SYMMETRIC GROUP CHARACTERS AS SYMMETRIC FUNCTIONS

ROSA ORELLANA AND MIKE ZABROCKI

Abstract. We introduce a basis of the symmetric functions that evaluates to the (irreducible) characters of the symmetric group, just as the Schur functions evaluate to the irreducible characters of \( GL_n \) modules. Our main result gives three different characterizations for this basis. One of the characterizations shows that the structure coefficients for the (outer) product of these functions are the stable Kronecker coefficients. The results in this paper focus on developing the fundamental properties of this basis.

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1. Introduction

The ring of symmetric functions and the representation theory of both the symmetric group, $S_k$, and the general linear group, $GL_n$, are deeply connected. It is well-known that the ring of symmetric functions $\text{Sym}$ is the “universal” character ring of $GL_n$. In fact, irreducible polynomial $GL_n$ characters are obtained as evaluations of Schur functions, $s_\lambda$, at the eigenvalues of the elements in $GL_n$. For this reason, Schur functions serve as a tool to decompose $GL_n$-modules in terms of irreducible submodules. The main objective of this paper is to show that the embedding of the symmetric group, $S_n$, as permutation matrices in $GL_n$ gives rise to a basis $\{\tilde{s}_\lambda\}$ of the ring of symmetric functions which evaluate to the characters of $S_n$ in a similar manner.

Schur functions are also useful in computing the decomposition of the tensor product of polynomial irreducible representations of $GL_n$. It is well known that this tensor product corresponds to the (outer) product of Schur functions. More specifically, for any three partitions $\lambda, \mu$ and $\nu$ such that $|\nu| = |\lambda| + |\mu|$, the multiplicity of irreducible representation $W^\nu$ in $W^\lambda \otimes W^\mu$ are the Littlewood-Richardson coefficients (denoted $c^\nu_{\lambda\mu}$). These are the same integers that occur as the structure coefficients for the Schur basis, that is,

$$ s_\lambda s_\mu = \sum_{\nu} c^\nu_{\lambda\mu} s_\nu. $$

The multiplicities of the tensor product of irreducible modules of the symmetric group have some dependence on $n$, but for partitions $\lambda$, $\mu$ and $\nu$ and $n$ sufficiently large, the multiplicity of the irreducible representation $S^{(n-|\nu|,\nu)}$ in $S^{(n-|\lambda|,\lambda)} \otimes S^{(n-|\mu|,\mu)}$ is independent of $n$ and the values are known as the stable or reduced Kronecker coefficients (denoted $\gamma^\nu_{\lambda\mu}$, see for instance [BOR1, Gu, Mur2, Mur3]). One of our main results (Theorem 1 part (3)) is a characterization of symmetric functions $\tilde{s}_\lambda$ as a family of elements whose structure coefficients are the reduced Kronecker coefficients, that is,

$$ \tilde{s}_\lambda \tilde{s}_\mu = \sum_{\nu} \gamma^\nu_{\lambda\mu} \tilde{s}_\nu. $$

Our main theorem is the following characterizations of the basis $\{\tilde{s}_\lambda\}$ of inhomogeneous degree:

**Theorem 1.** There is a unique (in-homogeneous) basis of the symmetric functions, $\{\tilde{s}_\lambda\}$, that are characterized by any of the following three properties:

(1) For a fixed partition $\lambda$, $\tilde{s}_\lambda$ is the unique symmetric function with the property that for all $n \geq |\lambda| + \lambda_1$ and for all partitions $\gamma$ of $n$,

$$ \tilde{s}_\lambda(x_1, x_2, \ldots, x_n) = \chi^{(n-|\lambda|,\lambda)}(\gamma), $$

where $x_1, x_2, \ldots, x_n$ are the eigenvalues of a permutation matrix of cycle structure $\gamma$ and $\chi^{(n-|\lambda|,\lambda)}(\gamma)$ are the values of the irreducible characters of the symmetric group.

(2) The set $\{\tilde{s}_\lambda\}$ is the unique family of symmetric functions such that, for a sufficiently large $n$, if the multiplicity of the $S_n$ module $S^{(n-|\nu|,\nu)}$ in the decomposition of the
restriction of the $GL_n$ module $W^\lambda$ to $S_n$ is $r_{\lambda\mu}$, that is,

$$W^\lambda \downarrow_{S_n}^{GL_n} \cong \bigoplus_{\mu} (S^{(n-|\mu|,\mu)})^{r_{\lambda\mu}} ,$$

then $s_\lambda = \sum_{\mu} r_{\lambda\mu} \tilde{s}_\mu$.

(3) The set $\{\tilde{s}_\lambda\}$ is the unique family of symmetric functions such that $s_{1^r} = \tilde{s}_{1^r} + \tilde{s}_{1^{r-1}}$ and

$$\tilde{s}_\lambda \tilde{s}_\mu = \sum_{\nu} g^\nu_{\lambda\mu} \tilde{s}_\nu .$$

We call this new basis the irreducible character basis (or $\tilde{s}$-basis). The basis $\{\tilde{s}_\lambda\}$ plays the same role for the symmetric group as the Schur functions $\{s_\lambda\}$ do for $GL_n$.

If we use the notation $G(GL_n)$ (respectively, $G(S_n)$) to denote the Grothendieck ring of (polynomial) $GL_n$-representations (respectively, $S_n$), then the map $\text{ch} : G(GL_n) \to \text{Sym}$, can be defined via the trace function (see, for instance, [Sta] Prop. A2.3 for details). Our main theorem implies that there exists a map $\tilde{\text{ch}}$ which maps the irreducible representation, $S^{(n-|\lambda|,\lambda)}$ of $S_n$ to $\tilde{s}_\lambda \in \text{Sym}$ such that the following diagram commutes.

$$\begin{array}{ccc}
G(GL_n) & \xrightarrow{\text{ch}} & \text{Sym} \\
\text{Res} \downarrow & & \downarrow \\
G(S_n) & \xrightarrow{\tilde{\text{ch}}} & \text{Sym}
\end{array}$$

The $\tilde{\text{ch}}$ map can be defined as a lift of the trace map on the representations of $S_n$.

The expansion of a Schur function in terms of the $\tilde{s}$-basis, i.e, $s_\lambda = \sum_{\mu} r_{\lambda\mu} \tilde{s}_\mu$, is equivalent to what we call the “restriction problem” which asks for the decomposition of a (polynomial) irreducible representation of $GL_n$, $W^\lambda$ when restricted to the symmetric group. This problem has been studied in the literature, [BK Kin Lit Nis ST STW H HSW NPPS1 NPPS2]. The coefficient $r_{\lambda\mu}$ is the multiplicity of the irreducible $S_n$ module indexed by $(n - |\mu|, \mu)$ in the restriction of the irreducible representation $W^\lambda$ of $GL_n$. In [Lit], Littlewood gives a formula for this using plethysms and Scharf and Thibon [ST] give a proof using Hopf algebra techniques that the coefficients have the expression

$$r_{\lambda\mu} = \langle s_\lambda, s_{(n-|\mu|,\mu)}[1 + s_1 + s_2 + \cdots] \rangle$$

for an integer $n$ which is sufficiently large. Details for computing the plethysm $f[1 + s_1 + s_2 + \cdots]$ can be found in [Mac Sta]. Finding an explicit combinatorial expression for the coefficients $r_{\lambda\mu}$ remains a motivating open problem.

To study the restriction problem, Butler and King [BK] considered the embedding of $S_n$ in $GL_{n-1}$. In their paper they describe a method for writing symmetric group characters in terms of Schur functions. However, they fell short of defining symmetric group characters as a basis of symmetric functions. The embedding they considered would lead to symmetric functions that differ from ours by a plethystic substitution. Using our notation, Table II of [BK] is the Schur expansion of the symmetric functions $\tilde{s}_\lambda[X+1]$ for partitions $|\lambda| \leq 5$. 
The irreducible characters of \( S_k \), \( \chi^\lambda \), are well known to occur as change of basis coefficients when we write the power symmetric functions, \( p_\mu \), in terms of the Schur functions, \( (1) \)

\[
p_\mu = \sum_{\lambda \subseteq k} \chi^\lambda(\mu) s_\lambda
\]

where \( \chi^\lambda(\mu) \) is the value of \( \chi^\lambda \) at the conjugacy class indexed by \( \mu \).

In the early 1990s Martin \cite{Mar1, Mar2, Mar3, Mar4} and Jones \cite{Jo} introduced the partition algebra, \( P_k(n) \). Jones showed that if we identify \( S_n \) with the permutation matrices in \( GL_n \) and act diagonally on \( V^\otimes k \), then the centralizer algebra we obtain is isomorphic to \( P_k(n) \) when \( n \geq 2k \). This means that \( S_n \) and \( P_k(n) \) are Schur-Weyl duals of each other.

The development of the representation theory of the partition algebra \cite{BDE, BDO, BH, BHH, Hal, HR, Jo, Mar1, Mar2, Mar3, Mar4} has shown that there are close connections between the characters of the partition algebra and the characters of the symmetric group considered through the embedding using permutation matrices. In fact, we find that the characters of the partition algebra appear as the change of basis coefficients between the power sum basis and the irreducible character basis. That is, in analogy to (1) we have

\[
p_\mu = \sum_{\lambda} \chi^\lambda_{P_k(n)}(d_\mu) \tilde{s}_\lambda
\]

where \( \chi^\lambda_{P_k(n)}(d_\mu) \) denotes the irreducible character of \( P_k(n) \) indexed by \( \lambda \) and evaluated at an element \( d_\mu \) which is analogous to a conjugacy class representative (see \cite{Hal} for further details).

We remark that our new basis provides a unifying mantle for many different objects connected with Kronecker coefficients such as the representation theory of the symmetric group and that of the partition algebra, character polynomials, etc. In particular, in Section 6 we provide a symmetric function version of the Murnaghan-Nakayama rule for computing the characters of the partition algebra, these were computed in \cite{Hal} via different methods. We also show in Section 5 that character polynomials can be obtained as evaluations of our symmetric functions using results in \cite{GG}. Further, the study of our new basis has led to the introduction of new combinatorial objects that will be useful in developing algorithms involving symmetric group characters and partition algebra characters.

The reader interested in computing \( \tilde{s}_\lambda \) from the formulae in this paper is recommended to use Equation (23) because this expression provides an explicit expansion in terms of the power sum basis. The definition (see (4) and (8)) makes a combinatorial connection with multiset tableaux and can be used to compute the elements recursively.

In this paper, we also introduce another basis of symmetric functions \( \{ \tilde{h}_\lambda \} \) that evaluate to the induced trivial characters from a Young subgroup of the symmetric group to the full symmetric group. This basis has a close combinatorial connection with multiset partitions and provides a useful tool in developing the change of basis coefficients.

In the third characterization of Theorem 1, we saw that the structure coefficients of the \( \tilde{s} \)-basis are the stable Kronecker coefficients. Therefore, our new basis connects to the Grothendieck ring of the tensor category \( \text{Rep}(S_t) \) of Deligne \cite{Del, EA}, which has simple objects that have as structure coefficients the stable Kronecker coefficients. In addition, our
basis should provide a more compact way to encode the characters of FI-modules studied by Church, Ellenberg and Farb \cite{CF,CEF}.

Similar results have been obtained for other subgroups of $GL_n$. For example, Koike and Terada \cite{KT} developed symmetric function expressions for characters of orthogonal and symplectic groups.

Additional information about this basis can be found in two followup publications \cite{OZ1,OZ2}.

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2. Notation and Preliminaries

In this section we provide definitions of the combinatorial objects arising in this work and establish notation and conventions.

For non-negative integers $n$ and $\ell$, a partition of size $n$ and length $\ell$ is a sequence of positive integers, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$, such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_\ell = n$. The size of the partition is denoted $|\lambda| = n$ and the length of the partition is denoted $\ell(\lambda) = \ell$. We will often use the shorthand notation $\lambda \vdash n$ to indicate that $\lambda$ is a partition of $n$. The symbols $\lambda$ and $\mu$ will be reserved exclusively for partitions. Let $m_i(\lambda)$ represent the number of times that $i$ appears in the partition $\lambda$. Sometimes it will be convenient to represent our partitions in exponential notation $\lambda = (1^{m_1}2^{m_2}\cdots k^{m_k})$, where $m_i = m_i(\lambda)$ is the number of times that part $i$ occurs in $\lambda$. Using this notation the number of permutations with cycle structure $\lambda \vdash n$ is $n! z_{\lambda}$ where

$$z_{\lambda} = \prod_{i=1}^{\lambda_1} m_i(\lambda)!z_{m_i(\lambda)}.$$ \hfill (2)

The most common operation we use is that of adding a part of size $n$ to the beginning of a partition. This is denoted $(n, \lambda)$. If $n < \lambda_1$, this sequence will no longer be an integer partition and we will have to interpret the object appropriately.

The \textit{Young diagram} of a partition $\lambda$ are the set of points (or cells) $\{(i,j) : 1 \leq i \leq \lambda_j, 1 \leq j \leq \ell(\lambda)\}$. We will represent these cells as stacks of boxes in the first quadrant (following ‘French notation’). A \textit{tableau} is a map from the set of cells in the diagram of the partition to a set of labels. We represent a tableau by filling the cells of the diagram with the labels. In our case, we will encounter tableaux where only a subset of the cells are mapped to labels (some boxes will be empty). A tableau $T$ is \textit{column strict} if $T(i,j) \leq T(i+1,j)$ and $T(i,j) < T(i,j+1)$ for all the filled cells of the tableau. The \textit{content} of a tableau is the multiset containing the total number of occurrences of each number.
A multiset is a set where elements can be repeated. To differentiate multisets from sets we use double brackets to denote multisets, i.e., \([b_1, b_2, \ldots, b_r]\). Multisets will also be represented by exponential notation, in this case the multiset \([a_1, 2^{a_2}, \ldots, \ell^{a_\ell}]\) represents the multiset where the element \(i\) occurs \(a_i\) times.

