EXTREMAL SYMPLECTIC CONNECTIONS ON SURFACES

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ABSTRACT. M. Cahen and S. Gutt found the moment map for the action of the symplectomorphism group on the symplectic affine space of symplectic connections on a symplectic manifold. Here a symplectic connection is called extremal if it is critical for the $L^2$ norm of the Cahen-Gutt moment map with respect to arbitrary variations. This paper studies extremal and moment constant volume-preserving connections on surfaces. A symplectic connection is extremal if and only if the Hamiltonian vector field generated by its moment map image acts on it as an infinitesimal automorphism. In particular, moment constant connections are extremal. A projectively flat symplectic connection is moment flat. A preferred symplectic connection is extremal. On a compact surface the Levi-Civita connection of a Kähler structure is extremal symplectic if and only if it has constant curvature, and more generally it is shown that on a compact surface an extremal connection is either projectively flat or is somehow inherently nonmetric. On a compact surface of genus at least two an extremal symplectic connection is moment flat. On the plane there are constructed examples of extremal symplectic connections that are neither moment constant nor preferred. Similar examples are found on the sphere, but they not smooth at the poles. More precisely, it is shown that, for a compact surface, the map associating to an orbit of the connected component of the identity of the symplectomorphism group acting in the space of moment flat symplectic connections the cohomology class of the one-form obtained by contracting with the symplectic form the projective Cotton tensor of a representative connection is surjective onto the first cohomology of the surface. W. Goldman described the space of strictly convex flat real projective structures on a closed surface by symplectic reduction utilizing that the projective Cotton tensor is a moment map for the action of the diffeomorphism group on the space of such projective structures. The close relation between the Cahen-Gutt and Goldman moment maps for a surface equipped with a volume form is explained.

1. INTRODUCTION

The symplectomorphism group $\text{symp}$ of the symplectic manifold $(M, \Omega)$ acts on the symplectic affine space $S(M, \Omega)$ comprising symplectic connections (those torsion-free affine connections preserving $\Omega$), and M. Cahen and S. Gutt showed that the action is Hamiltonian, calculating a moment map denoted here by $\mathcal{K}(\nabla)$ for $\nabla \in S(M, \Omega)$. For connection $\nabla$ on a surface preserving the volume form $\Omega_{ij}$ and having Ricci tensor $R_{ij}$, $-2\mathcal{K}(\nabla)$ equals the symplectic divergence $\Omega^{ij} \nabla_i \rho_j$ of the symplectic trace $\rho_i = \Omega^{pq} C_{pqi}$ of the projective Cotton tensor $C_{ijk} = -2 \nabla_i R_{ijk}$ of the projective structure generated by the given connection, which was shown by W. Goldman to be a moment map for the action of the diffeomorphism group on the space of projective structures. Consequently, projectively flat volume-preserving connections are moment flat, but the converse generally fails.

The Hermitian scalar curvature of a Kähler metric is a moment map for the action of the group of Hamiltonian diffeomorphisms on the bundle of almost complex structures and extremal Kähler metrics are defined to be the critical points of the $L^2$-norm of this moment map with respect to variations within a given Kähler class. Here $\mathcal{K}$ is regarded as an analogue of the scalar curvature of a metric connection. This analogy suggests calling extremal symplectic a connection $\nabla \in S(M, \Omega)$ critical with respect to arbitrary compactly supported variations for the $L^2$ norm $\mathcal{E}(\nabla) = \int_M \mathcal{K}(\nabla)^2 \Omega$. Although the analogy is incomplete in that it does not extend to the classes of variations considered, there being no obvious analogue for symplectic connection of variation within
a Kähler class, the results obtained suggest that it provides a reasonable guide to expectations. For example, a Kähler metric is extremal if and only if the gradient of its scalar curvature is a real homorphic vector field, and, similarly, a symplectic connection $\nabla$ is extremal if and only if the Hamiltonian vector field generated by $\mathcal{K}(\nabla)$ is an infinitesimal automorphism of $\nabla$. In particular moment constant symplectic connections are extremal.

This article studies extremal symplectic connections on surfaces. The following incomplete list of the results obtained for volume-preserving connections on surfaces indicates the scope of the paper.

1. A projectively flat connection is moment flat. More precisely, the Cahen-Gutt moment map can be identified with the differential of a scalar multiple of the projective Cotton tensor.

2. On a compact surface there exist moment flat symplectic connections that are not projectively flat (Theorem 6.1). Precisely, for a compact surface, the map associating to an orbit of the connected component of the identity of the symplectomorphism group acting in the space of moment flat symplectic connections the cohomology class of the one-form obtained by contracting with the symplectic form the projective Cotton tensor of a representative connection is surjective onto the first cohomology of the surface (Theorem 2.6).

3. A preferred symplectic connection is extremal (Theorem 5.1). (The content of this claim is implicit in the paper [5] of F. Bourgeois and M. Cahen.)

4. On the plane there exist extremal symplectic connections that are not moment flat (see section 5). There is a family of examples (Theorem 5.3) that are extremal symplectic and complete but which are not preferred except for a special case found previously in [5].

5. On a compact surface of genus at least two an extremal symplectic connection is moment flat (Theorem 2.3). This is analogous to the theorem of E. Calabi that on a compact surface an extremal Kähler metric has constant curvature. (Whether the requirement that the genus be at least two is necessary has not been resolved.)

6. On the complement of two poles in the two sphere with its standard volume form there exists a one-parameter family of extremal symplectic connections that are neither moment flat nor preferred (Theorem 5.11) and which extend continuously but not differentiably at the poles.

7. On a compact surface, the Levi-Civita connection of a Kähler metric is extremal symplectic if and only if the metric has constant scalar curvature (Theorem 7.3).

8. On a compact oriented surface of genus at least one, a symplectic connection that differs from the Levi-Civita connection of a Riemannian metric by the real part of a cubic differential holomorphic with respect to the complex structure determined by the metric is moment flat if and only if it is projectively flat (Theorem 7.4).

The two key technical observations underlying structural results for extremal symplectic connections that are not moment flat are the following, explained in detail in Lemmas 5.3 and 5.6. First, for an extremal $\nabla \in \mathcal{S}(M,\Omega)$ there is a constant $\tau$ such that $\mathcal{K}^2 + \rho(H_{\mathcal{K}}) = \tau$, where $H_{\mathcal{K}}$ is the Hamiltonian vector field generated by $\mathcal{K}$. Second, the zero set of $\tau - \mathcal{K}^2$ is a union of isolated points and geodesic circles, and on its complement there are canonical action-angle coordinates where the action variable is simply $\mathcal{K}$ and the angle variable is a primitive of the closed one-form $\sigma = (\tau - \mathcal{K}^2)^{-1} \rho$. In the special setting of preferred connections, some form of both these observations plays a key role in [5], and this signaled their importance in the more general context of extremal symplectic connections.

In conjunction with [5, 7] and [5] can be interpreted as saying that extremal symplectic connections of metric origin are projectively flat. This contrasts with [2] that shows that moment flat connections that are not projectively flat abound. On the other hand, although a handful of the connections constructed are shown to be complete, most probably are not, and the issue has not been analyzed systematically. In general for affine connections useful criteria for determining the
completeness of a connection are not available, and, even in the case of flat affine connections, it is not clear that the completeness property has the same importance as it has in the metric setting.

While many results obtained here extend to higher dimensions, the close relation between projective and moment flatness, for example claim (2) of Theorem 2.1 or the conclusion of Lemma 2.2, has no obvious such extension. Although the two-dimensional case has some special features, the results suggest many interesting questions about extremal symplectic connections in higher dimensions that will be addressed in future work.

2. Overview and principal theorems

This section records the notation, conventions, and background needed throughout the paper. Additionally, the main results and their context are surveyed. Throughout, \((M, \Omega)\) is a connected boundaryless symplectic two-dimensional manifold oriented by the volume form \(\Omega\).

2.1. Background and basic definitions. Frequently, tensors are indicated using the abstract index notation. Indices are labels and do not indicate a choice of frame, although if one is fixed indices can be interpreted as indicating components with respect to it. Enclosure of indices in square brackets (resp. parentheses) indicates complete antisymmetrization (resp. symmetrization) over the enclosed indices. The up position is used to label contravariant tensors and the down position to label covariant tensors. The summation convention is always used in the following form. Indices are in either up position or down. A label appearing as both an up and a down index indicates the trace pairing. Indices are raised and lowered (respecting horizontal position) using the symplectic form \(\Omega\) subject to the conventions \(X_i = X^p\Omega_{pi}\) and \(X^i = \Omega^{ij}X_j\) (so that \(\Omega^{ij}\Omega_{pj} = -\delta_j^i\), where \(\delta_j^i\) is the canonical pairing between the tangent space and its dual).

A torsion-free affine connection on a symplectic manifold \((M, \Omega)\) is symplectic if \(\nabla_i\Omega_{jk} = 0\) (here, a symplectic connection is always torsion-free). Every symplectic manifold admits a symplectic connection, for if \(\nabla\) is any torsion-free affine connection then \(\nabla = \nabla + \frac{1}{4}\Omega^{kj}\Omega(i\nabla_k)p\) is symplectic. An almost complex structure \(J\) is compatible with \(\Omega\) if \(g_{ij} = -J_i^p\Omega_{pj} = -J_{ij}\) is symmetric and positive definite. Since \(M\) is a surface, any almost complex structure is integrable, and so \((g, J, \Omega)\) is a Kähler structure; to specify it, it suffices to specify any two of \(g, J, \Omega\). In particular, in this case the Riemannian volume element \(d\text{vol}_g\) equals \(\Omega\) and the Levi-Civita connection \(D\) of \(g\) is symplectic. For a nondegenerate covariant symmetric two-tensor \(g_{ij}\), \(g^{ij}\) always denotes the inverse symmetric bivector, and not the tensor obtained by raising indices with \(\Omega\). In any case, when \(g_{ij}\) is determined by a compatible complex structure, the two possible meanings for \(g^{ij}\) coincide.

Let \(C^\infty_c(M)\) denote compactly supported smooth functions. The group \(\text{Symp}\) of symplectomorphisms of \((M, \Omega)\) comprises compactly supported diffeomorphisms of \(M\) that preserve \(\Omega\). Its Lie algebra \(\text{symp}\) comprises compactly supported vector fields \(X\) such that \(\ell_X\Omega = 0\); equivalently the one-form \(X^\nu = \Omega(X, \cdot)\) is closed. Since the Hamiltonian vector field \(H_f = -df^\nu = \Omega^{\nu\rho}df_p\) of \(f \in C^\infty_c(M)\) satisfies \(H_{\{f, g\}} = [H_f, H_g]\) for the Poisson bracket \(\{f, g\} = H_fH_g\Omega_{ij} = -df^p dg_p = dg(H_f)\) of \(f, g \in C^\infty(M)\), the compactly supported Hamiltonian vector fields constitute a subalgebra \(\h\subset \text{symp}\). On a surface, \(\{f, g\}\Omega = df \wedge dg = d(fdg)\), so that \(\{f, g\}\Omega\) is always exact. Since if \(M\) is compact, \(\int_M \{f, g\}\Omega = 0\) for \(f, g \in C^\infty_c(M)\), the subspace \(\mathcal{C}_0^\infty(M) \subset C^\infty(M)\) comprising mean zero functions is a Lie subalgebra of \(C^\infty(M)\), isomorphic to \(\h\) via the map \(f \to H_f\). If \(M\) is noncompact, then the Lie algebra of compactly supported Hamiltonian vector fields \(\h_v\), is identified with the Lie algebra \(\mathcal{C}_v^\infty(M)\).

Let \(\text{Symp}_0\) be the path connected component of the identity in \(\text{Symp}\). The subgroup \(\h\subset \text{Symp}_0\) of Hamiltonian diffeomorphisms of \((M, \Omega)\) comprises symplectomorphisms of \((M, \Omega)\) that can be realized as the time one flow of a normalized time-dependent Hamiltonian on \(M \times [-1, 1]\), where normalized means mean-zero if \(M\) is compact and compactly supported if \(M\) is noncompact.
By a theorem of A. Banyaga, the infinitesimal generator of a flow by Hamiltonian diffeomorphisms is a Hamiltonian vector field, so that the Lie algebra of $\text{Ham}$.

For a smooth vector bundle $E \to M$, $S^k(E)$ denotes the $k$th symmetric power of $E$, and $\Gamma(E)$ denotes the vector space of smooth sections of $E$. For $\alpha \in \Gamma(S^k(T^*M))$, write $|\alpha| = k$. The symmetric bilinear pairing of $\alpha, \beta \in \Gamma(S^k(T^*M))$ defined by $\langle \alpha, \beta \rangle = \int_M \alpha_i \wedge \beta^i$ is graded symmetric in the sense that $\langle \alpha, \beta \rangle = (-1)^{|\alpha||\beta|} \langle \beta, \alpha \rangle$. If a function is regarded as a $0$-tensor, then $\langle \cdot, \cdot \rangle$ agrees with the $\text{Symp}$-invariant $L^2$ inner product $(f, g) = \int_M fg \Omega$ on $C^\infty(M)$. Using a compatible complex structure it is straightforward to show that the pairing $\langle \alpha, \beta \rangle$ is (weakly) nondegenerate in the sense that if $\langle \alpha, \beta \rangle = 0$ for all compactly supported $\beta \in \Gamma(S^q(T^*M))$ then $\alpha = 0$. The affine space $S(M, \Omega)$ of symplectic connections on $(M, \Omega)$ is modeled on the vector space $\Gamma(S^3(T^*M))$, for if $\nabla \in S(M, \Omega)$ and $\nabla = \nabla + \Pi_{ij}^k$, then $\Pi_{[ij]}^k = 0$, because $\nabla$ and $\nabla$ are torsion free, and, with $0 = \nabla_i \Omega_{jk} = -2 \Pi_{i[jk]}$, this implies $\Pi_{ij} = \Pi_{(ij)k}$. Write $T_\nabla S(M, \Omega) = \Gamma(S^3(T^*M))$ for the tangent space to $S(M, \Omega)$ at $\nabla$. The pairing $\langle \alpha, \beta \rangle$ on $\Gamma(S^3(T^*M))$ can be viewed as the symplectic form $\Omega$ on $S(M, \Omega)$ defined by $\Omega_{\nabla}(\alpha, \beta) = -\int_M \text{tr}(\alpha \wedge \beta) = \int_M \alpha_{ij} \beta^{ij} \Omega$ for $\alpha, \beta \in T_\nabla S(M, \Omega)$. The translation invariance, $\Omega_{\nabla+\Pi} = \Omega_{\nabla}$, for $\Pi \in T_\nabla S(M, \Omega)$ means $\Omega$ is parallel, so closed, and so $\nabla$ is a symplectic form.

The diffeomorphism group $\text{Diff}(M)$ of $M$ acts on the right, by pullback, on the affine space $\mathcal{A}(M)$ of torsion-free affine connections on $M$; for $\phi \in \text{Diff}(M)$, $\phi^*(\nabla) X = T\phi^{-1} (\nabla_{T\phi(x)} X \phi(y))$. Since the pullback of a symplectic connection via a symplectomorphism is again symplectic, the action of $\text{Symp}$ on $\mathcal{A}(M)$ preserves the subspace $S(M, \Omega)$ and the form $\Omega$. This suggests that interesting classes of symplectic connections should be identified in terms of the symplectic geometry of the symplectic affine space $(S(M, \Omega), \Omega)$ and the action on it of $\text{Symp}$ (and its subgroup $\text{Ham}$).

The curvature $R_{ijkl}$ of $\nabla \in \mathcal{A}(M)$ is defined by $2 \nabla_i [\nabla_j X^k] = R_{ijk}^p X^p$ for $X \in \Gamma(TM)$; the Ricci curvature is $R_{ij} = R_{pij}^p$. Sometimes, for readability, there will be written Ric or $\text{Ric}_{ij}$ instead of $R_{ij}$. If $\nabla \in S(M, \Omega)$, then, since $\Omega$ is $\nabla$-parallel, $\nabla$ has symmetric Ricci tensor, for $2R_{ij}^p = -R_{ij} = 0$. By the Ricci identity, $0 = 2 \nabla_i [\Omega_{jk}] = -R_{ijk}^p \Omega_p^q$, in which $R_{ijk}^p = R_{ij}^p + \Omega_{ij}^p \Omega_{ik}$ $\Omega_{ij}$ $\Omega_{kl}$. With the Bianchi identity this yields $R_{ip}^p = -2R_{ip}^{pq} = 2R_{ij}^p = 2R_{ij}$; it follows that every nontrivial trace of $R_{ijkl}$ is a constant multiple of $R_{ij}$. Since, on a surface, $2R_{ij} = \alpha_{ij} = \alpha_{ij}^p \Omega_{ij}$ for any two-form $\alpha_{ij}$,
\begin{equation}
2R_{ijkl} = \Omega_{ij} (R_{il} - \Omega_{j(k} R_{lj)} + \Omega_{ij} R_{kl}) = 2 \Omega_{ij} R_{kl},
\end{equation}
showing that the curvature is completely determined by the Ricci curvature.

2.2. The moment map and curvature one-form. Absent some auxiliary metric structure, there is no reasonable notion of scalar curvature for a symplectic connection, but the symplectic dual of the curvature one-form $\rho_i = \rho(\nabla)_i = 2 \nabla^k R_{ip}$ of $\nabla \in S(M, \Omega)$ is a reasonable analogue of the Hamiltonian vector field generated by the scalar curvature of a Kähler metric. By the traced differential Bianchi identity $2g^{pq} D_p R_{iq} = D_i \mathcal{R}_g$, for a Kähler structure $(\Omega, g, J)$ with Levi-Civita connection $D$ and scalar curvature $\mathcal{R}_g = g^{ij} R_{ij}$ there holds
\begin{equation}
\rho_i = -2\mathcal{R}_g^p D_p R_{iq} = -2g^{pq} D_p (J_q R_{iq}) = 2g^{pq} D_p (J_q R_{ba}) = J_q R_{bi} \mathcal{R}_g = -g_{ia} H^q_{bi}.
\end{equation}
Equivalently, the vector field $\rho^i$ metrically dual to $\rho_i$ is the negative of the Hamiltonian vector field generated by the scalar curvature: $\rho^i = g^{pq} \rho_p = -H^q_{bi}$. In particular, a Kähler metric has constant scalar curvature if and only if $\rho = 0$, and is extremal if and only if $\rho^i$ is a real holomorphic vector field. For a general symplectic connection the curvature one-form $\rho_i$ serves as a substitute for the rotated differential of the scalar curvature, and the vanishing of $\rho$ is a reasonable substitute for the condition of constant scalar curvature.

Two torsion-free affine connections are projectively equivalent if the image of each geodesic of one is contained in the image of a geodesic of the other. This is the case if and only if their difference tensor has the form $2 \gamma_i \delta_{ij}^k$ for some one-form $\gamma_i$. A projective structure $[\nabla]$ is a projective
equivalence class of torsion-free affine connections. The space of projective structures on $M$ is written $\mathcal{P}(M)$. The space $\mathcal{P}(\mathcal{V})$ of principal connections on the $\mathbb{R}^\infty$ principal bundle $\mathcal{V}$ obtained by deleting the zero section from $|\det T^*M|$ is an affine space modeled on $\Gamma(T^*M)$. The map sending $\nabla$ to $\langle[\nabla,\beta]\rangle$, where $\beta \in \mathcal{P}(\mathcal{V})$ is induced by $\nabla$, is an affine bijection between the affine space $\mathcal{A}(M)$ and $\mathcal{P}(M) \times \mathcal{P}(\mathcal{V})$ that is equivariant for the action of $\Gamma(T^*M)$ on $\mathcal{A}(M)$ generating projective equivalence, and the action of $\Gamma(T^*M)$ on $\mathcal{P}(M) \times \mathcal{P}(\mathcal{V})$ by (appropriately scaled) translations in the second factor; the quotient of $\mathcal{A}(M)$ by this action is $\mathcal{P}(\mathcal{V})$. The projective Cotton tensor $C_{ijk} = -2\nabla^i R_{ijk} - \frac{1}{2}R_{ij} R_{k}^i$, does not depend on the choice of representative $\nabla \in \mathcal{V}$. The projective structure $\langle[\nabla]\rangle$ is projectively flat if $C_{ijk} = 0$; this is equivalent to the existence of an atlas of charts with transition functions in $\text{PGL}(3,\mathbb{R})$. Given $\nabla \in \mathcal{S}(M,\Omega)$, the differential Bianchi identity shows $C_{ijk} = \nabla^p R_{ijkp}$, while differentiating (2.4) yields $2\nabla^p R_{ijkp} = \Omega_{ij} R_{kp}$, so that

$$2C_{ijk} = 2\nabla^p R_{ijkp} = \rho_k \Omega_{ij}. \tag{2.3}$$

Consequently, a projective structure $\langle[\nabla]\rangle$ on an orientable surface is projectively flat if and only if for some, and hence every, volume form $\Omega_{ij}$, there vanishes the curvature one-form $\rho_i$ of the unique representative $\nabla \in \mathcal{V}$ for which $\Omega_{ij}$ is parallel.

An action of a Lie group $G$ on a symplectic manifold $(M,\Omega)$ is symplectic if $G$ acts by symplectic diffeomorphisms. The Lie algebra homomorphism from the Lie algebra $\mathfrak{g}$ of $G$ to $\mathfrak{symp}$ defined by $x \to X^x_p = \frac{d}{dt}|_{t=0}\exp(-tx) \cdot p$ is Hamiltonian if there is a map $\mu : M \rightarrow \mathfrak{g}^*$, equivariant with respect to the action of $G$ on $M$ and the coadjoint action of $G$ on $\mathfrak{g}^*$, such that for each $x \in \mathfrak{g}$, the Hamiltonian vector field $H_{\mu(x)}$ equals $\mathcal{X}^x$; $\mu$ is called a moment map. In [7] (see also [2] or [19]), M. Cahen and S. Gutt showed that, for a symplectic manifold of arbitrary dimension, the map $\mathcal{K} : \mathcal{S}(M,\Omega) \rightarrow C^\infty(M)$ defined by

$$\mathcal{K}(\nabla) = \nabla^i \nabla^j R_{ij} - \frac{1}{2}R^{ij} R_{ij} + \frac{4}{3} R^{ijkl} R_{ijkl} = \frac{1}{2} \nabla^i \rho_i - \frac{1}{2} R^{ij} R_{ij} + \frac{1}{4} R^{ijkl} R_{ijkl} \tag{2.4}$$

is a moment map for the action of $\mathfrak{ham}$ on the symplectic affine space $(\mathcal{S}(M,\Omega),\Omega)$. The part of (2.4) quadratic in the curvature is a constant multiple of the contraction of $\Omega$ with $\Omega$ with the first Pontryagin form of the connection $\nabla$ and so vanishes identically on a symplectic 2-manifold. Hence, on a surface, $2\mathcal{K}(\nabla) = \nabla^p \rho_p$. Alternatively, (2.4) yields the identities $2R^{pq} R_{pjq} = -2R_{iq} R_j^p = -R^{pq} R_{pq} \Omega_{ij} = \frac{1}{2} R_{abc} R_{jabc}$, and $R^{ijkl} R_{ijkl} = 2R^2 R_{ij}$, and in (2.3) these yield $2\mathcal{K}(\nabla) = \nabla^p \rho_p$.

