Black hole solutions in Einstein–Maxwell–Yang–Mills–Gauss–Bonnet theory

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Abstract. We consider Maxwell and Yang–Mills (YM) fields together, interacting through gravity both in Einstein and in Gauss–Bonnet (GB) theories. For this purpose we choose two different sets of Maxwell and metric ansätze. In our first ansatz, asymptotically for \( r \to 0 \) (and \( N > 4 \)) the Maxwell field dominates over the YM field. In the other asymptotic region, \( r \to \infty \), however, the YM field becomes dominant. For \( N = 3 \) and 4, where the GB term is absent, we recover the well-known Bañados–Teitelboim–Zanelli and Reissner–Nordström metrics, respectively. The second ansatz corresponds to the case of constant radius function for the \( S^{N-2} \) part in the metric. This leads to the maximally symmetrical, everywhere regular, Bertotti–Robinson type of solutions for representing our cosmos both at large and small scales.

Keywords: black holes, extra dimensions, gravity
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## 1. Introduction

We introduce the Maxwell field alongside the Yang–Mills (YM) field in general relativity and present spherically symmetric black hole solutions in any higher dimension. These two gauge fields, one Abelian and the other non-Abelian, are coupled through gravity and to our knowledge they have not been considered together in a common geometry so far. As they are linear, a multitude of Maxwellian fields can be superimposed with equal ease, but for the YM field there is no such freedom. Our treatment of the YM field in this paper, like that of the Maxwell field, is completely classical (i.e. non-quantum). From a physics standpoint, electromagnetism has long range effects and dominates outside the nuclei of natural matter. The YM field on the other hand is confined to acting inside nuclei; however, the existence of exotic and highly dense matter in our universe encourages us to use the YM field in a broader sense. To obtain exact solutions, the Maxwell field is chosen purely electric while the YM field is purely magnetic. Historically starting from the Reissner–Nordström (RN) metric, the higher dimensional extensions, known as the Einstein–Maxwell (EM) black holes, are well known by now. Einstein–Yang–Mills (EYM) black holes in higher dimensions have also attracted interest in more recent works [1]–[3].
In this paper we combine these two foregoing black holes (i.e. EM and EYM) under the common title of EMYM black holes. Further, we add to this the Gauss–Bonnet (GB) term; overall we use the term EMYMGB black holes (see [17] and the references therein). Such black holes will be characterized by mass \( M \), Maxwell charge \( q \), YM charge \( Q \), GB parameter \( \alpha \) and cosmological constant \( \Lambda \). For a broad class of black hole solutions we investigate the thermodynamic s and other properties as manifestations of these physical parameters. All of these parameters naturally have imprints in planetary motion, gravitational lensing, ripping apart of stars and ultimately the accurate picture of our cosmos. It is remarkable that exact solutions to such a highly non-linear theory can be found for two different Maxwell and metric ansätze. Before adding the GB term we find the EMYM solutions in which the relative contributions from the Maxwell/YM charges can be compared. It is shown that for \( r \to 0 \) the Maxwell term dominates over the YM term, which is reminiscent of the asymptotic freedom from the YM charge. There is no need to remind ourselves that the latter is a quantum effect while our treatment is entirely classical here. For \( r \to \infty \) the YM term becomes dominant. Although this may sound contradictory to the short range of the YM field, we may attribute this to the decisive role played by the higher dimensionality of spacetime. Such behaviours, we speculate, have relevance in connection with the miniature (supermassive) black holes as well as asymptotically (anti-)de Sitter spacetimes. The solutions for \( N > 5 \) turn out to be distinct from the lower dimensional (i.e., \( N = 3, 4 \)-dimensional) cases. We show that in the latter cases of \( N = 3 \) and 4 the role of a YM charge is similar to that of the Maxwell charge as far as the spacetime metric is concerned. Namely, one is still the Bañados–Teitelboim–Zanelli (BTZ) metric, while the other is the RN metric. It is found that for \( N = 3 \) and 5 the metric function contains a logarithmic term, while in the other cases we have inverse power law dependence on \( r \). The unprecedented logarithmic potential leads to entirely different Keplerian orbits. Since black holes are scale invariant objects, i.e., they occur both at microscales and macroscales as miniature and supermassive black holes, the orbit revealed on one scale is valid on all scales. To show this, we investigate the Newtonian approximation (for \( N = 5 \)) as projected into the polar plane, which reveals the role of the YM charge in forming bound states with smaller orbits. Our second ansatz leads to, beside the black hole solution, a Bertotti–Robinson (BR) type of metric within the EMYMGB theory. It is known that the extremal RN case, which is a supersymmetric soliton solution for extended supergravity, interpolates between two maximally symmetric spacetimes, namely the flat space at infinity and BR at the horizon \( (r = 0) \). In other words, the near horizon geometry of an extremal RN black hole is identified as the BR spacetime. This idea can be extended to higher dimensions via p-branes. The topology of an \( N \)-dimensional BR type YM field is, as in the EM theory, \( adS_2 \times S^{N-2} \) (i.e., anti-de Sitter \( \times \) an \( (N-2) \) sphere), with a marked difference in their radii. The maximal symmetry of the BR spacetime, without singularity, makes it a favourable model for representing our homogeneous and isotropic universe in the absence of rotation. For the representative of the rotating BR universe and its cosmological implications for \( N = 4 \), we refer the reader to [19]. The interesting feature here is that the BR parameter consists of all the parameters of the theory, namely, \( M, q, Q, \Lambda \) and \( \alpha \). In other words, even the topological GB parameter serves for constructing a finely tuned cosmological BR model with maximal symmetry. It was shown by Gibbons and Townsend [1] that the extreme self-gravitating Yang type of monopole also yields an \( adS_2 \times S^{2N} \) topological vacuum in
the \((2N + 2)\)-dimensional EYM theory. It is our belief that our metrics will be useful both in string/supergravity theories and in cosmology. One immediate conclusion that we can draw is that we can generate an ‘effective cosmological constant \(\Lambda_{\text{eff}}\)’ to play the same role, in the absence of a real \(\Lambda\). To engender confidence, we verify also that our physical sources in higher dimensions satisfy all the weak, strong, dominant and causality conditions.

