SCHUNCK CLASSES OF SOLUBLE RESTRICTED LIE ALGEBRAS

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ABSTRACT. I set out the theory of Schunck classes and projectors for soluble restricted Lie algebras and investigate its links to the corresponding theory for ordinary soluble Lie algebras over a field $F$ of characteristic $p \neq 0$.

1. Introduction

Schunck classes and formations were originally defined in the context of finite soluble groups where they provide families of intravariant subgroups, the $\mathfrak{H}$-projectors. Development of the analogous theory for finite dimensional soluble Lie algebras was started by Barnes and Gastineau-Hills in [6] and extended by Stitzinger in [14] and [15]. For Lie algebras, the notion of intravariance has to be defined in terms of derivations rather than automorphisms. It was shown in Barnes [2], that the Lie algebra $\mathfrak{H}$-projectors are intravariant in this sense.

The theories of Schunck classes of soluble Lie algebras and of restricted Lie algebras over a field $F$ of characteristic $p$ do not fit together smoothly. The restrictable Lie algebras do not form a Schunck class. Every non-zero Schunck class contains all nilpotent algebras, but not every nilpotent Lie algebra is restrictable. The two theories however are linked by Theorem 6.4 of Barnes [5], which I quote here for convenience of reference.

Theorem 1.1. Let $(L, [p])$ be a restricted Lie algebra over the field $F$ of characteristic $p \neq 0$ and suppose that $z[p] = 0$ for all $z$ in the centre of $L$. Let $\mathfrak{H}$ be a saturated formation and suppose $S \neq 0$ is subnormal in $L$ and $S \in \mathfrak{H}$. Let $V$ be an irreducible $p$-module of $L$. Then $V$ is $S\mathfrak{H}$-hypercentral.

Developments in the theory of finite groups have led to changes in the terminology which have not so far been copied in the Lie algebra literature. In this paper, I follow the terminology which has become standard in finite group theory, set out in Doerk and Hawkes [9]. I construct a theory of Schunck classes, saturated formations and projectors within the category of finite-dimensional soluble restricted Lie algebras over a field $F$ of characteristic $p \neq 0$ and investigate its relationship to the theory for ordinary soluble Lie algebras over the same field. We shall see that if a restricted Lie algebra $(L, [p])$ is in the Schunck class $\mathfrak{H}$ which contains all abelian algebras and if $[p]'$ is another $p$-operation on $L$, then $(L, [p]')$ is also in $\mathfrak{H}$. Thus the theory becomes in effect, a theory of Schunck classes and saturated formations of restrictable soluble Lie algebras.

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The development in Doerk and Hawkes is based on operations on classes of groups. In whatever category we are working, by a class of objects, we mean a collection $\mathfrak{X}$ of objects of the category with the property that if $X \in \mathfrak{X}$ and $Y \simeq X$, then $Y \in \mathfrak{X}$. The foundational issue of operations defined on classes is easily avoided. Because our algebras are all finite-dimensional over a given field $F$, we can construct a set of representative algebras such that every object of our category is isomorphic to at least one of them. Our operations could then be defined in terms of these representatives, at the cost of clumsiness of terminology. I shall not bother with this.

The arguments used in Doerk and Hawkes [9], with the exception of those referring to conjugacy, are not specific to groups but are readily adapted to other types of algebras subject to the following conditions. Firstly, we require that there be an action of an algebra on itself such that the subalgebras stable under this action are precisely the kernels of homomorphisms. For groups, this action is conjugation. For Lie algebras, it is the adjoint action. Secondly, we require a finiteness condition, some measure of “size” of an algebra such that proper subalgebras and proper quotients have smaller size. Thirdly, we require the solubility condition that for the action of an algebra $A$ on a chief factor $B/C$, $B$ is contained in the kernel of that action. These conditions ensure a covering-avoidance property: if $M$ is a maximal subalgebra of $A$ and $B/C$ is a chief factor of $A$, then $M$ either covers $B/C$, that is, $M + C \supseteq B$ or avoids it, that is, $M \cap B \subseteq C$. A fourth requirement, related to solubility, is that if $B$ is precisely the kernel of the action of $A$ on the chief factor $B/C$, then $A/C$ splits over $B/C$.

In Sections 2 and 3, I set out some preliminary results. Once we have these, the theory can be developed much as the theory for finite groups or ordinary Lie algebras. This is done in Sections 4, 5 and 6. The intravariance of projectors is established in Section 7. In Section 8, we look at the relationship between the two theories. In Section 9, we consider $p$-envelopes as another link between them.

Finite dimensional restricted Lie algebras over a field $F$ of characteristic $p \neq 0$ and infinitesimal algebraic group schemes over $F$ of height $\leq 1$ are equivalent categories. (See Demazure and Gabriel [5, Chapter II, §7, Proposition 4.1].) Working in the context of group schemes, Voigt [16] has obtained some of the preliminary results given below. For most of these, the proofs given here are shorter and more elementary.

2. Preliminaries

All algebras considered will be finite-dimensional over a field $F$ of characteristic $p \neq 0$. A restricted Lie algebra, as well as the usual Lie algebra operations, has a $p$-operation $[p] : L \to L$ satisfying $\text{ad}(x)^p = \text{ad}(x^{[p]})$, $(\lambda x)^{[p]} = \lambda^p x^{[p]}$ and $(x + y)^{[p]} = x^{[p]} + y^{[p]} + S(x, y)$ where $S(x, y)$ is a function whose formal definition and basic properties are set out in Strade and Farnsteiner [13, Chapter 2]. In the special case where $xy = 0$, this simplifies to $(x + y)^{[p]} = x^{[p]} + y^{[p]}$. Knowledge of the formal definition is not needed for this paper. We use the basic properties of restricted Lie algebras set out in Strade and Farnsteiner [13, Chapter 2]. Construction of examples will be based on the following theorem of Jacobson (see Jacobson [11, Theorem V.11, p.190] or Strade and Farnsteiner [13, Theorem 2.3, p.71]), of which we shall make frequent use:
Theorem 2.1. Let \( \{a_1, \ldots, a_n\} \) be a basis of the Lie algebra \( L \) and let \( b_1, \ldots, b_n \) be elements of \( L \) such that \( \text{ad}(a_i)^p = \text{ad}(b_i) \) for all \( i \). Then there exists one and only one \( p \)-operation \([p] \) on \( L \) such that \( a_i^{[p]} = b_i \) for all \( i \).

A Lie algebra \( L \) is called restrictable if, for all \( x \in L \), \( \text{ad}(x)^p \) is an inner derivation. By Jacobson’s Theorem, if \( L \) is restrictable, then there exists at least one \( p \)-operation on \( L \). Further, if \( L \) is restrictable and \( A \) is an ideal of \( L \), then \( L/A \) is restrictable. We shall from time to time, have need to consider restrictable Lie algebras \((L,[p])\) and \((L,[p]^\prime)\) with the same underlying Lie algebra \( L \). Note that in this situation, \( x^{[p]} \) and \( x^{[p]^\prime} \) differ by an element of the centre \( Z(L) \). A subalgebra \( U \) of \( L \) is called a \([p]\)-subalgebra if \( U^{[p]} \subseteq U \). As the centre \( Z(L) \) of \((L,[p])\) and so all terms of the ascending central series are \([p]\)-ideals, there is no need to distinguish nilpotency of \((L,[p])\) from nilpotency of the underlying Lie algebra \( L \). There is likewise no need to distinguish solubility from that of the underlying algebra. That \((L,[p])\) has a \([p]\)-chief series with all quotients abelian if \( L \) is soluble follows from the following lemma.

Lemma 2.2. Let \((L,[p])\) be a restricted Lie algebra whose underlying algebra \( L \neq 0 \) is soluble. Then \((L,[p])\) has a non-zero abelian \([p]\)-ideal.

Proof. If \( Z(L) \neq 0 \), then it is the required non-zero abelian \([p]\)-ideal, so suppose \( Z(L) = 0 \). Let \( A \) be a minimal ideal of \( L \). For any \( a \in A \), \( \text{ad}(a)^p = 0 \), so \( a^{[p]} \in Z(L) = 0 \), and \( A \) is a \([p]\)-ideal. \( \Box \)

Note however, that the derived algebra of a restricted Lie algebra need not be a \([p]\)-ideal.

Example 2.3. Let \( L = \langle a, b, c \rangle \) with multiplication given by \( ab = b, ac = bc = 0 \). By Jacobson’s Theorem (Theorem 2.1), \( L \) has a \( p \)-operation with \( a^{[p]} = a, b^{[p]} = c \) and \( c^{[p]} = 0 \). Then \( L' = \langle b \rangle \) which is not a \([p]\)-ideal.

Example 2.4. Let \( N = \langle a, b, c, d \rangle \) with multiplication given by \( ab = c, ac = bc = ad = bd = cd = 0 \). Setting \( a^{[p]} = b^{[p]} = d^{[p]} = 0 = 0 \) and \( c^{[p]} = d \) defines a \( p \)-operation on the nilpotent algebra \( N \). Then \( N' = \langle c \rangle \) which is not a \([p]\)-ideal.

In the following, I shall define analogues for restricted Lie algebras of concepts used in the theory of ordinary Lie algebras. Where these refer to a class of restricted Lie algebras, I attach the prefix “\( p \)” to the name of the concept. Where it depends on a particular \( p \)-operation, I attach the \( p \)-operation as prefix. Thus, I shall refer to \( p \)-formations and, as above, to \([p]\)-subalgebras. Where the meaning is clear from the context, I shall often simplify notation by writing \( L \) rather than \((L,[p])\).

Much of the theory of Schunck classes and projectors relies on lemmas asserting that, under certain circumstances, the Lie algebra \( L \) splits over some abelian ideal \( A \), that is, that there exists a subalgebra \( U \) such that \( U + A = L \) and \( U \cap A = 0 \). We shall need lemmas giving the existence of a \([p]\)-subalgebra with these properties. Note that it is possible for the underlying Lie algebra \( L \) to split over a \([p]\)-ideal \( A \) without there being any \([p]\)-subalgebra which complements \( A \).

Example 2.5. Let \( L = \langle a, b \rangle \) with \( ab = 0 \) and \( a^{[p]} = 0, b^{[p]} = a \). Then \( A = \langle a \rangle \) is a \([p]\)-ideal which is complemented in the underlying algebra but which has no complementary \([p]\)-subalgebra.
Lemma 2.6. Let \((L, [p])\) be a restricted Lie algebra. Suppose \(Z(L) = 0\). Let \(\alpha\) be an automorphism of \(L\). Then \(\alpha(x)[p] = \alpha(x^p)\).

Proof. For \(x, y \in L\), we have

\[
\text{ad}(\alpha(x))^p\alpha(y) = \alpha(\text{ad}(x)^p y) = \alpha(x^p y) = \alpha(x)[p] \alpha(y).
\]

Thus \(\text{ad}(\alpha(x))^p = \text{ad}(\alpha(x)[p])\). As \(Z(L) = 0\), this implies \(\alpha(x)^p = \alpha(x[p])\).

Lemma 2.7. Suppose \(A\) is an abelian non-central minimal \([p]\)-ideal of \((L, [p])\) and that \(M\) is a subalgebra of \(L\) which complements \(A\). Then \(M\) is a maximal \([p]\)-subalgebra of \(L\).

Proof. We have to prove that \(M\) is a \([p]\)-subalgebra of \(L\). That it is maximal then follows. Let \(x \in M\). Then \(x^p\) is uniquely expressible in the form \(x^p = x' + a\) with \(x' \in M\) and \(a \in A\). For \(y \in M\), \(\text{ad}(x)^p y \in M\). But

\[
\text{ad}(x)^p y = x^p y = x'y + ay,
\]

so \(ay \in M\). But \(ay \in A\), so \(ay = 0\) for all \(y \in M\). Since \(A\) is abelian, \(ay = 0\) also for all \(y \in A\), so \(a \in Z(L) \cap A = 0\). Thus \(x^p \in M\).

The theory of Schunck classes of soluble Lie algebras makes use of primitive algebras. I set out here the properties of primitive restricted Lie algebras.

Definition 2.8. A soluble restricted Lie algebra \((L, [p])\) is called primitive if it has a minimal \([p]\)-ideal \(A\) with \(C_L(A) = A\). The minimal \([p]\)-ideal \(A\) is called the socle of \((L, [p])\) and denoted by \(\text{Soc}(L, [p])\).

If \(A\) is an \(L\)-module, we put \(A^L = \{ a \in A \mid xa = 0 \text{ for all } x \in L \}\). We use \(H^n(L, A)\) to denote the ordinary cohomology of \(L\) acting on \(A\). The following lemma was proved by Voigt in [10] Remark 2.12, p.93 using the theory of group schemes.

Lemma 2.9. Let \((L, [p])\) be a primitive soluble restricted Lie algebra and let \(A = \text{Soc}(L, [p])\). Then for all \(n\), we have \(H^n(L, A, A) = 0\) and there exists a subalgebra \(M\) which complements \(A\), all such are conjugate under automorphisms of the form \(\alpha_a = 1 + \text{ad}(a)\) for \(a \in A\) and are maximal \([p]\)-subalgebras of \((L, [p])\).

Proof. The result is trivial if \(A = L\), so suppose \(A \neq L\). Let \(B/A\) be a minimal \([p]\)-ideal of \(L/A\). Then \(A^{B/A} = 0\) and so \(H^\beta(B/A, A) = 0\) for all \(\beta\). Thus \(H^n(L/B, H^\beta(B/A, A)) = 0\) for all \(\alpha, \beta\) and we have \(H^p(L/A, A) = 0\) for all \(n\). It follows that \(L\) splits over \(A\) as ordinary Lie algebra and that all complements to \(A\) in \(L\) are conjugate as asserted. That the complements are maximal \([p]\)-subalgebras follows by Lemma [10].

There is only one minimal soluble Lie algebra, namely the 1-dimensional algebra with zero multiplication. Taking this with the zero \(p\)-operation gives a restricted Lie algebra. The following restricted Lie algebras appear in Hochschild [10], where they are called strongly abelian.

Definition 2.10. A restricted Lie algebra \((L, [p])\) is called null if it is abelian and \(L[p] = 0\).
The 1-dimensional null algebra is not the only minimal soluble restricted Lie algebra. For example, for $A = \langle a \rangle$, we can set $a^{[p]} = a$ giving a non-null 1-dimensional restricted Lie algebra. Depending on the field, there could be other abelian algebras with no proper $[p]$-subalgebras. I shall call any of these minimal objects an atom.

**Lemma 2.11.** Let $A/B$ be a $[p]$-chief factor of the soluble restricted Lie algebra $(L, [p])$. Then $A/B$ is abelian. Either $A/B$ is null, $A^{[p]} \subseteq B$, or $A/B$ is central and is an atom.

