Analytical Estimate of Atmospheric Newtonian Noise Generated by Acoustic and Turbulent Phenomena in Laser-Interferometric Gravitational Waves Detectors

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We present a theoretical estimate of the atmospheric Newtonian noise due to fluctuations of atmospheric mass densities generated by acoustic and turbulent phenomena and we determine the relevance of such noise in the laser-interferometric detection of gravitational waves. First, we consider the gravitational coupling of interferometer test-masses to fluctuations of atmospheric density due to the propagation of sound waves in a semispace occupied by an ideal fluid delimited by an infinitely rigid plane. We present an analytical expression of the spectrum of acceleration fluctuations of the test-masses of the interferometer in terms of the experimentally obtainable spectrum of pressure fluctuations. Second, we consider the gravitational coupling of interferometer test-masses to fluctuations of atmospheric density due to the propagation of sound waves generated in a turbulent Lighthill process. We present an analytical expression in the Fourier space of the spectrum of acceleration fluctuations of the test-masses of the interferometer. Finally, we discuss the relevance of these noise sources in the detection of gravitational waves by comparing the estimated spectral densities of Newtonian atmospheric noises considered here to the expected sensitivity curve of the VIRGO detector.

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I. INTRODUCTION

It is known that in the weak field approximation of Einstein’s General Relativity (GR), the linearized equations of GR are hyperbolic, thus implying propagation of gravitational waves with the speed of light [1]. The discovery by Hulse and Taylor of the binary pulsar PSR 1913 – 16, the measured orbital period of which decreases following the predictions of GR, constitutes an indirect observational confirmation of the existence of gravitational waves [2].

The direct detection of gravitational waves is an important goal of contemporary physics and modern technology allows to build ground and space based experiments sufficiently sensitive to detect, in a direct way, gravitational waves radiated by astrophysical objects [3]. We emphasize that the detection of gravity waves is also important to verify the consistency of alternative theories of gravity that are currently not ruled out from a pure theoretical standpoint [4, 5]. Detection of gravitational waves is performed by measuring the relative displacements of several nearly free masses which carry the mirrors defining a Michelson interferometer. The measured quantity is the time-dependent difference in the lengths of the two orthogonal arms of the interferometer. In principle, this form of antenna can be sensitive down to quite low frequencies. In practice, various noise sources will limit the useful bandpass. Several noise processes can generate spurious signals in the antenna, masking the effect induced by the gravitational wave. Some of these processes, such as seismic and thermal noise, induce displacements of the mirrors (“displacement noise”), while others, such as the noise induced by frequency fluctuations of the laser, affect the phase of the optical rays even if a real movement of the mirrors is not present (“phase noise”). In order to define the sensitivity of the antenna it is necessary to compare the real signal $h_{GW}(t)$ to each relevant fake signal $h_{Noise}(t)$. This comparison is usually expressed in terms of the so-called “linear spectral density” which is defined as the square root of the power spectrum of the signal [3]. In the case of dimensionless amplitudes, the linear spectral density can be therefore expressed in units of $\sqrt{\text{Hz}}$. Considering for instance the French-Italian interferometric gravitational wave detector VIRGO [6], it turns out that in the low frequency range (below few tens of Hz) the VIRGO sensitivity is limited by the thermal noise of the pendulum suspension. Between few tens of Hz and few hundreds of Hz, the dominant mechanism is the thermal noise of the mirrors internal modes. At higher frequencies the VIRGO sensitivity curve is limited by shot-noise (noise generated by Poisson statistical fluctuations of the number of photons in the light beam). Another important source of noise is the so-called gravity-gradient noise, a noise due to fluctuating Newtonian gravitational forces that induce motions in the test masses of an interferometric gravitational-wave detector. Gravity gradients are potentially important at the low end of the interferometer frequency range, $f \leq 20 \text{Hz}$. Another noise source that is important at these frequencies is the vibrational seismic noise, in which the ground’s ambient motions, filtered through the detector’s vibration isolation system, produce motions of the test masses. It should be possible and practical to isolate the test masses from these seismic vibrations down to frequencies as low as $f \sim 3 \text{Hz}$, but it does not seem practical to achieve significant isolation from the fluctuating gravity gradients. Thus, gravity gradients constitute an
ultimate low-frequency noise source; seismic vibrations do not. The Virgo Sensitivity Curve is obtained by summing up in an incoherent way the spectral noise densities of all the considered noises (seismic noise, shot noise, thermal noise, etc.). This incoherent sum implies that the sensitivity curve is obtained by adding together only quadratic terms, averaging to zero the "interference" terms. In other words, it is assumed that different sources of noise are not correlated among each other and therefore,

\[ |\tilde{h}_{\text{sensitivity}}|^2 \equiv \sum_k \langle \tilde{h}_{\text{noise}, k}, \tilde{h}_{\text{noise}, k}^* \rangle, \]  

where,

\[ \langle \tilde{h}_{\text{noise}, i}, \tilde{h}_{\text{noise}, j}^* \rangle = 0 \text{ if } i \neq j. \]  

Gravity gradients were first identified as a potential noise source in interferometric gravitational-wave detectors by Weiss [7]. The first quantitative analyses of such gravity-gradient noise were performed by Saulson [8] and Spero [9]. Improvements of Saulson’s works were carried out by Thorne and others [10]. Finally, T. Creighton at Caltech has revisited the relevance of gravity gradient noise due to atmospheric fluctuations [11].

In this article, we present a theoretical estimate of the atmospheric Newtonian noise generated by fluctuations of atmospheric mass densities generated by acoustic and turbulent phenomena and we determine the relevance of such noise in the laser-interferometric detection of gravitational waves [12]. The structure of this paper is as follows: in Section II, we briefly review Creighton’s work on gravity gradient noise generated by atmospheric mass fluctuations. In Section III, we present the general expression for the spectrum of an arbitrary Newtonian noise. In Section IV, we consider the gravitational coupling of interferometer test-masses to fluctuations of atmospheric density due to the propagation of sound waves in a semispace occupied by an ideal fluid delimited by an infinitely rigid plane. We present an analytical expression of the spectrum of acceleration fluctuations \( S_{\tilde{a}}(\vec{r}_1, \vec{r}_2; \omega) \) of the test-masses of the interferometer in terms of the experimentally obtainable spectrum of pressure fluctuations \( S_p(\vec{r}_1, \vec{r}_2; \omega) \). In Section V, we consider the gravitational coupling of interferometer test-masses to fluctuations of atmospheric density due to the propagation of sound waves generated in a turbulent Lighthill process. We present an analytical expression - in the Fourier space - of the spectrum of acceleration fluctuations \( \tilde{S}_{\tilde{a}}(\vec{k}_1, \vec{k}_2; \omega) \) of the test-masses of the interferometer. Finally, conclusions and final remarks are presented in Section VI.

II. GRAVITY GRADIENT NOISE DUE TO ATMOSPHERIC FLUCTUATIONS: CREIGHTON’S WORK

Mass density fluctuations in the atmosphere caused, for example, by acoustic pressure waves or by temperature perturbations induce a stochastic gravitational field that couples directly to the test masses of gravitational interferometers and produce noise, the so-called atmospheric Newtonian noise. The first to discuss the effects of such noise in a quantitative way was Saulson [8]. He considered the effects of background acoustic pressure waves on the one hand and the motion of massive bodies in the proximity of the interferometer on the other. Saulson concluded the atmospheric Newtonian noise he considered would be insignificant even when using advanced gravitational interferometric detectors. Later, T. Creighton [11] revisited the issue of atmospheric Newtonian noise. Among the possible sources of mass density fluctuations in the atmosphere, Creighton considered those that he thought potentially most significant. The sources he considered were:

- Low pressure acoustic waves.
- Fast moving massive bodies in the proximity of the interferometer.
- Transient atmospheric shock waves.
- Massive bodies colliding with the ground or the structures housing the interferometer test masses.
- Perturbations of atmospheric temperature in the vicinity of the detector.

The largest small scale atmospheric density perturbations are not caused by pressure waves but rather by temperature perturbations. When heat is transported through a convective atmospheric layer, convective turbulence mixes warm and cold air leading to temperature perturbations on all scales up to the order of millimeters. On the time scales of interest, these perturbations can be viewed as essentially "frozen" effects in the air mass, while the pressure variations are dispersed quickly via sound waves. Therefore, the fluctuations in air density are induced primarily from
temperature perturbations which are generally larger than pressure perturbations by several orders of magnitude. Although "frozen" in the air mass, these temperature perturbations can cause density fluctuations that vary quickly in time, \( d\rho = -\rho \frac{dT}{T} \), when the wind transports them in space. Indeed, this is the primary source of noise in optical astronomy. We remark that in this context, it is also important to study the noise generated by acoustic perturbations (density fluctuations) that are either completely absorbed or reflected by the ground and/or structures in the vicinity of the interferometer.

