A Coordinate-Free Construction for a Class of Integrable Hydrodynamic-Type Systems

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Using a (1,1)-tensor $L$ with zero Nijenhuis torsion and maximal possible number (equal to the number of dependent variables) of distinct, functionally independent eigenvalues we define, in a coordinate-free fashion, the seed systems which are weakly nonlinear semi-Hamiltonian systems of a special form, and an infinite set of conservation laws for the seed systems.

The reciprocal transformations constructed from these conservation laws yield a considerably larger class of hydrodynamic-type systems from the seed systems, and we show that these new systems are again defined in a coordinate-free manner, using the tensor $L$ alone, and, moreover, are weakly nonlinear and semi-Hamiltonian, so their general solution can be obtained by means of the generalized hodograph method of Tsarev.

Introduction

In the present paper we deal with the systems of first order quasi-linear PDEs of the form

$$u_t = A(u)u_x,$$

where $u = (u^1, \ldots, u^n)^T$, $A$ is an $n \times n$ matrix, the superscript $T$ indicates the transposed matrix. The systems \textsuperscript{11} are usually called hydrodynamic-type systems or dispersionless systems. More specifically, we shall restrict ourselves to considering the systems \textsuperscript{11} which are semi-Hamiltonian in the sense of Tsarev \textsuperscript{23} and weakly nonlinear\textsuperscript{1} \textsuperscript{10}.

Although the class of weakly nonlinear semi-Hamiltonian (WNSH) systems was extensively studied in the literature, see e.g. \textsuperscript{6, 10, 12, 13, 14, 17} and references therein, the results obtained so far were mostly presented in the distinguished coordinates, the so-called Riemann invariants. In particular, in these coordinates we have a complete description of WNSH hydrodynamic-type systems \textsuperscript{10, 6} and, moreover, the general solution in implicit form for any such system can be found. What is more, any WNSH system written in the Riemann invariants can be linearized using a suitably chosen reciprocal transformation \textsuperscript{10} (see e.g. \textsuperscript{15, 16, 19, 13} and references therein for a general theory of reciprocal transformations).

However, not much is known so far about how to construct or identify WNSH systems in a coordinate-free fashion or construct reciprocal transformations for such systems written in arbitrary coordinates.\textsuperscript{1}

\textsuperscript{1}Note that weakly nonlinear systems are also known as linearly degenerate, see e.g. \textsuperscript{21, 11, 13, 6}.
Even though there exists a coordinate-free version of conditions under which a given hydrodynamic-type system is weakly nonlinear and semi-Hamiltonian, the conditions in question written in the coordinate-free form are quite cumbersome, and constructing any reasonably large classes of WNSH systems in arbitrary coordinates using these conditions is a virtually impossible task even for low values of $n$ except for the simplest cases of $n = 2$ and $n = 3$. As for the case of arbitrary $n$, some results were obtained in [6] for a special class of the WNSH systems, namely, the seed systems, see below for details.

In the present paper we construct in a coordinate-free fashion fairly extensive classes of WNSH systems from the so-called seed systems. We start with a $(1,1)$-tensor $L$ with zero Nijenhuis torsion and maximal possible number (equal to the number of dependent variables) of distinct, functionally independent eigenvalues. Using this tensor we define, in a coordinate-free fashion, a class of WNSH hydrodynamic-type systems which we call the seed systems, see Section 1 below for details.

We then observe that the seed systems possess infinitely many nontrivial conservation laws of a special form that can be written in a coordinate-free fashion. Note that even though any semi-Hamiltonian system has infinitely many conservation laws [23], in general there is no way to write them down explicitly in arbitrary coordinates.

Using the above special conservation laws we construct the reciprocal transformations (10) for the seed systems and show that these transformations yield new large classes (21) of WNSH hydrodynamic-type systems a priori written in a coordinate-free fashion. Finally, using the explicit form of the resulting systems in the Riemann invariants, we write down general solutions for the systems in question using the technique from [10, 6], see Section 3 below for details.

