ON THE HOWSON PROPERTY OF
DESCENDING HNN-EXTENSIONS OF GROUPS

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Abstract. A group $G$ is said to have the Howson property (or to be a Howson group) if the intersection of any two finitely generated subgroups of $G$ is finitely generated subgroup. It is proved that descending HNN-extension is not a Howson group under some assumptions satisfied by the base group of HNN-extension. In particular, a result of the paper joined with a Burns – Brunner result (received in 1979) implies that any descending HNN-extension of non-cyclic free group does not have the Howson property.

1. Introduction. Main results

A group $G$ is said to have the Howson property (or to be a Howson group) if the intersection of any two finitely generated subgroups of $G$ is the finitely generated subgroup too. This denomination was introduced into practice after the work of A. G. Howson [1], where it was proved that any free group possesses this property. Then, generalizing this result, B. Baumslag [2] have shown that a free product of two Howson groups is a Howson group. On the other hand, it was noted in [3] that the direct product of free group of rank 2 and of infinite cyclic group does not have the Howson property. This observation was then extended by R. Burns and A. Brunner: they have proved in [4] that any extension of non-cyclic finitely generated free group by infinite cyclic group is not a Howson group. Since every extension by infinite cyclic group is splittable, any such group is a special case of descending HNN-extension of free group.

Recall that descending (or named by some authors as ascending) HNN-extension is, in turn, a special case of general construction of HNN-extension and can be defined as follows.

Let $G$ be a group and let $\varphi$ be an injective endomorphism of $G$. Then descending HNN-extension of (base) group $G$ with respect to endomorphism $\varphi$ is the group $G(\varphi) = (G, t; \ t^{-1}gt = g\varphi \ (g \in G))$ generated by generators of $G$ and by one more element $t$ and defined by all defining relations of $G$ and by all relations of form $t^{-1}gt = g\varphi$ where $g \in G$. It is obvious that if endomorphism $\varphi$ is, in addition, surjective (i.e. it is an automorphism of $G$) then the group $G(\varphi)$ turns out to be a splitting extension of group $G$ by infinite cyclic group with generator $t$. Therefore, the following assertion can be considered as a supplement to the Burns – Brunner above result:

Theorem 1. Let $G$ be non-cyclic finitely generated free group and let $\varphi$ be an injective but not surjective endomorphism of $G$. Then the descending HNN-extension $G(\varphi) = (G, t; \ t^{-1}gt = g\varphi \ (g \in G))$ is not a Howson group.
Thus, this result joined with the Burns – Brunner result implies that any descending HNN-extension of non-cyclic free group does not have the Howson property.

The assumption that the base group of the HNN-extension is non-cyclic is essential. Indeed, any HNN-extension of infinite cyclic group is an one-relator group $G_k = \langle a, t; t^{-1}at = a^k \rangle$ (where $k$ is a non-zero integer) belonging to the family of Baumslag – Solitar groups, and it was shown in [3] that all $G_k$ are Howson groups. It is relevant to mention that this result was generalized in [4] as follows:

The group $G = \langle a_1, a_2, \ldots, a_m, t; t^{-1}ut = v \rangle$, where $u$ and $v$ are non-identity elements of free group $F = \langle a_1, a_2, \ldots, a_m \rangle$, is Howson group provided that at least one of $u$ and $v$ is not a proper power in $F$.

One more family of one-relator Howson groups provides the result of work [5] asserting that the generalized free product of two free groups with cyclic amalgamated subgroup which is isolated at least in one of free factors is a Howson group.

On the other hand, many one-relator groups do not possess the Howson property. It was shown in [3] that if non-abelian one-relator group with non-trivial center is not isomorphic to group $G_{-1} = \langle a, t; t^{-1}at = a^{-1} \rangle$ then it is not a Howson group. It should be noted that this assertion turns out to be a consequence of the Burns – Brunner result since non-cyclic one-relator group with non-trivial center is an extension of non-cyclic finitely generated free group by infinite cyclic group [6]. Recently some new examples of one-relator groups without Howson property were given in [7], [8] and [9]. However, it is easy to see that all these groups are a descending HNN-extensions of non-cyclic free group. Thus, the impracticability of Howson property in all examples of one-relator non-Howson groups that we know up to now is in fact a consequence of our Theorem 1 and Burns – Brunner result.

Theorem 1 is a special case of the following somewhat more general result. Let us say that a subgroup $H$ of group $G$ is freely complemented if there exists a non-identity subgroup $K$ of $G$ such that subgroup generated by subgroups $H$ and $K$ is their free product $H \ast K$.

