Brief paper

Fixed-time control under spatiotemporal and input constraints: A Quadratic Programming based approach

Kunal Garg a,*, Ehsan Arabi b, Dimitra Panagou a

a Department of Aerospace Engineering, University of Michigan, Ann Arbor, MI, 48109, USA
b Research and Advanced Engineering, Ford Motor Company, Dearborn, MI 48121, USA

ARTICLE INFO

Article history:
Received 6 June 2021
Received in revised form 29 November 2021
Accepted 3 March 2022
Available online xxxx

Keywords:
Fixed-time stability
Constrained control
Nonlinear systems
Optimization-based control

ABSTRACT

This paper presents a control synthesis framework for a general class of nonlinear, control-affine systems under spatiotemporal and input constraints. First, we study the problem of fixed-time convergence in the presence of input constraints. The relation between the domain of attraction for fixed-time stability with respect to input constraints and the required time of convergence is established. It is shown that increasing the control authority or the required time of convergence can expand the domain of attraction for fixed-time stability. Then, we consider the problem of finding a control input that confines the closed-loop system trajectories in a safe set and steers them to a goal set within a fixed time. To this end, we present a Quadratic Programming (QP) formulation to compute the corresponding control input. We use slack variables to guarantee the feasibility of the proposed QP under input constraints. Furthermore, when strict complementary slackness holds, we show that the solution of the QP is a continuous function of the system states and establish the uniqueness of closed-loop solutions to guarantee forward invariance using Nagumo’s theorem. To corroborate our proposed methods, we present two case studies, an example of an adaptive cruise control problem and a two-robot motion planning problem.

© 2022 Elsevier Ltd. All rights reserved.

1. Introduction

Driving the state of a dynamical system to a given desired point or a desired set in the presence of constraints is a problem of major practical importance. Constraints requiring the system trajectories to evolve in some safe set at all times while visiting some goal set(s) are common in safety-critical applications. Constraints on convergence to a goal set within a fixed time often appear in time-critical applications, e.g., when a task must be completed within a given time interval. Spatiotemporal specifications impose spatial (state) as well as temporal (time) constraints on the system trajectories. Safety in dynamical systems is typically realized as establishing that the desired set of safe states, or safe set, is forward invariant under the system dynamics. The control objective reduces to designing a control law such that the closed-loop system trajectories always remain in the safe set. The approach in Tee, Ge, and Tay (2009) utilizes Lyapunov-like barrier functions to guarantee that the system output always remains inside a given set. More recently, in Ames, Xu, Grizzle, and Tabuada (2017), conditions using Zeroing Control Barrier Functions (ZCBF) are presented to ensure forward invariance of the desired set. Various approaches have been developed to achieve convergence of system trajectories to desired sets or points while satisfying control input constraints. Methods such as Model Predictive Control (MPC) (Grancharova, Grettli, Ho, & Johansen, 2015; Saska, Kasl, & Přeucil, 2014) as well as Control Lyapunov Functions (CLF) (Li, Wang, Pierpaoli, & Egerstedt, 2018; Srinivasan, Coogan, & Egerstedt, 2018) have been studied extensively in the literature. Quadratic Programming (QP)-based approaches have gained popularity for control synthesis, see for instance Ames et al. (2017), Li et al. (2018), Rauscher, Kimmel, and Hirche (2016), Shaw Cortez, Oetomo, Manzie, and Choong (2019), Srinivasan et al. (2018), as these methods are suitable for real-time implementation.

Concurrent forward invariance of a safe set and convergence to a goal set can be achieved via a combination of CLFs and Control Barrier Functions (CBFs), see, e.g., Ames et al. (2017) and Romdlony and Jayawardhana (2016). However, the underlying control synthesis problem becomes challenging in the presence of input constraints, such as actuator saturation, since the latter may
affect the region of safety and convergence of the system trajectories. Most of the contributions above address control design that achieves safety and convergence to the desired goal set (or point) but without explicitly considering control input constraints. Such constraints are considered in Ames et al. (2017), where performance and safety objectives are represented using CLFs and CBFs, respectively, along with control input constraints in a QP. Furthermore, most of the work mentioned above, except (Li et al., 2018; Srinivasan et al., 2018), deals with asymptotic or exponential convergence of the system trajectories to the desired goal point or goal sets. In contrast, Fixed-Time Stability (FxTS) is a concept that guarantees convergence within a fixed amount of time (Polyakov, 2012). For specifications involving temporal constraints and time-critical systems, the theory of finite- or fixed-time stability can be leveraged in the control design to guarantee that such specifications are met. It has also been shown that a faster rate of convergence generally implies that the closed-loop system has better disturbance rejection properties, which further motivates the study of finite- and fixed-time stable systems. The authors of Srinivasan et al. (2018) formulate a QP for finite-time convergence to the desired set, however, without considering input constraints. This limitation is removed in Li et al. (2018), where the authors consider a QP formulation incorporating input, safety, and convergence constraints. The authors in Lindemann and Dimarogonas (2019) use CBFS in a QP formulation to encode Signal-Temporal Logic (STL) specifications that impose reaching a goal set within a finite time. In this paper, we study the problem of reaching a given goal set $S_G$ within a fixed time $T_{sd}$ while remaining in a given safe set $S_S$ at all times, for a general class of nonlinear control-affine systems with input constraints. In the preliminary conference version (Garg & Panagou, 2019), a QP formulation is proposed to compute the control input for fixed-time convergence under input and safety constraints, yet without any guarantees on the feasibility of the proposed method. Per its definition, FxTS of an equilibrium point from arbitrary initial conditions presumes unbounded control authority. To address the problem of FxTS in the presence of input constraints, new Lyapunov conditions from Garg and Panagou (2021a) are utilized. When used in a QP, the new Lyapunov conditions introduce a slack term, which results in feasibility guarantees in the presence of input constraints. The contributions of the paper are as follows:

- New FxTS conditions are utilized in a QP, and Karush-Kuhn-Tucker (KKT) conditions are used to compute the closed-form expression for the optimal value of the slack term for the case when the control input is saturated. Then, the relation between the Fixed-Time Domain of Attraction (FxTDoA), the input bounds, and the fixed time of convergence is established for a 1-D control-affine system.
- A novel QP formulation that utilizes Fixed-Time (FxT) CLFs and CBFs is proposed to synthesize controllers for nonlinear, control-affine systems, so that forward invariance of a safe set and FxT convergence of the system trajectories to a goal set is guaranteed under input constraints.
- Conditions for continuity of the control input defined as the optimal solution of the QP are studied under milder conditions compared to prior work, and it is shown that the closed-loop solutions are uniquely determined so that forward invariance of the safe set can be established.

The QP-based approaches in the prior literature, e.g., Ames et al. (2017), Garg and Panagou (2019), Li et al. (2018), Lindemann and Dimarogonas (2019), Nguyen and Sreenath (2016), Srinivasan et al. (2018) and Wang, Ames, and Egerstedt (2016, 2017), do not provide feasibility guarantees for the underlying QP in the presence of input constraints. However, without the feasibility of the QP, it is not guaranteed that a control input can always be synthesized. The resulting input might not be realizable on a real-world platform without consideration of input bounds. In comparison to these prior studies, we consider control input constraints in addition to the safety and convergence requirements and guarantee the feasibility of the proposed QP. The proposed approach further advances the results in Li et al. (2018) and Srinivasan et al. (2018) in terms of the achieved time of convergence. In contrast to Garg and Panagou (2021b), where the continuity of the solution of the QP is assumed, in this paper, we provide a detailed analysis on continuity of the solution of the underlying QP. Furthermore, we generalize the results of Ames et al. (2017) and Morris, Powell, and Ames (2015), where the regularity properties of the solution of the QP are discussed in the absence of input constraints. We show continuity of the solution of the proposed QP under input constraints and milder regularity assumptions on the CLF, CBF, and the system dynamics, compared to the work mentioned above.

