INSTABILITY OF BÉZOUT’S THEOREM AND ARBITRARILY LARGE INTERSECTION BETWEEN SYMPLECTIC SURFACES

Michele Ancona and Antonio Lerario

Abstract. We investigate the failure of Bézout’s Theorem for two symplectic surfaces in $\mathbb{C}P^2$ (and more generally on an algebraic surface), by proving that every plane algebraic curve $C$ can be perturbed in the $C^1$--topology to an arbitrarily close smooth symplectic surface $C_\varepsilon$ with the property that the cardinality $\#C_\varepsilon \cap Z_d$ of the transversal intersection of $C_\varepsilon$ with an algebraic plane curve $Z_d$ of degree $d$, as a function of $d$ can grow arbitrarily fast. As a consequence we obtain that, although Bézout’s Theorem is true for pseudoholomorphic curves with respect to the same almost complex structure, it is “arbitrarily false” for pseudoholomorphic curves with respect to different (but arbitrarily close) almost–complex structures (we call this phenomenon “instability of Bézout’s Theorem”).

1. Introduction

Bézout’s Theorem says that if two algebraic curves $C_1$ and $C_2$ in $\mathbb{C}P^2$ intersect transversally, then they meet exactly in $\text{deg}(C_1) \cdot \text{deg}(C_2)$ points. Pseudoholomorphic curves also share this property. More precisely, let $J$ be an almost–complex structure on $\mathbb{C}P^2$ and $C_1$ and $C_2$ be two $J$–holomorphic curves of degree $d_1$ and $d_2$ (meaning that the homology classes $[C_1]$ and $[C_2]$ equals, respectively $d_1[\mathbb{C}P^1]$ and $d_2[\mathbb{C}P^1]$). Then, if $C_1$ and $C_2$ intersect transversally, the number of points in $C_1 \setminus C_2$ is exactly equal to $d_1 \cdot d_2$.

Pseudoholomorphic curves are a fundamental tool in symplectic geometry, since the work of Gromov [Gro85]. When the almost–complex structure $J$ is tamed by a symplectic form $\omega$ (meaning $\omega(v, Jv) > 0$, for any tangent vector $v \neq 0$), then a $J$–holomorphic curve is a symplectic surface. Conversely, given a symplectic surface $C \subset \mathbb{C}P^2$, there exists a tamed almost–complex structure $J$ for which $C$ is a $J$–holomorphic curve [MS12]. It is then natural to investigate whether some form of Bézout’s Theorem is still valid for symplectic surfaces.

Similar homological bounds are not valid in general for the intersection of two symplectic surfaces in $\mathbb{C}P^2$ without the assumption that both surfaces are pseudoholomorphic for the same almost–complex structure $J$. In this paper we will prove, in a strong sense, that one cannot expect any type of bound at all (this will be made clear below). We will say that in the symplectic framework Bézout’s Theorem is “arbitrarily false” (see Remark 1.1). This will be a consequence of the following theorem.

Theorem 1.1. Let $C \hookrightarrow \mathbb{C}P^2$ be a smooth algebraic curve and $\{a_d\}_{d \in \mathbb{N}}$ be a sequence of positive natural numbers. For every $\varepsilon > 0$ there exists a $C^\infty$–surface $C_\varepsilon \hookrightarrow \mathbb{C}P^2$, $\varepsilon$–close to $C$ in the $C^1$–topology, and a sequence $\{Z_d\}_{d \in \mathbb{N}}$ of smooth algebraic curves, with $d = \text{deg}(Z_d)$, such that for infinitely many $d \in \mathbb{N}$:

$$\#Z_d \cap C_\varepsilon \geq a_d.$$  

Moreover the sequence $\{Z_d\}_{d \in \mathbb{N}}$ can be chosen such that the intersection $Z_d \cap C_\varepsilon$ is transversal for every $d \in \mathbb{N}$. 


We collect now some comments and consequences of Theorem 1.1.

Remark 1.1. The standard complex structure of $\mathbb{C}P^2$ is tamed by the standard Fubini–Study symplectic form $\omega_{FS}$ and in particular the algebraic curves are symplectic surfaces. “Being symplectic” is an open condition for the $C^1$–topology and so a small $C^1$–perturbation of a symplectic surface remains a symplectic surface. Then, the $C^\infty$–surface $C_\epsilon$ given by Theorem 1.1 can be chosen to be symplectic. In particular, Theorem 1.1 says that Bézout is “arbitrarily false” for symplectic surfaces.

Remark 1.2. The perturbation $C_\epsilon$ of $C$ is constructed using a small parameter $\epsilon$. Varying this parameter yields a smooth family of $C^\infty$–surfaces that tend to $C$ in the $C^1$–topology when $\epsilon \to 0$. As explained in Remark 1.1, for $\epsilon$ small enough, $C_\epsilon$ is a symplectic surface and hence it is a $J_\epsilon$–holomorphic curve for an almost–complex structure $J_\epsilon$ tamed by $\omega_{FS}$. We then have $J_\epsilon \to J_0$ in the $C^1$–topology, where $J_0$ is the standard complex structure of $\mathbb{C}P^2$. Theorem 1.1 shows that, although for two $J$–holomorphic curves Bézout’s Theorem is true, this is “arbitrarily false” for two pseudoholomorphic curves with respect to two arbitrarily close almost–complex structures. We call this phenomenon “instability of Bézout theorem”, from which the title of the article.

