EXTENDED ELECTRODYNAMICS:
II. Properties and invariant characteristics of the non-linear vacuum solutions

S.Donev
Institute for Nuclear Research and Nuclear Energy, Bulg.Acad.Sci., 1784 Sofia, blvd.Tzarigradsko shausee 72
e-mail: sdonev@inrne.acad.bg
BULGARIA

M.Tashkova
Institute of Organic Chemistry with Center of Phytochemistry, Bulg.Acad.Sci., 1574 Sofia,
Acad. G.Bonchev Str., Bl. 9
BULGARIA

Abstract
This paper aims to consider the general properties of the non-linear solutions to the vacuum equations of Extended Electrodynamics [1]. The *-invariance and the conformal invariance of the equations are mentioned. It is also proved that all non-linear solutions have zero invariants: \( \frac{1}{2}F_{\mu\nu}F^{\mu\nu} = \frac{1}{4}(\ast F)_{\mu\nu}F^{\mu\nu} = 0 \). The three invariant characteristics of the non-linear solutions: amplitude, phase and scale factor are introduced and discussed.
1 General properties of the equations

Recall from [1] the basic relation of Extended Electrodynamics (EED)

\[ \nabla (\delta \Omega, \ast \Omega) = \nabla(\Phi, \ast \pi_1 \Omega) + \nabla(\Psi, \ast \pi_2 \Omega). \]  

(1)

An external field is called vacuum with respect to the \( EM \)-field \( \Omega \) if the right hand side of (1) is equal to zero. Then the left hand side of (1) will also be equal to zero, so we get the equations

\[ \nabla (\delta \Omega, \ast \Omega) = 0. \]  

(2)

From this coordinate free compactly written expression we obtain the following equations for the components of \( \Omega \) in the basis \((e_1, e_2)\):

\[ (\delta F \wedge \ast F) \otimes e_1 \lor e_1 + (\delta \ast F) \wedge \ast \ast F \otimes e_2 \lor e_2 + 
   + (\delta F \wedge \ast F + \delta \ast F \wedge \ast F) \otimes e_1 \lor e_2 = 0. \]  

(3)

So, the field equations, expressed through the operator \( \delta \) look as follows

\[ \delta F \wedge \ast F = 0, \ \delta \ast F \wedge \ast \ast F = 0, \ \delta \ast F \wedge \ast F - \delta F \wedge F = 0. \]  

(4)

These equations, expressed through the operator \( d \) have the form

\[ \ast F \wedge \ast d \ast F = 0, \ F \wedge \ast d F = 0, \ F \wedge \ast d \ast F + \ast F \wedge \ast d F = 0. \]  

(5)

Using the components \( F_{\mu\nu} \), we obtain for the equations (4)

\[ F_{\mu\nu}(\delta F)^\nu = 0, \ (\ast F)_{\mu\nu}(\delta \ast F)^\nu = 0, \ F_{\mu\nu}(\delta \ast F)^\nu + (\ast F)_{\mu\nu}(\delta F)^\nu = 0. \]  

(6)

In the same way, for the equations (5) we get

\[ (\ast F)^{\mu\nu}(d \ast F)_{\mu\nu} = 0, \ F^{\mu\nu}(d F)_{\mu\nu} = 0, \ (\ast F)^{\mu\nu}(d F)_{\mu\nu} + F^{\mu\nu}(d \ast F)_{\mu\nu} = 0. \]  

(7)

Now we give the 3-dimensional form of the equations in the same order:

\[ B \times \left( \text{rot}B - \frac{\partial E}{\partial \xi} \right) - EdivE = 0, \ E \cdot \left( \text{rot}B - \frac{\partial E}{\partial \xi} \right) = 0, \]  

(8)

\[ E \times \left( \text{rot}E + \frac{\partial B}{\partial \xi} \right) - BdivB = 0, \ B \cdot \left( \text{rot}E + \frac{\partial B}{\partial \xi} \right) = 0, \]  

(9)
\[
\left( \text{rot} E + \frac{\partial B}{\partial \xi} \right) \times B + \left( \text{rot} B - \frac{\partial E}{\partial \xi} \right) \times E + B \text{div} E + E \text{div} B = 0,
\]

\[
B \cdot \left( \text{rot} B - \frac{\partial E}{\partial \xi} \right) - E \cdot \left( \text{rot} E + \frac{\partial B}{\partial \xi} \right) = 0. \tag{10}
\]