A set partition of a set \(S\) is a set of pairwise disjoint subsets \(\{S_1, S_2, \ldots, S_\ell\}\) such that \(S_1 \subseteq S\) for \(1 \leq i \leq \ell\) and \(S_1 \cup S_2 \cup \cdots \cup S_\ell = S\). A multiset partition \(\pi = \{S_1, S_2, \ldots, S_\ell\}\) of a multiset \(S\) is a similar construction to a set partition, but now each \(S_i\) may be a multiset, and it is possible that two multisets \(S_i\) and \(S_j\) (with \(i \neq j\)) have non-empty intersection (and may even be equal). The length of a multiset partition is denoted by \(\ell(\pi) = \ell\). We will use the notation \(\pi \vdash S\) to indicate that \(\pi\) is a multiset partition of the multiset \(S\).

We will use \(\tilde{m}(\pi)\) to represent the partition of the integer \(\ell(\pi)\) consisting of the multiplicities of the multisets which occur in \(\pi\). For example, \(\tilde{m}(\{\{1, 1, 2\}, \{1, 1, 2\}, \{1, 3\}\}) = (2, 1)\) because \(\{1, 1, 2\}\) occurs 2 times and \(\{1, 3\}\) occurs 1 time.

For non-negative integers \(n\) and \(\ell\), a composition of size \(n\) is an ordered sequence of positive integers \(\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)\) such that \(\alpha_1 + \alpha_2 + \cdots + \alpha_\ell = n\). A weak composition is such a sequence with the condition that \(\alpha_i \geq 0\) (zeros are allowed). To indicate that \(\alpha\) is a composition of \(n\) we will use the notation \(\alpha \models n\) and to indicate that \(\alpha\) is a weak composition of \(n\) we will use the notation \(\alpha \models_w n\). For both compositions and weak compositions, \(\ell(\alpha) := \ell\) (the length of the sequence).

Remark 2. Multiset partitions of a multiset are isomorphic to objects called vector partitions which have previously been used to index symmetric functions in multiple sets of variables \([\text{Com}, \text{MacM}, \text{Ros}]\). Since multiset partitions are more amenable to tableaux we have used this alternate combinatorial description.

2.1. The ring of symmetric functions. For complete details on this topic we refer the reader to \([\text{Mac}, \text{Sag}, \text{Sta}, \text{Las}]\). The ring of symmetric functions will be denoted \(\text{Sym} = \mathbb{Q}[p_1, p_2, p_3, \ldots]\). The \(p_k\) are power sum generators and they will be thought of as functions which can be evaluated at values when appropriate by making the substitution \(p_k \rightarrow x_1^k + x_2^k + \cdots + x_n^k\) but they are used algebraically in this ring without reference to their variables.

The fundamental bases of \(\text{Sym}\) used in this paper (each indexed by the set of partitions \(\lambda\)) are power sum \(\{p_\lambda\}_\lambda\), homogeneous/complete \(\{h_\lambda\}_\lambda\), elementary \(\{e_\lambda\}_\lambda\), and Schur \(\{s_\lambda\}_\lambda\). For consistent notation, \(p_0 = h_0 = e_0 = 1\) in this ring. The Hall inner product is defined by declaring that the power sum basis is orthogonal, i.e.,

\[\langle p_\lambda, p_\mu \rangle = z_\lambda \delta_{\lambda=\mu},\]

where the symbol \(\delta_{\lambda=\mu}\) is the Kronecker delta function that is equal to 1 if \(\lambda = \mu\) and 0 otherwise. Under this inner product the Schur functions are orthonormal, \(\langle s_\lambda, s_\mu \rangle = \delta_{\lambda=\mu}\). We use this scalar product to represent values of coefficients by taking scalar products with dual bases. We will also refer to the irreducible character of the symmetric group indexed by the partition \(\lambda\) and evaluated at a permutation of cycle structure \(\mu\) as the coefficient \(\langle s_\lambda, p_\mu \rangle = \chi^\lambda(\mu)\).
For $k > 0$, define
\[ \Xi_k := 1, e^{2\pi i/k}, e^{4\pi i/k}, \ldots, e^{2(k-1)\pi i/k} \]
denote the set of eigenvalues of a permutation matrix of a $k$-cycle. Then for any partition $\mu$, let
\[ \Xi_\mu := \Xi_{\mu_1}, \Xi_{\mu_2}, \ldots, \Xi_{\mu_\ell(\mu)} \]
be the multiset of eigenvalues of a permutation matrix with cycle structure $\mu$. We will evaluate symmetric functions at these eigenvalues. The notation $f[\Xi_\mu]$ represents the complex number we obtain when we take $f \in \text{Sym}$ and replace $p_k$ in $f$ with $x_1^k + x_2^k + \cdots + x_{|\mu|}^k$ and then replacing the variables $x_i$ with the values in $\Xi_\mu$.

3. Symmetric group character bases of the symmetric functions

In this section we introduce two new (in-homogeneous) bases of the ring of symmetric functions, $\{\tilde{h}_\lambda\}$ and $\{\tilde{s}_\lambda\}$. The evaluations of these families of symmetric functions at roots of unity (the eigenvalues of a permutation matrix) will be the values of characters of the symmetric group. We have relegated a large part of the necessary buildup of these symmetric function bases to two appendices in Section 7 and 8. The reason for this is that it will make more clear the goals of this paper which are the introduction of these symmetric functions and the study of their properties. In Section 7 we prove the following fundamental result.

**Theorem 3.** For all partitions $\nu$ and $\mu$, let $H_{\nu,\mu} := \langle h_{|\mu|-|\nu|} h_{\nu}, p_\mu \rangle$. We have the evaluation,
\[ h_\lambda[\Xi_\mu] = \sum_{\pi \vdash \{1^{\lambda_1}, 2^{\lambda_2}, \ldots, \ell^{\lambda_\ell}\}} H_{\tilde{m}(\pi),\mu} \cdot \]

Note that $H_{\nu,\mu} = 0$ if $|\mu| - |\nu| < 0$.

**Definition 4.** We take as definition of symmetric function elements $\tilde{h}_\nu$ the equation
\[ h_\lambda = \sum_{\pi \vdash \{1^{\lambda_1}, 2^{\lambda_2}, \ldots, \ell^{\lambda_\ell}\}} \tilde{h}_{\tilde{m}(\pi)}. \]

This is a recursive definition for calculating this basis since there is precisely one multiset partition of $\{1^{\lambda_1}, 2^{\lambda_2}, \ldots, \ell^{\lambda_\ell}\}$ such that $\tilde{m}(\pi)$ is of size $|\lambda|$, hence
\[ \tilde{h}_\lambda = h_\lambda - \sum_{\pi \vdash \{1^{\lambda_1}, 2^{\lambda_2}, \ldots, \ell^{\lambda_\ell}\}} \tilde{h}_{\tilde{m}(\pi)} \cdot \]

Now by equation (3) and an induction argument, we conclude that for all partitions $\mu$,
\[ \tilde{h}_\lambda[\Xi_\mu] = H_{\lambda,\mu} = \langle h_{|\mu|-|\lambda|} h_\lambda, p_\mu \rangle \]
and this is the value of the character of the trivial representation induced from $S_{|\mu|-|\lambda|} \times S_{\lambda_1} \times S_{\lambda_2} \times \cdots \times S_{\lambda_\ell(\lambda)}$ to the full symmetric group $S_{|\mu|}$, i.e., $1 \uparrow S_{|\mu|-|\lambda|} \times S_{\lambda_1} \times S_{\lambda_2} \times \cdots \times S_{\lambda_\ell(\lambda)}$. 

We call these symmetric functions ‘characters’ because we can think of them as functions that can be evaluated on the eigenvalues of permutation matrices and these evaluations are equal to the character values of symmetric group representations. Therefore, we call the symmetric functions \( \{ \tilde{h}_\lambda \} \), as a basis of the symmetric functions, the \emph{induced trivial character basis} (or \( \tilde{h} \)-basis).

**Example 5.** To compute a small example: \( \tilde{h}_1 = h_1 \). Since \( \{1\}, \{1\} \) and \( \{1, 1\} \) are the two multiset partitions of \( \{1, 1\} \), then \( h_2 = \tilde{h}_1 + \tilde{h}_2 \). Therefore \( \tilde{h}_2 = h_2 - h_1 \).

It is well known that two polynomials of degree \( d \) that agree on \( d+1 \) values are equal. We use a multivariate formulation of this idea repeatedly in order to justify some of our symmetric function identities. The details of this proof are left to Appendix II (Section 8).

**Proposition 6.** (Proposition 39 and Corollary 41) Let \( f, g \in \text{Sym} \) be symmetric functions of degree less than or equal to some positive integer \( n \). Assume that

\[
 f[\Xi_\gamma] = g[\Xi_\gamma]
\]

for all partitions \( \gamma \) such that \( |\gamma| \leq n \) (respectively, \( |\gamma| \geq n \)), then

\[
 f = g
\]

as elements of \( \text{Sym} \).

Next we define the symmetric functions \( \tilde{s}_\lambda \) by using the Kostka coefficients (the change of basis coefficients between the induced trivial characters \( 1\uparrow_{S_n} S_{\mu_1} \times \cdots \times S_{\mu_\ell} \) and the irreducible characters \( \chi^\lambda \) are denoted \( K_{\lambda\mu} \)) as the change of basis coefficients with \( \tilde{h}_\lambda \) basis. Choose an \( n \geq 2|\mu| \). We define \( \tilde{s}_\lambda \) to be the unique symmetric function which satisfy

\[
(7) \quad \tilde{h}_\mu = \sum_{|\lambda| \leq |\mu|} K_{(n-|\lambda|,\lambda)(n-|\mu|,\mu)} \tilde{s}_\lambda.
\]

Alternatively, the coefficient of \( \tilde{s}_\lambda \) in \( \tilde{h}_\mu \) is equal to \( \sum_\gamma K_{\gamma\mu} \), where the sum is over partitions \( \gamma \) such that \( \gamma/\lambda \) is a horizontal strip (at most one cell in each row) of size \( |\mu| - |\lambda| \). This also implies that we can express \( \tilde{s}_\lambda \) in terms of the \( \tilde{h} \)'s as

\[
(8) \quad \tilde{s}_\lambda = \sum_{|\mu| \leq |\lambda|} K^{-1}_{(n-|\lambda|,\lambda)(n-|\mu|,\mu)} \tilde{h}_\mu,
\]

where \( n \) is any positive integer greater than or equal to \( 2|\lambda| \) and \( K_{\lambda\mu}^{-1} \) are the inverse Kostka coefficients. There is a combinatorial interpretation for the Kostka coefficients \( K_{\lambda\mu} \) as the number of column strict tableaux of shape \( \lambda \) and content \( \mu \). Using this interpretation we can show that \( K_{(n-|\lambda|,\lambda)(n-|\mu|,\mu)} \) is independent of the value of \( n \) as long as \( n \) is sufficiently large.

If \( n \) is smaller than \( |\lambda| - 1 \), then the change of basis coefficients are the same as those between the complete symmetric functions and a Schur function indexed by a composition \( \alpha = (|\mu| - |\lambda|, \lambda) \), namely the expression representing the Jacobi-Trudi matrix

\[
(9) \quad s_\alpha = \det [h_{\alpha_i+i-j}]_{1 \leq i, j \leq \ell(\lambda)+1}.
\]
Thus, it follows that the $\tilde{s}_\lambda$ are the (unique) in-homogeneous symmetric functions of degree $|\lambda|$ that evaluate to the irreducible characters of the symmetric group, that is

$$\tilde{s}_\lambda[\Xi_\gamma] = \sum_{|\mu| \leq |\lambda|} K_{(n-|\lambda|,\lambda)}^{-1} K_{(n-|\mu|,\mu)}^{-1} h_{\mu}[\Xi_\gamma]$$

$$= \sum_{|\mu| \leq |\lambda|} K_{(n-|\lambda|,\lambda)}^{-1} K_{(n-|\mu|,\mu)}^{-1} \langle h_{n-|\mu|}^{\lambda}, p_\gamma \rangle$$

$$= \langle s_{(n-|\lambda|,\lambda)}^{\lambda}, p_\gamma \rangle = \chi^{(n-|\lambda|,\lambda)}(\gamma).$$

If $n \geq |\lambda| + \lambda_1$ then this last expression is equal to the value of the irreducible character of the symmetric group indexed by $(n-|\lambda|,\lambda)$ evaluated at an element of cycle type $\gamma$.

This allows us to state a first characterization of the symmetric functions $\tilde{s}_\lambda$.

**Theorem 1.** (Part (1)) For a fixed partition $\lambda$, $\tilde{s}_\lambda$ is the unique symmetric function with the property that for all $n \geq |\lambda| + \lambda_1$ and for all partitions $\gamma$ of $n$,

$$\tilde{s}_\lambda(x_1, x_2, \ldots, x_n) = \chi^{(n-|\lambda|,\lambda)}(\gamma),$$

where $x_1, x_2, \ldots, x_n$ are the eigenvalues of a permutation matrix of cycle structure $\gamma$ and $\chi^{(n-|\lambda|,\lambda)}(\gamma)$ are the values of the irreducible characters of the symmetric group.

**Proof.** By equation (10) we have established that for all $n \geq |\lambda| + \lambda_1$ and for any partition $\gamma$ of $n$,

$$\tilde{s}_\lambda[\Xi_\gamma] = \chi^{(n-|\lambda|,\lambda)}(\gamma).$$

By Corollary 41, the only symmetric function with this property must be equal to $\tilde{s}_\lambda$. □

We call the basis $\{\tilde{s}_\lambda\}$ the characters of the irreducible representations of the symmetric group when the symmetric group is realized as permutation matrices. They are characters in the same way that the Schur functions are the characters of the irreducible representations of the general linear group. We therefore name $\{\tilde{s}_\lambda\}$ the irreducible character basis.

