SPECTRAL PROPERTIES OF THE NEUMANN-LAPLACE OPERATOR IN QUASICONFORMAL REGULAR DOMAINS

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Abstract. In this paper we study spectral properties of the Neumann-Laplace operator in planar quasiconformal regular domains $\Omega \subset \mathbb{R}^2$. This study is based on the quasiconformal theory of composition operators on Sobolev spaces. Using the composition operators theory we obtain estimates of constants in Poincaré-Sobolev inequalities and as a consequence lower estimates of the first non-trivial eigenvalue of the Neumann-Laplace operator in planar quasiconformal regular domains.

1. Introduction

We study the spectral problem for the Laplace operator with the Neumann boundary condition in planar quasiconformal regular domains $\Omega \subset \mathbb{R}^2$. The weak statement of this spectral problem is as follows: a function $u$ solves this problem iff

$$
\int_\Omega \nabla u(x) \cdot \nabla v(x) \, dx = \mu \int_\Omega u(x)v(x) \, dx
$$

for all $v \in W^{1,2}_2(\Omega)$.

We prove discreetness of the spectrum of the Neumann–Laplace operator in quasiconformal $\beta$-regular domains and obtain the lower estimates of the first non-trivial eigenvalue in the terms of quasiconformal geometry of domains:

**Theorem A.** Let $\Omega \subset \mathbb{R}^2$ be a $K$-quasiconformal $\beta$-regular domain. Then the spectrum of the Neumann–Laplace operator in $\Omega$ is discrete, and can be written in the form of a non-decreasing sequence:

$$
0 = \mu_0(\Omega) < \mu_1(\Omega) \leq \mu_2(\Omega) \leq \ldots \leq \mu_n(\Omega) \leq \ldots,
$$

and

$$
\frac{1}{\mu_1(\Omega)} \leq \frac{4K}{\sqrt{\pi}} \left( \frac{2\beta - 1}{\beta - 1} \right)^{\frac{2\beta - 1}{\beta - 1}} \| J_\varphi \|_{L^\beta(\mathbb{D})},
$$

where $\varphi : \mathbb{D} \to \Omega$ is the $K$-quasiconformal mapping.

**Definition 1.1.** Let $\Omega \subset \mathbb{R}^2$ be a simply connected planar domain. Then $\Omega$ is called a $K$-quasiconformal $\beta$-regular domain if there exists a $K$-quasiconformal mapping $\varphi : \mathbb{D} \to \Omega$ such that

$$
\int_\mathbb{D} |J(x, \varphi)|^\beta \, dx < \infty \quad \text{for some} \quad \beta > 1.
$$

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The domain $\Omega \subset \mathbb{R}^2$ is called a $K$-quasiconformal regular domain if it is a $K$-quasiconformal $\beta$-regular domain for some $\beta > 1$.

The notion of quasiconformal regular domains is a generalization of the notion of conformal regular domains was introduced in [8] and was used for study conformal spectral stability of the Laplace operator (see, also [9]).

Recall that a homeomorphism $\varphi : \Omega \to \tilde{\Omega}$ between planar domains $\Omega, \tilde{\Omega} \subset \mathbb{R}^2$ is called a $K$-quasiconformal mapping if it preserves orientation, belongs to the Sobolev class $W_{2,\text{loc}}^1(\Omega)$ and its directional derivatives $D_v$ satisfy the distortion inequality

$$\max_{v : |v| = 1} |D_v \varphi| \leq K \min_{v : |v| = 1} |D_v \varphi| \text{ a.e. in } \Omega.$$ 

Note, that class of quasiconformal regular domain includes the class of Gehring domains [3] and can be described in terms of quasihyperbolic geometry [25].

Remark 1.2. The notion of quasiconformal $\beta$-regular domains is more general then the notion of conformal $\alpha$-regular domains. Consider, for example, the unit square $Q \subset \mathbb{R}^2$. Then $Q$ is a conformal $\alpha$-regular domain for $2 < \alpha \leq 4$ ($\alpha = 2\beta$ [20] and is a quasiconformal $\beta$-regular domain for all $1 < \beta \leq \infty$ because the unit square $Q$ is quasisometrically equivalent to the unit disc $D$.

Remark 1.3. Because $\varphi : D \to \Omega$ is a quasiconformal mapping, then integrability of the derivative is equivalent to integrability of the Jacobian:

$$\int_D |J(x, \varphi)|^\beta dx \leq \int_D |D\varphi(x)|^{2\beta} dx \leq K^{\beta} \int_D |J(x, \varphi)|^\beta dx.$$ 

In 1961 G.Polya [28] obtained upper estimates for eigenvalues of Neumann-Laplace operator in so-called plane-covering domains. Namely, for the first eigenvalue:

$$\mu_1(\Omega) \leq 4\pi |\Omega|^{-1}.$$ 

The lower estimates for $\mu_1(\Omega)$ were known before only for convex domains. In the classical work [29] it was proved that if $\Omega$ is convex with diameter $d(\Omega)$ (see, also [11, 12, 31]), then

$$\mu_1(\Omega) \geq \frac{\pi^2}{d(\Omega)^2}.$$ 

In [20] we proved, that if $\Omega \subset \mathbb{R}^2$ be a conformal regular domain, then the spectrum of Neumann-Laplace operator in $\Omega$ is discrete and the first non-trivial eigenvalue depends on hyperbolic geometry of the domain. Because quasiconformal mappings represent a more flexible class of mapping in the present paper we suggest an approach to the Poincaré-Sobolev inequalities which is based on the quasiconformal mappings theory in connection with the composition operators theory on Sobolev spaces.

