THERMODYNAMICS OF THE TWO-DIMENSIONAL BLACK HOLE IN THE TEITELBOIM-JACKIW THEORY

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Abstract

The two-dimensional theory of Teitelboim and Jackiw has constant and negative curvature. In spite of this, the theory admits a black hole solution with no singularities. In this work we study the thermodynamics of this black hole using York’s formalism.

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1. Introduction

Thermal radiation from black holes via the Hawking process hints that gravity, quantum mechanics and thermodynamics are linked together. The analysis of quantum fields in a black hole background has first appeared in four dimensional (4D) general relativity. It was then extended to lower dimensions and other theories, following indications from string theory that these are important and useful to study.

Two dimensions (2D) has been of particular interest after a black hole in string theory has appeared [1, 2]. Hawking radiation and thermodynamics of this black hole has been analysed by several authors (e.g., [3, 4, 5, 6]). Another 2D theory which has been studied in some detail is the Teitelboim-Jackiw theory [7, 8]. Although in this theory the curvature is constant and negative, it has a black hole solution [9, 10, 11, 12, 13, 14]. The existence of a black hole implies a non-trivial causal structure which in turn generates interesting non-trivial thermodynamics. Hawking radiation of this black hole has been analysed in [12], and thermodynamics of a black hole in a version of the theory with electromagnetic fields has been studied in [15].

Here we study the black hole of the original Teitelboim-Jackiw theory using York’s formalism [16, 17]. In 2D this formalism has already been used in [5] to study the 2D black hole in string theory. The formalism uses the fact that for a system of fixed size and fixed temperature the canonical partition function $Z_c$ characterizes thermodynamic equilibrium in the canonical ensemble. The free energy $F$ and the partition function are linked through $-\beta F = \log Z_c$, where $\beta$ is the inverse temperature. On the other hand $Z_c$ can be represented by a path integral, through a relation with the Euclidean action $I_E$ given by $I_E = \beta F = -\log Z_c$. As a path integral, the partition function depends on quantities that are fixed in the functional integration such as the boundary data chosen from the fields of the system.

2. The Lorentzian Black Hole Solution

In the Teitelboim-Jackiw 2D theory the action is

$$I = \frac{1}{2\pi} \int d^2x \sqrt{-g} e^{\Phi} (R - 2\Lambda) + I_B,$$

where $g$ is the determinant of the metric, $R$ is the curvature scalar, $\Lambda$ is
the cosmological constant (sometimes written as $\Lambda = -2\lambda^2$), and $I_B$ is a boundary term to specify later. A 2D metric can always be written as

$$ds^2 = -A(dx^0)^2 + \frac{(dx^1)^2}{P},$$  \hspace{1cm} (2)

where $x^0$ and $x^1$ are the time and spatial coordinates, respectively, and $A$ and $P$ are metric functions. The action (1) has got a black hole solution given in the unitary gauge by

$$ds^2 = - \sinh^2(\alpha x)dt^2 + dx^2,$$  \hspace{1cm} (3)

$$e^\Phi = e^{\Phi_0} \cosh(\alpha x),$$  \hspace{1cm} (4)

where the range of $x$ is $-\infty < x < +\infty$, $\Phi_0$ is a constant, and $\alpha^2 \equiv -\Lambda$. Transforming to the Schwarzschild gauge, through $r = \sqrt{b} \cosh \alpha x$, with $b$ constant greater than zero, one obtains

$$ds^2 = -(\alpha^2 r^2 - b)dt^2 + \frac{dr^2}{\alpha^2 r^2 - b},$$  \hspace{1cm} (5)

$$e^\Phi = c\alpha r,$$  \hspace{1cm} (6)

where $c$ is a constant.

