THE FINE STRUCTURE OF THE KASPAROV GROUPS III: RELATIVE QuASIDIAGONALITY

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Abstract. Quasidiagonality was introduced by P.R. Halmos for operators and quickly generalized to $C^*$-algebras. D. Voiculescu asked how quasidiagonality (in its various forms) was related to topological invariants. N. Salinas systematically studied the topology of the Kasparov groups $KK_*(A, B)$ and showed that this topology is related to relative quasidiagonality. In this paper we identify $QD(A, B)$, the quasidiagonal classes in $KK_1(A, B)$, in terms of $K_*(A)$ and $K_*(B)$, and we use these results in various applications. Here is our central result. Let $\tilde{N}$ denote the category of separable nuclear $C^*$-algebras which satisfy the Universal Coefficient Theorem.

Theorem. Suppose that $A \in \tilde{N}$ and $A$ is quasidiagonal relative to $B$. Then there is a natural isomorphism

$$QD(A, B) \cong Pext^1_2(K_*(A), K_*(B)).$$

Thus for $A \in \tilde{N}$ quasidiagonality of $KK$-classes is indeed a topological invariant. We also settle a question raised by L.G. Brown on the relation between relative quasidiagonality and the kernel of the natural map

$$\theta^* : Ext^1_2(K_*(A), K_*(B)) \to Ext^1_2(K_*(A)_t, K_*(B)).$$

where $K_*(A)_t$ denotes the torsion subgroup of $K_*(A)$. Finally we establish a converse to a theorem of Davidson, Herrero, and Salinas, giving conditions under which the quasidiagonality of $A/K$ implies the quasidiagonality of the associated representation of $A$.

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1. Introduction: Quasidiagonality and $KK$-theory

This is the third in a series of papers in which the topological structure of the Kasparov groups, systematically studied first by Salinas, is developed and put to use. The first two papers [20, 21] are devoted to general structural results and serve as the theoretical background for the present work, which centers about quasidiagonality. From the point of view of [20, 21], this is an exploration of the closure of zero in the Kasparov groups, which we have termed the fine structure subgroup.

Quasidiagonality was defined by P.R. Halmos [12] in 1970. A bounded operator on Hilbert space is quasidiagonal if it is a compact perturbation of a block-diagonal operator. This soon was generalized to $C^*$-algebras. Quasidiagonality is thus a finite dimensional approximation property. It is not well understood.

L.G. Brown, R.G. Douglas, and P.A. Fillmore [6] first recognized that the study of quasidiagonality for operators and for $C^*$-algebras might be approached by topological methods. They topologized their functor $\mathcal{E}xt(X)$ (which is known now to be isomorphic to the Kasparov group $KK_1(C(X), \mathbb{C})$) and announced that the closure of zero corresponded to the quasidiagonal extensions. L.G. Brown pursued this theme, particularly in [4] (cf. §§6,7).

Salinas [15] studied the topology on the Kasparov groups $KK_\ast(A, B)$ and showed that this topology is related to relative quasidiagonality. The quasidiagonal classes $QD(A, B)$ (defined precisely in §2) constitute a certain subgroup of $KK$-theory:

$$QD(A, B) \subseteq KK_1(A, B).$$

If $A$ is in the category $\tilde{\mathcal{N}}$ of separable nuclear $1$ $C^*$-algebras which satisfy the Universal Coefficient Theorem$^2$ then more can be said. The UCT is a natural short exact sequence

$$0 \rightarrow \mathcal{E}xt^\ast_2(K_*(A), K_*(B)) \xrightarrow{\delta} KK_\ast(A, B) \xrightarrow{\gamma} Hom^\ast_\mathbb{Z}(K_*(A), K_*(B)) \rightarrow 0$$

$^1$We require nuclearity for two reasons. First, we need at least $K$-nuclearity so that the Kasparov product will be continuous (cf. 1.2). Second, we apparently need nuclearity in order to satisfy the hypotheses of Salinas’s result identifying the quasidiagonal elements with the closure of zero in the Kasparov groups (cf. 2.2). It seems possible that if one restricts attention to extensions that have additive inverses so that the identification

$$\mathcal{E}xt_\ast(A, B) \cong KK_\ast(A, B)$$

holds, then his result might generalize to the $K$-nuclear setting.

$^2$To be precise, $\tilde{\mathcal{N}}$ is the full subcategory of the category of separable nuclear $C^*$-algebras $A$ that have the property that the UCT is satisfied for every pair $(A, B)$, where $B$ ranges over all separable $C^*$-algebras $B$. Equivalently, we could require that $A$ is $KK$-equivalent to a commutative $C^*$-algebra. The category $\tilde{\mathcal{N}}$ includes the bootstrap category $\mathcal{N}$, by the UCT of Rosenberg and the author [14]. The bootstrap category $\mathcal{N}$ [16, 14] is the smallest full subcategory of separable nuclear $C^*$-algebra which contains all separable Type I $C^*$-algebras and which is closed under strong Morita equivalence, direct limits, extensions, and crossed products by $\mathbb{Z}$ and $\mathbb{R}$. We may also require that if $J$ is an ideal of $A$ and if $J, A \in \mathcal{N}$ then so is $A/J$, and if $A$ and $A/J$ are in $\mathcal{N}$ then so is $J$. The hope is that every separable nuclear $C^*$-algebra is $KK$-equivalent to a commutative $C^*$-algebra.
which splits unnaturally and thus computes $KK_*(A, B)$ in terms of $K_*(A)$ and $K_*(B)$. In particular, it identifies a canonical subgroup of $KK_*(A, B)$, namely

$$\text{Ext}^1_\mathbb{Z}(K_*(A), K_*(B)) \overset{\delta}{\cong} \text{Ker}(\gamma) \hookrightarrow KK_*(A, B)$$

Henceforth we generally suppress mention of the map $\delta$.

Salinas has shown [15, 5.1] that if $A$ is quasidiagonal relative to $B$ then

$$QD(A, B) \subseteq \text{Ker}(\gamma)$$

and in fact [15, Theorem 5.2a as reformulated by M. Dadarlat (private communication)] that if $A \in \tilde{\mathcal{N}}$ so that the UCT holds and identifies

$$\text{Ker}(\gamma) \cong \text{Ext}^1_\mathbb{Z}(K_*(A), K_*(B))$$

then

$$QD(A, B) \subseteq P\text{ext}^1_\mathbb{Z}(K_*(A), K_*(B))_0$$

where $P\text{ext}^1_\mathbb{Z}(G, H)$ is the subgroup of $\text{Ext}^1_\mathbb{Z}(G, H)$ consisting of pure extensions.

In the first two papers in this series [20, 21] we demonstrated the following facts:

1.1. There is a natural structure of a pseudopolonais group on $KK_*(A, B)$ [20, 6.2].

1.2. The Kasparov pairing is jointly continuous with respect to this topology, provided that all $C^*$-algebras which appear in the first variable are $K$-nuclear [20, 6.8].

1.3. If $K_*(A)$ is finitely generated then $KK_*(A, B)$ is polonais. [20, 6.2]

Note, though, that the map $\delta$ has degree one and so the elements of degree zero in the group $\text{Ext}^1_\mathbb{Z}(K_*(A), K_*(B))$, denoted by $\text{Ext}^1_\mathbb{Z}(K_*(A), K_*(B))_0$, are contained in $KK_1(A, B)$.

A subgroup $H \subseteq J$ is said to be pure if for each $n \in \mathbb{N}$,

$$nH = H \cap nJ$$

and an extension of abelian groups

$$0 \to H \to J \to G \to 0$$

is said to be pure if $H$ is a pure subgroup of $J$. If $H$ is a direct summand of $J$ then $H$ is a pure subgroup, but the most interesting cases involve non-split pure extensions. For example, $tJ$, the torsion subgroup of $J$, is always a pure subgroup of $J$ but it is not necessarily a direct summand of $J$. See [11, §53] and [22].

A topological space is polonais if it is separable, complete, and metric. If it is a topological group then we insist that the metric be invariant. A pseudopolonais group is a separable topological group with invariant pseudometric whose Hausdorff quotient group is polonais.
1.4. The index map

\[ \gamma: KK_*(A, B) \to \text{Hom}_\mathbb{Z}(K_*(A), K_*(B)) \]

is continuous. If \( \text{Im}(\gamma) \) is closed (e.g., if \( \gamma \) is onto), then \( \gamma \) is an open map. If \( \gamma \) is an algebraic isomorphism then it is an isomorphism of topological groups. [20, 7.4]

1.5. The Universal Coefficient Theorem short exact sequence is a sequence of pseudopolonais groups and each of the splittings of the UCT constructed in [14] is a topological splitting. [21, 4.5]

1.6. If \( A \in \tilde{\mathcal{N}} \) then there is a natural isomorphism

\[ Z_*(A, B) \cong \text{Pext}_2^1(K_*(A), K_*(B)) \]

where \( Z_*(A, B) \) denotes the closure of zero in the group \( KK_*(A, B) \). [21, 3.3]

1.7. If \( A \in \tilde{\mathcal{N}} \) then it has an associated \( KK \)-filtration diagram for \((A, B)\) which is functorial into the category of pseudopolonais groups [21].\(^6\) In particular, the Milnor and Jensen sequences take values in this category and are natural with respect to both \( A \) and \( B \).

The following theorem is the most important result in this paper; all of our applications flow from it. Salinas [15] proved the theorem under certain stringent assumptions on \( A \) and \( K_*(A) \) which we have removed.

