A NOTE ON THE DEFINABILITY OF GENUS FOR ZARISKI GEOMETRIES

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Abstract. In this work we propose a notion of genus in the context of Zariski geometries and we obtain natural generalizations of the Riemann-Hurwitz Theorem and the Hurwitz Theorem in the context of very ample Zariski geometries. As a corollary, we show that such notion of genus cannot be first-order definable in the full language of a Zariski geometry.

1. Introduction

Classification theorems play an important role in the development of several areas in mathematics. In Model Theory, one of the main classification results appeared in [6], where Boris Zilber proved that for any strongly minimal set \( M \), its geometry is either trivial, locally projective, or there is a pseudo-plane definable in \( M \). As pointed out by himself in [7], this result and the faithful belief that “mathematical perfect structures cannot be overlook in the natural progression of mathematics”, motivated Zilber to conjecture in [6] the following stronger classification:

**Trichotomy conjecture.** For any given strongly minimal set \( M \), the pregeometry \((M, \text{acl})\) corresponding to \( M \) and its algebraic closure operator lies in one of the following classes:

- **Trivial pregeometries.** For any \( A \subseteq M \), \( \text{acl}(A) = \bigcup_{a \in A} \text{acl}(\{a\}) \).
- **Locally modular pregeometries.** For any closed sets \( A, B \subseteq M \),
  \[ \dim(A \cup B) + \dim(A \cap B) = \dim(A) + \dim(B) \]
- **Non-locally modular pregeometries.**

In particular, if the first two cases do not occur, then there is an algebraically closed field \( K \) definable in \( M \) and the only structure induced on \( K \) from \( M \) is definable in the field structure itself. (This property is called *purity* of \( K \).)

The veracity of the Trichotomy conjecture would have resulted in a very strong classification theorem for strongly minimal structures. However, it was shown in [3] by Ehud Hrushovski that the Trichotomy conjecture is false in general. Indeed, Hrushovski proved that there are strongly minimal theories which are not locally modular but do not even interpret an infinite group. Nevertheless, in a joint work [2] Hrushovski and Zilber defined an important class of strongly minimal structures where the Trichotomy conjecture holds. These structures are called **Zariski geometries**. Roughly speaking, a Zariski geometry is a strongly minimal structure with some topological properties that resemble properties of the Zariski topology over a smooth algebraic curve.

An important goal in the early development of the subject was to provide finer classification theorems for Zariski geometries, beyond Zilber’s trichotomy. In [2] Zariski geometries are classified into the following classes: trivial Zariski geometries, locally modular Zariski geometries, non-locally modular Zariski geometries, ample Zariski geometries and, very ample Zariski geometries.
It can be shown that a Zariski geometry is non-locally modular if and only if it is ample (see, for example, [4, Lemma 3.1] for a proof). Moreover, very ample Zariski geometries are characterized as smooth curves over algebraically closed fields:

**Theorem 1.1** (Theorem A and Proposition 1.1 in [2]). Any smooth curve $C$ over an algebraically closed field may be seen as a very ample Zariski geometry, where the topology on each $C^n$ is the restriction of the Zariski topology $\tau^n_{Zar}$. Conversely, if $D$ is a very ample Zariski geometry, there exists an algebraically closed field $K$ and a smooth curve $C$ over $K$ such that the Zariski geometry corresponding to $C$ is isomorphic to $D$. Moreover, the curve $C$ and the field $K$ are unique up to a curve isomorphism and up to a field isomorphism.

The results of this paper are related to the notion of genus for Zariski geometries. In Section 2 we present all the necessary preliminaries and provide a recollection of the main results used to show the classification theorems for Zariski geometries. In particular, Proposition 2.11 is used to guarantee the uniqueness of the curve $C$ and the field $K$ given in Theorem 1.1, and we also recall the definition of $D$-manifolds for a Zariski geometry $D$. In Section 3 we prove a slight generalization of Proposition 2.11, and use Theorem 1.1 to define a notion of genus for Zariski geometries that agrees with the genus for smooth curves over algebraically closed fields. This notion gives rise to corresponding generalizations of the Riemann–Hurwitz and Hurwitz theorems for algebraic curves (cf. Proposition 3.5 and Theorem 3.7). Furthermore, these results are used to prove in Theorem 3.8 that the notion of genus in very ample Zariski geometries cannot be first-order definable in the so-called full language, where there is a predicate for every closed subset of the Zariski geometry.

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### 2. Preliminaries

In this section we will present all the necessary preliminaries in Zariski geometries that we will require in the later sections. We begin by introducing Zariski geometries and the classification results for Zariski geometries provided by Hrushovski and Zilber in [2].

