Fermion Masses, Mixing Angles and Supersymmetric SO(10) Unification

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Abstract

We reanalyse the problem of fermion masses in supersymmetric SO(10) grand unified models. In the minimal model, both low energy Higgs doublets belong to the same 10 representation of SO(10) implying the unification not only of the gauge but also of the third generation Yukawa couplings. These models predict large values of $\tan \beta \sim 50$. In this paper we study the effects of departing from the minimal conditions in order to see if we can find models with a reduced value of $\tan \beta$. In order to maintain predictability, however, we try to do this with the addition of only one new parameter. We still assume that the fermion masses arise from interactions of the spinor representations with a single 10 representation, but this 10 now only contains a part of the two light Higgs doublets. This enables us to introduce one new parameter $\omega = \lambda_b/\lambda_t$. For values of $\omega \ll 1$ we can in principle reduce the value of $\tan \beta$. In fact, $\omega$ is an overall factor which multiplies the down quark and charged lepton Yukawa matrices. Thus the theory is still highly constrained. We show that the first generation quark masses and the CP-violation parameter $\epsilon_K$ are sufficient to yield strong constraints on the phenomenologically allowed models. In the end, we find that large values of $\tan \beta$ are still preferred.

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1 Introduction

The Standard Model describes with a great degree of precision the observed elementary particle interactions. It provides, however, no answer to the fundamental questions about the origin of the gauge group $SU(3)_c \times SU(2)_L \times U(1)_Y$, the structure of fermion masses and mixing angles and their quantum numbers. Grand Unified Theories (GUTs) have the power to fill the gap between theory and experiment [1]. Indeed, within this framework the low energy group proceeds from the spontaneous breakdown of a single compact group. The simplest and most attractive grand unified theories are based on the unitary group $SU(5)$ or the orthogonal group $SO(10)$. Remarkably, all low energy fermion quantum numbers find a natural explanation within these theories. For instance, the fifteen Weyl fermions in a Standard Model family, with their correct quantum numbers under the standard model gauge group, are contained in a $10$ and a $5$ representation of $SU(5)$. Most notably, they are contained in a single spinor representation of $SO(10)$, the extra state having the quantum numbers of a right handed neutrino and leading therefore to the possibility of including neutrino masses in a natural way.

If the grand unified group breaks at very high energies to the standard model gauge group, an essential requirement is that the theory should be supersymmetric [2]. Not only does supersymmetry stabilize the hierarchy between the grand unified scale and the weak scale, but also the predictions coming from gauge coupling unification within supersymmetric theories are in remarkably good agreement with the precise measurements of the weak mixing angle performed at LEP [3]-[5]. Moreover, supersymmetry provides the natural framework for the construction of a theory of quantum gravity, and hence for the unification of all forces observed in nature. Supersymmetric grand unified theories provide also a simple theoretical framework for the understanding of fermion masses. The condition of bottom–tau Yukawa coupling unification implies, for instance, a large value of the top quark Yukawa coupling at the grand unification scale [6],[7], the low energy value of the top quark mass being governed, in general, by the infrared fixed point structure of the theory [8]-[10]. Hence, supersymmetric GUTs provide an understanding of the large value of the top quark mass [11],[12]. Moreover, in the minimal SO(10) model, the three Yukawa couplings of the third generation unify at the grand unification scale. This yields predictions not only for the top quark mass, but also for the ratio of Higgs vacuum expectation values, $\tan \beta$, which becomes naturally large [16]. Large values of $\tan \beta$ are also associated with large corrections to the bottom mass [13],[14], which depend on the supersymmetric spectrum and which should be computed in a consistent way in order to obtain phenomenologically correct predictions for the top quark mass [15].

2 Minimal SO(10) models

The hierarchy between the third and the first and second generation quark masses, as well as the inter–generation mixing angles, may be explained by assuming that only the third generation quarks couple to the $10$ of Higgs by renormalizable interactions, while the other mass terms are induced through higher order operators. A systematic search for this class of
models within the framework of minimal SO(10) was done in Ref. [17], under the assumption that the model includes only three spinor representations, containing the three low energy families, a few extra heavy spinor representations, the 10 Higgs multiplet and some 45’s, necessary for the correct breakdown of the gauge symmetry and for the generation of the fermion mass operators. All higher order operators are of the form

$$O_{ij} = 16_i \frac{M_G^{k+1} \cdot \cdot \cdot 45_{m}}{M_P^{l} 45^{m-l}_X} 10 \frac{M_G^{n} \cdot \cdot \cdot 45_{p}}{M_P^{q} 45^{p-q}_X} 16_j,$$

where the 45 vevs in the numerator can be in any of the 4 directions, X, Y, B – L, T_{3R} (discussed below) and the 45 in the denominator can only be in the X direction which breaks SO(10) down to the subgroup SU(5) × U(1)_{X}. This occurs at a scale M_{10} which is assumed to lie between the GUT scale M_{G} \sim 10^{16} GeV and the Planck scale M_{P}.