**Theorem 2.3.** Suppose that \( A \in \tilde{\mathcal{N}} \) and \( A \) is quasidiagonal relative to \( B \). Then there is a natural isomorphism of topological groups

\[ QD(A, B) \cong \text{Pext}_2^1(K_*(A), K_*(B))_0 \]

regarded as topological subgroups of \( KK_*(A, B) \) via the canonical inclusion \( \delta \) in the UCT.

Note as an immediate consequence of this theorem that quasidiagonality of extensions is a topological invariant for \( A \in \tilde{\mathcal{N}} \), answering the relative form of a question of D. Voiculescu [24, 25]. For instance (see Theorem 3.5), if

\[
\begin{array}{cccccc}
0 & \longrightarrow & B \otimes \mathcal{K} & \longrightarrow & E_\tau & \longrightarrow & A & \longrightarrow & 0 \\
\downarrow 1 & & \downarrow e_\tau & & \downarrow \tau & & \\
0 & \longrightarrow & B \otimes \mathcal{K} & \longrightarrow & \mathcal{L}(H_B) & \longrightarrow & Q(H_B) & \longrightarrow & 0
\end{array}
\]

is an essential extension classified by \( \tau \) such that \( A \in \tilde{\mathcal{N}} \) and \( A \) is quasidiagonal relative to \( B \), then \( e_\tau(E_\tau) \) is \( B \)-quasidiagonal if and only if both of the following topological conditions hold:

1. \( \gamma(\tau) = 0 : K_*(A) \to K_*(B) \), and

\(^6\)We proved 1.6 and 1.7 in [21] for \( A \in \mathcal{N} \). However, the results hold for \( A \in \tilde{\mathcal{N}} \) as may easily be seen by inspection.
\(\tau \in \bigcap_n n\text{Ext}_2^1(K_*(A), K_*(B))_0 = P\text{ext}_2^1(K_*(A), K_*(B))_0.\)

If \(K_*(A)\) is torsionfree then condition 2) is automatically satisfied, so \(e_\tau(E_\tau)\) is \(B\)-quasidiagonal if and only if \(\gamma(\tau) = 0\). Thus when \(A \in \tilde{N}\) with \(K_*(A)\) torsionfree, the index invariant \(\gamma\) is a complete obstruction to relative quasidiagonality.

The remainder of the paper is organized as follows.

In Section 2 the definitions of quasidiagonality are recalled and Theorem 2.3 is established.

Theorem 2.3 has several corollaries which are developed in §3. Here is one. Suppose that \(f : A \to B\) so that

\([f] \in KK_0(A, B) \xrightarrow{\beta^A} KK_1(SA, B)\)

where \(\beta^A\) denotes the Bott periodicity isomorphism in the first variable. When is \(\beta^A([f])\) a quasidiagonal class? It is easy to show that the following are necessary conditions:

1. The induced homomorphism \(f_* : K_*(A) \to K_*(B)\) is trivial; and,
2. The associated short exact sequence

\[0 \to K_*(SB) \to K_*(Cf) \to K_*(A) \to 0\]

is pure exact, where \(Cf\) denotes the mapping cone of \(f\).

It is shown that (for \(A \in \tilde{N}\)) these conditions are also sufficient. If \(K_*(A)\) is torsionfree then (2) is automatic, and so \(\beta^A([f])\) is a quasidiagonal class if and only if \(f_* = 0\). The section concludes with Theorem 3.5, which demonstrates the topological nature of relative quasidiagonality in a concrete manner.

The remaining four sections are devoted to applications.

Section 4 is devoted to the application of some of the theory of infinite abelian groups to obtain results on quasidiagonality.

In Section 5 we answer a question raised by L.G. Brown [4] concerning the relation between quasidiagonality and the kernel of the map

\[\theta^* : \text{Ext}_2^1(K_*(A), K_*(B)) \to \text{Ext}_2^1(K_*(A)_t, K_*(B)).\]

induced by the natural inclusion of \(K_*(A)_t\), the torsion subgroup of \(A\), into \(K_*(A)\).

Section 6 deals with another result of L.G. Brown. Brown constructed [4] an operator \(T\) which was not quasidiagonal but such that \(T \oplus T\) was quasidiagonal. In Section 7 we analyze such phenomena.
Section 7 presents a converse to a theorem of Davidson, Herrero, and Salinas which deals with conditions under which the quasidiagonality of \( A/\mathcal{K} \) implies the quasidiagonality of \( A \).

It is a pleasure to acknowledge helpful correspondence and conversations regarding quasidiagonality and abelian groups with L.G. Brown, N. Brown, M. Dadarlat, H. Lin, T. Loring, N. Salinas, and D. Voiculescu, with a special thanks to N. Salinas for his help and encouragement.

In this paper all \( C^* \)-algebras are assumed separable with the obvious exceptions of multiplier algebras \( \mathcal{M}(B \otimes \mathcal{K}) \) and their quotients. Whenever we speak of quasidiagonal classes in \( KK_1(A, B) \) it is understood that \( B \) is separable, \( B \otimes \mathcal{K} \) has a countable approximate unit consisting of projections,\(^7\) and \( A \) is quasidiagonal relative to \( B \).

An isomorphism of topological groups is an algebraic isomorphism which is a homeomorphism of topological spaces. We use the topologists’ notation for graded abelian groups. For example,

\[
P_{ext}^1(K_*(A), K_*(B))_0 = P_{ext}^1(K_0(A), K_0(B)) \oplus P_{ext}^1(K_1(A), K_1(B)).
\]

Note: This paper, and to a small extent [20, 21], replace and very substantially extend the preliminary preprint entitled “Continuity of the Kasparov pairing and relative quasidiagonality” which will not appear.

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\(^7\)Dadarlat calls this property *stably unital*. Note that if \( B \) is unital and \( \{p_i\} \subset \mathcal{K} \) is a countable approximate unit for \( \mathcal{K} \) consisting of projections, then \( \{1 \otimes p_i\} \) is a countable approximate unit for \( B \otimes \mathcal{K} \) consisting of projections. Thus if \( B \) is unital then \( B \otimes \mathcal{K} \) is stably unital.
2. Quasidiagonality

Our description of the various definitions of quasidiagonality leans heavily upon the remarkable survey paper of Nathaniel P. Brown [7] and from the paper of Marius Dadarlat [9]. We are most grateful to them for clarifying these issues.

Halmos [12] introduced the notion of a quasidiagonal operator. A bounded linear operator on a separable Hilbert space $T \in \mathcal{L}(H)$ is a block diagonal operator if there exists a countable approximate unit consisting of projections, that is, an increasing sequence of finite rank projections $P_1 \leq P_2 \leq P_3 \ldots$ converging to the identity in the strong operator topology, which is central with respect to $T$:

$$P_n T - TP_n = 0 \quad \forall n.$$ 

An operator $T \in \mathcal{L}(H)$ is quasidiagonal if there exists a countable approximate unit consisting of projections $\{P_n\}$ which is quasicentral with respect to $T$:

$$\lim_{n \to \infty} ||P_n T - TP_n|| = 0.$$ 

The sum of a compact operator and a block diagonal operator is quasidiagonal, and Halmos proved that in fact every quasidiagonal operator has this form.

The concept extends to $C^*$-algebras as follows. Suppose that $B$ is a separable $C^*$-algebra. Let $H_B = B \otimes H$. We write

$$\mathcal{L}(H_B) = \mathcal{M}(B \otimes \mathcal{K}) \quad \text{and} \quad Q(H_B) = \mathcal{M}(B \otimes \mathcal{K})/B \otimes \mathcal{K}.$$ 

A separable$^8$ subset $E \subset \mathcal{L}(H_B)$ is called a $B$-quasidiagonal set$^9$ if there exists a countable approximate unit $\{p_n\}$ of $B \otimes \mathcal{K}$ consisting of projections $p_1 \leq p_2 \leq p_3 \ldots$ which is quasicentral with respect to each $a \in E$:

$$\lim_{n \to \infty} ||p_n a - a p_n|| = 0 \quad \forall a \in E.$$ 

A representation $\rho : A \to \mathcal{L}(H_B)$ of a separable $C^*$-algebra $A$ is said to be a $B$-quasidiagonal representation if the set $\rho(A)$ is a $B$-quasidiagonal set. If a separable $C^*$-algebra $A$ has a faithful essential$^{10}$ $B$-quasidiagonal representation then $A$ is said to be a $B$-quasidiagonal $C^*$-algebra.

Note that a set is $\mathbb{C}$-quasidiagonal if and only if there exists a countable approximate unit consisting of projections in $\mathcal{K}$ such that each operator in the set is quasicentral with respect to the countable approximate unit. In other words, each operator must be quasidiagonal in the classical sense, and there must be a countable approximate unit consisting of projections which works for every operator. We write quasidiagonal rather than $\mathbb{C}$-quasidiagonal.

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$^8$This definition is not correct if $E$ is not separable; see [7] for the correct definition.

$^9$Dadarlat calls this “quasidiagonal” but we prefer to keep track of $B$.

