On some common misconceptions regarding the "Ergodic Hierarchy".

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The well-known ergodic hierarchy of sheerly ergodic, mixing, Kolmogorov and Bernoulli systems, with each next level supposedly encompassing the previous one, is shown to be too simplistic in its usual formulation. A K-system can be sheerly ergodic and sometimes may be reduced to a sheerly mixing system by some simple projection. More precise characterizations of ergodic properties of dynamical systems should start out from a consideration of the full Lyapunov spectrum.

I. INTRODUCTION

In ergodic theory it has been common to define a hierarchy of degrees of ergodicity \( \mathbb{1} \). The weakest property is sheer ergodicity. This implies that, under some given dynamics, the time average of any Lebesgue integrable function on phase space equals an average over some properly chosen subset \( S \) of phase space. If the dynamics leaves volumes in phase space unchanged, as is the case for Hamiltonian dynamics, this phase space average simply assigns weights proportional to phase space volume to various parts of this subset. If the dynamics is phase space contracting one has to choose an attractor as subset and use the SRB-measure for its weighting, i.e. each part of the subset has to be weighted by its asymptotic phase space contraction factor. The next property in the ergodic hierarchy is mixing. This implies that under the dynamics the image of any subset \( A \) of non-zero measure becomes uniformly distributed on \( S \), i.e. the measure of its intersection with an arbitrary other subset \( B \) of non-zero measure approaches \( \mu(A)\mu(B)/\mu(S) \) as time increases. Mixing implies ergodicity, but not the other way around; therefore mixing is a stronger property than sheer ergodicity. In fact one may distinguish between strong mixing, where the uniformity property holds pointwise in time, and weak mixing where it only holds on average over long stretches of time. The next property in the ergodic hierarchy is that of being a Kolmogorov system or K-system. The original definition of this is rather involved, but for my present purposes it suffices that K-systems are systems with positive Kolmogorov-Sinai entropy (or KS-entropy). For systems without escape, i.e. with dynamics mapping all points of \( S \) onto \( S \), according to Pesin’s theorem this is equivalent to the requirement that the system has at least one positive Lyapunov exponent, or in other words, is chaotic. It is commonly assumed that being a K-system implies mixing and therefore is a stronger property. An even stronger property, the strongest in the ergodic hierarchy, is for a system being a Bernoulli system. This is most easily defined for maps, rather than flows (for the latter one therefore may take recourse to time slices or to Poincaré maps). It implies that \( S \) may be decomposed into a discrete collection of subsets \( A_i \) with the following property: for any subset of all points on \( S \) with a given history, i.e. with a given sequence \( i_{-1}, i_{-2}, \ldots \) of subsets visited at previous mappings, the fraction of points ending up in \( A_i \) at the next mapping is simply proportional to the measure of this subset. In other words, without additional information on where the phase point is or came from, the outcome of the mapping is like determined by pure chance, irrespective of what happened before.

Here I will argue that in fact this hierarchy should not be taken too literally. A K-system may be non-mixing and even if it is mixing, its ergodic behavior in certain respects need not be stronger than that of a merely mixing system.

II. MIXED BEHAVIORS

A standard example of a map exhibiting the Bernoulli property is the baker map, defined on the unit square as

\[
B(x, y) = (2x \mod 1, y/2 + \text{int}(2x)),
\]

with \( \text{int}(x) \) the integer part of \( x \). This map is area conserving and has Lyapunov exponents \( \pm \log 2 \), with corresponding eigenvectors in tangent space \((1, 0)\) and \((0, 1)\). It satisfies a Bernoulli scheme with respect to the subdivision of the unit square into the two subsets \( 0 \leq x < 1/2 \) and \( 1/2 \leq x < 1 \). It may be extended to a map on the unit cube of the form

\[
M(x, y, z) = (2x \mod 1, y/2 + \text{int}(2x), f(z)),
\]

with \( f(z) \) mapping the unit interval onto itself. This map is still Bernoulli, as it still satisfies a Bernoulli scheme with respect to the subdivision, now of the unit cube, into the subsets \( 0 \leq x < 1/2 \) and \( 1/2 \leq x < 1 \). It still has a
Lyapunov exponent $\log 2$, so it obviously is still a K-system, but whether it is mixing depends on the nature of the function $f$. If $f$ is merely ergodic, e.g. a shift mod 1 over an irrational number $\alpha$, the map $M$ is not mixing. So the statement that the property of being a K-system implies mixing, obviously is not true. In this case one may say that $M$ combines the property of being Bernoulli when projected on the $x - y$ plane, with that of being merely ergodic when projected on the $z$-axis. Therefore, to me it also seems dubious to state in general that K-systems are more strongly ergodic than mixing systems. In the present example I would say that the system is no more ergodic than the one defined by the one-dimensional mapping $f$ alone.

One may object now that the present example is very atypical because it can be factorized into a two-dimensional mapping that is Bernoulli and a one-dimensional mapping that is merely ergodic. It is not hard though to extend this example into non-factorizable mappings, e.g.

$$M_1(x, y, z) = (2x + z \mod 1, y/2 + \text{int}(2x), z + \alpha + (2x + z \mod 1) + y/2 + \text{int}(2x) - x - y), \quad (3)$$

which acts like a shift map to which an $x$ and $y$ dependent constant is added. Going to higher dimensions one may, for example in $d=4$, devise mappings that asymptotically reduce to a baker map in the $x - y$ plane times a quasi-periodic map on some curve in the remaining coordinates. In this case the decoupling of the two maps may easily be designed to hold only asymptotically on the attractor. No doubt increasingly more complicated scenario’s may be designed with increasing dimensionality.

### III. DISCUSSION

For the general case of a mapping or flow in a high-dimensional phase space the ergodic properties of the dynamics on a given ergodic component will first of all depend on the spectrum of Lyapunov exponents. For simplicity, let us restrict ourself to symplectic maps or flows, for which the spectrum is odd (a Lyapunov exponent $\lambda$ always comes together with $-\lambda$). Then, if there are no zero exponents, in the case of maps, or, in the case of flows, just a single pair related to displacements in the direction of the flow, one may say the system fully has the character of a K-system. No projections are possible in directions where the dynamics is merely mixing or even merely ergodic. Whether the system is also Bernoulli is a separate problem that has to be decided case by case. If there are more zero Lyapunov exponents, one should decide first of all what is the character of the behavior of the dynamics in the corresponding directions. This may be hard to decide, especially if the dynamics in these directions cannot be decoupled from that in the remaining ones.

Without further specifications the usual ergodic hierarchy remains a qualitative characterization of increasingly randomizing and chaotic types of dynamics. As such it is extremely useful. But for fixing a rigorous sequence in which each next stage implies the previous one, obviously more precision is needed.

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[1] V.I. Arnol’d and A. Avez, *Ergodic Problems of Classical Mechanics*, (W. A. Benjamin, New York, 1968)