D1D5 System and Noncommutative Geometry

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Abstract

Supergravity on $AdS_3 \times S^3 \times T^4$ has a dual description as a conformal sigma-model with the target space being the moduli space of instantons on the noncommutative torus. We derive the precise relation between the parameters of this noncommutative torus and the parameters of the near-horizon geometry. We show that the low energy dynamics of the system of $D1D5$ branes wrapped on the torus of finite size is described in terms of the noncommutative geometry. As a byproduct, we give a prediction on the dependence of the moduli space of instantons on the noncommutative $T^4$ on the metric and the noncommutativity parameter. We give a compelling evidence that the moduli space of stringy instantons on $R^4$ with the $B$ field does not receive $\alpha'$-corrections. We also study the relation between the $D1D5$ sigma-model instantons and the supergravity instantons.

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1 Introduction.

The formalism of noncommutative geometry is useful in string theory when one studies certain special points in the moduli space of compactifications. In string theory, small size manifolds are related to finite size manifolds by T duality. However, the required T duality transformation usually does not have a nice limit when the size of the manifold goes to zero. The form of the T duality which brings the background with the small size manifold to the background with the finite size manifold depends very irregularly on the moduli of the small size manifold, such as the metric and the $B$ field. Therefore, one could naively expect that the physics also does not have a regular limit. But it turns out that there exists a smooth description of physics at very small distances in terms of the noncommutative geometry. This amazing fact was discovered in string perturbation theory in [2] and in the study of the Matrix Theory in [3]. The relation between the noncommutative geometry and the string theory has been rigorously derived and made more precise in [4, 5, 6, 7]. The authors of [8] have found the zero slope limit in which the noncommutative Yang-Mills theory becomes a valid description of the open string sector of the string theory, established the relation between the conventional and the noncommutative gauge fields and found the stringy interpretation of the Morita equivalence [39]. Unfortunately, all the successful applications of noncommutative geometry to string theory have so far been limited to flat backgrounds, although the natural prescription for constructing the noncommutative algebra from the background with the $B$-field is known in mathematical literature [9, 10].

Putting a field theory on the noncommutative space effectively introduces the non-locality. This non-locality should presumably lead to softening some of the field theory divergences (as discussed, for example, in [11]). One very important example of divergences affected by the noncommutativity is the divergences in perturbation theory; the study of the effects of noncommutativity in Feynman diagrams has been initiated in [12]. The other example, directly related to our paper, is the resolution of singularities in the instanton moduli spaces. It was pointed out in [13] that introducing noncommutativity for the field theory on $\mathbb{R}^4$ is equivalent from the point of view of the instanton moduli space to considering the ADHM construction with the nonzero value of the Hyper-Kähler moment map. (In other words, it introduces the Fayet-Illiopoulos terms in the ADHM sigma-model). This discovery has an important application in the study of the string theory on Anti-deSitter space. It was conjectured in [14], that this theory has a dual description in terms of the conformal field theory. More precisely, the Type IIB string theory on $AdS_3 \times S^3 \times X$, where $X$ is $K3$ or $T^4$, is dual to the conformal sigma-model whose target space is the moduli space of instantons on $X$. This proposal, suggested in [14] and further studied in [15, 16, 17], is valid at the specific subspace of the moduli space. On this subspace, the fluxes of the NSNS $B$ field through the two-cycles of $X$ are zero. It was found in [18], that the non-zero fluxes of $B$ correspond to the non-zero value of the moment map in the ADHM construction. Therefore, the string theory on
$AdS_3 \times S^3 \times X$ with the nonzero $B$ field corresponds to the conformal sigma-model on the moduli space of instantons on the noncommutative $X$.

The purpose of this paper is to fix the precise relation between the parameters of the noncommutative SYM theory and the parameters of the near-horizon geometry. We will proceed in the following way. Wrapping $Q_4$ $D5$ branes on $X$, $Q_2$ $D3$ branes on the $i$th two-cycle of $X$, and adding $Q_0$ D-strings unwrapped, we end up having a black string in the $5+1$-dimensional flat space. The background $AdS_3 \times S^3 \times X$ arises in Type IIB as the near-horizon limit of this black string. The corresponding supergravity solution looks in a neighborhood of a given point in the $5+1$-dimensional space as $R^{5+1} \times X$, but the moduli of $X$ depend on the distance from the black string. For such a solution, let us consider the geometry of $X$ near the horizon (which we denote $X^h$) and $X$ far away from the string (we denote it $X^\infty$). Given $X^\infty$, one can find $X^h$, using the supergravity equations of motion. It turns out that $X^h$ cannot be arbitrary $K3$ (or $T^4$); the moduli of $X^h$ should satisfy certain attractor conditions [19]. On the other hand, the moduli of $X^\infty$ can be arbitrary. Therefore, the correspondence $X^\infty \mapsto X^h$ is not one-to-one; there is a whole family of $X^\infty$ flowing to a given $X^h$.

The CFT dual to the near-horizon supergravity can be found using the procedure suggested by Maldacena [14]. We look at the worldsheet theory of the corresponding system of branes. In the low energy limit, it becomes a conformal theory. According to the Maldacena conjecture, this is the conformal theory dual to $AdS$ supergravity. The worldsheet effective theory feels the flat background, the one which would be there without branes. The geometry of this background is $R^{1+5} \times X^\infty$. Therefore the moduli of the conformal theory depend on $X^\infty$. However, as we have already mentioned there are many possible $X^\infty$ which give the same near-horizon $X^h$. The low energy theories on the worldvolume of the black string should be equivalent for all the backgrounds $X^\infty$ flowing to the same $X^h$. This is a necessary condition for $AdS/CFT$ correspondence to be self-consistent. It was argued in [14], that the equivalence of the conformal theories corresponding to different $X^\infty$ flowing to the same $X^h$ follows from supersymmetry.

We want to ask the following question: given the family of flat backgrounds $X^\infty$ flowing to the same near-horizon background, can we find a ”canonical” representative, the one which gives the simplest description of the $D1D5$ worldsheet theory? At first sight, the most appealing possibility would be to find $X^\infty$ of a very large size. If we could do that, we would get the description of our CFT as the sigma model with the target space being the moduli space of Yang-Mills instantons. But it turns out that most of the attractors $X^h$ cannot be obtained as the near-horizon limit of the solutions with very large $X^\infty$. (In fact, there is an upper limit on the volumes of $X^\infty$ flowing to a given $X^h$). What we can do instead is to find a representative with very small $X^\infty$. For such backgrounds the low-energy effective action is formulated rather explicitly in terms of the noncommutative geometry. We can use the results of [8] relating the shape of $X^\infty$ and the $B$ field fluxes to the parameters of the noncommutative manifold which we will denote $\tilde{X}^\infty$. According to [8], the effective low-energy theory for the bound state of $D1$,
$D3$ and $D5$ is supersymmetric Yang-Mills theory on $\tilde{X}^{\infty} \times \mathbf{R}^{1+1}$. Therefore, the effective low-energy theory for the black string is 1 + 1-dimensional sigma-model with the target space the moduli space of instantons on $X^{\infty}$. (In fact, there is a subtlety here which we will discuss later). This gives the dual description of the near-horizon supergravity.

Let us mention an interesting consequence of this correspondence. It turns out that even if we restrict ourselves to the small-size $X^{\infty}$, there is still the whole family of them flowing to the same $X^h$. This gives us the family of noncommutative manifolds having the same moduli space of instantons. Indeed, the corresponding sigma-models should be equivalent; the equivalence of (4, 4) sigma-models implies that they have the same target spaces, modulo some discrete symmetries; but the equivalence classes turn out to be connected, therefore the possibility of discrete symmetries is excluded. It has been proven in \cite{8} that instantons on $\mathbf{R}^4$ depend only on the self-dual part of the noncommutativity parameter. Our method allows us to formulate the analogue of this statement for instantons on $T^4$.

The relation between the near-horizon geometry and the moduli of the dual CFT has been studied in great detail in the paper by R. Dijkgraaf \cite{20}. The precise correspondence between the parameters of the attractor and the moduli space of the dual CFT has been given in that paper. Combined with our paper, this gives the description of the geometry of the moduli space of instantons on the noncommutative torus.

Another application of our formulas is the study of the low-energy dynamics of D-branes wrapped on the torus of the finite size (of the order $l_6$). The most straightforward way of doing it would consist of two steps: 1) study the low-energy worldvolume theory of $D5$ branes wrapped on $T^4$ (with the numbers of $D1$ and $D3$ specifying the topological sector of this theory); 2) compactify this low energy theory on $T^4$, to get the low energy theory of the black string. However, we think that performing the first step is very hard for the finite size torus. We do not know how to construct the low-energy theory describing the slow degrees of freedom of $N > 1$ branes of finite size in the nontrivial topological sector. (The Nonabelian Born-Infeld Theory \cite{32} is valid when the covariant derivatives of the field strength are small, and this condition is not satisfied in our case). But we can bypass the first step and go directly to the low energy dynamics of the black string. Indeed, our considerations show that it is equivalent to the low energy dynamics of branes wrapped on some very small torus. Therefore, the low energy Lagrangian for the black string can always be represented as the sigma-model on the moduli space of noncommutative instantons.

There is a subtlety in the relation between the instanton sigma model and the low-energy supergravity on $AdS_3$. The statement that the low-energy dynamics of the $D1D5$ system is described by the sigma-model on the moduli space of instantons follows from the dimensional reduction of the 1 + 5-dimensional classical Super Yang-Mills theory on the noncommutative torus. Strictly speaking, we can treat this 1 + 5-dimensional theory classically only if it is weakly coupled on the compactification scale. On the other hand, classical $AdS$ supergravity is valid only if the radius of curvature of $AdS$ space is
large enough. It turns out that these two conditions are incompatible: we cannot trust dimensional reduction in the regime where AdS supergravity is valid. However, we conjecture that the shape of the target-space of the sigma-model describing the dynamics of the $D1D5$ system is independent of the string coupling constant. We give two arguments confirming this conjecture. The first argument is based on considering the dependence of the sigma-model for the $D1D5$ system on the parameters of the background. Our considerations in Section 3 will imply that the structure of equivalence classes of backgrounds giving the same sigma-model does not depend on $g_{str}$. Although this fact does not necessarily imply that the structure of the target space does not dependent of $g_{str}$, we think that it supports the conjecture. The second argument is based on the relation between the world-sheet instantons of the $D1D5$ sigma model and the D-instantons of the six-dimensional supergravity. An example of this relation was given in [25]. Instantons in toroidally compactified string theories where considered in [22, 23, 24]. We argue that under certain conditions there is a correspondence between these two kinds of instantons, and the action of the world-sheet instantons is equal to the action of the supergravity instantons. The action of the worldsheet instantons is related to the Hyper-Kähler period map of the target space. And the action of the supergravity instantons can be found when $g_{str}$ is small from supergravity. The formula agrees with the period map for the target space conjectured in [20] and does not contain $N$. This suggests that the target space for small $g_{str}N$ and large $g_{str}N$ should have the same period map. Therefore the Hyper-Kähler structure of the target space is the same in the regime when we can use dimensional reduction and in the regime when we can trust supergravity.

It was claimed in [21] that the moduli space of the twisted little string theories on $T^3$ is the moduli space of noncommutative instantons on $T^4$. This result was derived by relating the theory on $NS5$ brane to the $D2D6$ system, with the $D6$-brane wrapped on the four-torus. Strictly speaking, $D2D6$ system on $T^4$ is described by the noncommutative Yang-Mills theory only when this four-torus is very small. However, the situation with the $D2D6$ system should be similar to what happens with $D1D5$: namely, the moduli space does not depend on some combination of the background fields, and there is a small torus in each equivalence class. Therefore, the moduli space is always the moduli space of noncommutative instantons.

The paper is organized as follows. In the second section, we give a brief review of the attractor equations and explain when the supergravity approximation can be trusted. In the third section, we give the argument for independence of the sigma-model on certain combinations of the background fields, based on supersymmetry. This section is auxiliary, and it is not necessary for understanding the rest of the paper. Our arguments based on supersymmetry should be closely related to the argument given in [30], but we have tried to make them more precise. In the fourth section, we derive the relation between the asymptotic background and the near-horizon (attractive) background. This section is somewhat technical. The main formula is (59), giving the condition for the two backgrounds to flow to the same attractor. The formula (63) expresses the near-horizon...
Ramond-Ramond fields in terms of the Ramond-Ramond fields at infinity. In the fifth section, we review the correspondence between the string theory on the small torus and the Yang-Mills theory on the noncommutative torus. Combining these results with the results from the fourth section, we construct the flows on the moduli space of the non-commutative tori, which leave the moduli space of instantons invariant. In Subsection 5.2 we make an observation which suggests that the moduli space of instantons on noncommutative $\mathbb{R}^4$ does not receive $\alpha'$-corrections. In the sixth section we discuss the relation between the world-sheet instantons and the supergravity instantons. In Appendix A, we review the correspondence between the moduli of the Type IIB compactification and the points of Grassmanian.

2 Attractor Equation.

2.1 Supergravity Solution.

The dependence of the moduli of K3 or $T^4$ on the distance from the horizon is given by the “attractor equation”. This equation can be found in many papers, for example in [26], although in somewhat implicit form. In this section, we will briefly review the attractor equations and present them in the form most convenient for our purposes.

We will concentrate on the case $X = K3$. First of all, we want to review some properties of the chiral $N = 4$ $D = 6$ supergravity. We will follow the original paper [27] but use slightly different language. This theory describes an interaction of $N = 4b$ gravity multiplet with $n$ tensor multiplets. For the compactification of Type IIB on $K3$, we have $n = 21$. The gravity multiplet contains a graviton, four left-handed gravitini and five antisymmetric tensors with selfdual field strength. The tensor multiplet consists of an antisymmetric tensor with antiselfdual field strength, four right-handed symplectic Majorana spinors and five scalars. To describe the interacting theory, we think of the scalars as parametrizing the coset

$$Gr(5, 5 + 21) = \frac{SO(5, 21)}{SO(5) \times SO(21)}$$

We represent this coset as the manifold of five-dimensional positive planes in the space

$$L = \mathbb{R}^{5+21}$$

For a given five-plane $W \subset L$, we denote $P_+$ the projector on this plane, and $P_-$ the projector on the orthogonal plane. We denote the $1+5$-dimensional Minkowski space $M$. The field strengths of the gravity multiplet and the tensor multiplets may be combined in a vector $H \in \Omega^5 M \otimes L$, with the constraint

$$* H = (P_+ - P_-) H$$

(1)
Instead of writing $P_+H$ and $P_-H$, we will often write $H_+$ and $H_-$. Let $S^{1/2}M$ be the bundle of antichiral spinors on $M$, and $S^{3/2}M$ the bundle of Rarita-Schwinger fields (i.e., chiral spinors with vector indices). Also, we denote $\mathcal{W}_+$ the “tautological” bundle on the Grassmanian $Gr(5, 5 + 21)$, that is the bundle whose fiber over the point represented by the plane $W$ is the plane $W$ itself. Similarly, $\mathcal{W}_-$ will denote the bundle whose fiber is $W^\perp$. The gravitino $\psi_\mu$ and the fermions $\chi$ from the tensor multiplets are the sections of the following bundles:

$$\psi \in \Gamma([S^{3/2}M \otimes S(\mathcal{W}_+)]_R)$$
$$\chi \in \Gamma([S^{1/2}M \otimes S(\mathcal{W}_+)]_R \otimes \mathcal{W}_-)$$

Here $S(\mathcal{W}_+)$ is the spinor bundle associated with the $Spin(5)$ vector bundle over $W_+$ (we will call its sections “internal spinors”). The subindices $R$ mean that some Majorana conditions are imposed. Let us explain the Majorana conditions for $\psi$. $(S^{3/2}_+M \otimes S(\mathcal{W}_+))_R$ is generated by the expressions of the form $\sum_a s_a \otimes \phi_a$, where $s_a \in S^{3/2}_+M$, $\phi_a \in S(\mathcal{W}_+)$, with the reality condition

$$\sum_a s_a \otimes \phi_a = \sum a C_{(1)} s_a^T \otimes C_{(2)} \bar{\phi}_a^T$$

We have denoted $C_{(1)}$ and $C_{(2)}$ the charge conjugation matrices for the space-time spinors and the internal spinors, respectively. The Majorana condition for $\chi$ is defined in the same way. The supersymmetry generators $\epsilon$ are space-time and internal spinors:

$$\epsilon \in \Gamma([S^{1/2}_+M \otimes S(\mathcal{W}_+)]_R)$$

The supersymmetry transformations for the gravitino and $\chi$ are:

$$\delta_\epsilon \psi_\mu = D_\mu \epsilon + \frac{1}{4} \kappa_0 (H_{+\mu\nu\rho}).(\gamma^{\mu\nu\rho} \epsilon),$$
$$\delta_\epsilon \chi = -\frac{1}{2} \gamma^\mu (\partial_\mu \phi). \epsilon - \frac{1}{12} \kappa_0 \gamma^{\mu\nu\rho} H_{-\mu\nu\rho} \otimes \epsilon$$

