Vector fields with big and small volume on $\mathbb{S}^2$

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Abstract

We search for minimal volume vector fields on a given Riemann surface, specialising on the case of $M^*$, this is, the 2-sphere with two antipodal points removed. We discuss the homology theory of the unit sphere tangent bundle $(SM^*, \partial SM^*)$ in relation with calibrations and a minimal volume equation.

We find a family $X_{m,k}$, $k \in \mathbb{N}$, called the meridian type vector fields, defined globally and with unbounded volume on any given open subset $\Omega$ of $M^*$. In other words, we have that $\forall \Omega, \lim_k \text{vol}(X_{m,k}|_{\Omega}) = +\infty$. These are the strong candidates to being the minimal volume vector fields in their homology class, since they satisfy great equations. We also show a vector field $X_\ell$ on a specific region $\Omega_1 \subset \mathbb{S}^2$ with volume smaller than any other known optimal vector field restricted to $\Omega_1$.

Key Words: vector field; minimal volume; homology; calibration.

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1 – Previous results

In this article we explore some ideas and new examples of the theory of the volume of vector fields. Our findings are in the continuation of those in [1].

Suppose we are given an oriented Riemann surface $M$, eventually with boundary. Let $X$ be a unit norm $C^2$ vector field on $M$. Let us recall the definition of the volume of $X$, cf. [6]:

$$\text{vol}(X) = \text{vol}(M, X^* g^S) = \int_M \sqrt{1 + \|\nabla_{e_0} X\|^2 + \|\nabla_{e_1} X\|^2} \text{vol}_M$$

(1)

where $g^S$ is the Sasaki metric on the unit tangent sphere bundle $SM \to M$ and $e_0, e_1$ is any local orthonormal frame on $M$.

In article [1] we discover a sufficient condition to have a minimal volume vector field. First, let $A = A_1 + \sqrt{-1} A_0 : M \to \mathbb{C}$ be the function defined by the components
of \( \nabla X \) in the direct orthonormal frame \( \{ X, Y \} \) on \( M \). In other words, let \( A_0, A_1 \) be the functions defined by

\[
A_0 = \langle \nabla_X X, Y \rangle \quad A_1 = \langle \nabla_Y X, Y \rangle. \tag{2}
\]

Then, if \( X \) satisfies the following differential equation in a conformal chart \( z \) of \( M \):

\[
\frac{\partial}{\partial z} \frac{A}{\sqrt{1 + |A|^2}} = 0, \tag{3}
\]

then \( X \) has minimal volume over its domain. We remark this equation is reminiscent of the minimal area surface graph equation.

Let us also remark that the Cauchy-Riemann system above is orientation invariant, because \( A \) transforms accordingly. Just as well as it is invariant if the role of \( X \) and \( Y \) is permuted. That is consistent with the integrand in (1) being orientation invariant and, respectively, all \( (\cos \alpha)X + (\sin \alpha)Y \) having the same volume, for every constant \( \alpha \in \mathbb{R} \). More importantly, as the reader may notice below, is that each CR-equation is orientation invariant.

The present article is concerned with the geometry of vector fields on the 2-sphere with canonical metric. It brings up some surprising results, in the continuation of [2, 3, 4, 5, 6, 10] and of course [1].

Let us start by informing about the equation in the case of a constant hyperbolic metric. In [1] we find a solution of (3) for \( M \) with constant sectional curvature \( K < 0 \). Actually, given any \( M \) and \( X \) such that \( |A| \) is constant, then \( A \) is constant and \( M \) has constant sectional curvature \( K = -|A|^2 \leq 0 \). It follows that \( \text{vol}(X) = \sqrt{1 - K} \text{vol}(M) \). The result applies to (a germ of) the Lie group of affine transformations \( \text{Aff}(\mathbb{R}) \) with left invariant metric; any left invariant unit vector field \( X \) is thus minimal. \( \text{Aff}(\mathbb{R}) \) is a negative constant Gauss curvature surface, as it is well known.

Equation (3) proves quite difficult to solve, even in isothermal coordinates given for the hyperbolic metric. Uniqueness of solutions (up to some rigid rotation) is an open question.

