GENERALIZED CONDITIONAL YEH-WIENER INTEGRALS FOR THE SAMPLE PATH-VALUED CONDITIONING FUNCTION

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Abstract. The purpose of this paper is to treat the generalized conditional Yeh-Wiener integral for the sample path-valued conditioning function. As a special case of our results, we obtain the results in [6].

1. Introduction

Let \( t = g(s) \) be a monotonically decreasing and continuous function on \([0, S]\) with \( g(S) > 0 \) and let \( \Omega = \{(s, t) \mid 0 \leq s \leq S, \ 0 \leq t \leq g(s)\} \). Let \( C(\Omega) \) be a space of all real continuous functions \( x \) on \( \Omega \) such that \( x(s, t) = 0 \) for all \((s, t)\) in \( \Omega \) satisfying \( st = 0 \).

In [3], the authors treated the generalized conditional Yeh-Wiener integral which includes the conditional Yeh-Wiener integral in [5] and the modified conditional Yeh-Wiener integral in [1]. In [5–8], Park and Skoug treated the conditional Yeh-Wiener integral for various kinds of conditioning functions including the sample path-valued conditioning function.

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The purpose of this paper is to treat the generalized conditional Yeh-Wiener integral for the sample path-valued conditioning function. We obtain the formula for the generalized conditional Yeh-Wiener integral and then evaluate it for two kinds of functionals. As a special case of our results, we obtain the results in [6].

2. Generalized Conditional Yeh-Wiener integrals for sample path-valued conditioning function

For a functional $F$ of $x$ in $C(\Omega)$, $E(F) = \int_{C(\Omega)} F(x) dm(x)$ is called a generalized Yeh-Wiener integral of $F$ if it exists ([3]). As a stochastic process, $\{(x(s, t))|(s, t) \in \Omega\}$ has a mean $E(x(s, t)) = 0$ and $E(x(s, t)x(u, v)) = \min\{s, u\}\min\{t, v\}$. Let $C[0, g(S)]$ denote the standard Wiener space with the Wiener measure and assume that $\psi$ is in $C[0, g(S)]$.

For a generalized Yeh-Wiener integrable function $F$ of $x$ in $C(\Omega)$, consider the generalized conditional Yeh-Wiener integral of the form

\[(2.1) \quad E(F(x)|x(S, (\cdot, \wedge T) = \psi((\cdot, \wedge T))\]

with $g(S) = T$ and $a \wedge b = \min\{a, b\}$. Here, $(\cdot, \wedge T)$ belongs to $[0, g(s)]$ for $0 \leq s \leq S$. Since two processes $x(S, t \wedge T)$ and $\{x(s, t) - (s/S)x(S, t \wedge T)|(s, t) \in \Omega\}$ are (stochastically) independent, we have

\[(2.2) \quad E(F(x)|x(S, (\cdot, \wedge T) = \psi((\cdot, \wedge T))\]

\[\quad = E(F(x(\cdot, \cdot)) - \frac{s}{S}x(S, (\cdot, \wedge T) + \frac{s}{S}\psi((\cdot, \wedge T)))\]

for almost all $\psi$ in $C[0, T]$. Here, for the notational convenience, we denote $\cdot = (\cdot)$.

Especially, if $g(s) = T$ for all $0 \leq s \leq S$, then $(\cdot, \wedge T) = (\cdot)$, which agrees with (2.2) in [6]. This means that our result (2.2) is a slight generalization of the result in [6].

Let $y(\cdot)$ be a tied-down Brownian motion, that is,

\[\{y(t) \mid 0 \leq t \leq T\} = \{w \in C[0, T] \mid w(T) = \xi\}.

Then, as is well known ([6]), $y(\cdot)$ can be expressed in terms of the standard Wiener process,

\[y(t) = w(t) - \frac{t}{T}w(T) + \frac{t}{T}\xi.\]
The following theorem is one of our main results, which is slightly different from Theorem 1 in [6].

**Theorem 2.1.** If $F \in L_1(C(\Omega), m)$, then we have

\begin{equation}
E_w(E(F(x) \mid x(S, (\cdot) \wedge T) = \sqrt{S}w((\cdot) \wedge T)) = E(F(x)),
\end{equation}

\begin{equation}
E(F(x) \mid x(S, T) = \sqrt{S} \xi)
= E_w \left\{ E \left( F(x) \mid x(S, (\cdot) \wedge T) = \sqrt{S} \left( w((\cdot) \wedge T) - \frac{(\cdot) \wedge T}{T}w((\cdot) \wedge T) + \frac{(\cdot) \wedge T}{T} \xi \right) \right) \right\}.
\end{equation}

**Proof.** (1) Using (2.2), we write

\begin{equation}
E_w(E(F(x) \mid x(S, (\cdot) \wedge T) = \sqrt{S} w((\cdot) \wedge T))
= E_w \left\{ E \left( F(x(s, t) - (s/S)x(S, t \wedge T) + (s/\sqrt{S})\psi(t \wedge T)) \right) \right\}.
\end{equation}

Let $y(s, t) = x(s, t) - (s/S)x(S, t \wedge T) + (s/\sqrt{S})\psi(t \wedge T)$ for all $(s, t)$ in $\Omega$. Then we have $E(y(s, t) = 0$ and $E(y(s, t)y(u, v)) = \min\{s, u\}\min\{t, v\}$. This means that $\{(y(s, t)|(s, t) \in \Omega\}$ is a generalized Yeh-Wiener process, and the right-hand side of (2.5) becomes $\int_{C(\Omega)} F(y)dm(y) = E(F(x))$. Thus, we obtain the formula (2.3).

