Statistical Inference for Large-dimensional Matrix Factor Model from Least Squares and Huber Loss Points of View

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In the article we focus on large-dimensional matrix factor models and propose estimators of factor loading matrices and factor score matrix from the perspective of minimizing least squares objective function. The resultant estimators turns out to be equivalent to the corresponding projected estimators in Yu et al. (2021), which enjoys the nice properties of reducing the magnitudes of the idiosyncratic error components and thereby increasing the signal-to-noise ratio. We derive the convergence rate of the theoretical minimizers under sub-Gaussian tails, instead of the one-step iteration estimators by Yu et al. (2021). Motivated by the least squares formulation, we further consider a robust method for estimating large-dimensional matrix factor model by utilizing Huber Loss function. Theoretically, we derive the convergence rates of the robust estimators of the factor loading matrices under finite fourth moment conditions. We also propose an iterative procedure to estimate the pair of row and column factor numbers robustly. We conduct extensive numerical studies to investigate the empirical performance of the proposed robust methods relative to the state-of-the-art ones, which show the proposed ones perform robustly and much better than the existing ones when data are heavy-tailed while perform almost the same (comparably) with the projected estimators when data are light-tailed, and as a result can be used as a safe replacement of the existing ones. An application to a Fama-French financial portfolios dataset illustrates its empirical usefulness.

Keyword: Huber Loss; Latent low rank; Least squares; Matrix factor model.

1 Introduction

Large-dimensional factor model has been a powerful tool of summarizing information from large datasets and draws growing attention in the era of “big-data” where more and more records of variables are available. The last two decades have seen many studies on large-dimensional approximate factor model, since the seminal work by Bai and Ng (2002) and Stock and Watson (2002). See, for example, the representative work by Bai (2003), Onatski (2009), Ahn and Horenstein (2013), Fan et al. (2013), Bai and Li (2012), Bai and Li (2016) and Trapani (2018). These works all require the fourth moments (or even higher moments) of factors and idiosyncratic errors exist and there also exist works on relaxing the restrictive moment conditions, see the endeavors by Yu et al. (2019), He et al. (2020) and Chen et al. (2021).
In the last few years, large-dimensional matrix factor model has drawn much attention in view of the fact that observations are usually well structured to be an array such as in macroeconomic and financial area, see Chen and Fan (2021) for further examples of matrix observations. The seminal work is the one by Wang et al. (2019), which propose the following formulation for matrix time series observations \( \{X_t, 1 \leq t \leq T\} \):

\[
X_t = R_{p_1 \times k_1} \times F_t_{k_1 \times k_2} \times C_{k_2 \times p_2}^{\top} + E_t_{p_1 \times p_2},
\]

where \( R \) is the row factor loading matrix exploiting the variations of \( X_t \) across the rows, \( C \) is the \( p_2 \times k_2 \) column factor loading matrix reflecting the differences across the columns of \( X_t \), \( F_t \) is the common factor matrix for all cells in \( X_t \) and \( E_t \) is the idiosyncratic components. Wang et al. (2019) propose estimators of the factor loading matrices and of numbers of the row and column factors based on the eigen-analysis of the auto-cross-covariance matrix, under the assumption that the idiosyncratic term \( E_t \) is white noise. Chen and Fan (2021) propose an \( \alpha \)-PCA method for inference of (1.1), which conducts eigen-analysis of a weighted average of the sample mean and the column (row) sample covariance matrix; Yu et al. (2021) proposed a projected estimation method which further improved the estimation efficiency of the factor loading matrices and the numbers of factors. He et al. (2021) proposed a strong rule to determine whether there is a factor structure of matrix time series and also propose a sequential procedure to determine the numbers of factors. Extensions and applications of the matrix factor model include the dynamic transport network in the context of international trade flows by Chen and Chen (2020), the constrained matrix factor model by Chen et al. (2020b), and the threshold matrix factor model in Liu and Chen (2020).

In the current work, we first propose least squares estimators for the matrix factor models. The most interesting finding is that the least squares estimators are equivalent to the projected estimators by Yu et al. (2021), i.e., the projected estimators minimizes the least squares loss function. This finding provides another rationale for the projected estimation procedure by Yu et al. (2021), which is initially proposed for reducing the magnitudes of the idiosyncratic error components and thereby increasing the signal-to-noise ratio. Motivated by the least squares formulation, we further propose a robust method for estimating large-dimensional matrix factor model, by substituting the least squares loss function with the Huber Loss function. The resultant estimators of factor loading matrices can be simply viewed as the eigenvectors of weighted sample covariance matrices of the projected data, which are easily obtained by an iterative algorithm. As far as we know, this is the first work on robust analysis of matrix factor model. As an illustration, we check the sensitivity of the \( \alpha \)-PCA method by Chen and Fan (2021) and Projected Estimation (PE) method (or least squares minimization method) by Yu et al. (2021) to the heavy-tailedness of the idiosyncratic errors with a synthetic dataset. We generate the the idiosyncratic errors from matrix-variate normal, matrix-variate \( t_3 \) distributions that will be described in detail in Section 4. Figure 1 depicts the boxplots of the factor loading matrices \( R \) and \( C \)'s estimation errors over 1000 replications. It can be clearly seen that the \( \alpha \)-PCA method and the PE method result in much bigger biases and higher dispersions as the distribution tails become heavier. The proposed Robust Matrix Factor Analysis (RMFA) method still works quite satisfactorily when the idiosyncratic errors are from matrix-variate \( t_3 \) distribution.

To do factor analysis, the first step is to determine the number of factors. As for the Matrix Factor Model (MFM), both the row and column factor numbers should be determined in advance. Wang et al. (2019) proposed an estimation method based on ratios of consecutive eigenvalues of auto-covariance matrices, similar to Lam and Yao (2012); Chen et al. (2020a) proposed an \( \alpha \)-PCA based eigenvalue-ratio method and Yu et al. (2021) further proposed a projection-based iterative eigenvalue-ratio method. All the methods above borrow
(a) the estimation of $R$

(b) the estimation of $C$

Figure 1: Boxplot of the distance between the estimated loading space and the true loading space by RMFA, PCA and PE methods under different distributions (matrix normal and matrix $t_3$). $p_1 = 20, p_2 = T = 50$. The left panel (a) is the distance between the estimated loading space $\hat{R}$ and true loading space $R$, and the right panel (b) is the distance between the estimated loading space $\hat{C}$ and true loading space $C$.

idea from Ahn and Horenstein (2013) and as far as we know, He et al. (2021) is the only work that determines the factor numbers of MFM from the perspective of sequential hypothesis testing and the authors also provide a strong rule to determine whether there is a row/column factor structure in the matrix time series. However, neither of the methods mentioned above takes the well-known heavytailedness of financial/macroeconomic data into account (see also Figure 2 in the real data section). In the current work, we also present a robust iterative eigenvalue-ratio method to estimate the numbers of factors following the Huber loss formulation. The Huber Loss function has been well known in robust statistics and was initiated by Peter Huber in his seminal work Huber (1964). Asymptotic properties of the Huber estimator have been studied thoroughly, see for example Huber (1973); He and Shao (2000). High-dimensional penalized Huber regression has also been studied recently, see for example Fan et al. (2017) and Sun et al. (2020).

The contributions of the current work lie in the following aspects. Firstly, we formulate the estimation of MFM from the least squares point of view and find that minimizing the least squares under the identifiability conditions naturally leads to the iterative projection method discussed in Yu et al. (2021) and thus enjoys the nice properties of the PE method. Secondly, we further propose a robust estimation method for MFM by considering the Huber loss in place of the least squares loss. We also propose an efficient iterative algorithm to solve the corresponding optimization problem. As far as we know, this is the first work on robust estimation for MFM. Thirdly, we also propose an iterative algorithm for estimating the row/column factor number robustly. Lastly, we not only derive the convergence rates of the theoretical minimizers of least squares, but also derive the convergence rates of the robust estimators under a more relaxed finite fourth moment condition.

The rest of the article goes as follows. In Section 2, we first formulate the estimation of factor loading matrices and factor score matrix by minimizing the least squares loss and provide solutions to the optimization problem, from which we can see its equivalence to the projected estimation method. Then we provide robust estimators by considering the Huber loss function and present detailed algorithm to obtain the minimizers. In Section 3, we investigate the theoretical minimizers of the least squares and Huber loss function under mild conditions. In Section 4, we conduct thorough numerical studies to illustrate the advantages of the
RMFA method and the robust iterative eigenvalue-ratio method over the state-of-the-art methods. In Section 5, we analyze a macroeconomic dataset to illustrate the empirical performance/usefulness of the proposed methods. We discuss possible future research directions and conclude the article in Section 6. The proofs of the main theorems are put in the Appendix and additional details are collected in the supplementary materials.

Before ending this section, we introduce the notations adopted throughout the paper. For any vector \( \mathbf{\mu} = (\mu_1, \ldots, \mu_p)^\top \in \mathbb{R}^p \), let \( \|\mathbf{\mu}\|_2 = (\sum_{i=1}^p \mu_i^2)^{1/2}, \|\mathbf{\mu}\|_\infty = \max_i |\mu_i| \). For a real number \( a \), denote \([a]\) as the largest integer smaller than or equal to \( a \), let \( \text{sgn}(a) = 1 \) if \( a \geq 0 \) and \( \text{sgn}(a) = -1 \) if \( a < 0 \). Let \( I(\cdot) \) be the indicator function. Let \( \text{diag}(a_1, \ldots, a_p) \) be a \( p \times p \) diagonal matrix, whose diagonal entries are \( a_1, \ldots, a_p \). For a matrix \( \mathbf{A} \), let \( A_{ij} \) (or \( A_{i,j} \)) be the \((i,j)\)-th entry of \( \mathbf{A} \), \( \mathbf{A}\top \) the transpose of \( \mathbf{A} \), \( \text{Tr}(\mathbf{A}) \) the trace of \( \mathbf{A} \), \( \text{rank}(\mathbf{A}) \) the rank of \( \mathbf{A} \) and \( \text{diag}(\mathbf{A}) \) a vector composed of the diagonal elements of \( \mathbf{A} \). Denote \( \lambda_j(\mathbf{A}) \) as the \( j \)-th largest eigenvalue of a nonnegative definitive matrix \( \mathbf{A} \), and let \( \|\mathbf{A}\| \) be the spectral norm of matrix \( \mathbf{A} \) and \( \|\mathbf{A}\|_F \) be the Frobenius norm of \( \mathbf{A} \). For two series of random variables, \( X_n \) and \( Y_n \), \( X_n \approx Y_n \) means \( X_n = O_p(Y_n) \) and \( Y_n = O_p(X_n) \). For two random variables (vectors) \( \mathbf{X} \) and \( \mathbf{Y} \), \( \mathbf{X} \overset{d}{=} \mathbf{Y} \) means the distributions of \( \mathbf{X} \) and \( \mathbf{Y} \) are the same. The constants \( c, C_1, C_2 \) in different lines can be nonidentical.

2 Methodology

In this section we introduce the main methodologies of the article. In Section 2.1, we discuss the relationship between minimizing the least squares under the identifiability conditions and the PE method by Yu et al. (2021). In Section 2.2, we propose a robust method by exploiting the Huber loss and provide an efficient algorithm to solve the optimization problem. In Section 2.3, we provide an iterative algorithm to estimate the column/row factor number robustly.

2.1 Least Squares and Projected Estimation

Let \( \mathbf{X}_t \) be a \( p_1 \times p_2 \) matrix observed at time point \( t \). The matrix factor model is:

\[
\mathbf{X}_t = \mathbf{RF}_t \mathbf{C}^\top + \mathbf{E}_t, \quad t = 1, 2, \cdots, T, \tag{2.1}
\]

where \( \mathbf{R} \) is the \( p_1 \times k_1 \) row factor loading matrix, \( \mathbf{C} \) is the \( p_2 \times k_2 \) column factor loading matrix, \( \mathbf{F}_t \) is the common factor matrix and \( \mathbf{E}_t \) is the idiosyncratic component and \( \mathbf{S}_t = \mathbf{RF}_t \mathbf{C}^\top \) are common component matrix. The loading matrices \( \mathbf{R} \) and \( \mathbf{R} \) in model (2.1) are not separately identifiable. Without loss of generality, we assume that \( \mathbf{R}^\top \mathbf{R}/p_1 = \mathbf{I}_{k_1} \) and \( \mathbf{C}^\top \mathbf{C}/p_2 = \mathbf{I}_{k_2} \) for identifiability, see also Chen and Fan (2021); Yu et al. (2021).

From (2.1), it is a natural idea to estimate \( \mathbf{R} \) and \( \mathbf{C} \) by minimizing the least squares loss under the identifiability condition:

\[
\min_{\{\mathbf{R}, \mathbf{C}, \mathbf{F}_t\}} L_1(\mathbf{R}, \mathbf{C}, \mathbf{F}_t) = \frac{1}{T} \sum_{t=1}^T \|\mathbf{X}_t - \mathbf{RF}_t \mathbf{C}^\top\|_F^2,
\]

s.t. \( \frac{1}{p_1} \mathbf{R}^\top \mathbf{R} = \mathbf{I}_{k_1}, \frac{1}{p_2} \mathbf{C}^\top \mathbf{C} = \mathbf{I}_{k_2} \). \( \tag{2.2} \)
The right hand side of (2.2) can be simplified as:

\[
\frac{1}{T} \sum_{t=1}^{T} \| X_t - RF_t C^\top \|_F^2 = \frac{1}{T} \sum_{t=1}^{T} \left[ \text{Tr}(X_t^\top X_t) - 2\text{Tr}(X_t^\top RF_t C^\top) + p_1 p_2 \text{Tr}(F_t^\top F_t) \right].
\]

The optimization is non-convex over \{R, C, F_t\}, but given the others, the loss function is convex over the remaining parameter. For instance, given (R, F_t), \(L_1(R, C, F_t)\) is convex over C. Then we first assume that (R, C) are given and solve the optimization problem on \(F_t\). For each \(t\), taking \(\partial L_1(R, C)/\partial F_t = 0\), we obtain

\[
F_t = \frac{1}{p_1 p_2} R^\top X_t C.
\]

Thus by substituting \(F_t = R^\top X_t C/(p_1 p_2)\) in the loss function \(L_1(R, C, F_t)\), we further have

\[
\min_{\{R, C\}} L_1(R, C) = \frac{1}{T} \sum_{t=1}^{T} \left[ \text{Tr}(X_t^\top X_t) - \frac{1}{p_1 p_2} \text{Tr}(X_t^\top RR^\top X_t CC^\top) \right],
\]

s.t. \(\frac{1}{p_1} R^\top R = I_{k_1}, \frac{1}{p_2} C^\top C = I_{k_2}\).

