Newton’s second law, radiation reaction and type II Einstein–Maxwell fields

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Abstract
Considering perturbations of the Reissner–Nordström metric while keeping the perturbations in the class of type II Einstein–Maxwell metrics, we perform a spherical harmonic expansion of all the variables up to the quadrupole term. This leads to rather surprising results. Referring to the source of the metric as a type II particle (analogous to referring to a Schwarzschild–Reissner–Nordström or Kerr–Newman particle), we see immediately that the Bondi momentum of the particle takes the classical form of mass times velocity plus an electromagnetic radiation reaction term, while the Bondi mass loss equation becomes the classical gravitational and electromagnetic (electric and magnetic) dipole and quadrupole radiation. The Bondi momentum loss equation turns into Newton’s second law of motion containing the Abraham–Lorentz–Dirac radiation reaction force plus a momentum recoil (rocket) force, while the reality condition on the Bondi mass aspect yields the conservation of angular momentum. Two things must be pointed out: (1) these results, (equations of motion, etc) take place, not in the spacetime of the type II metric but in an auxiliary space referred to as $\mathcal{H}$-space, whose physical meaning is rather obscure and (2) this analysis of the type II field equations is a very special case of a similar analysis of the general asymptotically flat Einstein–Maxwell equations. Although the final results are similar (though not the same), the analysis uses different equations (specifically, the type II field equations) and is vastly simpler than the general case. Without a great deal of the technical structures needed in the general case, one can see rather easily where the basic results reside in the type II field equations.

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1. Introduction

For a variety of reasons, the algebraically special Einstein or Einstein–Maxwell metrics and equations [1–8] have played a major role in general relativity (GR) for many years: they have contained some of the most studied and useful solutions of the Einstein–Maxwell equations.
(Schwarzschild, Reissner–Nordström, Kerr–Newman [9], the Robinson–Trautman metrics [3],
plane gravitational waves), they contain the beautiful Goldberg–Sachs theorem [2] with its emphasis
on the importance of shear-free null geodesic congruences (NGCs) and are the source of much attractive
mathematical research, e.g., existence theorems, CR manifold theory.

In this work, we will be concerned only with the type II field equations and their solutions—and some
very surprising physical results that have been hiding in them—results that have been overlooked for
many years. Although the field equations are for the metric variables, we will adopt a slightly
unconventional point of view. In analogy with referring to the source of a Schwarzschild, Reissner–Nordström
or Kerr–Newman metric as a Schwarzschild, Reissner–Nordström or Kerr–Newman particle, we will refer
to the source of the type II metrics as type II particles. The field equations are then interpreted as the
equations governing the dynamics (motion and multipole behavior) of the type II particles. The metric itself
is given here for completeness, but it and its immediate geometric properties are not used.

If one starts with the full Einstein–Maxwell equations and imposes the type II conditions,
many of the resulting equations (the radial equations) can be easily integrated, leaving four
(complex) reduced type II Einstein–Maxwell equations—nonlinear and rather intractable
looking—for four complex field variables, which are functions of time and two angles, plus
a reality condition. Two of the four dependent variables are spin-coefficient versions of the
Maxwell field; one of them is a spin-coefficient Weyl tensor component and the last is a
(geometric) direction field that specifies the direction (at null infinity) of the shear-free NGC
defined by the type II condition. The structure of the four field equations is as follows: the
first two equations and the third determine, respectively, the two Maxwell fields and the Weyl
tensor component (that contains the Bondi energy–momentum four vector) in terms of the
direction field. The last equation, basically a conservation law, is the dynamics for the direction
field.

Working within the algebraically special type II Einstein–Maxwell field equations, we
study perturbations from the Reissner–Nordström metric. Specifically, we consider linear
perturbations (for the first three equations) with spherical harmonic expansions up to and
including the $l = 2$ terms. The fourth of the field equations (the $l = 0, 1$ terms form the
Bondi energy–momentum conservation law) is intrinsically quadratic and is essentially empty
if linearized. With no inconsistency, the results obtained from the three linearized equations
are inserted into the fourth. (This is analogous to first solving the linear Maxwell equations
and then using these solutions in the quadratic stress–tensor conservation laws.) Our main results
are in the $l = 0$ and 1 harmonics of the last equation. The $l = 0$ is the energy conservation
law, containing both gravitational quadrupole radiation and electric and magnetic dipole and
quadrupole radiation in complete agreement with known classical results. The $l = 1$ terms yield
Newton’s second law of motion ($F = Ma$), with the force coming from both radiation reaction
terms and momentum recoil terms. With the momentum recoil terms it is a generalization of the
Abraham–Lorentz–Dirac equation. The $l = 2$ terms determine the evolution of the quadrupole
moments. It must be emphasized that these results do not involve any model building and do
not contain any mass renormalization procedure. They are simply results that have been sitting
there in the type II equations waiting to be seen.

In section 2, we first define and then discuss the type II metrics and Maxwell fields with
their relevant variables, their four differential equations and reality condition. In addition, we
review the theory of shear-free NGCs and their associated auxiliary space complex world
lines. It is the theory of NGCs that lead to the basic variables, the direction field as well as the
particle position vector. Section 3 is devoted to the approximations: initially the linearization
of the first three of the field equations, then their expansion in the three spherical harmonics,
\[ l = 0, 1, 2, \] and finally their integration. The physical results and their interpretation are discussed in sections 3 and 4.

