Morphing Planar Graph Drawings with Unidirectional Moves

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Abstract

Alamdari et al. [1] showed that given two straight-line planar drawings of a graph, there is a morph between them that preserves planarity and consists of a polynomial number of steps where each step is a linear morph that moves each vertex at constant speed along a straight line. An important step in their proof consists of converting a pseudo-morph (in which contractions are allowed) to a true morph. Here we introduce the notion of unidirectional morphing step, where the vertices move along lines that all have the same direction. Our main result is to show that any planarity preserving pseudo-morph consisting of unidirectional steps and contraction of low degree vertices can be turned into a true morph without increasing the number of steps. Using this, we strengthen Alamdari et al.’s result to use only unidirectional morphs, and in the process we simplify the proof.

1 Introduction

Intuitively a morph can be thought of as a continuous deformation between structures. Morphs have been of interest in computer graphics [11]. Morphing can be used in the areas of medical imaging and geographical information systems [5, 4] for reconstructing a surface given a sequence of parallel slices. Here we are interested in morphs restricted to graph drawings.

A morph between two planar drawings \( \Gamma_1 \) and \( \Gamma_2 \) of a graph \( G \) is a continuous family of drawings indexed by time \( t \in [0, 1] \) where the drawing at time \( t = 0 \) is \( \Gamma_1 \) and the drawing at time \( t = 1 \) is \( \Gamma_2 \). A morph preserves planarity if all intermediate drawings are planar. Of necessity, this means that \( \Gamma_1 \) and \( \Gamma_2 \) are topologically equivalent, i.e., have the same faces and the same outer face. A morph preserves straight-line planarity if all intermediate
drawings are straight-line planar, in which case the morph is a continuous movement of the vertices of the graph drawing.

In 1944 Cairns [9] proved the existence of a straight-line planarity preserving morph between any two topologically equivalent straight-line planar drawings of a triangulated graph. Cairns’ proof is constructive but the resulting morph takes an exponential number of steps. Thomassen [15] extended the result to general straight-line planar drawings by augmenting both drawings to isomorphic triangulations, later called “compatible” triangulations [3]. For a graph of \( n \) vertices, compatible triangulations can be found in polynomial time and have size \( O(n^2) \) and this bound is tight. Floater and Gotsman [10] gave a polynomial time algorithm using Tutte’s graph drawing algorithm [16], but in their morph the trajectories of the vertices are complicated.

Morphs preserving other aspects have also been studied. For example, the method in [10] was generalized by Gotsman and Surazhsky [12] to obtain a morph between two simple polygons that preserves simplicity. The existence of intersection-free morphs of maximal planar graphs has been established for certain types of sphere drawings by Kobourov and Landis [13]. Biedl et al. present an algorithm to morph between any two planar and orthogonal graph drawings while preserving planarity and orthogonality using a polynomial number of linear morphing steps in [8].

Lubiw and Petrick [14] showed that given two planar drawings of a graph there exists a planarity preserving morph that consists of polynomially many linear morphs where edges are allowed to bend. After this result was proven it was still unknown whether it was possible to morph between drawings of maximal planar graphs while preserving planarity in a polynomial number of steps. Recently, Alamdari et al. [1] gave an algorithm, based on Cairns’ approach, that solves the problem using \( O(n^2) \) linear morphs, a morph that moves each vertex along a straight line at uniform speed. Using compatible triangulations this gives a morph of \( O(n^4) \) steps for general planar graphs.

In this paper we improve the result of Alamdari et al. on morphing triangulations in two ways: (1) we give a simpler proof; and (2) our elementary steps are unidirectional morphs. A unidirectional morph is a linear morph where every vertex moves parallel to the same line, i.e. there is a line \( L \) with unit direction vector \( \ell \) such that each vertex \( v \) moves at constant speed from initial position \( v_0 \) to position \( v_0 + k_v \ell \) for some \( k_v \in \mathbb{R} \). Note that \( k_v \) may be positive or negative and that different vertices may move different amounts along direction \( \ell \). We call this an \( L \)-directional morph. Our main contribution is to show that any planar preserving pseudo-morph consisting of unidirectional steps and contraction of low degree vertices can be turned into a true morph without increasing the number of steps. Using this and following the approach of Alamdari et al. we obtain a morph which is simpler and requires the same number of morphing steps, namely \( O(n^2) \). Very recently, our result was used by Angelini et al. [2] to give an improved morphing algorithm that uses only \( O(n) \) unidirectional steps.