From the second equations of (8) and (9) the well known Poynting relation follows

\[
\text{div} (E \times B) + \frac{\partial E^2 + B^2}{2} = 0,
\]

and from the second equation of (10), if \( E \cdot B = g(x, y, z) \), we obtain the known from Maxwell theory relation

\[
B \cdot \text{rot} B = E \cdot \text{rot} E.
\]

The explicit form of equations (8) and (9) should not make us conclude, that the second (scalar) equations follow from the first (vector) equations. Here is an example: let \( \text{div} E = 0, \text{div} B = 0 \) and the time derivatives of \( E \) and \( B \) are zero. Then the system of equations reduces to

\[
E \times \text{rot} E = 0, \quad B \times \text{rot} B = 0, \quad B \cdot \text{rot} E = 0, \quad E \cdot \text{rot} B = 0.
\]

As it is seen, the vector equations do not require any connection between \( E \) and \( B \) in this case, therefore, the scalar equations, which require such connection, can not follow from the vector ones. The third equations of (5) and (6) determine (in equivalent way) the energy-momentum quantities, transferred from \( F \) to \( \ast F \), and reversely, in a unit 4- volume, with the expressions, respectively

\[
F_{\mu\nu}(\delta \ast F)^{\nu} = -(F)_{\mu\nu}(\delta F)^{\nu}, \quad F^{\mu\nu}(d \ast F)_{\mu\nu} = -(\ast F)^{\mu\nu}(d F)_{\mu\nu}.
\]

From these relations it is seen, that the 1-forms \( \delta F \) and \( \delta \ast F \) play the role of "external" currents respectively for \( \ast F \) and \( F \). In the same spirit we could say, that the energy-momentum quantities \( F_{\mu\nu}(\delta F)^{\nu} \) and \( (\ast F)_{\mu\nu}(\delta \ast F)^{\nu} \), which \( F \) and \( \ast F \) exchange with themselves, are equal to zero. And this corresponds fully to our former statements, concerning the physical sense of the equations for the components of \( \Omega \).
From the first two equations of (6) and from the expression for the divergence $\nabla_\nu Q^\nu_\mu$ of the Maxwell’s energy-momentum tensor $Q^\nu_\mu$ in CED

$$\nabla_\nu Q^\nu_\mu = \frac{1}{4\pi} \left[ F_{\mu\nu} (\delta F)^\nu + (\ast F)_{\mu\nu} (\delta \ast F)^\nu \right]$$

it is immediately seen that on the solutions of our equations (6) this divergence is also zero. In view of this we assume the tensor $Q^\nu_\mu$, defined by

$$Q^\nu_\mu = \frac{1}{4\pi} \left[ \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} \delta^\nu_\mu - F_{\mu\sigma} F^{\nu\sigma} \right] = \frac{1}{8\pi} \left[ -F_{\mu\sigma} F^{\nu\sigma} - (\ast F)_{\mu\sigma} (\ast F)^{\nu\sigma} \right].$$

(12)

to be the energy-momentum tensor in EED. We shall be interested in finding explicit time-stable solutions of finite type, i.e. $F_{\mu\nu}$ to be finite functions of the three spatial coordinates, therefore, if it turns out that such solutions really exist, then integral conserved quantities can be easily constructed and computed, making use of the 10 Killing vectors on the Minkowski spacetime. We recall that in CED such finite and time-stable solutions in the whole space are not allowed by the Maxwell’s equations.

We first note, that in correspondence with the requirement for general covariance, equations (2), given above and presented in different but equivalent forms, are written down in coordinate free manner. This requirement is universal, i.e. it concerns all basic equations of a theory and means simply, that the existence of real objects and the occurrence of real processes can not depend on the local coordinates used in the theory, i.e. on the convenient for us way to describe the local character of the evolution and structure of the natural objects and processes. Of course, in the various coordinate systems the equations and their solutions will look differently. Namely the covariant character of the equations allows to choose the most appropriate coordinates, reflecting most fully the features of every particular case. A typical example for this is the usage of spherical coordinates in describing spherically symmetric fields. Let’s not forget also, that the coordinate-free form of the equations permits an easy transfer of the same physical situation onto manifolds with more complicated structure and nonconstant metric tensor. Shortly speaking, the coordinate free form of the equations in theoretical physics reflects the most essential properties of reality, called shortly objective character of the real phenomena.