It is reasonable to ask if the action of $\mathfrak{symp}$ on $(\mathcal{S}(M,\Omega),\Omega)$ is also Hamiltonian. Because $\mathfrak{symp}$ is identified with closed one-forms, a moment map for $\mathfrak{symp}$ must take values in the space $\Lambda^1(M)/d\Lambda^2(M)$ of smooth one-forms modulo coexact one-forms, where the symplectic codifferential $d'$ of a $k$-form $\alpha$, defined by $d'\alpha_{i,\ldots,i-1} = -\nabla^p \alpha_{pi\ldots,i-1}$, does not depend on the choice of $\nabla \in \mathcal{S}(M,\Omega)$ and satisfies $d' \circ d' = 0$. Its codifferential should be $\mathcal{K}(\nabla)$, so that when consideration is restricted to $\mathfrak{ham}$, it recovers $\mathcal{K}(\nabla)$. The precise meaning of the dual of $\mathfrak{ham}$ depends on a choice of a topology that is not relevant here. Since integration against $f \Omega$ defines a linear functional on $\mathfrak{ham}$ for any $f \in C^\infty(M)$, whatever the dual space $\mathfrak{ham}^*$ means precisely, it contains a subspace that can be identified with the normalized Hamiltonian functions ($C^\infty_0(M)$ or $C^\infty_c(M)$ as $M$ is or is not compact). This condition makes sense because the codifferential of a coexact one-form is zero.

Theorem 2.1. Let $(M,\Omega)$ be a symplectic 2-manifold.

1. (M. Cahen, S. Gutt; [7]) The map $\mathcal{K} : \mathcal{S}(M,\Omega) \rightarrow \mathfrak{ham}^*$ defined by $2\mathcal{K}(\nabla) = \nabla^p \rho_p$ is a moment map for the action of $\mathfrak{ham}$ on $(\mathcal{S}(M,\Omega),\Omega)$, equivariant with respect to the natural actions of $\mathfrak{symp}$.

2. The map sending $\nabla \in \mathcal{S}(M,\Omega)$ to $-\frac{1}{2} \rho$ is a moment map for the action of the group $\mathfrak{symp}$ of volume-preserving diffeomorphisms on $(\mathcal{S}(M,\Omega),\Omega)$, equivariant with respect to the natural actions of $\mathfrak{symp}$. 

For the reader’s convenience, Theorem 2.1 is proved in section 3. While the proof given of (1) is organized differently than that given in [4] or [19], there is no real novelty.

2.3. Moment constant and extremal symplectic connections. A symplectic connection \( \nabla \) for which \( \mathcal{K}(\nabla) \) is constant or zero will be said to be moment constant or to be moment flat. Since the curvature tensor of a locally symmetric \( \nabla \in S(M, \Omega) \) is parallel, any scalar quantity formed from it and \( \nabla \) is constant, and so, in this case, \( \nabla \) is moment constant. More generally, the same is true if the group of automorphisms of \( \nabla \in \mathcal{S}(M, \Omega) \) acts on \( M \) transitively by symplectomorphisms. When \( M \) is compact \( \int_M \mathcal{K}(\nabla) \Omega = 0 \), so on a compact symplectic 2-manifold, \( \mathcal{K}(\nabla) \) is constant if and only if it is 0. Consequently, if \( \mathcal{K}(\nabla) \) is constant and nonzero then \( M \) must be noncompact.

**Lemma 2.2.** On a surface, a projectively flat symplectic connection is moment flat.

**Proof.** By (2.3) and

\[
d\rho_{ij} = 2\nabla_{[i}\rho_{j]} = \nabla_{\rho^p \Omega_{ij}} = -2\mathcal{K}(\nabla)\Omega_{ij},
\]

\( \mathcal{K}(\nabla) = 0 \) if and only if \( \rho \) is closed, or, equivalently, the vector field \( \rho^i \) is symplectic. \( \square \)

Examples of moment flat connections that are not projectively flat are given in sections 5 and 6. The basic question of which closed one-forms are cohomologous to the curvature one-form of a moment flat symplectic connection is discussed in detail below.

That \( \mathcal{K} \) is a moment map is similar to the statement (due to [13]) that the Hermitian scalar curvature of the associated Hermitian connection is a moment map for the action of \( \text{Hol}(\Omega) \) on the space of almost complex structures compatible with a given symplectic structure. The critical points with respect to variations within a fixed Kähler class of the Calabi functional, the squared \( L^2 \) norm of the scalar curvature, are the extremal Kähler metrics first studied in [9]. Given a symplectic 2-manifold \( (M, \Omega) \), define a functional \( \mathcal{E} : \mathcal{S}(M, \Omega) \to \mathbb{R} \) by

\[
\mathcal{E}(\nabla) = \int_M \mathcal{K}(\nabla)^2 \Omega.
\]

By the \( \text{Symp} \) equivariance of \( \mathcal{K} \), \( \mathcal{E} \) is constant along \( \text{Symp} \) orbits in \( \mathcal{S}(M, \Omega) \). A symplectic connection \( \nabla \in \mathcal{S}(M, \Omega) \) is extremal if it is a critical point of \( \mathcal{E} \) for arbitrary compactly supported variations. The terminology reflects the analogy with extremal Kähler metrics. In it \( \mathcal{K} \) plays the role of the Hermitian scalar curvature and \( \mathcal{E} \) plays the role of the Calabi functional. A Kähler metric is extremal if and only if the \((1, 0)\) part of the metric gradient of its scalar curvature is holomorphic. Lemma 4.1 shows that, analogously, a symplectic connection \( \nabla \) is extremal if the Hamiltonian vector field \( H_{\mathcal{K}(\nabla)} \) generated by its moment map image is an infinitesimal automorphism of \( \nabla \).

Since it is necessarily moment flat, a moment constant symplectic connection on a compact surface is an absolute minimizer of \( \mathcal{E} \). On a compact manifold, constant scalar curvature metrics are absolute minimizers of the Calabi functional. In fact, by Theorem 1.5 of [12], every extremal Kähler metric is an absolute minimizer of the Calabi functional. It is not clear to what extent the analogous statements for extremal symplectic connections should be expected to be valid because the space over which \( \mathcal{E} \) is varied is (in some imprecise sense) much larger than the fixed Kähler class over which the Calabi functional is varied. The second variation of \( \mathcal{E} \) is computed in Lemma 4.5 and while formally it resembles the second variation in the Kähler setting, it yields useful information less readily. By Corollary 4.4, the second variation is nonnegative at a moment constant connection. The consequence that on a compact surface moment constant symplectic connections are absolute minimizers is in any case obvious since \( \mathcal{E} \) is nonnegative. On the other hand, in section 5 there is constructed a one-parameter family \( \nabla(t) \) of extremal symplectic connections on \( S^2 \) that are not absolute minimizers because \( \mathcal{E}(\nabla(t)) \) takes on all possible positive real values, but these connections fail to be differentiable at two points of \( S^2 \).
In the Kähler setting the uniformization theorem classifies the constant scalar curvature metrics, while Calabi’s theorem shows that on compact surfaces the extremal Kähler metrics are simply constant curvature metrics (for simplicity the more complicated situation in the noncompact case is not discussed here). In the setting of symplectic connections, the work of Goldman,Labourie, and Loftin gives a good understanding of the convex flat projective structure on compact surfaces similar to that afforded by the uniformization theorem (although still not so complete), and the principal remaining questions are understanding the difference between moment flat connections and projectively flat connections, and understanding the difference between extremal symplectic connections and moment constant connections. For both questions relevant example are constructed and some substantial partial answers are obtained, but neither is completely resolved. The remainder of this section describes what has been obtained.

2.4. Extremal connections that are not moment flat. A symplectic connection is preferred if the symmetrized covariant derivative $\nabla_{(i}R_{jk)}$ of its Ricci tensor vanishes. These are the equations of the critical points of the functional $\int_M R^{ij}R_{ij}\Omega$ on $\mathcal{S}(M,\Omega)$ (see \(3.16\)). For surfaces preferred symplectic connections were studied in \([5]\), where they are called solutions to the field equations. Purely local computations yield Theorem \(5.1\) showing that a preferred symplectic connection is extremal. Since there are preferred symplectic connections that are not moment flat, this motivates studying extremal symplectic connections that are not moment constant.

The essential content of Theorem \(5.1\) that the Hamiltonian flow generated by $\mathcal{K}$ preserves $\nabla$, is contained in Proposition 6.1 of \([5]\), and its proof there uses $\mathcal{K}$, $\rho$, and identities relating them and their differentials that continue to be valid in the more general setting of extremal symplectic connections considered here. This motivated much of section 5 where the key technical results that facilitate the description of an extremal symplectic connection $\nabla$ are recorded.

Since $H_\mathcal{K}$ preserves $\nabla$, it preserves the curvature tensors associated to $\nabla$. From $\mathcal{L}_{H_\mathcal{K}}\rho = 0$ it follows that there is a constant $\tau$ such that $\mathcal{K}^2 + \rho^2d\mathcal{K} = \tau$. Using this identity it can be concluded that when $\mathcal{K}$ is not constant each connected component of its critical set is an isolated point or an isolated closed $\nabla$-geodesic. On the complement $\bar{M}$ of the set where $d\mathcal{K} \wedge \rho$ vanishes the one-form $\sigma = (\tau - \mathcal{K}^2)^{-1}\rho$ is closed and $\Omega = d\mathcal{K} \wedge \sigma$. This can be interpreted as saying that $\mathcal{K}$ and a local primitive of $\sigma$ constitute canonical action-angle coordinates on $\bar{M}$; precisely the action coordinate $\mathcal{K}$ is a moment map for the action of the flow of the symplectically dual vector field $\sigma^\wedge$.

These observations are detailed in Lemma \(5.3\). They are useful both for constructing examples and for proving that extremal implies moment constant. On the one hand, in trying to construct examples of extremal but not moment flat symplectic connections, it can be assumed that $\Omega$ has the standard Darbouc form $dx \wedge dy$, that $\mathcal{K}(\nabla)$ equals $x + a$ for some constant $a$, and hence that $\rho = (\tau - (x + a)^2)^{-1}dy$. The connection $\nabla$ can be written as $\partial + \Pi$ where $\partial$ is the standard flat affine connection preserving $dx$ and $dy$, and $\rho, \mathcal{K}$, and the equations $\mathcal{L}_{H_\mathcal{K}}\nabla = 0$ can be computed explicitly in terms of the components of $\Pi$. The results of such an approach are stated in Lemmas \(5.7\) and \(5.8\) and they are used to construct examples discussed further below.

In \([5]\) there is constructed a complete preferred symplectic connection on the plane that is not moment constant, and with Theorem \(5.1\) this yields an example of an extremal symplectic connection that is not moment constant (and so not projectively flat). Theorem \(5.9\) embeds these examples in a larger family of extremal symplectic connections that in general are neither preferred nor moment constant and that are complete for some choices of parameters. These examples are constructed finding explicit solutions of the equations obtained from Lemmas \(5.7\) and \(5.8\).

On the other hand, by Theorem 7.2 of \([5]\) a preferred symplectic connection on a compact surface has parallel Ricci tensor; in particular it has $\rho = 0$ so is projectively flat. Consequently, Theorem \(5.1\) (the extremality of preferred connections) yields no interesting examples of extremal symplectic connections on compact surfaces. By Theorem 3.1 of \([9]\), on a compact surface an extremal Kähler
metric has constant scalar curvature. These results both suggest that on a compact surface any extremal symplectic connection must be moment flat. Theorem 2.3 partly confirms this expectation.

**Theorem 2.3.** Let \( (M, \Omega) \) be a compact symplectic two-manifold. If the genus of \( M \) is at least two then any extremal \( \nabla \in \mathcal{E}(M, \Omega) \) is moment flat.

The canonical action-angle coordinates described above are used to show that, when \( M \) is compact, each connected component of \( \tilde{M} \) carries a complete flat Kähler structure preserved by \( H_{\mathcal{K}} \). This is enough to conclude that the connected components of \( \tilde{M} \) are diffeomorphic to cylinders and so \( \tilde{M} \) is obtained by gluing together disks and cylinders, so must have nonnegative Euler characteristic. This argument leaves open the possibility that on the sphere or torus there are extremal symplectic connections that are not moment flat.

The equations for \( \mathcal{K} \) and \( H_{\mathcal{K}} \) in Darboux coordinates recounted in Lemma 5.7 are sufficiently complicated that a complete analysis of them has not been made, but with various simplifying assumptions they yield tractable equations that yield several classes of examples. The first such simplifying assumption is to seek an extremal \( \nabla \) that satisfies additionally \( \nabla_{(i}\rho_{j)} = 0 \). This ansatz determines a family of extremal symplectic connections on \( \mathbb{R}^2 \), described precisely in Theorem 5.9 that are not moment constant, and that are geodesically complete for certain choices of parameters. Although in general these examples are not preferred, for certain parameter values they specialize to the preferred connections constructed in Proposition 11.4 of [5].

The observation that the preferred condition implies \( \nabla_{(i}\rho_{j)} = 0 \) is what suggested that the examples from [5] could be generalized to yield connections extremal but not preferred.

Attempts to patch together extremal symplectic connections on Darboux charts on a sphere or torus have failed, although on \( S^2 \) this approach yields an interesting example of singular extremal connections. Precisely, on \( S^2 \) there is (see Theorem 5.11 and the surrounding discussion) a one-parameter family \( \nabla(t) \) of rotationally symmetric extremal symplectic connections that are neither moment flat nor preferred. However, these connections only extend continuously at the poles (where \( \mathcal{K} = \tau \)) in the following sense. The connection \( \nabla(0) \) is the Levi-Civita connection of the round metric on \( S^2 \), and the difference tensor \( \nabla(t) - \nabla(0) \) extends continuously but not differentiably at the poles. These connections satisfying \( \mathcal{E}(\nabla(t)) = 3\pi t^2 \), so are not absolute minimizers of \( \mathcal{E} \) except when \( t = 0 \). These observations increase the suspicion that an extremal symplectic connection smooth on all of \( S^2 \) must be \( \nabla(0) \), but the situation remains unclear.

Finally, the explicit expressions for \( \mathcal{K} \) and \( \rho \) in Lemma 5.7 simplify considerably if the difference tensor \( \Pi_{ij}^k = \nabla - \partial \) is decomposable, meaning that \( \Pi_{ijk} = X_i X_j X_k \) for some one-form \( X_i \). If \( X \) is moreover closed, explicit expressions are obtained, and, when \( X = df \) is exact, \( \mathcal{K} \) and \( \rho \) are expressible in terms of \( f, df \), and the Hessian of \( f \). The conclusion, stated in Theorem 5.11 is that for a function \( f \in C^\infty(\mathbb{R}^2) \) the graph of which is an improper affine sphere, the connection \( \nabla = \partial + df_i df_j df_k \) is moment flat but not projectively flat. A conceptual explanation for these examples is lacking, but they suggest that in seeking examples it is useful to examine symplectic connections whose difference tensor \( \Pi \) with some particularly nice fixed reference symplectic connection (e.g. the Levi-Civita connection of a constant curvature metric) has a simple form, e.g. is decomposable, is the symmetric product of a fixed metric with a one-form, etc. As is explained in Section 2.5 the particular case where \( \Pi_{ijk} = X_{(i} g_{jk)} \) for a constant curvature metric \( g \) and a harmonic one-form \( X \) yields a general way of constructing moment flat connections that are not projectively flat.

**2.5. Moment flat connections that are not projectively flat.** Since a projectively flat connection is moment flat, a basic issue is to what extent the projectively flat and moment flat conditions differ. In [18], W. Goldman showed that the projective Cotton tensor is a moment map for the action of diffeomorphisms on the space \( \mathbb{P}(M) \) of projective structures on \( M \). Together (2.3) and (2.6) show that the moment map \( \mathcal{K} \) can be viewed as a sort of derivative of the projective Cotton tensor. The relation between these two moment maps is explained now in more detail with
the aim of addressing the following general questions. The first is to recall that projectively flat connections abound. This discussion is helpful in properly formulating the second question which regards characterizing when a moment flat symplectic connection must be projectively flat. In this regard, Theorems 2.3 and 7.4 discussed further in the introduction, show that a moment flat connection that is not projectively flat necessarily has (in a sense made precise in Theorem 7.4), a nonmetric character. When \( X \) vanishes \( \rho \) is closed so determines a cohomology class. The second question asks which cohomology classes are represented by the one-form \( \rho \) of some moment flat symplectic connection. This is resolved by Theorem 2.6 below. This depends on Theorem 6.1 that shows that from a hyperbolic surface and a harmonic one-form there can be constructed, in the manner indicated at the end of Section 2.4, a moment flat connection that is not projectively flat.

The difference tensor of two projective structures is by definition the difference tensor of their unique representatives corresponding to a principal connection \( \beta \in \mathcal{P}(V) \), which does not depend on the choice of \( \beta \) and is trace free. Hence \( \mathcal{P}(M) \) is an affine space modeled on the space of completely trace-free tensors satisfying \( \Pi^{-k}_{ij} = \Pi^k_{(ij)} \), and \( T_{\nabla} \mathcal{P}(M) = \{ \Pi^{-k}_{ij} \in \Gamma(S^2(T^*M) \otimes TM) : \Pi^p_{ij} = 0 \} \). If \( M \) is oriented, then for \( \alpha, \beta \in T_{\nabla} \mathcal{P}(M) \) at least one of which is compactly supported it makes sense to integrate the two-form \(-2\alpha^{pq}_{\beta} \sigma^p_{ij} \beta^j_q \) over \( M \). There results the symplectic form \( \Omega_{\nabla}^{\alpha \beta} = -2\int_M \alpha^{pq}_{\beta} \sigma^p_{ij} \beta^j_q \) on \( \mathcal{P}(M) \). The action of \( \text{Diff}(M) \) on \( \mathcal{A}(M) \) by pullback commutes with the action of \( \Gamma(T^*M) \), so induces an action on \( \mathcal{P}(M) \), also by pullback, \( \phi^*(\nabla) = [\phi^*(\nabla)] \). The bijection \( \mathcal{A}(M) \rightarrow \mathcal{P}(M) \times \mathcal{P}(V) \) is also \( \text{Diff}(M) \)-equivariant, where the action of \( \text{Diff}(M) \) on \( \mathcal{P}(M) \times \mathcal{P}(V) \) is the product action, and the action of \( \text{Diff}(M) \) on \( \mathcal{P}(V) \) is induced from pullback of densities on \( M \). The Lie derivative \( \mathcal{L}_X[\nabla] \) is the derivative of the difference tensor with \( [\nabla] \) of the pullback of \( [\nabla] \) by the flow of \( X \). This is just the completely trace-free part of \( \mathcal{L}_X \nabla \) for any \( \nabla \in [\nabla] \). Regarded as a functional on \( \mathcal{P}(M) \), the projective Cotton tensor \( C \) is evidently \( \text{Diff}(M) \)-equivariant in the sense that \( C(\phi^*[\nabla]) = \phi^*(C([\nabla])) \) for \( \phi \in \text{Diff}(M) \).

Let \( \text{vec}_c(M) \) denote the space of compactly supported vector fields on \( M \) which is the Lie subalgebra of \( \text{vec}(M) \) corresponding to the identity component of the group \( \text{Diff}_c(M) \) of compactly supported diffeomorphisms. Regard \( \mathcal{L}_X[\nabla] \) as the vector field on \( \mathcal{P}(M) \) generated by \( X \in \text{vec}(M) \). For \( X \in \text{vec}_c(M) \) and \( A \in \mathcal{C}^2(M) = \{ \sigma_{ijk} \in \Gamma(\otimes^3T^*M) : \sigma_{ijk} = \sigma_{ij}k \text{ and } \sigma_{ijk} = 0 \} \) (the notation \( \mathcal{C}^2(M) \) is explained in the appendix) integration determines a pairing \( \langle X, A \rangle = \int_M X^p A_{ijp} \) in which \( X^p A_{ijp} \) is regarded as a two-form, that identifies \( \mathcal{C}^2(M) \) with a subspace of the dual vector space \( \text{vec}_c(M)^* \). In particular, via this identification \( C([\nabla]) \) is regarded as taking values in this subspace.

**Theorem 2.4** (W. Goldman; [17], [18]). Let \( M \) be a smooth surface.

1. The projective Cotton tensor is a moment map for the action of \( \text{Diff}_c(M) \) on the space \( \mathcal{P}(M) \) of projective structures on the oriented surface \( M \). Precisely, for \( X \in \text{vec}_c(M), [\nabla] \in \mathcal{P}(M), \) and \( \Pi \in T_{\nabla} \mathcal{P}(M), \delta_{\Pi} C([\nabla]), X = \Omega_{\nabla}[\mathcal{L}_X[\nabla], \Pi], \) where the first variation \( \delta C \) at \( [\nabla] \) in the direction of \( \Pi \in T_{\nabla} \mathcal{P}(M) \) is defined by \( \delta_{\Pi} C([\nabla]) = \frac{d}{dt}_{|t=0} C([\nabla] + t\Pi) \) (see (A.3) for an explicit formula for \( \delta_{\Pi} C \)).

2. For a compact surface \( M \), the symplectic quotient of \( \mathcal{P}_0(M) = C^{-1}(0) \) by the connected component \( \text{Diff}(M)_0 \) of the identity of the group of diffeomorphisms of \( M \) is the deformation space of isotopy classes of flat real projective structures on \( M \).

3. If \( M \) is compact and \( \chi(M) < 0 \) then the deformation space \( \mathcal{R}\mathcal{P}^2(M) = \mathcal{P}_0(M)/\text{Diff}(M)_0 \) is a real analytic manifold of dimension \(-8\chi(M)\).
For \( f \in C^\infty(M) \), by (2.3) and (2.4),
\[ (2.7) \quad \langle C([\nabla]), H_f \rangle = \frac{1}{2} \langle \rho, \delta^* f \rangle = -\frac{1}{2} \langle \delta \rho, f \rangle = \langle \mathcal{K}(s_1([\nabla])), f \rangle, \]
which shows explicitly the relation between the two moment maps. Lemma 2.5 shows that \( \iota \) and \( s_1 \) descend to the quotients modulo the actions of the relevant groups.

**Lemma 2.5.** On a finite volume symplectic 2-manifold \((M, \Omega)\) be a symplectic 2-manifold, the maps \( \iota: S(M, \Omega) \to \mathbb{R}^2(M) \) and \( s_1: \mathbb{R}^2(M) \to S(M, \Omega) \) induce inverse bijections between \( S(M, \Omega)/\text{Symp}_0 \) and \( \mathbb{R}^2(M)/\text{Diff}_c(M)_0 \), where \( \text{Diff}_c(M)_0 \) is the path connected component of the identity in \( \text{Diff}_c(M) \).

**Proof.** If \( \nabla \in S(M, \Omega) \) and \( \phi \in \text{Symp}_0 \) then \( \iota(\phi^*([\nabla])) = \phi^*(\iota([\nabla])) \), so that the image under \( \iota \) of the \( \text{Symp}_0 \) orbit of \( \nabla \) is contained in a \( \text{Diff}_c(M)_0 \) orbit of \( \iota([\nabla]) \) and \( \iota \) descends to a well-defined and evidently surjective map \( \iota: S(M, \Omega)/\text{Symp}_0 \to \mathbb{R}^2(M)/\text{Diff}_c(M)_0 \). If \( \phi \in \text{Diff}_c(M)_0 \) then \( \phi^*(\Omega) \) and \( \Omega \) are equal outside of some compact set \( K \). Since \( \int_M \phi^*(\Omega) = \int_M \Omega \), by a theorem of J. Moser (25), see also section 1.5 of [2], there is a diffeomorphism \( \psi \), supported in \( K \) and smoothly isotopic to the identity such that \( \psi^* \circ \phi^*(\Omega) = \Omega \). Then \( \tau = \phi \circ \psi \in \text{Symp}_0 \) is smoothly isotopic to \( \phi \) and equal to \( \phi \) outside a compact set. Therefore, given \( \{ [\nabla] \in \mathbb{R}^2(M), s_1(\tau^*([\nabla])) \} \) preserves \( \tau^*([\nabla]) = \Omega \), so \( s_1(\tau^*([\nabla])) = \tau^*s_1([\nabla]) \). Hence, if the projective structures \( [\nabla] \) and \( [\nabla] \) generated by \( \nabla, \nabla \in S(M, \Omega) \) lie in the same \( \text{Diff}_c(M)_0 \) orbit, then there is a \( \tau \in \text{Symp}_0 \) such that \( [\nabla] = \tau^*([\nabla]) \) and \( \nabla = s_1([\nabla]) = \tau^*s_1([\nabla]) = \tau^*\nabla \), so \( \nabla \) and \( \nabla \) lie in the same \( \text{Symp}_0 \) orbit. This shows that \( \iota \) is a bijection with inverse induced by \( s_1 \).