An irreducible polynomial $GL_n$-module $W^\lambda$, where $\lambda$ is a partition, has character equal to $s_\lambda(x_1, x_2, \ldots, x_n)$. As we are considering the embedding of $S_n$ as a subgroup of $GL_n$ we may consider the decomposition of $W^\lambda$ into irreducible $S_n$ modules when we restrict from $GL_n$ to $S_n$. The coefficients of the expansion of $s_\lambda$ into irreducible character basis establishes a second characterization of the symmetric functions $\{\tilde{s}_\lambda\}$.

**Theorem 1.** (Part (2)) The set $\{\tilde{s}_\lambda\}$ is the unique family of symmetric functions such that, for a sufficiently large $n$, if the multiplicity of the $S_n$ module $S^{(n-|\mu|,\mu)}$ in the restriction of the $GL_n$ module $W^\lambda$ to $S_n$ is $r_{\lambda\mu}$, that is,

$$W^\lambda \downarrow_{S_n}^{GL_n} \cong \bigoplus_{\mu} (S^{(n-|\mu|,\mu)})^{r_{\lambda\mu}} \Xi_{\lambda\mu},$$

then

$$s_\lambda = \sum_{\mu} r_{\lambda\mu} \tilde{s}_\mu .$$
Corollary 41 we conclude that

By Theorem 1 (part (1)), we know that this implies

\[ r_{\lambda\mu} = \sum_{r \geq 0} \left\langle s_\lambda, s_{(n-|\mu|,\mu)}[s_1 + s_2 + \cdots] \right\rangle \]

then all the terms on the right hand side are equal to 0 unless \( r \geq n-|\lambda| > |\mu| \), otherwise the degree of \( s_{(n-|\mu|,\mu)}[s_1 + s_2 + \cdots] \) is larger than the degree of \( s_\lambda \). If \( r \geq n - |\lambda| \geq |\mu| \), then \( s_{(n-|\mu|,\mu)}[s_1 + s_2 + \cdots] = s_d[s_1 + s_2 + \cdots]s_\mu[s_1 + s_2 + \cdots] \) for some integer \( d = n - r - |\mu| \).

We also have that

\[ \langle s_\lambda, s_d[s_1 + s_2 + \cdots]s_\mu[s_1 + s_2 + \cdots] \rangle = 0 \]

for \( d > |\lambda| - |\mu| \), hence

\[ r_{\lambda\mu} = \sum_{r \geq 0} \left\langle s_\lambda, s_{(n-|\mu|,\mu)}[r][s_1 + s_2 + \cdots] \right\rangle = \sum_{d=0}^{\frac{|\lambda|-|\mu|}{2}} \left\langle s_\lambda, s_d[s_1 + s_2 + \cdots]s_\mu[s_1 + s_2 + \cdots] \right\rangle. \]

We conclude that the expression for \( r_{\lambda\mu} \) is independent of \( n \). Hence for \( n \) sufficiently large and for all partitions \( \gamma \) such that \( |\gamma| \geq n \),

\[ s_\lambda[\Xi_\gamma] = \sum_{\mu} r_{\lambda\mu} \delta^{\mu}(\gamma). \]

By Theorem 1 (part (1)), we know that this implies \( s_\lambda[\Xi_\gamma] = \sum_{\mu} r_{\lambda\mu} \delta_\mu[\Xi_\gamma] \) and so by Corollary 41 we conclude that \( s_\lambda = \sum_{\mu} r_{\lambda\mu} \delta_\mu \). These coefficients \( r_{\lambda\mu} \) are then the change of basis between the Schur functions and the irreducible character basis.

Recall that the Kronecker product is the bilinear product on symmetric functions defined on the power sum basis by

\[ \frac{p_\lambda}{z_\lambda} \ast \frac{p_\mu}{z_\mu} = \delta_{\lambda=\mu} \frac{p_\lambda}{z_\lambda}. \]

The symbol \( \delta_{\lambda=\mu} \) is the Kronecker delta function that is equal to 1 if \( \lambda = \mu \) and 0 otherwise. Since \( \left\langle \frac{p_\lambda}{z_\lambda}, \frac{p_\mu}{z_\mu} \right\rangle = \delta_{\lambda=\mu} \), we can verify the trivial calculation

\[ \left\langle \frac{p_\lambda}{z_\lambda} \ast \frac{p_\mu}{z_\mu}, p_\gamma \right\rangle = \delta_{\lambda=\mu}\delta_{\gamma=\lambda} = \delta_{\lambda=\gamma}\delta_{\mu=\gamma} = \left\langle \frac{p_\lambda}{z_\lambda}, p_\gamma \right\rangle \left\langle \frac{p_\mu}{z_\mu}, p_\gamma \right\rangle. \]

Since our product and scalar product are bilinear, we have for any symmetric functions \( f \) and \( g \),

\[ \langle f \ast g, p_\gamma \rangle = \langle f, p_\gamma \rangle \langle g, p_\gamma \rangle. \]
By work of Murnaghan [Mur2, Mur3], for arbitrary partitions λ, µ, and ν, there exists coefficients \( \bar{g}_{\lambda \mu}^\nu \) with the property that
\[
s_{(n-|\lambda|,\lambda)} \ast s_{(n-|\mu|,\mu)} = \sum_\nu \bar{g}_{\lambda \mu}^\nu s_{(n-|\nu|,\nu)}
\]
for all sufficiently large \( n \). The \( \bar{g}_{\lambda \mu}^\nu \) are usually referred to as ‘reduced’ or ‘stable’ Kronecker coefficients (see for example [EA, BOR, Gu, Kly]).

As we show next, when we write \( \tilde{s}_\lambda \tilde{s}_\mu \) as a linear combination of the \( \tilde{s}_\nu \), we find the structure constants are the stable Kronecker coefficients.

**Theorem 7.** For partitions \( \lambda \) and \( \mu \),
\[
\tilde{s}_\lambda \tilde{s}_\mu = \sum_{|\nu| \leq |\lambda| + |\mu|} \bar{g}_{\lambda \mu}^\nu \tilde{s}_\nu .
\]

**Proof.** We begin by evaluating the product of these functions at the eigenvalues of a permutation matrix.
\[
\tilde{s}_\lambda[\Xi_\gamma] \tilde{s}_\mu[\Xi_\gamma] = \langle s_{(|\gamma|-|\lambda|,\lambda)}, p_\gamma \rangle \langle s_{(|\gamma|-|\mu|,\mu)}, p_\gamma \rangle = \langle s_{(|\gamma|-|\lambda|,\lambda)} \ast s_{(|\gamma|-|\mu|,\mu)}, p_\gamma \rangle = \sum_{|\nu| \leq |\lambda| + |\mu|} \bar{g}_{\lambda \mu}^\nu \langle s_{(|\gamma|-|\nu|,\nu)}, p_\gamma \rangle = \sum_{|\nu| \leq |\lambda| + |\mu|} \bar{g}_{\lambda \mu}^\nu \tilde{s}_\nu[\Xi_\gamma] .
\]

Since this expression is an identity for all \( \gamma \) of sufficiently large size, we conclude by Corollary 41 that the theorem holds. \( \square \)

The symmetric group character basis are not the only symmetric functions which will have the \( \bar{g}_{\lambda \mu}^\nu \) coefficients as their structure coefficients. Indeed, any algebra isomorphism applied to the irreducible character basis will have the same coefficients in their product expansion. However, if we also specify what the symmetric functions are equal to for a family of generators then this determines the functions uniquely. In the following characterization we choose to specify the elements \( \tilde{s}_1^r \) for \( r \geq 1 \) (since this is the most compact expression we could identify).

**Theorem 1.** (Part (3)) The set \( \{ \tilde{s}_\lambda \} \) is the family of symmetric functions such that for \( r \geq 1 \), \( s_1^r = \tilde{s}_1^r + \tilde{s}_1^{r-1} \) (or, equivalently, \( \tilde{s}_1^r = \sum_{i=0}^r (-1)^i e_{r-i} \)) and
\[
\tilde{s}_\lambda \tilde{s}_\mu = \sum_\nu \bar{g}_{\lambda \mu}^\nu \tilde{s}_\gamma .
\]

**Proof.** Assume that \( \tilde{s}_\lambda \) is a family of symmetric functions that satisfy equation (17) and such that \( s_1^r = \tilde{s}_1^r + \tilde{s}_1^{r-1} \).

It is known that the coefficients \( \bar{g}_{\lambda \mu}^\nu = e_{\lambda \mu}^\nu \) for \( |\lambda| + |\mu| = |\nu| \) (see for instance [BOR, Lit]). Let \( \gamma \) be the partition \( \lambda \) with the first column removed (that is \( \gamma = (\lambda_1-1, \lambda_2-1, \ldots, \lambda_{\ell(\lambda)}-1) \).
1). It follows that $s_{1^i(\lambda)} \bar{s}_{\gamma}$ is equal to $s_{\lambda}$ plus other terms which are indexed by partitions which are either larger than $\lambda$ in dominance order or of smaller size than $|\lambda|$. This implies that there is an order where $\bar{s}_{\lambda}$ is uni-triangularly related to the elements of the form $s_{1^{\lambda_1'}} \bar{s}_{1^{\lambda_2'}} \cdots \bar{s}_{1^{\lambda_{\ell(\lambda)}}}$ (where $\lambda_i'$ is the length of the $i^{th}$ column of $\lambda$) and hence $\bar{s}_{\lambda}$ are a linearly independent set of elements which are determined by products of $s_{1^r}$. A choice of the initial condition of the value of $s_{1^r}$ as an element of the symmetric functions for $r \geq 0$ determines the embedding of the basis $\bar{s}_{\lambda}$ as elements of the symmetric functions.

The fact that $s_{1^r} = s_{1^r} + s_{1^{r-1}}$ follows by computing the character of $\bigwedge^r(V_n)$ (the $r^{th}$ exterior product of the $S_n$ permutation module $V_n$) both as a $GL_n$ character and as a $S_n$ character. This initial condition determines the family of symmetric functions for all partitions.

**Remark 8.** For this third characterization of the irreducible character basis we could have equally defined it to be the basis which satifies equation (17) and which satisfies an initial condition on almost any set of generators of the algebra of symmetric functions (e.g. $\bar{s}_r$ for $r \geq 0$). We chose to state it in terms of $s_{1^r}$ because the expression was the most compact to state.

4. **The irreducible character expansion of a complete symmetric function**

The main result of this section is an expansion of the complete homogeneous functions $h_{\mu}$ in terms of the $\bar{s}$-basis. The coefficients are described combinatorially by combining the notion of multiset partition of a multiset and column strict tableau. To work with column strict tableaux on sets or multisets we need to establish a total order on these objects. We remark that, with few restrictions, we can do this with almost any total order and so we will use lexicographic if we read the entries of the multiset in increasing order. This may mean that the tableaux we work with will have content which is not a partition, but this is typical with column strict tableaux.

If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{\ell(\lambda)})$ is a partition, then use the notation $\overline{\lambda} = (\lambda_2, \lambda_3, \ldots, \lambda_{\ell(\lambda)})$ to represent the partition with the first part removed. For a tableau $T$, let $shape(T)$ denote the partition of the outer shape of the tableau.

**Theorem 9.** For a partition $\mu$,

\begin{equation}
\tag{18}
h_{\mu} = \sum_T \bar{s}_{shape(T)}
\end{equation}

the sum is over all skew-shape column strict tableaux of shape $\lambda/(\lambda_2)$ for some partition $\lambda$ where the cells are filled with non-empty multisets of labels such that the total content of the tableau is $\{1^{\mu_1}, 2^{\mu_2}, \ldots, \ell^{\mu_\ell}\}$.

**Proof.** From equation (4), we know the expansion of $h_{\mu}$ in terms of multiset partitions of a multiset and by (7) we know the expansion of $\bar{h}_{\overline{\mu}(\pi)}$ in the $\bar{s}_{\lambda}$ basis terms of column strict tableaux. Combining these two expansions we have that for an $n$ sufficiently large,

\begin{equation}
\tag{19}
h_{\mu} = \sum_{\pi \in \{1^{\mu_1}, 2^{\mu_2}, \ldots, \ell^{\mu_\ell}\}} \sum_{\lambda \vdash n} K_{\lambda(n-\ell(\pi), \overline{\mu}(\pi))} \bar{s}_{\lambda}.
\end{equation}
Now we note that for every multiset partition \( \pi \) and column strict tableaux of shape \( \lambda \) and content given by the partition \((n - \ell(\pi), m(\pi))\) we can create a skew-shaped tableau whose entries are multisets by replacing the \( n - \ell(\pi) \) labels with a 1 with a blank so that it is of skew shape \( \lambda/(n - \ell(\pi)) \) and the other labels by their corresponding multiset in \( \pi \). The value of \( n \) needs to be chosen so that \( n - \ell(\pi) \) is larger than the size of the first part of \( \lambda \).

