \frac{i}{\hbar} \delta_{\alpha \beta} \delta^3(\vec{x} - \vec{y}).
\] (31)
Consequently the raising and lowering operators satisfy
\[ [a^R_\vec{k}, a^{A\dagger}_\vec{k}'] = [a^A_\vec{k}, a^R_\vec{k}'] = \delta_{\vec{k},\vec{k}}. \] (32)

The other commutators vanish: \([a^R_\vec{k}, a^{A\dagger}_\vec{k}'] = [a^A_\vec{k}, a^R_\vec{k}'] = 0\) and \([a^R_\vec{k}, a^A_\vec{k}'] = 0\). This implies \([\phi_R(x), \phi_R(y)] = 0\) and \([\phi_A(x), \phi_A(y)] = 0\). Because the time dependence of \(\phi_R\) and \(\phi_A\), if these two field commutators did not vanish they would have to depend on both \(x_0\) and \(y_0\) (rather than \(x_0 - y_0\)) and would thus violate invariance under time translations. The only non-vanishing commutator of two fields is \([\phi_A(x), \phi_R(y)]\) and it automatically satisfies translational invariance.

### C. Hamiltonian \(\hat{H}_0\)

The Hamiltonian \(\hat{H}_{00} = \int d^3x \left[ \pi^R \dot{\phi}_R + \pi^A \dot{\phi}_A \right] - \hat{L}_{00}\) describes free quasiparticles. In terms of canonical variables it is
\[
\hat{H}_{00} = \int d^3x d^3x' \left[ \pi_A(x') \delta(x - x') \pi_R(x') \right. \\
\left. - \frac{1}{2} \pi_A(x') \Gamma(x - x') \phi_A(x') \right. \\
\left. + \frac{1}{2} \phi_R(x') \Gamma(x - x') \pi_R(x') \right. \\
\left. + \phi_R(x') E^2(x - x') \phi_A(x') \right].
\]
The kernel \(E^2(x - x')\) is the Fourier transform of \(E^2(\vec{k})\) analogous to (17). When the plane wave expansions (30) are substituted the Hamiltonian reduces to
\[
\hat{H}_{00} = \sum \vec{k} \left[ E (a^R_{\vec{k}} a^{A\dagger}_{\vec{k}} + \frac{1}{2}) + E^* (a^{A\dagger}_{\vec{k}} a^R_{\vec{k}} + \frac{1}{2}) \right],
\] (33)
which is manifestly self-adjoint. The complete analysis of this Hamiltonian is rather lengthy. The results of that analysis will be summarized here but not proven. The essential feature is that there are two parallel sets of eigenstates. One set is built upon a ground state \(|0_R\rangle\); the other, on a ground state \(|0_A\rangle\). The retarded vacuum state \(|0_R\rangle\) is defined by the conditions
\[
a^A_{\vec{k}} |0_R\rangle = 0 \quad a^{A\dagger}_{\vec{k}} |0_R\rangle = 0. \] (34)
(Recall that \(a^A\) and \(a^{A\dagger}\) commute.) The retarded vacuum is an eigenstate of \(\hat{H}_{00}\). (Since \(\hat{H}_{00}\) is not normal ordered, the eigenvalue is not zero.) The simplest excitations of the vacuum are the addition of one quasiparticle, \(a^R_{\vec{k}} |0_R\rangle\), or the addition of one quasihole, \(a^{R\dagger}_{\vec{k}} |0_R\rangle\). From the commutation relations (32) these operators shift the energy by \(E\) or \(-E^*\), respectively:
\[
[H_{00}, a^R_{\vec{k}}] = E a^R_{\vec{k}} \quad [H_{00}, a^{R\dagger}_{\vec{k}}] = -E^* a^{R\dagger}_{\vec{k}}.
\]
Note that both \(E\) and \(-E^*\) have the same negative imaginary part. The next excitations are those with two quasiparticles, \(a^R_{\vec{k}} a^R_{\vec{k}'} |0_R\rangle\), with one quasiparticle and one quasihole, \(a^R_{\vec{k}} a^{R\dagger}_{\vec{k}'} |0_R\rangle\), or with two quasiholes, \(a^{R\dagger}_{\vec{k}} a^{R\dagger}_{\vec{k}'} |0_R\rangle\). Every state of the form \((a^R) m |0_R\rangle\) is of the retarded type and has an energy with a negative imaginary part. (The real part of the energy may be of either sign.)