Here $\kappa_0$ is the asymptotic value of the six-dimensional string coupling constant, $\kappa_0 \sim \frac{g_{str}}{\sqrt{V_K}}$ (we will not need the precise expression for it). In the transformation law for $\psi$, the vector $H_{+\mu\nu\rho} \in \mathcal{W}_+$ acts on $S(\mathcal{W}_+)$ index of $\epsilon$ as gamma-matrices act on spinors. $D_\mu$ is a covariant derivative, which includes the natural connection on $\mathcal{W}_+$. In the expression for $\delta_\epsilon \chi$, $\phi$ denotes the point in the moduli space of scalars. Its derivative $\partial_\mu \phi$ acts on $\epsilon$ as follows. There is a natural isomorphism between the tangent bundle to $Gr(5, 5 + 21)$ and $\mathcal{W}_- \otimes \mathcal{W}_+$. Indeed, a tangent vector to $Gr(5, 5 + 21)$ is an infinitesimal rotation $\delta W$ of the plane $W$, and it is completely specified by saying what is $\delta w \in W^\perp$ for each vector $w \in W$. This gives us a map from $W$ to $W^\perp$, or a section of $\mathcal{W}_- \otimes \mathcal{W}_+^*$. Since $L$ is

\[\text{The moduli space of scalars is topologically trivial (the global coordinates may be found, for example, in [28]). Therefore, there is no difference between the Spin(5) bundles and the SO(5) bundles.}\]
equipped with a metric, we can identify $W_+^*$ with $W_+$. This means, that we can think of $d\phi$ as an element of $\mathcal{W}_- \otimes \mathcal{W}_+ \otimes \Omega^1 \mathcal{M}$. Then, to act on $\epsilon \in \Gamma([S^1_{1/2} \otimes S(\mathcal{W}_+)]_{\mathbb{R}})$ by $\partial_\mu \phi \in \mathcal{W}_- \otimes \mathcal{W}_+$, we use the action of $\mathcal{W}_+$ on $S(\mathcal{W}_+)$ (gamma-matrices act on spinors) and tensor multiplication by $\mathcal{W}_-$. After acting by the space-time gamma-matrix $\gamma^\mu$ we get a section of the bundle $(S^1_{1/2} \otimes S(\mathcal{W}_+))_{\mathbb{R}} \otimes \mathcal{W}_-$, which is where $\chi$ lives.

We want to find a supersymmetric background, corresponding to our black string. This means that for some parameters $\epsilon$, we should have $\delta_\epsilon \psi = 0$ and $\delta_\epsilon \chi = 0$. For our solution, there will be eight linearly independent $\epsilon$ satisfying these equations. We will try the following ansatz for the metric:

$$ds^2 = e^{2U(r)}(dt^2 - dx^2) - e^{-2U(r)}(dr^2 + r^2 d\Omega^2_3) \quad (5)$$

Let us introduce the vielbein $\{e^t, e^x, e^r, e^1, e^2, e^3\}$, so that

$$ds^2 = (e^t)^2 - (e^x)^2 - (e^r)^2 - 3 \sum_{i=1}^{3} (e^i)^2$$

We use the following ansatz for the three-form field strength:

$$H_{ijk} = \frac{e^{3U}}{r^3} \epsilon_{ijk} Z,$$

$$H_{\pm xtr} = \pm \frac{e^{3U}}{r^3} Z_{\pm} \quad (6)$$

The vector $Z$ should be integer-valued, $Z \in \Gamma^{5,21} \subset L$, because the fluxes of $H$ through $S^3$ are integers. This vector should be identified with the charge of the black string. We look for supersymmetry with the parameter $\epsilon$ which depends only on $r$. Also, we assume that the scalars depend only on $r$. First, let us write explicitly the transformations of various components of $\psi_\mu$. We will denote $\nabla_\mu$ the covariant derivative on $\mathcal{W}_+$. We do not need to know explicitly what is the spin connection in $\mathcal{W}_+$, only the space-time spin-connection will be important for us. We have the following conditions for $\delta \psi = 0$ (with $U' = \frac{dU}{dr}$):

$$\delta \psi_t = \frac{1}{2} U' e^U \gamma^t \gamma^r \epsilon + \frac{1}{2} \kappa_0 \frac{e^{3U}}{r^3} \gamma^t \gamma^r Z_{+..} \epsilon = 0$$

$$\delta \psi_x = -\frac{1}{2} U' e^U \gamma^x \gamma^r \epsilon - \frac{1}{2} \kappa_0 \frac{e^{3U}}{r^3} \gamma^t \gamma^r Z_{+..} \epsilon = 0$$

$$\delta \psi_r = \nabla_r \epsilon + \frac{1}{2} \kappa_0 \frac{e^{3U}}{r^3} \gamma^t \gamma^r Z_{+..} \epsilon = 0 \quad (7)$$

$$\delta \psi_i = \frac{1}{2} U' e^U \gamma^i \gamma^r \epsilon + \frac{1}{4} \kappa_0 \frac{e^{3U}}{r^3} \epsilon_{ijk} \gamma^j \gamma^k Z_{+..} \epsilon = 0$$

Let us denote

$$\hat{Z}_+ = \frac{Z_+}{\sqrt{||Z_+||^2}}$$
Then, it follows from the first equation in (7), that
\begin{equation}
U' = \frac{e^{2U}}{r^3} \sqrt{\kappa_0^2 |Z_+|^2}, \\
(\hat{Z}_+ - \gamma^x \gamma^t)\epsilon = 0
\end{equation}
(8)
The second row in (7) follows from (8), and the third gives the equation for the radial dependence of \( \psi \):
\begin{equation}
\nabla_r \psi(r) = \frac{1}{2} \kappa_0 \frac{e^{3U}}{r^3} \psi(r)
\end{equation}
(9)
This is consistent with \((\hat{Z}_+ - \gamma^x \gamma^t)\epsilon = 0\) because for our solution \( \hat{Z}_+ \) will be covariantly constant (that is, \( \nabla_r \hat{Z}_+(r) = 0 \)). The last equation in (7) follows from (8) if one takes into account that \( \epsilon \) is chiral.

Now we turn to the variation of the fermions from tensor multiplets. We have:
\begin{equation}
\delta \chi = - \frac{1}{2} e^U \gamma^r (\partial_r \phi).\epsilon - \frac{e^{3U}}{r^3} \kappa_0 \gamma^1 \gamma^2 \gamma^3 Z_- \otimes \epsilon = 0
\end{equation}
(10)
Taking into account the chirality condition \( \gamma^{123} \epsilon = - \gamma^{txr} \epsilon \) and equations (8) one can see that \( \delta \chi = 0 \) if \( \phi \) satisfies the following equation:
\begin{equation}
d\phi = -2dU \frac{1}{||Z_+||^2} Z_- \otimes Z_+
\end{equation}
(11)
This equation has a very clear geometrical meaning. Remember that we represent tangent vectors to the moduli space of scalars as linear maps from \( W \) to \( W^\perp \), giving the variations of vectors in \( W \). Suppose that we took a vector \( w \in W \) which is orthogonal to \( Z \). Then \( (Z_+ \cdot w) = 0 \) and (11) tells us that the variation of \( w \) is zero. Therefore, the subspace \( W \cap Z^\perp \subset W \) remains constant. We can represent \( W = (W \cap Z^\perp) \oplus \mathbb{R}Z_+ \). When \( r \) changes (\( W \cap Z^\perp \)) remains constant and the one-dimensional subspace \( \mathbb{R}Z_+ \) gets rotated in the plane \( \mathbb{R}Z_+ \oplus \mathbb{R}Z_- \):
In particular, we see that the near-horizon background is represented by \((W \cap Z^\perp) \oplus \mathbb{R}Z\).

This is a qualitative picture. Now we want to write the explicit formula for \(\phi(r)\). Let us introduce the hyperbolic angle \(\theta\) between the vector \(Z\) and the plane \(W\). Let \(\hat{Z}_+ = (||Z_+||^2)^{-1/2}Z_+\) and \(\hat{Z}_- = (-||Z_-||^2)^{-1/2}Z_-\). Then,

\[
\hat{Z} = \cosh \theta \hat{Z}_+ + \sinh \theta \hat{Z}_- \tag{12}
\]

(this is the definition of \(\theta\)) and

\[
d\hat{Z}_+ = -2 \tanh \theta dU \hat{Z}_- \tag{13}
\]

(this is \(d\phi\) from \((11)\) applied to the vector \(\hat{Z}_+ \in W_+\)). Combining these two equations, we have

\[
d\theta = 2 \tanh \theta dU \tag{14}
\]

This equation together with the second equation from \((8)\) determines the dependence of \(U\) and \(\theta\) on \(r\):

\[
U' = \frac{e^{2U}}{r^3} \sqrt{\kappa_0^2 ||Z||^2} \cosh \theta, \\
\theta' = 2 \frac{e^{2U}}{r^3} \sqrt{\kappa_0^2 ||Z||^2} \sinh \theta \tag{15}
\]

Adding and subtracting these two equations, we get:

\[
\frac{d}{dr}(2U \pm \theta) = 2 \sqrt{\kappa_0^2 ||Z||^2} \frac{e^{2U \pm \theta}}{r^3} \tag{16}
\]
The result of integration depends on two constants, $c_+$ and $c_-$:

\[
e^{-2U} = \left( c_+ + \frac{\kappa_0 \sqrt{||Z||^2}}{r^2} \right)^{1/2} \left( c_- + \frac{\kappa_0 \sqrt{||Z||^2}}{r^2} \right)^{1/2}
\]

\[
e^\theta = \left( \frac{\kappa_0 \sqrt{||Z||^2 + c_- r^2}}{\kappa_0 \sqrt{||Z||^2 + c_+ r^2}} \right)^{1/2}
\]

(17)

We want $U = 0$ at spatial infinity, therefore $c_+ c_- = 1$. The ratio $c_+/c_-$ is related to the asymptotic value of moduli. This gives:

\[c_\pm = e^{\mp \theta_\infty}\]  

(18)

**Two Examples.**

1) The system of $Q_0$ $D$-strings and $Q_4$ $D5$-branes with $B = 0$. (The corresponding supergravity solution has been found in [29]). The plane $W$ is generated by three vectors orthogonal to $Z$ and the vector $(1, 0, V)$. In this situation only the volume of $K3$ changes with $r$, the shape remains fixed. We have:

\[
\left( \frac{1}{\sqrt{V}}, \sqrt{V} \right) = \cosh \theta \left( \sqrt{\frac{Q_4}{Q_0}}, \sqrt{\frac{Q_0}{Q_4}} \right) + \sinh \theta \left( \sqrt{\frac{Q_4}{Q_0}}, -\sqrt{\frac{Q_0}{Q_4}} \right)
\]

(19)

Therefore,

\[e^{\theta(r)} = \sqrt{\frac{Q_4 V(r)}{Q_0}}, \quad c_+ = \sqrt{\frac{Q_4 V_\infty}{Q_0}}\]

(20)

Now, (17) gives

\[
e^{-2U(r)} = \left( 1 + \sqrt{2V_\infty \kappa_0 \frac{Q_4}{r^2}} \right)^{1/2} \left( 1 - \sqrt{2V_\infty \kappa_0 \frac{Q_0}{r^2}} \right)^{1/2}
\]

\[V(r) = V_\infty \frac{1 + \sqrt{2V_\infty \kappa_0 \frac{Q_0}{r^2}}}{1 + \sqrt{2V_\infty \kappa_0 \frac{Q_4}{r^2}}}\]

(21)

2) Although we will not consider adding symmetric branes and fundamental strings in this paper, we will now give the corresponding solution as an example. Consider a system of $q_0$ fundamental strings and $q_4$ $NS5$ branes wrapped on $X$. Using the definition (12) of $\theta$ and formula (125) from Appendix A, we have the equation for $\theta(r)$:

\[
\left( \kappa(r), \frac{1}{\kappa(r)} \right) = \cosh \theta(r) \left( \sqrt{\frac{q_4}{q_0}}, \sqrt{\frac{q_0}{q_4}} \right) + \sinh \theta(r) \left( \sqrt{\frac{q_4}{q_0}}, -\sqrt{\frac{q_0}{q_4}} \right)
\]

(22)
This gives \( c_+ = \frac{1}{\kappa_0} \sqrt{\frac{2q_4}{q_0}} \) and the following dependence of the metric and the coupling constant on \( r \):

\[
e^{-2U(r)} = \left( 1 + \frac{\sqrt{2}q_4}{r^2} \right)^{1/2} \left( 1 + \kappa_0^2 \sqrt{2q_0} \right)^{1/2}
\]

\[
\kappa^2(r) = \frac{\kappa_0^2}{r^2 + \sqrt{2q_0^2 q_0}}
\]

\[(23)\]

**Lagrangian Interpretation.**

The attractor equations have a Lagrangian interpretation. Indeed, let us introduce the “time” \( \tau = -\frac{1}{r^2} \). The attractor equations (15) imply that:

\[
\frac{d^2 U}{d\tau^2} = \frac{1}{2} \kappa_0^2 ||Z||^2 e^{4U} (\cosh^2 \theta + \sinh^2 \theta),
\]

\[
\frac{d^2 \theta}{d\tau^2} = 2 e^{4U} \kappa_0^2 ||Z||^2 \sinh \theta \cosh \theta
\]

These second order equations may be derived from the following Lagrangian:

\[
\mathcal{L} = \frac{1}{4} \left( \frac{d\theta}{d\tau} \right)^2 + \left( \frac{dU}{d\tau} \right)^2 + \frac{\kappa_0^2 ||Z||^2}{4} e^{4U} (\cosh^2 \theta + \sinh^2 \theta)
\]

\[(24)\]

The “potential energy” is

\[
E_{pot} = -\frac{1}{4} \kappa_0^2 e^{4U} (||P_+ Z||^2 - ||P_- Z||^2)
\]

\[(26)\]

The expression \( ||P_+ Z||^2 - ||P_- Z||^2 = 2||P_+ Z||^2 - ||Z||^2 \) is, up to the coefficient and the constant term, the square of the tension of the black string. At the attracting point, that is when \( P_+ Z = Z \), this tension reaches its minimum. Also, we can see from our solution \((17)\), that \( e^{4U} \) is zero at the horizon. Therefore, the “motion” in \( \tau \), corresponding to the geometry of the black string, may be considered as “climbing” up the potential. The initial velocity should be adjusted in such a way that getting to the top requires infinite time.

In what follows, we will not need the full solution for \( U(r) \) and \( W(r) \). The only thing important for us is that \( W \) changes with \( r \) in such a way that \( W(r) \cap Z^\perp \) remains constant, and also that the near-horizon \( W \) is \( W^h = (W \cap Z^\perp) \oplus \mathbb{R}Z \).

