Bell’s inequality for a single spin-1/2 particle and Quantum Contextuality

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Abstract

We argue that for a single particle Bell’s inequality is a consequence of noncontextuality and is incompatible with statistical predictions of quantum mechanics. Thus noncontextual models can be empirically falsified, independent of locality condition. For this an appropriate entanglement between disjoint Hilbert spaces pertaining to translational and spin degrees of freedom of a single spin-1/2 particle is invoked.

Bell’s inequality (henceforth BI) is derived from “Einstein Locality” (henceforth EL) which requires that the result of a measurement on a system be unaffected by measurement on its distant correlated partner[1]. In this Letter we bring out the significance of BI in a hitherto unexplored setting where EL is not relevant. This involves joint measurements of commuting observables pertaining to translational and spin degrees of freedom of a single particle prepared in an appropriate nonfactorisable state. A basic step is to formulate BI as a consequence of a condition more general than EL, viz. the hypothesis of noncontextuality (henceforth HNC) that may be characterized as follows[2]:

An individual measured value of a dynamical variable, say, $O_1$ is assumed to be specified corresponding to a definite premeasurement state of the particle. An outcome of a measurement is thus assumed not to depend on the experimental context. In particular, a measured value of $O_1$ is taken to be same, irrespective of any observable (commuting with $O_1$) measured with it. Note that EL is a special case of HNC when the measured commuting dynamical variables pertain to spatially separated and mutually non-interacting systems. However, if a model is contextual, it is not necessarily nonlocal.

In quantum mechanics noncontextuality is ensured to be satisfied in terms of the expectation value of a dynamical variable which is fixed by a wave function and is thus independent of the measuring arrangement. The apparently natural extension of such “context-independence” from statistical distributions to individual outcomes is what motivates HNC. That this is incompatible with the formalism of QM has been argued by a variety of no-go
theorems [3, 4, 5, 6, 7, 8]. However, all these theorems are based on models of HNC which share some features with the formalism of QM. This is in contrast to BI derived from EL entirely independent of QM [1].

The relevance of these no-go proofs has been questioned [9, 10, 11] on the ground that such proofs rely on ascribing outcomes to dynamical variables which are measured with infinite precision in the required experimental alignments. However, in practice no experimental arrangement can be aligned to measure, say, spin projections along coordinate axes that are specified with more than certain (necessarily finite) precision. On the other hand, it has been shown possible to specify noncontextually [9, 10, 11] outcomes of measurements of all observables in a dense subset whose closure contains observables whose noncontextual value assignments is prohibited by the no-go proofs. Since finite precision measurements (in the sense of “imprecision” in actually what is being measured) cannot distinguish between a dense subset and its closure, it is thus claimed that any model based on HNC cannot be experimentally discriminated from QM.

This contention has in turn prompted a number of rejoinders [12]. In order to settle this debate decisively, it is necessary that the incompatibility between QM and HNC be demonstrated in terms of statistical predictions of HNC (obtained independent of the formalism of QM) that not only conflict with QM but can also be subjected to experimental scrutiny by taking into account the inevitable imprecisions. A scheme to this end is what the present paper suggests by invoking a BI that is derived from HNC. It is applied to a single spin 1/2 particle by considering its mutually commuting spin and position degrees of freedom. Quantum mechanical predictions for the relevant joint probabilities are shown to violate such a BI. A testable conflict between QM and HNC is thus demonstrated in a situation where EL is not the issue. Note that while all experiments to date on EL use photons, our example is in terms of particles like neutrons.

The fact that QM violates BI in our scheme by a finite amount enables HNC to be empirically discriminated from QM even if the actual measurements are inevitably imprecise. The key point is that if HNC is a valid proposition satisfying the bound given by the relevant BI for ideal measurements, the fraction of runs in actual imprecise measurements that violate Bell’s inequality would become smaller (approaching the limit zero) as misalignments are minimized [13]. On the other hand, if HNC is ruled out, the fraction of runs corroborating QM predicted violation of BI would become larger (approaching the limit unity) as the alignments in actual measurements are made more precise. Thus HNC is empirically discriminable from QM in the same sense as EL and QM are discriminated.