**Theorem 2.** Let $G$ be a finitely generated group, let $\varphi$ be an injective but not surjective endomorphism of $G$ and $H = G\varphi$. If subgroup $H$ of group $G$ is freely complemented then the descending HNN-extension $G(\varphi)$ is not a Howson group.

In order to deduce Theorem 1 from Theorem 2 it is enough to note that if $G$ is a non-cyclic finitely generated free group and $\varphi$ is an injective but not surjective endomorphism of $G$ then subgroup $H = G\varphi$ is freely complemented. In fact, since rank of subgroup $H$ is equal to rank of $G$ and $H$ is a proper subgroup of $G$, the Schreier’s formula implies that $H$ is of infinite index in $G$. Therefore, it follows from the Hall – Burns Theorem (see e. g. [10, proposition I.3.10]) that $H$ is freely complemented.

One more application of Theorem 2 is

**Corollary.** Let a finitely generated group $G$ is the free product of non-identity groups $A \ast B$. If $\varphi$ is an injective but not surjective endomorphism of $G$ such that $A\varphi \subseteq A$ and $B\varphi \subseteq B$ then $G(\varphi)$ is not a Howson group.

In this case subgroup $H = G\varphi$ is generated by subgroups $A\varphi$ $B\varphi$ (and is a free product of them) and since $H\varphi \neq G$ then $A\varphi \neq A$ or $B\varphi \neq B$. Therefore, $H$ is of infinite index in $G$ and hence (see e. g. [11, p. 27]) subgroup $H$ is freely complemented.
The similar assertion is fulfilled for group that is decomposable into the direct product:

**Theorem 3.** Let group $G$ be a direct product of non-identity groups $A$ and $B$ and let $\varphi$ be an injective but not surjective endomorphism of $G$ such that $A\varphi \subseteq A$, $B\varphi \subseteq B$. If $A\varphi \neq A$, $B\varphi \neq B$ and at least one of subgroups $A$ and $B$ is finitely generated then $G(\varphi)$ is not a Howson group.

Theorem 3 implies, in particular, that if $G$ is a free abelian finitely generated group and $\varphi$ is injective endomorphism of $G$ such that the matrix of $\varphi$ in some free base of $G$ is of block-diagonal form where determinant of at least two diagonal blocks is not equal to $\pm 1$ then $G(\varphi)$ is not a Howson group. The problem of complete characterization of those descending HNN-extensions of free abelian groups that are a Howson groups is still open.

2. The proof of Theorem 2

Let $\varphi$ be an injective but not surjective endomorphism of finitely generated group $G$, let $H = G\varphi$ and $K$ be an non-identity subgroup of $G$ such that subgroup $L$ generated by subgroups $H$ and $K$ is their free product, $L = H \ast K$. It is obvious that we can assume subgroup $K$ to be finitely generated.

For any integer $n$ let $K_n = t^{-n}Kt^n$. Let also $N$ denote the subgroup of group $G(\varphi)$ that is generated by all subgroups $K_n$ and $M$ denote the subgroup of group $G(\varphi)$ that is generated by all subgroups $K_n$ with $n \geq 0$. Remark that for $n \geq 0$ we have $K_n = K\varphi^n$ and therefore subgroup $M$ is contained in the base group $G$ of HNN-extension $G(\varphi)$.

**Lemma 1.** Subgroup $N$ is the free product of family subgroups $K_n$, $n \in \mathbb{Z}$. Hence subgroups $N$ and $M$ are not finitely generated.

In order to prove Lemma 1 it is enough to prove that any subgroup generated by a finite family of subgroups $K_n$ is the free product of these subgroup, and to this end, it is enough to prove that for any integer $r \geq 1$ subgroup $M_r$ generated by subgroups $K_0 = K\varphi^0$, $K_1 = K\varphi$, $\ldots$, $K_r = K\varphi^r$ is the free product of these subgroups.

When $r = 1$ this is obvious since $L = H \ast K = H \ast K_0$ and $K\varphi \leq H$. Let us assume that for some $r \geq 1$ subgroup $M_r$ is the free product of subgroups $K_0$, $K_1$, $\ldots$, $K_r$, $K_{r+1}$. Then since the mapping $\varphi$ is an isomorphism of group $G$ on the group $H$ and for any $i \geq 0$ $K_i\varphi = K_{i+1}$ subgroup $M_r\varphi$ is the free product of subgroups $K_1$, $K_2$, $\ldots$, $K_{r+1}$. Since subgroup $M_{r+1}$ is generated by subgroups $K_0$ and $M_r\varphi$ and $M_r\varphi \leq H$ this implies that subgroup $M_{r+1}$ is the free product of subgroups $K_0$, $K_1$, $\ldots$, $K_{r+1}$. The proof of Lemma 1 is complete.