To explain why this is equal to the description stated in the theorem where there are precisely \( \lambda_2 \) blank cells in the first row, we note that \( K_{\lambda(n-\ell(\pi), m(\pi))} = K_{(n'-|\lambda|, \lambda_2)(n'-\ell(\pi), m(\pi))} \) as long as \( n' - \ell(\pi) \geq \lambda_2 \). This is because there is a bijection between these two sets of tableaux by inserting or deleting 1s in the first row of each tableau in the set. In particular, we may choose \( n' - \ell(\pi) = \lambda_2 \) and the description of the tableaux are precisely those that are column strict of skew of shape \((n' - |\lambda|, \lambda_2)\) and whose entries are the multisets in \( \pi \). □

**Example 10.** Consider the following 20 column strict tableaux of content \( \{1^2, 2\} \).

\[
\begin{array}{cccc}
1 & 1 & 2 & 2 \\
1 & 2 & 1 & 1 \\
2 & 1 & 1 & 2 \\
1 & 2 & 1 & 1 \\
1 & 2 & 1 & 1 \\
1 & 2 & 1 & 1 \\
1 & 2 & 1 & 1 \\
1 & 2 & 1 & 1 \\
1 & 2 & 1 & 1 \\
1 & 2 & 1 & 1 \\
1 & 2 & 1 & 1 \\
1 & 2 & 1 & 1 \\
1 & 2 & 1 & 1 \\
1 & 2 & 1 & 1 \\
1 & 2 & 1 & 1 \\
1 & 2 & 1 & 1 \\
1 & 2 & 1 & 1 \\
\end{array}
\]

Theorem 9 states then that

\[
h_{21} = \tilde{s}_3 + \tilde{s}_21 + 4\tilde{s}_2 + 3\tilde{s}_11 + 7\tilde{s}_1 + 4\tilde{s}_(). \tag{20}
\]

**Example 11.** Let us compute the decomposition of \( V \otimes^4 \) where \( V = \mathbb{C}\{x_1, x_2, x_3, \ldots, x_n\} \) as an \( S_n \) module with the diagonal action. The module \( V \) has character equal to \( h_1 = h_1 \). Therefore to compute the decomposition of this character into \( S_n \) irreducibles we are looking for the expansion of \( h_{14} \) into the irreducible character basis.

Using Sage [sage, sage-combinat] we compute that it is

\[
h_{14} = 15\tilde{s}_() + 37\tilde{s}_1 + 31\tilde{s}_11 + 10\tilde{s}_111 + \tilde{s}_{1111} + 31\tilde{s}_2 \\
+ 20\tilde{s}_{21} + 3\tilde{s}_{211} + 2\tilde{s}_{22} + 10\tilde{s}_3 + 3\tilde{s}_{31} + \tilde{s}_4. \tag{21}
\]

If \( n \geq 6 \) then the multiplicity of the irreducible \((n - 3, 3)\) will be 10. The combinatorial interpretation of this value is the number of column strict tableaux with entries that are multisets (or in this case sets) of \( \{1, 2, 3, 4\} \) of skew-shape \((4, 3)/(3)\) or \((3, 3)/(3)\). Those tableaux are

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
1 & 2 & 4 & 3 \\
1 & 3 & 4 & 2 \\
2 & 3 & 4 & 1 \\
1 & 2 & 3 & 4 \\
1 & 3 & 2 & 4 \\
1 & 2 & 3 & 4 \\
1 & 2 & 4 & 3 \\
1 & 2 & 3 & 4 \\
1 & 2 & 3 & 4 \\
1 & 2 & 3 & 4 \\
\end{array}
\]

What is interesting about this example is that the usual combinatorial interpretation for the repeated Kronecker product \( \chi^{(n-1, 1)} + \chi^{(n)} \) is stated in other places in the literature in terms of oscillating tableaux. □

This special case of
Theorem 9 gives a new combinatorial description of the multiplicities in terms of set valued tableaux.

5. Character polynomials and the irreducible character basis

Following the notation of [GG], a character polynomial is a multivariate polynomial $q(\lambda)(x_1, x_2, x_3, \ldots)$ in the variables $x_i$ such that for integer values $x_i = m_i \in \mathbb{Z}$ with $m_i \geq 0$ and $n = \sum_{i \geq 1} i m_i$,

\[
q(\lambda)(m_1, m_2, m_3, \ldots) = \chi^{(n-|\lambda|, \lambda)}(1^{m_1}2^{m_2}3^{m_3} \ldots)
\]

as long as $n$ is larger than or equal to $|\lambda| + \lambda_1$.

Character polynomials were first used by Murnaghan [Mur]. Much later, Specht [Sp] gave determinantal formulas and expressions in terms of binomial coefficients for these polynomials. They are treated as an example in Macdonald’s book [Mac, ex. I.7.13 and I.7.14]. More recently, Garsia and Goupil [GG] gave an umbral formula for computing them. We will show in this section that character polynomials are a transformation of character symmetric functions and this will allow us to give an expression for character polynomials as long as $n$ is larger than or equal to $|\lambda| + \lambda_1$.

As a consequence of Lemma 10 and Proposition 9 we have the following relationship between the character polynomials $q(\lambda)(x_1, x_2, \ldots)$ and character basis $s_\lambda$.

**Proposition 12.** For $n \geq 0$ and $\lambda \vdash n$,

\[
q(\lambda)(x_1, x_2, x_3, \ldots) = \left. \tilde{s}_\lambda \right|_{p_k \rightarrow \sum d_k \ dx_d}
\]

and

\[
\tilde{s}_\lambda = \left. q(\lambda)(x_1, x_2, x_3, \ldots) \right|_{x_k \rightarrow \frac{1}{k} \sum d_k \mu(k/d) p_d}
\]

where $a_i \rightarrow b_i$ means that we are replacing $a_i$ with the expression $b_i$.

In [GG], the character polynomials are computed algorithmically. If we make an additional substitution, i.e., $x_k$ by $\frac{1}{k} \sum d_k \mu(k/d) p_d$, in their algorithm, then we obtain $\tilde{s}_\lambda$ using the following steps.

1. Expand the Schur function $s_\lambda$ in the power sums basis $s_\lambda = \sum_\gamma \frac{\chi^\lambda(\gamma)}{z_\gamma} p_\gamma$.
2. Replace each power sum $p_i$ by $i x_i - 1$.
3. Expand each product $\prod_i (i x_i - 1)^{a_i}$ as a sum $\sum g \prod_i x_i^{g_i}$.
4. Replace each $x_k^{g_k}$ by $(x_k)^{g_k} = x_k(x_k - 1) \cdots (x_k - g_k + 1)$.
5. Replace each $x_k$ by $\frac{1}{k} \sum d_k \mu(k/d) p_d$.

**Example 13.** To compute $\tilde{s}_3$ we follow the steps to obtain:

1. $s_3 = \frac{1}{6}(p_1^3 + 3 p_21 + 2 p_3)$
2. $\frac{1}{6}(p_1^3 + 3 p_21 + 2 p_3) \rightarrow \frac{1}{6}((x_1 - 1)^3 + 3(2 x_2 - 1)(x_1 - 1) + 2(3 x_3 - 1))$
3. $\frac{1}{6}((x_1 - 1)^3 + 3(2 x_2 - 1)(x_1 - 1) + 2(3 x_3 - 1)) = \frac{1}{6}x_1^3 - \frac{1}{2}x_1^2 + x_1x_2 - x_2 + x_3$
4. $q_3 = \frac{1}{6}(x_1)^3 - \frac{1}{2}(x_1)^2 + x_1 x_2 - x_2 + x_3$
(5) \( \hat{s}_3 = \frac{1}{6} (p_1)_3 - \frac{1}{2} (p_1)_2 + p_1 \frac{p_2 - p_1}{2} - \frac{p_2 - p_1}{2} + \frac{p_3 - p_1}{3} \)

As an important consequence, we derive a power sum expansion of the irreducible character basis by following the algorithm stated above.

**Theorem 14.** For \( n \geq 0 \) and \( \lambda \vdash n \),

\[
\hat{s}_\lambda = \sum_{\gamma \vdash n} \chi^\lambda(\gamma) \frac{p_\gamma}{z_\gamma}
\]

that is, the linear map defined by \( \Gamma(s_\lambda) = \hat{s}_\lambda \) has the property that \( \Gamma(p_\gamma) = p_\gamma \) where

\[
p_r = \sum_{k=0}^{r} (-1)^{r-k} \binom{r}{k} \left( \frac{1}{i} \sum_{d|i} \mu(i/d)p_d \right)
\]

and \( (x)_k \) denotes the \( k \)-th falling factorial.

**Proof.** The proof of this proposition is exactly the steps outlined in the result of \([GG]\) and then add the additional step of replacing \( x_i \) with \( \frac{1}{i} \sum_{d|i} \mu(i/d)p_d \) in step (4). The Schur function has a power sum expansion given by

\[
s_\lambda = \sum_{\gamma \vdash |\lambda|} \chi^\lambda(\gamma) \frac{p_\gamma}{z_\gamma} = \sum_{\gamma \vdash |\lambda|} \chi^\lambda(\gamma) \frac{1}{z_\gamma} \prod_{i=1}^{\ell(\gamma)} (p_i)^{m_i(\gamma)}
\]

Then in the next step we replace \( p_i \) with \( ix_i - 1 \) and expand the expression. The part of the expression \( (p_i)^r \) becomes

\[
(p_i)^r \bigg|_{p_i \to ix_i - 1} = (ix_i - 1)^r = \sum_{k=0}^{r} (-1)^{r-k} \binom{r}{k} x_i^k
\]

In the last step the \([GG]\) algorithm replaces \( x_i^k \) with \( (x_i)_k \) and that is the expression for the character polynomial. To recover the irreducible character function \( \hat{s}_\lambda \), we use equation (69) and replace \( x_i \) with \( \frac{1}{i} \sum_{d|i} \mu(i/d)p_d \). Replacing \( x_i^k \) in (26) with \( \left( \frac{1}{i} \sum_{d|i} \mu(i/d)p_d \right)_k \)

means that \( (p_i)^r \) will be replaced by the expression in equation (24).

It follows that the composition of these steps changes \( s_\lambda \) to \( \hat{s}_\lambda \) and the power sum expansion of the Schur function to the right hand side of equation (23).

We also present a similar formula for the expansion of the induced trivial character basis in the power sum basis. To do this we introduce a basis which acts like an indicator function at the roots of unity. Define

\[
\overline{p}_r = \hat{r}^r \left( \frac{1}{i} \sum_{d|i} \mu(i/d)p_d \right)_r \quad \text{and} \quad \overline{p}_\gamma := \prod_{i \geq 1} \overline{p}_{i^{m_i(\gamma)}}
\]

**Lemma 15.** For partitions \( \gamma \) and \( \mu \) such that \( |\mu| < |\gamma| \), then \( \overline{p}_\gamma[\Xi_\mu] = 0 \). Moreover, if \( |\mu| = |\gamma| \), then

\[
\overline{p}_\gamma[\Xi_\mu] = z_\gamma \delta_{\gamma=\mu}
\]
\begin{proof}
Since \( p_d[\Xi_\mu] = \sum_{d\mid d'} d' m_{d'}(\mu) \) and
\begin{equation}
\frac{1}{i} \sum_{d\mid i} \frac{\mu(i/d)}{d} \left( \sum_{d'\mid d} d' m_{d'}(\mu) \right) = m_i(\mu),
\end{equation}
then plugging into (27) we see
\begin{equation}
\mathbf{p}_\gamma[\Xi_\mu] = i^r(m_i(\mu))_r.
\end{equation}
and hence \( \mathbf{p}_\gamma[\Xi_\mu] = \prod_{i\geq 1} i^{m_i(\gamma)}(m_i(\mu))^{m_i(\gamma)}. \)

Now if \(|\mu| < |\gamma|\) or \(||\mu| = |\gamma|\) and \( \mu \neq \gamma \), then there exists at least one \( i \) such that \( m_i(\mu) < m_i(\gamma) \) and for that value \( i, (m_i(\mu))^{m_i(\gamma)} = 0 \) and hence \( \mathbf{p}_\gamma[\Xi_\mu] = 0. \)

If \( \mu = \gamma \), then \( \mathbf{p}_\gamma[\Xi_\gamma] = \prod_{i\geq 1} i^{m_i(\gamma)}(m_i(\gamma))^{m_i(\gamma)} = \prod_{i\geq 1} i^{m_i(\gamma)}m_i(\gamma)! = z_\gamma. \)
\end{proof}

This basis can then be used to give a formula for the induced trivial character basis.

\begin{proposition}
For \( n \geq 0 \) and \( \lambda \vdash n \),
\begin{equation}
\tilde{h}_\lambda = \sum_{\gamma \vdash n} \langle h_\lambda, p_\gamma \rangle \frac{\mathbf{p}_\gamma}{z_\gamma},
\end{equation}
that is, the linear map defined by \( \Theta(h_\lambda) = \tilde{h}_\lambda \) has the property that \( \Theta(p_\gamma) = \mathbf{p}_\gamma. \)
\end{proposition}

\begin{proof}
For any partition \( \mu \) such that \(|\mu| < |\lambda|\), then \( \tilde{h}_\lambda[\Xi_\mu] = \langle h_{|\mu|-|\lambda|} h_\lambda, p_\mu \rangle = 0 \) because \( |\mu| - |\lambda| < 0 \). In addition, by Lemma 15 \( \sum_{\gamma\vdash |\lambda|} \langle h_\lambda, p_\gamma \rangle \frac{\mathbf{p}_\gamma[\Xi_\mu]}{z_\gamma} = 0. \) If \( |\mu| = |\lambda| \), then \( \langle h_{|\mu|-|\lambda|} h_\lambda, p_\mu \rangle = \langle h_\lambda, p_\mu \rangle \) and
\begin{equation}
\sum_{\gamma\vdash |\lambda|} \langle h_\lambda, p_\gamma \rangle \frac{\mathbf{p}_\gamma[\Xi_\mu]}{z_\gamma} = \langle h_\lambda, p_\mu \rangle = \tilde{h}_\lambda[\Xi_\mu].
\end{equation}

We can conclude since we have equality for all evaluations at \( \Xi_\mu \) for \(|\mu| \leq |\lambda|\), then by Proposition 39 Equation (31) holds at the level of symmetric functions.
\end{proof}

\begin{remark}
After finding this formula for \( \tilde{h}_\lambda \) in terms of the elements \( \mathbf{p}_\gamma \) we noticed that a similar expression appears in Macdonald’s book [Mac] on page 121. He defines polynomials for each partition \( \rho \) in variables \( a_1, a_2, \ldots \) as \( (a_\rho) := \prod_{r\geq 1} (m_r(\rho)). \) We noticed that if \( a_i = \frac{1}{i} \sum_{d\mid i} \frac{\mu(i/d)}{d} \) then \( \frac{a_\rho}{\rho} = \mathbf{p}_\rho. \) The interested reader can translate Equation (4) from Example 14 on page 123 for the character polynomial to conclude that
\begin{equation}
\tilde{s}_\lambda = \sum_{\sigma, \rho} (-1)^{l(\sigma)} \langle s_\lambda, p_\rho p_\sigma \rangle \frac{\mathbf{p}_\rho}{z_\rho},
\end{equation}
summed over all partitions \( \rho \) and \( \sigma \) such that \(|\rho| + |\sigma| = |\lambda|\). We can also translate Equation (5) from the same example on page 124 to show that
\begin{equation}
\tilde{s}_\lambda = \sum_{\mu} (-1)^{|\lambda| - |\mu|} \sum_{\gamma\vdash |\mu|} \langle s_\mu, p_\gamma \rangle \frac{\mathbf{p}_\gamma}{z_\gamma}
\end{equation}

\end{remark}
where the outer sum is over partitions $\mu$ such that $\lambda/\mu$ is a vertical strip (no more than one cell in each row of the skew partition).

