A nonlinear differential approach to the

Saffman-Taylor finger

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Abstract

Nonlinear time-dependent differential equations for the Hele-Shaw, Saffman-Taylor problem are derived. The equations are obtained using a separable ansatz expansion for the stream function of the displaced fluid obeying a Darcian flow. Suitable boundary conditions on the stream function, provide a potential term for the nonlinear equation. The limits for the finger widths derived from the potential and boundary conditions are \(1 > \lambda > \frac{1}{\sqrt{5}}\), in units of half the width of the Hele-Shaw cell, in accordance with observation. Stationary solutions with no free phenomenological parameters are found numerically. The dependence of asymptotic finger width on the physical parameters of the cell compares satisfactorily with experiment. The correct dispersion relation for the instabilities is obtained from the time dependent equation.

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1 Introduction

The Hele-Shaw cell experiment of a less viscous fluid displacing by a more viscous one is the paradigm of fingering phenomena\[1\]. The time-honored, work of Saffman and Taylor\[2\] showed both experimentally and theoretically that fingers with beautifully regular profiles arise in the Hele-Shaw cell.

In the intervening years a large body of works has added to our knowledge of the fingering phenomenon in various branches of the dynamics of continuous media, such as dendritic growth, directional solidification, diffusion- initiated aggregation, flame front propagation, electromigration, as well as fingering in porous media. The porous medium problem prompted the initial research of fingering phenomena. The topic is more than ever now, of the utmost importance in problems of transport in saturated and unsaturated soils, groundwater pollution, etc.\[3\] Moreover, research in the field is now evolving into more complicated geometries, like that of a spherical shape cell.\[4\]

Despite considerable efforts, the fingering phenomenon remains in many aspects uncharted territory. As Tanveer\[5\] described it very lucidly, the phenomenon defies intuition. In the words of Tanveer\[5\] , the disproportionally large influence of small effects, like local inhomogeneities, thin film effects, and the crucial role of surface tension make the theoretical description very difficult. So crucial is the surface tension in the fingering phenomenon, that the problem is ill-posed without it\[3\]. The original work of Saffman and Taylor\[2\] observed fingers in a cell with a less viscous

\[1\]The website http://www.maths.ox.ac.uk/~howison/Hele-Shaw/helearticles.bib , cited in ref.\[3\] carries an extensive (more than 600 papers), up to date list of references on the Hele-Shaw problem.
fluid (oil, air, water), penetrating into a more viscous fluid (oil, glycerine). An initial
instability develops into a finger or several competing fingers. Saffman and Taylor\[2\]
compared the experimentally photographed finger profiles to a theoretical prediction.
The formula was obtained by taking advantage of the analytical properties of the
complex fluid potential (potential and stream function). Reasonable boundary condi-
tions, and neglect of the surface tension, gave good agreement with the experimental
profiles only in the limit of very large capillary number (or alternatively small sur-
face tension). A drawback of this theoretical approach lies in the absence of surface
tension that, causes mathematical ill-posedness of the problem. This fact is reflected
in the absence of a limitation for the asymptotic size of the finger. Contrarily to the
measured profiles, that were always found to be limited by approximately one half of
the cell width, the theoretical expression showed no such lower bound.

McLean and Saffman\[7\] addressed the problem including surface tension. Simi-
larly to Saffman and Taylor they emphasized the role of analyticity. Tanveer\[5\] has
shown that the approach of McLean and Saffman\[7\], is equivalent to an expansion
in a parameter related to the finger width, capillary number and aspect ratio. The
expansion fails at the tail of the finger and higher order terms are needed. The re-
results of McLean and Saffman\[7\] nevertheless yielded profiles that matched very well
the front (nose) part of the finger. Also they found a limit of $\lambda > \frac{1}{2}$ to the finger
width. Their curve for the dependence of the finger width on the capillary number
show the right trend, albeit below the measured values. There appear to be, however,
other branches of solutions (two at least)\[8\], that compete with the branch found by
McLean and Saffman. These branches are presumably unstable as compared to the one found by McLean and Saffman.

More accurate experiments with many different aspect ratios (the ratio of width of the channel with respect to its thickness) showed that the limit is more likely around $\lambda > 0.45$ of the cell width.

Amongst the phenomenological approaches to the problem, the remarkable work of Pitts, took advantage of the observed dependence of the curvature of the finger on the angle, to obtain an analytical scale covariant expression for the finger shape that fitted measured values (by Saffman and Taylor and by Pitts himself) extremely well, especially for finger widths smaller than $\lambda \approx 0.8$. For wider fingers, the profile function was found to miss the measured finger by a small amount. To this day, there is no other stationary finger analytical expression in the literature that fits the data. Pitts derived also an expression for the dependence of the finger’s asymptotic width on the capillary number that includes a phenomenological (fitted) parameter taking into account the fluid films left behind by the passing finger. The fit was also very successful. The theoretical basis for the phenomenological assumptions used by Pitts are nevertheless unclear.

DeGregoria and Schwartz used a boundary integral Cauchy technique to investigate the production and propagation of fingers in the Hele-Shaw cell. Tryggvason and Aref used a vortex in line method to determine the relationship between finger width and flow parameters. Both works, as well as the stationary calculation of McLean and Saffman, give similar results concerning the finger width dependence.
on capillary number. A marked improvement is found in the numerical calculations of Reinelt [13].

The intention of the present work, is to offer yet another approach to the fingering phenomenon. We will derive a nonlinear differential equation for the finger in the Hele-Shaw cell. The equation resembles the Korteweg-de Vries (KdV) soliton equation [14]. It will be quite rewarding to find several features emerging from this method, like the ability to reproduce stationary finger profiles accurately, and the possibility to deal with transient phenomena such as the development and competition between fingers and the treatment of tip splitting. In the present work we will focus on the justification of the nonlinear soliton-like equation, and its stationary solutions.