Notice the equation gives a sufficient condition for minimality. A necessary condition is deduced in [6] p. 538. Due to the works of O. Gil-Medrano and E. Llinares-Fuster, we now know that a minimal vector field satisfies the Euler-Lagrange equation

\[
X \left( A_0 / \sqrt{1 + |A|^2} \right) + Y \left( A_1 / \sqrt{1 + |A|^2} \right) = 0. \tag{4}
\]

We then have the following reassuring result.

**Proposition 1.** *Cauchy-Riemann equation (3) implies Euler-Lagrange equation (4).*

**Proof.** If we have a holomorphic function, then its differential vanishes in the direction of \( X + \sqrt{-1}Y \); the imaginary part of \( d(A/\sqrt{1 + |A|^2})(X + \sqrt{-1}Y) = 0 \) yields the result. \[\blacksquare\]
Clearly (4) alone is far from giving the Cauchy-Riemann equations. The real part,
\[ X(A_1/\sqrt{1+|A|^2}) - Y(A_0/\sqrt{1+|A|^2}) = 0, \] should also be a necessary condition for minimality, but this remains uncertain. Convincing the reader that this may be so is also what motivates this article.

On the manifold \( S^2 \) with the round metric, punctured at two antipodal points, it is known that a minimum of vol is attained, with the solution being a certain \( X_0 \), cf. Proposition [4]. One easily checks that \( X_0 \) does not satisfy our equation (5). This is coherent with the theory, since we have found a vector field, though in a smaller region of \( S^2 \), which has even less volume than \( X_0 \) in that region. We present it later, below.

In a conformal chart we may improve the study a bit further, cf. [1]. A complex coordinate \( z = x + \sqrt{-1}y \) corresponds with isothermal coordinates, i.e. a chart such that the Riemannian metric is given by \( \lambda|dz|^2 \) for some function \( \lambda > 0 \).

The real vector field \( X \) is given by
\[ X = a\partial_x + b\partial_y = f\partial_z + \overline{f}\partial_{\overline{z}} \] where \( f = a + \sqrt{-1}b \). If \( Z = h\partial_z + \overline{h}\partial_{\overline{z}} \) is also a vector field, then
\[ \langle X, Z \rangle = \frac{\lambda}{2} (f\overline{h} + \overline{f}h). \] In particular \( \|X\|^2 = \lambda|f|^2 \). Note that \( Y = \sqrt{-1}(f\overline{\partial_z} - \overline{f}\partial_z) = \overline{Y} \).

The Levi-Civita connection is given by \( \nabla_z \partial_z = \Gamma \partial_z \), where \( \Gamma = \frac{1}{\lambda} \frac{\partial \lambda}{\partial z} \), \( \nabla_z \partial_{\overline{z}} = \nabla_{\overline{z}} \partial_z = \frac{1}{\lambda} \frac{\partial \lambda}{\partial \overline{z}} \partial_z \). In particular we have \( R(\partial_z, \partial_{\overline{z}}) \partial_z = -\frac{\partial \Gamma}{\overline{z}} \partial_z \) and hence
\[ K = \frac{\langle R(\partial_z, \partial_{\overline{z}}) \partial_z, \partial_z \rangle}{\langle \partial_z, \partial_{\overline{z}} \rangle^2} = -\frac{2}{\lambda} \frac{\partial \Gamma}{\overline{z}} \partial_z = -\frac{2}{\lambda} \frac{\partial^2 \log \lambda}{\partial z \partial \overline{z}}. \] We have proved in [1] that
\[ A = -2\lambda f^2 \frac{\partial \overline{f}}{\partial z} = 2(\Gamma f + \frac{\partial f}{\partial z}). \] We have \( |A| = 2|f|^2 \) and so a holomorphic unit vector field is just a parallel vector field, as it is well known.

2 – Topological invariants on the boundary

We have tried to find the equations of a 2-form calibration \( \varphi \in \Omega^2_{\mathbb{S}^3} \) having the minimal vector fields as calibrated submanifolds. Towards some calculus of variations insight, our approach proves useful when we have some guess of what \( X \) should be.