It is important to stress that, as shown in Section 2 below, for writing down the reciprocal transformations in question it suffices to know the tensor $L$ alone. Thus, the coordinate-free construction of weakly nonlinear semi-Hamiltonian hydrodynamic-type systems laid out in the present paper works for any $(1,1)$-tensor with zero Nijenhuis torsion and maximal possible number of distinct, functionally independent eigenvalues. Moreover, as shown in Section 1 below, this tensor always admits an infinite family of metrics for which it is an $L$-tensor in the sense of [3, 4, 5].

1 The seed systems

Consider an $n$-dimensional manifold $M$ endowed with a tensor $L$ of type $(1,1)$, i.e., with one covariant and one contravariant index, with zero Nijenhuis torsion and $n$ distinct, functionally independent eigenvalues. It can be shown that a tensor with these properties always is an $L$-tensor [3, 4, 5], also known as a special conformal Killing tensor of trace type [9].

Following [3, 5, 7], consider the following set of tensors of type $(1,1)$ on $M$:

$$K_1 = I, \quad K_r = \sum_{k=0}^{r-1} \rho_k L^{r-1-k}, \quad r = 2, \ldots, n,$$

(2)

where $I$ is the $n \times n$ unit matrix, and $\rho_i$ are coefficients of the characteristic polynomial of the tensor $L$, i.e.,

$$\det(\xi I - L) = \sum_{i=0}^{n} \rho_i \xi^{n-i}.$$  
(3)

For the sake of brevity in what follows we shall use the term ‘tensor’ instead of ‘tensor field’. 

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Now consider a vicinity $U \subset M$ with local coordinates $u^1, \ldots, u^n$, and a set of hydrodynamic-type systems of the form

$$K_1^{-1}u_{t_1} = K_2^{-1}u_{t_2} = \cdots = K_n^{-1}u_{t_n}$$

where let $u = (u^1, \ldots, u^n)^T$, and the superscript $T$ refers to the matrix transposition, $t_i$ are independent variables, $K_i^{-1}$ are tensors of type $(1,1)$ such that $K_iK_i^{-1} = I$, $i = 1, \ldots, n$.

For any fixed $j \in \{1, \ldots, n\}$ we can rewrite (4) as

$$u_{t_i} = K_iK_j^{-1}u_{t_j}, \quad i = 1, \ldots, n, \quad i \neq j.$$  

(5)

Notice that in (5) the variable $t_j$ plays the role of a space variable while the remaining times $t_i$ should be considered as evolution parameters. Moreover $K_iK_j^{-1}$ again is a tensor of type $(1,1)$. It is important to stress that the set (4) (or (5)) of hydrodynamic-type systems is covariant under arbitrary changes of local coordinates on $M$, and in fact the systems in question are well-defined on the whole of $M$.

We shall refer to the systems (4) or (5) with $K_i$ given by (2) as to the seed systems. In fact, these systems belong to a broader class of the so-called dispersionless Killing systems [8]. It can be shown [6] that the seed systems are weakly nonlinear and semi-Hamiltonian.

It is immediate from (2) that if we choose the eigenvalues $\lambda^i$, $i = 1, \ldots, n$, of $L$ for the local coordinates on $U$ (this is possible because the Nijenhuis torsion of $L$ vanishes and the eigenvalues in question are simple and functionally independent), the quantities (2) will be diagonal in these coordinates, and thus the eigenvalues in question will provide the Riemann invariants for the seed systems (5). As will be shown in Section 5, the solution for these systems, when expressed using the Riemann invariants, has the form (46). Note that the Lax representations for these systems also appear in the context of the so-called universal hierarchy [1, 2].

Interestingly enough, for the seed systems we have [8] an infinite set of conservation laws that can be constructed in a coordinate-free fashion.

In order to write this set down we need the so-called basic separable potentials $V_r^{(k)}$ that can be defined using the tensor $L$ via the following recursion relation [7]:

$$V_r^{(k)} = V_{r+1}^{(k-1)} - \rho_r V_1^{(k-1)}, \quad k \in \mathbb{Z},$$

with the initial condition

$$V_r^{(0)} = -\delta^n_r, \quad r = 1, \ldots, n.$$  

(6)

(7)

Here and below we tacitly assume that $V_r^{(k)} \equiv 0$ for $r < 1$ or $r > n$.

