Algorithms and Complexity for Functions on General Domains

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Abstract

Error bounds and complexity bounds in numerical analysis and information-based complexity are often proved for functions that are defined on very simple domains, such as a cube, a torus, or a sphere. We study optimal error bounds for the approximation or integration of functions defined on $D_d \subset \mathbb{R}^d$ and only assume that $D_d$ is a bounded Lipschitz domain. Some results are even more general. We study three different concepts to measure the complexity: order of convergence, asymptotic constant, and explicit uniform bounds, i.e., bounds that hold for all $n$ (number of pieces of information) and all (normalized) domains.

It is known for many problems that the order of convergence of optimal algorithms does not depend on the domain $D_d \subset \mathbb{R}^d$. We present examples for which the following statements are true:

1. Also the asymptotic constant does not depend on the shape of $D_d$ or the imposed boundary values, it only depends on the volume of the domain.

2. There are explicit and uniform lower (or upper, respectively) bounds for the error that are only slightly smaller (or larger, respectively) than the asymptotic error bound.

1 Introduction

We study optimal error bounds for the approximation or integration of functions $f : D_d \rightarrow \mathbb{R}$, where $D_d \subset \mathbb{R}^d$ is a bounded domain. We assume that $f \in F(D_d)$ where $F(D_d)$ is a unit ball with respect to some norm. Algorithms $A_n$ may use $n$ function values of $f$, this is called standard information and denoted by $\Lambda^{\text{std}}$. 
or $n$ values of general linear functionals. This is called general information and denoted by $\Lambda^{\text{all}}$. We discuss the worst case error of optimal algorithms and use common notation such as $e_n(F(D_d), \text{APP}_\infty, \Lambda^{\text{std}})$ and $e_n(F(D_d), \text{INT}, \Lambda^{\text{std}})$ and $e_n(F(D_d), \text{APP}_2, \Lambda^{\text{all}})$. These problems are linear and we know that

$$e_n(F(D_d), \text{APP}_\infty, \Lambda^{\text{std}}) = \inf_{x_1, \ldots, x_n \in D_d} \sup_{f \in F(D_d), f(x_i) = 0} \|f\|_\infty$$

and

$$e_n(F(D_d), \text{INT}, \Lambda^{\text{std}}) = \inf_{x_1, \ldots, x_n \in D_d} \sup_{f \in F(D_d), f(x_i) = 0} \int_{D_d} f(x) \, dx.$$ 

For these problems it is enough to consider linear algorithms. Linear algorithms are also optimal for $L_2$ approximation if $F(D_d)$ is a unit ball of a Hilbert space and in this case

$$e_n(F(D_d), \text{APP}_2, \Lambda^{\text{all}}) = \inf_{L_1, \ldots, L_n} \sup_{f \in F(D_d), L_i(f) = 0} \|f\|_2 = \sigma_{n+1}$$

coinsides with the approximation numbers (linear widths) or singular values of the embedding of $F(D_d)$ into $L_2$. Here the $L_i$ can be arbitrary linear functionals. Readers who are not familiar with optimal recovery or information-based complexity may read the formulas (1)-(3) as definitions; for more background see [30], in particular Section 4.2.

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Much is known about the order of convergence of the numbers $e_n$, in particular for simple domains $D_d$, such as the cube or the torus. General bounded Lipschitz domains are studied in [7, 10, 11, 12, 26, 27, 29, 38, 39, 40, 42, 43]. In many cases, the optimal order of the $e_n$ is of the form

$$e_n \asymp n^{-\alpha} (\log n)^{\beta},$$

where $\alpha$ and $\beta$ do not depend on the domain $D_d$. It is interesting to know also the exact asymptotic constant

$$C := \lim_{n \to \infty} e_n \, n^\alpha (\log n)^{-\beta},$$

if it exists. The value of $C$ is known only in rare cases (unless $d = 1$, we do not discuss the univariate case in detail), and usually only for very special domains,
like the cube, see [6, 20, 23, 24]. Quite remarkable are recent results of Mieth [26],
to be discussed later, since they hold for general domains $D_d$.

