Linear Convergence Rate Analysis of a Class of Exact First-Order Distributed Methods for Time-Varying Directed Networks and Uncoordinated Step Sizes

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Abstract
We analyze a class of exact distributed first order methods under a general setting on the underlying network and step-sizes. In more detail, we allow simultaneously for time-varying uncoordinated step sizes and time-varying directed weight-balanced networks, jointly connected over bounded intervals. The analyzed class of methods subsumes several existing algorithms like the unified Extra and unified DIGing (Jakovetic, 2019), or the exact spectral gradient method (Jakovetic, Krejic, Krklec Jerinkic, 2019) that have been analyzed before under more restrictive assumptions. Under the assumed setting, we establish R-linear convergence of the methods and present several implications that our results have on the literature. Most notably, we show that the unification strategy in (Jakovetic, 2019) and the spectral step-size selection strategy in (Jakovetic, Krejic, Krklec Jerinkic, 2019) exhibit a high degree of robustness to uncoordinated time-varying step sizes and to time-varying networks.

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1 Introduction

We consider a set of \( n \) computational agents and the following unconstrained optimization problem

\[
\min_{\mathbb{R}^d} f(y) \quad f(y) = \sum_{j=1}^{n} f_j(y) \tag{1}
\]

where each \( f_j \) is a real-valued function of \( \mathbb{R}^d \) and is held privately by one of the agents, and the agents can communicate according to a given network. Problems of this form arise in many practical applications such as sensor networks [11], distributed control [12], distributed learning [3] and many others. Several distributed methods [15, 18, 27, 13, 20, 9, 6, 7, 8] have been proposed in literature for the solution of (1) that achieve exact convergence to the minimizer with fixed step-size, when the objective function is convex and Lipschitz-differentiable. In [15] and [18] two exact gradient-based methods where proposed in literature for the solution of (1) that achieve exact convergence to the minimizer with fixed step-size, when the objective function is convex and Lipschitz-differentiable. In [15] and [18] two exact gradient-based methods where proposed, and the convergence was proved for the case where the underlying network is undirected, connected, and remains constant through the entire execution of the algorithm. In [9] a unified analysis of a class of first-order distributed methods is presented. In [20] the convergence of several first-order methods was generalized to the case of a time-varying network, provided that the network is connected at each iteration, while in [13] the convergence analysis of [15] is extended to the time-varying and directed case, assuming joint-connectivity of the sequence of networks and weight-balance of each graph\(^1\). In [16] an accelerated gradient-based method for the time-varying directed case is proposed, with weaker assumptions over the underlying networks. In [5] the authors considered the problem of minimizing \( f(y) + G(y) \) over a closed and convex set \( K \), where \( f \) is a possibly non-convex function as in (1) and \( G \) is a convex non-separable term, and they propose a gradient-tracking method that achieves convergence in the case

\(^1\)That is, we assume that it is possible to define a doubly stochastic consensus matrix associated with each of the networks.
of time-varying directed jointly-connected networks for diminishing synchronized step sizes. In [17] the method proposed in [5] is extended with constant step-sizes to a more general framework while in [19] $R$-linear convergence is proved for [17] with strongly convex $f(y)$. A unifying framework of these methods is presented in [26] and, for the case of constant and undirected networks, in [1]. In all the above methods the sequence of the step-sizes is assumed to be fixed and coordinated among all the agents. In [14], [27], [24], [25], and [28] the case of uncoordinated time-constant step sizes is considered, that is, each node has a different step-size but these step sizes are constant in all iterations. In [10] a modification of [15] is proposed, with step-sizes varying both across nodes and iterations, and it is proved that there exist suitable safeguards for the steps, depending on the regularity properties of the objective function and the network, such that $R$-linear convergence of the generated sequence to the solution of (1) holds. This results is obtained for undirected and stationary network. In [21] and [22] asynchronous modifications of [17] are proposed.

In this paper, we establish $R$-linear convergence of a class of exact distributed first-order methods under the general setting of time-varying directed weight-balanced networks, without the requirement of network connectedness at each iteration, and time-varying uncoordinated step sizes. While there have been several existing studies of exact distributed methods under general settings, our study implies several new contributions to the literature; these contributions cannot be derived from existing works and are novelties of this paper.

- We prove that the methods proposed in [9], referred to here (and also in [20]) as the unified Extra and the unified DIGing are robust to time-varying directed networks and time-varying uncoordinated step sizes, i.e., they converge $R$-linearly in this setting. Up to now, it is only known that these methods converge under static undirected networks [9] or time-varying networks where the network is connected at each iteration [20]. These methods have been previously considered only for time-invariant coordinated step sizes.

- We prove that the method proposed in [10] is robust to time-varying directed networks. Before the current paper, the method was only known to converge for static, undirected networks.
• It is shown in [20] that the Extra method [18] may diverge over time-varying networks, even when the network is connected at every iteration. On the other hand, as we show here, the unified Extra, a variant of Extra proposed in [9], is robust to time-varying networks. Hence, our results reveal that the unified Extra can be considered as a mean to modify Extra and make it robust.

• We provide a thorough numerical study and an analytical study for a special problem structure that demonstrates that the unification strategy in [9] and the spectral gradient-like step-size selection strategy in [10] exhibit a high degree of robustness to time-varying networks and uncoordinated time-varying step-sizes. More precisely, we show that these strategies converge, when working on time-varying networks, for wider step-size ranges than commonly used strategies such as constant coordinated step-sizes and DIGing algorithmic forms. In addition, we show by simulation that actually a combination of the unification and the spectral step-size strategies further improves robustness.

Technically, while considering weight-balanced digraphs instead of undirected graphs does not lead to a significant analysis difference, major technical differences here with respect to prior work correspond to the analysis of the unification strategy [9] under time-varying networks and time-and-node-varying step-sizes and spectral strategies [10] under time-varying networks.

This paper is organized as follows. In Section 2 we describe the computational framework that we consider and we present the methods that we analyse. In Section 3 we recall a few preliminary results from the literature and we prove a convergence theorem for the methods introduced in Section 2. In Section 4, we show analytically and by simulation that the unification and spectral step-size selection strategies increase robustness of the methods to time-varying networks and uncoordinated step-sizes. Finally, in Section 5, we conclude the paper and outline some future research directions.