Even though temperature perturbations transported via the wind are the dominant source of atmospheric density fluctuations, they do not produce significant Newtonian noise at very high frequencies. This is due to the long time intervals that packets of warm and cold air stay in the vicinity of the interferometer test masses. A possible exception exist when the air flow forms vortices around the structure in which the interferometer is housed. Since such flows can produce a noise spectrum that has a maximum around the frequencies of the typical vortex circulating in the proximity of the test mass, the presence of temperature fluctuations in the atmosphere transported by the wind near the detector can give rise to a non-trivial source of density fluctuations. Unexpected variations of pressure caused by atmospheric shock waves can be potential sources of transient signals that can be detected by the experimental apparatus, provided such phenomena occurs in the proximity of the detector. These "shocks" are relevant because they can produce significant pressure variations on time scales smaller than 0.1 sec. This time scale corresponds to the smallest value of the band-pass of the majority of current interferometric detectors of gravitational waves. Such shocks are essentially transient phenomena that may produce spurious signals in the detector, rather than raise the noise threshold. It would be useful to know the signal to noise ratio (SNR) that various shocks could produce.

Although atmospheric shock waves are potential sources of spurious signals in gravitational wave detectors, they are readily treatable using environmental detectors. If such sensors record a pressure variation larger than some millibar on time scales of 50 – 100 milliseconds, then we could expect spurious signals with (adimensional) amplitude of the order of \( 10^{-22} \) in the range of frequencies 10 – 20 Hz. An example of atmospheric shocks is "sonic booms" generated by supersonic bodies (for example a supersonic airplane flying over the space around the detector); they could obscure the detection of gravitational waves. Even though such events are rare, if not entirely non-existent, they accentuate the potential seriousness of shock waves with respect to other weaker acoustic sources or from acoustic sources originating at greater distances from the experimental apparatus.

Another potential source of spurious signals in the interferometer is the Newtonian noise caused by the motion of a single massive body in the proximity of the detector, or the collision of such a body with the experimental apparatus. This last possibility is particularly worrisome, since the deceleration of the body can produce high frequency signals.

Next we discuss one mechanisms that generate atmospheric Newtonian noise from background acoustic pressure waves. Consider a pressure plane wave with frequency \( f \), propagating at the speed of sound characteristic of the medium, \( c_s \). To estimate the relevance of this noise requires comparison of the spectral density of the acoustic noise signal to the sensitivity curve of the interferometric detector. The spectral density of the particular form of noise being considered is given by [11]:

\[
S_h(f) = \left( \frac{G\rho c}{4\pi^2\gamma l} \right)^2 \frac{1}{3fp^2} \sum_{i=1}^{4} C \left( \frac{2\pi f r_{t_{\min}}^{(i)}}{c_s} \right) S_p^{(i)}(f),
\]

(3)

where the indices \( i \) identify the various test masses of the interferometer, \( r_{t_{\min}}^{(i)} \) is the dead air radius about the \( i \)th test mass, and \( S_p^{(i)}(f) \) is the acoustic noise spectrum measured outside the building enclosing the \( i \)th test mass. Moreover, \( l \) is the length of the interferometer arm, \( \rho \) is the ambient air density, \( p \) is the ambient air pressure and \( \gamma = \frac{c_p}{c_v} \) is the ratio of heat capacities at constant pressure and temperature of the air at room temperature. In what follows we briefly reiterate the important points that led to (3).

1. One supposes that the relative fluctuation of air pressure is small: \( \frac{\delta p}{p} \ll 1 \).

2. Only sound waves in proximity of the interferometer are considered. Since the interferometer is sensitive only to movement of test-masses parallel to the arms of the detector, the gravitational acceleration produced by pressure waves on the test masses is reduced by a factor \( \cos \theta \). Here \( \theta \) is the angle between the direction of propagation of the pressure wave and the arm of the interferometer.

3. The test masses of the interferometer is inside a structure that in principle can be used to eliminate noise within a characteristic distance \( r_{\text{min}} \) from the test masses. In order to take this into account, a function \( C(x) \) is introduced. The function \( C(x) \) depends on the shape of the structure, the ways in which it reflects sound waves, and on many other factors.

4. The interferometer is located on the ground, not in an ideal homogenous empty space. It is assumed that the waves are almost completely reflected from the ground.
Exploiting these four points, the gravitational acceleration produced by sound waves in the propagation direction \( z \) is:

\[
g_z(t) = G \int z \frac{\delta \rho(t)}{r^3} dV = G \frac{\rho c_s}{\gamma pf} \cos \theta \cdot C \left( \frac{2\pi f r_{\text{min}}(i)}{c_s} \right) \delta \rho \left( t + \frac{1}{4f} \right). \tag{4}
\]

On the other hand, the gravitational wave signal \( h(t) \) in the interferometer is related to the acceleration of one of the test masses according to

\[
d^2 h \frac{dt^2}{dt^2} = g(t) \frac{1}{l}. \tag{5}
\]

In frequency space this relation becomes

\[
\tilde{h}(f) = \frac{1}{(2\pi f)^2} \frac{\tilde{g}(f)}{l}. \tag{6}
\]

Combining (4) with (6) we obtain

\[
\tilde{h}(f) = \frac{G \rho c_s}{4\pi^2 \gamma pf^3} \cos \theta \cdot C \left( \frac{2\pi f r_{\text{min}}(i)}{c_s} \right) \hat{i} \delta \tilde{p}(f). \tag{7}
\]

Assuming the noise is stationary and that the directions and amplitudes of the wave modes are uncorrelated, we have

\[
S_h(f) = \left[ \frac{G \rho c_s}{4\pi^2 \gamma pf^3} C \left( \frac{2\pi f r_{\text{min}}(i)}{c_s} \right) \right]^2 \langle \cos^2 \theta \rangle \frac{S_p(f)}{p^2}, \tag{8}
\]

where \( \langle \cdots \rangle \) indicates an average over the modes of the plane wave that contribute to the noise. Finally, we assume that the test masses in the interferometer are at a distance equal to several coherent lengths such that the noise can be considered uncorrelated and therefore, can be summed linearly. Finally using \( \langle \cos^2 \theta \rangle = 1/3 \) we obtain (8).

Creighton \[11\] concludes that the acoustic background and the temperature fluctuations produce Newtonian noise that is below the sensitivity threshold of even the most advanced interferometric detectors even though temperature perturbations transported along streamlines of non-laminar flux could produce noise within one order of magnitude from the maximum sensitivity region at 10 Hz.

A more comprehensive study of such phenomena requires the use of more sophisticated models. Additionally, shock waves in the atmosphere could produce potentially significant spurious signals for an advanced interferometer. These signals could be monitored from acoustic sensors placed outside the interferometer structure. In order to avoid the non-negligible noise of massive objects transported by the wind and to preserve the assumption of linear additivity of noise (hypothesis of uncorrelated noise), it would be necessary to construct barriers to keep such massive bodies at a secure distance from the test masses. One of the main purposes of the Virgo project is that of achieving good sensitivity at low frequencies (around 4 – 10 Hz). In this frequency band the thermal and seismic Newtonian noise represent the dominant sources of noise. If the thermal noise could be reduced at low frequencies using cryogenic techniques or by using high Q materials, the seismic Newtonian noise would represent the sensitivity limit at these frequencies \[3\].

The objective of this article is to present an analytical estimate of the atmospheric Newtonian noise generated by fluctuations of atmospheric mass densities and to judge the relevance of this noise in the detection of gravitational waves using laser-interferometric techniques. With the development of very refined and sensitive experimental techniques it is possible to study directly fluctuation phenomena in various areas of physics. In our view, fluctuation phenomena are related in a natural way to transport or convection phenomena. Diffusion phenomena and irreversible thermodynamics are based on results derived from the theory of fluctuations. We consider non-quantum fluctuations \( (\hbar \omega \ll k_B T) \) \[13\], which is the conventional hypothesis employed in fluid mechanics.

Additionally, we suppose that the viscosity coefficients and thermal conductivity of the fluid (atmosphere) are non-dispersive, that is, they are independent of the oscillations of fluctuation \( \omega \). The phenomena that cause such fluctuations of atmospheric mass density are divided into two groups:

1. Fluctuations generated from Acoustic Phenomena.
2. Fluctuations generated from Turbulent Phenomena.

We study theoretical models corresponding to each of these groups and present estimates of the spectral density of the atmospheric Newtonian noise for each case.
III. CALCULATION OF NEWTONIAN NOISE

Before studying these physical phenomena, we first calculate some relevant quantities that will enable us to quantitatively describe the shape of the Newtonian noise. The effect of atmospheric density fluctuations $\delta \rho(\vec{x}, t)$ on each one of the four test-masses that are suspended on the suspension tower of the interferometer, can be evaluate using Newton’s force law $\vec{F} = m \vec{a}$ with

$$\vec{a}(\vec{x}, t) = -G \int d^3\vec{x}' \delta \rho(\vec{x}', t) \frac{\vec{x} - \vec{x}'}{||\vec{x} - \vec{x}'||^3}$$  (9)

where $\vec{x}$ is the effective position of the test-mass. The test-masses are arranged so as to form the two arms of the interferometer where these arms are oriented along the $x$ and $y$ directions. Furthermore, since only variations of the relative length of the light path are detected, we are only interested in the $x$ and $y$ components of the acceleration.