The topology of the domain and of a vector field determines the class of its volume; this is what we may verify and was claimed in [7] concerning \( S^3 \), hence with no worry
about singularities, and followed in \cite{9} where one vector field $W$ with singularity on a hypersphere is evenly associated to a certain homology class.

There is now some quest for the analytic theory of calibrations to further illuminate the bridge between homology and minimal volume.

Let us develop these ideas, arguing first in any dimension. Let $M$ be a compact Riemannian $(n + 1)$-manifold eventually with boundary. Then we have the following isomorphism of Poincaré-Lefschetz duality, cf. \cite[Section 3.3]{8}, with integer coefficients:

$$H_k(SM, \partial SM) \simeq H^{2n+1-k}(SM)(\simeq H^k(SM)).$$

The second isomorphism may be Hodge duality, recalling that de Rham and singular cohomologies coincide for manifolds. However, we do not have a precise statement for Hodge duality for manifolds with boundary. (On the other hand, there exists a connecting homomorphism in the middle degree $k = n + 1$.)

Let $M^*$ denote a given closed manifold $M$ with a finite number of points $p_1, \ldots, p_N$ removed. Let $M_\epsilon$ denote $M \setminus \bigcup_i B_\epsilon(p_i)$, where the open geodesic balls are non-intersecting. Since $M_\epsilon \subset M_{\epsilon_1}$ for $\epsilon_1 < \epsilon$, the cohomology rings $H^j(SM_{\epsilon_1})$ are well-defined and isomorphic between them, $\forall \epsilon_1$. By \cite[Proposition 3.33]{8}, there is the inductive limit

$$\lim_{\to} H^{2n+1-k}(SM_\epsilon) \simeq H^k(SM^*).$$

By the above, we have:

$$H_k(SM^*, \partial SM^*) := \lim_{\to} H_k(SM_\epsilon, \partial SM_\epsilon) \simeq H^k(SM^*).$$

So in fact we overcome the uncertainty of Hodge duality with boundary.

We return to dimension 2. It is clear that $\partial SM_\epsilon$ is a disjoint union of $N$ 2-torus. Using parallel translation to a base point $S^1$-fibre along each circle, any $C^2$ unit vector field $X$ defined on $M^*$ certainly has its degree in $\mathbb{Z}$ as it restricts to a map $\partial B_\epsilon(p_i) = S^1 \to S^1$. The field $X$ is not singular at the $p_i$ in the sense of having a zero; though it still has an index, $I_X(p_i)$, independent of $\epsilon$. The sum $\sum_i N_i I_X(p_i)$ is the Euler characteristic of the surface (Poincaré-Hopf Theorem).

The $X$ determine a class in the relative homology $H_2(SM_\epsilon, \partial SM_\epsilon)$ in principle dependent on the various indices at the $p_i$, $i = 1, \ldots, N$ and nothing else. Notice $X(M^*)$ sits in $SM^*$ transversely to $\partial SM^* \subset SM$. So the homology class of the field is the main invariant; the relative homology must not complicate much more.

What one would finally expect from calibrations is that each $[\varphi] \in H^2(SM^*)$ determines a class $[X(M^*)]$ of minimal volume in (relative) homology. On the other hand, Theorem 1 in \cite{1} has led to quite demanding solutions. Besides the case for hyperbolic space.

Extending the theory to complex vector fields $X \in \Gamma(M; TM \otimes \mathbb{C})$ should give us perhaps a way to find the necessary and sufficient condition for the optimal vector field.
We may also consider a Berger metric dilation on the unit tangent sphere bundle, i.e. the usual metric on $SM$ with a weight on the direction of the spray $e_0$. Thus we assume the vector field $\tilde{e}_0 = \lambda e_0$ on $SM$, with $\lambda \in C^\infty_M(\mathbb{R}^+)$, has unit norm and its orthogonal plane remains the same. The problem of finding minimal vector fields for $\text{vol}_X$ would depend on being able to optimize volume with this squashed metric. In other words, to be certain of minimizing through the right weight function $\lambda > 0$ and the minimal vector field of $\int_M \frac{1}{2} \sqrt{1 + A_1^2 + \lambda^2 A_0^2} \text{vol}_M$. This strategy is equivalent to that referred in [3, Remark in Section 3] and [6] for spheres.