The order of convergence and the asymptotic constant are not really relevant
for the applications, where we only can use a “small” number of $n$; see [32] for a
drastic example. We need explicit (upper and lower) bounds for finite $n \in \mathbb{N}$. It
is remarkable that some bounds hold uniformly, i.e., for all $D_d$ of a given size or
volume. Explicit lower bounds can be used to prove the curse of dimension and
explicit upper bounds can be used to prove the tractability of certain problems.
We refer to [30, 31, 33], where mainly simple domains, usually the cube or the
torus, are studied.

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We discuss the approximation and integration of Lipschitz and Hölder functions
in Section 2. Using results of Hlawka, Kolmogorov and Tikhomirov, Sukharev and
Chernaya we see that, asymptotically, the error behaves like $c \cdot n^{-1/d}$ and $c$ only
depends on the volume $\lambda^d(D_d)$ of the domain $D_d$. We also prove explicit uniform
error bounds that hold for every $n \in \mathbb{N}$ and every domain $D_d$ (with a given
volume). In Section 3 we study functions with a higher smoothness and present,
in particular, an open problem concerning $C^2$ functions.

In Section 4 we use the class $\Lambda_{\text{all}}$ and mainly present results of Mieth concern-
ing $L_2$ approximation of functions from the Sobolev spaces $H^r(D_d)$. Again the
asymptotic constants do not depend on $D_d$ and, with the work of Kröger and
Li and Yau, one can obtain explicit uniform bounds that are very close to the
asymptotic bounds.

Along the way, we present several open problems.

# 2 Approximation of Hölder functions

## 2.1 $L_\infty$ Approximation

Assume that $(D, \rho)$ is a bounded metric space and consider the class

$$F^\omega(D) = \{f : D \to \mathbb{R} \mid \omega(f, h) \leq \omega(h)\}.$$ 

Here

$$\omega(f, h) = \sup\{|f(x) - f(y)| \mid \rho(x, y) \leq h\}$$
is the modulus of continuity of \( f \) and \( \omega: \mathbb{R}^+ \to \mathbb{R}^+ \) is assumed to be nondecreasing, continuous, subadditive with \( \lim_{h \to 0} \omega(h) = 0 \). We also need the covering numbers

\[
c_n = \inf_{x_1, \ldots, x_n \in D} \sup_{x \in D} \min_i \rho(x, x_i).
\]

We start with a result of Sukharev \[36\] that can also be found in Novak \[28\].

**Proposition 1.**

\[
e_n(F^\omega(D), \text{APP}_\infty, \Lambda^{\text{std}}) = \omega(c_n)
\]

**Example 1.** Consider the metric \( \rho(x, y) = \|x - y\|_\infty \) on \( \mathbb{R}^d \) and subsets \( D_d \subset \mathbb{R}^d \). For \( D_d = [0, 1]^d \) one gets \( c_n = \frac{1}{2} m^{-1} \) for \( n = m^d \) till \( n = (m + 1)^d - 1 \) and hence, for \( \omega(h) = h \),

\[
e_n(F^\omega([0, 1]^d), \text{APP}_\infty, \Lambda^{\text{std}}) \approx \frac{1}{2} n^{-1/d}.
\]

For general bounded sets \( D_d \) that contain an interior point we have

\[
e_n(F^\omega(D_d), \text{APP}_\infty, \Lambda^{\text{std}}) \asymp n^{-1/d}.
\]

For the existence of an asymptotic constant one needs stronger assumptions. If \( D_d \) is Jordan measurable with \( \lambda^d(D_d) > 0 \), then

\[
e_n(F^\omega(D_d), \text{APP}_\infty, \Lambda^{\text{std}}) \approx \frac{1}{2} \lambda^d(D_d)^{1/d} \cdot n^{-1/d}.
\]

Hence the asymptotic constant only depends on the volume of the domain. Moreover, the explicit uniform lower bound

\[
(4) \quad \inf_{D_d} e_n(F^\omega(D_d), \text{APP}_\infty, \Lambda^{\text{std}}) \cdot \lambda^d(D_d)^{-1/d} \geq \frac{1}{2} \cdot n^{-1/d}
\]

holds and this uniform lower bound fits nicely to the asymptotic result. Upper bounds for the covering numbers \( c_n \) and hence for the optimal error bounds \( e_n(F^\omega(D_d), \text{APP}_\infty, \Lambda^{\text{std}}) \) are known for particular sets \( D_d \subset \mathbb{R}^d \).

