Developing analytical solutions for transverse, matrix strain magnification and fibre strain reduction in uniaxially aligned continuous fibre reinforced composites, based on the principle of conservation of strain energy and the Reuss rule

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Abstract. Analytical solutions for transverse matrix strain magnification and transverse fibre strain reduction in uniaxially aligned continuous fibre reinforced composites developed here for a Representative Volume Element (RVE), based on the Reuss rule and the principle of constant strain energy. The results obtained show respective matrix and fibre transverse strain magnification and reduction that increase with increasing volume fraction of the reinforcing fibre for both square and hexagonal fibre packing geometries. The square arrays register higher values of matrix and fibre transverse strain magnification and reduction, respectively, than hexagonal arrays. The values of transverse stress ratio between the fibre and matrix in the RVE central sub-region, and between the two and the RVE central sub-region composite are seen to vary with the volume fraction of reinforcing fibre. The results obtained here denote a dependency of respective matrix and fibre transverse strain magnification and reduction as well a stress ratio on the reinforcing fibre packing geometry and volume fraction.

1. Background

Existing literature on the phenomenon of matrix transverse strain magnification is very scanty and is mainly based on the simple rule developed by Kies J. L in 1962 [1-3]. Moreover, and to the best knowledge of the author, very little work has been done on the effect of transverse matrix strain magnification on the transverse strain reduction in the reinforcing fibre. This creates a need to undertake rigorous studies of these two phenomena.

The objective of this work is to use the strain energy method based on an RVE to develop analytical solutions for use in studying the variation of matrix transverse strain magnification and transverse fibre strain reduction with volume fraction of reinforcing fibre, in uniaxially aligned continuous fibre reinforced composites. The developed solutions are also expected to help in studying the dependence of the two phenomena on the reinforcing fibre packing geometry, while providing insight into the applicability of the Reuss rule in studying the phenomena. The term, matrix strain magnification, as used in this paper refers to the ratio between the transverse strain in the matrix adjacent to the reinforcing fibres and the transverse strain in the bulk matrix. The term, fibre strain reduction, as used in this paper refers to the ratio between the strains in the fibre, taking into account
the effect of matrix strain magnification, to the strain in the bulk fibre in the absence of this consideration.

The properties of composite materials, both natural and manufactured, are dependent on the relative proportions and properties of their constituents, as well as the quality of the interface between constituents [1-4]. The various micromechanical rules that have used to try to predict the mechanical properties of composite materials can group into the four categories of phenomenological, elasticity, semi-empirical and homogenisation [5]. The Rule of Mixtures (ROM) also referred to as the iso-strain or Voigt rule and the Rule of Hybrid Mixtures for (RoHM) are known to give good predictions for the longitudinal composite mechanical properties of strength, stiffness and of the Poisson’s ratios (ν12 & ν23) for two phase materials and hybrid composites, respectively [5]. The Inverse Rule of Mixtures (IRoM) also referred to as iso-stress or Reuss rule for both transverse and shear properties however challenged in accuracy, and is only good as a first estimate [2-5]. Recourse is therefore sought in the Modified Rule of Mixtures (MRoM) corrected to increase the accuracy of predicting the values of transverse and shear moduli [5], in the semi-empirical rules of Halpin –Tsai and separately Chamis C. C. for dual phase materials [1-5] and also in the modified Halpin-Tsai semi-empirical rules for hybrid composite systems [6]. The properties of composite materials can also be determined through experimental testing, using for instance the very useful off-axis test that generates values of longitudinal, transverse, and shear stiffness and strength, as well as Poisson’s ratios of composites [3, 7].

The results arising from the aforementioned theoretical and semi-empirical models, as well as experimental testing represent composites well at a macro scale but are unable to account for micro-variation of properties that may exist, which is particularly important for the interfacial bond as well as transverse stresses and strains.

The Kies model showed transverse strain magnification in the matrix in between the fibres to be a function of the reinforcing fibre volume fraction, matrix/fibre stiffness ratio, fibre diameter and geometric arrangement of the reinforcing fibres [2, 4].

The loss of stiffness, decrease in fatigue strength and initiation of weepage degradation in composite pipes arising from transverse failure of composites, because of low transverse properties of fibre-reinforced composites noted to severely limit use of composite materials [8]. A transition of glass fibre/epoxy composite failure mode from adhesive rupture at the fibre/matrix interface to cohesive rupture in the matrix, at a global strain of 1.15%, has also observed in the same work. This low failure strain is likely to be a result of matrix transverse strain magnification.

The transverse tensile strength of unidirectional composites observes to be much lower than the tensile strength of the pure matrix material because of transverse strain magnification related stress concentration arising from the introduction of reinforcing fibres in the matrix [9]. Additionally, it was further observed that failure of composites under compression did not occur along a line orthogonal to the direction of application of load, as was the case in tension, but along a line inclined to the direction of application load.

It was observed that despite the Kies transverse strain magnification model predicting lower failure strains for higher reinforcing fibre volume fraction, failure in polyester glass composites with a fibre volume fraction between 37 - 43% occurred at strains of 0.13 to 0.6% [10]. It was further observed in the same work that failure of epoxy glass fibre composites with higher values of fibre volume fraction between 55-60%, occurred at considerably much higher strains of 0.55 to 0.60. It is noted here that the elastic moduli for the two matrices ranges between 2 – 4.5 and 3 – 6 GPa, respectively. This from Kies transverse strain magnification equation would tend to lower the transverse strain magnification effect for the polyester resin, which has a lower stiffness range thus raising the strain at failure. Clearly, there must have been other factors contributing to failure in this reported study other than that of transverse strain magnification. Whether this phenomenon is peculiar to the reported work or is universally applicable to composite materials is not clear.

The importance of the matrix transverse strain magnification phenomenon is evident from the foregoing review and advises the attempts made here to develop analytical expressions that can be
used to determine respective values of matrix and fibre transverse strain/stress magnification and reduction in continuous fibre reinforced composites. The analytical models developed here are also intended for use in studying their variation of the phenomenon with the volume fraction of reinforcing fibre. Due to the noted shortcomings of the Resuss rule and the lack of other alternative theoretical models for determining the transverse elastic properties of composite materials, recourse is sought in the RVE used by of Gibson [3] in order to develop expressions for the transverse elastic modulus of longitudinally aligned continuous fibre reinforced composites.