Finally the Lagrangian function is introduced as follows:

\[
\min_{\{R, C\}} L_1(R, C) + \text{Tr} \left[ \Theta \left( \frac{1}{p_1} R^\top R - I_{k_1} \right) \right] + \text{Tr} \left[ \Lambda \left( \frac{1}{p_2} C^\top C - I_{k_2} \right) \right],
\]

where the Lagrangian multipliers \(\Theta\) and \(\Lambda\) are symmetric matrices.

According to KKT condition, let

\[
\frac{\partial L_1}{\partial R} = -\frac{1}{T} \sum_{t=1}^{T} \frac{2}{p_1 p_2} X_t C C^\top X_t^\top R + \frac{2}{p_1} R \Theta = 0,
\]

\[
\frac{\partial L_1}{\partial C} = -\frac{1}{T} \sum_{t=1}^{T} \frac{2}{p_1 p_2} X_t^\top RR^\top X_t C + \frac{2}{p_2} C \Lambda = 0,
\]

respectively. Naturally, the following equations hold:

\[
\left\{ \begin{array}{l}
\left( \frac{1}{T p_2} \sum_{t=1}^{T} X_t C C^\top X_t^\top \right) R = R \Theta, \\
\left( \frac{1}{T p_1} \sum_{t=1}^{T} X_t^\top RR^\top X_t \right) C = C \Lambda,
\end{array} \right.
\]

or

\[
\left\{ \begin{array}{l}
M_c R = R \Theta, \\
M_r C = C \Lambda,
\end{array} \right.
\]

(2.4)

where

\[
M_c = \frac{1}{T p_2} \sum_{t=1}^{T} X_t C C^\top X_t^\top, \quad M_r = \frac{1}{T p_1} \sum_{t=1}^{T} X_t^\top RR^\top X_t.
\]

We denote the first \(k_1\) eigenvectors of \(M_c\) as \(\{r_1, \ldots, r_{k_1}\}\) and the corresponding eigenvalues as \(\{\theta_1, \ldots, \theta_{k_1}\}\). Similarly, we denote the first \(k_2\) eigenvectors of \(M_r\) as \(\{c_1, \ldots, c_{k_2}\}\) and the corresponding eigenvalues as \(\{\lambda_1, \ldots, \lambda_{k_2}\}\). From (2.4), it’s quite clear that \(R = (r_1, \ldots, r_{k_1}), C = (c_1, \ldots, c_{k_2}), \Theta = \text{diag}(\theta_1, \ldots, \theta_{k_1}), \Lambda = (\lambda_1, \ldots, \lambda_{k_2})\) satisfies the KKT condition. However, \(M_c\) relies on the unknown column factor loading \(C\).
while $M_r$ relies on the unknown row factor loading $R$, which motivates us to consider a iterative procedure to get the estimators. This turns out to be the same as the projected estimation procedure by Yu et al. (2021). We summarized the algorithm in Algorithm 1 and the initial estimators $\hat{R}^{(0)}$ and $\hat{C}^{(0)}$ are simply selected as the estimators by $\alpha$-PCA with $\alpha = 0$.

**Algorithm 1** Least squares method for estimating matrix factor spaces

**Input:** Data matrices $\{X_t\}_{t \leq T}$, the pair of row and column factor numbers $k_1$ and $k_2$

**Output:** Factor loading matrices $\hat{R}$ and $\hat{C}$

1: obtain the initial estimators $\hat{R}^{(0)}$ and $\hat{C}^{(0)}$ by $\alpha$-PCA with $\alpha = 0$;
2: project the data matrices to lower dimensions by defining: $\hat{Y}_t = X_t\hat{C}^{(0)}$ and $\hat{Z}_t = X_t^\top \hat{R}^{(0)}$;
3: given $\hat{Y}_t$ and $\hat{Z}_t$, define $\hat{M}_r = (T_p_1)^{-1} \sum_{t=1}^T \hat{Y}_t \hat{Y}_t^\top$ and $\hat{M}_c = (T_p_2)^{-1} \sum_{t=1}^T \hat{Z}_t \hat{Z}_t^\top$, and obtain the leading $k_1$ eigenvectors of $\hat{M}_r$, denote as $\{\hat{r}_1, \ldots, \hat{r}_{k_1}\}$ and the the leading $k_2$ eigenvectors of $\hat{M}_c$, denoted as $\{\hat{c}_1, \ldots, \hat{c}_{k_2}\}$; Then update $\hat{R}$ and $\hat{C}$ as $\hat{R}^{(1)} = (\hat{r}_1, \ldots, \hat{r}_{k_1})$ and $\hat{C}^{(1)} = (\hat{c}_1, \ldots, \hat{c}_{k_2})$.
4: repeat step 2 and 3 until convergence and output the estimators from the last step and denoted as $\hat{R}$ and $\hat{C}$.

### 2.2 Robust Matrix Factor Analysis

Standard statistical procedures that are based on the method of least squares often behave poorly in the presence of heavy-tailed data. The observed data are often heavy-tailed in areas such as finance and macroeconomics. To deal with heavy-tailed data, we are motivated to propose a robust method for inferring matrix factor model. The natural idea is to replace the least squares loss function with the Huber loss function, i.e., we consider the following optimization problem:

$$
\min_{\{R,C,F_t\}} L_2(R, C, F_t) = \frac{1}{T} \sum_{t=1}^T H_\tau \left( \sqrt{|X_t - RF_t^\top C^\top|}_F \right),
$$

s.t. $\frac{1}{p_1} R^\top R = I_{k_1}, \frac{1}{p_2} C^\top C = I_{k_2}$.

where the Huber loss $H_\tau(x)$ is defined as

$$
H_\tau(x) = \begin{cases} 
\frac{1}{2} x^2, & |x| \leq \tau, \\
\tau |x| - \frac{1}{2} \tau^2, & |x| > \tau.
\end{cases}
$$

For some fixed time point $t$, the Huber loss $H_\tau \left( \sqrt{|X_t - RF_t^\top C^\top|}_F \right)$ can be further expressed as

$$
\begin{align*}
&\left\{ \frac{1}{2} \left[ \text{Tr}(X_t^\top X_t) - 2 \text{Tr}(X_t^\top RF_t^\top C^\top) + p_1 p_2 \text{Tr}(F_t^\top F_t) \right] \right., & |X_t - RF_t^\top C^\top|_F^2 \leq \tau^2, \\
&\tau \sqrt{\text{Tr}(X_t^\top X_t) - 2 \text{Tr}(X_t^\top RF_t^\top C^\top) + p_1 p_2 \text{Tr}(F_t^\top F_t)} - \frac{1}{2} \tau^2, & |X_t - RF_t^\top C^\top|_F^2 > \tau^2.
\end{align*}
$$

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Similarly as the case for least squares, for each time point \( t \), take \( \frac{\partial H_r}{\partial F_t} = 0 \), we obtain \( F_t = R^T X_t C / (p_1 p_2) \). Thus by substituting \( F_t = R^T X_t C / (p_1 p_2) \) in \( H_r \), we further have

\[
H_r \left( \sqrt{\|X_t - RF_t C^T\|^2_F} \right) = \begin{cases} 0 & \|X_t - RF_t C^T\|^2_F = \tau^2, \\ \frac{1}{2} \left[ \text{Tr}(X_t^T X_t) - \frac{1}{p_1 p_2} \text{Tr}(X_t^T RR^T X_t CC^T) \right] & \|X_t - RF_t C^T\|^2_F \leq \tau^2, \\ \tau \sqrt{\|X_t^T X_t\| - \frac{1}{p_1 p_2} \text{Tr}(X_t^T RR^T X_t CC^T)} - \frac{1}{2} \tau^2 & \|X_t - RF_t C^T\|^2_F > \tau^2. \end{cases}
\]

For the case that \( \|X_t - RF_t C^T\|^2_F \leq \tau^2 \), we have

\[
\frac{\partial H_r}{\partial R} = -\frac{1}{p_1 p_2} X_t CC^T X_t^T R, \quad \frac{\partial H_r}{\partial C} = \frac{1}{p_1 p_2} X_t^T RR^T X_t C.
\]

For the case that \( \|X_t - RF_t C^T\|^2_F > \tau^2 \), we have

\[
\frac{\partial H_r}{\partial R} = -\frac{\tau}{p_1 p_2} \frac{X_t CC^T X_t^T R}{\sqrt{\text{Tr}(X_t^T X_t) - \frac{1}{p_1 p_2} \text{Tr}(X_t^T RR^T X_t CC^T)}}, \quad \frac{\partial H_r}{\partial C} = -\frac{\tau}{p_1 p_2} \frac{X_t^T RR^T X_t C}{\sqrt{\text{Tr}(X_t^T X_t) - \frac{1}{p_1 p_2} \text{Tr}(X_t^T RR^T X_t CC^T)}}.
\]

Then the Lagrangian function is introduced as follows:

\[
\min_{(R, C)} L_2(R, C) = L_2(R, C) + \text{Tr} \left[ \Theta \left( \frac{1}{p_1} R^T R - I_{k_1} \right) \right] + \text{Tr} \left[ \Lambda \left( \frac{1}{p_2} C^T C - I_{k_2} \right) \right],
\]

where the Lagrangian multiples \( \Theta \) and \( \Lambda \) are symmetric matrices. According to the KKT condition, we have

\[
\frac{\partial L_2}{\partial R} = -AR + \frac{2}{p_1} R \Theta = 0, \quad \frac{\partial L_2}{\partial C} = -BC + \frac{2}{p_2} CA = 0,
\]

where

\[
A = \frac{1}{T} \sum_{t=1}^{T} w_t^i X_t C^T X_t^T, \quad B = \frac{1}{T} \sum_{t=1}^{T} w_t^i X_t^T RR^T X_t.
\]

and the weights \( w_t^i \) are

\[
w_t^i = \begin{cases} \frac{1}{p_1 p_2}, & \sqrt{\|X_t - RF_t C^T\|^2_F} \leq \tau, \\ \frac{\tau}{p_1 p_2} \frac{\sqrt{\text{Tr}(X_t^T X_t) - \frac{1}{p_1 p_2} \text{Tr}(X_t^T RR^T X_t CC^T)}}}{\sqrt{\|X_t - RF_t C^T\|^2_F}}, & \sqrt{\|X_t - RF_t C^T\|^2_F} > \tau. \end{cases}
\]
By reorganizing the notations, we further have

\[ \mathbf{R} \mathbf{\Theta} = \mathbf{M}^w \mathbf{R} \quad \text{and} \quad \mathbf{C} \mathbf{A} = \mathbf{M}^r \mathbf{C}, \]

where

\[ \mathbf{M}^w = \frac{1}{T_p} \sum_{t=1}^{T_p} w_t \mathbf{X}_t \mathbf{C} \mathbf{C}^\top \mathbf{X}_t^\top, \quad \mathbf{M}^r = \frac{1}{T_p} \sum_{t=1}^{T_p} w_t \mathbf{X}_t^\top \mathbf{R} \mathbf{R}^\top \mathbf{X}_t, \]

and the new weights are

\[ w_t = \begin{cases} \frac{1}{2} - \frac{1}{\sqrt{\text{Tr}(\mathbf{X}_t^\top \mathbf{X}_t) - \frac{1}{p_1p_2} \text{Tr}(\mathbf{X}_t^\top \mathbf{R} \mathbf{R}^\top \mathbf{X}_t \mathbf{C} \mathbf{C}^\top)})}, & \frac{||\mathbf{X}_t - \mathbf{RF}_t \mathbf{C}^\top||^2_F}{2} \leq \tau, \\ \frac{\tau}{2} & \frac{||\mathbf{X}_t - \mathbf{RF}_t \mathbf{C}^\top||^2_F}{2} > \tau. \end{cases} \]

By (2.5), we can see that the \( \mathbf{M}^w \) and \( \mathbf{M}^r \) are indeed weighted versions of the \( \mathbf{M}_c \) and \( \mathbf{M}_r \) introduced in the last section. We denote the first \( k_1 \) eigenvectors of \( \mathbf{M}^w_c \) as \( \{\mathbf{r}^w_1, \ldots, \mathbf{r}^w_{k_1}\} \) and the corresponding eigenvalues as \( \{\lambda^w_1, \ldots, \lambda^w_{k_1}\} \). Similarly, we denote the first \( k_2 \) eigenvectors of \( \mathbf{M}^w_r \) as \( \{\mathbf{c}^w_1, \ldots, \mathbf{c}^w_{k_2}\} \) and the corresponding eigenvalues as \( \{\lambda^w_1, \ldots, \lambda^w_{k_2}\} \). From (2.5), we clearly see that \( \mathbf{R}^w = (\mathbf{r}^w_1, \ldots, \mathbf{r}^w_{k_1}), \mathbf{C}^w = (\mathbf{c}^w_1, \ldots, \mathbf{c}^w_{k_2}), \Theta^w = \text{diag}(\theta^w_1, \ldots, \theta^w_{k_1}), \Lambda^w = (\lambda^w_1, \ldots, \lambda^w_{k_2}) \) satisfies the KKT condition. Both \( \mathbf{M}^w_c \) and \( \mathbf{M}^w_r \) rely on the unknown row/column factor loading (see the weights \( w_t \)). We propose an iterative procedure to get the estimators, which turns out to be slightly different from the iterative procedure in the last section, as we need to update \( \mathbf{R}^w \) and \( \mathbf{C}^w \) simultaneously to update the weights \( w_t \), while in Algorithm 1 we can either proceed from \( \mathbf{R} \) or \( \mathbf{C} \). We summarized the algorithm in Algorithm 2 and the initial estimators \( \hat{\mathbf{R}} \) and \( \hat{\mathbf{C}} \) can also be chosen as the estimators by \( \alpha \)-PCA with \( \alpha = 0 \).

