RADIUS OF CONVEXITY OF PARTIAL SUMS OF ODD FUNCTIONS IN THE CLOSE-TO-CONVEX FAMILY

SARITA AGRAWAL AND SWADESH KUMAR SAHOO*

ABSTRACT. We consider the class of all analytic and locally univalent functions $f$ of the form $f(z) = z + \sum_{n=2}^{\infty} a_{2n-1} z^{2n-1}$, $|z| < 1$, satisfying the condition

$$\text{Re} \left( 1 + \frac{z f''(z)}{f'(z)} \right) > -\frac{1}{2}$$

We show that every section $s_{2n-1}(z) = z + \sum_{k=2}^{n} a_{2k-1} z^{2k-1}$, of $f$, is convex in the disk $|z| < \sqrt{2}/3$. We also prove that the radius $\sqrt{2}/3$ is best possible, i.e. the number $\sqrt{2}/3$ cannot be replaced by a larger one.

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1. Introduction and Main Result

Let $A$ denote the class of all normalized analytic functions $f$ in the open unit disk $D := \{ z \in \mathbb{C} : |z| < 1 \}$, i.e. $f$ has the Taylor series expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n.$$  \hfill (1)

The Taylor polynomial $s_n(z) = s_n(f)(z)$ of $f$ in $A$, defined by,

$$s_n(z) = z + \sum_{k=2}^{n} a_k z^k$$

is called the $n$-th section/partial sum of $f$. Denote by $S$, the class of univalent functions in $A$. A function $f \in A$ is said to be locally univalent at a point $z_0 \in D \subset \mathbb{C}$ if it is univalent in some neighborhood of $z_0$; equivalently $f'(z_0) \neq 0$. A function $f \in A$ is called convex if $f(D)$ is a convex domain. The set of all convex functions are denoted by $C$. The functions $f \in C$ are characterized by the well-known fact

$$\text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) > 0, \quad |z| < 1.$$  \hfill (1)

In this article, we mainly focus on a class, denoted by $L$, of all locally univalent odd functions $f$ satisfying

$$\text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) > -\frac{1}{2}, \quad z \in D.$$  \hfill (2)
Clearly, a function $f \in \mathcal{L}$ will have the Taylor series expansion $f(z) = z + \sum_{n=2}^{\infty} a_{2n-1} z^{2n-1}$. The function $f_0(z) = z / \sqrt{1 - z^2}$ plays the role of an extremal function for $\mathcal{L}$; see for instance [16, p. 68, Theorem 2.6i]. This article is devoted to finding the largest disk $|z| < r$ in which every section $s_{2n-1}(z) = z + \sum_{k=2}^{n} a_{2k-1} z^{2k-1}$, of $f \in \mathcal{L}$, is convex; that is, $s_{2n-1}$ satisfies
\[
\text{Re} \left( 1 + \frac{z s''_{2n-1}(z)}{s'_{2n-1}(z)} \right) > 0.
\]

Our main objective in this article is to prove

**Main Theorem.** Every section of a function in $\mathcal{L}$ is convex in the disk $|z| < \sqrt{2}/3$. The radius $\sqrt{2}/3$ cannot be replaced by a greater one.

This observation is also explained geometrically in Figure 1 by considering the third partial sum, $s_{3,0}$, of the extremal function $f_0$. We next discuss some motivational background of our problem.

![Figure 1](image_url)

**Figure 1.** The first figure shows convexity of the image domain $s_{3,0}(z)$ for $|z| < \sqrt{2}/3$ and the second figure shows non-convexity of the image domain $s_{3,0}(z)$ for $|z| < 2/3 =: r_0$ ($r_0 > \sqrt{2}/3$).

Considering odd univalent functions and studying classical problems of univalent function theory such as (successive) coefficient bounds, inverse functions, etc. are quite interesting and found throughout the literature; see for instance [8, 12, 15, 35]. In fact, an application of the Cauchy-Schwarz inequality shows that the conjecture of Robertson: $1 + |c_3|^2 + |c_5|^2 + \cdots + |c_{2n-1}|^2 \leq n$, $n \geq 2$, for each odd function $f(z) = z + c_3 z^3 + c_5 z^5 + \cdots$ of $\mathcal{S}$, stated in 1936 implies the well-known Bieberbach conjecture [25]; see also [3]. In our knowledge, studying radius properties for sections of odd univalent functions are new (as we do not find in the literature).

