SHARP HARDY-SOBOLEV-MAZ’YA, ADAMS AND HARDY-ADAMS INEQUALITIES ON QUATERNIONIC HYPERBOLIC SPACES AND THE CAYLEY HYPERBOLIC PLANE

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Abstract. Though Adams and Hardy-Adams inequalities can be extended to general symmetric spaces of noncompact type fairly straightforwardly by following closely the systematic approach developed in our early works on hyperbolic spaces [43], [44], [45], [38], [39] and more recently on complex hyperbolic spaces in [46], higher order Poincaré-Sobolev and Hardy-Sobolev-Maz’ya inequalities are more difficult to establish. The main purpose of this goal is to establish the Poincaré-Sobolev and Hardy-Sobolev-Maz’ya inequalities on quaternionic hyperbolic spaces and the Cayley hyperbolic plane. A crucial part of our work is to establish appropriate factorization theorems on these spaces which are of their independent interests. To this end, we need to identify and introduce the “Quaternionic Geller’s operators” and “Octonionic Geller’s operators” which have been absent on these spaces. Combining the factorization theorems and the Geller type operators with the Helgason-Fourier analysis on symmetric spaces, the precise heat and Bessel-Green-Riesz kernel estimates and the Kunze-Stein phenomenon for connected real simple groups of real rank one with finite center, we succeed to establish the higher order Poincaré-Sobolev and Hardy-Sobolev-Maz’ya inequalities on quaternionic hyperbolic spaces and the Cayley hyperbolic plane. The kernel estimates required to prove these inequalities are also sufficient for us to establish, as a byproduct, the Adams and Hardy-Adams inequalities on these spaces. This paper, together with earlier works [43], [44], [45], [46], [38] and [39], completes our study of the factorization theorems, higher order Poincaré-Sobolev, Hardy-Sobolev-Maz’ya, Adams and Hardy-Adams inequalities on all rank one symmetric spaces of noncompact type. The factorization theorems and higher order Poincaré-Sobolev and Hardy-Sobolev-Maz’ya inequalities on general higher rank symmetric spaces of noncompact type will be studied in a forthcoming paper.

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References

1. Introduction

Let $G$ be a simple Lie group of real rank one. That is, $G$ is one of the four groups $SO(n,1)$, $SU(n,1)$, $Sp(n,1)$ and $F_4$. Let $K$ be a maximal compact subgroup of $G$ and denote by $X = G/K$. Then $X$ is a rank one symmetric space of non-compact type, which is known as the real, complex and quaternionic hyperbolic spaces, and the Cayley hyperbolic plane, which we denote them by $H_n^R$, $H_n^C$, $H_n^Q$ and $H_2^O$. Throughout this paper, we let $\Delta_X$ be the Laplace-Beltrami operator of $X$ and $\rho_X$ be the half-sum of the positive roots of $X$. We note that

$$\rho_X = \begin{cases} \frac{n-1}{2}, & X = H_n^R; \\ n, & X = H_n^C; \\ 2n+1, & X = H_n^Q; \\ 11, & X = H_2^O. \end{cases}$$

and $\rho_X^2$ is the spectral gaps of $-\Delta_X$.

Our main object of study is the sharp higher order Poincaré-Sobolev and Hardy-Sobolev-Maz’ya inequalities and their borderline case, Adams and Hardy-Adams inequalities, on $X$. The Hardy-Sobolev-Maz’ya inequalities, studied firstly by Maz’ya [47], combine the Hardy and Sobolev inequalities into a single inequality and we state it as follows:

$$\int_{\mathbb{R}^n_+} |\nabla u|^2 dx - \frac{1}{4} \int_{\mathbb{R}^n_+} \frac{u^2}{x_1} dx \geq C_n \left( \int_{\mathbb{R}^n_+} x_1^\gamma |u|^p dx \right)^{\frac{2}{p}}, \quad u \in C_{0}^\infty (\mathbb{R}^n_+), n \geq 3,$$

where $2 < p \leq \frac{2n}{n-2}$, $\gamma = \frac{(n-2)p}{2} - n$, $\mathbb{R}^n_+ = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_1 > 0\}$ and $C_n$ is a positive constant which is independent of $u$. In terms of half-space model of real hyperbolic spaces, one can see such inequality is equivalent to the Poincaré-Sobolev inequality on $H_n^R$. The borderline case when $n = 2$ has been studied by Wang and Ye in [53] and the second and third authors [42]. The higher order inequalities of such type, namely the so-called Hardy-Adams inequalities have been established by the second and third authors and Li (see [38, 39, 45]).

1.1. The case $X = H_n^R$. We firstly recall the Poincaré half space model and ball model of $H_n^R$. The Poincaré half space model is given by $\mathbb{R}_+ \times \mathbb{R}^{n-1} = \{(x_1, \ldots, x_n) : x_1 > 0\}$ equipped with the Riemannian metric $ds^2 = \frac{dx_1^2 + \cdots + dx_n^2}{x_1^2}$. The induced Riemannian measure can be written as $dV = \frac{dx}{x_1}$, where $dx$ is the Lebesgue measure on $\mathbb{R}^n$. The ball model is given by the unit ball

$$\mathbb{B}^n = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid |x| < 1\}$$
equipped with the usual Poincaré metric
\[ ds^2 = \frac{4(dx_1^2 + \cdots + dx_n^2)}{(1 - |x|^2)^2}. \]

The factorization theorem on \( H^n_\mathbb{R} \) is given by:

- ball model. (see [40].)
  \( \left( \frac{1 - |x|^2}{2} \right)^{k+\frac{\alpha}{2}} (-\Delta)^k \left[ \frac{(1 - |x|^2)^{k-\frac{\alpha}{2}}}{2} \right] f = P_k f, \)

- half space model. (see [43].)
  \[ x_1^{-\alpha+k}(-\Delta)^k(x_1^{k-\frac{\alpha}{2}} f) = P_k f, \]

where \( f \in C^\infty(H^n_\mathbb{R}), \Delta \) is the Laplacian on Euclidean space, \( P_1 = -\Delta_\mathbb{R} - \frac{n(n-2)}{4} \) and
\[ P_k = P_1(P_1 + 2) \cdots (P_1 + k(k - 1)) \]
is the GJMS operators of order \( 2k \) on \( H^n_\mathbb{R} \). (see [25], [18], [34].) On the other hand, the Poincaré-Sobolev inequalities reads as
\[ \int_{H^n_\mathbb{R}} (\zeta^2 - \rho_X^2 - \Delta_X)^s(-\rho_X^2 - \Delta_X)^{\alpha/2} u \cdot udV \geq C \|u\|_{L^p(H^n_\mathbb{R})}^2, \]
where \( 0 < \alpha < 3, 0 < \zeta \) and \( u \in C^\infty_0(H^n_\mathbb{R}). \) Therefore, in terms of the Poincaré half space model and ball model of \( H^n_\mathbb{R}, \) we have the following Hardy-Sobolev-Maz’ya inequalities for higher order (see [43]).

**Theorem A.** Let \( 2 \leq k < \frac{n}{2} \) and \( 2 < p \leq \frac{2n}{n-2k}. \) There exists a positive constant \( C \) such that for each \( u \in C^\infty_0(\mathbb{R}^n_+), \)
\[ \int_{\mathbb{R}^n_+} |\nabla^k u|^2 dx - \prod_{i=1}^k \frac{(2i - 1)^2}{4} \int_{\mathbb{R}^n_+} u^2 \frac{x^n}{x^{2k}} dx \geq C \left( \int_{\mathbb{R}^n_+} x_1^\gamma |u|^p dx \right)^{\frac{2}{p}}, \]
where \( \gamma = \frac{(n-2k)p}{2} - n. \)

In terms of the Poincaré ball model \( \mathbb{B}^n, \) such inequality can be written as
\[ \int_{\mathbb{B}^n} |\nabla^k u|^2 dx - \prod_{i=1}^k (2i - 1)^2 \int_{\mathbb{B}^n} u^2 \frac{(1 - |x|^2)^{2k}}{(1 - |x|^2)^{2k}} dx \geq C \left( \int_{\mathbb{B}^n} (1 - |x|^2)^\gamma |u|^p dx \right)^{\frac{2}{p}}. \]

We mention in passing that the best constant in the above Hardy-Sobolev-Maz’ya inequalities when \( k = 1 \) and \( n = 3 \) is the same as the Sobolev constant (see [9]) and is strictly smaller than the Sobolev constant (see [28]). In the higher order derivative cases, it was proved in all the cases of \( n = 2k + 1, \) the best constants are the same as the Sobolev constants [44] (see also [30]).

In the borderline case, there holds the Hardy-Adams inequality and we state it as follows (see [44], [38], [39]).

**Theorem B.** Let \( n \geq 3, \zeta > 0 \) and \( 0 < s < 3/2. \) Then there exists a constant \( C_{\zeta,n} > 0 \) such that for all \( u \in C^\infty_0(H^n_\mathbb{R}) \)
\[ \int_{H^n_\mathbb{R}} (\zeta^2 - \rho_X^2 - \Delta_X)^s(-\rho_X^2 - \Delta_X)^{n/2} u \cdot udV \leq 1. \]
there holds
\[ \int_{H^n_\mathbb{R}} (e^{\beta_0(n/2,n)u^2} - 1 - \beta_0(n/2,n)u^2) dV \leq C_{\zeta,n}, \]
where
\[ \beta_0(\alpha, n) = \frac{n}{\omega_{n-1}} \left[ \frac{\pi^{n/2} 2^{\alpha} \Gamma(\alpha/2)}{\Gamma\left(\frac{n-\alpha}{2}\right)} \right]^{1/\alpha}, \quad 0 < \alpha < n, \]
is the best Adams’ constant on \( \mathbb{R}^n \) and \( \omega_{n-1} \) is the area of the surface of the unit \( n \)-ball.

In terms of the ball model, we have the following Hardy-Adams inequalities on \( \mathbb{R}^n \) (see [53], [45], [38].)

**Theorem C.** There exists a constant \( C > 0 \) such that for all \( u \in C_0^\infty(\mathbb{R}^n) \) with
\[
\int_{\mathbb{R}^n} |\nabla^2 u|^2 \, dx - \sum_{k=1}^{n/2} (2k - 1)^2 \int_{\mathbb{R}^n} \frac{u^2}{1 - |x|^2} \, dx \leq 1,
\]
there holds
\[
\int_{\mathbb{R}^n} e^{\beta_0(n/2,n) u^2} - 1 - \beta_0 \left( \frac{n}{2}, n \right) u^2 \frac{dx}{(1 - |x|^2)^{n/2}} \leq C.
\]

1.2. **The case** \( \mathcal{X} = H_\mathbb{C}^n \). The complex hyperbolic space is a simply connected complete Kaehler manifold of constant holomorphic sectional curvature \(-4\). There are two models of complex hyperbolic space, the Siegel domain model \( \mathcal{U}^n \) and the ball model \( \mathbb{B}_\mathbb{C}^n \). The Siegel domain \( \mathcal{U}^n \subset \mathbb{C}^n \) is defined as
\[
\mathcal{U}^n := \{ z \in \mathbb{C}^n : \varrho(z) > 0 \},
\]
where
\[
\varrho(z) = \text{Im} z_n - \sum_{j=1}^{n-1} |z_j|^2. \tag{1.1}
\]
The Bergman metric on \( \mathcal{U}^n \) is the metric with Kaehler form \( \omega = \frac{i}{2} \partial \bar{\partial} \log \frac{1}{\varrho} \). Its boundary \( \partial \mathcal{U}^n := \{ z \in \mathbb{C}^n : \varrho(z) = 0 \} \) can be identified with the Heisenberg group \( \mathbb{H}^{2n-1} \), which is a nilpotent group of step two with the group law
\[
(z, t) \circ (z', t') = (z + z', t + t' + 2 \text{Im}(z, z')),
\]
where \( z, z' \in \mathbb{C}^{n-1} \) and \( (z, z') \) is the Hermite inner product
\[
(z, z') = \sum_{j=1}^{n} z_j \bar{z}'_j.
\]
Set \( z_j = x_j + iy_j (1 \leq j \leq n - 1) \) and define \( X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t} \) for \( j = 1, \ldots, n - 1, \quad T = \frac{\partial}{\partial t} \).

The \( 2n - 1 \) vector fields \( X_1, \ldots, X_{n-1}, Y_1, \ldots, Y_{n-1}, T \) are left-invariant and form a basis for Lie algebra of \( \mathbb{H}^{2n-1} \). Let
\[
\mathcal{L}_0 = \sum_{j=1}^{n-1} (X_j^2 + Y_j^2)
\]
be the sub-Laplacian on \( \mathbb{H}^{2n-1} \). Then the Laplace-Beltrami operator is given by
\[
\Delta_X = 4\varrho [\varrho (\partial_{\varrho} + T^2) + \mathcal{L}_0 - (n - 1) \partial_{\varrho}].
\]

The ball model is given by the unit ball
\[
\mathbb{B}_\mathbb{C}^n = \{ z = (z_1, \ldots, z_n) \in \mathbb{C}^n : |z| < 1 \}
\]
equipped with the Kaehler metric
\[
ds^2 = -\partial \bar{\partial} \log (1 - |z|^2).
The Laplace-Beltrami operator is given by
\[ \Delta_X = 4(1 - |z|^2) \sum_{j,k=1}^{n} (\delta_{jk} - z_j \bar{z}_k) \frac{\partial^2}{\partial z_j \partial \bar{z}_k}, \]
where
\[ \delta_{j,k} = \begin{cases} 1, & j = k; \\ 0, & j \neq k. \end{cases} \]

The Geller’s operator \( \Delta_{\alpha,\beta} \) is defined by (see [22])
\[ \Delta_{\alpha,\beta} = 4(1 - |z|^2) \left[ \sum_{j,k=1}^{n} (\delta_{jk} - z_j \bar{z}_k) \frac{\partial^2}{\partial z_j \partial \bar{z}_k} + \alpha \sum_{j=1}^{n} z_j \frac{\partial}{\partial z_j} + \beta \sum_{j=1}^{n} z_j \frac{\partial}{\partial z_j} - \alpha \beta \right]. \tag{1.2} \]

Denote by
\[ R = \sum_{j=1}^{n} z_j \frac{\partial}{\partial z_j}, \quad \overline{R} = \sum_{j=1}^{n} \bar{z}_j \frac{\partial}{\partial \bar{z}_j}. \]
Then we have
\[ \Delta_{\alpha,\beta} = 4(1 - |z|^2) \left( 1 - \frac{1}{|z|^2} R \overline{R} - \frac{1}{|z|^2} \mathcal{L}_0' + \frac{n-1}{2} \frac{1}{|z|^2} (R + \overline{R}) + \alpha R + \beta \overline{R} - \alpha \beta \right), \]
where \( \mathcal{L}_0' \) is the Folland-Stein operator [19] on CR sphere defined as follows:
\[ \mathcal{L}_0' = -\frac{1}{2} \sum_{j<k} (M_{jk} \overline{M}_{jk} + \overline{M}_{jk} M_{jk}), \text{ where } M_{jk} = z_j \partial z_k - \bar{z}_k \partial z_j. \]

For simplicity, we set \( \Delta_{\alpha,\beta} = \frac{1}{4(1 - |z|^2)} \Delta_{\alpha,\beta} \).

The factorization theorem involving Geller’s operators on the complex hyperbolic space play an important role in establishing the higher order Poincaré-Sobolev and Hardy-Sobolev-Maz’ya inequalities on the complex hyperbolic spaces and can be stated as follows (see [46]).

**Theorem D.** Let \( a \in \mathbb{R} \) and \( k \in \mathbb{N} \setminus \{0\} \). In terms of the Siegel domain model, we have, for \( u \in C^\infty(U^n) \),
\[ \prod_{j=1}^{k} \left[ \partial \partial_{\bar{e}} + a \partial_{\bar{e}} + \bar{e} T^2 + \mathcal{L}_0 - i(k + 1 - 2j)T \right] (\partial \frac{k-n-a}{2} u) \]
\[ = 4^{-k} \partial^{k-n-a} \prod_{j=1}^{k} \left[ \Delta_X + n^2 - (a - k + 2j - 2) \right] u. \tag{1.3} \]

In terms of the ball model, we have, for \( f \in C^\infty(B^n) \),
\[ \prod_{j=1}^{k} \left[ \Delta_{j=n-k,n-rac{1}{2}} + \frac{(k + 1 - 2j)^2}{4} - \frac{k + 1 - 2j}{2} (R - \overline{R}) \right] [(1 - |z|^2) \frac{k-n-a}{2} f] \]
\[ = 4^{-k} (1 - |z|^2)^{-\frac{k+n+a}{2}} \prod_{j=1}^{k} \left[ \Delta_X + n^2 - (a - k + 2j - 2) \right] f. \tag{1.4} \]

We note the left sides of (1.3) and (1.4) are closely related to the CR invariant differential operators on the Heisenberg group and CR sphere, respectively.
We also have the following Poincaré-Sobolev inequalities on $H^p_C$:

$$\int_{H^p_C} (\zeta^2 - \rho^2_X - \Delta_X)^s (-\rho^2_X - \Delta_X)^{\alpha/2} u \cdot udV \geq C\|u\|_{L^p(H^p_C)}^2,$$

where $0 < \alpha < 3$, $\zeta > 0$ and $u \in C_0^\infty(H^p_C)$. Therefore, in terms of two models of $H^p_C$, we have the following Hardy-Sobolev-Maz’ya inequalities:

**Theorem E.** Let $a \in \mathbb{R}$, $1 \leq k < n$ and $2 < p \leq \frac{2n}{n-k}$. In terms of the Siegel domain model, there exists a positive constant $C$ such that for each $u \in C_0^\infty(\mathcal{U}^n)$ we have

$$\int_{\mathcal{U}^{2n-1}} u \prod_{j=1}^k \left[ -\vartheta \partial_{\vartheta^1} - a_\vartheta \vartheta - \vartheta T^2 - L_0 + i(k + 1 - 2j)T \right] u \frac{dzdt\vartheta}{\varrho^{1-a}}$$

$$\geq C \int_{\mathcal{U}^{2n-1}} |u|^p \vartheta^\gamma dzdt\vartheta,$$

where $\gamma = \frac{(n-k+a)p}{2} - n - 1$. In terms of the ball model, we have for $f \in C_0^\infty(B^n_C)$,

$$\int_{B^n_C} f \prod_{j=1}^k \left[ \Delta_{1-\frac{a}{n-1}} \frac{1}{2} + \frac{(k + 1 - 2j)^2 - k + 1 - 2j}{2} (R - \overline{R}) \right] f \frac{dz}{(1 - |z|^2)^{1-a}}$$

$$\geq C \left( \int_{B^n_C} |f|^p (1 - |z|^2)\gamma dz \right)^{\frac{1}{p}}.$$

In the borderline case, there holds the Hardy-Adams inequality and we state it as follows.

**Theorem F.** Let $n \geq 3$, $\zeta > 0$ and $0 < s < 3/2$. Then there exists a constant $C_{\zeta,n} > 0$ such that for all $u \in C_0^\infty(H^p_C)$ with

$$\int_{H^p_C} (\zeta^2 - \rho^2_X - \Delta_X)^s (-\rho^2_X - \Delta_X)^{\alpha/2} u \cdot udV \leq 1,$$

there holds

$$\int_{H^p_C} (e^{\beta_0(n,2n)u^2} - 1 - \beta_0(n,2n) u^2) dV \leq C_{\zeta,n}.$$

Furthermore, in terms of the Siegel domain model, we have that for all $u \in C_0^\infty(\mathcal{U}^n)$ with

$$4^n \int_{\mathcal{U}^{2n-1}} u \prod_{j=1}^n \left[ -\vartheta \partial_{\vartheta^1} - a_\vartheta \vartheta - \vartheta T^2 - L_0 + i(k + 1 - 2j)T \right] u \frac{dzdt\vartheta}{\varrho^{1-a}}$$

$$- \prod_{j=1}^n (a - n + 2j - 2)^2 \int_{\mathcal{U}^{2n-1}} u \frac{2}{\varrho^{n+1-a}} dzdt\vartheta \leq 1,$$

there holds

$$\int_{\mathcal{U}^{2n-1}} e^{\beta_0(n,2n)u^2} - 1 - \beta_0(n,2n) \varrho^2 u^2 \frac{dzdt\vartheta}{\varrho^{n+1}} \leq C.$$
In terms of the ball model, we have that for all \( u \in C_0^\infty(\mathbb{B}_C^m) \) with

\[
4^n \int_{\mathbb{B}_C^m} \frac{f}{(1 - |z|^2)^3} \left( \Delta X - \frac{n + 1 - 2j}{2} (R - \bar{R}) + \frac{(n + 1 - 2j)^2}{4} \right) dz \leq 1,
\]

there holds

\[
\int_{\mathbb{B}_C^m} \frac{e^{\beta_0(n, 2n)(1 - |z|^2)u^2} - 1 - \beta_0(n, 2n)(1 - |z|^2)u^2}{(1 - |z|^2)^{n+1}} dz \leq C.
\]

1.3. Our Main Results. In this paper, we will consider higher order Poincaré-Sobolev and Hardy-Sobolev-Maz’ya inequalities on the remaining two rank one symmetric spaces of non-compact type, i.e., the quaternionic hyperbolic spaces \( H^n_Q \) and the Cayley hyperbolic plane \( H^2_O \). The first main result is the factorization theorems. We shall use the NA group model (or Damek-Ricci space) and the ball model. We note Damek-Ricci space (see [16] [17] [5]) is a solvable Lie group with a left invariant Riemannian structure which include all the rank one symmetric spaces of non-compact type.

We need citations for Damek-Ricci spaces. The Damek-Ricci space \( NA \) is a semi-direct product of \( A \cong \mathbb{R} \) with a group of Heisenberg type \( N \). Let \( n \) be a Lie algebra of \( N \), \( \mathfrak{j} \) be the centre of \( n \) and \( \mathfrak{h} \) its orthogonal complement. Denote by \( Q = \frac{1}{2} \dim \mathfrak{h} + \dim \mathfrak{j} \) the homogeneous dimension of \( N \). We parameterize the elements in \( N = \exp n \) by \((X, Z)\), for \( X \in \mathfrak{h} \) and \( Z \in \mathfrak{j} \). Then the group law is given by

\[
(X, Z)(X', Z') = (X + X', Z + Z' + \frac{1}{2}[X, X']).
\]

Thus the multiplication in \( S = NA \) is given by

\[
(X, Z, a)(X', Z', a') = (X + a^{1/2}X', Z + aZ' + \frac{1}{2}a^{1/2}[X, X'], aa'), \ a, a' > 0.
\]

Let \( \Delta_Z \) denote the Euclidean Laplacian on the center of \( N \) and let \( \mathcal{L}_0 \) denote the sub-Laplacian on \( N \). Let \( \varrho \) denote the \( A \)-coordinate of a general point in \( S \), and let \( \partial_{\varrho} \) denote the unit vector in the Lie algebra of \( A \). Then the Laplace-Beltrami operator \( \Delta_S \) on \( S \) is given by

\[
\Delta_S = 4\varrho [\varrho (\partial_{\varrho} + \Delta_Z) + \mathcal{L}_0 - (Q - 1)\partial_{\varrho}]
\]

and that the bottom of the spectrum of \(-\Delta_S\) is \( Q^2 \).

Firstly, we establish the factorization theorem on a Damek-Ricci space from which the factorization theorems on the quaternionic hyperbolic spaces and the Cayley hyperbolic plane follow naturally. We state it as follows.