3 – New vector fields on the sphere

We consider the radius $r$ sphere with two antipodal points removed

$$M^* = S^2 \setminus \{p_S, p_N\}$$

(13)

endowed with the round metric (let $x = x(r, \theta)$)

$$\langle , \rangle = r^2 (d\theta)^2 + r^2 \sin^2 \theta (d\phi)^2 = r^2 \sin^2 \theta ((dx)^2 + (d\phi)^2)$$

(14)

where $r > 0$ is constant, and $(\theta, \phi) \in D = ]0, \pi[ \times ]0, 2\pi[$. Letting $i = \sqrt{-1}$ and continuing as in (7), we have $z = x + i\phi$ and then $\sin \theta dx = d\theta$ and $dz = dx + i d\phi$. Hence

$$\lambda = r^2 \sin^2 \theta$$

(15)

and

$$\partial_x = \sin \theta \partial_\theta, \quad \partial_z = \frac{1}{2} (\partial_x - i \partial_\phi).$$

(16)

In particular the volume form is given by $\lambda dx \wedge d\phi =$

$$\frac{i}{2} \lambda dz \wedge d\bar{z} = r^2 \sin \theta d\theta \wedge d\phi.$$ 

(17)

We have $\Gamma = \frac{1}{\lambda} \partial_z \lambda = \frac{1}{2 \sin \theta} \partial_\theta \sin^2 \theta = \cos \theta$, which verifies

$$K = \frac{2}{\lambda} \frac{\partial \Gamma}{\partial \bar{z}} = - \frac{\sin \theta}{r^2 \sin^2 \theta} \partial_\theta \cos \theta = \frac{1}{r^2}.$$ 

(18)

We also require the Levi-Civita connection in real coordinates

$$\nabla_\theta \partial_\theta = 0, \quad \nabla_\theta \partial_\phi = \nabla_\phi \partial_\theta = \cotg \theta \partial_\phi, \quad \nabla_\phi \partial_\phi = - \cos \theta \sin \theta \partial_\theta.$$ 

(19)

Guessing from the first equation above on the case of the unit norm vector field $\frac{1}{r} \partial_\theta$, we endeavour to look for those unit vector fields which are parallel along every meridian $p_S p_N$. 

Let \( X = \frac{a}{r} \partial_\theta + \frac{b}{r \sin \theta} \partial_\phi \) with \( a, b \) real valued \( C^2 \) functions on \( D \). Then we find
\[
\nabla_\theta X = a' \partial_\theta + \frac{b'}{\sin^2 \theta} \partial_\theta + \frac{b}{\sin \theta} \cot \theta \partial_\phi = a' \partial_\theta + \frac{b'}{\sin \theta} \partial_\phi.
\]

Hence
\[
\nabla_\theta X = 0 \iff \begin{cases} a' = 0, \\ b' = 0 \end{cases}.
\]

Since for unit \( X \) we must have \( a^2 + b^2 = 1 \), there exists \( \zeta = \zeta(\phi) \), function only of \( \phi \), such that
\[
X = \frac{\cos \zeta}{r} \partial_\theta + \frac{\sin \zeta}{r \sin \theta} \partial_\phi.
\]

Let \( Y = -\frac{\sin \zeta}{r} \partial_\theta + \frac{\cos \zeta}{r \sin \theta} \partial_\phi \) be the unique vector field such that \( X, Y \) is a direct orthonormal frame. Routine computations yield:
\[
A_0 = \langle \nabla X, Y \rangle = \sin \zeta \frac{\zeta' + \cos \theta}{r \sin \theta},
\]
\[
A_1 = \langle \nabla Y, X \rangle = \cos \zeta \frac{\zeta' + \cos \theta}{r \sin \theta}.
\]

Thus
\[
\frac{A_1 + iA_0}{\sqrt{1 + |A|^2}} = \frac{e^{i\zeta}(\zeta' + \cos \theta)}{\sqrt{r^2 \sin^2 \theta + (\zeta' + \cos \theta)^2}}.
\]

Notice that even with \( r = 1 \) and \( \zeta = 0 \), we get \( \partial_\theta \cos \theta = -\frac{1}{2} \sin^2 \theta \). (Of course, a vector field which has \( A_0 = 0 \) is not a good candidate as a solution of equation (3), unless it is parallel.) One would expect that \( \partial_\theta \) has minimal volume; and this is true globally, as it was proved by [4] and we shall soon recall.