We emphasize that the analysis and results described here can be considered to be a very severe specialization of a similar analysis of the general asymptotically flat Einstein–Maxwell equations. Our present type II case is vastly simpler than that of the general case where in the analysis first there must be a detailed discussion of certain difficult technical issues. By a simple look at the type II field equations, one can (of course, with hindsight) easily see where our results are residing. (We actually guessed the results by just visually looking at one of the type II equations.) The simplicity of the type II case thus yields a very useful example for the study of the general case. The main reason for the vast simplification lies in the following observation. In the general case, we first must consider the set of all asymptotically shear-free NGCs and then by a lengthy rather complicated procedure chose one specific congruence from the set that defines a complex center of mass. This, in turn, then leads to our equations of motion. For the type II case, this lengthy procedure is completely avoided by the realization that for type II spacetimes, among all the asymptotically shear-free NGCs there is a unique one that is not only asymptotically shear free but, via the Goldberg–Sachs theorem, is totally shear free aside from its caustics. This congruence is already built into the type II field equations thereby avoiding the lengthy search. An added advantage of studying the type II metrics, rather than the general case, is that one can see explicitly that the caustics of the shear-free congruence act as the source of the field and that from a dual perspective all the information about the equations of motion are hidden in the caustics.

2. Background

2.1. Notation

Much of our analysis takes place on the future null infinity, \( \mathcal{I}^+ \), which is coordinatized by Bondi coordinates: the null generators, by the complex stereographic coordinates, \((\zeta, \overline{\zeta})\) and the slices of \( \mathcal{I}^+ \) by \( u \). Although we are interested only in real results, we must consider, along the way, the complexification of \( \mathcal{I}^+ \). In that case, we allow \( u \) to take complex values and consider \( \zeta \) to be close to but independent of the complex conjugate of \( \zeta \).

Later we will switch from the Bondi \( u \) to \( \tau = \sqrt{2}c^{-1}u \) with the notation change \( \partial_u W = W = \sqrt{2}c^{-1} \partial_\tau W \equiv \sqrt{2}c^{-1} W' \).

We use the Lorentzian tetrad for the angular behavior of the field variables:

\[ \hat{\rho} = \frac{\sqrt{2}}{2(1 + \zeta \overline{\zeta})} (1 + \zeta \overline{\zeta}, \zeta + \overline{\zeta}, -i(\zeta - \overline{\zeta}), -1 + \zeta \overline{\zeta}) = \left( \frac{\sqrt{2}}{2}, \frac{1}{2} Y^0_1 \right), \quad (1) \]

\[ \hat{m}^a = \frac{\sqrt{2}}{2(1 + \zeta \overline{\zeta})} (0, 1 - \zeta^2, -i(1 + \zeta^2), 2\zeta) = (0, -Y^1_1), \quad (2) \]

\[ \overline{m}^a = \frac{\sqrt{2}}{2(1 + \zeta \overline{\zeta})} (0, 1 - \zeta^2, i(1 + \zeta^2), 2\overline{\zeta}) = (0, -Y^{-1}_1), \quad (3) \]

\[ \hat{n}^a = \frac{1}{2(1 + \zeta \overline{\zeta})} (1 + \zeta \overline{\zeta}, -(\zeta + \overline{\zeta}), i(\zeta - \overline{\zeta}), 1 - \zeta \overline{\zeta}) = \left( \frac{\sqrt{2}}{2}, -\frac{1}{2} Y^0_1 \right). \quad (4) \]
2.2. Algebraic types

In every Lorentzian spacetime, there are four null vectors (principal null vectors, (pnv), $L^a$) per spacetime point that are determined by solutions to the algebraic equation [1]

$$L^b L_{[c} L_{d]} L^a = 0,$$

(5)

with $C_{abcd}$ being the Weyl tensor. In the case, when two or more of these vectors are degenerate (or coincide) the associated metric is referred to as algebraically special. We consider only the double degeneracy case referred to as type II. (Other types are called types III, IV and D.) In the Newman–Penrose (NP) formalism, this condition translates to the requirement that a null tetrad system exists such that the spin-coefficient Weyl tensor components satisfy $\psi_0 = \psi_1 = 0$ and the Maxwell component $\phi_0 = 0$. It then follows from the beautiful Goldberg–Sachs [2] theorem that the degenerate pnv's are the tangent vectors to a NGC that is shear-free. These conditions greatly simplify the integration of the Einstein–Maxwell equations.