**Theorem 1.1.** Let \( a \in \mathbb{R} \) and \( f \in C^\infty(U) \). There holds

\[
\varrho^{\frac{k+Q-2}{2}} \prod_{j=1}^k \left[ \varrho \partial_{\varrho} + a \varrho + \varrho \Delta_Z + \mathcal{L}_0 - i(k + 1 - 2j)\sqrt{-\Delta_Z} \right] \left( \varrho^{\frac{k-Q-2}{2}} f \right)
\]

\[
\prod_{j=1}^k \left[ \varrho [\varrho (\partial_{\varrho} + \Delta_Z) + \mathcal{L}_0 - (Q - 1)\partial_{\varrho}] + \frac{Q^2}{4} - \frac{(a - k + 2j - 2)^2}{4} \right] f.
\]

To state the factorization theorem on the ball model of \( H^n_Q \), we need to introduce some conventions. First recall that the quaternionic space \( \mathbb{Q}^m \) may be identified with \( \mathbb{C}^{2m} \) by the identification

\[
\mathbb{Q}^m \ni q = (q_1, \ldots, q_m) \leftrightarrow \mathbb{C}^{2m} \ni z = (z_1, \ldots, z_{2m}),
\]

where \( z_j = q_{2j-1} + q_{2j}i \), \( j = 1, \ldots, m \).
where \( q_j = z_j + z_{m+j}i_2 \). This allows us to write \( \Delta \) in terms of the complex coordinates \( z \):
\[
\Delta_{\alpha} f(z) = 4(1 - |z|^2) \sum_{i,j=1}^{m} \left( (\delta_{ij} - z_i \bar{z}_j - \bar{z}_m z_{m+j}) \frac{\partial^2 f}{\partial z_i \partial \bar{z}_j} f 
+ (\bar{z}_i z_{m+j} - z_{m+i} \bar{z}_j) \frac{\partial^2 f}{\partial z_{m+i} \partial \bar{z}_j} + (\bar{z}_m z_{i+j} - z_i \bar{z}_{m+j}) \frac{\partial^2 f}{\partial z_i \partial \bar{z}_{m+j}} 
+ (\delta_{ij} - \bar{z}_i z_j - z_i \bar{z}_{m+j}) \frac{\partial^2 f}{\partial z_{m+i} \partial \bar{z}_{m+j}} \right) + R + \bar{R}
\]
where now
\[
R = \sum_{j=1}^{2m} z_j \frac{\partial}{\partial z_j} \quad \text{and} \quad \bar{R} = \sum_{j=1}^{2m} \bar{z}_j \frac{\partial}{\partial \bar{z}_j}.
\]

We introduce the following “Quaternionic Geller operators”: given \( \alpha \in \mathbb{C} \), define the quaternionic Geller operator
\[
\Delta_{\alpha} f(z) = 4(1 - |z|^2) \sum_{i,j=1}^{m} \left( (\delta_{ij} - z_i \bar{z}_j - \bar{z}_m z_{m+j}) \frac{\partial^2 f}{\partial z_i \partial \bar{z}_j} f 
+ (\bar{z}_i z_{m+j} - z_{m+i} \bar{z}_j) \frac{\partial^2 f}{\partial z_{m+i} \partial \bar{z}_j} + (\bar{z}_m z_{i+j} - z_i \bar{z}_{m+j}) \frac{\partial^2 f}{\partial z_i \partial \bar{z}_{m+j}} 
+ (\delta_{ij} - \bar{z}_i z_j - z_i \bar{z}_{m+j}) \frac{\partial^2 f}{\partial z_{m+i} \partial \bar{z}_{m+j}} \right) + (1 + \alpha)(R + \bar{R}) - \alpha(\alpha + 1) \}
\]
In particular, \( \Delta_0 = \Delta_X \), and if we set
\[
\Delta'_{\alpha} = \frac{1}{4(1 - |z|^2)} \Delta_{\alpha},
\]
then
\[
\Delta'_{\alpha} = \Delta_0' + \alpha(R + \bar{R}) - \alpha(\alpha + 1).
\]

We emphasize the analogy between \( \Delta_\alpha \) and \( D_{\alpha, \beta} \) by pointing out the following intertwining relationships: for \( u \in C^\infty(\mathbb{B}_n^\mathbb{C}) \) and \( s \in \mathbb{R} \), there holds
\[
\Delta_{s-n, n} \left[(1 - |z|^2)^{s-n} u \right] = 4^{-1}(1 - |z|^2)^{s-n} [\Delta_{0,0} + 4s(n-s)] u \quad \text{on} \quad \mathbb{B}_n^\mathbb{C}
\]
and, for \( u \in C^\infty(\mathbb{B}_n^\mathbb{Q}) \) and \( s \in \mathbb{R} \), there holds
\[
\Delta_{s-2m-1} \left[(1 - |z|^2)^{s-2m-1} u \right] = (1 - |z|^2)^{s-2m-1} [\Delta_0 + 4s(2m + 1 - s)] \quad \text{on} \quad \mathbb{B}_n^\mathbb{Q}.
\]

Recall that the spectral gaps of \(-\Delta_{0,0}\) and \(-\Delta_0\) are \((2m + 1)^2\) and \(n^2\), respectively. Similarly, we can also define the Geller’s operators \( \Delta_\alpha \) on \( H^2_0 \) through the intertwining relationships in term of ball model
\[
\Delta_\alpha \left[(1 - |x|^2)^{s-11} u \right] = (1 - |x|^2)^{s-11} [\Delta_X + 4s(11 - s)],
\]
where 11 is the spectral gaps of \(-\Delta_X\) on \( H^2_0 \). Now we can state the factorization theorem on the ball model of \( H^m_n \).

**Theorem 1.2.** Let \( a \in \mathbb{R} \) and \( k \in \mathbb{N}_{>0} \). Set \( \Gamma = (R - \bar{R})^2 - 2D_1 \bar{D}_1 - 2 \bar{D}_1 D_1 \), where
\[
D_1 = \sum_{a=1}^{n} \left\{ \bar{z}_a \frac{\partial}{\partial \bar{z}_{n+a}} - \bar{z}_{n+a} \frac{\partial}{\partial \bar{z}_a} \right\},
\]
\[
\bar{D}_1 = \sum_{a=1}^{n} \left\{ z_a \frac{\partial}{\partial z_{n+a}} - z_{n+a} \frac{\partial}{\partial z_a} \right\}.
\]
Then, in the ball model, for all \( f \in C^\infty(B^m_Q) \), there holds
\[
4^k \left( 1 - |z|^2 \right)^{k+1/(2m+1)} \prod_{j=1}^k \left[ \Delta'_{(a-k+2j-2)/2} + \left( \frac{k+1-2j}{4} - i \frac{k+1-2j}{2} \sqrt{\Gamma+1} \right) f \right] = \prod_{j=1}^k \left[ \Delta_X + (2m+1)^2 - (a-k+2j-2)^2 \right] \left( 1 - |z|^2 \right)^{k+1/(2m+1)} f.
\]

The factorization theorem on \( H^2_Q \) in terms of the ball model is more complex than that in \( H^m_Q \) and \( H^2_Q \) and involve rather involved computations. We shall address it in a forthcoming paper.

The second main result is the higher order Poincaré-Sobolev inequality. Using precise Bessel-Green-Riesz and heat kernel estimates, we obtain the following:

**Theorem 1.3.** Let \( 0 < \gamma < 3, \ 0 < \gamma', \ 2 < p \) and \( 0 < \zeta \). Denote by \( N = \dim X \). If \( 0 < \gamma' < N - \gamma \), suppose further that \( 2 < p \leq \frac{2N}{N - (\gamma + \gamma')} \). Then there exists a constant \( C > 0 \) such that, for all \( u \in C^\infty_0 (X) \), there holds
\[
\| u \|_p \leq C \left\| \left( -\Delta_X - \rho_X^2 + \zeta^2 \right)^{\frac{\gamma'}{2}} \left( -\Delta - \rho_X^2 \right)^{\frac{\gamma}{2}} u \right\|_2.
\] (1.5)

Using Theorem 1.3 and the factorization Theorems 1.1 and 1.2, we obtain the following Hardy-Sobolev-Maz'ya inequalities on \( X \). Here we state only for \( H^m_Q \).

**Theorem 1.4.** Let \( a \in \mathbb{R}, \ 1 \leq k < 2m, \ 2 < p < \frac{4m}{2m-k} \) and \( \lambda \leq \prod_{j=1}^k \frac{(a-k+2j-2)^2}{4} \). Then there exists a constant \( C > 0 \) so that, for all \( u \in C^\infty_0 (U^m_Q) \), there holds
\[
\int_{H^m_Q} \int_0^\infty u \prod_{j=1}^k \left[ -\varrho \partial_{\varrho}^2 - \varrho \partial_\varrho - \varrho \Delta_0 - i(k+1-2j)\sqrt{-\Delta_0} \right] u \frac{dxzd\varrho}{\varrho^{1+a}} \geq C \left( \int_{H^m_Q} \int_0^\infty |u|^p \frac{(2m+1-k+2j)^p}{4} (2m-2) dxzd\varrho \right)^{\frac{2}{p}},
\]
where \( U^m_Q \) is the quaternionic Siegel domain and \( H^{m-1}_Q \) is the quaternionic Heisenberg group. In terms of the ball model, for all \( f \in C^\infty(B^m_Q) \), there holds
\[
\int_{B^m_Q} \prod_{j=1}^k \left[ \Delta'_{(a-k+2j-2)/2} + \left( \frac{k+1-2j}{4} - i \frac{k+1-2j}{2} \sqrt{\Gamma+1} \right) f \right] \frac{dz}{(1 - |z|^2)^{1-a}} = \lambda \int_{B^m_Q} \frac{f^2}{(1 - |z|^2)^{k+1-a}} dz \geq C \left( \int_{B^m_Q} |f|^p \frac{(2m+1-k+a)^p}{4} (2m-2) dz \right)^{\frac{2}{p}}.
\]

In the limiting case, we can establish the Adams inequalities on \( X \).

**Theorem 1.5.** Let \( 0 < \alpha < 3 \) and \( \zeta > 0 \). Then there exists a constant \( C > 0 \) such that for all \( u \in C^\infty_0 (X) \) with
\[
\| (-\Delta_X - \rho_X^2 + \zeta^2)^{(2n-a)/4} (-\Delta_X - \rho_X^2)^{\alpha/4} u \|_2 \leq 1,
\]
there holds
\[
\int_X \left(e^{\beta_0(N/2,N)u^2} - 1 - \beta_0(N/2,N)u^2\right) dV \leq C.
\]

As an application of Theorem 1.5 and the factorization theorem, we have the following Hardy-Adams inequalities on $X$. We also state only for $H^m_0$.

**Theorem 1.6.** Let $a \in \mathbb{R}$. There exists a constant $C > 0$ such that for all $u \in C_0^\infty(B^m_0)$ with
\[
4^m \int_{B^m_0} u^2 \prod_{j=1}^{2m} \left[ \frac{\Delta + (a - (2m + 1))}{4} + \frac{2m + 1 - 2j}{2}\sqrt{1 + 1} \right] u \frac{dz}{(1 - |z|^2)^{1-a}} - \prod_{j=1}^{2m} (a - 2m + 2j - 2)^2 \int_{B^m_0} \frac{u^2}{(1 - |z|^2)^{2m+1-a}} dz \leq 1,
\]
there holds
\[
\int_{B^m_0} e^{\beta_0(2m,4m)(1-|z|^2)^{2m+1-a}} u^2 \frac{dz}{(1 - |z|^2)^{2m+2}} - 1 - \beta_0(2m,4m)(1 - |z|^2)^{2m+1} u^2 dz \leq C.
\]

In terms of the Siegel domain model, we have for all $u \in C_0^\infty(U^m_0)$ with
\[
4^m \int_{U^m_0} u^2 \prod_{j=1}^{n} \left[ -\varrho \partial_{\varrho} - a \varrho \partial_{\varrho} - \varrho \Delta_z - L_0 + i(k + 1 - 2j)\sqrt{-\Delta_z} \right] u \frac{dz d\varrho}{\varrho^{1-a}} - \prod_{j=1}^{2m} (a - n + 2j - 2)^2 \int_{U^m_0} \frac{u^2}{\varrho^{2m+1-a}} dxdzd\varrho \leq 1,
\]
there holds
\[
\int_{U^m_0} \int_0^\infty e^{\beta_0(2m,4m)\varrho^a u^2} - 1 - \beta_0(2m,4m) \varrho^a u^2 dxdzd\varrho \leq C.
\]

Finally, we set up some Adams type inequalities on Sobolev spaces $W^{\alpha,\frac{N}{\alpha}}(X)$ on $X$ with dimension $N$ for arbitrary positive fractional order $\alpha < N$. More precisely, we have the following

**Theorem 1.7.** Let $N \geq 2$, $0 < \alpha < N$ be an arbitrary real positive number, $p = N/\alpha$ and $\zeta$ satisfies $\zeta > 0$ if $1 < p < 2$ and $\zeta > \rho_X(\frac{1}{2} - \frac{1}{p})$ if $p \geq 2$. Then for measurable $E$ with finite Riemannian volume measure in $X$, there exists $C = C(\zeta, \alpha, N, |E|)$ such that
\[
\frac{1}{|E|} \int_E \exp(\beta_0(\alpha, N)|u|^{p'}) dV \leq C
\]
for any $u \in W^{\alpha,p}(X)$ with $\int_X |(-\Delta_X - \rho_X^2 + \zeta^2)^{\frac{1}{2}} u|^p dV \leq 1$. Here $p' = \frac{p}{p-1}$. Furthermore, this inequality is sharp in the sense that if $\beta_0(\alpha, N)$ is replaced by any $\beta > \beta_0(\alpha, N)$, then the above inequality can no longer hold with some $C$ independent of $u$.

**Theorem 1.8.** Let $N \geq 2$, $0 < \alpha < N$ be an arbitrary real positive number, $p = N/\alpha$ and $\zeta$ satisfies $\zeta > 2\rho_X\left|\frac{1}{2} - \frac{1}{p}\right|$. Then there exists $C = C(\zeta, \gamma, n)$ such that
\[
\int_X \Phi_p(\beta_0(\alpha, N)|u|^{p'}) dV \leq C
\]
hold simultaneously for any \( u \in W^{\alpha,p}(\mathbb{X}) \) with \( \int_{\mathbb{X}} |(-\Delta_X - \rho_X^2 + \zeta^2)^{\frac{2}{p}} u|^p dV \leq 1 \). Here
\[
\Phi_p(t) = e^t - \sum_{j=0}^{j_p-2} \frac{t^j}{j!}, \quad j_p = \min\{j \in \mathbb{N} : j \geq p\}.
\]
Furthermore, this inequality is sharp in the sense that if \( \beta_0(\alpha,N) \) is replaced by any \( \beta > \beta(2n,\alpha) \), then the above inequality can no longer hold with some \( C \) independent of \( u \).

Notice that \( |\frac{1}{2} - \frac{1}{p}| < \frac{1}{2} \) provided \( p > 1 \). Choosing \( \zeta = \rho_X \) in Theorem 1.8, we have the following

**Corollary 1.1.** Let \( N \geq 2, 0 < \alpha < N \) be an arbitrary real positive number and \( p = N/\alpha \). There exists \( C = C(\alpha,n) \) such that
\[
\int_{\mathbb{X}} \Phi_p(\beta_0(\alpha,N)|u|^p) dV \leq C
\]
hold simultaneously for any \( u \in W^{\alpha,p}(\mathbb{X}) \) with \( \int_{\mathbb{X}} |(-\Delta_X)^{\frac{\alpha}{2}} u|^p dV \leq 1 \).

The organization of the paper is as follows: In Section 2, we recall some necessary preliminary facts of quaternionic hyperbolic spaces and the Cayley hyperbolic plane. We shall prove the factorization theorem, namely Theorem 1.1 and 1.2, in Section 3. In section 4, we recall some necessary facts of Funk-Hecke formulas for \( Sp(m) \times Sp(1) \) and \( Spin(9) \) and use them to compute some integrals in term of hypergeometric function. Sharp estimates of Bessel-Green-Riesz kernels and their rearrangement estimates are given in Section 5 and Section 6, respectively. We shall prove the higher order Hardy-Sobolev-Maz’ya inequalities, namely Theorem 1.3 and 1.4, in Section 7. In Section 8, we prove the Hardy-Adams inequality, namely Theorem 1.5 and 1.6. In Section 9, we show the Adams type inequality, namely Theorem 1.7 and 1.8.

2. Preliminaries

We begin by setting up notations and then recall proper definitions shortly after.

Let \( \mathbb{Q} \) and \( \mathbb{Ca} \) denote, respectively, the quaternions and the Cayley algebra (i.e., octonions). Let \( H^m_\mathbb{Q} \) denote the quaternionic hyperbolic space of real dimension \( 4m \), and let \( H^m_\mathbb{Ca} \) denote the Cayley plane of real dimension \( 16m \). In general, we will use \( \mathbb{F} \) to denote any of the three normed division algebras \( \{\mathbb{C}, \mathbb{Q}, \mathbb{Ca}\} \) and \( H^m_\mathbb{F} \) to denote the corresponding hyperbolic space with \( \mathbb{F} \)-dimension \( m \). We recall that \( H^m_\mathbb{F} \) is a Riemannian symmetric space and that, as homogeneous spaces, there hold \( H^m_\mathbb{Q} = Sp(m,1)/Sp(m) \times Sp(1) \) and \( H^m_\mathbb{Ca} = F_4/Spin(9) \). Since there is only one Cayley plane, we shall often remove dimensional superscript and subscript decorations whenever specifying \( \mathbb{F} = \mathbb{Ca} \); e.g., \( H^m_\mathbb{F} \) with \( \mathbb{F} = \mathbb{Ca} \) shall be written simply as \( H^m_\mathbb{Ca} \).

We will also use \( \mathbb{B}^m_\mathbb{F} \subset \mathbb{F}^m \) and \( H^m_\mathbb{F} \) to denote \( H^m_\mathbb{F} \) when realized, respectively, in the Beltrami-Klein ball model and Siegel domain model. Let \( S^{4m-1} = \partial \mathbb{B}^m_\mathbb{Q} \) and \( S^{15} = \partial \mathbb{B}^m_\mathbb{Ca} \) denote, respectively, the quaternionic and octonionic spheres, and let \( d\sigma \) denote the round measure (i.e., the standard surface measure endowed from the ambient Euclidean space). Note that \( \mathbb{B}^m_\mathbb{Ca} \subset \mathbb{Ca}^2 = \mathbb{R}^{16} \).

Next, we let \( H^m_\mathbb{Q} \) denote the Heisenberg group over \( \mathbb{F} \in \{\mathbb{C}, \mathbb{Q}, \mathbb{Ca}\} \) and let \( Z = Z(H^m) \) denote the center of \( H^m_\mathbb{Q} \). We make the identifications \( \mathbb{H}^m_\mathbb{F} = \mathbb{R}^{2m} \times \mathbb{R}, \mathbb{H}^m_\mathbb{Q} = \mathbb{R}^{4m} \times \mathbb{R}^3 \) and \( \mathbb{H}^m_\mathbb{Ca} = \mathbb{R}^8 \times \mathbb{R}^7 \) and note \( Z(\mathbb{H}^m_\mathbb{F}) = \mathbb{R}, Z(\mathbb{H}^m_\mathbb{Q}) = \mathbb{R}^3 \) and \( Z(\mathbb{H}^m_\mathbb{Ca}) = \mathbb{R}^7 \). The homogeneous dimension of \( \mathbb{H}^m_\mathbb{F} \) is given by \( Q = \dim_{\mathbb{R}} \mathbb{H}^m_\mathbb{F} + \dim_{\mathbb{R}} \text{Im} \mathbb{F} \). In particular, the homogeneous dimensions for \( \mathbb{H}^m_\mathbb{F}, \mathbb{H}^m_\mathbb{Q} \) and \( \mathbb{H}^m_\mathbb{Ca} \) are, respectively, \( 2n + 2, 4n + 6 \) and \( 22 \).
Recalling that the boundary of $H^m_F$ has a natural group structure given by $\mathbb{H}^{m-1}_F$, we shall choose the normalization of the metric on $H^m_F$ and sign convention on $\Delta_X$ so that
\[
\text{spec}(-\Delta_X) = \left[\frac{Q^2}{4}, \infty\right).
\]
We recall that $Q/2$ also has the interpretation as $\rho_X$, the half sum of positive roots of $H^m_F$ counted with multiplicities. In particular, on $H^m_C$, $H^m_Q$ and $H_C$ we evaluate $Q/2$ to be, respectively, $m$, $2m + 1$, and $11$.

For the convenience of the reader, we include a short dictionary of the Laplacians considered in this paper:

- $\Delta \leftrightarrow$ Laplace-Beltrami operator on $H^m_F$ when $F = Q$ or $C$
- $\Delta_{H^m_F} \leftrightarrow$ Laplace-Beltrami operator on $H^m_F$ for a specified $n$
- $\Delta_Z \leftrightarrow$ Euclidean Laplacian on the center $Z = Z(\mathbb{H}^{m-1}_F)$
- $\Delta_b \leftrightarrow$ The sub-Laplacian on $H^m_F$.

In the ball model, the Riemannian volume forms on $H^m_Q$ and $H_C$ are given, respectively, by
\[
dV = \frac{dz}{(1 - |z|^2)^{2m+2}},
\]
and
\[
dV = \frac{dx}{(1 - |x|^2)^{2}},
\]
where $dz$ and $dx$ denote, respectively, the Lebesgue measure on $\mathbb{C}^m$ and $\mathbb{R}^{16}$.

2.1. Automorphisms and Convolution. In this section, we recall a family of automorphisms on $B^m_Q$ which are isometries and which are used to define convolution on $B^m_Q$. Analogous automorphisms are also defined for $B_C$ but require more notation and thus we direct the reader to [52, pg. 56] for formal definitions.

Following [52], we define for each $w \in B^m_Q$ the automorphism $\varphi_w : B^m_Q \to B^m_Q$ given by
\[
\varphi_w(z) = (1 - \langle z, w \rangle_Q)^{-1} \left( w - P_w(z) - \sqrt{1 - |w|^2} Q_w(z) \right)
\]
where
\[
P_w(z) = \begin{cases} 
\langle z, w \rangle_Q |w|^2 w & \text{if } w \neq 0 \\
0 & \text{if } w = 0
\end{cases}
\]
and
\[
Q_w(z) = z - P_w(z).
\]

We recall some properties of these automorphisms in the following proposition (see [52]). Note that property (iv) is not present in [52], but it is straightforward to prove.

**Lemma A.** For each $w \in B^m_Q$, $\varphi_w$ satisfies the following properties:

(i) $\varphi_w(0) = w$ and $\varphi_w(w) = 0$;
(ii) for $z \in B^m_Q$, there holds
\[
1 - |\varphi_w(z)|^2 = \frac{(1 - |w|^2)(1 - |z|^2)}{1 - \langle z, w \rangle_Q^2};
\]
(iii) $\varphi_w$ is an involutory isometry of $B^m_Q$;
(iv) for $z \in B^m_Q$, there holds
\[
\sinh (\rho(\varphi_w(z))) = \frac{|\varphi_w(z)|}{\sqrt{1 - |\varphi_w(z)|^2}}
\]
We will use \( \varphi_w \) to also denote the analogous automorphisms on \( \mathbb{B}_C \). We record in the following lemma the analogues to the properties recorded in the preceding lemma. In preparation, if \( z = (z_1, z_2), w = (w_1, w_2) \in \mathbb{B}_C \subset \mathbb{C}^2 \), then let

\[
\Psi_C(z, w) = \begin{cases} 
1 - (\bar{z}_1 w_2)(w_2^{-1} w_1) - z_2 \bar{w}_2 & \text{if } w_2 \neq 0 \\
1 - \bar{z}_1 w_1^2 & \text{if } w_2 = 0.
\end{cases}
\]

We also have \( \Psi_C(z, w) = \Phi_C(z, w) - 2\langle z, w \rangle + 1 \), where

\[
\Phi_C(z, w) = |z_1|^2 |w_1|^2 + |z_2|^2 |w_2|^2 + 2 \text{Re} \left( (z_1 z_2)(\bar{w}_1 \bar{w}_2) \right),
\]

and \( \langle \cdot, \cdot \rangle \) is the Euclidean inner product on \( \mathbb{R}^4 \). We also remark that \( \Phi_C(z, w) \) is an analogue of the form \( |\langle z, w \rangle|_F^2 \), and \( \Psi_C(z, w) \) is an analogue of the form \( |1 - \langle z, w \rangle|_F^2 \), where \( F \in \{ \mathbb{R}, \mathbb{C}, \mathbb{Q} \} \). We point out that \( \Psi_C(z, w) \leq |z|^2 |w|^2 \).

**Lemma B.** For each \( w \in \mathbb{B}_C \), \( \varphi_w \) satisfies the following properties:

(i) \( \varphi_w(0) = w \) and \( \varphi_w(w) = 0 \);

(ii) for \( z \in \overline{\mathbb{B}_C} \), there holds

\[
1 - |\varphi_w(z)|^2 = \frac{(1 - |w|^2)(1 - |z|^2)}{\Psi_C(z, w)};
\]

(iii) \( \varphi_w \) is an involutory isometry of \( \mathbb{B}_C \);

(iv) for \( z \in \overline{\mathbb{B}_C} \), there holds

\[
\sinh (\rho (\varphi_w(z))) = \frac{|\varphi_w(z)|}{\sqrt{1 - |\varphi_w(z)|^2}} = \left( \frac{\Psi_C(z, w) - (1 - |z|^2)(1 - |w|^2)}{(1 - |w|^2)(1 - |z|^2)} \right) \frac{1}{2}
\]

\[
\cosh (\rho (\varphi_w(z))) = \frac{1}{\sqrt{1 - |\varphi_w(z)|^2}} = \frac{\sqrt{\Psi_C(z, w)}}{(1 - |w|^2)(1 - |z|^2)}.
\]

With these automorphisms defined, we may introduce the following convolution on \( \mathbb{B}_F^m \):

for two functions \( f, g \) on \( \mathbb{B}_F^m \), let

\[
(f * g)(z) = \int_{\mathbb{B}_F^m} f(\varphi_w(z)) g(w) dV(w),
\]

whenever this is well-defined. It is easy to see that, if \( f \) is radial, then \( f * g = g * f \), when defined.
2.2. Helgason-Fourier Transform on Quaternionic Hyperbolic Spaces and Cayley Plane. In this section, we recall the Helgason-Fourier transforms on the quaternionic hyperbolic spaces and Cayley plane, as well as the resulting Plancherel and inversion formulas. (see [26, 27, 51].) Given a function \( f \) on \( \mathbb{B}_Q^m \), the Helgason-Fourier transform \( \hat{f} \) is defined by the formula

\[
\hat{f}(\lambda, \varsigma) = \int_{\mathbb{B}_Q^m} f(z) e^{-\lambda, \varsigma}(z) dV,
\]

for \( \lambda \in \mathbb{R} \) and \( \varsigma \in S^{4m-1} \), provided this integral exists. Here,

\[
e_{\lambda, \varsigma}(z) = \left( \frac{1 - |z|^2}{|1 - \langle z, \varsigma \rangle_Q|^2} \right)^{(2m+1)+i\lambda/2},
\]

defined for \( z \in \mathbb{B}_Q^m \), \( \lambda \in \mathbb{R} \) and \( \varsigma \in S^{4m-1} \), are eigenfunctions of \( \Delta \) with eigenvalue \(-(2m+1)^2 - \lambda^2\). Note that, for \( z \in \mathbb{B}_Q^m \) and \( \varsigma \in S^{4m-1} \), the function

\[
\left( \frac{1 - |z|^2}{|1 - \langle z, \varsigma \rangle_Q|^2} \right)^{2m+1},
\]

is the Poisson kernel on \( \mathbb{B}_Q^m \).

