A NEW SUBCONVEX BOUND FOR GL(3) L-FUNCTIONS IN THE $t$-ASPECT

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ABSTRACT. We revisit Munshi’s proof of the $t$-aspect subconvex bound for GL(3) $L$-functions, and we are able to remove the ‘conductor lowering’ trick. This simplification along with a more careful stationary phase analysis allows us to improve Munshi’s bound to,

$$L(1/2 + it, \pi) \ll_{\pi, \epsilon} (1 + |t|)^{3/4 - 3/40 + \epsilon}.$$ 

1. INTRODUCTION AND STATEMENT OF RESULT

Let $\pi$ be a Hecke cusp form of type $(\nu_1, \nu_2)$ for $SL(3, \mathbb{Z})$. Let the normalized Fourier coefficients be given by $\lambda(m_1, m_2)$ (so that $\lambda(1,1) = 1$). The $L$-series associated with $\pi$ is given by

$$L(s, \pi) = \sum_{n \geq 1} \lambda(1, n) n^{-s}, \quad \text{for} \ Re(s) > 1.$$ 

In this paper, we follow Munshi [13] but remove the ‘conductor lowering’ trick. This simplification makes the stationary phase analysis cleaner, which improves certain estimates and allows us to improve the subconvex bound exponent. We prove the following theorem.

**Theorem 1.1.** Let $\pi$ be a Hecke-Maass cusp form for $SL(3, \mathbb{Z})$. Then for any $\epsilon > 0$

$$L(1/2 + it, \pi) \ll_{\pi, \epsilon} t^{3/4 - 3/40 + \epsilon}.$$ 

We should mention that one can modify our approach in [1] for a $t$-aspect subconvex bound for GL(2) $L$-functions to remove the ‘conductor lowering’ trick. Indeed, it suffices to replace the delta method used in [1] by an averaged version of the following uniform partition of the circle to detect $n = 0$

$$\delta(n = 0) = \frac{1}{q} \sum_{a \mod q} e\left(\frac{an}{q}\right) \int_0^1 e\left(\frac{nx}{q}\right) dx, \quad \text{for} \ q \in \mathbb{N}.$$ 

By choosing $q$ to be of size approximately $t^{1/3}$, one can get the Weyl bound for GL(2) $L$-functions in $t$-aspect. This circle method seems insufficient to obtain a GL(3) $t$-aspect subconvex bound, so we use Kloosterman’s version of the circle method (Lemma 1.2).

A $t$-aspect bound for self-dual GL(3) $L$-functions was first established by Li [8], and improved upon by McKee, Sun and Ye [10], Sun and Ye [19] and Nunes [17]. A $t$-aspect bound for a general $SL(3, \mathbb{Z})$ $L$-function was proved by Munshi by a completely different approach. We revisit Munshi’s proof and improve upon his result. Although the bound obtained here is weaker than that in [10, 17, 19], it holds for any Hecke-Maass cusp form for $SL(3, \mathbb{Z})$, and not just the self-dual forms. We note that we get the same exponent as Sun and Zhao [20], whose work is on bounding twists of GL(3) $L$-functions in depth aspect. They use Kloosterman’s version of circle method, along with a ‘conductor lowering’ trick appropriate for the depth aspect. On the other hand, our result in $t$-aspect doesn’t need a ‘conductor lowering’ trick. Other works that deal with the subconvex bound problem for degree three $L$-functions include [2, 5, 9, 12–16, 19].

We start with applying the approximate functional equation (Lemma 3.1)

$$L(1/2 + it, \pi) \ll \sup_{1 \leq N < t^{3/2 + \epsilon}} \frac{S(N)}{N^{1/2}},$$ 

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where
\[ S(N) = \sum_{n=1}^{\infty} \lambda(n) n^{-it} V \left( \frac{n}{N} \right). \] (1)

\( V \) is a smooth function with bounded derivatives and supported in \([1, 2]\), that is allowed to depend on \( t \). An application of Cauchy-Schwarz inequality applied to \( S(N) \) followed by the Ramanujan bound on average (Lemma 3.4) gives the trivial bound \( S(N) \ll N^{1+\varepsilon} \). Therefore it suffices to beat the trivial bound \( O(N^{1+\varepsilon}) \) of \( S(N) \) for \( N \) in the range \( t^{3/2 - \delta} < N < t^{3/2 + \varepsilon} \) and some \( \delta > 0 \).

The next step is to separate the oscillations by writing
\[ S(N) = \sum_{n=1}^{\infty} \sum_{r=1}^{\infty} \lambda(n) n^{-it} U \left( \frac{n}{N} \right) V \left( \frac{r}{N} \right) \delta(n-r=0), \]
where \( U \) is a smooth function compactly supported on \([1/2, 5/2]\) and \( U(x) = 1 \) for \( x \in [1, 2] \). We note that \( U(x)V(x) = V(x) \). The separation of oscillations is done by using Kloosterman’s version of the circle method.

**Lemma 1.2** (Kloosterman circle method). Let \( Q \geq 1 \) and \( n \in \mathbb{Z} \). Then
\[ \delta(n=0) = 2 \text{Re} \int_{0}^{1} \sum_{1 \leq q \leq Q} \sum_{(a,q)=1} \frac{1}{q} e \left( \frac{na}{q} - \frac{nx}{aq} \right) dx, \]
where \( e(x) = e^{2\pi ix} \) and \( * \) on the summation denotes the coprimality condition \((a,q) = 1\).

**Proof.** See [7, Section 20.3]. \( \square \)

The next steps are to apply dual summation formulas to the \( n \) and the \( r \) sums, followed by stationary phase analysis of the oscillatory integrals, and a final application of Cauchy-Schwarz and Poisson summation to the \( n \)-sum. We give detailed heuristics of these calculations in Section 2. We prove the following main proposition.

**Proposition 1.3.** Let \( S(N) \) be given by equation (1) and \( t^{1+\varepsilon} < N < t^{3/2+\varepsilon} \). Let \( Q \) be a parameter satisfying \( N^{1+\varepsilon}/t < Q < N^{1/2} \). For any \( \varepsilon > 0 \), we have
\[ S(N) \ll \frac{N^{3/2+\varepsilon}}{Q^{3/2}} + QN^{1/4} t^{1/2+\varepsilon} \quad \text{if} \quad t^{1+\varepsilon} < N < t^{3/2+\varepsilon}. \]

By choosing \( Q = N^{1/2}/t^{1/5} \), we obtain the bound of \( S(N) \ll N^{3/4} t^{3/10+\varepsilon} \). Comparing this with the trivial bound of \( S(N) \ll N^{1+\varepsilon} \), the optimal range of \( N \) where the above proposition gives a better than the trivial estimate is \( t^{6/5} < N < t^{3/2+\varepsilon} \). For \( N \ll t^{6/5} \), we use the trivial bound \( S(N) \ll N^{1+\varepsilon} \). Taking the supremum of \( S(N) \) over the range \( N \ll t^{3/2+\varepsilon} \), the proposition implies Theorem 1.1.

**Notations.** In the rest of the paper, we use the notation \( e(x) = e^{2\pi ix} \). Let \( a \) and \( b \) be two positive real numbers. We denote \( a \sim b \) to mean \( k_1 < a/b < k_2 \) for some absolute constants \( k_1, k_2 > 0 \). We use \( \varepsilon \) to denote an arbitrarily small positive constant that can change depending on the context. We denote \( a \asymp b \) to mean \( t^{-\varepsilon} b \ll a \ll t^{\varepsilon} b \) asymptotically as \( |t| \to \infty \). We must add that for brevity of notation, we assume \( t > 0 \). Indeed, the same analysis holds for \( t < 0 \) by replacing \( t \) with \(-t\) appropriately.

2. Outline of the Proof

In this section, we give detailed heuristics of the proof. We use Kloosterman’s version of the circle method to separate the \( n \) and the \( r \) variables in \( S(N) \). Roughly
\[ S(N) \approx \int_{0}^{1} \sum_{1 \leq q \leq Q} \sum_{(a,q)=1} \frac{1}{aq} \sum_{r=1}^{N} r^{-it} e(\pi r/q) e(-rx/aq) \sum_{n=1}^{N} A(n) e(-n\pi/q)e(-nx/aq) dx, \] (2)
where \( A(n) = \lambda(1, n) \). Trivial bound gives \( S(N) \ll N^2 \). So we need to save \( N \) and a bit more. We start by applying dual summation formulas to the \( r \)-sum and the \( n \)-sum.
2.0.1. Poisson to the r-sum. The conductor of the r-sum is \( q(t + N/\alpha q) \). If \( Q \ll \sqrt{N/t} \), then \( t < N/\alpha q \), so that \( q(t + N/\alpha q) = N/\alpha \). The dual length after Poisson summation is then \( 1/Q \ll 1 \). If \( Q \) has size, that is if \( Q > t' \), only \( r = 0 \) contributes. In this case the congruence condition \( \delta(\tau \equiv -r \mod q) \) implies \( q = 1 \) and \( \alpha = [Q + 1] \). That is, both the \( q \) and \( \alpha \) sums vanish, and all the contributions of additive twists due to the circle method become trivial. We are therefore not able to improve upon the convexity bound. We choose \( Q \) so that \( q(t + N/\alpha q) = qt \) for some \( q \ll Q \). Let \( U \) be a smooth bump function controlling \( r \sim N \). Poisson summation transforms the r-sum as

\[
\sum_{r \geq 1} e(\tau r/q) r^{-it} e(-rx/aq) U(r/N) \leftrightarrow N^{1-it} \sum_{|r| \ll q/N} \delta(\tau \equiv -r \mod q) U \left( \frac{N(ra-x)}{aq}, 1-it \right). \tag{3}
\]

Thus \( a \) is determined mod \( q \) and \( a \sim Q \). \( U^\dagger(v, w) \) is a highly oscillatory integral that is negligible unless \( |v| = |w| \). We also observe that only \( r = 0 \) exists if \( Q < N/t^{1-\varepsilon} \). So we choose \( Q > N/t^{1-\varepsilon} \).

