Beurling–Ahlfors extension by heat kernel, $A_\infty$-weights for VMO, and vanishing Carleson measures

Huaying Wei and Katsuhiko Matsuzaki

Abstract

We investigate a variant of the Beurling–Ahlfors extension of quasisymmetric homeomorphisms of the real line that is given by the convolution of the heat kernel, and prove that the complex dilatation of such a quasiconformal extension of a strongly symmetric homeomorphism (that is, its derivative is an $A_\infty$-weight whose logarithm is in VMO) induces a vanishing Carleson measure on the upper half-plane.

1. Introduction

Beurling and Ahlfors [2] characterized the boundary value of a quasiconformal homeomorphism of the upper half-plane $U$ onto itself as a quasisymmetric homeomorphism $f$ of the real line $\mathbb{R}$. Here, an increasing homeomorphism $f: \mathbb{R} \to \mathbb{R}$ is quasisymmetric if there is a constant $\rho > 1$ such that $|f(2I)| \leq \rho |f(I)|$ for any bounded interval $I \subset \mathbb{R}$, where $| \cdot |$ is the Lebesgue measure and $2I$ denotes the interval of the same center as $I$ with $|2I| = 2 |I|$. They proved that any quasisymmetric homeomorphism of $\mathbb{R}$ extends continuously to a quasiconformal homeomorphism $F: U \to U$ in a certain explicit way. This is called the Beurling–Ahlfors extension.

Let $\phi(x) = 1_{[-1,1]}(x)/2$ and $\psi(x) = r(1_{[-1,0]}(x) - 1_{[0,1]}(x))/2$ for some $r > 0$, where $1_{E}$ denotes the characteristic function of $E \subset \mathbb{R}$. For any function $\varphi(x)$ on $\mathbb{R}$ and for $t > 0$, we set $\varphi_t(x) = \frac{1}{t} \varphi(\frac{x}{t})$. Then, for a quasisymmetric homeomorphism $f$, the Beurling–Ahlfors extension $F(x,t) = (U(x,t), V(x,t))$ for $(x, t) \in U$ is defined by the convolutions

$$U(x,t) = (f * \phi_t)(x), \quad V(x,t) = (f * \psi_t)(x).$$

The parameter $r$ may change when we consider a problem of estimating the maximal dilatation of the Beurling–Ahlfors extension $F$ in terms of the quasisymmetry constant of $f$ related to the doubling constant $\rho$. In particular, when we investigate the asymptotic conformality of possible quasiconformal extensions $F(x,t)$ of $f$ as $t \to 0$, the Beurling–Ahlfors extension for $r = 2$ gives a powerful tool, as is shown in Carleson [3].

Modification and variation to the Beurling–Ahlfors extension have been made by replacing the functions $\phi$ and $\psi$. These methods are particularly effective for a study of relevant problems in harmonic analysis. A locally integrable function $h$ on $\mathbb{R}$ is of BMO (denoted by $h \in \text{BMO}(\mathbb{R})$) if

$$\|h\|_{\text{BMO}} = \sup_{I \subset \mathbb{R}} \frac{1}{|I|} \int_I |h(x) - h_I| dx < \infty,$$

where the supremum is taken over all bounded intervals $I$ on $\mathbb{R}$ and $h_I$ denotes the integral mean of $h$ over $I$. Semmes [7] took $\phi$ and $\psi$ in $C^\infty(\mathbb{R})$ supported on $[-1,1]$ such that $\phi$ is an even
function with $\int_\mathbb{R} \phi(x) dx = 1$ and $\psi$ is an odd function with $\int_\mathbb{R} x \psi(x) dx = -1$. It was proved that if a quasisymmetric homeomorphism $f : \mathbb{R} \to \mathbb{R}$ is locally absolutely continuous and $\log \omega$ for $\omega = f'$ is in $\text{BMO}(\mathbb{R})$ with the norm $\| \log \omega \|_{\text{BMO}}$ small, then this modified Beurling–Ahlfors extension $F$ is quasiconformal such that $\frac{1}{t} |\mu_F(x, t)|^2 dx dt$ is a Carleson measure on $U$, where $\mu_F = \partial F/\partial F$ is the complex dilatation of $F$. Here, a measure $\lambda(x, t) dx dt$ on $U$ is called a Carleson measure if

$$
\| \lambda \|_c = \sup_{I \subset \mathbb{R}} \frac{1}{|I|} \int_0^{|I|/2} \int_I \lambda(x, t) dx dt < \infty,
$$

where the supremum is also taken over all bounded intervals $I$. The Carleson norm $\| \cdot \|_c$ of $\frac{1}{t} |\mu_F(x, t)|^2 dx dt$ is estimated in terms of $\| \log \omega \|_{\text{BMO}}$. The arguments rely on the John–Nirenberg inequality for BMO functions, so the assumption on the smallness of the BMO norm is needed for a single application of the Beurling–Ahlfors extension.

In the paper by Fefferman, Kenig and Pipher [5], a variant of the Beurling–Ahlfors extension was also utilized, where $\phi(x) = \frac{1}{\sqrt{\pi}} e^{-x^2}$ and $\psi(x) = \frac{1}{\sqrt{\pi} t} e^{-x^2/t}$. In this case, the $x$-derivative $U_x(x, \sqrt{t}) = (\omega * \phi_{\sqrt{t}})(x)$, for example, is the solution of the heat equation having $\omega = f'$ as the initial state, which is represented by the heat kernel $\phi_{\sqrt{t}}(x) = \frac{1}{\sqrt{\pi} t} e^{-x^2/t}$. We see from their arguments that if $f : \mathbb{R} \to \mathbb{R}$ is locally absolutely continuous and the derivative $\omega = f'$ is an $A_\infty$-weight introduced by Muckenhoupt (see [4]), which implies that $\log \omega \in \text{BMO}(\mathbb{R})$, then this variant of the Beurling–Ahlfors extension $F$ is quasiconformal and also induces a Carleson measure $\frac{1}{t} |\mu_F(x, t)|^2 dx dt$. No assumption on the BMO norm is necessary.

In this present paper, in view of the importance of the arguments in [5], we give a rather detailed proof of the aforementioned results by picking up related parts from the original paper and complementing necessary arguments between the sentences in it. Sections 2 and 3 are devoted to these arrangements of the theorems in [5]. Then in Section 4, we adapt the arguments involving the BMO norm in [7] to the variant of the Beurling–Ahlfors extension $F$ of $f$ given by the heat kernel. To this end, we generalize the proof in [7] for $\phi$ and $\psi$ of compact supports to those rapidly decreasing functions of non-compact supports, which is a novelty in this paper. As a result, we obtain an estimate of the Carleson norm of $\frac{1}{t} |\mu_F(x, t)|^2 dx dt$ in terms of the BMO norm of $\log \omega$ when it is small. This is valid even if the smallness is localized as in the case mentioned next.

It is said that $h \in \text{BMO}(\mathbb{R})$ is of $\text{VMO}$ if

$$
\lim_{|I| \to 0} \frac{1}{|I|} \int_I |h(x) - h_I| dx = 0.
$$

Correspondingly, a Carleson measure $\lambda(x, t) dx dt$ is vanishing if

$$
\lim_{|I| \to 0} \frac{1}{|I|} \int_0^{|I|/2} \int_I \lambda(x, t) dx dt = 0.
$$

Thus, we can show that if $\log \omega \in \text{VMO}(\mathbb{R})$ for an $A_\infty$-weight $\omega = f'$, then the variant of the Beurling–Ahlfors extension $F$ of $f$ by the heat kernel yields that the Carleson measure $\frac{1}{t} |\mu_F(x, t)|^2 dx dt$ is vanishing. This is a problem asked by Shen [8] in his study of the VMO Teichmüller space on the real line.

2. Heat equation for $A_\infty$-weights

This section is an exposition of a part of Section 3 of Fefferman, Kenig and Pipher [5].

