Mutual transformation among bound, virtual and resonance states in one-dimensional rectangular potentials

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Abstract. A detailed analysis has been made by R. Zavin and N. Moiseyev (2004 \textit{J. Phys. A: Math, Gen,} \textbf{37} 4619) for the change of bound states into resonance states via coalescence of virtual states in a one-dimensional symmetric rectangular attractive potential as it becomes shallow, with convergent wave functions of virtual and resonance states by the complex scaling method. As a complement to such an analysis, we discuss some global features of the pole spectrum of the $S$-matrix by using a complex extension of the real potential $V^{(\text{real})}$ to $e^{i\alpha}V^{(\text{real})}$ with a real phase $\alpha$. We show the structures of trajectories of poles developed for the change of $\alpha$ in the complex momentum plane, which is useful to understand the mutual transformation among the bound, virtual and resonance states.

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1. Introduction

In the quantum mechanics the dynamical properties of a system is determined by its potential. The bound states, virtual (antibound) states, resonance states with conjugate complex-virtual (antiresonance) states produced by the potential form a basis characterizing the system. These states transform mutually as the potential strength changes.

Recently it has been discussed by Zavin and Moiseyev [1] how the bound states change into the virtual states and subsequently into the resonance states as the potential depth becomes shallow for a one-dimensional symmetric rectangular well potential. The authors studied the problem about solutions of the Schrödinger equation with the complex scaling method; this procedure gives convergent wave functions for the virtual and resonance states, which otherwise are divergent.

The poles of the $S$-matrix in the complex energy plane generated by the potential correspond to the bound, virtual, resonance and complex virtual states. An extensive study of the poles of the $S$-matrix element of a rectangular potential was done previously by Nussenzveig [2]. The main interest of these analyses [1, 2] seems to be in the movements of various states or poles with the change of the potential depth. As a complement to these analyses, it will be useful to explore a whole spectrum of the poles for a given potential and to examine the change of spectrum with the potential depth. The approach taken by Nussenzveig, however, is very intricate and is not suitable for showing a global picture of the pole spectrum.

We examine this problem by a systematic way of locating the resonance states as well as the bound and virtual states for a given potential and show the change of the pole spectral chart for the one-dimensional rectangular potential. This approach uses a complex extension of the real potential. It provides a simple way of presenting an overall view of the pole spectrum and is applicable to other potentials.

There have been also many investigations which involve complex extensions of potentials, developed and widely used in calculations of resonance properties in various fields of physics [3, 4, 5, 6]. It seems, however, that no attempt has been made to use the complex extension for exploring some global feature of the pole spectrum as will be given in this article.

2. $S$-matrix

2.1. Some relations of $S$-matrix elements

The properties of $S$-matrix for a one-dimensional system with real potential have been extensively studied [7]. There seems, however, practically no article explicitly referring to those of one-dimensional system with complex potential to the knowledge of the present authors, though some detailed studies have been made for the basic properties of poles of the $S$-matrix for the spherical symmetric complex potential in three dimensional space [3]. We, therefore, briefly derive some properties of the $S$-matrix for the one-dimensional complex potential.

We consider a particle of mass $m$ in a complex potential $V(x)$ in one dimension. Here the potential is a finite-range potential given by

$$V(x) = \gamma V^{(\text{real})}(x) = \begin{cases} 0 & x < -a \\ \gamma v(x) & -a \leq x \leq a \\ 0 & x > a \end{cases} \quad \text{(region I)},$$

$$\gamma v(x) = \begin{cases} 0 & x < -a \\ \gamma v(x) & -a \leq x \leq a \\ 0 & x > a \end{cases} \quad \text{(region II)},$$

$$x > a \quad \text{(region III)},$$

(1)
where \( v(x) \) is a real function of \( x \) and \( \gamma \) is a complex constant. The time-independent Schrödinger equation for the wave function \( \psi(x) \) of the particle is in units of \( \hbar = 1 \) for a given energy \( E = k^2/2m \)

\[
-\frac{1}{2m} \frac{d^2}{dx^2} \psi(x) + V(x)\psi(x) = \frac{k^2}{2m} \psi(x),
\]

where \( k \) is the momentum of the particle in the exterior interaction free regions I and III.