The recursion (6) can be reversed. The inverse recursion is given by

$$V_r^{(k)} = V_{r-1}^{(k+1)} - \frac{\rho_{r-1}}{\rho_n} V_n^{(k+1)}, \quad k \in \mathbb{Z}, \quad r = 1, \ldots, n.$$  

(8)

Hence, the first nonconstant potentials are $V_r^{(n)} = \rho_r$ for $k > 0$ and $V_r^{(-1)} = \frac{\rho_{r-1}}{\rho_n}$ for $k < 0$, respectively.

The conservation laws in question read [8]

$$D_{t_i}(V_j^{(k)}) = D_{t_j}(V_i^{(k)}), \quad i, j = 1, \ldots, n, \quad i \neq j, \quad k \in \mathbb{Z},$$

(9)

where $D_{t_i}$ are total derivatives computed by virtue of (4). These conservation laws are obviously non-trivial for all integer $k \neq 0, \ldots, n - 1$.  

3
2 Reciprocal transformations and more

Using (9) we can define a large class of reciprocal transformations for the seed systems. Using these transformations we construct extensive new classes (21) of WNSH hydrodynamic-type systems. Most importantly, these transformed systems, just like their seed counterparts, possess an infinite set of non-trivial conservation laws that can be constructed in a coordinate-free fashion, and the general solution of any of the transformed systems (21) written in the Riemann invariants takes the form (47) and (48).

The reciprocal transformation in question is defined for the whole set (4) of the seed systems and reads [20] as follows:

\[ dt_{s_i} = - \sum_{j=1}^{n} V_j^{(\gamma_i)} dt_j, \quad i = 1, \ldots, k, \]
\[ t_m = t_m, \quad m = 1, 2, \ldots, n, \quad m \neq s_a \quad \text{for any} \quad a = 1, \ldots, k. \] (10)

Here \( 1 \leq k \leq n; \) the numbers \( s_a, a = 1, \ldots, k, \) are a \( k \)-tuple of distinct integers from the set \( \{1, \ldots, n\} \), and \( \gamma_j \) are arbitrary positive integers that satisfy the following conditions:

\[ \gamma_1 > \gamma_2 > \cdots > \gamma_k > n - 1. \] (11)

The choice of numbers \( k \in \{1, \ldots, n\}, s_a, \) and \( \gamma_a \) that satisfy the above conditions uniquely determines the transformation (10). Using (9) we can readily check that (10) is a well-defined reciprocal transformation.

The inverse of (10) has the form

\[ dt_{s_i} = - \sum_{j=1}^{n} \tilde{V}_j^{(n-s_i)} dt_j, \quad i = 1, \ldots, k, \]
\[ t_l = \tilde{t}_l, \quad q = 1, 2, \ldots, n, \quad l \neq s_a \quad \text{for any} \quad a = 1, \ldots, k. \] (12)

Here \( \tilde{V}_j^{(m)} \) are deformed separable potentials defined for all integer \( m \) as follows:

1) for \( j = s_1, \ldots, s_k \) we define \( \tilde{V}_{s_i}^{(m)} \) by means of the relations

\[ V_{s_i}^{(m)} + \sum_{p=1}^{k} \tilde{V}_{s_p}^{(m)} V_{s_i}^{(\gamma_p)} = 0, \] (13)

whence

\[ \tilde{V}_{s_i}^{(m)} = - \det W_i^{(m)} / \det W, \] (14)

where \( W \) is a \( k \times k \) matrix of the form

\[ W = \begin{vmatrix} V_{s_1}^{(\gamma_1)} & \cdots & V_{s_1}^{(\gamma_k)} \\ \vdots & \ddots & \vdots \\ V_{s_k}^{(\gamma_1)} & \cdots & V_{s_k}^{(\gamma_k)} \end{vmatrix}, \] (15)

and \( W_i^{(m)} \) are obtained from \( W \) by replacing \( V_{s_j}^{(\gamma_j)} \) by \( V_{s_j}^{(m)} \) for all \( j = 1, \ldots, k; \)