For an $A_\infty$-weight $\omega$ on the real line $\mathbb{R}$, we define

$$
u(x, t) = (\omega * \Phi_t)(x) \quad (x \in \mathbb{R}, \ t > 0),$$

where \( \Phi_t(x) \) is the heat kernel given by

\[
\Phi_t(x) = \frac{1}{\sqrt{\pi t}} e^{-\frac{x^2}{t}}.
\]

We remark that this \( \Phi_t \) comes from \( \Phi(x) = \frac{1}{\sqrt{\pi}} e^{-\frac{x^2}{\sqrt{\pi}}} \) and the definition of \( \varphi_t(x) = \frac{1}{\sqrt{\pi t}} \varphi\left(\frac{x}{\sqrt{t}}\right) \) for a general function \( \varphi \) in this section is slightly different from that in the other sections. This satisfies \( H\Phi_t(x) = 0 \) for

\[
H = \frac{\partial}{\partial t} - \frac{\partial^2}{4\partial x^2},
\]

and hence \( Hu(x,t) = 0 \).

The solution \( u \) for the heat equation with the initial state \( \omega \) satisfies the following:

**Lemma 2.1.** There is a constant \( c > 0 \) such that

\[
cu(x,t) \lesssim \frac{1}{\sqrt{t}} \int_{|x-y| < \sqrt{t}} \omega(y)dy \lesssim e^{\sqrt{\pi}}u(x,t)
\]

for any \( x \in \mathbb{R} \) and \( t > 0 \).

**Proof.** We decompose the integral for the convolution as

\[
u(x,t) = \int_{|x-y| < \sqrt{t}} \omega(y)\Phi_t(x-y)dy + \sum_{n=1}^{\infty} \int_{2^{n-1}\sqrt{t} \leq |x-y| < 2^n \sqrt{t}} \omega(y)\Phi_t(x-y)dy.
\]

Then, the second inequality in the statement is given by

\[
u(x,t) \geq \Phi_t(\sqrt{t}) \int_{|x-y| < \sqrt{t}} \omega(y)dy \geq \frac{1}{e^{\sqrt{\pi}} \sqrt{t}} \int_{|x-y| < \sqrt{t}} \omega(y)dy.
\]

For the first inequality, we use the doubling property of \( \omega \): there is a constant \( \rho > 1 \) such that

\[
\int_{2I} \omega(x)dx \leq \rho \int_I \omega(x)dx
\]

for any bounded interval \( I \subset \mathbb{R} \). Then, we have

\[
\sum_{n=1}^{\infty} \int_{2^{n-1}\sqrt{t} \leq |x-y| < 2^n \sqrt{t}} \omega(y)\Phi_t(x-y)dy \leq \sum_{n=1}^{\infty} \Phi_t(2^{n-1}\sqrt{t}) \int_{|x-y| < 2^n \sqrt{t}} \omega(y)dy
\]

\[
\leq \sum_{n=1}^{\infty} \frac{\rho^n}{e^{4n-1} \sqrt{\pi t}} \int_{|x-y| < \sqrt{t}} \omega(y)dy.
\]

Hence, for \( c^{-1} = (1 + \sum_{n=1}^{\infty} \frac{\rho^n}{e^{4n-1} \sqrt{\pi}}) / \sqrt{\pi} < \infty \), we obtain the first inequality. \( \square \)

We consider the spatial derivative \( u'(x,t) = \frac{\partial}{\partial x} u(x,t) = (\omega * (\Phi_t)')(x) \), where

\[
(\Phi_t)'(x) = -\frac{2}{t \sqrt{\pi t}} e^{-\frac{x^2}{t}} = -\frac{2}{t} \Phi_t(x).
\]

**Lemma 2.2.** There is a constant \( C_1 > 0 \) such that

\[
|u'(x,t)| \lesssim \frac{C_1}{\sqrt{t}} u(x,t)
\]

for any \( x \in \mathbb{R} \) and \( t > 0 \).
This yields to

\[ \sum_{n=1}^{\infty} \int_{|x-y|<\sqrt{t}} \omega(y)(\Phi_t)'(x-y)dy + \sum_{n=1}^{\infty} \int_{2^{n-1}\sqrt{t} \leq |x-y| < 2^{n}\sqrt{t}} \omega(y)(\Phi_t)'(x-y)dy. \]

The first term is estimated as

\[ \left| \int_{|x-y|<\sqrt{t}} \omega(y)(\Phi_t)'(x-y)dy \right| \leq \int_{|x-y|<\sqrt{t}} \omega(y)|(\Phi_t)'(x-y)dy \]

\[ \leq \frac{\sqrt{2}}{\sqrt{\pi} t} \int_{|x-y|<\sqrt{t}} \omega(y)dy. \]

Here, we used the fact that \( \max_{|x|<\sqrt{t}} |(\Phi_t)'(x)| = \sqrt{2}/(\sqrt{\pi} t) \) attained at \( x = \sqrt{t}/\sqrt{2} \). The remainder terms are estimated in the same way as before:

\[ \sum_{n=1}^{\infty} \int_{2^{n-1}\sqrt{t} \leq |x-y| < 2^{n}\sqrt{t}} \omega(y)(\Phi_t)'(x-y)dy \leq \sum_{n=1}^{\infty} |(\Phi_t)'(2^{n-1}\sqrt{t})| \int_{|x-y|<2^n\sqrt{t}} \omega(y)dy \]

\[ \leq \sum_{n=1}^{\infty} (2\rho)^n e^{4^{n-1}\sqrt{\pi} t} \int_{|x-y|<\sqrt{t}} \omega(y)dy. \]

Then, by Lemma 2.1, we have

\[ |u'(x,t)| \leq \left( \frac{\sqrt{2}}{\sqrt{\pi}} + \sum_{n=1}^{\infty} (2\rho)^n e^{4^{n-1}\sqrt{\pi} t} \right) \frac{1}{t} \int_{|x-y|<\sqrt{t}} \omega(y)dy \leq \frac{C_1}{\sqrt{t}} u(x,t) \]

for an appropriate constant \( C_1 > 0 \). \( \square \)

We prove the following necessary condition for a weight \( \omega \) to be in \( A_\infty(\mathbb{R}) \). This corresponds to [5, Theorem 3.4].

**Theorem 2.3.** The solution \( u \) for an initial state \( \omega \in A_\infty(\mathbb{R}) \) satisfies that

\[ \frac{1}{t} \int_0^t \int_{|x-x_0|<t} \frac{u'(x,s)^2}{u(x,s)^2} dxds \]

is uniformly bounded for any \( x_0 \in \mathbb{R} \) and \( t > 0 \).

**Proof.** A simple computation using \( Hu = 0 \) shows that

\[ H \log u(x,t) = \frac{\partial}{\partial t} \log u(x,t) - \frac{\partial^2}{4\partial x^2} \log u(x,t) \]

\[ = \frac{Hu(x,t)}{u(x,t)} + \frac{u'(x,t)^2}{4u(x,t)^2} = \frac{u'(x,t)^2}{4u(x,t)^2}. \]

This yields

\[ \frac{1}{4} \int_0^t \int_{|x-x_0|<t} \frac{u'(x,s)^2}{u(x,s)^2} dxds = \int_0^t \int_{|x-x_0|<t} H \log u(x,s)dxds \]

\[ = \int_{|x-x_0|<t} \int_0^t \frac{\partial}{\partial s} \log u(x,s)dsdx - \int_0^t \int_{|x-x_0|<t} \frac{\partial^2}{4\partial x^2} \log u(x,s)dxds \]

\[ = \int_{|x-x_0|<t} (\log u(x,t^2) - \log \omega(x))dx - \frac{1}{4} \int_0^t \left( \frac{u'(x_0 + t,s)}{u(x_0 + t,s)} - \frac{u'(x_0 - t,s)}{u(x_0 - t,s)} \right) ds. \]
Here, by Lemma 2.2, the second term of (1) is bounded by
\[
\frac{2}{4} \int_0^t \frac{C_1}{\sqrt{s}} ds = C_1 t.
\]
For the estimate of the first term of (1), we use the following result. □

**Lemma 2.4.** Any weight \( \omega \in A_{\infty}(\mathbb{R}) \) satisfies that
\[
\frac{1}{t} \int_{|x-x_0|<t} (\log u(x, t^2) - \log(1) - \log(\omega(x))) dx
\]
is uniformly bounded for any \( x_0 \in \mathbb{R} \) and \( t > 0 \).