The maximal analytical extension of (3) (or (5)) is represented in the Penrose diagram of figure 1. It is clear from that diagram and the metric given in equation (3) that the radius $r = \frac{\sqrt{b}}{\alpha}$ (or $x = 0$) can represent a horizon. However, it can also be a coordinate trick. Indeed the curvature scalar of the solution is $R = -\alpha^2$ which is a constant. Therefore, spacetime has constant negative curvature and, in principle, is anti-de Sitter spacetime. Now, anti-de Sitter spacetime has, in the unitary gauge, a metric given by

$$ds^2 = -\cosh^2(\alpha x)dt^2 + dx^2,$$  \hspace{1cm} (7)

and dilaton,

$$e^\Phi = e^{\Phi_0} \sinh(\alpha x).$$  \hspace{1cm} (8)

To transform to the Schwarzschild gauge we put $\tau = \frac{\sqrt{b}}{\alpha} \sinh \alpha x$ and obtain

$$ds^2 = -(\alpha^2 \tau^2 + b)dt^2 + \frac{d\tau^2}{(\alpha^2 \tau^2 + b)}.$$  \hspace{1cm} (9)
\[ e^\Phi = \tau \alpha \tau \]  

(10)

where \( \tau \) is a constant, and \( b > 0 \) is also a constant which in this case can always be set to one, \( \bar{b} = 1 \). The maximal analytical extension of (9) (or (7)) is given by the usual anti-de Sitter extension [18]. It is then clear that \( r \) and \( \tau \) are totally different coordinates. However, a set of transformations can indeed be found [12] as one might expect, since spacetime in both coordinates has constant negative curvature. Thus, in what sense can (9)-(10) be interpreted as a black hole? Or, in other words, in what sense are (9)-(10) and (3)-(4) different physical solutions?

The interpretation of (3)-(4) as a black hole comes from theories in 3D and 4D. It was shown in [10] that action (1) comes from dimensional reduction of 3D general relativity. 3D general relativity admits a static black hole solution with circular symmetry [19]. Solution (3)-(4) gives the corresponding 2D black hole. In the 3D theory \( e^\Phi \) is the circumference radius. On the other hand, it was also shown in [11] that (1) comes from dimensional reduction of a low energy 4D action of heterotic string theory, but now \( e^\Phi \) represents instead the string coupling. This 4D action admits near-extremal magnetic black holes which in turn generate the ansatz for the dimensional reduction process. In both 3D and 4D theories it does not make sense in physical terms to have a negative \( e^\Phi \). Thus, when (1) is used to model 3D and 4D black holes (as, for instance, s-wave scattering models in quantum evaporation of black holes), one has to cut the 2D spacetime at \( r = 0 \). In both cases, it is the dilaton the field which sets this boundary condition. Therefore, solutions with the same local metric properties as in (3)-(4) and (9)-(10) are in fact topologically different. There is an extremal black hole solution given when the parameter \( b = 0 \), see figure 3.

There is also the possibility of interpreting the solution (3)-(4) as a black hole without having to resort to higher dimensions. The idea in [13] is that the line \(-\infty < x < \infty\) (defined in (3)-(4)) corresponds to the segment \( \sqrt{\frac{b}{\alpha}} < r < \infty \). Each pair of space inverted points \((-x, x)\) degenerates into one \( r \). A slice at constant (Penrose) time in the diagram of figure 1 is shown in figure 3. There is a horizon at \( r = \sqrt{\frac{b}{\alpha}} \), i.e., \( x = 0 \). Observers at each end of the line \( x \to \pm \infty \) can only communicate if they enter through \( x = 0 \). The \( x = 0 \) segment is a null line, and test particles in timelike geodesics in one of the ends of the world \( (x \to \pm \infty) \) will cross this horizon in a finite time. There is a problem in this interpretation. As figure 4 indicates, there is a
cusp (i.e., a singularity) at the junction \( x = 0 \). Observers (or particles) when entering a new world have to decide which end (positive or negative \( x \)) they will join.