2 Basic equations

The basic equations governing the fluid flow are described in many works [1, 5]. We here reproduce them for the sake of completeness, and add our view on the problem. The Hele-Shaw creeping flow equation that neglects inertia effects and integrates over the transverse dimension of the cell is [2]

\[ \vec{v} = -\frac{b^2}{12\mu} \vec{\nabla} p \] (1)

where \( \vec{v} \) is the viscous fluid velocity in the plane of the cell, \( b \) is the thickness of the cell, \( \mu \) the viscosity, and \( p \), the pressure in the fluid. Even at this level, there are hidden assumptions that are not clearly valid in all situations. The integration over
the third dimension indeed helps in reducing the problem to a more manageable set of equations. However, the true and real problem is inherently three-dimensional\cite{12}. The existence of films of fluid left behind the finger, puts this aspect in evidence. The finger thickness is not constant comparing tail and nose. It is thinner at the nose and thicker at the tail. We are therefore already reducing dramatically the complexity, and richness, of the phenomenon. The price is paid at the time of comparing the measured finger widths as a function of the parameters of the flow to the theoretical values. As demonstrated by Reinelt\cite{13}, a careful treatment of films of fluid by means of a three (or four) domain splitting of the Hele-Shaw cell, changes dramatically the agreement between theory and experiment.

Originally eq.(1) was supposed to imitate Darcy’s law for the flow in porous media\cite{2}. The phenomenon is nevertheless of a general nature. It belongs to a large class of curved front propagation phenomena spanning such diverse topics as dendritic solidification and flame combustion.

The displacing fluid also obeys a Darcian law as above, however, and due to the lower viscosity, it is usually ignored and the pressure is assumed constant inside the finger. This assumption is also not perfectly justified, but, seems an economical working hypothesis, especially when the displacing fluid is a gas.

The above equation has to be augmented by boundary conditions at the interface. Here too the court is open to a variety of conditions, mainly due to the ignorance of the third dimension. Basically, the main boundary condition is the one of the pressure difference between both sides of the interface. Without corrections of thin
films [13, 13] the condition reads [6]

$$\Delta p = \frac{2}{b} T \cos \gamma + \frac{T}{R}$$

(2)

where, \( T \) is the surface tension parameter of the Young-Laplace formula, \( R \), the radius of curvature in the plane of the cell and \( \gamma \), the contact angle between the finger and the cell wall in the transverse \( b \) direction. Many corrections were proposed to the above equation. Park and Homsy [15] and Reinelt [13], developed a systematic expansion of the effective surface tension parameter as a function of the capillary number \( \text{Ca} = \frac{\mu V}{T} \), where \( V \) is the asymptotic velocity of the fluid far ahead of the finger that is conventionally related by flux conservation (and without considering Poisseuille flow, or else no-slip condition) to the velocity of the finger \( U \) by \( V = U \lambda \).

Here \( \lambda \) is half the width of the finger tail after it has fully developed and other competing fingers dissolved.

The term in (2) depending on \( b \) is usually dropped, because the flow depends on the gradient of the pressure and a constant term does not matter. However, this is not valid either. Two effects spoil the constancy, the variation of contact angle \( \gamma \) and, the change of finger thickness as one proceeds from nose to tail, or vice versa.

It is easy to see that the latter effect is very significant due to the smallness of \( b \) as compared to the width of the cell. A tiny change in the thickness of the finger carries over to a large modification to the pressure. However, the two-dimensional approach can not deal with this matter consistently.

The contribution of the thinning out of the finger when coming from the tail to
the nose adds up to the effect of the surface tension in the plane of the cell. One can then hope that an effective surface tension might somehow lump up the effect. However, this expectation seems to be too optimistic.

Pitts[10] found the need to introduce a phenomenological factor to account for the difference in pressure inside the fluid between tail and nose that reflects the thinning out of the finger. We will not delve here into the issue, but we do recognize that without a reliable treatment of this matter, only the trend of the curve of finger width versus capillary number is of interest. In particular we will follow the same path that ignores the first term in eq.(2) and use instead an effective (unknown) surface tension. For large surface tension, i.e. fingers that cover almost the whole cell the treatment will be correct. The narrower the finger as compared to the channel width, the stronger the effect of films of fluid left behind[13]. Therefore, without a three-dimensional treatment we have to be satisfied with results for wide fingers and small capillary numbers. Some comments on a possible parametrization that might account for the loss of information from the third dimension will be provided later.

Equation (2) reduces then to

\[
\Delta p = \frac{\tilde{T}}{R}
\]  

(3)

with \( \tilde{T} \) an effective surface tension parameter.

To complete the set of equations we assume the fluids to be immiscible. This means that there is no interpenetration. The displaced fluid does not cross the boundary. Its velocity is tangential to the boundary. The boundary is then along a constant
stream-line. For a frame of reference with the finger at rest it becomes

\[ \alpha(\vec{v}) = \tan^{-1}(\eta') \]  \hspace{1cm} (4)

where \( \alpha \) is the angle between the fluid velocity and the axis of propagation of the finger (x axis) and \( \eta' \) is the derivative to the finger (y axis) with respect to the x axis. The above condition is equivalent to the much simpler one of \( \Psi = \text{constant} \) on the finger, where \( \Psi \) is the stream function for the viscous fluid[2, 7], as well as to the condition of continuity of the velocity in the normal direction to the interface for a moving finger. Note also that the treatment of Reinelt[13] being structured in three different regions, with the front inside an intermediate zone, does allow for the crossing of fluid through the front. As it is our aim to provide an alternative differential approach, we will not deal with such complications at this stage.

To the above equations one has to add the continuity equation for an incompressible fluid of constant density, \( \nabla \cdot \vec{v} = 0 \).