The relative difference of the arm length is given by

$$h(t) = \frac{1}{L} \left[ (x_2^{(1)} - x_2^{(2)}) - (x_1^{(4)} - x_1^{(3)}) \right],$$  (10)

where $L$ is the length of the arms of the interferometer. Therefore, we can link the temporal second derivative of $h(t)$ to the Newtonian acceleration of the mass test,

$$\frac{d^2h(t)}{dt^2} = \frac{1}{L} \left[ (\tilde{a}_2^{(1)} - \tilde{a}_2^{(2)}) - (\tilde{a}_1^{(4)} - \tilde{a}_1^{(3)}) \right].$$  (11)

Since we are interested in the spectral amplitude of the atmospheric Newtonian noise, let us consider the Fourier transform of $h(t)$, expressed formally as

$$\tilde{h}(\omega) = \int dt h(t) e^{i\omega t}.$$  (12)

In the Fourier space, (11) becomes

$$-\omega^2 \tilde{h}(\omega) = \frac{1}{L} \left[ (\tilde{a}_2^{(1)} - \tilde{a}_2^{(2)}) - (\tilde{a}_1^{(4)} - \tilde{a}_1^{(3)}) \right].$$  (13)

Due to the stochastic nature of the density fluctuations, we are concerned with the ensemble average $S_h(\omega)$ defined as

$$S_h(\omega) = \langle h(\omega) \tilde{h}^*(\omega) \rangle.$$  (14)

Equation (14) represents the spectrum of Newtonian noise. Using (13), (14) becomes

$$S_h(\omega) = \frac{1}{L^2 \omega^4} \left\langle \left( \tilde{a}_2^{(1)} - \tilde{a}_2^{(2)} \right) - \left( \tilde{a}_1^{(4)} - \tilde{a}_1^{(3)} \right) \right\rangle^2.$$  (15)

We point out that (15) will be simplified in a convenient manner in the cases we consider by assuming conditions of homogeneity (invariance under translations in a statistical sense) and isotropy (invariance under rotation in a statistical sense) of the correlation functions describing density fluctuations and therefore, the fluctuations of acceleration.

IV. ACOUSTIC PHENOMENA

It is the interplay between the compressibility and inertia of the fluid that supports the propagation of sound waves (the oscillating motion of small amplitudes in an incompressible fluid is called sound waves) in the medium. We work with the linear theory of acoustics since we consider perturbations that are negligible in the equations of motion. We consider exclusively the compressibility and inertia of the fluid, but no other property of the fluid. We obtain the linearized equations of the acoustic theory in their simplest non-trivial form. We neglect the influence on the propagation of sound waves from viscosity, heat conduction and inhomogeneity at the boundaries. We consider the propagation of waves in an ideal fluid, with reference to so-called “small motions”. That is to say, we consider a linearized approximation of the Euler equation

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} = -\frac{1}{\rho} \nabla p.$$  (16)
This procedure is legitimate when the perturbations in pressure and density ($\delta p$ and $\delta \rho$, respectively) are sufficiently small compared to the equilibrium state ($p_0$ and $\rho_0$, respectively). We proceed to linearize (16) together with the continuity equation

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0$$

and the adiabatic condition

$$\frac{ds}{dt} = 0$$

($s$ it is the entropy for unit mass). Letting $p = p_0 + \delta p$, $\rho = \rho_0 + \delta \rho$ where $\rho_0$ and $p_0$ are the density and pressure of the fluid at equilibrium respectively, $\delta \rho$ and $\delta p$ are the perturbations of density and pressure satisfying $\delta \rho \ll \rho_0$ and $\delta p \ll p_0$. Defining the velocity potential function $\phi$ such that $\vec{\nabla} \phi = \vec{v}$ (in the reasonable hypothesis that the velocity field is irrotational, $\vec{\nabla} \times \vec{v} = 0$), we obtain the sound wave equation

$$
\left( \Delta - \frac{1}{c_s^2} \frac{\partial^2}{\partial t^2} \right) \phi (\vec{x}, t) = 0
$$

where $c_s = \sqrt{(\partial p/\partial \rho)_s}$ is the speed of sound and can be expressed as $c_s = 1/\sqrt{\rho \chi_s}$, where $\chi_s = \rho^{-1} (\partial p/\partial \rho)_s$ is the adiabatic compressibility of the medium. It is straightforward to verify that $\delta \rho$ and $\delta p$ satisfy the same wave equation once we recognize the relationship between the fluctuation of pressure and density of a compressible medium is given by $\delta p = c_s^2 \delta \rho$. We emphasize that the wave equation introduced for $\phi$, $\delta r$ or $\delta p$ are linear approximations. We will not consider non-linear wave propagation phenomena. We recall however, that non-linearities may produce "distortion" of the initial signal or even the emergence of discontinuities (shock waves). At this juncture we highlight the two fundamental assumptions that enable us to obtain the linearized equations:

1. The necessary condition for applying the linearized equations of motion to describe the propagation of sound waves is that the speed of the fluid particles is small compared to the speed of sound, $v \ll c_s$.

2. The propagation of sound waves in an ideal fluid is considered adiabatic.

It is reasonable to assume that the process of compression and rarefaction occurring locally is adiabatic even though as a consequence of this thermal inhomogeneities arise in the medium. Such inhomogeneities would lead to hypothesize the existence of heat exchange between adjacent regions at different temperature. It turns out however, that the time scale over which appreciable heat exchange occurs (locally) is much longer than the time scale over which the causes (local pressure and temperature gradients) that generate the thermal exchange are sustained. In other words, we assume

$$T_{\text{oscillation}} \ll \tau_{\text{conduction}}$$

where $T_{\text{oscillation}}$ represents the period of the sound wave, while $\tau_{\text{conduction}}$ represents the characteristic time scale over which a significant heat exchange in the medium can occur. These two time scales can be numerically estimated, yielding

$$\tau_{\text{conduction}} \simeq \frac{(\lambda/2)^2}{\chi} = \frac{c_s^2}{\chi f^2} \simeq 10^{16} \left( \frac{1 \text{Hz}}{f} \right)^2 \text{sec.}$$

where $\lambda$ is the wavelength considered and $\chi$ is the thermometric conductivity of the air. Clearly the conduction time $\tau_{\text{conduction}}$ is much greater than any oscillation time $T_{\text{oscillation}}$ of interest to us.

A. Propagation of Acoustic Waves in a Semispace

As a first problem concerning the generation of atmospheric Newtonian noise, let us consider the gravitational coupling of the interferometer test-masses to fluctuations of atmospheric density due to the propagation of sound waves in a semispace occupied by a compressible ideal fluid (atmosphere), where the semispace is delimited by an infinitely rigid plane (ground). It is necessary to integrate (19) with the boundary condition

$$\frac{\partial \phi (x, y, z = 0, t)}{\partial z} = 0.$$
This condition expresses the fact that the velocity of the ideal fluid on the surface of separation is purely tangential. The general solution of this problem is given by

\[ \phi(\vec{x}, t) = \int d^3\vec{k} f(\vec{k}) u_{\vec{k}}(\vec{x}, t) \]  \hspace{1cm} (23)

where \( u_{\vec{k}} \) are plane waves defined as

\[ u_{\vec{k}}(\vec{x}, t) = \exp \left[ i(\vec{k} \cdot \vec{x} - \omega t) \right] . \]  \hspace{1cm} (24)

The function \( f(\vec{k}) \) is defined such that it satisfies \( f(k_x, k_y, k_z) = f(k_x, k_y, -k_z) \) (to ensure normalization) as well as the boundary conditions of the problem. In order to find expressions for the acoustic Newtonian noise it is necessary to calculate the correlation function associated with the velocity potential \( \phi \) relative to fluctuations of acceleration of the test-masses. From fluid mechanics \[13\] we find

\[ \delta \rho(\vec{x}, t) = -\frac{\rho_0}{c_s^2} \frac{\partial \phi(\vec{x}, t)}{\partial t} . \]  \hspace{1cm} (25)

In Fourier space (26) becomes

\[ \delta \tilde{\rho}(\vec{x}, \omega) = \frac{i\omega \rho_0}{c_s^2} \tilde{\phi}(\vec{x}, \omega) . \]  \hspace{1cm} (26)