On the radius \( r \) sphere, we have:
\[
\text{vol}(X) = r \int_D \sqrt{r^2 \sin^2 \theta + (\zeta' + \cos \theta)^2} \, d\theta \wedge d\phi.
\]

\( X \) is well defined and continuous on \( M^* \) if it satisfies \( \lim_{\phi \to 2\pi} \zeta(\phi) = \zeta(0) + 2k\pi \), for some \( k \in \mathbb{Z} \). We may further assume that \( \zeta \) is \( C^2 \) on a neighborhood of \([0, 2\pi]\). This way the field \( X \) becomes also \( C^2 \).

The latter is the case when we fix \( k \in \mathbb{Z}, \phi_0 \in \mathbb{R} \) and take
\[
\zeta = k\phi + \phi_0.
\]

We define these as the vector fields of meridian type \((r = 1)\):
\[
X_{m,k} = \cos(k\phi + \phi_0) \partial_\theta + \frac{\sin(k\phi + \phi_0)}{\sin \theta} \partial_\phi.
\]

\( k \) is thus the number of times that \( X_{m,k} \) winds around itself when it goes around a so-called parallel or circle-of-latitude.

Now we are rewarded with a remarkable result.
**Proposition 2.** Every vector field of meridian type satisfies the Euler-Lagrange equation (4) for minimal vector fields.

**Proof.** An easy way to see this is to note from (25) that \( A/\sqrt{1 + |A|^2} \) equals \( e^{i\zeta} f \) with \( f = f(\theta) \) and \( \zeta = k\phi + \phi_0 \). Now, from (22),

\[
X + iY = \frac{\cos \zeta}{r} \partial_\theta + \frac{\sin \zeta}{r \sin \theta} \partial_\phi - i \frac{\sin \zeta}{r} \partial_\theta + i \frac{\cos \zeta}{r} \partial_\theta
\]

\[
= \frac{e^{-i\kappa}}{r} \partial_\theta + i \frac{e^{-i\kappa}}{r \sin \theta} \partial_\phi.
\]

Hence

\[
d(e^{i\zeta} f)(X + iY) = \frac{e^{-i\kappa}}{r} \partial_\theta(e^{i\zeta} f) + i \frac{e^{-i\kappa}}{r \sin \theta} ike^{i\zeta} f
\]

\[
= \frac{1}{r} f_\theta - \frac{k}{r \sin \theta} f.
\]

In particular, the imaginary part vanishes. (The real part does not.) 

A radius \( r \neq 1 \) also brings stability into discussion. This was first and foremost observed in [2, 5] in general, hence we assume \( r = 1 \) from now on for the meridian type vector fields.

Let us recall now the unit vector field defined on an \( n \)-dimensional punctured sphere, studied by S. Pedersen in [9] and denoted there by \( W \). On \( S^2 \setminus \{p_S\} \), the field \( W \) is defined as the parallel transport along the meridians of one given unit tangent vector at the North pole. Now, the following result becomes geometrically evident.

**Proposition 3.** On \( M^* \) the vector field \( X_{m,1} \) coincides with \( W \).

**Proof.** Both vector fields rotate, once and uniformly, while they go around the parallels, i.e. the curves \( \theta = \text{constant} \). Since the two fields are defined by parallel transport along the meridians, they must be the same, cf. Figure [1].

**Article [9] is mostly concerned with** \( W \), referring it as a Pontryagin cycle in the context of the homology theory of the orthogonal Lie groups and Stiefel manifolds. [3] also refers to \( W \) as an example of a certain Pontryagin vector field.

Taking the cases \( k = 0,1 \) with the above point of view, we have not found a reference for the vector fields \( X_{m,k} \).