**2.2 When can we use the supergravity approximation?**

We want to be able to use supergravity equations until we reach small enough values of \( r \), where the moduli are close to their fixed values. The low energy effective action does not receive any quantum corrections because of the supersymmetry. Therefore, the supergravity approximation is valid when we can neglect higher derivative corrections. We
will assume that the near-horizon string coupling is weak, therefore the main source of higher derivative corrections is the string perturbation theory. These corrections behave like a positive power of $\frac{l_{str}}{r}$ where $r$ is the typical radius of curvature. Therefore, the condition that supergravity is applicable is that the size of the AdS region is much larger than $l_{str}$. Let us explain what we mean by the “size of the AdS region”. The metric for the $D1D5$ solution contains factors of the form

$$\left(1 + \kappa_0 e^{-\theta_{\infty}} \sqrt{||Z||^2} \right) \left(1 + \kappa_0 e^{\theta_{\infty}} \frac{||Z||^2}{r^2} \right)$$

We are in AdS regime when we can approximate this expression as $\frac{\kappa_0^2 ||Z||^2}{r^4}$. Assuming $\theta_{\infty} > 0$, we see that the AdS regime starts at

$$r^2 < r_{cr}^2 = \kappa_0 e^{-\theta_{\infty}} \sqrt{||Z||^2}$$

(27)

The condition for supergravity to be valid is that $r_{cr} \gg l_{str}$. Now let us estimate $l_{str}$ in the near-horizon region. It is convenient to start with estimating the size of the core of the $D$-string. For the typical attractor, all the moduli of the near-horizon torus are of the order one, therefore all the black $D$-strings (for example, $D1$, $D3$ wrapped on a two-cycle and $D5$ wrapped on $T^4$) have approximately the same string length and the same size of the core. For the typical D-string in the typical background the size of the core is

$$r_{core, D}^2 \sim g_{str} l_{str}^2$$

(28)

As a typical D-string, let us choose the string with the charge $\frac{Z}{\sqrt{||Z||^2}}$. This charge is usually not integer, but we need only the supergravity solution. It has the metric with the characteristic factor

$$\left(1 + \kappa_0 e^{-\theta_{\infty}} \left[ \frac{\sqrt{||Z||^2}}{r^2} + \frac{1}{(r^2 - \bar{r}_0)^2} \right] \right) \left(1 + \kappa_0 e^{\theta_{\infty}} \left[ \frac{\sqrt{||Z||^2}}{r^2} + \frac{1}{(r^2 - \bar{r}_0)^2} \right] \right)$$

(29)

One can see from this expression that the size of the core for the D-string at the coordinate distance $r_0$ from the origin is of the order

$$r_{core, D}^2 \sim r_0^2 \frac{1}{\sqrt{||Z||^2}}$$

(30)

This equation together with (28) implies

$$l_{str}^2 (r_0) \sim \frac{r_0^2}{\sqrt{||Z||^2} g_{str} (r_0)}$$

(31)
Since $\kappa = \text{const}$ for the D-string, we have

$$g_{\text{str}}(r_0) = \sqrt{\frac{V(r_0)}{V_\infty}} g_{\text{str}} \sim \frac{g_{\text{str}}^\infty}{\sqrt{V_\infty}}$$

This implies that for $r_0 \sim r_{cr}$ we have:

$$\frac{r_{cr}^2}{l_{\text{str}}^2(r_{cr})} \sim \frac{g_{\text{str}}^\infty \sqrt{||Z||^2}}{\sqrt{V_\infty}}$$

(Remember that we measure all the lengths for the torus in string units). The right hand side of this expression has a clear physical meaning. It is the t’Hooft coupling constant for the $1+5$-dimensional Super Yang-Mills on $\mathbb{R}^{1,1} \times \tilde{T}^4$ on the scale of the size of $\tilde{T}^4$. (Here $\tilde{T}^4$ is the noncommutative torus “dual” to the torus $T^4$ at infinity as explained in Section 5). Our estimate (33) tells us that in the regime when we can trust the supergravity solution in the AdS region, the six-dimensional noncommutative Yang-Mills describing the $D1D5$ system Moduli pois necessarily strongly coupled on the compactification scale. Therefore, we cannot trust the classical dimensional reduction. We conjecture that the shape of the target space of the sigma-model on $\mathbb{R}^{1,1}$ which is the dimensional reduction of this six-dimensional theory actually does not depend on the six-dimensional coupling constant. We will give some evidence confirming this conjecture in Section 6.

3 Argument based on supersymmetry.

In this section we will prove using supersymmetry that two backgrounds flowing to the same attractor give the same low energy effective theory on the worldvolume of the black string. Our argument is very close to the one used in [31] to prove independence of the hypermultiplet metric on the scalars in the vector multiplet.

We will study some auxiliary brane configuration. It consists of the system of $N$ parallel black strings, all having the same charge $Q$. This system has a Coulomb branch. Let us consider the submanifold of this Coulomb branch, where $N - 1$ black strings sit at one point (the origin), and one is moving around:
We will call that black string which is moving around the “single black string”. We want to look at the worldsheet theory of this single black string. The low-energy worldsheet theory is some sigma model. Consider the target space $\hat{\mathcal{M}}$ of this sigma-model. It can be parametrized by the position in the transversal $\mathbb{R}^4$ plus the internal degrees of freedom, which specify the instanton on $T^4$. Notice that the size of the moduli space of instantons on $T^4$ is of the order $\sqrt{V_{\text{inst}}}$ and does not depend on $N$, while the characteristic distances in the transversal $\mathbb{R}^4$ are growing with $N$. (From our solution, we see that $R^2 \sim N \sqrt{||Z||^2}$.) Therefore, in the limit $N \to \infty$, we can consider the internal degrees of freedom as “fast”, and the motion along $\mathbb{R}^4$ as “slow”. Moreover, in this limit the single black string sitting at some point $x \in \mathbb{R}^4$ does not feel that the background changes with the distance from the origin. Therefore, the configuration space of the internal degrees of freedom should be approximately the same Hyper-Kähler manifold as for the black string in flat space. In other words, in the limit $N \to \infty$ we can think of $\hat{\mathcal{M}}$ as a bundle with the base $\mathbb{R}^4$ and the fiber the moduli space $\mathcal{M}(x)$ of internal degrees of freedom of $D1D5$ wrapped on $T^4(x)$:

$$\hat{\mathcal{M}} \xrightarrow{\sim} \mathbb{R}^4$$

The dependence of $T^4(x)$ on the point $x \in \mathbb{R}^4$ (and therefore the dependence of $\mathcal{M}(x)$ on $x$) is dictated by the supergravity solution for $N - 1$ black strings sitting at the origin.

Our arguments will go as follows. We will prove, using supersymmetry, that although the moduli of the torus $T^4$ depend on the distance from the origin, the fiber $\mathcal{M}$ does not. But in the limit $N \to \infty$, the fiber $\mathcal{M}(x_0)$ over the given point $x_0 \in \mathbb{R}^4$ is the same as the target space $\mathcal{M}_0$ for the $D1D5$ in the flat background with the torus $T^4(x) \equiv T^4(x_0)$. Therefore, independence of $\mathcal{M}(x_0)$ on $x_0$ implies that the target space for $D1D5$ is the same for all the backgrounds $T^4 = T^4(x_0), x_0 \in \mathbb{R}^4$. Since $T^4(x_0)$ depends only on the distance $r$ of $x_0$ from the origin, we had proven that the moduli space of $D1D5$ is the same for the backgrounds from the given one-parametric family $T^4(r), r \in [0, \infty]$. In particular, it is the same for $T^4(r = 0)$ and $T^4(r = \infty)$. This implies, that all the backgrounds at infinity $r = \infty$ which flow to the same near-horizon $T^4(r = 0)$ give the same moduli space.

Let us proceed with the proof. We want to understand what kind of restrictions the supersymmetry imposes on the geometry of our target space $\hat{\mathcal{M}}$. It is useful to start with considering the “flat” sigma-model, which describes the dynamics of $D1D5$ in flat space (without $N - 1$ strings at the origin). In this case, we would have:

$$\hat{\mathcal{M}}^0 = \mathbb{R}^4 \times \mathcal{M}_0$$

where $\mathcal{M}_0$ does not depend on the point of the base. Both $\mathbb{R}^4$ and $\mathcal{M}_0$ are Hyper-Kähler, and therefore we get the system with $(4, 4)$ supersymmetry. In fact, there are two Hyper-Kähler structures on $\mathbb{R}^4$: one with the self-dual Hyper-Kähler forms, and the other with anti-self-dual. Closer examination of the supersymmetry transformations shows that they are different in the left- and right-moving sectors: the left-moving $(4, 0)$ supersymmetry
uses self-dual complex structures on $\mathbb{R}^4$, and the right-moving one uses anti-self-dual complex structures.

When we introduce $N-1$ black strings at the origin the flat target space gets corrected; for example, the metric on $\mathbb{R}^4$ is not flat anymore. Nevertheless, our manifold $\hat{\mathcal{M}}$ should still support a sigma-model with $(4,4)$ supersymmetry. This follows from the fact that our single black string preserves the same supersymmetry as those $N-1$ black strings which we have added. Therefore, introducing these $N-1$ black strings does not break any more supersymmetry. The $(4,4)$ supersymmetry implies the existence of six complex structures, three for the left-moving sector, and three for the right-moving \cite{31}. The sigma-model action can include torsion, which is an antisymmetric tensor $b$ with $db \neq 0$.

The lagrangian is:

$$L = g_{\alpha\beta} (dX^\alpha \cdot dX^\beta) + 2g_{aa} (dX^\alpha \cdot dX^a) + g_{ab} (d\Phi^a \cdot d\Phi^b) +$$
$$+ b_{\alpha\beta} dX^\alpha \wedge dX^\beta + 2b_{aa} dX^\alpha \wedge d\Phi^a + b_{ab} d\Phi^a \wedge d\Phi^b \quad (35)$$

Here we denote $X^\alpha$ the coordinates in $\mathbb{R}^4$, and $\Phi^a$ the coordinates in the fiber (the moduli space of instantons). The complex structures should be covariantly constant, but with respect to the modified connection, with the torsion $T = db$ added. The sign of the torsion is opposite for the left and the right connections. We will write this condition explicitly for the corresponding Kähler forms:

$$\nabla^I \omega^I_{\mu\nu} \pm T^\sigma_{\mu\nu} \omega^I_{\sigma\mu} \pm T^\sigma_{\mu\nu} \omega^I_{\sigma\nu} = 0 \quad (36)$$

(the index $I = 1, 2, 3$ distinguishes between the three different Kähler forms, and the sign $\pm$ distinguishes between the right and the left sectors). In our situation, we know something about these six complex structures from comparison to the “flat” case. We know that in the vicinity of a given fiber, the Kähler forms are:

$$\omega^I = \omega^I_{(0)ab} dX^\alpha \wedge dX^\beta + \omega^I_{(0)ab} d\Phi^a \wedge d\Phi^b + \text{small corrections} \quad (37)$$

Here $\omega^I_{(0)ab}$ are the basic self-dual (with plus) or anti-self-dual (with minus) forms on $\mathbb{R}^4$, and $\omega^I_{(0)ab}$ are the Kähler forms on flat $\mathcal{M}_0$. The corrections are small, because locally our background is almost flat\footnote{The left Kähler forms on $\mathcal{M}_0$ coincide with the right Kähler forms, therefore it might seem strange that we are using the superscript $\pm$ to distinguish between $\omega^I_{(0)ab}$ and $\omega^I_{(0)\alpha\beta}$. The reason why we distinguish them is the following. The forms $\omega^I_{(0)ab}$ should be covariantly constant (with modified connection). In particular, they should be covariantly constant along the radial direction in $\mathbb{R}^4$. Although the three-dimensional space generated by $\omega^I_{(0)ab}$ and $\omega^I_{(0)\alpha\beta}$ coincide, we do not apriori exclude the possibility that the covariantly constant bases in these spaces are different in left and write sectors.}

We also have some information about the torsion. First of all, the torsion is proportional to the background Ramond-Ramond fields. Indeed, the corresponding term in the black string effective action violates parity. In the brane picture, parity turns a black
string into an anti-black string. It is the same as changing the sign of the Ramond-Ramond fields. Therefore, the torsion should be zero if the Ramond-Ramond fields are turned off. Also, we know that in flat case the torsion is zero \[.\] Now we want to see what happens to the torsion when we put \(N-1\) black strings at the origin in \(\mathbb{R}^4\). First of all, the components of the torsion along \(\mathbb{R}^4\) become nonzero. For example, \(D5\) branes from the pile at the origin create \(B_{RR}\), which couples to \(D1\) from the single black string. Closer examination shows that \(T_{ijk} \sim N||Z||^2\epsilon_{ijk}\) and the other components in \(\mathbb{R}^4\) are zero. Also, we should admit the possibility that the component \(T_{abr}\) is not zero any more. Indeed, the Ramond-Ramond fields along \(T^4\) change when we move in the radial direction. This implies that theta-angles on \(\mathcal{M}\) may change when we move in \(\mathbb{R}^4\). This precisely means that \(T_{abr} \neq 0\). On the other hand, we expect that \(T_{abc}\) is still zero, or at least very small when \(N \to \infty\). Indeed, we have seen that \(T_{abc}\) is zero when we put our black string in flat background. The non-flatness of the background can be locally described by the gradient of the background fields, which we can schematically denote \(\partial_\theta\) (\(\theta\) denotes the background fields). From the rotational symmetry, and assuming the analytic dependence of the torsion on the background fields, we estimate:

\[
T_{abc} \sim g^{\alpha\beta} \partial_\alpha \theta \partial_\beta \theta
\]

In other words, it is of the second order in the gradient of the background fields. Therefore we will neglect it. Also, the other components of the torsion (for example, \(T_{jia}\), where \(i\) and \(j\) are indices from the tangent space to \(S^3\)) are zero because of the rotational symmetry of \(\mathbb{R}^4\).

Now we want to consider the variation of the period map (the integrals of the Kähler forms over the two-cycles in \(\mathcal{M}\)), when we move in the radial direction. Let us consider two points, one at the distance \(r_1\) from the origin, and the other at the distance \(r_2\) (both on the same line with the origin). Consider the cycle \(c(r_1)\) in \(\mathcal{M}\) at the first point, and the same cycle \(c(r_2)\) in \(\mathcal{M}\) at the second point. We have, according to the Stokes theorem,

\[
\int_{c(r_2)} \omega^I_{\pm} - \int_{c(r_1)} \omega^I_{\pm} = \pm \int_{\partial \mathcal{I}[r_1,r_2]} dx^\lambda \wedge dx^\mu \wedge dx^\nu T_{[\lambda\mu\nu]^\pm}^\rho \]

The leading term on the right hand side is

\[
\pm \int dr \int_{c(r)} d\Phi^a \wedge d\Phi^b T_{r[a\omega^I_{\pm}]}^c \]

We know that the space of periods over the fiber of \(\omega^I_{\pm}\) coincides with that of \(\omega^I^-\), modulo small corrections. On the other hand, we see from \((\ref{periods})\) that the change in the periods

\[\text{Indeed, the components of the torsion along } \mathbb{R}^4 \text{ are not allowed in flat case because of } SO(4) \text{ invariance. The components of the torsion along } \mathcal{M}_0, T_{abc}, \text{ are also zero for the following reason. In the flat case, the only Ramond-Ramond fields are the constant fields along the torus. We know how they couple to our } D1D5 \text{ system from supergravity. They couple to the topological numbers. Therefore, the corresponding contribution to the black string effective action cannot change under the small deformations of the fields. In other words, these constant Ramond-Ramond fields generate theta-angles in our sigma-model, but not a torsion.} \]
of $\omega^J$ and the periods of $\omega^J$ as we move along $r$ is opposite. The only possibility to reconcile these two observations is that $T^c_{r[a} \omega^i_{b]c}$ is either zero or some combination $A^I_J \omega^J$. In any case, it follows that the space of periods does not depend on $r$.

The condition

$$T^c_{r[a} \omega^i_{b]c} = \text{linear combination of } \omega^J$$

is a restriction on the torsion. This restriction implies that the Hyper-Kähler structure of the fiber does not depend on $r$. It is automatically satisfied when $T^c_r$ is zero — in other words, when the theta-angles on $\mathcal{M}(x)$ do not depend on the position in $x \in \mathbb{R}^4$. We believe that this is what actually happens:

$$T^b_r = 0$$

Indeed, our analysis of supergravity solution shows that of the eight Ramond-Ramond fields on the torus, there are seven linear combinations which do not depend on $r$. It is natural that the seven worldsheet theta-angles depend precisely on these seven combinations of the Ramond-Ramond fields. We will check it in the case when the size of the torus is very small at the end of Section 5, using the methods of noncommutative geometry.

4 Near horizon geometry.

In this section we will study the correspondence between the background fields far away from the black string (we will call them “asymptotic background”) and the background fields at the horizon. The answer is given by the formulas (52) and (54). We will explain under which conditions two different asymptotic backgrounds flow to the same near-horizon background (equations (56)). We will show that for each background there exists a background with the small size torus which flows to the same near-horizon geometry. To simplify the discussion, we will first turn off the Ramond-Ramond fluxes and study only the “perturbative” moduli, parametrized by $Gr(4, 4 + 20)$. We will include the Ramond-Ramond fields at the end of this section. The correspondence between the Ramond-Ramond fluxes at infinity and at the horizon is given by (63).

4.1 Moduli of string perturbation theory.

We will use the correspondence between the string theory backgrounds and the planes $W$, which was first discussed in [33] and then made more precise in [34] and [20]. The plane $W$ is generated by the vectors $\vec{v}$ and $\nu^0$, given by:

$$\vec{v} = (0, \vec{\omega}, -B \cdot \vec{\omega}),$$

$$\nu^0 = (1, B, V - \frac{1}{2} B \cdot B)$$

17
Here $\varpi = (\omega^1, \omega^2, \omega^3)$ are the three Kähler forms, $V$ is the volume and $B$ is the $B$ field. Now suppose that we are given the charge vector $Z$:

$$Z = (Q_4, Q_2, Q_0)$$  \hspace{1cm} (44)

According to the previous section, the near-horizon geometry corresponds to the plane $W^h = (W \cap Z^\perp) \oplus RZ$. Let us describe this $W^h$ explicitly.

