NEW q-EULER NUMBERS AND POLYNOMIALS
ASSOCIATED WITH p-ADIC q-INTEGRALS

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ABSTRACT. The purpose of this paper is to construct new q-Euler numbers and polynomials. Finally we will consider the Witt’s type formula associated with these q-Euler numbers and polynomials, and construct q-partial zeta functions and p-adic q-l-functions which interpolate new q-Euler numbers and polynomials at negative integers.

1. Introduction

The constants $E_k$ in the Taylor series expansion

$$\frac{2}{e^t+1} = \sum_{n=0}^{\infty} \frac{E_n t^n}{n!}$$

are known as the Euler numbers. The first few are $1, -\frac{1}{2}, 0, \frac{1}{4}, -\frac{1}{2}, \ldots$ and $E_{2k} = 0$ for $k = 1, 2, \ldots$. Those numbers play an important role in number theory. For example, the Euler zeta-function essentially equals a Euler numbers at negative integer:

$$\zeta_E(-k) = E_k \quad \text{for } k \geq 0,$$

where

$$\zeta_E(s) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^s}, \quad s \in \mathbb{C} \text{ (see [1, 2, 3, 4, 5, 6, 7, 8]).}$$

Recently the $q$-extensions of those Euler numbers and polynomials have been studied by many authors, cf. [1,2,3].

2000 Mathematics Subject Classification. 11S80, 11B68, 11M99.
In [8, 9], Ozden and Simsek have studied \((h, q)\)-extensions of Euler numbers and polynomials by using \(p\)-adic \(q\)-integral on \(\mathbb{Z}_p\). From their \((h, q)\)-extensions of Euler numbers and polynomials, they have derived \((h, q)\)-extensions of Euler zeta function and they also gave some interesting relations between their \((h, q)\)-Euler numbers and \((h, q)\)-Euler zeta functions, see [8, 9]. Thought this paper \(\mathbb{Z}, \mathbb{Z}_p, \mathbb{Q}_p\) and \(\mathbb{C}_p\) will denote the ring of integers, the ring of \(p\)-adic rational integers, the the field of \(p\)-adic rational numbers and completion of the algebraic closure of \(\mathbb{Q}_p\), respectively. Let \(v_p\) be the normalized exponential valuation of \(\mathbb{C}_p\) with \(|p|_p = p^{-v_p(p)} = \frac{1}{p}\). When one talks of \(q\)-extension, \(q\) is variously considered as an indeterminate, a complex number \(q \in \mathbb{C}\), or a \(p\)-adic number \(q \in \mathbb{C}_p\). If \(q \in \mathbb{C}_p\), then we normally assume \(|1 - q|_p < 1\). If \(q \in \mathbb{C}\), then we assume that \(|q| < 1\). In this paper we use the following notations:

\[
[x]_q = \frac{1 - q^x}{1 - q} \quad \text{and} \quad [x]_{-q} = \frac{1 - (q)^x}{1 + q}, \quad \text{cf. [3]}
\]

Let \(d\) be a fixed integer, and let

\[
X = X_d = \lim_{N \to \infty} \left(\mathbb{Z}/dp^N\mathbb{Z}\right), \quad X^* = \bigcup_{0 < a < dp} a + dp\mathbb{Z}_p,
\]

\[
a + dp^N\mathbb{Z}_p = \{x \in X \mid x \equiv a \pmod{dp^N}\},
\]

where \(a \in \mathbb{Z}\) lies in \(0 \leq a < dp^N\). Let \(UD(\mathbb{Z}_p)\) be the space of uniformly differentiable function on \(\mathbb{Z}_p\). For \(f \in UD(\mathbb{Z}_p)\), the \(p\)-adic \(q\)-integral was defined by

\[
I_q(f) = \int_{\mathbb{Z}_p} f(x)d\mu_q(x) = \int_X f(x)d\mu_q(x)
\]

\[
= \lim_{N \to \infty} \frac{1}{[dp^N]_q} \sum_{x=0}^{dp^N-1} f(x)q^x \quad \text{for} \quad |1 - q|_p < 1.
\]

