Efficient Codes for Limited View Adversarial Channels

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Abstract—We introduce randomized Limited View (LV) adversary codes that provide protection against an adversary that uses their partial view of the communication to construct an adversarial error vector to be added to the channel. For a codeword of length $N$, the adversary selects a subset of $\rho_r N$ of the codeword components to “see”, and then “adds” an adversarial error vector of weight $\rho_w N$ to the codeword. Performance of the code is measured by the probability of the decoder failure in recovering the sent message. An $(N, q^k, \delta)$-limited view adversary code ensures that the success chance of the adversary in making decoder fail, is bounded by $\delta$ when the information rate of the code is at least $R$. Our main motivation to study these codes is providing protection for wireless communication at the physical layer of networks.

We formalize the definition of adversarial error and decoder failure, construct a code with efficient encoding and decoding that allows the adversary to, depending on the code rate, read up to half of the sent codeword and add error on the same coordinates. The code is non-linear, has an efficient decoding algorithm, and is constructed using a message authentication code (MAC) and a Folded Reed-Solomon (FRS) code. The decoding algorithm uses an innovative approach that combines the list decoding algorithm of the FRS codes and the MAC verification algorithm to eliminate the exponential size of the list output from the decoding algorithm. We discuss application of our results to Reliable Message Transmission problem, and open problems for future work.

I. INTRODUCTION

Shannon [18] formalized the study of reliable communication over noisy channels where transmitted symbols are changed according to a known fixed probability distribution. In adversarial channels corruption of transmitted symbols is adversarial: the adversary can corrupt any subset of the symbols as long as the size of the set is bounded and is a constant fraction of the transmitted sequence. Much less is known about adversarial channels. For example, although it is well known that the information capacity of a binary symmetric channel with crossover probability $\rho = 1 - H(\rho)$, the answer to the same question in the case of binary adversarial channels where the adversary corrupts a $\rho$ fraction of bits in unknown, although it is known that it is much less than $1 - H(\rho)$. Adversarial channels have received much attention in recent years [8][12][13] as they provide a powerful method of modelling communication channels where the channel behaviour is not known or varies over time.

In adversarial channels, one commonly assumes that the sent codeword is known, or even chosen (for example in randomized codes) by the adversary and that the adversary is allowed to corrupt a fraction of the sent symbols. For unique decoding the number of errors must be less than half the minimum distance of the code, and for higher fraction of errors, one needs to make extra assumptions such as a secret key shared by the sender and receiver in private codes [12], or bound on the computation of the adversary [14].

In this paper we consider an adversary with unlimited computation but assume that the adversary has a limited view of the transmitted codeword. That is we assume the adversary can see only a fraction of the sent codeword and can add errors to a fraction, possibly different, of the codeword. In other words the adversarial capability is specified by a pair of parameters $(\rho_r, \rho_w)$, meaning that the adversary can read $\rho_r N$ components of their choice, and corrupt $\rho_w N$ components of their choice. We do not assume any shared secret key.

A. Motivations

One of the motivations of our work is to model an online adversary in a wireless communication system, where the adversary can partially observe the communicated symbols before tampering with them [15].

We assume the encoded message is a $q$-ary vector and that the adversary can choose the positions that he would like to “see” (the remaining positions are not visible to the adversary) and then designs the tampering vector (noise) that is “added” to the encoded message. Our definition of limited view adversary codes aims to guarantee reliable authentic communication at the physical layer of communication channels and this means that the decoder will never output an incorrect (un-authentic) message, and with a very small probability fails to output the correct message. A somewhat similar scenario is has been considered in Algebraic Manipulation Detection Codes (AMD) [3] where the encoded message is stored in a secure storage and the adversary can only “add” errors to the codeword. In AMD codes the adversary cannot “see” the stored codeword and the aim of the code is to detect tampering with the message. We allow some partial information to be “leaked” to the adversary and the goal of the coding is to correctly recover the message. Note that because the code is randomized, recovering the message does not imply that the added noise can be found.
A second motivation for our model is to study 1-round \( \delta \)-Reliable Message Transmission (RMT) \([5]\) as a code and so establish the relationship between two seemingly different areas of communication over networks, and communication over noisy channels. Such relationship can enrich the tools and techniques developed in each area and result in better understanding and constructions in the two cases. In RMT scenario a sender is connected to a receiver through a set of \( N \) node disjoint communication paths, a subset of which is controlled by an adversary who can see what is sent on a controlled path and can replace it with a value of their choosing. Communication paths in RMT scenario are assumed end to end and unlike network coding \([1]\), nodes in the network do not take part in the communication protocol. In RMT the information processing is by the legitimate users (encoding and decoding) and happens at the ends of a path. The adversary interacts with the system by reading a subset of paths and changing the value sent over another subset of paths. When the two subsets are the same, the modification can be represented as adding an error vector. \( \delta \)-RMT protocols in general are multi-round and guarantee that message is correctly received with a probability at least \( 1 - \delta \). The bulk of research on \( \delta \)-RMT protocol assumes the adversary reads and modifies the same subset of paths.

**B. Our work**

We define and formalize randomized (stochastic) limited view adversary codes, with security against an adversary who can choose a fraction of positions of codeword to read and then add errors. For codewords of length \( N \), a \((\rho_r, \rho_w)\) adversary selects a subset of \( \rho_rN \) components to see, and then adds (component-wise addition over \( \mathbb{F}_q \)) an error vector of weight \( \rho_wN \) to the codeword. The decoder outputs either the correct message or a symbol \( \bot \), that shows the decoder failure. Performance of a code is measured by the probability of the decoder outputting \( \bot \); this is the success probability of the adversary in making the decoder fail. An \((N, M, \delta)\)-LV adversary code guarantees that the message can be correctly recovered against a \((\rho_r, \rho_w)\) adversary, and the success chance of the adversary in making the decoder to fail is upper-bounded by \( \delta \). The information rate of a code of length \( N \) with \( M \) codewords is \( \log_2 \frac{N}{M} \). A good code will have high information rate for high values of \( \rho_r \) and \( \rho_w \).

We construct an \((N, M, \delta)\)-LV adversary code that is non-linear, and uses two building blocks: a message authentication code and a Folded Reed-Solomon (FRS) code. To encode a message \( m \), the sender first chooses \( N \) appropriately constructed secret keys, uses the keys to construct \( N \) authentication tags for the message using the chosen MAC (See MAC Construction II for details), and appends the tags to the message. The tagged message is then encoded using an FRS code. The \( i \)-th component of the final codeword which is sent to the receiver consists of the corresponding component of the FRS code and the MAC key. The decoder recovers the correct message in a conceptually two step process: using the list decoding algorithm of the FRS code to construct a list of possible codewords and then applying the MAC verification algorithm to output either the correct message, or \( \bot \). This two step algorithm however can result in an exponential cost decoding because the output list of the FRS decoding algorithm can be of exponential size. A previous application of the general approach of using MACs and FRS codes for the construction of 1-round RMT \([16]\) has this shortcoming. The innovation in this paper is to combine the system of linear equations resulting from the algebraic list decoding algorithm \([9]\) of FRS codes, with a set of linear equations resulting from the verification algorithm of a specially constructed MAC, to have a single system of linear equation whose solution gives the correct message with a high probability. The MAC in this construction must be a key efficient MAC that can be used for different length messages and have appropriate verification algorithm suitable for efficient decoding. MAC Construction II satisfies these properties and could be of independent interest. The final decoder complexity is polynomial.