**Notation 1.** Let $M$ be a topological space and let $X, Y \subseteq M$. Throughout this work $X \subseteq_{cl} Y$ means that $X$ is a closed subset of $Y$; $\text{cl}(X)$ is the topological closure of $X$ and dimension means Krull dimension, which is the supremum of $n$ such that there exists an ascending chain of closed sets $X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n \subseteq X$, and it is well defined because the topological spaces we will consider are Noetherian. We write $\dim(X)$ for the dimension of $X$. For any set $C \subseteq X \times Y$ and $a \in X$ we write $C(a, Y)$ for the set $\{ y \in Y : (a, y) \in C \}$ and $|X|$ is the cardinal of $X$.

**Definition 2.1.** A Zariski geometry $\mathcal{D}$ consists of an infinite set $D$ together with a collection of Noetherian topologies $\mathcal{D} = \{ (D^n, \tau_n) : n \in \mathbb{N} \}$ such that the following axioms hold:

(Z0) Let $f_i : D^m \to D$ be a projection $f_i(\vec{x}) = x_{\sigma(i)}$ or a constant map $f_i(\vec{x}) = c$ (where $\sigma$ is a bijective function $\sigma : \{1, \ldots, m\} \to \{1, \ldots, m\}$), then the map $f : D^m \to D^n$ sending $\vec{x}$ to $(f_1(\vec{x}), \ldots, f_m(\vec{x}))$ is a continuous function. Also, the diagonal sets $\Delta_{ij} := \{ \vec{x} \in D^n : x_i = x_j \}$ are closed for all $1 \leq i < j \leq n$ for $n \in \mathbb{N}$.

(Z1) Let $C \subseteq_{cl} D^m$ and $\pi_{mn} : D^m \to D^n$ a projection map. Then there is a proper closed set $F \subseteq_{cl} \text{cl}(\pi_{mn}(C))$ such that $\pi_{mn}(C) \supseteq \text{cl}(\pi_{mn}(C)) - F$.

(Z2) $D$ is irreducible and uniformly one-dimensional, that is, if $C \subseteq_{cl} D^n \times D$, then there exists $N$ such that for all $\vec{a} \in D^n$ either $|C(\vec{a}, D)| \leq N$ or $C(\vec{a}, D) = D$.
(Z3) (Dimension Theorem) If $C$ is an irreducible closed set in $D^n$ and $\Delta^n_{ij}$ the $ij$-diagonal, then any component of $C \cap \Delta^n_{ij}$ has dimension $\geq \dim(C) - 1$.

The above definition is based on properties that have topological nature. Nevertheless, we are interested in the first-order properties of these structures, so it is necessary to introduce some languages in order to express such properties.

**Definition 2.2.** Let $\mathcal{D}$ be a Zariski geometry.

- The **full language** of $\mathcal{D}$ is denoted as $\mathcal{L}_{\text{full}}(\mathcal{D})$ and contains a $n$-ary relation symbol for each closed subset of $D^n$, for all $n \in \mathbb{N}$.
- The **natural language** of $\mathcal{D}$ is the sub-language $\mathcal{L}_{\text{nat}}(\mathcal{D})$ of $\mathcal{L}_{\text{full}}(\mathcal{D})$ consisting of all automorphism invariant subsets of $D^n$ for all $n$.

**Example 2.3** (Smooth curves). Let $C$ be an irreducible smooth algebraic curve over an algebraically closed field $K$. Consider on each power of $C$ the restriction of the Zariski topology $\tau^n_{\text{zar}}$. Then $\mathfrak{C} := \langle (C^n, \tau^n_{\text{zar}}) : n \in \mathbb{N} \rangle$ is a Zariski geometry. Indeed, (Z0) follows from the fact that the underlying functions are regular and regular functions are continuous (\cite[Remark 3.1.1]{[1]}). (Z1) follows from quantifier elimination for algebraically closed fields (\cite[pp. 109-110]{[4]}) and (Z2) follows from strong minimality of algebraically closed fields. Finally, (Z3) follows from Dimension Theorem for smooth curves (see e.g., \cite[Theorem I.7.2]{[1]}).

The following lemma summarizes the basic model-theoretical properties of Zariski geometries. The reader is referred to Sections 2 and 4 of the work \cite{[2]} for a proof.

**Lemma 2.4.** Let $\mathcal{D}$ be a Zariski geometry regarded as a $\mathcal{L}$-structure, where $\mathcal{L}$ is any of the languages $\mathcal{L}_{\text{nat}}(\mathcal{D})$ or $\mathcal{L}_{\text{full}}(\mathcal{D})$. Then the following is true about $\mathcal{D}$:

1. $\mathcal{D}$ eliminates quantifiers and is strongly minimal.
2. Any $\mathcal{L}$-elementary extension of $\mathcal{D}$ can be endowed with the structure of a Zariski geometry.
3. Any universal domain $\mathcal{D}^* \subseteq \mathcal{D}$ is a Zariski geometry.