Note that a subclass denoted by $\mathcal{F}$, of the class, $\mathcal{K}$, of close-to-convex functions, consisting of all locally univalent functions $f \in \mathcal{A}$ satisfying the condition (2) was considered in [22]. In this paper, we consider functions from $\mathcal{F}$ that have odd Taylor coefficients. Note that the following inclusion relations hold:

\[ \mathcal{L} \subset \mathcal{F} \subset \mathcal{K} \subset \mathcal{S}. \]
The fact that functions in $\mathcal{F}$ are close-to-convex may be obtained as a consequence of the result due to Kaplan (see [4, p. 48, Theorem 2.18]). In [22], Ponnusamy et al. have shown that every section of a function in the class $\mathcal{F}$ is convex in the disk $|z| < 1/6$ and the radius $1/6$ is the best possible. They conjectured that every section of functions in the family $\mathcal{F}$ is univalent and close-to-convex in the disk $|z| < 1/3$. This conjecture has been recently settled by Bharanedhar and Ponnusamy in [1, Theorem 1].

The problem of finding the radius of univalence of sections of $f$ in $\mathcal{S}$ was first initiated by Szegö in 1928. According to the Szegö theorem [4, Section 8.2, p. 243-246], every section $s_n(z)$ of a function $f \in \mathcal{S}$ is univalent in the disk $|z| < 1/4$; see [34] for the original paper. The radius $1/4$ is best possible and can be verified from the second partial sum of the Koebe function $k(z) = z/(1-z)^2$. Determining the exact (largest) radius of univalence $r_n$ of $s_n(z)$ ($f \in \mathcal{S}$) remains an open problem. However, many other related problems on sections have been solved for various geometric subclasses of $\mathcal{S}$, eg. the classes $\mathcal{S}^*$, $\mathcal{C}$ and $\mathcal{K}$ of starlike, convex and close-to-convex functions, respectively (see Duren [4, §8.2, p.241-246], [5, 20, 26, 27, 32] and the survey articles [6, 24]). In [13], MacGregor considered the class

$$\mathcal{R} = \{ f \in \mathcal{A} : \text{Re } f'(z) > 0, z \in \mathbb{D} \}$$

and proved that the partial sums $s_n(z)$ of $f \in \mathcal{R}$ are univalent in $|z| < 1/2$, where the radius $1/2$ is best possible. On the other hand, in [30], Ram Singh obtained the best radius, $r = 1/4$, of convexity for sections of functions in the class $\mathcal{R}$. The reader can refer to [21] for related information. Radius of close-to-convexity of sections of close-to-convex functions is obtained in [14].

By the argument principle, it is clear that the $n$-th section $s_n(z)$ of an arbitrary function in $\mathcal{S}$ is univalent in each fixed compact subdisk $\overline{\mathbb{D}}_r := \{ z \in \mathbb{D} : |z| \leq r \} (r < 1)$ of $\mathbb{D}$ provided that $n$ is sufficiently large. In this way one can get univalent polynomials in $\mathcal{S}$ by setting $p_n(z) = \frac{i}{r^2} s_n(r z)$. Consequently, the set of all univalent polynomials is dense in the topology of locally uniformly convergence in $\mathcal{S}$. The radius of starlikeness of the partial sums $s_n(z)$ of $f \in \mathcal{S}^*$ was obtained by Robertson in [20]; (see also [31, Theorem 2]) in the following form:

**Theorem A.** [20] If $f \in \mathcal{S}$ is either starlike, convex, typically-real, or convex in the direction of imaginary axis, then there is an $N$ such that, for $n \geq N$, the partial sum $s_n(z)$ has the same property in $\mathbb{D}_r := \{ z \in \mathbb{D} : |z| < r \}$, where $r \geq 1 - 3(\log n)/n$.