2 The Model and the Class of Considered Methods

We make the following regularity assumptions for the local cost functions \( f_i \).
Assumption A1.

- Each function \( f_i : \mathbb{R}^d \rightarrow \mathbb{R}, \ i = 1, \ldots, n, \) is twice continuously differentiable;
- There exists \( 0 \leq \mu_i \leq L_i \) such that for every \( i = 1, \ldots, n \) and every \( y \in \mathbb{R}^d, \)
  \[
  \mu_i I \preceq \nabla^2 f_i(y) \preceq L_i I \tag{2}
  \]

where we write \( A \preceq B \) if the matrix \( B - A \) is positive semi-definite. That is, we assume that each of the local functions is \( \mu_i \)-strongly convex, and has Lipschitz continuous gradient with constant \( L_i \). Denoting with \( L = \sum_{i=1}^{n} L_i \) and \( \mu = \sum_{i=1}^{n} \mu_i \), we have that the aggregate function \( f \) is \( \mu \)-strongly convex and \( \nabla f \) is Lipschitz-continuous with the constant \( L \).

Given \( x_1, \ldots, x_n \in \mathbb{R}^d \) we define

\[
  x := \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^{nd} \quad F(x) = \sum_{j=1}^{n} f_j(x_j). \tag{3}
\]

We denote with \( e \) the vector of length \( n \) with all components equal to 1.

For a matrix \( A \in \mathbb{R}^{n \times n} \) we denote with \( \nu_{\text{max}}(A) \) the largest eigenvalue of \( A \). Moreover, given a sequence of matrices \( \{M^k\}_k \) and \( m \in \mathbb{N} \), let

\[
  M_m^k := M^k M^{k-1} \ldots M^{k-m+1}, \quad M_0^k = I \tag{4}
\]

It is assumed that at iteration \( k \) the \( n \) agents are the nodes of a given network \( G^k = (\{1, \ldots, n\}, E_k) \), where \( E_k \) denotes the set of the edges of the network, and to each \( G^k \) we associate a consensus matrix \( W^k \in \mathbb{R}^{n \times n} \). The assumptions over the sequences \( \{G^k\} \) and \( \{W^k\} \), which are the same hypotheses considered in \[13\], are stated below.

Assumption A2.

For every \( k = 0, 1, \ldots, G^k = (\{1, \ldots, n\}, E_k) \) is a directed graph and \( W^k \) is an \( n \times n \) doubly stochastic matrix with \( w_{ij} = 0 \) if \( i \neq j \) and \( (i, j) \notin E_k \). Moreover, there exists a positive integer \( m \) such that \( \sup_{k=0:m} \nu_k < 1 \), where \( \nu_k = \nu_{\text{max}}(W_m^k - \frac{1}{n}ee^T) \).
Remark 2.1. Assumption A2 is weaker than requiring each graph $G^k$ to be connected. For example, it can be proved (see [13]) that in the case of undirected networks, if the sequence is jointly-connected then we can ensure Assumption A2 by taking $W^k$ as the Metropolis matrix, [23], associated with $G^k$.

We consider the following class of methods. Assume that at each iteration node $i$ holds two vectors $x^k_i$ and $u^k_i$ in $\mathbb{R}^d$ and that the global vectors $x^k$, $u^k \in \mathbb{R}^{nd}$, defined as in (3), are updated according to the following rules:

$$\begin{align*}
x^{k+1} &= W^k x^k - D^k (u^k + \nabla F(x^k)) \\
u^{k+1} &= u^k + (W^k - I)(\nabla F(x^k) + u^k - B^k x^k)
\end{align*}$$

where $W^k := (W^k \otimes I) \in \mathbb{R}^{nd \times nd}$, $D^k = \text{diag}(d^k_1 I, \ldots, d^k_n I)$ with $d^k_i$ being the step-size for node $i$ at iteration $k$ and $B^k$ is a symmetric $n \times n$ matrix that respects the sparsity structure of the communication network $G^k$ and such that for every $y \in \mathbb{R}^d$ we have $B^k (1 \otimes y) = c(1 \otimes y)$ for some $c \in \mathbb{R}$. Moreover, we assume that $x^0 \in \mathbb{R}^{nd}$ is an arbitrary vector and $u^0 = 0 \in \mathbb{R}^{nd}$. For $B^k = 0$ and appropriate choice of the step-sizes $d^k_i$ we get the method introduced in [10]. For $D^k = \alpha I$, if $B^k = bI$ or $B^k = bW$ we retrieve the class of methods analyzed in [9] while if $B_k = 0$ we retrieve the DIGing method proposed in [13, 15]. For $D^k = \alpha I$ and $B^k = bW$ with $b = \frac{1}{\alpha}$ we have the EXTRA method [18], but while this method can be described with this choice of the parameters in equation (5), it is not included in the class of methods we consider. Namely, the theoretical analysis that we carry out in Section 3 requires the parameter $b$ to be independent on the step-sizes, thus ruling out the choice $b = \frac{1}{\alpha}$ that yields EXTRA method. This is in line with [20] that shows that EXTRA may not converge in general for time-varying networks.

In our analysis, we consider the case $B^k = bI$ and $B^k = bW^k$ with $b$ non-negative constant and $d_{\min} \leq d^k_i \leq d_{\max}$ for every $k$ and every $j = 1, \ldots, n$ for appropriately chosen safeguards $0 < d_{\min} < d_{\max}$.

