(2+1)-dimensional gravity in Weyl integrable spacetime

J E Madriz Aguilar¹, C Romero², J B Fonseca Neto², T S Almeida² and J B Formiga³

¹ Departamento de Matemáticas, Centro Universitario de Ciencias Exatas e Ingenierías (CUCEI), Universidad de Guadalajara (UdG), Av. Revolución 1500, 44430, Guadalajara, Jalisco, México
² Departamento de Física, Universidade Federal da Paraíba, Caixa Postal 5008, 58059-970 João Pessoa, PB, Brazil
³ Centro de Ciências da Natureza, Universidade Estadual do Piauí, Caixa Postal 381, 64002-150 Teresina, PI, Brazil

E-mail: jemadriz@fisica.ufpb.br, madriz@mdp.edu.ar, cromero@pq.cnpq.br, cromero@fisica.ufpb.br and jfonseca@fisica.ufpb.br

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Abstract
We investigate (2+1)-dimensional gravity in a Weyl integrable spacetime (WIST). We show that, unlike general relativity, this scalar-tensor theory has a Newtonian limit for any dimension $n \geq 3$ and that in three dimensions the congruence of world lines of particles of a pressureless fluid has a non-vanishing geodesic deviation. We present and discuss a class of static vacuum solutions generated by a circularly symmetric matter distribution that for certain values of the parameter $\omega$ corresponds to a spacetime with a naked singularity at the center of the matter distribution. We interpret all these results as being a direct consequence of the spacetime geometry.

Keywords: lower dimensional gravity, Weyl geometry, alternative theories of gravity

1. Introduction
During the past three decades a great deal of effort has gone to the investigation into gravity theories in (2+1) spacetime dimensions [1]. It seems that part of this interest has been motivated by the fact that in this dimensionality Einstein gravity presents some odd peculiarities. First, it has no propagating gravity modes, which implies that its quantum version contains no gravitons. This is due to the fact that in (2+1) dimensions spacetime is flat outside sources. For point sources, gravity manifests itself as global topological defects rather than a
local geometrical curvature. Second, in three-dimensional spacetime Einstein theory does not reduce to Newtonian gravity in the static weak-field regime [2]. Another interest in (2+1)-dimensional gravity comes from the fact that studying classical models in lower dimensions has often been helpful to understand their quantum version [3].

The failure of (2+1)-dimensional Einstein gravity to provide a relativistic generalization of two-dimensional Newtonian gravity has led some authors to investigate the same problem in other theories of gravity. It has been proved that at least in two distinct gravitational theories the Newtonian limit may be recovered. These are the Brans–Dicke theory and the teleparallel gravity, a metric theory in which gravity is purely due to torsion [4]. Another approach to address this question is to consider (2+1)-dimensional gravity as obtained through a dimensional reduction of four-dimensional Einstein gravity [5]. The basic motivation in many of these attempts is to seek alternative ways beyond the scope of general relativity to gain more insight into some problems that do not seem to have a satisfactory solution in the context of Einstein theory. With the same idea in mind we consider the subject in the context of another theory of gravity, namely, the Weyl integrable spacetime theory (WIST) [6]. In this approach, the geometry of spacetime is not Riemannian, but corresponds to what is called a Weyl integrable geometry, a particular version of the geometry developed by Weyl in 1918 in connection with his gravitational theory [7]. We show that, in addition to leading to a Newtonian limit, WIST in (2+1) dimensions present some interesting properties that are not shared by Einstein theory, such as geodesic deviation between particles in a dust distribution and the existence of naked singularities.

The gravitational theory developed by Weyl is one of the first attempts to unify gravity and electromagnetism. Its geometrical structure is based on a simple generalization of Riemannian geometry. However, Weyl gravity, although admirably elegant and ingenious, turned out to be problematic as a physical theory. The first objections to the theory came from Einstein, who argued that in a non-integrable Weyl geometry the existence of sharp spectral lines in the presence of an electromagnetic field would not be possible since atomic clocks would depend on their past history [8]. Later, it was found that a particular version of Weyl geometry, known as Weyl integrable geometry, is not subject to the well-known criticism raised by Einstein, and since then has attracted the attention of some cosmologists [6]. In fact, in the opinion of some authors, Weyl geometrical theory ‘contains a suggestive formalism and may still have the germs of a future fruitful theory’ [9].

The paper is organized as follows. In sections 2 and 3 we give a brief account of the Weyl geometry and introduce the formalism of a specific theory of gravity known as Weyl integrable spacetime (WIST). The (2+1)-dimensional version of this theory is presented in section 4. In section 5, we show that WIST has a Newtonian limit for any dimension \( n \geq 3 \). We proceed in section 6 to prove that, unlike (2+1)-dimensional general relativity, in three-dimensional Weyl gravity the congruence of world lines of particles of a pressureless fluid has a non-vanishing geodesic deviation. In section 7, we present and discuss a class of static vacuum solutions generated by a circularly symmetric matter distribution. Finally, we conclude with some remarks in section 8.

