Some Properties of an Infinite Family of Deformations of the Harmonic Oscillator

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Abstract

In memory of Marcos Moshinsky, who promoted the algebraic study of the harmonic oscillator, some results recently obtained on an infinite family of deformations of such a system are reviewed. This set, which was introduced by Tremblay, Turbiner, and Winternitz, consists in some Hamiltonians $H_k$ on the plane, depending on a positive real parameter $k$. Two algebraic extensions of $H_k$ are described. The first one, based on the elements of the dihedral group $D_{2k}$ and a Dunkl operator formalism, provides a convenient tool to prove the superintegrability of $H_k$ for odd integer $k$. The second one, employing two pairs of fermionic operators, leads to a supersymmetric extension of $H_k$ of the same kind as the familiar Freedman and Mende super-Calogero model. Some connection between both extensions is also outlined.

Keywords: quantum Hamiltonians; superintegrability; exchange operators; supersymmetry

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1 INTRODUCTION

During his long and prolific career, the study of the harmonic oscillator has been one of the prominent topics dealt with by Marcos Moshinsky [1, 2], with whom I have enjoyed the privilege of collaborating for many years including on this subject [3, 4, 5]. As a token of reminiscence and gratitude for all what I learnt from him, in this paper dedicated to his memory I will review some recent results on a related problem.

In [6], Tremblay, Turbiner and Winternitz (TTW) indeed introduced an infinite family

\[ H_k = -\frac{1}{r} \frac{\partial^2}{\partial r^2} - \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2} + \omega^2 r^2 + \frac{k^2}{r^2} \left[ a(a-1) \sec^2 k \varphi + b(b-1) \csc^2 k \varphi \right], \]

\[ 0 \leq r < \infty, \quad 0 \leq \varphi < \frac{\pi}{2k}, \]

which may be considered as deformations of the harmonic oscillator on a plane and reduce for \( k = 1, 2, 3 \) to those of the familiar Smorodinsky-Winternitz [7, 8], BC2 model [9] and Calogero-Marchioro-Wolfes [10, 11] systems. They showed that for any real \( k \), \( H_k \) is integrable with \( X_k = -\frac{\partial^2}{\partial \varphi^2} + k^2 [a(a-1) \sec^2 k \varphi + b(b-1) \csc^2 k \varphi] \) an integral of motion. They also conjectured (and actually proved for \( k = 1, 2, 3, 4 \)) that it is superintegrable for any integer \( k \), the second integral of motion \( Y_{2k} \) being some 2\( k \)th-order differential operator. Later on, this was established by myself for odd \( k \) [12], then by Kahlins, Kress and Miller for integer (or even rational) \( k \) [13].

In the following, two extensions of \( H_k \) will be considered. The first one [14], based on the elements of the dihedral group \( D_{2k} \), was used in the superintegrability proof of Ref. [12] while the second one [15], employing two pairs of fermionic operators, led to a supersymmetric extension of the same kind as the familiar Freedman and Mende super-Calogero model [16]. Some connection between both extensions [17] will also be reviewed.

2 DIHEDRAL GROUP EXTENSION

The dihedral group \( D_{2k} \) has \( 4k \) elements \( R^i \) and \( I^i \), \( i = 0, 1, \ldots, 2k - 1 \), satisfying the relations

\[ R^{2k} = I^2 = 1, \quad IR = R^{2k-1} I, \quad R^\dagger = R^{2k-1}, \quad I^\dagger = I. \]
They are realizable on the Euclidean plane as the rotation operator through angle $\pi/k$, $R = \exp \left( \frac{\pi}{k} \partial_\varphi \right)$, and the operator changing $\varphi$ into $-\varphi$, $I = \exp(i\pi \varphi \partial_\varphi)$ [14].

