Out of Equilibrium Quantum Field Theory
— Perturbation Theory and Generalized Boltzmann Equation

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Abstract

This paper describes perturbative framework, on the basis of the closed-time-path formalism, in terms of quasiparticle picture for studying quasiequilibrium relativistic quantum field systems near equilibrium and nonequilibrium quasistationary systems. Two calculational schemes are introduced, the one is formulated on the basis of the initial-particle distribution function and the one is formulated on the basis of the “physical”-particle distribution function. It is shown that both schemes are equivalent and lead to a generalized kinetic or Boltzmann equation. Concrete procedure of computing a generic amplitude is presented.
1 Introduction

Since mid-fifties, efforts have been made to incorporate quantum field theory with nonequilibrium statistical mechanics\[1, 2, 3, 4\]. Necessity of this incorporation originated from the field of solid-state physics. Since then, rapid progress of the studies of the early universe and the quark-gluon plasma, which is expected to be produced in heavy-ion collisions and to have existed in the early universe, have further activated this field of research (cf., e.g., Refs. 5) and 6)

The standard formalism of nonequilibrium statistical quantum-field theory is broadly classified into two frameworks, the one is the closed-time-path (CTP) formalism \[1, 2, 3, 7, 8\] and the one is nonequilibrium thermo field dynamics.\[†\] In this paper, we employ the former. Throughout this paper, we are interested in quasuniform systems near equilibrium or nonequilibrium quasistationary systems, which we simply refer to as out-of-equilibrium systems. A brief discussion is made in §5 on the relation with thermo field dynamics.

The out-of-equilibrium systems are characterized by two different spacetime scales: microscopic or quantum-field-theoretical and macroscopic or statistical. The first scale, the microscopic-correlation scale, characterizes the range of radiative correction to reactions taking place in the system while the second scale measures the relaxation of the system. For a weak-coupling theory, in which we are interested in this paper, the former scale is much smaller than the latter scale. A well-known intuitive picture (cf., e.g., Ref. 7) for dealing with such systems is to separate spacetime into many “cells” whose characteristic size, \(L^\mu (\mu = 0, ..., 3)\), is in between the microscopic and macroscopic scales. It is assumed that the correlation between different cells is negligible in the sense that microscopic or elementary reactions can be regarded as taking place in a single cell. On the other hand, in a single cell, relaxation phenomena are negligible.

Above intuitive picture may be implemented as follows. Let \(\Delta(x, y)\) be a generic propagator. For an out-of-equilibrium system, \(\Delta(x, y)\), with \(x - y\) fixed, does not change appreciably in the region \(|X^\mu - X_0^\mu| \lesssim L^\mu\), where \(X^\mu \equiv (x^\mu + y^\mu)/2\) is the mid-point and \(X_0^\mu\) is an arbitrary spacetime point. A self-energy part \(\Sigma(x, y)\) enjoys similar property. Thus, \(X^\mu\) may be used as a label for the spacetime cells and is called the macroscopic

\[†\]The nonequilibrium thermo field dynamics is initiated by Arimitsu and Umezawa\[9\] and has developed into sophisticated form.\[1, 11\]
spacetime coordinates. On the other hand, relative spacetime coordinates \( x^\mu - y^\mu \) are responsible for describing microscopic reactions taking place in a single spacetime cell.

An inverse Fourier transformation with respect to the relative coordinates \( x - y \) yields

\[
\Delta(X; P) \equiv \int d^4(x - y) e^{iP \cdot (x - y)} \Delta(x, y) \tag{1.1}
\]

(with \( P^\mu = (p^0, \mathbf{p}) \)) together with a similar formula for \( \Sigma \). Above observation shows that, for \( |P^\mu| \gtrsim 1/L^\mu \), \( P^\mu \) in (1.1) is regarded as the momentum of the quasiparticle participating in the microscopic reaction under consideration.

Microscopic reactions discussed above cause change in the density matrix that characterizes the ensemble of the system, through which the number densities of quasiparticles change with macroscopic spacetime \( X^\mu \). Dealing with this is the subject of the “next stage,” where (weak) \( X^\mu \)-dependence of \( \Delta(X; P) \) and \( \Sigma(X; P) \) are explicitly taken into account.

The purpose of this paper is to clarify the structure of perturbative out-of-equilibrium quantum-field theory through introducing two mutually equivalent calculational schemes. The one (bare-\( N \) scheme) is formulated in terms of the initial- or “bare”-number-density function (§2 - §3) and the one (physical-\( N \) scheme) is formulated in terms of the “renormalized”-number-density function (§4). We demonstrate the equivalence of both schemes through deriving a generalized Boltzmann equation. Concrete procedure of computing a generic amplitude is presented in terms of Feynman rules supplemented with the derivative expansion and the generalized Boltzmann equation (§4). The bare-\( N \) scheme is the scheme which is directly deduced from first principles. To the best knowledge of the present author, this scheme has never been comprehensively dealt with. The physical-\( N \) scheme is the scheme which is employed in the literature. Within the CTP formalism, however, no self-consistent deduction of the scheme is available. We deduce this scheme from first principles in an unambiguous manner. Comparison with related work is made in §5.

Throughout this paper, for the sake of definiteness, we take up a self-interacting complex-scalar quantum field theory. Generalization to other theories is straightforward.
2 Closed-time-path formalism

To make the paper self-contained, in this section, we give a brief review of the CTP formalism.

2.1 Preliminaries

The Lagrangian (density) reads

$$L = -\phi^\dagger (\partial^2 + m^2) \phi - \frac{\lambda}{4} (\phi^\dagger \phi)^2.$$  (2.2)

The CTP formalism is constructed through introducing an oriented closed-time path $C (= C_1 \oplus C_2)$ in a complex-time plane, which goes from $-\infty$ to $+\infty (C_1)$ and then returns back from $+\infty$ to $-\infty (C_2)$. The time argument $x_0$ of the fields, $\phi(x)$, is on the time path $C$. The perturbative scheme for computing time-path-ordered Green functions is summarized as a set of Feynman rules (cf. Ref. 2).

Throughout this paper, we assume, for simplicity, that the density matrix $\rho$ commutes with the charge operator $Q$: $[\rho, Q] = 0$. (Generalization to the case $[\rho, Q] \neq 0$ is straightforward.) Building blocks of Feynman rules are the propagator, the vertex and the so-called initial-correlations. The propagator $\Delta$ is defined by

$$\Delta(x, y) = -i \text{Tr} \left[ T_C \left( \phi(x) \phi^\dagger(y) \right) \rho \right] (\equiv -i \langle T_C \left( \phi(x) \phi^\dagger(y) \right) \rangle)$$  (2.3)

and the initial correlations are

$$C_{2n} \equiv i (-)^n \text{Tr} \left[ : \phi(x_1) ... \phi(x_n) \phi^\dagger(y_1) ... \phi^\dagger(y_n) : \rho \right] (n \geq 2).$$  (2.4)

Here $T_C$ is the time-ordering operator along the time-path $C$ and $:\text{...}:$ indicates to take normal ordering with respect to the creation and annihilation operators in vacuum theory. Throughout this paper, we do not deal with $C_{2n} (n \geq 2)$. (In Appendix A, we discuss a condition, under which $C_{2n} (n \geq 2)$ may be ignored when compared to the $2n$-point Green function.)

A field $\phi(x_0, x)$ with $x_0 \in C_1 [C_2]$ is called a type-1 [type-2] field, which we write $\phi_1(x) [\phi_2(x)]$. The type-1 field is also called a physical field. A classical contour action may be written in the form,

$$\int_C dx_0 \int d^3 x L(\phi(x), \phi^\dagger(x)) = \int_{-\infty}^{+\infty} dx_0 \int d^3 x \left[ L(\phi_1(x), \phi_1^\dagger(x)) - L(\phi_2(x), \phi_2^\dagger(x)) \right]$$
\[ \hat{L}(x) = \int d^4x \hat{\mathcal{L}}(x). \] (2.5)

Following Refs. 4), 10) and 11), we call \( \hat{\mathcal{L}}(x) \) the hat-Lagrangian (density). One can choose any value for the time \( x_0 = T_m \), when interaction-picture fields and Heisenberg-picture ones coincide. As in, e.g., Refs. 2) and 3), we choose \( T_m = -\infty \), which we call the initial time. It is well known that the theory is ultraviolet (UV) renormalizable. As a matter of fact, in general, introduction of renormalization counter terms in vacuum theory is sufficient for the out-of-equilibrium theory to be renormalized. Thus, in the following, we do not write down explicitly the UV-renormalization counter terms. The free hat-Lagrangian \( \hat{\mathcal{L}}_0 \), being bilinear in the fields, is obtained from (2.2) and (2.5):

\[ \hat{\mathcal{L}}_0 = -\hat{\phi}^\dagger (\partial^2 + m^2) \hat{\tau}_3 \hat{\phi}, \] (2.6)

where \( \hat{\tau}_3 \) is the third Pauli matrix and

\[ \hat{\phi}^\dagger = \left( \phi_1^\dagger, \phi_2^\dagger \right), \quad \hat{\phi} = \left( \phi_1, \phi_2 \right). \]

Then, we are naturally lead to a two-component theory: The propagator, the vertex and the self-energy part take \( 2 \times 2 \) matrix form. We use \( \hat{B} \) to denote the \( 2 \times 2 \) matrix whose \((i, j)\) component is \( B_{ij} \). From (2.5), we see that the vertex matrix \( \hat{V} \) has simple structure, \( \hat{V} = \text{diag}(v, -v) \) with \( v \) the vertex factor in vacuum theory. The propagator (2.3) is translated into the propagator matrix:

\[ \hat{\Delta}(x, y) = -i\langle T_C (\hat{\phi}(x) \hat{\phi}^\dagger(y)) \rangle, \] (2.7)

where \( T_C \) acts as rearranging the fields in time-path order by recalling the above definition of the type-1 and type-2 fields. In this and the next sections, taking \( \hat{\mathcal{L}}_0 \) (Eq. (2.6)) as the free hat-Lagrangian, we construct a perturbative scheme, which we call the bare-N(umber) scheme.

Fourier transforming the propagator \( \hat{\Delta}(x, y) \) [the self-energy part \( \hat{\Sigma}(x, y) \)] on \( x - y \), we have \( \hat{\Delta}(X; P) \left[ \hat{\Sigma}(X; P) \right] \) (cf. Eq. (1.1)). From the setup in §1, \( \hat{\Delta}(X; P) \) and \( \hat{\Sigma}(X; P) \) vary slowly in \( X \). Then, we employ the derivative expansion:

\[ \hat{G}(X; P) = \left[ 1 + (X - Y)^\mu \partial_{\nu} + \frac{1}{2} (X - Y)^\mu (X - Y)^\nu \partial_{\gamma} \partial_{\tau} + \ldots \right] \hat{G}(Y; P), \] (2.8)
where $\hat{G}$ stands for $\hat{\Delta}$ or $\hat{\Sigma}$. Throughout this paper, unless otherwise stated, we keep only first two terms in (2.8) (gradient approximation):

$$\hat{G}(X; P) \simeq \hat{G}(Y; P) + (X - Y) \cdot \partial_Y \hat{G}(Y; P),$$

where and in the following, ‘$\simeq$’ is used to denote the gradient approximation.

### 2.2 Quasiparticle representation for the propagator

In this subsection, pigeonholing the standard approach,[2, 4, 7, 8, 10] we introduce a quasiparticle representation for the propagator.