2. Weyl geometry

The basic assumption of Weyl geometry is that, unlike what is assumed in Riemannian geometry, the covariant derivative of the metric tensor \( g \) is not zero, and is given by
where $\sigma_a$ denotes the components of a one-form field $\sigma$ in a local coordinate basis. This is a generalization of the Riemannian condition of compatibility between the connection $\nabla$ and $g$, which is equivalent to the requirement that the length of a vector is not altered by parallel transport [8]. If $\sigma$ is an exact form, i.e., $\sigma = d\phi$, where $\phi$ is a scalar field, then we say that we have an integrable Weyl geometry. A differentiable manifold $M$ endowed with both a metric $g$ and a Weyl scalar field $\phi$ will constitute a Weyl frame. It is important to note here that the Weyl condition (1) does not change when we go from one frame $(M, g, \phi)$ to another frame $(M, \bar{g}, \bar{\phi})$ by performing the following transformations in $g$ and $\phi$:

$$\bar{g} = e^f g,$$

$$\bar{\phi} = \phi + f,$$

where $f$ is an arbitrary scalar function defined on $M$.

In close analogy to Riemann geometry, it is easy to show that the condition (1) completely determines the Weyl connection $\bar{\nabla}$ (which is assumed to be symmetric) in terms of the metric $g$ and the Weyl one-form field $\sigma$. Indeed, a simple calculation shows that one can express the components of the affine connection with respect to an arbitrary vector basis completely in terms of the components of $g$ and $\sigma$:

$$\Gamma^\alpha_{\beta\lambda} = \left\{ \alpha \right\} - \frac{1}{2} \delta^{\alpha\mu} \left[ g_{\beta\lambda} \sigma_\mu + g_{\mu\lambda} \sigma_\beta - g_{\beta\mu} \sigma_\lambda \right].$$

where $\left\{ \alpha \right\}$ represents the Christoffel symbols.

The geometric properties of Weyl parallel transport become more clear from the following proposition. Let $M$ be a differentiable manifold with an affine connection $\nabla$, a metric $g$ and a Weyl field of one-forms $\sigma$. If $\nabla$ is compatible with $g$ in the sense of Weyl geometry, that is, if (1) holds, then for any curve $\alpha = \alpha(\lambda)$ and any pair of parallel transported vector fields $V$ and $U$ along $\alpha$, we have

$$\frac{d}{d\lambda} g(V, U) = \Gamma^\alpha_{\beta\lambda} \frac{d\alpha}{d\lambda} g(V, U),$$

where $\frac{d}{d\lambda}$ denotes the vector tangent to $\alpha$.

If we integrate this equation along the curve $\alpha$, starting from a point $P_0 = \alpha(\lambda_0)$, we obtain [8]

$$g(V(\lambda), U(\lambda)) = g(V(\lambda_0), U(\lambda_0)) e^{\int_{\lambda_0}^{\lambda} \Gamma^\alpha_{\beta\lambda} \frac{d\alpha}{d\lambda} d\lambda}.$$  

Setting $U = V$ and denoting by $L(\lambda)$ the length of the vector $V(\lambda)$ at a point $P = \alpha(\lambda)$ of the curve, it is easy to check that in a local coordinate system $\{x^\alpha\}$ equation (5) becomes

$$\frac{dL}{d\lambda} = \frac{\sigma_\alpha}{2} \frac{dx^\alpha}{d\lambda} L.$$

Let us consider now the set of all closed curves $\alpha : [a, b] \to M$, i.e., with $\alpha(a) = \alpha(b)$. Then, it follows that

$$g(V(b), U(b)) = g(V(a), U(a)) e^{\int_a^b \frac{d}{d\lambda} d\lambda}.$$

Throughout this paper our convention is that Greek indices take values from 0 to $n - 1$, $n$ being the spacetime dimension.
We thus see that it is the presence of the integral \( \int_0^1 \sigma(\frac{d}{d\lambda})d\lambda \) that is responsible for the difference between the readings of two identical clocks, according to general relativity, which follows different paths. From Stokes’ theorem, if \( \sigma \) is an exact form, there exists a scalar function \( \phi \), such that \( \sigma = d\phi \), which then implies

\[
\oint s l l \sigma \bigg( \frac{d}{d\lambda} \bigg)\lambda = 0
\]

for any loop. Therefore, in this case we conclude that the integral \( e^{-m(\frac{d}{d\phi})d\psi} \) does not depend on the path of integration. Because it is this integral that regulates the way clocks run, this particular version of Weyl geometry is not subject to the aforementioned objection put forward by Einstein \([8]\). In the next section, we shall consider a theory of gravity (WIST) formulated in a Weyl integrable manifold.

### 3. Weyl integrable spacetime theory in \( n \) dimensions

In Weyl integrable spacetime theory it is assumed that, in \( n \) dimensions, the dynamics of the gravitational field is given by the following action \([6]\):

\[
(n)\mathcal{S} = \int d^n x \sqrt{|g|} \left[ \mathcal{R} + \omega \phi \phi^{\alpha} + \kappa_n \epsilon^{\alpha \beta / 2} L_m \right]
\]

