Smooth projective horospherical varieties with nef tangent bundles

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Abstract

We show that smooth projective horospherical varieties with nef tangent bundles are rational homogeneous spaces.

Keywords: Horospherical varieties; Campana-Peternell conjecture; VMRT.

1 Introduction

We work over the field of complex numbers. A famous conjecture of Hartshorne proved by Mori [16] in 1979 shows that projective spaces are the only smooth projective varieties with ample tangent bundles. A natural question is whether there is a similar characterization for smooth projective varieties with some semipositive conditions. Demailly, Peternell and Schneider [7] proved that if $X$ is a compact Kähler manifold with nef tangent bundle, then there is a finite étale cover $	ilde{X}$ such that the Albanese map $\alpha : \tilde{X} \to A(\tilde{X})$ is a smooth fibration with fibers being Fano manifolds with nef tangent bundles. The question above will be answered if the following conjecture of Campana and Peternell [5] holds.

Conjecture 1.1 (Campana-Peternell Conjecture). Smooth projective Fano varieties with nef tangent bundles are rational homogeneous spaces.

From now on, we simply call a Fano manifold with nef tangent bundle a CP-manifold. Demailly, Peternell and Schneider [7] confirm Campana-Peternell Conjecture for Fano manifolds with dimension at most three. Through the works of Campana and Peternell [5], Mok [15], Hwang [9], Watanabe [24, 25], Kanemitsu [10, 11], Campana-Peternell Conjecture has been proved for Fano manifolds $X$ with $\dim(X) \leq \rho(X) + 4$, where $\rho(X)$ is the Picard number of $X$. The case when all the elementary Mori contractions of the manifold are smooth $\mathbb{P}^1$-fibrations have been solved by the series work of Muñoz, Occhetta, Solá Conde, Watanabe and Wiśniewski in [17] and [19]. In particular, [19] characterizes complete flag variety as the Fano manifolds all of whose elementary Mori contractions are smooth $\mathbb{P}^1$-fibrations.

In higher dimension very little is known except for the complete flag varieties cases. In fact, it is not known whether Campana-Peternell conjecture is true for quasi-homogeneous varieties. The aim of this paper is to verify the Campana-Peternell Conjecture for an important class of quasi-homogeneous varieties. We will show the following

Theorem 1.2. Smooth projective horospherical varieties with nef tangent bundles are rational homogeneous spaces.
The set of colors of $X$ is called $G$-horospherical variety if there exists a point $x_0 \in X$ such that $G \cdot x_0$ is an open $G$-orbit on $X$ and the isotropy group $G_{x_0} \supseteq R_u(B)$. In this situation, we also say $X$ is a horospherical $G/H$-embedding, where $H = G_{x_0}$. Irreducible $B$-stable divisors having nonempty intersection with the open $G$-orbit are called colors of $X$. Denote by $\mathfrak{D}(G/H)$ the set of colors of $X$.

Let $X$ be a $G$-horospherical variety. Let $x_0 \in X$ and $H \subseteq G$ be as in the definition above. Then the normalizer $N_G(H)$ of $H$ is a parabolic subgroup of $G$ (see [20, Proposition 2.2]). Denote by $I$ the subset of $S$ such that $N_G(H) = P_I$. From the Bruhat decomposition on $G/P_I$, we can get a natural bijective correspondence $S \setminus I \rightarrow \mathfrak{D}(G/H)$, $\alpha \mapsto D_{\alpha}$. When there is no confusions, we do not distinguish $S \setminus I$ and $\mathfrak{D}(G/H)$.

Denote by $M_{G/H} = \mathcal{X}(P_I/H)$. The character group $M_{G/H}$ is naturally a subgroup of $\mathcal{X}(B)$, and it is isomorphic to $\mathbb{C}(X)^{(B)}/\mathbb{C}^*$. Let $N_{G/H} = \text{Hom}(M_{G/H}, \mathbb{Z})$. For any $G$-orbit $Z$ on $X$, denote by $\mathfrak{C}_Z$ the associated cone in $(N_{G/H})$, and $\mathfrak{D}_Z = \{ D \in \mathfrak{D}(G/H) \mid Z \subseteq D \}$. Denote by $\mathcal{F}_X$ the associated colored fan of $X$, i.e., $\mathcal{F}_X = \{ (\mathfrak{C}_Z, \mathfrak{D}_Z) \mid Z \text{ is a } G\text{-orbit on } X \}$. Denote by $\mathfrak{D}_X = \bigcup \mathfrak{D}_Z$, where $Z$ runs over the set of $G$-orbits on $X$.

The following lemma is collected from [1, Proposition 2.4] and [13, Corollary 7.8].
Lemma 2.2. Let $X$ be a horospherical $G/H$-embedding. Assume $H \supseteq R_u(B)$. Denote by $I$ the subset of $S$ such that $P_I = N_G(H)$. Take a $G$-orbit $Z$ on $X$. Then the following hold.

(a) $M_{\xi z} \subseteq \mathcal{X}(P_I \cup D_z)$, where $M_{\xi z} = \{ \chi \in M_{G/H} \mid \forall v \in \mathcal{E}_z, \langle \chi, v \rangle = 0 \}$.

(b) $Z$ is $G$-equivariantly to $G/K$, where $K = \text{Ker}_{P_I \cup D_z} M_{\xi z} \supseteq H \supseteq R_u(B)$.

(c) $M_{\xi z} = M_{G/K}$ and $N_G(K) = P_I \cup D_z$.

(d) $\dim(Z) = \text{rank}(M_{\xi z}) + \dim(G/P_I \cup D_z)$.