**Lemma 2.** $N \cap G = M$.

Since the inclusion $M \subseteq N \cap G$ is trivial it is enough to prove the opposite inclusion. Any non-identity element $u$ of subgroup $N$ can be written in the form $u = v_1v_2 \cdots v_r$, where $r \geq 1$, for any $i = 1, 2, \ldots, r$ $v_i$ is non-identity element from some subgroup $K_{n_i}$, $v_i = t^{-n_i}g_it^{n_i}$ for some non-identity element $g_i \in K$, and if $r > 1$ then for any $i = 1, 2, \ldots, r - 1$ $v_i \neq 1$. 


We shall show that if at least one of the numbers $n_1, n_2, \ldots, n_r$ is negative, then element $u$ does not enter in subgroup $G$. Since otherwise the inclusion $u \in M$ is evident by that the proof of Lemma will be complete.

So, let us suppose that for some $i$, $1 \leq i \leq r$, we have $n_i < 0$. If $r = 1$ then since element $g_1$ does not belong to subgroup $H$, the expression $u = t^{-n_1}g_1t^{n_1}$ is reduced in $HNN$-extension $G(\varphi)$ and therefore $u \notin G$ Britton's Lemma.

Now, let $r > 1$ and $n$ denote the smallest from integers $n_1, n_2, \ldots, n_r$. Suppose by the contrary that element $u$ belongs to subgroup $G$. Then since $n \leq -1$ element $t^n ut^{-n} = w\varphi^{-n}$ belongs to subgroup $H$.

On the other hand since $n - n_i \leq 0$ for any $i = 1, 2, \ldots, r$, we have for every such number $i$
\[ t^{n_i}v_i t^{-n} = t^{-n_i}g_i t^{-(n-n_i)} = g_i\varphi^{-n_i} \in K\varphi^{-n_i} = n. \]

Therefore, since for any $i = 1, 2, \ldots, r - 1$ integers $n_i - n$ $n_i+1 - n$ are different, the following expression of element $t^n ut^{-n}$,
\[ t^{-n} ut^n = g_1\varphi^{-n_1} \cdot g_2\varphi^{-n_2} \cdot \ldots \cdot g_r\varphi^{-n_r}, \]
is reduced in decomposition of group $M$ into free product in Lemma 1.

By the choice of integer $n$ there exists at least one number $i$ such that $n_i - n = 0$; let $i_1 < i_2 < \cdots < i_s$ be all numbers of those syllables $g_i\varphi^{-n_i}$ for which this equality is satisfied. The rest syllables in this expression of element $t^n ut^{-n}$ belong to subgroup $H$ and by join all such consecutive syllables we obtain the expression of element $t^n ut^{-n}$ of form
\[ t^n ut^{-n} = w_0 g_{i_1} w_{i_2} g_{i_3} \cdots w_{i_{s-1}} g_{i_s} w_s, \]
where all $w_j$ are elements of subgroup $H$ that are not equal to identity except for, may be, $w_0, w_s$. In any case this expression is reduced in free decomposition $L = H \ast K$ of subgroup $L$ and since at least one syllable of it belongs to subgroup $K$, this contradicts to inclusion $t^n ut^{-n} \in H$. Lemma 2 is proved.

Now we can complete the proof of Theorem 2. Let $F$ be subgroup of group $G(\varphi)$ generated by subgroup $K$ and element $t$. We shall show that $F \cap G = M$. Since subgroups $F$ and $G$ are finitely generated while subgroup $M$ (by Lemma 1) is not finitely generated, this will imply that the group $G(\varphi)$ is not a Howson group.