Analogously, if \( f \) is a function on \( \mathbb{B}_{Ca}^m \), then its Helgason-Fourier transform \( \hat{f} \) is defined by the formula

\[
\hat{f}(\lambda, \varsigma) = \int_{\mathbb{B}_Q^m} f(z) e^{-\lambda, \varsigma}(z) dV,
\]

for \( \lambda \in \mathbb{R} \) and \( \varsigma \in S^{4m-1} \), provided this integral exists, where now

\[
e_{\lambda, \varsigma}(z) = \left( \frac{1 - |z|^2}{\Psi_{Ca}(z, \varsigma)} \right)^{11+i\lambda/2},
\]

defined for \( z \in \mathbb{B}_{Ca}^m \), \( \lambda \in \mathbb{R} \) and \( \varsigma \in S^{15} \), are eigenfunctions of \( \Delta \) with eigenvalue \(-121 - \lambda^2\). Note that, for \( z \in \mathbb{B}_Q^m \) and \( \varsigma \in S^{4m-1} \), the function

\[
\left( \frac{1 - |z|^2}{\Psi_{Ca}(z, \varsigma)} \right)^{11},
\]

is the Poisson kernel on \( \mathbb{B}_{Ca}^m \).

The Helgason-Fourier transform enjoys the following properties:

(i) For \( f, g \in C_0^\infty(\mathbb{B}_F^m) \) and \( g \) radial, there holds

\[
\hat{f} \ast g = \hat{f} \cdot \hat{g}.
\]

(ii) For \( f \in C_0^\infty(\mathbb{B}_F^m) \), there holds the inversion formula:

\[
f(z) = C_m \int_{-\infty}^{\infty} \int_{S_F^1} \hat{f}(\lambda, \varsigma) e_{\lambda, \varsigma}(z) |\varsigma(\lambda)|^{-2} d\lambda d\sigma(\varsigma), \tag{2.1}
\]

where \( C_m \) is a positive constant and \( \varsigma(\lambda) \) is the Harish-Chandra \( \varsigma \)-function; see [26, pg. 436] for an explicit formula.

(iii) For \( f \in C_0^\infty(\mathbb{B}_F^m) \), there holds the Plancherel formula:

\[
\int_{\mathbb{B}_F^m} |f(z)|^2 dV = C_m \int_{-\infty}^{\infty} \int_{S_F^1} |\hat{f}(\lambda, \varsigma)|^2 |\varsigma(\lambda)|^{-2} d\lambda d\sigma(\varsigma). \tag{2.2}
\]
(iv) For \( f \in C_0^\infty(\mathbb{R}^m) \), there holds
\[
\tilde{\Delta}f(\lambda, \varsigma) = -\left( \lambda^2 + \frac{Q^2}{4} \right) f(\lambda, \varsigma).
\]

3. Factorization Theorems for the Operators on \( \mathbb{X} \): proof of Theorem 1.1 and Theorem 1.2

3.1. The Factorization Theorem on Damek-Ricci space.

Lemma 3.1. Let \( a \in \mathbb{R} \) and \( f \in C^\infty(U) \). There holds
\[
[\theta \partial_{\theta \theta} + a \partial_{\theta} + \theta \Delta_Z + L_0] \left( \theta^{1-\frac{Q-\beta}{2}} f \right)
= \theta^{\frac{Q}{2}-1} \left\{ \theta \left[ \theta (\partial_{\theta \theta} + \Delta_Z) + L_0 - (Q-1)\theta + \frac{Q^2}{4} - \frac{(a-1)^2}{4} \right] f \right\}.
\]

Proof. For reference, we provide explicit computations as follows. Observing that, for any \( \beta \in \mathbb{R} \), there holds
\[
\theta^{\beta+1} [\theta \partial_{\theta \theta} + a \partial_{\theta} + \theta \Delta_Z + L_0] (\theta^{-\beta} f)
= \theta \left[ \theta (\partial_{\theta \theta} + \Delta_Z) + L_0 - (2\beta - a)\theta + \beta(\beta + 1 - a) f \right],
\]
we may choose \( \beta = \frac{Q-1+a}{2} \) to obtain
\[
\theta^{\frac{Q}{2}-1} \left\{ \theta \left[ \theta (\partial_{\theta \theta} + \Delta_Z) + L_0 - (Q-1)\theta + \frac{Q^2}{4} - \frac{(a-1)^2}{4} \right] f \right\}.
\]
The desired result follows. \( \square \)

Lemma 3.2. Let \( \beta \in \mathbb{R} \). There holds
\[
[\theta \partial_{\theta \theta} + (a + \beta) \partial_{\theta} + \theta \Delta_Z + L_0] \left\{ [\theta \partial_{\theta \theta} + (a - 1)\partial_{\theta} + \theta \Delta_Z + L_0]^2 + (\beta - 1)^2 \Delta_Z \right\}
= \left\{ [\theta \partial_{\theta \theta} + a \partial_{\theta} + \theta \Delta_Z + L_0]^2 + \beta^2 \Delta_Z \right\} [\theta \partial_{\theta \theta} + (a + \beta - 2) \partial_{\theta} + \theta \Delta_Z + L_0].
\]

Proof. Since
\[
\partial_{\theta} [\theta \partial_{\theta \theta} + (a - 1)\partial_{\theta} + \theta \Delta_Z + L_0] = [\theta \partial_{\theta \theta} + a \partial_{\theta} + \theta \Delta_Z + L_0] \partial_{\theta} + \Delta_Z.
\]
we have
\[
\partial_{\theta} [\theta \partial_{\theta \theta} + (a - 1)\partial_{\theta} + \theta \Delta_Z + L_0]^2
= [\theta \partial_{\theta \theta} + a \partial_{\theta} + \theta \Delta_Z + L_0] \partial_{\theta} [\theta \partial_{\theta \theta} + (a - 1)\partial_{\theta} + \theta \Delta_Z + L_0]
+ [\theta \partial_{\theta \theta} + (a - 1)\partial_{\theta} + \theta \Delta_Z + L_0] \Delta_Z
= [\theta \partial_{\theta \theta} + a \partial_{\theta} + \theta \Delta_Z + L_0]^2 \partial_{\theta} + [\theta \partial_{\theta \theta} + a \partial_{\theta} + \theta \Delta_Z + L_0] \Delta_Z
+ [\theta \partial_{\theta \theta} + (a - 1)\partial_{\theta} + \theta \Delta_Z + L_0] \Delta_Z
= [\theta \partial_{\theta \theta} + a \partial_{\theta} + \theta \Delta_Z + L_0]^2 \partial_{\theta} + 2 [\theta \partial_{\theta \theta} + a \partial_{\theta} + \theta \Delta_Z + L_0] \Delta_Z - \Delta_Z \partial_{\theta}.
\]
Similarly,
\[
[\theta \partial_{\theta \theta} + (a - 1)\partial_{\theta} + \theta \Delta_Z + L_0]^2
= [\theta \partial_{\theta \theta} + a \partial_{\theta} + \theta \Delta_Z + L_0] [\theta \partial_{\theta} + (a - 1)\partial_{\theta} + \theta \Delta_Z + L_0]
- \partial_{\theta} [\theta \partial_{\theta \theta} + (a - 1)\partial_{\theta} + \theta \Delta_Z + L_0]
= [\theta \partial_{\theta \theta} + a \partial_{\theta} + \theta \Delta_Z + L_0]^2 - 2 [\theta \partial_{\theta \theta} + a \partial_{\theta} + \theta \Delta_Z + L_0] \partial_{\theta} - \Delta_Z.
Combining these two computations, we obtain
\[ [g\partial_{ee} + (a + \beta)\partial_e + g\Delta_z + L_0] \{ [g\partial_{ee} + (a - 1)\partial_e + g\Delta_z + L_0]^2 + (\beta - 1)^2\Delta_z \} \]
\[ = [g\partial_{ee} + a\partial_e + g\Delta_z + L_0] \{ [g\partial_{ee} + (a - 1)\partial_e + g\Delta_z + L_0]^2 + (\beta - 1)^2\Delta_z \} \]
\[ + \beta \partial_e \{ [g\partial_{ee} + (a - 1)\partial_e + g\Delta_z + L_0]^2 + (\beta - 1)^2\Delta_z \} \]
\[ = [g\partial_{ee} + a\partial_e + g\Delta_z + L_0] \cdot \{ [g\partial_{ee} + a\partial_e + g\Delta_z + L_0]^2 - 2 [g\partial_{ee} + a\partial_e + g\Delta_z + L_0] \partial_e + \beta(\beta - 2)\Delta_z \} \]
\[ + \beta \{ [g\partial_{ee} + a\partial_e + g\Delta_z + L_0]^2 \partial_e + 2 [g\partial_{ee} + a\partial_e + g\Delta_z + L_0] \Delta_z + \beta(\beta - 2)\Delta_z \partial_e \} \]
\[ = \{ [g\partial_{ee} + a\partial_e + g\Delta_z + L_0]^2 + 2\beta^2\Delta_z \} [g\partial_{ee} + (a + \beta - 2)\partial_e + g\Delta_z + L_0] . \]

This provides the desired identity. \qed

**Lemma 3.3.** For \( k \in \mathbb{N} \setminus \{0\} \), there holds
\[ [g\partial_{ee} + (a + 2k)\partial_e + g\Delta_z + L_0] \prod_{j=1}^{k} \{ [g\partial_{ee} + (a - 1)\partial_e + g\Delta_z + L_0]^2 + (2j - 1)^2\Delta_z \} \]
\[ = (g\partial_{ee} + a\partial_e + g\Delta_z + L_0) \prod_{j=1}^{k} \{ [g\partial_{ee} + (a - 1)\partial_e + g\Delta_z + L_0]^2 + 4j^2\Delta_z \} , \]
and
\[ [g\partial_{ee} + (a + 2k)\partial_e + g\Delta_z + L_0] \]
\[ \left\{ (g\partial_{ee} + a\partial_e + g\Delta_z + L_0) \prod_{j=1}^{k-1} \{ [g\partial_{ee} + (a - 1)\partial_e + g\Delta_z + L_0]^2 + 4j^2\Delta_z \} \right\} \]
\[ = \prod_{j=1}^{k} \{ [g\partial_{ee} + (a - 1)\partial_e + g\Delta_z + L_0]^2 + (2j - 1)^2\Delta_z \} . \]

**Proof.** By Lemma 3.2, we have
\[ [g\partial_{ee} + (a + 2k)\partial_e + g\Delta_z + L_0] \prod_{j=1}^{k} \{ [g\partial_{ee} + (a - 1)\partial_e + g\Delta_z + L_0]^2 + (2j - 1)^2\Delta_z \} \]
\[ = [g\partial_{ee} + (a + 2k)\partial_e + g\Delta_z + L_0] \prod_{j=1}^{k-1} \{ [g\partial_{ee} + (a - 1)\partial_e + g\Delta_z + L_0]^2 + (2j - 1)^2\Delta_z \} \]
\[ \cdot \prod_{j=1}^{k-1} \{ [g\partial_{ee} + (a - 1)\partial_e + g\Delta_z + L_0]^2 + (2j - 1)^2\Delta_z \} \]
\[ = \{ [g\partial_{ee} + a\partial_e + g\Delta_z + L_0]^2 + 4k^2\Delta_z \} \]
\[ \cdot [g\partial_{ee} + (a + 2k - 2)\partial_e + g\Delta_z + L_0] \prod_{j=1}^{k-1} \{ [g\partial_{ee} + (a - 1)\partial_e + g\Delta_z + L_0]^2 + (2j - 1)^2\Delta_z \} . \]

By repeating this process, we get the first identity in the lemma. The second identity is similarly obtained. \qed

**Proof of Theorem 1.1** It is sufficient to show
\[ \prod_{j=1}^{k} \left[ g\partial_{ee} + a\partial_e + g\Delta_z + L_0 - i(k + 1 - 2j)\sqrt{-\Delta_z} \right] \left( \theta^{k-\frac{q-n}{2}} f \right) \]
If we make the substitution

\[
\Delta_i f(z) = 4 \left(1 + |z|^2 \right) \left\{ \sum_{i,j=1}^m \left( \delta_{ij} - z_i \bar{z}_j - \bar{z}_{m+i} z_{m+j} \right) \frac{\partial^2 f}{\partial z_i \partial \bar{z}_j} + \left( \bar{z}_i z_{m+j} - z_{m+i} \bar{z}_j \right) \frac{\partial^2 f}{\partial z_{m+i} \partial \bar{z}_j} \right. \\
\left. + \left( \bar{z}_i z_{m+j} - z_{m+i} \bar{z}_j \right) \frac{\partial^2 f}{\partial z_i \partial \bar{z}_m} \right. \\
\left. + \left( \bar{z}_i z_{m+j} - z_{m+i} \bar{z}_j \right) \frac{\partial^2 f}{\partial z_i \partial \bar{z}_{m+j}} \right)
\]

We shall prove the lemma by induction. We have by Lemma 3.1, the identity above is valid for \( k = 1 \). Now assume it is valid for \( k = l \), i.e.,

\[
\prod_{j=1}^l \left[ \varrho \partial_{ee} + a \partial_o + \varrho \Delta Z + \mathcal{L}_0 - i(l + 1 - 2j) \sqrt{-\Delta_Z} \right] \left( \varrho^{-\frac{Q-3}{2}} f \right) 
\]

making the substitution \( a \to a - 1 \), we obtain

\[
\prod_{j=1}^l \left[ \varrho \partial_{ee} + (a - 1) \partial_o + \varrho \Delta Z + \mathcal{L}_0 - i(l + 1 - 2j) \sqrt{-\Delta_Z} \right] \left( \varrho^{-\frac{Q-3}{2}} f \right) 
\]

If \( l \) is even, then Lemma 3.3 gives us

\[
\varrho \partial_{ee} + (a + l) \partial_o + \varrho \Delta Z + \mathcal{L}_0 \prod_{j=1}^l \left[ \varrho \partial_{ee} + (a - 1) \partial_o + \varrho \Delta Z + \mathcal{L}_0 - i(l + 2 - 2j) \sqrt{-\Delta_Z} \right] 
\]

Therefore, by Lemma 3.1, there holds

\[
\prod_{j=1}^l \left[ \varrho \partial_{ee} + (a - 1) \partial_o + \varrho \Delta Z + \mathcal{L}_0 \right] 
\]

We shall prove the lemma by induction. We have by Lemma 3.1, the identity above is valid for \( k = 1 \). Now assume it is valid for \( k = l \), i.e.,

\[
\prod_{j=1}^l \left[ \varrho \partial_{ee} + a \partial_o + \varrho \Delta Z + \mathcal{L}_0 - i(l + 1 - 2j) \sqrt{-\Delta_Z} \right] \left( \varrho^{-\frac{Q-3}{2}} f \right) 
\]

making the substitution \( a \to a - 1 \), we obtain

\[
\prod_{j=1}^l \left[ \varrho \partial_{ee} + (a - 1) \partial_o + \varrho \Delta Z + \mathcal{L}_0 - i(l + 1 - 2j) \sqrt{-\Delta_Z} \right] \left( \varrho^{-\frac{Q-3}{2}} f \right) 
\]

If \( l \) is even, then Lemma 3.3 gives us

\[
\varrho \partial_{ee} + (a + l) \partial_o + \varrho \Delta Z + \mathcal{L}_0 \prod_{j=1}^l \left[ \varrho \partial_{ee} + (a - 1) \partial_o + \varrho \Delta Z + \mathcal{L}_0 - i(l + 1 - 2j) \sqrt{-\Delta_Z} \right] 
\]

Therefore, by Lemma 3.1, there holds

\[
\prod_{j=1}^l \left[ \varrho \partial_{ee} + (a - 1) \partial_o + \varrho \Delta Z + \mathcal{L}_0 \right] 
\]

We shall prove the lemma by induction. We have by Lemma 3.1, the identity above is valid for \( k = 1 \). Now assume it is valid for \( k = l \), i.e.,

\[
\prod_{j=1}^l \left[ \varrho \partial_{ee} + a \partial_o + \varrho \Delta Z + \mathcal{L}_0 - i(l + 1 - 2j) \sqrt{-\Delta_Z} \right] \left( \varrho^{-\frac{Q-3}{2}} f \right) 
\]

making the substitution \( a \to a - 1 \), we obtain

\[
\prod_{j=1}^l \left[ \varrho \partial_{ee} + (a - 1) \partial_o + \varrho \Delta Z + \mathcal{L}_0 - i(l + 1 - 2j) \sqrt{-\Delta_Z} \right] \left( \varrho^{-\frac{Q-3}{2}} f \right) 
\]
Furthermore, if $\Delta = \Delta + (\alpha - \beta)(R + \bar{R}) + (\beta - \alpha)(\beta + \alpha + 1)$.  

Denote by $r = |z|$ and 

$$\rho = \frac{1}{2} \ln \frac{1 + r}{1 - r}.$$ 

Then 

$$\cosh \rho = \frac{1}{\sqrt{1 - r^2}}, \quad \sinh \rho = \frac{r}{\sqrt{1 - r^2}}, \quad \partial_p = (1 - r^2)\partial_r.$$  

Furthermore, if $f = f(\rho)$, then 

$$\Delta f(\rho) = \partial^2_{\rho} f + ((4m - 1)\coth \rho + 3 \tanh \rho)\partial_{\rho} f.$$ 

By using the identity $\Delta (fg) = g\Delta f + 2(\nabla f, \nabla g) + f\Delta g$ and (3.3), we have 

$$\Delta[(\cosh \rho)^a f] = \partial^2_{\rho} f + (4m + a + 2)\alpha(\cosh \rho)^{a^2} \partial_{\rho} f + 2a(\cosh \rho)^{a-1} \sinh \rho \partial_{\rho} f + (\cosh \rho)^a \Delta f \quad (\because \langle \nabla \rho, \nabla f \rangle = \partial_{\rho} f).$$ 

i.e.

$$\Delta - (4m + a + 2)\alpha][(\cosh \rho)^a f] = \Delta - (4m + a + 2)\alpha][(1 - |z|^2)^{\frac{a}{2}} f]$$ 

$$= (\cosh \rho)^{a-2} \left[ (\cosh \rho)^2 \Delta + 2a \tanh \rho \partial_{\rho} f - a(a + 2) \right] f$$ 

$$= (\cosh \rho)^{a-2} \left[ 4\Delta_0 + 2ar \partial_r - a(a + 2) \right] f$$ 

$$= 4(\cosh \rho)^{a-2} \Delta_0 f.$$ 

We are now ready to give the **Proof of Theorem 1.2**. It suffices to show the following

$$4^k (\cosh \rho)^{-k-a-(2m+1)} \prod_{j=1}^{k} \left[ \Delta'_{1-a-(2m+1)} + \frac{(k + 1 - 2j)^2}{4} - i \frac{k + 1 - 2j}{2} \sqrt{\Gamma + 1} \right] f$$ 

$$= \prod_{j=1}^{k} \left[ \Delta + (2m + 1)^2 - (a - k + 2j - 2)^2 \right] \left[ (\cosh \rho)^{k-a-(2m+1)} f \right].$$ 

We shall prove it by induction. For $k = 1$, we have, by (3.4), 

$$\left[ \Delta + (2m + 1)^2 - (a - 1)^2 \right] \left[ (\cosh \rho)^{1-a-(2m+1)} f \right]$$ 

$$= 4(\cosh \rho)^{1-a-(2m+1)} \Delta_0 f.$$ 

Assume it holds for $k$, replacing $a$ by $a - 1$, we have 

$$4^k (\cosh \rho)^{-k+1-a-(2m+1)} \prod_{j=1}^{k} \left[ \Delta'_{1-a-(2m+1)} + \frac{(k + 1 - 2j)^2}{4} - i \frac{k + 1 - 2j}{2} \sqrt{\Gamma + 1} \right] f$$ 

$$= \prod_{j=1}^{k} \left[ \Delta + (2m + 1)^2 - (a - 1 - k + 2j - 2)^2 \right] \left[ (\cosh \rho)^{k+1-a-(2m+1)} f \right].$$
Then for $k + 1$, we have, by using (3.5) and (3.6),

$$
\prod_{j=1}^{k+1} \left[ \Delta + (2m + 1)^2 - (a - 1 - k + 2j - 2)^2 \right] \left[ (\cosh \rho)^{k+1-a-(2m+1)} f \right] = \left[ \Delta + (2m + 1)^2 - (a - 1 + k)^2 \right]$$

$$
\left\{ 4^k (\cosh \rho)^{-k+1-a-(2m+1)} \prod_{j=1}^{k} \left[ \Delta_{2-a-(2m+1)}' + \frac{(k + 1 - 2j)^2}{4} - i \frac{k + 1 - 2j}{2} \sqrt{\Gamma + 1} \right] f \right\}
$$

$$= 4^{k+1} (\cosh \rho)^{-k+1-a-(2m+1)}$$

$$\Delta_{k-a-(2m+1)/2} \prod_{j=1}^{k} \left[ \Delta_{2-a-(2m+1)}' + \frac{(k + 1 - 2j)^2}{4} - i \frac{k + 1 - 2j}{2} \sqrt{\Gamma + 1} \right] f. \quad (3.7)
$$

The rest of the proof is similar to that given in [46] by using Lemma 3.5 and we omit it. The proof of Theorem 1.2 is thereby completed.

Before the proof of Lemma 3.5, we need the following:

**Lemma 3.4.** There holds

$$[\Delta_{0'} [R + \bar{R}]] = \Delta_0' - \frac{1}{2} (R + \bar{R}) + \frac{1}{4} (R + \bar{R})^2 - \frac{1}{4} \Gamma.$$

**Proof.** We compute

$$D_1 \bar{D}_1 = (\bar{z}_j \partial_{m+i} - \bar{z}_{m+j} \partial_j) \left( z_i \bar{\partial}_{m+i} - z_{m+i} \bar{\partial}_i \right)$$

$$= z_i \bar{z}_j \partial_{m+i} \partial_{m+j} - z_i \bar{\partial}_i \partial_{m+j} - z_{m+i} \bar{\partial}_i \partial_{m+j} - \bar{z}_{m+j} z_i \bar{\partial}_i \partial_{m+i} + \bar{z}_{m+j} \partial_{m+i} \partial_{m+j}$$

and so

$$-2 D_1 \bar{D}_1 - 2 \bar{D}_1 D_1 = 2 (R + \bar{R}) - 4 \sum_{i,j=1}^{m} \left( z_i \bar{z}_j \partial_{m+i} \partial_{m+j} + z_{m+i} \bar{z}_{m+j} \partial_i \partial_j \right)$$

$$+ 4 \sum_{i,j=1}^{m} \left( z_i \bar{z}_{m+j} \bar{\partial}_{m+i} \partial_j + z_{m+i} \bar{z}_j \bar{\partial}_i \partial_{m+j} \right)$$

A straightforward computation provides

$$\frac{1}{2} \left[ \Delta_0', R + \bar{R} \right] = \Delta_0' - (R + \bar{R}) + R \bar{R} + \sum_{i,j=1}^{m} \bar{z}_{m+i} z_{m+j} \partial_i \partial_j + \bar{z}_i z_j \partial_{m+i} \bar{\partial}_{m+j}$$

$$- \sum_{i,j=1}^{m} \bar{z}_i \bar{z}_{m+j} \partial_{m+i} \bar{\partial}_j + \bar{z}_{m+i} \bar{z}_j \partial_i \partial_{m+j}$$

$$= \Delta_0' - (R + \bar{R}) + R \bar{R} + \frac{1}{4} (2D_1 \bar{D}_1 + 2 \bar{D}_1 D_1 + 2 (R + \bar{R}))$$

$$= \Delta_0' - \frac{1}{2} (R + \bar{R}) + R \bar{R} + \frac{1}{2} (D_1 \bar{D}_1 + \bar{D}_1 D_1)$$

$$= \Delta_0' + R \bar{R} - \frac{1}{2} (R + \bar{R}) + \frac{1}{4} ((R - \bar{R})^2 - \Gamma).$$

The results follows.
By Lemma 3.4, it is easy to check

$$\left[ \Delta'_\alpha, \Delta'_\beta \right] = (\alpha - \beta)[R + \bar{R}, \Delta_0] = 2(\beta - \alpha) \left( \Delta'_0 - \frac{1}{2}(R + \bar{R}) + \frac{1}{4}(R + \bar{R})^2 - \frac{1}{4}\Gamma \right).$$

(3.8)

We shall frequently use the fact

$$[\Gamma, \Delta'_\alpha] = \Gamma \Delta'_\alpha - \Delta'_\alpha \Gamma = 0.$$

Lemma 3.5. There holds

$$\Delta'_{1-k-a} \left\{ \left[ \Delta'_{2-a} + \frac{(k-1)^2}{4} \right]^2 - \frac{(k-1)^2}{4} \{\Gamma + 1 \} \right\} \right\} f$$

$$= \left\{ \left[ \Delta'_{1-a} + \frac{k^2}{4} \right]^2 - \frac{k^2}{4} (\Gamma + 1) \right\} \Delta'_{1-k-a} f.$$

