Analytical approach to the nonlinear free vibration of a conservative oscillator

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Abstract
This paper applies the VIM-Pade technique for solving a nonlinear free vibration of a conservative oscillator. It is a combined method based on the variational iteration method, Laplace transformation and the Pade approximation. An approximated solution with extremely high accuracy can be obtained with ease. Runge-Kutta method is adopted to verify the efficiency of the technique.

Keywords
Variational iteration method, Pade approximation, Laplace transformation, oscillator, approximation

Introduction
We consider the equation of motion

\[
(1 + 3ezu^2) \frac{d^2u}{dt^2} + 6ez\left( \frac{du}{dt} \right)^2 + \omega^2u + e\omega^2u^3 = 0
\]

(1)

with the following initial conditions

\[
u(0) = A, \quad \frac{du}{dt} = 0
\]

(2)

where \(e, z, \omega, \) and \(A\) are given constants. This nonlinear system results from a free vibration of a conservative oscillator.\(^1\) It can be used for modeling the motion of a mass grounded by linear and nonlinear springs in series connection over a frictionless contact surface as shown in Figure 1. Here, \(m\) is the mass, \(k_1\) is the stiffness of linear spring, \(k_2\) and \(\beta\) are the coefficients of linear and nonlinear parts of nonlinear spring, respectively. The parameters \(e, z, \) and \(\omega\) are, respectively, defined by

\[
e = \frac{\beta}{k_2}, \quad \nu = \frac{k_2}{k_1}, \quad z = \nu \frac{1}{1 + \nu}, \quad \omega = \sqrt{\frac{k_2}{m(1 + \nu)}}
\]

(3)

As shown in the literature,\(^1,2\) the deflection of linear spring is given by

\[
y_1 = \nu u + e\nu u^3
\]

(4)
The displacement of attached mass $y_2$ is constructed by the deflection of linear and nonlinear springs as follows

$$y_2 = u + y_1$$  \hspace{1cm} (5)

Recently, some numerical and analytical methods were proposed for solving this nonlinear oscillator, including Lindstedt method and harmonic balance method,\(^1\) the modified armonic balance method,\(^2\) the homotopy analysis method,\(^3\) He’s iteration perturbation method,\(^4\) the variational iteration method (VIM),\(^5\) Hamiltonian approach,\(^6\) the global residue harmonic balance method,\(^7\) He’s frequency formulation\(^8\) and other methods.\(^9\) As a classical method, the VIM has been paid much attention,\(^10–13\) due to its wide application for solving the linear and nonlinear differential equations. In order to reduce the computational cost of VIM, Abassy et al.\(^14\) and Lu\(^15\) proposed a modified VIM. Anjum and He applied Laplace transform to identify the Lagrange multiplier involved in the iteration algorithm.\(^16\) A reliable algorithm based upon the homotopy perturbation method was applied to the strongly nonlinear oscillators.\(^17–21\) Motivated by these improvements, we consider an analytical approach (VIM-Pade´) based on the VIM, Laplace transformation and the Pade´ approximation to solve the nonlinear oscillator (equation (1)). Two numerical examples will be presented to show its efficiency.

**The VIM-Pade´ technique for nonlinear oscillator**

We apply the VIM-Pade´ technique based on VIM, Laplace transformation and Padé approximation\(^22,23\) to the equation of motion (1). By VIM,\(^10–13\) a correct functional can be constructed as follows

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda \left( \frac{\partial^2 u_n}{\partial \zeta^2} + 3\varepsilon\varepsilon u_n(\zeta) \right)^2 + c_\varepsilon \frac{\partial^2 u_n}{\partial \zeta^2} + 6\varepsilon\varepsilon u_n(\zeta) \left( \frac{\partial u_n}{\partial \zeta} \right)^2 + \omega^2 u_n(\zeta) + \varepsilon(\omega^2 u_n(\zeta)^3) \right) \, d\zeta$$  \hspace{1cm} (6)

where $\lambda$ is a general Lagrange multiplier, and $\delta u_n = 0$. The variational theory\(^24–28\) can be used to identify the Lagrange multiplier $\lambda$. We should specially point out that He and Wu presented a number of variational iteration formulae for solving various kinds of nonlinear equations.\(^10\) We make the correct functional (equation (6)) stationary, and obtain the following stationary conditions

$$\begin{align*}
\left. \frac{\partial^2 \lambda}{\partial \zeta^2} \right|_{\zeta = t} &= 0 \\
1 - \left. \frac{\partial \lambda}{\partial \zeta} \right|_{\zeta = t} &= 0 \\
\left. \lambda \right|_{\zeta = t} &= 0
\end{align*}$$