The solutions developed here assume isotropy and homogeneity of the composite and its constituents. It is understood however, that variations from this idealisation do exist such as the known anisotropy of carbon and aramid fibres [2, 3, 5] as well as polyethylene fibres [5]. This is clear from the values shown in Table 1.

| Fibres        | $E_{11}$ (GPa) | $E_{22}$ (GPa) | $G_{12}$ (GPa) | $\nu_{12}$ | $\nu_{23}$ |
|---------------|----------------|----------------|----------------|------------|------------|
| E-Glass       | 73.1           | 73.1           | 29.95          | 0.22       | 0.22       |
| Carbon        | 232            | 15             | 24             | 0.279      | 0.49       |
| Polyethylene  | 60.4           | 4.68           | 1.65           | 0.38       | 0.55       |
| Matrix        | $E_m$ (GPa)    | $G$ (GPa)      | $\nu_m$        |            |            |
| Epoxy         | 3.45           | 3.45           | 1.28           | 0.35       | 0.35       |
| Epoxy         | 5.35           | 5.35           | 1.97           | 0.354      | 0.354      |
| Epoxy         | 5.5            | 5.5            | 1.28           | 0.37       | 0.37       |

It is evident from Table 1 [5] that elastic modulus and Poisson’s ratio of some types of epoxy are higher and lower than the transverse elastic modulus of polyethylene fibres, which raises interesting possibilities with respect to the subject of this study. In the foregoing table, the subscripts “11” and “22”, refer to the longitudinal and orthogonal directions of the reinforcing fibres, respectively, while the subscripts “12” and “23” refer to the 12 and 13 planes, respectively. The symbols $E$, $G$ and $\nu$ in the table refer to the elastic modulus, shear modulus and Poisson’s ratio, respectively.

The theory of RVEs and their definition are covered well in literature and are discussed here in brief. RVEs are selected based on use, statistical representativeness of the whole (thus the same elastic constants, stresses and strains, and fibre volume fraction for the case of composites) and periodic repetitiveness. They must also be sufficiently larger than a typical structural macroscopic dimension, while being sufficiently smaller than the microscopic size. This makes them useful in simplifying analysis by focusing on a small item with known geometric and material properties including grains, flaws, inclusions, voids and fibres [1, 11-15]. Based on the statistical nature of the microstructure, statistical properties of the RVE, and based on homogenisation of material properties without reference to their statistical fluctuations over finite domains, the definitions of RVEs can classify into three broad areas [14]. An RVE can define as a volume of heterogeneous material that is large enough to include an effective sampling of all microstructural heterogeneities in the material and is therefore statistically representative of the whole, and which is small enough to be considered as an element of a continuum mechanics model [13]. An RVE can also been defined as, “the smallest material volume element of the composite for which the usual spatially constant (overall modulus) macroscopic constitutive representation is a sufficiently accurate model to represent mean constitutive response” [16]. A good overarching definition of an RVE is that of an element that can be used to determine the properties of a homogenised macroscopic model, with size between the macroscopic body and the microstructural size, and which has enough information about the microstructure and is a good representation of the continuum of a material [11].

Based on the principle of conservation of strain energy, it was demonstrated using E-glass fibre/epoxy resin that the assumption of constant transverse stress through the matrix and reinforcing
fibre depth, which is used in arriving at the Reuss rule was wrong and that the stresses in the two constituent components of a matrix were different [3]. He then adapted the Hopkins and Chamis model of the RVE applied to a square array of reinforcing fibres to develop an alternative expression for the transverse elastic modulus of a fibre reinforced composite. However, it is noted that the iso-stress or Reuss rule was still used to determine the value of transverse stiffness in sub-region (B) of the RVE. This raises concerns of the same constraints of the Reuss rule pointed out before, though at the much smaller scale of a single fibre/surrounding matrix cell.

2. Developing analytical solutions
In this section, the principle of constant strain energy is applied first, followed by the iso-stress rule, Hooke’s law and finally the assumptions of isotropy and homogeneity.

2.1. Applying the principle of constant strain energy to an RVE
Consider the RVE shown in the left end diagram of Figure 1, and its derivatives going to the right of the figure. The thick arrow in Figure 1 shows the direction of transverse deformation. The RVE is considered here to be the smallest volume element whose properties are representative of the macroscopic properties of the material. Figure 1b shows the reinforcing fibre of circular cross-section in Figure 1a, represented by a reinforcing fibre of square cross-section with the same effective cross-sectional area. Figure 1c consists of two sub-regions; the central one with the reinforcing fibre that has a square cross-sectional, and the two regions on either side of it.

\[ \sqrt{(1)} \]

The reinforcing fibre volume fractions \( v_f^{sq} \) and \( v_f^{hex} \) of square and hexagonal arrays of reinforcing fibre, respectively, with fibres of radius (r) and fibre center to centre distance (R), are given by the expressions [2,3].

\[ v_f^{sq} = \frac{\pi}{4} \left( \frac{r}{R} \right)^2 \quad \text{(2a)} \]

\[ v_f^{hex} = \frac{\pi}{2\sqrt{3}} \left( \frac{r}{R} \right)^2 \quad \text{(2b)} \]
\( \nu_f' \) of the reinforcing fibre in sub-region (B) of the right end diagram is the volume fraction (given by the expression:

\[
\nu_f' = \frac{s_f \times s_f'}{s_f \times 2R} = \frac{s_f}{2R}
\]  \hspace{1cm} (3)

Substituting Equation 1 into Equation 3 changes it to:

\[
\nu_f' = \sqrt[2]{\left(\frac{r}{R}\right)} = \sqrt[2]{\nu_f'}
\]  \hspace{1cm} (4)

Substituting Equation 4 into each one of the two expressions in Equation 2 generates the following expressions for the reinforcing fibre volume fraction in sub-region (B) for the two fibre geometries:

\[
\nu_f'|_{sq} = \sqrt{\nu_f'}_{sq}
\]  \hspace{1cm} (5a)

\[
\nu_f'|_{hex} = \sqrt[3]{\left(0.5\nu_f'ight)}_{hex}
\]  \hspace{1cm} (5b)

For the central sub-region (B) in Figure 1c, geometric compatibility of deformation in the transverse direction implies that:

\[
\epsilon_{t_2}' = \epsilon_f' \nu_f' + \epsilon_m' \nu_m'
\]  \hspace{1cm} (6)

