Minimal and maximal lengths from position-dependent non-commutativity

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Abstract

Fring et al (2010 J. Phys. A: Math. Theor. 43 345401) have introduced a new set of noncommutative space-time commutation relations in two space dimensions. It had been shown that any fundamental objects introduced in this space-space noncommutativity are string-like. Taking this result into account, we generalize the seminal work of Fring et al to the case that there is also a maximal length from position-dependent noncommutativity and a minimal momentum arising from generalized versions of Heisenberg’s uncertainty relations. The existence of maximal length is related to the presence of an extra, first order term in particle’s length that provides the basic difference of our analysis with theirs. This maximal length breaks up the well known singularity problem of space time. We establish different representations of this noncommutative space and finally we study some basic and interesting quantum mechanical systems in these new variables.

Keywords: deformed algebras, minimal length, maximal length, noncommutative quantum mechanics

1. Introduction

One of the oldest open problems in modern physics is the unification of general relativity (GR) and quantum theory (QT). The problem of finding a quantum formulation of the Einstein equation in GR still does not have a consistent and satisfactory solution. The difficulty arises since GR deals with the events which define the world-lines of particles, while quantum mechanics do not allow the definition of trajectory. Nevertheless, one of the most active candidate theories to address this problem, string theory, predicted that this unification should occur at the Planck scale and should give birth to quantum gravity [2, 3]. Thus, the minimal measurement of quantum scale allows a measurement of Planck order \( l_p = 10^{-35} \text{ m} \). This
value is extremely small; its experimental search lies beyond the energies currently accessible in the laboratory.

In the theoretical framework, the observational search for such existence of a minimal length can be derived from the so called generalized (Gravitational) uncertainty principle (GUP) [4]

\[ \Delta x \Delta p \geq \frac{\hbar}{2} \left[ 1 + \beta (\Delta p)^2 \cdots \right], \]

(1)

by deforming the Heisenberg algebra as follows

\[ [\hat{x}, \hat{p}] = i\hbar (1 + \beta \hat{p}^2 \cdots). \]

(2)

This latter implies a minimal position uncertainty \( \Delta x_{\text{min}} \) [4–8]. Moreover, the emergence of this minimal length in non-relativistics quantum mechanics introduces many consequences such as the deformation of the Heisenberg algebra, the loss of the localization of particles in the position representation, the deformation of the structures of the Hilbert space, the noncommutation in position space [4] etc. In quantum geometry as in quantum gravity, this minimal length induces an addition to the previous consequences observed in the Hilbert space, the violation of the Lorentz invariance [9, 10] and an intriguing mixing between the ultraviolet and the infrared [11]. It leads to a generalized Hawking temperature [12, 13] and removal of the Chandrasekhar limit in cosmology [14] etc.

Since the appearance in quantum mechanics, many alternative approaches to improve this minimal length had been introduced [15–18] which propose higher modifications to GUP. In this sense, a new set of noncommutative space-time commutation relations in two dimensional configuration space has been recently introduced [1]. The space-space commutation relations are deformations of the standard flat noncommutative space-time relations that have position dependent structure constants. These deformations lead to minimal lengths and it has been found that any object in this two dimensional space is string-like, in the sense that having a fundamental length beyond which a resolution is impossible. Some extensions of this work have been done in [1, 19–22] and the model of gravitational quantum well have been solved in these new variables [23].

In this paper, we are going to generalize the result of Fring et al [1] to the case that the existence of a maximal length is considered too. In this seminal work, one notes that a simultaneous measurement in position space-time leads to a minimal length in \( X \)-direction as well as a minimal length in \( Y \)-direction when informations are given-up in one of the directions. Here we just consider the case where for a simultaneous measurement, the lost of particle’s localization in \( X \)-direction leads to its maximal localization in \( Y \)-direction. Then, both minimal momentum and maximal length arise from the generalized versions of Heisenberg’s uncertainty relations for a simultaneous measurement in \( Y,P_y \)-directions. This proposal agrees with a similar perturbative approaches predicted by doubly special relativity theories (DSR) [24, 25] and by the seminal result of Nozari and Etemadi [26]. The existence of maximal length related to the presence of an extra, first order term in particle’s length, brings a lot of new features to the Hilbert space representation of quantum mechanics at the Planck scale. Moreover, the presence of minimal uncertainties in the representation of this algebra, allows us to work with the position \( Y \)-space representation. In this manner, we explore the quantum physical implications and Hilbert space representation in the presence of minimal measurable uncertainties and a maximal measurement length. Eventually, in order to avoid the ambiguity of the meaning of wavefunction due to the existence of minimal measurable uncertainties, we propose another representation of operators \( \hat{X}, \hat{Y}, \hat{P}_x, \hat{P}_y \) in terms of standard Heisenberg
operators $\hat{x}, \hat{y}, \hat{p}_x, \hat{p}_y$, through approximations in first order of deformed parameters $\theta, \tau$. This realization makes an effective noncommutative space and the whole phase space structure of the Lie-algebraic type is related to $\kappa$-like deformations of space and deformed Heisenberg algebra $[27–31]$. In the present paper we study some interesting quantum mechanics systems in two-dimensional position dependent noncommutative spaces and we determine how the Schrödinger equation in the reduced noncommutative algebra can be solved exactly or perturbatively. The paper is organized as follows. In section 2, we review the Heisenberg algebra and its deformation in two-dimensional quantum mechanics with theirs corresponding consequences. In section 3, we introduce the new set of position-dependent noncommutative space and we derive minimal uncertainties and a maximal length resulting from this space and the representations of wavefunction. In section 4, we study some simple models formulated in terms of our new set of variables such as the free particle, the particle in a box and the harmonic oscillator. The conclusion is given in section 5.

2. Heisenberg algebra and its deformation

Let $\mathcal{H} = L^2(-L, L)$ be the Hilbert space of square integrable functions $\psi(x)$ in one-dimensional interval $[-L, L]$. The scalar product on $\mathcal{H}$ is defined

$$\langle \phi | \psi \rangle = \int_{-L}^{+L} dx \phi(x)^* \psi(x). \quad (3)$$

We denote the elements of this Hilbert space by $\psi(x) \equiv |\psi\rangle$ and the elements of its dual by $\langle \psi |$, which maps elements of $L^2(-L, L)$ onto complex numbers by $\langle \psi | \phi \rangle = \langle \psi | \phi \rangle$. The corresponding norm is given as usual by $||\psi|| = \sqrt{\langle \psi | \psi \rangle}$. Since the function $\psi(x)$ is confined to the interval to $[-L, L]$, it vanishes at the boundary $\psi(-L) = 0 = \psi(L)$ $[32]$. Let also consider a physical observable represented by a Hermitian or self-adjoint operator $\hat{A}$ defined on its domain $\mathcal{D}(\hat{A})$ maximal dense on $\mathcal{H}$ and $\hat{A}^\dagger$ its adjoint defined on $\mathcal{D}(\hat{A}^\dagger)$ such as

$$\langle \phi | \hat{A} | \psi \rangle = \langle \hat{A}^\dagger | \phi | \psi \rangle \quad \text{and} \quad \mathcal{D}(\hat{A}) = \mathcal{D}(\hat{A}^\dagger) \quad (4)$$

where $|\phi\rangle \in \mathcal{D}(\hat{A}^\dagger)$ and $|\psi\rangle \in \mathcal{D}(\hat{A})$. The only fact that $\hat{A} = \hat{A}^\dagger$ ensures the expectation value $\langle \psi | \hat{A} | \psi \rangle$ is real, the inner products of wavefunctions in $\mathcal{H}$ have a positive norm and that the time evolution operator is unitary. Since we are in finite dimensional Hilbert space, the self-adjointness of the operator $\hat{A}$ coincides with its symmetry. But let us recall that, this situation is not always satisfy in finite dimensional Hilbert space $[32]$. In case of certain types of operators namely unsharp operators, Kempf showed in his elegant paper $[32]$ that, these operators are not self-adjoint but are only symmetric and are even able to generate the entire unitary group of the Hilbert space.

Moreover, for simultaneous measurement of two observables $\hat{A}$ and $\hat{B}$ in the state $|\psi\rangle$, the uncertainty satisfies the inequality

$$\Delta A \Delta B \geq \frac{\hbar}{2} \left| \langle \psi | [\hat{A}, \hat{B}] | \psi \rangle \right|, \quad (5)$$

where $\Delta A$ and $\Delta B$ are respectively, the dispersions defined as $\Delta A^2 := \langle \psi | [\hat{A}, \hat{A}] | \psi \rangle - \langle \psi | \hat{A} | \psi \rangle^2$ and $\Delta B^2 := \langle \psi | [\hat{B}, \hat{B}] | \psi \rangle - \langle \psi | \hat{B} | \psi \rangle^2$. From the equation (5), we deduce the following relation, that is
\[ \left\| \left( \hat{A} - \langle \hat{A} \rangle + \frac{\langle [\hat{A}, \hat{B}] \rangle}{2\Delta B^2} \left( \hat{B} - \langle \hat{B} \rangle \right) \right) \psi \right\| \geq 0. \]  

(6)

The Fourier transform of the wavefunction \( \psi(x) \) denoted by \( \psi(p) \) in the localized domain \([-L, L]\) is given by

\[ \psi(p) = \frac{1}{(2\pi \hbar)^{1/2}} \int_{-L}^{+L} \psi(x)e^{-i\hat{p}x}dx, \]  

(7)

and the inverse transform is given by

\[ \psi(x) = \frac{1}{(2\pi \hbar)^{1/2}} \int_{-L}^{+L} \psi(p)e^{i\hat{p}x}dp. \]  

(8)

Now, let us start with the general case with the following definition:

**Definition 2.1.** In \( d \)-dimensional space, a unitary representation of the Heisenberg algebra is

\[ [\hat{x}_i, \hat{x}_j] = 0, \quad [\hat{x}_i, \hat{p}_j] = i\hbar \delta_{ij}, \quad [\hat{p}_i, \hat{p}_j] = 0, \quad i, j = 1, 2 \cdots d, \]  

(9)

where \( \hat{x}_i \) and \( \hat{p}_j \) are Hermitian operators acting on \( \mathcal{H}_d = L^2(\mathbb{R}^d) \).