There is another set of states that are excitations of the advanced vacuum state \(|0_A\rangle\), which is defined by the conditions
\[
a^R_{\vec{k}} |0_A\rangle = 0 \quad a^{R\dagger}_{\vec{k}} |0_A\rangle = 0. \] (35)
The advanced representation of the one quasiparticle excitation is \(a^{A\dagger}_{\vec{k}} |0_A\rangle\) and of the one quasihole excitation, \(a^A_{\vec{k}} |0_A\rangle\). From the commutation relations (32) these operators shift the energy by \(E^*\) or \(-E\), respectively:
\[
[H_{00}, a^{A\dagger}_{\vec{k}}] = E^* a^{A\dagger}_{\vec{k}} \quad [H_{00}, a^A_{\vec{k}}] = -E a^A_{\vec{k}}.
\]
Both \(E^*\) and \(-E\) have the same positive imaginary part. The advanced representation of a general excitation is \((a^{A\dagger}) m |0_A\rangle\) and has an energy with a positive imaginary part.

As explained in Appendix A, the Hamiltonian does not act in a Hilbert space. None of the retarded states \(|\Psi_R\rangle\) have finite norm; none of the advanced states \(|\Psi_A\rangle\) have finite norm. However the scalar product between retarded and advanced states, \(|\Psi_A\rangle |\Psi_R\rangle\), is always finite. For example, \(|0_A\rangle a^A_{\vec{k}} a^{R\dagger}_{\vec{k}} |0_R\rangle = 1\) and \(|0_A\rangle a^{A\dagger}_{\vec{k}} a^R_{\vec{k}} |0_R\rangle = -1\). Physical applications always involve matrix elements of this form. The completeness relation,
\[
1 = \sum \Psi_R \langle \Psi_A |,
\] (36)
is useful in evaluating matrix elements of the form \(|\langle 0_A | X Y | 0_R \rangle\) as shown in the next section.

### VI. SAMPLE RESULTS

It is not possible in the allotted space to give a full discussion of this formalism. One of the important features that will be used below is that all the temperature-dependence is contained in the full Lagrangian (26). Consequently propagators, Green functions, and the free energy are computed as vacuum matrix elements; there are no thermal traces.

#### A. Free propagators for \(\phi_{R/A}\)

The quasiparticle pole in the propagator comes directly from the eigenstate of the Hamiltonian \(\hat{H}_{00}\). The free retarded propagator is defined as a time-ordered product:
only one time-ordering survives on the second line because \(\langle 0_A|\phi_B(0)\phi_A(x)|0_R\rangle = 0\). When the plane wave solutions (30) are inserted, the result is
\[
D_{AR}(x) = -i\theta(x_0)\langle 0_A|\phi_A(x)\phi_R(0)|0_R\rangle.
\]
(37)

The four-dimensional Fourier transform is
\[
D_{AR}(k) = \frac{1}{(k_0 - E)(k_0 + E^*)}.
\]
(38)

Thus the operator theory produces the quasiparticle propagator as desired. Similarly, the free advanced quasiparticle propagator is
\[
D_{RA}(x) = -i\theta(-x_0)\langle 0_A|\phi_A(0)\phi_R(x)|0_R\rangle
\]
(39)

Comparison with (37) shows that \(D_{RA}(x) = D_{AR}(-x)\). Thus \(D_{RA}(k) = D_{AR}(-k)\). Since \(\langle 0_A|\phi_R\phi_R|0_R\rangle = 0\) and \(\langle 0_A|\phi_A\phi_A|0_R\rangle = 0\) the matrix propagator in the retarded/advanced basis is
\[
D_{\alpha\beta}(k) = \begin{pmatrix} 0 & D_{RA}(k) \\ D_{AR}(k) & 0 \end{pmatrix}
\]
(40)
in agreement with Eq. (19).

### B. Free propagators for \(\phi_\pm\)

Although the complex quasiparticle energy \(\mathcal{E}(\vec{k})\) is temperature-dependent, there are no Bose-Einstein functions in the retarded or advanced propagators (40). However, the Bose-Einstein functions should appear in the propagators for \(\phi_\pm\) in order to agree with Eq. (8). The \(\phi_\pm\) propagators are vacuum matrix elements:
\[
G_{ab}(x-y) = -i\langle 0_A|T(\phi_a(x)\phi_b(y))|0_R\rangle,
\]
(41)

where \(a, b = \pm\). To compute this properly requires quantizing not \(\hat{L}_0\) but \(\hat{\Phi}\) given in Eq. (28). I will provide the essential feature of that quantization at the appropriate point below.