#### 2.2.1 Preliminary

From (2.6) and (2.7), we have

$$(\partial_x^2 + m^2) \hat{\Delta}(x, y) = (\partial_y^2 + m^2) \hat{\Delta}(y, x) = -\hat{\tau}_3 \delta^4(x - y)$$

and from (2.7), we can readily obtain

$$\begin{align*}
\Delta_{11} + \Delta_{22} &= \Delta_{12} + \Delta_{21}, & (2.9) \\
\Delta_R &= \Delta_{11} - \Delta_{12} = -i\theta(x_0 - y_0) \left[\phi(x), \phi^\dagger(y)\right], & (2.10) \\
\Delta_A &= \Delta_{11} - \Delta_{21} = -i\theta(y_0 - x_0) \left[\phi(x), \phi^\dagger(y)\right], & (2.11) \\
\Delta_c &= \Delta_{12} + \Delta_{21} = -i \langle \phi(x)\phi^\dagger(y) + \phi^\dagger(y)\phi(x) \rangle, & (2.12)
\end{align*}$$

where $\Delta_R$, $\Delta_A$ and $\Delta_c$ are retarded, advanced and correlation functions, respectively. Equation (2.9) tells us that out of four elements $\Delta_{ij}$ ($i, j = 1, 2$), only three are independent. For independent quantities, we choose $\Delta_R$, $\Delta_A$ and $\Delta_c$, which satisfy

$$\begin{align*}
(\partial_x^2 + m^2) \Delta_{R(A)}(x, y) &= (\partial_y^2 + m^2) \Delta_{R(A)}(y, x) = -\delta^4(x - y), \\
(\partial_x^2 + m^2) \Delta_c(x, y) &= (\partial_y^2 + m^2) \Delta_c(x, y) = 0. & (2.13)
\end{align*}$$

It is to be noted that (2.9) is valid also for self-energy-part resummed propagators (see Appendix B).

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*Note that $[\phi(x), \phi^\dagger(y)]$ is a c-number function.*
2.2.2 Quasiparticle representation

Now we introduce a short-hand notation $F \cdot G$, which is the function whose \( (x, y) \) component is
\[
[F \cdot G](x, y) = \int d^4z F(x, z)G(z, y).
\]

Motivated by equilibrium thermal field theory (ETFT),\[3, 4, 11, 12\] we introduce new functions\[2, 4, 10\]
\[
f(\pm)B(x, y) \equiv \delta(x_0 - y_0) g(\pm)B(x, y; x_0)
\]
(with \('B'\) for short of Bare-N scheme) and write $\Delta_{c}(x, y)$ \( (\equiv \Delta_{c}^{(+)} + \Delta_{c}^{(-)}) \) in the form,
\[
\Delta_{c}(\pm) \equiv \Delta_{R}^{(\pm)} \cdot (1 + 2f_{B}^{(\pm)}) - (1 + 2f_{B}^{(\pm)}) \cdot \Delta_{A}^{(\pm)},
\]
(2.14)

where $\Delta^{(+)}$'s \( [\Delta^{(-)}$'s] stand for the positive \( [\text{negative}] \) frequency part of $\Delta$'s (cf. (1.1)) and \('1'\) is the function whose \( (x, y) \) component” is $\delta^4(x - y)$. (The representation of $\Delta_{c}$ in terms of a vacuum-theory kit is given in Appendix C.) As in ETFT, in the case where $\rho$ is diagonal in momentum space, $\hat{\Delta}(x, y) = \hat{\Delta}(x - y)$ and
\[
\int d^3x e^{-i\tau p \cdot x} g^{(\tau)}_{B}(x) = \tau n_+(-\tau) \quad (\tau = \pm)
\]
with $n_+$ the number density of the quasiparticle and $n_-$ the anti-quasiparticle (cf. Appendix C). In the present out-of-equilibrium case, $g^{(\pm)}_{B}(x, y; x_0)$ depends on $X = (x + y)/2$ and on $x_0$ only weakly. Then, in what follows, for $g^{(\pm)}_{B}(x, y; x_0)$, we use the gradient approximation not only for $X = (x + y)/2$ but also for $x_0$.

It is worth noting that $\hat{\Delta}$ may be written as (cf. (2.9) - (2.12) and (2.14))
\[
\hat{\Delta} = \sum_{\tau = \pm} \hat{B}_{L}(f_{B}^{(\tau)}) \cdot \hat{\Delta}_{diag}^{(\tau)} \cdot \hat{B}_{R}(f_{B}^{(\tau)}),
\]
(2.15)
\[
\hat{\Delta}_{diag} = \sum_{\tau = \pm} \hat{\Delta}_{diag}^{(\tau)} = \text{diag}(\Delta_{R}, -\Delta_{A}),
\]
\[
\hat{B}_{L}(f_{B}^{(\tau)}) = \begin{pmatrix} 1 & f_{B}^{(\tau)} \\ f_{B}^{(\tau)} & 1 + f_{B}^{(\tau)} \end{pmatrix}, \quad \hat{B}_{R}(f_{B}^{(\tau)}) = \begin{pmatrix} 1 + f_{B}^{(\tau)} & f_{B}^{(\tau)} \\ f_{B}^{(\tau)} & 1 \end{pmatrix}.
\]
(2.16)

This representation may be interpreted in terms of “retarded/advanced-quasiparticle picture.” We introduce “retarded/advanced-quasiparticle fields,” $\hat{\phi}_{RA}$ and $\hat{\phi}_{RA}^\dagger$, through transformations (with obvious notation),
\[
\hat{\phi} = \sum_{\tau = \pm} \hat{\phi}^{(\tau)} = \sum_{\tau = \pm} \hat{B}_{L}(f_{B}^{(\tau)}) \cdot \hat{\phi}^{(\tau)}_{RA}, \quad \hat{\phi}^\dagger = \sum_{\tau = \pm} \hat{\phi}^{(\tau)\dagger} = \sum_{\tau = \pm} \hat{\phi}^{(\tau)\dagger}_{RA} \cdot \hat{B}_{R}(f_{B}^{(\tau)}).
\]
(2.17)
From (2.16), we see that
\[ \hat{B}_L(f_B^{(\tau)}) \cdot 1\hat{\tau}_3 \cdot \hat{B}_R(f_B^{(\tau)}) = 1 \hat{\tau}_3, \]
which guarantees the canonical commutation relation to be preserved:
\[ \left[ \hat{\phi}(x), \hat{\phi}^\dagger(y) \right] \delta(x_0 - y_0) = \left[ \hat{\phi}_{RA}(x), \hat{\phi}^\dagger_{RA}(y) \right] \delta(x_0 - y_0) = i\hat{\tau}_3 \delta^4(x - y). \]

Thus, the ETFT counterpart of (2.17) is called the (thermal) Bogoliubov transformation. The statistical average of
\[ T_C \left( \hat{\phi}_{RA} \hat{\phi}^\dagger_{RA} \right) \quad (cf. (2.7)) \]
assumes the form
\[ \langle T_C \left( \hat{\phi}_{RA}(x) \hat{\phi}^\dagger_{RA}(y) \right) \rangle = i\Delta_{\text{diag}}(x - y). \tag{2.18} \]

Thus the fields \( \hat{\phi}_{RA} \) and \( \hat{\phi}^\dagger_{RA} \) describe well-defined propagating modes in the system. It is to be noted that \( \hat{\phi}^\dagger_{RA} \) is not the hermitian-conjugate of \( \hat{\phi}_{RA} \), which is a characteristic feature of the theory of this type[4, 10]. Substituting (2.17) and (2.18) into (2.7), (2.9) - (2.12) with (2.14) are reproduced.

Incidentally, nonequilibrium thermo field dynamics[4, 10] is formulated by taking (2.17) as one of the starting hypothesis.

Now let us go to the \( P \)-space (cf. (1.1)). \( \Delta_R \) and \( \Delta_A \) go to (cf. (2.10) and (2.11))
\[ \Delta_{R(A)}(P) = \frac{1}{P^2 - m^2 \pm i\epsilon(p_0)0^\pm}. \tag{2.19} \]

It is straightforward to show that (2.14) goes to
\[ \Delta_c(X; P) \simeq -2\pi i\epsilon(p_0) \left[ 1 + 2g^{(e(p_0))}B(X; p) \right] \delta(P^2 - m^2) \]
\[ + 2i \partial_{X^\mu} g^{(e(p_0))}B(X; p) \partial_{P^\mu} \frac{P}{P^2 - m^2}, \tag{2.20} \]

where
\[ g^{(\pm)}B(X; p) = \int d(x - y)e^{-ip\cdot(x-y)}g^{(\pm)}B(x, y; X_0) \]
with \( X = (x + y)/2 \). One can easily show that (2.13) with (2.14) yields
\[ \partial_{X_0} g^{(\pm)}B(X; p) = p \cdot \nabla x g^{(\pm)}B(X; p) = 0. \tag{2.21} \]

Then, the last term in (2.20) vanishes\[ § \]
\[ \Delta_c(X; P) \simeq -2\pi i\epsilon(p_0) \left[ 1 + 2g^{(e(p_0))}B(X; p) \right] \delta(P^2 - m^2). \tag{2.22} \]

\[ § \] As has been mentioned in §1, (2.22) is meaningful in the region \( |P^\mu| \geq 1/L^\mu \), where (2.14) makes
2.2.3 Particle-number density

In order to find out the physical meaning of $g_B^{(\pm)}$, let us compute the “free” or nonperturbative part of the current density:

$$j^\mu(x) \equiv \frac{i}{2} \left[ \phi^\dagger(x) \tilde{\partial}^\mu \phi(x) - \phi(x) \tilde{\partial}^\mu \phi^\dagger(x) \right],$$

where $\tilde{\partial} \equiv \partial - \tilde{\partial}$. Taking the statistical average of these quantities, we obtain

$$\langle j^\mu(x) \rangle = \frac{1}{2} (\partial_{\mu x} - \partial_{\mu y}) \Delta_c(x, y) \bigg|_{x=y}. \quad (2.23)$$

Straightforward manipulation using (2.22) yields

$$\langle j^0(x) \rangle - \langle 0 | j^0(x) | 0 \rangle = \int \frac{d^3p}{(2\pi)^3} \left[ g_B^{(+)}(x; p) + \{1 + g_B^{(-)}(x; p)\} \right],$$

$$\langle j^\mu(x) \rangle - \langle 0 | j^\mu(x) | 0 \rangle = \int \frac{d^3p}{(2\pi)^3} \frac{p}{E_p} \left[ g_B^{(+)}(x; p) - \{1 + g_B^{(-)}(x; p)\} \right], \quad (2.24)$$

where $E_p = \sqrt{p^2 + m^2}$. Equation (2.24) is to be compared with

$$\langle j^\mu(x) \rangle - \langle 0 | j^\mu(x) | 0 \rangle = \int \frac{d^3p}{(2\pi)^3} v_B^\mu \left[ N_B^{(+)}(x; p) - N_B^{(-)}(x; p) \right], \quad (2.25)$$

where $v_B^\mu = (1, p/E_p)$ is the four-velocity and $N_B^{(+)}(x; p)$ [$N_B^{(-)}(x; p)$] is the number density (function) of the quasiparticle [anti-quasiparticle] with momentum $p$. Comparing (2.24) and (2.25), we have

$$g_B^{(+)}(x; p) = N_B^{(+)}(x; p), \quad g_B^{(-)}(x; p) = -1 - N_B^{(-)}(x; -p). \quad (2.26)$$