2.3. Type II metric and field equations

Although it plays no role in what follows other than to define our dependent variables, for completeness we display the type II metric [8, 6]:

$$g = 2 \left( l^* n^* + n^* m^* - m^* \bar{m}^* \right),$$

(6)

with

$$l^* = du - P^{-1}(L d\xi + L \bar{d}\bar{\xi}),$$
$$m^* = -P^{-1}[r - i \Sigma] d\bar{\xi} = P^{-1}\bar{\rho}^{-1} d\bar{\xi},$$
$$\bar{m}^* = -P^{-1}[r + i \Sigma] d\xi = P^{-1}\rho^{-1} d\xi,$$
$$n^* = dr + P^{-1}[\bar{K} + L \bar{\rho}^{-1}] d\xi + (\bar{K} + \bar{L} \rho^{-1}) \bar{d}\bar{\xi}$$
$$+ \left[ \frac{1}{2}(\rho \psi_0^0 + \bar{\rho} \bar{\psi}_0^0) + (1 + \frac{1}{2}(0 \bar{L} + \bar{L} L + \bar{L} L - L \bar{L}) - \frac{3}{2}(0 \bar{L} + \bar{L} L - L \bar{L} \bar{L} - L \bar{L} L) \right] n^*,$$

(7)

where

$$\rho = -\frac{1}{r + i \Sigma},$$

(8)

$$i \Sigma(u, \xi, \bar{\xi}) \equiv \frac{1}{2}(0 \bar{L} + L \bar{L} - \bar{L} L - L \bar{L}).$$

(9)

$$K = L + L \bar{\Sigma} + \frac{1}{2}(0 \bar{L} + L \bar{L} + L [0 \bar{L} - \bar{L} L + \bar{L} L - L \bar{L}]);$$

(10)

$$P = 1 + \xi \bar{\xi},$$

(11)

with $\rho$ being the complex divergence of the shear-free NGC and $\Sigma$ its twist. The coordinates are $(r, u, \xi, \bar{\xi})$, with $r$ being the radial coordinate (affine, along the type II NGC) and $(u, \xi, \bar{\xi})$ the Bondi coordinates on future null infinity, $J^+$. The spin-coefficient components of the associated Maxwell field in the $l^*, n^*, m^*, \bar{m}^*$ tetrad are

$$F^a_{\bar{b} \bar{c} l^*} m^{\bar{a} \bar{b}} = \phi_0^* = 0,$$

(12)

$$\frac{1}{2} F^a_{\bar{b} \bar{c} l^*} m^{\bar{a} \bar{b}} m^{\bar{a} \bar{b}} = \phi_1^* = \frac{\phi_1^{\psi_0^0}}{r^2} + O(r^{-3}),$$

(13)
$F_{00} \equiv \phi_2^0 = \frac{\phi_2^0}{r^2} + O(r^{-2}). \quad (14)$

Of relevance to us is that our 'particle' variables are seen to be the complex $L(u, \zeta, \bar{\zeta})$, $\psi_2^0 (u, \zeta, \bar{\zeta})$ coming from the metric and the two Maxwell variables, $\phi_1^0 (u, \zeta, \bar{\zeta})$, $\phi_2^0 (u, \zeta, \bar{\zeta})$.

It has been shown [6, 7] that they satisfy the following set of four field equations:

$$\overline{\partial} \phi_1^{0} + 2L \phi_1^{0} + L(\phi_1^{0}) = 0, \quad (15)$$

$$\overline{\partial} \phi_2^{0} + \partial \psi_2^{0} + (L \phi_2^{0}) = 0, \quad (16)$$

$$\overline{\partial} \psi_2^{0} = -L(\psi_2^{0}) - 3L \psi_2^{0} + 2k \phi_1^{0} \phi_2^{0}, \quad (17)$$

$$\Psi = \sigma(\overline{\sigma}) + k \phi_2 \phi_2^{0}, \quad (18)$$

with the mass aspect given as follows.

**Definition 1.**

$$\Psi \equiv \psi_2^{0} + 2L \partial \overline{\partial} (\overline{\sigma}) + L^2 (\overline{\sigma}) + \overline{\partial} \overline{\partial} (\overline{\sigma}) + \sigma (\overline{\sigma}), \quad (19)$$

(with the very important reality condition),

$$\Psi = \overline{\Psi} \quad (20)$$

and

$$\sigma \equiv \partial L + LI, \quad (21)$$

$$k = 2Ge^{-4}, \quad (22)$$

$$L \equiv \frac{\partial L}{\partial u}, \quad \text{etc.} \quad (23)$$

Note that we can switch back and forth between the mass aspect $\Psi$ and the Weyl component, $\psi_2^{0}$, via equation (19) as needed.

In section 3, we will take these equations, partially linearize them, expand them in spherical harmonics and then analyze and display their physical content.

### 2.4. Shear-free null geodesic congruences and complex world lines

The evolution of generic asymptotically flat Einstein–Maxwell spacetimes (described in Bondi coordinates and tetrad, $l, n, m, \overline{m}$) is driven by the freely given asymptotic shear, $\sigma(u, \zeta, \overline{\zeta})$, of the associated Bondi NGC, (the generators of the Bondi null surfaces) and Maxwell radiation field. In addition to the Bondi NGC, other NGCs, defined by the rotation of the tangent vector of the Bondi NGC, can be introduced. They in turn come with an associated transformed shear. More specifically, at each point $(u, \zeta, \overline{\zeta})$ of $\mathcal{J}^+$, we consider its past light cone and its sphere of null direction with the angles given by the pair of complex stereographic coordinates, $(L, \overline{L})$.