We begin with introducing a useful notation. The space $\Lambda_+^2 T^4$ of self-dual forms on our torus is generated by $\omega^i$. Let us consider the two-dimensional subspace $\Lambda_C = \{\omega \in \Lambda_+^2 T^4 | (Q_2 - Q_4 B) \cdot \omega = 0\}$  \hspace{1cm} (45)

We have denoted this space $\Lambda_C$ for the following reason. As explained in [33], the choice of the two-dimensional subspace in $H^2(X, R)$ (where $X$ is $K3$ or $T^4$) is equivalent to the choice of the complex structure up to a sign (to fix a sign, we have to orient the two-plane). Therefore, the charge vector $Z$ gives our torus a preferred complex structure modulo sign, $\Lambda_C$. The orthogonal space to $\Lambda_C$ in $\Lambda_+^2 X$ is one-dimensional. Therefore, there are two ways to choose the Kähler form, which differ by a sign. The choice of the sign of the Kähler form also fixes the sign of the complex structure. (In the case of torus this may be understood as follows: we need the value of the Kähler form on any tangent bivector to any holomorphic surface to be positive). Our preferred complex structure has a clear geometrical meaning in the case when $B = 0$. In this case, the bound state of $D1D5$ may be described as an instanton configuration on $U(Q_4)$ gauge theory with the topological charges $(Q_2, Q_0)$. The corresponding field strength is of the type $(1, 1)$ in the complex structure selected by the condition $(Q_2 \cdot \omega_C) = 0$. Our definition (45) is the generalization of this complex structure for $B \neq 0$.

Now we describe $W \cap Z^\perp$. This space contains a two-dimensional subspace $W_C = \{(0, \omega, -(B \cdot \omega)) | \omega \in \Lambda_C\}$  \hspace{1cm} (46)

Since $W \cap Z^\perp$ is three-dimensional, we need one more vector. Choose $\omega_R \in \Lambda_+^2 T^4$, which satisfies two conditions:

1) $\omega_R \perp \Lambda_C$

2) $||\omega_R||^2 = 1$  \hspace{1cm} (47)

The two-form $\sqrt{V} \omega_R$ is the Kähler form up to sign. The vector

$$v'_R = (1, B + s \omega_R, V - \frac{1}{2}B \cdot B - s B \cdot \omega_R)$$  \hspace{1cm} (48)

where

$$s = \frac{1}{2Q_4} \frac{||Q_2 - Q_4 B||^2 - ||Z||^2 - 2Q_4^2 V}{((Q_2 - Q_4 B) \cdot \omega_R)}$$  \hspace{1cm} (49)
belongs to $W \cap Z \perp$. Moreover, it is orthogonal to $W_C$. Therefore, $W \cap Z \perp$ is generated by the two-dimensional space $W_C$ and the vector $v'_R$. The near-horizon $W^h$ is generated by this $W \cap Z \perp$ and $Z$. Let us introduce the following linear combinations of $v'_R$ and $Z$:

$$
\begin{align*}
v^h_R &= \frac{Q_4 v'_R - Z}{\sqrt{||Z||^2 + Q_4^2 (2V + s^2)}} \\
v^h_0 &= \frac{||Z||^2 v'_R + Q_4 (2V + s^2) Z}{||Z||^2 + Q_4^2 (2V + s^2)}
\end{align*}
$$

They are useful, because they are orthogonal to each other and $W_C$, $v^h_R$ is of the form $(0, \ast, \ast)$ and $||v_R||^2 = 1$, and $v^h_0$ is of the form $(1, \ast, \ast)$. It follows from the relation (43) between the background fields and the plane $W$ that the near-horizon Kähler form, $B$ field and volume are related to the components of $v^h_R$ and $v^h_0$ in the following way:

$$
\begin{align*}
v^h_R &= (0, \omega^h_R, \ast) \\
v^h_0 &= (1, B^h, \ast) \\
||v^h_0||^2 &= 2V^h
\end{align*}
$$

The near-horizon background as a function of the background at infinity.

This gives the following expressions for the near-horizon background fields in terms of the asymptotic background fields:

$$
\begin{align*}
\Lambda^h_C &= \Lambda_C \\
\omega^h_R &= \frac{Q_4 (B + s \omega_R) - Q_2}{\sqrt{||Z||^2 + Q_4^2 (2V + s^2)}} \\
B^h &= \frac{||Z||^2 (B + s \omega_R) + (2V + s^2) Q_4 Q_2}{||Z||^2 + Q_4^2 (2V + s^2)} \\
2V^h &= \frac{||Z||^2 (2V + s^2)}{||Z||^2 + Q_4^2 (2V + s^2)}
\end{align*}
$$

The sign of the square root is not fixed (as we have explained before, the choice of the sign of the Kähler form corresponds to the choice of the sign of the complex structure). One can see that the near-horizon background fields satisfy the attractor conditions:

$$
Q_4 B^h - Q_2 = \sqrt{||Z||^2 - 2Q_4^2 V^h \omega^h_R}
$$

Which backgrounds at infinity flow to the given near-horizon background?

Let us now invert (52) and find the backgrounds flowing to the given attractor. The condition (53) tells us that the near-horizon $B$-field is expressed in terms of the near-horizon volume and Kähler form. Therefore, it is natural to parametrize attractors by
\((V^h, \omega^h, \Lambda^h_C)\). There is a twenty-parameter family of backgrounds flowing to \((V^h, \omega^h, \Lambda^h_C)\). They may be parametrized by a real number \(s\) and the Kähler form \(\omega_R\). The \(B\)-field, the volume and the complex structure are determined from the following equations:

\[
\begin{align*}
\Lambda_C &= \Lambda^h_C \\
B + s\omega_R &= \frac{Q_2}{Q^4} + \frac{||Z||^2}{Q^4\sqrt{||Z||^2 - 2Q_2^4V^h}} \omega^h_R \\
V + \frac{s^2}{2} &= \frac{||Z||^2}{||Z||^2 - 2Q_2^4V^h} V^h
\end{align*}
\]

From the last of these relations we get a restriction on \(V\):

\[
V \leq \frac{||Z||^2}{||Z||^2 - 2Q_2^4V^h} V^h
\]

Therefore, for any given attractor there is an upper limit on the volumes of the possible tori at infinity.

**When are two backgrounds at infinity equivalent?**

We will call two backgrounds equivalent if they flow to the same background at the horizon. Our equivalence classes are 20-dimensional subspaces in \(Gr(4, 4 + 20)\) in the case of \(K3\), or 4-dimensional subspaces in \(Gr(4, 4 + 4)\) in the case of \(T^4\). Indeed, given a space \(W \cap Z^\perp\), we need to specify a single line in \(R^{4,20}/(W \cap Z^\perp)\), in order to specify \(W\). Therefore, the equivalence class is itself a Grassmanian manifold, \(Gr(1, 1 + 20)\). The space of equivalence classes is also a Grassmanian, \(Gr(3, 3 + 20)\). It follows from (52) or (54) that for the background with parameters \(B', \omega^h\) to be equivalent to the background \(B, \omega\), the following conditions should be satisfied:

\[
\begin{align*}
1) &\quad \text{Complex structures are the same,} \\
2) &\quad B' + s'\omega^h_R = B + s\omega_R, \\
3) &\quad V' + \frac{(s')^2}{2} = V + \frac{s^2}{2}
\end{align*}
\]

(Apriori, the complex structures are the same modulo sign. But since the equivalence classes are connected, it is natural to choose the sign to be the same.) The third condition follows from the first two and the definition (53) of \(s\), therefore we have in total 40+20 = 60 conditions for \(X = K3\), or 8+4 = 12 for \(T^4\). (This is the dimension of \(Gr(3, 3 + 20)\) or \(Gr(3, 3 + 4)\), respectively.)

**Two Examples.**

1) As a consistency check, let us see what happens when \(V >> 1\). In this case, the near-horizon plane \(W^h\) is generated by the vectors:

\[
\begin{align*}
u_C^h &= (0, \omega_C, 0), \quad v_R^h \simeq (0, \omega_R, 0), \quad v_0^h \simeq (1, Q_2/Q_4, Q_0/Q_4)
\end{align*}
\]
This plane does not depend on $B$. Therefore, for the large volume torus the moduli space of instantons does not depend on $B$. This is what we expect. Indeed, in this limit we expect that the system is described by the modified six-dimensional Yang-Mills action:

$$S_B = \text{const} + \frac{1}{\alpha' g_{str}} \int \text{tr} (F - B)^2 d^6 x$$

The corresponding equations of motion do not depend on $B$, and the moduli space of solutions is the same as for $B = 0$.

2) Now consider the equivalence of backgrounds with very small torus, $V = V' = 0$. Then, we have simply $s = s'$. This allows us to rewrite the equivalence conditions in somewhat simpler form. Namely, the two small tori are equivalent if and only if:

1) They have the same complex structure

2) They have the same two-form

$$\omega_R - 2 \frac{(\omega_R \cdot (Q_2 - Q_4 B))}{||Q_2 - Q_4 B||^2 - ||Z||^2} (Q_2 - Q_4 B)$$

Each equivalence class contains small tori.

Given the torus of the finite size, it is easy to find the corresponding family of small tori. Let us denote the background fields for the finite size torus $(B, \omega_R, \Lambda^0_C)$, and the background fields for the small tori $(B^0, \omega_R^0, \Lambda^0_C)$. From the last of equations (56), we express $s^0$ for the family of small tori in terms of the volume and $s$ for our finite size torus:

$$s^0 = \sqrt{2V + s^2}$$

Our family of small tori can be parametrized by $\omega_R^0$, which is constrained to be orthogonal to $\Lambda^0_C = \Lambda_C$. The field $B^0$ is found from the second equation in (56) which tells us that $B^0 + s^0 \omega_R^0$ is equal to $B + s \omega_R$. (If $s^0$ is related to $s$ by the last of eqs. (56) and $(B^0, \omega_R^0)$ satisfy the second of (56), and $s$ is related to $(\omega_R, B, V)$ by the formula (49), then $s^0$ is also related to $(\omega_R^0, B^0, V^0 = 0)$ by the same formula (49)). We see that, indeed, there are small tori in each equivalence class. This allows us to describe the dynamics of branes wrapped on a finite torus in terms of branes wrapped on a small torus. The latter is related to the noncommutative geometry, as reviewed in the next section.

4.2 Turning on the Ramond-Ramond fields.

Now we want to include the Ramond-Ramond fluxes. We will use the notation for the vectors in \( \mathbf{R}^{5,21} \) which is explained at the end of Appendix A. Notice that turning on the Ramond-Ramond fields does not change the near-horizon values of the perturbative moduli, calculated in the previous subsection. Indeed, the four-plane specifying the perturbative moduli is given by \( (W \cap v^+) / v \) where \( v = [(0, 0, 0), (0, 1)] \). But \( v \in Z^\perp \), which means that the reduction to \( v^\perp / v \) commutes with the operation \( W \rightarrow (W \cap Z^\perp) \oplus \mathbf{R}Z \) used to compute the near-horizon moduli. We want to answer the question: what are
the near-horizon RR fluxes, given the RR fluxes at infinity? Consider the plane \( W^\infty \) corresponding to the background at infinity. It is generated by five vectors the fifth of which is:

\[
v^5 = [(-C_0, C_2, C_4), (1, \kappa^2 - \frac{1}{2}||C||^2)]
\]  

(61)

Let us denote \( W_4 = W \cap v^\perp \), \( v = [(0, 0, 0), (0, 1)] \). In particular \( W^\infty_4 \) is the plane generated by the first four vectors \( v^0, \ldots, v^3 \). Consider the vector

\[
v^5_h = v^5 - \frac{(v^5 \cdot Z)}{||P_{W^\infty_4}Z||^2} P_{W^\infty_4}Z
\]

(62)

We claim that \( v^5_h = [(-C_0^h, C_2^h, C_4^h), (1, \kappa_h^2 - \frac{1}{2}||C||^2)] \), where \( C_i^h \) and \( \kappa_h \) are the near-horizon values of the Ramond-Ramond fluxes and the six-dimensional string coupling constant. Let us prove it. The subspace \( W^\infty_4 \subset W^\infty \) consists of the vectors of the form: \([(*, *, *), (0, *))\]. We can characterize \( v^5 \) as the vector of the form \([(*, *, *), (1, *)] \), with the conditions \( v^5 \in W \) and \( v^5 \perp W_4 \). The vector \( v^5_h \) defined in (62) belongs to \( W^\infty \cap Z^\perp \). Since \( W \cap Z^\perp \) is constant, \( v_5 \) also belongs to the near-horizon five-plane \( W^h \). Because of the attractor condition, \( W^h \) contains \( Z \). Since we do not consider fivebranes and fundamental strings, the last two components of \( Z \) are zero. Therefore \( Z \) belongs to the near-horizon four-plane, \( Z \in W^h_4 \). Also, notice that \( W^\infty_4 \cap Z^\perp \) is three-dimensional. Now it follows that the near-horizon \( W^h_4 \) is generated by \( Z \) and \( W^\infty_4 \cap Z^\perp \). Besides being orthogonal to \( Z \), our vector \( v^5_h \) is also orthogonal to \( W^\infty_4 \cap Z^\perp \) (indeed, \( v^5 \) is orthogonal to \( W^\infty_4 \) and \( P_{W^\infty_4}Z \) is orthogonal to \( W^\infty_4 \cap Z^\perp \)). Therefore, \( v^5_h \) is orthogonal to \( W^h_4 \). Also, it is clearly of the form \([(*, *, *), (1, *)] \). Because of our characterization of \( v^5 \), these properties imply that the vector \( v^5_h \) defined in (62) is, indeed, the near-horizon \( v^5 \).

Straightforward computation using (62) gives (when \( V = 0 \)):

\[
\begin{align*}
C_0^h &= C_0 + t, \\
C_2^h &= C_2 - tB, \\
C_4^h &= C_4 + \frac{t}{2}||B||^2
\end{align*}
\]

(63)

where we have denoted:

\[
t = \frac{-Q_0C_0 + Q_4C_4 + (Q_2, C_2)}{Q_0 + (Q_2 \cdot B) - \frac{1}{2}Q_4||B||^2}
\]

(64)

One can check that the total flux of the near-horizon RR fields through our branes is zero:

\[
-Q_0C_0^h + Q_4C_4^h + (Q_2, C_2^h) = 0
\]

(65)

This is the attractor condition for the Ramond-Ramond fields. The relation (63) between the asymptotic and the near-horizon values of the Ramond-Ramond fields may be rewritten in terms of \( C = C_0 + C_2 + C_4 \):

\[
C^h = C + te^{-B}
\]

(66)
Notice that the expressions (63) for \( C_h \) do not involve \( \omega_R \) (when \( V = 0 \)). It has been shown in [20] that the near-horizon values of the Ramond-Ramond fluxes correspond to the fluxes of the \( B \)-fields in the sigma-model on the moduli space of instantons. In other words, they are the theta-angles of the instanton sigma-model. When \( B = 0 \), (63) tells us that these theta-angles depend only on \( C_2 \) and \( C_4 \). Intuitively this is what one would expect. Indeed, after \( T \) duality \( C_4 \) becomes \( C'_0 \) — the new RR zero-form, and \( C_2 \) becomes \( (C_2')^\vee \). Our \( Q_4 \) becomes instanton number, and \( Q_0 \) becomes the rank of the gauge group.

The interaction with the RR fluxes for the 1 + 5 dimensional theory is of the form:

\[
C_0 \int_{\mathbb{R}^2} \text{tr} \mathcal{F} + \int_{T^4 \times \mathbb{R}^2} (C_2' \wedge \text{tr} \mathcal{F} \wedge \mathcal{F} + C_4 \text{tr} \mathcal{F} \wedge \mathcal{F} \wedge \mathcal{F}) \tag{67}
\]

where \( \mathcal{F} \) is the six-dimensional field strength. Since these expressions are topological, they give theta-angles after the dimensional reduction to 1 + 1 dimensions. Notice that the Ramond-Ramond zero-form \( C_0 \) couples only to the \( U(1) \) gauge field, which decouples from the other fields and does not participate in the dual description of the AdS supergravity.

We explain the meaning of (63) when \( B \neq 0 \) in the next section.