2.0.2. Voronoi to the n-sum. The conductor of the n-sum is \( q^3(N/\alpha q)^3 \), so the new length after Voronoi formula is \( N^2/Q^3 \). Let \( V \) be a smooth bump function controlling \( n \sim N \). Voronoi summation transforms the n-sum as

\[
\sum_{n \geq 1} A(n) e(nr/q) e(nx/aq) V(n/N) \leftrightarrow N^{1/2} \sum_{n \leq N/2} A^*(n) \frac{S(\tau, n; q)}{\sqrt{N}} \int_{-N/q}^{N/q} \frac{1}{q} \left( \frac{Nn}{q^2} \right)^{-it} \gamma(-1/2 + it) \times V \left( Nx/aq, 1/2 + it \right) dr. \tag{4}
\]

where we have assumed \( Q^2 < N \) and \( A^*(n) = \lambda(n, 1) \). Combining (3) and (4), (2) transforms into

\[
S(N) = N^{3/2} \sum_{n \leq N/2} A^*(n) \frac{1}{\sqrt{n}} \sum_{1 \leq q \leq Q} \frac{1}{aq} \sum_{|r| \ll \sqrt{N}} S(\tau, n; q) \frac{1}{\sqrt{q}} I(n, r, q), \tag{5}
\]

where \( I(n, r, q) \) is a highly oscillatory double integral over \( x, \tau \).

2.0.3. Analysis of the integrals. A major part of the paper is performing a robust stationary phase analysis of \( I(n, r, q) \). We expect a square root saving of ‘the size of the oscillation’. \( U^\dagger \) in (3) has oscillation of size \( t \). Thus

\[
U \left( \frac{N(ra-x)}{aq}, 1-it \right) = \frac{1}{\sqrt{t}} \left( \frac{N(ra-x)}{aqt} \right)^it \times \text{smooth fn}. \tag{6}
\]

Similarly, \( V^\dagger \) in (4) has oscillation of size \( \tau = Nx/Qq \). Thus

\[
V \left( Nx/aq, 1/2 + it \right) = \frac{1}{\sqrt{\tau}} \left( \frac{Nx}{aq \tau} \right)^i \tau \times \text{smooth fn}. \tag{7}
\]

From (6) and (7), the x-integral in \( I(n, r, q) \) is therefore

\[
\int_0^1 (ra-x)^it x^i \tau \times \text{smooth fn} = \frac{1}{\sqrt{N/Qq}} \times \tau \text{-oscillations},
\]

where the above asymptotic comes from the observation that the x-oscillation is of size \( \tau \), which is at most \( N/Qq \). The biggest contribution comes from the range \( \tau = N/Qq \). Therefore (5) is

\[
S(N) = \frac{N^{1/2}}{t^{1/2}} \sum_{n \leq N/2} A^*(n) \frac{1}{\sqrt{n}} \sum_{1 \leq q \leq Q} \frac{1}{aq} \sum_{|r| \ll \sqrt{N}} \frac{S(\tau, n; q)}{\sqrt{q}} \int_{|r| = N/Qq} g(q, r, \tau) n^{-i\tau} dr, \tag{8}
\]

where \( g(q, r, \tau) = O(1) \) and has \( \tau \)-oscillations. Recalling the restriction \( Q^2 < N \), the above gives the bound \( S(N) \ll \frac{N^{3/2} t^{1/2}}{Q^{1/2}} \) \( = N^{3/4} t^{1/2} \). Since \( N \ll t^{3/2} \), this gives us the bound of \( L(1/2 + it, \pi) \ll t^{7/8+\varepsilon} \). We therefore need to save \( t^{1/8} \) and a bit more.
2.0.4. Cauchy-Schwarz and Poisson to the n-sum. The biggest contribution in (8) comes from the largest \( n \), that is \( n = N^2/Q^3 \). Applying Cauchy-Schwarz inequality with the n-sum outside, followed by the Ramanujan bound on average (Lemma 3.4)

\[
S(N) \ll \frac{N^{1/2}}{t^{1/2}} \left( \sum_{n = q^2} \left| \sum_{1 \leq q \leq Q} \sum_{|\tau| \leq \frac{N}{q}} \frac{S(\tau, n; q)}{\sqrt{q}} \int_{-N/4q}^{N/4q} g(q, r, \tau) n^{-it} \, d\tau \right|^2 \right)^{1/2}.
\]

Opening the absolute value squared, the n-sum inside the square root is

\[
\sum_{n = q^2} \frac{S(\tau_1, n; q_1) S(\tau_2, n; q_2)}{\sqrt{q_1} \sqrt{q_2}} n^{-i(\tau_2 - \tau_1)} U \left( \frac{n}{N^2/Q^3} \right).
\]

Application of Poisson summation to the n-sum gives the new length, \( \frac{q^2(N/qQ)}{N^2/Q^3} = Q^2q/N \), where \( q_1, q_2 \sim q \). The above sum transforms into

\[
\frac{N^2}{Q^3} \sum_{|n| \leq q^2} \frac{1}{q_1 q_2} C U^\uparrow \left( \frac{N^2 n}{q_1 q_2 Q^3}, 1 - i(\tau_1 - \tau_2) \right),
\]

where \( C \) is a character sum. When \( n = 0 \), \( C \) vanishes unless \( q_1 = q_2 \) and \( r_1 = r_2 \), in which case \( C \ll q_1 q_2 \). Moreover, \( U^\uparrow \) is negligible unless \( \tau_1 = \tau_2 \). In that case, \( U^\uparrow(0, \bullet) \ll 1 \). Therefore the contribution of \( n = 0 \) towards \( S(N) \) is

\[
S^0(N) \ll \frac{N^{1/2}}{t^{1/2}} \left( \sum_{1 \leq q_1, q_2 \leq Q} \sum_{|r_1| \leq N} \frac{|C| N^2}{q_1^2 q_2^3} \int_{-N/4q}^{N/4q} |g(q, r, \tau)|^2 \, d\tau \right)^{1/2} \ll \frac{N^{3/2}}{Q^{1/2}}.
\]

When \( n \neq 0 \), \( C \ll \sqrt{q_1 q_2} \), and the \( U^\uparrow \) as given in (9) is asymptotic to

\[
U^\uparrow \left( \frac{N^2 n}{q_1 q_2 Q^3}, 1 - i(\tau_1 - \tau_2) \right) \approx \left( \frac{q_1 q_2 Q^3}{N^2 n} \right)^{1/2} h(\tau_1, \tau_2),
\]

where \( h(\tau_1, \tau_2) \) is an oscillatory function of size 1. We again expect square root cancellation in the \( \tau_1, \tau_2 \)-integral. That is

\[
\int_{|\tau_1| = N/4q} g(q_1, r_1, \tau_1) g(q_2, r_2, \tau_2) h(\tau_1, \tau_2) \, d\tau_1 \, d\tau_2 \ll \frac{N}{Q \sqrt{q_1 q_2}}
\]

Therefore the contribution of the \( n \neq 0 \) terms towards \( S(N) \) is

\[
S^2(N) \ll \frac{N^{1/2}}{t^{1/2}} \left( \sum_{1 \leq q_1, q_2 \leq Q} \sum_{|r_1| \leq N} \frac{|C| N^2}{q_1 q_2 Q^3} \sum_{0 \neq |n| \leq q^2 q/4} \frac{Q^{3/2} \sqrt{q_1 q_2}}{n^{3/2}} \right)^{1/2} \ll QN^{1/4} t^{1/2}.
\]

Matching up the contributions of \( S^0(N) \) and \( S^2(N) \), the optimal choice of \( Q \) is \( Q = N^{1/2}/t^{1/5} \). In this case, we get \( S(N) \ll N^{3/4} t^{3/10 + \epsilon} \). Since \( N \ll t^{3/2 + \epsilon} \), we get the bound \( L(1/2 + it, \pi) \ll t^{3/4 - 3/40 + \epsilon} \).

We compare this with Munshi’s heuristics. Munshi chooses \( Q = \sqrt{N/K} \) with \( t^{1/4} < K < t^{1/2} \) for some parameter \( K \). Since we choose \( Q = N^{1/2}/t^{1/5} \), our choice of \( Q \) satisfies Munshi’s restriction. We should also mention that our improvement comes in the step of stationary phase analysis of \( I(n, r, q) \), which allows us to improve the bound for \( S^3(N) \) as given in (10). This is mentioned in Remark 8.1. Indeed, our bound for \( S^3(N) \) matches that of Munshi.

Moreover, we observe that one needs \( Q \) to be bigger than \( N/t \) after Poisson summation, and smaller than \( N^{1/2} \) after Voronoi summation for this approach to work. In the \( GL(n) \) \( t \)-aspect problem, say we choose \( Q = N^{1/2}/t^{\alpha_n} \) for some \( \alpha_n > 0 \). The restriction \( N/t < Q < N^{1/2}/t^{\alpha_n} \) implies our analysis is valid in the range \( N < t^{2 - 2\alpha_n} \). Since \( N \ll t^{n/2} \) we are forced to choose \( \alpha_n \leq 0 \) for \( n \geq 4 \). Therefore this approach doesn’t give a subconvex bound result for \( n \geq 4 \).
Let \( \pi \) be a Maass form of type \((\nu_1, \nu_2)\) for \( \text{SL}(3, \mathbb{Z}) \), which is an eigenfunction for all the Hecke operators. Let the Fourier coefficients be \( \lambda(n_1, n_2) \), normalized so that \( \lambda(1, 1) = 1 \). The Langlands parameter \((\alpha_1, \alpha_2, \alpha_3)\) associated with \( \pi \) are \( \alpha_1 = -\nu_1 - 2\nu_2 + 1, \alpha_2 = -\nu_1 + \nu_2, \alpha_3 = 2\nu_1 + \nu_2 - 1 \). The dual cusp form \( \tilde{\pi} \) has Langlands parameters \((-\alpha_3, -\alpha_2, -\alpha_1)\). \( L(s, \pi) \) satisfies a functional equation

\[
\gamma(s, \pi)L(s, \pi) = \gamma(s, \tilde{\pi})L(1 - s, \tilde{\pi}),
\]

where \( \gamma(s, \pi) \) and \( \gamma(s, \tilde{\pi}) \) are the associated gamma factors. We refer the reader to Goldfeld’s book on automorphic forms for \( \text{GL}(n) \) [4] for the theory of automorphic forms on higher rank groups.

### 3.1. Approximate functional equation and Voronoi summation formula

We are interested in bounding \( L(s, \pi) \) on the critical line, \( \text{Re}(s) = 1/2 \). For that, we approximate \( L(1/2 + it, \pi) \) by a smoothed sum of length \( t^{3/2+\varepsilon} \). This is known as the approximate functional equation and is proved by applying Mellin transformation to \( f \) followed by using the above functional equation.