sense. (In the region $|P^\mu| \lesssim 1/L^\mu$, $\Delta^{(\pm)}$ in (2.14) “mixes” with $\Delta^{(\mp)}_{R(A)}$.) For dealing with the region $|P^\mu| \lesssim 1/L^\mu$, one should return back to the fundamental form (2.12) or rather to (2.13) in Appendix C. Nevertheless, in most practical cases, one can use the form (2.22) for whole range of $P^\mu$. This can be seen as follows. Let $\mathcal{T}$ be a typical scale(s) of the system under consideration. In the case of thermal-equilibrium system, $\mathcal{T}$ is the temperature of the system. Due to interactions, an effective mass $M_{ind}(X)$ is induced. In the case of $m >> \sqrt{\mathcal{T}}$, $M_{ind}(X)$ is not much different from $m$ and, for $m \lesssim \sqrt{\mathcal{T}}$, a tadpole diagram induces $M_{ind}(X)$ of $O(\sqrt{\mathcal{T}})$. [Since $\sqrt{\mathcal{T}}$ (or even $\lambda \mathcal{T}$) is the scale that characterizes microscopic reactions, the setup in §1 shows that $1/L^\mu \ll \sqrt{\mathcal{T}}$.] Since most amplitudes, when computed in perturbation theory, are insensitive to the region $|P^\mu| \ll M_{ind}(X)$ of $\Delta_c(X; P)$, the precise form of $\Delta_c(X; P)$ in this region is irrelevant. In the following, we keep this in mind. Incidentally, in the case of equilibrium thermal QED or QCD ($m = 0$), there are some quantities that diverge at infrared limits to leading order of hard-thermal-loop resummation scheme[13, 12]. For analyzing such quantities, one should use the fundamental formula (2.12) in the region $|P^\mu| \lesssim 1/L^\mu$. \footnote{See also Appendix C.}
Let us write $N^{(\pm)}_B(X, \pm \mathbf{p}) = N^{(\pm)}_B(X; E_p, \pm \hat{\mathbf{p}})$, where $\hat{\mathbf{p}} \equiv \mathbf{p}/|\mathbf{p}|$. In the case of ETFT, $N^{(\pm)}_B = N^{(\pm)}_B(E_p)$. From (2.22), we see that $g^{(\pm)}_B(X, \mathbf{p})$ appears in combination with $\delta(P^2 - m^2) = \delta(p_0^2 - E_p^2)$. Then, at first sight, it seems that no difference arises between $N^{(\pm)}_B(X; E_p, \pm \hat{\mathbf{p}})$ and $N^{(\pm)}_B(X; \pm p_0, \pm \hat{\mathbf{p}})$. It is well known in ETFT that this is not the case, since, in general, $\Delta_c(P)$ is to be multiplied by the functions that are singular at $|p_0| = E_p$. The correct choice has been known\[14\] for some time now to be $N^{(\pm)}_B(\pm p_0) (= N^{(\pm)}_B(|p_0|))$ — the “$|p_0|$ prescription.” On the basis of this observation, we assume\[14\] that the $|p_0|$ prescription should be adopted:

\[

g^{(+)\pm}_B(X; \mathbf{p}) = N^{(+)\pm}_B(X; E_p, \pm \hat{\mathbf{p}}) \to N^{(+)\pm}_B(X; p_0, \pm \hat{\mathbf{p}}) \equiv f^{(+)\pm}_B(X; P),
\]

\[
g^{(-)\pm}_B(X; \mathbf{p}) = -1 - N^{(-)\pm}_B(X; E_p, -\hat{\mathbf{p}}) \to -1 - N^{(-)\pm}_B(X; -p_0, -\hat{\mathbf{p}}) \equiv f^{(-)\pm}_B(X; P),
\]

\[
f_B(X; P) = \theta(p_0) f^{(+)\pm}_B(X; P) + \theta(-p_0) f^{(-)\pm}_B(X; P). \tag{2.27}
\]

It is to be noted that the translation into the $|p_0|$ prescription is formally achieved by (cf. (2.14) - (2.16)),

\[
\Delta_c = \Delta_R \cdot (1 + 2f_B) - (1 + 2f_B) \cdot \Delta_A,
\]

\[
\hat{\Delta} = \hat{B}_L(f_B) \cdot \hat{\Delta}_{diag} \cdot \hat{B}_R(f_B), \tag{2.28}
\]

where $f_B(x, y)$ is defined by

\[
f_B(x, y) = \int \frac{d^4P}{(2\pi)^4} e^{-iP \cdot (x-y)} f_B(X; P) \quad (X = (x+y)/2)
\]

with $f_B(X; P)$ as in (2.27). Now (2.21) turns out to

\[
P \cdot \partial_X f_B(X; P) = 0. \tag{2.29}
\]

This is a continuity equation for $f_B$ along the “flow line” in a four-dimensional $X^\mu$-space. Equation (2.29) may be solved as

\[
f_B(X; P) = f_B(X_0 = T_{in}, X = (X_0 - T_{in})\mathbf{p}/p_0; P), \tag{2.30}
\]

\[\text{It can be shown that, as in ETFT, (2.27) is consistent with mass-derivative formula\[10, 13\] at least up to the terms with second-order $X^\mu$-derivative.}\]
where $T_{in}$ ($= -\infty$) is the initial time (cf. §2.1). The propagator $\hat{\Delta}(X; P)$ with this $f_B(X; P)$ takes the form (cf. (2.22)),

$$\Delta_c(X; P) = -2\pi i \epsilon(p_0) [1 + 2f_B(X; P)] \delta(P^2 - m^2). \quad (2.31)$$

It is to be noted that (2.29) guarantees that all higher-order terms in the derivative expansion (cf. (2.20)) vanish and, in this sense, (2.31) is exact.

3 Bare-$N$ scheme

Interactions among the fields give rise to reactions taking place in a system, which, in turn, causes a nontrivial change in the number density of quasiparticles. In this section, within the bare-$N$ scheme, we analyze the self-energy-part resummed propagator and derive a generalized Boltzmann equation.

3.1 Self-energy part

We start with noticing that

$$\sum_{i,j=1}^{2} \Sigma_{ij}(x, y) = 0,$$

a proof of which is given in Appendix B. Using this property and (2.16), we obtain

$$\hat{\Sigma} = \hat{B}_R(f_B) \cdot \hat{\Sigma} \cdot \hat{B}_L(f_B) = \begin{pmatrix} \Sigma_R & \Sigma_{off} \\ 0 & -\Sigma_A \end{pmatrix},$$

where $\Sigma_R$ and $\Sigma_A$ are called the retarded and advanced self-energy parts, respectively. Going to momentum space, we obtain

$$\Sigma_{off}(X; P) \simeq i \{ f_B(X; P), \text{Re}\Sigma_R(X; P) \} + i\tilde{\Gamma}^{(p)}(X; P),$$

where

$$\{ A(X; P), B(X; P) \} = \frac{\partial A(X; P)}{\partial X_\mu} \frac{\partial B(X; P)}{\partial P^\mu} - \frac{\partial A(X; P)}{\partial P^\mu} \frac{\partial B(X; P)}{\partial X_\mu}, \quad (3.36)$$

$$i\tilde{\Gamma}^{(p)}(X; P) \equiv (1 + f_B(X; P))\Sigma_{12}(X; P) - f_B(X; P)\Sigma_{21}(X; P). \quad (3.37)$$

In deriving (3.35), use has been made of the relation $\Sigma_A(X; P) = [\Sigma_R(X; P)]^*$, a proof of which is given in Appendix B.
3.2 Self-energy-part resummed propagator

Let us compute the self-energy-part resummed propagator, with the aid of the Schwinger-Dyson equation,

\[
\hat{G} = \hat{\Delta} + \hat{\Delta} \cdot \hat{\Sigma} \cdot \hat{G} = \hat{\Sigma} \cdot \hat{\Delta} + \hat{\Delta}.
\] (3.38)

Using (2.28) and (3.32), we obtain

\[
\hat{G} = \hat{B}_L(f_B) \cdot \left( \begin{array}{cc} G_R & -G_R \cdot \Sigma_{off} \cdot G_A \\ 0 & -G_A \end{array} \right) \cdot \hat{B}_R(f_B),
\] (3.39)

\[
G_{R(A)} = \Delta_{R(A)} \cdot \left[ 1 + \Sigma_{R(A)} \cdot G_{R(A)} \right].
\] (3.40)

As a matter of course, G’s obtained from (3.39) are written in the form (2.9) - (2.12) with \(G_R(G_A)\) for \(\Delta_R(\Delta_A)\) and, for \(\Delta_c\),

\[
G_c = G_R \cdot (1 + 2f_B) - (1 + 2f_B) \cdot G_A - 2G_R \cdot \Sigma_{off} \cdot G_A.
\] (3.41)

Let us compute \(G(X; P)\)’s (\(X = (x + y)/2\), the Fourier transform of \(G(x, y)\)’s on \(x - y\), to the gradient approximation. In ETFT, \(\Sigma_{off} = 0\). Furthermore, as will be shown below, \(\Sigma_{off}\) is proportional to \(\partial N^{(\pm)} / \partial X^\mu\), where \(N^{(\pm)}\) are (the contribution to) the physical number densities. Then, in the derivative expansion for \(\Sigma_{off}(x, y)\) (cf. (2.8)), we keep only zeroth order (no-derivative) term. Thus, we obtain

\[
G_{R(A)}(X; P) \simeq \frac{1}{P^2 - m^2 - \Sigma_{R(A)}(X; P) \pm i\epsilon(p_0)0^+},
\] (3.42)

\[
G_c(X; P) \simeq G_c^{(0)}(X; P) + G_c^{(1)}(X; P) + G_c^{(2)}(X; P),
\] (3.43)

where

\[
G_c^{(0)}(X; P) = [1 + 2f_B(X; P)] [G_R(X; P) - G_A(X; P)],
\] (3.44)

\[
G_c^{(1)}(X; P) = - \{ f_B(X; P), Im \Sigma_R(X; P) \} \left( G_R^2(X; P) - G_A^2(X; P) \right)
- i\tilde{\Gamma}^{(p)}(X; P) \left( G_R^2(X; P) + G_A^2(X; P) \right),
\] (3.45)

\[
G_c^{(2)}(X; P) = \Sigma_{off}(X; P) (G_R(X; P) - G_A(X; P))^2.
\] (3.46)

It is to be noted that (3.42) is exact to the gradient approximation, i.e., no term including first derivative (with respect to \(X^\mu\)) arises.

In narrow-width approximation, \(Im \Sigma_R = -Im \Sigma_A \rightarrow -\epsilon(p_0)0^+\), \(G_RG_A\) involved in \(G_c^{(1)}\) develops pinch singularity in a complex \(p_0\)-plane, while \(G_c^{(0)}\) and \(G_c^{(1)}\) turn out to the well-defined distributions.

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For clarifying the physical meaning of $G_{c_2}^{(1)}$, we compute the contribution to the physical-number density through analyzing the contribution to the statistical average of the current density $\langle j^\mu \rangle$, (2.23) with $G_c$ for $\Delta_c$. $G_{c_1}^{(1)}$ as well as $G_{c_2}^{(0)} - \Delta_c$ lead to perturbative corrections to $\langle j^\mu \rangle$ in (2.24) due to quantum and medium effects, while $G_{c_2}^{(1)}$ yields the large contribution,

$$\langle \delta j^\mu (X) \rangle = i \int \frac{d^4P}{(2\pi)^4} P^\mu \Sigma_{off}(X; P) (G_R(X; P) - G_A(X; P))^2,$$

(3.47)

which diverges in the narrow-width approximation. In the next subsection, we inspect this large contribution more closely.

### 3.3 Generalized Boltzmann equation

The contribution of $G_{c_2}^{(1)}$ to the physical number density, $\delta N^{(\pm)}$, is obtained from (3.47) (cf. (2.24)), which we carry out in Appendix D1:

$$\delta N^{(\pm)}(X; \pm \omega_\pm, \pm \hat{p}) = \frac{Z_\pm}{2} \left| \sum_{\pm} \right| \left| \frac{i \Sigma_{off}}{Im \Sigma_R} \right|_{p_0 = \pm \omega_\pm} + \frac{1}{2\omega_\pm} \frac{\Gamma_\pm^{(p)}}{\Gamma_\pm^{(p)}} + \left( f_B, Re \Sigma_R \right),$$

(3.48)

Here $\omega_\pm (= \omega_\pm(X; \pm \hat{p}))$, Eq. (D.1) in Appendix D, is the energy of the $\pm$ (quasiparticle/anti-quasiparticle) mode with momentum $\pm \hat{p}$, $Z_\pm$ (Eq. (D.3)) is the wave-function renormalization factor and $\tau_\pm \equiv 2\omega_\pm/|Im \Sigma_R(X; \pm \omega_\pm, \hat{p})|$ is the time during which the $\pm$ mode damps. $\Gamma_\pm^{(p)}$, Eq. (3.49), has come from $\tilde{\Gamma}_\pm^{(p)} \in -i \Sigma_{off}$ in (3.34), is the net production rate of the quasiparticle (anti-quasiparticle) of momentum $\hat{p}$ ($-\hat{p}$). In fact, $\Gamma_\pm^{(p)}$ is the difference between the production rate and the decay rate, so that $\Gamma_\pm^{(p)}$ is the net production rate. As mentioned above, in the case of equilibrium system, $\Gamma_\pm^{(p)} = 0$ (detailed balance formula). Note that (3.48) diverges in the narrow-width limit, $\tau_\pm \rightarrow \infty$.

