Maintaining Optimality in Assignment Problem against Weight Updates around Vertices

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Abstract

We consider a dynamic situation in the weighted bipartite matching problem in which edge weights are repeatedly updated and each update occurs around a single vertex. Our objective is to maintain an optimal matching at any moment. A trivial approach is to compute an optimal matching from scratch each time an update occurs, but this seems inefficient. In this paper, we show that, if we simultaneously maintain a dual solution, then it suffices to perform Dijkstra’s algorithm only once per update. As an application of our result, we provide a faster implementation of the envy-cycle procedure for finding an envy-free allocation of indivisible goods. Our algorithm runs in $O(mn^2)$ time, while the known bound of the original one is $O(mn^3)$, where $n$ and $m$ denote the numbers of agents and items, respectively.

Keywords Assignment problem, Matchings in bipartite graphs, Primal-dual update, Envy-free allocation of indivisible goods (EF1/EFX).

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1 Introduction

The weighted matching problem in bipartite graphs is a fundamental combinatorial optimization problem. Initiated by the so-called Hungarian method [8], a variety of efficient algorithms for the problem have been developed [3, 5–7, 11].

In this paper, we investigate a dynamic situation in this problem. Imagine an advertising agency that allocates advertisement slots (electronic billboards, banners on websites, etc.) to advertisers. Each edge connects an advertiser and a slot, and its weight represents the profit per unit time of the agency gained by matching them. This weight depends on the advertiser’s evaluation of the slot and the slot’s terms of sale. Each advertiser can change its evaluation of the slots when, for example, it releases a new product or its financial situation changes. In addition, the terms of sale for each slot can be changed when needed. Any of such changes is represented as weight updates around a vertex; that is, the weights of the edges incident to a vertex are updated simultaneously. The goal of the advertising agency is to adaptively change the matching in response to repeated weight updates so that it always maximizes the total profit. A trivial approach is to compute a maximum-weight matching from scratch each time the weights are updated. This simple method, however, seems inefficient; for example, if we use a typical efficient algorithm, it requires $O(n(m + n \log n))$ time per update, where $n$ and $m$ denote the numbers of vertices and edges, respectively. In a large market with frequent weight updates, the time required for updates may have a significant impact on profits.

In this paper, we propose a more efficient method for maintaining an optimal matching against weight updates around vertices. We show that, if we maintain a dual solution simultaneously, then it suffices to perform Dijkstra’s algorithm only once for updating both primal and dual solutions. This requires $O(m + n \log n)$ time per update.

We first describe our idea in the setting of the assignment problem, i.e., a special case in which the input bipartite graph is balanced and we are asked to find an optimal perfect matching. Suppose that the edge weights are updated around a vertex $s$. In the primal side, we can obtain a new optimal matching from the old one by exchange along a minimum-weight cycle intersecting $s$ in the auxiliary graph. Such a cycle can be found by computing shortest paths from $s$ in the modified auxiliary graph that is obtained by flipping the matching edge incident to $s$. In the dual side, from the shortest path length, we can obtain a new potential with respect to the new optimal matching. Since the modified auxiliary graph has no negative cycle, these primal/dual updates can be done in $O(mn)$ time by the Bellman–Ford algorithm. Our main observation is that, as the weights are updated around a single vertex, we can easily adjust the old potential to the modified auxiliary graph; this enables us to apply Dijkstra’s algorithm, which runs in $O(m + n \log n)$ time.

We then discuss cases in which the input bipartite graph is unbalanced or the perfectness of a matching is not imposed. Although such cases reduce to the assignment problem by adding dummy vertices and edges, a naïve reduction may essentially increase the input size. We show that, by avoiding an explicit reduction, the same computational time bound can be obtained.

We also show that our technique is useful for obtaining a faster implementation of the envy-cycle procedure of Lipton et al. [9] for finding an EF1 (envy-free up to one good) allocation of indivisible goods. We observe that the envy-cycle procedure can be modified so that the part of cancelling envy cycles is formulated as the assignment problem whose input graph is updated around a single vertex in each iteration. Our primal-dual update algorithm fits this situation. While the well-known running time bound of the original envy-cycle procedure is $O(mn^3)$, our variant runs in $O(mn^2)$ time, where $n$ and $m$ denote the numbers of agents and items, respectively.
The remainder of the paper is organized as follows. In Section 2, we provide necessary definitions and basic facts on the assignment problem. In Section 3, we describe our algorithm for updating primal/dual solutions against weight update around a vertex in the assignment problem, and discuss equivalent formulations of the weighted bipartite matching problem. In Section 4, we present an improved variant of the envy-cycle procedure.

2 Preliminaries

In the assignment problem, we are given an edge-weighted bipartite graph and asked to find a perfect matching of the minimum total weight. We briefly review the theory behind typical primal-dual approaches for this problem (see [10] for more details). Throughout the paper, except for Section 3.2, we assume that the input graph is simple and has a perfect matching.

The input is a pair \((G, w)\) of a bipartite graph \(G = (V^+, V^-; E)\) with vertex set \(V\) partitioned into \(V^+\) and \(V^-\) and edge set \(E\), where \(|V^+| = |V^-| = n\) and \(|E| = m\), and edge weights \(w: E \to \mathbb{R}^\pm\). We also regard \(G\) as a directed graph in which each edge is directed from \(V^+\) to \(V^-\).