Theorem A is based on the Poincaré–Sobolev inequalities in quasiconformal regular domains:

**Theorem B.** Let $\Omega \subset \mathbb{R}^2$ be a $K$-quasiconformal $\beta$-regular domain. Then:

1. the embedding operator

$$i_\Omega : W_{2}^1(\Omega) \hookrightarrow L_s(\Omega)$$
is compact for any $s \geq 1$;
(2) for any function $f \in W^1_2(\Omega)$ and for any $s \geq 1$, the Poincaré–Sobolev inequality
$$\inf_{c \in \mathbb{R}} \left( \int_{\Omega} |f(y) - c|^s dy \right)^{\frac{1}{s}} \leq B_{s,2}(\Omega) \left( \int_{\Omega} |\nabla f(y)|^2 dy \right)^{\frac{1}{2}}$$
holds with the constant
$$B_{s,2}(\Omega) \leq K^\frac{1}{2} B_{\frac{2s}{\beta s},2}(\mathbb{D}) J_{\phi} |L_{\beta}(\mathbb{D})|^\frac{1}{2}.$$ 

Here $B_{r,2}(\mathbb{D}) \leq (2^{-r} \pi)^{\frac{(r+2)}{2r}}$, $r = \beta s/(\beta - 1)$ is the exact constant in the Poincaré-Sobolev inequality for unit disc $\mathbb{D}$
$$\inf_{c \in \mathbb{R}} \left( \int_{\mathbb{D}} |g(x) - c|^r dx \right)^{\frac{1}{r}} \leq B_{r,2}(\mathbb{D}) \left( \int_{\mathbb{D}} |\nabla g(x)|^2 dx \right)^{\frac{1}{2}}.$$ 

The description of compactness of Sobolev embedding operators in the terms of capacity integrals was obtained in [27]. In the present work we give sufficient conditions of compactness of Sobolev embedding operators in the terms of quasi-conformal geometry of domains.

The suggested method is based on the theory of composition operators [30, 36] and its applications to the Sobolev type embedding theorems [15, 16].

The following diagram illustrates this idea:

$$W^1_2(\Omega) \xrightarrow{\phi^*} W^1_2(\mathbb{D})$$
$$\downarrow \quad \downarrow$$
$$L_s(\Omega) \xrightarrow{(\phi^{-1})^*} L_r(\mathbb{D}).$$

Here the operator $\phi^*$ defined by the composition rule $\phi^*(f) = f \circ \phi$ is a bounded composition operator on Sobolev spaces induced by a homeomorphism $\phi$ of $\mathbb{D}$ and $\Omega$ and the operator $(\phi^{-1})^*$ defined by the composition rule $(\phi^{-1})^*(f) = f \circ \phi^{-1}$ is a bounded composition operator on Lebesgue spaces. This method allows to transfer Poincaré-Sobolev inequalities from regular domains (for example, from the unit disc $\mathbb{D}$) to $\Omega$.

In the recent works we study composition operators on Sobolev spaces defined on planar domains in connection with the conformal mappings theory [17]. This connection leads to weighted Sobolev embeddings [18, 19] with the universal conformal weights. Another application of conformal composition operators was given in [8] where the spectral stability problem for conformal regular domains was considered.

2. Composition operators and quasiconformal mappings

In this section we recall basic facts about composition operators on Lebesgue and Sobolev spaces and also the quasiconformal mappings theory. Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a domain. For any $1 \leq p < \infty$ we consider the Lebesgue space $L_p(\Omega)$ of measurable functions $f : \Omega \to \mathbb{R}$ equipped with the following norm:
$$\|f|L_p(\Omega)\| = \left( \int_\Omega |f(x)|^p dx \right)^{\frac{1}{p}} < \infty.$$
The following theorem about composition operators on Lebesgue spaces is well known (see, for example [36]):

**Theorem 2.1.** Let \( \varphi : \Omega \rightarrow \tilde{\Omega} \) be a weakly differentiable homeomorphism between two domains \( \Omega \) and \( \tilde{\Omega} \). Then the composition operator

\[
\varphi^* : L_r(\tilde{\Omega}) \rightarrow L_s(\Omega), \quad 1 \leq s \leq r < \infty,
\]

is bounded, if and only if \( \varphi^{-1} \) possesses the Luzin \( N \)-property and

\[
\left( \int_\Omega |J(y, \varphi^{-1})|^{r/s} \, dy \right)^{s/r} = K < \infty, \quad 1 \leq s < r < \infty,
\]

\[
\text{ess sup}_{y \in \tilde{\Omega}} |J(y, \varphi^{-1})|^{1/s} = K < \infty, \quad 1 \leq s = r < \infty.
\]

The norm of the composition operator \( \| \varphi^* \| = K \).

We consider the Sobolev space \( W^{1,p}_p(\Omega) \), \( 1 \leq p < \infty \), as a Banach space of locally integrable weakly differentiable functions \( f : \Omega \rightarrow \mathbb{R} \) equipped with the following norm:

\[
\| f \|_{W^{1,p}_p(\Omega)} = \left( \int_\Omega |f(x)|^p \, dx \right)^{1/p} + \left( \int_\Omega |\nabla f(x)|^p \, dx \right)^{1/p}.
\]

Recall that the Sobolev space \( W^{1,1}_p(\Omega) \) coincides with the closer of the space of smooth functions \( C^\infty(\Omega) \) in the norm of \( W^{1,1}_p(\Omega) \).

We consider also the homogeneous seminormed Sobolev space \( L^{1,p}_p(\Omega) \), \( 1 \leq p < \infty \), of locally integrable weakly differentiable functions \( f : \Omega \rightarrow \mathbb{R} \) equipped with the following seminorm:

\[
\| f \|_{L^{1,p}_p(\Omega)} = \left( \int_\Omega |\nabla f(x)|^p \, dx \right)^{1/p}.
\]

Recall the notion of the \( p \)-capacity of a set \( E \subset \Omega \). Let \( \Omega \) be a domain in \( \mathbb{R}^n \) and a compact \( F \subset \Omega \). The \( p \)-capacity of the compact \( F \) is defined by

\[
\text{cap}_p(F; \Omega) = \inf \{ \| f \|_{L^p(\Omega)}^p, \ f \geq 1 \text{ on } F, \ f \in C_0(\Omega) \}.
\]

By the similar way we can define \( p \)-capacity of open sets.

