Centrally essential torsion-free rings of finite rank

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Abstract
It is proved that centrally essential rings, whose additive groups of finite rank are torsion-free groups of finite rank, are quasi-invariant but not necessarily invariant. Torsion-free Abelian groups of finite rank with centrally essential endomorphism rings are faithful.

Keywords  Centrally essential ring · Quasi-invariant ring · Faithful Abelian group

Mathematics Subject Classification  16R99 · 20K30

1 Introduction
We consider only associative rings with non-zero identity elements. For a ring $R$, we denote by $C(R)$, $J(R)$ and $N(R)$ the center, the Jacobson radical and the upper radical of the ring $R$, respectively. For Abelian groups, we use the additive notation.
1.1 Centrally essential rings

A ring $R$ is said to be centrally essential if for every non-zero element $a \in R$, there exist non-zero central elements $x, y$ with $ax = y$.\(^1\)

Centrally essential rings with non-zero identity elements are studied in Markov and Tuganbaev (2018, 2019a, b, c, 2020a, b, c). Every centrally essential semiprime ring with $1 \neq 0$ is commutative; see Markov and Tuganbaev (2018, Proposition 3.3). Examples of non-commutative group algebras over fields are given in Markov and Tuganbaev (2018). In addition, the Grassman algebra over a three-dimensional vector space over the field of order 3 also is a finite non-commutative centrally essential ring; see (Markov and Tuganbaev 2019a). In Markov and Tuganbaev (2019c), there is an example of a centrally essential ring $R$ whose factor ring with respect to the prime radical of $R$ is not a PI ring. Abelian groups with centrally essential endomorphism rings are considered in Lyubimtsev and Tuganbaev (2020).

1.2 Torsion-free rings of finite rank and faithful Abelian groups

A ring $R$ is called a torsion-free ring of finite rank (a tffr ring) if the additive group $(R, +)$ of $R$ is an Abelian torsion-free group of finite rank.

A ring $R$ is said to be right invariant (resp., right quasi-invariant) if every right ideal (resp., maximal right ideal) of $R$ is an ideal of $R$. The words such as “an invariant ring” (resp., “a quasi-invariant ring”) mean “a right and left invariant ring” (resp., “right and left quasi-invariant ring”).

An Abelian group $A$ is said to be faithful if $IA \neq A$ for every proper right ideal $I$ of the endomorphism ring $\text{End} A$ of the group $A$.

It is known that torsion-free groups of finite rank with commutative endomorphism rings are faithful (see Arnold 1982, Theorem 5.9). Faticoni got a stronger result, he proved that torsion-free groups of finite rank with right invariant endomorphism rings also are faithful groups (see Faticoni 1988, Lemma 3.1). Detailed information on faithful Abelian groups is contained in Krylov et al. (2003, Chapter VI, §33, 34).

The main results of this paper are Theorems 1.3 and 1.4.

1.3 Theorem Every centrally essential torsion-free ring of finite rank is a quasi-invariant ring which is not necessarily right or left invariant.\(^2\)

1.4 Theorem An Abelian torsion-free group of finite rank with centrally essential endomorphism ring is a faithful group.

We give some definitions used in the paper. A right ideal $I$ of the ring $R$ is said to be essential if $I \cap J \neq 0$ for every non-zero right ideal $J$ in $R$. In this case, one says that $R$ is an essential extension of $I$.

A right ideal $B$ is said to be closed if it coincides with any right ideal which is an essential extension of $B$.

\(^1\) It is clear that a ring $R$ with center $C$ is centrally essential if and only if the module $RC$ is an essential extension of the module $C_C$.

\(^2\) In Example 2.4 of Sect. 2 of this paper, we give an example of a centrally essential torsion-free ring of finite rank which is not a right or left invariant ring.
For a right ideal \( I \) of the ring \( R \), every right ideal \( J \) of \( R \), which is maximal with respect to the property \( I \cap J = 0 \), is called a \( \cap \)-complement for \( I \). It is easy to verify that \( I \bigoplus J \) is an essential right ideal in \( R \) in this case (e.g., see Goodearl 1976).

An element \( r \) of the ring \( R \) is called right regular or a left non-zero-divisor if the relation \( rx = 0 \) implies the relation \( x = 0 \) for every \( x \in R \).

A subgroup \( B \) of the Abelian group \( A \) is said to be pure if the equation \( nx = b \in B \), which is solvable in the group \( A \), is solvable in \( B \).

### 2 Properties of ideals of centrally essential rings

#### 2.1 Proposition
Let \( R \) be a centrally essential ring with center \( C(R) \). If there exists a maximal right ideal \( M \) of the ring \( R \) which is not an ideal, then \( C(R) \cap (\bigcap_{n \geq 1} M^n) \neq 0 \).

