Conservative discontinuous Galerkin scheme of a gyro-averaged Dougherty collision operator

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Abstract
A conservative discontinuous Galerkin scheme for a non-linear Dougherty collision operator in full-$f$ long-wavelength gyrokinetics is presented. Analytically this model operator has the advective-diffusive form of Fokker-Planck operators, it has a non-decreasing entropy functional, and conserves particles, momentum and energy. Discretely these conservative properties are maintained exactly as well, independent of numerical resolution. In this work the phase space discretization is performed using a novel version of the discontinuous Galerkin scheme, carefully constructed using concepts of weak equality and recovery. Discrete time advancement is carried out with an explicit time-stepping algorithm, whose stability limits we explore. The formulation and implementation within the long-wavelength gyrokinetic solver of \textsc{Gkeyll} are validated with relaxation tests, collisional Landau-damping benchmarks and the study of 5D gyrokinetic turbulence on helical, open field lines.

Keywords: gyrokinetic, collisions, turbulence, discontinuous Galerkin, collisional Landau damping, conservative, Helimak

1. Introduction

Many phenomena in plasma physics require kinetic treatment, meaning one must solve for the time-evolution of the particle distribution function in position-velocity phase space. For example, in collisionless astrophysical shocks the electrons and ions do not thermalize with each other nor with themselves via collisions on the time-scales of interest. Such systems are best modeled by the Vlasov-Fokker-Planck (Vlasov-FP) equation for the particle probability distribution $f_s(t,x,v)$:

$$\frac{\partial f_s}{\partial t} + \nabla \cdot (vf_s) + \nabla_v \cdot (a_s f_s) = 0,$$

where $a_s = (q_s/m_s)(E + \mathbf{v} \times \mathbf{B})$ is the acceleration due to the Lorentz force, and $q_s$ and $m_s$ the charge and mass of species $s$, respectively. One must simultaneously solve Maxwell’s equations to obtain the fields $\mathbf{E}$ and $\mathbf{B}$. For most plasmas of interest, the cumulative effect of frequent small-angle collisions dominates over that of rare ballistic, ‘large-angle’ collisions, which are more common in neutral gases or fluids. Then the effect of binary encounters can be modeled by the Fokker-Planck operator (FPO), which in Rosenbluth form \cite{1} is

$$\left( \frac{\partial f_s}{\partial t} \right)_c = -\frac{\partial}{\partial v_i} \langle \Delta v_i \rangle_s f_s + \frac{1}{2} \frac{\partial^2}{\partial v_i \partial v_j} \langle \Delta v_i \Delta v_j \rangle_s f_s.$$  \hspace{1cm} (2)

Here $\langle \Delta v_i \rangle_s$ indicates the average increments per unit time of the $i$th component of the velocity of species $s$. Computation of such increments involves integrals of the distribution functions in order to solve for the potentials $h_i$ and $g$:

$$\langle \Delta v_i \rangle_s = \Gamma_s \frac{\partial h_i}{\partial v_i}, \ h_i = \sum_r \frac{m_r}{m_s} \int_{-\infty}^{\infty} d^3v' f_r(v') |v - v'|^{-1},$$  \hspace{1cm} (3)
indicating that (2) is a non-linear integro-differential operator. In the above equations \( \Gamma \), \( 4 \pi q_i q_i \ln \Lambda \), \( \Lambda \), \( n_i \) is the Coulomb logarithm, and the index \( r \) labels the species that species \( s \) collides with.

Equations (1)–(2) and the accompanying Maxwell equations pose a formidable six-dimensional problem spanning a wide range of spatial and temporal scales, which is often numerically intractable. For magnetized environments and phenomena that evolve on a timescale much slower than the rapid particle gyration around the magnetic field, one can use a gyrokinetic reduction to five dimensions by systematically averaging over the fast gyromotion [2–4]. This formulation results in a change of phase-space variables from the particle variables \((x,v)\) to the gyrocenter variables \((R,v,\mu)\), where \( R \) is the gyrocenter coordinate, \( v \) is the velocity component parallel to the background magnetic field, and \( \mu = m_v v / 2B \) is the magnetic moment. In direct numerical simulations, the gyrokinetic equation permits the use of a larger time-step and coarser grids than the six-dimensional Vlasov-FP equation since the timestep is limited by turbulence time scales rather than the plasma period and the minimum length scale of interest is the gyroradius rather than the Debye length.

Some gyrokinetic models assume a constant background equilibrium gyrocenter distribution and only evolve fluctuating quantities, i.e. \( \delta f \). These models are challenged in problems with large-amplitude fluctuations that are on the order of background quantities. In order to model such scenarios we consider full-\( f \) gyrokinetic equations that do not separate the background and fluctuating components. We thus evolve the gyrocenter distribution function \( f(t,R,v,\mu) \) with the gyrokinetic equation [5]:

\[
\frac{\partial f}{\partial t} + \nabla \cdot (f \mathbf{v}) = \frac{\partial}{\partial v} \left( f \mathbf{J} \right) = \mathcal{J} C[f],
\]

where \( \mathcal{J} = B^* \) is the gyrokinetic phase-space Jacobian, \( B^* \) is the parallel component of the effective magnetic field \( B^* = B + \left( m_v v / q_i \right) \mathbf{\nabla} \times \mathbf{b} \), \( B = \mathbf{B} \) is the background magnetic field, and \( C[f] \) represents the gyrokinetic non-linear FP0. The phase-space advection velocities \( \mathbf{R} = \{R,H\} \) and \( \mathbf{v} = \{v_H\} \) are defined in terms of the Poisson bracket

\[
\{F,G\} = \frac{B^*}{m_e B} \left( \nabla F \frac{\partial F}{\partial v} = \nabla G \right) - \frac{1}{4 \pi q_i B} \mathbf{b} \cdot \nabla \times \nabla G,
\]

and the Hamiltonian

\[
H = \frac{1}{2} m_e v_H^2 + \mu B + q_i \langle \phi \rangle, \tag{7}
\]

where \( \langle \phi \rangle \) is the gyroaveraged electrostatic potential. To complete the gyrokinetic system, one may solve for the electrostatic potential using the gyroaveraged Poisson equation

\[
\boxed{- \epsilon_0 \nabla^2 \phi = \sum_s q_s \int f_s(R - x + \rho) dR \, dv \, d\mu \, d\alpha \tag{8}
+ \nabla \cdot \left( \sum_s \frac{n_s m_s}{B^2} \nabla \phi \right),}
\]

where \( \rho \) is the gyroradius vector. The first term on the right side of (8) contains the contributions from the guiding center densities, while the second represents the polarization densities due to the plasma’s response to the cross-field electric field.

The gyrokinetic set of equations (5)–(8) neglects some higher order terms [3, 6], uses a linearized polarization density (i.e. \( n_\alpha \) is constant in (8)) and is limited to electrostatics. Some of these limitations will be addressed in future work, as our focus here is the formulation and implementation of a DG scheme for the collision term. Such scheme can be incorporated in electromagnetic gyrokinetics [7]; in that case the inductive component of the parallel electric field (\( \partial A / \partial t \)) enters in the \( \partial / \partial v \) term of the kinetic equation (5) and necessitates the use of Ampere’s equation for computing \( A \). We make the further simplification of neglecting \( O(\rho^4) \) corrections to the lowest-order \( B^*_0 \) so that \( B^*_a \approx B \) (\( \rho^* = \rho / L \), \( L \) is the gyroradius and \( L \) is the length scale of profile variation). Lastly, we restrict ourselves to the long-wavelength limit of the above gyrokinetic system [5], which is sometimes called the drift-kinetic limit of gyrokinetics. In this limit we neglect the gyroaveraging in the Hamiltonian and that which ensues from the guiding center density integral in the Poisson equation (8), as well as the Debye shielding term (left side of (8)). Therefore, our electrostatic long-wavelength gyrokinetic system will employ the Hamiltonian

\[
H = \frac{1}{2} m_e v_H^2 + \mu B + q_i \phi, \tag{9}
\]

and the (quasineutrality) equation

\[
- \nabla \cdot \sum_s \frac{n_s m_s}{B^2} \nabla \phi = \sum_s \frac{2 \pi q_i}{m_s} \int f_s \, dv \, d\mu. \tag{10}
\]

The long-wavelength and linearized polarization approximations are consistent such that (5)–(6) and (9)–(10) conserve particle number, momentum and energy density. Note that although we have neglected most finite Larmor radius (FLR) effects, some FLR contributions must be retained in order to obtain the polarization density term in equation (10) which permits the calculation of the electrostatic potential.

The focus of this manuscript is the collision term, \( \mathcal{J} C[f] \), which is ideally provided by the gyrokinetic FP0. The full non-linear FP0 has been used in several non-turbulence codes [8–11], but it is less frequently found among turbulence solvers. The XGC particle-in-cell code [12], for example, is one of the few turbulence solvers that can employ the full non-linear FP0. On the other hand continuum gyrokinetic turbulence has on many occasions resorted to linear collision operators. These linear operators were developed for \( \delta f \) studies [13–15] and preserved key properties of the FP0, like conservation (of particles, momentum and energy) and non-decreasing entropy. More recently, enhanced linear gyrokinetic collision models that retain additional physics, such as
velocity-dependent collisionalities and FLR effects, have been proposed and incorporated in several codes [16–18]. Note that instead of using model operators it is also possible to use the exact linearized gyrokinetic FPO [19, 20], which has been implemented in the GENE code [21]. For those interested in full-f continuum gyrokinetic turbulence, as is our case, there also exists a recent formulation of the fully non-linear gyrokinetic FPO [22].

Adoption of the full-f non-linear gyrokinetic FPO can make the cost of turbulence simulations prohibitive. It can also be challenging to implement such complex operator using novel numerical methods with which we have relatively less experience. For those reasons there is strong interest in developing simple models that capture some of the important physics, yet can be efficiently implemented in numerical simulation. With the exception of a few reports [17, 23], there are few simplified collision models in full-f gyrokinetic turbulence modeling. Unfortunately there is relatively little background on the discretization of such models or the full non-linear gyrokinetic FPO using the discontinuous Galerkin (DG) methods we wish to utilize. Consequently our motivation for focusing on a simple model operator is two-fold: to further advance the formulation of effective and efficient full-f gyrokinetic collision models, and gain experience with the DG discretization thereof.

We present here the DG implementation of a non-linear full-f continuum gyroaveraged model collision operator. We build upon the formulation and implementation of similar algorithms for a Vlasov-Maxwell model operator presented in [24] (overview of DG schemes and background on various FPO solvers also appears therein), and employ the same concepts of weak equivalence, boundary correction and recovery. These algorithms are implemented in the DG gyrokinetic model within the GKEVIL computational plasma physics framework [25, 26]: download instructions as well as directions for running input files associated with this work can be found in A. In section 2, we present the Dougherty collision operator and its properties in continuous, infinite-velocity space. These features motivate the formulation of the discrete operator and its implementation, presented in section 3. In section 4, we discuss a suitable time-stepping algorithm and some of the challenges associated with the explicit finite-time integration of the collisional gyrokinetic equation. Finally, section 5 presents a number of relaxation tests showing the accuracy of the time evolution and the steady state solution with this operator, as well as a collisional Landau damping benchmark, and a complex, five-dimensional gyrokinetic simulation of open helical field-line turbulence. Additional discussion and concluding remarks are found in section 6.

2. The continuous gyroaveraged Dougherty operator

The full FPO in (2) can be simplified considerably while keeping its advective-diffusive structure. There is one particularly simple choice that was first introduced phenomenologically in order to preserve the conservation and entropic properties of the FPO, consisting of \(\langle \Delta v_i \rangle = -\nu_{ts} (v_i - u_{is})\) and \(\langle \Delta v_i \Delta v_j \rangle = 2\nu_{ts} v_{is}^2 \delta_{ij}\), that is the velocity increments of a particle colliding with frequency \(\nu_{ts}\) in a fluid of mean velocity \(u_{is}\) and thermal speed \(v_{ts} = \sqrt{\pi/\nu_{ts}}\) [27]. In this work we limit ourselves to collisions between particles of the same species, and make the additional simplification that the collision frequency is velocity-independent. In reality \(\nu\) should decrease as \(v^{-3}\) such that the high energy tail of the distribution is increasingly collisionless.