To prove the lower bound \( (4) \) it is enough to estimate the volume of all \( x \in D_d \) with \( \min_i \rho(x, x_i) \leq \varepsilon \). This volume is at most \( n \) times the volume of a \( \rho \)-ball with radius \( \varepsilon \) and so we obtain the inequality

\[
e_n(F^\omega(D_d), \text{APP}_\infty, \Lambda^{\text{std}}) \cdot \lambda^d(D_d)^{-1/d} \geq \frac{1}{2} \cdot n^{-1/d}.
\]

This inequality is sharp, as can be seen if we take \( D_d \) as the disjoint union of \( n \) \( \rho \)-balls with the same radius. \( \Box \)
From now on we do not any more consider arbitrary bounded metric spaces: We assume that $D_d \subset \mathbb{R}^d$ is bounded and the metric is induced by a norm in $\mathbb{R}^d$. We denote by $B$ the unit ball and also write $\| \cdot \|_B$ for the norm. To simplify the formulas we consider only Lipschitz functions, hence $\omega(h) = h$. We denote the respective space by $F^B(D_d)$; it contains all functions $f : D_d \to \mathbb{R}$ with

$$|f(x) - f(y)| \leq \|x - y\|_B.$$  

**Theorem 1.** Assume that $D_d \subset \mathbb{R}^d$ is a bounded Jordan domain with an interior point. Then the asymptotic constant is given by

$$e_n(F^B(D_d), \text{APP}_\infty, \Lambda^{\text{std}}) \approx \Theta_B^{1/d} \left( \frac{\lambda^d(D_d)}{\lambda^d(B)} \right)^{1/d} \cdot n^{-1/d},$$

where $\Theta_B$ is the covering constant of $\mathbb{R}^d$ with respect to $B$ and

$$1 \leq \Theta_B \leq d \log d + d \log \log d + 5d.$$

Moreover,

$$\inf_{D_d} e_n(F^B(D_d), \text{APP}_\infty, \Lambda^{\text{std}}) \cdot \lambda^d(D_d)^{-1/d} = \lambda^d(B)^{-1/d} \cdot n^{-1/d}$$

**Proof.** This is mainly a summary of known results: Hlawka [18] and Kolmogorov and Tikhomirov [19] proved (independently) sharp results about the covering numbers $c_n$ that yield, together with Proposition 1 of Sukharev [36], the asymptotic formula (5). The bound on $\Theta_B$ is from Rogers [34]. Only the explicit uniform lower bound (6) cannot be found in these papers. The inequality follows again from a simple volume estimate and the sharpness of the bound follows again by the example from above: take $D_d$ as the disjoint union of $n$ balls $\delta B$ with the same radius $\delta$.

**Remark 1.** Uniform upper bounds do not make sense for this problem since

$$\sup_{D_d} e_n(F^B(D_d), \text{APP}_\infty, \Lambda^{\text{std}}) \cdot \lambda^d(D_d)^{-1/d} = \infty.$$  

Let us consider the sub-class

$$F^B_0(D_d) = \{ f \in F^\omega(D_d) \mid f = 0 \text{ on } \partial D_d \}$$

of functions that vanish on the boundary of $D_d$. Then similar results hold as in Theorem [1] in particular

$$e_n(F^B_0(D_d), \text{APP}_\infty, \Lambda^{\text{std}}) \approx \Theta_B^{1/d} \left( \frac{\lambda^d(D_d)}{\lambda^d(B)} \right)^{1/d} \cdot n^{-1/d}.$$
For the lower bound we apply (5) to the sets \(D_\varepsilon = \{x \in D_d \mid d(x, \partial D_d) > \varepsilon\}\) and observe that \(\lim_{\varepsilon \to 0} \lambda^d(D_\varepsilon) = \lambda^d(D_d)\). Now, instead of (6), we obtain a uniform upper bound. It is easy to prove

\[
(7) \quad \sup_{D_d} e_n(F_0^B(D_d), \text{APP}_{\infty}, \Lambda^{\text{std}}) \cdot \lambda^d(D_d)^{-1/d} \leq 2 \cdot \lambda^d(B)^{-1/d} \cdot n^{-1/d}.
\]