Another 2D interpretation can be given to (5)-(6). One can notice that metric (5) represents a portion of the 2D anti-de Sitter spacetime in accelerated coordinates. Indeed, a stationary observer with \( r = \text{constant} \) in spacetime given by (5) has four acceleration \( a^\mu \) with magnitude \( a = \sqrt{a^\mu a_\mu} \) given by

\[
a = \frac{\alpha^2 r}{\sqrt{\alpha^2 r^2 - b}},
\]

with \( b > 0 \). The radius \( r = \sqrt{\frac{2}{\alpha}} \), where the acceleration is infinite, corresponds to the trajectory of a light ray. Thus, observers held at \( r = \text{constant} \) see this light ray as a horizon, they will never see events beyond this ray. They are accelerated observers and can see only a portion of anti-de Sitter spacetime. In this sense, region II in figure 1, can be considered a black hole for region I accelerated observers. Note that for anti-de Sitter, \( r = \text{constant} \) trajectories are straight vertical lines in the corresponding Penrose diagram \cite{18}. In these coordinates the acceleration is \( a = \frac{\alpha^2 r}{\sqrt{\alpha^2 r^2 + b}} \), \( b > 0 \). There is no infinite acceleration for such observers. The situation is analogous to the relation that Rindler and Minkowski 2D spacetimes bear with each other. However, here, there is an extra field, the dilaton.

Thus, equations (5)-(6) represent a black hole in several different physical interpretations. In view of this it is interesting to show that this black hole solution has non-trivial thermodynamics. We use here the formalism developed by York \cite{16, 17} to understand the thermal behavior of the black hole, (for other types of formalism see \cite{12, 15}).

The mass of the black hole of equation (5) can be calculated by the standard procedures \cite{14} and is given by,

\[
M = \frac{\alpha c}{2} b.
\]

3. The Euclidean Black Hole and its Reduced Action

We now follow \cite{17, 6} to find the reduced action of the system. We assume that there is a black hole inside a cavity with boundary \( B \). Now, the
Euclideanized form of the metric (2) can be written as \((\eta = ix^0, \rho = x^1)\),

\[
\begin{align*}
\ds_E^2 &= A \, d\eta^2 + \frac{d\rho^2}{P}.
\end{align*}
\]

(13)

Here \(\eta\) is a periodic coordinate running from 0 to \(2\pi\) and \(\rho\) runs from 0 at the horizon to \(\rho_B\) at the boundary. The values of the metric function \(A\) and dilaton \(\Phi\) at the boundary are denoted by \(A_B\) and \(\Phi_B\). The inverse temperature \(\beta\) at the boundary is related to the proper length of the boundary circle \(S^1\) through the relation,

\[
\beta = \int_0^{2\pi} \sqrt{A_B} \, d\eta = 2\pi \sqrt{A_B}.
\]

(14)

The regularity conditions of the metric and dilaton fields at the horizon imply,

\[
\sqrt{P} \left(\sqrt{A}\right)'_{\rho=0} = 1
\]

(15)

and

\[
P \Phi'^2 \big|_{\rho=0} = 0,
\]

(16)

where \(\prime \equiv \frac{\partial}{\partial \rho}\).

The Euclidean action can be obtained from (1),

\[
I_E = -\frac{1}{2} \int_V d^2 x \sqrt{g} e^\Phi (R + \alpha^2) - \int_{\partial V} d\rho \sqrt{h} e^\Phi (K - K^0),
\]

(17)

where the surface term is required to make the variational procedure self-consistent, which is important in analysing the thermodynamics, \(h\) is the induced metric on the boundary, \(K\) is the extrinsic curvature and \(K^0\) is a term necessary to choose the background (the zero point energy). As before, \(\alpha^2 = 2\lambda^2 = -\Lambda\). The equations of motion derived from (17) are,

\[
e^{\Phi} T_{ab} \equiv \frac{1}{2} D_a \Phi D_b \Phi + \frac{1}{2} D_a D_b \Phi - \frac{1}{2} g_{ab} D_c \Phi D^c \Phi + \frac{1}{2} g_{ab} D_c \Phi D^c \Phi - \frac{1}{2} g_{ab} \alpha^2 = 0.
\]

(18)

Then the \(T_{00}\) constraint, \(T_{00} = 0\), gives,

\[
\left[ (P \Phi'^2 - \alpha^2) e^{2\Phi} \right]' = 0,
\]

(19)

whose solution is

\[
P \Phi'^2 - \alpha^2 = -\alpha^2 be^{-2\Phi},
\]

(20)
where we have chosen the constant of integration as $-\alpha^2 b$ appropriately.