2) for \( j \neq s_1, \ldots, s_k \) we set

\[ \tilde{V}_j^{(m)} = V_j^{(m)} + \sum_{p=1}^{k} \tilde{V}_{s_p}^{(m)} V_j^{(\gamma_p)}, \] (16)
or equivalently
\[ \tilde{V}_j^{(m)} = \det \hat{W}_j^{(m)} / \det W, \]
where \( \hat{W}_j^{(m)} \) is a \((k + 1) \times (k + 1)\) matrix of the form
\[ \hat{W}_j^{(m)} = \begin{vmatrix} V_j^{(m)} & V_j^{(\gamma_1)} & \ldots & V_j^{(\gamma_k)} \\ V_{s1}^{(m)} & V_{s_1}^{(\gamma_1)} & \ldots & V_{s_1}^{(\gamma_k)} \\ \vdots & \vdots & \ddots & \vdots \\ V_{s_k}^{(m)} & V_{s_k}^{(\gamma_1)} & \ldots & V_{s_k}^{(\gamma_k)} \end{vmatrix}. \]

It can be shown that the above definition of \( \tilde{V}_i^{(j)} \) is equivalent to the one given in [20].

In order to find out how Eq. (4) transforms under (10), we temporarily rewrite the former as
\[ u_{i_1} = K_i Y, \quad i = 1, \ldots, n, \]
where \( Y \) is an arbitrary vector field on \( M \).

The transformation (10) sends the set (19) of seed systems into the following set:
\[ u_{i_1} = \tilde{K}_i Y, \quad i = 1, 2, \ldots, n, \]
which upon elimination of \( Y \) can be written in the form similar to (4):
\[ \tilde{K}_1^{-1} u_{i_1} = \tilde{K}_2^{-1} u_{i_2} = \cdots = \tilde{K}_n^{-1} u_{i_n}, \]
and can be further rewritten like (5)
\[ u_{i_1} = \tilde{K}_i \tilde{K}_j^{-1} u_{i_j}, \quad i = 1, \ldots, n, \quad i \neq j, \]
for any fixed \( j \in \{1, \ldots, n\} \). As will be shown in Section 5, the general solution for these systems is given by (17) and (48).

Using (12) and the chain rule we find, after a straightforward but tedious computation, that
\[ \tilde{K}_{s_i} = - \sum_{j=1}^{k} \tilde{V}_{s_i}^{(n-s_j)} K_{s_j} M^{-1}, \quad i = 1, \ldots, k, \]
\[ \tilde{K}_m = K_m M^{-1} - \sum_{l=1}^{k} \tilde{V}_m^{(n-s_l)} K_{s_l} M^{-1}, \quad m = 1, 2, \ldots, n, \quad m \neq s_a \text{ for any } a = 1, \ldots, k, \]
where
\[ M = - \det W_{s_1} / \det W, \]
\( W \) is given by (15), and \( W_{s_1} \) is obtained from \( W \) by replacing \( V_{s_j}^{(\gamma_j)} \) by \( K_{s_j} \) for all \( j = 1, \ldots, k \). Here \( \det W_{s_1} \) is a formal determinant with matrix-valued entries of the same kind as in [7].

Likewise, from (10) we infer that
\[ K_{s_i} = - \sum_{j=1}^{k} V_{s_i}^{(\gamma_j)} \tilde{K}_{s_j} \tilde{M}^{-1}, \quad i = 1, \ldots, k, \]
\[ K_m = \tilde{K}_m \tilde{M}^{-1} - \sum_{l=1}^{k} V_m^{(\gamma_l)} \tilde{K}_{s_l} \tilde{M}^{-1}, \quad m = 1, 2, \ldots, n, \quad m \neq s_a \text{ for any } a = 1, \ldots, k. \]
Here
\[ M = -\det \tilde{W}_{s_1} / \det \tilde{W}, \]  
(26)

\( \tilde{W} \) is a \( k \times k \) matrix of the form
\[ \tilde{W} = \begin{vmatrix} \tilde{V}_s^{(\gamma_1)} & \cdots & \tilde{V}_s^{(\gamma_k)} \\ \vdots & \ddots & \vdots \\ \tilde{V}_s^{(\gamma_1)} & \cdots & \tilde{V}_s^{(\gamma_k)} \end{vmatrix}, \]  
(27)

and \( \tilde{W}_{s_1} \) is obtained from \( \tilde{W} \) by replacing \( \tilde{V}_s^{(\gamma_j)} \) by \( \tilde{K}_s \) for all \( j = 1, \ldots, k \).