Proof: Assume that the disjoint balls \(B_\varepsilon(x_i) \subset D_d\) with \(i = 1, \ldots, n\) form a complete packing of \(D_d\), i.e., there is no room for another ball with radius \(\varepsilon\). Then there cannot exist an \(x \in D_d\) with a distance of more than \(2\varepsilon\) from \(\partial D_d \cup \{x_1, \ldots, x_n\}\) and hence the radius of information, using the function values at \(x_1, \ldots, x_n\), is at most \(2\varepsilon\). Then the statement again follows from a simple volume estimate since

\[n \cdot \lambda^d(B_\varepsilon) \leq \lambda^d(D_d).
\]

The upper bound (7) is almost optimal since

\[
\lambda^d(B)^{-1/d} \cdot (n + 1)^{-1/d} \leq \sup_{D_d} e_n(F_0^B(D_d), \text{APP}_{\infty}, \Lambda^{\text{std}}) \cdot \lambda^d(D_d)^{-1/d}.
\]

For this inequality it is enough to consider the case of \(n + 1\) disjoint balls with the same radius. \(\square\)

**Remark 2.** Formula (5) shows that the asymptotic constant does not depend on \(D_d\), it only depends on the volume of the domain. Moreover, the asymptotic constant is only by a factor \(\Theta_B^{1/d}\) larger than the uniform lower bound (6). This factor is very close to 1, in particular if \(d\) is large, \(\lim_{d \to \infty} \Theta_B^{1/d} \to 1\).

After a suitable normalization, when we put \(\lambda^d(D_d) = \lambda^d(B)\), we even obtain

\[e_n(F^B(D_d), \text{APP}_{\infty}, \Lambda^{\text{std}}) \approx \Theta_B^{1/d} \cdot n^{-1/d},\]

and

\[\inf_{D_d} e_n(F^B(D_d), \text{APP}_{\infty}, \Lambda^{\text{std}}) = n^{-1/d}.
\]

This means that the asymptotic constant very mildly depends on \(B\) and the explicit uniform lower bound does not depend on \(B\) at all.

For known results on the covering constants \(\Theta_B\) see the recent survey [8]. \(\square\)

**Remark 3.** A special metric is given by the standard norm in \(\ell^d_p\), i.e., \(B = B^d_p\) is the unit ball in \(\ell^d_p\). Then we write \(\| \cdot \|_p\) instead of \(\| \cdot \|_B\) and \(F^p(D_d)\) instead of \(F^B(D_d)\). It is the space of all functions \(f: D_d \to \mathbb{R}\) with

\[|f(x) - f(y)| \leq \|x - y\|_p.
\]

Of course we can apply Theorem 1 in this case and we may use the formula

\[\lambda^d(B^d_p)^{1/d} \approx 2\Gamma(1 + 1/p)(pe)^{1/p} \cdot d^{-1/p} = c_p \cdot d^{-1/p}.
\]
Assume again that $D_d \subset \mathbb{R}^d$ is a bounded Jordan domain with an interior point. Then the asymptotic constant is given by

$$e_n(F^p(D_d), \text{APP}_\infty, \Lambda^{\text{std}}) \approx \Theta_{B^d_p}^{1/d} \left( \frac{\lambda^d(D_d)}{\lambda^d(B^d_p)} \right)^{1/d} \cdot n^{-1/d},$$

and we obtain from (6) a uniform lower bound of the form

$$e_n(F^p(D_d), \text{APP}_\infty, \Lambda^{\text{std}}) \geq c'_p \ d^{1/p} \lambda^d(D_d)^{1/d} \cdot n^{-1/d}.$$  

This bound heavily depends on the parameter $p$ since $\lambda^d(B^d_p)^{1/d}$ depends on $p$.

**Remark 4.** For large $d$ and $p = 2$ and $\lambda(D_d) = 1$, formula (5) takes the form

$$e_n(F^2(D_d), \text{APP}_\infty, \Lambda^{\text{std}}) \approx (2\pi e)^{-1/2} \ d^{1/2} \ n^{-1/2} \approx 0.24197 \ d^{1/2} \ n^{-1/2}.$$  

If we take, instead of optimal sample points $x_1, x_2, \ldots, x_n$, a regular grid, then the error is $\frac{1}{2}d^{1/2} \ n^{-1/2}$. Hence we only loose a factor of roughly $1/2$ by taking the simplest possible function values instead of the optimal sample points.