Introducing parameters \( a \) and \( b \) to relate the strains in the central sub-region composite, to the strains in the fibre and matrix produces the expression:

\[
\epsilon_f' = a \epsilon_{c_2}'
\]  \hspace{1cm} (7a)

\[
\epsilon_m' = b \epsilon_{c_2}'
\]  \hspace{1cm} (7b)

Substituting Equations 7a and 7b into Equation 6 gives rise to:

\[
\epsilon_{c_2}' = a \epsilon_{c_2}' \nu_f' + b \epsilon_{c_2}' \nu_m'
\]  \hspace{1cm} (8)

Equation 8 can be rewritten as:

\[
1 = a \nu_f' + b \nu_m'
\]  \hspace{1cm} (9)

It is evident from Equation 9 that:

\[
a = \frac{1-b \nu_m'}{\nu_f'}
\]  \hspace{1cm} (10a)

\[
b = \frac{1-a \nu_f'}{\nu_m'}
\]  \hspace{1cm} (10b)

Assuming that the strain energy (U) in the composite is equal to the sum of the strain energies in the reinforcing fibre and matrix constituents of the composite [3], thus:

\[
U_{c_2}' = U_f' + U_m'
\]  \hspace{1cm} (11)

Equation 11 can be rewritten in terms of stresses and strains as:

\[
0.5 \sigma_{c_2} \epsilon_{c_2}' V_{c_2}' = 0.5 \sigma_f' a^2 \epsilon_f' V_f' + 0.5 \sigma_m' a^2 \epsilon_m' V_m'
\]  \hspace{1cm} (12)

Dividing out the entire expression by 0.5 and substituting in the known relationship between stress and strain for linear elastic deformation transforms the foregoing equation to:

\[
E_{c_2} \epsilon_{c_2}' V_{c_2}' = E_f' \epsilon_f' V_f' + E_m' \epsilon_m' V_m'
\]  \hspace{1cm} (13)
Substituting Equations 7a and 7b into Equation 13 gives rise to:
\[ E'_{c2} e'_{c2} V'_{c2} = E'_f a^2 e'_{c2} V'_{f} + E'_m b^2 e'_{c2} V'_{m} \]  
(14)

Dividing out Equation 14 with the term \( e'_{c2} V'_{c2} \) leads to:
\[ E'_{c2} = E'_f a^2 V'_{f} + E'_m b^2 V'_{m} \]  
(15)

Substituting Equations 10a and 10b into Equation 15 converts it to:
\[ E'_{c2} = E'_f a^2 V'_{f} + E'_m \left( \frac{1-aV'_{f}}{V'_{m}} \right) V'_{m} \]  
(16)

Equation 16 can be rewritten in the form of a quadratic equation in terms of the parameter \( a \) as follows:
\[ a^2 \left( \frac{E'_f V'_{f}}{V'_{m}} \right) + \frac{E'_m V'_{f}}{V'_{m}} - \frac{2aE'_m V'_{f}}{V'_{m}} + \left( \frac{E'_f}{V'_{m}} - E'_{c2} \right) = 0 \]  
(17)

The solutions for parameter \( a \) in Equation 17 are obtained from the general solution for a quadratic equation thus:
\[ a = \frac{2E'_m V'_{f} \sqrt{\left( \frac{2E'_m V'_{f}}{V'_{m}} \right)^2 - 4 \left( \frac{E'_f}{V'_{m}} - E'_{c2} \right)}}{2 \left( \frac{E'_f}{V'_{m}} + \frac{E'_m V'_{f}}{V'_{m}} \right)} \]  
(18)

Following a similar procedure, the solutions for parameter \( b \) are determined by the equation:
\[ b = \frac{2E'_m V'_{f} \sqrt{\left( \frac{2E'_m V'_{f}}{V'_{m}} \right)^2 - 4 \left( \frac{E'_m V'_{f}}{V'_{m}} + \frac{E'_m V'_{f}}{V'_{m}} \right)}}{2 \left( \frac{E'_m V'_{f}}{V'_{m}} + \frac{E'_m V'_{f}}{V'_{m}} \right)} \]  
(19)

An examination of Equations 18 and 19 shows them to contain the parameter \( E'_{c2} \) with unknown values. Therefore, this parameter is rewritten in terms of the known parameters by introducing the Hookean stress/strain relationship into Equation 6 for geometric compatibility, thus:
\[ \frac{\sigma'_{c2}}{E'_{c2}} = \frac{\sigma'_{f}}{E'_f} v'_{f} + \frac{\sigma'_{m}}{E'_m} v'_{m} \]  
(20)

Equation 20 may be rewritten as:
\[ \frac{1}{E'_{c2}} = \left( \frac{\sigma'_{f}}{\sigma'_{c2}} \right) v'_{f} + \left( \frac{\sigma'_{m}}{\sigma'_{c2}} \right) v'_{m} \]  
(21)

Making parameter \( E'_{c2} \) the subject converts Equation 21 into:
\[ E'_{c2} = \frac{E'_m E'_f}{E'_m \left( \frac{\sigma'_{f}}{\sigma'_{c2}} \right) v'_{f} + E'_f \left( \frac{\sigma'_{m}}{\sigma'_{c2}} \right) v'_{m}} \]  
(22)

Equation 22 can be rewritten as:
\[ E'_{c2} = \frac{E'_m}{\left( 1-v'_{f} \right) \left( \frac{\sigma'_{m}}{E'_{c2}} \right) v'_{f} + v'_{m} \left( \frac{\sigma'_{m}}{E'_{c2}} \right) v'_{m}} \]  
(23)

Equation 23 can be rewritten as:
Substituting Equations 5a and 5b into Equation 24 gives rise to the following two equations:

\[ E'_{c2} = \frac{E'_m}{(\frac{\sigma'_m}{E'_m} - \sqrt{1 - \frac{\sigma'_f}{E'_f}} + \frac{\sigma'_f}{E'_f} \frac{\sigma'_m}{E'_m} \frac{\sigma'_f}{E'_f})} \]  
(25a)

\[ E'_{c2|\text{hex}} = \frac{E'_m}{(\frac{\sigma'_m}{E'_m} - \sqrt{1 - \frac{\sigma'_f}{E'_f}} + \frac{\sigma'_f}{E'_f} \frac{\sigma'_m}{E'_m} \frac{\sigma'_f}{E'_f})} \]  
(25b)