The paper is organized as follows. In section 2 we introduce our action, metric, Maxwell, YM ansatz and the field equations. Section 3 follows with the solution of the field equations for different dimensionalities. The geometry outside a five-dimensional black hole with its Newtonian approximation is investigated in section 4. In section 5 we add the Gauss–Bonnet (GB) term and present the most general solution. Section 6 deals with a new set of ansätze for the Maxwell field and the metric function for which we study the black hole properties and BR classes inherent in them. Energy and causality conditions are discussed in section 7. The paper ends with concluding remarks in section 8.

2. Action, field equations and our ansätze (RN type)

The action which describes Einstein–Maxwell–Yang–Mills gravity with a cosmological constant in \(N\) dimensions reads

\[
I_G = \frac{1}{2} \int_M \sqrt{-g} \left( R - \frac{(N-1)(N-2)}{3} \Lambda - F_{\mu\nu} F^{\mu\nu} - \text{Tr}(F_{\mu\nu} F^{(\alpha)\mu\nu}) \right),
\]

where in this context we use the following abbreviation:

\[
\text{Tr}(\cdot) = \sum_{a=1}^{(N-1)(N-2)/2} (\cdot).
\]

Here, \(R\) is the Ricci scalar while the YM and Maxwell fields are defined respectively by

\[
F_{\mu\nu}^{(a)} = \partial_\mu A_{\nu}^{(a)} - \partial_\nu A_{\mu}^{(a)} + \frac{1}{2\sigma} C_{(b)(c)}^{(a)} A_{\mu}^{(b)} A_{\nu}^{(c)},
\]

\[
F_{\mu\nu} = \partial_\mu A_{\nu} - \partial_\nu A_{\mu},
\]

in which \(C_{(a)(b)(c)}\) stands for the structure constant of the \((N-1)(N-2)/2\) parameter Lie group \(G\) and \(\sigma\) is a coupling constant. \(A_{\mu}^{(a)}\) are the SO\((N-1)\) gauge group YM potentials while \(A_{\mu}\) represents the usual Maxwell potential. We note that the internal indices \(\{a, b, c, \ldots\}\) do not differ between covariant and contravariant forms. Variation of the action with respect to the spacetime metric \(g_{\mu\nu}\) yields the field equations

\[
G_{\mu\nu} + \frac{(N-1)(N-2)}{6} \Lambda g_{\mu\nu} = T_{\mu\nu},
\]

where the stress–energy tensor is the superposition of the Maxwell and YM parts, namely

\[
T_{\mu\nu} = \left( 2 F_{\mu}^{\lambda} F_{\nu\lambda} - \frac{1}{2} F_{\lambda\sigma} F^{\lambda\sigma} g_{\mu\nu} \right) + \text{Tr} \left[ 2 F_{\mu}^{(a)} F_{\nu\lambda}^{(a)} - \frac{1}{2} F_{\lambda\sigma}^{(a)} F^{(a)\lambda\sigma} g_{\mu\nu} \right].
\]
Variation with respect to the gauge potentials $A^{(a)}_{\mu}$ and $A_{\mu}$ yields the respective YM and Maxwell equations
\[ F^{(a)\mu\nu} + \frac{1}{\sigma} C^{(a)}_{(b)(c)} A_{\mu}^{(b)} F^{(c)\mu\nu} = 0, \quad F^{\mu\nu}_{,\mu} = 0, \tag{6} \]
whose integrability equations, in order, are
\[ * F^{(a)\mu\nu} + \frac{1}{\sigma} C^{(a)}_{(b)(c)} A_{\mu}^{(b)} * F^{(c)\mu\nu} = 0, \quad * F_{\mu\nu}^{\mu\nu} = 0, \tag{7} \]
in which * means duality [4]. The $N$-dimensional spherically symmetric line element is chosen as
\[ ds^2 = -f(r) \, dt^2 + \frac{dr^2}{f(r)} + r^2 \, d\Omega^2_{N-2}, \tag{8} \]
in which the $S^{N-2}$ line element will be expressed in the standard spherical form
\[ d\Omega^2_{N-2} = d\theta^2_1 + \sum_{i=2}^{N-1} \prod_{j=1}^{i} \sin^2 \theta_j \, d\theta_i^2, \tag{9} \]
where
\[ 0 \leq \theta_{N-2} \leq 2\pi, \quad 0 \leq \theta_i \leq \pi, \quad 1 \leq i \leq N - 3. \]
For the YM field we employ the magnetic Wu–Yang ansatz [3,5,6] where the potential 1-forms are expressed as
\[ A^{(a)} = \frac{Q}{r^2} (x_i \, dx_j - x_j \, dx_i), \quad Q = \text{charge}, \quad r^2 = \sum_{i=1}^{N-1} x_i^2, \tag{10} \]
\[ 2 \leq j + 1 \leq i \leq N - 1, \quad \text{and} \quad 1 \leq a \leq (N - 1) (N - 2)/2, \]
The Maxwell potential 1-form is chosen as
\[ A = \begin{cases} \frac{q}{r^{N-3}} \, dt, & N \geq 4, \\ q \ln(r) \, dt, & N = 3, \end{cases} \tag{11} \]
for the electric charge $q$. The following sections will be devoted to the EMYM equations (i.e. equation (4)) and their solutions for all dimensionalities $N \geq 3$. In the last two sections we add GB theory (for $N \geq 5$) into our formalism and find new combined solutions for different Maxwell and metric ansätze. The energy–momentum tensor for the Maxwell and YM fields for $N \geq 4$ (the three-dimensional case will be studied separately) are given by [3]
\[ T_{\text{Max}}^{ab} = - \frac{(N - 3)^2 q^2}{r^2 (N - 2)} \text{diag} [1, 1, -1, -1, \ldots, -1], \tag{12} \]
\[ T_{\text{YM}}^{ab} = - \frac{(N - 3)(N - 2)Q^2}{2 r^4} \text{diag} [1, 1, \kappa, \kappa, \ldots, \kappa], \tag{13} \]
\[ \kappa = \frac{N - 6}{N - 2}. \]
3. **EMYM solution for** $N \geq 6$