There is, however, also a different interpretation of the same result: To obtain a given error $\varepsilon$, one needs more than $2^d$ times more function evaluations if one uses a grid instead of optimal function evaluations.

**Remark 5.** One may use (8) or (9) to prove the curse of dimension as follows: Assume that the volume $\lambda^d(D_d)$ is one and we consider functions on $D_d$ with

$$|f(x) - f(y)| \leq d^{-1/p} \|x - y\|_p.$$  

Then we need, for small $\varepsilon$, at least $C\varepsilon^{-d}$ function values to reach the error $\varepsilon$, i.e., the problem suffers from the curse of dimension.

### 2.2 Integration and $L_1$ approximation

Now we study the problem of $L_1$ approximation or numerical integration and again we assume that $D_d \subset \mathbb{R}^d$ is Jordan measurable with $0 < \lambda^d(D_d) < \infty$. Asymptotic formulas, even for more general (weighted) integration problems, where proved by Chernaya [5] and by Gruber [9]. We add an upper bound for the asymptotic constant $\xi_B$ and see that it is very close to the lower bound, in particular for large $d$. Also the explicit uniform lower bound is new.
Theorem 2. Assume that $D_d \subset \mathbb{R}^d$ is a bounded Jordan domain with an interior point. Then

\[
e_n(F^B(D_d), \text{INT}, \Lambda^{\text{std}}) \approx \xi_B \lambda^d(D_d) \left( \frac{\lambda^d(D_d)}{\lambda^d(B)} \right)^{1/d} \cdot n^{-1/d},
\]

where $\xi_B$ is a constant that depends on the norm and

\[
\frac{d}{d+1} \leq \xi_B \leq \frac{d}{d+1} \Theta_B^{1/d} \leq \frac{d}{d+1} (d \log d + d \log \log d + 5d)^{1/d}.
\]

Moreover,

\[
\inf_{D_d} e_n(F^B(D_d), \text{INT}, \Lambda^{\text{std}}) \cdot \lambda^d(D_d)^{- (d+1)/d} = \frac{d}{d+1} \lambda^d(B)^{-1/d} \cdot n^{-1/d}.
\]

Proof. The asymptotic formula is from Chernaya [5], see also Gruber [9]. Also the lower bound on $\xi_B$ is contained in [5]. To prove the upper bound on $\xi_B$ we compare with (5) and take the case $\lambda^d(D_d) = 1$. We obtain $\xi_B \leq \Theta_B^{1/d}$, but even more is true: Assume that an information mapping is given and fixed, $N_n(f) = (f(x_1), \ldots, f(x_n))$. Then we can define the radius of information for the two problems APP\_\infty and INT by $r(N_n, \text{APP\_\infty}) = \sup_{\|f\| \leq 1, N_n(f) = 0} \|f\|_{\infty}$ and $r(N_n, \text{INT}) = \sup_{\|f\| \leq 1, N_n(f) = 0} \int_{D_d} f(x) \, dx$ and obtain

\[
r(N_n, \text{INT}) \leq \frac{d}{d+1} r(N_n, \text{APP\_\infty})
\]

by the exact formulas for the radius, and the statement follows.

Similarly, we obtain the bound

\[
\inf_{D_d} e_n(F^B(D_d), \text{INT}, \Lambda^{\text{std}}) \cdot \lambda^d(D_d)^{- (d+1)/d} \leq \frac{d}{d+1} \lambda^d(B)^{-1/d} \cdot n^{-1/d}.
\]

To prove equality it is again enough to consider $D_d$ as the disjoint union of $n$ balls $B + y_i$ with equal radius.

Remark 6. The results for $L_1$ approximation (or integration) and $L_\infty$ approximation are very similar. For normed problems with $\lambda^d(D_d) = 1$ the optimal error bounds differ at most by a factor $K_d \approx 1$. Therefore we do not study $L_p$ approximation for $1 < p < \infty$ in detail.