**Remark 3.2.** On the critical line, Stirling’s approximation to \( \gamma(s, \pi) \) followed by integration by parts to the integral representation of \( V_{1/2+it}(n) \) gives arbitrary saving for \( n \gg (1 + |t|)^{3/2+\varepsilon} \).

One of the main tools in our proof is a Voronoi type formula for \( \text{GL}(3) \). Let \( h \) be a compactly supported smooth function on \((0, \infty)\), and let \( \tilde{h}(s) = \int_0^\infty h(x)x^{s-1} \, dx \) be its Mellin transform. For \( \sigma > -1 + \max\{-\text{Re}(\alpha_1), -\text{Re}(\alpha_2), -\text{Re}(\alpha_3)\} \) and \( \ell = 0, 1 \), define

\[
\gamma_\ell(s) = \frac{\pi^{-3s-3/2}}{2} \prod_{i=1}^3 \frac{\Gamma\left(\frac{1+s+\alpha_i+\ell}{2}\right)}{\Gamma\left(\frac{1+s+\alpha_i}{2}\right)}.
\]

Further set \( \gamma_\pm(s) = \gamma_0(s) \mp i\gamma_1(s) \) and let

\[
H_\pm(y) = \frac{1}{2\pi i} \int_{(\sigma)} y^{-s} \gamma_\pm(s) \tilde{h}(-s) \, ds.
\]

We need the following Voronoi type formula (See [8, 11]).

**Lemma 3.3.** Let \( h \) be a compactly supported smooth function on \((0, \infty)\). We have

\[
\sum_{n=1}^\infty \lambda(1, n)e(an/q)h(n) = q \sum_{\pm} \sum_{n_1, n_2=1}^\infty \frac{\lambda(n_2, n_1)}{n_1n_2} S(\sigma, \pm n_2; q/n_1) H_\pm(n_1^2n_2/q^3),
\]

where \((a, q) = 1 \) and \( a\sigma \equiv 1 \) mod \( q \).

Stirling approximation of \( \gamma_\pm(s) \) gives \( \gamma_\pm(s+i\tau) \ll 1 + |\tau|^{3\sigma+3/2} \). Moreover on \( \text{Re}(s) = -1/2 \)

\[
\gamma_\pm(-1/2+i\tau) = (|\tau|/e\tau)^{3i\tau} \Phi_\pm(\tau), \quad \text{where } \Phi_\pm(\tau) \ll |\tau|^{-1}.
\]

(11)

We will also use the Ramanujan bound on average which follows from the Rankin-Selberg theory.

**Lemma 3.4** (Ramanujan bound on average). We have

\[
\sum\sum |\lambda(n_2, n_1)|^2 \ll_{\pi, \varepsilon} x^{1+\varepsilon}.
\]
4. Stationary phase analysis

We need to use stationary phase analysis for oscillatory integrals. Let $\mathcal{J}$ be an integral of the form

$$\mathcal{J} = \int_a^b g(x)e(f(x))dx,$$

where $f$ and $g$ are real valued smooth functions on $\mathbb{R}$. The fundamental estimate for integrals of the form (12) is the $r^{th}$-derivative test

$$\mathcal{J} \ll \left( \max_{[a,b]} g(x) \right) / \left( \min_{[a,b]} |f^{(r)}(x)|^{1/r} \right).$$

We will however need sharper estimates and will use the stationary phase analysis as given by Huxley [6] to analyze $\mathcal{J}$. Moreover, in the case when the stationary point lies far enough from the interval $[a, b]$, we use Lemma 8.1 of Blomer, Khan and Young [3] on stationary phase analysis to show that $\mathcal{J}$ is arbitrarily small. For completeness, we state the results here.

The following estimate are in terms of the parameters $\Theta_f, \Omega_f$ and $\Omega_g$ for which the derivatives satisfy

$$f^{(i)}(x) \ll \frac{\Theta_f}{\Omega_f^i}, \quad g^{(j)}(x) \ll \frac{1}{\Omega_g^j}.$$  

For the second assertion of the following lemma, we also require

$$f''(x) \geq \frac{\Theta_f}{\Omega_f^2}.$$

Lemma 4.1. Suppose $f$ and $g$ are real valued smooth functions satisfying (14) for $i = 2, 3$ and $j = 0, 1, 2$. Let $\Omega_f \gg (b - a)$.

1. Suppose $f'$ and $f''$ do not vanish on the interval $[a, b]$. Let $\Lambda = \min_{x \in [a, b]} |f'(x)|$. Then we have

$$\mathcal{J} = \frac{g(b)e(f(b))}{2\pi i f''(b)} - \frac{g(a)e(f(a))}{2\pi i f''(a)} + O\left( \frac{\Theta_f}{\Omega_f^2} \Lambda^2 \left( 1 + \frac{\Omega_f}{\Omega_g} + \frac{\Lambda}{\Omega_g^2 \Theta_f/\Omega_f} \right) \right).$$

2. Suppose that $f'(x)$ changes sign from negative to positive at $x = x_0$ with $a < x_0 < b$. Let $\kappa = \min\{b - x_0, x_0 - a\}$. Further suppose that bound in equation (14) holds for $i = 4$ and $5$. Then we have the following asymptotic expansion

$$\mathcal{J} = \frac{g(x_0)e(f(x_0) + 1/8)}{\sqrt{f''(x_0)}} + \frac{g(b)e(f(b))}{2\pi i f''(b)} - \frac{g(a)e(f(a))}{2\pi i f''(a)} + O\left( \frac{\Omega_f^4}{\Theta_f^3} \kappa^3 + \frac{\Omega_f^2}{\Theta_f^3/2} + \frac{\Omega_g^2}{\Theta_f^3/2} \right).$$

The above result is presented in Theorem 1 and Theorem 2 of [6]. We shall also use the following lemma from [3] when the unique point $x_0$ lies away from $(a, b)$.

Lemma 4.2. [3, Lemma 8.1] Let $\Theta_f \geq 1$, $\Omega_f, \Omega_g, \Lambda > 0$, and suppose that $f$ and $g$ are smooth real valued functions on the interval $[a, b]$, with $g$ supported on $[a, b]$. Let $f$ and $g$ satisfy (14) for $i = 2$ and $j = 0$. Moreover, let $\Lambda = \min_{x \in [a, b]} |f'(x)|$. Then $\mathcal{J}$ satisfies

$$\mathcal{J} \ll [b - a][\sqrt{(\Omega_f \Lambda/\sqrt{\Theta_f})}^{-A} + (\Lambda \Omega_g)^{-A}].$$

We shall also use the following estimates on exponential integrals in two variables. Let $f(x, y)$ and $g(x, y)$ be two real valued smooth functions on the rectangle $[a, b] \times [c, d]$. We consider the exponential integral in two variables given by

$$\int_a^b \int_c^d g(x, y)e(f(x, y))dx \, dy.$$

Suppose there exist parameters $p_1, p_2 > 0$ such that

$$\frac{\partial^2 f}{\partial x^2} \gg p_1^2, \quad \frac{\partial^2 f}{\partial y^2} \gg p_2^2, \quad \frac{\partial^2 f}{\partial x \partial y} \gg \left[ \frac{\partial^2 f}{\partial x \partial y} \right]^2 \gg p_1^2 p_2^2,$$

for all $x, y \in [a, b] \times [c, d]$. Then we have (See [18, Lemma 4])
\[ \int_a^b \int_c^d e(f(x,y))dxdy \leq \frac{1}{p_1p_2}. \]

Further suppose that \( \text{Supp}(g) \subset (a,b) \times (c,d) \). The total variation of \( g \) equals

\[ \text{var}(g) = \int_a^b \int_c^d \left| \frac{\partial^2 g(x,y)}{\partial x \partial y} \right| dxdy. \]

We have the following result (see [18, Lemma 5]).

**Lemma 4.3.** Let \( f \) and \( g \) be as above. Let \( f \) satisfies the conditions given in equation (16). Then we have

\[ \int_a^b \int_c^d g(x,y)e(f(x,y))dxdy \leq \frac{\text{var}(g)}{p_1p_2}, \]

with an absolute implied constant.

### 4.1. A Fourier-Mellin transform.

Let \( U \) be a smooth real valued function supported on the interval \([a,b] \subset (0,\infty)\) and satisfying \( U^{(j)} \ll_{a,b,j} 1 \). Let \( r \in \mathbb{R} \) and \( s = \sigma + i\beta \in \mathbb{C} \). We consider the following integral transform

\[ U^1(r,s) := \int_0^\infty U(x)e(-rx)x^{s-1}dx. \]

We are interested in the behaviour of this integral in terms of the parameters \( \beta \) and \( r \). The integral \( U^1(r,s) \) is of the form given in equation (12) with functions

\[ g(x) = U(x)x^{\sigma-1} \quad \text{and} \quad f(x) = \frac{1}{2\pi} \beta \log x - rx. \]

We shall use the following lemma.

