Rotating Parker wind

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ABSTRACT

We reconsider the structure of thermally driven rotating Parker wind. Rotation, without magnetic field, changes qualitatively the structure of the subsonic region: solutions become non-monotonic and do not extend to the origin. Contrary to previous claims, solutions never have two critical points. There is also a critical angular velocity of the central star \( \Omega_{\text{crit}} = \frac{GM_s}{(2\sqrt{2}c_sR_b^2)} \) (where \( M_s \) and \( R_b \) are mass and radius of the central star, \( c_s \) is the sound speed in the wind) beyond which the flow does not have a critical point. For disk winds (when the base of the wind rotates with Keplerian velocity) launched equatorially the coronal sound speed should be smaller than \( \approx 0.22v_K \) in order to connect to the critical curve (\( v_K \) is the Keplerian velocity at a given location on the disk).

1. Introduction

The Parker model of Solar wind (Parker 1965; Bondi 1952) as well as its MHD extension (Weber & Davis 1967) are at the core of the stellar wind theory (Lamers & Cassinelli 1999). Of particular interests to us here are thermally launched winds from rotating central objects - a star or a disk.

This is a classical topic in stellar wind theory, that has not been considered to the best of our knowledge. Previously, a number of large scale 2D models of thermally-driven winds were constructed (e.g. Fukue & Okada 1990, Clarke & Alexander 2016, Waters & Proga 2012), but the basic equatorial flow of rotating thermally driven winds has not been considered. Goossens (2003) (see also Keppens & Goedbloed 1999) provided incorrect interpretation of the results, that there are two critical points in flow. In fact, there is one or none critical points, as we demonstrate below. Skinner & Ostriker (2010) calculated numerically the structure of the rotating Parker winds - our analytical results seem to be with agreement with their work. Our analytical results also provide quantitative estimates for various wind regimes.
2. Rotating Parker wind

Consider a rotating star that launches thermally driven wind from its surface. The surface of a star may not have a clear physical definition - for mathematical purpose we define a surface at \( R_b \) as the base of the wind, which rotates with a given angular velocity \( \Omega \). In the frame rotating with the star, in the equatorial plane, and assuming axially symmetric flow, the governing equations are the Euler equation

\[
v_r v_r' - 2\Omega v_\phi - \frac{v_\phi^2}{r} = r\Omega^2 + \Phi'(r) - c_s^2 \partial_r \ln \rho
\]

\[
2\Omega r + (v_\phi r)' = 0 \tag{1}
\]

and mass conservation

\[
r^2 \rho v_r = \text{Constant} \tag{2}
\]

Above \( \Phi = GM_*/r \) is the gravitational potential and other notations are standard. We assumed isothermal equation of state - polytropic equations of state introduce only mild modifications to the structure of the solutions for most polytropic indices of interest (e.g. Lamers & Cassinelli 1999).

In the rotating frame, on the surface of the star \( v_\phi(R_b) = 0 \), hence

\[
v_\phi = \frac{\Omega R_b^2}{r} - r\Omega \tag{3}
\]

(In the observer frame only the first term remains, the main equation (1) remains unchanged.)

The radial component becomes

\[
\frac{v_r'}{v_r} = \frac{2c_s^2 r^2 + r^3 \Phi'(r) + R_0^4 \Omega^2}{v_r^2 - c_s^2} \tag{4}
\]

This is the generalization of Parker-Bondi equation, it is the main equation to be studied in the present paper. It is of critical point-type behavior. It differs from the classical Parker-Bondi case by the term \( \Omega^2 \) in the numerator. With somewhat different notations, it agrees with Goossens (2003), Eq. (6.44), see also Friend & Abbott (1986).

Introducing radial Mach number \( M_s = v_r/c_s \), Parker-Bondi radius \( r_0 = GM_*/(2c_s^2) \) and \( R_\Omega = \sqrt{2R_0^2 \Omega}/c_s \), Eq. (4) becomes

\[
(1 - M_s^2) \partial_r \ln M_s = -\frac{2r(r - r_0) + R_\Omega^2}{r^3} \tag{5}
\]

Judging by the fact that the righthand side is a quadratic equation, Goossens (2003) concluded that there are two critical points. Below we show that this is not the case. The solution has either 1 or no critical points.