Since the left hand sides of the equations are linear combinations of the first derivatives of the unknown functions with coefficients, depending linearly
on these unknown functions, (6) present a special type system of \textit{quasilinear first order partial differential equations}. The number of the unknown functions \(F_{\mu\nu} = -F_{\nu\mu}\) is 6, and in general, the number of the equations is \(3.4 = 12\), but the number of the independent equations depends strongly on if the two invariants \(I_1 = \frac{1}{2}F_{\mu\nu}F^{\mu\nu}\) and \(I_2 = \frac{1}{2}(\star F)_{\mu\nu}F^{\mu\nu}\) are equal to zero or not equal to zero. If \(I_2 \neq 0\) then \(\text{det}(F_{\mu\nu}) \neq 0\) and the first two equations of (6) are equivalent to \(\delta F = 0\) and \(\delta \star F = 0\), which automatically eliminates the third equation of (6), i.e. in this case our equations reduce to Maxwell’s equations.

It is clearly seen from the (7) form of the equations, that the metric tensor essentially participates (through the \(\star\)-operator applied to 2-forms only) in the equations. If we use the \(\delta\)-operator, then the metric participates also through the \(\star\)-operator, applied to 3-forms, but this does not lead to more complicated coordinate form of the equations. It worths to note that in nonlinear coordinates the metric tensor will participate with its derivatives, therefore, the very solutions will depend strongly on the metric tensor chosen. This may cause existence or non-existence of solutions of a given class, e.g. soliton-like ones. In our framework such additional complications do not appear because of the opportunity to work in global coordinates with constant metric tensor.

We note 2 important invariance (symmetry) properties of our equations.

\textbf{Property 1.} \textit{The transformation }\(F \rightarrow \star F\text{ does not change the system.}\)

The proof is obvious, in fact, the first two equations interchange, and the third one is kept the same. In terms of \(\Omega\) this means that if \(\Omega\) is a solution, then \(\star \Omega\) is also a solution, which means, in turn, that equations (2) are equivalent to the equations

\[
\nabla (\Omega, \star d\Omega) = 0.
\]

\textbf{Property 2.} \textit{Under conformal change of the metric the equations do not change.}

The proof of this property is also obvious and is reduced to the notice, that as it is seen from (7), the \(\star\)-operator participates only with its reduction to 2-forms, and as it is well known, \(\star_2\) is conformally invariant.

Summing up the first two equations of (8) and (9) we obtain how the classical Poynting vector changes in time in our more general approach:
\[ \frac{\partial}{\partial \xi} (E \times B) = \text{Ediv}E + \text{Bdiv}B - E \times \text{rot}E - B \times \text{rot}B. \]

In CED the first and the second terms on the right are missing.

Here is an example of static solutions of (2), which are not solutions to Maxwell’s equations.

\[ E = (a \sin \alpha z, a \cos \alpha z, 0), \quad B = (b \cos \alpha z, -b \sin \alpha z, 0), \]

where \( a, b \) and \( \alpha \) are constants. We obtain
\[ \text{rot}E = (a \alpha \sin \alpha z, a \alpha \cos \alpha z, 0), \quad \text{rot}B = (b \alpha \cos \alpha z, -b \alpha \sin \alpha z, 0) \]

Obviously, \( E \times \text{rot}E = 0, \quad B \times \text{rot}B = 0, \quad E \cdot \text{rot}B = 0, \quad B \cdot \text{rot}E = 0. \) For the Poynting vector we get \( E \times B = (0, 0, -ab) \), and for the energy density \( w = \frac{1}{2}(a^2 + b^2) \). Considered in a finite volume, these solutions could model some standing waves, but we do not engage ourselves with such interpretations, since we do not accept seriously that static \( EM \)-fields may really exist.

