ON EIGENVALUES OF DOUBLE BRANCHED COVERS

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ABSTRACT. For a given knot, we study the minimal number of positive eigenvalues of the double branched cover over spanning surfaces for the knot. The value gives a lower bound for various genera, the dealternating number and the alternation number of knots, and we prove that Batson’s bound for the non-orientable 4-genus gives an estimate of the value. In addition, we use the value to give a necessary condition for being quasi-alternating.

1. INTRODUCTION

Throughout this paper, all manifolds are assumed to be smooth and compact unless otherwise stated.

For a surface $F$ properly embedded in $B^4$, let $M_F$ denote the double branched cover of $B^4$ over $F$, and $b_2^+(M_F)$ ($b_2^-(M_F)$) the number of positive (resp. negative) eigenvalues of the intersection form of $M_F$. Then we can define the knot invariants
\[ b^+(K) := \min \{ b_2^+(M_F) \mid F \text{ is a surface in } S^3 \text{ with } \partial F = K \} \]
and the knot concordance invariants
\[ b^\pm(K) := \min \{ b_2^\pm(M_F) \mid F \text{ is a surface in } B^4 \text{ with } \partial F = K \} . \]

Here, we obtain $M_F$ for a surface $F$ in $S^3$ by pushing the interior of $F$ into the interior of $B^4$ and taking the double branched cover of the resulting surface. Recently, Greene [7] gives the following characterization of alternating knots.

**Theorem 1** (Greene, [7, Theorem 1.1]). A knot $K$ is alternating if and only if $b^+(K) = b^-(K) = 0$.

This theorem implies that alternating knots can be thought of as the trivial knot in terms of $b^\pm(K)$, while knots concordant to an alternating knot are like slice knots in terms of $b_2^\pm(K)$. In this paper, we study the invariants $b^\pm$ and $b_2^\pm$ and their relationship to various genera, the dealternating number [1], the alternation number [8] and quasi-alternating knots [13].

Here we mention two results of this work. The first result is an observation of Batson’s bound of the non-orientable 4-genus [3]. Here the non-orientable 4-genus $\gamma_4(K)$ of a knot $K$ is the minimal first Betti number of non-orientable surfaces in $B^4$ with boundary $K$. Using the Heegaard Floer correction term of the ($-1$)-surgery along $K$ (denoted $d(S^3_{-1}(K))$) and the knot signature $\sigma(K)$, Batson gives the inequality
\[ \gamma_4(K) \geq \frac{\sigma(K)}{2} - d(S^3_{-1}(K)) \]
and prove that $\gamma_4$ can be arbitrarily large. Batson’s bound is strong enough to prove $\gamma_4(T_{2n, 2n-1}) = n - 1$ for any integer $n > 1$ where $T_{2n, 2n-1}$ denotes the $(2n, 2n - 1)$-torus knot, while the bound becomes a trivial inequality for any alternating knot.
We found the reason of this gap; Batson’s bound is essentially a lower bound for $b^+(K)$.

**Theorem 2.** For any knot $K$, we have

$$\gamma_4(K) \geq b^+_4(K) \geq \frac{\sigma(K)}{2} - d(S^3_1(K)).$$

Theorem 2 implies that $b^+_4(K)$ can be arbitrarily large.

**Corollary 1.** $b^+_4(T_{2n,2n-1}) = n - 1$ for any $n \in \mathbb{Z}_{>0}$. In particular, $b^+_4$ can be arbitrarily large.

The second result is the following necessary condition for being quasi-alternating.

**Theorem 3.** If a knot $K$ is quasi-alternating, then $b^+_4(K) = b^-_4(K) = 0$.

In light of Theorem 1 we say that a knot $K$ is 4-dimensionally alternating if $b^+_4(K) = b^-_4(K) = 0$. Theorem 3 says that if a knot $K$ is quasi-alternating, then $K$ is 4-dimensionally alternating. Then, is the inverse also true? The answer is no; we classify 4-dimensionally alternating knots up to 10 crossings, and give 4-dimensionally alternating knots which are not concordant to any quasi-alternating knot.