**Lemma 4.4.** [13, Lemma 5] Let \( U \) be a smooth real valued function with \( \text{supp}(U) \subset [a,b] \subset (0,\infty) \) that satisfies \( U^{(j)}(x) \ll_{a,b,j} 1 \). Let \( r \in \mathbb{R} \) and \( s = \sigma + i\beta \in \mathbb{C} \). We have

\[ U^1(r,s) = \sqrt{2\pi}e(1/8) \frac{\beta^{i\beta}}{2\pi r} U_0\left(\sigma, \frac{\beta}{2\pi r}\right) + O_{a,b,\sigma} \left(\min\{|\beta|^{-3/2}, |r|^{-3/2}\}\right), \]

where \( U_0(\sigma,x) := x^\sigma U(x) \). Moreover, we have the bound

\[ U^1(r,s) = O_{a,b,\sigma,j} \left(\min\left\{\left(\frac{1 + |\beta|}{|r|}\right)^{j}, \left(\frac{1 + |r|}{|\beta|}\right)^{j}\right\}\right). \quad (17) \]

### 5. Application of circle method and dual summation formulas

In this paper, we present calculation for Hecke-Maass cusp forms, parallel to Munshi [13]. Let \( \pi \) be a Hecke-Maass cusp form for \( \text{GL}(3) \). We detect \( r = n \) by using the circle method in Lemma 1.2 to write \( S(N) = S_+(N) + S_-(N) \), where

\[ S^\pm(N) = \sum_{1 \leq q \leq Q < a} \sum_{\substack{1 \leq a < q \leq Q \leq \lambda(1,a)c \left(\frac{n}{q} - \frac{nx}{aq}\right) \left(\frac{n}{q} - \frac{nx}{aq}\right) V\left(\frac{n}{N}\right) dx. \]

We analyze \( S_+(N) \) and observe that the same bounds follow for \( S_-(N) \). We start with an application of dual summation formulas to the \( n \)-sum and the \( r \)-sum.
5.1. **Application of Poisson summation to the r-sum.** The r-sum in above is

\[ \sum_{r \neq 1} r^{-it} e \left( \frac{r\theta}{q} \right) e \left( -\frac{rx}{aq} \right) U \left( \frac{r}{N} \right). \]

Breaking the r-sum modulo q by changing variables \( r \to \beta + rq \), the above equals

\[ \sum_{r \in \mathbb{Z} \beta \mod q} (\beta + rq)^{-it} e \left( \frac{\beta \theta}{q} \right) e \left( -\frac{(\beta + rq)x}{aq} \right) U \left( \frac{\beta + rq}{N} \right). \]

Applying Poisson summation to the r-sum and changing variables, the above sum transforms into

\[ N^{1-it} \sum_{r \in \mathbb{Z}} \delta(\bar{\alpha} \equiv -r \mod q) \int_{\mathbb{R}} u^{-it} e \left( -\frac{Nu(ra - x)}{aq} \right) U(u) du = N^{1-it} \sum_{\pi \equiv -r \mod q} U^\dagger \left( \frac{N(ra - x)}{aq}, 1-it \right). \]

Repeated integration by parts to the \( u \)-integral gives arbitrary saving unless \( |r| \ll qt^{1+\varepsilon}/N \). The congruence condition determines \( a \mod q \). We further note that since \( (a, q) = 1 \), the congruence forces \( (r, q) = 1 \). Therefore \( r = 0 \) occurs only for \( q = 1 \), the contribution of which is negligible. For non-zero \( r \) to appear, we need \( q > N/t^{1-\varepsilon} \). From now on, we assume \( Q > N/t^{1-\varepsilon} \).

5.2. **Application of Voronoi formula to the n-sum.** The n-sum in \( S^+(N) \) is

\[ \sum_{n \geq 1} \lambda(1, n) e \left( \frac{nr}{q} \right) e \left( \frac{nx}{aq} \right) V \left( \frac{n}{N} \right). \]

Application of Voronoi summation formula as given in Lemma 3.3 transforms the above sum into

\[ q \sum_{\pm} \sum_{n_1 | q} \lambda(n_2, n_1) S(\tau, \pm n_2; q/n_1) J_\pm \left( \frac{n_2^2 n_2}{q^3}, x/aq \right), \]

where

\[ J_\pm \left( \frac{n_2^2 n_2}{q^3}, x/aq \right) = \frac{1}{2\pi i} \int_{(\sigma)} \left( \frac{n_2^2 n_2 N}{q^3} \right)^{-s} \gamma_\pm(s) V^\dagger(Nx/aq, -s) ds. \]

Let \( s = \sigma + i\tau \). Using Stirling’s approximation

\[ \gamma_\pm(s) \ll 1 + |\tau|^{3\sigma + 3/2} \]

for \( \sigma \geq -1/2 \). Moreover, the bound (17) of Lemma 4.4 implies

\[ V^\dagger(Nx/aq, -s) \ll_{\sigma, j} \min \left\{ 1, \left( 1 + \frac{|Nx/aq|}{|\tau|} \right)^j, \left( \frac{1 + |\tau|}{|Nx/aq|} \right)^j \right\}. \]

(18)

Shifting the line of integration to \( \sigma = M \) for large \( M \) and taking \( j = 3M + 3 \), we get arbitrary saving for \( n_1^2 n_2^2 \gg q^3(N/aq)^{3j}/N \sim N^2 t'/Q^3 \). For the smaller values of \( n_1^2 n_2^2 \), we move the contour to \( \sigma = -1/2 \) to write

\[ J_\pm \left( \frac{n_2^2 n_2}{q^3}, x/aq \right) = \frac{1}{2\pi} \left( \frac{n_2^2 n_2 N}{q^3} \right)^{1/2} \int_{\mathbb{R}} \left( \frac{n_2^2 n_2 N}{q^3} \right)^{-i\tau} \gamma_\pm(-1/2 + i\tau) V^\dagger(Nx/aq, 1/2 - i\tau) d\tau. \]

Due to the bounds on \( V^\dagger(Nx/aq, 1/2 - i\tau) \), we get arbitrary saving for \( |\tau| \gg N t'/aq \). Moreover, for \( 0 \leq x < aq t'/N \), we get arbitrary saving for \( |\tau| \gg t'^2 \) and for \( aq t'/N \leq x \leq 1 \), we get arbitrary saving for \( |\tau| < 1 \). These observations will be used later to effectively bound certain error terms. We smoothen the \( \tau \)-integral by introducing a partition of unity like Munshi [13]. Let \( J \) be a collection of \( O(\log t) \) many real numbers in the interval \([-N t'/aq, N t'/aq] \), containing 0. For each \( J \in \mathcal{J} \), we have a smooth function \( W_J(x) \) satisfying \( x^k W_J^{(k)}(x) \ll 1 \) for \( k \geq 0 \). Moreover, \( W_0(x) \) is supported in \([-1, 1]\) and satisfies the stronger bound \( W_0^{(k)}(x) \ll 1 \). For each \( J > 0 \) (resp. \( J < 0 \), \( W_J \) is supported in \([J, 4J/3] \) (resp. \([4J/3, J] \)). Finally, we require that

\[ \sum_{J \in \mathcal{J}} W_J(x) = 1 \text{ for } x \in [-N t'/aq, N t'/aq]. \]
The precise definition of $W_J$ is not needed. We break the $q$-sum into dyadic segments $C \leq q < 2C$ with $N/t^{1-\epsilon} \leq C \ll Q$ to write

$$S^+(N) = N \sum_{\substack{N/t^{1-\epsilon} \leq C \ll Q \ dyadic}} S(N, C) + O(t^{-2019})$$

where $S(N, C)$ is the following expression obtained after the above applications of dual summation formulas

$$S(N, C) = \frac{N^{1/2-it}}{2\pi} \sum_{\substack{J \in J \ n_1^2 n_2 \leq N^{2+e}/Q^2}} \sum_{\substack{n_2 \leq N/2}} \lambda(n_2, n_1) \frac{1}{n_2^{1/2}} \sum_{\substack{C < q \leq 2C, (r, q) = 1 \ \text{dyadic}}} \sum_{1 \leq |q| \leq q^{1+e}/N} S(\tau, \pm n_2; q/n_1) \frac{S(q, r, n_1^2 n_2)}{aq^{3/2}},$$

$$J_\pm(q, r, n_1^2 n_2) = \int_{|\tau| \leq N^{1/2}C} \left( \frac{n_1^2 n_2 N}{q^3} \right)^{-it} \gamma_\pm(-1/2 + it) W_J(\tau) J_\pm(q, r, \tau) d\tau,$$

and

$$J_\pm(q, r, \tau) = \int_{0}^{1} V^1(N x/aq, 1/2 - it) V^0 \left( \frac{N(ra - x)}{aq}, 1 - it \right) dx.$$ 

Next, we analyze the above integrals using stationary phase analysis.

6. Analysis of the integrals

Using Lemma 4.4, we get the asymptotic estimate

$$U^1 \left( \frac{N(ra - x)}{aq}, 1 - it \right) = \frac{\sqrt{2\pi e(1/8)}}{\sqrt{t}} \left( \frac{-taq}{2\pi e N(ra - x)} \right)^{-it} V_0 \left( 1, \frac{-taq}{2\pi e N(ra - x)} \right) + O(t^{-3/2+\epsilon}).$$

We recall that we get arbitrary saving for $0 < x < aqt^{x}/N$ and $|\tau| > t^{2\epsilon}$. Using the above asymptotic and the bound (18) for $V^1(N x/aq, 1/2 - it)$ in this range, we get

$$\int_{0}^{aqt^{x}/N} V^1(N x/aq, 1/2 - it) U^1 \left( \frac{N(ra - x)}{aq}, 1 - it \right) dx \ll \frac{QCt^{x}}{N^{1/2}} \left( \frac{t^x}{1 + |\tau|} \right)^j.$$ 

For $aqt^{x}/N < x \leq 1$ (so that $|\tau| > t^{x/2}$, otherwise we get arbitrary saving), we use Lemma 4.4 to write

$$V^1 \left( \frac{N x}{aq}, 1/2 - it \right) = \frac{\sqrt{2\pi e(1/8)}}{\sqrt{t}} \left( \frac{-taq}{2\pi e N x} \right)^{-it} V_0 \left( \frac{1}{2}, \frac{-taq}{2\pi e N x} \right) + O \left( \min \left\{ \left| \frac{N x}{aq} \right|^{-3/2}, |\tau|^{-3/2} \right\} \right).$$

Therefore for an absolute constant $c_1$,

$$J_\pm(q, r, \tau) = \frac{c_1}{\sqrt{\tau}} \left( \frac{-taq}{2\pi e N} \right)^{-it} \left( \frac{-taq}{2\pi e N} \right)^{-it} \int_{aqt^{x}/N}^{1} (ra - x)^{it} x^{it} U_0 \left( 1, \frac{-taq}{2\pi e N (ra - x)} \right) V_0 \left( \frac{1}{2}, \frac{-taq}{2\pi e N x} \right) dx$$

$$+ O \left( E^{**} + t^{-3/2+\epsilon} + \frac{QCt^{x}}{N^{1/2}} \left( \frac{t^x}{1 + |\tau|} \right)^j \right), \quad (19)$$

where

$$E^{**} = \frac{1}{t^{3/2}} \int_{aqt^{x}/N}^{1} \min \left\{ \left| \frac{N x}{aq} \right|^{-3/2}, |\tau|^{-3/2} \right\} dx.$$
and the main term and the error term \( E^{**} \) occur only for \( |\tau| > 1 \). We observe that \( E^{**} \) can be bounded as

\[
E^{**} = \left( 1 - \frac{1}{t^{1/2}} \right) \int_{|\tau| < 1} |x|^{-3/2} dx + \left( 1 - \frac{1}{t^{1/2}} \right) \int_{|\tau| < 1} \left( \frac{Nt}{a} \right)^{-3/2} \delta(|\tau|a |N < 1) dx
\]

\[
\ll \left( 1 - \frac{1}{t^{1/2}} \right) \min \left\{ 1, \left( \frac{|\tau|N}{a} \right) \right\} + \left( 1 - \frac{1}{t^{1/2}} \right) \frac{|\tau|N}{a} \delta(|\tau|a |N < 1) \ll \left( 1 - \frac{1}{t^{1/2}} \right) \min \left\{ 1, \left( \frac{|\tau|N}{a} \right) \right\} .
\]