## 2 General properties of the solutions

It is quite clear that the solutions of our equations are naturally divided into two classes: linear and nonlinear. The first class consists of all solutions to Maxwell’s vacuum equations, where the name linear comes from. These solutions are well known and won’t discussed here. The second class, called nonlinear, includes all the rest solutions. This second class is naturally divided into two subclasses. The first subclass consists of all nonlinear solutions, satisfying the conditions
\[ \delta F \neq 0, \quad \delta \ast F \neq 0, \quad (14) \]
and the second subclass consists of those nonlinear solutions, satisfying one of the two couples of conditions:
\[ \delta F = 0, \quad \delta \ast F \neq 0; \quad \delta F \neq 0, \quad \delta \ast F = 0. \]

Further we assume the conditions (14) fulfilled, i.e. the solutions of the second subclass will be considered as particular cases of the first subclass.
Our purpose is to show explicitly, that among the nonlinear solutions there are soliton-like ones, i.e. the components $F_{\mu \nu}$ of which at any moment are finite functions of the three spatial variables with connected support. We are going to study their properties and to introduce corresponding characteristics. First we shall establish some of their basic features, proving three propositions.

**Proposition 1.** All nonlinear solutions have zero invariants:

$$I_1 = \frac{1}{2} F_{\mu \nu} F^{\mu \nu} = 0, \quad I_2 = \frac{1}{2} (\ast F)_{\mu \nu} F^{\mu \nu} = 2 \sqrt{\det(F_{\mu \nu})} = 0.$$

**Proof.** Recall the field equations in the form (2.8):

$$F_{\mu \nu}(\delta F)^{\nu} = 0, \quad (\ast F)_{\mu \nu}(\delta \ast F)^{\nu} = 0, \quad F_{\mu \nu}(\delta \ast F)^{\nu} + (\ast F)_{\mu \nu}(\delta F)^{\nu} = 0.$$

It is clearly seen that the first two groups of these equations may be considered as two linear homogeneous systems with respect to $\delta F_{\mu}$ and $\delta \ast F_{\mu}$ respectively. In view of the nonequalities (2.14) these homogeneous systems have non-zero solutions, which is possible only if $\det(F_{\mu \nu}) = \det(\ast F_{\mu \nu}) = 0$, i.e. if $I_2 = 2E.B = 0$. Further, summing up these three systems of equations, we obtain

$$(F + \ast F)_{\mu \nu}(\delta F + \delta \ast F)^{\nu} = 0.$$

If now $(\delta F + \delta \ast F)^{\nu} \neq 0$, then

$$0 = \det(F + \ast F)_{\mu \nu} = \left[\frac{1}{2} (F + \ast F)_{\mu \nu} (F + \ast F)^{\mu \nu}\right]^2 = \frac{1}{4} [-2F_{\mu \nu} F^{\mu \nu}]^2 = (I_1)^2.$$

If $\delta F^\nu = -(\delta \ast F)^{\nu} \neq 0$, we sum up the first two systems and obtain $(\ast F - F)_{\mu \nu}(\delta \ast F)^{\nu} = 0$. Consequently,

$$0 = \det(\ast F - F)_{\mu \nu} = \left[\frac{1}{2} (\ast F - F)_{\mu \nu} (-F - \ast F)^{\mu \nu}\right]^2 = \frac{1}{4} [2F_{\mu \nu} F^{\mu \nu}]^2 = (I_1)^2.$$

This completes the proof.

Recall that in this case the energy-momentum tensor $Q_{\mu \nu}$ has just one isotropic eigen direction and all other eigen directions are space-like. Since all eigen directions of $F_{\mu \nu}$ and $\ast F_{\mu \nu}$ are eigen directions of $Q_{\mu \nu}$ too, it is clear that $F_{\mu \nu}$ and $(\ast F)_{\mu \nu}$ can not have time-like eigen directions. But the first
two systems of (2.8) require $\delta F$ and $\delta \ast F$ to be eigen vectors of $F$ and $\ast F$ respectively, so we obtain

$$(\delta F) . (\delta F) \leq 0, \ (\delta \ast F) . (\delta \ast F) \leq 0. \quad (15)$$

**Proposition 2.** All nonlinear solutions satisfy the conditions

$$(\delta F) \mu (\delta \ast F) \mu = 0, \ |\delta F| = |\delta \ast F| \quad (16)$$

**Proof.** We form the inner product $i((\delta \ast F)(\delta F \land \ast F)) = 0$ and get

$$(\delta \ast F) \mu (\delta F) \nu (\ast F) \nu \ast F dx') - (\delta \ast F)^2 (\ast F) - \delta F \land (\delta \ast F) \mu (\ast F) \mu \nu dx' = 0.$$  

Because of the obvious nulification of the second term the first term will be equal to zero (at non-zero $\ast F$) only if $(\delta F) \mu (\delta \ast F) \nu = 0$.

