Minors for alternating dimaps

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Abstract

We develop a theory of minors for alternating dimaps — orientably embedded digraphs where, at each vertex, the incident edges (taken in the order given by the embedding) are directed alternately into, and out of, the vertex. We show that they are related by the triality relation of Tutte. They do not commute in general, though do in many circumstances, and we characterise the situations where they do. The relationship with triality is reminiscent of similar relationships for binary functions, due to the author, so we characterise those alternating dimaps which correspond to binary functions. We give a characterisation of alternating dimaps of at most a given genus, using a finite set of excluded minors. We also use the minor operations to define simple Tutte invariants for alternating dimaps and characterise them. We establish a connection with the Tutte polynomial, and pose the problem of characterising universal Tutte-like invariants for alternating dimaps based on these minor operations.

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1 Introduction

The minor relation is one of the most important order relations on graphs. A graph $H$ is a minor of a graph $G$ if it can be obtained from $G$ by some sequence of deletions and contractions of edges. Many important classes of graphs can be characterised by the exclusion of some finite set of minors. These include forests, series-parallel graphs [12, 16], outerplanar graphs [10], planar graphs [26, 37] — and, in fact, any minor-closed class of graphs, by Robertson and Seymour’s proof of Wagner’s conjecture [32]. Minors also play a central role in enumerative graph theory: the Tutte-Whitney polynomials, which contain information on a great variety of counting problems on graphs or matroids, satisfy recurrence relations using deletion and contraction (see, e.g., [7, 18, 21, 38]).

The theory of minors derives much of its richness and beauty from the fact that the deletion and contraction operations are dual (in the sense of planar graph duality or, more generally, matroid duality [31]) and commute.

In this paper, we introduce and study a minor relation on alternating dimaps. An alternating dimap is a directed graph without isolated vertices, 2-cell-embedded in a disjoint union of orientable 2-manifolds, where each vertex has even degree and, for each vertex $v$, the edges incident with $v$ are directed alternately into, and out of, $v$ (when considered in the order in which they appear around $v$ in the embedding). An alternating dimap may have loops and/or multiple edges, but cannot have a bridge. We allow the empty alternating dimap with no vertices, edges or faces.

For alternating dimaps, we have three minor operations, instead of two. We show in §2 that they are related by a triality relation of Tutte [34], in a manner analogous to the duality between deletion and contraction. The form of the relationship is the same as that found by the author for some other combinatorial objects (binary functions) on which minors and triality can be defined [23].

One property of ordinary minor operations (and also of the minor operations in [23]) is that they commute. We show in §3 that minor operations on alternating dimaps do not commute in general, although they do in most circumstances, and we determine exactly when they do.

Although duality, with associated minors, appears in many forms for many different kinds of objects, there are far fewer settings with natural minor operations related by triality. Two of these are binary functions [23] and alternating dimaps. It is natural, then, to ask what relationship there may be between these settings. Some alternating dimaps certainly cannot be represented by binary functions, or by transition matroids, since alternating dimap minor operations may not commute, unlike those in the other settings. In §4 we determine those alternating dimaps that can be represented faithfully by binary functions. (Other settings with natural minor operations and triality are multimatroids (including isotropic systems) [4, 5] and the related transition matroids [33, pp. 8, 10]. Connections between them and alternating dimaps are a topic for future work, though the settings seem to differ significantly since, in these other settings, the minor operations commute.)
As seen in the first paragraph, two of the main themes of the classical theory of minors are excluded minor characterisations and Tutte invariants. The remainder of this paper takes these themes up for alternating dimaps and their minor operations. In §5 we give an excluded minor characterisation of alternating dimaps of at most a given genus, using a finite set of excluded minors in every case. In §6 we define simple Tutte invariants for alternating dimaps, and show that there are only a few of them and that they do not contain much information, in contrast to the situation for graphs, matroids and binary functions. We then define extended Tutte invariants and raise the question of how many and varied they might be. We show that they are much richer than the simple Tutte invariants, as they include, in a sense, the Tutte polynomial of a planar graph.

§§3–6 may be read independently of each other.

1.1 History and related work

Plane alternating dimaps were studied by Tutte [34, 35]. He showed that they come in triples, with the three members of a triple being derivable from a single larger structure, a bicubic map (see below). The relationship among the members of such a triple is called triality [34] or trinity [35], and each is trial or trine to the others. This relationship extends ordinary duality. Tutte’s original motivation was to determine when equilateral triangles can be tesselated by smaller equilateral triangles of different sizes or orientations. He also proved his Tree Trinity Theorem, on spanning arborescences of such maps. In the second paper [35] he noted the possibility of extending this theory to other surfaces. Berman [2] showed explicitly how to construct the trial of an alternating dimap without reference to the bicubic map from which three trial maps are derived, and gave alternative proofs of some of Tutte’s results. Tutte reviewed some aspects of his theory in [36].

It is interesting to note that this stream of research, first seen in Tutte’s 1948 paper [34], can be traced back to the same source that eventually gave rise to Tutte’s work on minor operations and his eponymous polynomial. Historically, the source of both streams was the famous paper on “squaring the square” [6]. The 1948 paper extended the theory to “triangulating the triangle” (where all triangles are equilateral) and introduced triality, among other things. However, this stream has not previously seen the development of minor operations or Tutte-like invariants for alternating dimaps.

Tutte showed that a “triangulated triangle” gives rise to a plane bipartite cubic graph (a plane bicubic map), which in turn has, as its dual, an Eulerian plane triangulation. The triple of trial alternating dimaps is derived from this bicubic map. Although bicubic maps are plane in Tutte’s work, they may more generally be taken to be embedded in some orientable surface, so each has a genus.

A separate stream of research concerns latin bitrades, which are pairs of partial latin squares of the same shape and with the same symbol set in each row and column. These may also be given a genus. Cavenagh and Lisoněk [8] established a
correspondence between spherical latin bitrades and 3-connected planar Eulerian triangulations (dually, 3-connected plane bicubic maps), while the relationship between spherical latin bitrades and triangulated triangles is described in [13, 14] (see also [9]). Batagelj [11] introduced two operations on plane Eulerian triangulations by means of which larger such maps can be generated from smaller ones. These operations have been used and extended in several papers (via the aforementioned correspondences) to generate latin bitrades [14, 24]. The inverse of one of these operations, translated to alternating dimaps, corresponds to (technically, restricted versions of) our minor operations. This inverse also appears as a reduction on plane bicubic maps due to Jaeger [25].

1.2 Definitions and notation

If \( G \) is an alternating dimap then \( kG \) is the disjoint union of \( k \) copies of \( G \). Each of these copies is regarded as being embedded in a different surface, with all these \( k \) surfaces being disjoint from each other.

An edge \( e \) from \( u \) to \( v \) is sometimes written \( e(u, v) \).

Let \( G \) be an alternating dimap, viewed topologically as embedded in a surface. Let \( C \subseteq E(G) \) be a circuit of \( G \), and let \( S \) be the connected surface in which the component of \( G \) containing \( C \) is embedded. Then the sides of \( C \) are the components of \( S \setminus C \). A side is planar if it is homeomorphic to the open unit disc.

The genus \( \gamma(G) \) of an alternating dimap \( G \) is given by

\[
|V(G)| - |E(G)| + |F(G)| = 2(k(G) - \gamma(G)),
\]

where \( F(G) \) is the set of faces of \( G \) and \( k(G) \) is the number of components of \( G \).

The edges around a face all go in the same direction, and we say the face is clockwise or anticlockwise according to the direction of the edges around it. We identify a clockwise (respectively, anticlockwise) face with its cyclic sequence of edges, and call it a c-face (resp., a-face) for short. Observe that the (edge sets of the) c-faces partition \( E(G) \), as do the a-faces. So every edge \( e \) belongs to one c-face, denoted by \( C(e) = C_G(e) \), and one a-face, denoted by \( A(e) = A_G(e) \). If two faces share a common edge, then one of the faces is clockwise and the other is anticlockwise. The left successor (respectively, right successor) of an edge \( e \) is the next edge along from \( e \), going around \( A_G(e) \) (resp., \( C_G(e) \)) in the direction given by \( e \). (This direction is anticlockwise for the left successor, and clockwise for the right successor.) Often, c-faces and a-faces are simple cycles, but this is not always the case. If \( v \) is a cutvertex of \( G \), then one face incident with \( v \) consists of two or more edge-disjoint cycles.

The numbers of clockwise and anticlockwise faces of \( G \) are denoted by \( cf(G) \) and \( af(G) \), respectively.

An in-star is the set of all edges directed into some vertex. So in-stars are in one-to-one correspondence with vertices. The in-star of edges directed into vertex \( v \)

\[\text{I thank Tony Grubman and Ian Wanless for pointing out this link.}\]
is denoted by $I(v) = I_G(v)$. Observe that the in-stars partition $E(G)$, so every edge $e$ also belongs to one in-star, denoted by $I(e) = I_G(e)$ (overloading notation slightly).

If an alternating dimap is disconnected, then we treat its components as being embedded in separate, disjoint surfaces.

Throughout, we set $\omega = \exp(2\pi i/3)$.

Alternating dimaps extend ordinary embedded graphs, in that replacing each edge of an embedded graph by a pair of directed edges, forming a clockwise face of size 2, gives an alternating dimap \cite{35}.

An alternating dimap $G$ defines three permutations $\sigma_{G,1}, \sigma_{G,\omega}, \sigma_{G,\omega^2}: E(G) \to E(G)$ (abbreviated $\sigma_1, \sigma_\omega, \sigma_{\omega^2}$ where $G$ is clear from the context), as follows. For each $e \in E(G)$, its image under $\sigma_{G,1}, \sigma_{G,\omega}$ and $\sigma_{G,\omega^2}$ is the next edge in clockwise order around $I_G(e)$, $A_G(e)$ and $C_G(e)$, respectively. So the left successor of $e$ is $\sigma_{G,\omega}(e)$, while the right successor of $e$ is $\sigma_{G,\omega^2}(e)$. Note that, in going around an in-star in clockwise order, we skip outgoing edges at the vertex as these do not belong to the in-star.

Any two of these permutations determine the other. This follows from the relation $\sigma_1 \sigma_\omega \sigma_{\omega^2} = \text{id}_{E(G)}$, the identity permutation on $E(G)$.

Let $S$ be the set of all triples $(\sigma_0, \sigma_1, \sigma_2)$ of permutations, all acting on the same set $E$, such that $\sigma_0 \sigma_1 \sigma_2 = \text{id}_E$. This is called a 3-constellation or a hypermap \cite{27}. When one of the permutations is an involution, we may take these involutions to correspond to undirected edges (using the aforementioned representation of embedded graphs by alternating dimaps), and we have a standard combinatorial representation of an orientably embedded graph (see, e.g., \cite[§2.2]{3}). In the general case, we have an equivalence with alternating dimaps which seems to be well known (see, e.g., \cite{11}) although I have not seen it stated explicitly.

**Proposition 1** The map $\{\text{alternating dimaps}\} \to S$ given by $G \mapsto (\sigma_{G,1}, \sigma_{G,\omega}, \sigma_{G,\omega^2})$ is a bijection. \qed

If $G$ is an alternating dimap, then the trial $G^\omega$ of $G$ is defined as follows. Its vertices represent the $c$-faces of $G$. We denote the vertex of $G^\omega$ representing $c$-face $C$ by $v_C$. (Think of $v_C$ being placed inside $C$ in the embedding.) Edges of $G^\omega$ are constructed as follows. Suppose two $c$-faces $C_1$ and $C_2$ of $G$ share a vertex $v$, and that there is an $a$-face $A$ containing edges $e$ and $f$ going into and out of $v$ respectively, with $e$ and $f$ also belonging to $C_1$ and $C_2$ respectively. (See Figure 1.) We do not require $e$ and $f$ to be distinct, or $C_1$ and $C_2$ to be distinct.) Then we put an edge $e^\omega$ from $v_{C_2}$ to $v_{C_1}$ in $G^\omega$. (Think of $e^\omega$ as being drawn by a curve from $v_{C_2}$, inside $C_2$, to its destination $v_{C_1}$ inside $C_1$, in such a way that it crosses $f$ in its “first half” (i.e., closer to its start than its end) and crosses $e$ in its “last half”.) These edges of $G^\omega$ are ordered around $C_1$ according to the order of the edges $f$ around $C_1$. Similarly, they are ordered around $C_2$ according to the order of the edges $e$ around $C_2$. It is routine to show that the map $\bullet^\omega: E(G) \to E(G^\omega)$, $e \mapsto e^\omega$ is a bijection, and that the $c$-faces, $a$-faces and in-stars of $G^\omega$ are the $a$-faces, in-stars and $c$-faces, respectively, of $G$. We can also express this relationship in the language of the permutation triples.
Proposition 2

\[ \sigma_{G,1}(e^\omega) = \sigma_{G^\omega,\omega}(e^\omega) \]  \hspace{1cm} (1)
\[ \sigma_{G,\omega}(e^\omega) = \sigma_{G^\omega,\omega^2}(e^\omega) \]  \hspace{1cm} (2)
\[ \sigma_{G,\omega^2}(e^\omega) = \sigma_{G^\omega,1}(e^\omega) \]  \hspace{1cm} (3)

If \( G \) is represented by \((\sigma_{G,1}, \sigma_{G,\omega}, \sigma_{G,\omega^2})\), then its trial \( G^\omega \) is represented by \((\sigma_{G,\omega}, \sigma_{G,\omega^2}, \sigma_{G,1})\).

It is clear that Proposition 2 still holds if \( \sigma \) is replaced by \( \sigma^{-1} \) throughout.

The trial operation on a component of \( G \) is independent of the other components.

![Construction of trial map.](image)

We write \( G^{\omega^2} \) for \((G^\omega)^\omega\). From the way triality changes c-faces to in-stars to a-faces to c-faces, we find that \( G^{\omega^3} = (G^{\omega^2})^\omega = G^1 = G \).

1.3 Loops and semiloops

A (standard) loop is just an edge of \( G \) which is a loop in the undirected version of \( G \). If it is a separating circuit of the embedding, then it divides its component of the embedding surface into two sides, its clockwise side and its anticlockwise side. If it is non-separating, then it has just one side, which we take to be both its clockwise and anticlockwise side.

A 1-loop is an edge whose head has degree 2. This is an edge whose left and right successors are identical, and in such cases we can refer unambiguously to its successor. An \( \omega \)-loop is an edge forming a single-edge a-face. An \( \omega^2 \)-loop is an edge forming a single-edge c-face. For any alternating dimap \( G \), an edge \( e \) is a 1-loop in \( G \) if and only if \( e^\omega \) is an \( \omega \)-loop in \( G^\omega \), which in turn holds if and only if \( e^{\omega^2} \) is an \( \omega^2 \)-loop in \( G^{\omega^2} \).

An triloop is an edge which is a \( \mu \)-loop for some \( \mu \in \{1, \omega, \omega^2\} \).

An ultraloop is a triloop which (together with its vertex) constitutes a component of the graph. It has two faces, a c-face and an a-face, and its vertex has degree 2, so
it is simultaneously a 1-loop, an $\omega$-loop and an $\omega^2$-loop. In fact, if an edge is a $\mu$-loop for $\mu$ equal to any two of $\{1, \omega, \omega^2\}$, then it is an ultraloop.

Note that $\omega$- and $\omega^2$-loops are also standard loops, but the converse is not necessarily true, as we will see when considering 1-semiloops shortly. Also, a 1-loop is typically not a standard loop, since its vertices do not have to coincide. A 1-loop is only a standard loop if it is an ultraloop.

A $\mu$-loop is a proper $\mu$-loop if it is not also an ultraloop. In such a case, it is not a $\nu$-loop for any $\nu \in \{1, \omega, \omega^2\} \setminus \{\mu\}$.

A 1-semiloop is just a standard loop. An $\omega$-semiloop is an edge $e$ such that $e$ and its right successor $\sigma_{\omega^2}(e)$ are equal, or (if distinct) either they form a cutset of $G$ or deleting both increases its genus. (After deletion, we no longer have an alternating dimap in general; we are really referring to the underlying undirected embedded graph here, rather than $G$ itself.) This latter condition, on cutset/genus, may be written: $k(G \setminus \{e, \sigma_{\omega^2}(e)\}) - \gamma(G \setminus \{e, \sigma_{\omega^2}(e)\}) > k(G) - \gamma(G)$. Similarly, an $\omega^2$-semiloop is an edge $e$ such that $e$ and its left successor $\sigma_{\omega^2}^{-1}(e)$ together have this same property.

For each $\mu \in \{1, \omega, \omega^2\}$, a proper $\mu$-semiloop is a $\mu$-semiloop that is not a triloop. A proper 1-semiloop $e$ either gives a non-contractible closed curve in the embedding, or each of its two sides contains an edge other than $e$ from the same component as $e$.

We make some basic observations about semiloops.

An edge is a $\mu$-semiloop in $G$ if and only if it is a $\mu\omega$-semiloop in $G^\omega$.

An edge is both a $\mu_1$-semiloop and a $\mu_2$-semiloop if and only if it is a $(\mu_1\mu_2)^{-1}$-loop.

2 Minors

Let $G$ be an alternating dimap and $e = e(u, v) \in E(G)$.

If $e$ is not a loop, then the graph $G[1]e$ is formed by deleting the edge $e$ and identifying its endpoints, while preserving the order of the edges and faces around vertices. If $e$ is an $\omega$-loop or an $\omega^2$-loop, then $G[1]e$ is formed just by deleting $e$. If $e$ is a 1-semiloop, then $G[1]e$ is formed as follows. Let the edges incident with $v$, in cyclic order around $v$ starting with $e$ directed into $v$, be $e, a_1, b_1, \ldots, a_k, b_k, e, c_1, d_1, \ldots, c_l, d_l$. Here, each $a_i$ and each $d_i$ is directed out of $v$, while each $b_i$ and $c_i$ is directed into $v$. We replace $v$ by two new vertices, $v_1$ and $v_2$, and reconnect the edges $a_i, b_i, c_i, d_i$ as follows. The tail of each $a_i$ and the head of each $b_i$ becomes $v_1$ instead of $v$, while the head of each $c_i$ and the tail of each $d_i$ becomes $v_2$ instead of $v$. The edge $e$ is deleted. The cyclic orderings of edges around $v_1$ and $v_2$ are those induced by the ordering around $v$. Everything else is unchanged. Observe that if $e$ is a separating 1-semiloop then, in effect, the shrinking of the loop severs the graph in its clockwise side from that in its anticlockwise side. If $e$ is not separating then, in effect, shrinking the loop cuts one of the handles of the surface component in which it is embedded, so reducing the genus by 1.