With their use, one can extend the partial derivatives $\partial_r$ and $\partial_\varphi$ into some differential-difference operators,

$$D_r = \partial_r - \frac{1}{r} (aR + b) \left( \sum_{i=0}^{k-1} R^{2i} \right) I,$$

$$D_\varphi = \partial_\varphi + a \sum_{i=0}^{k-1} \tan \left( \varphi + \frac{i\pi}{k} \right) R^{k+2i} I - b \sum_{i=0}^{k-1} \cot \left( \varphi + \frac{i\pi}{k} \right) R^{2i} I,$$  \hspace{0.5cm} (3)

similar to Dunkl operators [18]. They satisfy more complicated relations than $\partial_r$ and $\partial_\varphi$, namely

$$D_r^\dagger = -D_r - \frac{1}{r} \left[ 1 + 2(aR + b) \left( \sum_{i=0}^{k-1} R^{2i} \right) \right] I, \quad D_\varphi^\dagger = -D_\varphi,$$

$$RD_r = D_r R, \quad ID_r = D_r I, \quad RD_\varphi = D_\varphi R, \quad ID_\varphi = -D_\varphi I,$$  \hspace{0.5cm} (4)

$$[D_r, D_\varphi] = -\frac{2}{r} (aR + b) \left( \sum_{i=0}^{k-1} R^{2i} \right) ID_\varphi.$$

In (3), $k$ may be any odd integer. For even $k$, $D_r$ and $D_\varphi$ assume a different form, which we will not consider here.

The TTW Hamiltonian $H_k$ and its integral of motion $X_k$ can be generalized by incorporating the elements of $D_{2k}$, as well as the new operators $D_r$ and $D_\varphi$. The resulting $D_{2k}$-extended operators

$$H_k = -D_r^2 - \frac{1}{r} \left[ 1 + 2(aR + b) \left( \sum_{i=0}^{k-1} R^{2i} \right) I \right] D_r - \frac{1}{r^2} D_\varphi^2 + \omega^2 r^2$$

$$= -\partial_r^2 - \frac{1}{r} \partial_r - \frac{1}{r^2} \left[ D_r^2 - k(a^2 + b^2 + 2abR) \sum_{i=0}^{k-1} R^{2i} \right] + \omega^2 r^2$$  \hspace{0.5cm} (5)

and

$$X_k = -D_\varphi^2 = -\partial_\varphi^2 + \sum_{i=0}^{k-1} \sec^2 \left( \varphi + \frac{i\pi}{k} \right) a(a - R^{k+2i} I)$$

$$+ \sum_{i=0}^{k-1} \csc^2 \left( \varphi + \frac{i\pi}{k} \right) b(b - R^{2i} I) - k(a^2 + b^2 + 2abR) \sum_{i=0}^{k-1} R^{2i}$$  \hspace{0.5cm} (6)

turn out to be invariant under $D_{2k}$ and to give back $H_k$ and $X_k$ after projection in the $D_{2k}$ identity representation, i.e., by letting $R$ and $I$ go to 1.
2.1 Application to the superintegrability problem of $H_k$ for odd $k$

The formalism considered above can still be enlarged by introducing a set of $k$ (dependent) pairs of modified boson creation and annihilation operators

\[ A_i = \frac{1}{\sqrt{2\omega}} \left[ \cos \left( \varphi + \frac{i\pi}{k} \right) (\omega r + D_r) - \frac{1}{r} \sin \left( \varphi + \frac{i\pi}{k} \right) D_\varphi \right], \]

\[ A_i^\dagger = \frac{1}{\sqrt{2\omega}} \left[ \cos \left( \varphi + \frac{i\pi}{k} \right) (\omega r - D_r) + \frac{1}{r} \sin \left( \varphi + \frac{i\pi}{k} \right) D_\varphi \right], \]

where $i = 0, 1, \ldots, k - 1$. One can show that $A_i^\dagger$ is the Hermitian conjugate of $A_i$ and that the $2k$ operators $A_i$ and $A_i^\dagger$ satisfy the modified commutations relations

\[ [A_i, A_j] = [A_i^\dagger, A_j^\dagger] = 0, \]