The basic field equation that incorporates all relevant expressions is given by

$$\frac{r^3}{N-3} \left( f' + \frac{\Lambda}{3} (N-1) r \right) + (f-1) r^2 + Q^2 + \frac{2q^2(N-3) r^{2(4-N)}}{(N-2)} = 0,$$

where $f' \equiv (df/dr)$, which admits the integral

$$f(r) = 1 - \frac{2M}{r^{N-3}} - \frac{\Lambda}{3} r^2 + \frac{(N-3) q^2}{(N-2) r^{2(N-3)}} - \frac{(N-3) Q^2}{(N-5) r^2},$$

for the constant of integration $M$ as the mass parameter.

It is observed that this solution is not valid for $N = 5$ and 3; for these particular cases therefore, different solutions will be found. It is valid, however, for $N \geq 6$, which implies that the signs of the Maxwell and YM terms are opposite. This may have interesting consequences pertaining to the confinement of a system that possesses both types of charge. We write (for $\Lambda = 0$)

$$f(r) = 1 + 2 \Phi(r),$$

where $\Phi(r)$ stands for the Newtonian-like potential, which we identify as

$$\Phi(r) = \frac{M}{r^{N-3}} + \frac{(N-3) q^2}{(N-2) r^{2(N-3)}} - \frac{(N-3) Q^2}{2(N-5) r^2}.$$

We can define the active force through $F = -\nabla \Phi$, which yields

$$F(r) = -\frac{M(N-3)}{r^{N-2}} + \frac{2q^2(N-3)^2}{(N-2) r^{2N-5}} - \frac{(N-3) Q^2}{(N-5) r^3}.$$

The signs of the Maxwell and YM terms reveal that while the former is repulsive, the latter becomes attractive. One can easily show that for $r \to \infty$ the YM term dominates (let $\Lambda = 0$); namely

$$\lim_{r \to \infty} f(r) \to 1 - \frac{(N-3) Q^2}{(N-5) r^2}.$$  \hspace{1cm} (19)

For $r \to 0^+$ we have the opposite case:

$$\lim_{r \to 0^+} f(r) \to 1 + \frac{(N-3) q^2}{(N-2) r^{2(N-3)}},$$

which may be interpreted as an ‘asymptotic independence’ from one type of charge (or the other) in different limits. For the miniature black holes this has the striking effect that the Hawking temperature depends only on the electric charge.

To find the radius of possible horizon(s), we equate the metric function $f(r)$ to zero, which leads to the following equation:

$$6(N-3) (N-5) q^2 - 6 M (N-2) (N-5) r^{N-3} - \Lambda (N-2)(N-5) r^{2(N-2)}$$
$$+ 3 (N-2) (N-5) r^{2(N-3)} - 3 (N-2) (N-3) r^{2(N-4)} = 0.$$  \hspace{1cm} (21)
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Figure 1. Radius of the event horizon of the five-dimensional EMYM black hole in terms of the Maxwell and YM charges, and specific values for $M$ and $\Lambda$. The non-black hole region is shown non-shaded.

This equation in six-dimensional spacetime, without the cosmological constant and for zero mass, has an exact real solution:

$$r_h = \left[ \frac{1}{6} \left( \frac{1}{2} \sqrt[3]{\Delta} + \frac{2\tilde{Q}^4}{\sqrt[3]{\Delta}} + Q^2 \right) \right],$$  \hspace{1cm} (22)

where

$$\Delta = 8\tilde{Q}^6 - 108\tilde{q}^2 + 12\sqrt{3q^2 \left( 27q^2 - 4\tilde{Q}^6 \right)},$$

$$\tilde{Q}^2 = 6Q^2, \quad \tilde{q}^2 = 12q^2, \quad \tilde{q}^2 \geq \frac{4\tilde{Q}^6}{27}.$$  \hspace{1cm} (23)

3.1. The case $N = 5$

For $N = 5$, the master equation (14) admits the solution

$$f(r) = 1 - \frac{2M}{r^2} - \frac{\Lambda}{3} r^2 + \frac{4q^2}{3r^4} - \frac{2Q^2 \ln(r)}{r^2},$$

which involves an unusual logarithmic term. This is an asymptotically flat black hole solution in which finding the exact radius of the horizon of the possible black holes leads us to the following non-algebraic equation:

$$\frac{\Lambda}{3} r^6 - r^4 + 2Mr^2 + 2Q^2 r^2 \ln r - \frac{4}{3} q^2 = 0.$$  \hspace{1cm} (25)

This equation, even without the cosmological constant, does not give an analytical solution. By using a numerical method, we plot the root ($r_h$) of the above equation in figure 1, to show the contribution of the Maxwell and YM charges to the construction of such possible black holes. The Hawking temperature $T_H$ can be written as

$$T_H = \frac{\kappa}{2\pi} = \frac{1}{4\pi} \left| f'(r_h) \right| = \frac{1}{4\pi} \left| \frac{1}{r_h} - \frac{2}{3} \Lambda r_h - \frac{4}{3} \frac{q^2}{r_h^5} - \frac{Q^2}{r_h^3} \right|,$$  \hspace{1cm} (26)
where $r_h$ is the radius of the event horizon and $\kappa$ stands for the surface gravity. The corresponding Newtonian-like YM force term in this case has the form

$$\text{force} \sim \frac{Q^2}{r^3} (1 - 2 \ln r),$$

(27)

which implies that it is attractive (repulsive) for $r > \sqrt{e}$ ($r < \sqrt{e}$). The Maxwell term remains always repulsive. We notice also that for $r \to \infty$ (for $\Lambda = 0$) the YM term dominates over the Maxwell one:

$$\lim_{r \to \infty} f(r) \to 1 - \frac{2Q^2 \ln(r)}{r^2}.$$ 

(28)

