A FEW PROJECTS IN STRING THEORY

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Abstract

In these lectures I discuss various unsolved problems of string theory and their relations to quantum gravity, 3d Ising model, large N QCD, and quantum cosmology. No solutions are presented but some new and perhaps useful approaches are suggested.
1 Introduction

I consider string theory as a set of unifying concepts and methods in physics. Technically speaking it consists of conformal field theories coupled to the self induced two dimensional gravitational field. Physically, it describes random surfaces with different extra degrees of freedom, immersed into different ambient spaces.

Such objects are frequent in nature - phase boundaries in critical phenomena or the surfaces of biological membranes are notable examples.

In a somewhat more subtle way random surfaces appear in confining gauge theories. Here the lines of the color - electric flux form a closed string. The transition amplitudes, which are given by sums over histories, are expressed now in terms of random surfaces, formed by propagating strings.

Probably the most fascinating application of the string ideas is the theory of many dimensional gravity. Here one views a graviton as a tiny closed string. Under certain conditions this approach gives the only consistent theory of quantum gravity known today.

There are several other applications of the techniques of string theory. For example, conformal field theory describes most of two- dimensional statistical physics, the Kondo problem, polymers branching and perhaps two dimensional turbulence.

It is important, when working with strings, to keep in mind all these complementary aspects of the subject.

In these lectures I describe several projects concerning different parts of physics but united by their relations to strings. My choices are determined by personal tastes and there is no pretence for “objectivity” whatever it means. Even within this limited perspective I omitted the fascinating subject of conformal turbulence, since I have discussed it recently elsewhere. I concentrate on the open problems which are rarely considered in the literature.

2 Quantum Gravity

In the conventional theory of quantum gravity one begins with the notion of continuous and curved space-time. One believes, that there exists a fluctuating metric tensor $G_{\mu\nu}(x)$ which in the vacuum has non-zero Minkowskian expectation value.

In the classical limit this value satisfies the Einstein equations. The size of the quantum corrections at the energy $E$ is of the order of $\frac{E^2}{M^2}$ where $M$ is the Planck mass. Let us critically discuss this situation.

First of all, should space- time be continuous? It seems to me that the answer is no. Imagine the world which consists of the tangled network of world-lines. Events in this world consist of the crossing points of these lines. Our measuring of time is based on
counting the number of crossings, separating two given events. If so, the lapse of the time, when “nothing happens” is a meaningless notion.

So, most probably, the continuity of space-time is just an approximation similar to the one we use in condensed matter or hydrodynamics.

Of course, just the idea of discreteness, even together with the correspondence principle is not enough to build the theory. It is somewhat ironic that the discreteness, while being the most elementary of all possible notions, makes the theory apparently intractable. That is why we will concentrate on the continuous theories. Let us notice, however, that in two dimensions there exists an attractive discrete theory of gravity based on the matrix models, with the correct continuous limit.

Passing to the continuous theory let us discuss if we can use the Einstein action as a starting point for the quantum theory. The problem here is non-renormalizability: the quantum corrections, diverge at very high energies (and the energies of the virtual particles can be arbitrary high).

A possible way out is to conjecture that the coupling constant in gravity is scale dependent and tends to zero at small distances. This “antiscreening” of the gravitational interaction is quite natural since the larger is the cloud of the virtual particles, the stronger the gravitational force is.

It is instructive to compare the situation with the one in other non-renormalizable theories. Let us consider a non-linear $\sigma$ - model

$$L = \frac{1}{2\kappa_0} \left( \partial_\mu \vec{n} \right)^2, \quad \vec{n}^2 = 1, \quad \vec{n} = \left( \sqrt{1 - \vec{\pi}^2}, \vec{\pi} \right)$$  \hspace{1cm} (1)

The $\pi$ - meson field is the analogue of the gravitation. In four dimensions, perturbative interaction of pions as well as of the gravitons leads to the scale-dependence of the effective coupling $\kappa(p)$ :

$$\frac{\kappa(p)}{\kappa_0} = 1 + \text{const} \left( \kappa_0 p^2 \right) + \cdots$$  \hspace{1cm} (2)

(where $p$ is the momentum) formally similar to gravity.

The theory has a phase transition at some coupling. In order to reach the continuum limit, one has to force the coupling $\kappa_0$ to be close to the critical value. In this case the theory depends on one mass scale $M$ and the effective interaction behaves as:

$$\kappa \left( p^2 \right) \propto p^{-2}$$  \hspace{1cm} (3)

at ultra high energy ($p^2 >> M^2$) while in the low energy limit we simply have:

$$\kappa \sim M^{-2}$$  \hspace{1cm} (4)

To be precise, there are also logarithmic corrections to these formulas which we disregarded.
The physics in the above two regions is very different. Consider an expectation value of the field \( \vec{n} \). If we examine a domain of size smaller than the correlation length we find that the broken and unbroken phases are indistinguishable, while at low energies we have pions due to the symmetry breaking.

## 3 Phases in gravity

Let us return to gravity. Can we have a similar picture? Is it possible, that the effective Newton constant behaves as (3)? What is the meaning of the unbroken phase in gravity? There are no ready answers to these questions. I shall only make a few comments on them.