Remember that in a Noetherian space any closed set $X$ may be written (uniquely) as the union of irreducible sets, called the (irreducible) components of $X$.

If $X$ is irreducible, we say that a property $P$ holds generically on $X$, if $P$ holds for all elements of $X$ outside a proper closed subset or, equivalently, a generic point of $X$ satisfies the property $P$.

The following two definitions are the key in the classification theorems for Zariski geometries provided in \cite{[2]}. These definitions allow the existence of families of curves enriched with some geometrical structure.

For the rest of this work $\mathcal{D} = \langle (D^n, \tau_n) : n \in \mathbb{N} \rangle$ is an arbitrary Zariski geometry.

**Definition 2.5.** A plane curve in $D^2$ is an irreducible one-dimensional subset of $D^2$. A family of plane curves is a pair $(C, E)$ where $E \subseteq_{\text{cl}} D^n$ is irreducible closed; $C \subseteq_{\text{cl}} E \times D^2$ is such that $C(\vec{e})$ is a plane curve for generic $\vec{e} \in E$.

**Definition 2.6.** We say that $\mathcal{D}$ is **ample** if there exists a family of plane curves $(C, E)$ such that for generic $\vec{a}, \vec{b} \in D^2$ there is $\vec{e} \in E$ such that the curve $C(\vec{e})$ passes through both $\vec{a}$ and $\vec{b}$.

We say that $\mathcal{D}$ is **very ample** if there exists a family of plane curves $(C, E)$ making $\mathcal{D}$ an ample Zariski geometry and such that for any $\vec{a}, \vec{b} \in D^2$ there is a $C(\vec{e})$ passing through just one of $\vec{a}, \vec{b}$.

**Example 2.7** (\cite{[4]}, pp. 115). Let $C$ be a quasiprojective smooth algebraic curve in $P^n_K$, where $K$ is an algebraically closed field and consider on each power of $C$ the restriction of the Zariski topology, $\tau^n_{\text{zar}}$. As it was mentioned before $\mathfrak{C} = \langle (C^n, \tau^n_{\text{zar}}) : n \in \mathbb{N} \rangle$ is a Zariski geometry. We claim that $\mathfrak{C}$ is very ample. Indeed, the product $C \times C$ may be embedded in some projective space and let us choose $N$ to be the smallest positive integer such that there is an embedding $f : C \times C \to P^n_K$. Let us define the projective variety Grass$(1, N) = \{ H \subseteq P^n_K : H$ is a hyperplane $\}$ and $D = \{ (H, \vec{x}) : \vec{x} \in f^{-1}(H) \}$. For each $H \in \text{Grass}(1, N)$ we have $\dim(H \cap f(C \times C)) \leq 1$. Thus $(D, \text{Grass}(1, N))$ defines a very ample family of curves. Indeed, by Bertini’s Theorem (see e.g., \cite[pp. 179]{[1]}) it follows that for generic
Lemma 2.8. We will find finer classification results for Zariski geometries in a geometrical sense by the introduction of Zariski geometries. Results of this kind are the classical ones, and we will now state them. (Later the conjecture holds. In this sense the Trichotomy Conjecture gives a classification result for the class of classifying (via geometric stability theory) the models of strongly minimal theories and when it holds.)

2.1. Classical classification theorems. Zilber’s trichotomy conjecture was introduced as a way of classifying (via geometric stability theory) the models of strongly minimal theories and when it was refuted by E. Hrushovski the question remaining was: for which structures does the Trichotomy Conjecture hold? Zariski geometries turned out to be a large class of mathematical structures for which the conjecture holds. In this sense the Trichotomy Conjecture gives a classification result for the class of Zariski geometries. Results of this kind are the classical ones, and we will now state them. (Later we will find finer classification results for Zariski geometries in a geometrical sense by the introduction of a notion of “genus” for very ample Zariski geometries.)

Lemma 2.8 (Lemma 3.1 in [4]). A Zariski geometry is ample if and only if it is non-locally modular.

Lemma 2.9 (Trichotomy Theorem, Lemma 6.11 and Proposition 7.10 in [2]). If $\mathcal{D}$ is an ample Zariski geometry, then there exists an algebraically closed field $K$ interpretable in $\mathcal{D}$. Moreover, the only structure induced from $\mathcal{D}$ on $K$ is the field structure itself.