Equations 25a and 25b can be rewritten as:

\[ E'_{c2|\text{sq}} = \frac{E'_m}{1 - \sqrt{1 - \frac{\sigma'_f}{E'_f}} - \sqrt{1 - \frac{\sigma'_f}{E'_f}}} \]  
(26a)

\[ E'_{c2|\text{hex}} = \frac{E'_m}{1 - \sqrt{1 - \frac{\sigma'_f}{E'_f}} - \sqrt{1 - \frac{\sigma'_f}{E'_f}}} \]  
(26b)

2.1.1. Introducing the Reuss rule. Introducing the iso-stress rule for the central sub-region, which is represented by the relationship, \( \sigma'_c = \sigma'_f = \sigma'_m \), into the two foregoing equations converts them to:

\[ E'_{c2|\text{sq}} = \frac{E'_m}{1 - \sqrt{1 - \frac{\sigma'_f}{E'_f}} \frac{\sigma'_f}{E'_f}} \]  
(27a)

\[ E'_{c2|\text{hex}} = \frac{E'_m}{1 - \sqrt{1 - \frac{\sigma'_f}{E'_f}} \frac{\sigma'_f}{E'_f}} \]  
(27b)

Equation 27a is familiar as the equation that developed by Gibson [2] for the central sub-region of the right hand diagram in Figure 1, adapted for an isotropic reinforcing fibre. It is a special case of the Chamis equation for predicting the transverse stiffness of a composite [5] for isotropic reinforcing fibres.

Substituting the Equation 27a and 27b into Equations 18 and 19 converts them to a form solely dependent on known values of the parameters of the composite constituent components.

Substituting the known relationship for the volume fractions of the composite constituents, \( v'_m = 1 - v'_f \), into the expressions arising from this substitution produces the following expressions:

\[ a_{sq} = \frac{2v'_m + 2v'_f \sqrt{1 - v'_f}}{1 - v'_f} \sqrt{1 - v'_f} - 4 \left( \frac{2v'_m \sqrt{1 - v'_f}}{1 - v'_f} \right)^2 \left( \frac{E'_m}{1 - v'_f} \frac{v'_f}{E'_f} - \frac{E'_m \sqrt{1 - v'_f}}{1 - v'_f} \frac{v'_f}{E'_f} \right) \]

\[ \frac{E'_m \sqrt{1 - v'_f} - \frac{E'_m}{1 - v'_f} \frac{v'_f}{E'_f} - \frac{E'_m}{1 - v'_f} \sqrt{1 - v'_f} \frac{v'_f}{E'_f}}{2 \left( \frac{E'_m \sqrt{1 - v'_f}}{1 - v'_f} + \frac{E'_m \sqrt{1 - v'_f}}{1 - v'_f} \right)} \]  
(28)
\[ b_{sq} = \frac{2E_f\left(1-\nu_f\right)\left(2E_f\left(1-\nu_f\right)ight)^2}{E_f\left(1-\nu_f\right)^2 + \frac{E_m\left(1-\nu_f\right)^2}{\nu_f}} - 4\left(\frac{E_m\left(1-\nu_f\right)}{\nu_f}\left(1-\nu_f\right)\right)^2 \left(\frac{E_f\left(1-\nu_f\right)}{E_f} - \frac{E_m\left(1-\nu_f\right)}{E_f}\right) \]  

(29)

\[ a_{hex} = \frac{\left(2\nu_r\nu_s\nu_t\right)^2}{\left(\nu_r\nu_s\nu_t\right)^2} - 4\left(\frac{E_m\left(1-\nu_f\right)}{\nu_f}\left(1-\nu_f\right)\right)^2 \left(\frac{E_f\left(1-\nu_f\right)}{E_f} - \frac{E_m\left(1-\nu_f\right)}{E_f}\right) \]  

(30)

\[ b_{hex} = \frac{\left(2\nu_r\nu_s\nu_t\right)^2}{\left(\nu_r\nu_s\nu_t\right)^2} - 4\left(\frac{E_m\left(1-\nu_f\right)}{\nu_f}\left(1-\nu_f\right)\right)^2 \left(\frac{E_f\left(1-\nu_f\right)}{E_f} - \frac{E_m\left(1-\nu_f\right)}{E_f}\right) \]  

(31)

2.1.2. Introducing isotropy and homogeneity of the composite constituents. Assuming isotropy and homogeneity of the composites constituents converts Equations 28 and 31 to:

\[ a_{sq} = \frac{2E_m\left(1-\nu_f\right)\left(2E_m\left(1-\nu_f\right)\right)^2}{E_f\left(1-\nu_f\right)^2 + \frac{E_m\left(1-\nu_f\right)^2}{\nu_f}} - 4\left(\frac{E_f\left(1-\nu_f\right)}{\nu_f}\left(1-\nu_f\right)\right)^2 \left(\frac{E_m\left(1-\nu_f\right)}{E_f} - \frac{E_m\left(1-\nu_f\right)}{E_f}\right) \]  

(32)

\[ b_{sq} = \frac{2E_f\left(1-\nu_f\right)\left(2E_f\left(1-\nu_f\right)\right)^2}{E_f\left(1-\nu_f\right)^2 + \frac{E_m\left(1-\nu_f\right)^2}{\nu_f}} - 4\left(\frac{E_m\left(1-\nu_f\right)}{\nu_f}\left(1-\nu_f\right)\right)^2 \left(\frac{E_f\left(1-\nu_f\right)}{E_f} - \frac{E_m\left(1-\nu_f\right)}{E_f}\right) \]  