The code allows the adversary to, depending on the code rate, read up to half of the codeword and adds error on the same number of coordinates.

**RMT Construction:** One of the motivations for defining LV adversary codes is to cast the 1-round \( \delta \)-RMT construction as a coding problem. Our construction of LV adversary code can be immediately used to give an optimal 1-round \( \delta \)-RMT construction (See Section II-B for definitions,) whose parameters match the best known RMT constructions \([16]\). It is interesting to note that the LV adversary code parameters provide a more refined set of parameters for the evaluation of RMT. In particular, a 1-round \( \delta \)-RMT is optimal if transmission rate is \( O(1) \). Noting that transmission rate in RMT is the inverse of the information rate (See Section II-B) in LV adversary codes, any LV adversary code with non-zero information rate immediately results in an optimal 1-round \( \delta \)-RMT. For LV adversary codes however the rate of information communication is a key efficiency parameter and the goal is to maximize this rate (with other parameters fixed). LV adversary code view of 1-round \( \delta \)-RMT allows comparison of optimal systems in terms of their information rate. In addition to providing efficient decoding, the LV adversary code construction in this paper allows the parameters of the 1-round \( \delta \)-RMT code to be chosen such that the protocol achieves maximum information rate.

LV adversarial channels and codes open many new open questions. Finding general bounds and relationship among the information rate \( R \), observation and corruption ratios, \( \rho_r \) and \( \rho_w \) respectively, and finding the highest information rate (capacity) of LV adversary codes remain important research questions. Also construction of good codes by refining our approach here (combining message authentications codes and list decodable codes), or using new approaches, are interesting open problems.

**C. Related work**

In a previous submission \([17]\) we introduced deterministic LV adversary codes and gave a deterministic construction of such codes. Deterministic encoding enforces restrictions on
$p_r$ and $p_w$, that can be overcome by the randomized codes. The definition of decoder error in this paper follows the same approach as deterministic codes, but is in terms of probabilities instead of the combinatorics of the code. This is needed because of the randomize nature of the code removes restrictions that are dictated by the deterministic (one message, one codeword) nature of the code. In the same submission we also showed how to adapt a 1-round RMT protocol in [15] to construct a randomized construction for limited view codes. Decoding complexity of this construction was exponential and no security model and proof was provided for the code.

Protection against message manipulation was first considered in [2] and later formalized as message authentication codes in [19]. As noted earlier message authentication codes require shared secret key and provide protection against a powerful adversary who can completely replace a sent coded message with another one. The security guarantee for these codes is definition of manipulation.

Adversarial tampering by an adversary that does not “see” the encoded message, has been considered in [3]. AMD codes do not need a secret key but tampering is only by adding an adversarial noise. LV adversary codes do not require shared secret and aim at recovering the message. They limit manipulation to adding the noise but allow adversary to partially see the codeword before designing their adversarial noise vector.

Adversarial channels have been widely studied in the literature [4], [11]. Our model of adversarial channel has similarity with the model in [13] where binary oblivious channels are introduced. In oblivious channels the adversary sees the codeword, and depending on the level of obliviousness, can use one of the limited number of distributions on the error vectors that are available to them. A $\gamma$-oblivious adversary can emply at most $2^{1-\gamma}$ error distributions for corrupting the codewords. In these codes each codeword is associated with one error distributions. By limiting the adversary’s reading capability, our limited view adversary also effectively limits the number of distributions that the adversary can use. However each codeword can have more than one error distributions.

Organization.

In Section 2, we give the background for Folded Reed-Solomon code, 1-round $\delta$-RMT codes and message authentication codes. In Section 3, we introduce the randomized limited view adversary code and give new constructions for MAC. In Section 4, we present an efficient construction for randomized limited view adversary code. Section 5 discusses our results, open problems and future works.

II. BACKGROUND

We give an overview of the main building blocks and definitions required in this paper.

A. Folded Reed-Solomon code

Error correcting codes are used for reliable data transmission over noisy channels. Let the message space be a set $\mathcal{M}$ with probability distribution $Pr(m)$.

**Definition 1**: An $[N, q^{RN}]$ error correcting code $C$ with information rate $R$, is a set of $q^{RN}$ code vectors $C = \{c_1, \cdots , c_{q^N}\}$ where $c_i \in F_q^N$. The code has two algorithms: an encoding and a decoding algorithm. The encoding algorithm $Enc : \mathcal{M} \rightarrow C$ maps a message from $\mathcal{M}$ to a codeword in $C$ that is sent over the channel. The decoding algorithm $Dec : F_q^N \rightarrow \mathcal{M} \cup \{\perp\}$ is a deterministic algorithm that takes any vector in $F_q^N$ and outputs a message in $\mathcal{M}$ or fails, outputting a symbol $\perp$. A decoder error occurs if $\text{Dec}(\text{Enc}(m, r)) \neq m$.

The Hamming weight of a vector $e \in F_q^N$ is denoted by $wt(e)$ and is the number of non-zero components of $e$. For a vector $y \in F_q^N$ and an integer $r$, let $B(y, r)$ be the Hamming ball of radius $r$ centred at $y$. Let $\rho$ denote the fraction of errors (the number of errors divided by the length of the codeword) that can be corrected by the decoder.

**Definition 2**: A Bounded Distance Decoding (BDD) algorithm $\text{Dec}(y)$ takes a received word $y = (y_1, \cdots , y_N)$ and outputs $m \in \mathcal{M}$ if $m$ is the unique message of the codeword(s) that are at distance at most $wt(e)$ from $y$. The decoder outputs $\perp$ otherwise.

For deterministic codes, the above definition implies that the decoder outputs $m$, if $\text{Enc}(m)$ is the only codeword in $B(y, wt(e))$. In randomized codes however, $B(y, wt(e))$ may contain more than one encoding of $m$.

Using bounded distance decoding, the receiver $R$ outputs either a message $m$ or the fail symbol $\perp$, that is $\text{Dec}(y) \in \{\mathcal{M}, \perp\}$.

The above decoding is a unique decoding algorithm and requires that the output is a single message, or the fail symbol. For this decoding, correct decoding can be guaranteed if $\rho$ is less than half of the minimum distance of the code, that is $\rho \leq \frac{1-R}{2}$. Reed-Solomon code has an efficient unique decoding algorithm that can correct at most a fraction $\rho = \frac{1-R}{2}$ errors.

