A CHARACTERIZATION OF THE SINGULAR TIME OF THE MEAN CURVATURE FLOW

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Abstract. In this note we investigate the behaviour at finite-time singularities of the mean curvature flow of compact Riemannian submanifolds \( M^{m} \hookrightarrow (N^{m+n}, h) \). We show that they are characterized by the blow-up of a trace \( A = H \cdot \Pi \) of the square of the second fundamental form.

1. Introduction

It is well known that the mean curvature flow \( \partial_t F = H \) of submanifolds \( F_t : M^m \hookrightarrow \mathbb{R}^{m+n} \) has finite-time singularities characterized by the blowup of the second fundamental form \( \Pi \):

**Theorem 1.1** (Huisken [4]). Suppose \( T < \infty \) is the first singular time for a compact mean curvature flow. Then \( \max_M |\Pi| \to \infty \) as \( t \to T \).

We will prove that in fact it suffices to consider the tensor \( A_{ij} = H^\alpha h_{ij\alpha} \), where \( H = \text{tr} \Pi \) is the mean curvature and \( h \) are the components of \( \Pi \):

**Theorem 1.2.** Let \( (N, h) \) be a Riemannian manifold with bounded geometry. Suppose \( T < \infty \) is the first singular time for a mean curvature flow of compact submanifolds of \( (N, h) \). Then \( \max_M |A| \to \infty \) as \( t \to T \).

By slightly modifying the proof of Theorem 1.2, we also obtain

**Theorem 1.3.** Suppose that along the flow, \( |\Pi(x, t)|^p(T - t) \leq C \) for some \( p > 1 \). Then \( \max_M |H| \to \infty \) as \( t \to T \).

2. Preliminaries

First we recall some evolution equations for the flow. We use indices \( 1 \leq i, j, k, l \leq m, \ m + 1 \leq \alpha, \beta, \gamma \leq m + n \). Let \( h_{ij\alpha} \) denote the \( \alpha \)th component of \( \Pi(\partial_i, \partial_j) \), let \( H^\alpha \) denote the \( \alpha \)th component of the mean curvature \( H \), and let \( g_{ij} \) denote the induced metric on \( M \). Also, let \( \mathbf{R} \) with four indices denote the Riemannian curvature of \( (N, h) \), and let \( \mathbf{R} \) with two indices denote the Ricci curvature of \( (N, h) \). Moreover, let \( \nabla_i \) denote the tangential covariant derivative in the direction...
Lemma 2.1 (Huisken [4], Wang [9]). Along a mean curvature flow \( M_t \hookrightarrow (N, h) \), we have

\[
\begin{align*}
(1) & \quad \partial_t g_{ij} = -2H^\alpha h_{ij\alpha}, \\
(2) & \quad \partial_t \text{dvol} = -|H|^2 \text{dvol}, \\
(3) & \quad \partial_t h_{ij\alpha} = \Delta h_{ij\alpha} + (\nabla_k R)_{aij} + (\nabla_j R)_{ak\alpha} - 2R_{ijkl}h_{\alpha kj} \\
& \quad + 2\hat{R}_{\alpha\beta\eta k}h^\eta_i + 2\hat{R}_{\alpha\beta\gamma kj}h^\beta_j - \hat{R}_{\eta kj} h^\eta_i \alpha - \hat{R}_{\eta kj} h^\gamma_i \alpha + \hat{R}_{\alpha\beta\lambda}h^\lambda_{ij\beta} \\
& \quad + h_{\delta\alpha}(h_{\eta kj} - h_{\alpha kj} H^\eta) + h_{\delta\alpha}(h_{\eta ij} - h_{\alpha ij} H^\eta) \\
& \quad + h_{\delta\beta\alpha}(h_{\eta ij} - h_{\alpha ij} H^\eta) - h_{\alpha\beta\lambda}h^\lambda_{ij\beta}H^\beta + h_{ij\beta}(\epsilon_{\beta\alpha}, \nabla H e_\alpha).
\end{align*}
\]

By integrating the evolution equation for \(|\nabla^\ast \Pi|\) and using the Hölder and Morrey inequalities, one can obtain

Theorem 2.2 (Huisken [4]). Along the mean curvature flow, \( \sup_{M \times [0, T]} |\nabla^\ast \Pi| \) is bounded in terms of \( \sup_{M \times [0, T]} |\Pi| \) and the ambient geometry bounds.