Using the definition (23) of the \(\phi_\pm\) as a sum of the first N-1 time derivatives of \(\phi_R\) and \(\phi_A\) gives for \(G_{ab}\)
\[
G_{ab}(x-y) = -i\sum_{\ell, m=0}^{N-1} \langle 0_A|T(\frac{\partial^\ell \phi_A}{\partial x^\ell_0} \frac{\partial^m \phi_A}{\partial y^m_0})|0_R\rangle C_{\beta\alpha} C_{\gamma\beta}^m
\]

When \(\phi_{R/A}\) are quantized according to \(\hat{L}_0\) the time derivatives may be taken outside of the time-ordering so that
\[
G_{ab}(x-y) = \sum_{\ell, m=0}^{N-1} \frac{\partial}{\partial x^\ell_0} \frac{\partial}{\partial y^m_0} G_{\beta\gamma}(x-y) C_{\beta\alpha} C_{\gamma\beta}^m.
\]

The Fourier transform is
\[
G_{ab}(k) = \sum_{\ell, m=0}^{N-1} (-ik_0)^\ell C_{\beta\alpha} \left[ G_{\beta\gamma}(k) \right] \sum_{m=0}^{N-1} (ik_0)^m C_{\gamma\beta}^m
\]

As the number of derivatives goes to infinity the identity (25) allows the moments to be summed:
\[
\lim_{N \to \infty} G_{ab}(k) = V_{ab} \langle 0_A|T(\phi_b^2(x)\phi_b^2(y))|0_R\rangle.
\]

This is exactly the result (13) for the matrix propagator of quasiparticles. The Bose-Einstein functions are contained in \(V(k_0)\).

### C. One-loop self-energy

There is a pleasant surprise when we examine the interacting propagator. The one-loop self-energy, for example, is
\[
\Pi_{bc}(x-y) = -\frac{i\gamma^2}{4} \sigma_\alpha \sigma_c \langle 0_A|T(\phi_b^2(x)\phi_c^2(y))|0_R\rangle.
\]
(42)

One way to compute this is to use Wick’s theorem to express the matrix element as \([D_{bc}(x-y)]^2\). Computing \(\Pi_{bc}(k)\) then requires integrating the one-loop diagram Fig. 1 with quasiparticle propagators. The result is already given in Eq. (11).

The operator expression (42) can be evaluated in another manner that explains all the branch cuts displayed in Eq. (11). The quasiparticle Hamiltonian determines the time-dependence of the field operators:
\[
\phi_0(x) = e^{i\hat{H}_{00} x_0} \phi_0(\vec{x}) e^{-i\hat{H}_{00} x_0}
\]
as can be easily verified. For either of the two time-orderings in Eq. (42) a complete set of states may be inserted between the operators using the completeness relation (36). The states are energy eigenstates satisfying \(\hat{H}_{00}|\Psi_R\rangle = |\Psi_R\rangle \mathcal{E}_R^\Psi\) and \(\hat{H}_{00}|\Psi_A\rangle = |\Psi_A\rangle \mathcal{E}_A^\Psi\) with \(\mathcal{E}_A^\Psi = \mathcal{E}_R^\Psi\). The matrix element in Eq. (42) has the value
\[
\sum_{\Psi} \langle \theta(y_0 - x_0)|0_A|\phi_b^2(\vec{x})|\Psi_R\rangle \langle \Psi_A|\phi_b^2(\vec{y})|0_R\rangle e^{-i\mathcal{E}_R^\Psi (x_0-y_0)} + \theta(y_0 - x_0)|0_A|\phi_b^2(\vec{y})|\Psi_R\rangle \langle \Psi_A|\phi_b^2(\vec{x})|0_R\rangle e^{i\mathcal{E}_R^\Psi (x_0-y_0)}.
\]