In the remainder of this section we describe the high-level idea of our result.

The existence proof of Cairns works by successively contracting a vertex of degree at most 5 to a neighbour, i.e. moving the vertex along one of its incident edges until it reaches
the other endpoint of the edge. (The relevance of low degree is that a vertex of degree at most 5 always has a neighbour to which it can be contracted while preserving planarity.) Each such step is a unidirectional morph, for the trivial reason that only one vertex moves. The number of steps is exponential. The result is not a true morph since vertices become coincident, but Cairns argues that each vertex can be moved close to, but not coincident with, the target vertex. This fix causes a further exponential increase in the number of steps.

Alamdari et al. improved the number of morphing steps to a polynomial number using the same two-phase approach. The first phase finds a pseudo-morph which is defined as a sequence of the following kinds of steps:

- a linear morph
- a contraction of a vertex $p$ to another vertex, followed by a pseudo-morph between the two reduced drawings, and then an “uncontraction” of $p$.

The number of steps in a pseudo-morph is defined to be the number of linear morphs plus the number of contractions and uncontractions. Alamdari et al. give a pseudo-morph of $O(n^2)$ steps.

In the second phase they convert the pseudo-morph to a true morph that avoids coincident vertices. This requires a somewhat intricate geometric argument that instead of contracting a vertex $p$ to a neighbour, it is possible to move $p$ close to the neighbour and keep it close during subsequent morphing steps without increasing the number of steps.

Here we use a different approach for the second phase, which results in a simpler proof and uses only unidirectional morphs. This is in Section 3. To obtain our strengthened version of Alamdari et al.’s result, we use essentially the same pseudo-morph for the first stage. We must verify that unidirectional morphs suffice. This is described in Section 2.

We use the following notation. If $\Gamma_1, \ldots, \Gamma_k$ are straight-line planar drawings of a graph, then $\langle \Gamma_1, \ldots, \Gamma_k \rangle$ denotes the morph that consists of the $k - 1$ linear morphs from $\Gamma_i$ to $\Gamma_{i+1}$ for $i = 1, \ldots, k - 1$.

## 2 A pseudo-morph with unidirectional morphing steps

Alamdari et al. [1] give a pseudo-morph of $O(n^2)$ steps to go between any two topologically equivalent straight-line planar drawings of a triangulated graph on $n$ vertices. In this section we show that their pseudo-morph can be implemented with unidirectional morphs. They show that the only thing that is needed is a solution to the following problem using $O(n)$ linear morphs:

**PROBLEM 3.2. (4-GON CONVEXIFICATION)** Given a triangulated graph $G$ with a triangle boundary and a 4-gon $abcd$ in a straight-line planar drawing of $G$ such that neither $ac$ nor $bd$ is an edge outside of $abcd$ (i.e., $abcd$ does not have external chords), find a pseudo-morph so that $abcd$ becomes convex.

The main idea is to use Cairns’ approach: find a low-degree vertex, contract it to a neighbour, and recurse on the resulting smaller graph. Each such contraction is a unidirectional
morph. This approach works so long as there is a low-degree vertex that is not a problematic vertex $p$ defined as follows:

1. $p$ is a vertex of the boundary triangle $z_1, z_2, z_3$.
2. $p$ is a vertex of the 4-gon $abcd$ and is not on the boundary.
3. $p$ is outside the 4-gon, is not on the boundary, has degree at most 5, and is adjacent to both $a$ and $c$, and either $a$ or $c$ is in the kernel of the polygon formed by the neighbours of $p$. (In this case contracting $p$ to $a$ or $c$ would create the edge $ac$ outside the 4-gon.)

