For linear infinite systems the approximate controllability problem by control constraints is considered. Controllability conditions represented via system parameters are obtained.

Partial differential control systems and control systems with delays are considered as an example.

Let $X, Y, U$, be complex Banach spaces. Consider the abstract evolution equation

$$\dot{x}(t) = Ax(t) + Bu(t),$$

$$x(0) = x_0,$$

where $x(t) \in X$ is a current state, $x_0 \in X$ is an initial state; $u(t) \in U, u(.) \in L_2([0, t_1], U)$ is a control; $A : X \to X$ is a linear unbounded closed operator whose domain $D(A)$ is dense in $X$; $B : U \to X$ is a linear bounded operator.

We assume the problem (1)-(2) to be uniformly well-posed [5]. It follows from this assumption that $A$ generates a strongly continuous semigroup $S(t)$ on $X$ in the class $C_0 [3]$. We consider only weak solutions of the above equation.

As usual $X^*$ is a dual space, $A^*$ denotes an adjoint operator for the operator $A$. If $x \in X$ and $f \in X^*$, we will write $(x, f)$ instead of $f(x)$.

For any set $K \subset X$ we denote by $\overline{K}$ the closure of $K$ with respect to the norm of $X$.

Let $K^\perp$ be the set $\{y \in X^* : (x, y) = 0, \forall x \in K\}$.

As usual we denote by $\mathbb{R}$ the set of real numbers and by $\mathbb{R}^n$ the $n$-dimensional vector space.

We assume in the sequel that $u(t) \in \Omega, t \geq 0$, where $\Omega$ is a closed convex cone.

The attainable set $K(t)$ for equation (1)-(2) is defined by the formula:
\[ K(t) = \{ x \in X : \exists u(\cdot) \in L_2([0, t_1], U), x = x(t_1) \}, \]  

where \( x(t), x(0) = 0 \) is a weak solution of the equation (1) corresponding to the control \( u(\cdot) \).

Together with the set (3) we will use the set

\[ K_\Omega(t) = \{ x \in X : \exists u(\cdot) \in L_2([0, t_1], \Omega), x = x(t_1) \}. \]

We assume \( A \) to have the properties:

(i) The domain \( D(A^*) \) of the operator \( A^* \) is dense in \( X^* \).

(ii) The operator \( A \) has a purely point spectrum \( \sigma \) which is either finite or has no finite limit points and each \( \lambda \in \sigma \) has a finite multiplicity.

(iii) Let the numbers \( \lambda_i \in \sigma, i = 1, 2, \ldots \) be enumerated in the order of non-decreasing real parts, let \( \alpha_i \) be a multiplicity of \( \lambda_i \in \sigma \), let \( \varphi_{ij}, i = 1, 2, \ldots, j = 1, 2, \ldots, \beta_i, \beta_i \leq \alpha_i, A\varphi_{ij} = \lambda_i \varphi_{ij} \) be generalized eigenvectors of the operator \( A \), and let \( \psi_{kl}, k = 1, 2, \ldots, l = 1, 2, \ldots, \beta_k \), be generalized eigenvectors of the adjoint operator \( A^* \), such that

\[(\varphi_{pj\beta_j-l+1}, \psi_{jk}) = \delta_{pj}\delta_{lk}, \ p, j = 1, 2, \ldots, l = 1, \ldots, \beta_p, \ k = 1, \ldots, \beta_j.\]

We suppose that \( \lim_{i \to \infty} \text{Re} \lambda_i = -\infty \), and there exists a moment \( T, T \geq 0 \) such that for each \( x \in X, \alpha \in \mathbb{R} \)

\[ S(t)x = \sum_{j=0}^{N_\alpha} \exp(\lambda_j t) \sum_{l=0}^{\beta_j-1} \frac{t^l}{l!} \sum_{k=l+1}^{\beta_k} \varphi_{jk-li}(x, \psi_{jk}) + O(\exp(\alpha t)), \]

where \( N_\alpha \) is a natural number such that \( \text{Re} \lambda_j < \alpha, j = 1, 2, \ldots, N_\alpha \).

(iv) If \( x \in X, g \in X^* \) and \( (S(t)x, g) \equiv O(\exp(\alpha t)) \) for any \( \alpha \in \mathbb{R} \) then \( (S(t)x, g) \equiv 0 \) for each \( t > T \).