The wave function \( \psi_1(x, k, \gamma) \) for a unit incident beam along the positive direction of the \( x \) axis is expressed in regions I and III as

\[
\psi_1(x, k, \gamma) = \left\{ \begin{array}{ll}
 e^{ikx} + S_{21}(k, \gamma) e^{-ikx} & x < -a \quad \text{(region I)}, \\
 S_{11}(k, \gamma) e^{ikx} & x > a \quad \text{(region III)},
\end{array} \right.
\]

and another wave function \( \psi_2(x, k, \gamma) \) along the negative direction is

\[
\psi_2(x, k, \gamma) = \left\{ \begin{array}{ll}
 S_{22}(k, \gamma) e^{-ikx} & x < -a \quad \text{(region I)}, \\
 e^{-ikx} + S_{12}(k, \gamma) e^{ikx} & x > a \quad \text{(region III)}.
\end{array} \right.
\]

A matrix with its elements \( S_{ij} \) \((i, j = 1, 2)\) is the \( S \)-matrix of the present problem.

The momentum \( k \) is real so far and we have to make an analytic continuation of the \( S \)-matrix in the complex momentum plane. We prove the following relations about the \( S \)-matrix for the complex potential of finite range

\[
S^i(k, \gamma) S(-k, \gamma) = S(k, \gamma) S^*(-k, \gamma) = 1, \tag{5a}
\]

\[
S^i(k, \gamma) S(k^*, \gamma^*) = 1, \tag{5b}
\]

\[
S^*(-k^*, \gamma^*) = S(k, \gamma). \tag{5c}
\]

For the real potential the information of the \( S \)-matrix in the first quadrant of the complex \( k \)-plane is enough to determine the \( S \)-matrix in the entire complex plane. This is not enough for the complex potential as the relations (5a) and (5c) show.

2.1.1. Relation (5a) The Wronskian of two linearly-independent solutions of the homogeneous second order ordinary linear differential equation with no first order term is constant. Using this property of the Wronskian for solutions of (2), we derive those properties of the \( S \)-matrix.

First we consider the Wronskian of \( \psi_1(x, k, \gamma) \) and \( \psi_2(x, k, \gamma) \)

\[
W[\psi_1(x, k, \gamma), \psi_2(x, k, \gamma)] = \left| \begin{array}{cc}
 \psi_1(x, k, \gamma) & \psi_2(x, k, \gamma) \\
 \psi_1'(x, k, \gamma) & \psi_2'(x, k, \gamma)
\end{array} \right|. \tag{6}
\]

This gives

\[
W[\psi_1(x, k, \gamma), \psi_2(x, k, \gamma)]_{x < -a} = -2ik S_{22}(k, \gamma) \tag{7}
\]

and

\[
W[\psi_1(x, k, \gamma), \psi_2(x, k, \gamma)]_{x > a} = -2ik S_{11}(k, \gamma). \tag{8}
\]

Since these two values of the Wronskian are the same, we have a relation

\[
S_{11}(k, \gamma) = S_{22}(k, \gamma). \tag{9}
\]

The wave functions \( \psi_1(x, -k, \gamma) \) and \( \psi_2(x, -k, \gamma) \) are also solutions of (2). We calculate the Wronskians involving these solutions. From \( W[\psi_1(x, k, \gamma), \psi_1(x, -k, \gamma)] \) we have

\[
S_{11}(k, \gamma) S_{11}(-k, \gamma) + S_{21}(k, \gamma) S_{21}(-k, \gamma) = 1, \tag{10}
\]
from $W[\psi_1(x, k, \gamma), \psi_2(x, -k, \gamma)]$
\[ S_{11}(k, \gamma) S_{12}(-k, \gamma) + S_{21}(k, \gamma) S_{22}(-k, \gamma) = 0, \]  \hspace{1cm} (11)
and from $W[\psi_2(x, k, \gamma), \psi_2(x, -k, \gamma)]$
\[ S_{12}(k, \gamma) S_{12}(-k, \gamma) + S_{22}(k, \gamma) S_{22}(-k, \gamma) = 1. \] \hspace{1cm} (12)
Combining (10), (11) and (12) gives
\[ S(k, \gamma) S(-k, \gamma) = 1. \] \hspace{1cm} (13)
Further (9) implies
\[ S(k, \gamma) S(-k, \gamma) = 1. \] \hspace{1cm} (14)
This completes the relation (5).

2.1.2. Relations (5\textit{b}) and (5\textit{c}) First we derive the relation (5\textit{c}) and note on the relation (5\textit{b}) at the end.