Arbitrary element $f \in F$ can be written in the form
\[ f = g_0 t^{n_1} g_1 t^{n_2} \cdots t^{n_r} g_r \]
where $g_0, g_1, \ldots, g_r$ are some elements from subgroup $K$ and $n_1, n_2, \ldots, n_r$ are some integers. The factorization of group $G(\varphi)$ by the normal closure of subgroup $G$ shoes evidently that if element $f$ belongs to subgroup $G$ then $n_1 + n_2 + \cdots + n_r = 0$ and therefore $f \in N$. Thus, we have inclusion $F \cap G \subseteq N$ and this with taking into account of Lemma 2 and obvious inclusion $M \subseteq F$ implies that
\[ F \cap G = F \cap G \cap N = F \cap M = M. \]
3. The proof of Theorem 3

Let $G = A \times B$ and let $\varphi$ be an injective endomorphism of group $G$ such that $A\varphi \subseteq A$ and $B\varphi \subseteq B$. Suppose also that $A\varphi \neq A$, $B\varphi \neq B$ and subgroup $A$ is finitely generated. Let the restriction of mapping $\varphi$ on subgroup $B$ be denoted by $\varphi$ too and let $B(\varphi) = (B,t; t^{-1}bt = b\varphi (b \in B))$ be corresponding descending HNN-extension of group $B$.

It is easy to see that there exists a homomorphism $\rho$ of group $G(\varphi)$ to the group $B(\varphi)$ which sends the stable letter of group $G(\varphi)$ onto stable letter of group $B(\varphi)$ and action of which on subgroup $G$ coincides with action of projection $\pi : G \to B$. We claim that the kernel of $\rho$ is equal to subgroup $U = \bigcup_{k=0}^{\infty} t^k At^{-k}$.

Indeed, since $A\varphi = A\pi = 1$ the inclusion $U \subseteq \ker \rho$ is evident. Backwards, arbitrary element $v$ from $\ker \rho$ (just as any element of group $G(\varphi)$) can be written in form $v = t^mgt^{-n}$ for some integers $m \geq 0$ and $n \geq 0$ and some element $g \in G$. Let $g = ab$ where $a \in A$ and $b \in B$. Then $u\rho = t^mbt^{-n}$ and therefore in group $B(\varphi)$ we have the equality $t^mbt^{-n} = 1$. Since in any HNN-extension the stable letter generates subgroup that intersects the base group trivially then $b = 1$ and $m = n$. Thus, $v = t^m at^{-m} \in U$ and the proof of equality $\ker \rho = U$ is complete.

Remark that since $A\varphi \neq A$ subgroup $U$ is the union of strictly increasing sequence of subgroups and therefore is not finitely generated.

Let $C$ denote subgroup of group $G(\varphi)$ generated by subgroup $A$ and element $t$ and let $D$ denote subgroup generated by subgroup $A$ and element $tb$ where $b \in B \setminus B\varphi$.

It is evident that $U \subseteq C$ and it is easy to see that subgroup $U$ also is contained in $D$. In fact, we have $A \subseteq D$. If for some $k \geq 0$ subgroup $t^kAt^{-k}$ is contained in $D$ then $D$ contains subgroup $(tb)t^kAt^{-k}(tb)^{-1}$. But since $bt^k = t^k b\varphi^k$ we have $(tb)t^kAt^{-k}(tb)^{-1} = t^{k+1}At^{-(k+1)}$.

Thus, subgroup $U$ is contained in intersection of subgroups $C \cap D$. We shall prove now that, in fact, $C \cap D = U$. Since subgroups $C$ and $D$ are finitely generated and subgroup $U$ is not finitely generated then the proof of Theorem 3 will be complete.

The image of subgroup $C$ under homomorphism $\rho$ of group $G(\varphi)$ on group $B(\varphi)$ is the cyclic subgroup generated by element $t$ and the image of subgroup $D$ is the cyclic subgroup generated by element $tb$. If the intersection $C\rho \cap D\rho$ of these subgroups would be non-trivial then for some non-zero integers $m$ and $n$ in group $B(\varphi)$ must be fulfilled the equation $t^m = (tb)^n$. The passage to the quotient of group $B(\varphi)$ by normal closure of subgroup $B$ shows that $m = n$. Consequently, in group $B(\varphi)$ the equation $t^m = (tb)^m$ is fulfilled, where the integer $m$ may be supposed to be positive. Since

\[(tb)^m = t^m \cdot t^{-(m-1)}bt^{m-1}t^{-(m-2)}bt^{m-2} \cdots t^{-1}btb = t^m \cdot b\varphi^{m-1}b\varphi^{m-2} \cdots b\varphi b,
\]

we have the equality $b\varphi^{m-1}b\varphi^{m-2} \cdots b\varphi b = 1$. This implies the inclusion $b \in B\varphi$ which contradicts to the choice of element $b$.

So, $C\rho \cap D\rho = 1$ and therefore $(C \cap D)\rho = 1$. Since the kernel of $\rho$ coincides with subgroup $U$ this implies the required inclusion $C \cap D \subseteq U$.

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