The Four Qubits Deletion Code is the First Quantum Insertion Code

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Abstract: The classical insertion codes were discovered at the same time as the classical deletion codes. The quantum deletion code was presented at IEICE Communications Express last year, but quantum insertion codes were never constructed. In this article, we provide the first quantum insertion code. Its encoding is the same as the four-qubits deletion code, known as the theoretically shortest deletion code.

Keywords: Quantum Error-Correction, Insertion Error, Deletion Error

Classification: Fundamental theories for communications

References

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1 Introduction

Insertion errors (insertions) and deletion errors (deletions) have been introduced as a synchronization error model in a communication channel. Levenshtein developed fundamental theory for insertion/deletion error-correcting codes [1]. One of his remarkable results is the equivalence of “single deletion error correctable” and “single insertion error correctable” for a classical code space. In other words, a classical code is a single deletion error-correcting
code by a certain decoding algorithm if and only if the code is a single insertion error-correcting code by a certain decoding algorithm. Whether the equivalence holds in quantum coding theory is an open problem.

Nakayama found the first quantum single deletion error-correcting codes (qSDCs) [2]. Hagiwara found the shortest length qSDC [3]. Nakayama proposed combinatorial conditions for constructing qSDC [4]. Shibayama obtained instances that satisfy Nakayama’s condition [5]. However, there was no quantum insertion error-correcting code.

This manuscript shows that the shortest length qSDC is single insertion error-correctable. In other words, the shortest code becomes the first quantum single insertion error-correcting code. Hence this is an example result of the open problem.

2 Preliminaries: the Four Qubits Code

Recall that the encoding of the four qubits code is

$$\alpha \ket{0} + \beta \ket{1} \mapsto \alpha \ket{\zeta} + \beta \ket{\tau},$$

where $\alpha$ and $\beta$ are complex numbers, $\ket{0}$ and $\ket{1}$ are an orthonormal basis of a two-dimensional Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and

$$\ket{\zeta} = \frac{1}{\sqrt{2}} (\ket{0000} + \ket{1111}),$$

$$\ket{\tau} = \frac{1}{\sqrt{6}} (\ket{0011} + \ket{0101} + \ket{0110} + \ket{1001} + \ket{1010} + \ket{1100}).$$

In other words, $\ket{0}$ is encoded to the sum of quantum states of Hamming weight 0 (mod 4). $\ket{1}$ is encoded to the sum of quantum states of Hamming weight 2. Note that these Hamming weights are all even.

The four qubits code above is known as the shortest length qSDC [3]. This manuscript presents that the code is also quantum single insertion error-correctable. Therefore, the four qubits code is quantum single insertion/deletion error-correctable.

3 Quantum Single Insertion

This section defines the quantum single insertion. Let $p_1, p_2, \ldots, p_n$ be particles whose states are quantum states of level 2. This implies that the quantum state for each $p_i$ is represented by a 2-by-2 density matrix. Furthermore, the quantum state of $p_1, p_2, \ldots, p_n$ is represented by a $2^n$-by-$2^n$ density matrix.

A quantum single insertion is defined as that changes the $n$ particles $p_1, p_2, \ldots, p_n$ to $n + 1$ particles $p_1, p_2, \ldots, p_i, q, p_{i+1}, \ldots, p_n$ with the single particle $q$. Hence, its quantum state is represented by a $2^{n+1}$-by-$2^{n+1}$ density matrix. Note that the partial trace operation for $q$ provides the original state. Similar to the classical case, the position of $q$ is assumed to be unknown when a quantum single insertion has occurred over a quantum communication channel. If the position is known, it is easy to recover the original state by deleting $q$, i.e., the partial trace.
The next three sections propose the decoding procedure for a single quantum insertion to the four qubits code. The reader will see that the proposed decoding can correct the single insertion but cannot determine the insertion position. In particular, the decoding is not a partial trace for the insertion position.

4 The First Part of Error-Correction: Projection Measurement

Briefly speaking, the decoding procedure consists of three parts: 1) Projection Measurement, 2) Unitary Transformation, and 3) Deletion Operation.