We recall a few lemmas about one-parameter families of Riemannian metrics:

Lemma 2.3 (Glickenstein [2]). Suppose a one-parameter family of complete Riemannian manifolds \( (M, g(t)) \) is uniformly continuous in \( t \); that is, for any \( \epsilon > 0 \) there exists \( \delta > 0 \) so that for any \( t_0, (1 - \epsilon)g(t_0) \leq g(t) \leq (1 + \epsilon)g(t_0) \) for \( t \in [t_0, t_0 + \delta] \). Then for any \( p \in M \), \( r > 0 \), the metric balls centred at \( p \) satisfy

\[
B_{g(t_0)}(p, \frac{r}{\sqrt{1 + \epsilon}}) \subseteq B_{g(t)}(p, r) \subseteq B_{g(t_0)}(p, \frac{r}{\sqrt{1 - \epsilon}}).
\]

Proof. Let \( p, q \in M \). Let \( \gamma : [0, S] \to M \) be a minimizing geodesic from \( p \) to \( q \) for the metric \( g(t_0) \). Then the distance \( d_{g(t_0)}(p, q) \) in the metric \( g(t_0) \) satisfies

\[
d_{g(t_0)}(p, q) = \int_0^S |\gamma|_{g(t_0)}(s)ds \\
\geq \frac{1}{\sqrt{1 + \epsilon}} \int_0^S |\gamma|_{g(t)}(s)ds \\
\geq \frac{1}{\sqrt{1 + \epsilon}} d_{g(t)}(p, q)
\]

so that \( \frac{1}{\sqrt{1 + \epsilon}} d_{g(t)}(p, q) \leq d_{g(t_0)}(p, q) \). This immediately implies

\[
B_{g(t_0)}(p, \frac{r}{\sqrt{1 + \epsilon}}) \subseteq B_{g(t)}(p, r).
\]

The other inclusion is analogous. \( \square \)

Lemma 2.4 (Hamilton [3]). Let \((M, g(t))\) be a one-parameter family of compact Riemannian manifolds defined for \( t \in [0, T) \). Suppose that

\[
\int_0^T \max_{M_t} |\frac{\partial g}{\partial t}|_{g(t)} dt < \infty.
\]
Then the metrics \( g(t) \) are uniformly equivalent and converge pointwise as \( t \to T \) to a continuous positive-definite metric \( g(T) \).

3. II AND THE INJECTIVITY RADIUS

We will prove Theorems 1.2 and 1.3 by a blow-up argument. In particular we will use the Cheeger-Gromov convergence theorem to extract a limit of some submanifolds \( F_j : M \to \mathbb{R}^{m+n} \), thought of as Riemannian manifolds \((M, F^*_j\,dx^2)\). We therefore need the following relationship between injectivity radius and the second fundamental form.

**Theorem 3.1.** Let \( F : M^m \to \mathbb{R}^{m+n} \) be an immersion with \( |\Pi| \leq C \). Then
\[
\text{inj}(M, F^*\,dx^2) \geq \frac{1}{2\sqrt{C}}.
\]

We begin by considering the case of the graph of a map \( \psi : \mathbb{R}^m \to \mathbb{R}^n \), as in [7]. We need to compare the standard square-norm of certain objects, e.g. \( |D^2\psi|^2 = \sum_{1 \leq i,j \leq m} \left( \frac{\partial^2 \psi}{\partial x_i \partial x_j} \right)^2 \), with the norms of the tensors \( \Pi \) and \( \nabla^2 \Pi \) in the metric \( g \) induced by the immersion. To keep the norms straight, in this section we use \(|\cdot|\) for the standard square-norm and \(|\cdot|_g\) for the norm in \( g \):
\[
\begin{align*}
|\Pi|_g^2 &= h_{ij\alpha}h_{kl\beta}g^{\alpha\beta}g^{ij}g^{kl}, \\
|\nabla^2 \Pi|_g^2 &= \nabla_i h_{j\alpha k\beta} \nabla_p h_{qr\beta\gamma} g^{\alpha\gamma} g^{kr} g^{\alpha\beta}.
\end{align*}
\]

**Lemma 3.2.** Let \( \psi : D^m \to \mathbb{R}^n \) be a \( C^2 \) function on the disc of radius \( r \). Then
\[
|D^2\psi|^2 \leq (1 + |D\psi|^2)^3 |\Pi|_g^2
\]
where \( \Pi \) is the second fundamental form of the graph of \( \psi \).