Proof. We compute, by using (3.1) and Lemma 3.4

$$\begin{align*}
\Delta'_{1-k-a} \left[ \Delta'_{2-a} + \frac{(k-1)^2}{4} \right] & = \left( \Delta'_{1-k-a} + \frac{k^2}{4} - \frac{k}{2}(R + \bar{R}) + \frac{k}{2}(2 - a - k) \right) \left( \Delta'_{1-a} + \frac{k^2}{4} + \frac{1}{2}(R + \bar{R}) + \frac{a - 2 - k}{2} \right) \\
& = \left( \Delta'_{1-k-a} + \frac{k^2}{4} \right)^2 + \frac{1}{2} \left( \Delta'_{1-k-a} + \frac{k^2}{4} \right)(R + \bar{R}) - \frac{k}{2}(R + \bar{R}) \left( \Delta'_{1-k-a} + \frac{k^2}{4} \right) \\
& \quad + \frac{k^2}{2}(1 - a)k + a - 2 \left( \Delta'_{1-a} + \frac{k^2}{4} \right) + \frac{k(2 - a)}{2}(R + \bar{R}) + \frac{k(2 - a - k)(a - 2 - k)}{4} \\
& = \left( \Delta'_{1-k-a} + \frac{k^2}{4} \right)^2 + \frac{1}{2} \left( \Delta'_{1-k-a} + \frac{k^2}{4} \right)(R + \bar{R}) - \frac{k}{2} \left[ \Delta'_0 + \frac{k^2}{4}, R + \bar{R} \right] - \frac{k}{4}(R + \bar{R})^2 \\
& \quad + \frac{k^2}{2}(1 - a)k + a - 2 \left( \Delta'_{1-a} + \frac{k^2}{4} \right) + \frac{k(2 - a)}{2}(R + \bar{R}) + \frac{k(2 - a - k)(a - 2 - k)}{4} \\
& = \left( \Delta'_{1-k-a} + \frac{k^2}{4} \right)^2 + \frac{k(1 - a)k + a - 2}{2} \left( \Delta'_{1-a} + \frac{k^2}{4} \right) + \frac{k(2 - a)}{2}(R + \bar{R}) + \frac{k(2 - a - k)(a - 2 - k)}{4} \\
& = \left( \Delta'_{1-k-a} + \frac{k^2}{4} \right)^2 + \frac{1}{2} \left( \Delta'_{1-k-a} + \frac{k^2}{4} \right)(R + \bar{R}) + k \left( \Delta'_0 - \frac{1}{2}(R + \bar{R}) - \frac{1}{4}\Gamma \right) \\
& \quad + \frac{k^2}{2}(1 - a)k + a - 2 \left( \Delta'_{1-a} + \frac{k^2}{4} \right) + \frac{k(2 - a)}{2}(R + \bar{R}) + \frac{k(2 - a - k)(a - 2 - k)}{4} \\
& = \left( \Delta'_{1-k-a} + \frac{k^2}{4} \right)^2 + \frac{1}{2} \left( \Delta'_{1-k-a} + \frac{k^2}{4} \right)(R + \bar{R}) + k \left( \Delta'_{1-a} + \frac{k^2}{4} \right) - \frac{k}{4}(\Gamma + 1) \\
& \quad + \frac{k^2}{2}(1 - a)k + a - 2 \left( \Delta'_{1-a} + \frac{k^2}{4} \right) \\
& = \left( \Delta'_{1-k-a} + \frac{k^2}{4} \right)^2 + \frac{1}{2} k \left( \Delta'_{1-k-a} + \frac{k^2}{4} \right)(R + \bar{R}) - \frac{k}{4}(\Gamma + 1)
\end{align*}$$
\[
+ \frac{-k^2 + (3 - a)k + a - 2}{2} \left( \Delta_{\frac{k}{2}}^a + \frac{k^2}{4} \right)
= \left( \Delta_{\frac{k}{2}}^a + \frac{k^2}{4} \right)^2 + \frac{1 - k}{2} \left( \Delta_{\frac{k}{2}}^a + \frac{k^2}{4} \right) (R + \bar{R}) - \frac{k}{4}(\Gamma + 1)
\]
\[
- (k - 1)(k + a - 2) \left( \Delta_{\frac{k}{2}}^a + \frac{k^2}{4} \right)
= \left( \Delta_{\frac{k}{2}}^a + \frac{k^2}{4} \right)^2 + \frac{1 - k}{2} \left( \Delta_{\frac{k}{2}}^a + \frac{k^2}{4} \right) (R + \bar{R} + k + a - 2) - \frac{k}{4}(\Gamma + 1).
\]

Therefore, we have
\[
\Delta_{\frac{k}{2}}^a \left[ \Delta_{\frac{k}{2}}^a + \frac{(k - 1)^2}{4} \right] - \left( \Delta_{\frac{k}{2}}^a + \frac{k^2}{4} \right)^2 \Delta_{\frac{k}{2}}^a
= \left( \Delta_{\frac{k}{2}}^a + \frac{k^2}{4} \right)^2 \left( \Delta_{\frac{k}{2}}^a + \frac{(k - 1)^2}{4} - \Delta_{\frac{k}{2}}^a \right)
+ \frac{1 - k}{2} \left( \Delta_{\frac{k}{2}}^a + \frac{k^2}{4} \right) (R + \bar{R} + k + a - 2) \left[ \Delta_{\frac{k}{2}}^a + \frac{(k - 1)^2}{4} \right]
- \frac{k}{4}(\Gamma + 1) \left[ \Delta_{\frac{k}{2}}^a + \frac{(k - 1)^2}{4} \right]
= \frac{k - 1}{2} \left( \Delta_{\frac{k}{2}}^a + \frac{k^2}{4} \right)^2 (R + \bar{R} + k + a - 4)
- \frac{k}{4}(\Gamma + 1) \left[ \Delta_{\frac{k}{2}}^a + \frac{(k - 1)^2}{4} \right].
\]

On the other hand,
\[
\left( \Delta_{\frac{k}{2}}^a + \frac{k^2}{4} \right) (R + \bar{R} + k + a - 4) - (R + \bar{R} + k + a - 2) \left[ \Delta_{\frac{k}{2}}^a + \frac{(k - 1)^2}{4} \right]
= \left[ \Delta_{\frac{k}{2}}^a + \frac{k^2}{4}, R + \bar{R} + k + a - 4 \right] + (R + \bar{R} + k + a - 4) \left( \Delta_{\frac{k}{2}}^a + \frac{k^2}{4} \right)
- (R + \bar{R} + k + a - 2) \left[ \Delta_{\frac{k}{2}}^a + \frac{(k - 1)^2}{4} \right]
= [\Delta_0, R + \bar{R}] + (R + \bar{R} + k + a - 2) \left( \Delta_{\frac{k}{2}}^a + \frac{k^2}{4} - \Delta_{\frac{k}{2}}^a - \frac{(k - 1)^2}{4} \right) - 2 \left( \Delta_{\frac{k}{2}}^a + \frac{k^2}{4} \right)
= [\Delta_0, R + \bar{R}] - 2\Delta' \left[ R + \bar{R} \right] - \frac{1}{2}(R + \bar{R})^2 + (R + \bar{R}) - \frac{1}{2} = - \frac{1}{2}(\Gamma + 1).
\]

Combining both above inequalities yields
\[
\Delta_{\frac{k}{2}}^a \left[ \Delta_{\frac{k}{2}}^a + \frac{(k - 1)^2}{4} \right]^2 - \left( \Delta_{\frac{k}{2}}^a + \frac{k^2}{4} \right)^2 \Delta_{\frac{k}{2}}^a
= \frac{\Gamma + 1}{4} \left[ -(k - 1) \left( \Delta_{\frac{k}{2}}^a + \frac{k^2}{4} \right) - k \left( \Delta_{\frac{k}{2}}^a + \frac{(k - 1)^2}{4} \right) \right]
\]
\[ \frac{\Gamma + 1}{4} \left[ (k - 1)^2 \Delta'_{1-k-a} - k^2 \Delta'_{3-k-a} \right]. \]

The proof of Lemma 3.5 is thereby completed. \qed

4. Funk-Hecke Formulas

4.1. The Funk-Hecke formula for the quaternionic hyperbolic space. We note that the Funk-Hecke formula on the CR sphere was established by Frank and Lieb [20]. The main source for the following is [32, 33, 11, 10] where they extend Frank and Lieb’s formula. We begin by recalling the Funk-Hecke formulas for the quaternionic case. We recall that \( L^2(\mathbb{S}^{4m-1}) \) may be decomposed into the \( U(2m) \)-irreducibles decomposition

\[ L^2(\mathbb{S}^{4m-1}) = \bigoplus_{j,k \geq 0} \mathcal{H}_{j,k}, \]

where \( \mathcal{H}_{j,k} \) consists of the Euclidean harmonic homogeneous polynomials in the complex variables \((z, \bar{z})\) and of bidegree \((j, k)\). Recalling that \( H^m_C = \text{Sp}(m, 1)/\text{Sp}(m) \times \text{Sp}(1) \), the appropriate irreducible decomposition is into \( \text{Sp}(m) \times \text{Sp}(1) \)-irreducibles, and is given as follows:

\[ L^2(\mathbb{S}^{4m-1}) = \bigoplus_{j \geq k \geq 0} V_{j,k}, \tag{4.1} \]

where \( V_{j,k} \subset \mathcal{H}_{j,k} \) are the so-called \((j, k)\)-bispherical harmonic spaces generated by the \( \text{Sp}(m) \times \text{Sp}(1) \) action on a zonal harmonic polynomial (see [33, Theorem 3.1 (4)]).

We recall the following quaternionic Funk-Hecke formula of Christ, Liu and Zhang ([10, Lemma 5.4]). In the following, \( P^{\alpha, \beta}_{m}(t) \) denotes a Jacobi polynomial of degree \( k \).

\textbf{Theorem G.} Let \( K \) be an \( L^1 \) integrable function on the unit ball \( B^1_Q \) in \( \mathbb{Q} \). Then, any integral operator on \( S^{4n+3} \) with kernel given by \( K(\langle \zeta, \bar{\eta} \rangle_Q) \) is diagonal with respect to the decomposition (4.1), and the eigenvalue \( \lambda_{j,k}(K) \) on \( V_{j,k} \) is given by

\[ \lambda_{j,k}(K) = \frac{2\pi^{2n}k!}{(j - k + 1)!(k + 2n - 1)!} \int_0^{\pi/2} (\sin \theta)^{4n-1} (\cos \theta)^{j-k+3} P^{(2m-1,j-k+1)}_{k}(cos \theta) d\theta \]

\[ \times \int_{S^3} K(\cos \theta u) \sin (j - k + 1) \phi \sin \phi \, du, \tag{4.2} \]

where \( \text{Re} u = \cos \phi \, (\phi \in [0, \pi]) \) and \( du \) is the round measure on \( S^3 = \partial B^1_Q \).

Using Theorem G and taking inspiration from the proof of Lemma 5.5 of [10], we obtain the following integral formula which will be used later.

\textbf{Proposition 4.1.} If \(-\frac{1}{2} < \alpha < \infty\) and \( 0 < r < 1 \), then

\[ \int_{S^{4n+3}} \frac{1}{|1 - (r\xi, \zeta)_Q|^2} d\sigma(\eta) = \frac{2\pi^{2n+2}}{(2n+1)!} \binom{2a}{\alpha-1} \, 2F_1(\alpha, \alpha-1; 2n+2; r^2). \tag{4.3} \]

\textbf{Proof.} Define the kernel \( K_r(\eta) = |1 - r \eta|^{-2\alpha} \) on \( \mathbb{B}^1_Q \) and observe that (4.3) may be understood as an integral operator on \( S^{4n+3} \) with kernel \( K_r(\langle \zeta, \bar{\eta} \rangle_Q) \) applied to the constant...
function $1 \in V_{0,0}$. Therefore, we may apply the Funk-Hecke formula (4.2) to $K_r$ with $j = k = 0$ to obtain

$$\lambda_{0,0}(K_r) = \frac{8\pi^{2n+1}}{(2n-1)!} \int_0^\pi \sin^{4n-1} \theta \cos^3 \theta P_0^{2n-1,1} (\cos 2\theta) d\theta$$

$$\times \int_0^\pi (1 + r^2 \cos^2 \theta - 2r \cos \phi \cos \theta)^{-\alpha} \sin^2 \phi d\phi,$$

where we have used

$$|1 - r q|^2 = 1 + r^2|q| - 2 \text{Re} q \implies K(\cos \theta u) = |1 + r^2 \cos^2 \theta - 2r \cos \theta \cos \phi|^{-\alpha}$$

and

$$du = \sin^2 \phi \sin \phi' d\phi d\phi'^{\prime}, \quad \phi, \phi' \in [0, \pi], \phi'' \in [0, 2\pi]$$

$$\int_0^{2\pi} \int_0^\phi \sin \phi' d\phi' d\phi'' = 4\pi.$$ 

Note also $P_0^{(2n-1,1)}(1) = 1$. Using the cosine integral (see [20], (5.11))

$$\int_{-\pi}^{\pi} (1 - 2r \cos \phi + r^2)^{-\alpha} e^{it\phi} d\phi = \frac{2\pi}{\Gamma^2(\alpha)} \sum_{\mu \geq 0} r^{\ell+2\mu} \frac{\Gamma(\alpha + \mu) \Gamma(\alpha + \ell + \mu)}{\mu!(\ell + \mu)!}$$

for $\ell \in \mathbb{N}$, that the integrand in even and that $\sin^2 \phi = \frac{1}{2}(1 - \cos 2\phi)$, we have

$$\int_0^\pi (1 + r^2 \cos^2 \theta - 2r \cos \theta \cos \phi)^{-\alpha} \sin^2 \phi d\phi$$

$$= \frac{\pi}{2\Gamma^2(\alpha)} \sum_{\mu \geq 0} r^{2\mu} \cos^{2\mu} \theta \frac{\Gamma^2(\alpha)}{(\mu!)^2} - r^{2+2\mu} \cos^{2+2\mu} \theta \frac{\Gamma(\alpha)\Gamma(2 + \alpha)}{\mu!(\mu + 2)!}.$$ 

Consequently, there holds

$$\lambda_{0,0}(K_r) = \frac{4\pi^{2n+2}}{(2n-1)\Gamma^2(\alpha)} \sum_{\mu \geq 0} \frac{\Gamma(\mu + \alpha)}{\mu!} \left( r^{2\mu} \frac{\Gamma(\mu + \alpha)}{\mu!} \int_0^\pi \sin^{4n-1-\ell} \theta \cos^{2\mu} \theta d\theta \right)$$

$$- r^{2+2\mu} \frac{\Gamma(\mu + \alpha + 2)}{(2 + \mu)!} \int_0^\pi \sin^{4n-1} \theta \cos^{5+2\mu} \theta d\theta.$$ 

Letting $t = \cos 2\theta$ and observing $dt = -4\sin \theta \cos \theta d\theta$, $\cos^2 \theta = \frac{1}{2}(1 + t)$ and $\sin^2 \theta = \frac{1}{2}(1 - t)$, we find

$$\int_0^\pi \sin^{4n-1} \theta \cos^{\ell+2\mu} \theta d\theta = \frac{1}{4} \int_{-1}^{1} (\sin^2 \theta)^{2n-1} (\cos^2 \theta)^{\frac{\ell+1+\mu}{2}} dt$$

$$= 2^{-2-2n-\mu-\ell} \int_{-1}^{1} (1 + t)^{\frac{\ell+2+\mu-1}{2}} (1 - t)^{2n-1} dt$$

where $\ell \in \mathbb{N}$.
\[
\lambda_{0,0}(K_r) = \frac{4\pi^{2n+2}}{(2n-1)!\Gamma^2(\alpha)} \sum_{\mu \geq 0} \frac{\Gamma(\mu + \alpha)}{\mu!} \left( \frac{\Gamma(\mu + \alpha) \Gamma(2 + \mu) \Gamma(2n)}{2\Gamma(2 + \mu + 2n)} \right) \\
- r^{2\mu+2} \frac{\Gamma(\mu + \alpha + 2) \Gamma(3 + \mu) \Gamma(2n)}{2\Gamma(3 + \mu + 2n)} \right) \\
= 2\pi^{2n+2} \left( \sum_{\mu \geq 1} \left[ \frac{\Gamma^2(\mu + \alpha)(\mu + 1)!}{(\mu!)^2(\mu + 1 + 2n)!} - \frac{\Gamma(\mu + 1 + \alpha)\Gamma(\mu + 1 + 1)!}{(\mu - 1)!\Gamma(\mu + 1 + 2n)!} \right] r^{2\mu} + \frac{\Gamma^2(\alpha)}{(2n + 1)!} \right) \\
= (\alpha - 1) \frac{2\pi^{2n+2}}{\Gamma^2(\alpha)} \sum_{\mu \geq 0} \frac{\Gamma(\mu + \alpha)\Gamma(\mu + 1 + \alpha)}{\mu!(\mu + 1 + 2n)!} r^{2\mu} \\
= \frac{2\pi^{2n+2}}{(2n + 1)!} \sum_{\mu \geq 0} \frac{(\alpha)(\alpha + 1)\mu r^{2\mu}}{(2n + 2)^2 \Gamma(\mu)!} \\
= \frac{2\pi^{2n+2}}{(2n + 1)!} F_1(\alpha, \alpha - 1; 2n + 2; r^2).
\]

This is the desired identity. \(\square\)

4.2. **The Funk-Hecke formula for the Cayley hyperbolic plane.** We now discuss the Funk-Hecke formula for the octonionic case. We recall that \(L^2(S^{15})\) may be decomposed into the Spin(9)-irreducible decomposition

\[
L^2(S^{15}) = \bigoplus_{j \geq k \geq 0} W_{j,k} \tag{4.4}
\]

where \(W_{j,k}\) is the so-called \((j, k)\)-bispherical harmonic subspace, which is a finite dimensional space spanned by elements from the cyclic action of Spin(9) on zonal harmonics \(Z_{j,k}(\zeta)\) (see [33] or [11, eq. 2.12] for precise formula).

We point out that the Funk-Hecke formula given in [11] assumes the kernel function \(K\) is of the form \(K(\zeta \cdot \eta)\), where, if \(\zeta = (\zeta_1, \zeta_2), \eta = (\eta_1, \eta_2) \in \mathbb{C}a^2\), then \(\zeta \cdot \eta = \zeta_1\eta_1 + \zeta_2\eta_2\). Considering kernel functions of this form are due to their consideration of the natural distance function \(|1 - \zeta \cdot \eta|\) on the sphere \(S^{15}\). However, taking into consideration the geometry of the Cayley plane \(H\) and the non-associativity of \(\mathbb{C}a\), it is more appropriate for our purposes to consider kernels of the form \(K(\Phi_{\mathbb{C}a}(\zeta, \eta))\) or \(K(\Psi_{\mathbb{C}a}(\zeta, \eta))\) since \(\Phi_{\mathbb{C}a}(\zeta, \eta)\) and \(\Psi_{\mathbb{C}a}(\zeta, \eta)\) are octonionic analogues of \(|\langle \cdot, \cdot \rangle_F|^2\) and \(|1 - \langle \cdot, \cdot \rangle_F|^2\), respectively. As a result, we will establish the following Funk-Hecke formulas which are more suitable for our purposes.

**Theorem 4.1.** Suppose \(K(\Phi_{\mathbb{C}a}(\zeta, \eta))\) is such that the following integral exists. Then the integral operator with kernel \(K(\Phi_{\mathbb{C}a}(\zeta, \eta))\) is diagonal with respect to the bispherical decomposition harmonic decomposition (4.4), and the eigenvalue on \(W_{j,k}\) is given by

\[
\lambda_{j,k}(K) = \frac{15\pi^4 k!}{(k + 3)!} \int_0^{\frac{\pi}{2}} \cos^{j-k+7} \theta \sin^7 \theta F_k^{(3,3+j-k)}(\cos 2\theta) d\theta \int_S K(\Psi_{\mathbb{C}a}((1, 0), (\bar{u} \cos \theta, 0))) \\
\times (a_{j,k}^0 \cos(j - k) \phi + a_{j,k}^1 \cos(j - k + 2) \phi + a_{j,k}^2 \cos(j - k + 4) \phi)
\]
+a_{j,k}^3 \cos(j - k + 6) \phi \, du,

where \( \text{Re} \, u = \cos \phi \) (\( \phi \in [0, \pi] \)), \( du \) is the standard surface measure on \( S \), the unit sphere in \( \mathbb{C}a \), \( P^{(3,3+j-k)}_k(z) \) is the Jacobi polynomial of order \( k \) associated to the weight \((1 - z)^3(1 + z)^{3+j-k}\) and

\[
\begin{align*}
    a_{j,k}^0 &= \frac{1}{8j - k + 3} - \frac{1}{4j - k + 2} + \frac{1}{8j - k + 1}, \\
    a_{j,k}^1 &= \frac{3}{8j - k + 3} - \frac{1}{4j - k + 4} + \frac{1}{8j - k + 1}, \\
    a_{j,k}^2 &= \frac{1}{8j - k + 3} + \frac{1}{4j - k + 2} + \frac{1}{8j - k + 5}, \\
    a_{j,k}^3 &= \frac{1}{8j - k + 3} + \frac{1}{4j - k + 4} - \frac{1}{8j - k + 5}.
\end{align*}
\]

**Proof.** Since the latter half of the proof is the same as the proof of Lemma 3.3 in [11], we shall only point out the needed adaptation.

We have from Schur’s lemma and the irreducibility of the \( W_{j,k} \) that the integral operator with kernel \( K(\Psi_{Ca}(\zeta, \eta)) \) is diagonal. Let \( \lambda_{j,k} \) denote the eigenvalue corresponding to the subspace \( W_{j,k} \). Letting \( Y_{j,k}^\mu, 1 \leq \mu \leq \dim W_{j,k} \), be a normalized orthogonal basis of \( W_{j,k} \), we then have

\[
\int_{S^{15}} K(\Psi_{Ca}(\zeta, \eta))Y_{j,k}^\mu(\eta)d\sigma = \lambda_{j,k} Y_{j,k}^\mu(\zeta).
\]

Letting

\[
Z_{j,k}(\zeta, \eta) = Z_{j,k}(\zeta \cdot \bar{\eta}) = \sum_{\mu=1}^{\dim W_{j,k}} Y_{j,k}^\mu(\zeta) \overline{Y_{j,k}^\mu(\eta)}
\]

be the reproducing kernel of the projection onto \( W_{j,k} \), we have

\[
\int_{S^{15}} K(\Psi_{Ca}(\zeta, \eta))Z_{j,k}(\eta \cdot \bar{\zeta})d\eta = \lambda_{j,k} Z_{j,k}(1).
\]

Here \( Z_{j,k}(1) \) denotes the aforementioned zonal harmonic \( Z_{j,k}(\zeta) \) evaluated at \( \zeta = 1 \). All that is needed now is to observe that \( K(\Psi_{Ca}(\zeta, \eta)) \) and \( Z_{j,k}(\eta, \zeta) \) are invariant under the action of \( \text{Spin}(9) \). Indeed, if this were the case, then we would obtain

\[
\lambda_{j,k} = Z_{j,k}(1)^{-1} \int_{S^{15}} K(\Psi_{Ca}(\zeta, \eta))Z_{j,k}(\eta \cdot \bar{\zeta})d\sigma = Z_{j,k}(1)^{-1} \int_{S^{15}} K(\Psi_{Ca}((1,0), \eta))Z_{j,k}((1,0), \eta)d\eta,
\]

The remainder of the proof follows as the proof of Lemma 3.3 in [11].

That \( K(\Psi_C(\zeta, \eta)) \) is \( \text{Spin}(9) \)-invariant follows from the \( \text{Spin}(9) \)-invariance of \( \Psi_{Ca}(\zeta, \eta) \). Therefore,

\[
\int_{S^{15}} Z_{j,k}(A\zeta, A\eta)Y_{j,k}^\mu(\eta)d\eta = \int_{S^{15}} Z_{j,k}(A\zeta, \eta)Y_{j,k}^\mu(A^{-1}\eta)d\eta = Y_{j,k}^\mu(\zeta),
\]

which shows that \( Z_{j,k}(A\zeta, A\eta) = Z_{j,k}(\zeta, \eta) \) by the uniqueness of the representation of a linear functional. \( \square \)

Lastly we state and prove the octonionic analogue of Proposition 4.1.
Proposition 4.2. If \(-\frac{1}{2} < \alpha < \infty\) and \(0 < r < 1\), then
\[
\int_{S^{4n+3}} \frac{1}{\Psi_{Ca}(r^\xi, \zeta)^{\alpha}} d\sigma(\eta) = \frac{2^{\pi^2}}{\pi!} F_1 \left( \alpha, \alpha - 3; r^2 \right).
\]

Proof. The proof follows similarly to the proof of Proposition 4.1 by applying Theorem 4.1 to the kernel \(\Psi_{Ca}(r^\xi, \eta)^{-\alpha}\). It should be pointed out that
\[
\Psi_{Ca}((r, 0), (\bar{u} \cos \theta, 0)) = 1 - 2r \cos \phi \cos \theta + r^2 \cos^2 \theta
\]
since \(\text{Re} u = \cos \phi\). □

5. Kernel Estimates

We recall that the heat kernel \(e^{t\Delta}\) on \(H^m\) is given by the following formula:
\[
e^{t\Delta} = c_m t^{-\frac{1}{2}} e^{-(2m+1)^2 t} \int_{\rho}^{\infty} \sinh 2r \sqrt{\cosh 2r - \cosh 2\rho} \left( -\frac{1}{\sinh 2r} \frac{\partial}{\partial r} \right)^2 \left( -\frac{1}{\sinh 2r} \frac{\partial}{\partial r} \right)^{2m-2} e^{-r^2} dr,
\]
where \(c_m = 2^{-2m+1} \pi^{-2m+\frac{1}{2}}\). The heat kernel \(e^{t\Delta}\) on \(H_{Ca}\) is given by
\[
e^{t\Delta} = c_o t^{-\frac{1}{2}} e^{-t^2} \int_{\rho}^{\infty} \sinh 2r \sqrt{\cosh 2r - \cosh 2\rho} \left( -\frac{1}{\sinh 2r} \frac{\partial}{\partial r} \right)^4 \left( -\frac{1}{\sinh 2r} \frac{\partial}{\partial r} \right)^{4m-2} e^{-r^2} dr,
\]
where \(c_o = 2^{-2m+1} \pi^{-2m+\frac{1}{2}}\). Letting \(h_t(\rho, 2\tilde{m} + 1)\) denote the heat kernel on the odd dimensional real hyperbolic space \(H^{2\tilde{m}+1}_{\mathbb{R}}\), we recall also that
\[
h_t(\rho, 2\tilde{m} + 1) = b_{\tilde{m}} t^{-\frac{1}{2}} e^{-\tilde{m}^2 t} \left( -\frac{1}{\sinh \rho} \frac{\partial}{\partial \rho} \right)^{\tilde{m}} e^{-\frac{\rho^2}{4}},
\]
where \(b_{\tilde{m}} = 2^{-\tilde{m}-1} \pi^{-\tilde{m}-\frac{1}{2}}\). See for example [4] and [41] for these formulas.