This operation is called 1-reduction or contraction. See Figure 2(a),(b). It just adapts the usual contraction operation for surface minors to the alternating dimap context.
Figure 2: Minor operations: $G$ and reductions (black, solid edges, filled vertices), with their trials, which equal $G^\omega$ and reductions (blue, dashed edges, open vertices).

(a) $G$ and $G^\omega$

(b) $G[1]e$ and $(G[1]e)^\omega = G^\omega[\omega^2]e^\omega$

(c) $G[\omega]e$ and $(G[\omega]e)^\omega = G^\omega[1]e^\omega$

(d) $G[\omega^2]e$ and $(G[\omega^2]e)^\omega = G^\omega[\omega]e^\omega$
Let \( f = f(v, w_1) \) be the right successor of \( e \), and let \( g = g(v, w_2) \) be the left successor of \( e \). (It is possible that \( f = g \), in which case \( w_1 = w_2 \). This occurs when \( v \) has indegree = outdegree = 1, i.e., when \( e \) is a 1-loop.)

The graph \( G[\omega]e \) is formed by deleting \( e \) and \( g \) and, if \( e \neq g \), changing \( g \) so that it now joins \( u \) to \( w_2 \). The revised edge \( g \) replaces \( e \) and \( g \) in \( A(e) \), and it replaces \( g \) in \( I(w_2) \). If \( \text{deg} v = 2 \), then \( v \) and \( I(v) \) no longer exist in \( G[\omega]e \). If \( \text{deg} v \neq 2 \), then the in-star at \( v \) in \( G[\omega]e \) is \( I(v) \setminus \{e\} \). The c-face containing \( e' \) in \( G[\omega]e \) is \( (C(e) \setminus \{e\}) \cup C(g) \). This operation is called \( \omega \)-reduction. See Figure 2(a),(c).

The graph \( G[\omega^2]e \) is formed by deleting \( e \) and, if \( e \neq f \), changing \( f \) so that it now joins \( u \) to \( w_1 \). The revised edge \( f \) replaces \( e \) and \( f \) in \( C(e) \), and it replaces \( f \) in \( I(w_1) \). If \( \text{deg} v = 2 \), then \( v \) and \( I(v) \) no longer exist in \( G[\omega^2]e \). If \( \text{deg} v \neq 2 \), then the in-star at \( v \) in \( G[\omega^2]e \) is \( I(v) \setminus \{e\} \). The a-face containing \( e' \) in \( G[\omega^2]e \) is \( (A(e) \setminus \{e\}) \cup A(f) \). This operation is called \( \omega^2 \)-reduction. See Figure 2(a),(d).

For all \( \mu \in \{1, \omega, \omega^2\} \), \( \mu \)-reduction of a proper \( \mu^{-1} \)-semiloop either increases the number of connected components or decreases the genus.

Each of these three operations is a reduction or a minor operation.

The operations of \( \omega \)-reduction and \( \omega^2 \)-reduction are special cases of lifting (see, e.g., [28]), used in the immersion relation on graphs [30], here restricted to cases where the two incident edges are consecutive in a face.

An alternating dimap obtained from \( G \) by a sequence of minor operations is a minor of \( G \).

These constructions are illustrated in Figure 2.

If \( e \) is a triloop, then \( G[1]e = G[\omega]e = G[\omega^2]e \). We sometimes write \( G[*]e \) for the common result of the three reductions in this case, in order to avoid being unnecessarily specific.

If \( X = (x_1, \ldots, x_k) \) is a sequence of edges, then we write \( G[\mu]X \) as a shorthand for \( G[\mu]x_1[\mu]x_2[\mu]x_k \).

It is straightforward to translate the the above constructions for the minor operations into the language of permutation triples.
Theorem 3  If $G$ is an alternating dimap with permutation triple $(\sigma_1, \sigma_\omega, \sigma_{\omega^2})$, then the permutation triples for the three minors of $G$ are as given in the following table.

|          | $G[1]e$ | $G[\omega]e$ | $G[\omega^2]e$ |
|----------|---------|--------------|----------------|
| $\sigma_{G[1]e,1}(\sigma_1^{-1}(e))$ | $\sigma_{\omega^2}^{-1}(e)$ | $\sigma_{G[\omega]e,1}(\sigma_1^{-1}(e))$ | $\sigma_1(e)$ |
| $\sigma_{G[1]e,\omega}(\sigma_\omega^{-1}(e))$ | $\sigma_\omega(e)$ | $\sigma_{G[\omega]e,\omega}(\sigma_\omega^{-1}(e))$ | $\sigma_\omega(e)$ |
| $\sigma_{G[1]e,\omega^2}(\sigma_{\omega^2}^{-1}(e))$ | $\sigma_{\omega^2}(e)$ | $\sigma_{G[\omega]e,\omega^2}(\sigma_{\omega^2}^{-1}(e))$ | $\sigma_{\omega^2}(e)$ |
| $\sigma_{G[1]e,1}(\sigma_\omega(e))$ | $\sigma_1(e)$ | $\sigma_{G[\omega]e,\omega^2}(\sigma_1(e))$ | $\sigma_{\omega^2}(e)$ |

Otherwise:

|          | $G[1]e$ | $G[\omega]e$ | $G[\omega^2]e$ |
|----------|---------|--------------|----------------|
| $\sigma_{G[1]e,\mu}(f)$ | $\sigma_{\mu}(f)$ | $\sigma_{G[\omega]e,\mu}(f)$ | $\sigma_{\mu}(f)$ |
| $\sigma_{G[\omega]e,\mu}(f)$ | $\sigma_{\mu}(f)$ | $\sigma_{G[\omega^2]e,\mu}(f)$ | $\sigma_{\mu}(f)$ |

We will refer to a specific equation in the table by “Theorem 3(r, c)”, where $r$ and $c$ index the row and column in which the equation appears. For example, Theorem 3(2,1) is $\sigma_{G[1]e,\omega}(\sigma_\omega^{-1}(e)) = \sigma_\omega(e)$, and Theorem 3(5,2) is $\sigma_{G[\omega^2]e,\mu}(f) = \sigma_{\mu}(f)$ which holds when the pair $\mu, f$ is not covered by any of the previous entries in that column.

We are now in a position to establish the relationship between minors and triality.

Theorem 4  If $e \in E(G)$ and $\mu, \nu \in \{1, \omega, \omega^2\}$ then

$$G^\mu[\nu]e^\mu = (G[\mu\nu]e)^\mu.$$  

Proof.  We first prove that

$$G^\omega[1]e^\omega = (G[\omega]e)^\omega. \quad (4)$$

If $e$ is an ultraloop, then so is $e^\omega$, and both sides of (4) are empty, so the equation is true. So suppose that $e$ is not an ultraloop.

$$\sigma_{G^\omega[1]e^\omega,1}(\sigma_{G^\omega[1]e^\omega}^{-1}(e^\omega)) = \sigma_{G^\omega[\omega,1]e^\omega}^{-1}(e^\omega) \quad \text{(by Theorem 3(1,1))}$$

$$= \sigma_{G^\omega[\omega,\omega^2]e^\omega}^{-1}(e^\omega) \quad \text{(by Theorem 3(2,1))}$$

$$= \sigma_{G^\omega[\omega,\omega^2]e^\omega}^{-1}(e^\omega)^\omega \quad \text{(by Theorem 3(3,2))}$$

Similarly, we have

$$\sigma_{G^\omega[1]e^\omega,\omega}(\sigma_{G^\omega[1]e^\omega}^{-1}(e^\omega)) = \sigma_{G^\omega[\omega,\omega]e^\omega}^{-1}(e^\omega) = \sigma_{G[\omega,1]e,1}(\sigma_{G[\omega,1]e}^{-1}(e)) = \sigma_{G[\omega]e,\omega^2}^{-1}(e^\omega)$$
Equivalently, in these cases, we have
\[ \sigma_G^{-1}(e^\omega) = \sigma_G^{-1}(\sigma_G^{-1}(e^\omega)) = \sigma_G^{-1}(\sigma_G^{-1}(e^\omega)) \]

or in other words, \( \sigma_G^{-1}(e^\omega) = \sigma_G^{-1}(\sigma_G^{-1}(e^\omega)) \).

It follows that
\[ \sigma_G^{-1}(e^\omega) = \sigma_G^{-1}(\sigma_G^{-1}(e^\omega)) \]

or in other words, \( \sigma_G^{-1}(e^\omega) = \sigma_G^{-1}(\sigma_G^{-1}(e^\omega)) \).

Similar arguments show that
\[ G^\omega \cdot e^\omega = (G[\omega]e)^\omega, \]

From these it follows that
\[ G^\omega [1] e^\omega = (G[\omega]e)^\omega, \]

or in other words, \( G^\omega [1] e^\omega = (G[\omega]e)^\omega \).

We have now shown that, for all \( \mu \in \{1, \omega, \omega^2\} \) and \( f \in E(G) \),
\[ \sigma_G^{-1}(e^\omega) = \sigma_G^{-1}(\sigma_G^{-1}(e^\omega)) \]

It follows that
\[ (\sigma_G^{-1}(e^\omega), \sigma_G^{-1}(\sigma_G^{-1}(e^\omega), \sigma_G^{-1}(e^\omega)) = (\sigma_G^{-1}(e^\omega), \sigma_G^{-1}(e^\omega), \sigma_G^{-1}(e^\omega)) \]

or in other words, \( G^\omega [1] e^\omega = (G[\omega]e)^\omega \).

Similar arguments show that
\[ G^\omega [\omega] e^\omega = (G[\omega]e)^\omega, \]

From these it follows that
\[ G^\omega [1] e^\omega = (G[\omega]e)^\omega, \]

or in other words, \( G^\omega [1] e^\omega = (G[\omega]e)^\omega \).

It remains to consider the cases where
\[ (\mu, f^\omega) \notin \{ (1, \sigma_G^{-1}(e^\omega)), (\omega, \sigma_G^{-1}(e^\omega)), (\omega^2, \sigma_G^{-1}(e^\omega)), (1, \sigma_G^{-1}(e^\omega)) \}. \]

Equivalently,
\[ (\mu \omega, f^\omega) \notin \{ (1, \sigma_G^{-1}(e^\omega)), (\omega, \sigma_G^{-1}(e^\omega)), (\omega^2, \sigma_G^{-1}(e^\omega)), (1, \sigma_G^{-1}(e^\omega)) \}. \]

In these cases, we have
\[ \sigma_G^{-1}(e^\omega) = (\sigma_G^{-1}(e^\omega))^\omega \]

(by Theorem 3.1, using (5))

or in other words, \( \sigma_G^{-1}(e^\omega) = (\sigma_G^{-1}(e^\omega))^\omega \).

We have now shown that, for all \( \mu \in \{1, \omega, \omega^2\} \) and \( f \in E(G) \),
\[ \sigma_G^{-1}(e^\omega) = (\sigma_G^{-1}(e^\omega))^\omega, \]

It follows that
\[ (\sigma_G^{-1}(e^\omega), \sigma_G^{-1}(\sigma_G^{-1}(e^\omega)), \sigma_G^{-1}(e^\omega^2)) = (\sigma_G^{-1}(e^\omega), \sigma_G^{-1}(e^\omega), \sigma_G^{-1}(e^\omega^2)) \]

or in other words, \( G^\omega [1] e^\omega = (G[\omega]e)^\omega \).

Similar arguments show that
\[ G^\omega [\omega] e^\omega = (G[\omega]e)^\omega, \]

From these it follows that
\[ G^\omega [1] e^\omega = (G[\omega]e)^\omega, \]

or in other words, \( G^\omega [1] e^\omega = (G[\omega]e)^\omega \).

\[ \square \]
Figure 3: The relationship between the ordinary duality and minor operations.

Figure 4: The relationship between triality and the three minor operations. The diagram wraps around at its left and right sides.
Theorem 4 extends the classical relationship between duality and minors:

\[ G \setminus e = (G/e)^*, \quad G/e = (G \setminus e)^*. \]

The classical relationship is illustrated in Figure 3 while Theorem 4 is illustrated in Figure 4.

The mappings \( \sigma_{H,\mu} \), where \( H \) ranges over all minors of \( G \) and \( \mu \in \{1, \omega, \omega^2\} \), together generate (under composition) an inverse semigroup, which we denote by IS(\( G \)).

**Problem**

Describe IS(\( G \)), and classify it among known types of inverse semigroup.

## 3 Non-commutativity

Deletion and contraction are well known to commute, in the sense that, for any graph \( G \) and any distinct \( e, f \in E(G) \), we have

\[ G \setminus e \setminus f = G \setminus f \setminus e, \quad G/e/f = G/f/e, \quad G \setminus e/f = G/f \setminus e. \]

The variants of these operations for embedded graphs, where deletion/contraction of an edge is accompanied by appropriate modifications to the embedding, also commute [29, p. 103].

Perhaps surprisingly, the reductions we have introduced for alternating dimaps do not always commute, though they do in most situations. In this section we investigate the circumstances under which the reductions do or do not commute.

We first show that the reductions always commute if one of the edges involved is a triloop.

**Lemma 5** If \( f \) is a triloop, then for any \( \nu \in \{1, \omega, \omega^2\} \),

\[ G[1]e[\nu]f = G[\nu]f[1]e. \]

**Proof.** If \( f \) is an \( \omega \)-loop or an \( \omega^2 \)-loop, then \( \nu \)-reduction of \( f \) just amounts to deletion of \( f \). So

\[ G[1]e[\nu]f = G/e \setminus f = G \setminus f/e = G[\nu]f[1]e, \]

where the middle equality follows from the fact that deletion and contraction commute. (These deletion and contraction operations are for embedded graphs, and give surface minors, and so are not the usual deletion and contraction operations for abstract graphs. But it is still easy to see that they commute when \( f \) is a loop that is contractible in the surface.)
If $f$ is a 1-loop, then $\nu$-reduction of $f$ just amounts to contraction of $f$. So
\[ G[1]e[\nu]f = G/e/f = G/f/e = G[\nu]f[1]e. \]

**Theorem 6** If $f$ is a triloop and $\mu, \nu \in \{1, \omega, \omega^2\}$ then
\[ G[\mu]e[\nu]f = G[\nu]f[\mu]e. \]

**Proof.**
\[
G[\mu]e[\nu]f = \left( G^\mu[1]e^\mu[\nu^{-1}]f^\mu\right)^{\mu^{-1}} \quad \text{(by Theorem 4)}
= \left( G^\mu[\nu^{-1}]f^\mu[1]e^\mu\right)^{\mu^{-1}} \quad \text{(by Lemma 5)}
= G[\nu]f[\mu]e \quad \text{(by Theorem 4)}.
\]

We next show that two reductions of the same type always commute.

**Theorem 7** For all $\mu \in \{1, \omega, \omega^2\}$,
\[ G[\mu]e[\mu]f = G[\mu]f[\mu]e. \]

**Proof.** We show that
\[
G[1]e[1]f = G[1]f[1]e, \quad \text{(7)}
\]
which takes up most of the proof, and then use triality to complete it.

To show (7), we will show that, for all $\mu \in \{1, \omega, \omega^2\}$ and all $g \in E(G) \setminus \{e, f\}$,
\[
\sigma_{G[1]e[1]f, \mu}(g) = \sigma_{G[1]f[1]e, \mu}(g). \quad \text{(8)}
\]

We first do this for $\mu \in \{\omega, \omega^2\}$, which we now assume. Most situations are covered by the following reasoning:
\[
\sigma_{G[1]e[1]f, \mu}(g) = \begin{cases} 
\sigma_{G[1]e, \mu}(g) & \text{if } g \neq \sigma_{G[1]e, \mu}^{-1}(f) \\
\sigma_{G, \mu}(g) & \text{if } g \neq \sigma_{G, \mu}^{-1}(e) \\
\sigma_{G[1]f, \mu}(g) & \text{if } g \neq \sigma_{G[1]f, \mu}^{-1}(f) \\
\sigma_{G[1]f[1]e, \mu}(g) & \text{if } g \neq \sigma_{G[1]f[1]e, \mu}^{-1}(e)
\end{cases}
\]
by four applications of Theorem 5, since the conditions on $g$ ensure that cases (2,1) and (3,1) (according as $\mu = \omega$ or $\mu = \omega^2$) of that Theorem do not apply.

We now deal with situations where the above conditions on $g$ are not met. We have, apparently, four exceptional values of $g$. We consider each in turn.

Firstly, suppose $g = \sigma_{G[1]e, \mu}^{-1}(e)$.

In this case, we must assume that $f \neq \sigma_{G[1]f, \mu}^{-1}(e)$, else $g = f$ and $g \notin \text{dom } \sigma_{G[1]e[1]f, \mu}$. Consider $\sigma_{G[1]e[1]f, \mu}(g)$.
If \( f = \sigma_{G,\mu}(e) \) then \( \sigma_{G,\mu}^{-1}(f) = \sigma_{G,\mu}^{-1}(e) \), by Theorem \( \box{2,1},(3,1) \). This justifies the first step in the following:

\[
\sigma_{[1][1],\mu}(\sigma_{G,\mu}^{-1}(e)) = \sigma_{[1][1],\mu}(\sigma_{G,\mu}^{-1}(f)) = \sigma_{G,\mu}(f) \quad \text{(by Theorem \( \box{2,1} \) or (3,1))}
\]

On the other hand, if \( f \neq \sigma_{G,\mu}(e) \), then \( \sigma_{G,\mu}^{-1}(f) = \sigma_{G,\mu}^{-1}(e) \), by Theorem \( \box{3,1} \). This in turn does not equal \( \sigma_{G,\mu}^{-1}(e) \), since \( e \neq f \) and \( \sigma_{G,\mu}^{-1} \) is a bijection. So

\[
\sigma_{[1][1],\mu}(\sigma_{G,\mu}^{-1}(e)) = \sigma_{[1][1],\mu}(\sigma_{G,\mu}^{-1}(e)) = \sigma_{G,\mu}(e),
\]

by Theorem \( \box{3,1} \) then (2,1) or (3,1).

Now consider \( \sigma_{G,\mu}^{-1}(f) \). Observe that \( \sigma_{G,\mu}^{-1}(e) = \sigma_{G,\mu}^{-1}(f) \), by Theorem \( \box{3,1} \), since \( f \neq \sigma_{G,\mu}^{-1}(e) \). Therefore

\[
\sigma_{[1][1],\mu}(\sigma_{G,\mu}^{-1}(e)) = \sigma_{[1][1],\mu}(\sigma_{G,\mu}^{-1}(e)) = \sigma_{G,\mu}(e)
\]

by Theorem \( \box{2,1} \) or (3,1), and (5,1).

So \( \sigma_{G,\mu}^{-1}(g) = \sigma_{G,\mu}^{-1}(g) \) when \( g = \sigma_{G,\mu}^{-1}(e) \).