Since \(\text{Im} \mathcal{E}_R^\Psi < 0\) the time dependence is exponentially falling in both terms. The temporal Fourier transform to \(k_0\) at fixed \(\vec{x}\) and \(\vec{y}\) is
\[
\Pi_{bc} = \frac{q^2}{4} \int_{\sigma_c} \sum_{\Psi} \left( \frac{\langle 0_A|\phi_b^2(\vec{x})|\Psi_R\rangle \langle \Psi_A|\phi_b^2(\vec{y})|0_R\rangle}{k_0 - \mathcal{E}_R^\Psi} - \frac{\langle 0_A|\phi_b^2(\vec{y})|\Psi_R\rangle \langle \Psi_A|\phi_b^2(\vec{x})|0_R\rangle}{k_0 + \mathcal{E}_R^\Psi} \right).
\]
(43)
The operators $\phi^0_k$ only connect the vacuum with states containing two excitations (i.e., two quasiparticles, one quasiparticle and one quasihole, or two quasiholes). Thus the possibilities are

$$\begin{align*}
|\Psi_R\rangle & \quad \mathcal{E}^R_\Psi \\
\alpha^R_{k_1} \alpha^R_{k_2} |0_R\rangle & \quad \mathcal{E}^R_{k_1} + \mathcal{E}^R_{k_2} \\
\alpha_{k_1} \alpha_{k_2} |0_R\rangle & \quad \mathcal{E}^R_{k_1} - \mathcal{E}^R_{k_2} \\
\alpha^R_{k_1} \alpha_{k_2} |0_R\rangle & \quad -\mathcal{E}^R_{k_1} + \mathcal{E}^R_{k_2} \\
\alpha_{k_1} \alpha^R_{k_2} |0_R\rangle & \quad -\mathcal{E}^R_{k_1} - \mathcal{E}^R_{k_2} 
\end{align*} \tag{44}$$

The four terms in $\Pi_{bc}$ with denominators $k_0 - \mathcal{E}^R_\Psi$ all have branch cuts in the lower-half of the complex $k_0$ plane. These produce the four branch cuts previously found in Eq. (11). The retarded self-energy is the particular linear combination of the $\Pi_{bc}$ which contains only these four cuts. The four terms in Eq. (43) with denominators $k_0 + \mathcal{E}^R_\Psi$ all have branch cuts in the upper-half of the complex $k_0$ plane and produce the advanced self-energy.

The numerators in Eq. (43) may be evaluated using relations (23) and (25) and result in the correct Bose-Einstein functions $n(\mathcal{E})$ and $n(\mathcal{E}^*)$ in agreement with Eq. (11).

### VII. DISCUSSION

The treatment presented here is based on the fact that field theories at finite temperature inevitably have poles in the full propagator at complex, temperature-dependent energies $\mathcal{E}(\vec{k})$. To consistently include quasiparticles in Feynman diagram computations it is only necessary to use the quasiparticle propagator (8) and the counter-term (9). To go further and incorporate the quasiparticle effects into the operator structure requires the temperature-dependent Lagrangian (26). With this Lagrangian the questions posed at the beginning of Sec. IV may all be answered affirmatively.

The logical necessity of this development is rather tight. As shown in Sec. II, higher loop corrections will result in a breakdown of perturbation theory unless the free propagators are chosen to have the correct quasiparticle poles. The free quasiparticle propagator shown in Eq. (8) must then contain the off-shell Bose-Einstein function $n(k_0)$ in order to satisfy the KMS condition. (With quasiparticle poles, neither the quasiparticle energy $\mathcal{E}(\vec{k})$ nor the T=0 energy $E_0(\vec{k}) = (k^2 + m^2)^{1/2}$ can be the argument of the Bose-Einstein function.) The off-shell $n(k_0)$ leads to the time-dependent relation (22), or the more precise relation (23), between the original fields and $\phi_{R/A}$. That fixes the Lagrangian to be (26).

The standard operator formulation of finite temperature field theory is thermofield dynamics. The unperturbed propagators in thermofield dynamics have poles at $k^2 = m^2$. The matrix propagators contain the Bose-Einstein function with the T=0 energy as argument: $n(E_0)$. Consequently the transformation from the original fields $\phi_\pm$ to the TFD fields $\phi$ and $\bar{\phi}$ depends on space rather than on time. The transformation is automatically canonical. See Appendix B of [5]. (When TFD is extended to nonequilibrium situations, the transformation is time-dependent.)

In both formulations the number of degrees of freedom is doubled. In thermofield dynamics the operators act in a Hilbert space. The basis states are built out of two commuting raising operators, $a^A$ and $a^\dagger$. In the approach described here, operators do not act in a Hilbert space, as discussed in Appendix A. Furthermore, there are two parallel sets of basis states $|\Psi_R\rangle$ and $|\Psi_A\rangle$. The former are built from commuting raising operators $a^R$ and $a^\dagger$ acting on $|0_R\rangle$; the latter from $a^A$ and $a^\dagger$ acting on $|0_A\rangle$.