A possible choice for uncoordinated and time-varying step-sizes was proposed in [10] where we have $d^k_i = (\sigma^k_i)^{-1}$ with $\sigma^k_i$ given by

$$\sigma^k_i = \mathcal{D}_{\sigma_{\min}, \sigma_{\max}} \left( \frac{(s^k_i)^T y^k_i (s^k_i)^{-1}}{(s^k_i)^T s^k_i} + \sum_{j=1}^{n} w^k_{ij} \left( 1 - \frac{(s^k_i)^T s^k_{ij}}{(s^k_i)^T s^k_i} \right) \right)$$

6
where \( s_{i}^{k-1} = x_{i}^{k} - x_{i}^{k-1} \) and \( y_{i}^{k-1} = \nabla f_{i}(x_{i}^{k}) - \nabla f_{i}(x_{i}^{k-1}) \). Here, \( \mathcal{P}_{U} \) denotes the projection onto the closed set \( U \), \( \sigma_{min} = 1/d_{max} \), and \( \sigma_{max} = 1/d_{min} \). We refer to [10] for details on the derivation and intuition behind this step-size choice. For static networks, this step size choice incurs no communication overhead per iteration; see [10]. However, for time-varying networks, the communication and storage protocol to implement this step size needs to be adapted. One way to ensure at node \( i \) and iteration \( k \) the availability of \( s_{j}^{k-1} \) for \((i,j) \in E^{k}\), is that node \( i \) receives \( s_{j}^{k-1} \) for all \( j \) such that \((i,j) \in E^{k}\). That is, each node \( j \) per iteration additionally broadcasts one \( d \)-dimensional vector \( s_{j}^{k} \) to all its current neighbors. Therefore the method described by equation (5) combines [9] and [10] into a more general method.

3 Convergence Analysis

We now study the convergence of the method described in (5). Specifically, denoting with \( y^{*} \) the solution of (1) and defining

\[
\begin{pmatrix}
  y^{*} \\
  \vdots \\
  y^{*}
\end{pmatrix} \in \mathbb{R}^{nd}
\]

we prove that, if Assumptions A1 and A2 hold, there exist \( 0 < d_{min} < d_{max} \) such that the sequence \( \{x^{k}\} \) generated by (5) converges to \( x^{*} \).

Given a vector \( v \in \mathbb{R}^{nd} \), denote with \( \bar{v} \) the average \( \bar{v} = \sum_{j=1}^{n} v_{j} \in \mathbb{R}^{d} \) and with \( J \) the \( n \times n \) matrix \((I - \frac{1}{n} ee^{T})\), where \( e^{T} = (1, \ldots, 1) \in \mathbb{R}^{n} \). Recalling the definition of \( x^{k} \) and \( u^{k} \) given in (5), we define the following quantities, which will be used further on:

\[
\begin{align*}
\bar{x}^{k} &= x^{k} - e\bar{x}^{k} \in \mathbb{R}^{nd}, \\
\bar{u}^{k} &= u^{k} + \nabla F(x^{*}) \in \mathbb{R}^{nd}, \\
\bar{q}^{k} &= x^{k} - x^{*} = \bar{x}^{k} + e\bar{q}^{k} \in \mathbb{R}^{nd}.
\end{align*}
\]

To simplify the notation, in the rest of the section we assume that \( d = 1 \), but the same results can be proved analogously in the general case.

A few results listed below will be needed for the convergence result presented
in this paper. Since $W^k$ is doubly stochastic, we have that $\frac{1}{n}ee^T(W^k- I) = 0$. Using this equality and the definition of $u^{k+1}$ we get

$$\bar{u}^{k+1} = \frac{1}{n}ee^T\bar{u}^{k+1} = \frac{1}{n}ee^T\bar{u}^k + \frac{1}{n}ee^T(W^k- I)(u^k + \nabla F(x^k) - B^k x^k) = \bar{u}^k$$

and by the initialization $u^0 = 0$, we have that

$$\bar{u}^k = 0. \quad (9)$$

Directly by the definition of $\bar{u}^k$ and (9) we get

$$\frac{1}{n}ee^T\bar{u}^k = \frac{1}{n}ee^T\bar{u}^k + \frac{1}{n}\nabla f(y^*) = 0. \quad (10)$$

From Assumption A1, for every $k$ there exists a matrix $H_k \preceq LI$ such that

$$\nabla F(x^k) - \nabla F(x^*) = H_k (x^k - x^*). \quad (11)$$

Lemma 1. [13] If the matrix sequence $\{W^k\}_k$ satisfies assumption A2, then for every $k \geq m$ we have

$$\|JW^k y\| \leq \nu_k \|Jy\|$$

Lemma 2. [2] If the function $f$ satisfies assumption A1 and $0 < \alpha < \frac{1}{L}$, then

$$\|y - \alpha \nabla f(y) - y^*\| \leq \tau \|y - y^*\|$$

where $\tau = \max\{|1 - \alpha \mu|, |1 - \alpha L|\}$

Following the idea presented in [13], our convergence result relies on the Small Gain Theorem [4], which we now briefly recall. Denote by $a := \{a^k\}$ an infinite sequence of vectors, $a^k \in \mathbb{R}^d$ for $k = 0, 1, \ldots$. For a fixed $\lambda \in (0, 1)$ we define

$$\|a\|^{\lambda, K} = \max_{k=0,1,\ldots,K} \left\{ \frac{1}{\lambda_k} \|a^k\| \right\}$$

$$\|a\|^{\lambda} = \sup_{k \geq 0} \left\{ \frac{1}{\lambda_k} \|a^k\| \right\}.$$
Theorem 1. \[4\]. Let \( a = \{a^k\} \) and \( b = \{b^k\} \) be two vector sequences, with \( a^k, b^k \in \mathbb{R}^d \). If there exists \( \lambda \in (0, 1) \) such that for all \( K = 0, 1, \ldots \), the following inequalities hold
\[
\|a\|^{\lambda,K} \leq \gamma_1 \|b\|^{\lambda,K} + w_1, \\
\|b\|^{\lambda,K} \leq \gamma_2 \|a\|^{\lambda,K} + w_2,
\]
with \( \gamma_1 \cdot \gamma_2 \in [0,1) \), then
\[
\|a\|^{\lambda} \leq \frac{1}{1 - \gamma_1 \gamma_2} (w_1 \gamma_2 + w_2).
\]
and
\[
\lim_{k \to \infty} a^k = 0 \quad \text{R-linearly.}
\]

We will use the following technical Lemma to show that the sequences \( \|\bar{q}^k\| \) and \( \|\tilde{x}^k\| \) satisfy the hypotheses of Theorem 1.

