Gravitational waves from stellar encounters

Salvatore Capozziello¹, Mariafelicia De Laurentis²

¹Dipartimento di Scienze Fisiche, Università di Napoli “Federico II", INFN Sez. di Napoli, Compl. Univ. di Monte S. Angelo, Edificio G, Via Cinthia, I-80126, Napoli, Italy
²Dipartimento di Fisica, Politecnico di Torino and INFN Sez. di Torino, Corso Duca degli Abruzzi 24, I-10129 Torino, Italy

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The emission of gravitational waves from a system of massive objects interacting on elliptical, hyperbolic and parabolic orbits is studied in the quadrupole approximation. Analytical expressions are then derived for the gravitational wave luminosity, the total energy output and gravitational radiation amplitude. A crude estimate of the expected number of events towards peculiar targets (i.e. globular clusters) is also given. In particular, the rate of events per year is obtained for the dense stellar cluster at the Galactic Center.

Keywords: theory of orbits, gravitational radiation, quadrupole approximation.

I. INTRODUCTION

Gravitational-wave (GW) science has entered a new era. Experimentally, several GW ground-based-laser-interferometer detectors (10⁻¹ kHz) have been built in the United States (LIGO) [1], Europe (VIRGO and GEO) [2, 3] and Japan (TAMA) [4], and are now taking data at design sensitivity. Advanced optical configurations capable of reaching sensitivities slightly above and even below the so-called standard-quantum-limit for a free test-particle, have been designed for second [5] and third generation [6] GW detectors. A laser-interferometer space antenna (LISA) [7] (10⁻⁴ ~ 10⁻² Hz) might fly within the next decade. Resonant-bar detectors (~1 kHz) [8] are improving more and more their sensitivity, broadening their frequency band. At much lower frequencies, ~10⁻¹⁷ Hz, future cosmic microwave background (CMB) probes are devoted to detect GWs by measuring the CMB polarization [9]. Millisecond pulsar timing can set interesting upper limits [10] in the frequency range 10⁻⁹ ~ 10⁻⁸ Hz. In this frequency range, the large number of millisecond pulsars which will be detectable with the square kilometer array [11], would provide an ensemble of clocks that can be used as multiple arms of a GW detector.

From a theoretical point of view, recent years have been characterized by numerous major advances due, essentially, to the development of numerical gravity. Concerning the most promising sources to be detected, the GW generation problem has improved significantly in relation to the dynamics of binary and multiple systems of compact objects as neutron stars and black holes.

Besides, the problem of non-geodesic motion of particles in curved spacetime has been developed considering the emission of GWs [12, 13]. Solving this problem is of considerable importance in order to predict the accurate waveforms of GWs emitted by extreme mass-ratio binaries, which are among the most promising sources for LISA [14]. To this aim, searching for criteria to classify the ways in which sources collide is of fundamental importance. A first rough criterion can be the classification of stellar encounters in collisional as in the globular clusters and in collisionless as in the galaxies [25]. A fundamental parameter is the richness and the density of the stellar system and so, obviously, we expect a large production of GWs in rich and dense systems.

Systems with these features are the globular clusters and the galaxy centers. In particular, one can take into account the stars (early-type and late-type) which are around our Galactic Center, e.g. Sagittarius A* (Sgr A*) which could be very interesting targets for the above mentioned ground-based and space-based detectors.

In recent years, detailed information has been achieved for kinematics and dynamics of stars moving in the gravitational field of such a central object. The statistical properties of spatial and kinematical distributions are of particular interest (see e.g. [13, 15, 16]). Using them, it is possible to give a quite accurate estimate of the mass and the size of the central object: we have (2.61 ± 0.76) × 10⁶Msolar concentrated within a radius of 0.016pc (about 30 light-days) [15, 16]. More precisely, in [18], it is described a campaign of observations where velocity measurements in the central arcsec² are extremely accurate. Then from this bulk of data, considering a field of resolved stars whose proper motions are accurately known, one can classify orbital motions and deduce, in principle, the rate of production of GWs according to the different types of orbits. This motivates this paper in which, by a classification of orbits in accordance with the above mentioned ground-based detectors.

In this paper, we investigate the GW emission by binary systems in the quadrupole approximation considering bounded (circular or elliptical) and unbounded (parabolic or hyperbolic) orbits. Obviously, the main parameter is the approaching energy of the stars in the system (see also [23] and references therein). We
expect that gravitational waves are emitted with a "peculiar" signature related to the encounter-type: such a signature has to be a "burst" wave-form with a maximum in correspondence of the periastron distance. The problem of bremsstrahlung-like gravitational wave emission has been studied in detail by Kovacs and Thorne [24] by considering stars interacting on unbounded and bounded orbits. In this paper, we face this problem discussing in detail the dynamics of such a phenomenon which could greatly improve the statistics of possible GW sources.

The paper is organized as follows: in Sec. II, the main features of stellar encounters and orbit classification are reviewed. Sec. III is devoted to the emission and luminosity of GWs from binary systems in the different kinds of orbits assuming the quadrupole approximation. A discussion of the wave-form dependence from the orbital parameters is given in Sec. IV. In Sec. V, we derive the expected rate of events assuming the Galactic Center as a target. Section VI is devoted to concluding remarks.

## II. ORBITS IN STELLAR ENCOUNTERS

Let us take into account the Newtonian theory of orbits since stellar systems, also if at high densities and constituted by compact objects, can be usually assumed in Newtonian regime. We give here a self-contained summary of the well-known orbital types in order to achieve below a clear classification of the possible GW emissions. We refer to the text books [25, 27] for a detailed discussion.