Denote by

$$\mathcal{D}_0(G/H) = \{ D \in \mathcal{D}(G/H) \mid \forall f \in \mathcal{C}(X)^{(B)}, \nu_D(f) = 0 \}.$$ 

The following lemma is collected from Proposition 4.6, Theorem 4.7, Lemma 7.17, and Corollary 7.23 in [13]. Note that for any parabolic subgroup $P$ of $G$ containing $B$, the intersection $P \cap P^-$ is a connected reductive algebraic group.

Lemma 2.3. Let $X$ be a smooth projective horospherical $G/H$-embedding, where $H \supseteq R_u(B)$ and $N_G(H) = P_I$.

(i) Set $D_0 = \sum_{D \in \mathcal{D}_0(G/H)} D$, and $J$ the subset of $S$ corresponding to $\mathcal{D}_0(G/H)$. Then the linear system $|D_0|$ is base point free, and it induces a $G$-equivariant morphism $\pi : X \to G/P_{S \backslash J}$.

(ii) The Mori cone $\mathcal{N}(X) = F \times F'$, where $F$ is the extremal face corresponding to the Mori contraction $\pi$ and $F'$ is another extremal face.

(iii) Set $L = P_{S \backslash J} \cap P_{S \backslash J}^\circ$. Then all fibers of $\pi$ are isomorphic to a same irreducible $L$-horospherical variety $\mathcal{Y}$.

(iv) If all effective divisors on $X$ are nef, then $S$ is the disjoint union of $I$, $J$ and $\mathcal{D}_X$, and $Y$ is $L$-equivariantly isomorphic to $\prod_{i=1}^m X_i$, where each $X_i$ is a smooth projective $L$-horospherical variety of Picard number one.

2.2 Reduction to Picard number one cases

The following results on manifolds with nef tangent bundles are basically due to [14] Proposition 3.7, Theorem 5.2. See also [23] Theorem 4.4 and [18] Theorem 3.3 for modification of proofs of the conclusion (ii).

Proposition 2.4. Let $X$ be a CP-manifold. Then the following hold.

(i) All effective divisors on $X$ are nef.

(ii) Let $f : X \to Y$ be the contraction of an extremal ray. Then $Y$ and $f$ are smooth. Moreover, $Y$ and the fibers of $f$ are CP-manifolds.

Now we can state and prove the main result in this section. This proposition reduces the proof of Theorem 1.2 to the case when the horospherical CP-manifold is of Picard number one.

Proposition 2.5. Let $X$ be a smooth projective $G$-horospherical variety with nef tangent bundle. Then $X$ is isomorphic to $(\prod_{i=1}^m X_i) \times G/P$, where $P$ is a parabolic subgroup of $G$, and each $X_i$ is a smooth projective horospherical variety of Picard number one with nef tangent bundle.
Proof. Step 1. In this step, we will construct two Mori contractions on $X$.

Horospherical varieties are rational. Hence, $X$ is simply connected and all finite étale cover over $X$ is trivial (see [6, Corollary 4.18]). By considering the Albanese map (see [7, Theorem 3.14] or the statement of this result before Conjecture 1.1 in our introduction), $X$ is a CP-manifold. Hence, all effective divisors on $X$ are nef by Proposition 2.4(i). Thus, we can apply Lemma 2.3. There is a Mori contraction $\pi : X \to G/P_{S\setminus J}$ with fibers isomorphic to a same irreducible $L$-horospherical variety $\prod_{i=1}^{m} X_i$, where $J = \mathcal{D}(G/H) \subseteq S$, and each $X_i$ is a smooth projective $L$-horospherical variety of Picard number one.

The fact $X$ is a CP-manifold implies that $Y$ and all $X_i$ are CP-manifolds by Proposition 2.4(ii). So we only need to prove that $X \sim Y \times (G/P_{S\setminus J})$.

By Lemma 2.3(ii), the Mori cone $\overline{NE}(X) = F \times F'$, where $F$ is the extremal face corresponding to the Mori contraction $\pi$ and $F'$ is another extremal face.

Denote by $\Psi : X \to W$ the Mori contraction associated with the extremal face $F'$. For the existence of such $\Psi$, see [3, Theorem 3.1(i)]. From the same theorem, we know that $W$ is a projective $G$-horospherical variety and $\Psi$ is a $G$-equivariant morphism. Moreover, $\Psi$ is a smooth fibration by Proposition 2.4(ii). Then for any point $w \in \Psi(Z)$,

$$\dim \Psi^{-1}(w) = \dim X - \dim W. \tag{2.1}$$

By the description of Mori contractions on horospherical varieties in terms of colored fans (see [3, Proposition 3.4]) and Lemma 2.2, we can get the conclusions (a)(b)(c) as follows.

(a) Take any $G$-orbit $Z$ on $X$, $\Psi^{-1}(\Psi(Z)) = Z$, and $\mathcal{D}_{\Psi(Z)} = \mathcal{D}_Z$.

(b) There exists horospherical subgroups $H'$ and $K'$ of $G$ such that $K' \supseteq H' \supseteq H$, and $Z$ and $\Psi(Z)$ are $G$-equivariantly isomorphic to $G/H'$ and $G/K'$ respectively.

(c) $N_G(H') = P_{I \cup J \cup Z}$, $N_G(K') = P_{I \cup J \cup \mathcal{D}_Z}$, and $P_{I \cup J \cup \mathcal{D}_Z}/H' \cong P_{I \cup J \cup \mathcal{D}_Z}/K'$.

In summary, there is a commutative diagram as follows:

$$
\begin{array}{ccc}
Z & \longrightarrow & G/H' \longrightarrow G/P_{I \cup J \cup \mathcal{D}_Z} \\
\downarrow & & \downarrow \\
\Psi(Z) & \longrightarrow & G/K' \longrightarrow G/P_{I \cup J \cup \mathcal{D}_Z}.
\end{array}
$$

Step 2. The aim of this step is to show a claim on the Dynkin diagram of $G$.