Remark 1.3. As explained earlier, our goal is to study Bézout’s Theorem for symplectic surfaces in $\mathbb{C}P^2$. In order to construct symplectic surfaces that do not respect Bézout we start from smooth plane algebraic curves and apply a small smooth perturbation, as explained in Remark 1.1. However, as will be seen in the proof, the construction of the perturbation $C_\epsilon$ of $C$ is local and it is done on a small disk on $C$. The global smoothness of the algebraic curve $C$ is then not necessary and Theorem 1.1 is still valid if we start from a singular algebraic curve $C$: it is enough to apply the procedure described in Section 2 around a smooth point of $C$.

Remark 1.4. As it will become clear from the proof, for every $k \geq 0$ we can also require that the surfaces $C_\epsilon$ are of class $C^k$ and subanalytic (this is due to the existence of $C^k$– and subanalytic bump functions, see Remark 2.2). In particular, no quantitative estimate can be expected even in the tame world. For a “real” counterpart of this problem, see [BLN19, GKP99].

Another way to look at Bézout’s Theorem is from the point of view of positivity of intersection: two plane algebraic curves (or two $J$–holomorphic curves) always intersect positively and then the number of intersection points coincides with the intersection product in homology, which is the product of their degrees. This point of view allows us to extend this problem on any algebraic surface $X$: two algebraic curves on $X$ positively intersect and thus the number of intersection points coincides with the intersection product in homology. This positivity property remains valid for $J$–holomorphic curves in $X$, if we fix an almost–complex structure $J$ on $X$. In this sense, Theorem 1.1 is a special case of the following theorem.

Theorem 1.2. Let $X$ be an algebraic surface, $H$ be a very ample divisor, $C \hookrightarrow X$ be an algebraic curve and $\{a_d\}_{d \in \mathbb{N}}$ be a sequence of positive numbers. For every $\epsilon > 0$ there exists a $C^\infty$–surface $C_\epsilon \hookrightarrow X$, $\epsilon$–close to $C$ in the $C^1$–topology, and a sequence of algebraic curves $\{Z_d\}_{d \in \mathbb{N}}$, with $[Z_d] = d[H]$, such that for infinitely many $d \in \mathbb{N}$:

$$\#Z_d \cap C_\epsilon \geq a_d.$$  

Moreover the sequence $\{Z_d\}_{d \in \mathbb{N}}$ can be chosen such that the intersection $Z_d \cap C_\epsilon$ is transversal for every $d \in \mathbb{N}$.

It is worth noting that all previous Remarks 1.1–1.4 remain valid for any algebraic surface (not just for $\mathbb{C}P^2$) and can be applied verbatim to Theorem 1.2.
1.1. Structure of the paper. In Section 2 we prove Theorem 1.1 in the case of $\mathbb{CP}^1 \hookrightarrow \mathbb{CP}^2$. The main technical ingredient is Proposition 2.1, whose proof is given in Section 2.1. The proof of the general case of Theorem 1.1 is given in Section 3. There the main ingredient is Proposition 3.1, which combines the ideas from Proposition 2.1 with a gluing technique, in order to reduce the case of an algebraic curve to the case of $\mathbb{CP}^1$. Finally in Section 4 we prove Theorem 1.2, which uses a projective embedding and reduces to Proposition 3.1 using the local structure of a projective manifold as a graph of a holomorphic map on its tangent space.

2. Proof of Theorem 1.1: the case $C = \mathbb{CP}^1$

For the sake of exposition, we first give the proof of Theorem 1.1 in the special case $C = \mathbb{CP}^1$, and then extend the construction to include the case of an algebraic curve $C \hookrightarrow \mathbb{CP}^2$. The perturbation $C_\epsilon$ will be the image of an embedding $\eta_\epsilon : \mathbb{CP}^1 \rightarrow \mathbb{CP}^2$ which on $\mathbb{CP}^1 \setminus D$ equals the inclusion (here $D$ will be a disk), and on $D$ it will be a small perturbation of the inclusion.

Let us start by fixing some notation. We denote by $[z_0, z_1, z_2]$ the homogeneous coordinates on $\mathbb{CP}^2$. In these coordinates we have $\mathbb{CP}^1 = \{z_2 = 0\}$. We denote by $H := \{z_0 = 0\}$ the hyperplane at infinity and by $\phi : \mathbb{CP}^2 \setminus H \rightarrow \mathbb{C}^2$ the analytic affine chart

$$\phi : [z_0, z_1, z_2] \mapsto \left(\frac{z_1}{z_0}, \frac{z_2}{z_0}\right).$$

We have $\phi(\mathbb{CP}^1 \setminus H) = \phi(\mathbb{CP}^1 \setminus [0, 1, 0]) = \{(w_1, w_2) \in \mathbb{C}^2 | w_2 = 0\}$ and $\phi([1, 0, 0]) = (0, 0)$.

Let $B_r \subset \mathbb{C}$ be the disk of radius $r > 0$ centered at the origin. Denote by $D_\epsilon := \phi^{-1}(B_r \times \{0\})$ (this is a neighborhood of $[1, 0, 0]$ in $\mathbb{CP}^1$), so that $\phi$ induces a biholomorphism of triples:

$$(\mathbb{CP}^2 \setminus H, \mathbb{CP}^1 \setminus [0, 1, 0], D_\epsilon) \simeq (\mathbb{C}^2, \mathbb{C} \times \{0\}, B_r).$$

The proof of Theorem 1.1 relies on the following key proposition.