Eq. (21) possesses the following infinite set of nontrivial conservation laws similar to (9):
\[ D_{\tilde{t}_i}(\tilde{V}_m^{(m)}) = D_{\tilde{t}_j}(\tilde{V}_m^{(m)}), \quad i, j = 1, \ldots, n, \quad i \neq j, \]  
(28)

where the derivatives \( D_{\tilde{t}_i} \) are computed by virtue of (21).

As we have already mentioned above, any tensor \( L \) of type (1,1) with zero Nijenhuis torsion and \( n \) distinct, functionally independent eigenvalues always is an \( L \)-tensor for some family of metrics on \( M \).

Using the results of [7] it can be shown that the quantities \( \tilde{K}_i \) are Killing tensors of type (1,1) for a contravariant metric \( MG \), where \( G \) is any contravariant metric from the family (44). Thus Eq. (21) (or equivalently Eq. (22)) indeed defines a set of dispersionless Killing systems, and the systems (22) are weakly nonlinear and semi-Hamiltonian. Note that the weak nonlinearity of (21) can also be inferred from the general result of Ferapontov (Proposition 3.2 of [11]) stating that reciprocal transformations of hydrodynamic-type systems preserve weak nonlinearity. Alternatively, one can readily verify weak nonlinearity and semi-Hamiltonicity of (21) in the coordinate frame associated with the eigenvalues \( \lambda^i \), \( i = 1, \ldots, n \), of \( L \). Moreover, in the next section we show how to construct a general solution for any system (21) in this coordinate frame using the method from [10, 6].

3 Weakly nonlinear semi-Hamiltonian systems in Riemann invariants: general solution from separation relations

Consider a hydrodynamic-type system written in the Riemann invariants:
\[ \lambda^i_t = v^i(\lambda)\lambda^i_x, \quad i = 1, \ldots, n, \]  
(29)

where \( \lambda = (\lambda^1, \ldots, \lambda^n) \), and there is no sum over \( i \).

The system (29) is said to be weakly nonlinear (or linearly degenerate, see e.g. [10, 21] and references therein) if
\[ \partial v^i / \partial \lambda^i = 0, \quad i = 1, \ldots, n, \]  
(30)

and is said to be semi-Hamiltonian [23] if
\[ \frac{\partial}{\partial \lambda_j} \left( \frac{\partial v^i / \partial \lambda^k}{v^k - v^i} \right) = \frac{\partial}{\partial \lambda_k} \left( \frac{\partial v^i / \partial \lambda^j}{v^j - v^i} \right), \quad i, j, k = 1, \ldots, n, \quad i \neq j \neq k \neq i. \]  
(31)
It is natural to ask which is the most general weakly nonlinear semi-Hamiltonian (WNSH) hydrodynamic-type system (29) written in the Riemann invariants, or, in other words, which is the most general form of \( v^i \) that satisfy (30) and (31).

It turns out [10, 6] that any WNSH hydrodynamic-type system (29) admits \( n - 1 \) commuting flows of the same kind, so we actually have a set of commuting WNSH hydrodynamic-type systems just like (5).

In complete analogy with (5), this set can be written in a symmetric form as

\[
\frac{\lambda^i_t}{v^i_t} = \cdots = \frac{\lambda^n_t}{v^n_t}, \quad i = 1, \ldots, n, \tag{32}
\]

where \( v^i_t \equiv v^i, i = 1, \ldots, n \). The most general form of such a set of WNSH hydrodynamic-type systems is given by the formulas [10, 6]

\[
v^i_r = (-1)^{r+1} \frac{\det \Phi_{ir}}{\det \Phi_{11}}, \tag{33}\]

Here \( \Phi \) is a matrix of the form [10, 6]

\[
\Phi = \begin{pmatrix}
\Phi_1^1(\lambda^1) & \Phi_1^2(\lambda^1) & \cdots & \Phi_1^{n-1}(\lambda^1) & \Phi_1^n(\lambda^1) \\
\vdots & \vdots & \cdots & \vdots & \vdots \\
\Phi_n^1(\lambda^n) & \Phi_n^2(\lambda^n) & \cdots & \Phi_n^{n-1}(\lambda^n) & \Phi_n^n(\lambda^n)
\end{pmatrix}, \tag{34}\]

where \( \Phi_j^i(\lambda^i) \) are arbitrary functions of the corresponding variables; \( \Phi_{ik} \) is the \((n - 1) \times (n - 1)\) matrix obtained from \( \Phi \) by removing its \( i \)th row and \( k \)th column. Note that we can, without loss of generality, impose the normalization \( \Phi_{i1}^i = 1, i = 1, \ldots, n \), but we shall not use this normalization below.