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### APPENDIX A: COMPLEX EIGENVALUES FROM A SELF-ADJOINT HAMILTONIAN

States in a Hilbert space have finite norms and consequently self-adjoint operators cannot have complex eigenvalues. The state space for quasiparticles is not a Hilbert space. A retarded eigenstate of the quasiparticle Hamiltonian satisfies

$$\hat{H}_{00}|\Psi_R\rangle = |\Psi_R\rangle \mathcal{E}^R.$$ \tag{A1}

Since $\hat{H}_{00}$ is self-adjoint, the adjoint of this equation is $\langle \Psi_R|\hat{H}_{00} = \mathcal{E}^{R*}\langle \Psi_R|$. The inner product of this with $|\Psi_R\rangle$ would force $\mathcal{E}^R$ to be real. If the norm $\langle \Psi_R|\Psi_R\rangle$ existed. As shown below, this norm does not exist. Advanced eigenstates satisfy $\hat{H}_{00} |\Phi_A\rangle = |\Phi_A\rangle \mathcal{E}^A$. The scalar product of this with $\langle \Psi_R|\Phi_A\rangle$ does exist and implies

$$\langle \Psi_R|\Phi_A\rangle \mathcal{E}^{R*} = \langle \Psi_R|\Phi_A\rangle \mathcal{E}^A.$$ \tag{A2}

Thus if $\mathcal{E}^{R*} \neq \mathcal{E}^A$ then the states are orthogonal.

A concrete realization of the quasiparticle algebra is illuminating. The four operators $a^A$, $a^A\dagger$, $a^R$, $a^R\dagger$ may be represented in a two-dimensional real space as linear combinations of $x$, $\partial/\partial x$, $y$, $\partial/\partial y$ as follows

$$\begin{align*}
a^A + a^A\dagger &= x - i \frac{\partial}{\partial y} \\
a^A - a^A\dagger &= \frac{\partial}{\partial x} + iy \\
a^R + a^R\dagger &= x + i \frac{\partial}{\partial y} \\
a^R - a^R\dagger &= \frac{\partial}{\partial x} - iy \tag{A3}
\end{align*}$$
These satisfy the commutation relations (32). The free quasiparticle Hamiltonian (33) in this representation is

\[ \hat{H}_{00} = \frac{E}{2} \left( -\frac{\partial^2}{\partial x^2} + x^2 + \frac{\partial^2}{\partial y^2} - y^2 \right) + \frac{\Gamma}{2} \left( \frac{\partial^2}{\partial x \partial y} + xy \right). \]

The retarded vacuum state, defined by the conditions

\[ a^A|0_R\rangle = a^A|0_R\rangle = 0, \]

has the representation

\[ \langle \vec{x}|0_R\rangle = \exp(-ixy)/\sqrt{\pi}, \]  

which is not square integrable. The wave function is an eigenstate of \( H_{00} \) with eigenvalue \(-\Gamma/2\). Application of the quasiparticle creation operator \( a^R \) and the quasihole creation operator \( a^R \) give excited state wave functions

\[ \langle \vec{x}|(a^R)^l(a^R)^m|0_R\rangle = \int dx dy \exp(-\sqrt{\pi} iyx) H_l(x+iy) H_m(x-iy), \]

where \( H_l \) and \( H_m \) are Hermite polynomials. None of these are square integrable. These are eigenstates of \( H_{00} \) with eigenvalue \( \mathcal{E}(l+1/2) - \mathcal{E}(m+1/2). \)

The advanced vacuum state is defined by \( a^R|0_A\rangle = a^R|0_A\rangle = 0 \) and has the representation

\[ \langle \vec{x}|0_A\rangle = \exp(2\pi iy)/\sqrt{\pi}. \]  

It is not square integrable. However the scalar product of the advanced and retarded vacuum states is finite:

\[ \langle 0_A|0_R\rangle = \int dx dy \frac{\exp(-2\pi iy)}{\pi} = 1. \]  

The excited state wave functions are

\[ \langle \vec{x}|(a^A)^l(a^A)^m|0_A\rangle = \int dx dy \frac{\exp(2\pi iy)}{\pi} H_l(x-iy) H_m(x+iy). \]

All scalar products \( \langle \Phi_A|\Psi_R\rangle \) are finite.

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