By Lemma 2.5 the space of equivalence classes of symplectomorphic projectively flat symplectic connections is identified with \( \mathbb{R}^2(M) \). The tangent space of \( \mathbb{R}^2(M) \) is identified with the first cohomology of the projective deformation complex. A simple explicit construction of this complex is given in the appendix. In Theorem A.3, there is constructed a fine resolution of the projective deformation complex, and it follows that the tangent space to \( \mathbb{R}^2(M) \) is identified with the first Cech cohomology \( H^1(M, \mathcal{P}) \) where \( \mathcal{P} \) is the sheaf of projective Killing fields. This is motivated by the analogous statement for constant curvature metrics due to Calabi in [8] (see also [3]). While the existence of this resolution was stated by Hangan in [20] and [21], it seems the proof was never published, and while it can be deduced from the general BGG machinery for parabolic geometries, it is included because it seems useful to record the simple direct argument and its presentation makes it possible to discuss possible parallels with the more general setting of moment flat symplectic connections. In particular, it is not obvious if there is a parallel construction in the more general context of moment flat symplectic connections. The key ingredient in the description of the space of flat projective structure via symplectic reduction is the mentioned fine resolution of the sheaf of projective Killing fields. This construction uses in a fundamental way that a flat projective structure can be locally represented by a flat affine connection. While there is a corresponding complex for moment flat symplectic connections, it is not clear that it yields a resolution. The problem is precisely that a local geometric interpretation of the vanishing of \( \mathcal{K} \) is lacking.

If \( \nabla \in \mathcal{K}^{-1}(0) \), then, by (2.3), \( \rho([\nabla]) \) is closed so the de Rham cohomology class \([\rho]\) is defined. If \( \phi_t \) is the flow of \( X \in \text{Symp}_0 \) and \( \nabla \in \mathcal{K}^{-1}(0) \), then \( \rho(\phi_t([\nabla])) - \rho([\nabla]) = \phi_t^*(\rho([\nabla])) - \rho([\nabla]) \) is homotopic to the zero form, so exact, and hence the cohomology class \([\rho([\nabla])]\) is preserved by the action of \( \text{Symp}_0 \) on \( \mathcal{K}^{-1}(0) \). Hence \( \rho([\nabla \cdot \text{Symp}_0]) = [\rho([\nabla])] \) defines a map
\[ (2.8) \quad \rho: \mathcal{K}^{-1}(0)/\text{Symp}_0 \to H^1(M, \mathbb{R}), \]
the quotient \( \mathcal{K}^{-1}(0)/\text{Symp}_0 \) of the space of moment flat symplectic connections by the action of the identity component of the symplectomorphism group to the first cohomology of \( M \). It is natural to ask if \( \rho \) is surjective, that is whether for a given symplectic form a given de Rham cohomology class \([\alpha]\) in \( H^1(M, \mathbb{R}) \) can be represented by \( \rho([\nabla]) \) for some \( \nabla \in \mathcal{K}^{-1}(0) \). For compact surfaces of nonzero Euler characteristic the answer is affirmative.
Theorem 2.6. Let \((M, \Omega)\) be a compact symplectic 2-manifold with nonzero Euler characteristic. Let \([\alpha] \in H^1(M, \mathbb{R})\) be a de Rham cohomology class. There exists \(\nabla \in \mathcal{S}(M, \Omega)\) such that \(\mathcal{K}(\nabla) = 0\) and \(\rho(\nabla) \in [\alpha]\). That is the map \(\rho\) of \(\mathcal{P}\) is surjective.

Remark 2.7. It is not clear whether the connection \(\nabla\) constructed in the proof of Theorem 2.6 is complete. That is, the question whether every class in \(H^1(M, \mathbb{R})\) is represented by the curvature one form of some complete symplectic connection remains unresolved.

Since, by Lemma 2.5, the fiber \(\varphi^{-1}(0) \subset \mathcal{K}^{-1}(0)/\text{Symp}_0\) over the trivial cohomology class contains a subset identified with \(\mathbb{P}_0(M)/\text{Diff}(M)_0\) and this last space has dimension \(8 \dim H^1(M, \mathbb{R})\), the fibers of \(\varphi\) can be quite large. It would be interesting to know if the fibers of \(\varphi\) are finite-dimensional. A reformulation of this question yields the following. For a fixed cohomology class \([\alpha] \in H^1(M, \mathbb{R})\) let \(\mathcal{S}_{[\alpha]}\) comprise those \(\nabla \in \mathcal{K}^{-1}(0)\) such that \(\rho(\nabla)\) represents \([\alpha]\). What can be said about the structure of the quotient space \(\mathcal{S}_{[\alpha]}/\text{Symp}_0 = \varphi^{-1}([\alpha])\)?

2.6. Nonmetricity. The analogy with the Kähler setting suggests the question of when the Levi-Civita connection of a Kähler structure is extremal symplectic. Theorem 7.3, proved in section 7, shows that on a compact surface the Levi-Civita connection of a Kähler structure is extremal symplectic. That is, the question whether every class in \(H^1(M, \mathbb{R})\) represented by the 

3. Variation of the moment map

For a linear operator \(\mathcal{P} : \Gamma(S^q(T^\ast M)) \to \Gamma(S^p(T^\ast M))\) write \([\mathcal{P}] = p - q\). Define the (formal) adjoint \(\mathcal{P}^* : \Gamma(S^q(T^\ast M)) \to \Gamma(S^p(T^\ast M))\) of \(\mathcal{P}\) by \(\langle \mathcal{P} \alpha, \beta \rangle = (-1)^{|\alpha||\beta|}\langle \alpha, \mathcal{P}^* \beta \rangle\). The sign is dictated by the rule of signs. By definition, \((\mathcal{P}^*)^* = \mathcal{P}\) and \((\mathcal{PQ})^* = (-1)^{|\mathcal{P}|}|\mathcal{Q}| \mathcal{Q}^* \mathcal{P}^*\).

Define the symplectic divergence operator \(\delta\) by \(\delta \alpha_{i_1...i_{k-1}} = (-1)^{k-1} \nabla_p \alpha_{i_1...i_{k-1}} \rho\). For example, \(\rho = 2\delta \text{Ric}\) and \(-2\mathcal{K} = \delta \rho\). The formal adjoint \(\delta^*\) is given by \(\delta^* \alpha_{i_1...i_{k+1}} = -\nabla_{(i_1} \alpha_{i_2...i_{k+1})}\).
Straightforward computations show that, for $\alpha \in \Gamma(S^k(T^*M))$,
\begin{align}
(3.1) \quad \nabla_i \alpha_{i_1...i_k} &= -\delta^* \alpha_{i_1...i_k} + (-1)^{k+1} \sum_{i=i_1}^{i_k} \Omega(i_1, \delta \alpha_{i_2...i_k}), \\
(3.2) \quad (k+1)\delta^* \alpha_{i_1...i_k} + k^* \delta \alpha_{i_1...i_k} &= (-1)^k (k+1) R_{i_1...i_k} \alpha_{i_2...i_k},
\end{align}
The Lie derivative of $\alpha \in \Gamma(S^k(T^*M))$ along $X \in \Gamma(TM)$ equals $\mathfrak{L}_X \alpha_{i_1...i_k} = X^p \nabla_p \alpha_{i_1...i_k} + k \alpha_{p(i_1...i_k)} \nabla_{i_1} X^p$. Differentiating the pullback of $\nabla \in \mathfrak{L}(M)$ along the flow of $X \in \text{vec}(M)$ defines the Lie derivative $\mathfrak{L}_X \nabla$ of $\nabla$ along $X$. Explicitly,
\begin{align}
(3.3) \quad (\mathfrak{L}_X \nabla)_{ij} &= \nabla_i \nabla_j X^k + X^p \nabla_p \delta^* \alpha_{ij},
\end{align}
in which the last equality holds because in two dimensions $R_{ijk} = 2 \delta_{i}^{[l} R_{jk]}$. Tracing (3.3) and using the Ricci identity shows that
\begin{align}
(3.4) \quad \nabla_p (\mathfrak{L}_X \nabla)_{ij} = (\mathfrak{L}_X \text{Ric})_{ij} + \nabla_i \nabla_j X^p + 2 R_{ijp} \nabla_i X^p + 2 X^p \nabla_i R_{kj}.
\end{align}
For a symplectic connection, using (2.1), (3.3) simplifies to give
\begin{align}
(3.5) \quad (\mathfrak{L}_X \nabla)_{ij} &= \nabla_i \nabla_j X^k + X^p \delta^* \alpha_{ij} = \nabla_i \nabla_j X^k + X_i R_{jk},
\end{align}

Lemma 3.1. Let $(M, \Omega)$ be a two-dimensional symplectic manifold and $\nabla \in \mathfrak{L}(M, \Omega)$. If a compactly supported vector field $X$ satisfies $\mathfrak{L}_X \nabla = 0$ then $X \in \text{symp}$. Proof. Contracting (3.5) yields $0 = \nabla_i \nabla_p X^p$, so that $\nabla_p X^p$ is constant. Since $X$ has compact support, if $M$ is noncompact the constant must be 0. If $M$ is compact, then $0 = \int_M dX^p = \int_M \nabla_p X^p \Omega_{ij} = \nabla_p X^p \text{vol}_M$ implies $\nabla_p X^p = 0$. In either case, $dX^p = 0$, so $X \in \text{symp}$. □

Given $\nabla \in \mathfrak{L}(M, \Omega)$, define an operator $\mathfrak{H} : \Gamma(T^*M) \to T^*\mathcal{S}(M, \Omega)$ by $\mathfrak{H}(\nabla)_{ij} = (\mathfrak{L}_X \nabla)_{ij}$ for $X \in \Gamma(TM)$. Using its complete symmetry, $\mathfrak{H}$ can be rewritten as follows.
\begin{align}
(3.6) \quad \mathfrak{H}(\alpha)_{ij} &= \nabla_i \nabla_j \alpha_k + \alpha^p \nabla_p \delta^* \alpha_{ij} = \nabla_i \nabla_j \alpha_k + \alpha^p \nabla_p \delta^* \alpha_{ij} \\
&= \nabla_i \nabla_j \alpha_k + \alpha^p \nabla_p \delta^* \alpha_{ij}.
\end{align}
The induced action of $\text{Ham}$ on $\mathfrak{L}(M, \Omega)$ is given by the differential operator $\mathfrak{H} : C^\infty(M) \to T^*\mathcal{S}(M, \Omega)$ defined by
\begin{align}
(3.7) \quad \mathfrak{H}(f)_{ij} &= (\mathfrak{L}_f \nabla)_{ij} = \mathfrak{H}(-df)_{ij} = -\nabla_i \nabla_j df - df \nabla_i R_{jk} \\
&= -\nabla_i \nabla_j df - df \nabla_i R_{jk}.
\end{align}
The tensors $\mathfrak{L}(X^\gamma)$ and $\mathfrak{H}(f)$ can be viewed as the vector fields on $\mathfrak{L}(M, \Omega)$ generated by the actions of the flows of $X$ and $H_f$. The images of $\mathfrak{L}$ and $\mathfrak{H}$ are subbundles of $T^*\mathcal{S}(M, \Omega)$, respectively. It is convenient to abuse terminology and regard $\mathfrak{L}$ as a subspace of $\Gamma(TM)$ via symplectic duality. Lemma 3.1 shows that when restricted to compactly supported vector fields, $\text{ker} \mathfrak{L}$ can be viewed as a subspace of $\text{symp}$. Since $\mathfrak{L}(X, Y) = \mathfrak{L}_X \mathfrak{L}(Y) - \mathfrak{L}_Y \mathfrak{L}(X)$ for $X, Y \in \text{symp}$, the subspaces $\text{ker} \mathfrak{L}$ and $\text{ker} \mathfrak{H}$ are the Lie subalgebras of $\text{symp}$ and $\mathfrak{ham}$, respectively, comprising infinitesimal automorphisms of $\nabla$. Since the group of automorphisms of an affine connection is a finite-dimensional Lie group (see [22]), $\text{ker} \mathfrak{L}$ and $\text{ker} \mathfrak{H}$ are finite dimensional. Specializing the identity (3.3) for $\nabla \in \mathfrak{L}(M, \Omega)$ and $X \in \text{symp}$ yields
\begin{align}
(3.8) \quad \delta \mathfrak{L}(X^\gamma) = \mathfrak{L}(\text{Ric}), \quad \delta \mathfrak{H}(f) = \mathfrak{L}(\text{Ric}).
\end{align}

Lemma 3.2. For $\nabla \in \mathfrak{L}(M, \Omega)$ and $\beta \in T^*\mathcal{S}(M, \Omega)$, define $S(\beta) = \beta^{abc} R_{abc} = -\beta_{[a}^{ab} R_{bb}$. Then
\begin{align}
(3.9) \quad \mathfrak{L} \beta = -\delta \beta - S(\beta), \quad \mathfrak{H}(\beta) = (\delta \mathfrak{L} \beta)^* = \delta \mathfrak{L} \beta = -\delta^3 - \delta S.
\end{align}
Proof. For $\alpha \in \Gamma(T^*M)$ and $\beta \in T^*\mathcal{S}(M, \Omega)$,
\begin{align}
(3.10) \quad \langle \alpha, \mathfrak{L} \beta \rangle = \langle \mathfrak{L}(\alpha), \beta \rangle = (\delta \beta)^2 \alpha, \beta - \langle \alpha, S(\beta) \rangle = \langle \alpha, -\delta^2 \beta - S(\beta) \rangle,
\end{align}
which shows the first equality of (3.10), while the second follows from $\mathfrak{H}(f) = \mathfrak{L}(-df) = \mathfrak{L} \delta^2 f$. □
Although the explicit formula for $\mathcal{H}^*$ is not needed, it is recorded here for reference.

$$\mathcal{H}^*(\beta) = \nabla_i \nabla_j \nabla_k \beta^{ijk} + \nabla_k (R_{ijk} \beta^{ijk}) = \nabla_i \nabla_j \nabla_k \beta^{ijk} + R_{ijk} \nabla_k \beta^{ijk} + \beta^{ijk} \nabla (R_{ijk})$$

(3.11)

The first variation $\delta F(\nabla)$ of a (scalar or tensor valued) functional $F$ on $\mathcal{S}(M, \Omega)$ at $\nabla \in \mathcal{S}(M, \Omega)$ in the direction of $\Pi \in T_{\nabla} \mathcal{S}(M, \Omega)$ is defined by $\delta F(\nabla) = \frac{d}{dt} \langle F(\nabla + t \Pi), \Pi \rangle_{\mathcal{S}}$.

**Lemma 3.3.** Let $(M, \Omega)$ be a symplectic 2-manifold. For $\nabla \in \mathcal{S}(M, \Omega)$ and $\Pi_{ijk} \in T_{\nabla} \mathcal{S}(M, \Omega)$,

$$\rho(\nabla + t \Pi) = \rho(\nabla) - 2t \mathcal{L}^* (\Pi) - 2t^2 (\delta B(\Pi) + \Pi_{jq} \delta \Pi^{pq} - 2t^3 T(\Pi)_j),$$

(3.12)

$$\mathcal{H}(\nabla + t \Pi) = \mathcal{H}(\nabla) + t \mathcal{H}^*(\Pi) + \frac{1}{2} t^2 (\delta \Pi^{pq} + t^3 \delta T(\Pi)),$$

(3.13)

where

$$B(\Pi)_{ij} = \Pi_{ip} \Pi_{jq}^p, \quad (\Pi^{*})_{ij} = 3 \delta \Pi(\Pi)_{ij} - \Pi_{abc} \nabla_i \Pi_{abc}, \quad T(\Pi)_{ij} = \Pi_{ia} \Pi_{b} \Pi_{ik}.$$  

(3.14)

For $f \in C^\infty(M)$,

$$\langle \mathcal{H}(\nabla + t \Pi), f \rangle = \langle \mathcal{H}(\nabla), f \rangle + t \langle \mathcal{H}(f), \Pi \rangle + \frac{1}{2} t^2 \langle \mathcal{H}_f, \Pi \rangle + O(t^3).$$

(3.15)

In particular, the first variations $-2^{-1} \delta \rho_{\mathcal{V}}$ and $\delta \mathcal{H}_{\mathcal{V}}$ of $\rho(\nabla)$ and $\mathcal{H}(\nabla)$ equal the adjoints $\mathcal{L}^*$ : $T_{\nabla} \mathcal{S}(M, \Omega) \to \mathfrak{ve}(M)$ and $\mathcal{H}^*$ : $T_{\nabla} \mathcal{S}(M, \Omega) \to C^\infty(M)$ of the operators $\mathcal{L}$ and $\mathcal{H}$ with respect to the pairing $\langle \cdot, \cdot \rangle$; that is, $-2 \mathcal{L}^*(\Pi) = \delta \rho_{\mathcal{V}}$ and $\mathcal{H}^*(\Pi) = \delta \mathcal{H}_{\mathcal{V}}$.

**Proof.** Let $\tilde{R}_{ij} \Pi$ be the curvature of $\tilde{\nabla} = \nabla + t \Pi_{ij}$ and label with $\cdot$ the tensors derived from it, e.g. $\tilde{R}_{ij}$ is the Ricci curvature of $\nabla$. Then

$$\tilde{R}_{ij} = R_{ij} + t \nabla_p \Pi_{ij} \rho - t^2 \Pi_{ij} \Pi_{pq} \rho = R_{ij} + t \delta \Pi_{ij} - t^2 B(\Pi)_{ij}.$$ (3.16)

Using (3.10) there results

$$\nabla_i \tilde{R}_{jk} = \nabla_i R_{jk} + t \left( \nabla_i \delta \Pi_{jk} - 2 \Pi_{i(p} \delta R_{jk)} \right) - t^2 \left( \nabla_i B(\Pi)_{jk} + 2 \Pi_{i(p} \delta \Pi_{kp)} \right) + 2 t^3 \Pi_{ij(p} \delta \Pi_{kp)}.$$ (3.17)

Contracting (3.17) yields (3.12). Taking the exterior derivative of (3.12), contracting with $\Omega^{ij}$, and using (2.5) yields

$$\mathcal{H}(\nabla + t \Pi) = \mathcal{H}(\nabla) + t \mathcal{L}^*(\Pi) + t^2 \left( \delta^2 B(\Pi) + \delta \Pi^{pq} \delta \Pi_{pq} - \Pi^{abc} \delta^* \delta \Pi_{abc} \right) + t^3 \left( \Pi^{abc} \delta^* B(\Pi)_{abc} - B(\Pi)_{pq} \delta \Pi_{pq} \right).$$ (3.18)

There remains to simplify (3.18) to obtain (3.13). Upon simplifying $\delta \Pi^* \Pi$ using

$$\nabla^P \Pi^{abc} \nabla_p \Pi_{abc} - \nabla^P \Pi^{abc} \nabla_a \Pi_{bcp} = 2 \nabla^P \Pi^{abc} \Omega_{ap} \delta \Pi_{bc} = \delta \Pi^{pq} \delta \Pi_{pq},$$

there results

$$\delta \Pi^* \Pi = 2 \delta^2 B(\Pi) + \nabla^P \Pi_{abc} \nabla_a \Pi_{bcp} + \delta \Pi^{pq} \delta \Pi_{pq} + 3 R^{pq} B(\Pi)_{pq}.$$ (3.20)

On the other hand, $B(\Pi)_{ij} = \Pi_{ij} \Pi_{ab} \Pi_{abc} + \Pi^{abc} \nabla_a \Pi_{bci}$ and a bit of computation shows

$$\delta^2 B(\Pi) = \delta \Pi^{pq} \delta \Pi_{pq} - 2 \Pi^{abc} \delta^* \delta \Pi_{abc} - 3 R^{pq} B(\Pi)_{pq} - \nabla^P \Pi^{abc} \nabla_a \Pi_{bcp}.$$ (3.21)

Comparing (3.20) and (3.21) yields

$$\delta \Pi^* \Pi = 2 \left( \delta^2 B(\Pi) + \delta \Pi^{pq} \delta \Pi_{pq} - \Pi^{abc} \delta^* \delta \Pi_{abc} \right),$$ (3.22)

and substituted in (3.18), along with (3.9), this yields (3.13). If the map $X \to \mathcal{L}_X \Pi$ is viewed as an operator, its formal adjoint applied to $\Pi$ yields $\Pi^* \Pi$. That is, $\langle \mathcal{L}_X \Pi, \Pi \rangle = \langle X, \Pi^* \Pi \rangle$. For $f \in C^\infty(M)$, there follows $\langle \delta (\Pi^* \Pi), f \rangle = -\langle \Pi^* \Pi, \delta^* f \rangle = \langle \mathcal{L}_f, \Pi \rangle$, and so (3.19) follows from (3.13).
Lemma 3.5. For a symplectic structure $\Omega$ on a manifold $M$, if $L \in \mathfrak{symp}$ such that $S(L) \in \mathfrak{symp}$, then for $f, g \in C^\infty(M)$ it follows from (3.13) and (3.12) of Lemma 3.3 that
\begin{equation}
-2\Omega(\mathcal{L}(X), \Pi) = -2\Omega(X^\ast, \mathcal{L}^\ast(\Pi)) = \delta_{\Pi} \Omega(X, \rho),
\end{equation}
\begin{equation}
\Omega(\mathcal{H}(f), \Pi) = \Omega((f, \mathcal{H}^\ast(\Pi)) = \delta_{\Pi} \Omega(f, \mathcal{H}(\nabla)),
\end{equation}
showing that $\rho$ and $\mathcal{K}$ are moment maps. $\square$

Lemma 3.6. Let $\Pi \in T\mathfrak{symp}(M, \Omega)$, $X, Y \in \mathfrak{symp}$, and $f, g \in C^\infty(M)$.
\begin{equation}
\mathcal{L}^\ast(\mathcal{L}(X)) = -\frac{1}{2} \mathcal{L}_X \rho(\nabla), \quad \mathcal{K}^\ast \mathcal{H}(f) = \{f, \mathcal{H}(\nabla)\},
\end{equation}
\begin{equation}
\langle \mathcal{L}(X), \mathcal{L}(Y) \rangle = \langle \Omega(X, Y), \mathcal{K}(\nabla) \rangle, \quad \mathcal{H}(f), \mathcal{H}(g) = \langle \{f, g\}, \mathcal{K}(\nabla) \rangle = \langle f, g, \mathcal{K}(\nabla) \rangle.
\end{equation}

Proof. Differentiating the relations $\rho(\phi^t) = \phi^t \rho(\nabla)$ and $\mathcal{K}(\phi^t) = \mathcal{K}(\nabla) \circ \phi_t$ along the flow $\phi_t$ of $X \in \mathfrak{symp}$, and using Lemma 3.3 yields
\begin{equation}
-2\mathcal{L}^\ast(\mathcal{L}(X)) = \frac{d}{dt}|_{t=0} \phi^t(X) = \frac{d}{dt}|_{t=0} \phi^t \rho(\nabla) = \mathcal{L}_X \rho(\nabla),
\end{equation}
\begin{equation}
\mathcal{K}^\ast \mathcal{L}(X) = \frac{d}{dt}|_{t=0} \mathcal{K}(\phi^t) = \frac{d}{dt}|_{t=0} \mathcal{K}(\nabla) \circ \phi_t = d\mathcal{K}(\nabla)(X) = \mathcal{H}(X, \mathcal{K}(\nabla)).
\end{equation}
Taking $X = H_f$ in (3.24) gives the second equality in (3.24). The third equality in (3.24) follows from (3.25) and (2.5). The penultimate equality of (3.24) is a special case of the preceding, while the final equality follows from the general identity $\langle \{a, b\}, c \rangle = \langle a, \{b, c\} \rangle$, valid for any $a \in C^\infty(M)$ and $b, c \in C^\infty(M)$ (because $\{a, bc\} \Omega$ is exact). $\square$

Lemma 3.5. For a symplectic 2-manifold and $\nabla \in S(M, \Omega)$ consider the orbit $\text{Symp} \cdot \nabla \subset S(M, \Omega)$.