The multiplier $\lambda$ can be easily obtained as $\lambda = \xi - t$. We then have the following iteration formula

$$u_{n+1}(t) = u_n(t) + \int_0^t (\xi - t) \left( 1 + 3\varepsilon u_n(\xi)^2 \right) \frac{\partial^2 u_n}{\partial \zeta^2} + 6\varepsilon\varepsilon u_n(\xi) \left( \frac{\partial u_n}{\partial \zeta} \right)^2 + \omega^2 u_n(\xi) + \varepsilon(\omega^2 u_n(\xi)^3) \right) \, d\zeta$$  \hspace{1cm} (7)

To begin with an initial approximation $u_0(t) = A$, it is easy to obtain the VIM solution by equation (7). Obviously, the VIM solutions are expressed in series form,\(^29\) which may result in the deviation from the exact

**Figure 1.** Nonlinear system of a mass with serial linear and nonlinear springs over a frictionless contact surface.
solution of equation (1). For improving the accuracy of the VIM solutions, we use Laplace transformation and Padé approximation to \( u_{n+1}(t) \). Laplace transformation is used to transform the VIM solution and then Padé approximation is applied to the transformed solution, finally the approximated solution can be obtained by the inverse Laplace transformation. We briefly illustrate the idea of Padé approximation. Suppose that the transformed solution of \( u_{n+1}(t) \) is defined by a series solution \( v_{n+1}(t) = \sum_{k=0}^{\infty} q_k t^k \), we approximate it by a rational function as follows

\[
F[L, M] = \frac{P_L(t)}{Q_M(t)} \quad (8)
\]

where

\[
P_L(t) = p_0 + p_1 t + p_2 t^2 + \ldots + p_L t^L,
\]

\[
Q_M(t) = 1 + q_1 t + q_2 t^2 + \ldots + q_M t^M
\]

By using the normalization condition \( Q_M(0) = 1 \), the coefficients of \( P_L(t) \) and \( Q_M(t) \) can be given by linear equations with respect to \( p_0, p_1, \ldots, p_L \) and \( q_1, \ldots, q_M \). For clarity, the corresponding solution \( F[L, M] \) is called as \([L, M]\) Padé approximation to \( v_{n+1}(t) \).

**Numerical example**

In this section, we will consider two initial value problems of the nonlinear oscillator (equation (1)) to show the efficiency of VIM-Padé technique. We will compare it with Runge-Kutta method, and consider the sensitivity of the parameter \( A \). All the numerical computations are performed by a mathematical software on PC with an Intel Core 2 Duo CPU, 2.4 GHz, and 8 GB RAM.

We first consider the nonlinear oscillator (equation (1)) with \( m = 1, k_1 = 50, k_2 = 5, \beta = 2.5 \) and \( A = 1 \). By setting the initial approximation \( u_0(t) = A \), we have the following VIM iteration formula

\[
\begin{align*}
\frac{d^3 u_n}{dt^3} + \frac{2}{11} u_n(\xi) & + \frac{20}{11} u_n(\xi) \frac{d^2 u_n}{d\xi^2} + \frac{50}{11} u_n(\xi) \frac{du_n}{d\xi} + 25 u_n(\xi)^3 = 0 \\
\end{align*}
\]

(9)

By the above iteration (9), it follows the following approximations

\[
\begin{align*}
u_1 &= 1 - 3.4090909091 t^2, \\
u_2 &= 1 - 3.007607062 t^2 + 1.381913765 t^4 + 0.001726814 t^6 + 1.607959272 t^8, \\
u_3 &= 1 - 3.007607062 t^2 + 1.381913765 t^4 + 0.001726814 t^6 - 0.820536178 t^8 + 1.883873915 t^{10} - 2.609787903 t^{12} + \ldots, \\
u_4 &= 1 - 2.998962673 t^2 + 1.426165384 t^4 - 0.209372553 t^6 + 0.264159459 t^8 - 1.136788349 t^{10} + 2.983021729 t^{12} + \ldots
\end{align*}
\]

To improve the accuracy of the approximation \( u_4 \), the [4,4] Padé approximation will be constructed by the VIM-Padé technique. For simplicity, we denote \( u_4 \) by

\[
u_4 = 1 - 2.998962673 t^2 + 1.426165384 t^4 - 0.209372553 t^6 + 0.264159459 t^8 + O(t^{10})
\]

The Laplace transformation is applied to the fourth-order approximation \( u_4 \), which results in

\[
L[u_4] = \frac{1}{s} - \frac{5.997925347}{s^3} + \frac{34.22796214}{s^5} - \frac{150.74823817}{s^7} + \frac{10650.909372317}{s^9} + \ldots
\]
Letting \( t = \frac{1}{3} \), it follows the transformed solution

\[
L[\mu_4] = t - 5.997925347t^3 + 34.227969214t^5 - 150.74823817t^7 + 10650.909372317t^9 + \ldots
\]