$^{10}$That is, the induced homomorphism $A \xrightarrow{\rho} \mathcal{L}(H_B) \to Q(H_B)$ is faithful, or, equivalently, $\rho(A) \cap (B \otimes \mathcal{K}) = \{0\}$. 
Salinas [15, 4.3] shows that if $A$ is quasidiagonal (say via a quasidiagonal representation $\rho$) and $B$ has a countable approximate unit consisting of projections then $A$ is $B$-quasidiagonal, since we may easily construct the requisite countable approximate unit in $B \otimes \mathcal{K}$ which is quasicentral with respect to the representation

$$A \xrightarrow{\rho} \mathcal{L}(H) \cong \mathcal{M}(\mathcal{K}) \longrightarrow \mathcal{M}(B \otimes \mathcal{K}) \cong \mathcal{L}(H_B).$$

Every commutative $C^*$-algebra is quasidiagonal - this is a consequence of the spectral theorem. It is easy to show that any AF algebra is also quasidiagonal. As subalgebras of quasidiagonal algebras are obviously quasidiagonal, it follows that any $C^*$-algebra which embeds in an AF algebra is itself quasidiagonal. For example, this implies that the irrational rotation $C^*$-algebras are quasidiagonal.

The unitalization of a quasidiagonal $C^*$-algebra is quasidiagonal, as is the product and minimal tensor product of quasidiagonal $C^*$-algebras. Quasidiagonality does not pass to quotients in general. We return to this point in §7.

Any quasidiagonal Fredholm operator must have Fredholm index zero. Halmos used this fact to show that the unilateral shift is not quasidiagonal. Then any $C^*$-algebra containing the shift cannot be quasidiagonal. More generally, quasidiagonal $C^*$-algebras must be stably finite - they and matrix rings over them may not contain proper isometries.

Voiculescu proved [25] that if $A$ and $B$ are homotopy equivalent $C^*$-algebras then if one of them is quasidiagonal then the other must be as well. Thus for instance $CA$, the cone on any $C^*$-algebra, is quasidiagonal, being homotopy equivalent to 0, and the suspension $SA$ is quasidiagonal, as it is a subalgebra of $CA$. Salinas extended this to the $B$-quasidiagonality setting. We state his result formally as part of Proposition 2.1.

Blackadar and Kirchberg [3, 4] introduce the class of NF algebras and demonstrate that this class coincides with the class of separable nuclear quasidiagonal $C^*$-algebras.

Suppose that an injection

$$\tau : A \longrightarrow \mathcal{Q}(H_B)$$

classifies an extension of $C^*$-algebras. Taking pullbacks yields the corresponding extension

$$0 \rightarrow B \otimes \mathcal{K} \rightarrow E_\tau \rightarrow A \rightarrow 0$$

together with the canonical faithful representation

$$e_\tau : E_\tau \longrightarrow \mathcal{L}(H_B).$$

We say that this extension is a quasidiagonal extension if $e_\tau$ is a $B$-quasidiagonal representation. Equivalently, the extension is quasidiagonal if there is an approximate unit consisting of projections in $B \otimes \mathcal{K}$ which is quasicentral in $E_\tau$. Salinas shows that this property depends only upon the equivalence class

$$[\tau] \in KK_1(A, B)$$
and hence it makes sense to speak of a \textit{quasidiagonal class} in $KK_1(A,B)$.

Suppose given an essential extension

$$0 \to B \otimes \mathcal{K} \to E_\tau \to A \to 0$$

together with the canonical faithful representation $e_\tau : E_\tau \to \mathcal{L}(H_B)$, and suppose given a $*$-homomorphism $f : A' \to A$ which is an injection. Then the induced extension $f^* \tau$ is obtained by the pullback diagram

$$
\begin{array}{cccccc}
0 & \longrightarrow & B \otimes \mathcal{K} & \longrightarrow & E_{f^* \tau} & \longrightarrow & A' & \longrightarrow & 0 \\
\downarrow & & \downarrow \tilde{f} & & \downarrow f & & \\
0 & \longrightarrow & B \otimes \mathcal{K} & \longrightarrow & E_{\tau} & \longrightarrow & A & \longrightarrow & 0
\end{array}
$$

with $\tilde{f}$ also an injection. Then $e_{f^* \tau} : E_{f^* \tau} \to \mathcal{L}(H_B)$ is given by the composition

$$E_{f^* \tau} \xrightarrow{\tilde{f}} E_\tau \xrightarrow{e_\tau} \mathcal{L}(H_B).$$

If $e_\tau$ is a $B$-quasidiagonal representation then it is clear from the construction that the canonical faithful representation $e_{f^* \tau}$ is also $B$-quasidiagonal.\(^{11}\)

A separable nuclear $C^*$-algebra $A$ is said to be \textit{quasidiagonal relative to $B$} if the class of the trivial extension $0 \in KK_1(A,B)$ is quasidiagonal. If $A$ is quasidiagonal relative to $B$ then trivial extensions are quasidiagonal, obviously, but there may be no other quasidiagonal classes.\(^{12}\)

N. Brown remarks [7, 8.2] that it is possible to have an extension

$$0 \to B \otimes \mathcal{K} \to E \to A \to 0$$

with $E$ quasidiagonal relative to $B$ without either $B$ or $A$ being quasidiagonal. However, given a separable $C^*$-algebra $D \subset \mathcal{L}(H)$ then $D$ is a quasidiagonal set if and only if the extension

$$0 \to \mathcal{K} \to C^*\{D, \mathcal{K}\} \to A \to 0$$

is a quasidiagonal extension, where $C^*\{D, \mathcal{K}\}$ denotes the $C^*$-algebra generated by $D$ and by $\mathcal{K}$ in $\mathcal{L}(H)$. We expand upon this remark as follows.

\(^{11}\)Using Salinas’s result (last part of Proposition 2.1) we can see that it is not necessary to stipulate that $f$ be an injection, once it is established that the homomorphism

$$f^* : KK_* (A,B) \to KK_* (A', B)$$

is continuous.

\(^{12}\)For example, if $K_*(A)$ is a direct sum of cyclic groups then Theorem 2.3 implies that every quasidiagonal extension is trivial. We discuss such matters in some length in later sections.
Proposition 2.1. Let $A$ and $B$ be separable nuclear $C^*$-algebras. Then the following are equivalent:

(1) $A$ is quasidiagonal relative to $B$.

(2) $A$ is $B$-quasidiagonal.

If these hold then they also hold for $*$-subalgebras of $A$. In addition, if $A$ is $B$-quasidiagonal and $A$ is homotopy equivalent to $A'$ then $A'$ is $B$-quasidiagonal.

Proof. Suppose first that $A$ is quasidiagonal relative to $B$. Then there exists an essential extension $\tau$ representing $0 \in KK_1(A, B)$ and a commuting classifying diagram

$$
\begin{array}{cccccc}
0 & \longrightarrow & B \otimes \mathcal{K} & \longrightarrow & E_\tau & \longrightarrow & A & \longrightarrow & 0 \\
& & \downarrow 1 & & \downarrow e_\tau & & \downarrow \tau & \\
0 & \longrightarrow & B \otimes \mathcal{K} & \longrightarrow & \mathcal{L}(H_B) & \longrightarrow & Q(H_B) & \longrightarrow & 0
\end{array}
$$

and the map $\tau$ lifts to a $*$-homomorphism $\tilde{\tau} : A \to \mathcal{L}(H_B)$ since $[\tau] = 0$. By assumption $e_\tau$ is a $B$-quasidiagonal representation, and hence $e_\tau(E_\tau)$ is a $B$-quasidiagonal set. The representation $\tilde{\tau}$ has range contained in $e_\tau(E_\tau)$ and hence $\tilde{\tau}(A)$ is a $B$-quasidiagonal set. Thus $\tilde{\tau}$ is an essential $B$-quasidiagonal representation of $A$. So $A$ is $B$-quasidiagonal.

In the other direction, suppose that $A$ is $B$-quasidiagonal. Then there exists an essential $B$-quasidiagonal representation $\tilde{\tau} : A \to \mathcal{L}(H_B)$. Let

$$E = C^*\{\tilde{\tau}(A), B \otimes \mathcal{K}\}$$

and let $e : E \to \mathcal{L}(H_B)$ denote the natural inclusion. Then $e$ is a $B$-quasidiagonal representation and

$$0 \to B \otimes \mathcal{K} \to E \to A \to 0$$

is an essential quasidiagonal extension which is split by the map $\tilde{\tau}$. Thus $A$ is quasidiagonal relative to $B$.

If the properties hold then one uses the fact pointed out previously that the restriction of a $B$-quasidiagonal extension to a subalgebra of $A$ is again a $B$-quasidiagonal extension. The final statement was established by Salinas [15, Theorem 4.8]. □

Let $QD(A, B)$ denote the set of quasidiagonal classes of $KK_1(A, B)$. This set is non-empty if and only if $A$ is quasidiagonal relative to $B$ by [15] Theorem 4.4. Salinas has shown that the quasidiagonal elements may be described in terms of the topology on $KK_1(A, B)$. For $B = \mathcal{K}$ the following theorem was established by L.G. Brown [4, p. 63, Remark 1.]

\[ \text{End of Document} \]
**Theorem 2.2** (Salinas [15, Theorem 4.4]). If $A$ is quasidiagonal relative to $B$ then there is a natural isomorphism

$$Z_1(A, B) \cong QD(A, B).$$

Of course this implies that $QD(A, B)$ is a subgroup of $KK^*_1(A, B)$.

Here is the principal result of this paper.

**Theorem 2.3.** Suppose that $A \in \tilde{\mathcal{N}}$, $B$ is a separable $C^*$-algebra, and $A$ is quasidiagonal relative to $B$. Then there is a natural isomorphism of topological groups

$$QD(A, B) \cong P\text{ext}^1_\mathbb{Z}(K_*(A), K_*(B))_0$$

regarded as subgroups of $KK_1(A, B)$ via the canonical inclusion $\delta$ in the UCT.