5 Relation to the noncommutative torus.

5.1 Small torus and noncommutative torus (a very brief review).

There is a beautiful relation between the noncommutative geometry and the compactification of the string theory on the torus of very small size. We will not review it here, since it is thoroughly explained in [2, 3, 8]. But we will give a very brief description of this correspondence, in order to fix our notations. Consider the compactification of the Type IIA string theory on the small torus \( T^4 \) with the metric \( g_{ij} \) and the B-field \( B_{ij} \). Consider \( N \) D0 branes in this background. Suppose that the size of the torus is much less than the string length. Consider first the case \( B_{ij} = 0 \). Making \( T \) duality in all the four directions of the torus, we get the theory of \( N \) D4 branes on the dual torus (which has very large size). According to [35], the low energy worldsheet theory for these \( D4 \) branes is the \( N = 4 \) supersymmetric Yang-Mills theory. Now let us turn on some very small \( B \) field. (In our notations, \( B \) field is dimensionless; for the string worldsheet wrapping the cycle \( (ij) \) of the torus, the \( B \) field gives the phase factor \( e^{2\pi i B_{ij}} \) in the path integral). Since \( B \) is small, we expect that it does not considerably change the dynamics of \( D_0 \) branes. The effect of the small \( B \) field is some small correction to \( N = 4 \) SYM on \( T^4 \). It turns out, that this correction is precisely turning on the noncommutativity parameter. Moreover, this result holds for arbitrary \( B_{ij} \), not necessarily small.

\[\text{The map from the worldsheet } \mathbb{R}^2 \text{ to the moduli space of instantons on } T^4 \text{ implies the choice of the } U(N)\text{-bundle over } T^4 \times \mathbb{R}^2. \text{ The theta-angles assign a phase to such a map, which is the linear combination of the Chern classes specified in (63).}\]
The precise correspondence goes as follows. The noncommutative torus is defined in terms of the algebra of functions on it (see [38] and references therein for details). This is the noncommutative deformation of the algebra of functions on the usual torus. If we denote $\phi_i$ the coordinates on the torus, then the algebra of functions is generated by $e^{i\phi_j}$ for $j = 1, \ldots, 4$. The algebra of functions on the noncommutative torus may also be thought of as generated by $e^{i\phi_j}e^{i\phi_k}$, but instead of $e^{i\phi_j}e^{i\phi_k} = e^{i\phi_k}e^{i\phi_j}$ we have:

$$e^{i\phi_j}e^{i\phi_k} = e^{2\pi i \Theta_{jk}}e^{i\phi_k}e^{i\phi_j}$$

The bivector $\Theta_{ij}$ is called the noncommutativity parameter. This defines the noncommutative torus “as a manifold”. To define the Yang-Mills functional, we have also to specify the metric. It is done in the following way. One introduces a four-dimensional abelian algebra, acting on the algebra of functions as derivations. This algebra is the analogue of the algebra of constant vector fields on the commutative torus. It remains abelian in the noncommutative case. The metric on the noncommutative torus is defined as the metric $G^{ij}$ on this algebra (as on the linear space). We have the following correspondence between the parameters of the string theory torus $T^4$ and the parameters of the noncommutative torus $\tilde{T}^4$:

$$\Theta_{ij} = B_{ij}, \quad G^{ij} = (g^{-1})^{ij}$$

(68)

The relation between these formulas and the effective open string metric/noncommutativity parameter of $\tilde{T}$ is the following. The effective open string metric and noncommutativity parameter are given by Eq. (2.5) of $\tilde{T}$:

$$G^{ij} = \left( \frac{1}{g + B} \right)^{ij}_S, \quad \Theta^{ij} = \left( \frac{1}{g + B} \right)^{ij}_A$$

(69)

Suppose that we are given the background with the metric $g_{ij}$ and the B-field $B_{ij}$. First, we do the T-duality $g + B \rightarrow \frac{1}{g + B}$. Then, we use (69) to get $G = \frac{1}{g}$ and $\Theta = B$.

The vector bundle over the noncommutative torus is defined as a module $E$ over the algebra of functions. The gauge field is defined as the set of covariant derivatives $\nabla_1, \ldots, \nabla_d$ acting in $E$ in a way that agrees with the action of the algebra of functions (see $\tilde{T}$ for details). The curvature $F_{ij} = [\nabla_i, \nabla_j]$ is an element of $\text{End} E$. To understand the correspondence between the numbers of branes and the topological numbers of the noncommutative Yang-Mills theory we have to review the noncommutative analogue of the Chern classes. In conventional (commutative) geometry, we can think of $\text{tr} e^{2\pi i F}$ as a differential form; its cohomology class is the Chern character of our bundle. In the noncommutative case we cannot think of $\text{tr} e^{2\pi i F}$ as a differential form, in particular because the operation of integration cannot be separated from the operation of taking the trace. We will think of it in the following way. Given the noncommutative torus $T^d$, consider a commutative torus $U(1)^d$, consisting of the authomorphisms of the algebra $e^{i\phi_j}e^{i\phi_k} = e^{2\pi i \Theta_{jk}}e^{i\phi_k}e^{i\phi_j}$ of the following form:

$$e^{i\phi_j} \rightarrow e^{i\alpha_j}e^{i\phi_j}, \quad \alpha_j \in [0, 2\pi]$$

(70)
Consider a trivial bundle over this torus with the fiber being our module \( \mathcal{E} \) (the infinite-dimensional space). Then, \( F_{ij} = [\nabla_i, \nabla_j] \) is a two-form on this commutative torus with values in \( \text{End} \mathcal{E} \). Let us consider \( \text{tr} \frac{F}{2\pi i} \) as a (non-homogeneous) constant form on \( U(1)^d \). This form by itself does not have any integrality properties. However, it is known from mathematical literature \([14]\) that the following form on \( U(1)^d \):

\[
\mu = e^{\iota(\Theta)} \text{tr} \frac{F}{2\pi i}
\]

is an integer form. It is the noncommutative version of the Chern character for bundles on a torus. The correspondence between the numbers of branes and the topological numbers of the noncommutative Yang-Mills theory goes as follows. For a two-form \( \nu = \frac{1}{2} \nu_{ij} dx^i \wedge dx^j \), let us denote \( \nu^\vee \) the two-form on the dual torus with the components \((\nu^\vee)^{ij} = \frac{1}{2} \epsilon^{ijkl} \nu_{kl}\). Then, for \( d = 4 \) we have the following relation between the Chern character and the numbers of branes:

\[
\mu = \mu_0 + \mu_2 + \mu_4 \text{ vol}, \quad \mu_0 = Q_0, \quad \mu_2 = Q_2^\vee, \quad \mu_4 = -Q_4 \quad (72)
\]

Indeed, to fix the relation between the components of \( \mu \) and the charge \( Z \), we can consider the case \( B = 0 \). After T duality, \( C = C_0 + C_2 + C_4 \) goes to

\[
C' = -C_4 + C_2^\vee - C_0 \quad (73)
\]

(it follows from \((125)\) and \((126)\); T duality acts on \( \Gamma^{4,4} \) as \((x, v, y) \rightarrow (y, v^\vee, x)\)). We use the coupling to the Ramond-Ramond fields to find \( \mu \):

\[
\int C' \wedge \mu = -Q_0 C_0 + (Q_2 \cdot C_2) + Q_4 C_4 \quad (74)
\]

This implies \((72)\). In particular, we have the following relations:

\[
\text{tr} \mathbf{1} = Q_0 + (Q_2 \cdot B) - \frac{1}{2} Q_4 \| B \| ^2 \quad (75)
\]

\[
\text{tr} \frac{F}{2\pi i} = \mu_2 - \mu_4 \Theta^\vee = Q_2^\vee + Q_4 B^\vee
\]

Also notice that for the geometrically dual torus, the Kähler forms are related to the Kähler forms of the original torus as follows:

\[
\omega^i \mapsto (\omega^i)^\vee
\]

In our situation, we have Type IIB \( Q_0 \ D1, Q_2 \ D3 \) and \( Q_4 \ D5 \) branes. In this case, we get the five-dimensional \( U(Q_0) \) theory on \( \mathbb{R}^{1,1} \times \tilde{T}^4 \), where \( \mathbb{R}^{1,1} \) is commutative. (We may consider this as a particular case of the six-dimensional noncommutative Yang-Mills, with the noncommutativity bivector \( \Theta_{\mu\nu} \) vanishing when one of its indices is in the tangent
space to $\mathbb{R}^{1,1}$.) Dimensional reduction of this theory on the four-torus gives the sigma-model with the target space the moduli space of instantons on $\tilde{T}^4$. We will not discuss the details of the dimensional reduction in this paper. Let us explain why the resulting metric on the moduli space of noncommutative instantons is Hyper-Kähler. The argument goes precisely as for the conventional (commutative) instantons [41]. Consider the infinite-dimensional space of all connections $\nabla_i$ in the module $\mathcal{E}$ over the algebra of functions on $\tilde{T}^4$. This space is endowed with the flat metric $ds^2 = tr d\nabla_i d\nabla^i$ and the sphere worth of Kähler forms $\Omega[\omega] = \varepsilon^{ijkl} tr \omega_{ij} d\nabla^k \wedge d\nabla^l$, $\omega$ being a Kahler form on $\tilde{T}^4$. There is an infinite-dimensional group of gauge transformations $\nabla_i \rightarrow g \nabla_i g^{-1}$ preserving the metric and the complex structures. Choosing a particular Kahler form $\Omega[\omega]$ as a symplectic form, we can think of the gauge transformation with the infinitesimal parameter $\xi$ as generated by the Hamiltonian $H_{\xi}[\omega] = \varepsilon^{ijkl} \omega_{ij} tr F_{kl}$. The anti-self-duality condition may be written as the condition that for any Kahler form $\omega$

$$H_{\xi}[\omega] = (\omega \cdot \omega^+) tr \xi$$

where $\omega^+$ is the constant two-form on the right hand side of the instanton equations $F^+ = \omega^+$. Therefore, the moduli space of the anti-self-dual connections is the Hyper-Kähler reduction of the space of all connections. According to Theorem 3.2 in [42], the natural metric on the Hyper-Kähler quotient is Hyper-Kähler.

As an example of how this correspondence works, let us compute the tension of the black string in the limit of the small size torus. From the solution (17), we see that the tension is:

$$T = \frac{1}{g_{\text{str}} V^\infty} \sqrt{||Z||^2}$$

(76)

At small $V^\infty$, we have:

$$||Z_+||^2 = (Z \cdot \bar{v})^2 + \frac{(Z \cdot v^0)^2}{||v^0||^2} \simeq ||(Q_2 - Q_4 B)_+||^2 + \left( \frac{||Z||^2 - ||Q_2 - Q_4 B||^2}{8 Q_4^2 V^\infty} \right)^2 + \frac{1}{2} \left( ||Z||^2 - ||Q_2 - Q_4 B||^2 \right)$$

(77)

(we have neglected the terms of the order $V^\infty$). Then, the mass formula (76) gives:

$$T = \frac{1}{\sqrt{2} g_{YM}^2 M^2} \left\{ \frac{\tilde{V}}{2 Q_4} \left| \left| Z \right| \right|^2 - \left| \left| Q_2 - Q_4 B \right| \right|^2 \right\}$$

$$\pm Q_4 \left[ 1 + 2 \frac{\left| \left| (Q_2 - Q_4 B)_+ \right| \right|^2}{\left| |Z||^2 - ||Q_2 - Q_4 B||^2 \right|} \right] + O(\tilde{V}^{-1})$$

(78)

where $\pm$ is the sign of $||Z||^2 - ||Q_2 - Q_4 B||^2$ and $\tilde{V} = (V^\infty)^{-1}$. The first term is leading when $V^\infty \rightarrow 0$. The second term after identifications (68) and (72) is proportional to the action of the noncommutative instanton, found in [37].
Now we are in a position to give a formula relating the near-horizon geometry to the noncommutative torus. First, we look at which small size tori at infinity flow to the given torus at the horizon. We use (54) with $V = 0$. The third of relations (54) expresses $s$ in terms of $V^h$. Substituting this $s$ to the second equation, we get the relation between $B$ and $\omega_R$. We have:

$$\Lambda_C = \Lambda_C^h,$$

$$B = \frac{Q_2}{Q_4} + \frac{||Z||^2}{Q_4 \sqrt{||Z||^2 - 2Q_4^2V^h}} \omega_R^h - \frac{\sqrt{2V^h||Z||^2}}{\sqrt{||Z||^2 - 2Q_4^2V^h}} \omega_R^\vee$$  

(79)

These relations determine the family of small tori, which flow to our attractor. This family is parametrized by $\omega_R$. Each of these small tori determines the noncommutative torus, which is geometrically dual to it, with the noncommutativity parameter $\Theta = B$. Therefore, to the given attractor we associate the family of noncommutative tori. This family is characterized by the fixed complex structure and the following relation between the Kähler form and the noncommutativity parameter:

$$\Theta = \frac{\mu^\vee_2}{\mu_4} + \frac{||\mu||^2}{\mu_4 \sqrt{||\mu||^2 - 2\mu_4^2V^h}} \omega_R^h - \frac{\sqrt{2V^h||\mu||^2}}{\sqrt{||\mu||^2 - 2\mu_4^2V^h}} \Omega^\vee_R$$  

(80)

(Here we have used $\mu$ instead of $Z$). The formula (80) gives the answer to the question: which noncommutative torus corresponds to the $AdS$ background with parameters $(\Lambda_C^h, \omega_R^h, V^h, B^h)$? We see that the answer is not unique. It follows that all the tori from the family (80) have the same moduli space of noncommutative instantons.

Since the parameters of the attractor are unambiguously determined by the parameters of the small torus $(B, \omega_R, \Lambda_C)$, we can rewrite the equations (80) for the family of noncommutative tori so that it does not contain the parameters of the attractor. Indeed, rewriting (59) in terms of $\mu$ and $(\Theta, \Omega^\vee)$, we get:

1) $\Omega_C = \text{const (modulo phase)}$

2) $\Omega_R - 2 \frac{(\Omega_R \cdot (\mu_2 - \mu_4 \Theta^\vee))}{||\mu_2 - \mu_4 \Theta^\vee||^2 - ||\mu||^2}(\mu_2 - \mu_4 \Theta^\vee) = \text{const}$  

(81)

Different constants give different families of noncommutative tori. The form $\mu_2 - \mu_4 \Theta^\vee$ is expressed in terms of $\text{tr} F$ in (75). It would be interesting to prove in noncommutative geometry, that all the noncommutative tori from the family (81) have the same moduli space of instantons with instanton number $\mu$.

Notice that the Equations (81) imply that the self-dual part of the form

$$\frac{\mu_2 - \mu_4 \Theta^\vee}{||\mu_2 - \mu_4 \Theta^\vee||^2 - ||\mu||^2}$$  

(82)
is a covariantly constant section of the restriction of the “universal” bundle \( \Lambda_+^2 \tilde{T}^4 \) of self-dual forms on the family of equivalent tori.

Notice that for small noncommutativity and \( \mu_2 = 0 \), the second equation (81) implies that: 1) \( \Omega_R \) is constant and 2) the self-dual part of \( \Theta^\nu \) is constant. This is what we intuitively expect. Indeed, in the limit of almost commutative torus there are large regions in the moduli space, where some instantons become very small. But very small instantons on zero volume \( T^4 \) is almost the same as instantons on \( R^4 \). According to [8], the moduli space of instantons on \( R^4 \) depends only on the self-dual part of the noncommutativity parameter. Some properties of supersymmetric configurations on \( R^4 \) are discussed in the next subsection.

5.2 Supersymmetric configurations on \( R^4 \).

It was conjectured in [8, 8] that the supersymmetric configurations on \( R^4 \) with nonzero \( B \) field are noncommutative instantons. This has been proven in [8] in the zero-slope limit, that is when \( \alpha' \to 0 \). We want to show that this is true beyond the zero slope limit. The zero slope limit may be characterized by:

\[
(\alpha')^2 g^{ik} g^{jl} B_{ij} B_{kl} \simeq \frac{1}{\epsilon} \gg 1
\]

Notice that

\[
(\alpha')^2 g^{ik} g^{jl} B_{ij} B_{kl} = \frac{1}{(\alpha')^2} G^{ik} G_{jl} \Theta^{ij} \Theta^{kl}
\]

(this follows from the formula (69) for the effective open string metric and noncommutativity parameter).