\[ [A_i, A_j^\dagger] = [A_j, A_i^\dagger] = \cos \left( \frac{j - i}{k} \pi \right) + 2a \sum_l \cos \left( \frac{l - i}{k} \pi \right) \cos \left( \frac{l - j}{k} \pi \right) R^{k+2l} \]

\[ + 2b \sum_l \sin \left( \frac{l - i}{k} \pi \right) \sin \left( \frac{l - j}{k} \pi \right) R^{2l}, \]

for $i, j = 0, 1, \ldots, k - 1$. Here all summations over $l$ run from 0 to $k - 1$. The $A_i$ also fulfil the exchange relations

\[ R A_i R^{-1} = A_{i+1}, \quad i = 0, 1, \ldots, k - 2, \quad R A_{k-1} R^{-1} = -A_0, \]

\[ I A_0 I^{-1} = A_0, \quad I A_i I^{-1} = -A_{k-i}, \quad i = 1, 2, \ldots, k - 1. \]

These modified boson operators can be used to define $k$ modified oscillator Hamiltonians

\[ H_i = \frac{1}{2} \{ A_i^\dagger, A_i \}, \quad i = 0, 1, \ldots, k - 1, \]

which transform among themselves under $D_{2k}$:

\[ R H_i R^{-1} = H_{i+1}, \quad i = 0, 1, \ldots, k - 2, \quad R H_{k-1} R^{-1} = H_0, \]

\[ I H_0 I^{-1} = H_0, \quad I H_i I^{-1} = H_{k-i}, \quad i = 1, 2, \ldots, k - 1. \]
From the explicit expressions of the $H_i$'s, namely

$$
2 \omega H_i = -\cos^2 \left( \varphi + \frac{i\pi}{k} \right) D_r^2 + \frac{1}{r} \sin \left( \varphi + \frac{i\pi}{k} \right) \cos \left( \varphi + \frac{i\pi}{k} \right) (D_r D_\varphi + D_\varphi D_r)
$$

$$
- \frac{1}{r^2} \sin^2 \left( \varphi + \frac{i\pi}{k} \right) D_\varphi^2
$$

$$
- \frac{1}{r} \left[ \sin^2 \left( \varphi + \frac{i\pi}{k} \right) + 2a \sum_l \cos^2 \left( \frac{l - i}{k} \pi \right) R^{k+2l} + 2b \sum_l \sin^2 \left( \frac{l - i}{k} \pi \right) R^{2l} \right] D_r
$$

$$
+ \frac{1}{r^2} \left[ -2 \sin \left( \varphi + \frac{i\pi}{k} \right) \cos \left( \varphi + \frac{i\pi}{k} \right) - 2a \sum_l \sin \left( \frac{l - i}{k} \pi \right) \cos \left( \frac{l - i}{k} \pi \right) R^{k+2l} \right] D_\varphi
$$

$$
+ 2b \sum_l \sin \left( \frac{l - i}{k} \pi \right) \cos \left( \frac{l - i}{k} \pi \right) R^{2l} \right] D_\varphi + \omega^2 r^2 \cos^2 \left( \varphi + \frac{i\pi}{k} \right),
$$

it can be shown that such operators are connected with the $D_{2k}$-extended Hamiltonian through the relation

$$
2 \omega \sum_{i=0}^{k-1} H_i = \frac{k}{2} H_k.
$$

Hence $H_k$ may be considered as a modified boson oscillator Hamiltonian.

Next, it can be proved that all the $H_i$'s commute with $H_k$ and are therefore integrals of motion for the latter. From them, one can form two $D_{2k}$ invariants, namely their sum proportional to $H_k$ and their symmetrized product

$$
\mathcal{Y}_{2k} = (2\omega)^k \sum_p H_p(0) H_p(1) \cdots H_p(k-1),
$$

where the summation runs over all $k!$ permutations of 0, 1, ..., $k - 1$.