In 2-dimensions of this algebra, we have:

**Proposition 2.1.** Let \( \mathcal{H}_s = L^2(\mathbb{R}^2) \) be the Hilbert space that defined the algebra of linear operators in 2D commutative space

\[ [\hat{x}_x, \hat{y}_y] = 0, \quad [\hat{x}_x, \hat{p}_y] = i\hbar \mathbb{1}, \quad [\hat{y}_y, \hat{p}_x] = i\hbar \mathbb{1}, \]

\[ [\hat{p}_x, \hat{p}_y] = 0, \quad [\hat{x}_x, \hat{p}_x] = 0, \quad [\hat{y}_y, \hat{p}_y] = 0. \]  

(10)

where the operators \( \hat{x}_x, \hat{y}_y, \hat{p}_x, \hat{p}_y \) are operators acting on the space of square integrable function of \( \mathcal{H}_s \).

These commutation relations lead to the standard uncertainty relations

\[ \Delta x_x \Delta p_x \geq \frac{\hbar}{2}, \quad \Delta y_y \Delta p_y \geq \frac{\hbar}{2}. \]  

(11)

Consequently, the Schrödinger representation of the algebra in (10) is

\[ \hat{x}_x \psi(x_x, y_y) = x_x \cdot \psi(x_x, y_y), \quad \hat{y}_y \psi(x_x, y_y) = y_y \cdot \psi(x_x, y_y), \]  

(12)

\[ \hat{p}_x \psi(x_x, y_y) = -i\hbar \frac{\partial}{\partial x_x} \psi(x_x, y_y), \quad \hat{p}_y \psi(x_x, y_y) = -i\hbar \frac{\partial}{\partial y_y} \psi(x_x, y_y), \]  

(13)

where \( \psi(x_x, y_y) \in \mathcal{H}_s \). The above 2D Heisenberg algebra will be now replaced by the non-commutative Heisenberg algebra.

**Proposition 2.2.** Let \( \mathcal{H}_0 = L^2(\mathbb{R}^2) \) be the Hilbert space that describes the ordinary 2D noncommutative space. The Hermitian operators that act on this space satisfy the following relations

\[ [\hat{x}_0, \hat{y}_0] = i\theta \mathbb{1}, \quad [\hat{x}_0, \hat{p}_x] = i\hbar \mathbb{1}, \quad [\hat{y}_0, \hat{p}_y] = i\hbar \mathbb{1}, \]

\[ [\hat{p}_x, \hat{p}_y] = 0, \quad [\hat{x}_0, \hat{p}_y] = 0, \quad [\hat{y}_0, \hat{p}_x] = 0. \]  

(14)
where $\theta \in \mathbb{R}_+^*$ [1], is the noncommutative parameter which has the length square dimension. If $\theta$ is set to zero, we obtain the standard Heisenberg commutations relations (10).

The noncommutation relations (14) lead to an additional uncertainty due to the noncommutativity of the position operators

$$\Delta x_0 \Delta y_0 \gg \frac{|\theta|}{2}, \quad \Delta x_0 \Delta p_x \gg \frac{\hbar}{2}, \quad \Delta y_0 \Delta p_y \gg \frac{\hbar}{2}.$$  \hspace{1cm} (15)

Based on the fact that $\theta$ has dimension of (length)$^2$, then $\sqrt{\theta}$ defines a fundamental scale of length which characterizes the minimum uncertainty possible to achieve in measuring this quantity.

The action of these operators on the square integrable wavefunctions $\psi(x_0, y_0) \in \mathcal{H}_0$ can be realized as follows

$$\hat{x}_0 \psi(x_0, y_0) = x_0 \psi(x_0, y_0), \quad \hat{y}_0 \psi(x_0, y_0) = y_0 \psi(x_0, y_0),$$

$$\hat{p}_x \psi(x_0, y_0) = -i\hbar \frac{\partial}{\partial x_0} \psi(x_0, y_0), \quad \hat{p}_y \psi(x_0, y_0) = -i\hbar \frac{\partial}{\partial y_0} \psi(x_0, y_0),$$  \hspace{1cm} (16)

where $\ast$ denotes the so-called star product, defined by

$$(f \ast g)(x, y) = \exp \left(\frac{i}{\hbar} \theta \frac{\partial}{\partial x} \frac{\partial}{\partial y} \right) f(x)g(y),$$  \hspace{1cm} (17)

where $f$ and $g$ are two arbitrary infinitely differentiable functions on $\mathbb{R}^2$ is real and antisymmetric i.e $\theta_{ij} = \epsilon_{ij} \theta$ ( $\epsilon_{ij}$ a completely antisymmetric tensor with $\epsilon_{1,2} = 1$).

One possible way of implementing algebra equations (14) is to construct the noncommutative operators $\hat{x}_0, \hat{y}_0, \hat{p}_x, \hat{p}_y$ from the commutative operators $\hat{x}, \hat{y}, \hat{p}_x, \hat{p}_y$ by means of a linear transformation namely Bopp-shift denoted by $B_\theta$. In the literature [18, 28], there are many versions of the Bopp-shift such as the asymmetric Bopp-shift

$$B_\theta^{-a} : \begin{cases} \hat{x}_0 = \hat{x} - \frac{\theta}{\pi} \hat{p}_y, \\ \hat{y}_0 = \hat{y}, \\ \hat{p}_x = \hat{p}_x, \\ \hat{p}_y = \hat{p}_y, \end{cases} \quad \text{or} \quad B_\theta^{-s} : \begin{cases} \hat{x}_0 = \hat{x}, \\ \hat{y}_0 = \hat{y} + \frac{\theta}{\pi} \hat{p}_x, \\ \hat{p}_x = \hat{p}_x, \\ \hat{p}_y = \hat{p}_y, \end{cases}$$  \hspace{1cm} (18)

and the symmetric Bopp-shift

$$B_\theta^s : \begin{cases} \hat{x}_0 = \hat{x} - \frac{\theta}{\pi} \hat{p}_y, \\ \hat{y}_0 = \hat{y} + \frac{\theta}{\pi} \hat{p}_x, \\ \hat{p}_x = \hat{p}_x, \\ \hat{p}_y = \hat{p}_y, \end{cases}$$  \hspace{1cm} (19)

There are some advantages in using the asymmetric Bopp shift such as the decoupling of the operators in some of the problems and some simplifications of expressions. In fact $B_\theta^{-a}$ and $B_\theta^{-s}$ do not always lead to the same results for the same problems. For that reason the symmetrical Bopp shift $B_\theta^s$ is often used [33]. In the present work, some of these transformations will be used in the forthcoming development according to our purposes. With the transformations (18) and (19), it is easy to verify that the operators $\hat{x}_0, \hat{y}_0, \hat{p}_x, \hat{p}_y$ are Hermitian as we mentioned in proposition 2.2. Taking the transformation $B_\theta^s$ for example, one changes in the Schrödinger’s representations (38), the star product by the usual product of field such as
\[ \dot{x}_0 \psi(x_s, y_s) = x_s \cdot \psi(x_s, y_s) + \frac{i \theta}{2} \frac{\partial}{\partial y_s} \psi(x_s, y_s); \quad \dot{p}_0 \psi(x_s, y_s) = -i \hbar \frac{\partial}{\partial x_s} \psi(x_s, y_s), \]

\[ \dot{y}_0 \psi(x_s, y_s) = y_s \cdot \psi(x_s, y_s) - \frac{i \theta}{2} \frac{\partial}{\partial x_s} \psi(x_s, y_s); \quad \dot{p}_0 \psi(x_s, y_s) = -i \hbar \frac{\partial}{\partial y_s} \psi(x_s, y_s). \]

The equations (20) and (21) are a realization for the deformed Heisenberg algebra in the case of Moyal noncommutativity.

3. Measurement lengths from position dependent noncommutative space

3.1. Position dependent noncommutative algebra and uncertainty measurements

This section addresses the construction of a new set of noncommutative space by introducing new operators \( \hat{X}, \hat{Y}, \hat{P}_x, \hat{P}_y \) and by converting the constant \( \theta \) of the algebra (14) into a function \( \theta(X, Y) \) depending on the operators \( X, Y \) defined by \( \theta(X, Y) := \theta(1 - \tau Y + \tau^2 Y^2) \). We start with the following proposition.

**Proposition 3.1.** Given new set of Hermitian operators \( \hat{X}, \hat{Y}, \hat{P}_x, \hat{P}_y \) defined on \( \mathcal{H}_k = L^2(\mathbb{R}^2) \) satisfy the following commutations relations and all possible permutations of the Jacobi identities (see appendix)

\[
\begin{align*}
[\hat{X}, \hat{Y}] &= i\theta(1 - \tau \hat{Y} + \tau^2 \hat{Y}^2), \quad [\hat{X}, \hat{P}_x] = i\hbar(1 - \tau \hat{Y} + \tau^2 \hat{Y}^2), \\
[\hat{Y}, \hat{P}_y] &= i\hbar(1 - \tau \hat{Y} + \tau^2 \hat{Y}^2), \quad [\hat{P}_x, \hat{P}_y] = 0, \\
[\hat{Y}, \hat{P}_x] &= 0, \quad [\hat{X}, \hat{P}_y] = i\theta \tau(2\tau \hat{Y} \hat{X} - \hat{X}) + i\theta \tau(2\tau \hat{Y} \hat{P}_y - \hat{P}_y),
\end{align*}
\]

where \( \tau \in \mathbb{R}^+ \) is the deformed parameter [1]. By taking \( \tau \to 0 \), we obviously recover the algebra (14).