For any subset $A$ of $S$, denote by $\Gamma_A$ the full subdiagram of the Dynkin diagram $\Gamma$ of $G$ with vertices $A$.

Claim: Each element in $S \setminus (I \cup J)$ and each element in $J$ lie in different connected components of $\Gamma$.

Take any $G$-orbit $Z$ on $X$. Then by the conclusions (a)(b)(c) in Step 1, for any point $w \in \Psi(Z)$,

$$\dim \Psi^{-1}(w) = \dim P_{I \cup J \cup \mathcal{D}_Z} - \dim P_{I \cup \mathcal{D}_Z}. \tag{2.2}$$

By equations (2.1) and (2.2),

$$\dim P_{I \cup J \cup \mathcal{D}_Z} - \dim P_{I \cup \mathcal{D}_Z} = \dim X - \dim W. \tag{2.3}$$
Now apply formula (2.3) to the open $G$-orbit. Then for any $G$-orbit $Z$ on $X$,
\[
\dim(P_{l \cup J \cup D_Z}) - \dim(P_{l \cup D_Z}) = \dim(P_{l \cup J}) - \dim(P_l).
\]

(2.4)

Note that the union $I \cup J \cup D_Z$ is a disjoint union. By formula (2.4), for any $G$-orbit $Z$ on $X$, each element in $D_Z$ and each element in $J$ lie in different connected components of $\Gamma_{l \cup J \cup D_Z}$.

By Lemma 2.3(w), $S$ is the disjoint union of $I$, $J$ and $D_X$. Recall that $D_X = \bigcup D_Z$, where $Z$ runs over the set of $G$-orbits on $W$. Thus, the claim holds.

Step 3. In this step, we will combine the two Mori contractions to get the isomorphism.

Recall that $NE(X) = F \times F'$, and $\pi$ and $\Psi$ are the contractions corresponding to $F$ and $F'$ respectively. Hence, the natural morphism

\[
\Phi : X \to W \times (G/P_{S\setminus J}) \quad x \mapsto (\Psi(x), \pi(x))
\]

is finite onto the image. On the other hand, by the Claim in Step 2,
\[
\dim(G/P_l) = \dim(G/P_{S\setminus J}) + \dim(G/P_{l \cup J}).
\]

(2.5)

Denote by $G/K = \Psi(G/H)$ the open $G$-orbit on $W$ with $K \supseteq H$. Then by the conclusion (c) in Step 1, $\nu_G(H) = P_l$, $\nu_G(K) = P_{l \cup J}$, $P_l/H \cong P_{l \cup J}/K$. Hence,
\[
\dim(X) = \dim(G/P_l) + \dim(P_l/H) = \dim(G/P_{S\setminus J}) + \dim(G/P_{l \cup J}) + \dim(P_{l \cup J}/K) = \dim(G/P_{S\setminus J}) + \dim(W).
\]

Thus, the morphism $\Phi : X \to W \times (G/P_{S\setminus J})$ is surjective.

Take any point $x \in X$. Denote by $w = \Psi(x) \in W$. Then $\Psi^{-1}(w)$ is naturally a $G_w$-variety. By the conclusion (a) in Step 1, $\Psi^{-1}(w)$ is $G_w$-homogeneous, and it is $G_w$-equivariantly isomorphic to $G_w/G_x$. By the conclusions (a)(b)(c) in Step 1, $G_w/G_x \cong P_{l \cup J \cup D_{G_w}/P_{l \cup D_{G_w}}}$ and $D_{G_w} = D_{G_x}$. Thus, $\Psi^{-1}(w)$ is irreducible, it is isomorphic to $G^o_w/G_x \cap G^o_w$, and $G_x \cap G^o_w$ is a parabolic subgroup of the linear algebraic group $G^o_w$, where $G^o_w$ is the connected component of $G_w$ that contains the identity. Hence, the finite surjective $G^o_w$-equivariant morphism $\pi_{|\Psi^{-1}(w)} : \Psi^{-1}(w) \to G/P_{S\setminus J}$ is an isomorphism. So $\Phi : X \to W \times (G/P_{S\setminus J})$ is an isomorphism. By considering the Mori contraction $\pi$, we know $W$ is isomorphic to $Y$. Then the conclusion follows.

\[\square\]

3 \ Indices of smooth projective horospherical varieties of Picard number one

In this section, we turn to the cases of Picard number one. There is a classification of smooth projective horospherical varieties of Picard number one due to Pasquier.

**Proposition 3.1.** (\cite{21} Theorem 0.1, Lemma 1.19) Let $X$ be a smooth projective $G$-horospherical variety of Picard number one. Assume that $X$ is not homogeneous. Then the following hold.

1. $G$ acts on $X$ with three orbits, one open orbit and two closed orbits. Identify them with $G/H$, $G/P_1$ and $G/P_2$ respectively, where $P_1$ and $P_2$ are parabolic subgroups of $G$ containing...
B. If necessary, reorder $P_1$ and $P_2$. Then the automorphism group $\text{Aut}(X)$ acts on $X$ with two orbits, $G/H \cup G/P_1$ and $G/P_2$.

(2) The variety $X$ is uniquely determined by the triple $(G, P_1, P_2)$. The triple $(G, P_1, P_2)$ is one of the following list:

(i) $(B_m, P(\omega_{m-1}), P(\omega_m))$ with $m \geq 3$;
(ii) $(B_3, P(\omega_1), P(\omega_3))$;
(iii) $(C_m, P(\omega_{k+1}), P(\omega_k))$ with $m \geq 2$ and $1 \leq k \leq m - 1$;
(iv) $(F_4, P(\omega_2), P(\omega_3))$;
(v) $(G_2, P(\omega_2), P(\omega_1))$.

where $G = B_m$ (resp. $C_m$, $F_4$, $G_2$) means $G$ is a simple group of that type.