**Proposition 2.1.** Given $r > 0$ let $D_\epsilon := \phi^{-1}(B_r \times \{0\}) \subset \mathbb{CP}^1 \subset \mathbb{CP}^2$. Let $j_{r,0} : D_r \rightarrow \mathbb{CP}^2$ be the inclusion. For every sequence $\{a_d\}_{d \geq 1}$ of natural numbers there exists a family of smooth embeddings $j_{r,\epsilon} : D_\epsilon \rightarrow \mathbb{CP}^2$ with the property that for every $\epsilon > 0$

1. the embedding $j_{r,\epsilon}|D_\epsilon \setminus D_\frac{2r}{\epsilon}$ equals $j_{r,0}|D_r \setminus D_\frac{2r}{\epsilon}$;
2. the embedding $j_{r,\epsilon}|D_\frac{2r}{\epsilon}$ is holomorphic;
3. as $\epsilon \rightarrow 0$ the embedding $j_{r,\epsilon} \rightarrow j_{r,0}$ in the $C^1$–topology;
4. for every $m \geq 1$ the intersection $j_{r,\epsilon}(D_\epsilon) \cap \bar{Z}_{d_m}$ is transversal;
5. for every $m \geq 1$ the intersection $j_{r,\epsilon}(D_\frac{2r}{\epsilon}) \cap \bar{Z}_{d_m}$ consists of $a_{d_m}$ points, all of which are positively oriented.

Moreover there exists a subsequence $\{a_{d_m}\}_{m \geq 1}$ and, for every $\epsilon > 0$, a sequence of plane algebraic curves $\{\tilde{Z}_{d_m}\}_{m \geq 1}$ with $\deg(\tilde{Z}_{d_m}) = d_m$, such that:

We will prove Proposition 2.1 in the Section 2.1, we now show how Theorem 1.1, in the case $C = \mathbb{CP}^1$, follows from it.

**Proof of Theorem 1.1: case $C = \mathbb{CP}^1$.** Using the notation of Proposition 2.1, fix $r = 1$ and let $D = D_1 \subset \mathbb{CP}^1$. Define now $C_\epsilon$ to be

$$C_\epsilon := (\mathbb{CP}^1 \setminus D_\frac{2r}{\epsilon}) \cup j_{1,\epsilon}(D),$$
and consider the sequence \( \{ \tilde{Z}_{d_m} \}_{m \geq 1} \) of algebraic curves given by the same proposition.

For every \( \epsilon > 0 \) the set \( C_\epsilon \) is a well defined smooth surface in \( \mathbb{C}P^2 \) by point (1) of Proposition 2.1 and by point (5) we can make it arbitrarily close to \( \mathbb{C}P^1 \) in the \( \mathcal{C}^1 \)-topology. Point (3) and (4) of the proposition give now for every \( m \geq 1 \):

\[
\# \tilde{Z}_{d_m} \cap C_\epsilon \geq a_{d_m},
\]

with transversal intersection in \( j_{1,\epsilon}(D) \). The curves \( \{Z_{d_m}\}_{m \geq 1} \) are defined now by taking a small perturbation of the polynomials defining the \( \{\tilde{Z}_{d_m}\}_{m \geq 1} \), making them smooth and with the intersection which is transversal on the whole \( C_\epsilon \). This concludes the proof. □

**Remark 2.1.** The dependence on \( r > 0 \) of the ingredients from Proposition 2.1 did not play a role in the proof of Theorem 1.1 in the case \( C = \mathbb{C}P^1 \), but it will play a role in the general case.

**Corollary 2.2.** With the above notations, if \( \epsilon > 0 \) is small enough then \( C_\epsilon \hookrightarrow \mathbb{C}P^2 \) is symplectic.

**Proof.** This is immediate, since \( \mathbb{C}P^1 \hookrightarrow \mathbb{C}P^2 \) is symplectic and “being symplectic” is an open property in the \( \mathcal{C}^1 \)-topology. □

2.1. **Proof of Proposition 2.1.**

2.1.1. **Definition of the embedding \( j_{r,\epsilon} \).** The definition of the embedding \( j_{r,\epsilon} \) depends on the choice of a smooth map \( \varphi_r : B_r \to \mathbb{C} \), which we explicitly construct in Section 2.1.2. Once the map \( \varphi_r \) has been fixed, the embedding \( j_{r,\epsilon} \) will be defined as follows. Let \( \rho_r : \mathbb{C} \to [0,1] \) be a smooth bump function such that

\[
\rho_r|_{\mathbb{C}\setminus B_{\frac{2}{3}r}} = 0, \quad 0 < \rho_r|_{B_{\frac{2}{3}r}\setminus B_{\frac{1}{3}r}} < 1 \quad \text{and} \quad \rho_r|_{B_{\frac{1}{3}r}} = 1.
\]

**Remark 2.2.** If in this step we take \( \rho_r \) to be a \( \mathcal{C}^k \) and subanalytic bump function\(^1\), this will result in a \( \mathcal{C}^k \) and subanalytic perturbation of \( C \).

Given \( \epsilon > 0 \) let \( \varphi_{r,\epsilon} := \epsilon \rho_r \cdot \varphi_r : B_r \to \mathbb{C} \) and define \( j_{r,\epsilon} : D \to \mathbb{C}P^2 \) by (see Figure 1):

\[
j_{r,\epsilon}(x) := \phi^{-1}(\phi(j_{r,0}(x)), \varphi_{r,\epsilon}(\phi(j_{r,0}(x)))).
\]

\(^1\)Such functions exist. Of course they are not \( \mathcal{C}^\infty \).
Notice already that a map $j_{r,ε}$ defined in this way, with $ϕ_r$ smooth, verifies the properties (1) and (5) from Proposition 2.1.