**Algorithm 2 Robust Matrix Factor Analysis**

**Input:** Data matrices \( \{\mathbf{X}_t\}_{t \leq T} \), the row factor number \( k_1 \), the column factor number \( k_2 \)

**Output:** Factor loading matrices \( \hat{\mathbf{R}}^w \) and \( \hat{\mathbf{C}}^w \)

1. Obtain the initial estimators \( \hat{\mathbf{R}}^{(0)} \) and \( \hat{\mathbf{C}}^{(0)} \) by \( \alpha \)-PCA with \( \alpha = 0 \);
2. Compute the weights \( \{w_t\}, t = 1, \ldots, T \);
3. Using \( \{w_t\} \) and \( \hat{\mathbf{R}}^{(0)} \) and \( \hat{\mathbf{C}}^{(0)} \) to compute \( \mathbf{M}^w_c \) and its corresponding first \( k_1 \) eigenvectors \( \{\mathbf{r}^w_1, \ldots, \mathbf{r}^w_{k_1}\} \).
   Update \( \hat{\mathbf{R}}^{(1)} \) as \( (\mathbf{r}^w_1, \ldots, \mathbf{r}^w_{k_1}) \).
4. Using \( \{w_t\} \) and \( \hat{\mathbf{R}}^{(0)} \) and \( \hat{\mathbf{C}}^{(0)} \) to compute \( \mathbf{M}^w_r \) and its corresponding first \( k_2 \) eigenvectors \( \{\mathbf{c}^w_1, \ldots, \mathbf{c}^w_{k_2}\} \).
   Update \( \hat{\mathbf{C}}^{(1)} \) as \( (\mathbf{c}^w_1, \ldots, \mathbf{c}^w_{k_2}) \).
5. Repeat steps 2-4 until convergence and output the estimators from the last step and denoted as \( \hat{\mathbf{R}}^w \) and \( \hat{\mathbf{C}}^w \).

Once the factor loading matrices are estimated, the factor matrix \( \mathbf{F}_t \) can be estimated by \( \hat{\mathbf{F}}^w_t = \hat{\mathbf{R}}^w_t \hat{\mathbf{X}}_t \hat{\mathbf{C}}^w_t / (p_1p_2) \) and thus the common component matrix \( \mathbf{S} \) can be estimated by \( \hat{\mathbf{S}}^w = \hat{\mathbf{R}}^w \hat{\mathbf{F}}^w_t \hat{\mathbf{C}}^w_t \).

**2.3 Robust Estimation of the Pair of Factor Numbers**

We first need to determine the pair of factor numbers \( k_1 \) and \( k_2 \) prior to the matrix factor analysis. In this work, we focus on providing robust estimators for the pair of factor numbers \( (k_1, k_2) \), by borrowing the eigenvalue-ratio idea from Lam and Yao (2012) and Ahn and Horenstein (2013). In detail, \( k_1 \) and \( k_2 \) are
For optimization problem (2.2), let

$$\hat{k}_1 = \arg \max_{j \leq k_{\max}} \frac{\lambda_j(M_w^u)}{\lambda_{j+1}(M_w^u)}, \quad \hat{k}_2 = \arg \max_{j \leq k_{\max}} \frac{\lambda_j(M_c^u)}{\lambda_{j+1}(M_c^u)},$$

(2.6)

where $k_{\max}$ is a predetermined value larger than $k_1, k_2$.

Algorithm 3 Robust iterative algorithm to estimate numbers of factors

**Input:** Data $X_t$, maximum number $k_{\max}$, maximum iterative step $m$

**Output:** Numbers of row and column factors $\hat{k}_1$ and $\hat{k}_2$

1: initialization: $\hat{k}_1^{(0)} = k_{\max}, \hat{k}_2^{(0)} = k_{\max}$;
2: With $\hat{k}_1^{(0)} = k_{\max}, \hat{k}_2^{(0)} = k_{\max}$, estimate $R$ and $C$ by $\alpha$-PCA and denote the estimators as $\hat{R}^{(0)}, \hat{C}^{(0)}$;
3: for $t = 1, 2, \ldots, m$, given $\hat{k}_1^{(t-1)}, \hat{k}_2^{(t-1)}$, calculate $M_v^{u(t)}$ using $\hat{R}^{(t-1)}$ and $\hat{C}^{(t-1)}$, then obtain $\hat{k}_1^{(t)}$ by (2.6);
4: given $\hat{k}_1^{(t)}$, update $\hat{R}^{(t)}$ by (2.5), and calculate $M_v^{\pi(t)}$ using $\hat{R}^{(t)}$ and $\hat{C}^{(t-1)}$, then obtain $\hat{k}_2^{(t)}$ by (2.6);
5: given $\hat{k}_2^{(t)}$, update $\hat{C}^{(t)}$ by (2.5);
6: repeat steps 3-5 until $\hat{k}_1^{(t)} = \hat{k}_1^{(t-1)}$ and $\hat{k}_2^{(t)} = \hat{k}_2^{(t-1)}$ or reach the maximum iterative step.

If the common factors are sufficiently strong, the leading $k_1$ eigenvalues of $M_v^{u}$ ($M_v^{w}$) are well separated from the others, and the eigenvalue ratios in equation (2.6) will be asymptotically maximized exactly at $j = k_1$ ($j = k_2$). To avoid vanishing denominators, we can add an asymptotically negligible term to the denominator of equation (2.6). The remaining problem is that to calculate $M_v^{u}$ or $M_v^{w}$, both $R$ and $C$ must be predetermined, which further indicates that $k_1, k_2$ must be given in advance. However, both $k_1$ and $k_2$ are unknown empirically. To circumvent the problem, we propose an iterative algorithm to determine the pair of factor numbers, see Algorithm 3. Thus our method is an Robust iterative Eigenvalue-Ration based procedure, and thus briefly denoted as Rit-ER henceforth.

3 Theoretical results

3.1 The Asymptotic Property from the Least Square point of View

For optimization problem (2.2), let $\theta = (R, F_1, \ldots, F_T, C)$ and $\theta_0 = (R_0, F_0, \ldots, F_0, C_0)$ be the true parameters, where $R = (r_1, \ldots, r_p)^T, C = (c_1, \ldots, c_{p_2})^T$. In this section, we present the asymptotic properties of the theoretical minimizers $\hat{\theta}$, defined as

$$\hat{\theta} = (\hat{R}, \hat{F}_1, \ldots, \hat{F}_T, \hat{C}) = \arg \min_{\theta \in \Theta} \frac{1}{T} \sum_{t=1}^T \|X_t - RF_tC^\top\|_F^2,$$

where

$$\Theta = \left\{ \theta : R \in \mathcal{A} \subset \mathbb{R}^{p_1 \times k_1}, C \in \mathcal{B} \subset \mathbb{R}^{p_2 \times k_2}, F_t \in \mathcal{F} \subset \mathbb{R}^{k_1 \times k_2} \text{ for all } t, \frac{1}{p_1} R^\top R = I_{k_1}, \frac{1}{p_2} C^\top C = I_{k_2} \right\}.$$

To obtain the theoretical properties of $\hat{\theta}$, we assume that the following three assumptions hold.

**Assumption 1:** $\mathcal{A}, \mathcal{B}$ and $\mathcal{F}$ are compact sets and $\theta_0 \in \Theta$. $\|R_0\|_F / \sqrt{p_1}$ and $\|C_0\|_F / \sqrt{p_2}$ are bounded.
The factor matrix satisfies
\[
\frac{1}{T} \sum_{t=1}^{T} F_t F_t^\top \to \Sigma_1 \quad \text{and} \quad \frac{1}{T} \sum_{t=1}^{T} F_t^\top F_t \to \Sigma_2,
\]
where \(\Sigma_i, i=1,2\) is a \(k_i \times k_i\) positive definite matrix with bounded eigenvalues.

**Assumption 2:** Given \(\{F_{0t}, t = 1, \ldots, T\}, \{e_{ij,t}\}\) are independent across \(i,j\) and \(t\).

**Assumption 3:** The idiosyncratic component \(e_{ij,t}\) given \(\{F_{0t}\}\) is sub-Gaussian random variable and \(E(e_{ij,t}|\{F_{0t}\}) = 0\).

Assumptions 1 is standard in large factor models, and we refer, for example, to Chen and Fan (2021) and He et al. (2021). Assumption 2 assumes that the error terms are i.i.d conditional on the matrix factors, but they may not be i.i.d unconditionally. Although this condition seems to be restrictive, it is an exchange for simplicity, see for example Chen et al. (2021). Assumptions 3 assumes that the idiosyncratic errors have sub-Gaussian tails. Assumptions 3 can be further relaxed to finite fourth moment condition when we consider the Huber loss function.

The following theorem illustrates the asymptotic property of \(\hat{R}, \hat{C}\) and \(\hat{F}_t, t = 1, \ldots, T\).

**Theorem 3.1.** Let \(\hat{S}_1 = \text{sgn}\left(\frac{1}{T} \sum_{t=1}^{T} (\hat{F}_t F_{0t}^\top)\right)\) and \(\hat{S}_2 = \text{sgn}\left(\frac{1}{T} \sum_{t=1}^{T} (\hat{F}_t^\top F_{0t})\right)\). Then under Assumptions 1-3,
\[
\frac{1}{\sqrt{p_1}} \|\hat{R} - R_0 \hat{S}_1\|_F = O_p \left(\frac{1}{L}\right), \quad \frac{1}{\sqrt{p_2}} \|\hat{C} - C_0 \hat{S}_2\|_F = O_p \left(\frac{1}{L}\right) \quad \text{and} \quad \frac{1}{T} \|\sum_{t=1}^{T} (\hat{F}_t - \hat{S}_1 F_{0t} \hat{S}_2)\|_F = O_p \left(\frac{1}{L}\right),
\]
where \(L = \min\{\sqrt{p_1}, \sqrt{p_2}, T\}\).

In Theorem 3.1 the sign matrix \(\hat{S}_1, \hat{S}_2\) appears above due to the intrinsic sign indeterminacy of factors and loadings, i.e., the factor structure remains the same when a factor and its loading are both multiplied by -1. The convergence rate derived for the theoretical minimizers is almost of the same order with the \(\alpha\)-PCA method derived in Chen and Fan (2021).

### 3.2 The Asymptotic Property from the Huber loss point of View

In this section, we present the asymptotic properties of the theoretical minimizers \(\hat{\theta}_H\) from Huber loss, defined as
\[
\hat{\theta}_H = (\hat{R}_H, \hat{F}_{1H}, \ldots, \hat{F}_{TH}, \hat{C}_H) = \arg\min_{\theta \in \Theta} \frac{1}{T} \sum_{t=1}^{T} H_t \left(\sqrt{\|X_t - RF_t C^\top\|^2_F}\right),
\]
where
\[
\Theta = \left\{ \theta : R \in A \subset \mathbb{R}^{p_1 \times k_1}, C \in B \subset \mathbb{R}^{p_2 \times k_2}, F_t \in \mathcal{F} \subset \mathbb{R}^{k_1 \times k_2} \text{ for all } t, \frac{1}{p_1} R^\top R = I_{k_1}, \frac{1}{p_2} C^\top C = I_{k_2} \right\}.
\]

To obtain the theoretical properties of \(\hat{\theta}_H\), we further relax the condition of the idiosyncratic errors in Assumption 3.

**Assumption 3’:** The idiosyncratic component \(e_{ij,t}\) given \(\{F_{0t}\}\) has finite fourth moments and \(E(e_{ij,t}|\{F_{0t}\}) = 0\).
The sub-Gaussian tails requirement in Assumption 3, albeit being convenient for theoretical analysis, is not realistic in practical applications. Assumption 3′ only requires the fourth moment of the idiosyncratic component exist, which indicates the robust procedure would perform better when the error are heavy-tailed.

The following theorem illustrates the asymptotic property of $\hat{R}_H, \hat{C}_H$ and $\hat{F}_{tH}, t = 1, \cdots, T$.

**Theorem 3.2.** Let $\hat{S}_1 = \text{sgn}\left(\frac{1}{T} \sum_{t=1}^{T} (\hat{F}_{tH} \hat{F}_{0t})^\top\right)$ and $\hat{S}_2 = \text{sgn}\left(\frac{1}{T} \sum_{t=1}^{T} (\hat{F}_{tH} \hat{F}_{0t})\right)$. Then under Assumptions 1-2 and Assumption 3′,

$$\frac{1}{\sqrt{p_1}}\|\hat{R}_H - R_0\hat{S}_1\|_F = O_p(\frac{1}{L}), \frac{1}{\sqrt{p_2}}\|\hat{C}_H - C_0\hat{S}_2\|_F = O_p(\frac{1}{L}) \text{ and } \frac{1}{T} \sum_{t=1}^{T} (\hat{F}_{tH} - \hat{S}_1 F_{0t} \hat{S}_2)\|_F = O_p(\frac{1}{L}),$$

where $L = \min\{\sqrt{p_1}, \sqrt{p_2}, T\}$.

The convergence rate derived for the theoretical minimizers of the Huber loss is the same with that in Theorem 3.1, though under a more relaxed condition on the idiosyncratic components.

## 4 Simulation Study

### 4.1 Data Generation

In this section, we introduce the data generation mechanism of the synthetic dataset in order to verify the performance of the proposed Robust-Matrix-Factor-Analysis (RMFA) method.

