The First Quantum Error-Correcting Code for Single Deletion Errors

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Abstract: A quantum error-correcting code for single deletion errors is provided. To the authors’ best knowledge, this is the first code for deletion errors.

Keywords: Quantum error-correcting codes, quantum deletion errors, deletion codes

Classification: Fundamental theories for communications

References

[1] V. I. Levenshtein, “Binary codes capable of correcting deletions, insertions, and reversals,” Soviet physics doklady, pp. 707–710, 1966.

1 Introduction

In the classical coding theory, deletion error-correcting codes have been studied for synchronization of communication since the pioneer work by Levenshtein [1]. However, no quantum code for deletion error has been constructed yet. This letter provides the first quantum code for deletion errors that are defined as partial trace operations. In particular, an encoding and a decoding are described.

2 Single Quantum Deletion Error

For a square matrix $A$ over a complex field $\mathbb{C}$, $\mathrm{Tr}(A)$ denotes the sum of the diagonal elements of $A$. Set $|0\rangle, |1\rangle \in \mathbb{C}^2$ as $|0\rangle := (1,0)^T, |1\rangle := (0,1)^T$ respectively. We denote the set of all density matrices of order $n$ by $S(\mathbb{C}^n)$. A density matrix is used for representing a quantum state and is called a quantum message.

For an integer $1 \leq i \leq n$ and a square matrix $A = \sum_{x,y \in \{0,1\}^n} a_{x,y} \cdot |x_1\rangle \langle y_1| \otimes \cdots \otimes |x_n\rangle \langle y_n|$ with $a_{x,y} \in \mathbb{C}$, define the map $\mathrm{Tr}_i$ as $\mathrm{Tr}_i(A) := \sum_{x,y \in \{0,1\}^n} a_{x,y} \cdot \mathrm{Tr}(|x_i\rangle \langle y_i| \cdot |x_1\rangle \langle y_1| \otimes \cdots \otimes |x_{i-1}\rangle \langle y_{i-1}| \otimes |x_{i+1}\rangle \langle y_{i+1}| \otimes \cdots \otimes |x_n\rangle \langle y_n|$. The map $\mathrm{Tr}_i$ is called the partial trace.
**Definition 1.** For an integer $1 \leq i \leq n$, we call $D_i$ a single deletion error $E_i$, i.e.,

$$D_i(\rho) := \text{Tr}_i(\rho),$$

where $\rho \in S(\mathbb{C}^{2 \otimes n})$ is a quantum state.

### 3 Our Quantum Error-Correcting Code for Single Deletion Errors

#### 3.1 Encoding

For a quantum message $\sigma := |\psi\rangle \langle \psi| \in S(\mathbb{C}^2)$ with unit vector $|\psi\rangle := \alpha |0\rangle + \beta |1\rangle \in \mathbb{C}^2$, we encode $\sigma$ to $\rho := |\Psi\rangle \langle \Psi|$, where

$$|\Psi\rangle := \frac{\alpha}{\sqrt{2}}(|00001001\rangle + |01101111\rangle) + \frac{\beta}{\sqrt{2}}(|00011111\rangle + |01101001\rangle).$$

Remark that this encoding can be expressed by neither any CSS codes nor any stabilizer codes.

#### 3.2 Quantum States After the Deletion Errors

All the states $D_i(\rho)$ after single deletion errors for $1 \leq i \leq 8$ are following:

$$D_1(\rho)$$

$$= \frac{\alpha}{2} |0001001\rangle (\overline{\alpha} |0001001\rangle + \overline{\beta} |0001111\rangle + \overline{\beta} |1101001\rangle + \overline{\alpha} |1101111\rangle)$$

$$+ \frac{\beta}{2} |0001111\rangle (\overline{\alpha} |0001001\rangle + \overline{\beta} |0001111\rangle + \overline{\beta} |1101001\rangle + \overline{\alpha} |1101111\rangle)$$

$$+ \frac{\beta}{2} |1101001\rangle (\overline{\alpha} |0001001\rangle + \overline{\beta} |0001111\rangle + \overline{\beta} |1101001\rangle + \overline{\alpha} |1101111\rangle)$$

$$+ \frac{\alpha}{2} |1101111\rangle (\overline{\alpha} |0001001\rangle + \overline{\beta} |0001111\rangle + \overline{\beta} |1101001\rangle + \overline{\alpha} |1101111\rangle),$$

$$D_2(\rho) = D_3(\rho)$$

$$= \frac{\alpha}{2} |0001001\rangle (\overline{\alpha} |0001001\rangle + \overline{\beta} |0001111\rangle)$$

$$+ \frac{\beta}{2} |0001111\rangle (\overline{\alpha} |0001001\rangle + \overline{\beta} |0001111\rangle)$$

$$+ \frac{\beta}{2} |0101001\rangle (\overline{\beta} |0101001\rangle + \overline{\alpha} |0101111\rangle)$$

$$+ \frac{\alpha}{2} |0101111\rangle (\overline{\beta} |0101001\rangle + \overline{\alpha} |0101111\rangle),$$