2. Problem formulation and preliminaries

We use $\mathbb{R}$ to denote the set of real numbers and $\mathbb{R}_+$ to denote the set of non-negative real numbers. We use $\| \cdot \|$ to denote the $p$-norm, and $\| \cdot \|$ to denote the Euclidean norm. For a set $S \subset \mathbb{R}^n$, we use $|x|_S := \inf_{y \in S} \|x - y\|$ to denote the distance of the point $x \in \mathbb{R}^n \setminus S$ from the set $S$. We write $\delta S$ for the boundary of a closed set $S \subset \mathbb{R}^n$, $\text{int}(S) := S \setminus \delta S$ for its interior. We use $C^k$ to denote the set of $k$ times continuously differentiable functions. The Lie derivative of a $C^1$ function $V : \mathbb{R}^n \to \mathbb{R}$ along a vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ at a point $x \in \mathbb{R}^n$ is denoted as $L_f V (x) := \frac{d}{dt} V (tx)\bigg|_{t=0}$. A continuous function $\alpha : \mathbb{R}^+ \to \mathbb{R}$ is compact, and the sets $S_G$ is guaranteed under input constraints.

Consider the nonlinear, control-affine system

\[
\dot{x} = f(x) + g(x)u, \quad x(0) = x_0,
\]

where $x \in \mathbb{R}^n$ is the state vector, $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^{n \times m}$ are system vector fields, continuous in their arguments, and $u \in U \subset \mathbb{R}^m$ is the control input vector where $U$ is the input constraint set. In addition, consider a safe set $S_S := \{ x \mid h_S(x) \leq 0 \}$ to be rendered forward invariant under the closed-loop dynamics of (1), and a goal set $S_G := \{ x \mid h_G(x) \leq 0 \}$ to be reached by the fixed time of convergence $T_{sd} > 0$, where $h_S, h_G : \mathbb{R}^n \to \mathbb{R}$ satisfy the following assumption.

Assumption 1. The functions $h_S, h_G \in C^1, S_G \cap S_S \neq \emptyset$, the set $S_G$ is compact, and the sets $S_S$ and $S_G$ have non-empty interiors. There exists a class-$K_{\infty}$ function $\alpha_C$ such that $h_G(x) \geq \alpha_C(x|S_G)$, for all $x \notin S_G$.

Note that the boundary and the interior of the set $S_S$ (and similarly, of the set $S_G$) are given as $\partial S_S := \{ x \mid h_S(x) = 0 \}$ and $\text{int}(S_G) := \{ x \mid h_G(x) < 0 \}$, respectively. Next, we define the notion of fixed-time domain of attraction for a compact set $S \subset \mathbb{R}^n$.

Definition 1 (FxT-DoA). For a compact set $S \subset \mathbb{R}^n$, the set $D_S \subset \mathbb{R}^n$, satisfying $S \subset D_S$, is a Fixed-Time Domain of Attraction (FxT-DoA) with time $T > 0$ for the closed-loop system (1) under $u$, if

(i) for all $x(0) \in D_S, x(t) \in D_S$ for all $t \in [0, T]$, and
(ii) there exists $T_0 \in [0, T]$ such that $\lim_{t \to T_0} x(t) \in S$. 

K. Garg, E. Arabi and D. Panagou

Automatica 141 (2022) 110314
In plain words, an FxT-DoA for a set $S \subset \mathbb{R}^n$ is a set $D_S \supset S$ such that it is forward-invariant and starting from any point within the set $D_S$, the system trajectories reach the set $S$ within a fixed-time $T$. We can now state the main problem considered in this paper.

**Problem 1.** Design a control input $u \in \mathcal{U} := \{v \in \mathbb{R}^m \mid A_n v \leq b_n\}$ and compute $D \subset \mathbb{R}^n$, so that for all $x_0 \in D \subseteq S_S$, the closed-loop trajectories $x(t)$ of (1) satisfy $x(t) \in S_S$ for all $t \geq 0$, and $x(T_{sd}) \in S_S$, where $T_{sd} > 0$ is a user-defined fixed time and $D$ is a FxT-DoA for the set $S_S$.

Input constraints of the form $\mathcal{U} = \{v \in \mathbb{R}^m \mid A_n v \leq b_n\}$ are very commonly considered in the literature (Ames et al., 2017; Shaw Cortez et al., 2019).

Problem 1 requires that the closed-loop system trajectories of (1) stay in the set $S_S$ at all times, i.e., the set $S_S$ is forward-invariant. A set $S \subset \mathbb{R}^n$ is forward invariant for the system (1) if $x_0 \in S$ implies that $x(t) \in S$ for all $t \geq 0$. Nagumo’s theorem (Blanchini, 1999) is commonly used for guaranteeing forward invariance of the set $S_S$ for the control system (1). The interested reader is referred to Blanchini (1999, Section 3.1) for a detailed discussion on forward invariance of sets. We make the following assumption to guarantee that the safe set $S_S$ can be rendered forward invariant for (1).

**Assumption 2.** For each $x \in \partial S_S$, there exists a control input $u(x) \in \mathcal{U}$ such that $L_fh_S(x) + L_hg_S(x)u(x) \leq 0$.

Similar assumptions have been used in literature, either explicitly (e.g., Romdlony & Jayawardhana, 2016) or implicitly (e.g., Ames et al., 2017). In this work, we use the following notion of ZCBFs, introduced in Ames et al. (2017), to ensure forward invariance of the safe set $S_S$. One special case of the ZCBF in Ames et al. (2017) is $\inf_{u \in \mathcal{U}} L_fh_S(x) + L_hg_S(x)u \leq -\rho h_S(x)$, with $\rho > 0$. We will use this special form to guarantee forward invariance of the safe set $S_S$.

The authors in Polyakov (2012) define the origin to be an FxT equilibrium of (1) if it is Lyapunov stable and $\lim_{t \to \infty} x(t) = 0$ where the time of convergence $T = T(x(0))$ is uniformly bounded for all $x(0), i.e., \sup_{x(0) \in \mathbb{R}^n} T(x(0)) < \infty$. The following sufficient conditions for FxTS of the origin are adapted from Parsegov, Polyakov, and Scherbakov (2012).

**Lemma 1.** Let $V : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable, positive definite, radially unbounded function, satisfying $\inf_{u \in \mathcal{U}} L_fh_S(x) + L_hg_S(x)u \leq -\alpha_1 V(x)^{\gamma_1} - \alpha_2 V(x)^{\gamma_2}$, for all $x \in \mathbb{R}^n \setminus \{0\}$, with $\alpha_1, \alpha_2 > 0$, $\gamma_1 = 1 + \frac{1}{\mu}$ and $\gamma_2 = 1 - \frac{1}{\mu}$ for some $\mu > 1$. Then, the origin $0$ is an FxT of $S_S$ with $T$ that satisfies $T \leq \frac{\mu \pi}{2 \sqrt{\alpha_1 \alpha_2}}$.