Multiplying \[26\] for its complex conjugate and averaging (ensemble average) we obtain,

\[ \mathcal{S}_{\delta \rho}(\vec{x}_1, \vec{x}_2; \omega) = \langle \delta \tilde{\rho}(\vec{x}_1, \omega) \delta \tilde{\rho}^*(\vec{x}_2, \omega) \rangle = \frac{\omega^2 \rho_0^2}{c_s^2} \overline{\langle \tilde{\phi}(\vec{x}_1, \omega) \tilde{\phi}^*(\vec{x}_2, \omega) \rangle} . \]  \hspace{1cm} (27)

Furthermore, since

\[ a_i(\vec{x}, t) = -G \int d^3\vec{x}' \delta \rho(\vec{x}', t) \frac{x_i - x_i'}{||\vec{x} - \vec{x}'||^3} , \]  \hspace{1cm} (28)

we find, proceeding as in \[26\],

\[ (\mathcal{S}_a)_{ij}(\vec{x}_1, \vec{x}_2; \omega) = \langle a_i(\vec{x}_1, \omega) a_j^*(\vec{x}_2, \omega) \rangle \]

\[ = G \int d^3\vec{x}_1' \int d^3\vec{x}_2' \frac{(\vec{x}_1' - \vec{x}_1)(\vec{x}_2' - \vec{x}_2)}{||\vec{x}_1' - \vec{x}_1||^3 ||\vec{x}_2' - \vec{x}_2||^3} \mathcal{S}_{\delta \rho}(\vec{x}_1', \vec{x}_2' ; \omega) \]  \hspace{1cm} (29)

where \( a_i(\vec{x}, t) \) is the \( i \)th component of the fluctuation of acceleration exerted on the test-mass located at \( \vec{x} \). This fluctuation of acceleration is generated by fluctuations of atmospheric mass density occupying volume \( V \). For economy of notation we define the quantity

\[ K_{ij}(\vec{x}, \vec{y}) = \frac{x_i y_j}{||\vec{x}||^3 ||\vec{y}||^3} . \]  \hspace{1cm} (30)

Replacing \[27\] and \[30\] in \[29\], we obtain

\[ \langle a_i(\vec{x}_1, \omega) a_j^*(\vec{x}_2, \omega) \rangle = \frac{G^2 \omega^2 \rho_0^2}{c_s^4} \int d^3\vec{x}_1' \int d^3\vec{x}_2' K_{ij}(\vec{x}_1' - \vec{x}_1, \vec{x}_2' - \vec{x}_2) \overline{\langle \tilde{\phi}(\vec{x}_1', \omega) \tilde{\phi}^*(\vec{x}_2', \omega) \rangle} . \]  \hspace{1cm} (31)

Therefore the main problem is that of evaluating the expression for \( \mathcal{S}_\phi(\vec{x}_1, \vec{x}_2, \omega) \),

\[ \mathcal{S}_\phi(\vec{x}_1, \vec{x}_2; \omega) = \overline{\langle \tilde{\phi}(\vec{x}_1, \omega) \tilde{\phi}^*(\vec{x}_2, \omega) \rangle} = \int d^3\vec{k}_1 \int d^3\vec{k}_2 \tilde{u}_{\vec{k}_1}(\vec{x}_1, \omega) \tilde{u}_{\vec{k}_2}^*(\vec{x}_2, \omega) \overline{\langle f(\vec{k}_1) f^*(\vec{k}_2) \rangle} \]  \hspace{1cm} (32)

or in other words, that of evaluating the correlation function \( \overline{\langle f(\vec{k}_1) f^*(\vec{k}_2) \rangle} \). Assuming this correlation function is homogenous and isotropic, that is to say, assuming invariance (in a statistical sense) under translations and rotations, we have

\[ \overline{\langle f(\vec{k}_1) f^*(\vec{k}_2) \rangle} = (2\pi)^3 \delta(3) \overline{\langle \vec{k}_1 - \vec{k}_2 \rangle} \overline{\langle \left| f(\vec{k}_1) \right|^2 \rangle} . \]  \hspace{1cm} (33)
Including the constants of normalization in the definition of \( f(\vec{k}) \) and substituting \( \delta \rho \) into \( \delta \dot{\rho} \) we arrive at

\[
\langle \dot{\phi} (\vec{x}_1, \omega) \dot{\phi}^* (\vec{x}_2, \omega) \rangle = \int d^3 k \hat{\rho}_k (\vec{x}_1, \omega) \hat{\rho}_k^* (\vec{x}_2, \omega) \left\langle |f(\vec{k})|^2 \right\rangle.
\]

(34)

The problem has now been reduced to the calculation of the quantity \( \left\langle |f(\vec{k})|^2 \right\rangle \). From the general expression for \( f \) and using \( \vec{v} = \nabla \phi \), we get

\[
\nabla \cdot \vec{v} = - \int d^3 k k^2 f(\vec{k}) u_k (\vec{x}, t).
\]

(35)

The continuity equation \( \rho \) expressed in frequency space reads

\[
\delta \rho (\vec{x}, \omega) = -\frac{i \rho_0 \omega}{s^3} \hat{\rho}_k \cdot \vec{v}.
\]

(36)

Assuming fixed \( \omega \) and using \( \delta \rho \), equation \( \delta \dot{\rho} \) becomes

\[
\delta \dot{\rho} (\vec{x}, \omega) = \frac{i \rho_0 \omega^3}{c_s^4} \int d\Omega_k f(\vec{k}) u_k (\vec{x})
\]

and therefore

\[
\langle \delta \dot{\rho} (\vec{x}_1, \omega) \delta \dot{\rho}^* (\vec{x}_2, \omega) \rangle = \frac{\rho_0^2 \omega^6}{c_s^4} \int d\Omega_k \int d\Omega_p u_k^* (\vec{x}_1) u_p (\vec{x}_2) \left\langle f(\vec{k}) f^*(\vec{p}) \right\rangle.
\]

(38)

Using the invariance under rotation around the \( z \) axis and the constraint conditions on the amplitudes \( f(\vec{k}) \), it is found that \( \delta \rho \) becomes

\[
S_{\delta \rho} (\vec{x}_1, \vec{x}_2; \omega) = \frac{2 \rho_0^2 \omega^6}{c_s^4} \int \xi \mathcal{F} (\xi, \omega) J_0 \left( \omega \xi \sqrt{1 - \xi^2} \right) \left\{ \cos \left[ \omega \xi (z_1 - z_2) \right] + \cos \left[ \omega \xi (z_1 + z_2) \right] \right\},
\]

(39)

where \( r = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \), \( \mathcal{F} (\xi, \omega) = \left\langle |f(\xi, \omega)|^2 \right\rangle \) and \( J_0 (z) \) is the Bessel function of the first kind \( [24] \). Assuming \( x_1 = \vec{x}_2 = (0, 0, z) \), equation \( \delta \rho \) simplifies and becomes

\[
S_{\delta \rho} (z, \omega) = \frac{2 \rho_0^2 \omega^6}{c_s^4} \int d\xi \mathcal{F} (\xi, \omega) \left[ 1 + \cos \left( \frac{2 \omega \xi z}{c_s} \right) \right].
\]

(40)

Since \( c_s^4 S_{\delta \rho} = S_{\rho} \), we obtain

\[
S_{\rho} (z, \omega) = \frac{2 \rho_0^2 \omega^6}{c_s^4} \int d\xi \mathcal{F} (\xi, \omega) \left[ 1 + \cos \left( \frac{2 \omega \xi z}{c_s} \right) \right].
\]

(41)

From \( \delta \rho \) we get

\[
\int_0^\infty dz \cos (\alpha z) S_{\rho} (z, \omega) = \frac{2 \rho_0^2 \omega^6}{c_s^4} \int d\xi \mathcal{F} (\xi, \omega) \int_0^\infty dz \cos \alpha z \left[ 1 + \cos \left( \frac{2 \omega \xi z}{c_s} \right) \right],
\]

(42)

that is

\[
\int_0^\infty dz \cos (\alpha z) S_{\rho} (z, \omega) = \frac{\pi \rho_0^2 \omega^6}{c_s^4} \mathcal{F} \left( \frac{c_s}{2 \omega \alpha}, \omega \right).
\]

(43)
We therefore conclude,

\[
F \left( \frac{c_s}{2\omega}, \alpha, \omega \right) = \frac{c_s^4}{\pi \rho_0 \omega^6} \int_0^\infty dz \cos (\alpha z) S_p (z, \omega). (44)
\]