For arbitrary set \( E \subset \Omega \) we define a inner \( p \)-capacity as

\[
\text{cap}_p(E; \Omega) = \sup \{ \text{cap}_p(e; \Omega), \ e \subset E \subset \Omega, \ e \text{ is a compact} \},
\]

and a outer \( p \)-capacity as

\[
\overline{\text{cap}}_p(E; \Omega) = \inf \{ \text{cap}_p(U; \Omega), \ E \subset U \subset \Omega, \ U \text{ is an open set} \}.
\]

A set \( E \subset \Omega \) is called \( p \)-capacity measurable, if \( \overline{\text{cap}}_p(E; \Omega) = \underline{\text{cap}}_p(E; \Omega) \). The value

\[
\text{cap}_p(E; \Omega) = \underline{\text{cap}}_p(E; \Omega) = \overline{\text{cap}}_p(E; \Omega)
\]

is called the \( p \)-capacity of the set \( E \subset \Omega \).

By the standard definition functions of the class \( L^1_p(\Omega) \) are defined only up to a set of measure zero, but they can be redefined quasi-everywhere i. e. up to a set of conformal capacity zero. Indeed, every function \( u \in L^1_p(\Omega) \) has a unique quasi-continuous representation \( \tilde{u} \in L^1_p(\Omega) \). A function \( \tilde{u} \) is termed quasi-continuous if
for any $\varepsilon > 0$ there is an open set $U_\varepsilon$ such that the conformal capacity of $U_\varepsilon$ is less than $\varepsilon$ and the function $\tilde{u}$ is continuous on the set $\Omega \setminus U_\varepsilon$ (see, for example [24, 27]).

Let $\Omega$ and $\tilde{\Omega}$ be domains in $\mathbb{R}^n$. We say that a homeomorphism $\varphi : \Omega \to \tilde{\Omega}$ induces by the composition rule $\varphi^*(f) = f \circ \varphi$ a bounded composition operator

$$\varphi^* : L^1_p(\tilde{\Omega}) \to L^1_q(\Omega), \quad 1 \leq q \leq p \leq \infty,$$

if the composition $\varphi^*(f) \in L^1_q(\Omega)$ is defined quasi-everywhere in $\Omega$ and there exists a constant $K_{p,q}(\Omega) < \infty$ such that

$$\|\varphi^*(f)\|_{L^1_q(\Omega)} \leq K_{p,q}(\Omega) \|f\|_{L^1_p(\tilde{\Omega})}$$

for any function $f \in L^1_p(\tilde{\Omega})$ [37].

Let $\Omega \subset \mathbb{R}^n$ be an open set. A mapping $\varphi : \Omega \to \mathbb{R}^n$ belongs to $L^1_{p,\text{loc}}(\Omega)$, $1 \leq p \leq \infty$, if its coordinate functions $\varphi_j$ belong to $L^1_{p,\text{loc}}(\Omega)$, $j = 1, \ldots, n$. In this case the formal Jacobi matrix $D\varphi(x) = \left(\frac{\partial \varphi_j}{\partial x_i}(x)\right)$, $i, j = 1, \ldots, n$, and its determinant (Jacobian) $J(x, \varphi) = \det D\varphi(x)$ are well defined at almost all points $x \in \Omega$. The norm $\|D\varphi(x)\|$ of the matrix $D\varphi(x)$ is the norm of the corresponding linear operator $D\varphi(x) : \mathbb{R}^n \to \mathbb{R}^n$ defined by the matrix $D\varphi(x)$.

Let $\varphi : \Omega \to \tilde{\Omega}$ be weakly differentiable in $\Omega$. The mapping $\varphi$ is the mapping of finite distortion if $|D\varphi(z)| = 0$ for almost all $x \in \mathbb{R^n \setminus \{0\}}$.

A mapping $\varphi : \Omega \to \tilde{\Omega}$ possesses the Luzin $N$-property if a image of any set of measure zero has measure zero. Note that any Lipschitz mapping possesses the Luzin $N$-property.

The following theorem gives the analytic description of composition operators on Sobolev spaces:

**Theorem 2.2.** [30, 36] A homeomorphism $\varphi : \Omega \to \tilde{\Omega}$ between two domains $\Omega$ and $\tilde{\Omega}$ induces a bounded composition operator

$$\varphi^* : L^1_p(\tilde{\Omega}) \to L^1_q(\Omega), \quad 1 \leq q \leq p \leq \infty,$$

if and only if $\varphi \in W^1_{1,\text{loc}}(\Omega)$, has finite distortion, and

$$K_{p,q}(\Omega) = \left(\int_{\Omega} \left(\frac{|D\varphi(x)|^p}{|J(x, \varphi)|^{\frac{p-q}{q}}}ight)^\frac{q}{p-q} \, dx\right)^\frac{p-q}{p} < \infty.$$

Recall that a homeomorphism $\varphi : \Omega \to \tilde{\Omega}$ is called a $K$-quasiconformal mapping if $\varphi \in W^1_{1,\text{loc}}(\Omega)$ and there exists a constant $1 \leq K < \infty$ such that

$$|D\varphi(x)|^n \leq K|J(x, \varphi)|$$

for almost all $x \in \Omega$.

Quasiconformal mappings have a finite distortion, i.e., $D\varphi(x) = 0$ for almost all points $x$ that belong to set $Z = \{x \in \Omega : J(x, \varphi) = 0\}$ because any quasiconformal mapping possesses Luzin $N$-property and an inverse mapping is also quasiconformal.

If $\varphi : \Omega \to \tilde{\Omega}$ is a $K$-quasiconformal mapping then $\varphi$ is differentiable almost everywhere in $\Omega$ and

$$|J(x, \varphi)| = J_\varphi(x) := \lim_{r \to 0} \frac{|\varphi(B(x, r))|}{|B(x, r)|}$$

for almost all $x \in \Omega$. 

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For any planar $K$-quasiconformal homeomorphism $\varphi : \Omega \to \tilde{\Omega}$, the following sharp results is known: $J(x, \varphi) \in L_{p, \text{loc}}(\Omega)$ for any $p < K/(K-1)$ [2].

If $K \equiv 1$ then 1-quasiconformal homeomorphisms are conformal mappings and in the space $\mathbb{R}^n$, $n \geq 3$, are exhausted by Möbius transformations.