**Proof.** We have isomorphisms of topological groups

$$QD(A, B) \cong Z_1(A, B)$$

by 2.2

$$\cong P\text{ext}^1_\mathbb{Z}(K_*(A), K_*(B))_0$$

by Theorem 1.6. This completes the proof. □

**Corollary 2.4.** Suppose that $A \in \tilde{\mathcal{N}}$ and that $A$ is quasidiagonal. Then

$$QD(A, \mathcal{K}) \cong P\text{ext}^1_\mathbb{Z}(K_0(A), \mathbb{Z}).$$

□

Theorem 2.3 and Corollary 2.4 may be used readily since much is known about computing the $P\text{ext}$ groups shown; see [22].

**Remark 2.5.** The group $QD(A, B)$ may well have torsion. For instance, suppose that $A \in \tilde{\mathcal{N}}$, $A$ is quasidiagonal and $K_*(A)$ is torsionfree but not free. Then

$$QD(A, \mathcal{K}) \cong P\text{ext}^1_\mathbb{Z}(K_0(A), \mathbb{Z}) \cong Ext^1_\mathbb{Z}(K_0(A), \mathbb{Z})$$

which is always an uncountable, divisible group (cf. [22, Theorem 9.7], due to C.U. Jensen.)

\[13\] since

$$P\text{ext}^1_\mathbb{Z}(G, H) \cong Ext^1_\mathbb{Z}(G, H)$$

whenever $G$ is torsionfree.
Now choose \( A \in \tilde{N} \) with \( K_0(A) = \mathbb{Z}_p \), the integers localized at \( p \). Then
\[
\text{QD}(A, K) \cong \mathbb{Q}^{N_0} \oplus \mathbb{Z}(p^\infty).
\]

where \( \mathbb{Z}(p^\infty) \) denotes the \( p \)-torsion subgroup of \( \mathbb{Q}/\mathbb{Z} \). This point was overlooked in [15, Cor. 5.4]. In that paper the primary interest was the case with \( K_*(B) \) torsion-free and \( K_*(A) \) finitely generated. If \( G \) is finitely generated then \( \text{Ext}^1_Z(G, H) = 0 \) for all \( H \), so if \( A \in \tilde{N} \) with \( K_*(A) \) finitely generated then every quasidiagonal extension of the form
\[
0 \to B \otimes K \to E \to A \to 0
\]
is trivial in \( KK_1(A, B) \).

**Corollary 2.6.** Suppose that \( A \in \tilde{N} \), \( A \) is quasidiagonal relative to \( B \), and either

1. \( K_*(A) \) is a direct sum of (finite and/or infinite) cyclic groups.
2. \( K_*(B) \) is algebraically compact\(^{14}\).

Then any essential quasidiagonal extension
\[
0 \to B \otimes K \to E \to A \to 0
\]
is a trivial extension.

*Proof.* If \( G \) is a direct sum of cyclic groups then \( \text{Ext}^1_Z(G, H) = 0 \) for all groups \( H \). Dually, if \( H \) is algebraically compact then \( \text{Ext}^1_Z(G, H) = 0 \) for all groups \( G \).

The result is then immediate from Theorem 2.3. \( \square \)

**Corollary 2.7.** Suppose that \( A \in \tilde{N} \), \( A \) is quasidiagonal relative to \( B \), \( K_*(A) \) is torsionfree, and we are given an essential extension
\[
\tau : \quad 0 \to B \otimes K \to E \to A \to 0.
\]

Suppose further that the connecting homomorphism
\[
K_*(A) \to K_{*-1}(B)
\]

\(^{14}\)A group is algebraically compact if it is a direct summand in every group that contains it as a pure subgroup. Equivalently, it is algebraically compact if it is algebraically a direct summand in a group which admits a compact topology. Examples include compact groups, divisible groups, and bounded groups. A group is algebraically compact if and only if it is of the form
\[
D \oplus \prod_p D^p
\]
where \( D \) is divisible and for each prime \( p \) the group \( D^p \) is the completion in the \( p \)-adic topology of the direct sum of cyclic \( p \)-groups and groups of \( p \)-adic integers. For any sequence of abelian groups \( \{G_i\} \) the group \( \prod G_i/\oplus G_i \) is algebraically compact and its structure is known.
is trivial. Then the extension is quasidiagonal. If $K_*(A)$ is free then the extension is trivial.

Proof. The fact that the connecting homomorphism vanishes implies that the class of $\tau$ lies in the group $\text{Ext}^1_{\mathbb{Z}}(K_*(A), K_*(B))$. Since $K_*(A)$ is torsionfree we know [22] that

$$\text{Ext}^1_{\mathbb{Z}}(K_*(A), K_*(B)) = \text{PExt}^1_{\mathbb{Z}}(K_*(A), K_*(B))$$

and then we have

$$[\tau] \in \text{PExt}^1_{\mathbb{Z}}(K_*(A), K_*(B)) = QD(A, B)$$

as required. If $K_*(A)$ is free then $\text{PExt}^1_{\mathbb{Z}}(K_*(A), H) = 0$ for any abelian group $H$ and so $Q(A, B) = 0$ and the extension is trivial. □

Remark 2.8. The group $\text{PExt}^1_{\mathbb{Z}}(G, H)$ is either zero or huge. For example,

1. (Warfield) If both $G$ and $H$ are countable groups with $\text{PExt}^1_{\mathbb{Z}}(G, H) \neq 0$ and $H$ torsionfree, then $\text{PExt}^1_{\mathbb{Z}}(G, H)$ has $\mathbb{Q}^{\aleph_0}$ as a direct summand. It also may have torsion.

2. (R. Baer) If $G$ is a torsionfree group and $H$ is a countable torsion group then

$$\text{PExt}^1_{\mathbb{Z}}(G, H) = \mathbb{Q}^n$$

with either $n = 0$ or $n \geq \aleph_0$.

For proofs and references to these facts and for many more examples, please see [22].
3. Homomorphisms and Split Morphisms

In this section the identification of the quasidiagonal elements of Theorem 2.3 is made concrete. For instance, we answer a simple question. Suppose that $f : A \to B$. Then when is the canonically associated class $\beta^A[f] \in KK_1(SA, B)$ a quasidiagonal class? There are certain easy algebraic necessary conditions, and we show that (for $A \in \tilde{\mathcal{N}}$) these conditions are sufficient. Finally, we demonstrate how to deal with an extension given explicitly.

In applications many of the most interesting classes in $KK$-theory come from $*$-homomorphisms $f : A \to B$. The class $[f] \in KK_0(A, B)$ determines the structure of $KK_*(A, B)$ as a module over the ring $KK_*(A, A)$ and hence has special importance.

Let

$$\tau_C : 0 \to C \otimes K \to T \to SC \to 0$$

denote the universal Toeplitz extension which generates the group

$$KK_1(SC, C) \cong \mathbb{Z}.$$

We may tensor the extension with $B$ to obtain the extension

$$\tau_B : 0 \to B \otimes K \to B \otimes T \to SB \to 0.$$ 

Note that

$$\gamma(\tau_B) = 1 : K_*(B) \to K_*(B)$$

and hence

$$\gamma(\tau_B) = 0 \quad \text{if and only if} \quad K_*(B) = 0,$$

which would imply that $KK_*(B, B) = 0$ if $B \in \tilde{\mathcal{N}}$. Thus if $B \in \tilde{\mathcal{N}}$ then the extension $\tau_B$ is $B$-quasidiagonal if and only if it is trivial, and this only happens when $K_*(B) = 0$.

There is a commuting diagram

$$
\begin{array}{ccc}
KK_0(C, C) & \xrightarrow{\beta^C} & KK_1(SC, C) \\
\downarrow{\iota_B} & & \downarrow{\iota_B} \\
KK_0(B, B) & \xrightarrow{\beta^B} & KK_1(SB, B)
\end{array}
$$

where $\beta^A$ represents the Bott isomorphism in the first variable of $KK$ and $\iota_B$ the canonical structural map. Then

$$\tau_B = \iota_B(\tau_C)$$

by construction

$$= \iota_B \beta^C([1_C])$$
since this is the universal Toeplitz extension

\[ = \beta^B \iota_B([1_c]) \]

since the diagram commutes

\[ = \beta^B([1_B]). \]

The diagram

\[
\begin{array}{ccc}
KK_0(B, B) & \xrightarrow{\beta^B} & KK_1(SB, B) \\
\downarrow f^* & & \downarrow (Sf)^* \\
KK_0(A, B) & \xrightarrow{\beta^A} & KK_1(SA, B)
\end{array}
\]

commutes by the naturality of the Bott isomorphism, and hence we conclude that

\[ \beta^A([f]) = \beta^A f^*([1_B]) = (Sf)^* \beta^B([1_B]) = (Sf)^* \tau_B \in KK_1(SA, B). \]

Thus we may represent the class \[ \beta^A([f]) \] as the pullback of the extension \[ \tau_B \] by \[ Sf \], namely

\[
\begin{array}{ccccccc}
0 & \longrightarrow & B \otimes K & \longrightarrow & E & \longrightarrow & SA & \longrightarrow & 0 \\
& & \downarrow \cong & & \downarrow & & \downarrow Sf & & \\
0 & \longrightarrow & B \otimes \mathcal{K} & \longrightarrow & B \otimes T & \longrightarrow & SB & \longrightarrow & 0
\end{array}
\]

(If \[ Sf \] is not mono then as usual we add on a trivial extension so that the resulting Busby map classifying the pullback is mono.) So - here is a canonical extension associated to the map \[ f \]. The \( C^* \)-algebra \[ SA \] is always quasidiagonal and so the trivial extension added on to make \[ Sf \] mono was quasidiagonal. Thus it makes sense to ask when the class \[ \beta^A([f]) \] is quasidiagonal.