**Proposition 1.** The knots $10_{139}, 10_{152}, 10_{154}$ and $10_{161}$ in Rolfsen’s table are 4-dimensionally alternating but not concordant to any quasi-alternating knot. For any other knot $K$ with up to 10 crossings, $K$ is 4-dimensionally alternating if and only if $K$ is quasi-alternating or slice.

**Remark.** If $K^*$ is the mirror of $K$, then $b^-_4(K) = b^+_4(K^*)$ and $b^-_4(K) = b^+_4(K^*)$. Hence we only need to study $b^+_4(K)$ and $b^+_4(K)$, but we often use $b^-_4(K)$ and $b^-_4(K)$ for convenience.

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2. Relationship to various genera of knots

In this section, we study the relationship of $b^\pm_4(K)$ and $b^+_4(K)$ to genera of knots. We start from orientable genera of knots. The 3-genus $g_3(K)$ (the 4-genus $g_4(K)$) of a knot $K$ is the minimal number of the genus of any orientable surface in $S^3$ (resp. in $B^4$) with boundary $K$. Then Gordon-Litherland’s theorem gives the following inequalities;

**Proposition 2.** For any knot $K$, we have

$$b^\pm_4(K) \leq g_3(K) \pm \frac{\sigma(K)}{2},$$

and

$$b^+_4(K) \leq g_4(K) \pm \frac{\sigma(K)}{2}.$$

**Proof.** In [6], Gordon and Litherland prove that for any orientable surface $F$ in $B^4$ with boundary $K$, we have

$$\sigma(M_F) = \sigma(K),$$
where $\sigma(M_F) = b_2^+(M_F) - b_2^-(M_F)$. In addition, let $b_i$ denote the $i$-th Betti number, and then we can verify that

$$2g(F) = b_1(F) = b_2(M_F) = b_2^+(M_F) + b_2^-(M_F).$$

By these equalities, we have

$$b_i^\pm(M_F) = \frac{1}{2}(b_2(M_F) \pm \sigma(M_F)) = g(F) \pm \frac{\sigma(K)}{2}.$$ 

Hence, if $F$ is in $S^3$ and has genus $g_3(K)$, then we have

$$b_i^\pm(K) = b_i^\pm(M_F) = g_3(K) \pm \frac{\sigma(K)}{2}.$$ 

Similarly, if $F$ is in $B^4$ and has genus $g_4(K)$, we have

$$b_i^\pm(K) = b_i^\pm(M_F) = g_4(K) \pm \frac{\sigma(K)}{2}.$$ 

Next we consider non-orientable genera of knots. The non-orientable 3-genus $\gamma_3(K)$ of a knot $K$ is the minimal number of the first Betti number of any non-orientable surface in $S^3$ with boundary $K$. Then we have the following:

**Proposition 3.** For any knot $K$, we have

$$b^+(K) + b^-(K) \leq \gamma_3(K),$$

and

$$b^+(K) + b^-(K) \leq \gamma_4(K).$$

**Proof.** For any surface $F$ in $B^4$ with boundary $K$, we can verify that

$$b_1(F) = b_2(M_F) = b_2^+(M_F) + b_2^-(M_F).$$

Hence if $F$ is a non-orientable surface in $S^3$ with $\partial F = K$ and $b_1(F) = \gamma_3(K)$, then we have

$$\gamma_3(K) = b_1(F) = b_2^+(M_F) + b_2^-(M_F) \geq b^+(K) + b^-(K).$$

Similarly, we can prove that $\gamma_4(K) \geq b^+(K) + b^-(K)$. 

3. Dealternating number and alternation number

We next consider the dealternating number and the alternation number. We first recall the definition of these invariants. A knot diagram is $n$-almost alternating if $n$ crossing changes in the diagram turn the diagram into an alternating knot diagram. We say that a knot $K$ has the dealternating number $n$ if $K$ has an $n$-almost alternating diagram and no $k$-almost alternating diagram for any $k < n$. We denote the dealternating number of $K$ by $\text{dalt}(K)$. The alternation number $\text{alt}(K)$ of a knot $K$ is the minimal number of the Gordian distance between $K$ and any alternating knot. Then we have the following inequalities.