**Proof** We assume the contrary. Then there exist two elements \( m \in M \) and \( a \in R \) with \( am \notin M \). Since \( M \) is a maximal right ideal, there exist elements \( b \in R \) and \( m' \in M \) such that \( 1 = amb + m' \). Since \( am \notin M \), we have \( a \neq 0 \). Since \( R \) is a centrally essential ring, there exist non-zero elements \( c, d \in C(R) \) such that \( ac = d \neq 0 \). Then

\[
   c = (amb + m')c = (ac)mb + m'c = mbd + m'c \in M,
\]

\((ac)mb \in M^2 \) and \( m'c \in M^2 \). Therefore, \( c = (ac)mb + m'c \in M^2 \) and \( (ac)mb, m'c \in M^3 \). Then \( c \in M^3 \). By repeating a similar argument, we obtain that \( 0 \neq c \in C(R) \cap \left( \bigcap_{n=1}^{+\infty} M^n \right) \). \( \square \)

#### 2.2 Proposition
Let \( R \) be a centrally essential ring. Then every minimal right ideal in \( R \) is contained in the center \( C(R) \) of the ring \( R \). In particular, every minimal right ideal of the ring \( R \) is an ideal in \( R \).

**Proof** Let \( I \) be a non-zero minimal right ideal. Since the ring \( R \) is centrally essential, we have that for every non-zero element \( a \in I \), there exist two non-zero central elements \( c, d \in R \) such that \( ac = d \neq 0 \). Then \( d \in C(R) \cap I \). Consequently, \( C(R) \cap I \neq 0 \). In addition, \( C(R) \cap I = K \) is an ideal in \( R \), \( K \subseteq I \). Since \( I \) is a minimal right ideal, \( I = K \). \( \square \)

#### 2.3 Proposition
If a closed right ideal of a centrally essential ring contains a right regular element, then the right ideal is an ideal.

**Proof** Let \( J \) be a closed right ideal of the centrally essential ring \( R \) and \( r \in R \). If \( 0 \neq X \leq rJ + J \), then \( X \cap C(R) \neq 0 \). We take a non-zero element \( x \in X \cap C(R) \). We have \( x = ra + b \) for some \( a, b \in J \). By assumption, \( J \) contains a right regular element \( y \). In this case, we have

\[
0 \neq xy = yra + yb \in J \cap X.
\]

Consequently, the right ideal \( rJ + J \) is an essential extension of \( J \). Since \( J \) is a closed ideal, \( J = rJ + J \) and \( J \) is an ideal. \( \square \)
2.4 Example We consider the subring $\mathcal{R}$ in the ring $M_7(R)$ of all $7 \times 7$ matrices over the commutative domain $R$ consisting of the matrices $A$ of the form

$$A = \begin{pmatrix}
    \alpha & a & b & c & d & e & f \\
    0 & \alpha & 0 & b & 0 & 0 & d \\
    0 & 0 & \alpha & 0 & 0 & 0 & e \\
    0 & 0 & 0 & \alpha & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & \alpha & a & b \\
    0 & 0 & 0 & 0 & 0 & \alpha & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & \alpha
  \end{pmatrix}.$$

Let $A' \in \mathcal{R}$ with $a' = a + 1$ and let the remaining components of $A'$ coincide with corresponding components of the matrix $A$. Then $AA' \neq A'A$ if $a \neq 0$ and $b \neq 0$. Thus, the ring $\mathcal{R}$ is not commutative. It is easy to see that $C(\mathcal{R})$ consist of matrices

$$C = \begin{pmatrix}
    \alpha & 0 & 0 & c & d & e & f \\
    0 & \alpha & 0 & 0 & 0 & 0 & d \\
    0 & 0 & \alpha & 0 & 0 & 0 & e \\
    0 & 0 & 0 & \alpha & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & \alpha & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & \alpha & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & \alpha
  \end{pmatrix}.$$

Let $A \in \mathcal{R}$ and let $a \neq 0$ or $b \neq 0$. We take the matrix $B \in C(\mathcal{R})$ with $d = a$, $e = b$ and zeroes on the remaining positions. Then $0 \neq AB \in C(\mathcal{R})$. Thus, $\mathcal{R}$ is a centrally essential ring.

