Statistical Analysis of Joint Type-I Generalized Hybrid Censoring Data from Burr XII Lifetime Distributions

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The quality of the products coming from different lines of production requires some tests called comparative life tests. For lines having the same facility, the lifetime of the product is distributed by Burr XII, the lifetime distribution, and units are tested under type-I generalized hybrid censoring scheme. The observed censoring data are used under maximum likelihood and the Bayes method to estimate the model parameters. The theoretical results are discussed and assessed through data analysis and Monte Carlo simulation study. Finally, we reported some brief comments obtained from numerical computation.

1. Introduction

Statistical inference for the life products needs to put some units of product under test to get more information about the life products. Then, we design the life experiments to obtain the required data. Under consideration time and cost, data obtained may be complete or censored. The concept of complete data is used when the failure time of all units under the test is obtained. So far, the concept of censoring data is used when some but not all failure time of units are obtained. Censoring is available in different forms, the oldest ones are called type-I censoring scheme as well as type-II censoring scheme. When we need to run the experiments to prefixed time and the number of units fail is random, the type-I censoring scheme is a suitable scheme. But, when running the experiment to obtain a prefixed number of failure and the total test time is random, type-II censoring scheme is applied. The experimenter in some cases needs to run the experiment under joint cases of type-I and type-II censoring schemes, statistically known with hybrid censoring scheme (HCS).

Let the test time is denoted by $\tau^*$ and number of failure units needed to statistical inference is denoted by $m$, the experiment is removed in hybrid censoring scheme at the only one time of $(\tau^*, T_m)$. The HCS is combined with type-I and type-II censoring schemes to define type-I and type-II HCS. In type-I HCS, the experiment is removed from the test at the min $(\tau^*, T_m)$ [1, 2], but in type-II HCS, the experiment is removed from the test at the max $(\tau^*, T_m)$ [3]. All of these censoring schemes do not allow terminating units from the
are denoted by merits of life products has considerable attention through of production, the problem of determining the relative schematic diagram described by Figure 2. To overcome this problem [6], the two types of censoring schemes are generalized in generalized hybrid censoring scheme (GHCS) which is described as follows. Type-I GHCS, for \( n \) tested units, suppose prior integers \( s \) and \( m \) that satisfy \( 1 \leq s < m \leq n \) and prior ideal test time \( \tau^* \in (0, \infty) \). The three cases are considered. If \( T_{s} < \tau^* \), the test is terminated at \( \min (\tau^*, T_{m}) \), and in other cases, if \( \tau^* < T_{s} < T_{m} \), the test is terminated at \( T_{s} \), but if \( T_{m} < \tau^* \), the test is terminated at \( T_{m} \). The data in type-I GHCS satisfy the minimum number \( s \) needing for statistical inference, and the data are summarized as follows:

\[
\begin{align*}
& (1) \quad \ell = (t_{1s} < t_{1m} < \ldots < t_{xm}), \quad \text{if } \tau^* < T_{s} \\
& (2) \quad \ell = (t_{1s} < t_{1m} < \ldots < t_{rk} < \ldots < t_{rn}), \quad \text{if } \tau^* > T_{s} \text{ and } t_{m} < \tau^* \\
& (3) \quad \ell = (t_{1s} < t_{1m} < \ldots < t_{mn}), \quad \text{if } t_{m} < \tau^*
\end{align*}
\]

The scheme of the type-I GHCS can be formulated with the schematic diagram described by Figure 1.

Type-II GHCS, for \( n \) tested units, suppose prior times \( \tau_1^* \) and \( \tau_2^* \in (0, \infty) \), such that \( \tau_2^* < \tau_1^* \) and integer \( m \) satisfies that \( 1 \leq m \leq n \). The three cases are considered. If \( T_{m} < \tau_1^* \), the test is terminated at \( \tau_1^* \), and in second case, if \( \tau_1^* < T_{m} < \tau_2^* \), the test is terminated at \( T_{m} \). If \( \tau_2^* < T_{m} \), the test is terminated at \( \tau_2^* \). The data in type-II GHCS satisfy the maximum time \( \tau_2^* \), and the data are summarized as follows:

\[
\begin{align*}
& (1) \quad \ell = (t_{1s} < t_{1m} < \ldots < t_{xm}), \quad \text{if } T_{s} < \tau_1^* \text{ or } \tau_2^* < T_{m} \\
& (2) \quad \ell = (t_{1s} < t_{1m} < \ldots < t_{mn}), \quad \text{if } \tau_1^* > T_{m} < \tau_2^*
\end{align*}
\]

The scheme of the type-II GHCS can be formulated with the schematic diagram described by Figure 2.

For manufactured products coming from different lines of production, the problem of determining the relative merits of life products has considerable attention through last view years. Practices, suppose two lines of production are denoted by \( \Gamma_1 \) and \( \Gamma_2 \) in competing duration and let two independent samples with sizes \( N_1 \) and \( N_2 \), respectively. The joint sample of \( N_1 \) and \( N_2 \) is put under life testing. This experiment is restricted under consideration of time and cost to terminate after fixed time or number of failure. The data obtained from type of censoring are called joint samples discussed early in [7, 8]. The exact likelihood inference with bootstrap technique under joint sample is presented in [9]. For progressive joint sample, refer studies by Rasouli A and Balakrishnan [10, 11] and recently by B. N. Al-Matrafi and G. A. Abd-Elmougod[12]. Also, for the accelerate model of Rayleigh distribution, refer studies by Faten A. Momenkhan and Abd-Elmougod [13, 14].

Type-I GHCS can save time and minimum number needing in statistical inference. So this study aims at development of statistical inference for life products in competing duration under considering type-I GHCS with jointly censoring scheme. Therefore, first, the model formulation under lifetime Burr XII distribution is under jointly type-I GHCS scheme. So far, parameters estimation of Burr XII distributions is carried out when jointly type-I GHCS samples are available. The maximum likelihood and Bayes methods are applied for the parameters estimation. The developed theoretical method assessed through the simulation study as well as illustration is reported with data analysis.

