Temporal-Spatial Interference Pattern During Solitons Interaction

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Interference patterns associated with soliton-soliton interaction are investigated in detail. We find that the temporal and spatial interference patterns exhibit quite different characteristics. The period of the spatial interference pattern is determined by the relative velocity of the solitons, and the temporal pattern behavior is determined by the peak amplitudes and the kinetic energy of the solitons. Analytical expressions for the periods of the interference patterns are obtained. A method for classifying the nonlinearity of many nonlinear systems is proposed. As an example, we discuss possibilities to observe these properties of solitons in a nonlinear planar waveguide.

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I. INTRODUCTION

Since Zabusky and Kruskal introduced the concept of soliton at the first time in 1965\textsuperscript{1}, studies on dynamics of soliton have been done in many fields, including hydrodynamics\textsuperscript{2–4}, quantum field theory\textsuperscript{5}, plasma physics\textsuperscript{6–9}, nonlinear optics\textsuperscript{10–13}, and Bose-Einstein condensate (BEC)\textsuperscript{10–13}. Nowadays, it is well known that solitons collide elastically, and there are some phase shift appearing after the collision\textsuperscript{14–18}, which demonstrate the particle-like properties of soliton. These just provide us the knowledge about the properties of solitons before or after their collision. However, less attentions have been paid on the process of solitons’ interaction. Considering soliton is fundamentally a wave packet, one could expect that they can interfere with each other during their interaction process\textsuperscript{19, 20}. Notably, solitons’ interaction is a nonlinear process, since there is no linear superposition principle for solitons\textsuperscript{1, 18}. Then, are there any definite laws in the nonlinear process? It deserves further research for soliton application. Furthermore, it is known that one can measure many physical quantities through light interference. What can we get through solitons’ nonlinear interference pattern?

In this paper, we focus on the interference behavior in the process of soliton-soliton interaction. We find that there are two types of periodic behavior during the nonlinear interaction of solitons, namely, separate temporal and spatial interference patterns. We derive the period expressions of interference pattern analytically and exactly. Based on the precise expressions of the two periods, one can manipulate the interference pattern precisely through changing the shape and velocities of the solitons. It is shown that a nonlinear parameter could be obtained from interference pattern with properties of solitons known.

The paper is organized as follows. In Section II, we study on nonlinear interference pattern between solitons’ interaction process in detail. Definite scaling laws are found and the corresponding physics are discussed. In Section III, possibilities to observe interference pattern are discussed in a planar waveguide. A method for measuring the nonlinearity of the nonlinear systems is proposed. A conclusion is given in Section IV.

II. THE TWO-SOLITON SOLUTION AND THEIR INTERFERENCE PATTERNS

Since nonlinear Schrödinger (NLS) equation can describe dynamics of soliton in many physical systems, such as nonlinear fiber\textsuperscript{21}, Bose-Einstein condensate\textsuperscript{22}, plasma system\textsuperscript{23}, and even water wave tank\textsuperscript{24}, we start with the well-known simplified NLS equation

\[ i\frac{\partial U(x,t)}{\partial t} + \frac{\partial^2 U(x,t)}{\partial x^2} + 2g|U(x,t)|^2U(x,t) = 0, \quad (1) \]

which has been studied widely. The coefficient \(g\) is the nonlinear parameter. For a BEC system, it is related to scattering length, and in nonlinear optics it is related to the nonlinear Kerr effect. The interaction between solitons can be studied based on multi-soliton solution which can be obtained by Bäcklund transformation\textsuperscript{25–27}. Here, for simplicity and without loss of generality, we will discuss two solitons’ interaction based on the two-soliton solution. The two-soliton solution can be presented as follows through nonlinear superposition principle given by Bäcklund transformation\textsuperscript{10, 25}