**Definition 3**: An $(N, k)$ Reed-Solomon code with block length $N(< q)$ and dimension $k$ over field $F_q$ is a linear code with encoding and decoding described below. A message block of length $k$ defines a polynomial $f(x)$ of degree at most $k-1$ over $F_q$. The codeword corresponding to this message block is the vector obtained by the evaluation of this polynomial at $N$ distinct values $\alpha_1, \cdots , \alpha_N$, where $\alpha_i \in F_q$, $i = 1 \cdots N$. That is the codeword is $\{f(\alpha_1), \cdots , f(\alpha_N)\}$.

For higher error ratios, one can use list decoding [3] where the decoder outputs a list of possible codewords (messages).

**Definition 4**: Let $(N, q^{RN})$ code be to a code with length $N$ and information rate $R$. A code $C$ is $(\rho, L)$-list decodable if the number of codewords within distance $\rho N$ of any received word is at most $L$. That is for every word $y \in q^N$, there are at most $L$ codewords at distance $\rho N$ or less from $y$. List decodable codes can potentially correct up to $1-R$ fraction of errors. This is twice that of unique decoding and is called the list decoding capacity of the code.

Construction of good codes with efficient list decoding algorithms is an important research question. An explicit construction of list decodable code that achieves the list decoding capacity $\rho = 1-R - \varepsilon$ is given by Guruswami et al. [9].
code is called Folded Reed-Solomon codes (FRS codes) and has polynomial time encoding and decoding algorithms.

Definition 5: A \( u_1 \)-folded Reed-Solomon code is a code with block length \( N = n/u_1 \) over \( F_q^{u_1} \) with \( |F_q| > n \). We represent the message by a polynomial \( f(x) \) of degree at most \( k \) over \( F_q^{u_1} \). The FRS codeword is over \( F_q^{u_1} \) and each of its components is a \( u_1 \)-tuple \((f(\gamma^{u_1}j), f(\gamma^{u_1j+1}), \ldots, f(\gamma^{u_1j+u_1-1}))\), for \( 0 \leq j < N \), where \( \gamma \) is a generator of \( F_q^* \). In other words a codeword of a \( u_1 \)-folded Reed Solomon code of length \( N \) is in one-to-one correspondence with a codeword \( c \) of a Reed Solomon code of length \( u_1N \), and is obtained by grouping together \( u_1 \) consecutive components of \( c \).

\[
\begin{bmatrix}
    f(1) & f(\gamma^{u_1}) & \cdots & f(\gamma^{u_1(N-1)}) \\
    f(\gamma) & f(\gamma^{u_1+1}) & \cdots & f(\gamma^{u_1(N+1)}) \\
    \vdots & \vdots & \ddots & \vdots \\
    f(\gamma^{u_1-1}) & f(\gamma^{2u_1-1}) & \cdots & f(\gamma^{u_1N-1}) 
\end{bmatrix}
\]

We denote the encoding algorithm of FRS code by \( \text{Enc}_{\text{FRS}} \). \( u_1 \) is called the folding parameter of the FRS code.

There are a number of efficient list decoding algorithms for FRS codes. We will use the linear algebraic FRS decoding algorithm \[9\]. The algorithm reduces the list decoding problem of the code to solving a set of linear equations. This algorithm, although not the best in terms of the number of corrected errors, but asymptotically achieves the list decoding capacity. The structure of the decoding algorithm of the FRS code makes it possible to combine it with the new MAC verification algorithm, to obtain an efficient decoding algorithm for the LV adversary code. The following Theorem gives the decoding capability of linear algebraic FRS code.

**Lemma 1:** \[9\] For the Folded Reed-Solomon code of block length \( N \) and rate \( R = \frac{1}{u_1N} \), the following holds for all integers \( 1 \leq v \leq u_1 \). Given a received word \( y \in (F_q^{u_1})^N \), in \( O((N u_1 \log q)^2) \) time, one can find a basis for a subspace of dimension at most \( v - 1 \) that contains all message polynomials \( f \in F_q^{u_1}[x] \) of degree less than \( k \) whose FRS encoding agree with \( y \) in at least a fraction,

\[
N - \rho N > N \left( \frac{1}{v+1} + \frac{v}{v+1} \frac{u_1R}{u_1 - v + 1} \right)
\]

of \( N \) codeword positions. The algorithm outputs a list of size at most \( q^{v-1} \).

The decoding algorithm of FRS code is in appendix \[A\].

**B. Reliable Message Transmission**

In a 1-round \( \delta \)-RMT problem, the sender \( S \) and the receiver \( R \) are connected by \( N \) node disjoint paths. The goal is to enable \( S \) to send a message \( m \), drawn from message space \( \mathcal{M} \) to \( R \) such that \( R \) receives the message reliably. The adversary \( A \) has unlimited computational power and in threshold RMT, can corrupt any subset of at most \( t \) out of the \( N \) paths which is unknown to \( S \) and \( R \): the adversary can eavesdrop, block or modify communication that is sent over the corrupted wires. \( S \) uses the encoding algorithm of the RMT protocol to encode the message \( m \) into transcript that is sent to \( R \). The transcript may be corrupted by \( A \) and is received by \( R \) who uses the decoding algorithm of the RMT protocol to output a message \( m \), or output \( \perp \).

**Definition 6:** An RMT protocol between \( S \) and \( R \) is 1-round \( \delta \)-reliable message transmission (\( \delta \)-RMT) protocol if \( R \) correctly receives the message \( m \) with probability \( \geq 1 - \delta \), and outputs \( \perp \) with probability \( \leq \delta \). The receiver never outputs an incorrect message:

\[
\Pr[ R \text{ outputs } \perp ] \leq \delta
\]

The transmission efficiency is measured by the transmission rate which is the ratio of the total number of bits transmitted from \( S \) to \( R \) to the length of the message in bits. Protocols whose transmission rate asymptotically matches the lower bounds are called optimal. Optimal 1-round \( \delta \)-RMT protocols must have transmission rates \( O(1) \).

**Computational efficiency** is measured by the computational complexity of the encoding and the decoding, as a function of \( N \). Efficient scheme needs polynomial (in \( N \)) computation of both encoding and decoding algorithm.

**C. Message authentication codes**

A message authentication code (MAC) is a cryptographic primitive that allows a sender who shares a secret key with the receiver to send an information block over a channel that is tampered by an adversary, enabling the receiver to verify the integrity of the received message. We follow the terminology of \[19\] and refer to the information block as source state, and to the authenticated message that is sent over the channel as, the message. A message authentication code consists of two algorithms (\( \text{MAC}; \text{Ver} \)) that are used for tag generation and verification, respectively. The sender of a source state \( x \) computes an authentication tag, or simply a tag \( y = MAC(k; x) \), and forms the message \( (x, y) \) to be sent over the channel. The receiver accepts the pair \((x, y)\) if \( \text{Ver}(x, y, k) = 1 \). Security of a 1-time MAC is by requiring, \( \Pr[ (x', y'), \text{Ver}(k, (x', y')) = 1 | (x, y), y = MAC(k; x) ] \leq \epsilon \).
Definition 7: A \((\rho_r, \rho_w)\) limited view adversary, or a \((\rho_r, \rho_w)\) LV adversary for short, has two capabilities: reading and writing. For a codeword of length \(N\), these capabilities are:

- Reading: Adversary reads a subset \(S_r\) of size \(\rho_r N\), of the components of the sent codeword \(c\) and learns, \((c_1, \cdots, c_{\rho_r N})\).
- Writing: Adversary adds (component wise and over \(N\)) Decoding error: \(\{\mathbf{c} - \mathbf{c}_c\}_i\) in positions \(\{c_1, \cdots, c_{\rho_r N}\}\) in those positions is

\[
\delta(C(c_1, \cdots, c_{\rho_r N})) = \Pr[Enc(m, r) \in C | c_1, \cdots, c_{\rho_r N}]
\land Dec(Enc(m, r) + e) = \perp | C(c_1, \cdots, c_{\rho_r N})]
\]

The decoding algorithm fails, that is \(Dec(Enc(m, r) + e) = \perp\), if and only if there exist \(c' \in C \setminus C^m\) and \(c' \in B(c + e, \rho_r N)\).