The weak solution \( x(t) \) of the equation (1)-(2) is evaluated by the following variation of parameters formula (5):

\[ x(t) = S(t)x_0 + \int_0^t S(t - \tau)Bu(\tau)d\tau. \]

**Definition 1** The equation (1) is said to be approximately \( \Omega \)-controllable, if for any \( x_0, x_1 \in X \) and \( \varepsilon > 0 \) there exists \( u(\cdot) \in L_2^\infty([0, +\infty), \Omega) \) and a moment \( t_1, 0 < t_1 < +\infty \), for which the corresponding solution \( x(t), x(0) = 0 \) of the equation (2) is such that \( \|x_1 - x(t_1)\| < \varepsilon \).

**2 Main results**

Let \( \Omega \subseteq U \) be a closed convex cone.
Definition 2 The linear functional \( g_1 \in U^* \) is said to be not greater then the linear functional \( g_2 \in U^* \) with respect to \( \Omega \), if
\[
(u, g_1) \leq (u, g_2), \forall u \in \Omega.
\] (7)
We will denote the inequality (7) by
\[
g_1 \leq_\Omega g_2.
\] (8)

Theorem 1 The equation (1) is approximately \( \Omega \)-controllable, if and only if the inequality
\[
B^*S^*(t)g \leq_\Omega 0, \text{ a.e. on } [0, +\infty)
\] (9)
implies
\[
g = 0.
\]

Proof. Sufficiency. Assume the equation (1) be not approximately \( \Omega \)-controllable, i.e. \( K_\Omega \neq X \). As \( K_\Omega \) is a convex cone also, the origin is a boundary point of \( K_\Omega \), so either there exists a plane of support containing the origin or for each \( \epsilon > 0 \) there exists an element \( x_\epsilon \in K_\Omega \setminus \text{int}K_\Omega, \|x\| < \epsilon \), contained in a plane of support for \( K_\Omega \). It means that for any \( \epsilon > 0 \) there exists \( g \in X^*, g \neq 0 \), such that
\[
(x, g) \leq \epsilon, \forall x \in K_\Omega(t_1), \forall t_1 > 0.
\] (10)

Using (3) with \( x_0 = 0 \) in (1) and (1) in (10) we obtain
\[
\int_0^{t_1}(S(t_1 - \tau)Bu(\tau), g)d\tau < \epsilon
\] (11)
\[
\forall t_1 > 0, \forall u(\cdot) \in L_2([0, t_1], \Omega)
\]

Let there exist \( u_0 \in \Omega \) and \( t_1 > 0 \) such that the set
\[
\Delta = \{ t \in [0, t_1] : (S(t_1 - \tau)Bu_0, g) > 0 \},
\]
has a positive measure. Let
\[
u^0(t) = \begin{cases}
0, & \text{if } t \notin \Delta, \\
Lu_0, & \text{if } t \in \Delta,
\end{cases}
\] (12)
where \( u_0 \in \Omega, L > 0 \). Obviously, \( \nu^0(\cdot) \in L_2([0, t_1], \Omega), \forall L > 0 \). Substituting (12) to (1), using the abstract integration by parts and Euler-Lagrange Lemma [9] we obtain
\[
L \int_\Delta T(S(t_1 - \tau)Bu_0, g)d\tau < \epsilon, \forall L > 0.
\] (13)

However the inequality (13) cannot be true for any \( L > 0 \), if the measure of \( \Delta \) is positive. Hence the inequality (3) holds with \( g \neq 0 \) for a.e. \( t \in [0, t_1] \) and for arbitrary \( u \in \Omega \), and this contradicts to the condition of the theorem. This proves the sufficiency.
Necessity. Assume that the inequality (11) holds for some $g \neq 0$, for a.e. $t \in [0, +\infty)$. As shown above, this inequality is equivalent to the inequality (10) which holds with $g \neq 0$, that is $K_{\Omega} \neq X$. This completes the proof of the theorem. □

There exists a lot of equations (1) such that the operator $S(t)$ is injective for all $t \geq 0$. However in the general case it is impossible to assure that $S(t)$ is necessarily injective for each $t \geq 0$; in this case there exists $\zeta > 0$ and $x \in X$, $x \neq 0$, such that $S(t)x = 0$, $t \geq \zeta$, and the same is true for the operator $S^*(t)$. Let $h_g = \min\{ t : t \geq 0, S^*(t)g = 0 \}$. Obviously, $h_0 = 0; h_g > 0$ for each $g \neq 0$. We assume $h_g = +\infty$ if $S^*(t)g \neq 0$ for any $t \geq 0$. If $S^*(t)$ is injective for any $t \geq 0$, then we have $h_g = +\infty$ for any $g \neq 0$.