For smooth projective nonhomogeneous horospherical varieties of Picard number one, we have a formula as follows.

**Proposition 3.2.** Keep notations $X, G, P_1, P_2$ as in Proposition 3.1. Denote by $r_X$ the Fano index of $X$, and set $P = P_1 \cap P_2$. Then

$$r_X = \dim(P_1/P) + \dim(P_2/P) + 2. \quad (3.1)$$

**Proof.** Denote by $X_i$ the closed $G$-orbit on $X$ that is identified with $G/P_i$, and $U$ the open $G$-orbit on $X$. Consider the following diagram

$$
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\Phi} & G/P \\
\downarrow & & \downarrow \\
X, & & \\
\end{array}
$$

where $\Phi : \tilde{X} \to X$ is the morphism of blowing-up $X$ along $X_1 \cup X_2$, and the existence of the morphism $\pi$ follows from the fact that $X$ is a smooth horospherical variety. Moreover, both $\Phi$ and $\pi$ are $G$-equivariant morphisms, and $\pi$ is a $\mathbb{P}^1$-bundle. Denote by $E_i = \Phi^{-1}(X_i)$. Then $\pi$ sends both $E_1$ and $E_2$ $G$-equivariantly isomorphically to $G/P$.

Set the colored fan of $X$ to be $\mathcal{F}_X = \{(U, \emptyset), (X_1, D_2), (X_2, D_1)\}$, where $D_1$ and $D_2$ are the colors of $X$. Denote by $D_i \subseteq \tilde{X}$ the proper transform of $D_i$. Let $D_{ip} = \pi(D_i)$.

Now we will try to express the anticanonical divisor $-K_{G/P}$ using the information of $X$.

Since $X_1 \nsubseteq D_1$ and $X_2 \subseteq D_1$, we know that $\Phi^*(D_1) = \tilde{D}_1 + a_2E_2$ in $\text{Pic}(\tilde{X})\mathbb{Q}$ for some $a_2 \in \mathbb{Q}$. Now consider the following commutative diagram:

$$
\begin{array}{ccc}
E_2 & \xrightarrow{\pi|_{E_2}} & G/P \\
\downarrow \Phi|_{E_2} & & \downarrow p_2 \\
X_2 & \xrightarrow{\psi_2} & G/P_2,
\end{array}
$$

where $p_2$ is the natural morphism induced by the inclusion $P \subseteq P_2$, and $\psi_2$ is a $G$-equivariant isomorphism. Since $\Phi : \tilde{X} \to X$ is the blow-up morphism of $X$ along $X_1 \cup X_2$, the restriction $\Phi|_{E_2} : E_2 \to X_2$ is a $\mathbb{P}^{c_2-1}$-bundle, where $c_2$ is the codimension of $X_2$ in $X$. Thus,

$$\dim(P_2/P) = c_2 - 1. \quad (3.4)$$
and each fiber of \( p_2 \) is isomorphic to a projective space. Similarly, we know that
\[
\dim(P_1/P) = c_1 - 1,
\]
where \( c_1 \) is the codimension of \( X_1 \) in \( X \).

By classical results on intersection theory on rational homogeneous spaces, we know that
\[
\text{Pic}(G/P) = \mathbb{Z}(D_{1p}) \oplus \mathbb{Z}(D_{2p}),
\]
where \( \mathbb{Z}(D_{ip}) \) is the Picard group of \( G/P \) for \( i = 1, 2 \). Moreover, \( p_2 \) is induced by the linear system \( |D_{2p}| \). Hence, any line \( l \) in an arbitrary fiber \( F \) of \( p_2 \) is numerically equivalent to \( l_1 \). In particular, \( D_{1p} \cdot l = 1 \) and \( D_{1p}|_F \) is a generator of \( \text{Pic}(F) \).

Denote by \( \tilde{l} = (\pi|_{E_2})^{-1}(l) \). Since \( \pi : \tilde{X} \to G/P \) is a \( \mathbb{P}^1 \)-bundle, \( \tilde{D}_i = \pi^*D_{ip} \) for \( i = 1, 2 \). Thus, \( \Phi^*(D_1) \cdot \tilde{l} = (\tilde{D}_1 + a_2E_2) \cdot \tilde{l} = D_{1p} \cdot l + a_2E_2 \cdot \tilde{l} = 1 - a_2 \). Since \( \Phi \) contracts \( \tilde{l} \), \( \Phi^*(D_1) \cdot \tilde{l} = 0 \). Hence, \( a_2 = 1 \) and
\[
\Phi^*(D_1) = \tilde{D}_1 + E_2 \quad \text{in Pic}(\tilde{X}).
\]

Similarly, we have
\[
\Phi^*(D_2) = \tilde{D}_2 + E_1 \quad \text{in Pic}(\tilde{X}).
\]
Since \( D_1 = D_2 \) in \( \text{Pic}(X) \), we have
\[
\tilde{D}_1 - \tilde{D}_2 = E_1 - E_2 \quad \text{in Pic}(\tilde{X}).
\]
The fact \( E_1 \cap E_2 = \emptyset \) implies that
\[
E_1 \cdot E_2 \quad \text{is a zero cycle on} \quad \tilde{X}.
\]
The anticanonical divisor
\[
-K_X = r_XD_1 = r_XD_2 \quad \text{in Pic}(X).
\]
Since \( \Phi \) is the blow-up morphism, the anticanonical divisor
\[
-K_{\tilde{X}} = \Phi^*(-K_X) - (c_1 - 1)E_1 - (c_2 - 1)E_2 \quad \text{in Pic}(\tilde{X}).
\]

Moreover, the anticanonical divisor
\[
-K_{E_1} = (-K_{\tilde{X}} - E_1)|_{E_1} \quad \text{in Pic}(E_1).
\]