**Definition 2.3.** We call a bounded domain $\Omega \subset \mathbb{R}^2$ as $(r, q)$-Poincaré domain, $1 \leq q, r \leq \infty$, if the Poincaré–Sobolev inequality

$$\inf_{c \in \mathbb{R}} \|g - c| L_r(\Omega)\| \leq B_{r,q}(\Omega) \|\nabla g| L_q(\Omega)\|$$

holds for any $g \in L^q_0(\Omega)$ with the Poincaré constant $B_{r,q}(\Omega) < \infty$. The unit disc $D \subset \mathbb{R}^2$ is an example of the $(r, 2)$-embedding domain for all $r \geq 1$.

The following theorem gives a characterization of composition operators in the classical Sobolev spaces (see, for example [15, 16, 20]):

**Theorem 2.4.** Let $\Omega \subset \mathbb{R}^n$ be an $(r, q)$-Poincaré domain for some $1 \leq q \leq r \leq \infty$ and a domain $\tilde{\Omega}$ has finite measure. Suppose that a homeomorphism $\varphi : \Omega \to \tilde{\Omega}$ induces a bounded composition operator $\varphi^* : L^1_p(\tilde{\Omega}) \to L^1_q(\Omega)$, $1 \leq q \leq p < \infty$, and the inverse homeomorphism $\varphi^{-1} : \tilde{\Omega} \to \Omega$ induces a bounded composition operator $(\varphi^{-1})^* : L_r(\Omega) \to L_s(\tilde{\Omega})$, $1 \leq s \leq r < \infty$, for some $p \leq s \leq r$.

Then $\varphi : \Omega \to \tilde{\Omega}$ induces a bounded composition operator $\varphi^* : W^1_p(\tilde{\Omega}) \to W^1_q(\Omega)$, $1 \leq q \leq p < \infty$.

This theorem allows us to obtain compactness of the Sobolev embedding operator in quasiconformal regular domains.

### 3. Poincaré-Sobolev Inequalities

**Weighted Poincaré-Sobolev inequalities.** Let $\Omega \subset \mathbb{R}^2$ be a planar domain and let $v : \Omega \to \mathbb{R}$ be a real valued function, $v > 0$ a.e. in $\Omega$. We consider the weighted Lebesgue space $L^p_p(\Omega, v)$, $1 \leq p < \infty$, of measurable functions $f : \Omega \to \mathbb{R}$ with the finite norm

$$\|f| L^p_p(\Omega, v)\| := \left( \int_{\Omega} |f(x)|^p v(x) dx \right)^{\frac{1}{p}} < \infty.$$ 

It is a Banach space for the norm $\|f| L^p_p(\Omega, v)\|$. The following lemma gives connection between composition operators on Sobolev spaces and the quasiconformal mappings theory [33].

**Lemma 3.1.** A homeomorphism $\varphi : \Omega \to \tilde{\Omega}$ is a $K$-quasiconformal mapping if and only if $\varphi$ generates by the composition rule $\varphi^*(f) = f \circ \varphi$ an isomorphism of Sobolev spaces $L^1_n(\Omega)$ and $L^1_n(\tilde{\Omega})$:

$$\|\varphi^*(f)\ L^1_n(\Omega)\| \leq K \|f\ L^1_n(\tilde{\Omega})\|$$

for any $f \in L^1_n(\tilde{\Omega})$. 

On the base of this lemma we prove the universal weighted Poincaré-Sobolev inequality which is correct for any simply connected planar domain with non-empty boundary.

**Theorem 3.2.** Suppose that $\Omega \subset \mathbb{R}^2$ is a simply connected domain with non-empty boundary and $h(y) = |J(y, \varphi^{-1})|$ is the quasiconformal weight defined by a $K$-quasiconformal mapping $\varphi : \mathbb{D} \to \Omega$. Then for every function $f \in W^1_2(\Omega)$, the inequality

$$\inf_{c \in \mathbb{R}} \left( \int_{\Omega} |f(y) - c|^r h(y) dy \right)^{\frac{1}{r}} \leq B_{r,2}(\Omega, h) \left( \int_{\Omega} |\nabla f(y)|^2 dy \right)^{\frac{1}{2}}$$

holds for any $r \geq 1$ with the constant

$$B_{r,2}(\Omega, h) \leq K^\frac{1}{r} \cdot B_{r,2}(\mathbb{D}) \leq (2^{-1} \pi)^{\frac{2}{r}} (r + 2)^{\frac{1}{2}} K^\frac{1}{r}.$$

Here $B_{r,2}(\mathbb{D})$ is the best constant in the (non-weight) Poincaré-Sobolev inequality in the unit disc $\mathbb{D} \subset \mathbb{R}^2$ with the upper estimate (see, for example, [14, 20]):

$$B_{r,2}(\mathbb{D}) \leq (2^{-1} \pi)^{\frac{2}{r}} (r + 2)^{\frac{1}{2}}.$$

**Proof.** By [1], there exists a $K$-quasiconformal homeomorphism $\varphi : \mathbb{D} \to \Omega$. Then by Lemma 3.1 the inequality

$$(3.1) \quad ||\nabla (f \circ \varphi) | L_2(\Omega)|| \leq K^\frac{1}{2} ||\nabla f | L_2(\Omega)||$$

holds for every function $f \in L^2_2(\Omega)$.

Let $f \in L^2_2(\Omega) \cap C^1(\Omega)$. Then the function $g = f \circ \varphi$ is defined almost everywhere in $\mathbb{D}$ and belongs to the Sobolev space $L^2_2(\mathbb{D})$ [34]. Hence, by the Sobolev embedding theorem $g = f \circ \varphi \in W^{1,2}(\mathbb{D})$ [27] and the classical Poincaré-Sobolev inequality,

$$(3.2) \quad \inf_{c \in \mathbb{R}} ||f \circ \varphi - c | L_r(\mathbb{D})|| \leq B_{r,2}(\mathbb{D}) ||\nabla (f \circ \varphi) | L_2(\mathbb{D})||$$

holds for any $r \geq 1$.