The [4/4] Padé approximation to \( L[\mu_4] \) can be given by

\[
\left[ \frac{4}{4} \right] = \frac{t + 25.223722248t^3}{1 + 31.221647595t^2 + 153.037142258t^4}
\]

We rewrite the [4/4] diagonal approximation as

\[
\left[ \frac{4}{4} \right] = \frac{s^3 + 25.223722248s}{s^4 + 31.221647595s^2 + 153.037142258}
\]

By using the inverse Laplace transformation to the [4/4] Padé approximation, we obtain the following VIM-Padé solution

\[
u_{[4/4]}(t) = 1.004794808\cos(2.467637277t) - 0.004794808\cos(5.013223899t)
\]

Figure 2. Numerical results of VIM-Padé solutions and Runge-Kutta solutions with \( \beta = 1 \), (a) \( u(t) \), (b) \( y_1 \) and (c) \( y_2 \).
Similarly, the rest VIM-Padé solutions can be given by the previous procedure. Based on the approximation $u_4$, the [6/6] VIM-Padé solution reads as

$$u_{[6/6]}(t) = 1.00100528\cos(2.457914597t) - 0.001007615\cos(7.055489902t) - 2.336921219 \cdot 10^{-6}\cos(16.893834096t)$$

In order to show the efficiency of the VIM-Padé method, we provide the numerical comparisons of the VIM-Padé method and Runge-Kutta method for solving nonlinear oscillator (equation (1)). Figure 2(a) to (c) shows the numerical behaviors of the approximations to $u(t)$, $y_1$ and $y_2$, respectively. The VIM-Padé method works well for this initial value problem. The VIM-Padé solutions agree well with the approximated solutions given by Runge-Kutta method. We remark that the accuracy of [4/4] or [6/6] VIM-Padé solutions can be improved further by considering the diagonal Padé approximation of higher order.

We then consider the sensitivity of the parameter $A$. We will further consider the initial value problem associated with the nonlinear system (equation (1)) with a different $A = 0.5$. The VIM iteration formula can be represented as

$$u_{n+1}(t) = u_n(t) + \int_0^t (\xi - t) \left\{ \left(1 + \frac{3}{22}u_n(\xi)^2\right) \frac{d^2u_n}{d\xi^2} + \frac{3}{11}u_n(\xi)\left(\frac{du_n}{d\xi}\right)^2 + \frac{50}{11}u_n(\xi) + 25u_n(\xi)^3 \right\} d\xi$$

with an initial approximation $u_0(t) = 0.5$.

![Figure 3. Comparisons of VIM-Padé solutions and Runge-Kutta solutions with $A = 0.5$. (a) $u(t)$, (b) $y_1$ and (c) $y_2$.](image-url)
By iteration (10), it follows the fourth-order approximation
\[
 u_4 = 0.5 - 1.236262066t^2 + 0.521962409t^4 - 0.104835325t^6 + 0.021499521t^8 - 0.003561883t^{10} + 0.000041529t^{12} + \ldots
\]

We can obtain the \([4/4]\) and \([6/6]\) VIM-Padé solutions as follows
\[
 u_{[4/4]}(t) = 0.499775768\cos(2.219904941t) + 0.000242432\cos(6.335170804t),
\]
\[
 u_{[6/6]}(t) = 0.499776346\cos(2.220040632t) + 0.000223879\cos(6.469422276t) - 2.256027449 \cdot 10^{-7}\cos(12.258017653t)
\]

We show the numerical comparisons of the VIM-Padé solutions and the Runge-Kutta solutions in Figure 3(a) to (c), respectively. The numerical results show that the VIM-Padé technique also performs well for the oscillator (equation (1)) with a different \(A\). The approximated solutions obtained by VIM-Padé technique are expressed by a series of cosine functions. We note that the two frequencies are approximately multiplied, and the multiplier relationship will be improved further by considering more iteration steps of the VIM. In sum, the VIM-Padé technique can be seen as an efficient method for solving this nonlinear oscillator.

Conclusions
In this paper, we focused on the nonlinear free vibration of a conservative oscillator by a combined VIM-Padé technique. Numerical examples associated with two initial value problems were considered to illustrate the effectiveness of this method. Comparisons of the VIM-Padé technique and Runge-Kutta method were provided, showing that the VIM-Padé method performs well without linearization or perturbation. However, there are also two open problems, one is that how to obtain the approximated period solutions given by a series of cosine functions with a frequency and its multiples, the other is that how to choose the order of VIM-Padé approximation and the initial approximation such that the total computational cost can be optimal. We will focus on these two topics in our future work and extend the VIM and its modification to the nonlinear systems with fractal or fractional derivatives.29–37

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