The decoding error for the decoder is,

\[
\delta = \max_{\mathbf{c}} \max_{e} \delta(C(c_1, \cdots, c_{\rho_r N}))
\]

Definition 9: An \((N, M, \delta)\) randomized LV adversary code with protection against \((\rho_r, \rho_w)\) adversary, ensures that the probability of the decoding failure defined above, is no more than \(\delta\).

B. Randomized limited view adversary code

By observing the values \(\{c_1, \cdots, c_{\rho_r N}\}\), the adversary can determine a subset of possible sent codewords (those that match the seen positions). Let \(C(c_1, \cdots, c_{\rho_r N})\) denote the set of codewords that have \(\{c_1, \cdots, c_{\rho_r N}\}\) in positions \(S_r = \{i_1, \cdots, i_{\rho_r N}\}\).

1) Decoding error: Decoder uses bounded distance decoding with radius \(\rho_w N\): for a received vector \(y\), it considers all codewords that are in \(B(y, \rho_w N)\) and if it finds encodings of a unique message, it outputs that message; Otherwise it outputs \(\perp\). The error vector \(e\) is of weight \(w_H(e) \leq \rho_w N\) and is chosen by the adversary after reading \(\{c_1, \cdots, c_{\rho_r N}\}\). The adversary can find the failure probability of the decoder for any error vector \(e\), and choose the “best” one; this is the \(\epsilon\) that results in the highest failure probability for the decoder.

Definition 8: Consider an additive error \(e\) with \(w_H(e) = \rho_w N\). The decoding error \(\delta(C(c_1, \cdots, c_{\rho_r N}))\) for a message \(m\) and an error \(e\) if adversary chooses to read a \(S_r\) and see \(\{c_1, \cdots, c_{\rho_r N}\}\) in those positions is

\[
\delta(C(c_1, \cdots, c_{\rho_r N})) = \Pr[Enc(m, r) \in C | c_1, \cdots, c_{\rho_r N}]
\land Dec(Enc(m, r) + e) = \perp | C(c_1, \cdots, c_{\rho_r N})]
\]

The MAC function consists of three types of terms. For a message symbol \(x_m\) with index \(m\), one of the three types, as defined below, is calculated. The final MAC is the sum of all the calculated terms.

1) \(x_m r_m\), for \(1 \leq m \leq l\);
2) \(x_m r_i r_j\), for \(d + 1 \leq m \leq l\) where \(m = id + j - \frac{(i - 1)}{2}, 1 \leq i \leq j \leq d;\)
3) \(r_{d+1}\), which is independent of message symbols.

For \(d + 1 \leq m \leq l\), the algorithm works as follows.

1. Consider the message symbols \(m_{d+1}, m_{d+2}, \cdots, m_l\) as a sequence;
2. Construct a key sequence using the product of a pair of key symbols \(r_i, r_j\) as follows: start from the smallest \(i = 1, j = 1\); increase \(j\) by one from \(i\) to \(d\); then increase \(i\) by one and repeat to reach the highest values of the two indexes.
3. Find the product of \(x_m\) and the element of the key sequence constructed above, that corresponds with position \(m\).

It can be seen that for a given pair \(i, j\) and \(m\) will satisfy \(m = id + j - \frac{(i - 1)}{2}\).

Lemma 2: The probability that a computationally unlimited adversary can forge a message \((x', \text{tag}')\) with \(x' \neq x\), that passes the verification test is no more than \(2^{-q}\). We omit the security proof because of space and that it is essentially the same as the proof of Construction II.

2) MAC Construction II: We introduce a MAC that can be seen as a different representation of Construction I above, that will be used in the construction of efficient randomized LV adversary code. The MAC can be described by a set of equations over \(F_q\). The source state of the MAC is a vector of length \(N l\) over \(F_q\),

\[
x = [x_1, \cdots, x_{N - 1}]^T
\]

The key for the MAC is a vector of length \(Nd + 3N - 2\) over \(F_q\) where \(d\) is the smallest integer satisfies \(\frac{d(d + 3)}{2} \geq l\),

\[
r = [r_{1, 0}, \cdots, r_{1, N - 1}, \cdots, r_{d, 0}, \cdots, r_{d, N - 1}, r_{d + 1, 0}, \cdots, r_{d + 1, 3N - 3}]^T
\]
We write the key in the form of an \((3N - 2) \times (Nl + 1)\) matrix:

\[
R = \begin{bmatrix} R_1 & \cdots & R_d & R_{d+1} & \cdots & R_l & R_{l+1} \end{bmatrix}
\]

where \(R_m\) is a matrix that, depending on the value of the index \(m\), can take the following forms. For \(1 \leq m \leq d\),

\[
R_m = \begin{bmatrix} r_{m,0} & 0 & \cdots & 0 \\
0 & r_{m,0} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
r_{m,N-1} & r_{m,N-2} & \cdots & r_{m,0} \\
0 & r_{m,N-1} & \cdots & r_{m,1} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\end{bmatrix}
\]

For \(d + 1 \leq m \leq l\),

\[
R_m = \begin{bmatrix} r_{i,j,0} & 0 & \cdots & 0 \\
r_{i,j,1} & r_{i,j,0} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
r_{i,j,N-1} & r_{i,j,N-2} & \cdots & r_{i,j,0} \\
r_{i,j,N} & r_{i,j,N-1} & \cdots & r_{i,j,1} \\
\vdots & \vdots & \ddots & \vdots \\
r_{i,j,2N-1} & r_{i,j,2N-2} & \cdots & r_{i,j,N-1} \\
r_{i,j,2N} & r_{i,j,2N-1} & \cdots & r_{i,j,N} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & r_{i,j,2N-1} \\
\end{bmatrix}
\]

where \(m\) is written as a pair of integers \(i\) and \(j\), similar to the description of Construction I, and we have \(r_{i,j,k} = \sum_{a_1 + a_2 = k} r_{a_1,a_2} r_{j,a_2}\) for \(0 \leq k \leq 2N - 1\).

Finally, \(R_{l+1} = [r_{d+1,0}, \ldots, r_{d+1,3N-3}]^T\).