3. **Fixed-time stability under input constraints**

Consider, for the sake of illustration, a 1-dimensional control-affine system

$$\dot{x} = f(x) + g(x)u,$$

where $f, g : \mathbb{R} \to \mathbb{R}$ are continuous functions. Suppose that the control objective is to drive the closed-loop trajectories of (3) to a set $S_0 := \{x \mid V(x) \leq 0\}$ within a user-defined fixed time $T_{sd}$, where $V : \mathbb{R} \to \mathbb{R}$ is continuously differentiable, and positive definite with respect to the set $S_0$. Additionally, consider the input constraints $u_m \leq u \leq u_M$ where $u_m < u_M$. To this end, following the work in Ames et al. (2017) and Nguyen and Sreenath (2016) and using the FxTS conditions from Lemma 1, a QP can be formulated as follows:

$$\min_{u} \frac{1}{2} u^2$$

s.t.

$$\begin{bmatrix} 1 & -1 \end{bmatrix} u \leq \begin{bmatrix} u_M \\ -u_m \end{bmatrix}.$$ (4a)

$$L_f V(x) + L_g V(x)u \leq -\alpha_1 V(x)^{\gamma_1} - \alpha_2 V(x)^{\gamma_2},$$ (4c)

for all $x \notin S_0$, where $\alpha_1, \alpha_2, \gamma_1, \gamma_2$ are chosen as $\alpha_1 = \alpha_2 = \frac{\mu \pi}{2 \sqrt{\alpha_1 \alpha_2}}$, $\gamma_1 = 1 + \frac{1}{\mu}$ and $\gamma_2 = 1 - \frac{1}{\mu}$ with $\mu > 1$. The issue with the QP in (4) is that it might not be feasible for all $x \in \mathbb{R}^n \setminus S_0$ due to the presence of the input constraints. To address the issue of infeasibility of a QP under multiple constraints, the authors in Ames et al. (2017) introduce a slack variable in the CLF constraint. Inspired from this, the new FxTS conditions are presented next.

3.1. **New FxTS Lyapunov conditions**

**Lemma 2** (Garg & Panagou, 2021a), Let $V : \mathbb{R}^n \to \mathbb{R}$ be $c^1$, positive definite, radially unbounded function, satisfying

$$\inf_{u \in \mathcal{U}} L_fh_S(x) + L_hg_S(x)u \leq -\alpha_1 V(x)^{\gamma_1} - \alpha_2 V(x)^{\gamma_2},$$ (5)

for all $x \in \mathbb{R}^n \setminus \{0\}$, with $\alpha_1, \alpha_2 > 0$, $\gamma_1 = 1 + \frac{1}{\mu}$ and $\gamma_2 = 1 - \frac{1}{\mu}$ for some $\mu > 1$. Then, $D \subset \mathbb{R}^n$ is a FxT-DoA of the origin of (1) with time $T > 0$, where

$$D = \left\{ x \mid V(x) \leq k_1 \left( \frac{\mu \pi}{4 \alpha_1 \alpha_2 \sqrt{2}} \right)^\mu \right\},$$ (6)

$$T \leq \begin{cases} \frac{\mu \pi}{4 \alpha_1 \alpha_2 \sqrt{2}} \gamma_1 & \text{for } 0 \leq r < 1, \\ \frac{\mu \pi}{4 \alpha_1 \alpha_2 \sqrt{2}} \gamma_2 & \text{for } r \geq 1, \end{cases}$$ (7)

where $r = \frac{\gamma_1}{\gamma_2}$. Inspired by Lopez-Ramirez, Efimov, Polyakov, and Perruquetti (2019), we define a class of CLF for the system (1), which is used to encode the convergence of the system trajectories to a compact set $S \subset \mathbb{R}^n$ within a user-defined, fixed time $T_{sd}$.

**Definition 2** (FxTS CLF-S). A continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}$ is called FxTS CLF-S for (1) with parameters $\alpha_1, \alpha_2 > 0$, $\gamma_1 = 1 + \frac{1}{\mu}$, $\gamma_2 = 1 - \frac{1}{\mu}$ with $\mu > 1$, if $V$ is positive definite w.r.t. set $S$ and the following holds:

$$\inf_{u \in \mathcal{U}} L_fh_S(x) + L_hg_S(x)u \leq -\alpha_1 V(x)^{\gamma_1} - \alpha_2 V(x)^{\gamma_2},$$ (2)

for all $x \in \mathbb{R}^n \setminus S$, where the time of convergence $T$ satisfies $T \leq \frac{\mu \pi}{2 \sqrt{\alpha_1 \alpha_2}} \leq T_{sd}$.

In comparison to Lemma 1, Lemma 2 allows an additional (possibly, positive) term $\delta_i V$ in the upper bound of the time derivative of the Lyapunov function. It is worth noting that the time of convergence is directly proportional to the parameter $\mu$. In the limit when $\mu = 1$, the time of convergence is minimized. However, at the same time, the right-hand side of (5) becomes discontinuous for $\mu = 1$ (since $\gamma_2 = 0$ when $\mu = 1$) which might lead to chattering due to a sliding-mode behavior. On the other hand, in the limit $\mu \to \infty$, we recover the conditions for the exponential CLF, and thus, the time of convergence is infinity.
Thus, the parameter $\mu$ can be chosen based on the required time of convergence for a specific problem. The main idea is to use (5) in place of the constraint (9c), with the parameters $\alpha_1, \alpha_2, \mu$ chosen such that $R_{\text{tol}} = T_{\text{tol}}$, and with $\delta_1$ being a free, slack variable so that feasibility of the QP can be guaranteed. Then, the value of $\delta_1$ would dictate the FxT-DoA $D$. To see how the condition (5) can be used to guarantee FxTS in the presence of control input constraints, a new QP can be formulated as follows:

$$\min_{u, \delta_1} \frac{1}{2} u^T \mathbf{M}_x u + \frac{1}{2} \delta_1^2 + c \delta_1$$

s.t. $$\begin{bmatrix} 1 & 0 \end{bmatrix} u \leq \begin{bmatrix} u_m \end{bmatrix},$$

$$L_f V(x) + L_g V(x) u \leq \delta_1 V(x) - \alpha_1 V(x)^{\gamma_1} - \alpha_2 V(x)^{\gamma_2},$$

(8)

(8b)

(8c)

for $x \notin S_V$, where $c > 0$, and $\alpha_1, \alpha_2, \gamma_1, \gamma_2$ are chosen similarly as in (4). Note that when $\delta_1$ takes negative values, then out of (7) the bound on the time of convergence satisfies $T \leq \frac{\delta_1}{\alpha_1 \delta_1} = T_{\text{tol}}$, and therefore, convergence within the user-defined time $T_{\text{tol}}$ can be achieved. Thus, the linear term $c \delta_1$ is introduced in the cost function with $c > 0$ in order to penalize non-positive values of $\delta_1$. Here, the term $\delta_1 V(x)$ in (8c) is a slack term, allowing for satisfaction of the constraint (8c) in the presence of input constraints (8b) as shown below.

**Lemma 3.** For each $x \notin S_V$, there exist $u(x) \in \mathbb{R}, \delta_1(x) \in \mathbb{R}$ satisfying (8b)–(8c), i.e., the QP (8) is feasible for all $x \notin S_V$.

The proof is straightforward and is omitted here for the sake of brevity. Note that in prior work, e.g., Ames et al. (2017) and Nguyen and Sreenath (2016), the slack term is used as $L_f V(x) + L_g V(x) u \leq \delta - CV(x)$, for some $c > 0$. While this condition helps guarantee feasibility of the underlying QP, it does not guarantee that the function $V$ reaches its zero sub-level set when $\delta > 0$. Per Lemma 2, it holds that the system trajectories reach the zero sub-level set of the function $V$ even when $\delta > 0$. This is a unique feature of the new FxTS condition in Lemma 2.