However, Ruscheweyh in [29] proved a stronger result by showing that the partial sums $s_n(z)$ of $f$ are indeed starlike in $\mathbb{D}_{1/4}$ for functions $f$ belonging not only to $\mathcal{S}$ but also to the closed convex hull of $\mathcal{S}$. Robertson [26] further showed that sections of the Koebe function $k(z)$ are univalent in the disk $|z| < 1 - 3n^{-1} \log n$ for $n \geq 5$, and that the constant $3$ cannot be replaced by a smaller constant. However, Bshouty and Hengartner [2] pointed out that the Koebe function is not extremal for the radius of univalence of the partial sums of $f \in \mathcal{S}$. A well-known theorem by Ruscheweyh and Sheil-Small [28] on convolution allows us to conclude immediately that if $f$ belongs to $\mathcal{C}$, $\mathcal{S}^*$, or $\mathcal{K}$, then its $n$-th section is respectively convex, starlike, or close-to-convex in the disk $|z| < 1 - 3n^{-1} \log n$, for $n \geq 5$. Silverman in [31] proved that the radius of starlikeness for sections of functions in the convex family $\mathcal{C}$ is $(1/2n)^{1/n}$ for all $n$. We suggest readers refer to [22, 27, 32, 34] and
recent articles \[17 \, \cite{18} \, \cite{19} \, \cite{20}\] for further interest on this topic. It is worth recalling that radius properties of harmonic sections have recently been studied in \[\cite{7} \, \cite{9} \, \cite{10} \, \cite{23}\].

2. Preparatory results

In this section we derive some useful results to prove our main theorem.

**Lemma 2.1.** If \( f(z) = z + \sum_{n=2}^{\infty} a_{2n-1} z^{2n-1} \in \mathcal{L} \), then the following estimates are obtained:

(a) \(|a_{2n-1}| \leq \frac{(2n-2)!}{2^{n-2}(n-1)!} r^{-n+1}\) for \( n \geq 2 \). The equality holds for

\[
\frac{z}{\sqrt{1-z^2}}
\]
or its rotation.

(b) \( \left| \frac{zf''(z)}{f'(z)} \right| \leq \frac{3r^2}{1-r^2} \) for \(|z| = r < 1\). The inequality is sharp.

(c) \( \frac{1}{(1+r)^{3/2}} \leq |f'(z)| \leq \frac{1}{(1-r)^{3/2}} \) for \(|z| = r < 1\). The inequality is sharp.

(d) If \( f(z) = s_{2n-1}(z) + \sigma_{2n-1}(z) \), with \( \sigma_{2n-1}(z) = \sum_{k=n+1}^{\infty} a_{2k-1} z^{2k-1} \), then for \(|z| = r < 1\) we have

\[ |\sigma_{2n-1}'(z)| \leq A(n, r) \quad \text{and} \quad |z\sigma_{2n-1}''(z)| \leq B(n, r), \]

where

\[
A(n, r) = \sum_{k=n+1}^{\infty} \frac{(2k-1)!}{2^{k-2}(k-1)!^2} r^{2k-2} \quad \text{and} \quad B(n, r) = \sum_{k=n+1}^{\infty} \frac{(2k-2)(2k-1)!}{2^{2k-2}(k-1)!^2} r^{2k-2}.
\]

The ratio test guarantees that both the series are convergent.

**Proof.** (a) Set

\[
p(z) = 1 + \frac{2}{3} \left( \frac{zf''(z)}{f'(z)} \right).
\]

Clearly, \( p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \) is analytic in \( \overline{D} \) and \( \text{Re} p(z) > 0 \) there. So, by Carathéodory Lemma, we obtain that \(|p_n| \leq 2\) for all \( n \geq 1 \). Putting the series expansions for \( f'(z), \ f''(z) \) and \( p(z) \) in \( \left(3\right) \) we get

\[
\sum_{n=2}^{\infty} (2n-1)(2n-2) a_{2n-1} z^{2n-1} = \frac{3}{2} \sum_{n=2}^{\infty} \left( \sum_{k=1}^{n-1} p_{2k-1}(2n-2k-1)a_{2n-2k-1} \right) z^{2n-2}
\]

\[
+ \frac{3}{2} \sum_{n=2}^{\infty} \left( \sum_{k=1}^{n-1} p_{2k}(2n-2k-1)a_{2n-2k-1} \right) z^{2n-1}.
\]

Equate the coefficients of \( z^{2n-1} \) and \( z^{2n-2} \) on both sides, we obtain

\[
\sum_{k=1}^{n-1} p_{2k-1}(2n-2k-1)a_{2n-2k-1} = 0
\]
and

\begin{equation}
(2n - 1)(2n - 2)a_{2n-1} = \frac{3}{2} \sum_{k=1}^{n-1} p_{2k}(2n - 2k - 1)a_{2n-2k-1}, \quad \text{for all } n \geq 2.
\end{equation}