where \( \omega \) is an arbitrary coupling constant, \( \phi_\alpha \) denotes the derivative \( \frac{\partial \phi}{\partial x^\alpha} \) of the Weyl scalar field \( \phi \), \( \mathcal{R} \) is the Weylian Ricci scalar, \( L_m \) is the Lagrangian of matter, \( \kappa_n \) denotes the Einstein constant in \( n \) dimensions and, as usual, \( |g| \) indicates the absolute value of the determinant of the metric tensor. It is important to note here that \( L_m \) is constructed by following the so-called Weyl minimal coupling prescription \([10]\). This means that it will be assumed that \( L_m \) depends on \( \phi, g_{\mu \nu} \) and the matter fields, here generically designated by \( \xi \), its form being obtained from the special theory of relativity through the ‘minimum coupling’ prescription \( \eta_{\mu \nu} \rightarrow e^{-\phi} g_{\mu \nu} \), and \( \partial_{\mu} \rightarrow \nabla_{\mu} \), where \( \nabla_\mu \) denotes the covariant derivative with respect to the Weyl affine connection. If we designate the Lagrangian of the matter fields in special relativity by \( L_m^{sr} = L_m(\eta, \xi, \nabla \xi) \), then the form of \( L_m \) will be given by the rule \( L_m(g, \phi, \xi, \nabla \xi) \equiv L_m^{sr}(e^{-\phi} g, \xi, \nabla \xi) \). As can be easily seen, these rules also ensure the invariance under Weyl transformations of part of the action that is responsible for the coupling of matter with the gravitational field, and, at the same time, reproduce the principle of minimal coupling adopted in general relativity when we set \( \phi = 0 \), that is, when we go to the Riemann frame by a Weyl transformation.

It is interesting to rewrite the above action in Riemannian terms. This is done by expressing the Weyl Ricci scalar \( \mathcal{R} \) in terms of the usual Ricci scalar \( R \), which gives

\[
\mathcal{R} = R + \frac{(n - 1)(n - 2)}{4} \phi_\alpha \phi^{\alpha} - (n - 1) \Box \phi,
\]

where the symbol \( \Box \) denotes the \( n \)-dimensional D’Alembertian operator with respect to the Riemannian connection. Then, it is easy to show that by inserting \( R \) as given by \([8]\) into equation \((7)\) and using Gauss’ theorem to neglect divergence terms in the integral, we obtain

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5 Throughout this paper we shall adopt the following convention in the definition of the Riemann and Ricci tensors:

\[ R_{\mu \nu \rho \sigma} = \Gamma_{\mu \nu \rho}^{\sigma} - \Gamma_{\mu \nu \tau}^{\sigma} \Gamma_{\rho \tau}^{\rho} + \Gamma_{\mu \tau}^{\sigma} \Gamma_{\rho \nu}^{\rho} - \Gamma_{\mu \tau}^{\rho} \Gamma_{\rho \nu}^{\sigma} \]

as \( R_{\mu \nu} - \frac{1}{2}R g_{\mu \nu} = -\kappa T_{\mu \nu} \), with \( \kappa = \frac{8 \pi G}{c^4} \)
where \( R \) represents the scalar curvature evaluated with respect to the Riemannian connection. At this point, we should remark that the form of the action (9) is formally identical to the action of general relativity with a massless scalar field non-minimally coupled to matter. However, we would like to recall that while the action looks the same in both cases, the geodesics are different. In the WIST approach, the motion of particle and light rays are postulated to follow affine Weyl geodesics, whereas in general relativity the geodesics are assumed to be Riemannian, i.e., defined in terms of the Levi-Civita connection.

Let us now recall how the energy-momentum tensor \( T_{\mu\nu}^{(g)}(\phi, g, \xi, \nabla \xi) \) is defined in WIST gravity. In an arbitrary Weyl frame \( T_{\mu\nu}^{(g)}(\phi, g, \xi, \nabla \xi) \) is defined by the formula

\[
\delta \int d^n x \sqrt{|g|} e^{-\nu/2} L_m \left( g_{\mu\nu}, \phi, \xi, \nabla \xi \right) = \int d^n x \sqrt{|g|} e^{-\nu/2} T_{\mu\nu} \left( \phi, g_{\mu\nu}, \xi, \nabla \xi \right) \delta \left( e^\phi g^{\mu\nu} \right),
\]

where the variation on the left-hand side must be carried out simultaneously with respect to \( g_{\mu\nu} \) and \( \phi \). It is not difficult to see that the above definition makes sense. First, it must be clearly understood that the left-hand side of the equation (10) can always be written in the same form as the right-hand side of the same equation. This can be seen from the fact that

\[
\delta L_m = \left[ \frac{\partial}{\partial g_{\mu\nu}} g^{\mu\nu} + \frac{\partial}{\partial \phi} \phi \right] \delta \left( e^\phi g^{\mu\nu} \right) \text{ and that } \delta \left( \sqrt{|g|} e^{-\nu/2} \right) = -\frac{\nu}{2} \sqrt{|g|} e^{-\nu(1+n)/2} g_{\mu\nu} \delta \left( e^\phi g^{\mu\nu} \right).
\]

Varying the action \( (n)S \) with respect to the metric \( g_{\alpha\beta} \) and to the Weyl scalar field \( \phi \), we obtain, respectively, the following equations:

\[
R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \frac{(n-1)(n-2)}{4} \left[ \phi \phi_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \phi_{,\alpha} \phi_{,\alpha} \right] = -\kappa_n T_{\mu\nu} e^{(1-n/2)\phi},
\]

\[
\Box \phi = \frac{2\kappa_n}{(n-1)(n-2)} + \frac{\nu}{4} e^{(1-n/2)\phi} T
\]

where the symbol \( \Box \) denotes the \( n \)-dimensional D’Alembertian operator with respect to the Riemannian connection, and \( T = g^{\mu\nu} T_{\mu\nu} \).