A matching \(\mu \subseteq E\) in \(G\) is an edge set in which every vertex appears at most once as an endpoint, and \(\mu\) is called perfect if all the vertices appear exactly once. We also regard a perfect matching \(\mu\) as a bijection from \(V^+\) to \(V^-\) and vice versa. That is, we write \(\mu(u) = v\) and \(\mu^{-1}(v) = u\) if \(e = (u, v) \in \mu\). We define the weight of a matching \(\mu\) as \(w(\mu) = \sum_{e \in \mu} w(e)\).

Let \(\mu\) be a matching in \(G = (V, E) = (V^+, V^-; E)\). The auxiliary graph for \(\mu\) is a directed graph \(G_\mu = (V, E_\mu)\) obtained from \(G\) by flipping the direction of the edges in \(\mu\), i.e.,

\[
E_\mu := (E \setminus \mu) \cup \bar{\mu} = \{e \mid e \in E \setminus \mu\} \cup \{\bar{e} \mid e \in \mu\},
\]

where \(\bar{e}\) denotes the reverse edge of \(e\). We also define the auxiliary weight \(w_\mu: E_\mu \to \mathbb{R}\) by

\[
w_\mu(e) := \begin{cases} w(e) & (e \in E \setminus \mu), \\
-w(\bar{e}) & (e \in \mu). \end{cases}
\]

We define the weight of a path or a cycle in \(G_\mu\), say \(P\), as \(w_\mu(P) = \sum_{e \in P} w_\mu(e)\). We call a path/cycle shortest if it is of the minimum weight (under some specified condition), and negative if it is of negative weight.

Let \(V^+_\mu\) and \(V^-_\mu\) denote the sets of vertices not matched by \(\mu\) in \(V^+\) and in \(V^-\), respectively. An augmenting path in \(G_\mu\) is a directed path starting in \(V^+_\mu\) and ending in \(V^-_\mu\). By exchanging the edges in \(E \setminus \mu\) and in \(\mu\) along an augmenting path or a cycle in \(G_\mu\), say \(P\), one can obtain another matching, which we denote by \(\mu \triangle P\). Note that \(w(\mu \triangle P) = w(\mu) + w_\mu(P)\) by definition.

A celebrated feature of the minimum-weight matchings utilized in many algorithms is as follows.

Lemma 2.1. Let \(\mu\) be a matching of size \(k\). Then,

- if \(\mu\) is a minimum-weight matching in \(G\) subject to \(|\mu| = k\), then \((G_\mu, w_\mu)\) has no negative cycle, and these conditions are equivalent when \(k = n\);
- if \(\mu\) is a minimum-weight matching in \(G\) subject to \(|\mu| = k\) and \(P\) is a shortest augmenting path in \((G_\mu, w_\mu)\), then \(\mu' := \mu \triangle P\) is a minimum-weight matching in \(G\) subject to \(|\mu'| = k + 1\).

\(^1\)It is often assumed that the input graph is a complete bipartite graph. The present formulation reduces to that setting by adding all the absent edges with sufficiently large weight. This reduction, however, does not preserve the input size if the original graph is not dense, i.e., \(m = o(n^2)\). In this sense, we strictly evaluate computational times.
By Lemma 2.1, one can augment a minimum-weight matching \( \mu \) of size \( k \) to one \( \mu' \) of size \( k + 1 \) by solving once the shortest path problem in directed graphs with no negative cycle. This is done in \( O(nm) \) time by the Bellman–Ford algorithm. Furthermore, this is done in \( O(m + n \log n) \) time by Dijkstra’s algorithm (with the aid of the Fibonacci heap) if we simultaneously maintain a potential (i.e., a dual solution) defined as follows.

For a matching \( \mu \) and a function \( p : V \to \mathbb{R} \), we define the reduced weight \( w_{\mu,p} : E_\mu \to \mathbb{R} \) by

\[
w_{\mu,p}(e) := w_\mu(e) + p(u) - p(v) \quad (e = (u, v) \in E_\mu).
\]

We say that \( p \) is a potential for \( (G_\mu, w_\mu) \) if \( w_{\mu,p}(e) \geq 0 \) for every \( e \in E_\mu \).

We here summarize the Hungarian method with this speeding up. First, we set \( \mu_0 \leftarrow \emptyset \) and

\[
p_0(v) \leftarrow \begin{cases} 0 & (v \in V^+) \\ \min \{ w(e) \mid e \in \delta_G(v) \} & (v \in V^-)
\end{cases}
\]

where \( \delta_G(v) \) denotes the set of edges incident to \( v \) in \( G \). Then, for each \( k = 0, 1, \ldots, n - 1 \), we successively compute a minimum-weight perfect matching \( \mu_{k+1} \) of size \( k + 1 \) and a potential \( p_{k+1} \) for \( (G_{\mu_{k+1}}, w_{\mu_{k+1}}) \) from \( \mu_k \) and \( p_k \) as follows.