One can recover the algebra (22) by setting these operators in terms of the Hermitian operators \( \hat{x}_0, \hat{y}_0, \hat{p}_x, \hat{p}_y \) by using the following representation

\[
R_\tau : \begin{cases}
\hat{X} = \hat{x}_0 - \tau \hat{y}_0 \hat{x}_0 + \tau^2 \hat{y}_0^2 \hat{x}_0, \\
\hat{Y} = \hat{y}_0, \\
\hat{P}_x = \hat{p}_x, \\
\hat{P}_y = \hat{p}_y - \tau \hat{y}_0 \hat{p}_y + \tau^2 \hat{y}_0^2 \hat{p}_y.
\end{cases}
\]

From this representation follows immediately that some of the operators involved are no longer Hermitian and symmetric. We observe

\[
\hat{X}^\dagger = \hat{X} - i\theta \tau(1 + \tau \hat{Y}), \quad \hat{Y}^\dagger = \hat{Y}, \quad \hat{P}_x^\dagger = \hat{P}_x, \quad \hat{P}_y^\dagger = \hat{P}_y + i\hbar(1 - 2\tau \hat{Y}).
\]

As is apparent, the operators \( \hat{X} \) and \( \hat{P}_y \) are neither Hermitian, neither symmetric. This observation seems at first to be counterintuitive with the proposition 3.1. But, as in various studies of quantum gravity, this situation is widely expected and has been pointed out in the seminal paper of Kempf as the unsharp degrees of freedom which describe the space time at short distance [32]. Let us clarify that, as rule in quantum mechanics, the operators that act on square integrable functions are essentially Hermitian. There are exceptions to the rule. This
is because the basic quantization requirement that operators whose expectation values are real do not strictly require these operators be Hermitian. Indeed, the symmetry property is a sufficient condition to ensure that all expectation values are real. As will be shown in the next subsections, the symmetry of the operators $\hat{X}$ and $\hat{P}_x$ requires the introduction of special deformed completeness relations.

The parameter $\tau$ is assumed to be a small positive and can be compared to the deformed parameter $\beta = \frac{\hbar^2}{l_p^2}$ of [4, 26] such as $\Delta x = \hbar \sqrt{\beta}$, the minimal length of quantum gravity below which spacetime distances cannot be resolved as predicted by string theory [2]. Such a feature is expected to be a candidate theory of quantum gravity, since gravity itself is characterized by the Planck length $l_p$. In the present case this parameter manifests as deformation of the non-commutative space (14) by quantum gravity. The proposal (22) is consistent with the similar prediction of DSR [24, 25] and by the seminal result of Nozari and Etemadi [26]. From the representation (23), one can interpret $\hat{x}_0, \hat{y}_0, \hat{p}_x, \hat{p}_y$ as the set of operators at low energies which has the standard representation in position space and $\hat{X}, \hat{Y}, \hat{P}_x, \hat{P}_y$ as the set of operators at high energies, where they have the generalized representation in position space.

In comparison with the Fring et al noncommutative space [1], here there is an extra, first order term in particle’s length which will be the origin of the existence of a maximal length. The presence of this term is the source of differences between our set of algebra representation (22) and Fring et al’s algebra [1]. From these commutation relations (22), an interesting features can be observed through the following uncertainty relations:

\begin{align*}
\Delta X \Delta Y &\geq \frac{\theta}{2} \left(1 - \tau \langle \hat{Y} \rangle + \tau^2 \langle \hat{Y}^2 \rangle \right), \\
\Delta Y \Delta P_y &\geq \frac{\hbar}{2} \left(1 - \tau \langle \hat{Y} \rangle + \tau^2 \langle \hat{Y}^2 \rangle \right), \\
\Delta X \Delta P_x &\geq \frac{\hbar}{2} \left(1 - \tau \langle \hat{Y} \rangle + \tau^2 \langle \hat{Y}^2 \rangle \right).
\end{align*}

(i) In the situation of uncertainty relation (25), using $\langle \hat{Y}^2 \rangle = \Delta Y^2 + \langle \hat{Y} \rangle^2$, this relation can be rewritten as a second order equation for $\Delta Y$. The solution for $\Delta Y$ are as follows

\begin{equation}
\Delta Y = \Delta X \frac{\theta}{\theta \tau^2} \pm \sqrt{\left(\Delta X \frac{\theta}{\theta \tau^2}\right)^2 - \frac{\langle \hat{Y} \rangle}{\tau} \left(\tau \langle \hat{Y} \rangle - 1\right) - \frac{1}{\tau^2}}.
\end{equation}

The reality of solutions gives the following minimum value for $\Delta X$

\begin{equation}
\Delta X_{\min} = \theta \tau \sqrt{1 - \tau \langle \hat{Y} \rangle + \tau^2 \langle \hat{Y}^2 \rangle}. \tag{29}
\end{equation}

Therefore, these equations lead to the absolute minimal uncertainty $\Delta X_{\min}^{abs}$ in X direction and the absolute maximal uncertainty $\Delta Y_{\max}^{abs}$ in Y direction for $\langle \hat{Y} \rangle = 0$, such as:

\begin{align*}
\Delta X_{\min}^{abs} &= \theta \tau, \\
\Delta Y_{\max}^{abs} &= l_{\max} = \frac{1}{\tau}. \tag{31}
\end{align*}
In comparison with Fring et al formalism [1], where a simultaneous measurement in \(X\) and \(Y\) spaces leads to a minimal length for \(\hat{X}\) or for \(\hat{Y}\) when informations are given-up in one direction, here a simultaneous measurement leads to a minimal measurement in \(\hat{X}\) which introduces a lost of localization in \(X\)-direction and a maximal measurement in \(\hat{Y}\) which conversely allows maximal localization in \(Y\)-direction in the bounded interval \([-l_{\text{max}}, +l_{\text{max}}]\). Let us remark that, less is the value of the parameter \(\tau\), larger is the interval of localization.

(ii) Repeating the same calculation and argumentation in the situation of uncertainty relation (26) for simultaneous \(\hat{Y}, \hat{P}_y\)-measurement, we find the absolute maximal uncertainty \(\Delta Y_{\text{max}}\) (31) and an absolute minimal uncertainty momentum \(\Delta P^\text{abs}_{y_{\text{min}}}\) for \(\langle \hat{Y} \rangle = 0\), such
\[
\Delta P^\text{abs}_{y_{\text{min}}} = \hbar \tau. \tag{32}
\]

(iii) Finally, for the uncertainty relation (27), a simultaneous \(\hat{X}, \hat{P}_x\)-measurement does not present any minimal/maximal length or minimal momentum. However, one can wonder about a simultaneous measurement of \(\hat{X}\) and \(\hat{P}_y\)? Let say that, a simultaneous \(\hat{X}, \hat{P}_y\)-measurement is less straightforward since terms of the type \(\langle \hat{Y} \hat{X} \rangle\) and \(\langle \hat{Y} \hat{P}_y \rangle\) are encountered which cannot be treated in the same manner. Furthermore, since the behaviour of \(\hat{X}\) and \(\hat{P}_y\) is linear on both sides of the inequality in both cases, we do not expect a minimal/maximal length or a minimal momentum to arise in this circumstance.

3.2. Hilbert space representation with uncertainty relations

As we mentioned in the previous subsection, the emergence of minimal length \(\Delta X_{\text{abs}}\) and minimal momentum \(\Delta P^\text{abs}_{x_{\text{min}}}\) lead to the lost of representation of the wavefunctions in \(X\) and \(P_x\) directions respectively, except in \(Y\) direction where the representation of wavefunction of quantum gravity is accessible in a bounded interval. Therefore, we will restrict our study from the infinite two dimensional Hilbert space \(L^2(\mathbb{R}^2)\) into the finite one dimensional Hilbert space \(L^2(-l_{\text{max}}, +l_{\text{max}})\).

3.2.1. Representation with maximal length and minimal momentum. In the case of the uncertainty relation (26) that predicts a maximal length and a minimal momentum, deduced from the relation \([\hat{Y}, \hat{P}_y] = i\hbar(1 - \tau \hat{Y} + \tau^2 \hat{Y}^2)\) can be defined by the operators
\[
\hat{Y} = \hat{y}_0, \tag{33}
\]
\[
\hat{P}_y = (1 - \tau \hat{y}_0 + \tau^2 \hat{y}_0^2)\hat{p}_y, \tag{34}
\]
where \(\hat{p}_y = -i\hbar \partial_y\). Then by operating on position space wave function \(\psi(y_0)\), we have
\[
\hat{Y}\psi(y_0) = y_0 \star \psi(y_0), \tag{35}
\]
\[
\hat{P}_y\psi(y_0) = -i\hbar(1 - \tau y_0 + \tau^2 y_0^2)\partial_y \psi(y_0). \tag{36}
\]
By utilizing the asymmetric Bopp-shift \(B^\theta_0\), these equations become
\[
\hat{Y}\psi(y_x) = y_x \psi(y_x), \tag{37}
\]
\[
\hat{P}_y\psi(y_x) = -i\hbar(1 - \tau y_x + \tau^2 y_x^2)\partial_y \psi(y_x). \tag{38}
\]
Evidently, as we pointed out in the above subsection, the position operator is self-adjoint and symmetric while the momentum operator is not. Thus, the symmetry requirement of the momentum operator leads to the following proposition:

**Proposition 3.2.** For the given completeness relation on the complete basis \( \{ y_s \} \) such as

\[
\int_{-l_{\text{max}}}^{+l_{\text{max}}} \frac{dy_s}{(1 - \tau y_s + \tau^2 y_s^2)} |y_s\rangle \langle y_s| = \mathbb{I},
\]  

we have

\[
\langle \phi | \hat{P}_y \psi \rangle = \langle \hat{P}_y^\dagger \phi | \psi \rangle,
\]  

such as

\[
\mathcal{D}(\hat{P}_y) = \{ \psi, \psi' \in \mathcal{L}^2(-l_{\text{max}}, l_{\text{max}}); \psi(-l_{\text{max}}) = \psi'(l_{\text{max}}) = 0 \},
\]  

\[
\mathcal{D}(\hat{P}_y^\dagger) = \{ \phi, \phi' \in \mathcal{L}^2(-l_{\text{max}}, l_{\text{max}}) \}.
\]

**Proof.** Let consider \( \psi \in \mathcal{D}(\hat{P}_y) \) and \( \phi \in \mathcal{D}(\hat{P}_y^\dagger) \)

\[
\langle \phi | \hat{P}_y \psi \rangle = \int_{-l_{\text{max}}}^{+l_{\text{max}}} \frac{dy_s}{(1 - \tau y_s + \tau^2 y_s^2)} \phi^*(y_s) \left[ -i\hbar(1 - \tau y_s + \tau^2 y_s^2) \partial_y \psi(y_s) \right].
\]  