6. The partition algebra and a Murnaghan-Nakayama rule

The partition algebra was independently defined in the work of Martin [Mar1, Mar2, Mar3, Mar4] and Jones [Jo]. Jones showed that the partition algebra is the centralizer algebra of the diagonal action of the symmetric group on tensor space. In other words he described a Schur-Weyl duality between the symmetric group and the partition algebra. Later, Halverson [Hal] described the analogues of the Frobenius formula and the Murnaghan-Nakayama rule to compute the characters of the partition algebra. In this section we describe a connection between the partition algebra characters and our irreducible character basis.

If $V$ is the $r$-dimensional defining representation of $S_r$, then the centralizer of the diagonal action on $V^\otimes n$ depends on the two parameters $n$ and $r$ and is denoted by $P_n(r)$. The irreducible characters are indexed by partitions $(r - |\lambda|, \lambda)$, where $\lambda$ is a partition of size less than or equal to $n$. Halverson described conjugacy class analogues and denoted the representatives of this classes by $d_\mu$, where $\mu$ is a partition of size less than or equal to $n$. Using these notations, the irreducible partition algebra character values are denoted by $\chi_{P_n(r)}^{(r-|\lambda|,\lambda)}(d_\mu)$.

Corollary 4.2.3 of [Hal] states the following properties of the partition algebra characters.

**Corollary 18.** If $|\lambda| \leq n$ and $\mu$ is a composition of size less than or equal to $n$, then

1. $\chi_{P_n(r)}^{(r-|\lambda|,\lambda)}(d_\mu) = \begin{cases} r_n - |\mu| \chi_{P_n(r)}^{(r-|\lambda|,\lambda)}(d_\mu) & \text{if } |\mu| \geq |\lambda|, \\ 0 & \text{if } |\mu| < |\lambda| \end{cases}$

2. If $|\mu| = |\lambda| = n$, then $\chi_{P_n(r)}^{(r-|\lambda|,\lambda)}(d_\mu) = \chi_{S_n}(\mu)$

3. If $r \geq 2n$ and $|\mu| = n$, then $\chi_{P_n(r)}^{(r-|\lambda|,\lambda)}(d_\mu)$ is independent of $r$.

For a positive integer $n$ and $\mu \vdash n$, the usual Frobenius formula for the symmetric functions is a consequence of the classical Schur-Weyl duality between $S_n$ and $GL_r$, it states

$$p_\mu = \sum_{\lambda \vdash n} \chi_{S_n}(\mu)s_\lambda$$

where $s_\lambda$ (as symmetric functions) are the irreducible $GL_r$ characters.

If we restrict the diagonal action of $GL_r$ to the symmetric group, $S_r$, realized by the permutation matrices, we obtain the Schur-Weyl duality between the symmetric group $S_r$ and the partition algebra $P_n(r)$. A decomposition of $V^\otimes n$ as a $(P_n(r), S_r)$-module into irreducibles yields the analogue of the Frobenius formula for $P_n(r)$ and the symmetric group. See equation (3.2.1) and Theorem 3.2.2 of [Hal] where we assume that $r \geq 2|\mu|$, and $\gamma \vdash r$,

$$p_\mu[\Xi_\gamma] = \sum_{|\lambda| \leq |\mu|} \chi_{P_n(r)}^{(r-|\lambda|,\lambda)}(d_\mu)s_\lambda[\Xi_\gamma].$$
Since this identity holds for all \( r \) greater than a fixed value and all partitions \( \gamma \vdash r \), then by Corollary 41, this expression is a symmetric function identity and we have

\[
p_\mu = \sum_{|\lambda| \leq |\mu|} \chi^{(r-|\lambda|,\lambda)}_{P_\mu(r)} (d_\mu) \tilde{s}_\lambda.
\]

Our next result is a statement which is equivalent to the Murnaghan-Nakayama rule for the computation of the irreducible symmetric group characters. To state how this relation appears in the irreducible character basis, we first introduce a little notation.

For partitions \( \lambda \) and \( \nu \) such that \( \nu \subseteq \lambda \), we will say that \( \lambda \) differs from \( \nu \) by a \( k \)-border strip (abbreviated \( \lambda/\nu \in B_k \)) if the skew partition \( \lambda/\nu \) consists of \( k \) cells which are connected and do not contain a \( 2 \times 2 \) sub-configuration of cells. When we think of these partitions as having an extra long row, we will write \( \lambda/\nu \in B_k \) if \( \lambda/\nu \in B_k \) or if \( \lambda = \nu \) (in which case we think of the skew partition as \( (r,\lambda)/(r-k,\lambda) \) for a sufficiently large \( r \)).

Let \( h_t(\lambda/\nu) \) equal to the number of rows occupied by the skew partition minus 1 and, in particular, \( h_t(\lambda/\lambda) = 0 \).

**Lemma 19.** For \( n,k > 0 \) and \( \lambda \vdash n \), let \( \mu \vdash 2n+k \).

\[
\tilde{s}_\lambda[\Xi_{(k,\mu)}] = \sum_{\nu: \lambda/\nu \in B_k} (-1)^{h_t(\lambda/\nu)} \tilde{s}_\nu[\Xi_\mu] = \tilde{s}_\lambda[\Xi_\mu] + \sum_{\nu: \lambda/\nu \in B_k} (-1)^{h_t(\lambda/\nu)} \tilde{s}_\nu[\Xi_\mu].
\]

**Proof.** Recall that the Murnaghan-Nakayama rule says that

\[
p_k s_\lambda = \sum_{\nu: \lambda/\nu \in B_k} (-1)^{h_t(\nu/\lambda)} s_\nu
\]

and similarly by duality,

\[
p_k^\perp s_\lambda = \sum_{\lambda/\nu \in B_k} (-1)^{h_t(\lambda/\nu)} s_\nu.
\]

where \( p_k^\perp \) denotes the adjoint to \( p_k \) with respect to the inner product.

Next we calculate by translating the evaluation of \( \tilde{s}_\lambda[\Xi_{(k,\mu)}] \) to a symmetric function scalar product.

\[
\tilde{s}_\lambda[\Xi_{(k,\mu)}] = \chi^{(|\mu|+k-|\nu|,\lambda)}(k,\mu) = \langle s_{(|\mu|+k-|\nu|,\lambda)}, P_\mu P_k \rangle = \langle P_k^\perp s_{(|\mu|+k-|\nu|,\lambda)}, P_\mu \rangle = \sum_{\nu: \lambda/\nu \in B_k} (-1)^{h_t(\lambda/\nu)} \langle s_{(|\mu|-|\nu|,\lambda)}, P_\mu \rangle = \sum_{\nu: \lambda/\nu \in B_k} (-1)^{h_t(\lambda/\nu)} \tilde{s}_\nu[\Xi_\mu].
\]

Now to complete the statement of the lemma, we note that a \( k \)-border strip that starts in the first row of \( (|\mu|+k-|\nu|,\nu) \) lies only in the first row because we assume that \( |\mu| > 2|\nu| \). Hence one of the terms where \( \lambda/\nu \) is a \( k \) border strip is \( \nu = \lambda \). All the other partitions
such that $\lambda//\nu$ is a $k$ border strip will have the $k$ border strip start in the second row or higher and in this case $\lambda/\nu$ will be a $k$-border strip. Therefore equation (41) is equal to
\begin{equation}
\tilde{s}_{\lambda}[\Xi_{\mu}] + \sum_{\nu: \lambda//\nu \in B_k} (-1)^{ht(\lambda//\nu)} \tilde{s}_{\nu}[\Xi_{\mu}] .
\end{equation}

Let $\lambda$ and $\nu$ be partitions and choose $r > \text{max}(|\lambda| + \lambda_1, |\nu| + \nu_1)$. Then $(r - |\nu|, \nu)$ are partitions and we may compute
\begin{equation}
\sum_{\mu \vdash r} \frac{1}{z_{\mu}} \tilde{s}_{\lambda}[\Xi_{\mu}] \tilde{s}_{\nu}[\Xi_{\mu}] = \sum_{\mu \vdash r} \frac{1}{z_{\mu}} \chi^{(r-|\lambda|, \lambda)}(\mu) \chi^{(r-|\nu|, \nu)}(\mu) = \frac{1}{r!} \sum_{\sigma \in S_r} \chi^{(r-|\lambda|, \lambda)}(\sigma) \chi^{(r-|\nu|, \nu)}(\sigma) .
\end{equation}

By the orthogonality of symmetric group characters, this sum is equal to 1 if $\lambda = \nu$ and 0 otherwise.

This computation can be used to obtain a single coefficient of an $\tilde{s}_{\lambda}$ in a symmetric function expression $f$ which we rephrase as the following lemma.

**Lemma 20.** Let $r$ be a positive integer such that $r > \text{deg}(f)$. The coefficient of $\tilde{s}_{\lambda}$ in $f$ is equal to $\sum_{\mu \vdash r} \frac{1}{z_{\mu}} \tilde{s}_{\lambda}[\Xi_{\mu}] f[\Xi_{\mu}] .$

**Proof.** If $f = \sum_{\lambda} c_{\lambda} \tilde{s}_{\lambda}$, then for all $\lambda$ in the support of $f$, $(r - |\lambda|, \lambda)$ will always be a partition. Therefore since Equation (43) is equal to 1 if $\lambda = \nu$ and 0, otherwise, then by linearity $\sum_{\mu \vdash r} \frac{1}{z_{\mu}} \tilde{s}_{\lambda}[\Xi_{\mu}] f[\Xi_{\mu}] = c_{\lambda}$. □

We will apply this lemma to obtain the following expression which is equivalent to Halverson’s [Hal] Murnaghan-Nakayama rule for partition algebra characters.

**Theorem 21.** For $k > 0$ and a partition $\lambda$,
\begin{equation}
p_k \tilde{s}_{\lambda} = \sum_{\nu} \left( \sum_{d|k} \sum_{\alpha} (-1)^{ht(\lambda//\alpha) + ht(\nu//\alpha)} \tilde{s}_{\nu} \right) \tilde{s}_{\alpha}
\end{equation}

where the inner sum is over all partitions $\alpha$ such that both $\lambda//\alpha$ and $\nu//\alpha$ are border strips of size $d$ and the outer sum is over all partitions $\nu$ of size less than or equal to $k + |\lambda|$.

**Proof.** Our proof follows the computation of Halverson [Hal], but in the language of symmetric functions using the irreducible character basis. We can apply Lemma 20 to take the coefficient of $\tilde{s}_{\nu}$ in $p_k \tilde{s}_{\lambda}$. To begin, we note that $p_k[\Xi_{\mu}] = \sum_{d|k} dm_d(\mu)$ and choose an $r$ sufficiently large. We compute the coefficient by the expression
\begin{equation}
\sum_{\mu \vdash r} \frac{1}{z_{\mu}} \tilde{s}_{\nu}[\Xi_{\mu}] p_k[\Xi_{\mu}] \tilde{s}_{\lambda}[\Xi_{\mu}] = \sum_{d|k} \sum_{\mu \vdash r} \frac{dm_d(\mu)}{z_{\mu}} \tilde{s}_{\nu}[\Xi_{\mu}] \tilde{s}_{\lambda}[\Xi_{\mu}] .
\end{equation}
Now the non-zero terms of the sum over \( \mu \) occur when \( m_d(\mu) > 0 \) and in this case
\[
\frac{d m_d(\mu)}{z_\mu} = \frac{1}{z_{\mu - (d)}}.
\]
This is equivalent to summing over all partitions \( \mu \) of size \( r - d \) and
\( \mu = (d; \mu) \).

Therefore by applying Lemma 19,
\[
\sum_{\mu \vdash r} 1 \frac{z_{\mu}}{z_{\mu}} \sum_{d \mid k} \sum_{\nu} \sum_{\alpha} \frac{(-1)^{ht(\lambda/\alpha) + ht(\nu/\beta)}}{z_{\mu}} \hat{s}_\lambda[\Xi_{\mu}] \hat{s}_\alpha[\Xi_{(d; \nu)\mu}] = \sum_{d \mid k} \sum_{\nu} \sum_{\alpha} \frac{(-1)^{ht(\lambda/\alpha) + ht(\nu/\beta)}}{z_{\mu}} \hat{s}_\alpha[\Xi_{(d; \nu)\mu}] \hat{s}_\beta[\Xi_{\mu}]
\]
\[
= \sum_{d \mid k} \sum_{\nu} \sum_{\alpha} \frac{(-1)^{ht(\lambda/\alpha) + ht(\nu/\beta)}}{z_{\mu}} \hat{s}_\alpha[\Xi_{\mu}] \hat{s}_\beta[\Xi_{\mu}]
\]
(46)

where the sum over \( \alpha \) is such that \( \lambda/\alpha \) is a border strip of size \( d \) and the sum over \( \beta \) is
\( \nu/\beta \) is a border strip of size \( d \). The last equality holds by the orthogonality relations on
the symmetric group characters.

An expansion of Theorem 21 using the second equality in Lemma 19 yields the following
alternate expression for the Murnaghan-Nakayama rule for the irreducible character basis.