Let us take the complex conjugate of (2). The wave function $\psi^*_1(x, k, \gamma)$ satisfies the Schrödinger equation (2) with a potential $\gamma^* v(x)$ for energy $(k^2)^*/2m$. The exterior part of this wave function is given by
\[ \psi^*_1(x, k, \gamma) = \begin{cases} e^{-ikx} + S_{21}^*(k, \gamma) e^{+i(k^*)x} & x < a, \\ S_{11}^*(k, \gamma) e^{-i(k^*)x} & x > a, \end{cases} \] \hspace{1cm} (15)
which is written as
\[ \psi^*_1(x, k, \gamma) = \begin{cases} e^{+i(k^*)x} + S_{21}^*(k, \gamma) e^{-i(k^*)x} & x < -a, \\ S_{11}^*(k, \gamma) e^{i(k^*)x} & x > a. \end{cases} \] \hspace{1cm} (16)
This expression is seen to be equivalent to the solution $\psi_1(x, -k^*, \gamma^*)$ given by
\[ \psi_1(x, -k^*, \gamma^*) = \begin{cases} e^{+i(k^*)x} + S_{21}(-k^*, \gamma^*) e^{-i(k^*)x} & x < -a, \\ S_{11}(-k^*, \gamma^*) e^{i(k^*)x} & x > a. \end{cases} \] \hspace{1cm} (17)
Since such a solution is unique, we have relations
\[ S_{21}(k, \gamma) = S_{21}(-k^*, \gamma^*) \quad \text{and} \quad S_{11}^*(k, \gamma) = S_{11}(-k^*, \gamma^*). \] \hspace{1cm} (18)
By the same arguments for $\psi^*_2(k, \lambda)$ we have
\[ S_{12}^*(k, \gamma) = S_{12}(-k^*, \gamma) \quad \text{and} \quad S_{22}^*(k, \gamma) = S_{22}(-k^*, \gamma^*). \] \hspace{1cm} (19)
Hence we have the relation (5\textit{b}).

Finally we take the relation (5\textit{a}). It is obvious that this relation is obtainable from the relations (5\textit{a}) and (5\textit{c}).

2.2. The $S$-matrix elements for a symmetric rectangular potential

For the symmetric rectangular well, the potential is given by $v(x) = -U$ where $U$ is a positive constant. The general solution $\psi_{11}$ in the region II is given by
\[ \psi_{11} = A e^{iKx} + B e^{-iKx}, \] \hspace{1cm} (20)
where $K = \sqrt{k^2 + 2m\gamma^* U}$. The arbitrary constants $A$ and $B$ are determined by the continuity conditions of the wave functions and their space derivatives at $x = \pm a$ for each of the exterior wave functions (3) and (11). This gives the $S$-matrix as
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\[ S = \frac{2 \exp(-2ika) kK}{2kK \cos 2K\alpha - i(k^2 + K^2) \sin 2K\alpha} \times \begin{pmatrix} 1 & -i(k^2 - K^2) 2kK \sin 2K\alpha \\ -i(k^2 - K^2) 2kK \sin 2K\alpha & 1 \end{pmatrix}. \] (21)

The poles of the \( S \)-matrix elements come from the zeros of their common denominator. It is easy to observe the condition giving the zeros of the denominator is just the same as that obtained by the eigenvalue equation for the bound-state energy.

Since the present potential is symmetric for the space inversion \( x \rightarrow -x \), it is better to discuss the symmetric and antisymmetric states for the space inversion separately. The exterior wave functions \( \psi_{\pm}(x, k, \gamma) \) of the symmetric and antisymmetric states are given by

\[ \psi_{\pm}(x, k, \gamma) = \begin{cases} e^{ikx} \pm S_{\pm}(k, \gamma) e^{-ikx} & x < -a \text{ (region I)}, \\ S_{\pm}(k, \gamma) e^{ikx} \pm e^{-ikx} & x > a \text{ (region III)}. \end{cases} \] (22)

The \( S \)-matrix \( \hat{S} \) on the basis of the symmetric and antisymmetric states is given by

\[ \hat{S} = \begin{pmatrix} S_+ & 0 \\ 0 & S_- \end{pmatrix}. \] (23)

with

\[ S_+ = \exp(-2ika) \frac{K \cos K\alpha + iK \sin K\alpha}{K \cos K\alpha - iK \sin K\alpha}, \] (24)

\[ S_- = \exp(-2ika) \frac{K \cos K\alpha + ik \sin K\alpha}{K \cos K\alpha - ik \sin K\alpha}. \] (25)