The initial proposition of such simplified model operator did not offer more than an intuitive explanation for the choice of \(\langle \Delta v_i \rangle\) and \(\langle \Delta v_i \Delta v_j \rangle\). However one can formally arrive at such operator by taking the scattering particles to be locally Maxwellian, meaning the use of a shifted Maxwellian distribution

\[
f(v') = \frac{n_s}{(2\pi v_{ts})^{3/2}} \exp \left[-\frac{(v' - u_s)^2}{2v_{ts}^2}\right]
\]

in computing the Rosenbluth potentials through (3)–(4). The ensuing error functions are expanded in \((v - u_s)/\sqrt{2v_{ts}^2} \ll 1\), and after keeping the first couple of terms one obtains the three-dimensional Dougherty operator

\[
\left(\frac{\partial f}{\partial t}\right)_{c,\text{Dougherty}} = \nu_{ts} \frac{\partial}{\partial v} \left( (v - u_s) f_s + v_{ts}^2 \frac{\partial f_s}{\partial v} \right),
\]

with the collision frequency \(\nu_{ts}\) being arbitrary at this point. The Dougherty operator in \(v\) coordinates in (12) can then be gyroaveraged, the long-wavelength limit of which is what in this work we refer to as the gyroaveraged Dougherty operator (GkLBO):

\[
\mathcal{J}[f_s] = \nu_{ts} \left\{ \frac{\partial}{\partial \|v\|} \left[ (v_{ts} - u_{is}) \mathcal{J} f_s + v_{ts}^2 \frac{\partial \mathcal{J} f_s}{\partial v} \right] + \frac{\partial}{\partial \mu} \left[ 2\mu \mathcal{J} f_s + \frac{2m v_{ts}^2}{B} \frac{\partial \mathcal{J} f_s}{\partial \mu} \right] \right\},
\]

where \(u_{is} = b \cdot u_i\). The Jacobian \(\mathcal{J} = B\) only varies in configuration-space and can be incorporated into the collision operator.

The operator in equation (13) is frequently called the Lenard-Bernstein operator, and since we intend to use it with the (long-wavelength) gyrokinetic solver we refer to it as the GkLBO. Due to its advective-diffusive similarity to the FPO, we intend to employ the concepts and algorithms laid forth here for the full FPO, but in that case the velocity increments would be determined from the Rosenbluth potentials. For simplicity the species subscript ‘s’ will henceforth be assumed. The ‘primitive’ moments \(m_i\) and \(v_i\) are calculated in terms of the moments of the distribution function \(f(t, R, v_i, \mu)\):

\[
M_0 = (2\pi/m) \int \mathcal{J}[f(t, R, v_i, \mu)] \, dv_i \, d\mu,
\]

\[
M_1 = (2\pi/m) \int v_i \mathcal{J}[f(t, R, v_i, \mu)] \, dv_i \, d\mu,
\]
\[ M_2 = \frac{2\pi}{m} \int (v_{\|}^2 + 2\mu B/m) J\dot{f}(t, R, v_{\|}, \mu) \, dv_{\|} \, d\mu, \]  
\[ \text{(16)} \]

With the first three moments of the distribution function, the primitive moments (mean velocity and thermal speed) are then calculated using the relations
\[ u_{||} M_0 = M_1, \]  
\[ \text{(17)} \]
\[ u_{||} M_1 + 3v_{\|}^2 M_0 = M_2. \]  
\[ \text{(18)} \]

The gyrokinetic model as a whole conserves particle number, total momentum and total energy. To show this one must integrate the gyrokinetic equation (5) over all phase space. But the continuous GkLBO analytically conserves particle number, momentum and energy of each species, and to demonstrate such properties it suffices to take velocity moments of the collision operator alone. Particle number conservation of the GkLBO is evident in its (velocity) divergence form and assuming that the argument of the divergence vanishes at infinity. Momentum conservation,
\[ \frac{\partial}{\partial t} \frac{2\pi}{m} \int m v_{\|} J\dot{C}[f] \, dv_{\|} \, d\mu = 0, \]  
\[ \text{(19)} \]
can be satisfied as long as the definition
\[ -(2\pi/m) \int m v (v_{||} - u_{||}) J\dot{f} \, dv_{||} \, d\mu = 0. \]  
\[ \text{(20)} \]
is obeyed, which leads to (17). Similarly, conservation of total particle energy
\[ \frac{\partial}{\partial t} \frac{2\pi}{m} \int (m v_{\|}^2/2 + \mu B) J\dot{C}[f] \, dv_{\|} \, d\mu = 0 \]  
\[ \text{(21)} \]
will also be preserved as long as one satisfies the following relation
\[ -(2\pi/m) \int \nu [m v_{\|}(v_{||} - u_{||}) + 2\mu B - 3 m v_{\|}^2] J\dot{f} \, dv_{||} \, d\mu = 0, \]  
\[ \text{(22)} \]
and this is equivalent to (18). Arriving at the above properties and conditions requires that drag terms be integrated by parts once and the diffusion terms twice. One must also use the fact that \( f(t, R, v_{\|}, \mu) \to 0 \) faster than any power of the velocity as \( v_{\|}, \mu \to \infty \). Although these features, and their proofs, are discussed in various other texts, we summarize them to illustrate the nature of the constraints on the discrete GkLBO in order to arrive at a conservative scheme. Other important features of a good collision operator we would like to retain discretely are a non-decreasing entropy functional, and that the steady-state (or maximum-entropy) solution is a Maxwellian distribution. Defining the entropy as \( S = -\int_{-\infty}^{\infty} \ln f \, d^3 v \), this means
\[ \frac{\partial S}{\partial t} = \frac{2\pi}{m} \int \frac{\partial f}{\partial t} (\ln f + 1) \, dv_{\|} \, d\mu \geq 0. \]  
\[ \text{(23)} \]
One can show that the GkLBO obeys such relation by writing the operator as
\[ J\dot{C}[f] = \frac{\partial F_{\parallel}}{\partial v_{||}} + \frac{\partial F_{\mu}}{\partial \mu} \]  
\[ \text{(24)} \]
where
\[ F_{\parallel} = \nu(v_{||} - u_{||}) J\dot{f} + \nu v_{\|}^2 \frac{\partial J\dot{f}}{\partial v_{||}} \]  
\[ \text{(25)} \]
\[ F_{\mu} = 2\nu \mu J\dot{f} + \nu \left( \frac{m v^2}{2} \right) \frac{\partial J\dot{f}}{\partial \mu}. \]  
\[ \text{(26)} \]
Substitute these definitions into (23) and integrate by parts to get
\[ \frac{\partial S}{\partial t} = \frac{2\pi}{m} \int \frac{\partial f}{\partial t} \left( F_{\parallel} \frac{\partial f}{\partial v_{||}} + F_{\mu} \frac{\partial f}{\partial \mu} \right) \, dv_{\|} \, d\mu, \]  
\[ \text{(27)} \]
assuming that \( F_{\parallel} \to 0 \) as \( v_{||} \to \pm \infty \) and \( F_{\mu} \to 0 \) as \( v_{||} \to \infty \) faster than the logarithmic singularity from the \( \ln f \) term. Eliminate the partial derivatives in (27) using (25)–(26), resulting in
\[ \frac{\partial S}{\partial t} = \frac{2\pi}{m v_{\|}^2 \nu} \int \frac{1}{J\dot{f}} \left( F_{\parallel}^2 + \frac{B}{2 m \nu} F_{\mu} \right) \]  
\[ \text{(28)} \]
\[ -\nu(v_{||} - u_{||}) F_{\parallel} - \nu \frac{B}{m} F_{\mu} \, dv_{||} \, d\mu. \]

With the definitions of \( F_{\parallel} \) and \( F_{\mu} \), the second and third terms in (28) become
\[ \frac{2\pi}{m v_{\|}^2 \nu} \int \nu \left[ \left( v_{||}^2 + \frac{2B\mu}{m} \right) f \right. \]  
\[ \left. + (2v_{||}u_{||} - u_{||}^2 + 3v_{\|}^2) \right] dv_{\|} \, d\mu = 0, \]  
\[ \text{(29)} \]
after integration by parts and using definitions of the moments. Therefore, since \( \mu \geq 0 \), (28) indicates that
\[ \frac{\partial S}{\partial t} = \frac{2\pi}{m v_{\|} \nu} \int_{-\infty}^{\infty} \frac{1}{J\dot{f}} \left( F_{\parallel}^2 + \frac{B}{2 m \nu} F_{\mu} \right) \, dv_{||} \, d\mu \geq 0 \]  
\[ \text{(30)} \]
as long as \( f > 0 \).

The definition of entropy and the fact that it is a non-decreasing function can be used to show that the maximum entropy solution to the GkLBO is the Maxwellian given by
\[ f_M(n, u_{||}, v_{||}) = \frac{n}{(2\pi v_{\|}^2)^{3/2}} \exp \left[ -\frac{(v_{||} - u_{||})^2 + 2B\mu/m}{2v_{\|}^2} \right], \]  
\[ \text{(31)} \]
where \( n \) is the zeroth moment or particle number density, \( n = M_0 \). Such distribution arises from maximizing the entropy \( S \) subject to the constraint that density, momentum and energy do not change during the evolution. In other words, it is the result of finding the extrema of
\[ S = -\frac{2\pi}{m} \int_{-\infty}^{\infty} \ln f \, dv || \, d\mu \]
\[ + \lambda_0 \left[ \frac{2\pi}{m} \int J_f \, dv || \, d\mu - M_0 \right] \]
\[ + \lambda_1 \left[ \frac{2\pi}{m} \int v || J_f \, dv || \, d\mu - M_1 \right] \]
\[ + \lambda_2 \left[ \frac{2\pi}{m} \int (v^2 + 2\mu B/m) J_f \, dv || \, d\mu - M_2 \right], \tag{32} \]

where \( \lambda_0, \lambda_1 \) and \( \lambda_2 \) are Lagrange multipliers. Varying this Lagrangian and applying the constraints to determine the Lagrange multipliers, leads to the Maxwellian. Because the entropy is monotonically increasing, the Maxwellian maximizes the entropy. These are textbook observations of a good correction operator, yet in section 5 we will see that the meaning of a discrete maximum entropy solution must be examined carefully. A final property of the GkLBO, which certain numerical schemes would also benefit from, is its self-adjointness. This means that for arbitrary functions \( g(t, R, v ||, \mu), J_f(t, R, v ||, \mu) \) the GkLBO satisfies

\[ \langle g, C[f] \rangle = \langle C[g], f \rangle \tag{33} \]

with the inner product defined as

\[ \langle f, g \rangle = \frac{2\pi}{m} \int J_f g \, dv || \, d\mu \tag{34} \]

where \( f_M \) is the Maxwellian that satisfies \( C[f_M] = 0 \). Self-adjointness is demonstrated integrating (33) by parts and using (25) and (26) in order to arrive at

\[ \langle g, C[f] \rangle = -\frac{2\pi}{m} \int \left( F_{||} \frac{\partial}{\partial v ||} + F_{\mu} \frac{\partial}{\partial \mu} \right) \left( \frac{g}{f_M} \right) dv || \, d\mu. \tag{35} \]

Then use following the identities

\[ \nu v_i^2 f_M \frac{\partial}{\partial v ||} \left( \frac{J_f}{f_M} \right) = \nu (v || - u ||) J_f + \nu v_i \frac{\partial J_f}{\partial v ||} = F_{v ||} \tag{36} \]

and the ensuing equation as

\[ \langle g, C[f] \rangle = -\frac{2\pi \nu v_i^2}{m} \int J_f f_M \left\{ \frac{\partial}{\partial v ||} \left( \frac{f}{f_M} \right) \frac{\partial}{\partial v ||} \left( \frac{g}{f_M} \right) \right\} dv || \, d\mu \]
\[ + \frac{2 m}{B} \mu \frac{\partial}{\partial \mu} \left( \frac{f}{f_M} \right) \frac{\partial}{\partial \mu} \left( \frac{g}{f_M} \right) \}

This is symmetric in \( f \) and \( g \), and the self-adjoint property follows. The self-adjoint property indicates that all eigenvalues of the operator are real and the solution is damped, a characteristic certain numerical schemes benefit from. This \( 1/f_M \) weighting in the definition of the inner product is standard in kinetic theory, further discussion is found in [24].