Hence,

\begin{equation}
|a_{2n-1}| \leq \frac{3}{(2n - 1)(2n - 2)} \sum_{k=1}^{n-1} (2k - 1)|a_{2k-1}|.
\end{equation}

For \( n = 2 \), we can easily see that \( |a_3| \leq 1/2 \), and for \( n = 3 \), we have

\begin{equation*}
|a_5| \leq \frac{3}{20}(1 + 3|a_3|) \leq \frac{3}{8}.
\end{equation*}

Now, we can complete the proof by method of induction. Therefore, if we assume \( |a_{2k-1}| \leq \frac{(2k-2)!}{2^{2k-2}(k-1)!^2} \) for \( k = 2, 3, \ldots, n - 1 \), then we deduce from (5) that

\begin{equation*}
|a_{2n-1}| \leq \frac{3}{(2n - 1)(2n - 2)} \sum_{k=1}^{n-1} \frac{(2k - 1)!}{2^{2k-2}(k-1)!^2}.
\end{equation*}

Induction principle tells us to show that

\begin{equation*}
|a_{2n-1}| \leq \frac{(2n - 2)!}{2^{2n-2}(n-1)!^2}.
\end{equation*}

It suffices to show that

\begin{equation*}
\frac{3}{(2n - 1)(2n - 2)} \sum_{k=1}^{n-1} \frac{(2k - 1)!}{2^{2k-2}(k-1)!^2} = \frac{(2n - 2)!}{2^{2n-2}(n-1)!^2}
\end{equation*}

or,

\begin{equation*}
\sum_{k=1}^{n-1} \frac{3(2k - 1)!}{2^{2k-2}(k-1)!^2} = \frac{(2n - 2)(2n - 1)}{2^{2n-2}(n-1)!^2}.
\end{equation*}

Again, we prove this by method of induction. It can easily be seen that for \( k = 1 \) it is true. Assume that it is true for \( k = 2, 3, \ldots, n - 1 \), then we have to prove that

\begin{equation*}
\sum_{k=1}^{n} \frac{3(2k - 1)!}{2^{2k-2}(k-1)!^2} = \frac{(2n)(2n + 1)}{2^{2n}(n)!^2},
\end{equation*}

which is easy to see, since

\begin{equation*}
\sum_{k=1}^{n} \frac{3(2k - 1)!}{2^{2k-2}(k-1)!^2} = \frac{(2n - 2)(2n - 1)!}{2^{2n-2}(n-1)!^2} + \frac{3(2n - 1)!}{2^{2n-2}(n-1)!^2} = \frac{(2n)(2n + 1)!}{2^{2n}(n)!^2}.
\end{equation*}

Hence, the proof is complete. For equality, it can easily be seen that

\begin{equation*}
f_0(z) = \frac{z}{\sqrt{1 - z^2}} = z + \sum_{n=2}^{\infty} \frac{(2n - 2)!}{2^{2n-2}(n-1)!^2} z^{2n-1}
\end{equation*}

belongs to \( \mathcal{L} \).

The image of the unit disk \( \mathbb{D} \) under \( f_0 \) is shown in Figure 2 which indicates that \( f_0(\mathbb{D}) \) is not convex.
(b) We see from the definition of $\mathcal{L}$ that
\[ 1 + \frac{zf''(z)}{f'(z)} \prec \frac{1 + 2z^2}{1 - z^2}, \]
i.e.,
\[ \frac{zf''(z)}{f'(z)} \prec \frac{3z^2}{1 - z^2} =: h(z), \]
where $\prec$ denotes the usual subordination. The proof of (b) now follows easily.

(c) Since
\[ \frac{zf''(z)}{f'(z)} \prec h(z), \]
it follows by the well-known subordination result due to Suffridge [33] that
\[ f'(z) \prec \exp \left( \int_0^z \frac{h(t)}{t} \, dt \right) = \exp \left( 3 \int_0^z \frac{t}{1 - t^2} \, dt \right) = \frac{1}{(1 - z^2)^{3/2}}. \]
Hence, the proof of (c) follows.