**Proof.** The graph of \( \psi \) has immersion map \( F(x_1, \ldots, x_m) = (x_1, \ldots, x_m, \psi_1, \ldots, \psi_n) \). We use the following tangent and normal frames, where \( 1 \leq i \leq m \) and \( 1 \leq \alpha \leq n \):
\[
\begin{align*}
e_i &= (0, \ldots, 0, 1, 0, \ldots, 0, \frac{\partial \psi_1}{\partial x_i}, \ldots, \frac{\partial \psi_n}{\partial x_i}) = (0, \ldots, 0, 1, 0, \ldots, 0, D_i \psi), \\
\nu_\alpha &= (-\frac{\partial \psi_\alpha}{\partial x_1}, \ldots, -\frac{\partial \psi_\alpha}{\partial x_m}, 0, \ldots, 0, 1, 0, \ldots, 0) = (-D\psi_\alpha, 0, \ldots, 0, 1, 0, \ldots, 0).
\end{align*}
\]
These choices induce the metric on the tangent bundle of the graph, which we denote by \( g \) with Latin indices:
\[
g_{ij} = e_i \cdot e_j = \delta_{ij} + D_i \psi \cdot D_j \psi.
\]
We also get a metric on the normal bundle, which we denote by \( g \) with Greek indices:
\[
g_{\alpha\beta} = \nu_\alpha \cdot \nu_\beta = \delta_{\alpha\beta} + D\psi_\alpha \cdot D\psi_\beta.
\]
We will use \( g^{ij} \) to denote the inverse matrix to \( g_{ij} \) and \( g^{\alpha\beta} \) to denote the inverse to \( g_{\alpha\beta} \). We compute the second fundamental form. Note that \( D^2 F = (0, D^2 \psi) \). So we have
\[
\Pi(e_i, e_j) = \text{proj}^\perp (D^2 F(e_i, e_j))
\]
\[
= (D^2_{ij} F \cdot \nu_\beta) g^{\alpha\beta} \nu_\alpha
\]
\[
= \frac{\partial^2 \psi_\beta}{\partial x_i \partial x_j} g^{\alpha\beta} \nu_\alpha.
\]
In components, 

\[ h_{ij} = \frac{\partial^2 \psi_{\alpha}}{\partial x_i \partial x_j} \]

Then the norm-squared of the second fundamental form is

\[
|\Pi|^2_g = \frac{\partial^2 \psi_{\alpha}}{\partial x_i \partial x_j} \frac{\partial^2 \psi_{\beta}}{\partial x_k \partial x_l} g^{\alpha \beta} g^{ij}.
\]

(3.10)

We can think of \(|\Pi|^2_g\) as the norm-squared of \(D^2 \psi\) in the metric \(g\) as opposed to the standard metric. We will compare \(g^{\alpha \beta}\) and \(g^{ij}\) to the standard metric by giving estimates for the eigenvalues of \(g^{\alpha \beta}\) and \(g^{ij}\). To do this we estimate the eigenvalues of \(g_{ij}\) and \(g_{\alpha \beta}\).

Each eigenvalue \(\lambda\) of \(g_{\alpha \beta}\) has the form \(\lambda = \lambda(X, X) = g_{\alpha \beta} X^\alpha X^\beta\) for some eigenvector \(X \in \mathbb{R}^n\) with \(|X|^2 = \sum_\alpha (X^\alpha)^2 = 1\). Then

\[
\lambda = (\delta_{\alpha \beta} + D \psi_\alpha \cdot D \psi_\beta)(X^\alpha X^\beta)
\]

\[
= |X|^2 + (X^\alpha D \psi_\alpha) \cdot (X^\beta D \psi_\beta)
\]

\[
= 1 + |X^\alpha D \psi_\alpha|^2.
\]

(3.7)

Thus \(1 \leq \lambda \leq 1 + |D \psi|^2\). Similarly for an eigenvalue \(\mu\) of \(g_{ij}\), we have

\[
\mu = (\delta_{ij} + D \psi_i \cdot D \psi_j)X^i X^j
\]

\[
= |X|^2 + |D_X \psi|^2
\]

\[
= 1 + |D_X \psi|^2.
\]

(3.8)

So \(1 \leq \mu \leq 1 + |D \psi|^2\).

