Light Deflection on de-Sitter Space

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We study the light deflection on de-Sitter Space with negative cosmological constant.

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1 The light deflection

The most general covariant second-order equation for the gravitation field generated by a given (covariant) energy-matter distribution on the space-time is given by the famous Einstein field equation with a cosmological constant $\Lambda$ [with dimension (length)$^{-2}$([1]), namely.

$$\left( R_{\mu \nu}(g) - \frac{1}{2} g_{\mu \nu} R + \Lambda g_{\mu \nu} \right)(x) = 8\pi G T_{\mu \nu}(x) \quad (1)$$

where $x$ belongs to a space-time local chart.

It is well-known that studies on the light deflection by a gravitational field generated by a massive point-particle with a pure time-like geodesic trajectory (a rest particle “sun” for a three-dimensional spatial space-time section observer!) is always carried out by considering $\Lambda \equiv 0([2],[3])$.

Our purpose in this short note is to understand the light deflection phenomena in the presence of a non vanishing cosmological term in Einstein equation (1), at least on a formal mathematical level of solving trajectory motion equations.
Let us, thus, look for a static spherically symmetric solution of eq. (1) in the standard isotropic form ([2], [3])

$$(ds)^2 = B(r)(dt)^2 - A(r)(dr)^2 - r^2[(d\theta)^2 + \text{sen}^2 \theta(d\phi)^2]$$ (2)

In the space-time region $r = + |\vec{x}|^2 > 0$, where the matter-energy tensor vanishes identically, we have that the Einstein equation takes the following form

$$R_{\mu\nu}(g)(x) = -\Lambda(g_{\mu\nu}(x))$$ (3)

In the above cited region, the Ricci tensor is given by

$$-\Lambda g_{\mu\nu} = \begin{pmatrix} -\Lambda B(r) & 0 & 0 & 0 \\ 0 & \Lambda A(r) & 0 & 0 \\ 0 & 0 & \Lambda r^2 & 0 \\ 0 & 0 & 0 & \Lambda r^2 \text{sen}^2 \theta \end{pmatrix} = \begin{pmatrix} R_{tt} & 0 & 0 & 0 \\ 0 & R_{rr} & 0 & 0 \\ 0 & 0 & R_{\theta\theta} & 0 \\ 0 & 0 & 0 & R_{\phi\phi} \end{pmatrix}$$ (4)

we have, thus, the following set of ordinary differential equations in place of Einstein Partial Differential eq. (1)

$$R_{tt} = -\frac{B''}{2A} + \frac{1}{4} \left( \frac{B'}{A} \right) \left( \frac{A'}{A} + \frac{B'}{B} \right) - \frac{1}{r} \left( \frac{B'}{A} \right) = -\Lambda B$$ (5)

$$R_{rr} = \frac{B''}{2B} - \frac{1}{4} \left( \frac{B'}{B} \right) \left( \frac{A'}{A} + \frac{B'}{B} \right) - \frac{1}{r} \left( \frac{A'}{A} \right) = \Lambda A$$ (6)

$$R_{\theta\theta} = -1 + \frac{r}{2A} \left( -\frac{A'}{A} + \frac{B'}{B} \right) + \frac{1}{A} = \Lambda r^2$$ (7)

$$R_{\phi\phi} = \text{sen}^2 \theta R_{\theta\theta} = \Lambda(\text{sen}^2 \theta) r^2$$ (8)

At this point we note that

$$\frac{R_{tt}}{B(r)} + \frac{R_{rr}}{A(r)} = 0$$ (9)

or equivalently

$$A(r) = \frac{\alpha}{B(r)}$$ (10)

where $\alpha$ is an integration constant.

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Since $R_{\theta\theta} = -1 + \frac{\alpha}{\alpha} B' + \frac{B}{\alpha} = \Lambda r^2$, we get the following expression for the $B(r)$ function

$$B(r) = \frac{\alpha \Lambda r^2}{3} + \alpha + \frac{\beta}{r}$$

(11)

which $\beta$ denoting another integration constant.