The transformation of the Hele-Shaw cell problem to a two-dimensional situation permits the use of analytic functions techniques. Various authors took advantage of this feature either to expand the complex flow function in terms of the complex coordinate \( z = x + i y \) or vice versa[2], or use analyticity conditions to approximate the solution[7, 11]. The potential \( \Phi \) is defined by \( \vec{v} = \nabla \Phi \) and the stream function by \( \vec{\psi} = \nabla_x \nabla \bar{\psi} \). A single component of this vector is needed for flows averaged over the cell thickness, \( \Psi_b \), where \( b \) denotes the direction transverse to the cell. Moreover and in accordance with the Hele-Shaw approach, this function has to be independent
of the transverse coordinate. The Cauchy-Riemann equations for the potential and stream function, imply that the complex potential \( \omega = \Phi + i \Psi_b \), is analytic in \( z \).

Instead of working with the full complex potential and derive restrictions based on analyticity, we here treat the problem by means of the stream function. The stream function is a vector, whose curl determines the fluid flow. The incompressibility of the flow is then guaranteed. Although the velocity potential \( \Phi \) is a harmonic function of the coordinates \( x, y \), as can be readily obtained by applying the incompressibility condition to Darcy’s law of eq.(1), the stream function \( \vec{\Psi} \) does not obey the equation \( \nabla^2 \vec{\Psi} = 0 \) automatically. The stream function approach and the potential one differ. The use of the velocity potential \( \vec{v} = \vec{\nabla} \Phi \) assumes a vanishing vorticity, as the curl of a gradient vanishes. It is our claim, that this assumption is wrong. The stream function approach is superior, as it does not suppose such a neglect of vorticity. The velocity potential \( \Phi \) misses a crucial part of the problem. Therefore, it should not be considered as a trustworthy tool to find the finger equation of motion. In this respect the use of the full potential \( \omega = \Phi + i \Psi_b \), is also not valid. Hence, the analyticity conditions derived from it implying a harmonic equation for \( \Psi \) also, do not reflect the full reality of the problem.

For example, if we consider \( \Psi_b \), that depends on \( x \) and \( y \) only, we have \( \vec{\nabla} \cdot \vec{\Psi} = 0 \), because \( \Psi_b \) is the \( z \) component of the stream vector and it does not depend on \( z \). \( \Psi_b \) satisfies the incompressibility condition, by definition of the velocities. Using \( \vec{\nabla} \times \vec{\nabla} \times \vec{\Psi} = \vec{\nabla} \vec{\nabla} \cdot \vec{\Psi} - \nabla^2 \vec{\Psi} \), it is clear that, \( \Psi_b \) will obey the harmonic equation if the vorticity, \( \vec{\omega} = \frac{1}{2} \vec{\nabla} \times \vec{\nabla} \times \vec{\Psi} \), vanishes. The vorticity may be ignored, except at
the interface. However, it is there that we need the most the value of the stream function. Therefore we will not use the harmonic equation in the present work.

The boundary conditions alone will suffice for the determination of the stream function on the front, and, in turn, allow us to find the evolution equations for the finger. However, we will not be able to find the stream function explicitly everywhere in the fluid.

3 Stream function approach

The stream function will be determined by resorting to a perturbation expansion. The expanded stream function will be made to obey the equations of the flow as well as the boundary conditions. The outcome will be a stream function that is known solely at the boundary. This method will enable us to deduce the nonlinear differential equations.

We first renormalize the axes to $x \rightarrow \tilde{x} = \frac{x}{w/2}$, $y \rightarrow \tilde{y} = \frac{y}{w/2}$. As a consequence of this rescaling, the finger transverse dimension becomes less than one, and serves as an expansion parameter.

Two boundary conditions constrain the stream field. The first one is the impenetrability of the cell wall and, the second the immiscibility of the fluids. The latter amounts to considering the interface between the fluids as a free surface. Dropping the suffix $b$ and the tilde, we find
\[ \frac{\partial \Psi}{\partial x} = 0, \text{ at } y = \pm 1 \] (5)

and

\[ \frac{\partial \eta}{\partial t} + \frac{\partial \Psi}{\partial y} \frac{\partial \eta}{\partial x} = -\frac{\partial \Psi}{\partial x}, \text{ at } y = \eta \] (6)

where \( \eta(x, t) \) denotes the interface curve. Eq.(6) may be easily derived by writing the differential of \( \eta \) as

\[ d\eta = \frac{\partial \eta}{\partial t} \, dt + \frac{\partial \eta}{\partial x} \, dx \]

and, using the definitions

\[ v_x = \frac{dx}{dt} = \frac{\partial \Psi}{\partial y} \]

\[ v_y = \frac{dy}{dt} = -\frac{\partial \Psi}{\partial x} \] (7)

For the stationary finger, in its rest frame we have \( \frac{\partial \eta}{\partial t} = 0 \).

With this substitution, eq.(6) becomes

\[ \frac{\partial \Psi}{\partial y} \frac{\partial \eta}{\partial x} = -\frac{\partial \Psi}{\partial x}, \text{ at } y = \eta \] (8)

The differential of \( \Psi \)

\[ d\Psi = \frac{\partial \Psi}{\partial y} \, dy + \frac{\partial \Psi}{\partial x} \, dx, \text{ together with eq.(8) determine the unique solution for the stream function in the stationary case to be} \]

\[ \Psi_0 = \text{constant} \] (9)
Without loss of generality, this constant may be taken to be equal to zero, because
the stream function enters the calculations only through its derivatives.

For a finger in motion along the \( x \) axis, even with a time dependent velocity, but
still in the creeping flow approximation that neglects kinetic terms, we can write the
solution to be formally

\[
\Psi(x, y, t) = \Psi_0(x, y) - \int dx \frac{\partial \eta}{\partial t}
\]  

Eq.(10) solves eq.(6), with \( \Psi_0 \) given by the appropriate solution of eq.(8).