It will be useful to write \(e^{t\Delta}\) in terms of \(h_t\), and this can be done as follows. We consider \(H^m_{\mathbb{Q}}\) first. Observe that, if \(\tilde{m} = 2m - 2\), then
\[
e^{-(2m+1)^2 t} = e^{(-12m+3)t} e^{-\tilde{m}^2 t},
\]
and so
\[
e^{t\Delta} = \frac{c_m}{b_{2m-2}} \int_{\rho}^{\infty} \sinh 2r \sqrt{\cosh 2r - \cosh 2\rho} \left( -\frac{1}{\sinh 2r} \frac{\partial}{\partial r} \right)^2 \left( -\frac{1}{\sinh 2r} \frac{\partial}{\partial r} \right)^{2m-3} h_t(r, 4m - 3) dr. \tag{5.2}
\]

Similarly, on \(H_{Ca}\), there holds (by setting \(\tilde{m} = 4\))
\[
e^{t\Delta} = \frac{c_o}{b_1} \int_{\rho}^{\infty} \sinh 2r \sqrt{\cosh 2r - \cosh 2\rho} \left( -\frac{1}{\sinh 2r} \frac{\partial}{\partial r} \right)^4 \left( -\frac{1}{\sinh 2r} \frac{\partial}{\partial r} \right)^{8m-5} e^{-105t} h_t(r, 9) dr. \tag{5.3}
\]

We now recall the Bessel-Green-Riesz functions. For the sake of notational convenience, we write
\[
k_{\xi, \gamma} = \left(-\Delta - \frac{Q^2}{4} + r^2\right)^{-\frac{\gamma}{2}} \quad \text{for } 0 < \gamma < \dim_{\mathbb{R}} H^m_{\mathbb{R}} \quad \text{and} \quad \zeta > 0.
\]
\[
k_{\gamma} = \left(-\Delta - \frac{Q^2}{4}\right)^{-\frac{\gamma}{2}} \quad \text{for } 0 < \gamma < 3.
\]
In [3, page 1083, (iii)], Anker and Ji established the following asymptotics for $k_{\xi,\gamma}$ and $k_{\gamma}$:

\begin{align*}
  k_{\xi,\gamma} &\sim \rho^{\frac{\gamma+2}{2}} e^{-\xi \rho - \frac{Q}{2} \rho} \quad \text{for } \rho \geq 1 \\
  k_{\gamma} &\sim \rho^{\gamma-2} e^{-\frac{Q}{2} \rho} \quad \text{for } \rho \geq 1.
\end{align*}

We will need several technical lemmas to obtain small distance estimates of $k_{\xi}$. We state them now. The first estimate is a small distance estimate for the Bessel-Green-Riesz kernel on the real hyperbolic space $H^k_{\mathbb{R}}$ (see [39, Lemma 3.2]).

**Lemma C.** Let $k \geq 3$ and $0 < \gamma < 3$. If $0 < \rho < 1$, then

\[
  \left( -\Delta_{H^k_{\mathbb{R}}} - \left( \frac{k-1}{2} \right)^2 \right) \frac{1}{\gamma_k(\gamma)} \frac{1}{\rho^{k-\gamma}} + O \left( \frac{1}{\rho^{k-\gamma-1}} \right),
\]

where

\[
  \gamma_k(\gamma) = \frac{\pi^{\frac{1}{2}} \Gamma \left( \frac{\gamma}{2} \right)}{\Gamma \left( \frac{k-\gamma}{2} \right)}.
\]

The second lemma is an exact evaluation of a hyperbolic trigonometric integral (see [46, Lemma 3.2]).

**Lemma D.** Let $\beta > 0$ and $\rho > 0$. Then

\[
  \int_0^{\infty} \frac{\cosh r}{(\sinh r)^\beta} \frac{1}{\sqrt{\cosh 2r - \cosh 2\rho}} \, dt = \frac{\Gamma \left( \frac{1}{2} \right) \Gamma \left( \frac{\beta}{2} \right)}{2 \sqrt{2} \Gamma \left( \frac{1+\beta}{2} \right)} \frac{1}{(\sinh \rho)^\beta}.
\]

The last lemma pertains to controlling higher order derivatives of $\frac{\rho^\alpha}{\sinh r}$ for large $r$ (see also [39, Lemma 3.1] and [5, Corollary 5.14]).

**Lemma 5.1.** Let $p, q \in \mathbb{N}_{\geq 0}$ and $0 < \gamma < 3$. If $0 < r$, then

\[
  \left( -\frac{1}{\sinh 2r} \frac{\partial}{\partial r} \right)^q \left( -\frac{1}{\sinh r} \frac{\partial}{\partial r} \right)^p \frac{r^\beta-2}{\sinh r} \lesssim r^{\beta-2} e^{-(p+2q+1)r}.
\]

**Proof.** Using

\[
  \frac{1}{\sinh r} = \frac{2e^{-r}}{1-e^{-2r}} = 2 \sum_{j=0}^{\infty} e^{-(2j+1)r},
\]

it is easy to see that

\[
  \left( -\frac{1}{\sinh r} \frac{\partial}{\partial r} \right)^p \frac{r^\beta-2}{\sinh r} \sim r^{\beta-2} \left[ e^{-(p+1)r} + e^{-(p+3)r} + \ldots \right],
\]

and, similarly, that

\[
  \left( -\frac{1}{\sinh 2r} \frac{\partial}{\partial r} \right)^q \left( -\frac{1}{\sinh r} \frac{\partial}{\partial r} \right)^p \frac{r^\beta-2}{\sinh r} \lesssim r^{\beta-2} e^{-(2q+p+1)r},
\]

as desired. \qed

In the following subsections, we will prove various kernel estimates for $k_\gamma, k_{\xi,\gamma}, k_\gamma * k_{\xi,\gamma}$ and $k_\gamma * k_{\xi,\gamma} * f$ for smooth compactly supported function on $\mathbb{B}_Q^m$ and $\mathbb{B}_C^a$. Along with the Fourier analysis on symmetric spaces (i.e., the Plancherel theorem and Kunze-Stein phenomenon) and factorization, these estimates form the ingredients of the proofs of the Poincaré-Sobolev and Hardy-Sobolev-Maz’ya inequalities on $H^m_{Q}$ and $H^m_{C}$. 


5.1. Convolution Estimates. In order to prove the kernel estimates, we will need asymptotics of certain convolutions. This is contained in Lemmas 5.2, 5.3, 5.4 and 5.5 below. Due to the appearance of \( \Psi \) in the automorphisms on \( \mathbb{B} \), separate considerations are needed for \( \mathbb{B} \) and so we state the convolution estimates for \( \mathbb{B}_m \) and \( \mathbb{B} \) separately. We mention that, when compared to the complex hyperbolic setting, the hypothesis \( \lambda_1 + \lambda_2 > \gamma + \gamma' - 4m + 2 \) differs from the reasonably expected \( \lambda_1 + \lambda_2 > \gamma + \gamma' - 4m \), and this has to do with the higher dimensional center of \( \mathbb{H}_m \). This is similar for the corresponding hypothesis in Lemma 5.3 for \( H \).

We will need the following convolution integral on Euclidean space (see [49]).

Lemma E. For \( 0 < \gamma, \gamma' < k \) and \( 0 < \gamma + \gamma' < k \), there holds
\[
\int \frac{|x|^{-k}|y-x|^{\gamma'-k}}{|x|^{\gamma}} \, dx = \frac{\gamma_k(\gamma)\gamma_k(\gamma')}{\gamma_k(\gamma + \gamma')} |y|^{\gamma + \gamma' - k}.
\tag{5.5}
\]
where
\[
\gamma_k(\gamma) = \frac{\pi^{\frac{k}{2}}2^{\gamma}\Gamma\left(\frac{\gamma}{2}\right)}{\Gamma\left(\frac{k-\gamma}{2}\right)}.
\]

We may now state the main convolution estimate lemma for small distances.

Lemma 5.2. Let \( 0 < \gamma < 4m, 0 < \gamma' < 4m, \) and \( \lambda_1 + \lambda_2 > \gamma + \gamma' - 4m + 2 \). If \( 0 < \gamma + \gamma' < 4m - 1 \) and \( 0 < \rho < 1 \), then on \( \mathbb{B}_m \) there holds
\[
\frac{1}{(\sinh \rho)^{4m-\gamma}(\cosh \rho)^{\lambda_1}} \frac{1}{(\sinh \rho)^{4m-\gamma'}(\cosh \rho)^{\lambda_2}} \leq \frac{\gamma_4(m)(\gamma)\gamma_4(m')(\gamma')}{\gamma_4(m + \gamma')(\gamma + \gamma')} \frac{1}{\rho^{4m-\gamma-\gamma'}} + O\left(\frac{1}{\rho^{4m-\gamma-\gamma'-1}}\right).
\tag{5.6}
\]

If \( 4m - 1 \leq \gamma + \gamma' < 4m, 0 < \epsilon < 4m - \gamma - \gamma' \) and \( 0 < \rho < 1 \), then on \( \mathbb{B}_m \) there holds
\[
\frac{1}{(\sinh \rho)^{4m-\gamma}(\cosh \rho)^{\lambda_1}} \frac{1}{(\sinh \rho)^{4m-\gamma'}(\cosh \rho)^{\lambda_2}} \leq \frac{\gamma_4(m)(\gamma)\gamma_4(m')(\gamma')}{\gamma_4(m + \gamma')(\gamma + \gamma')} \frac{1}{\rho^{4m-\gamma-\gamma'}} + O\left(\frac{1}{\rho^{4m-\gamma-\gamma'-\epsilon}}\right).
\tag{5.7}
\]

Proof. By Lemma A item (iv), and by \( dV = \frac{dz}{(1 - |z|^2)^m} \), we compute
\[
\int_{\mathbb{B}^m} \frac{1}{(\sinh \rho)^{4m-\gamma}(\cosh \rho)^{\lambda_1}} \frac{1}{(\sinh \rho)^{4m-\gamma'}(\cosh \rho)^{\lambda_2}} \frac{(1 - |w|^2)(1 - |z|^2)}{|z - w|^2 + |z, w|_Q^2 - |z|^2|w|^2} \frac{dz}{(1 - |z|^2)^{2m+2}} = (1 - |w|^2)^{4m-\gamma + \lambda_1} \int_{\mathbb{B}^m} \frac{1}{|z|^{4m-\gamma}} \frac{1}{|z - w|^2 + |z, w|_Q^2 - |z|^2|w|^2} \frac{dz}{|1 - \langle z, w \rangle_Q^{\lambda_2}|^{2m+2}}.
\]

\[
= (cosh \rho(w))^{-(4m-\gamma + \lambda_2)} (A_5 + A_6),
\]
where
\[ A_5 = \int_{\{ |z| < \frac{1}{2} \}} \cdots \text{ and } A_6 = \int_{\{ \frac{1}{2} \leq |z| < 1 \}} \cdots . \]

Note that, when \( \rho(w) < 1 \) and \( |z| \leq \frac{1}{2} \), there holds
\[ |1 - \langle z, w \rangle_Q|^2 \left( 1 - |z|^2 \right)^{\frac{4 + \gamma' + 4m - \lambda_1 - \lambda_2}{2}} = 1 + O \left( |z| \right) . \]

On the other hand, there holds
\[ |\langle z, w \rangle_Q|^2 - |z|^2 |w|^2 = |z|^2 + \langle z, w - z \rangle_Q|^2 - |z|^2 |w - z + z|^2 \]
\[ = |\langle z, w - z \rangle_Q|^2 - |z|^2 |w - z|^2 \]
\[ = |z|^2 |w - z|^2 \left[ \left| \frac{z}{|z|} \right| |w - z|_Q - 1 \right] , \]
and so
\[ |z - w|^2 + |\langle z, w \rangle_Q|^2 - |z|^2 |w|^2 = |z - w|^2 \left[ 1 + |z|^2 \left[ \left| \frac{z}{|z|} \right| |w - z|_Q - 1 \right] \right] \]
\[ = |z - w|^2 \left( 1 + O \left( |z|^2 \right) \right) . \]

Since \( 0 < \gamma + \gamma' < 4m - 1 \), we may use Lemma \( E \) to compute
\[ A_5 = \int_{\{ |z| \leq \frac{1}{2} \}} \frac{1}{|z|^{4m-\gamma}} \frac{1}{|z - w|^{4m-\gamma'}} \left( 1 + O \left( |z| \right) \right) dz \]
\[ \leq \int_{\mathbb{R}^{4m}} \frac{1}{|z|^{4m-\gamma}} \frac{1}{|z - w|^{4m-\gamma'}} dz + O \left( \int_{\mathbb{R}^{4m}} \frac{1}{|z|^{4m-\gamma-1}} \frac{1}{|z - w|^{4m-\gamma'}} dz \right) \]
\[ = \frac{\gamma_4m(\gamma')}{\gamma_4m(\gamma + \gamma')} \frac{1}{|w|^{4m-\gamma - \gamma'}} + O \left( \frac{1}{|w|^{4m-\gamma - \gamma' - 1}} \right) . \]

Similarly, if \( 0 < \epsilon < 4m - \gamma - \gamma' \), we obtain
\[ A_5 = \frac{\gamma_4m(\gamma')}{\gamma_4m(\gamma + \gamma')} \frac{1}{|w|^{4m-\gamma - \gamma'}} + O \left( \frac{1}{|w|^{4m-\gamma - \gamma' - \epsilon}} \right) . \]

We are left with estimating \( A_6 \): since
\[ \frac{4 + \gamma + \gamma' + 4m - \lambda_1 - \lambda_2}{2} < 1 \]
is equivalent to
\[ \gamma + \gamma' - 4m + 2 < \lambda_1 + \lambda_2 \]
we find
\[ A_6 = \int_{\{ \frac{1}{4} \leq |z| \leq 1 \}} \frac{1}{|z|^{4m-\gamma}} \left( \frac{1}{|z - w|^2 + |\langle z, w \rangle_Q|^2 - |z|^2 |w|^2} \right)^{\frac{4m-\gamma'}{2}} dz \]
\[ \times \int_{\{ \frac{1}{2} \leq |z| \leq 1 \}} \frac{1}{|z - w|^{4m-\gamma - \lambda_2}} \left( 1 - |z|^2 \right)^{\frac{4 + \gamma + \gamma' - 4m - \lambda_1 - \lambda_2}{2}} dz \]
\[ \sim \int_{\{ \frac{1}{2} \leq |z| \leq 1 \}} \frac{1}{|z - w|^{4m-\gamma - \lambda_2}} \left( 1 - |z|^2 \right)^{\frac{4 + \gamma + \gamma' - 4m - \lambda_1 - \lambda_2}{2}} dz \]
\[
\sim \int_{\frac{1}{2}}^{1} \frac{r}{(1 - r^2)^{4\gamma + \gamma' - 4m - \lambda_1 - \lambda_2}} dr < \infty.
\]

In conclusion, since \(\cosh r \sim 1\) in as \(r \to 0\), we find
\[
\frac{1}{(\sinh \rho)^{4m-\gamma}(\cosh \rho)^{\lambda_1}} * \frac{1}{(\sinh \rho)^{4m-\gamma'}(\cosh \rho)^{\lambda_2}} \leq \frac{\gamma_m(\gamma)\gamma_m(\gamma')}{\gamma_m(\gamma + \gamma')} \frac{1}{|w|^{4m-\gamma-\gamma'}} + O\left(\frac{1}{|w|^{4m-\gamma-\gamma'-1}}\right),
\]
and the result follows since
\[
\rho(w) = \frac{1}{2} \log \frac{1 + |w|}{1 - |w|} = |w| + O\left(|w|^3\right)
\]
as \(|w| \to 0\).

**Lemma 5.3.** Let \(0 < \gamma < 16\), \(0 < \gamma' < 16\), and \(\lambda_1 + \lambda_2 > \gamma + \gamma' - 10\). If \(0 < \gamma + \gamma' < 15\) and \(0 < \rho < 1\), then on \(B_{Ca}\) there holds
\[
\frac{1}{(\sinh \rho)^{16-\gamma}(\cosh \rho)^{\lambda_1}} * \frac{1}{(\sinh \rho)^{16-\gamma'}(\cosh \rho)^{\lambda_2}} \leq \frac{\gamma_{16}(\gamma)\gamma_{16}(\gamma')}{\gamma_{16}(\gamma + \gamma')} \frac{1}{\rho^{16-\gamma-\gamma'}} + O\left(\frac{1}{\rho^{15-\gamma-\gamma'}}\right).
\]  
(5.8)

If \(15 \leq \gamma + \gamma' < 16\), \(0 < \epsilon < 16 - \gamma - \gamma'\) and \(0 < \rho < 1\), then on \(B_{Ca}\) there holds
\[
\frac{1}{(\sinh \rho)^{16-\gamma}(\cosh \rho)^{\lambda_1}} * \frac{1}{(\sinh \rho)^{16-\gamma'}(\cosh \rho)^{\lambda_2}} \leq \frac{\gamma_{16}(\gamma)\gamma_{16}(\gamma')}{\gamma_{16}(\gamma + \gamma')} \frac{1}{\rho^{16-\gamma-\gamma'-\epsilon}} + O\left(\frac{1}{\rho^{15-\gamma-\gamma'-\epsilon}}\right).
\]  
(5.9)

**Proof.** By Lemma B item (iv), and by \(dV = \frac{dz}{(1 - |z|^2)^{\gamma'}}\), we compute
\[
\frac{1}{(\sinh \rho)^{16-\gamma}(\cosh \rho)^{\lambda_1}} * \frac{1}{(\sinh \rho)^{16-\gamma'}(\cosh \rho)^{\lambda_2}} = \int_{B_{Ca}^\prime} \Psi_{Ca}(z, w)^{\frac{\lambda_1}{2}} (1 - |z|^2)^{\frac{\lambda_1}{2}} \frac{(1 - |w|^2)(1 - |z|^2)}{(\Psi_{Ca}(z, w) - (1 - |z|^2)(1 - |w|^2))^{16-\gamma'}} dz
\]
\[
\times \left(\frac{1 - |w|^2}{\Psi_{Ca}(z, w)}\right)^{\frac{\lambda_2}{2}} (1 - |z|^2)^{12} dz
\]
\[
= (1 - |w|^2)^{16-\gamma + \lambda_2} \int_{B_{Ca}^\prime} \frac{1}{|z|^{16-\gamma}} \Psi_{Ca}(z, w)^{\frac{\lambda_1}{2}} (1 - |z|^2)^{12} (1 - |w|^2)^{16-\gamma'} dz
\]
\[
\times \frac{1}{\Psi_{Ca}(z, w)^{\frac{\lambda_2}{2}}} (1 - |z|^2)^{\frac{\gamma + \gamma' - \lambda_1 - \lambda_2 - 8}{2}} dz
\]
\[
= (\cosh \rho(w))^{-16-\gamma + \lambda_2} (A_5 + A_6),
\]
where
\[
A_5 = \int_{\{ |z| < \frac{1}{2} \}} \ldots \text{ and } A_6 = \int_{\{ \frac{1}{2} \leq |z| < 1 \}} \ldots
\]
Note that, when \( \rho(w) < 1 \) and \( |z| \leq \frac{1}{2} \), there holds
\[
\Psi_{Ca}(z, w)^{\frac{\lambda_2}{2}}(1 - |z|^2)^{\frac{\gamma + \gamma' - \lambda_1 - \lambda_2 - \lambda}{2}} = 1 + O(|z|).
\]
Next, we have
\[
\Psi_{Ca}(z, a) - (1 - |z|^2)(1 - |w|^2) = \Phi_{Ca}(z, a) - 2\langle z, a \rangle_\mathbb{R} + |z|^2 + |a|^2 - |z|^2|a|^2 = \Phi_{Ca}(z, a) + |z - a|^2 - |z|^2|a|^2 = |z - a|^2 \left( 1 + \frac{\Phi_{Ca}(z, a) - |z|^2|a|^2}{|z - a|^2} \right).
\]
Moreover, it is not hard to see that
\[
\frac{\Phi_{Ca}(z, w) - |z|^2|w|^2}{|z - w|^2} = O(|z|^2).
\]
Indeed, using invariance of distance \( \rho \), we can assume \( w = (w_1, w_2) \) with \( \text{Re} \, w_1 = c \in \mathbb{R} \) and all other components are zero. Then \( \Phi_{Ca}(z, a) - |z|^2|a|^2 = -c^2|z|^2 \) and clearly \( |z|^2/|z - a|^2 \) is bounded as \( z \to a \). Therefore, using also that \( \Phi_{Ca}(z, w) \leq |z|^2|w|^2 \), we obtain
\[
\Psi_{Ca}(z, a) - (1 - |z|^2)(1 - |w|^2) = |z - a|^2 \left( 1 + O(|z|^2) \right).
\]
The remainder of the proof is analogous to the proof of Lemma 5.2 and is thus omitted.