The study is planned as follows: the concept and model formulation are reported in Section 2. Estimation with maximum likelihood as point and interval estimators is discussed in Section 3. Bayesian approach for point and credible interval estimators with the help of the MCMC method is presented in Section 4. Data analysis is exposed in Section 5. The numerical computation is discussed through a simulation study in Section 6. Finally, some brief comments are reported in Section 7.

2. Model

Consider that the product comes from two different lines of production \( \Gamma_1 \) and \( \Gamma_2 \) that have the same facility. Suppose, independent two samples of sizes \( N_1 \) and \( N_2 \) selected from \( \Gamma_1 \) and \( \Gamma_2 \) have independent and identical distributed (i.i.d) lifetimes \( W_1, W_2, \ldots, W_{N_1} \) and \( Z_1, Z_2, \ldots, Z_{N_2} \), respectively. The independent samples distributed with populations have \( f_j(\cdot) \) probability density functions (PDFs) and \( F_j(\cdot) \) cumulative distribution functions (CDFs) for \( j = 1, 2 \). The lifetime experiment begins with prior integers and ideal test time given by \( (s, m, \tau^*) \). In the experiment, the unit failure time, and its type, means from \( \Gamma_1 \) and \( \Gamma_2 \) are recorded. Then, the experiment is continued until the failure is observed; if \( \tau^* < T_{s} \), then \( s = r \), but if \( \tau^* > T_{m} \), then \( r = m \), and in other cases, \( s < r < m \). The vector of ordered sample \( \{T_{2}, \delta_{1}, (T_{2}, \delta_{1}), \ldots, (T_{r}, \delta_{r})\} \) from the sample \( \{W_1, W_2, \ldots, W_{N_1}, Z_1, Z_2, \ldots, Z_{N_2}\} \) with \( r = N_{1r} + N_{2r} \), and the integer \( r \) is taken with \( s, m, \tau^* \). The joint likelihood function from observed joint type-I GHSC, \( r = (t_{2}, \delta_{1}, (t_{2}, \delta_{1}), \ldots, (t_{r}, \delta_{r})) \) and \( m_1 = \sum_{i=1}^{r} \delta_{i} \), \( m_2 = \sum_{r+1}^{m} (1 - \delta_{i}) \) is presented by
where
\[ D = \begin{cases} 
  t_{s}, & \text{if } \tau^* < T_s \\
  t_{m}, & \text{if } T_m < \tau^* \\
  \tau^*, & \text{if } T_s < \tau^* < T_m,
\end{cases} \]

and survival functions \( S_j(.) \), \( j = 1, 2 \).

Suppose that the lifetime \( T \) has two parameters Burr XII populations with PDFs given by

\[ f_i(t) = a_i b_i t_i^{b_i-1} \left(1 + t_i^{b_i}\right)^{-(a_i+1)}, \quad t > 0, \ (a_i, b_i > 0), i = 1, 2, \]

and been applied in areas of quality control, reliability studies, duration, and failure time modeling, see for example [15].

### 3. Estimations under the Maximum Likelihood Method

The joint likelihood function (1) under Burr XII populations (3) and (4) and observed joint type-I GHS sample \( t = \{(t_2, \delta_1), (t_2, \delta_1), (t_r, \delta_r)\} \) is formed by
\[
L(a_1, b_1, a_2, b_2 | t) \propto (a_1 b_1)^{m_1} (a_2 b_2)^{m_2} \exp \left\{ (b_1 - 1) \sum_{i=1}^{r} \delta_i \log t_i - (a_1 + 1) \sum_{i=1}^{r} \delta_i \right. \\
\times \log \left[ 1 + t_i^{b_1} \right] - (a_1 + 1)(N_1 - m_1) \log \left[ 1 + D^{b_1} \right] \\
\left. + (b_2 - 1) \sum_{i=1}^{r} (1 - \delta_i) \log t_i - (a_2 + 1) \sum_{i=1}^{r} (1 - \delta_i) \log \left[ 1 + t_i^{b_2} \right] \right] \\
\left. - (a_2 + 1)(N_2 - m_2) \log \left[ 1 + D^{b_2} \right] \right\}.
\]

(5)

The natural logarithms of (5) without a normalized constant reduce to

\[
\ell(a_1, b_1, a_2, b_2 | t) = m_1 \log(a_1 b_1) + m_2 \log(a_2 b_2) + (b_1 - 1) \sum_{i=1}^{r} \delta_i \log t_i - (a_1 + 1) \\
\times \sum_{i=1}^{r} \delta_i \log \left[ 1 + t_i^{b_1} \right] - (a_1 + 1)(N_1 - m_1) \log \left[ 1 + D^{b_1} \right] \\
+ (b_2 - 1) \sum_{i=1}^{r} (1 - \delta_i) \log t_i - (a_2 + 1) \sum_{i=1}^{r} (1 - \delta_i) \log \left[ 1 + t_i^{b_2} \right] \\
- (a_2 + 1)(N_2 - m_2) \log \left[ 1 + D^{b_2} \right].
\]

(6)

3.1. MLEs. Under partial derivative of the log-likelihood function with respect to model parameters and equating to zero, we obtain the likelihood equations as follows:

\[
\frac{\partial \ell(a_1, b_1, a_2, b_2 | t)}{\partial a_i} = 0, \quad i = 1, 2,
\]

(7) and

\[
a_1(b_1) = \frac{m_1}{\sum_{i=1}^{r} \delta_i \log \left[ 1 + t_i^{b_1} \right] + (N_1 - m_1) \log \left[ 1 + D^{b_1} \right]}
\]

(8)

\[
a_2(b_2) = \frac{m_2}{\sum_{i=1}^{r} (1 - \delta_i) \log \left[ 1 + t_i^{b_2} \right] + (N_2 - m_2) \log \left[ 1 + D^{b_2} \right]}
\]

(9)

Also, has presented that

\[
\frac{\partial \ell(a_1, b_1, a_2, b_2 | t)}{\partial b_i} = 0, \quad i = 1, 2,
\]

(10)

\[
\frac{m_1}{b_1} + \sum_{i=1}^{r} \delta_i \log t_i - (a_1 + 1) \left\{ \sum_{i=1}^{r} \delta_i \frac{t_i^{b_1} \log t_i}{1 + t_i^{b_1}} + (N_1 - m_1) \frac{D^{b_1} \log D}{1 + D^{b_1}} \right\} = 0
\]

