THE LUSTERNIK-SCHNIRELMANN-CATEGORY AND
THE FUNDAMENTAL GROUP

ALEXANDER N. DRANISHNIKOV

ABSTRACT. We prove that
\[ \text{cat}_{LS} X \leq \text{cd}(\pi_1(X)) + \left\lceil \frac{\dim X - 1}{2} \right\rceil \]
for every CW complex \( X \) where \( \text{cd}(\pi_1(X)) \) denotes the cohomological dimension of the fundamental group of \( X \).

1. INTRODUCTION

The Lusternik-Schnirelmann category (briefly LS-category) \( \text{cat}_{LS} X \) of a topological space \( X \) is the minimal number \( n \) such that there is an open cover \( \{ U_0, \ldots, U_n \} \) of \( X \) by \( n + 1 \) contractible in \( X \) sets. We note that sets \( U_i \) are not necessarily contractible. The Lusternik-Schnirelmann category proved to be useful in different areas of mathematics. In particular, the classical theorem of Lusternik and Schnirelmann [CLOT] proven in the 30s states that \( \text{cat}_{LS} M \) gives a low bound for critical points on \( M \) of any smooth not necessarily Morse function. For nice spaces as CW complexes it is an easy observation that \( \text{cat}_{LS} X \leq \dim X \). In the 50s Whitehead proved that for simply connected CW complexes \( \text{cat}_{LS} X \leq \dim X/2 \) [CLOT]. In a presence of the fundamental group as small as \( \mathbb{Z}_2 \) the Lusternik-Schnirelmann category can be equal the dimension. An example is \( \mathbb{R}P^n \).

Nevertheless, Yu. Rudyak conjectured that in the case of free fundamental group there should be Whitehead’s type inequality at least for closed manifolds. There were partial results towards Rudyak’s conjecture [DKR],[St] until it was settled in [Dr]. Also it was shown in [Dr] that Whitehead’s type estimate holds for complexes with the fundamental group of cohomological dimension \( \leq 2 \). We recall that free groups (and only them [Sta],[Swan]) have cohomological dimension one. In this paper we prove Whitehead’s type inequality for complexes with fundamental groups having finite cohomological dimension.

We conclude the introductory part by definitions and statements from [Dr] which are used in this paper. Let \( \mathcal{U} = \{ U_\alpha \}_{\alpha \in A} \) be a family
of sets in a topological space $X$. Formally, it is a function $U : A \rightarrow 2^X \setminus \{\emptyset\}$ form the index set to the set of nonempty subsets of $X$. The sets $U_\alpha$ in the family $U$ will be called elements of $U$. The multiplicity of $U$ (or the order) at a point $x \in X$, denoted $\text{Ord}_x U$, is the number of elements of $U$ that contain $x$. The multiplicity of $U$ is defined as $\text{Ord} U = \sup_{x \in X} \text{Ord}_x U$. A family $U$ is a cover of $X$ if $\text{Ord}_x U \neq 0$ for all $x$. A cover $U$ is a refinement of another cover $C$ ($U$ refines $C$) if for every $U \in U$ there exists $C \in C$ such that $U \subseteq C$. We recall that the covering dimension of a topological space $X$ does not exceed $n$, $\dim X \leq n$, if for every open cover $C$ of $X$ there is an open refinement $U$ with $\text{Ord} U \leq n + 1$.

**Definition 1.1.** A family $U$ of subsets of $X$ is called a $k$-cover, $k \in \mathbb{N}$ if every subfamily of $k$ elements forms a cover of $X$.

The following is obvious (see [Dr]).

**Proposition 1.2.** A family $U$ that consists of $m$ subsets of $X$ is an $(n + 1)$-cover of $X$ if and only if $\text{Ord}_x U \geq m - n$ for all $x \in X$.

The following theorem can be found in [Os].