Alamdari et al. show how to handle each type of problematic vertex. We must go through the cases and argue that unidirectional morphs suffice in each case.

Problematic vertices of type (2) and type (3) are handled (in their Sections 4.1 and 4.3) by moving a single vertex at a time either by contracting or by moving a vertex very close to another vertex. Moving a single vertex is a unidirectional morph so these cases are done.

It remains to consider problematic vertices of type (1) which they do in Section 4.2. To handle this case they use an operation where one vertex of a triangle moves along a straight line and the other vertices inside the triangle follow along linearly. We will show that the motion is in fact unidirectional. Because we will need it later on, we will consider a more general situation where all three vertices of the triangle undergo a unidirectional morph.

**Lemma 1.** Let $a, b, c$ be the vertices of a triangle and let $x$ be a point inside the triangle defined by the convex combination $\lambda_1 a + \lambda_2 b + \lambda_3 c$ where $\sum \lambda_i = 1$ and $\lambda_i \geq 0$. If $a$, $b$, and $c$ move linearly in the direction of the vector $\vec{d}$ then so does $x$.

**Proof.** Suppose the morph is indexed by $t \in [0, 1]$ and that the positions of the vertices at time $t$ are $a_t, b_t, c_t, x_t$. Suppose that $a$ moves by $k_1 \vec{d}$, and $b$ moves by $k_2 \vec{d}$, and $c$ moves by $k_3 \vec{d}$. Thus $a_t = a_0 + tk_1 \vec{d}$ and etc. Then

$$x_t = \lambda_1 a_t + \lambda_2 b_t + \lambda_3 c_t = \lambda_1 a_0 + \lambda_2 b_0 + \lambda_3 c_0 + t(\lambda_1 k_1 + \lambda_2 k_2 + \lambda_3 k_3) \vec{d} = x_0 + tk \vec{d}$$

where $k = \lambda_1 k_1 + \lambda_2 k_2 + \lambda_3 k_3$. Thus $x$ also moves linearly in direction $\vec{d}$. 

Alamdari et al. show that to handle problematic vertices of type (1) it suffices to handle the following two cases. We repeat their arguments, adding details of exactly how Lemma 1 gives unidirectional morphs.

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**Figure 1:** A 4-gon $abcd$. 

A. There is a boundary vertex, say $z_1$, of degree 3. See Figure 2(a).

Then $z_1$ must have a neighbour $y$ that is adjacent to $z_2$ and $z_3$. If $abcd$ lies entirely inside the triangle $T = yz_2z_3$ then we recursively morph the subgraph contained in $T$. Otherwise $abcd$ must include at least one triangle outside $T$. It cannot have both triangles outside $T$ because of the assumption that there is no edge $ac$. Thus we can assume without loss of generality that $abcd$ consists of triangle $z_1yz_2$ and an adjacent triangle inside $T$, see Figure 2(b). The solution is to move $y$ towards $z_1$ to directly convexify $abcd$. As $y$ is moved, the contents of triangle $T$ follow along linearly. By Lemma 1 this is a unidirectional morph.

![Figure 2: A boundary vertex of degree 3.](image)

B. All three boundary vertices have degree 4. See Figure 3(a).

In this case there must exist an internal triangle $T = y_1y_2y_3$ containing all the internal vertices with $y_i$ adjacent to $z_j$ for $j \neq i, i, j \in \{1, 2, 3\}$. If $abcd$ lies entirely inside $T$ then we recursively morph the subgraph contained in $T$. If $abcd$ lies entirely outside $T$ then without loss of generality $abcd$ is $z_2z_1y_2y_3$. The solution is to move $y_2$ towards $z_1$ to directly convexify the 4-gon. As $y_2$ is moved, the contents of triangle $T$ follow along linearly. By Lemma 1 this is a unidirectional morph.

