Electroweak Vacuum Instability and Renormalized Vacuum Field Fluctuations in Adiabatic or Non-adiabatic Cosmological Background

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In this work, we investigate the electroweak vacuum instability in the adiabatic or non-adiabatic cosmological background. In the general cosmological background, the vacuum field fluctuations \(\langle \delta \phi^2 \rangle\) grow in proportion to the cosmological scale. The large vacuum fluctuations of the Higgs field can destabilize the effective Higgs potential, or generate the catastrophic AdS domains or bubbles. These unwanted phenomena cause the catastrophic collapse of the Universe. By using the adiabatic (WKB) expansion or the adiabatic regularization methods, we obtain the exact renormalized vacuum fluctuations \(\langle \delta \phi^2 \rangle_{\text{ren}}\) of the Higgs field in the adiabatic and the non-adiabatic cosmological background. The non-adiabatic Higgs vacuum fluctuations generally cause the catastrophic phenomena. On the other hand, the adiabatic Higgs vacuum fluctuations have little effect on the Higgs vacuum stability. However, in the slowly-varying background by another scalar field \(S\), the adiabatic Higgs vacuum fluctuations can destabilize the effective Higgs potential and provide the upper bound of the mass of the background scalar field \(S\) as \(m_S \lesssim 10^{13}\) GeV where we assume the instability scale \(\Lambda_I \approx 10^{11}\) GeV.

I. INTRODUCTION

The Large Hadron Collider (LHC) experiments discovered the Higgs boson and established the Standard Model (SM) of particle physics. However, the currently central values of the Higgs boson mass \(m_h = 125.09 \pm 0.21\) (stat) \(\pm 0.11\) (syst) GeV [1–4] and the top quark mass \(m_t = 172.44 \pm 0.13\) (stat) \(\pm 0.47\) (syst) GeV [5] suggest that the effective Higgs potential develops an instability at the high scale \(\Lambda_I \approx 10^{11}\) GeV. Therefore, if there are no new physics to stabilize the Higgs field, the current electroweak vacuum is not stable and finally cause a catastrophic vacuum decay through quantum tunneling [6–8]. Fortunately, the vacuum decay timescale is longer than the age of our Universe [9–12], and therefore, it has been thought that the metastability of our electroweak vacuum does not cause cosmological problems to the observed Universe.

However, the recent investigations [13–27] reveal that the electroweak vacuum metastability is incompatible with large-field inflation models. It is well-known that the vacuum field fluctuations \(\langle \delta \phi^2 \rangle\) grow rapidly in de Sitter space. If the inflationary vacuum fluctuations of the Higgs field \(\langle \delta \phi^2 \rangle\) destabilize the effective Higgs potential \(V_{\text{eff}}(\phi)\) or generate the Anti-deSitter (AdS) domains (or bubbles), these unwanted phenomena trigger off a catastrophic vacuum transition to a negative Planck-energy true vacuum and cause an immediate collapse of the Universe. Furthermore, even at the end of the inflation, the large vacuum fluctuations of the Higgs field are generated via parametric resonance or tachyonic resonance, and becomes potentially catastrophic [25, 28–32]. Therefore, the electroweak vacuum metastability is disfavored with respect to the inflationary Universe.

The naturally arising question is whether the electroweak vacuum metastability is consistent with other background. For instance, in the Schwarzschild background, the decay of the metastable Higgs vacuum can be enhanced [33–37]. Thus, the existence of the small black holes does not favor the electroweak vacuum metastability. Yet previous research has been discussed assuming somewhat special background like de-Sitter or Schwarzschild background, and therefore, it is worth considering the Higgs vacuum metastability in the more general cosmological background. In fact, not only de Sitter background, but also the curved background can enlarge the vacuum field fluctuations \(\langle \delta \phi^2 \rangle\) in proportional to the curvature scale \(R\), which is given by the quantum field theory (QFT) in curved spacetime. Therefore, the electroweak vacuum metastability is sensitive to the curvature scale of the Universe.

However, it is essentially difficult task to obtain the exact vacuum field fluctuations \(\langle \delta \phi^2 \rangle\) in the curved background. As well-known facts in QFT, the two-point correlation function \(\langle \delta \phi^2 \rangle\) which we call the vacuum field fluctuations, have the ultraviolet (UV) divergences and therefore the regularization or the renormalization must be required. In the Minkowski spacetime, these UV divergences can be eliminated by standard renormalization method. However, in the curved spacetime, it is not so simple to obtain the renormalized vacuum field fluctuations \(\langle \delta \phi^2 \rangle_{\text{ren}}\) and some troublesome regularizations are necessary. Our previous work [26] dealt with these renormalization issues via the adiabatic regularization or the point-splitting regularization methods, and investigated the electroweak vacuum instability in de Sitter spacetime. In the present work, by using the adiabatic (WKB) expansion or the adiabatic regularization methods, we provide the rigid renormalized vacuum field fluctuations in the adiabatic or non-adiabatic cosmological background. We thoroughly investigate the electroweak vacuum insta-

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bility in the general cosmological background, especially the expanding Universe, and provide new descriptions of the Higgs vacuum phenomena.

The present paper is organized as follows. In Section II we derive the effective Higgs potential in curved spacetime by using the adiabatic (WKB) expansion method. In Section III we consider the renormalized vacuum fluctuations in the adiabatic regime where the mass of the Higgs field is generally larger than the curvature scale. In Section IV we discuss the renormalized vacuum fluctuations in the non-adiabatic regime and provide the detail calculations of the renormalized non-adiabatic vacuum fluctuations. In Section V we consider the renormalized vacuum fluctuations in the adiabatic regime where the mass of the Higgs field and the vacuum fluctuations of the Higgs field sway the cosmological destiny of the Universe. Finally, in Section VII we draw the conclusion of our work.

II. EFFECTIVE HIGGS POTENTIAL IN CURVED SPACETIME

Essentially, global dynamics of the Higgs field can be determined by the effective potential. The matters of the effective potential in curved background has been thoroughly investigated in the literature [38–55] and there are a variety of methods to derive the effective potential in curved spacetime. In this section, we discuss the effective Higgs potential in curved spacetime via the adiabatic (WKB) expansion method following the literature [38–40]. This method can clearly handle the UV divergences of the vacuum field fluctuations and simply derive the effective potential in curved spacetime.

Through this paper, we consider the Friedmann-Lemaître-Robertson-Walker (FLRW) background described by the following FLRW metric

$$g_{\mu\nu} = \text{diag} \left( -1, \frac{a^2(t)}{1-Kr^2}, a^2(t) r^2, a^2(t) r^2 \sin^2 \theta \right),$$

(1)

where $a = a(t)$ express the scale factor with the cosmic time $t$ and $K$ is the spatial curvature parameter, where positive, zero, and negative values are related with closed, flat, and hyperbolic spacetime. For the spatially flat spacetime, we can take $K = 0$ and the Ricci scalar is given as

$$R = 6 \left[ \left( \frac{\dot{a}}{a} \right)^2 + \left( \frac{\ddot{a}}{a} \right) \right] = 6 \left( \frac{a''}{a^3} \right),$$

(2)

where $\eta$ is the conformal time and defined by $d\eta = dt/a$. In the radiation dominated Universe, the scale factor becomes $a(t) = t^{1/2}$ and the Ricci scalar is expressed as $R = 0$. On the other hand, in the matter dominated Universe, the scale factor becomes $a(t) = t^{2/3}$ and the Ricci scalar is expressed as $R = 3H^2$. Finally, in the de Sitter Universe, the scale factor becomes $a(t) = e^{Ht}$ and the Ricci scalar is expressed as $R = 12H^2$.