(33)
Equations 32–35 can be simplified in this order to:

\[
\begin{align*}
\alpha_{hex} &= \frac{2Em}{1 - \sqrt{\beta_f}} \pm \frac{\left(\sqrt{\frac{2Em}{1 - \sqrt{\beta_f}}}\right)^2 - 4 \left(\frac{E_f \sqrt{\beta_f}}{1 - \sqrt{\beta_f}} + \frac{Em \sqrt{\beta_f}}{1 - \sqrt{\beta_f}}\right)^2}{2 \left(\frac{E_f}{1 - \sqrt{\beta_f}} + \frac{Em \sqrt{\beta_f}}{1 - \sqrt{\beta_f}}\right)} \\
\beta_{hex} &= \frac{2E_f}{\sqrt{\beta_f}} \pm \frac{\left(\sqrt{\frac{2E_f}{\sqrt{\beta_f}}}\right)^2 - 4 \left(\frac{E_m}{\sqrt{\beta_f}} + \frac{E_m \sqrt{\beta_f}}{\sqrt{\beta_f}}\right)^2}{2 \left(\frac{E_m}{\sqrt{\beta_f}} + \frac{E_m \sqrt{\beta_f}}{\sqrt{\beta_f}}\right)} \\
\alpha_{sq} &= \frac{2Em \sqrt{\beta_f}}{1 - \sqrt{\beta_f}} \pm \frac{\left(\sqrt{\frac{2Em \sqrt{\beta_f}}{1 - \sqrt{\beta_f}}}\right)^2 - 4 \left(\frac{E_f \sqrt{\beta_f}}{1 - \sqrt{\beta_f}} + \frac{Em \sqrt{\beta_f}}{1 - \sqrt{\beta_f}}\right)^2}{2 \left(\frac{E_f}{1 - \sqrt{\beta_f}} + \frac{Em \sqrt{\beta_f}}{1 - \sqrt{\beta_f}}\right)} \\
\beta_{sq} &= \frac{2E_f}{\sqrt{\beta_f}} \pm \frac{\left(\sqrt{\frac{2E_f}{\sqrt{\beta_f}}}\right)^2 - 4 \left(\frac{E_m}{\sqrt{\beta_f}} + \frac{E_m \sqrt{\beta_f}}{\sqrt{\beta_f}}\right)^2}{2 \left(\frac{E_m}{\sqrt{\beta_f}} + \frac{E_m \sqrt{\beta_f}}{\sqrt{\beta_f}}\right)}
\end{align*}
\]
Equations 36 and 37, as well as Equations 38 and 39 facilitate the variation of parameters \( a \) and \( b \) with volume fraction of the reinforcing fibre to be investigated for square and hexagonal arrays, respectively. Examination of Equations 7a and 7b show the parameters to represent the strain reduction and magnification of the central sub-region reinforcing fibre and matrix, respectively, with reference to the central sub-region composite. Looking at the overall deformation of the RVE consisting of the central sub-region as one material and the two side matrix regions as another, reduces the problem to one of the deformation of a matrix/central sub-region composite in the RVE’s longitudinal direction, which is the overall composite transverse direction. Since the iso-strain or Voigt rule applies in the longitudinal direction of a fibre reinforced composite, the strain in the central sub-region is equal to the strain in the RVE in its longitudinal direction, which is the transverse direction of the composite. The absolute strain reduction and magnification of the reinforcing fibre (\( \varepsilon_f / \varepsilon_m \)) and matrix (\( \varepsilon_m / \varepsilon_m \)), respectively, are therefore equal to the strain ratios (\( \varepsilon_f / \varepsilon_c \)) and (\( \varepsilon_m / \varepsilon_c \)), represented by the parameters \( a \) and \( b \), respectively. In the foregoing strain ratios the strain \( \varepsilon_m \) stands for the strain in the matrix that does not experience transverse strain magnification. These matrix regions are adjacent to the central sub-region in Figure 1.

Introducing the relationship between stress and strain for linear elastic deformation into Equations 7a and 7b converts them to:

\[
\varepsilon_f = a \varepsilon_c' \quad \Rightarrow \quad \frac{\sigma_f}{E_f} = a \left( \frac{\sigma_c}{E_c} \right)
\]

\[
\varepsilon_m' = b \varepsilon_c' \quad \Rightarrow \quad \frac{\sigma_m'}{E_m} = b \left( \frac{\sigma_c}{E_c} \right)
\]

Equations 40a and 40b can be rewritten as:

\[
\frac{\sigma_f}{\sigma_c} = a \frac{E_f}{E_c}
\]

\[
\frac{\sigma_m}{\sigma_c} = b \frac{E_m}{E_c}
\]

For isotropic and homogeneous reinforcing fibres and matrix, their respective elastic moduli are constant, which allows the superscript to be taken of the terms for elastic modulus. Substituting Equations 27a and 27b into the Equations 40a and 40b, in this order, assuming isotropy and homogeneity for both fibre and matrix, changes them to:

\[
\frac{\sigma_f}{\sigma_c} = a \frac{E_f}{E_m} \left( 1 - \sqrt{V_f} \left( 1 - \frac{E_m}{E_f} \right) \right)
\]

\[
\frac{\sigma_m}{\sigma_c} = b \left( 1 - \sqrt{V_f} \left( 1 - \frac{E_m}{E_f} \right) \right)
\]
\[
\frac{\sigma_f}{\sigma_{c2}} = a \frac{E_f}{E_m} \left( 1 - \sqrt[3]{\frac{0.5}{E_f}} \left( 1 - \frac{E_m}{E_f} \right) \right)
\]  
(43a)

\[
\frac{\sigma_m}{\sigma_{c2}} = \frac{b}{a} \left( 1 - \sqrt[3]{\frac{0.5}{E_f}} \left( 1 - \frac{E_m}{E_f} \right) \right)
\]  
(43b)

Equations 42a and 43a, and 42b and 43b are statements of the stress magnification in the central sub-region reinforcing fibre and matrix, respectively, with respect to the central sub-region composite stress. The ratio of stress between the reinforcing fibre and matrix of the central sub-region for both square and hexagonal arrays of reinforcing fibre is obtained by equating Equations 42a to 42b, and Equations 43a to 43b, respectively, thus:

\[
\frac{\sigma_f}{\sigma_m} = \frac{a}{b} \frac{E_f}{E_m}
\]  
(44)

It is important to note here that the values of the parameters \(a\) and \(b\) are different for the two reinforcing fibre geometries, as is evident from an examination of their defining equations 36 – 39.

3. Results, analysis and discussion

Results of strain and stress ratios are now presented, analysed and discussed, first for square arrays of reinforcing fibre and finally for hexagonal reinforcing fibre arrays.

