DOMINATED SPLITTING FOR EXTERIOR K-POWERS OF COCYCLES AND SINGULAR HYPERBOLICITY

VITOR ARAUJO AND LUCIANA SALGADO

Abstract. We relate dominated splitting for a linear multiplicative cocycle with dominated splitting for the exterior k-powers of this cocycle. For a $C^1$ vector field $X$ on a 3-manifold, we can obtain singular-hyperbolicity using only the tangent map $DX$ of $X$ and a family of indefinite and non-degenerate quadratic forms without using the associated flow $X_t$ and its derivative $DX_t$. In this setting, we also improve a result from [5]. As a consequence, we show the existence of adapted metrics for singular-hyperbolic sets for three-dimensional $C^1$ vector fields.

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1. Introduction

Let $M$ be a connected compact finite $n$-dimensional manifold, $n \geq 3$, with or without boundary. We consider a vector field $X$, such that $X$ is inwardly transverse to the boundary $\partial M$, if $\partial M \neq \emptyset$. The flow generated by $X$ is denoted by $\{X_t\}$.

A singularity for the vector field $X$ is a point $\sigma \in M$ such that $X(\sigma) = 0$ or, equivalently, $X_t(\sigma) = \sigma$ for all $t \in \mathbb{R}$. The set formed by singularities is the singular set of $X$ denoted $\text{Sing}(X)$. We say that a singularity is hyperbolic if the eigenvalues of the derivative $DX(\sigma)$ of the vector field at the singularity $\sigma$ have nonzero real part.

A hyperbolic set for a flow $X_t$ on $M$ is a compact invariant set $\Gamma$ with a continuous splitting of the tangent bundle, $T_\Gamma M = E^s \oplus E^c \oplus E^u$, where $E^X$ is the direction of the vector field, for which the subbundles are invariant under the derivative $DX_t$ of the flow $X_t$

$$DX_t \cdot E^*_x = E^*_{X_t(x)}; \quad x \in \Gamma, \quad t \in \mathbb{R}, \quad * = s, X, u;$$

and $E^s$ is uniformly contracted by $DX_t$ and $E^u$ is likewise expanded: there are $K, \lambda > 0$ so that

$$\|DX_t \cdot E^s_x\| \leq Ke^{-\lambda t}, \quad \|DX_t \cdot E^u_x\|^{-1} \leq Ke^{-\lambda t}, \quad x \in \Gamma, \quad t \in \mathbb{R}. \quad (1.2)$$

Very strong properties can be deduced from the existence of such hyperbolic structure; see for instance $[10, 11, 24, 15, 22]$.

Weaker notions of hyperbolicity (e.g. dominated splitting, partial hyperbolicity, volume hyperbolicity, sectional hyperbolicity, singular hyperbolicity etc) have been developed to encompass larger classes of systems beyond the uniformly hyperbolic ones; see $[9]$ and specifically $[26, 4, 7]$ for singular hyperbolicity and Lorenz-like attractors.

**Definition 1.** A dominated splitting over a compact invariant set $\Lambda$ of $X$ is a continuous $DX_t$-invariant splitting $T_\Lambda M = E \oplus F$ with $E_x \neq \{0\}$, $F_x \neq \{0\}$ for every $x \in \Lambda$ such that there are positive constants $K, \lambda$ satisfying

$$\|DX_t|_{E_x}\| \cdot \|DX_t|_{F_{X_t(x)}}\| < Ke^{-\lambda t}, \quad \text{for all } x \in \Lambda, \text{ and all } t > 0. \quad (1.3)$$

A compact invariant set $\Lambda$ is said to be partially hyperbolic if it exhibits a dominated splitting $T_\Lambda M = E \oplus F$ such that subbundle $E$ is uniformly contracted, i.e., there exists $C > 0$ and $\lambda > 0$ such that $\|DX_t|_{E_x}\| \leq Ce^{-\lambda t}$ for $t \geq 0$. In this case $F$ is the central subbundle of $\Lambda$. Or else, we may replace uniform contraction along $E^s$ by uniform expansion along $F$ (the right hand side condition in (1.2)).

We say that a $DX_t$-invariant subbundle $F \subset T_\Lambda M$ is a sectionally expanding subbundle if $\dim F_x \geq 2$ is constant for $x \in \Lambda$ and there are positive constants $C, \lambda$ such that for every $x \in \Lambda$ and every two-dimensional linear subspace $L_x \subset F_x$ one has

$$|\det(DX_t|_{L_x})| > Ce^{\lambda t}, \quad \text{for all } t > 0. \quad (1.4)$$

**Definition 2.** $[17, \text{Definition 2.7}]$ A sectional-hyperbolic set is a partially hyperbolic set whose singularities are hyperbolic and central subbundle is sectionally expanding.
This is a particular case of the so called *singular hyperbolicity* which definition we recall now. A $DX_t$-invariant subbundle $F \subset T_\Lambda M$ is said to be a *volume expanding* if in the above condition 1.5, we may write

$$|\det(DX_t|_{F_x})| > Ce^{\lambda t}, \text{ for all } t > 0. \tag{1.5}$$

**Definition 3.** [18, Definition 1] A *singular hyperbolic set* is a partially hyperbolic set whose singularities are hyperbolic and central subbundle is volume expanding.

We remark that, in the three-dimensional case, these notions are equivalent. This is a feature of the Lorenz attractor as proved in [25] and also a notion that extends hyperbolicity for singular flows, because sectional hyperbolic sets without singularities are hyperbolic; see [19, 4].

**Remark 1.1.** The properties of singular hyperbolicity can be expressed in the following equivalent forms; see [4]. There exists $T > 0$ such that

- $\|DX_t|_{E_x}\| < \frac{1}{2}$ for all $x \in \Gamma$ (uniform contraction); and
- $|\det(DX_t|_{E_x})| > 2$ for all $x \in \Gamma$.

Proving the existence of some hyperbolic structure as in (1.1) and (1.2), is in general a non-trivial matter, even in its weaker forms. We recall that the Lorenz attractor was shown to exist through a computer assisted proof only very recently in [25] and, even more recently, in [13] it was constructed a concrete example of a mechanical system modeled by an Anosov flow.

The “cone field technique” is the usual way to prove hyperbolicity even in some of its weaker forms; see e.g. [1, 2, 3, 21, 20]. Given a field of non-degenerate and indefinite quadratic forms $\mathcal{J} : TM \to \mathbb{R}$ with constant index, we define the negative cone $C_-(x)$ as the set of vectors $v \in T_x M$ such that $\mathcal{J}(v) < 0$ and, analogously, we define the positive cone $C_+(x)$ as the set of vectors $v \in T_x M$ such that $\mathcal{J}(v) > 0$. Lewowicz used this notion in his study of expansive homeomorphisms [16] and obtained an equivalence involving quadratic forms and uniform hyperbolicity for diffeomorphisms. He in fact proved that a diffeomorphism $f$ is Anosov (that is, the whole manifold is a hyperbolic set) if, and only, if there exists a field of non-degenerate and indefinite quadratic forms $\mathcal{J}$ on the whole manifold $M$ such that the quadratic forms $f^t \mathcal{J} - \mathcal{J}$ are everywhere positive definite, where $f^t \mathcal{J}$ denotes the pullback of the quadratic form by the derivative of $f$.

This idea was adapted for the study of Lyapunov exponents in [28], where a counterpart of the Lewowicz result was obtained using the notion of $\mathcal{J}$-monotonicity, and was also used to study stochastic properties of diffeomorphisms in [14].

In [5] the authors extended these results obtaining a necessary and sufficient condition for a maximal invariant set $\Gamma$, possibly with singularities, of a trapping region $U$, to be a partially hyperbolic set for a $C^1$ flow $X_t$.