Lemma 3. Given \( b, \mu, L \geq 0, \nu \in (0, 1) \) and \( n, m \in \mathbb{N} \), where we denote with \( \mathbb{N} \) the set of positive integers, there exists \( \lambda \in (0, 1) \) and \( 0 \leq d_{\text{min}} < d_{\text{max}} \) such that the following conditions hold:
\[
1. \ \nu < \lambda^m, \\
2. \ \frac{d_{\text{min}}}{n} < \frac{2}{L}; \\
3. \ 1 - \mu d_{\text{min}} + \Delta L < \lambda; \\
4. \ \gamma \beta_2 < 1; \\
5. \ \beta_3 < 1; \\
6. \ \beta_5 \frac{\lambda}{1 - \beta_3} < 1; \\
7. \ \frac{\beta_4 + \gamma \beta_2}{1 - \gamma \beta_2} \cdot \frac{\beta_3 + \gamma \beta_5}{1 - \beta_3 - \gamma \beta_5} < 1,
\]
where
\[
\gamma = \frac{(b + L)C}{\lambda^m - \nu}, \quad \beta_1 = \frac{Ld_{\text{max}}}{\lambda - 1 + \mu d_{\text{min}} - \Delta L}, \\
\beta_2 = \frac{\Delta}{Ld_{\text{max}}} \beta_1, \quad \beta_3 = \frac{\nu}{\lambda^m} + \beta_4, \\
\beta_4 = L \beta_5, \quad \beta_5 = \frac{Cd_{\text{max}}}{\lambda^m}, \\
\Delta = d_{\text{max}} - d_{\text{min}}, \quad C = \frac{\lambda(1 - \lambda^m)}{1 - \lambda}.
\]
Proof. Take $\lambda^m > \nu$ and $d_{\min} < \frac{2\nu}{L}$ so that $[1]$ and $[2]$ hold. For $d_{\max} > d_{\min}$ and close enough to $d_{\min}$ one can ensure that

$$\frac{d_{\max}}{d_{\min}} < 1 + \frac{\mu}{L} \quad (13)$$

holds. By the previous inequality, we have $1 - \mu d_{\min} + \Delta L < 1$ and therefore, for fixed $d_{\max}$ and $d_{\min}$ we can always take $\lambda \in (0, 1)$ such that $[3]$ is satisfied and $[1]$ still holds. Moreover, we can take $d_{\min}$ arbitrarily small and $d_{\max}$ arbitrarily close to $d_{\min}$ without violating conditions $[1]-[3]$. Notice that $C = \frac{\lambda(1-\lambda^m)}{1-\lambda}$ is an increasing function of $\lambda$.

Let us now consider condition $[4]$ given by

$$\frac{(b + L)C \Delta}{(\lambda^m - \nu)(\lambda - 1 + \mu d_{\min} - \Delta L)} < 1.$$ 

The left hand side expression is an increasing function of $\Delta$ and it is equal to 0 for $\Delta = 0$. Therefore, taking $d_{\max}$ close enough to $d_{\min}$, condition $[4]$ holds.

Condition $[5]$ holds for $d_{\max} < \frac{\lambda m - \nu}{\lambda m L C}$.

Consider now condition $[6],

$$\frac{(b + L)C^2d_{\max}}{(\lambda^m - \nu)(\lambda^m - \nu)(\lambda^m - \nu - Ld_{\max} C)} < 1$$

The left hand side expression is an increasing function of $d_{\max}$ and taking $d_{\max}$ small enough we conclude that the previous inequality holds. Since we need $d_{\max} > d_{\min}$, in order to be able to take $d_{\max}$ small, we need to take $d_{\min}$ small enough, but this can be done without violating the previous conditions.

By definition, $\frac{\beta_2 + \gamma \beta_4}{1 - \gamma \beta_4}$ and $\frac{\beta_6 + \gamma \beta_8}{1 - \beta_4 - \gamma \beta_6}$ are also increasing functions of $d_{\max}$ and $\Delta$. Thus, we can apply the same reasoning that we applied to $[4]$ and $[6]$ to get $\gamma_2 < 1$ and $\gamma_3 < 1$. In particular, we can take $d_{\min}$ and $d_{\max}$ such that condition $[7]$ holds.

\[\square\]

**Theorem 2.** Let $B^k$ be defined as $B^k = bW^k$ or $B^k = bI$ for a positive constant $b$, or $B^k = 0$. If Assumptions A1 and A2 hold then there exists $d_{\min} < d_{\max}$ such that the sequence $\{x^k\}$ generated by (5) converges $R$-linearly to $x^*$. 

10
Proof. Define $\nu = \sup_{k=0:m-1} \nu_k < 1$ where $\nu_k$, $m$ are given in assumption A2, and take $\lambda \in (0,1), 0 \leq d_{min} < d_{max}$ given by Lemma 3. We prove that $n^{1/2}\tilde{q}^k$ and $\tilde{x}^k$ satisfy inequalities (12), thus ensuring $R$-linear convergence by Theorem 1.

We have $B^k = bI$ or $B^k = bW^k$, in both cases, $B^kx^* = bx^*$, therefore $(W^k - I)B^kx^* = 0$ and thus

$$ (W^k - I)B^kx^k = (W^k - I)B^k(x^k - x^*) = (W^k - I)B^kq^k $$

For $k \geq m - 1$, using (5), the previous equality and (11), we get

$$ \tilde{u}^{k+1} = u^{k+1} + \nabla F(x^*) = $$

$$ = u^k + (W^k - I)(u^k + \nabla F(x^k) - B^kx^k) + \nabla F(x^*) = $$

$$ = W^k(u^k + \nabla F(x^*)) + (W^k - I)(\nabla F(x^k) - \nabla F(x^*)) + $$

$$ - (W^k - I)B^kx^k = $$

$$ = W^k\tilde{u}^k + (W^k - I)H_kq^k - (W^k - I)B^kq^k = $$

$$ = W_m^k\tilde{u}^{k-m+1} + \sum_{t=0}^{m-1} W_t^k(W^{k-t} - I)(H_{k-t} - B^{k-t})q^{k-t} $$