The general solution for (32) can be written as [10, 6]

\[
\sum_{j=1}^n \int \frac{\Phi_j^{n-r}(\xi)}{\varphi_j(\xi)} d\xi = t_r, \quad r = 1, \ldots, n, \tag{35}\]

where \( \varphi_j(\xi) \) are arbitrary functions of a single variable.

If we fix \( r, k \in \{1, \ldots, n\}, r \neq k \), and consider the system

\[
\lambda_{tk}^i = \frac{v^i_k}{v^i_r} \lambda^i_t, \quad i = 1, \ldots, n, \tag{36}\]

then the general solution of (36) is given by (35) with \( t_j = \text{const} \) for all \( j \neq r, k \). For any pair \((r, k)\) the system (36) represents (29), where \( t_k = t, t_r = x \), and \( v^i = v^i_k/v^i_r \) satisfy the conditions (30) and (31).

Note that to a given matrix \( \Phi \) (34), or, equivalently, to a set of \( n \) Killing tensors and a class of metrics that admit them, we can associate the so-called separation relations of the form

\[
\sum_{j=1}^n \Phi_j^i(\lambda^i) H_j = f_i(\lambda^i) \mu^2, \quad i = 1, \ldots, n, \tag{37}\]

where \( H_j \) are separable geodesic Hamiltonians and \( f_i(\xi), i = 1, \ldots, n \), are arbitrary functions of a single variable [22, 6, 20] which are related to their counterparts in (35) via the formula

\[
\varphi_i(\xi) = \left( f_i(\xi) \sum_{j=1}^n \Phi_j^i(\xi) a_j \right)^{1/2}, \tag{38}\]
where \( a_j \) are arbitrary constants.

From the point of view of separation relations (37) the matrix \( \Phi \) (34) is nothing but the Stäckel matrix related to the Hamiltonians \( H_i \). Moreover, the commutativity of \( H_i \) implies the commutativity of the associated flows (32), and the general solution (35) for (32) in fact can be obtained \([10, 6]\) from the general solution for the simultaneous equations of motion for \( H_i \) which, in turn, is found using the separation relations (37).

Let us briefly recall the rationale behind the separation relations (37). If we define the Hamiltonians

\[
H_i = H_i(\lambda, \mu), \quad i = 1, \ldots, n,
\]

where \( \mu = (\mu_1, \ldots, \mu_n)^T \), as solutions of the system (37) of linear algebraic equations then these Hamiltonians have the form

\[
H_i = \mu^T K_i G \mu = \sum_{r,s=1}^{n} \mu_r (K_i G)^{rs} \mu_s, \quad i = 1, \ldots, n,
\]

and are well-known to Poisson commute with respect to the canonical Poisson bracket \( \{\lambda^i, \mu_j\} = \delta^i_j \).

Quite naturally, \( K_i \) are Killing tensors of type \((1, 1)\) for a contravariant metric \( G \). However, it is important to stress that in general these \( K_i \) do not necessarily have the form (2).

We know from [20] that for the seed systems the separation relations (37) read

\[
\sum_{j=1}^{n} (\lambda^i)^{n-j} H_j = f_i(\lambda^i) \mu_i^2, \quad i = 1, \ldots, n,
\]

while for the systems (21) we have

\[
\sum_{j=1}^{k} (\lambda^i)^{\gamma_j} \tilde{H}_{sj} + \sum_{p=1, p \neq s_1, \ldots, s_k}^{n} (\lambda^i)^{n-p} \tilde{H}_p = f_i(\lambda^i) \mu_i^2, \quad i = 1, \ldots, n.
\]

Using (40) and (41) we can readily read off the functions \( \Phi^i_j \) from (34) associated with (4) for \( K_i \) given by (2), and with (21), and construct general solution for any given seed system from (5), and for the transformed systems (22), by the method of [10, 6], see below for details.