This section constructs the projection measurement for the four qubits code. This measurement is applied to the state after an insertion. Therefore the size of the matrix representation of projections is $2^5$-by-$2^5$. Let us define two projections $P_0$ and $P_1$ as follows:

$$P_i := \sum_{x \in \{0,1\}^5, \text{wt}(x) \equiv i \pmod{2}} |x\rangle \langle x|,$$

where $\text{wt}(x)$ is the Hamming weight of the classical bit sequence $x$. The projection measurement is defined as $\{P_0, P_1\}$ and is denoted by $\mathcal{P}$.

Let $i$ denote the outcome related to $P_i$. If the outcome is 0 (resp. 1), the quantum state is changed to a state which is a linear combination of even (resp. odd) Hamming weight.

Recall that five particles are received while the original was four particles. The five particles are measured by $\mathcal{P}$. The next section defines the unitary operation which is the 2nd part of the decoding procedure. Before moving to the next section, let us observe an example of quantum states after an insertion and the measurement.

Example 4.1. Assume that a single insertion changes particles $p_1, p_2, p_3, p_4$ to $q, p_1, p_2, p_3, p_4$.

If the outcome is 0, the density matrix after the measurement is $|y_0\rangle \langle y_0|$, where $|y_0\rangle$ is the following pure state

$$|y_0\rangle := \alpha |0\rangle \otimes |F\rangle + \beta |0\rangle \otimes |T\rangle.$$

If the outcome is 1, the density matrix is $|y_1\rangle \langle y_1|$ where

$$|y_1\rangle := \alpha |1\rangle \otimes |F\rangle + \beta |1\rangle \otimes |T\rangle.$$

5 The Second Part of Error-Correction: Unitary Transformation

After the measurement $\mathcal{P}$, a unitary transformation is performed. The unitary transformation depends on the outcome of the measurement. In this manuscript, the transformation for the outcome 0 is defined. Even if the insertion position is unknown, this transformation changes the state such that the first position state is the same as the original state $\alpha |0\rangle + \beta |1\rangle$.

The other unitary transformation for the outcome 1 can be defined similarly.
5.1 The state after the measurement

As preparation, define the following ten sets of bit sequences.
- \(Z_1 := \{00000, 01111\}\).
- \(T_1 := \{00011, 00101, 00110, 01001, 01010, 01100\}\).
- \(Z_2 := \{00000, 10111\}\).
- \(T_2 := \{00011, 00101, 00110, 10001, 10010, 10100\}\).
- \(Z_3 := \{00000, 11011\}\).
- \(T_3 := \{00011, 01001, 01010, 10001, 10010, 11000\}\).
- \(Z_4 := \{00000, 11101\}\).
- \(T_4 := \{00101, 01001, 01100, 10001, 10100, 11000\}\).
- \(Z_5 := \{00000, 11110\}\).
- \(T_5 := \{00110, 01010, 01100, 10010, 10100, 11000\}\).

If the insertion position is \(i\) and the outcome is 0, the state after the measurement \(\mathcal{P}\) is
\[
|y\rangle := \frac{\alpha}{\sqrt{2}} \sum_{z \in Z_i} |z\rangle + \frac{\beta}{\sqrt{6}} \sum_{t \in T_i} |t\rangle.
\] (1)

On the other hand, if the insertion position is \(i\) and the outcome is 1,
\[
|y\rangle := \frac{\alpha}{\sqrt{2}} \sum_{z \in \bar{Z}_i} |\bar{z}\rangle + \frac{\beta}{\sqrt{6}} \sum_{t \in \bar{T}_i} |\bar{t}\rangle,
\]
where \(\bar{f}\) is obtained by bit-flipping for all entries of \(f\).

5.2 Unitary Transformation in the Case of the Outcome 0

With the assumption where the outcome is 0, the unitary transformation is defined below.

As preparation, define the map \(\delta : \bigcup_{1 \leq i \leq 5} T_i \rightarrow \mathcal{R}\) as follows, where \(\mathcal{R} = \{0001, 0010, 0011, 0100, 0101, 0110, 0111, 1000, 1001, 1010, 1011, 1100\}\).
- \(\delta(00011) := 0001\).
- \(\delta(00101) := 0010\).
- \(\delta(00110) := 0011\).
- \(\delta(01001) := 0100\).
- \(\delta(01010) := 0101\).
- \(\delta(01100) := 0110\).
- \(\delta(10001) := 0111\).
\[ \delta(10010) := 1000. \]
\[ \delta(10100) := 1001. \]
\[ \delta(11000) := 1010. \]

Note that the domain of \( \delta \) is \( \{ x \in \{0,1\}^5 \mid \text{wt}(x) = 2 \} \). Next, define the unitary transformation \( R_0 \) as follows.