(The north and south poles of the sphere are fixed by the null generators of $\mathcal{J}^+$ and the Bondi vector $l$.) Any field of null directions on $\mathcal{J}^+$ (which automatically determines an interior NGC) can be given by some complex regular spin-weight-one function $L = L(u, \zeta, \overline{\zeta})$. From a given
Bondi tetrad \((l^a, n^a, m^a, \bar{m}^a)\) a second tetrad \((l^{a*}, n^{a*}, m^{a*}, \bar{m}^{a*})\) in the neighborhood of \(\mathcal{I}^+\) can be constructed by the null rotation, with arbitrary \(L(u, \zeta, \bar{\zeta})\):

\[
l^a \rightarrow l^{a*} = l^a - \bar{L} m^a - L \bar{m}^a + 0(r^{-2}),
\]

\[
m^a \rightarrow m^{a*} = m^a + \bar{L} n^a + 0(r^{-2}),
\]

\[
n^a \rightarrow n^{a*} = n^a + 0(r^{-2}).
\]

(24)

The asymptotic shear of the field of new tangent null vectors \(l^{a*}\) is given by \([10]\)

\[
\sigma^{a*} = \sigma^0 - \partial L - LL\cdot
\]

so that shear-free NGCs (from \(\sigma^{a*} = 0\)) are determined by functions \(L(u, \zeta, \bar{\zeta})\) that satisfy the differential equation

\[
\partial L + LL\cdot = \sigma^0.
\]

(28)

In other words, a function \(L(u, \zeta, \bar{\zeta})\) that satisfies equation (28) determines an asymptotically shear-free NGC.

It has been shown \([12, 13]\) that regular solutions to equation (28) are generated by arbitrary complex analytic curves in an auxiliary space referred to as \(\mathcal{H}\)-space \([11]\). One begins by finding solutions to the ‘good-cut’ equation

\[
\partial^2 Z = \sigma^0(Z, \zeta, \bar{\zeta})
\]

which are known to depend on four complex parameters, the coordinates of \(\mathcal{H}\)-space, \(z^a\). The solution (in general complex), written as

\[
u = Z(z^a, \zeta, \bar{\zeta}),
\]

(30)

is interpreted as describing a four-parameter family of slices (cuts) of complexified \(\mathcal{I}^+\), the so-called ‘good cuts’. By choosing an arbitrary complex analytic curve in \(\mathcal{H}\)-space, parametrized by complex \(\tau, \xi^a(\tau)\) we have a one-complex parameter family of slices:

\[
u = Z(\xi^a(\tau), \zeta, \bar{\zeta}) = G(\tau, \xi, \bar{\xi}).
\]

(31)

The regular solutions to equation (28), \(L(u, \zeta, \bar{\zeta})\), are the stereographic angle field determined from the null normals to these slices. The solutions have the parametric form as follows:

\[
L = \partial_{(\tau)} G(\tau, \zeta, \bar{\zeta}),
\]

(32)

\[
u = G(\tau, \zeta, \bar{\zeta}),
\]

(33)

with the subscript \((\tau)\) indicating that the derivative is taken at constant \(\tau\).

 shear-free and asymptotically shear-free NGCs that are regular (i.e. no generators on \(\mathcal{I}^+\)) are induced by arbitrary complex curves in \(\mathcal{H}\)-space \([12, 13]\).

These results are applied to our considerations of asymptotically flat type II metrics by the following observation. Among all the asymptotically shear-free NGCs that exist in a type II spacetime, there is a unique one, determined by the Goldberg–Sachs \([2]\) theorem for the degenerate principal null vector field, that is not only asymptotically shear free but also totally shear free (aside from caustic regions). It is this particular complex world line with its associated cut function (equation (31)) that governs—or acts as the backbone—to the solutions to the field equations, (15)–(18).
Anticipating the approximations of the next section, the spherical harmonic expansion of the function \( u = G(\tau, \zeta, \bar{\zeta}) \) and the derived \( \sigma^0(u, \zeta, \bar{\zeta}) \) and \( L(u, \zeta, \bar{\zeta}) \) (parametrically described) up to the \( l = 2 \) harmonic are

\[
\begin{align*}
u &= G(\tau, \zeta, \bar{\zeta}) = \hat{n}^a(\tau) \hat{M}^a(\zeta, \bar{\zeta}) + \xi^{ij}(\tau) Y_{ij}^0, \\
L(u, \zeta, \bar{\zeta}) &= \hat{n}_{ij}(\tau, \zeta, \bar{\zeta}) = \xi^{ij}(\tau) Y_{ij}^0 - 6 \xi^{ij}(\tau) Y_{ij}^2 + 24 \xi^{ij}(\tau) Y_{ij}^2,
\end{align*}
\]

(34) \hspace{1cm} (35) \hspace{1cm} (36)

The last of the field equations become the evolution equations for \( \xi^i(\tau) \) and \( \xi^{ij}(\tau) \).