Consider the following commutative diagram:

\[
\begin{array}{ccc}
E_1 & \xrightarrow{j_1} & \tilde{X} \\
\pi|_{E_1} & \searrow & \downarrow \pi \\
& & G/P \\
\end{array}
\]

where \( j_1 \) is the natural inclusion morphism. Since the diagram is commutative and \( \pi|_{E_1} \) is an isomorphism, we know that \( (\pi|_{E_1})_*j_1^*\Phi^*(D_{ip}) = D_{ip} \) in \( \text{Pic}(G/P) \). Recall that \( \Phi^*(D_{ip}) = \tilde{D}_i \). Hence,
\[
(\pi|_{E_1})_*(\tilde{D}_i|_{E_1}) = D_{ip} \quad \text{in Pic}(G/P), \quad \text{for} \ i = 1, 2.
\]

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Note that \( \pi|_{E_1} : E_1 \to G/P \) is an isomorphism. Combining with equations (3.6) – (3.12), we can get that
\[
-K_{G/P} = (r_X - c_1) D_{1p} + c_1 D_{2p} \quad \text{in Pic}(G/P).
\]
Similarly, by considering \( E_2 \) instead of \( E_1 \), we can get that
\[
-K_{G/P} = c_2 D_{1p} + (r_X - c_2) D_{2p} \quad \text{in Pic}(G/P).
\]
Since Pic(G/P) is freely generated by \( D_{1p} \) and \( D_{2p} \), we know that
\[
r_X = c_1 + c_2 = \dim(P_1/P) + \dim(P_2/P) + 2,
\]
where the second equality follows from (3.3) and (3.3). \( \square \)

**Proposition 3.3.** Keep notations as in Proposition 5.2. Assume moreover that \( X \) is a CP-manifold. Denote by \( r_i \) the Fano index of \( G/P_i \) for \( i = 1, 2 \). Then
\[
r_X = \dim(P_1/P) + \dim(P_2/P) + 2 \geq \max\{r_1, r_2\}.
\]
(3.13)

**Proof.** Keep notation as in the proof of Proposition 5.2. Let \( C_i \) be a line on \( X_i \cong G/P_i \), i.e. \( C_i \) is an irreducible curve on \( X_i \) with smallest anticanonical degree. Thus, \( C_i \) is a smooth rational curve. Denote by \( f_i : C_i \to X \) the natural inclusion. By assumption, the tangent bundle \( T_X \) is nef. Thus, \( N_{X_i/X}|C_i \), the restriction of the normal bundle, is nef. Consider the short exact sequence:
\[
0 \to T_{X_i}|C_i \to T_X|C_i \to N_{X_i/X}|C_i \to 0.
\]
By the nefness of \( N_{X_i/X}|C_i \),
\[
-K_X \cdot C_i \geq \deg(T_{X_i}|C_i) = \deg(T_X|C_i) = -K_{X_i} \cdot C_i = r_i.
\]
To complete the proof, we only need to show that \( D_i \cdot C_i = 1 \) for \( i = 1, 2 \). This is equivalent to show that \( D_i|_{X_i} \) is the ample generator of Pic(\( X_i \)), where the later is known to be true. For the convenience of the readers, we give a proof in detail as follows.

We consider the following commutative diagram:
\[
\begin{array}{ccc}
G/P & \xrightarrow{\pi|_{E_2}} & E_2 \\
\downarrow{p_2} & & \downarrow{j_2} \\
G/P_2 & \xrightarrow{\psi_2} & X_2 \\
\end{array}
\]
\[\begin{array}{ccc}
& & \xrightarrow{\Phi|_{E_2}} \\
& & \downarrow{\Phi} \\
& & \xrightarrow{i_2} X
\end{array}
\]
(3.14)

where \( i_2 \) and \( j_2 \) are natural inclusions.

Let \( \tilde{D}_2 = p_2(D_{2p}) \). Then \( \tilde{D}_2 \) is the ample generator of Pic(\( G/P_2 \)) and \( p_2^* \tilde{D}_2 = D_{2p} \). Thus,
\[
(\Phi|_{E_2})^* i_2^* D_2 = j_2^* \Phi^* D_2 = (\tilde{D}_2 + E_1)|_{E_2} = \tilde{D}_2|_{E_2}
\]
\[
= (\pi|_{E_2})^* D_{2p} = (\pi|_{E_2})^* p_2^* \tilde{D}_2 = (\Phi|_{E_2})^* \psi_2^* \tilde{D}_2,
\]
where the fourth equality follows from an analogue of the formula (3.12). Hence, \( i_2^* D_2 = \psi_2^* \tilde{D}_2 \), which implies that \( D_2|_{X_2} \) is the ample generator of Pic(\( X_2 \)). Similarly, \( D_1|_{X_1} \) is the ample generator of Pic(\( X_1 \)). Then the conclusion follows. \( \square \)
The inequality (3.13) can be checked for the five series of horospherical varieties (i)–(v) in Proposition 3.1(2). Now keep notations as above. Denote by $d_i = \dim(P_i/P)$. Then we have the following table. It should be noticed that in Case (i) of Proposition 3.1(2), we have $m \geq 3$.

| Case | $d_1$ | $d_2$ | $r_1$ | $r_2$ | Does (3.13) hold? |
|------|-------|-------|-------|-------|------------------|
| (i)  | 1     | $m-1$ | $m+1$ | $2m$  | No               |
| (ii) | 3     | 2     | 5     | 6     | Yes              |
| (iii)| $2m-2k-1$ | $k$   | $2m-k+1$ | $2m-k$ | Yes              |
| (iv) | 2     | 2     | 5     | 7     | No               |
| (v)  | 1     | 1     | 3     | 5     | No               |

As a direct consequence, we get the following

**Corollary 3.4.** Let $X$ be a CP-manifold as well as a horospherical variety of Picard one. Then $X$ is either a rational homogeneous space or a variety as in Proposition 3.1(2)(ii) or (iii).