In the limit $r \to 0^+$, on the other hand, we obtain

$$\lim_{r \to 0^+} f(r) \to 1 + \frac{4q^2}{3r^4},$$

(29)

which is in conformity with the behaviour (20) for $N \geq 6$.

### 3.2. The case $N = 4$

In the four-dimensional case the solution for the metric function $f(r)$ is given by

$$f(r) = 1 - \frac{2M}{r} - \frac{\Lambda}{3} r^2 + \frac{(q^2 + Q^2)}{r^2},$$

(30)

in which the Maxwell and YM charges have similar features and the metric is the well-known RN–de Sitter one. The extremal YM black hole, for instance, follows in analogy with the four-dimensional RN black hole. Both charges act repulsively against the attractive property of mass. The black hole solution for $f(r) = 0$ given in equation (30), without the cosmological constant, has two roots:

$$r_{\pm} = M \pm \sqrt{M^2 - (q^2 + Q^2)},$$

(31)

in which the mass and charges must satisfy the constraint $M^2 \geq (q^2 + Q^2)$ to have the event ($r_+$) and Cauchy ($r_-$) horizons. Here the thermodynamic properties of the solution (30) are exactly same as those of the four-dimensional RN black holes and therefore we just comment that the roles of the YM and Maxwell charges in the metric are not distinguished from each other.

### 3.3. The case $N = 3$

For three-dimensional spacetime we adopt the Maxwell potential 1-form [7]

$$A = q \ln(r) \, dt, \quad (q = \text{electric charge}),$$

(32)

and introduce the YM gauge potential 1-forms accordingly as

$$A^{(1)} = Q \cos(\phi) \ln(r) \, dt,$$

$$A^{(2)} = Q \sin(\phi) \ln(r) \, dt,$$

$$A^{(3)} = -Q \, d\phi, \quad (Q = \text{YM charge}),$$

(33)
which satisfy the Maxwell and YM equations, respectively. The corresponding EMYM equation for $f(r)$, independent from equation (14), becomes

$$3rf' + 2\Lambda r^2 + 6 \left( Q^2 + q^2 \right) = 0,$$

which is readily integrated as

$$f(r) = -M - \frac{1}{3} \Lambda r^2 - 2 \left( q^2 + Q^2 \right) \ln(r),$$

and the line element is

$$ds^2 = -f(r) \, dt^2 + \frac{dr^2}{f(r)} + r^2 \, d\theta^2.$$

A negative cosmological constant leads to a black hole solution [7]:

$$f(r) = -M + \frac{1}{3} |\Lambda| r^2 - 2 \left( q^2 + Q^2 \right) \ln(r),$$

in which the possible radii of the horizons are given by

$$r_+ = \exp \left[ -\frac{1}{2} \text{Lambert} \left( -1, -\frac{|\Lambda| e^{-M/(Q^2+q^2)}}{3 (Q^2 + q^2)} \right) - \frac{M}{2 (Q^2 + q^2)} \right],$$

$$r_- = \exp \left[ -\frac{1}{2} \text{Lambert} \left( 0, -\frac{|\Lambda| e^{-M/(Q^2+q^2)}}{3 (Q^2 + q^2)} \right) - \frac{M}{2 (Q^2 + q^2)} \right],$$

in which Lambert $W(k, x)$ stands for the Lambert function [8]. The energy density, given by

$$\epsilon = T^{\tau\tau} = \frac{Q^2 + q^2}{r^2},$$

may be used to calculate the total energy of the black hole. This shows that the energy diverges logarithmically. The surface gravity, $\kappa$, defined by

$$\kappa^2 = \left( -\frac{1}{16} g^{ij} g_{\tau i} g_{\tau j} \right)_{r=r_h} = \left( \frac{1}{2} f'(r) \right)^2_{r=r_+},$$

gives

$$\kappa = \left| \frac{|\Lambda|}{3} r_+ - \frac{Q^2 + q^2}{r_+} \right|.$$  

Finally we find the Hawking temperature at the event horizon as

$$T_H = \frac{\kappa}{2\pi} = \frac{1}{2\pi} \left| \frac{|\Lambda|}{3} r_+ - \frac{Q^2 + q^2}{r_+} \right|,$$

and a plot of $T_H$ is given in figure 2 in terms of the mass, $\Lambda$ and $(Q^2 + q^2)$.

By analogy with the $N = 4$ case, the squared charges are simply superposed, and the metric is still the BTZ metric. It is observed that for $q = 0 \neq Q$ the same (BTZ) metric describes an EYM black hole in three dimensions. Addition of rotation to the metric, which is beyond our scope here, may add new features that differ for the Maxwell and YM fields.
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Figure 2. A plot of $T_H$ for the three-dimensional EMYM black hole solution, in terms of $\Lambda$ and $(Q^2 + q^2)$. We set the mass of the black hole to be unity.