In each iteration, we compute a shortest path from \( V^+_{\mu_k} \) to each vertex \( v \in V \) in \( (G_{\mu_k}, w_{\mu_k}) \) by Dijkstra’s algorithm with the aid of the potential \( p_k \). In the primal side, let \( P \) be a shortest augmenting path, and set \( \mu_{k+1} \leftarrow \mu_k \triangle P \). In the dual side, for each vertex \( v \in V \), let \( d(v) \) denote the weight of a shortest path from \( V^+_{\mu_k} \) to \( v \), and set \( p_{k+1}(v) \leftarrow d(v) \) (where we set \( p_{k+1}(v) \leftarrow p_k(v) \) if \( v \) is not reachable from \( V^+_{\mu_k} \)).

As the bottleneck is clearly executing Dijkstra’s algorithm \( n \) times, the computational time of the whole algorithm is bounded by \( O(n(m + n \log n)) \).

3 Primal/Dual Update against Weight Update around a Vertex

In this section, we propose an \( O(m + n \log n) \)-time algorithm for updating an optimal matching and a corresponding potential when the edge weights are updated around a single vertex. We show this for the assignment problem in Section 3.1 and discuss other polynomially equivalent formulations of the weighted bipartite matching problem in Section 3.2.

3.1 Assignment Problem Case

The problem we consider here is as follows. Suppose we have a pair of primal and dual optimal solutions for an instance \( (G, w) \) of the assignment problem, i.e., we have a minimum-weight perfect matching \( \mu \) and a potential \( p \) for \( (G_\mu, w_\mu) \). Now the edge weights are updated arbitrarily but only around a single vertex \( s \in V \), and we are asked to find primal and dual optimal solutions in the new instance \( (G, \bar{w}) \). By symmetry, we can assume \( s \in V^+ \). In addition, for the sake of simplicity, we assume that \( \delta_G(s) = \{s\} \times V^- \) by adding at most \( n \) absent edges with sufficiently large weights.

Our algorithm works as follows. Let \( e' = (s, t) \) be the edge in \( \mu \) incident to \( s \) and let \( \mu' := \mu \setminus \{e'\} \).

Then, in the graph \( G_{\mu'} \), all the edges in \( \delta_G(s) = \{s\} \times V^- \) are directed from \( s \) to \( V^- \). We intend to search shortest paths from \( s \) in \( G_{\mu'} \) with respect to the new weights \( \bar{w} \). Note that \( p \) is a potential for \( (G_\mu, w_\mu) \) (i.e., \( w_{\mu,p}(e) \geq 0 \) for every \( e \in E_\mu \)), but may not be a potential for the current graph \( (G_{\mu'}, \bar{w}_{\mu'}) \). However, since \( (G_{\mu'}, \bar{w}_{\mu'}) \) differs from \( (G_\mu, w_\mu) \) only around the vertex \( s \), we can
recover the nonnegativity condition only by modifying the value of \( p \) on \( s \). Precisely, we define \( p' \) by
\[
p'(s) := -\min \{ \tilde{w}(e) - p(v) \mid e = (s, v) \in \delta_G(s) \} \quad \text{and} \quad p'(v) := p(v) \quad \text{for every} \ v \in V \setminus \{ s \}.
\]
Then, \( p' \) is indeed a potential for \( (G_{\mu'}, \tilde{w}_{\mu'}) \), and we can compute a shortest path from \( s \) to each vertex efficiently by Dijkstra’s algorithm. Using the obtained paths and their weights, we update the primal and dual solutions.

**Algorithm 1 Primal Dual Update (PDU)**

**Input:** A bipartite graph \( G = (V^+, V^-; E) \), edge weights \( w: E \to \mathbb{R} \), a minimum-weight perfect matching \( \mu \) in \( (G, w) \), a potential \( p \) for \( (G_{\mu}, w_{\mu}) \), a vertex \( s \in V^+ \) with \( \delta_G(s) = \{ s \} \times V^- \), and new edge weights \( \tilde{w}: E \to \mathbb{R} \) such that \( \tilde{w}(e) = w(e) \) for every edge \( e \in E \setminus \delta_G(s) \).

**Output:** A minimum-weight perfect matching \( \tilde{\mu} \) in \( (G, \tilde{w}) \) and a potential \( \tilde{p} \) for \( (G_{\tilde{\mu}}, \tilde{w}_{\tilde{\mu}}) \).

1. Let \( e' = (s, t) \) be the edge in \( \mu \) incident to \( s \) and let \( \mu' = \mu \setminus \{ e' \} \).
2. Set \( p'(s) = -\min \{ \tilde{w}(e) - p(v) \mid e = (s, v) \in \delta_G(s) \} \) and \( p'(v) = p(v) \) for every \( v \in V \setminus \{ s \} \). Compute a shortest path \( P_t \) from \( s \) to each \( v \in V \) in \( (G_{\mu'}, \tilde{w}_{\mu'}) \) by Dijkstra’s algorithm with the aid of the potential \( p' \).
3. Set \( \tilde{\mu} \leftarrow \mu' \triangle P_t \). For each \( v \in V \), let \( d(v) \) denote the weight of \( P_v \), and set \( \tilde{p}(v) \leftarrow d(v) \).
4. Return \( \tilde{\mu} \) and \( \tilde{p} \).