Thus the eigenvalues of the inverse matrices \(g^{\alpha \beta}\) and \(g^{ij}\) are bounded away from zero and infinity:

\[
1 \geq |\lambda^{-1}| \geq \frac{1}{1 + |D \psi|^2},
\]

\[
1 \geq |\mu^{-1}| \geq \frac{1}{1 + |D \psi|^2}.
\]

(3.9)

So we can estimate

\[
|\Pi|^2_g = \frac{\partial^2 \psi_{\alpha}}{\partial x_i \partial x_j} \frac{\partial^2 \psi_{\beta}}{\partial x_k \partial x_l} g^{\alpha \beta} g^{ij} g^{kl}
\]

\[
\geq \sum_{1 \leq \alpha \leq n \leq m} \left( \frac{\partial^2 \psi_{\alpha}}{\partial x_i \partial x_j} \right)^2 \frac{1}{(1 + |D \psi|^2)^2} \frac{1}{(1 + |D \psi|^2)^2}
\]

\[
= |D^2 \psi|^2 \frac{1}{(1 + |D \psi|^2)^3},
\]

which establishes our lemma.

\[
\square
\]

Our next step is to show that any immersed submanifold \(F : M \hookrightarrow \mathbb{R}^{m+n}\) can be written as a collection of graphs of functions \(\psi\) with small |\(D \psi\)|.

We introduce the following notation and notions, following \(\text{[5]}\). Given \(q \in M\), denote by \(A_q\) any Euclidean isometry which takes \(F(q)\) to the origin and \(T_{F(q)} F(M)\) to the plane \(\{(x_1, \ldots, x_m, 0)\}\). Let \(\pi\) be the projection of \(\mathbb{R}^{n+m}\) to the plane \(\{(x_1, \ldots, x_m, 0)\}\). Define \(U_{r,q} \subset M\) to be the component of \((\pi \circ A_q^{-1} \circ F)^{-1}(D_r)\) which contains \(q\). We call \(F : M \hookrightarrow \mathbb{R}^{n+m}\) a \((r, \alpha)\)-immersion if for each \(q \in M\) there is some \(\psi_q : D_r^m \to \mathbb{R}^n\) with \(|D \psi_q| \leq \alpha\) so that \(A_q^{-1} \circ F(U_{r,q}) = \text{graph}(\psi_q)\).
Lemma 3.3. Let $0 < \alpha \leq 1$. Then for any $C^2$-immersed submanifold $F : M^m \hookrightarrow \mathbb{R}^{n+m}$ and any $r$ satisfying
\[ r \leq \frac{\alpha}{(1 + \alpha^2)^{3/2}} \frac{1}{\sup_M |\Pi| g}, \]
$F$ is an $(r, \alpha)$-immersion.

Proof. Let $q \in M$ be arbitrary. Every submanifold is locally a graph over its tangent plane; thus $A_q(F(U_{r,q}))$ can be written as a graph over $D_r$ for small enough $r$. So we set $S_q = \sup\{|r|F(U_{r,q}) = \text{graph}(\psi_{r,q})\}$. For any large $K$, if $F(U_{r,q}) = \text{graph}(\psi_{r,q})$ and $|D\psi_{r,q}| \leq \frac{K}{2}$, we can extend $\psi_{r,q}$ to have a larger domain and still have $|D\psi| \leq K$. Thus we have
\[ \lim_{r \to S_q} \inf_{\psi} \sup_{D_r} |D\psi| = \infty \]
where the infimum is taken over all $\psi$ of which $F(U_{r,q})$ is a graph. Thus for our given $\alpha$ there exist some $r_q, \psi_q : D_{r_q} \to \mathbb{R}^n$ with $\sup_{D_{r_q}} |D\psi_q| = \alpha$. Now we use the fundamental theorem of calculus and Lemma 3.2 to get
\[ \alpha = \sup_{D_{r_q}} |D\psi_q| \leq r_q \sup_{D_{r_q}} |D^2\psi_q| \leq r_q (1 + \alpha^2)^{3/2} \sup_{D_{r_q}} |\Pi_{\psi_q}| g, \]
which implies that
\[ r_q \geq \frac{\alpha}{(1 + \alpha^2)^{3/2}} \frac{1}{\sup_{D_{r_q}} |\Pi_{\psi_q}| g} \geq \frac{\alpha}{(1 + \alpha^2)^{3/2}} \frac{1}{\sup_M |\Pi| g}. \]

So for $r$ less than the right-hand side of (3.13), there is some $\psi : D_r \to \mathbb{R}^{m+n}$ which makes $F(U_{r,q})$ a graph and $|D\psi| \leq \alpha$.