Note that

\[ \gamma(\beta^A[f]) = (\beta^A)^* \gamma(f) = (\beta^A)^* f_* \]

where

\[ (\beta^A)^* : \text{Hom}_Z(K_*(A), K_*(B)) \xrightarrow{\cong} \text{Hom}_Z(K_*(SA), K_*(B)) \]

and since \( (\beta^A)^* \) is an isomorphism (shifting parity, of course), we see that

\[ \gamma(\beta^A[f]) = 0 \quad \text{if and only if} \quad f_* = 0 : K_*(A) \to K_*(B). \]

In order to state the answer, we must first introduce the mapping cone of \( f \). Recall that the mapping cone \( Cf \) is the \( C^* \) algebra

\[ Cf = \{(\xi, a) \in B[0,1] \oplus A : \xi(0) = 0, \, \xi(1) = f(a)\} \]
with associated mapping cone sequence (cf. [17])

\[ 0 \to SB \to Cf \to A \to 0. \]

that has equivalence class

\[ \beta_B[f] \in KK_1(A, SB) \]

where \( \beta_B \) represents Bott periodicity in the second variable of \( KK \). Note that the diagram

\[
\begin{array}{ccc}
KK_0(A, B) & \xrightarrow{\beta^A} & KK_1(A, SB) \\
\uparrow f_* & & \uparrow f_* \\
KK_0(A, A) & \xrightarrow{\beta^A} & KK_1(A, SA)
\end{array}
\]

commutes and hence

\[ \beta_B([f]) = \beta_B f_*([1_A]) = (Sf)_* \beta_A([1_A]) \]

where \( \beta_A([1_A]) \) is the class of the mapping cone of the identity map \( 1 : A \to A \) which has the form

\[ 0 \to SA \to CA \to A \to 0. \]

This class is \( KK \)-invertible with \( KK \)-inverse the Toeplitz class \( \tau_A \in KK_1(SA, A) \).

Proposition 3.1. Suppose that \( A \in \~N \). Let \( f : A \to B \) be a \(*\)-homomorphism. Then the class \( \beta^A[f] \in KK_1(SA, B) \) is quasidiagonal if and only if both of the following conditions hold:

1. The induced homomorphism \( f_* : K_*(A) \to K_*(B) \) is trivial; and,
2. The associated short exact sequence

\[ 0 \to K_*(SB) \to K_*(Cf) \to K_*(A) \to 0 \]

is pure exact.

If \( K_*(A) \) is torsionfree then condition (2) is automatically satisfied, so that \( \beta^A[f] \) is quasidiagonal if and only if \( f_* = 0 \).

Proof. By virtue of Theorem 2.2 it suffices to determine when the class \( \beta^A[f] \) is in \( Z_1(SA, B) \). The Bott map is a homeomorphism, by (1.2), and hence it is equivalent to ask when the class \([f] \in KK_0(A, B)\) lies in the subgroup \( Z_0(A, B) \). By Theorem 2.3 it suffices to show that

\[ [f] \in Pext_1^Z(K_*(A), K_*(B))_0 \]

\[ ^{15} \text{This fact is the core case in the proof of Bott periodicity in the } KK \text{-context.} \]
if and only if conditions 1) and 2) hold. Condition 1) is necessary and sufficient for
\[ [f] \in Ext^1_\mathbb{Z}(K_*(A), K_*(B)) \]
by the UCT, and condition 2) picks out those elements of \( Ext^1_\mathbb{Z}(K_*(A), K_*(B)) \) which lie in \( Pext^1_\mathbb{Z}(K_*(A), K_*(B)) \). If \( K_*(A) \) is torsionfree then
\[ Pext^1_\mathbb{Z}(K_*(A), K_*(B)) = Ext^1_\mathbb{Z}(K_*(A), K_*(B)) \]
and the corollary follows. \( \square \)

Suppose given an extension of \( C^* \)-algebras
\[ 0 \to B \xrightarrow{i} E \xrightarrow{p} A \to 0 \]
which is split by a \( * \)-homomorphism \( s : A \to E \). Then there is a unique class \( \pi_s \in KK_0(E, B) \) called the splitting morphism with the property that
\[ i_*(\pi_s) = [1] - [sp] \in KK_0(E, E). \]
Note that \( \pi_s \) induces a homomorphism
\[ \gamma(\pi_s) : K_*(E) \to K_*(B) \]
but this homomorphism generally does \textit{not} arise from a \( * \)-homomorphism
\[ E \to B \otimes \mathcal{K}. \]

Every \( KK \)-class may be represented as the \( KK \)-product of a class induced by a homomorphism and by a splitting morphism [1, \( \S\S 17.1.2, 17.8.3 \)]. So we wish to know when a splitting morphism corresponds to a quasidiagonal class. Here is the answer.

**Proposition 3.2.** Suppose that \( A \in \tilde{\mathcal{N}} \). Further, suppose given an extension of \( C^* \)-algebras
\[ (4.3) \quad 0 \to B \xrightarrow{i} E \xrightarrow{p} A \to 0 \]
which is split by a \( * \)-homomorphism \( s : A \to E \). Let \( \pi_s \in KK_0(E, B) \) be the associated splitting morphism. Consider the class
\[ \beta^E(\pi_s) \in KK_1(SE, B). \]
Then the following are equivalent:

(1) \( \beta^E(\pi_s) \) is a quasidiagonal class.
\( K_*(B) = 0. \)

(3) \( KK_*(A, B) = 0. \)

(4) \( \pi_s = 0. \)

Proof. The implications 3) \( \rightarrow \) 4) \( \rightarrow \) 1) are obvious. We shall show that 1) \( \rightarrow \) 2) \( \rightarrow \) 3).

Suppose first that \( \beta^E(\pi_s) \) is a quasidiagonal class. Then \( \gamma(\beta^E(\pi_s)) = 0 \) by Theorem 2.3, which implies that \( \gamma(\pi_s) = 0. \) Then

\[
1 - s*p_\ast = \gamma([1] - [sp]) = \gamma(i_\ast(\pi_s)) = i_\ast(\gamma(\pi_s)) = 0
\]

so that \( p_\ast : K_*(E) \rightarrow K_*(A) \) is an isomorphism and thus \( K_*(B) = 0 \) as required. Thus 1) \( \rightarrow \) 2). If \( K_*(B) = 0 \) then \( KK_*(E, B) = 0 \) by the UCT. Thus 2) \( \rightarrow \) 3). \( \Box \)

Proposition 3.1 tells when a class \( \beta^A[f] \) is quasidiagonal. Proposition 3.2 tells when the class \( \beta^A(\pi_s) \) is quasidiagonal. The following theorem describes when a class with factorization

\[
x = [f] \otimes_D \pi_s = f^*(\pi_s)
\]

corresponds to a quasidiagonal class \( \beta^A(x) \). Every \( KK \)-class has this form. Thus this theorem gives a complete solution to the quasidiagonality of the associated class \( \beta^A(x) \in KK_1(SA, B) \). We state the theorem for the class \( x \) for simplicity, remembering that \( \beta^A(x) \) is a quasidiagonal class if and only if \( x \in Z_0(A, B) \cong Ext^1_D(K_*(A), K_*(B))_1 \).

**Theorem 3.3.** Suppose that \( A \in \tilde{N}. \) Let \( x \in KK_0(A, B) \) with factorization

\[
x = [f] \otimes_D \pi_s = f^*(\pi_s)
\]

with respect to the map

\[
f : A \rightarrow D
\]

and extension

\[
0 \rightarrow B \otimes K \rightarrow D \xrightarrow{p} A \rightarrow 0
\]

with splitting \( s : A \rightarrow D \) and splitting morphism \( \pi_s \in KK_0(D, B) \).

Then:

(a) The following conditions are equivalent:

(1)

\[
\gamma(x) = 0 \in Hom_Z(K_*(A), K_*(B))_0.
\]

(2)

\[
Im(f_\ast : K_*(A) \rightarrow K_*(D)) \subseteq Ker(\gamma(\pi_s) : K_*(D) \rightarrow K_*(B)) = Im(s_\ast : K_*(A) \rightarrow K_*(D)).
\]
(b) Suppose that \( \gamma(x) = 0 \), so that

\[
x \in \text{Ext}_Z^1(K_*(A), K_*(B))_1 \subseteq KK_0(A, B).
\]

Then

\[
(*) \quad x = z \otimes_D \pi_s
\]

for some

\[
z \in \text{Ext}_Z^1(K_*(A), K_*(D))_1 \subseteq KK_0(A, D).
\]

The element \( z \) is unique modulo the subgroup

\[
s_*(\text{Ext}_Z^1(K_*(A), K_*(A)))_1.
\]

Conversely, any element \( x \) of form \( (*) \) is in the group \( \text{Ext}_Z^1(K_*(A), K_*(B))_1 \).

(c) Suppose that \( x \in \text{Pext}_Z^1(K_*(A), K_*(B))_1 \subseteq KK_0(A, B) \). Then

\[
(**) \quad x = z \otimes_D \pi_s
\]

for some

\[
z \in \text{Pext}_Z^1(K_*(A), K_*(D))_1 \subseteq KK_0(A, D).
\]

The element \( z \) is unique modulo the subgroup

\[
s_*(\text{Pext}_Z^1(K_*(A), K_*(A)))_1.
\]

Conversely, any element of form \( (** \) is in the group \( \text{Pext}_Z^1(K_*(A), K_*(B))_1 \).