Further we form the inner product $i((\delta \ast F)(\delta F \land F - \delta \ast F \land \ast F)) = 0$ and obtain

$$(\delta \ast F) \mu (\delta F) \nu (\ast F) \nu + \delta \ast F \land (\delta \ast F) \mu (\ast F) \mu \nu dx' = 0.$$  

Clearly, the first and the last terms are equal to zero. So, the inner product by $\delta F$ gives

$$(\delta F)^2 (\delta \ast F) \mu (\ast F) \mu \nu dx' - [(\delta F)^\mu (\delta \ast F) \nu (\ast F) \nu \mu ] \delta F + (\delta \ast F)^2 (\delta F)^\mu (\ast F) \mu \nu dx' = 0.$$  

The second term of this equality is zero. Besides, $(\delta \ast F) \mu (\ast F) \mu \nu dx' = -(\delta F)^\mu (\ast F) \mu \nu dx'$. So,

$$[ (\delta F)^2 - (\delta \ast F)^2 ] (\delta F) \mu (\ast F) \mu \nu dx' = 0.$$  

Now, if $(\delta F)^\mu (\ast F) \mu \nu dx' \neq 0$, then the relation $|\delta F| = |\delta \ast F|$ follows immediately. If $(\delta F)^\mu (\ast F) \mu \nu dx' = 0 = -(\delta \ast F)^\mu F_{\mu \nu} dx' \ast F$ according to the third equation of (6), we shall show that $(\delta F)^2 = (\delta \ast F)^2 = 0$. In fact, forming the inner product $i(\delta F)(\delta F \land \ast F) = 0$, we get

$$(\delta F)^2 \ast F - \delta F \land (\delta F) \mu (\ast F) \mu \nu dx' = (\delta F)^2 \ast F = 0.$$  

In a similar way, forming the inner product $i(\delta \ast F) \delta \ast F \land F = 0$ we have

$$((\delta \ast F)^2 F - \delta (\ast F) \land (\delta \ast F) \mu F_{\mu \nu} dx' = (\delta \ast F)^2 F = 0.$$  

8
This completes the proof.

We just note that in this last case the isotropic vectors $\delta F$ and $\delta \ast F$ are eigen vectors of $Q_{\mu \nu}$ too, and since $Q_{\mu \nu}$ has just one isotropic eigen direction, we conclude that $\delta F$ and $\delta \ast F$ are colinear.

In order to formulate the third proposition, we recall [2] that at zero invariants $I_1 = I_2 = 0$ the following representation holds:

$$F = A \wedge \zeta, \quad \ast F = A^* \wedge \zeta,$$

where $\zeta$ is the only (up to a scalar factor) isotropic eigen vector of $Q_{\nu}^{\nu}$ and the relations $A.\zeta = 0$, $A^*.\zeta = 0$. Having this in view we shall prove the following

**Proposition 3.** All nonlinear solutions satisfy the relations

$$\zeta^\mu(\delta F)_\mu = 0, \quad \zeta^\mu(\delta \ast F)_\mu = 0. \quad (17)$$

**Proof.** We form the inner product $i(\zeta)(\delta F \wedge \ast F) = 0$:

$$[\zeta^\mu(\delta F)_\mu] \ast F - \delta F \wedge (\zeta^\mu(\ast F))_{\mu \nu} dx^\nu =$$

$$= [\zeta^\mu(\delta F)_\mu] A^* \wedge \zeta - (\delta F \wedge \zeta) \zeta^\mu(A^*)_\mu + (\delta F \wedge A^*)\zeta^\mu \zeta_\mu = 0.$$  

Since the second and the third terms are equal to zero and $\ast F \neq 0$, then $\zeta^\mu(\delta F)_\mu = 0$. Similarly, from the equation $(\delta \ast F) \wedge F = 0$ we get $\zeta^\mu(\delta \ast F)_\mu = 0$. The proposition is proved.