The final possibility is that $abcd$ consists of one triangle outside $T$ and one triangle inside $T$. We may assume without loss of generality that $abcd$ consists of triangle $z_1y_3z_2$ and an adjacent triangle inside $T$, see Figure 3(b). We will convexify the 4-gon $z_1y_2y_1y_3$, which will necessarily also convexify $abcd$. The solution is to move $y_1$ towards $z_2$ to convexify the 4-gon while the contents of triangle $T$ follow along linearly. By Lemma 1 this is a unidirectional morph.

This completes the argument that the pseudo-morph of Alamdari et al. can be implemented with unidirectional morphs.
Figure 3: All three boundary vertices of degree 4.

3 Avoiding coincident vertices

In this section we describe our key lemma for converting a pseudo-morph to a true morph that avoids coincident vertices.

Lemma 2. Let $\mathcal{M}$ be a pseudo-morph between drawings $\Gamma_1$ and $\Gamma_2$ consisting of $k$ planar unidirectional steps that acts on a triangulated planar graph $G$. If only vertices of degree at most 5 are contracted in $\mathcal{M}$, then there exists a planar morph consisting of $k$ unidirectional steps from $\Gamma_1$ to $\Gamma_2$.

We now outline the proof of Lemma 2 by again following the approach of Alamdari et al. Suppose the pseudo-morph consists of the contraction of a non-boundary vertex $p$ of degree at most 5 to a neighbour $a$, followed by a pseudo-morph $\mathcal{M}$ of the reduced graph and then an uncontraction of $p$. The pseudo-morph $\mathcal{M}$ consists of unidirectional morphing steps and by induction we can convert it to a morph $M$ that consists of the same number of unidirectional morphing steps. We will show how to modify $M$ to $M^p$ by adding $p$ and its incident edges back into each drawing of the morph sequence. To obtain the final morph, we replace the contraction of $p$ to $a$ by a unidirectional morph that moves $p$ from its initial position to its position at the start of $M^p$, then follow the steps of $M^p$, and then replace the uncontraction of $p$ by a unidirectional morph that moves $p$ from its position at the end of $M^p$ to its final position. The result is a true morph that consists of unidirectional morphing steps and the number of steps is the same as in the original pseudo-morph.

Thus our main task is to modify a morph $M$ to a morph $M^p$ by adding a vertex $p$ of degree at most 5 and its incident edges back into each drawing of the morph sequence, maintaining the property that each step of the morph sequence is a unidirectional morph. It suffices to look at the polygon $P$ formed by the neighbours of $p$. We know that $P$ has a vertex $a$ that remains in the kernel of $P$ throughout the morph. We will place $p$ near $a$. We separate into the cases where $P$ has 3 or 4 vertices, which are quite easy, and the case where $P$ has 5 vertices, which is more involved. The following two lemmas handle these two cases, and together strengthen Lemma 5.2 of [1] by adding the unidirectional condition.
**Lemma 3.** Let $P$ be a $\leq 4$-gon and let $\Gamma_1, \ldots, \Gamma_k$ be straight-line planar drawings of $P$ such that each morph $\langle \Gamma_i, \Gamma_{i+1} \rangle, i = 1, \ldots, k - 1$ is unidirectional and planar, and vertex $a$ of $P$ is in the kernel of $P$ at all times during the whole morph $\langle \Gamma_1, \ldots, \Gamma_k \rangle$. Then we can augment each drawing $\Gamma_i$ to a drawing $\Gamma_i^p$ by adding vertex $p$ at some point $p_i$ inside the kernel of the polygon $P$ in $\Gamma_i$ and adding straight line edges from $p$ to each vertex of $P$ in such a way that each morph $\langle \Gamma_i^p, \Gamma_{i+1}^p \rangle$ is unidirectional and planar.

**Proof.** If $P$ is a triangle then by Lemma 1 we can place $p$ at a fixed convex combination of the triangle vertices in all the drawings $\Gamma_i$.