6.1. Analysis of the \( x \)-integral. To analyze the integral in (19), we set \( I = \int_{|\tau|_t |N < 1} e(f(x)) g(x) dx \) with

\[
f(x) = \frac{t \log(ra - x) + \tau \log x}{2\pi},
\]

\[
g(x) = U_0 \left( \frac{1}{2}, \frac{-taq}{2\pi N(ra - x)} \right) V_0 \left( \frac{1}{2}, \frac{-taq}{2\pi Nx} \right).
\]

Due to the bounds \( x^j U^{(j)}(x) \ll_j 1 \) and \( x^j V^{(j)}(x) \ll_j 1 \), and the definitions of \( U_0 \) and \( V_0 \) as given in Lemma 4.4, we have

\[
g^{(j)}(x) \ll_j 1.
\]

Since \( \text{supp} V(t, 1, 2) \subseteq [1, 2] \), the support of \( g(x) \) lies inside \([-\tau aq/4\pi N, -\tau aq/2\pi N]\). This lies inside \([aq t^3/N, 1]\) for \(-\tau \in [4t^3, 2\pi N/aq]\), while \( g(x) = 0 \) for \( x \in [aq t^3/N, 1] \) and \(-\tau \geq 4\pi N/aq\). For \(-\tau < 4\pi t^3\), we use the second derivative test (13). For \(-\tau \in [2\pi aq/4\pi N, aq t^3/aq]\), we need to be a little more careful in the analysis of integrals since the \( r \)-th derivative test (13) is not sufficient. For \( j \geq 1 \),

\[
2\pi f^{(j)}(x) = \frac{-t(j - 1)!}{(ra - x)^j} + \frac{\tau(j - 1)!}{x^j}.
\]

In the support of the integral, \( x - ra \sim taq/2\pi N \) and \( x \sim -\tau aq/2\pi N \), where \( a \sim b \) means \( k_1 < a/b < k_2 \) for some constants \( k_1, k_2 > 0 \). Since \( Q > N/t^{1-\tau} \), we have

\[
f^{(j)}(x) \sim -\tau N/\tau aq \] for \( j \geq 2 \)

(20)

where \( a \sim_j b \) means that the constants \( k_1, k_2 \) depend on \( j \). Therefore \( |f''(x)| \sim -\tau(N/\tau QC)^2 \). The point \( x_0 \) where \( f'(x_0) = 0 \) is called a stationary point. We have

\[
x_0 = \frac{ra \tau}{\tau + t}.
\]

We recall that due to (18), we have \( I = O(t^{-2020}) \) for \( x > aq t^3/N \) and \( |\tau| < t^{1/2} \). For \(-\tau \in [t^{1/2}, 4\pi t^3]\), we apply the second derivative bound (13) and the estimate (20) for \( j = 2 \) along with the observation that \( \text{Var}_{x \in [aq t^3/N, 1]} g(x) \ll t^\epsilon \) to bound

\[
I \ll t^{Q/C t^\epsilon}/N.
\]

(21)

For larger \( |\tau| \), we start by writing the Taylor expansion of \( f'(x) \) around \( x = x_0 \),

\[
f'(x) = (x - x_0) f''(x_0) + O \left( (x - x_0)^2 \tau \left( \frac{N}{\tau QC} \right)^3 \right).
\]

(22)

The error term follows from the estimate (20) for \( j \geq 3 \).

We now analyze the case \(-\tau \in [4\pi t^3, 2\pi N/aq] \), for which \( g(aq t^3/N) = g(1) = 0 \). We can therefore change the limits of the integral \( I \) to write

\[
I = \int_{-\tau aq/4\pi N}^{\tau aq/2\pi N} g(x)e(f(x)) dx.
\]

In case \( x_0 \) lies inside the interval \([-\tau aq/4\pi N, -\tau aq/2\pi N]\), one can expand the interval of integration to \([-\tau aq/8\pi N, -\tau aq/\pi N]\) without changing \( I \). Then \( \kappa = \min\{x_0 + \tau aq/8\pi N, -\tau aq/\pi N - x_0\} \gg |\tau QC/N| \).

Applying the second statement of Lemma 4.1 with

\[
\Theta_f = |\tau|, \quad \Omega_f = |\tau QC/N|, \quad \kappa = |\tau QC/N|, \quad \Omega_g = 1,
\]
so that the hypothesis $\Omega_f \gg (b - a)$ of Lemma 4.1 is satisfied, we obtain

$$J = \frac{g(x_0)e(f(x_0) + 1/8)}{\sqrt{f''(x_0)}} + O\left(\frac{QC}{|\tau|^{1/2}N}\right).$$  \hfill (23)

In case $x_0$ does not lie inside the interval $I = [-\tau q/4\pi N, -\tau q/2\pi N]$, (22) implies

$$\Lambda = \min_{x \in I} |f'(x)| \sim \min |x - x_0|N^2/|\tau|Q^2C^2.$$

In the case $\min_{x \in I} |x - x_0| > t^*|\tau|QC/N$, we use the above estimate for $\min_{x \in I} |f'(x)|$ and apply Lemma 4.2 to obtain $J = O(t^{-2020})$. While in the case $\min_{x \in I} |x - x_0| < t^*\sqrt{|\tau|QC}/N$, we expand the interval $I$ to $[-\tau q/8\pi N, -\tau q/\pi N]$ without changing $J$, so that $x_0$ now lies in the expanded interval with $\kappa \gg |\tau|QC/N$. The analysis now is the same as in the previous case. Putting together the estimates (21) and (23), we obtain that for $1 \leq |\tau| \leq 2\pi N/aq$,

$$J = \frac{g(x_0)e(f(x_0) + 1/8)}{\sqrt{f''(x_0)}} + O\left(\frac{QC}{|\tau|^{1/2}N}\right).$$  \hfill (24)

Next we analyze the case $-\tau \in [2\pi N/aq, 4\pi N/aq]$. We start by observing that in this case, $g(aqt^*/N) = 0$ but $g(1) \neq 0$. We will therefore have to divide our analysis depending on the size of $f'(1)$. In view of (22), this translates to the size of $|x - x_0|$. Let $\kappa := x_0 - 1$ (so that $x_0$ is outside $[aqt^*/N, 1]$ if $\kappa > 0$). Since $x_0 = r\tau/(l + \tau)$ and $ra = -taq/N$ (due to the support of $U_0(1, x)$ being in $x \in [1/2, 5/2]$), we have $x_0 \sim -aq\tau/N$. Therefore $|x_0 - aqt^*/N| \gg 1$.

We observe that $x_0 = r\tau/(l + \tau)$ implies $\tau = x_0/(ra - x_0)$. In particular, $x_0 = 1$ for $\tau_0 := t/(ra - 1)$. Since $x_0 = 1 + \kappa$ by definition,

$$\tau = \frac{(1 + \kappa)t}{ra - (1 + \kappa)} = \left(\frac{1 + \kappa}{ra - 1}\right)O\left(\frac{(1 + \kappa)t\kappa}{(ra - 1)^2}\right) = \tau_0 + \tau_0\kappa\left(1 + O\left(\frac{N}{tQC}\right)\right).$$  \hfill (25)

Let $\kappa_0 := t^*\sqrt{QC/N}$. If $|\kappa| < \kappa_0$, we apply the second derivative bound (13) and the estimate (20) for $j = 2$ along with the observation that $\text{Var}_{x \in [aqt^*/N, 1]} g(x) \ll t^*$ to obtain

$$J \ll \frac{QC\sqrt{|\tau|t^*}}{N}.$$

Since $|\tau| \sim N/QC$, $g(x) \ll 1$ and $f''(x) \sim N^2/Q^2C^2|\tau|$ in the support of $g(x)$, the above bound can be replaced by

$$J \ll \frac{g(x_0)e(f(x_0) + 1/8)}{\sqrt{f''(x_0)}} + t^*\sqrt{\frac{QC}{N}}.$$

Next, if $\kappa > \kappa_0$ (so that $x_0 > 1 + \kappa_0$ and therefore does not lie in $I = [aqt^*/N, 1]$), we have $\min_{x \in I} |f'(x)| = N\kappa/QC$ in view of (22). Applying the first statement of Lemma 4.1 with

$$\Lambda = \frac{N\kappa}{QC}, \quad \Theta_f = \frac{N}{QC}, \quad \Omega_f = \Omega_g = 1,$$

along with the observations $\kappa > \kappa_0 = t^*\sqrt{QC/N}$ and $|f'(1)| \gg N\kappa/QC$, we obtain

$$J \ll \frac{QC}{N\kappa}.$$  \hfill (27)

Finally, when $\kappa < -\kappa_0$ (so that $x_0$ lies inside $I = [aqt^*/N, 1]$), we apply the second statement of Lemma 4.1 with $\Theta_f = N/QC$, $\Omega_f = \Omega_g = 1$, and use $|\kappa| > t^*\sqrt{QC/N}$ and $|f'(1)| \gg N\kappa/QC$ to obtain

$$J = \frac{g(x_0)e(f(x_0) + 1/8)}{\sqrt{f''(x_0)}} + O\left(\frac{QC}{N|\kappa|}\right).$$  \hfill (28)

Putting together (26), (27) and (28), we obtain that for $-\tau \in [2\pi N/aq, 4\pi N/aq]$,

$$J = \frac{g(x_0)e(f(x_0) + 1/8)}{\sqrt{f''(x_0)}} + O\left(\min\left\{\frac{QC}{N|\kappa|}, \kappa_0\right\}\right).$$  \hfill (29)

In view of the change of variable (25), the estimate $N/tQC = o(1)$ and $\tau_0 \sim N/QC$, we can replace the above error term by $O(\min(|\tau - \tau_0|^{-1}, \kappa_0))$. 