We have already mentioned that Zariski geometries can be classified via Zilber’s trichotomy (Lemma 2.9). The following theorem states that in essence, the very ample Zariski geometries are just the smooth curves over algebraically closed fields.

Theorem 2.10 (Theorem A in [2]). Let $\mathcal{D}$ be a very ample Zariski geometry. Then there is a smooth algebraic curve $C$ over an algebraically closed field $K$ such that the Zariski geometries $\mathcal{D}$ and $\mathcal{C} = \langle (C^n, \gamma^n_{Zar}) : n \in \mathbb{N} \rangle$ are isomorphic. That is, there exists a homeomorphism $h : D \rightarrow C$ such that the induced map $h^n : D^n \rightarrow C^n$ defines a homeomorphism for each $n \in \mathbb{N}$.

The curve and the field mentioned in Theorem 2.10 are unique up to a curve isomorphism and a field isomorphism. The precise statement is contained in the following result:

Proposition 2.11 (Proposition 1.1 in [2]). Let $C$ be a smooth curve over an algebraically closed field $F$ and $C'$ a smooth curve over a field $F'$. Suppose $h : \mathcal{C} \rightarrow \mathcal{C}'$ is an isomorphism of Zariski geometries. Then there is a field isomorphism $h_F : F \rightarrow F'$ such that $h$ is an isomorphism of algebraic varieties with respect to the identification of fields $h_F$. (In other words, $h_F$ induces a field isomorphism $h^n_F : F(C) \rightarrow F'(C')$ of the fields of rational functions over $C$ and over $C'$.)

2.2. Manifolds. In general it is not true that Axiom (Z3) holds for arbitrary closed subsets of $D^n$ for the various $n \in \mathbb{N}$. In order to find a large collection of sets where (Z3) remains true it is necessary to define special imaginary sorts, and to identify the regular subsets (were (Z3) is preserved). It can be proved that in Zariski geometries imaginaries can be eliminated down to special sorts and that an analogue of Weil’s group chunk is valid in the context (see [2, Lema 5.11]).

Let $\mathcal{D}$ be a Zariski geometry, $H \subseteq S_n$ and let $H$ act on $D^n$ by permuting coordinates. The quotient $S_H := D^n / H$ is called the special imaginary sort corresponding to $H$.

Definition 2.12. Let $\Delta_C$ be the diagonal of $C \times C$ for $C \subseteq D^n$ closed irreducible with $\dim(C) = k$.

1. A point $\bar{p} \in C$ is regular on $C$ if there exists a closed irreducible set $G \subseteq D^n \times D^n$ with $\dim(D^n \times D^n) - \dim(G) = \dim(C)$ such that $\Delta_C$ is the unique irreducible component of $G \cap (C \times C)$ passing through $(\bar{p}, \bar{p})$.

2. If $C$ is any closed set in $D^n$, $\bar{p}$ is regular if $\bar{p}$ lies in a unique component of $C$, whose dimension is equal to $\dim(C)$ and is regular on that component.

3. Let $S_H$ be a special imaginary sort, $\pi : D^n \rightarrow S_H$ the quotient projection, and let $C$ be a closed subset of $S_H$. A point $\bar{p} \in C$ is regular on $C$ if $\bar{p} = \pi(\bar{p}^{\bar{k}})$ for some regular $\bar{p}^{\bar{k}}$ on $\pi^{-1}(C)$. 

Example 2.13. Let $C$ be a smooth plane curve over an algebraically closed field $K$, and let $\bar{p}$ be a non-singular point of $C$. Then $\bar{p}$ is a regular point of the Zariski geometry $\mathcal{C} = (\mathbb{C}^n, \tau_{\text{Zar}}^n : n \in \mathbb{N})$ corresponding to $C$. (See [4, pp. 113-114] for the details.)

Definition 2.14. Let $\mathcal{D}$ be a Zariski geometry. A $D$-manifold consist of a set $X$ together with a finite number of injective maps $\langle \phi_i : U_i \to X : 1 \leq i \leq n \rangle$ such that

- The set $X$ is covered by the $\phi(U_i)$’s, that is, $X = \phi_1(U_1) \cup \cdots \cup \phi_n(U_n)$.
- Each $U_i$ is an irreducible regular subset of some special sort.
- For each $1 \leq i, j \leq n$, the set $\{ (x, y) \in U_i \times U_j : \phi_i(x) = \phi_j(y) \}$ is a closed irreducible subset of $U_i \times U_j$ and projects onto nonempty open subsets of both $U_i, U_j$.

The most important relation between Zariski geometries and manifolds is given by the following lemma, asserting that any field interpretable in a Zariski geometry has a manifold structure where the field operations are morphisms (i.e., the graph of the map is irreducible closed), and such field is a pure field.