At the front, and only there, we can rewrite the equation as

\[
\Psi(t) = \Psi_0 - \int dy \frac{\partial \eta}{\partial t} / \frac{\partial \eta}{\partial x}
\]  

Where \( \eta(x, t) \) is understood as a function of \( x \) and \( t \) only. In the general case, \( \Psi \)
is no longer a constant on the curved front.

Consider the stream function in the finger rest frame. The finger is traveling in
the positive \( x \) direction at constant speed \( U \). Moreover, at long distances ahead of
the finger the fluid is assumed to flow with a constant velocity \( V \). Usually, the no-slip
condition cannot be imposed in the Hele-Shaw cell and consequently the velocity \( V \)
is considered to be the same all over the cell including at the lateral boundary. This
could be a good assumption provided the viscosity of the fluid is small, or else the
lateral edge is not considered as the boundary. However, for large capillary numbers,
or large viscosities, this procedure seems unreasonable. Boundary layers next to
surfaces are patent in common phenomena even for moderately viscous fluids. We opt for a more conservative approach and demand the stream line next to the lateral edge to carry a velocity $V$ that is not the asymptotic velocity of the fluid ahead of the finger. It is a parameter that depends on the flow properties. For capillary number tending to infinity it has to be equal to zero. Large capillary numbers may be implemented by using very viscous fluids. For such fluids, the no-slip condition is a must. Therefore in the limit of infinite capillary number $V=0$. In the opposite limit, i.e. no viscosity at all (finger width going to the full cell width) it is equal to the finger velocity. Hence we can assert $0 < V < U$. Reinelt’s equations\[13\] carry an extra term at the lateral edge that allows the velocity $V$ to be variable.

To our knowledge there have not been any experimental investigations of the fluid flow as a function of distance from the cell edge. In light of the previous discussion, this measurement could be of relevance.

Expanding the only component of the stream function, with a separable ansatz, antisymmetric in $y$, so that the finger is symmetric, and in the finger rest frame, we find (dropping from now on the tilde on $y$, i.e. we work with rescaled coordinates)

$$
\Psi(x, y, t) = y \left( V(x, t) - U(x, t) + A(x, t) + B(x, t) y^2 + C(x, t) y^4 + \ldots \right) \quad (12)
$$

with $A$, $B$, $C$ unknown functions. We will find below that the expansion will effectively be in terms of both $y$ and $1 - y^2$. It is then expected to be a reasonable approximation in all the cell gap, except perhaps around $y \approx 0.5$
Eq. (5), as well as the constancy of the fluid velocities, for the stationary finger in its rest frame imply

\[
\begin{align*}
    v_y &= -\frac{\partial \Psi}{\partial x} = 0, \text{ at } y = \pm 1 \\
    v_x &= \frac{\partial \Psi}{\partial y} = V - U, \text{ at } y = \pm 1
\end{align*}
\]  

(13)

Inserting eq. (13) in eq. (12), we are able to fix the functions \( B \) and \( C \) in terms of \( A \). Explicitly

\[
\begin{align*}
    0 &= A + 3B + 5C \\
    0 &= A' + B' + C'
\end{align*}
\]  

(14)

primes denoting derivatives with respect to \( x \).

Recalling that constants are irrelevant for the stream function, the unique solution to this order becomes

\[
\begin{align*}
    B &= -2A \\
    C &= A
\end{align*}
\]  

(15)

The stream function of eq. (12) with the conditions of eqs. (14, 13) reads

\[
\Psi(x, y) = (V - U) y + A(x) y (1 - y^2)^2 + ...
\]  

(16)

Eq. (16) demonstrates that the expansion is a dual one in \( y \) and, \( 1 - y^2 \), as stated above. In eq. (16), we have neglected higher order terms and also ignored possible
contributions to the stream function from the dependence of the velocities $V, U$ on distance. This is clearly not a good approximation for large capillary number as well as for the nonstationary problem.

If we expand the stream function to the next order

$$
\Psi(x, y, t) = y (V(x, t) - U(x, t) + A(x, t) + B(x, t) y^2 + C(x, t) y^4 + D(x, t) y^6 + ...) \quad (17)
$$

we need an extra condition to determine the function $D$. One possible constraint to fix this function, may arise from a model of the vortex structure on the curved front. The simplest one - although not the most realistic one-, consists in assuming a constant vorticity on the interface, with the vorticity defined by $\omega = \frac{1}{2} \nabla x \nabla x \Psi$. This procedure introduces a new parameter, the vorticity value $\omega$.

In order to stay as low as possible in the number of free parameters, we prefer to remain within the expansion to $O(y^6)$ with $y < 1$. To this order the two conditions of eq. (14), are sufficient. The constraints we take advantage of, depend on the stream function and its first derivative. Higher order expansions as in eq. (17) and further constraints, as the vorticity, amount to an expansion in second and higher order derivatives of the stream function. Due to the fact that, the even polynomials in $y$ span a complete set in the rescaled variable $0 < y < 1$ for the stationary finger case, this expansion presumably converges. The sixth order term is expected to be smaller than the fourth order one by factors of the form $\lambda^2$, with $\lambda < 1$. Although the exact dependence in $\lambda^2$ is not known at this stage, it appears a sound working hypothesis to truncate the expansion to the sixth order and work with the expression
of eq. (16). The quality of the finger shapes obtained below gives us more confidence in this hypothesis.