The problem with the running Newton constant is essentially that we have no calculational procedures to deal with the Einstein action. Some hope here may lie in the fact, that the structure of the operator algebra should be simplified due to the general covariance. It might consist of the few primary operators, plus infinite sets of their deformations caused by the diffeomorphisms. This is the picture we have in conformal field theories in which such a structure allows to obtain exact solutions. In the Einstein theory we are quite far from that, but it is worthwhile to keep in mind this possibility. It seems plausible that general covariance drastically simplifies any field theory.

The question of the unbroken phase in gravity presents not only computational, but also some conceptual difficulties.

In the most naive form, we would say that in the broken phase:

\[
< G_{\mu\nu} > = \delta_{\mu\nu} \tag{5}
\]

while in the unbroken phase:

\[
< G_{\mu\nu} > = 0 \tag{6}
\]

In some sense (5) means that we have non-zero density of the “space-time substance”.

The problem with this definition is that (5) is a gauge-dependent statement. It would be nice to have some gauge invariant criterion. One possibility is to consider a quantity:

\[
A(\mu) = < \exp - \mu \oint_C \sqrt{G_{\mu\nu}(x)} dx^\mu dx^\nu > \tag{7}
\]

This represents a particle forming a closed loop in our manifold. This amplitude is independent of the shape of the loop \( C \) since we average with respect to all possible geometries.

The test of the geometry can be performed by sending \( \mu \to \infty \). In order to avoid some trivial renormalization problems it is more convenient to look at the equivalent object, which is a Laplace transform of (7):

\[
B(L) = < \exp - \int_0^L G_{\mu\nu}(x(s)) \dot{x}^\mu(s) \dot{x}^\nu(s) ds > \tag{8}
\]
This quantity is some analogue of the Wilson loops in gauge theory. Its $L$ - independence at small $L$ measures effective dimensionality of the space-time, which is of course a scale dependent notion.

If the metric is flat, $B(L)$ has the standard, “heat kernel” behavior:

$$B(L) \sim L^{-D/2}$$

(9)

On the other hand in the regime (6) metric $G$ and $\lambda G$ may enter with the same weight (the Weyl symmetry being unbroken) in which case we expect:

$$B(L) \sim \text{constant}$$

(10)

This means that the effective dimensionality is zero. Of course, there could exist phases with intermediate dimensionality.

Unfortunately, in general we don’t have methods to study such questions. However this can be done in the case of 2d gravity.

4 Self-tuning universe

There is another serious problem with quantum gravity. It seems that we need a hyperfine tuning for the cosmological constant. Actually it is even worse than that, we know that very many different constants in physics conspire so as to allow a non-trivial universe. It takes a tiny change in the fundamental constants to stop nucleosynthesis, extinguish the sun etc. See recent discussion in [1].

All that may imply that the fundamental constants are in fact some self-tuning fields, and the really good universe is an attractor. At the present level of our knowledge, the best we can do is to play with some toy models which have self-tuning phenomenon.

Today we know several different mechanisms for self-tuning. One of them is based on the idea of uncontrollable emissions of tiny “baby-universes” [2]. This mechanism has been extensively discussed in the literature. The problem with it is the absence of reasons for the locality of coupling of the baby universes to the main one. Still it is an interesting possibility.

Another idea, which may work for the cosmological constant is the following [3]. The cosmological term in the Einstein action does not contain any derivatives of $G_{\mu\nu}$. In this case one would expect strong infrared interaction, at distances much larger than the Planck length.

For example, if one considers conformal fluctuations of the metric,$G_{\mu\nu} = \phi^2 \delta_{\mu\nu}$, the cosmological term is just a $\phi^4$ - interaction.

Such interactions tend to be self-screened, due to the “zero charge” phenomenon. Thus starting from the “bare” cosmological constant $\Lambda_0$ one gets the physical $\Lambda \to 0$. In the
“logarithmic” approximation, one gets:

$$\Lambda \sim \frac{1}{\log MR} \approx 0$$  \hspace{1cm} (11)$$

where $M$ is the Planck mass and $R$ is the inverse local curvature. Unfortunately this approximation is not adequate, but the “infrared screening” idea may be valid anyway. As we will discuss later, one should add the dilaton field and consider infrared graviton-dilaton fluctuations together. This has not been done yet.

A third self-tuning mechanism appears in the $c = 1$ models for string theory. In this case one has an infinite tower of tensor fields $B_{\mu_1 \ldots \mu_N} (x)$ representing higher string excitations.

However, string gauge invariance leads to the following. The most part of $B (x)$ is just a gauge artefact. By a gauge transformation one can replace the functions $B_{\mu_1 \ldots \mu_N} (x)$ by the set of constants $b_{\mu_1 \ldots \mu_N}$. Thus, instead of the full-fledged fields, in this case we will be dealing with the set of varying coupling constants $b_{\mu_1 \ldots \mu_N}$ which are just the remnants of $B - s$. A simple example of this phenomenon occurs with the two-dimensional vector quanta. They are described by the polarization $A_{\mu} (p)$, satisfying Lorentz condition:

$$p_{\mu}A_{\mu} (p) = 0$$

and defined modulo gauge transformations:

$$A_{\mu} (p) \rightarrow A_{\mu} (p) + p_{\mu}\phi (p)$$

For $p \neq 0$ no states remain, but for $p = 0$ the gauge transformation and the constraint disappear, leaving us with the $p = 0$ physical state.