4. The geometry outside the five-dimensional EMYM black hole

It is evident from solution (24) that $\xi^\alpha = (1, 0, 0, 0, 0)$ and $\eta^\alpha = (0, 0, 0, 0, 1)$ are two of the Killing vectors associated with the symmetry under displacements in the direction $t$, with rotation angle $\psi$. Accordingly the conserved quantities may be written as

$$e = -\xi \cdot u = -g_{\alpha\beta} \xi^\alpha u^\beta = f(r) u^t,$$

$$\ell = \eta \cdot u = g_{\alpha\beta} \eta^\alpha u^\beta = r^2 \sin^2(\theta) \sin^2(\phi) u^\psi,$$

where $u = (u^t, u^r, u^\theta, u^\phi, u^\psi)$ is the 5-velocity while $e$ and $\ell$ are the energy density and angular momentum per unit mass, respectively. We restrict the particle to staying on the plane $\theta = \pi/2, \phi = \pi/2$ with $u^\theta = u^\phi = 0$, such that by applying $u \cdot u = -1$ for the timelike geodesics one obtains

$$g_{\alpha\beta} u^\alpha u^\beta = -f(r) (u^t)^2 + \frac{1}{f(r)} (u^r)^2 + r^2 (u^\psi)^2 = -1,$$

in which $u^t = dt/d\tau$, $u^r = dr/d\tau$, $u^\theta = d\theta/d\tau$, $u^\phi = d\phi/d\tau$, $u^\psi = d\psi/d\tau$ and $\tau$ is the proper time measured by the observer moving with the particle. Putting (44) and (45) into (46), one gets

$$\frac{1}{2} \left( \frac{dr}{d\tau} \right)^2 + \frac{1}{2} \left[ \left( \frac{\ell^2}{r^2} + 1 \right) f(r) - 1 \right] = \frac{e^2}{2} - 1,$$

or equivalently

$$\frac{1}{2} \left( \frac{dr}{d\tau} \right)^2 + V_{\text{eff}}(r) = \mathcal{E},$$

where $\mathcal{E}$ is the density of the total energy per unit mass, $V_{\text{eff}}(r)$ is the effective potential for radial motion of the particle and $f(r)$ is given in (24). In figure 3 we plot $V_{\text{eff}}(r)$ in terms of $r$ for different values of $q, Q$ and $l$ but the zero value for $M$. Since the particle
is restricted to remaining in a plane in which only $r$ and $\psi$ would change, we rewrite equation (48) in terms of $r$ and $\psi$ as follows:

$$
\frac{1}{2} \left( \frac{d\psi}{d\tau} \right)^2 + \frac{1}{2} \left( \frac{dr}{d\psi} \right)^2 + V_{\text{eff}}(r) = \mathcal{E},
$$

(49)

or equivalently, from equation (49),

$$
\frac{dr}{d\psi} = \pm \sqrt{\frac{2r^4}{\ell^2} (\mathcal{E} - V_{\text{eff}}(r))},
$$

$$
V_{\text{eff}}(r) = \frac{1}{2} \left[ \left( \frac{\ell^2}{r^2} + 1 \right) f(r) - 1 \right],
$$

(50)

in which $\pm$ depends to the initial direction of the motion and we set it to be positive. In section 4.1, instead of studying the exact geodesic equation (50), we shall restrict ourselves to the relatively simple Newtonian approximation.

4.1. Newtonian motion

In the weak field limit (and setting $\Lambda$ to be zero), one may use $f(r) = 1 + 2\Phi(r)$ which implies

$$
\Phi(r) = -\frac{M}{r^2} + \frac{2q^2}{3r^4} - \frac{Q^2 \ln(r)}{r^2},
$$

(51)

and consequently the radial force on a unit mass particle is given by

$$
F_r = -\frac{d}{dr} \Phi(r) = -\left( \frac{2M}{r^3} - \frac{8q^2}{3r^5} - \frac{Q^2}{r^3} + \frac{2Q^2 \ln(r)}{r^3} \right).
$$

(52)
The radial equation of motion, therefore, may be written as

\[
\frac{d^2 r}{dt^2} = \frac{\ell^2}{r^3} - \left( \frac{2M}{r^3} - \frac{8q^2}{3r^3} - \frac{Q^2}{r^3} + \frac{2Q^2 \ln(r)}{r^3} \right),
\]

(53)

where \( \ell \) is the angular momentum per unit mass. As usual, one may start with the following substitution:

\[
u = \frac{1}{r},
\]

(54)

to reduce the last equation to the form

\[
\frac{d^2 \nu}{d\psi^2} + \left( 1 + \frac{Q^2 - 2M}{\ell^2} \right) \nu + \frac{2Q^2}{\ell^2} \nu \ln \nu - \frac{8q^2}{3\ell^2} \nu^3 = 0.
\]

(55)

As a particular case we set \( 1 + (Q^2 - 2M/\ell^2) = 0 \), \( \tilde{Q} = \sqrt{2Q/\ell} \) and \( \tilde{q} = \sqrt{8/3}(q/\ell) \) to get

\[
\frac{d^2 \nu}{d\psi^2} + \tilde{Q}^2 \nu \ln \nu - \tilde{q}^2 \nu^3 = 0,
\]

(56)

such that the inverse of the solution of this equation is as plotted in figure 4 (i.e., \( r = 1/\nu \) versus \( \psi \)). Obviously, from the figure, one observes that the roles of Maxwell and YM charges outside the black hole contrast with each other. Next, we consider equation (55) and set the mass of the black hole to zero. By adjusting charges in terms of the angular momentum, we express the equation in the form

\[
\frac{d^2 \nu}{d\psi^2} + \left( 1 + \frac{\tilde{Q}^2}{2} \right) \nu + \tilde{Q}^2 \nu \ln \nu - \tilde{q}^2 \nu^3 = 0.
\]

