MULTIPLICATION MAPS OF LINEAR SYSTEMS ON SMOOTH PROJECTIVE TORIC SURFACES

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Abstract. Let $X$ be a smooth projective toric surface and $L$ and $M$ two line bundles on $X$. If $L$ is ample and $M$ is generated by global sections, then we show that the natural map $H^0(X, L) \otimes H^0(X, M) \to H^0(X, L \otimes M)$ is surjective. We also consider a generalization to the case when $M$ is arbitrary line bundle with $h^0(X, M) > 0$.

In this note we shall prove the following results:

Theorem 1. Let $X$ be a smooth projective toric surface, $L$ an ample line bundle on $X$, and $M$ a line bundle on $X$ which is generated by global sections. Then the multiplication map $H^0(X, L) \otimes H^0(X, M) \to H^0(X, L \otimes M)$ is surjective.

Theorem 2. Let $X$ be a smooth projective toric surface and $L$ an ample line bundle on $X$. Then there exists a constant $C(L)$ such that for all line bundles $M$ on $X$ with $h^0(X, M) \neq 0$, $\dim(\text{coker}[H^0(X, L) \otimes H^0(X, M) \to H^0(X, L \otimes M)]) \leq C(L)$.

We do not know if either of these results holds if $\dim(X) > 2$ or if $\dim(X) = 2$ and $X$ is singular. Similar questions have been raised by Oda [4].

1. We refer the reader to [1] or [3] for basic facts about toric varieties.

From now on $X = X(\Delta)$ will always be a smooth projective toric surface associated to a fan $\Delta$ in $\mathbb{N}R$, where $M$ (resp. $N$) is the lattice of characters (resp. co-characters) of a 2-dimensional algebraic torus. Any divisor on $X$ is linearly equivalent to a divisor $D = \sum_{i=1}^{n} a_i D_i$ with $a_i \in \mathbb{Z}$, and the $D_i$’s are divisors on $X$ invariant under the torus action; these are in 1-1 correspondence with the rays in $\Delta$. (We assume that the indices are chosen so that $D_i$ and $D_{i+1}$ correspond to rays forming the boundary of a cone in $\Delta$. Here, and in what follows, we assume that subscripts are considered modulo $n$.). Moreover, if $D$ is effective then we may assume that all $a_i \geq 0$ and we denote by $PD$ the corresponding polygon in $M_R$.

We let $v_i$ be the minimal lattice vector in the ray corresponding to $D_i$.

The proof of Theorem 1 is easily reduced to combinatorial statements about convex polygons, using the well-known dictionary relating equivariant divisors $D$ with $O(D)$ generated by sections and lattice polygons. We may assume that $L$ and $M$ are of the form $O(D)$ and $O(E)$ for some equivariant divisors $D = \sum_{i=1}^{n} a_i D_i$ and $E = \sum_{i=1}^{n} b_i D_i$. Then it suffices to prove that $(PD \cap M) + (PE \cap M) = (PD + PE) \cap M$.

For a divisor $D$ as above with $O(D)$ generated by sections, let $\sigma_i(D) = PD \cap L_i$ where $L_i = \{ u \in M_R | \langle u, v_i \rangle = -a_i \}$. If $D$ is ample then $\sigma_i(D)$ is always an edge of $PD$ but in general it could also be a vertex.
Lemma 1. Let $D, E$ be as above and assume that $D$ is ample and $\mathcal{O}(E)$ is generated by sections. Then $\sigma_i(D + E) = \sigma_i(D) + \sigma_i(E)$ for all $i \in [1, n]$.

Proof. This follows easily from the fact that $P_{D+E} = P_D + P_E$. □

The proof of the following lemma is also left to the reader.

Lemma 2. Let $[a_1, b_1]$ and $[a_2, b_2]$ be closed intervals of $\mathbb{R}$. Suppose $[a_1, b_1] \cap \mathbb{Z} \neq \emptyset$ and $a_2, b_2 \in \mathbb{Z}$. Then any element $z$ of $[a_1 + a_2, b_1 + b_2] \cap \mathbb{Z}$ is of the form $c_1 + c_2$ for some $c_i \in [a_i, b_i] \cap \mathbb{Z}$, $i = 1, 2$.

To prove Theorem 1 we will first reduce to the case where $P_E$ is a triangle of a special kind, and then explicitly prove the equality in this case.

Proof of Theorem 1. We first dispose off the trivial cases: If $P_E$ is a point then the statement is obvious and if $P_E$ is 1-dimensional, hence a line segment, then the proof is elementary and we leave it to the reader (use Lemma 2).