Recalling the notion of the spacetime cells (cf. §1), in Appendix D2, we derive from (3.48) a generalized Boltzmann equation for $N^{(\pm)} \equiv N^{(\pm)}_B(X; \pm \omega_\pm, \pm \hat{p}) + \delta N^{(\pm)}(X; \pm \omega_\pm, \pm \hat{p})$ with $N^{(\pm)}_B$ the bare number density in §2:

$$\frac{\partial N^{(\pm)}_B}{\partial X_\mu} + v_\pm \cdot \nabla X N^{(\pm)} \simeq \left( Z_\pm \Gamma_\pm^{(p)} \pm \frac{\partial \omega_\pm}{\partial X_\mu} \frac{\partial N^{(\pm)}_B}{\partial P^\mu} \right)_{p_0 = \pm \omega_\pm}.$$

(3.50)
The first term on the right-hand side (RHS) is the collision term, while the last term represents the effect due to a change of “mass.” Discussion of (3.50) will be postponed until §4.

The physical-number-density $N(\pm)$ may be written as

$$N(\pm) = N(\pm) + \Delta N(\pm). \tag{3.51}$$

Here, as mentioned above, $\Delta N(\pm)$, being the contribution from $G_c^{(0)} + G_c^{(1)} - \Delta_c (\in G_c)$, is a “regular correction.”

### 3.4 Absence of large contributions

In perturbatively computing some quantity, $G_c(X; P)$ appears in the form,

$$i \int \frac{d^4P}{(2\pi)^4} G_c(X; P) F(P), \tag{3.52}$$

where $F(P)$ is the function that is regular at $p_0 = \pm \omega_\pm$. $G_c^{(0)} + G_c^{(1)} \in G_c$, Eqs. 3.43 - (3.43), leads to a “regular contribution” to (3.52). Equation (3.52) with $G_c^{(0)} + G_c^{(1)}$ for $G_c$ reads

$$i \int \frac{d^4P}{(2\pi)^4} \left[ G_c^{(0)}(X; P) + G_c^{(1)}(X; P) \right] F(P), \tag{3.53}$$

which diverges in the narrow-width limit. Using (D.4) (in Appendix D) in $G_c^{(1)} (= (G_R - G_A)^2 \Sigma_{off})$, we obtain

$$\text{Eq. (3.53)} = \sum_{\tau = \pm} \int \frac{d^3p}{(2\pi)^3} \frac{Z_\pm}{2\omega_\pm} \left[ \pm[1 + 2f_B^{(\pm)}(X; \pm \omega_\pm, p)] - \frac{i\Sigma_{off}^{(\pm)}}{|Im \Sigma_R^{(\pm)}|} \right] F(\pm \omega_\pm, p) + \ldots \tag{3.54}$$

$$= \sum_{\tau = \pm} \int \frac{d^3p}{(2\pi)^3} \frac{Z_\pm}{2\omega_\pm} \left[ 1 + 2 \left( Z_\pm^{-1} N^{(\pm)} + (1 - Z_\pm^{-1}) N_B^{(\pm)} \right) \right] F(\pm \omega_\pm, p) + \ldots, \tag{3.55}$$

where ‘…’ stands for “regular contributions.” In obtaining (3.54), we have assumed that the quasiparticles here are well defined, i.e., $Im \Sigma_R^{(\pm)} = \mp |Im \Sigma_R^{(\pm)}|$. Thus, in terms of $N^{(\pm)}$, there does not appear the contribution that diverges in the narrow-width limit. The “regular contributions” (3.53) are a functional of $f_B$, which, if one wants, can be rewritten in terms of $N^{(\pm)}$ or $N^{(\pm)}$ by using the solution to (3.50) and (3.51). Computational procedure, within the bare-$N$ scheme, of a generic amplitude is similar to the one within the physical-$N$ scheme to be explained in §4.
4 Physical-$N$ scheme

4.1 Preliminary

Introducing new functions $f^{(\pm)}(x, y) \equiv \delta(x_0 - y_0) g^{(\pm)}(x, y; x_0)$, we redefine the fields, $\hat{\phi}$ and $\hat{\phi}^\dagger$, by (2.17) with $\hat{B}_L(R)(f^{(\tau)})$ for $\hat{B}_L(R)(f_B^{(\tau)})$ (cf. (2.16)). Then, the fields in this scheme are different from the fields in the previous sections. In fact, that $f$ does not satisfy (2.21) or (2.29) means that (2.13) does not hold. Then, the interaction-picture field $\hat{\phi}$ and then also $\hat{\phi}^\dagger$ do not obey the Klein-Gordon equation, $(\partial^2 + m^2)\phi \neq 0$, which means that $\hat{L}_0$ in (2.6) is not the free hat-Lagrangian. For the purpose of finding the correct free hat-Lagrangian $\hat{L}_0'$, we proceed as follows: Compute $(\partial^2 + m^2)\hat{\Delta}(x, y)$ with $\hat{\Delta}(x, y)$ as in (2.15) with (2.16) (with $f_B^{(\tau)} \to f^{(\tau)}$). The result may be written in the form,

$$ \sum_{\tau = \pm} \hat{D}_\tau \cdot \hat{\Delta}^{(\tau)} \simeq -1, $$

$$ \hat{D}_\tau(x, z) \equiv \hat{\tau}_3(\partial^2 + m^2) \delta^4(x - z) + \hat{A} \left[ \hat{g}^{(\tau)}(x, z; x_0) + 2\hat{g}^{(\tau)}(x, z; x_0) \frac{\partial}{\partial x_0} - (\nabla_x^2 - \nabla_z^2)g^{(\tau)}(x, z; x_0) \right] \delta(x_0 - z), $$

where $\hat{g}(x, z; x_0) \equiv \partial g(x, z; x_0) / \partial x_0$, etc., and $\hat{A}$ is a matrix with $A_{ij} = (-)^{i+j}$ ($i, j = 1, 2$). This means that the free hat-Lagrangian $\hat{L}_0'$ is

$$ \hat{L}_0' = - \int d^4y \sum_{\tau = \pm} \hat{\phi}^{(\tau)\dagger}(x) \hat{D}_\tau(x, y) \hat{\phi}^{(\tau)}(y). \quad (4.56) $$

Starting with $(\partial^2 + m^2)\hat{\Delta}(x, y)$, we are led to the same $\hat{L}_0'$ as (4.56) above, as it should be. Since $\hat{L}_0' \neq \hat{L}_0$, Eqs. (1.56) and (2.6), the counter term $\hat{L}_c = \hat{L}_0 - \hat{L}_0'$ appears in the hat-Lagrangian, which yields the two-point vertex function,

$$ i\hat{V}(x, y) = i\hat{A} \sum_{\tau = \pm} \left[ \hat{g}^{(\tau)}(x, y; x_0) + 2\hat{g}^{(\tau)}(x, y; x_0) \frac{\partial}{\partial x_0} \right. $$

$$ - (\nabla_x^2 - \nabla_y^2)g^{(\tau)}(x, y; x_0) \left. \right] \delta(x_0 - y_0) $$

$$ = 2 \int \frac{d^4P}{(2\pi)^4} e^{-iP \cdot (x-y)} \sum_{\tau = \pm} \theta(\tau p_0) \left[ P \cdot \partial_X g^{(\tau)}(X; P) \right] \hat{A}, $$

where $X = (x + y)/2$. Going to the $|p_0|$ prescription, we have (cf. (2.27)),

$$ i\hat{V}(x, y) \equiv i\hat{V}(x, y) \hat{A} \to 2 \int \frac{d^4P}{(2\pi)^4} e^{-iP \cdot (x-y)} \left[ P \cdot \partial_X f(X; P) \right] \hat{A}. \quad (4.57) $$

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It should be emphasized that this result is exact, not relying on the gradient approximation.

The propagator $\hat{\Delta}(X; P)$ takes the form as in §2, provided that $\Delta_c(X; P)$ is given in (2.20) with $g_B^{(\pm)} \to g^{(\pm)}$. Going to the $|p_0|$ prescription, it reads

$$\Delta_c(X; P) \simeq -2\pi i\epsilon(p_0)[1 + 2f(X; P)]\delta(P^2 - m^2) - 2iP \cdot \partial_X f(X; P) \left[\Delta_R^2(P) + \Delta_A^2(P)\right]$$

\[
\equiv \Delta_c^{(0)} + \Delta_c^{(1)}. \tag{4.58}
\]

Now, $\Sigma = \hat{B}_R \cdot \hat{\Sigma} \cdot \hat{B}_L$ takes the form (3.32) with

$$\Sigma_{off} = \Sigma_{12} \cdot f - \Sigma_{21} \cdot f + \Sigma_A \cdot f - f \cdot \Sigma_A + \mathcal{V}. \tag{4.59}$$

Fourier transformation leads to

$$i\Sigma_{off}(X; P) \simeq \{f(X; P), P^2 - m^2 - Re\Sigma_R(X; P)\} - \tilde{\Gamma}^{(p)}(X; P). \tag{4.60}$$

The $\hat{\Sigma}$-resummed propagator is written in the form (3.42) - (3.46) with $f_B \to f$.

So far $f$ has not been specified.

### 4.2 “Renormalization condition” and the generalized Boltzmann equation

We now define $f$, such that the number densities

$$N^{(+)}(X; \omega_+, \hat{p}) = f(X; p_0 = \omega_+, \hat{p}),$$

$$N^{(-)}(X; \omega_-, -\hat{p}) = -1 - f(X; p_0 = -\omega_+, \hat{p}), \tag{4.61}$$

are as close as the physical-number densities $N^{(\pm)}$. As seen in §3, $G_{c2}^{(1)}$ in (3.46), which is proportional to $\Sigma_{off}$, yields a large contribution to the physical-number density. Then, as the determining equation for $f$, we adopt $\Sigma_{off}(x, y) = 0$ or

$$\Sigma_{off}(X; P) = 0, \tag{4.62}$$

which we refer to as the “renormalization condition” for the number density. It should be emphasized here that this condition is by no means unique. $\Sigma_{off}$ is a “functional”
of $f$, $\Sigma_{off}[f]$. We can adopt $\Sigma_{off}[\tilde{f}] = 0 (f \neq \tilde{f})$, provided that $\tilde{f}(X; p_0 = \pm \omega, p) = f(X; p_0 = \pm \omega, p)$.

In the present scheme with (4.62), transformed self-energy part $\hat{\Sigma}$ (cf. Eq. (3.32)), is diagonal:

$$\hat{B}_R(f) \cdot \hat{\Sigma} \cdot \hat{B}_L(f) = \left( \begin{array}{cc} \Sigma_R & 0 \\ 0 & -\Sigma_A \end{array} \right),$$

which is in conformity with the quasiparticle picture. Thus, the present scheme is similar in structure to ETFT: Fourier transform of $\hat{\Delta} \cdot \hat{\Sigma} \cdot \hat{\Delta}$, etc., are free from pinch singularity and then no pinch singularity appears in perturbative calculations. Similarly, $G_+(X; p)$ and then also $\hat{G}(X; p)$ are free from pinch singularity in the narrow-width limit (cf. above after (3.46)).