**Proof.** By Lemma 2.12, the $[p]$-chief factors of $(L, [p])$ are abelian. As we can work in $L/B$, we may suppose $B = 0$. Let $A_0 \subseteq A$ be a minimal ideal of $L$. Suppose $a \in A_0$ and that $a^{[p]} \neq 0$. Since $\text{ad}(a)^2 = 0$, $a^{[p]} \in \mathcal{Z}(L)$. Thus $A \cap \mathcal{Z}(L) \neq 0$ and it follows that $A \subseteq \mathcal{Z}(L)$. Any $[p]$-subalgebra of $A$ is a $[p]$-ideal of $L$, so $A$ is an atom. So, if $A \not\subseteq \mathcal{Z}(L)$, we have $a^{[p]} = 0$ for all $a \in A_0$, $A_0$ is a null $[p]$-ideal and $A_0 = A$. □

Atoms are primitive restricted Lie algebras. Any non-abelian primitive restrictable Lie algebra has trivial centre and so has only one $p$-operation.

**Lemma 2.12.** Let $(L, [p])$ be a restricted Lie algebra and let $A$ be an abelian ideal of the underlying algebra $L$. Then there exists a $p$-operation $[p]'$ on $L$ such that $A$ is a null $[p]'$-ideal of $L$.

**Proof.** Take a basis $a_1, \ldots, a_r$ of $A$ and extend with elements $b_1, \ldots, b_s$ to a basis of $L$. Since $\text{ad}(a_i)^2 = 0$, by Theorem 2.11 there exists a $p$-operation $[p]'$ on $L$ with $a_i^{[p]'} = 0$ and $b_j^{[p]'} = b_j^{[p]}$. Then $A$ is a null $[p]'$-ideal of $L$. □

**Corollary 2.13.** Let $K$ be an ideal of the soluble restrictable Lie algebra $L$. Then $L/K$ is restrictable.

**Proof.** If $K = 0$, then the result holds, so suppose $K \neq 0$. Let $A \subseteq K$ be a minimal ideal of $L$. By Lemma 2.12, $A$ is a $[p]'$-ideal of $L$ for some $p$-operation $[p]'$ on $L$. Thus $L/A$ is restrictable. By induction, $L/K$ is restrictable. □

**Lemma 2.14.** Let $(L, [p])$ be a soluble restricted Lie algebra and let $L/K$ be a non-abelian primitive quotient of $L$. Then $K$ is a $[p]$-ideal of $L$.

**Proof.** We may suppose $K \neq 0$. Let $A \subseteq K$ be a minimal ideal of $L$. By Lemma 2.12, $A$ is a $[p]'$-ideal of $L$ for some $p$-operation $[p]'$ on $L$. By induction, $K/A$ is a $[p]'$-ideal of $L/A$. But $K \supseteq \mathcal{Z}(L)$ since $L/K$ is non-abelian primitive. For $k \in K$, we have $k^{[p]} - k^{[p]'} \in \mathcal{Z}(L)$ and $k^{[p]'} \in K$. Thus $k^{[p]} \in K$ and $K$ is a $[p]$-ideal of $L$. □

**Lemma 2.15.** Suppose every $[p]$-chief factor of $(L, [p])$ is non-null. Then $L$ is abelian.

**Proof.** Since every $[p]$-chief factor is central, $L$ is nilpotent. Let $A$ be a minimal $[p]$-ideal of $L$. Then $A$ is central and, by induction over $\dim(L)$, $L/A$ is abelian. For all $x \in L$, we have $\text{ad}(x)^2 = 0$ and so $x^{[p]} \in \mathcal{Z}(L)$. But $(L^{[p]}) = L$, so $\mathcal{Z}(L) = L$. □

**Lemma 2.16.** Let $A$ be a null minimal $[p]$-ideal of $L$. Suppose that every chief factor of $L/A$ is non-null. Then there exists a maximal $[p]$-subalgebra $M$ which complements $A$. 

Lemma 2.7, \[ M \times \] for all \( L \).

Proof. By Lemma 2.15, \[ \text{Definition 3.1.} \]

The \([E] \text{ intersection } \] of the maximal \( p \)-subalgebras of \( (L, [p]) \).

If \( \Psi \) is a \([p] \text{-ideal } \] of \( (L, [p]) \), then the result holds, so suppose there exists a maximal \([p] \text{-subalgebra } M \) which does not contain any minimal \([p] \text{-ideal } \). Then \( M \cap A = 0 \) and \( M + A = L \). If \( C_L(A) \neq A \), then \( M \cap C_L(A) \) is a \([p] \text{-ideal } \), so \( C_L(A) = A \). Thus \( Z(L) = 0 \), \( A \) is a minimal ideal of \( L \) and \( L \) is primitive. The complements to \( A \) are \([p] \text{-subalgebras, so } \Psi(L) = \Phi(L) = 0 \). □

3. The \([p] \text{-Frattini subalgebra} \]

Definition 3.1. The \([p] \text{-Frattini subalgebra } \Psi(L, [p]) \text{ of } (L, [p]) \) is the intersection of the maximal \([p] \text{-subalgebras of } (L, [p]) \).

The following is Voigt [10] Theorem 2.88, p.247).

Lemma 3.2. If \( (L, [p]) \) is soluble then \( \Psi(L, [p]) \) is a \([p] \text{-ideal } \) of \( L \).

Proof. Trivially, \( \Psi \) is a \([p] \text{-subalgebra. We have to prove it an ideal. We use induction over } \dim(L) \). Let \( A \) be a minimal \([p] \text{-ideal of } L \). By induction, the intersection \( N \) of the maximal \([p] \text{-subalgebras which contain } A \) is an ideal. If every maximal \([p] \text{-subalgebra contains some minimal \([p] \text{-ideal, then the result holds, so suppose there exists a maximal \([p] \text{-subalgebra } M \) which does not contain any minimal \([p] \text{-ideal } \). Then } M \cap A = 0 \text{ and } M + A = L \). If \( C_L(A) \neq A \), then \( M \cap C_L(A) \) is a \([p] \text{-ideal, so } C_L(A) = A \). Thus \( Z(L) = 0 \), \( A \) is a minimal ideal of \( L \) and \( L \) is primitive. The complements to \( A \) are \([p] \text{-subalgebras, so } \Psi(L) = \Phi(L) = 0 \). □

For an element \( a \) of any finite-dimensional Lie algebra \( L \), the Engel subalgebra \( E_L(a) \) is the Fitting null space \( \ker(\text{ad}(a)^n) \) of \( \text{ad}(a) \) for sufficiently large \( n \). It is a subalgebra with the property that any subalgebra containing \( E_L(a) \) is self-normalising. Also \( L \) is the vector space direct sum

\[ \text{im}(\text{ad}(a)^n) \oplus \ker(\text{ad}(a)^n) \]

for sufficiently large \( n \).

Lemma 3.3. Let \( (L, [p]) \) be a restricted Lie algebra and let \( a \in L \). Then \( E_L(a) \) is a \([p] \text{-subalgebra of } (L, [p]) \).

Proof. If \( b \in E = E_L(a) \), then \( ab^{[p]} = -b^{[p]}a = -\text{ad}(b)^{[p]}a \in E \) since \( E \) is a subalgebra. But then \( \text{ad}(a)^n(ab^{[p]}) = 0 \) for some \( n \), that is, \( b^{[p]} \in E \). □

The following result was proved by Voigt [10] Corollary 2.92, p. 253] for \( F \) algebraically closed.

Corollary 3.4. Let \( (L, [p]) \) be a restricted Lie algebra. Suppose every maximal \([p] \text{-subalgebra of } (L, [p]) \) is an ideal. Then \( (L, [p]) \) is nilpotent.

Proof. The \([p] \text{-subalgebra } E_L(a) \) is not contained in any maximal \([p] \text{-subalgebra. Hence } E_L(a) = L \) for all \( a \in L \) and \( L \) is nilpotent by Engel’s Theorem. □

Definition 3.5. A \([p] \text{-subalgebra } S \) of \( (L, [p]) \) is called \([p] \text{-subnormal } \) if there exists a chain

\[ S = S_0 \subseteq S_1 \subseteq \cdots \subseteq S_n = L \]

where each \( S_i \) is a \([p] \text{-ideal of } S_{i+1} \).

A weak form of the following is given in Voigt [10] Theorem 2.86, p. 246]. (A stronger result is given in Theorem 4.25 below.)
Lemma 3.6. Let \((L, [p])\) be a (not necessarily soluble) restricted Lie algebra and let \(A\) be a \([p]\)-subnormal subalgebra of \((L, [p])\). Let \(B \subseteq A \cap \Psi(L, [p])\) be a \([p]\)-ideal of \(A\). Suppose \(A/B\) is nilpotent. Then \(A\) is nilpotent.

**Proof.** For \(a \in A\) and sufficiently large \(n\), we have \(\text{im} (\text{ad}(a))^{n} \subseteq B\) since \(A\) is subnormal and \(A/B\) is nilpotent. Therefore \(B + E_{L}(a) = L\). But \(B \subseteq \Psi(L, [p])\), so \(E_{L}(a) = L\). By Engel’s Theorem, \(A\) is nilpotent. \(\square\)

Corollary 3.7. Let \((L, [p])\) be a soluble restricted Lie algebra. Then \(\Psi(L, [p])\) is nilpotent.

**Proof.** By Lemma 3.6, \(\Psi(L, [p])\) is a \([p]\)-ideal. \(\square\)

**Lemma 3.8.** Let \((L, [p])\) be a soluble restricted Lie algebra. Then

\[ \Psi(L, [p]) \supseteq \Phi(L). \]

**Proof.** We use induction over \(\dim(L)\). Let \(A\) be a minimal \([p]\)-ideal of \((L, [p])\). Then the intersection of the maximal \([p]\)-subalgebras containing \(A\) contains the intersection of the maximal subalgebras of the underlying algebra \(L\) which contain \(A\). It follows that the intersection of all maximal \([p]\)-subalgebras contains the intersection of all maximal subalgebras unless there exists a maximal \([p]\)-subalgebra \(M\) which contains no minimal \([p]\)-ideal. But then \(M\) complements the minimal \([p]\)-ideal \(A\). Since \(C_{n}(A)\) is a \([p]\)-ideal, we must have \(C_{n}(A) = A\). \((L, [p])\) is primitive and \(\Psi(L, [p]) = 0\). As \(A\) is also a minimal ideal of \(L\), \(L\) is primitive and \(\Phi(L) = 0\). \(\square\)

Note that \(\Psi(L, [p])\) can be strictly greater than \(\Phi(L)\), as is the case in Examples 2.3, 2.4.

4. Schunck Classes and Projectors

Following the notations of Doerk and Hawkes, I denote the class of all soluble restricted Lie algebras by \(\mathcal{S}_{p}\), the class of nilpotent restricted Lie algebras by \(\mathcal{N}_{p}\), the class of abelian restricted Lie algebras by \(\mathcal{A}_{p}\), and the class of primitive restricted Lie algebras by \(\mathcal{P}_{p}\), while \(\mathcal{S}, \mathcal{N}, \mathcal{A}\) and \(\mathcal{P}\) denote the corresponding class of ordinary Lie algebras. For a class \(\mathfrak{X}\), I define

- \(\mathcal{Q}\mathfrak{X} = \{ (L/K, [p]) \mid (L, [p]) \in \mathfrak{X} \}\)
- \(\mathcal{R}\mathfrak{X} = \{ (L, [p]) \mid \exists \ [p]\)-ideals \(K_{i}\) of \((L, [p])\) with \((L/K_{i}) \in \mathfrak{X}\) and \(\cap_{i} K_{i} = 0\}\)
- \(\mathcal{E}_{\Phi}\mathfrak{X} = \{ (L, [p]) \mid \exists [p]\)-ideal \(K\) of \((L, [p])\) with \(K \leq \Psi(L, [p])\) and \((L/K, [p]) \in \mathfrak{X}\}\)
- \(\mathcal{P}\mathfrak{X} = \{ (L, [p]) \mid \exists (L, [p]) \cap \mathcal{P}_{p} \subseteq \mathfrak{X} \}\).

Thus \(\mathcal{Q}\mathfrak{X}\) is the class of quotients of restricted Lie algebras in \(\mathfrak{X}\), \(\mathcal{R}\mathfrak{X}\) is the class of subdirect sums and \(\mathcal{E}_{\Phi}\mathfrak{X}\) the class of Frattini extensions of algebras in \(\mathfrak{X}\), while \(\mathcal{P}\mathfrak{X}\) is the class of all algebras whose primitive quotients are in \(\mathfrak{X}\).

**Definition 4.1.** A non-empty class \(\mathfrak{X}\) of soluble restricted Lie algebras which is \(\mathcal{Q}\)-closed, that is, \(\mathcal{Q}\mathfrak{X} = \mathfrak{X}\), is called a \(p\)-homomorph. An \(R\)-closed \(p\)-homomorph is called a \(p\)-formation. A non-empty class which is \(\mathcal{E}_{\Phi}\)-closed is called saturated. A non-empty class \(\mathfrak{X}\) satisfying \(\mathcal{P}\mathfrak{X} = \mathfrak{X}\) is called a \(p\)-Schunck class.

These definitions differ from those of Doerk and Hawkes by the inclusion of the requirement, convenient for the theory of restricted Lie algebras but not for that of finite groups, that the classes be non-empty. Note also that saturation had a different meaning, explained below, in the older terminology. Clearly, a \(p\)-Schunck
class is a saturated $p$-homomorph. If $\mathfrak{X}$ is a non-empty class, then $p\mathfrak{X}$ is a $p$-Schunck class, and if $\mathfrak{X}$ is a $p$-homomorph, it is the smallest $p$-Schunck class containing $\mathfrak{X}$.

**Lemma 4.2.** Let $\mathfrak{X}$ be a $p$-homomorph which is not a $p$-formation. Then there exists a restricted Lie algebra $(L, [p])$ with $[p]$-ideals, $K_1, K_2$ such that $(L/K_i, [p]) \in \mathfrak{X}$, $i = 1, 2$, but $(L, [p]) \not\in \mathfrak{X}$.

**Proof.** Doerk and Hawkes [9, Proposition 2.5, p.272].

**Definition 4.3.** Let $\mathfrak{X}$ be a class of restricted Lie algebras. A $[p]$-subalgebra $U$ of $(L, [p])$ is called $\mathfrak{X}$-maximal in $(L, [p])$ if it is maximal in the set of those $[p]$-subalgebras of $(L, [p])$ which are in $\mathfrak{X}$.