The tag for a source state is a vector of length \(3N - 2\),

\[ t = [t_0, \ldots, t_{3N-3}]^T \]

A source state \(x\) is encoded to the message \((x, t)\) using the MAC algorithm,

\[
MAC(x, r) = \sum_{1 \leq m \leq d} x_j R_j + \sum_{d+1 \leq m \leq l} x_m R_m + R_{l+1}
\]

\[ = \begin{bmatrix} R_1 & \cdots & R_l & R_{l+1} \end{bmatrix} \begin{bmatrix} x_1 \left[ R_{l+1} - R_{l} \right] \end{bmatrix} \]

\[ = [t] \]

The verification algorithm \(Ver(r, (x', t'))\) for a key \(r\) is by calculating \(MAC'(x', r)\), and comparing it with the received \(t'\).

**Lemma 3:** The probability that a computationally unlimited adversary can forge a message \((x', t')\) with \(x' \neq x\), that passes the verification is no more than \(\frac{2}{q^3}\).

**Proof:** Appendix [H-C2]

**IV. CONSTRUCTION OF LV ADVERSARY CODE**

In this section we describe the construction of an LV adversary code that uses the MAC algorithm in Section [H-C2] together with an FRS code with appropriately chosen parameters.

A. \((N, q^{NuR}, \delta)\) randomized limited view adversary code

We assume the adversary reads \(\rho N\) positions and adds errors to the same positions. Let \(N\) and \(\hat{R}\) denote the code length and information rate, respectively.

The LV adversary code is over \(F_q^u\). The sender \(S\) wishes to send the message \(m = (m_0, \ldots, m_{NuR-1})\), \(m_i \in F_q\), to the receiver.

**Randomized LV adversary code:**

- \(m = (m_0, \ldots, m_{NuR-1})\)
- \(x = (m, 0)\)
- \(t_i = MAC(x, r_i)\)
- \((x, t_1, \ldots, t_N)\)

**FRSEnc(x, t_1, \ldots, t_N)**

The LV adversary code is constructed over \(F_q^u\) where \(u = u_1 + u_2\). The FRS code is over \(F_q^u\) and the randomness \(r_i\) has length \(u_2\). We set the parameters of MAC Construction II to be \(l = \lceil uR \rceil\) and \(d = \lfloor \sqrt{2u_1} \rfloor\).

We have \(u_2 = NuR + 3N - 2 = N\lfloor \sqrt{2u_1} \rfloor + 3N - 2\) and \(u = u_1 + N\lfloor \sqrt{2u_1} \rfloor + 3N - 2\).

**Encoding algorithm performed by the sender \(S\):**

**Step 1:** Append vector \(0 \in F_q^{NuR} \) to message \(m = (m_0, \ldots, m_{NuR-1})\), and form the vector \(x = (m, 0)\) of length \(NuR\).

**Step 2:** Generate random keys \(r_i, 1 \leq i \leq N\), for the MAC Construction II. Each key is written as a \((3N-2) \times (NuR+1)\) matrix,

\[ R_i = \begin{bmatrix} R_{i,1} & \cdots & R_{i,l} & R_{i,l+1} \end{bmatrix} \]
**Step 3: Use MAC Construction II to generate tags** $t_i = MAC(x, R_i)$, $i = 1, \ldots, N$.

The FRS code is of dimension $k = Nl + N(3N - 2)$. The message block for the FRS code is,

$$m^{FRS} = (x, t_1 \cdots t_N)$$

**Step 4: Use the FRS encoding algorithm to encode** $m^{FRS}$ to the codeword $c^{FRS} = Enc_{FRS}(m^{FRS})$.

The $i$th component of $c$, the codeword of the limited view adversary code, is obtained by appending the randomness $r_i$ to $c_i^{FRS}$, the $i$th component of the FRS code.

$$c_i = (c_i^{FRS}, r_i)$$

**Decoding algorithm performed by the receiver $R$:**

**Step 1:** Receive a corrupted word $y$ with the $i$th component $y_i = (y_i^{FRS}, \tilde{r}_i)$. Here $y_i^{FRS}$ and $\tilde{r}_i$ are the $i$th component of the FRS code and the randomness in corrupted form, respectively.

**Step 2:** Use the FRS decoding algorithm to decode the FRS codeword $y_i^{FRS}$ and obtain the system of linear equations.

**Step 3:** Generate $N$ systems of linear equations, each system obtained from the set of linear equations generated from the FRS decoding algorithm and one MAC key $r_i$. The $i$th system of linear equation is of the form,

$$[B_0 \ B_1 \ \cdots \ B_i \ \cdots \ B_N] \times \begin{bmatrix} x \\ t_1 \\ \vdots \\ t_i \\ \vdots \\ t_N \end{bmatrix} = \begin{bmatrix} -a' \\ -R_{i,d+1} \\ \vdots \\ 0 \end{bmatrix}$$

(3)

The first $Nl + N(3N - 2)$ equations are generated by the FRS decoding algorithm of Eq. 6 the first $Nl$ columns of the matrix of coefficients of these equations form $B_0$, and for $1 \leq i \leq N$, columns $(Nl + i - 1)(3N - 2)$ to $(Nl + i(3N - 2) - 1)$ of this matrix specify $B_i$. Finally, $-a'$ is the right hand side vector of Eq. 6. The last $3N - 2$ equations are from MAC Construction II using key $r_i$, with $R'_i = [R_{i,1} \ \cdots \ R_{i,i}]$, and $I$ is identity matrix.

**Step 4:** Solves each of the $N$ systems of linear equations. Let $x_i$ denote, the first $Nl$ components of a solution output by the $i$th system of linear equation. The $i$th system of linear equation is considered to have output output $x_i$, if $x_i$ is the unique output of this system. Otherwise $R$ marks the output of the $i$th system, as NULL. If there is a unique $x$ output by a set of the $N - \rho N$ systems of linear equations, $R$ outputs the first $NuR$ components of that $x$ as $m$. Otherwise outputs $\perp$.

**B. Adversary’s reading and writing capability**

**Theorem 1:** The $(N, q^{RN}, \delta)$ randomized limited view adversary code over $F_q^u$ above, can correctly decode if the adversary reads and writes on the same set of size $\rho N$ of a codeword.

$$\rho \leq \min\left(\frac{1}{2} - \frac{1}{2N}, \frac{v}{v + 1} - \frac{uR + 3N}{v + 1 N^2 + u - N(\sqrt{N^2 + 2u} + 3) - v}\right)$$

**Proof:** Firstly, $\rho < 1/2$: If the adversary can read and write on half of the components of a codeword $c$, they can choose any other codeword $c'$ and add appropriate error vector to replace components of $c$ on the controlled positions to obtain $y$ which is equal to $c'$ on the controlled components, and equal to $c$ on the remaining ones. The decoder can not decode $y$ and fail.