Let us inspect the physical implication of the “renormalization condition” (4.62). We write $Re \Sigma_R(X; p) = M^2(X) + Re \Sigma'_R(X; p)$, where $M^2(X)$ is the contribution from a tadpole diagram. Note that the tadpole diagram does not contribute to $\tilde{\Gamma}(p)$. Equation (4.62) with (4.60) becomes

$$2P \cdot \partial_X f(X; p) \simeq \tilde{\Gamma}(p)(X; p) + \frac{\partial f(X; p)}{\partial X^\mu} \frac{\partial Re \Sigma'_R(X; p)}{\partial P_\mu} - \frac{\partial f(X; p)}{\partial P_\mu} \frac{\partial [M^2(X) + Re \Sigma'_R(X; p)]}{\partial X^\mu}. \tag{4.63}$$

Recalling that $N^{(+)} = N^{(+)}(X; p_0 = \omega, \hat{p}) [N^{(-)} = N^{(-)}(X; -p_0 = \omega, -\hat{p})]$ in (4.61) is the number density of the quasiparticle [anti-quasiparticle] with momentum $p [-p]$, it is straightforward to show that (4.63) becomes, on the mass-shell, $p_0 = \pm \omega, \pm \hat{p}$,

$$\frac{\partial N^{(+)}(X; p_0 = \pm \omega, \hat{p})}{\partial X_0} + v_\pm \cdot \nabla_X N^{(\pm)}(X; p_0 = \pm \omega) \frac{\partial \omega_\pm}{\partial X_\mu} \bigg|_{p_0 = \pm \omega} = \frac{dN^{(\pm)}(X; \omega_\pm, p)}{dX_0} \pm \frac{d\omega_\pm}{d\hat{p}} \cdot \frac{dN^{(\pm)}(X; \omega_\pm, \hat{p})}{dX_0} \frac{\partial \omega_\pm}{\partial \hat{p}}. \tag{4.64}$$

which is (almost) identical to (3.50) in the bare-$N$ scheme. $N^{(\pm)} = N^{(\pm)}(X; \omega_\pm, \pm \hat{p})$ here is essentially (the main portion of) the relativistic Wigner function and (4.64) is the generalized kinetic or Boltzmann equation (cf. Ref. 17)).
4.3 Procedure of solving (4.63)

Equation (4.63) determines $f(X; P)$ self-consistently. One can also approximately solve (4.63), order by order, through the following iterative procedure. At the RHS of (4.63), three small quantities are involved: the coupling constant $\lambda$, $\tilde{\Gamma}(p)$ and $\partial f/\partial X^\mu$. $\tilde{\Gamma}(p)$ measures how the system is far from equilibrium. $P^\mu \partial X f$ has been determined by (4.63) at lower orders (than the order under consideration) but the other components of $\partial X f$ reflect the form of initial $f|_{X_0 = T_{in}} = f_B|_{X_0 = T_{in}}$. For simplifying the following presentation, we assumed that the interplay of all these small quantities yields three terms, on the RHS of (4.63), which are of the same order of magnitude. \([For other cases, (trivial) modification of the following procedure is necessary.\]

Let us write the RHS of (4.63) $R[\Delta_c(f)]$, which is a functional of $f$, and expand $f$, $\Delta_c$ and $R$ (with respect to the small quantity): $f = \sum_{i=0} f^{(i)}$, $\Delta_c = \sum_{i=0} \Delta_c^{(i)}$, $R = \sum_{i=1} R^{(i)}$. Here $\Delta_c^{(i)}$ are as in (4.58) and $f^{(0)}$ is nothing but $f_B$ in §2 - §3 and satisfies (2.29), $P \cdot \partial X f^{(0)} = 0$, whose solution is given in (2.30).

\[\text{Zeroth order: Equation (4.63) reduces to (2.29): } P \cdot \partial X f^{(0)} = 0.\]

\[\text{First order: Equation (4.63) becomes } 2P \cdot \partial X f^{(1)}(X; P) = R^{(1)}[\Delta_c^{(0)}(f^{(0)})].\]

Solve (4.63) under the condition $f^{(1)}|_{X_0 = T_{in}} = 0$.

\[\text{Second order: Compute } \Sigma_{off}(X; P) \text{ up to second } X^\mu \text{-derivative terms in the derivative expansion, which yields the additional term on the RHS of (4.63),}\]

\[\mathcal{R}'[\Delta_c(f)] = \frac{i}{8} \left[ \frac{\partial^2 (\Sigma_{12} - \Sigma_{21})}{\partial X^\mu \partial X^\nu} \frac{\partial^2 f}{\partial P^\mu \partial P^\nu} + \frac{\partial^2 (\Sigma_{12} - \Sigma_{21})}{\partial P^\mu \partial P^\nu} \frac{\partial^2 f}{\partial X^\mu \partial X^\nu} \right].\]

Then, we have

\[P \cdot \partial X f^{(2)} = \mathcal{R}^{(2)}[\Delta_c^{(0)}(f^{(0)})] + \{ \mathcal{R}^{(1)}[\Delta_c^{(0)}(f^{(0)} + f^{(1)}) + \Delta_c^{(1)}(f^{(0)})] \]

\[- \mathcal{R}^{(1)}[\Delta_c^{(0)}(f^{(0)})] \} + \mathcal{R}'[\Delta_c^{(0)}(f^{(0)})].\]

\[\text{Note that } \partial Re \Sigma_{R}/\partial X \text{ is a functional of } \partial f/\partial X.\]

\[\text{For the case of } m^2 \lesssim \lambda T^2 \text{ (with } T \text{ the typical scale(s) of the system) the tadpole self-energy-part resummed propagator } \text{ should be substituted for the bare propagator } \Delta.\]
Since $P \cdot \partial_X f^{(0)} = 0$, $\Delta_c^{(1)}[f^{(0)}] = 0$ (cf. (4.58)). Solve (4.66) under the condition $f^{(2)}|_{x_0 = t_{in}} = 0$.

**Higher orders:** Proceed similarly as above. Further “improvements” of (4.58) and (4.63) are necessary.

More efficient way is to construct $\hat{\Sigma}$-resummed propagator, Eqs. (3.42) - (3.45) ($\Sigma_{off} = 0$), at each order and use it for yet higher-order calculations (cf. below).

### 4.4 Procedure of perturbative calculation

Perturbative computation, to $N$th order, of an $n$-point amplitude $A(P_1, \ldots, P_n)$,

$$A(P_1, \ldots, P_n) = \int \prod_{i=1}^n d^4x_i \ e^{-i \sum P_i \cdot x_i} \tilde{A}(x_1, \ldots, x_n),$$

(4.67)
goes as follows.

1) Draw all relevant skeleton diagrams. Contribution to $\tilde{A}$ of each skeleton diagram may be written in the form,

$$\int \prod_{i=1}^j d^4z_i \tilde{S}(x_1, \ldots, x_n; z_1, \ldots, z_j).$$

(4.68)

Here $x_1, \ldots, x_n$ are the external-vertex (spacetime) points and $z_1, \ldots, z_j$ are the internal-vertex (spacetime) points, which we collectively write $\xi_1, \ldots, \xi_{n+j}$.

2) For the propagators involved in each skeleton diagram, use the self-energy-part ($\hat{\Sigma}$) resummed propagator $\hat{G}$, with $\hat{\Sigma}$ and $f$ computed to appropriate orders. [For the skeleton diagram that is already of $N$th order, it is sufficient to use the bare propagators and the “bare” $f$’s.]

3) To analyze $\tilde{S}(\xi_1, \ldots, \xi_{n+j})$, as (not necessarily independent) variables, employ the center-of-mass coordinates of $\tilde{A}$, $X (= \sum_{i=1}^n x_i/n)$, and relative coordinates $\xi_{i_k} - \xi_{j_k}$ ($k = 1, \ldots, m$ with $m$ a number of propagators). Here $\xi_{i_k}$ and $\xi_{j_k}$ in $\xi_{i_k} - \xi_{j_k}$ are the points that are connected by the propagator $\hat{G}(\xi_{i_k}; \xi_{j_k}) (\equiv \hat{G}_k)$,

$$\hat{G}_k = \hat{G} \left( \frac{\xi_{i_k} + \xi_{j_k}}{2}; \xi_{i_k} - \xi_{j_k} \right) = \int \frac{d^4Q_k}{(2\pi)^4} e^{-iQ_k \cdot (\xi_{i_k} - \xi_{j_k})} \hat{G} \left( \frac{\xi_{i_k} + \xi_{j_k}}{2}; Q_k \right),$$

whose form is known above. [An external as well as an internal point appears, in general, in several relative coordinates.] It can easily be shown that $(\xi_{i_k} + \xi_{j_k})/2$ may
be “decomposed” as

$$\frac{\xi_{ik} + \xi_{jk}}{2} = X + \sum_{l(\neq k)} c_{kl}(\xi_{il} - \xi_{jl}),$$  \hspace{1cm} (4.69)

where $c$’s are numerical coefficients. The decomposition (4.69) is not unique, which is not the matter for our purpose. Carry out the derivative expansion:

$$\hat{G}_{k} = \int \frac{d^4 Q_k}{(2\pi)^4} e^{-iQ_k \cdot (\xi_{ik} - \xi_{jk})} \left[ 1 + \sum_{l(\neq k)} c_{kl}(\xi_{il} - \xi_{jl}) \cdot \partial_X + \ldots \right] \hat{G}(X; Q_k).$$  \hspace{1cm} (4.70)

Pick up the term with $(\xi_{il} - \xi_{jl})$. Multiplication of $\hat{G}_{l} (\in \tilde{S})$ yields

$$c_{kl}(\xi_{il} - \xi_{jl}) \cdot \partial_X \hat{G}(X; Q_k) \hat{G}_{l}$$

$$= c_{kl} \partial_{X^\mu} \hat{G}(X; Q_k) \int \frac{d^4 Q_l}{(2\pi)^4} e^{-iQ_l \cdot (\xi_{il} - \xi_{jl})} \left( \frac{1}{i \partial Q_l^\mu} \right) \hat{G} \left( \frac{\xi_{il} + \xi_{jl}}{2}; Q_l \right).$$

The higher $X^\mu$-derivative terms ‘...’ in (4.70) may be dealt with similarly. Deal with all the propagators in the above manner. For $f(X; Q_l)$’s, substitute the solution (of appropriate order) to (4.62).

4) Carry out the integrations over $z$’s in (4.68), which yields $j$ momentum-conservation $\delta$-functions at the internal points, $z$’s.

5) Carry out the integrations over $j$ $Q$’s. Add the contributions of all the diagrams to obtain $\tilde{A}(x_1, \ldots, x_n)$, which, as is obvious from the above procedure, is of the form $\tilde{A}(X; x_1 - x_2, \ldots, x_{n-1} - x_n)$.

6) Substitute $\tilde{A}$ thus obtained into (4.67) to obtain

$$A(P_1, \ldots, P_n) = \int d^4 X A(X; P_1, \ldots, P_n),$$  \hspace{1cm} (4.71)

$$A(X; P_1, \ldots, P_n) = \int \prod_{i=1}^{n-1} d^4(x_{i+1} - x_i) e^{-i \sum_{i=1}^{n-1} P_i^\mu (x_{i+1} - x_i)} \times \tilde{A}(X; x_1 - x_2, \ldots, x_{n-1} - x_n),$$  \hspace{1cm} (4.72)

where $P_i'$ is the linear combination of $P_1, \ldots, P_n$. Carry out the integration over $x_{i+1} - x_i$ ($i = 1, \ldots, n - 1$) to obtain $A(X; P_1, \ldots, P_n)$. Note that the momentum conservation $\sum_{i=1}^{n} P_i = 0$ holds (cf., however, the footnote in §2.2.2). Note also that $A$ depends weakly on $X$ through $f(X; Q_i)$’s. Were it not for this $X$-dependence, integration over $X$ in (4.71) would yield $(2\pi)^4 \delta^4(\sum_i P_i)$.
5 Comparison with related work

As has been mentioned in §1, the physical-N scheme is employed in the literature.

