In-medium Pion and Partial Restoration of Chiral Symmetry

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Abstract

The partial restoration of chiral symmetry in nuclear medium is investigated in a model independent way by exploiting operator relations in QCD. An exact sum rule is derived for the quark condensate valid for all density. This sum rule is simplified at low density to a new relation $\langle \bar{q}q \rangle^*/\langle \bar{q}q \rangle = (F_{\pi}^t/F_{\pi})Z_\pi^*/2\pi$ with the in-medium quark condensate $\langle \bar{q}q \rangle^*$, in-medium pion decay constant $F_{\pi}^t$ and in-medium pion wave-function renormalization $Z_\pi^*$. Calculating $Z_\pi^*$ at low density from the iso-scalar pion-nucleon scattering data and relating $F_{\pi}^t$ to the isovector pion-nucleus scattering length $b_1^*$, it is concluded that the enhanced repulsion of the s-wave isovector pion-nucleus interaction observed in the deeply bound pionic atoms directly implies the reduction of the in-medium quark condensate. The knowledge of the in-medium pion mass $m_{\pi}^*$ is not necessary to reach this conclusion.

1. Introduction

Exploring possible evidence of partial restoration of chiral symmetry in the nuclear medium is one of the most important and challenging problems in hadron physics. Experimental studies along this line have been carried out by the spectroscopy of deeply bound pionic atoms \cite{1}, by low energy pion-nucleus scatterings \cite{2}, and by the production of di-pions in hadron-nucleus and photon-nucleus reactions \cite{3,4,5}. Important experimental observations are (i) a repulsive enhancement of the in-medium $\pi^-$-nucleon interaction in the pionic atoms and pion-nucleus scattering, and (ii) an attractive enhancement of the scalar-isoscalar $\pi^-$-$\pi$ interaction in nuclei. In the theoretical side, it was suggested that the reduction of the temporal part of the pion decay constant in the nuclear medium $F_{\pi}^t$ is intimately related to the phenomenon (i) \cite{6}. It was also argued that the reduction of $F_{\pi}^t$ is responsible for the phenomenon (ii) \cite{8}.
The purpose of this paper is to present theoretical basis of the direct connection between $F_π^2$ and the in-medium quark condensate $\langle \bar{q}q \rangle^*$ (the order parameter of the chiral phase transition) only by using the current commutation relations, spectral decomposition and the linear density approximation.\footnote{A preliminary account of this work has been reported in \cite{9}.} We derive an exact sum rule relating the in-medium quark condensate and hadronic matrix elements valid for all density in the chiral limit. We also derive following model-independent relations at low density: The in-medium Glashow–Weinberg (GW) relation $F_π^2 G_π^2 = -\langle \bar{q}q \rangle^*$ in the chiral limit, the in-medium Tomozawa–Weinberg (TW) relation $T(\omega) \simeq \omega/(2F_π^2)$ in the chiral limit, and the in-medium Gell-Mann–Oakes–Renner (GOR) relation $(F_π^2)^2 m_π^2 = -2m_q \langle \bar{q}q \rangle^*$ away from the chiral limit. Here $G_π^2$ is the pseudo-scalar pion coupling constant, $T(\omega)$ the isovector pion-nucleus scattering amplitude at zero spacial momentum and $m_π^*$ is the isospin averaged pion mass. In particular, the in-medium GW relation, which has not been discussed in previous literatures, gives a new way to estimate $\langle \bar{q}q \rangle^*$ from the experimental data of the pionic atoms and the pion-nucleus and pion-nucleon scatterings without recourse to the information of $m_π^*$ as will be shown in this paper.

The organization of this paper is as follows: In section 2 we derive the exact sum rule for the quark condensate valid for all density in the chiral limit. In section 3 we show that the exact sum rule leads to the in-medium GW relation and the in-medium GOR relation which are valid at low density. In section 4 we evaluate the reduction of $G_π^2$ by relating it to the isoscalar $\pi N$ scattering amplitude. In section 5 we derive the in-medium TW relation in the chiral limit. In section 6 utilizing all the results derived, the in-medium quark condensate is related to the observables in $\pi$-nucleus and $\pi N$ scattering data. In section 7 we discuss how the exact sum rule is expressed off the chiral limit, based on the PCAC relation. In section 8 the relation of the present argument to the in-medium chiral perturbation theory is discussed. Section 9 is devoted to summary and concluding remarks.