A NEW SUBCONVEX BOUND FOR GL(3) L-FUNCTIONS IN THE t-ASPECT 11
Putting together the estimates (24) and (29), we obtain that for \(1 \leq |\tau| \ll Nt'/QC\),
\[
J = \frac{g(x_0)q(f(x_0) + 1/8)}{\sqrt{f''(x_0)}} + O\left(\frac{QC}{|\tau|^{1/2}N} + \delta(|\tau| \sim N/QC) \min\{|\tau - \tau_0|^{-1}, \kappa_0\}\right).
\]

Explicit computations show that
\[
2\pi f(x_0) = t \log(t/\tau) + (t + \tau) \log \left(\frac{ra}{t + \tau}\right)
\]
\[
2\pi f''(x_0) = \frac{(t + \tau)^3}{t \tau^2 a^2}, \quad \text{and}
\]
\[
g(x_0) = U_0 \left(1, \frac{-q(t + \tau)}{2\pi Nr}\right) V_0 \left(\frac{1}{2}, \frac{-q(t + \tau)}{2\pi Nr}\right) = V_0 \left(\frac{3}{2}, \frac{-q(t + \tau)}{2\pi Nr}\right).
\]

We recall that \(U(x) \equiv V(x)\), therefore \(U_0(r_1, x) V_0(r_2, x) = V_0(r_1 + r_2, x)\). The above calculations therefore show that the expression in (19) is asymptotic to
\[
c_2 \frac{ra}{(t + \tau)^{3/2}} \left(-\frac{(t + \tau)q}{2\pi e Nr}\right)^{-i(t + \tau)} V_0 \left(\frac{3}{2}, \frac{-q(t + \tau)}{2\pi Nr}\right),
\]
for some constant \(c_2\). We collect these calculations in the following lemma.

**Lemma 6.1.** Let \(N/t^{1-\varepsilon} < Q\). Then
\[
J^{**}(q, r, \tau) = J_1(q, r, \tau) + J_2(q, r, \tau)
\]
with,
\[
J_1(q, r, \tau) = c_2 \frac{ra}{(t + \tau)^{3/2}} \left(-\frac{(t + \tau)q}{2\pi e Nr}\right)^{-i(t + \tau)} V_0 \left(\frac{3}{2}, \frac{-q(t + \tau)}{2\pi Nr}\right),
\]
and \(J_2(q, r, \tau) = O(B(C, \tau))\) where
\[
B(C, \tau) = \frac{QC}{Nt^{1/2}} \left(\frac{t'}{1 + |\tau|}\right)^{10} + t^{-3/2+\varepsilon} + \left(E^{**} + \frac{QC}{t^{1/2} |\tau| N}\right) \delta(|\tau| > 1)
\]
\[
+ \delta(|\tau| \sim N/QC) \frac{1}{t^{1/2} |\tau|^{1/2}} \min\{|\tau - \tau_0|^{-1}, \kappa_0\}.
\]

We observe that since \(N/t^{1-\varepsilon} < Q < N^{1/2}\),
\[
\int_{|\tau| \ll Nt'^{1/2}/QC} B(C, \tau) \ll t' \sqrt{QC/Nt}.
\]

**Remark 6.2.** We observe that \(J_1(q, r, \tau) \ll QT^{1/2}/N^{1/2}\). This is slightly better than the corresponding trivial bound of \(1/t^{1/2} K\) obtained by Munshi in Lemma 8 of [13]. This comparison is made by substituting \(Q = (N/K)^{1/2}\). This will finally help us to improve upon Munshi’s bound.

Using Lemma 6.1, we have the following decomposition of \(S(N,C)\).

**Lemma 6.3.**
\[
S(N, C) = \sum_{J \in \mathcal{J}} S_{1,J}(N, C) + S_{2,J}(N, C)
\]
where
\[
S_{t,J}(N, C) = \frac{N^{1/2-it}}{2\pi} \sum_{n_1, n_2} \sum_{n_1^2 n_2 < N^{2t'/Q^3}} \frac{\lambda(n_2, n_1)}{n_2} \sum_{C < q \leq 2C(t, \tau)} \sum_{1 \leq |\tau| \leq q^{t'/N}} \frac{S(\tau, \pm n_2; q/n_1)}{aq^{3/2}} J_{t,J,(q, r, n_1^2 n_2)}
\]
and
\[
J_{t,J,(q, r, n)} = \int_{\mathbb{R}} \left(\frac{n N}{q^3}\right)^{-i\tau} \gamma_{\pm}(-1/2 + i\tau) J_{t}(q, r, \tau) W_{J}(\tau) d\tau,
\]
where \(J_{t}(q, r, \tau)\) is defined in Lemma 6.1.
7. Cauchy-Schwarz and Poisson summation- First Application

In this section, we analyze

\[ S_2(N, C) := \sum_{j \in J} S_{2,j}(N, C). \]

Breaking the \( n \)-sum into dyadic segments of length \( L \)

\[
S_2(N, C) \ll t' N^{1/2} \sum_{|r| < N^{1/2}} \sum_{dyadic} \sum_{1 \leq L \leq N^{2t'}/Q^3} \left| \lambda(n_2, n_1) \right| U \left( \frac{n_2^2}{L} \right) \times \left| \sum_{C < q \leq 2C, (r,q) = 1} \sum_{1 \leq |r| < q^{1+\epsilon}/N_{n_1/q}} \frac{S(r, \pm n_2; q/n_1)}{aq^{3/2-3\epsilon}} J_2(q, r, \tau) \right| d\tau.
\]

Next we apply Cauchy-Schwarz inequality with the \( n_1, n_2 \)-sums outside and apply the Ramanujan bound on average (Lemma 3.4) to write

\[
S_2(N, C) \ll t' N^{1/2} \sum_{|r| < N^{1/2}} \sum_{dyadic} L^{1/2} [S_{2,\pm}(N, C, L, \tau)]^{1/2} d\tau,
\]

where

\[
S_{2,\pm}(N, C, L, \tau) = \sum_{n_1, n_2} \frac{1}{n_2} U \left( \frac{n_2^2}{L} \right) \left| \sum_{C < q \leq 2C, (r,q) = 1} \sum_{1 \leq |r| < q^{1+\epsilon}/N_{n_1/q}} \frac{S(r, \pm n_2; q/n_1)}{aq^{3/2-3\epsilon}} J_2(q, r, \tau) \right|^2.
\]

The analysis of \( S_{2,-}(N, C, L, \tau) \) is similar to that of \( S_{2,+}(N, C, L, \tau) \), so we consider only \( S_{2,+}(N, C, L, \tau) \). Opening the absolute value squared the expression of \( S_{2,+}(N, C, L, \tau) \) is

\[
\sum_{n_1 \in 2C} \sum_{C < q_1 \leq 2C, (r_1, q_1) = 1} \sum_{1 \leq |r_1| < q_1^{1+\epsilon}/N_{n_1/q_1}} \sum_{n_2 \in 2C} \sum_{C < q_2 \leq 2C, (r_2, q_2) = 1} \sum_{1 \leq |r_2| < q_2^{1+\epsilon}/N_{n_1/q_2}} \frac{1}{a q_1 q_2^{3/2-3\epsilon}} J_2(q_1, r_1, \tau) J_2(q_2, r_2, \tau) \tau
\]

where we temporarily set \( \tau \) to include the \( n_2 \)-sum

\[
\tau = \sum_{n_2} \frac{1}{n_2} U \left( \frac{n_2^2}{L} \right) S(r_1, n_2, q_1/n_1) S(r_2, n_2, q_2/n_1).
\]

Let \( \tilde{q}_1 = q_1/n_1 \) and \( \tilde{q}_2 = q_2/n_1 \). Breaking the \( n_2 \)-sum modulo \( \tilde{q}_1 \tilde{q}_2 \) and applying Poisson summation formula as Munshi does, \( \tau \) transforms into

\[
\tau = \frac{n_1^2}{\tilde{q}_1 \tilde{q}_2} \sum_{\beta \mod \tilde{q}_1 \tilde{q}_2} \mathcal{C} U^\dagger(n_2L/q_1q_2, 0),
\]

where \( \mathcal{C} \) is the sum

\[
\mathcal{C} = \sum_{\beta \mod \tilde{q}_1 \tilde{q}_2} S(r_1, \beta, q_1/n_1) S(r_2, \beta, q_2/n_1) e(\beta n_2/\tilde{q}_1 \tilde{q}_2).
\]

Bounds on \( U^\dagger(n_2L/q_1q_2, 0) \) give arbitrary saving for \( |n_2| \gg C^2 t'/L \). Recalling that \( a \) is of size \( Q \), the expression of \( S_{2,+}(N, C, L, \tau) \) in (32) is bounded by

\[
\frac{B(C, \tau)^2}{Q^2 C^2} \sum_{n_1 \in 2C} \sum_{C < q_1 \leq 2C, (r_1, q_1) = 1} \sum_{1 \leq |r_1| < q_1^{1+\epsilon}/N_{n_1/q_1}} \sum_{n_2 \in 2C} \sum_{C < q_2 \leq 2C, (r_2, q_2) = 1} \sum_{1 \leq |r_2| < q_2^{1+\epsilon}/N_{n_1/q_2}} n_2^2 \sum_{|n_2| \leq C^2 t'/L} |\mathcal{C}| + O(t^{-2019}).
\]

We use Lemma 13 of [13] to bound the character sum \( \mathcal{C} \). For completeness, we state it here.
Lemma 7.1. We have
\[ \mathcal{C} \ll q_1^2(\hat{q}_1, r_1 - r_2). \]
Moreover for \( n_2 = 0 \), we get that \( \mathcal{C} = 0 \) unless \( \hat{q}_1 = \hat{q}_2 \), in which case
\[ \mathcal{C} \ll q_1^2(\hat{q}_1, r_1 - r_2). \]