3. The field equations

3.1. Linearization and harmonic expansion

We begin with the field equations (15)–(20) [6]

\[
\begin{align*} \\
\phi^{(0)} &= -2L \phi^{(0)} - L(\phi^{(0)}), \\
(\phi^{(0)}) &= - \partial \phi^{(0)} - (L \phi^{(0)}), \\
\psi^{(0)} &= -L(\psi^{(0)}) - 3L \psi^{(0)} + 4Gc^{-4} \phi^{(0)} \bar{\psi}^{(0)}, \\
\psi &= \bar{\psi} \equiv \psi^{(0)} + 2L \bar{\psi}^{(0)} + L^2(\bar{\psi}) + \partial^2 \bar{\psi} + \sigma(\bar{\psi}), \\
\sigma &= \bar{\psi} L + L L, \\
\end{align*}
\]

(37) \hspace{1cm} (38) \hspace{1cm} (39) \hspace{1cm} (40) \hspace{1cm} (41) \hspace{1cm} (42)

and first note that the Reissner–Nordström solution is given by

\[
\begin{align*} \\
\phi^{(0)} &= q, \\
\phi^{(0)} &= \phi^{(0)} = L = 0, \\
\psi^{(0)} &= \Psi \equiv \psi^{(0)} = \frac{-M_B}{c^2},
\end{align*}
\]

(43) \hspace{1cm} (44) \hspace{1cm} (45)

with the zero-order terms, \( q \) and \( M_B \), being the charge and Bondi mass.

The linearization of the first four equations (37)–(40) leads to

\[
\begin{align*} \\
\bar{\partial} \phi^{(0)} &= -2qL, \\
(\phi^{(0)}) &= - \partial \phi^{(0)} + 4Gc^{-4} q \phi^{(0)} \bar{\psi}^{(0)}, \\
\psi &= \bar{\psi} \equiv \psi^{(0)} + \partial^2 \bar{L}. \\
\end{align*}
\]

(46) \hspace{1cm} (47) \hspace{1cm} (48) \hspace{1cm} (49)

By eliminating \( \psi^{(0)} \) in terms of \( \Psi \), via (49), inserting \( c \) explicitly and rescaling \( u \) (to obtain retarded time) by \( u = \sqrt{2} c^3, \) (i.e. define \( k' = \sqrt{2} c^{-1} k = \sqrt{2} c^{-1} K' \), we obtain our three linear equations:

\[
\begin{align*} \\
\bar{\partial} \phi^{(0)} &= 2q \sqrt{2} c^{-1} L',
\end{align*}
\]

(50)
Using the spherical harmonic expansions

\begin{align}
L(u, \zeta, \bar{\zeta}) &= \delta_{i(c)} G(\tau, \zeta, \bar{\zeta}) = \xi^i(\tau) Y^i_1, \\
\phi^{(0)}_1 &= q + \phi^{(0)}_1 Y^0_1 + \phi^{(0)ij}_1 Y^0_{2j}, \\
\phi^{(0)}_2 &= -2c^{-2} D^{''}_{ekm} Y^{-1}_1 - \sqrt{2} c^{-3} Q^{''}_{ekm} Y^{-1}_{2j}, \\
\psi &= \psi^0 + \psi^i Y^0_i + \psi^{ij} Y^0_{ij},
\end{align}

(53) (54) (55) (56)

(where \(D^{''}_{ekm} \equiv D^{''}_{\text{elec}} + i D^{''}_{\text{mag}}\) and \(Q^{''}_{ekm} \equiv Q^{''}_{\text{elec}} + i Q^{''}_{\text{mag}}\) are respectively the complex dipole and complex quadrupole moments of the source, defined from the radiation field, i.e. the \(r^{-1}\) part of \(F^{\text{el}}\)), we find, from (50) and (51), with equation (55), the full asymptotic Maxwell field

\begin{align}
\phi^{(0)}_0 &= q + \sqrt{2} \psi Y^0_1 + \frac{1}{12} c^{-1} q Y^0_{2j}, \\
\psi &= \psi^0 + \psi^i Y^0_i + \psi^{ij} Y^0_{ij},
\end{align}

(57)

with

\begin{align}
D^{''}_{ekm} &= q \xi^i, \\
Q^{''}_{ekm} &= -24 \sqrt{2} c q \xi^{ij}.
\end{align}

(58) (59)

Note that they are given in terms of the harmonic components of \(L(u, \zeta, \bar{\zeta})\).

By substituting these results into the last of the linear equations, (52), we obtain for the \(l = 1, 2\) harmonic coefficients:

\begin{align}
\psi^i &= \psi^i = \frac{3\sqrt{2}}{2} c^{-1} \psi^0 \xi^i + 4 Ge^{-5} q D^{''}_{ekm}, \\
\psi^{ij} &= \psi^{ij} = -24 \sqrt{2} \psi^0 c^{-1} \xi^{ij} + G q e^{-5} \frac{\sqrt{2} \psi^{''}_{ekm}}{18},
\end{align}

(60) (61)

From the definition of the Bondi energy–momentum vector \((M_B c^2, P^i)\),

\begin{align}
\psi^0 &= -M_B \frac{2 \sqrt{2} G}{c^2}, \\
\psi^i &= -6 G \frac{c^2}{c^2} P^i,
\end{align}

(62) (63)

the reality conditions and the decompositions,

\begin{align}
\xi^a &= \xi^a_R + i \xi^a_I, \\
\xi^{ij} &= \xi^{ij}_R + i \xi^{ij}_I,
\end{align}

(64) (65)

we find from equation (60)

\begin{align}
P^i &= M_B \xi^{i}_R - \frac{2}{3} c^{-3} q \xi^{''}_{R} \\
0 &= M_B \xi^{ii}_R + \frac{2}{3} c^{-3} q \xi^{''}_{I}.
\end{align}

(66) (67)
Remark. These relations have immediate physical interpretation. Equation (66) defines the Bondi momentum with the standard $Mv$ term plus the radiation reaction term that later becomes the well-known radiation reaction force and equation (67) is the conservation of angular momentum law.