Before concluding this section, let us recall the well-known mathematical fact which says that for \( n > 3 \) the Riemann tensor \( R_{\mu\nu} \) can be decomposed in terms of the Ricci tensor \( R_{\mu\nu} \), the scalar curvature \( R \) and the Weyl tensor \( W_{\mu\nu\rho\sigma} \). However, if \( n = 3 \), then, because \( W_{\mu\nu\rho\sigma} \) vanishes identically, we have the following expression for \( R_{\mu\nu} \) [11]:

\[
R_{\mu\nu} = g_{\mu\lambda} R_{\lambda\nu} - g_{\lambda\nu} R_{\mu\lambda} + g_{\lambda\rho} R_{\mu\rho} - g_{\rho\lambda} R_{\mu\rho} - \frac{R}{2} \left( g_{\mu\nu} g_{\lambda\kappa} - g_{\mu\lambda} g_{\nu\kappa} \right).
\]

After some straightforward algebra it is not difficult to verify that we can express (13) as

\[
R_{\lambda\mu\nu} = \kappa_n e^{-\phi/2} \left[ g_{\lambda\kappa} T_{\kappa\mu\nu} + g_{\lambda\nu} T_{\kappa\mu\kappa} - g_{\lambda\mu} T_{\kappa\nu\kappa} - g_{\lambda\nu} T_{\mu\kappa\kappa} + \Psi_{\lambda\mu\nu} \right],
\]

where

\[
\Psi_{\lambda\mu\nu} = \frac{\nu}{4} e^{(1-n/2)\phi} \left[ g_{\mu\nu} T_{\lambda\kappa\kappa} + g_{\mu\kappa} T_{\nu\lambda\kappa} - g_{\nu\kappa} T_{\mu\lambda\kappa} - g_{\nu\kappa} T_{\mu\kappa\lambda} + \left( g_{\mu\nu} g_{\lambda\kappa} - g_{\lambda\mu} g_{\nu\kappa} \right) T \right].
\]

\[
R_{\mu\nu} = \frac{n(n-1)}{2} \phi_{,\mu} \phi_{,\nu} + \frac{n(n-2)}{4} \phi_{,\mu}^2 + \frac{(n-1)(n-2)}{2} \phi_{,\mu} R_{,\nu} - \frac{(n-1)(n-2)}{2} \phi_{,\nu} R_{,\mu} + \frac{R}{4} \left( g_{\mu\nu} - 2 \phi_{,\mu} \phi_{,\nu} \right).
\]

In order to keep the kinetic term in the Lagrangian we shall assume throughout the paper that \( \omega = -\frac{(n-1)(n-2)}{2} \).
where
\[
\Psi_{\mu\nu\kappa\lambda} = \frac{1}{2} (2\omega + 1) \left[ g_{\mu\nu} \phi_{\rho\kappa} \phi_{\sigma} + g_{\mu\kappa} \phi_{\rho\lambda} \phi_{\sigma} - g_{\kappa\lambda} \phi_{\rho\mu} \phi_{\sigma} - g_{\mu\lambda} \phi_{\rho\kappa} \phi_{\sigma} \right] \\
+ \frac{1}{4} (2\omega + 1) (g_{\mu\kappa} g_{\nu\sigma} - g_{\mu\nu} g_{\kappa\sigma}) \phi_{\rho\alpha} \phi^{\alpha}.
\]

(15)

4. Weyl integrable theory of gravity in (2+1)-dimensional spacetime

Let us now consider WIST in a (2+1)-dimensional spacetime in the absence of matter. In this case, if \( n = 3 \), (11) and (12) become

\[
R_{\mu\nu} = \frac{1}{2} g_{\mu\nu} R + \frac{(1 + 2\omega)}{2} \left[ \phi_{\rho\alpha} \phi^{\rho\alpha} - \frac{1}{2} g_{\rho\alpha} \phi^{\rho\alpha} \right] = 0,
\]

(16)

\[ \square \phi = 0, \]

(17)

recalling that \( \omega = -\frac{1}{2} \). On the other hand, if we take the trace of equation (11) with \( T_{\mu\nu} = 0 \), we shall get

\[
R = -\frac{1}{2} (2\omega + 1) \phi_{\rho\alpha} \phi^{\rho\alpha}.
\]

(18)

Substituting (18) into equation (16) leads to

\[
R_{\mu\nu} = -\frac{1}{2} (2\omega + 1) \phi_{\rho\alpha} \phi^{\rho\alpha}.
\]

(19)

By taking into account (18) and (19) we can express (13) as

\[
R_{\mu\nu\kappa\lambda} = \frac{1}{2} (2\omega + 1) \left[ g_{\mu\nu} \phi_{\rho\kappa} \phi_{\sigma} + g_{\mu\kappa} \phi_{\rho\lambda} \phi_{\sigma} - g_{\kappa\lambda} \phi_{\rho\mu} \phi_{\sigma} - g_{\mu\lambda} \phi_{\rho\kappa} \phi_{\sigma} \right] \\
+ \frac{1}{4} (2\omega + 1) (g_{\mu\kappa} g_{\nu\sigma} - g_{\mu\nu} g_{\kappa\sigma}) \phi_{\rho\alpha} \phi^{\alpha}.
\]