**Definition 4.4.** Let $\mathfrak{X}$ be a $p$-homomorph. A $[p]$-subalgebra $U$ of $(L, [p])$ is called an $\mathfrak{X}$-projector of $(L, [p])$ if, for every $[p]$-ideal $K$ of $L$, $U + K/K$ is $\mathfrak{X}$-maximal in $L/K$.

**Definition 4.5.** Let $\mathfrak{X}$ be a $p$-homomorph. A $[p]$-subalgebra $U$ of $(L, [p])$ is called an $\mathfrak{X}$-covering subalgebra of $(L, [p])$ if, whenever $V$ is a $[p]$-subalgebra of $L$ containing $U$ and $K$ is a $[p]$-ideal of $V$ with $V/K \in \mathfrak{X}$, we have $U + K = V$.

Thus, an $\mathfrak{X}$-covering subalgebra $U$ of $(L, [p])$ is an $\mathfrak{X}$-projector of every $[p]$-subalgebra of $L$ which contains $U$. We denote the (possibly empty) set of $\mathfrak{X}$-projectors of $(L, [p])$ by $\text{Proj}_\mathfrak{X}(L, [p])$ and the set of $\mathfrak{X}$-covering subalgebras by $\text{Cov}_\mathfrak{X}(L, [p])$.

**Lemma 4.6.** Let $\mathfrak{X}$ be a $p$-homomorph and let $A$ be a minimal $[p]$-ideal of $(L, [p])$. Suppose $L/A \in \mathfrak{X}$, $L \not\in \mathfrak{X}$ and that $U \in \text{Proj}_\mathfrak{X}(L)$. Then $U$ complements $A$ in $L$.

**Proof.** We have $U + A = L$. As $U \cap A$ is a $[p]$-ideal of $L$, $U \cap A = 0$.

Our next lemma requires a condition which is automatic for ordinary Lie algebras.

**Lemma 4.7.** Suppose $\mathfrak{X}$ is a $p$-homomorph which contains all atoms. Suppose $H \in \text{Cov}_\mathfrak{X}(L)$ and that $H$ is contained in the $[p]$-subalgebra $U$ of $L$. Then $N_L(U) = U$.

**Proof.** $N = N_L(U)$ is a $[p]$-subalgebra of $L$ and $U$ is a $[p]$-ideal of $N$. If $N \not= U$ then there exists a $[p]$-subalgebra $K$ of $N$ such that $K/U$ is an atom. Since $K/U \in \mathfrak{X}$, $H + U = K$. But $H \subseteq U$, so $K \subseteq U$ contrary to the choice of $K$.

**Lemma 4.8.** Let $\mathfrak{X}$ be a $p$-homomorph and let $(L, [p])$ be a primitive algebra not in $\mathfrak{X}$ but with $L/\text{Soc}(L) \in \mathfrak{X}$. Then $\text{Cov}_\mathfrak{X}(L) = \text{Proj}_\mathfrak{X}(L)$ and is the set of all complements to $\text{Soc}(L)$ in $L$.

**Proof.** Clearly, the projectors are those maximal $[p]$-subalgebras which do not contain $\text{Soc}(L)$. As $\text{Soc}(L)$ is the only minimal $[p]$-ideal of $L$, these maximal $[p]$-subalgebras are easily seen to be $\mathfrak{X}$-covering subalgebras.

**Lemma 4.9.** Let $\mathfrak{X}$ be a $[p]$-homomorph. Let $A$ be an abelian $[p]$-ideal of $(L, [p])$. Suppose $L/A \in \mathfrak{X}$ and that $U_1, U_2 \in \text{Cov}_\mathfrak{X}(L)$. Then there exists $a \in A$ such that $\alpha_a(U_1) = U_2$.

**Proof.** Suppose first that $A$ is a minimal $[p]$-ideal. If $L \in \mathfrak{X}$, then $U_1 = U_2 = L$. Suppose $L \not\in \mathfrak{X}$. Then $U_1, U_2$ complement $A$ in $L$. The result holds if $L$ is primitive.
If \( L \) is not primitive, there exists a minimal \([p]\)-ideal \( B \subseteq C_{U_2}(A) \) of \( L \). Since \((U_1 + B)/B \in \text{Cov}_X(L/B)\), by induction, there exists \( a \in A \) such that
\[
\alpha_a(U_1 + B) = U_2 + B = U_2.
\]
It follows that \( \alpha_a(U_1) = U_2 \) since \( U_1 \simeq L/A \simeq U_2 \).

Now suppose that \( A_1 \) is a minimal \([p]\)-ideal contained in \( A \). By induction, the result holds in \( L/A_1 \), so by replacing \( U_1 \) by a suitable \( \alpha_a(U_1) \), we may suppose that \( A_1 + U_1 = A_1 + U_2 \). Either \( A_1 + U_1 \subseteq L \) and the result holds by induction, or \( L/A_1 \in \mathfrak{X} \) and we have the case already proved.

Lemma 4.10. Let \( \mathfrak{X} \) be a \( p \)-homomorph. Let \( K \) be a \([p]\)-ideal of \((L,[p])\). Suppose \( V/K \in \text{Proj}_X(L/K) \) and \( U \in \text{Proj}_X(V) \). Then \( U \in \text{Proj}_X(L) \).

Proof. Doerk and Hawkes [9, Proposition 3.7, p.290].

Lemma 4.11. Let \( \mathfrak{X} \) be a \( p \)-homomorph. Let \( K \) be a \([p]\)-ideal of \((L,[p])\). Suppose \( V/K \in \text{Cov}_X(L/K) \) and \( U \in \text{Cov}_X(V) \). Then \( U \in \text{Cov}_X(L) \).

Proof. Doerk and Hawkes [9, Proposition 3.7, p.290].

Lemma 4.12. Let \( \mathfrak{X} \) be a \([p]\)-homomorph which contains all atoms. Let \( A \) be a minimal \([p]\)-ideal of \((L,[p])\). Suppose \( L/A \in \mathfrak{X} \), \( L \not\in \mathfrak{X} \) and \( \text{Cov}_X(L) \neq \emptyset \). Then \( \text{Cov}_X(L) \) is the set of all complements to \( A \) in \( L \) and \( H^1(L/A,A) = 0 \).

Proof. Let \( U_1 \in \text{Cov}_X(L,[p]) \). By Lemma 4.10 \( U_1 \) complements \( A \) in \( L \). If \( A \) is central, then \( U_1 \) is a \([p]\)-ideal of \( L \) and \( L/U_1 \in \mathfrak{X} \) since \( \mathfrak{X} \) contains all atoms. Therefore \( A \) is not central in \( L \). Thus \( A \) is null and is a minimal ideal of \( L \). Let \( U_2 \) be another complement to \( A \) in \( L \). Then \((U_2,[p]) \simeq (L/A,[p]) \simeq (U_1,[p])\), so \( U_2 \in \mathfrak{X} \). We have to prove that \( U_2 \in \text{Cov}_X(L) \).

Suppose \( \mathcal{Z}(L) = 0 \). There is an automorphism \( \alpha \) of the underlying algebra \( L \) which maps \( U_1 \) onto \( U_2 \). By Lemma 2.10 \( \alpha \) is an automorphism of \((L,[p])\), and it follows that \( U_2 \in \text{Cov}_X(L) \). Now suppose \( \mathcal{Z}(L) \neq 0 \). Let \( Z \subseteq \mathcal{Z}(L) \) be a minimal \([p]\)-ideal of \( L \). Since \( U_1 \) is not an ideal of \( L \), we have \( U_1 \supset Z \). By induction, \( U_2/Z \in \text{Cov}_X(L/Z) \). As \( U_2 \in \mathfrak{X} \), by Lemma 4.11 \( U_2 \in \text{Cov}_X(L) \).

By Lemma 1.9 it now follows for every complement \( U_2 \) to \( A \) in \( L \), that there exists \( a \in A \) such that \( U_2 = \alpha_a(U_1) \). This is equivalent to \( H^1(L/A,A) = 0 \).

Definition 4.13. A \( p \)-homomorph \( \mathfrak{X} \) is called projective if, for every soluble restricted Lie algebra \((L,[p])\), we have \( \text{Proj}_X(L) \neq \emptyset \). It is called a Gaschütz class if, for every soluble restricted Lie algebra \((L,[p])\), \( \text{Cov}_X(L) \neq \emptyset \).

In the older terminology, what are here called \( \mathfrak{X} \)-covering subalgebras were called \( \mathfrak{X} \)-projectors, and what are here called Gaschütz classes were called saturated homomorphs.

Theorem 4.14. Let \( \mathfrak{X} \) be a \( p \)-homomorph. Then the following are equivalent:

(a) \( \mathfrak{X} \) is a Gaschütz class.

(b) \( \mathfrak{X} \) is a projective class.

(c) \( \mathfrak{X} \) is a \( p \)-Schunck class.

Proof. If \( \mathfrak{X} \) is a Gaschütz class, it is clearly a projective class. Suppose \( \mathfrak{X} \) is projective. To show that it is a \( p \)-Schunck class, we have to show that, if \((L,[p]) \in p \mathfrak{X} \), the \((L,[p]) \in \mathfrak{X} \). Let \((L,[p]) \) be a minimal counterexample and let \( A \) be a minimal \([p]\)-ideal of \( L \). Then \( L/A \in \mathfrak{X} \). Since \( \mathfrak{X} \) is projective, there exists an \( \mathfrak{X} \)-projector \( U \)
of \( L \). If there exists a minimal \([p]\)-ideal \( B \) of \( L \) contained in \( U \), then \( L/B \in \mathcal{X} \) contrary to \( U \) being an \( \mathcal{X} \)-projector. Therefore \( \mathcal{C}_L(A) = A \) and \( L \) is primitive. But by assumption, all primitive quotients of \( L \) are in \( \mathcal{X} \), so \( L \in \mathcal{X} \) contrary to assumption.

Now suppose that \( \mathcal{X} \) is a \( p \)-Schunck class. We use induction over the dimension to show that every soluble restricted Lie algebra \((L,[p])\) has an \( \mathcal{X} \)-covering subalgebra. We may suppose \( L \not\in \mathcal{X} \). Thus there exists a \([p]\)-ideal with \( P = L/K \) primitive, not in \( \mathcal{X} \) but with \( P/\text{Soc}(P) \in \mathcal{X} \). If \( K \neq 0 \), then by induction, there exists an \( \mathcal{X} \)-covering subalgebra \( U/K \) of \( L/K \). As \( U \neq L \), by induction, we have there exists an \( \mathcal{X} \)-covering subalgebra \( V \) of \( U \). By Lemma 4.11, \( V \in \text{Cov}_\mathcal{X}(L) \). If \( K = 0 \), then \( L \) is primitive and the complements to \( \text{Soc}(L) \) are \( \mathcal{X} \)-covering subalgebras of \( L \) by Lemma 4.18.

**Theorem 4.15.** Let \( \mathfrak{H} \) be a \( p \)-Schunck class which contains all atoms. Let \((L,[p]) \in \mathfrak{H} \) and let \([p]’\) be another \( p \)-operation on \( L \). Then \((L,[p]’) \in \mathfrak{H} \).

**Proof.** Let \((L,[p]) \) be a minimal counterexample. Let \( A \) be a minimal \([p]\)-ideal of \( L \). Then \( U + A/A \) is a \( p\mathfrak{H} \)-covering subalgebra of \((L/A)\). If \( U + A \subset L \), then \( U \) is a \( p\mathfrak{H} \)-covering subalgebra of \( U + A \) and so also of \( L \) by Lemma 4.11. Hence \( U + A = L \) and \( U \) complements \( A \) in \( L \). As this holds for every minimal \([p]\)-ideal, \( U \) contains no non-trivial \([p]\)-ideal of \( L \) and it follows that \( \mathcal{C}_L(A) = A \). As \( L \) is primitive and not in \( \mathfrak{H} \), \( L \not\in p\mathfrak{H} \). But \( L/A \in p\mathfrak{H} \) and the complements to \( A \) in \( L \) are \( p\mathfrak{H} \)-covering subalgebras.\[\square\]

Our next lemma is a slightly modified version of Doerk and Hawkes [9] Lemma 3.14, p.295.

**Lemma 4.16.** Let \( \mathfrak{H} \) be a \( p \)-homomorph. Suppose \( U \) is an \( \mathfrak{H} \)-covering subalgebra of \((L,[p])\). Then \( U \) is a \( p\mathfrak{H} \)-covering subalgebra of \((L,[p])\).

**Proof.** Let \((L,[p]) \) be a minimal counterexample. Let \( A \) be a minimal \([p]\)-ideal of \( L \). Then \( U + A/A \) is a \( p\mathfrak{H} \)-covering subalgebra of \((L/A)\). If \( U + A \subset L \), then \( U \) is a \( p\mathfrak{H} \)-covering subalgebra of \( U + A \) and so also of \( L \) by Lemma 4.11. Hence \( U + A = L \) and \( U \) complements \( A \) in \( L \). As this holds for every minimal \([p]\)-ideal, \( U \) contains no non-trivial \([p]\)-ideal of \( L \) and it follows that \( \mathcal{C}_L(A) = A \). As \( L \) is primitive and not in \( \mathfrak{H} \), \( L \not\in p\mathfrak{H} \). But \( L/A \in p\mathfrak{H} \) and the complements to \( A \) in \( L \) are \( p\mathfrak{H} \)-covering subalgebras.\[\square\]

**Theorem 4.17.** Let \( \mathcal{X} \) be a \( p \)-Schunck class. Let \( N \) be a nilpotent \([p]\)-ideal of \((L,[p])\) and let \( U \) be an \( \mathcal{X} \)-maximal \([p]\)-subalgebra of \( L \) such that \( L = U + N \). Then \( U \in \text{Cov}_\mathcal{X}(L) \).

**Proof.** We use induction on the dimension of \( L \). If \( L \not\in \mathcal{X} \), the result holds, so suppose \( L \in \mathcal{X} \). Then there exists a \([p]\)-ideal \( K \) of \( L \) such that \( P = L/K \) is primitive, not in \( \mathcal{X} \) but with \( P/\text{Soc}(P) \in \mathcal{X} \). Then \( N \not\subseteq K \) since \( L/N \simeq U/(U \cap N) \in \mathcal{X} \). Hence \( N + K/K \) is a non-zero nilpotent \([p]\)-ideal of \( P \) and so \( N + K/K = \text{Soc}(P) \). But \( U + K/K \) is a \([p]\)-subalgebra of \( P \) complementing \( N + K/K \), so \( U + K/K \in \text{Cov}_\mathcal{X}(L/K) \) by Lemma 4.11. Now \( (U + K) \cap N \) is a nilpotent \([p]\)-ideal of \( U + K \). Also \( U + ((U + K) \cap N) = U + K \) by the modular law for subspaces. As \( U + K \neq L \), by induction, we have that \( U \in \text{Cov}_\mathcal{X}(U + K) \). By Lemma 4.11, \( U \in \text{Cov}_\mathcal{X}(L) \).\[\square\]

**Theorem 4.18.** Let \( \mathcal{X} \) be a \( p \)-Schunck class. Let \( U \in \text{Proj}_\mathcal{X}(L,[p]) \). Then \( U \in \text{Cov}_\mathcal{X}(L,[p]) \).