Equation (67), rewritten as
\[ J^i = M_B c \xi^i R + \frac{2}{3} c^{-2} q^2 \xi^i_R, \]  
\[ J^\nu = 0, \]  
(68)

generalizes the spin-angular momentum of the Kerr–Newman metric, $J^i = M_B c \xi^i R$, and its conservation. In addition, the electric and magnetic dipole moments are given, via equation (58), by
\[ D^i_e = q \xi_i R, \]  
\[ D^i_m = q \xi_i I, \]  
(70)

The real and imaginary parts of $\Psi^{ij}$ (with the reality condition), from (61), yield
\[ \Psi^{ij} = -24 \xi_R^{ij} - 3 \sqrt{2} \Psi^0 c^{-1} \xi_R^{ij} - \frac{1}{2} G c^{-6} \xi_R^{ij}, \]  
\[ 0 = 24 \xi_R^{ij} - 3 \sqrt{2} \Psi^0 c^{-1} \xi_R^{ij} + \frac{1}{2} G c^{-6} \xi_R^{ij}. \]  
(72)

Note. Returning to the issue of the structure of our equations, it can be seen that our linear equations have determined three of the four dependent variables, i.e. $\phi^0_1$, $\phi^0_2$ and $\psi_1$ (or $\psi_2$), in terms of $L$ (or $\xi$ and $\xi^{ij}$). The evolution of the imaginary parts of $\xi^i$ and $\xi^{ij}$ (arising from the reality condition) is determined by equations (69) and (61). The evolution of the real parts will be determined by the nonlinear equation (41).

3.2. The quadratic field equation and conservation laws

Since the linearization of equation (41) yields the trivial, $\Psi' = 0$, we can, with no inconsistency, consider its full quadratic version
\[ \sqrt{2} \Psi' = 20 L' \partial L + 2 G c^{-4} \phi^{ijm0}_1 \phi^{ijm0}_2, \]  
(74)

where the linear results, with the Bondi energy–momentum relations, are inserted. The procedure, essentially straightforward, (utilizing the Clebsch–Gordon expansion of spherical harmonic products), yields for the $l = 0, 1, 2$ harmonics:

(I) $l = 0$.

\[ E' \equiv M_B c^2 = - \frac{1}{5} c^{-2} G \Omega^{ijm0}_1 \Omega^{ijm0}_1 - \frac{2}{5} c^{-3} D^{ijm0}_e D^{ijm0}_e - \frac{1}{180} c^{-5} Q^{ijm0}_e Q^{ijm0}_e. \]  
(75)

The first term is the classical Bondi gravitational quadrupole energy loss, while the second and third terms are the classical electromagnetic dipole and quadrupole radiation losses [14]. Note that they contain both the electric and magnetic dipole and quadrupole radiation and furthermore each of these expressions is given in terms of the angle fields $L$ via
\[ Q^{ijm0}_g = 12 \sqrt{2} G c^{-2} \xi^{ij} \]  
\[ D^{ijm0}_e = q \xi_i, \]  
\[ Q^{ijm0}_e = -24 \sqrt{2} c q \xi_i. \]  
(76)
(II) \( l = 1 \).

\[ P^\mu = -i\epsilon_{ijk} \left( \frac{2}{15c^6} GQ_{\text{Grav}}^{\mu} \Theta^{m} + \frac{4}{3c^4} D^m_{\text{ekm}} \Theta^{n} + \frac{2}{135c^6} O_{\text{ekm}}^{\mu} \Theta^{m} \right) \]

\[-2\sqrt{\frac{2}{15c^3}} (Q_{\text{ekm}}^m \Theta^{n} + D^m_{\text{ekm}} \Theta^{n}) \right), \quad (79)\]

All the terms on the right-hand side are real and involve the products of electric- and magnetic-type moments. Presumably (though we did not verify it), the second and third terms arise from the Poynting vector hidden inside the asymptotic Einstein–Maxwell equations. We refer to the right-hand side as the (field) momentum recoil terms. By replacing the \( P \), from equation (66), \( P = M_{\mu} \xi_{\mu} = \frac{1}{2} c^{-2} q^2 \xi_{\mu} \), into equation (79), we obtain Newton’s second law:

\[ M_{\mu} \xi_{\mu} = F^i, \quad (80) \]

\[ F^i = \frac{2}{3c^4} q^2 \xi_{R}^i - M_{\mu} \xi_{\mu}^i \frac{2\sqrt{2}}{15c^3} (Q_{\text{ekm}}^m \Theta^{n} + D^m_{\text{ekm}} \Theta^{n}) \]

\[-i\epsilon_{ijk} \left( \frac{2}{15c^3} GQ_{\text{Grav}}^{\mu} \Theta^{m} + \frac{4}{3c^4} D^m_{\text{ekm}} \Theta^{n} + \frac{2}{135c^6} O_{\text{ekm}}^{\mu} \Theta^{m} \right) \right). \quad (81)\]

The force consists of three parts: the classical radiation reaction term, a rocket-like momentum loss term and the momentum recoil terms from the fields. Note that one can have non-vanishing recoil arising from the interaction of the electric quadrupole with an electric dipole, i.e. one need not have magnetic and spin terms in order to have a field recoil force.