(20)

Equation (20) means that even in the absence of matter, except for \( \omega = -\frac{1}{2} \), the spacetime is not necessarily Riemann flat, as it depends on the Weyl scalar field. Likewise, the curvature tensor \( R_{\mu\nu\kappa\lambda} \) calculated with the Weyl connection \( \Gamma^{\alpha}_{\beta\lambda} \) does not vanish in the absence of matter as it is given by

\[
R_{\mu\nu\kappa\lambda}^{\alpha} = g^{\alpha\lambda} R_{\mu\nu\kappa} - e^{-\phi} g_{\mu\lambda} \delta^{[\alpha}_{\beta} Q^{\lambda]}_{\nu},
\]

(21)

with \( Q^{\alpha}_{\beta\lambda} = 4 e^{\phi/2} (e^{\phi/2})_{\beta\lambda} g^{\alpha\lambda} - 2 (e^{\phi/2})_{\nu} g^{\mu\omega} \delta^{\alpha}_{\omega} \), and \( Q^{\alpha}_{\beta\lambda} \) denotes antisymmetrization with respect to both upper and lower indices. Thus, unlike general relativity in (2+1) dimensions, in WIST gravity the gravitational field does not necessarily vanish outside the sources. In the next section, we shall investigate the Newtonian limit of Weyl gravity in the weak-field regime.

5. The Newtonian limit

A metric theory of gravity is said to possess a Newtonian limit in the non-relativistic weak-field regime if one can derive Newton’s second law from the geodesic equations as well as
Poisson’s equation from the gravitational field equations. Let us now proceed to examine whether Weyl gravity fulfills these requirements. The method we shall employ here to treat this problem is standard and can be found in most textbooks on general relativity (see, for instance, [9]).

Since in Newtonian mechanics the space geometry is Euclidean, a weak gravitational field in a geometric theory of gravity should manifest itself as a metric phenomenon through a slight perturbation of the Minkowskian spacetime metric. Thus we consider a time-independent metric tensor of the form

\[ g_{\mu\nu} = \eta_{\mu\nu} + \epsilon h_{\mu\nu}, \]  

where \( \eta_{\mu\nu} \) denotes the Minkowski metric tensor, \( \epsilon \) is a small parameter and the term \( \epsilon h_{\mu\nu} \) represents a very small time-independent perturbation due to the presence of some matter configuration. Since we are working in the non-relativistic regime we shall suppose that the velocity \( V \) of a particle moving along a geodesic is much less than \( c \), so that the parameter \( \beta = \frac{V}{c} \) will be regarded as very small; hence in our calculations only first-order terms in \( \epsilon \) and \( \beta \) will be kept. The same kind of approximation will be assumed to hold with respect to the Weyl scalar field \( \phi \), which will be supposed to be static and very small, i.e., of the same order as \( \epsilon \), and to emphasise this fact we shall write \( \phi = \epsilon \varphi \), where \( \varphi \) is finite.

If we adopt the Galilean coordinates of special relativity we can write the line element defined by (22) as

\[ ds^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 + \epsilon h_{\mu\nu}dx^\mu dx^\nu, \]

which leads, in our approximation, to

\[ \left( \frac{dx}{dt} \right)^2 \cong c^2 (1 + \epsilon h_{00}). \]  \hspace{1cm} (23)

We now apply the same approximation to the geodesic equations

\[ \frac{d^2x^\mu}{dt^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} = 0, \]  \hspace{1cm} (24)

recalling that the symbol \( \Gamma^\mu_{\alpha\beta} \) represents the components of the Weyl affine connection. From (4) it is easy to see that, to first order in \( \epsilon \), we have

\[ \Gamma^\alpha_{\mu\nu} = \frac{\epsilon}{2} \eta^{\alpha\lambda} \left[ \eta_{\mu,\nu} + \eta_{\nu,\mu} - \eta_{\mu\nu,\lambda} + n_{\mu\nu,\lambda} - n_{\mu,\nu,\lambda} - n_{\lambda,\mu,\nu} \right]. \]  \hspace{1cm} (25)

It is not difficult to see that, unless \( \mu = \nu = 0 \), the product \( \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} \) is of the order of \( \epsilon \beta \) or higher. In this way, the geodesic equations (24) become, to first order in \( \epsilon \) and \( \beta \),

\[ \frac{d^2x^\mu}{dt^2} + \Gamma^\mu_{\alpha0} \left( \frac{dx^\alpha}{ds} \right)^2 = 0. \]  \hspace{1cm} (26)

By taking into account (23) the above equation may be written as

\[ \frac{d^2x^\mu}{dt^2} + c^2 \Gamma^\mu_{00} = 0. \]  \hspace{1cm} (27)

Clearly for \( \mu = 0 \) equation (27) reduces to an identity. On the other hand, if \( \mu \) is a spatial index, a simple calculation gives us \( \Gamma^0_{00} = -\frac{\epsilon}{2} \eta^0_\nu \frac{\partial}{\partial x^\nu} (\epsilon h_{00} - \varphi) \), hence the geodesic equation in this approximation becomes, in three-dimensional vector notation,

\[ \frac{d^2x^\mu}{dt^2} = c^2 \Gamma^\mu_{00} = 0. \]