Next, we will state and prove the main convolution lemma for large distances. In preparation, we recall some properties and definitions of certain special functions. First, recall the generalized hypergeometric function
\[
_{p}F_{q}(a_1, \ldots, a_p; b_1, \ldots, b_q; z) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k z^k}{(b_1)_k \cdots (b_q)_k k!}.
\]
Second, we also recall the following hypergeometric integral (see [23, Equation 7.512.5]): supposing the complex parameters \( \alpha, \beta, \gamma, \rho, \sigma \) satisfy
\[
\text{Re} \, \rho > 0, \quad \text{Re} \, \sigma > 0, \quad \text{Re} \, (\gamma + \sigma - \alpha - \beta) > 0,
\]
there holds
\[
\int_{0}^{1} x^{\rho - 1}(1 - x)^{\sigma - 1} _{2}F_{1}(\alpha, \beta; \gamma; x)dx = \frac{\Gamma(\rho)\Gamma(\sigma)}{\Gamma(\rho + \sigma)} _{3}F_{2}(\alpha, \beta, \gamma; \rho, \rho; \sigma, 1). \tag{5.10}
\]

**Lemma 5.4.** Let \( 0 < \gamma < 4m, \ 0 < \gamma' < 4m, \) and \( \lambda_1 + \lambda_2 > \gamma + \gamma' - 4m + 2 \). If \( \lambda_2 - \gamma' < \lambda_1 - \gamma \) and \( 1 \leq \rho \), then on \( \mathbb{R}^{m}_0 \) there holds
\[
\frac{1}{(\sinh \rho)^{4m-\gamma}} \frac{1}{(\cosh \rho)^{\lambda_1}} \sim e^{-(4m-\gamma'+\lambda_2)\rho}.
\]

**Proof.** By the proof of Lemma 5.2, we have
\[
\frac{1}{(\sinh \rho)^{4m-\gamma}} \frac{1}{(\cosh \rho)^{\lambda_1}} \sim \frac{1}{(\cosh \rho)^{4m-\gamma'+\lambda_2}} \int_{\mathbb{R}^{m}_0} \frac{1}{|z|^{4m-\gamma}} \left( \frac{1}{|z-w|^2 + |\langle z, w \rangle|^{2} - |z|^2|w|^2} \right)^{\frac{4m-\gamma'}{2}}.
\]
Consequently, there holds

\[
\lim_{|w|\to 1} \int_{S^{4m-1}} \frac{1}{|z|^{4m-\gamma}} \left( \frac{1}{|z-w|^2 + |\langle z, w \rangle_Q|^2 - |z|^2|w|^2} \right) \frac{4m-\gamma'}{2} \times \frac{1}{|1-\langle z, w \rangle_Q|^{\lambda_2} (1-|z|^2)^{4\lambda + \lambda' - 4m - \lambda_1 - \lambda_2}} dz.
\]

Setting

\[
F(w) = \int_{S^{4m-1}} \left( \frac{1}{|z-w|^2 + |\langle z, w \rangle_Q|^2 - |z|^2|w|^2} \right)^{\frac{4m-\gamma'}{2}} \frac{1}{|1-\langle z, w \rangle_Q|^{\lambda_2}} d\sigma,
\]

we see that \( F(w) = F(|w|) \). Moreover, by Proposition 4.1, we find

\[
\lim_{|w|\to 1^-} F(w) = \lim_{|w|\to 1^-} \int_{S^{4m-1}} \frac{1}{|z-w|^2 + |\langle z, w \rangle_Q|^2 - |z|^2|w|^2} \frac{4m-\gamma'}{2} \times \frac{1}{|1-\langle z, w \rangle_Q|^{\lambda_2}} d\sigma.
\]

Consequently, there holds

\[
\lim_{|w|\to 1^-} \int_{S^{4m-1}} \frac{1}{|z|^{4m-\gamma}} \left( \frac{1}{|z-w|^2 + |\langle z, w \rangle_Q|^2 - |z|^2|w|^2} \right) \frac{4m-\gamma'}{2} \times \frac{1}{|1-\langle z, w \rangle_Q|^{\lambda_2} (1-|z|^2)^{4\lambda + \lambda' - 4m - \lambda_1 - \lambda_2}} dz
\]

\[
= \frac{2\pi^{2m}}{\Gamma(2m)} \int_0^1 r^{\gamma-1} (1-r^2)^{-\left(\frac{4\lambda + \lambda' - 4m - \lambda_1 - \lambda_2}{4}\right)} \times {}_2F_1 \left( \frac{4m - \gamma' + \lambda_2}{2}, \frac{4m - \gamma' + \lambda_2 - 2}{2}; 2m; r^2 \right) dr
\]

\[
= \frac{2\pi^{2m}}{\Gamma(2m)} \int_0^1 t^{\gamma-1} (1-t)^{-\left(\frac{4\lambda + \lambda' - 4m - \lambda_1 - \lambda_2}{4}\right)} \times {}_2F_1 \left( \frac{4m - \gamma' + \lambda_2}{2}, \frac{4m - \gamma' + \lambda_2 - 2}{2}; 2m; t \right) dt,
\]

where the change of variable \( r^2 = t \) was used in the last equality. Now, using (5.10), we have

\[
\lim_{|w|\to 1^-} \int_{S^{4m-1}} \frac{1}{|z|^{4m-\gamma}} \left( \frac{1}{|z-w|^2 + |\langle z, w \rangle_Q|^2 - |z|^2|w|^2} \right) \frac{4m-\gamma'}{2} \times \frac{1}{|1-\langle z, w \rangle_Q|^{\lambda_2} (1-|z|^2)^{4\lambda + \lambda' - 4m - \lambda_1 - \lambda_2}} dz
\]

\[
= \frac{\pi^{2m}}{\Gamma(2m)} \frac{\Gamma \left( \frac{4m+\lambda_1+\lambda_2-\gamma'-2}{2} \right)}{\Gamma \left( \frac{4m+\lambda_1+\lambda_2-\gamma'-2}{2} \right)} \times {}_3F_2 \left( \frac{4m - \gamma' + \lambda_2}{2}, \frac{4m - \gamma' + \lambda_2 - 2}{2}, \frac{\gamma}{2}; 2m, \frac{4m + \lambda_1 + \lambda_2 - \gamma' - 2}{2}; 1 \right).
\]

At last, using \( \cosh r \sim e^r \) for \( 1 \leq r \), we have proved

\[
\frac{1}{(\sinh \rho)^{4m-\gamma}} \frac{1}{(\sinh \rho)^{4m-\gamma}} \sim (\cosh \rho)^{-(4m-\gamma' + \lambda_2}.
\]
\[ \sim e^{-(4m'-\gamma+\lambda_2)\rho}. \]

\[ \square \]

**Lemma 5.5.** Let \( 0 < \gamma < 16, \ 0 < \gamma' < 16, \) and \( \lambda_1 + \lambda_2 > \gamma + \gamma' - 10. \) If \( \lambda_2 - \gamma' < \lambda_1 - \gamma \) and \( 1 \leq \rho, \) then

\[ \frac{1}{(\sinh \rho)^{16-\gamma}} (\cosh \rho)^{\lambda_1} \cdot \frac{1}{(\sinh \rho)^{16-\gamma'}} (\cosh \rho)^{\lambda_2} \sim e^{-(16-\gamma'+\lambda_2)\rho}. \]

**Proof.** By the proof of Lemma 5.3, we have

\[ \frac{1}{(\sinh \rho)^{16-\gamma}} (\cosh \rho)^{\lambda_1} \cdot \frac{1}{(\sinh \rho)^{16-\gamma'}} (\cosh \rho)^{\lambda_2} \]

\[ = (\cosh \rho(w))^{-(16-\gamma' + \lambda_2)} \int_{S^m} \frac{1}{|z|^{16-\gamma}} \left( \frac{1}{\Psi_{Ca}(z,w) - (1 - |z|^2)(1 - |w|^2)} \right)^{\frac{16-\gamma'}{2}} \]

\[ \times \frac{1}{\Psi_{Ca}(z,w)^{\lambda_2}} (1 - |z|^2)^{\frac{\gamma + \gamma' - \lambda_1 - 2\gamma_2}{2}} dz. \]

Setting

\[ F(w) = \int_{S^m} \left( \frac{1}{\Psi_{Ca}(z,w) - (1 - |z|^2)(1 - |w|^2)} \right)^{\frac{16-\gamma'}{2}} \frac{1}{\Psi_{Ca}(z,w)^{\lambda_2}} d\sigma, \]

we see that \( F(w) = F(|w|). \) Moreover, by Proposition 4.2, we find

\[ \lim_{|w| \to 1^-} F(w) = \lim_{|w| \to 1^-} \int_{S^m} \Psi_{Ca}(z,w)^{-\frac{16-\gamma'+\lambda_2}{2}} d\sigma \]

\[ = \frac{2\pi^8}{7!} 2F1 \left( \frac{16 - \gamma' + \lambda_2}{2}, \frac{10 - \gamma' + \lambda_2}{2}; 8; r^2 \right). \]

Consequently, there holds

\[ \lim_{|w| \to 1^-} \int_{B_{Ca}} \frac{1}{|z|^{16-\gamma}} \left( \frac{1}{\Psi_{Ca}(z,w) - (1 - |z|^2)(1 - |w|^2)} \right)^{\frac{16-\gamma'}{2}} \]

\[ \times \frac{1}{\Psi_{Ca}(z,w)^{\lambda_2}} (1 - |z|^2)^{\frac{\gamma + \gamma' - \lambda_1 - 2\gamma_2}{2}} dz \]

\[ = \frac{2\pi^8}{7!} \int_0^1 t^{\gamma-1} (1 - t^2)^{-\left(\frac{\gamma + \gamma' - \lambda_1 - 2\gamma_2}{2}\right)} \]

\[ \times 2F1 \left( \frac{16 - \gamma' + \lambda_2}{2}, \frac{16 - \gamma' + \lambda_2 - 6}{2}; 8; r^2 \right) dt \]

\[ = \frac{2\pi^8}{7!} \int_0^1 t^{\gamma-1} (1 - t)^{-\left(\frac{\gamma + \gamma' - \lambda_1 - 2\gamma_2}{2}\right)} \]

\[ \times 2F1 \left( \frac{16 - \gamma' + \lambda_2}{2}, \frac{10 - \gamma' + \lambda_2}{2}; 8; r^2 \right) dt. \]
As mentioned above, we only need to prove the estimate for \( 0 \leq \rho \leq 1 \). Proof. \( H_n \) denotes the heat kernel on \( \mathbb{R} \) (resp. \( \mathbb{R}^n \)) and let
\[
\Psi_{Ca}(z, w) = \frac{1}{\pi^2} \int_{\mathbb{R}^n} \frac{1}{|z|^2} \frac{1}{|w|^2} e^{-\frac{|x-y|^2}{4}} \, dx.
\]
where the change of variable \( r^2 = t \) was used in the last equality. Now, using (5.10), we have
\[
\lim_{|w| \to 1^-} \int_{\mathbb{R}^n} \frac{1}{|z|^{16-\gamma}} \left( \Psi_{Ca}(z, w) - (1 - |z|^2)(1 - |w|^2) \right) \left( 1 - (1 - |z|^2)(1 - |w|^2) \right)^{16 - \gamma'}\frac{dz}{\Psi_{Ca}(z, w) - (1 - |z|^2)^{16 - \gamma'}d^2z} = \pi^8 \left( \frac{2}{7} \right)^{16} \Gamma \left( \frac{10 + \lambda_1 + \lambda_2 - \gamma'}{2} \right) \times 3F2 \left( \frac{16 - \gamma' + \lambda_2}{2}, \frac{10 - \gamma' + \lambda_2}{2}, \frac{10 + \lambda_1 + \lambda_2 - \gamma'}{2}; 1 \right).
\]
At last, using \( \cosh r \sim e^r \) for \( 1 \leq r \), we have proved
\[
\frac{1}{(\sinh \rho)^{16-\gamma}} (\cosh \rho)^{\lambda_1} \frac{1}{(\sinh \rho)^{16-\gamma'}} (\cosh \rho)^{\lambda_2} \sim (\cosh \rho)^{-(16-\gamma'+\lambda_2)} \sim e^{-(16-\gamma'+\lambda_2)\rho}.
\]
\( \Box \)

5.2. Estimates for \( k_\gamma \). In this subsection, we obtain the asymptotics for \( k_\gamma \). Note that the large distance asymptotics \((1 \leq \rho)\) are already contained in (5.4).

**Lemma 5.6.** Let \( 0 < \gamma < 3 \) and let \( N = \dim_{\mathbb{R}} H^m_{F} \). If \( 0 < \rho < 1 \), then
\[
k_\gamma \leq \frac{1}{\gamma N(\gamma)} \frac{1}{\rho^{N-\gamma}} + O \left( \frac{1}{\rho^{(N-\gamma-1)}} \right).
\]
If \( 1 \leq \rho \), then
\[
k_\gamma \sim \rho^{\gamma-2} e^{-\frac{2}{\gamma} \rho}.
\]

**Proof.** As mentioned above, we only need to prove the estimate for \( 0 < \rho < 1 \).
By using (5.2), we will write \( k_\gamma \) in terms of a Bessel-Green-Riesz kernel on \( H^m_{\mathbb{R}} \), where \( n = 2\tilde{m} + 1 \) and \( \tilde{m} = 2m - 2 \) if \( F = \mathbb{Q} \) and \( \tilde{m} = 4 \) if \( F = \mathbb{C}a \). Also recall that \( h_4(\rho, n) \) denotes the heat kernel on \( H^m_{\mathbb{R}} \) (see (5.1)). Lastly let \( c \) denote \( c_m \) (resp. \( c_o \)) from (5.2) (resp. (5.3)) and let \( \mu = 2 \) (resp. \( \mu = 4 \)) when \( F = \mathbb{Q} \) (resp. \( F = \mathbb{C}a \)).

Then, by the Mellin transform and (5.2) and (5.3), we have
\[
k_\gamma(\rho) = \frac{1}{\Gamma \left( \frac{2}{7} \right)} \int_0^\infty t^{\frac{2}{7} - 1} e^{t \left( \Delta + \frac{\rho^2}{4} \right)} \, dt
\]
\[
= \frac{c}{b_{\tilde{m}}} \int_0^\infty \frac{\sinh 2r}{\cosh 2r - \cosh 2\rho} \left( - \frac{1}{\sinh 2r} \frac{\partial}{\partial r} \right)^\mu
\]
\[
\times \frac{1}{\Gamma \left( \frac{2}{7} \right)} \int_0^\infty t^{\frac{2}{7} - 1} e^{\frac{\rho^2}{2} t} e^{-\tilde{m}^2 t} h_4(r, n) dr dt
\]
\[
= \frac{c}{b_{\tilde{m}}} \int_0^\infty \frac{\sinh 2r}{\cosh 2r - \cosh 2\rho} \left( - \frac{1}{\sinh 2r} \frac{\partial}{\partial r} \right)^\mu \left( -\Delta h^m - \left( \frac{n-1}{2} \right)^2 \right)^{\frac{\gamma}{2}} dr
\]
\[
= A_1 + A_2,
\]
where
\[ A_1 = \int_1^\rho 1 \cdots \text{ and } A_2 = \int_1^\infty 1 \cdots . \]

We begin by estimating \( A_1 \). Using Lemma C, it is easy to see that, for \( 0 < r < 1 \), there holds
\[
\left( -\frac{1}{\sinh 2r \partial r} \right)^2 \left( -\Delta_{H^m_r} - \left( n - \frac{1}{2} \right)^2 \right)^\frac{\gamma}{2} = \left( -\frac{1}{\sinh 2r \partial r} \right)^2 \left( \frac{1}{\gamma_n(\gamma)} \frac{1}{r^{n-\gamma}} + O \left( \frac{1}{r^{n-\gamma-1}} \right) \right)
\]
= \frac{1}{\gamma_n(\gamma)} \frac{(n-\gamma)(n+2-\gamma)}{4} \frac{1}{r^{n+4-\gamma}} + O \left( \frac{1}{r^{n+3-\gamma}} \right),
\]
and similarly
\[
\left( -\frac{1}{\sinh 2r \partial r} \right)^4 \left( -\Delta_{H^m_r} - \left( n - \frac{1}{2} \right)^2 \right)^\frac{\gamma}{2} = \frac{1}{\gamma_n(\gamma)} \frac{(n-\gamma)(n+2-\gamma)(n+4-\gamma)(n+6-\gamma)}{16} \frac{1}{r^{n+8-\gamma}} + O \left( \frac{1}{r^{n+\gamma-\gamma}} \right).
\]
Consequently, in the quaternionic case, there holds
\[
\sinh 2r \left( -\frac{1}{\sinh 2r \partial r} \right)^2 \left( -\Delta_{H^m_r} - \left( n - \frac{1}{2} \right)^2 \right) = \frac{1}{\gamma_n(\gamma)} \frac{(n-\gamma)(n+2-\gamma)}{2} \frac{1}{r^{n+3-\gamma}} + O \left( \frac{1}{r^{n+2-\gamma}} \right)
\]
and, in the octonionic case, there holds
\[
\sinh 2r \left( -\frac{1}{\sinh 2r \partial r} \right)^4 \left( -\Delta_{H^m_r} - \left( n - \frac{1}{2} \right)^2 \right) = \frac{1}{\gamma_n(\gamma)} \frac{(n-\gamma)(n+2-\gamma)(n+4-\gamma)(n+6-\gamma)}{8} \frac{1}{r^{n+7-\gamma}} + O \left( \frac{1}{r^{n+6-\gamma}} \right).
\]
Now, using Lemma D, we compute in the quaternionic case that
\[
A_1 = \frac{c_m}{b_{2m-2}} \frac{(n-\gamma)(n+2-\gamma)}{2\gamma_n(\gamma)} \int_1^\rho \frac{1}{\sqrt{\cosh 2r - \cosh 2\rho}} \left[ \frac{1}{r^{n+3-\gamma}} + O \left( \frac{1}{r^{n+2-\gamma}} \right) \right] dr
\]
\leq \frac{c_m}{b_{2m-2}} \frac{(n-\gamma)(n+2-\gamma)}{2\gamma_n(\gamma)} \int_1^\rho \frac{\cosh r}{\sqrt{\cosh 2r - \cosh 2\rho}} \left[ \frac{1}{(\sinh r)^{n+3-\gamma}} + O \left( \frac{1}{(\sinh r)^{n+2-\gamma}} \right) \right] dr
\]
= \frac{c_m(n-\gamma)(n+2-\gamma)}{2\gamma_n(\gamma)b_{2m-2}} \frac{\Gamma \left( \frac{1}{2} \right) \Gamma \left( \frac{n+3-\gamma}{2} \right)}{2\sqrt{2\Gamma \left( \frac{n+4-\gamma}{2} \right) (\sinh \rho)^{n+3-\gamma}}} + O \left( \frac{1}{(\sinh \rho)^{n+2-\gamma}} \right)
\]
= \frac{1}{\gamma_{4m}(\gamma)} \frac{1}{(\sinh \rho)^{4m-\gamma}} + O \left( \frac{1}{(\sinh \rho)^{4m-\gamma-1}} \right),
\]
where we have computed
\[
\frac{c_m(n-\gamma)(n+2-\gamma)}{2\gamma_n(\gamma)b_{2m-2}} \frac{\Gamma \left( \frac{1}{2} \right) \Gamma \left( \frac{n+3-\gamma}{2} \right)}{2\sqrt{2\Gamma \left( \frac{n+4-\gamma}{2} \right) (\sinh \rho)^{n+3-\gamma}}} = \frac{1}{\gamma_{4m}(\gamma)}.
\]
Similarly, we have in the octonionic case that
\[
A_1 = \frac{1}{\gamma_{16}(\gamma)} \frac{1}{(\sinh \rho)^{16-\gamma}} + O \left( \frac{1}{(\sinh \rho)^{15-\gamma}} \right).
Concerning estimating $A_2$, it is clear from Lemma 5.1 that $A_2 \lesssim 1$ for both the quaternionic and octonionic cases, and so

$$k_{\gamma}(\rho) = A_1 + A_2$$

$$\leq \frac{1}{\gamma N(\gamma)} \frac{1}{\rho^{\gamma - 1}} + O\left(\frac{1}{\rho^{N - \gamma - 1}}\right),$$

as desired.

5.3. **Estimate for** $k_{\zeta,\gamma}$. In this subsection, we obtain the asymptotics for $k_{\zeta,\gamma}$ for $0 < \gamma < 4m$ and $0 < \zeta$. Note that the large distance asymptotics ($1 \leq \rho$) are already contained (5.4).

**Lemma 5.7.** Let $N = \dim_{\mathbb{R}} H_m^0$, $0 < \gamma < N$, $0 < \zeta < \epsilon < \min\{1, N - \gamma\}$. If $0 < \rho < 1$, then

$$k_{\zeta,\gamma} \leq \frac{1}{\gamma N(\gamma)} \frac{1}{\rho^{\gamma - 1}} + O\left(\frac{1}{\rho^{N - \gamma - \epsilon}}\right).$$

If $1 \leq \rho$, then

$$k_{\zeta,\gamma} \sim \rho^{\frac{2\gamma - 2}{2}} e^{-\zeta \rho - \frac{Q^2}{4}.}$$

**Proof.** As mentioned above, we only need to prove the estimate for $0 < \rho < 1$.

As before, let $n = 2\hat{m} + 1$ with $\hat{m}$ as above, and choose $\hat{\gamma}$ and $\ell$ such that $0 < \hat{\gamma} < 3$, $0 \leq \ell < n - 1$ and $\gamma = \hat{\gamma} + \ell$. Then

$$k_{\zeta,\gamma} = k_{\zeta,\hat{\gamma}} * k_{\zeta,\ell}.$$

Using Lemmas 5.2 and 5.3, it will be sufficient to estimate $k_{\zeta,\hat{\gamma}}$ and $k_{\zeta,\ell}$ separately.

To estimate $k_{\zeta,\hat{\gamma}}$, note that, by Lemma 5.6, there holds

$$k_{\zeta,\hat{\gamma}} = \left(-\Delta - \frac{Q^2}{4} + \zeta^2\right)^{-\frac{\hat{\gamma}}{2}}$$

$$= \frac{1}{\Gamma\left(\frac{\hat{\gamma}}{2}\right)} \int_{0}^{\infty} t^{\frac{\hat{\gamma}}{2} - 1} e^{\left(\Delta + \frac{Q^2}{4} - \zeta^2\right)t} dt$$

$$\leq \frac{1}{\Gamma\left(\frac{\hat{\gamma}}{2}\right)} \int_{0}^{\infty} t^{\frac{\hat{\gamma}}{2} - 1} e^{t\left(\Delta + \frac{Q^2}{4}\right)} dt$$

$$= \left(-\Delta - \frac{Q^2}{4}\right)^{-\frac{\hat{\gamma}}{2}}$$

$$= k_{\hat{\gamma}}$$

$$\leq \frac{1}{\gamma N(\hat{\gamma})} \frac{1}{\rho^{\gamma - 1}} + O\left(\frac{1}{\rho^{N - \gamma - 1}}\right).$$

We see that, if $\ell = 0$, then we are done, and so we assume without loss of generality that $0 < \ell$.

We now estimate $k_{\zeta,\ell}$. As in the previous proof, let $\mu = 2$ for the quaternionic case and $\mu = 4$ for the octonionic case, and let $c$ denote $c_m$ or $c_o$ in the respective cases. We compute

$$k_{\zeta,\ell} = \frac{1}{\Gamma\left(\frac{\ell}{2}\right)} \int_{0}^{\infty} t^{\ell - 1} e^{t\left(\Delta + \frac{Q^2}{4} - \zeta^2\right)} dt$$
\[
\frac{c}{b_n} \int_{\rho}^{\infty} \frac{\sinh 2r}{\sqrt{\cosh 2r - \cosh 2\rho}} \left( -\frac{1}{\sinh 2r \partial r} \right)^2 \left( -\Delta_{H^\gamma} - \left( \frac{n-1}{2} \right)^2 + \zeta^2 \right)^{-\frac{1}{2}} dr
\]

= \int_{\rho}^{1} \cdots \text{ and } \int_{1}^{\infty} = A_7 + A_8.

where

\[ A_7 = \int_{\rho}^{1} \cdots \text{ and } A_8 = \int_{1}^{\infty} \cdots . \]

From [38, Proposition 2.5], we have that

\[ \left( -\Delta - \left( \frac{n-1}{2} \right)^2 + \zeta^2 \right)^{-\frac{1}{2}} = \frac{1}{\gamma_n(\gamma)} \frac{1}{\rho^{n-\ell}} + O \left( \frac{1}{\rho^{n-\ell-1}} \right), \]

and by similar computations to those given in the proof of Lemma 5.6, we have

\[
\sinh 2r \left( -\frac{1}{\sinh 2r \partial r} \right)^2 \left( -\Delta_{H^\gamma} - \left( \frac{n-1}{2} \right)^2 \right) = \frac{(n-\ell)(n+2-\ell)}{2\gamma_n(\gamma)} \frac{1}{\rho^{n+3-\ell}} + O \left( \frac{1}{\rho^{n+2-\ell}} \right),
\]

and

\[
\sinh 2r \left( -\frac{1}{\sinh 2r \partial r} \right)^4 \left( -\Delta_{H^\gamma} - \left( \frac{n-1}{2} \right)^2 \right) = \frac{(n-\ell)(n+2-\ell)(n+4-\ell)(n+6-\ell)}{8\gamma_n(\gamma)} \frac{1}{\rho^{n+7-\ell}} + O \left( \frac{1}{\rho^{n+6-\ell}} \right).
\]

Consequently, using \( 1 \leq \cosh r \) and Lemma D, we find for the quaternionic case that

\[
A_7 \leq \frac{c_m}{b_{2m-2}} \frac{(n-\ell)(n+2-\ell)}{2\gamma_n(\gamma)} \int_{\rho}^{\infty} \frac{\sinh 2r}{\cosh 2r - \cosh 2\rho} \left[ \frac{1}{(\sinh r)^{n+3-\ell}} + O \left( \frac{1}{(\sinh r)^{n+2-\ell}} \right) \right] dr
\]

= \frac{1}{\gamma_{4m}(\ell)} \frac{1}{(\sinh \rho)^{4m-\ell}} + O \left( \frac{1}{(\sinh \rho)^{4m-\ell-1}} \right),

where we have computed

\[
\frac{c_m(n-\ell)(n+2-\ell) \Gamma \left( \frac{1}{2} \right) \Gamma \left( \frac{n+3-\ell}{2} \right)}{2\gamma_n(\ell) b_{2m-2} 2\sqrt{\Gamma \left( \frac{n+4-\ell}{2} \right)}} = \frac{1}{\gamma_{4m}(\ell)}.
\]

Similarly, we have for that octonionic case that

\[
A_7 \leq \frac{1}{\gamma(\ell) \gamma(\ell)} \frac{1}{(\sinh \rho)^{16-\ell}} + O \left( \frac{1}{(\sinh \rho)^{15-\ell}} \right).
\]

Again, using Lemma 5.1, we have that \( A_8 \leq 1 \), and so we have proved to two estimates

\[ k_{\zeta,\ell} \leq \frac{1}{\gamma_N(\ell)} \frac{1}{(\sinh \rho)^{-\ell-1}} + O \left( \frac{1}{(\sinh \rho)^{N-\ell-1}} \right) \]

\[ k_{\zeta,\gamma} \leq \frac{1}{\gamma_N(\gamma)} \frac{1}{(\sinh \rho)^{-\gamma}} + O \left( \frac{1}{(\sinh \rho)^{N-\gamma-1}} \right). \]

Now, using (5.4), we have, for any \( 0 < \zeta' < \zeta \), \( 0 < \alpha \) and \( 1 \leq \rho \), there holds

\[ k_{\zeta,\alpha} \sim \rho^{\frac{\alpha-2}{2}} e^{-\zeta \rho - \frac{\alpha}{2}} \lesssim e^{-\zeta' \rho - \frac{\alpha}{2} \rho}. \]
Therefore, using this estimate and that \( \cosh r \sim e^r \) and \( \sinh r \sim e^r \) for \( r > 1 \), and \( \sinh r \sim r \) and \( \cosh r \sim 1 \) for \( 0 < r < 1 \), we obtain the following global estimates (i.e., for \( 0 < \rho < 1 \)):

\[
\begin{align*}
k_{\zeta, \ell} &\leq \frac{1}{\gamma N(\ell)} \frac{(\cosh \rho)^{N-\frac{2}{2}-\ell-\gamma'}}{(\sinh \rho)^{N-\ell}} + O \left( \frac{(\cosh \rho)^{N-\frac{2}{2}-\ell-\gamma'-1}}{(\sinh \rho)^{N-\ell-1}} \right), \\
k_{\zeta, \hat{\gamma}} &\leq \frac{1}{\gamma N(\hat{\gamma})} \frac{(\cosh \rho)^{N-\frac{2}{2}-\hat{\gamma}-\gamma'}}{(\sinh \rho)^{N-\hat{\gamma}}} + O \left( \frac{(\cosh \rho)^{N-\frac{2}{2}-\hat{\gamma}-\gamma'-1}}{(\sinh \rho)^{N-\hat{\gamma}-1}} \right).
\end{align*}
\]

At last, using Lemmas 5.2 and 5.3 and letting \( 0 < \epsilon < \min \{1, N - \gamma\} \), we obtain

\[
k_{\zeta, \gamma} = k_{\zeta, \ell} * k_{\zeta, \hat{\gamma}}
\]

\[
\leq \frac{1}{\gamma N(\ell + \hat{\gamma})} \frac{1}{\rho^{N-\gamma-\ell}} + O \left( \frac{1}{\rho^{N-\gamma-\gamma'-\epsilon}} \right),
\]

which gives the desired estimate since \( \gamma = \ell + \hat{\gamma} \).

\[\square\]

5.4. Estimates for \( k_{\gamma} * k_{\zeta, \gamma'} \). In this subsection, we obtain the asymptotics for \( k_{\gamma} * k_{\zeta, \gamma} \) for \( 0 < \gamma < 3 \), \( 0 < \gamma' < N - \gamma \) and \( 0 < \zeta \).