Here \( \hat{S} \) is related to \( S \) of (21) by a similarity transformation \( \hat{S} = USU^{-1} = USU^t \) with \( U \) given as

\[ U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}. \] (26)

For representing the pole structures, the complex momentum \( k \) plane gives a simpler picture than the complex energy \( E \) plane which has cut structure, though the energy variable may be more familiar than the momentum variable at interface with experiments. Here we note on the cut structure of the \( S \)-matrix in the complex \( E \) plane. There is a cut starting from the branch point \( E = 0 \) to \( \infty \). We denote the Riemann sheet given by \( \text{Im} k > 0 \) as \( \text{I} \) and the sheet for \( \text{Im} k < 0 \) as \( \text{II} \), which are called the physical and the unphysical sheet, respectively.

In the complex momentum \( k \) plane the bound state appears on the positive imaginary axis and the virtual state on the negative imaginary axis, while the resonance state emerges in the lower half-plane with positive real part (the fourth quadrant of the complex plane) and the complex virtual state (antiresonance state) with negative real part (the third quadrant).

There appear an infinite number of poles for finite range potential for the rectangular potential [7], with an infinite number of resonance states and a finite number of bound and virtual states.
Since the S-matrix elements are given in the explicit analytical form (24) and (25) for the rectangular potential, it is simple to find the pole locations of the bound and virtual states for a given well depth. As they lie on the imaginary momentum axis, we can find zeros of the denominators in the expressions of the $S_+$ and $S_-$ by numerical calculations with or without the aid of a standard graphical method which is often very helpful.

The resonance-state poles, however, are less easy to be located, as they are scattered in the complex momentum plane. One way is to trace each of virtual states descending on the imaginary momentum axis and finally changing into a resonance pole by coalescence with another virtual state ascending on the imaginary axis as the potential becomes shallow, as done by Zavin and Moiseyev [1]. This requires some work to draw a whole pole chart in the complex momentum plane for a given value $U$, since we need a tracing of one virtual state for each of resonances starting from a deeper well. There is a simple way to draw the spectrum chart of resonance poles which is given in the following.

3. Pole charts

3.1. Method for locating poles

3.1.1. Procedures

Although we study the one-dimensional problem in this paper, the present approach is not restricted to it.

(i) First we search for poles on the imaginary axis for a given real potential depth ($\gamma = 1$). We do the same for the sign reversed potential, i.e., the repulsive rectangular barrier with its height equivalent to the depth of the well ($\gamma = -1$). This can be easily calculated numerically if the S-matrix elements are given analytically.

(ii) Next step is to trace each of poles on the imaginary momentum axis for making a complex rotation of the potential $V(x)$ by taking

$$\gamma = e^{i\alpha},$$

with a real phase parameter $\alpha$ starting from $\alpha = 0 (\pi)$ for the poles of the well (barrier). This is performed numerically by changing $\alpha$ with a small step. The well and the barrier are analytically related by the present complex extension of the potential and their pole spectra have close connections.

These procedures produce either a closed trajectory or an open trajectory of infinite length in the complex momentum plane for each of poles on the imaginary momentum axis. The trajectories obtained in this way present a simple picture for the global structures of the pole spectrum of the system.

Relation (26) implies that each of the trajectories is mirror symmetric about the imaginary momentum axis or has a mirror partner.

If the location of a resonance pole is found in some way, it is possible to draw a pole trajectory starting from this pole with the complex extension $e^{i\alpha}$. In general, it is much simpler to locate the virtual-state and bound-state poles emerging on the imaginary momentum axis rather than the resonance-state poles in the complex plane. This is certainly one merit of the present approach, but the most interesting point is the trajectory structures exposed by the phase rotation (27).

There appear bound states only for the well, while resonance and virtual states appear for both of the well and the barrier. Hence, we call the states appearing for the well “attractive” and those for the barrier “repulsive” for simplicity.
3.1.2. Some comments on the present approach

Transformations with the change of phase $\Delta \alpha = \pm \pi$ can be considered as a mapping of the set of all the attractive poles to the set of repulsive poles, and with $\Delta \alpha = \pm 2\pi$ a self-mapping onto itself.

For the self-mapping transformation, there appear invariant subsets for which the mapping closes within each of subsets. A closed trajectory corresponds to a finite invariant subset which consists of a finite number of attractive and repulsive poles on the trajectory, while an open trajectory to an infinite invariant subset. Tracing a pole of the invariant subset by continuously changing $\alpha$ implies repeating a self-mapping for every change of the phase by $\Delta \alpha = \pm 2\pi$, therefore, it provides a systematic way of scanning all elements of the subset. The self-mapping is essentially a cyclic permutation among the elements of a finite invariant subset and a sequential transposition of the elements for an infinite invariant subset along its trajectory.