Now, using,

$$\sqrt{g} R = -\left(\frac{\sqrt{PA'}}{\sqrt{A}}\right)' , \quad \sqrt{g} = \frac{\sqrt{A}}{P} , \quad \sqrt{h} = \sqrt{A} , \quad K = \frac{1}{2} \frac{\sqrt{PA'}}{A},$$

we can transform (17) into the following:

$$I_E = -\frac{1}{2} \int d\eta d\rho e^\Phi \left(\frac{\sqrt{PA'}}{\sqrt{A}} \Phi' + \sqrt{AP} \alpha^2\right) - \frac{1}{2} \int d\eta e^\Phi \left[\frac{\sqrt{P}}{\sqrt{A}} A'\right]_{\rho=0} + I_0, \quad (22)$$

where $I_0 \equiv \int_{\partial \mathcal{V}} d\rho \sqrt{h} e^\Phi K^0$ is an important term for choosing the background. Then, integrating (22) and using the constraints and boundary conditions we find,

$$I(h^{-1}) = -(G^{-1})\beta e^{\Phi_B} \alpha \sqrt{1 - e^{2(\Phi_H - \Phi_B)}} - (h^{-1})2\pi e^\Phi + (G^{-1})\beta e^{\Phi_B} \alpha, \quad (23)$$

where $\Phi_H$ is the value of $\Phi$ at the horizon and $I_0 \equiv \beta e^{\Phi_B} \alpha$ was chosen appropriately. In (23) we have put back Newton’s constant $G$ and Planck’s constant $h$ (still putting Boltzmann’s constant and the velocity of the light equal to one). Note that in 2D we use the following units for the constants:

$[G] = LM^{-1}T^{-1}$ and $[h] = MT^{-1}$. As in 4D [17], one sees that a quantum term has appeared in the action, namely the term $2\pi e^\Phi$, which is associated with the entropy of the system. Equation (23) is thus the reduced action $I = I(\beta, \Phi_B; \Phi_H)$ which yields the important thermodynamic quantities.

4. Temperature and the Canonical Boundary Conditions

To find the temperature we have to obtain the stationary point of the reduced action, by differentiating $I(\beta, \Phi_B; \Phi_H)$ with respect to $\Phi_H$. Setting the resulting equation to zero, i. e., $\frac{\partial I}{\partial \Phi_H} = 0$, we find,

$$\beta = \frac{2\pi}{\alpha^2} \sqrt{W_B} e^{-(\Phi_H - \Phi_B)}, \quad (24)$$

where,

$$W_B = \alpha^2(1 - e^{2(\Phi_H - \Phi_B)}). \quad (25)$$
Equation (24) gives the inverse of the temperature ($\beta = \frac{1}{T}$) of the 2D black hole.

Now, a thermal equilibrium configuration in the canonical ensemble, has to yield $\Phi_H$ as a function of $\beta$. Indeed, inverting (24) gives

$$\Phi_H = \Phi_B - \frac{1}{2} \ln(1 + \frac{\alpha^2 \beta^2}{4\pi^2})$$

or in terms of the Schwarzschild gauge of equation (3) (where, $e^{\Phi_H} = \alpha r_H = \sqrt{b} = \sqrt{\frac{2M}{\alpha c}}$) we find from (26),

$$\frac{2M}{\alpha c} = \frac{\alpha^2 r_H^2}{1 + \frac{\alpha^2 \beta^2}{4\pi^2}}.$$  (27)

Thus as $T \to 0$ we have $M \to 0$. As $T \to \infty$ we have a maximum mass $M_{\text{max}} = \frac{1}{2} \alpha^3 c r_B^2$ for the BH in the thermal bath. That is, for a given $r_B$ the mass of the hole cannot be larger than the one which gives a horizon radius equal to $r_B$. There is nothing like the instanton solution of the Schwarzschild bath in 4D.

In figure 5 we draw the graph, $r_H$ as a function of $r_B$. We see that, at equilibrium, for $T \to \infty$ one has $r_H = r_B$ for any $r_B$, while for $T \to 0$ one has that $r_H$ is very small in relation to $r_B$. This means that for very high temperatures, the boundary is located at the horizon, precisely. At low temperatures the boundary has to be far from the horizon radius.