4 Stationary and time-dependent nonlinear differential equations for the finger

On the curved front, in the finger rest frame, with the finger nose located at the center of coordinates, the condition $\Psi = \text{constant}$ of eq. (9) becomes

$$\Psi = 0, \quad \text{at} \; y = \eta \quad (18)$$

Inserting eq. (18) in eq. (16), $A(x)$ is determined at the front to be

$$A(x) = \frac{U - V}{(1 - \eta^2)^2}, \; \text{at} \; y = \eta(x) \quad (19)$$

Using eqs. (7, 16, 19), the velocity of the fluid at the static finger, the only place where we can determine $A$ in closed form, becomes

$$v_{x,\text{static}} = (V - U) + A(x) \left(1 - \eta^2\right) \left(1 - 5 \eta^2\right)$$

$$v_{y,\text{static}} = -\frac{\partial A(x)}{\partial x} \eta \left(1 - \eta^2\right)^2$$

$$A(x) = \frac{U - V}{(1 - \eta^2)^2} \quad (20)$$
with $\eta$, a function of $x$. The fluid still flows around the finger in its rest frame, hence the name static.

The rescaled curvature in eq. (3) is

$$\frac{1}{R} = \frac{\eta''}{(1 + \eta'^2)^{\frac{3}{2}}}$$  \hspace{1cm} (21)

Where primes denote derivatives with respect to $x$.

The sign of the curvature is the appropriate one. This can be seen by using eq. (1). The left hand side of the equation is the positive velocity of the finger, chosen here to move from left to right, therefore the curvature has to decrease along $x$. In the upper part of the finger the curvature is negative and becomes more so as we proceed along $x$. Therefore the right hand side of eq. (1) is positive with the positive sign in eq. (21).

Eq. (21) is the standard expression for the curvature, and, it is easily derivable from the definition of the arclength and the corresponding angle for differential increments.

Equations (1, 3, 21) after rescaling imply

$$v_x = -\frac{\bar{T} b^2}{3 w^2 \mu \partial x} \left[ \frac{\eta''}{(1 + \eta'^2)^{\frac{3}{2}}} \right]$$  \hspace{1cm} (22)

The stationary finger equation is obtained now by evaluating the derivative in eq. (22), with

$$v_{x, static} = (V - U) \frac{4 \eta^2}{1 - \eta^2}$$  \hspace{1cm} (23)
from eq.(20); and transforming back to the rest frame of the cell, with the finger in motion, \( v_x = v_{x,\text{static}} + U \).

After some straightforward algebraic manipulations we find

\[
0 = \eta_{xxx} - 3 \frac{\eta_x^2 \eta_x}{1 + \eta_x^2} + (1 + \eta_x^2)^{\frac{3}{2}} \frac{W(\eta)}{4B}
\]

\[
W(\eta) = \frac{4 \epsilon \eta^2 + 1 - 5 \eta^2}{1 - \eta^2}
\] (24)

where the suffix indicates differentiation with respect to \( x \).

In terms of the arclength measured from the tail of the finger, the equation becomes

\[
0 = \eta_{sss} + \frac{\eta_s^2 \eta_s}{1 - \eta_s^2} + (1 - \eta_s^2) \frac{W(\eta)}{4B}
\]

(25)

In eqs.(24,25), \( \epsilon = \frac{V}{U} \). As was found in previous works\[7,9\] the only parameter entering the equation is \( \frac{1}{B} = \frac{12\mu U w^2}{T b^2} \) with \( w \) being the width of the Hele-Shaw cell and all the variables are scaled to units of half this length. The existence of a single parameter is clearly insufficient as demonstrated by Reinelt[13]. Customarily \( \epsilon \) of eqs.(24,25) is equated to \( \lambda \), half the finger width. We here take it as a parameter as explained in section (3). Asymptotically far back at the tail we must have \( W(\eta) = 0 \), or, \( \epsilon = \frac{5 \lambda^2 - 1}{4 \lambda^2} \). The parameter \( \epsilon \), is then determined by the solutions to the equations and is not a phenomenological parameter.
The potential $W$ guides the propagation of the finger. The equation is of third order in the spatial derivatives as the Korteweg-deVries equation[14]. Third order differential equations for interface propagation are well known in the literature. Some notorious examples are: Landau and Levich[16], Bretherton[17], and Park and Homsy[15].

The third order equation (24) can be transformed to a more manageable second order one in terms of the angle tangent to the curve, with $\eta$ obtained by integration.

For $ds$ starting at the tail, where $\theta \approx \pi$, $ds \cos(\theta) = -dx$.

Equation (25) now reads

$$0 = \frac{\partial^2 \theta}{\partial s^2} - \cos(\theta) \frac{W(\eta)}{4B}$$

$$\eta = \lambda - \int ds \sin(\theta)$$

(26)

With $W(\eta)$ defined in eq.(24).

The time dependent equation is obtained by using eq.(11), and recalling the definition of $v_x$ in terms of the stream function of eq.(7). For the velocity in the $x$ direction, we now have

$$v_x = -\frac{\eta}{\eta_x} + U + v_{x,static}$$

(27)

Rescaling as before the length coordinates $\eta$, and $x$ by $\frac{w}{2}$, and the time by
\( t \to \frac{t \cdot U}{w/2} \), and inserting eq.(27) in eq.(22), we obtain the time dependent differential equation

\[
0 = \eta_{xxx} - 3 \frac{\eta_{xx} \eta_x}{1 + \eta_x^2} + (1 + \eta_x^2) \frac{1}{4B} \left[ W(\eta) - \frac{\eta_t}{\eta_x} \right] \tag{28}
\]

In terms of the arclength eq.(28) becomes

\[
0 = \eta_{sss} + \frac{\eta_{ss} \eta_s}{1 - \eta_s^2} + (1 - \eta_s^2) \frac{1}{4B} \left[ W(\eta) - \sqrt{1 - \eta_s^2} \frac{\eta_t}{\eta_s} \right] \tag{29}
\]