We set $k_1 = k_2 = 3$, draw the entries of $R$ and $C$ independently from uniform distribution $\mathcal{U}(-1,1)$, and let

$$\text{Vec}(F_t) = \phi \times \text{Vec}(F_{t-1}) + \sqrt{1 - \phi^2} \times \epsilon_t, \epsilon_t \sim N(0, I_{k_1 \times k_2}),$$

$$\text{Vec}(E_t) = \psi \times \text{Vec}(E_{t-1}) + \sqrt{1 - \psi^2} \times \text{Vec}(U_t),$$

where $\phi$ and $\psi$ controls the temporal and cross-sectional correlations, and $U_t$ are generated either from the matrix normal distribution or matrix t-distribution respectively. In detail, when $U_t$ is from a matrix normal distribution $MN(O, U_E, V_E)$, then $\text{Vec}(U_t) \sim N(0, V_E \otimes U_E)$. When $U_t$ is from a matrix $t$-distribution $t_{p_1, p_2}(\nu, M, U_E, V_E)$, $U_t$ has probability density function

$$f(U_t; \nu, M, U_E, V_E) = K \times \left| I_{p_1} U_E^{-1}(U_t - M)V_E^{-1}(U_t - M)^\top \right|^{\nu + p_1 + p_2 - 1} 2$$

where $K$ is the regularization parameter. In our simulation study, we set $M = 0$ and let $U_E$ and $V_E$ be matrices with ones on the diagonal, and the off-diagonal entries of which are $1/p_1$ and $1/p_2$ respectively. For matrix $t$-distribution, we resort to the R package “MixMatrix” to generate random samples. The pair of factor numbers is given except in Section 4.4, where we investigate the empirical performance of Algorithm 3 to estimate the numbers of factors. All the simulation results reported hereafter are based on 1000 replications. In our simulation study, the parameter $\tau$ is set as the median of $\{\|X_t - \hat{R}\hat{F}_t\hat{C}_t^\top\|_F, t = 1, \cdots, T\}$, where $\hat{R}$ and $\hat{C}$ are the initial estimators by the $\alpha$-PCA method with $\alpha = 0$. 

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4.2 Verifying the convergence rates for loading spaces

In this section, we will compare the performances of RMFA method with those of the $\alpha$-PCA method by Chen and Fan (2021) and the PE method by Yu et al. (2021) in terms of estimating the factor loadings spaces. We consider the following two settings:

- **Setting A**: $p_1 = 20, T = p_2 \in \{20, 50, 100, 150, 200\}, \phi = \psi = 0.1$.
- **Setting B**: $p_2 = 20, T = p_2 \in \{20, 50, 100, 150, 200\}, \phi = \psi = 0.1$.

We first introduce a metric between two factor spaces as the factor loading matrices $R$ and $C$ are not identifiable. For two column-wise orthogonal matrices $(Q_1)_{p \times q_1}$ and $(Q_2)_{p \times q_2}$, we define

$$D(Q_1, Q_2) = \left(1 - \frac{1}{\max(q_1, q_2)} \text{Tr}(Q_1 Q_1^\top Q_2 Q_2^\top)\right)^{1/2}.$$  

By the definition of $D(Q_1, Q_2)$, we can easily see that its value lies in the interval $[0, 1]$, which measures the distance between the column spaces spanned by $Q_1$ and $Q_2$. The column spaces spanned by $Q_1$ and $Q_2$ are the same when $D(Q_1, Q_2) = 0$, while are orthogonal when $D(Q_1, Q_2) = 1$. The Gram-Schmidt orthonormal transformation can be applied when $Q_1$ and $Q_2$ are not column-orthogonal matrices.

Table 1 shows the averaged estimation errors with standard errors in parentheses under Settings A and B for matrix normal distribution and matrix-variate $t_3$ distribution. Yu et al. (2021)’s simulation study showed that for $\alpha$-PCA method, the performances corresponding to $\alpha \in \{-1, 0, 1\}$ are comparable, thus we only report the simulation results for $\alpha$-PCA with $\alpha = 0$ here. First we focus on the case that the idiosyncratic errors are from matrix-normal distribution. All the methods benefit from large dimensions, and when $p_1$ is small, RMFA and PE methods always show advantage over $\alpha$-PCA in terms of estimating $R$. When $p_2$ is small, RMFA and PE methods always show advantage over $\alpha$-PCA in terms of estimating $C$, which is consistent with the findings by Yu et al. (2021). What we want to emphasize is that the RMFA and PE method performs comparably under this normal idiosyncratic error case, which is also clearly seen from Figure 1 in the Introduction section. This is expected as RMFA (PE) is based on eigen-analysis of weighted (unweighted) sample covariance matrix of the projected data, both share the benefits brought by projection and under the normal idiosyncratic error case, the weights in RMFA tends to be equal and thus RMFA and PE perform almost the same. Then we move to the case when the idiosyncratic errors are from heavy-tailed matrix $t_3$ distribution, where the picture is completely different. Though all the methods benefit from large dimensions, the RMFA method shows great advantage over both the $\alpha$-PCA and PE methods in all scenarios. The estimation errors by PE and $\alpha$-PCA methods are at least two times of those by RMFA method, which indicates that the weights of the sample covariance matrix of the projected data involved in RMFA method play important role when outliers exist. The simulation results for matrix $t_4$ distribution are put in Table D1 in the supplement and similar conclusions can be drawn as for matrix $t_3$ distribution, though the advantage of RMFA over PE and $\alpha$-PCA weakens. In a word, the RMFA performs robustly and much better than PE/$\alpha$-PCA when data are heavy-tailed and performs almost the same (comparably) with the PE method when data are light-tailed, and as a result can be used as a safe replacement of the $\alpha$-PCA and PE methods.
Table 1: Averaged estimation errors and standard errors of $\mathcal{D}(\hat{\mathbf{R}}, \mathbf{R})$ and $\mathcal{D}(\hat{\mathbf{C}}, \mathbf{C})$ for Settings A and B under Matrix Normal distribution and Matrix $t_3$ distribution over 1000 replications. “RMFA”: proposed robust matrix factor analysis method. “$\alpha$-PCA”: $\alpha$-PCA with $\alpha = 0$. “PE”: projection estimation method.

| Evaluation | $T$ | $p_1$ | $p_2$ | RMFA | $\alpha$-PCA | PE |
|------------|-----|-------|-------|------|--------------|----|
| **Matrix Normal Distribution** |     |       |       |      |              |    |
| $\mathcal{D}(\hat{\mathbf{R}}, \mathbf{R})$ | 20  | 20    | 20    | 0.0915(0.0160) | 0.1117(0.0314) | 0.0919(0.0164) |
|           | 50  | 50    | 0.0356(0.0052) | 0.0594(0.0233) | 0.0356(0.0052) |
|           | 100 | 100   | 0.0176(0.0024) | 0.0467(0.0193) | 0.0176(0.0023) |
|           | 150 | 150   | 0.0117(0.0016) | 0.0432(0.0196) | 0.0116(0.0016) |
|           | 200 | 200   | 0.0088(0.0012) | 0.0421(0.0201) | 0.0088(0.0012) |
| $\mathcal{D}(\hat{\mathbf{C}}, \mathbf{C})$ | 20  | 20    | 20    | 0.0930(0.0162) | 0.1140(0.0313) | 0.0935(0.0167) |
|           | 50  | 50    | 0.0567(0.0060) | 0.0590(0.0066) | 0.0569(0.0061) |
|           | 100 | 100   | 0.0398(0.0032) | 0.0404(0.0034) | 0.0399(0.0033) |
|           | 150 | 150   | 0.0323(0.0025) | 0.0325(0.0025) | 0.0324(0.0025) |
|           | 200 | 200   | 0.0261(0.0021) | 0.0282(0.0021) | 0.0282(0.0022) |
| $\mathcal{D}(\hat{\mathbf{R}}, \mathbf{R})$ | 20  | 20    | 20    | 0.0915(0.0160) | 0.1117(0.0314) | 0.0919(0.0164) |
|           | 50  | 50    | 0.0356(0.0053) | 0.0589(0.0225) | 0.0356(0.0053) |
|           | 100 | 100   | 0.0177(0.0025) | 0.0479(0.0216) | 0.0177(0.0025) |
|           | 150 | 150   | 0.0118(0.0017) | 0.0437(0.0246) | 0.0118(0.0017) |
|           | 200 | 200   | 0.0088(0.0012) | 0.0428(0.0204) | 0.0088(0.0012) |
| **Matrix $t_3$ Distribution** |     |       |       |      |              |    |
| $\mathcal{D}(\hat{\mathbf{R}}, \mathbf{R})$ | 20  | 20    | 20    | 0.1454(0.1416) | 0.3752(0.1713) | 0.2830(0.2004) |
|           | 50  | 50    | 0.0433(0.0738) | 0.2959(0.1716) | 0.1437(0.1943) |
|           | 100 | 100   | 0.0171(0.0315) | 0.2657(0.1620) | 0.0830(0.1630) |
|           | 150 | 150   | 0.0111(0.0179) | 0.2488(0.1575) | 0.0684(0.1571) |
|           | 200 | 200   | 0.0088(0.0225) | 0.2382(0.1513) | 0.0581(0.1475) |
| $\mathcal{D}(\hat{\mathbf{C}}, \mathbf{C})$ | 20  | 20    | 20    | 0.1440(0.1407) | 0.3709(0.1733) | 0.2783(0.2008) |
|           | 50  | 50    | 0.0613(0.0732) | 0.2154(0.1853) | 0.1740(0.1941) |
|           | 100 | 100   | 0.0362(0.0347) | 0.1387(0.1651) | 0.1197(0.1720) |
|           | 150 | 150   | 0.0288(0.0166) | 0.1125(0.1588) | 0.0986(0.1621) |
|           | 200 | 200   | 0.0253(0.0241) | 0.0992(0.1535) | 0.0892(0.1581) |
| $\mathcal{D}(\hat{\mathbf{R}}, \mathbf{R})$ | 20  | 20    | 20    | 0.1454(0.1416) | 0.3752(0.1713) | 0.2830(0.2004) |
|           | 50  | 50    | 0.0598(0.0660) | 0.2143(0.1837) | 0.1734(0.1925) |
|           | 100 | 100   | 0.0359(0.0237) | 0.1362(0.1578) | 0.1142(0.1621) |
|           | 150 | 150   | 0.0293(0.0231) | 0.1120(0.1539) | 0.0972(0.1572) |
|           | 200 | 200   | 0.0247(0.0176) | 0.0992(0.1447) | 0.0815(0.1473) |
| $\mathcal{D}(\hat{\mathbf{C}}, \mathbf{C})$ | 20  | 20    | 20    | 0.1440(0.1407) | 0.3709(0.1733) | 0.2783(0.2008) |
|           | 50  | 50    | 0.0418(0.0669) | 0.2992(0.1742) | 0.1442(0.1922) |
|           | 100 | 100   | 0.0168(0.0231) | 0.2644(0.1584) | 0.0802(0.1578) |
|           | 150 | 150   | 0.0115(0.0251) | 0.2572(0.1571) | 0.0652(0.1518) |
|           | 200 | 200   | 0.0082(0.0153) | 0.2459(0.1559) | 0.0521(0.1408) |

### 4.3 Estimation error for common components

In this section, we will compare the performances of RMFA method with those of the $\alpha$-PCA method and the PE method in terms of estimating the common component matrices. We evaluate the performance of
different methods by the mean squared error, i.e.,
\[
\text{MSE} = \frac{1}{T p_1 p_2} \sum_{t=1}^{T} \| \hat{S}_t - S_t \|,
\]
where the \( \hat{S}_t \) refers to an arbitrary estimate and \( S_t \) is the true common component matrix at time point \( t \).

Table 2 shows the averaged MSEs with standard errors in parentheses under Settings A and B for matrix normal distribution and matrix-variate \( t_3, t_4 \) distribution. From Table 2, we can see that the RMFA and PE methods perform comparably under the normal case, and both performs better than the \( \alpha \)-PCA method. This attributes to the projection technique of both the RMFA and PE methods. In contrast, under the heavy-tailed \( t_3, t_4 \) cases, the RMFA performs much better than both PE and \( \alpha \)-PCA methods in terms of estimating the common components. This is consistent with the conclusions drawn for the factor loadings in the last subsection.

| Distribution | T | RMFA | \( \alpha \)-PCA | PE |
|--------------|---|------|----------------|----|
| normal       | Setting A: \( p_1 = 20, T = p_2 \) | | | |
| 20           | 0.0136(0.0021) | 0.0188(0.0042) | 0.0137(0.0022) | |
| 50           | 0.0040(0.0004) | 0.00258(0.0041) | 0.0040(0.0004) | |
| 100          | 0.0018(0.0001) | 0.00032(0.0010) | 0.0018(0.0002) | |
| 150          | 0.0011(0.0001) | 0.00025(0.0009) | 0.0011(0.0001) | |
| 200          | 0.0008(0.0001) | 0.00022(0.0009) | 0.0008(0.0001) | |
| \( t_3 \)    | 20 | 0.0408(0.0743) | 0.1762(0.1242) | 0.1228(0.1354) |
| 50           | 0.0100(0.0499) | 0.1181(0.1371) | 0.0745(0.1403) | |
| 100          | 0.0030(0.0389) | 0.0890(0.1232) | 0.0499(0.1269) | |
| 150          | 0.0013(0.0124) | 0.0766(0.1152) | 0.0420(0.1177) | |
| 200          | 0.0014(0.0178) | 0.0708(0.1127) | 0.0381(0.1136) | |
| \( t_4 \)    | 20 | 0.0102(0.0250) | 0.0519(0.0631) | 0.0270(0.0589) |
| 50           | 0.0019(0.0004) | 0.0219(0.0499) | 0.0101(0.0444) | |
| 100          | 0.0008(0.0001) | 0.0098(0.0264) | 0.0028(0.0229) | |
| 150          | 0.0005(0.0001) | 0.0083(0.0273) | 0.0023(0.0257) | |
| 200          | 0.0004(0.0001) | 0.0070(0.0228) | 0.0018(0.0216) | |
| normal       | Setting B: \( p_2 = 20, T = p_1 \) | | | |
| 20           | 0.0136(0.0021) | 0.0188(0.0042) | 0.0137(0.0022) | |
| 50           | 0.0040(0.0004) | 0.0058(0.0014) | 0.0040(0.0004) | |
| 100          | 0.0018(0.0002) | 0.0033(0.0011) | 0.0018(0.0002) | |
| 150          | 0.0011(0.0001) | 0.0025(0.0013) | 0.0011(0.0001) | |
| 200          | 0.0008(0.0001) | 0.0022(0.0009) | 0.0008(0.0001) | |
| \( t_3 \)    | 20 | 0.0408(0.0743) | 0.1762(0.1242) | 0.1228(0.1354) |
| 50           | 0.0087(0.0418) | 0.1160(0.1291) | 0.0723(0.1341) | |
| 100          | 0.0021(0.0163) | 0.0843(0.1121) | 0.0447(0.1136) | |
| 150          | 0.0017(0.0178) | 0.0776(0.1137) | 0.0392(0.1139) | |
| 200          | 0.0010(0.0120) | 0.0707(0.1129) | 0.0349(0.1149) | |
| \( t_4 \)    | 20 | 0.0102(0.0250) | 0.0519(0.0631) | 0.0270(0.0589) |
| 50           | 0.0020(0.0005) | 0.0187(0.0394) | 0.0077(0.0344) | |
| 100          | 0.0008(0.0001) | 0.0112(0.0282) | 0.0041(0.0300) | |
| 150          | 0.0005(0.0001) | 0.0092(0.0299) | 0.0029(0.0269) | |
| 200          | 0.0004(0.0001) | 0.0072(0.0241) | 0.0017(0.0216) | |
4.4 Estimating the numbers of factors

It’s well known that accurate estimation of the numbers of factors is of great importance to do matrix factor analysis (Yu et al., 2021). In this subsection, we compare the empirical performances of the proposed Rit-ER method with those of the α-PCA based ER method (α-PCA-ER) by Chen and Fan (2021), and the IterER method by Yu et al. (2021) in terms of estimating the numbers of factors.