(57)
One notices that, if $\dot{Q} = 0$, this reduces to a Duffing type of equation:

$$\frac{d^2u}{d\psi^2} + u - \tilde{q}^2 u^3 = 0,$$

which can be considered, for small $\tilde{q}$, as a perturbed simple harmonic oscillator. In our work, however, we are interested in considering both terms (i.e., Maxwell and YM terms). From equation (55) we observe that $u = 1$ forms a circular orbit provided that the condition (with $M \neq 0$)

$$Q^2 - \frac{8}{3} q^2 = 2M - \ell^2$$

holds. We investigate the stability of this orbit by choosing

$$u = 1 + a \cos \beta \psi,$$

in which $\beta$ and $a$ are constants such that $a \ll 1$. By substitution we obtain

$$\beta = \pm \sqrt{\frac{2}{\ell}} \sqrt{Q^2 - \frac{8}{3} q^2},$$

so stability of the circular orbits is attained provided $Q^2 > (8/3)q^2$. Thus, a dominating YM charge gives rise to a deeper well and stable orbits. The foregoing argument can easily be extended to cover elliptical orbits as well; this will not be repeated here. Let us note that the possibilities involved in a complete analysis of equation (55) may reveal different behaviours as well.

5. EMYM black holes in GB gravity

In this section we use our previous ansätze and find solutions with the GB term. The new action is modified now to

$$I_G = \frac{1}{2} \int_M \mathcal{L}_G \sqrt{-g} \left( R + \alpha \mathcal{L}_{GB} - \frac{(N - 1)(N - 2)}{3} \Lambda - F_{\mu\nu} F^{\mu\nu} - \text{Tr} \left( F_{\mu\nu}^{(a)} F^{(a)\mu\nu} \right) \right),$$

where the new term $\mathcal{L}_{GB} = R_{\mu
u\alpha\beta} R^{\mu
u\alpha\beta} - 4 R_{\alpha\beta} R^{\alpha\beta} + R^2$ is the GB Lagrangian with the constant GB parameter $\alpha$. The Maxwell and YM ansatz are chosen as in section 4. The EMYMGB equation that helps us to determine $f(r)$ is given by

$$\left( \Delta - \frac{\Lambda}{3} (N - 1) r^4 \right) (N - 2) r^{(2N-6)} - 2(N - 3)^2 q^2 r^4 = 0,$$

$$\Delta = (-r^3 + 2\tilde{\alpha} r (f(r) - 1)) f'(r) + \tilde{\alpha} (N - 5) (f(r) - 1)^2 - r^2(N - 3) (f(r) - 1) - Q^2 (N - 3).$$

The solution for $f(r)$ follows as

$$f_{\pm}(r) = \begin{cases} 
1 + \frac{r^2}{4\tilde{\alpha}} \pm \Psi, & N = 5, \\
1 + \frac{r^2}{2\tilde{\alpha}} (1 \pm \Upsilon), & N \geq 6,
\end{cases}$$

where $\Psi$ and $\Upsilon$ are determined from the boundary conditions.

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Figure 5. Radius of the event horizon of the five-dimensional EMYMGB black hole in terms of Maxwell and YM charges, and specific values for $M$, $\alpha$ and $\Lambda$. The non-black hole region is shown non-shaded. (These plots may be compared with those in figure 1.)

where

$$\Psi = \sqrt{1 + \frac{M}{2\alpha} + \left(\frac{\Lambda}{3} + \frac{1}{8\alpha}\right) \frac{r^4}{2\alpha} + \frac{Q^2\ln(r)}{\alpha} - \frac{2q^2}{3\alpha r^2}}, \quad (65)$$

$$\Upsilon = \sqrt{1 + \frac{4\tilde{\alpha}\Lambda}{3} + \frac{4\tilde{\alpha}Q^2(N - 3)}{(N - 5)r^4} + \frac{8M\tilde{\alpha}}{r^{N-1}} - \frac{8(N - 3)q^2\tilde{\alpha}}{(N - 2)r^{2(N-2)}}, \quad (66)$$

in which $\tilde{\alpha} = (N - 3)(N - 4)\alpha$ and $M$ is a constant of integration, to represent mass. In the limit $\alpha \to 0$ the expression for $f_{-}(r)$ reduces to the ones in section 4, as it should do, and for the positive branch $\alpha$ cannot be zero. For $Q = 0$ our result reduces to the one known before for the EMGB theory [9]. Similarly for $q = 0$ we recover the results obtained previously [10]–[12]. In figure 5 we plot the radius of the event horizon of the five-dimensional EMYMGB solution in terms of Maxwell and YM charges. We emphasize the different effects of these charges. For $N \geq 6$ it can easily be seen that the Maxwell and YM terms have opposite signs. It is remarkable to observe that the solutions (64) behave in asymptotically dS (adS) manners, such that the effective cosmological constant may be written as

$$\Lambda_{\text{eff}} \doteq \begin{cases} 
-\frac{3}{4\alpha} \left(1 \pm \sqrt{1 + \frac{8\Lambda}{3}\tilde{\alpha}}\right), & N = 5, \\
-\frac{3}{2\alpha} \left(1 \pm \sqrt{1 + \frac{4\Lambda}{3}\tilde{\alpha}}\right), & N \geq 6,
\end{cases} \quad (67)$$

in which in the limit of $\tilde{\alpha} \to 0$ the negative branch admits $\Lambda_{\text{eff}} \to \Lambda$, and the positive branch in the limit of $\Lambda = 0$ gives $\Lambda_{\text{eff}} = -(3/\tilde{\alpha})$.

For the case $N = 5$, on the other hand, the range of $r$ determines the sign of the YM term. Although the $\pm$ signs determine the roles of both terms, we prefer the choice
(-) under which in the limit $\alpha \to 0$ we recover the EMYM black holes. In conclusion, by studying the Maxwell and YM fields together we see that these fields compete for dominance for dimensions $N \geq 5$. We observe on the other hand that for lower dimensions ($N = 3, 4$) their roles remain indistinguishable. The asymptotic solutions reveal that the physical results are independent of our choice of charge. It is our belief that this may be helpful in understanding the problem of confinement (i.e., accretion, collapse) versus the electric and YM charges.