Lemma 2.15 (Lemma 5.12 and Proposition 7.10 in [2]). If a field $K$ is definably interpretable in $\mathcal{D}$, then $K$ may be endowed with a manifold structure such that the field operations are morphisms. Moreover, $V$ is a $K$-manifold, i.e., an algebraic variety over $K$, if and only if it is a $\mathcal{D}$-manifold. □

3. Results

The main lemma of this paper consists of a strengthening of Proposition 2.11 in the case where $F = F'$. Our proof is a straightforward adaptation of the proof of Proposition 1.1 in [2], but we included it for sake of completeness.

3.1. Main Lemma.

Lemma 3.1. Let $C, C'$ be a smooth quasi-projective algebraic curves over an algebraically closed field $F$. Let $h_C : \mathcal{C} \to \mathcal{C}'$ be an injective morphism of the corresponding Zariski geometries. Then there is a field homomorphism $h_F : F \to F$ such that $h_C$ induces a rational morphism $h_F^\# : C \to C'$ of algebraic varieties with respect to $h_F$. In other words, $h_F$ induces a field homomorphism $h : F(C') \to F(C)$.

Proof. Since the Zariski geometries $\mathcal{C}$ and $\mathcal{C}'$ are ample, by Lemma 2.9 there are algebraically closed fields $K$ and $K'$ such that $K$ is interpretable in $\mathcal{C}$ and $K'$ is interpretable in $\mathcal{C}'$. Due to the fact that $C$ and $C'$ are algebraic curves over $F$, there exist $F$-definable isomorphisms $g : F \to K$ and $g' : F \to K'$. By Lemma 2.15, $K$ has a $C$-manifold structure making the field operations morphisms. In particular, $K$ is also an $F$-manifold and $g$ defines an isomorphism of manifolds. Since $h_C$ is injective, there is a partial morphism $h_C^{-1} : \mathcal{C}' \to \mathcal{C}$ that extends to a morphism $(\mathcal{C}', K') \to (\mathcal{C}, K)$, by taking $h_K$ as the map $g \circ g'^{-1} : K' \to K$.

Let $p$ be the characteristic of the fields involved and consider the map given by $\text{Frob}(x) = x^p$. Let $F(C)$ and $F(C')$ be the fields of rational functions over $C$ and $C'$, respectively, considered as functions over the projective line $\mathbb{P}^1_F$, and let $F_p(C)$ and $F_p(C')$ be their corresponding perfect closures. For any given $m \in \mathbb{Z}^+$, we can consider the map $h^{[m]}_F : F \to F$ defined by $h^{[m]}_F := g'^{-1} \circ \text{Frob}^m \circ h^{-1}_K \circ g$. During this proof we will view $h_m = (h^{[m]}_F, h^{-1}_C, h_K)$ as a map from the triple $(F, C', K')$ to the triple $(F, C, K)$.

Claim 1: For every $m \in \mathbb{N}$, the morphism $h_m$ maps $F_p(C')$ into $F_p(C)$.

Proof of Claim 1: The map $h^{[m]}_F$ may be seen as a map from $\mathbb{P}^1_F$ to $\mathbb{P}^1_F$ by sending $\infty \mapsto \infty$. Let us take $f : C' \to \mathbb{P}^1_F$ in $F_p(C')$. Then, the set $\Gamma_f$ (the graph of $f$) is Zariski-closed and irreducible in $C' \times \mathbb{P}^1_F$. We now show that $h_m(f) \in F_p(C)$, and for this we will prove that $h_m(\Gamma_f)$ is irreducible and Zariski-closed in $C \times \mathbb{P}^1_F$ and the defining function $h_m(f) : C \to \mathbb{P}^1_F$ is not constantly infinite.
The map \( g' \) is a morphism. Since the composition of morphisms is a morphism, it follows that the map \( g' \circ f : C' \to K \) is a morphism. Thus \( F_{g' \circ f} \) (the graph of \( g' \circ f \)) is irreducible and Zariski-closed in \( C' \times K \). By assumption, \( h_m \) is a morphism so \( h_m(F_{g' \circ f}) \) is irreducible and Zariski-closed in \( C \times K \). Furthermore, \( h_m(F_{g' \circ f}) \) is the graph of the map \( \text{Frob}^{-m} \circ h_K \circ g' \circ h_m(f) \). Since \( \text{Frob} \) and \( g' \) are morphisms of Zariski geometries, \( \Gamma_{\text{Frob}} \) and \( \Gamma_{g'} \) are closed irreducible sets. Thus \( \Gamma_{h(f)} \) must be also irreducible and closed. Now, if \( h_m(f) \) is constantly \( \infty \), then its graph is a component of \( C \times F \times K \) which is impossible since this product is irreducible. This shows that \( h_m(\Gamma_f) \) is an irreducible Zariski-closed subset of \( C \times P^1_k \) and that \( h(f) \) is not constantly \( \infty \), so \( h_m(f) \in F_p(C) \). Therefore, \( h_m \) maps \( F_p(C') \) to \( F_p(C) \), and we have finished the proof of Claim 1.

