Soft-photon exponentiation beyond the quenched approximation in QED$_{2+1}$

Yuichi Hoshino

Kushiro National College of Technology,
Otanoshike Nishi 2-32-1, Kushiro, Hokkaido, 084-0916, Japan

Abstract

We discuss the infrared and ultraviolet behavior of the fermion propagator in (2+1)-dimensional QED based on spectral representation. If we choose soft-photon exponentiation to include all orders of soft-photon emission by electron, its spectral function may be written as $e^{F}$, where $F$ is a model independent spectral function of the lowest order in the coupling constant. We evaluate the function $F$ in an analytic way and show its short and long distance behavior with an infrared cut-off $\mu$. At short distance function $F$ has linear and logarithmic infrared divergence. However in the long distance limit function $F$ vanishes. So that only short distance part of $e^{F}$ is modified from unity. We may avoid the linear divergence by the choice of the gauge $d = -1$, where $d$ is a covariant gauge fixing parameter. In this gauge the spectral function vanishes in the limit of zero bare photon mass $\mu$. We overcome this difficulty by adding continuous spectrum of massive fermion loop to photon spectral function $\rho(\mu^2)$, where $\mu$ has the role of invariant mass for fermion-antifermion pair and is larger than $2m$. So that unquenched fermion spectral function survives. For the application of chiral symmetry breaking we carefully studied the position space propagator $S_F(0)$. At least for weak coupling these values agree quite well with that obtained in Dyson-Schwinger equation with proper vertex correction. We also study these parameter as a function of the flavour number $N$ and t’Hooft coupling $\alpha = e^2N/8\pi$ for strong coupling case.
Contents

I. Introduction 103

II. Spectral representation of the propagator 104
   A. Fermion 104
   B. Evaluation of the spectral function 107
      1. general property of spectral function 107
      2. non-perturbative spectral function 110
      3. perturbative spectral function 113
   C. Photon 115

III. Analysis in position space 116
   A. Quenched case 116
   B. Unquenched case 118

IV. Renormalization constant and order parameter 120
   A. by Spectral function 120
   B. by Dyson-Schwinger equation 122

V. Summary 123

VI. Acknowledgement 125

VII. References 125

VIII. Appendices 125
   A. Evaluation of specral function of one-photon state 125
   B. Feynman-Dyson perturbative spectral function 128
I. INTRODUCTION

About 30 years ago it was pointed out that high temperature limit of the field theory is described by the same theory with less-dimension and it suffers from severe infrared divergences with dimensionful coupling constant [1,2]. From this results we may feel that the asymptotic field does not exist for QED by severe infrared singularities. In this work we discuss the infrared behaviour of the massive fermion propagator by so called soft photon exponentiation. For the S matrix and the transition probability soft-photon exponentiation is familiar by the work of S. Weinberg. It has been famous to prove the cancellation of infrared divergences between real photon process and virtual photon process in the transition probability in QED[3]. There is a renormalization group analysis as well as Bloch-Nordsieck model on the infrared behaviour of the electron propagator. These works are known to lead one particle singularity of the propagator, where infrared anomalous dimension modifies the pole structure of electron[8]. It has been shown that approximate propagator by soft photon exponentiation leads the same infrared bahavior as the renormalization group analysis[3]. Following soft-photon exponentiation we discuss the femion propagator in (2+1)-dimensional QED. In (2+1)-dimensional QED, the leading infrared divergence for $O(e^2)$ spectral function $F(x, \mu)$ is linear as $1/\mu$, where $\mu$ is a bare photon mass. This term turned out to be proportional to $(d + 1)$, we may choose gauge parameter $d = -1$ to avoid linear infrared divergences. These soft photon contributions may be expressed as an extra factor as $e^{F(x, \mu)}$. In this way we obtain the fermion position space full propagator by $S^0_F(x) e^{F(x, \mu)}$. From this form of the propagator we show only short distance part of the propagator is modified while the long distance part is kept as free one by soft-photon correction with finite infrared cut-off $\mu$. This is the main feature in our approximate solution to the propagator in the previous work[4]. In this case we have a serious problem: the vanishment of the propagator in the limit $\mu$ equals to zero. In this limit $e^F$ vanishes as $(\mu|x|)^{\gamma}$ for small $|x|$, as $\exp(-\gamma m/\mu \exp(-\mu|x|))$ for large $|x|$ where $\gamma = e^2/8\pi m$. These fact show a severe infrared confinement in quenched case. To obtain finite propagator with zero infrared cut-off we may replace bare photon propagator with full propagator including the vacuum polarization of massive fermion loops. In this case imaginary part of photon propagator provides photon spectral function $\rho(\mu^2)$ with $2m \leq \mu \leq \infty$. In section II we introduce spectral representation of fermion and photon. Perturbative and non-perturbative spectral functions based on its definition are given too. Section III is devoted to
the analysis in position space propagator. We evaluate the full propagator for quenched case with bare photon mass and improve it in unquenched case. In section IV using full propagator we show the vacuum expectation value $\langle \bar{\psi} \psi \rangle$ as a function of coupling constant and numbers of flavor $N_f$, renormalization constant $Z_2$, and the results obtained by Dyson-Schwinger equation with vertex correction which satisfy Ward-Takahashi-identity. Section V is devoted to Summary.

II. SPECTRAL REPRESENTATION OF THE PROPAGATOR

A. Fermion

In this section we show how to evaluate the fermion propagator non-perturbatively by the spectral representation which preserves unitarity analyticity and CPT invariance [2,3,4,7]. Assuming parity conservation we adopt 4-component spinor. The spectral function of the fermion in (2+1) dimension is defined

$$\langle 0 | T(\psi(x) \bar{\psi}(y) | 0) = i \int \frac{d^3 p}{(2\pi)^3} e^{-ip \cdot (x-y)} \int_0^\infty ds \frac{\gamma \cdot p \rho_1(s) + \rho_2(s)}{p^2 - s + i\epsilon},$$

$$\rho(p) = \frac{1}{\pi} \text{Im} iS_F(p) = \gamma \cdot p \rho_1(p) + \rho_2(p)$$

$$= (2\pi)^2 \sum_n \delta^{(3)}(p - p_n) \langle 0 | \psi(0) | n \rangle \langle n | \bar{\psi}(0) | 0 \rangle.$$  \hspace{1cm} (1)

In the quenched approximation the state $| n >$ stands for a fermion and arbitrary numbers of photons,

$$| n > = | r; k_1, ..., k_n >, r^2 = m^2.$$  \hspace{1cm} (2)

In deriving the matrix element $\langle 0 | \psi(0) | n \rangle$ we must take into account the soft photon emission vertex which is written in the textbook for the scattering of charged particle by external electromagnetic field or collision of charged particles. Based on low-energy theorem the most singular contribution for the matrix element $T_n = \langle \Omega | \psi | r; k_1, ..., k_n \rangle$ is known as the soft photons attached to external line. We want to consider $T_n$ for $k_n^2 \neq 0$, hence we continue off the photon mass shell by a Lehmann-Symanzik-Zimmermann (LSZ) formula. First we notice the Fourier expansion of $A_{in}(x)$ as that of free fields.

$$A_{in}^T(x) = \int d^2 k [a_{in}^T(k) A_k^T(x) + a_{in}^{\dagger T}(k) A_k^{*T}(x).$$

104
with
\[ A_k^T(x) = \frac{1}{(2\pi)^2 k_0} e^{-ik \cdot x} \epsilon^T(k) \] (4)
and, upon inversion,
\[ a_{in}(k, \lambda) = i \int d^2 x A_k^T(x) \cdot \overrightarrow{\partial_0} A_{in}^T(x) \]
\[ = -i \int d^2 x A_{k,\lambda}^T(x) \cdot \overrightarrow{\partial_0} A_{in}(x)^\mu \] (5)
where \( \overrightarrow{\partial_0} \) is defined by
\[ f \overrightarrow{\partial_0} g = f \partial_0 g - (\partial_0 f) g. \] (6)

In developing the reduction formula for photon, there is a minor change from transverse photon to arbitrary state
\[ \frac{1}{\sqrt{Z_3}} A_k^T(x)_{\square_{x_i}} \cdot \langle 0 | \cdots A^T(x_i) \cdots | 0 \rangle \]
\[ = - \frac{1}{\sqrt{Z_3}} A_{k,\lambda}^\mu(x_i)_{\square_{x_i}} \langle 0 | \cdots A_\mu(x_i) \cdots | 0 \rangle . \] (7)
The additional minus sign in (5), (6) comes from the space-like nature of the polarization unit vector
\[ \epsilon^\mu \epsilon^\mu = -\epsilon \cdot \epsilon = -1. \]
\[ \epsilon^\mu T_\mu = \langle \Omega | T \psi e_n^\mu A_n^\mu(x) | r; k_1, k_2, \ldots, k_{n-1} \rangle \]
\[ = -i \lim_{t_i \to -\infty} \int d^2 x e^{ik_n \cdot x} \overrightarrow{\partial_0} \langle \Omega | T \psi e_n^\mu A_n^\mu(x) | r; k_1, k_2, \ldots, k_{n-1} \rangle \]
\[ = -i \int d^3 x \partial_0 [e^{ik_n \cdot x} \overrightarrow{\partial_0} \langle \Omega | T \psi e_n^\mu A_n^\mu(x) | r; k_1, k_2, \ldots, k_{n-1} \rangle]. \] (8)