(2) We use Theorem 2 in [5] to have

\begin{equation}
E(F(x) \mid x(S, T) = \sqrt{S} \xi)
= E \left\{ F(x(s, t) - \frac{\cdot \wedge T}{S}x(S, T) + \frac{\cdot \wedge T}{\sqrt{S}} \psi((::) \wedge T) \right) \right\}.
\end{equation}

We can rewrite the right-hand side of (2.6) as the following form:

\begin{equation}
E \left\{ F(x(s, t) - \frac{\cdot \wedge T}{S}x(S, (\cdot) \wedge T)
+ \frac{\cdot \wedge T}{S}x(S, (\cdot) \wedge T) - \frac{\cdot \wedge T}{T}x(S, T) + \frac{\cdot \wedge T}{T} - \sqrt{S} \xi \right) \right\}.
\end{equation}

We use $E(x(s, t)x(u, v)) = \min\{s, u\}\min\{t, v\}$ to show that two processes $x(s, (\cdot) - (s/S)x(S, (\cdot) \wedge T)$ and $x(S, (\cdot) \wedge T) - ((\cdot) \wedge T)x(S, T)/T)$ are stochastically independent. Furthermore, $\sqrt{S} \left( w((\cdot) \wedge T)(w(T)/T)\right)$ and $x(S, (\cdot) \wedge T) - ((\cdot) \wedge T)x(S, T)/T)$ are equivalent processes, where $w((\cdot)$ is the standard Wiener process. Thus, (2.7) becomes
Therefore, we get the formula (2.4). \hfill \square

For the special case \( g(s) = T \) for \( 0 \leq s \leq S \), we have the same result of Theorem 1 in [6]. In a certain sense, our result is a slight generalization of the result in [6].

In [6], Park and Skoug treated the rectangle \( Q \), but we treat the more general region \( \Omega \). Let \( \Omega \) be the region given by

\[ \Omega = \{(s, t) \mid 0 \leq s \leq S, \ 0 \leq t \leq g(s) \} \]

where \( t = g(s) \) is a monotonically decreasing and continuous function on \([0, S]\) with \( g(S) = T > 0 \). In the following two theorems we evaluate the generalized conditional Yeh-Wiener integral for the sample path-valued conditioning function.

**Theorem 2.2.** Let \( F \) be a functional on \( C(\Omega) \) given by \( F(x) = \int_{\Omega} x(s, t) ds dt \). Then we have

\[
E(F(x) \mid x(S, (\cdot) \wedge T) = \psi((\cdot) \wedge T)) = \frac{S}{2} \int_0^T \psi(t) dt + \frac{\psi(T)}{S} \int_0^S \int_0^{g(s)} s dt ds.
\]

**Proof.** By (2.2) and Fubini theorem, we have

\[
E(F(x) \mid x(S, (\cdot) \wedge T) = \psi((\cdot) \wedge T)) = \int_{\Omega} E\left( x(s, t) - \frac{s}{S} x(S, (\cdot) \wedge T) + \frac{s}{S} \psi((\cdot) \wedge T) \right) ds dt.
\]

\[
= \int_{\Omega} \frac{s}{S} \psi((\cdot) \wedge T) ds dt.
\]
The right hand side of the last equality in (2.10) comes from the fact that $E(x(s, t)) = 0$ and $m(C(\Omega)) = 1$. By the straightforward calculation, we have

$$(2.11) \quad E(F(x) \mid x(S, (\cdot) \wedge T)) = \psi((\cdot) \wedge T)$$

$$= \int_0^S \int_0^{g(s)} \frac{s}{S} \psi((\cdot) \wedge T) dtds$$

$$= \frac{S}{2} \int_0^T \psi(t) dt + \frac{\psi(T)}{S} \int_0^S \int_T^{g(s)} s dt ds,$$

which is our desired result.

**Theorem 2.3.** Let $F$ be a functional on $C(\Omega)$ given by $F(x) = \int_\Omega x^2(s, t) ds dt$ and $g(S) = T$. Then we have

$$(2.12) \quad E(F(x) \mid x(S, (\cdot) \wedge T)) = \psi((\cdot) \wedge T)$$

$$= \frac{S^2 T^2}{12} + \frac{S}{3} \int_0^T \psi^2(t) dt + \int_0^S \int_T^{g(s)} \left( st - \frac{s^2}{S} T + \frac{s^2}{S^2} \psi^2(T) \right) dt ds.$$

**Proof.** By (2.2) and Fubini theorem, we have

$$(2.13) \quad E(F(x) \mid x(S, (\cdot) \wedge T)) = \psi((\cdot) \wedge T)$$

$$= \int_\Omega E \left( \left( x(s, t) - \frac{s}{S} x(S, (\cdot) \wedge T) + \frac{s}{S} \psi((\cdot) \wedge T) \right)^2 \right) ds dt$$

$$= \int_\Omega \left\{ st - \frac{s^2}{S} (t \wedge T) + \frac{s^2}{S^2} \psi^2(t \wedge T) \right\} dt ds.$$

The right hand side of the last equality in (2.13) comes from the fact that $E(x(s, t)) = 0$, $E(x(s, t)x(u, v)) = \min\{s, u\} \min\{t, v\}$ and $m(C(\Omega)) = 1$. By the straightforward calculation, the right hand side of the last equality in (2.13) becomes

$$(2.14) \quad \frac{S^2 T^2}{12} + \frac{S}{3} \int_0^T \psi^2(t) dt$$

$$+ \int_0^S \int_T^{g(s)} \left( st - \frac{s^2}{S} T + \frac{s^2}{S^2} \psi^2(T) \right) dt ds,$$

which is our desired result. □
Remark 2.4. In Theorem 2.2 and Theorem 2.3, we have the extra terms which does not appear in Example 1 and Example 2 of [6]. This means that Park and Skoug’s examples in [6] are the special case of our results for the rectangle Ω.

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