To derive (27) from (24) we just write \( \frac{dx}{dt} = \frac{1}{c^2 + \epsilon h_{00}} \frac{ds}{dt} \) from (23), and then retain only first-order terms in \( \epsilon \) and \( \beta \).
\[
\frac{d^2X}{dt^2} = -\frac{\epsilon e^2}{2}V(h_{00} - \varphi),
\]
which is simply Newton’s equation of motion in a classical gravitational field provided we identify the scalar gravitational potential as
\[
U = \frac{\epsilon e^2}{2}(h_{00} - \varphi). \tag{28}
\]

It is interesting to note here the presence of the Weyl field \(\varphi\) in the equation above. It is the combination \(h_{00} - \varphi\) that makes up the Newtonian potential.

Let us now turn our attention to the Newtonian limit of the field equations. As we have seen previously, in \(n\) dimensions the field equations of Weyl gravity in the presence of matter are given by (11) and (12). It is now convenient to recast equation (11) in the form
\[
R_{\mu\nu} = -\kappa_n e^{(1-\frac{n}{2})\phi} \left( T_{\mu\nu} - g_{\mu\nu} \frac{T}{n-2} \right) - \frac{(n-1)(n-2) + 4\omega}{4} \phi \phi_{\mu\nu} \tag{29}
\]

In the weak-field approximation, i.e., when \(g_{\mu\nu} = \eta_{\mu\nu} + \epsilon h_{\mu\nu}\), it is easy to show that to first order in \(\epsilon\), we have \(R_{00} = -\frac{1}{2}\nabla^2 \epsilon h_{00}\), where \(\nabla^2\) denotes the Laplacian operator in \(n\)-dimensional flat spacetime. On the other hand, because we are assuming a static regime \(\phi_0 = 0\), equation (29) for \(\mu = \nu = 0\) now reads
\[
\nabla^2 h_{00} = -\kappa_n \left( T_{00} - \frac{800 T}{n-2} \right) e^{(1-\frac{n}{2})\phi}.
\]

For a perfect fluid configuration, according to our minimal coupling prescription, we have \(T_{\mu\nu} = e^{-2\phi}[(\rho c^2 + p)V_\mu V_\nu - pg_{\mu\nu}]\), where \(\rho\), \(p\) and \(V^\mu\) denote, respectively, the rest mass density, pressure and velocity field of the fluid. Since in this approximation \(\rho\) and \(\phi = \epsilon \varphi\) are small quantities, we can write \(e^{-2\phi} \simeq 1 - 2\epsilon \varphi\), and thus, to first order in \(\epsilon\), we have \(T_{\mu\nu} \simeq [(\rho c^2 + p)V_\mu V_\nu - pg_{\mu\nu}]\). On the other hand, in a non-relativistic regime we can neglect \(p\) with respect to \(\rho\), which then implies \(T \simeq \rho c^2\). Since in this approximation \(\rho\) and \(\phi = \epsilon \varphi\) are small quantities, we have \(T e^{(1-\frac{n}{2})\phi} \simeq \rho c^2(1 - \frac{2}{n} \epsilon \varphi) \simeq \rho c^2\). Thus we have
\[
\frac{\epsilon}{2} \nabla^2 h_{00} = \left( \frac{n-3}{n-2} \right) \kappa_n \rho c^2. \tag{30}
\]

In the same approximation (11) becomes
\[
\epsilon \nabla^2 \varphi = \frac{-4\kappa_n \rho c^2}{(n-1)(n-2) + 4\omega}. \tag{31}
\]

From (28), (30) and (31) we finally get the \(n\)-dimensional Poisson’s equation
\[
\nabla^2 U = -K_n \rho \tag{32}
\]
where \(K_n = \kappa_n e^4 \left( \frac{n-3}{2n-2} + \frac{2}{(n-1)(n-2) + 4\omega} \right)\) plays the role of the gravitational constant in \(n\) dimensions. At this point we recall that in \(n\)-dimensional general relativity the equation that corresponds to (32) is (see, for instance, [2])
\[
\nabla^2 U = \left( \frac{n-3}{n-2} \right) K_n \rho c^4. \tag{33}
\]
For $n = 3$ the right-hand side of the above equation vanishes, hence the linearized Einstein theory fails to produce Newtonian gravity. However, due to the presence of the scalar field, $K_n \neq 0$. Therefore, WIST has a Newtonian limit for any $n \geq 3$.

6. Geodesic deviation

An aspect of the strange behaviour of three-dimensional general relativity, first pointed out by Giddings et al [2], is the prediction that world lines of dust particles do not deviate. This is equivalent to saying that even if the spacetime is allowed to have curvature these particles do not feel the gravitational interaction. Let us now investigate the same phenomenon in light of the Weyl integrable spacetime theory.