2. Exact sum rule in the chiral limit at all density

Let us first discuss how the in-medium quark condensate is expressed in terms of hadronic quantities. We consider the following correlation function in isospin symmetric nuclear matter:

$$\Pi_5^{ab}(q) = \int d^4x \ e^{iq\cdot x} \partial^\mu \langle \Omega| T[A^\mu_a(x)\phi_b^5(0)]|\Omega\rangle, \quad (1)$$

where $\phi_5$ is the pseudoscalar density $\phi_5^5 \equiv \bar{\psi}i\gamma_5(\tau^a/2)\psi$ with the Pauli matrix $\tau^a$ and the quark field $\psi = (u, d)^T$; $A^\mu_a$ denotes the axial vector current associated with the SU(2) chiral transformation. The ground state of the isospin symmetric nuclear matter denoted by $|\Omega\rangle$ in Eq. (1) is normalized as $\langle \Omega|\Omega\rangle = 1$ and is specified by the baryon density $\rho$.

Invoking an operator relation $\partial^\mu T[A^\mu_a(x)\phi_b^5(0)] = T[\partial^\mu A^\mu_a(x)\phi_b^5(0)] + \delta(x_0) [A^\mu_a(x), \phi_b^5(0)]$ with the conservation of the axial current $\partial^\mu A^\mu_a = 0$ in the chiral limit, $\Pi_5^{ab}(0)$ is written in terms of the in-medium quark condensate $\langle \bar{q}q \rangle^*$ as

$$\Pi_5^{ab}(0) = \langle \Omega| [Q_5^a, \phi_b^5]|\Omega\rangle = -i\delta^{ab} \langle \bar{q}q \rangle^*,$$  \quad (2)
where $Q^a_\mu(t) = \int d^3x A^a_\mu(t,x)$ and $\langle \bar q q \rangle^* \equiv \langle \Omega | \phi | \Omega \rangle$ with the scalar density $\phi$ defined by $\phi \equiv \frac{1}{4} \bar \psi \psi = \frac{1}{4} (\bar u u + \bar d d)$.

The hadronic contributions to $\Pi^b_0(q)$ can be read off from the spectral representation:

$$\Pi^b_0(q) = -i q^\mu \int_0^\infty d\omega' \left( \frac{\sigma^\mu_+ (\omega', q)}{\omega - \omega' + i \epsilon} - \frac{\sigma^\mu_- (\omega', q)}{\omega + \omega' - i \epsilon} \right), \quad (3)$$

where $q = (\omega, q)$ with the energy $\omega$ measured from the nuclear matter ground state. The spectral functions $\sigma^\mu_\pm (\omega, q)$ give the strength of the hadronic excitations with the energy $\omega$ and momentum $q$ and are expressed in terms of the matrix elements of $A^a_\mu$ and $\phi^a_\ell$:

$$\sigma^\mu_+ (\omega, q) = i \sum_\ell \langle \Omega | A^a_\mu | \Omega_\ell \rangle \langle \phi^b_\ell | \Omega \rangle \frac{\delta(q, p_\ell) \delta(\varepsilon_\ell - \omega)}{2 \varepsilon_\ell}, \quad (4)$$

$$\sigma^\mu_- (\omega, q) = i \sum_\ell \langle \Omega | \phi^b_\ell | \Omega \rangle \langle A^a_\mu | \Omega_\ell \rangle \frac{\delta(q, p_\ell) \delta(\varepsilon_\ell - \omega)}{2 \varepsilon_\ell}, \quad (5)$$

where $| \Omega_\ell \rangle$ are the eigenstates of the QCD Hamiltonian normalized as $\langle \Omega_\ell | \Omega_\ell' \rangle = 2 \varepsilon_\ell (2\pi)^3 V \delta_{\mu\nu} \delta(p_\ell - p_{\ell'})$ with the eigenvalue $\varepsilon_\ell$ measured from the ground state and the spatial volume $V$. In the infinite volume limit, the summation should be replaced as $\sum_\ell \to \int d\varepsilon n(\varepsilon)$ with the density of states $n(\varepsilon)$.

Since the states $| \Omega \rangle$ and $| \Omega_\ell \rangle$ have isospin 0 and 1, respectively, we put the isospin label $a (= 1, 2, 3)$ explicitly to the latter state as $| \Omega^a_\ell \rangle$ for later convenience. In the soft limit $q_\mu \to 0$, we have zero modes denoted by $\ell = 0$ with the property $\varepsilon_0 \to 0$ as $q \to 0$, and the non-zero modes denoted by $\ell = 1, 2, 3$ with the property $\varepsilon_\ell \neq 0$ at $q = 0$.