Noting that \( \partial_0 (f \overrightarrow{\partial_0} g) = f \overrightarrow{\partial_0} \partial_0 g \) and \( \partial_0^2 e^{ik_n \cdot x} = \nabla^2 e^{ik_n \cdot x} \), we find
\[ \epsilon^\mu T_\mu = \frac{-i}{\sqrt{Z_3}} \int d^3 x e^{ik_n \cdot x} \square_{x} \langle \Omega | T \psi e_n^\mu A_n^\mu(x) | r; k_1, k_2, \ldots, k_{n-1} \rangle \]
\[ = \frac{-i}{\sqrt{Z_3}} \int d^3 x e^{ik_n \cdot x} \langle \Omega | T \psi e_n^\mu j_\mu(x) + (1 - d) \epsilon^\mu \partial_\mu \partial \cdot A(x) | r; k_1, k_2, \ldots, k_{n-1} \rangle . \] (9)

provided
\[ \square_x T \psi A_\mu(x) = T \psi \square_x A_\mu(x) = T \psi (j_\mu(x) + (1 - d) \partial_\mu \partial \cdot A(x)), \] (10)
where the electromagnetic current is
\[ j_\mu(x) = \overline{\psi}(x) \gamma_\mu \psi(x) \] (11)
and $d$ is a covariant gauge fixing parameter. From the definition (9) $T_n$ is seen to satisfy the following Ward-Takahashi identity

$$k^n_\mu T^n_\mu(r;k_1,k_2,...k_n) = eT_{n-1}(r;k_1,k_2,...k_{n-1}), r^2 = m^2,$$

(12)

provided by current conservation

$$\partial_\mu j_\mu(x) = 0,$$

$$ik_n e^{ik_n x} T(\psi_j(x)) = \partial_\mu (e^{ik_n x} T(\psi_j(x))) - ie^{ik_n x} \partial_\mu T(\psi_j(x)), \quad (13)$$

equal time commutation relations

$$\partial_\mu T(\psi_j(x)) = -e\psi(x),$$

$$\partial_\mu T(\bar{\psi}_j(x)) = e\bar{\psi}(x), \quad (14)$$

and

$$\Box \partial \cdot A(x) = 0. \quad (15)$$

Here we impose on the physical state

$$\partial_\mu A^{\mu(+)}(x)|_{phys} = 0,$$

(16)

to drop the gauge dependent part. We have an approximate solution of equation (12)

$$T_n|_{k_n^2=0} = e \frac{\gamma}{\gamma \cdot (r+k_n) - m} T_{n-1}. \quad (17)$$

From this relation the $n$-photon matrix element is replaced by the products of $T_1$

$$T_n T^+_n \gamma_0 \rightarrow \prod_{j=1}^n T_1(k_j) T^+_1(k_j) \gamma_0 / n!.$$

(18)

by one-photon state matrix element

$$T_1 = \langle 0|\psi(0)|r,k\rangle = -i \int \langle 0^m|T[\psi(0), e \int d^3 x \bar{\psi}^m(x) \gamma_\mu \psi^m(x) A^m_\mu(x)]|r;k\rangle \

= -ie \int d^3 x S_F(0-x) \gamma_\mu \langle 0|\psi(x)|r\rangle \langle 0|A_\mu(x)|k\rangle \\n
= \frac{-ie}{\gamma \cdot (r+k) - m + ie} \gamma_\mu \epsilon_\lambda(k) U S(r) \sqrt{\frac{m}{E_r}} \frac{1}{\sqrt{2k_0}}. \quad (19)$$
where $U_S(r)$ is a free particle spinor [5]. Spectral function $\rho$ is written symbolically as

$$\rho(p^2) = \int \frac{d^3x e^{-ip\cdot x}}{(2\pi)^3} \int (m + i\gamma \cdot \partial_r) d^3 r \delta(r^2 - m^2)e^{ir\cdot x}$$

$$\times \sum_{n=0}^{\infty} \left( \int \frac{d^3k}{(2\pi)^2} e^{ik\cdot x} \delta(k^2) \theta(k_0) \right) \sum_{\lambda} T_\lambda \bar{T}_\lambda,$$  \hspace{1cm} (20)

where we replaced the $\delta$ function$$\delta^{(3)}(p - r - k) \rightarrow \int \frac{d^3x}{(2\pi)^3} e^{-i(p - r - k)\cdot x}.$$ \hspace{1cm} (21)

The polarization sum for photon is given

$$\sum_{\lambda} \epsilon^\mu_\lambda(k) \epsilon^\nu_\lambda(k) = -[g^\mu\nu + (d - 1) \frac{k^\mu k^\nu}{k^2}].$$ \hspace{1cm} (22)

Since we do not fix the numbers of photon $n$ we sum up from $n$ equals zero to infinity. The function

$$F = \int \frac{d^3k}{(2\pi)^2} \delta(k_0) \delta(k^2) 2k_0 \frac{E_r}{m} \sum_{\lambda, S} T_\lambda \bar{T}_\lambda e^{-ik\cdot x},$$ \hspace{1cm} (23)

is a phase space integral of one photon intermediate state which may be exponentiated in (20) as $e^F$. The soft photon extra factor is well known in the evaluation of S matrix with soft photon emission [3]. Accordingly spectral function $\rho$ becomes

$$\rho(p^2) = \int \frac{d^3x e^{-ip\cdot x}}{(2\pi)^3} \int \frac{(m + i\gamma \cdot \partial_r) d^2r}{2E_r} (e^{ir\cdot x} - e^{-ir\cdot x})e^F,$$ \hspace{1cm} (24)

which may be explained in the next section.

**B. Evaluation of the spectral function**

1. **general property of spectral function**

Here we consider the fermion spectral function. The vacuum expectation value of the anticommutator has the form [7]

$$S'(x, y) = i \langle 0 | \{ \psi(x), \bar{\psi}(y) \} | 0 \rangle$$

$$= i \sum_n \left[ \langle 0 | \psi(0) | n \rangle \langle n | \bar{\psi}(0) | 0 \rangle e^{-ip_n \cdot (x-y)} + \langle 0 | \bar{\psi}(0) | n \rangle \langle n | \psi(0) | 0 \rangle e^{ip_n \cdot (x-y)} \right].$$ \hspace{1cm} (25)
We introduce the spectral amplitude by grouping together in the sum over \( n \) all states of given three-momentum \( q \)

\[
\rho_{\alpha\beta}(q) = (2\pi)^2 \sum_n \delta^{(3)}(p_n - q) \langle 0 | \psi_\alpha(0) | n \rangle \langle n | \bar{\psi}_\beta(0) | 0 \rangle
\]  

(26)

and set out to construct its general form from invariance arguments. \( \rho(q) \) is a \( 4 \times 4 \) matrix and may be expanded in terms of 16 linearly independent products of \( \gamma \) matrices. Under the assumptions of Lorentz invariance and Parity transformation it reduces to the form

\[
\rho(q)_{\alpha\beta} = \rho_1(q) \gamma \cdot q + \rho_2(q) \delta_{\alpha\beta}.
\]  

(27)

Second term in (25) can be related directly to (26) with the aid of PCT invariance of the theory\([7]\). For definiteness we define \( \gamma \)-matrices by Dirac representation:

\[
\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}
\]  

(28)

where \( i = 1, 2, 3 \), \( I \) is \( 2 \times 2 \) identity matrix and the \( \sigma^i \)'s are the Pauli matrices

\[
\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]  

(29)

Parity, Charge conjugation and Time reversal transformation are defined

\[
P \psi(t, x, y) P^{-1} = \gamma^1 \gamma^5 \psi(t, -x, y),
\]  

(30)

\[
PA^0(t, x, y) P^{-1} = A^0(t, -x, y),
\]  

(31)

\[
PA^1(t, x, y) P^{-1} = -A^1(t, -x, y),
\]  

(32)

\[
PA^2(t, x, y) P^{-1} = A^2(t, -x, y),
\]  

(33)

\[
C \psi(x) C^{-1} = C \gamma^0 \psi^* = C \bar{\psi}^T = \psi_c,
\]  

(34)

\[
C = i\gamma^2 \gamma^0, \quad C^{-1} \gamma^\mu C = -\gamma^{\mu T},
\]  

(35)

\[
CA^\mu(t, x, y) C^{-1} = -A^\mu(t, x, y),
\]  

(36)

\[
T \psi(t, x, y) T^{-1} = i\gamma^1 \psi(-t, x, y), \quad T = i\gamma^1,
\]  

(37)

\[
TA^0(t, x, y) T^{-1} = A^0(-t, x, y),
\]  

(38)

\[
TA^{1, 2}(t, x, y) T^{-1} = -A^{1, 2}(-t, x, y).
\]  