(11)

\[
\frac{m_2}{b_2} + \sum_{i=1}^{r} (1 - \delta_i) \log t_i - (a_2 + 1) \left\{ \sum_{i=1}^{r} (1 - \delta_i) \frac{t_i^{b_2} \log t_i}{1 + t_i^{b_2}} + (N_2 - m_2) \frac{D^{b_2} \log D}{1 + D^{b_2}} \right\} = 0.
\]

(12)
Then, the nonlinear, equations (11) and (12) are reduced after replacing \( a_1 \) and \( a_2 \) from (8) and (9) to

\[
\begin{align*}
m_1 & + \sum_{i=1}^{r} \delta_i \log t_i - \left( \sum_{i=1}^{r} \delta_i \log \left[ \left( 1 + t_i^{b_i} \right) \right] + (N_1 - m_1) \log \left[ 1 + D^{b_1} \right] + 1 \right) \\
\left( \sum_{i=1}^{r} \delta_i t_i^{b_i} \log t_i / (1 + t_i^{b_i}) + (N_1 - m_1) D^{b_1} \log D / (1 + D^{b_1}) \right) &= 0
\end{align*}
\]

(13)

\[
\begin{align*}
m_2 & + \sum_{i=1}^{r} (1 - \delta_i) \log t_i - \left( \sum_{i=1}^{r} (1 - \delta_i) \log \left[ \left( 1 + t_i^{b_i} \right) \right] + (N_2 - m_2) \log \left[ 1 + D^{b_2} \right] + 1 \right) \\
\left( \sum_{i=1}^{r} (1 - \delta_i) t_i^{b_i} \log t_i / (1 + t_i^{b_i}) + (N_2 - m_2) D^{b_2} \log D / (1 + D^{b_2}) \right) &= 0
\end{align*}
\]

(14)

The two nonlinear equations presented by (13) and (14) present the likelihood equations of parameters \( b_1 \) and \( b_2 \), which are more simple to solve with Newton–Raphson or with fixed point iteration. After obtaining the values \( b_1 \) and \( b_2 \) from (13) and (14), the values \( \tilde{a}_1 \) and \( \tilde{a}_2 \) are obtained from (8) and (9). In some cases, if \( m_1 = 0 \) or \( m_2 = 0 \), the parameter values \( a_1 \) and \( b_1 \) or \( a_2 \) and \( b_2 \), respectively, are difficult to obtain [16].

3.2. Approximate Interval Estimation. The second partial derivatives of log-likelihood function (6) with respect to parameters vector \( \omega = (a_1, b_1, a_2, b_2) \) are given by

\[
\begin{align*}
\frac{\partial^2 \ell(a_1, b_1, a_2, b_2 | t)}{\partial a_i^2} &= \frac{-m_i}{a_i^2}, \quad i = 1, 2, \\
\frac{\partial^2 \ell(a_1, b_1, a_2, b_2 | t)}{\partial a_i \partial b_i} &= \frac{-m_i}{b_i} -(a_1 + 1) \left\{ \sum_{i=1}^{r} \delta_i t_i^{b_i} (\log t_i)^2 / (1 + t_i^{b_i})^2 + (N_1 - m_1) D^{b_i} (\log D)^2 / (1 + D^{b_i})^2 \right\}, \\
\frac{\partial^2 \ell(a_1, b_1, a_2, b_2 | t)}{\partial b_i^2} &= \frac{-m_2}{b_2^2} -(a_2 + 1) \left( \sum_{i=1}^{r} (1 - \delta_i) t_i^{b_i} (\log t_i)^2 / (1 + t_i^{b_i})^2 + (N_2 - m_2) D^{b_i} (\log D)^2 / (1 + D^{b_i})^2 \right) \\
\frac{\partial^2 \ell(a_1, b_1, a_2, b_2 | t)}{\partial a_1 \partial b_1} &= \frac{\partial^2 \ell(a_1, b_1, a_2, b_2 | t)}{\partial a_1 \partial b_1} = - \sum_{i=1}^{r} \delta_i t_i^{b_i} \log t_i / (1 + t_i^{b_i}) - (N_1 - m_1) D^{b_i} \log D / (1 + D^{b_i}), \\
\frac{\partial^2 \ell(a_1, b_1, a_2, b_2 | t)}{\partial a_1 \partial b_2} &= \frac{\partial^2 \ell(a_1, b_1, a_2, b_2 | t)}{\partial a_2 \partial a_1} = \frac{\partial^2 \ell(a_1, b_1, a_2, b_2 | t)}{\partial a_1 \partial b_2} = \frac{\partial^2 \ell(a_1, b_1, a_2, b_2 | t)}{\partial b_2 \partial a_1} = 0,
\end{align*}
\]

(15)

The Fisher information matrix is defined by minus expectation of second partial derivative of the log-likelihood function. Practice, under a large sample, the Fisher information matrix can be approximate with approximate
information matrix. Let η denote the Fisher information matrix defined by
\[
\eta = -E \left( \frac{\partial^2 \ell(a_1, b_1, a_2, b_2 | T)}{\partial \omega_i \partial \omega_j} \right), \quad i, j = 1, 2, 3, 4, \quad (16)
\]
where \( \omega = (a_1, b_1, a_2, b_2) \). Then, the approximate information matrix of η denoted by \( \eta_0 \) is defined as
\[
\eta_0 = \left( \frac{\partial^2 \ell(a_1, b_1, a_2, b_2 | T)}{\partial \omega_i \partial \omega_j} \right)_{\omega = (\bar{a}_1, \bar{b}_1, \bar{a}_2, \bar{b}_2)}, \quad i, j = 1, 2, 3, 4.
\]
(17)

Hence, under asymptotic normality distribution of MLEs \( (\bar{a}_1, \bar{b}_1, \bar{a}_2, \bar{b}_2) \) with mean \( (a_1, b_1, a_2, b_2) \) and variance covariance matrix \( \eta_0^{-1} \), 100(1−2\( \alpha \))% confidence intervals from the model parameter \( \omega = (a_1, b_1, a_2, b_2) \) are given by
\[
\tilde{a}_1 \pm z\sqrt{c_{11}},
\tilde{b}_1 \pm z\sqrt{c_{22}},
\tilde{a}_2 \pm z\sqrt{c_{33}},
\tilde{b}_2 \pm z\sqrt{c_{44}}.
\]
(18)