**Proof.** Let \( A \) be a minimal \([p]\)-ideal of \( L \) and put \( V = U + A \). Then \( V/A \in \text{Proj}_\mathcal{X}(L/A) \), so by induction, \( V/A \in \text{Cov}_\mathcal{X}(L/A) \). But \( U \) is \( \mathcal{X} \)-maximal in \( V \), \( A \) is a nilpotent \([p]\)-ideal of \( V \) and \( U + A = V \). By Lemma 4.17, \( U \in \text{Cov}_\mathcal{X}(V) \). By Lemma 4.11, \( U \in \text{Cov}_\mathcal{X}(L) \).\[\square\]
Lemma 4.19. Let $\mathfrak{F}$ be a $p$-formation and let $A$ be a minimal $[p]$-ideal of $L$. Suppose $L/A \in \mathfrak{F}$, $L \notin \mathfrak{F}$ and that $H$ complements $A$ in $L$. Then $H \in \text{Cov}_\mathfrak{F}(L)$.

Proof. $H \in \mathfrak{F}$ and is a maximal $[p]$-subalgebra of $L$. Thus we only have to prove that $K$ a $[p]$-ideal of $L$, $L/K \in \mathfrak{F}$ implies $H + K = L$. Since $\mathfrak{F}$ is a $p$-formation, $L/K \cap A \in \mathfrak{F}$. Since $A$ is minimal and $L \notin \mathfrak{F}$, this implies $K \supseteq A$ and $H + K \supseteq H \cup A = L$.

Taking $\mathfrak{F}$ to be the $p$-homomorph of null algebras of dimension at most 1 and $L$ the 2-dimensional null algebra shows that the assumption that $\mathfrak{F}$ is a $p$-formation, not merely a $p$-homomorph, cannot be omitted from Lemma 4.19.

Lemma 4.20. Let $\mathfrak{H}$ be a $p$-homomorph. A necessary and sufficient condition for $\mathfrak{H}$ to be a $p$-Schunck class is that $L \notin \mathfrak{H}$, $A$ a minimal $[p]$-ideal of $L$ and $L/A \in \mathfrak{H}$ implies $\mathfrak{H}(L) \neq \emptyset$.

Proof. The condition is trivially necessary. Suppose $\mathfrak{H}$ satisfies the condition. We use induction over $\dim(L)$ to prove for all $(L, [p])$ that $\text{Cov}_\mathfrak{H}(L) \neq \emptyset$. Let $A$ be a minimal $[p]$-ideal of $L$. Then there exists a $[p]$-subalgebra $U \supseteq A$ such that $U/A \in \text{Cov}_\mathfrak{H}(L/A)$.

If $U \subseteq L$, then by induction, there exists $H \in \text{Cov}_\mathfrak{H}(U)$ and then $H \in \text{Cov}_\mathfrak{H}(L)$ by Lemma 4.11. If $U = L$, then $L/A \in \mathfrak{H}$. Either $L \in \mathfrak{H}$ or $L \notin \mathfrak{H}$ and $\mathfrak{H}(L) \neq \emptyset$ by hypothesis.

Corollary 4.21. Let $\mathfrak{F}$ be a $p$-formation. Then $\mathfrak{F}$ is a $p$-Schunck class if and only if $\mathfrak{F}$ is saturated.

Proof. Suppose $\mathfrak{F}$ is a $p$-Schunck class. Suppose $L/\Psi(L) \in \mathfrak{F}$. We have to prove that $L \in \mathfrak{F}$. We may suppose $\Psi(L) \neq \emptyset$. Let $A \subseteq \Psi(L)$ be a minimal $[p]$-ideal of $L$. By induction, $L/A \in \mathfrak{F}$. Let $U \in \text{Proj}_\mathfrak{F}(L)$. If $L \notin \mathfrak{F}$, then by Lemma 4.10 $U$ complements $A$ in $L$ contrary to $A \subseteq \Psi(L)$. Therefore $L \notin \mathfrak{F}$.

Suppose $\mathfrak{F}$ is saturated. Suppose $(L, [p]) \notin \mathfrak{F}$ and that $A$ is a minimal $[p]$-ideal of $L$ with $L/A \in \mathfrak{F}$. Since $\mathfrak{F}$ is saturated, $A \not\subset \Psi(L)$, so there exists a maximal $[p]$-subalgebra $U$ complementing $A$ in $L$. Suppose $K$ is a $[p]$-ideal of $L$ and that $U + K/K$ is not $\mathfrak{F}$-maximal in $L/K$. Then we must have $K \supseteq U$ an $L/K \in \mathfrak{F}$. But then $L/(K \cap A) \in \mathfrak{F}$ since $\mathfrak{F}$ is a $p$-formation, that is, $L \in \mathfrak{F}$.

Lemma 4.22. Let $\mathfrak{H}$ be a $p$-Schunck class. Then

1. $L, M \in \mathfrak{H}$ implies $L \oplus M \in \mathfrak{H}$, and
2. $L/\Psi(L) \in \mathfrak{H}$ implies $L \in \mathfrak{H}$.

Proof. As for Barnes and Gastineau-Hills [10 Lemma 3.5].

Corollary 4.23. Let $\mathfrak{H}$ be a $p$-Schunck class which contains all atoms. Suppose $A$ is a central $[p]$-ideal of $L$ and $L/A \in \mathfrak{H}$. Then $L \in \mathfrak{H}$.

Proof. We may suppose $A$ minimal. If $L \notin \mathfrak{H}$, then $L$ splits over $A$ and we have $L \simeq (L/A) \oplus A$.

If a $p$-Schunck class is non-zero, it must contain some atom. However, it need not contain every atom.

Example 4.24. Let $\mathfrak{A}$ be the class of all those soluble restricted Lie algebras all of whose $[p]$-chief factors are non-null. Then $\mathfrak{A}$ is clearly a formation. By Lemma 4.19 $\mathfrak{A}$ is saturated. By Lemma 4.20 every algebra in $\mathfrak{A}$ is abelian.
The following result, which generalises Lemma 3.6 is the analogue of Barnes and Newell [7, Theorem 4.3]. Omitting reference to the $p$-operation gives an improved proof of that result.

**Theorem 4.25.** Let $\mathfrak{H}$ be a $p$-Schunck class which contains all atoms. Let $A$ be $[p]$-subnormal in the (not necessarily soluble) restricted Lie algebra $(L,[p])$ and let $B$ be a $[p]$-ideal of $A$. Suppose $B \subseteq \Psi(L)$ and $A/B \in \mathfrak{H}$. Then $A \in \mathfrak{H}$.

**Proof.** Let $(L,[p])$ be a minimal counterexample. If $C$ is any non-zero $[p]$-ideal of $L$, then $A + C/C \in \mathfrak{H}$. We prove first that there exists a null minimal $[p]$-ideal $C$ of $L$ with $C \subseteq A$. If $Z(L) \neq 0$, then $A/A \cap Z(L) \in \mathfrak{H}$ and it follows by Corollary 4.23 that $A \in \mathfrak{H}$. So $Z(L) = 0$. Since $A \not\subseteq \mathfrak{H}$, $A$ is not nilpotent. By Schenkman’s Theorem (see [12]), the nilpotent residual $A_{\mathfrak{N}}$ of $A$ is an ideal of $L$. Let $C = A_{\mathfrak{N}}$ be a minimal ideal of $L$. By Lemma 3.6 $B$ is nilpotent, so $A$ is soluble. Hence $C$ is abelian. As $\text{ad}(e)^2 = 0$ for all $e \in C$, $e^{[p]} \in Z(L) = 0$. Thus $C$ is a null $[p]$-ideal of $L$.

We have $A/C \in \mathfrak{H}$ but $A \not\subseteq \mathfrak{H}$, so there exists a $[p]$-ideal $K$ of $A$ such that $P = A/K$ is primitive and not in $\mathfrak{H}$. As $A/C \in \mathfrak{H}$, $C \not\subseteq K$. Thus $C + K/K$ is a non-zero abelian $[p]$-ideal of $A/K$. As $A/K$ is primitive, $C + K/K$ is its minimal $[p]$-ideal. Working with the underlying algebras, we have $M = C \cap K$ is a maximal $A$-submodule of $C$, $A$ acts non-trivially on $C/M$, $A/M$ splits over $C/M$ and, by Lemma 1.12 $H^1(A/C,C/M) = 0$. By Barnes and Newell [7, Lemma 4.2], $L$ splits over $C$. Let $U$ complement $C$ in $L$. By Lemma 4.7 $U$ is a maximal $[p]$-subalgebra of $L$. Since $B \subseteq \Psi(L)$, $U \supseteq B$. Put $D = M + B$.

Then $D$ is a $[p]$-ideal of $A$, $B + C/D$ is a complemented $[p]$-chief factor of $A$ isomorphic to $C/M$. Thus $A/D$ has a primitive quotient isomorphic to $P$. But this is a primitive quotient of $A/B \in \mathfrak{H}$, contrary to $P \not\subseteq \mathfrak{H}$. □

**Definition 4.26.** Let $(L,[p])$ be a restricted Lie algebra and let $V$ be an $L$-module giving the representation $\rho$. $V$ is called a $[p]$-module for $L$ or an $(L,[p])$-module and $\rho$ a $[p]$-representation of $L$ if $\rho(x^{[p]}) = \rho(x)^p$ for all $x \in L$.

Clearly, an $L$-submodule of an $(L,[p])$-module is an $(L,[p])$-submodule. Thus, an irreducible $(L,[p])$-module is also irreducible as $L$-module.

If $A$ is an $(L,[p])$-module, we can form the split extension of $A$ by $(L,[p])$. This is the ordinary split extension $X$ of $A$ by $L$ with $p$-operation coinciding with the given operation on $L$ and null on $A$. By Theorem 2.4 there is one and only one such $p$-operation. Sometimes we are given a $p$-operation on the module $A$ as well as on $L$, for example if $A$ is a $[p]$-ideal of $L$, and require the $p$-operation on $X$ to
agree with these. For \( a \in A \), \( \text{ad}_X(a)^2 = 0 \), so for this to be possible we must have \( a[p] \in A^L \). If this condition is satisfied, then we have \( \text{ad}_X(a)p = \text{ad}(a[p]) \). It then follows by Theorem 2.1 of [3] that the given \( p \)-operation \( [p] \) on \( L \) and \( A \) has a unique extension to a \( p \)-operation on \( X \).

**Definition 4.27.** A \( p \)-homomorph \( \mathfrak{H} \) is said to be **split** if \( L \in \mathfrak{H} \) and \( A \) an abelian \( [p] \)-ideal of \( L \) imply that the split extension of \( A \) by \( L/A \) is also in \( \mathfrak{H} \).

**Lemma 4.28.** Let \( \mathfrak{F} \) be a \( p \)-formation. Then \( \mathfrak{F} \) is split.

**Proof.** As for Barnes and Gastineau-Hills [3] Lemma 1.16.

**Lemma 4.29.** Let \( \mathfrak{H} \) be a \( p \)-Schunck class which is split. Then \( \mathfrak{H} \) is a \( p \)-formation.

**Proof.** As for Barnes and Gastineau-Hills [3] Theorem 2.8.

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**5. \( \mathfrak{F} \)-HYPERCENTRAL MODULES**

**Definition 5.1.** Let \( \mathfrak{F} \) be a saturated \( p \)-formation and let \( V \) be an irreducible \( (L, [p]) \)-module. \( V \) is called \( \mathfrak{F} \)-central if the split extension of \( V \) by \( L/C_L(V) \in \mathfrak{F} \) and \( \mathfrak{F} \)-eccentric otherwise. An \((L, [p])\)-module \( V \) is called \( \mathfrak{F} \)-hypercentral if every composition factor of \( V \) is \( \mathfrak{F} \)-central. An \((L, [p])\)-module \( V \) is called \( \mathfrak{F} \)-hypereccentric if every composition factor of \( V \) is \( \mathfrak{F} \)-eccentric.

If \( A, B \) are \([p]\)-ideals of \((L, [p])\) and \( A/B \) is a null \([p]\)-chief factor of \( L \), then \( A/B \) is an irreducible \((L, [p])\)-module and it may be classified as \( \mathfrak{F} \)-central or \( \mathfrak{F} \)-eccentric as above. I extend the definitions to apply to any \([p]\)-chief factor.

**Definition 5.2.** Let \( \mathfrak{F} \) be a saturated \( p \)-formation which contains the null atom. A \([p]\)-chief factor \( A/B \) of \((L, [p])\) is called \( \mathfrak{F} \)-central if the split extension of \( A/B \) by \( L/C_L(A/B) \in \mathfrak{F} \) and \( \mathfrak{F} \)-eccentric otherwise.

**Lemma 5.3.** Let \( \mathfrak{F} \) be a saturated \( p \)-formation which contains all atoms. Then \((L, [p]) \in \mathfrak{F}\) if and only if every \([p]\)-chief factor of \( L \) is \( \mathfrak{F} \)-central.

**Proof.** Suppose \((L, [p]) \in \mathfrak{F}\). Let \( A/B \) be a \([p]\)-chief factor. By Lemma 4.28 the split extension \( X \) of \( A/B \) by \( L/A \) is in \( \mathfrak{F} \). Let \( U \sim L/A \) be a complement to \( A/B \) in \( X \) and let \( C = C_U(A/B) \). Then \( C \) is a \([p]\)-ideal of \( X \) and \( X/C \) is the split extension of \( A/B \) by \( L/C_L(A/B) \) and is in \( \mathfrak{F} \).

Suppose conversely, that every \([p]\)-chief factor of \((L, [p])\) is \( \mathfrak{F} \)-central. Let \( A \) be a minimal \([p]\)-ideal of \( L \). By induction, we may suppose that \((L/A, [p]) \in \mathfrak{F}\). If \((L/A, [p]) \notin \mathfrak{F}\), then by Corollary 4.21 there exists a \([p]\)-subalgebra \( M \) of \( L \) which complements \( A \). Let \( C = C_M(A) \). Then \( L/C \) is the split extension of \( A \) by \( L/C_L(A) \) and is in \( \mathfrak{F} \). Thus \( L/C \) and \( L/A \) are in \( \mathfrak{F} \), so \( L/(C \cap A) \notin \mathfrak{F}\). But \( C \cap A = 0 \).