(39)
Effects of PCT transformation on $\bar{\psi}(y)\psi(x)$ is

\[ PCT\psi^A_\alpha(t, x, y)T^{-1}C^{-1}P^{-1} = (\gamma^5 \alpha^a \gamma^0)_{\alpha\gamma} \bar{\psi}^T_\gamma(-t, -x, y), \]  
\[ PCT\bar{\psi}^B_\beta(t, x, y)T^{-1}C^{-1}P^{-1} = (\gamma^5 \alpha^a \gamma^0)_{\lambda\beta} \psi^T_\lambda(-t, -x, y), \]  
\[ PCT\bar{\psi}^A_\alpha(t', x', y')\psi^B_\beta(t, x, y)T^{-1}C^{-1}P^{-1} = (\gamma^0 \alpha^a \gamma^5)_{\alpha\gamma} \bar{\psi}^A_\gamma(-t', -x', y')(\gamma^0 \alpha^a \gamma^5)_{\lambda\beta}. \]  

(42)

Inserting (42) along with (27) into (25) and using $\gamma^{2T} = -\gamma^2, \gamma^{5T} = \gamma^5$, we obtain finally

\[ S'_{\alpha\beta}(x - y) = i \int \frac{d^3q}{(2\pi)^3} \theta(q_0)([\gamma \cdot \rho_1(q^2) + \rho_2(q^2)]_{\alpha\beta} e^{-i(q - y)} + \{\gamma^0 \alpha^a \gamma^5[\gamma \cdot \rho_1(q^2) + \rho_2(q^2)]_{\alpha\beta} e^{iq(x - y)}\} = i \int \frac{d^3q}{(2\pi)^2} \theta(q_0)[\rho_1(q^2)i\gamma \cdot \partial_x + \rho_2(q^2)]_{\alpha\beta} (e^{-i(q - y)} - e^{iq(x - y)}) = i \int \frac{d^3q}{(2\pi)^2} [\theta(q_0)\gamma \cdot \rho_1(q^2) + \epsilon(q_0)\rho_2(q^2)]_{\alpha\beta} e^{-i(q - y)}, \]  

(43)

where $x' = (-t, -x_1, x_2)$. Since $\rho$ vanishes for space-like $q^2$, we may also write this as an integral over mass spectrum by introducing

\[ \rho(q^2) = \int_0^\infty \rho(s)\delta(q^2 - s)ds. \]  

(44)

We find

\[ S'(x - y) = - \int ds [i\rho_1(s)\gamma \cdot \partial + \rho_2(s)]\Delta(x - y; \sqrt{s}) = \int ds \{\rho_1(s)S'(x - y; \sqrt{s}) + [\sqrt{s}\rho_1(s) - \rho_2(s)]\Delta(x - y; \sqrt{s})\} \]  

(45)

where invariant $\Delta$ function is given

\[ \Delta'(x, y) \equiv -i \langle 0|\phi(x), \phi(y)|0 \rangle \]  
\[ = -i \sum_n \langle 0|\phi(0)|n \rangle \langle n|\phi(0)|0 \rangle (e^{-iP_n \cdot (x - y)} - e^{iP_n \cdot (x - y)}) = \frac{-i}{(2\pi)^2} \int d^3q \rho(q^2)\theta(q_0)(e^{-i(q - y)} - e^{iq \cdot (x - y)}) = \frac{-i}{(2\pi)^2} \int_0^\infty ds \rho(s) \int d^3q \delta(q^2 - s)\epsilon(q_0)e^{-i(q - y)} = \int_0^\infty ds \rho(s)\Delta(x - y, \sqrt{s}). \]  

(48)
From the above representation of propagator $S'(x - y)$ we have the relation $\rho_2(s) = \sqrt{s}\rho_1(s).$ The above spectral representation goes through unchanged for the vacuum expectation value of the time-ordered product of Dirac field; it is necessary only to replace the $S$ and $\Delta$ by the Feynman propagator $S_F$ and $\Delta_F$.

2. non-perturbative spectral function

Here we return to our case. In our approximation, propagator is a product of free one and the exponentiation of one photon state $e^F.$ First we set $\rho_1(r^2) = \delta(r^2 - m^2), \rho_2(r^2) = \sqrt{r^2}\delta(r^2 - m^2).$ Second using on-shell momentum expansion, its Fourier transformation is given

$$\rho(p^2) = \int \frac{d^3x}{(2\pi)^3} \int \frac{d^2r}{2E_r} \frac{e^{-ir\cdot x}}{2E_r} \left( e^{-ir\cdot x} - e^{ir\cdot x} \right) e^{F(x)}. \quad (49)$$

We evaluate the above in the $p$ rest frame, by writing the integral in the form

$$\rho(p^2) = \int_{-\infty}^{\infty} \frac{dt}{2\pi} e^{-ip_0 t} \int d^2x \int \frac{d^2r}{2E_r} \frac{e^{-ir\cdot x - iE_r t}}{(2\pi)^2} \left( e^{-ir\cdot x} - e^{ir\cdot x} \right) e^{F(x)} \quad (50)$$

We expand

$$\frac{1}{2E_r} e^{iE_r t} e^{F(x)}$$

in powers of $r$ keeping only the first terms in $\rho$ as

$$\frac{1}{2E_r} e^{iE_r t} e^{F(x)} = e^{imt} \frac{2m}{e^{F(it)}}.$$

Performing $r$ integral gives

$$\rho_1(p^2) = \int_{-\infty}^{\infty} \frac{dt}{2\pi} \frac{1}{2m} \left( e^{i(m-p_0)t} + e^{-i(m+p_0)t} \right) \int d^2x e^{F(it)}, \quad (51)$$

$$\rho_2(p^2) = \int_{-\infty}^{\infty} \frac{dt}{2\pi} \frac{p_0}{2m} \left( e^{i(m-p_0)t} + e^{-i(m+p_0)t} \right) e^{F(it)}. \quad (52)$$
\( \rho_1(p^2), \rho_2(p^2) \) are normalized to \( \delta(p^2 - m^2) \) and \( \sqrt{p^2} \delta(p^2 - m^2) \) when \( F(t) = 0 \). In this way we obtain

\[
\rho_1(s) = \int_{-\infty}^{\infty} \frac{dt}{2\pi} e^{-i(s-1)m^2t} e^{F(it)}, \quad s = p^2/m^2, \tag{53}
\]

\[
\rho_2(s) = \int_{-\infty}^{\infty} \frac{dt}{2\pi} m \sqrt{s} e^{-i(s-1)m^2t} e^{F(it)}. \tag{54}
\]

Now we evaluate the spectral function. First we examine the four dimensional scalar case as a guide. It has been shown in [3] that the function \( F \) in four-dimension was derived in the same way in three dimension as

\[
D^+(x) = \frac{1}{(2\pi)^{d/2}} \int d^4k \, e^{i k \cdot x} \theta(k^0) \delta(k^2 - m^2). \tag{55}
\]

\[
D^+(x) = \frac{1}{4\pi} \epsilon(x^0) \delta(x^2) - \frac{mi}{8\pi \sqrt{x^2}} \theta(x^2) \left[ N_1(m\sqrt{x^2}) - i\epsilon(x^0) J_1(m\sqrt{x^2}) \right] + \frac{mi}{4\pi^2 \sqrt{-x^2}} \theta(-x^2) K_1(m\sqrt{-x^2}). \tag{56}
\]

In the neighborhood of the light cone \( D^+ \) has the form

\[
D^+(x) \simeq \frac{1}{4\pi} \epsilon(x^0) \delta(x^2) + \frac{i}{4\pi^2 x^2} - \frac{im^2}{8\pi^2} \ln\left(\frac{m\sqrt{x^2}}{2}\right) - \frac{m^2}{16\pi} \epsilon(x^0) \theta(x^2). \tag{57}
\]

To derive the function \( F \), it is enough to use only one term

\[
D^+(x) \simeq \frac{i}{4\pi^2 x^2}, \tag{58}
\]

with infrared cut-off

\[
D^+(x) = \frac{i \exp(-\mu x)}{4\pi^2 x^2}. \tag{59}
\]

Using the formula in the appendix A

\[
F = ie^2m^2 \int_0^{\infty} \alpha d\alpha D_+(x + \alpha r, \mu) - e^2 \int_0^{\infty} d\alpha D_+(x + \alpha r, \mu) - ie^2(d-1) \frac{\partial}{\partial \mu^2} D^+(x + \alpha r, \mu), \tag{60}
\]

we obtain

\[
F = \frac{e^2}{4\pi^2} + \frac{e^2}{16\pi^2} [3 - d] [\ln\left(\frac{1}{4\mu^2 |x^2|}\right) + 2C] + \frac{e^2}{4\pi^2} \left( \frac{1}{|x|} \ln(2) \right), \tag{61}
\]

where \( C \) is a Euler constant. Neglecting regular contribution in the infrared, we have the spectral function for infinitesimal cut-off \( \mu \)

\[
\rho(p^2) = \frac{1}{m^2} \left( \frac{m}{\mu} \right)^{d/2} \epsilon^{\alpha/\pi} e^{\alpha/\pi - C \beta} (s - 1 + \epsilon)^{\beta-1}, \tag{62}
\]

\[
\beta = \alpha(d - 3)/2\pi, \tag{63}
\]