4. Bayesian Approach

In this section, we discuss the Bayesian estimations of model parameters, point, and credible interval. Bayesian approach needs prior information about the model parameters, which we considered as independent gamma prior, described as follows:
\[
\omega \propto \omega_i^{\alpha-1} \exp|v_i|, \quad i = 1, 2, 3, 4,
\]
(19)
where \( \omega = (a_1, b_1, a_2, b_2) \) is the model parameter. The joint prior distribution is given by
\[
\Psi (a_1, b_1, a_2, b_2) \propto \prod_{i=4} \omega_i^{\alpha-1} \exp|v_i|, \quad i = 1, 2, 3, 4.
\]
(20)

Generally, the posterior distribution from the model parameters \( \Psi (a_1, b_1, a_2, b_2 | T) \) is given by
\[
\Psi (a_1, b_1, a_2, b_2 | T) = \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \Psi (a_1, b_1, a_2, b_2) \times L(a_1, b_1, a_2, b_2 | T) \mathrm{da}_1 \mathrm{db}_1 \mathrm{da}_2 \mathrm{db}_2.
\]
(21)

And the Bayes estimate under squared error loss function (SEL) of function \( \Omega (a_1, b_1, a_2, b_2) \) is given by
\[
\hat{\Omega}_B = \int_0^\infty \int_0^\infty \int_0^\infty \Omega (a_1, b_1, a_2, b_2) \Psi (a_1, b_1, a_2, b_2 | T) \mathrm{da}_1 \mathrm{db}_1 \mathrm{da}_2 \mathrm{db}_2.
\]
(22)

The equation (22) has ratio of two integrals, generally cannot be obtained in a closed form. Then, numerical approximation will be used to solve this problem. One way is called numerical integration, and other way used can be called Lindley approximate. The important method which has considerable attention in the last year called the MCMC method is discussed as follows.

The proportional form of joint posterior distribution (21) with joint prior distribution (20) and the likelihood function (6) is given by
\[
\Psi (a_1, b_1, a_2, b_2 | T) \propto a_1^{m_1+\rho_1-1} b_1^{m_1+\rho_1-1} a_2^{m_2+\rho_2-1} b_2^{m_2+\rho_2-1} \exp[-v_{1a} - v_{2b} - v_{3a} - v_{4b}]
\]
\[
+ (b_1 - 1) \sum_{i=1}^r \delta_i \log t_i - (a_1 + 1) \sum_{i=1}^r \delta_i \log [1 + t_i^{b_1}] - (a_1 + 1) \times (N_1 - m_1) \log [1 + D^{b_1}]
\]
\[
+ (b_2 - 1) \sum_{i=1}^r (1 - \delta_i) \log t_i - (a_2 + 1) \times \sum_{i=1}^r (1 - \delta_i) \log [1 + t_i^{b_2}] - (a_2 + 1) \times (N_2 - m_2) \log [1 + D^{b_2}].
\]
(23)
The joint posterior distribution (23) reduced with the full-conditional probability distributions PDF's is given as follows:

\[
\Psi_1(a_1|b_1,a_2,b_2,t) \propto \exp \left\{ -a_1 \left( v_1 + \sum_{i=1}^{r} \delta_i \log \left[ 1 + t_i^{b_1} \right] + (N_1 - m_1) \log \left[ 1 + D_1^{b_1} \right] \right) \right\} \\
\times a_1^{m_1+p_1-1},
\]

\[
\Psi_2(a_2|a_1,b_1,b_2,t) \propto \exp \left\{ -a_2 \left( v_2 + \sum_{i=1}^{r} (1 - \delta_i) \log \left[ 1 + t_i^{b_2} \right] + (N_2 - m_2) \log \left[ 1 + D_2^{b_2} \right] \right) \right\} \\
\times a_2^{m_2+p_2-1},
\]

\[
\Psi_3(b_1|a_1,a_2,b_2,t) \propto b_1^{m_1+p_2-1} \exp \left\{ -v_1 b_1 + b_1 \sum_{i=1}^{r} \delta_i \log t_i - (a_1 + 1) \right\} \\
\times \sum_{i=1}^{r} \delta_i \log \left[ 1 + t_i^{b_1} \right] (a_1 + 1) (N_1 - m_1) \log \left[ 1 + D_1^{b_1} \right]
\]

\[
\Psi_4(b_2|a_1,a_2,b_1,t) \propto b_2^{m_1+p_2-1} \exp \left\{ -v_2 b_2 + b_2 \sum_{i=1}^{r} (1 - \delta_i) \log t_i - (a_2 + 1) \right\} \\
\times \sum_{i=1}^{r} (1 - \delta_i) \log \left[ 1 + t_i^{b_2} \right] (a_2 + 1) (N_2 - m_2) \log \left[ 1 + D_2^{b_2} \right].
\]

(2) The values \(a_1^{(x)}\) and \(a_2^{(x)}\) are generated from gamma distributions (24)

(3) MH algorithms with \(N(b_1^{(x-1)}, \epsilon_{22})\) and \(N(b_2^{(x-1)}, \epsilon_{44})\) proposal distributions are used to generate \(b_1^{(x)}\) and \(b_2^{(x)}\), respectively, as follows

(i) Begin with \(\bar{b}_1\) and \(\bar{b}_2\) as an arbitrary starting point \(b_1^{(0)}\) and \(b_2^{(0)}\) for (24) and (25)

(ii) At time \(k\), sample a candidate point or proposal \(b_1^{*}\), from \(N(b_1^{(k-1)}, \epsilon_{22})\) and \(b_2^{*}\), from \(N(b_2^{(k-1)}, \epsilon_{44})\), the proposal distributions

(iii) Calculate the acceptance probability

\[
\rho_1(b_1^{(k-1)}, b_1^{*}) = \min \left[ 1, \frac{\Psi_3(b_1^{*}|a_1^{(k-1)}, a_2^{(k-1)}, b_2^{(k-1)}, t)}{f(b_1^{(k-1)}|x)} \right]
\]

\[
\rho_2(b_2^{(k-1)}, b_2^{*}) = \min \left[ 1, \frac{\Psi_4(b_2^{*}|a_1^{(k-1)}, a_2^{(k-1)}, b_1^{(k-1)}, t)}{f(b_2^{(k-1)}|x)} \right].
\]
Bayes estimates as well as asymptotic confidence interval and credible interval are summarized in Table 2. The chen in MCMC methods is reported for 11000 iterations that contain the first 1000 samples as burn-in. Usually, it is not hard to construct a Markov chain with the desired properties. To determine how many steps are needed to converge to the stationary, more difficult problem is distribution within an acceptable error. Then, we can test if stationary distribution is reached quickly starting from an arbitrary position. The plot for the simulation number of the model parameters and the corresponding histogram shown in Figures 3–6 can be used to describe the convergence results in MCMC methods.