\section{3. The discrete gyroaveraged Dougherty operator}

This work is primarily concerned with the discontinuous Galerkin (DG) discretization of the GkLBO in (5). The schemes are presented assuming three dimensions \( (x, v ||, \mu) \), but they can be easily extended to higher dimensions. We wish to find the numerical solution \( f \) defined on a phase-space domain \( \Omega \equiv [v_{min}, v_{max}] \times [v ||, \mu_{min}, \mu_{max}] \) which is discretized by the structured rectangular mesh defined as \( \Omega_{i,j,k} \equiv [x_{i-1/2}, x_{i+1/2}] \times [v ||, j-1/2, v ||, j+1/2] \times [\mu_{k-1/2}, \mu_{k+1/2}] \). The velocity extents of this mesh (except for \( \mu_{min} = 0 \)) are typically far enough from zero that the distribution function \( f \) has decreased by many orders of magnitude, although we will see below that for exact conservation we cannot assume it to be zero there. On each cell we select a set of orthogonal basis functions \( \psi_{\ell}(x, v ||, \mu) \) for \( \ell = 1, \ldots, N_p \), such that

\[ \int_{\Omega_{i,j,k}} \psi_{\ell} \, \psi_{pm} \, dv || \, d\mu = \delta_{\ell m} \frac{\Delta v ||}{2} \frac{\Delta \mu}{2}, \tag{39} \]

where \( \Delta v ||, \Delta \mu \) are the cell lengths in each direction. In GKEVLL we typically employ Serendipity bases constructed by choosing and orthonormalizing monomials from the polynomial space \( \mathcal{Y}_{\ell} = \{ x^m v ||^n \mu^p | \deg(x^m v ||^n \mu^p) \leq p \} \) of order \( p \) and dimension 3 [28], though the algorithm presented here is general to other orthonormal basis sets (deg. refers to the sum of all monomial powers that appear superlinearly).

We build a DG scheme for the GkLBO leveraging the concept of weak equality. This in turn yields a conservative GkLBO scheme that is also alias-free. This section describes weak equality, presented in [24] and reproduced here for completeness, and its role in the recovery DG approach to second order derivatives, followed by the conservative DG discretization of the GkLBO.

\subsection{3.1. Weak equality and recovery DG}

For some (phase) space interval \( I \) and some basis \( \psi_{\ell} \), with \( k \in [1, \ldots, N_p] \), spanning the function space \( \mathcal{P} \), two functions \( f \) and \( g \) are weakly equal if

\[ \int_{I} (f - g) \psi_{\ell} \, dx = 0. \tag{40} \]

That is, the projections of these functions on a given basis are equal. In finite-element theory and applied mathematics weak equality is referred to as weak equivalence or a weak solution to \( f = g \) [29]. We denote a weak equality with \( f \equiv g \), and in section 5 we also describe how weak equalities lead to the proper spectral decomposition of DG signals. Although it is presented here in the context of DG, it is general to the use of finite-dimensional vector spaces.

The discrete form of a quantity expanded in the basis \( \psi_{\ell} \) is given in terms of a weak definition. For example, the discrete forms of the first three moments of the distribution function are defined as

\[ M_0 \equiv (2\pi/m) \int J_f \, dv || \, d\mu \tag{41} \]
\[
M_1 \doteq (2\pi/m) \int \mathbf{v}_\| \mathcal{J} f \mathbf{v}_\| d\mu
\]  
(42)

\[
M_2 \doteq (2\pi/m) \int (\mathbf{v}_\|^2 + 2B\mu/m) \mathcal{J} f \mathbf{v}_\| d\mu.
\]  
(43)

The primitive moments, \( u_\| \) and \( v_\| \), on the other hand must be computed using a combination of what we call weak multiplication and division in order for them to lie in \( \mathcal{L}_P \). To illustrate these operations consider the definition of the mean velocity \( u_\| \) given by the relation
\[
u_\| \mathbf{M}_0 \doteq M_1.
\]  
(44)

Using (40) and \( u_\| = \sum_\ell u_\| \varphi_\ell(x) \) we can express this weak operation as a system of linear equations
\[
\sum_\ell u_\| \mathbf{M}_0 \varphi_\ell \mathbf{dx} = \int_\mathcal{M} M_1 \varphi_\ell \mathbf{dx},
\]  
(45)

where \( M_0 \) and \( M_1 \) also have expansions in the configuration-space basis set, \( \varphi_\ell(x) \). The inversion of this system to compute the \( u_\| \mathbf{M}_0 \doteq M_1 \) coefficients needs to take place in each cell of the configuration-space grid and is referred to as weak-division. Having obtained the expansion of the mean parallel velocity, one can then perform the weak multiplication \( u_\| M_1 \doteq K \) to obtain the kinetic energy \( K \); this is needed to compute the thermal velocity via the weak analogue of (18):
\[
u_\| M_1 + 3v_\|^2 M_0 \doteq M_2.
\]  
(46)

However, weak division needs to be limited for numerical stability. As the function \( M_0 \) becomes too steep \( u_\| \) begins to diverge. In order to avoid this we limit weak division by performing cell-average division only (e.g. for \( p = 1 \) \( u_\|,0 = M_{1,0}/M_{0,0} \) and \( u_\| = 0 \) when \( |M_{0,1}| < M_{0,0}/\sqrt{3} \)). For more information see [24]. The framework provided by weak equalities also leads to a natural formulation of recovery DG (RDG) for higher order derivatives and, more generally, recovering a continuous function from a discontinuous one. Suppose we wish to compute the second derivative \( g = \partial^2 f/\partial x^2 = f_{xx} \). Integration by parts in cell \( I_j = [x_{j-1/2},x_{j+1/2}] \) gives
\[
g_k = \psi_k \mathcal{J} f_{1/2} - \int \frac{\partial \psi_k}{\partial x} f_{x} \mathbf{dx},
\]  
(47)

Recall that \( f \) has its own expansion, but since it is generally discontinuous from one cell to the next we need a way to compute its derivative at the cell boundaries. We could instead replace \( f_{x} \) in the boundary term with \( \tilde{f}_{k} \), where \( \tilde{f} \) is the recovery polynomial constructed such that
\[
\begin{align*}
\tilde{f} &\doteq f_L \quad x \in I_L \quad \text{on} \quad \mathcal{P}_L, \\
\tilde{f} &\doteq f_R \quad x \in I_R \quad \text{on} \quad \mathcal{P}_R.
\end{align*}
\]  
(48)

One needs to perform two recoveries, one at \( x_\| = 1/2 \) and another at \( x_\| = 1/2 \). At \( x_\| = 1/2 \), \( f_L \) refers to the function \( f \) on the element \( I_L = I_{j-1} = [x_{j-3/2},x_{j-1/2}] \), and \( f_R \) is \( f \) on \( I_j \). Each of these is defined on the respective function spaces \( \mathcal{P}_L \) and \( \mathcal{P}_R \). The equalities in (48) establish the projections of \( f \) on \( I_L \) and \( I_R \), but to determine it uniquely we can use the \( 2N_p \) pieces of information (\( N_p \) coefficients from each of \( f_L \) and \( f_R \)) and assume \( \tilde{f} \) is the maximal-order polynomial:
\[
\tilde{f}(x) = \sum_{k=0}^{2N_p-1} \tilde{f}_k x^k.
\]  
(49)

Replacing this definition into (48) leads to a linear system of \( 2N_p \) equations in the \( \tilde{f}_k \) unknowns. Figure (1) illustrates an example recovery polynomial. An alternative RDG we follow here is to integrate (47) a second time to arrive at
\[
g_k = \left( \psi_k \mathcal{J} f_{1/2} - \frac{\partial \psi_k}{\partial x} f_{x} \right)_{x_\| = 1/2} + \int \frac{\partial^2 \psi_k}{\partial x^2} f_{xx} \mathbf{dx}.
\]  
(50)

The system in (48) is only inverted once to obtain the ensuing stencil for \( f \) and \( f_{x} \) evaluated at the boundaries. RDG schemes of this kind were first proposed over a decade ago [30] as an alternative to the traditional local DG (LDG) approach to diffusion terms [31]. This RDG has better convergence of both cell averages and slopes upon grid refinement [32] and leads to a conservative discrete GKLBO, which we prove in the next section.

The maximal-order recovery polynomial in (49) has fourth-order convergence (see figure 1(b)), but it does not guarantee positivity. For systems in which \( f \) must remain positive, using the highest order polynomial possibly may lead to incursions below zero when \( f \) is small. An example of this, due to both diffusion and advection, is illustrated in section 5. We explored computing second derivatives with lower order polynomials in order to see if positivity problems are minimized. The test function \( f = \sin x \) in \( x \in [-1,1] \) was discretized with piecewise polynomial bases \( (p = 1, N_p = 2) \) which allows for a cubic maximal-order polynomial. We could also, instead of using (49), request a quadratic or a linear \( f \). In these two cases (48) leads to an over determined system that we solved by least squares. The error norms in the computation of \( \partial^2 (\sin x)/\partial x^2 \) for all three recoveries are given in figure 1(b) as a function of resolution. Unfortunately the convergence of these lower-order, least squares methods is inferior (closer to second order) and, in advection-diffusion problems, they were less stable. Also note that the quadratic least-squares \( f \) (dashed orange in figure 1(b)) did not do any better than the linear \( f \) (dotted green in figure 1(b)); a better procedure is to construct \( f \) by matching the cell-averages in each cell, and seeking the least-squares solution that tries to match the slopes in both cells (orange dash-dot in figure 1(b)). In what follows, we use the maximal-order recovery polynomial and take other measures to decrease the likelihood of \( f < 0 \).

3.2. Discrete GKLBO scheme

In order to discretize the GKLBO we project (13) onto the phase-space basis by multiplying by a test function \( w \in \mathcal{V}_3^p \) and
integrating over all phase-space:

\[
\int_{Ω_{ij,k}} w \frac{\partial J_f}{\partial τ} \, dx \, dv_∥ \, dμ = \\
ν \int_{Ω_{ij,k}} w \left\{ \frac{\partial}{\partial τ} \left[ (v_j - u_j) J_f + v_i^2 \frac{\partial J_f}{\partial v_j} \right] + \frac{\partial}{\partial μ} \left[ 2μ J_f + \frac{2mv_i^2}{B} \frac{\partial J_f}{\partial μ} \right] \right\} \, dx \, dv_∥ \, dμ,
\]

where we neglect the \(2πl\) integration factor to simplify notation. Integrate by parts once to give

\[
\nu \int_{Ω_{ij,k}} \left\{ \frac{\partial w}{\partial v_j} \right\} \left[ (v_j = u_j) J_f + v_i^2 \frac{\partial J_f}{\partial v_j} \right] \, dx \, dv_∥ \, dμ \]

\[
- ν \int_{Ω_{ij,k}} \left\{ \frac{\partial w}{\partial μ} \right\} \left[ v_j^2 \frac{∂^2 J_f}{∂μ^2} \right] \, dx \, dv_∥ \, dμ
\]

\[
+ \nu \int_{Ω_{ij,k}} \left\{ \frac{∂w}{∂μ} \right\} \left[ 2μ J_f + \frac{2mv_i^2}{B} \frac{\partial J_f}{\partial μ} \right] \, dx \, dv_∥ \, dμ.
\]

The quantities \(G_{v_j}(f_L,f_R)\) and \(G_μ(f_L,f_R)\) are numerical fluxes chosen to preserve, in addition to conservation, other properties like stability and positivity. For example, the Lax-Friedrichs (LF) penalty fluxes are

\[
G_{v_j}(f_L,f_R) = \frac{1}{2}(v_j - u_j)(J_fR + J_fL) - \frac{τ_{v_j}}{2} (J_fL - J_fR)
\]

\[
G_μ(f_L,f_R) = μ(J_fR + J_fL) - \frac{τ_μ}{2} (J_fL - J_fR)
\]

where \(τ_{v_j} = \max(|v_j - u_j|)\) and \(τ_μ = \max(2μ)\). The notation \(f_L\) and \(f_R\) refer to the distribution function in the left and right cells of a boundary, respectively. Aside from the \(v_j^2\) terms and the fact that the penalization factors \(τ_{v_j}\) and \(τ_μ\) are based on global maxima, this discretization follows the treatment of the collisionless terms \([7]\). In section 4 we also discuss pure upwind fluxes which are more beneficial for positivity. Notice that the contribution to the numerical flux arising from the diffusion term is computed using the recovery polynomial described in section 3.1.