We consider the right ideal $I$ of $\mathcal{R}$ consisting of the matrices of the form

$$B = \begin{pmatrix}
    0 & 0 & b & 0 & 0 & 0 & f \\
    0 & 0 & 0 & b & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & b \\
    0 & 0 & 0 & 0 & 0 & 0 & 0
  \end{pmatrix}.$$

It is directly verified that $I$ is not an ideal in $\mathcal{R}$. In addition, $I$ is a closed right ideal. Indeed, the ideal of $\mathcal{R}$, which has only $c$ as a non-zero component, is a $\cap$-complement for $I$. 
At the same time, the closed left ideal \( J \) of \( R \) consisting of elements are the matrices

\[
D = \begin{pmatrix}
0 & a & 0 & 0 & 0 & f \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & a \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}.
\]

is not an ideal. The ideal which has only \( c \) as a non-zero component, is also a \( \cap \)-complement for \( J \).

The above example can be generalized to the case of matrices of any order \( n \). Namely, the subring \( R \) of the ring \( M_n(R) \) of all \( n \times n \) matrices over the commutative ring \( R \) consisting of matrices \( A \) of the form

\[
A = \begin{pmatrix}
\alpha & a_{12} & a_{13} & a_{14} & a_{15} & \ldots & a_{1n-2} & a_{1n-1} & a_{1n} \\
0 & a & 0 & a_{13} & 0 & \ldots & 0 & 0 & a_{1n-2} \\
0 & 0 & \alpha & 0 & 0 & \ldots & 0 & 0 & a_{1n-1} \\
0 & 0 & 0 & \alpha & 0 & \ldots & 0 & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & 0 & 0 & \ldots & \alpha & 0 & a_{12} \\
0 & 0 & 0 & 0 & 0 & \ldots & 0 & \alpha & a_{13} \\
0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & \alpha \\
\end{pmatrix},
\]

is non-commutative centrally essential ring which is not invariant.

3 The proof of Theorem 1.3

For any two elements \( a, b \) of the ring \( R \), their commutator \( ab - ba \) is denoted by \([a, b]\).

3.1 Proposition Let \( R \) be a centrally essential tffr ring. Then the ring \( R/N(R) \) is commutative.

Proof Since \( R \) is a centrally essential ring, \( \mathbb{Q}R = \mathbb{Q} \otimes R \) also is a centrally essential ring; see (Lyubimtsev and Tuganbaev 2020, Proposition 3.1). In addition, the ring \( \mathbb{Q}R \) is Artinian, since it is a finite-dimensional \( \mathbb{Q} \)-algebra. It is well known that \( N(\mathbb{Q}R) = J(\mathbb{Q}R) \); e.g., see (Arnold 1982, Proposition 9.1(c)). Then the ring \( \mathbb{Q}R/N(\mathbb{Q}R) \) is commutative (Markov and Tuganbaev 2019b, Theorem 1.5). Let \( a, b \in R \). By considering (Arnold 1982, Proposition 9.1(c)), we have

\[
[a, b] \in N(\mathbb{Q}R) \cap R = N(R).
\]

Consequently, the ring \( R/N(R) \) is commutative. \( \square \)
3.2 Corollary If $R$ is a centrally essential tffr ring, then the ring $R/J(R)$ is commutative.

Proof Indeed, we have for $a, b \in R$:

$$[a, b] \in N(R) \subseteq J(R).$$

3.3. The completion of the proof of Theorem 1.3

Let $R$ be a centrally essential ring, $r \in R$, and let $M$ be a maximal right ideal of the ring $R$. We have to prove that $rM \subseteq M$. We assume the contrary. Then $rm \notin M$ for some element $m \in M$. Since $N(R) \subseteq M$, we have $r, m \notin N(R)$. Consequently, $\bar{r} = r + N(R)$ and $\bar{m} = m + N(R)$ are non-zero elements of the commutative ring $R/N(R)$. Then $\bar{r}\bar{m} = \bar{m}\bar{r}$; this implies that $[r, m] \in N(R)$. Since $mr \in M$, we have $rm \in M$. This is a contradiction.

3.4 Remark If we take a tffr ring as the ground ring $R$ in Example 2.4, then the ring $R$ also is a tffr ring. This implies that a tffr ring is not necessarily right or left invariant.

4 The proof of Theorem 1.4

4.1 Lemma (Arnold 1982, Proposition 9.1(c)) Let $R$ be a tffr ring and $nr \in C(R)$ for some $n \in \mathbb{Z}, r \in R$. Then $r \in C(R)$. Thus, $C(R)$ is a pure subgroup of $(R, +)$.

Proof We assume the contrary: $r \notin C(R)$. Then $rx \neq xr$ for some $r \in R$. This implies that $n(rx - xr) \neq 0$ and $(nr)x \neq x(nr)$. This is a contradiction.