Let $s : M_\mathbb{R} \times M_\mathbb{R} \to M_\mathbb{R}$ be the sum map $(x, y) \mapsto x + y$. Then $P_{D+E} = P_D + P_E = s(P_D \times P_E)$. Let $p \in P_{D+E} \cap M$ and let $Q = s^{-1}(p) \cap (P_D \times P_E)$. This is a convex polygon (possibly degenerate) in $M_\mathbb{R} \times M_\mathbb{R}$ and we let $Q_i = \pi_i(Q)$, where $\pi_i, i = 1, 2$, are the two projections. So $Q_1 \subset P_D$ and $Q_2 \subset P_E$ are also convex polygons.

Let $(q_1, q_2) \in Q$ be such that $q_2$ is a vertex of $Q_2$. If $q_2$ is in the interior of $P_E$ then $q_1 \in Q_1$ must be a vertex of $P_D$ (sic), hence $q_1 \in M$. Then $q_2 \in M$ and $p = q_1 + q_2$, so we are done. Otherwise, since $Q_2$ must have at least one vertex, it follows that there exists a point $q \in Q_2$ which lies on the boundary of $P_E$. If $q \in M$, then we are done so we may assume that $q$ lies in the interior of an edge $\sigma$ of $P_E$. We let $m_1, m_2$ be the two end points of $\sigma$.

Recall that $P_E = \{u \in M_\mathbb{R} | \langle u, v_i \rangle \geq -b_i \text{ for all } i \}$. We may assume that the edge $\sigma$ corresponds to $v_1$, so $\langle m_1, v_1 \rangle = -b_1$, $i = 1, 2$. Now for each $i = 1, \ldots, n$, let $c_i = \min\{c \in \mathbb{Z} | \langle m_j, v_i \rangle \geq -c \text{ for } j = 1, 2\}$. Then $c_1 = b_1$ and $c_i \leq b_i$ for all $i$. Let $P = \{u \in M_\mathbb{R} | \langle u, v_i \rangle \geq -c_i \text{ for all } i\}$; by construction $P \subset P_E$ and $m_1$ and $m_2$ are vertices of $P$. If $P = \sigma$ then we are done, so we may assume that $P$ is 2-dimensional. Without loss of generality we may assume that $\langle m_1, v_2 \rangle > \langle m_2, v_2 \rangle$ and we let $k = \max\{i \in [2, n] | \langle m_1, v_i \rangle > \langle m_2, v_i \rangle\}$. So for $i \in [1, k]$, $c_i = -\langle m_2, v_i \rangle$ and for $i \in [k+1, n]$, $c_i = -\langle m_1, v_i \rangle$. By our assumption that $P$ is not 1-dimensional it follows that $\langle m_1, v_i \rangle \neq \langle m_2, v_i \rangle$ for all $i \in [2, n]$. Then $P = \{u \in M_\mathbb{R} | \langle u, v_i \rangle \geq -c_i \text{ for } i = 1, k, k+1\}$, hence $P$ a triangle. Since $X$ is smooth it follows that the third vertex is also in $M$, so $P$ corresponds to an equivariant divisor on $X$ whose associated line bundle is generated by sections.

Since $q \in P$ by construction, by replacing $P_E$ by $P$ we have reduced to the case when $P_E$ is a triangle with the further property that there exists an $i \in [1, n]$ such that $\sigma_i(E)$ and $\sigma_{i+1}(E)$ are both (non-degenerate) edges of $P_E$. By using the basis of $M$ dual to $\{v_i, v_{i+1}\}$, and after possible translation by elements of $M$ (which does not affect the hypotheses or the conclusion), we have the following picture: $P_E$ is the convex span of the points $(0, 0), (a, 0), (0, b)$, for some $a, b > 0$, $P_D$ is entirely contained in the first quadrant, and $(0, 0)$ is also a vertex of $P_D$ (consequently $P_D$ must also have edges along the positive $x$ and $y$ axes).

We shall now complete the proof of the theorem by analysing this case. Decompose the region $P_D + P_E \setminus P_D$ as a union of the three regions, $A, B$ and $C$, as illustrated in the figure below — to see that this is correct we use Lemma 2. Note that $A$ or $B$ may be empty; this happens precisely when $k = 2$ or $k = n - 1$. 
We claim that any lattice point in the region $A$ is of the form $m + (x, 0)$ where $m \in P_D \cap M$ and $0 \leq x \leq a$. This is because the trapezium, two of whose sides are the base of the triangle $P$ and the edge $U$ of $P_D$, is contained in $P_D$ and both these sides contain at least two lattice points each. Thus each horizontal line which contains a lattice element of $A$ also contains a lattice element of $P_D$, so the claim follows by Lemma 2. By a symmetric argument, any lattice point in the region $B$ is of the form $m + (0, y)$ where $m \in P_D \cap M$ and $0 \leq y \leq b$.