In general, these remnants of higher string excitations must be described by a topological field theory. It is still to be found.

The above mechanism is known to operate in two dimensions only. It is unclear, whether it is generalizable to higher dimensions.

To summarize this part - we probably need some self-tuning mechanism in our universe and we have few toy models for it, but a realistic theory is absent.

## 5 The Dilaton

The dilaton may play an important role in such a theory. It appears naturally in any string-theoretic description of gravity. Let us try to explain its origin in the most general terms.

The Einstein theory describes a massless tensor field $G_{\mu\nu} (x)$ with the maximal possible gauge group, consisting of the diffeomorphisms on a given manifold, $(DiffM)$. This group removes all negative norm states of this field and leaves us with the physical gravitons.
Let us now try a different logic. Instead of postulating the maximal gauge group, let us look for the minimal one needed for elimination of the negative norms. Negative norms carried by tensor fields arise due to the Minkowskian signature of space-time. For example, the vector potential, describing photons has time-like components, corresponding to oscillators with negative norms. In QED gauge invariance guarantees that these ghost states are never emitted in real processes. The gauge transformations in this case contain one arbitrary function. Hence, the “size” of the gauge group matches the number of negative fields. In gravity the negative states are produced by the components \( G_{0k}(\text{where } k \text{ is a space-like index}) \). Hence we need a gauge group containing \( d-1 \) arbitrary functions (where \( d \) is a dimension of the manifold). It is easy to see that the simplest choice -the group of volume preserving diffeomorphisms - eliminates all unwanted states. Since it is smaller than the Einstein group of all diffeomorphisms, it gives us an extra physical state -the dilaton. This is easy to see from the invariant action:

\[
S = \int \left( a \left( \sqrt{G} \right) R + b \left( \sqrt{G} \right) + \ldots \right) d^4x
\]

Here \( G = \det(G_{\mu\nu}) \), while \( a \) and \( b \) are some unknown functions. This action can be “covariantized” by adding the dilaton field, multiplying the determinant of the metric. The above form, however is helpful for finding low energy theorems since it relates dilaton emission to the reduced covariance.

It is quite clear that in the theories with dilatons all fundamental couplings become dilaton - dependent. That means that if the dilaton remains massless, it provides us with another self - tuning mechanism. A big question is whether this masslessness is consistent with the known cosmology and the equivalence principle. To answer this question one needs to know the low energy effective action to all orders in the string coupling. This is another big question. This action must be local, since all non-local terms arising from the massless exchanges contain high powers of momenta. A possible clue to the problem is the restricted covariance mentioned above, since in string theory it appears through the anomaly in the complete covariance and thus might be tractable. I believe that the popular opinion that the dilaton must be massive is unwarranted before these big questions are answered.

6 The Big Bang

Talking about cosmology, I would like to mention the unusual picture of Big Bang in string theories, which has to modify our views on inflation. Let us consider Friedman ansatz:

\[
ds^2 = -dt^2 + a^2(t) d\tilde{n}^2
\]

Here \( \tilde{n}^2 = 1 \) represents a space element with constant curvature.

In string theory one has to add the dilaton field \( \Phi(t) \) and to consider (in the one loop approximation) the \( \beta = 0 \) equations \([4]\) which replace the Einstein equation. Just as in
the usual case, it gives a Big Bang singularity, represented by the vanishing of \( a^2(t) \) at some moment of time.

In order to understand the origin of this result it is helpful to take a different view on the string theory in the background. Namely, let us introduce \( t = i\phi \) and consider a problem of the non-linear \( \sigma \)-model, described by the \( \vec{n} \)-field coupled to the two-dimensional gravitational field, described (in the conformal gauge) by the Liouville field \( \phi \). In this interpretation, the field \( \phi \) defines a scale, while \( a^2 = a^2(\phi) \) is an inverse running coupling constant for this model. It is easy to write the standard \( \beta \)-function equations for \( a^2(\phi) \) and the dilaton \( \Phi(\phi) \):

\[
\begin{align*}
da' a\Phi' + a''a + (N - 2)(a')^2 &= N - 2 \quad (N - 1)\frac{a''}{a} + \Phi'' &= 0
\end{align*}
\]

(14)

Now, the Big Bang singularity which appear in these equations is nothing but the standard logarithmic pole signifying asymptotic freedom. Of course, this pole is somewhat dressed by gravity. The theory of gravitational dressing is a fascinating subject by itself, which I hope to discuss elsewhere. For the present purposes it is sufficient to know that in one loop approximation gravity doesn’t introduce qualitative changes of the formulas of asymptotic freedom.

The undressed formula for \( a^2 \) is given by [5]:

\[
a^2(\lambda) = a^2(\lambda_0) - \frac{N - 2}{2\pi} \log \frac{\lambda}{\lambda_0}
\]

(15)

Here \( \lambda \) is the scale and is related to the Liouville field. This relation is somewhat complicated in general, but for large \( \phi \) and \( \lambda \) it simplifies to:

\[
\phi \propto \log \lambda
\]

(16)

The solution of eq. (15) will give the same vanishing of \( a^2 \) as a function of \( \phi \) except that it will not be a simple zero in terms of \( \phi \).