We now study some thermodynamic quantities in this canonical ensemble formulation. We also analyse thermodynamic stability.

5. Thermodynamical Quantities

The entropy is defined through the equation

$$S_H = \beta \left( \frac{\partial I}{\partial \beta} \right)_{\Phi_B} - I.$$  (28)

Using (23) we find,

$$S_H = 2\pi e^{\Phi_H}.$$  (29)
which has the same functional expression as the one found in [5]. In the Schwarzschild gauge it gives,

$$S_H = 2\pi \sqrt{\frac{2M}{\alpha c}}. \tag{30}$$

It is interesting to note that the functional dependence given in (29) is the same for all black holes having a simple 2D Brans-Dicke action [20]. Note also that the extreme case ($M = 0$) has zero entropy.

The thermodynamic energy $E$ is defined by

$$E \equiv \left( \frac{\partial I}{\partial \beta} \right)_{\Phi_B}. \tag{31}$$

Then from (23) we obtain

$$E = \alpha e^{\Phi_B} - \alpha e^{\Phi_B} \sqrt{1 - e^{2(\Phi_H - \Phi_B)}}, \tag{32}$$

which, in the Schwarzschild gauge, can be put in the form

$$E = c\alpha^2 r_B \left(1 - \sqrt{1 - \frac{r_H^2}{r_B^2}}\right). \tag{33}$$

We see here that the zero point was chosen so that when there is no mass ($r_H = 0$) the thermal energy is zero. Since $r_H^2 = \frac{2M}{\alpha c}$ we can invert expression (33) to yield

$$\frac{1}{\alpha r_B} M = E - \frac{1}{\alpha^2 c^2 r_B} E^2, \tag{34}$$

which relates the ADM mass and the thermal energy. The ADM mass (the mass at infinity) is equal to the thermal energy times the length (in intrinsic units) of the reservoir minus a self-energy thermal term. Expression (34) is the closest one can get to the Schwarzschild expression found in [16] for the Schwarzschild mass, i.e., $M = E - \frac{1}{2} \frac{E^2}{r_B}$.

Now, we want to find the Euler relation for this thermodynamic system. From (24) we obtain the temperature $T = \frac{1}{\beta}$,

$$T = \frac{\alpha r_H}{2\pi r_B} \frac{1}{\sqrt{1 - \frac{r_H^2}{r_B^2}}}. \tag{35}$$
We define a linear pressure by

\[ p = -\frac{\partial E}{\partial r_B} = \left( 1 - \sqrt{1 - \frac{r_H^2}{r_B^2}} \right). \quad (36) \]

Then, using (30), (34), (35) and (36) we obtain

\[ dE = TdS - pdr_B. \quad (37) \]

After integration we obtain the Euler relation

\[ E = TS - pr_B. \quad (38) \]

Upon scaling, \( r_B \to lr_B \) and \( r_H \to lr_H \) or \( S \to lS \) one has \( E \to lE \). Thus, \( E \) is homogeneous of degree 1 in \( S \) and \( r_B \).

To analyse thermodynamic stability we first find the heat capacity. For 2D black holes it is defined by

\[ C_{r_B} \equiv T \left( \frac{\partial S}{\partial T} \right)_{r_B}. \quad (39) \]

Using the expressions (29) for \( S_H \) we find

\[ C_{r_B} = 2\pi c\alpha \frac{r_H}{r_B^2} (r_B^2 - r_H^2). \quad (40) \]

Thus the heat capacity is positive always, since \( r_B \geq r_H \). Therefore, one has thermal stability always. The root-mean-square energy fluctuations \( \Delta E \) are given by

\[ < (\Delta E)^2 > = C_{r_B} T^2 = \frac{\alpha^3 c r_H^3}{2\pi r_B^2}. \quad (41) \]

When \( r_B \to r_H \) we have \( \Delta E \) finite and given by \( \sqrt{< (\Delta E)^2 >} = \sqrt{\frac{\alpha^3 c}{2\pi} r_H} \).