In [2, 6] the bosonic integral was considered from a more physical point of view to the bosonic limit \(q \to 1\) as follows:

\[
I_1(f) = \lim_{q \to 1} I_q(f) = \int_{\mathbb{Z}_p} f(x)d\mu_1(x) = \lim_{N \to \infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x).
\]

Furthermore, we can consider the fermionic integral in contrast to the conventional “bosonic.” That is,

\[
I_{-1}(f) = \int_{\mathbb{Z}_p} f(x)d\mu_{-1}(x), \quad \text{see [6].}
\]
From this, we derive

\[ I_{-1}(f_1) + I_{-1}(f) = 2f(0), \]

where \( f_1(x) = f(x + 1). \) Also we have

\[ I_{-1}(f_n) + (-1)^{n-1}I_{-1}(f) = 2 \sum_{x=0}^{n-1} (-1)^{n-1-x} f(x), \]

where \( f_n(x) = f(x+n) \) and \( n \in \mathbb{Z}^+. \) For \( |1-q|_p < 1, \) we consider fermionic \( p \)-adic \( q \)-integral on \( \mathbb{Z}_p \) which is the \( q \)-extension of \( I_{-1}(f) \) as follows:

\[ I_{-q}(f) = \int_{\mathbb{Z}_p} f(x)d\mu_{-q}(x) = \lim_{N \to \infty} \left[ \frac{1}{dpN} \right]_{-q} \sum_{x=0}^{pN-1} f(x)(-q)^x, \]

see [1-11]. By using (1), Ozen and Simsek studied twisted \((h, q)\)-Euler numbers and polynomials and twisted generalized \((h, q)\)-Euler numbers attached to \( \chi, \) see [8, 9].

In this paper, we consider \( q \)-Euler numbers and polynomials which are different the \( q \)-Euler numbers and polynomials of Ozen-Simsek. Finally, we will give some relations between these \( q \)-Euler numbers and polynomials, and construct \( q \)-partial zeta functions and \( p \)-adic \( q \)-functions which interpolate new \( q \)-Euler numbers and polynomials at negative integers.

\section{\textbf{q-extensions of Euler numbers and polynomials}}

From [6], we can derive the following formula:

\[ qI_{-q}(f_1) + I_{-q}(f) = [2]_q f(0), \]

where \( f_1(x) \) is translation with \( f_1(x) = f(x + 1). \)

If we take \( f(x) = e^{tx}, \) then we have \( f_1(x) = e^{t(x+1)} = e^{tx}e^t. \) From (4), we derive

\[ (qe^t + 1)I_{-q}(e^{tx}) = [2]_q. \]

Hence we obtain

\[ I_{-q}(e^{tx}) = \int_{\mathbb{Z}_p} e^{tx}d\mu_{-q}(x) = \frac{[2]_q}{qe^t + 1}. \]
We now define

\[ [2]_q \frac{q e^t + 1}{qe^t + 1} = \sum_{n=0}^{\infty} E_{n,q} \frac{t^n}{n!}. \]

By (5) and (6), we see that

\[ \int_{\mathbb{Z}_p} x^n d\mu_{-q}(x) = E_{n,q}. \]

From (4), we also note that

\[ \int_{\mathbb{Z}_p} e^{(x+y)t} d\mu_{-q}(y) = \frac{[2]_q}{qe^t + 1} e^{xt}. \]

In view of (7), we can consider $q$-extension of Euler polynomials as follows:

\[ \frac{[2]_q}{qe^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{n!}. \]

By (5), (6), (7) and (8), we obtain the following theorem:

**Theorem 1.** (Witt’s formula) For $q \in \mathbb{C}_p$ with $|1 - q|_p < 1$,

\[ \int_{\mathbb{Z}_p} x^n d\mu_{-q}(x) = E_{n,q} \quad \text{and} \quad \int_{\mathbb{Z}_p} (x+y)^n d\mu_{-q}(y) = E_{n,q}(x). \]

Note that $\lim_{q \to 1} E_{n,q} = E_n$ and $\lim_{q \to 1} E_{n,q}(x) = E_n(x)$, where $E_n$ and $E_n(x)$ are Euler numbers and polynomials.