From (26), we have:

\[
\text{vol}(X_{m,k}) = \iint_D \sqrt{1 + k^2 + 2k \cos \theta} \, d\theta \wedge d\phi.
\]  

(29)

We notice the cases \( k \) and \( -k \) yield the same volume on \( M^* \), as expected.
Theorem 1 (Big volume everywhere). The sequence of unit vector fields $(X_{m,k})_{k \in \mathbb{N}}$ defined on $M^*$ is such that, for every open subset $\Omega \subset M^*$ corresponding to a domain $\bar{D} \subset D$, we have

$$(k-1)\text{vol}_Euc(\bar{D}) < \text{vol}(X_{m,k}|_{\Omega}) < (k+1)\text{vol}_Euc(\bar{D}).$$

(30)

In particular, for every open sets $\Omega \subset \Omega_1 \subset M^*$

$$\sup\{\text{vol}(X|_{\Omega}) : X \text{ is a unit vector field on } \Omega_1\} = +\infty.$$  

(31)

Proof. The result follows as a particular case of a similar statement for the meridian type vector fields defined on the radius $r^2$-sphere. From (26), we have for instance that the following elliptic integral on the right hand side satisfies

$$r \int \int_{\bar{D}} \sqrt{r^2 \sin^2 \theta + k^2 - 2k + \cos^2 \theta} \, d\theta d\phi < \text{vol}(X)$$

and also has an obvious upper bound. 

Theorem 1 shows that we can have big volume everywhere. Formula (29) also yields the computational part, with $k = 0$ and $k = 1$, of the next result.

Proposition 4 ([3 4 9]). The meridian vector fields $X_{m,0} = \cos \phi_0 \partial_\theta + \frac{\sin \phi_0}{\sin \theta} \partial_\phi$ and $X_{m,1}$ have minimal volume in the respective homology classes $M^*$ and $S^2\setminus \{p_S\}$, when $r = 1$. Moreover, $\text{vol}(X_{m,0}) = 2\pi^2 \approx 6.28\pi$ and $\text{vol}(X_{m,1}) = 8\pi$.

Let us just show that

$$\text{vol}(X_{m,1}) = \int_0^{2\pi} \int_0^\pi \sqrt{2 + 2 \cos \theta} \, d\theta d\phi = 2\sqrt{2}\pi \int_0^\pi \sqrt{1 + \cos \theta} \, d\theta$$

$$= -2\sqrt{2}\pi \int_{-1}^1 \frac{\sqrt{1 + t}}{\sqrt{1 - t^2}} \, dt = -2\sqrt{2}\pi \int_{-1}^1 \frac{1}{\sqrt{1 - t}} \, dt$$

$$= -4\sqrt{2}\pi \left[\sqrt{1 - t}\right]_{-1}^1 = 8\pi$$
is consistent with [9, Theorem 10]. The minimality in the homology class is proved in [3].

**Theorem 2.** For the meridian vector field \( X = X_{m,k} \) with \( k \geq 0 \), we may fix an orientation on \( M^* \) such that

\[
I_X(p_N) = 1 - k \quad I_X(p_S) = 1 + k.
\]

**Proof.** We may fix an orientation on \( D \) and deduce that \( \partial_\theta \), or any other \( X_{m,0} \), has index 1 at \( p_N \). Indeed, fixing a trivialization of \( \mathbb{S}^2 \) in a neighbourhood of \( p_N \), along any directed circle-of-latitude (col) around \( p_N \) the vector field \( \partial_\theta \) describes another entire circle identically, ie. describes the identity map of \( \mathbb{S}^1 \) after parallel transport to a base point of the col, which therefore gives a degree 1 self-map of \( \mathbb{S}^1 \). Taking a neighborhood of \( p_S \), the field \( \partial_\theta \) moves in the same way as before, even though the direction in any col close to \( p_S \) must be the opposite of the previous homotopic col. Either indices of \( \partial_\theta \) or \(-\partial_\theta\) are 1 at \( p_S \).

For \( X_{m,1} = W \), we have that \( p_N \) is a smooth point. Indeed, by definition, \( W \) extends as a unit vector to that point (notice it does not extend continuously to the South pole). The index at \( p_N \) is thus 0, and this follows also because, along any directed col around \( p_N \), the vector field \( W \) describes in its range a new circle rotating ‘clockwise’ once, ie. it gives a constant valued self-map of \( \mathbb{S}^1 \) after parallel transport to a base point of the col as above, which henceforth gives a degree 0 map (compare with \( \partial_\theta \)). Conversely, we reach the South pole with a degree 2 map. Indeed, conforming with orientation, one would have to rotate \( W \) twice ‘anti-clockwise’ to draw the symmetry with the North pole; hence the index at \( p_S \) is 2.