**Corollary 22.** For \( k > 0 \),
\[
p_k \hat{s}_\lambda = \text{divisors}(k) \hat{s}_\lambda + \sum_{d \mid k} \sum_{\nu} \sum_{\alpha} \frac{(-1)^{ht(\lambda/\alpha) + ht(\nu/\beta)}}{z_{\mu}} \hat{s}_\lambda[\Xi_{\nu}]
\]
\[
+ \sum_{d \mid k} \sum_{\nu} \sum_{\lambda} \frac{(-1)^{ht(\lambda/\nu)}}{z_{\mu}} \hat{s}_\lambda[\Xi_{\nu}] + \sum_{d \mid k} \sum_{\nu} \sum_{\lambda} \frac{(-1)^{ht(\nu/\lambda)}}{z_{\mu}} \hat{s}_\lambda[\Xi_{\nu}]
\]

where \text{divisors}(k) \) is equal to the number of divisors of \( k \).

7. **Appendix I: Evaluations of symmetric functions at roots of unity**

We refer readers to [Las, Lemma 5.10.1] for the following result which we will use as
our starting point for evaluations of symmetric functions. In the following expressions, the
notation \( r \mid n \) is shorthand for “\( r \) divides \( n \).”

**Proposition 23.** For \( r \geq 0 \),
\[
h_0[\Xi_r] = e_0[\Xi_r] = p_0[\Xi_r] = 1.
\]
In addition, for \( n > 0 \),
\[
(47) \quad h_n[\Xi_r] = \delta_{r \mid n}, \quad p_n[\Xi_r] = r \delta_{r \mid n}, \quad e_n[\Xi_r] = (-1)^{r-1} \delta_{r = n}.
\]

We will need to take this further and give a combinatorial interpretation for the evaluation
of \( h_\lambda[\Xi_\mu] \) in order to make a connection with character symmetric functions.

**Proposition 24.** For a nonnegative integer \( n \) and a partition \( \mu \), \( h_n[\Xi_\mu] \) is equal to the
number of weak compositions \( \alpha \) of size \( n \) and length \( \ell(\mu) \) such that \( \mu_i \) divides \( \alpha_i \) for all \( i \).
Proof. The alphabet addition formula for $h_n$ says
\[
    h_n[X_1, X_2, \cdots, X_r] = \sum_{\alpha \vdash w n \atop \ell(\alpha) = r} h_{\alpha_1}[X_1]h_{\alpha_2}[X_2] \cdots h_{\alpha_r}[X_r]
\]
therefore evaluating at $\Xi_\mu$, we have
\[
    h_n[\Xi_\mu] = \sum_{\alpha \vdash w n \atop \ell(\alpha) = \ell(\mu)} \prod_{i=1}^{\ell(\mu)} h_{\alpha_i}[\Xi_{\mu_i}]
\]
where the sum is over all weak compositions of $n$ (that is, 0 parts are allowed) such that the length of the composition (including the 0 parts) is equal to the length of the partition $\mu$. Since $h_{\alpha_i}[\Xi_{\mu_i}] = 0$ unless $\mu_i | \alpha_i$, the product contributes 1 to the sum if and only if the sum of the $\alpha_i$ is $n$ and $\mu_i$ divides $\alpha_i$. Therefore this expression represents the number of weak compositions of size $n$ and length $\ell(\mu)$ such that $\mu_i | \alpha_i$ for all $i$. □

Example 25. To evaluate $h_2[\Xi_\mu]$, we can reduce the computation by expressing it in terms of $h_n[\Xi_r]$.
\[
    h_2[\Xi_\mu] = \sum_{i=1}^{\ell(\mu)} h_2[\Xi_{\mu_i}] + \sum_{1 \leq i < j \leq \ell(\mu)} h_1[\Xi_{\mu_i}]h_1[\Xi_{\mu_j}].
\]
Now we know from Proposition 23 that $h_2[\Xi_{\mu_i}] = 1$ if and only if $\mu_i = 1$ or 2, hence
\[
    \sum_{i=1}^{\ell(\mu)} h_2[\Xi_{\mu_i}] = m_1(\mu) + m_2(\mu).
\]
In addition, Proposition 23 implies $h_1[\Xi_{\mu_i}] = 1$ if and only if $\mu_i = 1$ hence
\[
    \sum_{1 \leq i < j \leq \ell(\mu)} h_1[\Xi_{\mu_i}]h_1[\Xi_{\mu_j}] = \binom{m_1(\mu)}{2}.
\]
Alternatively, we can see this evaluation in terms of its computation of the combinatorial interpretation. We have that this is the number of weak compositions of length $\ell(\mu)$ of the form $(0^{i-1}, 2, 0^{\ell(\mu)-i})$ where the 2 is in the position $i$ where $\mu_i = 1$ or 2 or of the form $(0^{i-1}, 1, 0^{j-i-1}, 1, 0^{\ell(\mu)-j})$ where $\mu_i = \mu_j = 1$. Clearly there are $m_1(\mu) + m_2(\mu)$ of the first type and $\binom{m_1(\mu)}{2}$ of the second.

Equation 49 allows us to give a combinatorial interpretation for $h_\lambda[\Xi_\mu]$ in terms of sequences of compositions.

Definition 26. Define the set $C_{\lambda,\mu}$ to be the sequences $\alpha^{(s)} = (\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(\ell(\lambda))})$ of weak compositions $\alpha^{(i)} \vdash w \lambda_i$ such that $\ell(\alpha^{(i)}) = \ell(\mu)$ and $\mu_j$ divides $\alpha^{(i)}_j$ for each $1 \leq i \leq \ell(\lambda)$.
Example 27. To compute $C_{(3,1),(3,2,2,1)}$ we count pairs $(\alpha, \beta)$ where $\alpha$ is a weak composition of 3 of length 5 and $\beta$ is a weak composition of 1 of length 5 such that $\mu_i$ divides $\alpha_i$ and $\beta_i$ where $\mu = (3,3,2,2,1)$. There are 5 ways of doing this given by the following pairs of compositions

$$(((00003), (00001)), ((00021), (00001)), ((00021), (00001))$$
$$(((03000), (00001)), ((30000), (00001)))$$

Proposition 28. For partitions $\lambda$ and $\mu$, 

$$h_{\lambda}[\Xi_{\mu}] = |C_{\lambda, \mu}|.$$  

Proof. Since $h_r[\Xi_{\mu}]$ is equal to the number of weak compositions $\alpha$ of length $\ell(\mu)$ and size $r$ such that $\mu_j|\alpha_j$ (that is, it is equal to $|C_{(r), \mu}|$) hence, by the multiplication principle, the expression

$$h_{\lambda}[\Xi_{\mu}] = h_{\lambda_1}[\Xi_{\mu}]h_{\lambda_2}[\Xi_{\mu}] \cdots h_{\lambda_{(\lambda)}}[\Xi_{\mu}],$$

is equal to the number of sequences of compositions $\alpha = (\alpha^{(1)}, \alpha^{(2)}, \cdots, \alpha^{(\ell(\lambda))})$ such that $\alpha^{(i)}$ is a weak composition of $\lambda_i$ and of length $\ell(\mu)$ with $\mu_j$ divides $\alpha_j^{(i)}$ for each $1 \leq j \leq \ell(\mu)$. \qed

This combinatorial interpretation for $h_{\lambda}[\Xi_{\mu}]$ is only the starting point. We will give an expression for this quantity in terms of symmetric function coefficients.

We can state this as the following combinatorial result.

Proposition 29. For partitions $\nu$ and $\mu$, $H_{\nu, \mu} := \langle h_{\nu} h_{|\mu|-|\nu|}, p_{\mu} \rangle$ is equal to the number of ways that some of the cells of the diagram of $\mu$ can be filled with the labels $\{1, 2, \ldots, \ell(\nu)\}$ such that the cells in the same row all have the same label and in total $\nu_j$ cells are labeled with the integer $j$ for $1 \leq j \leq \ell(\nu)$.

Proof. Since $H_{\nu, \mu}$ is equal to the coefficient of $\frac{p_{\mu}}{z_{\mu}}$, it is also equal to $z_{\mu}$ times the coefficient of $p_{\mu}$ in the symmetric function $h_{\nu_1}h_{\nu_2} \cdots h_{\nu_{(\nu)}}h_{|\mu|-|\nu|}$. Since $h_{\nu_i} = \sum_{\gamma \vdash \nu_i} z_{\gamma} \frac{p_{\gamma}}{z_{\gamma}}$ then one way that we can express this is

$$H_{\nu, \mu} = \sum_{\gamma^{(*)} \subseteq \mu, \nu} z_{\nu} z_{\gamma_{(1)}} z_{\gamma_{(2)}} \cdots z_{\mu \setminus \gamma_{(1)}}$$

where the sum is over all sequences of partitions $\gamma^{(*)} = (\gamma^{(1)}, \gamma^{(2)}, \cdots, \gamma^{(\ell(\nu))})$ with $\gamma^{(i)}$ a partition of $\nu_i$ for $1 \leq i \leq \ell(\nu)$ and such that the parts of $\bigcup_i \gamma^{(i)} = \gamma^{(1)} \cup \gamma^{(2)} \cup \cdots \cup \gamma^{(\ell(\nu))}$ are a subset of the parts of $\mu$. The last partition that appears in this denominator is $\mu \setminus \bigcup_i \gamma^{(i)}$ and this is a partition of size $|\mu| - |\nu|$ of all the parts of $\mu$ which are not used in $\gamma^{(1)} \cup \gamma^{(2)} \cup \cdots \cup \gamma^{(\ell(\nu))}$. We are using the convention that $h_{-r} = 0$ for $r > 0$, hence $H_{\nu, \mu} = 0$ if $|\nu| > |\mu|$ since we will have $h_{|\mu|-|\nu|} = 0$. 

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Now recall that $z_\mu = \prod_{i=1}^{\mu_1} i^{m_i(\mu)} m_i(\mu)!$. This implies that

$$
\frac{z_\mu}{z_{\gamma(1)} z_{\gamma(2)} \cdots z_{\gamma(\ell(\nu))} z_\mu \setminus \bigcup_i \gamma(i)} = \frac{1}{\prod_{i=1}^{\mu_1} m_i(\gamma(1))! m_i(\gamma(2))! \cdots m_i(\gamma(\ell(\nu)))! m_i(\mu \setminus \bigcup_i \gamma(i))!} \prod_{i=1}^{\mu_1} \left( m_i(\gamma(1)), m_i(\gamma(2)), \ldots, m_i(\gamma(\ell(\nu))) \right).
$$

(56)

We have established by Equations (55) and (56) that

$$
H_{\nu, \mu} = \langle h_{[\mu]-[\nu]} h_\nu, p_\mu \rangle = \sum_{\gamma^{(*)}} \prod_{i=1}^{\mu_1} \left( m_i(\gamma(1)), m_i(\gamma(2)), \ldots, m_i(\gamma(\ell(\nu))) \right)
$$

(57)

where the sum is over all sequences of partitions $\gamma^{(*)} = (\gamma(1), \gamma(2), \ldots, \gamma(\ell(\nu)))$ with $\gamma(j)$ a partition of $\nu_j$ for $1 \leq j \leq \ell(\nu)$ and such that the parts of $\bigcup_i \gamma(i) = \gamma(1) \cup \gamma(2) \cup \cdots \cup \gamma(\ell(\nu))$ are a subset of the parts of $\mu$.

Take a filling of some of the rows of the partition $\mu$ such that the all the cells in a row are assigned the same label and the total number of cells with label $j$ is equal to $\nu_j$. The rows of $\mu$ that are labeled with $j$ determine a partition $\gamma(j)$ of size $\nu_j$. This determines the sequence $\gamma^{(*)}$.

Now among the $m_i(\mu)$ parts of the partition $\mu$ that are of size $i$, $m_i(\gamma(j))$ parts are labeled with label $j$. There are precisely

$$
\binom{m_i(\mu)}{m_i(\gamma(1)), m_i(\gamma(2)), \ldots, m_i(\gamma(\ell(\nu)))}
$$

ways this can be done. In other words, there is a sequence of positions which determines where the $m_i(\gamma(j))$ parts of size $i$ are chosen from the $m_i(\mu)$ parts of $\mu$ of size $i$.

The reverse bijection is found by taking a sequence of partitions $\gamma^{(*)}$ and sequence of positions which tells us which of the $m_i(\mu)$ parts of size $i$ of the partition which are filled with the labels $j$.

**Example 30.** To calculate $H_{(4),(3,3,2,2,1,1)}$ we need to label some of the rows of the partition with just one label such that the number of cells labeled is equal to 4. The following diagrams are the only ways that this can be done.

Therefore $H_{(4),(3,3,2,2,1,1)} = 7$. Similarly, to compute $H_{(3,1),(3,3,2,2,1,1)}$, we fill the rows of diagram of the partition $(3,3,2,2,1,1)$ with labels of three 1’s and 2 such that the whole
Example 31. In the multiset \(\{1^2, 2^7, 3^2\}\) we have an example multiset partition
\[
\pi = \{\{1, 2\}, \{1, 2\}, \{1, 1\}, \{1, 1\}, \{1, 1\}, \{1, 2, 3\}, \{1, 2, 3\}, \{1\}\}
\]
The parts \(\{1, 2\}\) occur first because they occur 3 times, \(\{1, 1, 1\}\) and \(\{1, 2, 3\}\) each occur twice so they are each second and \(\{1, 1, 1\}\) is before \(\{1, 2, 2, 3\}\) because \(111 <_{\text{lex}} 1223\). Finally \(\{1\}\) only occurs once, hence it is last.

Definition 32. Let \(\mathcal{P}_{\lambda, \mu}\) be the set of pairs \((\pi, T)\) where \(\pi\) is a multiset partition of the multiset \(\{1^{\lambda_1}, 2^{\lambda_2}, \ldots, \ell^{\lambda_\ell}\}\) and \(T\) is a filling of some of the cells of the diagram of the partition \(\mu\) with content \(\gamma = \tilde{m}(\pi)\) and all cells in the same row have the same label.