3.1. Square arrays of reinforcing fibre

Plots of fibre strain reduction and matrix strain magnification in the central sub-region, with respect to the central sub-region composite represented by the respective parameters \(a\) and \(b\). And plots of the ratio \((b/a)\) for stress magnification represented by the ratios shown in Equations 42 (a & b) and 44, for various values of reinforcing fibre volume, are shown in Figures 2-9 for square arrays of reinforcing.

Figures 2-3 show the variation of the transverse strain in the matrix with reinforcing fibres with volume fraction of the reinforcing fibres. Two roots (the term roots are used throughout the paper to denote solutions to a polynomial) were obtained with different magnitudes and trends and are plotted separately in Figures 2 and 3. The roots are seen to have magnitudes that are significantly different, with ratios between the two that increase at an increasing rate with increasing volume fraction of the reinforcing fibre (\(v_f\)), and which vary between 10.31 at \(v_f = 1\%\) and 1329.89 at \(v_f = 99\\%\). The different curve profiles in the two figures denote different trends of the phenomena of matrix strain magnification with increasing volume fraction of the reinforcing fibre. The two values of root ratios given above for two different volume fractions of reinforcing fibre emphasises the change in the scale of the difference of the roots with increasing volume fraction of the reinforcing fibre.

![Figure 2](image_url)

**Figure 2.** Variation of the magnification of transverse strain in the matrix with reinforcing fibre volume fraction of a square array– 1st root (Equation 37).
Figure 3. Variation of the reduction of transverse strain in the matrix with reinforcing fibre volume fraction of a square array – 2nd root (Equation 37).

Figure 4. Variation of the magnification / reduction of transverse strain in the in the matrix with reinforcing fibre volume fraction for a square array – 1st and 2nd roots (Equation 37).

The curve showed in Figure 2 displays a magnification of transverse strain in the matrix that increases at an increasing rate with increasing volume fraction of the reinforcing fibre, with sample values of 2.0142 at $\nu_f = 1\%$, 12.2361 at $\nu_f = 78\%$, to 20.8064 at $\nu_f = 91\%$. The physical meaning of the 1st root is plotted in Figures 2 and 4, which easily explained from the fact that the matrix with its elastic modulus is smaller than that of the reinforcing fibre deforms more than the reinforcing fibre under the same applied load. Therefore, the resulting strains are expected to be greater for lower volume fractions of the matrix.

The curves showed in Figures 3 and 4 manifests a reduction of transverse strain in the matrix whose magnitude of reduction increases at a decreasing rate with increasing volume fraction of the reinforcing fibre, with sample values of 0.1952 at $\nu_f = 1\%$, 0.0295 at $\nu_f = 78\%$, and 0.0274 at
\( v_f = 91\% \). The values and trend of the 2\textsuperscript{nd} root shown in Figure 3 do not have any physical meaning that is evident to the authors. Be it in terms of the magnitudes (which wrongly impute a higher elastic modulus for the matrix than the reinforcing fibre) and trend (which implies an increasing reduction of strain with increasing fibre volume fraction which is converse to the expectation) and can therefore be ignored.

Figure 5. Variation of the reduction of transverse strain in the fibre with reinforcing fibre volume fraction of a square array – 1\textsuperscript{st} and 2\textsuperscript{nd} roots (Equation 36).

Figure 5 shows two superimposed curves that exhibit a strain reduction from 0.0578 at \( v_f = 1\% \), 0.3210 at \( v_f = 78\% \), and 0.5453 at \( v_f = 91\% \) that decreases at an increasing rate with increasing volume of the reinforcing fibre. This is consistent with the fact of reduced matrix content with its attendant higher strains because of its lower stiffness, leading to a transfer of load to the reinforcing fibres with their higher stiffness. The effect of the increase in volume fraction of reinforcing fibres is an increase of strain in the fibres, at a reducing rate. A high strain reduction of 0.321 at \( v_f = 91\% \) is seen to prevail near the maximum theoretically possible fibre reinforcing volume fraction of 78.5\% for square arrays. This is consistent with the fact that as the matrix content reduces, the response of the composite tends to that of the reinforcing fibres, which because of the higher stiffness of fibres translates to lower values of strain when compared to that of the unreinforced matrix.

Figure 6. Variation of the fibre 1\textsuperscript{st} and 2\textsuperscript{nd} roots / matrix 1\textsuperscript{st} root transverse stress ratio with reinforcing fibre volume fraction for a square array (Equation 44).

The fibre 1\textsuperscript{st} and 2\textsuperscript{nd} roots/matrix 2\textsuperscript{nd} root transverse stress ratio are seen in Figure 6 to decrease at a decreasing rate with increasing volume fraction of the reinforcing fibre from 0.5484 at \( (v_f = 1\%) \),
0.5012 at ($\nu_f = 78\%$) and 0.5007 at ($\nu_f = 91\%$). The fact that the curve in Figures 6 is not horizontal and equal to unity implies that the stresses in the matrix and reinforcing fibre are different for all volume fractions of reinforcing fibre, which contradicts the assumption of iso-stress.

**Figure 7.** Variation of the fibre 1st and 2nd roots/matrix 2nd root transverse stress ratio with the volume fraction of the reinforcing fibre for a square array (Equation 44).

The curve in Figure 7 should be ignored as it is based on values of the 2nd root of the matrix strain, which was recommended earlier to be ignored because of a lack of physical meaning to justify the values of the root and trend.

Analysis of the curves in Figures 6 & 7 may also be approached from considerations of the linear elastic relationship between stress and strain, the curves in Figures 6 and 7 represent the relationship, $(\sigma_f/\sigma_m) = (\epsilon_f/\epsilon_m)(E_f/E_m)$, which may be rewritten as, $(\sigma_f/\sigma_m) = (\epsilon_f/\epsilon_m)(E_f/E_m)$. Noting that the values of stiffness of the fibre and matrix are constants, their ratio may be represented by a constant $(E_R)$, which then converts the foregoing relationship to, $(\sigma_f/\sigma_m) = E_R(\epsilon_f/\epsilon_m)$. It is evident from this relationship that the curve in Figure 6 implies that the values of fibre strain are less than those in the matrix for all volume fractions of reinforcing fibre, and further that the rate of increase of strain in the fibre with increasing volume fraction of the reinforcing is greater than in the matrix. The curve in Figure 7 indicates values of strain and a rate of increase of strain in the fibre that is greater than that in the matrix at all volume fractions of the reinforcing fibre, which is inconsistent with the expected behaviour of the composite given the relative values of stiffness of the matrix and fibre.
Figure 8. Variation of the fibre and matrix / central sub-region composite transverse stress ratios with the volume fraction of the reinforcing fibre for a square array (Equations 42a & 42b).