By (10) and Lemma 1,

$$ \|W_m^k\tilde{u}^{k-m+1}\| = \|W_m^kJ\tilde{u}^{k-m+1}\| \leq \nu\|J\tilde{u}^{k-m+1}\| = $$

$$ = \nu\|\tilde{u}^{k-m+1}\| $$

and by (11), the definition of $B^k$ and the fact that $W^k$ is doubly stochastic, we get

$$ \|W_t^k(W^{k-t} - I)(H_{k-t} - B^{k-t})q^{k-t}\| \leq (L + b)\|q^{k-t}\|. $$

Taking the norm in (14) and using the two previous inequalities, we have that for $k \geq m - 1$

$$ \|\tilde{u}^{k+1}\| \leq \nu\|\tilde{u}^{k-m+1}\| + (b + L)\sum_{t=0}^{m-1} \|q^{k-t}\|. $$

Notice that the above inequality also holds for the third case considered, i.e. for $B^k = 0$, taking $b = 0$. Multiplying times $\frac{1}{\chi + \tau}$, taking the maximum for
\( k = -1 : \bar{k} - 1 \), and defining
\[
\tilde{\omega}_1 = \max_{k = -1 : m - 1} \left\{ \frac{1}{\lambda^{k+1}} \| \tilde{u}^{k+1} \| \right\}
\]
we get
\[
\| \tilde{u} \|= \lambda^\bar{k} = \max_{k = -1 : m - 1} \left\{ \frac{1}{\lambda^{k+1}} \| \tilde{u}^{k+1} \| \right\}
\leq \frac{\nu}{\lambda^m} \max_{k = m : \bar{k}} \left\{ \frac{1}{\lambda^{k-m+1}} \| \tilde{u}^{k-m+1} \| \right\} + (b + L) \sum_{t=0}^{m-1} \frac{1}{\lambda^t} \max_{k = m : \bar{k}} \left\{ \frac{1}{\lambda^{k-t}} \| q^{k-t} \| \right\} + \tilde{\omega}_1 \leq \frac{\nu}{\lambda^m} \| \tilde{u} \|= \lambda^\bar{k} + (b + L) \frac{\lambda(1 - \lambda^m)}{\lambda m} \| q \| \lambda^\bar{k} + \tilde{\omega}_1.
\]

Since by condition 1. in Lemma 3 we have \( \nu < \lambda^m \), reordering the terms in the previous inequality and using the fact that \( q^k = \tilde{x}^k + e\bar{q}^k \), we get
\[
\| \tilde{u} \|= \lambda^\bar{k} \leq \gamma_1 \| q \|= \lambda^\bar{k} + \omega_1 \leq \gamma_1 \| \tilde{x} \|= \lambda^\bar{k} + \gamma_1 n^{1/2} \| q \|= \lambda^\bar{k} + \omega_1
\]
with
\[
\gamma_1 = \frac{(b + L)\lambda(1 - \lambda^m)}{(1 - \lambda)(\lambda^m - \nu)}, \quad \omega_1 = \frac{\lambda^m}{\lambda^m - \nu} \tilde{\omega}_1.
\]

Let us now consider \( \bar{q}^k \).
\[
\bar{q}^{k+1} = \bar{x}^{k+1} - y^* = \frac{1}{n} e e^T x^{k+1} - y^* = \]
\[
= \frac{1}{n} e e^T (W^k x^k - D^k (u^k + \nabla F(x^k))) - y^* = \]
\[
= \bar{x}^k - y^* - \frac{d_{\min}}{n} \nabla F(\bar{x}^k) + \]
\[
+ \frac{d_{\min}}{n} \sum_{j=1}^{n} (\nabla f_j(\bar{x}^k) - \nabla f_j(y^*)) + \]
\[- \frac{1}{n} \sum_{j=1}^{n} (d_j^k - d_{\min})(\nabla f_j(x_j^k) - \nabla f_j(y^*)) + \]
\[+ \frac{1}{n} \sum_{j=1}^{n} (d_{\min} - d_j^k) \bar{a}_j^k.
\]
Taking the norm, by Lipschitz continuity of the gradient and denoting with $\Delta = d_{max} - d_{min}$, we have
\[
\|\bar{q}^{k+1}\| = \left\| x^k - y^* - \frac{d_{min}}{n} \nabla F(\bar{x}^k) \right\| + \frac{Ld_{min}}{n} \| x^k - y^* \|_1 + \\
+ \frac{L\Delta}{n} \| x^k - x^* \|_1 + \frac{\Delta}{n} \| \bar{a}^k_j \|_1.
\]
Now $\frac{d_{min}}{n} < \frac{2}{L}$, thus Lemma 2 gives a bound for the first term in the right hand side of the last inequality, and we get
\[
n^{1/2}\|\bar{q}^{k+1}\| \leq n^{1/2} \tau \| x^k - y^* \| + Ld_{min} \| x^k - y^* \| + \\
+ L\Delta \| x^k - x^* \| + \Delta \| \bar{a}^k_j \| \leq \\
\leq n^{1/2}(\tau + \Delta L) \| \bar{q}^k \| + Ld_{max} \| \bar{x}^k \| + \Delta \| \bar{u}^k \|.
\]
Multiplying with $\frac{1}{\lambda}$ and taking the maximum for $k = -1 : \bar{k} - 1$ we get
\[
n^{1/2}\|\bar{q}\|^{\lambda \bar{k}} \leq \frac{\tau}{\lambda} n^{1/2} \| \bar{q} \|^{\lambda \bar{k}} + \frac{Ld_{max}}{\lambda} \| \bar{x} \|^{\lambda \bar{k}} + \frac{\Delta}{\lambda} \| \bar{u} \|^{\lambda \bar{k}}.
\]
By Lemma 3 we have $\tau = 1 - \mu d_{min}$ and $\tau + \Delta L < \lambda$, thus reordering and using (15), we get
\[
n^{1/2}\|\bar{q}\|^{\lambda \bar{k}} \leq + \frac{Ld_{max}}{\lambda - \tau - \Delta L} \| \bar{x} \|^{\lambda \bar{k}} + \frac{\Delta}{\lambda - \tau - \Delta L} \| \bar{u} \|^{\lambda \bar{k}} \leq \\
\leq (\beta_1 + \gamma_1 \beta_2) \| \bar{x} \|^{\lambda \bar{k}} + \gamma_1 \beta_2 n^{1/2} \| \bar{q} \|^{\lambda \bar{k}} + \beta_2 \omega_1
\]
where $\beta_1$ and $\beta_2$ are defined in Lemma 3. Take
\[
\gamma_2 = \frac{\beta_1 + \gamma_1 \beta_2}{1 - \gamma_1 \beta_2}, \quad \omega_2 = \frac{\beta_2 \omega_1}{1 - \gamma_1 \beta_2}.
\]
From 4. in Lemma 3 we get
\[
n^{1/2}\|\bar{q}\|^{\lambda \bar{k}} \leq \gamma_2 \| \bar{x} \|^{\lambda \bar{k}} + \omega_2.
\] (16)