While for the angle tangent to the front, the time-dependent equation (28) reads

\[
0 = \frac{\partial^2 \theta}{\partial s^2} - \cos(\theta) \frac{1}{4B} \left[ W(\eta) + \cos(\theta) \frac{\partial \theta}{\partial \theta} \right] \frac{\partial}{\partial s} \left( \frac{\partial \theta}{\partial s} \right)
\]

\[
\eta = \lambda - \int ds \sin(\theta) \tag{30}
\]

To gain confidence in the equations, we proceed to show that the time dependent equation (28) gives the correct dispersion relation for the perturbation of a planar front. We will find that the instability parameter in time, here called \( \sigma \), will obey a dispersion relation of the form \( \sigma = |\alpha| (1 - \lambda \alpha^2) \), with \( \alpha \) related to the wavenumber of the perturbation and \( \lambda \) a parameter proportional to the surface tension. The velocity, or pressure gradient, destabilizes and the surface tension stabilizes. We will connect to Chuoke et al.\[13\] in the derivation.

We will consider a perturbation protruding from a moving front with velocity U. Eq.(28) is firstly rewritten around \( y \approx 0 \)
\begin{align*}
0 &= \eta_{xxx} - 3 \frac{\eta_{xx}^2 \eta_x}{1 + \eta_x^2} + (1 + \eta_x^2)^{\frac{3}{2}} \frac{1}{4B} \left(1 - \frac{\eta_t}{\eta_x}\right) \quad (31)
\end{align*}

This equation is equivalent to

\begin{align*}
\frac{\partial}{\partial x} \chi &= 0 \\
\chi &= 4B \frac{\eta_{xx}}{1 + \eta_x^2}^{\frac{3}{2}} + x - \int dx \frac{\eta_t}{\eta_x} \quad (32)
\end{align*}

Only around \( y \approx 0 \), the equation can be integrated once analytically.

Now we need an ansatz for the solution. Following Chuoke et al.\cite{19} we use the expression

\begin{align*}
x(y, t) &= \beta(\alpha) e^\phi \\
\phi &= \alpha y + \sigma t \quad (33)
\end{align*}

\( \beta \) is the amplitude of the perturbation, that is taken as dependent on \( \alpha \) due to the nonlinearity of the equation, as Chuoke et al.\cite{19} assumed. (See below their equation \[4\]).

The integral in eq.\((32)\) may be rewritten as

\begin{align*}
I = \int dx \frac{\eta_t}{\eta_x} = \int x_y x_t \, dy \quad \text{at} \quad y = \eta \quad (34)
\end{align*}

Evaluating the integral \((34)\) with eq.\((33)\) we find
\[ I = \frac{\sigma}{2} \beta^2 e^{2\sigma t} (e^{2\alpha y} - 1) \]
\[ \approx \sigma \beta^2 \alpha y \quad (35) \]

The curvature may be calculated now by using

\[
\frac{\eta_{xx}}{(1 + \eta_x^2)^{\frac{3}{2}}} = -\frac{x_{yy}}{(1 + x_y^2)^{\frac{3}{2}}} \\
\approx -x_{yy} \\
= -\alpha^2 x \quad (36)
\]

Using eq.(36), the integral of eq.(35), and, \(x\) of eq.(33), at \(t = 0\) to \(O(y^2)\), as we are assuming a perturbative expansion, \(\chi\) of eq.(32), becomes

\[
\chi_0 \approx (-4B \alpha^2 \beta + \beta) (1 + \alpha y) - \sigma \beta^2 \alpha y \quad (37)
\]

The constant piece of \(\chi_0\) in eq.(37), \(-4B \alpha^2 \beta + \beta\), is irrelevant. This is the reason we do not need to specify the lower bound of the integral in eq.(34). We took it as \(y = 0\). Any other choice will merely change the unimportant constant in \(\chi\).

To satisfy eq.(32), to the lowest order in \(y\), eq.(37) has to obey the algebraic condition

\[
-4B \alpha^3 \beta + \beta \alpha - \sigma \beta^2 \alpha = 0 \quad (38)
\]
The third term in eq.(38) is quadratic in \( \beta \). Independence from the initial perturbation amplitude, requires \( \beta \) to be a function of \( \alpha \), as assumed in eq.(33). We can now determine this dependence to be \( \beta \alpha = k \), with \( k \), a constant. \( \beta \) is a positive number, therefore, \( k \) has to be positive for positive \( \alpha \), or negative for negative \( \alpha \).

Equivalently, the proportionality above can be written as

\[
\beta = \frac{|k|}{|\alpha|}. \tag{39}
\]

Inserting eq.(39) into eq.(38) we find

\[
-4B\alpha^3\beta + \beta\alpha - \sigma\beta\alpha \frac{|k|}{|\alpha|} = 0 \tag{40}
\]

Finally, redefining \(|\alpha| \rightarrow \frac{|\alpha|}{|k|}\), \( \lambda = 4Bk^2 \) we find the expected dispersion relation

\[
\sigma = |\alpha| \left( 1 - \lambda \alpha^2 \right) \tag{41}
\]

The time-dependent equation was developed on the basis of an antisymmetric stream function, eq.(16), the solutions, may, nevertheless, lack any predetermined symmetry. This phenomenon may be related to the physical concept of hidden symmetries and spontaneous symmetry breaking\cite{20}. Although the equations are symmetric under a certain transformation, the solutions can break the symmetry. Boundary conditions and initial conditions are key factors in determining the symmetries
of the solutions. Such is the case of an initial asymmetric perturbation to the front. On the experimental side, a symmetric finger develops only asymptotically at long times. The transient behavior is patently asymmetric.