7.1. Diagonal contribution. We first consider the contribution of \( n_2 = 0 \), which we denote by \( S_{2,+}^1(N, C, L, \tau) \). Using the second statement of Lemma 7.1

\[ S_{2,+}^1(N, C, L, \tau) \ll \frac{B(C, \tau)^2}{Q^2C^5} \sum_{n_1 \leq 2C} \left( \sum_{C < q_1 < 2C} \sum_{1 \leq |r_1| < \pi q^1 t^{1+\epsilon}/N} \frac{C^5 \tau t + C^5 q^2}{n_1^3 N} \right), \]

where the first term is the contribution from terms with \( r_1 = r_2 \) and the second term is the contribution from \( r_1 \neq r_2 \). Since we will choose \( N > t^{1+\epsilon} \), the first term dominates and we get

\[ S_{2,+}^1(N, C, L, \tau) \ll \frac{t^{1+\epsilon} B(C, \tau)^2}{N^2} \] \quad (34)

7.2. Off-diagonal contribution. We now bound the contribution of terms with \( n_2 \neq 0 \), which we denote by \( S_{2,+}^2(N, C, L, \tau) \). Using the first statement of Lemma 7.1,

\[ S_{2,+}^2(N, C, L, \tau) \ll \frac{B(C, \tau)^2}{Q^2C^5} \sum_{n_1 \leq 2C} \left( \sum_{C < q_1 < 2C} \sum_{1 \leq |r_1| < \pi q^1 t^{1+\epsilon}/N} \frac{n_1^2}{n_1^3 N} \right) \]

\[ \ll \frac{t^{2+\epsilon} C^3}{Q^2 N^2 L} B(C, \tau)^2. \] \quad (35)

7.3. Estimating \( S_2(N, C) \). Using the bounds (34) and (35) in (31),

\[ S_2(N, C) \ll t^{1/2} \int_{|\tau| < \frac{Q^1}{6}} \frac{B(C, \tau) \sum \sum_{dyadic \ n_1 \leq 2C} \left( \frac{L^{1/2} t^{1/2}}{N^{1/2} Q} + \frac{t C^{3/2}}{N Q} \right) d\tau. \]

Using the estimate (30) in above

\[ S_2(N, C) \ll t^{1/2} C^{1/2} \left( \frac{N^{1/2}}{Q^{5/2}} + \frac{t^{1/2} C^{3/2}}{N Q} \right). \]

Multiplying by \( N \) and summing over \( C \ll Q \) dyadically, the contribution of \( S_2(N, C) \) towards \( S^+(\mathcal{N}) \) is

\[ t^{1/2} \frac{N^{1/2}}{Q^{5/2}} + \frac{t^{1/2} C^{1/2}}{N}. \] \quad (36)

8. Cauchy-Schwarz and Poisson summation - Second application

We now analyze the sum \( S_{1,J}(N, C) \). We need to get further cancellations in the \( \tau \)-integral to get a subconvexity bound. For notational simplicity, we consider only the positive \( J \) with \( -J \gg t' \). The same analysis holds for negative values of \( J \) with \(-J \gg t' \). For \( |J| \ll t' \), the analysis is done as before since we do not need to get cancellation in the \( \tau \)-integral. We recall that

\[ S_{1,J}(N, C) = \frac{N^{1/2 - it}}{2\pi} \sum_{n_1} \sum_{n_2 \ll N^{\epsilon}} \frac{\lambda(n_2, n_1)}{n_1^{1/2}} \sum_{C < q_1 < 2C} \sum_{1 \leq |r_1| < \pi q^1 t^{1+\epsilon}/N} \frac{S(\tau, \pm n_2; q/n_1)}{aq^3/2} \mathcal{J}_{1, \pm, J}(q, r, n_1^2 n_2) \]

where

\[ \mathcal{J}_{1, \pm, J}(q, r, n) = \int_{\mathbb{R}} \left( \frac{n_1^2 n_2 N}{q^3} \right)^{-it} \gamma_{\pm}(-1/2 + i\tau) \mathcal{J}_1(q, r, \tau)W_J(\tau) d\tau. \]
Taking absolute values while keeping the $\tau$-integral inside,
\[
S_{1,J}(N, C) \ll t^e N^{1/2} \sum_{\pm 1 \leq L \leq N^{2t^e}/Q^3} \sum_{n_1, n_2} \frac{\lambda(n_2, n_1)}{n_2^{1/2}} U\left(\frac{n_1^2 n_2}{L}\right) \times \left| \int_{\mathbb{R}} (n_1^2 n_2 N)^{-i\tau} \gamma_+ (-1/2 + i\tau) \sum_{C \leq q \leq 2C, (r, q) = 1} \frac{S(\tau, n_2; q/n_1)}{aq^{3/2 - 3i\tau}} J_1(q, r, \tau) W_j(\tau) d\tau \right|.
\]

Applying Cauchy-Schwarz inequality by pulling the $n_1, n_2$-sums outside along with the Ramanujan bound on average (Lemma 3.4),
\[
S_{1,J}(N, C) \ll t^e N^{1/2} \sum_{\pm 1 \leq L \leq N^{2t^e}/Q^3} L^{1/2} [S_{1, \pm, J}(N, C, L)]^{1/2}
\]
where
\[
S_{1, \pm, J}(N, C, L) = \sum_{n_1, n_2} \frac{1}{n_2} U\left(\frac{n_1^2 n_2}{L}\right) \left| \int_{\mathbb{R}} (n_1^2 n_2 N)^{-i\tau} \gamma_+ (-1/2 + i\tau) \sum_{C \leq q \leq 2C, (r, q) = 1} \frac{S(\tau, n_2; q/n_1)}{aq^{3/2 - 3i\tau}} J_1(q, r, \tau) W_j(\tau) d\tau \right|^2.
\]

Like earlier, we consider only $S_{1, +, J}(N, C, L)$. Opening absolute value squared,
\[
S_{1, +, J}(N, C, L) = \sum_{n_1 \leq 2C} \int_{\mathbb{R}^2} (n_1^2 N)^{-i(\tau_2 - \tau_1)} \gamma_+ (-1/2 + i\tau_1) \gamma_+ (-1/2 + i\tau_2) W_j(\tau_1) W_j(\tau_2)
\]
\[
\times \sum_{C \leq q \leq 2C, (r_2, q_2) = 1} \frac{1}{q_1 q_2} \sum_{1 \leq |r_1| \leq C t^{1+e}/N} \sum_{n_1 | q_1} \sum_{n_2 | q_2} 1 a_1 a_2 q_1^{3/2 - 3i\tau_1} q_2^{3/2 - 3i\tau_2} J_1(q_1, r_1, \tau_1) J_1(q_2, r_2, \tau_2) T d\tau_1 d\tau_2,
\]
where we temporarily set
\[
T = \sum_{n_2 \leq 2C} \frac{1}{n_2} U\left(\frac{n_1^2 n_2}{L}\right) S(\tau_1, n_2; q_1/n_1) S(\tau_2, n_2; q_2/n_1).
\]

Like before, breaking the $n_2$-sum mod $q_1 q_2 / n^2$ and applying Poisson summation to it
\[
T = \frac{n_1^2}{q_1 q_2} \left(\frac{L}{n_1^2}\right)^{-i(\tau_2 - \tau_1)} \sum_{n_2 | q_2} c U^\dagger(n_2 L / q_1 q_2, i(\tau_2 - \tau_1)).
\]

Here $c$ is the character sum as given in (33). Since $|\tau_1| \ll N t^e / CQ$, $U^\dagger$ gives arbitrary saving for $|n_2| \gg N C t^{e}/QL$. Recalling that $a \sim Q$, $S_{1, +, J}(N, C, L)$ is bounded by
\[
\frac{t^e}{Q^2 C^2} \sum_{n_1 \leq 2C} \sum_{C \leq q \leq 2C, (r_1, q_1) = 1} \sum_{1 \leq |r_1| \leq C t^{1+e}/N} \sum_{n_1 | q_1} n_1^2 \sum_{|n_2| \ll N C t^{e}/QL} |c| |\mathfrak{K}| + O(t^{-2019}),
\]
where
\[
\mathfrak{K} = \int_{\mathbb{R}^2} (\gamma_+(-1/2 + i\tau_1) \gamma_+(-1/2 + i\tau_2) W_j(\tau_1) W_j(\tau_2) \frac{(LN)^i(\tau_2 - \tau_1)}{q_1^{3i\tau_1} q_2^{3i\tau_2}}
\]
\[
\times J_1(q_1, r_1, \tau_1) J_1(q_2, r_2, \tau_2) U^\dagger(n_2 L / q_1 q_2, i(\tau_2 - \tau_1)) d\tau_1 d\tau_2.
\]
8.1. Analysis of the integral $\Phi$. Using the expression for $\mathcal{I}_1(q, r, \tau)$ as given in Lemma 6.1, we have

$$\Phi = c_2 \int_{\mathbb{R}^2} r_1 r_2 q_1 q_2 \frac{r_1 q_1}{t^2} \gamma_+(-1/2 + i r_1) \gamma_+(-1/2 + i r_2) W_f(q_1, r_1, \tau) W_f(q_2, r_2, \tau) \frac{e^{(r_2 - r_1) i}}{q_1^{3r_1} q_2^{3r_2}}$$

$$\times \left( \frac{-(t + \tau) q_1}{2\pi e N r_1} \right)^{-i(t + \tau)} \left( \frac{-(t + \tau) q_2}{2\pi e N r_2} \right)^{-i(t + \tau)} U^n(2L/t, q \tau - \tau, -r_2)$$

where

$$W_f(q, r, \tau) = \frac{t}{(t + \tau)^{3/2}} W_f(\tau) V_0 \left( \frac{3}{2}, \frac{2}{2\pi N r} \right).$$

Since $|\tau| \ll t^{1-\epsilon}$,

$$\frac{\partial}{\partial \tau} W_f(q, r, \tau) \ll \frac{1}{t^{1/2}|\tau|}.$$ 

When $n_2 = 0$, the bounds on $U^n(0, i(\tau_2 - \tau_1))$ gives arbitrary saving if $|\tau_2 - \tau_1| \gg t$. Recalling $a_i \sim Q$, $|r_1| \ll C t^{1+\epsilon}/N$ and $|\tau_1| \ll N t^{1}/QC$, the bound on $\Phi$ in this case is

$$\Phi \ll \frac{QC}{N t^{\epsilon}}.$$

Next, we estimate $\Phi$ when $n_2 \neq 0$. Applying [13, Lemma 5]

$$U^n \left( \frac{n_2 L}{q_1 q_2}, i(\tau_2 - \tau_1) \right) = c_3 \left( \frac{\tau_2 - \tau_1}{\tau_1 - \tau_1} \right)^{1/2} U \left( \frac{(\tau_2 - \tau_1) q_1 q_2}{2\pi n_2 L} \right)^{i(\tau_2 - \tau_1)}$$

$$+ O \left( \min \left\{ \frac{1}{|\tau_2 - \tau_1|^{3/2}}, \left( \frac{|n_2 q_2|}{L} \right)^{3/2} \right\} \right),$$

for some constant $c_3$ (which depends on the signs of $n_2$ and $(\tau_2 - \tau_1)$). The contribution of this error term towards $\Phi$ is

$$O \left( t^{\epsilon} \frac{Q^2 C^2}{N t^2} \int_{|J, L|/3} \int \min \left\{ \frac{1}{|\tau_2 - \tau_1|^{3/2}}, \left( \frac{|n_2 q_2|}{L} \right)^{3/2} \right\} d\tau_1 d\tau_2 \right).$$