Proof. For Part (a) we compute:

\[
\gamma(x) = \gamma([f] \otimes_D \pi_s) = \gamma(\pi_s)f_*
\]

and hence \( \gamma(x) = 0 \) if and only if \( \text{Im}(f_*) \subseteq \text{Ker}(\gamma(\pi_s)) \). The identification \( \text{Ker}(\gamma(\pi_s)) = \text{Im}(s_*) \) is immediate from the definition of \( \gamma(\pi_s) \). This proves Part (a).

In order to prove Parts (b) and (c) the following Lemma is required. It uses the notation and assumptions of Theorem 3.3.
Lemma 3.4. The diagram

\[
\begin{array}{ccc}
\text{Ext}_1^1(K_*(A), K_*(D))_{* -1} & \xrightarrow{\delta_D} & KK_*(A, D) \\
\downarrow \gamma(\pi_s)^* & & \downarrow (\cdot) \otimes D \pi_s \\
\text{Ext}_1^1(K_*(A), K_*(B))_{* -1} & \xrightarrow{\delta_B} & KK_*(A, B)
\end{array}
\]

commutes, where the maps \(\delta\) are the inclusion maps from the UCT.

Proof. This does not follow immediately from the naturality of the UCT, since the map \((\cdot) \otimes D \pi_s\) is not induced by a map of \(C^*\)-algebras. We argue as follows. Expand the diagram to the diagram

\[
\begin{array}{ccc}
0 & 0 & \\
\downarrow & & \\
\text{Ext}_1^1(K_*(A), K_*(A))_{* -1} & \xrightarrow{\delta_A} & KK_*(A, A) \\
\downarrow s_* & & \downarrow s_* \\
\text{Ext}_1^1(K_*(A), K_*(D))_{* -1} & \xrightarrow{\delta_D} & KK_*(A, D) \\
\downarrow \gamma(\pi_s)^* & & \downarrow (\cdot) \otimes D \pi_s \\
\text{Ext}_1^1(K_*(A), K_*(B))_{* -1} & \xrightarrow{\delta_B} & KK_*(A, B) \\
\downarrow & & \downarrow \\
0 & 0 & \\
\end{array}
\]

Each column is split exact, since \(s\) is a splitting, and the upper square commutes by the naturality of the UCT (since the map \(s_*\) is induced by the map \(s : A \to D\)) and it is easy to see that the map \(\delta_B\) is the quotient map, making the lower square commute. □

Returning to the proof of Part (b), we see from Lemma 3.4 that the restriction of the map \((\cdot) \otimes_D \pi_s\) to the map

\[
(\cdot) \otimes_D \pi_s : \text{Ext}_1^1(K_*(A), K_*(D)) \to \text{Ext}_1^1(K_*(A), K_*(B))
\]

is just the split surjection \(\gamma(\pi_s)^*\). This implies that each element of the topological group \(\text{Ext}_1^1(K_*(A), K_*(B))\) is of the requisite form and the indeterminacy is as stated.

For Part (c), consider the commuting diagram
The lower horizontal map is just the map $\gamma(\pi_s)_*$ by the Lemma, and so the upper horizontal map (which is a further restriction, of course) is just the map $\gamma(\pi_s)_*$ as well. Then the conclusion of the Theorem is an easy consequence of previous results and the fact that both horizontal maps are split surjections with known kernels. □

Here is our solution to the relative quasidiagonality problem, applied to a concrete extension.

**Theorem 3.5.** Suppose that $A \in \tilde{N}$ is quasidiagonal relative to $B$. Suppose given an essential extension

$$0 \to B \otimes K \to E_\tau \to A \to 0$$

representing $\tau \in KK_1(A, B)$ with corresponding representation

$$e_\tau : E_\tau \to \mathcal{L}(H_B).$$

Then the representation $e_\tau$ is $B$-quasidiagonal if and only if both of the following conditions hold:

1. $\gamma(\tau) = 0$, or, equivalently, the boundary homomorphism $K_*(A) \to K_{*-1}(B)$ is trivial; and

2. $\tau \in \bigcap_n n\text{Ext}_2^1(K_*(A), K_{*-1}(B))_0$

or, equivalently,

$$\tau \in \text{Ker} [\varphi : \text{Ext}_2^1(K_*(A), K_*(B))_0 \to \text{Ext}_2^1(K_*(A), K_*(B))_0^\wedge]$$

where $G^\wedge$ denotes the $\mathbb{Z}$-adic completion of $G$.

If $K_*(A)$ is torsionfree then condition (2) is satisfied automatically, so that $e_\tau$ is a $B$-quasidiagonal representation if and only if the boundary homomorphism $K_*(A) \to K_{*-1}(B)$ is trivial.

**Proof.** Conditions 1) and 2) are exactly the conditions that guarantee that $\tau$ lies in the subgroup $QD(A, B)$. □
4. Purity and Quasidiagonality

In this section we take advantage of standard results in infinite abelian groups to deduce results on quasidiagonality.

**Proposition 4.1.**

(a) Suppose that $H$ is a countable abelian group. Then the following are equivalent:

1. $\text{Ext}^1_\mathbb{Z}(G, H) = 0$ for all countable abelian groups $G$.
2. $\text{Ext}^1_\mathbb{Z}(G, H) = 0$ for the groups $G = \mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$.
3. $H$ is algebraically compact.

(b) Suppose that $G$ is a countable abelian group. Then the following are equivalent:

1. $\text{Ext}^1_\mathbb{Z}(G, H) = 0$ for all countable abelian groups $H$.
2. $\text{Ext}^1_\mathbb{Z}(G, H) = 0$ for all countable direct sums of cyclic groups $H$.
3. $G$ is the direct sum of cyclic groups.

**Proof.** First concentrate on Part (a). Of course (a1) implies (a2). The implication (a3) implies (a1) is immediate from [11, 53.4]. The implication (a2) implies (a3) is the least obvious, but it is also found in [11, page 232.]

Turning to part (b), the implication (b1) implies (b2) is trivial, and the implication (b3) implies (b1) follows from [11, 53.4] as well. For the following argument that (b2) implies (b3) I am indebted to John Irwin. Suppose that $G$ satisfies condition (b2). Let $\tilde{G}$ be the free abelian group on the (countable) set

$$\{[g] : g \in G\}$$

modulo the relations given by

$$n[g] = 0$$

if $g \in G$ has order $n$. There is an obvious surjection $\tilde{G} \to G$ and hence a short exact sequence

$$\Theta : 0 \to K \to \tilde{G} \to G \to 0.$$ 

This sequence is pure, since every torsion element of $G$ lifts to a torsion element of $\tilde{G}$ of the same order. The group $\tilde{G}$ is countable by construction, hence $K$ is countable. Further, $\tilde{G}$ is a direct sum of cyclic groups. The group $K$ is a subgroup of a direct sum of cyclic groups and by Kulikov’s theorem (cf. [11, 20.1]) $K$ itself is a direct sum of cyclic groups. Thus

$$\Theta \in \text{Ext}^1_\mathbb{Z}(G, K) = 0$$

and hence the extension $\Theta$ must be split. This implies that $G$ is isomorphic to a subgroup of a direct sum of cyclic groups and hence (by Kulikov’s theorem) is itself a direct sum of cyclic groups. \( \square \)
The following theorem is the $KK$-version of the preceding, purely algebraic results. First a bit of notation. For $G$ any countable abelian group, let $C_G$ be a separable commutative $C^*$-algebra such that for each $j = 0, 1,$

$$K_j(C_G) = G.$$ Note that $C_G$ exists by geometric realization [16] and is unique up to $KK$-equivalence by the UCT. Each $C_G$ is quasidiagonal, since it is commutative, and hence $C_G$ is $B$-quasidiagonal for all separable $B.$

**Theorem 4.2.**

(a) Suppose that $B$ is a separable $C^*$-algebra. Then the following are equivalent:

1. For each $A \in \tilde{N}$ with $A$ quasidiagonal relative to $B,$

$$QD(A, B) = 0.$$

2. For $G = \mathbb{Q}$ and $G = \mathbb{Q}/\mathbb{Z},$

$$QD(C_G, B) = 0.$$

3. $K_*(B)$ is algebraically compact.

(b) Suppose given a quasidiagonal $C^*$-algebra $A \in \tilde{N}$. Then the following are equivalent:

1. For each separable $C^*$-algebra $B,$

$$QD(A, B) = 0.$$

2. For $H$ any direct sum of cyclic groups,

$$QD(A, C_H) = 0.$$

3. $K_*(A)$ is the direct sum of cyclic groups.

**Proof.** First consider (a). The implication (a1) implies (a2) is immediate. The implication (a3) implies (a1) is elementary, since if $K_*(B)$ is algebraically compact then

$$(*) \quad Pext^1_2(G, K_*(B)) = 0$$

for all groups $G$ by 4.1(1). If $A \in \tilde{N}$ with $A$ quasidiagonal relative to $B$ then

$$QD(A, B) \cong Pext^1_2(K_*(A), K_*(B))$$

If desired we may choose $C_G = C^0_G \oplus C^1_G$ where $K_i(C^j_G) = G$ if $i = j$ and $K_i(C^j_G) = 0$ if $i \neq j.$ Then we could use the $C^j_G$ separately in parts (2) and (3) of Theorem 4.2.
by Theorem 2.3, and

\[ P_{ext}^1_2(K_*(A), K_*(B)) = 0 \]

by (*), completing the argument.