Now, we come to the crucial point.

As was explained in [5] this pole is just an artefact of the one-loop approximation. What happens in reality is that the mass gap is formed and the growth of interaction stops. In terms of \( a^2(\phi) \) it means that before the Big Bang we enter a novel regime with the massive \( \vec{n} \)-field. It seems that the finite mass gap of the \( \vec{n} \)-field implies that the metric tensor and coordinates lose their classical meaning. In a very early universe space-time does not exist. Instead we have some string Hilbert space description of the universe which in some approximation gives the metric tensor, but the approximation breaks at the logarithmic pole. No singularities are present and the “big bang” was not that big after all.
The physics is hidden in the amplitudes:
\[ A(l_1m_1 \ldots l_nm_n) = \langle V_{l_1m_1} \ldots V_{l_nm_n} \rangle \]
\[ V_{lm} = \int d^2\xi Y_{lm}(\vec{n}(\xi))\psi_l(\phi(\xi)) \] (17)

Here \( Y_{lm} \) - are the spherical functions and the Liouville eigenfunctions \( \psi_l \) are determined from the conformal invariance.

In the WKB limit it must be possible to reconstruct the space-time metric from the amplitudes (17), but beyond that we should view (17), as a set of numbers characterizing the state of the universe, its whole history.

When passing to Minkowskian regime, we have, perhaps, to change simultaneously a sphere \( \vec{n} \) to hyperboloid in order to have vanishing of \( a^2(t) \) at real time. But this requires further investigations.

In any case in this unusual situation one has to reconsider all the standard problems (horizon etc.) which led to the inflation scenario. It is clear that the argument that the homogeneity of the Friedman universe requires acausal connections at the early times is not very restrictive in our picture. Indeed, these early times are described by the wave functional of the non-linear \( \sigma \) - model, and there are acausal connections of the Einstein-Podolsky -Rosen type. May be inflation (whatever it means in this context) is simply unnecessary, although the “flatness” and the “monopoles” problems are still to be resolved.

Of course, the above sketch is not a complete theory but a tempting program for future research.

7 Fuzzy strings and 3d Ising model

Some years ago I have suggested that the three-dimensional Ising model can be reduced to the theory of free strings [6]. This suggestion has been further developed in a number of papers [7]. My basic argument was the following. Consider first the 2d - Ising model. In this case we have the order parameters \( \sigma_{\vec{x}} \) and disorder parameters \( \{\mu_{\vec{x}}\} \), each of which satisfy complicated equations. It is possible, however to form fermionic fields \( \psi_a(\vec{x}) \) where \( \vec{x} \) is a point of the lattice and \( a = 1,..4 \) denotes one of the four possible directions from \( \vec{x} \) to the center of the adjacent plaquette.

It is common knowledge by now that \( \psi_a(\vec{x}) \) satisfy linear equations, which in the continuous limit become 2d Dirac equations:
\[ i\vec{\tau} \frac{\partial}{\partial \vec{x}} \chi = m\chi \] (18)

with \( \chi_\alpha(\vec{x}) \), \( \alpha = 1,2 \) being \( spin_\frac{1}{2} \) component of \( \psi_a \) while \( spin_2 \) component is not propagating (see e.g. [8] for details).
In three dimensions we have order variables $\sigma_{\vec{x}a}$ attached to each link (we treat the model as $Z_2$ gauge theory) and the disorder variables placed at the centers of the cubes. Let us consider the objects in the loop space:

$$\Psi_{a_1...a_L}(C) = \prod_C \sigma_{\vec{x}a} \prod \mu_{\vec{x}a}$$

(19)

(where $C$ is a closed loop of the length $L$, and $a = 1,..,4$ denotes one of four possible cubes adjacent to the link $(\vec{x}\alpha)$).

It is easy to see that this object satisfies a linear equation in the loop space (modulo contact terms). The precise form of these equations is given in [7]. They somewhat simplify in the hamiltonian version of this theory. Namely, if, by squeezing the lattice in the “time” direction we pass to the continuous time, we obtain planar loops with arrows looking inside or outside the loop. Correspondingly we introduce the object $\Phi_{\alpha_1...\alpha_L}(C; t_1...t_L)$ with $C$ being a planar loop of the length $L$ while $\alpha = 1,2$ describes the direction of the arrow. By simple manipulations one gets the equations

$$\frac{\partial}{\partial t_s} \Phi (C) = u \tau_s^x \Phi (C) - v \Phi (C + \tau_s^z \Pi)$$

(20)

Here $\tau - s$ are Pauli matrices acting on the index number $s$, the symbol in the last argument means that the link number $s$ is replaced by the outside looking letter $\Pi$ if $\alpha_s = 1$ and by the inside looking $\Pi$ otherwise. In this form (4) is true only for convex contours. Otherwise, there are some local sign changes. Quantities $u$ and $v$ are related to the parameters of the Ising hamiltonian.

Equations (20) are the counterpart of the 2d Ising equations. They have a very simple physical meaning. Namely they imply that each bit of the 3d Ising string moves as a spinning particle. To see it, compare (20) with the lattice Dirac equation:

$$\frac{\partial}{\partial t} \psi = u \tau^x \psi (x) - v \psi (x + \tau^z)$$

(21)

Here again the last argument means the jump to the left or to the right, depending on the direction of spin.