Finally, let us consider $\bar{x}^k$. For $k \geq m - 1$, we have
\[
\bar{x}^{k+1} = J(W^k x^k - D^k (u^k + \nabla F(x^k)) = \\
= JW^k W^{k-1} x^{k-1} - JW^k D^{k-1} (u^{k-1} + \nabla F(x^{k-1})) - \\
- JD^k (u^k + \nabla F(x^k)) = \\
= JW_{m+1}^{k-m+1} - J \sum_{t=0}^{m-1} W_t^k D^{k-t} (u^{k-t} + H_{k-t} \bar{q}^{k-t}).
\]
Taking the norm, applying Lemma 1 and (11), we get
\[ \|\tilde{x}^{k+1}\| \leq \nu \|\tilde{x}^{k-m+1}\| + d_{\text{max}} \sum_{t=0}^{m-1} (\|\tilde{u}^{k-t}\| + L\|q^{k-t}\|). \]

Multiplying with $\frac{1}{\lambda_{k+1}}$ and taking the maximum for $k = -1 : \bar{k} - 1$ we get
\[ \|\tilde{x}\|_{\lambda_{\bar{k}}} \leq \nu \lambda_{m} \|\tilde{x}^{k-m+1}\| + d_{\text{max}} \lambda_{m}(1 - \lambda) \|\tilde{u}\|_{\lambda_{k}} + \lambda(1 - \lambda_{m}) \|q\|_{\lambda_{k}} \tilde{\omega}_{3} \leq \beta_{3} \|\tilde{x}^{k-m+1}\| + \beta_{4} n^{1/2} \|\tilde{q}\|_{\lambda_{k}} + \beta_{5} \|\tilde{u}\|_{\lambda_{k}} \tilde{\omega}_{3}. \]

where $\tilde{\omega}_{3} = \max_{k=-1:m-1} \{ \frac{1}{\lambda_{k+1}} \|\tilde{x}^{k+1}\| \}$ and $\beta_{3}, \beta_{4}, \beta_{5}$ are defined in Lemma 3. In particular, we have $\beta_{3} < 1$, and can rearrange the terms of the previous inequality to get
\[ \|\tilde{x}\|_{\lambda_{k}} \leq \beta_{4} n^{1/2} \|\tilde{q}\|_{\lambda_{k}} + \beta_{5} \|\tilde{u}\|_{\lambda_{k}} + \tilde{\omega}_{3}. \]

Now, applying $|15|$ and 6. from Lemma 3 we obtain
\[ \|\tilde{x}\|_{\lambda_{k}} \leq \gamma_{3} n^{1/2} \|\tilde{q}\|_{\lambda_{k}} + \omega_{3} \]

with
\[ \gamma_{3} = \frac{\beta_{4} + \beta_{5} \gamma_{1}}{1 - \beta_{3} - \gamma_{1} \beta_{5}}, \quad \omega_{3} = \frac{\tilde{\omega}_{3} + \beta_{5} \omega_{1}}{1 - \beta_{3} - \gamma_{1} \beta_{5}}. \]

We thus proved
\[ n^{1/2} \|\tilde{q}\|_{\lambda_{k}} \leq \gamma_{2} \|\tilde{x}\|_{\lambda_{k}} + \omega_{2} \]
\[ \|\tilde{x}\|_{\lambda_{k}} \leq \gamma_{3} n^{1/2} \|\tilde{q}\|_{\lambda_{k}} + \omega_{3} \]

with $\gamma_{2} \gamma_{3} < 1$ by condition 7. in Lemma 3. By the Small Gain Theorem, we have that $\|q^{k}\|$ and $\|\tilde{x}^{k}\|$ converge to 0, and thus $\|q^{k}\|$ converges to zeros, which gives the thesis.
4 Analytical and Numerical Studies of Robustness of the Methods

Theorem 2 and Lemma 3 ensure convergence of the considered class of methods. Namely, they establish existence of bounds $d_{\text{min}} < d_{\text{max}}$ such that the methods converge $R$-linearly under the given assumptions. However, they do not provide any information about the difference $\Delta = d_{\text{max}} - d_{\text{min}}$ and thus about how much the steps employed by different nodes and at different iterations can differ. In this section, we try to address this issue by investigating in practice the length of the interval of admissible step-sizes. First, we show a particular example where the method converges without any upper bound $d_{\text{max}}$, then we present a set of numerical results that show how the step bounds influence the convergence and the performance of the methods.

We consider the same framework considered in [10] (section 4.2) and we prove that even if we allow the consensus matrix to change from iteration to iteration, the method converges. Consider the following objective function

$$f(y) = \sum_{i=1}^{n} f_i(y) \quad \text{with} \quad f_i(y) = \frac{1}{2} (y - a_i)^2 \quad \text{and} \quad a \in \mathbb{R}^n \quad (17)$$

and assume that at iteration $k$ the consensus matrix is given by

$$W_k = (1 - \theta_k I) + \theta_k J, \quad \text{with} \quad \theta \in (0, 1).$$

**Lemma 4.** Assume that $\theta_k \in (\frac{1}{3}, \frac{3}{4})$ for every $k$, and that $\{x^k\}$ is the sequence generated by $\{5\}$ with $b = 0$, $e^T x_0 = e^T a$ and $e^T (u_0 + \nabla F(x_0)) = 0$.

If $d^k_i = \alpha$ for every $i = 1, \ldots, n$ and for every $k$, then the method converges $R$-linearly to the solution of $(17)$ if $\alpha_{\text{min}} \leq \alpha \leq \frac{2}{3}$ and $\alpha_{\text{min}} > 0$ small enough. On the other hand, for any $\alpha > 2$, there exists a sequence $\{\theta_k\}$, $k = 0, 1, 2, \ldots$ that satisfies the assumptions of the Lemma such that the method diverges, i.e., $\|x^k\| \to \infty$.