**Proof.** Lemma 2.1 implies that
\[
u(x, t^2) \asymp \frac{1}{t} \int_{|x-y|<t} \omega(y) dy
\]
uniformly for all \( x \in \mathbb{R} \) and \( t > 0 \) (the notation \( \asymp \) is used in this sense hereafter). Moreover, the doubling property of \( \omega \) implies that if \( x \) satisfies \( |x - x_0| < t \) for a fixed \( x_0 \), then
\[
\int_{|x-y|<t} \omega(y) dy \asymp \int_{|x_0-y|<t} \omega(y) dy.
\]
Hence, if \( |x - x_0| < t \), then
\[
u(x, t^2) \asymp \frac{1}{t} \int_{|x_0-y|<t} \omega(y) dy
\]
independently of \( t > 0 \). This shows that there is a constant \( C_2 > 0 \) such that
\[
\int_{|x-x_0|<t} \log u(x, t^2) dx \leq 2t \log \left( \frac{1}{2t} \int_{|x_0-y|<t} \omega(y) dy \right) + tC_2.
\]
It is known that \( \omega \in A_{\infty}(\mathbb{R}) \) if and only if
\[
\frac{1}{|I|} \int_I \omega(x) dx \asymp \exp \left( \frac{1}{|I|} \int_I \log \omega(x) dx \right)
\]
for every bounded interval \( I \subset \mathbb{R} \) (see [6]). This implies that there is a constant \( C_3 > 0 \) such that
\[
2t \log \left( \frac{1}{2t} \int_{|x_0-y|<t} \omega(y) dy \right) - \int_{|x_0-y|<t} \log(\omega(y) dy) \leq C_3 t.
\]
Combining this with the previous estimate, we have
\[
\int_{|x-x_0|<t} (\log u(x, t^2) - \log(\omega(x))) dx \leq (C_2 + C_3) t,
\]
which proves the required inequality. □

**Proof of Theorem 2.3 continued.** By the above estimates for the last line of (1), we obtain that
\[
\frac{1}{4} \int_0^t \int_{|x-x_0|<t} u'(x,s)^2 \frac{u(x,s)}{u(x,s)} dx ds \leq (C_1 + C_2 + C_3) t,
\]
and thus proves the statement. □
We can obtain a similar result to Theorem 2.3 concerning the spatial second derivative \( u''(x,t) = \frac{\partial^2}{\partial x^2} u(x,t) = (\omega \ast (\Phi_t)''(x) \). As this result is necessary for the proof of Theorem 3.4 in the next section, we formulate this especially and give a proof for it in our paper. We apply the argument in [5, Lemma 3.2].

**Theorem 2.5.** The solution \( u \) for an initial state \( \omega \in A_\infty(\mathbb{R}) \) satisfies that

\[
\frac{1}{t} \int_0^t \int_{|x-x_0|<t} s \frac{u''(x,s)}{u(x,s)^2} \, dx \, ds
\]

is uniformly bounded for any \( x_0 \in \mathbb{R} \) and \( t > 0 \).

**Proof.** First, we will find an appropriate function \( \eta \) on \( \mathbb{R} \) that satisfies \( \Phi = \eta \ast \Phi_\frac{1}{2} \). We use the Fourier transformation

\[
\mathcal{F}(h)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(x) e^{-i\xi x} \, dx
\]

of a function \( h \) on \( \mathbb{R} \). Then, the desired function \( \eta \) should satisfy that

\[
i\xi \mathcal{F}(\Phi)(\xi) = \mathcal{F}(\Phi')(\xi) = \sqrt{2\pi} \mathcal{F}(\eta)(\xi) \cdot \mathcal{F}(\Phi_\frac{1}{2})(\xi),
\]

and hence

\[
\mathcal{F}(\eta)(\xi) = \frac{i}{\sqrt{2\pi}} \frac{\mathcal{F}(\Phi')(\xi)}{\mathcal{F}(\Phi_\frac{1}{2})(\xi)} = \frac{i}{\sqrt{2\pi}} \frac{\frac{1}{\sqrt{2\pi}} e^{-\frac{\xi^2}{4}}}{\frac{1}{\sqrt{2\pi}} e^{-\frac{\xi^2}{8}}} = \frac{i}{\sqrt{2\pi}} e^{-\frac{\xi^2}{4}}.
\]

Therefore, by the inverse Fourier transformation

\[
\mathcal{F}^{-1}(\tilde{h})(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{h}(\xi) e^{i\xi x} \, d\xi,
\]

we have that

\[
\eta(x) = \mathcal{F}^{-1} \left( \frac{i}{\sqrt{2\pi}} \frac{e^{-\frac{\xi^2}{4}}}{e^{-\frac{\xi^2}{8}}} \right)(x) = -\frac{4\sqrt{2}}{\sqrt{\pi}} x e^{-2x^2}.
\]

We represent \( u''(x,t) = (\omega \ast (\Phi_t)''(x) \) by using \( \eta_t(x) = \frac{1}{\sqrt{t}} \eta(\frac{x}{\sqrt{t}}) \). We note that

\[
(\Phi_t)'(x) = \frac{1}{\sqrt{t}} (\Phi)'(\frac{x}{\sqrt{t}}) = \frac{1}{\sqrt{t}} (\eta \ast \Phi_{\frac{1}{2}})(\frac{x}{\sqrt{t}}) = \frac{1}{\sqrt{t}} (\eta_t \ast \Phi_{\frac{1}{2}})(x).
\]

Hence

\[
u''(x,t) = (\omega \ast (\Phi_t)''(x) = \frac{1}{\sqrt{t}} (\omega \ast \eta_t \ast (\Phi_{\frac{1}{2}})'')(x).
\]

Using this, we obtain that

\[
u''(x,t)^2 = \frac{1}{t} \left( \int_{-\infty}^{\infty} \eta_t(x-y)(\omega \ast (\Phi_{\frac{1}{2}})'')(y) \, dy \right)^2
\]

\[
\leq \frac{1}{t} \left( \int_{-\infty}^{\infty} |\eta_t(x-y)| \, dy \right) \left( \int_{-\infty}^{\infty} |\eta_t(x-y)| (\omega \ast (\Phi_{\frac{1}{2}})'')(y)^2 \, dy \right)
\]

\[
\leq \frac{2}{t} \int_{-\infty}^{\infty} |\eta_t(x-y)| u''(y,t/2)^2 \, dy,
\]

where we used \( \int |\eta_t(y)| \, dy = \int |\eta(y)| \, dy = 2\sqrt{2} / \sqrt{\pi} < 2 \) in the last inequality.
To dominate the integrand on question, we use an inequality $u(x, t/2)^2 \leq D_1 u(x, t)^2$ for some constant $D_1 > 0$, which is obtained by Lemma 2.1. Then

$$
\frac{u''(x, t)}{u(x, t)} = \frac{u(x, t/2)^2}{u(x, t)^2} \frac{u''(x, t)}{u(x, t/2)^2} \leq D_1 \frac{u''(x, t)}{u(x, t/2)^2}.
$$

Therefore,

$$
\frac{1}{t}\int_0^t \int_{|x-x_0|<t} s \frac{u''(x, s)}{u(x, s)^2} dx ds
\leq \frac{2D_1}{t} \int_0^t \int_{|x-x_0|<t} \left( \int_{-\infty}^{\infty} |\eta_s(x-y)| u'(y, s/2)^2 dx \right) \frac{1}{u(x, s/2)^2} dx ds
\leq \frac{4D_1}{t} \int_0^t \int_{|x-x_0|<t} \left( \int_{|x-y|<\sqrt{s}} |\eta_s(x-y)| u(y, s)^2 \frac{u'(y, s)^2}{u(x, s)^2} dx dy \right) ds
\leq \frac{4D_1}{t} \int_0^t \int_{|x-x_0|<t} \sum_{k=1}^{\infty} \left( \int_{\sqrt{2k} \leq |x-y| < 2k \sqrt{s}} |\eta_s(x-y)| u(y, s)^2 \frac{u'(y, s)^2}{u(x, s)^2} dx dy \right) ds.
$$