Additionally we impose the following boundary conditions on the numerical fluxes:

\[
G_{v_j}(f_L(v_{j,\min},f_R(v_{j,\min})) = G_{v_j}(f_L(v_{j,\max},f_R(v_{j,\max}))) = 0,
\]

\[
G_μ(f_L(0),f_R(0)) = G_μ(f_L(μ_{\max},f_R(μ_{\max}))) = 0.
\]

The discrete form in (52)–(54) does not conserve momentum, shown by substituting \(w = v_j\) into (52) and summing over \(v_j\) space (j index). The diffusion term in the volume integral can be integrated by parts again, yielding a surface term with non-vanishing jumps that breaks momentum conservation. A similar issue arises with energy conservation. The conservation of \(M_0\), \(M_1\) and \(M_2\) can be guaranteed if one integrates by parts twice and evaluates the additional surface terms using the recovered distribution function \(J\). Therefore the conservative GKLBO DG scheme follows

\[
\int_{Ω_{ij,k}} w \frac{\partial J_f}{\partial τ} \, dx \, dv_∥ \, dμ = ν \int_{Ω_{ij,k}} \left\{ \frac{∂w}{∂v_j} \left[ (v_j - u_j) J_f + v_i^2 \frac{∂J_f}{∂v_∥} \right] \right\} \, dx \, dv_∥ \, dμ \]

\[
- ν \int_{Ω_{ij,k}} \left\{ \frac{∂w}{∂μ} \right\} \left[ v_j^2 \frac{∂^2 J_f}{∂μ^2} + \frac{∂w}{∂μ} \frac{2mv_i^2}{B} \frac{∂J_f}{∂μ} \right] \, dx \, dv_∥ \, dμ
\]

\[
- 2 \frac{mv_i^2}{B} \left( \frac{∂^2 w}{∂μ^2} + \frac{∂w}{∂μ} \right) \frac{∂J_f}{∂μ} \, dx \, dv_∥ \, dμ.
\]
3.2.1. Number density conservation. Scheme 55 conserves number density:

\[
\frac{d}{dt} \sum_{j,k} \int_{\Omega_{i,j,k}} v_{\|} J f dx dv_{\|} d\mu = 0,
\]

(56)

In order to show this use \( w = 1 \) in (55) and sum over all velocity space cells. The sum need not be over configuration space as the configuration space gradients only occur in the collisionless terms of the gyrokinetic equation. The numerical flux is continuous across interior cell surfaces, so those contributions cancel. Only the fluxes at the boundaries of velocity space remain, but those are zero given the boundary conditions in (54).

3.2.2. Discrete momentum conservation. Scheme 55 conserves momentum:

\[
\frac{d}{dt} \sum_{j,k} \int_{\Omega_{i,j,k}} v_{\|} J f dx dv_{\|} d\mu = 0,
\]

(57)

if the following weak-equality relation is satisfied:

\[
u_\| M_0 - v_\| \sum_k \int_{\mu_{k,-1/2}}^{\mu_{k,+1/2}} \left( J f(v_{\|,\max}) - J f(v_{\|,\min}) \right) d\mu = 0.
\]

One can show momentum conservation and arrive at this constraint using \( w = v_\| \) in (55) and summing over all velocity space cells to get

\[
\frac{d}{dt} \sum_{j,k} \int_{\Omega_{i,j,k}} v_{\|} J f dx dv_{\|} d\mu =
\]

\[
- \nu \sum_{j,k} \int_{\mu_{k,-1/2}}^{\mu_{k,+1/2}} v_{\|}^2 J f dx dv_{\|} \bigg|_{v_{\|,\min}}^{v_{\|,\max}}
\]

\[
- \nu \sum_{j,k} \int_{\Omega_{i,j,k}} (v_{\|} - u_{\|}) J f dx dv_{\|} d\mu
\]

(59)

The contributions from the numerical fluxes \( G_{\nu_\|} \) and \( G_{\mu_\|} \) drop out due to continuity and boundary conditions. In the first term all interface contributions from \( v_{\|} \) will cancel except the first and last. Combined with the definition of the discrete moments in the second term leads to the constraint

\[
\int_{\mu_{k,-1/2}}^{\mu_{k,+1/2}} \left[ \sum_k \int_{\mu_{k,-1/2}}^{\mu_{k,+1/2}} v_{\|}^2 \left( J f(v_{\|,\max}) - J f(v_{\|,\min}) \right) dx \right] d\mu + M_1 - u_{\|} M_0 = 0.
\]

(60)

Using the definition of weak equality, this implies that the momentum will be conserved if (58) is satisfied. Notice that \( f \) was replaced by \( f \) because at the outer velocity boundaries there is no ‘outside’ cell to allow for recovery of a continuous distribution function.

The weak-equality constraint (58) is stronger than what is required by (60). However, ensuring that the weak-equality constraint is satisfied automatically ensures that momentum conservation is preserved. For simplicity we assume a unit mass \( m = 1 \) in what follows, with no loss of generality.

3.2.3. Discrete energy conservation. Scheme 55 conserves energy

\[
\frac{d}{dt} \sum_{i,j,k} \int_{\Omega_{i,j,k}} \left( \frac{1}{2} v_{\|^2} + \mu B \right) J f dx dv_{\|} d\mu = 0,
\]

(61)

if the following weak-equality relation is satisfied:

\[
u_\| M_0 + v_\|^2 \left[ 3 M_0 - \sum_k \int_{\mu_{k,-1/2}}^{\mu_{k,+1/2}} (v_{\|,\max} J f(v_{\|,\max}) - v_{\|,\min} J f(v_{\|,\min})) d\mu \right] + \sum_j \int_{v_{\|,j,-1/2}}^{v_{\|,j,+1/2}} \notag
\]

\[
2 \left( \mu_{\max} J f(\mu_{\max}) - \mu_{\min} J f(\mu_{\min}) \right) dv_{\|} \notag \rangle = M_2.
\]

(62)

Assuming that \( p \geq 2 \) such that \( v^2 \in V_{\nu_\|}^2 \), the above two equations follow from replacing \( w = v_{\|^2}/2 + \mu B \) in (55) and summing over all velocity space cells, which yields

\[
\frac{d}{dt} \sum_{i,j,k} \int_{\Omega_{i,j,k}} \frac{1}{2} v_{\|^2} J f dx dv_{\|} d\mu =
\]

\[
- \nu \sum_{i,j,k} \int_{\mu_{i,-1/2}}^{\mu_{i,+1/2}} \left( \int_{\mu_{i,-1/2}}^{\mu_{i,+1/2}} v_{\|^2} J f \right) dx dv_{\|} d\mu
\]

\[
+ \int_{v_{\|,i,-1/2}}^{v_{\|,i,+1/2}} \notag 2 v_{\|^2} \mu J f \left| \mu_{i,-1/2} \right| dv_{\|} dx
\]

\[
- \nu \sum_{i,j,k} \int_{\Omega_{i,j,k}} \left( v_{\|} - u_{\|} \right) J f dx dv_{\|} d\mu.
\]

(63)

All contributions from interior cell interfaces cancel in the first term. Velocity-space integrals in the second term can be written in terms of discrete moments, leading to the following constraint in order to have energy conservation:

\[
\int_{\mu_{i,-1/2}}^{\mu_{i,+1/2}} \sum_k \int_{\mu_{i,-1/2}}^{\mu_{i,+1/2}} v_{\|^2} J f(v_{\|,\max}) - v_{\|^2} J f(v_{\|,\min}) d\mu + \sum_j \int_{v_{\|,j,-1/2}}^{v_{\|,j,+1/2}} 2 v_{\|^2} \left( \mu_{\max} J f(\mu_{\max}) - \mu_{\min} J f(\mu_{\min}) \right) dv_{\|} + M_2 - u_{\|} M_1 - 3 v_{\|^2} M_0
\]

dx = 0.

(64)

Using the definition of weak equality this implies that the energy will be conserved if (62) is satisfied.

Thus, exact \( p \geq 2 \) conservation of momentum and energy leads to the following set of weak-equality relations:

\[
v_{\|} M_0 - v_{\|}^2 \int_{\mu_{i,-1/2}}^{\mu_{i,+1/2}} J f dv_{\|} d\mu = M_1,
\]

\[
\int_{\Omega_{i,j,k}} \left( \frac{1}{2} v_{\|^2} + \mu B \right) J f dx dv_{\|} d\mu = 0.
\]
\[ u_t M_1 + v_r^2 \left[ 3M_0 - \frac{2\pi}{m} \sum_k \int_{\mu_{k-1}}^{\mu_{k+1}} v_{||} |\mathcal{J}_f| v_{||,\text{max}} d\mu \right. \]
\[ \left. - \frac{2\pi}{m} \sum_j \int_{v_{||,j-1/2}}^{v_{||,j+1/2}} 2\mu |\mathcal{J}_f|_{v_{||,\text{max}}} dv_{||} \right] \equiv M_2, \tag{65} \]

where we restored the \(2m/m\) factors for clarity. This is a weak linear system of equations that needs to be inverted in every cell of the configuration-space grid to compute the parallel drift velocity \(u_t\) and the thermal speed \(v_r^2\). Without the boundary corrections presented above, the errors in the conserved quantities are several orders of magnitude higher. Also, instabilities can be observed when velocity-grid extents are too low and \(f\) is appreciable at the boundary.

3.2.4. Discrete \(p=1\) energy conservation. The above energy conservation theorem applied to \(p \geq 2\) basis functions which span the quadratic term in the test function \(v_{||}^2/2 + \mu B\). For a piecewise linear basis \((p = 1)\), conservation can be maintained in the sense that we can conserve the projection of the second moment \(M_2\) onto the piecewise linear basis. This property can be ensured if the quadratic term in the test function is replaced by its projection onto the basis functions, \(\bar{v}_{||}^2 \in \mathcal{V}_1\), which is weakly equivalent to \(v_{||}^2\) in this basis:

\[ \bar{v}_{||}^2 \equiv v_{||}^2 \quad \text{on} \quad \mathcal{V}_1. \tag{66} \]

It is straightforward to show that \(\bar{v}_{||}^2\) is continuous. Also, in a weak-equality sense, the definition of particle energy is the same, whether we use the original quadratic expression or its projection. Therefore we can show that scheme (55) satisfies

\[ \frac{d}{dt} \sum_{j,k} \int_{\Omega_{j,k}} \left( v_{||}^2 / 2 + \mu B \right) |\mathcal{J}_f| dv_{||} d\mu = 0, \tag{67} \]

with a piecewise linear basis as long as the following weak equality is satisfied:

\[ u_t M_1^* + v_r^2 \left[ M_0^* + 2M_0 - \sum_k \int_{\mu_{k-1}}^{\mu_{k+1}} \left( \bar{v}_{||,\text{max}} |\mathcal{J}_f|_{\bar{v}_{||,\text{max}}} \right) d\mu - \sum_j \int_{v_{||,j-1/2}}^{v_{||,j+1/2}} 2\left( |\mathcal{J}_f|_{\mu_{\text{max}}} \right) d\mu \right] \equiv M_2^*. \tag{68} \]

where \(\bar{v}_{||,j} = \left( v_{||,j+1/2} + v_{||,j-1/2} / 2, \Delta v_{||,j} = \bar{v}_{||,j+1} - \bar{v}_{||,j}\) and the ‘star’ moments are defined as

\[ M_0^* \equiv \sum_{j \neq \text{max}} \sum_k \int_{\mu_{k-1}}^{\mu_{k+1}} \Delta v_{||,j} |\mathcal{J}_f|_{\mu_{\text{max}}} d\mu, \]
\[ M_1^* \equiv \sum_{j} \int_{v_{||,j-1/2}}^{v_{||,j+1/2}} \bar{v}_{||,j} |\mathcal{J}_f| dv_{||}, \]
\[ M_2^* \equiv \sum_{j} \int_{v_{||,j-1/2}}^{v_{||,j+1/2}} \left( \bar{v}_{||,j} v_{||}^2 + 2\mu B \right) |\mathcal{J}_f| dv_{||} d\mu. \tag{69} \]