\[ U = \frac{4F_1}{\sqrt{gF_2}}, \quad |U|^2 = \frac{|\ln(F_2)|_{xx}}{g}, \quad (2) \]
where
\[
F_1 = \left\{ i a_1 \left[ (b_1 - b_2)^2 + a_1^2 - a_2^2 \right] \cosh(2X_2) \\
+ 2a_1a_2 (b_2 - b_1) \sinh(2X_2) \right\} e^{2i \psi_1} \\
+ \left\{ i a_2 \left[ (b_1 - b_2)^2 + a_2^2 - a_1^2 \right] \cosh(2X_1) \\
+ 2a_1a_2 (b_1 - b_2) \sinh(2X_1) \right\} e^{2i \psi_2},
\]

\[
F_2 = \left[ (a_1 + a_2)^2 + (b_1 - b_2)^2 \right] \cosh A_1 \\
+ \left[ (a_1 - a_2)^2 + (b_1 - b_2)^2 \right] \cosh A_2 - 4a_1a_2 \cos A_3,
\]

\[
X_1 = a_1 (x - 4 b_1 t) + c_1, \quad Y_1 = b_1 x + 2 \left( a_1^2 - b_1^2 \right) t + d_1,
\]

\[
X_2 = a_2 (x - 4 b_2 t) + c_2, \quad Y_2 = b_2 x + 2 \left( a_2^2 - b_2^2 \right) t + d_2,
\]

\[
A_1 = 2 \left( a_1 - a_2 \right) x + 8 \left( a_2 b_2 - a_1 b_1 \right) t + 2 \left( c_1 - c_2 \right),
\]

\[
A_2 = 2 \left( a_2 + a_1 \right) x - 8 \left( a_1 b_1 + a_2 b_2 \right) t + 2 \left( c_2 + c_1 \right),
\]

\[
A_3 = 2 \left( b_1 - b_2 \right) x + 4 \left( a_1^2 - a_2^2 + b_1^2 - b_2^2 \right) t \\
+ 2 \left( d_1 - d_2 \right),
\]

\[a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2\] are arbitrary real numbers. When the related parameters are chosen, the solution will present us the dynamics of two solitons directly. The collision of them can be observed conveniently. As usual, one can observe that the collision is elastic, and there are phase shifts appearing. These particle-like characteristics have been studied in [14–18], which just provide us the knowledge about the properties of solitons before or after their collision. Then, what about the interaction process of two solitons’ collision? Interestingly, we find that interference pattern can emerge during the process, which demonstrates the wave properties of solitons.

A. Interference phenomena

We find that there are two periodic behaviors in the process of soliton interaction, one on space distribution and one on time evolution, which are called by spatial and temporal interference patterns separately. When the difference between their square velocities or density peaks is large and the relative velocity (RV) is large (how to define large or small will be discussed in next section), the temporal-spatial pattern will appear, such as Fig. 1(a). When they collide with different shapes and large RV, the interference pattern will change in the collision process, such as Fig. 1(b). Then, one could be curious about whether there are any definite laws for the nonlinear interference behavior.

Especially, when solitons collide with identical shapes and square velocities, the temporal pattern will disappear, and just the spatial interference patterns emerge in Fig. 2. Keeping solitons’ shapes and relative phase unchanged in Fig. 2, we study the relation between the pattern and RV. It is found that the highest peak is a constant, and has no relation with RV. However, as the RV increases, the spatial period of interference pattern will decrease. Their quantitative relation will be discussed in the next section.

\[\text{FIG. 1: (color online) (a) The temporal-spatial interference pattern for solitons with same shapes and large kinetic energy difference. The coefficients are } a_1 = 1.5, b_1 = 1, g = 0.25, a_2 = 1.5, b_2 = 20.5, c_1 = d_1 = 0, \text{ and } c_2 = d_2 = 0. \text{ (b) The interference pattern appears in the collision process of solitons with different shapes and large RV. The coefficients are } a_1 = 0.6, b_1 = 0, g = 0.1, a_2 = 0.8, b_2 = 6, c_1 = d_1 = 0, \text{ and } c_2 = d_2 = 0.\]