The bare (unrenormalized) action for the Higgs field with the potential $V(\phi)$ in curved spacetime is given by

$$S[\phi] = -\int d^4x \sqrt{-g} \left( \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + V(\phi) \right),$$

(3)

where we assume the simple form for the Higgs potential with bare parameters as

$$V(\phi) = \frac{1}{2} (m^2 + \xi R) \phi^2 + \frac{\lambda}{4} \phi^4. $$

(4)

Thus, the Klein-Gordon equation for the Higgs field are written as

$$\Box \phi + m^2 \phi + \xi R \phi + \lambda \phi^3 = 0, $$

(5)

where $\Box$ expresses the generally covariant d’Alembertian operator, $\Box = g^{\mu\nu} \nabla_\mu \nabla_\nu = 1/\sqrt{-g} \partial_\mu (\sqrt{-g} \partial^\mu)$ and $\xi$ is the non-minimal Higgs-gravity coupling constant.

In the quantum field theory, we treat the Higgs field $\phi$ as the field operator acting on the ground states, then the Higgs field $\phi$ is decomposed into a classical field and a quantum field as

$$\phi = \phi(\eta, x) + \delta \phi(\eta, x),$$

(6)

where we assume that the vacuum expectation value of the Higgs field is $\langle \phi \rangle = 0$ and $\langle 0 \rangle = 0$. By introducing the renormalized parameters and the counterterms as $m^2 = m^2(\mu) + \delta m^2$, $\xi = \xi(\mu) + \delta \xi$ and $\lambda = \lambda(\mu) + \delta \lambda$, we can obtain the Klein-Gordon equations in the one-loop approximation as

$$\Box \phi + (m^2(\mu) + \delta m^2) \phi + (\xi(\mu) + \delta \xi) R \phi + 3(\lambda(\mu) + \delta \lambda) \phi^3 = 0, $$

(7)

$$\Box + (m^2(\mu) + \xi(\mu) R + 3\lambda(\mu) \phi^2) \delta \phi = 0. $$

(8)

The quantum Higgs field $\delta \phi$ is decomposed into each $k$ modes as,

$$\delta \phi(\eta, x) = \int d^3k \left( a_k \delta \phi_k(\eta, x) + a_k^* \delta \phi_k^*(\eta, x) \right),$$

(9)

where

$$\delta \phi_k(\eta, x) = \frac{e^{ik \cdot x}}{(2\pi)^{3/2} \sqrt{C(\eta)}} \delta \chi_k(\eta),$$

(10)

with $C(\eta) = a^2(\eta)$. Now, we can build a complete set of mode functions, which are orthonormal with respect to the scalar product in the curved spacetime

$$\langle \delta \phi_k, \delta \phi_{k'} \rangle = i \int d^2y \sqrt{-g_{\Sigma}} \delta \phi_k^*(\partial_\mu \delta \phi_k) - (\partial_\mu \delta \phi_k^*) \delta \phi_k \delta \phi_{k'},$$

(11)
where \( d\Sigma^\mu = n^\mu d\Sigma \) is expressed by the a unit timelike vector \( n^\mu \) and the volume element \( d\Sigma \). These orthonormal mode solutions satisfy

\[
(\delta \phi_k, \delta \phi_{k'}) = \delta (k - k')
\]  

(12)
The creation and annihilation operators of \( \delta \phi_k \) are required to satisfy the commutation relations

\[
[a_k, a_k^\dagger] = [a_k^\dagger, a_k] = 0, \quad [a_k, a_{k'}^\dagger] = \delta (k - k')
\]  

(13)
where the in-vacuum state \( |0\rangle \) is defined as \( a_k |0\rangle = 0 \) and depends on the boundary conditions of the mode functions \( \delta \phi_k \). Different boundary conditions of \( \delta \phi_k \) corresponds to different initial state of the quantum vacuum. The vacuum field fluctuations \( \langle \delta \phi^2 \rangle \) of the Higgs field can be written as

\[
\langle 0 | \delta \phi^2 | 0 \rangle = \int d^3 k |\delta \phi_k (\eta, x)|^2,
\]  

(14)

\[
= \frac{1}{2\pi C (\eta)} \int_0^\infty dk k^2 |\delta \chi_k|^2,
\]  

(15)
where \( \langle \delta \phi^2 \rangle \) has ultraviolet (quadratic and logarithmic) divergences, which require a regularization, e.g. cut-off regularization or dimensional regularization, and must be cancelled by the counterterms of the couplings.

From Eq. (8), the Klein-Gordon equation for the quantum rescaled field \( \delta \chi \) is written by

\[
\delta \chi''_k + \Omega_k^2 \delta \chi_k = 0,
\]  

(16)
where

\[
\Omega_k^2 (\eta) = k^2 + C (\eta) (m^2 + 3\lambda \phi^2 + (\xi - 1/6) R).
\]  

(17)
The orthonormal condition of Eq. (12) for the mode functions \( \delta \chi \) can be given by

\[
\delta \chi_k \delta \chi'^*_{k'} - \delta \chi'^*_{k} \delta \chi_{k'} = i
\]

(18)
which is the normalization of the mode function \( \delta \chi (\eta) \). Eq. (16) is consistent with the differential equation of the harmonic oscillator with time-dependent mass. Thus, we can rewrite the mode function \( \delta \chi \) by the two complex function \( \alpha_k (\eta) \) and \( \beta_k (\eta) \) as

\[
\delta \chi_k (\eta) = \frac{1}{\sqrt{2\Omega_k (\eta)}} \{ \alpha_k (\eta) \delta \varphi_k (\eta) + \beta_k (\eta) \delta \varphi_k^* (\eta) \},
\]  

(19)
where \( \delta \varphi_k (\eta) \) are given by

\[
\delta \varphi_k (\eta) = \exp \left\{ -i \int^\eta \Omega_k (\eta_1) \, d\eta_1 \right\}
\]  

(20)
From Eq. (16), we can obtain the relations for \( \alpha_k (\eta) \) and \( \beta_k (\eta) \) as the following

\[
\alpha_k' = \frac{1}{2 \Omega_k} \beta_k \delta \varphi_k^* (\eta), \quad \beta_k' = \frac{1}{2 \Omega_k} \alpha_k \delta \varphi_k (\eta).
\]  

(21)
The Wronskian condition can be written by

\[
|\alpha_k (\eta)|^2 - |\beta_k (\eta)|^2 = 1
\]  

(22)
The initial conditions for \( \alpha_k (\eta_0) \) and \( \beta_k (\eta_0) \) corresponds to the choice of the in-vacuum. From Eq. (19), the vacuum field fluctuations \( \langle \delta \phi^2 \rangle \) of the Higgs field can be given by

\[
\langle \delta \phi^2 \rangle = \frac{1}{4\pi^2 C (\eta)} \int_0^\infty dk k^2 \Omega_k^{-1} \left\{ 1 + 2|\beta_k|^2 + \alpha_k \beta_k \delta \varphi_k^2 + \alpha_k^* \beta_k^* \delta \varphi_k^* \right\}
\]

(23)
where the number density of created particles and the corresponding energy density are given by

\[
N = \frac{1}{2\pi^2 a^3 (\eta)} \int_0^\infty dk k^2 |\beta_k|^2
\]  

(24)
\[
\rho = \frac{1}{2\pi^2 a^3 (\eta)} \int_0^\infty dk k^2 \Omega_k |\beta_k|^2
\]  

(25)
For simplicity, we define \( n_k \) and \( z_k \) as the following

\[
n_k = |\beta_k|^2, \quad z_k = \alpha_k \beta_k \delta \varphi_k^2.
\]  

(26)
From Eq. (21), \( n_k \) and \( z_k \) satisfy the following differential equations

\[
n_k' = \frac{\Omega_k}{\Omega_k} \Re z_k, \quad z_k' = \frac{\Omega_k}{\Omega_k} \left( n_k + \frac{1}{2} \right) - 2i \Omega_k z_k.
\]  

(27)
To solve Eq. (27), we must take adequately the initial conditions. For simplicity, we choose the following condition

\[
n_k (\eta_0) = z_k (\eta_0) = 0
\]  

(28)
which is equivalent to \( \alpha_k (\eta_0) = 1, \beta_k (\eta_0) = 0 \) and corresponds to the conformal vacuum at the time \( \eta_0 \). The quantity \( n_k = |\beta_k (\eta)|^2 \) can be interpreted as the number density created in the curved spacetime. By using \( n_k \) and \( z_k \), we obtain the following expression of the vacuum field fluctuations as

\[
\langle \delta \phi^2 \rangle = \frac{1}{4\pi^2 C (\eta)} \int_0^\infty dk k^2 \Omega_k^{-1} \left\{ 1 + 2n_k + 2\Re z_k \right\}
\]  