The argument leading to (67)–(69) begins with using the projection \(v_{||}^2/2\) (66) in showing conservation of the energy, since \(v_{||}^2/2 \notin \mathcal{V}_1\). This means setting \(\varepsilon = v_{||}^2/2 + \mu B\) in (55) and summing over all velocity space cells. Aided by the fact that

\[ \frac{\partial}{\partial v_{||}^2} \left( \frac{1}{2} \bar{v}_{||}^2 \right) = \frac{1}{2} \left( v_{||,j+1/2} + v_{||,j-1/2} \right) \equiv \bar{v}_{||,j}, \tag{70} \]

one then arrives at

\[ \frac{d}{dt} \sum_{j,k} \int_{\Omega_{j,k}} \left( v_{||}^2 / 2 + \mu B \right) |\mathcal{J}_f| dv_{||} d\mu = \]
\[ -\nu \sum_{j,k} \int_{\mu_{k-1}}^{\mu_{k+1}} \left( \int_{\mu_{k-1}}^{\mu_{k+1}} \bar{v}_{||,j} v_{||}^2 |\mathcal{J}_f|_{v_{||,\text{max}}} d\mu \right) dv_{||} \]
\[ + \int_{v_{||,j-1/2}}^{v_{||,j+1/2}} 2\nu^2 \mu |\mathcal{J}_f|_{v_{||,\text{max}}} dv_{||} \]
\[ - \nu \sum_{j} \int_{\Omega_{j,k}} \left[ \bar{v}_{||,j} (v_{||} - v_{||}) + 2\mu B - 2v_{||}^2 \right] |\mathcal{J}_f| dv_{||} d\mu. \tag{71} \]

The terms containing the numerical flux \(G_{||}\) drop out since \(\bar{v}_{||}^2/2\) is continuous and we are enforcing zero-flux boundary conditions in velocity-space (and so does the \(G\mu\) term). However, as \(\bar{v}_{||,j}\) is not continuous the contribution from the first term in (71) does not drop out. This term can be written as

\[ \int_{\mu_{k-1}}^{\mu_{k+1}} \int_{v_{||,j-1/2}}^{v_{||,j+1/2}} \nu^2 \left( \bar{v}_{||,\text{max}} |\mathcal{J}_f|_{\bar{v}_{||,\text{max}}} \right) - \bar{v}_{||,\text{min}} |\mathcal{J}_f|_{\bar{v}_{||,\text{min}}} \right) \]
\[ - \nu^2 \sum_{j \neq \text{max}} \Delta v_{||,j} |\mathcal{J}_f|_{\mu_{\text{max}}} \] d\mu.

Utilizing the star moments in (69) and the definition of weak equality, this last relation implies that the energy will be conserved in the \(p = 1\) case if (68) is satisfied. In summary, for piecewise linear basis the drift velocity and thermal speed must be determined using the following set of linear weak-equality relations

\[ u_t M_0 + v_r^2 \frac{2\pi}{m} \sum_k \int_{\mu_{k-1}}^{\mu_{k+1}} |\mathcal{J}_f|_{v_{||,\text{max}}} d\mu \equiv M_1, \]
\[ u_t M_1^* + v_r^2 \left\{ M_0^* + 2M_0 - \frac{2\pi}{m} \sum_k \int_{\mu_{k-1}}^{\mu_{k+1}} \left( \bar{v}_{||,\text{max}} |\mathcal{J}_f|_{\bar{v}_{||,\text{max}}} \right) \right\} \equiv M_2^*. \tag{73} \]
We have again reinstated the $2\pi/m$ factors for completeness. Notice that this weak system requires computing the first two regular moments ($M_0$ and $M_1$) and the three star moments.

A final remark we wish to make is that we have not yet proven that our discrete Dougherty operator is self-adjoint, or that it has a non-decreasing entropy functional. There are several challenges associated with demonstrating such properties; these include the projection of the logarithm $\ln f$ onto the piecewise discontinuous polynomial basis, the definition of the discrete Maxwellian that enters the definition of the self-adjointness inner product, and the manipulation of the numerical fluxes. Discrete self-adjointness and entropy production is a topic of ongoing research.

4. Time-stepping and stability

A high-order, conservative DG scheme for the GkLBO must be accompanied by a suitable time-stepping scheme. In this section we complement the spatial discretization of the gyrokinetic-GkLBO equation presented in sections 2-3, and that of the Vlasov-Maxwell-Dougherty system presented in [24], with a description of the time-stepping algorithm and its stability. As a preliminary, recall that one can determine the appropriate time-step ($\Delta t$) for a linear problem $df/dt = L[f]$ by estimating the eigenvalues $\lambda$ of the operator $L$. Then the time-step is chosen such that $\lambda \Delta t$ is within the region of numerical stability for a particular time-stepping algorithm for all eigenvalues of the operator [33].

Purely damped modes, those for which $df/dt = \lambda f$ and $\lambda < 0$, will be stable when using an individual Euler step if $|\lambda|\Delta t < 2$ because $f^{n+1} = (1 + \lambda \Delta t)^n f^n$ (where the $n$ exponent labels the $n$th time step). In GKEYLL we instead use an explicit third-order Strong Stability Preserving (SSP) Runge-Kutta (SSP-RK3) with convex combinations of individual Euler steps that has a combined stability limit of $|\lambda|\Delta t \lesssim 2.512$ for purely damped modes, as can also be seen in figure 2.4 in [33]. One may instead wish to ensure the stability of each Euler stage and avoid ‘overdamped’ solutions that oscillate around zero (instead of just damping with the same sign) by using the more conservative limit $|\lambda|\Delta t < 1$.

Although the GkLBO is non-linear we can use these ideas to estimate stability limits of the drag and diffusion terms separately. We will then combine these into a single rule for each DG cell that effectively provide a finer mesh within each DG cell. We will assume, as is true for our domain accounts for the ($\Delta x$) and ($\Delta x$) are fit perfectly by the formula $e^{\Delta x}$, and calculate the spectrum of eigenvalues of the DG discretization of the $d/dt$ operator as a function of wavenumber $k$, from $k = 0$ up to $k = k_{max} = (p+1)\pi/(\Delta x)_{cell}$, where the extended $k$ domain accounts for the ($p+1$) degrees of freedom within each DG cell that effectively provide a finer mesh than the cell width. We will assume, as is true for $p = 2$, that the stability limit is set by the Nyquist mode, $k = k_{max}$. In the $j$th cell the Nyquist mode for $p = 1$ has a mean value of 0 (so $f_0 = 0$) and a linear slope $f_j(x, t) = f_j(t)\psi_1(x - x_j)$, where $\psi_1(x) = \sqrt{3x}/(\Delta x/2)$ is an orthonormal basis function. Then the evolution of the DG representation of this mode is

$$\frac{\partial f_j}{\partial t} = -v \left( \psi_1, \frac{\partial f_j}{\partial x} \right) = -\frac{v\sqrt{3}}{\Delta x} (\hat{f}(\Delta x/2) + \hat{f}(-\Delta x/2))$$

$$= -6\frac{v}{\Delta x} f_j,$$

where $\hat{f}$ is the upward numerical flux at the cell boundaries. In this case, the eigenvalue is

$$\lambda_{max} = 6\frac{v}{\Delta x}.$$

Equations (75) and (77) are fit perfectly by the formula $\lambda_{max} = 2(2p+1)v/(\Delta x)_{cell}$. However, for piecewise quadratic basis functions ($p = 2$), one finds that $\lambda_{max} \approx 11.9v/(\Delta x)_{cell}$. (This is done by solving for the $\lambda$ spectrum of the DG discretization of $\partial / \partial x$ which, for $p = 2$, requires solving a $3 \times 3$ linear system.) The resulting general formula for the maximum eigenvalue for advection is

$$\lambda_{adv} = 2C_{adv,p}(2p+1)\max(v/(\Delta x)_{cell}),$$

where the advection coefficient is $C_{adv,p} = \{1, 1, 1, 2\}$ for $p = \{0, 1, 2\}$, respectively. For RK3 $\lambda_{max}\Delta t < 2.512$ gives
a time step limit of $\nu \Delta t / \Delta x_{\text{cell}} < 1.256, 0.418, 0.209$ for $p = \{0, 1, 2\}$, which is within 3% of the empirically determined numbers in table 2.2 of [35]. Note that an equivalent finite-difference/finite-volume mesh with the same number of degrees of freedom would have an effective grid spacing $\Delta x_{\text{cell}} = \Delta x_{\text{cell}}/(p + 1)$, so the stability limit on the time step in terms of an effective Courant number is $\nu \Delta t / \Delta x_{\text{cell}} < 1.256(p + 1)/(C_{\text{adv,p}}(2p + 1)) = \{1.256, 0.836, 0.627\}$ for $p = \{0, 1, 2\}$, which does not drop as quickly at higher $p$ as a Courant number $\nu \Delta t / \Delta x_{\text{cell}}$ expressed in terms of cell width.

4.2. Stability of a DG diffusion operator

Consider the diffusion equation $\partial f / \partial t = D \partial^2 f / \partial x^2$. In second-order centered finite-difference (equivalent to finite-volume) discretizations of this operator the largest magnitude eigenvalue is $\lambda_{\text{max}} = -4D/\Delta x^2$ when using a forward Euler step. This is the $p = 0$ limit of DG, and one might attempt to employ the same formula but with the effective DG cell length $\Delta x = \Delta x_{\text{cell}}/(p + 1)$ that was used for advective terms. For a parabolic term (diffusion) it turns out that $\Delta x = \Delta x_{\text{cell}}/(p + 1)$ is more accurate, van Leer and Nomura calculate the eigenmodes and eigenvalues of a diffusion operator using RDG with $p = 1$ and $p = 2$ [30]. Their equation 79 and figure 1 give $\lambda_{\text{max}} = 15/\Delta x_{\text{cell}}^2$ for $p = 1$, while their figure 3 gives $\lambda_{\text{max}} \approx 33/\Delta x_{\text{cell}}^2$ for $p = 2$. These results can be fit with the expression

$$\lambda_d = -4C_{\text{diff,p}}D \left( \frac{p + 1}{\Delta x_{\text{cell}}} \right)^2, \quad (79)$$

where $C_{\text{diff,p}} = \{1, 0.94, 0.92\}$ for $p = \{0, 1, 2\}$, respectively.

4.3. Stability of the non-linear model-Fokker-Planck operator

The spatial discretization of the Vlasov-Dougherty equation discussed in [24] can use the same estimates for the SSP-RK3 $\Delta t$ as those for the one-velocity-dimension GkLBO. Such limit of (13) is

$$\frac{\partial f}{\partial t} = \mathcal{J}f = \left[ \nu (v^l - u^l) \mathcal{J}f + \nu v^l \frac{\partial \mathcal{J}f}{\partial v^l} \right]. \quad (80)$$

The first term looks like an advection term, so one might think that it gives an imaginary part to the eigenvalues. However, the eigenvalues of the full collision operator are not necessarily a simple sum of the separate eigenvalues of the diffusion and advection terms. We have already noted in section 2 that the combined drag and diffusion terms in the continuous GkLBO have a set of eigenmodes that are all purely damped (real $\lambda < 0$).