**Theorem 3.1.** Algorithm 1 correctly finds \( \tilde{\mu} \) and \( \tilde{p} \) in \( O(m + n \log n) \) time.

**Proof.** The algorithm clearly runs in \( O(m + n \log n) \) time. We now prove the correctness.

We first confirm that the function \( p' \) in Step 2 is indeed a potential for \( (G_{\mu'}, \tilde{w}_{\mu'}) \). Since \( p \) is a potential for \( (G_{\mu}, w_{\mu}) \), we have \( \tilde{w}_{\mu'}(e) = w_{\mu, p}(e) \geq 0 \) for every edge \( e \in E_{\mu'} \setminus \delta_G(s) \subseteq E_{\mu} \). Also, as every edge \( e = (s, v) \in \delta_G(s) \) is directed from \( s \) to \( v \) in \( G_{\mu'} \), we have
\[
\tilde{w}_{\mu'}(e) = \tilde{w}(e) + p'(s) - p'(v) = \tilde{w}(e) - p(v) - \min \{ \tilde{w}(e') - p(v') \mid e' = (s, v') \in \delta_G(s) \} 
\geq 0.
\]

We next show the correctness of the primal update. Since the edge \( e' = (s, t) \) itself forms an \( s-t \) path in \( G_{\mu'} \), the weight of \( P_t \) is at most \( \tilde{w}(e') \), and hence \( \tilde{w}(\tilde{\mu}) \leq \tilde{w}(\mu') + \tilde{w}(e') = \tilde{w}(\mu) \). Suppose to the contrary that there is a perfect matching \( \mu^* \) in \( G \) with \( \tilde{w}(\mu^*) < \tilde{w}(\tilde{\mu}) \leq \tilde{w}(\mu) \). Then, the symmetric difference \( \mu^* \triangle \mu \) contains a negative cycle \( C \) in \( (G_{\mu}, \tilde{w}_{\mu}) \) (where each edge in \( \mu \) is flipped). Since \( (G_{\mu}, w_{\mu}) \) has no negative cycle by Lemma 2.1 and \( \tilde{w} \) differs from \( w \) only around \( s \), the cycle \( C \) must intersect \( s \). This implies that \( C \) is the only negative cycle in \( \mu^* \triangle \mu \) and hence \( \tilde{w}_{\mu}(C) \leq \tilde{w}(\mu^*) - \tilde{w}(\mu) \). In addition, as \( e' \) is the only edge entering \( s \) in \( G_{\mu} \), it is contained in \( C \). By removing \( e' \) from \( C \), we obtain an \( s-t \) path \( P_t^* \) in \( (G_{\mu}, \tilde{w}_{\mu}) \) such that \( \tilde{w}_{\mu'}(P_t^*) = \tilde{w}_{\mu}(C) - \tilde{w}_{\mu}(e') \leq \tilde{w}(\mu^*) - \tilde{w}(\mu) + \tilde{w}(e') < \tilde{w}(\tilde{\mu}) - \tilde{w}(\mu') = \tilde{w}(\mu')(P_t) \), which contradicts the fact that \( P_t \) is shortest.

Finally, we show that \( \tilde{p} \) is a potential for \( (G_{\tilde{\mu}}, \tilde{w}_{\tilde{\mu}}) \). Since \( \tilde{p}(v) \) is the weight of the shortest \( s-v \) path \( P_v \) in \( (G_{\mu}, \tilde{w}_{\mu}) \) for any \( v \in V \), the function \( \tilde{p} \) is clearly a potential for \( (G_{\mu'}, \tilde{w}_{\mu'}) \). That is, we have \( \tilde{w}_{\mu'}(e) = \tilde{w}_{\mu'}(e) + \tilde{p}(u) - \tilde{p}(v) \geq 0 \) for each \( e = (u, v) \in E_{\mu'} \). In particular, the equality holds for every edge \( e \) used in the shortest \( s-v \) path \( P_v \) for some \( v \in V \). Observe that \( G_{\tilde{\mu}} \) is obtained from \( G_{\mu'} \) by flipping the direction of the edges on \( P_t \), whose \( w_{\mu', \tilde{p}} \) values are 0. Therefore, \( \tilde{w}_{\mu', \tilde{p}}(e) \geq 0 \) holds for every edge \( e \in E_{\tilde{\mu}} \), and hence \( \tilde{p} \) is a potential for \( (G_{\tilde{\mu}}, \tilde{w}_{\tilde{\mu}}) \).
Using Algorithm\[1\] one can design a simple, $O(n(m+n \log n))$-time algorithm for the assignment problem as follows. We first set $w'(e) \leftarrow 0$ for each $e \in E$, find any perfect matching $\mu$ in $G$, and initialize $p(v) \leftarrow 0$ for each $v \in V$. This part is done in $O(nm)$ time by a naïve augmenting path algorithm (or faster, e.g., by the Hopcroft–Karp algorithm). Clearly, $\mu$ is a minimum-weight perfect matching in $(G, w')$ and $p$ is a potential for $(G_\mu, w'_\mu)$. We then repeatedly make $w'$ close to the actual input weights $w$ by updating around a single vertex $s \in V^+$ in any order.