Proof of Theorem 3.1. Now choose $\alpha = 1$, and let $r$ be given by Lemma 3.3. It is clear that $B(q, r) \subset U_{r,q}$, since $A_q F(U_{r,q})$ is a graph over a disc of radius $r$. Thus $B(q, r)$ can be written as a graph over the tangent plane, and in particular $\text{inj}(q) \geq r$.

Since $q$ was arbitrary, we have $\text{inj}(M) \geq r > 0$.

4. THE TENSOR $A$ BLOWS UP

We will prove Theorem 1.2 by contradiction. To this end, assume $\max_{M} |A(t)| \leq C$ for all $t \in [0, T)$ and that the flow has a singularity at $T < \infty$.

In particular, we have $|H|^4 = (\text{tr} A)^2 \leq n|A|^2 \leq nC^2$. So $|H|$ is also bounded along the flow. We will use $C$ to denote this bound as well.

By Theorem 1.1 we know that as $t \to T$, $\max_{M} |\Pi(t)| \to \infty$. Let $(p_j, t_j)$ be a sequence in $M \times [0, T]$ with $t_j \to T$ and $\max_{t \leq t_j} |\Pi| = |\Pi(p_j, t_j)| = Q_j \to \infty$.

Since $|\partial_t F| = |H| \leq C$, we know that $F_t(M)$ is contained in the $CT$ tubular neighborhood of $F_0(M)$. Thus the $F(p_j, t_j)$ accumulate. Passing to a subsequence, we have $F(p_j, t_j) \to p_0$. For any $R > 0$, we may choose a $j_0$ so that $F(p_j, t_j)$ lies in the ambient ball of radius $R$ about $p_0$, $B^N(R)$ for all $j \geq j_0$. In particular, we will take $R$ to be less than the injectivity radius of $(N, h)$.

Consider the flows given by scaling the ambient metric by $Q_j$ and time by $Q_j^{-2}$:
\[ F_j(p, t) = F(p, t_j + \frac{t}{Q_j^2}) : M \hookrightarrow (B^N, Q_j^2 h). \]
Lemma 4.1. Each $F_j$ is a mean curvature flow on $M \times [-Q_j^2 t_j, 0]$. The second fundamental form of $F_j$ is bounded:

\[
\max_{t \leq 0} |\Pi_j| = |\Pi_j(p_j, 0)| = 1.
\]

Proof. Clearly $\partial_t F_j = Q_j^{-2} \partial_t F$. We need to show that by scaling the ambient metric, we induce the same scaling in $H$. By definition

\[
H_{Q_j^2 h}(F_j) = \text{tr}_{Q_j^2 h}(\Pi_{Q_j^2 h}(F_j))
\]

\[
= (Q_j^2 h)^{pq}(\partial_p \partial_q F_j)^{\perp Q_j^2 h}
\]

where $(Q_j^2 h)^{pq}$ is the inverse matrix of $(Q_j^2 h)(\partial_p F_j, \partial_q F_j)$ and $\perp Q_j^2 h$ is the projection onto the normal bundle induced from $Q_j^2 h$.

$Q_j^2 h$ induces the same splitting into tangent and normal bundles as $h$, so we have

\[
H_{Q_j^2 h}(F_j) = Q_j^{-2} [h^{pq}(\partial_p \partial_q F_j)^{\perp h}]
\]

\[
= Q_j^{-2} \text{tr}_h \Pi_h(F_j)
\]

\[
= Q_j^{-2} H_h(F_j).
\]

So $H$ scales as required.