Secondly, suppose \( g = \sigma_{G,\mu}^{-1}(f) \). This can be treated the same as the first case, except that \( e \) and \( f \) are swapped throughout.

Thirdly and fourthly, the remaining two exceptional values of \( g \), namely \( \sigma_{G,\mu}^{-1}(f) \) and \( \sigma_{G,\mu}^{-1}(e) \), are really nothing new, for application of Theorem \( \box{3} \) gives

\[
\sigma_{G,\mu}^{-1}(f) = \begin{cases} 
\sigma_{G,\mu}^{-1}(e), & \text{if } f = \sigma_{G,\mu}(e), \\
\sigma_{G,\mu}^{-1}(f), & \text{otherwise};
\end{cases}
\]


\[
\sigma_{G,\mu}^{-1}(e) = \begin{cases} 
\sigma_{G,\mu}^{-1}(e), & \text{if } f = \sigma_{G,\mu}(e), \\
\sigma_{G,\mu}^{-1}(f), & \text{otherwise}.
\end{cases}
\]

Thus, in any event, each of these two values of \( g \) actually falls into one of the first two cases. This completes the treatment of the exceptional values of \( g \) (apparently four in number, but really just two).

We have now proved \( \box{8} \) for \( \mu \in \{\omega,\omega^2\} \). But it then follows immediately for \( \mu = 1 \) too, since \( \sigma_{H,1} = \sigma_{H,\omega} \circ \sigma_{H,\omega}^{-1} \) for any \( H \). So \( \box{8} \) holds for all \( \mu \) and all \( g \), which establishes \( \box{7} \).

Now that we know contractions commute, we can use triality to show that any two reductions of the same type commute. For any \( \mu \in \{1,\omega,\omega^2\} \),

\[
G[\mu]e[\mu]f = (G[1]e[1]f[\mu])^{1-\mu} = (G[1]f[1]e[\mu])^{1-\mu} = G[\mu]f[\mu]e.
\]
However, it is not always the case that two reductions commute. Figure 5 illustrates the fact that, in general, if $f = \sigma_{G,\omega}(e)$ then $G[1]e[\omega]f \neq G[\omega]f[1]e$. By triality, it follows that if $f = \sigma_{G,\omega^2}(e)$ then in general $G[\omega^2]e[1]f \neq G[1]f[\omega^2]e$, and if $f = \sigma_{G,1}(e)$ then in general $G[\omega^2]e[\omega^2]^2 f \neq G[\omega^2]f[\omega]e$.

Most of the remainder of this section is devoted to showing that these exceptional cases are the only situations where reductions do not commute.

We will need some lemmas.

**Lemma 8**

(a) $\sigma^{-1}_{G[1]e,\omega}(h) = \begin{cases} 
\sigma^{-1}_{G,\omega}(e), & \text{if } h = \sigma_{G,\omega}(e); \\
\sigma^{-1}_{G,\omega}(h), & \text{otherwise.} 
\end{cases}$

(b) $\sigma^{-1}_{G[\omega]f,\omega}(h) = \begin{cases} 
\sigma^{-1}_{G,\omega}(f), & \text{if } h = \sigma_{G,\omega}(f); \\
\sigma^{-1}_{G,\omega}(h), & \text{otherwise.} 
\end{cases}$

(c) $\sigma^{-1}_{G[1]e,1}(h) = \begin{cases} 
\sigma_{G,\omega}(e), & \text{if } h = \sigma_{G,1}(e); \\
\sigma^{-1}_{G,\omega}(e), & \text{if } h = \sigma^{-1}_{G,\omega^2}(e); \\
\sigma_{G,1}(e), & \text{if } h = \sigma_{G,1}(e); \\
\sigma^{-1}_{G,1}(h), & \text{otherwise.} 
\end{cases}$
(d)  \[ \sigma_{G[\omega]f_{1}}^{-1}(h) = \begin{cases} \sigma_{G,1}^{-1}(f), & \text{if } h = \sigma_{G,1}(f); \\ \sigma_{G,1}^{-1}(h), & \text{otherwise.} \end{cases} \]

(e)  \[ \sigma_{G[1]e,\omega^{2}}^{-1}(h) = \begin{cases} \sigma_{G,\omega^{2}}^{-1}(e), & \text{if } h = \sigma_{G,\omega^{2}}(e); \\ \sigma_{G,\omega^{2}}^{-1}(h), & \text{otherwise.} \end{cases} \]

(f)  \[ \sigma_{G[\omega]f_{\omega^{2}}}^{-1}(h) = \begin{cases} \sigma_{G,\omega^{2}}^{-1}(f), & \text{if } h = \sigma_{G,\omega}^{-1}(f); \\ \sigma_{G,1}(f), & \text{if } h = \sigma_{G,\omega^{2}}(f); \\ \sigma_{G,\omega^{2}}^{-1}(h), & \text{otherwise.} \end{cases} \]

Proof. Immediate from: (a) Theorem 3(2,1),(5,1); (b) Theorem 3(2,2),(5,2); (c) Theorem 3(1,1),(4,1),(5,1); (d) Theorem 3(1,2),(5,2); (e) Theorem 3(3,1),(5,1); (f) Theorem 3(3,2),(4,2),(5,2).

Lemma 9 If \( f \neq \sigma_{G,\omega}(e) \) then

\[ \sigma_{G[1]e[\omega]f_{\omega}} = \sigma_{G[\omega]f[1]e[\omega]}. \]

Proof. The proof has some similarities to that of Theorem 7 but is significantly more complicated.

We prove that

\[ \sigma_{G[1]e[\omega]f_{\omega}}(g) = \sigma_{G[\omega]f[1]e[\omega]}(g) \]

for all \( g \in E(G) \setminus \{e, f\} \).

If \( g \notin \{\sigma_{G,\omega}^{-1}(e), \sigma_{G,\omega}^{-1}(f), \sigma_{G[1]e,\omega}^{-1}(f), \sigma_{G[\omega]f_{\omega}}^{-1}(e)\} \) then

\[ \sigma_{G[1]e[\omega]f_{\omega}}(g) = \sigma_{G[1]e,\omega}(g) = \sigma_{G,\omega}(g) = \sigma_{G[\omega]f_{\omega}}(g) = \sigma_{G[\omega]f[1]e,\omega}(g), \]

by Theorem 3(5,1),(5,2).

This leaves four special cases for \( g \), which we will consider in turn, after noting some facts which we will use repeatedly.

Observe that the Lemma’s condition, \( f \neq \sigma_{G,\omega}(e) \), implies

\[ \sigma_{G[1]e,\omega}^{-1}(f) \neq \sigma_{G,\omega}^{-1}(e), \]

by Lemma 8(a).

Also, by Lemma 8(b),

\[ e = \sigma_{G,\omega}(f) \iff \sigma_{G[\omega]f_{,\omega}}^{-1}(e) = \sigma_{G,\omega}^{-1}(f). \]

Case 1: \( g = \sigma_{G,\omega}^{-1}(e) \).
\[ \sigma_{G[1]e[\omega]f,\omega}(\sigma_{G,\omega}^{-1}(e)) = \sigma_{G[1]e,\omega}(\sigma_{G,\omega}^{-1}(e)) \] 
(by Theorem 3(5,2) applied to \( G[1]e \), using (9))

\[ = \sigma_{G,\omega}(e) \] 
(by Theorem 3(2,1)).

On the other hand, if \( e \neq \sigma_{G,\omega}(f) \) then

\[ \sigma_{G[\omega]f[1]e,\omega}(\sigma_{G,\omega}^{-1}(e)) = \sigma_{G[\omega]f[1]e,\omega}(\sigma_{G[\omega]f,\omega}^{-1}(e)) \] 
(by Lemma 8(b))

\[ = \sigma_{G[\omega]f,\omega}(e) \] 
(by Theorem 3(2,1)) applied to \( G[\omega]f \)

\[ = \sigma_{G,\omega}(e) \] 
(by Theorem 3(5,2), using (9)).

while if \( e = \sigma_{G,\omega}(f) \) then \( \sigma_{G,\omega}^{-1}(e) = f \) which is not in the domain of \( \sigma_{G[\omega]f[1]e,\omega} \) so this situation does not arise.

So, in any event, \( \sigma_{G[1]e[\omega]f,\omega}(g) = \sigma_{G[\omega]f[1]e,\omega}(g) \) in this case.

**Case 2:** \( g = \sigma_{G,\omega}^{-1}(f) \).

\[ \sigma_{G[1]e[\omega]f,\omega}(\sigma_{G,\omega}^{-1}(f)) = \sigma_{G[1]e[\omega]f,\omega}(\sigma_{G[1]e,\omega}^{-1}(f)) \] 
(by Lemma 8(a), using \( f \neq \sigma_{G,\omega}(e) \))

\[ = \sigma_{G[1]e,\omega}(f) \] 
(by Theorem 3(2,2))

\[ = \begin{cases} 
\sigma_{G,\omega}(f) & \text{if } f \neq \sigma_{G,\omega}^{-1}(e), \\
\sigma_{G[1]e,\omega}(\sigma_{G,\omega}^{-1}(e)) & \text{if } f = \sigma_{G,\omega}^{-1}(e),
\end{cases} \]

\[ = \begin{cases} 
\sigma_{G,\omega}(f), & \text{if } f \neq \sigma_{G,\omega}^{-1}(e), \\
\sigma_{G,\omega}(e), & \text{if } f = \sigma_{G,\omega}^{-1}(e).
\end{cases} \]

Now consider \( \sigma_{G[\omega]f[1]e,\omega}(\sigma_{G,\omega}^{-1}(f)) \).

If \( e \neq \sigma_{G,\omega}(f) \), we have

\[ \sigma_{G[\omega]f[1]e,\omega}(\sigma_{G,\omega}^{-1}(f)) = \sigma_{G[\omega]f[1]e,\omega}(\sigma_{G,\omega}^{-1}(f)) \] 
(by Theorem 3(5,1) and (10))

\[ = \sigma_{G,\omega}(f) \] 
(by Theorem 3(2,2)).

If \( e = \sigma_{G,\omega}(f) \) then

\[ \sigma_{G[\omega]f[1]e,\omega}(\sigma_{G,\omega}^{-1}(f)) = \sigma_{G[\omega]f[1]e,\omega}(\sigma_{G[\omega]f,\omega}^{-1}(f)) \] 
(by Lemma 8(b) and (10))

\[ = \sigma_{G[\omega]f,\omega}(e) \] 
(by Theorem 3(2,1))

\[ = \sigma_{G,\omega}(e) \] 
(by Theorem 3(5,2), using \( e \neq \sigma_{G,\omega}(f) \)).

**Case 3:** \( g = \sigma_{G[1]e,\omega}^{-1}(f) \)

\[ \sigma_{G[1]e[\omega]f,\omega}(\sigma_{G[1]e,\omega}^{-1}(f)) = \sigma_{G[1]e,\omega}(f) \] 
(by Theorem 3(2,2))

\[ = \begin{cases} 
\sigma_{G,\omega}(f) & \text{if } e \neq \sigma_{G,\omega}(f), \\
\sigma_{G[1]e,\omega}(\sigma_{G,\omega}^{-1}(e)) & \text{if } e = \sigma_{G,\omega}(f),
\end{cases} \]

\[ = \begin{cases} 
\sigma_{G,\omega}(f), & \text{if } e \neq \sigma_{G,\omega}(f), \\
\sigma_{G,\omega}(e), & \text{if } e = \sigma_{G,\omega}(f) \text{ (by Theorem 3(2,1))}.
\end{cases} \]
If \( e \neq \sigma_{G,\omega}(f) \) then
\[
\sigma_{G[\omega],f[1]e,\omega}(\sigma_{G[1]e,\omega}^{-1}(f)) = \sigma_{G[\omega],f[1]e,\omega}(\sigma_{G,\omega}^{-1}(f)) \quad \text{(by Lemma 8(a))}
\]
\[
= \sigma_{G[\omega],f(\sigma_{G,\omega}^{-1}(f))} \quad \text{(by Theorem 3(3,1) and (10))}
\]
\[
= \sigma_{G,\omega}(f) \quad \text{(by Theorem 3(2,2))}.
\]

If \( e = \sigma_{G,\omega}(f) \) then
\[
\sigma_{G[\omega],f[1]e,\omega}(\sigma_{G[1]e,\omega}^{-1}(f)) = \sigma_{G[\omega],f[1]e,\omega}(\sigma_{G,\omega}^{-1}(f)) \quad \text{(by Lemma 8(a), using \( f \neq \sigma_{G,\omega}(e) \))}
\]
\[
= \sigma_{G[\omega],f[1]e,\omega}(\sigma_{G,\omega}^{-1}(f)) \quad \text{(by Lemma 8(b) and (10))}
\]
\[
= \sigma_{G,\omega}(f) \quad \text{(by Theorem 3(2,1))}
\]
\[
= \sigma_{G,\omega}(e) \quad \text{(by Theorem 3(5,2), using \( e \neq \sigma_{G,\omega}^{-1}(f) \)).}
\]

Case 4: \( g = \sigma_{G[\omega],f,\omega}^{-1}(e) \)

This case can be proved in a manner similar to the previous cases. But in fact this is not necessary, since we have shown the permutations \( \sigma_{G[1]e,\omega}, \omega, \omega \)
agree on every element of their common domain except one, so they must agree on this last element too.

\[\square\]

Lemma 10 If \( f \neq \sigma_{G,\omega}(e) \) then
\[
\sigma_{G[1]e,\omega},f,1 = \sigma_{G[\omega],f[1]e,1}.
\]

Proof. This proof is more complicated again than that of Lemma 9.

We may suppose that neither \( e \) nor \( f \) is a triloop, since we have already established commutativity in such cases, in Lemma 5. So \( \sigma_{G,\mu}(e) \neq e \) and \( \sigma_{G,\mu}(f) \neq f \), for \( \mu \in \{1, \omega, \omega^2\} \).

We prove that
\[
\sigma_{G[1]e,\omega},f,1(g) = \sigma_{G[\omega],f[1]e,1}(g) \tag{11}
\]
for all \( g \in E(G) \setminus \{e, f\} \).

Observe that
\[
\sigma_{G[1]e,\omega},f,\omega^2(g) = \sigma_{G[1]e,\omega}(g) \quad \text{if} \ g \not\in \{\sigma_{G[1]e,\omega}^{-1}(f), \sigma_{G[1]e,1}(f)\}, \text{by Theorem 3(5,2)}
\]
\[
= \sigma_{G,\omega}(g) \quad \text{if} \ g \neq \sigma_{G,\omega}^{-1}(e), \text{by Theorem 3(5,1)}
\]
\[
= \sigma_{G[\omega],f,\omega^2}(g) \quad \text{if} \ g \not\in \{\sigma_{G,\omega}^{-1}(f), \sigma_{G,1}(f)\}, \text{by Theorem 3(5,2)}
\]
\[
= \sigma_{G[\omega],f[1]e,\omega}(g) \quad \text{if} \ g \neq \sigma_{G[\omega],f,\omega^2}(e), \text{by Theorem 3(5,1)}.
\]

It follows that if \( g \not\in \{\sigma_{G,1}(f), \sigma_{G,\omega}^{-1}(f), \sigma_{G,\omega}(e), \sigma_{G[\omega],f,\omega^2}(e), \sigma_{G[1]e,\omega}(f), \sigma_{G[1]e,1}(f)\} \) then
\[
\sigma_{G[1]e,\omega},f,1(g) = \sigma_{G[1]e,\omega}(\sigma_{G[1]e,\omega},f,\omega^2(g)) \quad \text{(using} \sigma_1 \circ \sigma_\omega \circ \sigma_\omega^2 = \text{identity})
\]
\[
= \sigma_{G[1]e,\omega}(\sigma_{G[\omega],f[1]e,\omega}(g)) \quad \text{(by the previous paragraph)}
\]
\[
= \sigma_{G[\omega],f[1]e,\omega}(\sigma_{G[\omega],f[1]e,\omega}(g)) \quad \text{(by Lemma 9)}
\]
\[
= \sigma_{G[\omega],f[1]e,1}(g) \quad \text{(using} \sigma_1 \circ \sigma_\omega \circ \sigma_\omega^2 = \text{identity, again}).
\]
We now consider in turn how to deal with the exceptional values of \( g \), apparently six in number.

Case 1: \( g = \sigma_{G,1}(f) \).

We must have \( \sigma_{G,1}(f) \neq e \), else \( g = e \) which is forbidden.

Theorem \( 3(5,1) \) tells us that

\[
\sigma_{G[1]e,1}(f) = \sigma_{G,1}(f),
\]

(12) since \( f \neq \sigma_{G,1}^{-1}(e) \) (by the previous paragraph) and \( f \neq \sigma_{G,\omega}(e) \) (by hypothesis), so Theorem \( 3(1,1) \) and \( 4(1,1) \) do not apply.

We have

\[
\begin{align*}
\sigma_{G[1]e[\omega],1}(\sigma_{G,1}(f)) & = \sigma_{G[1]e[\omega],1}(\sigma_{G[1]e,1}(f)) \\
& = \sigma_{G[1]e,1}(f) \quad \text{(by Lemma \( 8(d) \), first case)} \\
& = \begin{cases} 
\sigma_{G,\omega}(e), & \text{if } f = \sigma_{G,1}(e), \\
\sigma_{G,1}^{-1}(e), & \text{if } f = \sigma_{G,1}(e), \\
\sigma_{G,1}(f), & \text{otherwise,}
\end{cases}
\end{align*}
\]

by Lemma \( 8(c) \).

Now consider \( \sigma_{G[\omega]f[1]e,1}^{-1}(\sigma_{G,1}(f)) \).

If \( f = \sigma_{G,1}(e) \) then \( \sigma_{G[\omega]f,1}(e) = \sigma_{G[\omega]f,1}(\sigma_{G,1}(f)) = \sigma_{G,1}(f) \), with the second equality following from Theorem \( 3(1,2) \). This justifies the first step of the following.

\[
\begin{align*}
\sigma_{G[\omega]f[1]e,1}(\sigma_{G,1}(f)) & = \sigma_{G[\omega]f[1]e,1}(\sigma_{G[\omega]f,1}(e)) \\
& = \sigma_{G[\omega]f,1}(e) \quad \text{(by Lemma \( 8(c) \))} \\
& = \sigma_{G,\omega}(e) \quad \text{(using our hypothesis, } e \neq \sigma_{G,\omega}(f)\text{).}
\end{align*}
\]

If \( f = \sigma_{G,\omega}^{-1}(e) \), i.e., \( e = \sigma_{G,\omega}(f) \), then \( \sigma_{G[\omega]f,1}(e) = \sigma_{G,1}(f) \), by Lemma \( 8(f) \) (second case). This justifies the first step of the following.

\[
\begin{align*}
\sigma_{G[\omega]f[1]e,1}(\sigma_{G,1}(f)) & = \sigma_{G[\omega]f[1]e,1}(\sigma_{G[\omega]f,1}(e)) \\
& = \sigma_{G[\omega]f,1}(e) \quad \text{(by Lemma \( 8(c) \))} \\
& = \sigma_{G,1}(f) \quad \text{(by Lemma \( 8(d) \), using } e \neq \sigma_{G,1}(f)\text{).}
\end{align*}
\]

Suppose, then, that \( f \neq \sigma_{G,1}(e) \) and \( f \neq \sigma_{G,\omega}^{-1}(e) \).