Secondly, we find a bound on $\rho$ when $\rho < 1/2$. The code dimension for the FRS code is $k = NuR$, and each component is in $F_q^u$. Note that only the FRS code, which is over $F_q^u$, contains the information message. Hence, $k = NuR_1$. Let $R_{FRS}$ be the information rate of the FRS code. The decoding algorithm of LV adversary code need to satisfy the decoding condition of FRS code. According to Lemma 11, the FRS code with length $N$ and information rate $R_{FRS}$ can decode $\rho N$ adversary errors if satisfying the condition:

$$N - \rho N \geq N\left(\frac{1}{v + 1} + \frac{v}{v + 1} \frac{u_1 R_{FRS}}{u_1 - v + 1}\right)$$

(4)

The equation is satisfied if,

$$N - \rho N \geq \frac{N}{v + 1} + \frac{v}{v + 1} (N(u_1 R_1 + 1) + N(3N - 2))$$

The maximum error that the adversary can add is,

$$\rho \leq \frac{v}{v + 1} - \frac{v}{v + 1} \frac{(u_1 R_1 + 3N - 1)}{u_1 - v + 1}$$

The LV adversary code is over $F_q^u$ and $u = u_1 + \lceil \sqrt{2u_1} \rceil N + 3N - 2$. So we have,

$$u_1 \geq N^2 + u - 3N + 1 - N \sqrt{N^2 + 2u - 2(3N - 1)}$$

The decoding condition of FRS code is satisfied if the following inequality is met:

$$\rho \leq \frac{v}{v + 1} - \frac{v}{v + 1} \frac{uR + 3N - 1}{N^2 + u - 3N + 2 - N \sqrt{N^2 + 2u - 2(3N - 1)} - v + 1}$$
This is equivalent to,
\[ \rho \leq \frac{v}{v + 1} - \frac{v}{v + 1} N^2 + u - N(\sqrt{N^2 + 2u - 3} - v) \]

C. Decoding error

The adversary reads \( \rho N \) components of a corrupted codeword and adds errors to the same positions using the knowledge of the components that are read.

Lemma 4: If the adversary does not choose the \( i^{th} \) position for read and write, the probability that the \( i^{th} \) system of linear equations (Eqs. 3) does not produce the unique solution that contains the correct message \( m \) is at most \( \frac{1}{q^{N - v + 1}} \). This is equivalent to,

\[ Pr[d_H(\epsilon^{F_R}, y^{FRS}) \leq \rho N, t'_i = MAC(x', r_i)|C[c_1, \cdots, c_{v_N}]] \]

with \( \epsilon^{FRS} = Enc_{FRS}(m^{FRS}) \) and \( m^{FRS} = (x', t'_1, \cdots, t'_N) \) and \( (x' \neq x) \).

Proof: Firstly, because the correct message is always contained in the decoded list of the FRS decoding algorithm, the correct \( x = \{m, 0\} \) will be in the solution space of the system of linear Eq. [3]. Also because the key \( r_i \) has not been modified, the solution will be contained in the solution space of the equations generated by the MAC. Hence the solution space of the Eqs. [3] must contain the correct message \( m \).

Secondly, a solution \( x' \), where \( x' \neq x \), of the system of linear Eqs. [6] resulting from the FRS decoding algorithm, with probability at most \( \frac{1}{q^{N - v + 1}} \) will be a solution of the system of linear Eqs. [3]. Now assume \( x' \neq x \) is a solution of Eqs. [3]. This means that it must satisfy the equations generated by MAC:

\[ [R_i' - I] \times [x'_i] = [-R_{i,v+2}] \] (5)

Using lemma [3] the probability that \( MAC(x', r_i) = t'_i \) is at most \( \frac{1}{q^{N - v + 1}} \).

Finally, the system of linear equations Eq. [6] generated by the decoding algorithm of the FRS code produces a list of at most \( q^{N - v + 1} \) solutions, \( \epsilon^{FRS} : d_H(\epsilon^{FRS}, y^{FRS}) \leq \rho N \), where each codeword represents a message of the form \( m^{FRS} = (x', t'_1, \cdots, t'_N) \). The first \( N \) components of each solution gives one solution for \( x' \). By union the probability of the solutions \( x' \neq x \) of Eqs. [6] that are also the solution of Eqs. [3] the Eqs. [3] has more than one solution with probability no more than \( \frac{1}{q^{N - v + 1}} \).

The adversary has no information of \( r_i \). After observing \( \{c_1, \cdots, c_{v_N}\} \), the probability that there exist \( \epsilon^{FRS} : d_H(\epsilon^{FRS}, y^{FRS}) \leq \rho N \) and the message passing MAC verification \( MAC(x', r_i) = t'_i \) is still equal to \( \frac{1}{q^{N - v + 1}} \).

Theorem 2: The decoding error of the \((N, q^{RN}, \delta)\) randomized limited view adversary code is at most \( \delta \leq \frac{2N}{q^{N - v + 1}} \).

Proof: Let \( y = Enc(m, r) + \epsilon \) be the corrupted word, and \( I_3 = S_i = S_i' \) denote the positions that are read and modified by the adversary. For a codeword \( c' = (c^{F_R}, r'_1, \cdots, r'_N) \) with \( c^{F_R} = Enc_{FRS}(m^{FRS}) \) and \( m^{FRS} = (x', t'_1, \cdots, t'_N) \) and \( x' \neq x \), let \( I'_a = \{ i : c'_i = y \} \) and \( I'_b = \{ i : MAC(x', r'_i) = t'_i \} \).

According to definition [3] the probability of decoding failure for an encoding of a message \( m \) that satisfies the observation set \( \{c_1, \cdots, c_{v_N}\} \) is,

\[ Pr[B(Enc(m, r) + e, \rho N) \cap \{ C \setminus C_m \} \neq \emptyset | C[c_1, \cdots, c_{v_N}]] \]

This is the probability that for a codeword \( c' \in C \setminus C_m \), there exists two subsets \( I'_a \) and \( I'_b \) such that, \( |I'_a| \geq N - \rho N \), \( |I'_b| = 0 \) and \( |I'_a \cap I'_b| \geq N - \rho N \). The latter two conditions imply \( |I'_a \cap I'_b| \geq N + \rho N + 1 \) if \( \rho < \frac{1}{2} \), which can be written as, \( |\{\{I \}\} \cap I'_1 \cap I'_2| = 1 \). Note that \( |I'_1| \geq N - \rho N \) implies \( d_H(c'^{F_R}, y^{FRS}) \leq \rho N \), and \( \{|\{I\} \cap I'_1 \cap I'_2| = 1 \} \) implies existence of \( r' \) such that \( r' \in \{I'_1 \cap I'_2\} \) and \( r' \in \{N \setminus I_3\} \).