We recall that a *trapping region* $U$ for a flow $X_t$ is an open subset of the manifold $M$ which satisfies: $X_t(U)$ is contained in $U$ for all $t > 0$, and there exists $T > 0$ such that $X_t(U)$ is contained in the interior of $U$ for all $t > T$. We define $\Gamma(U) = \Gamma_X(U) := \cap_{t > 0} X_t(U)$ to be the *maximal positive invariant subset in the trapping region* $U$. 
We say that a compact invariant subset $\Lambda$ is non-trivial if
- either $\Lambda$ does not contain singularities;
- or $\Lambda$ contains at most finitely many singularities, $\Lambda$ contains some regular orbit and is connected.

Two subspaces $E$ and $F$ of a vector space are said to be almost orthogonal with respect to an inner product $\langle \cdot, \cdot \rangle$ if, given $\varepsilon > 0$, $|\langle u, v \rangle| < \varepsilon$, for all $u \in E, v \in F$, with $\|u\| = 1 = \|v\|$.

**Theorem 1.2.** [5, Theorem A] A non-trivial compact invariant subset $\Gamma$ is a partially hyperbolic set for a flow $X_t$ if, and only if, there is a $C^1$ field $J$ of non-degenerate and indefinite quadratic forms with constant index, equal to the dimension of the stable subspace of $\Gamma$, such that $X_t$ is a non-negative strictly $J$-separated flow on a neighborhood $U$ of $\Gamma$.

Moreover $E$ is a negative subspace, $F$ a positive subspace and the splitting is almost orthogonal.

Here strict $J$-separation corresponds to strict cone invariance under the action of $DX_t$ and $\langle \cdot, \cdot \rangle$ is a Riemannian inner product in the ambient manifold.

We note that the condition stated in Theorem 1.2 allows us to obtain partial hyperbolicity checking a condition at every point of the compact invariant set that depends only on the tangent map $DX$ to the vector field $X$ together with a family $J$ of quadratic forms without using the flow $X_t$ or its derivative $DX_t$. This is akin to checking the stability of singularity of a vector field using a Lyapunov function.

In addition, we presented a criterion for partial hyperbolicity through infinitesimal Lyapunov functions based on the space derivative $DX$ of the vector field $X$ only. We assume that coordinates are chosen locally adapted to $J$ in such a way that $J(v) = \langle J_x(v), v \rangle$, $v \in T_xM, x \in U$, and $J_x : T_xM \to \mathbb{R}$ is a self-adjoint linear operator having diagonal matrix with $\pm 1$ entries along the diagonal.

We say that a $C^1$ family $J$ of indefinite and non-degenerate quadratic forms is compatible with a continuous splitting $E \oplus F = E\Gamma$ of a vector bundle over some compact subset $\Gamma$ if $E_x$ is a $J$-negative subspace and $F_x$ is a $J$-positive subspace for all $x \in \Gamma$.

**Proposition 1.3.** [5, Proposition 1.3] A $J$-non-negative vector field $X$ on $U$ is strictly $J$-separated if, and only if, there exists a compatible family $J_0$ of forms and there exists a function $\delta : U \to \mathbb{R}$ such that the operator $\tilde{J}_{0,x} := J_0 \cdot DX(x) + DX(x)^\ast \cdot J_0$ satisfies

$$\tilde{J}_{0,x} - \delta(x)J_0 \text{ is positive definite, } x \in U,$$

where $DX(x)^\ast$ is the adjoint of $DX(x)$ with respect to the adapted inner product.

The results leading to Theorem 1.2 and Proposition 1.3, in the more general case of linear multiplicative cocycles, were proved by the authors in [5], and then the general cocycle can be replaced by the derivative cocycle $DX_t$ of the flow $X_t$ with infinitesimal generator $DX$.

Building on this, in [5, Corollary B and Proposition 1.4] it was obtained a necessary and sufficient condition for the set $\Gamma$, possibly with hyperbolic singularities, to be a sectional-hyperbolic set for a $C^1$ flow $X_t$ involving a stronger condition than the strict $J$-separation for the Linear Poincaré Flow of $X$ over all compact invariant subsets $\Gamma_0$ without singularities of $\Gamma$. 
A characterization of $\mathcal{J}$-monotonicity of the Linear Poincaré Flow similar to the one in Proposition 1.3 was also obtained in [5, Proposition 1.4] involving the space derivative $DX$ of the field $X$, the field of forms $\mathcal{J}$ and the projection $\Pi DX$ on the normal bundle to $X$ away from singularities. However, dealing with the Linear Poincaré Flow near singularities is prone to numerical instability and the projection $\Pi DX$ does not extend to the singularities.

Here we provide an alternative way to obtain singular hyperbolicity for three-dimensional flows using the same expression as in Proposition 1.3 applied to the infinitesimal generator of the exterior square $\wedge^2 DX_t$ of the cocycle $DX_t$. This infinitesimal generator can be explicitly calculated through the infinitesimal generator $DX$ of the linear multiplicative cocycle $DX_t$ associated to the vector field $X$.

In a number of situations dealing with mathematical models from the physical, engineering or social sciences, it is the vector field that is given and not the flow. Thus we expect that the results here presented to be useful to check some weaker forms of hyperbolicity.

Indeed, we are able to explicitly prove that the geometrical Lorenz attractor is singular-hyperbolic in a straightforward way using this technique; see Section 1.3.

As a consequence of these ideas we show the existence of adapted metrics for singular-hyperbolic subsets for general $C^1$ three-dimensional vector fields.

1.1. Statements of results. Let $U$ be a trapping region for a $C^1$ vector field $X$ on a compact, $n$-dimensional manifold $M$, which is non-negative strictly $\mathcal{J}$-separated, and whose singularities are hyperbolic in $U$. We write $\overline{A}$ for the topological closure of the set $A \subset M$ in what follows. Let $\Gamma = \Gamma(U) := \cap_{t \in \mathbb{R}} X_t(U)$ be the maximal invariant set of $X$ in $U$.

Sectional-hyperbolicity deals with area expansion along any two-dimensional subspace of a vector subbundle. It is then natural to consider the linear multiplicative cocycle $\wedge^2 DX_t$ over the flow $X_t$ of $X$ on $U$, that is, for any pair $u, v$ of vectors in $T_xM, x \in U$ and $t \in \mathbb{R}$ such that $X_t(x) \in U$ we set

$$((\wedge^2 DX_t) \cdot (u \wedge v)) = (DX_t \cdot u) \wedge (DX_t \cdot v),$$

see [8, Chapter 3, Section 2.3] or [27] for more details and standard results on exterior algebra and exterior products of linear operator.

Given a partially hyperbolic splitting $T_t M = E_t \oplus F_t$ over the compact $X_t$-invariant subset $\Gamma$, the bundle of bivectors $\wedge^2 T_t M$ admits also a partially hyperbolic splitting, and $T_t M$ has a sectional hyperbolic splitting if, and only if, $\wedge^2 T_t M$ has a partial hyperbolic splitting of a specific kind. This can in fact be extended to arbitrary $k$th exterior powers.

We note that if $E \oplus F$ is a $DX_t$-invariant splitting of $T_t M$, with $\{e_1, \ldots, e_{\ell}\}$ a family of basis for $E$ and $\{f_1, \ldots, f_{\ell}\}$ a family of basis for $F$, then $\widetilde{F} = \wedge^k F$ generated by $\{f_{i_1} \wedge \cdots \wedge f_{i_k}\}_{1 \leq i_1 < \cdots < i_k \leq \ell}$ is naturally $\wedge^k DX_t$-invariant by construction. In addition, $\widetilde{E}$ generated by $\{e_{i_1} \wedge \cdots \wedge e_{i_k}\}_{1 \leq i_1 < \cdots < i_k \leq \ell}$ together with all the exterior products of $i$ basis elements of $E$ with $j$ basis elements of $F$, where $i + j = k$ and $i, j \geq 1$, is also $\wedge^k DX_t$-invariant and, moreover, $\widetilde{E} \oplus \widetilde{F}$ gives a splitting of the $k$th exterior power $\wedge^k T_t M$ of the subbundle $T_t M$. One of our results is the following.