Leaving the discussion on the physical content of these states to the next section, we focus on the symmetry aspect in the nuclear medium.

With the vector $n_\mu$ characterizing Lorentz frame of the nuclear matter, the matrix elements of $\phi^a_\ell$ and $A^a_\mu$ for the states $| \Omega \rangle$ and $| \Omega^a_\ell \rangle$ are generically written as

$$\langle \Omega^a_\ell | \phi^b_\ell (x) | \Omega \rangle = \delta^{ab} G^a_\ell^{1/2} e^{ik \cdot x}, \quad (6)$$

$$\langle \Omega | A^a_\mu (x) | \Omega^a_\ell \rangle = i \delta^{ab} [n_\mu (n \cdot k) N^*_\ell + k_\mu F^*_\ell] e^{-ik \cdot x}. \quad (7)$$

Here the space-time independent constants, $G^a_\ell$ (pseudo-scalar coupling constant), and $N^*_\ell$ and $F^*_\ell$ (axial-vector coupling constants) are the functions of $k \cdot n$ and $k^2$. The factor $n \cdot k$ in front of $N^*_\ell$ is introduced to make $N^*_\ell$ and $F^*_\ell$ have the same dimension. Because of the conservation law of the axial current $\partial \cdot A = 0$ in the chiral limit, $N^*_\ell$ and $F^*_\ell$ are not independent: $(k \cdot n)^2 N^*_\ell + k^2 F^*_\ell = 0$. Taking the rest frame of the nuclear matter where $n_\mu = (1, 0, 0, 0)$, we have

$$\varepsilon_\ell^2 (N^*_\ell + F^*_\ell) - k^2 F^*_\ell = 0, \quad (8)$$

with $k = (\varepsilon_\ell, k)$. According to this relation, the axial-vector coupling of the non-zero modes $N^*_\ell + F^*_\ell$ should vanish in the limit $k \to 0$.

The correlator (3) can be calculated with the matrix elements (6) and (7): For example, the spectral function $\sigma^\mu_\pm$ defined in (6) is evaluated to be

\[ Strictly speaking, we should introduce a small quark mass $m_q$ and take the limit $m_q \to 0$ after $V \to \infty$ to make the whole procedure well-defined. \]
$i\delta^{ab}\sum_{\ell} \frac{\omega \varepsilon_{\ell} N^*_\ell + (\omega \varepsilon_{\ell} - q^2) F^*_\ell}{2\varepsilon_{\ell} (\omega - \varepsilon_{\ell} + i\epsilon)} G^*_{\ell} \\
\xrightarrow{q_{\mu} \to 0} \frac{i}{2} \delta^{ab} \sum_{\alpha} (N^*_\alpha + F^*_\alpha) G^*_{\alpha}^{1/2}, \quad (9)$

where we have used Eq.(8) together with the fact that $N^*_\ell + F^*_\ell$ can have finite values only for the zero-modes ($\ell = \alpha$).

Note that the result is independent of how to take the soft limit, $q_{\mu} \to 0$. The contribution from $\sigma^-_{\mu}$ is found to be the complex conjugate of $\sigma^+_{\mu}$. Thus the hadronic contribution to the correlator (1) in the soft limit becomes

$\Pi_{ab}^\mu(0) = i\delta^{ab} \sum_{\alpha} \text{Re} \left[ (N^*_\alpha + F^*_\alpha) G^*_{\alpha}^{1/2} \right]. \quad (10)$

where the summation is taken for all the zero modes.

Equating Eqs.(2) and (10), we finally obtain an exact sum rule in the chiral limit which is valid for all densities,

$\sum_{\alpha} \text{Re} \left[ (N^*_\alpha + F^*_\alpha) G^*_{\alpha}^{1/2} \right] = -\langle \bar{q} q \rangle^*. \quad (11)$

At zero density, only the pion contributes to the sum rule and Eq. (11) reduces to the well-known Glashow-Weynberg relation, $F^s G^s_{1/2} = -\langle \bar{q} q \rangle [10]$. On the other hand, at finite density, the left hand side of Eq. (11) may receive various contributions not only from the in-medium pion but also from other collective excitations in nuclear matter, so that full dynamical treatment of the many body system would be necessary. However, the discussion becomes much simpler when the linear density approximation is adopted as will be shown in the next section.