In the 2d Ising case it is straightforward to take the continuum limit of (20) which gives the Dirac equation (18).

However, any reliable technique to deal with the loop space equations (19) will perhaps appear in the next century. In the present paper I shall try to guess what will be the result of these future considerations. Of course, it is dangerously easy in these circumstances to chose an inappropriate universality class, or in plain language, to go astray. The only consolation in this case is that the universality class may be interesting enough by itself to deserve the study.

It is obvious, first of all, that we are dealing with a string theory, which carries spin density on its world sheet and that the motion of the string is correlated with the directions of the spin i.e. we have spin-orbit interaction. Also, as was explained above,
each bit of the string moves as in a plane orthogonal to the corresponding link as a
two-dimensional fermion. Therefore in a thoughtless continuous limit (which would work
nicely for particles) we should replace (20) by the following:

\[ \tau^x (s) \frac{\delta \Psi (C)}{\delta t (s)} + \tau^z (s) \frac{\delta \Psi (C)}{\delta x_\perp (s)} = m \Psi (C) \]  

(22)

Here the sign \( \perp \) means the functional derivative in the direction orthogonal to \( \frac{d}{ds} \vec{r} \).

If we wish this equation to make sense, we have to specify, what we mean by \( \vec{\tau} (s) \), or what is the string generalization of the Dirac matrices. One possible answer to this
question is well known. It is beautiful construction of Ramond [9] and Neveu-Schwartz [10]. These authors generalized the Dirac equation by introducing anti-commutativity
condition for \( s \neq s' \) among Dirac matrices, or, in other words, by replacing them by new
fermion fields. That lead to the super-symmetric string action of NSR. It seems, however,
that 3D-Ising wave equation (22) requires a different generalization of the Dirac equation.

The reason is simple. If we take \( s \) and \( s' \) far apart, matrices \( \tau (s) \) and \( \tau (s') \) become
independent and commute, rather then anticommute. There are no Jordan-Wigner
factors in the equation (22).

After this is understood, it immediately comes to mind to replace \( \vec{\tau} (s) \) by the SO(3)
current algebra with, perhaps, a central extension.

As we shall see, this is close to the truth but not quite correct since there are com-
plicated renormalizations. The problem now is to find a continuous string theory which
incorporates the field \( \vec{\tau} \) and couples it to the string field \( \vec{x} \). If we forget about the \( \vec{x} \)- field
for a moment, we can easily write the action, which describes fluctuations of \( \vec{\tau} \). It is well
known [11], that the one-dimensional Heisenberg antiferromagnet is described by the \( \vec{n} \)-
field with the \( \theta \)- term:

\[ S_1 = \int \left( \frac{1}{2\alpha} (\partial_a \vec{n})^2 + i \pi \varepsilon^{ab} \vec{n} [\partial_a \vec{n} \partial_b \vec{n}] \right) d^2 \xi \]  

(23)

and the \( \vec{n} \)-field can be identified with the Heisenberg spins \( \vec{\tau}_s \). The model is gapless and
is exactly soluble by different methods.

Now, at the critical point of the Ising model, equation (22) can be interpreted as an
orthogonality of \( \vec{\tau} \) to the direction of string propagation. A natural covariant formulation
of this condition is:

\[ \vec{n} (\xi) \partial_a \vec{x} (\xi) = 0 \]  

(24)

or in other words, we require that the spins, living on the string must be orthogonal
to the world sheet.

This is not a correct effective action yet. As we will see in a moment, the constraint
(24) does not survive in renormalization. Indeed, let us add to \( S_1 \) a Lagrange multiplier
ensuring the constraint (24):

\[ S_2 = \int \lambda^a (\vec{n} \partial_a \vec{x}) \, d^2 \xi \]  

(25)
One can compute then the induced action \( S_3(\lambda^a) \). It can be done directly by expanding it in powers of \( \lambda \). Since \( \lambda \) has dimension one, we expect and get logarithmic counterterms to the induced action. They have the form

\[
S_3 \propto \log \Lambda \int (\lambda^a)^2 \, d^2 \xi
\]  

(26)

As usual, it means that to keep the theory renormalizable one has to introduce a new coupling constant in front of (26) and to examine the two couplings renormalization group. If we do this and integrate out the field \( \lambda \), we obtain:

\[
S_4 = f \int (\vec{n} \partial_a \vec{x})^2 \, d^2 \xi
\]  

(27)

where \( f \) is a new coupling. The geometrical meaning of the replacement of the constraint (24) by the action (27) is transparent: as we renormalize and go to the block-variables the constraint gets smeared and what was the normal becomes a tilted vector on the surface. We can now write the effective action, describing the string with spin density and spin-orbit interaction:

\[
S = \int \left( \frac{1}{2} \sqrt{g} g^{ab} \partial_a \vec{x} \partial_b \vec{x} + \mu \sqrt{g} + \frac{1}{2\alpha} \sqrt{g} g^{ab} \partial_a \vec{n} \partial_b \vec{n} + f \sqrt{g} g^{ab} (\vec{n} \partial_a \vec{x}) (\vec{n} \partial_b \vec{x}) \right) d^2 \xi \\
+ i\pi \int \varepsilon^{ab} \vec{n} [\partial_a \vec{n} \partial_b \vec{n}] \, d^2 \xi
\]  

(28)

The last term in this expression is not an Euler character of the surface, since \( \vec{n} \) is not an exact normal. One can say that renormalization destroys Gauss-Bonnet theorem.