Systems with delays in an argument provide a number of non-trivial examples of operators $S(t)$, where $S^*(t)$ are not injective for some $h > 0$.

The following result obtained by means of the Theorem 1 provides an approximate $\Omega$-controllability criterion, represented via parameters of the equation (1).

**Theorem 2** For the equation (1) to be approximately $\Omega$-controllable, it is necessary and sufficient, that

1. 
   \[
   \text{range}\{\lambda I - A, B\} = X, \quad \forall \lambda \in \sigma; \tag{14}
   \]

2. the conditions
   \[
   S^*(h_g)g = 0, \tag{15}
   \]
   \[
   B^*S^*(\tau)g \leq \Omega_0, 0 \leq \tau \leq h_g < +\infty
   \]
   hold if and only if $g = 0$;

3. the operator $A^*$ has no real eigenvector $\eta$ such that
   \[
   B^*\eta \leq \Omega_0. \tag{16}
   \]

**Proof. Sufficiency.** Let the conditions (1) and (14)–(15) hold. We will prove that if $g \neq 0$, then for a sufficiently large $\alpha$ the set

\[
J_\alpha = \{ j : 1 \leq j \leq N_\alpha, \exists \gamma_j, 1 \leq \gamma_j \leq \beta_j \text{ such that } (\varphi_{jk}, g) = 0, k = 1, ..., \gamma_j - 1, (\varphi_j g_j, g) \neq 0 \}
\]

is not empty.

Let

\[
(\varphi_{jk}, g) = 0, j = 1, 2, ..., k = 1, ..., \beta_j. \tag{17}
\]

It follows from (17) and (3) that

\[
(S(t)x, g) = O(\exp(\alpha t)), \forall x \in X, \forall \alpha \in \mathbb{R}. \tag{18}
\]

By (iv) and (18) we obtain

\[
S^*(T)g = 0. \tag{19}
\]
Moreover, \( h_g \leq T \). It follows from (19) and (9) that
\[
S^*(h_g)S^*(T - h_g)g = 0, \tag{20}
\]
\[
B^*S^*(\tau)S^*(T - h_g)g \leq \Omega 0, 0 \leq \tau \leq h_g. \tag{21}
\]
So we obtain from (20), (21), and (13) that \( S^*(T - h_g)g = 0 \). Continuing this process by the similar way, we will obtain after a finite number of steps that \( g = 0 \), i.e. we have a contradiction.

Using \( Bu \) instead of \( x \) in (3) and (4) in the bilinear form \( (S(t)Bu, g) \), we obtain
\[
(S(t)Bu, g) = \sum_{j=0}^{N_a} \exp(\lambda_j t) \sum_{l=0}^{\beta_j - 1} \frac{t^l}{l!} \sum_{k=l+1}^{\beta_j} (\varphi_{jk-l}, g)(Bu, \psi_{jk}) + O(\exp(\alpha t)). \tag{22}
\]

Let \( J_1 = \{ j \in J_\alpha : \lambda_j \text{ is real}\}; J_2 = \{ j \in J_\alpha : \lambda_j \text{ is complex}\}. \) Apparently, \( J_\alpha = J_1 \cup J_2, J_1 \cap J_2 = \emptyset \).

We use the following notations:
\[
\begin{align*}
\mu_1 &= \max\{\lambda_j, j \in J_1\}, I_1 = \{ j \in J_1, \lambda_j = \mu_1\}, \\
\mu_2 &= \max\{\beta_j - \gamma_j, j \in I_1\}, I_2 = \{ j \in J_1, \beta_j - \gamma_j = l_1\}; \\
l_2 &= \max\{\Re\lambda_j, j \in J_2\}, I_3 = \{ j \in J_2, \Re\lambda_j = \mu_2\}, \\
l_3 &= \max\{\beta_j - \gamma_j, j \in I_2\}, I_4 = \{ j \in I_2, \beta_j - \gamma_j = l_2\}.
\end{align*}
\]