The adjoint 45’s may be labelled according to the direction of their vacuum expectation values. There are four special directions [17]. The X direction, necessary for the breakdown of SO(10) to SU(5) × U(1)_{X} at the scale M_{10}. The 45_{X} in the denominator can arise when integrating out heavy 16 and \overline{16} states with mass from the 45_{X} vev. Of course, this only makes sense if M_{10} > M_{G}. Other directions are the Y and B – L, which break SU(5) to the Standard Model gauge group. The presence of the latter is required for a natural solution of the doublet–triplet splitting problem in this theory [18]. Finally, there is another, linearly dependent direction, T_{3R}, which, as we shall explain below, may be useful to achieve low values of tan β within this model.

Taking into account the experimental constraints on the lowest generation fermion masses and the Cabibbo-Kobayashi-Maskawa (CKM) mixing angles, the authors of Ref. [17] identified nine potentially acceptable models, in which the up and down quark and lepton mass matrices are of the form,

$$\lambda_a = \begin{pmatrix}
0 & z_a' C & 0 \\
za C & ya E e^{i \phi} & x_a' B \\
0 & x_a B & A 
\end{pmatrix},$$

where za, z’a, xa, x’a and ya are Clebsch factors, while A, B, C, E and φ are arbitrary parameters, which respect the hierarchy A \gg B, E \gg C and must be adjusted in order to obtain predictions in agreement with the present data. The Higgs sector provides an additional free parameter, which is the ratio of vacuum expectation values, tan β. Using the presently best known low energy parameters m_e, m_μ, m_τ, m_c, m_b and |V_{cd}| as input, the values of M_{t}, tan β, |V_{cb}|, |V_{ub}|, m_u, m_d, m_s and the CP-odd Jarlskog invariant J [14] are predicted (we shall denote physical and running masses by capital and small letters, respectively). This leads, hence, to eight low energy predictions, which should be compared with the present experimental values.

There are several properties, which are shared by all these models. First of all, they maintain the Georgi–Jarlskog relation [20] of the ya Clebsch factors: |y_e| : |y_d| : |y_u| \equiv 3 : 1 : 0. This relation of Clebsch factors appears in a natural way, for example, through the operator

$$O_{22} = 16_2 \frac{45_X}{M} 10 \frac{45_{B-L}}{45_X} 16_2,$$
and it is important in order to derive correct predictions for the first two generations of
quark masses. In fact,

$$\frac{m_s}{m_d} \simeq \left( \frac{y_d}{y_e} \right)^2 \frac{m_\mu}{m_e} \left| \frac{z_e z'_e}{z_d z'_d} \right|.$$  \hspace{1cm} (4)

Hence, as long as the equality $z_e z'_e = z_d z'_d$ holds, the ratio of lepton and quark masses is in
good quantitative agreement with the observed experimental values.

Another important property of these models is the unification of the three Yukawa cou-
plings of the third generation and, in particular, the unification of the bottom and top Yukawa
couplings, which requires large values of $\tan \beta$. Such large values of $\tan \beta$ are associated with
three effects:

1. potentially large corrections to the down quark mass matrix (these radiative corrections
   are discussed in detail in the appendix);

2. with some fine-tuning of GUT scale soft SUSY breaking parameters in order to obtain
   radiative electroweak symmetry breaking at the weak scale. The range of parameters
   which satisfy the second constraint (when universal scalar masses are imposed at $M_G$),
   in fact, requires the corrections to down quark masses to be large, and

3. the proton decay rate resulting from dimension 5 baryon violating interactions is en-
   hanced.

It has recently been shown that the first two consequences of large $\tan \beta$ are ameliorated
when the constraint of universal scalar masses is removed\textsuperscript{[26]}. The corrections to the down
quark masses can be small and the amount of fine-tuning is greatly reduced. The problem
of an enhanced proton decay rate is unaffected. On the other hand, these strong constraints
become weaker for smaller values of $\tan \beta$. It becomes an important question whether the
prediction for large $\tan \beta$ can be altered without destroying the predictability of the theory.

\section{Trying to reduce $\tan \beta$ in minimal SO(10) models}

Lower values of $\tan \beta$ can easily be achieved by assuming that only one $\mathbf{10}$ of Higgs couples to
fermions, but this $\mathbf{10}$ contains only a piece of the two Higgs doublets, the other components
coming, for instance, from an additional $\mathbf{10}$. The overall effect is to multiply the down and
lepton mass matrices by a factor $\omega$, which is the ratio of the relative components of the two
Higgs doublets in the $\mathbf{10}$ which couples to fermions. The minimal model would hence be
obtained for $\omega = 1$.

Such a situation can come about as follows: Consider the superpotential

$$W = \mathbf{10} \, 45_{B-L} \mathbf{10}' + \left[ M_1 \mathbf{10}' + (M_2 + 45_X) \mathbf{10} \right] \mathbf{10}''$$  \hspace{1cm} (5)

where $M_1$ and $M_2$ are of order $M_{GUT}$; $\mathbf{10}$, $\mathbf{10}'$ and $\mathbf{10}''$ are decouplets and only $\mathbf{10}$ participates
in the fermion mass operators.