Continuing this way, for \( k \geq 2 \), we will find \( I(p_N) = 1 - k \). And, still with \( k > 0 \), proceeding to find the referred symmetry, achieved by unwinding our vector field 2k times, we find \( I(p_S) = 1 - (k - 2k) = 1 + k \). And the result follows. \( \blacksquare \)

Notice we have \( \chi(\mathbb{S}^2) = 2 \) as predicted by the Theorem of Poincaré-Hopf.

Let us recall from [4, Theorem 1.1] that any vector field \( X \) on \( M^* \) with radius 1 satisfies:

\[
(\pi + |I_X(p_S)| + |I_X(p_N)|) - 2)2\pi \leq \text{vol}(X).
\]

We have thus verified the ‘big index, big volume’ precept.

In the continuation of Proposition [4] the minimum volume \( 2\pi^2 \) is attained with \( k = 0 \), ie. with \( X_{m,0} \) in its domain \( M^* \). Minimality of \( X_{m,1} \) in its domain and homological class is conjectured in [9] and proven by [3]. From (33), we only get \( (\pi + 2 + 0 - 2)2\pi = 2\pi^2 < 8\pi \). Minimality depends on the topology of the domain and the vector fields. How these classes effectively rule challenges our understanding.

For any \( k > 0 \), we have \( (\pi + 1 + k - 1 - 2)2\pi = (\pi + 2k - 2)2\pi \leq \text{vol}(X_{m,k}) \).

Notice with \( k \geq 4 \), we get \( (\pi + 2k - 2)2\pi < (k - 1)2\pi^2 \leq \text{vol}(X_{m,k}) \) by Theorem [1].

Up to now the relative homology class of a vector field on \( M^* \) is determined by a unique index, namely, the integer \( k \). Based on the known cases \( k = 0,1 \) and Proposition [2] we may finally and firmly state a conjecture.
Conjecture 1. For each \( k \in \mathbb{Z}^+ \), the meridian type vector field \( X_{m,k} \) realizes minimal volume in its (relative) homology class.

4 – And new vector fields on the sphere

Finding solutions of the holomorphic equation (3) in \( M^* \) remains a local question. We shall see there are vector fields in an open region of \( S^2 \) with even less volume than the above.

We now consider the equations for unit vector fields \( Z \) which are parallel along the parallels; the latter being also known as the circles-of-latitude.

We return to the radius \( r \) 2-sphere.

Let \( Z = \frac{a}{r} \partial_\theta + \frac{b}{r \sin \theta} \partial_\phi \) with \( a^2 + b^2 = 1 \). Applying (19), the desired condition on \( Z \) translates into

\[
\nabla_\phi Z = 0 \iff \begin{cases}
a' \cos \theta - b \cos \theta = 0 \\
b' + a \cos \theta = 0
\end{cases} .
\]

(34)

The general solution follows:

\[
a = \sin \eta, \quad b = \cos \eta
\]

(35)

where

\[
\eta(\theta, \phi) = \phi \cos \theta + \phi_0, \quad \phi_0 \in \mathbb{R}.
\]

(36)

Since \( \eta(\theta, 2\pi) - \eta(\theta, 0) = 2\pi \cos \theta \notin 2\pi \mathbb{Z} \), it is only possible to have the vector field \( Z \) defined on \( M^* \setminus \{ \phi = 0 \} \), ie. \( M^* \) with one meridian removed.