(III) \( l = 2 \). For completeness we give, in an outline form, the \( l = 2 \) harmonic equation, a rather nasty nonlinear relation for \( \xi^{ij} \), coupled to \( \xi^i \). Its explicit form does not shed much light on the physics (the details just need Clebsch–Gordon expansions on the right side):

\[ \xi^{ij}_{R} = \frac{2GM_{\mu}}{c^3} \xi^m + \frac{G\xi^2}{9c^7} \xi_{R}^m = -\frac{1}{24} \left( 2\partial L / \partial L + Gc^{-4} \phi_2^m \phi_2^m \right)_{|_{l=2}}, \quad (82) \]

with

\[ L(u, \xi, \bar{\xi}) = \xi^{i} Y_{li} - 6\xi^{i} Y_{2j}, \quad (83) \]

\[ \phi_2^m = -2D^m_{\text{ekm}} Y_{1i} - \frac{\sqrt{2}}{12} G_{\text{ekm}}^m Y_{2j}. \quad (84) \]

4. Summary and discussion

4.1. Summary

We have considered perturbations of the Reissner–Nordström metric that remain in the class of algebraically special type II metrics. After all radial integrations had been performed (already described in the literature), there remained four relevant field equations (with a reality condition) for four dependent variables, two Maxwell fields (\( \phi_1^0, \phi_2^0 \)), a Weyl tensor component \( \psi_2^2 \) (or the mass aspect, \( \Psi \)) and an angle field \( L \). The angle field describes the direction that the shear-free NGC associated with the degenerate type II principal null vector field intersects future null infinity, \( \mathcal{I}^+ \). The independent variables are the Bondi coordinates of \( \mathcal{I}^+ \), the time variable \( u \) and the stereographic angles \( (\zeta, \bar{\zeta}) \).
The four field variables were expanded in spherical harmonics:

\[
L(u, \zeta, \bar{\zeta}) = \bar{\partial}_i G(\tau, \zeta, \bar{\zeta}) = \xi^i(\tau)Y^0_{1i} - 6\xi^i(\tau)Y^0_{2ij},
\]

\[
\phi_1^{s0} = q + \phi_1^{s0}(\tau)Y^0_{1i} + \phi_1^{sij}(\tau)Y^0_{2ij},
\]

\[
\phi_2^{s0} = -2c^{-2}D^{\mu\nu}_{e\text{km}}(\tau)Y^{-1}_{1i} = \frac{\sqrt{2}}{12}c^{-3}Q^{ijm}_{e\text{km}}(\tau)Y^{-1}_{2ij},
\]

\[
\Psi = \Psi^0(\tau) + \Psi^i(\tau)Y^0_{1i} + \Psi^{ij}(\tau)Y^0_{2ij},
\]

(with the \(u\) behavior given parametrically by \(u = G(\tau, \zeta, \bar{\zeta}) = \tau - \frac{1}{2}\xi^i(\tau)Y^0_{1i}(\zeta, \bar{\zeta}) + \xi^{ij}(\tau)Y^0_{2ij}\) and inserted into the first three of the field equations. The first two determine the electric and magnetic dipole and quadrupole moments in terms of \(\Psi^{\mu}\): \(\Psi^{\mu}\):

\[
D^{\mu\nu}_{e\text{km}} = D^{\mu}_{e} + iD^{\nu}_{m} = q(\xi^{\mu}_{1i} + i\xi^{\mu}_{2ij}),
\]

\[
Q^{ijm}_{e\text{km}} = Q^{j}_{e} + iQ^{m}_{m} = -24\sqrt{2}\epsilon c^{2}\xi^{ij},
\]

while the third yields, from the real part, the Bondi mass and, from the imaginary part, the conservation of angular momentum:

\[
P^{\mu} = M_B\xi^{\mu}_{R} - \frac{3}{2}c^{-2}q^{2}\xi^{\mu}_{R},
\]

\[
J^{\mu} = 0,
\]

\[
J^{I} = M_B\xi^{I}_{I} + \frac{3}{2}c^{-1}q^{2}\xi^{I}_{I}.
\]