(d) By (a), we see that
\[ |\sigma'_{2n-1}(z)| \leq \sum_{k=n+1}^{\infty} (2k - 1)|a_{2k-1}|r^{2k-2} \leq A(n, r). \]
and
\[ |z\sigma''_{2n-1}(z)| \leq \sum_{k=n+1}^{\infty} (2k - 1)(2k - 2)|a_{2k-1}|r^{2k-2} \leq B(n, r). \]
The proof of our lemma is complete. \qed
Radius of convexity of partial sums of odd functions

3. PROOF OF THE MAIN THEOREM

For an arbitrary \( f(z) = z + \sum_{n=2}^{\infty} a_{2n-1} z^{2n-1} \in \mathcal{L} \), we first consider its third section \( s_3(z) = z + a_3 z^3 \) of \( f \). Simple computation shows

\[
1 + \frac{z s''_3(z)}{s'_3(z)} = 1 + \frac{6a_3 z^2}{1 + 3a_3 z^2}.
\]

By using Lemma 2.1(a), we have \(|a_3| \leq 1/2\) and hence

\[
\text{Re} \left( 1 + \frac{z s''_3(z)}{s'_3(z)} \right) \geq 1 - \frac{6|a_3||z|^2}{1 - 3|a_3||z|^2} \geq 1 - \frac{3|z|^2}{1 - \frac{3}{2}|z|^2}
\]

which is positive for \(|z| < \sqrt{2}/3\). Thus, \( s_3(z) \) is convex in the disk \(|z| < \sqrt{2}/3\). To show that the constant \( \sqrt{2}/3 \) is best possible, we consider the function \( f_0(z) \) defined by

\[
f_0(z) = \frac{z}{\sqrt{1 - z^2}}.
\]

We denote by \( s_{3,0}(z) \), the third partial sum \( s_3(f_0)(z) \) of \( f_0(z) \) so that \( s_{3,0}(z) = z + (1/2)z^3 \) and hence, we find

\[
1 + \frac{z s''_{3,0}(z)}{s'_{3,0}(z)} = \frac{2 + 9z^2}{2 + 3z^2}.
\]

This shows that

\[
\text{Re} \left( 1 + \frac{z s''_{3,0}(z)}{s'_{3,0}(z)} \right) = 0
\]

when \( z^2 = (-2/9) \) or \((-2/3)\) i.e., when \(|z|^2 = (2/9) \) or \((2/3)\). Hence, the equality occurs.

Next, let us consider the case \( n = 3 \). Our aim in this case is to show that

\[
\text{Re} \left( 1 + \frac{z s''_3(z)}{s'_3(z)} \right) = \text{Re} \left( \frac{1 + 9a_3 z^2 + 25a_5 z^4}{1 + 3a_3 z^2 + 5a_5 z^4} \right) > 0
\]

for \(|z| < \sqrt{2}/3\). Since the real part \( \text{Re} [(1+9a_3 z^2 + 25a_5 z^4)/(1+3a_3 z^2 + 5a_5 z^4)] \) is harmonic in \(|z| \leq \sqrt{2}/3\), it suffices to check that

\[
\text{Re} \left( \frac{1 + 9a_3 z^2 + 25a_5 z^4}{1 + 3a_3 z^2 + 5a_5 z^4} \right) > 0
\]

for \(|z| = \sqrt{2}/3\). Also we see that

\[
\text{Re} \left( \frac{1 + 9a_3 z^2 + 25a_5 z^4}{1 + 3a_3 z^2 + 5a_5 z^4} \right) = 3 - \text{Re} \left( \frac{2 - 10a_5 z^4}{1 + 3a_3 z^2 + 5a_5 z^4} \right) \geq 3 - \left| \frac{2 - 10a_5 z^4}{1 + 3a_3 z^2 + 5a_5 z^4} \right|
\]

and, so by considering a suitable rotation of \( f(z) \), the proof reduces to \( z = \sqrt{2}/3 \); this means that it is enough to prove

\[
\frac{3}{2} > \left| \frac{81 - 20a_5}{81 + 54a_3 + 20a_5} \right|.
\]

From (11), we have

\[
a_3 = \frac{p_2}{4} \quad \text{and} \quad a_5 = \left( \frac{3}{40} \right) \left( \frac{3}{4} p_2^2 + p_4 \right).
\]
Since $|p_2| \leq 2$ and $|p_4| \leq 2$, it is convenient to rewrite the last two relations as

$$a_3 = \frac{\alpha}{2} \quad \text{and} \quad a_5 = \frac{3}{40}(3\alpha^2 + 2\beta)$$

for some $|\alpha| \leq 1$ and $|\beta| \leq 1$.

