EXTENDED CESÁRO OPERATORS ON ZYGMUND SPACES IN THE UNIT BALL

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ABSTRACT. Let \( g \) be a holomorphic function of the unit ball \( B \) in the \( n \)-dimensional space, and denote by \( T_g \) and \( I_g \) the induced extended Cesáro operator and another integral operator. The boundedness and compactness of \( T_g \) and \( I_g \) acting on the Zygmund spaces in the unit ball are discussed and necessary and sufficient conditions are given in this paper.

1. Introduction

Let \( f(z) \) be a holomorphic function on the unit disc \( D \) with Taylor expansion
\[
f(z) = \sum_{j=0}^{\infty} a_j z^j,
\]
the classical Cesáro operator acting on \( f \) is
\[
C[f](z) = \sum_{j=0}^{\infty} \left( \frac{1}{j+1} \sum_{k=0}^{j} a_k \right) z^j.
\]

In the past few years, many authors focused on the boundedness and compactness of extended Cesáro operator between several spaces of holomorphic functions. It is well known that the operator \( C \) is bounded on the usual Hardy spaces \( H^p(D) \) for \( 0 < p < \infty \) and Bergman space, we recommend the interested readers refer to [10, 12, 8, 2, 13]. But the operator \( C \) is not always bounded, in [15], Shi and Ren gave a sufficient and necessary condition for the operator \( C \) to be bounded on mixed norm spaces in the unit disc. Recently, Siskakis and Zhao in [14] obtained sufficient and necessary conditions for Volterra type operator, which is a generalization of \( C \), to be bounded or compact between \( BMOA \) spaces in the unit disc. It is a natural question to ask what are the conditions for higher dimensional case.

Let \( dv \) be the Lebesgue measure on the unit ball \( B \) of \( \mathbb{C}^n \) normalized so that \( v(B) = 1 \), and \( dv_\beta = c_\beta (1 - |z|^2)^\beta dv \), where \( c_\beta \) is a normalizing constant so that \( dv_\beta \) is a probability measure. The class of all holomorphic functions on \( B \) is defined by \( H(B) \). For \( f \in H(B) \) we write
\[
Rf(z) = \sum_{j=1}^{n} z_j \frac{\partial f}{\partial z_j}(z).
\]

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A little calculation shows $C[f](z) = \frac{1}{z} \int_0^z f(t)(\log \frac{1}{1-t})' dt$. From this point of view, if $g \in H(B)$, it is natural to consider the extended Cesàro operator (also called Volterra-type operator or Riemann-Stieltjes type operator) $T_g$ on $H(B)$ defined by

$$T_g(f)(z) = \int_0^1 f(tz) Rg(tz) \frac{dt}{t}.$$ 

It is easy to show that $T_g$ take $H(B)$ into itself. In general, there is no easy way to determine when an extended Cesàro operator is bounded or compact.

Motivated by [15], Hu and Zhang [6, 7, 17] gave some sufficient and necessary conditions for the extended $C$ to be bounded and compact on mixed norm spaces, Bloch space as well as Dirichlet space in the unit ball.

Another natural integral operator is defined as follows:

$$I_g(f)(z) = \int_0^1 Rf(tz) g(tz) \frac{dt}{t}.$$ 

The importance of them comes from the fact that

$$T_g(f) + I_g(f) = M_g f - f(0) g(0)$$

where the multiplication operator is defined by

$$M_g(f)(z) = g(z) f(z), f \in H(B), z \in B.$$ 

Now we introduce some spaces first. Let $H^\infty$ denote the space of all bounded holomorphic functions on the unit ball, equipped with the norm $\|f\|_\infty = \sup_{z \in B} |f(z)|$.

The Bloch space $B$ is defined as the space of holomorphic functions such that

$$\|f\|_B = \sup \{(1 - |z|^2)|Rf(z)| : z \in B\} < \infty.$$ 

It is easy to check that if $f \in B$ then

$$|f(z)| \leq C \log \frac{2}{1 - |z|^2} \|f\|_B.$$ 

We define weighted Bloch space $B_{log}$ as the space of holomorphic functions $f \in H(B)$ such that

$$\|f\|_{B_{log}} = \sup \{(1 - |z|^2)|Rf(z)| \log \frac{2}{1 - |z|^2} : z \in B\} < \infty.$$ 

The Zygmund space $Z$ [18] in the unit ball consists of those functions whose first order partial derivatives are in the Bloch space.