If \( V, W \) are \((L, [p])\)-modules, then \( V \otimes_F W \) and \( \text{Hom}_F(V, W) \) are also \((L, [p])\)-modules. The following results are proved exactly as the corresponding results for ordinary Lie algebras.

**Theorem 5.4.** Let \( \mathfrak{F} \) be a saturated \( p \)-formation and let \( V, W \) be \( \mathfrak{F} \)-hypercentral \((L, [p])\)-modules. Then \( V \otimes_F W \) and \( \text{Hom}_F(V, W) \) are also \( \mathfrak{F} \)-hypercentral.

**Proof.** The argument for Theorem 2.1 of [3] applies.
If $S$ is a $[p]$-subalgebra of $(L, [p])$ and $V$ is an $(L, [p])$-module, then it is also a $(S, [p])$-module. Let $F$ be a saturated $p$-formation. We say that $V$ is $S\mathcal{F}$-hypercentral if it is $\mathcal{F}$-hypercentral as $S$-module and $S\mathcal{F}$-hyperexcentric if it is $\mathcal{F}$-hyperexcentric as $S$-module.

**Theorem 5.5.** Let $(L, [p])$ be a (not necessarily soluble) restricted Lie algebra. Let $\mathcal{F}$ be a saturated $p$-formation. Suppose $S$ is $[p]$-subnormal in $(L, [p])$ and that $S \in \mathcal{F}$. Let $V$ be a finite-dimensional $(L, [p])$-module. Then $V$ is the $L$-module direct sum $V = V_0 \oplus V_1$ where $V_0$ is $S\mathcal{F}$-hypercentral and $V_1$ is $S\mathcal{F}$-hyperexcentric.

*Proof.* The argument for Lemma 1.1 of [5] applies. \hfill \Box

**Theorem 5.6.** Let $(L, [p])$ be a (not necessarily soluble) restricted Lie algebra and suppose that $z^|[p] = 0$ for all $z \in Z(L)$. Let $\mathcal{F}$ be a saturated $p$-formation and suppose $S$ is $[p]$-subnormal in $L$, $S \neq 0$ and that $S \in \mathcal{F}$. Let $V$ be an irreducible $(L, [p])$-module. Then $V$ is $S\mathcal{F}$-hypercentric.

*Proof.* The argument for Theorem 6.4 of [5] applies. \hfill \Box

6. CONSTRUCTIONS AND EXAMPLES

Starting with any non-empty class $\mathcal{X}$, we can construct a $p$-Schunck class by forming the class $\Psi(\mathcal{X})$ of all soluble restricted Lie algebras whose primitive quotients are quotients of algebras in $\mathcal{X}$. In this section, I give some constructions for $p$-Schunck classes with the extra property of being formations.

**Definition 6.1.** Let $\mathcal{R}$ be a saturated $p$-formation which contains all atoms and let $\mathcal{F}$ be a $p$-formation. The $p$-formation *residually defined* by $\mathcal{R}$ and $\mathcal{F}$ is the class

$$\mathcal{R} \cdot \mathcal{F} = \{(L, [p]) \mid L_\mathcal{F} \in \mathcal{R}\}.$$  

This is not the product $\mathcal{R}\mathcal{F}$ as defined in Doerk and Hawkes [9, Definition 1.3, p. 263]. An algebra in $\mathcal{R} \cdot \mathcal{F}$ is an extension of an algebra in $\mathcal{R}$ by an algebra in $\mathcal{F}$. If $\mathcal{R}$ is $[p]$-ideal closed as is, for example, the class $p\mathcal{R}$ of nilpotent restricted algebras, then every restricted algebra in $\mathcal{R}\mathcal{F}$, that is, every extension of an algebra in $\mathcal{R}$ by one in $\mathcal{F}$, is in $\mathcal{R} \cdot \mathcal{F}$.

**Theorem 6.2.** Let $\mathcal{R}$ be a saturated $p$-formation which contains all atoms and let $\mathcal{F}$ be a $p$-formation. Then $\mathcal{R} \cdot \mathcal{F}$ is a saturated formation.

*Proof.* Suppose $(L, [p]) \in \mathcal{R} \cdot \mathcal{F}$ and let $K$ be a $[p]$-ideal of $L$. Then

$$(L/K)_\mathcal{F} = (L_\mathcal{F} + K)/K \simeq L_\mathcal{F}/(L_\mathcal{F} \cap K) \in \mathcal{R},$$

so $(L/K, [p]) \in \mathcal{R} \cdot \mathcal{F}$. Now suppose $A_1, A_2$ are $[p]$-ideals of $(L, [p])$ and that $(L/A_1, [p]) \in \mathcal{R} \cdot \mathcal{F}$. We have to show that $L/(A_1 \cap A_2) \in \mathcal{R} \cdot \mathcal{F}$. We may suppose $A_1 \cap A_2 = 0$. We have $L_\mathcal{F}/(L_\mathcal{F} \cap A_i) \simeq (L_\mathcal{F} + A_i)/A_i \in \mathcal{R}$. Thus $L_\mathcal{F} \in \mathcal{R}$ and so $\mathcal{R} \cdot \mathcal{F}$ is a formation.

Now suppose $A$ is a minimal $[p]$-ideal of $L$, $A \subseteq \Psi(L)$ and that $L/A \in \mathcal{R} \cdot \mathcal{F}$. We have to prove that $L_\mathcal{F} \in \mathcal{R}$. If $A \not\subseteq L_\mathcal{F}$, then $L_\mathcal{F} \simeq (L_\mathcal{F} + A)/A \not\in (L/A)_\mathcal{F} \in \mathcal{R}$.

Suppose $A \subseteq L_\mathcal{F}$. Then $L_\mathcal{F}/A \in \mathcal{R}$. If $A \subseteq Z(L)$, then $L_\mathcal{F} \in \mathcal{R}$, so we may suppose that $A$ is null and therefore is a minimal ideal of the underlying algebra $L$. Suppose $L_\mathcal{F} \not\in \mathcal{R}$. Since $A$ is irreducible as $L$-module and $L_\mathcal{F}$ is an ideal of $L$, all composition factors of $A$ as $L_\mathcal{F}$-module are isomorphic. Let

$$A = A_0 \supset A_1 \supset \cdots \supset A_n = 0$$
be a composition series of $A$ as $L_3$-module. Since $L_3 \notin \mathcal{R}$, there exists $r$ such that $L_3/A_r \notin \mathcal{R}$ but $L_3/A_{r+1} \notin \mathcal{R}$. Therefore $L_3/A_{r+1}$ splits over $A_r/A_{r+1}$ and it follows that the split extension of $A_r/A_{r+1}$ by $L_3/A$ is not in $\mathcal{R}$. But $\mathcal{R}$ is a saturated formation, so this implies that the split extension is the only extension, so $H^2(L_3, A_r/A_{r+1}) = 0$. We also have $H^0(L_3, A_r/A_{r+1}) = H^1(L_3, A_r/A_{r+1}) = 0$. Since all the $A_i/A_{i+1}$ are isomorphic, we have $H^n(L_3, A_i/A_{i+1}) = 0$ for all $i$ and $n \leq 2$. It follows that $H^n(L_3, A) = 0$ for $n \leq 2$. By the Hochschild-Serre spectral sequence, we have $H^n(L, A) = 0$ for $n \leq 2$. Thus $L$ splits over $A$. By Lemma 2.7, the complements are maximal $[p]$-subalgebras contrary to $A \subseteq \Psi(L)$.

In particular, the class $p\mathfrak{C}$ of completely soluble restricted Lie algebras, that is the algebras with nilpotent derived algebras, is a saturated $p$-formation since the $[p]$-closure $(L')[[p]]$ of $L'$ is the $[p]$-abelian residual and, by Strade and Farnsteiner [19] Proposition 1.3(3), p. 66], $(L')[[p]]$ is nilpotent if $L'$ is nilpotent.

Likewise, by induction over $k$, it is easily seen that $p\mathfrak{H}$, the $p$-formation of algebras of nilpotent length at most $k$, is saturated.

Since every $[p]$-chief factor of a soluble restricted Lie algebra $(L, [p])$ is either a chief factor of the underlying algebra $L$ or is central, the intersection of the centralisers of the $[p]$-chief factors of $(L, [p])$ is the intersection of the centralisers of the chief factors of $L$, that is, the nil radical $N(L)$. Let $\mathfrak{F}$ be a $p$-formation. The $p$-formation $p\text{Loc}(\mathfrak{F})$ $p$-locally defined by $\mathfrak{F}$ is the class of all $(L, [p])$ such that, for every $[p]$-chief factor $C$ of $(L, [p])$, $(L/C_{L}(C), [p]) \in \mathfrak{F}$, that is, all $(L, [p])$ with $(L/N(L), [p]) \in \mathfrak{F}$ or equivalently, $L_3 \in p\mathfrak{H}$. Thus $p\text{Loc}(\mathfrak{F}) = p\mathfrak{H}$ and is saturated.

**Definition 6.3.** The algebra $(L, [p])$ is called $p$-supersoluble if every $[p]$-chief factor of $(L, [p])$ is an atom.

This is equivalent to the definition given in Voigt [10] Definition 2.1, p.77]. Voigt proves ([10] Satz 2.17, p.117]), for a soluble restricted Lie algebra $(L, [p])$ over an algebraically closed field $F$, that $(L, [p])$ is supersoluble if and only if every maximal $[p]$-subalgebra has codimension 1. The assumption that $F$ is algebraically closed, made necessary by the existence of atoms of dimension greater than 1, can be dropped if we replace “dimension” by “[p]-dimension”, defined for a soluble restricted Lie algebra as the length of a [p]-composition series. Voigt’s Theorem 2.18, that a soluble restricted Lie algebra is supersoluble if and only if all maximal chains of $[p]$-subalgebras have the same length, also follows without the assumption of algebraic closure.

As every $[p]$-chief factor is either a chief factor of $L$ or central, $(L, [p])$ is $p$-supersoluble if and only if $L$ is supersoluble. We denote the class of $p$-supersoluble restricted Lie algebras by $p\mathfrak{M}$.

**Lemma 6.4.** $p\mathfrak{M}$ is a saturated $p$-formation.

**Proof.** $p\mathfrak{M}$ is obviously a $p$-formation. Suppose $A$ is a minimal $[p]$-ideal of $(L, [p])$, $(L/A, [p]) \in p\mathfrak{M}$ and $A$ is not an atom. Then $A$ is not central, so is a minimal ideal of $L$. As $L/A$ is supersoluble and $A$ has dimension greater than 1, $L$ splits over $A$ and the result follows.

**Definition 6.5.** Let $\Lambda$ be an $F$-subspace of the algebraic closure $\bar{F}$. We say that $\Lambda$ is $p$-normal if $\lambda \in \Lambda$ implies that every conjugate of $\lambda$ is in $\Lambda$ and that $\lambda^p \in \Lambda$. 
If $F'$ is a normal extension field of $F$, then $F'$ is a $p$-normal subspace of $\bar{F}$. If $p > 2$ and $k \in F$ has no square root in $F$, then the space $\langle \sqrt{k} \rangle$ is another example. If $\lambda^p = k \in F$, $\lambda \notin F$, then $\langle \lambda \rangle$ is a normal subspace of $\bar{F}$ which is not $p$-normal. $F$ being perfect does not ensure that a normal space will be $p$-normal. The following example is based on an idea provided by G. E. Wall.

**Example 6.6.** Let $F$ be the field of $p^n$ elements. Let $q > p$ be a prime dividing $p^n - 1$. Then $F$ contains a primitive $q$-th root $\xi$ of unity and also has an element $c$ which has no $q$-th root in $F$. Let $u \in \bar{F}$ be a root of the polynomial $f(t) = t^q - c$ and let $\Lambda$ be the space spanned by the conjugates of $u$. Then the $u\xi^i$ for $i = 0, 1, \ldots, q-1$ are the roots of $f(t)$. If $m(t) = t^k + a_1 t^{k-1} + \cdots + a_k$ is the minimum polynomial of $u$, then $m_i(t) = t^k + a_1 \xi t^{k-1} + \cdots + a_k (\xi)^k$ is the minimum polynomial of $u\xi^i$. Each of these divides $f(t)$. As $f(t)$ is a product of irreducible polynomials of the same degree $k$, the degree of $f(t)$ is divisible by $k$. As the degree $q$ is prime, it follows that $f(t)$ is irreducible and is the minimal polynomial of $u$. We therefore have $\Lambda = \langle u, \xi u, \ldots, \xi^{q-1} u \rangle = \langle u \rangle$ since $\xi \in F$. If $u^p \in \Lambda$, then $u^p = bu$ for some $b \in F$, we have $u^{p-1} - b = 0$. But $t^{p-1} - b$ is a polynomial of degree less than that of the minimum polynomial of $u$. Hence $u^p \notin \Lambda$.

Such $p, n, q$ do exist, for example, $p = 2, n = 2, q = 3$ and $p = 3, n = 3, q = 13$. Indeed, for any $p$, there exist such $n$ and $q$.

**Lemma 6.7.** Let $p$ be prime. Then there exists $n$ and a prime $q > p$ which divides $p^n - 1$.

**Proof.** I show that the set $S$ of primes $q$ which divide $1 + p + \cdots + p^{k-1}$ for some $k$ is infinite and so has a member greater than $p$. For each $q \in S$, there is a least $k$ for which $q$ divides $1 + p + \cdots + p^{k-1}$. Consider $n > k$. Then $1 + \cdots + p^{n-1} = (1 + \cdots + p^{k-1}) + p^k(1 + \cdots + p^{k-1}) + \cdots + p^r (1 + \cdots + p^{s-1})$ where $r$ is the largest integer less than $n/k$ and $s = n - rk$. As $q$ divides every term except possibly the last, $q$ divides $p^n - 1$ if and only if it divides the last term, that is, if and only if $k$ divides $n$. By taking $n$ prime, we ensure that no prime $q$ which divides $1 + p + \cdots + p^{k-1}$ for $k < n$ also divides $1 + p + \cdots + p^{n-1}$. Thus $S$ is infinite. 

A minor modification to Example 6.6 provides further examples of $p$-normal spaces.