(29)
where we must solve adequately Eq. (27) and insert \( n_k \) and \( z_k \) into Eq. (29) in order to obtain the vacuum field fluctuations \( \langle \delta \phi^2 \rangle \) of the Higgs field. It is difficult to solve analytically Eq. (27), and therefore, we generally use the adiabatic (WKB) expansion method, which is valid in large mass, large momentum mode or slowly-varying background as follows

\[
|\alpha_k (\eta)|^2 \ll 1
\]

(30)
By using the adiabatic (WKB) expansion method, $n_k$ and $z_k$ can be approximated as follows:

$$n_k = n_k^{(2)} + n_k^{(4)} + \cdots, \quad (31)$$

$$Rez_k = Rez_k^{(2)} + Rez_k^{(4)} + \cdots \quad (32)$$

where superscripts $(i)$ express the adiabatic order and the second order expressions are given by

$$n_k^{(2)} = \frac{1 \Omega^2_k}{16 \Omega^4_k}, \quad (33)$$

$$Rez_k^{(2)} = \frac{1 \Omega''_k}{8 \Omega^2_k} - \frac{1 \Omega^2_k}{4 \Omega^4_k} \quad (34)$$

The forth order adiabatic expressions are given by

$$n_k^{(4)} = -\frac{\Omega'_k \Omega''''_k}{32 \Omega^6_k} + \frac{\Omega''^2_k}{64 \Omega^6_k} + \frac{5 \Omega''^2_k \Omega''_k}{32 \Omega^6_k} - \frac{45 \Omega^4_k}{256 \Omega^6_k}, \quad (35)$$

$$Rez_k^{(4)} = -\frac{\Omega'''_k \Omega'''_k}{32 \Omega^6_k} + \frac{11 \Omega'_k \Omega''''_k}{32 \Omega^6_k} - \frac{115 \Omega''^2_k \Omega''_k}{64 \Omega^6_k} + \frac{7 \Omega^2_k}{32 \Omega^4_k} + \frac{45 \Omega^4_k}{32 \Omega^6_k} \quad (36)$$

By using the adiabatic (WKB) expansion method, we can obtain the following approximation of the vacuum field fluctuations of the Higgs field as

$$\langle \delta \phi^2 \rangle = \langle \delta \phi^2 \rangle^{(0)} + \langle \delta \phi^2 \rangle^{(2)} + \langle \delta \phi^2 \rangle^{(4)} + \cdots \quad (37)$$

where

$$\langle \delta \phi^2 \rangle^{(0)} = \frac{1}{4 \pi^2 C(\eta)} \int_0^\infty dk k^2 \Omega_k^{-1} \quad (38)$$

$$\langle \delta \phi^2 \rangle^{(2n)} = \frac{1}{4 \pi^2 C(\eta)} \int_0^\infty dk k^2 \Omega_k^{-1} \left\{2n_k^{(2n)} + 2Rez_k^{(2n)} \right\} \quad (39)$$

Although the higher order approximation can become finite, the lowest order approximation have UV (quadratic and logarithmic) divergences. However, the divergences in the lowest order expression are the same as the divergences in the Minkowski spacetime. Thus, we can regularize the divergence integral via the cut-off regularization or the dimensional regularization and offset the divergences by the counterterms of the couplings.

By using the dimensional regularization, we obtain the following lowest order expression as

$$\langle \delta \phi^2 \rangle^{(0)} = \frac{M^2(\phi)}{16\pi^2} \left[ \ln \left( \frac{M^2(\phi)}{\mu^2} \right) - \frac{1}{\epsilon} - \log 4\pi - \gamma - \frac{3}{2} \right], \quad (40)$$

with

$$M^2(\phi) = m^2(\mu) + 3\lambda(\mu)\phi^2 + (\xi(\mu) - 1/6)R, \quad (41)$$

where $\mu$ is the renormalization scale and $\gamma$ is the Euler-Mascheroni constant. The counterterms $\delta m^2$, $\delta\xi$ and $\delta\lambda$ must cancel these divergences and are calculated to be given by

$$\delta m^2 = \frac{3\lambda(\mu)}{16\pi^2} \left( \frac{1}{\epsilon} + \log 4\pi + \gamma \right) \quad (42)$$

$$\delta\xi = \frac{3\lambda(\mu)}{16\pi^2} \left( \xi(\mu) - \frac{1}{6} \right) \left( \frac{1}{\epsilon} + \log 4\pi + \gamma \right) \quad (43)$$

$$\delta\lambda = \frac{9\lambda(\mu)}{16\pi^2} \left( \frac{1}{\epsilon} + \log 4\pi + \gamma \right) \quad (44)$$

Thus, the renormalized vacuum field fluctuations of the Higgs field of the lowest order can be given by

$$\langle \delta \phi^2 \rangle^{(0)}_{\text{ren}} = \frac{M^2(\phi)}{16\pi^2} \left[ \ln \left( \frac{M^2(\phi)}{\mu^2} \right) - \frac{3}{2} \right] \quad (45)$$

where the above expression corresponds to the renormalized vacuum field fluctuations of the Minkowski spacetime. From the renormalized Higgs vacuum field fluctuations of Eq. (45), we can construct the one-loop effective evolution equation as follows:

$$\ddot{\phi} + 3H \dot{\phi} + \frac{\partial V_{\text{eff}}(\phi)}{\partial \phi} = 0, \quad (46)$$

where the one-loop effective potential in curved spacetime is given by

$$V_{\text{eff}}(\phi) = \frac{1}{2} m^2(\mu) \phi^2 + \frac{1}{2} \xi R \phi^2 + \frac{\lambda}{4} \phi^4 + \frac{M^2(\phi)}{16\pi^2} \left[ \ln \left( \frac{M^2(\phi)}{\mu^2} \right) - \frac{3}{2} \right] \quad (47)$$

with

$$M^2(\phi) = m^2(\mu) + 3\lambda(\mu)\phi^2 + (\xi(\mu) - 1/6)R \quad (48)$$

From the one-loop effective potential of Eq. (47), the one-loop $\beta$ functions in curved spacetime are given by

$$\beta_{\lambda} = \frac{d\lambda}{d\ln\mu} = \frac{18\lambda^2}{(4\pi)^2}, \quad (49)$$

$$\beta_{\xi} = \frac{d\xi}{d\ln\mu} = \frac{6\lambda}{(4\pi)^2} (\xi - 1/6), \quad (50)$$

$$\beta_{m^2} = \frac{dm^2}{d\ln\mu} = \frac{6\lambda m^2}{(4\pi)^2} \quad (51)$$

The dynamics of the homogeneous Higgs field $\phi(t)$ can be determined by the effective evolution equation of Eq. (120) and the excursion of the Higgs field $\phi(t)$ to the Planckian vacuum state leads to the catastrophic collapse of the Universe. However, the effective evolution equation of Eq. (120) is approximately valid on the adiabatic regime of Eq. (30), strictly speaking, we must count the high-order vacuum field fluctuations to express the particle production effects of the curved spacetime.
III. RENORMALIZED VACUUM FIELD FLUCTUATIONS IN ADIABATIC REGIME

The lowest-order (Minkowskian) vacuum field fluctuations contract the one-loop effective Higgs potential. However, the higher-order adiabatic vacuum field fluctuations appears as a result of the particle production effects of the curved background, and therefore, provide a significant contribution to the global evolution of the Higgs field. In order to obtain the exact one-loop evolution equation in the adiabatic background, we must count up to the higher adiabatic order of the vacuum field fluctuations. From Eq. (31), (32), and (39), the second (adiabatic) order expressions of the vacuum field fluctuations are given by [38]

\[
\langle \delta\phi^2 \rangle^{(2)} = \frac{1}{16\pi^2 C(\eta)} \int_0^\infty dk k^2 \Omega_k^{-1} \left\{ \frac{\Omega_k''}{\Omega_k} + \frac{3}{2} \right\}
\]

with

\[
\Omega_k^2 = k^2 + C(\eta) \left( m^2 + 3\lambda \phi^2 + (\xi - 1/6) R \right).
\]