An important feature of the action (28) is its renormalizability. There are highly non-trivial trajectories of all couplings involved, which up to now I explored only in the one loop approximation. However, the theory may be exactly solvable. Indeed, its “\( n \)” part can be reduced to the \( k = 1 \) Wess-Zumino-Novikov-Witten model, and as expected represents Kac-Moody currents, coupled to the world sheet. Of course, one should also couple it to the 2d gravity described by \( g_{ab} \), so all in all it is not an easy problem. But if solved, it has chances to describe 3d Ising model and 3d critical phenomena in general.

8 The QCD string

From the very early years of QCD, on the basis of \( 1/N \) expansion [12], and strong coupling expansion [13] it was suspected that it is somehow related to string theory. Later equations in the loop space, satisfied by the phase factors have been examined [6, 14, 15]. I had a hope to find some string lagrangian, such that the sum over surfaces satisfied these loop equations, much in the same way in which sums over paths satisfied the Klein-Gordon equations.

In other words, the hope was for an exact duality between the gauge field representation and the string representation of QCD. In spite of many subsequent efforts [16, 17] the problem remains unsolved.
In this section I will try to summarize where we stand now. The main object in the theory is the phase factor (the Wilson loop):

\[ \Phi (C) = \langle Tr P \exp \oint_C A_\mu dx^\mu \rangle \]  

(29)

where \( P \) - is an ordering sign and the brackets mean averaging in QCD vacuum. This quantity, together with more complicated correlations:

\[ \Phi (C_1 \ldots C_n) = \langle Tr P \exp \oint_{C_1} A_\mu dx^\mu \ldots Tr P \exp \oint_{C_n} A_\mu dx^\mu \rangle \]  

(30)

satisfy the chain equations in loop space. In the large \( N \) limit we have a relation:

\[ \Phi (C_1 \ldots C_n) \approx \Phi (C_1) \ldots \Phi (C_n) \]  

(31)

and hence one obtains a closed equation for \( \Phi (C) \). Let us explain the structure of this equation and its possible solutions.

The basic idea is to find such an operator in the loop space, that being applied to \( \Phi (C) \) it will give on the right hand side an expression proportional to the Yang-Mills equations of motion. It is easy to see, that the required operator has the form:

\[ \frac{\partial^2}{\partial x^2 (s)} = \lim_{a \to 0} \int_{-a}^{a} dt \frac{\delta^2}{\delta x (s + \frac{1}{2}) \delta x (s - \frac{1}{2})} \]  

(32)

and that the equation for \( \Phi (C) \) has the form:

\[ \frac{\partial^2}{\partial x^2 (s)} \Phi (C) = 0 \text{ (modulo contact terms)} \]  

(33)

where the contact terms, described in the old papers [16, 17] are nonzero only for self-intersecting loops.

Our aim is to find sum over surfaces representation which satisfies equations (33), in the same way, in which Feynman’s sums over paths satisfy the Klein - Gordon equation.

At this point we see the problem. All conventional string theories have string wave equations, which contain an operator \( \frac{\delta^2}{\delta x^2 (s)} \) instead of (32). The reason for that is the presence of the \( (\partial x)^2 \) term in the string action, which is essentially the Nambu term. This term, containing the square of the time derivative of \( x \) unavoidably leads to the second order wave equation. At the same time the operator (32) is of the first order. Indeed, for any two functionals, \( A \) and \( B \) we have:

\[ \frac{\partial^2}{\partial x^2 (s)} (AB) = A \frac{\partial^2}{\partial x^2 (s)} B + \left( \frac{\partial^2}{\partial x^2 (s)} A \right) B \]  

(34)

We come to the conclusion that we have to find a string theory in which the Nambu term is absent or irrelevant, and try to adjust the action so that the wave operator takes
the form (33). The second requirement of course is the correct form of the contact term. In what follows, I will attempt to resolve the first part, while the second one remains unclear.

In the standard string theory we often introduce background fields—the metric \( G_{\mu\nu} \), the antisymmetric field \( B_{\mu\nu} \), etc. It is clear from the above that in order to avoid the problem with the second derivative we have to make the following radical assumption about the background:

\[
G_{\mu\nu} = 0, \tag{35}
\]

while retaining the \( B_{\mu\nu} \) term, which contains the time derivatives in the first power.