This means that we have,

\[ Pr[B(Enc(m, r) + e, \rho N) \cap \{ C \setminus C_m \} \neq \emptyset | C[c_1, \cdots, c_{v_N}]] \]

\[ \leq Pr[|i' \in \{N \setminus I_3\}, (i' \in \{I'_1 \cap I'_2\}),
\]

\[ (d_H(c'^{F_R}, y^{FRS}) \leq \rho N) | C[c_1, \cdots, c_{v_N}]\]

\[ \leq (N - \rho N) Pr[(i' \in \{I'_1 \cap I'_2\}), (i' \in \{N \setminus I_3\}), (d_H(c'^{F_R}, y^{FRS}) \leq \rho N) | C[c_1, \cdots, c_{v_N}]]\]

\[ = (N - \rho N) Pr[(i' \notin \{I_3\}), (MAC(x', r'_i) = t'_i),
\]

\[ (d_H(c'^{F_R}, y^{FRS}) \leq \rho N) | C[c_1, \cdots, c_{v_N}]]\]

\[ \leq \frac{2N}{q^{N - v + 1}} \]

The last inequality is correct because of lemma [4].

If we choose \( v = \frac{1}{2}, u = \frac{2}{q^N} + \frac{2N}{q^N} \) where \( q > 0 \) is a small value, the decoding capability \( \rho \) can be approximated is \( \rho \leq \min \left( \frac{1}{2}, \frac{N}{2}\right) \), \( 1 - (N \epsilon^2) R - N \epsilon^2 - N^2 \epsilon^6 \), and the decoding error will be given by \( \delta \leq q^{N - v} \). The field size \( q \) can be chosen as the smallest prime \( q > N \). The encoding algorithm is polynomial in \( N \). For decoding algorithm, the computational complexity of solving any \( i^{th} \) system of linear equation Eq. [3]is \( \mathcal{O}((uN + N^2) \log q)^2 \) and there are \( N \) systems of linear equations. So the computational time of decoding algorithm is polynomial in \( \mathcal{O}(N((uN + N^2) \log q)^2) \).

Corollary 1: Assume the adversary is allowed to read (at most) \( \rho \) fraction of a codeword and can write on the same set. The \((N, q^{RN}, \delta)\) randomized limited view adversary code over \( \mathbb{F}_q^{1+\frac{1}{2}} \) with,

\[ \rho \leq \min \left( \frac{1}{2}, \frac{1}{2N}, 1 - (1 + N \epsilon^2) R - N \epsilon^2 - N^2 \epsilon^6 \right) \]

can correctly decode the errors and the decoding error \( \delta \to 0 \) if \( N \to \infty \). The computational time is polynomial in \( N \).

The construction above can be immediately used to construct an optimal 1-round \( \delta \)-RMT, by using the encoding algorithm of the LV adversary code with appropriate length, to construct a
codeword for the message, and simply send the $i^{th}$ component of the codeword on path $i$ in the RMT setting. The decoding error in LV adversary codes is equivalent to the strongest definition of reliability in RMT scenario where the adversary can choose the message, and so $\delta$ in RMT will be at most equal to the decoder failure in LV adversary codes. The optimality follows from the constant (non-zero) rate of the LV adversary code.

Corollary 2: The construction of the randomized LV adversary code give an optimal 1-round $\delta$-$\text{RMT}$, where $\delta$ is the same as the decoding error in LV adversary codes.

V. CONCLUDING REMARKS

We introduced randomized limited view adversary codes and gave an efficient construction that with appropriate choice of parameters, can correct close to $N/2$ errors and will have information rate close to $1/2$. Although in general the observation and corruption sets can be different, in our construction we assumed they are the same. Giving a construction without this assumption will be our future work. In our construction the field size is a function of $N$ and so small $\delta$ can be obtained for large field sizes. Finding good LV adversary codes with fixed field size, and/or information rate approaching $\frac{1}{2}$-$\text{RMT}$ and $(0, \delta)$-$\text{SMT}$, ACNS, pp. 344–362, 2012.

REFERENCES

[1] R. Ahlswede, N. Cai, S. Li, and R. Yeung, “Network Information Flow”, IEEE Transactions on Information Theory, vol. 44(6), pp. 2148–2177, 1998.

[2] L. Carter and M. Wegman, “Universal Classes of Hash Functions”, SIAM Journal on Computing, vol. 18(2), pp. 143-154, 1977.

[3] R. Cramer, Y. Dodis, S. Fehr, C. Padró, D. Wichs, “Detection of Algebraic Manipulation with Applications to Robust Secret Sharing and Fuzzy Extractors”, EUROCRYPT, pp. 471–488, 2008.

[4] I. Csiszar and P. Narayan, “The Capacity of the Arbitrarily Varying Channel Revisited: Positivity, Constraints”, IEEE Transaction Information Theory, vol. 34, pp. 181-193, 1988.

[5] D. Dolev, C. Dwork, O. Waarts, and M. Yung, “Perfectly Secure Message Transmission”. In Journal of the ACM, vol. 40(1), pp. 17–47, 1993.

[6] P. Elias, “List Decoding for Noisy Channels”, MIT Research Lab of Electronics, Technical Report 335, 1957.

[7] M. Franklin and R. Wright, “Secure Communication in Minimal Connectivity Models”, Journal of Cryptology, vol. 13(1), pp. 9–30, 2000.

[8] V. Guruswami, A. Smith, “Codes for Computationally Simple Channels: Explicit Constructions with Optimal Rate”, FOCS, pp. 723–732, 2010.

[9] V. Guruswami, “Linear Algebraic List Decoding of Folded Reed-Solomon Codes”, IEEE Conference on Computational Complexity, pp. 77–85, 2011.

[10] H. Krawczyk, “New Hash Functions For Message Authentication”, EUROCRYPT, pp. 301–311, 1995.

[11] A. Lapidoth and P. Narayan, “Reliable Communication Under Channel Uncertainty”, IEEE Transaction Information Theory, vol. 44(6), pp. 2148–2177, 1998.

[12] M. Langberg, “Private Codes or Succinct Random Codes That Are (Almost) Perfect”, FOCS, pp. 325–334, 2004.

[13] M. Langberg, “Oblivious Communication Channels and Their Capacity”, IEEE Transaction Information Theory, Vol. 54(1), pp. 424–429, 2008.

[14] S. Micali, C. Peikert, M. Sudan, D. Wilson, “Optimal Error Correction Against Computationally Bounded Noise”, TCC, pp. 1–16, 2005.

[15] C. Pöpper, N. Tippenhauer, B. Danev, S. Capkun, “Investigation of Signal and Message Manipulations on the Wireless Channel”, ESORICS, pp. 40–59, 2011.

[16] R. Safavi-Naini, M. Tuinh, P. Wang, “A General Construction for 1-round $\delta$-$\text{RMT}$ and $(0, \delta)$-$\text{SMT}$, ACNS, pp. 344–362, 2012.

[17] R. Safavi-Naini, P. Wang, “Codes for Limited View Adversarial Channels”, Submission to ISIT 2013.

[18] C. Shannon, “A Mathematical Theory of Communication”, Bell System Technical Journal, vol. 27, pp. 379–423 and 623–656, Jul. and Oct. 1948.

[19] G. Simmons, “Authentication theory/coding theory”, CRYPTO, pp. 411–432, 1984.

APPENDIX

A. Decoding algorithm of FRS code

Linear algebraic list decoding [9] has two main steps: interpolation and message finding as outlined below.