**Theorem A.** Let $T_t M = E_t \oplus F_t$ be a $DX_t$-invariant splitting over the compact $X_t$-invariant subset $\Gamma$ such that $\dim F = c \geq 2$. For any given $2 \leq k \leq c$, let $\widetilde{F} = \wedge^k F$ be the
∧^k DX^t-invariant subspace generated by the vectors of F and \( \tilde{E} \) be the \( \wedge^k DX^t \)-invariant subspace such that \( \tilde{E} \oplus \tilde{F} \) is a splitting of the \( k \)th exterior power \( \wedge T^*_\Gamma M \) of the subbundle \( T^*_\Gamma M \).

Then \( E \oplus F \) is a dominated splitting if, and only if, \( \tilde{E} \oplus \tilde{F} \) is a dominated splitting for \( \wedge^k DX^t \).

As a consequence of this last result we can obtain sectional hyperbolicity without use any condition over the linear Poincaré flow.

**Corollary 1.4.** Assume, in the statement of Theorem A, that \( E \) is uniformly contracted by \( DX^t \). Then \( E \oplus F \) is a sectional-hyperbolic splitting for \( DX^t \) if, and only if, \( \tilde{E} \oplus \tilde{F} \) is a partially hyperbolic splitting for \( \wedge^2 DX^t \) such that \( \tilde{F} \) is uniformly expanded by \( \wedge^2 DX^t \).

**Remark 1.5.** A similar statement to Theorem A is true for discrete dynamical systems, that is, replacing \( DX^t \) in the statement of Theorem A and in (1.1), (1.2), (1.3) and also (1.5) by the tangent map \( Df \) to a diffeomorphism \( f \) of a compact manifold \( M \).

We note that it is not clear how to derive, from the knowledge of \( J \) in a general situation, a field of indefinite non-degenerate quadratic forms \( \tilde{J} \) defined on \( \wedge^2 T^*_\Gamma M \) such that \( \wedge^2 DX^t \) is strictly \( \tilde{J} \)-separated; see Example 1 in the comments Section 1.3 below.

However, in a 3-manifold, we show that singular-hyperbolicity corresponds to strict \( J \)-separation for \( DX^t \) together with strict \(( -J )\)-separation for \( \wedge^2 DX^t \) plus a condition on the rate function \( \delta \), so the same field of quadratic forms can be used to obtain both partial hyperbolicity and singular-hyperbolicity.

In a three-dimensional manifold, let \( (u, v, w) \) be an orthonormal base with positive orientation on \( T_x M \) for a given \( x \in U \). Since we can identify \( \wedge \) with the cross-product \( \times \), then for all \( t \in \mathbb{R} \) we can make the identification

\[
\wedge^2 DX^t \cdot w = (DX^t u) \times (DX^t v).
\] (1.6)

Now the meaning of Theorem A is clear: for an orthogonal vector \( w \) to the two-dimensional central direction \( E^c \), the variation of the size of \( \wedge^2 DX^t \cdot w \) corresponds to the variation of the area of the parallelogram with sides \( DX^t(x)u, DX^t(x)v \). Hence, we have uniform expansion of area along \( E^c \) if, and only if, \( \wedge^2 DX^t \) uniformly expands the size of \( w \).

The area under the function \( \delta \) provided by Proposition 1.3 allows us to characterize different dominated splittings with respect to linear multiplicative cocycles on vector bundles (the natural generalizations of the object \( DX^t \) over \( T^*_\Gamma M \) and \( \wedge^2 DX^t \) over \( \wedge^2 T^*_\Gamma M \)). For this, define the function

\[
\Delta^b_a(x) := \int_a^b \delta(X_s(x)) \, ds, \quad x \in \Gamma, \, a, b \in \mathbb{R}.
\] (1.7)

The next result, proved in [5], gives us this characterization.

**Theorem 1.6.** [5, Theorem 2.23] Let \( \Gamma \) be a compact invariant set for \( X^t \) admitting a dominated splitting \( E^c = F_- \oplus F_+ \) for \( A_t(x) \), a linear multiplicative cocycle over \( \Gamma \) with values in \( E \). Let \( J \) be a \( C^1 \) family of indefinite quadratic forms such that \( A_t(x) \) is strictly \( J \)-separated. Then
(1) $F_\ominus \oplus F_\oplus$ is partially hyperbolic with $F_\ominus$ not uniformly contracting and $F_\oplus$ uniformly expanding if, and only if, $\Delta^t(x) \xrightarrow{(t-s) \to +\infty} +\infty$ for all $x \in \Gamma$.

(2) $F_\ominus \oplus F_\oplus$ is partially hyperbolic with $F_\ominus$ uniformly contracting and $F_\oplus$ not uniformly expanding if, and only if, $\Delta^t(x) \xrightarrow{(t-s) \to +\infty} -\infty$ for all $x \in \Gamma$.

(3) $F_\ominus \oplus F_\oplus$ is uniformly hyperbolic if, and only if, there exists a compatible family $J_0$ of quadratic forms in a neighborhood of $\Gamma$ such that $J_0'(<\tilde{J}_x v, v>) > 0$ for all $v \in \mathcal{E}_x$ and all $x \in \Gamma$.

Above we write $J'(v) = <\tilde{J}_x v, v>$, where $\tilde{J}_x$ is given in Proposition 1.3.

We use Theorem 1.6 to obtain a sufficient condition for a flow $X_t$ on a 3-manifold to have a $\Lambda^2 DX_t$-invariant one-dimensional uniformly expanding direction orthogonal to the two-dimensional center-unstable bundle.

If $\Lambda^2 DX_t$ is strictly separated with respect to some family $\mathcal{J}$ of quadratic forms, then there exists the function $\delta_2$ as stated in Proposition 1.3. We set

$$\tilde{\Delta}_a^b(x) := \int_a^b \delta_2(X_s(x)) \, ds$$

the area under the function $\delta_2 : U \to \mathbb{R}$ given by Proposition 1.3 with respect to $\Lambda^2 DX_t$ and its infinitesimal generator.

We recall that the index of a field quadratic forms $\mathcal{J}$ on a set $\Gamma$ is the dimension of the $\mathcal{J}$-negative space.

**Theorem B.** Suppose that $X$ is three-dimensional vector field on a compact manifold and $X_t$ is non-negative strictly $\mathcal{J}$-separated over a non-trivial subset $\Gamma$, where $\mathcal{J}$ has index 1. Let also $\delta : \Gamma \to \mathbb{R}$ be the function provided by Proposition 1.3.

Then $\Lambda^2 DX_t$ is strictly $(-\delta)$-separated and $\delta_2 = 2 \text{tr}(DX) - \delta$. Moreover, $\Gamma$ is a singular hyperbolic set if, and only if, either one of the following properties holds true:

1. $\tilde{\Delta}_a^b$ satisfies condition (1) of Theorem 1.6.
2. $\mathcal{J}' - 2 \text{tr}(DX)\mathcal{J} > 0$ on $\Gamma$.

This result provides a useful characterization, in dimension three, of singular-hyperbolicity using only one family of quadratic forms $\mathcal{J}$ and its space derivative $DX$, without need to check cone invariance and contraction/expansion conditions for the flow $X_t$ generated by $X$ on a neighborhood of $\Gamma$; see the examples in Section 1.3 below.

The definition of hyperbolicity is clearly independent of the Riemannian metric on the manifold $M$. By a recent result from [12], there exists an adapted metric on $M$ for $X_t$, which means that the constants in the above expressions (1.2) and (1.3) become 1.

This result is used in the proof of Theorem 1.2. As a consequence of the proof of Theorem B we show that for $C^1$ flows having a singular-hyperbolic set $\Gamma$ there exists a metric adapted to the partial hyperbolicity and the area expansion, as follows.