Table 3: The frequencies of exact estimation and underestimation of the numbers of factors under Settings A and B over 1000 replications. “Rit-ER”: the proposed robust iterative eigenvalue-ration based method. “α-PCA-ER”: α-PCA based eigenvalue-ration method with α = 0. “IterER”: iterative eigenvalue-ration based method.

| Distribution | $T$ | Rit-ER | α-PCA-ER | IterER |
|--------------|-----|--------|----------|--------|
| Setting A: $p_1 = 20, T = p_2$ |   |        |          |        |
| normal       | 20  | 0.992(0.008) | 0.647(0.353) | 0.992(0.008) |
|              | 50  | 1.000(0.000) | 0.891(0.109) | 1.000(0.000) |
|              | 100 | 1.000(0.000) | 0.880(0.120) | 1.000(0.000) |
|              | 150 | 1.000(0.000) | 0.912(0.088) | 1.000(0.000) |
|              | 200 | 1.000(0.000) | 0.901(0.099) | 1.000(0.000) |
| $t_4$        | 20  | 0.774(0.226) | 0.310(0.690) | 0.720(0.280) |
|              | 50  | 0.937(0.063) | 0.622(0.378) | 0.864(0.136) |
|              | 100 | 0.986(0.014) | 0.665(0.335) | 0.933(0.067) |
|              | 150 | 0.992(0.008) | 0.655(0.345) | 0.955(0.045) |
|              | 200 | 0.991(0.009) | 0.651(0.349) | 0.953(0.047) |
| $t_3$        | 20  | 0.360(0.640) | 0.097(0.903) | 0.307(0.693) |
|              | 50  | 0.704(0.296) | 0.249(0.751) | 0.579(0.421) |
|              | 100 | 0.810(0.190) | 0.264(0.736) | 0.643(0.357) |
|              | 150 | 0.855(0.145) | 0.276(0.724) | 0.691(0.309) |
|              | 200 | 0.873(0.127) | 0.260(0.740) | 0.695(0.305) |
| Setting B: $p_2 = 20, T = p_1$ |   |        |          |        |
| normal       | 20  | 0.992(0.008) | 0.647(0.353) | 0.992(0.008) |
|              | 50  | 1.000(0.000) | 0.895(0.105) | 1.000(0.000) |
|              | 100 | 1.000(0.000) | 0.907(0.093) | 1.000(0.000) |
|              | 150 | 1.000(0.000) | 0.909(0.091) | 1.000(0.000) |
|              | 200 | 1.000(0.000) | 0.903(0.097) | 1.000(0.000) |
| $t_4$        | 20  | 0.774(0.226) | 0.310(0.690) | 0.720(0.280) |
|              | 50  | 0.926(0.074) | 0.643(0.357) | 0.871(0.129) |
|              | 100 | 0.950(0.050) | 0.672(0.328) | 0.913(0.087) |
|              | 150 | 0.963(0.037) | 0.652(0.348) | 0.928(0.072) |
|              | 200 | 0.968(0.032) | 0.650(0.350) | 0.937(0.063) |
| $t_3$        | 20  | 0.360(0.640) | 0.097(0.903) | 0.307(0.693) |
|              | 50  | 0.624(0.376) | 0.223(0.777) | 0.551(0.449) |
|              | 100 | 0.716(0.284) | 0.277(0.723) | 0.631(0.369) |
|              | 150 | 0.723(0.277) | 0.270(0.730) | 0.640(0.360) |
|              | 200 | 0.775(0.225) | 0.268(0.732) | 0.703(0.297) |

Table 3 presents the frequencies of exact estimation and underestimation over 1000 replications under Setting A and Setting B by different methods. We set $k_{max} = 8$ for IterER and α-PCA-ER and Rit-ER. Under the normal case, we see that the IterER and Rit-ER performs comparably and both performs better than α-PCA-ER , i.e., have higher estimation accuracy and lower underestimation risk. Even small $T = 50$ is sufficient for perfect estimation by IterER and Rit-ER. However, as the tails of the idiosyncratic errors become heavier (from normal to $t_4, t_3$), it can be seen that all the methods’s performance deteriorates,
especially for \( \alpha \)-PCA-ER method. The proposed Rit-ER method performs robustly and always performs the best for heavy-tailed data, and it can also be seen that as the sample size \( T \) grows, the proportion of exact estimation by Rit-ER has the tendency to converge to 1, which is consistent with our theoretical analysis.

5 Real data example

In this section, we illustrate the empirical performance of our proposed methods by analyzing a financial portfolio dataset, which was also studies by both Wang et al. (2019) and Yu et al. (2021). The financial portfolio dataset is composed of monthly returns of 100 portfolios, well structured into a \( 10 \times 10 \) matrix at each time point, with rows corresponding to 10 levels of market capital size (denoted as S1-S10) and columns corresponding to 10 levels of book-to-equity ratio (denoted as BE1-BE10). The dataset collects monthly returns from January 1964 to December 2019 covering totally 672 months and the details of which are available at the website http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html.

![Histogram of the sample kurtosis for 100 portfolios and the red dashed line is the theoretical kurtosis of \( t_5 \) distribution.](image)

Figure 2: Histogram of the sample kurtosis for 100 portfolios and the red dashed line is the theoretical kurtosis of \( t_5 \) distribution.

Following the same preprocessing strategy by Wang et al. (2019) and Yu et al. (2021), we first subtracted the corresponding monthly market excess returns for the original return series and then standardized each of the series. As for missing values, we take the imputation strategy according to the factor-model-based method by Xiong and Pelger (2019). The result of augmented Dickey-Fuller test indicates the stationarity of the return series. The histogram of the sample kurtosis for 100 portfolios is presented in Figure 2, from which we can deduce that the data are heavy-tailed and the robust methods may be more appropriate. We
first perform the strong rule by He et al. (2021) to see whether there is a genuine matrix factor structure in this dataset. For the parameters involved in the test, we consider \( \alpha = 0.01, M = 100, S \in \{200, 300, 400\}, f_1(S) = 1 - \alpha - \sqrt{2 \ln S / S}, f_2(S) = 1 - \alpha - S^{-1/3}, f_3(S) = 1 - \alpha - S^{-1/4}, f_4(S) = 1 - \alpha - S^{-1/5}, f_5(S) = (1 - \alpha)/2 \). The Results for all test specifications shows that there is overwhelming evidence in favour of a matrix structure in the dataset, which corresponds to not rejecting the null hypotheses that \( k_1, k_2 \geq 1 \). Thus it’s reasonable/appropriate to adopt a matrix factor model for this real dataset.

For the preprocessed monthly returns dataset, IterER method suggests that \((k_1, k_2) = (2, 1)\) while the Rit-ER suggests that \((k_1, k_2) = (1, 2)\). The difference between the estimates by IterER method and Rit-ER method may be attributed to the heavy-tailedness of the returns of 100 portfolios. As overestimation is better than underestimation, and for better illustration, we take \((k_1, k_2) = (2, 2)\).

The estimated front and back loading matrices after varimax rotation and scaling are reported in Table 4.

Table 4: Loading matrices for Fama–French data set, after varimax rotation and scaling by 30. “RMFA” stands for the robust matrix factor analysis method, “PE” stands for the projected estimation method by Yu et al. (2021), \(\alpha\)-PCA represents the method in Chen et al. (2020a) with \(\alpha = 0\) and “ACCE” refers to the approach proposed by Wang et al. (2019).

| Size     | Method | Factor | S1  | S2  | S3  | S4  | S5  | S6  | S7  | S8  | S9  | S10 |
|----------|--------|--------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
|          | RMFA   | 1      | -16 | -15 | -13 | -11 | -8  | -6  | -3  | 0   | 4   | 6   |
|          |        | 2      | -6  | -2  | 2   | 5   | 8   | 10  | 12  | 14  | 15  | 10  |
|          | PE     | 1      | -16 | -15 | -12 | -10 | -8  | -5  | -3  | -1  | 4   | 7   |
|          |        | 2      | -6  | -1  | 3   | 5   | 8   | 11  | 12  | 13  | 15  | 10  |
|          | \(\alpha\)-PCA | 1 | -14 | -14 | -13 | -11 | -9  | -7  | -4  | -2  | 3   | 7   |
|          |        | 2      | -4  | -2  | 1   | 3   | 6   | 9   | 12  | 13  | 16  | 14  |
|          | ACCE   | 1      | -12 | -14 | -12 | -13 | -10 | -6  | -3  | -1  | 4   | 9   |
|          |        | 2      | -1  | -1  | -1  | 2   | 5   | 10  | 11  | 18  | 15  | 11  |

| Book-to-Equity | Method | Factor | BE1 | BE2 | BE3 | BE4 | BE5 | BE6 | BE7 | BE8 | BE9 | BE10 |
|----------------|--------|--------|-----|-----|-----|-----|-----|-----|-----|-----|-----|------|
|                | RMFA   | 1      | 6   | 1   | -3  | -6  | -9  | -11 | -12 | -13 | -12 | -11  |
|                |        | 2      | 19  | 17  | 12  | 9   | 5   | 3   | 0   | -1  | -1  | 0    |
|                | PE     | 1      | 6   | 1   | -4  | -7  | -10 | -11 | -12 | -12 | -10 | -10  |
|                |        | 2      | 20  | 17  | 11  | 8   | 4   | 2   | 0   | -1  | -1  | 0    |
|                | \(\alpha\)-PCA | 1 | 6   | 2   | -4  | -7  | -10 | -11 | -12 | -13 | -12 | -11  |
|                |        | 2      | 19  | 18  | 12  | 8   | 4   | 2   | 0   | -1  | -1  | -1   |
|                | ACCE   | 1      | 6   | -1  | -4  | -8  | -8  | -9  | -10 | -13 | -15 | -12  |
|                |        | 2      | 21  | 15  | 11  | 6   | 5   | 2   | 1   | -2  | -3  | 1    |

From the table, we observe that the PE, \(\alpha\)-PCA and Auto-Cross-Correlation Estimation (ACCE) method by Wang et al. (2019) lead to very similar estimated loadings. From the perspective of size, the small size portfolios load heavily on the first factor while the large size portfolios load mainly on the second factor. From the perspective of book-to-equity, the small BE portfolios load heavily on the second factor while the large BE portfolios load mainly on the first factor.

To further compare these methods, we use a rolling-validation procedure as in Wang et al. (2019). For each year \(t\) from 1996 to 2019, we repeatedly use the \(n\) (bandwidth) years observations before \(t\) to fit the matrix-variate factor model and estimate the two loading matrices. The loadings are then used to estimate
the factors and corresponding residuals of the 12 months in the current year. Specifically, let $Y^i_t$ and $\hat{Y}^i_t$ be the observed and estimated price matrix of month $i$ in year $t$, denote $\bar{Y}_t$ as the mean price matrix, and further define

$$\text{MSE}_t = \frac{1}{12 \times 10 \times 10} \sum_{i=1}^{12} \| \hat{Y}^i_t - Y^i_t \|_F^2,$$

as the mean squared pricing error and unexplained proportion of total variances, respectively. During the rolling-validation procedure, the variation of loading space is measured by $v_t := D(\hat{C}_t \otimes \hat{R}_t, \hat{C}_t \otimes \hat{R}_t - 1)$. Table 5 reports the means of $\text{MSE}$, $\bar{\rho}$ and $v$ by PE, ACCE, $\alpha$-PCA and a conventional PCA estimation applied to the vectorized data. Diverse combinations of bandwidth $n$ and numbers of factors ($k_1 = k_2 = k$) are compared. On the one hand, the pricing errors of PE, $\alpha$-PCA, and the vector model are very close especially for large $n$ and $k$, but lower than the ACCE method. On the other hand, in terms of estimating the loading space, PE always performs much more stably compared with the other two methods. Financial data are usually heavily-tailed with outliers, so the more robust PE method is preferred to control transaction costs and reduce risks.

6 Discussion

The current work study the large-dimensional matrix factor model from the least squares and Huber Loss points of view. Viewed from the least squares, the KKT conditions of minimizing the residual sum of squares under the identifiability condition naturally motivates one to adopt the iterative projection estimation algorithm by Yu et al. (2021). For the Huber loss, the corresponding KKT conditions motivates one to adopt a similar iterative algorithm which is based on the eigen-analysis of weighted sample covariance matrix of the projected data. We investigates the theoretical properties of the robust estimators obtained by minimizing the Huber loss under the identifiability condition. We also propose robust estimators of the pair of factor numbers and prove their consistency under mild assumptions. Numerical studies show the proposed RMFA (Rit-ER) methods perform robustly and much better than the existing PE/\(\alpha\)-PCA (IterER/\(\alpha\)-PCA-ER).
when data are heavy-tailed while perform almost the same (comparably) with the PE (IterER) when data are light-tailed, and as a result can be used as a safe replacement of the existing ones.