That stresses in the matrix and fibre within the central sub-region are different is clear from the three curves in Figure 8. It is evident from the curves in this figure that the ratio of the central sub-region matrix $1^{st}$ root stress to the central sub-region composite stress is greater than 1.82 for all volume fractions of reinforcing fibre. The ratio is seen in the figure to increase at a decreasing rate with increasing volume fraction of the reinforcing fibre, tending to a ratio of two. The curve for the $2^{nd}$ root for the matrix may be ignored for reasons noted previously in this work. Moreover, the trend of the curve towards zero with increasing volume fraction of reinforcing fibre would indicate stresses in the matrix and therefore strains that tend to zero as well, which is inconsistent with the prevailing physical situation. Furthermore, the stress ratios that are less than unity, seen in the curve for the $2^{nd}$ root for the matrix, denote a reduction in stress and therefore strain in the matrix. This is contrary to the known fact of the existence of transverse strain magnification in the matrix. The reinforcing fibre / central sub-region composite transverse stress ratio is seen to be equal to unity for all volume fractions of reinforcing fibre. It is evident from Figure 8 that the stresses in the reinforcing fibre and the matrix within the composite central sub-region are different at all volume fractions of the reinforcing fibre. This is contrary to the assumption of iso-stress made earlier in the paper.

3.1.1. Hexagonal reinforcing fibre arrays. Plots of fibre strain reduction and matrix strain magnification in the central sub-region fibre, with respect to the central sub-region composite represented by the respective parameters a and b. And plots of the ratio (b/a) for stress magnification represented by the ratios shown in Equations 43 (a & b), and 44 for various values of reinforcing fibre volume, are shown in Figures 10-17 for hexagonal reinforcing arrays.
The two roots for Equation 44 in this case are equal therefore the overlaid curves shown in Figure 9. The curves exhibit a magnification of strain in the matrix that increases with increasing volume fraction of the reinforcing fibre from 1.0967 at $\nu_f = 1\%$, 4.52219 at $\nu_f = 78\%$, to 6.2999 at $\nu_f = 91\%$. This increase in magnification is explainable based on “diminishing” matrix as the volume fraction of reinforcing fibre increases, coupled with the higher ductility of the matrix compared to that of the reinforcing fibre.

The curves in Figure 10 and Figure 11 both manifest strain reductions in the reinforcing fibre that decrease at increasing rates with increasing volume fractions of the reinforcing fibre.

**Figure 9.** Variation of the magnification of transverse strain in the matrix with reinforcing fibre volume fraction of a hexagonal array – 1st and 2nd roots (Equation 39).

**Figure 10.** Variation of the reduction of transverse strain in the fibre with reinforcing fibre volume fraction of a hexagonal array – 1st root (Equation 38).
The values of reinforcing fibre strain reduction shown in the curve plotted in Figure 10 for the 1st root vary from 0.0619 at $v_f = 1\%$, 0.4235 at $v_f = 78\%$, and 0.7186 at $v_f = 91\%$. The values of reinforcing fibre strain reduction for the 2nd root that are shown in Figure 11 vary from 0.0535 at $v_f = 1\%$, 0.1643 at $v_f = 78\%$, and 0.2140 at $v_f = 91\%$. The variations in the two curves are consistent with expectations arising from the known relative values of stiffness of the composite constituents and the increasing volume fraction of the reinforcing fibre. The 2nd root predicts higher strain reductions than the 1st root.

The basis for choice between the curves for the 1st and 2nd root shown in Figures 10 – 12 is not evident from the information contained in the figures. It is notable however, that the strain reduction exhibited by the 2nd root of 0.2140 at $v_f = 91\%$, which is near the maximum theoretically possible volume fraction of reinforcing fibre of 90.7% for hexagonal arrays, is considerably much higher than the value of 0.7186 at $v_f = 91\%$ for the 1st root. This is significant in light of the fact that the strain in the composite is expected to converge to the value for the reinforcing fibre as the content of matrix diminishes, which is expected to be lower than that of the unreinforced matrix because of the higher stiffness for the reinforcing fibre. This argument tips the balance of choice to the 2nd root.

**Figure 11.** Variation of the reduction transverse strain in the fibre with reinforcing fibre volume fraction of a hexagonal array – 2nd root (Equation 38).
Figure 12. Variation of the magnification / reduction of transverse strain in the fibre with reinforcing fibre volume fraction of a hexagonal array – 1\textsuperscript{st} and 2\textsuperscript{nd} roots (Equation 38).

Figure 13. Variation of the fibre 1\textsuperscript{st} and 2\textsuperscript{nd} roots / matrix root transverse stress ratio of a hexagonal array with the volume fraction of the reinforcing fibre (Equation 44).

The curves shown in Figure 13 are both different in magnitudes and trends. The curve for the 1\textsuperscript{st} root show values of fibre/matrix stress ratio that are greater than unity for all volume fractions of reinforcing fibre, while those for the 2\textsuperscript{nd} root show values that are less than unity over the same range. While the curve for the 1\textsuperscript{st} root exhibits a fibre/matrix stress ratio that increases at an increasing rate with increasing volume fraction of reinforcing fibre, the curve for the 2\textsuperscript{nd} root shows a ratio that decreases at a more or less constant rate. From the previously noted relationship, \((\sigma_f/\sigma_m) = E_R(\varepsilon_f/\varepsilon_m)\), it is evident that a higher fibre/matrix stress ratio is indicative of higher fibre/matrix strain ratio, which is contrary to the prevailing physical state in the transverse direction of a fibre reinforced composite. Moreover, an increase in the fibre/matrix stress ratio with increasing volume fraction of the reinforcing fibre is contrary to the expected trend as the fibre content increase, given they are stiffer than the matrix. These considerations move the balance of choice between the two curves for fibre strain reduction to the curve for the 2\textsuperscript{nd} root, a choice that ties in with the one made purely on considerations of transverse strain magnification. The fact that none of the two curves is equal to unity at any point implies that the iso-stress state does not exist at any volume fraction of the reinforcing fibre.
Figure 14. Variation of the fibre / central sub-region composite transverse stress ratio of a hexagonal array with the volume fraction of the reinforcing fibre (Equation 43a).