It is well known that (Theorem 7.11 in [18]) $f \in Z$ if and only if $Rf \in B$, and $Z$ is a Banach space with the norm

$$\|f\| = |f(0)| + \|Rf\|_B.$$ 

The purpose of this paper is to discuss the boundedness and compactness of extended Cesàro operator $T_g$ and another integral operator $I_g$ on the Zygmund space in the unit ball.
In the following, we will use the symbol $C$ to denote a finite positive number which does not depend on variable $z$ and $f$.

In order to prove the main results, we will give some Lemmas first.

**Lemma 1.** Assume $f \in Z$, then we have

$$||f||_{\infty} \leq C||f||$$

**Proof.** Since $f \in Z$ implies that $Rf \in B$, it follows from (2) that

$$|f(z)| \leq C \log \frac{2}{1 - |z|^2} ||Rf||_B \leq C \log \frac{2}{1 - |z|^2} ||f||.$$  

Furthermore by $\lim_{|z| \to 1} (1 - |z|^2) \log \frac{2}{1 - |z|^2} = 0$ we have

$$|Rf(z)| \leq C (1 - |z|^2) \log \frac{2}{1 - |z|^2} ||f|| < \infty,$$

so $f \in B$. It follows from Theorem 2.2 in [18] that

$$Rf(z) = \int_B \frac{Rf(w) dv_\beta(w)}{(1 - z, w)^{n+1+\beta}}$$

where $\beta$ is a sufficiently large positive constant. Since $Rf(0) = 0$,

$$f(z) - f(0) = \int_0^1 \frac{Rf(tz)}{t} dt = \int_B Rf(w) L(z, w) dv_\beta(w)$$

where the kernel

$$L(z, w) = \int_0^1 \frac{1}{(1 - t < z, w >)^{n+1+\beta}} - 1 \frac{dt}{t}$$

satisfies

$$|L(z, w)| \leq \frac{C}{|1 - z, w |^{n+\beta}}$$

for all $z$ and $w$ in $B$. Note that $t^{1/2} \log \frac{2}{t} \leq \frac{2}{1 - \log 2}$ for all $t \in (0, 1]$, then

$$|f(z) - f(0)| = C \int_B \frac{(1 - |w|^2) |Rf(w)| dv_{\beta-1}(w)}{|1 - z, w |^{n+\beta}}$$

$$\leq C \int_B \frac{(1 - |w|^2) \log \frac{2}{1 - |w|^2} ||f|| dv_{\beta-1}(w)}{|1 - z, w |^{n+\beta}}$$

$$\leq C \int_B \frac{(1 - |w|^2)^{1 - 1/2} ||f|| dv_{\beta-1}(w)}{|1 - z, w |^{n+\beta}}$$

$$\leq C||f||.$$

The last inequality holds since $\int_B \frac{(1 - |w|^2)^t dv(w)}{|1 - z, w |^{n+1+\beta}}$ is bounded for $c < 0$. This completes the proof of Lemma 1.

By Lemma 1, Montel theorem and the definition of compact operator, the following lemma follows.
Lemma 2. Assume that \( g \in H(B) \). Then \( T_g \) (or \( I_g \) : \( \mathbb{Z} \rightarrow \mathbb{Z} \) is compact if and only if \( T_g \) (or \( I_g \)) is bounded and for any bounded sequence \( (f_k)_{k \in \mathbb{N}} \) in \( \mathbb{Z} \) which converges to zero uniformly on \( \mathbb{B} \) as \( k \to \infty \), \( ||T_g f_k|| \to 0 \) (or \( ||I_g f_k|| \to 0 \)) as \( k \to \infty \).

Lemma 3. If \( (f_k)_{k \in \mathbb{N}} \) is a bounded sequence in \( \mathbb{Z} \) which converges to zero uniformly on compact subsets of \( B \) as \( k \to \infty \), then \( \limsup_{k \to \infty, z \in B} |f_k(z)| = 0 \).