By performing a partial integration, we have

\[
\langle \phi | \hat{P}_y \psi \rangle = \int_{-l_{\text{max}}}^{+l_{\text{max}}} \frac{dy_s}{1 - \tau y_s + \tau^2 y_s^2} \left[ -i\hbar(1 - \tau y_s + \tau^2 y_s^2) \partial_y \phi(y_s) \right] ^* \psi(y_s)
\]  

\[
+ \left[ -i\hbar \phi^*(y_s) \psi(y_s) \right] \int_{-l_{\text{max}}}^{+l_{\text{max}}} \frac{dy_s}{1 - \tau y_s + \tau^2 y_s^2}.
\]  

where \( \psi(y_s) \) vanishes at \( \pm l_{\text{max}} \) then \( \phi^*(y_s) \) can attain any arbitrary value at the boundaries. The above equation implies that \( \hat{P}_y \) is symmetric but it is not a self-adjoint operator. The situation is that, \( \hat{P}_y \) is a derivative operator on an interval with Dirichlet boundary conditions and all the candidates for the eigenfunctions of \( \hat{P}_y \) are not in the domain of \( \hat{P}_y \) because they obey no longer the Dirichlet boundary conditions [35]. In fact, the domain of \( \hat{P}_y^\dagger \) is much larger than that of \( \hat{P}_y \), so \( \hat{P}_y \) is indeed not self-adjoint.

Consequently, the scalar product between two states \( |\Psi\rangle \) and \( |\Phi\rangle \) and the orthogonality of position eigenstate become

\[
\langle \Phi | \Psi \rangle = \int_{-l_{\text{max}}}^{+l_{\text{max}}} \frac{dy_s}{1 - \tau y_s + \tau^2 y_s^2} \Phi^*(y_s) \Psi(y_s),
\]  

\[
\langle y_s | y_s' \rangle = (1 - \tau y_s + \tau^2 y_s^2) \delta(y_s - y_s').
\]  

For \( \tau \to 0 \), we recover the usual completeness and orthogonality relations of bounded space \( \mathcal{L}^2(-l_{\text{max}}, l_{\text{max}}) \).

In order to give an explicit expression of the eigenfunction \( \psi(y_s) \), one solves the eigenvalue problem
By solving the following differential equation

$$-i\hbar(1 - \tau y_s + \tau^2 y_s^2) \frac{\partial \psi_\zeta(y_s)}{\partial y_s} = \zeta \psi_\zeta(y_s),$$

we obtain the position eigenvectors in the form

$$\psi_\zeta(y_s) = \psi(0) \exp \left( i \frac{2\zeta}{\tau \hbar \sqrt{3}} \left[ \arctan \left( \frac{2\tau y_s - 1}{\sqrt{3}} \right) + \arctan \left( \frac{1}{\sqrt{3}} \right) \right] \right).$$

Then by normalization, $\langle \psi_\zeta | \psi_\zeta \rangle = 1$, we have

$$1 = \int_{-l_{\text{max}}}^{+l_{\text{max}}} \frac{dy_s}{(1 - \tau y_s + \tau^2 y_s^2)} \psi^*_\zeta(y_s) \psi_\zeta(y_s)$$

so, we find

$$\psi(0) = \sqrt{\frac{\tau \sqrt{3}}{2}} \left[ \arctan \left( \frac{2\tau l_{\text{max}} - 1}{\sqrt{3}} \right) + \arctan \left( \frac{2\tau l_{\text{max}} + 1}{\sqrt{3}} \right) \right]^{-\frac{1}{2}}$$

Substituting this equation (51) into the equation (49), we have

$$\psi_\zeta(y_s) = \sqrt{\frac{\tau \sqrt{3}}{\pi}} \exp \left( i \frac{2\zeta}{\tau \hbar \sqrt{3}} \left[ \arctan \left( \frac{2\tau y_s - 1}{\sqrt{3}} \right) + \arctan \left( \frac{1}{\sqrt{3}} \right) \right] \right).$$

This is the generalized position space eigenstate of the position operator in the presence of both minimal momentum and maximal length. In comparison with the seminal result of Nozari and Etemadi [26] done on momentum space, our result slightly fits with theirs.

However, our goal in this framework is to determine the generalized Fourier transform corresponding to the position representation (52). The situation is that, the appearance of the minimal momentum given by equation (32) leads to a loss of the notion of localized momentum states since we cannot probe the momentum space with a resolution less than the minimal momentum. So, to treat this problem in a realistic manner, we are forced to introduce the maximal momentum localization states that let information on momentum space accessible.

Now we consider the maximal localization states denoted by $|\psi_\gamma^{\text{max}}\rangle$ defined as states localized around a momentum $\gamma$, such that we have

$$\langle \psi_\gamma^{\text{max}} | \hat{P}_y | \psi_\gamma^{\text{max}} \rangle = \gamma$$

and are solutions of the following equation:

$$\left( \hat{P}_y - \langle \hat{P}_y \rangle + \frac{\langle [\hat{Y}, \hat{P}_y] \rangle}{2\Delta Y^2} (\hat{Y} - \langle \hat{Y} \rangle) \right) |\psi_\gamma^{\text{max}}\rangle = 0.$$
Using equations (37) and (38), the differential equation in position space corresponding to (54) is in the following form

\[
-i\hbar(1-\tau y_s + \tau^2 y_s^2)\partial_{y_s} - (\hat{P}_y) + i\hbar \frac{1-\tau(\hat{Y}) + \tau^2 \Delta Y^2 + \tau^2(\hat{Y})^2}{2\Delta Y^2} (y_s - \langle \hat{Y} \rangle) \times \psi^\max(y_s) = 0. \tag{55}
\]

The solution to this equation is given by

\[
\psi^\max(y_s) = \Psi e^{-i\frac{\pi}{12}\left[\frac{8}{3\sqrt{3}} + \frac{\sqrt{3}}{i}\right] y_s (1-\tau y_s + \tau^2 y_s^2) \frac{1-\tau(\hat{Y}) + \tau^2 \Delta Y^2 + \tau^2(\hat{Y})^2}{4\tau^2 \Delta Y^2} + i\hbar \frac{1-\tau(\hat{Y}) + \tau^2 \Delta Y^2 + \tau^2(\hat{Y})^2}{2\Delta Y^2} (y_s - \langle \hat{Y} \rangle)} \tag{56}
\]

where

\[
\Psi = \psi^\max(0)(1-\tau y_s + \tau^2 y_s^2) \frac{1-\tau(\hat{Y}) + \tau^2 \Delta Y^2 + \tau^2(\hat{Y})^2}{4\tau^2 \Delta Y^2}. \tag{57}
\]

The states of absolutely maximal momentum localization are those with \(\langle \hat{P}_y \rangle = \gamma, \langle \hat{Y} \rangle = 0\) and if we restrict these states to the ones for which \(\Delta Y = \frac{1}{\tau}\), we obtain

\[
\psi^\max(y_s) = \psi^\max(0)(1-\tau y_s + \tau^2 y_s^2) \frac{1-\tau(\hat{Y}) + \tau^2 \Delta Y^2 + \tau^2(\hat{Y})^2}{4\tau^2 \Delta Y^2} \times e^{i\frac{\pi}{12}\left[\frac{8}{3\sqrt{3}} + \frac{\sqrt{3}}{i}\right] y_s (1-\tau y_s + \tau^2 y_s^2)} \tag{58}
\]

To determine \(\psi^\max(0)\), we normalize to unity, \(\langle \psi^\max|\psi^\max \rangle = 1\), we find

\[
1 = \int_{-\Delta Y}^{\Delta Y} dy_s \frac{1}{(1-\tau y_s + \tau^2 y_s^2)} \psi^\max(y_s) \psi^\max(y_s)
= \psi^\max(0) \int_{-\Delta Y}^{\Delta Y} dy_s e^{i\frac{\pi}{12}\left[\frac{8}{3\sqrt{3}} + \frac{\sqrt{3}}{i}\right] y_s (1-\tau y_s + \tau^2 y_s^2)}, \tag{59}
\]

which gives

\[
\psi^\max(0) = A^{-1/2} \times \left[ B (e^{\alpha_1} + e^{\alpha_2}) + C (e^{\alpha_2} \mathcal{F}^3 - e^{\alpha_1} \mathcal{F}^2) + \sqrt{2} (e^{-\alpha_2} - e^{\alpha_2} \mathcal{F}^3 - e^{\alpha_1} \mathcal{F}^2) \right]^{-1/2}, \tag{60}
\]

where

\[
A = \frac{\sqrt{3}}{2\tau(i\sqrt{2} - 2)}, \quad B = \frac{i}{\sqrt{3}(2i + \sqrt{2})}, \quad C = (2i + \sqrt{2}), \tag{61}
\]

\[
\alpha_1 = -\frac{\sqrt{2}}{3}, \quad \alpha_2 = \frac{\sqrt{2}}{6}, \quad \mathcal{F}^1 = 2 F_1(1, -\frac{i}{\sqrt{2}}, 1 - \frac{i}{\sqrt{2}}, e^{i\frac{\pi}{2}}), \tag{62}
\]

\[
\mathcal{F}^2 = 2 F_1(1, -\frac{i}{\sqrt{2}}, 1 - \frac{i}{\sqrt{2}}, e^{-i\frac{\pi}{2}}), \quad \mathcal{F}^3 = 2 F_1(1, -\frac{i}{\sqrt{2}}, -i, e^{i\frac{\pi}{2}}), \tag{63}
\]

\[
\mathcal{F}^4 = 2 F_1(1, -\frac{i}{\sqrt{2}}, 2 - \frac{i}{\sqrt{2}}, -e^{-i\frac{\pi}{2}}). \tag{64}
\]