Let \( J_1 = \emptyset \). Then \( J_2 \neq \emptyset \) and (22) can be written as
\[
(S(t)Bu, g) = \exp(\mu_2 t) t^{l_2} \psi(t, u) + O(\exp(\alpha t)), \tag{23}
\]
where
\[
\psi(t, u) = 2 \sum_{j \in I_4} (\Re((\varphi_{j\gamma_j}, g)(Bu, \psi_{j\beta_j}))) \cos \Im\lambda_j t - \Im((\varphi_{j\gamma_j}, g)(Bu, \psi_{j\beta_j})) \sin \Im\lambda_j t). \tag{24}
\]

Let \( k \in I_4 \). It follows from (14) and the definition of numbers \( \gamma_j \) that for each \( k \in N_\alpha \) that there exists \( u_k \) such that
\[
|(\varphi_{k\gamma_k}, g)(Bu, \psi_{j\beta})| \neq 0. \tag{25}
\]
Moreover, \( O(\exp(\alpha t)) = O(\exp(\mu_2 t)^{l_2}) \), because \( \mu_2 \leq \alpha \).

All the functions \( \cos(\Im\lambda_j t), \sin(\Im\lambda_j t) \) are linearly independent with zero mean values In virtue of (23) and the lemma on almost-periodic functions [3] there are a sequence \( t_v, \lim_{v \to \infty} \) and a number \( v_0 \) such that for an arbitrary \( v \geq v_0 \)
\[
(S(t_v)Bu_k, g) > 0. \tag{26}
\]

If \( J_1 \neq \emptyset \) and \( J_2 = \emptyset \), then (22) can be written as
\[
(S(t)Bu, g) = \exp(\mu_1 t) t^{l_1} \psi(t, u) + O(\exp(\alpha t)), \tag{27}
\]
where
\[ \psi(t, u) = (Bu, \sum_{j \in I_3} (\varphi_jg, \gamma_j) \psi_{j\beta_j}). \]

Let \( \psi_1 = \sum_{k \in I_3} (\varphi_jg, \gamma_j) \psi_{j\beta_k} \). The vectors \( \psi_{j\beta_k}, k \in I_3 \) are linearly independent, hence by the definition of numbers \( \gamma_j \) we obtain \( \psi_1 \neq 0 \), so vector \( \psi_1 \) is an eigenvector of the operator \( A \), corresponding to the real eigenvalue \( \mu_1 \), and in virtue of the third condition of the theorem there exists \( u_0 \in \Omega \) such that \( (Bu_0, \psi_1) > 0 \). We have also \( O(\exp(\alpha t)) = O(\exp(\alpha t)t^{l_1}) \), because \( \alpha \leq \beta \). Hence
\[ (S(t)Bu_0, g) = \exp(\alpha t)t^{l_1}(Bu_0, \psi_1) + O(\exp(\alpha t)t^{l_1}), \]
Since \( (Bu_0, \psi_1) > 0 \), there exists \( T_1 > 0 \) such that for arbitrary \( t \geq T_1 \)
\[ (S(t)Bu_0, g) > 0. \]

If \( J_1 \neq \emptyset \) and \( J_2 \neq \emptyset \), then arguing as in above cases we can write
\[ (S(t)Bu_0, g) = \exp(\alpha t)t^{l_1}(Bu_0, \psi_1) + \exp(\alpha t)t^{l_1}(t - u_0) + O(\exp(\alpha t)t^{l_1}) + O(\exp(\alpha t)t^{l_1}), \]
where \( (Bu_0, \psi_1) > 0; \)
\[ \psi(t, u_0) = 2 \sum_{j \in I_3} \alpha_{1j} \cos \delta_j t + \alpha_{2j} \sin \delta_j t, \]
where \( \mu_3 \leq \mu_2, l \leq l_2, I_3 \subseteq I_4; \delta_k \neq \delta_p, k, p \in I_5 \) and there exists \( v \in I_5 \) such that \( |a_{1v}| + |a_{2v}| \neq 0 \). Hence by the lemma on almost-periodic functions there exist a sequence \( t_v, \lim_{v \to \infty} t_v = \infty \) and a number \( v_1 \), such that \( \exp(\alpha t_v)t^{l_1}(t_v, u_0) + O(\exp(\alpha t_v)t^{l_1}) > 0 \) for any \( v \geq v_0 \). If \( v \geq v_1 \) such that \( t_v \geq T_1 \), then for the sufficiently small \( \eta \)
\[ (S(t)Bu_0, g) > 0, t_v \leq t \leq t_v + \eta. \]

The formula (31) shows that if \( g \neq 0 \) then (2) doesn’t hold so the sufficiency follows by Theorem L.