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APPENDIX: PROOF OF MAIN RESULTS AND ADDITIONAL SIMULATION RESULTS

A Proof of Theorem 3.1

**Notation.** At the beginning of this section, we define some definitions and notations. Let \( M = p_1k_1 + p_2k_2 + Tk_1k_2 \), the number of parameters to be estimated and

\[
M(\theta) = \frac{1}{Tn_1n_2} \sum_{t=1}^{T} \sum_{i=1}^{p_1} \sum_{j=1}^{p_2} (x_{ij,t} - r_i^T \mathbf{F}_t \mathbf{c}_j)^2 = \frac{1}{Tn_1n_2} \sum_{t=1}^{T} \| \mathbf{X}_t - \mathbf{R} \mathbf{F}_t \mathbf{C}^T \|^2_F.
\]

Thus minimizing \( M(\theta) \) is equivalent to minimizing the least squares loss \( L_1(\mathbf{R}, \mathbf{C}, \mathbf{F}_t) \). And \( \hat{\theta} \) is defined as

\[
\hat{\theta} = (\hat{\mathbf{R}}, \hat{\mathbf{F}}_1, \cdots, \hat{\mathbf{F}}_T, \hat{\mathbf{C}}) = \arg \min_{\theta \in \Theta} M(\theta).
\]

To prove the theorems, we introduce the semimetric \( d \) by

\[
d(\theta_a, \theta_b) = \sqrt{\frac{1}{Tn_1n_2} \sum_{t=1}^{T} \sum_{i=1}^{p_1} \sum_{j=1}^{p_2} (r_{i,t}^a \mathbf{F}_a \mathbf{c}_{aj} - r_{i,t}^b \mathbf{F}_b \mathbf{c}_{bj})^2} = \frac{1}{\sqrt{Tn_1n_2}} \sum_{t=1}^{T} \sum_{i=1}^{p_1} \sum_{j=1}^{p_2} |E[w_{ij,t}(r_i, \mathbf{F}_t, \mathbf{c}_j)]|,
\]

and let \( w_{ij,t}(r_i, \mathbf{F}_t, \mathbf{c}_j) = (x_{ij,t} - r_i^T \mathbf{F}_t \mathbf{c}_j)^2 - (x_{ij,t} - r_{i,t}^0 \mathbf{F}_t \mathbf{c}_j)^2 \),

\[
M^*(\theta) = \frac{1}{Tn_1n_2} \sum_{t=1}^{T} \sum_{i=1}^{p_1} \sum_{j=1}^{p_2} w_{ij,t}(r_i, \mathbf{F}_t, \mathbf{c}_j),
\]

\[
W(\theta) = M^*(\theta) - \tilde{M}^*(\theta) = \frac{1}{Tn_1n_2} \sum_{t=1}^{T} \sum_{i=1}^{p_1} \sum_{j=1}^{p_2} |E[w_{ij,t}(r_i, \mathbf{F}_t, \mathbf{c}_j)]|.
\]

Last, note that \( K_1, K_2, \cdots \) are some positive constants. And for any random variable \( X \), the Orlicz norm is

\[
\|X\|_{\psi} = \inf\{C > 0 : E\psi(|X|/C) \leq 1\}.
\]

Especially, when \( \psi(x) = e^{x^2} - 1 \), the norm is denoted as \( \|X\|_{\psi^2} \). To prove the main theorem, we first give some useful lemmas.

**Lemma A.1.** Under Assumptions 1-3, \( d(\hat{\theta}, \theta_0) = o_p(1) \) as \( p_1, p_2, T \to \infty \).

**Lemma A.2.** Under Assumptions 1-3 and for sufficiently small \( \delta > 0 \), for any \( \theta \in \Theta(\delta) = \{ \theta \in \Theta : d(\theta, \theta_0) \leq \delta \} \), it holds that

\[
\frac{1}{p_1} \| \mathbf{R} - \mathbf{R}_0 \mathbf{S}_1 \|^2_F + \frac{1}{p_2} \| \mathbf{C} - \mathbf{C}_0 \mathbf{S}_2 \|^2_F + \frac{1}{T^2} \sum_{t=1}^{T} \| \mathbf{F}_t - \mathbf{S}_t \mathbf{F}_0 \mathbf{S}_2 \|^2_F \leq K_0 \delta,
\]

where \( \mathbf{S}_1 = sgn(\frac{1}{T} \sum_{t=1}^{T} (\mathbf{F}_t^T \mathbf{F}_0 \mathbf{S}_2)) \) and \( \mathbf{S}_2 = sgn(\frac{1}{T} \sum_{t=1}^{T} (\mathbf{F}_t \mathbf{F}_0^T \mathbf{S}_2)) \).
Lemma A.3. Under Assumptions 1-3, we have that
\[ E[\sup_{\theta \in \Theta} |\mathbb{W}(\theta)|] \lesssim \frac{\delta}{L}, \]

where \( L = \min\{\sqrt{p_1}, \sqrt{p_2}, T\}. \)

The proof of Theorem 3.1: First, we divide the parameter space \( \Theta \) into \( S_j = \{ \theta \in \Theta : 2^{j-1} < L \cdot d(\theta, \theta_0) \leq 2^j \} \). If \( L \cdot d(\hat{\theta}, \theta_0) > 2^V \) for a certain \( V \), then \( \hat{\theta} \) is in one of the shells \( S_j, j > V \), where the infimum of \( \mathbb{M}^*(\theta) = \mathbb{M}(\theta) - \mathbb{M}(\theta_0) \) is nonpositive over this shell by the definition of \( \hat{\theta} \). Therefore, for every \( \eta > 0 \), we have
\[ P[L \cdot d(\hat{\theta}, \theta_0) > 2^V] \leq \sum_{j > V, 2^{j-1} \leq \eta L} P[\inf_{\theta \in S_j} \mathbb{M}^*(\theta) \leq 0] + P[d(\hat{\theta}, \theta_0) > \eta]. \]

According to Lemma A.1, for arbitrarily small \( \eta > 0 \), \( P[d(\hat{\theta}, \theta_0) > \eta] \) converges to 0 as \( T, p_1, p_2 \to \infty \).

By (B.1), for each \( \theta \) in \( S_j \) it holds that
\[ -\mathbb{M}^*(\theta) = -d^2(\theta, \theta_0) \leq -\frac{2^{2j-2}}{L^2}. \]

Then from \( \inf_{\theta \in S_j} \mathbb{M}^*(\theta) \leq 0 \), we have
\[ \inf_{\theta \in S_j} \mathbb{W}(\theta) \leq -\frac{2^{2j-2}}{L^2}. \]

So
\[ \sum_{j > V, 2^{j-1} \leq \eta L} P[\inf_{\theta \in S_j} \mathbb{M}^*(\theta) \leq 0] \leq \sum_{j > V, 2^{j-1} \leq \eta L} P[\sup_{\theta \in \Theta} |\mathbb{W}(\theta)| \geq \frac{2^{2j-2}}{L^2}] \]

By Lemma A.3 and Markov's inequality,
\[ P[\sup_{\theta \in \Theta} |\mathbb{W}(\theta)| \geq \frac{2^{2j-2}}{L^2}] \lesssim \frac{L^2}{2^{2j}} \cdot E[\sup_{\theta \in \Theta} |\mathbb{W}(\theta)|] \lesssim \frac{L^2}{2^{2j}} \cdot \frac{2^j}{L^2} = 2^{-j}, \]

then
\[ \sum_{j > V, 2^{j-1} \leq \eta L} P[\inf_{\theta \in S_j} \mathbb{M}^*(\theta) \leq 0] \lesssim \sum_{j > V} 2^{-j}. \]

As \( V \to \infty \), \( \sum_{j > V, 2^{j-1} \leq \eta L} P[\inf_{\theta \in S_j} \mathbb{M}^*(\theta) \leq 0] \) converges to 0, implying \( L \cdot d(\hat{\theta}, \theta_0) = O_p(1) \), e.g. \( d(\hat{\theta}, \theta_0) = O_p(1/L) \). By Lemma A.2, we have thus proved the theorem. \( \Box \)

B  The proof of Lemmas

The proof of Lemma A.1. First, for any \( R \in \mathcal{A}, C \in \mathcal{B}, F_t \in \mathcal{F} \), we have
\[ E[(x_{ij,t} - r'_t \epsilon_j)^2 - (x_{ij,t} - r_{0t}' F_0 \epsilon_j)^2] = (r'_t \epsilon_j - r_{0t}' F_0 \epsilon_j)^2. \]

Then for \( \forall \theta \in \Theta \), it holds that
\[ \overline{M}^*(\theta) = d^2(\theta, \theta_0). \] (B.1)
Second, according to the definition of $\hat{\theta}$, $M^*(\hat{\theta}) = M(\hat{\theta}) - M(\theta_0) \leq 0$, namely $\mathbb{W}(\hat{\theta}) + M^*(\hat{\theta}) = M^*(\hat{\theta}) \leq 0$. From (B.1), we have $0 \leq d^2(\hat{\theta}, \theta_0) = M^*(\hat{\theta}) \leq \sup_{\theta \in \Theta} |\mathbb{W}(\theta)|$.

Now, it is sufficient to prove that

$$\sup_{\theta \in \Theta} |\mathbb{W}(\theta)| = o_p(1).$$

Choose $K_1$ large enough such that $\|r_{i0}\|_2, \|c_{0j}\|_2, \|F_{0i}\|_F, \|r_i\|_2, \|r_{i\theta}\|_2, \|F_i\|_F \leq K_1$ for all $i, j, t$. Let $B_k(K_1)$ denote a Euclidean ball in $\mathbb{R}^d$ with radius $K_1$ ($i = k_1, k_2$), $B_{k_1 \times k_2}(K_1)$ denote a Euclidean ball in $\mathbb{R}^{k_1 \times k_2}$ with radius $K_1$. For any $\epsilon > 0$, let $r_{i(1)}, \ldots, r_{i(J)}$ be the maximal set of points in $B_k(K_1)$ such that $\|r_{i(i)} - r_{i(h)}\|_2 > \epsilon/K_1$ for all $i, h$. Let $\phi_i \subseteq \{c_{(i)} : k \leq J_1, \|r_{i(k)} - r_i\|_2 \leq \epsilon/K_1\}, c_{(i)}^j = \{c_{(j)} : k \leq J_2, \|c_{(j)} - c_{i}\|_2 \leq \epsilon/K_1\}$, $F_i = \{F_{(i)} : k \leq J_3, \|F_{(i)} - F_i\|_F \leq \epsilon/K_1\}$. Also $\mathbb{W}(\theta) = \mathbb{W}(\star) + \mathbb{W}(\theta) - \mathbb{W}(\star)$. It is easy to get that

$$0 \leq \sup_{\theta \in \Theta} |\mathbb{W}(\theta) - \mathbb{W}(\star)| \leq 3K_1 \epsilon.$$
Finally, by Markov’s inequality and (B.3), for any $\delta > 0$,

$$P[\sup_{\theta \in \Theta} |W(\theta)| > \delta] \leq P[\sup_{\theta \in \Theta} |W(\theta^\ast)| > \delta/2] + P[\sup_{\theta \in \Theta} |W(\theta) - W(\theta^\ast)| > \delta/2] \leq 2/\delta \cdot E[\sup_{\theta \in \Theta} |W(\theta^\ast)|] + P[3K_1K_2\epsilon > \delta/2].$$

Therefore, (B.2) is proved since $\epsilon$ is arbitrary. The lemma follows. □

**The proof of Lemma A.2.** First, let $U_1 \in \mathbb{R}^{k_1 \times k_1}, U_2 \in \mathbb{R}^{k_2 \times k_2}$ be a diagonal matrix whose diagonal elements are either 1 or $-1$, respectively. By assumptions, $\|F_0\|_F^2 \leq K_0$ for every $t$. It is easy to see that

$$\left\|\sum_{t=1}^{T} F_t - U_1(\sum_{t=1}^{T} F_0)U_2\right\|_F^2 \leq T\sum_{t=1}^{T} \|F_t - U_1F_0U_2\|_F^2,$$

and

$$\|F_t - U_1F_0U_2\|_F^2 \leq \frac{1}{p_1p_2} \|R(F_t - U_1F_0U_2)C^T\|_F^2.$$

Thus, we have

$$\left\|\sum_{t=1}^{T} F_t - U_1(\sum_{t=1}^{T} F_0)U_2\right\|_F^2 \leq \frac{T}{p_1p_2} \sum_{t=1}^{T} \|F_t - U_1F_0U_2\|_F^2,$$

where

$$\sum_{t=1}^{T} \|R_0F_0C_0^T - RU_1F_0U_2C^T\|_F^2 = \sum_{t=1}^{T} \|R_0F_0C_0^T - RU_1F_0U_2C^T\|_F^2.$$

Then it follows that

$$\frac{1}{T^2} \left\|\sum_{t=1}^{T} (F_t - U_1F_0U_2)\right\|_F^2 \leq 2d^2(\theta, \theta_0) + 2K_0 \frac{1}{p_1} \|R - R_0U_1\|_F^2 + 2K_0 \frac{1}{p_2} \|C - C_0U_2\|_F^2. \quad (B.5)$$

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Thus, it can be concluded from (B.5) that, for $\theta \in \Theta(\delta)$, we have

$$\frac{1}{p_1} \| R - R_0 U_1 \|_F^2 + \frac{1}{p_2} \| C - C_0 U_2 \|_F^2 + \frac{1}{T^2} \sum_{t=1}^T \| F_t - U_t F_0 U_2 \|_F^2 \leq 2d^2(\theta, \theta_0) + (2K_0 + 1) \frac{1}{p_1} \| R - R_0 U_1 \|_F^2 + (2K_0 + 1) \frac{1}{p_2} \| C - C_0 U_2 \|_F^2 \leq 2\delta^2 + (2K_0 + 1)(\frac{1}{p_1} \| R - R_0 U_1 \|_F^2 + \frac{1}{p_2} \| C - C_0 U_2 \|_F^2)$$

(B.6)

Then let us consider $\frac{1}{p_1} \| R - R_0 U_1 \|_F^2$ and $\frac{1}{p_2} \| C - C_0 U_2 \|_F^2$, seperately.