The spectrum of pressure \( S_p(z, \omega) \) can be obtained experimentally by use of an acoustic detector (microphone) directed along different directions (the parameter \( \alpha \) accounts for this directional variability of the acoustic detector). Hence, by integrating (44) in \( z \), the quantity \( F \left( \frac{c_s}{2\omega}, \alpha, \omega \right) \) can be readily calculated. Using \( F \left( \frac{c_s}{2\omega}, \alpha, \omega \right) \) it is possible to estimate the correlation function associated with fluctuations of acceleration of the test-masses. Alternatively, it is possible to determine any correlation starting from the correlations at the ground level. A simple way to verify this consists of applying a Fourier transform to the variables \( x, y, t \) in the wave equations and boundary conditions. The problem is then reduced to integration of an ordinary differential equation with initial conditions depending on the values of pressure at the ground level. Due to a lack of detailed measurements of the correlations of pressure, it is necessary to further simplify the model. For example, we can start again from the wave equation for pressure fluctuation and Fourier transform only the time variable,

\[
\begin{cases}
\left( \nabla^2 + \frac{\omega^2}{c_s^2} \right) \delta p \left( \vec{r}, \omega \right) = 0 \\
\left( \frac{\partial \delta p \left( \vec{r}, \omega \right)}{\partial z} \right)_{z=0} = 0 \\
\left[ \delta p \left( \vec{r}, \omega \right) \right]_{z=0} = \delta p_{\text{exp}}, (x, y, \omega)
\end{cases}
(45)
\]

In the boundary conditions appears measures of the fluctuations at ground level, \( \delta p_{\text{exp}} \). The solution of problem (45) can be written as the sum on modes at fixed frequency \( \omega \),

\[
\delta p \left( \vec{r}, \omega \right) = \int \frac{d^2 \vec{k}}{(2\pi)^2} \left( A \left( \vec{k}, \omega \right) e^{i\vec{k} \cdot \vec{r}} \cos \left( \gamma_{\vec{k}} z \right) \right)
(46)
\]

where \( \gamma_{\vec{k}} = \sqrt{\omega^2/c_s^2 - k^2} \) and the integration is extended to all values of \( \vec{k} = (k_x, k_y, 0) \) such that \( \gamma_{\vec{k}} \) is real. Then the correlations of pressure fluctuations can be written as

\[
S_p \left( \vec{r}_1, \vec{r}_2; \omega \right) = \int \frac{d^2 \vec{k}_1}{(2\pi)^2} \int \frac{d^2 \vec{k}_2}{(2\pi)^2} \left\langle A \left( \vec{k}_1, \omega \right) A^* \left( \vec{k}_2, \omega \right) \right\rangle e^{i\vec{k}_1 \cdot \vec{r}_1} e^{-i\vec{k}_2 \cdot \vec{r}_2} \cos \left( \gamma_{\vec{k}_1} z_1 \right) \cos \left( \gamma_{\vec{k}_2} z_2 \right)
(47)
\]

The simplifying hypothesis is that the correlations between the amplitudes of modes have the following form

\[
\left\langle A \left( \vec{k}_1, \omega \right) A^* \left( \vec{k}_2, \omega \right) \right\rangle = (2\pi)^2 \Lambda \left( \omega \right) \delta^{(2)} \left( \vec{k}_1 - \vec{k}_2 \right)
(48)
\]

that is to say, different modes are completely uncorrelated and depend only on the frequency \( \omega \). From such hypothesis we can rewrite the correlations of pressure fluctuations as

\[
S_p \left( \vec{r}_1, \vec{r}_2; \omega \right) = \frac{1}{2\pi} \left( \frac{\omega}{c_s} \right)^2 \Lambda \left( \omega \right) \int_0 \omega \eta J_0 \left( \frac{\omega \eta}{c_s} R_{12} \right) \cos \left( \frac{\omega z_1}{c_s} \sqrt{1-\eta^2} \right) \cos \left( \frac{\omega z_2}{c_s} \sqrt{1-\eta^2} \right)
(49)
\]

where \( R_{12} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \). At this point we consider a local measurement of the spectrum of pressure fluctuations given by,

\[
S_p \left( 0, 0; \omega \right) = \frac{1}{2\pi} \left( \frac{\omega}{c_s} \right)^2 \Lambda \left( \omega \right).
(50)
\]

Using (49) and (50) we obtain the result

\[
\frac{S_p \left( \vec{r}_1, \vec{r}_2; \omega \right)}{S_p \left( 0, 0; \omega \right)} = \int_0 \omega \eta J_0 \left( \frac{\omega \eta}{c_s} R_{12} \right) \cos \left( \frac{\omega z_1}{c_s} \sqrt{1-\eta^2} \right) \cos \left( \frac{\omega z_2}{c_s} \sqrt{1-\eta^2} \right).
(51)
\]
Finally, for the correlation between accelerations we find

\[ (S_a)_{ij}(\vec{r}_1, \vec{r}_2; \omega) = \frac{G^2}{c_s^4}S_\rho(0, 0; \omega) \int d^3\vec{r}_1 \int d^3\vec{r}_2 K_{ij}(\vec{r}_1 - \vec{r}_1^*, \vec{r}_2 - \vec{r}_2^*) \times \]

\[ \times \int_0^1 d\eta J_0 \left( \frac{\omega \eta}{c_s} \right) \cos \left( \frac{\omega \eta_1}{c_s} \sqrt{1 - \eta^2} \right) \cos \left( \frac{\omega \eta_2}{c_s} \sqrt{1 - \eta^2} \right), \]

where \( J_m(z) \) is the Bessel function of the first kind given by,

\[ J_m(z) \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k+|m|} k! (|m| + k)!} z^{2k+|m|}, \quad |m| \neq \frac{1}{2}. \]  

(53)

For \( m = 0 \), \( J_0(z) \) becomes [24],

\[ J_0(z) = \frac{1}{\pi} \int_0^\pi \exp(i z \cos \theta) d\theta = \sum_{k=0}^{\infty} (-1)^k \frac{(k^2)^k}{(k!)^2}. \]

(54)

Equation (52) allows to obtain an analytical estimate for the correlation between the Newtonian accelerations for any pair of points in terms of the spectrum of pressure fluctuations which can be experimentally determined. Unfortunately, we do not have direct measurements of \( S_\rho \) for the VIRGO detector. As evident from (52) and (53), \( (S_a)_{ij}(\vec{r}_1, \vec{r}_2; \omega) \) has a non-trivial dependence on \( \omega \) and therefore to compare the effect of this noise with the sensitivity curve of Virgo, it is crucial to focus on the proper frequency-band of the noise considered. Then, we compare the square root of the strain amplitude of the noise considered. In principle, we could use our analytical estimate together with experimental values extracted from the literature [14] and provide numerical evidence leading to a numerical estimate of the relevance of the noise considered.

V. TURBULENT PHENOMENA

Before considering our specific problem, it is useful to briefly discuss the main characteristics of turbulence. Consider a turbulent flow of an incompressible fluid. A turbulent flow is by definition unstable: a small perturbation will in general be amplified due to non-linearities appearing in the equations describing the flow. Furthermore, it is evident from a great amount of experimental data the turbulent flow of an incompressible fluid is rotational, that is to say, \( \vec{\omega} = \nabla \times \vec{v} \neq 0 \), at least in certain regions of the space. The set of equations that defines this physical system consists is the Navier-Stokes [12] equation

\[ \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \vec{v} \]

(55)

where \( \nu \) it is the kinematic viscosity of the fluid and the incompressible condition reads

\[ \nabla \cdot \vec{v} = 0. \]

(56)

The system should be integrated taking account of the chosen initial and boundary conditions. The Navier-Stokes equation encodes all we need to know about turbulence. However, it is essential to have data from experimental observations in order to properly understand the phenomena since it is a highly non-trivial task to analytically integrate the equation due to its inherent non-linearities. Among the parameters that characterize the turbulent flow, only the kinematic viscosity \( \nu = \frac{\rho}{\rho} \) (\( \eta \) is the dynamic viscosity) appears in the Navier-Stokes equations. The unknown quantities to be determined are \( \vec{v} \) and \( \vec{g} \). Moreover the flow depends - through the boundary conditions - on the shape and dimensions of the body being inserted into the fluid in order to break the irrotationality of the velocity field thereby giving rise to turbulence. Since generally the shape of the body is assumed given, the geometric properties are characterized by a typical linear dimension denoted \( l \). Let \( u \) be a typical speed of the principal flow of the fluid. Then every flow is specified by three parameters: \( \nu, u, \) and \( l \). The only adimensional quantity that can be constructed from these three parameters is the so-called Reynolds number \( R \) [13],

\[ R = \frac{ul}{\nu} \]

(57)
The numerator of \( \frac{\Delta \vec{v}}{\Delta t} \) represents the transport term (or inertial term), while the denominator is the viscosity term. When the Reynolds number is small \( (R \ll 1) \), it is permissible to neglect the inertial forces and therefore, the Navier-Stokes equations can be linearized. On the other hand, when \( R >> 1 \) (large Reynolds numbers), the inertial forces dominate those of viscosity. In such situations, instabilities develop that lead to chaotic motion (turbulence).

In order to clarify the mechanism that leads to the emergence of turbulence, we introduce the concepts of broken and restored symmetries. The concept of symmetry is central in the study of transition phenomena and fully developed turbulence. Transformation symmetries are represented by either continuous or discrete invariance groups associated with restored symmetries. The concept of symmetry is central in the study of transition phenomena and fully developed turbulence.