In each iteration, we set

$$\hat{w}(e) \leftarrow \begin{cases} w(e) & (e \in \delta_G(s)), \\ w'(e) & \text{(otherwise)}, \end{cases}$$

and find a minimum-weight perfect matching $\hat{\mu}$ in $(G, \hat{w})$ and a potential $\hat{p}$ for $(G_\hat{\mu}, \hat{w}_\hat{\mu})$ by applying Algorithm\[1\] to $(G, \hat{w}', \mu, p, s, \hat{w})$, where we temporarily add the absent edges around $s$ with sufficiently large weights if necessary. We then update $w' \leftarrow \hat{w}$, $\mu \leftarrow \hat{\mu}$, and $p \leftarrow \hat{p}$.

After the iterations, we have $w' = w$. The correctness and the computational time bound directly follow from Theorem \[3.1\]

### 3.2 Other Formulations

In this section, we discuss other polynomially equivalent formulations of our problem so that we can deal with various settings, including the advertisement allocation situation described in the introduction. In that situation, we are required to find a maximum-weight (not necessarily perfect) matching in an edge-weighted (not necessarily balanced) bipartite graph. First, by negating all the edge weights, the objective turns into minimizing the total weight of a matching.

Let $G = (V^+, V^−; E)$ be a bipartite graph with $|V^+| \geq |V^−| > 0$, and let $w: E \to \mathbb{R}$. Without loss of generality, we assume that $w$ is nonpositive, since any positive-weight edge can be excluded from an optimal matching. We say that a matching $\mu$ in $G$ is right-perfect if all the vertices in $V^−$ appear exactly once. By adding a vertex $v'$ to $V^+$ with an edge $(v', v)$ with weight 0 for each $v \in V^−$, any instance can be transformed into an instance that has a right-perfect optimal matching. Note that the input size does not essentially increase. In what follow, we let $(G, w)$ be the graph after this transformation, so we have $|V^+| > |V^−|$. Let $n = |V^+|$ and $m = |E|$.

We now consider the analogous problem: we have a minimum-weight right-perfect matching $\mu$ in an edge-weighted unbalanced bipartite graph $(G, w)$ and a potential $p$ for $(G_\mu, w_\mu)$, the edge weights $w$ change arbitrarily but only around a single vertex $s \in V$, and we are asked to find a minimum-weight right-perfect matching $\hat{\mu}$ in the new graph $(G, \hat{w})$ and a potential $\hat{p}$ for $(G_\hat{\mu}, \hat{w}_\hat{\mu})$.

By adding $|V^+| − |V^−|$ dummy vertices to $V^−$ that are adjacent to all the vertices in $V^+$ by edges with weight 0, this problem reduces to the problem solved in Section \[3.1\]. Let $(G', w')$ be the edge-weighted balanced bipartite graph obtained by this reduction, and let $V'$ and $E'$ denote its vertex set and edge set, respectively. We then have $|V'| = 2n$, but $|E'|$ may be $\omega(n + n \log n)$ if $G$ is sparse and far from balanced. Hence, this naïve reduction may worsen the computational time bound in Theorem \[3.1\].

We show that one can solve this problem by emulating Algorithm\[1\] for $(G', w')$ in $O(m + n \log n)$ time by avoiding an explicit construction of $(G', w')$. Let $\mu'$ and $p'$ be the corresponding optimal solutions and $(G', \hat{w}')$ be the corresponding new graph in the reduced problem.

**Lemma 3.2.** If $(G_\mu', \hat{w}_\hat{\mu}')$ contains a negative cycle, then there exists a shortest cycle that intersects at most one dummy vertex.
Proof. Let \( C \) be a shortest cycle in \((G_{\mu'}, \tilde{w}_{\mu'})\) that consists of as few edges as possible. We show that \( C \) intersects at most one dummy vertex. Suppose to the contrary that \( C \) intersects two distinct dummy vertices \( v_1 \) and \( v_2 \) in the extended \( V^- \). Then, \( C \) is divided into two paths \( P_1 \) from \( v_1 \) to \( v_2 \) and \( P_2 \) from \( v_2 \) to \( v_1 \). The negative cycle \( C \) must intersect \( s \) by Lemma 2.4, as \( s \) is not a dummy vertex, we assume that \( P_1 \) intersects \( s \) and \( P_2 \) does not without loss of generality.

Let \( u \in V^+ \) be the vertex just before \( v_1 \) on \( P_2 \). If \( \mu'(u) = v_2 \), then \( \tilde{w}_{\mu'}(P_2) = \tilde{w}(u, v_2) = 0 \) by definition. Otherwise, the graph \( G'_{\mu'} \) contains a directed edge \( e = (u, v_2) \), and one can obtain a cycle \( C' \) from \( P_2 \) by replacing the last edge \((u, v_1)\) with \( e \). The cycle \( C' \) does not contain \( s \), and hence \( \tilde{w}_{\mu'}(P_2) = \tilde{w}_{\mu'}(C') \geq 0 \) by Lemma 2.4. Thus, in either case, we have \( \tilde{w}_{\mu'}(P_2) \geq 0 \), which implies \( \mu'(P_1) < 0 \).