The last of the field equations (basically the evolution equation for \(L\)) contains the conservation of energy and momentum laws:

\[
E^{\prime} \equiv M_Bc^{2} = -\frac{1}{5}c^{-7}GQ^{ijm}_{Grav}\frac{Q^{ijm}_{Grav}}{G} - 2\frac{2}{3}c^{-3}D^{\mu}_{e\text{km}}\frac{D^{\mu}_{e\text{km}}}{G} - \frac{1}{180}c^{-5}Q^{ijm}_{e\text{km}}\frac{Q^{ijm}_{e\text{km}}}{G},
\]

\[
P^{\mu} = -2\sqrt{\frac{2}{15}c^{3}}(Q^{km}_{e\text{km}}\frac{D^{\mu}_{e\text{km}}}{G} + D^{km}_{e\text{km}}\frac{Q^{ijm}_{e\text{km}}}{G})
- i\epsilon_{ijk}\left(\frac{2G}{15c^{2}}Q^{km}_{Grav}\frac{O^{ijm}_{Grav}}{G} + \frac{4}{3c^{3}}\epsilon_{ijk}D^{\mu}_{e\text{km}}\frac{D^{\mu}_{e\text{km}}}{G} + \frac{2}{135c^{6}}Q^{ijm}_{e\text{km}}\frac{Q^{km}_{e\text{km}}}{G}\right).
\]

The momentum law is easily converted into Newton’s second law

\[
M_B\xi^{\mu}_{R} = F^{\mu},
\]

with

\[
F^{\mu} = \frac{2}{3c^{2}}q^{2}\xi^{\mu}_{R} - M_B^{\prime}\xi^{\mu}_{R} - 2\sqrt{\frac{2}{15}c^{3}}(Q^{km}_{e\text{km}}\frac{D^{\mu}_{e\text{km}}}{G} + D^{km}_{e\text{km}}\frac{Q^{ijm}_{e\text{km}}}{G})
- i\epsilon_{ijk}\left(\frac{2G}{15c^{2}}Q^{km}_{Grav}\frac{O^{ijm}_{Grav}}{G} + \frac{4}{3c^{3}}\epsilon_{ijk}D^{\mu}_{e\text{km}}\frac{D^{\mu}_{e\text{km}}}{G} + \frac{2}{135c^{6}}Q^{ijm}_{e\text{km}}\frac{Q^{km}_{e\text{km}}}{G}\right),
\]

the force consisting of the radiation reaction term and different types of recoil terms.
The question immediately arises: How seriously should one take this? One’s first reaction might be: ‘yes, this is important. One sees, in a relatively simple situation (type II metrics), so many of the fundamental relations of classical mechanics. They arise with no model building and no renormalization problems, just GR coupled to the Maxwell field and constrained to type II. The associated field equations are interpretable as the dynamics of a particle, yielding both its motion and the evolution of its internal multipole structure. These particles (or spacetimes) are just time-dependent generalizations (that remain in the algebraically special class) of the fundamental metrics: Schwarzschild, Reissner–Nordström, Kerr–Newman’. These observations certainly suggest that they should be taken seriously.

On the other hand, for a variety of reasons, one must remain skeptical of their physical importance. It is hard to imagine what type of physical object could be identified with these type II particles. One also has the immediate classical difficulty with the run-away behavior associated with the radiation reaction force. (Could the momentum recoil force damp out the run-away behavior? Unlikely?) One has the difficult conceptual issue of what is the physical space in which these particles are moving, i.e. how do we identify physically the complex position vector \( \xi^a(\tau) \), or, what is the physical or observational meaning to \( H \)-space and the \( H \)-space world line? On this last issue, there are attempts to give (observational) physical meaning to \( \xi^a(\tau) \) [15], which are partially but not totally satisfactory. More specifically, there is a point of view that is dual to that of the complex world line in \( H \)-space. By returning to the spacetime interpretation of the equations and considering the real shear-free, but twisting, NGC defined by the Goldberg–Sachs theorem via type II metrics, we can interpret the ‘type II particle’ as the caustics of the congruence, i.e. the spacetime points where \( \rho \to \infty \),

\[
\rho = -\frac{1}{r + i\Sigma},
\]

(97)

\[
i\Sigma(u, \xi, \bar{\xi}) = \frac{1}{2}\left(\partial L + LL - \bar{\partial}L - TL\right),
\]

(98)

or where both \( r \) and \( \Sigma \to 0 \). Perturbatively, these caustics have the form of a world tube, \((S^1 \times R)\), and can be considered as the source of the type II metrics since the Weyl curvature and the Maxwell field are singular on the tube. All the information contained in the complex picture of \( \mathcal{H} \)-space and its world line can be reconstructed via the real twisting congruence. Unfortunately, the geometry of the complex point of view is simpler and far more attractive than the real picture.

In spite of the difficulties, we find issues raised by these type II particles to be rather fascinating. If they have no connection to physics, why do they appear so connected; if they are connected, what is the precise connection? It is hard to believe that all this is just a strange accident or coincidence.

As a final comment, we note that the perturbations, with considerably more effort, can be and have been removed from their restriction to type II metrics [13]. Almost all asymptotically flat Einstein–Maxwell spacetimes have been considered. Although there are similarities in the final results (equations of motion, energy–momentum and angular momentum conservation laws, etc), some of the difficulties disappear, others remain. It should be emphasized that the results described here are from a very special example—vastly simpler and straightforward—of the more general asymptotic considerations.
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