Variable structure Copula model and its Comparison with static and time-varying Copula models

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Abstract. Copula model has been increasingly used in reliability modeling of correlation systems. In this paper, the static copula model, time-varying copula model and variable structure copula model are compared. Firstly, variable structure copula model is introduced. Secondly, the procedures of Bayes diagnosis method which is used to diagnose the variable structure points are introduced. Finally, the correctness of Bayes diagnosis method for finding variable structure points is verified by a simple numerical example. The comparison results illustrate that the staged correlation parameter estimated by variable structure Copula model is better than the constant correlation parameter estimated by static Copula model, and also it has a consistent trend with correlation parameter estimated by time-varying Copula model.

1. Introduction
In the traditional research on the reliability of system, it is often assumed that each component or failure mode in the system is independent of each other, which is not consistent with the actual engineering situation. In practical engineering, it is universal that correlations between multiple failure modes of mechanical parts or multiple components of the same system. In the process of reliability modeling, prediction, analysis and evaluation of complex systems, due to the complex composition, large scale, diverse failure modes and the increasingly close relationship and coupling between the failure modes, it is obviously inappropriate to treat them independently, and their correlation must be considered. The calculation of correlation reliability has always been a difficult problem for scholars.

Previously, Copula function was mainly used in the correlation analysis of finance, stock, exchange rate [1]. At present, Copula function is used to study the correlation between failure modes or subsystems. The life corresponding to each failure mode or each subsystem is regarded as the low-dimensional marginal distribution, the life of the whole system is regarded as the joint distribution function, which is connected by copula function, so as to solve the reliability modeling problem of the complex correlation system.

Lai et al. pointed out that copula function will play a great role on the correlation analysis in reliability modeling [2]. Rychlik connected the joint distribution of multivariate random variables with the marginal distribution through copula function, which provided an idea for the reliability research of component dependent systems [3]. Wang et al. proposed a system reliability evaluation method based on non-parametric copulas. The approximated joint probability distribution satisfying
the constraints specified by correlations had the maximal relative entropy with respect to the joint probability distribution of independent random variables. The estimation of the non-parametric copula parameters from Pearson linear correlation, Spearman rank correlation, and Kendall rank correlation were provided, respectively [4]. Fang et al. investigated a bivariate degradation model of a coherent system. To analyze the accelerated degradation data, a flexible class of bivariate stochastic processes were proposed to incorporate the effects of environmental stress variables and the dependency between two degradation processes was modeled by a copula function [5]. Jorge Navarro et al. obtained representations for the reliability function of the associated system residual lifetimes, based on the copula representation for the component lifetimes and the structure of the system. These representations can be applied to systems with generally dependent components. The obtained representations were also used to compare the respective system residual lifetimes under different stochastic orders [6]. Liu et al. constructed a mixed Clayton copula function of Markov transformation by using four kinds of Clayton copula, and the marginal distributions of variables were modeled by combining model, then the correlation and its corresponding risk measurement models were constructed based on mixed Clayton copula. Markov-switching mixed-Clayton copula model can achieve better effect of parameter estimation [7]. Xu et al. proposed a method of multivariate failure behavior modeling and reliability assessment based on vine-copula and accelerated degradation data. The method, considering the coupling modeling under accelerated stress and the relevant acceleration mechanism consistency test, can not only clearly describe the multivariate coupling relationship of the product, but also can carry out the reliability assessment of the product within the affordable time and cost. The failure behavior modeling method consisted of two parts which were degradation behavior and correlation relationship among variables [8]. Serkan Eryilmaz applied a multivariate copula based modeling methodology for dynamic reliability modeling of weighted-k-out-of-n systems [9]. Árpád Rózsás et al. qualitatively and quantitatively analyzed the impact of this copula assumption on failure probability. The calculations showed that the copula function has significant effect on failure probability [10]. Wang et al. proposed a general framework and corresponding methods to deal with the time-dependent reliability analysis of a mechanism effectively. The envelope function was used to estimate the finite expansion points, which converted the time-dependent reliability problem to the time-independent reliability problem under the general probability distribution. The covariance matrix was then derived with the vine-copula function to describe the correlation of the performance at the obtained expansion points. With the covariance matrix, the joint probability density function (PDF) was acquired based on the kernel density estimation (KDE) method. The time dependent reliability was finally estimated by the integral [11].

The above literatures show that static Copula model and time-varying Copula model are mainly studied at present, while the research on variable structure Copula model is relatively less. Next, we introduce the variable structure Copula model.

2. Variable Structure Copula Model

Besides time-varying copula model, dynamic copula model also includes variable structure copula model. Different from the time-varying copula model, the variable structure copula model will only change at some point. Of course, the changes are not only the correlation parameters in copula model, but also the copula functions describing the correlation, or the marginal distribution models and their parameters. Here, in order to facilitate the comparison with the time-varying copula model and static copula model, only the changes of the correlation parameters in the copula model is considered. In this paper, Staged modeling method is used to construct the variable structure copula model.