**Proof.** Assume \( ||f_k|| \leq M \). For any given \( \epsilon > 0 \), there exists \( 0 < \eta < 1 \) such that \( \frac{\sqrt{1-\eta}}{\eta} < \epsilon \). Note that \( t^{1/2} \log \frac{2}{t} \leq \frac{2}{e} \) (1-log 2) for all \( t \in (0,1] \), then when \( \eta < |z| < 1 \), it follows from (3) that

\[
|f_k(z) - f_k(\frac{\eta}{|z|}z)| = \left| \int_{\frac{\eta}{|z|}}^1 Rf_k(tz) \frac{dt}{t} \right| \leq C \int_{\frac{\eta}{|z|}}^1 \log \left( 1 - \frac{\eta^2}{|z|^2} \right) \frac{||f_k||}{t} \frac{dt}{t} \leq C \frac{|z|}{\eta} \int_{\frac{\eta}{|z|}}^1 \frac{||f_k||}{(1 - |tz|^2)^{1/2}} \frac{dt}{t} \leq CM \frac{(1-\eta)^{1/2}}{\eta} \leq C\epsilon.
\]

So we get \( \sup_{\eta < |z| < 1} |f_k(z)| \leq C\epsilon + \sup_{|w|=\eta} |f_k(w)| \). Thus, we have

\[
\limsup_{k \to \infty, z \in B} |f_k(z)| \leq \lim \left( \sup_{|z| \leq \eta} |f_k(z)| + \sup_{\eta < |z| < 1} |f_k(z)| \right) \leq C\epsilon.
\]

Now we finish the proof of this lemma.

Lemma 4. Let \( g \in H(B) \), then

\[
R[T_g f](z) = f(z) Rg(z)
\]

for any \( f \in H(B) \) and \( z \in B \).

**Proof.** Suppose the holomorphic function \( fRg \) has the Taylor expansion

\[
(fRg)(z) = \sum_{|\alpha| \geq 1} a_\alpha z^\alpha.
\]

Then we have

\[
R(T_g f)(z) = R \int_0^1 f(tz) R(tz) \frac{dt}{t} = R \int_0^1 \sum_{|\alpha| \geq 1} a_\alpha (tz)^\alpha \frac{dt}{t}
\]

\[
= R \left[ \sum_{|\alpha| \geq 1} a_\alpha z^\alpha \right] = \sum_{|\alpha| \geq 1} a_\alpha z^\alpha = (fRg)(z).
\]

3. Main Theorems

Theorem 1. Suppose \( g \in H(B) \), then the following conditions are all equivalent:

(a) \( T_g \) is bounded on \( \mathbb{Z} \);
(b) \( T_g \) is compact on \( \mathbb{Z} \);
(c) \( g \in \mathcal{Z} \).

**Proof.** \( b \implies a \) is obvious. For \( a \implies c \) we just take the test function given by \( f(z) \equiv 1 \).

We are going to prove \( c \implies b \). Now assume that \( g \in \mathcal{Z} \) and that \((f_k)_{k \in \mathbb{N}}\) is a sequence in \( \mathcal{Z} \) such that \( \sup_{k \in \mathbb{N}} ||f_k|| \leq M \) and that \( f_k \to 0 \) uniformly on \( \overline{B} \) as \( k \to \infty \). Now note that \( T_0g_k(0) = 0 \) and for every \( \epsilon > 0 \), there is a \( \delta \in (0, 1) \), such that

\[
(1 - |z|^2)(\ln \frac{2}{1 - |z|^2})^2 < \epsilon
\]

whenever \( \delta < |z| < 1 \). Let \( K = \{ z \in B : |z| \leq \delta \} \), it follows from Lemma 4 and (4) that

\[
||T_gf_k|| = \sup_{z \in B} (1 - |z|^2)|R(R(T_gf_k))|
\]

\[
= \sup_{z \in B} (1 - |z|^2)|Rf_k \cdot Rg + f_k \cdot R(Rg)|
\]

\[
\leq \sup_{z \in B} (1 - |z|^2)(|Rf_k \cdot Rg| + |f_k \cdot R(Rg)|)
\]

\[
\leq \sup_{z \in K} (1 - |z|^2)|Rf_k \cdot Rg| + \sup_{z \in B^c - K} (1 - |z|^2)(|Rf_k \cdot Rg|)
\]

\[
+ \sup_{z \in B}(1 - |z|^2)|f_k \cdot R(Rg)|
\]

\[
\leq C||g|| \sup_{z \in K} (1 - |z|^2)|Rf_k(z)||\log \frac{2}{1 - |z|^2}
\]

\[
+ C||f_k|| \cdot ||g|| \sup_{z \in B^c - K} (1 - |z|^2)(\log \frac{2}{1 - |z|^2})^2 + ||g|| \cdot \sup_{z \in B} |f_k(z)|.
\]