- For \( t \in \{0,1\}^5 \) of Hamming weight 2, \( R_0|t\rangle := |1\rangle \otimes |\delta(t)\rangle \)
- \( R_0|00000\rangle := \frac{1}{\sqrt{12}} \sum_{r \in R} |0\rangle \otimes |r\rangle \)
- \( R_0|01111\rangle := \frac{1}{\sqrt{12}} \sum_{r \in \delta(T_1)} |0\rangle \otimes |r\rangle \) - \( \frac{1}{\sqrt{12}} \sum_{r \in \delta(T_1 \setminus T_1)} |0\rangle \otimes |r\rangle \).
- \( R_0|01111\rangle := \frac{1}{\sqrt{12}} \sum_{r \in \delta(T_2)} |0\rangle \otimes |r\rangle \) - \( \frac{1}{\sqrt{12}} \sum_{r \in \delta(T_2 \setminus T_2)} |0\rangle \otimes |r\rangle \).
- \( R_0|11011\rangle := \frac{1}{\sqrt{12}} \sum_{r \in \delta(T_3)} |0\rangle \otimes |r\rangle \) - \( \frac{1}{\sqrt{12}} \sum_{r \in \delta(T_3 \setminus T_3)} |0\rangle \otimes |r\rangle \).
- \( R_0|11101\rangle := \frac{1}{\sqrt{12}} \sum_{r \in \delta(T_4)} |0\rangle \otimes |r\rangle \) - \( \frac{1}{\sqrt{12}} \sum_{r \in \delta(T_4 \setminus T_4)} |0\rangle \otimes |r\rangle \).
- \( R_0|11110\rangle := \frac{1}{\sqrt{12}} \sum_{r \in \delta(T_5)} |0\rangle \otimes |r\rangle \) - \( \frac{1}{\sqrt{12}} \sum_{r \in \delta(T_5 \setminus T_5)} |0\rangle \otimes |r\rangle \).

By the routine calculation, it is easy to check the following.

**Lemma 5.1.** For any even Hamming weight \( x_1, x_2 \in \{0,1\}^5 \),

\[
\langle R_0|x_1\rangle, R_0|x_2\rangle = \begin{cases} 
0 & x_1 \neq x_2 \\
1 & x_1 = x_2.
\end{cases}
\]

In other words, \( \{ R_0|x \rangle \mid \text{wt}(x) \text{ is even} \} \) is orthonormal.

For the remaining odd Hamming weight \( t \in \{0,1\}^5 \), \( R_0|t\rangle \) is defined to keep the unitarity. It is possible to define by the Gram-Schmidt process.

As we have seen, odd weight states do not appear in \( |y\rangle \) of (1).

**Theorem 5.1.** Let \( \alpha|0\rangle + \beta|1\rangle \) be the original state before the four qubits encoding. Assume that the single insertion occurs at the \( i \)th position and the outcome is 0 by the measurement \( P \).

The unitary transformation \( R_0 \) changes the state after the measurement \( P \) to

\[
(\alpha|0\rangle + \beta|1\rangle) \otimes \left( \frac{1}{\sqrt{6}} \sum_{r \in \delta(T_i)} |r\rangle \right). \tag{2}
\]

In particular, the first position state is the original state for any insertion and is independent of the state in the remaining positions.

Note that the choice of \( R \) and the definition of \( \delta \) are the key ideas of error-correction. The author feels that they seem nontrivial and curious.
6 The Third Part of Error-Correction: Deletion Operation

Similar to $R_0$, the reader can define the unitary transformation $R_1$ such that
$R_1$ changes the state after the measurement $\mathcal{P}$ to

$$
(\alpha|0\rangle + \beta|1\rangle) \otimes \left( \frac{1}{\sqrt{6}} \sum_{r \in \delta(T_i)} |\bar{r}\rangle \right).
$$

for the case where the insertion position is $i$ and the outcome is 1. For both
cases (2) and (3), by deleting the 2nd, 3rd, 4th, and 5th particles, the original
state $\alpha|0\rangle + \beta|1\rangle$ is obtained.

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