111
\[ \rho(s) = \int_{-\infty}^{\infty} \frac{dt}{2\pi} e^{i(s-1)m^2t}e^{F(it)}, \ s = p^2/m^2 \]

with a help of generalized function
\[ e^{i\gamma \pi/2} \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{dx e^{-ixy}}{(x + i\epsilon)^\gamma} = \frac{(y + i\epsilon)^{\gamma-1}}{\Gamma(\gamma)}. \]

The propagator in momentum space is given
\[ G(p) = \frac{Z}{(p^2 - m^2)^{1-\beta}}, \]
\[ Z = \mu^{-\beta}2^{\alpha/\pi}\pi^{-\alpha/\pi-C\beta}\Gamma(1 - \beta), \]

provided
\[ G(p) = \int \frac{d\rho(s)}{p^2 - m^2 s + i\epsilon}. \]

Here we return to three dimension case. For three dimensional Minkowski space, we have
\[ F = \frac{-e^2}{8\pi m} E_1(itm + 1) + \frac{(d + 1)e^2 e^{-i\mu t}}{16\pi \mu}. \]

Unfortunately we cannot evaluate the spectral function analytically as in four dimension. Using asymptotic behaviour of the function \( E_1(z) \) for small \( z \) and large \( z \)
\[ E_1(z) = -C - \ln(z), \ (|z| \ll 1), \]
\[ E_1(z) = -z \left\{ 1 - \frac{1}{z} + \frac{1}{z^2} \cdot \frac{1}{3} + \ldots \right\} (|arcz| < \frac{3\pi}{2}) \]

we approximate \( \exp(F) \) by the rough functional form in the following. In the gauge \( d = -1 \) for \( \exp(F(t)) = e^{C(\mu t)} \) for \( |t| \leq 1/\mu \), and \( \exp(F) = 1 \) otherwise for \( D = (e^2/8\pi m) = 1 \) and finite \( \mu \) case. We separate the integral in the three regions
\[
\rho_1(s) = \left[ \int_{-\Lambda}^{-1/\mu} \frac{dt}{2\pi} e^{-i(s-1)m^2t}dt + \int_{1/\mu}^{\Lambda} \frac{dt}{2\pi} e^{-i(s-1)m^2t}dt \right] + e^{C} \int_{-1/\mu}^{1/\mu} \frac{dt}{2\pi} e^{-i(s-1)m^2t}i\mu t \]
\[
= \frac{\sin((s-1)m^2\Lambda)}{\pi(s-1)m^2} \left( \frac{\mu \sin((s-1)m^2)}{(s-1)^2m^4} \right) + e^{C} \left[ \frac{-\cos((s-1)m^2)}{\pi(s-1)m^2} \left( \frac{\mu \sin((s-1)m^2)}{(s-1)^2m^4} \right) \right]
\]
\[
\to [\delta((s-1)m^2) - \frac{\sin((s-1)m^2)}{\pi(s-1)m^2}] + e^{C} \left[ \frac{-\cos((s-1)m^2)}{\pi(s-1)m^2} \frac{\sin((s-1)m^2)}{\pi(s-1)m^2} \right],
\]

\[
\rho_2(s) = m\sqrt{s} \rho_1(s).
\]

After calculation we have the limit \( \Lambda \to \infty \). From the above formula we see there is a pole at \( s = 1 \) and that the third and fourth term cancels at \( s = 1 \). Then these terms do not have a singularity at \( s = 1 \). In the limit \( \mu \to 0 \), spectral function \( \rho \) vanishes. The above result is one of the main goals in this work.

112
3. perturbative spectral function

Perturbative $O(e^2)$ spectral function can be obtained by the usual definition

$$\rho_1(p^2) = \int d^3x \frac{d^2r}{(2\pi)^2} \frac{d^2k}{(2\pi)^2} \sum_{\lambda, S} T_1 T_1 \left( e^{-i(p-k-r) \cdot x} - e^{i(p+k+r) \cdot x} \right). \quad (73)$$

If we integrate $x$ first, we obtain $(2\pi)^2 \delta^{(3)}(k + r - p)$ for energy-momentum conservation. In our case first we integrate $k$. After that we exponentiate the function $F$ and integrate $r$ in the non perturbative case. At that stage the results are position dependent. Finally if we integrate $x$ we obtain the desired spectral function $\rho_1(p^2)$. Here we evaluate $F$ from one-photon state matrix element $T_1(19)$ which appeared in section II A. Thanks to Ward-identity

$$\frac{1}{\gamma \cdot (r + k) - m} \frac{\gamma \cdot k U(r)}{= U(r)} = \frac{1}{\gamma \cdot (r + k) - m} \left[ \frac{\gamma \cdot (r + k) - m - (\gamma \cdot r - m)}{U(r)} \right] \quad (74)$$

for the $k_\mu k_\nu$ part of photon polarization sum, we obtain

$$- \sum_{\lambda, S} T_1 T_1 = e^2 \frac{m}{E_r} \frac{\gamma \cdot r + m}{2k_0} \left[ \frac{m^2}{(r \cdot k)^2} + \frac{1}{r \cdot k} + \frac{(d-1)}{k^2} \right]. \quad (75)$$

Since the propagator is normalized as $(\gamma \cdot r + m) \exp(F)$, we may drop the $(\gamma \cdot r + m)/2m$ as

$$F = -e^2 \int \frac{d^3k e^{ik \cdot x}}{(2\pi)^2} \theta(k^0) \delta(k^2) \frac{m}{E_r} \left[ \frac{m^2}{(r \cdot k)^2} + \frac{1}{r \cdot k} + \frac{(d-1)}{k^2} \right], \quad (76)$$

where we used covariant $d$ gauge photon propagator and $\delta(k^2)$ is read as the imaginary part of the free photon propagator.

$$-iD_6^{\mu \nu}(k) = \frac{1}{k^2 + i\epsilon} \left[ g^{\mu \nu} + (d-1) \frac{k_\mu k_\nu}{k^2} \right]. \quad (77)$$

In the appendix A explicit form of the function $F$ is given. In section III we discuss the position space propagator by using the function $F$. From (73), (75) in the Feynman gauge we obtain

$$(2\pi)^2 \rho^{(2)}(p^2) \equiv -e^2 \left[ \frac{4m^2}{(p^2 - m^2)^2} + \frac{2}{p^2 - m^2} \right] \int \frac{d^2r d^2k}{2E_r 2k_0} \delta^{(3)}(r + k - p), \quad (78)$$

where we omit the factor $(\gamma \cdot p + m)$. The phase space integral is evaluated in center-of-mass system, namely $p_i = 0, p_0 = p$

$$\int \frac{d^2r d^2k}{2E_r 2k_0} \delta^{(3)}(r + k - p) = \frac{\pi}{4p_0}. \quad (79)$$
where
\[ p_0 = \sqrt{r^2 + m^2} + \sqrt{k^2 + \mu^2}, \quad r + k = 0; \]
\[ \delta(\sqrt{r^2 + m^2} + \sqrt{k^2 + \mu^2} - p_0) = \frac{1}{f'(r_0)} \delta(r - r_0), \]
\[ \frac{d}{dr}(\sqrt{r^2 + m^2} + r - p_0) = \frac{r}{\sqrt{r^2 + m^2}} + \frac{r}{\sqrt{r^2 + \mu^2}} \]
\[ = \frac{r(\sqrt{r^2 + m^2} + \sqrt{r^2 + \mu^2})}{\sqrt{r^2 + m^2} \sqrt{r^2 + \mu^2}} = \frac{p_0 r}{E_r E_k}, \]
\[ r_0^2 = (p_0^2 - (m + \mu)^2)(p_0^2 - (m - \mu)^2) \]
Therefore, we have
\[ \frac{1}{4p_0} \int \frac{d\Omega r dr}{r} \delta(r - r_0) = \frac{\pi}{4p_0}. \] (80)
where \( p^2 = (r + k)^2 = s. \) Then the spectral function are given
\[ (2\pi)^2 \rho_1^{(2)}(s) = -e^2 \frac{\pi}{4\sqrt{s}} \frac{4m^2}{(s - m^2)^2} + \frac{2}{s - m^2} \theta(s - m^2), \] (81)
\[ \rho_2^{(2)}(s) = \sqrt{s} \rho_1^{(2)}(s), \] (82)
where \( p^2 = (r + k)^2 = s. \) Propagator at \( O(e^2) \) is expressed in the Lehmann representation
\[ S_F'(p) = \frac{i(\gamma \cdot p + m)}{p^2 - m^2 + i\epsilon} - \int_{(m+\mu)^2}^{\infty} \frac{i(\gamma \cdot p \pm \sqrt{s})\rho_1^{(2)}(s)}{p^2 - s + i\epsilon} ds, \] (83)
where \( \mu \) is an infrared cut-off. We get
\[ \int_{(m+\mu)^2}^{\infty} \frac{\gamma \cdot p \rho_1^{(2)}(s)}{p^2 - s + i\epsilon} ds = -\frac{e^2}{8\pi} \frac{1}{p^2 - m^2} \left[ \frac{1}{m + p^2} \ln\frac{m + p^2}{m + \sqrt{p^2}} \right] \]
\[ + \frac{2m}{(m^2 - p^2)^2} \left[ \ln\frac{m}{\sqrt{p^2}} + \ln\frac{m + \sqrt{p^2}}{m + \sqrt{p^2}} \right]. \] (84)
\[ \int_{(m+\mu)^2}^{\infty} \frac{\rho_2^{(2)}(s)}{p^2 - s + i\epsilon} ds = -\frac{e^2}{8\pi} \frac{1}{p^2 - m^2} \left[ \ln\frac{m^2 - p^2}{2m\mu} + \frac{m}{\mu} \right] \]
\[ + \frac{2m^2}{(m^2 - p^2)^2} \ln\left(\frac{m^2 - p^2}{2m\mu}\right). \] (85)
Next we evaluate the gauge dependent part.
\[ f(x) \equiv -(d - 1)e^2 \int \frac{d^3k \theta(k_0) \delta(k^2 - \mu^2)e^{ik \cdot x}}{(2\pi)^2(k^2 - \mu^2)} \]
\[ = -e^2(d - 1) \frac{\partial}{\partial \mu^2} \frac{e^{-\mu \sqrt{-x^2}}}{8\pi \sqrt{-x^2}} = \frac{e^2(d - 1)e^{-\mu \sqrt{-x^2}}}{16\pi \mu}. \] (86)
First we consider the $d = -1$ gauge case. Following (73) spectral function is directly given as