From \( e \neq \sigma_{G,1}^{-1}(f) \) we deduce that \( \sigma_{G[\omega]f,1}(e) = \sigma_{G,1}(f) \), by Theorem \( 3(5,2) \). Also, since \( e \neq \omega \) and \( \sigma_{G,1} \) is a bijection, we have \( \sigma_{G,1}(e) \neq \sigma_{G,1}(f) \). So \( \sigma_{G[\omega]f,1}(e) \neq \sigma_{G,1}(f) \).

From \( e \neq \sigma_{G,\omega}(f) \), and our hypothesis \( e \neq \sigma_{G,\omega}(f) \), we deduce from Lemma \( 8(f) \) that \( \sigma_{G[\omega]f,1}(e) = \sigma_{G,\omega}^{-1}(e) \). Our hypothesis \( e \neq \sigma_{G,\omega}(f) \) implies \( e \neq \sigma_{G,\omega}^{-1}(\sigma_{G,1}(f)) \), which in turn implies \( \sigma_{G,\omega}^{-1}(e) \neq \sigma_{G,1}(f) \). Combining the conclusions of the two previous sentences, we obtain \( \sigma_{G[\omega]f,1}(e) \neq \sigma_{G,1}(f) \).
The conclusions of the previous two paragraphs, together with Lemma 8(c), justify the first step in the following.

\[
\sigma_{G,1}^{-1}(\sigma_{G,1}(f)) = \sigma_{G,1}^{-1}(\sigma_{G,1}(f))
\]

We have shown, then, that \(\sigma_{G,1}^{-1}(\sigma_{G,1}(f))\) and \(\sigma_{G,1}^{-1}(\sigma_{G,1}(f))\) agree on \(g = \sigma_{G,1}(f)\), in all circumstances. This deals with the first of our exceptional values of \(g\).

Case 2: \(g = \sigma_{G,1}^{-1}(f)\).

We must have \(\sigma_{G,1}^{-1}(f) \neq e\), else \(g = e\) which is forbidden.

Firstly, observe that \(\sigma_{G,1}(f) \in \{\sigma_{G,1}^{-1}(e), \sigma_{G,1}(f)\}\), by Theorem 8(1,1),(5,1), using the hypothesis \(f \neq \sigma_{G,1}(e)\). Now, \(\sigma_{G,1}^{-1}(e) \neq \sigma_{G,1}^{-1}(f)\), since \(e \neq f\). Furthermore, \(\sigma_{G,1}(f) \neq \sigma_{G,1}^{-1}(f)\), since if \(\sigma_{G,1}(f) = \sigma_{G,1}^{-1}(f)\) then \(f = \sigma_{G,1}(f) = \sigma_{G,1}^{-1}(f)\), which means that \(f\) is a triloop, which we excluded at the start. So, whatever its value, we have \(\sigma_{G,1}^{-1}(f) \neq \sigma_{G,1}^{-1}(f)\).

Secondly, observe that \(\sigma_{G,1}^{-1}(e) = \sigma_{G,1}(f)\), by Lemma 8(c), using \(f \neq \sigma_{G,1}^{-1}(e)\) (see the start of this Case).

The conclusions of these two previous paragraphs justify the first two steps in the following.

\[
\sigma_{G,1}^{-1}(\sigma_{G,1}(f)) = \sigma_{G,1}^{-1}(\sigma_{G,1}(f))
\]

by Theorem 8(2,1),(5,2).

Now consider \(\sigma_{G,1}^{-1}(\sigma_{G,1}^{-1}(f))\).

If \(f = \sigma_{G,1}^{-1}(e)\), then \(\sigma_{G,1}^{-1}(f) = \sigma_{G,1}(e)\). Also, \(e \neq \sigma_{G,1}^{-1}(f)\), since if \(e = \sigma_{G,1}^{-1}(f)\) then \(f = \sigma_{G,1}^{-1}(\sigma_{G,1}(f)) = \sigma_{G,1}(f)\), so that \(f\) is a triloop, which we have excluded. So \(\sigma_{G,1}^{-1}(f) = \sigma_{G,1}(e) = \sigma_{G,1}^{-1}(f)\), with the first equality holding by Theorem 8(5,2).

This justifies the first step in the following.

\[
\sigma_{G,1}^{-1}(\sigma_{G,1}(f)) = \sigma_{G,1}^{-1}(\sigma_{G,1}(f))
\]

by Theorem 8(2,1),(5,2), since \(e \neq \sigma_{G,1}^{-1}(f)\) by hypothesis.

If \(f \neq \sigma_{G,1}(e)\), then \(\sigma_{G,1}^{-1}(f) \neq \sigma_{G,1}(e)\). Also, \(\sigma_{G,1}(f) \neq \sigma_{G,1}(f)\), else \(f\) is a triloop, as we saw early in this Case. So \(\sigma_{G,1}^{-1}(f) \not\in \{\sigma_{G,1}(e), \sigma_{G,1}(f)\}\). But \(\sigma_{G,1}(f) \in \{\sigma_{G,1}(e), \sigma_{G,1}(f)\}\). So \(\sigma_{G,1}(f) \neq \sigma_{G,1}(f)\).
Since \( e \neq \sigma_{G,\omega}^{-1}(f) \) by hypothesis, \( \sigma_{G[\omega]f,\omega^2}^{-1}(e) \in \{\sigma_{G,1}(f), \sigma_{G,\omega^2}(e)\} \). Now, as we have seen, \( \sigma_{G,\omega^2}^{-1}(f) \neq \sigma_{G,1}(f) \), else \( f \) is a triloop; also, \( \sigma_{G,\omega^2}^{-1}(f) \neq \sigma_{G,\omega^2}^{-1}(e) \), since \( e \neq f \). So \( \sigma_{G,\omega^2}^{-1}(f) \neq \sigma_{G[\omega]f,\omega^2}(e) \).

The conclusions of the previous two paragraphs, together with Lemma 8(c), justify the first step of the following.

\[
\sigma_{G[\omega]f[1]e,1}^{-1}(\sigma_{G,\omega^2}^{-1}(f)) = \sigma_{G[\omega]f,1}^{-1}(\sigma_{G,\omega^2}^{-1}(f)) = \sigma_{G,1}^{-1}(\sigma_{G,\omega^2}(f)) \quad \text{(by Lemma 8(d), using } \sigma_{G,\omega^2}^{-1}(f) \neq \sigma_{G,1}(f)\text{)}
\]

So \( \sigma_{G[1]e[\omega]f,1}^{-1} \) and \( \sigma_{G[\omega]f[1]e,1}^{-1} \) agree on \( g = \sigma_{G,\omega^2}^{-1}(f) \), in all circumstances. This deals with the second of our exceptional values of \( g \).

Case 3: \( g = \sigma_{G,\omega^2}^{-1}(e) \).

We must have \( \sigma_{G,\omega^2}^{-1}(e) \neq f \), else \( g = f \) which is forbidden.

Observe that \( \sigma_{G[1]e,1}(\sigma_{G,1}^{-1}(e)) = \sigma_{G,\omega^2}^{-1}(e) \), by Theorem 3(1,1).

So, if \( f = \sigma_{G,1}^{-1}(e) \) then

\[
\sigma_{G[1]e[\omega]f,1}^{-1}(\sigma_{G,\omega^2}^{-1}(e)) = \sigma_{G[1]e[\omega]f,1}^{-1}(\sigma_{G[1]e,1}(f)) = \sigma_{G,1}^{-1}(f) \quad \text{(by Lemma 8(c))}
\]

by \( \sigma_{G,1}^{-1}(f) \) with \( f \neq \sigma_{G,\omega^2}^{-1}(e) \).

On the other hand, if \( f \neq \sigma_{G,1}^{-1}(e) \) then \( \sigma_{G[1]e,1}(f) \neq \sigma_{G[1]e,1}(\sigma_{G,1}^{-1}(e)) \), since \( \sigma_{G[1]e,1} \) is a bijection. So \( \sigma_{G,\omega^2}^{-1}(e) \neq \sigma_{G[1]e,1}(f) \). We therefore have

\[
\sigma_{G[1]e[\omega]f,1}^{-1}(\sigma_{G,\omega^2}^{-1}(e)) = \sigma_{G[1]e[\omega]f,1}^{-1}(\sigma_{G,\omega^2}^{-1}(e)) \quad \text{(by Lemma 8(d))}
\]

So, in summary,

\[
\sigma_{G[1]e[\omega]f,1}^{-1}(\sigma_{G,\omega^2}^{-1}(e)) = \begin{cases} 
\sigma_{G,\omega}(e), & \text{if } f = \sigma_{G,1}^{-1}(e) \text{ and } f = \sigma_{G,1}(e), \\
\sigma_{G,1}^{-1}(f), & \text{if } f = \sigma_{G,1}^{-1}(e) \text{ and } f \neq \sigma_{G,1}(e), \\
\sigma_{G,1}(e), & \text{if } f \neq \sigma_{G,1}^{-1}(e).
\end{cases}
\]

Now consider \( \sigma_{G[\omega]f[1]e,1}^{-1}(\sigma_{G,\omega^2}^{-1}(e)) \).

Since \( e \neq \sigma_{G,\omega^2}(f) \), by hypothesis, and \( e \neq \sigma_{G,\omega^2}(f) \) (see start of this Case), Lemma 8(f) gives \( \sigma_{G,\omega^2}^{-1}(e) = \sigma_{G[\omega]f,\omega^2}^{-1}(e) \).
Consider, for a moment, the circumstances under which $\sigma_{G[\omega],1}^{-1}(e) = e$. Lemma \(\text{d})\) tells us that

$$
\sigma_{G[\omega],f,1}^{-1}(e) = \begin{cases} 
\sigma_{G[\omega]}^{-1}(f), & \text{if } e = \sigma_{G,1}(f), \\
\sigma_{G,1}^{-1}(e), & \text{if } e \neq \sigma_{G,1}(f).
\end{cases}
$$

If $e \neq \sigma_{G,1}(f)$ then $\sigma_{G[\omega],f,1}^{-1}(e) \neq e$, since otherwise $e = \sigma_{G,1}^{-1}(e)$, so that $e$ is a triloop, which we have excluded. Also, if $e \neq \sigma_{G,1}(f)$ then $e$ cannot equal either of the two possible expressions just given for $\sigma_{G[\omega],f,1}^{-1}(e)$ (using the triloop exclusion, again, for the second of these). So, again, $\sigma_{G[\omega],f,1}^{-1}(e) \neq e$. On the other hand, if $e = \sigma_{G,1}(f)$ and $e = \sigma_{G,1}^{-1}(f)$ then the first case above gives $\sigma_{G[\omega],f,1}^{-1}(e) = \sigma_{G,1}^{-1}(f) = e$. In this situation, applying Theorem \(\text{d})\) to $G[\omega]$, we cannot use case (1,1), since that would require $\sigma_{G[\omega],f,1}^{-1}(e) \neq e$. Similarly, we cannot use the inverse of case (1,1) to find $\sigma_{G[\omega],f,1}^{-1}(e) \neq e$; instead, we must use case (5,1).

If $e = \sigma_{G,1}(f)$ and $e = \sigma_{G,1}^{-1}(f)$, then, we have

$$
\sigma_{G[\omega],f[1],e,1}^{-1}(G,\omega) = \begin{cases} 
\sigma_{G[\omega],f[1],e,1}^{-1}(G,\omega^{-2}(e)), & \\
\sigma_{G[\omega],f[1],e,1}^{-1}(G,\omega^{-1}(e)), & \text{by Theorem } \text{d}(5,1)
\end{cases}
$$

Otherwise, we have

$$
\sigma_{G[\omega],f[1],e,1}^{-1}(G,\omega^{-1}(e)) = \begin{cases} 
\sigma_{G[\omega],f[1],e,1}^{-1}(G,\omega^{-2}(e)), & \\
\sigma_{G[\omega],f[1],e,1}^{-1}(G,\omega^{-1}(e)), & \text{by Lemma } \text{d}(c)
\end{cases}
$$

by Lemma \(\text{d})\).

So $\sigma_{G[\omega],f,1}^{-1}$ and $\sigma_{G[\omega],f[1],e,1}^{-1}$ agree on $g = \sigma_{G,\omega}^{-1}(e)$, in all circumstances. This deals with the third of our exceptional values of $g$.

Cases 4–6: $g \in \{\sigma_{G[\omega],f,1}^{-1}(G,\omega^{-2}(e)), \sigma_{G[\omega],f[1],e,1}^{-1}(G,\omega^{-2}(e)), \sigma_{G[\omega],f[1],e,1}^{-1}(G,\omega^{-1}(e))\}$.

Theorem \(\text{d})\) and Lemma \(\text{d})\) tell us that

$$
\sigma_{G[\omega],f,1}^{-1}(G,\omega^{-2}(e)) \in \{\sigma_{G,1}(f), \sigma_{G,\omega}^{-1}(e)\},
$$

$$
\sigma_{G[\omega],f[1],e,1}^{-1}(G,\omega^{-1}(e)) \in \{\sigma_{G,1}(f), \sigma_{G,\omega}^{-1}(e)\}.
$$

So these are not really new cases at all; they each take us back into one of Cases 1–3.

This completes our proof of \(\text{d})\), and hence of the Lemma.
Theorem 11  If \( f \neq \sigma_{G, \omega}(e) \) then
\[
G[1]e[\omega]f = G[\omega]f[1]e.
\]

Proof. In view of Lemmas 9 and 10, we know that \( \sigma_{G[1]e[\omega]f, \mu} = \sigma_{G[\omega]f[1]e, \mu} \) for \( \mu \in \{1, \omega\} \). But then it follows for \( \mu = \omega^2 \) too, since \( \sigma_{\omega^2} = \sigma_{\omega}^{-1} \circ \sigma_{1}^{-1} \).

Triality gives the following two corollaries.

Corollary 12  If \( f \neq \sigma_{G, 1}(e) \) then
\[
G[\omega]e[\omega^2]f = G[\omega^2]f[\omega]e.
\]

Corollary 13  If \( f \neq \sigma_{G, \omega^2}(e) \) then
\[
G[\omega^2]e[1]f = G[1]f[\omega^2]e.
\]

The results so far in this section (together with the fact of non-commutativity in general for the excluded cases for the previous three results) give us a complete description of when the \( \mu \)-reductions do, or do not, commute, in general.

But some interesting questions remain. Given that the excluded (generally non-commutative) cases are so specific, it is natural to ask for a characterisation of those alternating dimaps for which all reductions always commute.

Consider \( f = \sigma_{G, \omega}(e) \), illustrated in Figure 5. In this case, \( \bullet[1]e \) and \( \bullet[\omega]f \) do not commute in general, but we can still investigate when they do.

Proposition 14  If \( f = \sigma_{G, \omega}(e) \) then \( \bullet[1]e \) and \( \bullet[\omega]f \) commute if and only if at least one of \( e, f \) is a triloop.

Proof. If either \( e \) or \( f \) is a triloop, then they commute by Lemma 5. Suppose then that neither \( e \) nor \( f \) is a triloop. If \( e \) and \( f \) form an a-face of size 2, then it is routine to show that these reductions do not commute unless the head of \( f \) meets no other edge except \( f \), but that would make \( f \) a 1-loop. If \( e \) and \( f \) do not form such an a-face, then the endpoints of \( e \) and \( f \) — three in number — are all distinct. The situation is then exactly as in Figure 5 (except that the right-hand vertex might coincide with the tail of \( f \) or the head of \( e \), but that is immaterial). It is evident from the Figure that the only way the reductions can commute in this case is if the head of \( e \) has in-degree 1 (i.e., if the edges shown in green do not exist), which would make \( e \) a 1-loop.

Theorem 15  Every pair of reductions on \( G \) commutes if and only if the set of triloops of \( G \) includes at least one of each pair of edges that are consecutive in any in-star, a-face or c-face.
Proof. Use Proposition 14 and triality.

We pause now to introduce a graph derived from \( G \) which gives an alternative way of framing Theorem 15.

The trimedial graph \( \text{tri}(G) \) of the alternating dimap \( G \) has vertex set \( E(G) \) with two vertices of \( \text{tri}(G) \) being adjacent if their corresponding edges in \( G \) are consecutive in an a-face, a c-face, or an in-star of \( G \). The trimedial graph is always undirected and 6-regular, and may have loops and/or multiple edges. Its 6-regularity implies that, if it has no loops or multiple edges, then it is nonplanar even if \( G \) is plane (in contrast to the usual medial graph).

With this definition, we may rewrite Theorem 15.

**Corollary 16** Every pair of reductions on \( G \) commutes if and only if the set of triloops of \( G \) form a vertex cover of \( \text{tri}(G) \).  

So far, we have considered the usual kind of commutativity, where the order in which two operations are applied does not matter. We can also ask about stronger forms of commutativity. If a set of \( k \) reductions (each of the form \( \bullet [\mu] e \), where each \( \mu \in \{1, \omega, \omega^2\} \) and all the \( e \) are distinct) has the property that applying them in any order always gives the same result, then we say that it is \( k \)-commutative on \( G \). We say that \( G \) is \( k \)-reduction-commutative if every set of \( k \) reductions is \( k \)-commmutative on \( G \). It is totally reduction-commutative if it is \( k \)-reduction-commutative for every \( k \).

In this terminology, ordinary commutativity is 2-commutativity, in the sense that, if two particular reductions \( \bullet [\mu] e \) and \( \bullet [\nu] f \) commute, then the set \( \{ \bullet [\mu] e, \bullet [\nu] f \} \) is 2-commutative. Theorem 15 characterises alternating dimaps that are 2-reduction-commutative.

While total reduction-commutativity implies \( k \)-reduction-commutativity for any fixed \( k \), which in turn implies \( l \)-reduction-commutativity for any \( l < k \), the converses do not hold.

Consider how taking minors affects these properties.

**Proposition 17** If \( G \) is totally reduction-commutative, then so is any minor of \( G \).  

By contrast, 2-reduction-commutativity is not in general preserved by taking minors. To see this, let \( H \) be any alternating dimap with no triloops, and form \( G \) from it by inserting an \( \omega^2 \)-loop at each vertex of each anticlockwise face and an \( \omega \)-loop at each vertex of each clockwise face. Then \( H \) is a minor of \( G \), yet Theorem 15 tells us that \( G \) is 2-reduction-commutative yet \( H \) is not.