Again one may use (11) to prove the curse of dimension. We formulate the result as a corollary. It improves Proposition 3.2 of [15] and also similar results of Sukharev [37].
Corollary 1. Assume that $D_d \subset \mathbb{R}^d$ is Jordan measurable with $\lambda^d(D_d) = 1$ and consider functions on $D_d$ with

$$|f(x) - f(y)| \leq d^{-1/p}\|x - y\|_p.$$ 

Then we need, for the integration problem

$$S_d(f) = \int_{D_d} f(x) \, dx,$$

and small $\varepsilon > 0$, at least $C\varepsilon^{-d}$ function values to reach the error $\varepsilon$, i.e., the problem suffers from the curse of dimension. \[\Box\]

There are two directions to continue these studies:

- We may study other function spaces.
- We may allow general linear information instead of function evaluation.

We will follow both directions in the following.

3 Other function spaces

We start with a result from [16]. For any open set $D_d \subset \mathbb{R}^d$ with volume $\lambda^d(D_d) = 1$ we consider the set

$$C^r(D_d) = \{ f : D_d \to \mathbb{R} \mid \|D^\beta f\|_\infty \leq 1, |\beta|_1 \leq r \}.$$

Theorem 3. For all $r \in \mathbb{N}$ there exists a constant $c_r > 0$ such that for all $d, n \in \mathbb{N}$

$$e_n(C^r(D_d), \text{INT}, \Lambda^{\text{std}}) \geq \min\{1/2, c_r d n^{-r/d}\}.$$ 

Observe that the constant $c_r$ does not depend on $D_d$ or $d$, we have an explicit uniform lower bound; the same lower bound holds for the infimum over all $D_d$ (with volume 1).

Remark 7. 1) The lower bound cannot be improved since for cubes we have a similar upper bound. It would be good to know more on the constants $c_r$ and on extremal sets $D_d$, where $e_n(C^r(D_d), \text{INT}, \Lambda^{\text{std}})$ is small, for given $r$, $d$ and $n$.

For these function spaces there cannot be a meaningful explicit uniform upper bound since $\sup_{D_d} e_n(C^r(D_d), \text{INT}, \Lambda^{\text{std}}) = 1$. The supremum over $D_d$ makes sense if we impose boundary conditions such as $f(x) = 0$ for $x \in \partial D$. 

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2) It is known that the weak order
\[ e_n(\mathcal{C}^r(D_d), \text{INT}, \Lambda_{\text{std}}) \asymp n^{-r/d} \]
holds at least for every bounded Lipschitz domain. This follows from much more general results of [27, 29]. We guess that more is true and the asymptotic constant
\[ \lim_{n \to \infty} e_n(\mathcal{C}^r(D_d), \text{INT}, \Lambda_{\text{std}}) \cdot n^{r/d} = C_{D_d} \]
does not depend on \( D_d \) (for fixed \( d \)) if \( D_d \) is Jordan measurable with \( \lambda^d(D_d) = 1 \).

**Remark 8.** The problem APP\(_{\infty}\) for the same spaces and norms was studied by Krieg [21]. The lower bound \( d n^{-r/d} \) is now replaced by \( d^{r/2} n^{-r/d} \) if \( r \) is even and again this cannot be improved since the bound is sharp for the cube. If \( r \) is odd and \( r \geq 3 \) then the exact order is unknown, Krieg proved for the cube \( D_d = [0,1]^d \) the lower bound \( d^{r/2} n^{-r/d} \) and the upper bound \( d^{(r+1)/2} n^{-r/d} \). It follows that approximation is essentially more difficult then integration iff \( r \geq 3 \). The lower bound holds for \( \varepsilon < \varepsilon_r \) where \( \varepsilon_r \) is rather small and hence the case of large \( \varepsilon \) remains open.

**Remark 9.** The norm
\[ \max_{|\beta|_1 \leq r} \| D^\beta f \|_{\infty} \]
might be reasonable when we consider a cube \( D_d = [0,1]^d \) but it is not invariant with respect to rotation. The function \( f(x) = \sum x_i \) has a gradient with length \( d^{1/2} \) as \( g(x) = d^{1/2} \cdot x_1 \), but all partial derivatives of \( f \) are bounded by one. Therefore we also consider an orthogonal invariant norm. Since for this modified space \( \tilde{\mathcal{C}}^r(D_d) \) many problems are still open, we only discuss the case \( r = 2 \) in the following.