Substituting the values for $a_3$ and $a_5$, and applying the maximum principle in the last inequality, it suffices to show the inequality

$$\frac{3}{2} \left| 81 + 27\alpha + \frac{9\alpha^2}{2} + 3\beta \right| > \left| 81 - \frac{9\alpha^2}{2} - 3\beta \right|$$

for $|\alpha| = 1 = |\beta|$. Finally, by the triangle inequality, the last inequality follows if we can show that

$$9 \left| 9 + 3\alpha + \frac{\alpha^2}{2} \right| - 6 \left| 9 - \frac{\alpha^2}{2} \right| > 5$$

which is easily seen to be equivalent to

$$9 \left| 9\alpha + 3 \right| - 6 \left| 9\alpha - \frac{\alpha}{2} \right| > 5$$

as $|\alpha| = 1$. Write $\text{Re}(\alpha) = x$. It remains to show that

$$T(x) := 9 \sqrt{18x^2 + 57x + \frac{325}{4} - 6 \sqrt{\frac{361}{4} - 18x^2}} > 5$$

for $-1 \leq x \leq 1$.

It suffices to show

$$9 \sqrt{18x^2 + 57x + \frac{325}{4}} > 5 + 6 \sqrt{\frac{361}{4} - 18x^2}.$$  

Squaring both sides we have

$$2106x^2 + 4617x + \frac{13229}{4} > 60 \left( \sqrt{\frac{361}{4} - 18x^2} \right).$$
Again by squaring both sides we have
\[
\left(2106x^2 + 4617x + \frac{13229}{4}\right)^2 > 3600\left(\frac{361}{4} - 18x^2\right).
\]

After computing, it remains to show that \(\phi(x) > 0\), where
\[
\phi(x) = ax^4 + bx^3 + cx^2 + dx + e
\]
and the coefficients are
\[
a = 4435236, b = 19446804, c = 35311626, d = 30539146.5, e = 10613002.5625.
\]
Here we see that \(\phi^{iv}(x) = 24a > 0\). Thus the function \(\phi''''(x)\) is increasing in \(-1 \leq x \leq 1\) and hence \(\phi''''(-1) = 10235160 > 0\). This implies \(\phi'(x)\) is increasing. Hence \(\phi''''(x) \geq \phi''''(-1) = 7165260 > 0\). Consequently, \(\phi'(x)\) is increasing and we have \(\phi'(x) \geq \phi'(-1) = 515362.5 > 0\). Finally we get, \(\phi(x)\) is increasing and hence we have \(\phi(x) > \phi(-1) = 373914.0625 > 0\). This completes the proof for \(n = 3\).

We next consider the general case \(n \geq 4\). It suffices to show that
\[
\text{Re} \left(1 + \frac{zs''_{2n-1}}{s'_{2n-1}}\right) > 0 \quad \text{for} \quad |z| = r
\]
with \(r = \sqrt{2}/3\) for all \(n \geq 4\). From the maximum modulus principle, we shall then conclude that the last inequality holds for all \(n \geq 4\)
\[
\text{Re} \left(1 + \frac{zs''_{2n-1}}{s'_{2n-1}}\right) > 0
\]
for \(|z| < \sqrt{2}/3\). In other words, it remains to find the largest \(r\) so that the last inequality holds for all \(n \geq 4\).