4.2 Proposition Let $R$ be a centrally essential tffr ring. Then $R/pR$ is a centrally essential ring.

Proof Let $r + pR \in R/pR$, where $r \notin pR$. We prove that there exists an element $c' \in C(R)\setminus pR$ such that $0 \neq rc' = d' \in C(R)\setminus pR$.

Since $R$ is a centrally essential ring, there exists an element $c \in C(R)$ such that $0 \neq rc = d \in C(R)$. Let $c \in pR$ and let the $p$-height $h_p(c) = k, c = p^k c'$. It follows from Lemma 4.1 that $c' \in C(R)$. Then $rc = p^k rc' \in C(R)$. We again use Lemma 4.1 and obtain that $0 \neq rc' = d' \in C(R)\setminus pR$.

4.3. The completion of the proof of Theorem 1.4

It is well known that for an Abelian torsion-free group $A$ of finite rank, the endomorphism ring $\text{End} A$ is a tffr ring. We denote by $R$ the ring $\text{End} A$. In Lyubimtsev and Tuganbaev (2020), it is proved that an Abelian torsion-free group $A$ of finite rank can have non-commutative centrally essential endomorphism ring only if $A$ is a reduced strongly indecomposable group. Following (Faticoni 1988, Lemma 3.1), we assume that $IA = A$ for some maximal right ideal of the ring $R$. It follows from Faticoni (1988) that $pR \subseteq I$ for some prime integer $p$. By Theorem 1.3, the ring $R$ is quasi-invariant. Therefore, $I$ is a maximal ideal. Since $R/pR$ is a finite ring, the Jacobson
radical $J/pR$ of $R$ is nilpotent. We have $J \subseteq I$. Since $R/J$ is a semisimple Artinian ring, $I/J = \overline{e}(R/J)$ for some idempotent $1 \neq \overline{e} \in R/J$. Since the ideal $J/pR$ is nilpotent, we can lift $\overline{e}$ to some idempotent $1 \neq e \in R/pR$, i.e., $\overline{e} = e + J/pR$. It follows from Proposition 4.2 that the ring $R/pR$ is centrally essential. In a centrally essential ring, all idempotents are central (see Markov and Tuganbaev 2018, Lemma 2.3). Therefore, the right ideal $e(R/pR)$ is an ideal of $R/pR$. The remaining part of the proof repeat the Faticoni’s proof. We give it for completeness.

We remark that $I/pR = e(A/pA) + J/pR \neq R/pR$.

We have

$$IA/pA = (I/pR)(A/pA) = [e(R/pR) + J/pR](A/pA) = e(A/pA) + J/pR(A/pA),$$

by our choice of the idempotent $e$. Since $A/pA$ is a finite $R/pR$-module and $I/pR$, $e(R/pR)$ are ideals of the ring $R/pR$, we can apply the Nakayama lemma to the relation $IA/pA = e(A/pA) + J/pR(A/pA)$. As a result, we obtain $IA/pA = e(A/pA)$. Then the non-zero element $1 - e$ annihilates $IA/pA$. Since $A/pA$ is a faithful left $R/pR$-module, we have $IA/pA \neq A/pA$. Therefore, $IA \neq A$; this is a contradiction. Thus, $A$ is a faithful group.

In Example 2.4, we set $R = \mathbb{Z}$. Then the ring $\mathcal{R}$ is a countable ring whose additive group is a free group of finite rank $n$. Therefore, this ring is the endomorphism ring of some Abelian torsion-free group of rank $n$ (see Zassenhaus 1967). Therefore, we obtain Corollary 4.4.

### 4.4 Corollary

For every positive integer $n \geq 7$, there exists a faithful Abelian torsion-free group of rank $n$ with centrally essential endomorphism ring which is not right invariant.

## 5 Remarks and open questions

### 5.1 Open question

Is it true that every essential right ideal of a centrally essential torsion-free ring of finite rank is an ideal?

### 5.2

Let $A$ be a torsion-free group of finite rank which is a flat module over the endomorphism ring of $A$ (i.e., $A$ is an endo-flat group). In Faticoni (1988), it is proved that End $A$ is a right invariant ring if and only if $A$ is a faithful group and every $A$-generated subgroup of the group $A$ is fully invariant in $A$. In connection to this fact, we formulate open question 5.3.

### 5.3 Open question

Let $A$ be an endo-flat torsion-free strongly indecomposable faithful Abelian group of finite rank. Find a group-theoretical property which is is equivalent to the property that the endomorphism ring of $A$ is a centrally essential ring.
5.4 Open question  Is it true that there exists an Abelian group $A$ with centrally essential endomorphism ring $\text{End} A$ such that $\text{End} A$ is not a ring with polynomial identity?

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