Necessity. If (14) doesn’t hold, then there exists \( \lambda \in \sigma \) and eigenvector \( g_\lambda \) of the operator \( A^* \) such that \( B^*g_\lambda = 0 \). Therefore \( B^*S^*(t)g_\lambda = \exp(\lambda t)B^*g_\lambda = 0, \forall t \geq 0, g_\lambda \neq 0 \). This contradicts to the Theorem L.

Let there exists a vector \( g \in X^*, g \neq 0 \), such that \( h_g < +\infty \) and
\[ S^*(t)g = 0, \forall t \geq h_g, \]
\[ B^*S^*(t)g \leq \Omega 0, 0 \leq \tau \leq h_g. \]
Relations (32) imply
\[ B^*S^*(t)g \leq \Omega 0, \forall t \geq 0, \]
where \( g \neq 0 \). This contradicts to the Theorem L.

If there exists a real eigenvalue \( \lambda \) and a corresponding real eigenvector \( \eta_\lambda \) such that \( B^*\eta_\lambda \leq \Omega 0 \), we obtain from the last inequality, that \( B^*S^*(t)\eta_\lambda = \exp(\lambda t)B^*\eta_\lambda \leq \Omega 0, \forall t \geq 0, \eta_\lambda \neq 0 \).

The theorem follows by the contradiction to the Theorem L. □
**Corollary 1** Let $b \in U, b \neq 0$. Consider the cone $\Omega = b \mathbb{R}^+ = \{u \in U : u = b\alpha, \alpha \geq 0\}$. For the equation (1) to be approximately $\Omega$-controllable, it is necessary and sufficient that

1. The condition (14) holds.
2. The conditions (15) hold if and only if $g = 0$.
3. The operator $A$ has no real eigenvalues.

**Proof.** If $\Omega = b \mathbb{R}^+$, then the condition 2 of the Corollary is equivalent to the condition (16) of the Theorem 2.

## 3 Examples

**Example 1.** Let $H$ be a Hilbert space. Consider the equation (1) with the self-adjoint operator $A$ generating a $C_0$-semigroup. It is well-known that the spectrum $\sigma$ of $A$ is the sequence $\{\lambda_j, j = 1, 2, \ldots\}$ of real negative numbers; $\lim_{j \to \infty} \lambda_j = -\infty$; the operator $A$ has the properties (i)-(iv) with $T = 0$.

Let $\varphi_j, j = 1, 2, \ldots$ be eigenfunctions of $A$ corresponding to eigenvalues $\lambda_j, j = 1, 2, \ldots$, and let $I_j = \{k : \lambda_k = \lambda_j\}$. It is well-known, that the sequence $\{\varphi_j, j = 1, 2, \ldots\}$ is complete. One can prove, that for a self-adjoint operator $A$ the condition (14) is equivalent to the condition

$$\text{Ker}\{\lambda I - A\} \cap \text{Ker}\{B^*\} = \{0\}. \quad (33)$$

It is easily to show that the condition (33) is equivalent to the linear independence of vectors

$$B^*\varphi_k, k \in I_j, j = 1, 2, \ldots. \quad (34)$$

Since the sequence of eigenfunctions of $A$ is complete, we obtain that the operator $S^*(t_1)$ is injective for an arbitrary $t_1 \geq 0$, therefore the conditions (15) hold for each self-adjoint operator $A$. As all the eigenvalues of the operator $A$ are real, the condition (16) is equivalent to the following:

(v) there is no natural $j$ such that $B^*\varphi_j \leq_\Omega 0$.

Thus, we obtain the validity of the following theorem:

**Theorem 3** Let $A$ be a self-adjoint operator in a Hilbert space $H$. The equation (1) is approximately $\Omega$-controllable if and only if (33) and (v) hold.

Obviously, (v) does not hold, if $U = \mathbb{R}, \Omega = \mathbb{R}^+ = \{\alpha \in \mathbb{R}, \alpha \geq 0\}$, so the next corollary follows.