By Theorem 1, we easily see that $E_{n,q}(x) = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} E_{k,q}$. For $n \in \mathbb{Z}^+$, let $f_n(x) = f(x + n)$. Then we have

\[ q^n I_{-q}(f_n) + (-1)^{n-1} I_{-q}(f) = [2]_q \sum_{l=0}^{n-1} (-1)^n t^{n-1} q^l f(l), \text{ see } [2]. \]

If $n$ is odd positive integer, we have

\[ q^n I_{-q}(f_n) + I_{-q}(f) = [2]_q \sum_{l=0}^{n-1} (-1)^l q^l f(l). \]
Let \( \chi \) be a Dirichlet’s character with conductor \( d (= \text{odd}) \in \mathbb{Z}^+ \). If we take \( f(x) = \chi(x)e^{tx} \), then we have \( f_d(x) = f(x + d) = \chi(x)e^{td}e^{tx} \). From (10), we derive

\[
\int_X \chi(x)e^{tx}d\mu_q(x) = \frac{[2]q \sum_{a=1}^{d} (-1)^aq^a \chi(a)e^{ta}}{qd^e + 1}.
\]

In view of (11), we can also consider the generalized Euler numbers attached to \( \chi \) as follows:

\[
\sum_{a=1}^{d} (-1)^aq^a \chi(a)e^{ta} = \sum_{n=0}^{\infty} E_{n, \chi, q} t^n.
\]

From (11) and (12), we derive the following Witt’s formula:

**Theorem 2.** Let \( \chi \) be a Dirichlet’s character with conductor \( d (= \text{odd}) \in \mathbb{Z}^+ \). Then we have

\[
\int_X \chi(x)x^nd\mu_q(x) = E_{n, \chi, q}
\]

for \( n \geq 0 \).

### 2. \( q \)-extension of Euler zeta functions

For \( q \in \mathbb{C} \) with \( |q| < 1 \), let

\[
F_q(t, x) = \frac{[2]q e^{tx}}{qe^t + 1} = \sum_{n=0}^{\infty} \frac{E_{n, q}(x)}{n!} t^n
\]

for \( |t + \log q| < \pi \). Then we see that \( F_q(t, x) \) is an analytic function on \( \mathbb{C} \). From (14), we can derive the following expansion:

\[
\frac{[2]q e^{tx}}{qe^t + 1} = [2]q \sum_{n=0}^{\infty} q^n (-1)^n e^{(n+x)t}.
\]

Thus we have

\[
E_{k, q}(x) = \frac{d^k}{dt^k} F_q(t, x) \bigg|_{t=0} = [2]q \sum_{n=0}^{\infty} q^n (-1)^n (n + x)^k, \quad k \geq 0
\]

and

\[
E_{k, q} = \frac{d^k}{dt^k} F_q(t, 0) \bigg|_{t=0} = [2]q \sum_{n=0}^{\infty} q^n (-1)^n n^k, \quad k \geq 0.
\]
Definition 3. For \( s \in \mathbb{C} \), define

\[
\zeta_{q,E}(s, x) = [2]_q \sum_{n=0}^{\infty} \frac{(-1)^n q^n}{(n + x)^s}, \quad \zeta_{q,E}(s) = [2]_q \sum_{n=1}^{\infty} \frac{(-1)^n q^n}{n^s}.
\]

Note that \( \zeta_{q,E}(s, x) \) and \( \zeta_{q,E}(s) \) are analytic functions in the whole complex \( s \)-plane.