We now characterise alternating dimaps that are totally reduction-commutative.

A 1-circuit is an alternating dimap consisting of a single directed circuit, in which every edge is a 1-loop. An \( \omega \)-circuit (respectively, \( \omega^2 \)-circuit) consists of a single vertex together with a number of \( \omega \)-loops (resp., \( \omega^2 \)-loops) at it. A tricircuit is an alternating
dimap that can be constructed from a 1-circuit, an $\omega$-circuit and an $\omega^2$-circuit (any of which may have no edges), taking a single vertex in each, and identifying these three vertices in the natural way. This is done so as to preserve the alternating dimap property, and will entail having the $\omega$-circuit and $\omega^2$-circuit on opposite sides of the 1-circuit.

**Theorem 18** An alternating dimap $G$ is totally reduction-commutative if and only if each of its components is a tricircuit.

**Proof.** Suppose $G$ is totally reduction-commutative. Then it is certainly 2-reduction-commutative, so by Theorem 15 the set of triloops of $G$ includes at least one of each pair of edges that are consecutive in any in-star, a-face or c-face.

Consider those edges of $G$ which have distinct endpoints (i.e., the non-loops). Suppose two non-loop edges $e$ and $f$ share an endpoint $v$, so $e, f \in I(v)$. Since $e$ and $f$ are not loops, they do not come out of $v$. The number of half-edges going out of $v$ must be two greater than the number of half-edges other than $e$ and $f$ going into $v$. So there must be two half-edges going out of $v$ that do not match (i.e., are not part of the same edge as) any half-edge going into $v$. Let $g$ be an edge to which one of these half-edges belongs. Without loss of generality, suppose that $e, g, f$ occur in that order, going clockwise around $v$. Let the sequence of edges of $I(v)$ which are between $g$ and $e$ going anticlockwise be $h_1, \ldots, h_a$, and let the sequence of edges of $I(v)$ which are between $g$ and $f$ going clockwise be $i_1, \ldots, i_c$. Then the alternating dimap $G' := G[\omega^2](h_1, \ldots, h_a)[\omega](i_1, \ldots, i_c)$ is left with $e, g, f$ intact, still in this same order around $v$, and with no edges intervening between them any more. Then $e$ and $f$ are consecutive (clockwise) in the in-star at $v$ in $G'$. By Theorem 15 this implies non-commutativity of some reductions on $G'$, which in turn implies that $G$ is not totally reduction-commutative.

Now suppose two non-triloop non-loops $e$ and $f$ are head-to-tail: say, with $v = \text{head of } e = \text{tail of } f$. Since $e$ is not a 1-loop, there must be other edges at $v$. If all of those edges lie between $e$ and $f$ going clockwise, then $e$ and $f$ are consecutive around the clockwise face containing $e$, so Theorem 15 gives non-commutativity of some reductions, so $G$ is not totally reduction-commutative. Similarly, if those extra edges at $v$ all lie on the other side — between $f$ and $e$ going clockwise — then, again, $G$ is not totally reduction-commutative. So there are some edges on each side. Let the edges of $I(v)$ between $f$ and $e$ going anticlockwise be $h_1, \ldots, h_k$. Then $G' := G[\omega^2](h_1, \ldots, h_k)$ has $e$ and $f$ as consecutive edges in the anticlockwise face containing $e$. This gives some non-commutative reductions in $G'$, so $G$ is not totally reduction-commutative.

If non-triloop non-loops $e$ and $f$ belong to the same component of $G$, then let $P$ be the shortest path, in the underlying undirected graph, from one to the other. (Note, $e, f \not\in E(P)$, and $P$ meets $e$ and $f$ only at the endpoints of $P$, by its minimality.) If all the edges of $P$ are contracted, to give $G[1]E(P)$, then we have $e$ and $f$ sharing an endpoint and we are in one of the previous two paragraphs, so $G[1]E(P)$ is not totally reduction-commutative, so neither is $G$. 

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So each component of $G$ has at most one edge that is neither a triloop nor a loop. All the 1-loops in a component of $G$ must lie in a single directed circuit in that component. To see this, take any 1-loop $e = uv$. It has a unique successor, which cannot be a loop or $e$ would not be a 1-loop. So it must either be a 1-loop or the sole edge which is neither a triloop nor a loop. Now let us go back the other way. Consider the edges in $I(u)$. At least one of them must be a non-loop. But if $I(u)$ has two non-loops, then both of them are not 1-loops, and so this component has at least two edges that are neither a triloop nor a loop, which is a contradiction. So $I(u)$ has only one non-loop, which must either be a 1-loop or the sole non-triloop non-loop. We can follow 1-loops forwards and backwards in this way until we are forced to stop. This happens when we complete a circuit, which will either be a circuit consisting entirely of 1-loops — in which case it is an entire component of $G$ — or consisting of 1-loops except for the sole non-triloop non-loop, which we call $f = wx$. In the latter case, other edges may meet the head $x$ of that special edge, but cannot meet any other vertex on the circuit. The other edges at $x$ must all be loops, since if any is an outgoing non-loop then another must be an incoming non-loop which is then not a 1-loop either, a contradiction with the uniqueness of $f$. Furthermore, if any edge $g$ at $x$ is a proper 1-semiloop, then we can form a minor, by reduction of any $\omega$-loops or $\omega^2$-loops that get in the way, in which $f$ and $g$ form a configuration that allows non-commutativity. So those other edges at $x$ must all be $\omega$-loops or $\omega^2$-loops.

This description of the component of $G$, as a circuit whose edges are 1-loops with possibly one exception, and with the head of that exception holding $\omega$-loops and $\omega^2$-loops, identifies the component as a tricircuit. So every component of $G$ is in fact a tricircuit.

Conversely, if every component of $G$ is a tricircuit, then each component has at most one edge that is not a triloop, so any two reductions on $G$ commute, by Theorem 6. Therefore $G$ is totally reduction-commutative.

So far, we have considered commutativity (or otherwise) with respect to identity: reductions commute if and only if carrying them out in each possible order gives alternating dimaps that are identical. We could also define commutativity with respect to isomorphism.

**Problem**

Characterise alternating dimaps $G$ for which, for all $\mu_1, \mu_2 \in \{1, \omega, \omega^2\}$ and all $e, f \in E(G)$,

$$G[\mu_1]e[\mu_2]f \cong G[\mu_2]f[\mu_1]e.$$ 

**4 Connections with binary functions**

The relationship between triality and minor operations for alternating dimaps is reminiscent of properties of binary functions found by the author in [23]. In this section we briefly summarise those properties and compare the relationships found there with
those found here. We will determine those alternating dimaps which can be repre-
sented, in a certain faithful manner, by binary functions.

Let $E$ be a finite set, with $m = |E|$. A binary function with ground set $E$ and
dimension $m$ is a function $f : 2^E \rightarrow \mathbb{C}$ such that $f(\emptyset) = 1$. Equivalently, we regard it
as a $2^m$-element complex vector $f$ whose elements are indexed by the subsets of $E$ and
whose first element (indexed by $\emptyset$) is 1. (The restriction $f(\emptyset) = 1$ was not imposed as
part of the definition in earlier work [19, 20, 21, 22], but all scalar multiples of a binary
function are equivalent for our purposes, and we have always been most interested in
the cases where $f(\emptyset) \neq 0$.) We often represent a subset $X \subseteq E$ by its characteristic
vector $x \in \{0, 1\}^E$, with $x_e = 1$ if $e \in X$ and $x_e = 0$ otherwise. Since $x$ may be
thought of as a binary string, it may also be taken to be the binary representation of
a number $x$ such that $0 \leq x \leq 2^m - 1$. With this notation, $f(X)$ may also be written
$f_x$ or $f^x$. In particular, $f(\emptyset) = f_{(0, \ldots , 0)} = f_0 = 1$. We write $0_k$ for the sequence of $k$
0s, and sometimes drop the subscript $k$ when it is clear from the context.

The definition was motivated by indicator functions of linear spaces over $\text{GF}(2)$, especially of cutset spaces of graphs: if
$N$ is a matrix over $\text{GF}(2)$ whose columns are indexed by $E$ (such as the incidence matrix of a graph, or the matrix representation
of a binary matroid), then the indicator function of the rowspace of $N$ takes value 1
on a set $X \subseteq E$ if the characteristic vector of $X$ belongs to the rowspace of $N$, and
takes value 0 otherwise.

If $f, g : 2^E \rightarrow \mathbb{C}$ and there exists a constant $c \in \mathbb{C} \setminus \{0\}$ such that $f(X) = cg(X)$
for all $X \subseteq E$, then we write $f \simeq g$.

Define

$$M(\mu) := \frac{1}{2\sqrt{2}} \left( \begin{array}{cc} \sqrt{2} + 1 + (\sqrt{2} - 1)\mu & 1 - \mu \\ 1 - \mu & \sqrt{2} - 1 + (\sqrt{2} + 1)\mu \end{array} \right).$$

The $\mu$-transform of $f$, denoted by $L[\mu]f$, is given by

$$L[\mu]f := M(\mu)^{\otimes m}f,$$

where the $2^m \times 2^m$ matrix on the right is the $m$-th Kronecker power of $M(\mu)$.

When $\mu = 1$, we have the identity transform, while when $\mu = -1$, we have a scalar
multiple of the Hadamard transform. It is well known that the Hadamard transform
takes the indicator function of a linear space to a scalar multiple of the indicator
function of its dual, from which it follows that the indicator functions of the cutset
and circuit spaces of a graph are related by the Hadamard transform in the same way.
It was shown in [19] that general matroid duality is also described by the Hadamard transform.

It is easy to show that $M(\mu_1\mu_2) = M(\mu_1)M(\mu_2)$, see [23]. It follows (using
the mixed-product property for the Kronecker product) that composition of the $L[\mu]$
transforms corresponds to multiplication of their $\mu$ parameters: $L[\mu_1]L[\mu_2] = L[\mu_1\mu_2]$,
from [23 Theorem 2]. At this point, readers may ask: what happens when $\mu = \omega$?
We look at this shortly.

Suppose $E = \{e_0, \ldots , e_{m-1}\}$. 

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We use $f_\bullet$ as shorthand for the vector of length $2^{m-1}$, with elements indexed by subsets of $E \setminus e_0$, whose $X$-element is $f(X)$, if $b = 0$, or $f(X \cup \{e_0\})$, if $b = 1$ (for $X \subseteq E \setminus \{e_0\}$). We define $f_\bullet$ in the same way, except that we use $e_{m-1}$ instead of $e_0$ throughout. The vectors $f_0\bullet$ and $f_1\bullet$ give the top and bottom halves, respectively, of $f$, while $f_0$ and $f_1$ give the elements in even and odd positions, respectively, of $f$.

Let $I_l$ denote the $l \times l$ identity matrix. If $e \in E$, then the $[\mu]$-minor of $f$ by $e$ is the $2^{m-1}$-element vector $f\|_{[\mu]} e$, with entries indexed by subsets of $E \setminus \{e\}$, given by

$$f\|_{[\mu]} e_i := c \cdot \left( I_2^{\otimes i} \otimes \left( 1 + \frac{1 + \mu}{\sqrt{2} + 1 - (\sqrt{2} - 1)\mu} \right) \otimes I_2^{\otimes (m-i-1)} \right) f,$$

where $c$ is such that the $\emptyset$-element of $f\|_{[\mu]} e_i$ is 1.

Put $(\mu_0, \mu_1, \mu_2) = (1, \omega, \omega^2)$ and, for each $j \in \{0, 1, 2\}$,

$$\lambda_j := \frac{1 + \mu_j}{\sqrt{2} + 1 - (\sqrt{2} - 1)\mu_j}.$$

Then $f\|_{[\mu_j]} e_i$ is a scalar multiple of $(I_2^{\otimes i} \otimes \lambda_j I_2^{\otimes (m-i-1)}) f$.

When $f$ is the indicator function of the cutset space of a graph, the minor $f\|_{[\mu]} e$ amounts to deletion when $\mu = 1$ and contraction when $\mu = -1$. See [23, §2,§6], and also [20] for the first definition of generalised minor operations interpolating between deletion and contraction (albeit with a different parameterisation to that used here and in [23]). This work has its roots in [19], where deletion and contraction are expressed in terms of indicator functions of cutset spaces, and these operations are extended to general binary functions.

It is shown in [23, Theorem 9] that, for all $\mu_1 \in \mathbb{C} \setminus \{0\}$ and $\mu_2 \in \mathbb{C}$,

$$(L^{[\mu_1]} f)\|_{[\mu_2]\mu_1} e \simeq L^{[\mu_1]}(f\|_{[\mu_2]} e).$$

In particular, we have

$$(L^{[\omega]} f)\|_{[1]} e \simeq L^{[\omega]}(f\|_{[\omega]} e),$$

$$(L^{[\omega]} f)\|_{[\omega]} e \simeq L^{[\omega]}(f\|_{[\omega]} e),$$

$$(L^{[\omega]} f)\|_{[\omega^2]} e \simeq L^{[\omega]}(f\|_{[1]} e).$$

This relationship between the transform $L^{[\omega]}$ (called the trinity transform [23] or triality transform) and the minor operations for binary functions follows the same pattern as the relationships between triality and minors for alternating dimaps, given in Theorem[4]. It is natural to ask what connection there may be between the two.

For binary functions, the minor operations always commute [20, Lemma 4]. In fact, that result implies that every binary function is totally reduction-commutative (using the natural analogue of that definition for binary functions). But, as we saw in [3], the minor operations for alternating dimaps do not always commute. It follows that alternating dimaps, along with triality and minor operations, cannot
be represented faithfully by binary functions with their triality transform and minor operations described above.

Nonetheless, we can ask if there is a subclass of alternating dimaps which can be represented faithfully by binary functions in this way. For this to occur, this subclass must consist only of alternating dimaps that are totally reduction-commutative. Such alternating dimaps are disjoint unions of tricircuits, by Theorem [18].

Later we will give a definition of faithful representation by binary functions, and determine when such a representation is possible. To do the latter, it will help to characterise those binary functions for which any reduction, on any element of the ground set, gives the same given binary function.

To do this, we need some more notation.

Throughout, we write
\[ i = \left( \begin{array}{c} 1 \\ 0 \end{array} \right), \quad j = \left( \begin{array}{c} 0 \\ 1 \end{array} \right), \quad H = (h_0, \ldots, h_{k-1}) \in \{i,j\}^{0,\ldots,k-1}. \]

For each \( i \),
\[ H^{(i)} = (h_0, \ldots, h_{i-1}, h_{i+1}, \ldots, h_{k-1}) \]
is the sequence obtained from \( H \) by omitting the term indexed by \( i \).

For each \( H \), define the sequence \( G = G(H) = (g_0, \ldots, g_{k-1}) \) by
\[ g_i = \begin{cases} 0, & \text{if } h_i = i; \\ 1, & \text{if } h_i = j. \end{cases} \]
The sequence obtained from this by omitting the term indexed by \( i \) is \( G^{(i)} = (g_0, \ldots, g_{i-1}, g_{i+1}, \ldots, g_{k-1}) \).

The subsequence \( (g_{i_1}, \ldots, g_{i_2}) \) of \( G \) is denoted by \( G[i_1, i_2] \).

If \( b \in \{0,1\} \), then \( G : i \leftarrow b \) denotes the sequence obtained by inserting \( b \) between the \( i \)-th and \( (i + 1) \)-th elements of \( G \):
\[ G : i \leftarrow b = (g_0, \ldots, g_{i-1}, b, g_i, \ldots, g_{k-1}). \]
The two-element vector \( f_{G;i} \) is defined by
\[ f_{G;i} = \left( \begin{array}{c} f_{G;i\leftarrow 0} \\ f_{G;i\leftarrow 1} \end{array} \right). \]

Write \( u \) for a \( 2^k \)-element vector indexed by the numbers \( 0, \ldots, 2^{k-1} \) — or, equivalently, by vectors of \( k \) bits, or by subsets of \( \{0, \ldots, k - 1\} \).

For a given \( G \), we write \( u_G \) for the entry of \( u \) whose index has binary representation given by \( G \), i.e., whose index is \( \sum_{i=0}^{k-1} g_i 2^{k-1-i} \).

It is routine to show that, if \( m \geq 1 \) and \( u \) is a (vector representation of a) binary function with ground set of size \( m - 1 \), then
\[ u = \sum_H \left( h_0 \otimes \cdots \otimes h_{m-2} \right) u_G. \quad \tag{15} \]
If $m = 1$ then there is a single $H$ to sum over, consisting of the empty sequence, and the empty product $h_0 \otimes \cdots \otimes h_{m-2}$ is the trivial single-element vector $(1)$. Also $G$ is the empty bit-sequence, representing the number 0, and $u_G = 1$, so $u = (1)$, as expected.

**Lemma 19** Suppose $f$ and $u$ are binary functions with

$$f|_{\mu} e_i = u, \quad \text{for all } \mu \in \{1, \omega, \omega^2\}.$$  

Then for all $G \in \{0, 1\}^{\{0, \ldots, m-2\}}$ and all $b \in \{0, 1\}$,

$$f_G: i \leftarrow b = f_{0: i}.$$  

(16)

**Proof.** Let us write the hypothesis as a set of equations, using (13) and (14). If $f|_{\mu_j} e_i = u$ for all $j$, then, for each $j$, there exists $c_{ij}$ such that

$$\left( I_2^j \otimes (1 \lambda_j) \otimes I_2^{(m-i-1)} \right) f = c_{ij} u.$$  

(17)

Put

$$R = \begin{pmatrix} 1 & \lambda_0 \\ 1 & \lambda_1 \\ 1 & \lambda_2 \end{pmatrix} \quad \text{and} \quad c_i = \begin{pmatrix} c_{i0} \\ c_{i1} \\ c_{i2} \end{pmatrix}.$$  

The equations (17) may be written (using (15)),

$$\left( I_2^i \otimes R \otimes I_2^{(m-i-1)} \right) f = \sum_H \left( h_0 \otimes \cdots \otimes h_{i-1} \otimes c_i \otimes h_i \otimes \cdots \otimes h_{m-2} \right) u_G.$$  

(18)

We show by induction on $m$ that the solutions to this equation satisfy

$$R f_{G;i} = c_i u_G,$$  

(19)

for all $G \in \{0, 1\}^{\{0, \ldots, m-2\}}$.

For the inductive basis, let $m = 1$, so $i = 0$. Then we may write

$$f = \begin{pmatrix} 1 \\ f_1 \end{pmatrix}, \quad u = (1)$$  

(where $u_0 = 1$ since $u$ is a binary function), and our equations (18) are

$$R f = c_0.$$  

In this case, any $f$ will do, with appropriate choice of $c_0$. Equations (19) are satisfied in this case (with there being just a single sequence $G$, which is the empty sequence and represents the number 0). Observe that $f_{G;i} = f_{G:0} = f$.