Furthermore, observe that the symmetry breaking [16]. When the ground state of the system does not share the same symmetry of the Hamiltonian, the symmetry is said to be spontaneously broken. However, since the system can actually be in one and only one of these ground states, the symmetry appears broken. What actually happens is the manner in which the symmetry manifests itself. It is natural to assume that in the most perfect manifestation the ground state should be invariant under symmetry transformations. In many cases, the manifested symmetry at high temperatures is generally a property of the microscopic Hamiltonian of the system. As such, it cannot cease to exist even when the symmetries associated with the Hamiltonian appear to be violated. The question is, where did the symmetry go? For example the microscopic Hamiltonian of a ferromagnet is invariant under rotation. Lowering the temperature of the system certainly does not change this fact. What actually changes is the manner in which the symmetry manifests itself. It is natural to assume that in the most perfect manifestation of symmetry the ground state should be invariant under symmetry transformations. In many cases, the system has several equivalent ground states that can be mapped into each other via symmetry transformation. However, since the system can actually be in one and only one of these ground states, the symmetry appears broken. When the ground state of the system does not share the same symmetry of the Hamiltonian, the symmetry is said to be spontaneously broken.

Returning to the issue of turbulence, it is experimentally observed that when the Reynolds number increases in the fluid, we define the concept of spontaneous symmetry breaking [15].

It appears that macroscopic systems generally have a smaller degree of symmetry at low rather than high temperatures. The manifested symmetry at high temperatures is generally a property of the microscopic Hamiltonian of the system. As such, it cannot cease to exist even when the symmetries associated with the Hamiltonian appear to be violated. The question is, where did the symmetry go? For example the microscopic Hamiltonian of a ferromagnet is invariant under rotation. Lowering the temperature of the system certainly does not change this fact. What actually changes is the manner in which the symmetry manifests itself. It is natural to assume that in the most perfect manifestation of symmetry the ground state should be invariant under symmetry transformations. In many cases, the system has several equivalent ground states that can be mapped into each other via symmetry transformation. However, since the system can actually be in one and only one of these ground states, the symmetry appears broken. When the ground state of the system does not share the same symmetry of the Hamiltonian, the symmetry is said to be spontaneously broken.

Returning to the issue of turbulence, it is experimentally observed that when the Reynolds number increases, the symmetries permitted by the equations of motion and the boundary conditions are subsequently broken. However, for very high Reynolds numbers there appears to be a tendency to restore the symmetries (in a statistical sense) far away from the boundaries. That is to say, the symmetries are restored on average but not for a single realization of the velocity field of the system. Such turbulence is referred to as fully developed turbulence [15]. Fully developed turbulence is turbulence that is free to develop without any constraints. The only possible constraints are the boundaries, external forces or viscosity. It is observed that the structures of a flow that develops on scales comparable to the dimensions over which the fluid evolves cannot be properly defined as "developed". For this reason, no real fluid - even if has a high Reynolds number - can be "fully developed" on large energy scales. On smaller scales however, turbulence will be fully developed provided the viscosity does not play a direct role in the dynamics at such scales.

The turbulent flow at sufficiently high Reynolds numbers is characterized by an extremely irregular and disordered temporal variation of the velocity field at each point. There is experimental evidence that the velocity field of a turbulent flow is fractal in nature [17, 18]. Turbulent fluids seem to have fractal velocity fields in the sense that the increments of the velocity field are proportional to the power 1/3 of the increment of space. In a turbulent flow at very
large Reynolds numbers, the average quadratic increment of the velocity between two points separated by distance $l$ scales approximately with the power $2/3$ of this distance [18]

$$\langle |\delta v(l)|^2 \rangle \sim l^{2/3}. \tag{58}$$

We will not proceed further with regard to turbulence and will introduce new concepts and theories as required. From a mathematical point of view, the central problem of turbulence theory is that of obtaining statistical solutions of the Navier-Stokes equations. The standard techniques of fluid mechanics are not sufficiently powerful to study turbulence. In the second half of the last century a formal analogy has been found between the theory of turbulence and quantum field theory. In both cases a system of interacting fields is non-linear, in principle with an infinite number of degrees of freedom. From here follows the similarity of the mathematical apparatus used in both theories. For example, the method of Feynman diagrams [19] used to represent equations (transition amplitudes, etc.) may also be applied to turbulence theory (tree diagrams, etc.). Our approach to the study of turbulence is to treat it as an analytic statistical theory (Kraichnan-Orszag) that relies on dimensional considerations and similarity [20].

A. Incompressible Turbulence: Lighthill Process

Very weak turbulence can be described by linearized dynamical equations [21]. Let us consider the case in which turbulence is relatively weak, but not so much so that we are able to neglect the non-linear terms in the dynamical equations. In this context, we consider as the main quadratic effect the generation of sound by turbulence in a compressible medium. The production of sound due to the self-interaction of turbulent vortices (this is the main interaction that cause second order effects) happens only when the compressibility of the medium is taken into consideration. As an example concerning the generation of mass density fluctuations due to turbulent phenomena, let us consider the wave equation of linear acoustics with a source term due to turbulence [13]

$$\left( \nabla^2 - \frac{1}{c_s^2} \frac{\partial^2}{\partial t^2} \right) \delta \rho (\vec{x}, t) = -\rho_0 \frac{\partial^2}{\partial x_i \partial x_j} v'_i v'_j \tag{59}$$

Equation (59) describes the so-called Lighthill process. This phenomenon consists of the generation of acoustic noise due to the presence of turbulent fluid flow. In other words, turbulence (or more accurately, the fluctuations of turbulent velocity) generates sound [13]. We wish to emphasize two points that are implicit in (59).

1. Equation (59) has been obtained by assuming we are dealing with compressible turbulence. A fundamental difference between compressible and incompressible turbulence is the following: for compressible turbulence, variations in the velocity field imply variations in the local mass density of the fluid. Fluctuations in the mass density imply local variations of pressure that lead to the emission of sound waves. For incompressible turbulence instead, changes in pressure do not produce changes in density.

2. In the source term of (59) are present only the fluctuating parts of the turbulent velocity field ( $\vec{v} = \vec{u} + \vec{v}'$, where $\vec{u}$ is the average velocity field and $\vec{v}'$ is the fluctuating velocity field). The quantity $\rho c_s^2 v'_i v'_j$ is called the Reynolds strain. The laminar portion of the flow, if one exists, does not play any role in generating turbulent fluctuations that give rise to acoustic noise in the Lighthill process.

For reasons of analytical complexity, we will not solve (59). We will instead study the simplified equation

$$\left( \nabla^2 - \frac{1}{c_s^2} \frac{\partial^2}{\partial t^2} \right) \delta \rho (\vec{x}, t) = -\rho_0 \frac{\partial^2}{\partial x_i \partial x_j} v_i v_j \tag{60}$$

where the source term is due to incompressible turbulence, keeping the hypothesis of compressibility in the medium where the sound waves propagate. Equation (60) has been derived by neglecting the effects related to viscosity and thermal conductivity. Furthermore, it has been assumed that the incompressible velocity fluctuations $\vec{v}'$ are small compared to the average speed of sound $c_s$. Equation (60) can then be viewed as describing the generation of sound by turbulence with a small Mach number $M = U/c_s$ (where $U$ is the characteristic scale of the velocity of the system) and not simply by decaying turbulence (that is, turbulence approaching transition to laminar flow).

Equation (60) leads to a number of important consequences. For example the right hand side of the equation is a combination of second order derivatives of the field $\vec{v}'(\vec{x})$. This means that in absence of boundary conditions, the generation of sound waves from turbulence is equivalent to the radiation emitted from a set of acoustic quadrupoles (and not by the usual acoustic dipole sources). Therefore, it follows that if there are no boundaries, at small Mach
numbers the turbulence does not represent an efficient source of sound. Thus, we assume the “acoustic noise” is generated within a bounded region (“turbulent region”) of the fluid in which velocity fluctuations occur. We assume that the medium surrounding this volume is at rest (this is a much more extended region than the turbulent region and is called the “radiation region”). In order to solve (60) we apply analytic statistical theory of turbulence [21]. To simplify calculations we assume we are dealing with fully developed turbulence that is homogenous, isotropic and stationary.