6. Simulation Studies

The quality of estimators depend on some tolls or measures that are computed for a suitable numbers of generated samples from the populations with given parameter values is known by a simulation study. Then, we assess the theoretical estimation results of MLEs and Bayes estimators under discussing and computing average (AV) and MSEs for point estimate and coverage probability (CP) and average of interval length (AL) to the interval estimation. The simulation study is reported for different sample sizes (N1 + N2) and different effected sample sizes (m). Also, we consider different cases of min number s and different ideal test time (τ∗). Also, we study the effect of parameters change with considering two sets of populatons parameters, say ω = (a1, b1, a2, b2) ∈ ((1,0), (2.0, 2.0), (3.0, 0.4), (1.2, 1.5, 2.0)). The prior parameters are selected to satisfy Eψ, (ωi) = (ρi/υi). Hence, in our simulation study, we proposed different two cases from prior information; one of them is expressed to noninformative prior (prior0), in which the posterior distribution is proportional with likelihood function. The second case is expressed to informative prior (prior and prior1). The prior1 is (ρ1, ρ2, ρ3, ρ4) = (1, 3, 4, 5) and (τ1, τ2, τ3, τ4) = (1, 2, 1.5, 2). The prior2 is (ρ1, ρ2, ρ3, ρ4) = (1, 2, 1.5, 4) and (τ1, τ2, τ3, τ4) = (2.0, 2.0, 2.0, 1.5). For Bayesian approach, without loss of the generality for any loss function, all computations are reported under squared error loss function. The MCMC method is performed under 11000 chen with 1000 burn-in, and the results are reported in Tables 3–6.

7. Concluding Remarks

In the industrial field, the existing different lines of production have the same products under the same facility. The problem of measuring the relative merits of product in the competing duration has considerable attention in past view years. This problem has been discussed in this study for products distributed with Burr XII lifetime distribution. This problem presented in parameters estimation forms with ML and Bayesian estimations under joint type-I GHCS. Then, the developed method is assessed through the Monte Carlo simulation study. The results obtained from these studies show the following comments.
Table 1: The joint type-I GHS data with \((s, m) = (25, 35)\) and \(\tau = 0.5, d = 25\).

| \(s\) | \(m\) | \(0.0459775\) | \(0.0604831\) | \(0.156614\) | \(0.167157\) | \(0.268757\) | \(0.284369\) | \(0.293896\) | \(0.336897\) | \(0.337772\) | \(0.389696\) | \(0.397422\) | \(0.402109\) | \(0.434761\) | \(0.446852\) | \(0.466979\) | \(0.469356\) | \(0.470276\) | \(0.475754\) | \(0.499894\) | \(0.539457\) | \(0.548363\) | \(0.569382\) | \(0.584881\) | \(0.596894\) | \(0.604739\) | \(0.607685\) | \(0.640021\) | \(0.719833\) | \(0.747014\) | \(0.752982\) | \(0.760519\) | \(0.761898\) | \(0.766247\) | \(0.789178\) | \(0.824323\) |
|------|------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| 0    | 1    | 1           | 1           | 1           | 1           | 0           | 0           | 0           | 0           | 1           | 0           | 0           | 0           | 0           | 1           | 0           | 0           | 1           | 0           | 1           | 0           | 1           | 1           | 0           | 1           | 0           | 1           | 0           | 1           |
| 1    | 0    | 1           | 0           | 0           | 1           | 0           | 0           | 1           | 0           | 0           | 1           | 0           | 1           | 0           | 1           | 0           | 1           | 0           | 1           | 0           | 1           | 0           | 1           | 0           | 1           | 0           | 1           | 0           | 1           |
| 0    | 0    | 0           | 1           | 0           | 0           | 1           | 0           | 1           | 0           | 0           | 1           | 0           | 1           | 0           | 1           | 0           | 1           | 0           | 1           | 0           | 1           | 0           | 1           | 0           | 1           | 0           | 1           | 0           | 1           |
| 1    | 1    | 0           | 1           | 0           | 0           | 1           | 0           | 1           | 0           | 0           | 1           | 0           | 1           | 0           | 1           | 0           | 1           | 0           | 1           | 0           | 1           | 0           | 1           | 0           | 1           | 0           | 1           | 0           | 1           |
| 1    | 0    | 0           | 0           | 1           | 0           | 0           | 1           | 0           | 0           | 1           | 0           | 0           | 1           | 0           | 0           | 1           | 0           | 1           | 0           | 1           | 0           | 1           | 0           | 0           | 1           | 0           | 1           | 0           | 1           |

Table 2: The point and 95% confidence intervals (ACIs and CIs) of MLEs Bayes estimates.

| Pa.s \(a_1\), \(b_1\) | \(\hat{\alpha}_{ML}\) | \(\hat{\alpha}_{BMCMC}\) | 95% ACIs | Length | 95% CI | Length |
|----------------------|------------------------|------------------------|----------|--------|--------|--------|
| \(a_1 = 1.82\)       | 2.44789                | 1.86465                | (1.1898, 3.7060) | 2.5163 | (1.1225, 2.8007) | 1.6783  |
| \(b_1 = 1.7\)        | 2.44643                | 2.14852                | (1.5600, 3.3329) | 1.7729 | (1.4706, 2.9120) | 1.4413  |
| \(a_2 = 1.2\)        | 2.02458                | 1.39188                | (0.8110, 3.2382) | 2.4272 | (0.8109, 2.1501) | 1.3392  |
| \(b_2 = 1.92\)       | 1.56413                | 1.82341                | (0.8581, 2.2702) | 1.4121 | (1.1559, 2.5984) | 1.4425  |

Figure 3: Simulation number and the corresponding histogram of \(a_1\) generated by the MCMC method.