Let \( u' \in V^+ \) be the vertex just before \( v_2 \) on \( P_1 \). If \( \mu'(u') = v_1 \), then \( \tilde{w}_{\mu'}(P_1) = \tilde{w}(u', v_1) \) and \( \tilde{w}(u', v_2) = 0 \), a contradiction. Thus, there exists a directed edge \( e' = (u', v_1) \) in \( G'_{\mu'} \), and one can obtain a cycle \( C'' \) with \( \tilde{w}_{\mu'}(C'') \leq \tilde{w}_{\mu'}(C) \) and \( |C''| < |C| \) from \( C \) by replacing the subpath consisting of \((u', v_2)\) and \( P_2 \) with \( e' \). This contradicts the choice of \( C \), and we are done. \( \square \)

**Lemma 3.3.** For any vertex \( v \in V \), there exists a shortest \( s-v \) path in \((G'_{\mu'}, \tilde{w}_{\mu'})\) that intersects at most one dummy vertex. Moreover, for all the dummy vertices, the weights of the shortest paths are the same.

Proof. The first claim is shown by almost the same argument as the proof of Lemma 3.2. Pick a shortest \( s-v \) path \( P \) in \((G'_{\mu'}, \tilde{w}_{\mu'})\) that consists of as few edges as possible. If \( P \) intersects two distinct dummy vertices \( v_1 \) and \( v_2 \), then the subpath between \( v_1 \) and \( v_2 \) can be skipped without increasing the weight, which contradicts the minimality of \( P \).

To see the second claim, for two distinct dummy vertices \( v_1 \) and \( v_2 \), let \( u_1, u_2 \) be the vertices such that \((u_1, v_1), (u_2, v_2) \in \mu' \). Then, by definition, \((G'_{\mu'}, \tilde{w}_{\mu'})\) has four edges \((v_1, u_1), (v_2, u_2), (u_1, v_2)\), and \((u_2, v_1)\) of weight 0, and hence we have \( d(v_2) \leq d(v_1) \) and \( d(v_1) \leq d(v_2) \).

Lemmas 3.2 and 3.3 imply that, in order to emulate Algorithm 1 for \((G', \tilde{w}')\), it suffices to introduce only one dummy vertex \( x \) that is adjacent to all the vertices \( V^+ \) by edges of weight 0 and to modify the perfect matching constraint so that \( x \) appears exactly \(|V^+| - |V^-|\) times. Thus we are done.

### 4 An Improved Variant of Envy-Cycle Procedure

In this section, as an application of our algorithm, we show an improved variant of the envy-cycle procedure for finding an envy-free allocation of indivisible goods. We describe the problem setting and the envy-cycle procedure in Section 4.1, and demonstrate how to make the algorithm faster via the assignment problem in Section 4.2.

#### 4.1 Envy-Free Allocation of Indivisible Goods

Consider a situation in which \( n \) agents share \( m \) indivisible items. Let \( N = [n] := \{1, 2, \ldots, n\} \) and \( M = [m] \) be the sets of agents and items, respectively. Each agent \( i \in N \) evaluates each item \( k \in M \) by a nonnegative real \( v_{i,k} \), and the evaluation is additive. That is, each item subset \( X \subseteq M \), called a bundle, is evaluated by the sum \( v_i(X) := \sum_{k \in X} v_{i,k} \). An allocation is an \( N \)-indexed subpartition \((X_i)_{i \in N} \) of \( M \), i.e., \( X_i \subseteq M \) for each \( i \in N \) and \( X_i \cap X_j = \emptyset \) for distinct \( i, j \in N \).
In an allocation \((X_i)_{i \in N}\), an agent \(i \in N\) envies another agent \(j \in N\) if \(v_i(X_i) < v_i(X_j)\). Since each item is indivisible, it is usually impossible to achieve a completely envy-free allocation of the whole item set \(M\). The following describes two well-studied relaxations of envy-freeness [4,9].

**Definition 4.1.** Let \(\mathcal{X} = (X_i)_{i \in N}\) be an allocation.

- \(\mathcal{X}\) is envy-free up to one good, and EF1 for short, if, for every pair of agents \(i, j \in N\), we have \(v_i(X_i) \geq v_i(X_j)\) or \(v_i(X_i) \geq v_i(X_j \setminus \{k\})\) for some \(k \in X_j\).
- \(\mathcal{X}\) is envy-free up to any good, and EFX for short, if, for every pair of agents \(i, j \in N\), we have \(v_i(X_i) \geq v_i(X_j \setminus \{k\})\) for every \(k \in X_J\).

Clearly, an EFX allocation is EF1. It is known that an EF1 allocation always exists and can be found by a rather simple algorithm, the so-called envy-cycle procedure [9], while the existence of an EFX allocation is known for very restricted situations and is widely open in general (see, e.g., [1]).

The envy-cycle procedure is summarized as follows. We initialize \(\mathcal{X} \leftarrow (X_i)_{i \in N}\) with \(X_i = \emptyset\) for each \(i \in N\), and distribute the items one-by-one in any order. In each iteration, we consider the envy graph \(H(\mathcal{X}) = (N, F(\mathcal{X}))\) for the current allocation \(\mathcal{X}\), which is defined by

\[
F(\mathcal{X}) := \{ (i, j) \mid i, j \in N, v_i(X_i) < v_i(X_j) \}.
\]

While the graph \(H(\mathcal{X})\) contains a cycle, we update \(\mathcal{X}\) by exchanging the bundles along it; that is, pick a cycle \(C\) and update \(X_i \leftarrow X_j\) for all the edges \((i, j) \in C\) simultaneously.\(^2\) After that, as \(H(\mathcal{X})\) has no cycle, there exists an agent \(i \in N\) having no incoming edge. We distribute to such an agent \(i\) an unallocated item, say \(k \in M \setminus \bigcup_{i \in N} X_i\); that is, we set \(X_i \leftarrow X_i \cup \{k\}\).