**Proposition 4.** $b^+(K) + b^-(K) \leq \text{dalt}(K)$.

**Proposition 5.** $b_2^+(K) + b_2^-(K) \leq 2\lceil \frac{\text{alt}(K)}{2} \rceil$. 
The aim of this section is to prove the above two propositions. To prove Proposition 4, we use the Goeritz form for surfaces in $S^3$, which is introduced in [6]. For a surface $F$ in $S^3$, let $G_F$ denote the Goeritz form for $F$, $\sigma(G_F)$ the signature of $G_F$, and $e(F)$ the Euler number of $F$. Then it is proved in [6] that $\sigma(G_F) = \sigma(M_F) = \sigma(K) - \frac{e(F)}{2}$.

**Proof of Proposition 4.** Suppose that a diagram $D$ for a knot $K$ is deformed into an alternating diagram $D'$ for a knot $K'$ by $n$ crossing changes. Note that $D$ and $D'$ have the same projection. We choose an orientation of $K$ and a checkerboard coloring of $D$ arbitrarily, and choose those of $K'$ and $D'$ so that the orientation and coloring on the projection induced by $K'$ and $D'$ are equal to ones induced by $K$ and $D$. Let $B$ (and $W$) denote the spanning surface for $K$ in $S^3$ dedicated by the black regions (resp. white regions) on $D$. Similarly, we take the spanning surfaces $B'$ and $W'$ for $K'$ from the checkerboard coloring of $D'$ respectively. Here we note that since $D'$ is an alternating diagram, one of $G_{B'}$ and $G_{W'}$ is positive definite and the other is negative definite. We may assume that $G_{B'}$ is positive definite.

We consider the value of $\sigma(G_B)$ and $\sigma(G_W)$. On the diagrams $D$ and $D'$, we divide $n$ crossings performed crossing change into two types; Type I and Type II in Figure 1. In addition, we assign $+1$ or $-1$ to each crossing as shown in Figure 2, which is called the sign of a crossing. Let $n_1$ (and $n_2$) denote the number of Type I (resp. Type II) crossings. Then $n = n_1 + n_2$. Moreover, the Euler number of $B$ and $B'$ are computed by counting the sign of Type II crossings, and we see that

$$\frac{e(B')}{2} - \frac{e(B)}{2} \leq 2n_1.$$ 

Similarly, the Euler number of $W$ and $W'$ are computed by counting the sign of Type I crossings and we have

$$\frac{e(W')}{2} - \frac{e(W)}{2} \leq 2n_2.$$ 

Now, since $b_1(B) = b_1(B') = \sigma(G_{B'})$, we see that

$$\sigma(G_B) = \sigma(K) - \frac{e(B)}{2} \geq \sigma(K) - \frac{e(B')}{2} - 2n_1 = \sigma(K) - \sigma(K') + b_1(B) - 2n_1.$$ 

This implies that

$$n_1 \geq \frac{1}{2} (b_1(B) - \sigma(G_B)) + \frac{1}{2} (\sigma(K) - \sigma(K')) = b^-_2(M_B) + \frac{1}{2} (\sigma(K) - \sigma(K')).$$

Similarly, since $b_1(W) = b_1(W') = -\sigma(G_{W'})$, we see that

$$\sigma(G_W) \leq \sigma(K) - \sigma(K') - b_1(W) + 2n_2.$$ 

This implies that

$$n_2 \geq \frac{1}{2} (b_1(W) + \sigma(G_W)) + \frac{1}{2} (\sigma(K') - \sigma(K)) = b^+_2(M_W) + \frac{1}{2} (\sigma(K') - \sigma(K)).$$

Since $n = n_1 + n_2$, we have

$$n = n_1 + n_2 \geq b^+_2(M_W) + b^-_2(M_B) \geq b^+(K) + b^-(K).$$

This completes the proof. □
Next we prove Proposition 5.