Note that if \( p = 0 \), then \( F(C) \) and \( F(C') \) are perfect fields and the morphism \( h_0 \) maps \( F(C') \) to \( F(C) \). From now on, let us assume that \( p > 0 \) and consider the collection \( \mathcal{F} \) of all subfields of \( F_p(C) \) containing \( F(C) \) and finitely generated over \( F \) whose perfect closure is \( F_p(C) \), together with the collection \( \mathcal{F}' \) of all subfields of \( F_p(C') \) containing \( F(C') \) and finitely generated over \( F \) whose perfect closure is \( F_p(C') \). Notice that \( h_m \) maps fields in \( \mathcal{F}' \) to fields in \( \mathcal{F} \), and in particular \( h_m \) maps \( F(C') \) to some field \( L_0 \in \mathcal{F} \).

**Claim 2:** If \( L, L' \in \mathcal{F} \), there is some \( m_0 \in \mathbb{Z} \) such that \( L^{p^{m_0}} = L \), that is, \( \text{Frob}^{m_0}(L) = L' \).

**Proof of Claim 2:** Assume for a moment that \( p \neq 0 \) and notice that \( F_p(C) = \bigcup_{m \in \mathbb{N}} F(C)^{1/p^m} \), that is, \( F_p(C) \) is the smallest field containing all \( p^m \)-th roots of elements in \( F(C) \). Now, by the choice of \( \mathcal{F} \), for any \( L, L' \in \mathcal{F} \) it follows that their respective perfect closures \( L_p \) and \( L'_p \) satisfy

\[
L_p = \bigcup_{m \in \mathbb{N}} L^{1/p^m} = F_p(C) = \bigcup_{m \in \mathbb{N}} (L')^{1/p^m} = L'_p.
\]

Notice that \( F(C) \subseteq L \subseteq F_p(C) \). So, for each one of the finitely many generators \( \xi_i \) of \( L \) over \( F \), there is some \( n_i \in \mathbb{Z} \) such that \( \xi_i \in (L')^{1/p^{n_i}} \). By picking \( m_1 \) to be the product of the integers \( n_i \), we obtain \( L \subseteq (L')^{1/p^{m_1}} \). In other words, \( \text{Frob}^{m_1}(L) \subseteq L' \). A similar procedure shows that for some \( m_2 \in \mathbb{N} \) we have \( \text{Frob}^{m_2}(L') \subseteq L, \) or equivalently, \( L' \subseteq \text{Frob}^{−m_2}(L) \).

Hence, we have \( \text{Frob}^{m_1}(L) \subseteq L' \subseteq \text{Frob}^{−m_2}(L) \), and since \( [\text{Frob}^{−m_2}(L) : \text{Frob}^{m_1}(L)] = p^{m_1+m_2} \), there must exist some \( n \in \mathbb{N} \) such that \( [L' : \text{Frob}^{m_1}(L)] = p^n \). Therefore, by taking \( m = m_1 - n \in \mathbb{Z} \), we have that \( \text{Frob}^m(L) = \text{Frob}^{-n}(\text{Frob}^{m_1}(L)) = L' \). This concludes the proof of Claim 2.

Since \( h_0 \) is a map over \( F \) sending \( F_p(C') \) to \( F_p(C) \), \( L = h_0(F(C')) \) is a field in \( \mathcal{F} \), as well as \( L' = F(C) \). By Claim 2, there is \( m_0 \in \mathbb{Z} \) such that \( \text{Frob}^{m_0}(L) = L' \), and we have two cases:

(i) If \( m_0 \leq 0 \), then \( h_0(F(C')) = \text{Frob}^{-m_0}(F(C)) \subseteq F(C) \), so \( h_0 \) maps \( F(C') \) into \( F(C) \).

(ii) If \( m_0 > 0 \) we have

\[
F(C) = \text{Frob}^{m_0}(h_0(F(C'))) = (\text{Frob}^{m_0} \circ (g')^{-1} \circ h_K^{-1} \circ g)(F(C')) \\
= ((g')^{-1} \circ \text{Frob}^{m_0} \circ h_K^{-1} \circ g)(F(C')) \\
= h^{[m]}_F(F(C')).
\]

In any case, there is \( m \in \mathbb{N} \) such that \( h^{[m]}_F(F(C')) \subseteq F(C) \), and by Claim 1 we have obtained a morphism \( h = h_m \) that maps \( F(C') \) to \( F(C) \), satisfying the conclusion of the lemma.