Any point in the region $C$ is contained in $P + P_E$. Since $P$ and $P_E$ are similar triangles (i.e. are translates of multiples of the same triangle) one easily sees that any lattice point in $P + P_E$ is the sum of lattice points in $P$ and $P_E$.

The following lemma is the key to the deduction of Theorem 2 from Theorem 1.

**Lemma 3.** Let $D = \sum_{i=1}^{n} a_i D_i$ be an effective divisor on $X$. Then there exists integers $b_i$, $0 \leq b_i \leq a_i$, such that for $D' = \sum_{i=1}^{n} b_i D_i$, $\mathcal{O}(D')$ is generated by its sections and the natural map $H^0(X, \mathcal{O}(D')) \to H^0(X, \mathcal{O}(D))$ is an isomorphism.

**Proof.** Let $P_D$ be the polygon associated to $D$. Let $S = P_D \cap M$ and let $P$ be the convex hull of the points in $S$. For each $i$, $1 \leq i \leq n$, let $b_i = \min \{ c \in \mathbb{Z} \mid \langle s, v_i \rangle \geq -c \text{ for all } s \in S \}$. Since $0 \in S$ and $S \subseteq P_D$, it follows that $0 \leq b_i \leq a_i$. We claim that $P$ is equal to $P' := \{ u \in M_{\mathbb{R}} \mid \langle u, v_i \rangle \geq -b_i \text{ for all } i \}$.

Consider the lines $L_i = \{ u \in M_{\mathbb{R}} \mid \langle u, v_i \rangle = -b_i \}$. By construction, each of these lines contains a point $s_i$ of $S$ and all points of $S$ are contained in the “positive” half planes $H_i = \{ u \in M_{\mathbb{R}} \mid \langle u, v_i \rangle \geq -b_i \}$. It suffices to show that the point
$m_i = L_i \cap L_{i+1}$ is in $S$ for all $i$. Suppose not; then there exists $i$ and $j$ such that $\langle m_i, v_j \rangle < -b_j$. Since $\langle s_i, v_j \rangle \geq -b_j$ and $\langle s_{i+1}, v_j \rangle \geq -b_j$, it follows $L_j$ must intersect $L_i$ and $L_{i+1}$ along the segments $[s_i, m_i]$ and $[s_{i+1}, m_i]$. But looking at the normal rays, this says that the ray along $v_j$ must lie between the rays along $v_i$ and $v_{i+1}$. But we have assumed that the $v_i$’s are cyclically ordered, so this is a contradiction.

Since $P = P'$, it follows that $P$ corresponds to a divisor $D'$ on $X$ of the required form. $\square$

**Lemma 4.** There exists a constant $C$ depending only on $X$ with the following property: Let $D, D'$ be as in Lemma 3 and let $J = \{i \in [1, n] \mid b_i < a_i\}$. Then the number of lattice points on all the edges of $P'_D$ whose normal ray contains $v_j$ for some $j \in J$ is bounded by $C$.

**Proof.** This follows because there are only finitely many rays in $\Delta$, the fan of $X$. If we choose a Euclidean metric on $M_R$, then there are only finitely many possibilities for the angles at the vertices of any polygon associated to a divisor on $X$. Bounding the number of lattice points is also the same as bounding the lengths of edges. Suppose that the length of the edge corresponding to $v_j$ is not bounded. Let $D'' = \sum_{i=1}^{n} c_i D_i$ with $c_i = b_i$ for $i \neq j$ and $c_j = b_j + 1$. Then $P'_D \subset P'' \subset P_D$ and if the length of is sufficiently large, $P'_D$ would contain lattice points not contained in $P_D$, contradicting the defining property of $D'$. $\square$

The next lemma implies that even though there may be infinitely many divisors $D$ giving rise to the same $D'$ as in Lemma 3, up to translation by elements of $M$ there are only finitely many polygons occurring as connected components of $P_D \setminus P'_{D'}$, where $D$ ranges over all effective divisors on $X$.