1. The orbit $\text{Symp} \cdot \nabla$ is isotropic if $\mathcal{K}(\nabla) = 0$ or if $d\mathcal{K}(\nabla) = 0$ and $H^1(\mathbb{R}) = \{0\}$.

2. If the orbit $\text{Symp} \cdot \nabla$ is isotropic then $\mathcal{K}(\nabla)$ is constant. In particular, if $M$ is compact then the orbit $\text{Symp} \cdot \nabla$ is isotropic if and only if $\mathcal{K}(\nabla) = 0$.

Proof. The orbit $\text{Symp} \cdot \nabla$ is the image of the map $\Phi^V : \text{Symp} \rightarrow S(M, \Omega)$. If $\text{Symp} \cdot \nabla$ is compact, then $\mathcal{K}(\nabla) = 0$.

For a subspace $V$ of a symplectic vector space $(W, \Omega)$, $V^\perp$ denotes the subspace of $W$ comprising $w \in W$ such that $\Omega(v, w) = 0$ for all $v \in V$.

Lemma 3.6. Let $(M, \Omega)$ be a symplectic 2-manifold and let $\nabla \in \mathcal{K}^{-1}(0)$. Then $\ker \mathcal{K}$ comprises $\Pi \in T\mathfrak{symp}(M, \Omega)$ such that $\mathcal{L}^\ast(\Pi)$ is a closed one-form, and $\mathcal{L}(\mathfrak{symp})^\perp$ comprises $\Pi \in T\mathfrak{symp}(M, \Omega)$ such that $\mathcal{L}^\ast(\Pi)$ is an exact one-form. In particular,
\begin{equation}
\ker \mathcal{K}^\ast = \mathcal{K}(\text{hom}) \subset \mathcal{L}(\mathfrak{symp}) \subset \mathcal{L}(\mathfrak{symp})^\perp \subset \ker \mathcal{K} = \mathcal{K}(\text{hom})^\perp.
\end{equation}

As a consequence there is an injective linear map $\ker \mathcal{K}^\ast / \mathcal{L}(\mathfrak{symp})^\perp \rightarrow H^1(M, \mathbb{R})$ defined by $\Pi + \mathcal{L}(\mathfrak{symp}) \rightarrow [-2\mathcal{L}^\ast(\nabla)]$ and this map can be interpreted as the derivative at $\nabla$ of the map $\phi$ defined in (2.5).
Proof. That \( \ker \mathcal{H} = \mathcal{H}(\mathfrak{ham})^{\perp} \) follows from the strong nondegeneracy of the \( L^2 \) inner product on functions. Precisely, if \( \beta \in \mathcal{H}(\mathfrak{ham})^{\perp} \), then \( 0 = \langle \mathcal{H}(f), \beta \rangle = \langle f, \mathcal{H}^*(\beta) \rangle \) for all \( f \in \mathfrak{ham} \). If \( M \) is noncompact this means \( \mathcal{H}^*(\beta) = 0 \), while if \( M \) is compact it means \( \mathcal{H}^*(\beta) \) is constant, but since \( \mathcal{H}^*(\beta) \) is a divergence, this constant must be 0. Hence \( \mathcal{H}(\mathfrak{ham})^{\perp} \subseteq \ker \mathcal{H}^* \). On the other hand, if \( \mathcal{H}^*(\beta) = 0 \) then \( \langle \mathcal{H}(f), \beta \rangle = \langle f, \mathcal{H}^*(\beta) \rangle = 0 \) for all \( f \in \mathfrak{ham} \), so \( \beta \in \mathcal{H}(\mathfrak{ham})^{\perp} \), and thus \( \ker \mathcal{H}^* = \mathcal{H}(\mathfrak{ham})^{\perp} \). A one-form \( \alpha \) is closed if and only if \( \delta \alpha = 0 \). Since, \( \Pi \in \ker \mathcal{H}^* \) if and only if \( \delta \mathcal{L}^*(\Pi) = 0 \), this implies that \( \mathcal{L}^*(\Pi) \) is a closed one-form if and only if \( \Pi \in \ker \mathcal{H}^* \). If \( \Pi \in \mathcal{L}(\text{sym})^{\perp} \) then \( 0 = \langle \mathcal{L}(\alpha), \Pi \rangle = \langle \alpha, \mathcal{L}^*(\Pi) \rangle = \int_M \alpha \wedge \mathcal{L}^*(\Pi) \) for all closed compactly supported one-forms \( \alpha \) on \( M \). By Poincare duality this means \( \mathcal{L}^*(\Pi) \) is an exact one-form. That \( \mathcal{L}(\text{sym}) \subseteq \mathcal{L}(\text{sym})^{\perp} \) is immediate from (3.22).

Consider a path \( \nabla(t) \) in \( \mathcal{K}^{-1}(0)/\text{Sym}_1 \) such that \( q(\nabla(t)) \) is constant. Let \( \nabla(t) \) be a path in \( \mathcal{K}^{-1}(0) \) lifting \( \nabla(t) \). Then \( \rho(\nabla(t)) = \rho(\nabla(0)) \) is exact for all \( t \) and so, by (3.12) of Lemma 3.3, \( -2\mathcal{L}^*(\Pi) = \frac{d}{dt} \rho(\nabla(t)) \) is exact where \( \Pi = \frac{d}{dt} \rho(\nabla(t)) \). The preceding computations transform naturally under the action of \( \text{Sym}_1 \) on all the objects involved, and so this justifies regarding the map \( \Pi + \mathcal{L}(\text{sym}) \rightarrow [-2\mathcal{L}^*(\nabla)] \) as the derivative at \( \nabla \) of \( q \). □

In section 2 it is asked whether the fibers of the map \( q \) defined in (2.8) are finite-dimensional. Since, by (3.12), the derivative of \( q \) is \( -2\mathcal{L}^* \), this would follow if it could be shown that, when \( \mathcal{K}(\nabla) = 0 \), then \( \mathcal{L}(\text{sym})^{\perp}/\mathcal{L}(\text{sym}) \) is finite-dimensional.

**Lemma 3.7.** Let \((M, \Omega)\) be a symplectic 2-manifold and let \( \nabla \in \mathcal{S}(M, \Omega) \). The sequence
\[
0 \rightarrow \Gamma(T^*M) \xrightarrow{\xi} \Gamma(S^3(T^*M)) \xrightarrow{\mathcal{K}} \Gamma(T^*M) \rightarrow 0
\]
is a complex if and only if \( \rho(\nabla) = 0 \). The sequence
\[
0 \rightarrow H^0(M; \mathbb{R}) \xrightarrow{i} C^\infty(M) \xrightarrow{\mathcal{K}} C^\infty(M) \xrightarrow{\mathcal{K}} 0
\]
is a complex if and only if \( d\mathcal{K}(\nabla) = 0 \).

For compact \( M \) the second \( C^\infty(M) \) in (3.29) can be replaced by \( C^\infty_0(M) \).

**Proof.** The claim about the sequence (3.28) is immediate from Lemma A.1 and Remark A.2 in the appendix. By (3.24), if (3.29) is a complex then \( \mathcal{K}(\nabla) \) Poisson commutes with every \( f \in C^\infty(M) \), so \( d\mathcal{K}(\nabla) = 0 \). On the other hand, by (3.24), if \( d\mathcal{K}(\nabla) = 0 \) then \( \mathcal{K}(\mathfrak{ham}) \subseteq \ker \mathcal{H}^* \). □

### 4. Extremal symplectic connections

Recall that \( \nabla \in \mathcal{S}(M, \Omega) \) is extremal if it is a critical point with respect to compactly supported variations of the functional \( \mathcal{E}(\nabla) = \int_M \mathcal{K}(\nabla)^2 \Omega \). Additionally, \( \nabla \in \mathcal{S}(M, \Omega) \) is gauge extremal if it is a critical point of \( \mathcal{E} \) for all variations of the form \( \delta^* \alpha \) for compactly supported \( \alpha \in \Gamma(S^2(T^*M)) \). By definition an extremal symplectic connection is gauge extremal.

**Lemma 4.1.** A symplectic connection \( \nabla \in \mathcal{S}(M, \Omega) \)

1. is extremal if and only if \( \mathcal{K}(\mathcal{K}(\nabla)) = 0 \);
2. is gauge extremal if and only if \( \mathcal{L}_{\mathfrak{H}_K(\nabla)} \text{Ric} = 0 \).

**Proof.** Calculating the first variation \( \delta_{\Pi} \mathcal{E}(\nabla) \) among \( \Pi \in T_{\nabla} \mathcal{S}(M, \Omega) \) using Lemma 3.3 yields
\[
\delta_{\Pi} \mathcal{E}_\nabla = 2(\mathcal{K}(\nabla), \mathcal{K}^*(\Pi)) = 2\mathcal{L}_{\nabla}(\mathcal{K}(\mathcal{K}(\nabla))), \Pi, \Pi
\]
which yields (1). By (4.1) and (3.8),
\[
\delta_{\delta^* \alpha} \mathcal{E}_\nabla = 2(\mathcal{K}(\mathcal{K}(\nabla)), \delta^* \alpha) = 2(\delta^* \mathcal{H}(\mathcal{K}(\nabla)), \alpha) = 2(\mathcal{L}_{\mathcal{K}(\nabla)} \text{Ric}, \alpha),
\]
which yields (2). □
Claim (2) of Lemma 4.1 shows that $\nabla$ is gauge extremal if and only if the Hamiltonian vector field $H_{\nabla(\nabla)}$ generated by its moment map image preserves its Ricci tensor.

**Lemma 4.2.** On a compact symplectic 2-manifold $M$ of genus at least two, if a symplectic connection $\nabla$ is gauge extremal and $\mathcal{K}(\nabla)$ is not identically zero, then the Ricci tensor of $\nabla$ is degenerate somewhere on $M$.

**Proof.** A compact orientable surface supporting a mixed signature metric must be a torus, so if the Ricci tensor is nondegenerate, it must be definite because of the hypothesis on the genus. Then, by claim (2) of Lemma 4.1 $H_{\nabla(\nabla)}$ is a Killing field for a Riemannian metric, and so by Bochner’s theorem must be identically 0. Hence $\mathcal{K}(\nabla)$ is constant, so zero, because $M$ is compact. □

Because $\mathcal{S}(M,\Omega)$ is affine it carries a canonical flat connection $\delta$. Via the corresponding parallel translation elements $\alpha, \beta \in \Gamma(S^3(T^*M))$ can be viewed as constant, parallel vector fields on $\mathcal{S}(M,\Omega)$. The Lie algebra structure on vector fields on $\mathcal{S}(M,\Omega)$ is generated by its group of translations, and so, viewed as vector fields on $\mathcal{S}(M,\Omega)$, $\alpha$ and $\beta$ commute. The differential of $\mathcal{F}$ is $\delta_\mathcal{F}$, and $\delta_\mathcal{F}$ is its evaluation on the constant vector field $\alpha$. The Hessian of a twice differentiable functional is well defined at a critical point without the imposition of any extra structure. If there is fixed a connection, then the Hessian is well defined everywhere, where the Hessian of a functional $\mathcal{F}$ on $\mathcal{S}(M,\Omega)$ means its Hessian with respect to the flat connection $\delta$. The Hessian $\delta_\mathcal{F}$ at $\alpha$ and $\beta$ is $\delta_\mathcal{F}(\beta) = \delta_\mathcal{F}(\beta) - \delta_\mathcal{F} = \delta_\mathcal{F}$; the term $\delta_\mathcal{F}$ vanishes because $\delta_\mathcal{F}$ is $\delta_\mathcal{F}$. It follows that the Hessian of $\mathcal{F}$ at $\nabla \in \mathcal{S}(M,\Omega)$ in the directions $\alpha$ and $\beta$ is given by the first variation in the direction $\alpha$ of the first variation in the direction $\beta$ of $\mathcal{F}$, and so can be written simply $\delta_\mathcal{F}(\nabla)$. Define the Jacobi operator $\nabla: \Gamma(S^3(T^*M)) \to \Gamma(S^3(T^*M))$ associated to $\nabla \in \mathcal{S}(M,\Omega)$ by

$$J(\alpha) = \mathcal{H}(\alpha) + L_{H_{\nabla(\nabla)}}\alpha.$$ 

The name for $J$ is justified by the following observations. Let $f \in C^\infty(M)$. By the definition of $\mathcal{H}(f)$ and the fact that $H_f$ commutes with raising and lowering indices there holds

$$\delta_\mathcal{F}(\mathcal{H}(f)) = \frac{d}{dt}|_{t=0} \mathcal{H}(\nabla + tf) = \frac{d}{dt}|_{t=0} \mathcal{H}(\nabla + t\alpha)^{\nabla},$$

in which $(\alpha^i)^k = \alpha_{ij}^{-1}$ and $L_{H_f}(\nabla + t\alpha)^{\nabla}$ means $L_{H_f}(\nabla + t\alpha)^{ij}. Let \nabla(t)$ be a $C^1$ curve in the space of extremal symplectic connections depending smoothly on $t$ and such that $\frac{d}{dt}|_{t=0} \nabla(t) = \alpha_{ij}^{\nabla}$. By (3.13) and (4.4), differentiating $0 = \mathcal{H}(\nabla_{\nabla(\nabla)}) = \mathcal{H}(\nabla(\nabla))$ in $t$ yields $0 = J(\alpha)$, so that the kernel of $J$ describes the tangent space to the deformations of an extremal symplectic connection through extremal symplectic connections.

**Lemma 4.3.** The Hessian of $\mathcal{E}$ at $\nabla \in \mathcal{S}(M,\Omega)$ in the directions $\alpha, \beta \in T_{\nabla}\mathcal{S}(M,\Omega)$ is

$$\delta_\mathcal{F}(\mathcal{E}(\nabla)) = 2\mathcal{H}(\alpha, \beta) = 2\mathcal{H}(\alpha, \beta) + 2\mathcal{E}(\alpha, \beta).$$

**Proof.** By (3.1) and (4.4),

$$\delta_\mathcal{F}(\mathcal{E}(\nabla)) = 2\delta_\mathcal{F}(\mathcal{H}(\nabla), \beta) = 2\mathcal{H}(\alpha, \beta) + 2\mathcal{E}(\alpha, \beta).$$

An alternative proof is to expand $\mathcal{E}(\nabla + \nabla(\nabla))$ using (3.13) and simplify the result as in (3.15). □

**Corollary 4.4.** Let $\nabla \in \mathcal{S}(M,\Omega)$. If $\mathcal{K}(\nabla)$ is constant then $\delta_\mathcal{F}(\mathcal{E}(\nabla)) = 0$ with equality if and only if $\mathcal{H}(\alpha, \beta) = 0$. If $M$ is compact then a moment constant $\nabla$ is an absolute minimizer of $\mathcal{E}$.

**Proof.** Because $\mathcal{K}(\nabla)$ is constant, the last term in (4.5) vanishes, so $\delta_\mathcal{F}(\mathcal{E}(\nabla)) = \mathcal{H}(\alpha, \beta) = \mathcal{H}(\alpha, \beta) = 0$, with equality if and only if $\mathcal{H}(\alpha, \beta) = 0$. The final claim holds because, if $M$ is compact, then a moment constant connection is moment flat. □

**Lemma 4.5.** For $X \in \mathfrak{so}(\mathfrak{symp}, M, \nabla)$, $J\mathcal{L}(X^\nabla) = \mathcal{L}_X\mathcal{K}(\nabla)$, where $\mathcal{K} = \mathcal{K}(\nabla)$. In particular, if $\nabla$ is extremal then $\delta_\mathcal{F}(\mathcal{L}_X\mathcal{K}(\nabla)) = 0$ for all $X \in \mathfrak{so}(\mathfrak{symp}, M, \nabla)$. 

5. Structural results and examples

5.1. Preferred connections are extremal. Let $\nabla \in \mathcal{S}(M, \Omega)$. Since $\rho = 2\delta \text{Ric}$, by (3.2),

$$\delta^* \rho = 2\delta^* \text{Ric} = -3\delta^* \text{Ric}.$$ 

(5.1)

Since $-2\mathcal{K} = \delta \rho$, by (3.2),

$$2d\mathcal{K} = -2\delta^* \mathcal{K} = \delta^* \delta \rho = -2\delta^* \rho + 2R_{ip} \rho^p.$$ 

(5.2)

By (5.1),

$$\nabla \rho = -\delta^* \rho - 3\mathcal{K}, 
3\nabla_i R_{jk} = -3\delta^* \text{Ric}_{ijk} - \Omega_{ijkl} \rho_k.$$ 

(5.3)

Since $\delta d\mathcal{K} = 0$, by (5.2),

$$\delta^2 \delta^* \rho = \nabla_p (R_{q}^{\ p} \rho^q) = R^{pq} \delta^* \rho_{pq}.$$ 

Differentiating (5.2) and simplifying the result using (5.3), $\delta^2 \delta^* \rho = R^{pq} \delta^* \rho_{pq}$, and (3.2) yields

$$\nabla_i d\mathcal{X}_j = \frac{1}{6} \rho_i \rho_j - \mathcal{K} R_{ij} - \rho^p \delta^* \text{Ric}_{ijp} + 4R_{(i}^{\ p} \delta^* \rho_{j)p} - \frac{3}{2} \delta^* \delta \rho_{ij}.$$ 

(5.4)

Differentiating (5.4) yields an explicit expression for $\mathcal{H}(\mathcal{X})$. Without some further hypothesis, (5.4) is too complicated to be useful, but it simplifies considerably if $\delta^* \rho$ or $\delta^* \text{Ric}$ vanishes. A symplectic connection $\nabla$ is preferred if $-\delta^* \text{Ric}_{ijk} = \nabla_i (R_{jk}) = 0$. This notion was introduced and studied in detail for surfaces by F. Bourgeois and M. Cahen in [5].

**Theorem 5.1.** On a symplectic 2-manifold a preferred symplectic connection is extremal.

**Proof.** By (5.1), if $\nabla \in \mathcal{S}(M, \Omega)$ is preferred, then $\delta^* \rho = 0$, so (5.4) simplifies to

$$\nabla_i d\mathcal{X}_j = \frac{1}{6} \rho_i \rho_j - \mathcal{K} R_{ij}.$$ 

(5.5)

Differentiating (5.5) shows $\nabla_i \nabla_j d\mathcal{X}_k = -d\mathcal{X}_i R_{jk}$, so that $\mathcal{H}(\mathcal{X}) = 0$. □

**Remark 5.2.** The essential content of the conclusion of Theorem 5.1 that for a preferred symplectic connection the vector field $H_{\nabla(\mathcal{X})}$ is an infinitesimal affine automorphism of $\nabla$, is implicit in the proof of Proposition 6.1 of [5]. In particular, in sections 5 and 6 of [5], in the context of preferred symplectic connections, a key role is played by a function $\beta$ and one-form $u$ that are constant multiples of the specializations to this case of $\mathcal{X}(\nabla)$ and $\rho$. For example, the identity $3\nabla_i R_{jk} = -\Omega_{i(j} \rho_{k)}$ is in [5], and (6.1) of [5] is equivalent to (5.5).
5.2. **Structural lemmas.** Lemma 5.3 shows that given an extremal symplectic connection that is not moment constant there is a large Darboux coordinate chart on which $\mathcal{K}$ can be considered the action coordinate of action-angle variables. This information is useful both for proving nonexistence of such connections in Theorem 2.3 and for constructing examples.

For $f \in C^\infty(M)$ let $\mathfrak{crit}(f) = \{ p \in M : df_p = 0 \}$ be the set of the critical points of $f$. For $\nabla \in \mathcal{S}(M, \Omega)$ let $Z(\rho)$ be the set of points of $M$ where $\rho$ vanishes and let $\text{dg}(\nabla)$ be the set of points in $M$ where $d\mathcal{K} \wedge \rho$ vanishes. Note that $\text{dg}(\nabla) \supset \mathfrak{crit}(\mathcal{K}) \cup Z(\rho)$.

**Lemma 5.3.** Let $(M, \Omega)$ be a symplectic two-manifold and let $\nabla \in \mathcal{S}(M, \Omega)$ be extremal symplectic with moment map $\mathcal{K} = \mathcal{K}(\nabla)$.

1. If $\mathcal{K}$ is not constant then each connected component of $\mathfrak{crit}(\mathcal{K})$ is a point or a closed image of a geodesic of $\nabla$, and these connected components are isolated in the sense that among each there is an open neighborhood containing no other connected component of $\mathfrak{crit}(\mathcal{K})$.

2. There is a constant $\tau$ such that

$$\mathcal{K}^2 + \rho^2 d\mathcal{K}_\rho = \tau.$$  

Consequently $\text{dg}(\nabla) = \{ p \in M : \mathcal{K}(p)^2 = \tau \}$ and

(a) $\text{dg}(\nabla) = M$ if and only if $\mathcal{K}$ is constant.

(b) if $\text{dg}(\nabla)$ is empty then $M$ is noncompact.

(c) if $\tau$ is negative then $\text{dg}(\nabla)$ is empty.

(d) If $\text{dg}(\nabla)$ is nonempty, $\tau$ is nonnegative and at every point of $\text{dg}(\nabla)$, $|\mathcal{K}|$ equals $\sqrt{\tau}$.

(e) A maximal integral curve of the symplectically dual vector field $\rho^{\wedge^1} = \rho^{i}$ intersecting $\text{dg}(\nabla)$ lies entirely in $\text{dg}(\nabla)$ and $\mathcal{K}$ is constant along the curve, equal to one of $\pm \sqrt{\tau}$.

(f) If $\text{vol}(M) < \infty$, $\varepsilon(\nabla) < \infty$, and $d(\rho(H_\mathcal{K}))$ is integrable then

$$\tau \text{vol}(M) = 3\varepsilon(\nabla) + \int_M d(\mathcal{K}\rho).$$

In particular, if $M$ is compact then $\tau \text{vol}(M) = 3\varepsilon(\nabla)$.

3. If $\mathcal{K}$ is not constant, then on the nonempty open subset $\tilde{M} = M \setminus \text{dg}(\nabla)$ there hold:

(a) the one-form $\sigma = (\tau - \mathcal{K}^2)^{-1} \rho$ is closed;

(b) $d(\mathcal{K}\sigma) = d\mathcal{K} \wedge \sigma = \Omega$;

(c) the symplectically dual vector field $\sigma^{\wedge^1} = \sigma^{j}$ commutes with $H_\mathcal{K}$ and $\Omega(\sigma^{\wedge}, H_\mathcal{K}) = 1$;

(d) Around any point $p \in \tilde{M}$ at which $d\mathcal{K}$ does not vanish there are local coordinates $(x, y)$ such that $p$ corresponds to the origin, $\mathcal{K} - \mathcal{K}(p) = x$, $dy = (\tau - \mathcal{K}^2)^{-1} \rho$, and $\Omega = dx \wedge dy$.