### 3 Algebraic properties of the nonlinear solutions

Since all nonlinear solutions have zero invariants $I_1 = I_2 = 0$ we can make a number of algebraic considerations, which clarify considerably the structure and make easier the study of the properties of these solutions. As we mentioned earlier, all eigen values if $F$, $\ast F$ and $Q_{\mu \nu}$ in this case are zero, and the eigen vectors can not be time-like. There is only one isotropic direction, defined by the isotropic vectors $\pm \zeta$ and the representations $F = A \wedge \zeta$, $\ast F = A^* \wedge \zeta$ hold, moreover, we have $A.A^* = 0$, $A^2 = (A^*)^2 \leq 0, A.\zeta = A^*.\zeta = 0$. Recall that the two 1-forms $A$ and $A^*$ are defined up to isotropic additive factors, colinear to $\zeta$. The above representation of $F$ and $\ast F$ through
\( \zeta \) shows that these factors do not contribute to \( F \) and \( *F \), therefore, we assume further that, these additive factors are equal to zero.

We express now \( Q_{\mu\nu} \) through \( A, A^\ast \) and \( \zeta \). First we normalize the vector \( \zeta \). This is possible, because it is an isotropic vector, so its time-like component \( \zeta_4 \) is always different from zero. We divide \( \zeta_\mu \) by \( \zeta_4 \) and get the vector \( V = (V^1, V^2, V^3, 1) \), defining, of course, the same isotropic direction. Now we make use of the identity,

\[
\frac{1}{2} F_{\alpha\beta} G^{\alpha\beta} \delta_\mu^\nu = F_{\mu\sigma} G^{\nu\sigma} - (*G)_{\mu\sigma} (*F)^{\nu\sigma},
\]

where we put \( F_{\mu\nu} \) instead of \( G_{\mu\nu} \). Having in view that \( I_1 = \frac{1}{2} F_{\mu\nu} F^{\mu\nu} = 0 \), we obtain \( F_{\mu\sigma} F^{\nu\sigma} = (*F)_{\mu\sigma} (*F)^{\nu\sigma} \). So, the energy-momentum tensor looks as follows

\[
Q_{\mu\nu} = -\frac{1}{4\pi} F_{\mu\sigma} F^{\nu\sigma} = -\frac{1}{4\pi} (*F)_{\mu\sigma} (*F)^{\nu\sigma} =
\]

\[
= -\frac{1}{4\pi} (A)^2 V_\mu V^\nu = -\frac{1}{4\pi} (A^\ast)^2 V_\mu V^\nu. \tag{19}
\]

This choice of \( \zeta = V \) determines the following energy density \( 4\pi Q_4^4 = |A|^2 = |A^\ast|^2 \).

We consider now the influence of the conservation law \( \nabla_\nu Q^\nu_{\mu} = 0 \) on \( V \).

\[
\nabla_\nu Q^\nu_{\mu} = -A^2 V^\nu \nabla_\nu V_\mu - V_\mu \nabla_\nu \left( A^2 V^\nu \right) = 0.
\]

This relation holds for every \( \mu = 1, 2, 3, 4 \). We consider it for \( \mu = 4 \) and get \( V^\nu \nabla_\nu (1) = V^\nu \partial_\nu (1) = 0 \). Therefore, \( V_4 \nabla_\nu (A^2 V^\nu) = \nabla_\nu (A^2 V^\nu) = 0 \). Since \( A^2 \neq 0 \), we obtain that \( V \) satisfies the equation

\[
V^\nu \nabla_\nu V^\mu = 0,
\]

which means, that \( V \) is a geodesic vector field, i.e. the integral trajectories of \( V \) are isotropic geodesics, or isotropic straight lines. Hence, every non-linear solution \( F \) defines unique isotropic geodesic direction in the Minkowski space-time. This important consequence allows a special class of coordinate systems, called further \( F \)-adapted, to be introduced. These coordinate systems are defined by the requirement, that the trajectories of the unique \( V \), defined by \( F \), to be parallel to the \((z, \xi)\)-coordinate plain. In such a coordinate system we have \( V_\mu = (0, 0, \varepsilon, 1) \), \( \varepsilon = \pm 1 \). Further on, we shall work in such arbitrary chosen but fixed \( F \)-adapted coordinate system, defined by the corresponding \( F \) under consideration.
We write down now the relations $F = A \wedge \mathbf{V}$, $*F = A^* \wedge \mathbf{V}$ component-wise, take into account the values of $\mathbf{V}_\mu$ in the $F$-adapted coordinate system and obtain the following explicit relations:

\[
F_{12} = F_{34} = 0, \quad F_{13} = \varepsilon F_{14}, \quad F_{23} = \varepsilon F_{24},
\]

\[
(*F)_{12} = (*F)_{34} = 0, \quad (*F)_{13} = \varepsilon(*F)_{14} = -F_{24}, \quad (*F)_{23} = \varepsilon(*F)_{24} = F_{14},
\]

\[
A = (F_{14}, F_{24}, 0, 0), \quad A^* = (-F_{23}, F_{13}, 0, 0) = (-\varepsilon A_2, \varepsilon A_1, 0, 0). \quad (20)
\]