Now suppose that the claim is true for $m = k \geq 2$. We show that it is true for $m = k + 1$. 

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We wish to solve (18) when \( m = k + 1 \). Since \( k \geq 1 \), either \( i > 0 \) or \( k - i > 0 \) (or both). We treat these two cases in turn.

Suppose \( i > 0 \).

We can write

\[
f = i \otimes f_{i}\ast + j \otimes f_{1}\ast .
\]

This allows us to rewrite the left-hand side of (18):

\[
( I_{2}^{\otimes i} \otimes R \otimes I_{2}^{\otimes (m-i-1)} ) f
= ( I_{2} \otimes I_{2}^{\otimes (i-1)} \otimes R \otimes I_{2}^{\otimes (m-i-1)} ) ( i \otimes f_{i}\ast + j \otimes f_{1}\ast )
= I_{2} i \otimes ( I_{2}^{\otimes (i-1)} \otimes R \otimes I_{2}^{\otimes (m-i-1)} ) f_{i}\ast + I_{2} j \otimes ( I_{2}^{\otimes (i-1)} \otimes R \otimes I_{2}^{\otimes (m-i-1)} ) f_{1}\ast
= i \otimes ( I_{2}^{\otimes (i-1)} \otimes R \otimes I_{2}^{\otimes ((m-1)-(i-1)-1)} ) f_{i}\ast + j \otimes ( I_{2}^{\otimes (i-1)} \otimes R \otimes I_{2}^{\otimes ((m-1)-(i-1)-1)} ) f_{1}\ast
\]

On the other hand, the right-hand side of (18) may be rewritten:

\[
\sum_{H} ( h_{0} \otimes \cdots \otimes h_{i-1} \otimes c_{i} \otimes h_{i} \otimes \cdots \otimes h_{m-2} ) u_{G}
= \sum_{h_{0}, H^{(0)}} ( h_{0} \otimes h_{1} \otimes \cdots \otimes h_{i-1} \otimes c_{i} \otimes h_{i} \otimes \cdots \otimes h_{m-2} ) u_{G}
= \sum_{H^{(0)}} ( i \otimes h_{1} \otimes \cdots \otimes h_{i-1} \otimes c_{i} \otimes h_{i} \otimes \cdots \otimes h_{m-2} ) u_{0G^{(0)}} + \sum_{H^{(0)}} ( j \otimes h_{1} \otimes \cdots \otimes h_{i-1} \otimes c_{i} \otimes h_{i} \otimes \cdots \otimes h_{m-2} ) u_{1G^{(0)}}
= i \otimes \sum_{H^{(0)}} ( h_{1} \otimes \cdots \otimes h_{i-1} \otimes c_{i} \otimes h_{i} \otimes \cdots \otimes h_{m-2} ) u_{0G^{(0)}} + j \otimes \sum_{H^{(0)}} ( h_{1} \otimes \cdots \otimes h_{i-1} \otimes c_{i} \otimes h_{i} \otimes \cdots \otimes h_{m-2} ) u_{1G^{(0)}}.
\]

These rewritten forms of each side give a top half and a bottom half for each. Equating these tells us that (18) is equivalent to the following two simultaneous equations.

\[
( I_{2}^{\otimes (i-1)} \otimes R \otimes I_{2}^{\otimes ((m-1)-(i-1)-1)} ) f_{i}\ast = \sum_{H^{(0)}} ( h_{1} \otimes \cdots \otimes h_{i-1} \otimes c_{i} \otimes h_{i} \otimes \cdots \otimes h_{m-2} ) u_{0G^{(0)}},
\]

\[
( I_{2}^{\otimes (i-1)} \otimes R \otimes I_{2}^{\otimes ((m-1)-(i-1)-1)} ) f_{1}\ast = \sum_{H^{(0)}} ( h_{1} \otimes \cdots \otimes h_{i-1} \otimes c_{i} \otimes h_{i} \otimes \cdots \otimes h_{m-2} ) u_{1G^{(0)}}.
\]

Each of these equations is an instance of the same type of equation as (18), with dimension \( m \) and position \( i \) each reduced by one. So, by the inductive hypothesis, their solutions are

\[
R f_{0G^{(0)}; i} = c_{i} u_{0G^{(0)}},
R f_{1G^{(0)}; i} = c_{i} u_{1G^{(0)}}.
\]
Combining these gives (19), for all \( G \). This deals with the case \( i > 0 \).

If \( k - i > 0 \) then a similar argument can be used, peeling off the identity matrix from the right, rather than the left, of \( I^\otimes_2 \otimes R \otimes I^\otimes_{(m-i-1)} \) in (18), and using \( f = f_0 \otimes i + f_1 \otimes j \), and so on.

It follows by induction that the solutions to (18) satisfy (19).

When \( G = 0 \), (19) and \( u_0 = 1 \) give

\[
R f_{0;i} = c_i.
\]

Using this with (19) gives, for any \( G \),

\[
R f_{G;i} = R f_{0;i} u_G.
\]

Since \( R \) has rank 2 (because the \( \lambda_i \) are distinct),

\[
f_{G;i} = f_{0;i} u_G.
\]

Hence, for all \( b \in \{0, 1\} \) and all \( G \),

\[
f_{G;i-b} = f_{0;i-b} u_G.
\]

We now define our notion of faithful representation, and then determine when it is possible.

**Definition**

A **strict binary representation** of a minor-closed set \( \mathcal{A} \) of alternating dimaps is a triple \((F, \varepsilon, \nu)\) such that

(a) \( F : \mathcal{A} \to \{\text{binary functions}\} \)

(b) \( \varepsilon = (\varepsilon_G \mid G \in \mathcal{A}) \) is a family of bijections \( \varepsilon_G : E(G) \to E(F(G)) \);

(c) \( \nu \in \mathbb{C} \) with \( |\nu| = 1 \);

(d) \( F(G^{(\omega)}) \simeq L^{[\omega]} F(G) \) for all \( G \in \mathcal{A} \);

(e) \( F(G[\mu]e) \simeq F(G)[\nu|e] \varepsilon_G(e) \) for all \( G \in \mathcal{A}, e \in E(G) \) and \( \mu \in \{1, \omega, \omega^2\} \).

Let \( C_1 \) denote the ultraloop. We write \( \mathcal{U}_k = \{iC_1 \mid i = 0, \ldots, k\} \) and \( \mathcal{U}_\infty = \{iC_1 \mid i \in \mathbb{N} \cup \{0\}\} \), where 0\( C_1 \) is the empty alternating dimap.

**Theorem 20** If \( \mathcal{A} \) is a minor-closed class of alternating dimaps which has a strict binary representation then \( \mathcal{A} = \emptyset \), or \( \mathcal{A} = \mathcal{U}_k \) for some \( k \), or \( \mathcal{A} = \mathcal{U}_\infty \).
Proof. Suppose \((F, \varepsilon, \nu)\) is a strict binary representation of \(\mathcal{A}\).

The theorem is immediately true if \(\mathcal{A} = \emptyset\). So suppose \(\mathcal{A} \neq \emptyset\).

If \(|\mathcal{A}| \geq 1\) then, since it is minor-closed, it must contain the empty alternating dimap \(C_0\), and the image \(F(C_0)\), representing \(C_0\) as a binary function, must be the binary function \(f : 2^\emptyset \to C\) defined by \(f(\emptyset) = 1\).

So, if \(|\mathcal{A}| = 1\) then \(\mathcal{A} = \mathcal{U}_0\), and the previous paragraph gives a strict binary representation of \(\mathcal{A}\).

Similarly, if \(|\mathcal{A}| \geq 2\), then it must contain the ultraloop \(C_1\), since that is the only alternating dimap on one edge.

Claim 1: The image \(F(C_1)\) of the ultraloop \(C_1\) is given by
\[
F(C_1) = \begin{pmatrix} 1 \\ \sqrt{2} - 1 \end{pmatrix}.
\]

Proof:
\(F(C_1)\) must be some binary function \(f\) on a singleton ground set, \(E = \{e\}\) say, with \(f(\emptyset) = 1\) and \(f(\{e\}) = u\) for some \(u \in C\). Since \(C_1\) is self trial, so must \(f\) be (by (d) above). This means that its vector form \(f = \begin{pmatrix} 1 \\ u \end{pmatrix}\) must be an eigenvector for eigenvalue 1 of the matrix \(M(\omega)\). Now this matrix has eigenvalues 1 and \(\omega\), and the eigenvectors for the former are the scalar multiples of
\[
\begin{pmatrix} 1 \\ \sqrt{2} - 1 \end{pmatrix}.
\]
So this is \(F(C_1)\), and \(u = \sqrt{2} - 1\). So Claim 1 is proved.

If \(|\mathcal{A}| = 2\) then \(\mathcal{A}\) consists of just the empty alternating dimap and the ultraloop. The \(F\) given by Claim 1, together with appropriate identity maps \(\varepsilon\) (and, in fact, any \(\nu\)), gives a strict binary representation. So we are done in this case.

It remains to deal with \(|\mathcal{A}| \geq 3\), when \(\mathcal{A}\) contains at least one alternating dimap on two edges.

Claim 2: The only binary function \(f\) with the property that every reduction, on any of the elements of its ground set, gives \(u = F(C_1)^{\otimes k}\), is \(f = F(C_1)^{\otimes(k+1)}\).

Proof:
Observe that, by Claim 1, \(u = (u_G \mid G \in \{0, 1\}^E)\) where \(u_G = (\sqrt{2} - 1)^{|G|}\), where \(|E| = k\) and \(|G|\) is the number of 1s in \(G\).

Applying Lemma 19 for all \(i \in \{0, \ldots, k\}\), to \(u = F(C_1)^{\otimes k}\) gives
\[
f_{G:i-\leftarrow} = f_{0:i-\leftarrow} u_G.
\]
Hence, for each \(i\) and each \(G\),
\[
f_{G:i-\leftarrow = 0} = f_{0:i+1} u_G = u_G = (\sqrt{2} - 1)^{|G|}.
\]

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Now consider $f_{G;i←1}$. Put $j := 0$ if $i \neq 0$ and $j := 1$ otherwise (so $j \neq i$). Then
\[
f_{G;i←1} = f_{0;i←1} u_G = f_{0:k−1;i←1} u_G = f_{0:k−1;i←1} u_G = f_{0:k+1} u_{0:k−1;i←1} u_G = 1 \cdot (\sqrt{2} − 1) \cdot (\sqrt{2} − 1)^{|G|} = (\sqrt{2} − 1)^{|G|+1}.
\]
It follows that, for all $G' \in \{0, 1\}^{k+1}$,
\[
f_{G'} = (\sqrt{2} − 1)^{|G'|}.
\]
Therefore
\[
f = F(C_1)^{⊗(k+1)},
\]
proving the Claim.

Claim 3: If $k \geq 2$ and every reduction of $G$ is $kC_1$, then $G = (k + 1)C_1$.

Before proving the claim, consider the case $k = 1$, which it does not cover. Then every alternating dimap on two edges (of which there are four) has the claimed property. Of these, the only self-trial one is $2C_1$.

Proof:
Suppose $k = 2$. If $G$ is connected, then there must be some $e \in E(G)$ and some $\mu \in \{1, \omega, \omega^2\}$ such that $G[\mu]e = 2C_1$ and is therefore disconnected. The only way in which $\mu$-reducing a single edge can disconnect a connected alternating dimap is if the edge is a proper $\mu^{-1}$-semiloop. It is easily determined that the only alternating dimaps on three edges which have this property are those consisting of two triloops and a semiloop. These do not have three proper semiloops. So, although they have the specified property for one of their edges, they do not have it for all of their edges. So $G$ must be disconnected. Since $G$ has only three edges, some component of $G$ must be an ultraloop. But this disappears when reduced, so the rest of $G$ must be $2C_1$, so $G = 3C_1$.

Now suppose $k \geq 3$. It is impossible for $G$ to be connected, because no reduction of any edge of any connected alternating dimap can possibly break it up into three or more components. So consider the components of $G$. If any of these is not an ultraloop, then it has at least two edges, and also is left unchanged by reduction of any edge in any other component (of which there must be at least one), so we would have a reduction of $G$ that does not give $kC_1$, which is a contradiction. So every component of $G$ must be an ultraloop. Each of these just disappears on reduction, giving $kC_1$, as desired.

So Claim 3 is proved.

Claim 4: For all $k \geq 0$, either $A$ has no members with $k$ edges, or it has just one such member which is $kC_1$.

Proof:
We prove the claim by induction on \( k \).

We have seen that this is true already for \( k \leq 1 \).

Suppose \( k = 2 \). Every alternating dimap \( G_2 \) on two edges has the property that every reduction of it gives the ultraloop. Therefore, if \( G_2 \in \mathcal{A} \) then \( F(G_2) = F(C_1)^{\otimes 2} \). But \( F(C_1)^{\otimes 2} \) is self-trial, since \( F(C_1) \) is. Therefore \( G_2 \) must be self-trial too. But the only self-trial alternating dimap on two edges is \( 2C_1 \). So the only member of \( \mathcal{A} \) with two edges is \( 2C_1 \).

Now suppose it is true regarding members of \( \mathcal{A} \) with \( k - 1 \) edges, where \( k \geq 3 \). We show that it is true for \( k \) edges.

If \( \mathcal{A} \) has no members with \( k - 1 \) edges, then it can have no members with \( k \) edges either, since it is minor-closed.

If \( \mathcal{A} \) has at least one member with \( k - 1 \) edges, then by the inductive hypothesis it can have only one such member, and this must be \( (k - 1)C_1 \). We must show that, if \( \mathcal{A} \) has at least one member with \( k \) edges, then it can have only one, and it is \( kC_1 \).

Let \( G \) be a member of \( \mathcal{A} \) with \( k \) edges. Since \( \mathcal{A} \) is minor-closed and has \( (k - 1)C_1 \) as its only member with \( k - 1 \) edges, all reductions of \( G \) must give \( (k - 1)C_1 \). So, by the requirements of a strict binary representation, all reductions of \( F(G) \) must give \( F(C_1)^{\otimes (k-1)} \). This implies that \( F(G) = F(C_1)^{\otimes k} \), by Claim 3. This completes the proof of Claim 4.

It follows from Claim 4 that \( \mathcal{A} \) can only be one of the classes given in the statement of the theorem. It remains to establish that a strict binary representation is possible for each of those classes. This is routine, using

\[
F(kC_1) = \left( \frac{1}{\sqrt{2}} - 1 \right)^{\otimes k}
\]

for every \( k \) for which \( kC_1 \in \mathcal{A} \). Let \( \varepsilon \) consist just of identity maps. To show that this does indeed enable a strict binary representation, use Claims 1–3. The details are a routine exercise.

It is possible to develop broader definitions of binary representations of classes of alternating dimaps. For example, we could allow the edges of \( G \) to be represented by disjoint subsets of elements of the ground set of \( F(G) \) instead of just by distinct single elements.

**Problem**

Characterise those minor-closed classes of alternating dimaps that have binary function representations of a more general type, such as that suggested above.

### 5 Excluded minors for fixed genus

A posy, or \( k \)-posy, is an alternating dimap with one vertex, \( 2k + 1 \) edges (all loops), and two faces. Its genus is \( k \). Up to isomorphism, there is a single 0-posy, a single
1-posy and three 2-posies. The 0-posy is just a single ultraloop. The 1-posy and the three 2-posies are shown in Figure 6.

Figure 6: The 1-posy and the three 2-posies. In each posy, the two faces are coloured grey (clockwise) and white (anticlockwise).

**Theorem 21** A nonempty alternating dimap $G$ has genus $< k$ if and only if none of its minors is a disjoint union of posies of total genus $k$.

**Proof.** The forward implication is clear, since every such union of posies has genus $k$. 

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For the reverse implication, we prove by induction on $|E(G)|$ that, if $G$ is nonempty and has genus $\geq k$, then it has, as a minor, a disjoint union of posies of total genus $k$.

This is true for $|E(G)| = 1$, since a nonempty planar alternating dimap must have a directed cycle, hence a subnet contractible to a loop, hence a 0-posy minor.

Now suppose it is true for all alternating dimaps of $< m$ edges, where $m > 1$. Let $G$ be any alternating dimap with $m$ edges and genus $\geq k$. Let $e \in E(G)$. Now, $G[1]e$, $G[\omega]e$ and $G[\omega^2]e$ each have $m - 1$ edges, so by the inductive hypothesis, $G[\mu]e$ has as a minor a disjoint union of posies of total genus $\gamma(G[\mu]e)$, for each $\mu \in \{1, \omega, \omega^2\}$. Such a minor of $G[\mu]e$ is also a minor of $G$, so we see that $G$ has such a minor, for each such $\mu$. If $\gamma(G[\mu]e) = \gamma(G)$ for any such $\mu$, we are done. So it remains to consider the case where $\gamma(G[\mu]e) < \gamma(G)$ (in which case $\gamma(G[\mu]e) = \gamma(G) - 1$) for each $\mu$ and each $e \in E(G)$.

The condition $\gamma(G[1]e) < \gamma(G)$ implies that $e$ is a proper 1-semiloop, so already we know that every edge of $G$ is a loop that does not enclose its own face. Let $e$ be any edge and let $v$ be the vertex at which $e$ is a loop. The condition $\gamma(G[\omega^2]e) < \gamma(G)$ implies that $e$ is also a proper $\omega$-semiloop. Let $F$ be the face on the right side of $e$, and let $F'$ be the face on the right side of the left successor $e'$ of $e$ (see Figure 7). If faces $F$ and $F'$ were distinct, then $\omega$-reduction of $e$ would not reduce the genus and $e$ would not be an $\omega$-semiloop. So $F = F'$. Applying this same reasoning to the next edge (clockwise from $e$) in the in-star at $v$ (denoted by $f$ in Figure 7) shows that the face $F'$ is, in turn, identical to the second face beyond it (denoted by $F''$), continuing to go clockwise around $v$. Continuing in this manner we find that every second “face” around $v$ is really just part of one single face.

![Figure 7: Proof of Theorem 21](image)

Figure 7: Proof of Theorem 21 faces around $v$ when $e$ is a 1-semiloop and an $\omega$-semiloop.

In a similar manner, the condition $\gamma(G[\omega]e) < \gamma(G)$ implies that $e$ is also a proper $\omega^2$-semiloop, and we find that all the “faces” at $v$ which were not accounted for in
the previous paragraph (being every other second face around \( v \)) are, again, just one single face (but necessarily distinct from the face \( F \) we found there). So the component of \( G \) that consists of loops at \( v \) is a posy. So \( G \) itself is just a disjoint union of all these posies and has genus \( k \), so we are done.