The first term in $W$ implements the Dimopoulos-Wilczek mechanism \textsuperscript{[18]} and yields 4
light doublets: $\mathbf{2}$, $\tilde{\mathbf{2}}$, $\tilde{\mathbf{2}}'$ and $\tilde{2}$; the color triplets get a mass of order $M_{GUT}$. The second term
gives a mass to a linear combination of $2$ and $2'$ (by pairing it with $\bar{2}''$) and a different linear combination of $\bar{2}$ and $\bar{2}'$ (by pairing it with $2''$). Explicitly, the light states are given by

$$2_L = \frac{M_1 \frac{2 - (M_2 + v)}{2'}}{\sqrt{M_1^2 + (M_2 + v)^2}}$$

and

$$\bar{2}_L = \frac{M_1 \frac{2 - (M_2 - v)}{2'}}{\sqrt{M_1^2 + (M_2 - v)^2}},$$

where $<45_X> = v \times X$, with $X = +(-)$ when it acts on the 5 ($\bar{5}$) of a 10 representation, respectively. Since $2$ couples to the up quarks and $\bar{2}$ couples to the down quarks, in this example we have

$$h_t = \lambda \frac{M_1}{\sqrt{M_1^2 + (M_2 + v)^2}},$$

and

$$h_b = \lambda \frac{M_1}{\sqrt{M_1^2 + (M_2 - v)^2}},$$

Notice that, in this simple example, $M_1$ (or $M_2 \pm v$) cannot be too small, or else a pair of light triplets $3$ and $\bar{3}$ would appear in the spectrum, affecting the prediction for $\sin^2 \theta_W$. Hence, $\omega$ cannot become too small in this case.

From now on, we shall discuss the consequences of the departure from the minimal conditions, taking values of $\omega$ lower than one. Values of $\omega$ lower than one decreases the bottom to top Yukawa coupling ratio but still requires bottom–tau Yukawa coupling unification.

### 3.1 \textbf{SO}(10) models with moderate values of $\tan \beta$ — The second and third generations

We have introduced the parameter $\omega \leq 1$ in an attempt to lower $\tan \beta$. In this section we discuss the results for the second and third generations with the additional parameter $\omega$. In general, taking into account variations of $\omega$ and reasonable assumptions on radiative corrections to down quark masses, we find that, in order to avoid a very heavy top quark, with mass larger then 190 GeV, the value of $\tan \beta$ should be either larger than 20 or very close to 1.

As a general feature, in order to obtain unification of the bottom and tau Yukawa couplings, the third generation Yukawa couplings must partially compensate the strong gauge coupling renormalization group effects. For $\omega = 1$, this is partially achieved by large values of the bottom Yukawa coupling. Indeed, the relation between the bottom quark and tau masses is given by

$$\frac{m_b}{m_\tau} = G \exp(-I_t - 3I_b + 3I_\tau),$$

where $G$ is the gauge coupling constant.
where $G$ includes the $\omega$ independent, gauge coupling dependent factors, $I_a = \int (h_a/4\pi)^2 dt$ with $h_a$ the corresponding Yukawa coupling and $t = \ln(Q/M_Z)$. In the following, we shall always assume that the right handed neutrinos acquire large Majorana masses of order $M_{GUT}$ and hence decouple from the renormalization group equations. Although there is a partial cancellation of the bottom and tau Yukawa coupling contributions at scales close to the unification scale, due to the factor 3 and the relation $I_b > I_\tau$, the bottom contribution becomes important for $\omega = 1$. For values of $\omega < 1$, for which only the bottom and $\tau$ Yukawa couplings unify, the top Yukawa coupling must increase in order to compensate for the smaller contribution of the bottom Yukawa coupling. For smaller values: $\omega < 0.5$, associated with moderate or small values of $\tan \beta$, and in the absence of supersymmetric threshold corrections, the top quark Yukawa coupling must acquire large values at the grand unification scale, being driven towards its infrared fixed point value at low energies. The convergence of the top quark mass to its fixed point value is naturally weaker for $\omega \simeq 1$.

For $\tan \beta \geq 5$, the fixed point value of the pole top quark mass reads $M_t \simeq 190-210$ GeV, which is somewhat large in comparison to the current experimentally preferred value $M_t \simeq 180 \pm 12$ GeV [27]. The convergence to the fixed point for moderate values of $\omega$ may be softened by the presence of large bottom mass corrections, which become particularly relevant for values of $\tan \beta > 10$. For values of $\tan \beta \leq 5$, the bottom mass corrections are generically small, but the infrared fixed point value of the top quark mass, $M_t \simeq \sin \beta \times 200$ GeV, is lowered by the $\sin \beta$ factor (see Fig. 1). Indeed, values of $\tan \beta \lesssim 3$ are required, for the fixed point solution to be in the range of phenomenologically preferred values. As we shall discuss below, these small values of $\tan \beta$ demand very small values of $\omega$.