If $P$ is a 4-gon $abcd$ then the line segment $ac$ also stays in the kernel, so we can place $p$ at a fixed convex combination of $a$ and $c$ in all the drawings $\Gamma_i$ (using the degenerate version of Lemma 1 where the triangle collapses to a line segment).

**Lemma 4.** Let $P$ be a 5-gon and let $\Gamma_1, \ldots, \Gamma_k$ be straight-line planar drawings of $P$ such that each morph $\langle \Gamma_i, \Gamma_{i+1} \rangle, i = 1, \ldots, k - 1$ is unidirectional and planar, and vertex $a$ of $P$ is in the kernel of $P$ at all times during the whole morph $\langle \Gamma_1, \ldots, \Gamma_k \rangle$. Then we can augment each drawing $\Gamma_i$ to a drawing $\Gamma_i^p$ by adding vertex $p$ at some point $p_i$ inside the kernel of the polygon $P$ in $\Gamma_i$ and adding straight line edges from $p$ to each vertex of $P$ in such a way that each morph $\langle \Gamma_i^p, \Gamma_{i+1}^p \rangle$ is unidirectional and planar.

The proof of Lemma 4 is more involved. Let $P$ be the 5-gon $abcde$ labelled clockwise. We assume that vertex $a$ is fixed throughout the morph. This is not a loss of generality because if $a$ moves during an $L$-directional morph we can translate the whole drawing back in direction $L$ so that $a$ returns to its original position. An $L$-directional morph composed with a translation in direction $L$ is again an $L$-directional morph, and planarity is preserved since the relative positions of vertices do not change.

Observe that at any time instant $t$ during morph $\langle \Gamma_1, \ldots, \Gamma_k \rangle$ there exists an $\epsilon_t > 0$ such that the intersection between the disk $D$ centered at $a$ with radius $\epsilon_t$ and the kernel of polygon $P$ consists of a positive-area sector $S$ of $D$. This is because $a$ is a vertex of the kernel of $P$. Let $\epsilon = \min \epsilon_t$ be the minimum of $\epsilon_t$ among all time instants $t$ of the morph.

Fix $D$ to be the disk of radius $\epsilon$ centered at $a$. In case $a$ is a convex vertex of $P$, the sector $S$ is bounded by the edges $ab$ and $ae$ and we call it a positive sector. See Figure 4(a). In case $a$ is a reflex vertex of $P$, the sector $S$ is bounded by the extensions of edges $ab$ and $ae$ and we call it a negative sector. See Figure 4(b). More precisely, let $b'$ and $e'$ be points so that $a$ is the midpoint of the segments $bb'$ and $ee'$ respectively. The negative sector is bounded by the segments $ae'$ and $ab'$. Note that when an $L$-directional morph is applied to $P$, the points $b'$ and $e'$ also move at uniform speed in direction $L$.

The important property we use from now on is that any point in the sector $S$ lies in the kernel of polygon $P$. Let the sector in drawing $\Gamma_i$ be $S_i$ for $i = 1, \ldots, k$. Let the direction of the unidirectional morph $\langle \Gamma_i, \Gamma_{i+1} \rangle$ be $L_i$ for $i = 1, \ldots, k - 1$. In other words, $\langle \Gamma_i, \Gamma_{i+1} \rangle$ is an $L_i$-directional morph.

Our task is to choose for each $i$ a position $p_i$ for vertex $p$ inside sector $S_i$ so that all the $L_i$-directional morphs keep $p$ inside the sector at all times. A necessary condition is that
Figure 4: A disk $D$ centered at $a$ whose intersection with the kernel of $P$ (the lightly shaded polygonal region) is a non-zero-area sector $S$ (darkly shaded). (a) Vertex $a$ is convex and $S$ is a positive sector. (b) Vertex $a$ is reflex and $S$ is a negative sector.

the line through $p_ip_{i+1}$ be parallel to $L_i$. We will first show that this condition is in fact sufficient (see Lemma 7). Then we will show that such points $p_i$ exist.