By (15) and (16), we obtain the following:

**Theorem 4.** Let \( n \in \mathbb{Z}^+ \cup \{0\} \). Then we have

\[
\zeta_{q,E}(-n, x) = E_{n,q}(x), \quad \zeta_{q,E}(-n) = E_{n,q}.
\]

Let \( \chi \) be a primitive Dirichlet’s character with conductor \( d (= \text{odd}) \in \mathbb{Z}^+ \). Then the generalized \( q \)-Euler numbers attached to \( \chi \) are defined as

\[
(17) \quad F_{q,\chi}(t) = [2]_q \sum_{a=1}^{d} (-1)^a q^a \chi(a) e^{ta} \frac{t^n}{q^d e^{td} + 1} = \sum_{n=0}^{\infty} E_{n,\chi,q} \frac{t^n}{n!},
\]

where \( |t + \log q| < \frac{\pi}{d} \).

From (17), we note that

\[
(18) \quad F_{q,\chi}(t) = [2]_q \sum_{a=1}^{d} (-1)^a q^a \chi(a) e^{ta} \sum_{l=0}^{\infty} q^{ld} e^{ldt} (-1)^l
\]

By (17) and (18), we easily see that

\[
(19) \quad E_{k,\chi,q} = \left. \frac{d^k}{dt^k} F_{q,\chi}(t) \right|_{t=0} = [2]_q \sum_{n=0}^{\infty} (-1)^n q^n \chi(n) n^k.
\]

**Definition 5.** Let \( \chi \) be a primitive Dirichlet’s character with conductor \( d (= \text{odd}) \in \mathbb{Z}^+ \). Then we define the \( l_q \)-function as follows:

\[
l_q(s, \chi) = [2]_q \sum_{n=1}^{\infty} \frac{(-1)^n q^n \chi(n)}{n^s}, \quad s \in \mathbb{C}.
\]

Note that \( l_q(s, \chi) \) is an analytic function in the whole complex \( s \)-plane.

From (19) and Definition 5, we derive the following:
Theorem 6. For \( n \in \mathbb{Z}^+ \cup \{0\} \), we have

\[
I_q(-n, \chi) = E_{n, \chi, q}.
\]

Let us consider a partial \( q \)-zeta function as follows:

\[
H_q(s, a|F) = [2]_q \sum_{m\equiv a \pmod{F}, m>0} \frac{(-1)^m q^m}{m^s}
\]

\[
= [2]_q (-1)^a q^a \sum_{n=0}^{\infty} (-1)^n F q^n \frac{(n + \frac{a}{F})^s}{F^s}
\]

\[
= (-1)^a q^a \frac{[2]_q}{F^s} \zeta_F(s, \frac{a}{F}),
\]

where \( F (= \text{odd}) \) is positive integers with \( 0 < a < F \). Let \( \chi \neq 1 \) be the Dirichlet’s character with conductor \( F (= \text{odd}) \). Then we have

\[
l_q(s, \chi) = \sum_{a=1}^{F} \chi(a) H_q(s, a|F)
\]

for \( s \in \mathbb{C} \). The function \( H_q(s, a|F) \) is an analytic function in whole complex plane. For \( n \in \mathbb{Z}^+ \), we have

\[
H_q(-n, a|F) = (-1)^a q^a F^n \frac{[2]_q}{[2]_q F^s} E_{n, q^F}(\frac{a}{F}).
\]

Note that

\[
E_{n, q^F}(\frac{a}{F}) = \sum_{k=0}^{n} \binom{n}{k} \left( \frac{a}{F} \right)^{n-k} E_{k, q^F}.
\]