There are two points where the right hand side of Eq. (5) is zero:

\[
r_{\pm} = \frac{1}{2} \left( r_0 \pm \sqrt{r_0^2 - R_\Omega^2} \right) \tag{6}
\]
(tidy form of these relations motivated our choice of normalization of $R\Omega$). Only plus sign in (6) corresponds to the critical point, the sub-to-supersonic transition, $M_s = 1$, see Fig. 1 and Fig. 2.

Fig. 1.— Phase diagram for Parker wind with rotation. Plotted is sonic Mach number $M_s$ versus radius normalized to Parker-Bondi radius $r_0$; parameter $R\Omega = 0.75r_0$. Dashed line is the conventional isothermal Parker wind. Most important modifications in the rotating case is that the solution cannot extend to $r \to 0$. The large-$r$ asymptotic is not affected much. The value of $r_{min}$ is plotted in Fig. 3; the Mach numbers at $r_{min}$ are plotted in Fig. 4; evolution of density is plotted in Fig. 5.

General integral of (5) is (Bernoulli function)

$$B = \frac{M_s^2}{2} - \ln(M_s) - \frac{1}{2} \left( -\frac{R\Omega^2}{2r^2} + \frac{4r_0}{r} + 4\ln(r) \right)$$  \hspace{1cm} (7)

The differential of the Bernoulli function vanishes only at $r_+$ - the only critical point in the flow.
Fig. 2.— Same as Fig. 1 but for different $R_\Omega = 0.1, 0.2...1$.

The critical curves are given by

$$-\frac{M_s^2}{2} + \ln M_s = -\frac{1}{2} + \left(\frac{1}{r^2} - \frac{1}{r_+^2}\right) \frac{R_\Omega^2}{4} + 2 \left(\frac{1}{r_+} - \frac{1}{r}\right) r_0 + 2 \ln \left(\frac{r_+}{r}\right)$$

Near the critical point $M_s = 1$, $r = r_+$,

$$M_s = 1 \pm \sqrt{2 - \frac{r_0}{r_+} \left(1 - \frac{r}{r_+}\right)}$$

(the point $r_0 = 2r_+$ is when the critical point disappears).

At $r \gg r_+$ the Mach numbers evolve according to $M \approx 2\sqrt{\ln(r/r_+)}$ for supersonic branch and $M \approx (r/r_+)^{-2}$ for the subsonic branch, similar to the classical case of Parker wind.

The second point where the right hand side of Eq. (5) vanishes corresponds to points $r_-$, where $\partial_r M_s = 0$. At the points $r_-$ the Mach number $M_s \neq 1$, see Fig. 4.

Finally, setting $M_s = 1$ gives two roots: one is $r_+$ - the critical point of the flow; another
defines the minimal $r_{\text{min}}$ for which the model is applicable

$$-rac{R_{\Omega}^2}{4r_{\text{min}}^2} - 2\ln \left( \left( 1 + \sqrt{1 - \frac{R_{\Omega}^2}{r_0^2}} \right) \frac{r_0}{2r_{\text{min}}} \right) + \frac{2r_0}{r_{\text{min}}} + \frac{R_{\Omega}^2}{r_0^2 \left( 1 + \sqrt{1 - \frac{R_{\Omega}^2}{r_0^2}} \right)^2} - \frac{4}{1 + \sqrt{1 - \frac{R_{\Omega}^2}{r_0^2}}} = 0,$$

(10)

see Fig. 3. At $r_{\text{min}}$ we have $\partial_r M_s = \infty$. So, neither $r_-$ or $r_{\text{min}}$ are critical points (the phase curve is smooth and non-self-intersecting at those points).

For each given $R_{\Omega}$ the critical curve reaches at $r_-$ some maximal and minimal Mach numbers $M_{s,\text{max/min}}$, plotted in Fig. 4. Mach number at $r_-$ are not unity, except in the case $R_{\Omega} = r_0$, when points $r_\pm$ coincide.

Overall phase diagrams are plotted in Fig. 5. For any $0 < R_{\Omega} < r_0$ there are closed phase curves confined to $r_{\text{min}} < r < r_+$.

The density along the critical curves of Fig. 1 are plotted in Fig. 6.