Now we notice

\[
\begin{cases}
a'_\theta = -b \phi \sin \theta \\
b'_\theta = a \phi \sin \theta
\end{cases} .
\]

(37)

As usual, we consider the unit orthogonal \( Y = -\frac{b}{r} \partial_\theta + \frac{a}{r \sin \theta} \partial_\phi \). Recalling (20), we find

\[
\nabla_\theta Z = \frac{a'_\theta}{r} \partial_\theta + \frac{b'_\theta}{r \sin \theta} \partial_\phi = \phi \sin \theta Y.
\]

(38)

Hence

\[
A_0 = \langle \nabla Z, Y \rangle = \frac{a}{r} \langle \nabla_\theta Z, Y \rangle = \frac{a \phi \sin \theta}{r} , \quad A_1 = \langle \nabla Y, Z \rangle = -\frac{b \phi \sin \theta}{r}
\]

(39)

and

\[
\frac{A}{\sqrt{1 + |A|^2}} = -e^{-i\eta} \frac{\phi \sin \theta}{\sqrt{r^2 + \phi^2 \sin^2 \theta}}
\]

(40)

leading through easy computations to a conclusion.

Proposition 5. No integrability equation (4) or (5) is satisfied with \( X_\ell \).
Finally, by (1), the volume of a circles-of-latitude vector field is

$$\text{vol}(X_\ell) = r \int_D \sqrt{r^2 + \phi^2 \sin^2 \theta} \sin \theta \, d\phi d\theta = r \int_0^{2\pi} \int_0^\pi \sqrt{r^2 + y^2} \, dy \, d\theta. \quad (41)$$

Indeed, we call the above the circles-of-latitude or parallels type vector field ($r = 1$):

$$X_\ell = \sin(\phi \cos \theta + \phi_0) \partial_\theta + \frac{\cos(\phi \cos \theta + \phi_0)}{\sin \theta} \partial_\phi. \quad (42)$$

Finally a new minimum of the volume functional is achieved inside $D$.

**Theorem 3.** On the region $\Omega = \{(\theta, \phi) \in D : \phi \neq 0, \phi \sin^2 \theta < |\cos \theta|\}$, a circles of latitude type vector field has volume strictly lower than the minimal meridian type vector fields. More precisely, on $\Omega$,

$$\text{vol}(X_\ell) < \text{vol}(\partial_\theta) = \text{vol}_{\text{Euc}}(\Omega). \quad (43)$$

**Proof.** Recall from formula (29) with $k = 0$ that the volume of $\partial_\theta$ is equal to the Euclidean volume of the region of definition in $D$. Now the result is straightforward from (41) and $(1 + \phi^2 \sin^2 \theta) \sin^2 \theta = \sin^2 \theta + \phi^2 \sin^4 \theta < 1$.

With $\theta \in [0, \theta_0] \cup [\pi - \theta_0, \pi]$, where $\theta_0$ is the unique solution of $\frac{\sin \theta}{\sin^2 \theta} = 2\pi$, we do have $\phi \in [0, 2\pi[$. Here, the field $X_\ell$ almost draws a complete turn around itself when it goes around the parallels minus a point.

It is somehow surprising that there is just one volume functional concerning the $X_\ell$, unlike the case of $X_{m,k}$. They lack domain of definition and minimality equations, their volume is weak. Yet they are important.

A few important questions must be raised. Can one find a vector field with even less volume than $X_\ell$ in some equal volume subset of the 2-sphere? Is there a minimum
of $\text{vol}(X)$ per volume of its domain? What is the infimum and what does it depend on?

We end with a simple remark. It is possible to give a more concise definition involving the two types of vector fields $X_m, X_\ell$ briefly studied above.

Let the unit vector field $T = \frac{a}{r} \partial_\theta + \frac{b}{r \sin \theta} \partial_\phi$ with constant coefficients $a, b$ be defined on $M^*$. This is of course the case $k = 0$ meridian type vector field. The flow of $T = T_{a,b}$ integrates to a well-known family of curves, namely the so-called loxodromes or rhumb lines. Indeed, these curves go across every meridian with a constant angle $\angle(\partial_\theta, T)$. (Such is their original definition by Pedro Nunes in the XVIth century).

Now we define the vector fields of $T$-type as those $X_T$ which are parallel in the direction of $T_{a,b}$ for some $(a, b) \in \mathbb{S}^1$ fixed:

$$\nabla_T X_T = 0. \quad (44)$$

$T, X_{m,k}, X_\ell$ are particular cases of vector fields of this type. And from these we may define other just as well.

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