### 3.2. Relation of FxT-DoA with input constraints and time of convergence

As mentioned above, the slack term $\delta_1$ is used to guarantee the feasibility of the underlying QP. Now, the relation of this slack term with FxT-DoA, input constraints, and the fixed time of convergence is explored. In the particular case when $\alpha_1 = \alpha_2 = \alpha$ for some $\alpha > 0$, let us examine how the FxT-DoA $D$ in (6) is affected by the ratio $r^* = \frac{\delta_1}{\alpha_1 \delta_1}$, where $(u^*(x), \delta^*(x))$ is the optimal solution of the QP (8). Note that an estimate of the FxT-DoA $D$ of the set $S_V$ is given as the set of points where the right-hand side of the constraint (8c) takes negative values. Define $r_{\text{inf}} = \sup_{x \in S_V} r^*(x)$. For $r_{\text{inf}} < 1$, it holds that $D = \mathbb{R}^m$, which is the largest possible FxT-DoA. For $r_{\text{inf}} \geq 1$, it holds that $D = \left\{ x \mid V(x) \leq \inf_{x \in S_V} \mu_1 \left(r^*(x) - \sqrt{r^*(x)^2 - 1}\right) \right\}$. It can be readily verified that $\tilde{r} = \sqrt{r^2 - 1}$ is a monotonically decreasing function for $r > 1$ and therefore, it holds that $D = \left\{ x \mid V(x) \leq \inf_{x \in S_V} \mu_1 \left(r_{\text{inf}} - \sqrt{r_{\text{inf}}^2 - 1}\right) \right\}$. Fig. 1 plots the FxT-DoA $D$ for $V(x) = \frac{1}{2} \|x\|^2$. It can be concluded that the domain $D$ shrinks as $r_{\text{inf}}$ increases, which is also demonstrated in Fig. 1. Thus, the smaller the value of $r_{\text{inf}}$, the larger the FxT-DoA $D$. The following Lemma establishes the closed-form expression for the slack variable $\delta_1^*$ in the particular case when the control input is saturated for the QP (8).

**Lemma 4.** Consider the QP (8) and assume that $u_m \geq 0$. Then, for $x \in S_M$ where $S_M := \{ x \mid a(x) \mathbf{L}_a V(x) + u_m V(x)^2 < 0, a(x) > 0 \}$, where $a(x) := L_f V(x) + L_g V(x) u_m + c V(x) + \alpha_1 V(x)^{\gamma_1} + \alpha_2 V(x)^{\gamma_2}$, the optimal values of $u, \delta_1$ are given as $u(x) = u_m, \delta_1^*(x) = \frac{L_f V(x) + L_g V(x) u_m}{\alpha_1 V(x)^{\gamma_1} + \alpha_2 V(x)^{\gamma_2}}$.

The proof is provided in Appendix A. Recall that the QP (8) is defined for $x \notin S_V$. Note that for $x \in S_M \setminus S_V$, $L_f V(x) < 0$ (per definition of the set $S_M$) and $V(x) > 0$ (since $V$ is positive definite w.r.t. the set $S_V$). Define $r^*(x) = \frac{\delta_1^*(x)}{\alpha_1 \delta_1^*(x)}$, so that $r^*(x) = \frac{L_f V(x) + L_g V(x) u_m}{\alpha_1 V(x)^{\gamma_1} + \alpha_2 V(x)^{\gamma_2}} - \frac{1}{2} \frac{x^T H x + f^T z}{\alpha_1 \delta_1^*(x)} - \frac{1}{2} \frac{x^T H z + f^T z}{\alpha_1 \delta_1^*(x)}$.

In this section, a QP-based feedback synthesis approach is presented to address Problem 1. Define $z = [u^T \delta_1^* \delta_1]^T \in \mathbb{R}^{m+2}$, and consider the QP:

$$\min_{z \in \mathbb{R}^{m+2}} \frac{1}{2} z^T H z + f^T z$$

s.t. $A_n v \leq b_n,

L_f h_c(x) + L_g h_c(x) v \leq \delta_1 h_c(x) - \alpha_1 \max(0, h_c(x))^{\gamma_1}$

$$- \alpha_2 \max(0, h_c(x))^{\gamma_2}$$

$$L_f h_2(x) + L_g h_2(x) v \leq - \delta_2 h_2(x),$$

where $H = \text{diag}[w_1, \ldots, w_m, w_1, w_1]$ is a diagonal matrix consisting of positive weights $w_1, w_1 > 0, f = [0^T \ 0^T \ 0^T]$ with $q_1 > 0$ and $b_n \in \mathbb{R}^m$ a column vector consisting of zeros. The parameters $\alpha_1, \alpha_2, \gamma_1, \gamma_2$ are fixed, and are chosen as $\alpha_1 = \alpha_2 = \frac{\mu}{2 \mu_0}, \gamma_1 = 1 + \frac{1}{\mu}$, and $\gamma_2 = 1 - \frac{1}{\mu}$ with $\mu > 1$. The linear term $f^T z = q_2 \delta_1$ in the objective function of (9) penalizes the positive values of $\delta_1$ (see Theorem 2 for details on why $\delta_1$ being non-positive
could be useful). Constraint (9b) is imposed to ensure that the control input satisfies the control input constraints. Constraint (9c) is imposed for convergence of the closed-loop trajectories of (1) to the set $S_C$, and the constraint (9d) is imposed for forward invariance of the set $S_C$.

The slack terms corresponding to $\delta_1$, $\delta_2$ allow the upper bounds of the time derivatives of $h_1(x)$ and $h_2(x)$, respectively, to have a positive term for $x$ such that $h_1(x)<0$ and $h_2(x)>0$. This ensures the feasibility of the QP (9), as shown below.

**Lemma 5.** Under Assumptions 1–2, for each $x \in S_C \setminus S_C$, there exists $v(x) \in \mathcal{U}$, $\delta_1(x), \delta_2(x) \in \mathbb{R}$ satisfying (9b)–(9d), i.e., the QP (9) is feasible for all $x \in S_C \setminus S_C$.

**Proof.** Since $x \notin S_C$, it holds that $h_C(x) > 0$. Consider the following two cases separately: $h_1(x) = 0$ and $h_2(x) < 0$. First, let $x \in \text{int}(S_C)$ so that $h_1(x) < 0$. Since $\mathcal{U}$ is non-empty, there exists $v \equiv \bar{v} \in \mathcal{U}$ such that (9b) is satisfied. Choose $\delta_1(x) := \frac{\bar{v} h_C(x)}{\gamma(h_C(x))^2 + \lambda_C}$, so that (9d) is satisfied with equality. Also, for $x \in \text{int}(S_C) \setminus S_C$, it holds that $h_C(x) > 0$. Define $\delta_1(x) := \frac{h_1(x)}{\lambda_C + h_2(x)}$, so that (9c) holds with equality. Thus, for the case when $h_1(x) < 0$, there exists $(\bar{v}, \bar{\delta}_1, \bar{\delta}_2)$ such that (9b)–(9d) hold.

Next, let $x \in \partial S_C$ so that $h_1(x) = 0$. Per Assumption 2, it holds that there exists $v \equiv \bar{v} \in \mathcal{U}$ such that (9d) holds, since $h_1(x) = 0$. Any value of $\delta_1$ is feasible, and hence, one can choose $\delta_2 = 0$. Hence, the choice of $(\bar{v}, \delta_1, \delta_2) = (\bar{v}, \bar{\delta}_1, 0)$ satisfies (9b)–(9d). Thus, the QP (9) is always feasible.

One of the main novelties of the QP (9) is the way the slack variables are introduced in the FxT-CLF and the ZCBF constraints. Not only do these slack variables guarantee that the QP remains feasible under input constraints, but they also do not jeopardize forward-invariance of the set $S_C$ or convergence to the goal set $S_C$.