Therefore, the position space wave functions of the states that are maximally localized around a momentum \(\gamma\) are in the following form
\[ \psi^\text{max}_\gamma(y_s) = \psi^\text{max}_0 \sqrt{1 - \tau y_s + \tau^2 y_s^2} e^{\frac{\pi}{\sqrt{3}} \left( \arctan\left( \frac{2\tau y_s}{\sqrt{3}} \right) + \arctan(\frac{\tau}{\sqrt{3}}) \right)} \times e^{\frac{i}{\sqrt{3} \hbar} \left( \arctan\left( \frac{2\tau y_s}{\sqrt{3}} \right) + \arctan(\frac{\tau}{\sqrt{3}}) \right) \phi(y_s)}. \]  

By projecting arbitrary states onto this maximally localized states (86) we recover information about the localization around the momentum. This procedure is known as the concept of quasi representation wavefunction. We take \( |\chi\rangle \) as an arbitrary state, then the probability amplitude on maximal localization states around the momentum \( \gamma \) is \( \langle \psi^\text{max}_\gamma | \chi \rangle = \chi(\gamma) \) namely quasi-momentum wavefunction. Thus, the passing from the position-space wave function into its quasi representation wave function now would be

\[ \chi(\gamma) = \psi^\text{max}_0 \int_{-l_{\text{max}}}^{+l_{\text{max}}} \frac{dy_s}{(1 - \tau y_s + \tau^2 y_s^2)^{\frac{3}{2}}} e^{\frac{i}{\sqrt{3} \hbar} \left( \arctan\left( \frac{2\tau y_s}{\sqrt{3}} \right) + \arctan(\frac{\tau}{\sqrt{3}}) \right) \phi(y_s)} \times e^{-\frac{i}{\sqrt{3} \hbar} \left( \arctan\left( \frac{2\tau y_s}{\sqrt{3}} \right) + \arctan(\frac{\tau}{\sqrt{3}}) \right) \phi(y_s)} \chi(y_s). \]

This transformation that maps position space wave functions into quasi-momentum space wave functions is the generalization of the Fourier transformation. The inverse transformation is given by

\[ \chi(y_s) = \int_{-\infty}^{+\infty} \frac{d\gamma}{(1 - \tau y_s + \tau^2 y_s^2)^{\frac{3}{2}}} e^{-\frac{i}{\sqrt{3} \hbar} \left( \arctan\left( \frac{2\tau y_s}{\sqrt{3}} \right) + \arctan(\frac{\tau}{\sqrt{3}}) \right) \phi(y_s)} \times e^{\frac{i}{\sqrt{3} \hbar} \left( \arctan\left( \frac{2\tau y_s}{\sqrt{3}} \right) + \arctan(\frac{\tau}{\sqrt{3}}) \right) \phi(y_s)} \chi(\gamma). \]

3.2.2. Representation with maximal and minimal lengths.

\( \diamond \) Representation on position space

From the relation \([\hat{X}, \hat{Y}] = i\hbar(1 - \tau \hat{Y} + \tau^2 \hat{Y}^2)\) that predicts maximal and minimal lengths can be defined by the operators

\[ \hat{Y} = \hat{y}_0, \]

\[ \hat{X} = (1 - \tau \hat{y}_0 + \tau^2 \hat{y}_0^2) \hat{x}_0. \]

Using again the asymmetric Bopp-shift \( B_{\theta}^{\hat{y}_0} \) and acting these operators on the wave function \( \psi(y_s) \), we have

\[ \hat{Y}\phi(y_s) = y_s \phi(y_s), \]

\[ \hat{X}\phi(y_s) = (1 - \tau y_s + \tau^2 y_s^2) x_s \phi(y_s) + i\theta \left( 1 - \tau y_s + \tau^2 y_s^2 \right) \partial_{y_s} \phi(y_s). \]

Based on the equation (39), one can state the following proposition:

**Proposition 3.3.** The operator \( \hat{X} \) on the dense domain \( D(\hat{X}) \) is symmetric such as

\[ \langle \psi | \hat{X} \phi \rangle = \langle \hat{X}^\dagger \psi | \phi \rangle, \]
but is not self-adjoint
\[ D(\hat{X}) = \{ \phi, \phi' \in L^2(-l_{\text{max}}, l_{\text{max}}); \phi(-l_{\text{max}}) = \phi'(l_{\text{max}}) = 0 \}, \]
\[ D(\hat{X}^\dagger) = \{ \psi, \psi' \in L^2(-l_{\text{max}}, l_{\text{max}}) \}. \]

\[ \diamond \text{Position eigenfunction} \]

The position operator \( \hat{X} \) acting on the operator \( \hat{Y} \) eigenstates gives
\[ \hat{X}\phi_\lambda(y_s) = \lambda \phi_\lambda(y_s). \]  
By solving the following differential equation
\[ i\theta \left( 1 - \tau y_s + \tau^2 y_s^2 \right) \partial_x \phi_\lambda(y_s) = \left[ \lambda - (1 - \tau y_s + \tau^2 y_s^2) x_s \right] \phi_\lambda(y_s), \]  
we obtain
\[ \phi_\lambda(y_s) = \phi(0) \exp \left( -i \frac{2\lambda}{\tau\theta\sqrt{3}} \left[ \arctan \left( \frac{2\tau y_s - 1}{\sqrt{3}} \right) + \arctan \left( \frac{1}{\sqrt{3}} \right) \right] + i \frac{x_s}{\theta} y_s \right). \]  
Through the normalization of this function, we have
\[ \phi_\lambda(y_s) = \sqrt{\frac{\tau\sqrt{3}}{\pi}} e^{-i \left( \frac{x_s}{\theta} y_s \right) \arctan \left( \frac{2\tau y_s - 1}{\sqrt{3}} \right) + \arctan \left( \frac{1}{\sqrt{3}} \right) - \frac{2\lambda}{\tau\theta\sqrt{3}}}. \]

\[ \diamond \text{Maximal localization} \]

Now we consider \( |\phi_{\eta}^{\text{max}}\rangle \) the states of maximal localization around a position \( \eta \) such as
\[ \langle \phi_{\eta}^{\text{max}} | \hat{X} | \phi_{\eta}^{\text{max}} \rangle = \eta, \]  
and are solution of the equation
\[ \left( \hat{X} - \langle \hat{X} \rangle + \frac{[\hat{X}, \hat{Y}]}{2\Delta Y^2} \left( \hat{Y} - \langle \hat{Y} \rangle \right) \right) |\phi_{\eta}^{\text{max}}\rangle = 0. \]  
Using equations (70) and (71), the differential equation in position space corresponding to (80) is in the following form
\[ \begin{aligned} &\left( 1 - \tau y_s + \tau^2 y_s^2 \right) x_s \phi_{\eta}^{\text{max}}(y_s) \\ \ \ + \left( i\theta \left( 1 - \tau y_s + \tau^2 y_s^2 \right) \partial_x - \langle \hat{X} \rangle + i\theta \frac{1 - \tau(\hat{Y}) + \tau^2(\hat{Y})^2 + \tau^2 \Delta Y^2}{2\Delta Y^2} (y_s - \langle \hat{Y} \rangle) \right) \\ \ \ \times \phi_{\eta}^{\text{max}}(y_s) = 0. \end{aligned} \]  
We obtain the states of maximal localization as follows
\[ \phi_{\eta}^{\text{max}} = \Phi e^{i\psi} e^{-\frac{i}{\tau\sqrt{3}} \left( \frac{x_s}{\Delta Y} \arctan \left( \frac{2\tau y_s - 1}{\sqrt{3}} \right) + \arctan \left( \frac{1}{\sqrt{3}} \right) \right)} \]  
where
\[ \Phi = \phi_{\eta}^{\text{max}}(0) \left( 1 - \tau y_s + \tau^2 y_s^2 \right) - \frac{1 - \tau(\hat{Y}) + \tau^2(\hat{Y})^2 + \tau^2 \Delta Y^2}{\Delta Y^2} \frac{x_s}{\Delta Y}. \]
The states of absolutely maximal localization are those with \( \langle \hat{X} \rangle = \eta, \langle \hat{Y} \rangle = 0 \) and if we restrict these states to the ones for which \( \Delta Y = \frac{1}{\tau} \), we obtain
\[
\phi^\text{max}_\eta = \phi^\text{max}(0) \left( 1 - \tau y_s + \tau^2 y_s^2 \right)^{-\frac{1}{2}} e^{\frac{i}{\hbar} y_s} e^{-\frac{i}{\hbar} \left( \arctan \left( \frac{\eta \sqrt{2} y_s}{\theta} \right) + \arctan \left( \frac{\theta}{\eta} \right) \right)}
\times e^{-i \frac{2}{\pi \tau} \sqrt{\frac{\eta}{\theta}} \left( \arctan \left( \frac{\eta \sqrt{2} y_s}{\theta} \right) + \arctan \left( \frac{\theta}{\eta} \right) \right)}.
\]  
By normalization to unity, \( \langle \phi^\text{max}_\eta | \phi^\text{max}_\eta \rangle = 1 \), we find
\[
\phi^\text{max}(0) = \left( \frac{8 e^{-\frac{2}{\pi \tau} \sqrt{\frac{\eta}{\theta}}}}{21 \tau} - \frac{2 e^{\frac{4}{\pi \tau} \sqrt{\frac{\eta}{\theta}}}}{7 \tau} \right)^{-\frac{1}{4}}.
\]  
Therefore, the position space wave functions of the states that are maximally localized around a momentum \( \eta \) are in the following form
\[
\phi^\text{max}(y_s) = \frac{\phi^\text{max}(0)}{\left( 1 - \tau y_s + \tau^2 y_s^2 \right)^{1/2}} e^{\frac{i}{\hbar} y_s} e^{-\frac{i}{\hbar} \left( \arctan \left( \frac{\eta \sqrt{2} y_s}{\theta} \right) + \arctan \left( \frac{\theta}{\eta} \right) \right)}
\times e^{-i \frac{2}{\pi \tau} \sqrt{\frac{\eta}{\theta}} \left( \arctan \left( \frac{\eta \sqrt{2} y_s}{\theta} \right) + \arctan \left( \frac{\theta}{\eta} \right) \right)} \rho(y_s).
\]  
\[\diamond\] The generalization of the Fourier transformation and its inverse