Proof of Proposition 5. Suppose that a knot $K$ is deformed into an alternating knot $J$ after $n$ crossing changes. Then the regular homotopy dedicated by the crossing changes gives a self-transverse immersed annulus $A$ in $S^3 \times [0,1]$ such that the number of its self-intersections is $n$ and $(\partial(S^3 \times [0,1]), \partial A)$ is diffeomorphic to the disjoint union $(S^3, K) \amalg (S^3, J^*)$. By the argument in the proof of [9, Proposition 2.3], we can construct a (possibly non-orientable) cobordism $C$ in $S^3 \times [0,1]$ from $J$ to $K$ which satisfies $b_1(C) = 2\lceil \frac{n}{2} \rceil + 1$. Here, since $J$ is alternating, $J$ bounds two surfaces $P$ and $N$ in $B^4$ such that $M_P$ and $M_N$ are positive definite and negative definite respectively. Now, by gluing $C$ with $P$ and $N$ along $J$, we obtain two surfaces $C \cup P$ and $C \cup N$ in $B^4$ with boundary $K$. Since $M_{C \cup P} = M_C \cup M_P$ and $M_{C \cup N} = M_C \cup M_N$, it follows from elementary homology theory that

\begin{enumerate}
  \item $b^-_2(M_{C \cup P}) = b^-_2(M_C)$,
  \item $b^+_2(M_{C \cup N}) = b^+_2(M_C)$, and
  \item $b^+_2(M_C) + b^-_2(M_C) = b_1(C) - 1$.
\end{enumerate}

These imply that $2\lceil \frac{n}{2} \rceil = b^+_2(M_{C \cup N}) + b^-_2(M_{C \cup P}) \geq b^+_4(K) + b^-_4(K)$. \qed

4. Proof of main theorems

In this section, we prove Theorem 2 and Theorem 3. We start from Theorem 2

Proof of Theorem 2. The inequality $\gamma_4(K) \geq b^+_4(K)$ is given by Proposition 3. We prove $b^+_4(K) \geq \sigma(K)/2 - d(S^3_4(K))$. 

**Figure 1.** Type of crossings

**Figure 2.** Sign of crossings
Let $F$ be a surface in $B^4$ with boundary $K$. We first assume that $b_1(F)$ is odd. Then $F$ is non-orientable, and it follows from [3, Theorem 1.5] that
\[ b_1(F) \geq \frac{e(F)}{2} - 2d(S^3_{-1}(K)). \]
Moreover, as mentioned in Section 3, the equality
\[ \sigma(M_F) = \sigma(K) - \frac{e(F)}{2} \]
also holds. Combining them, we have
\[ \sigma(K) - 2d(S^3_{-1}(K)) \leq b_1(F) + \sigma(M_F) = 2b_2^+(M_F). \] (2)
Next we assume that $b_1(F)$ is even. Then, by taking the boundary connected sum of $F$ with a Möbius band in $B^4$ with boundary the unknot and Euler number $+2$, we have a non-orientable surface $F'$ in $B^4$ with boundary $K$ such that $b_1(F') = b_1(F) + 1$ and $e(F') = e(F) + 2$. By applying [3, Theorem 1.5] to $F'$, we have
\[ b_1(F) + 1 \geq \frac{e(F) + 2}{2} - 2d(S^3_{-1}(K)). \]
This inequality is equivalent to the inequality (1), and hence the inequality (2) also holds in this case. This completes the proof. \hfill \Box

Next, we prove Theorem 3. We first recall quasi-alternating links. For a link $L$, fix a diagram of $L$ and choose a crossing on the diagram. Then we obtain two links $L_0$ and $L_1$ by replacing the crossing by the two simplifications shown in Figure 3. We call the links $L_0, L_1$ a pair of resolutions for $L$. The set $Q$ of quasi-alternating links is the smallest set of links which satisfies the following properties:

1. the unknot is in $Q$
2. the set $Q$ is closed under the following operation. Suppose $L$ is any link which has a pair of resolutions $L_0, L_1$ with the following properties:
   - $L_0, L_1 \in Q$,
   - $\det(L_0), \det(L_1) \neq 0$,
   - $\det(L) = \det(L_0) + \det(L_1)$;
then $L \in Q$.