**Corollary 2** Let $A$ be a self-adjoint operator in the Hilbert space $H$. If $U = \mathbb{R}, \Omega = \mathbb{R}^+$, then the equation (1) is not approximately $\Omega$-controllable.
To illustrate above theorem consider the following simple example.
Consider the control system described by one-dimensional heat equation
\[
\frac{\partial x}{\partial t}(t, \xi) = \frac{\partial^2 x}{\partial \xi^2}(t, \xi) + B(\xi)u(t);
\] (35)
\[
x(0, \xi) = \varphi(\xi), 0 \leq \xi \leq \pi,
\] (36)
\[
x(t, 0) = 0, x(t, \pi) = 0, 0 \leq t < +\infty,
\] (37)
where \(x, \varphi \in \mathbb{R}, B(\xi) = \text{col}\{b_1(\xi), b_2(\xi)\}, \varphi(\cdot), b_j(\cdot) \in L_2[0, \pi], j = 1, 2\).

The equation (35) is a particular case of the problem (33)-(34), where
\[
X = L_2[0, \pi], U = \mathbb{R}^2, (Ax)(\xi) = \frac{d^2 x}{d\xi^2}(\xi)
\]
with the domain
\[
D(A) = \{x \in C^2[0, \pi] : x(0) = x(\pi) = 0\}.
\]

It is well-known that the operator \(A\) generates a compact self-adjoint \(C_0\)-semigroup; \(\sigma = \{-j^2, j = 1, 2, \ldots\}\); \(\alpha_j = \beta_j = 1, \varphi_j(\xi) = \sqrt{2}\sin(j\xi)\) are the eigenfunctions of the operator \(A\) corresponding to eigenvalues \(\lambda_j = -j^2, j = 1, 2, \ldots\); \(A\) is a self-adjoint operator; for each \(\varphi(\cdot) \in X\) the corresponding solution \(x(t, \xi)\) of equation (35)-(37) is expanded into the series
\[
x(t, \xi) = \sum_{j=1}^{\infty} \left( \int_0^\pi \varphi(\xi) \sin(j\xi)d\xi \right) \exp(-j^2 t),
\]
convergent uniformly for any segment \([0, h]\).

Let the closed convex cone \(\Omega\) be described by the set \(\Omega = \{(v_1, v_2) \in \mathbb{R}^2\}\), where
\[
v_1 = c_{11}u_1 + c_{12}u_2,
\]
\[
v_2 = c_{21}u_1 + c_{22}u_2,
\]
where \(u_1 \geq 0, u_2 \geq 0\).

The implementation of the Theorem (3) shows the validity of the following result:

**Theorem 4** The equation (33)-(34) is approximately \(\Omega\)-controllable if and only if there is no natural \(j\) such that
\[
\int_0^\pi (b_1(\xi)c_{11} + b_2(\xi)) \sin(j\xi)d\xi \leq 0,
\]
and
\[
\int_0^\pi (b_1(\xi)c_{11} + b_2(\xi)) \sin(j\xi)d\xi \leq 0.
\]
Example 2. Consider the linear hereditary system

\[ \frac{d}{dt} x(t) = A_0 x(t) + A_1 x(t - h) + B_0 u(t), \]

(38)

\[ x(0) = x_0, x(\tau) = \varphi(\tau), -h \leq \tau \leq 0, \]

(39)

where \( x, x_0 \in \mathbb{R}^n, u \in \mathbb{R}^r, A_0, A_1 \) are \( n \times n \) constant matrices, \( \varphi(.) \in L_2^2[-h, 0] \); \( B_0 \) is a \( n \times r \) constant matrix. It is known that the problem (38)-(39) is well-posed \(^3\). Hence the problem (38)-(39) is a particular case of the problem (1)-(2), where

\[ X = \mathbb{R}^n \times L_2^2[-h, 0], U = \mathbb{R}^r; \]

the corresponding operator \( A \) has the properties (i)-(ii) \(^3\); the properties (iii)-(iv) hold for \( T = nh \) \(^2, \(^4\); the operator \( B \) is defined by

\[ Bu = \{ B_0 u, 0 \}. \]

Let the closed convex cone \( \Omega \) is described by the set

\[ v = C u \leq 0, j = 1, ..., m, \]

where \( C \) is \( r \times m \) constant matrix, \( 1 \leq m \leq r, u \geq 0 \).