For now we use a conservative estimate of the eigenvalues of the DG discretized GkLBO based on a sum of contributions from the advection and diffusion terms. We use $\Delta v^l = \Delta v^l_{\text{cell}}/(2p + 1)$ for the advection term and $\Delta v^l = \Delta v^l_{\text{cell}}/(p + 1)$ for the diffusion term. Assuming a constant grid spacing, the estimated maximum eigenvalue of the GkLBO is

$$|\lambda_{\text{max}}| = 2\nu C_{\text{adv,p}} \max\left(\frac{|v^l - u^l|}{\Delta v^l_{\text{cell}}} \left( \frac{2p + 1}{\Delta v^l_{\text{cell}}} \right)^2 \right) \left( \frac{p + 1}{\Delta v^l_{\text{cell}}} \right)^2 + 4\nu C_{\text{diff,p}} v^l_1^2 \left( \frac{p + 1}{\Delta v^l_{\text{cell}}} \right)^2, \quad (81)$$

This can be generalized to higher dimensions without difficulty. For the 1X2V (one configuration-space dimension, and two velocity-space dimensions) GkLBO we use the following maximum eigenvalue estimate

$$|\lambda_{\text{max}}| = 2\nu C_{\text{adv,p}}(2p + 1) \left[ \frac{\max\left(\frac{|v^l - u^l|}{\Delta v^l_{\text{cell}}} \right)}{\Delta v^l_{\text{cell}}} + \frac{2\mu_{\text{max}}}{\Delta \mu_{\text{cell}}} \right] + 4\nu C_{\text{diff,p}} v^l_1^2 (p + 1)^2 \left( \frac{1}{\Delta v^l_{\text{cell}}} \right)^2 + \frac{m}{B} \left( \frac{2\mu_{\text{max}}}{\Delta \mu_{\text{cell}}} \right)^2 \quad (82)$$

This eigenvalue is computed every stage of the SSP-RK3 and used to calculate the time step according to $\Delta t|\lambda_{\text{GkLBO}}| < \text{CFL}$. The CFL number is close to unity, but in GKEYLL it can be modified as a user input. In order to illustrate the impact of these choices consider the spatially homogeneous relaxation problem in 1X1V posed by (80) with a bump-on-tail distribution function for its initial condition:

$$f(t = 0) = f_M(n, u^l, v^l) + f_M(n, u^l, v^l, b) \left( \frac{d_p}{(v^l - u^l, b)^2 + s^2_p} \right). \quad (83)$$

Here $f_M(n, u^l, v^l)$ is the one-velocity-space dimension Maxwellian

$$f_M(n, u^l, v^l) = \frac{n}{\sqrt{2\pi v^l}} \exp \left[ -\frac{(v^l - u^l)^2}{2v^l} \right], \quad (84)$$

and we employed the parameters $n = 1, u^l = 0, v^l = 1/3, a_b = \sqrt{0.1}, u^l, v^l, b = 6v^l/\sqrt{3}, v^l, b = 1.0$ and $s_p = 0.12$. This distribution is discretized in a $[0, 1] \times [-8v^l, 8v^l]$ domain using $2 \times 32$ cells and a piecewise linear basis ($p = 1$), or $2 \times 16$ and a piecewise quadratic basis ($p = 2$). Using a collisionality of $\nu = 0.01$ we show that by time $t = \nu^{-1}$ the GkLBO relaxes this initial condition to be close to a Maxwellian (figure 2). By gradually increasing the CFL for each test, we discovered that for piecewise linear basis functions ($p = 1$) the simulation begins to become unstable for $\text{CFL} \gtrsim 1.431$, which gives $\Delta t \approx 0.075 726$. Oscillations are observed close to or below zero in regions where $f$ is small (inset of figure 2(a)), and the simulation diverges at later time. Piecewise quadratic basis functions ($p = 2$) allowed CFL $\leq 0.251$, corresponding to $\Delta t \approx 0.062 479$ (figure 2). The fact that these two cases ($p = 1, 2$) were not stable all the way up to CFL $= \gamma_{\text{max}} \Delta t < 2.512$ (the stability limit for RK3 for damped modes) indicates that there are some inaccuracies in the approximations that led to (82), such as in treating advection and diffusion separately or in neglecting boundary conditions. Nevertheless, it captures the main scaling of the allowable time step with the parameters of the problem.
Similar oscillations are observed in 1X2V simulations. We projected (83) onto the 1X2V DG basis, using the Maxwellian $f_M(n, u||, v_i)$ defined in (31), with parameters $v_i = 1/\sqrt{2}$, $v_{ib} = 1/\sqrt{2}$, and $u_0 = 4v_i$. The bump-on-tail initial condition is relativistic by using local Lax-Friedrichs (LF) fluxes: instead of $u$.

As CFL increases, an instability begins to develop at CFL $(\mu = 3)$, where $\mu = 4v_i$. The domain in these plots is $12 \times 12 \times 12$, and the relaxation proceeds without a problem; this is the maximum entropy solution. In section (4.4), we commented on how the stability condition on each cell: $\alpha_{\parallel} = \left(\frac{\alpha}{\partial n_{\parallel}}\right) R_{\parallel}$, $\parallel$ acts on the flux $\alpha$ with a phase-space velocity $\parallel = (\pmb{R}, v_i)$. This is a non-linear advection equation, for which we can use (87) to estimate the stability condition on each cell:

$$\lambda_{\parallel} = 2 C_{adv, \parallel} (2p + 1) \sum_k \max(0, \alpha \cdot \hat{n}_k) \Delta z_k,$$

where $\parallel$ is the unit vector pointing out of the corresponding surface where the numerical flux is evaluated. We note that this is also how the (phase-space) advection numerical fluxes are computed for the collisionless terms $\parallel$. Since the up-winding is based on local values of the phase-space velocity we refer to this as the local LF fluxes, and it yields a steady state that avoids the negative incursion of the global LF fluxes (figure (4)).

### 4.4. Stability condition in the full gyrokinetic-GkLBO system

The (long wavelength) gyrokinetic system in (5)–(6) and (9)–(10) is also limited by the CFL constraints of the collisionless, or Hamiltonian, terms. We can estimate this condition by considering the kinetic equation

$$\frac{\partial f}{\partial t} + \nabla_z \cdot (\alpha f) = 0,$$

where the phase-space gradient $\nabla_z \equiv \left(\nabla, \frac{\partial}{\partial n_{\parallel}}\right)$ acts on the flux $\alpha$ with a phase-space velocity $\alpha \equiv (\pmb{R}, v_i)$. This is a non-linear advection equation, for which we can use (78) to estimate the stability condition on each cell:

$$\lambda_{\parallel} = 2 C_{adv, \parallel} (2p + 1) \sum_k \max(0, \alpha \cdot \hat{n}_k) \Delta z_k,$$

where $\sum_k$ is over all faces of the cell, $\hat{n}_k$ is the outward normal of the kth face, and $\Delta z_k$ is the grid spacing in the direction corresponding to the kth face. The form of the maximum

---

5 This system has no configuration-space variation so we could have used a $1 \times 32^2$ grid instead.
function guarantees that the sum is only over faces where there is an outgoing flux. Although the eigenvalues of the full collisional gyrokinetic equation are not a sum of the collisionless eigenvalues and the GkLBO eigenvalues, we follow this conservative approach and compute the time step according to

$$\Delta t \left( \lambda_H + \lambda_{\text{GkLBO}} \right) < \text{CFL}. \quad (89)$$

An example of what establishes $\lambda_H$ in the electrostatic limit is the electrostatic shear Alfvén or $\omega_H$ mode $[36, 37]$. A dispersion relation for this electrostatic instability can be derived by linearizing the collisionless form of (5) and (10). In the long-wavelength limit this becomes

$$\omega_H = \sqrt{\frac{n_e}{n_0}} \frac{|k_\parallel v_e|}{|k_\perp \rho_s|}. \quad (90)$$

where $n_0$ is the linear ion polarization density used in the Poisson equation. We seek an estimate for CFL so that $\omega_{H,\max} \Delta t < 1.73$, which is the stability limit for the RK3 time-stepping method. To estimate $\omega_{H,\max}$, assume $k_{\parallel,\max} \approx \Delta z = (2p + 1)/\Delta z_{\text{cell}}$ and $k_{\perp,\min} = \pi/L_x$, where $\Delta z$ is the cell spacing in $z$ and $L_x$ is the domain width in $x$. The initial time-step is set by the fastest parallel electron transit rate, $v_\parallel,\max$. The corresponding eigenvalue is $v_{\parallel,\max}/\Delta z = (2p + 1)v_{\parallel,\max}/\Delta z_{\text{cell}}$, giving a time-step estimate of

$$\Delta t = \frac{\Delta z_{\text{cell}} \text{CFL}}{(2p + 1)v_{\parallel,\max}}. \quad (91)$$

Combining with (90) and the RK3 stability limit gives

$$\omega_{H,\max} \Delta t = \text{CFL} \sqrt{\frac{n_{e,\max}}{n_0}} \frac{v_e L_x}{\pi v_{\parallel,\max} \rho_s} < 1.73, \quad (92)$$

which we use to set an appropriate value of CFL as an input parameter prior to run time. In the future, we plan to calculate $\omega_{H,\max}$ within the code to dynamically set the time-step limit due to the electrostatic shear Alfvén mode.

5. Benchmark problems

In this section, we present tests designed to further understand the discrete scheme and to verify the accuracy of the gyrokinetic-GkLBO system. Relaxation tests without the collisionless terms demonstrate properties of the discrete
GkLBO such as conservation, entropy and positivity. We also explore collisional Landau-damping to understand the physical implications of this model and compare it to analytic theory. Finally, simulations of 5D turbulence on helical, open field lines with collisions modeled by the GkLBO are presented. These last two benchmarks employed a DG treatment of the collisionless terms that was described in earlier publications [7, 37].

5.1. Relaxation tests of the GkLBO

When an initial distribution function is subjected to the GkLBO alone, without the Hamiltonian terms, it will relax to the maximum entropy solution. In the continuous sense, the maximum entropy solution is the Maxwellian in (31), but the discrete equilibrium solution, \( f_M \), is not necessarily the projection of (31) onto the DG basis. In principle, \( f_M \) could be derived by repeating the derivation of (31) but assuming the discrete form of the GkLBO and a finite velocity domain. This implies that if we project \( f_M \) onto the basis (e.g. using Gaussian quadrature) and use that as an initial condition, the system will not be static and will evolve some. Figure 5(a), for example, shows this initial, projected Maxwellian and its final state after one collisional period. At first sight they are indistinguishable, but the difference, shown in figure 5(b), shows that the projected Maxwellian was not in the kernel of the discrete \( C[f] \). These tests were carried out in a \([0,1] \times [-12 \nu_t, 12 \nu_t]\) domain with \(2 \times 96\) cells using a zero-drift Maxwellian with \( \nu_t = 1/\sqrt{2} \), \( \nu = 0.01 \) and piecewise linear bases.

A Maxwellian is equivalent to the (properly normalized) Gaussian-weighted zeroth Hermite basis [38]. This is also an eigenfunction of \( C[f] \), as we will show in section 5.2. Therefore, in a continuous infinite velocity-space, its Hermite spectrum should remain a Dirac delta function peaked at the maximum entropy solution. In the continuous sense, the GkLBO alone, without the Hamiltonian terms, it will relax to the discrete equilibrium solution,

\[
\frac{\partial f_m}{\partial t} = -\nu m f_m.
\]

We can test the analytic solution to this equation, \( f_m(t = 0) e^{-\nu mt} \), numerically by using \( f_{m=0} = f_{m=5} = f_{m=10} = 1 \) and zero for all other modes, rather than only initializing \( f_{m=0} = 1 \), as in figure (5). The time evolution of the three higher modes is shown in figure (6). Its agreement with analytic theory is very good in the \( t \in [0, \nu^{-1}] \) time window, and if the discrete Hermite basis functions were eigenfunctions of our discrete GkLBO, the three higher modes would simply decay exponentially indefinitely. However, when the amplitude of the \( m = 20 \) mode reaches the noise introduced by the evolution of \( m = 0 \), the spectral analysis of \( m = 20 \) begins to deviate from the analytic result. This noise we showed in figure 5(c) is at a \( |f_m|^2 \sim 10^{-18} \) level, and it is at that point that the green \( m = 20 \) line in figure (6) deviates from its analytic expectation. Were this spectral analysis to be carried out
with $m_{\text{max}} + 1 < 33$ the error in $m = 20$ would be slightly larger at $\nu t > 0.6$, but solving the least-squares problem in (93) with $m_{\text{max}} + 1 > 34$ yields errors of order of magnitude larger.