$$D_4(\rho)$$

$$= \frac{\alpha}{2} |0001001\rangle (\overline{\alpha} |0001001\rangle + \overline{\beta} |0001111\rangle + \overline{\beta} |0111001\rangle + \overline{\alpha} |0111111\rangle)$$

$$+ \frac{\beta}{2} |0001111\rangle (\overline{\alpha} |0001001\rangle + \overline{\beta} |0001111\rangle + \overline{\beta} |0111001\rangle + \overline{\alpha} |0111111\rangle)$$

$$+ \frac{\beta}{2} |0111001\rangle (\overline{\alpha} |0001001\rangle + \overline{\beta} |0001111\rangle + \overline{\beta} |0111001\rangle + \overline{\alpha} |0111111\rangle)$$

$$+ \frac{\alpha}{2} |0111111\rangle (\overline{\alpha} |0001001\rangle + \overline{\beta} |0001111\rangle + \overline{\beta} |0111001\rangle + \overline{\alpha} |0111111\rangle),$$
\[
D_5(\rho) = \frac{\alpha}{2} \sum_{\rho \in \mathbb{C}^{2 \times 2}} \rho = \frac{\alpha}{2} \left( \begin{array}{cc}
\langle 0000001 \mid \rho \mid 0000001 \rangle + \langle 0000001 \mid \rho \mid 0000111 \rangle + \langle 0000001 \mid \rho \mid 0110001 \rangle + \langle 0000001 \mid \rho \mid 0110111 \rangle
\end{array} \right)
\]

\[
+ \frac{\beta}{2} \sum_{\rho \in \mathbb{C}^{2 \times 2}} \rho = \frac{\beta}{2} \left( \begin{array}{cc}
\langle 0000111 \mid \rho \mid 0000001 \rangle + \langle 0000111 \mid \rho \mid 0000001 \rangle + \langle 0000111 \mid \rho \mid 0110001 \rangle + \langle 0000111 \mid \rho \mid 0110111 \rangle
\end{array} \right)
\]

\[
+ \frac{\beta}{2} \sum_{\rho \in \mathbb{C}^{2 \times 2}} \rho = \frac{\beta}{2} \left( \begin{array}{cc}
\langle 0110001 \mid \rho \mid 0000001 \rangle + \langle 0110001 \mid \rho \mid 0000001 \rangle + \langle 0110001 \mid \rho \mid 0110001 \rangle + \langle 0110001 \mid \rho \mid 0110111 \rangle
\end{array} \right)
\]

\[
+ \frac{\alpha}{2} \sum_{\rho \in \mathbb{C}^{2 \times 2}} \rho = \frac{\alpha}{2} \left( \begin{array}{cc}
\langle 0110111 \mid \rho \mid 0000001 \rangle + \langle 0110111 \mid \rho \mid 0000001 \rangle + \langle 0110111 \mid \rho \mid 0110001 \rangle + \langle 0110111 \mid \rho \mid 0110111 \rangle
\end{array} \right),
\]

\[
D_6(\rho) = D_7(\rho) = \ldots = D_8(\rho) = \ldots
\]

3.3 Decoding
Let \( P = \{P_1, P_2, \ldots, P_m\} \) be a set of complex square matrices of order \( 2^n \).
Let \( P \) be a set of complex square matrices of order \( 2^{2n} \). For a quantum state \( \rho \in S(\mathbb{C}^{2^n}) \), the probability that we obtain an outcome \( 1 \leq k \leq m \) by the measurement \( P \) is given by \( \text{Tr}(P_k \rho) \).

By the measurement \( P \), the outcome \( k \) is obtained.

Let us define a projection measurement \( P = \{P_1, P_2, \ldots, P_m\} \) as

\[
P_1 = \left( \begin{array}{cc}
0001001 \mid 0001001 \rangle + 0001111 \langle 0001111
\end{array} \right),
\]

\[
P_2 = \left( \begin{array}{cc}
1101001 \mid 1101001 \rangle + 1101111 \langle 1101111
\end{array} \right),
\]

\[
P_3 = \left( \begin{array}{cc}
0101001 \mid 0101001 \rangle + 0101111 \langle 0101111
\end{array} \right),
\]

\[
P_4 = \left( \begin{array}{cc}
0111001 \mid 0111001 \rangle + 0111111 \langle 0111111
\end{array} \right),
\]

\[
P_5 = \left( \begin{array}{cc}
0000111 \mid 0000111 \rangle + 0110111 \langle 0110111
\end{array} \right),
\]

\[
P_6 = \left( \begin{array}{cc}
0000001 \mid 0000001 \rangle + 0110001 \langle 0110001
\end{array} \right),
\]

\[
P_7 = \left( \begin{array}{cc}
0001011 \mid 0001011 \rangle + 0110101 \langle 0110101
\end{array} \right),
\]

\[
P_8 = \left( \begin{array}{cc}
0000100 \mid 0000100 \rangle + 0110100 \langle 0110100
\end{array} \right),
\]

\[
P_9 = I_7 - \sum_{1 \leq j \leq 8} P_j.
\]
Table I shows the probabilities corresponding to the single deletion $D_i(\rho)$ and the outcome $k$.