Before closing this section, let us consider a zero mode with a linear dispersion relation $\varepsilon_{\alpha} = v_{\alpha}|k|$ for small momenta. In this case, the matrix element (7) can be written in a simpler form. Let us introduce the temporal and spatial “decay constants” as

$\langle \Omega| A_0^a(0)|\Omega^b_{\alpha}(k) \rangle = i\delta^{ab}\varepsilon_{\alpha} F^s_{\alpha}, \quad (12)$

$\langle \Omega| A_0^b(0)|\Omega^a_{\alpha}(k) \rangle = i\delta^{ab} k_i F^s_{\alpha}. \quad (13)$

Then, by using the dispersion relation together with Eq. (8) and $n_{\mu} = (1, 0, 0, 0)$, we have

$F^s_{\alpha} = F^*_s_{\alpha}, \quad F^t_{\alpha} = N^*_\alpha + F^s_{\alpha} = F^s_{\alpha}/v^2_{\alpha}. \quad (14)$

Here it should be noticed that the relation $F^t_{\pi} = F^s_{\pi}/v^2_{\pi}$ for the pionic mode at finite temperature is derived in [11][12].

3. In-medium Glashow-Weinberg relation at low density

The exact sum rule in the chiral limit, Eq. (11), is reduced to a simple form at low density. The expectation value of an arbitrary operator $\mathcal{O}$ in nuclear matter up to the linear term in density reads

$\langle \Omega|\mathcal{O}|\Omega \rangle \simeq \langle 0|\mathcal{O}|0 \rangle + \rho\langle N|\mathcal{O}|N \rangle. \quad (15)$
Fig. 1. Diagrams for the $\pi N$ scattering amplitude induced by the external fields, $A_\mu^a$ and $\phi_5^b$. The solid, dotted and wavy lines denote nucleon, pion and external fields, respectively.

For $\Pi^{ch}_5(q)$ in Eq. (1), the first term in the right hand side of Eq. (15) is dominated by the massless pion in the chiral limit. The second term corresponds to the forward scattering amplitude of the external fields $(A_\mu^a$ and $\phi_5^b)$ with the nucleon multiplied by $q^\mu$.

As shown in Fig. 1, there are four different contributions to the forward amplitude: (a) the term with double $\pi$ pole where both $A_\mu^a$ and $\phi_5^b$ couple directly to the pion, (b,c) the term with single $\pi$ pole where either $A_\mu^a$ or $\phi_5^b$ couples directly to the pion, and (d) the term without $\pi$ pole where the external fields couple directly to the nucleon.

It is easy to see that the $p$-wave coupling of the pion or the external fields to the nucleon gives vanishing contribution to the forward amplitude in (a-d). On the other hand, the $s$-wave coupling in (a,b,c) can leave finite contributions even in the soft limit due to the pion pole(s). In particular, particle-hole excitations do not contribute to the sum rule in the linear-density approximation. This leads to the conclusion that only the pionic mode with possible medium modification contributes to the sum rule in the previous section at low density:

$$F_t^\pi G^{*1/2}_\pi = -\langle \bar{q}q \rangle^*.$$  \hspace{1cm} (16)

Note that Eq. (16) is only valid if both sides are expanded up to $O(\rho)$.

Taking the ratio of Eq. (16) and its counterpart at zero density, we find the scaling law:

$$\left(\frac{F_t^\pi}{F_\pi}\right) Z^{1/2}_\pi = \frac{\langle \bar{q}q \rangle^*}{\langle \bar{q}q \rangle},$$  \hspace{1cm} (17)

with the in-medium wave function renormalization $Z^*_\pi \equiv G^*_\pi/G_\pi$. As will be shown in the next section, in-medium change of $Z^*_\pi$ from 1 can be evaluated from the isosinglet pion-nucleon scattering amplitude, while $F_t^\pi/F_\pi$ is related to the pion-nucleus isovector scattering lengths through the in-medium TW relation. Therefore, Eq. (17) gives a direct link between the in-medium modification of the quark condensate and that of the pion decay constant\[5\].

Alternative relation between $\langle \bar{q}q \rangle^*$ and $F_t^\pi$ is obtained by taking the matrix element of the PCAC relation

$$\partial \cdot A^\mu(x) = 2m_\pi \phi_5^b(x)$$ slightly away from the chiral limit. With the matrix elements given in Eqs. (6) and (12) with $\alpha = \pi$, we have $(\varepsilon^2 - v_\pi^2 k^2)F_t^\pi = 2m_\pi G^{*1/2}_\pi$, where $F_t^\pi$ and $G^*_\pi$ are the values in the chiral and soft limit. Since the in-medium pion mass is given by $\varepsilon^2 = m_\pi^* + v_\pi^2 k^2 + O(k^4)$, we have $m_\pi^* F_t^\pi = 2m_\pi G^{*1/2}_\pi$. Combining this with Eq. (11), we obtain