A mass \( m_1 \) is moving in the gravitational potential \( \Phi \) generated by a second mass \( m_2 \). The vector radius and the polar angle depend on time as a consequence of the star motion, i.e. \( r = r(t) \) and \( \phi = \phi(t) \). With this choice, the velocity \( \mathbf{v} \) of the mass \( m_1 \) can be parameterized as

\[
\mathbf{v} = v_r \hat{r} + v_\phi \hat{\phi},
\]

where the radial and tangent components of the velocity are, respectively,

\[
v_r = \frac{dr}{dt}, \quad v_\phi = \frac{r d\phi}{dt}.
\]

In this case, the total energy and the angular momentum, read out

\[
\frac{1}{2} \mu \left( \frac{dr}{dt} \right)^2 + \frac{L^2}{2 \mu r^2} - \frac{\gamma}{r} = E
\]

and

\[
L = r^2 \frac{d\phi}{dt},
\]

respectively, where \( \mu = \frac{m_1 m_2}{m_1 + m_2} \) is the reduced mass of the system and \( \gamma = G m_1 m_2 \).

We can split the kinetic energy into two terms where, due to the conservation of angular momentum, the second one is a function of \( r \) only. An effective potential energy \( V_{\text{eff}} \),

\[
V_{\text{eff}} = \frac{L^2}{2 \mu r^2} - \frac{\gamma}{r}
\]

is immediately defined. The first term corresponds to a repulsive force, called the angular momentum barrier. The second term is the gravitational attraction. The interplay between attraction and repulsion is such that the effective potential energy has a minimum. Indeed, differentiating with respect to \( r \) one finds that the minimum lies at \( r_0 = \frac{L^2}{\gamma} \) and that

\[
V_{\text{eff}}^{\text{min}} = -\frac{\mu \gamma^2}{2 L^2}.
\]

Therefore, since the radial part of kinetic energy,

\[
K_r = \frac{1}{2} \mu \left( \frac{dr}{dt} \right)^2
\]

is non-negative, the total energy must be not less than \( V_{\text{eff}}^{\text{min}} \), i.e.

\[
E \geq E_{\text{min}} = -\frac{\mu \gamma^2}{2 L^2}.
\]
The equal sign corresponds to the radial motion. For $E_{\text{min}} < E < 0$, the trajectory lies between a smallest value $r_{\text{min}}$ and greatest value $r_{\text{max}}$ which can be found from the condition $E = V_{\text{eff}}$, i.e.

$$r_{\{\text{min, max}\}} = -\frac{\gamma}{2E} \pm \sqrt{\left(\frac{\gamma}{2E}\right)^2 + \frac{L^2}{2\mu E}}$$  \hspace{1cm} (9)$$

where the upper (lower) sign corresponds to $r_{\text{max}}$ ($r_{\text{min}}$). Only for $E > 0$, the upper sign gives an acceptable value; the second root is negative and must be rejected.

Let us now proceed in solving the differential equations (3) and (4). We have

$$\frac{dr}{dt} = \frac{d\phi}{dt} - \frac{L}{\mu v^2} \frac{dr}{d\phi} = -\frac{L}{\mu} \frac{d}{d\phi} \left(\frac{1}{r}\right)$$  \hspace{1cm} (10)$$

and defining, as standard, the auxiliary variable $u = 1/r$, Eq. (3) takes the form

$$u'' + u^2 - \frac{2\gamma\mu}{L^2} u = \frac{2\mu E}{L^2}$$  \hspace{1cm} (11)$$

where $u' = du/d\phi$ and we have divided by $L^2/2\mu$. Differentiating with respect to $\phi$, we get

$$u' \left(u'' + u - \frac{\gamma\mu}{L^2}\right) = 0$$  \hspace{1cm} (12)$$

hence either $u' = 0$, corresponding to the circular motion, or

$$u'' + u = \frac{\gamma\mu}{L^2}$$  \hspace{1cm} (13)$$

which has the solution

$$u = \frac{\gamma\mu}{L^2} + C \cos (\phi + \alpha)$$  \hspace{1cm} (14)$$

or, reverting the variable,

$$r = \left[\frac{\gamma\mu}{L^2} + C \cos (\phi + \alpha)\right]^{-1}$$  \hspace{1cm} (15)$$

which is the canonical form of conic sections in polar coordinates [26]. The constant $C$ and $\alpha$ are two integration constants of the second order differential equation (13). The solution (15) must satisfy the first order differential equation (11). Substituting (15) into (11) we find, after a little algebra,

$$C^2 = \frac{2\mu E}{L^2} + \left(\frac{\gamma\mu}{L^2}\right)^2$$  \hspace{1cm} (16)$$

and therefore, taking account of Eq. (8), we get $C^2 \geq 0$. This implies the four kinds of orbits given in Table I.

A. Circular Orbits

Circular motion corresponds to the condition $u' = 0$ by which one find $r_0 = L^2/\mu\gamma$ where $V_{\text{eff}}$ has its minimum. We also note that the expression for $r_0$ together with Eq. (8) gives

$$r_0 = -\frac{\gamma}{2E_{\text{min}}}$$  \hspace{1cm} (17)$$

Thus the two bodies move in concentric circles with radii, inversely proportional to their masses and are always in opposition.