In order to derive a testable consequence of HNC, Cabello and Garcia-Alcaine (CGA) [14] used a two particle two-state system. CGA considered sets of compatible propositions such that a joint measurement of a particular set of compatible observables would discriminate between QM and HNC. However, this type of non-statistical argument in terms of yes-no validity of propositions is contingent on the relevant dynamical variables being precisely specified and is thus affected by finite precision considerations in actual measurements. For particles with spin higher than 1/2, a scheme for using BI that can discriminate between QM and HNC has been suggested by Roy and Singh [15]. This approach is in terms of “stochastic noncontextuality” that requires ascribing probability distributions to “hidden variables”. On the other hand, our treatment does not require any assumption concerning distribution functions of hidden variables.

A spin 1/2 particle has remained unexplored for studying the conflict between QM and HNC because QM is compatible with HNC for a spin 1/2 particle described in a Hilbert space.
of dimension two $\mathbb{C}^2$. In our example, a spin 1/2 particle is described in terms of a tensor product Hilbert space $H = H_1 \otimes H_2$ where $H_1$ and $H_2$ are disjoint Hilbert spaces corresponding to spin and translational degrees of freedom. Hence here the total Hilbert space is of dimension greater than two (as discussed later, in our example, the Hilbert space is four dimensional). Thus there is no inconsistency with Gleason’s theorem $\cite{3}$. We shall now formulate the pertinent BI for our example.

Given an ensemble of identical systems specified by a wave function, the result of an individual measurement of an arbitrary dynamical variable is, in general, not uniquely fixed by the wave function which provides only a probabilistic prediction. The very notion of HNC thus hinges on assuming that if a wave function description is suitably supplemented, an individual outcome of measuring a dynamical variable can, in principle, be specified irrespective of what commuting variables are measured along with it.

Let $A_1, A_2$ and $B_1, B_2$ be two pairs of noncommuting dynamical variables pertaining to a spin-1/2 particle such that $A_i$’s (i=1,2) commute with $B_j$’s (j=1,2) where $A_i$’s and $B_j$’s belong to mutually disjoint Hilbert spaces corresponding to mutually commuting degrees of freedom (say, spin and position-momentum). We take each of $A_1, A_2$ and $B_1, B_2$ to be two valued (say, $\pm 1$). If one considers the outcomes of joint measurements of four commuting pairs $A_1B_1, A_1B_2, A_2B_1$, and $A_2B_2$, the following equality holds good

$$A_1B_1 + A_1B_2 + A_2B_1 - A_2B_2 = \pm 2$$  \hspace{1cm} (1)

Note that Eq. (1) pertains to measurements on a collection of particles assumed to be prepared in a common ‘completely specified’ state so that both the occurrences of, say, $A_1$ in Eq. (1) have the same value; this also holds good for $A_2, B_1$, and $B_2$ (input of HNC). Next, taking the ensemble averages, it follows from Eq. (1)

$$|\langle A_1B_1 \rangle + \langle A_1B_2 \rangle + \langle A_2B_1 \rangle - \langle A_2B_2 \rangle| \leq 2$$  \hspace{1cm} (2)

Thus Eq. (2) is a form of BI that can be viewed as a testable consequence of HNC, requiring no input from QM and is independent of EL (see also Ref. $\cite{4}$). Next, to demonstrate QM violation of the inequality (2) for a suitable entangled state of a single spin-1/2 particle, the first step is to construct an appropriate two dimensional Hilbert space $H_1$ disjoint to two dimensional Hilbert space $H_2$ involving the spin variables.

Consider a particle entering a Mach-Zehnder type interferometer (Fig. 1) through the beam splitter BS1. It can be in either of two possible mutually exclusive states (designated by, say, $\psi_1$ and $\psi_2$) corresponding to the transmitted and reflected channels. $\psi_1$ and $\psi_2$ are recombined at a second beam splitter BS2 coupled with a suitable phase shifting (PS) arrangement. The output channels from BS2 + PS are labelled by $\psi_3$ and $\psi_4$ that are registered at the detectors $D_3$ and $D_4$ respectively. The states $\psi_1, \psi_2, \psi_3, \psi_4$ are taken as eigenstates of projection operators pertaining to observables that represent the determination of ‘which channel’ the particle is in. For example, the detector $D_3$ registers whether a particle is in the channel $\psi_3$ or not. This corresponds to measuring the projection operator $P(\psi_3)$. Results of such a measurement with binary alternatives are designated by the eigenvalues of $P(\psi_3)$; the eigenvalue $+1(0)$ corresponding to the particle being found (not found) in the channel $\psi_3$. The description based on projection operators like $P(\psi_3)$ and $P(\psi_4)$ where $\langle \psi_3 | \psi_4 \rangle = 0$ generates a two dimensional Hilbert space $H_1$ which is isomorphic to the Hilbert space $H_2$ for spin 1/2. Similarly, the description using projection operators $P(\psi_1)$ and $P(\psi_2)$ also generates a two dimensional Hilbert space $H_1$ isomorphic to $H_2$, where $\langle \psi_1 | \psi_2 \rangle = 0$. 