$$F_t^\pi G^{*1/2}_\pi = -\langle \bar{q}q \rangle^*.$$

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$$F_t^\pi G^{*1/2}_\pi = -\langle \bar{q}q \rangle^*.$$
\[(F^\pi_\tau)^2 m^2_\tau = -2m_q (\bar{q}q)^*, \quad (18)\]
\[(F^\pi_\tau/F^\pi_\tau)^2 \left[ \frac{m^*_\pi}{m^2_\pi} \right] = \langle \bar{q}q \rangle^*/\langle \bar{q}q \rangle, \quad (19)\]

Eq. (18) is the in-medium generalization of the Gell-Mann–Oakes–Renner relation [14] and was derived before in the Nambu–Jona-Lasinio model [15] and in chiral perturbation theory [16,11]. Theoretically, Eq. (17) is equivalent to Eq. (19). Experimentally, the information on \(m^*_\pi\) is necessary to check the latter sum rule, which is relatively difficult.

4. In-medium wave-function renormalization

Let us consider the relation between the pion wave function renormalization constant \(Z^*\_\pi\) and the isospin singlet pion-nucleon scattering amplitude. For this purpose, we introduce the off-shell \(\pi N\) amplitude near the pion pole through the operator \(\phi^a_5(x)\), as is done in [17],

\[T^{ab}_\piN(\nu, \nu_B; m_\pi) = \delta^{ab} T^{(+)} + \frac{1}{2} \{r^a, r^b\} T^{(-)} \equiv \frac{i}{G^\pi} q^2 q'^2 \int d^4x e^{i\vec{q}\cdot\vec{x}} \langle N(P')|T[\phi^a_5(x)\phi^b_5(0)]|N(P)\rangle \quad (20)\]

with the in-coming (out-going) pion momentum \(q (q')\) and the kinematical variables defined as \(\nu \equiv P \cdot (q + q')/(2M_N)\) and \(\nu_B \equiv -q \cdot q'/(2M_N)\). In the forward limit \(q' \rightarrow q\) with \(q = 0\), the scattering amplitude is a function solely of \(\omega\). Thus, the isospin singlet amplitude for small \(\omega\) is written as

\[T^{(+)}(\omega; m_\pi) \simeq \alpha + \beta \omega^2 \quad (21)\]

Thanks to the special off-shell extrapolation given in Eq. (20), the coefficients \(\alpha\) and \(\beta\) can be evaluated as follows [17]: (i) At the off-shell Weinberg point, we have \(T^{(+)}(0; m_\pi) = \alpha = -\sigma_{\pi N}/F^2_\pi\) with the \(\pi N\) sigma term \(\sigma_{\pi N} \simeq 45\text{MeV}\) [15]. (ii) At the on-shell threshold, we have \(T^{(+)}(\omega = m_\pi; m_\pi) = 4\pi(1 + m_\pi/m_N)a_{\pi N}\) with the scattering length \(a_{\pi N} = (0.0016 \pm 0.0013)m_\pi^{-1}\) [19]. Combining these, we obtain

\[\beta \simeq \frac{\sigma_{\pi N}}{F^2_\pi m^2_\pi} + \left(1 + \frac{m_\pi}{m_N}\right) \frac{4\pi a_{\pi N}}{m^2_\pi}. \quad (22)\]

Numerically, the first term in the right hand side of Eq. (22) dominates over the second term and we find \(\beta = 2.17 \pm 0.04\text{ fm}^3\).

To obtain a relation between \(\beta\) and \(Z^*\_\pi\), we now consider the correlation of \(\phi^a_5\) in symmetric nuclear matter in the chiral limit expanded up to linear in density according to Eqs. (15), (20) and (21):

\[D^{ab}(q) = \int d^4x e^{i\vec{q}\cdot\vec{x}} \langle \Omega|T[\phi^a_5(x)\phi^b_5(0)]|\Omega\rangle \quad (23)\]

\[
\lim_{q \rightarrow 0} -\delta^{ab} G^\pi \left[ \frac{1}{\omega^2} - \frac{1}{\omega^2} T^{(+)}(\omega; 0) \rho \frac{1}{\omega^2} \right], \quad (24)
\]

\[= i\delta^{ab} G^\pi \left[ 1 - \beta \rho \frac{1}{\omega^2} \right] = i\delta^{ab} G^\pi \frac{1}{\omega^2}. \quad (25)\]
Then, we obtain

$$Z_{\pi}^{1/2} \equiv \left( \frac{G_{\pi}^*}{G_{\pi}} \right)^{1/2} = 1 - \frac{\rho}{\rho_0}, \quad (26)$$

with $\gamma = \beta \rho_0 / 2 \simeq 0.184$. Notice that the reduction of $Z_{\pi}^*$ in the nuclear medium given in Eq. (26) stems solely from the $s$-wave pion-nucleon interaction.