Considering the time series \( \{ x_t \} \), \( \{ y_t \} \), the variable structure Copula model is established by Staged modeling method as follows [12]:

\[ \]
\[
\begin{align*}
& (x_t, y_t) - C_1(F_1(x_t), G_1(y_t); k_1) \\
& (x_t, y_t) - C_2(F_2(x_t), G_2(y_t); k_2) \\
& \vdots \\
& (x_t, y_t) - C_K(F_K(x_t), G_K(y_t); k_K) \\
& \text{where } t = 1, \ldots, \tau_i \\
& \text{and } t = \tau_i + 1, \ldots, \tau_2 \\
& \text{for } i = 1, \ldots, K - 1 \\
& \text{and } t = \tau_{K-1} + 1, \ldots, T
\end{align*}
\]

Where \( F (\cdot) \) and \( G (\cdot) \) are the marginal distribution functions of variables \( X \) and \( Y \) respectively. \( C_k (\cdot, \cdot, \cdot) (k = 1, \ldots, K) \) is the copula function connecting two variables, and \( k_K \) is the parameter vector of Copula function. \( \tau_1, \ldots, \tau_{K-1} \) are the \( K-1 \) variable structure points of Copula function, and \( T \) is the sample size. Set \( \tau_0 = 0, \tau_K = T \), it is assumed that correlation parameters remain constant for a period of time between any two variable structure points \( \tau_j + 1, \tau_{j+1} \) \( (i = 0, \ldots, K - 1) \), namely the correlation structure between variables can be described by a Copula function with parameter vectors \( k_{i+1} \). In this paper, in order to compare with the time-varying copula model, the variable structure copula models only change the correlation parameters, and Copula functions remain unchanged, i.e. \( C_1 = C_2 = \cdots = C_K \).

Similar to financial events, the correlation structure changes among failure modes of complex equipment systems are complicated. It is difficult to determine the variable structure point of copula model by a certain event, the changes of performance index or failure mode. It is more accurate to find the variable structure points through statistical diagnosis procedures. Such as Bayes diagnosis method, is used to find out the points that make the changes in the failure mode sequence. The Bayes diagnosis method is introduced as follows.

3. Bayes diagnosis method

The procedures of Bayes diagnosis method which is used to diagnose the variable structure points of serial variance are as follows [12-14]:

(1) Stationarity test was performed on the variance of the original sample. Considering the following null hypothesis

\[
H_0 : \beta = \sigma_2^2 / \sigma_1^2 = 1
\]

where \( \sigma_2^2 \) and \( \sigma_1^2 \) are the variances of two implicit constitutions.

The test requires the calculation value of the unconditional probability \( (p^\prime) \) under the null hypothesis, and the stationarity test of the variance needs to be repeated as each data point is added to the initial sample until the value is less than a predetermined significance level (the null hypothesis is rejected).

(2) Determine the variable structure points of the variance. If the null hypothesis is rejected, there is a variable structure point of variance for the sample. The posterior probability of each possible two-system partition is then calculated for this sample. If a point corresponding to the constitution classification has the largest posterior probability, it is more likely to be a variable structure point of variance from the posterior perspective. The first variable structure point is written as \( \hat{\tau}_1 \). Repeat the above steps with the next data point (such as \( \hat{\tau}_1 + 1 \) point) as the starting point of the original sample. Two values in each test interval are need to be calculated by Bayes diagnosis method: unconditional \( p^\prime \) value and posterior distribution of variable structure points \( \tau \).

When HPD interval method is used to test \( H_0 : \beta = \sigma_2^2 / \sigma_1^2 = 1 \) for the sample \( \{y_i\}_{i=1}^n \) with a volume of \( n \), the calculation formula for unconditional \( p^\prime \) value is as follows
\[ p_{\beta,1} = \sum_{t} 2 [1 - F_{t-n-n-1}(\Theta(\beta))] \pi(\tau|y) \]  

(3)

where \( F_{t-n-n-1}(\cdot) \) is the cumulative distribution function of the \( F \) distribution with degree of freedom \((\tau-1, n-\tau-1)\), and

\[ \Theta(\beta) = \max \left\{ \beta (\hat{\sigma}_1^2 / \hat{\sigma}_2^2), \frac{1}{\beta} (\hat{\sigma}_1^2 / \hat{\sigma}_2^2) \right\} \]  

(4)

The posterior probability density function \( \pi(\tau|y) \) of \( \tau \) is expressed as follows

\[ \pi(\tau|y) \propto \pi(\tau)\tau^{-1/2} (n-\tau)^{1/2} \Gamma((n-\tau-1)/2) \times S_1^{-(\tau-1)/2} S_2^{-(n-\tau-1)/2} \]  

(5)

where \( S_1 = \sum_{i=1}^{\tau} (y_i - \bar{y}_1)^2 \), \( S_2 = \sum_{i=\tau+1}^{n} (y_i - \bar{y}_2)^2 \), \( \bar{y}_1 = \frac{1}{\tau} \sum_{i=1}^{\tau} y_i \), \( \bar{y}_2 = \frac{1}{n-\tau} \sum_{i=\tau+1}^{n} y_i \), \( \Gamma(\cdot) \) is gamma function, \( \pi(\tau) \) is prior distribution of \( \tau \).