As the solution relaxes onto the discrete maximum entropy solution, $f_{Mh}$, it also exhibits a physical non-decreasing entropy. We again project the bump-on-tail distribution of (83) onto the the 1X2V $(x,v_{\parallel},v_{\perp})$ DG basis, use a stable CFL = 1, and run to $\nu t = 10$. The norm of the difference between $f(t)$ and the maximum entropy solution, $f_{Mh} = f(t = 10\nu^{-1})$, decreases rapidly as shown in figure 7(a). Meanwhile, the entropy, $S(t) = -\int f(t) \ln f(t) \, dx \, dv_{\parallel} dv_{\perp}$, increases monotonically. The relative difference between initial entropy and $S(t)$ is given in figure 7(b). Although we have not yet proven an $H$-theorem for the discrete operator (or proved its self-adjointness), the entropy is seen to increase in the cases we have explored. Part of the challenge in proving self-adjointness of the operator lies in guaranteeing that $f$ remains positive. Positivity of $f$ is something we are able to build into the discretization of the GkLBO’s drag term (not presented here), but additional work is needed to ensure the diffusion term does not cause $f$ to go negative.

Figure 5. (a) An initial Maxwellian projected onto the DG basis and its final ($t = \nu^{-1}$) state after relaxation. (b) Difference between initial projection of the Maxwellian and final discrete equilibrium, i.e. $f_{Mh} = f(t = \nu^{-1})$. (c) Time evolution of the Hermite spectrum when initial state is a projected Maxwellian.

Figure 6. Time evolution of the (squared) Hermite expansion coefficients of the distribution function.

These 1X2V relaxation tests also confirm the conservative properties of our scheme and, although not shown here, conservation of particle number, momentum, and energy are also guaranteed in higher dimensions. For the case of the 1X2V bump-on-tail initial condition, figure (8) shows the norm of the relative difference in the momentum and energy densities, $M_1$ and $M_2$. Over ten collisional periods the relative change in these quantities remains within machine precision, consistent with sections 3.2.2-3.2.4. In this case machine precision accuracy refers to the fact that the relative error per time step in the momentum is $\sim 2 \times 10^{-12}/1400 \sim 10^{-13}$, where 1400 is the approximate number of time steps. The non-vanishing boundary contributions in the surface term of (55) and in the calculation of the primitive moments, $u_{||}$ and $v_{\perp}$, are necessary for exact conservation, even if $f$ is small at the boundaries. Neglecting these corrections gives errors in momentum and energy conservation that are orders of magnitude larger.

5.2. Collisional Landau damping of ion acoustic waves

The study of collisionless (Landau) damping of plasma waves due to velocity-space resonance dates back to the origins of plasma physics, and its modification due to the presence of collisions remains an important area of research. Ion sound waves suffer from this decay, and scientists have been constructing a theory of such phenomenon for decades. Consider that neutral sound waves are undamped in its highly (molecular) collisionality environment, so one may expect that as collisions become more frequent Landau damping of ion acoustic waves would weaken. The consensus, however, is that the actual trend depends on whether one considers self-species collisions, multi-species collisions, or both. The description of collisional Landau damping can also vary with the collision operator employed. An early study with a Krook operator [39] noted that under ion-ion collisions alone the damping rate ($\gamma = -i\omega$) monotonically decreases towards the regular sound wave limit ($\gamma \rightarrow 0$) as $\tau_{\|}$ increases if the temperature ratio $\tau = T_i/T_e = 1$. This was not limited to the simple Krook operator as numerical integration of the Vlasov-FPO equation also arrived at the same conclusion [40]. But these
adiabatic and will not collide with the ions (only ion
plasma in a curvature-free homogeneous magnetic field such
that the collisionality is low). The entropy $S$ increases monotonically, and so does the relative difference in $S$.

Landau damping of plasma sound waves is central to ion-
temperature gradient instabilities, ion acoustic instabilities
and other transport processes in astrophysical and laboratory
plasmas. As a commonplace ingredient in plasmas, it is
not only necessary to understand its collisional modifications
with the full FPO, but also with the simple models frequently
used by analytic and computational studies. The model-FPO
Dougherty operator considered here has been explored little in
the context of ion-acoustic waves. One of the few studies available
explored ion-acoustic instabilities in the presence of
self-species and multi-species collisions, and was only able to
do so at low collisionalities (i.e. $\nu_{ei} \ll 1$). The study of collisional Landau damping of ion-acoustic waves at
arbitrary collisionality here then serves as both documentation
of this process with the Dougherty operator, and also as validation
of our scheme and implementation within GKEYLL.

Consider a system consisting of a single-ion hydrogen plasma in a curvature-free homogeneous magnetic field such
that $\mathbf{J} = \mathbf{B} = \mathbf{b} \cdot \mathbf{B} = \hat{\mathbf{z}} \cdot \mathbf{B}$. The electrons will be assumed
adiabatic and will not collide with the ions (only ion
self-species collisions are included here), so we refer to the
collisionality $\nu_{ei} = \nu$. Then one only needs to evolve the ion
equation (5), which upon linearization about an equilibrium,$f = f_0 + f_1$ (we omit the ion subscript here for simplicity),
simply becomes

\[
\frac{\partial f_1}{\partial t} + v_1 \frac{\partial f_1}{\partial z} + \frac{1}{B} [\phi, f_0] - \frac{e}{m} \frac{\partial \phi}{\partial z} \frac{\partial f_0}{\partial v_0} = C[f_0, f_1]
\]

\[
= \nu \left\{ \frac{\partial}{\partial v_{||}} \left[ (v_{||} - u_{||,0}) f_1 - u_{||,0} f_0 + v_{i,2} \frac{\partial f_1}{\partial v_{||}} + \frac{\partial f_0}{\partial v_{||}} \right] + \frac{\partial}{\partial \mu} 2 \mu \left[ f_1 + \frac{m}{B} \left( v_{r,0}^2 \frac{\partial f_1}{\partial \mu} + v_{i,1}^2 \frac{\partial f_0}{\partial \mu} \right) \right] \right\}.
\]

The simplified Poisson bracket $[F, G] = b \cdot \nabla F \times \nabla G$ vanishes since $f_0$ is homogeneous in configuration space, and the first-
order primitive moments are defined as

\[
u_{\parallel,1} = \frac{2\pi B}{m n_0} \int v_{||} f_1 \, dv_{||} \, d\mu,
\]

\[
u_{\perp,0} = \frac{2\pi B}{m n_0} \int \left( \frac{2\mu}{m} + v_{\perp,0}^2 \right) f_1 \, dv_{||} \, d\mu,
\]

and $n_0$, $u_{\parallel,0}$ and $v_{\perp,0}$ are the number density, mean velocity and
thermal speed of $f_0$, respectively, although we have assumed

Figure 7. As a 1X2V a bump-on-tail distribution relaxes, the norm of the difference of $f$ and the discrete equilibrium solution
$f_{eq} = f(t = 10\omega^{-1})$ decreases (a). The entropy $S$ increases monotonically, and so does the relative difference in $S$.

Figure 8. 1X2V relaxation of a bump-on-tail distribution with $p = 1$ and $p = 2$. Relative norm of the difference in (a) momentum density
$M_1$ and (b) energy density $M_2$, as a function of time. Both are conserved to machine precision.
\[ u_{||,0} = 0. \] It is convenient to write the perturbed distribution as \( f_1 = f_0 \varphi \) with \( \varphi \ll 1 \). The linearized collision operator then becomes

\[
C[f_0,f_1] = u f_0 \left\{ -(v_{||} - u_{||,0}) \frac{\partial \varphi}{\partial v_{||}} + v_{t,0}^2 \frac{\partial^2 \varphi}{\partial v_{t,0}^2} + 2 \mu \frac{\partial \varphi}{\partial \mu} + \frac{mv_{t,0}^2}{B} \frac{\partial \varphi}{\partial \mu} + \frac{v_{t,0}^2}{v_{t,0}^2} \right\}.
\]

We have assumed there is no equilibrium component to the electrostatic potential \( \phi = \phi_0 \), and from here on we will use the normalized variables \( 2 \mu B/(mv_{t,0}^2) \to \mu \) and \( v_{||}/v_{t,0} \to v_{||} \). Assuming wave-like modes according to the ansatz \( f_1 = \tilde{f} \exp[i(kz - \omega t)] \), \( \phi = \phi \exp[i(kz - \omega t)] \), and employing the quasineutrality between adiabatic electrons and the equilibrium ion distribution (via Poisson’s equation), renders our kinetic equation into

\[
i \left( v_{||} - \Omega \right) \tilde{f} + i \pi v_{t,0}^3 \rho \frac{T_{10} - n_0}{T_0} \int d\mu \mu = -\varphi \chi = 0.
\]

We now refer to the normalized mode frequency, \( \Omega = \omega/(k_{||}v_{t,0}) \), and the normalized collisionality \( \eta = \nu/(k_{||}v_{t,0}) \), and \( \chi(\varphi) \) is the term between curly brackets in (98).

One can proceed by expanding in a set of Hermite-Laguerre polynomials [44] as

\[
\varphi = \sum_{m,n=0}^{\infty} a_{mn} \varphi_{mn} = \sum_{m,n=0}^{\infty} \frac{1}{\sqrt{m!}} H_m(v_{||}) L_n(\mu/2),
\]

which satisfy the orthogonality relation

\[
\langle \varphi_{m,n}, \varphi_{m,n'} \rangle = \frac{1}{\sqrt{2\pi}} \int \varphi_{m,n} \varphi_{m,n'} e^{-(v_{||}^2 + \mu^2)/2} dv_{||} d\mu = \delta_{n,m} \delta_{m,m'}.
\]

Together with the recursion relations

\[
H_{m+1}(v_{||}) = v_{||} H_m(v_{||}) - m H_{m-1}(v_{||}),
\]

\[
(\mu/2) L'_n(\mu/2) = n L_n(\mu/2) - n L_{n-1}(\mu/2)
\]

one may find the projection of the transformed linear kinetic equation (99) onto the basis \( \varphi_{mn} \). This projection, after some algebra, is

\[
\langle \varphi_{mn}, \text{equation (99)} \rangle = \begin{cases}
\Omega \alpha_{10} - a_{00} - \sqrt{2} a_{20} - \frac{T_{10} - n_0}{T_0} a_{00} = 0 & (m,n) = (1,0) \\
\Omega^2 a_{10} - \sqrt{3} a_{30} + i \eta \left( \frac{1}{2} a_{00} + \frac{2}{3} a_{01} \right) = 0 & (m,n) = (2,0) \\
\Omega a_{01} - a_{11} + i \eta \left( \frac{1}{2} a_{01} + \frac{2}{3} a_{20} \right) = 0 & (m,n) = (0,1) \\
\Omega + i \eta (m + 2n) \delta_{n,m} - \frac{a_{mn}}{\sqrt{m(\max - 1)n}} - \frac{\sqrt{m + 1} a_{(m+1)n}}{m + 1} = 0 & \text{all other } (m,n) \!
\end{cases}
\]

From the last of these equations one can show that for a physically realizable solution \( a_{m+1,n}/a_{mn} \to a_{mn}/\sqrt{m} \) as \( m \to \infty \) [45]. The presence of collisions limits the extent of the spectrum in \( m \), allowing us to truncate the expansion at an upper limit \( m_{\max} \). One can then use

\[
\Omega + i \eta (m_{\max} + 2n) a_{mn} - \sqrt{m_{\max}} a_{(m_{\max} - 1)n} = 0
\]

in conjunction with the last relation in (103) to iterate backwards from \( m_{\max} \) and find:

\[
a_{m0} = \frac{\sqrt{m}}{\Omega + i \eta (m + 2) - \cdots - \frac{m_{\max}}{\Omega + i \eta m_{\max}}} a_{(m-1)0}.
\]