5.1 CTP formalism

The derivation of the generalized Boltzmann equation (GBE) in Refs. 2), 7), 18) and 19) starts with the Schwinger-Dyson equation, Eq. (3.38),
\[
\left(\hat{\Delta}^{-1} - \hat{\Sigma}\right)\hat{G} = \hat{G} \left(\hat{\Delta}^{-1} - \hat{\Sigma}\right) = 1. \tag{5.73}
\]
Here we stress that this equation is nothing more than the equation that serves as resumming the self-energy part to makeup the resummed propagator, (3.40) and (3.41). Then in order to derive the GBE, an additional input or condition is necessary. Our condition is the “renormalization condition” \(\Sigma_{\text{off}}\) in (4.62).

We first see the derivation in Refs. 7), 18) and 19), in which the lowest nontrivial order is dealt with. The condition adopted there is essentially
\[
G_c(X; P) = \left[1 + 2f'(X; P)\right] \left[G_R(X; P) - G_A(X; P)\right]. \tag{5.74}
\]
[To distinguish from ours, we write \(f'\) for \(f\) here.] No counter Lagrangian is introduced, so that the solution for \(G_c\) to (5.73) is given by (3.41) with \(f_B \to f'\) (cf. the argument in §4):
\[
G_c = G_R \cdot (1 + 2f') - (1 + 2f') \cdot G_A - 2G_R \cdot \Sigma_{\text{off}} \cdot G_A, \tag{5.75}
\]
where \(\Sigma_{\text{off}}\) is as in (3.34) with \(f_B \to f'\). Note that the condition (5.74) is the leading term of the (Fourier transform of the) exact form (5.73) with \(\Sigma_{\text{off}}=0\).

Fourier transforming (5.73) and equating with (5.74), one obtains
\[
\{f'(X; P), G_R(X; P) + G_A(X; P)\} \simeq -2iG_R(X; P)\Sigma_{\text{off}}(X; P)G_A(X; P). \tag{5.76}
\]
To the lowest nontrivial order under consideration, the left-hand side (LHS) of (5.76) is approximated as
\[
\text{LHS of (5.76)} \sim -2iP \cdot \partial_X f'(X; P)\{G_R^2(X; P) + G_A^2(X; P)\}
= 4iP \cdot \partial_X f'(X; P)G_R(X; P)G_A(X; P) \quad (p_0 = \pm \omega). \tag{5.77}
\]
\[\]
Using (5.77) in (5.76) on the mass-shell, $p_0 = \pm \omega$, one gets the Boltzmann equation.

In Ref. 2) is given an all-order derivation of the GBE without introducing the counter Lagrangian. Although the correct GBE comes out, the derivation is not self consistent. In order to see this, we first note that, upon using (3.33), (3.34) and (3.40), the solution (5.75) may be rewritten as

$$G^{-1}_R G_c G^{-1}_A = \tilde{G}_1 + \tilde{G}_2,$$

$$\tilde{G}_1 = \Sigma_{11} + \Sigma_{22}$$

$$\tilde{G}_2 = 2 \left( f' \cdot \Delta_{\Delta}^{-1} - \Delta^{-1}_R \cdot f' \right).$$

(5.78)

(5.79)

(5.80)

At an intermediate step of solving Schwinger-Dyson equation, there appears the term proportional to $\Delta^{-1}_R \cdot \Delta_c \cdot \Delta^{-1}_R$, which turns out to $\tilde{G}_2$. In Ref. 2), $\tilde{G}_2$ is missing, and thus, when interaction is switched off, $G_c$ does not reduce to the free form (2.28). [It is to be noted that, in ETFT, Fourier transform of $\Delta^{-1}_R \cdot \Delta_c \cdot \Delta^{-1}_R$ is proportional to $(P^2 - m^2)^2 \delta(p^2 - m^2)$ and then vanishes.] As the additional condition, in place of (5.74), $G_c = G_R \cdot (1 + f') - (1 + f') \cdot G_A$ is adopted. This again contradicts with the exact solution (5.75) unless $\Sigma_{o_{off}} = 0$. By equating two relations above, the correct GBE comes about.

As has been shown in §4, for constructing a consistent physical-$N$ scheme, the counter Lagrangian $L_c$ should be introduced. As mentioned above, however, $L_c$ is not introduced in Refs. 2), 7), 18) and 19), so that no self-consistent perturbative calculational procedure is presented.

### 5.2 Thermo field dynamics (TFD)

In contrast to the CTP formalism, the TFD counterparts of $\phi_1$ and $\phi_2$ are independent fields and mutually commutable.

We first consider the equilibrium case. In spite of the fact mentioned above, as far as the perturbation theory is concerned, TFD is equivalent to CTP formalism. However, we should mention one point — the $|p_0|$ prescription (§2.2.4). In the CTF formalism, the $|p_0|$ prescription is deduced from first principles. The very existence of the vertical segments of the time-path in a complex time plane plays an important role. In TFD, however, there is no counterpart of the vertical segments, and then, on the basis of the consistency argument (with the spectral representation and KMS condition), the $|p_0|$ prescription is assumed.
The nonequilibrium TFD\cite{4,10} (NETFD) is formulated on the basis of (nonequilibrium generalization of) the so-called thermal-state condition. As a consequence of this, initial correlations (cf. (2.4) and Appendix A) are absent. Thus, when compared to the (CTP) formalism developed in this paper, NETFD may be applied to more restricted class of nonequilibrium systems, i.e., the systems which are described by suitably defined quasiparticles, in terms of which the generalized thermal-state condition holds. On the other hand, NETFD is a “single-(space)time” canonical formalism, without distinction between the microscopic- and macroscopic-(space)times. In this sense, NETFD has much wider applicability than the CTP formalism presented in this paper, the latter of which applies only for “out-of-equilibrium systems.” Incidentally, in NETFD, there is one parameter (s-parameter), which has no counterpart in the CTP formalism.

Our condition for determining the “physical-number density,” $N^{(\pm)}$, is that, with $N^{(\pm)}$, pinch singularities (in the narrow-width limit) disappear. On the other hand, NETFD, in which the (space)time representation is employed, imposes an “on-shell renormalization condition,” which results in the GBE. Since “the pinch singularity” is a notion in momentum space, it seems not to be immediately obvious how to translate this condition to the (space)time representation as adopted in NETFD. Nevertheless, closer inspection of the structure of both formalisms tells us that our condition is in accord with the “on-shell renormalization condition” in NETFD. Incidentally, how to reconcile the NETFD with the $|p_0|$ prescription, a notion in momentum space, remains to be an open question.

6 Summary and discussion

In this paper, we have dealt with perturbative framework of out-of-equilibrium relativistic complex-scalar-field systems. We have assumed the existence of two different spacetime scales, the microscopic and macroscopic. The first small scale characterizes the microscopic correlations and the second large scale is inherent in the relaxation phenomena.

We have proposed mutually equivalent two computational schemes, the bare-$N$ and physical-$N$ schemes. Both of them lead to the generalized relativistic kinetic or Boltzmann equation. Then, we have presented the procedure of perturbatively computing a
generic amplitude.

It is worth pointing out a similarity between two computational schemes presented in this paper and those in UV-renormalization in vacuum theory. Taking mass renormalization as an example, let us see this.

A “bare scheme” in vacuum theory takes \( L_0 = -\phi^\dagger (\partial^2 + m_B^2)\phi \), with \( m_B \) the bare mass, as the free Lagrangian. Then, the propagator reads, \( \Delta(P) = 1/(P^2 - m_B^2 + i0^+) \).

Perturbative computation of the renormalized mass \( m \) yields \( m = M(m_B) \), which may not necessarily be the physical mass \( M \). Perturbative computation of some quantity, e.g., the physical mass \( M \), yields the result as a function of \( m_B \), which may be rewritten, using \( m = M(m_B) \), in terms of \( m \). The correspondence of the “bare scheme” here to the bare-\( N \) scheme in \( \S 2 - \S 3 \) is as follows: \( L_0 \to \hat{L}_0 \) in (2.10), \( \Delta(P) \to \hat{\Delta}(X; P) \) (Eqs. (2.10) - (2.12) with (2.19) and (2.31)), \( m_B \to N_B^\pm \), \( m \to N^\pm \), \( m = M(m_B) \to Eq. (3.48) \), \( M \to N^\pm \).

A “physical scheme” in vacuum theory takes \( L'_0 = -\phi^\dagger (\partial^2 + m^2)\phi \), with \( m \) the renormalized mass, as the free Lagrangian and then there emerges the counter-Lagrangian, \( L_c = \phi^\dagger (m^2 - m_B^2)\phi \). The propagator is \( \Delta(P) = 1/(P^2 - m^2 + i0^+) \). \( m - m_B^2 \) is determined so that the perturbatively computed mass, \( M(m, m^2 - m_B^2) \), is equal to \( m \), \( m = M(m, m^2 - m_B^2) \). The correspondence of the “physical scheme” here to the physical-\( N \) scheme in \( \S 4 \) is as follows: \( L'_0 \to \hat{L}'_0 \) in (4.56), \( \Delta(P) \to \hat{\Delta}(X; P) \) (Eqs. (2.10) - (2.12) with (2.19) and (4.58)), \( L_c \to \hat{L}_c = \hat{L}_0 - \hat{L}'_0 \), \( m = M(m, m^2 - m_B^2) \to \Sigma_{off} = 0 \) in (4.62), \( m \to N^\pm \), \( M \to N^\pm \).

To summarize, the bare-\( N \) scheme is constructed in terms of the original bare-number density \( f_B(X; P) \) [respect. the bare mass \( m_B \)]. On the other hand, the physical-\( N \) scheme is constructed in terms of the “renormalized”-number density \( f(X; P) \) [respect. the renormalized mass \( m \)]. Both perturbative schemes are equivalent. The first scheme starts with the “bare quantity” and the “renormalization” is done at the end, while in the second scheme, the “renormalization” is done at the beginning by introducing the counter hat-Lagrangian \( \hat{L}_c \) [respect. \( L_c \)]. It is worth recalling that the renormalized mass \( m \) is defined so as to absorb UV divergences. However, there is arbitrariness in the definition of the “finite part” of \( m \), which is determined by imposing some condition. Under this condition, \( m \) is determined order by order in perturbation series. This is also the case in the present physical-\( N \) scheme. \( f \) (or \( N^\pm \)) is defined so as to absorb
large contributions, which diverge in narrow-width limit. As pointed out above after (4.62), there is arbitrariness in fixing the “finite part” of \( f \). In §4, we have imposed the condition for determining \( f \), under which \( f(X;P) \) turns out to be determined order by order in perturbation series.

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**Appendix A: Initial correlations**

Here we discuss to what extent, the initial correlations \( C_{2n} \), Eq. (2.4), may be ignored.

We first note that, in the two-component representation (cf. §2.1), \( C_n \) has \( 2^n \) components, all of which are identical. Then, from (2.10) and (2.11), we see that \( C_2 \), which constitutes the medium part of the propagator \( \hat{\Delta} \), does not appear in the retarded and advanced Green functions. This means that \( C_2 \) does not appear in the two-point correlation in the linear response theory. With the aid of the standard formulae (cf. Sections 2 and 5 of Ref. 2)), one can also show that \( C_{2n} \) does not appear in multi-point correlations in the nonlinear as well as the linear response theory.