### 4.1. Continuity of the solution of the QP

Guaranteeing forward invariance of the safe set $S_C$ using Nagumo's theorem requires the uniqueness of the system solutions. Traditionally, Lipschitz continuity of the right-hand side of (1) is utilized in order to guarantee existence and uniqueness of the solutions of (1), see, e.g., Ames et al. (2017), Lindemann and Dimarogonas (2019) and Xu, Tabuada, Grizzle, and Ames (2015). When the right-hand side of (1) is only continuous, existence and uniqueness of the solutions can be established using the results in Agarwal and Lakshmikantham (1993, Section 3.15-3.18) (see Lemma 6). To this end, first, it is shown that the control input $u$ as a solution of the QP (9) is continuous in its arguments. Define $A: \mathbb{R}^n \rightarrow \mathbb{R}^{2(m-2) \times 2(m-2)}$ and $b: \mathbb{R}^n \rightarrow \mathbb{R}^{2(m-2)}$ as

$$A(x) := \begin{bmatrix} A_u & 0_{2m} \\ L_1 h_C(x) & -L_2 h_0(x) \end{bmatrix}, \quad b(x) := \begin{bmatrix} b_u \\ 0_{2m} \end{bmatrix}$$

where

$$b_2(x) := \begin{bmatrix} 0 \\ L_1 h_0(x) \end{bmatrix}$$

and $L_1 \in \mathbb{R}^{2(m-2) \times 2m}$ is a column vector consisting of zeros. Also, define the functions $G_i(x,z) := A_i(x)z - b_i(x)$ where $A_i \in \mathbb{R}^{2(m-2) \times 2}$ is the ith row of the matrix $A$, and $b_i \in \mathbb{R}$ the ith element of $b$, so that the constraints (9b)–(9d) can be written as $G_i(x,z) \leq 0$ for $i = 1, \ldots, 2m + 1$. In (9a), $z^* \in \mathbb{R}^{2m+1}$ denotes the optimal solution of (9), and the corresponding optimal Lagrange multiplier, respectively. The following assumption is made to prove the main results of this section.

**Assumption 3.** The strict complementary slackness holds for (9) for all $x \in \text{int}(S_C) \setminus S_C$, i.e., for each $i \in \{1, 2, \ldots, 2m + 2\}$, it holds that either $\lambda_i^* > 0$ or $G_i(x, z^*) < 0$ for all $x \in \text{int}(S_C) \setminus S_C$.

Complementary slackness, i.e., $\lambda_i^* G_i(x, z^*) = 0$, for all $i = 1, \ldots, 2m + 2$, is a both necessary and sufficient condition for optimality of the solution for QPs (Boyd & Vandenberghe, 2004, Chapter 5). Note that this condition permits that for some $i$, both $\lambda_i^* = 0$ and $G_i(x, z^*) = 0$. Strict complementary slackness rules out this possibility, and requires that for each $i$, either $\lambda_i^*$ or $G_i(x, z^*)$ is non-zero.

**Theorem 1.** Under Assumptions 1 and 3, the solution $z^* \in \mathbb{R}^{m+2}$ of (9) is continuous on $\text{int}(S_C) \setminus S_C$.

The proof is provided in Appendix B. Note that the above result guarantees that the control input defined as $u = v^*(x)$ is continuous on $\text{int}(S_C) \setminus S_C$. The authors in Ames et al. (2017) assume that the functions $f$, $g$ and the Lie derivatives $L_i h_C, L_i h_0$ are locally Lipschitz continuous to show Lipschitz continuity of the solution of QP in the absence of control input constraints. Note that in the presented formulation, the only requirement is that the functions $f$, $g$ are continuous, and $h_C$, $h_0$ is continuously differentiable in $x$, which is a relaxation of the prior assumptions. Next, it is shown that closed-loop solution of (1) under $u = v^*(x)$ exists and is unique.

**Lemma 6.** Let Assumptions 1–3 hold. The solution of (9) be given as $z^* = \begin{bmatrix} x^T & \delta_1^T & \delta_2^T \end{bmatrix}$ and define $\delta_i := \max(0, h_i(x))^{1/2} \alpha_i$. Then, there exists a neighborhood $D$ of the set $S_C$ such that the closed-loop trajectory under $u = v^*(x)$ exists and is unique for all $t \geq 0$ and for all $x(0) \in D$. Furthermore, if $\delta_i \leq 1$, then the result holds with $D = S_C$.

**Proof.** The proof is based on Agarwal and Lakshmikantham (1993, Theorem 3.18.1). Using Agarwal and Lakshmikantham (1993, Theorem 3.15.1) and choosing a Lyapunov candidate $v = \frac{1}{2} |y|^2$, it can be shown that $\dot{y} = 0$ is the unique solution of $\dot{y} = 0$. Theorem 1 guarantees that the solution of the QP (9) is continuous, which implies continuity of the closed-loop system dynamics (1) when $u = v^*(x)$. Note that $h_C(x) = 0$ for $x \in \partial S_C$ and $h_0(x) > 0$ for $x \notin S_C$, i.e., the function $h_0$ is positive definite with respect to the set $S_C$. Define $\phi(y) := \alpha_1 \text{sign}(y) |y|^2 - \alpha_2 \text{sign}(y) |y|^2$. From Lemma 6, it holds that there exists a neighborhood $D_\tau \subset \mathbb{R}$ of the origin such that for all $y \in D_\tau$, $\phi(y) \leq 0$. Thus, there exists a function $g$ defined as $g(r, V) = 0$ that satisfies condition (i) of Agarwal and Lakshmikantham (1993, Theorem 3.18.1), the closed-loop dynamics of (1) satisfies the condition (ii), and there exists a function $V$ defined as $V(x) = h_C(x)$ that satisfies the condition (iii). Thus, using Agarwal and Lakshmikantham (1993, Theorem 3.18.1), there exists $\tau > 0$ such that the solution of the closed-loop system (1) exists and is unique for all $x(0) \in D = \{x \mid V(x) \in D_\tau\}$, and $t \in [0, \tau)$. Since the closed-loop solution $x(t)$ is bounded in the compact set $D$, the solution is complete (see Aubin & Cellina, 2012, Ch.2, Theorem 1), and thus, $\tau = \infty$.

Finally, when $\delta_1 \leq 1$, it holds that the $D_\tau = \mathbb{R}$, and thus, the result holds with $D = S_C$.

### 4.2. Safety and fixed-time convergence

Finally, it is shown that under some conditions, the solution of (9) solves Problem 1.

**Theorem 2.** Let Assumptions 1–3 hold. If the solution of (9), given as $z^* = \begin{bmatrix} x^T & \delta_1^T & \delta_2^T \end{bmatrix}$, satisfies $\delta_i(x) \leq 0$ for all $x \in \text{int}(S_C)$, then $u = v^*(x)$ solves Problem 1 for all $x(0) \in \text{int}(S_C)$, i.e., $D = \text{int}(S_C)$.

**Proof.** First, the convergence of the closed-loop trajectories $x(t)$ to the set $S_C$ within the user-defined time $T_{ad}$ is shown. Since
\[ \delta_i(x) \leq 0, \text{ per Lemma 2, it holds that the closed-loop trajectories of (1) with } u := v^*(x) \text{ reach the set } S_G \text{ within fixed time } T \leq \frac{\mu \kappa}{2v_0(x)^2}, \text{ i.e., within the user-defined time } T_{ud} \text{ for all } x(0) \in \text{int}(S_G). \]

Next, it is shown that the closed-loop trajectories of (1) satisfy \( x(t) \in \text{int}(S_G) \) for all \( t \leq T_{ud} \) under \( u := v^*(x) \). From Lemma 6, it holds that the closed-loop solution of (1) exists and is unique under \( u = v^* \) for all \( 0 \leq t \leq T_{ud} \) and for all \( x(0) \in \text{int}(S_G) \). Using the similar arguments as in Ames et al. (2017, Theorem 1), it can be shown that the set \( \text{int}(S_G) \) is forward-invariant. Therefore, the control input \( u := v^*(x) \) solves Problem 1 for all \( x(0) \in \text{int}(S_G) \).