Figure 4: Simulation number and the corresponding histogram of \(b_1\) generated by the MCMC method.
Table 3: The AVs and MSEs of Burr XII populations with $\omega = (1.0, 2.0, 2.0, 3.0)$.

| $r^*$ | $(N_1, N_2)$ | $(k, m)$ | Pa. | MLEs | Bayes (prior$_0$) | Bayes (prior$_1$) |
|-------|--------------|-----------|-----|------|------------------|------------------|
|       |              |           |     | AVs  | MSEs            | AVs  | MSEs            | AVs  | MSEs            | AVs  | MSEs            |
| 0.5   | (15, 20)     | (15, 30)  | $a_1$ | 1.232 | 0.324          | 1.241 | 0.327          | 1.210 | 0.210          |
|       |              |           |     | $b_1$ | 2.213          | 0.410 | 2.230          | 0.407 | 2.200          | 0.332 | 2.214          |
|       |              |           | $a_2$ | 2.188 | 0.433          | 2.201 | 0.399          | 2.214 | 0.341          |
|       |              |           |     | $b_2$ | 3.241          | 0.621 | 3.245          | 0.618 | 3.214          | 0.540 | 3.214          |
|       |              |           | $a_1$ | 1.201 | 0.300          | 1.222 | 0.309          | 1.202 | 0.198          |
|       |              |           |     | $b_1$ | 2.200          | 0.389 | 2.215          | 0.398 | 2.187          | 0.301 | 2.190          |
|       |              |           | $a_2$ | 2.174 | 0.414          | 2.198 | 0.401          | 2.201 | 0.317          |
|       |              |           |     | $b_2$ | 3.215          | 0.598 | 3.232          | 0.600 | 3.220          | 0.525 | 3.220          |
|       |              |           | $a_1$ | 1.187 | 0.265          | 1.182 | 0.263          | 1.179 | 0.154          |
|       |              |           |     | $b_1$ | 2.199          | 0.365 | 2.200          | 0.363 | 2.155          | 0.277 | 2.155          |
|       |              |           | $a_2$ | 2.142 | 0.401          | 2.151 | 0.395          | 2.188 | 0.298          |
|       |              |           |     | $b_2$ | 3.202          | 0.575 | 3.225          | 0.580 | 3.202          | 0.478 | 3.202          |
|       |              |           | $a_1$ | 1.160 | 0.230          | 1.177 | 0.225          | 1.161 | 0.115          |
|       |              |           |     | $b_1$ | 2.188          | 0.332 | 2.190          | 0.337 | 2.141          | 0.238 | 2.141          |
|       |              |           | $a_2$ | 2.150 | 0.384          | 2.135 | 0.377          | 2.175 | 0.265          |
|       |              |           |     | $b_2$ | 3.189          | 0.532 | 3.195          | 0.527 | 3.175          | 0.422 | 3.175          |
|       |              |           | $a_1$ | 1.144 | 0.218          | 1.152 | 0.221          | 1.149 | 0.100          |
|       |              |           |     | $b_1$ | 2.146          | 0.311 | 2.155          | 0.314 | 2.122          | 0.219 | 2.122          |
|       |              |           | $a_2$ | 2.122 | 0.373          | 2.130 | 0.369          | 2.165 | 0.240          |
|       |              |           |     | $b_2$ | 3.170          | 0.511 | 3.188          | 0.514 | 3.164          | 0.401 | 3.164          |
### Table 3: Continued.

| $r^*$ | $(N_1, N_2)$ | $(k, m)$ | Pa. | MLEs | Bayes (prior$_0$) | Bayes (prior$_1$) |
|-------|--------------|----------|-----|------|------------------|------------------|
|       |              |          |     | AVs  | MSEs            | AVs  | MSEs            | AVs  | MSEs            |
|       |              |          |     |      |             |      |             |      |             |
|       |              |          |     |      |             |      |             |      |             |
|       |              |          |     |      |             |      |             |      |             |
|       |              |          |     |      |             |      |             |      |             |
| 1.2   | (20, 20)     | (15, 30) |     |      |             |      |             |      |             |
|       |              |          |     |      |             |      |             |      |             |
|       |              |          |     |      |             |      |             |      |             |
|       |              |          |     |      |             |      |             |      |             |
|       |              |          |     |      |             |      |             |      |             |
|       |              |          |     |      |             |      |             |      |             |
|       |              |          |     |      |             |      |             |      |             |
| 0.5   | (25, 30)     | (25, 35) |     |      |             |      |             |      |             |
|       |              |          |     |      |             |      |             |      |             |
|       |              |          |     |      |             |      |             |      |             |
|       |              |          |     |      |             |      |             |      |             |
|       |              |          |     |      |             |      |             |      |             |
|       |              |          |     |      |             |      |             |      |             |
|       |              |          |     |      |             |      |             |      |             |

Table 4: The ALs and PCs of 95% CI of Burr XII populations with \( \omega = (1.0, 2.0, 2.0, 3.0) \).

| $r^*$ | $(N_1, N_2)$ | $(k, m)$ | Pa. | MLEs | Bayes (prior$_0$) | Bayes (prior$_1$) |
|-------|--------------|----------|-----|------|------------------|------------------|
|       |              |          |     |      |             |      |             |      |             |
|       |              |          |     |      |             |      |             |      |             |
|       |              |          |     |      |             |      |             |      |             |
|       |              |          |     |      |             |      |             |      |             |
|       |              |          |     |      |             |      |             |      |             |
|       |              |          |     |      |             |      |             |      |             |
|       |              |          |     |      |             |      |             |      |             |
|       |              |          |     |      |             |      |             |      |             |
|       |              |          |     |      |             |      |             |      |             |
|       |              |          |     |      |             |      |             |      |             |
|       |              |          |     |      |             |      |             |      |             |