(2)

We estimate the first term in the last line of (2). If $|x-y| < \sqrt{s}$, then by Lemma 2.1 and the doubling property of $\omega$ with the constant $\rho$, we have

$$
\frac{u(y, s)^2}{u(x, s)^2} \leq D_2 \left( \int_{|y-z|<\sqrt{s}} \omega(z) dz \right)^2 \leq D_2 \rho^2
$$

for some constant $D_2 > 0$. Moreover, if $|x-y| < \sqrt{s} \leq t/\sqrt{2}$ and $|x-x_0| < t$, then $|y-x_0| < 2t$. Hence, the integrand $I_0(s)$ by $ds$ is estimated as

$$
I_0(s) = \int_{|x-y| < \sqrt{s}} \left( \int_{|x-x_0|<t} |\eta_s(x-y)| u(y, s)^2 \frac{u'(y, s)^2}{u(x, s)^2} dx dy \right)
\leq \frac{(\max |\eta|) D_2 \rho^2}{\sqrt{2s}} \int_{|y-x_0| < 2t} u'(y, s)^2 dy ds \leq D_2 \rho^2 \int_{|y-x_0| < 2t} u'(y, s)^2 dy ds,
$$

where we used a fact that $\max_{x\in\mathbb{R}} |\eta(x)|$ is $2\sqrt{2}e^{-1/2}/\sqrt{\pi} < 1$ attained at $x = \pm 1/2$ in the last inequality. Then, by letting the uniform bound in Theorem 2.3 $C_0 > 0$, this theorem shows that

$$
\frac{4D_1}{t} \int_0^t I_0(s) ds \leq \frac{8D_1 D_2 \rho^2}{2t} \int_0^{(2t)^2} \int_{|y-x_0| < 2t} u'(y, s)^2 dy ds dyds \leq 8C_0 D_1 D_2 \rho^2.
$$

Next, we consider the second term in the last line of (2). If $|x-y| < 2^{k+1} \sqrt{s}$ for $k \geq 1$, then

$$
\frac{u(y, s)^2}{u(x, s)^2} \leq D_2 \rho^2(k+1).
$$
Moreover, if \( |x - y| < 2^k \sqrt{s} \leq 2^k t / \sqrt{2} \) and \( |x - x_0| < t \), then \( |y - x_0| < 2^{k+1} t \). Furthermore, if \( 2^{k-1} \sqrt{s} \leq |x - y| \) in addition, then
\[
|\eta_2(x - y)| = \frac{2\sqrt{2}}{\sqrt{s}} |x - y| e^{-\frac{2|x-y|^2}{s}} < \frac{2^{k+1}}{\sqrt{s}} e^{-4^{k-1}}.
\]

Hence, the integrand \( I_k(s) \) \((k \geq 1)\) by \( ds \) is estimated as
\[
I_k(s) = \iint_{2^{k-1} \sqrt{s} \leq |x - y| < 2^k \sqrt{s}} |\eta_2(x - y)| u(y,s)^2 u'(y,s)^2 dxdy
\leq D_2 \rho 2^{2(k+1)} \frac{2^{k+1}}{\sqrt{s}} e^{-4^{k-1}} \iint_{|y - x_0| < 2^{k+1} t} u'(y,s)^2 dxdy
\leq D_2 (2\rho) 2^{2(k+1)} e^{-4^{k-1}} \iint_{|y-x_0|<2^{k+1}t} u'(y,s)^2 dy.
\]

Then, we have
\[
\frac{4D_1}{t} \int_0^t I_k(s) ds
\leq 4D_1 D_2 (2\rho) 2^{2(k+1)} e^{-4^{k-1}} \frac{1}{t} \int_0^t \int_{|y-x_0|<2^{k+1}t} u'(y,s)^2 dyds
\leq D_1 D_2 (4\rho) 2^{2(k+1)} e^{-4^{k-1}} \left( \frac{1}{2^{k+1} t} \int_0^{(2^{k+1}) t} \int_{|y-x_0|<2^{k+1}t} u'(y,s)^2 dyds \right)
\leq C_0 D_1 D_2 (4\rho) 2^{2(k+1)} e^{-4^{k-1}},
\]
where the last inequality is also due to Theorem 2.3.

Combining these two estimates for (2), we obtain that
\[
\frac{1}{t} \int_0^t \int_{|x-x_0|<t} u''(x,s)^2 ds dxds \leq C_0 D_1 D_2 \sum_{k=0}^{\infty} (4\rho) 2^{2(k+1)} e^{-4^{k-1}} < \infty.
\]

Thus, we complete the proof of the theorem.

\[\square\]

3. The heat kernel variant of the Beurling–Ahlfors extension

This section is an exposition of a part of Section 4 of Fefferman, Kenig and Pipher [5].

Let \( \phi(x) = \frac{1}{\sqrt{\pi t}} e^{-x^2} \) and \( \psi(x) = \phi'(x) = -2x\phi(x) \). For any \( t > 0 \), we set \( \phi_t(x) = \frac{1}{t} \phi(\frac{x}{t}) \) and \( \psi_t(x) = \frac{1}{t} \psi(\frac{x}{t}) \). For a doubling weight \( \omega \) on \( \mathbb{R} \), we define a quasisymmetric homeomorphism \( f : \mathbb{R} \to \mathbb{R} \) by \( f(x) = \int_0^x \omega(y)dy \). Then, we extend \( f \) to the upper half-plane \( \mathbb{U} = \{(x, t) \mid t > 0\} \) by setting a differentiable map \( F(x, t) = (U, V) \) for \( U(x, t) = (f * \phi_t)(x) \) and \( V(x, t) = (f * \psi_t)(x) \).

In Section 2, we dealt with the heat kernel \( \Phi_t(x) = \frac{1}{\sqrt{\pi t}} e^{-\frac{x^2}{2t}} \) and the solution \( u(x, t) = (\omega * \Phi_t)(x) \) of the heat equation with the initial state \( \omega \). Then, the partial derivatives of \( U \) and

\[
\frac{dU}{dt}(x, t) = \frac{1}{2\sqrt{\pi t}} e^{-\frac{x^2}{2t}} (x^2 - 1) \quad \text{in} \quad \mathbb{U}.
\]

Moreover, the heat kernel variant of the Beurling–Ahlfors extension

\[
I_k(s) \leq D_2 (2\rho) 2^{2(k+1)} e^{-4^{k-1}} \iint_{|y-x_0|<2^{k+1}t} u'(y,s)^2 dy.
\]

Combining these two estimates for (2), we obtain that
\[
\frac{1}{t} \int_0^t \int_{|x-x_0|<t} u''(x,s)^2 ds dxds \leq C_0 D_1 D_2 \sum_{k=0}^{\infty} (4\rho) 2^{2(k+1)} e^{-4^{k-1}} < \infty.
\]