Here $\det(L)$ denotes the determinant of $L$; namely, if we denote the double branched cover of $S^3$ over $L$ by $\Sigma(L)$, then
\[ \det(L) := \begin{cases} |H_1(\Sigma(L); \mathbb{Z})| & \text{(if } H_1(\Sigma(L); \mathbb{Z}) \text{ is finite)} \\ 0 & \text{(otherwise)} \end{cases} \].

Figure 3. Resolutions for a link $L$
Figure 4.

For a quasi-alternating link $L = L_\emptyset$ and its resolutions $L_0$, $L_1$ satisfying the property (2), choose the checkerboard coloring of their diagrams as shown in Figure 4. For each $i \in \{\emptyset, 0, 1\}$, let $B_i$ (and $W_i$) denote the spanning surface for $L_i$ in $S^3$ dedicated by the black regions (resp. white regions). Then the differences $B_\emptyset - B_1$ and $W_\emptyset - W_0$ can be regarded as cobordisms in $S^3 \times [0, 1]$ from $L_0$ and $L_1$ respectively. We denote these cobordisms by $C_B$ and $C_W$ respectively. Theorem 3 follows from the following lemma.

Lemma 4. The double branched cover $M_{C_B}$ is positive definite, and $M_{C_W}$ is negative definite.

Proof. We first consider $M_{C_B}$. For a given 4-manifold $M$, let $Q_M$ denote the intersection form of $M$. Since $M_{C_B} = M_{B_1} \cup M_{C_B}$ and $\Sigma(L_1)$ is a rational homology 3-sphere, the bilinear form $Q_{M_{B_1}}$ is isomorphic to $Q_{M_{B_1}} \oplus Q_{M_{C_B}}$ over $\mathbb{Q}$. Since $b_2(M_{C_B}) = b_2(M_{B_1}) - b_2(M_{B_1}) = 1$, the 4-manifold $M_{C_B}$ is positive definite if and only if $\sigma(M_{C_B}) = \sigma(M_{B_1}) - \sigma(M_{B_1}) = 1$.

It is proved in [6] that for any surface $F$ in $S^3$, its Goeritz form $G_F$ is isomorphic to $Q_M$, and so we can prove $\sigma(M_{B_1}) - \sigma(M_{B_1}) = 1$ by studying the Goeritz forms $G_{B_1}$ and $G_{B_1}$. Let $g_{B_1}$ be a representation matrix for $G_{B_1}$. Note that $B_1$ is lying both in $B_\emptyset$ and in $B_0$, and there exist representation matrices $g_{B_\emptyset}$ and $g_{B_0}$ for $G_{B_\emptyset}$ and $G_{B_0}$ such that

$$g_{B_\emptyset} = \begin{pmatrix} a & b \\ b & g_{B_1} \end{pmatrix} \quad \text{and} \quad g_{B_0} = \begin{pmatrix} a - 1 & b \\ b^\tau & g_{B_1} \end{pmatrix}$$

for some integer $a$ and row vector $b$. Moreover, there exist $\mathbb{Q}$-coefficient square matrices $p$ and $q$ which satisfy

1. $\det p = \det q = 1$,
2. the product $p(g_{B_1})p^\tau$ is a diagonal matrix, and
3. $q(g_{B_\emptyset})q^\tau = \begin{pmatrix} a' & 0 \\ 0 & p(g_{B_1})p^\tau \end{pmatrix}$ and $q(g_{B_0})q^\tau = \begin{pmatrix} a' - 1 & 0 \\ 0 & p(g_{B_1})p^\tau \end{pmatrix}$

for some rational number $a'$.

This implies that $\sigma(M_{B_\emptyset}) - \sigma(M_{B_1}) = 1$ if and only if $a' > 0$. We compare the determinant of $g_{B_\emptyset}$, $g_{B_0}$ and $g_{B_1}$ to prove $a' > 0$. It is known that for a 4-manifold $M$, the determinant of any representation matrix for $Q_M$ is equal to the order of $H_1(\partial M; \mathbb{Z})$. Hence we see that

$$\det(L_i) = |\det(g_{B_i})|$$
and
\[ \det(g_{B_0}) = a' \cdot \det(g_{B_1}) = \det(g_{B_0}) + \det(g_{B_1}). \]

Since \( \det(L) = \det(L_0) + \det(L_1) \), the above equalities imply that \( \det(g_{B_0}) \) and \( \det(g_{B_1}) \) have the same sign, and so \( a' = \frac{\det(g_{B_0})}{\det(g_{B_1})} + 1 \) is positive. Similarly, we can prove that \( M_{C_W} \) is negative definite.