Figure 1 shows the dependence of the pole top quark mass on $\tan \beta$ (and also on $\omega$) for three different values of $\alpha_3(M_Z)$ and different values of the coefficient $K_c$ parametrizing the bottom mass corrections, $\delta m_b = -m_b K_c \tan \beta$. Values of $K_c \geq 0.005$ lead to significant corrections to the predicted top quark mass values and, as was shown in Ref [15], may appear in the presence of universal soft supersymmetry breaking mass parameters at the grand unification scale. We concentrate on positive values of $K_c$, since for negative and large $K_c$ either the bottom quark mass is above its experimentally preferred values or a Landau pole in the top quark Yukawa coupling appears at scales below $M_{GUT}$. In Figure 1 we have chosen a representative value of $m_b(m_b) = 4.15$ GeV. Larger (lower) values of $m_b$ within the experimentally allowed range $m_b = 4.25 \pm 0.25$ GeV, would lead to somewhat lower (larger) values of $M_t$ [5], without changing the general properties of the solutions.

It is interesting to note that, for large values of $K_c$ and low values of $\alpha_3(M_Z)$, the top quark mass predictions in model 6 differ from the ones obtained in model 9 for the same values of $\omega$. This reflects the effect of the mixing between the second and third generations on the predictions for the third generation masses. It is easy to prove that, although this effect is generically small, the $\tau$ mass in model 9 receives a significant correction due to the mixing, which for values of $\alpha_3(M_Z) = 0.115$ and $K_c = 0.006$ becomes of order 15%. Due to the condition of bottom-tau Yukawa unification, large tau mass corrections also imply large variations in the top quark mass predictions.

To summarize, we observe that depending on the size of the one loop supersymmetric corrections to the down quark masses, successful top quark mass predictions may be obtained for the minimal models with $\omega = 1$, but also for moderate and small values of $\tan \beta$ (associated
with moderate or very small values of $\omega$). It is hence important to know if the same is true for the first two generations of quark masses and mixing angles. Since the relation between the down quark and lepton masses is only weakly dependent on $\omega$, we should maintain the Georgi-Jarlskog relation even for values of $\omega$ different from one. Moreover, the values of $V_{cb}$ can also be successfully accommodated for lower values of $\omega$. This can be easily seen writing its dependence in terms of the top and charm quark masses,

$$|V_{cb}| \simeq \chi \sqrt{\frac{m_c}{\eta_{\alpha} m_t}} \exp \left( \frac{I_b - I_t}{2} \right)$$

(11)

where $\chi = |x_d - x_u|/\sqrt{|x_u x'_u|}$, and $\eta_{\alpha}$ are the $\omega$ independent, renormalization group factors relating the running masses at the scale $M_Z$ with the on–shell ones (for the $u$, $d$ and $s$ quarks, the scale of definition of the running masses is taken to be 1 GeV). For $\alpha_3(M_Z) \simeq 0.12$, for which the value of $\eta_c \simeq 2.2$, it follows that the phenomenologically preferred values of $|V_{cb}| = 0.040 \pm 0.005$, require values of $\chi < 1$ [23]. The different values of $\chi$ are the basis for the classification of models performed in Ref. [17], where the best fit to the data was achieved by two models: model 6, with $x_u = x'_u = -4$, $x_d = x'_d = -2/3$ and $x_e = x'_e = 6$, and model 9 with $x_u = x'_u = 1$, $x_d = 1/9$, $x_e = 9$ and $x'_d = x'_e = 1$. These models have $\chi = 5/6$ and $8/9$, respectively and both lead to somewhat large values of $V_{cb}$. Model 4, with $\chi = 2/3$, leads to a better prediction for $V_{cb}$, but yields insufficient CP-violation.

For lower values of $\omega$, the dependence of Eq. (11) on $I_b$ and $I_t$ is such, that the values of $V_{cb}$ tend to decrease. Moreover, in the absence of down quark mass matrix corrections, for $\tan \beta \geq 4$, the value of $V_{cb}$ decreases due to larger top quark mass values, which, as we discussed before, may become too large in comparison with the experimentally preferred ones. As shown in Fig. 1, lower values of the top quark mass may be obtained through large bottom mass corrections. Lowering the top quark mass enhances the value of $V_{cb}$, but the total effect of the down quark mass corrections on $V_{cb}$ cannot be determined a priori; it depends on the relative size of the gluino corrections, which affect the value of $V_{cb}$ due to their effect on the predicted top quark mass value, and the chargino corrections, which modify not only the top quark mass value through the bottom mass corrections, but they have also a direct effect on the CKM matrix elements [21] (see Appendix).

Fig. 2 shows the predictions for $V_{cb}$ for models 6 and 9, as a function of $\tan \beta$, for three different bottom mass corrections and three different values of $\alpha_3(M_Z)$, under the assumption that $(\delta m_b/m_b)^{\beta} = -3(\delta m_b/m_b)^{ch}$ (which is reasonable in view of the running of the soft breaking parameters and the structure of the bottom mass corrections when the squark mass matrices are approximately three by three block diagonal [15]). We see that, independently of the bottom mass corrections, the predictions for $V_{cb}$ may be significantly improved for moderate values of $\omega$. Indeed, apart from the solutions with $\tan \beta$ very close to one, $\omega = 1$ leads to the largest values of $V_{cb}$ for each fixed $\Delta m_b$ correction. Observe as well that for values of $\tan \beta \leq 2$, $V_{cb}$ increases, due to the lower values of $m_t$ appearing in this regime. Furthermore, for the present case, for any fixed value of $\omega$, there is an effective cancellation of the chargino and gluino-induced one loop corrections to $V_{cb}$ and the total effect of the down quark mass corrections on $V_{cb}$ is small. Consequently, since for a fixed value of $\omega$ large down quark mass corrections lower the value of $\tan \beta$, as can be seen from
Fig. 2, they also yield larger values of $V_{cb}$ for a given fixed value of $\tan \beta$. From Fig. 2 we also observe that the predictions for $V_{cb}$ improve for larger values of $\alpha_3(M_Z)$. In fact, for moderate values of $\omega$, if large values of $\alpha_3(M_Z)$ and large bottom mass corrections are present, the predictions for $V_{cb}$ in model 6 may actually be below the preferred experimental values, but these solutions are associated with values of $M_t$ which are generally too large.