**Remark 3.5.** varieties in Proposition 3.1(2)(ii) and (iii) have geometric explanations. Firstly, let $X$ be as in Case (ii). By Proposition 3.3 and the table above, $\dim(X) = 9$ and $r_X = 7$. Thus, $X$ is a Mukai variety. In fact, $X$ is isomorphic to a nonsingular section of the 10-dimensional spinor variety $S^{10} \subseteq \mathbb{P}^{15}$ (see also [8, Subsection 3.3]).

Now assume $X$ to be as in Case (iii). Then $X$ is isomorphic to the odd symplectic Grassmannian $G_\omega(k+1, 2m+1)$ with $m \geq 2$ and $1 \leq k \leq m-1$ (see [21, Proposition 1.12]). The odd symplectic Grassmannian is defined as follows. Let $V$ be a linear space of dimension $2m+1$. Equip a skew bilinear form $\omega$ of rank $2m$ on $V$. Then $G_\omega(i, 2m+1)$ is the variety of $i$-dimensional isotropic linear subspaces of $V$, where $1 \leq i \leq m+1$.

### 4 Lines on horospherical varieties of Picard number one

Let $X \subseteq \mathbb{P}^N$ be an $n$-dimensional (not necessarily irreducible or smooth) projective variety that are covered by lines. Denote by $F(X)$ the variety of lines in $X$. For any point $x \in X$, denote by $F(x, X) \subseteq \mathbb{P}(T_x X)$ the variety of lines in $X$ passing through $x$, and we call it the variety of minimal rational tangents (VMRT for short) of $X$ at $x$. If moreover $F(x, X) \subseteq \mathbb{P}^{n-1}$ is covered by lines, then for any $[l] \in F(x; X)$, where $[l]$ stands for a line $l$ in $X$ passing through $x$, we denote $F(x, l, X) = F([l], F(x, X))$.

**Remark 4.1.** (i) The VMRT has the definition in a more general version. But we only need this restrictive version of definition here.

(ii) Let $X$ be a smooth projective horospherical variety of Picard number one. Then the ample generator of Pic$(X)$, denoted by $\mathcal{O}_X(1)$, is very ample. Denote by $\Phi$ the closed embedding defined by the linear system $|\mathcal{O}_X(1)|$. Then the image $\Phi(X)$ is covered by lines. We always identify $X$ with the image $\Phi(X)$. In particular, the VMRTs are naturally defined. Sometimes we also regard $X$ as a closed subvariety of a natural projective variety $Y$ (for example, in Proposition 4.2). If we can show $F(x, X) \neq \emptyset$ for some $x \in X \subseteq Y$, then the inclusion $X \subseteq Y$ is compatible with the closed embedding induced by $|\mathcal{O}_X(1)|$.

The aim of this section is to show the following

**Proposition 4.2.** Let $X$ be a variety as in Case (ii) or (iii) in Proposition 3.1(2). Then there exists a point $x \in X$ such that $F(x, X)$ is singular.
Remark 4.3. If $X$ is a CP-manifold, then the nefness of the tangent bundle implies that any irreducible rational curve on $X$ is free. Hence, for any $x \in X$, $F(x, X)$ is smooth. Then by Proposition 4.2, varieties in Proposition 3.1(2)(ii) and (iii) do not have nef tangent bundles. This completes the proof of Theorem 1.2.

Let $V$ be an $n$-dimensional linear space. Denote by $\mathcal{F}(k_1, \ldots, k_m; V)$ the flag variety parameterizing the sequences $(V_{k_1}, \ldots, V_{k_m})$ such that $V_{k_1} \subseteq V_{k_2} \subseteq \ldots \subseteq V_{k_m}$ and $V_{k_i}$ is a $k_i$-dimensional linear subspace of $V$. We also denote the Grassmannian $G(k, n) = G(k, V) = \mathcal{F}(k; V)$.

Remark 4.4. Keep notation as above. By [12, Theorem 4.9], $F(G(k, V)) = \mathcal{F}(k - 1, k + 1; V)$ and the VMRT of $G(k, V) \subseteq \mathcal{F}(A^k V)$ is isomorphic to $\mathbb{P}^{k - 1} \times \mathbb{P}^{n - k - 1} \subseteq \mathbb{P}^{k(n - k) - 1}$. Thus, the natural projection $\phi : \mathcal{F}(k - 1, k, k + 1; V) \rightarrow \mathcal{F}(k - 1, k + 1; V)$ is the universal family of lines on $G(k, V)$, and the natural projection $\psi : \mathcal{F}(k - 1, k + 1; V) \rightarrow G(k, V)$ is the evaluation morphism. Each line $l \subseteq G(k, V)$ corresponds to a point $x = (V_{k-1}, V_{k+1}) \in \mathcal{F}(k - 1, k + 1; V)$, $l = \psi(\phi^{-1}(x))$ and $\phi^{-1}(x) = G(1, V_{k+1}/V_{k-1})$. So we can regard $l$ as a line in $G(1, V/V_{k-1})$. In particular, for any $k$-dimensional linear subspace $V_k$ of $V$, the VMRT $F([V_k], G(k, V))$ admits a morphism

$$
\Pi : F([V_k], G(k, V)) \rightarrow G(k - 1, V_k)
$$

$$
[l] \mapsto [V_{k-1}],
$$

where $[l] = (V_{k-1}, V_{k+1}) \in \mathcal{F}(k - 1, k + 1; V)$. The morphism $\Pi$ is a trivial $\mathbb{P}^{n - k - 1}$-bundle and for each $(k - 1)$-dimensional linear subspace $V_{k-1}$ of $V_k$, the fiber

$$
\Pi^{-1}([V_{k-1}]) = F([V_k/V_{k-1}], G(1, V/V_{k-1})).
$$

Let $G = SO(7)$. Consider the following diagram:

$$
\begin{array}{ccc}
G/(P(\omega_1) \cap P(\omega_3)) & \xrightarrow{p_1} & G/P(\omega_1) \\
\downarrow{p_3} & & \downarrow{p_3} \\
G/P(\omega_3). & & 
\end{array}
$$