We decompose \( \phi(x) \) as in vacuum theory:

\[
\phi(x) = \int \frac{d^3q}{\sqrt{2E_q(2\pi)^3}} \left[ a(q)e^{-iQ\cdot x} + b^\dagger(q)e^{iQ\cdot x} \right],
\]

where \( q_0 = E_q = \sqrt{q^2 + m^2} \). \( a(q) \) \( [b^\dagger(q)] \) in (A.1) is the annihilation [creation] operator of a particle [antiparticle] in vacuum theory: \( [a(q), a^\dagger(q')] = [b(q), b^\dagger(q')] = \delta(q - q') \).

We restrict our concern to \( C_4 \). The argument below may be generalized to the case of general \( C_{2n} \) \( (n \geq 3) \). We are concerned only with the particle part of \( \phi \), the part with \( a(q) \) in (A.1), and of \( \phi^\dagger \), the hermitian conjugate of \( \phi \). Other parts may be dealt with similarly and the same conclusion will be obtained.

Substituting (A.1) and its hermitian conjugate to \( C_4 \) in (2.4) and carrying out the
Fourier transformation, we obtain, up to an numerical factor,

$$\tilde{C}_4 \propto \frac{1}{\prod_{j=1}^2 \sqrt{E_{p_j} E_{q_j}}} \prod_{j=1}^2 \left[ \delta(p_j^0 - E_{p_j}) \delta(q_j^0 - E_{q_j}) \right] \langle q_1, q_2; p_1, p_2 \rangle_c$$

(A.2)

$$\langle q_1, q_2; p_1, p_2 \rangle_c \equiv \langle q_1, q_2; p_1, p_2 \rangle - \langle q_1; p_1 \rangle \langle q_2; p_2 \rangle - \langle q_1; p_2 \rangle \langle q_2; p_1 \rangle,$$

where $$\langle q_1, q_2; p_1, p_2 \rangle \equiv \langle a^\dagger(q_1)a^\dagger(q_2)a(p_1)a(p_2) \rangle$$ and $$\langle q; p \rangle \equiv \langle a^\dagger(q)a(p) \rangle$$, etc. The setup in §1, $$\mathcal{C}_4$$ does not change appreciably in a single spacetime cell, leads to approximate momentum-conservation:

$$\tilde{C}_4 \simeq 0 \text{ for } \sum_{j=1}^2 (P_j^\mu - Q_j^\mu) \gtrsim 1/L^\mu.$$  

(A.3)

Thus, we may write

$$\tilde{C}_4 = \Delta^4 \left( \sum_{j=1}^2 (P_j - Q_j) \right) \prod_{j=1}^2 \left[ \delta_+(P_j^2 - m^2) \delta_+(Q_j^2 - m^2) \right] \mathcal{A},$$

(A.4)

where $$\Delta(P^\mu)$$ is the function whose width [height] is of $$O(1/L^\mu)$$ [$$O(L^\mu)$$] and satisfies $$\int \Delta(P^\mu) dP^\mu = 1$$, and $$\delta_+(P^2 - m^2) \equiv \theta(p_0)\delta(P^2 - m^2)$$.

A transition probability or a rate of a microscopic reaction is related to an on-shell amplitude. (In the case of equilibrium thermal field theories, this relation is settled in Ref. 20.) Let us now analyze the on-shell ($$P_j^2 = m^2$$, etc.) amplitudes. As can be seen from (A.2), $$\mathcal{C}_4$$ per se does not contribute to the on-shell amplitudes. Then, $$\mathcal{C}_4$$ or $$\tilde{C}_4$$ appears as a part(s) of an on-shell amplitude. Let us see the structure of a connected amplitude $$\mathcal{S}$$, which includes a single $$\mathcal{C}_4$$. We may write, with obvious notation,

$$\mathcal{S} = \int \prod_{j=1}^2 [d^4P_j d^4Q_j] \, \tilde{C}_4 \mathcal{F}(Q_1, Q_2; P_1, P_2)$$

$$\propto \int \frac{d^3p_1 d^3p_2 d^3q_1}{E_{p_1} E_{p_2} E_{q_1} E_{[p_1 + p_2 - q_1]}} \Delta(E_{p_1} + E_{p_2} - E_{q_1} - E_{[p_1 + p_2 - q_1]})$$

$$\times \mathcal{A} \mathcal{F}(P_1, P_2; Q_1, Q_2 = P_1 + P_2 - Q_1).$$

(A.5)

$$\tilde{C}_4$$ is to be compared with the four-point Green functions $$\tilde{G}_4$$’s. For the sake of comparison, we take the $$n$$th-order contribution, $$\tilde{G}_4^{(n)}$$, whose four legs are bare propagator with on-shell $$\delta$$-function:

$$\tilde{G}_4^{(n)} \propto \lambda^{n+1} \delta^4 \left( \sum_{j=1}^2 (P_j - Q_j) \right) \prod_{j=1}^2 \left[ \delta(p_j^2 - m^2) \delta(q_j^2 - m^2) \right] \mathcal{B},$$

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The counterpart of $S$, Eq. (A.5), is
\[ \tilde{S}^{(n)} = \lambda^{n+1} \int \frac{d^3p_1 d^3p_2 d^3q_1}{E_{p_1} E_{p_2} E_{q_1} E_{|p_1+p_2-q_1|}} \delta(E_{p_1} + E_{p_2} - E_{q_1} - E_{|p_1+p_2-q_1|}) \times B \mathcal{F}(P_1, P_2; Q_1, Q_2 = P_1 + P_2 - Q_1). \] (A.6)

From (A.5) and (A.6), we obtain, with obvious notation,
\[ \frac{S}{\tilde{S}^{(n)}} = \frac{\langle A \mathcal{F} \rangle}{\lambda^{n+1} \langle B \mathcal{F} \rangle}. \] (A.7)

Noticing that $A$ and $B$ are dimensionless, we assume that $A$ and $B$ are of $O(1)$. Equation (A.7) tells us that, unless $\langle A \mathcal{F} \rangle / \langle B \mathcal{F} \rangle \ll 1/\lambda^{n+1}$, $C_4$ may not be ignored.

Equation (C.2) in Appendix C, together with the setup in §1, tells us that
\[ \langle a^\dagger (q) a(p) \rangle \simeq 0 \quad \text{for} \quad |q - p| \gtrsim 1/|L|. \] (A.8)

Motivated with this relation, we consider a case, where $\langle q_1, q_2; p_1, p_2 \rangle_c$ has the property of factorizability or short-range correlation in momentum space:
\[ \langle q_1, q_2; p_1, p_2 \rangle_c \simeq 0 \quad \text{for} \quad |p_1 - q_1| \gtrsim 1/|L| \quad \text{and} \quad |p_1 - q_2| \gtrsim 1/|L|. \] (A.9)

In this case, using (A.9) in (A.2), we see that (A.3) becomes, with obvious notation,
\[ S \propto \frac{1}{V} \int \frac{d^3p_1 d^3p_2}{E_{p_1} E_{p_2}} \Delta(E_{p_1} + E_{p_2} - E_{q_1} - E_{|p_1+p_2-q_1|}) A \mathcal{F} \bigg|_{p_1 \simeq q_1 \oplus p_2 \simeq q_1}, \] (A.10)

where $V = L^1 L^2 L^3$. On the basis of the observation made at the footnote in §2.2.2, we assume that $\tilde{S}^{(n)}$ in (A.6) is insensitive to the region $|q_1| \lesssim 1/|L|$. Then, in (A.6), the integration over $q_1$ yields an $O(T^3)$ quantity, with $T$ the typical scale(s) of the system, which characterizes the microscopic reaction and is much larger than $1/|L|$. Since we have assumed that $A$ and $B$ are of $O(1)$, we obtain
\[ \frac{S}{\tilde{S}^{(n)}} = O \left( \frac{1}{\lambda^{n+1} T^3 V} \right). \]

Thus, $S/\tilde{S}^{(n)} \ll 1$ and the initial correlation $C_4$ may be ignored. There are, however, several cases, in which this observation does not apply:

- Dimensionless quantity $A$ in (A.10) is large.
- $\mathcal{F}$ in (A.10) is extremely large at $q_1 \simeq p_1$ and/or $q_1 \simeq p_2$.
- In massless theory ($m = 0$), some quantities are sensitive to the infrared region (cf. the footnote in §2.2.2). If $S$ is (A.5) is such a quantity, more elaborate analysis is necessary to see if $C_4$ may be ignored or not.
Appendix B: Some properties of the propagator and the self-energy part

Self-energy-part ($\hat{\Sigma}$) inserted propagator is obtained by perturbatively computing the formula for the full propagator:

$$\hat{G}(x, y) \equiv -i \langle T \hat{C} (\hat{\phi}(x) \hat{\phi}^\dagger(y) e^{i \int d^4z \hat{L}_{int}(z)} \rangle, \tag{B.1}$$

where $\hat{L}_{int} (= \hat{L} - \hat{L}_0)$ is the interaction hat-Lagrangian. From this, we obtain $G_{11} + G_{22} = G_{12} + G_{21}$, of which (2.9) is a special case. Applying this to the single-$\hat{\Sigma}$ inserted propagator ($\hat{G} \rightarrow \hat{\Delta} \cdot \hat{\Sigma} \cdot \hat{\Delta}$) and using (2.9), we obtain

$$\sum_{i, j=1}^{2} \Sigma_{ij}(x, y) = 0.$$ 

Taking complex-conjugate of (B.1), we see that

$$i G_{11}(x, y) = [i G_{22}(y, x)]^*, \quad i G_{12(21)}(x, y) = [i G_{12(21)}(y, x)]^*.$$ 

Using this for $\hat{G} \rightarrow \hat{\Delta}$ and $\hat{G} \rightarrow \hat{\Delta} \cdot \hat{\Sigma} \cdot \hat{\Delta}$, we obtain

$$-i(\Sigma(x, y))_{11} = [-i(\Sigma(x, y))_{22}]^*, \quad -i(\Sigma(x, y))_{12(21)} = [-i(\Sigma(y, x))_{12(21)}]^*.$$ 

Fourier transformation on $x - y$ yields

$$(\Sigma(X; P))_{11} = -(\Sigma(X; P))_{22},$$

$$(\Sigma(X; P))_{12}, \quad (\Sigma(X; P))_{21} : \text{pure imaginary.}$$

Using this in (3.33), we obtain $\Sigma_A(X; P) = [\Sigma_R(X; P)]^*$.

In the case of physical-$N$ scheme, $\hat{L}_c$ in §4 contributes to $\hat{\Sigma}$, $-\hat{\nu}(x, y) \in \hat{\Sigma}(x, y)$ (cf. (4.57)). This “additional” contribution does not invalidate the above properties.

Appendix C: Representation of $\Delta_c$ in terms of a vacuum-theory kit

Here obtain the expression for $\Delta_c$ and the number density $N_B^{(\pm)}$ in terms of the quantities in vacuum theory.
Straightforward but a bit lengthy calculation using (A.1) and its hermitian conjugate yields

\[
i\Delta_c(X; P) = \int d^4(x-y)e^{iP\cdot(x-y)}\langle \phi(x)\phi(y) + \phi^\dagger(y)\phi(x) \rangle
\]

\[
= i\Delta_c^{(0)}(P) + i\Delta_c^{(1)}(X; P),
\]

\[
i\Delta_c^{(0)}(P) = \frac{\pi}{E_p} [\delta(p_0 - E_p) + \delta(p_0 + E_p)]
\]

\[
= 2\pi\delta(P^2 - m^2).
\]

\[
i\Delta_c^{(1)}(X; P) = 2\pi \int d^3q \frac{1}{\sqrt{E_+E_-}} \times \left[ \langle a^\dagger(p_+)a(p_-) \rangle \delta(p_0 - (E_+ + E_-)/2)e^{i[(E_+ - E_-)X_0 - q\cdot X]}
\]

\[
+ \langle b^\dagger(-p_+)b(-p_-) \rangle \delta(p_0 + (E_+ + E_-)/2)e^{i[(E_+ - E_-)X_0 + q\cdot X]}
\]

\[
+ \frac{1}{8} \langle a^\dagger(p_+)b^\dagger(-p_-) \rangle \delta(p_0 - (E_+ - E_-)/2)e^{i[(E_+ + E_-)X_0 - q\cdot X]}
\]

\[
+ \frac{1}{8} \langle b(-p_-)a(p_+) \rangle \delta(p_0 - (E_+ - E_-)/2)e^{-i[(E_+ + E_-)X_0 - q\cdot X]} \right],
\]

where \( p_\pm \equiv q / 2 \) and \( E_\pm \equiv E_{p,\pm} \).