A restatement of Proposition 29 would be that
\[
\sum_{\pi \vdash \{1^{\lambda_1}, 2^{\lambda_2}, \ldots, \ell^{\lambda_\ell}\}} H_{\tilde{m}(\pi), \mu} = |\mathcal{P}_{\lambda, \mu}|.
\]

Example 33. The set \(\mathcal{P}_{(3,1),(3,3,2,2,1)}\) consists of the following 5 pairs of multiset partitions of the multiset \(\{1, 1, 1, 2\}\) and fillings of the diagram for \((3, 3, 2, 2, 1)\).

\[
\begin{align*}
(\{\{1, 1, 1, 2\}\}, \quad (\{\{1\}, \{1\}, \{1, 2\}\}, \quad (\{\{1\}, \{1\}, \{1, 2\}\}, \quad (\{\{1, 1\}, \{1, 1\}, \{1\}\}, \quad (\{\{1\}, \{1\}, \{1\}, \{2\}\}, \quad (\{\{1\}, \{1\}, \{1\}, \{2\}\}, \quad (\{\{1\}, \{1\}, \{1\}, \{2\}\}, \quad (\{\{1\}, \{1\}, \{1\}, \{2\}\}, \quad (\{\{1\}, \{1\}, \{1\}, \{2\}\}).
\end{align*}
\]
Definition 34. Let $\mathcal{T}_{\lambda,\mu}$ be the set of fillings of some of the cells of the partition $\mu$ with content $\lambda$ such that any number of labels can go into the same cell but all cells in the same row must have the same multiset of labels.

Example 35. The set $\mathcal{T}_{(3,1),(3,3,2,2,1)}$ consists of the following 5 fillings

where the $w$ in the first diagram is $w = 1112$ ($w$ represents the multiset of labels $\{1, 1, 2\}$).

The last two examples suggest the following proposition.

Proposition 36. For partitions $\lambda$ and $\mu$,

\begin{equation}
\sum_{\pi \vdash \{1^{\lambda_1}, 2^{\lambda_2}, \ldots, \ell^{\lambda_{\ell}}\}} H_{\tilde{m}(\pi), \mu} = |T_{\lambda,\mu}|.
\end{equation}

Proof. By Proposition 29 we have established that

\begin{equation}
\sum_{\pi \vdash \{1^{\lambda_1}, 2^{\lambda_2}, \ldots, \ell^{\lambda_{\ell}}\}} H_{\tilde{m}(\pi), \mu} = |P_{\lambda,\mu}|,
\end{equation}

hence it remains to show that there is a bijection between the elements of $P_{\lambda,\mu}$ and $T_{\lambda,\mu}$.

Take a pair $(\pi, T) \in P_{\lambda,\mu}$. Since the order on the parts of the multiset partition of $\pi$ puts them in weakly decreasing order dependent on the number of times that they occur, the label 1 in $T$ can be replaced by the first multiset which occurs in $\pi$, the 2 can be replaced by the second multiset which occurs, etc. The result will be a filling $T'$ which is an element of $T_{\lambda,\mu}$ because all rows of $T$ have the same labels (and so will be the case with $T'$) and the content of $T'$ will be the same as the multiset and so will be $\{1^{\lambda_1}, 2^{\lambda_2}, \ldots, \ell^{\lambda_{\ell}}\}$.

Example 37. Consider the pair $(\pi, T) \in P_{(12,7,2),(3,3,2,2,1)}$ where

\[
\pi = \{\{1, 2\}, \{1, 2\}, \{1, 2\}, \{1, 1, 1\}, \{1, 1, 1\}, \{1, 2, 2, 3\}, \{1, 2, 2, 3\}, \{1\}\}
\]

and $T$ is the filling

\[
\begin{array}{ccc}
4 & 2 & 2 \\
3 & 3 \\
1 & 1 & 1 \\
\end{array}
\]

This is mapped to the filling $T'$ equal to

\[
\begin{array}{ccc}
1 \\
111 & 111 \\
12 & 12 & 12 \\
\end{array}
\]
where $T' \in \mathcal{T}_{(12,7,2),(3,3,2,2,1)}$. The map is to replace the labels of $T$ with the parts of $\pi$.

As long as the order in which the multisets occur in $\pi$ is fixed, the map from pairs $(\pi, T) \in \mathcal{P}_{\lambda,\mu}$ to $T' \in \mathcal{T}_{\lambda,\mu}$ is invertible since we can recover the multiset partition of a multiset $\pi$ as the set of labels in $T'$ and the filling $T$ is just the filling $T'$ with each of the sets replaced by the integer order in which the set appears in $\pi$. □

We are now prepared to provide a proof of the following result which was first stated as Theorem 3 in Section 3 and used to provide a definition of the induced trivial character basis.

**Theorem 38.** For all partitions $\lambda$,

\[
(63) \quad h_\lambda[\Xi_\mu] = \sum_{\pi \vdash \{1^{\lambda_1}, 2^{\lambda_2}, \ldots, \ell^{\lambda_\ell}\}} H_{\tilde{\pi}(\pi), \mu}.
\]

**Proof.** We have already established in Proposition 28 that $h_\lambda[\Xi_\mu] = |C_{\lambda,\mu}|$ and in Proposition 36 that

\[
(64) \quad \sum_{\pi \vdash \{1^{\lambda_1}, 2^{\lambda_2}, \ldots, \ell^{\lambda_\ell}\}} H_{\tilde{\pi}(\pi), \mu} = |\mathcal{P}_{\lambda,\mu}| = |\mathcal{T}_{\lambda,\mu}|.
\]

In order to show this result we will provide a bijection between $C_{\lambda,\mu}$ and $\mathcal{T}_{\lambda,\mu}$.

We start with a filling $T \in \mathcal{T}_{\lambda,\mu}$ and define a list of weak compositions

\[
\alpha^{(s)} = (\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(\ell(\lambda))})
\]

where $\alpha^{(d)}_i$ is equal to the number of labels $d$ in the $i^{th}$ row of $T$. Since the content of $T$ is equal to the multiset $\lambda$, we know that $\alpha^{(d)}_i$ will be a weak composition of $\lambda_d$ and because in row $i$ the multiset of labels is the same for each cell in a row $\mu_i$, it must be that $\mu_i$ divides $\alpha^{(d)}_i$.

This procedure is reversible since starting with a sequence of weak compositions $\alpha^{(s)} \in C_{\lambda,\mu}$, we can place $\alpha^{(d)}_i/\mu_i$ labels of $d$ in each cell of the $i^{th}$ row of the diagram $\mu$ to recover the filling $T' \in \mathcal{T}_{\lambda,\mu}$.

We conclude that

\[
(65) \quad h_\lambda[\Xi_\mu] = |C_{\lambda,\mu}| = |\mathcal{T}_{\lambda,\mu}| = |\mathcal{P}_{\lambda,\mu}| = \sum_{\pi \vdash \{1^{\lambda_1}, 2^{\lambda_2}, \ldots, \ell^{\lambda_\ell}\}} H_{\tilde{\pi}(\pi), \mu}. \quad \square
\]

**8. Appendix II: Roots of unity and zeros of multivariate polynomials**

The goal of this section is to prove the following basic proposition.

**Proposition 39.** Let $f, g \in \text{Sym}$ be symmetric functions of degree less than or equal to some positive integer $n$. Assume that

\[
f[\Xi_\gamma] = g[\Xi_\gamma]
\]

for all partitions $\gamma$ such that $|\gamma| \leq n$, then

\[
f = g
\]
as elements of Sym.

Our plan for the proof is to reduce this proposition to the fact that a univariate polynomial of degree \( n \) has at most \( n \) zeros and hence a univariate polynomial of degree at most \( n \) with more than \( n \) zeros must be the 0 polynomial.

To begin, notice that \( p_k[\Xi_r] = r \) if \( r \) divides \( k \) and it is equal to 0 otherwise. In general, we can express any partition \( \gamma \mid n \) with more than \( n \) mial of degree \( \text{Sym} \) as elements of polynomial evaluates to 0 for all (66)

\[
p_k[\Xi_\gamma] = \sum_{d|k} dm_d.
\]

Hence any symmetric function \( f \) evaluated at some set of roots of unity is equal to a polynomial in values \( m_1, m_2, \ldots, m_n \) where

\[
f[\Xi_\gamma] = f \left|_{p_k \to \sum_{d|k} dm_d} = q(m_1, m_2, \ldots, m_n) \right.
\]

Moreover, if we know this polynomial \( q(m_1, m_2, \ldots, m_d) \) we can use Möbius inversion to recover the symmetric function since if \( p_k = \sum_{d|k} dm_d \), then \( km_k = \sum_{d|k} \mu(k/d)p_d \) where

\[
\mu(r) = \begin{cases} (-1)^d & \text{if } r \text{ is a product of } d \text{ distinct primes} \\ 0 & \text{if } r \text{ is not square free} \end{cases}
\]

Therefore, we also have

\[
q\left(p_1, \frac{p_2 - p_1}{2}, \frac{p_3 - p_1}{3}, \ldots, \frac{1}{n} \sum_{d|n} \mu(n/d)p_d\right) = f.
\]

To show that Proposition 39 is true, we will prove that if \( h[\Xi_\gamma] = 0 \) for all partitions \( |\gamma| \leq n \), then \( h = 0 \) as a symmetric function where \( h = f - g \). We will do this by considering \( h \) as \( q(x_1, x_2, \ldots, x_n) \) where \( x_i \) is replaced by \( \frac{1}{r} \sum_{d|r} \mu(r/d)p_d \) and that this multivariate polynomial evaluates to 0 for all \( (m_1, m_2, \ldots, m_n) \) where \( \gamma = (1^{m_1} 2^{m_2} \cdots n^{m_n}) \) is a partition with \( |\gamma| \leq n \). This is a consequence of the next lemma below.

Define the degree of a monomial so that \( \deg(x_i) = i \) and hence \( \deg(x_1^{a_1} x_2^{a_2} \cdots x_k^{a_k}) = a_1 + 2a_2 + 3a_3 + \cdots + ka_k. \)

**Lemma 40.** Let \( q(x_1, x_2, \ldots, x_n) \) be an element in a multivariate polynomial ring \( \mathbb{Q}[x_1, x_2, \ldots, x_n] \) with \( \deg(q(x_1, x_2, \ldots, x_n)) \leq d \) for some \( d \). If \( q(m_1, m_2, \ldots, m_n) = 0 \) for all sequences \( (m_1, m_2, \ldots, m_n) \) with \( m_i \geq 0 \) and \( m_1 + 2m_2 + 3m_3 + \cdots + nm_n \leq d \), then \( q(x_1, x_2, \ldots, x_n) = 0. \)

**Proof.** We argue by induction on the number of variables \( n \). First we note that if \( n = 1 \), then if \( q(x_1) \) is a polynomial of degree \( \leq d \) and \( q(0) = q(1) = \cdots = q(d) = 0 \), then \( q(x_1) = 0 \) because we know that a univariate polynomial of degree \( r \) can have at most \( r \) roots.
Let our induction assumption be that, if \( q(m_1, m_2, \ldots, m_{n-1}) = 0 \) for all sequences \((m_1, m_2, \ldots, m_{n-1})\) with \( m_i \geq 0 \) and \( m_1 + 2m_2 + 3m_3 + \cdots + (n-1)m_{n-1} \leq d \), then \( q(x_1, x_2, \ldots, x_{n-1}) = 0 \).

Now assume that our inductive hypothesis is true and consider a polynomial in \( n \) variables,

\[
(70) \quad q(x_1, x_2, \ldots, x_n) = \sum_{i=0}^{r} q^{(i)}(x_1, x_2, \ldots, x_{n-1}) x_i^n
\]

where \( r \) is the maximum power of the variable \( x_n \) in the polynomial \( q \) and \( 0 \leq r \leq d/n \) and the coefficient of \( x_i^n \) is \( q^{(i)}(x_1, x_2, \ldots, x_{n-1}) \) a multivariate polynomial of degree less than or equal to \( d - ni \).

We wish to show that \( q(x_1, x_2, \ldots, x_n) \) is in fact 0. Assume \( r \) is the largest exponent of \( x_n \) for which there is a non-zero coefficient, then fix \((m_1, m_2, \ldots, m_{n-1})\) such that \( m_1 + 2m_2 + \cdots + (n-1)m_{n-1} \leq d - rn \). Now

\[
(71) \quad q(m_1, m_2, \ldots, m_{n-1}, m_n) = 0
\]

for each \( m_n = 0, 1, 2, \ldots, r \), hence \( q(m_1, m_2, \ldots, m_{n-1}, x_n) = 0 \) because it is a polynomial of degree at most \( r \) with more than \( r \) roots and in particular the coefficient \( q^{(r)}(m_1, m_2, \ldots, m_{n-1}) \) is equal to 0. But now we have that \( q^{(r)}(x_1, x_2, \ldots, x_{n-1}) \) is a polynomial of degree \( \leq d - rn \) in \( n - 1 \) variables which vanishes at all \((x_1, x_2, \ldots, x_{n-1}) = (m_1, m_2, \ldots, m_{n-1})\) with \( m_1 + 2m_2 + \cdots + (n-1)m_{n-1} \leq d - rn \) and hence \( q^{(r)}(x_1, x_2, \ldots, x_{n-1}) = 0 \) by our induction hypothesis.

Hence, Proposition 39 follows as a corollary.

**Proof.** (of Proposition 39) We will show the equivalent statement, by setting \( h = f - g \), that if \( h \) is a symmetric function of degree less than or equal to \( n \) and \( h[\Xi_\gamma] = 0 \) for all \( |\gamma| \leq n \), then \( h = 0 \).