Denote by $h(y) := |J(y, \varphi^{-1})|$ quasiconformal weight in $\Omega$. Using the change of variable formula for quasiconformal mappings [34], the classical Poincaré-Sobolev inequality for the unit disc

$$\inf_{c \in \mathbb{R}} \left( \int_{\mathbb{D}} |g(x) - c|^r dx \right)^{\frac{1}{r}} \leq B_{r,2}(\mathbb{D}) \left( \int_{\mathbb{D}} |\nabla g(x)|^2 dx \right)^{\frac{1}{2}}$$

and inequality (3.1), we get

$$\inf_{c \in \mathbb{R}} \left( \int_{\Omega} |f(y) - c|^r h(y) dy \right)^{\frac{1}{r}} = \inf_{c \in \mathbb{R}} \left( \int_{\Omega} |f(y) - c|^r |J(y, \varphi^{-1})| dy \right)^{\frac{1}{r}}$$

$$= \inf_{c \in \mathbb{R}} \left( \int_{\Omega} |g(x) - c|^r dx \right)^{\frac{1}{r}} \leq B_{r,2}(\mathbb{D}) \left( \int_{\mathbb{D}} |\nabla g(x)|^2 dx \right)^{\frac{1}{2}} \leq K^\frac{1}{r} B_{r,2}(\mathbb{D}) \left( \int_{\Omega} |\nabla f(y)|^2 dy \right)^{\frac{1}{2}}.$$
Approximating an arbitrary function \( f \in W^1_2(\Omega) \) by smooth functions we have

\[
\inf_{c \in \mathbb{R}} \left( \int_{\Omega} |f(y) - c|^r h(y) \, dy \right)^{\frac{1}{r}} \leq B_{r,2}(\Omega, h) \left( \int_{\Omega} |\nabla f(y)|^2 \, dy \right)^{\frac{1}{2}}
\]

with the constant

\[
B_{r,2}(\Omega, h) = K^\beta \cdot B_{r,2}(\mathbb{D}) \leq \left( 2^{-\frac{1}{2}} (r + 2) \right)^{\frac{\beta}{2}} K^\frac{\beta}{2}.
\]

The property of the quasiconformal \( \beta \)-regularity implies the integrability of a Jacobian of quasiconformal mappings and therefore for any quasiconformal \( \beta \)-regular domain we have the embedding of weighted Lebesgue spaces \( L^r(\Omega, h) \) into non-weighted Lebesgue spaces \( L^s(\Omega) \) for \( s = \frac{\beta - 1}{\beta} r \).

**Lemma 3.3.** Let \( \Omega \) be a \( K \)-quasiconformal \( \beta \)-regular domain. Then for any function \( f \in L^r(\Omega, h) \), \( \beta/(\beta - 1) \leq r < \infty \), the inequality

\[
||f||_{L^s(\Omega)} \leq \left( \int_{\mathbb{D}} \left| J(x, \varphi)^\beta \right|^r \, dx \right)^{\frac{1}{r}} \left( ||f||_{L^r(\Omega, h)} \right)^{\frac{\beta - 1}{\beta}}
\]

holds for \( s = \frac{\beta - 1}{\beta} r \).

**Proof.** By the assumptions of the lemma these exists a \( K \)-quasiconformal mapping \( \varphi : \mathbb{D} \to \Omega \) such that

\[
\int_{\mathbb{D}} \left| J(x, \varphi)^\beta \right|^r \, dx < +\infty,
\]

Let \( s = \frac{\beta - 1}{\beta} r \). Then using the change of variable formula for quasiconformal mappings \( \frac{34}{34} \), Hölder’s inequality with exponents \((r, rs/(r - s))\) and the equality \( |J(y, \varphi^{-1})| = h(y) \), we obtain

\[
||f||_{L^s(\Omega)}
\]

\[
= \left( \int_{\Omega} \left| f(y) \right|^s \, dy \right)^{\frac{1}{s}} = \left( \int_{\Omega} \left| f(y) \right|^r \left| J(y, \varphi^{-1}) \right|^{\frac{r}{s}} \left| J(y, \varphi^{-1}) \right|^{-\frac{r}{s}} \, dy \right)^{\frac{1}{r}}
\]

\[
\leq \left( \int_{\Omega} \left| f(y) \right|^r \left| J(y, \varphi^{-1}) \right| \, dy \right)^{\frac{1}{r}} \left( \int_{\Omega} \left| J(y, \varphi^{-1}) \right|^{-\frac{r}{s}} \, dy \right)^{\frac{r}{s}}
\]

\[
\leq \left( \int_{\Omega} \left| f(y) \right|^r h(y) \, dy \right)^{\frac{1}{r}} \left( \int_{\mathbb{D}} \left| J(x, \varphi) \right|^{\frac{r}{s}} \, dx \right)^{\frac{s}{r}}
\]

\[
= \left( \int_{\Omega} \left| f(y) \right|^r h(y) \, dy \right)^{\frac{1}{r}} \left( \int_{\mathbb{D}} \left| J(x, \varphi) \right|^\beta \, dx \right)^{\frac{\beta - 1}{\beta}}.
\]

\[ \square \]
The following theorem is the main technical tool of this paper:

**Theorem B.** Let $\Omega \subset \mathbb{R}^2$ be a $K$-quasiconformal $\beta$-regular domain. Then:

1. the embedding operator 
   $$i_\Omega : W^{1,2}_2(\Omega) \hookrightarrow L^s(\Omega)$$
   is compact for any $s \geq 1$;
2. for any function $f \in W^{1,2}_2(\Omega)$ and for any $s \geq 1$, the Poincaré–Sobolev inequality
   $$\inf_{c \in \mathbb{R}} \left( \int_{\Omega} |f(y) - c|^sd\gamma \right)^\frac{1}{s} \leq B_{s,2}(\Omega) \left( \int_{\Omega} |\nabla f(y)|^2d\gamma \right)^\frac{1}{2}$$
   holds with the constant
   $$B_{s,2}(\Omega) \leq K^\beta B_{\frac{\beta}{\beta - 1}}(\Omega) \| J_{\phi} \| L^\beta(\mathbb{D}) \|^\frac{1}{2}.$$

**Proof.** Let $s \geq 1$. Since $\Omega$ is a $K$-quasiconformal $\beta$-regular domain then there exists a $K$-quasiconformal mapping $\phi : \mathbb{D} \to \Omega$ such that
   $$\int_{\mathbb{D}} |J(x, \phi)|^\beta dx < \infty \text{ for some } \beta > 1.$$

By Theorem 2.1, the composition operator 
   $$(\phi^{-1})^* : L^r_r(\mathbb{D}) \to L^s(\Omega)$$
   is bounded if
   $$\left( \int_{\mathbb{D}} |J(x, \phi)|^\frac{r-s}{s} dx \right)^\frac{s}{r-s} < \infty.$$

Because $\Omega$ is a $K$-quasiconformal $\beta$-regular domain this condition holds for $r/(r - s) = \beta$ i.e. for $r = \beta s/(\beta - 1)$.