**Theorem C.** Let $\Gamma$ be a compact invariant singular-hyperbolic set for a $C^1$ three-dimensional vector field $X$. Then there exists a Riemannian metric $\langle \cdot, \cdot \rangle$ inducing a norm $|\cdot|$ on $\mathcal{E}$
such that there exists $\lambda > 0$ satisfying for all $x \in \Gamma$ and $t > 0$ simultaneously
\[ |DX_t \cdot E_x| \cdot (|DX_t \cdot F_x|)^{-1} \leq e^{-\lambda t} \quad \text{and} \quad |DX_t \cdot E_x| \leq e^{-\lambda t} \]
and also $|\det(DX_t \cdot F_x)| \geq e^{\lambda t}$.

1.2. Organization of the text. The main definitions and results on linear multiplicative cocycles needed for our arguments here are proved in Section 2.

The proofs of Theorem A, Theorem B and Theorem C, presented in Subsections 3.1, 3.2 and 3.3 respectively, depend on several results about $\beta$-separation for linear multiplicative cocycles given in Section 2. In Section 4, we present some conjectures related to Theorem C.

1.3. Comments and examples of application. To geometrically understand Theorem B, let us consider a singular hyperbolic compact set $\Gamma$ with partial hyperbolic splitting $T_\Gamma M = E^s_\Gamma \oplus E^c_\Gamma$. Following [12], we can obtain a smooth Riemannian adapted metric to the partial hyperbolic splitting so that the decomposition becomes almost orthogonal. In this setting, it is clear that at each point $x \in \Gamma$ and with respect to this metric we have
\[ \hat{E}^u_\Gamma := (E^s_A)^\perp \approx E^s_\Gamma \quad \text{and} \quad \hat{E}^c_\Gamma = (E^c_A)^\perp \approx E^c_\Gamma \] (1.8)
where $\approx$ means that the subbundles are inside a cone of small width centered at one of them.

Hence, by definition of $\wedge^2 DX_t$, the decomposition $\hat{E}^u_\Gamma \oplus \hat{E}^c_\Gamma$ is also $\wedge^2 DX_t$-invariant. In addition, $\wedge^2 DX_t$ expands the length along the $\hat{E}^u_\Gamma$ direction, due to area expansion along the $E^c_\Gamma$ direction under the action of $DX_t$. Moreover, $\hat{E}^c_\Gamma$ is dominated by $\hat{E}^u_\Gamma$ since the area along $\hat{E}^c_\Gamma$ should be contracted under the action of $\wedge^2 DX_t$. This provides a partial hyperbolic splitting for $\wedge^2 DX_t$.

Therefore, by Theorem 1.2, there exists some family of quadratic forms such that $\wedge^2 DX_t$ is strictly separated. But to arrive at the right expansion and domination relations, we should have that $\hat{E}^u_\Gamma$ is now inside the positive cone, and $\hat{E}^c_\Gamma$ inside the negative cone, so that $\hat{E}^u_\Gamma$ dominates $\hat{E}^c_\Gamma$. By (1.8) this can precisely be achieved by taking $(-\beta)$ as our family of quadratic forms.

Example 1. In a higher dimensional setting, consider $\sigma$ a hyperbolic fixed point for a vector field $X$ in a 4-manifold such that $DX(\sigma)$ is diagonal with eigenvalues $\lambda_0 < \lambda_1 < \lambda_2 < 0 < \lambda_3$ along the coordinate axis, satisfying $\lambda_1 + \lambda_3 > 0$ (this is similar to the Lorenz singularity except for the extra contracting direction corresponding to $\lambda_0$). Consider also the quadratic form $\beta(\vec{x}) = -x_0^2 - x_1^2 + x_2^2 + x_3^2 = \langle J\vec{x}, \vec{x} \rangle$ with $J = \text{diag}\{-1, -1, 1, 1\}$ on $T_\sigma M$. It is standard to define a bilinear form on $\wedge^2 T_\sigma M$ using $\beta$ by
\[ (u_1 \wedge u_2, v_1 \wedge v_2) = \det \begin{pmatrix} \langle Ju_1, v_1 \rangle & \langle Ju_1, v_2 \rangle \\ \langle Ju_2, v_1 \rangle & \langle Ju_2, v_2 \rangle \end{pmatrix} \] (1.9)
on simple bivectors and then extend by linearity to the whole $\wedge^2 T_\sigma M$.

However, letting $e_0, e_1, e_2, e_3$ be the canonical base, $(e_i \wedge e_j, e_i \wedge e_j) = -1, i = 0, 1, j = 2, 3$; but $(e_1 \wedge e_2, e_1 \wedge e_2) = 1 = (e_2 \wedge e_3, e_2 \wedge e_3)$, and $e_1 \wedge e_2$ is contracted while $e_2 \wedge e_3$ is expanded.
by $\wedge^2 DX_t = \wedge^2 e^{tD}X(\sigma)$; likewise $e_1 \wedge e_2$ is contracted but $e_1 \wedge e_3$ is expanded. Thus we have mixed behavior with both positive and negative bivectors.

Hence, the standard way of building a quadratic form on $\wedge^2 T_\sigma M$ from a quadratic form on $T_\sigma M$ does not capture the the partial hyperbolic behavior on bivectors.

In the above example, the problem was caused by the increased dimension of the negative $\mathcal{J}$-subspace, as the following example shows.

**Example 2.** The case of codimension one: let us assume that $E \oplus F$ is a section-hyperbolic splitting over a compact invariant subset $\Gamma$ of a $C^1$ vector field $X$, where $E$ is one-dimensional and $F$ has arbitrary dimension. Then we have strict $\mathcal{J}$-separation for a certain family of quadratic forms which are given by $\mathcal{J}(u) = \langle J(u), u \rangle$, $u \in T_\Gamma M$ for a certain non-singular linear operator $J = J_x : T_x M$. We can now define an bilinear form on $\wedge^2 T_\Gamma M$ as in (1.9) and check that, due to the one-dimensional character of $E$, the new form gives positive values to bivectors $u \wedge v$ where both $u, v$ belong to $F$; and negative values to bivectors $u \wedge v$ where only $u$ belongs to $E$. These two classes of bivectors split $\wedge^2 T_\Gamma M = \tilde{E} \oplus \tilde{F}$ as in the statement of Theorem A.

Hence, in this codimension one setting, we may use the standard construction of a bilinear form on the external square of a vector space to obtain a quadratic form which is suitable to study domination and partial hyperbolicity, directly from the originally given $\mathcal{J}$-separating quadratic form.

We now present applications of these results. First a very simple but illustrative example.

**Example 3.** Let us consider a hyperbolic saddle singularity $\sigma$ at the origin for a smooth vector field $X$ on $\mathbb{R}^3$ such that the eigenvalues of $DX(\sigma)$ are real and satisfy $\lambda_2 < \lambda_3 < 0 < \lambda_1$. Through a coordinate change, we may assume that $D = DX(\sigma) = \text{diag}\{\lambda_1, \lambda_2, \lambda_3\}$ and $DX_t(\sigma) = e^{tD} = \text{diag}\{e^{\lambda_1 t}, e^{\lambda_2 t}, e^{\lambda_3 t}\}$. Then $\lambda_2 < \lambda_3 < 0$ ensures that condition (1) in Proposition 1.3 and Proposition 2.2 we have strict $\mathcal{J}$-separation and sectional-expansion along the $yz$-direction for $X_t$ (and so singular-hyperbolicity), since $\delta > 0$ ensures that condition (1) in Theorem B is true.
Now we show how our results simplify the checking of singular-hyperbolicity in the standard example of a geometric Lorenz flow.

**Example 4.** We now consider the geometric Lorenz example as constructed in [4, Chapter 3, Section 3]; see Figure 1. In the linear region around the origin we have $X(\xi) = D \cdot \xi$

and so $DX(\xi) = D$ for all points $\xi$ around the origin, where $D$ was defined in Example 3. Hence we can calculate, as in Example 3 with $\lambda_1 + \lambda_3 > 0$, and show that we have strict $\mathcal{J}$ separation with $2\lambda_2 < \delta < 2\lambda_3 < 0$, that is, $\delta$ is negative and bounded away from zero.