6. EMYMGB solution for a specific ansatz for $N \geq 4$ (BR type)

In this section we choose our metric ansatz as

$$ds^2 = -f(r)\,dt^2 + \frac{dr^2}{f(r)} + h^2\,d\Omega^2_{N-2}, \quad N \geq 4,$$

in which $h = \text{constant}$ is to be expressed in terms of the parameters of the theory. While the YM field will be chosen as before, our Maxwell field will be different. For the present purpose let our Maxwell 1-form be given by the choice

$$A = qr\,dt,$$

where the constant $q$ is related to the electric charge. This choice has the feature that the only non-vanishing electromagnetic field 2-form

$$F = -q\,dt \wedge dr$$

yields a uniform electric field. The non-vanishing Maxwell energy–momentum tensor components $T^a_{\text{Max}\,b}$ are

$$T^a_{\text{Max}\,b} = -q^2 \, \text{diag} [1, 1, -1, -1, \ldots, -1].$$

The non-zero YM energy–momentum components $T^a_{\text{YM}\,b}$ are

$$T^a_{\text{YM}\,b} = -\frac{(N-3)(N-2)Q^2}{2h^4} \, \text{diag} [1, 1, \kappa, \kappa, \ldots, \kappa], \quad N \geq 4,$$

$$\kappa = \frac{N-6}{N-2}.$$ (72)

The field equations (4) with the GB and $\Lambda$ terms, on the premise that $h = \text{constant}$, reduce to

$$[h^2 + 2\alpha (N-3) (N-4)]\,f'' - (N-3)(N-4) [1 + \alpha (N-5)(N-6)] + \frac{1}{3} (N-1)(N-2)\Lambda h^2 + \frac{(N-3)(N-6)}{h^2} Q^2 - 2q^2 h^2 = 0, \quad N \geq 4.$$ (73)

The solution for $f(r)$ can be expressed as

$$f(r) = C_1 r^2 + C_2 r + C_3.$$ (74)
where $C_2$ and $C_3$ are integration constants while $C_1$ is a constant depending on the parameters of the theory. Explicitly we have

$$C_1 = \left\{ \frac{1}{2} (N - 3) (N - 4) \left[ 1 + \alpha (N - 5) (N - 6) \right] - \frac{1}{6} (N - 2) (N - 1) \Lambda h^2 \right\} \left[ h^2 + 2 \alpha (N - 3) (N - 4) \right], \quad N \geq 4. \tag{75}$$

The constant $h^2$ is also expressible as

$$\frac{2}{3} h^2 = \frac{(N - 3) (N - 2) \pm \sqrt{(N - 3) (N - 2) [K (N - 2) + 8q^2 L]}}{6q^2 + (N - 1) (N - 2) \Lambda}, \tag{76}$$

in which we have used the abbreviation

$$K = \frac{4}{3} (N - 1) \left[ (N - 5) (N - 4) \alpha - Q^2 \right] \Lambda + (N - 3),$$
$$L = (N - 5) (N - 4) \alpha - Q^2, \quad N \geq 4, \tag{77}$$

and (+) and (−) signs are chosen in the Maxwell and YM limits, respectively. From these expressions we obtain the EMYM limit by setting $\alpha = 0$. Similarly the EMGB and EYMGB limits can be obtained by setting $Q = 0$ and $q = 0$, respectively. For $C_2 \neq 0 \neq C_3$ we can have the roots of $f(r) = 0$, which signals the horizons for black holes. The abundance of parameters in the EYMGB theory create a large class of possibilities admitting various black hole solutions, which we shall not pursue here. By choosing $C_2 = C_3 = 0$ and constraining $h^2 = (\beta / C_1)$, for a suitable constant $\beta (> 0)$, followed by a redefinition of time, we cast the line element into the form

$$ds^2 = \frac{h^2}{\beta} \left( -dt^2 + \frac{dr^2}{r^2} + \beta d\Omega^2_{N-2} \right), \tag{78}$$

which is of the static BR form. For $\alpha = 0 = \Lambda = q$ (as a limit) we arrive at [13, 14]

$$ds^2 = \frac{Q^2}{N - 3} \left( -dt^2 + \frac{dr^2}{r^2} + (N - 3) d\Omega^2_{N-2} \right). \tag{79}$$

In general, since $\beta \neq 1$, conformal flatness is not satisfied. Only for $N = 4$ do we have the exact BR case which is conformally flat. However in general we have shown that in the EYMGB theory we construct a metric which is of BR type, although it fails to satisfy conformal flatness. In the pure Maxwell limit, ($q \neq 0$), $Q = \Lambda = \alpha = 0$ and adopting $C_2 = C_3 = 0$, we obtain (after scaling)

$$ds^2 = \frac{1}{C_1} \left[ -dt^2 + \frac{dr^2}{r^2} + (N - 3)^2 d\Omega^2_{N-2} \right], \tag{80}$$

which is also of similar type, with the constant $C_1 = 2q^2((N - 3) / (N - 2))$. This is in agreement with the higher dimensional BR metric in the EM theory [13]. We recall that following the method of Ginsparg and Perry [15], the $N$-dimensional YM–BR solution can be expressed in the form

$$ds^2 = \frac{Q^2}{(N - 3)} \left[ -\sinh^2 \chi dT^2 + d\chi^2 + (N - 3) d\Omega^2_{N-2} \right]. \tag{81}$$
The transformation that takes us to this result in the limit $\epsilon \to 0$ is given by taking
\[ f(r) = \frac{N-3}{Q^2} (r + t) (r - t), \quad t = \frac{Q^2}{(N-3)\epsilon} T, \quad r = \epsilon \cosh \chi, \] (82)
while the magnetic type YM field remains unchanged.