Using above Theorem 2, one can prove the validity of the following result.

Theorem 5 For approximate \( \Omega \)-controllability of the system (38) it is necessary and sufficient, that

1. for any \( \lambda \in \sigma \)

\[ \text{rank}\{ \lambda I - A_0 - A_1 \exp(\lambda \tau), B_0 \} = n; \]

(40)

2. the system of equations and inequalities

\[ A_1 x = 0, \]

(41)

\[ B_0^T x \leq 0 \]

has only the trivial solution;

3. the system

\[ \dot{y}(t) = A_0^T y(t) + A_1^T y(t - h) \]

(42)

has no real eigenvectors \( g \) such that

\[ g^T B_0 C \leq 0. \]

(43)

Proof. The equivalence between (40) and (41) follows from the results of \(^3\).

1. Now we have to establish the equivalence between (41) and (43).

1.1. Let \( x = \{ x_0, \varphi(\cdot) \}, g = \{ y_0, \psi(\cdot) \} \) be arbitrary elements from \( X \) and \( X^* \) correspondingly, and let \( F(t) \) be a fundamental matrix of the system (38).
It follows from the results of [2], [3], [7], that

\[(S(t)x, g) = y^T(t)x_0 + \int_0^h y^T(t - \tau)A_1 \varphi(\tau - h)d\tau, \forall t \geq 0, \quad (44)\]

where

\[y(t) = F^T(t)y_0 + \int_{-h}^0 F^T(t + \tau)\psi(\tau)d\tau, t \geq 0. \quad (45)\]

It is easy to show that \(y(t)\) is a solution of the equation (42) for \(t \in [h, +\infty)\). Hence the equality \(S^*(h)g = 0\) is equivalent to

\[y(h) = 0, A_1^Ty(\tau) \overset{a.e.}{=} 0 \text{ on } [0, h]. \quad (46)\]

Further we are to evaluate \(B^*S^*(\tau)g, 0 \leq \tau \leq h\). Using in (44) \(x = Bu, u \in \mathbb{R}^r\), we obtain

\[(S(\tau)Bu, g) = y^T(\tau)B_0u, 0 \leq \tau \leq h, \forall u \in \mathbb{R}^r. \quad (47)\]

Therefore, the inequality \(B^*S^*(\tau)g \leq 0, 0 \leq \tau \leq h\) is equivalent to

\[B_0^Ty(\tau) \leq 0, 0 \leq \tau \leq h. \quad (48)\]

Thus the conditions (46) and (48) are equivalent to the conditions (15).

If (11) holds, then (46) and (48) imply

\[y(h) = 0, y(\tau) \overset{a.e.}{=} 0 \text{ on } [0, h]. \quad (49)\]

By (45), (49) we obtain

\[y(\tau) = \exp(A_0\tau)y_0 + \int_0^\tau \exp(A_0(\tau - \theta)\psi(-\theta)d\theta, 0 \leq \tau \leq h \quad (50)\]

Hence \(y(\tau), 0 \leq \tau \leq h\) is an absolutely continuous function and it satisfies the ordinary non-homogeneous differential equation

\[\dot{y}(\tau) = A_0y(\tau) + \psi(\tau), 0 \leq \tau \leq h. \quad (51)\]

The equations (49) and (51) imply

\[y_0 = 0, \psi(\tau) \overset{a.e.}{=} 0 \text{ on } [-h, 0]. \]

Therefore, the conditions (11) imply the conditions (15).

1.2. Now we will prove that the conditions (11) imply the condition (14).

Let (13) be true and let \(x \in \mathbb{R}^m\) be such that

\[A_1^Tx = 0, \quad B_0^Tx \leq 0. \quad (52)\]

\[\text{The superscript } ^T \text{ denotes a transposition.}\]
Consider
\[ y = \{0, x\theta\}, -h \leq \theta \leq 0. \] (53)

Obviously, \( y \in X = \mathbb{R}^n \times L^2_2[-h, 0] \). Let \( y(t) \) be the solution of equation (42) on \([h, t]\) with initial condition
\[ \xi = \{0, x \cdot (\theta - h)\}, 0 \leq \theta \leq h. \] (54)