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### Table 4: Continued.

| \( \tau^* \) | \((N_1, N_2)\) | \((k, m)\) | Pa. | MLEs | Bayes (prior\(_0\)) | Bayes (prior\(_1\)) |
|---|---|---|---|---|---|---|
| | | | | AVs | MSEs | AVs | MSEs | AVs | MSEs |
| \((15, 20)\) | \(a_1\) | 2.480 | (91) | 2.475 | (91) | 2.085 | (92) |
| | | \(b_1\) | 4.100 | (90) | 4.190 | (90) | 3.541 | (90) |
| | | \(a_2\) | 3.980 | (91) | 3.986 | (91) | 3.533 | (92) |
| | | \(b_2\) | 4.838 | (90) | 4.801 | (91) | 4.112 | (91) |
| \((20, 20)\) | | | | \((15, 30)\) | \(a_1\) | 2.443 | (91) | 2.441 | (93) | 2.052 | (95) |
| | | | | \(b_1\) | 4.072 | (93) | 4.066 | (92) | 3.471 | (93) |
| | | | | \(a_2\) | 3.961 | (91) | 3.955 | (93) | 3.512 | (93) |
| | | | | \(b_2\) | 4.790 | (95) | 4.727 | (93) | 4.000 | (95) |
| | | | | \((25, 30)\) | \(a_1\) | 2.370 | (92) | 2.381 | (93) | 1.744 | (94) |
| | | | | \(b_1\) | 4.012 | (93) | 4.007 | (93) | 3.441 | (92) |
| | | | | \(a_2\) | 3.922 | (92) | 3.905 | (93) | 3.471 | (95) |
| | | | | \(b_2\) | 4.790 | (95) | 4.727 | (93) | 4.000 | (95) |
| | | | | \((30, 30)\) | \(a_1\) | 2.352 | (94) | 2.344 | (92) | 1.652 | (97) |
| | | | | \(b_1\) | 3.871 | (94) | 3.865 | (93) | 3.400 | (93) |
| | | | | \(a_2\) | 3.721 | (91) | 3.718 | (94) | 3.003 | (94) |
| | | | | \(b_2\) | 4.733 | (93) | 4.725 | (93) | 3.842 | (95) |

### Table 5: The AVs and MSEs of Burr XII populations with \( \omega = (0.4, 1.2, 1.5, 2.0) \).

| \( \tau^* \) | \((N_1, N_2)\) | \((k, m)\) | Pa. | MLEs | Bayes (prior\(_0\)) | Bayes (prior\(_1\)) |
|---|---|---|---|---|---|---|
| | | | | AVs | MSEs | AVs | MSEs | AVs | MSEs |
| \((15, 20)\) | \(a_1\) | 0.589 | 0.121 | 0.585 | 0.122 | 0.564 | 0.091 |
| | | \(b_1\) | 1.444 | 0.371 | 1.440 | 0.365 | 1.399 | 0.277 |
| | | \(a_2\) | 1.465 | 0.432 | 1.522 | 0.370 | 1.539 | 0.309 |
| | | \(b_2\) | 4.731 | 0.088 | 0.555 | 0.090 | 0.534 | 0.068 |
| | | | | \((20, 20)\) | \(a_1\) | 1.413 | 0.315 | 1.417 | 0.324 | 1.329 | 0.227 |
| | | | | \(b_1\) | 1.514 | 0.340 | 1.538 | 0.338 | 1.530 | 0.282 |
| | | | | \(a_2\) | 2.310 | 0.399 | 2.299 | 0.395 | 2.231 | 0.281 |
| | | | | \(b_2\) | 4.731 | 0.088 | 0.555 | 0.090 | 0.534 | 0.068 |
| | | | | \((25, 30)\) | \(a_1\) | 0.522 | 0.071 | 0.531 | 0.068 | 0.531 | 0.049 |
| | | | | \(b_1\) | 1.455 | 0.311 | 1.402 | 0.307 | 1.299 | 0.211 |
| | | | | \(a_2\) | 1.535 | 0.318 | 1.531 | 0.325 | 1.537 | 0.269 |
| | | | | \(b_2\) | 2.301 | 0.362 | 2.299 | 0.368 | 2.228 | 0.268 |
| | | | | \((30, 30)\) | \(a_1\) | 0.489 | 0.061 | 0.499 | 0.067 | 0.488 | 0.035 |
| | | | | \(b_1\) | 1.381 | 0.277 | 1.388 | 0.269 | 1.290 | 0.210 |
| | | | | \(a_2\) | 1.525 | 0.303 | 1.521 | 0.301 | 1.545 | 0.255 |
| | | | | \(b_2\) | 2.280 | 0.355 | 2.275 | 0.338 | 2.015 | 0.261 |
| | | | | \((25, 35)\) | \(a_1\) | 0.471 | 0.044 | 0.460 | 0.041 | 0.466 | 0.022 |
| | | | | \(b_1\) | 1.358 | 0.264 | 1.366 | 0.261 | 1.271 | 0.181 |
| | | | | \(a_2\) | 1.527 | 0.299 | 1.536 | 0.300 | 1.536 | 0.256 |
| | | | | \(b_2\) | 2.272 | 0.344 | 2.266 | 0.351 | 2.225 | 0.251 |
### Table 5: Continued.