$$
\int d^3x \int \frac{d^3r}{(2\pi)^3} \epsilon(r_0) \delta(r^2 - m^2)e^{-i(p-r) \cdot x}\frac{-e^2 e^{-\mu |x|}}{8\pi \mu}
= \int d^3x e^{-ip \cdot x} e^{2e^{-(m+\mu) |x|}} \frac{1}{4\pi |x| 8\pi \mu} = \frac{e^2}{8\pi \mu p^2 + (m + \mu)^2}.
$$

(87)

In Minkowski space we have

$$
\rightarrow \frac{e^2}{8\pi \mu p^2 - m^2 + i\epsilon} = \frac{e^2}{8\pi \mu} \left( -P \left( \frac{1}{p^2 - m^2} \right) + i\pi \delta(p^2 - m^2) \right).
$$

(88)

Linear infrared divergence of wave function correction in (84),(85) cancels with this term. Therefore in $d = -1$ gauge linear divergence is absent. Adding this term to spectral function we may write

$$
(2\pi)^2 \rho^{(2)}(s) = -e^2 \pi \frac{4m^2}{4\sqrt{s} (s - m^2)^2} + \frac{2}{s - m^2} \theta(s - m^2) + \frac{(2\pi)^2(d - 1)e^2}{16\pi \mu} \delta(s - m^2).
$$

(89)

In $d = -1$ gauge logarithmic infrared divergence $\ln(2m/\mu)$ can be absorbed to the wave function renormalization constant $Z_2$. We have $Z_2 = 1 - e^2/16\pi m$.

C. Photon

For unquenched case we use the dressed photon with fermion loop with $N$ flavours. Spectral functions for dressed photon are given by vacuum polarizaiton [6]

$$
\Pi_{\mu\nu}(k) \equiv i e^2 N \int d^3p T r (\gamma_\mu \frac{1}{\gamma \cdot p - m} \gamma_\nu \frac{1}{\gamma \cdot (p - k) - m})
= -e^2 N \frac{T_{\mu\nu}}{8\pi} [\sqrt{k^2 + \frac{4m^2}{k^2}} \ln(\frac{2m + \sqrt{k^2}}{2m - \sqrt{k^2}}) - 4m],
$$

(90)

$$
T_{\mu\nu} = (g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}), \quad \frac{d^3p}{(2\pi)^3},
D_{\mu\nu}^{-1}(k) \equiv T_{\mu\nu}(k^2 + dk_\mu k_\nu) - \Pi_{\mu\nu}(k).
$$

(91)

Polarization function $\Pi(k)$ is

$$
\Pi(k) = \frac{-e^2}{8\pi} N [\sqrt{k^2 + \frac{4m^2}{k^2}} \ln(\frac{2m + \sqrt{k^2}}{2m - \sqrt{k^2}}) - 4m],
= \begin{cases}
-\frac{e^2}{8} Ni \sqrt{k^2} (k^2 > 0, m = 0), \\
-\frac{e^2 N}{6\pi m} k^2 + O(k^4)(k^2/m \ll 1).
\end{cases}
$$

(92)
Fermion mass is assumed to be generated dynamically. In quenched case it is shown that $m$ is proportional to $e^2|k^2/2|$. For massless case we have for number of $N$ fermion flavour

$$\rho_\gamma^D(k) = \frac{1}{\pi} \text{Im} D_F(k) = \frac{c\sqrt{k^2}}{k^2(c^2 + k^2)}, c = \frac{e^2}{8},$$  \hspace{1cm} (93)

$$D_{\mu\nu}(k) = -\int_0^\infty \frac{(g_{\mu\nu} - k_\mu k_\nu/k^2)\rho_\gamma(s)ds}{k^2 - s + i\epsilon} - d_R \frac{k_\mu k_\nu}{(k^2 + i\epsilon)^2},$$  \hspace{1cm} (94)

where $d_R = Z_3^{-1}d$. If we include massive fermion loop to photon spectral function $\rho_\gamma$ we have

$$\rho_\gamma(s) = \frac{1}{\pi} \text{Im} D_F(s) = Z_3\delta(s) + \sigma(s)\theta(s - 4m^2),$$  \hspace{1cm} (95)

where $\sigma$ is a imaginary part of the vacuum polarization function $\Pi(k^2)$ for $k^2 \geq 4m^2$. However real part of vacuum polarization $\Pi(k^2)$ affects the residue of the massless pole.

$$D_{\mu\nu}^F(k)' = -\frac{(g_{\mu\nu} - k_\mu k_\nu/(k^2 + i\epsilon))}{k^2 - \Pi(k) + i\epsilon} - d_R \frac{k_\mu k_\nu}{(k^2 + i\epsilon)^2}$$  \hspace{1cm} (96)

So that the renormalization constant $Z_3$ is defined by expanding $\Pi(k^2)$ at $k^2 = 0$

$$\lim_{k^2\to0} D_F(k) = \lim_{k^2\to0} \frac{-1}{k^2 - \Pi(k^2) + i\epsilon} = \frac{-Z_3}{k^2 + i\epsilon},$$  \hspace{1cm} (97)

$$Z_3^{-1} = 1 - \frac{\Pi(k^2)|_{k^2=0}}{k^2}.$$  \hspace{1cm} (98)

$$1 = Z_3 + \int_{4m^2}^\infty \sigma(s)ds,$$  \hspace{1cm} (99)

$$\sigma(s) = \frac{\text{Im}\Pi(s)}{\pi(-s + \text{Re}\Pi(s))^2 + (\text{Im}\Pi(s))^2}\theta(s - 4m^2).$$  \hspace{1cm} (100)

### III. ANALYSIS IN POSITION SPACE

#### A. Quenched case

To evaluate the function $F$ it is helpful to use the exponential cut-off (infrared cut-off)\[3,4\]. In the appendices the way to evaluate spectral function $F$ of one-photon state in the covariant $d$ gauge is given.

$$F = -\frac{e^2}{8\pi m}(m|x| + 1)E_1(\mu|x|) + \frac{(d + 1)e^2e^{-\mu|x|}}{16\pi\mu},$$  \hspace{1cm} (101)

where $|x| = \sqrt{-x^2}$ and $\mu$ is a bare photon mass and

$$E_1(x) = \int_x^\infty \frac{e^{-t}}{t}dt.$$  \hspace{1cm} (102)
Short distance behavior of $F$ has the following form

$$E_1(\mu|x|) \sim -\gamma - \ln(\mu|x|) + \mu|x|,$$
(103)

$$F \sim \frac{(d + 1)e^2}{16\pi\mu} + \frac{e^2}{8\pi m}[ -\mu|x| + (1 + m|x|)(\gamma \ln(\mu|x|)) - \frac{(d + 1)e^2}{16\pi}|x|, (\mu|x| \ll 1).$$
(104)

Long distance behavior is given by the asymptotic expansion of $E_1(\mu|x|)$

$$E_1(z) \sim e^{-z}\left\{1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{2} \cdot 3 \cdot z^2 + ...\right\}, \left(|\arg z| < \frac{3}{2}\pi\right),$$
(105)

therefore we have

$$F \sim \frac{-e^2}{8\pi m}(m|x| + 1)\frac{e^{-\mu|x|}}{\mu|x|}(1 - \frac{1}{\mu|x|}) + \frac{(d + 1)e^2e^{-\mu|x|}}{16\pi\mu},$$
(106)

$$= -\frac{e^2}{8\pi\mu}e^{-\mu|x|} + \frac{(d + 1)e^2e^{-\mu|x|}}{16\pi\mu}, \left(1 \ll \mu|x|\right),$$
(107)

where $\gamma$ is an Euler constant. In (104) linear term in $|x|$ is understood as the finite mass shift from the form of the propagator in position space, and $|x|\ln(\mu|x|)$ term is a position dependent mass

$$m_S = m + \frac{(d + 1)e^2}{16\pi},$$
(108)

$$m(x) = m - \frac{e^2}{8\pi}\ln(\mu|x|),$$
(109)

which has mass changing effects at short distance. We have seen the drastic change of the function $F(\mu,x)$ from short distance to long distance. For long distance $F$ vanishes and it may be a free particle for fixed $\mu$. The electron propagator in position space can be written approximately for arbitrary $D$

$$S'_F(x) = -(i\gamma \cdot \partial + m)\frac{1}{4\pi\sqrt{x^2}^2}
\left(\begin{array}{c}
e^{-m|x|A(\mu|x|)^{D+C|x|}(\mu|x| \leq 1)} \\