**Remark 1.** As pointed out in Ames et al. (2017), the conflict between safety and the convergence constraint require a non-zero slack term for satisfaction of (9c)–(9d) together. With this observation and keeping in mind the discussion in Section 3, one can readily conclude that if the control-input bounds or the user-defined time \( T_{ud} \) is sufficiently large, then it is possible to satisfy (9c) with \( \delta_1 \leq 0 \).

Next, some cases are listed when the solution of the QP (9) might not solve Problem 1 with the specified time constraint, and from all initial conditions, however, it still renders the closed-loop trajectories safe and convergent to the set \( S_G \) within some fixed time.

**Theorem 3.** Under Assumptions 1–3, the following hold:

1. Define \( \delta \equiv \sup_{x: x(0) \in S_G} \delta_i(x) \). If
\[ \delta < 2 \sqrt{\frac{\alpha}{\kappa}} \frac{1}{\omega}, \]
then, for all \( x(0) \in \text{int}(S_G), \) the closed-loop trajectories \( x(t) \) of (1) under \( u := v^*(x) \) reach the set \( S_G \) in a fixed time \( T_1 \leq \frac{\mu \kappa}{4v_0(x)^2} \left( \frac{\omega^2}{2} - \tan^{-1} k_2 \right) \), where \( k_1 := \sqrt{\frac{4v_0(x)^2}{2}} \) and \( k_2 := -\sqrt{\frac{\alpha}{\kappa}} \), while satisfying \( x(t) \in S_G \) for all \( t \geq 0, \text{ i.e., } D \equiv \text{int}(S_G). \)

2. If (10) does not hold, then for all \( x(0) \in D, \) the closed-loop trajectories satisfy \( x(t) \in \text{int}(S_G) \) for all \( t \geq 0 \) and reach the goal set \( S_G \) within a fixed time \( T_2 \leq \frac{\mu \kappa}{4v_0(x)^2} \left( \frac{1}{\omega^2} - \tan^{-1} k_2 \right) \), where \( D \) is the largest sub-level-set of the function \( h_c \) in the set \( D_G \cap \text{int}(S_G) \), with \( D_G = \{ x \in \text{int}(S_G) \} \).

**Proof.** In both cases, following the proof of Theorem 2, it holds that the closed-loop trajectories satisfy \( x(t) \in \text{int}(S_G) \) for all \( 0 \leq t \leq T \) for any \( T < \infty \). When (10) holds, using Lemma 2, it follows that the closed-loop trajectories of (1) under \( u := v^*(x) \) reach the set \( S_G \) within fixed time \( T_1 \) for all \( x(0) \in \text{int}(S_G) \) satisfying \( T_1 \leq \frac{\mu \kappa}{4v_0(x)^2} \left( \frac{\omega^2}{2} - \tan^{-1} k_2 \right) \), where \( k_1 := \sqrt{\frac{4v_0(x)^2}{2}} \) and \( k_2 := -\sqrt{\frac{\alpha}{\kappa}} \). Also, per (10), it holds that \( k_1 > 0 \) and so, \( T_1 < \infty \).

For the case when (10) does not hold, using Lemma 2, it holds that the closed-loop trajectories of (1) under \( u := v^*(x) \) reach the set \( S_G \) within time \( T_2 \) for all \( x(0) \in D_G \) where \( T_2 \leq \frac{\mu \kappa}{4v_0(x)^2} \left( \frac{1}{\omega^2} - \tan^{-1} k_2 \right) \) and \( D_G = \{ x \in \text{int}(S_G) \} \).

Since it is also required that \( x(0) \in \text{int}(S_G) \), define \( D \) as the largest sub-level-set of the function \( h_c \) in the set \( D_G \cap \text{int}(S_G) \), so that \( D \) is forward invariant (see Fig. 2). Therefore, for all \( x(0) \in D \), the closed-loop trajectories of (1) reach the set \( S_G \) within the fixed time \( T_2 \), while maintaining safety at all times.

In brief, the solution of the QP (9) always exists, is a continuous function of \( x \), and renders the set \( \text{int}(S_G) \) forward invariant, i.e., guarantees safety. Furthermore, the control input is guaranteed to yield fixed-time convergence of the closed-loop trajectories to the goal set \( S_G \). In the case when \( \delta_1 \leq 0 \), the convergence is guaranteed for all \( x(0) \in \text{int}(S_G) \), and within the user-defined fixed time \( T_{ud} \). If \( \delta_1 \) satisfies (10), then fixed-time convergence is guaranteed for all \( x(0) \in \text{int}(S_G) \) (i.e., \( D = \text{int}(S_G) \)), but the time of convergence \( T_1 \) may exceed the time \( T_{ud} \). Finally, if (10) does not hold, then fixed-time convergence is guaranteed for all \( x(0) \in D \subset \text{int}(S_G) \), however, the time of convergence \( T_2 \) may exceed the time \( T_{ud} \).

5. Numerical case studies

We present two case studies to illustrate the efficacy of the proposed method. We use Euler discretization to discretize the continuous-time dynamics, and the MATLAB function quadprog to solve the QP.

5.1. Adaptive cruise control problem

We consider an adaptive cruise control (ACC) problem with a following and a lead vehicle in this example. The objective for the following vehicle is to achieve the desired speed and maintain a safe distance from the lead vehicle. Considering that the two vehicles are modeled as point masses and traveling along a straight line, the system dynamics can be written as \( \dot{x} = f(x) + gu \) with \( f(x) = [-F(x)/M \quad a_k \quad f_0 \quad f_1] \), \( g = [1/M \ 0 \ 0] \), where \( u \in [-u_{\text{max}}, u_{\text{max}}] \) is the control input, \( x = [x_1, x_2, x_3] = [v, v_1, v_2] \in \mathbb{R}^3 \) is the system state with \( v_1 \) being the velocity of the following vehicle, \( v_2 \) being the velocity of the lead vehicle, and \( d \) being the distance between the two vehicles (see Ames et al., 2017 for more details). Here, \( M \) is the mass of the following vehicle, \( F_1(x) = f_0 + f_1 v_1 + f_2 v_2^2 \) is the drag force, and \( a_k \) is the gravitational acceleration. We define the goal and the safe sets, respectively, using the functions \( h_c(x) = (v_1 - v_d)^2, h_s(x) = t_	ext{d} (v_1 - d) \), where \( v_d \) is a desired fixed velocity and \( t_	ext{d} = 1.8 \) is the desired time headway. We set the maximum available control effort to \( u_{\text{max}} = 0.25M a_k \) with \( a_k = 9.81 \text{ m/s}^2 \) and \( M = 1650 \text{ Kg} \), the desired velocity to \( v_d = 22 \text{ m/s} \), the initial velocity of the lead vehicle to \( v_0 = 10 \text{ m/s} \), initial distance to \( d_0 = 150 \text{ m} \), \( f_0 = 0.1 \text{ N} \), \( f_1 = 5 \text{ Ns/m} \), \( f_2 = 0.25 \text{ Ns}^2/\text{m}^2 \), and \( a_k = 0.3 \). We implement the QP in (9) with \( T_{ud} = 10 \text{ sec} \), and \( \mu = 5 \) resulting in \( r_1 = 1.2, r_2 = 0.8 \). Fig. 3 illustrates the tracking performance of the resulting controller, where the solid lines represent the velocity of the following vehicle for different initial velocity of the following vehicle \( v_0(x) \in [17, 27] \text{ m/s} \). The desired speed is achieved when the trajectories are away from the boundaries of the safe set, while closer to the boundaries of the safe set the speed of the following vehicle is reduced to maintain safety.