Table I: Orbits in Newtonian regime classified by the approaching energy.

| $C = 0$ | $E = E_{\text{min}}$ | circular orbits |
| 0 < |$C$| < $\frac{\mu\gamma}{L^2}$ | $E_{\text{min}} < E < 0$ | elliptic orbits |
| | | $E = 0$ | parabolic orbits |
| | | $E > 0$ | hyperbolic orbits |

B. Elliptical Orbits

For 0 < |$C$| < $\mu\gamma/L^2$, $r$ remains finite for all values of $\phi$. Since $r(\phi + 2\pi) = r(\phi)$, the trajectory is closed and it is an ellipse. If one chooses $\alpha = 0$, the major axis of the ellipse corresponds to $\phi = 0$. We get

$$r|_{\phi=0} = r_{\text{min}} = \left[\frac{\mu\gamma}{L^2} + C\right]^{-1}$$

and

$$r|_{\phi=\pi} = r_{\text{max}} = \left[\frac{\mu\gamma}{L^2} - C\right]^{-1}$$

and since $r_{\text{max}} + r_{\text{min}} = 2a$, where $a$ is the semi-major axis of the ellipse, one obtains

$$a = r|_{\phi=0} = r_{\text{min}} = \frac{\gamma\mu}{L^2} \left[\left(\frac{\gamma\mu}{L^2}\right)^2 + C^2\right]^{-1}$$

$C$ can be eliminated from the latter equation and Eq. (16) and then

$$a = \frac{\gamma}{2E}$$

Furthermore, if we denote the distance $r|_{\phi=\pi/2}$ by $l$, the so-called semi-latus rectum or the parameter of the ellipse, we get

$$l = \frac{L^2}{\gamma\mu}$$

and hence the equation of the trajectory

$$r = \frac{l}{1 + \epsilon \cos \phi}$$

where $\epsilon = \sqrt{\frac{l - a}{a}}$ is the eccentricity of the ellipse.

C. Parabolic and Hyperbolic Orbits

These solutions can be dealt together. They correspond to $E \geq 0$ which is the condition to obtain unbounded orbits. Equivalently, one has |$C$| $\geq \gamma\mu/L^2$. 
The trajectory is
\[ r = l \left( 1 + \epsilon \cos \phi \right)^{-1} \]  
where \( \epsilon \geq 1 \). The equal sign corresponds to \( E = 0 \). Therefore, in order to ensure positivity of \( r \), the polar angle \( \phi \) has to be restricted to the range given by
\[ 1 + \epsilon \cos \phi > 0 \]  
This means \( \cos \phi > -1 \), i.e. \( \phi \in (-\pi, \pi) \) and the trajectory is not closed any more. For \( \phi \to \pm \pi \), we have \( r \to \infty \).

The curve (23), with \( \epsilon = 1 \), is a parabola. For \( \epsilon > 1 \), the allowed interval of polar angles is smaller than \( \phi \in (-\pi, \pi) \), and the trajectory is a hyperbola. Such trajectories correspond to non-returning objects.

### III. GRAVITATIONAL WAVE LUMINOSITY IN THE QUADRUPOLE APPROXIMATION

At this point, considering the orbit equations, we want to classify the gravitational radiation for the different stellar encounters. To this aim, let us start with a short review of the quadrupole approximation for GW radiation. We add this discussion for the sake of completeness, but we send the Reader to the References [28, 29, 30, 31] for a detailed exposition.

The Einstein field equations give a description of how the curvature of space-time is related to the energy-momentum distribution. In the weak field approximation, moving massive objects produce gravitational waves which propagate in the vacuum with the speed of light. In this approximation, we have
\[ g_{\mu\nu} = \delta_{\mu\nu} + \kappa h_{\mu\nu}, \quad (|h_{\mu\nu}| \ll 1), \]  
whit \( \kappa \) the gravitational coupling. The field equations are
\[ \Box h_{\mu\nu} = -\frac{1}{2} \kappa T_{\mu\nu} \]  
where
\[ \tilde{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} \delta_{\mu\nu} h_{\lambda\lambda}, \]  
and \( T_{\mu\nu} \) is the total stress-momentum-energy tensor of the source, including the gravitational stresses. A plane GW can be written as
\[ \tilde{h}_{\mu\nu} = h_{\mu\nu} = h e_{\mu\nu} \cos(\omega t - k \cdot x) \]  
where \( h \) is the amplitude, \( \omega \) the frequency, \( k \) the wave number and \( e_{\mu\nu} \) is a unit polarization tensor, obeying the conditions
\[ e_{\mu\nu} = e_{\nu\mu}, \quad e_{\mu\mu} = 0, \quad e_{\mu\nu} e_{\mu\nu} = 1. \]  
Let us assume a gauge in which \( e_{\mu\nu} \) is space-like and transverse; thus, a wave travelling in the \( z \) direction has two possible independent polarizations:
\[ e_1 = \frac{1}{\sqrt{2}}(\hat{x}\hat{x} - \hat{y}\hat{y}), \quad e_2 = \frac{1}{\sqrt{2}}(\hat{x}\hat{y} - \hat{y}\hat{x}). \]  
One can now search for wave solutions of (26) from a system of masses undergoing arbitrary motions, and then obtain the power radiated. The result, assuming the source dimensions very small with respect to the wavelengths (quadrupole approximation [27]), is that the power \( \frac{dE}{d\Omega} \) radiated in a solid angle \( \Omega \) with polarization \( e_{ij} \) is
\[
\frac{dE}{d\Omega} = \frac{G}{8 \pi c^5} \left( \frac{d^3 Q_{ij}}{dt^3} c_{ij} \right)^2 \tag{31}
\]

where \( Q_{ij} \) is the quadrupole mass tensor

\[
Q_{ij} = \sum_a m_a (3x_i^a x_j^a - \delta_{ij} r_a^2) , \tag{32}
\]