**Proof of Theorem 3** Here we prove Theorem 3 for all quasi-alternating links; namely, we prove that any quasi-alternating link bounds surfaces \( P \) and \( N \) in \( B^4 \) whose double branched cover \( M_P \) and \( M_N \) are positive definite and negative definite respectively. We prove this assertion by induction on \( \det(L) \).

If \( L \) is a quasi-alternating link with \( \det(L) = 1 \), then \( L \) is the unknot and obviously 4-dimensionally alternating. Suppose that \( \det(L) = n > 1 \) and Theorem 3 holds for any quasi-alternating link with determinant less than \( n \). Then there exists a pair of resolutions \( L_0, L_1 \) for \( L \), such that \( \det(L) = \det(L_0) + \det(L_1) \). Since \( \det(L_0), \det(L_1) < n \), we can take spanning surfaces \( N_0 \) and \( P_1 \) for \( L_0 \) and \( L_1 \) in \( B^4 \) whose double branched cover are negative definite and positive definite respectively. Therefore, it follows from Lemma 4 that \( C_W \cup N_0 \) (and \( C_B \cup P_1 \)) is a spanning surface for \( L \) in \( B^4 \) whose double branched cover is negative definite (resp. positive definite). This completes the proof.

### 5. Proof of Proposition 1

In this section, we prove Proposition 1. Note that any slice knot is obviously 4-dimensionally alternating, and it follows from Theorem 3 that any quasi-alternating knot is 4-dimensionally alternating. Therefore, Proposition 1 follows from the following proposition.

**Proposition 6.** The knots 10\(_{139}\), 10\(_{152}\), 10\(_{154}\) and 10\(_{161}\) are 4-dimensionally alternating but not concordant to any quasi-alternating knot. For any knot \( K \) with 10 or fewer crossings except for the above four knots, if \( K \) is neither quasi-alternating nor slice, then \( K \) is not 4-dimensionally alternating.

Moreover, it is described in [5] that a knot \( K \) with 10 or fewer crossings is not quasi-alternating if and only if \( K \) is one of the following 14 knots
\[
\begin{align*}
8_{19}, 9_{42}, 9_{46}, 10_{124}, 10_{128}, 10_{132}, 10_{136}, \\
10_{139}, 10_{140}, 10_{145}, 10_{152}, 10_{153}, 10_{154}, \text{ and } 10_{161},
\end{align*}
\]
where 9\(_{46}\), 10\(_{153}\) and 10\(_{140}\) are slice knots. Taking these facts into consideration, we decompose Proposition 6 into the following three lemmas.

**Lemma 5.** If \( K \) is one of 8\(_{19}^*, 9_{12}^*, 10_{128}^*, \text{ and } 10_{136}^* \), then \( \frac{\sigma(K)}{2} - d(S^3_1(K)) = 1 \), and hence \( K \) is not 4-dimensionally alternating.

**Lemma 6.** If \( K \) is one of 10\(_{124}\), 10\(_{132}\) and 10\(_{145}\), then \( \Sigma(K) \) cannot bound any positive definite 4-manifold, and hence \( K \) is not 4-dimensionally alternating.

**Lemma 7.** The knots 10\(_{139}\), 10\(_{152}\), 10\(_{154}\) and 10\(_{161}\) are 4-dimensionally alternating but not concordant to any quasi-alternating knot.