Lemma 5.8. Let \( 0 < \gamma < 3 \), \( 0 < \gamma' < N - \gamma \), \( 0 < \zeta \) and \( 0 < \epsilon < \min \{1, N - \gamma - \gamma', \frac{\zeta}{2}\} \). If \( 0 < \rho < 1 \), then

\[
k_{\gamma} * k_{\zeta, \gamma'} \leq \frac{1}{\gamma N(\gamma' + \gamma')} \frac{1}{\rho^{N-\gamma-\gamma'}} + O \left( \frac{1}{\rho^{N-\gamma-\gamma'-\epsilon}} \right).
\]

If \( 1 \leq \rho \), then

\[
k_{\gamma} * k_{\zeta, \gamma'} \lesssim e^{(\rho - \frac{\zeta}{2})\rho}.
\]

Proof. By Lemma 5.6, we have for \( 0 < \rho < 1 \) the estimate

\[
k_{\gamma} \leq \frac{1}{\gamma N(\gamma)} \frac{1}{(\sinh \rho)^{N-\gamma}} + O \left( \frac{1}{(\sinh \rho)^{N-\gamma-1}} \right),
\]

and, by (5.4), we have for any \( 0 < \epsilon \) and \( 1 \leq \rho \) the estimate

\[
k_{\gamma} \sim \rho^{-2} e^{-\frac{\rho}{2}} \lesssim e^{(\rho - \frac{\zeta}{2})\rho}.
\]

Consequently, we obtain the global estimates (i.e., for \( 0 < \rho \)):

\[
k_{\gamma} \leq \frac{1}{\gamma N(\gamma)} \frac{(\cosh \rho)^{N-\frac{2}{2}-\gamma+\epsilon}}{(\sinh \rho)^{N-\gamma}} + O \left( \frac{(\cosh \rho)^{N-\frac{2}{2}-\gamma+\epsilon-1}}{(\sinh \rho)^{N-\gamma-1}} \right).
\]

Similarly, we have for \( 0 < \rho \) the global estimates

\[
k_{\zeta, \gamma'} \leq \frac{1}{\gamma N(\gamma')} \frac{(\cosh \rho)^{N-\frac{2}{2}-\gamma'-\zeta+\epsilon}}{(\sinh \rho)^{N-\gamma'}} + O \left( \frac{(\cosh \rho)^{N-\frac{2}{2}+\epsilon-\gamma'-\zeta-1}}{(\sinh \rho)^{N-\gamma-1}} \right).
\]

Therefore, by Lemmas 5.2 and 5.3, there holds

\[
k_{\gamma} * k_{\zeta, \gamma'} \leq \frac{1}{\gamma N(\gamma + \gamma')} \frac{1}{\rho^{N-\gamma-\gamma'}} + O \left( \frac{1}{\rho^{N-\gamma-\gamma'-\epsilon}} \right)
\]

for \( 0 < \rho < 1 \).

Similarly, using Lemmas 5.4 and 5.5 we have

\[
k_{\gamma} * k_{\zeta, \gamma'} \lesssim e^{(\rho - \frac{\zeta}{2})\rho}.
\]

\[\square\]
Lemma 5.9. Let $0 < \gamma < 3$, $0 < \gamma' < N - \gamma$, $0 < \zeta$ and $0 < \zeta' < \zeta$. If $1 \leq \rho$, then

$$k_\gamma * k_{\zeta',\gamma'} \lesssim e^{-\frac{Q_\gamma}{2} \rho} + \rho^{-2} e^{-\frac{Q_\gamma}{2} \rho} * k_{\zeta',\gamma'}.$$  

Proof. Using (5.4), we have

$$k_\gamma * k_{\zeta',\gamma'} = \int_{\{z \in B_T^\rho : |z| < \frac{1}{2}\}} k_\gamma (\rho(z)) k_{\zeta',\gamma'} (\rho(z, w)) dV(z)$$

$$+ \int_{\{z \in B_T^\rho : \frac{1}{2} \leq |z| < 1\}} k_\gamma (\rho(z)) k_{\zeta',\gamma'} (\rho(z, w)) dV(z)$$

$$\lesssim \int_{\{z \in B_T^\rho : |z| < \frac{1}{2}\}} k_\gamma (\rho(z)) k_{\zeta',\gamma'} (\rho(z, w)) dV(z)$$

$$+ \int_{\{z \in B_T^\rho : \frac{1}{2} \leq |z| < 1\}} \rho(z)^{\gamma - 2} e^{-\frac{Q_\gamma}{2} \rho(z)} k_{\zeta',\gamma'} (\rho(z, w)) dV(z)$$

$$\leq \int_{\{z \in B_T^\rho : |z| < \frac{1}{2}\}} k_\gamma (\rho(z)) k_{\zeta',\gamma'} (\rho(z, w)) dV(z)$$

$$+ \rho^{-2} e^{-\frac{Q_\gamma}{2} \rho} * k_{\zeta',\gamma'}.$$  

Thus we need only show that, for $1 \leq \rho$, there holds

$$\int_{\{z \in B_T^\rho : |z| < \frac{1}{2}\}} k_\gamma (\rho(z)) k_{\zeta',\gamma'} (\rho(z, w)) dV(z) \lesssim e^{-\frac{Q_\gamma}{2} \rho}.$$  

By Lemma 5.6, we have that, for $\rho(z) < \frac{1}{2}$, there holds

$$k_\gamma (\rho(z)) \lesssim \frac{1}{\rho(z)^{N - \gamma}} \sim \frac{1}{|z|^{N - \gamma}}.$$  

Next, observing that $1 \leq \rho(w)$ and $\rho(z) < \frac{1}{2}$ imply $\frac{1}{2} \leq \rho(w) - \rho(z) \leq \rho(z, w)$, we have by (5.4) that, for $0 < \zeta' < \zeta$, there holds

$$k_{\zeta',\gamma'} (\rho(z, w)) \lesssim e^{-\zeta' \rho(z, w) - \frac{Q_\gamma}{2} \rho(z, w)} \sim (\cosh \rho(z, w))^{-\left(\zeta' + \frac{Q_\gamma}{2}\right)}.$$  

Combining these estimates with Lemma A, we compute

$$\int_{\{z \in B_T^\rho : |z| < \frac{1}{2}\}} k_\gamma (\rho(z)) k_{\zeta',\gamma'} (\rho(z, w)) dV(z)$$

$$\lesssim \int_{\{z \in B_T^\rho : |z| < \frac{1}{2}\}} \frac{1}{|z|^{4m - \gamma}} \left( \sqrt{(1 - |w|^2)(1 - |z|^2)} \right)^{2m + 1 + \zeta'} \left( \frac{1}{1 - |z|^2} \right)^{2m + 2} dz$$

$$\sim (1 - |w|^2)^{2m + 1 + \zeta'} \int_{\{z \in B_T^\rho : |z| < \frac{1}{2}\}} \frac{1}{|z|^{4m - \gamma}} dz$$

$$\sim (\cosh \rho)^{-2m + 1 + \zeta'}$$

$$\sim e^{-\left(\zeta' + 2m + 1\right) \rho}.$$
Similarly, 
\[ \int_{\{z \in B_C^a : \rho(z) < \frac{1}{2}\}} k_{\gamma} (\rho(z)) k_{\zeta', \gamma'} (\rho(z, w)) dV(z) \]
\[ \lesssim \int_{\{z \in B_C^a : \rho(z) < \frac{1}{2}\}} \frac{1}{|z|^{16 - \gamma}} \left( \frac{(1 - |w|^2)(1 - |z|^2)}{\Psi_C(z, w)} \right)^{\frac{11 + \zeta'}{2}} \left( \frac{1}{1 - |z|^2} \right)^{\frac{12}{2}} dz \]
\[ \sim (1 - |w|^2)^{\frac{11 + \zeta'}{2}} \int_{\{z \in B_C^a : \rho(z) < \frac{1}{2}\}} \frac{1}{|z|^{16 - \gamma}} dz \]
\[ \sim (\cosh \rho)^{-(11 + \zeta')} \]
\[ \sim e^{-\left(\zeta' + 11\right)} \rho \quad \square \]

6. Rearrangement Estimates

We firstly collect known results about nonincreasing rearrangements and Lorentz spaces on the hyperbolic spaces $\mathbb{X}$. These results will be used to prove estimates on $k_{\gamma} * k_{\zeta', \gamma'} * f$ for $f \in C_0^\infty(\mathbb{X})$.

To begin, let $f : \mathbb{X} \to \mathbb{R}$, and define 
\[ f^*(t) = \inf \{ s > 0 : \lambda_f(s) \leq t \} \]
\[ \lambda_f(s) = \left| \{ z \in \mathbb{X} : |f(z)| > s \} \right| \]
\[ = \int_{\mathbb{X} : |f(z)| > s} dV(z). \]

Next, for a domain $\Omega \subset \mathbb{X}$, we recall the Lorentz spaces $L^{p,q}(\Omega)$ consist of functions for which the following norm is finite:
\[ \|f\|_{L^{p,q}(\Omega)} = \begin{cases} \left\| \frac{1}{p} f^*(t) \right\|_{L^p(0,|\Omega|)} & 1 \leq q < \infty \\ \sup_{t>0} \left( \frac{1}{p} f^*(t) \right) & q = \infty \end{cases} \]

Define next $f^{**}(t) = \frac{1}{t} \int_0^t f^*(s) ds$ and
\[ \|f\|_{L^{p,q}(\Omega)} = \begin{cases} \left\| \frac{1}{p} f^{**}(t) \right\|_{L^p(0,|\Omega|)} & 1 \leq q < \infty \\ \sup_{t>0} \left( \frac{1}{p} f^{**}(t) \right) & q = \infty \end{cases} \]

Let $1 < r, p_1, p_2 < \infty$ and $1 \leq s, q_1, q_2 \leq \infty$ satisfy
\[ \frac{1}{p_1} + \frac{1}{p_2} - 1 = \frac{1}{r}, \quad \frac{1}{q_1} + \frac{1}{q_2} \geq \frac{1}{s}, \]
and assume $f \in L^{p_1, q_1}(\mathbb{X})$ and $g \in L^{p_2, q_2}(\mathbb{X})$. The generalized Young’s inequality (see [48, Theorem 2.6])
\[ \|f \ast g\|_{L^{r,s}} \leq C \|f\|_{L^{p_1, q_1}} \|g\|_{L^{p_2, q_2}}, \]
and norm equivalence (see [48] for $1 \leq r < \infty$ and [55, Theorem 3.4] for $0 < r < 1$)
\[ \|f \ast g\|_{L^{r,s}} \leq \frac{q}{q - 1} \|f \ast g\|_{L^{r,s}} \]
give the following lemma.

**Lemma F.** Let $1 < r, p_1, p_2 < \infty$ and $1 \leq s, q_1, q_2 \leq \infty$. If

$$\frac{1}{p_1} + \frac{1}{p_2} - 1 = \frac{1}{r}, \quad \frac{1}{q_1} + \frac{1}{q_2} \geq \frac{1}{s},$$

$f \in L^{p_1,q_1}(X)$ and $g \in L^{p_2,q_2}(X)$, then

$$\|f \ast g\|_{L^{r,s}} \leq C \|f\|_{L^{p_1,q_1}} \|g\|_{L^{p_2,q_2}}.$$

In this section, we collect the kernel estimates obtained above and state the corresponding estimates for their nonincreasing rearrangements. We also prove that the square integrability of the rearrangement $[k_\zeta * k_{\zeta,\gamma}]^*$ on any interval of the form $(c, \infty)$, $0 < c$. In preparation of obtaining the rearrangement estimates, we first estimate the volume of the geodesic ball $B_\rho$ centered at the origin and with radius $\rho$. For $H^m_Q$, we may use

$$|B_\rho| = \omega_{4m-1} \int_0^\rho (\sinh r)^{4m-1} (\cosh \rho)^3 dr,$$

to obtain

$$|B_\rho| = \frac{\omega_{4m-1}}{4m} \rho^{4m} + O(\rho^{4m+2}) \text{ if } 0 < \rho < 1$$

and

$$|B_\rho| \sim e^{(4m+2)\rho} \text{ if } 1 \leq \rho.$$

Similarly, for $H_{Ca}$, we may use

$$|B_\rho| = \omega_{15} \int_0^\rho (\sinh r)^{15} (\cosh \rho)^7 dr,$$

to obtain

$$|B_\rho| = \frac{\omega_{15}}{16} \rho^{16} + O(\rho^{18}) \text{ if } 0 < \rho < 1$$

and

$$|B_\rho| \sim e^{22\rho} \text{ if } 1 \leq \rho.$$

Next, we collect the kernel estimates established above. On $H^m_\mathbb{H}$ with $N = \dim_{\mathbb{R}} H^m_\mathbb{H}$, there holds

- Let $0 < \zeta$. If $0 < \gamma < N$, $0 < \epsilon < \min\{1, N - \gamma\}$ and $0 < \rho < 1$, then

  $$k_{\zeta,\gamma} \leq \frac{1}{\gamma N(\gamma)} \frac{1}{\rho^{N-\gamma}} + O\left(\frac{1}{\rho^{N-\gamma-\epsilon}}\right).$$

  If $0 < \gamma$ and $1 \leq \rho$, then

  $$k_{\zeta,\gamma} \sim \rho^{\frac{\gamma-2}{2}} e^{-(\gamma+\frac{\gamma}{2})\rho}.$$

- Let $\zeta = 0$. If $0 < \gamma < 3$ and $0 < \rho < 1$, then

  $$k_{\gamma} \leq \frac{1}{\gamma N(\gamma)} \frac{1}{\rho^{N-\gamma}} + O\left(\frac{1}{\rho^{N-\gamma-1}}\right).$$

  If $0 < \gamma < 3$ and $1 \leq \rho$, then

  $$k_{\gamma} \sim \rho^{\gamma-2} e^{-\frac{\gamma}{2}\rho}.$$
Let $0 < \zeta$. If $0 < \gamma < 3$, $0 < \gamma' < N - \gamma$, $0 < \epsilon < \min\{1, N - \gamma - \gamma', \frac{\zeta}{2}\}$ and $0 < \rho < 1$, then
\[
k_\gamma * k_{\zeta,\gamma'} \leq \frac{1}{\gamma_N(\gamma + \gamma')} \frac{1}{\rho^{N-\gamma-\gamma'}} + O\left(\frac{1}{\rho^{N-\gamma-\gamma'-\epsilon'}}\right).
\]
If $1 \leq \rho$, then
\[
k_\gamma * k_{\zeta,\gamma'} \lesssim e^{(\epsilon' - \frac{q}{2})\rho}.
\]
If $0 < \zeta' < \zeta$ and $1 \leq \rho$, then
\[
k_\gamma * k_{\zeta,\gamma'} \leq e^{-(\zeta' + \frac{q}{2})\rho} + \rho^{\gamma - \frac{q}{2}} e^{-\frac{q}{2}\rho} * k_{\zeta,\gamma'}.
\]
The corresponding estimates for their rearrangements are listed now.

Let $0 < \gamma$. If $0 < \gamma < N$, $0 < \epsilon < \min\{1, N - \gamma\}$ and $0 < t < 2$, then
\[
[k_{\zeta,\gamma}]^* \leq \frac{1}{\gamma_N(\gamma)} \left(\frac{N}{\omega_{N-1}} t\right)^{\frac{2-\gamma}{N}} + O\left(t^{\frac{2-\gamma}{N}}\right).
\]
If $0 < \gamma$ and $2 \leq t$, then
\[
[k_{\zeta,\gamma}]^* \sim t^{-\frac{1}{\gamma} + \frac{3}{2}\zeta} (\ln t)^{\frac{2-\gamma}{2}}.
\]
Let $\zeta = 0$. If $0 < \gamma < 3$ and $0 < t < 2$, then
\[
[k_{\gamma}]^* \leq \frac{1}{\gamma_N(\gamma)} \left(\frac{N}{\omega_{N-1}} t\right)^{\frac{2-\gamma}{N}} + O\left(t^{\frac{2-\gamma}{N}}\right).
\]
If $0 < \gamma < 3$ and $2 \leq t$, then
\[
[k_{\gamma}]^* \sim t^{-\frac{1}{\gamma}} (\ln t)^{-\gamma}.
\]
Let $0 < \zeta$. If $0 < \gamma < 3$, $0 < \gamma' < N - \gamma$, $0 < \epsilon < \min\{1, N - \gamma - \gamma', \frac{\zeta}{2}\}$ and $0 < t < 2$, then
\[
[k_\gamma * k_{\zeta,\gamma'}]^* \leq \frac{1}{\gamma_N(\gamma + \gamma')} \left(\frac{N}{\omega_{N-1}} t\right)^{\frac{2-\gamma'}{N}} + O\left(t^{\frac{2-\gamma'}{N}}\right).
\]
If $2 \leq t$, then
\[
[k_\gamma * k_{\zeta,\gamma'}]^* \leq t^{-\frac{q}{2}}.
\]
Moreover, using Lemma 5.9, we have, for $c > 0$,
\[
\int_c^\infty |[k_\alpha * k_{\zeta,\beta}]^*(t)|^2 dt < \infty.
\]

The proof of (6.3) is similar to that given in [39], Lemma 4.1 and we omit it.

6.1. Estimates for $k_\gamma * k_{\zeta,\gamma'} * f$. In this section, we prove and $L^p - L^{p'}$ inequality for $k_\gamma * k_{\zeta,\gamma'} * f$, which is dual to the Poincaré-Sobolev inequality. We will need to make use of the Kunze-Stein phenomenon. Kunze-Stein phenomenon is important in harmonic analysis (see [14], [12], [13], [31], [50]) and is closely related to geometric and functional inequalities (see Beckner [7]). We begin by recalling relevant results. The proofs of Lemmas G and H may be found in [46].

We begin by recalling that Cowling, Giulini and Meda(see [14], [12], [13]) established the following sharp version on Lorentz space ([29], [48]) of the Kunze-Stein phenomenon for connected real simple groups $G$ of real rank one with finite center:
\[
L^{p,q_1}(G) * L^{p,q_2} \subset L^{p,q_2}(G)
\]
provided $1 < p < 2$, $1 \leq q_1, q_2, q_3 \leq \infty$ and $1 + \frac{1}{q_3} \leq \frac{1}{q_1} + \frac{1}{q_2}$. In particular, this applies to $Sp(m, 1)$ and $F_4$, and by following [46], we can obtain similar phenomenon on $H^m_0$ and $H^2_0$. To be more precise, let $L^p(G)$ and $L^{p,q}(G)$ denote the usual Lebesgue and Lorentz spaces, respectively, and let $L^{p,q}(G/K)$, $L^{p,q}(K\backslash G)$ and $L^{p,q}(K\backslash G/K)$ denote the closed subspaces of $L^{p,q}(G)$ of the right-$K$-invariant, left-$K$-invariant and $K$-bi-invariant functions, respectively. Following [46], we can show

**Lemma G.** For $p \in (1, 2)$, there holds

$$L^p(K\backslash G) * L^p(G/K) \subset L^{p,\infty}(K\backslash G/K).$$

**Lemma H.** For $p \in (1, 2)$ and $p' = \frac{p}{p+1}$, there holds

$$L^{p',1}(K\backslash G/K) * L^p(G/K) \subset L^{p'}(G/K)$$

and, if $f \in L^{p,1}(K\backslash G/K)$ and $h \in L^p(G/K)$, then there exists a constant $C > 0$ such that

$$\|f \ast h\|_{L^{p'}(G/K)} \leq C \|f\|_{L^{p',1}(K\backslash G/K)} \|h\|_{L^p(G/K)}.$$

Using Lemma H, we prove the following estimate on $k_\gamma * k_{\zeta,\gamma'} * f$.

**Lemma 6.1.** Let $0 < \gamma < 3$, $0 < \gamma' < N - \gamma$, $0 < \zeta$ and $\frac{2N}{N+\gamma+\gamma'} \leq p < 2$. Then, for $f \in C_0^\infty(\mathbb{B}_x^m)$, there holds

$$\|k_\gamma * k_{\zeta,\gamma'} * f\|_{L^{p'}} \leq C \|f\|_{L^p}.$$

**Proof.** Define the cut off functions

$$\eta_1(\rho) = \begin{cases} k_\gamma * k_{\zeta,\gamma'} & 0 < \rho < 1 \\ 0 & 1 \leq \rho \end{cases}$$

$$\eta_2(\rho) = k_\gamma * k_{\zeta,\gamma'} - \eta_1(\rho).$$

By (??), there exists a $t_0 > 0$ such that, for $0 < t \leq t_0$, there holds

$$\eta_1^*(t) \lesssim t^{\frac{\gamma'}{N} - N},$$

and, for $t_0 < t$, there holds $\eta_1^*(t) = 0$. Next, by Lemma F, there holds

$$\|\eta_1 * f\|_{L^{p'}} = \|\eta_1 * f\|_{L^{p',p'}} \leq C \|\eta_1\|_{L^{p',\infty}} \|f\|_{L^{p'}}.$$

But

$$\|\eta_1\|_{L^{p',\infty}} = \sup_{0 < t < \infty} t^{\frac{\gamma'}{N}} \eta_1^*(t) \lesssim \sup_{0 < t < t_0} t^{\frac{\gamma'}{N} + \frac{\gamma + \gamma' - N}{N}} < \infty,$$

provided

$$\frac{2}{p'} + \frac{\gamma + \gamma' - N}{N} > 0,$$

which is equivalent to

$$p > \frac{2N}{\gamma + \gamma' + N},$$

as is assumed. Consequently, there holds

$$\|\eta_1 * f\|_{L^{p'}} \lesssim \|f\|_{L^{p'}}.$$
Next, by (6.2), there exists a $0 < t_0$ such that, for $0 < t \leq t_0$, there holds
\[ \eta_2(t) \lesssim 1, \]
and, for $t_0 < t$ and $0 < \epsilon < \min\{1, N - \gamma - \gamma', \frac{N}{2}\}$, there holds
\[ \eta_2^*(t) \lesssim t^{-\frac{\epsilon}{N}}. \]
Consequently, we find, for $0 < \epsilon < \frac{Q}{2} + \frac{N}{p}$, there holds
\[ \|\eta_2\|_{L^p, 1} = \int_0^\infty t^{\frac{1}{p} - 1} \eta_2^*(t) dt < \infty. \]

At last, Lemma H, we obtain
\[ \|\eta_2 \ast f\|_{L^p} \leq C \|f\|_{L^p}, \]
and therefore
\[ \|k_{\gamma} \ast k_{\zeta, \gamma'} \ast f\|_{L^p} \leq \|\eta_1 \ast f\|_{L^p} + \|\eta_2 \ast f\|_{L^p} \leq C \|f\|_{L^p}, \]
as desired. \qed

7. Proofs of Theorem 1.3 and 1.4

With all of the kernel estimates proved in Section 5, we are ready to prove the Poincaré-Sobolev inequality (Theorem 1.3) and Hardy-Sobolev-Maz’ya inequality (Theorem 1.4). For the reader’s convenience, we restate these theorems before their respective proofs.