(e) the complex structure $J$ defined on $\tilde{M}$ by $J(\sigma^{\wedge}) = H_\mathcal{K}$ and $J(H_\mathcal{K}) = -\sigma^{\wedge}$ is $H_\mathcal{K}$ and $\sigma^{\wedge}$ invariant;

(f) If $\tau > 0$, the function

$$T = \frac{1}{2\sqrt{\tau}} \log \left| \frac{\sqrt{\tau} + \mathcal{K}}{\sqrt{\tau} - \mathcal{K}} \right| = \begin{cases} \tau^{-1/2} \arctanh(\tau^{-1/2} \mathcal{K}) & \text{if } \tau > \mathcal{K}^2, \\ \tau^{-1/2} \text{arccoth}(\tau^{-1/2} \mathcal{K}) & \text{if } 0 < \tau < \mathcal{K}^2, \end{cases}$$

is smooth on $\tilde{M} \setminus \text{dg}(\nabla)$, while if $\tau < 0$ the function

$$T = \tau^{-1/2} \text{arccot}(\tau^{-1/2} \mathcal{K})$$

is smooth on all of $M$. With the complex structure $J$ defined by $J(H_T) = \rho^{\wedge}$ and $J(\rho^{\wedge}) = -H_T$ the flat Riemannian metric

$$(k_{ij} = dT_i dT_j + \sigma_i \sigma_j)$$

forms a Kähler structure with volume $(\tau - \mathcal{K}^2)^{-1} \Omega = dT \wedge \sigma = d(T\sigma)$. Moreover, $H_\mathcal{K}$ is a Killing field for $k_{ij}$. 
(g) If $\mathrm{dg}(\nabla)$ is nonempty, then $T$ maps each maximal integral manifold of the vector field $\rho^\lambda$ on $M$ diffeomorphically onto its image. Precisely, if $\phi : I \to M$ is a maximal integral curve of $\rho^\lambda$ such that $\phi(0) = p \in M$ then $T \circ \phi(t) = t + T(p)$.

(h) If $\mathrm{dg}(\nabla)$ is nonempty, and the restriction to $\tilde{M}$ of $\rho^\lambda$ is complete, then the metric $k$ of (5.10) is complete on $\tilde{M}$, and each connected component of $\tilde{M}$ with the Riemann surface structure $(k, J)$ is conformally equivalent to the complex plane or the punctured disk.

Proof. Each connected component of the zero set of an infinitesimal automorphism of a torsion-free affine connection is a closed totally geodesic submanifold (see p. 61 of [23]). In particular this applies to the zeros of $H_K$, that is to $\text{crit}(K)$. If a connected component of $\text{crit}(K)$ has nonempty interior then it must be all of $M$, in which case $K$ is constant. Hence if $K$ is nonconstant each connected component of $\text{crit}(K)$ is a closed totally geodesic submanifold of $M$ of codimension at least one, so is a closed image of a geodesic or a point. Suppose $K$ is nonconstant. Suppose $x \in \text{crit}(K)$ and every sufficiently small neighborhood of $x$ contains some point of $\text{crit}(K)$ distinct from $x$. Fix a geodesically convex open neighborhood $U$ of $x$ and choose $y$ in $\text{crit}(K) \cap U$ distinct from $x$. Let $L$ be the unique geodesic segment connecting $x$ to $y$ and contained in $U$. Since the flow of $H_K$ fixes $x$ and $y$ it fixes $L$ too, and so $L \subset \text{crit}(K)$. Hence $x$ and $y$ belong to the same connected component of $\text{crit}(K)$. It follows that the connected components of $\text{crit}(K)$ are isolated.

Since the flow of $H_K$ preserves $\nabla$ and $\Omega$ it preserves $\rho$, so, using (2.5),

\begin{equation}
0 = \mathfrak{L}_{H_K} \rho = \iota(H_K) d\rho + d(\rho(H_K)) = 2K d\kappa + d(\rho(H_K)) = d(\kappa^2 + \rho(H_K)),
\end{equation}

from which it follows that there is a constant $\tau$ satisfying (5.9). From (5.9) it is apparent that $\mathrm{dg}(\nabla)$ is the zero locus of $\tau - \kappa^2$. In particular, $K$ is constant if and only if $\mathrm{dg}(M) = \tilde{M}$. If $\mathrm{dg}(\nabla)$ is empty, then $d\kappa$ does not vanish, so $M$ must be noncompact.

By (5.9), if $q \in \mathrm{dg}(\nabla)$ then $\mathfrak{L}_{K}(q)^2 = \tau$, so if $\mathrm{dg}(\nabla)$ is nonempty, then $\tau$ is nonnegative and, at every point of $\mathrm{dg}(\nabla)$, $|K|$ equals $\sqrt{\tau}$. Let $q \in \mathrm{dg}(\kappa)$ and let $I \subset \mathbb{R}$ be a maximal open interval around 0 such that $\phi : I \to M$ is a smooth integral curve of $\rho^\lambda$ satisfying $\phi(0) = q$. By (5.9), $u(t) = K \circ \phi(t)$ solves $\dot{u} = \tau - u^2$. If $\phi(0) = q \in \mathrm{dg}(\nabla)$ then $u(0) = K(q) = \pm \sqrt{\tau}$ and so, by the uniqueness of the solution to the initial value problem for $\dot{u} = \tau - u^2$, $K \circ \phi(t) = u(t) = \pm \sqrt{\tau}$ for all $t \in I$, and hence $\phi(I) \subset \mathrm{dg}(\nabla)$.

If $\text{vol}_\Omega(M)$ is finite and $\mathcal{E}(\nabla)$ is finite, then integrating (5.9) and using (2.5) yields

\begin{equation}
\tau \text{vol}_\Omega(M) = \int_M \kappa^2 \Omega + \int_M \rho^\lambda dKp = \mathcal{E}(\nabla) + \int_M dK \wedge p = \mathcal{E}(\nabla) + \int_M dK \wedge p - \int_M \kappa \wedge dp + \int_M d(Kp) - 2\mathcal{E}(\nabla) + \int_M d(Kp),
\end{equation}

where the last expression makes sense provided $d(Kp)$ is integrable. This shows (5.12).

That $\sigma = (\tau - \kappa^2)^{-1} \rho$ is closed on $\tilde{M}$ follows from differentiating (5.9) and using (2.5) and

\begin{equation}
dK \wedge p = (\tau - \kappa^2) \Omega.
\end{equation}

That $dK \wedge \sigma = \Omega$ is immediate from (5.13). Since $\mathfrak{L}_{H_K} \rho = 0$, the vector fields $\rho^\lambda$ and $H_K$ commute. Since any function of $K$ is constant along the flow of $H_K$, this implies $\sigma^\lambda$ commutes with $H_K$. Because $\sigma^\lambda$, $H_K$, and $\Omega$ are all invariant under $H_K$ and $\sigma^\lambda$ so too is the complex structure $J$.

If $\nabla$ is extremal but not moment constant, then on the universal cover of each connected component of $\tilde{M}$ there are coordinates $x$ and $y$ such that $\Omega = dx \wedge dy$, $K = x$, and $\rho = (\tau - x^2)^{-1} dy$. Simply define $x = K$ and let $y$ be a global primitive of $\sigma$. This proves (5.11).

The claims of (5.12) are all verified by straightforward computations. Now suppose $\mathrm{dg}(\nabla)$ is a nonempty proper subset of $M$, so that $\tau > 0$. Given $p \in M$ let $\phi : I = (-a, b) \to M$ be a maximal integral curve of $\rho^\lambda$ such that $\phi(0) = p$. Since $dT_p \rho^\lambda = (\tau - \kappa^2)^{-1} dK_p \rho^\lambda = 1$, the function
Remark 5.4. Let $\bar{\phi}$ be globally defined. Let $C$ be a connected component of $M$. Then $C$ contains a connected component $S$ of $K^{-1}(0)$ and, because $T \circ \phi_t(p) = t$ for $p \in K^{-1}(0)$, the map $\Phi : \mathbb{R} \times S \to C$ defined by $\Phi(t, p) = \phi_t(p)$ is a diffeomorphism. Since $S$ is a connected one-manifold it is diffeomorphic to a circle $S^1$ or the line $\mathbb{R}$. Since $\phi_t'(\sigma) = \sigma$, the pullback $\Phi^\ast(\sigma)$ is a closed one-form on $S$ extended trivially to $\mathbb{R} \times S$, so is exact if $S$ is a line, or is a multiple of the generator $\partial \theta$ of the cohomology of $S^1$ if $S$ is a circle. It follows that the pullback via $\Phi$ of the metric $k$ of (5.10) is the flat metric on the plane $\mathbb{R} \times \mathbb{R}$ or on the infinite Euclidean cylinder $\mathbb{R} \times S^1$. The completeness of $k$ follows. In this case, each connected component of $M \setminus dg(\mathcal{K})$ carries a complete flat Kähler structure and a nontrivial holomorphic vector field preserving this Kähler structure. A Riemann surface with biholomorphism group that is not discrete is biholomorphic to one of the following: the Riemann sphere, the plane, the punctured plane, a torus, the unit disc, or an annulus of radii $1$ and $2$ where the only surfaces covered holomorphically by the plane are the plane, torus, and punctured disk. Since $\mathcal{K}$ is not constant and $dg(\nabla)$ nonempty, the torus cannot occur as the complement of $dg(\nabla)$. \hfill $\square$

Remark 5.5. Let $M$ and $\nabla$ be as in Lemma 5.3. Then the metric
\begin{equation}
 h_{ij} = |\tau - K^2|^{-1}dK_xdK_y + |\tau - K^2|\sigma_i\sigma_j = |\tau - K^2|k_{ij}
\end{equation}
on $\bar{M} = M \setminus \{dg(\nabla)\}$ has constant scalar curvature $R_h = 2$ where $\tau > K^2$ and constant scalar curvature $R_h = -2$ where $\tau < K^2$, its volume form $vol_h$ consistent with the orientation given by $\Omega$ equals $\Omega$, and $H_{\mathcal{K}}$ is a Killing field for $h$. However the restriction of $h$ to a connected component of $\bar{M}$ is generally not complete, so it is not clear how useful is $h$.

This can be proved as follows. Suppose $dg(\nabla)$ is nonempty and fix $p \in \bar{M} = M \setminus \{dg(\nabla)\}$. Since $\sigma$ is closed, there is a neighborhood of $p$ on which there is a smooth function $\phi$ such that $d\phi = \sqrt{\tau}\sigma$. If $p$ is contained in a connected component of $\bar{M}$ on which $\tau > K^2$ is positive, then, since $\mathcal{K} \in [-\sqrt{\tau}, \sqrt{\tau}]$, there can be defined on this neighborhood a smooth function $\theta$ such that $K = -\sqrt{\tau}\cos \theta$. In the coordinates $(\theta, \phi)$ around $p$ the metric $h$ takes the form $d\theta^2 + \sin^2 \theta d\phi^2$, which is one of the well known standard forms of the spherical metric. This shows $h$ has constant scalar curvature $2$. Its volume form is $\sin \theta d\theta \wedge d\phi = dK \wedge \sigma = \Omega$. If $p$ is contained in a connected component of $\bar{M}$ on which $\tau - K^2$ is negative, then setting $\mathcal{K} = \pm \sqrt{\tau}\cos \theta$ and $d\phi = \pm \sqrt{\tau}\sigma$ as $\mathcal{K}$ is positive or negative, there result $h = d\theta^2 + \sinh^2 \theta d\phi^2$, a standard form of the hyperbolic metric, and $\Omega = dK \wedge \sigma = \sinh \theta d\theta \wedge d\phi$. Since $H_{\mathcal{K}}$ preserves $K$, $dK$, $\Omega$, and $\sigma$, it preserves $h$.

Remark 5.5. For preferred symplectic connections, Lemma 5.3 specializes to the results of sections 5 and 6 of [5], and 4 of Lemma 5.3 specializes to Proposition 11.4 of [5].

Lemma 5.6 concludes from Lemma 5.3 that on a compact surface carrying an extremal symplectic connection the complement of the set of critical points of $\mathcal{K}$ is a union of parabolic Riemann surfaces.

Lemma 5.6. Let $(M, \Omega)$ be a compact symplectic two-manifold and let $\nabla \in \mathcal{S}(M, \Omega)$ be extremal symplectic with moment map $\mathcal{K} = \mathcal{K}(\nabla)$.

1. There holds $Z(\rho) \subset \text{crit}(\mathcal{K}) = \{dg(\nabla)\}$ and every critical point of $\mathcal{K}$ is a global extremum at which $|\mathcal{K}|$ equals $\sqrt{\tau}$. In particular, $\tau = 0$ if and only if $\mathcal{K}$ is identically zero.

2. If $|\mathcal{K}|$ is not constant, then:
   a. Each $r \in (-\sqrt{\tau}, \sqrt{\tau})$ is a regular value of $\mathcal{K}$ and the level set $L_r(\mathcal{K}) = \{q \in M : \mathcal{K}(q) = r\}$ is a disjoint union of smoothly embedded circles.
   b. For $r, s \in (-\sqrt{\tau}, \sqrt{\tau})$, the level sets $L_r(\mathcal{K})$ and $L_s(\mathcal{K})$ are diffeomorphic.
   c. The number of connected components of $\text{crit}(\mathcal{K})$ where $\mathcal{K}$ assumes its minimum equals the number of connected components of $\text{crit}(\mathcal{K})$ where $\mathcal{K}$ assumes its maximum.
(d) Each connected component of $M = M \setminus \text{crit}({\mathcal{K}})$ is diffeomorphic to the infinite Euclidean cylinder $\mathbb{R} \times S^1$, and the metric $k_{ij}$ of (5.10) is complete, isometric to the standard flat metric on $\mathbb{R} \times S^1$.

Proof. Because $M$ is compact, $\mathcal{K}$ has a maximum and a minimum, at which it takes the values $\pm \sqrt{\tau}$. Since it takes these same values at any point in $dg(\nabla)$, any point of $dg(\nabla)$ is a global extremum. In particular $dg(\nabla) = \text{crit}({\mathcal{K}})$. If $\mathcal{K}$ is not constant then for $r \in (-\sqrt{\tau}, \sqrt{\tau})$, by (5.10), along $L_r(\mathcal{K})$ there holds $\rho^\ast d\mathcal{X}_p = \tau - r^2 > 0$, so that $d\mathcal{X}$ is nonvanishing along $L_r(\mathcal{K})$. This suffices to show that $L_r(\mathcal{K})$ is a smoothly embedded one-dimensional submanifold. Since it is also closed, it must be a union of circles. If $-\sqrt{\tau} < r < s < \sqrt{\tau}$ then $\mathcal{K}^{-1}([r, s])$ is compact and contains no critical point of $\mathcal{K}$, so $L_r(\mathcal{K})$ is diffeomorphic to $L_s(\mathcal{K})$ (see Theorem 3.1 of [21]); moreover, the flow of $\rho^\ast$ maps $L_r(\mathcal{K})$ diffeomorphically onto $L_s(\mathcal{K})$. The claim about the equality of the numbers of components of $\text{crit}({\mathcal{K}})$ on which $\mathcal{K}$ assumes its minimum and maximum follows. Since $M$ is compact, $\rho^\ast$ is complete, and so (24) follows from (31) of Lemma 5.3. □

Note that in Lemma 5.6 the compactness is used in part to guarantee the existence of a global flow for $\rho^\ast$ and several of the conclusions are valid in greater generality provided such a global flow exists. The compactness is used in a more essential way to invoke the regular interval theorem to conclude that the level sets of $\mathcal{K}$ are unions of circles.

With Lemma 5.6 in hand the proof of Theorem 2.3 is straightforward.

Proof of Theorem 2.3. Suppose $\nabla$ is extremal and not moment flat. It will be shown that $M$ must be a sphere or a torus. As in the proof of Lemma 5.6 $\mathcal{K}^{-1}(I_\ast)$ is a disjoint union of cylinders, where $I_\ast = [-\sqrt{\tau} + \epsilon, \sqrt{\tau} - \epsilon]$. Each of the cylinders constituting $\mathcal{K}^{-1}(I_\ast)$ has a positive end and a negative end, as $\mathcal{K}$ tends to a maximum or a minimum at the end. For a suitably small $\epsilon$, the complement $M \setminus \mathcal{K}^{-1}(I_\ast)$ is a disjoint union of tubular neighborhoods of the connected components of $\text{crit}({\mathcal{K}})$; precisely, its connected components are cylindrical bands around the closed geodesics contained in $\text{crit}({\mathcal{K}})$ and disks around the points contained in $\text{crit}({\mathcal{K}})$. Each of these cylinders and disks is positive or negative as $\mathcal{K}$ has on the connected component of $\mathcal{K}$ that it contains a maximum or minimum. It follows that $M$ is the union of cylinders attached end to end and disks attached to cylinders in a manner compatible with the assignment of signs to the cylinders and disks. A closed compact surface obtained by attaching cylinders end to end and cylinders to disks has nonnegative Euler characteristic; since $M$ is oriented, it must be a sphere or a torus.

An alternative way of reaching the conclusion about the structure of the complement of $\text{crit}({\mathcal{K}})$ goes as follows. By (33) of Lemma 5.6 each connected component of $M \setminus \text{crit}({\mathcal{K}})$ carries a parabolic complex structure $J$ and a nontrivial holomorphic vector field preserving $J$. A Riemann surface with biholomorphism group that is not discrete is biholomorphic to one of the following: the Riemann sphere, the plane, the torus, the unit disc, the punctured unit disc, or an annulus of radii $r < 1$ and $1$. If these the only parabolic surfaces are the plane, torus, and punctured disk. If $\text{crit}({\mathcal{K}})$ is nonempty, closed compact surfaces cannot be obtained as connected components of its complement, and the plane is not the complement of a set containing at least two connected components. The remaining option is the punctured disk which is conformally equivalent to a cylinder, and the rest of the argument is as before. □

There remains the question of whether there exist on the torus or sphere extremal symplectic connections that are not moment flat. It was shown in [5] that a preferred symplectic connection on a compact surface must be locally symmetric. A key point in the argument is to show that $\mathcal{K}(\nabla)$ must have a nondegenerate critical point; this uses in an essential way the simplification of (5.3) to (5.5) available in this case, and it is not clear if this argument can be adapted to the more general setting of extremal symplectic connections considered here.
5.3. Symplectic connections in Darboux coordinates. Claim (3d) of Lemma 5.3 motivates calculating $\mathcal{H}(\mathcal{X})$ explicitly for a symplectic connection in Darboux coordinates.

**Lemma 5.7.** Let $x$ and $y$ be global affine coordinates on $\mathbb{R}^2$ with respect to the standard flat affine connection $\partial$ and let $\Omega = dx \wedge dy$ be the standard symplectic form. Write $X = \partial_x$ and $Y = \partial_y$ for the coordinate vector fields. Partial derivatives with respect to $x$ and $y$ will be indicated by subscripts. The most general $\nabla \in \mathfrak{S}(\mathbb{R}^2, \Omega)$ has the form $\nabla = \partial + \Pi$ where

$$\Pi(X, X) = AX + BY, \quad \Pi(X, Y) = \Pi(Y, X) = -DX - AY, \quad \Pi(Y, Y) = CX + DY,$$

for some $A, B, C, D \in C^\infty(\mathbb{R}^2)$. For $\nabla \in \mathfrak{S}(\mathbb{R}^2, \Omega)$ of the form $\nabla = \partial + \Pi$ with $\Pi$ as in (5.15), (5.16)

$$-\frac{1}{2} \rho = (2A_{xy} - B_{yy} - D_{xx} + (AD - BC)_x + 3(A^2 - BD)_y + 2AD_x - DA_y - BC_x)dx + (2D_{xy} + A_{yy} + C_{xx} + 3(AC - D^2)_x - (AD - BC)_y - 2DA_y + AD_B + CB_y)dy,$$

and, writing $\mathcal{K} = \mathcal{K}(\nabla)$,

$$\mathcal{K}(\mathcal{K}) = (\mathcal{K}_{xxx} - 3A\mathcal{K}_{xx} - 3B\mathcal{K}_{xy} - B_x\mathcal{K}_y)dx \otimes 3 + (\mathcal{K}_{yyy} - 3D\mathcal{K}_{xy} + C\mathcal{K}_y - C_y\mathcal{K}_x)dy \otimes 3 + 3(A\mathcal{K}_{xy} + 2D\mathcal{K}_{yy} + (AD - BC)_x + A_x\mathcal{K}_y - A_y\mathcal{K}_x)dx \otimes dx \otimes dy + 3(\mathcal{K}_{yy} - 2A\mathcal{K}_{yy} - C\mathcal{K}_{xx} + D\mathcal{K}_{xy} + D_y\mathcal{K}_x - D_x\mathcal{K}_y)dx \otimes dy \otimes dy,$$

where $\otimes$ denotes the symmetrized tensor product, so, for example, $2dx \otimes dy = dx \otimes dy + dy \otimes dx$.

**Proof.** By (2.5), differentiating (5.16) yields (5.17) (several terms cancel), so it suffices to check (5.16). Routine computation shows that the Ricci tensor of $\nabla$ is

$$\text{Ric} = (A_x + B_y + 2(BD - A^2)) dx \otimes dx + (C_x + D_y + 2(AC - D^2)) dy \otimes dy + 2(-A_y - D_x + AD - BC) dx \otimes dy.$$

Since $X$ and $Y$ constitute a symplectic frame,

$$-\frac{1}{2} \rho(Z) = (\nabla_x \text{Ric}(Z, Y) - (\nabla_Y \text{Ric})(Z, X),$$

for all $Z \in \Gamma(TM)$. Routine computations show that

$$(\nabla_X \text{Ric})(X, X) = A_{xx} + B_{xy} + 2(BD - A^2)_x - 2A(A_x + B_y) + 2B(A_y + D_x) - 6ABD + 4A^3 + 2B^2C,$$

$$(\nabla_Y \text{Ric})(Y, Y) = C_{xx} + D_{yy} + 2(AC - D^2)_y + 2C(A_y + D_x) - 2D(C_x + D_y) - 6ACD + 4D^3 + 2BC^2,$$

$$(\nabla_X \text{Ric})(X, Y) = -A_{xy} - D_{xx} + (AD - BC)_x - B(C_x + D_y) + D(A_x + B_y) + 4BD^2 - 2A^2D - 2ABC,$$

$$(\nabla_Y \text{Ric})(Y, X) = A_{yy} + D_{xx} + 2(AC - D^2)_x - 2D(A_y + D_x) + 2C(C_x + D_y) + 4A^2C - 2AD^2 - 2BCD,$$

$$(\nabla_X \text{Ric})(X, Y) = -B_{yy} - D_{xy} + (AD - BC)_y + A(C_x + D_y) - C(A_x + B_y) + 4A^2C - 2AD^2 - 2BCD.$$
Substituting (5.21) in (5.20), and observing that the terms involving no derivatives cancel yields
\[
-\frac{1}{2} \rho = (-2A_{xy} - B_{yy} - D_{xx} + (AD - BC)_x + 2(A^2 - BD)_y \\
- B(C_x + D_y) - D(A_x + B_y) + 2A(A_y + D_x)) \, dx \\
+ (2D_{xy} + C_{xx} + A_{yy} - (AD - BC)_y + 2(AC - D^2)_x \\
- 2D(A_y + D_x) + A(C_x + D_y) + C(A_x + B_y)) \, dy.
\] (5.22)

Simplifying (5.22) yields (5.16). From
\[
\nabla dx = -Addx \otimes 2 - Cdy \otimes 2 + 2Ddx \otimes dy, \quad \nabla dy = -Bdx \otimes 2 - Ddy \otimes 2 + 2Adx \otimes dy,
\] it follows straightforwardly that
\[
\nabla d\chi = (\chi_{xx} - AX_x - BX_y)dx \otimes 2 + (\chi_{yy} - C\chi_x - D\chi_y)dy \otimes 2 \\
+ (\chi_{xy} + D\chi_x + A\chi_y)(dx \otimes dy + dy \otimes dx).
\] (5.24)

Using (5.5), (5.19), (5.23), and (5.24) to compute \( \nabla^2 d\chi + d\chi \otimes \text{Ric} \), and symmetrizing the result yields (5.18).