If $d^k_i = (\sigma_i^k)^{-1}$ with

$$\sigma_i^{k+1} = \mathcal{P}_{[\sigma_{\text{min}}, \sigma_{\text{max}}]} \left( 1 + \sigma_i^k \sum_{j=1}^{n} w_{ij}^{k} \left( 1 - \frac{s_j^k}{s_i^k} \right) \right), \quad (18)$$

with $s_i^k = x_i^{k+1} - x_i^k$, $\sigma_{\text{min}} = 0$, $\sigma_{\text{max}} = 3/2$ and $\sigma_i^0 = \sigma \in (\sigma_{\text{min}}, \sigma_{\text{max}})$ for every $1, \ldots, n$, then $\{x^k\}$ converges $R$-linearly to the solution of $(17)$. 

15
Proof. In the case we are considering (5) is equivalent to
\[
\begin{align*}
x^{k+1} &= W^k x^k - D^k z^k \\
z^{k+1} &= W^k z^k + x^{k+1} - x^k
\end{align*}
\]
where $D^k = \text{diag}(d_1^k, \ldots, d_n^k)I$.

Let us consider the case with fixed step-size $d_i^k = \alpha$ and let us denote with $\xi^k$ the vector $(q^k, z^k) \in \mathbb{R}^{2n}$. We can see that for every $k$ we have $\xi^{k+1} = A_k \xi^k$ where the matrix $A_k$ is given by
\[
A_k = \begin{pmatrix} W_k - J & -\alpha I \\ W_k - I & W_k - \alpha I \end{pmatrix} \in \mathbb{R}^{2n \times 2n}.
\]

In order to prove the first part of the Lemma, it is enough to show that there exists $\mu < 1$ such that $\|A_k\|_2 < \mu$ for every iteration index $k$. That is, we have to prove that there exists $\mu < 1$ such that the spectral radius of $A_k^T A_k$ is smaller than $\mu$ for every $k$. Denoting with $\lambda_1^k, \lambda_2^k, \ldots, \lambda_n^k$ the eigenvalues of $W^k$, it can be proved that the eigenvalues of $A_k^T A_k$ are given by the eigenvalues of the $2 \times 2$ matrices $M_i^k$ defined as
\[
M_1^k = \begin{pmatrix} \alpha^2 & \alpha(\alpha - 1) \\ \alpha(\alpha - 1) & (\alpha - 1)^2 \end{pmatrix}
\]
\[
M_i^k = \begin{pmatrix} (\lambda_i^k)^2 + \alpha^2 & (\lambda_i^k)^2 - (1 + \alpha)\lambda_i^k + \alpha^2 \\ (\lambda_i^k)^2 - (1 + \alpha)\lambda_i^k + \alpha^2 & 2(\lambda_i^k)^2 - 2(1 + \alpha)\lambda_i^k + 1 + \alpha^2 \end{pmatrix}
\]
for $i = 2, \ldots, n$. By direct computation we can see that the eigenvalues of $M_i^k$ are given by 0 and $1 - \alpha < 1 - \alpha_{\text{min}}$ and therefore it is enough to take $\mu > 1 - \alpha_{\text{min}}$. Denoting with $p_i^k(t)$ the characteristic polynomial of $D_i^k$ we can see that with the values of $\theta_{\text{min}}, \theta_{\text{max}}$ and $\alpha_{\text{max}}$ given by the assumptions, we can always find $1 - \alpha_{\text{min}} < \mu < 1$ such that $p_i^k(\mu) > 0$ and $p_i^k(-\mu) > 0$ and thus such that the eigenvalues of $M_i^k$ belong to $(-\mu, \mu)$ for every $k$ and for every $i = 1, \ldots, n$. To prove that if $\alpha > 2$ the method is in general not convergent it is enough to consider the case when $\theta_k = \theta_0$ for every iteration index $k$. In this case we have that $A_k = A_0$ for every $k$ and thus $\xi^k = A_0^k c^0$. In this case we can see [10] that $1 - \alpha$ is an eigenvalue of $A_0$ an therefore if $\alpha > 2$ we have that $\rho(A_0) > 1$ and thus the sequence $\{\xi^k\}$ does not converge. This concludes the first part of the proof.
Assume now that the step-sizes are computed as in (18). Proceeding as in the proof of Proposition 4.3 in [10] we can prove that \( \sigma_{k+1} = \sigma_{k+1} \) for every \( i \) with \( \sigma_{k+1} \) given by

\[
\sigma_{k+1} = \begin{cases} 
\min\{\sigma_{\text{max}}, 1 + \sigma_{\text{max}}\theta_k\} & \text{if } \sigma_k = \sigma_{\text{max}} \\
\min\{\sigma_{\text{max}}, \hat{\sigma}_{k+1}\} & \text{otherwise}
\end{cases}
\]

where \( \hat{\sigma}_{k+1} = 1 + \theta_k + \theta_k \theta k - 1 + \cdots + \prod_{j=1}^{k} \theta_j + \sigma^0 \prod_{j=0}^{k} \theta_j \). By using the fact that \( \theta_k > 1/3 \) and \( \sigma_{\text{max}} = 3/2 \) we can prove that there exists \( \bar{k} \) such that \( \sigma^k = \sigma_{\text{max}} \) for every \( k > \bar{k} \). Therefore, for \( k > \bar{k} \) the step-size becomes the same for all nodes and equal to \( d_k^i = \sigma_{\text{max}}^{-1} = 2/3 \) and thus the method converges by the first part of the Lemma.

In particular Lemma 4 shows that in this particular framework we do not need an upper bound \( d_{\text{max}} \) to the step-sizes.

We now present some numerical results. We consider the problem of minimizing a logistic loss function with \( l_2 \) regularization, that is, we assume the local objective function \( f_i \) at node \( i \) is given by

\[
f_i(y) = \ln\left(1 + \exp(-b_i^T a_i y)\right) + \frac{1}{2} R \|y\|^2_2
\]

where \( a_i \in \mathbb{R}^d \), \( b_i \in \{-1, 1\} \) and \( R > 0 \). We compare 3 different choices of the matrix \( B \) in (5) and three different definitions of the step-sizes \( d_k^i \), resulting in nine methods. For increasing values of \( d_{\text{max}} \) we run each method on the given problem and we plot in Figure 1 the number of iterations necessary to arrive at convergence.

