A HILBERT SPACE ON LEFT-DEFINITE STURM-LIOUVILLE DIFFERENCE EQUATIONS

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Abstract
We investigate the discrete Sturm-Liouville problems

\[-\Delta(p\Delta u)(n - 1) + q(n)u(n) = lw(n)u(n),\]

where \(p\) is strictly positive, \(q\) is nonnegative and \(w\) may change sign. If \(w\) is positive, the \(\ell^2\)-space weighted by \(w\) is a Hilbert space and it is customary to use that space for setting the problem. In the present situation the right-hand-side of the equation does not give rise to a positive-definite quadratic form and we use instead the left-hand-side to definite such a form. We prove in this paper that this form determines a Hilbert space (such problems are called left-definite).

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1 Introduction
Let \(\mathbb{N}\) be the set of natural number. Define \(S(\mathbb{N})\) to be the set of all the sequences over \(\mathbb{N}\) which are complex valued. If \(u \in S(\mathbb{N})\)
then define $\triangle : S(\mathbb{N}) \to S(\mathbb{N})$ to be the first forward difference operator given by

$$(\triangle u)(n) = u(n + 1) - u(n).$$

Using this definition,

$$(\triangle (fg))(n) = g(n + 1)(\triangle f)(n) + f(n)(\triangle g)(n). \quad (1.1)$$

Also, using the fact $\sum_{i=j}^{k}(\triangle u)(i) = u(k + 1) - u(j)$, we get the summation by parts formula:

$$\sum_{n=j}^{N} g(n + 1)(\Delta f)(n) = (fg)(N + 1) - (fg)(j) - \sum_{n=j}^{N} f(n)(\Delta g)(n). \quad (1.2)$$

This equation implies

$$\sum_{n=1}^{N} (p\Delta u)(n)\Delta v(n)$$

$$= (p\Delta u)(N)v(N + 1) - (p\Delta u)(0)v(1) - \sum_{n=1}^{N} \Delta(p\Delta u)(n - 1)v(n). \quad (1.3)$$

We associate the term left-definite problem with an inner product associated with the left hand side of the equation $Lu = wf$.

The left-definite spectral problem was first raised by Weyl in his seminal paper $[10]$ and treated by him in $[9]$. There is now a large body of literature on the problem of determining spectral properties for such systems. We mention here for instance Niessen and Schneider $[7]$, Krall $[3, 4]$, Marletta and Zettl $[6]$, Littlejohn and Wellman $[5]$.

In this paper, we are interested in studying an inner product determined by the left-hand-side of the difference equation

$$-(\Delta(p\Delta u))(n - 1) + q(n)u(n) = \lambda w(n)u(n); \quad n \geq 2, \quad (1.4)$$

Some spectral properties were discussed in $[11]$ related to left-hand-side of the equation

$$-(\Delta^2 u)(n - 1) + q(n)u(n) = \lambda w(n)u(n); \quad n \geq 2. \quad (1.5)$$
Unlike the continuous case, the equation (1.4) cannot be transformed to (1.5).

Now, for the solutions $\phi$ and $\theta$ of the equation (1.4), we define the Wronskian, $W_{\phi, \theta}$, to be

$$W_{\phi, \theta}(n) = p(n)(\phi(n)(\Delta \theta)(n) - (\Delta \phi)(n)\theta(n)).$$

**Proposition 1.1.** $W_{\phi, \theta}(n)$ is constant for all $n \in \mathbb{N}$.

**Proof.** Using the product rule (1.1)

$$(\Delta W_{\phi, \theta})(n) = \phi(n + 1)(\Delta(p\Delta \theta))(n) + (\Delta \phi)(n)(p\Delta \theta)(n)$$

$$- (\theta(n + 1)(\Delta(p\Delta \phi))(n) + (p\Delta \phi)(n)(\Delta \theta)(n))$$

Using the fact that $\phi, \theta$ are solutions for (1.4), then

$$(\Delta W_{\phi, \theta})(n)$$

$$= \phi(n + 1)((q - \lambda w)\theta)(n + 1) - \theta(n + 1)((q - \lambda w)\phi)(n + 1) = 0.$$ 

Hence, the Wronskian is constant. \qed

Our main interest is studying the equation (1.4) where $\lambda$ is a complex parameter and where $q$ and $w$ are sequences with $q$ defined on $\mathbb{N}_0$ and assumes non-negative real values but is not identically equal to zero, $w$ is defined on $\mathbb{N}$ and real-valued, and $p$ is defined on $\mathbb{N}_0$ and assumes strictly positive real values.