The output states $\psi_3, \psi_4$ from BS2+PS are related to the input states $\psi_1, \psi_2$ by

$$
\begin{pmatrix}
\psi_3 \\
\psi_4
\end{pmatrix} =
\begin{pmatrix}
\sin \theta e^{i\phi} & \cos \theta e^{i\phi} \\
-\cos \theta & \sin \theta
\end{pmatrix}
\begin{pmatrix}
\psi_1 \\
\psi_2
\end{pmatrix}
$$

(3)

where $\sin^2 \theta, \cos^2 \theta$ are the reflection and transmission probabilities; $\phi$ denotes phase shift introduced after BS2. The arrangement constituting beam splitter BS2 and phase shifter PS is characterized by the parameters $\theta$ and $\phi$. Thus for an incident given linear combination of $\psi_1$ and $\psi_2$, by varying $\theta$ and $\phi$, one can generate at the output various linear combinations of $\psi_1$ and $\psi_2$, which in turn correspond to different probability amplitudes of finding a particle in the channels $\psi_3$ and $\psi_4$.

For our purpose it is convenient to consider the following dichotomic observable defined in $H_1$: $A = P(\psi_3) - P(\psi_4)$. The eigenvalues $\pm 1$ of $A$ correspond to the particle being found in a channel corresponding to either $\psi_3$ or $\psi_4$. The expectation value $\langle A \rangle$ can be determined from the counts registered at $D_3$ and $D_4$. Changing BS2 + PS (i.e., by varying $\theta$ and $\phi$) one can thus construct different observables $A$’s (corresponding to different relative counts at $D_3$ and $D_4$). In particular, if $\psi_1, \psi_2$ are taken to correspond to spin-up and spin-down states along, say, $z$-axis, measuring $A$’s for different $\theta, \phi$ is analogous to measuring spin components along directions differently oriented with respect to the $z$-axis. It is this correspondence which is crucial to showing QM violation of the HNC inequality (2) for a single spin 1/2 particle.

We now consider the required experimental arrangement which is shown in Fig. 2, similar to that indicated in Fig. 1 with the following differences: (a) A spin-flipper (SF) and a phase-shifter (PS1) are placed along the channels $\psi_1$ and $\psi_2$ respectively. (b) The detectors $D_3$ and $D_4$ are coupled with suitably oriented similar Stern-Gerlach (SG) devices measuring the relevant spin component; i.e., each of $D_3, D_4$ is connected to channels of each SG device so that $D_3$ registers the combined counts of $D'_3, D''_3$ and $D_4$ registers the combined counts of $D'_4, D''_4$. Note that while counts at the unprimed detectors correspond to measuring an observable $A$, those at the primed detectors correspond to measuring a spin observable $B$. Thus an observable like $A$ and a spin observable $B$ are measured jointly.

Let a spin-1/2 particle with spin polarized along, say, $+z$ axis be incident on BS1 with transmission and reflection probabilities being given by $|a|^2, |b|^2$ respectively. The state subsequently incident on BS2 is of the EPR-Bohm entangled type given by

$$
\Psi = a \langle \uparrow \rangle_p \otimes \langle \downarrow \rangle_z + b e^{i\delta} \langle \downarrow \rangle_p \otimes \langle \uparrow \rangle_z
$$

(4)

where $\langle \downarrow \rangle_z, \langle \uparrow \rangle_z$ denote states corresponding to spin components $\sigma_z = -1, +1$ respectively, and $\psi_1, \psi_2$ are denoted by $|\uparrow \rangle_p$ and $|\downarrow \rangle_p$ (“up” and “down” channel states in the position space) that are analogous to spin-up and spin-down states along $z$-axis.