**Theorem 1.3 (Kolmogorov-Ostrand).** A metric space $X$ is of dimension $\leq n$ if and only if for each open cover $C$ of $X$ and each integer $m \geq n + 1$, there exist $m$ disjoint families of open sets $U_1, \ldots, U_m$ such that their union $\bigcup U_i$ is an $(n + 1)$-cover of $X$ and it refines $C$.

Let $f : X \rightarrow Y$ be a map and let $X' \subset X$. A set $U \subset X$ is fiber-wise contractible to $X'$ if there is a homotopy $H : U \times [0, 1] \rightarrow X$ such that $H(x, 0) = x$, $H(U \times \{1\}) \subset X'$, and $f(H(x, t)) = f(x)$ for all $x \in U$.

We refer to [Dr] for the proof of the following

**Theorem 1.4.** Let $U = \{U_0, \ldots, U_n\}$ be an open cover of a normal topological space $X$. Then for any $m = n, n + 1, \ldots, \infty$ there is an open $(n + 1)$-cover of $X$, $U_m = \{U_0, \ldots, U_m\}$ extending $U$ such that for $k > n$, $U_k = \bigcup_{i=0}^{n} V_i$ is a disjoint union with $V_i \subset U_i$.

**Corollary 1.5.** Let $f : X \rightarrow Y$ be a continuous map of a normal topological space and let $U = \{U_0, \ldots, U_n\}$ be an open cover of $X$ by sets fiber-wise contractible to $X'$. Then for any $m = n, n + 1, \ldots, \infty$ there is an open $(n + 1)$-cover of $X$, $U_m = \{U_0, \ldots, U_m\}$ by sets fiber-wise contractible to $X'$.

2. Generalization of Ganea’s fibrations

Let $A \subset Z$ be a closed subset. By $P_A Z$ we denote the space of paths issued from $A$, i.e. the space of continuous maps $\phi : [0, 1] \rightarrow Z$
with \( \phi(0) \in A \). The we define a map \( p_A : P_A Z \to Z \) by the formula \( p(\phi) = \phi(1) \). Clearly, \( p_A \) is a Hurewicz fibration. Let \( F \) be its fiber.

**Proposition 2.1.** There is a Hurewicz fibration \( \pi : F \to A \) with the fiber \( \Omega Z \), the loop space on \( Z \).

**Proof.** The map \( q' : P_A Z \to A \times Z \) that sends a path to the end points is a Hurewicz fibration as a pull-back of the Hurewicz fibration \( q : Z^{[0,1]} \to Z \times Z \) [Sp]. The fiber of \( q \) is the loop space \( \Omega Z \). Since \( p_A = pr_2 \circ q' \), the fiber \( F = \pi^{-1}(x) = (q')^{-1}pr_2^{-1}(x) = q^{-1}(A) \) is the total space of a Hurewicz fibration \( q \) over \( A \) with the fiber \( \Omega Z \). \( \Box \)

We define the \( k \)-th generalized Ganea’s fibration \( p_k : E_k(Z, A) \to Z \) over a path connected space \( Z \) with a fixed closed subset \( A \) as the fiber-wise join product of \( k + 1 \) copies of the fibrations \( p_A : P Z \to Z \). Since \( p_A \) is a Hurewicz fibration and the fiber-wise join of Hurewicz fibrations is a Hurewicz fibration, so are all \( p_k \) [Sv]. Note that the fiber of \( p_k \) is the join product \( \ast^{k+1} F \) of \( k + 1 \) copies of \( F \) (see [CLOT] for more details). Also we note that for \( A = \{ z_0 \} \) the fibration \( p_k \) is the standard Ganea fibration. The following is a generalization of the Ganea-Švarc theorem.

**Theorem 2.2.** Let \( A \subset X \) be a subcomplex contractible in \( X \). Then \( \text{cat}_{LS}(X) \leq k \) if and only if the generalized Ganea fibration

\[ p_k : E_k(Z, A) \to Z \]

admits a section.