**Example 6.8.** Let $F$ be the field of $p^n$ elements. Let $q$ be a prime dividing $p - 1$. As before, we have $\xi \in F$ a primitive $q$-th root of unity and take $u$ a root of $f(t) = t^q - c$ for some $c \in F$ which has no $q$-th root in $F$. As before, the space $\Lambda$ spanned by the conjugates of $u$ is $\langle u \rangle$. We have $u^q = c$. We have $u^{p-1} = (u^q)^k = c^k$, where $k = (p - 1)/q$, and $u^p = c^k u \in \Lambda$. It follows that $\Lambda^p = \Lambda$.

**Definition 6.9.** Let $\Lambda$ be a $p$-normal subspace of $\bar{F}$. The *eigenvalue defined $p$-formation* $\rho \text{Ev}(\Lambda)$ is the class of all restricted soluble Lie algebras $(L, [p])$ such that every eigenvalue of $\text{ad}(x)$ is in $\Lambda$ for all $x \in L$.

The class $\rho \text{Ev}(\Lambda)$ is clearly a $p$-formation. We shall show that it is saturated. For this, we need the following lemmas.

**Lemma 6.10.** Let $L \in \text{Ev}(\Lambda)$ be a soluble Lie algebra and let $V$ be an irreducible $L$-module giving the representation $\rho$. Let $M$ be a maximal ideal of $L$. Suppose all
eigenvalues of \( \text{ad}(b) \) are in \( \Lambda \) for all \( b \in M \), but that for some \( a \in L - M \), \( \text{ad}(a) \) has an eigenvalue not in \( \Lambda \). Then every eigenvalue of \( \text{ad}(a) \) is outside \( \Lambda \).

**Proof.** The characteristic polynomial \( \chi(t) \) of \( \rho(a) \) can be expressed as a product \( \chi(t) = f(t)g(t) \) where every root of \( g(t) \) is in \( \Lambda \) while no root of \( f(t) \) is in \( \Lambda \). Put \( V_f = \{ v \in V \mid f(\rho(a))v = 0 \} \) and \( V_g = \{ v \in V \mid g(\rho(a))v = 0 \} \). Then \( V = V_f \oplus V_g \).

We want to prove that \( V_g = 0 \). As \( V \) is irreducible, this follows if we prove that \( V_g \) is an \( L \)-submodule of \( V \). Consider \( L \) and \( V \) as \( \langle \cdot \rangle \)-modules. We work over the algebraic closure, with the algebra \( \bar{L} = \bar{F} \otimes L \) and module \( \bar{V} = \bar{F} \otimes V \). Then \( \bar{V}_g = \bar{F} \otimes V_g \).

Let \( \lambda \) be an eigenvalue of \( \rho(a) \). The weight space \( \bar{V}_\lambda \) is the space

\[
\{ v \in \bar{V} \mid (\rho(a) - \lambda)^nv = 0 \text{ for some } n \}
\]

and \( \bar{V}_g = \sum_{\lambda \in \Lambda} \bar{V}_\lambda \). Likewise, \( \bar{L} \) is a sum of weight spaces. If \( b \in \bar{L}_\mu \) for the eigenvalue \( \mu \) of \( \text{ad}(a) \), and \( v \in \bar{V}_\lambda \), then \( bv \in \bar{V}_{\lambda + \mu} \). For \( \lambda, \lambda + \mu \in \Lambda \) since all eigenvalues \( \mu \) of \( \text{ad}(a) \) are in \( \Lambda \). It follows that \( b\bar{V}_g \subseteq \bar{V}_g \) for all \( b \in \bar{L} \) and so, that \( V_g \) is a submodule of \( V \).

\( \square \)

**Lemma 6.11.** Let \( (L, [p]) \in p\text{Ev} (\Lambda) \) and let \( V \) be an irreducible \( (L, [p]) \)-module giving the representation \( \rho \). Suppose for some \( x \in L \), \( \rho(x) \) has an eigenvalue not in \( \Lambda \). Then \( H^n(L, V) = 0 \) for all \( n \).

**Proof.** Since any \( L \)-submodule of \( V \) is an \( (L, [p]) \)-submodule, \( V \) is irreducible as \( L \)-module. We forget the \( p \)-operation and prove the result for ordinary Lie algebras. We use induction over \( \dim(L) \). The result holds if \( \dim(L) = 1 \). Let \( M \) be a maximal ideal of \( L \) and let \( W \) be a composition factor of \( V \) as \( M \)-module. If there exists an element \( b \) of \( M \) for which \( \rho(b) \) has an eigenvalue outside \( \Lambda \), then \( \rho(b) | W \) has an eigenvalue outside \( \Lambda \) since all \( M \)-composition factors of \( V \) are isomorphic. By induction, \( H^n(M, W) = 0 \) for all \( n \) and all composition factors \( W \). Therefore \( H^n(M, V) = 0 \) for all \( n \).

By the Hochschild-Serre spectral sequence, \( H^n(L, V) = 0 \).

We may therefore suppose that every eigenvalue of \( \rho(b) \) is in \( \Lambda \) for all \( b \in M \). By Lemma 6.11, we have \( a \in L - M \) with every eigenvalue of \( \text{ad}(a) \) outside \( \Lambda \). Every eigenvalue of the action of \( a \) on the degree \( q \) component \( E^q(M) \) of the exterior algebra on \( M \) is in \( \Lambda \). Thus every eigenvalue of its action on \( \text{Hom}(E^q(M), V) \) is outside \( \Lambda \). As \( H^q(M, V) \) is a subquotient of \( \text{Hom}(E^q(M), V) \), every eigenvalue of the action of \( a \) on \( H^q(M, V) \) is outside \( \Lambda \). Hence \( H^n(L/M, W) = 0 \) for every composition factor \( W \) of \( H^q(M, V) \). Thus \( H^p(L/M, H^q(M, V)) = 0 \) for all \( p, q \) and the result follows by the Hochschild-Serre spectral sequence.

\( \square \)

**Theorem 6.12.** Let \( \Lambda \) be a \( p \)-normal subspace of the algebraic closure \( \bar{F} \) of \( F \). Then the eigenvalue defined \( p \)-formation \( p\text{Ev}(\Lambda) \) is saturated.

**Proof.** Suppose \( A \) is a minimal \( [p] \)-ideal of \( (L, [p]), (L/A, [p]) \in p\text{Ev}(\Lambda) \) but \( (L, [p]) \not\in p\text{Ev}(\Lambda) \). We have to prove that \( L \) splits over \( A \). But this holds by Lemma 6.11. \( \square \)

In characteristic 0, every saturated formation is an eigenvalue defined formation. If we restrict attention to completely soluble algebras, this also holds in characteristic \( p \neq 0 \) and for restricted algebras.

Let \( \lambda \in \bar{F} \) have minimal polynomial \( m(t) \) over \( F \). There exists a vector space \( V \) and linear transformation \( a : V \rightarrow V \) with characteristic polynomial \( m(t) \). Let \( A \) be the Lie subalgebra \( \langle a, a^p, a^{2p}, \ldots \rangle \) of \( \text{Hom}(V, V) \). Then the split extension of \( V \) by \( A \) is a primitive completely soluble restricted Lie algebra which we denote...
by $P_\lambda$. It is the smallest algebra for which $\lambda$ appears as an eigenvalue. (For the case of ordinary Lie algebras, we use the ordinary primitive algebra $dP_\lambda$ the split extension of $V$ by $A = \langle a \rangle$.)

**Lemma 6.13.** Let $\mathfrak{F}$ be a saturated $p$-formation of completely soluble algebras which contains all atoms. Let $(L, [p]) \in \mathfrak{F}$ and suppose the element $a \in L$ has $\lambda$ as an eigenvalue of $\text{ad}(a)$. Then $P_\lambda \in \mathfrak{F}$.

**Proof.** The element $a$ has $\lambda$ as an eigenvalue on some $[p]$-chief factor $V/W$ of $(L, [p])$. As $V/W$ is $\mathfrak{F}$-central, the split extension of $V/W$ by $L/C_L(V/W)$ is in $\mathfrak{F}$. As $L$ is completely soluble, $L' \subseteq C_L(V/W)$. We may thus suppose $W = 0$ and that $L$ is the split extension of $V$ by an abelian algebra $M$ and that $M$ is a faithful irreducible $\mathfrak{F}$-central $M$-module. Let $\rho$ be the representation of $M$ on $V$. We have $a \in M$ for which $\rho(a)$ has $\lambda$ as an eigenvalue. Note that every eigenvalue of $\rho(a)$ is a conjugate of $\lambda$. Let $A = \langle a, a^p, a^p^2, \ldots \rangle$. We construct the subdirect sum $M^*$ of two copies of $M$ by setting

$$M^* = \{(x, y) \in M \oplus M \mid (x + A) = (y + A)\}.$$ 

It is a restricted algebra with the $p$-operation $(x, y)^{[p]} = (x^p, y^p)$. We have projections $\pi_1 : M^* \to M$ given by $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$. Then ker $\pi_1 = A_1 = \{(0, y) \mid y \in A\}$ and ker $\pi_2 = A_2 = \{(x, 0) \mid x \in \}$. Since $M^*/A_i \simeq M$ and $A_1 \cap A_2 = 0$, $M^* \in \mathfrak{F}$. The diagonal subalgebra $D = \{(x, x) \mid x \in M\}$ is a $[p]$-ideal since $M^*$ is abelian, and we have $M^* = A_i \oplus D$.

We construct $M^*$-modules $V_1, V_2$ from copies of $V$ with action defined via $\pi_1, \pi_2$, that is, $(x, y)v_1 = xv_1$ and $(x, y)v_2 = yv_2$ for $(x, y) \in M^*$ and $v_1 \in V_1$, $v_2 \in V_2$. Then $C_{M^*}(V_i) = A_i$ and $V_i$ is $\mathfrak{F}$-central. The $M^*$-module $W = \text{Hom}(V_1, V_2)$ is $\mathfrak{F}$-hypercentral by Theorem 5.3. If $f : V_1 \to V_2$ is an $M$-module homomorphism, then $((x, x)f)(v_1) = x \cdot f(v_1) - f(xv_1) = 0$. Thus $(x, x)f = 0$ for all $(x, x) \in D$ and $W^D \neq 0$. But $W_D$ is an $A_1$-module and is $\mathfrak{F}$-hypercentral. Take any irreducible $A_1$-submodule $K$ of $W^D$ and form the split extension $P$ of $K$ by $A_1$. Then $P \in \mathfrak{F}$. Every eigenvalue of $(a, 0)$ on $V_1$ is a conjugate of $\lambda$, while $(a, 0)V_2 = 0$. Thus every eigenvalue of $(a, 0)$ on $K$ is a conjugate of $\lambda$. As $K$ is irreducible under the action of $a$, $P \simeq P_\lambda$. \hfill \Box

**Lemma 6.14.** Let $\mathfrak{F}$ be a saturated $p$-formation of completely soluble algebras which contains all atoms. Let $(A, [p])$ be abelian and let $V$ be a faithful irreducible $(A, [p])$-module giving the representation $\rho$. Let $P$ be the split extension of $V$ by $A$. Suppose that for all $a \in A$ and every eigenvalue $\lambda$ of $\rho(a)$, we have $P_\lambda \in \mathfrak{F}$. Then $P \in \mathfrak{F}$.

**Proof.** Take a basis $a_1, \ldots, a_n$ of $A$. Put $A_i = \langle a_i, a_i^p, a_i^p^2, \ldots \rangle$. Let $\lambda_i$ be an eigenvalue of $\rho(a_i)$ and let $W_i$ be an $A_i$-module isomorphic to an irreducible $A_i$-submodule of $V$. The split extension of $W_i$ by $A_i$ is isomorphic to $P_{\lambda_i}$ and so, by assumption, is in $\mathfrak{F}$. Put $A^* = \bigoplus A_i$. Then $W_i$ is an $A^*$-module with the summand $A_j$ for $j \neq i$ acting trivially. It is $\mathfrak{F}$-central. We put $W = \bigoplus W_i$. Then $W$ is an $\mathfrak{F}$-hypercentral $A^*$-module by Theorem 5.3.

We have a homomorphism $\phi : A^* \to A$ given by $\phi(b_1, \ldots, b_n) = b_1 + \cdots + b_n$ for $b_i \in A_i$. Now choose $v \in V$, $v \neq 0$ and $w_i \in W_i$, $w_i \neq 0$. Put $\rho_i = \rho(a_i)$ and let $m_i(t)$ be the minimal polynomial of $\rho_i$. Any element of $W_i$ can be expressed in the form $f_i(\rho_i)w_i$ for some polynomial $f_i(t)$ determined up to multiples of $m_i(t)$. We
have a map $\psi : W \to V$ defined by
\[
\psi(f_1(\rho_1)w_1 \otimes \cdots \otimes f_n(\rho_n)w_n) = f_1(\rho_1) \cdots f_n(\rho_n)v.
\]
This is independent of the choices of the $f_i(t)$ as the $\rho_i$ commute and $m_i(\rho_i)v = 0$.
Now $V$ is also an $A^*$-module via $\phi$ and $\psi$ is an $A^*$-module homomorphism. As $\psi$
 is surjective, $V$ also is $\mathfrak{g}$-hypercentral as $A^*$-module. It is irreducible, so $\mathfrak{g}$-central
and the split extension of $V$ by $A^*/C_{A^*}(V)$ is in $\mathfrak{g}$. But $C_{A^*}(V) = \ker \phi$. Thus
the split extension is $P$.

**Theorem 6.15.** Let $\mathfrak{g}$ be a saturated $p$-formation of completely soluble restricted
Lie algebras which contains all atoms. Then $\mathfrak{g} = p\Ev(\Lambda) \cap p\mathfrak{C}$ for some $p$-normal
subspace $\Lambda$ of $\hat{F}$.

**Proof.** Let $\lambda$ be the set of all eigenvalues of all elements of all algebras $(L, [p]) \in \mathfrak{g}$. Then
clearly $\mathfrak{g} \subseteq p\Ev(\Lambda) \cap p\mathfrak{C}$. Suppose $(L, [p]) \in p\Ev(\Lambda) \cap p\mathfrak{C}$. If $\lambda$ is an eigenvalue
of $\ad(x)$ for some $x \in L$, then $\lambda \in \Lambda$ and by the definition of $\Lambda$, is an eigenvalue
of $\ad(A)$ for some element of an algebra in $\mathfrak{g}$. By Lemma 6.13 $P_\lambda \in \mathfrak{g}$. By Lemma
6.14 every primitive quotient of $(L, [p])$ is in $\mathfrak{g}$. Therefore $(L, [p]) \in \mathfrak{g}$. □

7. **Intravariance of projectors**

**Definition 7.1.** The $[p]$-subalgebra $U$ of the restricted Lie algebra $(K, [p])$ is said to be
invariant in $(K, [p])$ if every derivation of $K$ is expressible as the sum of an inner derivation and a derivation which stabilises $U$.