Thus, we can obtain the following expression as

\[
\langle \delta\phi^2 \rangle^{(2)} = \frac{1}{16\pi^2 C(\eta)} \int_0^\infty dk k^2 \Omega_k^{-1} \left\{ \left( M M'' + M^2 \right) \Omega_k \right\}
\]

with

\[
M^2 = C(\eta) M^2(\phi)
\]

Now, we must perform the integral of Eq. (54). As already pointed out, the high-order (adiabatic) expressions as \(\langle \delta\phi^2 \rangle^{(2)}\) are UV finite and therefore there is no need to renormalize the high-order vacuum field fluctuations. The corresponding integrals converge to the finite values as the following

\[
F(\alpha) \equiv \int_0^\infty dk k^2 (k^2 + M^2)^{-\alpha} = \frac{M^{1-2\alpha}}{2} \Gamma(3/2) \Gamma(\alpha - 3/2) \quad (57)
\]

where the above expression is valid for \(\alpha > 3/2\). By using Eq. (57), the second (adiabatic) order of the vacuum field fluctuations \(\langle \delta\phi^2 \rangle^{(2)}\) are given as follows

\[
\langle \delta\phi^2 \rangle^{(2)} = \frac{1}{16\pi^2 C(\eta)} \left\{ \left( M M'' + \dot{M}^2 \right) F\left( \frac{5}{2} \right) \right\}
\]

Thus, the renormalized vacuum field fluctuations of the Higgs field via the adiabatic (WKB) expansion method in curved background are given by

\[
\langle \delta\phi^2 \rangle^{(2)} = \frac{M^2}{16\pi^2} \left[ \ln \left( \frac{M^2}{\mu^2} \right) - \frac{3}{2} \right] + \frac{1}{48\pi^2 C(\eta)} \frac{M''}{M} + \cdots
\]

where the first term express the (Minkowskian) renormalized vacuum field fluctuations and the second term describes the dynamical contribution of the renormalized vacuum field fluctuations. Now, we can obtain the second (adiabatic) order expression of the proper time \(t\) as

\[
\langle \delta\phi^2 \rangle^{(2)} = \frac{1}{48\pi^2} \left\{ \frac{\dot{a}^2}{a^2} + \frac{3}{2} \frac{\dot{a}}{a} M + \frac{\dot{M}}{M} \right\}
\]

\[
\approx \frac{1}{48\pi^2} \left\{ \frac{3}{2} \frac{\dot{a}}{a} M + \frac{\dot{M}}{M} \right\}.
\]

For simplicity, we consider the conformal coupling case \(\xi = 1/6\) and we can obtain the following expression

\[
\langle \delta\phi^2 \rangle^{(2)} = \frac{1}{48\pi^2} \left\{ \frac{3}{2} \frac{\dot{a}}{a} M + \frac{\dot{M}}{M} \right\}
\]

\[
\approx \frac{1}{48\pi^2} \left\{ \frac{3}{2} \frac{\dot{a}}{a} M + \frac{\dot{M}}{M} \right\}.
\]

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\[
\langle \delta\phi^2 \rangle^{(2)} = \frac{1}{48\pi^2} \left\{ \frac{3}{2} \frac{\dot{a}}{a} M + \frac{\dot{M}}{M} \right\}
\]

\[
\approx \frac{1}{48\pi^2} \left\{ \frac{3}{2} \frac{\dot{a}}{a} M + \frac{\dot{M}}{M} \right\}.
\]

For nearly constant Higgs field, the time-derivative terms are negligible and the second (adiabatic) order expressions of the vacuum field fluctuations are simplified as

\[
\langle \delta\phi^2 \rangle^{(2)} = \frac{R}{288\pi^2}
\]

Therefore, in the massive conformal coupling case (\(\xi = 1/6\) and \(m \gtrsim H\)), we have the high-order renormalized vacuum field fluctuations corresponding to the particle production effects as follows:

\[
\langle \delta\phi^2 \rangle^{(2)}_{\text{ren}} = \frac{R}{288\pi^2} + O(R^2) + \cdots.
\]
In the radiation dominated Universe, the Ricci scalar becomes $R = 0$ and, in the matter dominated Universe, the Ricci scalar becomes $R = 3H^2$. On the other hand, in the de Sitter Universe, the Ricci scalar becomes $R = 12H^2$. Thus, we summarize the renormalized vacuum field fluctuations in the massive conformal coupling case ($\xi = 1/6$ and $m \gtrsim H$) as follows:

$$\langle \delta \phi^2 \rangle_{\text{ren}} \simeq \begin{cases} 
0, & \text{(radiation Universe)} \\
H^2/96\pi^2, & \text{(matter Universe)} \\
H^2/24\pi^2, & \text{(de Sitter Universe)} 
\end{cases}$$

(64)

Note that the massive vacuum field fluctuations in curved spacetime are described by Eq. (63). However, the massless vacuum field fluctuations cannot satisfy the adiabatic (WKB) condition of Eq. (30) as the following expression in the massless case.

$$\Omega_k \simeq \frac{2H}{m} \ll 1,$$

(65)

where we assume $m = \text{const}$, and therefore, the adiabatic (WKB) expansion method does not provide the exact expression in the massless case. In small mass or rapid varying background, the vacuum field fluctuations are generally enlarged as

$$\langle \delta \phi^2 \rangle_{\text{ren}} \gg \mathcal{O}(H^2).$$

(66)

where the vacuum field fluctuations in the non-anatic regime are generally larger than the adiabatic vacuum field fluctuations. This situation cosmologically occurs during inflation for the massless scalar field or during preheating stage of the parametric resonance (see, e.g. Ref.[56]). In the next section, we discuss the non-adiabatic vacuum field fluctuations in the curved background.

IV. RENORMALIZED VACUUM FIELD FLUCTUATIONS IN NON-ADIABATIC REGIME

In the non-anatic regime, e.g. small mass or rapid varying background, we must usually solve the following equation with the suitable in-vacuum,

$$\langle \delta \phi^2 \rangle_{\text{ren}} = \frac{1}{4\pi^2 C(\eta)} \int_0^\infty dk \Omega_k \Omega_k^{-1} \{2n_k + 2\Re z_k\}$$

(67)

where

$$n_k = \frac{\Omega_k'}{\Omega_k} \Re z_k, \quad z_k' = \frac{\Omega_k'}{\Omega_k} \left(n_k + \frac{1}{2}\right) - 2i\Omega_k z_k.$$ 

(68)

However, it is hard task to calculate the non-adiabatic vacuum field fluctuations via the above equations. In fact, if we assume unspecified initial conditions or any initial vacuum, we obtain the following expression of $z_k$ as the following equation [38]

$$z_k(\eta) = \int_{\eta_0}^\eta d\eta' \Omega_k(\eta') \left(n_k(\eta') + \frac{1}{2}\right) \times \exp\left\{-2i \int_{\eta_0}^\eta d\eta' \Omega_k(\eta')\right\}$$

(69)

where we must solve Eq. (68) and inset into Eq. (67), and therefore, there is usually no other way except numerical calculations in order to obtain the non-adiabatic vacuum field fluctuations. However, if we analytically get the exact mode function of $\delta \phi$ from the Klein-Gordon equation of Eq. (67) with the suitable in-vacuum, we can obtain the renormalized vacuum fluctuations $\langle \delta \phi^2 \rangle_{\text{ren}}$ by removing the divergences of $\langle \delta \phi^2 \rangle$ via the adiabatic regularization or the point-splitting regularization.