Translate $L_i$ to go through point $a$ and distinguish the following two cases in the relationship between $L_i$ and $S_i$:

one-sided case Points $b_i$ and $e_i$ lie in the same closed half-plane determined by $L_i$. In this case, whether the sector $S_i$ is positive or negative, $L_i$ does not intersect the interior of $S_i$. See Figure 5 An $L_i$-directional morph keeps $b_i$ and $e_i$ on the same side of $L_i$ so if $S_i$ is positive it remains positive and if $S_i$ is negative it remains negative.

two-sided case Points $b_i$ and $e_i$ lie on opposite sides of $L_i$. In this case $L_i$ intersects the interior of the sector $S_i$. See Figure 6. During an $L_i$-directional morph the sector $S_i$ may remain positive, or it may remain negative, or it may switch between the two, although it can only switch once.

Figure 5: The one-sided case where $S_i$ lies to one side of $L_i$, illustrated for a positive sector $S_i$. (a) An $L_i$-directional morph to $S_{i+1}$. (b) $p$ remains inside the sector iff it remains inside $D$ and between the two lines $ba$ and $ea$.  

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Our main tool is the following lemma proving that “sidedness” on line $L$ is preserved in an $L$-directional morph.

**Lemma 5.** Let $L$ be a horizontal line and $x_0, x_1, y_0, y_1$ be points in $L$. Consider a point $x$ that moves at constant speed from $x_0$ to $x_1$ in one unit of time. If $y_i$ is to the right of $x_i$, $i = 0, 1$, and $y$ is a point that moves at constant speed from $y_0$ to $y_1$ in one unit of time then $y$ remains to the right of $x$ during their movement. Note that $x_0$ may lie to the right or left of $x_1$ and ditto for $y_0$ and $y_1$.

Proof. Let $x_i$ and $y_i$, $i = 0, 1$, be points as described above, see Figure 7. Denote by $x_t$ and $y_t$ the positions of $x$ and $y$ for $0 < t < 1$. First note that

$$y_i = x_i + \delta_i$$  \hspace{1cm} (1)

for $i = 0, 1$, with $\delta_i > 0$. Since $x$ and $y$ are moving at constant speed, we have $x_t = (1 - t)x_0 + tx_1$ and $y_t = (1 - t)y_0 + ty_1$. Now, using equation (1) in the expression for $y_t$ we have

$$y_t = (1 - t)(x_0 + \delta_0) + t(x_1 + \delta_1)$$ \hspace{1cm} (2)

$$= x_t + (1 - t)\delta_0 + t\delta_1,$$ \hspace{1cm} (3)

where $(1 - t)\delta_0 + t\delta_i > 0$. \hfill \Box
Corollary 6. Consider an L-directional morph acting on points p, r and s. If p is to the right of the line through rs at the beginning and the end of the L-directional morph, then p is to the right of the line through rs throughout the L-directional morph.

We are now ready to prove our main lemma about the relative positions of points $p_i$ and $p_{i+1}$.

Lemma 7. If point $p_i$ lies in sector $S_i$ and point $p_{i+1}$ lies in sector $S_{i+1}$ and the line $p_ip_{i+1}$ is parallel to $L_i$ then an $L_i$-directional morph from $S_i, p_i$ to $S_{i+1}, p_{i+1}$ keeps the point in the sector at all times.

Proof. We use the notation that $b$ moves from $b_i$ to $b_{i+1}$, $p$ moves from $p_i$ to $p_{i+1}$, etc.

First consider the one-sided case. Suppose $S_i$ is a positive sector (the case of a negative sector is similar). Observe that a point $p$ remains in the sector during an $L_i$-directional morph if and only if it remains in the disc $D$ and remains between the lines $ba$ and $ea$. See Figure 5(b). Because $p_i$ and $p_{i+1}$ both lie in disc $D$, thus the line segment between them lies in the disc, and $p$ remains in the disc throughout the morph. In the initial configuration, $p_i$ lies between the lines $b_i a$ and $e_i a$, and in the final configuration $p_{i+1}$ lies between the lines $b_{i+1} a$ and $e_{i+1} a$. Therefore by Corollary 6 $p$ remains between the lines throughout the $L_i$-directional morph. Thus $p$ remains inside the sector throughout the morph.