It is easy to observe that, at any point of this algorithm, \(\mathcal{X}\) is an EF1 allocation of the set of items distributed so far. The computational time is bounded by counting the number of edges in \(H(\mathcal{X})\), which can increase only when an item is distributed and must decrease when the bundles are exchanged along a cycle.

**Theorem 4.2** (Lipton et al. [9, Theorem 2.1]). The envy-cycle procedure correctly finds an EF1 allocation in \(O(mn^3)\) time.

It was pointed out in [2] that, if the valuations are in the same order for all the agents, this algorithm can be utilized for finding an EFX allocation.

**Theorem 4.3.** Suppose that \(v_{i,1} \geq v_{i,2} \geq \cdots \geq v_{i,m}\) for every agent \(i \in N\). Then, if the items are distributed in ascending order of \(k = 1, 2, \ldots, m\), the envy-cycle procedure correctly finds an EFX allocation in \(O(mn^3)\) time.

### 4.2 Improvement via Assignment Problem

First, we slightly modify the definition of an allocation by regarding it as a combination of “division” and “assignment.” Hereafter, an allocation is a pair \((\mathcal{X}, \mu)\) of an \(N\)-indexed subpartition \(\mathcal{X} = (X_i)_{i \in N}\) of \(M\) and a bijection \(\mu : N \rightarrow \mathcal{X}\), which means that the bundle \(\mu(i) \in \mathcal{X}\) is assigned to the agent \(i \in N\). We say that \((\mathcal{X}, \mu)\) is EF1 or EFX if this is true of the subpartition \((X'_i)_{i \in N}\) with \(X'_i = \mu(i) (i \in N)\) in the sense of the original definition (see Definition 4.1). The welfare of an allocation \((\mathcal{X}, \mu)\) is defined as \(W(\mathcal{X}, \mu) = \sum_{i \in N} v_i(\mu(i))\).

\(^2\)This exchange is usually repeated until some agent has no incoming edge, and it suffices. We employ the present form in order to make its behavior analogous to the improved variant given in the next section.
Definition 4.4. For an allocation \((\mathcal{X}, \mu)\) with the bundle set \(\mathcal{X} = (X_i)_{i \in N}\), the assign-envy graph \(G(\mathcal{X}, \mu) = (N, \mathcal{X}; E(\mathcal{X}, \mu))\) is a bipartite graph with edge weights \(w: E(\mathcal{X}, \mu) \rightarrow \mathbb{R}\) defined by

\[
E(\mathcal{X}, \mu) := \{(i, \mu(i)) \mid i \in N\} \cup \{(i, X_j) \mid i \in N, X_j \in \mathcal{X}, v_i(\mu(i)) < v_i(X_j)\},
\]

\[
w(e) := -v_i(X_j) \quad (e = (i, j) \in E(\mathcal{X}, \mu)).
\]

Note that the bijection \(\mu: N \rightarrow \mathcal{X}\) is regarded as a perfect matching in \(G(\mathcal{X}, \mu)\). Note also that, if we contract each edge \((i, \mu(i))\) in \(G(\mathcal{X}, \mu)\), then the resulting graph coincides with the envy graph \(H(\mathcal{X}')\) for \(\mathcal{X}' = (X'_i)_{i \in N}\) with \(X'_i = \mu(i)\) \((i \in N)\), which is defined in Section 4.1.

Lemma 4.5. Let \((G, w)\) be the assign-envy graph for an allocation \((\mathcal{X}, \mu)\), and \((G_{\mu}, w_{\mu})\) the auxiliary weighted graph for \(\mu\). If \(G_{\mu}\) has a cycle \(C\), then the weight \(w_{\mu}(C)\) is negative.

Proof. If an allocation is updated by exchanging bundles along a cycle in the envy graph, each agent in the cycle receives a more valuable bundle, and hence the welfare strictly increases. This is also true for the assign-envy graph, and by the definitions of \(w\) and \((G_{\mu}, w_{\mu})\), the weight \(w_{\mu}(C)\) is equal to \(W(\mathcal{X}, \mu) - W(\mathcal{X}, \mu') < 0\), where \(\mu'\) is the assignment after exchange along \(C\).

Lemma 4.6. Let \((G, w)\) be the assign-envy graph for an allocation \((\mathcal{X}, \mu)\), and \(\mu^*\) be a minimum-weight perfect matching in \((G, w)\). Let \((G^*, w^*) := G(\mathcal{X}, \mu^*)\) and let \((G^*_{\mu^*}, w^*_{\mu^*})\) be the auxiliary weighted graph for \(\mu^*\). Then, the directed graph \(G^*_{\mu^*}\) has no cycle.

Proof. Suppose to the contrary that \(G^*_{\mu^*}\) has a cycle \(C\). We then have \(w^*_{\mu^*}(C) < 0\) by Lemma 4.5. Let \(\mu'\) be the assignment obtained from \(\mu^*\) by exchange along \(C\). We then have \(W(\mathcal{X}, \mu') > W(\mathcal{X}, \mu^*)\) and \(v_i(\mu'(i)) > v_i(\mu^*(i))\) for every \(i \in N\) with \(\mu'(i) \neq \mu^*(i)\). The latter implies that all the edges in \(\mu'\) exist in \(G\), and thus the former contradicts that \(\mu^*\) is a minimum-weight perfect matching in \((G, w)\).