Since the mapping $\phi : \mathbb{D} \to \Omega$ induced a bounded composition operator 
   $$\phi^* : L^1_2(\Omega) \to L^1_2(\mathbb{D}),$$
then by Theorem 2.4, the composition operator 
   $$\phi^* : W^{1,2}_2(\Omega) \to W^{1,2}_2(\mathbb{D}),$$
   is bounded.

For the unit disc $\mathbb{D}$ the embedding operator
   $$i_\mathbb{D} : W^{1,2}_2(\mathbb{D}) \hookrightarrow L^r_r(\mathbb{D}),$$
   is compact (see, for example [27]) for any $r \geq 1$.

Therefore the embedding operator
   $$i_\Omega : W^{1,2}_2(\Omega) \to L^s(\Omega)$$
   is compact as a composition of bounded composition operators $\phi^*$, $(\phi^{-1})^*$ and the compact embedding operator $i_\mathbb{D}$. 

Let \( f \in W^1_2(\Omega) \). Then by Theorem 3.2 and Lemma 3.3 we obtain

\[
\inf_{c \in \mathbb{R}} \left( \int_{\Omega} |f(y) - c|^s dy \right)^{\frac{1}{s}} \\
\leq \left( \int_{\Omega} |J(x, \varphi)|^\beta dx \right)^{\frac{1}{\beta}} \inf_{c \in \mathbb{R}} \left( \int_{\Omega} |f(y) - c|^r h(y) dy \right)^{\frac{1}{r}} \\
\leq K^\frac{1}{\beta} B_{r,2}(\mathbb{D}) \left( \int_{\Omega} |J(x, \varphi)|^\beta dx \right)^{\frac{1}{\beta}} \left( \int \Omega |\nabla f(y)|^2 dy \right)^{\frac{1}{2}}
\]

for \( s \geq 1 \).

The following theorem gives compactness of the embedding operator in the case \( \beta = \infty \):

**Theorem 3.4.** Let \( \Omega \) is a \( K \)-quasiconformal \( \infty \)-regular domain. Then:

1. The embedding operator
   \[
i_\Omega : W^1_2(\Omega) \hookrightarrow L_2(\Omega),\]
   is compact.

2. For any function \( f \in W^1_2(\Omega) \), the Poincaré–Sobolev inequality
   \[
   \inf_{c \in \mathbb{R}} \left( \int_{\Omega} |f(y) - c|^2 dy \right)^{\frac{1}{2}} \leq B_{2,2}(\Omega) \left( \int \Omega |\nabla f(y)|^2 dy \right)^{\frac{1}{2}}
   \]
   holds.

3. The following estimate is correct: \( B_{2,2}(\Omega) \leq K^\frac{1}{2} B_{2,2}(\mathbb{D}) \|J_{\varphi}\|_{L_\infty(\mathbb{D})}\|^\frac{1}{2} \). Here \( B_{2,2}^2(\mathbb{D}) = 1/\mu_1(\mathbb{D}) \) is the exact for the Poincaré inequality in the unit disc.

**Proof.** Since \( \Omega \) is a \( K \)-quasiconformal \( \infty \)-regular domain then there exists a \( K \)-quasiconformal mapping \( \varphi : \mathbb{D} \to \Omega \) such that

\[
\|J_{\varphi}\|_{L_\infty(\mathbb{D})} = \text{ess sup}_{x \in \mathbb{D}} |J(x, \varphi)| < \infty.
\]

Hence by Theorem 2.1 the composition operator

\[
(\varphi^{-1})^* : L_2(\mathbb{D}) \to L_2(\Omega)
\]

is bounded.

Since the mapping \( \varphi : \mathbb{D} \to \Omega \) induced a bounded composition operator

\[
\varphi^* : L^1_2(\Omega) \to L^1_2(\mathbb{D}),
\]

then by Theorem 2.4 the composition operator

\[
\varphi^* : W^1_2(\Omega) \to W^1_2(\mathbb{D}),
\]

is bounded.

For the unit disc \( \mathbb{D} \), the embedding operator

\[
i_\mathbb{D} : W^1_2(\mathbb{D}) \hookrightarrow L_2(\mathbb{D}),
\]

is compact (see, for example [27]).
Therefore the embedding operator

\[ i_\Omega : W^1_2(\Omega) \to L^2(\Omega), \]

is compact as a composition of bounded composition operators \( \varphi^* \), \((\varphi^{-1})^*\) and the compact embedding operator \( i_D:\)

\[ i_D : W^1_2(\mathbb{D}) \hookrightarrow L^2(\mathbb{D}). \]

The first part of this theorem is proved.

For every function \( f \in W^1_2(\Omega) \cap C^1(\Omega) \) and \( g = f \circ \varphi \in W^1_2(\mathbb{D}) \), the following inequality are correct:

\[
\inf_{c \in \mathbb{R}} \left( \int_{\Omega} |f(y) - c|^2 dy \right)^\frac{1}{2} = \inf_{c \in \mathbb{R}} \left( \int_{\Omega} |f(y) - c|^2 |J(y, \varphi)\varphi^{-1}|^{-1} |J(y, \varphi^{-1})| dy \right)^\frac{1}{2} \leq \|J_{\varphi^{-1}} \|_{L^\infty(\Omega)}^{-\frac{1}{2}} \inf_{c \in \mathbb{R}} \left( \int_{\Omega} |f(y) - c|^2 |J(y, \varphi^{-1})| dy \right)^\frac{1}{2}.
\]