By using (22) and (23) we have

\[
H_q(-n, a|F) = (-1)^a q^a F^n \frac{[2]_q}{[2]_q F^s} \sum_{k=0}^{n} \binom{n}{k} \left( \frac{a}{F} \right)^{n-k} E_{k, q^F}.
\]
We now modify a partial \( q \)-zeta function as follows:

\[
H_q(s, a|F) = (-1)^aq^a a^{-s} \frac{[2]_q}{[2]_q F^a} \sum_{k=0}^{\infty} \binom{-s}{k} \left( \frac{F}{a} \right)^k E_{k,q^F}
\]

for \( s \in \mathbb{C} \). From (21) and (24'), we note that

\[
l_q(s, \chi) = \frac{[2]_q}{[2]_q F} \sum_{a=1}^{F} (-1)^a \chi(a) q^a a^{-s} \sum_{k=0}^{\infty} \binom{-s}{k} \left( \frac{F}{a} \right)^k E_{k,q^F}
\]

for \( s \in \mathbb{C} \). By (25), we easily see that

\[
l_q(s, \chi) = \frac{[2]_q}{[2]_q F} \sum_{a=1}^{F} (-1)^a \chi(a) q^a \left\{ a^{-s} + a^{-s} \sum_{k=1}^{\infty} \binom{-s}{k} E_{k,q^F} \left( \frac{F}{a} \right)^k \right\}.
\]

From the Taylor expansion at \( s = 0 \), we have

\[
l_q(0, \chi) = \frac{[2]_q}{[2]_q F} \sum_{a=1}^{F} (-1)^a q^a \chi(a),
\]

and

\[
l_q(1, \chi) = \frac{[2]_q}{[2]_q F} \sum_{a=1}^{F} (-1)^a \frac{q^a}{a} \chi(a) \left\{ 1 + \sum_{k=1}^{\infty} (-1)^k E_{k,q^F} \left( \frac{F}{a} \right)^k \right\}.
\]

3. \( p \)-adic interpolating function for \( q \)-Euler numbers

We shall consider the \( p \)-adic analogue of the \( l_q \)-functions which are introduced in the previous section. Throughout this section we assume that \( p \) is an odd prime. Let \( \omega \) be denoted as the Teichmüller character having conductor \( p \). For an arbitrary character \( \chi \), let \( \chi_n = \chi \omega^{-n} \), where \( n \in \mathbb{Z} \), in sense of the product of characters. Let \( \langle a \rangle = \omega^{-1}(a)a = \frac{q}{\omega(a)} \).

Let \( \chi \) be the Dirichlet’s character with conductor \( d (= \text{odd}) \) and let \( F \) be a positive integral multiple of \( p \) and \( d \). Now, we define the \( p \)-adic \( l_q \)-functions as follows:

\[
l_{p,q}(s, \chi) = \frac{[2]_q}{[2]_q F} \sum_{a=1}^{F} (-1)^a \chi(a) \langle a \rangle^{-s} \sum_{k=0}^{\infty} \binom{-s}{k} \left( \frac{F}{a} \right)^k E_{k,q^F}.
\]
Then \( l_q(s, \chi) \) is an analytic function in \( D = \{ s \in \mathbb{C}_p \mid |s|_p < p^{1 - \frac{1}{d}} \} \) since \( \langle a \rangle^{-s} \) and \( \sum_{k=0}^{\infty} \binom{-s}{k} \left( \frac{a}{F} \right)^k E_{k,q,F} \) are analytic functions in \( D \), cf. [5,6,7,9].

We set

\[
H_{p,q}(s, a|F) = (-1)^a q^a \langle a \rangle^{-s} \frac{[2]_q}{[2]_{q^d}} \sum_{k=0}^{\infty} \binom{-s}{k} \left( \frac{F \chi(a)}{a} \right)^k E_{k,q,F}.
\]

Thus, by (24), note that

\[
H_{p,q}(-n, a|F) = \omega^{-n}(a) H_q(-n, a|F).
\]

for \( n \in \mathbb{Z}^+ \). We also consider the \( p \)-adic analytic function which interpolates \( q \)-Euler number at negative integer as follows:

\[
l_{p,q}(s, \chi) = \sum_{\substack{a=1 \atop p|a}}^F \chi(a) H_{p,q}(s, a|F).
\]