**Theorem 1.3.** Let $0 < \gamma < 3$, $0 < \gamma'$, $2 < p$ and $0 < \zeta$. Denote by $N = \dim X$. If $0 < \gamma' < N - \gamma$, suppose further that $2 < p \leq \frac{2N}{N - (\gamma + \gamma')}$. Then there exists a constant $C > 0$ such that, for all $u \in C_0^\infty (X)$, there holds
\[ \|u\|_p \leq C \left\| \left( -\Delta_X - \rho_X^2 + \zeta^2 \right)^{\frac{\gamma'}{4}} \left( -\Delta - \rho_X^2 \right)^{\frac{\zeta}{4}} u \right\|_2. \] (7.1)

**Proof.** By Lemma 6.1, we have
\[ \left\| \left( -\Delta_X - \rho_X^2 + \zeta^2 \right)^{\frac{\gamma'}{4}} \left( -\Delta - \rho_X^2 \right)^{\frac{\zeta}{4}} u \right\|_{L^p} \leq C \|u\|_{L^p}. \] (7.2)
Consulting [6, Appendix Lemma], we have that (7.2) is equivalent to
\[ \|u\|_{L^p} \leq C \left\| \left( -\Delta_X - \rho_X^2 + \zeta^2 \right)^{\frac{\gamma'}{4}} \left( -\Delta - \rho_X^2 \right)^{\frac{\zeta}{4}} u \right\|_{L^2}, \]
thereby proving the theorem. \qed

**Proof of Theorem 1.4.** We need only prove the inequality in case
\[ \lambda = \prod_{j=1}^{k} \frac{(a - k + 2j - 2)^2}{4}. \]
We will use the factorization Theorem (Theorem 1.1), and so, we set
\[ u = g^\frac{k-(2m+1)-a}{2} f, \]
and obtain

\[ 4^k \int_{H_0^{m-1}} \int_0^\infty u \prod_{j=1}^k \left[ -\varphi \partial_{\varphi} - a \partial_{\varphi} - \varphi \Delta z - L_0 + i(k + 1 - 2j) \sqrt{-\Delta z} \right] u \frac{dxdzd\varphi}{\varphi^{1-a}} \]

\[ = \int_{H_0^{m-1}} \int_0^\infty f \prod_{j=1}^k \left[ -\Delta - (2m + 1)^2 + (a - k + 2j - 2)^2 \right] f \frac{dxdzd\varphi}{\varphi^{2m+2}} \]

\[ = 4 \int_{U_m} f \prod_{j=1}^k \left[ -\Delta - (2m + 1)^2 + (a - k + 2j - 2)^2 \right] f dV. \]

Next, using that \( \text{spec} (-\Delta) = [(2m + 1)^2, \infty) \), we have the following sharp inequality

\[ \int_{U_m} f \prod_{j=1}^k \left[ -\Delta - (2m + 1)^2 + (a - k + 2j - 2)^2 \right] f dV \leq \prod_{j=1}^k (a - k + 2j - 2)^2 \int_{U_m} f^2 dV. \]

Applying Plancherel’s theorem, there holds

\[ \int_{U_m} f \prod_{j=1}^k \left[ -\Delta - (2m + 1)^2 + (a - k + 2j - 2)^2 \right] f dV - \prod_{j=1}^k (a - k + 2j - 2)^2 \int_{U_m} f^2 dV \]

\[ = C_m \int_{-\infty}^\infty \int_{S^{4m-1}} \left[ \prod_{j=1}^k (\lambda^2 + (a - k + 2j - 2)^2) - \prod_{j=1}^k (a - k + 2j - 2)^2 \right] |\hat{f}(\lambda, \varsigma)|^2 |\epsilon(\lambda)|^{-2} d\lambda d\sigma(\varsigma). \]

Choosing \( 0 < \delta \) so that

\[ \prod_{j=1}^k (\lambda^2 + (a - k + 2j - 2)^2) - \prod_{j=1}^k (a - k + 2j - 2)^2 \geq \lambda^2 (\lambda^2 + \delta)^{k-1}, \]

applying Theorem 1.3, and applying the Plancherel theorem, we obtain

\[ \int_{U_m} f \prod_{j=1}^k \left[ -\Delta - (2m + 1)^2 + (a - k + 2j - 2)^2 \right] f dV - \prod_{j=1}^k (a - k + 2j - 2)^2 \int_{U_m} f^2 dV \]

\[ \geq C_m \int_{-\infty}^\infty \int_{S^{4m-1}} \lambda^2 (\lambda^2 + \delta)^{k-1} |\hat{f}(\lambda, \varsigma)|^2 |\epsilon(\lambda)|^{-2} d\lambda d\sigma(\varsigma) \]

\[ = \int_{U_m} f \left( -\Delta - (2m + 1)^2 \right) \left( -\Delta - (2m + 1)^2 + \delta \right)^{k-1} f dV \]

\[ \geq C \|f\|_{L_p}^2. \]

This proves the first inequality. The proof of the second inequality is similar and we omit it.

8. Proofs of Theorem 1.5 and 1.6

Proof of Theorem 1.5 Set \( \Omega(u) = \{ x \in \mathbb{H}_C^n : |u(x)| \geq 1 \} \). Then by Theorem 1.3, we have, for \( p > 2 \),

\[ |\Omega(u)| = \int_{\Omega(u)} dV \leq \int_{\mathbb{H}_C^n} |u|^p dV \lesssim 1. \]
Therefore, $|\Omega(u)| \leq \Omega_0$ for some constant $\Omega_0$ independent of $u$. We write

$$\int_{\mathbb{R}^2} (e^{\beta_0(N/2,N)u^2} - 1 - \beta_0(N/2,N)u^2) \, dV$$

$$= \int_{\Omega(u)} (e^{\beta_0(N/2,N)u^2} - 1 - \beta_0(N/2,N)u^2) \, dV + \int_{\mathbb{R}^2 \setminus \Omega(u)} (e^{\beta_0(N/2,N)u^2} - 1 - \beta_0(N/2,N)u^2) \, dV$$

$$\leq \int_{\Omega(u)} e^{\beta_0(N/2,N)u^2} \, dV + \int_{\mathbb{R}^2 \setminus \Omega(u)} (e^{\beta_0(n,2n)u^2} - 1 - \beta_0(N/2,N)u^2) \, dV.$$ 

(8.1)

The second part of right hand of (8.1) is bounded. In fact, we have

$$\int_{\mathbb{R}^2 \setminus \Omega(u)} (e^{\beta_0(N/2,N)u^2} - 1 - \beta_0(N/2,N)u^2) \, dV$$

$$= \int_{\mathbb{R}^2 \setminus \Omega(u)} \sum_{n=2}^{\infty} \frac{(\beta_0(N/2,N)u^2)^n}{n!} \, dV$$

$$\leq \int_{\mathbb{R}^2 \setminus \Omega(u)} \sum_{n=2}^{\infty} \frac{(\beta_0(N/2,N)u^2)^n}{n!} \, dV$$

$$\leq \sum_{n=2}^{\infty} \frac{(\beta_0(N/2,N))^n}{n!} \int_{\mathbb{R}^2} |u(x)|^4 \, dV \leq C.$$

Here we use the fact $|u(z)| < 1$, $z \in \mathbb{X} \setminus \Omega(u)$ and Theorem 1.3.

Next we shall show that $\int_{\Omega(u)} e^{\beta_0(N/2,N)u^2} \, dV$ is also bounded by some universal constant. Set

$$v = (-\Delta_{\mathbb{X}} - \rho_\mathbb{X}^2 + \xi^2)(2n-\alpha)^4(-\Delta_{\mathbb{X}} - \rho_\mathbb{X}^2)^{\alpha/4}u.$$ 

Then

$$\int_{\mathbb{X}} |v|^2 \, dV \leq 1$$ 

(8.2)

and we can write $u$ as a potential

$$u = (-\Delta_{\mathbb{X}} - \rho_\mathbb{X}^2 + \xi^2)^{-}(2n-\alpha)^4(-\Delta_{\mathbb{X}} - \rho_\mathbb{X}^2)^{-\alpha/4}v = v * \phi, $$

(8.3)

where $\phi = (-\Delta_{\mathbb{X}} - \rho_\mathbb{X}^2 + \xi^2)^{-}(2n-\alpha)^4(-\Delta_{\mathbb{X}} - \rho_\mathbb{X}^2)^{-\alpha/4} = k_{\xi}(N-\alpha/2) * k_{\alpha/2}$. By 6.1 and 6.3,

$$\phi^c(t) \leq \frac{1}{\gamma_\alpha(N/2)} \left( \frac{Nt}{\omega_{N-1}} \right)^{-\frac{1}{2}} + O \left( \frac{t^{-\frac{1}{2}}} {t^{\frac{N-1}{2}}} \right), \quad 0 < t < 2 \quad \text{and} \quad \int_c^\infty \phi^c(t)^2 \, dt < \infty, \forall c > 0.$$

Closely following the proof of Theorem 1.7 in [38], we have that there exists a constant $C$ which is independent of $u$ and $\Omega(u)$ such that

$$\int_{\Omega(u)} e^{\beta_0(N/2,N)u^2} \, dV = \int_0^{[\Omega(u)]} \exp(\beta_0(N/2,N)|u^*(t)|^2) \, dt \leq \int_0^{\Omega_0} \exp(\beta_0(N/2,N)|u^*(t)|^2) \, dt \leq C.$$

The proof of Theorem 1.5 is thereby completed.
**Proof of Theorem 1.6** It is enough to show that in term of ball model, for some $\zeta > 0$, there holds

\[
\|(\Delta_X - \rho_X^2 + \zeta^2)^{(2m-1)/2}(\Delta_X - \rho_X^2)^{1/2}[1 - |z|^2]^{\frac{a+1}{2}} u\|_2 
\leq 4^{2m} \int_{\mathbb{R}^n} u^{2m} \prod_{j=1}^{2m} \left[ \left( \frac{\Delta_z (\Delta_X - \rho_X^2)^{1/2}}{2} \right) + \frac{(2m + 1 - 2j)^2}{4} - \frac{i(2m + 1 - 2j)}{2} \sqrt{1 + 1} \right] u \frac{dz}{(1 - |z|^2)^{1-a}}
\]

and in term of Siegel domain,

\[
\|(\Delta_X - \rho_X^2 + \zeta^2)^{(2m-1)/2}(\Delta_X - \rho_X^2)^{1/2}[1 - \frac{a+1}{2}] u\|_2 
\leq 4^{2m} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} u^{2m} \prod_{j=1}^{2m} \left[ -\varphi_\varphi x - a\Delta_x - \varphi_\Delta Z + \mathcal{C}_0 + i(k + 1 - 2j)\sqrt{-\Delta Z} \right] u \frac{dxdzdz}{\varphi_\varphi x^{1-a}}
\]

The proof is similar to that given in the proof of 1.4 via Plancherel formula and we omit it.

9. **Appendix: Proofs of Theorem 1.7 and 1.8**

In this section, we will outline the proofs of Adams inequalities, namely Theorem 1.7 and 1.8 for the convenience of the reader. We refer the interested reader to [38], [39], [45], [46] for all the details.

**Proof of Theorem 1.7** Let $f = (-\Delta_X - \rho_X^2 + \zeta^2)^{\frac{a}{2}} u$. Then $\|f\|_p \leq 1$ and

\[
u = (-\Delta_X - \rho_X^2 + \zeta^2)^{-\frac{a}{2}} f = f * k_{\zeta, a}
\]

Using O’Neil’s lemma ([48]), we have for $t > 0$,

\[
u(t) \leq \frac{1}{t} \int_0^t f(s)ds \int_0^t k_{\zeta, a}(s)ds + \int_t^\infty f(s)k_{\zeta, a}(s)ds.
\]

Using the rearrangement estimates of $k_{\zeta, a}$, it is easy to check

\[
[k_{\zeta, a}]^*(t) \leq \frac{1}{\gamma_N(\alpha)} \left( \frac{Nt}{\omega_{2n-1}} \right)^{\frac{\alpha-N}{N}} + O \left( t^{\frac{\alpha-N}{N}} \right), \ 0 < t < 2;
\]

\[
\int_0^\infty \nu(t)dt < \infty, \ \forall c > 0.
\]

Closely following the proof of Theorem 1.13 in [39], we have that there exists a constant $C$ which is independent of $u$ such that

\[
\frac{1}{|E|} \int_E \exp(\beta_0(\alpha, N)|u|^p)dv \leq \frac{1}{|E|} \int_0^{[E]} \exp(\beta_0(\alpha, N)|u|^p)dt
\]

\[
\leq \frac{1}{|E|} \int_0^{[E]} \exp \left( \beta_0(\alpha, N) \left[ \frac{1}{7} \int_0^t f(s)ds \int_0^t k_{\zeta, a}(s)ds + \int_t^\infty f(s)k_{\zeta, a}(s)ds \right]^{p'} \right) dt \leq C.
\]

The sharpness of the constant $\beta_0(\alpha, N)$ can be verified by the process similar to that in [1, 35] and thus the proof of Theorem 1.7 is completed.
Using the symmetrization-free argument from the local inequalities to global ones developed by Lam and the second author [36, 37], we can conclude the

**Proof of Theorem 1.8** Let \( u \in W^{\alpha,p}(X) \) with \( \int_X \left| (-\Delta_X - \rho_X^2 + \zeta^2)^{\frac{\alpha}{2}} u \right|^p \, dV \leq 1 \). By Hörmander-Mikhlin type multiplier theorem (see [2]), we have

\[
\int_X |u|^p \, dV \lesssim \int_X \left| (-\Delta_X - \rho_X^2 + \zeta^2)^{\frac{\alpha}{2}} u \right|^p \, dV \leq 1
\]

provided \( \zeta > 2 \rho_X \left| \frac{1}{p} - \frac{1}{2} \right| \). Set \( \Omega(u) = \{ z \in X : |u(z)| \geq 1 \} \). Then we have

\[
|\Omega(u)| = \int_{\Omega(u)} dV \leq \int_X |u|^p \, dV \leq \Omega_0,
\]

where \( \Omega_0 \) is a constant independent of \( u \). We write

\[
\int_X \Phi_p(\beta(\alpha, N)|u|^{p'}) \, dV = \int_{\Omega(u)} \Phi_p(\beta(\alpha, N)|u|^{p'}) \, dV + \int_{X \setminus \Omega(u)} \Phi_p(\beta(\alpha, N)|u|^{p'}) \, dV.
\]

Notice that on the domain \( X \setminus \Omega(u) \), we have \( |u(z)| < 1 \). Thus,

\[
\int_{X \setminus \Omega(u)} \Phi_p(\beta(\alpha, N)|u|^{p'}) \, dV \leq \sum_{k=j-1}^{\infty} \sum_{n=2}^{\infty} \frac{\beta(\alpha, N)^k}{k!} \int_{X \setminus \Omega(u)} |u|^{p'k} \, dV
\]

\[
\leq \sum_{k=j-1}^{\infty} \frac{\beta(\alpha, N)^k}{k!} \int_{X \setminus \Omega(u)} |u|^p \, dV \leq C.
\]

Moreover, by Theorem 1.7, if \( \zeta \) satisfies \( \zeta > 0 \) if \( 1 < p < 2 \) and \( \zeta > 2n \left| \frac{1}{p} - \frac{1}{2} \right| \) if \( p \geq 2 \), then

\[
\int_{\Omega(u)} \Phi_p(\beta(\alpha, N)|u|^{p'}) \, dV \leq \int_{\Omega(u)} \exp(\beta(\alpha, N)|u|^{p'}) \, dV \leq C.
\]

Combining (9.1) and (9.2) yields

\[
\int_X \Phi_p(\beta(\alpha, N)|u|^{p'}) \, dV = \int_{\Omega(u)} \Phi_p(\beta(\alpha, N)|u|^{p'}) \, dV + \int_{X \setminus \Omega(u)} \Phi_p(\beta(\alpha, N)|u|^{p'}) \, dV \leq C
\]

provided that \( \zeta \) satisfies \( \zeta > 2 \rho_X \left| \frac{1}{p} - \frac{1}{2} \right| \).

The sharpness of the constant \( \beta(\alpha, N) \) can be verified by the process similar to that in [39].

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**References**

[1] D.R. Adams, A sharp inequality of J. Moser for higher order derivatives, Ann. of Math. 128 (2) (1988) 385–398. 47
[2] J.-P. Anker, \( L_p \) Fourier multipliers on Riemannian symmetric spaces of the noncompact type. Ann. of Math., 132(3) (1990) 597–628. 48
[3] J.-P. Anker, L. Ji, Heat kernel and Green function estimates on noncompact symmetric spaces, Geom. Funct. Anal., 9 (1999) 1035–1091. 27
[4] J.-P. Anker and P. Ostellari. The heat kernel on noncompact symmetric spaces. In *Lie groups and symmetric spaces*, volume 210 of Amer. Math. Soc. Transl. Ser. 2, pages 27–46. Amer. Math. Soc., Providence, RI, 2003. 26
[5] J.-P. Anker, E. Damek, and C. Yacoub. Spherical analysis on harmonic AN groups. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 23(4):643–679 (1997), 1996, 7, 27.
[6] W. Beckner, On the Grushin operator and hyperbolic symmetry, Proc. Amer. Math. Soc., 129(2001), 1233-1246. 44.
[7] W. Beckner, On Lie groups and hyperbolic symmetry—from Kunze-Stein phenomena to Riesz potentials. *Nonlinear Anal.* 126 (2015), 394-414. 42.
[8] W. Beckner, Symmetry in Fourier Analysis: Heisenberg Group to Stein-Weiss Integrals (2021), J. Geom. Anal. (published online).
[9] R. Benguria, R. Frank, and M. Loss, The sharp constant in the Hardy-Sobolev-Maz'ya inequality in the three dimensional upper half-space, Math. Res. Lett. 15 (2008), no. 4, 613-622. 3.
[10] M. Christ, H. Liu, and A. Zhang. Sharp Hardy-Littlewood-Sobolev inequalities on quaternionic Heisenberg groups. *Nonlinear Anal.*, 130:361–395, 2016. 22.
[11] M. Christ, H. Liu, and A. Zhang. Sharp Hardy-Littlewood-Sobolev inequalities on the octonionic Heisenberg group. *Calc. Var. Partial Differential Equations*, 55(1):Art. 11, 18, 2016. 22, 24, 25.
[12] M. Cowling, Unitary and uniformly bounded representations of some simple Lie groups, in Harmonic Analysis and Group Representations, Liguori, Naples, 1982, 49-128. 42.
[13] M. Cowling, S. Giulini, S. Meda, $L^p - L^q$ estimates for functions of the Laplace-Beltrami operator on noncompact symmetric spaces. I, Duke Math. J., 72 (1993), 109-150. 42.
[14] M. Cowling, Herz’s “principe de majoratio” and the Kunze-Stein phenomenon, in: Harmonic Analysis and Number Theory, Montreal, 1996, in: CMS Conf. Proc., vol. 21, Amer. Math. Soc., 1997, 73-88. 42.
[15] E. B. Davies, N. Mandouvalos, Heat kernel bounds on hyperbolic space and Kleinian groups, Proc. London Math. Soc. (3) 52 (1988), 182-208.
[16] Damek, E.; Ricci, F. A class of nonsymmetric harmonic Riemannian spaces. Bull. Amer. Math. Soc. 27 (1992), 139-142. 7.
[17] Damek, E.; Ricci, F. Harmonic analysis on solvable extensions of H-type groups. J. Geom. Anal. 2 (1992), 213-248. 7.
[18] C. Fefferman, Charles; R. Graham, Juhl’s formulae for GJMS operators and Q-curvatures. J. Amer. Math. Soc. 26 (2013), 1191-1207. 3.
[19] G. Folland, E. Stein, Estimates for the $\partial$ complex and analysis on the Heisenberg group, Comm. Pure Appl. Math. 27 (1974) 429-522. 5.
[20] R. Frank, E. H. Lieb, Sharp constants in several inequalities on the Heisenberg group, Ann. Math., 176(2012), 349-381. 22, 23.
[21] R. Gangolli, V. Varadarajan, Harmonic analysis on spherical functions on real reductive groups, Springer-Verlag (1988).
[22] D. Geller, Some results in $H^p$ theory for the Heisenberg group, Duke Math. J. 47(1980) 365-390. 5.
[23] I. S. Gradshteyn, L. M. Ryzhik, Table of Integrals, Series, and Products. 7th edition. Translation edited and with a preface by Alan Jeffrey and Daniel Zwillinger. Academic Press, Inc., San Diego, CA, 2007. Reproduction in P.R.China authorized by Elsevier (Singapore) Pte Ltd. 31.
[24] C.R. Graham, Compatibility operators for degenerate elliptic equations on the ball and Heisenberg group, Math. Z. 187 (3) (1984) 289-304.
[25] C.R. Graham, R. Jenne, L.J. Mason, and G.A.J. Sparling, Conformally invariant powers of the Laplacian. I. Existence. Journal of the London Mathematical Society, (2)46 (1992), 557-565. 3.
[26] S. Helgason, Groups and geometric analysis. Integral geometry, invariant differential operators, and spherical functions. Pure and Applied Mathematics. 113 Academic Press, 1984. 14.
[27] S. Helgason, Geometric analysis on symmetric spaces. Second edition. Mathematical Surveys and Monographs, 39. American Mathematical Society, Providence, RI, 2008. 14.
[28] E. Hebey, Nonlinear Analysis on Manifolds: Sobolev Spaces and Inequalities, Courant Lect. Notes Math., vol. 5, American Mathematical Society, Providence, RI, 1999. 3.
[29] R. A. Hunt, On $L(p, q)$ spaces, Enseignement Math. (2)12(1966), 249-276. 42.
[30] Q. Hong, Sharp constant in third-order hardy-sobolev-maz'ya inequality in the half space of dimension seven, Int. Math. Res. Not. IMRN 2021, no. 11, 8322-8336.
[31] A.D. Ionescu, An endpoint estimate for the Kunze-Stein phenomenon and related maximal operators, Ann. of Math. 152 (1) (2000), 259-275. 3.
[32] K.D. Johnson, Composition series and intertwining operators for the spherical principal series. II, Trans. Amer. Math. Soc. 215 (1976) 269-283. 22.
[33] K.D. Johnson, N.R. Wallach, Composition series and intertwining operators for the spherical principal series. I, Trans. Amer. Math. Soc. 229 (1977) 137-173.

[34] A. Juhl, Explicit formulas for GJMS-operators and Q-curvatures, Geom. Funct. Anal. 23(2013), 1278-1370.

[35] H. Kozono, T. Sato, H. Wadade, Upper bound of the best constant of a Trudinger-Moser inequality and its application to a Gagliardo-Nirenberg inequality, Indiana Univ. Math. J. 55 (6) (2006), 1951-1974.

[36] N. Lam, G. Lu, Sharp Moser-Trudinger inequality in the Heisenberg group at the critical case and applications, Adv. Math. 231 (6) (2012), 3259-3287.

[37] N. Lam, G. Lu, A new approach to sharp Moser-Trudinger and Adams type inequalities: a rearrangement-free argument, J. Diff. Equa. 255(2013), 298-325.

[38] J. Li, G. Lu, Q. Yang, Fourier analysis and optimal Hardy-Adams inequalities on hyperbolic spaces of any even dimension, Adv. Math. 333(2018), 350-385.

[39] J. Li, G. Lu, Q. Yang, Sharp Adams and Hardy-Adams inequalities of any fractional order on hyperbolic spaces of all dimensions, Trans. Amer. Math. Soc. 373(5)(2020), 3483-3513.

[40] G. Liu, Sharp higher-order Sobolev inequalities in the hyperbolic space \( \mathbb{H}^n \), Calc. Var. Partial Differ. Equations 47, no. 3-4, (2013), 567-588.

[41] N. Lohoué and T. Rychener. Die Resolvente von \( \Delta \) auf symmetrischen Räumen vom nichtkompakt Typ. Comment. Math. Helv., 57(3):445–468, 1982.

[42] G. Lu, Q. Yang, A sharp Trudinger-Moser inequality on any bounded and convex planar domain, Calc. Var. Partial Differ. Equ. 55: 153, 1-16(2016).

[43] G. Lu, Q. Yang, Paneitz operators on hyperbolic spaces and high order Hardy-Sobolev-Maz’ya inequalities on half spaces, Amer. J. Math. 141 (2019), no. 6, 1777-1816.

[44] G. Lu, Q. Yang, Green’s functions of Paneitz and GJMS operators on hyperbolic spaces and sharp Hardy-Sobolev-Maz’ya inequalities on half spaces, arXiv:1903.10365.

[45] G. Lu, Q. Yang, Sharp Hardy-Adams inequalities for bi-Laplacian on hyperbolic space of dimension four, Advances in Mathematics, 319 (2017), 567-598.

[46] G. Lu, Q. Yang, Sharp Hardy-Sobolev-Maz’ya, Adams and Hardy-Adams inequalities on the Siegel domains and complex hyperbolic spaces, arxiv.org 1, 2, 4, 47.

[47] V.G. Maz’ya, Sobolev Spaces, Springer-Verlag, Berlin, 1985.

[48] R. O’Neil, Convolution operators and \( L(p,q) \) spaces, Duke Math. J. 30(1963), 129-142.

[49] E. M. Stein, Singular integrals and differentiability properties of functions. Princeton University Press, Princeton, New Jersey, 1970.

[50] E. M. Stein, Some problems in harmonic analysis suggested by symmetric spaces and semi-simple groups, in: Actes Congrèss Intern. Math. (Nice, 1970). Tome 1, Gauthier-Villars, 1971, pp. 173-189.

[51] A. Terras. Harmonic analysis on symmetric spaces and applications. I. Springer-Verlag, New York, 1985.

[52] Valery V. Volchkov, Vitaly V. Volchkov, Harmonic analysis of mean periodic functions on symmetric spaces and the Heisenberg group, Springer-Verlag, 2009.

[53] G. Wang, D. Ye, A Hardy-Moser-Trudinger inequality, Adv. Math. 230 (2012), 294-320.

[54] Q. Yang, Hardy-Sobolev-Maz’ya inequalities for polyharmonic operators, Annali di Matematica (2021). https://doi.org/10.1007/s10231-021-01091-9.

[55] L. Y. H. Yap, Some remarks on convolution operators and \( L(p,q) \) spaces, Duke Math. J., 36(1969), 647-658.

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