In the literature situation, ([2], [3]), one always consider the case $\Lambda \neq 0$ in a pure classical mathematical vacuum situation context, the so called de-Sitter vacuum pure gravity. However in our case it becomes physical to consider that our solution depends analytically on the cosmological constant. In other words, if the parameter $\Lambda \to 0$ in our solution, it must converges to the usual Schwarzschild solution with a mass singularity at the origin $r = 0$. That is our boundary condition hypothesis imposed on our solution.

As a consequence, one gets our proposed Schwarzschild-de-Sitter solution

$$\begin{align*}
(ds)^2 &= \left( \frac{\Lambda r^2}{3} + 1 - \frac{2MG}{r} \right) (dt^2) \\
&\quad - \left( \frac{\Lambda r^2}{3} + 1 - \frac{2MG}{r} \right)^{-1} (dr)^2 - r^2 [(d\theta)^2 + (\sin^2 \theta)(d\phi)^2] 
\end{align*}$$

(12)

At this point let us comment that for the space-time region exterior to the spatial sphere $r > (\frac{3mG}{\Lambda})^{1/3}$, the field gravitation approximation leads to the anti-gravity (a repulsion gravity force) if $\Lambda < 0$, so explain from this Einstein Gravitation theory of ours the famous “Huble accelerating Universe expansion”.

In what follows we are going to consider a non-vanishing $\Lambda < 0$ and study the path of a light ray on such negative cosmological constant Einstein manifold eq. (12).

We have the following null-geodesic equation for light propagating in $\theta = \pi/2$ plane (Einstein hypothesis) for light propagation on the presence of the sun (Section 2 for the related formulae)

$$0 = B(r) - \frac{1}{B(r)} \left( \frac{dr}{dt} \right)^2 - r^2 \left( \frac{d\phi}{dt} \right)^2$$

(13)

At this point we note that

$$\left( \frac{d\phi}{dt} \right) = \left( \frac{B(r)J}{r^2} \right)$$

(14)

where $J$ is a integration constant.
After substituting eq. (14) into eq. (13) we have the following differential equation for the light trajectory as a function of the deflection angle \( \phi \)

\[
\left( \frac{dr}{d\phi} \right)^2 + \frac{J^2 B^3(r)}{r^2} - B^2(r) = 0 \tag{15}
\]

which is exactly integrable

\[
d\phi = \frac{dr}{r^2 \sqrt{\frac{1}{r^2} - \frac{B(r)}{r^2}}} \tag{16}
\]

By supposing a deflection point \( r_m \) where \( \frac{dr}{dt} = 0 \) and, thus, \( J = r_m/\sqrt{B(r_m)} \), we get the deflection angle.

\[
\Delta_1 \phi = \int_{r_m}^\infty \frac{dr}{r^2 \left[ \frac{B(r_m)}{r_m} - \frac{B(r)}{r^2} \right]^{1/2}} = \int_0^1 \frac{dU}{\left[ \left( U_m^2 - U^2 \right) - 2MG(U_m^3 - U^3) \right]^{1/2}} \tag{17}
\]

which is exactly that one given in the pure \((\Lambda = 0)\) Schwarzschild famous case. However, if one supposes that there is no deflection (a continuous monotone trajectory \( r = r(\phi) \)), the total deflection angle now depends on the cosmological constant and is given formally by the expression below.

\[
\Delta_2 \phi = \int_0^{r_m} dr \left\{ \frac{1}{r^2 \sqrt{\frac{1}{r^2} + \frac{2MG}{r^3}}} \left[ \frac{1}{\sqrt{1 + \left[ \frac{r^3(1-\Lambda J^2)}{2MG-r} \right]^{1/2}}} \right] \right\} \neq \Delta_1 \phi \tag{18}
\]

As a general conclusion of our note we claim that the usual light-deflection experimented test does not make difference between the usual non-cosmological Schwarzschild case and our case Eq.(12), and, thus, it should be not considered as a definitive physical support for Einstein General Relativity without cosmological constant.
2 The Trajectory Motion Equations