Similarly scaling the ambient metric by $Q_j^2$ scales $|\Pi|$ by $Q_j^{-1}$, so we have

\[
\max_{t \leq 0} |\Pi_j| = \max_{t \leq t_j} Q_j^{-1}|\Pi|
\]

\[
= Q_j^{-1}|\Pi(p_j, t_j)| = 1.
\]

It is clear that the $(B^N, Q_j^2 h, p_0)$ converge in the Cheeger-Gromov sense to $(\mathbb{R}^{m+n}, dx^2, 0)$, where $dx^2$ is the Euclidean metric. In particular, we have a monotone exhausting sequence of open sets $V_j \subset \mathbb{R}^{m+n}$ and embeddings $\psi_j$: $(V_j, 0) \rightarrow (B^N, p_0)$, such that $\psi_j^*(Q_j^2 h) \rightarrow dx^2$.

Let $s_0 = -Q_j^2 t_1$. After passing to a smaller spatial region $\tilde{M} \subset M$, we can assume $F_j(M \times [s_0, 0]) \subset \psi_j(V_j)$. We restrict our argument to this smaller region and write $M$ without confusion.

Define $\tilde{F}_j : M \times [s_0, 0] \rightarrow \mathbb{R}^{m+n}$ by $\tilde{F}_j = \psi_j^{-1} F_j$. Each $\tilde{F}_j$ is a mean curvature flow with respect to the metric $\psi_j^*(Q_j^2 h)$.

The second fundamental forms $\tilde{\Pi}_j$ of the $\tilde{F}_j$ are uniformly bounded, so Theorem 2.2 gives uniform bounds on the covariant derivatives of the $\tilde{\Pi}_j$.

Since $\partial_t \tilde{F}_j = \tilde{H}_j$, we get bounds on the time derivative of $\tilde{F}_j$. In fact the evolution of $H$ gives a bound

\[
|\partial^t_\tilde{F}_j| = |\partial_t \tilde{H}_j| \leq |\Delta \tilde{H}_j| + C_1 |\nabla \tilde{H}_j| + C_2 |\tilde{H}_j||\tilde{\Pi}_j|^2
\]

\[
\leq |\nabla^2 \tilde{\Pi}_j| + C_1 |\nabla \tilde{\Pi}_j| + C_2 |\tilde{\Pi}_j|^3.
\]

Similarly, any iterated time derivative $\partial^r_\tilde{F}_j = \partial^r_\tilde{H}_j$ is controlled in terms of $|\nabla^r \tilde{\Pi}_j|$ for $r \leq 2(s - 1)$. The mixed derivatives $\partial^l_\tilde{H}_j \nabla^r \tilde{\Pi}_j$ are similarly controlled by $|\nabla^l \tilde{\Pi}_j|$ for $l \leq 2r + s$.

Since $Q_j^2 h \rightarrow dx^2$ in $C^k$ for any $k$, our bounds on the $|\nabla^r \tilde{\Pi}_j|$ give bounds on $|\nabla^r \Pi_j|$, where $\nabla$ and $\Pi_j$ are the connection and second fundamental form of $\tilde{F}_j$ with respect to the metric $dx^2$. 


Let $t \in [s_0, 0]$. Theorem 3.1 gives $\text{inj}(M, \tilde{F}_j(t)^* dx^2) \geq \frac{1}{2\sqrt{j}}$. The Gauss equation guarantees that the Riemannian curvature and its covariant derivatives of $(M, \tilde{F}_j(t)^* dx^2, p_j)$ are bounded uniformly in $j$. Thus by Cheeger-Gromov there is a limit Riemannian manifold $(M_\infty, g_\infty(t), p_\infty)$.

Adapting the ideas of [8], we consider the growth of balls in $(M_\infty, g_\infty(0))$. We will write $g_\infty$ for $g_\infty(0)$.

**Proposition 4.2.** $(M, g_\infty)$ has Euclidean volume growth about $p_\infty$.

**Proof.** Let us use the following conventions for balls and volume forms. $B_\infty(p)$ will denote the metric ball in $g_\infty$ centered at $p_\infty$; $B_j(p)$ will denote the metric ball in $F_j^*(Q_j^2 h)$ centred at $p_j$; $B_t^j(p)$ will denote the metric ball in $F_t^j h$ centred at $p_j$.