As stated before, there is no guarantee for the existence of the solution of the proposed QP in Ames et al. (2017) when there is
a control input constraint. For the specific problem of adaptive cruise control as in this example, the authors in Ames et al. (2017) introduced two control barrier functions, namely optimal and conservative CBFs, based on the simplified system dynamics with no drag effect $F_s(x)$ to ensure the feasibility of the solution. Due to conservatism, the newly constructed safe sets $h_O^T(x)$ and $h_C^T(x)$ for the optimal and conservative CBFs are violated initially for a large initial velocity of the following vehicle. However, the actual safe set $\{x | \tau y - d \leq 0\}$ is not violated, and the problem can be still feasible.

Figs. 4 and 5 compare the tracking performance of the proposed approach and the results with optimal and conservative CBFs with $\nu y(0) = 18$ m/s. Since we are solving the QP directly and without the conservatism mentioned above, one can see from Fig. 4 that our proposed control approach tracks the desired goal speed of 22 m/s for a longer duration before departure from this speed for maintaining safety.

5.2. Multi-agent motion planning

In the second scenario, we present a two-agent motion planning example under spatiotemporal specifications, where the robot dynamics are modeled under constrained unicycle dynamics as:

$$\dot{x}_i = u_i \cos(\theta_i), \quad \dot{y}_i = u_i \sin(\theta_i), \quad \dot{\theta}_i = \omega_i$$  \hspace{1cm} (11)$$

where $[x_i, y_i]^T \in \mathbb{R}^2$ is the position vector of the agent $i$ for $i \in \{1, 2\}, \theta_i \in \mathbb{R}$ its orientation and $[u_i, \omega_i]^T \in \mathbb{R}^2$, the control input vector comprising of the linear speed $u_i \in [0, u_{\text{max}}]$ and angular velocity $|\omega_i| \leq \omega_{\text{max}}$. The closed-loop trajectories for the agents, starting from $[x_i(0), y_i(0)]^T \in C_1 = \{ z \in \mathbb{R}^2 | \|z - [1.5, 1.5]^T\|_\infty \leq 0.5 \}$ and $[x_i(0), y_i(0)]^T \in C_2 = \{ z \in \mathbb{R}^2 | \|z - [1.5, 1.5]^T\|_\infty \leq 0.5 \}$, respectively, are required to reach to sets $C_2$ and $C_1$, while staying inside the blue rectangle $[z \in \mathbb{R}^2 | \|z\|_\infty \leq 2]$, and outside the red-dotted circle $[z \in \mathbb{R}^2 | \|z\|_2 \leq 1.5]$, as shown in Fig. 6. The agents are also required to maintain an inter-agent distance $d_{\text{min}} > 0$ at all times.

Note that the sets $C_i$ are not overlapping with each other, and the corresponding functions $h_i$ are not continuously differentiable. Thus, to satisfy Assumption 2 and use the QP (9), we construct auxiliary sets $\tilde{S}_1 = \{ [x \ y] | \| x - y_2 \|_\infty + \| y - y_1 \|_\infty \leq 1 \}$ (orange circle), $\tilde{S}_2 = \{ [x \ y] | \| x - y_2 \|_\infty + \| y - y_1 \|_\infty \leq 1 \}$ (blue ellipse) and $\tilde{S}_3 = \{ [x \ y] | \| x - y_2 \|_\infty + \| y - y_1 \|_\infty \leq 1 \}$ (gray circle) as shown in Fig. 6. We choose the barrier functions as $h(x) = d_m - \|x_1 - x_2\|_\infty$, $h_1(x, y) = 1.5 - \|x_y\|_\infty$, $h_2(x, y) = [x - y_2]_+ \|x - y_1\|_\infty$ and $\phi = \angle([x \ y])$, where $P = \begin{bmatrix} 1/1.5^2 & 0 \\ 0 & 1/0.5^2 \end{bmatrix}$ and $\phi = \angle([x \ y])$ is the angle of the position vector $[x \ y]$ from the x-axis. The functions $h_1$ and $h_2$, along with $h_0$, help keep the agent inside the set $S_i$ and outside the red-dotted circle, respectively, as in Fig. 6. We choose the Lyapunov function as $V_1 = (x - x_2)^T + (y - y_2)^T - 0.5^2$, $V_2 = (\theta - \phi_2)^2 - 0.1^2$, where $[x_2 \ y_2]^T = [1.5 \ 1.5]^T$ is the goal location and $\phi(x) = \angle([x \ y])$ is the angle between the x-axis and the vector that is defined from the agent’s location to the goal point. These functions help steer the agent towards the goal location. We choose $\alpha = 2, \omega = 5, T = 2, \mu = 5$, so that $\gamma_1 = 1.2, \gamma_2 = 0.8, \alpha_1 = \alpha_2 = \frac{5}{T}$. The safety distance is chosen as $d_{\text{min}} = 0.1$. Fig. 6 plots the closed-loop trajectories of the agent.
and shows that the agent visit the required sets, while remaining inside the safe region and maintaining the safe distance with each other at all times. This is also evident from Fig. 7, which plots the pointwise maximum of all the barrier functions (i.e., $h_0, h_1, h_2, h_3$) plotted at all times, it implies that the both agents satisfy the safety requirements at all times. Fig. 8 plots the individual inputs of the two agents. It is evident from the figure that the input constraints for the agents are satisfied at all times. Furthermore, note that the linear speeds $u_1, u_2$ go to zero before $t = 2$, implying that the agents reach their respective goal sets within the user-defined time $T = 2$.

6. Conclusions

In this paper, we considered the problem of satisfying spatiotemporal constraints requiring that the closed-loop trajectories of a class of nonlinear, control-affine systems remain in a safe set at all times and reach a goal set within a fixed time in the presence of control input constraints. We established the relation between the domain of attraction for fixed-time stability, the input bounds, and the time of convergence, showing that relaxing the time constraint or increasing the input bound results in a larger FT-DtA. Then, we proposed a novel QP formulation, proved its feasibility under the assumption of the existence of a control input that renders the safe set forward invariant, and showed continuity of the solution of the proposed QP. In the future, we would like to study the spatiotemporal control synthesis for large-scale multi-agent systems with concurrent consideration of switching in the dynamics or the system states. It will be interesting to see how the proposed method extends to systems with non-smooth dynamics and formulate efficient optimization methods under such spatiotemporal constraints.

Appendix A. Proof of Lemma 4

Proof. Consider the Lagrangian of the QP in (8) given as $L := \frac{1}{2}u^T + \frac{1}{2}x^T S x + c_0^T x + c_0^T (u - u_m) + \lambda^T (L_x V + L_z V) x - \frac{1}{2}x^T V x + \alpha V x - \frac{1}{2}x^T V x$. Now, in order to see the effect of how input constraints affect $\delta_1$, the case when the constraint $u = u_m$ is active is studied under the assumption that $u_m > 0$. Lemma 3 guarantees feasibility of the QP in (8) for all $x \notin S_C$. Thus, the Slater’s condition holds and the KKT conditions are both necessary and sufficient for optimality (see e.g., Boyd & Vandenberghe, 2004, Chapter 5). Using the KKT conditions, it follows that $\delta_1^*(x) = -c + \lambda_1^*(x) V x$, $u^*(x) = -\lambda_2^*(x) + \lambda_3^*(x) L_x V x$, $\lambda_1^*(x) \geq 0$, $\lambda_2^*(x) \geq 0$, $\lambda_3^*(x) \geq 0$, for any $x \notin S_C$.