Second, note that

$$\frac{1}{p_1} \| R - R_0 U_1 \|_F^2 \leq 2 \frac{p_2}{p_1} \| R_0 U_1 - \frac{1}{p_1} R R' R_0 U_1 \|_F^2 + 2 \frac{p_1}{p_1} \| RR' R_0 U_1 - R \|_F^2.$$

Further, we have

$$\| R_0 U_1 - \frac{1}{p_1} RR' R_0 U_1 \|_F^2 = \| R_0 - \frac{1}{p_1} R R' R_0 \|_F^2 = \| (I_k - P_R) R_0 \|_F^2 = \| M_R R_0 \|_F^2,$$

and

$$\| \frac{1}{p_1} RR' R_0 U_1 - R \|_F^2 = \| R ( \frac{1}{p_1} R' R_0 U_1 - I_k) \|_F^2 = p_1 \| R' R_0 - U_1 \|_F^2,$$

where $P_A = A(A'A)^{-1}A'$ and $M_A = I - P_A$. Therefore, it can easily be shown that

$$\frac{1}{p_1} \| R - R_0 U_1 \|_F^2 \leq 2 \frac{p_2}{p_1} \| M R R_0 \|_F^2 + 2 \frac{p_1}{p_1} \| R' R_0 - U_1 \|_F^2.$$

(B.7)

Third, it follows that

$$\frac{1}{T p_1 p_2} \sum_{t=1}^T \| M_R(R F_t C_0^T - R_0 F_{ot} C_0^T) \|_F^2 \leq \frac{1}{T p_1 p_2} \sum_{t=1}^T \| \text{rank}[ M_R(R F_t C_0^T - R_0 F_{ot} C_0^T) ] \|_F^2 \| R F_t C_0^T - R_0 F_{ot} C_0^T \|_2^2 \leq \frac{1}{T p_1 p_2} \sum_{t=1}^T \| R F_t C_0^T - R_0 F_{ot} C_0^T \|_F^2 = d^2(\theta, \theta_0).$$

(B.8)

And

$$\frac{1}{T p_1 p_2} \sum_{t=1}^T \| M_R(R F_t C_0^T - R_0 F_{ot} C_0^T) \|_F^2 = \frac{1}{T p_1 p_2} \sum_{t=1}^T \| M_R R_0 F_{ot} C_0^T \|_F^2 = \frac{1}{T p_1} \sum_{t=1}^T tr(F_0 F_{ot}^T R_0) M_R R_0 \geq \frac{1}{T p_1} \lambda_{\min}(F_0 F_{ot}^T) tr(R_0^T M_R R_0) \geq \frac{K_1}{p_1} \| M_R R_0 \|_F^2.$$

(B.9)
Thus from (B.8) and (B.9), we conclude

$$
\frac{1}{p_1} \| M_R R_0 \|^2_F \lesssim \frac{1}{K_9} d^2(\theta, \theta_0).
$$

(B.10)

Similarly, it can be shown that

$$
\frac{1}{p_1} \| M_{R_0} R \|^2_F \lesssim \frac{1}{\sum_{t=1}^T \lambda_{\min}(F_t F_t^T)} d^2(\theta, \theta_0).
$$

(B.11)

Fourth, we have

$$
\frac{1}{T p_1 p_2} \sum_{t=1}^T \| P_R (R F_t C_t^T - R_0 F_{0t} C_{0t}^T) \|^2_F \leq \frac{1}{T p_1 p_2} \sum_{t=1}^T \| P_R \|^2_F \| R F_t C_t^T - R_0 F_{0t} C_{0t}^T \|^2_F
$$

$$
= \frac{k_1}{T p_1 p_2} \sum_{t=1}^T \| R F_t C_t^T - R_0 F_{0t} C_{0t}^T \|^2_F \lesssim d^2(\theta, \theta_0).
$$

(B.12)

Thus

$$
\frac{1}{T p_1 p_2} \sum_{t=1}^T \| P_R (R F_t C_t^T - R_0 F_{0t} C_{0t}^T) \|^2_F = \frac{1}{T p_1 p_2} \sum_{t=1}^T \| R F_t C_t^T - P_R R_0 F_{0t} C_{0t}^T \|^2_F
$$

$$
= \frac{1}{T p_1 p_2} \sum_{t=1}^T \| R F_t C_t^T - \frac{1}{p_1} R R^T R_0 F_{0t} C_{0t}^T \|^2_F
$$

$$
\leq \frac{1}{T p_2} \sum_{t=1}^T \| F_t C_t^T - \frac{1}{p_1} R^T R_0 F_{0t} C_{0t}^T \|^2_F
$$

$$
\lesssim d^2(\theta, \theta_0).
$$

(B.13)

Similarly,

$$
\frac{1}{T p_2} \sum_{t=1}^T \| F_{0t} C_{0t}^T - \frac{1}{p_1} R_0^T R F_t C_t^T \|^2_F \lesssim d^2(\theta, \theta_0).
$$

(B.14)

Fifth, let us define $A = \frac{1}{p_1} R^T R_0$, then $R A = \frac{1}{p_1} R R^T R_0 = P_R R_0$. Clearly,

$$
I_{k_1} = \frac{1}{p_1} R_0^T R_0 = A^T (\frac{1}{p_1} R_0^T R_0) A + \frac{1}{p_1} R_0^T R_0 - A^T (\frac{1}{p_1} R_0^T R_0) A
$$

$$
= A^T A + \frac{1}{p_1} R_0^T R_0 - \frac{1}{p_1} R_0^T R_0 A - \frac{1}{p_1} R_0^T R_0 A - A^T (\frac{1}{p_1} R_0^T R_0) A
$$

$$
= A^T A + \frac{1}{p_1} R_0^T (R_0 - RA)
$$

$$
= A^T A + \frac{1}{p_1} M_R R_0.
$$

(B.15)

where

$$
\frac{1}{p_1} R_0^T R_0 A - A^T (\frac{1}{p_1} R_0^T R_0) A = \frac{1}{p_1} (R_0 - RA)^T R A = 0.
$$
By the same way as (B.15),

\[
I_{k_1} = AA^T + R^T \frac{1}{p_1}(R - R_0A^T) = AA^T + \frac{1}{p_1}R^TM_{R_0}R
\]  \hspace{1cm} (B.16)

Moreover, we have

\[
\frac{1}{T_{p_2}} \sum_{i=1}^{T} F_{oi}C_0^T(F_{oi}C_0^T)^T = A^T(\frac{1}{T_{p_2}} \sum_{i=1}^{T} F_iC^T(F_iC^T)^T)A + \frac{1}{T_{p_2}} \sum_{i=1}^{T} (F_{oi}C_0^T(F_{oi}C_0^T)^T - A^TF_iC^T(F_iC^T)^T)A
\]

\[
= A^T(\frac{1}{T_{p_2}} \sum_{i=1}^{T} F_iC^T(F_iC^T)^T)A + \frac{1}{T_{p_2}} \sum_{i=1}^{T} (F_{oi}C_0^T - A^TF_iC^T)(F_{oi}C_0^T)^T + \frac{1}{T_{p_2}} \sum_{i=1}^{T} A^TF_iC^T(F_{oi}C_0^T - A^TF_iC^T)^T.
\]

Furthermore,

\[
\frac{1}{T_{p_2}} \sum_{i=1}^{T} F_{oi}C_0^T(F_{oi}C_0^T)^T = A^T(\frac{1}{T_{p_2}} \sum_{i=1}^{T} F_iC^T(F_iC^T)^T)(A^T)^{-1}A^T + \frac{1}{T_{p_2}} \sum_{i=1}^{T} (F_{oi}C_0^T - A^TF_iC^T)(F_{oi}C_0^T)^T
\]

\[
+ \frac{1}{T_{p_2}} \sum_{i=1}^{T} A^TF_iC^T(F_{oi}C_0^T - A^TF_iC^T)^T
\]

\[
= A^T(\frac{1}{T_{p_2}} \sum_{i=1}^{T} F_iC^T(F_iC^T)^T)(A^T)^{-1} + A^T(\frac{1}{T_{p_2}} \sum_{i=1}^{T} F_iC^T(F_iC^T)^T)(A^T)^{-1}(A^TA - I_{k_1})
\]

\[
+ \frac{1}{T_{p_2}} \sum_{i=1}^{T} (F_{oi}C_0^T - A^TF_iC^T)(F_{oi}C_0^T)^T + \frac{1}{T_{p_2}} \sum_{i=1}^{T} A^TF_iC^T(F_{oi}C_0^T - A^TF_iC^T)^T.
\]  \hspace{1cm} (B.17)

In addition,

\[
(\frac{1}{T_{p_2}} \sum_{i=1}^{T} F_{oi}C_0^T(F_{oi}C_0^T)^T + D_1)A^T = A^T(\frac{1}{T_{p_2}} \sum_{i=1}^{T} F_iC^T(F_iC^T)^T),
\]

where

\[
D_1 = A^T(\frac{1}{T_{p_2}} \sum_{i=1}^{T} F_iC^T(F_iC^T)^T)(A^T)^{-1}(I_{k_1} - A^TA) - \frac{1}{T_{p_2}} \sum_{i=1}^{T} (F_{oi}C_0^T - A^TF_iC^T)(F_{oi}C_0^T)^T
\]

\[
- \frac{1}{T_{p_2}} \sum_{i=1}^{T} A^TF_iC^T(F_{oi}C_0^T - A^TF_iC^T)^T
\]

\[
= A^T(\frac{1}{T_{p_2}} \sum_{i=1}^{T} F_iC^T(F_iC^T)^T)(A^T)^{-1}(\frac{1}{p_1}R_0^TM_{R_0}R_0) - \frac{1}{T_{p_2}} \sum_{i=1}^{T} (F_{oi}C_0^T - A^TF_iC^T)(F_{oi}C_0^T)^T
\]

\[
- \frac{1}{T_{p_2}} \sum_{i=1}^{T} A^TF_iC^T(F_{oi}C_0^T - A^TF_iC^T)^T.
\]
From (B.10), (B.13) and (B.14), it can be proved that

\[
\|D_1\|_F^2 \leq 3\|A^T (\frac{1}{Tp_2} \sum_{t=1}^{T} F_t C^T (F_t C^T)^T) (A^T)^{-1} (\frac{1}{p_1} R_0^T M R_0)\|_F^2 + 3\|\frac{1}{Tp_2} \sum_{t=1}^{T} (F_{0t} C_0^T - A^T F_t C^T) (F_{0t} C_0^T)^T\|_F^2
\]

\[
+ 3\|\frac{1}{Tp_2} \sum_{t=1}^{T} A^T F_t C^T (F_{0t} C_0^T - A^T F_t C^T)^T\|_F^2 \lesssim d^2(\theta, \theta_0).
\]

According to the Bauer-Fike theorem (Golub and Van Loan (2013)),

\[
|\lambda_{\min}(\frac{1}{Tp_2} \sum_{t=1}^{T} F_t C^T (F_t C^T)^T) - \lambda_{\min}(\frac{1}{Tp_2} \sum_{t=1}^{T} F_{0t} C_0^T (F_{0t} C_0^T)^T)|
\]

\[
= |\lambda_{\min}(\frac{1}{T} \sum_{t=1}^{T} F_t F_t^T) - \lambda_{\min}(\frac{1}{T} \sum_{t=1}^{T} F_{0t} F_{0t}^T)|
\]

\[
\leq \|D_1\|_F^2 \leq \|D_1\|_F^2 \lesssim d^2(\theta, \theta_0).
\]

(B.19)

Meanwhile, according to assumption and the perturbation theory for eigenvectors,

\[
\|A^T V_R S_1 - I_{k_i}\|_F^2 = \|A^T V_R - S_1\|_F^2 \lesssim d^2(\theta, \theta_0).
\]

(B.20)

where \(V_R = \text{diag}((A_1 A_1')^{-1/2}, \ldots, (A_{k_i} A_{k_i}')^{-1/2})\), \(A_j'\) is the \(j\)th column of \(A^T\).

From (B.19), we have \(\lambda_{\min}(\sum_{t=1}^{T} F_t F_t^T / T)\) is bounded by a positive constant. Therefore \(d^2(\theta, \theta_0) / \sum_{t=1}^{T} \lambda_{\min}(F_t F_t^T)\) is bounded by Weyl’s theorem. Then

\[
\frac{1}{p_1} \|M_{R_0} R\|_F^2 \lesssim d^2(\theta, \theta_0)
\]

(B.21)

from (B.11). According to the triangular inequality,

\[
\|A^T - S_1\|_F^2 \leq 2\|A^T V_R - S_1\|_F^2 + 2\|A^T V_R - A^T\|_F^2 \leq 2\|A^T V_R - S_1\|_F^2 + 2\|A\|_F^2 \|V_R - I_{k_i}\|_F^2.
\]

(B.22)

From (B.15) and (B.21), it holds that

\[
\|V - I_{k_i}\|_F^2 \lesssim \|A A^T - I_{k_i}\|_F^2 = \|\frac{1}{p_1} R^T M R_0 \|_F^2 \leq \frac{1}{p_1} \|R\|_F^2 \|M_{R_0} R\|_F^2 = \frac{1}{p_1} \|M_{R_0} R\|_F^2 \lesssim d^2(\theta, \theta_0).
\]

(B.23)

For small enough \(d^2(\theta, \theta_0)\), it can be easily seen from (B.20), (B.22) and (B.23) that

\[
\frac{1}{p_1} \|R^T R_0 - S_1\|_F^2 = \|A - S_1\|_F^2 \lesssim d^2(\theta, \theta_0).
\]

(B.24)

By setting \(U_1 = S_1\), from (B.7), (B.10) and (B.24), it follows that for sufficiently small \(d^2(\theta, \theta_0)\),

\[
\frac{1}{p_1} \|R - R_0 U_1\|_F^2 \lesssim d^2(\theta, \theta_0).
\]

(B.25)
In a similar way,
\[
\frac{1}{p^2} \|C - C_0 U_2\|_F^2 \lesssim d^2(\theta, \theta_b). \tag{B.26}
\]

Thus we have derived the conclusion from (B.6), (B.25) and (B.26).