Now, choosing BS1 such that the reflectivity/transmittivity is 50% and adjusting PS1 so that $\delta = \pi$, the state given by Eq. (4) becomes maximally entangled, given by

$$
\Psi = \frac{1}{\sqrt{2}} (|\uparrow \rangle_p \otimes |\downarrow \rangle_z - |\downarrow \rangle_p \otimes |\uparrow \rangle_z)
$$

(5)

Subsequently we consider measurements of appropriate spin observables (say, $B_1$ and $B_2$) along with the observables $A_1, A_2$ whose eigenstates are suitable linear combinations of $\psi_3$ and $\psi_4$. Now note that BS2 + PS in Fig. 2 is viewed as a part of the arrangement making measurements on $\Psi$ of Eq. (5), prepared by the setup preceding BS2 + PS. Hence in view of
the isomorphism between $H_1$ and $H_2$, the parameters $\theta, \phi$ may be varied to make appropriate choices of $A_1, A_2$ along with suitably oriented SG devices which measure the spin components $B_1, B_2$ so that the HNC inequality (2) is violated by QM predictions for an entangled state (4). The magnitude of such violation is finite.

To spell out explicitly the correspondence between actual measurements in our scheme and quantities occurring in the HNC inequality (2), consider any one pair, say, $A_1B_1$. Registered counts at the respective detectors are denoted by $N_3, N_3', N_4, N_4'$ and $N_4''$. Then $\langle A_1 \rangle = N_3 - N_4; \langle B_1 \rangle = (N_3' - N_3'') + (N_4' - N_4'')$; and $\langle A_1B_1 \rangle = (N_3 - N_4) \left[ (N_3' - N_3'') + (N_4' - N_4'') \right]$. Similar correspondence also holds good for $\langle A_1B_2 \rangle, \langle A_2B_1 \rangle$ and $\langle A_2B_2 \rangle$. Now, if the HNC inequality (2) is empirically violated, it would mean that an individual outcome of measuring a spin component of a spin-1/2 particle depends on the preceding choice of BS2 and PS. Thus varying the parameters characterizing “which channel” measurement pertaining to translational degrees of freedom would affect the outcome of individual spin measurement.

For an experimental probing of the present scheme, neutrons seem to be particularly suitable since absorption of a neutron beam splitter is extremely small ($<0.001$), detector efficiency is very high ($\sim0.999$) and it could be possible to change the reflectivity of a beam splitter in a controlled way over a sufficiently large range to show violation of the HNC inequality [17].

Although wave packets spread while neutrons travel, it does not affect the isomorphism between $H_1$ and $H_2$ (which relies on dichotomy of the relevant observables and orthogonality between $\psi_3$ and $\psi_4$) because of unitarity which ensures that the inner product between $\psi_3$ and $\psi_4$ be unchanged. Imprecisions resulting from wave packet spreading can be minimised by suitably choosing the separation between SG devices and their distances from BS2 + PS. For a typical wave packet in a neutron interferometer, initial width $\sim0.1mm$, mean velocity $\sim2\times10^3ms^{-1}$ so that after traversing $\sim1m$, the final spread $\sim(0.1+0.001)mm$ which shows that the effect is indeed quite small.

To conclude, quantum entanglement between disjoint Hilbert spaces of a single spin-1/2 particle can be used to show that any model of quantum mechanics should be inherently contextual. This in our example entails a mutual dependence between individual measurements on “path” and spin degrees of freedom. The precise nature of such contextuality and its contrast with nonlocality calls for further studies.

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Note Added: This work was reported in a preprint quant-ph/9907030, after which an experiment using similar idea but involving photons has been reported by M. Michler et al, Phys. Rev. Lett, 84, 5457 (2000). However, our present scheme enables to test quantum contextuality for spin 1/2 systems. Thus a related experiment using particles such as neutron, electron is called for.

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Figure Captions

Figure 1.
A particle (say, a neutron) entering this Mach-Zehnder type interferometer through the beam splitter BS1 can be in a channel corresponding to either $\psi_1$ or $\psi_2$. $\psi_1$ and $\psi_2$ are then recombined at the beam splitter BS2 coupled with a suitable phase shifting arrangement (PS). Neutrons at the output channels $\psi_3$ and $\psi_4$ are registered at the detectors $D_3$ and $D_4$ respectively.

Figure 2.
A spin-polarized particle, say, a neutron passing through BS1 is prepared in an entangled state of the form given by Eq. (7) or Eq. (8). By adjusting the parameters $\theta$ and $\phi$ of BS2+PS and by suitably orienting the Stern-Gerlach (SG) devices, appropriate measurements of the observables $A_1, A_2$ and $B_1, B_2$ are performed. Each of the detectors $D_3, D_4$ is coupled with detectors along channels of the respective SG device so that an observable like $A_1$ or $A_2$ and the relevant spin observable $B_1$ or $B_2$ can be measured jointly.