The body trajectory \((t(p), r(p), \theta(p), \varphi(p))\) on the presence of the gravitational field generated by the metric eq.(2)–eq.(10) is described by the following geodesic equations

\[
\frac{d^2t}{dp^2} + \frac{B'}{B} \left( \frac{dr}{dp} \right) \left( \frac{dt}{dp} \right) = 0
\]

\[
\frac{d^2r}{dp^2} + \frac{A'}{2A} \left( \frac{dr}{dp} \right)^2 - \frac{r}{A} \left( \frac{d\theta}{dp} \right)^2 - r \sin^2 \theta \left( \frac{d\phi}{dp} \right)^2 + \frac{B'}{2A} \left( \frac{dt}{dp} \right)^2 = 0
\]

\[
\frac{d^2\theta}{dp^2} + \frac{2d\theta dr}{r \, dr \, dp} - \sin \theta \cdot \cos \theta \left( \frac{d\phi}{dp} \right)^2 = 0
\]

\[
\frac{d^2\varphi}{dp^2} + \frac{2d\phi dr}{r \, dp \, dp} + 2\cot \theta \left( \frac{d\phi}{dp} \right) \frac{d\phi}{dp} = 0
\]

At this point we remark that by multiplying eq.(19) by \(B(r(dp))\), it reduces to the exactly integral form relating the Einstein proper-time (physical evolution parameter) \(p\) with the geometrical dependent coordinate Newtonian time \(t\):

\[
\frac{dt}{dp} = \frac{1}{B(r)}
\]

We remark either that eq.(22) can be rewritten in the form

\[
\frac{d}{dp} \left( \ell n \frac{d\phi}{dp} + \ell n r^2 + 2\ell n \sin \theta \right) = 0
\]

which reduces to the following form

\[
\left( \frac{d\phi}{dp} r^2(p) \sin^2(\theta(p)) \right) = J
\]

where \(J\) is a integration constant.

By substituting eq.(23) and eq.(25) into equations (20) and (21) we obtain the full set of equations describing the body trajectory in relation to the \((r, \theta)\) variables

\[
\frac{d^2r}{dp^2} - \frac{B'}{2B} \left( \frac{dr}{dp} \right)^2 - rB \left( \frac{d\theta}{dp} \right)^2 - \frac{J^2 B}{r^3} + \frac{B'}{2B} = 0
\]

\[
\frac{d^2\theta}{dp^2} + \frac{2d\theta dr}{r \, dr \, dp} - \frac{\cos \theta \, J^2}{\sin^2 \theta \, r^4} = 0
\]
For Einstein hypothesis of light propagation on the plane $\theta = \pi/2$, eq.(27) vanishes and eq.(26) takes the form

$$\frac{d^2 r}{d^2 p} - \frac{B'}{2B} \left( \frac{dr}{dp} \right)^2 - \frac{J^2 B}{r^3} + \frac{B'}{2B} = 0$$

(28)

or in a more manageable alternative form after multiplying eq.(28) by $\frac{2}{B} \left( \frac{dr}{dp} \right)$ and by using eq.(23) for exchange the geometrical parameter $p$ by the time manifold coordinate

$$\left( \frac{dr}{dt} \right)^2 \frac{1}{B^3} + \frac{J^2}{r^2} - \frac{1}{B} + E = 0$$

(29)

where $E$ denotes another integration constant.

By writing $r$ is a function of $\phi$ and using eq.(25) $\left( \frac{d\phi}{dt} r^2 = J \right)$, we get our final trajectory equation

$$\frac{dr}{d\phi} = \pm r^2 \left[ \frac{1}{J^2} - \frac{B}{r^2} - \frac{BE}{J^2} \right]^{1/2}$$

(30)

which leads to the body trajectory geometric form

$$\phi = \pm \int \frac{dr}{r^2B^{1/2}\left[ \frac{1}{J^2B} - \frac{E}{J^2} - \frac{1}{r^2} \right]^{1/2}}$$

(31)

Note that for light propagation the integration constant $E$ always vanishes, a result used on the text by means of eq. (16).