With the uniform convergence of \( f_k \) to 0 and the Cauchy estimate, the conclusion follows by letting \( k \to \infty \).

**Theorem 2.** Suppose \( g \in H(B) \), \( I_g : \mathcal{Z} \to \mathcal{Z} \). Then \( I_g \) is bounded if and only if \( g \in H^\infty \cap \mathcal{B}_{\log} \).

**Proof.** First we assume that \( g \in H^\infty \cap \mathcal{B}_{\log} \). Notice that \( I_gf(0) = 0 \) and \( R(I_gf) = fRg \), it follows from (4) that

\[
(1 - |z|^2)||RR(I_gf)(z)|| = (1 - |z|^2)||R(Rf(z) \cdot g(z))||
\]

\[
= (1 - |z|^2)||R(Rf(z)) \cdot g(z) + Rf(z) \cdot Rg(z)||
\]

\[
\leq ||Rf(z)||_B ||g||_\infty + |Rf(z)|(1 - |z|^2)||Rg(z)||
\]

\[
\leq C||f|| \cdot ||g||_\infty + C||f|| \cdot Rg(z)||\log \frac{2}{1 - |z|^2}
\]

\[
\leq C||f|| \cdot ||g||_\infty + C||f|| \cdot ||g||_{\mathcal{B}_{\log}}.
\]

The boundedness of \( I_g \) follows.

Conversely, assume that \( I_g \) is bounded, then there is a positive constant \( C \) such that

\[
||I_gf|| \leq C||f||
\]
for every \( f \in \mathcal{Z} \). Setting
\[
h_a(z) = (\log \frac{2}{1 - |a|^2})^{-1}(< z, a > - 1)[(1 + \log \frac{2}{1 - < z, a >})^2 + 1]
\]
for \( a \in B \) such that \( |a| \geq \sqrt{1 - 2/e} \), then
\[
Rh_a(z) = < z, a > (\log \frac{2}{1 - < z, a >})^2(\log \frac{2}{1 - |a|^2})^{-1}
\]
and
\[
RRh_a(z) = \{ < z, a > (\log \frac{2}{1 - < z, a >})^2 + 2 < z, a >^2 \log \frac{2}{1 - < z, a >} \} (\log \frac{2}{1 - |a|^2})^{-1}
\]
It is easy to check that \( M = \sup_{\sqrt{1 - 2/e} \leq |a| < 1} ||h_a|| < \infty \). Therefore, we have that
\[
\infty > \|I_g\| ||h_a|| \geq ||I_g h_a|| \geq \sup_{z \in B} (1 - |z|^2) |RRh_a(z) \cdot g(z) + Rh_a(z) \cdot Rg(z)|
\]
\[
\geq (1 - |a|^2)|\frac{2|a|^4}{1 - |a|^2} g(a) + |a|^2 \log \frac{2}{1 - |a|^2} g(a) + |a|^2 Rg(a) \log \frac{2}{1 - |a|^2}|
\]
\[
\geq -(2|a|^4 + |a|^2 \frac{2}{1 - \log 2} |g(a)| + |a|^2 (1 - |a|^2) |Rg(a)| \log \frac{2}{1 - |a|^2}
\]
(7) \[
\geq -(2 + \frac{2}{e} (1 - \log 2)) |a|^2 + |a|^2 (1 - |a|^2) |Rg(a)| \log \frac{2}{1 - |a|^2}.
\]
Next let
\[
f_a(z) = h_a(z) - \int_0^1 < z, a > \log \frac{2}{1 - t < z, a >} dt
\]
then
\[
Rf_a(z) = < z, a > \{ (\log \frac{2}{1 - < z, a >})^2(\log \frac{2}{1 - |a|^2})^{-1} - \log \frac{2}{1 - < z, a >} \}
\]
\[
RRf_a(z) = RRh_a(z) - < z, a > \log \frac{2}{1 - < z, a >} - < z, a >^2
\]
and consequently \( N = \sup_{\sqrt{1 - 2/e} \leq |a| < 1} ||f_a|| < \infty \). Note that \( Rf_a(a) = 0 \) and
\[
RRf_a(a) = \frac{|a|^4}{1 - |a|^2}, \text{ we have}
\]
\[
\infty > \|I_g\| \cdot ||f_a|| \geq ||I_g f_a|| \geq \sup_{z \in B} (1 - |z|^2) |RRf_a(z) \cdot g(z) + Rf_a(z) \cdot Rg(z)|
\]
(8) \[
\geq (1 - |a|^2)|RRf_a(a) g(a) + Rf_a(a) Rg(a)| = |a|^4 |g(a)|.
\]
From the maximum modulus theorem, we get \( g \in H^\infty \). So it follows from (7) and (8) that
\[
\sup_{\sqrt{1 - 2/e} \leq |a| < 1} (1 - |a|^2)|Rg(a)| \log \frac{2}{1 - |a|^2} < \infty.
\]
On the other hand, we have
\[ \sup_{|a| \leq \sqrt{1-2/e}} (1 - |a|^2)|Rg(a)| \log \frac{2}{1-|a|^2} \leq \frac{2}{e} \cdot (1 - \log 2) \max_{|a| = \sqrt{1-2/e}} |Rg(a)| \]
\[ \leq \frac{2}{\sqrt{1-2/e}} (1 - |a|^2)|Rg(a)| \log \frac{2}{1-|a|^2} < +\infty. \]