### 3.2. The genus of a very ample Zariski geometry.

In Algebraic Geometry, the notion of *genus* is presented as a numerical invariant for one-dimensional algebraic varieties over an algebraically closed field \( K \). In this subsection we will show that using Theorem 2.10 a similar notion could be derived for very ample Zariski geometries.
Definition 3.2. We say that a very ample Zariski geometry $\mathcal{D}$ has genus $g$, denoted $\text{gen}(\mathcal{D}) = g$, if $\mathcal{D}$ is isomorphic to the Zariski geometry of a smooth curve $C$ of genus $g$. (We will also write gen($C$) for the genus of the curve $C$.)

According to Theorem 2.10 the genus of a Zariski geometry is always defined and it follows from Proposition 2.11 that it is invariant under isomorphisms of Zariski geometries.

Lemma 3.3 (Riemann–Hurwitz Theorem). Let $C_1$ and $C_2$ be smooth quasi-projective algebraic curves over an algebraically closed field $K$. If there exists a non-constant morphism of curves $f : C_1 \to C_2$, then $g_2 \leq g_1$, where $g_i$ is the genus of $C_i$, for $i = 1, 2$.

For a further discussion about Lemma 3.3 we recommend [1] from pp. 299 on. Together Lemma 3.1 and Lemma 3.3 show the following:

Proposition 3.4. Let $C_1$ and $C_2$ be smooth quasi-projective algebraic curves over an algebraically closed field $F$. Suppose that there exists an injective morphism $f : \mathcal{C}_1 \to \mathcal{C}_2$ of very ample Zariski geometries. Then there exists a non-constant morphism of curves $f_\ast : C_1 \to C_2$. In particular, $\text{gen}(C_2) \leq \text{gen}(C_1)$.

Proposition 3.4 is relating the two structures considered over the curves $C_1, C_2$, namely, the structure of curve and that of Zariski geometry.

Theorem 3.5 (Riemann–Hurwitz for very ample Zariski geometries). Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be very ample Zariski geometries and assume that there is an algebraically closed field $K$ interpretable in both geometries. Also assume that there exists an injective morphism $f : \mathcal{D}_1 \to \mathcal{D}_2$ of Zariski geometries, then $\text{gen}(\mathcal{D}_2) \leq \text{gen}(\mathcal{D}_1)$.

Proof. By Theorem 2.10 we can find smooth curves over $K$, say $C_1$ and $C_2$, such that considered as Zariski geometries $\mathcal{C}_1$ is isomorphic to $\mathcal{D}_1$ and $\mathcal{C}_2$ is isomorphic to $\mathcal{D}_2$. Let $i : \mathcal{D}_1 \to \mathcal{C}_1$ and $j : \mathcal{D}_2 \to \mathcal{C}_2$ be isomorphisms of Zariski geometries. Then we have an injective morphism of Zariski geometries $j \circ f \circ i^{-1} : \mathcal{C}_1 \to \mathcal{C}_2$. By Lemma 3.1 we may find a non-constant morphism of curves $\iota : C_1 \to C_2$. By Lemma 3.3 we have that $\text{gen}(C_2) \leq \text{gen}(C_1)$ so $\text{gen}(\mathcal{D}_2) \leq \text{gen}(\mathcal{D}_1)$. □

Corollary 3.6. Let $\mathcal{D}$ be a very ample Zariski geometry and let $K$ be an algebraically closed field interpretable in $\mathcal{D}$. Suppose that there exists an injective morphism of Zariski geometries $f : \mathbb{P}_K^1 \to \mathcal{D}$. Then $\text{gen}(\mathcal{D}) = 0$.

Proof. According to Theorem 3.5, $\text{gen}(\mathcal{D}) \leq \text{gen}(\mathbb{P}_K^1) = 0$. □

We now see one more application of Proposition 2.11.

Theorem 3.7 (Hurwitz Theorem). Let $\mathcal{D}$ be a very ample Zariski geometry interpreting an algebraically closed field $K$ of characteristic 0. Assume that $g = \text{gen}(\mathcal{D}) \geq 2$. Then there exists at most $84(g - 1)$ automorphisms of $\mathcal{D}$.