Clearly, the 1-forms $A$ and $-A^*$ can be interpreted as electric and magnetic fields respectively. Only 4 of the components $Q^\nu_\mu$ are different from zero, namely:

\[
Q^4_4 = -Q^3_3 = \varepsilon Q^3_4 = -\varepsilon Q^4_3 = |A^2|.
\]

Introducing the notations $F_{14} \equiv u$, $F_{24} \equiv p$, we can write

\[
F = \varepsilon u dx \wedge dz + u dx \wedge d\xi + \varepsilon p dy \wedge dz + p dy \wedge d\xi,
\]

\[
*F = -p dx \wedge dz + \varepsilon p dx \wedge d\xi + u dy \wedge dz + \varepsilon u dy \wedge d\xi.
\]

In the important for us spatially finite case, i.e. when the functions $u$ and $p$ are finite with respect to the spatial variables $(x, y, z)$, for the integral energy $W$ and momentum $p$ we obtain

\[
W = \int Q^4_4 dx dy dz = \int (u^2 + p^2) dx dy dz < \infty,
\]

\[
p = \left(0, 0, \varepsilon \frac{W}{c}\right), \quad \rightarrow c^2|p|^2 - W^2 = 0. \quad (21)
\]

Now we show how the nonlinear solution $F$ defines at every point a pseudoorthonormal basis in the corresponding tangent and cotangent spaces. The nonzero 1-forms $A$ and $A^*$ are normed to

\[
A = A/|A|, \quad A^* = A^*/|A^*|.
\]

Two new unit 1-forms $R$ and $S$ are introduced through the equations:

\[
R^2 = -1, \quad A^\nu R_\nu = 0, \quad (A^*)^\nu R_\nu = 0, \quad V^\nu R_\nu = \varepsilon, \quad S = V + \varepsilon R.
\]

The only solution of the first 4 equations is $R_\mu = (0, 0, -1, 0)$. Then for $S$ we obtain $S_\mu = (0, 0, 0, 1)$. Clearly, $R^2 = -1$ and $S^2 = 1$. This pseudoorthonormal (co-tangent) basis $(A, A^*, R, S)$ is carried over to a (tangent) pseudoorthonormal basis by means of the pseudometric $\eta$.

We proceed further to introduce the concepts of amplitude and phase in a coordinate-free manner. First, of course, we look at the invariants, we have: $I_1 = I_2 = 0$. But in our case we have got another invariant, namely, the
module of the 1-forms $A$ and $A^*$: $|A| = |A^*|$. Let’s begin with the amplitude, which shall be denoted by $\phi$. As it’s seen from the above obtained expressions, the magnitude $|A|$ of $A$ coincides with the square root of the energy density in any $F$-adapted coordinate system. As we noted before, this is the sense of the quantity amplitude. So, we define it by the module of $|A| = |A^*|$. We give now two more coordinate-free ways to define the amplitude.

Recall first, that at every point, where the field is different from zero, we have three bases: the pseudoorthonormal coordinate basis $(dx, dy, dz, d\xi)$, the pseudoorthonormal basis $\chi^0 = (A, \varepsilon A^*, R, S)$ and the pseudoorthogonal basis $\chi = (A, \varepsilon A^*, R, S)$. The matrix $\chi_{\mu\nu}$ of $\chi$ with respect to the coordinate basis is

$$
\chi_{\mu\nu} = \begin{pmatrix}
    u & -p & 0 & 0 \\
    p & u & 0 & 0 \\
    0 & 0 & -1 & 0 \\
    0 & 0 & 0 & 1 
\end{pmatrix}.
$$