B. Theoretical analysis

Believing that some terms related to interference pattern should appear in the evolution of the two-soliton density, we calculate the density of the two-soliton solution exactly. The periodic functions appear in the above solution, which are \(\cos(\theta)\) and \(\sin(\theta)\), where \(\theta = 2(b_2 - b_1)x + 4(a_2^2 - a_1^2 + b_1^2 - b_2^2)t\). This means that there are two periodic behaviors in the evolution of two-soliton, which refer to spatial and temporal interference patterns separately. The spatial-period and temporal-
and their full width at half maximum \( W_j = \frac{1}{2a_j} \ln (3 + 2\sqrt{2}) \), which keep unchanged after collision. The velocity of solitons can be given as

\[ v_j = 4b_j. \]

Obviously, \( a_j \) and \( b_j \) determine soliton’s peak and velocity respectively. One can observe interaction between arbitrary two solitons through varying the parameters. From the asymptotic analysis, as usual, one can know that the solitons keep their shapes after the collision, and the “phase shift” emerge. Furthermore, we present definite properties of the nonlinear interference period during solitons’ interaction process, through combining the density calculation and asymptotic analysis technic. From Eq. (3), we can know the relative velocity determines the property of spatial interference pattern, namely

\[ D = \frac{4\pi}{v_2 - v_1}. \]

From physical viewpoint, soliton could be seen as a particle. In two-soliton circumstance, one soliton can be chosen as the reference, the other soliton’s velocity will become the RV between the two solitons. Then, the second soliton’s matter wave length will be determined by RV, based on matter wavelength theory. When RV increases, the soliton’s matter wavelength will decrease. Spatial interference pattern can be observed when the RV is large enough that soliton’s matter wavelength is smaller than scale of solitons, such as Fig. 1(a). When the matter wavelength is not smaller than the scale of soliton, namely, solitons’ relative velocity is small, one could not observe the interference pattern. This is the reason why spatial interference pattern cannot been seen in the most previous works.

From Eq. (4), Eq. (5) and Eq. (6), we can know that the temporal-period is determined by both the peaks and velocities of solitons, and the nonlinear coefficient, namely,

\[ T = \frac{2\pi}{g(P_j^2 - P_1^2) + \frac{1}{4}(v_1^2 - v_2^2)}, \]

where \( P_j = 4a_j^2/g \) \((j = 1, 2)\) denote peaks of solitons.

From the expressions (Eq. (7) and Eq. (8)) which describe the periods in space and time, one can manipulate the pattern precisely through changing the peaks and RV of solitons. When solitons’ peaks and square velocities are identical, the temporal interference pattern will disappear. Therefore, we can show the two-soliton just with space-period in Fig. 2. When the velocities of solitons are identical, the spatial interference pattern will disappear. These results provide us particular ways to observe spatial or temporal interference pattern separately. When the spatial period less than solitons’ scale and temporal period less than the time scale of collision, we can observe the temporal-spatial interference pattern in Fig. 1. It should be pointed that the asymptotic analysis based on \( b_1 \neq b_2 \), namely, the two solitons have different velocities. When two solitons are relatively static, the above analyzes will fail.

III. APPLICATION INTO NONLINEAR PLANAR WAVEGUIDE

Experimentally, one can choose spatial optical soliton system to observe the interference pattern conveniently, due to the coordinates are both spatial coordinates for spatial optical soliton. Namely, \( t \rightarrow Z \) and \( x \rightarrow X \), which are the transverse and propagation coordinates respectively.
FIG. 3: (color online) The incident angle $\theta_j$ is defined as the rotation angle from the incident direction to $+Z$ direction.