Thus, we complete the proof of the theorem.
$V$ are represented as follows:

$$U_x = \frac{\partial U}{\partial x} = (\omega * \phi_t)(x) = (\omega * \Phi_1^*)(x) = u(x, t^2);$$

$$V_x = \frac{\partial V}{\partial x} = (\omega * \psi_t)(x) = t(\omega * (\Phi_1^*))'(x) = tu'(x, t^2);$$

$$U_t = \frac{\partial U}{\partial t} = \left( f * \frac{\partial \phi_t}{\partial t} \right)(x) = \frac{1}{2} (\omega * \psi_t)(x) = \frac{1}{2} V_x;$$

$$V_t = \frac{\partial V}{\partial t} = \left( f * \frac{\partial \psi_t}{\partial t} \right)(x) = U_x + \frac{t^2}{2} (\omega * (\phi_t)'')(x) = 2(\omega * \tilde{\phi}_t)(x)$$

for $\tilde{\phi}_t(x) = \frac{t^2}{\pi} \phi_t(x)$. We note here that

$$\frac{\partial \phi_t}{\partial t}(x) = \frac{t}{2} \frac{\partial^2 \phi_t}{\partial x^2}(x) = \frac{1}{2} (\psi_t)'(x);$$

$$\frac{\partial \psi_t}{\partial t}(x) = \frac{\partial \phi_t}{\partial x}(x) + \frac{t^2}{2} \frac{\partial^3 \phi_t}{\partial x^3}(x) = 2(\tilde{\phi}_t)'(x).$$

**Proposition 3.1.** $|V_x(x, t)| = 2|U_t(x, t)| \leq C_1 U_x(x, t)$ whereas $|V_t(x, t)| \leq U_x(x, t)$.

**Proof.** Lemma 2.2 implies

$$\frac{|V_x(x, t)|}{U_x(x, t)} = \frac{2|U_t(x, t)|}{U_x(x, t)} = \frac{t|u'(x, t^2)|}{u(x, t^2)} \leq C_1.$$

Since $V_t(x, t) = 2(\omega * \tilde{\phi}_t)(x)$ and $U_x(x, t) = (\omega * \phi_t)(x)$, the latter statement follows from the next lemma. \qed

**Lemma 3.2.** For a doubling weight $\omega$ on $\mathbb{R}$, we have

$$\int_{\mathbb{R}} \omega(y) \phi_t(x - y) dy \leq \frac{1}{t} \int_{\mathbb{R}} \omega(y)|x - y| \phi_t(x - y) dy \leq \frac{1}{t^2} \int_{\mathbb{R}} \omega(y)|x - y|^2 \phi_t(x - y) dy$$

uniformly for any $x \in \mathbb{R}$ and $t > 0$.

**Proof.** Lemma 2.1 implies

$$\int_{\mathbb{R}} \omega(y) \phi_t(x - y) dy \geq \frac{1}{t} \int_{|x - y| < t} \omega(y) dy.$$

An inequality

$$\int_{\mathbb{R}} \omega(y)|x - y| \phi_t(x - y) dy \leq C \int_{|x - y| < t} \omega(y) dy$$

for some $C > 0$ is essentially given in Lemma 2.2. By a similar argument of the decomposition of the integral, we can also show that

$$\frac{1}{t} \int_{\mathbb{R}} \omega(y)|x - y|^2 \phi_t(x - y) dy \leq C \int_{|x - y| < t} \omega(y) dy$$

for some $C > 0$ possibly different. Indeed, $\max_{|x| < t} x^2 \phi_t(x) = \frac{1}{e \sqrt{\pi}}$ implies

$$\int_{|x - y| < t} \omega(y)|x - y|^2 \phi_t(x - y) dy \leq \frac{t}{e \sqrt{\pi}} \int_{|x - y| < t} \omega(y) dy,$$
and the remainder terms are estimated as
\[
\sum_{n=1}^{\infty} \left| \int_{2^{n-1}t \leq |x-y| < 2^n t} \omega(y)|x-y|^2 \phi_t(x-y)dy \right| \leq \sum_{n=1}^{\infty} \frac{4^{n-1}t}{\pi n} \int_{|x-y| < 2^n t} \omega(y)dy
\]
\[
\leq \sum_{n=1}^{\infty} \frac{(4\rho)^n}{4\pi^{n+1}} \int_{|x-y| < t} \omega(y)dy.
\]
Hence, we have
\[
\frac{1}{t} \int_{\mathbb{R}} \omega(y)|x-y|^2 \phi_t(x-y)dy \leq \left( \frac{1}{e\pi} + \sum_{n=1}^{\infty} \frac{(4\rho)^n}{4\pi^{n+1}} \right) \int_{|x-y| < t} \omega(y)dy.
\]
For the converse inequalities, by using the doubling constant \( \rho > 1 \) for \( \omega \) again, we have
\[
\int_{|x-y| < t} \omega(y)dy \leq \int_{|x-y| < 4t} \omega(y)dy \leq \rho \int_{t \leq |x-y| < 3t} \omega(y)dy.
\]
Moreover, trivial estimates show that
\[
\int_{\mathbb{R}} \omega(y)|x-y|\phi_t(x-y)dy \geq \int_{t \leq |x-y| < 3t} \omega(y)|x-y|\phi_t(x-y)dy \geq \frac{3}{e\pi} \int_{t \leq |x-y| < 3t} \omega(y)dy;
\]
\[
\int_{\mathbb{R}} \omega(y)|x-y|^2\phi_t(x-y)dy \geq \int_{t \leq |x-y| < 3t} \omega(y)|x-y|^2\phi_t(x-y)dy \geq \frac{9t}{e\pi} \int_{t \leq |x-y| < 3t} \omega(y)dy.
\]
Therefore,
\[
\int_{\mathbb{R}} \omega(y)|x-y|\phi_t(x-y)dy \approx \int_{|x-y| < t} \omega(y)dy \approx \frac{1}{t} \int_{\mathbb{R}} \omega(y)|x-y|^2\phi_t(x-y)dy,
\]
which proves the statement. \( \square \)

The heat kernel variant of the Beurling–Ahlfors extension can be stated as follows. If we start with a given quasisymmetric homeomorphism \( f \) of \( \mathbb{R} \), the only requirement for \( f \) in this theorem is that \( f \) is locally absolutely continuous. This theorem corresponds to [5, Lemma 4.4]

**Theorem 3.3.** For a doubling weight \( \omega \) on \( \mathbb{R} \), the differentiable map \( F : U \to \mathbb{U} \) is a quasiconformal homeomorphism that extends continuously to the quasisymmetric homeomorphism \( f \) of \( \mathbb{R} \).

**Proof.** For the complex dilatation \( \mu_F = \partial F/\overline{\partial F} \), we consider
\[
K_F(x,t) = \frac{1 + |\mu_F|^2}{1 - |\mu_F|^2} = \frac{|\partial F|^2 + |\overline{\partial F}|^2}{|\partial F|^2 - |\overline{\partial F}|^2} = \frac{U_x^2 + U_t^2 + V_x^2 + V_t^2}{2(U_x V_t - U_t V_x)},
\]
and prove that this is uniformly bounded. Proposition 3.1 implies
\[
U_x^2 + U_t^2 + V_x^2 + V_t^2 \asymp U_x^2.
\]
The Cauchy–Schwarz inequality implies
\[
U_x V_t = \frac{2}{t^2} \int_{\mathbb{R}} \omega(y)\phi_t(x-y)dy \int_{\mathbb{R}} \omega(y)(x-y)^2\phi_t(x-y)dy
\]
\[
\geq \frac{2}{t^2} \left( \int_{\mathbb{R}} \omega(y)|x-y|\phi_t(x-y)dy \right)^2.
\]
Then,
\[ U_x V_t - U_t V_x \geq \frac{2}{t^2} \left( \int_R \omega(y)|x-y|\phi_t(x-y)dy \right)^2 - \frac{2}{t^2} \left( \int_R \omega(y)(x-y)\phi_t(x-y)dy \right)^2. \]

We set
\[ I_1(x, t) = \int_{x-y \geq 0} \omega(y)(x-y)\phi_t(x-y)dy; \quad I_2(x, t) = \int_{x-y \leq 0} \omega(y)|x-y|\phi_t(x-y)dy. \]

Then, by similar arguments to those in Lemmas 2.1, 2.2 and 3.2 using the doubling property of \( \omega \), we have
\[ I_1(x, t) \asymp \int_{0 \leq x-y < t} \omega(y)(x-y)\phi_t(x-y)dy \asymp \int_{-t < x-y \leq 0} \omega(y)|x-y|\phi_t(x-y)dy \asymp I_2(x, t). \]