The $p$-operation plays no part in this definition, so $U$ is invariant in $(K, [p])$
if and only if it is invariant in the underlying Lie algebra $K$. By [2] Lemma 1.2, the
invariant $[p]$-subalgebras of $(K, [p])$ are precisely those $[p]$-subalgebras $U$ with
the property that, if $K = [p]$-ideal of $(L, [p])$, then $L = K + \mathcal{N}_L(U)$. That a Cartan
subalgebra $U$ of $(K, [p])$ (that is, a nilpotent $[p]$-subalgebra with $\mathcal{N}_K(U) = U$) is
invariant is immediate from [2] Theorem 2.1 without any requirement that $K$
be soluble. The analogue of [2] Theorem 2.2 also holds with minor modification to
cope with the existence of non-null atoms.

**Theorem 7.2.** Let $\mathfrak{f}$ be a $p$-homomorph of soluble restricted Lie algebras which
contains all atoms. Let $K$ be a soluble $[p]$-ideal of the restricted Lie algebra $(L, [p])$
and let $S$ be an $\mathfrak{f}$-covering subalgebra of $K$. Then $L = K + \mathcal{N}_L(S)$.

**Proof.** The result is trivial if $K = L$ or $S = K$. Let $(L, [p])$ be a counterexample
of least possible dimension. Let $A$ be a minimal $[p]$-ideal of $(L, [p])$ contained in $K$.
Then $S + A/A$ is an $\mathfrak{f}$-covering subalgebra of $K/A$. By induction, we have
\[
K/A + \mathcal{N}_L(A) = K/A,
\]
that is, $K + \mathcal{N}_L(S + A) = L$. Put $N = \mathcal{N}_L(S + A)$. Then $K \cap N$ is a $[p]$-ideal of $N$
and $S$ is an $\mathfrak{f}$-covering subalgebra of $K \cap N$. If $N < L$, then by induction we have
$(K \cap N) + \mathcal{N}_N(S) = N$. But then
\[
K + \mathcal{N}_L(S) \supseteq K + (K \cap N) + \mathcal{N}_U(S) = K + N = L
\]
contrary to $(L, [p])$ being a counterexample. Therefore $K_1 = S + A$ is a $[p]$-ideal
of $(L, [p])$. It is clearly sufficient to prove $K_1 + \mathcal{N}_L(S) = L$. As $K_1$ satisfies the
conditions required of $K$, we may replace $K$ with $K_1$, so we may suppose $S + A = K$.
We then have $K/A \simeq S/S \cap A \in \mathfrak{f}$.
By Corollary 4.10 we may suppose that $\mathfrak{H}$ is saturated. If $A \subseteq \Psi(L, [p])$, then $K \in \mathfrak{H}$ by Theorem 4.22 and $S = K$. Therefore there exists a maximal $[p]$-subalgebra $U$ which complements $A$ in $L$. Put $V = K \cap U$, $B = A \cap S$ and $T = B + V$.

Since $A$ is abelian and $A + S = K$, $B$ is a $[p]$-ideal of $K$. Both $S$ and $T$ complement $A/B$ in $K/B$. Since $S/B$ is an $\mathfrak{H}$-covering subalgebra of $K/B$, so is $T/B$. By Lemma 4.9, $T = \alpha_a(S)$ for some $a \in A$. But $\alpha_a = 1 + \text{ad}(a)$ is an automorphism of $(L, [p])$, so we may suppose $S = T = B + V$.

If $B = 0$, then $S = V$, $N_L(S) \supseteq U$ and $K + N_L(S) = L$. Therefore $B \neq 0$. Let $B_1$ be a maximal $K$-submodule of $B$. Then $S/B_1$ is the split extension of $B/B_1$ by $V$ and is in $\mathfrak{H}$. Let $A_1$ be a maximal $K$-submodule of $A$ containing $B$. Since $A$ is irreducible as $L$-module and $K$ is an ideal of $L$, all $K$-composition factors of $A$ are isomorphic by Zassenhaus [17, Lemma 1]. Since both $K/A_1$ and $S/B_1$ are split extensions of composition factors of $A$ by $V$, they are isomorphic and $K/A_1 \in \mathfrak{H}$. But $S + A \neq K$ contrary to $S$ being an $\mathfrak{H}$-covering subalgebra of $K$. Thus no minimal counterexample exists.

8. Comparisons

In this section, I investigate the relationship between Schunck classes of soluble Lie algebras and $p$-Schunck classes of soluble restricted Lie algebras over the same field $F$ of characteristic $p$. For this, I define a function $\text{Res}$ classes of Lie algebras to classes of restricted Lie algebras and a function $\text{Under}$ from classes of restricted Lie algebras to classes of Lie algebras.

**Definition 8.1.** Let $\mathfrak{H}$ be a class of soluble Lie algebras over $F$. We define $\text{Res}(\mathfrak{H})$ to be the class of all restricted Lie algebras $(L, [p])$ with $L \in \mathfrak{H}$. For a class $\mathfrak{K}$ of soluble restricted Lie algebras, we define $\text{Under}(\mathfrak{K})$ to be the class of underlying algebras of members of $\mathfrak{K}$.

**Lemma 8.2.** Let $\mathfrak{H}$ be a homomorph. Then $\text{Res}(\mathfrak{H})$ is a $p$-homomorph. If $\mathfrak{H}$ is a formation, then $\text{Res}(\mathfrak{H})$ is a $p$-formation.

*Proof.* If $(L, [p]) \in \mathfrak{H}$ and $A$ is a $[p]$-ideal of $L$, then $L/A \in \mathfrak{H}$, so $(L/A, [p]) \in \text{Res}(\mathfrak{H})$. If $\mathfrak{H}$ is a formation and $(L, [p])$ is a restricted Lie algebra with $p$-ideals $A, B$ such that $(L/A, [p])$ and $(L/B, [p])$ are in $\text{Res}(\mathfrak{H})$, then $L/A$ and $L/B$ are in $\mathfrak{H}$. So $L/(A \cap B) \in \mathfrak{H}$ and $(L/(A \cap B), [p]) \in \text{Res}(\mathfrak{H})$. □
Note that in the above, we had a restricted Lie algebra \((L, [p])\) and quotients \(L/A, L/B\) in the formation \(\mathfrak{R}\). For a Lie algebra \(L\), having restrictable quotients \(L/A, L/B\) does not imply \(L/(A \cap B)\) restrictable.

**Example 8.3.** Let \(U = \langle a, b, c \rangle\) with \(ab = b.ac = bc = 0\). Let \(V = \langle v_0, \ldots, v_{p-1} \rangle\) be the \(U\)-module with the action \(av_i = iv_i, bv_i = v_{i+1}\), where the indices are integers mod \(p\). Let \(W = \langle w_0, w_1 \rangle\) be the \(U\)-module with action \(aw_0 = 0, aw_1 = w_1, bw_0 = -w_1, bw_1 = 0, cw_i = w_i\). Let \(X\) be the split extension of \(V\) by \(U\). Putting \(a[p] = a, b[p] = c[p] = c, v_i[p] = 0\) makes \((X, [p])\) a restricted Lie algebra. Let \(Y\) be the split extension of \(W\) by \(U\). Putting \(a[p'] = a, b[p'] = 0, c[p'] = c, w_i[p'] = 0\) makes \((Y, [p'])\) a restricted Lie algebra. Since \(X\) and \(Y\) have trivial centres, \([p]\) and \([p']\) are the only \(p\)-operations on them. Let \(L\) be the split extension of \(V \oplus W\) by \(U\). Any \(p\)-operation on \(L\) would have to agree with \([p]\) on \(L/W\) and with \([p']\) on \(L/Y\). But \([p]\) and \([p']\) do not agree on \(U = L/(V + W)\).

From Example 8.3 we see that \(\mathfrak{R}\) being a \(p\)-formation which contains with any \((L, [p])\), also \((L, [p']\) for any \(p\)-operation \([p']\) on \(L\) does not ensure that \(\text{Under}(\mathfrak{R})\) is a formation.

**Lemma 8.4.** Let \(\mathfrak{R}\) be a \(p\)-homomorph with the property that, if \((L, [p]) \in \mathfrak{R}\) and \([p']\) is another \(p\)-operation on \(L\), then \((L, [p]) \in \mathfrak{R}\). Then \(\text{Under}(\mathfrak{R})\) is a homomorph.

**Proof.** It is sufficient to prove that if \(A\) is a minimal ideal of \(L \in \text{Under}(\mathfrak{R})\), then \(L/A \in \text{Under}(\mathfrak{R})\). By Lemma 2.12, \(A\) is a \([p']\)-ideal for some \(p\)-operation \([p]') on \(L\), so \(L/A \in \text{Under}(\mathfrak{R})\).

However, \(\mathfrak{R}\) a \(p\)-Schunck class does not ensure that \(\text{Under}(\mathfrak{R})\) is a Schunck class. If the \(p\)-Schunck class \(\mathfrak{R}\) contains the algebras \((X, [p])\) and \((Y, [p'])\) of Example 8.3, then \(\text{Under}(\mathfrak{R})\) contains \(X\) and \(Y\). If it is a Schunck class, then it contains any algebra whose primitive quotients are quotients of \(X\) or \(Y\). In particular, it must contain \(L\) which, being not restrictable, cannot be in \(\text{Under}(\mathfrak{R})\). I introduce another function which does give Schunck classes.

**Definition 8.5.** Let \(\mathfrak{R}\) be a \(p\)-Schunck class. We define \(\text{Ord}(\mathfrak{R}) = \mathfrak{P}(\text{Under}(\mathfrak{R}))\).

As every non-zero Schunck class contains the class \(\mathfrak{R}\) of nilpotent Lie algebras, and every \(p\)-Schunck class containing all atoms contains the class \(p\mathfrak{R}\), we restrict attention to \((p)\)-Schunck classes containing \((p)\mathfrak{R}\). We denote the set of all non-zero Schunck classes by \(\text{Sch}\) and the set of all \(p\)-Schunck classes which contain all atoms by \(p\text{Sch}\). They are partially ordered by inclusion. We make them into lattices by defining the sum of \((p)\)-Schunck classes.

**Definition 8.6.** Let \(\mathfrak{S}\) be a \((p)\)-homomorph. We define the skeleton \(\text{Skel}(\mathfrak{S})\) of \(\mathfrak{S}\) to be the class of all primitive algebras in \(\mathfrak{S}\). We define the sum or join of two \((p)\)-Schunck classes \(\mathfrak{S}_1, \mathfrak{S}_2\) by

\[
\mathfrak{S}_1 + \mathfrak{S}_2 = \mathfrak{P}(\text{Skel}(\mathfrak{S}_1) \cup \text{Skel}(\mathfrak{S}_2)).
\]

For a \((p)\)-Schunck class \(\mathfrak{S}\), we clearly have \(\mathfrak{P}(\text{Skel}(\mathfrak{S})) = \mathfrak{S}\) and \(\mathfrak{S}_1 + \mathfrak{S}_2\) is the smallest \((p)\)-Schunck class containing both \(\mathfrak{S}_1\) and \(\mathfrak{S}_2\). The \((p)\)-Schunck classes are in one-one correspondence with their skeleta and the operations \(\cap, +\) on them correspond to the set-theoretic operations \(\cap, \cup\) on the skeleta. Thus \(\text{Sch}\) and \(p\text{Sch}\) are lattices.
Lemma 8.7. Sch and pSch, partially ordered by inclusion, are complete distributive lattices.

Proof. If $L$ is in some infinite sum of Schunck classes, it has only finitely many chief factors, so only finitely many isomorphism types of primitive quotients. It thus is in the sum of a finite subset of the Schunck classes. Thus

$$\sum \mathfrak{H}_i = \text{p}(\bigcup_i \text{Skel}(\mathfrak{H}_i)),$$

and the result for Sch follows. The result for pSch follows similarly. \hfill $\Box$

Note that Sch and pSch have greatest (the classes $\mathfrak{S}, p\mathfrak{S}$ of all soluble algebras) and least elements (the classes $\mathfrak{N}, p\mathfrak{N}$ of all nilpotent algebras).

The intersection of two saturated $p$-formations is a saturated $p$-formation, however their sum need not be a $p$-formation.

Example 8.8. Let $\Lambda_1, \Lambda_2$ be $p$-normal subspaces of $\tilde{F}$, neither of which contains the other (for example, normal extension fields of $F$ of relatively prime degrees). There exist $\lambda_1 \in \Lambda_i$ for which $\lambda_1 + \lambda_2$ is in neither space. We have the primitive algebras $P_{\lambda_i} \in p\text{Ev}(\Lambda_i)$. Any saturated $p$-formation which contains both $P_{\lambda_1}$ and $P_{\lambda_2}$ must also contain $P_{\lambda_1 + \lambda_2}$. But $P_{\lambda_1 + \lambda_2} \not\in p\text{Ev}(\Lambda_1) + p\text{Ev}(\Lambda_2)$.

The above construction is not possible over an algebraically closed field. To provide an example over an arbitrary field, I use the standard examples of soluble algebras with non-nilpotent derived algebras. Let $N = (a, b, c)$ be the nilpotent Lie algebra, $ab = c, ac = bc = 0$ and let $K = \langle k_0, \ldots, k_{p-1} \rangle$, where the indices are integers mod $p$, be the $N$-module with $ak_i = ik_{i-1}$, $bk_i = k_{i+1}$ and $ck_i = k_i$. We set $a_{[p]} = 0, b_{[p]} = c$ and $c_{[p]} = c$. Then $K$ is an $(N, [p])$-module. Let $P$ be the split extension of $K$ by $N$. Let $S = \langle x, y \rangle$ be the Lie algebra with $xy = y$. Let $V = \langle v_0, \ldots, v_{p-1} \rangle$ be the $S$-module with $xv_i = iv_i$ and $yv_i = v_{i+1}$ and let $Q$ be the split extension of $V$ by $S$. Then $Q$ is not restrictable, so we also use the algebra $Q^*$ which is the split extension of $V$ by $S^* = S \oplus \langle z \rangle$ with $zv_i = v_i$.