If the potential depth varies, it occurs degeneracy of two virtual states or of a pair of resonance and complex virtual states. At this point the mapping is indefinite and the degeneracy induces the change of two virtual states into a pair of resonance and complex virtual states, or in reverse order. This implies the loss of one-to-one correspondence between the two virtual-state poles and the pair of resonance-state and complex virtual-state poles and causes rearrangement between two invariant subsets. It is added that the rearrangement between the invariant subsets happens also when the corresponding trajectories mutually contact off the imaginary momentum axis.

In the one-dimensional rectangular potential the degeneracy of states occurs always on the negative imaginary momentum axis at the fixed point $k_c = -i/a$, which is the only double zero point of the denominator of the $S$-matrix elements, common to $S_+$ and $S_-$. It could happen that some closed trajectories do not cross the imaginary momentum axis as the well depth varies for the assumed strength of the potential. These trajectories consist of only either resonance-state poles or complex virtual-state poles. In fact, such trajectories appear for the $d$-wave states in the spherical rectangular potential. This, however, does not cause a serious problem in the present approach for the pole search, as the concerned resonance poles can be easily located on the trajectory starting from the imaginary axis by the change of the potential depth.

In the following we show for the pole spectrum of the rectangular potential how it changes with the potential depth. The rectangular potential produces only one open trajectory and a finite number of closed trajectories for a finite value of $U$. The open trajectory passes all the attractive resonance poles and almost all of the repulsive resonance poles for the change of $\alpha$ from the starting value on the imaginary axis to $-\infty$ and corresponding conjugate complex virtual poles to $\infty$ in the present choice of the phase factor, while the closed trajectory completes with the periodicity of $\Delta \alpha = 2\pi$ or $4\pi$. The open trajectory implies the existence of an infinite number of resonance-state and complex virtual-state poles and its appearance is very restricted to the case of some “singular” potentials as the present rectangular potential.

3.2. Symmetric states: poles of $S_+$

Here we take $m = 1$ and $a = 1.5$ in the atomic units. The pole charts of the symmetric states of $S$-matrix element $S_+$ are given in figure 1, where the full circles indicate the attractive poles and the open circles the repulsive poles. The lines connecting the circles are the paths of poles drawn by continuously changing the phase $\alpha$. The relation (5) implies that the complex virtual-state poles emerge always in pair with
Figure 1. The trajectories of the poles of $S_+$ for the symmetric states of the one-dimensional symmetric rectangular potential. The full circles (•) indicate the attractive poles and the open circles (○) the repulsive poles. The arrows given along the curves indicate the directions of the movement of poles as the phase parameter $\alpha$ increases.

the resonance-state poles at the symmetrical points about the imaginary momentum axis. Any trajectory is symmetric about the imaginary momentum axis, if it crosses
the imaginary axis. If it does not cross the axis, it has a mirror symmetric partner.

For small $U$, there appear one bound-state and two repulsive virtual-state poles on the imaginary momentum axis (figure 1(a) $U = 0.09$). Starting from these poles, two trajectories are given; one is the closed $2\pi$-periodic trajectory which has the bound state and one of the two repulsive virtual states and the other is the open trajectory having all of the resonance and complex virtual states as well as the other repulsive virtual state.

Necessity of repulsive virtual states for the present approach lies in the fact that some trajectories cross the imaginary momentum axis at the repulsive virtual states; without these repulsive virtual states it is not possible to complete the pole chart for the corresponding attractive potential by the present procedure starting from the poles on the imaginary momentum axis.

As $U$ increases, two repulsive virtual states approach; the upper one going down and the lower going up. These repulsive virtual states finally collide at the momentum $k_c = -i/a \approx -0.667i$ and change into a pair of a repulsive resonance and its conjugate complex virtual state, causing the fusion of the closed trajectory and the open trajectory into one open trajectory (figure 1(b) $U = 0.1$).