\( G \) being the Newton constant, \( r_a \) the modulus of the vector radius of the \( a \)-th particle and the sum running over all masses \( m_a \) in the system. It has to be noted that the result is independent of the kind of stresses which are present into the dynamics. If one sums (31) over the two allowed polarizations, one obtains

\[
\sum_{\text{pol}} \frac{dE}{d\Omega} = \frac{G}{8 \pi c^5} \left[ \frac{d^3 Q_{ij}}{dt^3} \frac{d^3 Q_{ij}}{dt^3} - 2n_i \frac{d^3 Q_{ij}}{dt^3} n_k \frac{d^3 Q_{kj}}{dt^3} - \frac{1}{2} \left( \frac{d^3 Q_{ii}}{dt^3} \right)^2 \right. \\
+ \left. \frac{1}{2} \left( n_i n_j \frac{d^3 Q_{ij}}{dt^3} \right)^2 + \frac{d^3 Q_{ii}}{dt^3} n_j n_k \frac{d^3 Q_{jk}}{dt^3} \right] \tag{33}
\]

where \( \hat{n} \) is the unit vector in the radiation direction. The total radiation rate is obtained by integrating (33) over all directions of emission; the result is

\[
\frac{dE}{dt} = -\frac{G \langle Q_{ij}^{(3)} Q_{ij}^{(3)} \rangle}{45 \pi c^5} \tag{34}
\]

where the index \((3)\) represents the differentiation with respect to time, the symbol \(</>\) indicates that the quantity is averaged over several wavelengths. This crucial point is linked with the difficulties of localizing gravitational energy so the right hand side of Eq. (34) cannot be viewed as an instantaneous quantity. This problem has been already faced for circular and elliptical orbits in [14, 32, 33]. For hyperbolic and parabolic orbits, it is crucial to estimate the quantity in the right hand side of Eq. (34) in the zone where stars are slightly changing their trajectories, that means at peri-astron, while we expect no emission in asymptotic regime of stars approaching to or going away from this region. For a detailed discussion in the hyperbolic case, see [20].

With this formalism, it is possible to estimate the amount of energy emitted in the form of GWs from a system of massive objects interacting among them [21, 22]. In this case, the components of the quadrupole mass tensor in the equatorial plane \((\theta = \pi/2)\) are

\[
\begin{align*}
Q_{xx} &= \mu r^2 (3 \cos^2 \phi - 1) , \\
Q_{yy} &= \mu r^2 (3 \sin^2 \phi - 1) , \\
Q_{zz} &= -\mu r^2 , \\
Q_{xz} &= Q_{zx} = 0 , \\
Q_{yz} &= Q_{zy} = 0 , \\
Q_{xy} &= Q_{yx} = 3 \mu r^2 \cos \phi \sin \phi ,
\end{align*} \tag{35}
\]

where the masses \( m_1 \) and \( m_2 \) have polar coordinates \( \{ r_i \cos \theta \cos \phi, r_i \cos \theta \sin \phi, r_i \sin \theta \} \) with \( i = 1, 2 \). The origin of the motions is taken at the center of mass. Such components can be differentiated with respect to the time as in Eq. (34). In doing so, we can use some useful relations derived in the previous Section.
A. GW luminosity from circular and elliptical orbits

Using Eq. (22), let us derive the angular velocity equation

\[ \dot{\phi} = \sqrt{\frac{G l (m_1 + m_2) (\epsilon \cos \phi + 1)^2}{l^2}} \]  \hspace{1cm} (36)

and then, from (35), the quadrupolar components for the elliptical orbits

\[ \frac{d^3 Q_{xx}}{dt^3} = \beta (24 \cos \phi + \epsilon (9 \cos 2\phi) + 11)) \sin \phi \]  \hspace{1cm} (37)

\[ \frac{d^3 Q_{yy}}{dt^3} = -\beta (24 \cos \phi + \epsilon (13 + 9 \cos 2\phi)) \sin \phi \]  \hspace{1cm} (38)

\[ \frac{d^3 Q_{zz}}{dt^3} = -2\beta \epsilon \sin \phi \]  \hspace{1cm} (39)

\[ \frac{d^3 Q_{xy}}{dt^3} = \beta (24 \cos \phi + \epsilon (11 + 9 \cos 2\phi)) \sin \phi \]  \hspace{1cm} (40)

where

\[ \beta = \frac{Gl(m_1 + m_2)^{3/2} \mu (\epsilon \cos \phi + 1)^2}{l^4}. \]  \hspace{1cm} (41)

The total power radiated is given by

\[ \frac{dE}{dt} = \frac{G^3}{45c^5l^7} f(\phi) \]  \hspace{1cm} (42)

where

\[ f(\phi) = \left[(m_1 + m_2)^3 \mu^2 (1 + \epsilon \cos \phi)^4 \left(415\epsilon^2 + 3(8 \cos \phi + 3\epsilon \cos 2\phi) \right) \left(72 \cos \phi + \epsilon(70 + 27 \cos 2\phi)\right)\sin \phi^2 \right] \]  \hspace{1cm} (43)