**Proof.** When \( A \) is a point this statements turns into the classical Ganea-Švarc theorem ([CLOT], [Sv]). Since for \( z_0 \in A \), the fibration \( p_k : E_k(Z, z_0) \to Z \) is contained in \( p_k : E_k(Z, A) \to Z \), the classical Ganea-Švarc theorem implies the only if direction.

The barycentric coordinates of a section to \( p_k \) define an open cover \( U_0, \ldots, U_k \) of \( U_i \) with each \( U_i \) contractible to \( A \). Since \( A \) is contractible in \( Z \), all sets \( U_i \) are contractible in \( Z \). \( \Box \)

We call a map \( f : X \to Y \) a stratified locally trivial bundle (with two strata) with fiber \( (Z, A) \) if there \( X' \subset X \), such that \( (f^{-1}(y), g^{-1}(y)) \cong (Z, A) \) for all \( y \in Y \), where \( g = f|_{X'} \), and there is an open cover \( U = \{ U \} \) of \( Y \) such that \( (f^{-1}(U), g^{-1}(U)) \) is homeomorphic as a pair to \( (Z \times U, A \times U) \) by means of a fiber preserving homeomorphism. Such a bundle is called a trivial stratified bundle if one cant take \( U \) consisting of one element \( U = Y \).
Now let \( f : X \to Y \) be a stratified locally trivial bundle with a subbundle \( g : X' \to Y \) and a fiber \((Z,A)\). We define a space
\[
E_0 = \{ \phi \in C(I,X) \mid f\phi(I) = f\phi(0), \ \phi(0) \in g^{-1}(f\phi(0)) \}
\]
to be the space of all paths \( \phi \) in \( f^{-1}(y) \) for all \( y \in Y \) with the initial point in \( g^{-1}(y) \). The topology in \( E_0 \) is inherited from \( C(I,X) \). We define a map \( \xi_0 : E_0 \to X \) by the formula \( \xi_0(\phi) = \phi(1) \). Then \( \xi_k : E_k \to X \) is defined as the fiber-wise join of \( k+1 \) copies of \( \xi_0 \). Formally, we define inductively \( E_k \) as a subspace of the join \( E_0 \ast E_{k-1} \):
\[
E_k = \bigcup \{ \phi \ast \psi \in E_0 \ast E_{k-1} \mid \xi_0(\phi) = \xi_{k-1}(\psi) \}
\]
which is the union of all intervals \([\phi, \psi] = \phi \ast \psi \) with the endpoints \( \phi \in E_0 \) and \( \psi \in E_{k-1} \) such that \( \xi_0(\phi) = \xi_{k-1}(\psi) \). There is a natural projection \( \xi_k : E_k \to X \) that takes all points of each interval \([\phi, \psi]\) to \( \phi(0) \).

Note that when \( f : X = Z \times Y \to Y \) is a trivial stratified bundle with the subbundle \( g : A \times Y \to Y \), \( A \subset Z \), then \( E_k = E_k(Z,A) \times Y \) and \( \xi_k = p_k \times 1_Y \) where \( p_k : (E_k, A) \to Z \) is the generalized Ganea fibration.

**Lemma 2.3.** Let \( f : X \to Y \) be a stratified locally trivial bundle between paracompact spaces with a fiber \((Z,A)\) in which \( A \) is contractible in \( Z \). Then

i. For each \( k \) the map \( \xi_k : E_k \to X \) is a Hurewicz fibration.

ii. The fiber of \( \xi_k \) is the join of \( k+1 \) copies of the fiber \( F \) of \( p_A : P_AZ \to Z \).

iii. If the projection \( \xi_k \) has a section, then \( X \) has an open cover \( U = \{U_0, \ldots, U_k\} \) by sets each of which admits a fiber-wise deformation into \( X' \) where \( g : X' \to Y \) is the subbundle.