If $U$ increases further, poles generally go upwards, with the attractive resonance states approaching and with the repulsive ones leaving the imaginary axis (figure 1(c) $U = 1$). Then a pair of attractive resonance-state and complex virtual-state poles approach the imaginary momentum axis (figure 1(d) $U = 1.95$). For deepening the well, this pair of poles collide at $k_c$ on the imaginary momentum axis, then turn into two attractive virtual states with a formation of one closed $4\pi$-periodic trajectory from the open trajectory (figure 1(e) $U = 2$). As $U$ increases, the attractive virtual-state pole on the open trajectory becomes an excited bound-state pole (figure 1(f) $U = 3$).

3.3. Antisymmetric states: poles of $S_-$

The pole charts of the antisymmetric states of $S$-matrix element $S_-$ are given in figure 2.

For the antisymmetric states, we have only one attractive virtual-state pole for small $U$, and there is only one open trajectory which passes all the resonance-state and complex virtual-state poles (figure 2(a) $U = 0.02$). As $U$ increases, all the poles move upwards (figure 2(b) $U = 0.2$) with the attractive resonance-state and complex virtual-state poles approaching and the repulsive ones leaving the imaginary axis as in the symmetric states. If $U$ becomes larger than $U_0^- = \pi^2/8ma^2 \approx 0.548$, the lowest bound state of the antisymmetric state appears as the first excited state of the rectangular potential. Even in this case there is only one open trajectory (figure 2(c) $U = 2$).

As $U$ increases further, a pair of attractive resonance-state and complex virtual-state poles mutually approach (figure 2(d) $U = 4.7$) and collide on the imaginary momentum axis. Then, these poles turn into two attractive virtual-state poles, one going up and the other going down, forming one $4\pi$-periodic closed and one open trajectories (figure 2(e) $U = 4.8$). As the potential becomes deeper, the attractive virtual-state pole of the open trajectory becomes a bound-state pole (figure 2(f) $U = 5$). As noted before, the collision of attractive resonance pole with its conjugate complex virtual pole occurs at the point $k_c$ which is the common colliding spot for all the pairs of resonance-state and complex virtual-state poles.

The movements of the poles of $S_-$ are very similar to those of $S_+ \text{ except the change}
of the two repulsive virtual-state poles into a pair of resonance-state and complex virtual-state poles in the latter.
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It is well-known that the $s$-wave pole spectrum of the quantum system with the spherical rectangular potential with range $a$ in three-dimensional space is completely the same as that of the antisymmetric states of the system with one-dimensional one.

3.4. Applications to other potentials

The present procedures can be applied to any of potentials if the analytical expressions of the $S$-matrix elements are given explicitly. For example, let us take the exponential [9] and the Hulthén [10] potential in three-dimensional space. These two potentials show very similar behaviours to each other for the attractive case with an infinite number of virtual state poles for shallow potential depth. As the potential depth increases, these virtual-state poles go upwards on the imaginary momentum axis and successively emerge on the positive imaginary momentum axis as bound-state poles. The spectra for the repulsive case, however, are different. The Hulthén potential gives an infinite number of virtual-state poles which are simply going downwards as the potential strength increases, while the repulsive virtual states for the exponential turn successively into resonance-state and complex virtual-state poles. All the trajectories of the Hulthén potential are closed $2\pi$-periodic ones, while the exponential potential gives a finite number of $4\pi$-periodic trajectories and an infinite number of $2\pi$-periodic ones. These can be easily observed by the present approach. Some details of the pole trajectories for the three-dimensional spherical symmetric potentials including the exponential and Hulthén potentials will be given elsewhere.

4. Some Remarks

The bound states, the virtual states and the resonance states including the conjugate complex virtual states have the same dynamical origin and are mutually transforming as the potential strength varies.

In practice, with currently available accessibility to computing facilities, the resonance-pole research does not require laborious work even in the approach by Nussenzveig [2], as far as the analytical expressions of the $S$-matrix are given as in the case of the rectangular potential. The method of complex extension of potential presented in this article affords a simple and systematic way of making a global pole chart for a given potential and of observing the change of spectrum as the potential strength varies.

We have shown the trajectory structures of the pole spectrum for the attractive rectangular potential. Here the repulsive resonance-state and complex virtual-state poles play an important role as mediators connecting attractive poles. Even without considering such role of the repulsive resonance states, it seems necessary to pay more attention to these states, especially of the $s$-wave ones, which seem to appear commonly for the most of repulsive potentials, as observed for a group of potentials having the exponential tail. These poles could appear very near the real energy axis at low energies.

The present approach relates two physical systems having the potentials of the same shape but with opposite signatures and will be useful to consider the quantum system from an extended global aspect. The physical and mathematical implications of the pole-trajectory structures need further investigations.
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