For $u^*(x) = u_m$, it is required that $\lambda_2^*(x) > 0$. Since $u^*(x) = u_m$ and $u_m < u_m$ (it follows $u^*(x) > u_m$ (i.e., the lower-bound constraint is inactive) and so $\lambda_2^*(x) = 0$. It follows that $\lambda_2^*(x) = -u_m - \lambda_1^*(x) L_x V x$. When $u^*(x) = u_m$, the constraint (8c) must be active. Otherwise, we have $\lambda_1^*(x) = 0$, which implies that $\lambda_2^*(x) \geq 0$. Thus, for $\lambda_2^*(x) > 0$ when $u^*(x) = u_m$, it is essential that the constraint (8c) is active, and it follows that the optimal value of $\delta_1$ is given as $\delta_1^*(x) = \frac{1}{2}V x + L_x V u_m + \alpha V x^2/2$. Using this, and the definition of function $a_i$, it follows that $\lambda_2^*(x) = \frac{a_i}{a_i^2}$. The latter function is a positive definite function. Using this, and the fact that the QP (9) is feasible, it holds that the second-order sufficient conditions for optimality hold (see e.g., Robinson, 1974, Section 2.3). Note that $\lambda_2^*(x) = 0$, and thus, the optimal values are given as $\delta_1^*(x) = -\frac{U x}{V} u_m + a_2 V x^2 - u^*(x) = u_m$.

Appendix B. Proof of Theorem 1

Proof. Denote by $I(x)$, the indices of rows of matrix $A(x)$ corresponding to the active constraints, i.e., $I(x)$ implies $A_i(x)_x^*(x) = b_i(x)$, where $A_i$, $b_i \in \mathbb{R}^{n \times m}$ is the $j$th row of the matrix $A$ and $b_j \in \mathbb{R}$ the $j$th element of $b$. Define matrix $A_{ac}$ and $b_{ac}$ by collecting $A_i$ and of $b_i$, respectively, so that $A_{ac}(x)x^*(x) = b_{ac}(x)$. Since at most one of the input constraints $u_i \leq u_m$ or $u_m \leq u_i$ can be active at any given time, the matrix $A_{ac}(x)$ has $k$ rows from $A_m \{0_m, 0_m\}$, where $k \leq m$ which are linearly independent. Furthermore, it has $p$ rows from $\{L h_c - h_c 0\}$, $L h_c - h_c 0 0$, where $p \leq 2$. Since $h_c, h_c \neq 0$ for $x \in \text{int}(S_C) \setminus S_C$, these $p + k$ rows are linearly independent. Thus, the matrix $A_{ac}$ is full row-rank, i.e., the gradients of the active constraints $[A_{ac}(x)]$, where $A_{ac}(x)$ is the $i$-th row of matrix $A_{ac}(x)$, are linearly independent.

The second derivative of the Lagrangian defined as $L(z, x, \lambda) := \frac{1}{2}z^2 H z + F^T z + \lambda^T (A(x) z - b(x))$, with respect to $z$ is $H$, which is a positive definite matrix. Using this, and the fact that the QP (9) is feasible, it holds that the second-order sufficient conditions for optimality hold (see e.g., Robinson, 1974, Section 2.3). Note that $\lambda_{ij}^*(x)$ is a set of eigenvalues of matrix $L_{ij} + L_{ij} + L_{ij}$, and the constraint functions $G_i(x)$ are linear in $u$, the second derivative of these functions are independent of $x$, and thus, satisfy this condition trivially. Finally, the strict complementary slackness condition is satisfied per Assumption 3. Thus, all the conditions of Robinson (1974, Theorem 2.1) are satisfied. Therefore, for each $x \in \text{int}(S_C) \setminus S_C$, there exists an open neighborhood $x \in \text{int}(S_C) \setminus S_C$ of $x$ such that the solution $z^*$ is continuous in $x$. Since this holds for all $x \in \text{int}(S_C) \setminus S_C$, it follows that the solution $z^*$ is continuous on $\text{int}(S_C) \setminus S_C$.

References

Agarwal, Ravi P., & Lakshminantham, V. (1993). Uniqueness and non-uniqueness criteria for ordinary differential equations, vol. 6. World Scientific.

Ames, Aaron D., Xu, Xiangru, Grizzle, Jessy W., & Tabuada, Paulo (2017). Control barrier function based quadratic programs for safety critical systems. IEEE Transactions on Automatic Control, 62(8), 3861-3876.

Aubin, J.-P., & Cellina, Arrigo (2012). Differential inclusions: Set-valued maps and viability theory, vol. 264. Springer Science & Business Media.

Blanchini, Franco (1999). Set invariance in control. Automatika, 35(11), 1747-1767.

Boyd, Stephen, & Vandenberghe, Lieven (2004). Convex optimization. Cambridge University Press.

Garg, Kunal, & Panagou, Dimitra (2019). Control-Lyapunov and control-barrier functions based quadratic program for Spatio-temporal specifications. In 58th IEEE conference on decision and control (pp. 1422-1429). IEEE.

Garg, Kunal, & Panagou, Dimitra (2021a). Characterization of domain of fixed-time stability under control input constraints. In 2021 American control conference (pp. 2268-2273).

Garg, Kunal, & Panagou, Dimitra (2021b). Robust control barrier and control Lyapunov functions with fixed-time convergence guarantees. In 2021 American control conference (pp. 2292-2297).
Dimitra Panagou received the Diploma and Ph.D. degrees in Mechanical Engineering from the National Technical University of Athens, Greece, in 2006 and 2012, respectively. Since September 2014, she has been an Assistant Professor with the Department of Aerospace Engineering from the Indian Institute of Technology, Mumbai, India. Prior to joining the University of Michigan, she was a postdoctoral research associate with the Coordinated Science Laboratory, University of Illinois, Urbana-Champaign (2012–2014), a visiting research scholar with the GRASP Lab, University of Pennsylvania (June 2013, fall 2010) and a visiting research scholar with the University of Delaware, Mechanical Engineering Department (spring 2009). Her research interests include the fields of multi-agent planning, coordination, control, and estimation, with applications in safe and resilient unmanned aerial systems, robotic networks, and autonomous multi-vehicle systems (ground, marine, aerial, space). She is a recipient of the NASA Early Career Faculty Award, the AFOSR Young Investigator Award, the NSF CAREER Award, and a Senior Member of the IEEE and the AIAA.

Kunal Garg received his Bachelor of Technology degree in Aerospace Engineering from the Indian Institute of Technology, Mumbai, India in 2016. He received his Master of Science in Engineering and Ph.D. degrees in the Department of Aerospace Engineering from the University of Michigan in 2019 and 2021, respectively. He is currently a postdoctoral scholar at the University of California Santa Cruz. His research interests include finite- and fixed-time stability of dynamical systems with applications to control synthesis for spatiotemporal specifications and continuous-time optimization, robust multi-agent path planning, switched and hybrid system-based analysis, and control synthesis for multi-agent coordination. He is a Member of the IEEE.

Kunal Garg, E. Arabi and D. Panagou

Automatica 141 (2022) 110314

Xu, Xiangru, Tabuada, Paulo, Grizzle, Jessy W., & Ames, Aaron D. (2015). Robustness of control barrier functions for safety critical control. IFAC-PapersOnLine, 48(27), 54–61.