3 On the topology of the Euclidean Space-Time

One of the most interesting aspects of Einstein gravitation theory is the question of the non-existence of “holes” in the space-time $C^2$-manifold from the view point of a mathematical observer situated on the Euclidean space $R^9$ associated to the ”minimal” Whitney imbedding theorem of $M$ on Euclidean spaces ([5]).

In order to conjecture the validity of such topological space-time property, let us suppose that $M$ is a $C^2$ manifold and the analytically continued (Euclidean) matter distribution tensor generating the (Euclidean) gravitation field on $M$ allows a well-defined Euclidean metric tensor (solution of Euclidean Einstein equation) ([6]).
At this point we note that $M$ must be always orientable in order to have a well-defined theory of integration on $M$ and, thus, the validity of the rule of integration by parts: Stoke’s-theorem is always needed in order to construct matter tensor energy momentum. Since Euclidean Einstein’s equations says simply that the sum of sectional curvatures is a measure of the (classical) matter energy density generating gravity, which must be always considered positive, it will be natural to expect that the positivity of the Euclidean Energy-Momentum of the matter content leads to the result that the associated sectional curvatures are positive individually. Since $M$ is even-dimensional (four), the famous Synge’s theorem ([4]) leads to the result that $M$ is simply connected (note that this topological property is obviously independent of the metric structure being Lorentzian or Euclidean!) and as a direct consequence of this result, any physical geodesic (particles trajectory) on $M$ can be topologically deformed to a point, and, thus, $M$ does not posseses “holes” from the point of view of the Whitney imbedding extrinsic minimal space $R^9$.

Finally, let us argument out that the existence of a (symmetric) energy-momentum tensor on $M$ is associated to the “General Relativity” description of the space-time manifold $M$ by means of charts (the Physics is invariant under the action of the diffeomorphism group of $M$ ) which by its turn; leads to the existence of the matter energy-momentum tensor by means of Noether theorem (a metric-independent result) applied to the matter distribution Lagrangean (a scalar function defined on the tangent bundle of $M$).

As a consequence, let us conjecture again that the introduction of a cosmological term on Einstein equation spoils the physical results presented on the hole topology and the physical requirement of positivity of the Matter-Energy Universe moments tensor, given, thus, a plausible topological argument for the vanishing of the cosmological constant at the level of the global-topological aspects of the Space-Time Manifold.

Finally, let us show the mathematical formulae associated to our ideas and conjectures above written.

Let $e_0, \ldots, e_3$ be an orthonormal frame at a point of $M$ (Euclidean). It is well known
that the Ricci quadratic form can be expressed in terms of sectional curvatures

$$\text{Ric}(e_i, e_i) = \sum_{j \neq i} K(e_i \wedge e_j)$$

(32)

and the Einstein tensor is defined by

$$G_{ij} = R_{ij} - \frac{1}{2} g_{ij} R$$

(33)

Since the Einstein equation reads in term of quadratic forms associated to sectional curvatures as

$$G(e_p, e_p) = + \sum K(e^\perp_p) = T(e_p, e_p)$$

(34)

with $T_{ij}$ being the matter energy tensor and $e^\perp_p$ is the basis 2-plane orthogonal to $e_p$, one can is principle write the sectional curvatures $K(e_p \wedge e_q)$ in terms of the quadratic Energy-momentum sectional curvatures $T_{ij}(e_p, e_q)$ at least for “short-time” cylindrical geometro dynamical space-time configurations as expected in a Quantum theory of gravitation (see ref. [1]). For the two-dimensional case this assertive is straightforward as one can see from the relations below

$$G(e_0, e_0) = K(e_1 \wedge e_1)$$

(35)

$$G(e_1, e_1) = K(e_0 \wedge e_0)$$

(36)

As a consequence, one should conjectures that the positivity of the Energy-momentum tensor $T(e_p, e_q)$ leads to the individual positivity of the sectional curvatures set $K(e_r, e_s)$ on basis of eq.(34), namely

$$G(e_p, e_q) = T(e_p, e_q) = \text{Ric}(e_p, e_q) - \delta_{pq} \left[ \sum_{i,j,i \neq j} K(e_i \wedge e_j) \right]$$

(37)
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