Proof. There exists a smooth quasi-projective algebraic curve over $K$ which considered as Zariski geometry is isomorphic to $\mathcal{D}$ and $\text{gen}(\mathcal{D}) := \text{gen}(C)$. By Proposition 2.11 any automorphism of $\mathcal{D}$ induces an automorphism of $C$ (as algebraic curve), but by Hurwitz Theorem for algebraic curves (see [1, pp. 305]) $C$ has at most $84(g - 1)$ automorphisms, so $\mathcal{D}$ has at most the same number of automorphisms. □

3.3. Non-definability of genus. We wonder if there exists a first order sentence $\Phi^g$ such that a very ample Zariski geometry $\mathcal{D}$ satisfies $\Phi^g$ if and only if $\mathcal{D}$ has genus $g$. The following theorem answers the question in the negative for several cases.

Theorem 3.8 (Non-definability of genus). Let $\mathcal{D}$ be a very ample Zariski geometry of genus $g \geq 2$ interpreting an algebraically closed field of characteristic 0. Thus if $\mathcal{L}$ is one of the languages $\mathcal{L}_{\text{nat}}(\mathcal{D})$ or $\mathcal{L}_{\text{all}}(\mathcal{D})$, there is no $\mathcal{L}$-sentence $\Phi^g$ ensuring that $\mathcal{D}$ has genus $g$, thus the genus of $\mathcal{D}$ is not definable.

We need the following lemma (see [2, §6] for a proof):
Lemma 3.9. Let $\mathcal{D}$ be a very ample Zariski geometry. Then for some elementary extension $\mathcal{D}^*$ of $\mathcal{D}$, we may find an array $f = \langle f_{ij} : i, j < \omega \rangle$ such that $f$ is indiscernible. That is, for any $n, m < \omega$ the first order type of any $(n, m)$-rectangle from $f$ is constant.

Proof of Theorem 3.8 (The proof running is valid in both, the natural and the full languages of $\mathcal{D}$.) Let $\mathcal{D}$ be a very ample Zariski geometry interpreting an algebraically closed field of characteristic zero and with genus $g \geq 2$. Assume that there is a first order formula $\Phi^g$ asserting that $\mathcal{D}$ has genus $g$. By Hurwitz Theorem 3.7 $\mathcal{D}$ has at most a finite number of automorphisms and this number is bounded above by $84(g - 1)$. According to the previous lemma, if we extend to some sufficiently large elementary extension of $\mathcal{D}$, say $\mathcal{D}'$, there we can find an indiscernible array of elements $f = \langle f_{ij} : i, j \in \mathbb{N} \rangle$. By definition, for any $(i, j)$ fixed we have $\text{tp}(f_{ij}) = \text{tp}(f_{nm})$ for any $n, m \in \mathbb{N}$. If we assume $\mathcal{D}'$ to be a universal domain of $\mathcal{D}$, it follows from homogeneity that may find an automorphism of $\mathcal{D}'$, say $\sigma_{ij}$ sending $f_{ij}$ to $f_{nm}$. This argument shows that there exists at least $\aleph_0$ distinct automorphisms of $\mathcal{D}$. However, since $\mathcal{D} \preceq \mathcal{D}'$, $\mathcal{D}' \models \Phi^g$ so $\mathcal{D}'$ also has genus $g$. Thus the number of automorphisms of $\mathcal{D}'$ is bounded by above $84(g - 1)$, a contradiction. \hfill $\Box$

In the proof of Theorem 3.8 it is implicit that for a Zariski geometry of genus $g \geq 2$ interpreting an algebraically closed field of characteristic zero, there exists an elementary extension of $\mathcal{D}$, say $\mathcal{D}'$ such that $\text{gen}(\mathcal{D}) \neq \text{gen}(\mathcal{D}')$. Therefore, the genus of $\mathcal{D}$ cannot be definable even by a set of $\mathcal{L}$-sentences, where $\mathcal{L}$ is either $\mathcal{L}_{\text{nat}}(\mathcal{D})$ or $\mathcal{L}_{\text{full}}(\mathcal{D})$.

4. Final remarks

We are not aware of a definition of genus that is intrinsic to the definition of Zariski geometries, without appealing to Theorem 2.10. This question remains open so far. Nevertheless, we have shown that for any given definition of the genus of a very large Zariski geometry, which agrees with the notion of genus of smooth curves, the Riemann–Hurwitz and Hurwitz theorems must be valid, and Theorem 3.8 must hold.

So, even though the genus can be presented as a natural geometric invariant for Zariski geometries, we have shown that it is not definable by any collection of sentences in the language $\mathcal{L}_{\text{full}}(\mathcal{C})$ or $\mathcal{L}_{\text{nat}}(\mathcal{C})$. This kind of phenomenon occurs even for more elementary theories such as the theory of $\mathbb{Q}$-vector spaces in the language $\mathcal{L} = \{0, +, (\lambda_q : q \in \mathbb{Q})\}$ where the dimension is not first-order definable.

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