Consider the operator on the left-hand side of (1.4) by $L$, i.e.,

$$(Lu)(n) = -\Delta(pu\Delta)(n - 1) + (qu)(n), \quad n \in \mathbb{N}.$$ 

Note that $L$ operates from $\mathbb{C}^{\mathbb{N}_0}$ to $\mathbb{C}^{\mathbb{N}}$.

### 2 Main Result

Due to the fact that the sign of $w$ is indefinite it is not convenient to phrase the spectral and scattering theory in the usual setting of a weighted $\ell^2$-space, since it is not a Hilbert space. Instead the requirement that $q$ is non-negative but not identically equal to zero allows us to define an inner product associated with the left hand
side of the equation $Lu = wf$ giving rise to the term left-definite problem. To do so define the set

$$H_1 = \{ u \in C^{N_0} : \sum_{n=0}^{\infty} (p(n)|\Delta u(n)|^2 + q(n)|u(n)|^2) < \infty \}$$

and introduce the scalar product

$$< u, v > = \sum_{n=0}^{\infty} (p(n)|\Delta u(n)|\Delta v(n) + q(n)|u(n)v(n)|).$$

The associated norm is denoted by $\| \cdot \|$. We will also use the norm in $\ell^2(N_0)$ which we denote by $\| \cdot \|_2$. We claim $H_1$ with this norm is a complete space. Such a result plays a role in studying the spectral properties of (1.4).

We start with the following sequence of lemmas:

**Lemma 2.1.** If $m \geq n$, then for $u \in S(N)$

$$|u(m)| \leq |u(n)| + \sum_{l=1}^{\infty} p(l)(\Delta u(l))^2)^{1/2}(\sum_{l=n}^{m-1} \frac{1}{p(l)})^{1/2}. \tag{2.1}$$

**Proof.**

\[|u(m)| - |u(n)| \leq |u(m) - u(n)|,\]

and

\[|u(m) - u(n)| = |\sum_{l=n}^{m-1} (\Delta u(l))| \leq \sum_{l=n}^{m-1} |\Delta u(l)|.\]

Now, the inequality of Cauchy-Schwarz gives that

\[\sum_{l=n}^{m-1} |\Delta u(l)| = \left(\sum_{l=n}^{m-1} \sqrt{p(l)}|\Delta u(l)| \left(\frac{1}{\sqrt{p(l)}}\right)\right) \leq \left(\sum_{l=n}^{m-1} p(l)(\Delta u(l))^2\right)^{1/2}(\sum_{l=n}^{m-1} \frac{1}{p(l)})^{1/2}.\]

By combining the previous inequalities, we get:

\[|u(m)| - |u(n)| \leq \left(\sum_{l=n}^{m-1} p(l)(\Delta u(l))^2\right)^{1/2}(\sum_{l=n}^{m-1} \frac{1}{p(l)})^{1/2}.,\]
this inequality implies the required result since
\[
\left( \sum_{l=m}^{m-1} p(l) |\Delta u(l)|^2 \right)^{1/2} \leq \left( \sum_{l=1}^{\infty} p(l) |\Delta u(l)|^2 \right)^{1/2}.
\]

\[\square\]

**Lemma 2.2.** If \( r \) satisfies \( \sum_{n=1}^{r} q(n) > 0 \), then for \( 1 \leq n \leq m \leq r < \infty \) and \( u \in S(N) \)

\[
|u(m)| \sum_{n=1}^{r} q(n) \leq \left( \sum_{n=1}^{r} q(n) \right)^{1/2} \left( \sum_{n=1}^{r} q(n) |u(n)|^2 \right)^{1/2} + C_r \left( \sum_{l=1}^{\infty} p(l) |\Delta u(l)|^2 \right)^{1/2} \sum_{n=1}^{r} q(n),
\]

where \( C_r = \left( \sum_{l=1}^{r} \frac{1}{p(l)} \right)^{1/2} \).