From the discussion above, we see that the second and third generation fermion masses and mixing angles can be consistently described within an SO(10) GUT with $\omega \leq 1$. However, as we shall show in the following, the constraint coming from the predictions for the first generation quark masses rule out values of $\omega \leq 0.5$ within the minimal model. In section 4 we show how to overcome this difficulty at the expense of adding one new operator and 2 more parameters, in addition to $\omega$.

3.2 The first generation

The operator $O_{12}$ is necessary to achieve acceptable predictions for the lowest generation quark masses. Indeed, within the minimal model, there is a “unique” operator,

$$O_{12} = 16_1 \left( \frac{45_X}{M_p} \right)^n 10 \left( \frac{45_X}{M} \right)^m 16_2,$$

with $n = m = 3$, which yields acceptable ratios for the masses of the up, down and strange quarks. This operator determines the equality of the Clebsch factors $z_a$ and $z'_a$ and the ratios of the Clebsch factors $z_d/z_u = 27$ and $z_d/z_c = 1$ (the ratio of Clebsch factors $z_d/z_u$ increases by a factor 3 for each power of $45_X$). In addition the ratio, appearing in Eq. (12),

$$\frac{(z_u z'_u)/(z_d z'_d)}{\eta_u \eta_c \eta_d \eta_s} = 1.$$ 

The above operator is of dimension ten, meaning that the absence of any lower dimensional operators should be insured by some symmetry of the theory.

For $\omega = 1$, one might think that the large ratio of Clebsch factors, $z_d/z_u$, arising from the above relation, Eq. (12), is necessary in order to compensate the $\tan \beta$ ($\approx m_t/m_b$) dependence of the up–type quark masses with respect to the down–type quark ones. It is interesting to investigate then if lower values of $\omega$, and hence of $\tan \beta$, can serve to relax the restrictions on the Clebsch factors and hence, to lower the dimensionality of the above $O_{12}$ operator. This, however, is not the case, as can be easily shown considering the relation

$$\frac{m_u}{m_d} = \frac{m_u}{m_c} \left( \frac{m_t}{m_b} \right)^2 \frac{\eta_u \eta_c \eta_d \eta_s}{\eta_u \eta_c \eta_d \eta_s} \left| \frac{z_u z'_u}{z_d z'_d} \right| \exp 4(I_t - I_b).$$

From Eq. (13) it follows that independent of the source of the hierarchy between the top and bottom quark masses, large ratios of Clebsch factors are necessary in order to obtain the phenomenologically preferred values for the ratio of the up to down quark masses, $0.2 \leq m_u/m_d \leq 0.8$. The additional dependence on the integral factors $I_b$ and $I_t$ does not help to lower this ratio. On the contrary, since for lower values of $\omega$, the integral factor $I_b$ decreases, while $I_t$ changes only slightly, for the same values of the second and third generation quark masses the ratio of the up to the down quark masses increases. This means that, in order to keep phenomenologically allowed values of the first generation quark masses, the ratio of the Clebsch factors $z_u/z_c$ should actually increase, implying that the dimensionality of the operator $O_{12}$ should correspondingly increase for lower values of $\omega$. 

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It is therefore apparent that, keeping the same operator structure as before, the range of possible values of $\omega$, that is to say of $\tan \beta$, will be restricted. Indeed, Figure 3 shows the dependence of the ratio $m_u/m_d$ as a function of $\tan \beta$ ($\omega$) for models 6 and 9 and for different values of the down quark mass corrections, under the same assumptions discussed for Fig. 1. It follows that the down quark mass corrections help only marginally in getting phenomenologically allowed values for $m_u/m_d$, and values of $\omega \lesssim 0.5$ are disfavoured for all these models. Indeed, for larger values of $\alpha_3(M_Z) \geq 0.12$, even larger values of $\omega$ are necessary in order to achieve good predictions for the first generation masses.

Increasing by one the dimensionality of the operator $O_{12}$ keeps the equality $z_e z'_e = z_d z'_d$, necessary to achieve the proper ratio of first and second generation quark and lepton masses, Eq. (4), but leads to wrong predictions for the Cabibbo angle [17]. Indeed, ignoring small factors, the Cabibbo angle is approximately given by

$$s_c \simeq \sqrt{\frac{m_d}{m_s}} | \frac{z_d}{z'_d} |.$$

Relaxing the equality $z_d = z'_d$ by increasing the power of one of the $45_X$ in $O_{12}$ would change $s_c$ by a factor $\sqrt{3}$, what would lead to predictions in conflict with present data.