Projection in the $D_{2k}$ identity representation then leads to $H_k$, on one hand, and to an integral of motion $Y_{2k}$ of the latter, on the other hand. From its definition, it is clear that $Y_{2k}$ is a differential operator of order $2k$. It is also functionally independent of $X_k$ because it can be established that one of the highest-order terms in $[X_k, Y_{2k}]$ does not vanish and remains nonvanishing after projection in the $D_{2k}$ identity representation, thereby proving that $[X_k, Y_{2k}] \neq 0$.

We conclude that for any odd $k$, the operators $X_k$ and $Y_{2k}$ provide us with a set of two functionally independent integrals of motion of $H_k$, which is therefore superintegrable as claimed in the TTW conjecture.
3 \( \mathcal{N} = 2 \) supersymmetric extension

By using two independent pairs of fermionic creation and annihilation operators \((b^\dagger_x, b_x)\) and \((b^\dagger_y, b_y)\), the TTW Hamiltonian \(H_k\) can be extended into a supersymmetric Hamiltonian \(H^s\) \cite{15}. This can be carried out in the framework of an \(osp(2/2, \mathbb{R})\) superalgebra with even generators \(K_0, K_\pm, Y\) (closing the \(sp(2, \mathbb{R}) \times so(2)\) Lie algebra) and odd generators \(V_\pm, W_\pm\) (which are two \(sp(2, \mathbb{R})\) spinors). The corresponding (nonvanishing) commutation or anticommutation relations and Hermiticity properties are given by

\[
\begin{align*}
[K_0, K_\pm] &= \pm K_\pm, & [K_+, K_-] &= -2K_0, \\
[K_0, V_\pm] &= \pm \frac{1}{2} V_\pm, & [K_0, W_\pm] &= \pm \frac{1}{2} W_\pm, \\
[K_\pm, V_\mp] &= \mp V_\pm, & [K_\pm, W_\mp] &= \mp W_\pm, \\
[Y, V_\mp] &= \frac{1}{2} V_\pm, & [Y, W_\mp] &= -\frac{1}{2} W_\pm,
\end{align*}
\]

\(\{V_\pm, W_\mp\} = K_\pm, \quad \{V_\pm, W_\mp\} = K_0 \mp Y\) (15)

and

\[
\begin{align*}
K^\dagger_0 &= K_0, & K^\dagger_\pm &= K_\mp, & Y^\dagger &= Y, & V^\dagger_\mp &= W_\pm,
\end{align*}
\]

(16)

respectively.

Standard supersymmetric quantum mechanics \cite{19}, with supersymmetric Hamiltonian \(H^s\) and supercharges \(Q, Q^\dagger\) such that

\[
[H^s, Q] = [H^s, Q^\dagger] = 0, \quad \{Q, Q^\dagger\} = H^s,
\]

is realized by the three operators

\[
H^s = 4\omega(K_0 + Y), \quad Q = 2\sqrt{\omega} W_+, \quad Q^\dagger = 2\sqrt{\omega} V_-
\]

(18)

generating a \(sl(1/1)\) subsuperalgebra of \(osp(2/2, \mathbb{R})\).

To get simpler expressions of the generators, it is useful to introduce new ‘rotated’ fermionic operators defined by

\[
\bar{b}^\dagger_x = b^\dagger_x \cos \varphi + b^\dagger_y \sin \varphi, \quad \bar{b}^\dagger_y = -b^\dagger_x \sin \varphi + b^\dagger_y \cos \varphi
\]