By the same setting of \(f(z)\) as in Lemma 2.11(d), it follows easily that
\[
1 + \frac{zs''_{2n-1}}{s'_{2n-1}} = 1 + \frac{zf''(z)}{f'(z)} = 1 + \frac{zf''(z)}{f'(z)} + \frac{zf''(z)}{f'(z)}\frac{\sigma'_{2n-1}(z) - z\sigma''_{2n-1}(z)}{f'(z) - \sigma'_{2n-1}(z)}
\]
or,
\[
\text{Re} \left(1 + \frac{zs''_{2n-1}}{s'_{2n-1}}\right) \geq 1 - \frac{|zf''(z)|}{|f'(z)|} - \frac{|zf''(z)|}{|f'(z)|}\frac{|\sigma'_{2n-1}(z)| + |z\sigma''_{2n-1}(z)|}{|f'(z)| - |\sigma'_{2n-1}(z)|}.
\]
Then by using Lemma 2.11 we obtain
\[
\text{Re} \left(1 + \frac{zs''_{2n-1}}{s'_{2n-1}}\right) \geq 1 - \frac{3r^2}{1 - r^2} - \frac{\left(\frac{3r^2}{1 - r^2}\right) A(n, r) + B(n, r)}{1 - (1 + r^2)A(n, r)}.
\]
Thus, we conclude that
\[
\text{Re} \left(1 + \frac{zs''_{2n-1}}{s'_{2n-1}}\right) > 0
\]
provided
\[
\frac{1 - 4r^2}{1 - r^2} - \frac{(1 + r^2)^{3/2}}{1 - r^2} \left(\frac{3r^2A(n, r) + (1 - r^2)B(n, r)}{1 - (1 + r^2)^{3/2}A(n, r)}\right) > 0,
\]
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or, equivalently
\[
(1 + r^2)^{3/2} \left( \frac{3r^2 A(n, r) + (1 - r^2) B(n, r)}{1 - (1 + r^2)^{3/2} A(n, r)} \right) < 1 - 4r^2.
\]
We show that the above relation holds for all \(n \geq 4\) with \(r = \sqrt{2}/3\). The choice \(r = \sqrt{2}/3\) brings the last inequality to the form
\[
\left( \frac{11}{9} \right)^{3/2} \left( \frac{\frac{2}{3} A(n, \frac{\sqrt{2}}{3}) + \frac{7}{9} B(n, \frac{\sqrt{2}}{3})}{1 - (\frac{11}{9})^{3/2} A(n, \frac{\sqrt{2}}{3})} \right) < \frac{1}{9}.
\]
Set
\[
C \left( n, \frac{\sqrt{2}}{3} \right) := 1 - \left( \frac{11}{9} \right)^{3/2} A \left( n, \frac{\sqrt{2}}{3} \right).
\]
We shall prove that \(C \left( n, \frac{\sqrt{2}}{3} \right) > 0\) for \(n \geq 4\) i.e.,
\[
A \left( n, \frac{\sqrt{2}}{3} \right) < \frac{27}{(11)^{3/2}}
\]
and
\[
A \left( n, \frac{\sqrt{2}}{3} \right) + B \left( n, \frac{\sqrt{2}}{3} \right) < \frac{27}{7 \times (11)^{3/2}} \quad \text{for } n \geq 4.
\]
If the last inequality is proved, then automatically the previous one follows. Hence, it is enough to prove the last inequality. Now,
\[
A(n, r) + B(n, r) = \sum_{k=n+1}^{\infty} \frac{(2k - 1)(2k - 1)!}{2^{2k-2}(k - 1)!^2} (r^2)^{k-1}
\]
\[
\leq \sum_{k=5}^{\infty} \frac{(2k - 1)(2k - 1)!}{2^{2k-2}(k - 1)!^2} (r^2)^{k-1}
\]
\[
= \sum_{k=1}^{\infty} \frac{(2k - 1)(2k - 1)!}{2^{2k-2}(k - 1)!^2} (r^2)^{k-1} - \sum_{k=1}^{4} \frac{(2k - 1)(2k - 1)!}{2^{2k-2}(k - 1)!^2} (r^2)^{k-1}
\]
\[
= \frac{1 + 2r^2}{(1 - r^2)^{5/2}} - \left( 1 + \frac{9}{2} r^2 + \frac{75}{8} r^4 + \frac{245}{16} r^6 \right).
\]
Substituting the value \(r = \sqrt{2}/3\), we obtain
\[
A \left( n, \frac{\sqrt{2}}{3} \right) + B \left( n, \frac{\sqrt{2}}{3} \right) \leq 0.076 \cdots < 0.105 \cdots = \frac{27}{7 \times (11)^{3/2}}.
\]
This completes the proof of our main theorem. □

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Radius of convexity of partial sums of odd functions

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Sarita Agrawal, Discipline of Mathematics, Indian Institute of Technology Indore, Simrol, Khandwa Road, Indore 452 020, India
E-mail address: saritamath44@gmail.com

Swadesh Kumar Sahoo, Discipline of Mathematics, Indian Institute of Technology Indore, Simrol, Khandwa Road, Indore 452 020, India
E-mail address: swadesh@iiti.ac.in