Let us first show that the contribution of the third and fourth terms of the bracketed quantities in (C.2) is negligibly small. From the setup in §1, \( \Delta_c^{(1)}(X; P) \) varies slowly in \( X \), which means that \( \langle a^\dagger(p_+)b^\dagger(-p_-) \rangle \sim \langle b(-p_-)a(p_+) \rangle \sim 0 \) for \( |q| \gtrsim 1/|L| \) and \( E_+ + E_- \gtrsim 1/L^0 \), where \( L^0 \) is the size of a spacetime cell (see §1). This can be realized only if \( m \lesssim 1/L^0 \). However, as mentioned at the second footnote in §4.3, in the case of \( m \lesssim 1/L^0 \), the bare propagator should be replaced with the tadpole self-energy-part resummed propagator. The tadpole induces the mass of \( O(\sqrt{\lambda T}) \) with \( T \) the typical scale(s) of the system, so that \( E_+ + E_- >> 1/L^0 \).

Let us turn to the first and second terms in (C.2). Same reasoning as above leads to \( \langle a^\dagger(p_+)a(p_-) \rangle \sim \langle b^\dagger(-p_+)b(-p_-) \rangle \sim 0 \) for \( |q| \gtrsim 1/|L| \). We shall consider the “hard region” \( |P^\mu| > O(1/L^\mu) \). From the first term, we pick out

\[
\delta\left(p_0 - \frac{E_+ + E_-}{2} \right)e^{-i q \cdot x}
\]

\[
= \left[ \delta(p_0 - E_p) - \frac{E_p^2q^2 - (p \cdot q)^2}{8E_p^3}\delta'(p_0 - E_p) + \ldots \right] e^{-i q \cdot x}
\]

\[
= \left[ \delta(p_0 - E_p) + \delta'(p_0 - E_p) \frac{E_p^2\nabla_x^2 - (p \cdot \nabla_x)^2}{8E_p^3} + \ldots \right] e^{-i q \cdot x}.
\]
Then, to the gradient approximation, we have
\[ \delta(p_0 - (E_+ + E_-)/2) \simeq \delta(p_0 - E_p). \]
Similarly, we have \( E_+ E_- \simeq E_p^2 \). The second term in (C.2) may be dealt with similarly.

After all this, for \( |P^\mu| > O(1/L^\mu) \), we finally obtain
\[ i \Delta_c(X; P) \simeq 2\pi \left[ \theta(p_0) \{1 + 2N_B^+(X; p)\} + \theta(-p_0) \{1 + 2N_B^-(X; -p)\} \right] \delta(P^2 - m^2), \quad (C.3) \]
where
\[ N_B^+(X; p) \equiv \int d^3q \langle a^\dagger(p + q/2)a(p - q/2) \rangle \exp \left[ i \left( \frac{p \cdot q}{E_p}X_0 - q \cdot X \right) \right], \]
\[ N_B^-(X; -p) \equiv \int d^3q \langle b^\dagger(-p - q/2)b(-p + q/2) \rangle \exp \left[ i \left( \frac{p \cdot q}{E_p}X_0 + q \cdot X \right) \right]. \quad (C.4) \]
are the number densities of the quasiparticle (+) and the anti-quasiparticle (−). Comparison of (C.3) with (2.22) shows that
\[ g_B^+(X; p) = N_B^+(X; p), \]
\[ g_B^-(X; p) = -1 - N_B^-(X; -p). \]

For \( \Delta_c(X; P) \) with \( |P^\mu| \lesssim 1/L^\mu \), one should use the fundamental formula (C.2).

Incidentally, in the case where \( \rho \) is diagonal in momentum space, \( \langle a^\dagger(p)a(q) \rangle = \delta(p - q)\tilde{n}_+(p) \) and \( \langle b^\dagger(p)b(q) \rangle = \delta(p - q)\tilde{n}_-(p) \), we obtain \( N_B^{(+)}(X; p) = \tilde{n}_+(p) \), where \( \tilde{n}_\pm \) are the number densities of the (anti)particle in vacuum theory.

**Appendix D Derivation of (3.48) and (3.50)**

D1 Derivation of (3.48)

Let us start with computing the piece of (3.47), which diverges in the narrow-width limit, \( \text{Im} \Sigma_R \to -\epsilon(p_0)0^+ \). The relevant region (of integration) is where \( \text{Re}(G_R^{-1}(X; P)) = \text{Re}(G_A^{-1}(X; P)) = P^2 - m^2 - \text{Re}\Sigma_R(X; P) \sim 0 \). We define “on the mass-shell” \( p_0 = \pm \omega_\pm(X; \pm p) \) \((\equiv \pm \omega_\pm)\) through
\[ [P^2 - m^2 - \text{Re}\Sigma_R(X; P)] \bigg|_{p_0 = \pm \omega_\pm} = 0, \quad (D.1) \]
from which we obtain
\[\pm Z_{\pm}^{-1} \omega_{\pm} v_{\pm} = p + \frac{1}{2} \left. \frac{\partial Re \Sigma_R}{\partial p} \right|_{p_0 = \pm \omega_{\pm}},\]
\[\left. \frac{\partial Re \Sigma_R}{\partial X^\mu} \right|_{p_0 = \pm \omega_{\pm}} = 2 Z_{\pm}^{-1} \omega_{\pm} \frac{\partial \omega_{\pm}}{\partial X^\mu}.\]  \hspace{1cm} (D.2)

Here \(v_{\pm} \equiv \pm \frac{\partial \omega_{\pm}}{\partial p}\) are the velocity of the \(\pm\) mode with momentum \(\pm p\) and
\[Z_{\pm}^{-1} \equiv 1 \mp \frac{1}{2} \omega_{\pm},\]  \hspace{1cm} (D.3)
are wave-function renormalization factors. To extract the “diverging” piece, we can make following approximations;
\[P^{\mu} \Sigma_{off} \sim \theta(p_0) (\omega_+ , p)^{\mu} \Sigma_{off}^{(+)} + \theta(-p_0) (-\omega_- , p)^{\mu} \Sigma_{off}^{(-)},\]
\[-i \left(2 \omega_{\pm} Z_{\mp}^{-1} (p_0 - \tau \omega_{\mp})\right)^2 + \left(Im \Sigma_R^{(\tau)}\right)^2.\]  \hspace{1cm} (D.4)

where
\[\Sigma_{off}^{(\pm)} = \Sigma_{off}(X; p_0 = \pm \omega_{\pm}, p), \quad \Sigma_R^{(\pm)} = \Sigma_R(X; p_0 = \pm \omega_{\pm}, p).\]  \hspace{1cm} (D.5)

Using (D.4) in (3.47), we obtain
\[\langle \delta j^\mu(X) \rangle \simeq \frac{1}{2} \int \frac{d^3 p}{(2\pi)^3} \left[ \hat{P}^{\mu}_+ Z_+ \frac{-i \Sigma_{off}^{(+)} / |Im \Sigma_R^{(\tau)}|}{|Im \Sigma_R^{(\tau)}|} - \hat{P}^{\mu}_- Z_- \frac{-i \Sigma_{off}^{(-)} / |Im \Sigma_R^{(-)}|}{|Im \Sigma_R^{(-)}|} \right],\]  \hspace{1cm} (D.6)

where \(\hat{P}^{\mu}_\pm \equiv (1, \pm p / \omega_{\pm})\). In deriving (D.6), we have assumed that \(|Im \Sigma_R^{(\pm)}| << \omega_{\pm}\), under which (D.4) is a good approximation. Referring to (2.25), we can extract from (D.6) the contribution to the physical-number density:
\[\delta N^{(\pm)}(X; \pm \omega_{\pm}, \pm p) = \frac{Z_\pm}{2} \frac{-i \Sigma_{off}^{(\pm)} / |Im \Sigma_R^{(\pm)}|}{|Im \Sigma_R^{(\pm)}|}.\]

**D2 Derivation of (3.50)**

We note that \(\Gamma^{(p)}_{\pm} \tau_{\pm}/2\) in (3.48) is the change in the “physical”-number density, during the time interval \(\tau_{\pm}/2\), due to the net production rate. In §1, we have introduced the spacetime cells, whose size is \(L^\mu\) \((\mu = 0, 1, 2, 3)\). It is quite natural to take \(\tau_{\pm} = L_0,\)
where $L_0$ is the size of the time direction of the spacetime cell (including the spacetime point $X^\mu$ in (3.48)). [Strictly speaking, in general, $\tau_+ \neq \tau_-$. However, in the present crude argument, we ignore this difference.] It is interesting to note that $\Gamma^{(p)}_{\pm}/2$ in (3.48) is half of the net production probability, due to the reactions, during the time interval $\tau_\pm = L_0$. [In this respect, cf. Ref. 21.)]

Noticing that we are concerned about the quasiparticle mode with momentum $p$ [anti-quasiparticle mode with momentum $-p$], we see that (3.48) leads to the relations,

$$
\delta N_\pm (X_0 + L_0, X + v_\pm L_0; \pm \omega_\pm, \pm \hat{p}) - \delta N_\pm (X_0, X; \pm \omega_\pm, \pm \hat{p}) \approx L_0 Z_\pm - i \Sigma_{\text{eff}} (X; \pm \omega_\pm, p) \frac{2 \omega_\pm}{2^\omega_\pm},
$$

(D.7)

from which we obtain

$$
\frac{\partial \delta N_\pm (X; \pm \omega_\pm, \pm \hat{p})}{\partial X_0} + v_\pm \cdot \nabla_X \delta N_\pm \approx Z_\pm \left[ \Gamma^{(p)}_{\pm} + \left\{ f_B, Re \Sigma_R \right\} \right]_{p_0 = \pm \omega_\pm}.
$$

(D.8)

Using (D.2) with (D.3), we get

$$
\left. \left\{ f_B, Re \Sigma_R \right\} \right|_{p_0 = \pm \omega_\pm} = -2 Z_{\omega_\pm}^{-1} \left( \frac{p_\pm}{\omega_\pm} \right) \frac{\partial N^{(\pm)}_B (X; \pm \omega_\pm, \pm \hat{p})}{\partial X} + \frac{\partial \omega_\pm}{\partial X_\mu} \frac{\partial N^{(\pm)}_B (X; \pm \omega_\pm, \pm \hat{p})}{\partial P_\mu} \bigg|_{p_0 = \pm \omega_\pm}.
$$

where use has been made of (2.29) and (2.27). Substituting this into (D.8) and adding (2.29), we finally obtain

$$
\frac{\partial N^{(\pm)} (X_0)}{\partial X_0} + v_\pm \cdot \nabla_X N^{(\pm)} \approx \left[ Z_\pm \Gamma^{(p)}_{\pm} + \frac{\partial \omega_\pm}{\partial X_\mu} \frac{\partial N^{(\pm)}_B (X; \pm \omega_\pm, \pm \hat{p})}{\partial P_\mu} \right]_{p_0 = \pm \omega_\pm},
$$

where $N^{(\pm)}_B (X; \pm \omega_\pm, \pm \hat{p}) = N^{(\pm)} (X; \pm \omega_\pm, \pm \hat{p}) + \delta N_\pm (X; \pm \omega_\pm, \pm \hat{p}).$

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