Also, $\text{vol}_\infty$ will denote the volume form of $g_\infty$; $\text{vol}_j$ will denote the volume form of $F_j^*(Q_j^2 h)$; $\text{vol}_t^j$ will denote the volume form of $F_t^j h$. Note that

$$B_j(p) = B_t^j(\frac{p}{Q_j}),$$

$$\text{vol}_j = Q_j^m \text{vol}_t^j.$$

We have, for any $r > 0$,

$$\frac{\text{vol}_\infty(B_\infty(r))}{r^m} = \lim_{j} \frac{\text{vol}_j(B_j(r))}{r^m} = \lim_{j} \frac{\text{vol}_t^j(B_t^j(\frac{r}{Q_j}))}{(\frac{r}{Q_j})^m}.$$

The evolution of $g$ is

$$\partial_t g_{ij} = -2H^o h_{ij} = -2A_{ij},$$

so we have $|\partial_t g| \leq C$, and in particular $g$ is uniformly continuous in time in the sense of Lemma 2.3.

Thus we may apply Lemma 2.3 to estimate the metric balls at any time $t_j$ by the metric ball at time $t_{j_0}$, so long as $t_j - t_{j_0} \leq \delta$. Since $t_j \to T$, we can pick a $j_0$ so that this condition holds for all $j \geq j_0$. So we can estimate (4.8) by

$$\lim_{j} \frac{\text{vol}_j(B_t^j(\frac{r}{Q_j}))}{(\frac{r}{Q_j})^m} \leq \lim_{j} \frac{\text{vol}_t^j(B_{t_{j_0}}(\frac{r}{\sqrt{1-\epsilon}Q_j}))}{(\frac{r}{\sqrt{1-\epsilon}Q_j})^m}.$$

The evolution of the volume form shows that the flow is pointwise volume-reducing. So $\text{vol}_t^j \leq \text{vol}_{t_{j_0}}$ for $j \geq j_0$. Thus we can estimate (4.10) by

$$\lim_{j} \frac{\text{vol}_t^j(B_{t_{j_0}}(\frac{r}{\sqrt{1-\epsilon}Q_j}))}{(\frac{r}{\sqrt{1-\epsilon}Q_j})^m} \leq \lim_{j} \frac{\text{vol}_t^j(B_{t_{j_0}}(\frac{r}{Q_j}))}{(\frac{r}{Q_j})^m} = (1 - \epsilon)^{-\frac{m}{2}} \lim_{j} \frac{\text{vol}_t^j(B_{t_{j_0}}(\frac{r}{Q_j}))}{(\frac{r}{Q_j})^m}.$$

The only dependence of the right-hand side on $j$ is in the $Q_j$.

The limit on the right-hand side of (4.11) is the local volume comparison at $p_{j_0}$ for the Riemannian manifold $(M, F_{t_{j_0}}^j h)$. It is well-known that this limit is $\omega_m$, the volume of the Euclidean unit $m$-ball. Therefore we have

$$\frac{\text{vol}_\infty(B_\infty(r))}{r^m} \leq (1 - \epsilon)^{-\frac{m}{2}} \omega_m.$$
Since $\epsilon$ was arbitrary, we have shown $\text{vol}_\infty(B_\infty(r)) \leq \omega_mr^m$.

To show the reverse inequality, we make a similar argument starting from (4.8), this time using the first inclusion of Lemma 2.3. We now seek to estimate $\text{vol}_t$ below by $\text{vol}_{t_0}$. Since we have assumed $|H| \leq C$, the evolution of $\text{vol}$ implies that

$$\text{vol}_t \geq e^{-C^2(t-t_0)} \text{vol}_{t_0},$$

and taking $j_0$ large enough we may ensure that $e^{-C^2(t_j-t_0)} \geq 1-\epsilon$. Then we can estimate (4.8) by

$$\lim_j \frac{\text{vol}_j(B_j(\frac{r}{Q_j}))}{(\frac{r}{Q_j})^m} \geq \lim_j (1-\epsilon)(1+\epsilon)^{-\frac{m}{2}} \frac{\text{vol}_{j_0}(B_{j_0}(\frac{r}{\sqrt{1+\epsilon Q_j}}))}{(\frac{r}{\sqrt{1+\epsilon Q_j}})^m}.$$ 

Again we can take the limit in $j$ to get

$$\frac{\text{vol}_\infty(B_\infty(r))}{r^m} \geq (1-\epsilon)(1+\epsilon)^{-\frac{m}{2}} \omega_m.$$ 

Since $\epsilon$ was arbitrary, we have shown $\text{vol}_\infty(B_\infty(r)) \geq \omega_mr^m$. □

Proof of Theorem 1.2. To finish the proof of the theorem, we want to use the volume growth of $(M_\infty,g_\infty)$ to obtain a contradiction. The expansion for the volume of balls about $p$ in $r$ is

$$\text{vol}_\infty(B_\infty(r)) = \omega_mr^m(1 - \frac{R_\infty(p)}{6(m+2)}r^2 + O(r^3))$$

where $R_\infty(p_\infty)$ is the scalar curvature [1]. So Proposition 4.2 immediately implies that $R_\infty(p_\infty) = 0$.