This solution is defined by the formula \[ y(t) = \int_{h}^{2h} F^T(t - \tau)A^T_1 x \cdot (\tau - 2h)d\tau, t \geq h. \] (55)

It follows from (55) that \( y(t) \equiv 0, t \geq h. \)

Define the function \( p(t), 0 \leq t \leq h, \) by the formula:
\[ p(-t) = x - A^T_0 x \cdot (t - h)^2. \] (56)

The function \( y(t) \) is a solution of the non-homogeneous system
\[ \dot{y}(t) = A^T_0 y(t) + A^T_1 y(t - h) + q(t), t \geq 0 \] (57)

with the initial condition
\[ y_0 = x \cdot (-h), y(\tau) \equiv 0, -h \leq \tau \leq 0, \]

where
\[ q(t) = \begin{cases} 0, t > 0, \\ p(-t), 0 \leq t \leq h. \end{cases} \]

Therefore \[ y(t) = F^T(t) x \cdot (-h) + \int_{-h}^{0} F^T(t + \tau)p(\tau)d\tau. \] (58)

The formulas (44), (58) and \( y(t) \) a.e. = 0 on \([h, +\infty)\) imply
\[ S^*(2h)g = 0, \] (59)

where \( g = \{x \cdot (-h), p(\cdot)\} \). The condition \( y(t) \stackrel{a.e.}{=} 0 \) on \([h, +\infty)\) and (45) imply
\[ B^*S^*(\tau)g \leq 0, 0 \leq \tau \leq 2h, \] (60)

In account of (45) it follows from (59) and (60) that \( g = 0 \), i.e. \( x = 0 \). Hence, (41) holds, as required.

3. It remains to establish the equivalence between (43) and (13). We have
\[ B^*g = B^*_0 g_0, \forall g = \{g_0, g(\cdot)\} \in \mathbb{R}^n \times L^2_2[-h, 0]. \]

\[ ^2 \]In accordance with the definition of the function \( y(t) \) we have \( \dot{y}(t) \equiv x, 0 \leq t \leq h. \)
If $\eta = \{\eta_0, \eta(\cdot)\}$ is an eigenvector of the operator $A^*$ corresponding to an eigenvalue $\lambda$ then

$$
\eta(\tau) = \eta_0 \exp(-\lambda \tau), -h \leq \tau \leq 0, \quad (61)
$$

$$
\eta_0^T (\lambda I - A_0 - A_1 \exp(-\lambda h)) = 0. \quad (62)
$$

Hence, if $\eta = \{\eta_0, \eta(\cdot)\}$ is an eigenvector of the operator $A^*$ such that (16) holds, then it follows from (61)-(62) that $C^T B_0^T \eta_0 \leq 0$, where $g_0$ is an eigenvalue of the system (12), as required. $\Box$

Theorem 5 is a particular case of the theorem proven in [7].

4 Conclusion

We have obtained the necessary and sufficient conditions for the approximate controllability by control constraints for an abstract operator differential equation. These results allow us to obtain the appropriate controllability conditions for various known classes of distributed control systems by the unified manner. Partial differential control systems and differential-difference control systems have been considered as an example.

References

[1] A. V. Balakrishnan, Applied Functional Analysis. Springer-Verlag, New York Heidelberg Berlin, 1976.

[2] R. Bellman and K. Cook, Differential-Difference Equations. New York, Academic Press, London, 1963.

[3] J. Hale, Theory of Functional Differential Equations, Springer-Verlag, New York Heidelberg Berlin, 1977.

[4] D. Henry, Small solutions of linear autonomous functional differential equations, Journal of Differential Equations, 8(1970), pp. 494–501.

[5] E. Hille and R.S. Phillips, Functional analysis and semigroup, Amer. Math. Soc., Providence, 1957.

[6] V. Korobov, A. Marinich and E.N. Podol’skii, Controllability of linear autonomous systems with restrictions on the control, Differential Equations, 11(1975), No 11, pp. 1967–1979.

[7] B. Shklyar, Approximate controllability of systems with time-lag in the positive control class, Differential Equations, 21(1985), No 12, pp. 1403–1411.

[8] B. Shklyar, Controllability of linear systems with distributed parameters, Differential Equations, 27(1991), No 3, pp. 326–335.

[9] L. Yang, Lectures on the calculus of variations and optimal control theory. W.B. Saunders Company, Philadelphia London Toronto, 1969.