For the same reasons given in the analysis of the curves shown in Figure 13, the choice of curves in Figure 14 is for the 2nd root. The stresses in the reinforcing fibre are seen in Figure 14 to be less than those in the central sub-region composite and to reduce with increasing volume fraction of the reinforcing fibre. The fact that none of the two curves in this figure is equal to unity implies that the iso-stress state does not exist at any volume fraction of the reinforcing fibres.

Figure 15. Variation of the matrix / central sub-region composite transverse stress ratio of a hexagonal array with the volume fraction of the reinforcing fibre (Equation 43b).

The curve shown in Figure 15 depicts equality of the stress in the central sub-region composite and matrix at all volume fractions of the reinforcing fibre. This implies a direct, constant relationship between the product of the transverse strain and stiffness of the central sub-region composite and the transverse strain and stiffness of the matrix, since for, \((\sigma_{c2} = k\sigma_m)\), then\((E_c\epsilon_c E_m\epsilon_m)\).
It is noted here that whilst the stiffness of the matrix remains constant, composite theory shows that the stiffness of the central sub-region composite increases with increasing volume fraction of reinforcing fibre. The foregoing equality therefore implies that the transverse strain of the central sub-region composite increases at a lower rate than that of the surrounding matrix.

![Figure 16](image-url)

**Figure 16.** Comparison of the variation of matrix transverse strain magnification based on the displacement and strain energy methods for both square and hexagonal arrays of reinforcing fibre.

The values of matrix transverse strain magnification for the 1st and 2nd roots of the strain energy method for the hexagonal arrays are the same and their curves in Figure 16 are therefore coincident. With an exception of the curve for the 2nd root of the square array which shows a decreasing trend with increasing volume fraction of the reinforcing fibre (whose values were recommended earlier in this paper to be ignored), all other curves in Figure 16 exhibit increasing values of matrix transverse strain magnification with increasing volume fraction of the reinforcing fibre. With the same exception, all the curves tend to converge and diverge with decreasing and increasing volume fraction of the reinforcing fibre, respectively. The curves for the deformation method [17] applied to square and hexagonal arrays and the curves for the two roots of the strain energy method applied to hexagonal arrays are closer to one another in magnitude when compared with the other two curves, up to a volume fraction of reinforcing fibre of about 40%. The lowest values of matrix transverse strain magnification for all volume fractions of the reinforcing fibre, with the exception of the curve noted here, occur for the deformation method [17] applied to the hexagonal array. The highest values of matrix transverse strain magnification on the other hand occur for the 1st root of the strain energy method applied to the square array up until a volume fraction of reinforcing fibre just above 70%, after
which the deformation method [17] applied to the square array shows the highest values. For each one of the two types of fibre geometries the values of matrix transverse strain magnification are lower for the deformation method [15] than for the strain energy method, over most of the reinforcing the fibre volume fraction.

![Figure 17](image)

**Figure 17.** Comparison of the variation of transverse fibre strain reduction based on the displacement and strain energy methods for both square and hexagonal arrays of reinforcing fibre.

The values transverse fibre strain reduction for the 1st and 2nd roots of the strain energy method for the square array are the same and their curves in Figure 17 are therefore coincident. The curves all exhibit strain reductions that decrease with increasing volume fraction of the reinforcing fibre. It is noted from the figure that, the curve for the 1st root of the strain energy method applied to hexagonal arrays coincides with that of the deformation method [17], applied to square arrays, up to a volume fraction of the reinforcing fibre of about 42%. It is recalled that the balance of choice between the 1st and 2nd root of the strain energy method applied to hexagonal arrays was argued in the second paragraph below Figure 11 to tip towards the 2nd root. The curve for the 2nd root of the strain energy method applied to hexagonal arrays coincides with that of the deformation method [17] applied to the hexagonal array up to a volume fraction of the reinforcing fibre of about 20%. The curves all tend to converge and diverge with reducing and increasing volume fraction of the reinforcing fibre, respectively. The curves for the square arrays are lower for the strain energy method while those for the hexagonal arrays are lower for the deformation method [17].

4. Conclusions

It can be concluded from the foregoing work that in the transverse direction of uniaxially aligned fibre reinforced composites:
- The matrix lying in between the reinforcing fibres experiences transverse strain magnification whose magnitude increases non-linearly (at an increasing rate) with the increasing volume fraction of reinforcing fibres.
• The matrix transverse strain magnification for the square arrays is greater than that for the hexagonal arrays based on the strain energy method, as was the case in the method of direct deformation.
• The reinforcing fibres experience strain reduction whose magnitude decreases non-linearly (at an increasing rate) with the increasing volume fraction of reinforcing fibres.
• The fibre/matrix stress ratio is less than unity and decreases with increasing volume fraction of the reinforcing fibres for both reinforcing fibre geometries, being higher for hexagonal than for square arrays of reinforcing fibre.
• The curves for the 2nd root of matrix transverse strain magnification for the square arrays of reinforcing fibre and the 1st root of fibre strain reduction for the hexagonal arrays of reinforcing fibre should be investigated further, in order to determine the reasons for their inconsistency with expected trends and limits, respectively.
• The approach of direct deformation generates respective solutions of matrix and fibre transverse strain magnification and reduction that are very similar in magnitude to those arising from the more rigorous strain energy methods referred to the RVE.
• In the transverse direction, the respective strain reduction and magnification of the reinforcing fibre and matrix with respect to the central sub-region are equal to the respective strain reduction and magnification of the reinforcing fibre and matrix with respect to the matrix in between the reinforcing fibres in the longitudinal direction.
• This strain reduction is in turn is equal to the strain in the overall composite that does not experience strain magnification; the matrix sub-regions (A) in Figure 1c.
• The assumption of iso-stress (the Reuss rule) is shown not to apply for the fibre and matrix, fibre and central sub-region composite or matrix and central sub-region composite.

5. Recommendations
• Solutions should be sought that are independent of the iso-stress rule in order to quantify the magnitude of errors arising from use of the Reuss rule in the present work based on the strain energy method and that of the deformation method.
• Expressions should be developed for the solutions presented here with reference to the (s/r) ratio in order to facilitate comparison of these solutions with others in literature that have the same reference.

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