2.1.2. The construction of the map $ϕ_r$. The construction depends on the given sequence $\{a_d\}_{d≥1}$.
In the process we will also construct the sequence $\{d_m\}_{m≥1}$ of the degrees and consequently the subsequence $\{a_{d_m}\}_{m≥1}$. We will search for a map $ϕ_r$ given by a series of polynomials, $ϕ_r = \sum_{m≥1} Q_m$, converging on $B_r$. (While constructing the polynomials $\{Q_m\}_{m≥1}$ we will need auxiliary polynomials $\{q_m\}_{m≥1}$. These families of polynomials also depend on $r > 0$, but we omit this dependence in the notation.)

Let $d_1 = 1$ and $q_1(w) = Q_1(w)$ be the zero polynomial.

Proceeding iteratively, for $m ≥ 2$ let $q_m(w)$ be a polynomial in one complex variable with precisely $a_{d_{m-1}}$ zeroes on $B^*_r$, all of which are simple. Define:

$$Q_m(w) := c_m(w - λ)^{2d_{m-1}} q_m(w),$$

where $λ \notin B_r$ and $c_m > 0$ is some positive constant that we will fix later\(^2\). Define $d_m := \deg(Q_m)$. Since $Q_m$ has only nondegenerate zeroes on $B^*_r$, there exists $δ_m > 0$ such that for every function $h \in ℂ^1(B_r, ℂ)$ with $\|h\|_{C^1(B_r, ℂ)} ≤ δ_m$ we have

$$(2.2) \quad \#Z(Q_m) ∩ B_r = \#Z(Q_m + h) ∩ B_r,$$

where all zeroes are still nondegenerate.

We choose now the sequence $\{c_m\}_{m≥1}$ such that for every $m ≥ 1$

$$(2.3) \quad \|Q_m\|_{C^1(B_r, ℂ)} < \frac{1}{2m} \min\{δ_1, \ldots, δ_{m-1}\},$$

and such that the series $\sum_{m≥1} Q_m$ of holomorphic functions converges on $B_r$. We finally define:

$$ϕ_r(w) := \sum_{m≥1} Q_m(w).$$

Notice that we have also obtained the sequences $\{d_m\}_{m≥1}$ and $\{a_{d_m}\}_{m≥1}$.

2.1.3. The construction of the curves $Z_{d_m}$. The construction from this section is inspired by [GKP99, BLN19].

We still need to construct the sequence of curves $\{Z_{d_m}\}_{m≥1}$. Each curve $Z_{d_m}$ will be the zero set of a homogeneous polynomial $\overline{p}_m ∈ ℂ[z_0, z_1, z_2]_{(d_m)}$. We will first construct some auxiliary families of polynomials, letting the definition of $\overline{p}_m$ be the final step of this process. (Once again, these families of polynomials depend on $r > 0$, but we omit the dependence in the notation.)

For $m ≥ 1$ and $ε > 0$ define the polynomial

$$(2.4) \quad P_{m,ε}(w_1, w_2) := w_2 - ε \sum_{j=1}^m Q_j(w_1).$$

Notice that $\deg(P_{m,ε}) = d_m$. Moreover, on $B_r × ℂ$ the system of equations

$$(2.5) \quad \{w_2 - εϕ_r(w_1) = 0, P_{m,ε}(w_1, w_2) = 0\}$$

\(^2\)The factor $(w − λ)^{2d_{m-1}}$ has two purposes: the power $2d_{m-1}$ makes sure that there are no cancellations in (2.4) and that $\deg(P_{m,ε}) = d_m$ and $λ$ is chosen to be outside $B_r$ to make sure that we do not introduce extra zeroes in $B_r$.}
is equivalent to

\begin{equation}
\begin{cases}
w_2 - \epsilon \varphi_r(w_1) = 0, 
Q_{m+1}(w_1) + \sum_{j \geq m+2} Q_j(w_1) = 0
\end{cases}
\end{equation}

In fact, subtracting the second equation in (2.5) from the first equation in (2.5), we get the second equation in (2.6). The solutions of (2.6) with \((w_1, w_2) \in B_r \times \mathbb{C}\) are all on the graph of \(\varphi_{r, \epsilon} := \epsilon \varphi_r\) (moreover, they are all positively oriented, since \(\varphi_{r, \epsilon}|_{B_r}\) is holomorphic). In other words, every solution to this system is of the form \((u, \varphi_{r, \epsilon}(u))\) with \(u \in Z(Q_{m+1} + \sum_{j \geq m+2} Q_j)\). By (2.3) it follows that

\[
\left\| \sum_{j \geq m+2} Q_j \right\|_{C^1(B_r, \mathbb{C})} \leq \sum_{j \geq m+2} \frac{1}{2j} \min\{\delta_1, \ldots, \delta_{j-1}\} < \delta_{m+1}.
\]

In particular, from (2.2) the number of solutions of (2.5) with \((w_1, w_2) \in B_r \times \mathbb{C}\) is the same as the number of solutions of (2.6). Moreover, all these solutions are nondegenerate.

Let now \(\tilde{p}_m(z_0, z_1, z_2)\) be the homogenization of \(P_{m, \epsilon}(z_1, z_2)\) and set \(\tilde{Z}_{d_m} := Z(\tilde{p}_m)\). Notice that \(\deg(\tilde{p}_m) = d_m\). Let now (see Figure 2)

\[
\Gamma_{r, \epsilon} := \text{graph}(\varphi_{r, \epsilon}|_{B_{\tilde{r}}}) \subset \phi^{-1}(j_{r, \epsilon}(D)) \quad ^3
\]

and observe that \(\phi\) maps the set \(Z(\tilde{p}_m) \cap \phi^{-1}(\Gamma_{r, \epsilon})\) to the set of solutions of (2.5). In particular, since \(\phi\) is a biholomorphism, by construction,

\[
\#\left( Z(\tilde{p}_m) \cap \phi^{-1}(\Gamma_{r, \epsilon}) \right) = a_{d_m},
\]

with the intersection \(Z(\tilde{p}_m) \cap \phi^{-1}(\Gamma_{r, \epsilon})\) which is transversal.