Next, we review the adiabatic regularization [57–64] which is the extremely powerful method to obtain the renormalized vacuum fluctuations even in the non-adiabatic regime. The adiabatic regularization is not the mathematical method of regularizing divergent integrals as dimensional regularization or cut-off regularization. As previously discussed, the divergences of $\langle \delta \phi^2 \rangle$ come from the lowest-order adiabatic mode, and therefore, we can remove these divergences by subtracting the lowest-order adiabatic (Minkowskian) vacuum field fluctuations $\langle \delta \phi^2 \rangle^{(0)}$ from $\langle \delta \phi^2 \rangle$. Thus, we can obtain the renormalized expression of the adiabatic or the non-adiabatic vacuum fluctuations as the following

$$\langle \delta \phi^2 \rangle_{\text{ren}} = \langle \delta \phi^2 \rangle - \langle \delta \phi^2 \rangle^{(0)}$$

(70)

$$= \frac{1}{4\pi^2 C(\eta)} \int_0^\infty dk \frac{k^2}{2} \Omega_k^{-1} \{2n_k + 2\Re z_k\}$$

where we must get the exact mode function of $\delta \chi_k$ with appropriate in-vacuum. This method is equivalent to the point-splitting regularization which regularizes divergences via the point separation in the two-point function.

As a concrete example how to use the adiabatic regularization, we consider the vacuum field fluctuations of the massless minimally coupling scalar field ($\xi = 0$ and $m = 0$) in the de Sitter spacetime (for the details see e.g. Ref.[62, 63]). In this case, the mode function $\delta \chi_k(\eta)$ can be exactly given by

$$\delta \chi_k(\eta) = \frac{1}{\sqrt{2k}} \left\{\alpha_k \delta \varphi_k(\eta) + \beta_k \delta \varphi_k^* (\eta)\right\},$$

(71)

where

$$\delta \varphi_k(\eta) = e^{-ik\eta} \left(1 + \frac{1}{ik\eta}\right).$$

(72)
In the massless minimally coupled case, the vacuum field fluctuations \( \langle \delta \phi^2 \rangle \) have not only ultraviolet divergences but also infrared divergences. Thus, we assume that the Universe changes from the radiation-dominated spacetime to the de Sitter spacetime in order to avoid the infrared divergences

\[
a(\eta) = \begin{cases} 
2 - \frac{2}{\eta_0}, & (\eta < \eta_0) \\
\frac{2}{\eta_0}, & (\eta > \eta_0)
\end{cases}
\] (73)

where \( \eta_0 = -1/H \) and we choose the mode function as the in-vacuum state

\[
\delta \chi_k = e^{-i k \eta} / \sqrt{2k}.
\] (74)

during the radiation-dominated region \( (\eta < \eta_0) \). By requiring the conditions \( \delta \chi_k (\eta) \) and \( \delta \chi_k (\eta) \) at the matching time \( \eta = \eta_0 \), we obtain the corresponding coefficients of the mode function as follows

\[
\alpha_k = 1 + \frac{H}{\sqrt{k}} - \frac{H^2}{2k^2},
\]

\[
\beta_k = -\frac{H^2}{2k^2} e^{2i k} = \alpha_k + \frac{2i k}{3H} + O \left( \frac{k^2}{H^2} \right).
\] (75)

By using the above coefficients of \( \alpha_k \) and \( \beta_k \), we obtain the suitable mode function of \( \delta \chi_k \). For small \( k \) modes in the de Sitter Universe \( (\eta > \eta_0) \), we can approximate the mode function as the following

\[
|\delta \chi_k|^2 = \frac{1}{2k} \left[ \left( \frac{2}{3H \eta} + 2 + \frac{H^2 \eta^2}{6} \right)^2 + O \left( \frac{k^2}{H^2} \right) + \cdots \right].
\] (77)

where there is no infrared divergence because \( k^2 |\delta \chi_k|^2 \approx O(k) \). For large \( k \) modes, we can obtain the following expression

\[
|\delta \chi_k|^2 = \frac{1}{2k} \left[ 1 + \frac{1}{k^2 \eta^2} - \frac{H^2}{k^2} \cos \left( 2k (1/H + \eta) \right) + O \left( \frac{H^3}{k^3} \right) + \cdots \right].
\] (78)

Here, we must require the cut-off of \( k \) mode form the the adiabatic (WKB) condition \( \Omega_k^2 > 0 \), i.e. \( k > \sqrt{2/|\eta|} = \sqrt{2aH} \). Therefore, we can obtain the renormalized vacuum fluctuations form Eq. (70) as follows:

\[
\langle \delta \phi^2 \rangle_{\text{ren}} = \lim_{\Lambda \to \infty} \frac{1}{4\pi^2 C(\eta)} \left[ \int_0^\Lambda 2k^2 |\delta \chi_k|^2 dk - \int_{\sqrt{2/|\eta|}}^{\Lambda} dk k^2 \Omega_k^{-1} \right]
\]

\[
= \lim_{\Lambda \to \infty} \frac{1}{4\pi^2 C(\eta)} \left[ \int_0^\Lambda 2k^2 |\delta \chi_k|^2 dk - \int_{\sqrt{2/|\eta|}}^{\Lambda} k^2 \left( \frac{1}{\sqrt{k^2 - 2/\eta^2}} \right) dk \right]
\] (79)

\[
= \lim_{\Lambda \to \infty} \frac{1}{4\pi^2 C(\eta)} \left[ \int_0^\Lambda 2k^2 |\delta \chi_k|^2 dk - \int_{\sqrt{2/|\eta|}}^{\Lambda} \left( k + \frac{1}{k^2 \eta^2} + \cdots \right) dk \right].
\] (80)

For large \( k \) modes, we can use Eq. (78) and subtract the UV divergences as the following

\[
\lim_{\Lambda \to \infty} \frac{1}{4\pi^2 C(\eta)} \left[ \int_{\sqrt{2/|\eta|}}^{\Lambda} \left( k + \frac{1}{k^2 \eta^2} \right) dk \right] = 0.
\] (81)

Thus, we obtain the following expression of the renormalized vacuum field fluctuations as

\[
\langle \delta \phi^2 \rangle_{\text{ren}} = \frac{1}{2\pi^2 C(\eta)} \left[ \int_{\sqrt{2/|\eta|}}^{\Lambda} k^2 |\delta \chi_k|^2 dk \right.
\]

\[
+ \frac{\eta^2 H^2}{4\pi^2} \int_{\sqrt{2/|\eta|}}^{\infty} \left( -\frac{H^2}{k^2} \cos \left( 2k (1/H + \eta) \right) - \frac{H^2}{k^2} \right) dk
\]

\[
+ \frac{1}{9\pi^2} \int_0^H k dk + \frac{H^2}{4\pi^2} \int_{\sqrt{2/|\eta|}}^{\pi} \frac{1}{k} dk.
\] (82)

At the late cosmic-time \( (\eta \approx 0 \text{ corresponds to } N_{\text{tot}} = Ht \gg 1) \), we have the following approximation

\[
\langle \delta \phi^2 \rangle_{\text{ren}} \approx \frac{\eta^2 H^2}{2\pi^2} \int_{\sqrt{2/|\eta|}}^{\Lambda} k^2 |\delta \chi_k|^2 dk,
\]

\[
\approx \frac{1}{9\pi^2} \int_0^H k dk + \frac{H^2}{4\pi^2} \int_{\sqrt{2/|\eta|}}^{\pi} \frac{1}{k} dk
\] (83)

where we approximate the mode function \( \delta \chi_k \) from Eq. (77) and Eq. (78) as the following

\[
|\delta \chi_k|^2 = \begin{cases} 
\frac{1}{2k} \left( \frac{2}{3H \eta} + 2 + \frac{H^2 \eta^2}{6} \right)^2 & (0 \leq k \leq H) \\
\frac{1}{2k} \left( 1 + \frac{1}{k^2 \eta^2} \right) & (H \leq k \leq \sqrt{2/|\eta|})
\end{cases}
\] (84)

Therefore, we can finally obtain the well-know expression as follows

\[
\langle \delta \phi^2 \rangle_{\text{ren}} \approx \frac{H^2}{18\pi^2} + \frac{H^2}{4\pi^2} \left( \frac{1}{2} \log 2 + Ht \right)
\]

\[
\approx \frac{H^3}{4\pi^2} t,
\] (85)
which grows as cosmic-time proceeds.