Combining (3d) of Lemma 5.3 and Lemma 5.7 yields equations for extremal symplectic connections that can be solved. Let \( \nabla \in S(M, \Omega) \) be extremal with \( \chi \) nonconstant and suppose \( p \not\in \text{crit}(\chi) \). Let \( \tau = \rho(H_\chi) + \chi^2 \). By (3d) of Lemma 5.3 in an open neighborhood of \( p \) there may be taken as Darboux coordinates \( x = \chi - a \) where \( a = \chi(p) \), and \( y \) where \( dy = (\tau - \chi^2)^{-1} \rho = (\tau - (x + a)^2)^{-1} \rho \). Hence \( \chi = x + a \) and \( \rho = (\tau - (x + a)^2)dy \). Let \( \nabla \) have the form \( \partial + \Pi \) where \( \Pi \) is as in (5.15) and \( \partial \) is the standard flat affine connection in the coordinates \( (x, y) \). By (5.18),
\[
0 = -3\chi(\chi) = B_y dx \otimes 3 - 3A_y dx \otimes dx \otimes dy + 3D_y dx \otimes dy \otimes y - C_y dy \otimes 3,
\] (5.25)
so that \( A, B, C, \) and \( D \) depend only on \( x \). This can be seen in another way as follows. The vector field \( H_\chi = \partial_y \) is Killing for both the flat affine connection \( \partial \) and \( \nabla \), so the Lie derivative along \( H_\chi \) of the difference tensor \( \Pi = \nabla - \partial \) vanishes.

Comparing \( \rho = (\tau - (x + a)^2)dy \) and (5.16) shows that \( A, B, C, \) and \( D \) satisfy the equations
\[
(C_x + 3(AC - D^2))_{xx} = x + a, \quad D_{xx} = (AD - BC)_x + 2AD_x - DA_x - BC_x.
\] (5.26)
The preceding is summarized in Lemma 5.8.

**Lemma 5.8.** Let \( \partial \) be the standard flat affine connection in the coordinates \( (x, y) \) and let \( \Omega = dx \wedge dy \). The connection \( \nabla = \partial + \Pi \), where \( \Pi \) is defined as in (5.15), is extremal symplectic if \( A, B, C, \) and \( D \) are functions of \( x \) alone and satisfy the equations (5.25). In this case \( \chi(\nabla) = x + a \) for some constant \( a \).

Computing \( \delta^* \text{Ric} \) and \( \delta^* \rho \) using (5.21) and (5.23) yields
\[
-3\delta^* \text{Ric} = (A_{xx} + 2B_x D + 4BD_x - 3(A^2)_x + 4A^3 - 6ABD + 2B^2 C) \, dx \otimes 3 \\
+ (2CD_x - 2C_x D - 6ACD + 2BC^2 + 4D^3) \, dy \otimes 3 \\
+ (-2D_{xx} + 6A_x D - 4BC_x - 2B_x C - 6ABC + 12BD^2 - 6A^2 D) \, dx \otimes dx \otimes dy \\
+ (C_{xx} + 6AC_x - 3(D^2)_x + 12A^2 C - 6AD^2 - 6BCD) \, dx \otimes dy \otimes dy,
\] (5.27)
\[
\delta^* \rho = (\tau - (x + a)^2)(Bdx \otimes 2 + Ddy \otimes 2 - 2Adx \otimes dy) + 2(x + a)dx \otimes dy.
\] (5.28)

This suggests choosing \( A, B, \) and \( D \) so that \( \delta^* \rho = 0 \). Then \( B = D = 0 \), \( A = (x + a)(\tau - (x + a)^2)^{-1} \), and, by (5.20), \( (C_x + 3AC)_x = x + a \). For any constants \( p \) and \( q \) the function \( C = -6^{-1}(\tau - (x + a)^2)((x + a)^2 + p(x + a) + q) \) yields a solution defined at least on the region where \( \tau \neq (x + a)^2 \). Substituting into (5.27) yields \( 9\delta^* \text{Ric} = (\tau - q)dx \otimes dy \otimes dy \). If \( q = \tau \) then the resulting connection
is preferred. The family of connections corresponding to \( q = \tau \) (with \( p \) arbitrary) was found in Proposition 11.4 of [5], where it is shown that every preferred symplectic connection on \( \mathbb{R}^2 \) that is not symmetric is equivalent to one of these connections with \( q = \tau < 0 \) and \( p \) arbitrary. On the other hand, if \( q \neq \tau \), then the resulting connections are extremal but not preferred.

**Theorem 5.9.** Let \( \partial \) be the standard flat affine connection in the coordinates \( (x, y) \) and let \( \Omega = dx \wedge dy \). For any choices of constants \( a, p, q, \) and \( \tau \), the connection \( \nabla = \partial + \Pi \) where \( \Pi \) is defined as in (5.15) with \( B = 0 = D, A = (x + a)(\tau - (x + a)^2)^{-1} \) and \( C = -6^{-1}(\tau - (x + a)^2)((x + a)^2 + p(x + a) + q) \) satisfies:

1. On each connected component of \( \{(x, y) \in \mathbb{R}^2 : \tau \neq (x + a)^2\} \), \( \nabla \) is an extremal symplectic connection satisfying \( \mathcal{K}(\nabla) = x + a \) and \( \delta^* p = 0 \).
2. \( \nabla \) is preferred if and only if \( q = \tau \).
3. If \( \tau < 0 \) then \( \nabla \) is defined on all of \( \mathbb{R}^2 \) and is geodesically complete.

**Proof.** There remains only to prove that \( \nabla \) is complete when \( \tau < 0 \). In the case \( q = \tau \) this is claimed in section 11 of [5]. The equations of the geodesics of \( \nabla \) as in Lemma 5.7 are

\[
\ddot{x} + A \dot{x}^2 - 2D \dot{x} \dot{y} + C \dot{y}^2 = 0, \quad \ddot{y} + B \dot{x}^2 - 2A \dot{x} \dot{y} + D \dot{y}^2 = 0.
\]

No generality is lost by supposing \( a = 0 \). Then (5.29) becomes

\[
\ddot{x} + \frac{x}{\tau - x^2} \dot{x}^2 - \frac{1}{6}(\tau - x^2)(x^2 + px + q) \dot{y}^2 = 0, \quad \ddot{y} - \frac{2x}{\tau - x^2} \dot{x} \dot{y} = 0.
\]

From (5.30) it follows that along a solution \( (\tau - x^2) \dot{y} \) equals some constant \( \tau \). (This follows from the constancy of \( \rho(\gamma) \) for any \( \nabla \)-geodesic \( \gamma \), which is an immediate consequence of \( \delta^* p = 0 \).) Hence

\[
\ddot{x} + \frac{x}{\tau - x^2} \dot{x}^2 - \frac{r^2(x^2 + px + q)}{6(\tau - x^2)} = 0, \quad \ddot{y} = \frac{r}{\tau - x^2}.
\]

Suppose \( \tau < 0 \) and let \( x = \sqrt{-\tau} \sinh u \). Then \( u \) solves the conservative equation

\[
\ddot{u} = f(u) = -\frac{r^2}{6} \frac{-\tau \sinh^2 u + p\sqrt{-\tau} \sinh u + q}{(\sqrt{-\tau} \cosh u)^2}.
\]

Along each solution there is a constant \( E \) such that \( \dot{u}^2 + g(u) = E \) where \( g(u) - g(u_0) = -2 \int_{u_0}^u f(v) \, dv \). It is straightforward to see that there is a constant \( C > 0 \) such that \( |f(u)| \leq C/(4 \cosh(u)) \) for \( u \in \mathbb{R} \). This bound implies \( |g(u) - g(u_0)| \leq C |\arctan(e^{ru}) - \arctan(e^{ru_0})| \leq C r \). Consequently, if \( u(t) \) solves (5.32) with initial conditions \( u(0) = u_0 \) and \( \dot{u}(0) = v_0 \), then

\[
\ddot{u}(t)^2 = E - g(u(t)) \leq E - g(u(0)) \leq E - g(u(0)) + C\pi = E - g(u(0)) + C \pi = v_0^2 + C \pi.
\]

Hence there is a constant \( K > 0 \) depending on \( v_0 \) such that \( |\dot{u}(t)| \leq K \) for all \( t \). This implies \( |u(t)| \leq |u_0| + K |t| \), so that \( u(t) \) does not blow up in finite time. Thus for any \( u_0 \) and \( v_0 \) there is a unique solution \( u \) of (5.32) defined for all time, so \( x \) and \( y \) are as well. Hence \( \nabla \) is complete.

**Remark 5.10.** In the examples of Theorem 5.9 the condition \( B = 0 \) can be dropped and solutions will still be obtained. With \( D = 0 \), the solutions are as above, but with \( B_x C = -2BC_x \).

5.4 Examples on the two sphere. Other examples can be constructed using Lemma 5.8 but in general it is difficult to find solutions that yield complete connections. For example, let \( P(x) = x^4/24 + ax^3/6 + bx^2/2 + cx + d \). By (5.17), (5.18), and Lemma 5.8 the \( \nabla \) defined by taking \( A = B = D = 0 \) and \( C = P(x) \) in (5.15) has \( \mathcal{K}(\nabla) = x + a \) and \( \mathcal{H}(\mathcal{K}(\nabla)) = 0 \) so is extremal symplectic but not moment constant. These connections are not generally complete. By (5.29) a geodesic satisfies \( y(t) = pt + q \) and \( x(t) \) solves \( \dot{x} = -p^2 P(x) \). For a polynomial \( Q(x) \) with derivative equal to \( P(x) \), \( \dot{x}^2 + 2p^2 Q(x) \) is constant along a solution. Since the constant term of \( Q \) is arbitrary, the constant of integration can be absorbed into \( Q \), and there can be written \( \dot{x}^2 + 2p^2 Q(x) = 0 \),
where the choice of primitive $Q$ depends on the particular solution curve. In general solutions of such an equation blow up in finite time.

The following addresses the question of whether, for an appropriate choice of $P$, the connection so obtained can be extended to a symplectic connection on the two sphere with its usual volume form. The strange result is the construction of a family $\nabla(t)$ of extremal symplectic connections defined on the complement in $S^2$ of two antipodal points, such that $\nabla(0)$ is the Levi-Civita connection of the round metric, and, for $t \neq 0$, the difference tensor $\nabla(t) - \nabla(0)$ extends continuously but not differentiably at the two excluded antipodal points. Moreover, $\mathcal{E}(\nabla(t))$ is a multiple of $t^2$, so while these connections are extremal they are not minimizers of $\mathcal{E}$ except when $t = 0$.

The round two-dimensional sphere $S^2$ of volume $4\pi$ is the subset \{$(\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \in \mathbb{R}^3 : \theta \in [0, \pi], \phi \in [0, 2\pi]$\} of $\mathbb{R}^3$ with the induced metric. In the coordinates $(x, y)$ defined on the complement of the poles by $x = -\cos \theta \in (-1, 1)$ and $y = \phi$ the standard round metric $g = d\theta^2 + \sin^2 \theta d\phi^2$ takes the form $g = (1 - x^2)^{-1} dx^2 + (1 - x^2) dy^2$, and the volume form $\Omega = \sin \theta d\theta \wedge \phi \phi$ of $g$ is the Darboux form $dx \wedge dy$. Let $\partial$ be the flat connection in the coordinates $(x, y)$. The Levi-Civita connection $\nabla$ of $g$ has the form $\nabla = \partial + \Lambda$ where $\Lambda(x, X) = x(1 - x^2)^{-1} \Lambda(x, X) = -x(1 - x^2)^{-1} Y$, and $\Lambda(Y, Y) = x(1 - x^2)^{-1} X$. Define $\nabla = \nabla = \partial + \Pi$ where the components $A, B, C$, and $D$ of $\Pi = \Pi + \Lambda$ are as in (5.13). Taking $\Lambda = x(1 - x^2)^{-1} = D = 0$, and $C = x(1 - x^2) + P(x)$, where $P$ is a quartic polynomial in $x$, yields a connection $\nabla$ with $\nabla(\nabla) = C_{xxx} + 3(AC)_{xx} - 6ax$, so if $a = -1/6$, the resulting connection satisfies $\nabla(\nabla) = \nabla$. So far this $\nabla$ has been defined only away from the poles in $S^2$. There remains to determine whether $b$ and $c$ can be chosen so that it extends smoothly at the poles.

As is explained following its proof, Theorem 5.11 gives an intrinsic construction of a special case of the connections just described. Although its statement appears more general, Theorem 5.11 is interesting mainly in the special case $D$ is the Levi-Civita connection of a round metric on $M = S^2$.

**Theorem 5.11.** Let $(M, \Omega)$ be a symplectic two-manifold and let $D \in \Omega(M, \Omega)$ have parallel Ricci tensor $\mathfrak{R}_{ij}$. Suppose $Z_i$ is a nontrivial one-form satisfying $D_i Z_j = 0$. Let $\gamma = D_p Z^p$. Then

\[ \nu = \gamma^2 + 4Z^a Z^b \mathfrak{R}_{ab} \]

is constant. On $\hat{M} = \{ p \in M : (\gamma(p)^2 \neq \nu \}$ define $\Gamma_{ijk} = (\gamma - \gamma^2)^{-1} Z_i Z_j Z_k$. The symplectic connection $\nabla(t)$ is $D + t \Gamma_{ij}^k$ satisfies

\[ R(t)_{ij} = \mathfrak{R}_{ij} + t \gamma(\nu - \gamma^2)^{-1} Z_i Z_j, \]

\[ -\delta^\nu_{\nabla(t)} \text{Ric}(\nabla(t))_{ijk} = \nabla(t)_{(i} R(t)_{j)k} = -4t(\nu - \gamma^2)^{-2} Z_i Z_j \mathfrak{R}_{jk} p Z^p, \]

\[ \rho(\nabla(t)) = \frac{1}{2} t(\nu - 3\gamma^2)(\nu - \gamma^2)^{-1} Z_i, \]

\[ \mathcal{K}(\nabla(t)) = \frac{1}{4} \gamma^2, \]

\[ \tau(\nabla(t)) = (3t/16) \nu \]

where $R(t)_{ij} = \text{Ric}(\nabla(t))_{ij}$ and $\tau$ is the constant of (5.0) of Lemma 5.3. Moreover, $\mathfrak{R}_{\nabla(t)}(\gamma) = 0$, so $\nabla(t)$ is extremal. If $4\nu(\nu - \gamma^2)^{-2} Z_i Z_j \mathfrak{R}_{jk} p Z^p$ is nonzero somewhere on $\hat{M}$, then $\nabla(t)$ is not preferred for $t \neq 0$. Finally, $\mathfrak{R}_{\nabla(t)}(\tau) = 0$ for all $t \in \mathbb{R}$.

**Proof.** Let $\nabla = \nabla(1)$. For $t \neq 0$ it suffices to check all the claims for $\nabla$ because $t \Gamma_{ijk}$ is obtained from $t Z_i$ in the same way as $\Gamma_{ijk}$ is obtained from $Z_i$; that is the connection $\nabla$ corresponding to $tZ$ in place of $Z$ is simply $\nabla(t)$.

By assumption $2D_i Z_j = 2D_i Z_j = \gamma \Omega_{ij}$. By the Ricci identity,

\[ 2d\gamma_i = 2D_i D^p Z^p = 2D^p D_i Z^p - 2\mathfrak{R}_{ip} Z^p = D_p (\gamma \delta_i^p) - 2\mathfrak{R}_{ip} Z^p = d\gamma_i - 2\mathfrak{R}_{ip} Z^p, \]
so

\[ d\gamma_i = -2R_{ij}Z^i, \]

Since \( R_{ij} \) is parallel, \( D_t(Z^aZ^b{\mathcal R}_{ab}) = 2Z^a{\mathcal R}_{ab}D_tZ^b = \gamma R_{ij}Z^i \). Combined with (5.41) this shows that \( d(\gamma^2 + 4Z^aZ^b{\mathcal R}_{ab}) = 0 \), so there is a constant \( \nu \) satisfying (5.34). From (5.41) and (5.34) there follows

\[ Z^p d\gamma_p = -2Z^aZ^b{\mathcal R}_{ab} = -(\nu - \gamma^2)/2. \]

Computing using \( 2D_tZ_j = \gamma \Omega_{ij} \) and (5.42) shows

\[ \delta \Gamma_{ij} = D_p \Gamma_{ij}^p = \gamma (\nu - \gamma^2)^{-1} Z_i Z_j. \]

Since \( \Gamma_{ip}^{-1} \Gamma_{jk}^p = 0 \) and \( \Gamma_{ip}^p \), (5.35) follows from (5.43). A bit of computation using (5.35) shows

\[ \nabla_i R_{jk} = D_i R_{jk} - 2\Gamma_{ij}^p R_{kp} = D_i (\gamma(\nu - \gamma^2)^{-1} Z_i Z_j) - 2(\nu - \gamma^2)^{-1} Z_i Z_j R_{ip} Z^p Z^k \]

\[ = -2((\nu - \gamma^2)^{-1} + 2\gamma^2(\nu - \gamma^2)^{-2}) Z^p R_{ip} Z_j Z_k \]

\[ - 2(\nu - \gamma^2)^{-1} Z_i Z_j R_{ip} Z^p + 2\gamma^2(\nu - \gamma^2)^{-1} \Omega_{ij} Z_k, \]

from which (5.36) follows. Contracting (5.44) and simplifying using (5.42) yields (5.37). Alternatively, (5.37) follows from (5.42) coupled with the observations that, by (5.43), \( 4L_D^* (\Gamma)_i = (\nu - \gamma^2)^{-1} (3\gamma^2 - \nu) Z_i \) and that in (3.12) the terms of order at least two in \( \Gamma \) all vanish. Since \( Z^p \rho(\nabla)_p = 0, \nabla_i \rho(\nabla)_j = D_i \rho(\nabla)_j \), so to compute \( \mathcal{K}(\nabla) \) it suffices to compute \( D_i \rho(\nabla)_j \) and contract. There results (5.38). As for (5.37), (5.38) can also be computed using (3.13) and

\[ \mathcal{H}^*_D(\Gamma) = \delta_D L_D^* (\Gamma) = -(3/4) \gamma. \]

Differentiating (5.41) yields \( D_t d\gamma_{ij} = -\gamma R_{ij} \), and, with (5.42), there follow

\[ \nabla_i d\gamma_{jk} = D_i d\gamma_{jk} - 2(\nu - \gamma^2)^{-1} Z_i Z_j Z^p \nabla_k d\gamma_p = -d\gamma_i R_{jk} + 2\gamma^2(\nu - \gamma^2)^{-1} Z_i Z_j Z^p \nabla_k d\gamma_p \]

\[ = -d\gamma_i R_{jk} + \frac{1}{2} \Omega_{ij} Z_k + 2\gamma^2(\nu - \gamma^2)^{-1} Z_i Z_j Z^p R_{ip} Z^k \]

Using (5.35), (5.41), and (5.46), and (5.47) yields

\[ \mathcal{H}(\nabla) = \nabla_i \nabla_j d\gamma_{jk} + d\gamma_i R_{jk} = \nabla_i \nabla_j d\gamma_{jk} + d\gamma_i R_{jk} + \gamma (\nu - \gamma^2)^{-1} d\gamma_i Z_j Z_k \]

\[ = \frac{1}{2} \Omega_{ij} Z_k + \gamma (\nu - \gamma^2)^{-1} (d\gamma_i Z_j Z_k), \]

\[ = \frac{1}{2} \Omega_{ij} Z_k + \gamma (\nu - \gamma^2)^{-1} (d\gamma_i Z_j Z_k), \]

in which the final equality follows from \( 4d\gamma_i Z_j = 2Z^p d\gamma_p \Omega_{ij} = (\nu - \gamma) \Omega_{ij} \), which uses (5.42).

It follows from (5.35) that \( L_{\nabla} (\Gamma)_ij = L_{\nabla}^* (\Gamma)_{ij} \), and from (5.43) and (5.46) that \( \mathcal{H}(\nabla) = \mathcal{H}^* (\Gamma) = \delta_{\nabla} L_{\nabla} (\Gamma) = -\delta_D L_{\nabla}^* (\Gamma) = -(3/4) \gamma \). Then

\[ \mathcal{J}(\Gamma) = \mathcal{H} \mathcal{H}^* (\Gamma) + \mathcal{L}_{\mathcal{H}(\nabla)} (\Gamma) = \mathcal{H} (\frac{3}{4}) \Gamma - (3/4) \mathcal{L}_{\mathcal{H}} \Gamma = -(3/4) \mathcal{L}_{\mathcal{H}} \Gamma, \]

where \( X^i = -d\gamma^i \). Since

\[ D_p \Gamma_{ijk} = \frac{2\gamma}{2} (\nu - \gamma^2)^{-1} \mathcal{L}_{\mathcal{H}(\nabla)} (\Gamma) + 2\gamma (\nu - \gamma^2)^{-2} d\gamma_p Z_i Z_j Z_k, \]

and \( D_t d\gamma_{ij} = -\gamma R_{ij} \),

\[ \mathcal{L}_{\mathcal{H}} (\Gamma)_{ij} = -d\gamma_p D_p \Gamma_{ijk} - 3D_i (d\gamma_p \Gamma_{jk}) \]

\[ = \frac{2\gamma}{2} (\nu - \gamma^2)^{-1} d\gamma_p Z_i Z_j Z_k + 3\gamma (\nu - \gamma^2)^{-2} Z^p R_{ip} Z_j Z_k = 0, \]

the last equality by (5.41), it follows from (5.48) that \( \mathcal{J}(\Gamma) = 0. \) While the preceeding argument does not show that \( \mathcal{J}(\Gamma) = 0 \), this is easily checked directly. \( \square \)
If $D$ is the Levi-Civita connection of a Riemannian metric then the hypothesis of Theorem 5.11 means that $Z_i$ is metrically dual to a Killing field, so if $M$ is compact and $Z$ is nontrivial then $M$ must be a sphere or a torus. Since by hypothesis the Ricci tensor is parallel, in the case $M$ is a torus, the metric must be flat and any Killing field is parallel; in this case $\nu = 0$ and $\gamma = 0$ and the set $\tilde{M}$ of Theorem 5.11 is empty, so the construction is vacuous.

In the case $M = S^2$, the vector field $Y = \partial_y$, which is rotation around the axis through the deleted antipodal poles, is a nontrivial Killing field for the round metric $g$ for which the metrically dual one-form $Z = \iota(Y)g = (1-x^2)dy$ has the properties required in Theorem 5.11. (This example motivated the theorem.) Here the notations are as in the paragraphs preceding the statement of Theorem 5.11. Precisely, $Z$ satisfies $DZ = -x\Omega$, and for an appropriate choice of $P$, the difference tensor $\Gamma$ is

\[(1/4)(1-x^2)^{-1}Z_iZ_jZ_k = (1/4)(1-x^2)^2dy\otimes dy\otimes dy.\]

Since $\gamma = -2\pi$ and $Z^aZ^b\mathcal{R}_{ab} = |Z|^2 = (1-x^2)$, the constant $\nu$ equals 4. Theorem 5.11 shows that in this setting the resulting $\nabla$ is extremal but not preferred. However, while the tensor $\Gamma_{ijk}$ extends continuously at the poles, vanishing there, its extension is not differentiable at the poles. The behavior at the poles is seen most easily in different coordinates. A convenient choice is $u = \tan(\theta/2)\cos y = \sqrt{1-x^2}\cos y$ and $v = \tan(\theta/2)\sin y = \sqrt{1-x^2}\sin y$.