Because quasiconformal mappings possess the Luzin \( N \)-property, then

\[
\frac{1}{|J(y, \varphi^{-1})|} = |J(x, \varphi)| \text{ for almost all } x \in \mathbb{D} \text{ and for almost all } y \in \Omega.
\]

Hence

\[
\inf_{c \in \mathbb{R}} \left( \int_{\Omega} |f(y) - c|^2 dy \right)^\frac{1}{2} \leq \|J_{\varphi} \|_{L^\infty(\mathbb{D})}^{\frac{1}{2}} \inf_{c \in \mathbb{R}} \left( \int_{\Omega} |f(y) - c|^2 |J(y, \varphi^{-1})| dy \right)^\frac{1}{2}.
\]
Using the change of variable formula for quasiconformal mappings \[34\], the Poincaré inequality in the unit disc and the inequality (3.1) finally we obtain

\[
\inf_{c \in \mathbb{R}} \left( \int_{\hat{\Omega}} |f(y) - c|^2 \, dy \right)^{\frac{1}{2}} \leq \|J_\varphi | L_\infty(D)\|^{\frac{1}{2}} \inf_{c \in \mathbb{R}} \left( \int_{\Omega} |f(y) - c|^2 |J(y, \varphi^{-1})| \, dy \right)^{\frac{1}{2}}
\]

\[
= \|J_\varphi | L_\infty(D)\|^{\frac{1}{2}} \inf_{c \in \mathbb{R}} \left( \int_{\Omega} |g(x) - c|^2 \, dx \right)^{\frac{1}{2}}
\]

\[
\leq \|J_\varphi | L_\infty(D)\|^{\frac{1}{2}} B_{2,2}(D) \left( \int_{\hat{\Omega}} |
abla g(x)|^2 \, dx \right)^{\frac{1}{2}}
\]

\[
\leq K^{\frac{1}{2}} B_{2,2}(D) \|J_\varphi | L_\infty(D)\|^{\frac{1}{2}} \left( \int_{\hat{\Omega}} |
abla f(y)|^2 \, dy \right)^{\frac{1}{2}}.
\]

4. **Eigenvalue Problem for Neumann-Laplacian**

The eigenvalue problem for the free vibrating membrane is equivalent to the corresponding spectral problem for the Neumann–Laplace operator. The classical formulation of the spectral problem for the Neumann–Laplace operator in smooth domains in the following:

\[
\left\{ \begin{array}{ll}
-\Delta u = \mu u & \text{in } \Omega \\
\frac{\partial u}{\partial n} = 0 & \text{on } \partial \Omega.
\end{array} \right.
\]

Because quasiconformal regular domain are not necessary smooth, the weak statement of the spectral problem for the Neumann–Laplace operator is convenient: a function \(u\) solves the previous problem iff \(u \in W^1_2(\Omega)\) and

\[
\int_{\Omega} \nabla u(x) \cdot \nabla v(x) \, dx = \mu \int_{\Omega} u(x)v(x) \, dx
\]

for all \(v \in W^1_2(\Omega)\).

By the Min–Max Principle \[10\], the inverse to the first eigenvalue is equal to the square of the exact constant in the Poincaré inequality:

\[
\inf_{c \in \mathbb{R}} \|f - c | L_2(\Omega)\| \leq B_{2,2}(\Omega)\|\nabla f | L_2(\Omega)\|, \quad f \in W^1_2(\Omega).
\]

**Theorem A.** Let \(\Omega \subset \mathbb{R}^2\) be a \(K\)-quasiconformal \(\beta\)-regular domain. Then the spectrum of the Neumann–Laplace operator in \(\Omega\) is discrete, and can be written in the form of a non-decreasing sequence:

\[
0 = \mu_0(\Omega) < \mu_1(\Omega) \leq \mu_2(\Omega) \leq \ldots \leq \mu_n(\Omega) \leq \ldots,
\]
and

\[
\frac{1}{\mu_1(\Omega)} \leq KB \frac{2\beta}{\pi r^2} (\mathbb{D}) \left( \int_{\mathbb{D}} |J(x, \varphi)|^\beta \, dx \right)^{\frac{1}{\beta}} \leq \frac{4K}{\sqrt{\pi}} \left( \frac{2\beta - 1}{\beta - 1} \right)^{\frac{2\beta - 1}{2\beta}} \|J_\varphi | L_\beta(\mathbb{D})\|,
\]

where \( \varphi : \mathbb{D} \to \Omega \) is the \( K \)-quasiconformal mapping.

**Proof.** By Theorem B in the case \( s = 2 \), the embedding operator

\[
i_\Omega : W^{1,2}_2(\Omega) \to L^2(\Omega)
\]

is compact.

Therefore the spectrum of the Neumann–Laplace operator is discrete and can be written in the form of a non-decreasing sequence.

By the same theorem and the Min-Max principle we have

\[
\inf_{c \in \mathbb{R}} \left( \int_\Omega |f(y) - c|^2 \, dy \right) \leq B^2_{2,2}(\Omega) \int_\Omega |\nabla f(y)|^2 \, dy,
\]

where

\[
B_{2,2}(\Omega) \leq K^2 B_{r,2}^2(\mathbb{D}) \left( \int_{\mathbb{D}} |J(x, \varphi)|^\beta \, dx \right)^{\frac{1}{\beta}}.
\]

Hence

\[
\frac{1}{\mu_1(\Omega)} \leq KB^2_{r,2}(\mathbb{D}) \left( \int_{\mathbb{D}} |J(x, \varphi)|^\beta \, dx \right)^{\frac{1}{\beta}}.
\]

By the upper estimate of the Poincaré constant in the unit disc (see, for example, [14, 20])

\[
B_{r,2}(\mathbb{D}) \leq \left( \frac{2}{\beta - 1} \right)^{\frac{2\pi}{2\beta}} (r + 2)^{\frac{2\pi}{2\beta}}.
\]

Recall that by Theorem B, \( r = 2\beta/(\beta - 1) \). In this case

\[
B_{2\beta/(\beta - 1),2}(\mathbb{D}) \leq 2\pi^{-\frac{1}{2\beta}} \left( \frac{2\beta - 1}{\beta - 1} \right)^{\frac{2\beta - 1}{2\beta}}.
\]