The total energy emitted in the form of gravitational radiation, during the interaction, is given by

\[ \Delta E = \int_0^\infty \left| \frac{dE}{dt} \right| dt. \]  \hspace{1cm} (44)

From Eq. (4), we can adopt the angle \( \phi \) as a suitable integration variable. In this case, the energy emitted for \( \phi_1 < \phi < \phi_2 \) is

\[ \Delta E(\phi_1, \phi_2) = \frac{G^3}{45c^5l^7} \int_{\phi_1}^{\phi_2} f(\phi) \, d\phi. \]  \hspace{1cm} (45)

and the total energy can be determined from the previous relation in the limits \( \phi_1 \to 0 \) and \( \phi_2 \to \pi \). Thus, one has

\[ \Delta E = \frac{G^4 \pi (m_1 + m_2)^3 \mu^2}{l^5c^5} F(\epsilon) \]  \hspace{1cm} (46)

where \( F(\epsilon) \) depends on the initial conditions only and is given by

\[ F(\epsilon) = \frac{(13824 + 102448\epsilon^2 + 59412\epsilon^4 + 2549\epsilon^6)}{2880}. \]  \hspace{1cm} (47)

In other words, the gravitational wave luminosity strictly depends on the configuration and kinematics of the binary system.
B. GW luminosity from parabolic and hyperbolic orbits

Also in this case, we use Eq. (23) and Eq. (35) to calculate the quadrupolar formula for parabolic and hyperbolic orbits. The angular velocity is

\[ \dot{\phi} = \ell^2 L (\epsilon \cos \phi + 1)^2 \]  

(47)

and the derivative are

\[ \frac{d^3 Q_{xx}}{dt^3} = \rho (24 \cos \phi + \epsilon (9 \cos 2\phi + 11)) \sin \phi \]  

(48)

\[ \frac{d^3 Q_{yy}}{dt^3} = -\rho (24 \cos \phi + \epsilon (13 + 9 \cos 2\phi)) \sin \phi \]  

(49)

\[ \frac{d^3 Q_{zz}}{dt^3} = -2 \rho \epsilon \sin \phi \]  

(50)

\[ \frac{d^3 Q_{xy}}{dt^3} = -\frac{3}{2} \rho (\epsilon \cos \phi + 1)^2 (5 \epsilon \cos \phi + 8 \cos 2\phi + 3 \epsilon \cos 3\phi) \]  

(51)

where

\[ \rho = \frac{\ell^4 L^3 \mu (\epsilon \cos \phi + 1)^2}{120 c^5} \]  

(52)

The radiated power is given by

\[ \frac{dE}{dt} = -G \rho^2 \left( [314 \epsilon^2 + (1152 \cos(\phi + 187 \epsilon \cos 2\phi) - 3(30 \cos 3\phi + 11 \epsilon \cos 2\phi + 48 \cos 5\phi + 9 \epsilon \cos 6\phi)) \epsilon - 192 \cos 4\phi + 576] \right) \]  

then

\[ \frac{dE}{dt} = -\frac{Gl^8 L^6 \mu^2}{120 c^5} f(\phi) \]  

(53)

where \( f(\phi) \), in this case, is

\[ f(\phi) = (314 \epsilon^2 + (1152 \cos(\phi + 187 \epsilon \cos 2\phi) - 3(30 \cos 3\phi + 11 \epsilon \cos 2\phi + 48 \cos 5\phi + 9 \epsilon \cos 6\phi)) \epsilon - 192 \cos 4\phi + 576) \]  

(54)

Then using Eq. (34), the total energy emitted in the form of gravitational radiation during the interaction as a function of \( \phi \) is given by

\[ \Delta E(\phi_1, \phi_2) = -\frac{G l^8 L^6 \pi (1271 \epsilon^6 + 24276 \epsilon^4 + 34768 \epsilon^2 + 4608) \mu^2}{480 c^5} \ d\phi , \]  

(55)

and the total energy can be determined from the previous relation in the limits \( \phi_1 \to -\pi \) and \( \phi_2 \to \pi \) in the parabolic case. Thus, one has

\[ \Delta E = -\frac{G l^8 L^6 \pi \mu^2}{480 c^5} F(\epsilon) , \]  

(56)

where \( F(\epsilon) \) depends on the initial conditions only and is given by

\[ F(\epsilon) = (1271 \epsilon^6 + 24276 \epsilon^4 + 34768 \epsilon^2 + 4608) \right) . \]  

(57)

In the hyperbolic case, we have that the total energy is determined in the limits \( \phi_1 \to -\frac{3\pi}{4} \) and \( \phi_2 \to -\frac{3\pi}{4} \), i.e.

\[ \Delta E = -\frac{G l^8 L^6 \mu^2}{201600 c^5} F(\epsilon) , \]  

(58)
where \( F(\epsilon) \) depends on the initial conditions only and is given by
\[
F(\epsilon) = [315\pi (1271\epsilon^6 + 24276\epsilon^4 + 34768\epsilon^2 + 4608) + \\
+ 16\pi \epsilon^2 (926704\sqrt{2} - 7\epsilon (3319\epsilon^2 - 32632\sqrt{2}\epsilon + 55200) - 383460) + 352128\sqrt{2}] .
\] (59)

As above, the gravitational wave luminosity strictly depends on the configuration and kinematics of the binary system.