**Example 2.** Let us discuss the integration problem for the class
\[ \tilde{\mathcal{C}}^2(D_d) = \{ f \in \mathcal{C}^2(D_d) \mid \| f \|_{\infty} \leq 1, \text{Lip}(f) \leq d^{-1/2}, \text{Lip}(D^\Theta f) \leq d^{-1} \}, \]
as in [14, 15, 17]. Here \( D^\Theta f \) denotes any directional derivative in a direction \( \Theta \in S^{d-1} \) and
\[ \text{Lip}(g) = \sup_{x,y \in D_d} \frac{|g(x) - g(y)|}{\| x - y \|_2}. \]
Again we assume that \( D_d \subset \mathbb{R}^d \) is a domain with \( \lambda^d(D_d) = 1 \). We conjecture that there exists a constant \( C > 0 \) (independent on \( d \) and \( D_d \)) such that
\[ (12) \quad e_n(\tilde{\mathcal{C}}^2(D_d), \text{INT}, \Lambda_{\text{std}}) \geq n^{-2/d}. \]
Some comments are in order:

1) This would be another explicit uniform lower bound and, because of known upper bounds for the cube, it certainly cannot be improved.

2) The lower bound is, so far, not even known for the cube $D_d = [0, 1]^d$. Nevertheless there are some partial results or signs that this could be true. We mention a few.

3) It is known that the integration problem for $\tilde{C}^2(D_d)$ and certain $D_d$ suffers from the curse of dimensionality. This is true if $D_d$ has a small radius, see [17] for the best known results. For example, the curse is known if $D_d$ is a $\ell_p^d$ ball and $p \geq 2$. It is not known for $p$ balls and $p < 2$.

4) It is easy to see that the lower bound (12) is true if $D_d$ is a disjoint union of $n$ euclidean balls of the same size. But, of course, this domain is not extremal for the given norm.

5) Assume that the $x_i$ form a grid. Then, for $d = 1$, one may take a quadratic spline $f_1$ as a fooling function and for $d > 1$ one can take a fooling function of the form

$$f_d(x) = \frac{1}{d} \sum_{i=1}^{d} f_1(x^i),$$

where $x = (x^1, \ldots, x^d)$. Hence the conjecture is true if we restrict the information to grids and we conjecture that the error for grid information can only be improved by a constant, independent of $d$.

\[\square\]

**Remark 10.** Optimal recovery for $C^2$ functions on general domains was also studied in [1]. The authors use function values and values of the gradient as information and prove asymptotic results that, again, only depend on the size of $D_d$.

**Remark 11.** Consider the compact embedding of $W^r_p(D_d)$ into $L_q(D_d)$ for a bounded Lipschitz domain $D_d$. For $\Lambda^{\text{all}}$ it is known that the rate of convergence for the approximation numbers (error of optimal linear algorithms) and the Gelfand numbers (up to a factor two the error of optimal algorithms) do not depend on $D_d$. The same is known for $\Lambda^{\text{std}}$ and then the optimal order is the same for linear and nonlinear algorithms. The optimal order is

$$e_n(W^r_p(D_d), \text{APP}_q, \Lambda^{\text{std}}) \approx n^{-r/d+(1/p-1/q)/2},$$

for all bounded Lipschitz domains, see [27, 29], whenever the Sobolev space is embedded into $C$. There are many common equivalent norms for the Sobolev
space, we may take
\[ \|f\|_{W^{p}_p(D)}^p = \sum_{|\alpha| \leq r} \|D^\alpha f\|_{L^p}^p. \]

Open Problem: Is the asymptotic constant independent on the shape of \( D_d \) and only depends on the volume of \( D_d \)?

We may ask the same question for \( \Lambda^{\text{all}} \) and also may distinguish between all algorithms and the class of linear algorithms. See the next section for the case \( p = q = 2 \). \( \square \)

## 4 Arbitrary linear information

Weyl [46] proved that the asymptotic constant for the size of the eigenvalues of the Dirichlet Laplacian and also of the Neumann Laplacian do not depend on the shape of \( D_d \), it only depends on the volume of \( D_d \); see [35] for an accessible proof. The results of Weyl were extended by many authors, see the surveys by Birman and Solomjak [3, 4]. The papers by Birman and Solomjak [2] and Tulovsky [41] are important for the asymptotic constant of more general differential equations and boundary value problems. These results can be used, as explained in Mieth [26], to compute the asymptotic constant for Sobolev embeddings in the Hilbert space case, i.e., \( p = q = 2 \).