The generalized Fourier transform obtained from the passing of the position-space wave function into quasi representation wave function \( \langle \phi^\text{max}_\eta | \rho \rangle = \rho(\eta) \) is given by
\[
\rho(\eta) = \phi^\text{max}(0) \int_{-\infty}^{+\infty} dy_s \frac{(1 - \tau y_s + \tau^2 y_s^2)^{1/2}}{\pi \theta \phi^\text{max}(0)} e^{\frac{i}{\hbar} y_s} e^{-\frac{i}{\hbar} \left( \arctan \left( \frac{\eta \sqrt{2} y_s}{\theta} \right) + \arctan \left( \frac{\theta}{\eta} \right) \right)} \rho(y_s)
\]  
and the inverse transformation is given by
\[
\rho(y_s) = \int_{-\infty}^{+\infty} dy \frac{(1 - \tau y_s + \tau^2 y_s^2)^{1/2}}{\pi \theta \phi^\text{max}(0)} e^{\frac{i}{\hbar} y_s} e^{-\frac{i}{\hbar} \left( \arctan \left( \frac{\eta \sqrt{2} y_s}{\theta} \right) + \arctan \left( \frac{\theta}{\eta} \right) \right)} \rho(\eta).
\]  
3.3. Decoupled and reduction into commutative space

Another possibility of representation of wave functions is to decouple directly the set of operators \( \hat{X}, \hat{Y}, \hat{P}_x, \hat{P}_y \) in terms of operators \( \hat{x}_s, \hat{y}_s, \hat{p}_{sx}, \hat{p}_{sy} \) using the transformations \( R_e \) and \( B'_e \). We find
\[
\hat{X} = \hat{x}_s - \frac{\theta}{2 \hbar} \hat{p}_{sx} - \tau \hat{y}_s \hat{x}_s + \frac{\tau \theta}{2 \hbar} (\hat{y}_s \hat{p}_{sx} - \hat{p}_{sy} \hat{x}_s) + \frac{\tau \theta^2}{4 \hbar^2} \hat{p}_{sx} \hat{p}_{sy} + \tau^2 \hat{y}_s^2 \hat{x}_s + \frac{\theta^2 \tau^2}{2 \hbar^2} (\hat{y}_s^2 \hat{P}_x - \hat{P}_y \hat{x}_s) + \frac{\theta^2 \tau^2}{4 \hbar^2} (\hat{P}_x^2 \hat{x}_s - 2 \hat{y}_s \hat{p}_{sx} \hat{p}_{sy}),
\]
\[
\hat{Y} = \hat{y}_s + \frac{\theta}{2 \hbar} \hat{p}_{sy}.
\]
\[ \hat{P}_x = \hat{p}_x, \]  
\[ \hat{P}_y = \hat{p}_y - \frac{\tau \theta}{2\hbar} \hat{p}_x \hat{p}_y - \frac{\tau^2 \theta}{\hbar} \hat{y}_s \hat{p}_x \hat{p}_y + \frac{\tau^2 \theta^2}{4\hbar^2} \hat{p}_x^2 \hat{p}_y. \]  
\[ \hat{P}_z = \hat{p}_z, \quad \hat{P}_\tau = \hat{p}_\tau - \tau \hat{y}_s \hat{p}_\tau. \]

From these representations, one still observes that the operators \( \hat{X} \) and \( \hat{P}_y \) are no longer Hermitian and symmetric in the space in which the operators \( \hat{x}_s, \hat{y}_s, \hat{p}_x, \hat{p}_y \) are Hermitian. An immediate consequence is that Hamiltonian of models formulated in terms of these operators will in general also be neither Hermitian, neither symmetric. However, to guarantee the symmetry of \( \hat{X} \) and \( \hat{P}_y \), we proceed by approximations in a first order of parameters \( \theta \) and \( \tau \) that we assume very small. Therefore we obtain through the approximations of these operators an effective noncommutative space given by

\[ \hat{X} = \hat{x}_s - \frac{\theta}{2\hbar} \hat{p}_y \hat{x}_s, \quad \hat{Y} = \hat{y}_s + \frac{\theta}{2\hbar} \hat{p}_x, \]
\[ \hat{P}_x = \hat{p}_x, \quad \hat{P}_y = \hat{p}_y - \tau \hat{y}_s \hat{p}_y. \]

It is easy to verify that these operators are symmetric except the operator \( \hat{P}_y \) that one needs to symmetrize in order to guarantee the complete symmetry of this space.

**Proposition 3.4.** For the given completeness relation

\[ \int_{-\infty}^{+\infty} \frac{dx \, dy}{(1 - \tau y)^c} |x, y\rangle \langle x, y| = I, \]

with \( |x_s, y_s\rangle \) elements of the domain of \( \hat{P}_y \) maximally dense in \( \mathcal{L}^2(\mathbb{R}^2) \), we have

\[ \hat{P}_y = \hat{P}_{y}^{-1}. \]

The realizations (89)–(92) and especially (93) connected to \( \kappa \)-like realisations and to the deformed Heisenberg algebra [27–31].

From the actions of operators (93) on the wave function \( \psi(x_s, y_s) \), we can thus obtain the following differential representations

\[ \hat{X} \psi(x_s, y_s) = (x_s + i\theta/2\partial_{y_s} - \tau y_s x_s) \psi(x_s, y_s), \]
\[ \hat{Y} \psi(x_s, y_s) = (y_s - i\theta/2\partial_{x_s}) \psi(x_s, y_s), \]
\[ \hat{P}_x \psi(x_s, y_s) = -i\hbar \partial_{x_s} \psi(x_s, y_s), \]
\[ \hat{P}_y \psi(x_s, y_s) = -i\hbar (1 - \tau y_s) \partial_{y_s} \psi(x_s, y_s), \]

and the corresponding maximal domains

\[ \mathcal{D}(\hat{X}) = \{ \psi(x_s, y_s) \in \mathcal{L}^2(\mathbb{R}^2) : (x_s + i\theta/2\partial_{y_s} - \tau y_s x_s) \psi(x_s, y_s) \in \mathcal{L}^2(\mathbb{R}^2) \}, \]
\[ \mathcal{D}(\hat{Y}) = \{ \psi(x_s, y_s) \in \mathcal{L}^2(\mathbb{R}^2) : (y_s - i\theta/2\partial_{x_s}) \psi(x_s, y_s) \in \mathcal{L}^2(\mathbb{R}^2) \}, \]
\[ \mathcal{D}(\hat{P}_x) = \{ \psi(x_s, y_s) \in \mathcal{L}^2(\mathbb{R}^2) : -i\hbar \partial_{x_s} \psi(x_s, y_s) \in \mathcal{L}^2(\mathbb{R}^2) \}, \]
\[ \mathcal{D}(\hat{P}_y) = \{ \psi(x_s, y_s) \in \mathcal{L}^2(\mathbb{R}^2) : -i\hbar (1 - \tau y_s) \partial_{y_s} \psi(x_s, y_s) \in \mathcal{L}^2(\mathbb{R}^2) \}. \]
\[ \mathcal{D}(\hat{P}_y) = \{ \psi(x_s, y_s) \in \mathcal{L}^2(\mathbb{R}^2) : -i\hbar (1 - \tau y_s) \partial_{y_s} \psi(x_s, y_s) \in \mathcal{L}^2(\mathbb{R}^2) \}. \]  

(103)

From the solutions of the above differential equations, one can straightforwardly deduce the corresponding Fourier transforms. We leave this part to the reader to determine these transformations.

Notice that the set of deformed operators (93) is less restrictive than the representation (23) because the latter leads to the minimal uncertainty measurements while the representation (93) does not present any ambiguity in the meaning of wavefunction. It now depends on our choice to treat models in the representation of preference. In what follows, we use the representation (93) to illustrate the study of some simple models in quantum mechanics.

4. Models in position dependent noncommutative space

The models of interest are the free particle, the particle in a box and the harmonic oscillator. We start by formulating them in terms of operators \( \hat{X}, \hat{Y}, \hat{P}_x, \hat{P}_y \) and then determine how to solve the Schrödinger equation exactly or perturbately. Now, let consider \( \hat{H} \) the Hamiltonian of a system of mass \( m \) defined as follows

\[ \hat{H}(\hat{P}_x, \hat{P}_y, \hat{X}, \hat{Y}) := \frac{1}{2m}(\hat{P}_x^2 + \hat{P}_y^2) + V(\hat{X}, \hat{Y}), \]  

(104)

where \( V \) is the potential energy of the system. Using the relations (93), this Hamiltonian is decoupled in terms of the following Hamiltonians

\[ \hat{H} = \hat{H}_s + \hat{H}_\theta + \hat{H}_\tau \]  

(105)

where \( \hat{H}_s \) is the non-pertubated Hamiltonian, \( \hat{H}_\theta \) and \( \hat{H}_\tau \) are respectively the \( \tau \)-perturbation and \( \theta \)-perturbation Hamiltonians. Let stress that the Hamiltonians (104) and (105) are just different points of view to describe the same type of physics and in what follows, we will use the form (105) to solve the eigenvalue problems.

