Extension operators on balls and on spaces of finite sets

Antonio Avilés, joint work with Witold Marciszewski

Universidad de Murcia, Author supported by MINECO and FEDER under project MTM2014-54182-P

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Extension Operators

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**Theorem (Tietze)**

Every \( f \in C(K) \) extends to a function in \( C(L) \).
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**Theorem (Tietze)**

Every $f \in C(K)$ extends to a function in $C(L)$.

An extension operator is an operator $E : C(K) \rightarrow C(L)$ that sends every $f \in C(K)$ to an extension.
Let $M(K) = C(K)^*$ be the regular Borel measures on $K$, with weak* topology.
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Having an extension operator $E$ is all the same as having a continuous $E^*: L \rightarrow M(K)$ such that $E^*(x) = \delta_x$ for $x \in K$. 
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$$E(f)(x) = \int f \ dE^*(x)$$
The Borsuk-Dugundji extension theorem

Theorem (Borsuk, Dugundji)

If $K$ is metric, then there exists a positive extension operator $E : C(K) \to C(L)$ with $\|E\| = 1$. 

In the non-metric case, we define

$$\eta(K, L) = \inf \{\|E\| : E : C(K) \to C(L) \text{ is an extension operator} \}$$

which might be $\infty$ if there is no such $E$ exists.
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Balls in Hilbert space:

$$rB(\Gamma) = \{x \in \ell_2(\Gamma) : \|x\| \leq r\}$$

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$$\{1,2,3\}, \{1,2,4\}, \{1,2,5\}, \ldots \longrightarrow \{1,2\}$$
Our main results

Theorem (Corson, Lindenstrauss 65)

1. $rB(\Gamma)$ is not a retract of $sB(\Gamma)$ for $\Gamma$ uncountable and $r < s$. 

Theorem (A., Marciszewski)

There is no extension operator from $rB(\Gamma)$ to $sB(\Gamma)$ for $\Gamma$ uncountable and $r < s$. 

Theorem (A., Marciszewski)

$\eta(\sigma_m(\Gamma), \sigma_n(\Gamma))$ is an odd integer that depends on $m$, $n$, and $|\Gamma|$. It takes values:

1. If $|\Gamma| \leq \aleph_0$.
2. $n - 2m + 1$, if $|\Gamma| = \aleph_1$.

$\sum_{m \leq k} (n^k)(n - k - 1)^{m-k}$, if $|\Gamma| \geq \aleph_\omega$. 
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1. $1$, if $|\Gamma| \leq \aleph_0$.
2. $2n - 2m + 1$, if $|\Gamma| = \aleph_1$.
3. $\sum_{k=0}^{m} \binom{n}{k} \binom{n-k-1}{m-k}$, if $|\Gamma| \geq \aleph_\omega$. 
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The function $\{x < q\} \mapsto q\delta\{p\} + 1q\delta\{q\}$ gives an extension operator of norm 1 when $\Gamma = N$. The function $\{x, y\} \mapsto \delta\{x\} + \delta\{y\} - \delta/0$ gives an extension operator of norm 3. This is optimal for sizes $\geq \aleph_1$. 
Let us think of $m = 1$, $n = 2$. We need to associate to each set of cardinality $\leq 2$, a measure (a formal linear combination) on the sets of cardinality $\leq 1$. 

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Let us call a natural extension operator between $\sigma_m$ and $\sigma_n$ to a family of extension operators $E_\Gamma : C(\sigma_m(\Gamma)) \rightarrow C(\sigma_n(\Gamma))$ such that all diagrams

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commute for $\Delta \subset \Gamma$. 

**Theorem (A., Marciszewski)**

$\eta(\sigma_m(\aleph_\omega), \sigma_n(\aleph_\omega))$ equals the least norm of a natural extension operator from $\sigma_m$ to $\sigma_n$. 

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### Theorem (A., Marciszewski)

$\eta(\sigma_m(\aleph_\omega), \sigma_n(\aleph_\omega))$ equals the least norm of a natural extension operator from $\sigma_m$ to $\sigma_n$. 
There is essentially a unique formula for a natural extension operator from $\sigma_m$ to $\sigma_n$:

$$A \mapsto \sum_{B \in [A] \leq m} (-1)^{m-|B|} \left( \binom{|A| - |B| - 1}{m - |B|} \right) \delta_B$$
Getting free sets

Suppose $|\Gamma| \geq \aleph_n$. 
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Suppose $|\Gamma| \geq \aleph_n$. Let $F$ be a function that sends each finite subset of $\Gamma$ to another disjoint finite subset of $\Gamma$. Then, there exists $Z \subset \Gamma$ with $|A| = n + 1$ such that $F(A) \cap Z \neq \emptyset$ for all $A \subset Z$.

Getting free sets (case $n = 1$)

Suppose $|\Gamma| \geq \aleph_1$. Let $F$ be a function that sends each finite subset of $\Gamma$ to another disjoint finite subset of $\Gamma$. Then, there exists $Z \subset \Gamma$ with $|Z| = 2$ such that $F(A) \cap Z \neq \emptyset$ for all $A \subset Z$.

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Another related Banach space problem

Problem (Enflo, Rosenthal 73)

Does a nonseparable $L_p(\mu)$ have an unconditional basis when $p \neq 2$?

They get a negative answer for density at least $\aleph_{\omega}$.

The combinatorics behind it are again free sets.

The new free set property in $\aleph_1$ is not strong enough to solve this problem.
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Open Problem: A non-separable Miljutin theorem?

Is $C(B(\Gamma))$ isomorphic to $C(\sigma_1(\Gamma)^\mathbb{N})$?