\end{array}\right),$$

$$= (S_V(x) + S_S(x)),$$
(110)

where

$$A = \exp\left(\frac{(d + 1)e^2}{16\pi\mu}\right), C = \frac{e^2}{8\pi}, D = \frac{e^2}{8\pi m}.$$  

(111)

Since $\lim_{x \to 0}|x|^{|x|} = 1$, we see that $C|x|$ term dose not affect the short distance behavior. $A$ is a violent linear infrared divergent factor. Hereafter we choose $d = -1$ gauge to avoid linear
infrared divergence. In this gauge, for $D > 1$, in the limit $\mu \to 0$ for finite $|x|$ scalar part of the propagator $S'_F(x)$ vanishes as

$$\frac{m e^{-m|x|}}{4\pi \sqrt{x^2}} \exp(-\frac{e^2}{8\pi \mu} e^{-\mu|x|}) \to 0(1 \ll \mu|x|).$$

Therefore fermion is confined in quenched case for $D > 1$. There is no way to determine the value $D$ in principle. So that we need extra physical condition such as the finiteness of the vacuum expectation value of chiral order parameter which is proportional to $-itr(S'_F(x))$ in the case of dynamical chiral symmetry breaking. In this sense we see that $S'_F(0)$ is finite and the physical mass equals to $m = e^2/8\pi$ for $D = 1$ with finite value of $\mu$. In the next section we consider the way to avoid the vanishment of the propagator by including vacuum polarization. We will discuss unquenched case with finite $\mu$ in numerical analysis in Minkowski space in the next section.

**B. Unquenched case**

In our approximation quenched propagator vanishes in the limit of zero bare photon mass $\mu$. So that we may apply the spectral function of photon $\rho_\gamma(\mu)$ to evaluate the unquenched fermion propagator. In that case we simply integrate the function $e^{F(x,\mu)}\rho_\gamma(\mu)$ for $\mu$, where $\mu$ is an invariant mass of two fermion-antifermion intermediate state. Spectral function of photon with massless fermion loop is derived from (92) and we have

$$\rho_\gamma^0(\mu) = \frac{2c}{\pi(\mu^2 + c^2)}, c = \frac{e^2}{8}, Z_3 = \int_0^\infty d\mu \rho_\gamma^0(\mu) = 1,$$

(113)

where photon has not a simple pole at $k^2 = 0$. In this case the spectral function of the fermion in position space is given

$$\tilde{\rho}(x) = \int_0^\infty d\mu \rho_\gamma^0(\mu) e^{F(x,\mu)}.$$  

(114)

However massless loop are not able to suppress large $\mu$ region as $1/\mu^3$. In fact integral which appears at short distance part of $\exp(F(x,\mu))$

$$\int_0^\infty \rho_\gamma^0(\mu)(\mu|x|)^D d\mu$$

(115)

is logarithmically divergent for $D = 1$. So that we may consider spectral function with the massive fermion loop.
In terms of real and imaginary part of polarization function $\Pi(k)$

$$\Im \Pi(k) = -\frac{e^2}{8}[\sqrt{k^2} + \frac{4m^2}{\sqrt{k^2}}] \theta(k^2 - 4m^2),$$

$$\Re \Pi(k) = -\frac{e^2}{8\pi}[(\sqrt{k^2} + \frac{4m^2}{\sqrt{k^2}}) \ln \left| \frac{2m + \sqrt{k^2}}{2m - \sqrt{k^2}} - 4m \right|].$$

(116)

spectral function of photon is given.

$$\rho_\gamma(\mu^2) = \frac{-\Im \Pi(\mu)}{\pi[(\mu^2 - \Re \Pi(\mu))^2 + (\Im \Pi(\mu))^2]},$$

(118)

This spectral function damps as $1/\mu^3$ at large $\mu$ by (100) in section II.C. Small $|x|$ region of the fermion propagator is modified with $N_F$ massive fermion loop for photon spectral function

$$S'_F(x) = -(i\gamma \cdot \partial + m) e^{-m\sqrt{x^2}} \frac{4\pi}{4\pi \sqrt{x^2}} \tilde{\rho}(x),$$

(119)

$$\tilde{\rho}(x) = \int_{4m^2}^{\infty} d\mu^2 \sigma(\mu^2) e^{F(x,\mu)},$$

(120)

where all bare photon propagators in $\exp(F)$ are dressed by $\rho_\gamma$. The above result is the second goal in our work. In the massive fermion case renormalization constant $Z_3$ is given by equations (92) and (98) in section II.C

$$Z_3^{-1} = (1 - \frac{\Pi(k)}{k^2}|_{k^2=0}) = (1 + \frac{Ne^2}{6\pi m}) = \frac{7}{3} (N = 1)$$

(121)

with our choice of physical mass $m = e^2/8\pi$. In Fig.1 we see fermion loop effects lead infrared finite spectral function $\tilde{\rho}(x)$ with dynamical fermion mass $m = \alpha/N$, $e^2 = \alpha/N$ from $N = 1$(lower) to 3(upper) with $d = -1$, $\alpha = e^2N/8\pi$ is fixed to unity. Next we evaluate the spectral function in $d = -1$ gauge.

$$F = \frac{-e^2}{8\pi m} E_1(i\mu t)(itm + 1).$$

(122)

$$\rho_1(s) = \int_{-\infty}^{\infty} \frac{dt}{2\pi} e^{-i(s-1)m^2 t - e|t|} \int_{4m^2}^{\infty} d\mu^2 \sigma(\mu^2) e^{F(i\mu t)}, s = p^2/m^2.$$  

(123)

In Fig.2 we show $\rho_1(s)$. At $s = 1$ there seems to be a pole like singularity which is expected by long distance behavior of the function $e^F \sim 1$. The height of the peak at $s = 1$ is consistent with $\Lambda/\pi$ from the approximate formula $\delta(s-1) = \sin(\Lambda(s-1))/\pi(s-1)$, $\Lambda = 6.17 \times 10^5$. 

119
FIG. 1: $\bar{\rho}(x)$ for $\alpha = 1, N = 1$ (blue)..3 (red).

FIG. 2: $\rho_1(s)$ for $e^2 = 1, N = 1, s = 0.025l$.

IV. RENORMALIZATION CONSTANT AND ORDER PARAMETER

A. by Spectral function

In this section we consider the renormalization constant and chiral order parameter in our model. It is easy to evaluate the renormalization constant by the equation

$$\lim_{\gamma \cdot p \to m} -iS'_F(p) = \frac{Z_2}{\gamma \cdot p - m}. \quad (124)$$
where \( Z_2 \) is defined for one particle state in the theory. In quenched approximation with finite cut-off we have a \( \delta(s-1) \) term in \( \rho_1 \) which implies

\[
Z_2 \neq 0. \tag{125}
\]

Formally renormalization constant \( Z_2 \) is defined in our approximation

\[
Z_2 = \int_1^\infty m^2 \rho_1(s) ds, \tag{126}
\]

\[
\int_1^\infty m^2 \rho_1(s) ds = \int_{-\infty}^\infty \frac{m^2 dt}{2\pi} \int_1^\infty ds e^{-i(s-1)m^2t-\epsilon s} \int_0^\infty \rho_\gamma(\mu^2) d\mu^2 e^{F}(i\mu t)
\]

\[
= \int_{-\infty}^\infty \frac{dt}{2\pi} \frac{-e^{-i(s-1)m^2t-\epsilon s}}{it} \bigg|_{s=\infty}^{-1} \int_0^\infty \rho_\gamma(\mu^2) d\mu^2 e^{F}(i\mu t)
\]

\[
= \int_{4m^2}^\infty d\mu^2 \sigma(\mu^2) \int_{-\infty}^\infty \frac{dt}{2\pi} \frac{e^{F}(i\mu t)}{it}, \tag{127}
\]

where we add the convergence factor \( \exp(-\epsilon s) \) to define the definite integral with infinitesimal \( \epsilon \). Following the above formula we evaluated \( Z_2 \) numerically for \( d = -1 \) gauge with \( e^2 = 1 \). We have \( Z_2 = 0.283, 0.400 \) for \( N = 1 \) and 3 respectively. If we consider \( N_F \) flavor of massless four component fermion, there is a chiral like symmetry \( U(2N_F) \) which breaks down to \( SU(N_F) \otimes SU(N_F) \otimes U(1) \otimes U(1) \) by fermion mass generation[9,10]. Chiral like symmetry is realized by \( \gamma_3, \gamma_5 \) transformation in place of \( \gamma_5 \) in four dimension. There appears a pair of Goldstone boson like Pion in QCD. Chiral order parameter for each flavour \( \langle \bar{\psi}\psi \rangle \) is given

\[
\langle \bar{\psi}\psi \rangle = -itrS_F'(x). \tag{128}
\]