**Proof.** The equation (2.1) gives that

\[
|u(m)| \sum_{n=1}^{r} q(n) \leq \left( \sum_{n=1}^{r} q(n) \right)^{1/2} \left( \sum_{n=1}^{r} q(n) |u(n)|^2 \right)^{1/2} + C_r \left( \sum_{l=1}^{\infty} p(l) |\Delta u(l)|^2 \right)^{1/2} \sum_{n=1}^{r} q(n),
\]

Multiplying by \( q(n) \) and taking the sum from 1 to \( r \) with respect to \( n \) give

\[
|u(m)| \sum_{n=1}^{r} q(n) \leq \left( \sum_{n=1}^{r} q(n) \right)^{1/2} \left( \sum_{n=1}^{r} q(n) |u(n)|^2 \right)^{1/2} + C_r \left( \sum_{l=1}^{\infty} p(l) |\Delta u(l)|^2 \right)^{1/2} \sum_{n=1}^{r} q(n),
\]

Now, the inequality of Cauchy-Schwarz gives

\[
\sum_{n=1}^{r} q(n) |u(n)| \leq \left( \sum_{n=1}^{r} q(n) \right)^{1/2} \left( \sum_{n=1}^{r} q(n) |u(n)|^2 \right)^{1/2}.
\]

Then (2.2) becomes

\[
|u(m)| \sum_{n=1}^{r} q(n) \leq \left( \sum_{n=1}^{r} q(n) \right)^{1/2} \left( \sum_{n=1}^{r} q(n) |u(n)|^2 \right)^{1/2} + C_r \left( \sum_{l=1}^{\infty} p(l) |\Delta u(l)|^2 \right)^{1/2} \sum_{n=1}^{r} q(n).
\]

\[\square\]
We are ready to prove the following lemma:

**Lemma 2.3.** For any $N \in \mathbb{N}$, there exists $C_N$ such that

$$|u(m)| \leq C_N \|u\|_{H_1},$$

for any $m$ such that $1 \leq m \leq N$ and any $u \in H_1$.

**Proof.** For any $N \in \mathbb{N}$ there exists $r \geq N$ such that $\sum_{n=1}^{r} q(n) > 0$. Now, Lemma 2.2 implies

$$|u(m)| \sum_{n=1}^{r} q(n) \leq \|u\|_{H_1} \sum_{n=1}^{r} q(n)^{1/2} + C_r \sum_{n=1}^{r} q(n),$$

or

$$|u(m)| \leq C_N \|u\|_{H_1},$$

where

$$C_N = C_r + \left( \sum_{n=1}^{r} q(n) \right)^{-1/2}.$$

The following lemma gives some properties for the Cauchy sequences in $H_1$.

**Lemma 2.4.** Let $n \mapsto u_n(\cdot)$ be a Cauchy sequence in $H_1$, then

1. There exists $v(\cdot)$ such that $(\sqrt{p} \Delta u_n)(\cdot) \rightarrow v(\cdot)$ in $l^2(\mathbb{N})$ as $n \rightarrow \infty$.

2. $\sqrt{q(\cdot)} u_n(\cdot) \rightarrow \sqrt{q(\cdot)} u(\cdot)$ in $l^2(\mathbb{N})$, where $u(k) = \lim_{n \rightarrow \infty} u_n(k)$ in $C$ for all $k \in \mathbb{N}$.