| $r^*$ | $(N_1, N_2)$ | $(k, m,)$ | Pa. | MLEs AVs | MLEs MSEs | Bayes (prior$_0$) AVs | Bayes (prior$_0$) MSEs | Bayes (prior$_1$) AVs | Bayes (prior$_1$) MSEs |
|-------|-------------|------------|-----|----------|----------|----------------------|----------------------|----------------------|----------------------|
|       |             |            |     |          |          |                      |                      |                      |                      |
|       | (15, 20)    | (15, 30)   |     | $a_1$    | 0.562    | 0.085    | 0.555                | 0.082                | 0.532                | 0.061                |
|       |             |            |     | $b_1$    | 1.421    | 0.331    | 1.412                | 0.328                | 1.385                | 0.238                |
|       |             |            |     | $a_2$    | 1.524    | 0.345    | 1.511                | 0.341                | 1.518                | 0.284                |
|       |             |            |     | $b_2$    | 2.312    | 0.410    | 2.315                | 0.418                | 2.010                | 0.299                |
|       |             |            |     | $a_1$    | 0.541    | 0.076    | 0.538                | 0.072                | 0.520                | 0.049                |
|       |             |            |     | $b_1$    | 1.399    | 0.302    | 1.402                | 0.307                | 1.318                | 0.211                |
|       | (15, 20)    | (20, 20)   |     | $a_1$    | 1.522    | 0.321    | 1.521                | 0.318                | 1.500                | 0.265                |
|       |             | (15, 30)   |     | $b_1$    | 2.291    | 0.384    | 2.288                | 0.378                | 0.213                | 0.272                |
|       |             |            |     | $a_1$    | 0.501    | 0.057    | 0.505                | 0.058                | 0.518                | 0.028                |
|       |             | (25, 30)   |     | $b_1$    | 1.378    | 0.287    | 1.400                | 0.290                | 1.287                | 0.199                |
|       |             | (25, 30)   |     | $a_2$    | 1.517    | 0.300    | 1.519                | 0.298                | 1.513                | 0.251                |
|       |             | (25, 30)   |     | $b_2$    | 2.288    | 0.345    | 2.278                | 0.339                | 0.211                | 0.247                |
|       |             | (25, 30)   |     | $a_1$    | 0.479    | 0.042    | 0.480                | 0.038                | 0.477                | 0.017                |
|       |             | (25, 35)   |     | $b_1$    | 1.366    | 0.260    | 1.370                | 0.255                | 1.277                | 0.175                |
|       |             | (30, 30)   |     | $a_1$    | 1.512    | 0.298    | 1.511                | 0.285                | 1.530                | 0.241                |
|       |             | (30, 30)   |     | $b_1$    | 2.676    | 0.332    | 2.240                | 0.338                | 0.210                | 0.247                |
|       |             | (25, 35)   |     | $b_1$    | 0.455    | 0.037    | 0.445                | 0.029                | 0.454                | 0.012                |
|       |             | (25, 50)   |     | $a_1$    | 1.510    | 0.284    | 1.530                | 0.277                | 1.520                | 0.241                |
|       |             | (25, 50)   |     | $b_1$    | 2.267    | 0.332    | 2.240                | 0.338                | 0.210                | 0.239                |

### Table 6: The ALs and PCs of 95% CI of Burr XII populations with $\omega = (0.4, 1.2, 1.5, 2.0)$.

| $r^*$ | $(N_1, N_2)$ | $(k, m,)$ | Pa. | MLEs ALs | MLEs PCs | Bayes (prior$_0$) ALs | Bayes (prior$_0$) PCs | Bayes (prior$_1$) ALs | Bayes (prior$_1$) PCs |
|-------|-------------|------------|-----|----------|----------|----------------------|----------------------|----------------------|----------------------|
|       |             |            |     |          |          |                      |                      |                      |                      |
|       | (15, 20)    | (15, 30)   |     | $a_1$    | 1.752    | (89)      | 1.744                | (89)                 | 1.610                | (90)                 |
|       |             |            |     | $b_1$    | 3.245    | (90)      | 3.260                | (90)                 | 2.452                | (91)                 |
|       |             | (25, 30)   |     | $a_1$    | 3.745    | (88)      | 3.749                | (89)                 | 3.541                | (90)                 |
|       |             | (25, 30)   |     | $b_1$    | 4.452    | (89)      | 4.445                | (90)                 | 4.223                | (90)                 |
|       | (20, 20)    | (15, 30)   |     | $a_1$    | 1.700    | (90)      | 1.702                | (89)                 | 1.564                | (91)                 |
|       |             |            |     | $b_1$    | 3.210    | (90)      | 3.212                | (92)                 | 2.403                | (92)                 |
|       |             | (25, 30)   |     | $a_1$    | 3.700    | (90)      | 3.703                | (90)                 | 3.500                | (92)                 |
|       |             | (25, 30)   |     | $b_1$    | 4.411    | (91)      | 4.402                | (90)                 | 4.185                | (92)                 |
|       |             |            |     | $a_1$    | 1.640    | (92)      | 1.652                | (90)                 | 1.511                | (93)                 |
|       |             | (25, 35)   |     | $b_1$    | 3.153    | (92)      | 3.280                | (92)                 | 2.379                | (92)                 |
|       |             | (30, 30)   |     | $a_1$    | 3.640    | (92)      | 3.632                | (92)                 | 3.470                | (94)                 |
|       |             | (25, 35)   |     | $b_1$    | 4.382    | (91)      | 4.379                | (93)                 | 4.162                | (93)                 |
|       |             | (25, 50)   |     | $a_1$    | 3.122    | (92)      | 3.145                | (92)                 | 2.344                | (96)                 |
|       |             | (30, 30)   |     | $a_1$    | 3.611    | (92)      | 3.608                | (96)                 | 3.444                | (94)                 |
|       |             | (25, 50)   |     | $b_1$    | 4.324    | (94)      | 4.333                | (93)                 | 4.141                | (94)                 |
|       |             |            |     | $a_1$    | 1.514    | (95)      | 1.502                | (94)                 | 1.432                | (93)                 |
|       |             | (25, 50)   |     | $b_1$    | 3.100    | (92)      | 3.104                | (92)                 | 2.312                | (96)                 |
|       |             |            |     | $a_2$    | 3.581    | (92)      | 3.575                | (96)                 | 3.402                | (94)                 |
|       |             | (25, 50)   |     | $b_2$    | 4.284    | (94)      | 4.279                | (93)                 | 4.113                | (94)                 |
All results obtained in Tables 3–6 show that the developed method works well in all cases under joint type-I GHCS for Burr XII lifetime products. The results under MLE and noninformative Bayes estimation are close to itself. The Bayes method performs better than the ML method under informative prior. The MSEs and interval length are reduced under increases in effective sample size \((s, m)\). The results perform better for the large value of test time \(\tau^*\). Under parameters change, the results of the simulation study show the validity of the results of all parameters chosen.

### Data Availability

The data used to support the findings of this study are included within the article.

### Conflicts of Interest

The authors declare that they have no conflicts of interest.

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