Therefore, for low values of $\omega$, if the dimension of the operator $O_{12}$, Eq. (12), is changed, to obtain correct values for the ratio of the first and second generation quark masses, it should be increased by two units. Once more, however, the variation in the dimensionality of this operator has an additional effect, which is related to the behaviour of the Jarlskog CP-odd invariant $J = \text{Im} [V_{ud}V_{tb}V_{td}V_{ub}^*]$. Ignoring again small, inessential factors, it is straightforward to show that

$$J \simeq \chi^2 \left| \frac{z_u z_d}{z_e z'_e} \right| \frac{|y_e|}{|y_d|} \frac{m_e}{m_\tau} \exp (2I_t + 2I_b - 3I_\tau).$$

Thus, increasing the dimension of the operator $O_{12}$ in two units implies a decrease in the CP–odd invariant $J$ in a factor three. Since the observed CP–violation in the $K$ system is well described by models 6 and 9 before the modification of the operator $O_{12}$, a factor three suppression of the Jarlskog invariant would imply that the amount of CP–violation associated with the Cabibbo-Kobayashi-Maskawa is insufficient to explain the experimental data. The possible variations of $V_{cb}$ (or equivalently of $V_{td}$) due to supersymmetric threshold corrections in the down quark sector, which we have discussed above, are not sufficiently large to compensate this type of effect. Numerically, we observe the effect of increasing the dimension of the operator $O_{12}$ through the prediction for the bag parameter $B_K$

$$B_K \simeq \epsilon_K \frac{|z_e z'_e|}{\sqrt{|z_d z'_d| z_u}} \frac{m_\tau}{m_e} \frac{m_d \exp (2I_t + 2I_b - 3I_\tau)}{\chi^2 \sin \phi}.$$

which tends to be larger than one in all models, and, hence, unacceptable since the phenomenologically preferred values are $B_K = 0.8 \pm 0.2$ [22,23].

The general conclusion of this study is that, keeping the same operator structure as in Ref. [17], values of $\omega \leq 0.5$ cannot be accommodated, without spoiling the predictions for either the first generation quark masses, the Cabibbo angle or the CP-violation sector of the
theory. Hence, the preferred value of $\omega \simeq 1$ restricts us to be close to the minimal $SO(10)$ model and the values of $\tan \beta$ and $M_h$ which lead to acceptable predictions are also quite restricted (see the discussion in section 3.1). In the next section we show that it is still possible to obtain acceptable predictions for the first generation with small values of $\tan \beta$. However, this solution requires the addition of one new operator and thus one more complex parameter in addition to the free parameter $\omega$ discussed above.

4 Extending minimal $SO(10)$ and $\tan \beta \sim 1$

One could think of improving the agreement between the theoretical predictions and the experimentally observed values of the first generation masses, or the $\epsilon_K$ parameter, by assuming very large supersymmetric threshold corrections to these variables. In Fig. 3 we have shown that if the down quark mass corrections have the structure which naturally appears when the squark matrices are block diagonal (see Appendix), only slight changes of the predictions for the first generation masses are obtained through such threshold corrections. In the general case, however, the squark mass matrices may be far from being three by three block diagonal and first generation down quark mass corrections, proportional to the second or even third generation masses, as shown in the appendix (Eq. (19)), may be present.

Pursuing this direction however opens up a pandoras box of new possibilities and new problems. It is interesting to note that, if the supersymmetry breaking is transferred to the observable sector through gravitational effects, a nontrivial inter-generation squark mixing, generated through renormalization effects at scales of the order of the grand unification scale, is unavoidable [28], [29], [30], [31]. A reliable computation of this effect demands, however, the knowledge of the precise physics beyond the grand unification scale. In general, a large squark mixing would also involve large flavor changing neutral current effects. Barring unnatural cancellations, large flavor violations in the fermion sector can only be consistent with the experimental constraints on flavor changing neutral currents and the neutron electric dipole moment if the characteristic scale of the squark masses is larger or of order 1 TeV. A large squark mixing also implies significant couplings of these heavy squarks to the Higgs sector of the theory (unless the third generation squarks do not mix with the first and second generation ones), this will in turn imply a significant fine tuning in order to preserve the stability of the weak scale. The presence of large supersymmetric corrections to the $\epsilon_K$ parameter have similar consequences. In this work, we assume the presence of a super GIM mechanism and avoid the discussion of non-universal squark and slepton masses at the GUT scale. Note that in order to reduce the fine-tuning and large corrections associated with large values of $\tan \beta$ it is only necessary to have non universal Higgs masses.