We will argue that the moduli space of supersymmetric configurations depends only on certain combination of metric and \( B \) field, and that for the background with generic \( g \) and \( B \) we can always find the background which gives the same moduli space of supersymmetric configurations and satisfies the zero slope condition. Let us prove it. Suppose that we are given the generic metric and \( B \) field on \( R^4 \), not necessarily satisfying (83). The moduli space of supersymmetric configurations should not change significantly if we replace our \( R^4 \) by sufficiently large torus. (We will make this statement more precise below). This torus will have a very large volume and a very large \( B \)-field; in fact, for the generic background the \( B \)-field will scale like the square root of the volume. The target space of the sigma-model describing the low energy dynamics of the D1D5 system wrapped on this torus is the moduli space of supersymmetric configurations on this torus. For finite values of the charges, it has a region corresponding to supersymmetric configurations on \( R^4 \). Indeed, for a very large torus we expect to have such field configurations that the field strength is localized in the small region of the torus which may be thought of as \( R^4 \) with a nontrivial \( B \) field. We will call such field configurations “localized”. Since we have \( g_{ij} \simeq B_{ij} \), these field configurations do not satisfy the condition of the zero slope
limit. We want to argue, however, that our large torus is equivalent to the torus with $|g_{ij}| << |B_{ij}|$, for which the localized supersymmetric configurations do satisfy the zero slope condition. Let us show it. It is very convenient to do first the T duality $g + B \rightarrow \frac{1}{g + B}$ which gives us the torus with small $g_{ij}$ and $B_{ij}$. The metric and the B field are, still, of the same order of magnitude. In other words, we have:

\[ ||B||^2 \approx V << 1, \quad Q_0 \simeq Q_4 \]  

(and we put $Q_2 = 0$). Although the volume is small, it is finite. As we have argued in the previous section, we can always find an equivalent torus with zero volume. The Kähler form $\omega^0_R$ and the B-field $B^0$ for that equivalent torus satisfy:

\[ B^0 + s^0 \omega^0_R = B + s \omega_R \]  

where

\[ s^0 = \sqrt{2V + s^2}, \quad s \simeq \frac{Q_0}{Q_4 (B \cdot \omega_R)} \]  

(we have used an equation (19) for $s$ in the regime (85)). Since $s >> 1$ and $V << 1$ we have:

\[ s^0 = s + O(V/s) \]  

Then, (86) is satisfied by:

\[ \omega^0_R = \omega_R, \quad B^0 = B + O(V/s) \simeq B \]  

(the correction for $B$ is much smaller then $B$ itself). We see that the torus with small $V$ and $B$ satisfying (85) is equivalent to the zero volume torus, which has the same shape and the same (small) $B$ field. (Of course, we can take any other zero volume torus with the same shape and the same self-dual part of the $B$ field). In terms of the T-dual torus $(g_{ij}, B_{ij})$ with large $g_{ij}$ and $B_{ij}$ (the one we have started with) this means that it is equivalent to the zero size torus with large B-field $\bar{B}_{ij}$. However, the relation between the shape of this equivalent torus and the shape of $g_{ij}$, as well as the relation between $\bar{B}$ and $B$ is more complicated. In fact, it is very simple in terms of the effective open string metric and the noncommutativity parameter. Namely, the noncommutative torus $(G_{ij}, \Theta^{ij})$ with small $\Theta^{ij}$ and $G_{ik}G_{jl}\Theta^{ij}\Theta^{kl} \simeq (\alpha')^2$ is equivalent to the very large noncommutative torus $(G_{ij} = \infty)$ with the same shape and the same noncommutativity parameter. The corresponding moduli space is just the moduli space of instantons on large noncommutative $T^4$ and the region in the moduli space corresponding to localized field configurations is the moduli space of instantons on the noncommutative $R^4$.

Let us summarize what we have. We have started with the torus of the large size and large B-field. We have argued that there are localized field configurations, which are the same as supersymmetric configurations on $R^4$ with the B field. On the other hand, we have proven that the moduli space of supersymmetric configurations on this
torus is the same as the moduli space of supersymmetric configurations on the small size torus with some other $B$ field (which is also large). This moduli space also has a region corresponding to localized configurations. It is a natural conjecture (we did not prove it) that the localized supersymmetric field configurations on the large torus correspond to the localized configurations on the equivalent small torus. But the localized configurations on the small torus are described by the zero slope limit, and therefore they are instantons on the noncommutative $\mathbb{R}^4$.

This shows that the moduli space of supersymmetric configurations on $\mathbb{R}^4$ does not depend on $\alpha'$. 

### 5.3 Wilson Lines.

The moduli space of anti-self-dual connections on $T^4$ has an obvious $U(1)^4$ symmetry:

$$\nabla_i \rightarrow \nabla_i + ia_i$$

(90)

where $a_i$ are real numbers. This symmetry does not have fixed points. Therefore, the moduli space has a structure of a bundle with the fiber four-torus $\bar{T}^4$. In the commutative case (when $B = 0$) we could identify this $T^4$ with the dual to the torus on which our Yang-Mills fields live. In the noncommutative case we expect more subtle relation between these tori. One way to see it is to notice that the periodic identifications $a_i \equiv a_i + 2\pi$ result from the gauge transformations, which are modified for the noncommutative torus. The gauge transformation with the parameter $e^{2\pi i \phi_i}$ would not just change $a_i \rightarrow a_i + 2\pi i$, but will instead add some nontrivial function of $\phi_1$ and $\phi_2$ to $\nabla_i$. Shifting $a_i$ by constant will require more complicated gauge transformation, and the periodicity properties of the Wilson lines will be changed. We expect that the torus $\bar{T}^3$ parametrizing the Wilson lines will be different from the geometrically dual to $T^4$. Notice that in the presence of the $B$ field the shape of the torus $T^3$ depends on the representative in the equivalence class of backgrounds. Therefore, if $T^4$ was just geometrically dual to $T^4$, we would get non-equivalent sigma-models for equivalent backgrounds. In fact, we know from Section 4 that $\omega_C$ and $B + s\omega_R$ are invariants of the equivalence relation. Therefore, it is natural to conjecture that the Hyper-Kähler structure on the moduli space of flat connections in the bundle with the given $(Q_4, Q_2, Q_0)$ is specified by the forms $\text{Re} \Omega_C$, $\text{Im} \Omega_C$ and $\frac{\bar{\Omega}_R}{\sqrt{|\Omega_R|^2}}$, where $\bar{\Omega}_R$ is given by the second line in (81). We do not know how to prove it in noncommutative geometry.

### 5.4 Possible singularities in the moduli space.

If the instanton moduli space for one of the tori from the family (81) has singularities, then all the tori from this family have singular instanton moduli spaces. Singularities of the instanton moduli spaces are related in string theory to the possibility for the system of
branes to split into two or more subsystems preserving the BPS condition. The conditions for the possibility of the decay of the black string have been worked out in [36]. The answer is the following. For the black string with the charge $Z$ in the background $W$ to be able to decay into two strings, one with the charge $Z_1$ and the other with the charge $Z - Z_1$, it is necessary that the projection $(Z_1)_+$ of $Z_1$ on $W$ and the projection $Z_+$ of $Z$ on $W$ are collinear vectors. This means, that $W \cap Z_1^\perp = W \cap Z^\perp$. In other words, $W \cap Z^\perp$ should be orthogonal to $Z_1$. Therefore, singularity or nonsingularity depends only on $Z^\perp \cap W$.

But our definition of equivalent tori is precisely that they have the same $Z^\perp \cap W$.

We get the simplest example of the singular moduli space, if we take $Q_2 = 0$, $B$ antiselfdual, and turn off the Ramond-Ramond fields. In this situation, $(B \cdot \omega) = 0$ (this is the definition of $B$ being antiselfdual), and the plane $W \cap Z^\perp$ is generated by the three vectors $[(0, \omega^i, 0), (0, 0)]$ together with the fourth vector $[(0, 0), (1, \kappa^{-2})]$. We see that the whole lattice $\Gamma^{1,1}$ generated by the integer vectors of the form $[(\ast, 0, \ast), (0, 0)]$ is orthogonal to $W \cap Z_1^\perp$. Therefore, our $(Q_4, Q_0)$ system may decay into $(Q_4', Q_0')$ and $(Q_4 - Q_4', Q_0 - Q_0')$, provided that $Q_4'Q_0' > 0$ and $(Q_4 - Q_4')(Q_0 - Q_0') > 0$. The corresponding singularity in the moduli space of instantons is due to small instantons. Notice that if we turn on the generic Ramond-Ramond fields, then the separation of branes becomes impossible. Therefore, although the moduli space of instantons remains geometrically singular, the conformal sigma-model is nonsingular. The reason is, of course, that we have turned on theta-angles.

It is not true that an arbitrary background with singular instanton moduli space is related to this example by dualities. Indeed, it can happen that the sublattice orthogonal to $W \cap Z^\perp$ is not $\Gamma^{1,1}$ (and has dimension higher than two). Then, the corresponding singularity cannot be related to the small instantons by the chain of dualities.

## 5.5 Theta-angles.

The target space $\mathcal{M}$ of the $D1D5$ sigma-model has a nontrivial second cohomology group. There is a natural map $\theta_Z$ from the cohomology lattice of $T^4$ to the second cohomology group of the moduli space of noncommutative instantons with the charge $Z$. Let us remind how this map is constructed. Given a cohomology class $\nu \in H^2(T^4, \mathbb{Z})$, we want to describe how to compute $\theta_Z(\nu)$ on a cycle $\Sigma \subset \mathcal{M}$. We describe the embedding $\Sigma \rightarrow \mathcal{M}$ as a connection on a vector bundle $\hat{\mathcal{E}}$ over $\Sigma \times T^4$. Then, the value of $\theta_Z(\nu)$ on $\Sigma$ is the highest component of $\mu(\hat{\mathcal{E}}) \wedge \nu$. (This is the noncommutative generalization of $\int_{\Sigma \times T^4} ch(\hat{\mathcal{E}}) \wedge \nu$.)

Although $\theta_Z$ is naturally defined over $\mathbb{Z}$ we can consider it over $\mathbb{R}$. Consider the fluxes of the Ramond-Ramond field through the cycles of $T^4$ as an element of $H^*(T^4, \mathbb{Z})$. We want to show that the theta-terms in the $D1D5$ effective action are the values of $\theta_Z$ on the near-horizon Ramond-Ramond fields:

$$\text{Theta-angles} = \theta_Z(C^h)$$

(91)
In particular, the theta terms do not depend on the choice of the representative in the equivalence class of the backgrounds.

To derive this formula, we will use the following trick. We first consider the case of Euclidean D-branes wrapped on $T^6$. We will take this $T^6$ to be a product of $T^2$ and a small $T^4$, and go to the dual noncommutative $\tilde{T}^4$. In this case, we know that the Ramond-Ramond fields couple to the winding numbers of various D-branes on this six-torus:

$$S_{RR} = C' \wedge \mu_{T^2 \times \tilde{T}^4}$$ (92)

where $\mu_{T^2 \times \tilde{T}^4}$ is the integer cohomology class of $T^2 \times \tilde{T}^4$ specifying the number of branes wrapped on various cycles. Then we look at (92) from the perspective of the noncommutative geometry. We can explicitly express $\mu_{T^2 \times \tilde{T}^4}$ in terms of the field strength of the Yang-Mills field on $T^2 \times \tilde{T}^4$ using the Elliott’s formula (71):

$$S_{RR} = C' \wedge \mu_{T^2 \times \tilde{T}^4} = C' \wedge e^{i\Theta} \left[ \text{tr} \exp \left( \frac{1}{2\pi i} F \right) \right] =$$

$$= (e^{i\Theta} \wedge C')' \wedge \text{tr} \exp \left( \frac{1}{2\pi i} F \right) = A' \wedge \text{tr} \exp \left( \frac{1}{2\pi i} F \right)$$ (93)

(the forms $A$ and $C$ are defined in Appendix A, and $C'$ is given by (73).) Now we want to pass to $R^2 \times \tilde{T}^4$ by making $T^2$ very large, much larger then $\tilde{T}^4$. The low energy theory is a dimensional reduction of the six-dimensional Yang-Mills on $\tilde{T}^4$. This theory lives on an ordinary, commutative two-torus. However, it does remember, in a way, that it was obtained from the reduction on the noncommutative manifold; it turns out that the integral along this commutative two-torus of $\text{tr} F$ is not integer. This follows from the formula

$$\text{tr} e^{\frac{1}{2\pi i} F} = e^{-i\Theta} \mu$$ (94)

derived in [40]. For the worldsheet instanton $\mu$ has two indices along $T^2$ and two or four indices along $\tilde{T}^4$. Its contraction with the noncommutativity parameter gives nontrivial (and noninteger) flux of $\text{tr} F$ along the two-torus. This flux contributes to the formula (93) for the coupling to the RR fields.

The appearance of noninteger $\int \text{tr} F_{zz}$ may seem strange and we want to make a comment about it. Let us consider the following example. Take our noncommutative torus $T^4$ to be a product of two noncommutative tori $T^2_{(1)} \times T^2_{(2)}$, with the noncommutativity parameters $\theta_1$ and $\theta_2$. Let us consider an instanton with $\int_{T^2 \times T^2_{(1)}} F \wedge F \neq 0$ (the first $T^2$ in the product $T^2 \times T^2_{(1)}$ is the commutative torus). In the adiabatic limit (i.e., when the size of $T^2$ is much larger then the size of $T^2_{(1)}$) this configuration may be thought of as wrapping the large (and commutative) $T^2$ on the nontrivial two-cycle in the moduli space of flat connections in $T^2_{(1)}$ (see [42]). Naively it seems that we can just put $A_z$ and $A_{\bar{z}}$ equal to zero, which would clearly give $F_{zz} = 0$, and no flux. However, there is a subtlety here. In fact, we have to consider flat connections modulo the gauge transformations.
And it is not true that for our map from $T^2$ to the moduli space of flat connections on $T^2_{(1)}$ we can globally choose the representative in the gauge equivalence class smoothly depending on the point in $T^2$. Therefore, we should really cover our $T^2$ with patches and construct separately the family of flat connections on $T^2_{(1)}$ over each patch, and then glue them together by the appropriate gauge transformations. These gauge transformations can make it impossible to choose $A_z = 0$ and $\bar{A}_z = 0$. In the case if the small torus is noncommutative, we have additional complications due to the fact that our fields $A$ are not really gauge fields, but noncommutative gauge fields. That is, they have non-standard gauge transformations. In fact, non-commutative gauge fields can be expressed in terms of the ordinary gauge fields, as explained in [8]. The formula to the first order in the noncommutativity parameter is:

$$\tilde{A}_i = A_i - \frac{1}{4} \Theta^{kl} (A_k (\partial_l A_i + F_{li}) + (\partial_l A_i + F_{li}) A_k)$$  \hfill (95)

The corresponding noncommutative field strength is, in our case:

$$\tilde{F}_{z \bar{z}} = F_{z \bar{z}} + \frac{1}{4} \theta^{ij} [2 (F_{zi} F_{\bar{j} \bar{z}} + F_{\bar{j} \bar{z}} F_{zi}) - (A_i (\nabla_j F_{z \bar{z}} + \partial_j F_{z \bar{z}}) + (\nabla_j F_{z \bar{z}} + \partial_j F_{z \bar{z}}) A_i)]$$  \hfill (96)

This expression shows that $\tilde{F}_{z \bar{z}}$ cannot be taken to be zero. The noncommutative instanton is the deformation of the configuration with $\int F \wedge F = 8\pi^2 n_{\text{inst}}$ and $\int_{T^2} F = 0$. In this topological sector, we can take $F_{\bar{z}z} = 0$. However, $\tilde{F}_{z \bar{z}} \neq 0$:

$$\int_{T^4} \text{tr} \tilde{F}_{z \bar{z}} = \frac{1}{2} \int_{T^4} \theta^{ij} F_{zi} F_{\bar{j} \bar{z}} \simeq \theta_4 n_{\text{inst}}$$  \hfill (97)

which is in agreement with (94).