(19)

and similarly for \(\bar{b}_x, \bar{b}_y\). Such a transformation, however, breaks the commutativity of bosonic and fermionic degrees of freedom (e.g., \([\partial_x, \bar{b}_x] = \bar{b}_y\)).
The resulting even and odd \( osp(2/2, \mathbb{R}) \) generators can be expressed as

\[
K_0 = K_{0,B} + \Gamma, \quad K_\pm = K_{\pm,B} - \Gamma,
\]

\[
K_{0,B} = \frac{1}{4\omega} H_k, \quad K_{\pm,B} = \frac{1}{4\omega} [-H_k + 2\omega^2 r^2 \mp 2\omega (r \partial_r + 1)]
\]

\[
\Gamma = \frac{k}{2\omega r^2} \{ a [\tilde{b}^+_x \tilde{b}^+_y - \tan k \varphi (\tilde{b}^+_x \tilde{b}^+_y + \tilde{b}^+_y \tilde{b}^-_x)] + (k \sec^2 k \varphi - 1) \tilde{b}^+_y \tilde{b}^-_y \}
\]

\[
Y = \frac{1}{2} [\tilde{b}^+_x \tilde{b}^-_y - k(a + b) - 1]
\]

and

\[
V_\pm = \frac{1}{2\sqrt{\omega}} \left[ \tilde{b}^+_x \left( \mp \partial_r + \omega r \pm \frac{k(a + b)}{r} \right) \mp \tilde{b}^+_y \frac{1}{r} (\partial_\varphi + ka \tan k \varphi - kb \cot k \varphi) \right],
\]

\[
W_\pm = \frac{1}{2\sqrt{\omega}} \left[ \tilde{b}^+_x \left( \mp \partial_r + \omega r \mp \frac{k(a + b)}{r} \right) \mp \tilde{b}^+_y \frac{1}{r} (\partial_\varphi - ka \tan k \varphi + kb \cot k \varphi) \right],
\]

respectively.

On starting from the wavefunctions of the TTW Hamiltonian \( H_k \)

\[
\Psi_{N,n}(r, \varphi) = \mathcal{N}_{N,n} Z_N^{(2n+a+b)}(z) \Phi_n^{(a,b)}(\varphi),
\]

\[
Z_N^{(2n+a+b)}(z) = \left( \frac{z}{\omega} \right)^{(n+a+b)/2} L_N^{(2n+a+b)}(z) e^{-\frac{1}{2}z}, \quad z = \omega r^2,
\]

\[
\Phi_n^{(a,b)}(\varphi) = \cos^a k \varphi \sin^b k \varphi P_n^{(a-\frac{1}{2}, b-\frac{1}{2})}(\xi), \quad \xi = -\cos 2k \varphi,
\]

\( N, n = 0, 1, 2, \ldots \),

defined in terms of Laguerre and Jacobi polynomials and such that

\[
H_k \Psi_{N,n}(r, \varphi) = E_{N,n} \Psi_{N,n}(r, \varphi), \quad E_{N,n} = 2\omega [2N + (2n + a + b)k + 1],
\]

\[
\int_0^\infty dr r \int_0^{\pi/(2k)} d\varphi |\Psi_{N,n}(r, \varphi)|^2 = 1,
\]

one gets eigenstates of the supersymmetrized TTW Hamiltonian \( \mathcal{H}^* \) after multiplication by the fermionic vacuum state \( |0\rangle \). The corresponding eigenvalues are \( E_{N,n} = E_{N,n} - E_{0,0} = 4\omega (N + nk) \). Such extended states are also eigenstates of the \( osp(2/2, \mathbb{R}) \) weight generators \( K_0 \) and \( Y \), corresponding to the eigenvalues

\[
\tau = \left( n + \frac{a + b}{2} \right) k + \frac{1}{2}, \quad q = -\frac{1}{2} [(a + b)k + 1],
\]

respectively.
For each value of $n \in \{0, 1, 2, \ldots\}$ (specifying the angular wavefunctions of $H_k$ as well as the eigenvalues of the first integral of motion $X_k$), it is possible to construct an $osp(2/2, \mathbb{R})$ irreducible representation (irrep) characterized by $(\tau, q)$. Its nature, however, depends on the value assumed by $n$.