**Lemma 5.** There exists a constant $C_2$ depending only on $X$ such that for $D, D'$ and $J$ as in Lemma 3 and $J' = [j_1, j_2]$ any subinterval of $[1, n]$ contained in $J$, exactly one of the following holds:
1) $[1, n] \setminus J'$ contains at most one element. In this case all the edges of $P_{D'}$ have length $\leq C_1$, and there exists $J'' \subset [1, n]$ such that $a_j \leq C_2$ for $j \in J''$ and such that the polyhedron $P(J'') = \{u \in M_R \mid \langle u, v_j \rangle \geq -a_j \text{ for all } j \in J''\}$ is bounded.
2) $[1, n] \setminus J'$ contains at least two elements, so $j_1 - 1$ and $j_2 + 1$ are distinct elements of $[1, n]$. Consider the lines $L_{j_1-1}$ and $L_{j_2+1}$. Then either
2a) The lines intersect in a point $p$ such that any line segment joining $p$ and $\sigma_j$ for any $j \in J'$ does not contain any point of $P_{D'}$ except for an endpoint. Or
2b) The two lines are parallel. Then there exists $j \in J'$ such that $a_j - b_j \leq C_2$. Or
2c) There exists a subset $J''$ of $J$ such that $a_j - b_j \leq C_2$ for $j \in J''$ and such that the region $P(J'') = \{u \in M_R \mid \langle u, v_j \rangle \geq -a_j \text{ for all } j \in J'' \cup \{j_1 - 1, j_2 + 1\}\}$ is bounded.

**Proof.** The lemma is essentially a consequence of Lemma 4. Since there are only finitely many possibilities for the lengths of the edges of $P_{D'}$ corresponding to $j \in J$, the number of possible configurations (upto translation) of the subset of the boundary of $P_{D'}$ which is the union of the edges corresponding to the $j$’s in $J'$ is also finite. (In case 1), even the number of possible $D'$ is finite.) So it is enough to
find a constant $C$ which works in each case separately, since we can then let $C_2$ be the maximum of all these.

First assume that we are in case 1). Let $\Sigma(J')$ be the collection of subsets $J''$ of $J'$ such that the rays of $\Delta(X)$ corresponding to $j \in J''$ give rise to a complete fan i.e. any open half-space in $N_\mathbb{R}$ must contain one of these rays; so these are precisely the subsets $J''$ for which $P(J'')$ is always bounded. Suppose the conclusions in case 1) do not hold. Since there are only a finite number of possible $D'$, we may consider each of them separately, so we may assume that there is no constant which works for some fixed $D'$. Since $J'$ is a finite set it follows that there exists a sequence of divisors $D^l = \sum_{i=1}^n a_i^l D_i$ with $D^l = D'$ and a subset $J'''$ of $J'$ such that $a_i^l \to \infty$ as $j \to \infty$ for all $i \in J'''$. Furthermore $J''' \cap J'' \neq \emptyset$ for all $J'' \in \Sigma(J')$. It follows that $\cup_l P_{D^l} \supset \{u \in M_\mathbb{R} | (u, v_i) \geq -b_i \text{ for all } i \in J' \setminus J''\}$. But $J' \setminus J'''$ is not in $\Sigma(J')$ so $\cup_l P_{D^l}$ contains an unbounded polyhedron and hence must contain infinitely many elements of $M$. But this contradicts the assumption that $D^l = D'$ for all $l$.

Case 2) is handled in an analogous manner, the remarks at the beginning of the proof allowing us to consider essentially one $D'$ at a time. For 2a) there is nothing to prove and 2b) is elementary. For 2c) we let $S(J')$ be as above except that we require that $J'' \cup \{j_1, j_2\}$ give rise to a complete fan; it follows by assumption that $S(J') \neq \emptyset$. The reason for the $a_j - b_j$ occurring here, instead of just the $a_j$ in case 1), is because we only have finiteness of possible configurations up to translation. (Note that the lengths of the edges corresponding to $i \notin J$ have no effect, since the claim is “local” around a given $J'$.)

Proof of Theorem 3. Let $C = O(D)$ and $M = O(E)$. Let $E'$ be the divisor associated to $E$ using Lemma 3. By Theorem 3 it follows that the map $H^0(X, O(D)) \otimes H^0(X, O(E')) \to H^0(X, O(D) \otimes O(E'))$ is surjective or equivalently the map $(P_D \cap M) + (P_{E'} \cap M) \to (P_{D+E'} \cap M)$ is surjective. By combining the previous three lemmas, it follows that that there are only finitely many possibilities for the connected components of $P_{D+E'} \setminus P_{D+E'}$, up to translation by lattice points. This is because $D$ is fixed, so the lengths of the edges of $P_{D+E'}$ corresponding to $j \in J$ are bounded independently of $E$ (use Lemma 3). The number of lattice points in $P_{D+E'} \setminus P_{D+E'}$ can thus be bounded by a constant depending only on $D$, whence the theorem.

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