Given the velocity and density, Eq. 6, one can calculate the mass loss rate. It cannot be
Fig. 4.— Maximal and minimal values of Mach number in the subsonic region along the critical curve as function of $R_\Omega$.

compared simply to the case of non-rotating winds. Usually, mass loss rate is calculated for a given base radius $R_b$ and local density - then the critical curve fixes the velocity. For rotating case, first, there is a limit on the radius $R_b > r_{\text{min}}$, so such a procedure may not work, and, second, the similar procedure will give some different launching velocity at the same radius.

3. Constraints on the parameters

3.1. Existence of the critical point

There are several constraints on the parameters. Let’s introduce two dimensionless parameters

$$\eta_\Omega = \frac{\Omega}{\Omega_K} \leq 1$$
Fig. 5.— Phase curves on $r - M_s$ diagram for $R_\Omega = 0.9r_0$ (left panel) and $R_\Omega = r_0$ (right panel). Critical curves are highlighted.

Fig. 6.— Density along the critical curves (plotted is the value of $r_0^2/(M_s r^2)$) for $R_\Omega = 0.75r_0$ and $R_\Omega = r_0$. At large radii larger densities correspond to the subsonic branch, where density reached a constant (since in that regime $M_s \propto r^{-2}$).

\[ \eta_s = \frac{c_s}{R_0 \Omega K} \]  
\[ \frac{R_\Omega}{r_0} = 2\sqrt{2} \eta_R \eta_s \]  
\[ c_s = \frac{\eta_s v_K}{\sqrt{2}} \]
For the points $r_\pm$ to exist, it is required that $R_\Omega < r_0$, which translates to

$$\Omega \leq \Omega_{\text{crit},1} = \frac{1}{2\sqrt{2}} \frac{GM}{c_s R_b^2} = \frac{1}{2\sqrt{2}} \frac{R_b \Omega_K^2}{c_s} = \frac{1}{2\sqrt{2} \eta_s} \Omega_K$$

$$2\sqrt{2} \eta_0 \eta_s \leq 1$$

(13)

$\Omega_K$ is the break-up frequency at the equator. Thus, there is a range $1/(2\sqrt{2}) < \eta_s < 1$ where $\Omega_{\text{crit}} < \Omega_K$. In physical units, this requires

$$c_s \geq \frac{1}{2\sqrt{2}} \sqrt{\frac{GM}{R_b}}$$

(14)

This is four times less that the escape velocity without rotation $\sqrt{2GM/R_b}$. Value $\eta_0 = \sqrt{\frac{1}{2}}$ corresponds to the hydrodynamic energy parameter (HEP) of Waters & Proga (2012).

For larger sound speed there is no critical point and the flow always remains supersonic or subsonic depending on the conditions at the launch location $R_b$.

For $R_\Omega = r_0$ we have

$$(1 - M_s^2) \frac{\partial_r M_s}{M_s} = -\frac{(2r - r_0)^2}{2r^3}$$

$$\ln (M_s) - \frac{M_s^2}{2} = \frac{5}{2} + \frac{r_0^2}{4r_{\text{min}}^2} - \frac{2r_0}{r_{\text{min}}} + 2 \ln \left( \frac{r_0}{2r_{\text{min}}} \right)$$

(15)

Thus, the numerator does not change sign - the outflow must start sonic at the critical point $r = r_0/2$. For large $R_\Omega$ the wind must start supersonic outside of the critical point in order to be supersonic at infinity, see Fig. 5, right panel.

### 3.2. Connection to the base

The radius of the star $R_b$ cannot be smaller than $r_{\text{min}}$ for the model to be applicable. If the radius of the star $R_b$ is larger than $r_-$, then the wind continuously accelerate. The value of $r_{\text{min}}$ is very close to analytical $r_-$. The requirement $R_b \geq r_-$ translates to $\Omega < \Omega_{\text{crit},2}$,

$$\Omega_{\text{crit},2} = \sqrt{\frac{\Omega_K^2}{2} - 2(c_s/R_b)^2}$$

$$\frac{c_s}{R \Omega_K} \leq \frac{1}{\sqrt{2}} \sqrt{1 - (\Omega/\Omega_K)^2}$$

(16)

where $\Omega_K = \sqrt{GM_s/R_b^3}$ is Keplerian angular velocity at equator, see Fig. 7.