IV. GRAVITATIONAL WAVE AMPLITUDE

Direct signatures of gravitational radiation are its amplitude and its wave-form. In other words, the identification of a GW signal is strictly related to the accurate selection of the shape of wave-forms by interferometers or any possible detection tool. Such an achievement could give information on the nature of the GW source, on the propagating medium, and, in principle, on the gravitational theory producing such a radiation [32].

It is well known that the amplitude of GWs can be evaluated by
\[
h^{jk}(t, R) = \frac{2G}{Rc^2} Q^{jk} ,
\] (61)

\( R \) being the distance between the source and the observer and \( \{ j, k \} = 1, 2 \). Let us now derive the GW amplitude in relation to the orbital shape of the binary systems.

A. GW amplitude from elliptical orbits

Considering a binary system and the single components of eq. (61), it is straightforward to show that
\[
h^{11} = -\frac{2G}{Rc^2} \frac{\mathcal{G}(m_1 + m_2)\mu}{2t} (13\epsilon \cos \phi + 12 \cos 2\phi + \epsilon (4\epsilon + 3 \cos 3\phi)) ,
\]
\[
h^{22} = \frac{2G}{Rc^2} \frac{\mathcal{G}(m_1 + m_2)\mu}{2t} (17\epsilon \cos \phi + 12 \cos 2\phi + \epsilon (8\epsilon + 3 \cos 3\phi)) ,
\]
\[
h^{12} = h^{21} = -\frac{2G}{Rc^2} \frac{\mathcal{G}(m_1 + m_2)\mu}{2t} (15\epsilon \cos \phi + 12 \cos 2\phi + \epsilon (4\epsilon + 3 \cos 3\phi)) ,
\] (62)

so that the expected strain amplitude \( h \simeq (h^{11}_t + h^{22}_t + 2h^{12}_t)^{1/2} \) turns out to be
\[
h = \frac{G^3 (m_1 + m_2) \mu^2}{c^4 R^2} (3(13\epsilon \cos \phi + 12 \cos 2\phi + \epsilon (4\epsilon + 3 \cos 3\phi))^2 + (17\epsilon \cos \phi + 12 \cos 2\phi + \epsilon (8\epsilon + 3 \cos 3\phi))^2)^{1/2} ,
\] (63)

which, as before, strictly depends on the initial conditions of the stellar encounter. A remark is in order at this point. A monochromatic gravitational wave has, at most, two independent degrees of freedom. In fact, in the TT gauge, we have \( h_+ = h^{11} + h^{22} \) and \( h_\times = h^{12} + h^{21} \) (see e.g. [32]). As an example, the amplitude of gravitational wave is sketched in Fig. 1 for a stellar encounter close to the Galactic Center. The adopted initial parameters are typical of a close impact and are assumed to be \( b = 1 \) AU and \( v_\theta = 200 \) Kms\(^{-1}\), respectively. Here, we have fixed \( M_1 = M_2 = 1.4M_\odot \). The impact parameter is defined as \( L = bv \) where \( L \) is the angular momentum and \( v \) the incoming velocity. We have chosen a typical velocity of a star in the galaxy and we are considering, essentially, compact objects with masses comparable to the Chandrasekhar limit (\( \sim 1.4M_\odot \)). This choice is motivated by the fact that ground-based experiments like VIRGO or LIGO expect to detect typical GW emissions from the dynamics of these objects or from binary systems composed by them (see e.g. [30]).

B. GW amplitude from parabolic and hyperbolic orbits

In this case the single components of Eq. (61) for a parabolic and hyperbolic orbits, are
\[
h^{11} = -\frac{G^2 L_\mu^2}{Rc^2} (13\epsilon \cos \phi + 12 \cos 2\phi + \epsilon (4\epsilon + 3 \cos 3\phi)) ,
\]
\[
h^{22} = \frac{G^2 L_\mu^2}{Rc^2} (17\epsilon \cos \phi + 12 \cos 2\phi + \epsilon (8\epsilon + 3 \cos 3\phi)) ,
\]
\[
h^{12} = h^{21} = -\frac{3G^2 L_\mu^2}{Rc^2} (4\cos \phi + \epsilon (4\cos 2\phi + 3)) \sin \phi ,
\] (64)
Figure 1: The gravitational wave-forms from elliptical orbits shown as function of the polar angle \( \phi \). We have fixed \( M_1 = M_2 = 1.4 M_\odot \). \( M_2 \) is considered at rest while \( M_1 \) is moving with initial velocity \( v_0 = 200 \text{ Kms}^{-1} \) and an impact parameter \( b = 1 \text{ AU} \). The distance of the GW source is assumed to be \( R = 8 \text{ kpc} \) and the eccentricity is \( \epsilon = 0.2, 0.5, 0.7 \).

and then the expected strain amplitude is

\[
h = \frac{2l^4 L^4 \mu^2}{c^3 R} \left( 10 \epsilon^4 + 9 \epsilon^3 \cos 3 \phi + 59 \epsilon^2 \cos 2 \phi + 59 \epsilon^2 + (47 \epsilon^2 + 108) \epsilon \cos \phi + 36 \right)^{1/2},
\]

which, as before, strictly depends on the initial conditions of the stellar encounter. We note that the gravitational wave amplitude has the same analytical expression for both cases and differs only for the value of \( \epsilon \) which is \( \epsilon = 1 \) if the motion is parabolic and the polar angle range is \( \phi \in (\pi, \pi) \), while it is \( \epsilon > 1 \) and \( \phi \in (-\pi, \pi) \) for hyperbolic orbits. In these cases, we have non-returning objects.