\(\square\)

**The proof of Lemma A.3.** According to Lemma A.1, it is easy to see that for any \(\theta_a, \theta_b \in \Theta\),
\[
\|\sqrt{T} p_1 p_2 |W(\theta_a) - W(\theta_b)|\|_{\psi_2} \lesssim \frac{1}{\sqrt{T}} d(\theta_a, \theta_b),
\]
equivalently,
\[
\|\sqrt{T^2 p_1 p_2} W(\theta_a) - W(\theta_b)\|_{\psi_2} \lesssim d(\theta_a, \theta_b). \tag{B.27}
\]

Since the process \(W(\theta)\) is separable, we have that
\[
E[\sup_{\Theta} \sqrt{T^2 p_1 p_2} |W(\theta)|] \lesssim \sup_{\Theta} \sqrt{T^2 p_1 p_2} |W(\theta)|\|_{\psi_2} \lesssim \int_0^\delta \sqrt{\log D(\epsilon, d, \Theta(\delta))} d\epsilon
\]
from the Theorem 2.2.4 of Van der vaart and Wellner (1996).

From Lemma A.2, it is clear that
\[
\Theta(\delta) \subset \bigcup_{U_1 \in S_1, U_2 \in S_2} \Theta(\delta; U_1, U_2),
\]
where \(S_1 = \{U_1 \in \mathbb{R}^{k_1 \times k_1} : U_1 = \text{diag}(u_1, \cdots, u_{k_1}), u_j \in \{-1, 1\} \text{ for } j = 1, \cdots, k_1\}, S_2 = \{U_2 \in \mathbb{R}^{k_2 \times k_2} : U_1 = \text{diag}(u_1, \cdots, u_{k_2}), u_j \in \{-1, 1\} \text{ for } j = 1, \cdots, k_2\} \) and \(\Theta(\delta, U_1, U_2) = \{\theta \in \Theta : \frac{1}{p_1} \|R - R_0 U_1\|_F^2 + \frac{1}{p_2} \|C - C_0 U_2\|_F^2 + \frac{1}{T^2} \|\sum_{t=1}^T (F_t - U_1 F_0 U_2)\|_F^2 \leq K_0 \delta\} \). Because there are \(2^{k_1}\) elements in \(S_1\) and there are \(2^{k_2}\) elements in \(S_2\), we need only study the property of \(\int_0^\delta \sqrt{\log D(\epsilon, d, \Theta(\delta); U_1, U_2)} d\epsilon\) for each \(U_1 \in S_1\) and \(U_2 \in S_2\). Without loss of generality, we set \(U_1 = I_{k_1}\) and \(U_2 = I_{k_2}\).

For any \(\theta_a, \theta_b \in \Theta\), we have
\[
d(\theta_a, \theta_b) = \frac{1}{\sqrt{T} p_1 p_2} \left\{ \sum_{t=1}^T \|R_a F_{at} C_a^T - R_b F_{bt} C_b^T\|_F^2 \right\}^{1/2}
\]
\[
= \frac{1}{\sqrt{T} p_1 p_2} \left\{ \sum_{t=1}^T \|R_a F_{at} C_a^T - R_b F_{at} C_a^T + R_b F_{at} C_a^T - R_b F_{bt} C_b^T\|_F^2 \right\}^{1/2}
\]
\[
\leq \frac{1}{\sqrt{T} p_1 p_2} \left\{ \sum_{t=1}^T \|2(R_a - R_b) F_{at} C_a^T\|_F^2 + 2 \|R_b (F_{at} C_a^T - F_{bt} C_b^T)\|_F^2 \right\}^{1/2}
\]
\[
\leq \frac{1}{\sqrt{T} p_1 p_2} \left\{ 2 \sum_{t=1}^T \|R_a - R_b\|_F^2 C_a^T\|_F^2 + \frac{1}{\sqrt{T} p_1 p_2} \left( \sum_{t=1}^T \|R_b (F_{at} C_a^T - F_{bt} C_b^T)\|_F^2 \right) \right\}^{1/2}.
\]
For the first item,\
\[ \frac{1}{\sqrt{T_p_1 p_2}} \sqrt{2 \sum_{t=1}^{T} \| (R_a - R_b) F_{a t} C_a^\top - F_{b t} C_b^\top \|_F^2} \leq \sqrt{2} \left( \frac{T_p_1 p_2}{1} \right)^{-\frac{1}{2}} \frac{\| R_a - R_b \|_F^2}{\sqrt{T_p_1 p_2}} \sum_{t=1}^{T} \| F_{a t} \|_F^2 \| C_a \|_F^2 \frac{1}{T_p_1 p_2} \sqrt{\| R_a - R_b \|_F} \leq K \frac{1}{\sqrt{p_1}} \| R_a - R_b \|_F, \]

and for the second item,\
\[ \frac{1}{\sqrt{T_p_1 p_2}} \sqrt{2 \sum_{t=1}^{T} \| R_b (F_{a t} C_a^\top - F_{b t} C_b^\top) \|_F^2} \leq \sqrt{2} \frac{T_p_1 p_2}{T_p_1 p_2} \sum_{t=1}^{T} \| F_{a t} C_a^\top - F_{b t} C_b^\top \|_F^2 \]
\[ \leq \sqrt{2} \frac{T_p_1 p_2}{T_p_1 p_2} \sum_{t=1}^{T} \| F_{a t} C_a^\top - F_{a t} C_b^\top + F_{a t} C_b^\top - F_{b t} C_b^\top \|_F^2 \]
\[ \leq \sqrt{2} \frac{T_p_1 p_2}{T_p_1 p_2} \sum_{t=1}^{T} \| F_{a t} C_a^\top - F_{a t} C_b^\top \|_F^2 \| F_{a t} C_b^\top - F_{b t} C_b^\top \|_F^2 \]
\[ \leq \sqrt{2} \frac{\sum_{t=1}^{T} \| F_{a t} \|_F^2}{T} \| C_a - C_b \|_F + 2 \frac{1}{\sqrt{p_2}} \sum_{t=1}^{T} \| F_{a t} - F_{b t} \|_F^2 \]
\[ \leq K \frac{1}{\sqrt{p_2}} \| C_a - C_b \|_F + K \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \| F_{a t} - F_{b t} \|_F^2. \]

Therefore,
\[ d(\theta_a, \theta_b) \leq K \left( \frac{1}{\sqrt{p_1}} \| R_a - R_b \|_F + \frac{1}{\sqrt{p_2}} \| C_a - C_b \|_F + \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \| F_{a t} - F_{b t} \|_F^2 \right). \]

Then we define
\[ d^*(\theta_a, \theta_b) = 3K \sqrt{\frac{1}{p_1} \| R_a - R_b \|_F^2 + \frac{1}{p_2} \| C_a - C_b \|_F^2 + \frac{1}{T} \sum_{t=1}^{T} \| F_{a t} - F_{b t} \|_F^2}. \]

It is easy to prove that \( d(\theta_a, \theta_b) \leq d^*(\theta_a, \theta_b). \) And we have \( \Theta(\delta; I_{k_1}, I_{k_2}) \subset \Theta^*(\delta) = \{ \theta \in \Theta : d^*(\theta, \theta_0) \leq K \delta \}. \)
Then
\[ D(\epsilon, d, \Theta(\delta; I_{k_1}, I_{k_2})) \leq D(\epsilon, d^*, \Theta(\delta; I_{k_1}, I_{k_2})) \leq D(\epsilon/2, d^*, \Theta^*(\delta)) \leq C(\epsilon/4, d^*, \Theta^*(\delta)). \] (B.28)

Next we find an upper bound for \( C(\epsilon/4, d^*, \Theta^*(\delta)) \). Let \( \eta = \epsilon/4, \theta_1^*, \ldots, \theta_J^* \) be the maximal set in \( \Theta^*(\delta) \) such that \( d^*(\theta_j^*, \theta_j^*) > \eta, \forall j \neq l \). Set \( B(\theta, c) = \{ \gamma \in \Theta : d^*(\gamma, \theta) \leq c \} \). Then \( B(\theta_1^*, \eta), \ldots, B(\theta_J^*, \eta) \) cover \( \Theta^*(\delta) \) and \( C(\epsilon/4, d^*, \Theta^*(\delta)) \leq J \). Also \( B(\theta_1^*, \eta/4), \ldots, B(\theta_J^*, \eta/4) \) are disjoint and
\[
\bigcup_{j=1}^J B(\theta_j^*, \eta/4) \subset \Theta^*(\delta + \eta/4).
\]

The volume of a ball defined by \( d^* \) with radius \( c \) is equal to \( h_M c^M \), where \( h_M \) is a constant. So
\[
J \cdot h_M \left( \frac{\eta}{4} \right)^M \leq h_M (K(\delta + \eta/4))^M.
\]

Then
\[
J \leq \left( \frac{K(4\delta + \eta)}{\eta} \right)^M = \left( \frac{K(16\delta + \epsilon)}{\epsilon} \right)^M \leq \left( \frac{K\delta}{\epsilon} \right)^M
\] (B.29)
for \( \epsilon \leq \sqrt{\delta} \). From (B.29) and (B.28), we have
\[
\int_0^\delta \sqrt{\log D(\epsilon, d, \Theta(\delta; I_{k_1}, I_{k_2}))} d\epsilon \leq \int_0^\delta \sqrt{\log C(\epsilon/4, d^*, \Theta^*(\delta))} d\epsilon
\]
\[
\leq \sqrt{p_1 k_1 + p_2 k_2 + T k_1 k_2} \int_0^\delta \sqrt{\log(K \sqrt{\delta}/\epsilon)} d\epsilon
\]
\[
\leq \sqrt{p_1 p_2 k_1 + T^2 p_2 k_2 + T_2 p_2 k_1 k_2} \int_0^\delta \sqrt{\log(K \delta/\epsilon)} d\epsilon
\]
Thus
\[
E[\sup_{\Theta} |W(\theta)|] \lesssim \int_0^\delta \sqrt{\log D(\epsilon, d, \Theta(\delta))} / \sqrt{T^2 p_1 p_2} \lesssim \frac{\delta}{T}
\]
This completes the proof. \( \square \)

C Proof of Theorem 3.2

The proof is similar to the proof of Theorem 3.1 and thus we omit it.
## D Additional Simulation Results

Table D1: Averaged estimation errors and standard errors of $\mathcal{D}(\hat{\mathbf{R}}, \mathbf{R})$ and $\mathcal{D}(\hat{\mathbf{C}}, \mathbf{C})$ for Settings A and B under matrix $t_4$ distribution over 1000 replications. "RMFA": proposed robust matrix factor analysis method. "PCA": $\alpha$-PCA with $\alpha = 0$. "PE": projection estimation method.

| Evaluation | $T$ | $p_1$ | $p_2$ | RMFA | PCA | PE |
|------------|-----|-------|-------|------|-----|----|
| $\mathcal{D}(\hat{\mathbf{R}}, \mathbf{R})$ | 20  | 20    | 20    | 0.0739(0.0545) | 0.1865(0.1309) | 0.1154(0.1173) |
|           | 50  | 50    |       | 0.0249(0.0045) | 0.1191(0.0972) | 0.0408(0.0730) |
|           | 100 | 100   | 0.0124(0.0020) | 0.0911(0.0688) | 0.0160(0.0315) |
|           | 150 | 150   | 0.0081(0.0013) | 0.0850(0.0668) | 0.0111(0.0351) |
|           | 200 | 200   | 0.0060(0.0010) | 0.0784(0.0521) | 0.0084(0.0296) |
| $\mathcal{D}(\hat{\mathbf{C}}, \mathbf{C})$ | 20  | 20    | 20    | 0.0737(0.0557) | 0.1879(0.1339) | 0.1141(0.1159) |
|           | 50  | 50    | 0.0395(0.0058) | 0.0781(0.0835) | 0.0592(0.0759) |
|           | 100 | 100   | 0.0273(0.0030) | 0.0401(0.0395) | 0.0339(0.0363) |
|           | 150 | 150   | 0.0223(0.0021) | 0.0304(0.0361) | 0.0272(0.0353) |
|           | 200 | 200   | 0.0192(0.0018) | 0.0257(0.0322) | 0.0233(0.0310) |
| $\mathcal{D}(\hat{\mathbf{R}}, \mathbf{R})$ | 20  | 20    | 20    | 0.0739(0.0545) | 0.1865(0.1309) | 0.1154(0.1173) |
|           | 50  | 50    | 0.0400(0.0061) | 0.0733(0.0694) | 0.0558(0.0642) |
|           | 100 | 100   | 0.0274(0.0029) | 0.0421(0.0509) | 0.0360(0.0488) |
|           | 150 | 150   | 0.0223(0.0021) | 0.0326(0.0464) | 0.0285(0.0416) |
|           | 200 | 200   | 0.0191(0.0018) | 0.0259(0.0337) | 0.0223(0.0313) |
| $\mathcal{D}(\hat{\mathbf{C}}, \mathbf{C})$ | 20  | 20    | 20    | 0.0737(0.0557) | 0.1879(0.1339) | 0.1141(0.1159) |
|           | 50  | 50    | 0.0252(0.0046) | 0.1151(0.0883) | 0.0376(0.0619) |
|           | 100 | 100   | 0.0123(0.0020) | 0.0918(0.0735) | 0.0183(0.0478) |
|           | 150 | 150   | 0.0082(0.0013) | 0.0879(0.0626) | 0.0125(0.0409) |
|           | 200 | 200   | 0.0060(0.0009) | 0.0787(0.0557) | 0.0084(0.0308) |