In position space we evaluate directly from (119),(120)

\[
\langle \bar{\psi}\psi \rangle = -4 \lim_{|x| \to 0} \int_{2m}^\infty d\mu \frac{\alpha e^{-m|x|}e^{F}(x,\mu)}{N \cdot 8\pi |x|}, \tag{129}
\]

where \( m = \alpha/N \). Since \( S_F'(x - x') \) may be expressed as the sum of even and odd function in \( (x - x')[7] \), odd part vanishes at \( x = x' \). Then we have a half value of the propagator at the origin as the vacuum expectation value. From the above equation we see that \( \langle \bar{\psi}\psi \rangle \neq 0 \) and is finite only if \( D = \alpha/Nm = 1 \). At least for weak coupling we get the value of order parameter \( \langle \bar{\psi}\psi \rangle \) as \(-3.26 \times 10^{-3}e^4 \) for \( N = 1, e^2 = 1 \) case. We used infrared cut-off for quenched part \( \mu = .05 \) which is smaller than \( 2m_f = 1/4\pi = .079 \). Contribution of quenched part to the condensation is \( \langle \bar{\psi}\psi \rangle_Q = -.56 \times 10^{-3}e^4 \). This is the third goal in our work. For large \( N, m \) is \( O(\alpha/N) \) and mass generation is suppressed which are shown in Fig.3. In Fig.4 coupling dependence of order parameter at fixed \( N \) is shown. Since fermion mass is proportional to \( \alpha \), fermion loop effect is suppressed at large \( \alpha \). There may be a critical value of \( \alpha \).
FIG. 3: $-\langle \bar{\psi}\psi \rangle$ for $N, \alpha = 1$.

FIG. 4: $-\langle \bar{\psi}\psi \rangle$ for $\alpha, N = 1$.

**B. by Dyson-Schwinger equation**

We have a similar solution of the propagator at short distance which is known by the analysis of Dyson-Schwinger equation[10]. The value of order parameter $\langle \bar{\psi}\psi \rangle$ as $-3.26 \times 10^{-3} e^4, N = 1, d = -1$ case for weak coupling may be compared with $-3.21 \sim -3.5 \times 10^{-3} e^4$ for quenched case with the gauge parameter range $d = 0 \sim 1$ and vertex correction of Ball-Chiu(BC) or Curtis-Pennington(CP) ansatz for the transverse vertex. The BC vertex
ansatz which satisfies Ward-Takahashi-identity

\[(p - q)_{\mu}\Gamma_{\mu}(p, q) = S_{F}^{-1}(q) - S_{F}^{-1}(p) \]  \hspace{1cm} (130)

is known as

\[\Gamma_{BC}^{\mu}(p, q) = \frac{A(p^2)}{2p^2} + \frac{A(q^2)}{2q^2} \gamma_{\mu} + \frac{\Delta A}{p^2} \gamma_{\mu} + \frac{\Delta B(p + q)}{p^2 - q^2}, \]  \hspace{1cm} (131)

The Dyson-Schwinger equation for the fermion propagator with BC vertex ansatz is written in the Landau gauge

\[A(p) = 1 - \frac{e^2}{4\pi^2p^2} \int_{0}^{\infty} \frac{dq^2}{A(q^2)q^2 + B(q^2)} \{[\Delta A(p, q)A(q) + \Delta B(p, q)B(q)]I_1(p, q)\}, \]  \hspace{1cm} (133)

\[B(p) = \frac{e^2}{4\pi^2} \int_{0}^{\infty} \frac{dq^2}{A(q^2)q^2 + B(q^2)} \{[(A(p) + A(q))B(q)I_0(p, q) + (\Delta A(p, q)B(q) - \Delta B(p, q)A(q))]I_1(p, q)\}, \]  \hspace{1cm} (134)

where

\[I_0(p, q) = -\frac{1}{2pq} \ln\left(\frac{(p - q)^2}{(p + q)^2}\right), \]  \hspace{1cm} (135)

\[I_1(p, q) = -\frac{p^2 + q^2}{2pq} \ln\left(\frac{(p - q)^2}{(p + q)^2}\right) - 2. \]  \hspace{1cm} (136)

Here we notice that if we drop the \(\Delta B\) term in the BC vertex, value of order parameter is equal to the case of bare vertex in the Landau gauge \((2.1 \times 10^{-3} e^4)\) and we have \(A(p) \simeq 1\) except for the low momentum region. In the chiral symmetric case with \(B(p) = 0\), thanks to Ward-Takahashi identity we have \(A(p) = 1\). With BC vertex the value of \(\langle \bar{\psi}\psi \rangle\) is larger by \((1.6 - 1.8)\) times than that for bare vertex in the Dyson-Schwinger equation.

V. SUMMARY

We evaluated the fermion propagator in three dimensional QED with dressed photon by the dispersion method and Dyson-Schwinger equation with vertex correction. In the \(O(e^2)\) fermion position space spectral function \(F\) we obtain finite mass shift, wave function renormalization and position dependent mass. However there remains infrared divergences such as...
linear and logarithmic ones which were regularized by bare photon mass $\mu$. In our approximation the linear divergence is absent in covariant $d = -1$ gauge. After exponentiation of $F$ we have position space spectral function proportional to $(\mu \sqrt{-x^2})^\beta$ for short distance and $\exp(-e^2/(8\pi \mu) \exp(-\mu \sqrt{-x^2}))$ for long distance, $\beta = e^2/(8\pi m)$. These facts imply fermion to be free at low energy. This feature is not given in the Dyson-Schwinger equation. In the limit $\mu \to 0$, the quenched spectral function vanishes for both short and long distance. If we include massive fermion loop to vacuum polarization for photon, imaginary part of the photon propagator gives us a modified photon spectral function $\rho(\mu^2)$ in place of quenched $\delta(\mu^2)$ function and we may avoid the vanishment of the propagator in unquenched case. Our analysis is an extension of dispersion like method applied in the determination of one-particle singularity in scalar QED[3]. In Minkowski space the spectral function of fermion is evaluated numerically and $\rho_1(s)$ shows the simple pole structure near $p^2 = m^2$. This fact is consistent with a finite renormalization constant $Z_2$ that was obtained in the previous section. In our approximation only short distance behavior is modified in position space. So that electron is free at low energy which is suitable for the application of the model in the condensed matter physics in (2+1) dimension. It has been shown that there is no infrared divergences in non-covariant gauges[3]. In (2+1)-dimensional case our approximation leads results similar to non-covariant gauge case in the infrared, where there may be no unphysical degrees of freedom for photon polarization. As far as we know only Gauge Technique allows us to have a confining solution with infrared anomalous dimension provided by massless loop[6]. Generally speaking propagator or self-energy obtained by Dyson-Schwinger is gauge dependent especially in the infrared. So that we have not a definite conclusion about confinement or existence of asymptotic field. On the other hand our approximation with soft-photon exponentiation is gauge invariant in the infrared, it strongly suggests the existence of asymptotic fields in (2+1)-dimension. For the case of chiral symmetry breaking, if we set the anomalous dimension to be unity at short distance we obtain finite vacuum expectation value $\langle \bar{\psi} \psi \rangle$. This value agrees quite well with that provided by quenched Dyson-Schwinger equation. In our approximation vacuum expectation value and physical mass are not so sensitive to flavor number $N$ which may be linear in $1/N$ in comparison with Dyson-Schwinger analysis with massless fermion loop. In the strong coupling with fixed $N$ order parameter vanishes by suppression of heavy fermion loop, where fermion mass is proportional to $e^2$. If we neglect fermion mass our results may reproduce the results in an earlier work of Templeton, where fermion mass is assumed.
to be neglected at high temperature[2].

VI. ACKNOWLEDGEMENT

The author would like to thank to Prof.Robert Delbourgo at University of Tasmania for his introduction of Gauge Technique. He also thanks to Prof.Roman Jackiw at MIT to recommend to apply his dispersion like method to our problem.

VII. REFERENCES

[1] R. Jackiw, S. Templeton, Phys. Rev. D. 23 (1981) 2291.
[2] Stephen. Templeton, Phys. Rev. D. 24 (1981) 3134.
[3] R. Jackiw, L. Soloviev, Phys. Rev. 137.3 (1968) 1485; S. Weinberg, Phys. Rev. 140.2B (1965) 516.
[4] Yuichi Hoshino, JHEP0409:048, 2004.
[5] K. Nishijima, Fields and Particles, W.A. BENJAMIN, INC (1969).
[6] A. B. Waites, R. Delbourgo, Int. J. Mod. Phys. A7 (1992) 6857.
[7] James D. Bjorken and Sidney D. Drell, Relativistic Quantum Fields, McGraw-Hill Book Company.
[8] N. N. Bogoliubov and D. V. Shirkov, Introduction to the Theory of Quantized Fields, WILEY-INTERSCIENCE.
[9] C. J. Burden, Nuclear Physics B 387 (1992) 419-446.
[10] C. S. Fischer, R. Alkofer, T. Dahm, P. Maris, Phys. Rev. D70, 073007 (2004): [arXiv:hep-th/0407014].