Now let us see what happens when we replace $T^2$ by $R^2$, which gives us our original configuration of $D1D5$ wrapped on $T^4$ with two common noncompact directions. In this case, the flux of $\text{tr} F$ through $R^2$ is not determined by the topology. The configuration with zero $\text{tr} F_{z \bar{z}}$ minimizes the action. Indeed, we can add to the gauge field $A_z$ the piece constant in $T^4$: $A_z \to A_z + a(z, \bar{z})$, $A_{\bar{z}} \to A_{\bar{z}} + \bar{a}(z, \bar{z})$, $a \in \mathbb{C}$. The only modification to the action will be (remember that $\text{tr}$ includes integration along $\tilde{T}^4$):

$$\int_{T^2} d^2 z \text{ tr} F_{z \bar{z}}^2 \to \int_{T^2} d^2 z \text{ tr} (F - da)_{z \bar{z}}^2$$

Varying with respect to $a$, we get $(da)_{z \bar{z}} \text{ tr} 1 = \text{ tr} F_{z \bar{z}}$. This means that the configurations with minimal action have $\text{tr} F_{z \bar{z}} = 0$. In particular, the flux of $\text{tr} F$ through $R^2$ is zero. This implies that the coupling to the Ramond-Ramond fields becomes different from (94). In fact, putting $\text{tr} F_{z \bar{z}} \to 0$ in (93) is equivalent to replacing the Ramond-Ramond fields $(C_0, C_2, C_4)$ by their near-horizon value $(C^h_0, C^h_2, C^h_4)$ given by the formula (63). Let us prove it. We will consider the formula (83) valid for the theory on the six-torus, but
put the diagonal part of the components of $F$ along $T^2$ equal to zero. In other words, replace $F$ with $F - \left( \frac{1}{\text{Tr}} \int_{T^2} \text{tr} F \right) \sigma$ where $\sigma$ is the fundamental cohomology class of $T^2$, $\int_{T^2} \sigma = 1$. We will get:

$$\int_{T^2} d^2 z \left\{ (e^{\Theta} \wedge C)' \wedge \text{tr} \exp \left[ \frac{1}{2\pi i} \left( F - \frac{\int_{T^2} \text{tr} F}{\text{tr} 1} \sigma \right) \right] \right\} =$$

$$= \exp \left[ -\frac{\int_{T^2} \text{tr} F}{2\pi i \text{tr} 1} \right] \wedge C' \wedge \int_{T^2} d^2 z \left\{ e^{\Theta} \text{tr} \exp \left[ \frac{F}{2\pi i} \right] \right\} =$$

$$= \exp \left[ -\frac{\int_{T^2} \text{tr} F}{2\pi i \text{tr} 1} \right] \wedge C' \wedge \mu_{T^2 \times \bar{T}^4} =$$

$$= C' \wedge \mu_{T^2 \times \bar{T}^4} - C' \wedge \mu_{T^2 \times \bar{T}^4} \wedge \left( \frac{\int_{T^2} \text{tr} F}{2\pi i \text{tr} 1} \sigma \right) =$$

$$= C' \wedge \mu_{T^2 \times \bar{T}^4} - \left( C' \cdot \mu_{T^2 \times \bar{T}^4} \right) \int_{T^2} \text{tr} \frac{F}{2\pi i}$$

Let us compare this expression with what we would get by replacing $C \rightarrow C^h$ in (92):

$$\int_{T^2} (e^{\Theta} \wedge (C - t e^{-\Theta}))' \wedge \text{tr} \exp \left[ \frac{1}{2\pi i} F \right] =$$

$$= C' \wedge \mu_{T^2 \times \bar{T}^4} - t \int_{T^2} \text{tr} \frac{F}{2\pi i}$$

(99)

Substituting $t$ from (64) and taking into account (75) we see that (98) is equal to (99). In other words, the coupling of the sigma-model to the Ramond-Ramond fields is given by the formula (92) with integer $\mu_{T^2 \times \bar{T}^4}$ but $C$ replaced with $C^h$. This means that the sigma-model on $R^2$ couples to the Ramond-Ramond fields only through their near-horizon values.

### 6 Sigma-model Instantons and Supergravity Instantons.

In this section we want to discuss the supergravity meaning of the period map. The period map associates to each Hyper-Kähler manifold the space of integrals (periods) over its two-cycles of the three Kähler forms. Given the period map of the target space, one can immediately compute the action of any sigma-model instanton. Indeed, the corresponding map $f : R^2 \rightarrow \mathcal{M}$ from the wordsheet to the target space should be holomorphic in some complex structure and the action is given by

$$S = \sum_{I=1}^{3} \left( \int_{R^2} f^* \omega_I \right)^2$$

(100)
But the integrals over the worldsheet of the pullbacks of the Kähler forms is given in terms of the period map and the topology of the instanton. More exactly, let us introduce the cohomology lattice \( H^2(\mathcal{M}, \mathbb{Z}) \). There is a natural nondegenerate pairing on this lattice (the construction is reviewed in [20]). The topology of the sigma-model instanton can be specified by giving a vector \( z \in H^2(\mathcal{M}, \mathbb{Q}) \), so that the integral of \( p \in H^2(\mathcal{M}, \mathbb{Z}) \) over the worldsheet equals \((z \cdot p)\). The period map gives us a three-plane \( W^\sigma_{\text{model}} \in \mathbb{R} \otimes H^2(\mathcal{M}, \mathbb{Z}) \), and the action of the instanton is

\[
S = \sqrt{||P_{W^\sigma_{\text{model}}} z||^2}
\]

We want to interpret this formula in supergravity. Our sigma-model arises on the world-volume of the \( D1D5 \) system wrapped on \( T^4 \). Let us consider the \( D1D5 \) system with the worldvolume not \( \mathbb{R}^2 \), but some compact Riemann surface \( \Sigma \) (for example, \( S^2 \)). Of course, this system will not be BPS and therefore will presumably collapse and decay into gravitons and other fields of six-dimensional supergravity. But let us consider the case when we have nontrivial instanton charge on \( \Sigma \). This instanton charge will couple to the Ramond-Ramond background fields, and therefore it cannot decay into the states from the perturbative string spectrum. In fact it decays into some perturbative state plus one of the D-instantons of the six-dimensional supergravity, such as Type IIB D-instanton, or Type IIB Euclidean D-string wrapped on the two-cycle in \( T^4 \), or Type IIB Euclidean D3 brane wrapped on \( T^4 \), or a combination of them. Therefore, we can think of D-instantons as black strings with compact worldsheet wrapping a nontrivial cycle in the target space.

Do we expect the action of these supergravity instantons to be equal to the action of the corresponding instantons in our sigma-model? For generic backgrounds, we do not. One reason is that we expect the sigma-model description to break down for small enough worldsheets. But even if we forget about the possible breakdown of the low energy description, there is still a reason why the action of supergravity instantons is different. As we have explained in section 5.3, the sigma-model instanton on \( \mathbb{R}^2 \) usually has nonzero abelian gauge field strength on \( \mathbb{R}^2 \), determined from minimizing the action. In generic situation, the flux of this gauge field through \( \mathbb{R}^2 \) is non-integer. If we want to compactify \( \mathbb{R}^2 \) to \( S^2 \) or any other Riemann surface, we have to change the two-dimensional abelian gauge field so that the flux is integer. This will increase the action. Therefore, we expect that in the generic background, the actions of the supergravity instantons are different from the actions of the sigma-model instantons. However, for some special backgrounds and instanton charges, the flux of the abelian field happens to be zero. The condition for it is that the coupling of the instanton to \( (C_0, C_2, C_4) \) is the same as the coupling to \( (C_0^h, C_2^h, C_4^h) \). In other words, the charge \( \chi \) of the supergravity instanton, defined so that the phase factor is \( e^{2\pi i (C \cdot \chi)} \), satisfies:

\[
(P_{W_4} Z \cdot \chi) = 0
\]

We will see that this condition has a natural interpretation in supergravity. We will argue
that if this condition is satisfied (together with some other conditions) then the action of the supergravity instanton is equal to the action of the sigma-model instanton.

6.1 Can supergravity instantons be “absorbed” by branes?

We will argue that under certain conditions (including (102)) there exists the solution of the Euclidean six-dimensional supergravity, corresponding to the black string with the worldsheet $\mathbb{R}^2$ and the D-instanton at some distance from it. This solution preserves four real supercharges. It has moduli corresponding to the motion of the D-instanton in the space transverse to the black string. The action does not depend on the position of the D-instanton. It is possible that such a D-instanton can approach our black string and “dissolve” into it, becoming a sigma-model instanton. We conjecture that the action is not changed in this process. For example, we may consider the black string which is obtained by wrapping $N > 1$ Type IIB $D3$ branes on some two-cycle of the torus. If we turn off the B-field on the torus, then introducing a $D_{-1}$-brane will leave eight supersymmetries unbroken. The $D_{-1}$-brane can move in the directions transversal to $N D3$, or it can become an instanton on the worldvolume of $N D3$. From the point of view of the Yang-Mills theory on the worldvolume of the threebranes, we can describe this process as follows. The $D_{-1}$ brane on top of $N D3$ branes may be thought of as point-like instanton. This point-like instanton corresponds to the special point in the moduli space of instantons of $SU(N)$ theory. Moving in the moduli space, we deform it to the finite size instanton, which can be described as a classical supersymmetric field configuration on the worldsheet.

We should stress, however, that it is not always true that the supersymmetric configuration of the supergravity instanton and the D-brane can be represented as a classical instanton on the worldvolume of the D-brane. For example, the configuration consisting of a single $D3$ brane and a D-instanton on top of it is supersymmetric; nevertheless, there are no finite size instantons in the $U(1)$ gauge theory living on the worldvolume of the single $D3$ brane. For our configuration consisting of the black string and the instanton, we do not know whether the instanton can really be represented by the smooth worldsheet field configuration\textsuperscript{6}. We can probably say that there is always an “ideal” instanton represented by the singular fields, but the precise meaning of this statement is not very clear.

To prove the connection between the supergravity instantons and the sigma-model instantons, we should answer two questions. The first question is whether the supergravity instanton sitting on top of the brane can be deformed to the smooth supersymmetric solution of the six-dimensional Yang-Mills equations. In other words, whether the cor-

\textsuperscript{6} Supersymmetric solutions of Yang-Mills equations on six-dimensional manifolds have been studied in \cite{45, 46, 47, 48} and references therein. For these solutions, the field strength should belong to $su(3) \subset su(4) = so(6)$ modulo constant. (The variation of fermion with nonzero constant field strength can be compensated by the $\eta^*$ transformation, as explained in \cite{43}.)
responding singular field configuration is just a point on the boundary space of smooth configurations. The second question is whether this smooth six-dimensional solution (if it exists) has a good limit when we shrink $T^4$ to zero size; if yes, then it should correspond to the instanton of the $D1D5$ sigma-model.

We did not prove that the answers to these two questions are positive. We want to give an example which shows that at least the first assumption is true for some instantons. Some supersymmetric solutions of the six-dimensional Yang-Mills theory on Calabi-Yau threefold were obtained (in the language of complex geometry) in [19]. Our example is analogous to the solutions found in that paper.

Consider Type IIB compactified on a six-torus $T^6 = T^2 \times T^4$. Introduce the following brane configuration. Take $N$ D3 branes wrapped on $T^4$ and one D3 brane wrapped on $T^2 \times \alpha$ where $\alpha \in H_2(T^4, \mathbb{Z})$, $||\alpha||^2 > 0$. The four-torus $T^4$ with a given metric admits a family of complex structures parametrized by $S^2$. We can choose one of these complex structures so that the cohomology class Poincare-dual to $\alpha$ (which we will call $[\alpha]$) is of the type $(1,1)$. Then, there are holomorphic line bundles on $T^4$ whose Chern class is $[\alpha]$. Let us choose one of them and call it $L$. The other bundles with the same Chern class may be obtained from this one by shifts in $T^4$. It is proven in Chapter 2.6 of [50] that

$$\dim H^0(T^4, \mathcal{L}) = \frac{||\alpha||^2}{2}$$

Zeroes of the global sections of $\mathcal{L}$ are holomorphic curves representing $\alpha$. Fixing one particular bundle corresponds to fixing two out of $\frac{||\alpha||^2}{2} + 1$ complex moduli of such holomorphic curves. Let us parametrize the global sections of $\mathcal{L}$ by the vectors $v \in \mathbb{C}^{\frac{||\alpha||^2}{2}} = H^0(M, \mathcal{L})$. We will denote the corresponding sections $\phi[v](x_1, x_2)$, where $x_1$ and $x_2$ are the local complex coordinates on $T^4$. Our configuration of D3 branes can be represented by the following equation in $T^6 = T^2 \times T^4$:

$$F_0(x_1, x_2, x, y) \overset{def}{=} \phi[v_0](x_1, x_2)(a_0 + a_1 x + a_2 y + \ldots + a_N x^{n/2}) = 0 \quad (103)$$

Here $(x, y)$ is the point of $T^2$ described by the Weierstrass equation $y^2 = x^3 + ax + b$. If $N$ is odd, then the last term is $x^{n-2} y$. This equation represents a union of $N$ D3 branes wrapping $T^4$ and the single D3 brane wrapping $T^2 \times \alpha$. It preserves $\frac{1}{8}$ of supersymmetries. One can think of this equation as specifying a single complex surface in $T^6$, but then this surface is singular. To make it nonsingular, we consider a deformation of (103) specified by the choice of

$$[u_0, \ldots, u_N] \in \mathbb{P}(\mathbb{C}^{N+1} \otimes H^0(T^4, \mathcal{L})) \quad (104)$$

This deformation is:

$$F[u_0, \ldots, u_N](x_1, x_2, x, y) = \phi[u_0](x_1, x_2) + \phi[u_1](x_1, x_2)x + \phi[u_2](x_1, x_2)y + \ldots = 0 \quad (105)$$
In particular, \( F[a_0 v_0, \ldots, a_N v_N] = F_0 \). Unlike the initial equation (103), it represents a smooth four-surface \( X[u_0, \ldots, u_N] \) in \( T^6 \). This smooth four-surface belongs to the same homology class \( N[T^4] + \alpha \times [T^2] \) as the original singular surface (103). Let us now assume that \( N \) is even and \( \alpha \) is divisible by two. Then, the Chern class of the normal bundle to \( X \) is divisible by two, and \( X \) is a spin manifold. Therefore, we can wrap a Euclidean three-brane on it \([51, 52]\). Let us take \( n > 1 \) such \( D_3 \) branes on top of each other and put \( k \) \( SU(n) \) instantons on their world-volume (the corresponding smooth field configuration exists for large enough \( k \) \([53]\)).

After making T-duality in both directions of \( T^2 \), we get a system of \( nN \) \( D_5 \) branes with nontrivial field configuration on it. Since the initial field configuration was nonsingular, we should get nonsingular configuration after T-duality. The brane charge of this configuration is

\[
Nn[T^6] + \left( k - \frac{N||\alpha||^2}{8} \right)[T^4] + n[T^2] \wedge \alpha
\]

The term \( \frac{N||\alpha||^2}{8} \) appears because \( X \) has nontrivial topology, and therefore our \( D_3 \) brane couples to axion even if we turn off the world-volume gauge fields. Namely, the D-instanton charge induced by the nontrivial topology of \( X \) is

\[
-\hat{A}(TX)[X] = \frac{D_1(TX)}{24}[X] = -\frac{1}{24} \int_{T^6} [X] \wedge [X] \wedge [X] = -\frac{1}{8} N||\alpha||^2
\]

This example demonstrates that at least some configurations exist as smooth solutions on the brane worldvolume. However, the configuration which we considered is obviously not the most generic one. The most serious restriction in this example is that

\[
\int_{T^6} \text{tr} F \wedge F \wedge F = 0
\]

(this is what allowed us to reduce dimension from six to four by doing T duality). Also, we did not prove that turning on the \( B \) field will not destroy our configuration.

In the remaining part of this section we will just assume that supergravity instantons correspond to smooth field configurations and are well described by the \( D1D5 \) sigma-model, and see what this assumption implies.

### 6.2 Classification of instantons in six-dimensional supergravity.

The simplest instanton to consider is the \( D_{-1} \)-instanton, which is the dimensional reduction of the D-instanton in ten-dimensional Type IIB. Its action is

\[
S_{D_{-1}} = C_0 - \frac{i}{g_{str}} \tag{106}
\]

The real part of this action is the phase associated to the instanton. It is equal to the expectation value of the axion field. In the classical low energy supergravity we have a
symmetry $C_0 \rightarrow C_0 + \text{const.}$ This symmetry is a subgroup $\mathbb{R}^1$ of $SO(5,5,\mathbb{R})$. But the higher derivative corrections to the low energy action do not have such a symmetry. The reason is precisely the contribution of the $D_{-1}$-instantons, which carry a phase depending on $C_0$. Now we want to consider more general instantons. Notice that the subgroup $\mathbb{R}^1 \subset SO(5,5,\mathbb{R})$ consisting of the shifts of axion can be represented in terms of the light-like vector $v \in \mathbb{R} \otimes \Gamma^{5,5}$, and the orthogonal vector $\chi \in v^\perp/v$. Indeed, let us consider the following $v$ and $\chi$:

$$v = [(0,0,0),(0,1)]$$
$$\chi = [(1,0,0),(0,0)]$$

(107)

One can check that the one-parametric group of transformations $T_{v,\chi}(t) \in SO(5,5,\mathbb{R})$ acting on vectors $x \in \mathbb{R} \otimes \Gamma^{5,5}$ in the following way:

$$T_{v,\chi}(t)x = x + t \left[ (\chi \cdot x)v - (v \cdot x) \left( \chi + \frac{t}{2} ||\chi||^2 v \right) \right]$$

(108)

is precisely the group of shifts of $C_0$. We can consider arbitrary pairs $(v, \chi) \in \Gamma^{5,5}$, such that $v$ is lightlike and $\chi$ is orthogonal to it. Any pair $(v, \chi)$ specifies the $\mathbb{R}$-subgroup of $SO(5,5,\mathbb{R})$, and there is a corresponding instanton breaking this symmetry. To find the action of this instanton, we have to rewrite (106) in terms of $v$ and $\chi$:

$$S = \frac{(\chi \cdot P_Wv)}{||P_Wv||^2} - i \sqrt{\frac{||P_Wv\cap\chi||^2}{||P_Wv||^2}}$$

(109)

Invariance under U-duality group $SO(5,5,\mathbb{Z})$ implies that (109) gives the action of the six-dimensional instanton for an arbitrary pair $(v, \chi)$. Although the $D_{-1}$-instanton we have started with was somewhat special because the corresponding vector $\chi$ was a null-vector, the expression (109) is valid for all D-instantons. Indeed, let us consider the configuration consisting of $Q_3$ Euclidean D3-branes wrapped on $T^4$, and $Q_{-1}$ $D_{-1}$-instantons. If the $B$-field on the torus is anti-self-dual, this action for this configuration is the sum of the action of $Q_{-1}$ $D_{-1}$-instantons and $Q_3$ Euclidean D3-branes. This is in agreement with (109). The backgrounds with a generic $B$-field may be obtained from this one by the rotation by the element of $SO(4,4,\mathbb{R})$ preserving the vector $\chi = (Q_3,0,Q_{-1})$. The formula (109) is invariant under such rotations. This means that (109) is valid for an arbitrary $\chi$ of the form $(Q_3,0,Q_{-1})$. But any other primitive vector in the lattice $\Gamma^{4,4}$ is equivalent to one of these.