Suppose that as the source of the gravitational field we have a pressureless perfect fluid (‘dust’). In this case, the energy-momentum tensor of the fluid is given by

$$T^{\alpha\beta} = \rho u^\alpha u^\beta,$$

where $u^\alpha = u^\alpha (x)$ denotes the components of the 4-velocity field of the fluid particles. Let $\eta^\alpha$ denote the deviation vector of the congruence of geodesics determined by $u^\alpha$. The equation of geodesic deviation is given by

$$\frac{D^2 \eta^\alpha}{dx^2} = R^{\alpha}_{\mu
u\kappa\lambda} u^\mu u^\nu \eta^\kappa \eta^\lambda,$$

where the operator $D/\partial x$ stands for the absolute derivative along the geodesic congruence. Now, from (14) and the identity (21), the above equation may be written as

$$\frac{D^2 \eta^\alpha}{dx^2} = \left(1 + \frac{2\omega}{2}\right) \left(\delta_{\mu\nu} \phi^\mu \phi_{,\nu} \eta^\alpha - \phi^\mu \phi_{,\nu} u^\mu u^\nu \eta^\alpha - \frac{1}{2} \phi^\mu \phi_{,\nu} \phi_{,\mu} \phi_{,\nu} \eta^\alpha \right) + \phi^\mu \phi_{,\mu} u^\nu \eta^\nu - \frac{1}{2} \phi^\mu \phi_{,\mu} u^\nu u^\nu \eta^\alpha + \frac{1}{2} \phi^\mu \phi_{,\mu} \phi_{,\nu} \phi_{,\nu} \eta^\alpha.$$

(36)

We thus see that in three-dimensional Weyl gravity the congruence of world lines of particles of a pressureless fluid has a non-vanishing geodesic deviation, so that a pair of freely falling particles will exhibit a relative accelerated motion, revealing the presence of a gravitational field. It is interesting to note that in this case the gravitational field manifests itself only through the Weyl scalar field $\phi$.

7. A static and circularly symmetric solution

In this section, we consider the problem of determining the gravitational field generated by a circularly symmetric matter distribution in a region outside the source. By solving the field equations we obtain a vacuum static solution, which, as we shall see, unlike three-dimensional general relativity, is not flat in regions where matter is absent.

Let us start by writing the line element of a circularly symmetric spacetime in its most general form, which may be given by

$$ds^2 = e^{2N} dt^2 - e^{2P} dr^2 - r^2 d\theta^2,$$

(37)

where $N(r)$ and $P(r)$ are functions of the radial coordinate only. It is also assumed that the Weyl scalar field $\phi = \phi (r)$ also depends only on $r$. Then, the field equations (19) and (17) become
\( N'' + N'^2 - N'P' + \frac{N'}{r} = 0, \quad (38) \)

\( N'' + N'^2 - N'P' + \frac{P'}{r} = \lambda \phi'^2, \quad (39) \)

\( N' - P' = 0, \quad (40) \)

\( \phi'' + \phi' \left( N' - P' \right) + \frac{\phi'}{r} = 0, \quad (41) \)

where prime denotes the derivative with respect to \( r \) and \( \lambda = -\frac{1}{2}(1 + 2\omega) \).

By substituting (40) into (41) we get

\[ \phi'' + \frac{\phi'}{r} = 0, \]

whose general solution is given by

\[ \phi = \phi_0 + A \ln r, \]

where \( \phi_0 \) and \( A \) are integration constants. (Since in Weyl geometry the presence of an additive constant in the expression of the scalar field has no geometrical meaning it is convenient to set \( \phi_0 = 0 \).) On the other hand, also from (40), the equations (38) and (39) reduce, respectively, to

\( N'' + \frac{N'}{r} = 0 \)

and

\( N'' - \frac{N'}{r} = \lambda \phi'^2. \)

It is easily seen that the above equations yield

\( N = N_0 + B \ln r, \)

with \( B = -\frac{\lambda}{2} A^2 \), while (40) gives

\( P = P_0 + B \ln r, \)

where \( N_0 \) and \( P_0 \) are integration constants.

By rescaling the time coordinate \( t \) we can set \( N_0 = 0 \). On the other hand, if we assume that there is no conical singularity in the spacetime we can also take \( P_0 = 0 \). Finally, the line element (37) may be written as

\[ ds^2 = r^{2\theta} \, dr^2 - r^2 \, d\theta^2 - r^2 d\phi^2, \quad (42) \]

while the scalar field is given by

\[ \phi = A \ln r. \quad (43) \]

Let us now apply the weak-field limit to find the constant \( B \) in terms of the mass \( M \) of the matter distribution. To do this, we first note that when \( B = 0 \), (42) reduces to the metric of Minkowski spacetime. From (43) it seems natural to identify the parameter \( \epsilon \) of section 5 with the constant \( A \). For small values of \( B \) we can write \( r^{2\theta} = e^{\ln r^{2\theta}} \approx 1 + 2B \ln r \). At this point, let us recall that in two spatial dimensions the Newtonian gravitational potential \( U(r) \) of a circularly symmetric mass distribution is given by \( U(r) = G_2 M \ln r \), where \( G_2 \) denotes the
gravitational constant in this dimensionality. Now, from (28) and recalling that \( B = -\frac{3}{2}A^2 \) is a second-order term in \( e = A \) we obtain \( A = -\frac{2MG}{c^2} \) and \( B = -\frac{2MG^2}{c^4} = \frac{(1+2\omega)MG^2}{c^4} \).