For $n = 0$, one obtains a lowest-weight state (LWS) irrep based on the extended ground state $\Psi_{0,0}(r, \varphi)|0\rangle$. This state is indeed annihilated by all the lowering generators $K_-, V_-, W_-$ of $osp(2/2, \mathbb{R})$. The irrep is a so-called atypical one with $\tau = -q$ (which means that the vanishing Casimir operators $C_2$ and $C_3$ cannot specify the irrep). It contains only two $sp(2, \mathbb{R}) \times so(2)$ irreps: $(\tau)(q)$ and $(\tau + \frac{1}{2})(q + \frac{1}{2})$ (spanned by zero- and one-fermion states, respectively).

For any $n \neq 0$, one gets an $osp(2/2, \mathbb{R})$ irrep containing four $sp(2, \mathbb{R}) \times so(2)$ irreps: $(\tau)(q)$, $(\tau - \frac{1}{2})(q + \frac{1}{2})$, $(\tau + \frac{1}{2})(q + \frac{1}{2})$, and $(\tau)(q+1)$. The first one is spanned by zero-fermion states, the next two by a mixture of one-fermion states and the last one by two-fermion states. No state is annihilated by all the $osp(2/2, \mathbb{R})$ lowering generators.

The eigenvalues of the two Casimir operators of $osp(2/2, \mathbb{R})$ are given by

$$C_2 \rightarrow n(n + a + b)k^2, \quad C_3 \rightarrow -\frac{1}{2}(a + b)n(n + a + b)k^2,$$

which proves the above-mentioned result for $n = 0$.

The supersymmetric extension presented in this section is valid for any real value of $k$. For the special cases of $k = 1, 2, 3$, it gives back some known results related to the super-Calogero model [16] and to the supersymmetrization of other Calogero-like systems [20].

## 4 CONNECTION BETWEEN BOTH EXTENSIONS

To start with, it is possible to realize the elements $R^i$ and $R^iL$, $i = 0, 1, \ldots, 2k - 1$, of the dihedral group $D_{2k}$ in terms of two independent pairs of fermionic operators $(b_x^i, b_x^i)$ and
$$(b_y^\dagger, b_y) \text{[17].}$$ On starting from the definitions

$$R \equiv 1 + \left( \cos \frac{\pi}{k} - 1 \right) (b_x^\dagger b_x + b_y^\dagger b_y) + \sin \frac{\pi}{k} (b_x^\dagger b_y - b_y^\dagger b_x) + 2 \left( 1 - \cos \frac{\pi}{k} \right) b_x^\dagger b_x b_y^\dagger b_y, \tag{26}$$

$$I \equiv 1 - 2b_y^\dagger b_y = -[b_y^\dagger, b_y],$$

one can indeed show that for any $i = 0, 1, \ldots, 2k - 1$

$$R^i = 1 + \left( \cos \frac{i\pi}{k} - 1 \right) (b_x^\dagger b_x + b_y^\dagger b_y) + \sin \frac{i\pi}{k} (b_x^\dagger b_y - b_y^\dagger b_x) + 2 \left( 1 - \cos \frac{i\pi}{k} \right) b_x^\dagger b_x b_y^\dagger b_y, \tag{27}$$

and that such operators satisfy all defining relations of $D_{2k}$.

The next step consists in making the substitution (27) in the $D_{2k}$-extended TTW Hamiltonian, given in (5). As a result, the latter is mapped onto the difference between the supersymmetric TTW Hamiltonian $H_s$ and its purely fermionic term $4\omega Y$ provided the trigonometric identities

$$\sum_{i=0}^{k-1} \tan \left( \varphi + \frac{i\pi}{k} \right) \cos \frac{2i\pi}{k} = -k \frac{\sin[(k-2)\varphi]}{\cos k\varphi},$$

$$\sum_{i=0}^{k-1} \tan \left( \varphi + \frac{i\pi}{k} \right) \sin \frac{2i\pi}{k} = k \frac{\cos[(k-2)\varphi]}{\cos k\varphi} - \delta_{k,1} \tag{28}$$

are satisfied. A simple proof of these relations has been found, thereby establishing a connection between the $D_{2k}$ and the supersymmetric extensions of the TTW Hamiltonian.

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