Consider the following functional integral:

\[
Z = \int d\mu (B) \int D x (\xi) \exp \int d^2 \xi B_{\mu\nu} (x (\xi)) \varepsilon^{ab} \partial_a x^\mu \partial_b x^\nu \tag{36}
\]

Here \( Z \) is a partition function, \( x (\xi) \) describes a closed surface, and \( d\mu (B) \) is an unknown measure for the antisymmetric field. In order to use the loop equation (33) one has to consider an analogous expression for the surfaces with the boundary. Such expressions are even more difficult to define in the precise sense. However if one has the audacity to disregard multiple problems with the regularization, the result would be:

\[
\frac{\partial^2}{\partial x^2} \Phi (C) = \langle \partial_\mu B_{\mu\nu} (x (s)) \dot{x}^\nu (s) \rangle \tag{37}
\]

It is easy to adjust the B-measure so that the RHS of (37) is zero. This happens to be the case for the Gaussian measures. Unfortunately, under these naive operations the contact terms don’t come out right and the problem remains unsolved. The only additional comment which I can make concerns the relation between the above models and the theory of “rigid strings” [18]. For the gaussian B-fields they are connected via the formula:

\[
\int d\sigma^{\mu\nu} (\xi) d\sigma^{\mu\nu} (\eta) \delta (x (\xi) - x (\eta)) = \text{const} A + \int K^2 d^2 \xi \tag{38}
\]

Here \( A \) is the area of the surface, \( K \) is an extrinsic curvature and :

\[
d\sigma^{\mu\nu} = \varepsilon^{ab} \partial_a x^\mu \partial_b x^\nu d^2 \xi \tag{39}
\]

The constant in (38) is quadratically divergent and perhaps should be set to zero. Most probably, some type of the \( \vartheta \) -term will be needed for the theory to make sense. These terms for rigid strings serve to restore unitarity, apparently broken by the higher derivatives [18]. One of the ways to check the above representation may be to compare it with the recent results in two-dimensional QCD [19] which suggest “stringy” structure in this case. Encouraging signs for such a comparison are irrelevance of the folds and volume-preserving diffeomorphism invariance in both cases. But much more still has to be done.
9 Scale dependence in quantum gravity

One of the most important tools of quantum field theory is the renormalization group. It tells us how different quantities depend on the overall scale. In order to approach the problems described in the preceding sections we need to develop a similar apparatus in the presence of the two-dimensional quantum gravity. Here again we know only few elementary facts, while the problem as a whole is still mysterious.

First of all, there is some conceptual puzzle in the question of scale dependence. It is well known that in the usual theories scale transformations are generated by the trace of the energy-momentum tensor, $T_{aa}$. At the same time, in any theory of gravity, all components of the energy-momentum tensor are zero or, if one wishes to fix a gauge, are BRST commutators.

Let us first resolve this puzzle in the most general terms. The apparent vanishing of the energy-momentum tensor follows from the relation:

$$\int Dg_{ab} \frac{\delta}{\delta g_{ab}(\xi)} \exp(-S(g_{ab})) = 0 (?) \quad (40)$$

Here $S$ is the action, the $g_{ab}$ is a two-dimensional metric. We have used a somewhat symbolic notations in which we don’t fix the gauge explicitly. More careful definitions wouldn’t change the result and hence are not needed. Since:

$$T^{ab} = \frac{\delta S}{\delta g_{ab}} \quad (41)$$

we come to the above mentioned conclusion, that all matrix elements of the energy-momentum tensor are zero. This conclusion is wrong. The source of the error is the boundary in the space of all geometries, which produces non-zero boundary contribution to the integral (40). Indeed, if the metric $g_{ab}$ has two eigenvalues, $\lambda_{1,2}$ we must constrain the integration region in (40) by the condition:

$$\lambda_{1,2} \geq 0 \quad (42)$$

An interesting conclusion which we can draw from this consideration is that the scale dependence or the $\beta$ function in the theory of gravity must be determined by degenerate geometries, say by pinched spheres if the overall topology is spherical.

Let us first demonstrate how this idea works in the most trivial case - the theory of random paths. Consider the action for the path in the background field $G_{\mu\nu}$:

$$S = \int G_{\mu\nu}(x(\tau)) \dot{x}^{\mu} \dot{x}^{\nu} h^{-1}(\tau) d\tau \quad (43)$$

Here $h(\tau)$ is a one dimensional metric on the path. Let us consider now the partition function, defined by:

$$Z = \int (\exp -S) Dh(\tau) Dx(\tau) \quad (44)$$
The scales in the ambient space and in the internal space are related. Take the Weyl variation of $Z$:

$$G_{\mu\nu} \frac{\delta Z}{\delta G_{\mu\nu}} = \langle \int T(\tau) \delta (x - x(\tau)) h(\tau) d\tau \rangle$$  \hspace{1cm} (45)

Here $T(\tau) = h^{-2}G_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$ is the one dimensional metric tensor. Since we have to integrate over the metrics $h(\tau)$ the naive expectation would be that the RHS of (45) is zero. In fact it is described by the boundary in the space of all 1d geometries, which consists of the loops of zero (or better to say infinitesimal) length. These loops contribute to the RHS of (45) a term, which is local in $G_{\mu\nu}$ and is the standard conformal anomaly, derived in a nonstandard way. The locality of this term (while $Z$ itself is highly non-local) is the reflection of its "zero loop" origin.

When we attempt to generalize this consideration to strings many complications arise. It is not easy to describe the pinched geometries, mentioned above. Nevertheless some general statements can be made.