Now consider the two-sided case. Observe that a point $p$ remains in the sector during an $L_i$-directional morph if and only if it remains on the same side of the lines $bb'$ and $ee'$. Note that this is true even when the sector changes between positive and negative. See Figure 6(b). As in the one-sided case, $p$ remains in the disc throughout the morph. Also, $p$ is on the same side of the lines $bb'$ and $ee'$ in the initial and final situations, and therefore by Corollary 6 $p$ remains on the same side of the lines throughout the morph. Thus $p$ remains inside the sector throughout the morph.

With Lemma 7 in hand the only remaining issue is the existence of points $p_i$. We call the possible positions for $p_i$ inside sector $S_i$ the nice points, defined formally as follows:

- All points in the interior of $S_k$ are nice.
- For $1 \leq i \leq k - 1$, a point $p_i$ in the interior of $S_i$ is nice if there is a nice point $p_{i+1}$ in $S_{i+1}$ such that $p_ip_{i+1}$ is parallel to $L_i$.

By Lemma 7 it suffices to show that all the nice sets are non-empty. We will in fact characterize the sets. Given a line $L$, an $L$-truncation of a sector $S$ is the intersection of $S$ with an open slab that is bounded by two lines parallel to $L$ and contains all points of $S$ in a small neighbourhood of $a$. In particular, an $L$-truncation of a sector is non-empty.

Lemma 8. The set of nice points in $S_i$ is an $L_i$-truncation of $S_i$ for $i = 1, \ldots, k$.

Proof. Let $N_i$ denote the nice points in $S_i$. The proof is by induction as $i$ goes from $k$ to 1. All the points in the interior of $S_k$ are nice. Suppose by induction that $N_{i+1}$ is an $L_{i+1}$-truncation of $S_{i+1}$.
Consider the one-sided case. See Figure 8. The slab determining $N_i$ consists of all lines parallel to $L_i$ that go through a point of $N_{i+1}$. $L_i$ itself forms one boundary of the slab and the slab is non-empty since $N_{i+1}$ contains all of $S_{i+1}$ in a small neighbourhood of $a$. Thus the slab contains all points of $S_i$ in a neighbourhood of $a$, and thus $N_i$ is an $L_i$-truncation of $S_i$.

![Figure 8](image)

Figure 8: $N_i$ (lightly shaded) is an $L_i$-truncation of $S_i$ in the one-sided case. $N_{i+1}$ is darkly shaded. $L_i$ and $L'_i$ are the slab boundaries for $N_i$.

Consider the two-sided case. See Figure 9. The slab determining $N_i$ consists of all lines parallel to $L_i$ that go through a point of $N_{i+1}$. The slab contains $a$ in its interior and thus $N_i$ is an $L_i$-truncation of $S_i$.

![Figure 9](image)

Figure 9: $N_i$ (lightly shaded) is an $L_i$-truncation of $S_i$ in the two-sided case. $N_{i+1}$ is darkly shaded. $L'_i$ and $L''_i$ are the slab boundaries for $N_i$.

Lemma 8 implies in particular that the set of nice points is non-empty, which provides the last ingredient in the proof of Lemma 4.

Lemma 8
4 Concluding remarks

In this paper we considered the problem of morphing between two straight-line drawings of a planar triangulation. We showed that one can morph between these drawings in $O(n^2)$ steps, where each step is a unidirectional morph. However, the grid size of the intermediate drawings was not bounded. It is then a natural question to ask whether such problem can be solved while guaranteeing that each intermediate drawing is on a polynomially sized grid. A partial result has been obtained in [6, 7] where it is shown that for the class of Schnyder drawings we can morph between any two of them in $O(n^2)$ steps while preserving planarity and where each intermediate drawing is in a $6n \times 6n$ grid. However, the question of finding a bound on the grid size for morphs between drawings outside this class remains open.

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