Example 8.9. Let $\mathfrak{F}_1$ be the saturated $p$-formation generated by $P$ and let $\mathfrak{F}_2$ be the saturated $p$-formation generated by $Q^*$. Suppose $\mathfrak{F}$ is a saturated $p$-formation which contains both $P$ and $Q^*$. Then $N \oplus S^* \in \mathfrak{F}$. With $S^*$ acting trivially on $K$ and $N$ acting trivially on $V$, $K$ and $V$ are $\mathfrak{F}$-central $(N \oplus S^*)$-modules. By Theorem 4.21 $K \otimes V$ is $\mathfrak{F}$-hypercentral. It is a faithful irreducible $(N \oplus S^*)$-module, so the split extension $T$ is a primitive algebra in $\mathfrak{F}$. Denoting the $p$-formation of algebras which are nilpotent of class at most 2 by $p\mathfrak{N}_2$, we have $P \in \mathfrak{F}_1^p = p\text{Loc}(p\mathfrak{N}_2)$, so $\mathfrak{F}_1 \subseteq \mathfrak{F}^p_2$. Let $\mathfrak{M}$ be the $p$-formation of metabelian algebras with the property that every nilpotent subalgebra is abelian. Then $Q^* \in \mathfrak{F}^p_2 = p\text{Loc}(\mathfrak{M})$ and $\mathfrak{F}_2 \subseteq \mathfrak{F}^p_2$. But $T$ is primitive and not in either $\mathfrak{F}_1^p$, so $T \not\in \mathfrak{F}_1 + \mathfrak{F}_2$.

Lemma 8.10. Let $\mathfrak{H} \neq 0$ be a Schunck class of soluble Lie algebras over $F$. Then $\text{Res}(\mathfrak{H})$ is a $p$-Schunck class which contains every atom.

Proof. Every abelian algebra is in $\mathfrak{H}$, so every atom is in $\text{Res}(\mathfrak{H})$. Suppose every primitive quotient of $(L, [p])$ is in $\text{Res}(\mathfrak{H})$. Let $K$ be an ideal of $L$ with $L/K$ primitive. If $L/K$ is abelian, then $L/K \in \mathfrak{H}$. If $L/K$ is non-abelian, then by Lemma 2.2, $K$ is a $[p]$-ideal of $L$, so $(L/K, [p]) \in \text{Res}(\mathfrak{H})$ and $L/K \in \mathfrak{H}$. Thus $L \in p\mathfrak{H} = \mathfrak{H}$ and $(L, [p]) \in \text{Res}(\mathfrak{H})$. \hfill $\Box$
Theorem 8.11. \( \text{Res} : \text{Sch} \to p\text{Sch} \) and \( \text{Ord} : p\text{Sch} \to \text{Sch} \) are homomorphisms of complete lattices and, for any \( p \)-Schunck class \( \mathcal{R} \),

\[ \text{Res} (\text{Ord}(\mathcal{R})) = \mathcal{R}. \]

Proof. The lattice operations correspond to set operations on the skeleta and the result follows. \( \square \)

9. \( p \)-envelopes

We use the basic theory of \( p \)-envelopes as set out in Strade and Farnsteiner [13, pp. 94–97].

Lemma 9.1. Let \( A \) be an abelian ideal of the soluble Lie algebra \( U \). Then there exists a \( p \)-envelope \( (L, [p]) \) of \( U \) in which \( A \) is a null \( [p] \)-ideal.

Proof. Take any \( p \)-envelope \( (L, [p]) \) of \( U \). By Lemma 2.12, there exists a \( [p] \)-operation \( [p]' \) on \( L \) with \( A[p]' = 0 \). Replacing \( L \) by the \([p]' \)-closure \( U[p]' \) of \( U \) gives the required \( p \)-envelope. \( \square \)

Lemma 9.2. Suppose \( P \) is a primitive soluble Lie algebra and let \( (L, [p]) \) be a minimal \( p \)-envelope of \( P \). Then \( (L, [p]) \) is primitive.

Proof. If \( P \) is abelian, then it is 1-dimensional and so is any minimal \( p \)-envelope. Suppose \( P \) is non-abelian. By Strade and Farnsteiner [13, Theorem 5.8, p. 96], \( Z(L) \subseteq P \). But \( Z(P) = 0 \), so \( Z(L) = 0 \). Let \( A \) be the minimal ideal of \( P \). Then \( \text{ad}(a)^2 = 0 \) for all \( a \in A \), so \( A \) is a null \( [p] \)-ideal of \( (L, [p]) \). We have \( H^n(P/A, A) = 0 \) for all \( n \). Thus there exists a complement \( M \) to \( A \) in \( L \). Suppose \( M \) contains a minimal ideal \( C \) of \( L \). Then \( C \cap P = 0 \) as \( A \) is the only minimal ideal of \( P \). Therefore \( C \subseteq Z(L) \) by [13, Lemma 5.5, p. 94]. But \( Z(L) = 0 \). Therefore \( C_U(A) = A \). \( \square \)

Note that, by [13, Theorem 5.8, p. 96], any two minimal \( p \)-envelopes are isomorphic as Lie algebras. As the minimal \( p \)-envelopes of a non-abelian primitive algebra have trivial centre, they are isomorphic as restricted Lie algebras.

Lemma 9.3. Suppose \( (L, [p]) \) is a \( p \)-envelope of the non-abelian Lie algebra \( U \) and that \( (L, [p]) \) is primitive. Then \( U \) is primitive.

Proof. Let \( A \) be a minimal ideal of \( U \). Then \( A \) is an abelian ideal of \( L \), so \( A[p] \subseteq Z(L) = 0 \). Therefore \( A \) is the unique minimal \( [p] \)-ideal of \( L \). Thus \( C_L(A) = A \) and \( U \) is primitive. \( \square \)

Note that it is possible for a restricted Lie algebra to be the minimal \( p \)-envelope of distinct primitive algebras. If \( (L, [p]) \) is a minimal \( p \)-envelope of a non-restrictable primitive Lie algebra \( U \), then it is also a minimal \( p \)-envelope of the primitive algebra \( L \).

Definition 9.4. Let \( \mathcal{R} \) be a \( p \)-Schunck class of soluble restricted Lie algebras which contains all atoms. We define the enveloped class \( \text{Envd}(\mathcal{R}) \) of \( \mathcal{R} \) to be the class of all Lie algebras \( L \) having a \( p \)-envelope in \( \mathcal{R} \).

Lemma 9.5. Suppose \( \mathcal{R} \) is a \( p \)-Schunck class which contains all atoms. Suppose \( (L, [p]) \in \mathcal{R} \) is a \( p \)-envelope of \( U \). Then every minimal \( p \)-envelope of \( U \) is in \( \mathcal{R} \).
Proof. Let \( i : U \to L \) be the inclusion. Let \((M, [p])\) be a minimal \( p \)-envelope of \( U \) with \( i' : U \to M \) the inclusion. By Proposition 5.6, p. 95, there exists a homomorphism \( f : L \to M \) with \( f \circ i = i' \). Let \( K = \ker(f) \). Then \( K \) is an ideal of \( L \) and \( K \cap U = 0 \). By Lemma 5.5, p. 94, \( K \subseteq Z(L) \). Further, \( L/K \) with some \( p \)-operation \([p]''\) is a \( p \)-envelope of \( U \). But \( M \) has the least possible dimension for a \( p \)-envelope, so \( f(L) = M \). Now for some \( p \)-operation \([p]'\) on \( L \), \( K \) is a \([p]'\)-ideal. By Theorem 14.15, \((L, [p]') \in \mathcal{R} \), so \((L/K, [p]'') \in \mathcal{R} \), that is, \((M, [p])\) is a homomorphism \( f \). Then \( K \subseteq Z(L) \). Therefore also \((L, [p]) \in \mathcal{R} \).

Lemma 9.6. Suppose \( \mathcal{R} \) is a \( p \)-Schunck class which contains all atoms. Suppose \( U \) has a \( p \)-envelope in \( \mathcal{R} \). Then every \( p \)-envelope of \( U \) is in \( \mathcal{R} \).

Proof. By Lemma 9.6, \( U \) has a minimal \( p \)-envelope \((M, [p]) \in \mathcal{R} \). Let \((L, [p], i)\) be a \( p \)-envelope. There exists a homomorphism \( f : L \to M \) with \( f \circ i = i' \). Let \( K = \ker(f) \). Then \( f(L) = M \). \( K \) is central in \( L \) and is a \([p]'\)-envelope for some \( p \)-operation \([p]'\). We have \((L/K, [p]'') \in \mathcal{R} \) and so \((L, [p]') \in \mathcal{R} \) by Corollary 14.23. Therefore also \((L, [p]) \in \mathcal{R} \).

Lemma 9.7. Suppose \( \mathcal{R} \) is a \( p \)-Schunck class which contains all atoms. Then \( \text{Envd}(\mathcal{R}) \) is a homomorphism.

Proof. Suppose \( U \in \text{Envd}(\mathcal{R}) \). Let \( A \) be a minimal ideal of \( U \). By Lemmas 9.1 and 9.6, we can choose a \( p \)-envelope \((L, [p]) \in \mathcal{R} \) with \( A \) a \([p]\)-ideal. Then \((L/A, [p]) \in \mathcal{R} \) is a \( p \)-envelope of \( U/A \), so \( U/A \in \text{Envd}(\mathcal{R}) \). It follows that \( U/K \in \mathcal{R} \) for any ideal \( K \) of \( U \).

Theorem 9.8. Suppose \( \mathcal{R} \) is a \( p \)-Schunck class which contains all atoms. Then \( \text{Envd}(\mathcal{R}) \) is a \( p \)-Schunck class.

Proof. Suppose \( A \) is a minimal ideal of \( U \), \( U/A \in \text{Envd}(\mathcal{R}) \) and \( U \notin \text{Envd}(\mathcal{R}) \). Take a minimal \( p \)-envelope \((L, [p]) \) of \( U \) such that \( A \) is a \([p]\)-ideal. Now \((L/A, [p]) \in \mathcal{R} \) by Lemma 9.6, but \((L, [p]) \notin \mathcal{R} \). There exists a \( \mathcal{R} \)-projector \( M \) of \((L, [p]) \). \( M \) complements \( A \) in \( L \) and \( M \cap U \) complements \( A \) in \( U \). We have to show that \( M \cap U \) is an \( \text{Envd}(\mathcal{R}) \)-projector of \( U \).

Let \( K \) be an ideal of \( U \). We have to show that, if \( U/K \in \text{Envd}(\mathcal{R}) \), then \((M \cap U) + K = U \). This holds if \( K \subseteq M \), so we may suppose \( K \subseteq M \). Now \( K \) is an ideal of \( L \). Let \( B \subseteq K \) be a minimal ideal of \( L \). There exists a \( p \)-operation \([p]'\) on \( L \) which vanishes on the abelian ideal \( A + B \). From the minimality of \((L, [p]) \), \((L, [p]) \) is also a minimal \( p \)-envelope of \( U \), so we may assume that \( B \) is a \([p]\)-ideal. Since \( U/K \in \text{Envd}(\mathcal{R}) \), we have \((L/K, [p]) \in \mathcal{R} \) by Lemma 9.6. But \( M \) is a \( \mathcal{R} \)-projector of \( L \) and \( M + K = L \) contrary to \( K \subseteq M \).

Theorem 9.9. The class map \( \text{Envd} : p\text{Sch} \to \text{Sch} \) is a lattice homomorphism which sends \( p \)-formations to formations.

Proof. Let \( \mathcal{R}, \mathcal{R}' \in p\text{Sch} \). Suppose \( U \in \text{Envd}(\mathcal{R}) \cap \text{Envd}(\mathcal{R}') \). Then \( U \) has a \( p \)-envelope \((L, [p]) \in \mathcal{R} \) and a \( p \)-envelope \((L', [p]) \in \mathcal{R}' \). By Lemma 9.6, \((L, [p]) \in \mathcal{R} \), so \( U \in \text{Envd}(\mathcal{R} \cap \mathcal{R}') \). Trivially, if \( U \in \text{Envd}(\mathcal{R} \cap \mathcal{R}') \), then \( U \in \text{Envd}(\mathcal{R}) \cap \text{Envd}(\mathcal{R}') \).

Thus \( \text{Envd}(\mathcal{R} \cap \mathcal{R}') = \text{Envd}(\mathcal{R} \cap \mathcal{R}') \).

Suppose \( U \in \text{Envd}(\mathcal{R}) \cup \text{Envd}(\mathcal{R}') \). Let \((L, [p]) \) be a \( p \)-envelope of \( U \). Let \((L/K, [p]) \) be a primitive quotient of \((L, [p]) \). Then \((L/K, [p]) \) is a \( p \)-envelope of \( U + K/K \). If \( U + K/K \) is abelian, \((L/K, [p]) \) is an atom and by assumption, is in both \( \mathcal{R} \) and \( \mathcal{R}' \). Suppose \( U + K/K \) is non-abelian. Then by Lemma 14.8...
$U + K/K$ is primitive. But every primitive quotient of $U$ is in $\text{Envd}(\mathfrak{r})$ or $\text{Envd}(\mathfrak{r}')$ and it follows that $(L/K, [p])$ is in either $\mathfrak{r}$ or $\mathfrak{r}'$. Thus $(L, [p]) \in \mathfrak{r} \cap \mathfrak{r}'$ and $U \in \text{Envd}(\mathfrak{r} \cap \mathfrak{r}')$.

Suppose $U \in \text{Envd}(\mathfrak{r} \cup \mathfrak{r}')$. Let $K$ be an ideal of $U$ with $U/K$ primitive. If $K = 0$, then a minimal $p$-envelope of $U$ is primitive, is in $\text{Envd}(\mathfrak{r} \cup \mathfrak{r}')$ and so in $\mathfrak{r}$ or $\mathfrak{r}'$. We then have $U \in \text{Envd}(\mathfrak{r}) \cup \text{Envd}(\mathfrak{r}')$, so suppose $K \neq 0$. Let $A \subseteq K$ be a minimal ideal of $U$. By Lemma 9.1, there exists a $p$-envelope $(L, [p])$ of $U$ in which $A$ is a null $[p]$-ideal. It follows that $(L/A, [p]) \in \mathfrak{r} \cap \mathfrak{r}'$ is a $p$-envelope of $U/A$. By induction on $\dim(U)$, $U/K$ is in $\text{Envd}(\mathfrak{r})$ or in $\text{Envd}(\mathfrak{r}')$. Thus $U \in \text{Envd}(\mathfrak{r}) \cup \text{Envd}(\mathfrak{r}')$.

Now suppose $K \in p\text{Sch}$ is a $p$-formation. Suppose $A_1, A_2$ are minimal ideals of $U$ and that $U/A_i \in \text{Envd}(\mathfrak{r})$. We can choose a $p$-envelope $(L, [p])$ of $U$ such that $[p]$ vanishes on the abelian ideal $A_1 + A_2$. We then have $(L/A_i, [p])$ is a $p$-envelope of $U/A_i$ and so is in $\mathfrak{r}$. Therefore $(L, [p]) \in \mathfrak{r}$ and $U \in \text{Envd}(\mathfrak{r})$. □

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