To improve the agreement between the theoretical and experimental predictions for small values of $\tan \beta$, a possible alternative is the modification of the structure of the operators discussed above. Since, as shown section 3, low values of $\omega$ are perfectly consistent with the second and third generation quark and lepton masses and mixing angles, any modification should concentrate on the form of the “12” elements. In Ref. [17], it was argued that if $O_{12}$ proceeds from a single operator, its form is uniquely determined. This conclusion is based on the analysis of the associated Clebschcs and the relations given in Eqs. (1), (13) and (14).
However, since $O_{12}$ has a large dimension, the relaxation of the assumption that the “12” elements come from a single operator seems natural. If the effect of two operators had to add in an unnatural way in order to lead to the correct phenomenological predictions, the predictive power of the theory would be spoiled. Therefore, the additional operator should not modify the equality of $z_u z'_u$ and $z_d z'_d$ and should give no relevant corrections to the ratio of $z_d/z'_d$. On the other hand, we want to modify the ratio of $m_u/m_d$ without affecting the CP-odd sector in a relevant way. It is crucial to notice that there is a very important difference between the dependence of $J$ and that of $m_u/m_d$ on the Clebsch factors. While $m_u/m_d$ depends on the product $z_u z'_u$, the CP-odd invariant $J$ depends on $z_u$, but is independent of $z'_u$. Hence, we are searching for an operator which modifies $z'_u$, leaving $z_u$ invariant. There is only one combination of operators which fulfills all the above criteria, namely,

$$O_{12} = 16_t (45_X)^n 10 (45_X)^n 16_2 + K 16_t (45_X)^m (45_T_{3R})^l 10 (45_X)^m 16_2$$

(17)

where $K$ is of order one. The predictions for $z_u$ and $z'_u$ within this framework are:

$$
\begin{align*}
    z_u & = 1, \quad z'_u = 1 + f \\
    z_d & = (-3)^n, \quad z'_d = (-3)^n + (-1)^l (-3)^m f \\
    z_e & = (-3)^n, \quad z'_e = (-3)^n + (-1)^l (-3)^m f,
\end{align*}
$$

(18)

where $f$ is the coefficient characterizing the relative weights of the two contributions, and it is computable from $K$ and the vacuum expectation values above. For simplicity, we shall assume that $f$ is a real number. In that case, for values of $f$ of order one and values of $m$ smaller than $n$ by at least two units, it is easy to see that the only prediction which will be modified considerably is $m_u/m_d$. One can therefore achieve low values of $\tan \beta$ with a correct prediction for $m_u/m_d$. This demands very low values of $\omega$ and values of $f$ close to −1. For instance, for model 6, $m_b(m_b) \approx 4.2$ GeV, and $\alpha_3(M_Z) \approx 0.12$, the value of $\omega$ which leads to $\tan \beta \approx 1.5$ is as small as 0.004. In this case, values of $f \approx -0.8$, $n = 3$ and $m = 0$ lead to good predictions for the CKM matrix and the quark masses.

5 Conclusions

We have analysed the fermion mass problem within the context of supersymmetric $SO(10)$ unification, studying not only the minimal case, but also the departure from the minimal conditions assuming that the fermion masses arise from interactions of the spinor representations with a single 10 representation, but this 10 only contains a part of the two light Higgs doublets. Moreover, we studied the implications of the down quark mass corrections, under the assumption that, within a good approximation, a super GIM mechanism is in effect. We have shown that, for $\omega < 1$ (moderate values of $\tan \beta$), and considering the simplest operator structure, large bottom mass corrections are helpful in accommodating the experimentally preferred values for $M_t$, yielding also acceptable values for $V_{cb}$. However, moderate or low values of $\tan \beta$ lead to wrong predictions either for the first generation quark masses or for the CP-odd sector of the theory, a property that is not changed by the presence of supersymmetric threshold corrections. We have also shown that the operator structure may
be extended to yield proper values for all fermion masses and mixing angles for low values of $\tan \beta \leq 3$. This extension requires, however, the presence of additional 45 states in the theory, one new operator contributing to the first generation masses and another complex parameter.

6 Appendix – Supersymmetric threshold corrections

Let us discuss the down quark mass corrections induced by supersymmetric particle loops in more detail. The dominant corrections to the down quark mass matrix are given by chargino–up squark and gluino–down squark one loop contributions and they read,

\[
(\delta m_d)_{IL} = -\frac{2\alpha_3}{3\pi} M_{\tilde{g}} \sum_{j=2}^{6} \left[ D_{ij} D_{(L+3)j}^* (m_{\tilde{d}_j}^2 - m_{\tilde{d}_1}^2) I(m_{\tilde{d}_j}^2, m_{\tilde{d}_1}^2, M_{\tilde{g}}^2) \right] + \sum_{j=2}^{6} \sum_{\alpha=1}^{2} \left[ \frac{d_L}{16\pi^2} Z_{2\alpha} Z_{2\alpha} m_\alpha C_{KL} U_{Kj} u_M U_{(M+3)j} C_{MI} (m_{\tilde{u}_j}^2 - m_{\tilde{u}_1}^2) I(m_{\tilde{u}_j}^2, m_{\tilde{u}_1}^2, \mu^2) \right].
\] (19)