6 Tutte invariants

We now extend the notion of a Tutte invariant to alternating dimaps, and investigate what invariants of this type exist.

**Definition**

A *simple Tutte invariant* for alternating dimaps is a function \( F \) defined on every alternating dimap such that \( F \) is invariant under isomorphism, \( F(\text{empty alternating dimap}) = 1 \), and there exist \( w, x, y, z \) such that, for any alternating dimap \( G \),

1. for any ultraloop \( e \) of \( G \),
   \[
   F(G) = w F(G - e)
   \]  
\( (20) \)

2. for any proper 1-loop \( e \) of \( G \),
   \[
   F(G) = x F(G[1]e)
   \]  
\( (21) \)

3. for any proper \( \omega \)-loop \( e \) of \( G \),
   \[
   F(G) = y F(G[\omega]e)
   \]  
\( (22) \)

4. for any proper \( \omega^2 \)-loop \( e \) of \( G \),
   \[
   F(G) = z F(G[\omega^2]e)
   \]  
\( (23) \)

5. for any edge \( e \) of \( G \) that is not an ultraloop or a triloop,
   \[
   F(G) = F(G[1]e) + F(G[\omega]e) + F(G[\omega^2]e).
   \]  
\( (24) \)

**Lemma 22** Let \( F \) be a simple Tutte invariant of alternating dimaps. If \( w = 0 \) then \( F(G) = 0 \) for any nonempty \( G \).

**Proof.** Induction on \( |E(G)| \).

**Theorem 23** The only simple Tutte invariants of alternating dimaps are:

(a) \( F(G) = 0 \) for nonempty \( G \), with \( w = 0 \);

(b) \( F(G) = 3^{|E(G)|} \), with \( w = x = y = z = 3 \);

(c) \( F(G) = (-1)^{|V(G)|} \), with \( y = z = 1 \) and \( x = w = -1 \);
\((d)\) \(F(G) = (-1)^{cf(G)}\), with \(x = z = 1\) and \(y = w = -1\);
\((e)\) \(F(G) = (-1)^{af(G)}\), with \(x = y = 1\) and \(z = w = -1\).

Proof. Let \(F\) be a simple Tutte invariant of alternating dimaps, with \(w, x, y, z\) as in the definition.

If \(w = 0\) then \(F(G) = 0\) for nonempty \(G\), by Lemma 22. So suppose \(w \neq 0\).

For any \(k \geq 1\), let \(U_k\) be a disjoint union of \(k\) ultraloops. For any \(k \geq 2\), let \(L_{k,1}\) be the directed cycle of \(k\) vertices and \(k\) edges, which has one a-face, one c-face and \(k\) in-stars. For \(k \geq 2\) and \(\mu \in \{\omega, \omega^2\}\), let \(L_{k,\mu}\) be the alternating dimap consisting of a single vertex with \(k\) \(\mu\)-loops.

It is clear that
\[
F(U_k) = w^k, \tag{25}
\]
\[
F(L_{2,1}) = xw, \tag{26}
\]
\[
F(L_{2,\omega}) = yw, \tag{27}
\]
\[
F(L_{2,\omega^2}) = zw. \tag{28}
\]

Consider the alternating dimap \(L_{2,1} + e_{\omega^2}\) obtained by adding an \(\omega^2\)-loop \(e\) to a vertex \(v\) of \(L_{2,1}\). Let \(f\) (respectively, \(g\)) be the edge of \(L_{2,1}\) going out of (resp., into) \(v\). Observe that \(f\) is a 1-loop and \(g\) is not a triloop. We can calculate \(F(L_{2,1} + e_{\omega^2})\) by applying \((21)\) at \(f\) (or \((23)\) at \(e\)), obtaining \(xzw\). Alternatively, we can apply \((24)\) at \(g\), obtaining \((x + z + w)w\). Equating the results and using \(w \neq 0\), we obtain
\[
x + z + w = xz. \tag{29}
\]

Similar reasoning for the trials \((L_{2,1} + e_{\omega^2})^\omega\) and \((L_{2,1} + e_{\omega^2})^{\omega^2}\) gives
\[
F((L_{2,1} + e_{\omega^2})^\omega) = xyw, \tag{30}
\]
\[
F((L_{2,1} + e_{\omega^2})^{\omega^2}) = yzw, \tag{31}
\]
\[
x + y + w = xy, \tag{32}
\]
\[
y + z + w = yz. \tag{33}
\]

Now consider the alternating dimap obtained from \(L_{2,1}\) (with edges \(g, h\)) by adding, to the endpoint \(v\) of \(h\), a clockwise loop \(e\) within the anticlockwise face and an anticlockwise loop \(f\) within the clockwise face. Call it \(A\). Using the \(\omega^2\)-loop \(e\), or the \(\omega\)-loop \(f\), or the 1-loop \(g\), we find that
\[
F(A) = xyzw. \tag{34}
\]

But using \(h\), which is not a triloop, we have
\[
F(A) = F(L_{2,1} + e_{\omega^2}) + F((L_{2,1} + e_{\omega^2})^\omega) + F((L_{2,1} + e_{\omega^2})^{\omega^2})
= xzw + xyw + yzw
= (xz + xy + yz)w.
\]
Equating with (34), and using \( w \neq 0 \), we obtain
\[
xz + xy + yz = xyz.  \tag{35}
\]

From (29), (32), (33) we obtain
\[
w = xz - x - z = xy - x - y = yz - y - z.
\]
The second equality here gives \((x - 1)z = (x - 1)y\), so either \(x = 1\) or \(y = z\). Similarly, either \(y = 1\) or \(x = z\), and either \(z = 1\) or \(x = y\). Combining these, we have one of

(i) \(x = y = z\),

(ii) \(x = y = 1\) and \(z \neq 1\),

(iii) \(x = z = 1\) and \(y \neq 1\),

(iv) \(y = z = 1\) and \(x \neq 1\).

If (i) holds, then any of (29), (32), (33) gives
\[
w = x(x - 2). \tag{36}
\]

Also (35) gives \(3x^2 = x^3\), whence \(x = 3\) (since \(x = 0\) would imply \(w = 0\), by (36)) and \(w = 3\) (by (36)).

If (ii) holds, then (32) gives \(w = -1\). Similarly, cases (iii) and (iv) give \(w = -1\) too.

Also, (35) implies \(z = -1\) in case (ii), \(y = -1\) in case (iii) and \(z = -1\) in case (iv).

We now establish the form of \(F\) for each of cases (i)–(iv) in turn. The numbering of the claims indicates the case to which each applies.

Claim (i): \(F(G) = 3^{|E(G)|}\).

Proof of Claim (i): we use induction on \(|E(G)|\). If \(|E(G)| = 0\) then the claim is true by the definition of \(F\). Suppose \(|E(G)| = m > 1\). Let \(e \in E(G)\). If \(e\) is an ultraloop, then \(F(G) = wF(G - e) = 3F(G - e) = 3 \cdot 3^{m-1}\), by the inductive hypothesis, which equals \(3^m\). If \(e\) is a proper \(\mu\)-loop, with \(\mu \in \{1, \omega, \omega^2\}\), then \(F(G) = xF(G[\mu]e) = 3 \cdot 3^{m-1} = 3^m\). If \(e\) is neither an ultraloop or a trilooop, then \(F(G) = F(G[1]e) + F(G[\omega]e) + F(G[\omega^2]e) = 3^{m-1} + 3^{m-1} + 3^{m-1} = 3^m\).

Claim (ii): \(F(G) = (1)^{\text{aff}(G)}\).

Proof of Claim (ii): we use induction on \(|E(G)|\). If \(|E(G)| = 0\) then the claim is true by the definition of \(F\). Suppose \(|E(G)| = m > 1\). Let \(e \in E(G)\). If \(e\) is an ultraloop, then \(F(G) = wF(G - e) = -F(G - e) = -(-1)^{\text{aff}(G)} = (-1)^{\text{aff}(G)}\). Observe that the number of anticlockwise faces in an alternating di-map is unchanged by \(1\)- or \(\omega\)-reduction, but may be altered by \(\omega^2\)-reduction. If \(e\) is a proper \(\omega\)-loop, then \(F(G) = zF(G[\omega^2]e) = -F(G[\omega^2]e) = -(-1)^{\text{aff}(G)} = (-1)^{\text{aff}(G)}\). If \(e\) is a proper \(\mu\)-loop with \(\mu \in \{1, \omega, \omega^2\}\), then \(F(G) = F(G[\mu]e) = (-1)^{\text{aff}(G)} = (-1)^{\text{aff}(G)}\).

If \(e\) is a proper \(\omega\)-semiloop, then \(\text{aff}(G[\omega^2]e) = \text{aff}(G) + 1\), while if \(e\) is neither an ultraloop nor a trilooop nor an \(\omega\)-semiloop, then \(\text{aff}(G[\omega^2]e) = \text{aff}(G) - 1\). In any event,
if \( e \) is not a triloop, then
\[
F(G) = F(G[1]e) + F(G[\omega]e) + F(G[\omega^2]e) = (-1)^{\text{af}(G[1])} + (-1)^{\text{af}(G[\omega])} + (-1)^{\text{af}(G[\omega^2])} = (-1)^{\text{af}(G)} + (-1)^{\text{af}(G)} + (-1)^{\text{af}(G)+1} = (-1)^{\text{af}(G)}.
\]

Claim (iii): \( F(G) = (-1)^{\text{af}(G)} \).

Claim (iv): \( F(G) = (-1)^{|V(G)|} \).

The proofs of Claims (iii) and (iv) are similar to that of Claim (ii), and are left as an exercise. For Claim (iv), bear in mind that \(|V(G)|\) is the number of in-stars of \( G \). \( \Box \)

A Tutte invariant for alternating dimaps is defined as for a simple Tutte invariant, except that condition 5, with \((24)\), is replaced by a requirement that there exist nonzero \( a, b, c \) such that, for any alternating dimap \( G \) and any \( e \in E(G) \),

\[
F(G) = aF(G[1]e) + bF(G[\omega]e) + cF(G[\omega^2]e),
\]

**Theorem 24** The only Tutte invariants of alternating dimaps are:

(a) \( F(G) = 0 \) for nonempty \( G \), with \( w = x = y = z = 0 \)

(b) \( F(G) = 3|E(G)|\alpha V(G)|\beta f(G)|\gamma a f(G) = 3 \), \( \frac{a}{b} = \frac{b}{c} = 3 \), \( \frac{w}{abc} = 3 \).

(c) \( F(G) = a|V(G)|\beta f(G)(-c)^{af(G)} \), \( a = \frac{b}{c} = 1 \), \( \frac{c}{a} = -1 \), \( \frac{w}{abc} = -1 \).

(d) \( F(G) = a|V(G)|\beta f(G)c^{af(G)} \), \( \frac{a}{b} = \frac{b}{c} = 1 \), \( \frac{c}{a} = -1 \), \( \frac{w}{abc} = -1 \).

(e) \( F(G) = (-a)|V(G)|\beta f(G)c^{af(G)} \), \( \frac{a}{b} = \frac{b}{c} = 1 \), \( \frac{c}{a} = -1 \), \( \frac{w}{abc} = -1 \).

**Proof.** If

\[
F(G) = aF(G[1]e) + bF(G[\omega]e) + cF(G[\omega^2]e),
\]

then define

\[
F'(G) = \frac{F(G)}{a|V(G)|\beta f(G)c^{af(G)}}.
\]  \hspace{1cm} (37)

If \( e \) is an ultraloop, then

\[
F'(G) = \frac{wF(G-e)}{a|V(G)|\beta f(G)c^{af(G)}} = \frac{wF(G-e)}{a|V(G-e)+1\beta f(G-e)+1c^{af(G)+1}} = \frac{w}{abc} \cdot \frac{F(G-e)}{a|V(G-e)|\beta f(G-e)c^{af(G-e)}} = \frac{w}{abc} \cdot F'(G-e).
\]

If \( e \) is a 1-loop, then

\[
F'(G) = \frac{xF(G[1]e)}{a|V(G)|\beta f(G)c^{af(G)}}
\]
\[ xF(G[1]e) \]
\[ = \frac{xF(G[1]e)}{a|V(G[1]e)|+1|f(G[1]e)c_{af(G[1]e)}} \]
\[ = \frac{x}{a} \cdot \frac{F(G[1]e)}{F(G[1]e)} \]
\[ = \frac{x}{a} \cdot F'(G[1]e). \]

Similarly, if \( e \) is an \( \omega \)-loop, then
\[ F'(G) = \frac{y}{b} \cdot F'(G[\omega]e), \]
and if \( e \) is an \( \omega^2 \)-loop, then
\[ F'(G) = \frac{z}{c} \cdot F'(G[\omega^2]e). \]

If \( e \in E(G) \) is neither an ultraloop nor a triloop,
\[ F'(G) = \frac{aF(G[1]e) + bF(G[\omega]e) + cF(G[\omega^2]e)}{a|V(G)||f(G)c_{af(G)}} \]
\[ = \frac{F(G[1]e)}{a|V(G)|-1|f(G)c_{af(G)}} + \frac{F(G[\omega]e)}{a|V(G)|-1|f(G)c_{af(G)}} + \]
\[ = \frac{F(G[1]e)}{a|V(G)|-1|f(G)c_{af(G)}} + \frac{F(G[\omega]e)}{a|V(G)|-1|f(G)c_{af(G)}} + \]
\[ = F'(G[1]e) + F'(G[\omega]e) + F'(G[\omega^2]e). \]

Hence \( F' \) is a simple Tutte invariant.

By Theorem 23, we must have one of

- \( F'(G) = 0, w = x = y = z = 0 \)
- \( F'(G) = 3|E(G)|, \quad \frac{x}{a} = \frac{y}{b} = \frac{z}{c} = 3, \quad \frac{w}{abc} = 3. \)
- \( F'(G) = (-1)^{af(G)}, \quad \frac{x}{a} = \frac{y}{b} = 1, \quad \frac{z}{c} = -1, \quad \frac{w}{abc} = -1. \)
- \( F'(G) = (-1)^{df(G)}, \quad \frac{z}{a} = \frac{x}{c} = 1, \quad \frac{y}{b} = -1, \quad \frac{w}{abc} = -1. \)
- \( F'(G) = (-1)^{|V(G)|}, \quad \frac{y}{b} = \frac{z}{c} = 1, \quad \frac{x}{a} = -1, \quad \frac{w}{abc} = -1. \)

Use (37) to complete the proof.

Other definitions of Tutte invariants for alternating dimaps are possible.

Definition
An extended Tutte invariant for alternating dimaps is a function $F$ defined on every alternating dimap such that $F$ is invariant under isomorphism, $F(\text{empty alternating dimap}) = 1$, and there exist $w, x, y, z, a, b, c, d, e, f, g, h, i, j, k, l$ such that, for any alternating dimap $G$,

1. for any ultraloop $e$ of $G$,
   \[ F(G) = w F(G - e) \]  
   (38)
2. for any proper 1-loop $e$ of $G$,
   \[ F(G) = x F(G[1]e) \]  
   (39)
3. for any proper $\omega$-loop $e$ of $G$,
   \[ F(G) = y F(G[\omega]e) \]  
   (40)
4. for any proper $\omega^2$-loop $e$ of $G$,
   \[ F(G) = z F(G[\omega^2]e) \]  
   (41)
5. for any proper 1-semiloop $e$ of $G$,
   \[ F(G) = a F(G[1]e) + b F(G[\omega]e) + c F(G[\omega^2]e). \]  
   (42)
6. for any proper $\omega$-semiloop $e$ of $G$,
   \[ F(G) = d F(G[1]e) + e F(G[\omega]e) + f F(G[\omega^2]e). \]  
   (43)
7. for any proper $\omega^2$-semiloop $e$ of $G$,
   \[ F(G) = g F(G[1]e) + h F(G[\omega]e) + i F(G[\omega^2]e). \]  
   (44)
8. for any edge $e$ of $G$ that is not an ultraloop or a triloop or a semiloop,
   \[ F(G) = j F(G[1]e) + k F(G[\omega]e) + l F(G[\omega^2]e). \]  
   (45)

**Problem**

Characterise all extended Tutte invariants of alternating dimaps.

One basic extended Tutte invariant is

\[ F(G) = \alpha^{|E(G)|} \beta^{|V(G)|} \gamma^{af(G)} \delta^{df(G)}, \]

with $\alpha, \beta, \gamma, \delta \neq 0$. This satisfies the definition with $w = \alpha \beta \gamma \delta$, $x = \alpha \beta$, $y = \alpha \gamma$, $z = \alpha \delta$, $a = \alpha / \beta$, $f = \alpha / \delta$, $h = \alpha / \gamma$, $b = c = d = e = g = i = 0$, $j = \alpha \beta / 3$, $k = \alpha \gamma / 3$, and $l = \alpha \delta / 3$. 
Extended Tutte invariants are much richer than Tutte invariants, since they include the Tutte polynomial for planar graphs, in a sense we now explain.

The Tutte polynomial $T(G; x, y)$ of a graph $G$ has the following inductive definition. If $E(G) = \emptyset$ then $T(G; x, y) = 1$. Otherwise, for any $e \in E(G)$,

$$T(G; x, y) = \begin{cases} xT(G \setminus e; x, y), & \text{if } e \text{ is a coloop;} \\ yT(G/e; x, y), & \text{if } e \text{ is a loop;} \\ T(G/e; x, y) + T(G \setminus e; x, y), & \text{otherwise.} \end{cases}$$

To any orientably 2-cell-embedded (undirected) graph $G$, we can associate two alternating dimaps $\text{alt}_c(G)$ and $\text{alt}_a(G)$ as follows. For $\text{alt}_c(G)$ (respectively, $\text{alt}_a(G)$), replace each edge $e = uv \in E(G)$ by a pair of oppositely directed edges $(u, v)$ and $(v, u)$, forming a clockwise (resp., anticlockwise) face of size two. The faces of $G$ now all correspond to anticlockwise (resp., clockwise) faces in $\text{alt}_c(G)$ (resp., $\text{alt}_a(G)$).