Combining (9) and (10), we finish the proof of Theorem 2.

**Corollary** The multiplication operator \( M_g : \mathcal{Z} \rightarrow \mathcal{Z} \) is bounded if and only if \( g \in \mathcal{Z} \).

**Proof.** If \( M_g \) is bounded on \( \mathcal{Z} \), then setting the test function \( f \equiv 1 \), we have \( M_g f = g \in \mathcal{Z} \).

Conversely, if \( g \in \mathcal{Z} \), from Lemma 1 and (5), it is easy to see that \( g \in H_\infty \cap B_{\log} \), so by Theorems 1 and 2, both \( T_g \) and \( I_g \) are bounded, it follows from (1) that \( M_g \) is also bounded.

**Theorem 3.** Suppose \( g \in H(B) \), \( I_g : \mathcal{Z} \rightarrow \mathcal{Z} \). Then \( I_g \) is compact if and only if \( g = 0 \).

**Proof.** The sufficiency is obvious. We just need to prove the necessity. Suppose that \( I_g \) is compact, for any given sequence \((z_k)_{k \in \mathbb{N}}\) in \( B \) such that \( |z_k| \rightarrow 1 \) as \( k \rightarrow \infty \), if we can show \( g(z_k) \rightarrow 0 \) as \( k \rightarrow \infty \), then by the maximum modulus theorem we have \( g \equiv 0 \). In fact, setting
\[
f_k(z) = h_{z_k}(z) - (\log \frac{2}{1-|z_k|})^{-2} \int_0^1 < z, z_k > (\log \frac{2}{1-t < z, z_k>})^2 dt.
\]
Using the same way as in Theorem 2, we can show \( \sup_{k \in \mathbb{N}} \|f_k\| \leq C \) and \( f_k \) converges to 0 uniformly on compact subsets of \( B \). Since \( I_g \) is compact, we have \( \|I_g f_k\| \rightarrow 0 \) as \( k \rightarrow \infty \). Note that \( Rf_k(z_k) = 0 \) and \( RRf_k(z_k) = \frac{|z_k|^4}{1-|z_k|^2} \), it follows that
\[
|z_k|^4 |g(z_k)| \leq \sup_{z \in B} (1 - |z|^2)|RRf_k(z) \cdot g(z) + Rf_k(z) \cdot Rg(z)|
\leq \sup_{z \in B} (1 - |z|^2)|RR(I_g f_k)(z)| \leq \|I_g f_k\| \rightarrow 0
\]
as \( k \rightarrow \infty \). This ends the proof of Theorem 3.

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