On the other hand, tracing the Gauss equation twice gives that

$$R_\infty(p_\infty) = \lim_j R_j(p_j)$$

$$= \lim_j |H_j(p_j)|^2 - |\Pi_j(p_j)|^2$$

$$\leq \lim_j \frac{C}{Q_j^2} - 1$$

$$= -1.$$ 

This is the desired contradiction. □

5. A CONDITION FOR THE BLOW UP OF $H$

Definition 5.1. A singularity at time $T < \infty$ is of type I if $|\Pi(x,t)|^p(T-t) \leq C < \infty$ for all $x \in M_t$ and all $t \in [0,T)$.

This is the slowest possible rate of singularity formation and is attained in the case of a shrinking sphere or cylinder.

We can prove that the mean curvature blows up under a slightly more general condition, namely

Theorem 5.1. Suppose that along the flow, $|\Pi(x,t)|^p(T-t) \leq C$ for some $p > 1$. Then $\max_{M_t}|H| \to \infty$ as $t \to T$. 

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Proof. For the purpose of contradiction, suppose \(|H| \leq C\) all along the flow.

As in the proof of Theorem 1.2, we consider the parabolic rescales

\[
F_j(p, t) = F(p, t_j + \frac{t}{Q_j^2}) : M \hookrightarrow (N, Q_j^2 h)
\]

where \(Q_j = \max_{t \leq t_j} |II|\). Each \(F_j\) is a mean curvature flow, and we have

\[
|H_j| \leq \frac{C}{Q_j}.
\]

As in the proof of Theorem 1.2, we want to obtain a limit manifold \((M_\infty, g_\infty)\) whose volume growth yields a contradiction.

To proceed to a contradiction as in the proof of Theorem 1.2, we need to establish that the balls of the metrics \(g(t)\) can be compared as in the conclusion of Lemma 2.3.

Consider the evolution of the metric:

\[
\frac{\partial g}{\partial t} = 2|A| \leq 2|H||II| \leq 2C(T - t)^{-\frac{1}{p}}.
\]

The function \((T - t)^{-\frac{1}{p}}\) is integrable, so Lemma 2.4 guarantees that the metrics \(g(t)\) converge to a positive definite \(g(T)\) as \(t \nearrow T\). We thus have for any \(\varepsilon_1 > 0\) the existence of a \(\delta > 0\) such that \(t \geq T - \delta\) guarantees \(1 - \varepsilon_1 \leq \frac{g(t)}{g(s)} \leq 1 + \varepsilon_1\). If \(t, s \geq T - \delta\), then we have \(\frac{1 - \varepsilon_1}{1 + \varepsilon_1} \leq \frac{g(t)}{g(s)} \leq \frac{1 + \varepsilon_1}{1 - \varepsilon_1}\). Choosing the \(\delta\) which corresponds to \(\varepsilon_1 = \frac{\varepsilon_2}{2}\), then guarantees that \((1 - \varepsilon)g(s) \leq g(t) \leq (1 + \varepsilon)g(s)\).

We may then apply the proof of Lemma 2.3 to estimate balls in the metrics \(g(t_j)\) for \(j\) sufficiently large and proceed as in the proof of Theorem 1.2, starting from equation (4.10). \(\square\)

Remark 1. The tensors \(H\) and \(A\) are either both zero or both nonzero. A bound on \(A\) implies that \(H\) is bounded. Theorem 1.2 says that \(H\) cannot decay “too fast” relative to \(II\). The following question presents itself:

**Question 1.** Suppose \(T < \infty\) is the first singular time for a compact mean curvature flow. Is it generally true that \(\max_{M_t} |H| \to \infty\) as \(t \to T\)?

Remark 2. The blowup of the mean curvature in the type I case has been independently discovered by Le and Sàsàm [6].

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