Since $|\tau_1| \sim J \ll N t^{1}/QC$, the above is bounded by

$$t^{\epsilon} \frac{Q^2 C^2}{N t^2 (|n_2 q_2|/L)^{3/2}} \ll \frac{Q C^2}{N t^{1/2}} \frac{t^{\epsilon}}{(|n_2 q_2|/L)^{3/2}}.$$

We therefore set

$$B^*(C, 0) = \frac{Q C}{N t^{1/2}}, \quad \text{and} \quad B^*(C, n_2) = \frac{Q C^2}{N t^{1/2}} \frac{1}{(|n_2 q_2|/L)^{3/2}} \quad (\text{for } n_2 \neq 0). \quad (39)$$

Finally we analyze the main term contribution towards $\Phi$ and proceed exactly as Munshi [13] does. By Fourier inversion

$$\left( \frac{(\tau_2 - \tau_1) q_1 q_2}{2\pi n_2 L} \right)^{-1/2} U \left( \frac{(\tau_2 - \tau_1) q_1 q_2}{2\pi n_2 L} \right) = \int_{\mathbb{R}} U^n(y, 1/2) e \left( \frac{(\tau_2 - \tau_1) q_1 q_2}{2\pi n_2 L} y \right) dy$$

Using the above and the Stirling approximation (11) for $\gamma_+(s)$

$$\Phi = c_4 \frac{r_1 r_2 a_1 a_2}{t^2} \left( \frac{q_1 q_2}{n_2 L} \right)^{1/2} \int_{\mathbb{R}} \int_{\mathbb{R}} U^n(y, 1/2) g(\tau_1, \tau_2) e(f(\tau_1, \tau_2)) d\tau_1 d\tau_2 dy + O(B^*(C, n_2)), \quad (40)$$

where

$$2\pi f(\tau_1, \tau_2) = 3\tau_1 \log(|\tau_1|/e\pi) - 3\tau_2 \log(|\tau_2|/e\pi) - (\tau_1 - \tau_2) \log LN + 3\tau_1 \log q_1 - 3\tau_2 \log q_2$$

$$- (t + \tau_1) \log \left( \frac{(t + \tau_1) q_1}{2\pi e N r_1} \right) + (t + \tau_2) \log \left( \frac{(t + \tau_2) q_2}{2\pi e N r_2} \right)$$

$$- (\tau_1 - \tau_2) \log \left( \frac{(\tau_1 - \tau_2) q_1 q_2}{2\pi e n_2 L} \right) + \frac{(\tau_2 - \tau_1) q_1 q_2}{n_2 L}.$$
and
\[ g(r_1, r_2) = \Phi_+(r_1)\Phi_+(r_2)W_f(q_1, r_1, r_2)W_f(q_2, r_2, r_2). \]

Then
\[ 2\pi \frac{\partial^2}{\partial^2 r_1} f(r_1, r_2) = \frac{3}{r_1} - \frac{1}{t + r_1}, \quad 2\pi \frac{\partial^2}{\partial^2 r_2} f(r_1, r_2) = \frac{3}{r_2} + \frac{1}{t + r_2} + \frac{1}{r_2 - r_1}, \]
\[ 2\pi \frac{\partial^2}{\partial r_1 \partial r_2} f(r_1, r_2) = \frac{1}{r_1 - r_2}. \]

Therefore
\[ 4\pi \left[ \frac{\partial^2}{\partial^2 r_1} f(r_1, r_2) \frac{\partial^2}{\partial^2 r_2} f(r_1, r_2) - \left( \frac{\partial^2}{\partial r_1 \partial r_2} f(r_1, r_2) \right)^2 \right] = -\frac{6}{t_1 r_2} + O(t'/Jt). \]

We notice that \( \partial^2 f / \partial^2 r_1 = 0 \) for \( r_2 = (2t - 3r_1) r_1 / (3t - 4r_1) \), and is therefore small when \( r_1 = (2/3 + o(1)) r_2 \). We however recall that \( r_1 \in [J, 4J/3] \) (since \( W_f \) is supported there) and \( [2J/3, 8J/9] = \emptyset \). Therefore \( \partial^2 f / \partial^2 r_1 \gg 1/|r_1| \). The same argument justifies why \( \partial^2 f / \partial^2 r_2 \gg 1/|r_1| \). Therefore the conditions of Lemma 4.3 hold for above with \( p_1 = p_2 = 1/J^{1/2} \). Since \( \Phi'_+(\tau) \ll |\tau|^{-1} \) and \( W'_f(q, r, \tau) \ll t^{-1/2} |\tau|^{-1} \) (the derivative with respect to \( \tau \)), the total variation of \( g(r_1, r_2) \) is bounded as \( \text{var}(g) \ll t^{-1+\epsilon} \). By applying Lemma 4.3, the \( r_1, r_2 \)-integral is bounded by \( O(Jt^{-1+\epsilon}) \). Therefore the contribution of the leading term of (40) towards \( R \) is bounded by
\[ O\left( \frac{Q^2C^2}{N^2} \frac{C}{(|n_2|L)^{1/2}} \frac{Nt'}{QCt} \right) = O(B^*(C, n_2)), \]
where we have used \( J \ll Nt'/QC \). After this analysis, we bound the expression of \( S_{1,+,J}(N, C, L) \) as given in (38).

8.2. Diagonal contribution. We first consider the contribution of \( n_2 = 0 \), which we denote by \( S_{1,+,J}^0(N, C, L) \). Using the second statement of Lemma 7.1 and the bound of \( B^*(C, 0) \) as given in (39)
\[ S_{1,+,J}^0(N, C, L) \ll \frac{1}{Q^2C^8} \frac{QCt'}{Nt} \sum_{n_1 \leq 2C} \left\{ \frac{C^5 t}{n_1^4 N} + \frac{C^5 t}{n_1 N^2} \right\}, \]
where the first term is the contribution from terms with \( r_1 = r_2 \) and the second term is the contribution from \( r_1 \neq r_2 \). Since we will choose \( N > t \), the first term dominates and we get
\[ S_{1,+,J}^0(N, C, L) \ll \frac{Ct'}{N^2 Q}. \]  

Remark 8.1. The diagonal contribution as given in (41) improves over Munshi’s corresponding estimate in [13, Section 6.2]. Munshi estimates, \( S_{1,+,J}^0(N, C, L) \ll t'^{1/2} N^{3/2} K^{1/2} C \) where \( K = N/Q^2 \). Therefore, \( S_{1,+,J}^0(N, C, L) \ll Qt'/N^2 C \) with \( N/t^{1-\epsilon} < C < Q \). Therefore Munshi obtains
\[ S_{1,+,J}^0(N, C, L) \ll \frac{Q^{1+\epsilon}}{N^3}. \]

This bound is worse than the bound we obtain
\[ S_{1,+,J}^0(N, C, L) \ll \frac{Ct'}{N^2 Q} \ll \frac{t'}{N^2}. \]

This improvement helps us to improve upon the subconvex estimate of Munshi [13].
8.3. Off-diagonal contribution. We now bound the contribution of terms with $n_2 \neq 0$, which we denote by $S_{1,+J}(N, C, L)$. Using the first statement of Lemma 7.1 and the bound on $B^*(C, n_2)$ as given in (39)

$$
S_{1,+J}^d(N, C, L) \ll \frac{t^e}{Q^{3/2}} \sum_{n_1 < 2C} \sum_{C < q_1 \leq 2C} \sum_{(r_1, q_1) = 1} \sum_{1 \leq |r_1| \leq t^{1+\varepsilon}/N} \sum_{n_1|q_1} n_1^2 \sum_{1 \leq |n_2| \leq CN^e/QL} (q, n_2)B^*(C, n_2)
$$

$$
\ll \frac{C^{7/2}t^{1+\varepsilon}}{Q^{3/2}N^{5/2}L} \ll \frac{C^{2}t^{1+\varepsilon}}{N^{5/2}L}.
$$

(42)

8.4. Estimating $S_{1,J}(N, C)$. Using the bounds (41) and (42) in (37)

$$
S_{1,+J}(N, C) \ll t^e N^{1/2} \sum_{1 \leq L \leq N^{e+\varepsilon}/Q^2} \left( \frac{L^{1/2}C^{1/2}}{Q^{1/2}N} + \frac{Ct^{1/2}}{N^{5/4}} \right) \ll t^e N^{1/2} \left( \frac{C^{1/2}}{Q^2} + \frac{Ct^{1/2}}{N^{5/4}} \right).
$$

Multiplying by $N$ and summing dyadically over $J \in \mathcal{J}$ and $1 \leq C \leq Q$, the contribution of the above expression towards $S^+(N)$ is

$$
\frac{N^{3/2}t^e}{Q^{3/2}} + QN^{1/4}t^{1/2+\varepsilon}.
$$

(43)

We notice that the respective diagonal and off-diagonal estimates in the main term (43) are equal to or bigger than those of the error term (36) if $Q < N^{1/2}$. Under this assumption

$$
S(N) \ll \frac{N^{3/2}t^e}{Q^{3/2}} + QN^{1/4}t^{1/2+\varepsilon}.
$$

The optimum choice of $Q$ is therefore $Q = N^{1/2}/t^{1/5}$. Thus the condition $Q < N^{1/2}$ is satisfied. Finally, we observe that the condition $N/t < Q$ is the same as $N/t < N^{1/2}/t^{1/5}$, which is always satisfied since $N < t^{3/2+\varepsilon}$. Thus we get the final bound of $S(N) \ll N^{3/4}t^{3/10+\varepsilon}$, which proves Proposition 1.3.

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