5 Stationary finger solutions

Let us analyze the predicted finger half-widths $\lambda$, as a function of the parameter $B$. As discussed in section (4), we expect the parameter $\epsilon$ to be equal to one in the limit of $B \to \infty$, that is possible to implement with a displaced fluid of zero viscosity, but a density that still assures the instability and creation of a finger. In this limit the potential determines the finger half-width (location at which the derivatives of $\eta$ vanish) to be $\lambda = 1$, as expected. On the other end of $B \to 0$, or infinite viscosity limit, we argued that $\epsilon = 0$ in the spirit of the no-slip condition for viscous fluids. In this limit $\lambda = \frac{1}{\sqrt{n}} \approx 0.447$. Looking at the experimental work of Tabeling et al.\cite{9}, especially figure 8, it is seen that for $\frac{1}{B} \to \infty$ the finger width for various aspect ratios tends to approximately $\lambda = 0.45$.

Therefore we can be quite confident on the potential $W$. On the other hand, the dynamics as described by the third order differential equation linear in the parameter $B$ is certainly only approximately correct. In the same manner as the KdV equation is only an approximate equation derived by expanding to a certain order in the derivatives. The need of higher order terms in the curvature is apparent from the work of Brower et al.\cite{22}. As stressed in the introduction, a full three dimensional account of
the fluid flow is necessary in order to take care of the thinning out of the finger in the transverse direction. Alternatively we could view the equation as essentially correct, with the dependence on $B$ nonlinear and unknown. Consequently, we can trust the equation in the limit of small $\frac{1}{B}$.

The numerical solution was started from positive $y = \lambda$ and followed around the finger to $y = -\lambda$. We employed a fourth order Runge-Kutta algorithm. We opted for the mixed integrodifferential approach of eq.(26), that proved easier to handle numerically. The equation can be solved without this change of variables also with the full third order equation \([24]\).

For fixed $B$ we varied manually the value of the asymptotic half-width (thereby fixing $\epsilon$), the slope $\theta'$ and the value of $\theta$ less than $\theta = 180^\circ$. The equation is integrated both backwards and forwards and the parameters are fine tuned until the shape of the finger looks as symmetric as possible $x(-\lambda) \approx x(\lambda)$, and the nose tip shows up at exactly $90^\circ$. We inforced the symmetry condition to an accuracy of around 1%. Moreover, only solutions where the angle decreases smoothly from the tail forwards were accepted. We rejected meandering (even slightly so) solutions that could appear as a many prongued finger.

This method of tedious search proved to be reliable. Other algorithms, such as a shooting method, etc., ended up in unreasonable, asymmetric solutions.

The literature does not abound in theoretically calculated profiles of fingers. We could hardly find any such solutions besides those of McLean and Saffman\([7]\), the phenomenological solutions of Pitts\([10]\), and, the original ones of Saffman and Taylor\([2]\).
Figure 1: Rescaled finger profile for $\lambda = 0.93$ at $\frac{1}{B} = 12$. Numerical solution, solid line, analytical solution of Pitts [10], dashed line.

The latter obtained without the inclusion of surface tension effects. It appears then, that it is not an easy task to predict properly finger shapes.

We proceed to show 3 profiles found for finger half-widths $\lambda = 0.93, 0.82, 0.66$. They are compared with the analytical solution of Pitts[10] that fits the data extremely well for finger half-widths smaller than around $\lambda \approx 0.8$ and lies slightly below the data points for higher values of $\lambda$. The figures are depicted for rescaled coordinates in terms of $\lambda$ as has became customary[7, 10].
Figure 2: Rescaled finger profile for $\lambda = 0.82$ at $\frac{1}{B} = 20$. Numerical solution, solid line, analytical solution of Pitts [10], dashed line.
From figures 1-3 it is clear that the solutions match the phenomenological solutions of Pitts $\cos\left(\frac{\pi}{2} \frac{y}{\lambda}\right)e^{-\frac{\pi}{2}x}\lambda = 1$, and are even superior to it for large values of $\lambda$. The dependence of the asymptotic finger half-width $\lambda$ on $B$, is shown in figure 4. The present results improve upon the calculations of McLean and Saffman\cite{7} in the range of $B$ for which we were able to find reliable solutions, but still lie below the measured values that were obtained from averaging the results of Tabeling et al.\cite{9}, for various aspect ratios. Note however, that the results do not include finite gap corrections that lift the curve upwards, whereas the results of McLean and Saffman\cite{7}, shown in the figure do adjust for such corrections. Without these corrections the latter results fall much below the depicted curve. As stressed above, finite film effects are extremely important, as evidenced from the calculations made Reinelt\cite{13}.

We found numerically stationary finger profiles for $B < 110$. Beyond this value, the large contributions introduced by the potential in eq. (26), render the integration from the tail unstable. This is not unexpected in light of the arguments provided in section (2). The curve of $\lambda$ versus $\frac{1}{B}$ for large values of this parameter, can be fitted by a function of the form $B e^{k B}$, with $k$ a parameter. In future investigations higher order terms in the potential will be added, in the search of such a functional dependence in a self-consistent stream function approach.

Several attempts were made in the present study, to improve the algorithm by matching solutions run from the tail and from the nose. However, starting from the nose is very problematic, because of the singularity in the slope, that no matter how, creeps up in any scheme, even after transforming to better behaved variables.
Figure 3: Rescaled finger profile for $\lambda = 0.66$ at $\frac{1}{B} = 48$. Numerical solution, solid line, analytical solution of Pitts [10], dashed line.
Figure 4: Asymptotic half-width of the finger, $\lambda$ as a function of $\frac{1}{B}$. Present work, full line, theoretical results of McLean and Saffman [7] corrected for finite film effects, dashed line, and, experimental results of Tabeling et al. [9], dash-dot line with squares.
The equation is extremely sensitive to the slope and initial angle that must be close to 180°. Fortunately, it was found that for a fixed value of B, there corresponds a single value of λ; a unique finger solution. The selection of the finger half-width is univocous, with all the numerical methods employed. However, we cannot rule out completely the existence of other unstable branches not accessed by the numerical method, as found by Vanden-Broek\cite{8} for the McLean and Saffman\cite{7} set of integrodifferential equations.