For \( d (=\text{odd}) \in \mathbb{Z}^+ \), by (14) and (17), we note that

\[
\frac{1}{[2]_q} F_{q,\chi}(t) = \frac{1}{[2]_{q^d}} \sum_{a=1}^{d} (-1)^a q^a \chi(a) F_{q^a}(dt, \frac{a}{d})
\]

\[
= \sum_{n=0}^{\infty} \left[ d^n \frac{1}{[2]_{q^d}} \sum_{a=1}^{d} (-1)^a q^a \chi(a) E_{n,q^d}(\frac{a}{d}) \right] \frac{t^n}{n!}.
\]

Then we have

\[
\frac{1}{[2]_q} E_{n,\chi,q} = d^n \frac{1}{[2]_{q^d}} \sum_{a=1}^{d} (-1)^a q^a \chi(a) E_{n,q^d}(\frac{a}{d}).
\]

In particular, if \( F = dp \), then we have

\[
\frac{1}{[2]_q} E_{n,\chi,q} = F^n \frac{1}{[2]_{q^F}} \sum_{a=1}^{F} (-1)^a q^a \chi(a) E_{n,q^F}(\frac{a}{F}).
\]

(32')
Indeed, by (31), it is sufficient to show that

\[
\frac{1}{[2]q^F} \sum_{a=1}^{F} (-1)^a q^a \chi(a) F_{q^F}(Ft, a, F) = \sum_{a=1}^{F} (-1)^a q^a \chi(a) \frac{e^{t \chi(a)}}{q^F e^{Ft} + 1} \\
= \sum_{b=1}^{d} \sum_{c=0}^{p-1} (-1)^{b+cd} \chi(b + cd) q^{b+cd} \frac{e^{t (b+cd)}}{q^F e^{Ft} + 1} \\
= \sum_{b=1}^{d} (-1)^b q^b \chi(b) \frac{e^{tb}}{q^F e^{Ft} + 1} \sum_{c=0}^{p-1} (-1)^c (q^d e^{dt})^c \\
= \frac{1}{[2]q^d} \sum_{b=1}^{d} (-1)^b q^b \chi(b) F_{q^d} (dt, b, \frac{b}{d}).
\]

Thus, since \( \chi_n = \chi \omega^{-n} \), by (29) and (32'), we obtain

\[
lpq(-n, \chi) = \sum_{a=1}^{F} \chi(a) H_{p,q}(-n, a|F) \\
= \sum_{a=1}^{F} \chi_n(a) H_q(-n, a|F) \\
(33)
= F^n \frac{[2]q}{[2]q^F} \sum_{a=1}^{F} \chi_n(a)(-1)^a q^a E_{n,q^F} \left( \frac{a}{F} \right) \\
= F^n \frac{[2]q}{[2]q^F} \sum_{a=1}^{F} \chi_n(a)(-1)^a q^a E_{n,q^F} \left( \frac{a}{F} \right) \\
- F^n \frac{[2]q}{[2]q^F} \sum_{a=1}^{F} \chi_n(pa)(-1)^{pq} q^{pa} E_{n,q^F} \left( \frac{pa}{F} \right).
\]

From Theorem 2 and (32'), we see that

\[
\frac{1}{[2]q} E_{n,\chi,q} = \frac{1}{[2]q^F} F^n \sum_{a=1}^{F} (-1)^a q^a \chi(a) E_{n,q^F} \left( \frac{a}{F} \right) = \frac{1}{[2]q} \int_X \chi(x)x^n d\mu_q(x).
\]
and

\[
\left[\frac{E}{q}\right]_q E_{n,\chi,q^p} = \left(\frac{F}{p}\right)^n \left[\frac{E}{q}\right]_q \sum_{a=1}^{F} (-1)^a (q^p)^a \chi(a) E_{\left\lfloor \frac{a}{p} \right\rfloor, \chi, q^p} \left(\frac{a}{p}\right).
\]