2.1.4. \textit{End of the proof of Proposition 2.1.} Property (2) follows from (2.1) and the fact that \(\rho \varphi_{r, \epsilon}|_{D_{\tilde{r}}}\) is holomorphic. Properties (4) and (5) follows from (2.7) and the definition of the polynomials \(\tilde{p}_m\), whose zero sets are the \(\tilde{Z}_{d_m}\). Property (3) follows from the fact that \(\rho \varphi_{r, \epsilon} \to 0\) in the \(C^1\)-topology. This concludes the proof. \(\square\)

\(^3\)The “\(B_{\tilde{r}}\)” is not a typo: we indeed want \(\Gamma_{r, \epsilon}\) to be the restriction of the graph of \(\varphi_{r, \epsilon}\) to the smaller ball \(B_{\tilde{r}}\) (as opposed to the graph of \(\varphi_{\tilde{r}, \epsilon}\)).
3. Proof of Theorem 1.1: the general case

We continue now with the proof of Theorem 1.1 in the case of a general plane algebraic curve $C \to \mathbb{CP}^2$. The key ingredient is the following proposition, which is similar to Proposition 2.1.\footnote{We stress that Proposition 3.1 is not just a simple extension of Proposition 2.1, as in the proof of this new proposition we will use the previous one.}

Theorem 1.1 will follow immediately from Proposition 3.1.

Proposition 3.1. Let $j : D \to \mathbb{CP}^2$ an analytic embedding of a disk. There exists a disk $E$ with $\overline{E} \subset D$ such that for every sequence $\{a_d\}_{d \geq 1}$ of natural numbers there exists a family of smooth embeddings $j_{\epsilon} : D \to \mathbb{CP}^2$ with the property that for every $\epsilon > 0$

1. the embedding $j_{\epsilon}|_{D \setminus E}$ equals $j|_{D \setminus E}$;
2. there exists a disk $F_{\epsilon} \subset E$ such that the embedding $j_{\epsilon}|_{F_{\epsilon}}$ is holomorphic;
3. as $\epsilon \to 0$ the embedding $j_{\epsilon} \to j$ in the $C^1$-topology.

Moreover there exists a subsequence $\{a_{d_{m}}\}_{m \geq 1}$ and, for every $\epsilon > 0$, a sequence of plane algebraic curves $\{\tilde{Z}_{d_{m}}\}_{m \geq 1}$ with $\deg(\tilde{Z}_{d_{m}}) = d_{m}$, such that:

1. for every $m \geq 1$ the intersection $j_{\epsilon}(D) \cap \tilde{Z}_{d_{m}}$ is transversal;
2. for every $m \geq 1$ the intersection $j_{\epsilon}(F_{\epsilon}) \cap \tilde{Z}_{d_{m}}$ consists of $a_{d_{m}}$ points, all of which are positively oriented;

As before, we first show how Theorem 1.1 follows from Proposition 3.1, leaving the proof of the proposition to the sequel.

Proof of Theorem 1.1: general case. The proof is similar to the proof of Theorem 1.1. Pick a disk $D \subset C$ and consider the embedding $j : D \to \mathbb{CP}^2$. Let $\{a_d\}_{d \geq 1}$ be the given sequence of natural numbers and apply Proposition 3.1 to get $E \subset D$, the family of embeddings $j_{\epsilon} : D \to \mathbb{CP}^2$, a subsequence $\{a_{d_{m}}\}_{m \geq 1}$ and, for every $\epsilon > 0$, a sequence of plane algebraic curves $\{\tilde{Z}_{d_{m}}\}_{m \geq 1}$ with $\deg(\tilde{Z}_{d_{m}}) = d_{m}$. We define

$$C_{\epsilon} := (C \setminus E) \cup j_{\epsilon}(D).$$

We define now $\{Z_{d_{m}}\}_{m \geq 1}$ to be small perturbations of $\{\tilde{Z}_{d_{m}}\}_{m \geq 1}$ which are smooth, transverse to $C_{\epsilon}$ and preserve the cardinality of intersection $\#Z_{d_{m}} \cap E = \#\tilde{Z}_{d_{m}} \cap E$. \hfill \Box

3.1. Proof of Proposition 3.1. Pick a smooth point $p \in j(D)$ and assume that $\mathbb{CP}^1$ is tangent to $p$ at $j(D)$ (this is true up to a linear projective transformation, which does not change the geometry of the problem). Then there exists a neighborhood $W$ of $(0,0)$ in $\mathbb{C}^2$ of the form $W = W_1 \times W_2$ and a holomorphic map $\gamma : W_1 \to W_2$ with $\gamma(0) = 0$ and $\gamma'(0) = 0$ such that $\phi(j(D)) \cap W = \{(w_1, w_2) | w_2 = \gamma(w_1)\}$.

For every $r > 0$ such that $\{|w_1| < 2r\} \subset W_1$, we consider the disk $D_r = \phi^{-1}(B_r \times \{0\})$ as above. This disk is not on $j(D)$.