For any alternating dimap $G$, define $T_c(G; x, y)$ and $T_a(G; x, y)$ as follows. If $E(G) = \emptyset$, then $T_c(G; x, y) = T_a(G; x, y) = 1$. Otherwise, for any $e \in E(G)$,

$$T_c(G; x, y) = \begin{cases} T_c(G[\ast e]; x, y), & \text{if } e \text{ is an } \omega^2\text{-loop (including an ultraloop);} \\ xT_c(G[\omega^2 e]; x, y), & \text{if } e \text{ is an } \omega\text{-semiloop;} \\ yT_c(G[1 e]; x, y), & \text{if } e \text{ is a proper 1-semiloop or an } \omega\text{-loop;} \\ T_c(G[1 e]; x, y) + T_c(G[\omega^2 e]; x, y), & \text{if } e \text{ is not a semiloop.} \end{cases}$$

$$T_a(G; x, y) = \begin{cases} T_a(G[\ast e]; x, y), & \text{if } e \text{ is an } \omega\text{-loop (including an ultraloop);} \\ xT_a(G[\omega e]; x, y), & \text{if } e \text{ is an } \omega^2\text{-semiloop;} \\ yT_a(G[1 e]; x, y), & \text{if } e \text{ is a proper 1-semiloop or an } \omega^2\text{-loop;} \\ T_a(G[1 e]; x, y) + T_a(G[\omega e]; x, y), & \text{if } e \text{ is not a semiloop.} \end{cases}$$

**Theorem 25** For any plane graph $G$,

$$T(G; x, y) = T_c(\text{alt}_c(G); x, y) = T_a(\text{alt}_a(G); x, y).$$

**Proof.** For any vertex $v$, write $L^\omega(v)$ and $L^{\omega^2}(v)$ for an $\omega$-loop and an $\omega^2$-loop, respectively, at $v$. If such a loop is added to an alternating dimap, it must be placed within a $c$-face or an $a$-face, respectively.

Consider $\text{alt}_c(G)$. Observe that, for any $uv \in E(G)$,

$$\text{alt}_c(G)[1](u, v) = \text{alt}_c(G/uv) + L^{\omega^2}(u'),$$

$$\text{alt}_c(G)[\omega^2](u, v) = \text{alt}_c(G \setminus uv) + L^{\omega^2}(u'),$$

for some $u'$. (Mostly $u' = u$, except that a little more detail is needed if $(u, v)$ is a proper 1-semiloop, but the exact location of these extra triloops is not important.)

These observations can be used to prove, by induction on $|E(G)|$, that $T(G; x, y) = T_c(\text{alt}_c(G); x, y)$ for any plane graph $G$.

It is clear from the definitions that they are identical when $G$ is empty.

Suppose then that $T(G; x, y) = T_c(\text{alt}_c(G); x, y)$ when $|E(G)| < m$, where $m \geq 1$. 

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Let \( G \) be any orientably 2-cell-embedded graph on \( m \) edges, and let \( e = uv \in E(G) \).

If \( e \) is a coloop, then \( (u, v) \) and \( (v, u) \) are both \( \omega \)-semiloops in \( \text{alt}_c(G) \). (Conversely, if \( (u, v) \) and \( (v, u) \) are both proper \( \omega \)-semiloops in \( \text{alt}_c(G) \), then \( uv \) is a coloop in \( G \).

This does not hold in general if \( G \) is not plane, however.) We have

\[
T_c(\text{alt}_c(G); x, y) = xT_c(\text{alt}_c(G)[\omega^2](u, v); x, y) = xT_c(\text{alt}_c(G/uv) + L(\omega^2)(u); x, y)
= xT_c(\text{alt}_c(G/uv); x, y) = xT(G/uv; x, y) = T(G; x, y),
\]

where the penultimate equality uses the inductive hypothesis.

If \( e \) is a loop, then in \( \text{alt}_c(G) \) the directed versions \( (u, v) \) and \( (v, u) \) are both 1-semiloops. (This time, the converse holds even if \( G \) is not plane.) One of them may also be an \( \omega \)-loop, but neither is an \( \omega^2 \)-loop. In any case, we find that \( T_c(\text{alt}_c(G); x, y) = T(G; x, y) \) by a similar argument to that just used for coloops.

If \( e \) is neither a coloop nor a loop, we have

\[
T_c(\text{alt}_c(G); x, y) = T_c(\text{alt}_c(G)[1](u, v); x, y) + T_c(\text{alt}_c(G)[\omega^2](u, v); x, y)
= T_c(\text{alt}_c(G/uv) + L(\omega^2)(u'); x, y) + T_c(\text{alt}_c(G \setminus uv) + L(\omega^2)(u'); x, y)
= T_c(\text{alt}_c(G/uv); x, y) + T_c(\text{alt}_c(G \setminus uv); x, y)
= T(G/uv; x, y) + T(G \setminus uv; x, y) \quad \text{(by the inductive hypothesis)}
= T(G; x, y).
\]

We conclude by induction that \( T(G; x, y) = T_c(\text{alt}_c(G); x, y) \) holds for all \( G \).

The proof that \( T(G; x, y) = T_a(\text{alt}_a(G); x, y) \) follows the same line, with appropriate adjustments.

Having constructed alternating dimaps from embedded graphs, by replacing edges by \( c \)-faces, or by \( a \)-faces, of size 2, it is natural to ask about replacing edges by in-stars of size 2. To do this, for an embedded graph \( G \), first construct its medial graph, \( \text{med}(G) \), then turn it into an alternating dimap by directing the edges so as to ensure the alternating property. For each component of \( G \), there are two such ways of directing the edges in that component. There are therefore \( 2^{k(G)} \) different alternating dimaps constructible from \( G \) in this way, all with \( \text{med}(G) \) as the underlying embedded graph. We refer to any one of them as \( \text{alt}_i(G) \).

Write \( T_i(G; x) \) for any invariant of alternating dimaps that satisfies the following.

\[
T_i(G; x) = \begin{cases} 
1, & \text{if } G \text{ is empty;} \\
T_i(G[e]; x), & \text{if } e \text{ is a 1-loop (including an ultraloop);} \\
xT_i(G[\omega^2]e; x), & \text{if } e \text{ is a proper } \omega \text{-semiloop or an } \omega^2 \text{-loop;} \\
xT_i(G[\omega]e; x), & \text{if } e \text{ is a proper } \omega^2 \text{-semiloop or an } \omega \text{-loop;} \\
T_i(G[\omega]e; x) + T_i(G[\omega^2]e; x), & \text{if } e \text{ is not a semiloop.}
\end{cases}
\]

This is not a full definition of a unique \( T_i(G; x) \), since we have not specified what happens if \( e \) is a proper 1-semiloop. But it will turn out that, in the alternating dimaps of interest here, proper 1-semiloops do not arise.
Theorem 26  For any plane graph $G$, 
\[ T(G; x, x) = T_i(alt_i(G); x). \]

Proof. For any alternating dimap $H$, write $H^{(k)}$ for any alternating dimap obtained from $H$ by performing, $k$ times, a subdivision of an edge (by insertion in it of a new vertex of indegree = outdegree = 1, with the edge going into the new vertex being a proper 1-loop).

Let $G$ be a plane graph and fix any specific $alt_i(G)$. If $e \in E(G)$, write $e^+$ for either of the edges of $alt_i(G)$ that are directed into the vertex representing $e$.

Observe that, for any $e \in E(G)$,
\[ \{alt_i(G)[\omega]e^+, alt_i(G)[\omega^2]e^+\} = \{alt_i(G/e)^{(1)}, alt_i(G \setminus e)^{(1)}\}. \]

Therefore
\[ T_i(alt_i(G)[\omega]e^+; x) + T_i(alt_i(G)[\omega^2]e^+; x) = T_i(alt_i(G/e)^{(1)}; x) + T_i(alt_i(G \setminus e)^{(1)}; x). \]

If $e$ is either a coloop or a loop, then
\[ alt_i(G)[\omega^2]e^+ \in \{alt_i(G/e)^{(1)}, alt_i(G \setminus e)^{(1)}\}, \]
\[ alt_i(G)[\omega]e^+ \in \{alt_i(G/e)^{(1)}, alt_i(G \setminus e)^{(1)}\}. \]

We now prove the theorem by induction on $|E(G)|$. The base case is immediate from the definition. So suppose $G$ is an orientably 2-cell-embedded graph with $m$ edges, where $m \geq 1$.

If $e$ is a coloop or a loop, then $e^+$ is an $\omega$- or an $\omega^2$-semiloop in $alt_i(G)$, except that it is not a 1-loop. From our (partial) definition of $T_i(alt_i(G); x)$, we have
\[ T_i(alt_i(G); x) \in \{xT_i(alt_i(G)[\omega]e^+; x), xT_i(alt_i(G)[\omega^2]e^+; x)\} \]
\[ = \{xT_i(alt_i(G/e)^{(1)}; x), xT_i(alt_i(G \setminus e)^{(1)}; x)\} \]
\[ = \{xT_i(alt_i(G/e); x), xT_i(alt_i(G \setminus e); x)\} \]
\[ = \{xT(G/e; x, x), xT(G \setminus e; x, x)\}, \]
by the inductive hypothesis. But, for such an $e$, the graphs $G/e$ and $G \setminus e$ have isomorphic cycle matroids, so their Tutte polynomials are identical. Therefore
\[ T_i(alt_i(G); x) = x T(G/e; x, x) = x T(G \setminus e; x, x). \]

But these two quantities each equal $T(G; x, x)$, for such an $e$, so we are done in this case.

If $e$ is neither a loop nor a coloop, then
\[ T_i(alt_i(G); x) = T_i(alt_i(G)[\omega]e^+; x) + T_i(alt_i(G)[\omega^2]e^+; x) \]
\[ = T_i(alt_i(G/e)^{(1)}; x) + T_i(alt_i(G \setminus e)^{(1)}; x) \]
\[ = T_i(alt_i(G/e); x) + T_i(alt_i(G \setminus e); x) \]
\[ = T(G/e; x, x) + T(G \setminus e; x, x) \quad \text{(by the inductive hypothesis)} \]
\[ = T(G; x, x). \]
The result follows.

Observe that, since med\((G)\) is 4-regular, alt\(_i\)(\(G\)) has no proper 1-semiloops. Furthermore, the only minors of it we need to form do not require 1-reduction, so these minors are each 4-regular and so have no proper 1-semiloop too.

The Tutte polynomial evaluation \(T(G; x, x)\) is just the Martin polynomial of \(med(G)\) (see, e.g., [17]).

One reason that Tutte invariants of alternating dimaps are more limited than Tutte invariants for graphs is the non-commutativity of the minor operations. The definitions of such invariants for alternating dimaps require the stated recursive relations to hold for reduction of any edge of the stated type, which means that the invariant will need to be unperturbable by some variations of the order of operations.

These observations raise the possibility that better invariants may come from including an ordering of the edges in the object to which the invariant applies.

Definitions

An ordered alternating dimap is a pair \((G, <)\) where \(G\) is an alternating dimap and < is a linear order on \(E(G)\).

If \((G, <)\) is an ordered alternating dimap and \(\mu \in \{1, \omega, \omega^2\}\), then the \(\mu\)-reduction \((G, <)[\mu]\) of \((G, <)\) is the ordered alternating dimap \((G[\mu]e_0, <')\) where \(e_0\) is the first edge in \(E(G)\) under < and the order <' on \(E(G) \setminus \{e_0\}\) is obtained by simply removing \(e_0\) from the order <.

Tutte invariants and extended Tutte invariants are defined for these objects by modifying the definitions of such invariants for ordinary alternating dimaps as follows:

1. The definitions apply to ordered alternating dimaps, rather than just to alternating dimaps.

2. All references to \(G[\mu]e\) are replaced by \((G, <)[\mu]\), for each \(\mu\).

3. All universal quantification over edges is deleted (since there is no choice of which edge to reduce, since it is always the first edge in the ordering which must be reduced).

4. All reference to an edge \(e\) is replaced by reference to the first edge \(e_0\) in the ordering.

For example, the second condition in each of the definitions becomes: if \(e_0\) is a 1-loop, then \(F((G, <)) = xF((G, <)[1])\).

When \(G\) is a general plane alternating dimap, the extended Tutte invariants \(T_c(G; x, y)\) and \(T_a(G; x, y)\) we considered earlier actually depend on the order in which the edges are considered. So they pertain to ordered alternating dimaps. But, if \(G\) has the form alt\(_c\)(\(H\)) (with analogous remarks applying to alt\(_a\)(\(H\))), then the order in which the edges \(uv\) of \(H\) are considered does not matter, and each time a corresponding \((u, v)\) is reduced in \(G\), it leaves behind an \(\omega^2\)-loop which can be reduced at
any time. (Note also that, if we do not use $\omega$-reductions, we cannot encounter those situations of non-commutativity for two edges that are consecutive in an a-face or an in-star.) For such cases, the invariants are well-defined without having to specify an order on the edges at the beginning.

**Problem**

Characterise (a) Tutte invariants, and (b) extended Tutte invariants, of ordered alternating dimaps.

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**References**

[1] V. Batagelj, An improved inductive definition of two restricted classes of triangulations of the plane, in: Z. Skupień and M. Borowiecki (eds.), *Combinatorics and Graph Theory* (Warsaw, 1987), Banach Center Publ. 25, PWN, Warsaw, 1989, pp. 11–18.

[2] K. A. Berman, A proof of Tutte’s trinity theorem and a new determinant formula, *SIAM J. Alg. Disc. Meth.* 1 (1980) 64–69.

[3] C. P. Bonnington and C. H. C. Little, *The Foundations of Topological Graph Theory*, Springer, New York, 1995.

[4] A. Bouchet, Isotropic systems. *European J. Combin.* 8 (1987) 231–244.

[5] A. Bouchet, Multimatroids. II. Orthogonality, minors and connectivity, *Electron. J. Combin.* 5 (1998), Research Paper 8, 25pp.

[6] R. L. Brooks, C. A. B. Smith, A. H. Stone, and W. T. Tutte, The dissection of rectangles into squares. *Duke Math. J.* 7 (1940) 312–340.

[7] T. H. Brylawski and J. G. Oxley, The Tutte polynomial and its applications, in: N. White (ed.), *Matroid Applications*, Encyclopedia Math. Appl. 40, Cambridge Univ. Press, 1992, pp. 123–225.

[8] N. Cavenagh and P. Lisoněk, Planar Eulerian triangulations are equivalent to spherical Latin bitrades, *J. Combin. Theory Ser. A* 115 (2008), 193–197.

[9] N. J. Cavenagh and I. M. Wanless, Latin trades in groups defined on planar triangulations, *J. Algebraic Combin.* 30 (2009), 323–347.
[10] G. Chartrand and F. Harary, Planar permutation graphs, Ann. Inst. H. Poincaré Sect. B (N.S.) 3 (1967) 433–438.

[11] R. Cori and J. G. Penaud, The complexity of a planar hypermap and that of its dual, Ann. Discrete Math. 9 (1980) 53–62.

[12] G. A. Dirac, A property of 4-chromatic graphs and remarks on critical graphs, J. London Math. Soc. 27 (1952) 69–81.

[13] A. Drápal, On a planar construction of quasigroups. Czechoslovak Math. J. 41 (116) (1991), 538–548.

[14] A. Drápal, On elementary moves that generate all spherical Latin trades, Comment. Math. Univ. Carolin. 50 (2009) 477–511.

[15] A. Drápal, C. Hämäläinen and V. Kala, Latin bitrades, dissections of equilateral triangles, and abelian groups, J. Combin. Des. 18 (2010), 1–24.

[16] R. J. Duffin, Topology of series-parallel networks, J. Math. Anal. Appl. 10 (1965) 303–318.

[17] J. A. Ellis-Monaghan, Identities for circuit partition polynomials, with applications to the Tutte polynomial, Adv. in Appl. Math. 32 (2004) 188–197.

[18] J. A. Ellis-Monaghan and C. Merino, Graph polynomials and their applications I: The Tutte polynomial, in: M. Dehmer (ed.), Structural Analysis of Complex Networks, Birkhäuser/Springer, New York, 2011, pp. 219–255.

[19] G. E. Farr, A generalization of the Whitney rank generating function, Math. Proc. Camb. Phil. Soc. 113 (1993) 267–280.

[20] G. E. Farr, Some results on generalised Whitney functions, Adv. in Appl. Math. 32 (2004) 239–262.

[21] G. E. Farr, Tutte-Whitney polynomials: some history and generalizations, in: G. R. Grimmett and C. J. H. McDiarmid (eds.), Combinatorics, Complexity and Chance: A Tribute to Dominic Welsh, Oxford University Press, 2007, pp. 28–52.

[22] G. E. Farr, On the Ashkin-Teller model and Tutte-Whitney functions, Combin. Probab. Comput. 16 (2007) 251–260.

[23] G. E. Farr, Transforms and minors for binary functions, Ann. Combin. 17 (2013) 477–493.

[24] T. Grubman and I. Wanless, Growth rate of canonical and minimal group embeddings of spherical latin trades, J. Combin. Theory Ser. A, to appear.

[25] F. Jaeger, A new invariant of plane bipartite cubic maps, Discrete Math. 101 (1992) 149–164.
[26] C. Kuratowski, Sur le problème des courbes gauches en topologie, *Fund. Math.* **15** (1930) 217–283.

[27] S. K. Lando and A. K. Zvonkin, *Graphs on Surfaces and their Applications*, Springer, Berlin, 2004.

[28] W. Mader, A reduction method for edge-connectivity in graphs, in: *Advances in Graph Theory* (Cambridge Combinatorial Conf., Trinity College, Cambridge, 1977), *Ann. Discrete Math.* **3** (1978), 145–164.

[29] B. Mohar and C. Thomassen, *Graphs on Surfaces*, Johns Hopkins University Press, Baltimore, 2001.

[30] C. St. J. A. Nash-Williams, On well-quasi-ordering infinite trees, *Proc. Cambridge Philos. Soc.* **61** 1965 697–720.

[31] J. G. Oxley, *Matroid Theory*, Oxford University Press, New York, 1992.

[32] N. Robertson and P. D. Seymour, Graph minors. XX. Wagner’s conjecture, *J. Combin. Theory (Ser. B)* **92** (2004) 325–357.

[33] L. Traldi, The transition matroid of a 4-regular graph: an introduction, preprint, 2013, [http://arxiv.org/abs/1307.8097](http://arxiv.org/abs/1307.8097).

[34] W. T. Tutte, The dissection of equilateral triangles into equilateral triangles, *Proc. Cambridge Philos. Soc.* **44** (1948) 463–482.

[35] W. T. Tutte, Duality and trinity, in: *Infinite and Finite Sets (Colloq., Keszthely, 1973)*, Vol. III, Colloq. Math. Soc. Janos Bolyai, Vol. 10, North-Holland, Amsterdam, 1975, pp. 1459–1472.

[36] W. T. Tutte, Bicubic planar maps, *Symposium à la Mémoire de François Jaeger* (Grenoble, 1998), *Ann. Inst. Fourier (Grenoble)* **49** (1999) 1095–1102.

[37] K. Wagner, Über eine Eigenschaft der ebenen Komplexe, *Math. Ann.* **114** (1937) 570–590.

[38] D. J. A. Welsh, *Complexity: Knots, Colourings and Counting*, London Math. Soc. Lecture Note Series 186, Cambridge University Press, 1993.