We consider
\[ \left( \frac{\int_R \omega(y)(x-y)\phi_t(x-y)dy}{\int_R \omega(y)|x-y|\phi_t(x-y)dy} \right)^2 = \left( \frac{I_1 - I_2}{I_1 + I_2} \right)^2 = \frac{(1 - I_2/I_1)^2}{(1 + I_2/I_1)^2}. \]

If \( I_2 \leq I_1 \), then the ratio \( I_2/I_1 \leq 1 \) is bounded away from 0. Similarly, when \( I_1 \leq I_2 \), we consider \( I_1/I_2 \leq 1 \) instead. Hence, there is some constant \( \varepsilon > 0 \) such that
\[ \left( \int_R \omega(y)(x-y)\phi_t(x-y)dy \right)^2 \leq (1 - \varepsilon) \left( \int_R \omega(y)|x-y|\phi_t(x-y)dy \right)^2. \]

The above inequality implies
\[ U_x V_t - U_t V_x \geq \frac{2\varepsilon}{t^2} \left( \int_R \omega(y)|x-y|\phi_t(x-y)dy \right)^2 > 0. \]

Then, Lemma 3.2 shows that the middle term of this inequality is greater than \( \varepsilon' U_x^2 \) for some constant \( \varepsilon' > 0 \). This concludes that \( K_F(x, t) \) is uniformly bounded, and hence \( \|\mu_F\|_\infty < 1 \).

By the property of the heat kernel, we see that \( U(x, t) \to f(x) \) and \( V(x, t) \to 0 \) as \( t \to 0 \). This shows that \( F \) extends continuously to \( f \) on \( \mathbb{R} \). Moreover, \( F(x, t) \to \infty \) as \( (x, t) \to \infty \). Since the Jacobian determinant \( J_F = U_x V_t - U_t V_x \) is positive at every point as we have seen above, \( F \) is a local homeomorphism. Then, a topological argument deduces that \( F \) is an orientation-preserving global diffeomorphism of \( U \) onto itself. By \( \|\mu_F\|_\infty < 1 \), we see that \( F \) is quasiconformal.

If we further assume that \( \omega \) is an \( A_\infty \)-weight, that is, \( f \) is a strongly quasisymmetric homeomorphism, then we see that the complex dilatation \( \mu_F \) induces a Carleson measure on \( U \). This corresponds to [5, Theorem 4.2].

**Theorem 3.4.** For an \( A_\infty \)-weight \( \omega \) on \( \mathbb{R} \), the complex dilatation \( \mu_F \) of the quasiconformal diffeomorphism \( F : U \to \mathbb{R} \) satisfies that \( \frac{1}{2} |\mu_F(x, t)|^2 dxdt \) is a Carleson measure on \( \mathbb{U} \).

**Proof.** The complex dilatation \( \mu_F = \partial F/\partial F \) satisfies that
\[ |\mu_F|^2 = \frac{U_x^2 + U_t^2 + V_x^2 + V_t^2 - 2J_F}{U_x^2 + U_t^2 + V_x^2 + V_t^2 + 2J_F} \leq \frac{2U_t^2 + 2V_x^2 + (U_x - V_t)^2}{U_x^2}, \]
where $J_F = U_xV_t - U_tV_x$ is the Jacobian determinant of $F$. Here,

$$\frac{4U_t^2}{U_x^2} = \frac{V_t^2}{V_x^2} = t^2 \frac{u'(x, t)^2}{u(x, t^2)^2},$$

and by the change of the variables again, we have

$$\frac{1}{t} \int \int_{[x-x_0]<t} \left( \frac{s^2 u''(x, s^2)}{u(x, s^2)^2} \right) dxds = \frac{1}{t^2} \int \int_{[x-x_0]<t} \frac{u'(x, \sigma)^2}{u(x, \sigma)^2} dx\sigma.$$

By Theorem 2.3, this is uniformly bounded. Moreover,

$$\frac{(U_x - V_t)^2}{U_x^2} = t^4 \frac{u''(x, t^2)^2}{u(x, t^2)^2}$$

and by the change of the variables again, we have

$$\frac{1}{t} \int \int_{[x-x_0]<t} \left( \frac{s^2 u''(x, s^2)}{u(x, s^2)^2} \right) dxds = \frac{1}{t} \int \int_{[x-x_0]<t} \frac{u''(x, \sigma)^2}{u(x, \sigma)^2} dx\sigma.$$

By Theorem 2.5, this is also uniformly bounded. Combining these two estimates, we see that

$$\frac{1}{t} \int \int_{[x-x_0]<t} |\mu_F(x, s)|^2 \frac{dxds}{s}$$

is uniformly bounded, which shows that $\frac{1}{t} |\mu_F(x, t)|^2 dxdt$ is a Carleson measure on $U$. 

\[ \square \]

4. The quasiconformal extension of strongly symmetric homeomorphisms and vanishing Carleson measures

We assume that $\log \omega$ for an $A_{\infty}$-weight $\omega$ is in VMO($\mathbb{R}$), that is, $f(x) = \int_0^x \omega(y)dy$ is a strongly symmetric homeomorphism. Then, we prove that the complex dilatation $\mu_F$ of the quasiconformal diffeomorphism $F : U \to U$ examined in Theorems 3.3 and 3.4 induces a vanishing Carleson measure on the upper half-plane $U$. An idea of the argument comes from that by Semmes [7, Proposition 4.2]. This answers the question raised by Shen [8].

**Theorem 4.1.** For an $A_{\infty}$-weight $\omega$ on $\mathbb{R}$ with $\alpha = \log \omega \in \text{VMO}(\mathbb{R})$, the complex dilatation $\mu_F$ of the quasiconformal diffeomorphism $F : \mathbb{R} \to \mathbb{R}$ satisfies that $\frac{1}{t} |\mu_F(x, t)|^2 dxdt$ is a vanishing Carleson measure on $U$.

**Proof.** We use inequality (3) to show that

$$\frac{1}{t} \int \int_{[x-x_0]<t} |\mu_F(x, s)|^2 \frac{dxds}{s} \to 0$$

uniformly as $t \to 0$. Here, we note that $U_x(x, t) = (\omega \ast \phi_t)(x)$ and each of $U_t(x, t)$, $V_x(x, t)$, and $(U_x - V_t)(x, t)$ can be represented by $(\omega \ast \gamma_t)(x)$ explicitly for a certain $\gamma \in C^\infty(\mathbb{R})$ such that $\int_{\mathbb{R}} \gamma(x)dx = 0$, $|\gamma|$ is an even function, and $\gamma(x) = O(x^2 e^{-x^2})$ ($|x| \to \infty$). For instance, $V_x(x, t) = (\omega \ast \psi_t)(x)$ for $\psi(x) = -\frac{1}{2} xe^{-x^2}$. We set $I(x_0, t) = \{x \mid |x-x_0| < t\}$. Then, for the statement, it suffices to prove that

$$A(x_0, t) = \frac{1}{t} \int \int_{I(x_0, t)} \frac{(\omega \ast \gamma_t)(x)^2}{(\omega \ast \phi_t)(x)^2} \frac{dxds}{s} \to 0$$

uniformly as $t \to 0$. 