As an example, we consider continuous wave optical beams propagating inside a planar nonlinear waveguide with a refractive index

$$n = n_0 + \gamma I(x,z),$$

where $I(x,z)$ is the optical intensity, and $x, z$ the transverse coordinate and propagation distance respectively. Here the first term $n_0$ stands for the linear part of refractive index and the second term represents a Kerr-type nonlinearity. The Kerr coefficient $\gamma$ can be positive (negative) for nonlinear self-focusing (self-defocusing) medium. The nonlinear wave equation governing beam propagation in such a waveguide can be written as

$$i \frac{\partial u}{\partial z} + \frac{1}{2k_0} \frac{\partial^2 u}{\partial x^2} + k_0 \frac{\gamma}{n_0} |u|^2 u = 0,$$

where $k_0 = 2\pi n_0/\lambda_0$ is the wave number of the optical source generating the beam. After introducing normalized variables $U = u$, $X = \sqrt{2k_0} x$, $Z = z$, and $2g = \frac{\gamma n_0}{k_0}$, the Eq.(8) can be rewritten as

$$i \frac{\partial U}{\partial Z} + \frac{\partial^2 U}{\partial X^2} + 2g |U|^2 U = 0$$

In this system, the velocities of solitons will denote the tangent values of incident angles. The incident angle is defined as the rotation angle from the incident direction to $+Z$ direction, as shown in Fig. 3. The RV before will become the difference between the tangent values of incident angles. Therefore, the interference pattern could be observed much more conveniently in the nonlinear planar waveguide. For this system, the parameter $g$ is related to Kerr nonlinear parameter. We show that the nonlinear parameter can be derived from the interference pattern with some initial conditions of solitons. As an example, we study interference pattern in Fig. 1(b). The corresponding initial solitons are shown in Fig. 4(a). Their intensity values $P_j$ are assumed to be known, which can be measured more easily. The incident angle of one soliton is known to be zero, namely $\theta_1 = 0$. From the interference pattern, one can measure the periods along the two directions independently, as shown in Fig. 4(b) and (c). Then the other soliton’s incident angle can be known as

$$\theta_2 = \arctan \left( \frac{4\pi}{D} \right).$$

Furthermore, the parameter $g$ will be given by

$$g = \frac{2\pi}{T(P_2^2 - P_1^2)} + \frac{4\pi^2}{D^2(P_2^2 - P_1^2)}.$$

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FIG. 4: (color online) (a) The initial shape of the two solitons in Fig. 1(b) before collision. (b) The period $D$ on transverse direction. (c) The period $T$ on propagation direction. They are the cutaway view of the interference pattern in Fig. 1(b) at $Z = 0$ and $X = 0$ respectively.

Based on the relation, we can calculate $g$ from the period of longitudinal interference pattern and initial information of solitons before collision. Then, the Kerr nonlinear coefficient can be obtained easily from $\gamma = \frac{2P_1}{k_0}$, for $k_0$ can be known from the wave number of the optical source generating the beam, $n_0$ is very easy to be measured for it is linear refractive index. Similar discussion can be made on solitons in BEC, for the dynamic equations are identical fundamentally [10–13]. Namely, we find a possible way to measure nonlinear parameter through observing the interference pattern. It should be noted that the expressions will fail to predict the nonlinear parameter when the peak values of solitons are identical. However, the width and peak of solitons should be...
related in a certain way, \( g = \frac{\ln(3+2\sqrt{2})^2}{P_jW_j^2} \). Namely, one can calculate the nonlinear coefficient from the information of soliton’s shape too.

IV. CONCLUSION

It should be emphasized that the collision between solitons is still elastic. Here, we focus on the process of solitons’ collision. We find that temporal-spatial interference pattern can appear during the interaction of solitons. Period of spatial interference pattern is determined by the relative velocity of solitons. Spatial interference fringes can be just observed under the condition that soliton’s matter wavelength is shorter than the scale of solitons. Temporal interference pattern can be seen when two solitons are relatively static and the distance between them is not large. From the expressions which describe the period of the pattern, we know that interference pattern could be manipulated precisely through changing RV and intensity peaks of solitons. As an example, we show the possibilities to observe the patterns in a nonlinear planar waveguide. We find that the Kerr nonlinear coefficient can be read out from the interference pattern, with two solitons’ density peaks and incident angles known.

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