7. Energy and causality conditions

7.1. $N \geq 5$ dimensions (RN type)

The energy conditions (EC) of the matter associated with the energy–momentum tensor given by (12) and (13) for $N \geq 5$ dimensions, i.e.
\[ T^a_b = T^a_{\text{Max}} + T^a_{\text{YM} b}, \] (83)
can be studied by using the definition of the energy density of the matter $\rho$ [18]:
\[ \rho = -T^t_t = -T^r_r = \frac{(N-3)^2 q^2}{r^{2(N-2)}} + \frac{(N-3)(N-2)Q^2}{2r^4}, \] (84)
the principal pressures $p_i$:
\[ p_i = T^i_i \quad \text{(no sum convention)}, \] (85)
and the effective pressure:
\[ p_{\text{eff}} = \frac{1}{N-1} \sum_{i=1}^{N-1} p_i. \] (86)

7.1.1. Weak energy condition (WEC). The WEC may be expressed as
\[ \rho \geq 0 \quad \text{and} \quad \rho + p_i \geq 0 \quad (i = 1, 2, \ldots, N-1), \] (87)
which holds true in any number of dimensions $N \geq 4$, with the energy–momentum tensor given by (83).

7.1.2. Strong energy condition (SEC). The SEC states that
\[ \rho + \sum_{i=1}^{N-1} p_i \geq 0 \quad \text{and} \quad \rho + p_i \geq 0 \quad (i = 1, 2, \ldots, N-1), \] (88)
which for $4 < N \leq 6$ holds true, but for $N \geq 7$ the SEC is satisfied for $r \leq r_{\text{sec}}$ in which
\[ r_{\text{sec}} = \left( \frac{2(N-3)q^2}{(N-6)Q^2} \right)^{1/(2(N-4))}, \quad N \geq 7. \] (89)

7.1.3. Dominant energy conditions (DEC). In accordance with the DEC, which are given by
\[ \rho \geq |p_i| \quad (i = 1, 2, \ldots, N-1), \] (90)
our energy–momentum tensor satisfies these for any dimensions $N > 4$.  

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7.1.4. Causality condition (CC). We express the CC after finding that \( \rho > 0 \) as

\[
0 \leq p_{\text{eff}} < \rho, \tag{91}
\]

which is satisfied by the energy–momentum tensor given by (83) for \( N = 4, 5 \). For \( N \geq 6 \), (91) is satisfied if \( r < r_{cc} \) where

\[
r_{cc} = \left( \frac{2(N-3)^2 q^2}{(N-2)(N-5)Q^2} \right)^{1/(2(N-4))}, \quad N \geq 6. \tag{92}
\]

7.2. \( N = 3, 4 \) dimensions (RN type)

In four dimensions, the energy–momentum tensor simply reads

\[
T^a_b = -q^2 + Q^2 \frac{1}{r^4} \text{diag} [1, 1, -1, -1], \tag{93}
\]

in which the WEC, SEC, DEC and CC are all verified. In three dimensions also the energy–momentum tensor which is given by

\[
T^a_b = -q^2 + Q^2 \frac{1}{r^2} \text{diag} [1, 1, -1] \tag{94}
\]

satisfies all the energy and causality conditions.

7.3. \( N \geq 4 \) dimensions (BR type)

To investigate the energy conditions of solutions of the second type (BR type) we rewrite the energy–momentum tensor of the system in the form

\[
T^a_b = T^a_{\text{Max}} + T^a_{\text{YM}}, \tag{95}
\]

where

\[
T^a_{\text{Max}} = -q^2 \text{diag} [1, 1, -1, -1, \ldots, -1], \quad T^a_{\text{YM}} = -\tilde{Q}^2 \text{diag} [1, 1, \kappa, \kappa, \ldots, \kappa], \quad N \geq 4,
\]

\[
\kappa = \frac{N-6}{N-2}, \quad \tilde{Q}^2 = \frac{(N-3)(N-2)Q^2}{2h^4}. \tag{96}
\]

One can show that the WEC is satisfied for arbitrary \( N \geq 4 \). The SEC is also verified for \( N = 4, 5, 6 \), but for \( N \geq 7 \) only under the condition

\[
q^2 \geq \kappa \tilde{Q}^2 \tag{97}
\]

is the SEC satisfied. It is also easy to show that the DEC is satisfied for any number of dimensions. Finally, the CC is satisfied for \( N = 4, 5 \), but for \( N \geq 6 \) it becomes valid only with the additional condition

\[
q^2 \geq \frac{N-5}{N-3} \tilde{Q}^2. \tag{98}
\]
8. Conclusion

Our exact solution in the first part of the paper suggests that for $N \geq 5$, the Maxwell and YM fields compete for dominance in the asymptotic regions. That is, for $r \to 0$ ($r \to \infty$) the Maxwell charge $q$ (the YM charge $Q$) dominates. This may shed light on the problem of gravitational confinement (i.e., accretion, collapse) versus the Maxwell and YM charges. As a drawback of our model, the YM field is treated, by analogy with the Maxwell field, as entirely classical. The Newtonian approximation in the polar plane consisting of the coordinates $(r, \psi)$ reveals that the YM charge deepens the potential well to form bound states. In lower dimensions (i.e., for $N = 3, 4$), however, the roles of $q$ and $Q$ remain indistinguishable. The presence of a logarithmic term for $N = 3$ and 5 is a distinctive property as compared to the case for other dimensions. An effective cosmological constant can be defined from the GB parameters for $r \to \infty$. In the last part of the paper where we introduced a different ansatz, we present an exact solution for the EMYMGB theory which can represent a variety of black holes. Another possibility is, by the choice of parameters, to cast the metric into the static BR form which lacks conformal flatness but may be important in supergravity/string theory [16], as well as in cosmology. Finally, the validity of the energy/causality conditions is discussed for all solutions that are obtained in the paper.

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