By Lemma 4.6 instead of repeatedly exchanging the bundles along cycles in the envy graph, it suffices to find a minimum-weight perfect matching in the assign-envy graph once in each iteration of the envy-cycle procedure. Thus, we obtain its variant as shown in Algorithm 2.

Algorithm 2  Envy-Cycle Procedure via Assignment Problem

Input: A set \(N\) of agents, a set \(M\) of items, and valuations \(v_{i,k}\) \((i \in N, k \in M)\).

Output: An EF1 allocation \((\mathcal{X}, \mu)\) of \(M\).

1. Initialize \(\mathcal{X} \leftarrow (X_i)_{i \in N}\) with \(X_i = \emptyset\) for each \(i \in N\), and \(\mu(i) \leftarrow X_i\) for each \(i \in N\).

2. For each item \(k \in M\) (in any order), do the following.

   (a) Find a minimum-weight perfect matching \(\mu^*\) in \((G(\mathcal{X}, \mu), w)\), and update \(\mu \leftarrow \mu^*\).

   (b) Pick a bundle \(X_i \in \mathcal{X}\) that has only one incident edge in \(G(\mathcal{X}, \mu)\) (which is \((\mu^{-1}(X_i), X_i) \in \mu)\), and update \(X_i \leftarrow X_i \cup \{k\}\).

3. Return \(\mathcal{X}\).

The number of edges in \(G(\mathcal{X}, \mu)\) is always bounded by \(O(n^2)\). Hence, by using the Hungarian method in Step 2(a), this algorithm correctly finds an EF1 allocation in \(O(mn^3)\) time, which is the same bound as Theorem 4.2.
We improve this bound with the aid of Algorithm 1 given in Section 3.1. We deal with \( G(\mathcal{X}, \mu) \) as the complete bipartite graph by adding all the absent edges with sufficiently large weights. Then, in each iteration of Step 2, only the weights of edges around the bundle \( X_i \in \mathcal{X} \) chosen in Step 2(b) change, and hence we can correctly update the current assignment \( \mu \) and the corresponding potential in \( O(n^2) \) time by Algorithm 1. Thus, the total computational time is bounded by \( O(mn^2) \), which leads to the following theorems analogous to Theorems 4.2 and 4.3.

**Theorem 4.7.** Algorithm 2 correctly finds an EF1 allocation in \( O(mn^2) \) time.

**Theorem 4.8.** Suppose that \( v_{i,1} \geq v_{i,2} \geq \cdots \geq v_{i,m} \) for every agent \( i \in N \). Then, if the items are chosen in ascending order of \( k = 1, 2, \ldots, m \) in Step 2, Algorithm 2 correctly finds an EFX allocation in \( O(mn^2) \) time.

**Remark 4.9.** The computational time is actually bounded by using the number of edges in \( G(\mathcal{X}, \mu) \), which may almost always be \( \Omega(n^2) \) in the worst case. The algorithm, however, runs much faster than the theoretical bound in practice, and this is also true for the original envy-cycle procedure, because an acyclic graph tends to be sparse. In addition, in this special situation, without keeping a potential, we can update the current assignment \( \mu \) by solving the shortest path problem in directed acyclic graphs as follows; this slightly improves the running time when \( G(\mathcal{X}, \mu) \) is almost always sparse, with \( o(n \log n) \) edges in particular.

Let \((G, w)\) and \((G', w')\) be the assign-envy graphs for the allocations \((\mathcal{X}, \mu)\) just before and after Step 2(b), respectively. As \( \mu \) is a minimum-weight perfect matching in \((G, w)\), the auxiliary weighted graph \((G_\mu, w_\mu)\) has no cycle by Lemma 4.5. In addition, since \((G', w')\) coincides with \((G, w)\) except around \( X_i \in \mathcal{X} \) picked in Step 2(b), every cycle in \( G'_\mu \) intersects \( X_i \). This means that the auxiliary weighted graph \((G'_\mu, w'_\mu)\) to which we apply Dijkstra’s algorithm in Step 2 of Algorithm 1 is always acyclic, where \( \mu' = \mu \setminus \{(\mu^{-1}(X_i), X_i)\} \). Thus, Dijkstra’s algorithm with a potential is not necessary, and the elementary, linear-time dynamic programming is sufficient for computing a shortest path from \( \mu^{-1}(X_i) \) to \( X_i \).

**Remark 4.10.** As originally written in [9], Theorem 4.2 holds when the valuations of the agents are not necessarily additive but nonnegative and monotone. That is, each agent \( i \in N \) evaluates each bundle \( X \subseteq M \) by \( v_i(X) \) according to a set function \( v_i : 2^M \to \mathbb{R}_{\geq 0} \) such that \( v_i(X) \leq v_i(Y) \) if \( X \subseteq Y \subseteq M \), where we assume that each function \( v_i \) is given as an oracle in the algorithm. This is also true of Theorem 4.7 since we do not use the additivity in Definition 4.4, Lemmas 4.5 and 4.6, and Algorithm 2.

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