5. In-medium Tomozawa-Weinberg relation

In-medium pion properties are conventionally expressed in terms of the pion-nucleus optical potential. For instance, the $s$-wave potential for $\pi^-$ is parametrized as [6]

$$2m_{\pi}U_s = -4\pi \left[ 1 + \frac{m_{\pi}}{m_N} \right] (b_0^*(\rho) - b_1^*(\rho)\delta \rho), \quad (27)$$

with the isoscalar density $\rho = \rho_p + \rho_n$ and the isovector density $\delta \rho = \rho_p - \rho_n$. The parameters $b_0^*$ and $b_1^*$ represent the pion-nucleus scattering lengths in isoscalar and isovector channels, respectively. $T^{(+)}(\omega; m_{\pi})$ are the isoscalar and isovector pion-nucleus scattering amplitudes at zero spatial momentum. The value of $b_1^*$ extracted from the pionic atom and $\pi^-\text{-nucleus}$ scattering data is larger than that in the vacuum, which indicates an enhanced repulsion from nuclei [1,2]. This extra repulsion expressed by $b_1^*$ is interpreted as originating from the in-medium reduction of $F_{\pi}^t$ through the in-medium generalization of the Tomozawa-Weinberg relation [6].

Let us derive a relation between $b_1^*$ and $F_{\pi}^t$ in a way parallel to the derivation of $Z_{\pi}^*$ in the previous section using current commutation relations. We consider the following axial vector correlator in a slightly asymmetric nuclear matter $|\Omega\rangle$ in the chiral limit:

$$\Pi_{ab}^{\mu}(q) = \int d^4x \, e^{iq\cdot x} \partial^\mu \langle \Omega| T[A_\mu^a(x)A_\nu^b(0)]|\Omega\rangle. \quad (29)$$

Then using the current conservation $\partial \cdot A = 0$ and the relation $\int d^4x \, \partial^\mu T[A_\mu^a(x)A_\nu^b(0)] = [Q^a_{\nu}, A_\mu^b(0)] = i\epsilon^{abc}V_{\nu}^c(0)$ satisfied in the chiral limit, we obtain the sum rule

$$\Pi_0^{ab}(0) = i\epsilon^{abc}V_{\nu}^c(0) |\Omega\rangle \simeq i\epsilon^{abc} \frac{1}{2} \delta \rho. \quad (30)$$

On the other hand, Eq. (29) at the chiral and soft limits is saturated with linear in isovector density as

$$\Pi_0^{ab}(q) \longrightarrow q \longrightarrow i\omega \left[ \frac{\omega F_{\pi}^t}{\omega^2} \cdot T^{(-)}(\omega; 0) \delta \rho \cdot \frac{\omega F_{\pi}^t}{\omega^2} \right]. \quad (31)$$

Comparing Eqs. (30) and (31), we find

$$T^{(-)}(\omega; 0) \simeq \frac{\omega}{2(F_{\pi}^t)^2}. \quad (32)$$

This is an in-medium generalization of the Tomozawa-Weinberg relation [20] and was obtained before in Ref. [6] with chiral perturbation approach.
Using the definitions Eqs. (27,28) together with Eq. (32), we obtain a formula relating the in-medium change of the isovector scattering length and the in-medium change of the pion decay constant in the chiral limit:

\[ \frac{b_1}{b_1^*} = \left( \frac{F_t^*}{F_t} \right)^2. \]  

(33)

6. In-medium quark condensate

Now, inserting Eqs. (26) and (33) into Eq. (16), we arrive at one of the central results in this work,

\[ \frac{\langle \bar{q}q \rangle^*}{\langle \bar{q}q \rangle} \simeq \left( \frac{b_1}{b_1^*} \right)^{1/2} \left( 1 - \frac{\gamma \rho}{\rho_0} \right), \]  