In these coordinates $g$ has the standard form $4(1+u^2+v^2)^{-2}(du^2 + dv^2)$ and the origin corresponds to $x = -1$ (the pole at $x = 1$ can be handled similarly). Since $u^2 + v^2 = (1+x)/(1-x)$, $x = (u^2 + v^2 - 1)/(u^2 + v^2 + 1)$, and $dy = (u^2 + v^2)^{-1}(udv - vdu)$,

\[(5.51) \Gamma = \frac{4}{(1-u^2-v^2)^2(u^2+v^2)^3}(udv - vdu)^3.\]

The components of $\Gamma$ behave like a smooth multiples of $u^a v^{3-a}(u^2 + v^2)^{-1}$ for $a \in \{0,1,2,3\}$, so they remain continuous at the origin of $(u,v)$ coordinates, but do not extend differentiably there.

For the $\nabla(t)$ of Theorem 5.11 by (5.38), since $\mathcal{K}(\nabla(t))$ extends smoothly to all of $S^2$,

\[(5.52) \mathcal{E}(\nabla(t)) = \int_{S^2} \mathcal{K}(\nabla)^2 \Omega = (9/4)t^2 \int_0^{2\pi} \int_{-1}^1 x^2 \, dx \, dy = 3\pi t^2.\]

Since the Levi-Civita connection $D$ of the round metric on $S^2$ has $\mathcal{E}(D) = 0$, this shows that $\nabla(t)$ is not an absolute minimizer of $\mathcal{E}$ when $t \neq 0$, although it is extremal. In particular, there is a one-parameter family of extremal symplectic connections on the Darboux cylinder along which $\mathcal{E}$ takes on all nonnegative real values. Comparing (5.39) and (5.52) shows the necessity of the boundary term in (5.7). For $\nabla = \nabla(1)$, by (5.39), $\text{vol}_3(M) = 3\pi$, while, by (5.52), $\mathcal{E}(\nabla) = 3\pi$, rather than $9\pi$ as (5.7) would imply were the boundary term null. However, by (5.37) there holds $\rho = 2^{-1}(1 - 3x^2)dy$, and so $\mathcal{K}\rho = (3/4)x(1 - 3x^2)dy$, which does not extend differentiably at the poles $x = \pm 1$ since the one-form $dy$ does not extend differentiably. It follows that the boundary term $\int_M d(\mathcal{K}\rho)$ in (5.53) does not vanish. Indeed,

\[(5.53) \int_0^{2\pi} \int_{-1+\epsilon_1}^{1-\epsilon_2} d(\mathcal{K}\rho) = -\int_0^{2\pi} (\mathcal{K}\rho)_{x=1-\epsilon_2} + \int_0^{2\pi} (\mathcal{K}\rho)_{x=-1+\epsilon_1}
\quad = -2\pi(3/4)(1-\epsilon_2)(1-3(1-\epsilon_2)^2) + 2\pi(3/4)(\epsilon_1 - 1)(1-3(\epsilon_1 - 1)^2)
\quad \text{which tends to } 6\pi \text{ as } \epsilon_1 \rightarrow 0 \text{ and } \epsilon_2 \rightarrow 0. \]

This is what is needed to yield equality in (5.7).

5.5. Not projectively flat moment flat examples from improper affine spheres. For $\Pi$ as in (5.15), the components of $\Pi^\gamma(U,V,W) = \Omega(\Pi(U,V),W)$ are

\[(5.54) \Pi^\gamma(X,X,Y) = -B, \quad \Pi^\gamma(X,Y,X) = A, \quad \Pi^\gamma(Y,X,Y) = -D, \quad \Pi^\gamma(Y,Y,Y) = C.\]

The expressions (5.10) and (5.17) in Lemma 5.7 simplify considerably if $AD - BC = 0, A^2 - BD = 0,$ and $AC - D^2 = 0$. These are the conditions for the cubic form (5.54) to take values in the image of a rational normal curve, that is for $\Pi^\gamma$ to be decomposable in the sense that there is a one-form $\sigma$ such that $\Pi^\gamma = \sigma \otimes \sigma \otimes \sigma$. Theorem 5.11 shows what happens when $\sigma$ is metrically dual to a
metric Killing field. A different simplification is obtained by supposing that \(\sigma\) is closed. Lemma 5.12 shows how Lemma 5.11 simplifies when \(\Pi^Y\) is supposed to be the cube of a closed one-form, and Theorem 5.14 shows how it simplifies further when \(\Pi^Y\) is the cube of an exact one-form.

**Lemma 5.12.** Let \((M, \Omega)\) be a surface with a volume form, and let \(D \in \mathcal{S}(M, \Omega)\). Let \(\nabla = D + \Pi_{ij}^k \in \mathcal{S}(M, \Omega)\), where \(\Pi_{ij}^k = X_i X_j X_k\) and the one-form \(X_i\) is closed. Let \(R_{ij}\) and \(R_{ij}\) be the Ricci curvatures of \(\nabla\) and \(D\) and let \(\kappa = \det D X^j\). Then

\[
R_{ij} = R_{ij} + \delta_{ij} \Pi_{pq} = X_i + 2 X^p X_q (D_{ij}^k) X_p,
\]

\[
\rho(\nabla) = \rho(D) + 12 \kappa X_i - 6 X_i X^p X^q R_{pq} - 2 D_i (X^p X^q D_p X_q)
\]

(5.55)

\[
= \rho(D) + 8 \kappa X_i - 6 X_i X^p X^q R_{pq} - 2 X^p X^q D_i D_p X_q,
\]

(5.56)

\[
\mathcal{K}(\nabla) = \mathcal{K}(D) - 6 X^p d\kappa_p + 3 X^p D_p (X^a X^b \Omega_{ab}).
\]

(5.57)

**Proof.** Because \(2 D_{ij} X_j = D_p X^p \Omega_{ij}\), that \(X\) is closed means that \(D_p X^p = 0\). For any \((1,1)\)-tensor \(A_{ij}\) such that \(A_p^i = 0\) there holds \(A_p^i A_p^j = -\det(A) \delta_{ij}\), and so there hold \(A_p^i A_p^j = -2 \det(A)\), \(A_p^i A_p^j = 2 \det(A)\), and \(A_p^i A_p^j = 0\). Applying these observations to \(A_{ij} = D_i X^j\) yields

\[
D_i X^p D_p X_j = -\kappa \Omega_{ij}, \quad D^p X^q D_p X_q = 2 \kappa.
\]

(5.58)

By (5.55) and the Ricci identity,

\[
X^p X^q D_p D_q X_q = D_i (X^p X^q D_p X_q) - 2 X^p D_i X^q D_p X_q = D_i (X^p X^q D_p X_q) - 2 \kappa X_i.
\]

(5.59)

Since \(R_{ij}^p = 2 R_{ij}\), \(D^p D_p X_i = R_{ip} X^p\), and, by (2.1), \(R_{ijkl} X^j X^k X^l = -X_i X^p X^q R_{pq}\). Because \(X_i\) is closed, \(\delta_{ij} = D_p \Pi_{ij} = 2 X^p X^i (D_{ij}) X_q\), and (5.55) follows. Using (5.58), \(D^p D_p X_i = R_{ip} X^p\), the Ricci identity, and finally (5.59) there results

\[
\mathcal{L}^\ast(\Pi)_i = -\delta^2 \Pi_i - \Pi_{ipq} R_{pq} = -4 \kappa X_i + 2 X^p X^q R_{pq} X_i + X^p X^q D_p D_q X_i
\]

\[
= -4 \kappa X_i + 2 X^p X^q R_{pq} X_i + X^p X^q (D_i D_p X_q - \Omega_{pq} R_{pq} X_a)
\]

\[
= -4 \kappa X_i + 3 X^p X^q R_{pq} X_i + X^p X^q D_i D_p X_q
\]

\[
= -6 \kappa X_i + 3 X^p X^q R_{pq} X_i + D_i (X^p X^q D_p X_q).
\]

(5.60)

Since \(\Pi_{ip}^q \Pi_{jq}^p = 0\) and \(\Pi_{ipq} \delta_{pq} = 0\), by (5.12), \(\rho(\nabla) = \rho(D) - 2 \mathcal{L}^\ast(\Pi)\), and with (5.60) this yields (5.56). Since \(2 X(\nabla) = \nabla D \rho(\nabla)_p = D^p \rho(\nabla)_p\), applying \(D^i\) to (5.60) yields (5.57). \(\square\)

On a surface \(M\), let \((g_{ij}, J, j)\) be a constant curvature Kähler structure having Levi-Civita connection \(D\) and volume form \(\Omega_{ij} = J_{ij} g_{pq}\). For \(f \in C^\infty(M)\), the function \(M(f) = \det(D df)\) is the usual Hessian determinant. Precisely, the determinant \(\det\) of the covariant two-tensor \(\Omega\) is identified in a canonical way with the section \(\Omega^\otimes 2\) of the tensor square of the line bundle of two-forms. Consequently, as \(D^i df^j \Omega_{ij} = D_i df_j, M(f) \Omega^\otimes 2 = \det Df\). In the case \(D\) is a flat affine connection and \(\Omega = dx \wedge dy, M(f) = \det Df \otimes \Omega^\otimes -2\) is just the usual Hessian determinant. Since \(D^i df^p D_p df_j = M(f) \delta_j^i\), the tensor \(D^i df^j\) is the adjugate tensor of \(D_i df_j\). Define \(\mathcal{U}(f) = df^i df_j D^i df^j\), the contraction of the adjugate tensor of the Hessian \(\partial df\) of \(f\) with \(df \otimes df\).

**Corollary 5.13.** On a surface \(M\), let \((g_{ij}, J, j)\) be a constant curvature \(\Re\) Kähler structure having Levi-Civita connection \(D\) and volume form \(\Omega_{ij} = J_{ij} g_{pq}\). For \(f \in C^\infty(M)\), let \(M(f) = \det(D df)\). Then \(\nabla = D + df, df, df^k \in \mathcal{S}(M, \Omega)\) satisfies

\[
\rho(\nabla)_i = -3 R(df)_i^2 df^i + 12 M(f) df_i - 2 d \mathcal{U}(f)_i, \quad \mathcal{K}(\nabla) = 6 \{f, M(f)\} - \frac{3}{2} R(f, |df|_g^2).
\]

(5.61)

**Proof.** Take \(X_i = df_i\) in Lemma 5.12. \(\square\)

Corollary 5.13 yields moment flat symplectic connections that are not projectively flat.
Theorem 5.14. Let \( \partial \) be the standard flat connection on \( \mathbb{R}^2 \) equipped with the symplectic form \( \Omega = dx \wedge dy \). Define \( \nabla \in \mathcal{S}(\mathbb{R}^2, \Omega) \) by \( \nabla = \partial + \Pi \) where \( \Pi_{ijk} = df_i df_j df_k \) for \( f \in C^\infty(M) \). Then
\[
(5.62) \quad \rho = 12M(f)df - 2d\Omega(f), \quad \mathcal{K}(\nabla) = 6\{f, M(f)\}.
\]
If \( M(f) \) Poisson commutes with \( f \) then \( \mathcal{K}(\nabla) = 0 \), but \( \nabla \) is not projectively flat whenever \( d\Omega(f) - 6M(f)df \) is somewhere nonzero. In particular, if the graph of \( f \) is an improper affine sphere then \( \mathcal{K}(\nabla) = 0 \). There exist improper affine spheres for which the resulting \( \nabla \) is not projectively flat.

Proof. Except for the final claims, this is a special case of Corollary 5.13. The graph of \( f \) is an improper affine sphere if and only if \( M(f) \) equals a nonzero constant. In this case \( M(f) \) Poisson commutes with \( f \) so \( \mathcal{K}(\nabla) = 0 \).

If \( f \) has homogeneity \( 2 \), meaning \( xf_x + yf_x = 2f \), then \( \mathcal{U}(f) = 2fM(f) \), so that in this case, by (5.12), \( \rho = 12M(f)df - 4d(fM(f)) = 8M(f)df - 4d\Omega(f) \). If, moreover, \( M(f) \) is constant but \( f \) is not, then \( \rho = 8M(f)df \) is not zero, although \( \mathcal{K}(\nabla) \) vanishes. This occurs for any homogeneous quadratic polynomial, e.g. \( x^2 \pm y^2 \) or \( xy \). More interesting examples are obtained as follows. Let \( u(x) \) be any smooth function on the line. Then \( f = xy + u(x) \) satisfies \( M(f) = -1 \) and \( \mathcal{U}(F) = x^2u' - 2x(y + u') \), so that \( \rho = -2d\Omega(f) = (4y + 4u' - 2x^2u'')dx + 4xdy \). Since \( M(f) = -1 \) the graph of \( f \) is an improper (ruled) affine sphere, and since \( \rho \) never vanishes identically the resulting connection is not projectively flat.

\[
\Box
\]

Remark 5.15. Specializing (5.53) shows that the Ricci curvature of \( \nabla \) as in Theorem 5.14 has the form \( \text{Ric} = -\nabla_{H_f}(df \otimes df) = -\partial_{H_f}(df \otimes df) \). Hence \( \text{Ric}(H_f, H_f) = 0 \), so \( \text{Ric} \) degenerates along \( H_f \).

Horospheres in hyperbolic space are convex submanifolds, flat in the induced metric, and having constant Gaussian curvature, so are analogous to affine spheres. This motivates the next example.

Theorem 5.16. Let \( D \) be the Levi-Civita connection of the metric \( g \) on the hyperbolic plane and let \( \Omega \) be the corresponding volume form. Let \( \beta \) be a Busemann function normalized to take the value \(-\infty\) at a fixed point in the ideal boundary. Define \( \nabla \in \mathcal{S}(\mathbb{R}^2, \Omega) \) by \( \nabla = \partial + \Pi \) where \( \Pi_{ijk} = df_i df_j df_k \) and \( f = e^\beta \). Then \( \rho = 6f^2df = 2d(f^3) \), so \( \mathcal{K}(\nabla) = 0 \) but \( \nabla \) is not projectively flat.

Proof. In the upper half-space model of hyperbolic space, the Busemann function at the point at infinity is the negative of the logarithm of the vertical coordinate. Using this it is straightforward to check that \( Dd\beta + d\beta \otimes d\beta = g \), so that \( Ddf = Dde^\beta = e^\beta g = fg \). Hence \( Ddf^2 = -fJ_i^j \), so \( M(f) = \det(D, df^j) = f^2 \). As \( |d\beta|_g^2 = 1 \), there holds \( |df|_g^2 = e^{2\beta} = f^2 \). Hence \( \mathcal{U}(f) = df^2df^jD_i^j = f[df|_g^2 = f^3 \). Consequently \( \rho = -2d\Omega(f) + 12M(f)df = 6f^2df \).

\[
\Box
\]

6. Moment flat connections on compact surfaces

Theorem 6.1. On a surface \( M \), let \( (g_{ij}, J_i^j) \) be a Kähler structure having Levi-Civita connection \( D \), volume form \( \Omega_{ij} = J_i^p g_{pj} \), and scalar curvature \( \mathcal{R} \). Let \( X_i \) be a nontrivial harmonic one-form. Then, for \( \Pi_{ijk} = 3X_i(g_{jk}) \), the symplectic connection \( \nabla = D + \Pi_{ijk} \) satisfies \( \rho(\nabla)_i = \rho(D)_i - 6\mathcal{R}X_i \). In particular, if \( g \) has constant curvature, then \( \rho(\nabla) = -6\mathcal{R}X_i \) and \( \mathcal{K}(\nabla) = 0 \).

Proof. The difference \( \rho(\nabla) - \rho(D) \) is computed using (3.12). The operators \( \delta, \mathcal{L} \), etc. are those associated with \( D \). Since one customarily raises and lowers indices with the metric rather than the volume form, the notation used here can be confusing. For instance, since \( J_i^p \Omega_{pjk} = -g_{jk} \), \( J_i = -g_{ij} \) and \( g_i^j = -J_i^j \). That \( X \) is harmonic implies that \( D_i X_j = D_j X_i \), \( J_i^p \Omega_{pjk} = J_j^p D_j \).
and only if the Hamiltonian vector field

By definition,

Proof. Let \( \Pi_{ij} \) be the unique \( \Gamma \)-harmonic representative of \( [\alpha] \). By Theorem 6.1 the connection \( \nabla = D + \Pi_{ij} \frac{\partial}{\partial x^i} \), where \( \Pi_{ijk} = (6\mathcal{R}_g)^{-1} \alpha_i(\partial_j g_{k}) \), is symplectic and satisfies \( \mathcal{K}(\nabla) = 0 \) and \( \rho(\nabla) = \alpha \).

7. Extremal symplectic connections of metric origin are projectively flat

Let \( \mathcal{R}_g = g^{ij} \mathcal{R}_{ij} \) be the scalar curvature of a Kähler structure \( (g, J, \Omega) \) with Levi-Civita connection \( D \). Computing \( \Omega^{ab} D_a \rho_b \) using (2.2) shows that

\[
\delta \mathcal{R}_g = \Omega \mathcal{R}_g \quad (7.1)
\]

where \( \Delta_g f = g^{ij} D_i D_j f \). A Kähler structure \( (g, J, \Omega) \) is extremal symplectic if its Levi-Civita connection \( D \) is extremal symplectic.

Computing the squared \( L^2 \)-norm of \( (\mathcal{L}_X g)_{ij} = 2D_i X_j \) via an integration by parts using the identity \( (\mathcal{L}_X D)_{ij} g_{kp} = D_i (D_j \mathcal{L}_X g)_{jk} \) shows the well known Lemma 7.1.

Lemma 7.1. On an orientable manifold \( M \), a compactly supported vector field \( X \) is an infinitesimal automorphism of the Levi-Civita connection \( D \) of a Riemannian metric \( g \) if and only if it is \( g \)-Killing. That is \( (\mathcal{L}_X D)_{ij} = 0 \) if and only if \( (\mathcal{L}_X g)_{ij} = 0 \).

Lemma 7.2. On a compact surface \( M \), a Kähler structure \( (g, J, \Omega, D) \) is extremal symplectic if and only if the Hamiltonian vector field \( X' = -\Omega^p D_p \mathcal{K} \) generated by \( \mathcal{K}(D) \) is real holomorphic. In this case the metric gradient \( g^{ij} D_p \mathcal{K}(D) \) is also real holomorphic and \( \mathcal{K}(D) \) Poisson commutes with the scalar curvature \( \mathcal{R}_g \).

Proof. By definition, \( D \) is extremal symplectic if and only if \( \mathcal{L}_X D = 0 \). By Lemma 7.1 since \( M \) is compact, this is the case if and only if \( \mathcal{L}_X g = 0 \). On a compact Kähler manifold a (real) vector field is metric Killing if and only if it is real holomorphic and preserves the volume form. Since \( X \) is Hamiltonian it preserves the volume form, and so \( X \) is real holomorphic if and only if \( D \) is extremal symplectic. In this case, as the metric gradient \( g^{ij} D_p \mathcal{K}(D) \) equals \( J_p X \), it is also real holomorphic. Also, since \( X \) is Killing its flow preserves \( \mathcal{R}_g \), so \( 0 = g^{ij} \Delta_g(X) = \{ \mathcal{K}(D), \mathcal{R}_g \} \).

By Lemma 7.2 for an extremal symplectic Kähler structure on a compact surface the \( (1, 0) \) part of \( H\mathcal{K}(D) \) is holomorphic. This implies that a Kähler structure on a compact surface admitting no nontrivial holomorphic vector field is extremal symplectic if and only if \( \mathcal{K}(D) \) is constant, and so
necessarily 0. Since \(2\mathcal{K}(D) = \Delta_g \mathcal{R}_g = 0\), \(\mathcal{R}_g\) is constant by the maximum principle. In particular, since a compact surface of genus at least two has no nontrivial holomorphic vector field, on a compact orientable surface of genus at least two, the Levi-Civita connection \(D\) of a Kähler structure is extremal symplectic if and only if \(\mathcal{R}_g\) is constant. With a different argument, the restriction on the genus can be removed, yielding Theorem 7.3.

**Theorem 7.3.** On a compact, oriented surface the Levi-Civita connection of a Riemannian metric \(g\) is extremal symplectic with respect to the symplectic structure determined by \(g\) and the given orientation if and only if \(g\) has constant curvature.

**Proof.** There is a unique complex structure \(J_i\) such that \(J_i j^p J_j^q g_{pq} = g_{ij}\) and the symplectic form \(\Omega_{ij} = J_i^p J_j^q g_{pq}\) determines the given orientation. By the Levi-Civita connection \(D\) of \(g\) satisfies \(2\mathcal{K}(D) = \Delta_g \mathcal{R}_g\). That constant curvature implies extremal symplectic is immediate. If \(D\) is extremal symplectic, then by Lemma 7.2 the metric gradient \(X^i = g^{ij} D_j \mathcal{K}(D)\) is the real part of a holomorphic vector field. On a Riemann surface a vector field is real holomorphic if and only if it is conformal Killing. By Theorem II.9 of [6], on a compact Riemannian manifold \((M, g)\) with scalar curvature \(\mathcal{R}_g\) any conformal Killing vector field \(Y^i\) satisfies \(\int_M Y^p D_p \mathcal{R}_g \, d\text{vol}_g = 0\), and so, by integration by parts,

\[
(7.2) \quad 0 = \int_M X^i D_i \mathcal{R}_g \, d\text{vol}_g = \int_M g^{ij} D_j \mathcal{K}(D) D_i \mathcal{R}_g \, d\text{vol}_g = - \int M \mathcal{K}(D) \Delta_g \mathcal{R}_g \, d\text{vol}_g = -\mathcal{E}(D).
\]

Hence \(0 = 2\mathcal{K}(D) = \Delta_g \mathcal{R}_g\), and so \(\mathcal{R}_g\) is constant by the maximum principle.

Let \(M\) be a compact orientable surface. As explained in the proof of Theorem 2.6 as a consequence of Moser’s theorem, any volume form \(\Omega\) on \(M\) can be realized as the volume form of some Riemannian metric \(g\), so there is a distinguished subset of \(\mathcal{S}(M, \Omega)\) comprising the Levi-Civita connections of Riemannian metrics with volume \(\Omega\). Fixing \(g\) determines the unique complex structure \(J\) such that \(g_{ij} = J_j^p \Omega_{ip}\), so once \(g\) has been chosen it makes sense to speak of holomorphic cubic differentials. If the genus of \(M\) is at least one, then there are nontrivial holomorphic cubic differentials. Let \(D\) be the Levi-Civita connection of \(g\) and consider \(\nabla = D + \Pi_{ijk} \Omega^{kp}\) where \(\Pi\) is the real part of a holomorphic cubic differential. The following theorem shows that \(\nabla\) is moment flat if and only if it is projectively flat.

**Theorem 7.4.** Let \(M\) be an orientable compact surface of genus at least one. Let \(\Omega\) be the volume form of a Riemannian metric \(g\) with Levi-Civita connection \(D\) and compatible complex structure \(J\). Let \(\Pi_{ijk}\) be the real part of a holomorphic cubic differential. The following are equivalent.

1. The symplectic connection \(\nabla = D + \Pi_{ijk} \Omega^{kp}\) is moment flat.
2. The symplectic connection \(\nabla = D + \Pi_{ijk} \Omega^{kp}\) is projectively flat.
3. \(\mathcal{R}_g - |\Pi|^2_g\) is constant, where \(\mathcal{R}_g\) is the scalar curvature of \(g\) and \(|\Pi|_g|^2 = g^{ia} g^{jb} g^{ic} \Pi_{ijk} \Pi_{abc}\).