In the course of the numerical investigation we found solutions that correspond to various multiple finger-like structures. These structures originate from specific boundary conditions at \( y = \lambda \). The existence of such complicated patterns is quite encouraging, especially if we want to proceed further to the production, development, competition and splitting between fingers with the time dependent equation displayed above\footnote{Beautiful martingales similar to the twisting and whirling of plant branches were also found. See remarks on symmetry below eq.\,(28)}.

### 6 Conclusions

The contribution of the present work consists in the development of nonlinear differential equations for the Hele-Shaw Saffman-Taylor fingering problem. The approach is based upon the treatment of the stream function as a separable potential without any free phenomenological, adjustable parameters. The numerical stationary profiles, match extremely well the experimental ones, that are, in turn, quite accurately re-
produced by the phenomenological solution of Pitts [10]. The predicted dependence of the asymptotic finger half-width $\lambda$ on the physical parameters of the problem, is also satisfactory.

For the stationary case, the finger nonlinear equations are eqs. (24), and (25) the latter expressed in terms of the arclength, while eq. (26) is the corresponding integrodifferential equation, of second order, for the angle as a function of arclength. These are nonlinear and nonlocal equations.

The time-dependent equation is of the third order in space, and first order in time. Eqs. (28, 29), corresponds to $\eta$ in terms of $x$ and the arclength respectively, whereas eq. (30) is its integrodifferential equation for the angle as a function of the arclength also.

The equations found here, resemble the Korteweg-deVries (KdV) equation, that has been long recognized as a solitary wave equation [23]. This equation was obtained for the nonlinear propagation of shallow water waves in a channel, and it is of the same order in space and time, as our equations.

The KdV equation reads

$$c_0 \left( \frac{h^2}{6} - \frac{2}{\rho g} T h \right) \eta_{xxx} + \frac{3 c_0}{2 h} \eta \eta_x + \eta_x (c_0 + \frac{\eta_t}{\eta_x}) = 0$$

(42)

where $h$, is the depth of the channel in which the soliton propagates. $c_0 = \sqrt{g h}$ is the speed of linear propagation of waves in the channel, $g$ is the acceleration of gravity and $\eta$ denotes the soliton profile propagating in the channel along the $x$
axis with velocity $U = -\frac{\eta}{\eta_x}$, in a fluid with density $\rho$ and surface tension $T$. This is a third order equation nonlinear in $\eta$. The specific nonlinear term, and, the potential found in eq. (24), differ from the quadratic nonlinear term of the KdV equation (12). The nonlinear terms in eq. (24), and eq. (12) arise from the boundary conditions at the interface between the front and the displaced fluid as well as the surface tension pressure drop. Eq. (24) however, carries also the information of the lateral flow constraint that is absent in eq. (12). Moreover, eq. (24) is based upon the creeping flow assumption of eq. (1), while eq. (12) does not assume such a restricted version of the Navier-Stokes equation. Therefore the nonlinear terms differ.

Nevertheless, the equations are similar. Both are of third order in the shape variable, both are nonlinear, and are linear in time. In the derivation of eq. (24) we adhered to the separable assumption expansion used by Korteweg and deVries (23) and their techniques. It is then, perhaps, not surprising, that we found a similar equation - despite the differences in context-, as both equations describe the propagation of stationary and stable shape fronts in a fluid medium.

Links between soliton equations and the Hele-Shaw finger are already hinted in the work of Kadanoff (18), in which varieties of the so-called Harry-Dym equations were found to be in close relationship to finger development. The above equations belong to a broad class of flux-like partial differential equations, that are widely used in the literature. In the context of Darcian flow in the Hele-Shaw cell, Goldstein et al. (21) study instabilities and singularities by means of an equation of this type. The main difference between the equations we derived here, and the one of Goldstein et
al.\cite{21}, is, again, the appearance of a potential term $W(\eta)$ in eqs.\cite{28,30}.

The present method does not use any phenomenologically observed characteristic. The model results follow solely from a treatment in terms of the stream function, and the relaxation of the condition of $\lambda = \frac{U}{V}$, that amounts to an unphysical restriction on the Hele-Shaw cell walls especially for large viscosities, or small capillary numbers. A flat tail requires $\epsilon = \frac{5 \lambda^2 - 1}{4 \lambda^2}$. $\epsilon$, is then determined by $\lambda$, that is in turn determined by the existence of solutions. We assumed a constant, capillary number dependent $\epsilon$. Further progress requires a model for the dependence of $\epsilon$ on the capillary number.

The predicted dependence of finger half-width $\lambda$, on capillary number through the parameter $B$ of figure 4, achieves more than the integrodifferential approach of McLean and Saffman\cite{7} even without corrections of finite films left behind the propagating finger. The partial success in dealing with the problem, suggests that front propagation phenomena\cite{11} might have a corresponding soliton-like guiding equation, at least for restricted geometries.

For the Hele-Shaw cell, the cell provides the potential for the propagation of the finger, while the pressure difference determined by the surface tension provides the dynamics. In dendritic growth, the temperature gradient provides the dynamical guidance and the cell and substrates, the potential.

The obvious shortcoming of the current method, and other existing methods, is the neglect of the dynamics along the third dimension. One possible way to improve the equations would be to include vorticity. Even if in the bulk there is no vorticity, on the front there could be a sizable amount of it without spoiling the bulk properties.
of the fluid in the hydrodynamic regime, as in superfluid Helium[24]. It seems then, that a treatment including vorticity may be more appropriate.

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