|     | $D_1(\rho)$ | $D_2(\rho)$ | $D_3(\rho)$ | $D_4(\rho)$ | $D_5(\rho)$ | $D_6(\rho)$ | $D_7(\rho)$ | $D_8(\rho)$ |
|-----|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| $k = 1$ | 0.5 | 0.5 | 0.5 | 0.5 | 0 | 0 | 0 | 0 |
| $k = 2$ | 0.5 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $k = 3$ | 0 | 0.5 | 0.5 | 0 | 0 | 0 | 0 | 0 |
| $k = 4$ | 0 | 0 | 0 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 |
| $k = 5$ | 0 | 0 | 0 | 0 | 0.5 | 0.5 | 0.5 | 0.5 |
| $k = 6$ | 0 | 0 | 0 | 0 | 0 | 0.5 | 0.5 | 0 |
| $k = 7$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0.5 |
| $k = 8$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0.5 |
| $k = 9$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Table I: the probabilities corresponding to the single deletion $D_i(\rho)$ and the outcome $k$

Let us explain how to correct deletion error for each outcome $k$. Case $k = 1$: In this case, the obtained state $\tilde{\rho}$ is

$$\tilde{\rho} = \alpha |0011001\rangle \langle \bar{0}001001| + \beta |0001111\rangle \langle \bar{0}001111|$$

Let $U_1$ be a unitary matrix of order $2^7$ that satisfies

$$U_1 |0001001\rangle = |0000000\rangle , U_1 |0001111\rangle = |0000001\rangle .$$

Case $k = 2$: In this case, the obtained state $\tilde{\rho}$ is

$$\tilde{\rho} = \beta |1101001\rangle \langle \bar{0}1101001| + \alpha |1101111\rangle \langle \bar{0}1101111|$$

Let $U_2$ be a unitary matrix $U_2$ that satisfies

$$U_2 |1101111\rangle = |0000000\rangle , U_2 |1101001\rangle = |0000001\rangle .$$

Case $k = 3$: In this case, the obtained state $\tilde{\rho}$ is

$$\tilde{\rho} = \beta |0101001\rangle \langle \bar{0}0101001| + \alpha |0101111\rangle \langle \bar{0}0101111|$$

Let $U_3$ be a unitary matrix $U_3$ that satisfies

$$U_3 |0101111\rangle = |0000000\rangle , U_3 |0101001\rangle = |0000001\rangle .$$

Case $k = 4$: In this case, the obtained state $\tilde{\rho}$ is

$$\tilde{\rho} = \beta |0111001\rangle \langle \bar{0}0111001| + \alpha |0111111\rangle \langle \bar{0}0111111|.$$
Let $U_4$ be a unitary matrix that satisfies

$$U_4 |0111111\rangle = |0000000\rangle, U_4 |0111001\rangle = |0000001\rangle.$$ 

Case $k = 5$: In this case, the obtained state $\tilde{\rho}$ is

$$\tilde{\rho} = \beta |0000111\rangle \langle 0000111| + \alpha |0110111\rangle \langle 0110111|.$$ 

Let $U_5$ be a unitary matrix that satisfies

$$U_5 |0110111\rangle = |0000000\rangle, U_5 |0000111\rangle = |0000001\rangle.$$ 

Case $k = 6$: In this case, the obtained state $\tilde{\rho}$ is

$$\tilde{\rho} = \alpha |0000011\rangle \langle 0000011| + \beta |0110001\rangle \langle 0110001|.$$ 

Let $U_6$ be a unitary matrix that satisfies

$$U_6 |0000001\rangle = |0000000\rangle, U_6 |0110001\rangle = |0000001\rangle.$$ 

Case $k = 7$: In this case, the obtained state $\tilde{\rho}$ is

$$\tilde{\rho} = \alpha |0000101\rangle \langle 0000101| + \beta |0110101\rangle \langle 0110101|.$$ 

Let $U_7$ be a unitary matrix that satisfies

$$U_7 |0000101\rangle = |0000000\rangle, U_7 |0110101\rangle = |0000001\rangle.$$ 

Case $k = 8$: In this case, the obtained state $\tilde{\rho}$ is

$$\tilde{\rho} = \alpha |0000100\rangle \langle 0000100| + \beta |0110100\rangle \langle 0110100|.$$ 

Let $U_8$ be a unitary matrix that satisfies

$$U_8 |0000100\rangle = |0000000\rangle, U_8 |0110100\rangle = |0000001\rangle.$$ 

For each outcome $k$, we can use $U_k$ as a recovery operator:

$$\underbrace{\text{Tr}_1 \circ \cdots \circ \text{Tr}_1}_{6 \text{ times}} (U_k \tilde{\rho} U_k^\dagger) = \sigma.$$ 

Thus we can obtain the original quantum message $\sigma$.

4 Conclusion

This letter gave the quantum code for single deletions. Our code does not belong to previously known classes of quantum error-correcting codes. For future work, the authors like to construct a class of quantum codes for deletion errors.

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