We first prove Lemma 5.
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Proof of Lemma 5. Lemma 5 in the case of \(8 \ast 19\) and \(9 \ast 42\) are proved in [3] and [15] respectively. We prove the lemma for \(10 \ast 128\) and \(10 \ast 136\). It is easy to check that \(\sigma(10 \ast 128) = 6\) and \(\sigma(10 \ast 136) = 2\). To compute \(d(S^3_{3/1}(K))\), we use Ni-Wu’s \(V_k\)-sequence [10]. Here \(V_k\) is a \(\mathbb{Z}_{\geq 0}\)-valued concordance invariant for each \(k \in \mathbb{Z}_{\geq 0}\). In particular, it follows from [10, Proposition 1.6] that

\[
d(S^3_{3/1}(K)) = -d(S^3_{3/1}(K^*)) = 2V_0(K^*)
\]

We compute \(V_0\) of \(10 \ast 128\) and \(10 \ast 136\) by using the following proposition, which immediately follows from [14, Proposition 1.9]. Here we denote \(\mathbb{C}P^2\) with open 4-ball deleted by \(\text{punc}\, \mathbb{C}P^2\).

**Proposition 7.** Suppose that a knot \(K\) bounds a disk \(D\) in \(\text{punc}\, \mathbb{C}P^2\) such that \([D, \partial D] = n\gamma \in H_2(\text{punc}\, \mathbb{C}P^2, \partial(\text{punc}\, \mathbb{C}P^2); \mathbb{Z})\) for a generator \(\gamma\) and some odd integer \(n > 0\). Then we have

\[
V_0(K) = \frac{(n - 1)(n + 1)}{8}.
\]

As shown in Figure 5 and 6, we see that \(10 \ast 128\) and \(10 \ast 136\) bounds a disk in \(\text{punc}\, \mathbb{C}P^2\) with \(n = 3\) and \(n = 1\). By Proposition 7 we have \(V_0(10 \ast 128) = 1\) and \(V_0(10 \ast 136) = 0\). This completes the proof.

![Figure 5. 10\_128 bounds a disk in punc\, \mathbb{C}P^2 with n = 3](image)

![Figure 6. 10\_136 bounds a disk in punc\, \mathbb{C}P^2 with n = 1](image)

In order to prove Lemma 6 and Lemma 7 we use Akbulut’s method in [2] for describing a handle diagram for the double branched cover of \(B^4\) over any ribbon surface.

**Proof of Lemma 6.** We can verify that the boundary of the ribbon surfaces in Figure 7, Figure 8 and Figure 9 are \(10_1\), \(10_2\) and \(10_4\) respectively. By applying Akbulut’s method to these ribbon surfaces, we see that \(\Sigma(10_1) = S^3_{3/1}(3_1) = -S^3_{3/1}(3_1), \Sigma(10_2) = S^3_{5/2}(3_1) = -S^3_{5/2}(3_1)\) and \(\Sigma(10_4) = S^3_{-3}(5_2) = -S^3_{-3}(5_2)\).
It is proved in [11] that for any $r \in \mathbb{Q}_{>0}$, the manifold $S^3_r(3_1)$ bounds a negative definite definite 4-manifold if and only if $r \geq 4$. Hence neither $\Sigma(10_{124})$ nor $\Sigma(10_{132})$ can bound a positive definite 4-manifold.

Assume that $\Sigma(10_{145}) = -S^3(5_2)$ bounds a positive definite 4-manifold $M$, and let $C$ be the cobordism represented by the relative handle diagram in Figure 10. Since $C$ is negative definite and $\partial C = -S^3(5_2) \amalg S^3(3_1)$, the manifold $-M \cup C$ is a negative definite and has boundary $S^3(3_1)$. This contradicts the above result of [11], and hence $\Sigma(10_{145})$ cannot bound any positive definite 4-manifold. □

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure7.png}
\caption{a ribbon surface with boundary $10_{124}$}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure8.png}
\caption{a ribbon surface with boundary $10_{132}$}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure9.png}
\caption{a ribbon surface with boundary $10_{145}$}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure10.png}
\caption{a negative definite cobordism from $S^3(5_1)$ to $S^3(3_1)$}
\end{figure}
Finally, we prove Lemma 7. To prove Lemma 7, we use $H(2)$-move, which is a deformation of link diagram shown in Figure 11. If a diagram $D_1$ for a knot $K_1$ is deformed into a diagram $D_2$ for a knot $K_2$ by an $H(2)$-move, then we can naturally obtain a non-orientable cobordism $C$ from $K_1$ to $K_2$ in $S^3 \times [0, 1]$ such that $b_2(M_C) = 1$. Moreover, if we denote the writhe of a knot diagram $D$ by $w(D)$, then the equality $e(C) = -(w(D_2) - w(D_1))$ holds, and hence Gordon-Litherland’s formula induces the equality

$$\sigma(M_C) = (\sigma(K_2) - \sigma(K_1)) + \frac{1}{2}(w(D_2) - w(D_1)).$$

We say that an $H(2)$-move is positive (negative) if $\sigma(M_C) = 1$ (resp. $\sigma(M_C) = -1$).