6. Free Energies and the Ground State of the Canonical Ensemble

The Helmholtz free energy function for black holes, \( F_{BH} \), can be deduced from the action by the relation

\[ I_{BH} = \beta F_{BH}. \quad (42) \]
This free energy applies to the equilibrium value of the mass (or $r_H$) given in (27). From (23) we have in the Schwarzschild gauge the following free energy for the black hole,

$$F_{BH} = -\alpha e^{\phi_B} \frac{r_B - \sqrt{r_B^2 - r_H^2}}{\sqrt{r_B^2 - r_H^2}},$$  \hspace{1cm} (43)

which is non-positive for all $r_B$. Then, the action at equilibrium is given by the equation,

$$-I_{BH} = \alpha e^{\phi_B} \beta \frac{r_B - \sqrt{r_B^2 - r_H^2}}{\sqrt{r_B^2 - r_H^2}}. \hspace{1cm} (44)$$

But from (24) the inverse temperature is given by $\beta = \frac{2\pi}{\alpha} \sqrt{1 - \frac{r_H^2}{r_B^2}}$, which can be inverted to yield the relation, $\frac{r_B^2}{r_H^2} = \frac{1}{\frac{2\pi}{\alpha} + 1}$. Then (44) can be put in the form

$$-I_{BH}(r_H) = -\beta e^{\phi_B} \alpha + 2\pi e^{\phi_B} \sqrt{1 + \frac{\alpha^2 \beta^2}{4\pi^2}}. \hspace{1cm} (45)$$

We now find the free energy for hot anti-de Sitter space (HADS) in 2D. The local energy density, $\rho_0$, of radiation can be found to be

$$\rho(T) = \frac{\pi}{12} g T_{\text{local}}^2, \hspace{1cm} (46)$$

where $g$ is the number of massless spin states and where $T_{\text{local}}$ is the locally measured temperature. The energy-momentum tensor of a perfect fluid is

$$T_{ab} = \rho u_a u_b + p(g_{ab} + u_a u_b). \hspace{1cm} (47)$$

A perfect radiation fluid in 2D obeys the following equation of state

$$p = \rho. \hspace{1cm} (48)$$

Thus in 2D the energy-momentum tensor of radiation becomes,

$$T^a_b = \rho (\delta^a_b - 2\delta^a_0 \delta^0_b). \hspace{1cm} (49)$$

Therefore,

$$T^0_0 = -\rho = -\frac{\pi}{12} g T_{\text{local}}^2. \hspace{1cm} (50)$$
By the Tolman formula we have

\[ T_{\text{local}} = \frac{T}{\sqrt{-g_{00}}}, \quad (51) \]

where \( T \) is the temperature measured at infinity. Thus, (48) yields

\[ -T^0_0 = \rho = \frac{\pi}{12} g_2 \frac{T^2}{(-g_{00})}, \quad (52) \]

The Tolman energy for HADS can also be found,

\[ E_{\text{HADS}} = \int \rho dV = \int \rho \sqrt{-g} dx. \quad (53) \]

where \( V \) is the proper volume of the energy one wants to measure. Now, in the Schwarzschild gauge, anti-de Sitter spacetime has metric given by (4). Then, \( \sqrt{-g} = 1 \). Thus

\[ E_{\text{HADS}} = \int \rho dx = \int_{r_B}^{r_B} \frac{f(T)}{(-g_{00})} dr = f(T) \int_{r_B}^{r_B} \frac{dr}{\alpha^2 r^2 + 1} = f(T)\mathcal{V}. \quad (54) \]

Here, \( \mathcal{V} \) is the optical volume of radius \( r_B \), defined by,

\[ \mathcal{V} = \int_{-r_B}^{r_B} \frac{dr}{\alpha^2 r^2 + 1} = \frac{2}{\alpha} \arctan(\alpha r_B). \quad (55) \]

We see here that ADS spacetime behaves as an enclosure of finite volume. From (55) we have,

\[ E_{\text{HADS}} = f(T) \frac{2}{\alpha} \arctan(\alpha r_B) = \frac{\pi}{6\alpha} g T^2 \arctan(\alpha r_B). \quad (56) \]