The problem is generated as follows. The convergence analysis we carried out in Section 3 does not rely on any particular definition of the step-sizes \( d_k^i \), therefore we need to specify how each node chooses the step-size at each iteration. We consider here two cases. The first one, referred to as spectral in Figure 1, is the case where \( d_k^i = (\sigma_k^i)^{-1} \) with \( \sigma_k^i \) as in (18). The second case we consider is the one where each node performs local line search by employing a backtracking strategy starting at \( d_{\text{max}} \) to satisfy classical Armijo condition on the local objective function. That is, to satisfy

\[
f_i\left(\sum_{j=1}^{n} w_{ij}^k x_j^k - d_k^i z_i^k\right) \leq f(x_i^k) - c d_k^i \nabla f_i(x_i^k)^T z_i^k
\]
with \( c = 10^{-3} \) and \( z_{ki}^k = u_{ki}^k + \nabla f_i(x_{ki}^k) \). We refer to this method as line search. For comparison, we also consider the method with fixed step-size \( d_{ki}^k = d_{max} \) for every \( k \) and every \( i = 1, \ldots, n \). The choices of the matrix \( B^k \) are given by \( B^k = 0 \) (plot (a) in Figure 1), \( B^k = d_{max}^{-1} I \) (plot (b)) and \( B^k = d_{max}^{-1} W^k \) (plot(c)), where for the case \( B \neq 0 \) the choice is made following [9]. Notice that the case \( d_{ki}^k = d_{max} \) and \( B^k = 0 \) corresponds to [13, 15] with constant, coordinated step-sizes. We consider increasing values of \( d_{max} \) in \([\frac{1}{100}, \frac{10}{L}]\), while we fix \( d_{min} = 10^{-8} \) as, in the considered framework, we saw that its choice does not influence the performance of the methods significantly.

In Figure 1 we plot the results in the case where the underlying network is symmetric and time-varying, defined as follows: we consider a network \( G \) with \( n = 25 \) nodes undirected and connected, generated as a random geometric graph with communication radius \( \sqrt{n^{-1} \ln(n)} \), and we define the sequence of networks \( \{G^k\} \) by deleting each edge with probability \( \frac{1}{4} \). We carried out analogous tests in the cases where \( G \) is symmetric and constant and in the case where it is given by a directed ring. The obtained results were comparable to the ones that we present. We also observed in practice that double stochasticity of the underlying network appears to be essential for the convergence of the considered methods.

We set the dimension \( d \) as equal to 10 and we generate the quantities involved in the definition of the local objective functions (20) as follows. For \( i = 1, \ldots, n \) we define \( a_i = (a_{i1}, \ldots, a_{i,d-1}, 1)^T \) where the components \( a_{ij} \) are independent and come from the standard normal distribution, and \( b_i = \text{sign}(a_{i1}^T y^* + \epsilon_i) \) where \( y^* \in \mathbb{R}^d \) with independent components drawn from the standard normal distribution, and \( \epsilon_i \) are generated according to the normal distribution with mean 0 and standard deviation 0.4. Finally, we take the regularization parameter \( R = 0.25 \). The initial vectors \( x_{ki}^0 \) are generated independently, with components drawn from the uniform distribution on \([0, 1]\), and at each iteration we define the consensus matrix \( W_k \) as the Metropolis matrix [23].

We are interested in the number of iterations required by each method to reach a prescribed accuracy. More precisely, we evaluate the iteration number \( \bar{k} \) at which \( \max_{i=1,\ldots,n} \|x_{ki}^k - y^*\| < \varepsilon \), where \( \varepsilon = 10^{-5} \). In Figure 1, on the \( x \)-axis we show the upper bound \( d_{max} \) while on the \( y \)-axis we show \( \bar{k} \) for each method.

We can see from Figure 1 that for all considered choices of the matrix \( B \) the spectral method allows for maximum step-size that is at least 10 times
larger than the method with fixed step-size, while line search allows for maximum step-size equal to 2 and 3 times the maximum step-size allowed by the method with fixed steplength, for $B$ equal to 0 and $bI$ or $bW$ respectively. Moreover, we can see that choosing $B = bI$ seems to increase the maximum value of $d_{\max}$ that yields convergence for all the considered methods. Finally, for most of the tested values of $d_{\max}$ we can see that the spectral methods requires a smaller number of iterations than the method with fixed step-size, therefore in the considered framework, using uncoordinated time-varying step-sizes given by [10] helps to significantly improve the robustness of the method and also the performance. Notice also that the spectral step-size strategy exhibits a “stable”, practically unchanged, performance for a wide range of $d_{\max}$; hence, it is not sensitive to tuning of $d_{\max}$. This is in contrast with the constant step-size strategy that is very sensitive to the step-size choice $d_{\max}$. It is also worth noting that Theorem 2 requires a conservative upper bound on the step-size $d_{\max}$ and a conservative upper bound on step-size differences $\Delta$ and that both depend on multiple global system

Figure 1
parameters (Lemma 3). However, simulations presented here and other extensive numerical studies suggest that an apriori upper bound on $\Delta$ is not required for convergence. In addition, $d_{\min}$ can be set to a small value independent of system parameters, e.g., $d_{\min} = 10^{-8}$, and setting $d_{\max}$ requires only a coarse upper bound on quantity $1/L$.

5 Conclusions

We proved that a class of distributed first-order methods, including those proposed in [9, 10], is robust to time-varying and uncoordinated step-sizes and time-varying weight-balanced digraphs, wherein connectedness of the network at each iteration, unlike, e.g., the recent work [20], is not required. The achieved results provide a solid improvement in understanding of the robustness of exact distributed first-order methods to time-varying networks and uncoordinated time-varying step-sizes. Most notably, we show that the unification strategy in [9] and the spectral-like step-size selection strategy in [10], as well as combination of those, exhibits a high degree of robustness. This paper considers weight-balanced directed networks. Extensions to weight-imbalanced networks requires redefining the algorithmic class and the respective analysis, and represents an interesting future research direction.

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