This statement is directly equivalent to Lemma 40 since if we replace \( p_k \) in \( h \) with \( \sum_{d|k} dx_d \) then

\[
q(x_1, x_2, \ldots, x_n) = \left| h \right|_{p_k \rightarrow \sum_{d|k} dx_d}
\]

is a polynomial of degree at most \( n \) that has the property that \( q(m_1, m_2, \ldots, m_n) = h[\Xi_\gamma] = 0 \) for all \( \gamma = (1^{m_1} 2^{m_2} \cdots n^{m_n}) \) for all \(|\gamma| \leq n \). By Lemma 40, \( q(x_1, x_2, \ldots, x_n) = 0 \) and hence by equation (69), \( h = 0 \).

We considered the case in Proposition 39 that \( f[\Xi_\gamma] = g[\Xi_\gamma] \) implies \( f = g \) when \( \gamma \) is small, but more often we will know that \( f[\Xi_\gamma] = g[\Xi_\gamma] \) for all \( \gamma \) which are partitions of integers greater than or equal to some value \( n \). We can reduce the implication to the previous case in the following proposition.

**Corollary 41.** Let \( f, g \in \text{Sym} \) be symmetric functions of degree less than or equal to some positive integer \( n \). Assume that

\[
f[\Xi_\gamma] = g[\Xi_\gamma]
\]
for all partitions $\gamma$ such that $|\gamma| \geq n$, then
\[ f = g \]
as elements of $\text{Sym}$.

\textbf{Proof.} We reduce the conditions of this corollary to the previous case by considering partitions $\bar{\gamma}$ of size less than or equal to $n$ and then let $\gamma = (n + 1, \bar{\gamma})$. Since for all $k \leq n$, $p_k[\Xi_{n+1}] = 0$, then $p_k[\Xi_\gamma] = p_k[\Xi_{\bar{\gamma}}]$.

Since $f$ (and similarly $g$) are of degree less than or equal to $n$, then $f = \sum_{|\lambda| \leq n} c_\lambda p_\lambda$ for some coefficients $c_\lambda$ then $p_\lambda[\Xi_\gamma] = p_\lambda[\Xi_{\bar{\gamma}}]$ since $p_\lambda[\Xi_{n+1}] = 0$. Therefore $f[\Xi_\gamma] = f[\Xi_{\bar{\gamma}}]$ and $g[\Xi_\gamma] = g[\Xi_{\bar{\gamma}}]$ and hence $f[\Xi_\gamma] = g[\Xi_\gamma]$. By Proposition 39 we can conclude that $f = g$. \hfill $\square$

9. Appendix III: Using Sage to compute the character bases of symmetric functions

This section does not appear in the journal published version of this paper \cite{OZa} and is an expanded version of a tutorial written in an extended abstract \cite{OZb} that the authors uploaded to the arXiv so that code could be reviewed for Sage \cite{sage, sage-combinat}. The purpose of this section is to give examples of the use of the irreducible character basis and the induced trivial character basis as a computation tool in Sage.

9.1. Symmetric functions in Sagemath. Sage is an open source symbolic calculation program based on the computer language Python. It may be downloaded from the website:

\url{https://www.sagemath.org/}

Computations may be done online without downloading the program at

\url{https://sagecell.sagemath.org/} or at CoCalc.

The following two online tutorials will be useful to cover the basic functionality of symmetric functions in Sage:

- Symmetric Functions Tutorial
- Documentation: Symmetric functions, with their multiple realizations

This appendix will focus on the use of the character bases which are only superficially covered in other documentation and tutorials.

A large community of mathematicians participate in its support and add to its functionality \cite{sage, sage-combinat}. In particular, the built-in library for symmetric functions includes a large extensible set of functions which makes it possible to do calculations within the ring following closely the mathematical notation that we use in this paper. The language itself has a learning curve, but the contributions made by the community towards the functionality make that a barrier worth overcoming.

In version 6.10 or later of Sage (released January 2016) these bases are available as methods in the ring of symmetric functions. At the time of this writing the current version is 9.3 (released May 2021).
9.2. **Defining bases.** We demonstrate examples of some of the definitions and results in this paper by Sage calculations. The first step to using the symmetric function code is to define bases of the ring over the field of rational numbers.

```
sage: Sym = SymmetricFunctions(QQ)
sage: Sym
Symmetric Functions over Rational Field
sage: st = Sym.irreducible_symmetric_group_character(); st
Symmetric Functions over Rational Field in the irreducible character basis
sage: ht = Sym.induced_trivial_character(); ht
Symmetric Functions over Rational Field in the induced trivial symmetric group character basis
```

Instead of defining each basis one at a time, alternatively there is a command to define all of the bases of the symmetric functions that do not use a parameter with a single command:

```
sage: SymmetricFunctions(QQ).inject_shorthands('all', verbose=False)
```

Note that by setting `verbose=True` in that command will list all of the bases defined. For the commands in the examples below we will assume that all of these bases have been added to the namespace. A table with a list of the bases defined by this command (in Sage version 9.3) appears in Figure 9.2.

| mathematical name          | Sage shorthand | mathematical name                  | Sage shorthand |
|----------------------------|----------------|------------------------------------|----------------|
| Schur                      | s              | irreducible character              | st             |
| complete/homogeneous       | h              | induced trivial character          | ht             |
| elementary                 | e              | orthogonal                         | o              |
| monomial                   | m              | symplectic                         | sp             |
| power                      | p              | Witt                               | w              |
| forgotten                  | f              |                                    |                |

**Figure 1.** A list of bases defined by the `inject_shorthands('all')` command.

9.3. **Structure coefficients.** Theorem [1] part (3) states that the structure coefficients of the irreducible character basis are equal to the reduced Kronecker coefficients. We can compare the structure coefficients of the $s$-basis with the Kronecker product of Schur functions whose first part is sufficiently large. If the first row of the indexing partition is removed from the terms in the expression for the Kronecker product, there is a corresponding term in the product of the irreducible character basis.

```
sage: st[2]*st[2]
st[1] + st[1] + st[1, 1] + 2*st[2] + 2*st[2, 1] + st[2, 2] + st[3] + st[3, 1] + st[4]
sage: s[6,2].kronecker_product(s[6,2])
s[4, 2, 2] + s[4, 3, 1] + s[4, 4] + s[5, 1, 1, 1] + 2*s[5, 2, 1] + s[5, 3] + s[6, 1, 1] + 2*s[6, 2] + s[7, 1] + s[8]
```
9.4. **Change of bases.** If $f$ is an element of the ring of symmetric functions and $\text{basis}$ is a basis of the symmetric functions then $\text{basis}(f)$ expresses the element in $f$ in that basis. For example, the symmetric function in Example 10 and Example 11 are computed with the commands:

```python
sage: st(h[2,1]) # express $h_{21}$ in the st-basis
4*st[] + 7*st[1] + 3*st[1, 1] + 4*st[2] + st[2, 1] + st[3]
sage: st(h[1,1,1,1])
15*st[] + 37*st[1] + 31*st[1, 1] + 10*st[1, 1, 1] + st[1, 1, 1, 1] + 31*st[2] + 20*st[2, 1] + 3*st[2, 1, 1] + 2*st[2, 2] + 10*st[3] + 3*st[3, 1] + st[4]
```

The main result of [AS] is that the Schur expansion of the irreducible character basis is alternating in sign by degree. This can be observed in the following example calculation:

```python
sage: s(st[2,2])
-s[1] + 4*s[1, 1] + 2*s[2] - 2*s[2, 1] + s[2, 2] - s[3]
```

9.5. **Eigenvalues of a permutation matrix.** In Section 2.1 we introduce the operation of evaluating a symmetric function at the eigenvalues of a permutation matrix. The character bases are defined so that $\tilde{s}_\lambda(\Xi_\mu)$ is equal to the irreducible $S_{|\mu|}$ character indexed by the partition $(|\mu| - |\lambda|, \lambda)$ at a permutation of cycle type $\mu$. In Sage, elements of the ring of symmetric functions have the method `eval_at_permutation_roots` which represents this operation.

For instance, an example of Proposition 23 with $n = 8$ and for $1 \leq r \leq 10$, we compute the following list of values:

```python
sage: [h[8].eval_at_permutation_roots([r]) for r in range(1,11)]
[1, 1, 0, 1, 0, 0, 1, 0, 0]
sage: [p[8].eval_at_permutation_roots([r]) for r in range(1,11)]
[1, 2, 0, 4, 0, 0, 0, 8, 0, 0]
sage: [e[8].eval_at_permutation_roots([r]) for r in range(1,11)]
[0, 0, 0, 0, 0, 0, -1, 0, 0]
```

In particular the character bases have the property that they evaluate to certain values of characters of symmetric group modules.

```python
sage: ht[3,1].eval_at_permutation_roots([3,3,2,2,1])
2
sage: st[3,1].eval_at_permutation_roots([3,3,2,2,1])
-1
```

In Sage, these can be compared to the following coefficients of the power sum basis in the complete and Schur elements as follows:

```python
sage: h[7,3,1].scalar(p[3,3,2,2,1])
2
sage: s[7,3,1].scalar(p[3,3,2,2,1])
-1
```
9.6. A Frobenius character map. In the paper [OZ2, Equation (7)] we define the characteristic map [Sag, Section 4.7], [Mac, Section I.7, p. 113] (sometimes referred to as the ‘Frobenius map’) by the function which interprets an element \( f \in \text{Sym} \) as a symmetric group character and sends it to a symmetric function of degree \( n \) which is a generating function for the character values of \( S_n \),

\[
\phi_n(f) = \sum_{\mu \vdash n} f[\Xi_\mu] \frac{p_\mu}{z_\mu}.
\]

In Sage, symmetric function elements have the method \texttt{character_to_frobenius_image} which calculates the map \( \phi_n \) on the element.

The character map is the origin of our definitions since in the beginning of our investigations the \( \tilde{h} \) and \( \tilde{s} \) basis for us were the pre-images of the Schur and complete symmetric functions in the \( \phi_n \) map for \( n \) sufficiently large.

We have defined the \( \tilde{s} \) and \( \tilde{h} \) bases so that they have the property \( \phi_n(\tilde{s}_\lambda) = s_{(n-|\lambda|,\lambda)} \) and \( \phi_n(\tilde{h}_\lambda) = h_{(n-|\lambda|,\lambda)} \) respectively if \( n \geq |\lambda| + \lambda_1 \). If \( n < |\lambda| + \lambda_1 \), then the corresponding symmetric function will still be equivalent to a Schur function or complete symmetric function indexed by a list of integers.

\[
sage: s(st[3,2].character_to_frobenius_image(8))
\]
\[
s[3, 3, 2]
\]
\[
sage: h(ht[3,2].character_to_frobenius_image(6))
\]
\[
h[3, 2, 1]
\]
\[
sage: s[2,1].character_to_frobenius_image(6)
\]
\[
s[3, 2, 1] + 2*s[4, 1, 1] + 2*s[4, 2] + 3*s[5, 1] + s[6]
\]

Since the Schur function is the character of an irreducible \( Gl_n \) module, the last calculation indicates that

\[
\text{Res}^{Gl_6}_{S_6} V^{(32)} \simeq V^{(321)}_{S_6} \oplus (V^{(411)}_{S_6})^\oplus 2 \oplus (V^{(42)}_{S_6})^\oplus 2 \oplus (V^{(51)}_{S_6})^\oplus 3 \oplus V^{(6)}_{S_6}.
\]

This calculation can be compared with the expansion of the Schur function \( s_{21} \) in the irreducible character basis:

\[
sage: st(s[2,1])
\]
\[
st[1] + 3*st[1] + 2*st[1, 1] + 2*st[2] + st[2, 1]
\]

9.7. Plethysm. Since Sage also can be used to compute plethysms a single coefficient in this expansion can be (inefficiently) calculated using Littlewood’s formula

\[
r_{\lambda\mu} = \langle s_{\lambda}, s_{(n-|\mu|,\mu)}[1 + s_1 + s_2 + \cdots] \rangle
\]

for the restriction coefficient:

\[
sage: s[5,1](1+s[1]+s[2]+s[3]).scalar(s[2,1])
\]
\[
3
\]
9.8. Orthogonal group character basis. As mentioned in the introduction, two additional bases of the symmetric functions were defined by Koike and Terada [KT] which correspond to the irreducible characters of the orthogonal and symplectic groups coming from the Weyl character formula. In Sage, these bases are implemented and the orthogonal basis has the shorthand $\mathfrak{o}$ while the symplectic basis has the shorthand $\mathfrak{sp}$.

Since we have the containment of the orthogonal group in the general linear group and the symmetric group (as permutation matrices) in the orthogonal group, that is,

$$S_n \subseteq O_n \subseteq GL_n,$$

it follows that the Schur functions (characters of irreducible $GL_n$ modules) will have non-negative coefficients when expressed in the orthogonal basis and the orthogonal basis will have non-negative coefficients when expressed in the irreducible character basis.

For instance, we compute the examples:

```python
sage: o(s[2,2])
o[0] + o[2] + o[2, 2]
sage: st(o[2,2])
st[1] + st[1, 1] + 3*st[2] + 2*st[2, 1] + st[2, 2] + st[3]
```

This computation implies that for $n$ sufficiently large

$$\text{Res}_{GL_n}^{O_n} V^{(22)} \simeq V^{(22)} \oplus V^{(2)} \oplus V^{(1)}$$

and

$$\text{Res}_{O_n}^{S_n} V^{(22)} \simeq V^{(n-4,22)} \oplus V^{(n-3,3)} \oplus (V^{(n-3,21)} \oplus 2 \oplus (V^{(n-2,2)} \oplus 3 \oplus V^{(n-2,11)} \oplus V^{(n-1,1)}).$$

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DARTMOUTH COLLEGE, MATHEMATICS DEPARTMENT, HANOVER, NH 03755, USA
Email address: rosa.c.orellana@dartmouth.edu

DEPARTMENT OF MATHEMATICS AND STATISTICS, YORK UNIVERSITY, TORONTO, ONTARIO M3J 1P3, CANADA
Email address: zabrocki@mathstat.yorku.ca