If the genus of \(M\) is at least two then the conditions 1-3 are equivalent to the condition:

4. The symplectic connection \(\nabla = D + \Pi_{ijk} \Omega^{kp}\) is extremal.

**Proof.** In this proof the operators \(\delta, \mathcal{L}, \text{etc.}\) are those associated to \(D\). That \(\Pi\) be the real part of a holomorphic cubic differential is equivalent to the conditions that \(\Pi\) be \(g\)-trace free, \(g^{pa} \Pi_{ipa} = 0\), and that \(\Pi\) be \(D\)-divergence free, \(g^{pa} D_p \Pi_{ipa} = 0\), see Lemmas 3.3 and 3.5 of [16]. In this case \(2\Pi^{(3,0)} = \Pi - i \mathcal{L}(\Pi)\) where \(\Pi^{(3,0)}\) is the \((3, 0)\) part of \(\Pi\) and \(J(\Pi)_{ijk} = J_k^p \Pi_{ijp}\); see Lemma 3.4 of [16]. Since \(J(\Pi)\) is the real part of the holomorphic cubic differential \(i \Pi^{(3,0)}\), it is also completely symmetric, \(g\)-trace free, and \(D\)-divergence free. Since \(\delta \Pi_{ij} = \Omega^{pa} D_p \Pi_{ijp}\) is the \(D\)-divergence of \(J(\Pi)_{ijk} = \Pi_{ijp} J_k^p\), it vanishes. That is \(\mathcal{L}\Pi = 0\). Since the Ricci curvature of \(g\) equals \((\mathcal{R}_g/2) g_{ij}\) and \(\Pi\) is \(g\)-trace free, it follows from (3.3) that \(\mathcal{L}^* (\Pi) = 0\). Recall the notation used in Lemma 5.3. From the fact that \(J(\Pi)\) is completely \(g\)-trace free it follows that \(2 \mathcal{B}(\Pi) = 2 J(\Pi)_{ipa} g^{pa} J(\Pi)_{jpg} g^{pb} = |J(\Pi)|_g^2 g_{ij} = \).
Here the tensor norm is that given by complete contraction with the metric. Because \( \Pi \) is \( g \)-trace free, 2\( T(\Pi) = \Pi_{i\alpha} \Omega^{\alpha\beta} B(\Pi)_{\beta\gamma} \Omega^{\gamma} = \Pi_{i\alpha} \Omega^{\alpha\beta} g_{\beta\gamma} \Omega^{\gamma} = 0 \). In [5.12] the preceding shows that \( \rho(\nabla) = \rho(D) - 2\delta B(\Pi) = \rho(D) - J_i \nu D_p |\Pi|_g^2 \). This can be written more compactly as \( \rho(\nabla) = \rho(D) - 2\delta B(\Pi) = \rho(D) + *d|\Pi|_g^2 \), where \( * \) is the Hodge star operator. By (2.2), \( \rho(D) = - *dR_g \), so by (5.12),

\[
(7.3) \quad \rho(\nabla) = \rho(D) + *d|\Pi|_g^2 = *d(|\Pi|_g^2 - R_g).
\]

From (7.3) it is immediate that \( \nabla \) is projectively flat if and only if \( R_g - |\Pi|_g^2 \) is constant. By (7.3), \( *\rho(\nabla) \) is exact. If \( K(\nabla) = 0 \) then \( \rho(\nabla) \) is also closed, so \( *\rho(\nabla) \) is metrically coclosed and hence harmonic. Hence \( *\rho(\nabla) \) is an exact harmonic one-form. On a compact surface, an exact harmonic one-form is identically zero. Hence \( \rho(\nabla) = 0 \) and \( \nabla \) is projectively flat. Finally, by Theorem 2.3 (1) and (4) are equivalent when the genus of \( \sigma \) is identically zero. Hence \( \rho(\nabla) = 0 \) and \( \nabla \) is projectively flat. Theorems 7.3 and 7.4 have similar characters. They show that for symplectic connections that somehow have a metric origin that extremal or moment flat implies projectively flat.

**Remark 7.5.** Together Theorems 2.3 and 7.4 give an alternative proof of Theorem 7.3 for compact surfaces of genus at least two. If a Kähler structure on such a surface is extremal symplectic, then by Theorem 2.3 it is moment flat, while by Theorem 7.4 with \( \Pi = 0 \) it is projectively flat.

**Appendix A. Projective deformation complex**

The description that follows of the cohomology of the sheaf of projective Killing fields and its relation to the deformation space of flat projective structures is modeled on Calabi’s treatment in [8] of the cohomology of the sheaf of Killing fields on a constant curvature Riemannian manifold; see also [3] and [21].

For \( i \geq 0 \) define sheaves \( \mathcal{E}^i \) by \( \mathcal{E}^0(U) = \Gamma(TU), \mathcal{E}^i = \{0\} \) for \( i > 2 \), and

\[
(\mathcal{E}^1(U) = \{ \Pi_{ij}^k \in \Gamma(S^2(T^* U) \otimes TU) : \Pi_{ip}^p = 0 \},
(\mathcal{E}^2(U) = \{ \sigma_{ijk} \in \Gamma(\otimes^3 T^* U) : \sigma_{ijk} = \sigma_{[ijkl]} \} = \{0\}.
\]

where \( U \subset M \) is an open subset. The restriction homomorphisms are given by restriction in the ordinary sense. Define maps \( \mathcal{E}^i : \mathcal{E}^i \to \mathcal{E}^{i+1} \) by

\[
(\mathcal{E}^0 X)_{ij}^k = \mathcal{L}_X(\nabla)_{ij}^k, \quad (\mathcal{E}^1 \Pi)_{ijk} = \delta_{\Pi} C(\nabla))_{ijk}.
\]

A bit of computation shows 2\( \nabla \nu_i \nabla_{ij}^{kp} = 2\nabla_p \nabla_{ij}^{kp} \), and a bit more yields

\[
(C^1 \Pi)_{ijk} = \delta_{\Pi} C(\nabla))_{ijk} = -2 \nabla_i \nabla_{[ij]k}^p + 2\Pi_{k[i}^p R_{j|p]} + \frac{2}{3} \Pi_{ki}^p R_{|j|p} - \frac{2}{3} \Pi_{k[i}^p R_{|j|p]} - \frac{2}{3} \Pi_{k[i}^p R_{|j|p]},
\]

in which \( \nabla \in [\nabla] \) is arbitrary. Note that this shows that the right side of (A.3) is a projectively invariant differential operator, something tedious to check directly.

**Lemma A.1.** For \( [\nabla] \in \mathcal{F}(M) \) and \( X \in \Gamma(TM) \), there holds \( C^1 C^0(X) = \mathcal{L}_X C(\nabla) \). Moreover, \( [\nabla] \) is flat if and only if \( C^1 C^0 = 0 \). That is, the sequence

\[
0 \to \mathcal{E}^0 \to \mathcal{E}^1 \xrightarrow{C^1} \mathcal{E}^2 \to 0
\]

is a complex if and only if \( [\nabla] \) is flat.

**Proof.** Let \( \phi_t \) be the flow of \( X \). As \( (\nabla) + t\mathcal{L}_X(\nabla) = \phi_t^* \nabla \) has order at least two in \( t \),

\[
(\delta_{\mathcal{L}_X(\nabla))) = \frac{d}{dt}|_{t=0} C(\nabla) + \mathcal{L}_X C(\nabla) = \frac{d}{dt}|_{t=0} \mathcal{L}_X(\nabla) = \frac{d}{dt}|_{t=0} \phi_t^* C(\nabla) = \mathcal{L}_X C(\nabla),
\]

the penultimate equality by the diffeomorphism equivariance of the curvature, and the last equality by definition of the Lie derivative. This shows \( C^1 C^0(X) = \mathcal{L}_X C(\nabla) \). If \( [\nabla] \) is flat then \( C(\nabla) = 0 \),
so $C^4C^0(X) = \delta \varepsilon \chi_{[\nabla]}C(\nabla) = 0$. At any $p \in M$, there can be chosen $X$ which vanishes at $p$ and such that $\nabla X$ is the identity endomorphism on $T_pM$. For such an $X$ and any tensor $A_{ij}^{1 \ldots r}$ there holds $\varepsilon \chi_X A = (r - s)A$ at $p$, and so, if $r \neq s$ and $\varepsilon \chi_X A = 0$ for all $X \in \Gamma(TM)$, then $A$ is identically 0. Applying this with $A$ taken to be $C_{ijk}$, it follows from $C^4C^0(X) = \varepsilon \chi_X C(\nabla)$ that if $C^4C^0(X) = 0$ for all $X$, then, at every $p$, $C_{ijk} = 0$, so $\nabla$ is flat.  \[ \Box \]

**Remark A.2.** For any open $U \subset M$ the sequence \( (3.28) \) of Lemma \( 3.7 \) and the sequence \( (A.4) \) are isomorphic via symplectic duality, as indicated in the diagram \( (A.6) \).

\[
\begin{array}{cccccc}
0 & \longrightarrow & C^0(U) & \xrightarrow{c^0} & C^1(U) & \xrightarrow{c^1} & C^2(U) & \longrightarrow & 0 \\
X \xrightarrow{X_i} X & \xrightarrow{\sigma_{ij}^{k \to \sigma_{ijk}}} & \sigma_{ij}^{k \to \Omega_{ij}^{k \to \sigma_{ijk}}} & \xrightarrow{\sigma_{ijk} \to \Omega_{ijk}^{k \to \sigma_{ijk}}} & \sigma_{ijk} & \longrightarrow & \frac{1}{2}\sigma_{ijk} & \longrightarrow & 0 \\
0 & \longrightarrow & \Gamma(T^*U) & \xrightarrow{\varepsilon} & \Gamma(S^3(T^*U)) & \xrightarrow{\varepsilon} & \Gamma(T^*U) & \longrightarrow & 0 \\
\end{array}
\]

The conclusion of Lemma \( A.4 \) transported to the sequence \( (3.28) \) is stated in Lemma \( 3.7 \).

Given a flat $\nabla \in \mathcal{P}(M)$ define the presheaf of *projective Killing fields* on $M$ by

\[
\mathcal{P}(U) = \{ X \in \Gamma(U, TM) : \varepsilon \chi_X [\nabla] = 0 \}
\]

for an open set $U \subset M$. The restriction homomorphisms are given by ordinary restriction of vector fields. It is clear that $\mathcal{P}$ is a sheaf of Lie algebras. Let $i : \mathcal{P} \to C^0$ be the inclusion homomorphism.

**Theorem A.3.** If $\nabla \in \mathcal{P}(M)$ is flat then

\[
(A.8) \quad 0 \longrightarrow \mathcal{P} \xrightarrow{i} C^0 \xrightarrow{c^0} C^1 \xrightarrow{c^1} C^2 \longrightarrow 0
\]

is a fine resolution of the sheaf $\mathcal{P}$ of projective Killing fields and an elliptic complex. The cohomology of the complex $C^\bullet(M)$ of global sections is isomorphic to the Cech cohomology of the sheaf $\mathcal{P}$ of projective Killing fields. If $M$ is compact these cohomologies are finite-dimensional.

**Remark A.4.** That \( (A.8) \) is a fine resolution is stated without proof as Theorem 1 of T. Hangan's \[20\], and also as Theorem 2.1 of \[21\] where it is described in more detail, although also without proof. Presumably Hangan's proof was similar to that here; it seems that it was never published.

**Remark A.5.** By Theorem \( A.3 \) in the projectively flat case the complex of Lemma \( A.4 \) gives rise to a fine resolution of the sheaf of projective Killing fields. For moment flat connection an analogous claim based on Lemma \( 3.4 \) has not been proved, and it is not clear whether it could be correct.

**Proof.** Lemma \( A.1 \) implies that the sequence \( (C^\bullet, C^\bullet) \) is a complex. That the complex be elliptic means that the associated principal symbol complex is exact over the complement of the zero section. It suffices to check that if $\sigma_{ij}^{k \to \sigma_{ijk}} \in C^1$ satisfies $Z_pZ_i[\sigma_{ijk}]^p = 0$ for $Z \in T_xM \setminus \{0\}$, then there is $\tau \in T_xM$ such that $\sigma_{ij}^{k \to \sigma_{ijk}} = Z_i Z_j A^k - \frac{2}{3}Z_p A^p Z_i[\sigma_{ijk}]^k$. Because $Z_pZ_i[\sigma_{ijk}]^p = 0$ there is $\tau \in T_xM$ such that $Z_p[\sigma_{ijk}]^p = Z_i[\sigma_{ijk}]^p$. Then $Z_j \tau_i = \sigma_{ij}^{k \to \sigma_{ijk}} Z_k = \sigma_{ij}^{k \to \sigma_{ijk}} Z_k = Z_i[\sigma_{ijk}]^p$, so there is $c \in \mathbb{R}$ such that $\tau_i = cZ_i$. Choose linearly independent $X, Y \in T_xM$ such that $X^pZ_p = 1$ and $Y^pZ_p = 0$ and let $U_i \in T^*_xM$ be such that $X^pU_i = 0$ and $Y^pU_i = 1$. Since $Z_k[\sigma_{ijk}]^k - 3c(Z_i Z_j X^k - \frac{2}{3}Z_i[\sigma_{ijk}]^k) = 0$ there are constants $p, q, r \in \mathbb{R}$ such that

\[
(A.9) \quad \sigma_{ij}^{k \to \sigma_{ijk}} = 3c(Z_i Z_j X^k - \frac{2}{3}Z_i[\sigma_{ijk}]^k) + (p Z_i Z_j + 2qZ_i U_j + rU_i U_j) Y^k.
\]

Since $0 = \sigma_{ip}^{p \to \sigma_{ijk}} = qZ_i + rU_i$ and $Z$ and $U$ are linearly independent, it must be $q = 0 = r$. Setting $A^i = 3cX^i + pY^i$ there results $\sigma_{ij}^{k \to \sigma_{ijk}} = Z_i Z_j A^k - \frac{2}{3}Z_p A^p Z_i[\sigma_{ijk}]^k$, as claimed.

The sheaves $C^\bullet$ are fine because they are given by sections of smooth tensor bundles. That the sequence of sheaves \( (A.8) \) is exact at the 0 level is immediate from the definition of $\mathcal{P}$. To prove exactness at the level 1 it suffices to prove that if a tensor $\Pi_{ij}^{k \to \sigma_{ijk}} \in C^1(U)$ satisfies $\delta \varepsilon \chi_C(\nabla) = 0$ then
given any \( p \in U \) there is an open neighborhood \( V \) containing \( p \) and contained in \( U \) such that the restriction of \( \Pi \) to \( V \) is equal to \( \mathcal{L}_X[\mathcal{V}] \) for some \( X \in \Gamma(TV) \). Because \( \mathcal{V} \) is projectively flat, there can be chosen a neighborhood \( W \) of \( p \) and a representative \( \partial \) of the restriction to \( W \) of \( \mathcal{V} \) such that \( \partial \) is a flat affine connection. In the remainder of this proof the word \textit{locally} means \textit{restricting to a smaller open neighborhood of} \( p \) \((if necessary)\). Locally there is a \( \partial \)-parallel symplectic (volume) form \( \Omega_{ij} \), which will be used to raise and lower indices. Since any two-form \( \alpha_{ij} \) satisfies \( 2\alpha_{ij} = \alpha_{ip}^{\,\,\,	ext{业界}}\Omega_{ij} \), that \( \partial \mathcal{P}_p A_{p_{ij...i_k}} = 0 \) is equivalent to \( \partial_i A_{ij1...i_k} = 0 \) and so, by the usual Poincare lemma, implies that locally there is \( B_{ij...i_k} \) such that \( A_{ij1...i_k} = \partial_i B_{ij...i_k} \). This observation will be used several times. By the flatness of \( \Pi \) and the hypothesis \( C^4(\Pi) = 0 \), \( \partial_i \partial_j \Pi_{jk}^p = \partial_p \nabla_p [\Pi_{jk}^p] = 0 \). Hence locally there is a one-form \( A_i \) such that \( \partial_i \Pi_{ij}^p = \partial_i A_i \). Since \( 0 = \partial_p [\Pi_{ij}^p] = \partial_p A_{ij} \), again by the Poincare lemma, locally there is a function \( f \) such that \( \partial_p A_{ij} = \partial_i \partial_j f \). Locally there is a vector field \( X^i \) such that \( \partial_p X^p = f \). Then \( \partial_p (\Pi_{ij}^p - \partial_i \partial_j X^p) = 0 \) and so locally there is a tensor \( A_{ij} = A_{ij}^p \) such that \( \Pi_{ijk} = \partial_i \partial_j X_k + \partial_k A_{ij} \). Then \( 0 = \Pi_{ij}^p - \partial_i \partial_j X^p = \partial_p A_{ij}^p \) so that \( \partial_p (A_{ij}^p - \partial_i X^p) = 0 \). Hence there is a vector field \( V_i \) such that \( A_{ij}^p = \partial_i X_j + \partial_j Y_i \), and so \( \Pi_{ijk} = \partial_i \partial_j X_k + \partial_k \partial_i X_j + \partial_i \partial_j Y_i \). Since \( 0 = \partial_p A_{ij}^p = \partial_p (X^p - Y^p) \), locally there is a function \( g \) such that \( Y_i = X_i + g \). Hence \( \Pi_{ijk} = 3\pi_i (\partial_i X_j + \partial_j X_k) + \partial_i \partial_j g \). Using \( \partial_i \partial_j X_k = \partial_i g \partial_j X_k + 2\partial_i \partial_j X_k \), \( \partial_i \partial_j Y_i + \partial_i f \Omega_{jk} \) there results \( \Pi_{ijk} = 3\partial_i \partial_j X_k - 2\partial_i (f \Omega_{jk}) + \partial_i \partial_j g \). Since \( \partial_p X^p = f \), setting \( Z^i = 3X^i + \partial_i g \) there results
\[
\Pi_{ij}^k = \partial_i \partial_j Z^k - \frac{3}{2} \delta_{ij} \partial_j Z^k = \mathcal{L}_Z[\partial] - \mathcal{L}_Z[\mathcal{V}].
\]
This completes the proof of the local exactness of \( \mathcal{A}^\bullet \). By the abstract de Rham theorem the cohomology of the complex \( C^\bullet(M) \) of global sections is isomorphic to the Cecohomology of the sheaf \( \mathcal{P} \) of projective Killing fields. If \( M \) is compact, becase \( C^\bullet(M) \) is elliptic, its cohomology is finite-dimensional, by Proposition 6.5 of \( \Pi \).}

\begin{remark}
The sequence (complex) \((\mathcal{C}^\bullet, \mathcal{C}^\bullet)\) of \( \mathcal{A}^\bullet \) is a concrete realization of the generalized BGG sequence associated to the adjoint representation of \( \mathfrak{sl}(3, \mathbb{R}) \), although the arguments justifying this claim are omitted (consult [13] and [15] for discussion of BGG sequences in the context of projective structures). Theorem \( \mathcal{A}^\bullet \) can be obtained by specializing the main theorem about BGG sequences proved in either [10] or [11], although the demonstration of this claim requires too much space to be included here. Some (mostly formal) work is required, because to connect Theorem \( \mathcal{A}^\bullet \) with the parabolic geometry formalism of parabolic geometries there must be used a lifting construction based on the Thomas or tractor connection.
\end{remark}

\begin{thebibliography}{99}
\bibitem{1} M. F. Atiyah and R. Bott, \textit{A Lefschetz fixed point formula for elliptic complexes. I}, Ann. of Math. (2) 86 (1967), 374–407.
\bibitem{2} A. Banyaga, \textit{The structure of classical diffeomorphism groups}, Mathematics and its Applications, vol. 400, Kluwer Academic Publishers Group, Dordrecht, 1997.
\bibitem{3} L. Bérard-Bergery, J. P. Bourguignon, and J. Lafontaine, \textit{Déformations localement triviales des variétés Riemanniennes}, Differential geometry (Proc. Sympos. Pure Math., Vol. XXVII, Stanford Univ., Stanford, Calif., 1973), Part I, Amer. Math. Soc., Providence, R.I., 1975, pp. 3–32.
\bibitem{4} P. Bieliavsky, M. Cahen, S. Gutt, J. Rawnsley, and L. Schwachhöfer, \textit{Symplectic connections}, Int. J. Geom. Methods Mod. Phys. 3 (2006), no. 3, 375–420.
\bibitem{5} F. Bourgeois and M. Cahen, \textit{A variational principle for symplectic connections}, J. Geom. Phys. 30 (1999), no. 3, 233–265.
\bibitem{6} J.-P. Bourguignon and J.-P. Ezin, \textit{Scalar curvature functions in a conformal class of metrics and conformal transformations}, Trans. Amer. Math. Soc. 301 (1987), no. 2, 723–736.
\bibitem{7} M. Cahen and S. Gutt, \textit{Moment map for the space of symplectic connections}, Liber Amicorum Delanghe, F. Brackx and H. De Schepper eds., Gent Academia Press, 2005, pp. 27–36.
\bibitem{8} E. Calabi, \textit{On compact, Riemannian manifolds with constant curvature. I}, Proc. Sympos. Pure Math., Vol. III, American Mathematical Society, Providence, R.I., 1961, pp. 155–180.
\end{thebibliography}
9. ______, Extremal Kähler metrics, Seminar on Differential Geometry, Ann. of Math. Stud., vol. 102, Princeton Univ. Press, Princeton, N.J., 1982, pp. 259–290.
10. D. M. J. Calderbank and T. Diemer, Differential invariants and curved Bernstein-Gelfand-Gelfand sequences, J. Reine Angew. Math. 537 (2001), 67–103.
11. A. Čap, J. Slovák, and V. Souček, Bernstein-Gelfand-Gelfand sequences, Ann. of Math. (2) 154 (2001), no. 1, 97–113.
12. X. Chen, Space of Kähler metrics. III. On the lower bound of the Calabi energy and geodesic distance, Invent. Math. 175 (2009), no. 3, 453–503.
13. S. K. Donaldson, Remarks on gauge theory, complex geometry and 4-manifold topology, Fields Medallists’ lectures, World Sci. Ser. 20th Century Math., vol. 5, World Sci. Publ., River Edge, N.J., 1997, pp. 384–403.
14. M. Eastwood, Notes on projective differential geometry, Symmetries and overdetermined systems of partial differential equations, IMA Vol. Math. Appl., vol. 144, Springer, New York, 2008, pp. 41–60.
15. M. G. Eastwood and A. R. Gover, The BGG complex on projective space, SIGMA Symmetry Integrability Geom. Methods Appl. (2011), Paper 060, 18.
16. D. J. F. Fox, Einstein-like geometric structures on surfaces, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) XII (2013), no. 5, 499–585.
17. W. M. Goldman, Convex real projective structures on compact surfaces, J. Differential Geom. 31 (1990), no. 3, 791–845.
18. ______, The symplectic geometry of affine connections on surfaces, J. Reine Angew. Math. 407 (1990), 126–159.
19. S. Gutt, Remarks on symplectic connections, Lett. Math. Phys. 78 (2006), no. 3, 307–328.
20. T. Hangan, Complexes différentiels associés aux pseudo-groupes projectif et conforme, C. R. Acad. Sci. Paris Sér. I Math. 312 (1991), no. 11, 867–870.
21. ______, La résolution de Calabi et ses analogies projective et conforme, Geometry Seminars. Sessions on Topology and Geometry of Manifolds (Italian) (Bologna, 1990), Univ. Stud. Bologna, Bologna, 1992, pp. 99–112.
22. S. Kobayashi, Transformation groups in differential geometry, Springer-Verlag, New York, 1972, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 70.
23. S. Kobayashi and K. Nomizu, Foundations of differential geometry. Vol. II, Interscience Tracts in Pure and Applied Mathematics, No. 15 Vol. II, Interscience Publishers John Wiley & Sons, Inc., New York-London-Sydney, 1969.
24. J. Milnor, Morse theory, Based on lecture notes by M. Spivak and R. Wells. Annals of Mathematics Studies, No. 51, Princeton University Press, Princeton, N.J., 1963.
25. J. Moser, On the volume elements on a manifold, Trans. Amer. Math. Soc. 120 (1965), 286–294.

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