Since $\phi(x) \geq 1/(e \sqrt{\pi})$ for $x \in (-1, 1)$, we see that $\phi_t(x - y) \geq 1/(te \sqrt{\pi})$ if $|x - y| < t$. From this, we have

$$(\omega \ast \phi_t)(x) \geq \frac{1}{te \sqrt{\pi}} \int_{|x-y|<t} \omega(y)dy.$$ 

Moreover, the Cauchy–Schwarz inequality implies

$$\left( \frac{1}{2t} \int_{|x-y|<t} \omega(y)dy \right) \left( \frac{1}{2t} \int_{|x-y|<t} \omega(y)^{-1}dy \right) \geq \left( \frac{1}{2t} \int_{|x-y|<t} \omega(y)^{1/2}\omega(y)^{-1/2}dy \right)^2 = 1.$$ 

Therefore,

$$(\omega \ast \phi_t)(x)^{-2} \leq c \left( \frac{1}{2t} \int_{|x-y|<t} \omega(y)^{-1}dy \right)^2$$

for $c = e^2\pi/4$. Hence, $A(x_0, t)$ is estimated as follows:

$$A(x_0, t) \leq \frac{c}{t} \int_{I(x_0,t)} \int_0^t \left( \frac{1}{2s} \int_{|x-y|<s} \omega(y)^{-1}dy \right)^2 (\omega \ast \gamma_s)(x)^2 \frac{1}{s} ds dx$$

$$= \frac{c}{t} \int_{I(x_0,t)} \int_0^t \left( \frac{1}{2s} \int_{|x-y|<s} \omega(y)^{-1}1_{I(x_0,Nt)}(y)dy \right)^2 (\omega \ast \gamma_s)(x)^2 \frac{1}{s} ds dx$$

$$\leq \frac{c}{t} \int_{I(x_0,t)} \left( \sup_{s>0} \left\{ \frac{1}{2s} \int_{|x-y|<s} \omega(y)^{-1}1_{I(x_0,Nt)}(y)dy \right\} \right)^2 \left( \int_0^t (\omega \ast \gamma_s)(x)^2 \frac{1}{s} ds \right) dx$$

$$\leq \frac{c}{t} \left[ \int_{I(x_0,t)} \left( \sup_{s>0} \left\{ \frac{1}{2s} \int_{|x-y|<s} \omega(y)^{-1}1_{I(x_0,Nt)}(y)dy \right\} \right)^4 dx \right]^{1/2}$$

$$\times \left[ \int_{I(x_0,t)} \left( \int_0^t (\omega \ast \gamma_s)(x)^2 \frac{1}{s} ds \right)^2 dx \right]^{1/2}.$$ 

(4)

We note that if $x \in I(x_0, t)$ and $|x - y| < s \leq t$, then $y \in I(x_0, Nt)$ for any $N \geq 2$. Thus, for the equality in the middle line, we have replaced $\omega(y)^{-1}$ with $\omega(y)^{-1}1_{I(x_0,Nt)}(y)$ taking the product of the characteristic function.

The integrand of the first factor of (4) is the fourth power of the maximal function

$$M(\omega^{-1}1_{I(x_0,Nt)})(x) = \sup_{s>0} \left\{ \frac{1}{2s} \int_{|x-y|<s} \omega(y)^{-1}1_{I(x_0,Nt)}(y)dy \right\}.$$ 

The strong $L^4$-estimate of the maximal function implies

$$\int_{I(x_0,t)} M(\omega^{-1}1_{I(x_0,Nt)})(x)^4 dx \leq \int_{\mathbb{R}} M(\omega^{-1}1_{I(x_0,Nt)})(x)^4 dx$$

$$\leq C' \int_{\mathbb{R}} \omega(x)^{-4}1_{I(x_0,Nt)}(x)dx = C' \int_{I(x_0,Nt)} \omega(x)^{-4} dx$$

for some $C' > 0$.

We assume hereafter that $\int_{I(x_0,Nt)} \alpha(x)dx = 0$ because adding a constant to $\alpha = \log \omega$ corresponds to a dilation of $f$, which does not change the complex dilatation $\mu_F$. We remark
that once $I(x_0, Nt)$ is given, we can assume this only for $I(x_0, Nt)$ throughout the arguments. When $x_0$, $t$, or $N$ change, we regard that the assumption is renewed accordingly.

We denote the integral mean of $\alpha$ on a bounded interval $I \subset \mathbb{R}$ by $\alpha_I = |I|^{-1} \int_I \alpha(x)dx$. By the John–Nirenberg inequality, we have

$$\frac{1}{|I(x_0, Nt)|} \int_{I(x_0, Nt)} \omega(x)^{-4}dx \leq \frac{1}{|I(x_0, Nt)|} \int_{I(x_0, Nt)} \exp(4|\alpha(x) - \alpha_{I(x_0, Nt)}|)dx$$

$$= \frac{1}{|I(x_0, Nt)|} \int_{I(x_0, Nt)} (\exp(4|\alpha(x) - \alpha_{I(x_0, Nt)}|) - 1)dx + 1$$

$$= 4 \int_{0}^{\infty} \frac{e^{\lambda}}{|I(x_0, Nt)|} |\{x \in I(x_0, Nt) : |\alpha(x) - \alpha_{I(x_0, Nt)}| > \lambda\}|d\lambda + 1$$

$$\leq 4C_1 \int_{0}^{\infty} e^{4\lambda} \left( \frac{|\alpha|_{\text{BMO}(I(x_0, Nt))}}{\alpha_{\text{BMO}(I(x_0, Nt))}} \right)^{\lambda/2} + 1 = \frac{4C_1|\alpha|_{\text{BMO}(I(x_0, Nt))}}{C_2 - 4|\alpha|_{\text{BMO}(I(x_0, Nt))}} + 1$$

for some positive constants $C_1$ and $C_2$, where $|\alpha|_{\text{BMO}(I)}$ denotes the BMO norm of $\alpha$ on a bounded interval $I$. Thus, for a sufficiently small $t > 0$ with $N$ fixed, this is bounded; as a consequence, the integral by $dx$ over $I(x_0, t)$ in the first factor of (4) is bounded by $C'Nt$ for some uniform constant $C'>0$.

Next, we consider the integrand of the second factor of (4). Since $\int_{\mathbb{R}} \gamma(x)dx = 0$, we can replace the convolution $\omega \ast \gamma_s$ with $(\omega - 1) \ast \gamma_s$. For a sufficiently large $N > 0$, we decompose this convolution into the integrals on the interval $I(x_0, Nt)$ and on its complement $I(x_0, Nt)^c = \mathbb{R} \setminus I(x_0, Nt)$ and estimate the integrand as

$$\left( \int_{0}^{t} ((\omega - 1) \ast \gamma_s)(x)^2 \frac{1}{s} ds \right)^2$$

$$= \left( \int_{0}^{t} \left[ ((\omega - 1)1_{I(x_0, Nt)} \ast \gamma_s)(x) + ((\omega - 1)1_{I(x_0, Nt)^c} \ast \gamma_s)(x) \right]^2 \frac{1}{s} ds \right)^2$$

$$\leq 8 \left( \int_{0}^{t} ((\omega - 1)1_{I(x_0, Nt)} \ast \gamma_s)(x)^2 \frac{1}{s} ds \right)^2$$

$$+ 8 \left( \int_{0}^{t} ((\omega - 1)1_{I(x_0, Nt)^c} \ast \gamma_s)(x)^2 \frac{1}{s} ds \right)^2.$$ (5)

First, we consider the integral of the first term of (5) by $dx$ over $I(x_0, t)$. We utilize the Littlewood–Paley function defined by the rapidly decreasing function $\gamma$ with $\int_{\mathbb{R}} \gamma(x)dx = 0$:

$$S_\gamma((\omega - 1)1_{I(x_0, Nt)})(x) = \left( \int_{0}^{\infty} ((\omega - 1)1_{I(x_0, Nt)} \ast \gamma_s)(x)^2 \frac{1}{s} ds \right)^{1/2}.$$ (1)

The strong $L^4$-estimate of the Littlewood–Paley function (see [1, p. 363]) implies

$$\int_{I(x_0, t)} \left( \int_{0}^{t} ((\omega - 1)1_{I(x_0, Nt)} \ast \gamma_s)(x)^2 \frac{1}{s} ds \right)^2 dx$$

$$\leq \int_{I(x_0, t)} S_\gamma((\omega - 1)1_{I(x_0, Nt)})(x)^4 dx$$
for some $C'' > 0$. Here, applying the John–Nirenberg inequality again with the assumption $\int_{I(x_0,Nt)} \alpha(x)dx = 0$, we have

\[
\frac{1}{|I(x_0,Nt)|} \int_{I(x_0,Nt)} (\omega(x) - 1)^4 \, dx
= \frac{1}{|I(x_0,Nt)|} \int_{I(x_0,Nt)} (\exp(|\alpha(x)| - \alpha_{I(x_0,Nt)}) - 1)^4 \, dx
= 4 \int_0^\infty \frac{(e^\lambda - 1)^3 e^\lambda}{|I(x_0,Nt)|} \left| \{x \in I(x_0,Nt) : |\alpha(x) - \alpha_{I(x_0,Nt)}| > \lambda \} \right| \, d\lambda
\leq 4C1 \int_0^\infty e^{4\lambda} \exp\left(-\frac{C_2 \lambda}{\|\alpha\|_{BMO(I(x_0,Nt))}}\right) \, d\lambda = \frac{4C1 \|\alpha\|_{BMO(I(x_0,Nt))}}{C_2 - 4\|\alpha\|_{BMO(I(x_0,Nt))}}.
\]

Thus, for a sufficiently small $t > 0$ with $N$ fixed, the integral of the first term of (5) by $dx$ over $I(x_0,t)$ is bounded by $C''|Nt\|\alpha\|_{BMO(I(x_0,Nt))}$ for some uniform constant $C'' > 0$.