Thus

\[
\frac{1}{\mu_1(\Omega)} \leq KB \frac{2\beta}{\pi r^2} (\mathbb{D}) \left( \int_{\mathbb{D}} |J(x, \varphi)|^\beta \, dx \right)^{\frac{1}{\beta}} \leq \frac{4K}{\sqrt{\pi}} \left( \frac{2\beta - 1}{\beta - 1} \right)^{\frac{2\beta - 1}{2\beta}} \|J_\varphi | L_\beta(\mathbb{D})\|.
\]

\[\Box\]

In case \( K \)-quasiconformal \( \infty \)-regular domains we have:
Theorem 4.1. Let $\Omega \subset \mathbb{R}^2$ be a $K$-quasiconformal $\beta$-regular domain for $\beta = \infty$. Then the spectrum of the Neumann–Laplace operator in $\Omega$ is discrete, and can be written in the form of a non-decreasing sequence:

$$0 = \mu_0(\Omega) < \mu_1(\Omega) \leq \mu_2(\Omega) \leq \ldots \leq \mu_n(\Omega) \leq \ldots,$$

and

$$(4.1) \quad \frac{1}{\mu_1(\Omega)} \leq KB_{2,2}(\mathbb{D}) \| J_\varphi \| _{L_{\infty}(\mathbb{D})} = \frac{K}{j_1,1} \| J_\varphi \| _{L_{\infty}(\mathbb{D})},$$

where $j_1,1$ is the first positive zero the derivative of the Bessel function $J_1$, and $\varphi : \mathbb{D} \to \Omega$ is the $K$-quasiconformal mapping.

As an application of Theorem 4.1, we obtain the lower estimates of the first non-trivial eigenvalue on the Neumann eigenvalue problem for the Laplace operator in a non-convex domains with a non-smooth boundaries.

Example 1. The homeomorphism

$$w = (|z|^{k-1}z + 1)^2, \quad z = x + iy, \quad k \geq 1,$$

is $k$-quasiconformal and maps the unit disc $\mathbb{D}$ onto the interior of the cardioid

$$\Omega_\varepsilon = \{ (x, y) \in \mathbb{R}^2 : (x^2 + y^2 - 2x)^2 - 4(x^2 + y^2) = 0 \}.$$

We calculate the Jacobian of mapping $w$ by the formula

$$J(z, w) = |w_z|^2 - |w_\overline{z}|^2.$$

Here

$$w_z = \frac{1}{2} \left( \frac{\partial w}{\partial x} - i \frac{\partial w}{\partial y} \right) \quad \text{and} \quad w_\overline{z} = \frac{1}{2} \left( \frac{\partial w}{\partial x} + i \frac{\partial w}{\partial y} \right).$$

A straightforward calculation yields

$$w_z = (k + 1)|z|^{k-1}(|z|^{k-1}z + 1) \quad \text{and} \quad w_\overline{z} = (k - 1)|z|^{k-3}z^2(|z|^{k-1}z + 1).$$

Hence

$$J(z, w) = 4k|z|^{2k-2}(|z|^{2k} + |z|^{k-1}(z + \overline{z}) + 1).$$

Then by Theorem 4.1 we have

$$\frac{1}{\mu_1(\Omega_\varepsilon)} \leq \frac{K}{j_1,1} \sup_{|z| \leq 1} |J(z, w)| \leq \frac{16k^2}{j_1,1}.$$

Example 2. The homeomorphism

$$w = |z|^{k}z, \quad z = x + iy, \quad k \geq 0,$$

is $(k + 1)$-quasiconformal and maps the square

$$Q := \left\{ (x, y) \in \mathbb{R}^2 : -\frac{\sqrt{2}}{2} < x < \frac{\sqrt{2}}{2}, \ -\frac{\sqrt{2}}{2} < y < \frac{\sqrt{2}}{2} \right\}$$

onto star-shaped domains $\Omega_\varepsilon^*$ with vertices $(\pm \sqrt{2}/2, \pm \sqrt{2}/2), (\pm \varepsilon, 0)$ and $(0, \pm \varepsilon)$, where $\varepsilon = (\sqrt{2}/2)^{k+1}$.

We calculate the partial derivatives of mapping $w$

$$w_z = \frac{k}{2}(k + 1)|z|^{k-1} \quad \text{and} \quad w_\overline{z} = \frac{k}{2}|z|^{k-2}z^2.$$
Figure 4.1. Domains $\Omega^*_\varepsilon$ under $\varepsilon = \frac{1}{2\sqrt{2}}$ and $\varepsilon = \frac{1}{32}$.

Thus

$$J(z, w) = (k + 1)|z|^{2k}.$$

Because the square $Q$ is the quasiconformal $\infty$-regular domain, by Theorem 4.1 we have

$$\frac{1}{\mu_2(\Omega^*_\varepsilon)} \leq B_{2, 2}(Q) \cdot K \cdot \text{ess sup}_{|z| \leq 1} |J(z, w)| \leq \frac{2(k + 1)^2}{\pi^2}.$$

Here $B_{2, 2}(Q) = \sqrt{2}/\pi$ (see, for example, [26]) is the exact constant for the Poincaré inequality in the square $Q$.

In [32] (see, example 4.2) obtained estimates of constants in weighted Poincaré inequality for stars $\Omega^T_\varepsilon$:

$$C_{p,\lambda}(\Omega^T_\varepsilon) \leq 14\bar{C}_p \left[ \frac{8}{7} \left( \frac{1}{2} + \frac{p}{2\bar{C}_p} \right) \right]^{\frac{1}{p}},$$

where $p \geq 1$ and

$$\bar{C}_1 = \frac{1}{2}, \quad \bar{C}_2 = \frac{1}{\pi}, \quad \bar{C}_p \leq 2 \left( \frac{p}{2} \right)^{\frac{1}{p}}.$$

Note that estimate (4.2) under $0 \leq k < 6$ is a better by comparison with estimate (4.3) for $p = 2$ ($C_{2,\lambda}(\Omega^T_\varepsilon) \leq 4\sqrt{7(1 + 2\pi)}/\pi \approx 9, 09117$).

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