We define now the amplitude $\phi$ of the field by

$$
\phi = \sqrt{\det(\chi_{\mu\nu})}.
$$

We proceed further to define the phase of the nonlinear solution $F$. We shall need the matrix $\chi^0_{\mu\nu}$ of the basis $\chi^0$ with respect to the coordinate basis. We obtain

$$
\chi^0_{\mu\nu} = \begin{pmatrix}
    \frac{u}{\sqrt{u^2+p^2}} & \frac{-p}{\sqrt{u^2+p^2}} & 0 & 0 \\
    \frac{p}{\sqrt{u^2+p^2}} & \frac{u}{\sqrt{u^2+p^2}} & 0 & 0 \\
    0 & 0 & -1 & 0 \\
    0 & 0 & 0 & 1 
\end{pmatrix}.
$$
The trace of this matrix is

\[ tr(\chi^0_{\mu\nu}) = \frac{2u}{\sqrt{u^2 + p^2}}. \]

Obviously, the inequality \(|\frac{1}{2}tr(\chi^0_{\mu\nu})| \leq 1\) is fulfilled. Now, by definition, the quantity \(\varphi = \frac{1}{2}tr(\chi^0_{\mu\nu})\) will be called phase function of the solution, and the quantity

\[ \theta = \arccos(\varphi) = \arccos\left(\frac{1}{2}tr(\chi^0_{\mu\nu})\right) \]

will be called phase of the solution.

Making use of the amplitude \(\phi\) and the phase function \(\varphi\) we can write

\[ u = \phi \varphi, \quad p = \phi \sqrt{1 - \varphi^2}. \]

We note that the couple of 1-forms \(A = udx + pdy, \quad A^* = -pdx + udy\) defines a completely integrable Pfaff system, i.e. the following equations hold:

\[ dA \wedge A \wedge A^* = 0, \quad dA^* \wedge A \wedge A^* = 0. \]

In fact, \(A \wedge A^* = (u^2 + p^2)dx \wedge dy\), and in every term of \(dA\) and \(dA^*\) at least one of the basis vectors \(dx\) and \(dy\) will participate, so the above exterior products will vanish.

**Remark.** These considerations stay in force also for those linear solutions, which have zero invariants \(I_1 = I_2 = 0\). But Maxwell’s equations require \(u\) and \(p\) to be running waves, so the corresponding phase functions will be also running waves. As we’ll see further, the phase functions for nonlinear solutions are arbitrary bounded functions.

We proceed further to define the new and important concept of scale factor \(L\) for a given nonlinear solution. It is defined by

\[ L = \frac{|A|}{|\delta F|} = \frac{|A^*|}{|\delta^* F|}. \]

Clearly, \(L\) can not be defined for the linear solutions, and in this sense it is new and we shall see that it is really important.

From the corresponding expressions \(F = A \wedge V\) and \(*F = A^* \wedge V\) it follows that the physical dimension of \(A\) and \(A^*\) is the same as that of \(F\). We conclude that the physical dimension of \(L\) coincides with the dimension of the coordinates, i.e. \([L] = length\). From the definition it is seen that \(L\) is
an invariant quantity, and depends on the point, in general. The invariance of $L$ allows to define a time-like 1-form (or vector field) $f(L)\mathbf{S}$, where $f$ is some real function of $L$. So, every nonlinear solution determines a time-like vector field on $M$.

If the scale factor $L$, defined by the nonlinear solution $F$, is a finite and constant quantity, we can introduce a characteristic finite time-interval $T(F)$ by the relation

$$cT(F) = L(F),$$

as well, as corresponding characteristic frequency by

$$\nu(F) = 1/T(F).$$

In these ”wave” terms the scale factor $L$ acquires the sense of ”wave length”, but this interpretation is arbitrary and we shall not make use of it.

It is clear, that the subclass of nonlinear solutions, which define constant scale factors, factors over the admissible values of the invariant $f(L)$. This makes possible to compare with the experiment. For example, at constant scale factor $L$ if we choose $f(L) = L/c$, then the scalar product of $(L/c)\mathbf{S}$ with the integral energy-momentum vector, which in the $F$-adapted coordinate system is $(0, 0, \varepsilon W, W)$, gives the invariant quantity $W.T$, having the physical dimension of action, and its numerical value could be easily measured.
References:

[1]. Donev, S., Tashkova, M., **Extended Electrodynamics**: I. Basic Notions, Principles and Equations, submitted for publication.

[2]. Synge, J., *Relativity: the Special Theory*, North-Holland Publ.Comp., Amsterdam, 1958