Next, we consider the massive minimally coupled scalar field ($\xi = 0$ and $m \ll H$) in the de Sitter spacetime. This situation is cosmologically important in order to understand the origin of the primordial perturbations or the backreaction of the inflaton field in the inflationary Universe (see, e.g. Ref.[65, 66]). In this case, the mode function $\delta \chi_k (\eta)$ can be given by

$$\delta \chi_k (\eta) = \sqrt{\frac{\pi}{4}} \eta^{1/2} \left\{ \alpha_k H^{(2)}_\nu (k \eta) + \beta_k H^{(1)}_\nu (k \eta) \right\}, \tag{86}$$

with

$$\nu \equiv \sqrt{\frac{9 - \frac{m^2}{2H^2}}{3 - \frac{m^2}{3H^2}}} \approx 3 - \frac{m^2}{3H^2} \tag{87}$$

where $H^{(1,2)}_\nu (k \eta)$ are the Hankel functions. As previously mentioned, we assume the spacetime transition from the radiation-dominated stage to the de Sitter stage and require the matching conditions at $\eta = \eta_0$ to determine the Bogoliubov coefficients

$$\alpha_k = \frac{1}{2k} \sqrt{\frac{\pi \kappa \eta_0}{2}} \left( -i + \frac{H}{2k} \right) H^{(1)}_\nu (k \eta_0) - H^{(1\prime)}_\nu (k \eta_0) e^{ik/H} \tag{88}$$

$$\beta_k = - \frac{1}{2k} \sqrt{\frac{\pi \kappa \eta_0}{2}} \left( -i + \frac{H}{2k} \right) H^{(2)}_\nu (k \eta_0) - H^{(2\prime)}_\nu (k \eta_0) e^{ik/H} \tag{89}$$

Thus, the renormalized vacuum field fluctuations form Eq. (70) are given as follows:

$$\langle \delta \phi^2 \rangle_{\text{ren}} = \lim_{\Lambda \rightarrow \infty} \frac{1}{4\pi^2 C(\eta)} \left[ \int_0^\Lambda 2k^2 |\delta \chi_k|^2 dk - \int_{\sqrt{2}/|\eta|}^\Lambda dkk^2 \Omega_k^{-1} \right]$$

$$= \eta^2 \frac{H^2}{2\pi^2} \int_0^{H} k^2 |\delta \chi_k|^2 dk + \eta^2 \frac{H^2}{2\pi^2} \int_{H}^{\sqrt{2}/|\eta|} k^2 |\delta \chi_k|^2 dk. \tag{90}$$

The divergence part exactly cancel as previously discussed

$$\lim_{\Lambda \rightarrow \infty} \frac{1}{4\pi^2 C(\eta)} \left[ \int_{\sqrt{2}/|\eta|}^\Lambda 2k^2 |\delta \chi_k|^2 dk - \int_{\sqrt{2}/|\eta|}^\Lambda dkk^2 \Omega_k^{-1} \right] \tag{91}$$

where we must take the adiabatic mode cut-off as $k > \sqrt{2 - m^2/2H^2}/|\eta| \approx \sqrt{2}/|\eta|$. It is more difficult task in the massive case than in the massless case to obtain exactly the renormalized vacuum fluctuations from Eq. (88) and Eq. (89). However, by using the asymptotic behavior of the Hankel functions, we can easily get the renormalized vacuum fluctuations of $\langle \delta \phi^2 \rangle_{\text{ren}}$ via the adiabatic regularization method (for the details, see Ref.[62, 63]).

By using the following formula of the Hankel functions

$$H^{(1,2)}_\nu (k \eta_0) = H^{(1,2)}_\nu (k \eta_0) - \frac{\nu}{k \eta_0} H^{(1,2)}_\nu (k \eta_0), \tag{92}$$

and the Bessel function of the first kind defined by $J_\nu = (H^{(1)}_\nu + H^{(2)}_\nu)/2$, we obtain the following expression

$$|\alpha_k - \beta_k| = \sqrt{\frac{\pi}{2H}} \left| J_{\nu - 1} (k \eta_0) + \left( i - \frac{H}{2k} + \frac{\nu H}{k} \right) J_{\nu} (k \eta_0) \right| \tag{93}$$

For small $k$ modes, the the Bessel function and the Hankel function asymptotically behave as

$$J_{\nu} (k \eta_0) \simeq \frac{1}{\Gamma(\nu + 1)} \left( \frac{k \eta_0}{2} \right)^\nu \tag{94}$$

$$H^{(2)}_\nu (k \eta_0) \simeq -H^{(1)}_\nu (k \eta_0) \simeq \frac{i}{\pi} \Gamma (\nu) \left( \frac{k \eta_0}{2} \right)^{-\nu} \tag{95}$$

Thus, we obtain the following expression of the mode function

$$|\delta \chi_k|^2 \simeq \frac{2}{9k} (H |\eta|)^{1-2\nu} \quad (0 \leq k \leq H) \tag{96}$$

For large $k$ modes, we can approximate the Bogoliubov coefficients as $\alpha_k \simeq 1$ and $\beta_k \simeq 0$ and evaluate the mode function

$$\delta \chi_k (\eta) \simeq \sqrt{\frac{\pi}{4}} \eta^{1/2} H^{(2)}_\nu (k \eta) \tag{97}$$

Thus, we obtain the following expression

$$|\delta \chi_k|^2 \simeq \frac{|\eta|}{16} \left( \frac{k |\eta|}{2} \right)^{-2\nu} \left( H \leq k \leq \sqrt{2}/|\eta| \right). \tag{98}$$

From Eq. (96) and Eq. (98), the renormalized vacuum field fluctuations are given by

$$\langle \delta \phi^2 \rangle_{\text{ren}} \simeq \frac{(H |\eta|)^{3-2\nu}}{9\pi^2} \int_0^H kdk$$

$$\int_{H}^{\sqrt{2}/|\eta|} k^2 |\delta \chi_k|^2 dk + \frac{H^2 |\eta|^{3-2\nu}}{4\pi^2 \cdot 2^{2-2\nu}} \int_{H}^{\sqrt{2}/|\eta|} k^{2-2\nu} dk$$

$$\simeq \frac{H^2}{18\pi^2} e^{-\frac{2m^2}{3H^2}} + \frac{3H^2}{8\pi^2 m^2} \left( 1 - e^{-\frac{2m^2}{3H^2}} \right). \tag{99}$$

In the de Sitter spacetime, the renormalized vacuum fluctuations $\langle \delta \phi^2 \rangle_{\text{ren}}$ via the adiabatic regularization are summarized as follows [26]

$$\langle \delta \phi^2 \rangle_{\text{ren}} \simeq \begin{cases} H^3 t/4\pi^2, & (m = 0, \xi = 0) \\ 3H^4/8\pi^2 m^2, & (m \ll H, \xi \ll 1/6) \\ H^2/24\pi^2, & (m \gg H, \xi \gg 1/6) \end{cases} \tag{100}$$
The vacuum field fluctuations as described by Eq. (64) and Eq. (101) are equivalent to the quantum particle creation from the curved background, and therefore, once generated vacuum fluctuations remains on the cosmological timescale. However, if the created particles can decay into other particles, the created vacuum field fluctuations would disappear on the particle decay timescale.

V. RENORMALIZED VACUUM FIELD FLUCTUATIONS IN VARYING SCALAR FIELD BACKGROUND

In the general cosmological situations, the background Higgs field dynamically changes and does not stagnate for all times. The dynamical variation of the Higgs field or other scalar field coupled with the Higgs field provide the varying effective mass and leads to the real particle productions or the vacuum fluctuations of the Higgs field. Even in the slowly varying scalar field background, the generated vacuum field fluctuations are non-negligible. In this section, we consider the vacuum filed fluctuations in the (slowly) varying scalar field background following the literature [38].