Consider first a smooth function $\alpha : \mathbb{C} \to \mathbb{R}$ such that $\alpha|_{B_{\frac{1}{2}}} \equiv 1$, $0 < \alpha|_{B_{\frac{1}{2}} \setminus B_1} < 1$ and $\alpha|_{B_1} \equiv 0$.

We define now $\alpha_r(w) := \alpha(rw)$ and consider the function $\gamma_r := \alpha_r \gamma : B_{2r} \to \mathbb{C}$. This function coincides with $\gamma$ on $B_{2r} \setminus B_{\frac{4}{r}}$, and it is zero on $B_r$. We denote by $\Gamma \subset B_{2r} \times W_2$ the graph of $\gamma_r$ and by $\Gamma \subset B_{2r} \times W_2$ the graph of $\gamma_r$ and define the embedding $j_r : D \to \mathbb{CP}^2$ by:

$$j_r(x) = \phi^{-1}(\phi(j(x)), \gamma_r(\phi(j(x))))$$.
The embedded disk $\tilde{j}_r(D)$ contains now the disk $D_r$ and coincides with $j(D)$ outside the disk $\phi^{-1}(\Gamma)$. Nearby $D_r$, the disk $\tilde{j}_r(D)$ is $C^1$–close to $j(D)$ in the following sense (see Figure 3).

**Lemma 3.2.** As $r \to 0$ the embedding $\tilde{j}_r \to j$ in the $C^1$–topology.

**Proof.** Let $r_0 > 0$ be such that $D_{2r_0}$ is contained in the set $W_1$ defined above. For $r < 2r_0$, outside $D_{2r_0}$ the embedding $\tilde{j}_r$ already coincides with $j$; it is therefore enough to prove that $\tilde{j}_r|_{D_{2r_0}} \to j|_{D_{2r_0}}$ in the $C^1$–topology as $r \to 0$. Working in the affine chart $\phi$, this is equivalent to show that $\gamma_r|_{B_{2r_0}} \to \gamma|_{B_{2r_0}}$ in the $C^1$–topology as $r \to 0$.

Using the fact that $\alpha_r|_{\{|w| \leq \frac{4r}{3}\}} \equiv 1$ and the fact that $\gamma(0) = 0$, we have:

$$
\max_{|w| \leq 2r_0} |\gamma_r(w) - \gamma(w)| = \max_{|w| \leq 2r_0} |\alpha_r(w) - 1||\gamma(w)| \\
\leq \max_{\frac{4r}{3} \leq |w| \leq 2r_0} |\alpha_r(w) - 1||\gamma(w)| + \max_{|w| \leq \frac{4r}{3}} |\alpha_r(w) - 1||\gamma(w)| \\
= \max_{|w| \leq \frac{4r}{3}} |\alpha_r(w) - 1||\gamma(w)| \\
\leq \max_{|w| \leq \frac{4r}{3}} |\gamma(w)| \to 0 \text{ as } r \to 0.
$$
Figure 4. A picture of the embedding $\tau_{r,\epsilon} : D \to \mathbb{C}P^2$, in the chart $\phi : \mathbb{C}P^2 \setminus H \to \mathbb{C}^2$. We have already performed one perturbation so that $\Gamma_r$ contains $\phi(D_r) = B_r \times \{0\}$. The blue line represents the image of the further perturbation, which only happens on $B_r$.

For the derivative, again using $\alpha_r \{ |w| \geq \frac{r}{2} \} \equiv 1$ and the fact that $\gamma'(0) = 0$ we have:

$$\max_{|w| \leq 2r_0} \|d_w \gamma_r - d_w \gamma\| = \max_{|w| \leq 2r_0} \|(d_w \gamma)\alpha_r(w) + \gamma(w)(d_w \alpha_r) - d_w \gamma\|$$

$$\leq \max_{|w| \leq 2r_0} \|(d_w \gamma)\alpha_r(w) - d_w \gamma\| + \max_{|w| \leq 2r_0} \|\gamma(w)(d_r \alpha)\|$$

$$= \max_{\frac{r}{2} \leq |w| \leq 2r_0} |\alpha_r(w) - 1| \|d_w \gamma\| + \max_{|w| \leq 2r_0} \|\gamma(w)(d_r \alpha)\|$$

$$\leq \max_{\frac{r}{2} \leq |w| \leq 2r_0} |\alpha_r(w) - 1| \|d_w \gamma\| + \max_{|w| \leq 2r_0} \|\gamma(w)(d_r \alpha)\|$$

where in the last line we have used $\gamma'(0) = 0$. □

3.1.1. End of the proof of Proposition 3.1. We recall that the space of $C^1$ maps of a compact manifold into another one with the $C^1$-topology is metrizable (see [Hir76, Chapter 2]). Let now $r_E > 0$ be such that $D_{2r_E} \subset W_1$ and set:

$$E := j^{-1}(\phi^{-1}(D_{2r_E})).$$

Given $\epsilon > 0$, Lemma 3.2 provides us a small enough $0 < r_\epsilon < r_E$, such that $\text{dist}_{C^1}(\tilde{\iota}_r, j) \leq \frac{\epsilon}{2}$ for all $r \leq r_\epsilon$. For $t > 0$, let now $\tau_{r_{\epsilon},t} : D \to \mathbb{C}P^2$ be the embedding defined by (see Figure 4):
\[ \tau_{r,t}(x) := \begin{cases} \tilde{j}_{r,t} & x \in D \setminus \tilde{j}_{r,t}^{-1}(D_{r,t}) \\
 j_{r,t}(\tilde{j}_{r,t}(x)) & x \in \tilde{j}_{r,t}^{-1}(D_{r,t}) \end{cases}, \]

where \( j_{r,t} \) comes from Proposition 2.1. The embedding \( \tau_{r,t} \) coincides with the embedding \( \tilde{j}_{r,t} \) on \( D \setminus E \subset D \setminus \tilde{j}_{r,t}^{-1}(D_{r,t}) \) and it is the composition of \( j_{r,t} \) with the embedding \( j_{r,t} \) from Proposition 2.1 on \( \tilde{j}_{r,t}^{-1}(D_{r,t}) \) (observe that \( j_{r,t}(\tilde{j}_{r,t}(x)) \) for \( x \in \tilde{j}_{r,t}^{-1}(D_{r,t}) \) is well defined because \( D_{r,t} \) is the domain of definition of \( j_{r,t}, t \)).