The turbulence is said to be homogenous if all quantities constructed with a set of \( n \) points \( \vec{x}_1, \ldots, \vec{x}_n \) (at instants \( t_1, \ldots, t_n \)) are invariant under any translation of this set. In particular [22]

\[
\langle u_{\alpha_1} (\vec{x}_1, t_1) \cdots u_{\alpha_n} (\vec{x}_n, t_n) \rangle = \langle u_{\alpha_1} (\vec{x}_1 + \vec{y}, t_1) \cdots u_{\alpha_n} (\vec{x}_n + \vec{y}, t_n) \rangle
\]

where \( \langle \ldots \rangle \) is the usual ensemble expectation value. The turbulence is said to be stationary if all the average quantities involved in the \( n \) instants \( t_1, \ldots, t_n \) are invariant under any temporal translation. In particular [22]

\[
\langle u_{\alpha_1} (\vec{x}_1, t_1) \cdots u_{\alpha_n} (\vec{x}_n, t_n) \rangle = \langle u_{\alpha_1} (\vec{x}_1, t_1 + \tau) \cdots u_{\alpha_n} (\vec{x}_n, t_n + \tau) \rangle \]

Finally, the homogenous turbulence is said to be isotropic if all the average quantities concerning the set of \( n \) points \( \vec{x}_1, \ldots, \vec{x}_n \) (at instants \( t_1, \ldots, t_n \)) are invariant under any arbitrary rotation.

It could be argued that there is no turbulent flows that are homogenous or isotropic at large-scale. Isotropy and homogeneity can even be debatable at small scales. Nevertheless, these hypotheses enable us to easily exploit analytic statistical theories, thereby enormously simplifying the equations of motion. Such theories are quite powerful in the sense that they allow to deal with strong non-linearity when the deviation from the hypothesis of non-Gaussianity is not large. The point of view adopted in this article is that these (analytic-statistical) techniques describe in a satisfactory manner the dynamics of three-dimensional turbulent flows at small scales. The price of this simplification is the gap between the situation studied theoretically and that which can be realized in practice. The homogeneity hypothesis implies that turbulence is uniform in space and the concept of stationarity can be described as homogeneity in time. Isotropy implies there are no preferred directions in space. There cannot be any average velocity in an isotropic field since that would immediately imply a preferred direction. Turbulent isotropic and homogenous fields can be relatively simple but they are unphysical. In actual velocity fields, energy arises from some average gradient of pressure, temperature or mass, and therefore these fields must be anisotropic. Moreover, these fields will be subject to specific boundary conditions that imply they are necessarily inhomogeneous.

Despite these facts, we will consider in this work fully developed, homogenous, isotropic and stationary turbulence. That said, we recast (60) in frequency space

\[
\left(-\vec{k}^2 + \frac{\omega^2}{c_s^2}\right) \delta \bar{p}(\vec{k}, \omega) = \rho_0 k_i k_j \tilde{V}_{ij}(\vec{k})
\]

with

\[
\tilde{V}_{ij}(\vec{k}) = \int_V d^3 \vec{x} e^{i \vec{k} \cdot \vec{x}} v_i \left( \vec{x} \right) v_j \left( \vec{x} \right)
\]

where \( V \) is the volume occupied by the turbulent fluid and \( \rho_0 \) is the average density. We obtain

\[
\delta \bar{p}(\vec{k}, \omega) = \frac{\rho_0 c_s^2}{\omega^2 - c_s^2 \vec{k}^2} k_i k_j \tilde{V}_{ij}(\vec{k})
\]

and thus the noise spectrum \( \tilde{S}_p(\vec{k}, \omega) \) associated to the pressure fluctuation becomes, in the frequency space,

\[
\tilde{S}_p(\vec{k}, \omega) = \left\langle \delta \bar{p}(\vec{k}, \omega) \delta \bar{p}^*(\vec{k}, \omega) \right\rangle = \frac{\rho_0^2 c_s^4}{\left(\omega^2 - c_s^2 \vec{k}^2\right)^2} k_i k_j k_l \left\langle \tilde{V}_{ij}(\vec{k}) \tilde{V}_{lm}^*(\vec{k}) \right\rangle.
\]

Finally, the noise spectrum becomes

\[
\tilde{S}_p(\vec{k}, \omega) = \frac{\rho_0^2 c_s^4}{\left(\omega^2 - c_s^2 \vec{k}^2\right)^2} k_i k_j k_l C_{ijkl}(\vec{k}),
\]

where

\[
C_{ijkl}(\vec{k}) = \left\langle \delta \bar{p}(\vec{k}, \omega) \delta \bar{p}(\vec{k}, \omega) \right\rangle
\]
where

\[ C_{ijlm}(\mathbf{k}) = \left\langle \bar{V}_{ij}(\mathbf{k}) \bar{V}_{lm}(\mathbf{k}) \right\rangle. \] (68)

Our task now is to compute the quantity \( C_{ijlm}(\mathbf{k}) \).

\[ C_{ijlm}(\mathbf{k}) = \int d^3\mathbf{x} \int d^3\mathbf{x}' e^{-i\mathbf{k} \cdot (\mathbf{x}' - \mathbf{x})} B_{ij, lm}^{(4)}(\mathbf{x}, \mathbf{x}') \] (69)

with

\[ B_{ij, lm}^{(4)}(\mathbf{x}, \mathbf{x}') = \left\langle v_i(\mathbf{x}) v_j(\mathbf{x}) v_l(\mathbf{x}') v_m(\mathbf{x}') \right\rangle. \] (70)

The quantity \( B_{ij, lm}^{(4)}(\mathbf{x}, \mathbf{x}') \) appearing in (70) is known in statistical fluid mechanics as the tensorial statistical moment \[10\]. In the most general case, the stochastic moments of a stochastic vectorial field represent tensors of order \( k \) that have the form

\[ B_{ij...p}^{(k)}(\mathbf{x}_1,..., \mathbf{x}_n). \] (71)

Additional limiting conditions on such stochastic fields lead to new symmetry properties of such tensors. The conditions of homogeneity and isotropy are of particular interest since, in practice, they are the only symmetries considered. However, even without these special conditions, the expression of the statistical moment \( B_{ij...p}^{(k)} \) cannot be arbitrary since it must satisfy special tensorial transformations. The quantity \( B_{ij...p}^{(k)}(\mathbf{x}_1,..., \mathbf{x}_n) \) is a tensor of forth order representing a two-point statistical moment. As a working hypothesis we assume that the fields \( v_i(\mathbf{x}) \) are Gaussian, stochastic velocity fields. Under such hypothesis we obtain (in analogy to what is obtained by applying Wick’s theorem in quantum field theory \[10\])

\[ \left\langle v_i(\mathbf{x}) v_j(\mathbf{x}) v_l(\mathbf{x}') v_m(\mathbf{x}') \right\rangle = \left\langle v_i(\mathbf{x}) v_j(\mathbf{x}) \right\rangle \left\langle v_l(\mathbf{x}') v_m(\mathbf{x}') \right\rangle + \left\langle v_i(\mathbf{x}) v_l(\mathbf{x}') \right\rangle \left\langle v_j(\mathbf{x}) v_m(\mathbf{x}') \right\rangle + \right. \] (72)

\[ \left. + \left\langle v_i(\mathbf{x}) v_m(\mathbf{x}') \right\rangle \left\langle v_j(\mathbf{x}) v_l(\mathbf{x}') \right\rangle. \]

From turbulence theory it follows that \[23\]

\[ \left\langle v_i(\mathbf{x}) v_j(\mathbf{x}) \right\rangle = \frac{2}{3} \delta_{ij} \int_0^\infty dk \mathcal{E}(k) \] (73)

and

\[ \left\langle v_i(\mathbf{x}) v_j(\mathbf{x}') \right\rangle = \int \frac{d^3\mathbf{k}}{(2\pi)^3} e^{-i\mathbf{k} \cdot (\mathbf{x}' - \mathbf{x})} \left( \delta_{ij} - \frac{k_i k_j}{k^2} \right) \] (74)

where \( \mathcal{E}(k) \) is the Kolmogorov energy spectrum, that in the interval \( k_0 << k << k_\nu \) can be written as

\[ \mathcal{E}(k) = K_0 \varepsilon^{\frac{2}{3}} k^{-\frac{5}{3}}. \] (75)

The interval of validity of (75) is characterized by \( k_0 \sim 2\pi/L \) and \( k_\nu = R^{3/4}/L = (\varepsilon/\nu^3)^{1/4} \) where \( L \) is the linear dimension of the volume of the turbulent fluid and \( K_0 \) is the Kolmogorov constant. The quantity \( \varepsilon \) represents the total energy dissipated due to viscous forces and is given by

\[ \varepsilon = \int_0^\infty dk 2\nu \mathcal{E}(k). \] (76)

The quantity \( \mathcal{E}(k) \) in (75) represents the Kolmogorov energy spectrum in the so-called inertial range. This is permitted since we are considering a turbulence problem and such turbulence is characterized by inertial modes \((k_0 << k << k_\nu)\) while the motion of dissipative modes \((k_\nu << k << k_\nu', k_\nu' \) is the viscous scale) is always laminar (the function of
dissipative modes is to absorb energy from inertial modes and dissipate it). Therefore, using (72), (73), (74) and (75) the quantity \( C_{ijlm}(\vec{k}) \) becomes

\[
C_{ijlm}(\vec{k}) = \frac{4}{9} \delta_{ij} \delta_{lm} \left( \int_{k_0}^{k} dkE(k) \right)^2 \int \frac{d^3 \vec{x}}{V} \int \frac{d^3 \vec{x}'}{V} e^{-i\vec{k} \cdot (\vec{x}' - \vec{x})} + \]

\[
+ \frac{4}{9} (\delta_{il} \delta_{jm} + \delta_{im} \delta_{jl}) \int \frac{d^3 \vec{x}}{V} \int \frac{d^3 \vec{x}'}{V} e^{-i\vec{k} \cdot (\vec{x}' - \vec{x})} \left( \int \frac{dkE(k)}{k_0} \right) \frac{k}{kl} \]