There is also a special point where (13) and (16) match

$$\eta_0 = \frac{1}{\sqrt{2}}$$

$$\eta_s = \frac{1}{2}$$

$$R_\Omega = r_0, \Omega_{\text{max}} = \Omega_K/2, r_+ = r_0/2$$

(17)
Fig. 7.— Allowed regions in the $\eta_s - \eta_\Omega$ plane. Even for $\Omega = 0$ the values of $c_s$ (and $\eta_s$) are limited by the requirement of sub-Keplerian rotation.

In this case the wind starts sonically right from the surface $R_b$ and evolves according to

$$-\frac{M_s^2}{2} + \ln M_s = \frac{5}{2} - 4 \frac{R_b}{r} + \left( \frac{R_b}{r} \right)^2 + 2 \ln \frac{R_b}{r}$$

(18)

3.3. Equatorial disk winds

One of the possible applications of the model is for thermal winds launched by disks (e.g. [Weber & Davis 1967; Konigl & Pudritz 2000; Blandford & Payne 1982; Waters & Proga 2018]). Assuming thin disk, so that the flow starts from a Keplerian-moving base. In this case then $\Omega = \Omega_K$, $\eta_\Omega = 1$, $R_b$ means the local radius of the disk, and $c_s$ refers to the sound speed in the corona above the disk. Our parameters become

$$R_\Omega = \sqrt{2GM_sR_b/c_s} = 2\sqrt{2\eta_s}r_0$$
Thus, to have a critical point (to have a subsonic region) it is required that \( \eta_s < 1/(2\sqrt{2}) = 0.354 \), or in physical units,

\[
c_s \leq \frac{R_b \Omega_K}{2 \sqrt{2}} = \frac{v_K}{4},
\]

For larger coronal sound speed there is no critical point.

There is also the condition that \( r_{\text{min}} \) should be smaller than \( R_b \) for a flow to accelerate outwards. Here we cannot approximate \( r_{\text{min}} \) as \( \sim r_{-} \) (this would give only a trivial solution \( \eta_s = 0 \)). For the Keplerian disk \( R_b = R_{\text{b}}^2/(4r_0) \) and, using the calculation of \( r_{\text{min}} \), Fig. 3, we find that in order to start subsonically the coronal sound speed should satisfy

\[
\eta_s \leq 0.3190,
\]

or, in physical units

\[
c_s < 0.3190R_b \Omega_K = 0.225v_K
\]

(for larger coronal sound speeds the critical curve does not reach a given point on the disk).

4. Discussion

In this paper we consider a highly idealized problem of thermal wind launched from a rotating object, a star or a disk. Our analytical results seem to be in agreement with previous numerical works by Skinner & Ostriker (2010); Waters & Proga (2012). In particular, Waters & Proga (2012) argued for a single critical point and also found regimes of non-continuous accelerations, “enthalpy deficit regime”.

Our results differ from the case of radiation-driven rotating winds (Friend & Abbott 1986). In that case, e.g. the critical point moves outward due to rotation, while the terminal velocity is smaller. In our case the critical point moves inward, while at each radius the velocity is higher than in the non-rotating case. The critical conditions in line-driven winds are qualitatively different from the pressure-driven winds, (e.g. Lamers & Cassinelli 1999).

It is of interest to discuss the cases of high rotation rates/high sound speeds when the model breaks down. There are two constraints: (i) connection to the base, \( r_{\text{min}} \geq R_b \); (ii) existence of critical points, \( R_{\Omega} < r_0 \). If the condition (i) is broken, then the only way for a subsonic flow to connect to infinity is through unphysical “breeze” solution (it is subsonic, but typically has much larger pressure that prevents a smooth match to the interstellar medium). Similarly, if \( R_{\Omega} > r_0 \) and the flow is subsonic at the base, the breeze solution is the only choice. These cases are somewhat different from the classical Parker model, where the critical subsonic curve extends to \( M_s = 0 \) as
$r \to 0$. In our case all subsonic breeze solutions connect to supersonic non-critical solutions at $M_s = 1$. In this regimes, most likely, the flow either becomes non-stationary and/or may form shocks.

The generalization to the polytropic case should be straightforward and follow the classic Parker’s case: instead of continuous acceleration, a supersonic branch of the flow would reach a constant velocity.

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