Let us consider a partition function for non-critical string in some background graviton-dilaton field, $Z = Z(G_{\mu\nu}, \Phi)$. Let us stress that in this case, unlike critical strings, the background is not fixed by any equations of motion and can be chosen more or less arbitrarily. The functional $Z$ is given by a formula, similar to (44). In order to find its scale dependence, we have to devise such a transformation of the space-time quantities that it is reduced to the energy-momentum tensor on the world sheet, as was the case with (45). This is easily achieved if instead of the Weyl rescaling we consider the following change in the target space, which we will call the $\beta$ - symmetry:

$$\delta G_{\mu\nu}(x) = \varepsilon(x)\beta_{\mu\nu}(x); \delta \Phi(x) = \varepsilon(x)\beta^\Phi(x)$$  \hspace{1cm} (46)

It is clear now that under this symmetry the partition function will change in the following way:

$$\beta_{\mu\nu}(x) \frac{\delta Z}{\delta G_{\mu\nu}} + \beta^\Phi(x) \frac{\delta Z}{\delta \Phi} = \langle \int T_a(\xi) \delta (x - x(\xi)) \sqrt{g}d^2\xi \rangle = \text{anomaly}$$

$$\text{anomaly} = \Gamma^{IJ}(\lambda^K(x)) \frac{\delta Z}{\delta \lambda^I} \frac{\delta Z}{\delta \lambda^J} + f(G^{\mu\nu}(x), \Phi(x), \ldots)$$  \hspace{1cm} (47)

Here the fields $\lambda^I$ correspond to the different string states, the $\Gamma^{IJ}$ is some metric in the space of these states and $f$ is some local function of fields corresponding to these states; the world sheet is assumed to have spherical topology.

This formula follows from the fact that the RHS of (47) is entirely dominated by the degenerate metrics. The first term corresponds to the pinching of the world sheet at a given point, while the second local term comes from the complete collapse of the world sheet. Unfortunately it is not known in general how to calculate the quantities entering in (47). Nevertheless, this equation has an interesting interpretation from the point of view of critical string in one extra dimension (which, as usual, is interpreted as the Liouville coordinate. In this case we introduce background fields $G_{\mu\nu}(x, \phi), \Phi(x, \phi)$ etc. This time
those fields can not be arbitrary. They must satisfy equations of motion which follow from the vanishing of all $\beta$ - functions. It is well known that these equations follow from the least action principle for a certain effective action $W$ [4]. It can be shown that the background fields, introduced in the non-critical context are simply initial data for the critical background:

$$G_{\mu\nu}(x) = G_{\mu\nu}(x, \phi_0); \ldots$$

for some fixed $\phi_0$. Moreover the partition function $Z$ is simply the value of the classical action $W$ on the solution with the initial values (48):

$$Z = \min W$$

With this understanding, one can interpret (47) as a Hamilton-Jacobi equation for the classical action. Perhaps this is a simplest way to find the contribution of the pinched spheres, at least in a one loop approximation. Still it is very desirable to have direct methods for the computations.

It is clear that we need a much better understanding of the renormalization group in the presence of gravity. I believe that gravity should actually simplify the structure of the renormalization, since it decreases the number of physical states by means of decoupling of the descendant operators. It can be hoped that there are some very general rules governing the scale dependence. These rules are still to be found. Here I shall quote some partial results obtained together with I. Klebanov and I. Kogan. We showed that in the one loop approximation all the $\beta$ functions get changed by gravity, acquiring an extra factor $\frac{k+2}{k+1}$ where $k$ is the central charge of the gravitational $SL(2)$ algebra. What happens in the higher orders is a fascinating question.

10 Conclusions

It is obvious that we have a lot of interesting, difficult and important work ahead of us.

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REFERENCES

[1] L. Okun Usp. Fiz. Nauk 161, 9, (1991), 177
[2] S. Coleman Nucl. Phys. B310(1988)64
[3] A. Polyakov Sov. Phys. Usp. 25 (1982) 187
[4] C. Callan et al. Nucl. Phys. B262, (1985) 593
[5] A. Polyakov Phys. Lett. 59B (1975) 80
[6] A. Polyakov Phys. Lett. 82B (1979) 247
[7] E. Fradkin M. Srednicky L. Susskind Phys. Rev. D21 (1980) 2885;
    C. Itzykson, Nucl. Phys. B210, 477 (1982);
    A. Sedrakyan Phys. Lett. 137B (1984) 397;
    A. Casher, D. Föerster and P. Windey, Nucl. Phys. B251, 29 (1985);
    Vl. Dotsenko and A. Polyakov, in Advanced Studies in Pure Math. 15 (1987).
[8] A. Polyakov “Gauge fields and Strings” Harwood Academic Publishers (1987)
[9] P. Ramond Phys. Rev. D3(1971) 2415
[10] A. Neveu J. Schwarz Nucl. Phys. B31 (1971) 86
[11] F.D.M. Haldane Phys. Lett. 93 A (1983) 464.
[12] G. t’ Hooft Nucl. Phys. B72 (1974) 461
[13] K. Wilson Phys. Rev. D8(1974)2445
[14] J. Gervais A. Neveu Phys. Lett. 80B (1979) 255
[15] Y. Nambu Phys. Lett. 80B (1979) 372
[16] Yu. Makeenko, A Migdal Nucl. Phys. B188(1981) 269
[17] A. Polyakov Nucl. Phys. B164 (1980) 171
[18] A. Polyakov Nucl. Phys. B268 (1986) 406
[19] D. Gross W. Taylor Princeton preprint PUPT-1382 (1992)