A similar relation is obtained for \( n = 1 \). Since the recursion relation in (103) does not couple Laguerre moments together, one need only solve the system for \( a_{00}, a_{10}, a_{20} \) and \( a_{01} \). The coefficients \( a_{00} \) and \( a_{11} \) can be written in terms of continued fractions like (105), and one obtains a linear problem with the determinant of the mass matrix yielding the dispersion relation [44]

\[
\frac{T_{10}}{T_{10}-n_0} = \frac{8\eta^2 + 9F_1F_2}{8\eta^2 (\Omega^2 - 1) + 9 [1(\Omega^2 - 1) F_1 - 2F_2] F_2},
\]

where the functions \( F_1(\Omega, \eta) \) and \( F_2(\Omega, \eta) \) are
We set up an analogous scenario in GKEYLL using adiabatic electrons, $\tau = 1.0$, hydrogen mass ratio and perturbed the initial state using a wave mode with $k_\parallel\rho_i = 0.5$. These simulations were done on a domain $[-\pi/k_\parallel, \pi/k_\parallel] \times [-6\nu_{i\parallel}, 6\nu_{i\parallel}] \times [0, m_i(5\nu_{i\parallel})^2/(2B)]$ discretized with $64 \times 128 \times 16$ cells. This resolution and the time step stability constraints are probably conservative and were chosen to guarantee these results were well converged. In figure 9(a) the decay of the wave is displayed by the decrease in electrostatic energy over time, from which one can measure both the real part and the imaginary part of the wave frequency. The energy trace of three different collisionalities show that as ion-ion collisions alone become more frequent, the damping mechanism is progressively eroded. We scanned the entire collisional range and compared our results to the roots of the dispersion relation in (106). Figure 9(b) shows excellent agreement between theory and our implementation in GKEYLL. Consistent with intuition, when the mean-free-path becomes comparable to the wavelength, $\nu/(k_\parallel\nu_i) \sim 1$, fewer particles will be able to resonate with the wave before experiencing collisional scattering, thus considerably reducing damping. In the high-frequency limit, collisions maintain the plasma in a local thermodynamic equilibrium so the plasma behaves like an ideal gas that has undamped compressional oscillations. This test also confirms an earlier partial agreement between theory and simulation of Landau damping of electron (Langmuir) waves due to a disparity in the dimensionality of the two [24].

\[ F_1(\Omega, \eta) = \frac{3}{\Omega + 3i\eta} - \frac{4}{\Omega + 4i\eta} - \frac{5m_{\max}}{\Omega + 5i\eta + \cdots} \times \Omega + i\eta(m_{\max} + 2) \]

\[ F_2(\Omega, \eta) = \frac{2}{\Omega + 3i\eta} - \frac{1}{\Omega + 4i\eta} - \frac{3m_{\max}}{\Omega + 5i\eta + \cdots} \times \Omega + i\eta(m_{\max} + 2) \]

(107)

5.3. Helical open-field-line plasma turbulence

We now present a benchmark test of the full 5D (long-wavelength) gyrokinetic system, given by (5)–(6) and (9)–(10). With the moment-conserving GkLBO, we simulated plasma turbulence on helical, open field lines, using a nonorthogonal field-line-following coordinate system as in [46, 47]. In this coordinate system, $z$ is parallel to magnetic field lines, $x$ is the radial coordinate, and $y$ is the ‘bi-normal’ coordinate. To ensure numerical stability, we used (92) to determine that $CFL \approx 0.28$ was necessary to prevent the electrostatic shear Alfvén mode from becoming unstable. We set $CFL = 0.2$ as a conservative estimate. We used the same physical parameters as in [47] to simulate the Texas Helimak simple magnetized torus experiment and make direct comparison with those results. We calculated the ion and electron collision frequencies from [48] using background densities ($n_0$) and temperatures ($T_{i0}$) that are constant in space and time. It is important to note that simulations in [47] included collision frequencies with spatially- and time-varying densities and temperatures, as well as electron-ion collisions (but no ion-electron collisions). Those simulations were also carried out with an earlier, nodal DG scheme employing different algorithms than those presented here while still remaining conservative by correcting for the errors. We denote this by $\nu_s(x,t)$ to differentiate it from simulations presented here with the moment-conserving GkLBO, using the constant collision frequency $\nu_s$ and neglecting multi-species collisions. We also present results from a simulation with a reduced collision frequency, $0.1\nu_s$. All simulations were run to 16 ms. Calculated equilibrium profiles were averaged in time from 10 to 16 ms and in the bi-normal direction $y$.

Figure 10 shows snapshots of electron density, electron temperature, and plasma potential in the nonorthogonal field-line-following coordinate system at 10 ms. Turbulent structures and density levels are very similar to those presented in [47], though electron temperature and plasma potential values are slightly greater. Electron density profiles are compared in figure 11(a), with all three profiles being very similar. More differences are visible in figure 11(b), which compares the electron temperature profiles. Compared to the simulation with constant like-species collisionality only (dotted blue line in figure 11(b)), including electron-ion collisions and spatially varying collisionality (solid green line figure 11(b)) reduced the electron temperature. Given the inverse dependence of the interchange linear growth rate on the electron-ion collisionality [49] one may consider the possibility of cross-field transport increasing as $\nu_{ei}$ decreases; were this effect to be significant parallel transport would be less competitive against perpendicular fluxes and would not carry out heat through the sheath as efficiently, leading to a temperature increase across the plasma. However the linear analysis suggests that the interchange growth rate is only weakly dependent on $\nu_{ei}$ [49]. Instead, a contributing factor to the increase of $T_e$ when collisions strengthen is that due to pitch-angle scattering more electrons are carried to higher $v_\parallel$, where they are lost through...
Figure 9. (a) Field energy time trace and (b) damping rates as a function of collisionality for the ion acoustic wave.

Figure 10. Snapshots of electron density (left), electron temperature (middle), and plasma potential (right) in the $xy$-plane, from simulations of plasma turbulence on helical, open field lines in 5D with the moment-conserving GkLBO.

Figure 11. Comparison of (a) electron density and (b) electron temperature equilibrium profiles from simulations with different constant collision frequencies ($\nu_s, 0.1\nu_s$) to that with time- and spatially-varying collision frequencies and multi-species collisions ($\nu_{ss}(x, t)$).

Figure 12. Comparison of (a) plasma potential and (b) density fluctuation profiles from simulations with different constant collision frequencies ($\nu_s, 0.1\nu_s$) to that with time- and spatially-varying collision frequencies and multi-species collisions. A non-adiabatic electron response might explain the slight decrease in the plasma potential in the lower collisionality case. Turbulence levels in (b) are also reduced for the lower collisionality case.

The sheath. This effect can increase the heat loss rate, lowering the temperature of the remaining electrons. Such mechanism would also apply to the increase in $T_e$ seen in comparing the simulation using constant like-species collisions (dotted blue line in figure 11(b)) with a similar simulation which used a reduced collisionality (orange dash-dot line in figure 11(b)).

Plasma potential profiles are compared in figure 12(a). All three $\phi(R)$ profiles are relatively similar, and...
collisionality-induced changes to the $E \times B$ profile do not appear significant enough to indicate that shear stabilization would play a major role in the changes to the simulated profiles or the turbulence. It is however interesting that the constant like-species collisionality simulation with higher $T_e$ (orange dash-dot line in figure 11(b)) is actually the one with a lower potential, contrary to what we would expect from an adiabatic electron response $e\phi \sim AT_e$. One possibility is possible that the lower collisionality allows for an increasingly non-adiabatic response of the plasma potential. It is also possible that despite the increase in $T_e$, there is a stronger decrease in $\Lambda$: at low collisionality fewer electrons scatter above the sheath potential, so the sheath potential has to drop to allow more electrons to escape in order to match the ion flux into the sheath. Lastly, we highlight that density fluctuation levels are reduced for the lower collisionality case as compared with the other simulations (figure 12(b)).

A more in depth analysis of the physics of these simulations is possible but beyond the scope of this manuscript. In general the intention here is to demonstrate that the moment-conserving GkLBO presented in this work has been successfully incorporated into more complex 5D simulations, and that despite being limited to like-species collisions it produces results with reasonable agreement with previous simulations [47]. This agreement will likely improve by including features such as spatially-varying collision frequencies and multi-species collisions.

6. Discussion and summary

We have presented a gyroaveraged Lenard-Bernstein-Dougherty collision operator (GkLBO), including a novel formulation of the discrete discontinuous Galerkin form and its implementation in Gkeyll. Building upon [24], we use the concept of weak equality to formulate a recovery DG (RDG) algorithm for the diffusion term of the GkLBO. It also provides a rigorous means to compute the primitive moments, $u_0$ and $v_e$. If such calculations are carried out using point-wise or cell average-based operations, significant errors ensue, causing non-conservation and instability. This concept guarantees that our discrete operator retains conservation properties and leads to an energy-conserving scheme even in the case of piecewise linear basis functions, provided that we carefully consider quadratic quantities projected onto the $p=1$ basis. Weak equality is also crucial in the definition of spectral transforms of DG data.

The continuous GkLBO is self-adjoint and satisfies the $H$-theorem but we have not yet proven that the discrete operator retains such properties. This is challenging because the present discrete operator does not guarantee $f > 0$, though we have already implemented a positivity-preserving drag term (not presented here). Self-adjointness enhances the efficacy of some approaches to accelerate the time integration (e.g. super time-stepping [34]), which we eventually wish to implement in order to more efficiently model highly collisional plasmas. Guaranteeing positivity, self-adjointness and non-decreasing entropy in the DG discretization scheme is the objective of on-going work.

We analyzed the stability conditions for DG advection and diffusion problems, and used this to establish the time step stability criterion for the GkLBO. Satisfying these conditions helps to avoid some issues associated with negative values of the distribution function, since $f > 0$ is not currently guaranteed in our scheme. For the SSP-RK3 time integration in Gkeyll, we presented a conservative estimate of the largest, stable time step.

Relaxation tests of the pure GkLBO demonstrated the exact numerical conservation properties of our scheme. These systems evolved to a maximum entropy solution which, as shown through a Hermite spectral lens, is subtly different from a Maxwellian projected onto the DG basis. This makes the Hermite analysis of collisional DG data more complicated for the larger Hermite moments, since the high-$m$ noise associated with the evolution of the zeroth-order Hermite moment causes higher moments to deviate from the analytic solution. However, lower moments of the GkLBO evolve according to analytic theory, and the evolution of higher moments converges with resolution.

We performed tests of collisional Landau damping of ion acoustic waves. Using a Hermite-Laguerre basis, we obtained a dispersion relation whose least-damped roots agreed well with simulation results. We presented a more complicated test of 5D turbulence on open, helical field lines in the Texas Helimak device, which agreed well with previous simulations, even without the additional physics of multi-species collisions and spatially-varying collision frequencies. Multi-species collisions give rise to slightly different (discrete) conservation laws and requirements for the calculation of the velocities and thermal speeds in the cross-collision terms. More accurate gyrokinecitcon simulations of laboratory and astrophysical plasmas will include spatially-varying, and even velocity-dependent, collisionalities. These features are currently being developed and tested within the Gkeyll code.

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Appendix A. Accessing and running Gkeyll

The Gkeyll code (in binary and source format) and the input files to reproduce results presented here are available for download. Gkeyll installation instructions can be found on the Gkeyll website (http://gkeyll.readthedocs.io). The code can be installed on Unix-like operating systems (including Mac OS and Windows using the Windows Subsystem for Linux) either by installing the pre-built binaries using the conda package manager or building the code via sources. The input files for simulations presented here can be found at https://github.com/ammarhakim/gkyl-paper-inp/tree/master/GkL.BO.

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