![Figure 11. $H(2)$-move](image)

Proof of Lemma 7. Let $\tau$ be Ozsváth-Szabó’s $\tau$-invariant. It follows in [4] that the knots $10_{139}$, $10_{152}$, $10_{154}$ and $10_{161}$ do not satisfy $\tau(K) = -\frac{\sigma(K)}{2}$, and hence they are not concordant to any quasi-alternating knot. We prove that these knots satisfy $b^\pm(K) = 0$. Here we consider the case of $10_{139}$. It immediately follows from the lower diagrams in Figure 12 that $b^+(10_{139}) = 0$. Moreover, the knot $10_{139}$ is obtained from $3^*_1$ by the positive $H(2)$-move shown in the upper diagrams of Figure 12. Since $3^*_1$ bounds a surface in $B^4$ whose double branched cover is positive definite, we have $b^-(10_{139}) = 0$. Similarly, Figure 13, Figure 14 and Figure 15 shows that $10_{152}$, $10_{154}$ and $10_{161}$ satisfy $b^\pm(K) = 0$. □

References

[1] C. Adams, J. F. Brock, J. Bugbee, T. Comar, K. A. Faigin, A. M. Huston, A. M. Joseph and D. Pesikoff, Almost alternating links. Topology Appl. 46 (1992), no. 2, 151–165.
[2] S. Akbulut, 4-manifolds. Oxford Graduate Texts in Mathematics, 25. Oxford University Press, Oxford, 2016.
[3] J. Batson, Nonorientable slice genus can be arbitrarily large. Math. Res. Lett. 21 (2014), no. 3, 423–436.
[4] J. C. Cha and C. Livingston, KnotInfo: Table of Knot Invariants. http://www.indiana.edu/~knotinfo/ (September 17, 2017).
[5] A. Champanerkar and I. Kofman, Twisting quasi-alternating links. Proc. Amer. Math. Soc. 137 (2009), no. 7, 2451–2458.
[6] C. McA. Gordon and R. A. Litherland, On the signature of a link. Invent. Math. 47 (1978), 53–69.
[7] J. E. Greene, Alternating links and definite surfaces. Duke Math. J. 166 (2017), no. 11, 2133–2151.
[8] A. Kawauchi, On alternation numbers of links. Topology Appl. 157 (2010), no. 1, 274–279.
[9] H. Murakami and A. Yasuhara, Four-genus and four-dimensional clasp number of a knot. Proc. Amer. Math. Soc. 128 (2000), no. 12, 3693–3699.
[10] Y. Ni and Z. Wu, Cosmetic surgeries on knots in $S^3$. J. Reine Angew. Math. 706 (2015), 1–17.
[11] B. Owens and S. Strle, Dehn surgeries and negative-definite four-manifolds. Selecta Math. (N.S.) 18 (2012), no. 4, 839–854.
[12] P. Ozsváth and Z. Szabó, Knot Floer homology and the four-ball genus. Geom. Topol. 7 (2003), 615–639.
[13] P. Ozsváth and Z. Szabó, On the Heegaard Floer homology of branched double-covers. Adv. Math. 194 (2005), no. 1, 1–33.
[14] K. Sato, A full-twist inequality for the $\nu^+$-invariant. arXiv:1706.02820
[15] K. Sato and M. Tange, Non-orientable genus of a knot in punctured $CP^2$. Tokyo J. Math. 38 (2015), no. 2, 561–574.
**Figure 14.** ribbon surfaces with boundary $10_{154}$

**Figure 15.** ribbon surfaces with boundary $10_{161}$