In the \( \mathcal{C}^1 \)-topology, as \( t \to 0 \), since \( j_{r,t} \to j_{r,0} \) (point (5) of Proposition 2.1), then:

\[ j_{r,t} \circ \tilde{j}_{r,t} \to j_{r,0} \circ \tilde{j}_{r,t} = \tilde{j}_{r,t} \quad \text{on} \quad D_{r,t}. \]

In particular there exists \( t_\epsilon > 0 \) such that \( \text{dist}(j_{r,t} \circ \tilde{j}_{r,t}, \tilde{j}_{r,t}) \leq \frac{\epsilon}{2} \) for all \( t \leq t_\epsilon \) on \( D_{r,t} \).

We define

\[ j_\epsilon := \tau_{r,t_\epsilon}. \]

By construction we have:

\[ \text{dist}(j_\epsilon, j) \leq \text{dist}(\tau_{r,t}, \tilde{j}_{r,t}) + \text{dist}(\tilde{j}_{r,t}, j) \leq \epsilon. \]

Pick now the curves \( \{ \tilde{Z}_{d_m} \}_{m \geq 1} \) given by Proposition 2.1 and set

\[ F_\epsilon := \phi^{-1} \left( j_{r,\epsilon}(D_{\tilde{Z}}) \right). \]

Properties (1)–(5) of Proposition 3.1 follows now from the construction. This finishes the proof of Proposition 3.1.

4. Proof of Theorem 1.2

In this section we prove Theorem 1.2. The idea of the proof is as follows. First we embed \( X \) into some \( \mathbb{C}P^N \) using the very ample divisor \( H \). Then, we choose a projective plane \( \mathbb{C}P^2 \subset \mathbb{C}P^N \) tangent to \( X \) at some point \( x \in C \subset X \). In this way, a neighborhood of \( x \) in \( X \) is the graph of a holomorphic function \( \psi \) defined on an open set of \( \mathbb{C}P^2 \) and the curve \( C \subset X \) locally defines an analytic disk \( \hat{D} \) in \( \mathbb{C}P^2 \). We then apply Proposition 3.1 to the analytic disk \( \hat{D} \) in \( \mathbb{C}P^2 \) and we exploit the graph structure to transport the construction given in Proposition 3.1 from \( \hat{D} \subset \mathbb{C}P^2 \) to \( C \subset X \).

4.0.1. Embedding of \( X \). Let \( L \) be the very ample line bundle on \( X \) associated with the very ample divisor \( H \). The algebraic curves \( \{ Z_d \}_{d \in \mathbb{N}} \) with \( [Z_d] = d [H] \) will be constructed as zero sets of holomorphic sections of \( L^{\otimes d} \). Let \( \{s_0, \ldots, s_N\} \) be a basis of \( H^0(X, L) \) and consider the Kodaira embedding \( X \hookrightarrow \mathbb{C}P^N \) given by \( x \mapsto [s_0(x), \ldots, s_N(x)] \). Remark that \( \mathcal{O}_{\mathbb{C}P^N}(1)|_X = L \), so that the restriction to \( X \) of degree \( d \) homogeneous polynomials in \( N + 1 \) variables will give us holomorphic sections of \( L^{\otimes d} \).

Let \( p \) be a point in \( C \subset X \) and consider a projective plane \( \mathbb{C}P^2 \subset \mathbb{C}P^N \) tangent to \( X \) at \( p \). Let \( [z_0, \ldots, z_N] \) be the homogeneous coordinates of \( \mathbb{C}P^N \). Up to projective transformations, we can suppose that the point \( p \) has homogeneous coordinates \( [1, 0, \ldots, 0] \), and the projective plane we have chosen has equations \( \{ z_3 = \cdots = z_N = 0 \} \). Let \( \phi : \mathbb{C}P^N \setminus \{ z_0 = 0 \} \to \mathbb{C}^N \) be the analytic affine chart

\[ \phi : [z_0, z_1, \ldots, z_N] \mapsto \left( \frac{z_1}{z_0}, \ldots, \frac{z_N}{z_0} \right) := (w_1, \ldots, w_N). \]

In particular, \( \phi(p) \) is the origin in \( \mathbb{C}^N \) and \( \phi(\mathbb{C}P^2 \setminus \{ z_0 = 0 \}) = \mathbb{C}^2 \times \{ 0 \} \subset \mathbb{C}^2 \times \mathbb{C}^{N-2} \). Let \( U \subset \mathbb{C}^2 \) be a small neighborhood of \( (0, 0) \) in \( \mathbb{C}^2 \), so that we can find a holomorphic function
ψ : U → CN−2 such that the graph Γψ of ψ is a neighborhood of φ(p) in φ(X \ {z0 = 0}). We denote by (ψ1, . . . , ψN) the components of ψ. We define V := φ−1(Γψ), which is a small neighborhood of p in X. Up to shrinking U a bit, we can assume that V ∩ C is a small disk around p in C.