(34)

which directly relates the in-medium quark condensate with the observables related to the pion in nuclei. The deeply bound pionic atom data suggest the repulsive enhancement of \( b_1^* \). The \( \pi N \) scattering data tell that \( \gamma > 0 \). Thus, Eq. (34) implies that these experimental facts give a direct evidence of the reduction of the quark condensate in nuclear medium. Quantitatively, the experimental value of \( b_1/b_1^* \) is obtained as 0.79±0.05 at the effective density \( \rho \approx 0.6 \rho_0 \) in deeply bound pionic atoms [1]. With this value and \( \gamma = 0.184 \) estimated in Sec. 4 together with the linear density approximation, we find for the ratio of the quark condensates \( \langle \bar{q}q \rangle^*/\langle \bar{q}q \rangle \simeq 1 - 0.37 \rho/\rho_0 \). We also evaluate this ratio with \( b_1/b_1^* = 0.75 \) obtained in elastic \( \pi \)-nucleus scatterings [2] assuming the effective density \( \rho \approx \rho_0 \). The result is \( \langle \bar{q}q \rangle^*/\langle \bar{q}q \rangle \simeq 1 - 0.43 \rho/\rho_0 \). These numbers are consistent with that given by the formula expressed with the pion-nucleon \( \sigma \)-term (see footnote 3).

For further quantitative argument, one has to take into account explicit chiral symmetry breaking effects and higher density contributions.

Here we emphasize that the renormalization of the pion field is inevitable in describing the pion dynamics when partial restoration of chiral symmetry occurs. This is understood clearly in the context of the chiral effective theory. The chiral effective theory is based on a consistent decomposition of the field variables on the chiral manifold with the original symmetry to radial (\( \sigma \)) and angular (pionic, i.e., Nambu-Goldstone) ones. For dynamics in the vacuum, the relevant degree of freedom is the massless angular mode expressed by the dimensionless chiral field \( U \). Since the pion field has the dimension of energy and the order parameter of the dynamical symmetry breaking provides the only relevant energy scale in the chiral limit, the pion field should be normalized by the quark condensate. Therefore, when partial restoration of the chiral symmetry takes place, the pion field is necessarily renormalized according to the reduction of the quark condensate [8]; see also Sec. 8.

7. The sum rule beyond the chiral limit

Let us generalize the in-medium sum rule Eq. (11) in the chiral limit to the case with a finite quark mass \( m_q \). From the PCAC relation \( \partial \cdot A^a(x) = 2m_q \phi_5^a(x) \), Eq. (2) is easily generalized to

\[ \Pi_5^{ab}(0) - 2m_qD^{ab}(0) = -i\delta^{ab}\langle \bar{q}q \rangle^* \]  

(35)

where \( D^{ab}(q) \) is the correlation function of the pseudoscalar density \( \phi_5^a \) given in Eq. (23).
Also, the definition of the coupling constants, Eqs.(6) and (7), together with the PCAC relation lead to the generalization of Eq.(8),

\[ \varepsilon^2 \ell (N^* \ell + F^* \ell) - k^2 F^* \ell = 2m_q G^{*1/2}. \]  
\[ (36) \]

The hadronic matrix elements in the left hand side of Eq.(35) can be evaluated as before and we find the sum rule away from the chiral limit,

\[ \sum_{\ell} \text{Re} \left[ (N^* \ell + F^* \ell) G^{*1/2} \right] = -\langle \bar{q}q \rangle^*, \]  
\[ (37) \]

where the summation is taken over all states coupled to \( A_\mu \) and \( \phi_5 \), and the matrix elements are evaluated at \( k_\ell = (\varepsilon_\ell(k = 0),0) \). Modes \( \ell \) may be classified into would-be zero modes \( (\ell = \alpha) \) and would-be non-zero modes \( (\ell = \beta) \). Then, eq.(36) shows that, in the soft limit, \( N^* \alpha + F^* \alpha = \text{const.} + O(m_q) \) for would-be zero-modes, while \( N^*_\beta + F^*_\beta = O(m_q) \) for would-be non-zero modes. Namely, the quark mass correction to \( \langle \bar{q}q \rangle^* \) in the rhs of Eq.(37) receives both effects. To evaluate them, detailed hadronic model calculations such as given in Ref. [21] are needed.

8. Relation to the chiral effective theory

So far, we have derived the in-medium sum rule Eq.(11) based only on general operator relations in QCD without assuming any hadronic descriptions. The sum rule can be used for checking consistency of theoretical models with the fundamental symmetry of QCD, and also for experimental confirmations of partial restoration of chiral symmetry in nuclear medium, once the matrix elements of the currents are experimentally extracted, as discussed in section 6.