Second, we consider the integral of the second term of (5) by $dx$ over $I(x_0,t)$. For an estimate of the convolution, we use a fact that the weight $\omega + 1$ has the doubling property with some constant $\rho > 1$. We note that $|\gamma|$ is an even function. Let $n_0 = n_0(s,t,N) \in \mathbb{N}$ satisfy $2^{n_0 - 1} = (N - 1)t/s$ (we may adjust $N$ so that $n_0$ becomes an integer). Then, for $x \in I(x_0,t)$, we see that

\[
|((\omega - 1)1_{I(x_0,Nt)}) \ast \gamma_s)(x)| \leq \int_{|y-x_0| \geq Nt} (\omega(y) + 1) |\gamma_s(y - x)| \, dy
\leq \int_{|y-x_0| \geq (N-1)t} (\omega(y) + 1) |\gamma_s(y - x_0)| \, dy
= \sum_{n=n_0}^\infty \int_{2^{n-1}s \leq |y-x_0| < 2^n s} (\omega(y) + 1) |\gamma_s(y - x_0)| \, dy
\leq \sum_{n=n_0}^\infty \rho^n |\gamma_s(2^{n-1}s)| \int_{|y-x_0| < s} (\omega(y) + 1) \, dy.
\]

Here, by $|\gamma| = O(x^2 e^{-x^2})$ (|x| –> \infty), we have

\[
\rho^n |\gamma_s(2^{n-1}s)| \leq \frac{D_1}{s} \frac{(4\rho)^n}{e^{4\rho}}
\]

for some $D_1 > 0$. For $n \geq n_0(s,t,N)$, we may assume that $(4\rho)^n/e^{4\rho} \leq 1$. In fact, by $2^{n_0 - 1} \geq N - 1$, this holds when $N$ is sufficiently large. Moreover,

\[
\int_{|y-x_0| < s} (\omega(y) + 1) \, dy \leq 2s + \int_{I(x_0,Nt)} \omega(y) \, dy \leq D_2 Nt
\]

for some $D_2 > 0$. This estimate of the integral of $\omega$ over $I(x_0,Nt)$ is carried out in a similar way as before by using the John–Nirenberg inequality when $t$ is sufficiently small with $N$ fixed.
Therefore, we obtain that if \( x \in I(x_0, t) \), then
\[
|((\omega - 1)1_{I(x_0, Nt)} \ast \gamma_s)(x)| \leq \frac{D_1 D_2 N t}{s} \sum_{n = n_0}^{\infty} \frac{1}{e^{2^n - 1}} \leq \frac{D N t}{s} \exp \left( -\frac{N t}{2s} \right)
\]
for some uniform constant \( D > 0 \).

We will complete the estimate concerning the second term of (5). By the above inequality, we have
\[
\int_0^t \left( ((\omega - 1)1_{I(x_0, Nt)} \ast \gamma_s)(x)^2 \frac{1}{s} ds \right)^2 \leq (D N t)^2 \int_0^t \frac{\exp \left( -\frac{N t}{s} \right)}{s^3} ds \leq D^2 N^2 e^{-N}
\]
for \( x \in I(x_0, t) \). For the last inequality, we have used a fact that
\[
\max_{0 \leq s \leq t} \frac{\exp \left( -\frac{N t}{s} \right)}{s^3} = \frac{e^{-N}}{t^3}
\]
whenever \( N \geq 3 \). Hence,
\[
\int_{I(x_0, t)} \left( \int_0^t (\omega - 1)1_{I(x_0, Nt)} \ast \gamma_s)(x)^2 \frac{1}{s} ds \right)^2 dx \leq 2D^4 N^4 t e^{-2N}.
\]

Finally, we substitute what we have obtained into (4) and complete the proof. By replacing \( 16D^4 \) with \( \tilde{D} \), we conclude that
\[
A(x_0, t) \leq \frac{C}{t} \left( \tilde{\omega}'(N t)^{1/2} \right) \left( \tilde{\omega}''(N t) \| \alpha \|_{\text{BMO}(I(x_0, Nt))} \right) + \tilde{D} N^4 t e^{-2N})^{1/2}
\]
where we cleared up the last line by introducing the final constant \( C > 0 \). Now, for an arbitrary positive \( \varepsilon > 0 \), we choose a sufficiently large \( N > 0 \) that satisfies
\[
N^5 e^{-2N} \leq \frac{\varepsilon^2}{2C^2},
\]
and fix it. Then, for this fixed \( N \), we can find some \( \delta > 0 \) such that if \( t \leq \delta \), then
\[
N^2 \| \alpha \|_{\text{BMO}(I(x_0, Nt))} \leq \frac{\varepsilon^2}{2C^2}.
\]
This is because \( \alpha \in \text{VMO}(\mathbb{R}) \). Thus, if \( t \leq \delta \), then \( A(x_0, t) \leq \varepsilon \), independently of \( x_0 \). \( \square \)

**Remark.** Conversely, a quasiconformal homeomorphism \( F : \mathbb{R} \rightarrow \mathbb{R} \) with \( \frac{1}{t} |\mu_F(x, t)|^2 dx \) a vanishing Carleson measure extends continuously to \( f : \mathbb{R} \rightarrow \mathbb{R} \) as a strongly symmetric homeomorphism, which has been proved by Shen [8].

**References**

1. A. Benedek, A. P. Calderón and R. Panzone, 'Convolution operators on Banach space valued functions', Proc. Natl. Acad. Sci. USA 48 (1962) 356–365.
2. A. Beurling and L. V. Ahlfors, 'The boundary correspondence under quasiconformal mappings', Acta Math. 96 (1956) 125–142.
3. L. Carleson, 'On mappings, conformal at the boundary', J. Anal. Math. 19 (1967) 1–13.
4. R. R. Coifman and C. Fefferman, 'Weighted norm inequalities for maximal functions and singular integrals', Studia Math. 51 (1974) 241–250.
5. R. A. Fefferman, C. E. Kenig and J. Pipher, 'The theory of weights and the Dirichlet problems for elliptic equations', Ann. of Math. (2) 134 (1991) 65–124.
6. S. V. Hruščev, ‘A description of weights satisfying the $A_\infty$ condition of Muckenhoupt’, Proc. Amer. Math. Soc. 90 (1984) 253–257.
7. S. Semmes, ‘Quasiconformal mappings and chord-arc curves’, Trans. Amer. Math. Soc. 306 (1988) 233–263.
8. Y. Shen, ‘VMO-Teichmüller space on the real line’, Preprint.

Huaying Wei  
Department of Mathematics and Statistics  
Jiangsu Normal University  
101 Shanghai Rd, Tongshan District  
Xuzhou 221116  
PR China

dywei@jsnu.edu.cn

Katsuhiko Matsuzaki  
Department of Mathematics  
School of Education Waseda University  
Nishi-Waseda 1-6-1  
Shinjuku, Tokyo 169-8050  
Japan

matsuzak@waseda.jp