4.0.2. Construction of Ce. Let ˆD ⊂ CP2 ⊂ CPN be the holomorphic disk defined in affine coordinates by

\[ \phi(\hat{D}) = \{(w_1, w_2, 0) \in \mathbb{C}^2 \times \mathbb{C}^{N-2}, (w_1, w_2) \in U, (w_1, w_2, \psi_3(w_1, w_2), \ldots, \psi_N(w_1, w_2)) \in \phi(C)\}. \]

In other words, ˆD is obtained by projecting the disk V ∩ C around p in C into the projective plane CP2.

Let now D be an (abstract) disk and let j : D → CP2 be a holomorphic parametrization of ˆD, that is j : D → CP2 is a holomorphic embedding such that j(D) = ˆD. We can then apply (and use the notation of) Proposition 3.1 to produce a family of (non–holomorphic) embeddings

\[ \tilde{j}_r : D \hookrightarrow \mathbb{C}^2 \]

and a family \( \{p_m\}_{m \in \mathbb{N}} \) of homogenous polynomials in 3 variables of degree \ with \# \tilde{j}_r(D) ∩ Z(p_m) ≥ adm and such that \( \tilde{j}_r \) verifies the properties (1)–(5) of Proposition 3.1. Let us denote by \( D_\epsilon \) the image of D under \( \tilde{j}_r \).

We will now construct the global \( \mathcal{C}^\infty \)--surface \( C_\epsilon \) as in the statement of Theorem 1.2. Let \( D_\epsilon \) be the disk in X whose affine coordinates are

\[ \phi(D_\epsilon) = \{(w_1, w_2, \psi_3(w_1, w_2), \ldots, \psi_N(w_1, w_2)) \in \mathbb{C}^2 \times \mathbb{C}^{N-2}, \phi^{-1}(w_1, w_2, 0, \ldots, 0) \in \tilde{D}_{r, \epsilon}\}. \]

In this way the projection of \( D_\epsilon \) into the projective plane CP2 tangent to X at p equals \( \tilde{D}_\epsilon \).

Remark that, by point (1) of Proposition 3.1, the disks \( D_\epsilon \) and \( \tilde{D}_\epsilon \) coincide on a neighborhood of their boundaries. This implies that \( D_\epsilon \) and \( V \cap C \) coincide on a neighborhood of their boundaries. We can then define

\[ C_\epsilon := (C \setminus (V \cap C)) \cup \tilde{D}_\epsilon, \]

which is a \( \mathcal{C}^\infty \)--surface in X and which is \( \epsilon \)--close to C in the \( \mathcal{C}^1 \)--topology.

4.0.3. Construction of \( Z_d \). Let us now construct the algebraic curves \( \{Z_d\}_{d \geq 1} \) as in the statement of Theorem 1.2. For every \( \epsilon > 0 \), let \( \{\tilde{p}_m\}_{m \in \mathbb{N}} \) be the family homogenous polynomials in 3 variables with zero sets the \( \{\tilde{Z}_{d_m}\}_{m \geq 1} \) of Proposition 3.1, that is such that deg(\( \tilde{p}_m \)) = \( \tilde{d}_m \) and

\[ \#Z(\tilde{p}_{d_m}) \cap \tilde{D}_\epsilon \geq \tilde{a}_{d_m}. \]

For every \( m \geq 1 \), let \( h_m \) be the image of \( \tilde{p}_m \) under the natural inclusion

\[ \mathbb{C}[z_0, z_1, z_2]_{(d_m)} \hookrightarrow \mathbb{C}[z_0, \ldots, z_N]_{(d_m)} \]

(roughly speaking, we just consider a homogenous polynomial in 3 variables as a polynomial in \( N + 1 \) variables, that is \( h_m(z_0, z_1, z_2, \ldots, z_m) = \tilde{p}_m(z_0, z_1, z_2) \)).
Let us now study the intersection $Z(\tilde{h}_m) \cap \tilde{D}_\epsilon$. We have

$$Z(\tilde{h}_m) \cap \tilde{D}_\epsilon = \phi^{-1}\left\{(w_1, w_2, \ldots, w_N) \in \mathbb{C}^N, \ p_m(1, w_1, w_2, \ldots, w_N) = 0, \ (w_1, w_2) \in \phi(\tilde{D}_\epsilon), \ (w_3, \ldots, w_N) = \psi(w_1, w_2)\right\}$$

$$= \phi^{-1}\left\{(w_1, w_2, \ldots, w_N) \in \mathbb{C}^N, \ p_m(1, w_1, w_2) = 0, \ (w_1, w_2) \in \phi(\tilde{D}_\epsilon), \ (w_3, \ldots, w_N) = \psi(w_1, w_2)\right\}$$

$$= Z(\tilde{p}_m) \cap \tilde{D}_\epsilon.$$ 

By (4.1), we then have $\#Z(\tilde{h}_m) \cap \tilde{D}_\epsilon \geq a_d m$. As, by construction, the disk $\tilde{D}_\epsilon$ is included in $C_\epsilon$, we obtain the inequality $\#Z(\tilde{h}_m) \cap C_\epsilon \geq a_d m$. Then, an arbitrarily small perturbation $P_m$ of $\tilde{h}_m$ will give a smooth algebraic curve $Z_d_m := Z(P_m) \cap X$ in the class $d_m[H]$, intersecting $C_\epsilon$ transversally and with $\#Z_d_m \cap C_\epsilon \geq a_d m$, which proves Theorem 1.2.

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