**Proof.** 1. If $n \mapsto u_n(\cdot)$ is a Cauchy sequence in $H_1$, then for $\varepsilon > 0$, there exists $n_0$ such that for all $m, n \geq n_0$

$$\|u_m(\cdot) - u_n(\cdot)\|_{H_1} < \varepsilon \quad (2.3)$$

consequently,

$$\|(\sqrt{p} \Delta u_m)(\cdot) - (\sqrt{p} \Delta u_n)(\cdot)\|_{l^2(\mathbb{N})} < \varepsilon,$$

this means by the completeness of $l^2(\mathbb{N})$ that there exists $v(\cdot)$ such that, as $n \rightarrow \infty$,

$$(\sqrt{p} \Delta u_n)(\cdot) \rightarrow v(\cdot) \text{ in } l^2(\mathbb{N}). \quad (2.4)$$

Therefore,

$$(\sqrt{p} \Delta u_n)(k) \rightarrow v(k) \text{ in } C. \quad (2.5)$$
2. Lemma 2.3 gives $K_r$ such that if $k \leq r$

$$|u_m(k) - u_n(k)| \leq K_r \|u_m(\cdot) - u_n(\cdot)\|_{H_1} < K_r \varepsilon,$$

this means that $n \mapsto u_n(k)$ is a Cauchy sequence in $C$. The completeness of the complex numbers $C$ gives the existence of $u \in S(N)$ such that, as $n \to \infty$,

$$u_n(k) \to u(k) \text{ in } C \quad (2.6)$$

and hence

$$\sqrt{q(k)}u_n(k) \to \sqrt{q(k)}u(k) \text{ in } C. \quad (2.7)$$

Moreover, equation (2.3) gives

$$\|\sqrt{q(\cdot)u_m(\cdot)} - \sqrt{q(\cdot)u_n(\cdot)}\|_{l^2(N)} < \varepsilon.$$ 

Again by the completeness of $l^2(N)$ then there exists $\nu(\cdot)$ such that as $n \to \infty$, $\sqrt{q(\cdot)u_n(\cdot)} \to \nu(\cdot)$ in $l^2(N)$ this means

$$\sum_{k=1}^{\infty} |\sqrt{q(k)}u_n(k) - \nu(k)|^2 \to 0.$$ 

Hence, for any $k$,

$$\sqrt{q(k)}u_n(k) \to \nu(k) \text{ in } C,$$

which implies by (2.7) $\nu(k) = \sqrt{q(k)}u(k)$. 

\[ \Box \]

**Proposition 2.5.** The space $H_1$ is complete.

**Proof.** First, assume $n \mapsto u_n(\cdot)$ is a Cauchy sequence. Then using Lemma 2.4 there exist $u \in S(N)$ such that $u_n$ converges to $u$ pointwise and $\nu(\cdot) \in L^2(N)$ such that $(\Delta u_n)(\cdot) \to \nu(\cdot)$. This proves that $u(k) = u(1) + \sum_{j=1}^{k-1} v(j) \in H_1$. Also, this lemma implies $(\sqrt{p}\Delta u_n)(\cdot) \to (\sqrt{p}\Delta u)(\cdot)$ and $(\sqrt{q}u_n)(\cdot) \to (\sqrt{q}u)(\cdot)$ in $L^2(N)$.

Moreover, one can prove that $u_n(\cdot) \to u(\cdot)$ in $H_1$ as follows. Since

$$\|u_n(\cdot) - u(\cdot)\|_{H_1} = \sum_{k=1}^{\infty} p(k) |(\Delta u_n)(k) - (\Delta u)(k)|^2 + \sum_{k=1}^{\infty} q(k) |u_n(k) - u(k)|^2,$$

then

$$\|u_n(\cdot) - u(\cdot)\|_{H_1} = \sum_{k=1}^{\infty} \left| (\sqrt{p(k)}(\Delta u_n)(k) - v(k)) \right|^2.$$
\[ + \sum_{k=1}^{\infty} |(\sqrt{q(k)}(u_n(k) - u(k)))^2|. \]

Using Lemma 2.4 and the last equation, we get \( \|u_n(\cdot) - u(\cdot)\|_{\mathcal{H}_1} \rightarrow 0 \), which means \( u_n(\cdot) \rightarrow u(\cdot) \) in \( \mathcal{H}_1 \), and hence the Cauchy sequence in \( \mathcal{H}_1 \) is convergent, this means \( \mathcal{H}_1 \) is complete. \( \square \)

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