A. The Higgs field background

For convenience, we rewrite Eq. (67) in order to obtain the renormalized vacuum field fluctuations on the varying Higgs field background

\[ \langle \delta \phi^2 \rangle_{\text{ren}} = \frac{1}{4\pi^2 C(\eta)} \int_0^{\infty} dk k^2 \Omega_k^{-1} \{ n_k + 2 \text{Re} z_k \} \]  

(102)

where \( n_k \) and \( z_k \) are determined by the differential equations of Eq. (68) as follows

\[ n'_k = \frac{\Omega'_k}{\Omega_k} \text{Re} z_k, \quad z'_k = \frac{\Omega'_k}{\Omega_k} \left( n_k + \frac{1}{2} \right) - 2i \Omega_k z_k. \]  

(103)

Here, we assume the conformal vacuum satisfying the initial conditions \( n_k (\eta_0) = z_k (\eta_0) = 0 \). In this situation, we can obtain the following equations as

\[ n_k (\eta) = \int_{\eta_0}^{\eta} \left( \int_{\eta_0}^{\eta_2} \frac{d\eta_1}{\Omega_k (\eta_1)} \right) \cos \left( \frac{1}{2} n_k (\eta_2) \right) \]  

(104)

\[ \text{Re} z_k (\eta) = \int_{\eta_0}^{\eta} \left( \int_{\eta_0}^{\eta_2} \frac{d\eta_1}{\Omega_k (\eta_1)} \right) \cos \left( \frac{1}{2} \int_{\eta_0}^{\eta_2} d\eta_2 \Omega_k (\eta_2) \right) \]  

(105)

For simplicity, we assume the following condition as

\[ \int_{\eta_0}^{\eta} \frac{\Omega'_k (\eta_1)}{\Omega_k (\eta_1)} \ll 1 \]  

(106)

which corresponds to the small time-difference of \( \bar{M}^2 (\eta) \) as follows

\[ |M^2 (\eta) - \bar{M}^2 (\eta_0)| \ll 2M^2 (\eta) \text{ or } 2\bar{M}^2 (\eta_0) \]  

(107)

In this assumption, we can approximate these equations of Eq. (104) and Eq. (105) as follows

\[ n_k (\eta) \approx 0, \]  

(108)

\[ \text{Re} z_k (\eta) \approx \frac{1}{2} \int_{\eta_0}^{\eta} d\eta_1 \frac{\Omega'_k (\eta_1)}{\Omega_k (\eta_1)} \cos \left( 2 \int_{\eta_1}^{\eta} d\eta_2 \Omega_k (\eta_2) \right). \]  

Furthermore, we can approximate Eq. (108) as the following

\[ \text{Re} z_k (\eta) \approx \frac{1}{2} \int_{\eta_0}^{\eta} d\eta_1 \frac{\bar{M} (\eta_1) \bar{M}' (\eta_1)}{\Omega_k^2 (\eta_1)} \cos \left( 2 \int_{\eta_1}^{\eta} d\eta_2 \Omega_k (\eta_2) \right) \]  

\[ \times \cos \left( 2 \Omega_k (\eta_1) (\eta - \eta_1) \right) \]  

(109)

From Eq. (102), we can obtain the renormalized vacuum field fluctuations as

\[ \langle \delta \phi^2 \rangle_{\text{ren}} = \frac{1}{4\pi^2 C(\eta)} \int_0^{\infty} dk k^2 \Omega_k^{-1} \{ n_k + \text{Re} z_k \} \]  

\[ \approx \frac{1}{2\pi^2 C(\eta)} \int_0^{\infty} dk k^2 \Omega_k^{-3} \int_{\eta_0}^{\eta} d\eta_1 \bar{M} (\eta_1) \bar{M}' (\eta_1) \]  

\[ \times \cos \left( 2 \Omega_k (\eta_1) (\eta - \eta_1) \right) \]  

(110)

By using the partial integration, we can have the following expression

\[ \langle \delta \phi^2 \rangle_{\text{ren}} \approx \frac{\bar{M}^2 (\eta)}{8\pi^2 C(\eta)} \left( \bar{M}^2 (\eta_0) - \bar{M}^2 (\eta) \right) \int_0^{\infty} dk k^2 \Omega_k^{-3} \]  

\[ + \frac{1}{4\pi^2 C(\eta)} \int_0^{\infty} dk k^{-1} \Omega_k^{-1} \int_{\eta_0}^{\eta} d\eta_1 \]  

\[ \times \bar{M} (\eta_1) \bar{M}' (\eta_1) \cos \left( 2 \Omega_k (\eta_1) (\eta - \eta_1) \right) \]  

\[ + \frac{\bar{M}^2 (\eta)}{4\pi^2 C(\eta)} \int_0^{\infty} dk k^{-2} \int_{\eta_0}^{\eta} d\eta_1 \]  

\[ \times \left( \bar{M}^2 (\eta_1) - \bar{M}^2 (\eta_0) \right) \sin \left( 2 \Omega_k (\eta_1) (\eta - \eta_1) \right) \]  

(111)

which equivalent to the result by using the perturbation technique [67]. By performing the integration, we can obtain the following expression

\[ \langle \delta \phi^2 \rangle_{\text{ren}} \approx \frac{1}{8\pi^2 a^2 (\eta)} \left( \bar{M}^2 (\eta_0) - \bar{M}^2 (\eta) \right) \]  

\[ - \frac{1}{8\pi^2 a^2 (\eta)} \int_{\eta_0}^{\eta} d\eta_1 \bar{M} (\eta_1) \bar{M}' (\eta_1) \]  

\[ \times N_0 \left( 2 \bar{M} (\eta - \eta_1) \right) \]  

\[ + \frac{\bar{M}^2 (\eta)}{8\pi^2 a^2 (\eta)} \int_{\eta_0}^{\eta} d\eta_1 \left( \eta - \eta_1 \right) \]  

\[ \times \left( \bar{M}^2 (\eta_1) - \bar{M}^2 (\eta_0) \right) F \left( 2 \bar{M} (\eta - \eta_1) \right) \]  

(112)
where $N_0(x)$ is the Bessel function, $F(x)$ are combination of Bessel and Struve functions defined in Ref. [67] and $M(\eta)$ is described by the varying Higgs background field as $M(\eta) \approx 3\alpha(\eta)\phi^2(\eta)$. When the expansion of the Universe is slow and the background Higgs field $\phi(\eta)$ evolve quickly on the cosmological timescale, the vacuum field fluctuations enlarge in proportion to $M(\eta)$. The vacuum field fluctuations given by Eq. (112) approximately equal to the first-order adiabatic approximation of Eq. (37) where the odd-order adiabatic number density is zero as $n_{k(2n+1)} = 0$. As previously discussed, the second-order approximation of the vacuum field fluctuations of Eq. (58) are written by

$$\langle \delta\phi^2 \rangle_{\text{ren}} = \frac{1}{48\pi^2} \frac{\dot{M}''(\eta)}{M(\eta)}$$

with $\dot{M}(\eta) = \dot{a}(\eta) (m^2 + 3\lambda\phi^2 + (\xi - 1/6) R)$. Therefore, we can obtain the following expression of the second-order adiabatic vacuum fluctuations as

$$\langle \delta\phi^2 \rangle_{\text{ren}} = \frac{1}{48\pi^2} \left\{ \frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{3}{2} \frac{\dot{a}}{m^2 + (\xi - 1/6) R + 3\lambda\phi^2} \right. \right.$$

$$- \frac{1}{4} \left[ \left( (\xi - 1/6) \dot{R} + 6\lambda\phi\dot{\phi} \right)^2 \right.$$

$$\left. + \frac{1}{2} \left( (\xi - 1/6) \dot{R} + 6\lambda\phi\dot{\phi} + \phi^2 \right) \right\}.$$ (114)

For the large background Higgs field $\phi(t)$ where we can safely neglect the mass terms or the non-minimal curvature terms, the second-order adiabatic expression of the vacuum field fluctuations can be given by

$$\langle \delta\phi^2 \rangle_{\text{ren}} \approx \frac{1}{48\pi^2} \left\{ \frac{1}{6} + \frac{3H\phi}{\dot{\phi} + \phi} \right\}.$$ (115)

When the curvature effects of the spacetime are negligible and the Higgs background field evolves quickly as $\phi(t) \sim e^{-M(\phi)t}$ or $\phi(t) \sim \sin(M(\phi)t)$, from Eq. (114) and Eq. (115) the renormalized vacuum field fluctuations on the varying Higgs field background can be approximated by

$$\langle \delta\phi^2 \rangle_{\text{ren}} \approx \frac{M^2(\phi)}{48\pi^2}. \hspace{1cm} (116)$$

If the Higgs field has the large effective mass $M(\phi)$, the Higgs background field develops rapidly on the cosmological timescale and the vacuum field fluctuations of the Higgs field grow in proportional to $M(\phi)$.