In the same region we also get strict $(-\mathcal{J})$-separation for the exterior square of the cocycle $DX_t$ with $\delta_2 = 2 \text{tr}(D) - \delta$ between $2(\lambda_1 + \lambda_2)$ and $2(\lambda_1 + \lambda_3)$; thus $\delta_2$ can be taken positive and bounded away from zero by setting $\delta \gtrless 2\lambda_2$.

We are left to prove these properties in the lobes, where the flow is a combination of a rotation on the $xz$-plane, a dilation and a translation on the $y$-direction; and then check the singular-hyperbolic conditions on the transitional region between the lobes and the linear flow.

We can write the vector field in the interior of the lobes as $X'_i = A_i \cdot (X_i - C_i) + P_i$, where $C_i$ is the center of the rotation, $P_i$ is a vector representing a translation and

$$A_i = \begin{pmatrix} \rho \lambda_1 & 0 & -(1)^i \\ 0 & \zeta \lambda_2 & 0 \\ -(1)^i & 0 & \rho \lambda_1 \end{pmatrix} \text{ with } 0 < \rho, \zeta \ll 1, i = 1, 2.$$  

**Figure 1.** The geometric Lorenz flow.

Here $i = 1$ corresponds to the lobe starting with $x > 1$ and $i = 2$ to the other lobe. We observe that by an affine change of coordinates we can write the vector field as $Y' = A_i \cdot Y$.

Using the same quadratic form $\mathcal{J}$ we get $\mathcal{J}_i = J \cdot A_i + A_i^* \cdot J = \text{diag}\{2\rho \lambda_1, -2\zeta \lambda_2, 2\rho \lambda_1\}$ and so $\mathcal{J}_i$ is positive definite, $i = 1, 2$.

For the exterior square of derivative cocycle of the flow, we observe that $\delta_2$ becomes $2(2\rho \lambda_1 - \zeta \lambda_2) - \delta$ and $2\rho \lambda_1 - \zeta \lambda_2 > 0$. 


Hence, there exists $\delta_* > 0$ such that for all $\delta < \delta_*$ we have $J'_i - \delta J > 0$ and $\delta > 0$ on the lobes; so singular-hyperbolicity is also verified on the lobes.

We still have to check the transition between the linear region and the lobes.

We can find a smooth path $A$ from $D$ in the linear region to $A_i$ in the lobes made of symmetric matrices, ensuring that $J$ will remain diagonal: it is enough to take the line segment between $D$ and $A_i$ in $\mathbb{R}^{3 \times 3}$ and in this way the signs of the first and second elements of the diagonal of the corresponding matrix $J$ do not change. However, the last element of the diagonal goes from $2\lambda_3 < 0$ to $2\rho_1 > 0$.

Therefore, since the vector field $Z_i$ in the transitional region is defined as a linear combination $\mu X + (1 - \mu)X_i$ of the fields in the linear region and the lobes, $i = 1, 2$ and $0 \leq \mu \leq 1$, we can write $\tilde{J}$ in the transitional region as

$$\text{diag}\{2\lambda_1(\mu + \rho(1 - \mu)), -2\lambda_2(\mu + \zeta(1 - \mu)), 2(\lambda_3\mu + \rho\lambda_1(1 - \mu))\}.$$  

Hence $\tilde{J} - \delta J > 0$ subject to the condition $2\lambda_2 < \delta < 2\lambda_3$, as in Example 3. This provides partial hyperbolicity with the negative $J$-cone uniformly contracting on the geometric Lorenz attractor.

Finally, $\delta_2 = 2 \text{tr}(DZ_i) - \delta = 2 \text{tr}(D) \cdot \mu + 2 \text{tr}(A_i) \cdot (1 - \mu) - \delta$ and so, if we take $2\lambda_2 \leq \delta < 2\lambda_3 < 0$, then we again obtain $\delta_2 > 0$, because the condition on $\delta$ is compatible with all the previous conditions on the linear region and on the lobes.

In this way we have strictly negative $\delta$ and strictly positive $\delta_2$ on all points in a neighborhood of the geometric Lorenz attractor with respect to $J$ and, from Theorem B, this alone ensures that the geometric Lorenz attractor is a singular-hyperbolic set.

2. Some definitions and useful results

2.1. Fields of quadratic forms, positive and negative cones. Let $E_U$ be a finite dimensional vector bundle with inner product $\langle \cdot, \cdot \rangle$ and base given by the trapping region $U \subset M$. Let $\mathcal{J} : E_U \to \mathbb{R}$ be a continuous family of quadratic forms $\mathcal{J}_x : E_x \to \mathbb{R}$ which are non-degenerate and have index $0 < q < \text{dim}(E) = n$. The index $q$ of $\mathcal{J}$ means that the maximal dimension of subspaces of non-positive vectors is $q$. Using the inner product, we can represent $\mathcal{J}$ by a family of self-adjoint operators $J_x : E_x \subset E_x$ as $\mathcal{J}_x(v) = \langle J_x(v), v \rangle, v \in E_x, x \in U$.

We also assume that $(\mathcal{J}_x)_{x \in U}$ is continuously differentiable along the flow. The continuity assumption on $\mathcal{J}$ means that for every continuous section $Z$ of $E_U$ the map $U \ni x \mapsto \mathcal{J}(Z(x)) \in \mathbb{R}$ is continuous. The $C^1$ assumption on $\mathcal{J}$ along the flow means that the map $\mathbb{R} \ni t \mapsto \mathcal{J}_{X_t(x)}(Z(X_t(x))) \in \mathbb{R}$ is continuously differentiable for all $x \in U$ and each $C^1$ section $Z$ of $E_U$.

Using Lagrange diagonalization of a quadratic form, it is easy to see that the choice of basis to diagonalize $\mathcal{J}_x$ depends smoothly on $y$ if the family $(\mathcal{J}_x)_{x \in U}$ is smooth, for all $y$ close enough to a given $x$. Therefore, choosing a basis for $T_x$ adapted to $\mathcal{J}_x$ at each $x \in U$, we can assume that locally our forms are given by $\langle J_x(v), v \rangle$ with $J_x$ a diagonal matrix whose entries belong to $\{\pm 1\}$, $J^*_x = J_x, J^2_x = I$ and the basis vectors depend as smooth on $x$ as the family of forms $(\mathcal{J}_x)_x$. 

We let $\mathcal{C}_\pm = \{ C_\pm(x) \}_{x \in U}$ be the family of positive and negative cones associated to $\mathcal{J}$

$$C_\pm(x) := \{ v \in E_x : \pm \mathcal{J}_x(v) > 0 \} \quad x \in U$$

and also let $\mathcal{C}_0 = \{ C_0(x) \}_{x \in U}$ be the corresponding family of zero vectors $C_0(x) = \mathcal{J}_x^{-1}(\{0\})$ for all $x \in U$.

**2.2. Linear multiplicative cocycles over flows.** Let $A : E \times \mathbb{R} \to E$ be a smooth map given by a collection of linear bijections

$$A_t(x) : E_x \to E_{X_t(x)}, \quad x \in M, t \in \mathbb{R},$$

where $M$ is the base space of the finite dimensional vector bundle $E$, satisfying the cocycle property

$$A_0(x) = Id, \quad A_{t+s}(x) = A_t(X_s(x)) \circ A_s(x), \quad x \in M, t, s \in \mathbb{R},$$

with $\{X_t\}_{t \in \mathbb{R}}$ a smooth flow over $M$. We note that for each fixed $t > 0$ the map $A_t : E \to E, v_x \in E_x \mapsto A_t(x) \cdot v_x \in E_{X_t(x)}$ is an automorphism of the vector bundle $E$.