(77)

with \( l = |\vec{x}' - \vec{x}| \). For economy of notation, we define

\[
I^{(1)}_{k_0k_v}(\vec{k}) = \int \frac{d^3 \vec{x}}{V} \int \frac{d^3 \vec{x}'}{V} e^{-i\vec{k} \cdot (\vec{x}' - \vec{x})} \left( \int \frac{dkE(k)}{k_0} \right) \]

(78) and

\[
I^{(2)}_{k_0k_v}(\vec{k}) = \int \frac{d^3 \vec{x}}{V} \int \frac{d^3 \vec{x}'}{V} e^{-i\vec{k} \cdot (\vec{x}' - \vec{x})} \left( \int \frac{dkE(k)}{k_0} \right) \frac{\sin(kl)}{kl} .
\]

(79)

In terms of (78) and (79) the quantity \( C_{ijlm}(\vec{k}) \) in (77) can be written as

\[
C_{ijlm}(\vec{k}) = \frac{4}{9} \delta_{ij} \delta_{lm} I^{(1)}_{k_0k_v}(\vec{k}) + \frac{4}{9} (\delta_{il} \delta_{jm} + \delta_{im} \delta_{jl}) I^{(2)}_{k_0k_v}(\vec{k}),
\]

(80)

such that

\[
k_i k_j k_l k_m C_{ijlm}(\vec{k}) = \frac{4}{9} k^4 \left( I^{(1)}_{k_0k_v}(\vec{k}) + 2I^{(2)}_{k_0k_v}(\vec{k}) \right).
\]

(81)

Placing (81) in (77), the noise spectrum \( \tilde{S}_p(\vec{k}, \omega) \) associated to the pressure fluctuation becomes

\[
\tilde{S}_p(\vec{k}, \omega) = \frac{4}{9} \rho_0 c_s^4 \left( I^{(1)}_{k_0k_v}(\vec{k}) + 2I^{(2)}_{k_0k_v}(\vec{k}) \right) \frac{\vec{k}^4}{(\omega^2 - c_s^2 k^2)^2}.
\]

(82)

Performing a Fourier transform of \( (S_a)_{ij} (\vec{x}_1, \vec{x}_2; \omega) \) in (29) in the \( \vec{x} \) variable, we obtain

\[
\tilde{S}_a(\vec{k}, \omega) \equiv \langle \hat{a}_i (\vec{k}, \omega) \hat{a}_j^* (\vec{k}, \omega) \rangle = \left( \frac{4 \pi G}{c_s^2} \right)^2 \frac{k_i k_j}{k^4} \tilde{S}_p(\vec{k}, \omega).
\]

(83)

The quantity \( \tilde{S}_a(\vec{k}, \omega) \) in (83) represents the correlation function associated with the fluctuation of acceleration in the Fourier space in terms of the spectrum of pressure described in the frequency space. Substituting (82) into (83) we find

\[
\tilde{S}_a(\vec{k}, \omega) = \frac{4}{9} (4\pi \rho_0 G)^2 \left( I^{(1)}_{k_0k_v}(\vec{k}) + 2I^{(2)}_{k_0k_v}(\vec{k}) \right) \frac{k_i k_j}{k^4} \frac{\vec{k}^4}{(\omega^2 - c_s^2 k^2)^2}.
\]

(84)

At this point all that remains is to evaluate the integrals \( I^{(1)}(\vec{k}) \) and \( I^{(2)}(\vec{k}) \). For \( I^{(1)}(\vec{k}) \) we find the explicit functional form

\[
I^{(1)}_{k_0k_v}(\vec{k}) = \frac{9}{4} \vec{k}^2 \varepsilon^{\frac{3}{2}} \left( k_0^2 + \frac{2}{k^2} \right)^2 \frac{16\pi^2}{k^6} \left[ \sin(kL) - kL \cos(kL) \right]^2.
\]

(85)

Concerning \( I^{(2)}(\vec{k}) \), we notice it can be recast in the following form,

\[
I^{(2)}_{k_0k_v}(\vec{k}) = \frac{4 \pi}{k} \left( \frac{\vec{k} \cdot \vec{L}}{L} \right) \int \frac{d\alpha}{\alpha^2} \sin(kl) \left( \int \frac{dk}{k_{il}} \frac{\sin \alpha}{\alpha^2} \right)^2.
\]

(86)
The integral in $\alpha$ can be expressed exactly in terms of incomplete Gamma functions $\Gamma(a, z)$ \[24\],

$$\Gamma(a, z) \overset{\text{def}}{=} \int_z^\infty t^{a-1} \exp(-t) \, dt.$$  \hfill (87)

The remaining expression in (86) can be numerically integrated for a given choice of the parameters in play. The numerical values of the parameters used in our calculations is given in Table 1. The theoretical value of the Kolmogorov constant $K_0$ is borrowed from Reference \[25\].

| Parameter                                      | Symbol | Value [MKSA-units] |
|-----------------------------------------------|--------|--------------------|
| linear scale of turbulent region              | $L$    | 150                |
| Newton’s constant                             | $G$    | $6.67 \times 10^{-11}$ |
| kinematic viscosity of air at $T = 15^\circ C$ | $\nu_0$ | $1.8 \times 10^{-5}$ |
| air density                                   | $\rho_0$ | 1.3               |
| Reynolds number                               | $R$    | 3200              |
| Kolmogorov constant                           | $K_0$  | 9.85              |

Table 1

Our numerical estimate leads to the square root of the strain amplitude of the atmospheric Newtonian noise $\tilde{h}_{ANN}(f)$ given by,

$$\tilde{h}_{ANN}(f) \overset{\text{def}}{=} \sqrt{\left\langle \left| \tilde{h}_{ANN}(f) \right|^2 \right\rangle} = \frac{1}{(2\pi)^2 L f^2} \sqrt{\tilde{S}_a(k, \omega)}.$$  \hfill (88)

Recall that the sensitivity of the Virgo detector is quantified in terms of the square root of the strain amplitude $\tilde{h}_{rss}(f)$. For instance, at $f \approx 360\text{Hz}$ the best sensitivity at 50\% of efficiency is $\left[\tilde{h}_{50\%\,rss}(f)\right]_{f=360\text{Hz}} \approx 1.1 \times 10^{-20}/\sqrt{\text{Hz}}$ \[26\]. In our work, we have compared Virgo’s $\tilde{h}_{rss}(f)$ to $\tilde{h}_{ANN}(f)$ in the frequency range $f \in [4, 10]$. We have,

$$\left[\frac{\tilde{h}_{ANN}(f)}{\tilde{h}_{rss}(f)}\right]_{f\in[4, 10]\text{Hz}} \approx \frac{10^{-23}}{10^{-20}} = 10^{-3} \ll 1.$$  \hfill (89)

It turns out that the effect of acoustic noise generated in the Lighthill process considered here is at least three orders of magnitude below the sensitivity of the VIRGO interferometer.

VI. CONCLUSIONS

In this article, we presented a theoretical estimate of the atmospheric Newtonian noise generated by fluctuations of atmospheric mass densities due to acoustic and turbulent phenomena and we judge the relevance of such noise in the laser-interferometric detection of gravitational waves. First, we considered the gravitational coupling of interferometer test-masses to fluctuations of atmospheric density due to the propagation of sound waves in a semispace occupied by an ideal fluid delimited by an infinitely rigid plane. We presented an analytical expression of the spectrum of acceleration fluctuations $S_a(\vec{r}_1, \vec{r}_2; \omega)$ of the test-masses of the interferometer in terms of the experimentally determinable spectrum of pressure fluctuations $S_p(\vec{r}_1, \vec{r}_2; \omega)$. We do not have direct measurements of $S_a$ for the VIRGO detector. However, values extracted from the literature lead to conclude that the effect would be at least two orders of magnitude below the sensitivity curve. Second, we considered the gravitational coupling of interferometer test-masses to fluctuations of atmospheric density due to the propagation of sound waves generated in a turbulent Lighthill process. We presented an analytical expression, in the Fourier space, of the spectrum of acceleration fluctuations $\tilde{S}_a(\vec{k}_1, \vec{k}_2; \omega)$ of the test-masses of the interferometer. We estimated that the acoustic noise generated in the Lighthill process is three orders of magnitude below the sensitivity curve of the VIRGO interferometer.

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