**Proof.** i. First, we note that this statement holds true for trivial stratified bundles. By the assumption there is a cover \( U \) of \( Y \) such that \( f|_{f^{-1}(U)} : f^{-1}(U) \to U \) is a trivial stratified bundle and hence \( \xi_k \) is a Hurewicz fibration over \( f^{-1}(U) \) for all \( U \in U \). Then we apply Dold’s theorem [Do] to conclude that that \( \xi_k \) is a Hurewicz fibration over \( X \).

ii. We note that \( \xi_k \) over \( f^{-1}(y) \) coincides with the generalized Ganea fibration \( p_k \) for \((Z,A)\). Therefore, the fiber of \( \xi_k \) coincides with the fiber of \( p_k \). Then we apply Proposition 2.1

iii. Suppose \( \xi_k \) has a section \( \sigma : X \to E_k \). For each \( x \in X \) the element \( \sigma(x) \) of \( *^{k+1}\Omega F \) can be presented as the \((k+1)\)-tuple
\[
\sigma(x) = (\phi_0, t_0, \ldots, (\phi_k, t_k)) \mid \sum t_i = 1, t_i \geq 0.
\]
We use the notation \( \sigma(x)_i = t_i \). Clearly, \( \sigma(x)_i \) is a continuous function.
A section $\sigma : X \to E_k$ defines a cover $\mathcal{U} = \{U_0, \ldots, U_k\}$ of $X$ as follows:

$$U_i = \{x \in X \mid \sigma(x)_i > 0\}.$$

By the construction of $U_i$ for $i \leq n$ for every $x \in U_i$ there is a canonical path connecting $x$ with $g^{-1}f(x)$. These paths define a fiber-wise deformation of $U_i$ into $g^{-1}f(U_i) \subset X'$.

3. The main result

We recall that the homotopical dimension of a space $X$, $hd(X)$, is the minimal dimension of a CW-complex homotopy equivalent to $X$ [CLOT].

**Proposition 3.1.** Let $p : E \to X$ be a fibration with $(n-1)$-connected fiber where $n = hd(X)$. Then $p$ admits a section.

*Proof.* Let $h : Y \to X$ be a homotopy equivalence with the homotopy inverse $g : X \to Y$ where $Y$ is a CW complex of dimension $n$. Since the fiber of $p$ is $(n-1)$-connected, the map $h$ admits a lift $h' : Y \to E$. Let $H$ be a homotopy connecting $h \circ g$ with $1_X$. By the homotopy lifting property there is a lift $H' : X \times I \to E$ of $H$ with $H|_{X \times \{0\}} = h' \circ g$. Then the restriction $H|_{X \times \{1\}}$ is a section. $\square$

We recall that $\lceil x \rceil$ denotes the smallest integer $n$ such that $x \leq n$.

**Lemma 3.2.** Suppose that a stratified locally trivial bundle $f : X \to Y$ with a fiber $(F,A)$ is such that $F$ is $r$-connected, $A$ is $(r-1)$-connected, $A$ is contractible in $F$, and $Y$ is locally contractible. Then

$$\text{cat}_{LS} X \leq \dim Y + \left\lceil \frac{hd(X) - r}{r + 1} \right\rceil.$$

*Proof.* Let $\text{cat}_{LS} Y = m$ and $hd(X) = n$.

We note that the fiber $K$ of $p_k : E_k(F,A) \to F$ is the join product $*^{k+1}K_0$ of $k + 1$ copies of the fiber $K_0$ of the map $p_A : PF \to F$. By Proposition 2.1, $K_0$ admits a fibration $\phi : K_0 \to A$ with fibers homotopy equivalent to the loop space $\Omega F$. Since the base $A$ and the fibers are $(r-1)$-connected, $K_0$ is $(r-1)$-connected. Thus, $K$ is $(k + (k+1)r - 1)$-connected. By Proposition 3.1 there is a section $\sigma : X \to E_k$ whenever $k(r+1) + r \geq n$. The smallest such $k$ is equal to $\lceil \frac{n-r}{r+1} \rceil$.