If the unconditional \( p \) value in formula (3) is less than the given significance level, then the posterior probability of each sample point can be calculated by using equation (5). And the data point with the maximum posterior probability is selected as the variable structure point of the variance. In order to prevent pseudo variable structure points of variance, 10 data points should be selected at least in each constitution and 50 data points should be selected for the initial test samples. If there are too many sample points in the initial test sub-samples, it is possible to include the observed values from two constitutions.

4. Numerical examples

Example 1 A simple Monte Carlo simulation test is carried out to verify the correctness of Bayes diagnosis method for finding variable structure points. The details steps are as follows:

(1) Select the total sample number \( T=2000 \), and the actual variable structure points are at \( T= 500 \), \( T= 1000 \), and \( T= 1500 \).

(2) 2000 pairs of random sequences of correlation parameters \( \{\rho_t\}_{t=1}^{2000} \) are generated by Monte Carlo simulation, as shown in Figure 1. Among them, the first 500 pairs of random sequences are \( \{\rho_t\}_{t=1}^{500} = 0.3 + 0.01\Delta \rho ; 501 \) to 1000 pairs of random sequences are \( \{\rho_t\}_{t=501}^{1000} = 0.4 + 0.01\Delta \rho ; 1001 \) to 1500 pairs of random sequences \( \{\rho_t\}_{t=1001}^{1500} = 0.5 + 0.01\Delta \rho \); and 1501 to 2000 pairs of random sequences are \( \{\rho_t\}_{t=1501}^{2000} = 0.6 + 0.01\Delta \rho \). \( \Delta \rho \) follows the standard normal distribution.

(3) When \( p_{\beta,1} \) is less than the predetermined significance level 0.06, the variable structure points which are obtained by Bayes diagnosis method are \( T=510 \), \( T=1020 \), and \( T=1530 \), and they are very close to the actual variable structure points \( T=500 \), \( T=1000 \), and \( T=1500 \). The results prove the correctness of Bayes diagnosis method for finding the variable structure points.

Example 2 The original numerical sequences \( X \) and \( Y \) in reference [15] are taken as input data, as shown in Fig. 2. And Bayes diagnosis method is used to diagnose the variable structure points of the original numerical sequences \( X \) and \( Y \). Also variable structure Copula model is Compared with static and time-varying Copula model through this numerical example.

Firstly, correlation parameter sequence \( \{\rho_t\}_{t=1}^{2500} \) is obtained by time-varying bivariate normal copula model (Patton model) [15,16], as shown in figure 3. And then Bayes diagnosis method is used to diagnose the variable structure points of correlation parameter sequence \( \{\rho_t\}_{t=1}^{2500} \). When \( p_{\beta,1} \) is less than the predetermined significance level 0.06, the variable structure points which are obtained by Bayes diagnosis method are \( t=378 \), \( t=756 \), \( t=1134 \), \( t=1512 \), \( t=1890 \), \( t=2268 \).
Then, the stage correlation parameters are obtained as follows: \( \{ \rho_t \}_{t=1}^{1378} = 0.4367 \), \( \{ \rho_t \}_{t=739}^{1134} = 0.1667 \), \( \{ \rho_t \}_{t=1135}^{1512} = -0.0464 \), \( \{ \rho_t \}_{t=1513}^{1890} = 0.0861 \), \( \{ \rho_t \}_{t=1891}^{12668} = 0.2274 \), \( \{ \rho_t \}_{t=12668}^{2269} = 0.3123 \), \( \{ \rho_t \}_{t=2269}^{2500} = 0.0030 \). And the comparison of correlation parameters, which are obtained by static Copula, time-varying Copula model and variable structure Copula model, is shown in Figure 3.

As shown in Figure 3, the correlation parameter estimated by variable structure Copula model show stage changes, which is better than the constant correlation parameter estimated by static Copula model, and also it has a consistent trend with correlation parameter estimated by time-varying Copula model.

5. Conclusion
The correctness of Bayes diagnosis method for finding the variable structure points has been proved. The correlation parameter estimated by variable structure Copula model show stage changes, which is better than the constant correlation parameter estimated by static Copula model, and also it has a consistent trend with correlation parameter estimated by time-varying Copula model.

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![Figure 1. Random sequences of correlation parameters \( \rho_t \).](image-url)
Figure 2. Numerical sequences $X$.

Figure 3. Numerical sequences $Y$. 
Figure 4. Comparison with static(constant), time varying and variable structure Copula model.

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