(4.1)

By the classical results on rational homogeneous spaces, $G/P(\omega_1)$ and $G/P(\omega_3)$ are smooth quadric hypersurfaces of dimension 5 and 6 respectively, $p_1$ is a $\mathbb{P}^3$-bundle, and $p_3$ is a $\mathbb{P}^2$-bundle. For any point $x \in G/P(\omega_1)$, $p_3$ sends $p_1^{-1}(x)$ isomorphically to a 3-dimensional linear subspace contained in the 6-dimensional quadric hypersurface $G/P(\omega_3)$.

Lemma 4.5. Keep notations as above. Then there exists a line $l$ in $G/P(\omega_3)$ and a line $C$ in $G/P(\omega_1)$ such that for any $q \in C$, $p_3(p_1^{-1}(q))$ is a 3-dimensional linear space containing $l$.

Proof. Let $V$ be a 7-dimensional linear space equipped with a nondegenerate symmetric bilinear form. Then for $1 \leq i \leq 3$, the variety $G/P(\omega_i)$ can be identified with the variety of $i$-dimensional isotropic linear subspaces of $V$. Take $V_2$ to be a 2-dimensional isotropic linear subspace of $V$. Set

$${C} := G(1, V_2), \text{ and } {l} := \{[V_3] \in G(3, V) | V_2 \subseteq V_3 \subseteq V_3^\perp\}.$$
By regarding $G/P(\omega_1)$ as a subvariety of the Grassmannian $G(1, V)$, we can see that $C$ is a line in $G/P(\omega_1)$ (see Remark 3.3). Moreover, $l$ is a rational curve in $G/P(\omega_3)$.

Now consider the following diagram:

$$
\begin{array}{c}
G/(P(\omega_2) \cap P(\omega_3)) \\ p_2
\end{array} \longrightarrow
\begin{array}{c}
G/P(\omega_2) \\ p_3
\end{array} \longrightarrow
\begin{array}{c}
G/P(\omega_3).
\end{array}
$$

Then $l = p_3(p_2^{-1}([V_2]))$. By the intersection theory on rational homogeneous spaces, $l$ is a line in $G/P(\omega_3)$. Hence, for any $q \in C$, $p_3(p_2^{-1}(q))$ is a 3-dimensional linear space containing $l$.

**Proposition 4.6.** Let $X \subseteq \mathbb{P}^{14}$ be a nonsingular hyperplane section of the 10-dimensional spinor variety $S^{10} \subseteq \mathbb{P}^{15}$. Then there exists $x \in X$ such that $F(x, X)$ is not smooth.

**Proof.** Firstly, $X$ is a horospherical variety as in Proposition 3.1(2)(ii). The automorphism group $\text{Aut}(X)$ acts on $X$ with two orbits. Denote by $Z$ the closed orbit. Then $Z \subseteq X$ is a 6-dimensional smooth quadric hypersurface. Denote by $\Phi : \tilde{X} \rightarrow X$ the blow-up of $X$ along $Z$. Let $E$ be the exceptional divisor of $\Phi$. Then there is a morphism $\pi : \tilde{X} \rightarrow Q^5$ making $\tilde{X}$ to be a $\mathbb{P}^4$-bundle over a 5-dimensional smooth quadric hypersurface. In summary, we have the following commutative diagram:

$$
\begin{array}{c}
E' \\
\downarrow
\end{array} \longrightarrow
\begin{array}{c}
\tilde{X} \\
\downarrow \Phi
\end{array} \longrightarrow
\begin{array}{c}
Q^5,
\end{array}
$$

Moreover, the restriction $\pi|_E : E \rightarrow Q^5$ is a $\mathbb{P}^3$-bundle. Let $G = SO(7)$. Then $Z \cong G/P(\omega_3)$, $G/P(\omega_1) \cong Q^5$, $E \cong G/(P(\omega_1) \cap P(\omega_3))$, and the morphisms $\pi|_E : E \rightarrow Q^5$ and $\Phi|_E : E \rightarrow Z$ coincide with $p_2$ and $p_3$ in the diagram (4.1) respectively.

By Lemma 4.3 there exists a line $C \subseteq Q^5$ and a line $l \subseteq Z$ such that for each $q \in C$, $\Phi((\pi|_E)^{-1}(q)) \subseteq Z$ is a 3-dimensional linear subspace containing the line $l$.

Now take any point $q \in C$. Consider the following commutative diagram:

$$
\begin{array}{c}
(\pi|_E)^{-1}(q) \\
\downarrow
\end{array} \longrightarrow
\begin{array}{c}
\pi^{-1}(q) \\
\downarrow
\end{array} \longrightarrow
\begin{array}{c}
\Phi((\pi|_E)^{-1}(q)) \subseteq \Phi(\pi^{-1}(q)).
\end{array}
$$

By considering the diagram (4.4), we get that $\Phi$ sends $(\pi|_E)^{-1}(q)$ isomorphically to a 3-dimensional linear subspace containing the line $l$. Recall that $\pi^{-1}(q)$ is isomorphic to $\mathbb{P}^4$. Thus, $\Phi$ sends
is defined to be the hyperplane $G \subseteq V$. In particular, when $x \in X$, let 

$$\Pi : F \rightarrow \mathbb{P}^k.$$

Restrict $\Pi$ on $(x,M)$, we get a surjective morphism 

$$\Phi((\pi_{x,x'}(q')) \neq \Phi((\pi_{x,x'}(q))).$$

Thus, $\Phi(\pi^{-1}(q')) \neq \Phi(\pi^{-1}(q))$. Hence, there is a 1-dimensional family of planes contained in $F(z,l,X)$.