Next we show that (a2) implies (a3). The condition (a2) implies that

\[ P_{ext}^1_2(G, K_*(B)) = 0 \]

for \( G = \mathbb{Q} \) and \( G = \mathbb{Q}/\mathbb{Z} \) and then Theorem 4.1 implies that \( K_*(B) \) is algebraically compact. This completes the proof of part (a).

The proof of part (b) is quite similar, and we comment only on the deep implication (b2) implies (b3). Condition (b2) together with Theorem 1.2 imply that

\[ P_{ext}^1_2(K_*(A), H) = 0 \]

whenever \( H \) is a direct sum of cyclic groups, and then Theorem 4.1(1) implies that \( K_*(A) \) is itself a direct sum of cyclic groups. This completes the proof of Theorem 4.2.  \( \square \)
5. A problem of L.G. Brown

Let \( \theta : K_*(A)_t \to K_*(A) \) be the canonical inclusion of the torsion subgroup of \( K_*(A) \). L.G. Brown \[4, page 63\] showed that (with \( B = K \)) there is a relation between quasidiagonality and the kernel of the induced map

\[
\theta^* : \text{Ext}_2^1(K_*(A), K_*(B)) \to \text{Ext}_2^1(K_*(A)_t, K_*(B)).
\]

We generalize his result as follows. Let \( G_t \) denote the torsion subgroup of a group \( G \) and \( G_f = G/G_t \) denote the maximal torsionfree quotient of \( G \). Recall \[23\] that given \( A \), there is an associated extension of \( C^* \)-algebras

\[
0 \to A \otimes K \to A_f \to SA_t \to 0
\]

whose \( K \)-theory long exact sequence degenerates to the pure short exact sequence

\[
0 \to K_*(A)_t \xrightarrow{\theta} K_*(A) \to K_*(A)_f \to 0.
\]

In particular,

\[
K_*(A_t) \cong K_*(A)_t \quad \text{and} \quad K_*(A_f) \cong K_*(A)_f.
\]

Further, if \( A \in \tilde{N} \) then so are both \( A_t \) and \( A_f \).\(^{17}\)

**Theorem 5.1.** Suppose that \( A \in \tilde{N} \) and \( \delta \) is quasidiagonal relative to \( B \). Then:

1. There is a natural commutative diagram with exact columns:

\[
\begin{array}{cccc}
0 & 0 \\
\downarrow & \downarrow \\
\text{Ker}(Q\theta^*) & \cong & \text{Ker}(\theta^*) \\
\downarrow & \downarrow \\
QD(A, B) & \xrightarrow{\cong} & \text{Pext}_2^1(K_*(A), K_*(B))_0 \\
\downarrow & \downarrow & \downarrow \\
QD(A_t, B) & \xrightarrow{\cong} & \text{Pext}_2^1(K_*(A)_t, K_*(B))_0 \\
\downarrow & \downarrow & \downarrow \\
0 & 0 & 0
\end{array}
\]

2. There is a natural exact sequence

\[
\text{Hom}_2^1(K_*(A)_t, K_*(B))_0 \xrightarrow{\delta'} QD(A_f, B) \to \text{Ker}(\theta^*) \to 0
\]

where \( \delta' \) is the boundary map in the \( \text{Hom-Pext} \) long exact sequence associated to the pure short exact sequence \( (*) \).

\(^{17}\)We established this in \[23\], Theorem 1.1, for \( A \in N \) but once again it is clear by inspection of that proof that the statement holds for \( A \in \tilde{N} \).
If $\text{Im}(\delta') = 0$ then there is a natural isomorphism

$$QD(A_f, B) \cong \text{Ker}(\theta^*).$$

Proof. The short exact sequence (*) is pure exact and thus produces a six term $\text{Hom}$-$\text{Ext}$ sequence in the first variable. Identifying entries using Theorem 2.3, one obtains the left column below, which is exact. The six term $\text{Hom}$-$\text{Ext}$ exact sequence contributes the right column below, and this column is also exact.

$$\begin{array}{cccc}
\text{Hom}_\mathbb{Z}(K_* (A)_t, K_* (B))_0 & \xrightarrow{\cong} & \text{Hom}_\mathbb{Z}(K_* (A)_t, K_* (B))_0 \\
\downarrow \delta' & & \downarrow \delta \\
QD(A_f, B) & \xrightarrow{\cong} & \text{Ext}_\mathbb{Z}^1(K_* (A)_f, K_* (B))_0 \\
\downarrow & & \downarrow \\
QD(A, B) & \rightarrow & \text{Ext}_\mathbb{Z}^1(K_* (A), K_* (B))_0 \\
\downarrow Q\theta^* & & \downarrow \theta^* \\
QD(A_t, B) & \rightarrow & \text{Ext}_\mathbb{Z}^1(K_* (A)_t, K_* (B))_0 \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

The diagram commutes, the horizontal maps are injections, and the map

$$QD(A_f, B) \rightarrow \text{Ext}_\mathbb{Z}^1(K_* (A)_f, K_* (B))_0$$

is an isomorphism since $K_* (A)_f$ is torsionfree. An easy diagram chase shows that $\text{Ker}(Q\theta^*) \cong \text{Ker}(\theta^*)$ from which 1) is immediate. Part 2) follows from expressing $\text{Ker}(Q\theta^*)$ as the quotient of $QD(A_f, B)$ modulo the group $\text{Im}(\delta')$. □

---

18 This condition usually holds. For instance, it holds if $K_* (A)_t$ is a direct summand of $K_* (A)$ or if $K_* (B)$ is torsionfree, and of course it holds if $\text{Hom}_\mathbb{Z}(K_* (A)_t, K_* (B)) = 0$. 
6. Quasidiagonality and Torsion

One of the early applications of the Brown-Douglas-Fillmore theory was con-
tained in work of L.G. Brown [4]. He exhibited an example of a bounded operator
$T$ which was not quasidiagonal but such that $T \oplus T$ was quasidiagonal. In fact
Brown showed that $T \oplus T$ generated a trivial extension.

Here is an analysis of such behavior from our perspective.

**Theorem 6.1.** Suppose that $A \in \tilde{N}$, $A$ is quasidiagonal relative to $B$, and $x \in \text{KK}_1(A, B)$. Then:

1. If $\gamma(x) \neq 0$ then $x$ is not a quasidiagonal class.

2. If $\gamma(x)$ has infinite order in the group $\text{Hom}_\mathbb{Z}(K_*(A), K_*(B))$ then no mul-
tiple of $x$ is a quasidiagonal class.

3. If $\gamma(x) \neq 0$ and $\gamma(x)$ has order $n$ in the group $\text{Hom}_\mathbb{Z}(K_*(A), K_*(B))$ then
$k nx$ is a quasidiagonal class for each $k \in \mathbb{N}$.

4. If $\gamma(x) = 0$ then $x$ is a quasidiagonal class if and only if it is in the kernel
of the natural map

$$\varphi : \text{Ext}^1_\mathbb{Z}(K_*(A), K_*(B)) \to \text{Ext}^1_\mathbb{Z}(K_*(A), K_*(B)).$$

5. If $K_*(A)$ is torsionfree then $x$ is a quasidiagonal class if and only if $\gamma(x) = 0$.

**Proof.** This is all immediate from Theorem 2.3. \[\square\]

We apply Theorem 6.1 to the setting of $x \in \text{KK}_1(A, \mathbb{C})$.

**Theorem 6.2.** Suppose that $A \in \tilde{N}$ and $A$ is a quasidiagonal $C^*$-algebra. Suppose
given an essential extension

$$\tau : 0 \to K \to E_\tau \xrightarrow{p} A \to 0$$

representing $\tau \in \text{KK}_1(A, \mathbb{C})$ so that

$$\gamma(\tau) : K_1(A) \to K_0(K) \cong \mathbb{Z}.$$

Then:

1. If $\gamma(\tau) \neq 0$ then $\tau$ is not a quasidiagonal extension and no multiple of $\tau$ is
a quasidiagonal extension.

2. If $\gamma(\tau) = 0$ then $\tau$ is a quasidiagonal extension if and only if the short exact
sequence

$$0 \to \mathbb{Z} \to K_0(E_\tau) \xrightarrow{p_*} K_0(A) \to 0$$

is a pure short exact sequence.
(3) If $K_0(A)$ is torsionfree then $\tau$ is quasidiagonal if and only if $\gamma(\tau) = 0$.

(4) If $K_0(A)$ is a direct sum of (finite and/or infinite) cyclic groups then $\tau$ is a quasidiagonal extension if and only if it is a trivial extension.

Proof. The group $\text{Hom}_Z(K_1(A), \mathbb{Z})$ is torsionfree since $\mathbb{Z}$ is torsionfree. Thus 6.1(1 and 2) imply 1). Part 2) is a restatement of 6.1(4). Part 3) follows from 6.1(5). Part 4) holds since $\text{Ext}^1_Z(G, H) = 0$ whenever $G$ is a direct sum of cyclic groups. □

Remark 6.3. There are examples [19] where the surjection

$$\varphi : \text{Ext}^1_Z(K_*(A), K_*(B)) \longrightarrow \text{Ext}^1_Z(K_*(A), K_*(B))^\wedge$$

is not a split surjection. In fact, using the example of Christensen-Strickland cited there, it is possible to produce an extension $\tau \in KK_1(A, B)$ which satisfies all of the following conditions:

1. $\gamma(\tau) = 0$,
2. $\varphi(\tau) \neq 0$ so that $\tau$ is not a quasidiagonal class, and
3. For some $k \in \mathbb{N}$, $k\tau$ is a quasidiagonal class but is not trivial.