The amplitude of the gravitational wave is sketched in Figs. 2 and 3 for stellar encounters close to the Galactic Center. As above, we consider a close impact and assume \( b = 1 \text{ AU} \), \( v_0 = 200 \text{ Kms}^{-1} \) and \( M_1 = M_2 = 1.4 M_\odot \).

V. RATE AND EVENT NUMBER ESTIMATIONS

An important remark is due at this point. A galaxy is a self-gravitating collisionless system where stellar impacts are very rare [25]. From the GW emission point of view, close orbital encounters, collisions and tidal interactions should be dealt on average if we want to investigate the gravitational radiation in a dense stellar system as we are going to do now.

Let us give now an estimate of the stellar encounter rate producing GWs in some interesting astrophysical conditions like a typical globular cluster or towards the Galactic Center after we have discussed above the features distinguishing the various types of stellar encounters. Up to now, we have approximated stars as point masses. However, in dense regions of stellar systems, a star can pass so close to another that they raise tidal forces which dissipate their relative orbital kinetic energy. In some cases, the loss of energy can be so large that stars form binary or multiple systems; in other cases, the stars collide and coalesce into a single star; finally stars can exchange gravitational interaction in non-returning encounters.

To investigate and parameterize all these effects, we have to compute the collision time \( t_{\text{coll}} \), where \( 1/t_{\text{coll}} \) is the collision rate, that is, the average number of physical collisions that a given star suffers per unit time. For the sake of simplicity, we restrict to stellar clusters in which all stars have the same mass \( m \).

Let us consider an encounter with initial relative velocity \( v_0 \) and impact parameter \( b \). The angular momentum per unit mass of the reduced particle is \( L = bv_0 \). At the distance of closest approach, which we denote by \( r_{\text{coll}} \), the radial velocity must be zero, and hence the angular momentum is \( L = r_{\text{coll}} v_{\max} \), where \( v_{\max} \) is the relative speed at \( r_{\text{coll}} \). From the energy equation [25], we have
Figure 2: The gravitational wave-forms for a parabolic encounter as a function of the polar angle $\phi$. As above, $M_1 = M_2 = 1.4M_\odot$ and $M_2$ is considered at rest. $M_1$ is moving with initial velocity $v_0 = 200$ Kms$^{-1}$ with an impact parameter $b = 1$ AU. The distance of the GW source is assumed at $R = 8$ kpc. The eccentricity is $\epsilon = 1$.

Figure 3: The gravitational wave-forms for hyperbolic encounters as function of the polar angle $\phi$. As above, we have fixed $M_1 = M_2 = 1.4M_\odot$. $M_2$ is considered at rest while $M_1$ is moving with initial velocity $v_0 = 200$ Kms$^{-1}$ and an impact parameter $b = 1$ AU. The distance of the source is assumed at $R = 8$ kpc. The eccentricity is assumed with the values $\epsilon = 1.2, 1.5, 1.7$.

If we set $r_{coll}$ equal to the sum of the radii of two stars, then a collision will occur if and only if the impact parameter is less than the value of $b$, as determined by Eq. (66).

Let $f(v_a)d^3v_a$ be the number of stars per unit volume with velocities in the range $v_a + d^3v_a$. The number of encounters per unit time with impact parameter less than $b$ which are suffered by a given star is just $f(v_a)d^3v_a$ times the volume of the annulus with radius $b$ and length $v_0$, that is, 

$$b^2 = r_{coll}^2 + \frac{4Gmr_{coll}}{v_0^2}. \quad (66)$$
\[
\int f(v_a) \pi b^2 v_0 d^3 v_a
\]  

(67)

where \( v_0 = |v - v_a| \) and \( v \) is the velocity of the considered star. The quantity in Eq. (67) is equal to \( 1/t_{coll} \) for a star with velocity \( v \): to obtain the mean value of \( 1/t_{coll} \), we average over \( v \) by multiplying (67) by \( f(v)/v \), where \( \nu = \int f(v) d^3 v \) is the number density of stars and the integration is over \( d^3 v \). Thus

\[
\frac{1}{t_{coll}} = \frac{\nu}{8\pi^2 \sigma^6} \int e^{-(v^2 + v_a^2)/2\sigma^2} \left( r_{coll} |v - v_a| + \frac{4Gm r_{coll}}{v - v_a} \right) d^3 v d^3 v_a
\]  

(68)

We now replace the variable \( v_a \) by \( V = v - v_a \). The argument of the exponential is then

\[
-(v^2 + V^2)/2\sigma^2
\]

and if we replace the variable \( v \) by \( v_{cm} = v - \frac{1}{2} V \) (the center of mass velocity), then we have

\[
\frac{1}{t_{coll}} = \frac{\nu}{8\pi^2 \sigma^6} \int e^{-(v_{cm}^2 + V^2)/2\sigma^2} \left( r_{coll} V + \frac{4Gm r_{coll}}{V} \right) dV.
\]  

(69)

The integral over \( v_{cm} \) is given by

\[
\int e^{-v_{cm}^2/\sigma^2} d^3 v_{cm} = \pi^{3/2} \sigma^3.
\]  

(70)

Thus

\[
\frac{1}{t_{coll}} = \frac{\pi^{1/2} \nu}{2\sigma^3} \int_{\infty}^{0} e^{-V^2/4\sigma^2} \left( r_{coll} V + 4Gm V r_{coll} \right) dV.
\]  

(71)