VIII. APPENDICES

A. Evaluation of spectral function of one-photon state

In this section we evaluate the $O(e^2)$ spectral function $F$

$$F = -e^2 \int \frac{d^3 k}{(2\pi)^3} e^{i k \cdot x} \theta(k^0) [ \frac{m^2}{(r \cdot k)^2} + \frac{1}{(r \cdot k)} + \frac{(d-1)}{k^2} ] .$$

(137)
The gauge dependent term is an off-shell quantity in general. Here we introduce a small photon mass in the above expression to avoid infrared divergences. Therefore we have

\[
F = -e^2 \int \frac{d^3k e^{ik \cdot x}}{(2\pi)^2} \theta(k^0) \left[ \delta(k^2 - \mu^2) \left( \frac{m^2}{(r \cdot k)^2} + \frac{1}{(r \cdot k)} \right) - (d - 1) \frac{\partial}{\partial \mu^2} \delta(k^2 - \mu^2) \right].
\]  

(138)

If we use the parameter

\[
\lim_{\epsilon \to 0} \int_0^\infty d\alpha e^{-\alpha (\epsilon - i r \cdot r)} = i k \cdot r,
\]

\[
\lim_{\epsilon \to 0} \int_0^\infty d\alpha e^{-\alpha (\epsilon - i r \cdot r)} = -\frac{1}{(k \cdot r)^2},
\]

(139)

and the retarded propagator with bare mass \( \mu \)

\[
D^+(x) = \int \frac{d^3k}{i(2\pi)^2} \delta(k^2 - \mu^2) \theta(k^0) e^{ik \cdot x}
\]

\[
= \frac{1}{i(2\pi)^2} \int_0^\infty \frac{\pi k dk J_0(k |x|)}{2 \sqrt{k^2 + \mu^2}} = \frac{e^{-\mu |x|}}{8\pi i |x|},
\]

(140)

we obtain the function \( F \) as

\[
F = ie^2 m^2 \int_0^\infty d\alpha D_+(x + \alpha r) - e^2 \int_0^\infty d\alpha D_+(x + \alpha r),
\]

\[
= ie^2 m^2 F_1(x) - e^2 F_2(x)
\]

(141)

in the Feynman gauge. Soft photon divergence corresponds to the large \( \alpha \) region and \( \mu \) is an infrared cut-off. It is simple to evaluate the gauge dependent term in \( F \) by

\[
F_L = -\int \frac{d^3k}{(2\pi)^2} \theta(k^0) \frac{\partial}{\partial \mu^2} \delta(k^2 - \mu^2) e^{ik \cdot x}.
\]

(142)

Finally we get

\[
F_2 = \int_0^\infty d\alpha \frac{\exp(-\mu (x + \alpha r))}{8\pi (x + \alpha r)} = \frac{E_1(\mu|x|)}{8\pi \sqrt{r^2}},
\]

(143)

\[
F_1 = \int_0^\infty d\alpha \frac{\exp(-\mu (x + \alpha r))}{8\pi (x + \alpha r)} = \frac{\exp(-\mu |x|) - \mu |x| E_1(\mu|x|)}{8\pi r^2 \mu},
\]

(144)

\[
F_L = \frac{1}{16\pi} \frac{\exp(-\mu |x|)}{\mu},
\]

(145)

where \( r^2 = m^2 \) and

\[
E_1(z) = \int_z^\infty \frac{\exp(-t)}{t} dt.
\]

(146)
Series expansion and asymptotic expansion of $E_1(z)$ are

$$E_1(z) = -\gamma - \ln(z) - \sum_{n=1}^{\infty} \frac{(-1)^n z^n}{nn!}, (|\arg z| < \pi)$$

$$E_1(z) \sim \exp(-z) \left\{ 1 - \frac{1}{z} + \frac{1 \cdot 2}{z^2} - \frac{1 \cdot 2 \cdot 3}{z^3} + \ldots \right\}, (|\arg z| < \frac{3}{2} \pi).$$

From the above expressions, we have the short and long distance behaviour of $E_1(\mu|x|)$

$$E_1(\mu|x|) = -\gamma - \ln(\mu|x|) + \mu|x| + O(\mu^2)(\mu|x| \ll 1),$$

$$E_1(\mu|x|) = \exp(-\mu|x|) (1 - \frac{1}{\mu|x|} + \frac{2}{(\mu|x|)^2})(\mu|x| \gg 1).$$

For the leading order in $\mu$ we obtain

$$e^{2m^2}F_1 = \frac{e^2}{8\pi} (\frac{1}{\mu} + |x| (-1 + \ln(\mu|x|) + \gamma)) + O(\mu^3), \quad (147)$$

$$-e^2F_2 = \frac{e^2}{8\pi m}(\ln(\mu|x|) + \gamma - \mu|x|) + O(\mu), \quad (148)$$

$$(d-1)e^2F_L = \frac{(d-1)e^2}{16\pi} (\frac{1}{\mu} - |x|) + O(\mu). \quad (149)$$

At short distance we have

$$F = \frac{(d+1)e^2}{16\pi\mu} - \frac{e^2}{8\pi m}\mu|x| + \frac{e^2}{8\pi m}(1 + m|x|)(\ln(\mu|x| + \gamma))$$

$$- \frac{(d+1)e^2}{16\pi} |x|. \quad (150)$$

Finally we define the function $F$ in Minkowski space by analytic continuation from $\sqrt{-x^2} \to it$

$$F = -\frac{e^2}{8\pi m}E_1(i\mu t)(itm + 1) + \frac{(d+1)e^2}{16\pi \mu} e^{-i\mu t}. \quad (151)$$

If we expand in real time $t$, we obtain the short and long time behaviour

$$F = \frac{e^2}{8\pi m}(\gamma + \ln(i\mu t)) + \frac{(d+1)e^2}{16\pi \mu} + O(t), |\mu t| \ll 1, \quad (152)$$

$$F \simeq \frac{e^2}{8\pi m}\frac{me^{-i\mu t}}{\mu} - \frac{e^{-i\mu t}}{i\mu t} + \frac{(d+1)}{16\pi \mu} e^{-i\mu t}, 1 \ll |\mu t|, \quad (153)$$

provided

$$E_1(i\mu t) = \frac{-ie^{-i\mu t}}{\mu t} + O\left(\frac{1}{t^2}\right), |\mu t| \ll 1. \quad (154)$$
B. Feynman-Dyson perturbative spectral function

In the Feynman-Dyson theory \( O(e^2) \) propagator is given[5]

\[
S'_F(p) = S_F(p) - S_F(p) \sum (p) S_F(p),
\]

where

\[
\sum (p) = -\frac{e^2}{16\pi} \int_{-\infty}^{\infty} \frac{da}{\gamma \cdot p - a + i\epsilon} [\frac{4m}{a} + d(\frac{a + m^2}{a^2})] \theta(a^2 - m^2). \tag{156}
\]

From the above formula we obtain the self-energy

\[
\sum (p) = \frac{e^2}{8\pi} \frac{m}{\sqrt{p^2}} (2 + d) \ln(\frac{m - \sqrt{p^2}}{m + \sqrt{p^2}}) - d\gamma \cdot p(\frac{1}{2\sqrt{p^2}}(1 + \frac{m^2}{p^2}) \ln(\frac{m - \sqrt{p^2}}{m + \sqrt{p^2}}) + \frac{1}{m}). \tag{157}
\]

\[
-S_F(p) \sum (p) S_F(p) = \frac{de^2}{8\pi} \frac{\gamma \cdot p + m}{p^2 - m^2} (\frac{1}{2\sqrt{p^2}}(1 + \frac{m^2}{p^2}) \ln(\frac{m - \sqrt{p^2}}{m + \sqrt{p^2}}) + \frac{1}{m})
+ \frac{p^2 + 3m^2 + 2m\gamma \cdot p \epsilon^2 m}{(p^2 - m^2)^2} \frac{e^2}{8\pi} \frac{d}{2\sqrt{p^2}} (\frac{2}{\sqrt{p^2}} + \frac{d}{2\sqrt{p^2}}) \ln(\frac{m - \sqrt{p^2}}{m + \sqrt{p^2}}) + d. \tag{158}
\]

In this approximation \( O(e^2) \) spectral functions for vector and scalar part are given

\[
S_F(p) \sum (p) S_F(p) = \int_{(m+\mu)^2}^{\infty} \frac{ds(\rho_1(s)\gamma \cdot p + \rho_2(s))}{p^2 - s + i\epsilon}. \tag{159}
\]

\[
\rho_1(s) = -\frac{e^2}{8\pi} \frac{2m^2}{(s - m^2)^2} \frac{4 + 3d}{2\sqrt{s}} + \frac{d}{(s - m^2)2\sqrt{s}}(1 + \frac{m^2}{s}) \theta(s - m^2), \tag{160}
\]

\[
\rho_2(s) = -\frac{e^2}{8\pi} \frac{s + 3m^2}{s - m^2} \frac{m4 + 3d}{2} + \frac{md}{(s - m^2)2}(1 + \frac{m^2}{s}) \theta(s - m^2). \tag{161}
\]

It is now clear that the above spectral functions have different gauge dependences from that ones obtaind based on the definition (73) in section II B 3.In 4-dimension it is known that the Feynman-Dyson theory leads the same spectral function as we get based on its definition.Formal proof is given for simple scalar model in ref[5].