We will actually consider only D-instantons, therefore all that we need is (101) for $v = [(0,0,0),(0,1)]$. In this case, $||P_Wv||^2 = \kappa^2$, and (109) becomes:

$$S = (\mathcal{C} \cdot \chi) - \frac{i}{\kappa} \sqrt{||P_W\chi||^2}$$

(110)

(We denote $W_4 = W \cap v^\perp$).
6.3 When do supergravity instanton and black string preserve supersymmetry?

So far we have discussed instantons in flat six-dimensional space. Now let us introduce the black string with some charge vector $Z$. In the presence of this black string, the group $SO(5, 5, \mathbb{R})$ of symmetries of the classical supergravity is broken to the subgroup $SO(4, 5, \mathbb{R})$ preserving the vector $Z$. In particular, the subgroup of shifts $T_{v, \chi}$ defined in (108) preserves $Z$ if and only if $(Z \cdot \chi) = 0$ (we are considering only “D black strings”, that is $(Z \cdot v) = 0$). Therefore, if $(Z \cdot \chi) = 0$, it makes sense to consider the instantons “charged” under this symmetry. The condition $(Z \cdot \chi) = 0$ by itself does not guarantee that the instanton in the presence of the black string is a supersymmetric configuration. For example, let us consider the $D_{-1}$-instanton in the background of the $D3$-brane wrapped on the two-cycle of the torus. If there is a nonzero flux of the $B$ field through this two-cycle, then this configuration breaks all the supersymmetry. The condition for this configuration to be supersymmetric is that the flux of $B$ through the two-cycle on which we wrap our $D3$ brane is zero. This means that

$$(P_{W_4} \chi \cdot Z) = 0 \tag{111}$$

In general case, the condition (111) is a necessary condition for the instanton and the black string to be a supersymmetric configuration. One can see it in the following way. Let us consider the instanton in the presence of the large number $N$ of coinciding black strings, each having charge $Z$. At each point of the transverse space, the instanton feels approximately flat background. Therefore, its action is given approximately by (110) but with $W$ depending on the distance from the black string, according to the attractor equation. The conditions for the action (110) to be constant when $W$ changes according to the attractor equation with given $Z$ are precisely that $(P_{W_4} Z \cdot \chi) = 0$ and $(Z \cdot \chi) = 0$. We will prove in Appendix B that the configuration consisting of the black string and the instanton preserves supersymmetry if the following conditions are satisfied:

1) $(Z \cdot \chi) = 0$
2) $(Z \cdot P_{W_4} \chi) = 0$
3) $||\chi||^2 \geq 0 \tag{112}$

6.4 Supergravity instantons and period map.

When the condition (111) is satisfied, we can rewrite $||P_{W_4} \chi||^2$ as follows:

$$||P_{W_4} \chi||^2 = ||P_{W_4 \cap Z^\perp} \chi||^2 \tag{113}$$

It makes sense to think of $D$-instantons classically when their action is very large, that is $g_{str} << 1$. However, nothing prevents us from taking simultaneously $g_{str} \sqrt{||Z||^2} >> 1$. 

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Therefore, there is a regime where we can trust both the formula for the action of the D-instanton and the near-horizon AdS supergravity.

How can D-instantons help us to prove that the geometry of the target space does not receive string loop corrections? We know that the target space is a Hyper-Kähler manifold with the second cohomology group isomorphic to $\Gamma^4 \cap Z^\perp$. The period map can be described as a choice of the positive-definite three-plane $W_3^{\sigma-\text{model}} \subset Z^\perp \cap (R \otimes \Gamma^4)$. It was conjectured in [20] that

$$W_3^{\sigma-\text{model}} = Z^\perp \cap W_4$$

We want to check this formula by comparing the actions of the sigma-model instantons to the actions of the supergravity instantons. We have argued that for some special backgrounds the sigma-model instantons are related to the D-instantons in supergravity. The actions of these supergravity instantons are given by (110). We expect the action of the supergravity instanton with the charge $\chi \in Z^\perp$ to be equal to the action of the corresponding sigma-model instanton when $(\chi \cdot P W_4 Z) = 0$. In other words, when $\chi$ is orthogonal to the projection of $Z_+ = P W_4 Z$ on $Z^\perp$. Let $L \subset R \otimes \Gamma^4$ be the six-dimensional space of all vectors in $Z^\perp$ which are orthogonal to $P W_4 Z$:

$$L = (R \otimes \Gamma^4) \cap Z^\perp \cap (P W_4 Z)^\perp$$

Then, our condition on the instanton charge is just: $\chi \in L$. For generic background there are no such integer vectors $\chi$ which satisfy this condition. However, there is a dense set of such backgrounds that we can find $\chi \in L$. Moreover, for any vector $v \in L$ we can find an integer vector $V \in \Gamma^4$ and a number $\lambda \in R$ such that $\lambda V \approx v$. This means (assuming the smooth dependence of $W_3^{\sigma-\text{model}}$ on the background fields) that the action formula for the supergravity instantons gives us the angle between the plane $W_3^{\sigma-\text{model}}$ and an arbitrary vector in $L$. Namely, given $v \in L$, the action formula (110) implies that:

$$||P_{W_3^{\sigma-\text{model}}} v||^2 = ||P W_4 v||^2$$

For $v \in Z^\perp \cap W_4$ we have $||P_{W_3^{\sigma-\text{model}}} v||^2 = ||v||^2$. This implies (114). Therefore, the action formula for supergravity instantons is in agreement with the conjectured period map. In particular, we see that the periods do not depend on the string coupling constant.

We should notice, however, that the validity of this argument depends on the assumption made at the end of Section 6.1.

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A Moduli of $K3$ or $T^4$ as points of Grassmanian.

Here we will review the correspondence between the moduli and the points of the Grass-
manian manifold, following mostly [33]. We will concentrate on the case of $K3$, the case
of $T^4$ is similar.

Let us first turn off the RR fields and consider weak coupling limit. There is a one
to one correspondence between the points of the moduli space and the positive 4-planes
in $\mathbb{R}^{4,20}$. The moduli include the metric and the $B$ field. First, let us describe the shape
of $K3$. Let $\Pi \subset \mathbb{R}^{4,20}$ be a positive 4-plane. Choose $w \in \Gamma^{4,20}$ — a primitive light-like
vector. Notice that $(w \perp \cap \Gamma^{4,20})/w \simeq \Gamma^{3,19}$

\begin{equation}
(w \perp \cap \Gamma^{4,20})/w \simeq \Gamma^{3,19} \tag{117}
\end{equation}

We can consider the three-plane $\tilde{\Sigma} = \Pi \cap w \perp$ and $\Sigma = p(\tilde{\Sigma})$ where $p : w \perp \rightarrow w \perp /w$
is the projection. The 3-plane $\Sigma$ corresponds, via the period map, to the Hyper-Kähler
structure of our $K3$. Now we want to specify the volume and the $B$ field. To do this, let
us choose another primitive light-like vector, $w^*$, so that $(w \cdot w^*) = 1$. We represent $\Pi$ as
an orthogonal direct sum:

$$
\Pi = \tilde{\Sigma} \oplus \mathbb{R} \tilde{B}, \quad \tilde{B} \perp \tilde{\Sigma} \tag{118}
$$

with

$$
\tilde{B} = \alpha w + w^* + B \tag{119}
$$

and

$$
(B \cdot w) = (B \cdot w^*) = 0 \tag{120}
$$

We identify $B$ as the $B$ field, and $\alpha$ as $V - \frac{1}{2}(B \cdot B)$. We have an equivalence $B \approx B + \nu$, with $\nu \in H^2(K3, \mathbb{Z})$. Such a shift of $B$ can be accounted for as an ambiguity in the choice
of $w^*$. Indeed, let us identify the cohomology classes of $K3$ with the vectors in the lattice
$\Gamma^{3,19}$ which is the sublattice in $\Gamma^{4,20}$, orthogonal to both $w$ and $w^*$. Given such a vector $\nu$, let us consider the following automorphism of the lattice:

$$
w^* \mapsto (w^*)' = w^* - \nu - \frac{1}{2}w(\nu \cdot \nu),
\mu \mapsto \mu' = \mu + (\mu \cdot \nu)w,
\quad w \mapsto w
\tag{121}
$$

where $\mu$ is any vector in $\Gamma^{3,19}$. Then the new $w$ and $w^*$ are orthogonal to the new $\Gamma^{3,19}$.

If we also change $B$ and $\alpha$ as follows:

$$
B \mapsto (B + w(B \cdot \nu)) + (\nu + w(\nu \cdot \nu))
\alpha \mapsto \alpha + \frac{1}{2} [||B||^2 - ||B + \nu||^2] \tag{122}
$$

then the new $\tilde{B}$ coincides with the old $\tilde{B}$, therefore all that we have done was to pick
different $w^*$. The transformation laws (122) suggest that $B$ should be identified with the
$B$ field, and $\alpha$ should be identified as follows:

$$
\alpha = V - \frac{1}{2}(B \cdot B) \tag{123}
$$
As far as we know, this formula for $\alpha$ was first obtained in [34]. Following [20], we use the notation $(x, v, y)$ for the vector $xw^* + yw + v$, where $x, y \in \mathbb{R}$ and $v \in \mathbb{R}^{3,19}$.

Now let us include the Ramond-Ramond fields and the coupling constant. We enlarge our lattice from $\Gamma^{4,20}$ to $\Gamma^{5,21}$ by adding $\Gamma^{1,1}$. We will denote the vectors from $\mathbb{R}^{5,21}$ in the following way: $[(x, v, y), (x', y')]$ where $(x, v, y)$ is the vector from $\mathbb{R}^{4,20}$ and $(x', y')$ is the vector from $\mathbb{R}^{1,1}$. The scalar product is:

$$|| [(x, v, y), (x', y')] ||^2 = 2xy + ||v||^2 + 2x'y'$$

Now we are ready to describe the correspondence between the moduli and the points of Grassmanian. The plane $W$ is generated by the following five vectors. Four vectors are of the form

$$E^i = [v^i, (0, -(C \cdot v^i))]$$

with $v^i = (0, \omega^i, -(B \cdot \omega^i))$ and $v^0 = (1, B, V - \frac{1}{2}(B \cdot B))$, and the fifth vector is

$$E^5 = [C, \left(1, \frac{1}{\kappa^2} - \frac{||C||^2}{2}\right)]$$

Let us explain what is $\kappa$ and $C$. We define $C \in \mathbb{R}^{4,20}$ as follows:

$$C = (-C_0, C_2, C_4)$$

where $C_0, C_2, C_4$ are related to the Ramond-Ramond fluxes $A$ in the following way:

$$C = C_0 + C_2 + C_4 = (A_0 + A_2 + A_4)e^{-B}$$

Here we consider $A_0, C_0$ zero-forms, $A_2, C_2$ and $B$ two-forms and $A_4, C_4$ four-forms. The coupling of the fields on the brane world-volume to the Ramond-Ramond fluxes is:

$$\int A \wedge \text{tr exp} \left(\frac{1}{2\pi i} F - B\right)$$

The formula (127) can be verified by considering the transformations of $C$ under the element of the T-duality group which shifts the B-field. For the large volume torus (or $K3$) we have the following relation between the numbers of branes $(Q_4, Q_2, Q_0)$ and the parameters of the $1 + 5$-dimensional Yang-Mills:

$$(Q_4, Q_2, -Q_0) = \text{tr exp} \left(\frac{1}{2\pi i} F\right)$$

(For the large volume torus, the Yang-Mills theory lives on $D5$ branes, and the rank of the gauge group is $Q_4$. The sign of $Q_0$ may be understood in the following way. Let us consider the $D1D5$ system wrapped on $K3$. On one hand, to get BPS configuration we should have $Q_0Q_4 > 0$. On the other hand, D-strings should become anti-self-dual gauge fields on $K3$, so that $\int \text{tr } \left(\frac{1}{2\pi i} F\right)^2 < 0$).

The six-dimensional string coupling constant is related to the ten-dimensional $g_{str}$:

$$\kappa = \frac{g_{str}}{\sqrt{V}}$$

This $\kappa$ is invariant under T duality.
B An instanton in the presence of a string: an example.

Here we want to prove that the configuration consisting of the black string with the charge $Z$ and the D-instanton with the charge vector $\chi$ preserves $\frac{1}{8}$ of supersymmetry, if $(\chi \cdot Z) = 0$, $(\chi \cdot P_{W_4} Z) = 0$ and $||\chi||^2 \geq 0$. We will consider the configuration consisting of the black string with $Q_2 = 0$ (only $D1$ and $D5$, no $D3$), and the instanton obtained by wrapping the $D1$-string on the holomorphic cycle $\alpha$ in $T^4$. It is enough to consider this configuration, because any other pair of orthogonal vectors $\chi$ and $Z$ can be brought to this one by $SO(4, 4, \mathbb{R})$ transformations (we do not need this transformations to be defined over $\mathbb{Z}$, because we want just to understand whether supersymmetry is preserved, and this is the property of the classical supergravity solution).

Let us temporarily turn off the $B$ field. The Euclidean $Dp$-brane stretched along the coordinates $y^1, \ldots, y^{p+1}$ preserves the combination of supercharges, satisfying the condition:

$$\Gamma_1 \cdots \Gamma_{p+1} \epsilon_L = i \epsilon_R$$

(131)

For example, for the $D_{-1}$ instanton, $\epsilon_L = i \epsilon_R$. Let us denote $x^0$ and $x^5$ coordinates parallel to the black string, and $x^1, \ldots, x^4$ coordinates in the torus. The supersymmetries preserved by the $D1/D5$ system are:

$$\Gamma_0 \Gamma_5 \epsilon_L = i \epsilon_R$$

$$\Gamma_0 \Gamma_3 \Gamma_1 \cdots \Gamma_4 \epsilon_L = i \epsilon_R$$

(132)

Suppose that the cycle $\alpha$ on which we wrap our D-string to get an instanton is holomorphic in some complex structure of the torus. The corresponding Kähler form $\omega_\mathbb{R}$ can act on spinors (being contracted with gamma-matrices). We will denote the corresponding operator $\hat{\omega}_\mathbb{R}$. The supersymmetry preserved by the instanton is selected by the condition that $\epsilon_L$ is chiral in the tangent space of $T^4$ and $\epsilon_R$ is:

$$\hat{\omega}_\mathbb{R} \epsilon_L = i \epsilon_R$$

(133)

This equation, together with (132) implies that $\epsilon_L$ is chiral in the tangent space of $T^4$ and also subject to the condition $\hat{\omega}_\mathbb{R} \epsilon_L = \Gamma_0 \Gamma_5 \epsilon_L$. Then, $\epsilon_R$ is expressed in terms of $\epsilon_L$ by one of the equations (132). We see that our background preserves $\frac{1}{8}$ of the supersymmetry.

Our background was quite generic except that we have put $B = 0$. To turn on the $B$ field, we use the subgroup $SO(3, 3, \mathbb{R}) \subset SO(4, 4, \mathbb{R})$ which preserves both $Z = (Q_4, 0, Q_0)$ and $\chi = (0, \alpha, 0)$. This groups contains rotations in the plane generated by $(Q_4, 0, -Q_0)$ and $(0, \beta, 0)$, where $\beta \in H^2(T^4, \mathbb{R})$ is orthogonal to $\alpha$. One can see that infinitesimal rotations in such a plane transform our background with zero $B$ field to the background with the infinitesimal $B$ field $\delta B$ satisfying the condition $(\delta B \cdot \alpha) = 0$ and otherwise arbitrary. This means, that there is one and only one condition for the background with nonzero $B$ to be connected to the background with zero $B$ by the rotation orthogonal to both $Z$ and $\chi$. This condition is $(Z \cdot P_{W_4} \chi) = 0$. 

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