4.1. The free particle

The free particle Hamiltonian reads

\[ \hat{H}(\hat{X}, \hat{Y}, \hat{P}_x, \hat{P}_y) = \frac{1}{2m}(\hat{P}_x^2 + \hat{P}_y^2). \]  

(106)

In the form (105), this Hamiltonian reads as

\[ \hat{H}(\hat{x}_s, \hat{y}_s, \hat{p}_x, \hat{p}_y) = \frac{1}{2m}\hat{p}_x^2 + \frac{1}{2m} (1 - \tau y_s) \hat{p}_y^2 + \frac{i\hbar \tau}{2m} (1 - \tau y_s) \hat{p}_y. \]  

(107)

Apparently, this Hamiltonian seems at first to be non-Hermitian \( \hat{H} \neq \hat{H}^\dagger \). As we know, this observation is unacceptable partly because it may lead to complex energy spectrum and does not maintain the time evolution of this Hamiltonian unitary. However, it is known in the literature [36, 37] that, a certain class of non-Hermitian Hamiltonians may possess a real energy spectrum. As we will see by solving the Schrödinger equation, the Hamiltonian (107) may possess real energy spectra. Using the relation (94), one can also show that the symmetry of this Hamiltonian is guaranteed.
The Schrödinger equation is given by
\[ \hat{H} \psi(x_s, y_s) = E \psi(x_s, y_s). \] \hspace{1cm} (108)

As it is clearly seen, the system is decoupled and the solution to the eigenvalue equation (108) is given by
\[ \psi(x_s, y_s) = \psi_k(x_s) \psi_n(y_s), \quad E = E_k + E_n \] \hspace{1cm} (109)

where \( \psi_k(x_s) \) is the wave function in the \( x_s \)-direction and \( \psi_n(y_s) \) the wave function in the \( y_s \)-direction. Since the particle is free in the \( x_s \)-direction, the wave function is
\[ \psi_k(x_s) = \int_{-\infty}^{+\infty} dk g(k) e^{ikx_s}, \] \hspace{1cm} (110)

where \( g(k) \) determines the shape of the wave packet and the energy spectrum is continuous \[1, 23\]
\[ E_k = \frac{\hbar^2 k^2}{2m}. \] \hspace{1cm} (111)

In \( y_s \)-direction, we have to solve the following equation
\[ (1 - \tau y_s) \frac{d^2 \psi_n}{dy_s^2} - \tau (1 - \tau y_s) \frac{d \psi_n}{dy_s} + \frac{2m}{\hbar^2} E_n \psi_n = 0. \] \hspace{1cm} (112)

By setting \( (1 - \tau y_s) = e^z \), the above equation is reduced into
\[ \frac{d^2 \psi_n}{dz^2} + \lambda^2 \psi_n = 0. \] \hspace{1cm} (113)

This equation is the equation of free harmonic oscillations with \( \lambda^2 = \frac{2m}{\tau^2 \hbar^2} E_n \) the frequency of oscillation. The solution is given by
\[ \psi_n(y_s) = A \sin(\lambda z) + B \cos(\lambda z) = A \sin \left[ \lambda \ln(1 - \tau y_s) \right] + B \cos \left[ \lambda \ln(1 - \tau y_s) \right], \] \hspace{1cm} (114)

where \( A, B \) are constants and \( \tau \) is considered very smaller than one. If we assume that, the frequency of oscillation is quantized such as \( \lambda = 2\pi n \) with \( n \in \mathbb{N}^* \), therefore the eigenvalue \( E_n \) is given by
\[ E_n = \frac{2\pi^2 \tau^2 \hbar^2}{m} n^2. \] \hspace{1cm} (115)

As it is clearly obtained, the presence of the deformed parameter \( \tau \) in \( y_s \)-direction converted the free particle into harmonic oscillator. This fact comes to confirm the fundamental property of gravity which consists of contracting the matter.

4.2. Particle in a box

We consider the above free particle of mass \( m \) captured in a two-dimensional box of length \( a \) and height \( b \). The boundaries of the box are located at \( 0 \leq x_s \leq a \) and \( 0 \leq y_s \leq b \). The above Hamiltonian (107) is rewritten as follows
To solve the eigenvalue equation, we may resort to the perturbation theory to obtain some useful insight on the solutions. Thus, the eigenvalues and eigenfunctions of $H_2$ are given by [34]

$$E_i = \frac{\hbar^2 \pi^2}{2m} \left[ \frac{n_{x_i}^2}{a^2} + \frac{n_{y_i}^2}{b^2} \right],$$

and

$$\psi_i(x_i, y_i) = \frac{2}{\sqrt{ab}} \sin \left( \frac{n_{x_i} \pi x_i}{a} \right) \sin \left( \frac{n_{y_i} \pi y_i}{b} \right).$$

$n_{x_i}, n_{y_i} \in \mathbb{N}^*$ and $ab$ is just the area of the box. The wave functions satisfy the Dirichlet condition i.e. it vanishes at the boundaries $\psi_i(0) = \psi(a) = 0$ and $\psi_i(0) = \psi_i(b) = 0$.

Now, for the sake of simplicity we restrict the Hamiltonian $H_2$ to first order of the parameter $\tau$ which is given by

$$H_\tau = -\frac{\tau}{2m} \left( 2y_\tau \hat{p}_{y_\tau}^2 - i\hbar \hat{p}_{y_\tau} \right) + O(\tau).$$

Using the perturbation theory, we determine the effect of $E_\tau$ on the energy eigenvalues

$$E_\tau = \langle \psi_i | H_\tau | \psi_i \rangle = \frac{\hbar^2}{2m} \int_0^a \int_0^b \psi_i^*(x, y) \left( 2y_\tau \hat{p}_{y_\tau}^2 + \partial_{y_\tau} \right) \psi_i(x, y) dx dy = -\frac{\hbar^2 \pi^2 n_{y_\tau}^2}{2mb}.$$}

Comparing the $\tau$-corrections to the unperturbed energy term in the case where $a = b = L$ and $n_{x_\tau} = n_{y_\tau} = n$, we get

$$\frac{|E_\tau|}{E_i} = \frac{L}{2}.$$}

4.3. The harmonic oscillator

The Hamiltonian of a two dimensional harmonic oscillator is given by

$$\hat{H} = \frac{1}{2m} (\hat{p}_x^2 + \hat{p}_y^2) + \frac{1}{2} m\omega^2 (\hat{x}^2 + \hat{y}^2).$$

Using the representation (93), the corresponding Hamiltonian reads

$$\hat{H} = \begin{cases}
\hat{H}_0 = \frac{1}{2m} (\hat{p}_x^2 + \hat{p}_y^2) + \frac{m\omega^2}{2} (\hat{x}^2 + \hat{y}^2) \\
\hat{H}_\tau = -\frac{\tau}{2m} \hat{P}_x \\
\hat{H}_\theta = -\frac{m\omega^2}{2b^2} \hat{P}_y \\
\hat{H}_{\tau \theta} = 0
\end{cases}$$

where $\hat{L}_z = (\hat{x}\hat{p}_y - \hat{y}\hat{p}_x)$ is the angular momentum. It is important to remark that the $\theta$-perturbation introduced a dynamical $SO(2)$ rotations in the plane. Since $[\hat{H}_\tau, \hat{H}_\theta] = 0$, to determine
the corresponding basis which can diagonalize simultaneously these operators, we consider
the helicity Fock algebra generators as follows
\begin{equation}
a_{\pm} = \frac{m\omega}{2\hbar \sqrt{2}} \left[ (\hat{x}_s \pm i\hat{y}_s) + \frac{i}{m\omega} (\hat{p}_x \mp i\hat{p}_y) \right],
\end{equation}
\begin{equation}
a_{\pm}^\dagger = \frac{m\omega}{2\hbar \sqrt{2}} \left[ (\hat{x}_s \pm i\hat{y}_s) - \frac{i}{m\omega} (\hat{p}_x \mp i\hat{p}_y) \right],
\end{equation}
which satisfy
\begin{equation}
[a_{\pm}, a_{\pm}^\dagger] = I, \quad [a_{\pm}, a_{\mp}^\dagger] = 0.
\end{equation}
The associated orthonormalized helicity basis $|\psi_{n_+, n_-}\rangle$ are defined as follows
\begin{equation}
|\psi_{n_+, n_-}\rangle = \frac{1}{\sqrt{n_+! n_-!}} \left( a_{+}^\dagger \right)^{n_+} \left( a_{-}^\dagger \right)^{n_-} |\psi_0\rangle \quad \text{and}
\end{equation}
\begin{equation}
\langle \psi_{m_+, m_-} | \psi_{n_+, n_-}\rangle = \delta_{m_+, n_+} \delta_{m_-, n_-}, \quad \sum_{n_\pm = 0}^{+\infty} |\psi_{n_+, n_-}\rangle \langle \psi_{n_+, n_-}| = I.
\end{equation}
The action of these operators reads as
\begin{equation}
a_{\pm} |\psi_{n_\pm}\rangle = \sqrt{n_{\pm} |\psi_{n_{\pm}-1}\rangle},
\end{equation}
\begin{equation}
a_{\pm}^\dagger |\psi_{n_\pm}\rangle = \sqrt{n_{\pm} + 1} |\psi_{n_{\pm}+1}\rangle,
\end{equation}
\begin{equation}
a_{\pm}^\dagger a_{\pm} |\psi_{n_\pm}\rangle = n_{\pm} |\psi_{n_\pm}\rangle.
\end{equation}
Conversely, we have
\begin{equation}
\hat{x}_s = \frac{1}{2} \sqrt{\frac{\hbar}{m\omega}} \left[ a_+ + a_- - a_{+}^\dagger + a_{-}^\dagger \right], \quad \hat{y}_s = \frac{i}{2} \sqrt{\frac{\hbar}{m\omega}} \left[ a_+ - a_- - a_{+}^\dagger + a_{-}^\dagger \right],
\end{equation}
\begin{equation}
\hat{p}_x = -i \frac{m\omega}{2} \sqrt{\frac{\hbar}{m\omega}} \left[ a_+ + a_- - a_{+}^\dagger - a_{-}^\dagger \right],
\end{equation}
\begin{equation}
\hat{p}_y = \frac{m\omega}{2} \sqrt{\frac{\hbar}{m\omega}} \left[ a_+ - a_- + a_{+}^\dagger - a_{-}^\dagger \right].
\end{equation}
At first order of the parameters $\theta$ and $\tau$, the Hamiltonian is reduced into
\begin{equation}
\hat{H} = \hat{H}_s + \hat{H}_\theta + \hat{H}_\tau + O(\tau) + O(\theta)
\end{equation}
The energy eigenvalues for the Hamiltonian $\hat{H}_s$ and for the pertubated Hamiltonian $\hat{H}_\theta$ and $\hat{H}_\tau$ read as follows
\begin{equation}
E_s = \hbar \omega (n_+ + n_- + 1), \quad E_\theta = \frac{m\omega^2 \theta}{2\hbar} (n_- - n_+), \quad E_\tau = 0.
\end{equation}
These results show that, for the case $E_\tau = 0$, there is no contribution in $\tau$-deformed energy spectrum. To improve this result we look at the second order in $\tau$-perturbation, namely
\[ E_{\tau^2} = \sum_{k \neq n}^{\infty} \frac{\langle \psi_{n+} | [\hat{H}, \psi_{k+}] | \psi_{k+} \rangle \langle \psi_{k+} | [\hat{H}, \psi_{n+}] | \psi_{n+} \rangle}{E_{n+}^0 - E_{k+}^0}. \] (136)