For the special case of (40) the Killing tensors \( K_i \) in (39) are given by (2), and in the \( \lambda \)-coordinates \( L \) has the form

\[
L = \text{diag}(\lambda^1, \ldots, \lambda^n).
\]

On the other hand, from the separation relations (41) we find that

\[
\tilde{H}_i = \mu^T \tilde{K}_i \tilde{G} \mu = \sum_{r,s=1}^{n} \mu_r (\tilde{K}_i \tilde{G})^{rs} \mu_s, \quad i = 1, \ldots, n,
\]

where \( \tilde{K}_i \) are given by (23) and \( \tilde{G} = MG \) with \( M \) given by (24). Thus, the Hamiltonian \( H_i \) (resp. \( \tilde{H}_i \)) is naturally associated with the twice contravariant Killing tensor \( K_i G \) (resp. \( \tilde{K}_i \tilde{G} \)), and vice versa.

Now, the Hamiltonian \( H_1 \) associated with \( K_1 G = G \), i.e., with the original contravariant metric \( G \) itself, is the coefficient at the highest power of \( \lambda^i \) on the left-hand side of (40). Likewise, in view of (41) the coefficient at the highest power of \( \lambda^i \) on the left-hand side of (41) equals \( \tilde{H}_{s_1} \). This is the reason why it is natural to consider the contravariant metric \( \tilde{G} \) associated with \( \tilde{H}_{s_1} \) from (41) as a natural counterpart of the original contravariant metric \( G \), cf. [7].
Note also that the set of Hamiltonians $H_i$, $i = 1, \ldots, n$, in (40) is related to the set of $\tilde{H}_i$, $i = 1, \ldots, n$, in (41) via the so-called multiparameter generalized Stäckel transform of a special form, see [20] for further details, and this very fact uniquely determines the shape of the reciprocal transformation (10) and (12) relating (5) and (21).

Let us now apply the above results on general solutions to the seed systems (4) and their transformed counterparts (21) in the coordinates $\lambda^i$ being the eigenvalues of $L$. In the coordinates in question (42) holds by assumption.

Note that any metric $G$ that admits $L$ of the form (42) as an $L$-tensor in the $\lambda$-coordinates can be written in the form [7]

$$G = \text{diag} \left( \frac{f_1(\lambda_1)}{\Delta_1}, \ldots, \frac{f_n(\lambda_n)}{\Delta_n} \right),$$

where $\Delta_i = \prod_{j \neq i} (\lambda^i - \lambda^j)$. The class (44) with arbitrary functions $f_i(\xi)$ is precisely the class of the metrics that admit the set of Killing tensors given by (2) with $L$ of the form (42).

The joint general solution for the set of systems (4) written in the Riemann invariants, that is,

$$\frac{\lambda^i_1}{G^{ii}\partial \rho_1/\partial \lambda^i} = \ldots = \frac{\lambda^i_n}{G^{ii}\partial \rho_n/\partial \lambda^i}, \quad i = 1, \ldots, n,$$

or equivalently,

$$\frac{\lambda^i_1}{\partial \rho_1/\partial \lambda^i} = \ldots = \frac{\lambda^i_n}{\partial \rho_n/\partial \lambda^i}, \quad i = 1, \ldots, n,$$

where we used the formula $\sum_{j=0}^{r-1} (\lambda^i)^{r-1-j} \rho_j = \partial \rho_r/\partial \lambda^i$ (see e.g. [8]), reads

$$\sum_{j=1}^n \int \frac{\xi^{n-r}}{\varphi_j(\xi)} d\xi = t_r, \quad r = 1, \ldots, n.$$  

(46)

Notice that $\rho_i$ are nothing but the Viète polynomials in the variables $\lambda$.

Likewise, using (41) we see that the general solution of (21) in implicit form reads

$$\sum_{j=1}^n \int \frac{\xi^{q}}{\varphi_j(\xi)} d\xi = \tilde{t}_q, \quad q = 1, \ldots, k,$$

$$\sum_{j=1}^n \int \frac{\xi^{n-i}}{\varphi_j(\xi)} d\xi = \tilde{i}_i, \quad i = 1, \ldots, n, \quad i \neq s_q, \quad q = 1, \ldots, k.$$  

(47)

(48)

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