The integrals can be easily calculated and then we find

\[
\frac{1}{t_{coll}} = 4\sqrt{\pi \nu \sigma r_{coll}^2} + \frac{4\sqrt{\pi \nu Gm r_{coll}}}{\sigma}.
\]  

(72)

The first term of this result can be derived from the kinetic theory. The rate of interaction is \( \nu \Sigma \langle V \rangle \), where \( \Sigma \) is the cross-section and \( \langle V \rangle \) is the mean relative speed. Substituting \( \Sigma = \pi r_{coll}^2 \) and \( \langle V \rangle = 4\sigma/\sqrt{\pi} \) (which is appropriate for a Maxwellian distribution with dispersion \( \sigma \)) we recover the first term of (72). The second term represents the enhancement in the collision rate by gravitational focusing, that is, the deflection of trajectories by the gravitational attraction of the two stars.

If \( r_* \) is the stellar radius, we may set \( r_{coll} = 2r_* \). It is convenient to introduce the escape speed from stellar surface, \( v_* = \sqrt{\frac{2Gm}{r_*}} \), and to rewrite Eq. (72) as

\[
\Gamma = \frac{1}{t_{coll}} = 16\sqrt{\pi \nu \sigma r_*^2} \left( 1 + \frac{v_*^2}{4\sigma^2} \right) = 16\sqrt{\pi \nu \sigma r_*^2} (1 + \Theta),
\]  

(73)

where

\[
\Theta = \frac{v_*^2}{4\sigma^2} = \frac{Gm}{2\sigma^2 r_*^2}.
\]  

(74)

is the Safronov number [25]. In evaluating the rate, we are considering only those encounters producing gravitational waves, for example, in the LISA range, i.e. between \( 10^{-4} \) and \( 10^{-2} \) Hz (see e.g. [36]). Numerically, we have

\[
\Gamma \simeq 5.5 \times 10^{-10} \left( \frac{v}{10 \text{kms}^{-1}} \right) \left( \frac{\sigma}{UA^2} \right) \left( \frac{10 \text{pc}}{R} \right)^3 \text{yrs}^{-1} \quad \Theta \ll 1
\]  

(75)
\[ \Gamma \simeq 5.5 \times 10^{-10} \left( \frac{M}{10^5 M_\odot} \right)^2 \left( \frac{v}{10 \text{km s}^{-1}} \right) \left( \frac{\sigma}{UA^2} \right) \left( \frac{10 \text{pc}}{R} \right)^3 \text{yrs}^{-1} \quad \Theta >> 1 \] (76)

If \( \Theta >> 1 \), the energy dissipated exceeds the relative kinetic energy of the colliding stars, and the stars coalesce into a single star. This new star may, in turn, collide and merge with other stars, thereby becoming very massive. As its mass increases, the collision time is shortened and then there may be runaway coalescence leading to the formation of a few supermassive objects per clusters. If \( \Theta << 1 \), much of the mass in the colliding stars may be liberated and forming new stars or a single supermassive objects (see [37, 38]).

Note that when we have the effects of quasi-collisions in an encounter of two stars in which the minimum separation is several stellar radii, violent tides will raise on the surface of each star. The energy that excites the tides comes from the relative kinetic energy of the stars. This effect is important for \( \Theta >> 1 \) since the loss of small amount of kinetic energy may leave the two stars with negative total energy, that is, as a bounded binary system. Successive peri-center passages will dissipates more energy by GW radiation, until the binary orbit is nearly circular with a negligible or null GW radiation emission.

Let us apply these considerations to the Galactic Center which can be modelled as a system of several compact stellar clusters, some of them similar to very compact globular clusters with high emission in X-rays [37].

For a typical compact stellar cluster around the Galactic Center, the expected event rate is of the order of \( 2 \times 10^{-9} \) yrs\(^{-1} \) which may be increased at least by a factor \( \approx 100 \) if one considers the number of compact clusters in the whole Galaxy eventually passing nearby the Galactic Center. If the compact stellar cluster at the Galactic Center is taken into account and assuming the total mass \( M \approx 3 \times 10^6 M_\odot \), the velocity dispersion \( \sigma \approx 150 \) km s\(^{-1} \) and the radius of the object \( R \approx 10 \) pc (where \( \Theta = 4.3 \)), one expects to have \( \approx 10^{-5} \) open orbit encounters per year. On the other hand, if a cluster with total mass \( M \approx 10^6 M_\odot \), \( \sigma \approx 150 \) km s\(^{-1} \) and \( R \approx 0.1 \) pc is considered, an event rate number of the order of unity per year is obtained. These values could be realistically achieved by data coming from the forthcoming space interferometer LISA. As a secondary effect, the above wave-forms could constitute the "signature" to classify the different stellar encounters thanks to the differences of the shapes (see the above figures).

VI. CONCLUDING REMARKS

We have analyzed the gravitational wave emission coming from stellar encounters in Newtonian regime and in quadrupole approximation. In particular, we have taken into account the expected luminosity and the strain amplitude of gravitational radiation produced in tight impacts where two massive objects of \( 1.4 M_\odot \) closely interact at an impact distance of \( 1 AU \). Due to the high probability of such encounters inside rich stellar fields (e.g. globular clusters, bulges of galaxies and so on), the presented approach could highly contribute to enlarge the classes of gravitational wave sources (in particular, of dynamical phenomena capable of producing gravitational waves). In particular, a detailed theory of stellar orbits could improve the statistic of possible sources.

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