1. Find a polynomial, $Q(X,Y_1,\cdots, Y_v) = A_0(X) + A_1(X)Y_1 + \cdots + A_v(X)Y_v$ over $F_q$ such that $\deg(A_i(X)) \leq D$, for $i = 1, \cdots, v$, and $\deg(A_0(X)) \leq D - 1 + k - 1$, satisfying $Q(A_1, y_1, y_2, \cdots, y_v) = 0$ for $1 \leq i \leq n_0$, where $n_0 = (u_1 + v - 1)N$.

2. Find all polynomials $f(X) \in F_q[X]$ of degree at most $k - 1$, with coefficients $f_0, f_1, \cdots, f_{k-1}$, that satisfy, $A_0(X) + A_1(X)f(X) + A_2(X)f(\gamma) + \cdots + A_v(X)f(\gamma^{v-1})X = 0$, by solving linear equation system.

The two above requirements are satisfied if $f \in F_q[X]$ is a polynomial of degree at most $k - 1$ whose FRS encoding (Eq 10 agrees with the received word $y$ in at least $T$ components:

$$T > N \left( \frac{1}{v+1} + \frac{v}{v+1} \frac{u_1 R}{u_1 - v + 1} \right)$$

This means we need to find all polynomials $f(X) \in F_q[X]$ of degree at most $k - 1$, with coefficients $f_0, f_1, \cdots, f_{k-1}$, that satisfy,

$$A_0(X) + A_1(X)f(X) + A_2(X)f(\gamma) + \cdots + A_v(X)f(\gamma^{v-1})X = 0$$

Let us denote $A_i(X) = \sum_{j=0}^{D+k-1} a_{i,j} X^j$ for $0 \leq i \leq v$, $(a_{i,j} = 0$ when $i \geq 1$ and $j \geq \tilde{D})$. Define the polynomials,

$$B_0(X) = a_{1,0} + a_{2,0}X + a_{3,0}X^2 + \cdots + a_{v,0}X^{v-1}$$

$$B_{k-1}(X) = a_{1,k-1} + a_{2,k-1}X + a_{3,k-1}X^2 + \cdots + a_{v,k-1}X^{v-1}$$

We examine the condition that the coefficients of $X^i$ of the polynomial $Q(X) = A_0(X) + A_1(X)f(X) + A_2(X)f(\gamma) + \cdots + A_v(X)f(\gamma^{v-1})X = 0$ equals 0, for $i = 0 \cdots k - 1$. This
is equivalent to the following system of linear equations for $f_0 \cdots f_{k-1}$.

$$
\begin{bmatrix}
B_0(\gamma^0) & 0 & \cdots & 0 \\
B_1(\gamma^0) & B_0(\gamma^1) & \cdots & 0 \\
B_2(\gamma^0) & B_1(\gamma^1) & B_0(\gamma^2) & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
B_{k-1}(\gamma^0) & B_{k-2}(\gamma^1) & B_{k-3}(\gamma^2) & \cdots & B_0(\gamma^{k-1})
\end{bmatrix} 
\begin{bmatrix}
f_0 \\
f_1 \\
f_2 \\
\vdots \\
f_{k-1}
\end{bmatrix} = 
\begin{bmatrix}
-a_{0,0} \\
-a_{0,1} \\
-a_{0,2} \\
\vdots \\
-a_{0,k-1}
\end{bmatrix}
$$

(6)

The rank of the matrix of Eqs. 6 is at least $k-v+1$ because there are at most $v-1$ solutions of equation $B_0(X) = 0$ so at most $v-1$ of $\gamma^i$ that makes $B_0(\gamma^i) = 0$. The dimension of solution space is at most $v-1$ because the rank of matrix of Eqs. 6 is at least $k-v+1$. So there are at most $q^{v-1}$ solutions to Eqs. 6, and this determines the size of the list which is equal to $q^{-1}$.

B. Proof of lemma 3

Proof: We need to find the following probability:

$$
\Pr[(MAC(x', r) = t')|(MAC(x, r) = t)]
$$

The MAC function given by Eqs. 3 is equivalent to the MAC of the polynomial form in Eq. 2. For $0 \leq i \leq 3N-3$, the coefficients of $X^i$ in both sides of equation 7 form the same equation as the $i^{th}$ equation in the system of linear equations 2.

$$
t(X) = MAC(x, r) = \sum_{1 \leq m \leq d} x_m(X)r_m(X) + \sum_{d+1 \leq m \leq l} x_m(X)r_i(X)r_j(X) + r_{d+1}(X) \mod q
$$

(7)

where each polynomial is given below:

$$
x_m(X) = x_{m,0} + \cdots + x_{m,N-1}X^{N-1} \mod q, \ 1 \leq i \leq l
$$

$$
r_m(X) = r_{m,0} + \cdots + r_{m,N-1}X^{N-1} \mod q, \ 1 \leq m \leq d
$$

$$
r_m(X) = r_{i,j,0} + \cdots + r_{i,j,2N-2}X^{2N-2} =
$$

$$
r_i(X)r_j(X) \mod q, \ d+1 \leq m \leq l, \ m = id + j - \frac{i(i-1)}{2}
$$

$$
r_{d+1}(X) = r_{d+1,0} + \cdots + r_{d+1,3N-3}X^{3N-3} \mod q
$$

Finally, $t(X) = t_0 + \cdots + t_{3N-3}X^{3N-3} \mod q$.

So if we can prove that the adversary’s forging capability to the MAC in the form of Eq. 2 is no more than $\varepsilon$, then the adversary’s forging capability to MAC construction II (Eqs. 2) is also no more than $\varepsilon$.

Next we prove the adversary forging capability to MAC in the form of Eq. 2 is no more than $\frac{2}{q^N}$. Assume the adversary forges a message $(x', t')$ with $x' \neq x$, that passes the verification. We write the MAC in polynomial form.

$$
t'(X) = MAC(x', r) = \sum_{1 \leq m \leq d} x'_m(X)r_m(X) +
\sum_{d+1 \leq m \leq l} x'_m(X)r_i(X)r_j(X) + r_{d+1}(X) \mod q
$$

(8)

By subtracting the two equations we will have,

$$
\sum_{d+1 \leq m \leq l} \Delta x_m(X)r_i(X)r_j(X) + \sum_{1 \leq m \leq d} \Delta x_m(X)r_m(X) = \Delta t(X) \mod q
$$

The above equation has at most $2q^{N(d-1)}$ solutions for $(r_i(X), \cdots, r_d(X))$. This means that there are at most $2q^{N(d-1)}$ keys $r$ that satisfy $MAC(x, r) = t$, and $MAC(x', r) = t'$. However, there are $q^{Nd}$ possible values for $r$ satisfying $MAC(x, r) = t$. So the success probability of the forgery is,

$$
\Pr[(MAC(x', r) = t')|(MAC(x, r) = t)] = \frac{2q^{N(d-1)}}{q^{Nd}} = \frac{2}{q^N}
$$

\[\blacksquare\]