The natural example of a linear multiplicative cocycle over a smooth flow $X_t$ on a manifold is the derivative cocycle $A_t(x) = DX_t(x)$ on the tangent bundle $TM$ of a finite dimensional compact manifold $M$.

The following definitions are fundamental to state our main result.

**Definition 4.** Given a continuous field of non-degenerate quadratic forms $\mathcal{J}$ with constant index on the positively invariant open subset $U$ for the flow $X_t$, we say that the cocycle $A_t(x)$ over $X_t$ is

- **$\mathcal{J}$-separated** if $A_t(x)(C_+(x)) \subset C_+(X_t(x))$, for all $t > 0$ and $x \in U$ (simple cone invariance);
- **strictly $\mathcal{J}$-separated** if $A_t(x)(C_+(x) \cup C_0(x)) \subset C_+(X_t(x))$, for all $t > 0$ and $x \in U$ (strict cone invariance).

We say that the flow $X_t$ is (strictly) $\mathcal{J}$-separated on $U$ if $DX_t(x)$ is (strictly) $\mathcal{J}$-separated on $T_x M$.

**Remark 2.1.** If a flow is strictly $\mathcal{J}$-separated, then for $v \in T_x M$ such that $\mathcal{J}_x(v) \leq 0$ we have $\mathcal{J}_{X^{-t}(x)}(DX_{-t}(v)) < 0$ for all $t > 0$ and $x$ such that $X_{-s}(x) \in U$ for every $s \in [-t, 0]$. Indeed, otherwise $\mathcal{J}_{X^{-t}(x)}(DX^{-t}(v)) \geq 0$ would imply $\mathcal{J}_x(v) = \mathcal{J}_x(DX_t(DX^{-t}(v))) > 0$, contradicting the assumption that $v$ was a non-positive vector.

This means that a flow $X_t$ is strictly $\mathcal{J}$-separated if, and only if, its time reversal $X_{-t}$ is strictly ($-\mathcal{J}$)-separated.

A vector field $X$ is **$\mathcal{J}$-non-negative** on $U$ if $\mathcal{J}(X(x)) \geq 0$ for all $x \in U$, and **$\mathcal{J}$-non-positive** on $U$ if $\mathcal{J}(X(x)) \leq 0$ for all $x \in U$. When the quadratic form used in the context is clear, we will simply say that $X$ is non-negative or non-positive.
2.3. Properties of $\mathcal{J}$-separated linear multiplicative cocycles. We present some useful properties about $\mathcal{J}$-separated linear cocycles whose proofs can be found in [5].

Let $A_t(x)$ be a linear multiplicative cocycle over $X_t$. We define the infinitesimal generator of $A_t(x)$ by

$$D(x) := \lim_{t \to 0} \frac{A_t(x) - Id}{t}. \quad (2.1)$$

The following is the basis of our arguments leading to Theorem 1.2.

**Proposition 2.2.** [5, Proposition 2.7] Let $A_t(x)$ be a cocycle over $X_t$ defined on an open subset $U$ and $D(x)$ its infinitesimal generator. Then

1. $\mathcal{J}'(v) = \partial_t A_t(x)v = \langle \mathcal{J}_x(x)A_t(x)v, A_t(x)v \rangle$ for all $v \in E_x$ and $x \in U$, where
   $$\mathcal{J}_x := J \cdot D(x) + D(x)^* \cdot J \quad (2.2)$$
   and $D(x)^*$ denotes the adjoint of the linear map $D(x) : E_x \to E_x$ with respect to the adapted inner product at $x$;
2. the cocycle $A_t(x)$ is $\mathcal{J}$-separated if, and only if, there exists a neighborhood $V$ of $\Lambda$, $V \subset U$ and a function $\delta : V \to \mathbb{R}$ such that
   $$\mathcal{J}'(x) \geq \delta(x)\mathcal{J}_x \quad \text{for all } x \in V. \quad (2.3)$$
   In particular we get $\partial_t \log |\mathcal{J}(A_t(x)v)| \geq \delta(X_t(x)), v \in E_x, x \in V, t \geq 0$;
3. if the inequalities in the previous item are strict, then the cocycle $A_t(x)$ is strictly $\mathcal{J}$-separated. Reciprocally, if $A_t(x)$ is strictly $\mathcal{J}$-separated, then there exists a compatible family $\mathcal{J}_0$ of forms on $V$ satisfying the strict inequalities of item (2).
4. For a $\mathcal{J}$-separated cocycle $A_t(x)$, we have $\frac{\partial_t |A_t(x)v|}{|A_t(x)v|} \geq \exp \Delta^t_0(x)$ for all $v \in E_x$ and reals $t_1 < t_2$ so that $\mathcal{J}(A_t(x)v) \neq 0$ for all $t_1 \leq t \leq t_2$, where $\Delta^t_0(x)$ was defined in (1.7).
5. if $A_t(x)$ is $\mathcal{J}$-separated and $x \in \Gamma(U), v \in C_+(x)$ and $w \in C_-(x)$ are non-zero vectors, then for every $t > 0$ such that $A_t(x)w \in C_-(X_s(x))$ for all $0 < s < t$
   $$\frac{|\mathcal{J}(A_t(x)w)|}{\mathcal{J}(A_t(x)v)} \leq \frac{|\mathcal{J}(w)|}{\mathcal{J}(v)} \exp \left(2\Delta^t_0(x)\right). \quad (2.4)$$
6. we can bound $\delta$ at every $x \in \Gamma$ by $\inf_{v \in C_+(x)} \frac{\mathcal{J}'(v)}{\mathcal{J}(v)} \leq \delta(x) \leq \sup_{v \in C_-(x)} \frac{\mathcal{J}'(v)}{\mathcal{J}(v)}$.

**Remark 2.3.** We stress that the necessary and sufficient condition in items (2-3) of Proposition 2.2, for (strict) $\mathcal{J}$-separation, shows that a cocycle $A_t(x)$ is (strictly) $\mathcal{J}$-separated if, and only if, its inverse $A_{-t}(x)$ is (strictly) $-\mathcal{J}$-separated.

**Remark 2.4.** Item (2) above of Proposition 2.2 shows that $\delta$ is a measure of the “minimal instantaneous expansion rate” of $|\mathcal{J} \circ A_t(x)|$ on positive vectors; and item (5) shows in addition that $\delta$ is also a bound for the “instantaneous variation of the ratio” between $|\mathcal{J} \circ A_t(x)|$ on negative and positive vectors.
3. The exterior square of the cocycle

We consider the action of the cocycle $DX_t(x)$ on $k$-vector first and bivectors later, that is, the exterior square $\wedge^k DX_t$ of the cocycle acting on $\wedge^k T_t M$ with $k > 2$ and then $k = 2$, to deduce Theorem A and Corollary 1.4 first and then prove Theorem B and Theorem C.

3.1. Dominated splitting and the exterior cocycle. We denote by $\| \cdot \|$ the standard norm on bivectors induced by the Riemannian norm of $M$; see, e.g. [8]. We write $m = \dim M$.

Proof of Theorem A. We assume that $T_t M$ admits a dominated splitting $E_t \oplus F_t$ with $\dim E_t = s$ and $\dim F_t = c$. So there exists $\eta > 0$ such that, for any $X_t$-invariant probability measure $\mu$ supported on $\Gamma$, the Lyapunov exponents of $DX_t$ with respect to $\mu$ are (repeated according to multiplicity) $\lambda_1 \leq \cdots \leq \lambda_s \leq \lambda_{s+1} \leq \cdots \leq \lambda_m$ and satisfy $\lambda_{s+1} - \lambda_s > \eta$.