Also, by (34) and (35), we have

\[
\int_{pX} \chi(x)x^n d\mu_{q^p}(x) = \int_{X} \chi(px)(px)^n d\mu_{q^p}(px)
= \chi(p)p^n \left[\frac{E}{q}\right]_q F^n \sum_{a=1}^{F} (-1)^a (q^p)^a \chi(a) E_{\left\lfloor \frac{a}{p} \right\rfloor, \chi, q^p} \left(\frac{a}{F}\right)
= \chi(p)p^n \left[\frac{E}{q}\right]_q E_{n,\chi,q^p},
\]

since \(d\mu_{q^p}(px) = \left[\frac{E}{q}\right]_q d\mu_{q^p}(x)\). Therefore, we obtain the following theorem:

**Theorem 7.** Let \(F(=\text{odd})\) be a positive integral multiple of \(p\) and \(d = (d, \chi)\), and let

\[
l_{p,q}(s, \chi) = \left[\frac{E}{q}\right]_q \sum_{a=1}^{F} (-1)^a \chi(a) q^a \langle a \rangle^{-s} \sum_{k=0}^{\infty} \left(\frac{F}{k}\right) \left(\frac{a}{F}\right)^k E_{k,q^p}.
\]

Then we have

(a) \(l_{p,q}(s, \chi)\) analytic in \(D = \{s \in \mathbb{C} \mid |s|_p < p^{1 - \frac{1}{p - 1}}\}\).

(b) \(l_{p,q}(-n, \chi) = E_{n,\chi,q^p} - p^n \chi_n(p) \left[\frac{E}{q}\right]_q E_{n,\chi,q^p}\) for \(n \in \mathbb{Z}^+\).

(c) For \(n \in \mathbb{Z}^+\),

\[
l_{p,q}(-n, \chi) = \int_{X} \chi_n(x)x^n d\mu_{q^p}(x).
\]

**Corollary 8.** Let \(F(=\text{odd})\) be a positive integral multiple of \(p\) and \(d = (d, \chi)\), and let

\[
l_p(s, \chi) = \sum_{a=1}^{F} (-1)^a \chi(a) \langle a \rangle^{-s} \sum_{k=0}^{\infty} \left(\frac{F}{k}\right) \left(\frac{a}{F}\right)^k E_k, \text{ see } [10].
\]
Then we have

(a) \( l_p(s, \chi) \) analytic in \( D = \{ s \in \mathbb{C}_p \mid |s|_p < p^{1-p-1} \} \).

(b) \( l_{p,q}(-n, \chi) = E_{n,\chi_n} - p^n \chi_n(p)E_{n,\chi_n} \) for \( n \in \mathbb{Z}^+ \).

(c) \( l_p(s, \chi) = \int_{X^*} \chi_n(x)x^{-s}d\mu_{-1}(x) \). Observe that for \( n \in \mathbb{Z}^+ \),

\[
l_p(-n, \chi) = \int_{X^*} \chi_n(x)x^n d\mu_{-1}(x).
\]

**Remark 9.** In the recent paper (see [9]), Ozden and Simsek have studied the \((h, q)\)-extension of twisted Euler numbers. However, these \((h, q)\)-extension of twisted Euler numbers and generating function do not seem to be natural ones; in particular, these numbers cannot be represented as a nice Witt’s type formula for the \(p\)-adic invariant integral on \( \mathbb{Z}_p \) and the generating function does not seem to be simple and useful for deriving many interesting identities related to the extension of Euler numbers. By this reason, we consider new \(q\)-extensions of Euler numbers and polynomials which are different. Our \(q\)-extensions of Euler numbers and polynomials to treat in this paper can be represented by \(p\)-adic \(q\)-fermionic integral on \( \mathbb{Z}_p \) and this integral representation also can consider as Witt’s type formula for \(q\)-extensions of Euler numbers and polynomials.

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