The above expression has also been obtained in Ref. [21]. In the above, $U$ and $D$ are the unitary matrices diagonalizing the six by six up and down squark mass matrices, $(D_{ii}$ and $D_{ij}$ denote, for example, the component of the mass eigenstate $\tilde{d}_i$ in the left and right handed down squark, respectively), $Z_{\alpha\beta}$ are the unitary matrix which diagonalize the two by two chargino matrix, $m_\alpha$ are the chargino mass eigenstates, $C_{IJ}$ are the CKM matrix elements, $d_I$ and $u_I$ are the down and up quark Yukawa couplings, respectively, and $M_{\tilde{g}}$ is the gluino mass. The integral $I(a, b, c)$ is given by

\[
I(a, b, c) = \frac{ab \ln(a/b) + bc \ln(b/c) + ac \ln(c/a)}{(a - b)(b - c)(a - c)}. \] (20)

All indices denoted by capital letters run from 1 to 3 and a summation over the indices $K$ and $M$ is implicit. The state $\tilde{d}_1$ ($\tilde{u}_1$) denotes any particular eigenstate, which may be chosen, for example, as the heaviest one. A dependence of the above expression on the quark mass matrices is implicit in the necessary left right mixing term, which is only generated by terms proportional to the quark masses.

Furthermore, if the up and down squark matrices are three by three block diagonal, implying the existence of a super GIM mechanism in the theory, the following property is fulfilled

\[
D_{Kj} D_{(M+3)j}^* = \pm \delta_{KM} \frac{d_K (A_{dK} - \mu \tan \beta) v_1}{m_{\tilde{d}_K}^2 - m_{\tilde{d}_{(K+3)}}^2}, \quad U_{Kj} U_{(M+3)j}^* = \pm \delta_{KM} \frac{u_K (A_{uK} - \mu \cot \beta) v_2}{m_{\tilde{u}_K}^2 - m_{\tilde{u}_{(K+3)}}^2},
\] (21)

where $A_K$ are the conventionally defined trilinear soft supersymmetry breaking terms and $\mu$ is the supersymmetric Higgs mass parameter appearing in the superpotential. The positive sign in the above expression corresponds to the case $j = K$, while the negative sign corresponds to the case $j = K + 3$. Keeping the dominant terms in the large $\tan \beta$ regime, the down
quark mass corrections take, hence, a very simple form,

\[ (\delta m_d)_{IL} = \frac{2\alpha_3}{3\pi} \delta_{IL}(d_Lv_1) \tan \beta \mu M_3 I(m^2_{d_L}, m^2_{d_{(L+3)}}, M^2_3) \]

\[ + \sum_{j=2}^{6} \sum_{\alpha=1}^{2} \left[ \frac{(d_Lv_1) \tan \beta}{16\pi^2} Z^j_{2\alpha} Z^{j-\alpha} m_\alpha C^*_{ML} |u_M| A_{u_M} C_{MI}(m^2_{u_M}, m^2_{u_{(M+3)}}, m^2_\alpha) \right] \]

The above expression reproduces the one obtained in Ref. [21] under similar assumptions. In the present limit, the gluino corrections affect only the values of the mass eigenstates, while the chargino corrections give also corrections to the off–diagonal terms. Studying the renormalization group evolution of the soft supersymmetry breaking mass parameters one can show that the gluino contributions are generally dominant and opposite in sign to the chargino contributions [15]. Moreover, due to the hierarchy between the up quark masses, only the term proportional to \(|u_3|^2\) becomes important in the chargino contributions. Hence the chargino-induced corrections to the down and strange masses are very small. Furthermore, as has been shown in Ref. [21], the most relevant corrections to the CKM matrix elements are given by \(\delta V_{cb}/V_{cb} \simeq -(\delta m_b/m_b)^{ch} \simeq \delta V_{td}/V_{td}\), where \((\delta m_b/m_b)^{ch}\) represents only the chargino contributions to the total bottom mass corrections.

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Figure 1: a) The pole top quark mass as a function of $\tan \beta$ for a running bottom mass $m_b(m_b) = 4.15$ GeV and three different values of the strong gauge coupling, $\alpha_s(M_Z) = 0.115, 0.120$ and 0.125, respectively, for model 6. The coefficient $K_c$ parametrizing the down quark mass corrections takes values, $K_c = 0$ (dashed line), $K_c = -0.003$ (dotted line) and $K_c = -0.006$ (dot-dashed line). The solid lines represent, from right to left, values of $\omega = 1, 0.6, 0.2$ and 0.06, respectively. For large values of $\alpha_3(M_Z)$ the curves are cutted at the point at which the top Yukawa coupling becomes strong at high energy scales, $h_t^2(M_{GUT})/4\pi \geq 1$. 
Figure 1: b) The same as Fig. 1.a but for model 9.
Figure 2: a) The same as in Fig. 1.a but for the Cabibbo-Kobayashi-Maskawa matrix element $V_{cb}$ as a function of $\tan \beta$. 

$\alpha_s (M_Z) = 0.115$

$\alpha_s (M_Z) = 0.120$

$\alpha_s (M_Z) = 0.125$
Figure 2: b) The same as in Fig. 2.a but for model 9.
Figure 3: a) The same as in Fig. 1.a but for the ratio of the first generation masses $m_u/m_d$ as a function of $\tan \beta$. 
Figure 3: b) The same as in Fig. 3.a but for model 9.