Let us demonstrate here how the in-medium GW relation Eq.(16) is expressed in terms of the low energy constants in chiral perturbation theory. An effective Lagrangian for the pion in nuclear medium is obtained in the mean field approximation of the nucleon field [16]:

\[ \mathcal{L}^{\text{eff}} = \left( \frac{f^2}{4} + \frac{c_3}{2}\rho \right) \text{Tr}[D_\mu U D^\mu U^\dagger] + \frac{\tilde{c}_2}{2}\rho \text{Tr}[D_0 U D_0 U^\dagger] \]
\[ + \left( \frac{f^2}{4} + c_1\rho \right) \text{Tr}[U^\dagger \chi + \chi^\dagger U], \]  
\[ (38) \]

where \( \rho = \rho_p + \rho_n \) is the isoscalar density, \( c_i \) \( (i = 1,2,3) \) are the low energy constants with \( \tilde{c}_2 = c_2 - g_A^2/(8m_N) \) and \( f \) is the tree-level pion decay constant in the vacuum in the chiral limit. We also introduce the chiral vector currents, \( \ell_\mu \) and \( r_\mu \), in the covariant derivative \( D_\mu U = \partial_\mu U - ir_\mu U + iU\ell_\mu \), and the scalar and pseudoscalar external fields, \( s \) and \( p \), in \( \chi = 2B_0(s + ip) \) with a normalization constant \( B_0 \). The chiral field \( U \) is given by

\[ U = \exp[i\pi^a/\hat{f}]. \]  
\[ (39) \]

The \( f^* \) represents the normalization of the pion field \( \pi^a \) and is left as a parameter to be fixed so as to normalize the pion kinetic term properly [8]. Expanding the effective
Lagrangian [38] in terms of the pion field, we find the normalization in the linear-density approximation to be

\[ f^* = f \left[ 1 + \frac{\rho}{f^2} (c_3 + \tilde{c}_2) \right]. \tag{40} \]

The axial current and the pseudoscalar density in the effective Lagrangian are evaluated through the formulas

\[ A^\mu_a(x) = -(\partial \mathcal{L}_{\text{eff}} / \partial r^\mu_a - \partial \mathcal{L}_{\text{eff}} / \partial t^\mu_a)/2 \text{ and } \phi^a(x) = \partial \mathcal{L}_{\text{eff}} / \partial p^a. \]

Evaluating the matrix elements of the axial current and the pseudoscalar density for the in-medium one-pion state in the tree approximation, we obtain

\[ F^t_\pi = f^2 + \frac{2(c_3 + \tilde{c}_2)\rho}{f^*}, \quad G^{*1/2}_\pi = 2B_0 \frac{f^2 + 4c_1\rho}{f^*}. \tag{41} \]

Thus, combining Eq. (41) together with the pion field normalization (40), we obtain

\[ F^t_\pi G^{*1/2}_\pi = -2B_0(f^2 + 4c_1\rho), \tag{42} \]

in the linear density approximation. This is exactly the same formula as Eq. (16) for the medium modification of the quark condensate, since

\[ \langle \bar{q}q \rangle^* = \langle -\partial \mathcal{L}_{\text{eff}} / \partial s^0 \rangle^* = -2B_0(f^2 + 4c_1\rho). \]

To evaluate the sum rule beyond the linear density approximation, one has to calculate the matrix elements with taking into account of dynamical properties of the pion and nucleons in nuclear medium. For this purpose, the chiral effective theory is one of the powerful tool which gives systematic description in terms of density and chiral symmetry breaking off soft limit.

9. Summary

In this paper, we have derived various in-medium sum rules in a model-independent way by exploiting operator relations and chiral symmetry in QCD. We have found an exact sum rule Eq. (11) valid for any density in the chiral limit. Also, a new scaling law Eq. (17) as a generalization of the Glashow-Weinberg relation is deduced from the exact sum rule at low density, in which the in-medium quark condensate, the in-medium pion decay constant and the in-medium wave-function renormalization of the pion field are related. With the information of the πN scattering data and the in-medium Tomozawa-Weinberg relation Eq. (33), the quark condensate is related directly to the isovector π-nucleus scattering length as shown in Eq. (34).

Our formulas should be useful to check the consistency among in-medium pion properties at low density from the point of view of chiral symmetry, and to extract the in-medium quark condensate. Generalization of our approach to the pion-pion interaction inside a nuclear medium following the idea of [8] is one of the interesting future problems to be examined and may provide us with a further insight into the in-medium pions and possible partial restoration of chiral symmetry in nuclei. It is also an interesting and important task to estimate contributions beyond the linear density approximation [21,22,23].

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