We consider the numbers \( e_n(H^r(D_d), \text{APP}_2, \Lambda^{\text{all}}) \), i.e., the approximation numbers of Sobolev embeddings. In addition to \( H^r(D_d) \) we also consider the subspace \( H^r_0(D_d) \), the closure of \( C_0^\infty(D_d) \). We always assume that \( D_d \) is a bounded (nonempty) domain in \( \mathbb{R}^d \) and for the results concerning \( H^r_0(D_d) \) this assumption is enough. When the whole space \( H^r(D_d) \) is considered then one needs the extension property of \( D_d \); it is enough to assume that \( D_d \) is a bounded Lipschitz domain. We collect some results, most of them as in Mieth [26], and add a little bit using results of Birman and Solomjak [2].

**Theorem 4.** The asymptotic constant
\[
\lim_{n \to \infty} e_n(H^r(D_d), \text{APP}_2, \Lambda^{\text{all}}) \cdot n^{r/d} \cdot \lambda(D_d)^{-r/d} = C_{r,d}
\]
e exists and is independent of \( D_d \) and also coincides with the asymptotic constant
\[
\lim_{n \to \infty} e_n(H^r_0(D_d), \text{APP}_2, \Lambda^{\text{all}}) \cdot n^{r/d} \cdot \lambda(D_d)^{-r/d} = C_{r,d}
\]
for the subspace with zero boundary values. \( \square \)
Hence the asymptotic constant $C_{r,d}$ does not depend on the boundary conditions and does not depend on the shape of $D_d$. The norm in $H^r(D_d)$ can be given by $\|f\|_2 = \|f\|_2^2 + \sum_{|\alpha|=r} \|D^\alpha f\|_2^2$ (or similar) and there are explicit formulas for $C_{r,d}$, see [26].

**Remark 12.** Theorem [3] is for Sobolev embeddings in the Hilbert space case. Here the error bounds coincide with approximation numbers or singular values. It would be interesting to have similar results for embeddings of $W^r_p(D_d)$ into $L_q(D_d)$ and general $p$ and $q$. 

**Remark 13.** Mieth [26] used results of Kröger [22] and Li and Yau [25] to prove explicit uniform upper bounds for $e_n(H^r_0(D_d), \text{APP}_2, \Lambda^{\text{all}})$ and lower bounds for $e_n(H^r(D_d), \text{APP}_2, \Lambda^{\text{all}})$. An extended *Polya conjecture* for this case reads

$$e_n(H^r(D_d), \text{APP}_2, \Lambda^{\text{all}}) \cdot n^{r/d} \cdot \lambda(D_d)^{-r/d} \geq C_{r,d}$$

and

$$e_n(H^r_0(D_d), \text{APP}_2, \Lambda^{\text{all}}) \cdot n^{r/d} \cdot \lambda(D_d)^{-r/d} \leq C_{r,d}$$

for all $n \in \mathbb{N}$. The known results, see Mieth [26], are only slightly weaker than these conjectured ones.

We finish the paper with a remark on $C^\infty$ functions.

**Remark 14.** Functions from the class $C^\infty$ with the norm $\|f\| := \sup_{x \in \mathbb{R}^d} \|D^\alpha f\|_\infty$ are studied in [32, 45], see also Vybíral [44]. The curse is proved for domains $D_d$ of the form $D_d = [a, b]^d$, where $b - a > 0$ can be small but is independent of $d$. By the proof technique it is clear that all proved lower bounds also hold for larger domains. Nevertheless, the proof does not cover all $D_d$ with a size $\lambda^d(D_d) \geq \alpha^d$ and it would be interesting to know whether the curse also holds for these more general domains. If one assumes that all directional derivatives of all orders are bounded by one and $D_d = [0, 1]^d$ then one can prove the weak tractability of the integration problem using the Clenshaw-Curtis Smolyak algorithm, see [13].

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