For \( \alpha r_B \to \infty \) one has, \( E_{\text{HADS}}(r_B \to \infty) = \frac{\pi^2}{12\alpha} g T^2 \), which is the energy for the whole spacetime. The action for HADS can be taken from the expression, \( I_{\text{HADS}} = \int E_{\text{HADS}} d\beta \). Using (56) one obtains,

\[ -I_{\text{HADS}} = \frac{\pi}{6\alpha} g T \arctan(\alpha r_B). \quad (57) \]

The ground state is the state of least free energy. Since \( I = \beta F \), and \( \beta \geq 0 \), we can compare directly the reduced actions for HADS and the black hole. We find that HADS dominates whenever

\[ I_{\text{HADS}} \leq I_{\text{BH}}. \quad (58) \]
Then using equations (45) and (57) one obtains,

\[ T \geq \alpha \frac{12c}{g} \frac{\alpha r_B}{\arctan(\alpha r_B)} \sqrt{1 - \frac{g}{12\pi c} \frac{\arctan(\alpha r_B)}{\alpha r_B}}. \]  

(59)

Whenever the number of particle species is relatively large then HADS is favoured for sufficiently small \( r_B \). Indeed, if \( g > \frac{12\pi c}{\alpha r_B} \), then the quantity inside square brackets is negative up to some boundary radius given implicitly by \( \frac{\alpha r_B}{\arctan(\alpha r_B)} = \frac{12\pi c}{g} \). This means that up to this radius HADS dominates and for larger \( r_B \) HADS dominates if \( T \) obeys (59) (see figure 5, line (a)). If \( g < \frac{12\pi c}{\alpha r_B} \) then HADS is favoured only if \( T \) obeys (59) (see figure 5, line(c)). The case \( g = \frac{12\pi c}{\alpha r_B} \) says that for \( r_B \rightarrow 0 \) HADS is dominant (see figure 5, line (b)). Note that when the boundary \( r_B \rightarrow \infty \) one obtains that, for finite temperature, the black hole is the ground state.

It is also interesting to find the density of states, \( \nu(E) \). Following [16], one finds

\[ \nu(E) = \delta(E - A)e^{2\pi \Phi_H}. \]  

(60)

Thus the density of states is proportional to the entropy.

7. Conclusions

The Teitelboim-Jackiw theory has, in absence of matter, constant curvature spacetime solutions. Therefore the black hole solution of the theory has no singularities. In the first studies exploring this theory it was thought that such a black hole did not exist. However, solutions containing point particles and horizons were found [21] which also had some interesting thermodynamic properties. To establish the existence of the black hole in this theory one has to invoke topological arguments. This solution is special in the sense that to have a black hole one needs to add features which are not contained in the metric, i.e., one has to add boundary conditions.

We have then showed that this black hole yields non-trivial thermodynamics in York’s scheme. Through an analysis of the free energies of both the black hole solution and hot anti-de Sitter spacetime it was possible to infer that for small enough ambient temperature the black hole is the ground state.
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Figure Captions

Figure 1 - The Penrose diagram for metric (3).
Figure 2 - The Penrose diagram for the non-singular black hole.
Figure 3 - The Penrose diagram for the extreme black hole.
Figure 4 - The line \(-\infty < x < \infty\) has a junction at \(x = 0\) (or \(r = r_H\)). Observers on each side of the line can only communicate if they cross \(x = 0\). The time direction is vertical.
Figure 5 - The horizon radius is plotted as a function of the radius of the boundary for a given temperature, see equation (27). For each temperature the line is straight. It is also shown which regions favour hot anti-de Sitter spacetime and which favour the existence of a black hole, see equation (59). When \(g > 12\pi c\) (in the figure it was used \(\frac{g}{12\pi c} = 2\)), HADS is favoured to the left of line (a), (this case is represented in this figure). When \(g = 12\pi c\), HADS is favoured to the left of line (b). When \(g < 12\pi c\) (in the figure it was used \(\frac{g}{12\pi c} = \frac{1}{2}\)), HADS is favoured to the left of line (c). See text for more details.
Figure 1
Figure 2
Figure 4
Figure 5