The Lyapunov exponents of $\wedge^k DX_t$ are given by $\{\lambda_{i_1} + \cdots + \lambda_{i_k}\}_{1 \leq i_1 < \cdots < i_k \leq s + c}$; see e.g. [8, Chapter 3]. Hence, in our setting, we have that

$$(\lambda_{i_1} + \cdots + \lambda_{i_s}) + (\lambda_{h_1} + \cdots + \lambda_{h_m}) + k\eta < \lambda_{j_1} + \cdots + \lambda_{j_k}$$

for all $s + 1 \leq j_1 < \cdots < j_k \leq c$ and $1 \leq i_1 < \cdots < i_n \leq s$, $s + 1 \leq h_1 < \cdots < h_m \leq c$ with $m + n = k, m, n \geq 1$. This implies that for $\mu$-almost every $x \in \Gamma$

$$\lim_{t \to +\infty} \frac{1}{t} \log (\| \wedge^k DX_t \mid \tilde{E}_x \| \cdot \|(\wedge^2 DX_t \mid \tilde{F}_x)^{-1}\|)$$

$$= \lambda_{\max\{s-k,1\}} + \cdots + \lambda_{\max\{s-k,1\}+k} - (\lambda_{s+1} + \cdots + \lambda_{s+k}) \leq -\eta, \quad (3.1)$$

that is, the maximum rate of expansion along $\tilde{E}$ minus the minimum rate of expansion along $\tilde{F}$.

We now set $f_t(x) = \log (\| \wedge^k DX_t \mid \tilde{E}_x \| \cdot \|(\wedge^k DX_t \mid \tilde{F}_x)^{-1}\|)$ and, since we obtain (3.1) for an arbitrary $X_t$-invariant probability measure, we can apply the following result which is an improvement from [6, Proposition 3.4].

Lemma 3.1. [7, Corollary 4.2] Let $\{t \mapsto f_t : S \to \mathcal{R}\}_{t \in \mathcal{R}}$ be a continuous family of continuous functions which is subadditive and suppose that $\int f(x) d\mu < 0$ for every $\mu \in \mathcal{M}_X$, with $\tilde{f}(x) := \lim_{t \to +\infty} \frac{1}{t} f_t(x)$. Then there exist a $T > 0$ and a constant $\lambda < 0$ such that for every $x \in S$ and every $t \geq T$:

$$f_t(x) \leq \lambda t.$$

We thus have $f_t(x) \leq \kappa - \eta t, t \geq 0, x \in \Lambda$ for a constant $\kappa > 0$, as required for a dominated splitting with respect to $\wedge^k DX_t$. This proves sufficiency in the first part of Theorem A.

For necessity, we just have to observe that domination of $\tilde{E} \oplus \tilde{F}$ by the action of $\wedge^k DX_t$ ensures (3.1) holds for the Lyapunov spectrum of any given $X_t$-invariant probability measure $\mu$. Hence, in particular, we obtain

$$\lambda_s - \lambda_{s+1} = \lambda_s + \lambda_{s+2} + \cdots + \lambda_{s+k+1} - (\lambda_{s+1} + \lambda_{s+2} + \cdots + \lambda s + k + 1) < -\eta. \quad (3.2)$$
We now set $f_t(x) = \log (\|DX_t|_{E_x} \cdot \|(DX_t|_{F_x})^{-1}\|)$ and, since we obtain (3.2) for an arbitrary $X_t$-invariant probability measure, we can apply again Lemma 3.1 and deduce $f_t(x) \leq \kappa - \eta t, t \geq 0, x \in \Lambda$ for a constant $\kappa > 0$, proving domination with respect to $DX_t$. This completes the proof of Theorem A.

\textbf{Proof of Corollary 1.4.} For the Corollary 1.4, we assume that $T_\Gamma M$ admits a sectional hyperbolic splitting $E_\Gamma \oplus F_\Gamma$ with dim $E_\Gamma = s$ and dim $F_\Gamma = c$. Then if $x \in \Gamma$ and $B = \{e_1, \ldots, e_c\}$ is a basis for $F_x$, it is obvious that we can find $\lambda > 0$ such that $\|DX_t u \wedge DX_t v\| \geq Ce^\lambda t$ for $t > 0$ by definition of sectional hyperbolicity. Hence $\tilde{F} = \wedge^2 F$ is uniformly expanded by $\wedge^2 DX_t$.

The reciprocal statement is straightforward. Indeed, let us assume that $T_\Gamma M$ admits a $DX_t$-invariant partial hyperbolic splitting $E \oplus F$ with $E$ uniformly contracted, and $\wedge^2 T_\Gamma M$ admits a $\wedge^2 DX_t$-invariant and partial hyperbolic splitting $\tilde{E} \oplus \tilde{F}$ with $\tilde{F} = \wedge^2 F$ and $\tilde{F}$ uniformly expanded. Then clearly, given a basis $\{u, v\}$ of a two-dimensional subspace $G$ of $F$, we have that $\|\wedge^2 DX_t (u \wedge v)\|$ grows exponentially, and this means that the area along $G$ is uniformly expanded. Hence $E \oplus F$ is a sectional hyperbolic splitting.

This concludes the proof. \hfill \Box

3.2. The three-dimensional case. Here we prove Theorem B.

Now $M$ is a 3-manifold and $\Gamma$ is a compact $X_t$-invariant subset having a singular-hyperbolic splitting $T_\Gamma M = E_\Gamma \oplus F_\Gamma$. By Theorem A we have a $\wedge^2 DX_t$-invariant partial hyperbolic splitting $\wedge^2 T_\Gamma M = \tilde{E} \oplus \tilde{F}$ with dim $\tilde{F} = 1$ and $\tilde{F}$ uniformly expanded. Following the proof of Theorem A, if we write $e$ for a unit vector in $E_x$ and $\{u, v\}$ an orthonormal base for $F_x$, $x \in \Gamma$, then $\tilde{E}_x$ is a two-dimensional vector space spanned by $e \wedge u$ together with $e \wedge v$.

From Theorem 1.2 and the existence of adapted metrics (see e.g. [12]), there exists a field $\mathcal{J}$ of quadratic forms so that $X$ is $\mathcal{J}$-non-negative, $DX_t$ is strictly $\mathcal{J}$-separated on a neighborhood $U$ of $\Gamma$, $E_\Gamma$ is a negative subbundle, $F_\Gamma$ is a positive subbundle and these subspaces are almost orthogonal. In other words, there exists a function $\delta : \Gamma \to \mathbb{R}$ such that $\mathcal{J}_x = \delta(x) \mathcal{J}_x > 0, x \in \Gamma$ and we can locally write $\mathcal{J}(v) = \langle J(v), v \rangle$ where $J = \text{diag}\{-1, 1, 1\}$ with respect to the basis $\{e, u, v\}$ and $\langle \cdot, \cdot \rangle$ is the adapted inner product; see [5].

It is well-known that $A \wedge A = \det(A) \cdot (A^{-1})^*$ with respect to the adapted inner product which trivializes $\mathcal{J}$, for any linear transformation $A : T_x M \to T_y M$. Hence $\wedge^2 DX_t(x) = \det(DX_t(x)) \cdot (DX_{-t} \circ X_t)^*$ and a straightforward calculation shows that the infinitesimal generator $D^2(x)$ of $\wedge^2 DX_t$ equals $\text{tr}(DX(x)) \cdot Id - DX(x)^*$.

Therefore, using the identification between $\wedge^2 T_x M$ and $T_x M$ through the adapted inner product and Proposition 2.2

$$\tilde{\mathcal{J}}_x = \partial_t (-\mathcal{J})(\wedge^2 DX_t \cdot v)|_{t=0} = \langle -(J \cdot D^2(x) + D^2(x)^* \cdot J)v, v \rangle$$

$$= \langle [(\mathcal{J} \cdot DX(x) + DX(x)^* \cdot \mathcal{J}) - 2 \text{tr}(DX(x))\mathcal{J}]v, v \rangle$$

$$= (\mathcal{J}' - 2 \text{tr}(DX(x))\mathcal{J})(v).$$

(3.3)
To obtain strict (-\(\mathcal{J}\))-separation of \(\wedge^2 DX_t\) we search a function \(\delta_2 : \Gamma \to \mathbb{R}\) so that
\[
(\mathcal{J}' - 2 \text{tr}(DX)\mathcal{J}) - \delta_2(-\mathcal{J}) > 0 \quad \text{or} \quad \mathcal{J}' - (2 \text{tr}(DX) - \delta_2)\mathcal{J} > 0.
\]
Hence it is enough to make \(\delta_2 = 2 \text{tr}(DX) - \delta\). This shows that in our setting \(\wedge^2 DX_t\) is always strictly (-\(\mathcal{J}\))-separated.