By Lemma 2.3 a section $\sigma : X \to E_k$ defines a cover $\mathcal{U} = \{U_0, \ldots, U_k\}$ by the sets fiber-wise contractible to $X'$ where $X' \subset X$ is the first stratum. Let $\mathcal{U}_{m+k} = \{U_0, \ldots, U_{m+k}\}$ be an extension of $\mathcal{U}$ to a $(k+1)$-cover of $X$ from Corollary 1.5.
Let $\mathcal{V} = \{V_0, \ldots, V_{m+k}\}$ be an open $(m+1)$-cover of $Y$ such that for every $i$, $V_i$ is contractible to a point in $V'_i$ and $f$ is trivial stratified bundle over $V'_i$. Such $\mathcal{V}$ exists in view of Theorem 1.3. We show that the sets $W_i = f^{-1}(V_i) \cap U_i$ are contractible in $X$ for all $i \in \{0, 1, \ldots, m+k\}$. By Corollary 1.5 $U_i$ is fiber-wise contractible into $X'$ for $i \leq m+k$. Hence we can contract $f^{-1}(V_i) \cap U_i$ to $f^{-1}(V_i) \cap X' \cong V_i \times A$ in $X$. Then we apply a contractions to a point of $V_i$ in $V'_i$ and $A$ in $F$ to obtain a contraction to a point of $f^{-1}(V_i) \cap X' \cong V_i \times A$ in $f^{-1}(V'_i) = V'_i \times F$.

Next we show that $\{W_i\}_{i=0}^{m+k}$ is a cover of $X$. Since $\mathcal{V}$ is an $(m+1)$-cover, by Proposition 1.2 every $y \in Y$ is covered by at least $k+1$ elements $V_{i_0}, \ldots, V_{i_k}$ of $\mathcal{V}$. Since $U_{m+k}$ is a $(k+1)$-cover, $U_{i_0}, \ldots, U_{i_k}$ is a cover of $X$. Hence $W_{i_0}, \ldots, W_{i_k}$ covers $f^{-1}(y)$.

**Theorem 3.3.** For every CW complex $X$ with the following inequality holds true:

$$\text{cat}_{LS} X \leq \text{cd}(\pi_1(X)) + \left\lceil \frac{\text{hd}(X) - 1}{2} \right\rceil.$$ 

**Proof.** Let $\pi = \pi_1(X)$ and let $\tilde{X}$ denote the universal cover of $X$. We consider Borel’s construction

$$\begin{array}{cccccc}
\tilde{X} & \longrightarrow & \tilde{X} \times \pi E\pi & \longrightarrow & E\pi & \\
\downarrow & & \downarrow & & \downarrow & \\
X & \longleftarrow & \tilde{X} \times_\pi E\pi & \longleftarrow & f & \longrightarrow & B\pi.
\end{array}$$

The 1-skeleton $X^{(1)}$ of $X$ defines a $\pi$-equivariant stratification $\tilde{X}^{(1)} \subset \tilde{X}$ of the universal cover. This stratification allows us to treat $f$ as a stratified locally trivial bundle with the fiber $(\tilde{X}, \tilde{X}^{(1)})$. We note that all condition of Lemma 3.2 are satisfied for $r = 1$. Therefore,

$$\text{cat}_{LS} X \leq \dim B\pi + \left\lceil \frac{\text{hd}(\tilde{X} \times_\pi E\pi) - 1}{2} \right\rceil.$$ 

Since $g$ is a fibration with the homotopy trivial fiber, the space $\tilde{X} \times_\pi E\pi$ is homotopy equivalent to $X$. Thus, $\text{hd}(\tilde{X} \times_\pi E\pi) = \text{hd}(X)$. In view of the results of Eilenberg and Ganea [EG],[Br] we may assume that $\dim B\pi = \text{cd}(\pi)$ if $\text{cd}(\pi) > 2$. The case when $\text{cd}(\pi) \leq 2$ is treated in [Dr].

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Department of Mathematics, University of Florida, 358 Little Hall, Gainesville, FL 32611-8105, USA

E-mail address: dranish@math.ufl.edu