This sort of phenomenon can occur only when both $K_*(A)$ and $K_*(B)$ have $p$-torsion for some fixed prime $p$ and then only rarely.

Theorem 6.1 implies an early result [6] of Brown, Douglas, and Fillmore (BDF) on quasidiagonality. Recall that a bounded operator $T$ on a separable Hilbert space $H$ is essentially normal if $T^*T - TT^*$ is compact. The essential spectrum of $T$ is the spectrum of $\pi T \in \mathcal{Q}(H)$. We let $\text{ind}(T)$ denote the Fredholm index of the operator $T$.

Theorem 6.4 (BDF). Suppose that $T \in L(H)$ is an essentially normal operator. Let $X$ denote the essential spectrum of $T$. Then the following are equivalent:

1. $T$ is a quasidiagonal operator.
2. For each $\lambda \in \mathbb{C} - X$,

$$\text{ind}(T - \lambda I) = 0.$$

3. $T$ is of the form (normal) + (compact).

Proof. There is a natural extension

$$0 \longrightarrow K \longrightarrow C^*\{T, K\} \longrightarrow C(X) \longrightarrow 0$$

which gives rise to an element

$$[T] \in KK_1(C(X), \mathbb{C}).$$
The operator $T$ is quasidiagonal if and only if $[T]$ is a quasidiagonal class, by 2.2 and the remark preceding it. The fact that $X \subset \mathbb{C}$ implies that $K_*(C(X)) \cong K^*(X)$ is torsionfree. Hence $[T]$ is a quasidiagonal class if and only if $\gamma([T]) = 0$ by Theorem 6.1. So it suffices to compute $\gamma([T])$. BDF [6] show that there is a natural isomorphism

$$KK_1(C(X), \mathbb{C}) \xrightarrow{\gamma} Hom_{\mathbb{Z}}(K^1(X), \mathbb{Z}) \cong \tilde{H}^0(\mathbb{C} - X)$$

and this map takes $[T]$ to the function

$$\lambda \mapsto \text{ind}(T - \lambda I)$$

so that $[T] = 0$ if and only if $\text{ind}(T - \lambda I) = 0$ where defined. This completes the proof. \hspace{1cm} \Box
7. Lifting Quasidiagonality

In this section we present a converse to the following theorem of Davidson, Herrero, and Salinas.

**Theorem 7.1.** [10] Suppose that

\[ \tau : 0 \rightarrow K \rightarrow E_\tau \rightarrow A \rightarrow 0 \]  

is an essential extension with associated faithful representation

\[ e_\tau : E_\tau \rightarrow \mathcal{L}(H) \]

and suppose further that \( E_\tau \) is separable and nuclear. If \( e_\tau \) is a quasidiagonal representation then \( A \) is a quasidiagonal \( C^* \)-algebra.

There is an obvious obstruction to a converse to this theorem, known already to Halmos [12]. Suppose that \( S \) is the unilateral shift. Then there is a canonical associated extension

\[ \tau_S : 0 \rightarrow K \rightarrow C^*\{S, K\} \rightarrow A \rightarrow 0 \]

and the quotient \( A \cong C(S^1) \) is commutative, hence quasidiagonal. However the \( C^* \)-algebra \( C^*\{S, K\} \) itself is not quasidiagonal since it contains the unilateral shift \( S \), which is not quasidiagonal. Halmos demonstrates this by observing that \( S \) has non-trivial Fredholm index, whereas any Fredholm quasidiagonal operator must have trivial Fredholm index. In modern jargon, the index map

\[ \gamma : KK_1(C(S^1), K) \rightarrow Hom_{\mathbb{Z}}(K^1(S^1), \mathbb{Z}) \cong \mathbb{Z} \]

satisfies

\[ \gamma(\tau_S)(z) = -1 \neq 0 \]

which is an obstruction to quasidiagonality.

Here is the complete story, at least within the category \( \tilde{\mathcal{N}} \).

**Theorem 7.3.** Suppose given the essential extension (7.2) with \( A \in \tilde{\mathcal{N}} \) and quasidiagonal. Then the representation \( e_\tau : E_\tau \rightarrow \mathcal{L}(H) \) is a quasidiagonal representation if and only if the following two conditions hold:

1. \( \gamma(\tau) = 0 \), and
2. The resulting \( K \)-theory short exact sequence

\[ 0 \rightarrow \mathbb{Z} \rightarrow K_0(E_\tau) \rightarrow K_0(A) \rightarrow 0 \]

is a pure exact sequence.
If in addition $K_0(A)$ is torsionfree then condition 2) is automatically satisfied, so that $e_\tau$ is a quasidiagonal representation if and only if $\gamma(\tau) = 0$.

**Proof.** Theorem 2.3 reduces in this case to the identification

$$QD(A,K) \cong P\text{ext}^1_\mathbb{Z}(K_0(A),\mathbb{Z}).$$

Condition 1) is equivalent to

$$\tau \in \text{Ext}^1_\mathbb{Z}(K_0(A),\mathbb{Z})$$

and condition 2) is simply a statement that

$$\tau \in P\text{ext}^1_\mathbb{Z}(K_0(A),\mathbb{Z}).$$

If in addition $K_0(A)$ is torsionfree then

$$P\text{ext}^1_\mathbb{Z}(K_0(A),\mathbb{Z}) \cong \text{Ext}^1_\mathbb{Z}(K_0(A),\mathbb{Z}).$$

This completes the proof. \[\square\]

**Remark 7.4.** It is instructive to compare our Theorem 7.3 with a related result due to N. Brown and M. Dadarlat [5] where they overlap. Consider the essential extension

$$\tau : 0 \to K \to E_\tau \to A \to 0$$

with associated faithful representation

$$e_\tau : E_\tau \to \mathcal{L}(H)$$

and suppose that $A \in \tilde{N}$ with $A$ quasidiagonal. Brown and Dadarlat show ([5, Theorem 3.4] specialized to this case) that if $\gamma(\tau) = 0$ then $E_\tau$ is also quasidiagonal. We show that if $\gamma(\tau) = 0$ and the resulting $K$-theory short exact sequence is pure then the representation $e_\tau$ is a quasidiagonal representation. If $\gamma(\tau) = 0$ but the $K$-theory short exact sequence is not pure then we conclude that even though $E_\tau$ is quasidiagonal, the representation $e_\tau$ is not a quasidiagonal representation.

Here is an example. Let $G$ be any countable torsion group. Then

$$\text{Ext}^1_\mathbb{Z}(G,\mathbb{Z}) \cong \text{Hom}_\mathbb{Z}(G,\mathbb{Q}/\mathbb{Z}) = \mathcal{P}(G)$$

the Pontrjagin dual group of $G$.

\[19\] For example, if $G$ is finite then $\mathcal{P}(G) = G$. If $G = \bigoplus_1^\infty \mathbb{Z}/p$, the sum of countably many copies of the group $\mathbb{Z}/p$, then $\mathcal{P}(G) = \prod_1^\infty \mathbb{Z}/p$ which is, of course, uncountable. Note that $\mathcal{P}(\mathcal{P}(G)) = G$ by the Pontrjagin duality theorem, and hence if $G \neq 0$ then $\text{Ext}^1_\mathbb{Z}(G,\mathbb{Z}) \neq 0$.\[19\]
Choose a commutative $C^*$-algebra $A$ with $K_0(A) = G$ and $K_1(A) = 0$. This is always possible, and $A$ is unique up to $KK$-equivalence. Then

$$\text{Hom}_{\mathbb{Z}}(K_*(A), \mathbb{Z}) = 0$$

and hence the index map $\gamma$ is identically zero. Thus there is a natural isomorphism

$$KK_1(A, \mathcal{K}) \cong \text{Ext}^1_{\mathbb{Z}}(K_0(A), \mathbb{Z}) \cong \text{Ext}^1_{\mathbb{Z}}(G, \mathbb{Z}) \cong \mathcal{P}(G).$$

Using Brown and Dadarlat’s result, we conclude that if

$$0 \to \mathcal{K} \to E_\tau \to A \to 0$$

is any essential extension then the $C^*$-algebra $E_\tau$ is quasidiagonal. On the other hand,

$$QD(A, \mathcal{K}) \cong P\text{ext}^1_{\mathbb{Z}}(K_0(A), \mathbb{Z}) = P\text{ext}^1_{\mathbb{Z}}(G, \mathbb{Z}) = 0$$

since $G$ is a torsion group and $\mathbb{Z}$ is torsionfree, by [22, Theorem 9.1]. Thus among all of the various $\tau \in KK_1(A, \mathcal{K}) \cong \mathcal{P}(G)$ and associated representations

$$e_\tau : E_\tau \to \mathcal{L}(H),$$

the only representation $e_\tau$ that is quasidiagonal is the one corresponding to the trivial extension, where

$$[\tau] = 0 \in KK_1(A, \mathcal{K}).$$

This also illustrates the phenomenon discovered by L.G. Brown and discussed in Section 6, since any non-trivial extension $\tau \in KK_1(A, \mathcal{K})$ will have the property that it itself is not quasidiagonal, but when added to itself enough times it becomes quasidiagonal and trivial.
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