Finally, according to Theorem 1.6, to obtain the partial hyperbolic splitting of \(\wedge^2 DX_t\) which ensures singular-hyperbolicity, it is necessary and sufficient that either \(\Delta^b_a(x) = \int_a^b \delta_2(X_s(x)) \, ds\) satisfies item (1) of Theorem 1.6 or \(\mathcal{J}_x\) is positive definite, for all \(x \in \Gamma\). This amounts precisely to the necessary and sufficient condition in the statement of Theorem B and we are done.

3.3. Existence of adapted inner product for singular-hyperbolicity. Now we show how to prove Theorem C adapting the previous arguments.

From Theorem A we know that, for a singular-hyperbolic attracting set \(\Gamma\) for a three-dimensional vector field with a splitting \(E \oplus F\), we have a partially hyperbolic splitting \(F \oplus E\) for the action of \(\wedge^2 DX_t\), where \(E\) is uniformly expanded by \(\wedge^2 DX_t\). Hence, from [12, Theorem 1], there exists an adapted inner product \(\langle \cdot, \cdot \rangle\) for this cocycle. Let \(\| \cdot \|\) be the associated norm on \(T_\Gamma M\). Then there exists \(\lambda > 0\) such that \(\|(\wedge^2 DX_t) \mid E\| \geq e^{\lambda t}\) for all \(t > 0\).

We know that \(E, F\) are almost orthogonal with respect to this inner product and we can choose a continuous family of vectors \(\{e_x\}\), a unit basis of \(E_x\), and \(\{u_x, v_x\}\) an orthonormal basis of \(F_x, x \in \Gamma\). We define the linear operator \(J : T_x \mathcal{M} \ominus \) in the basis \(\{e_x, u_x, v_x\}\) such that its matrix is diag\{-1, 1, 1\}. Now the associated quadratic form \(\tilde{\mathcal{J}}_x(w) = \langle J(w), w \rangle\) is such that \(\wedge^2 DX_t\) is strictly (-\(\mathcal{J}\))-separated by construction; see [5, Section 2.5].

This means that there exists a continuous function \(\tilde{\delta} : \Gamma \to \mathbb{R}\) for which \(\tilde{\mathcal{J}} - \tilde{\mathcal{J}}(-\mathcal{J}) > 0\), where \(\tilde{\mathcal{J}}\) is given in (3.3). That is, we have \(\mathcal{J}' + (\tilde{\delta}(x) - 2 \text{tr}(DX(x)))\mathcal{J} > 0\). Hence, if we set \(\delta(x) = \tilde{\delta}(x) - 2 \text{tr}(DX(x))\), then we obtain strict \(\mathcal{J}\)-separation for \(DX_t\) over \(\Gamma\), as guaranteed by Proposition 2.2.

This ensures, in particular, that the norm \(|w| = \xi \sqrt{\mathcal{J}(w_E)^2 + \mathcal{J}(w_F)^2}\), \(w = w_E + w_F \in E_x \oplus F_x, x \in \Gamma\) is adapted to the dominated splitting \(E \oplus F\) for the cocycle \(DX_t\), where \(\xi\) is an arbitrary positive constant; see [5, Section 4.1]. This means that there exists \(\mu > 0\) such that \(|DX_t \mid E_x| \cdot |DX_{-t} \mid F_{X_t(x)}| \leq e^{-\mu t}\) for all \(t > 0\).

Moreover, from the definition of the inner product and the relation between \(\wedge\) and the cross-product \(\times\), it follows that \(|\det(DX_t \mid F_x)| = \|(\wedge^2 DX_t) \cdot (u \wedge v)\| = \|(\wedge^2 DX_t) \mid E\| \geq e^{\lambda t}\) for all \(t > 0\), so \(\cdot\) is adapted to the area expansion along \(F\).

To conclude, we are left to show that \(E\) admits a constant \(\omega > 0\) such that \(|DX_t \mid E| \leq e^{-\omega t}\) for all \(t > 0\). But since \(E\) is uniformly contracted, we know that \(X(x) \in F_x\) for all \(x \in \Gamma\).

**Lemma 3.2.** Let \(\Gamma\) be a compact invariant set for a flow \(X\) of a \(C^1\) vector field \(X\) on \(M\). Given a continuous splitting \(T_\Gamma M = E \oplus F\) such that \(E\) is uniformly contracted, then \(X(x) \in F_x\) for all \(x \in \Lambda\).

**Proof.** See [5, Lemma 3.3].
On the one hand, on each non-singular point $x$ of $\Gamma$ we obtain for each $w \in E_x$

$$e^{-\mu t} \geq \frac{|DX_t \cdot w|}{|DX_t \cdot X(x)|} \geq \frac{|DX_t \cdot v|}{\sup\{|X(z)| : z \in \Gamma\}} \geq |DX_t \cdot v|,$$

since we can always choose a small enough constant $\xi > 0$ in such a way that $\sup\{|X(z)| : z \in \Gamma\} \leq 1$. We note that the choice of the positive constant $\xi$ does not change any of the previous relations involving $| \cdot |$.

On the other hand, for $\sigma \in \Gamma$ such that $X(\sigma) = \vec{0}$, we fix $t > 0$ and, since $\Gamma$ is a non-trivial invariant set, we can find a sequence $x_n \to \sigma$ of regular points of $\Gamma$. The continuity of the derivative cocycle ensures $|DX_t |_{E_\sigma}| = \lim_{n \to \infty} |DX_t |_{E_{x_n}}| \leq e^{-\lambda t}$. Since $t > 0$ was arbitrarily chosen, we see that $| \cdot |$ is adapted for the contraction along $E_\sigma$.

This shows that $\omega = \mu$ and completes the proof of Theorem C.

4. Conjectures

We finish this work with some conjectures about the adapted metric as in Theorem C.

**Conjecture 1.** There exists an adapted metric for all sectional-hyperbolic sets for any $C^1$ vector field in any dimension.

Moreover, we should extend this to more general notions of sectional-expansion of area.

**Conjecture 2.** There exists an adapted metric for all compact invariant subsets of a $C^1$ vector field $X$ on a manifold $M$, which are partially hyperbolic with splitting $E \oplus F$, $E$ uniformly contracted and all $k$-subspaces of $F$ are volume-expanding.

In terms of exterior powers, the last condition on Conjecture 2 means that there are $C, \lambda > 0$ such that

$$\|(\wedge^k DX_t) \cdot (v_1 \wedge \cdots \wedge v_k)\| \geq C e^{\lambda t},$$

for all $t > 0$ and every $k$-frame $v_1, \ldots, v_k$ inside $F$, with $2 \leq k \leq \dim(F)$. In other words, we can find an adapted Riemannian metric for $M$ whose naturally induced norm in the $k$-exterior product of $TM$ satisfies the above inequality for some $\lambda > 0$ and $C = 1$.

This should also be true for discrete dynamical systems.

**Conjecture 3.** There exists an adapted metric for all compact invariant subsets of a $C^1$ diffeomorphism admitting a partially hyperbolic splitting $E \oplus F$, where $E$ is uniformly contracted and all $k$-subspaces of $F$ are volume-expanding.

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(V.A. & L.S.) Universidade Federal da Bahia, Instituto de Matemática, Av. Adhemar de Barros, S/N, Ondina, 40170-110 - Salvador-BA-Brazil

*E-mail address*: vitor.d.araujo@ufba.br & lsalgado@ufba.br