With respect to the spacetime corresponding to this solution, a few comments are in order. First, let us note that the basic invariants in this dimensionality are \( I_1 = e^2/R \) and \( I_2 = e^{2\nu}R_\nu\nu\). For the spacetime (42) we have \( I_1 = \frac{4B - c^4}{2}r^{A-2B-2} \) and \( I_2 = F(A, B) \) \( r^{2(A-2B-2)} \), where \( F(A, B) \) is a function of the constants \( A \) and \( B \).

It is not difficult to verify that that the solution (42) corresponds to a spacetime presenting a naked singularity at the center of the matter distribution for values of \( \omega > -\frac{1}{2} + \omega_0 \), with \( \omega_0 = \frac{A^2}{2c^2} \). We first note that the invariants \( I_1 \) and \( I_2 \) diverge at \( r = 0 \) for \( \omega > -\frac{1}{2} + \omega_0 \). Second, we can easily see that there is no event horizon in this spacetime. We start by noting that there is no change in the metric signature for any value of the radial coordinate. In addition, let us also consider the following argument. Suppose that at \( t = t_0 \) a static observer \( \mathcal{O} \) at \( r = r_0 \) sends a light signal radially towards the centre of the matter distribution at \( r = 0 \). We now wonder whether or not the emitted light signal can be reflected back to the observer in a finite time. Let us first recall that, according to the Weyl minimal coupling prescription, we must set \( \gamma_{\mu\nu} \rightarrow e^{-\nu}g_{\mu\nu} \), so the proper time \( \Delta\tau \) measured by \( \mathcal{O} \) will be given by the frame invariant formula \( \Delta\tau = \int_t^{t'} e^{-\nu} \left( g_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt} \right)^{\frac{1}{2}} dt \), where \( t_2 \) denotes the coordinate time corresponding to the reception of the light ray by the observer. For a light ray we have from (42) that \( dt = \pm dr \). Thus, it is not difficult to verify that \( \Delta\tau = 2(r_0)^{A+1-2} \), which is a finite value, indicating that we are in the presence of a naked singularity, for if we had an event horizon we would expect an infinite value for \( \Delta\tau \). This behaviour is in sharp contrast with the known BTZ general relativistic solution, which is a solution of Einstein equations in \((2+1)\) dimensions with a negative cosmological constant and corresponds to a black hole exhibiting properties very similar to black holes in \(3+1\) dimensions. However, since general relativity in \((2+1)\) dimensions has no Newtonian limit, it has been shown that the presence of a negative cosmological constant is a necessary condition for black holes to exist in Einstein’s gravity in this dimensionality [12]. It should also be mentioned that stationary black holes have been found in \((2+1)\) dimensions in gravity theories with a dilaton field [13]. Let us remark that, in fact, naked singularities are a common feature observed in \((3+1)\)-dimensional general relativity solutions with a massless scalar field, the most known of these being Wyman’s spherically symmetric spacetime [14], although non-static naked singularities have also been found recently [15].

8. Final remarks

Investigation in lower dimensional gravity has arisen essentially after the realization that the two- or three-dimensional versions of Einstein gravity are rather peculiar and, in some sense, unsuitable for a theory of the gravitational interaction (for instance, gravitational waves do not exist in three-dimensional general relativity). However, the current motivation for this kind of research is that lower dimensional models can provide useful insights and ideas to construct a successful quantum theory of gravity [3]. Apart from general relativity, other theories of gravity have been studied in this context. Of much interest is the study of how black holes may form in gravity theories formulated in lower dimensions. Because general relativistic spacetimes are locally flat outside matter sources, there are no black hole solutions, unless a negative cosmological constant is introduced, as was pointed out by Bañados et al some years ago [16]. On the other hand, different models of dilaton gravity also lead to the discovery of
black hole solutions. In this work, we have approached the subject from the standpoint of Weyl integrable geometry, in which a scalar field plays the role of a geometrical field. We have found out that this modification in the spacetime geometry leads to new features that are not present in three-dimensional Einstein gravity, such as the existence of a Newtonian limit and the non-vanishing geodesic deviation of the trajectories of freely falling particles. We also have shown that another effect of the geometrical scalar field is that the spacetime generated by a static circularly symmetric matter distribution corresponds, for certain values of the parameter $\omega$, to a curved spacetime with a naked singularity at the center of the distribution. To conclude, it seems natural, as a follow up to this work, to consider the case of $(1+1)$-dimensional gravity. In fact, a great motivation for further investigation in this dimensionality comes from the recently discovered interesting connection between Weyl symmetry and Liouville theory [17].

A final remark concerning the proper geometrical meaning of WIST is in order. One might plausibly argue that Weyl integrable spacetime is not a genuine Weyl manifold as the scalar field can be made to vanish in the Riemann frame. Nevertheless, we think we can take the alternative view that a Riemannian spacetime may be regarded as an element of an equivalence class in which Weyl’s compatibility condition between the metric and the affine connection would still hold, the Weyl vector field being replaced by the gradient of a scalar field. In fact, we think that this view could perhaps be useful to shed some light on the unresolved question of what is the physical frame in Brans–Dicke theory, the Jordan or the Einstein frame. Incidentally, in the present geometrical approach, this controversy would not arise, as the physical entities defined in the theory are naturally invariant under frame transformations [18].

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