For the sake of simplicity, this energy at the ground states \( n_{\pm} = 0 \) is evaluated at
\[
E_{\tau^2} = \frac{\tau^2}{4m^2} \left( \frac{5m\hbar^2}{12} + 0 + \frac{17m\hbar^2}{48} \right) = \frac{37\hbar}{384m} \tau^2. \] (137)

5. Conclusion remarks

We have introduced a new version of position dependent noncommutative space-time in two dimensional configuration spaces. This space-time that we provided, generalizes the set of noncommutative space-time recently introduced by Fring et al [1]. To construct this noncommutative space-time (22), we have considered the most used deformed commutative space-time (14) in such a way that at the limit \( \tau \to 0 \) we recovered this algebra (14). The interesting physical consequence we found is that, this noncommutative space-time leads to minimal and maximal lengths for simultaneous measurement in \( X, Y \)-directions. Then for a simultaneous measurement in \( Y, P_y \)-directions, this space also leads to a minimal momentum and a maximal length. The existence of this maximal length, which is the basic difference to the work of Fring et al, is related to the presence of an extra, first order term in particle’s length. It brings a lot of new features in the representation of this noncommutation space. Moreover, to escape the difficulties from dealing with this representation due to the presence of the minimal uncertainties, we propose another representation of operators obtained by approximations in first order of parameters \( \theta \) and \( \tau \). In this new representation, we provided the spectra of some fundamental quantum systems such as the free particle, the particle in a box and the Harmonic oscillator.

It is well known that the presence of both minimal length and minimal momentum raised the question of singularity of the space-time i.e the space is inevitably bounded by minimal quantities beyond which any further localization of particle is not possible [4]. With Fring et al noncommutative space-time, it is shown that any object in this space will be string like i.e a measurement in \( \hat{X} \) and \( \hat{Y} \) spaces leads to a minimal length for \( \hat{X} \) or for \( \hat{Y} \) when informations are given-up in one direction. In comparison with this work, my version of noncommutative space-time introduces a singularity in \( X \)-direction and a broken singularity in \( \hat{Y} \)-direction for simultaneous measurement in both directions. This means that, the lost of localization of particle in \( X \)-direction can be maximally recorved in \( Y \)-direction. Furthermore the singularity in momentum \( P_y \)-direction leads to the maximal localization in \( \hat{Y} \)-direction for a simultaneous measurement in both directions. If indeed the quantum gravity induces a minimal measurement length of Planck order which requires extremely high energies, the emergence of maximal length in this theory could bring down this high energies to possibly energies currently accessible through accelerators. Without attempting to draw deep philosophical predictions, this theory of maximal length could break up the big bang singularity which predicts the existence of the multiverse [38, 39]. Let us point out that these predictions stand for some ideas that could be realized up and could be matter of real experimentation in the laboratory with specific technical skills.

Moreover, from the representation \( \mathcal{R}_\tau \) which generates the algebra (22), follows immediately that some operators are no longer Hermitian in the space in which the operators \( \hat{x}_0, \hat{y}_0, \hat{p}_{\hat{x}_0}, \hat{p}_{\hat{y}_0} \) are Hermitian. In order to introduce the deformed complemenetess relation or to
use the approximation method to recover the symmetry of these operators, we may try to find a similarity transformation, i.e. a Dyson map [40] to restor the symmetry of these operators as was considered in the elegant paper of Fring and his colleagues [1]. This situation is currently under investigation and is the goal of my next work. Finally, referring to Fring et al’s work and this one, the position dependent noncommutative space-time can be generalized as

\[ [\hat{X}, \hat{Y}] = i hf(\hat{Y}), \quad [\hat{X}, \hat{P}_x] = i hf(\hat{Y}), \quad [\hat{Y}, \hat{P}_x] = i hf(\hat{Y}), \quad (138) \]

where \( f \) is called function of deformation and we assume that it is strictly positive (\( f > 0 \)). Based on these equations, one can ask the question: for what function of deformation \( f \) there exists nonzero minimal uncertainties or maximal uncertainties?

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**Appendix. Jacoby Identities**

In this appendix, we prove all the possible Jacoby identities of the proposition 3.1

\[
[ [\hat{X}, \hat{Y}], \hat{X} ] + [ [\hat{Y}, \hat{X}], \hat{X} ] + [ [\hat{X}, \hat{X}], \hat{Y} ] = 0, \quad (A.1)
\]

\[
[ [\hat{X}, \hat{Y}], \hat{Y} ] + [ [\hat{Y}, \hat{Y}], \hat{X} ] + [ [\hat{Y}, \hat{X}], \hat{Y} ] = 0, \quad (A.2)
\]

\[
[ [\hat{X}, \hat{Y}], \hat{P}_x ] + [ [\hat{Y}, \hat{P}_x], \hat{X} ] + [ [\hat{P}_x, \hat{X}], \hat{Y} ] = 0 \quad (A.3)
\]

\[
[ [\hat{X}, \hat{Y}], \hat{P}_x ] + [ [\hat{Y}, \hat{P}_x], \hat{X} ] + [ [\hat{P}_x, \hat{X}], \hat{Y} ] = 0 \quad (A.4)
\]

\[
[ [\hat{X}, \hat{P}_x], \hat{Y} ] + [ [\hat{P}_x, \hat{Y}], \hat{X} ] + [ [\hat{Y}, \hat{X}], \hat{P}_x ] = 0 \quad (A.5)
\]

\[
[ [\hat{X}, \hat{P}_x], \hat{X} ] + [ [\hat{P}_x, \hat{X}], \hat{X} ] + [ [\hat{X}, \hat{X}], \hat{P}_x ] = 0 \quad (A.6)
\]

\[
[ [\hat{X}, \hat{P}_x], \hat{P}_x ] + [ [\hat{P}_x, \hat{P}_x], \hat{X} ] + [ [\hat{P}_x, \hat{X}], \hat{P}_x ] = 0 \quad (A.7)
\]

\[
[ [\hat{X}, \hat{P}_x], \hat{P}_x ] + [ [\hat{P}_x, \hat{P}_x], \hat{X} ] + [ [\hat{P}_x, \hat{X}], \hat{P}_x ] = 0 \quad (A.8)
\]

\[
[ [\hat{Y}, \hat{P}_x], \hat{X} ] + [ [\hat{P}_x, \hat{X}], \hat{Y} ] + [ [\hat{X}, \hat{Y}], \hat{P}_x ] = 0 \quad (A.9)
\]

\[
[ [\hat{Y}, \hat{P}_x], \hat{Y} ] + [ [\hat{P}_x, \hat{Y}], \hat{Y} ] + [ [\hat{Y}, \hat{Y}], \hat{P}_x ] = 0 \quad (A.10)
\]

\[
[ [\hat{Y}, \hat{P}_x], \hat{P}_x ] + [ [\hat{P}_x, \hat{P}_x], \hat{Y} ] + [ [\hat{P}_x, \hat{Y}], \hat{P}_x ] = 0 \quad (A.11)
\]

\[
[ [\hat{Y}, \hat{P}_x], \hat{P}_x ] + [ [\hat{P}_x, \hat{P}_x], \hat{Y} ] + [ [\hat{P}_x, \hat{Y}], \hat{P}_x ] = 0 \quad (A.12)
\]

\[
[ [\hat{P}_x, \hat{P}_x], \hat{X} ] + [ [\hat{P}_x, \hat{X}], \hat{P}_x ] + [ [\hat{X}, \hat{P}_x], \hat{P}_x ] = 0 \quad (A.13)
\]
\[
\begin{align*}
[[\hat{P}_x, \hat{P}_y], \hat{Y}] + [[\hat{P}_y, \hat{Y}], \hat{P}_x] + [[\hat{Y}, \hat{P}_x], \hat{P}_y] &= 0 \\
[[\hat{P}_x, \hat{P}_y], \hat{P}_x] + [[\hat{P}_y, \hat{P}_y], \hat{P}_x] + [[\hat{P}_x, \hat{P}_x], \hat{P}_y] &= 0 \\
[[\hat{P}_x, \hat{P}_y], \hat{P}_x] + [[\hat{P}_y, \hat{P}_y], \hat{P}_x] + [[\hat{P}_x, \hat{P}_x], \hat{P}_y] &= 0 \\
[[\hat{Y}, \hat{P}_x], \hat{X}] + [[\hat{P}_x, \hat{X}], \hat{P}_y] + [[\hat{X}, \hat{P}_y], \hat{P}_x] &= 0 \\
[[\hat{Y}, \hat{P}_x], \hat{X}] + [[\hat{P}_x, \hat{X}], \hat{P}_y] + [[\hat{X}, \hat{P}_y], \hat{P}_x] &= 0, \\
[[\hat{Y}, \hat{P}_x], \hat{P}_y] + [[\hat{P}_x, \hat{P}_y], \hat{Y}] + [[\hat{P}_x, \hat{Y}], \hat{P}_x] &= 0, \\
[[\hat{X}, \hat{P}_y], \hat{X}] + [[\hat{P}_y, \hat{X}], \hat{X}] + [[\hat{X}, \hat{X}], \hat{P}_y] &= 0 \\
[[\hat{X}, \hat{P}_y], \hat{Y}] + [[\hat{P}_y, \hat{Y}], \hat{X}] + [[\hat{Y}, \hat{X}], \hat{P}_y] &= 0 \\
[[\hat{X}, \hat{P}_y], \hat{P}_x] + [[\hat{P}_y, \hat{P}_x], \hat{X}] + [[\hat{P}_x, \hat{X}], \hat{P}_y] &= 0 \\
[[\hat{X}, \hat{P}_y], \hat{P}_x] + [[\hat{P}_y, \hat{P}_x], \hat{X}] + [[\hat{P}_x, \hat{X}], \hat{P}_y] &= 0 \\
\end{align*}
\]

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