By the discussion in last paragraph, $F(z,l,X) = \mathbb{P}^1 \times \mathbb{P}^2 \subseteq \mathbb{P}^5$ and $F(z,X)$ is singular at the point $[l]$. 

Proposition 4.7. Let $X$ be the odd symplectic Grassmannian $G_{\omega}(k,2m+1)$ with $m \geq 2$ and $2 \leq k \leq m$. Regard it as a subvariety of the Grassmannian $G(k,2m+1) \subseteq \mathbb{P}(\wedge^k \mathbb{C}^{2m+1})$. Denote by $Z$ the closed orbit under the action of the automorphism group of $X$. Take a point $x \in X$.

Then there is a surjective morphism $\pi : F(x,X) \rightarrow \mathbb{P}^{k-1}$.

(i) If $x \notin Z$, then $\pi$ is a $\mathbb{P}^{2m-2k+1}$-bundle.

(ii) If $x \in Z$, then there exists a hyperplane $H \subseteq \mathbb{P}^{k-1}$ such that 

$$\pi^{-1}(v) \cong \begin{cases} \mathbb{P}^{2m-2k+2}, & v \in H, \\ \mathbb{P}^{2m-2k+1}, & v \in \mathbb{P}^{k-1} \setminus H. \end{cases}$$

In particular, when $x \in Z$, $F(x,X)$ is not smooth.

Proof. Let $V$ be a $(2m+1)$-dimensional linear space equipped with a skew bilinear form $\omega$ of rank $2m$. Denote by $M = G(k,2m+1) \subseteq \mathbb{P}(\wedge^k V)$. Let $V_k$ be the $k$-dimensional linear subspace of $V$ represented by $x \in X \subseteq M$. By Remark 4.4, there exists a $\mathbb{P}^{2m-k}$-bundle $\Pi : F(x,M) \rightarrow G(k-1,V_k)$ such that for each $(k-1)$-dimensional linear subspace $V_{k-1}$ of $V_k$, 

$$\Pi^{-1}([V_{k-1}]) = F([V_k/V_{k-1}], G(1,V/V_{k-1})).$$

Restrict $\Pi$ on $F(x,X)$, we get the morphism $\pi : F(x,X) \rightarrow G(k-1,V_k) = \mathbb{P}^{k-1}$.

For any linear subspace $W$ of $V$, define $W^\perp := \{v \in V \mid \omega(v,W) = 0\}$. Take an arbitrary $(k-1)$-dimensional linear subspace $V_{k-1}$ of $V_k$. The form $\omega$ induces a skew bilinear form $\tilde{\omega}$ on $V_{k-1}/V_{k-1}$. Moreover, 

$$V_{k-1}^\perp \supseteq V_k^\perp \supseteq V_k \supseteq V_{k-1},$$

$$X \cap G(1,V/V_{k-1}) = G_{\tilde{\omega}}(1,V_{k-1}^\perp/V_{k-1}),$$

$$\pi^{-1}([V_{k-1}]) = F([V_k/V_{k-1}], G_{\tilde{\omega}}(1,V_{k-1}^\perp/V_{k-1})).$$

Note that $R := V^\perp$ is a 1-dimensional linear subspace and the induced form on $V/R$ of the skew bilinear form $\omega$ is skew bilinear and nondegenerate. Hence, 

$$\dim(V_{k-1}^\perp) = \begin{cases} 2m - k + 3, & R \subseteq V_{k-1}, \\ 2m - k + 2, & R \not\subseteq V_{k-1}, \end{cases}$$

and 

$$G_{\tilde{\omega}}(1,V_{k-1}^\perp/V_{k-1}) = G(1,V_{k-1}^\perp/V_{k-1}) \cong \begin{cases} \mathbb{P}^{2m-2k+3}, & R \subseteq V_{k-1}, \\ \mathbb{P}^{2m-2k+2}, & R \not\subseteq V_{k-1}. \end{cases}$$

By [14] Proposition 4.3, $Z = \{[W] \in X \mid R \subseteq W\}$. Then the conclusion (ii) holds, where $H$ is defined to be the hyperplane $G(k-2,V_k/R)$ in $G(k-1,V_k) = \mathbb{P}^{k-1}$. Note that when $x \in Z,$

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$F(x, X)$ has two irreducible components with the same dimension and the intersection of them is not empty. So $F(x, X)$ is not smooth for $x \in Z$.

Now assume $x \not\in Z$. Then $\pi$ is a $\mathbb{P}^{2m-2k+1}$ fibration over $\mathbb{P}^{k-1}$. It is known that such fibration are projective bundles. So the conclusion (i) holds.

Finally, we summarize the proof of Theorem 1.2 as follows.

**Proof of Theorem 1.2.** Let $X$ be a smooth projective horospherical variety with nef tangent bundle. By Proposition 2.5, $X \cong (\prod_{i=1}^{m} X_i) \times G/P$ and each $X_i$ is a smooth projective horospherical variety of Picard number one as well as a CP-manifold.

It suffices to show that each $X_i$ is a rational homogeneous space. Now assume that there exists some $i_0$ such that $X_{i_0}$ is not homogeneous. By Corollary 3.4, $X_{i_0}$ is as in Proposition 3.1(ii) or (iii). By Proposition 4.2 in both cases we can find $x \in X_{i_0}$ such that the variety $F(x, X_{i_0})$ is singular. This contradicts Remark 4.3. Then the conclusion follows.

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