**B. The scalar (inflaton) field background**

On the other hand, if there are other (coherent or classical) scalar fields $S$ like the inflaton field which couple the Higgs field with $\lambda_{S\phi},$ the effective mass of the Higgs field can be generated as $m_S^2 = \lambda_{S\phi}S^2$. The above situation is cosmological realistic during or after inflation. In this case, the Higgs field acquire the effective mass as $M(\eta) = \dot{a}(\eta) (m^2 + 3\lambda\phi^2 + \lambda_{S\phi}S^2 + (\xi - 1/6) R)$ and the second-order adiabatic vacuum fluctuations can be given as

$$\langle \delta\phi^2 \rangle_{\text{ren}} = \frac{1}{48\pi^2} \left\{ \frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{3}{2} \frac{\dot{a}}{m^2 + (\xi - 1/6) R + 3\lambda\phi^2 + \lambda_{S\phi}S^2} \right.$$

$$- \frac{1}{4} \left[ \left( (\xi - 1/6) \dot{R} + 6\lambda\phi\dot{\phi} + 2\lambda_{S\phi}S\dot{S} \right)^2 \right.$$

$$\left. - \frac{1}{2} \left( (\xi - 1/6) \dot{R} + 6\lambda\phi\dot{\phi} + 2\lambda_{S\phi}S\dot{S} \right) \right\}.$$ (117)

For the large background scalar field $S(t)$, the second-order adiabatic vacuum fluctuations can be given by

$$\langle \delta\phi^2 \rangle_{\text{ren}} \approx \frac{1}{48\pi^2} \left\{ \frac{1}{6} + \frac{3H\dot{S}}{\dot{S} + \dot{S}} \right\}.$$ (118)

The evolution of the background scalar field $S(t)$ is determined by the effective scalar potential $V_{\text{eff}}(S)$. Therefore, the renormalized vacuum field fluctuations on the varying background scalar field can be given by

$$\langle \delta\phi^2 \rangle_{\text{ren}} \approx \frac{m_S^2}{48\pi^2}, \hspace{1cm} (119)$$

where $m_S$ is the effective mass of $S$ and defined by $V_{\text{eff}}''(S) = m_S^2$. The vacuum fluctuations of the Higgs field can become as large as the curvature scale $R$, the mass of the Higgs field $\phi$ or the mass of the scalar field $S$ in the general cosmological background.

**VI. ELECTROWEAK VACUUM INSTABILITY IN CURVED SPACETIME**

So far we have discussed the vacuum field fluctuations of the Higgs field in the adiabatic, non-adiabatic background or the varying scalar field background. In this section, we investigate the electroweak vacuum instability in the general cosmological background by using the results of Section III, Section IV and Section V. The fate of the electroweak false vacuum is determined by the dynamics of the background Higgs field $\phi(t)$ and the vacuum fluctuations $\langle \delta\phi^2 \rangle_{\text{ren}}$ of the Higgs field. As previously discussed in Section II, the one-loop effective evolution equation of the Higgs field can be given as follows:

$$\ddot{\phi} + 3H\dot{\phi} + \frac{\partial V_{\text{eff}}(\phi)}{\partial\phi} = 0, \hspace{1cm} (120)$$

where $V_{\text{eff}}(\phi)$ is the effective potential of the Higgs field.
where the one-loop standard model Higgs potential in curved spacetime can written as \[18\]
\[
V_{\text{eff}}(\phi) = \frac{1}{2} m^2(\mu) \phi^2 + \frac{1}{2} \xi(\mu) R \phi^2 + \frac{\lambda(\mu)}{4} \phi^4 + \sum_{i=1}^{9} \frac{n_i}{64 \pi^2} M_i^4(\phi) \left[ \log \frac{M_i^2(\phi)}{\mu^2} - C_i \right],
\]
with
\[
M_i^2(\phi) = \kappa_i \phi^2 + \kappa_i \langle \delta \phi^2 \rangle_{\text{ren}} + \theta_i R.
\]
where the coefficients \(n_i, \kappa_i, \kappa_i', \theta_i\) and \(C_i\) are given by Table I of Ref.[18]. The effective evolution equation and the one-loop effective potential in curved spacetime has been well known in the literature [38–55, 68, 69]. However, as the previous discussed, the additional contribution from the vacuum field fluctuations change the effective evolution equation of the Higgs field as follows
\[
\dot{\phi} + 3 H \phi + \frac{\partial V_{\text{eff}}(\phi)}{\partial \phi} + \lambda(\mu) \langle \delta \phi^2 \rangle_{\text{ren}} \phi = 0,
\]
where the vacuum fluctuations term provides the effective mass and this formulation was first given by Ref.[67]. However, this effective evolution equation and the one-loop effective potential in curved spacetime given by the literature, include some problem of the renormalization scale \(\mu\). In fact, we can appropriately take the renormalization scale \(\mu\) so as to suppress the high order logarithmic corrections of \(\log M_i^2(\phi)/\mu^2\). In the Minkowski space-time as \(R = 0\), we usually take the renormalization scale to be \(\mu \approx \phi\). The renormalization scale \(\mu\) corresponds to the effective mass of the Higgs field or the cosmological energy scale. It is clear that these log-corrections in Eq. (120) do not include the vacuum fluctuation terms of \(\langle \delta \phi^2 \rangle_{\text{ren}}\) and we can not chose the renormalization scale to be \(\mu^2 \approx \langle \delta \phi^2 \rangle_{\text{ren}}\) in this expression. Thus, we must improve the formulation of Eq. (120) and Eq. (121) to satisfy these facts. Here, we provide the simple solution of these problems where we shift the Higgs field \(\phi^2 \rightarrow \phi^2 + \langle \delta \phi^2 \rangle_{\text{ren}}\) so as to include the backreaction terms from the vacuum field fluctuations. Thus, the standard model Higgs effective potential in curved spacetime can be modified as follows:
\[
V_{\text{eff}}(\phi) = \frac{1}{2} m^2(\mu) \phi^2 + \frac{1}{2} \xi(\mu) R \phi^2 + \frac{\lambda(\mu)}{4} \phi^4 + \sum_{i=1}^{9} \frac{n_i}{64 \pi^2} M_i^4(\phi) \left[ \log \frac{M_i^2(\phi)}{\mu^2} - C_i \right] \approx \phi^2 + R + \langle \delta \phi^2 \rangle_{\text{ren}} \phi^2 + \lambda(\mu) \langle \delta \phi^2 \rangle_{\text{ren}} \phi^2 + \sum_{i=1}^{9} \frac{n_i}{64 \pi^2} M_i^4(\phi) \left[ \log \frac{M_i^2(\phi)}{\mu^2} - C_i \right] + \lambda(\mu) \langle \delta \phi^2 \rangle_{\text{ren}} \phi^2 + \sum_{i=1}^{9} \frac{n_i}{64 \pi^2} M_i^4(\phi) \left[ \log \frac{M_i^2(\phi)}{\mu^2} - C_i \right],
\]
where
\[
M_i^2(\phi) = \kappa_i \phi^2 + \kappa_i \langle \delta \phi^2 \rangle_{\text{ren}} + \kappa_i' + \theta_i R.
\]
Now, we can properly take the renormalization scale to be \(\mu^2 \approx \phi^2 + R + \langle \delta \phi^2 \rangle_{\text{ren}}\) in the effective Higgs potential. The running couplings of \(m^2(\mu), \xi(\mu)\) and \(\lambda(\mu)\) vary depending on the renormalization scale which corresponds to the energy scale of the phenomenological environment. In particular, the running Higgs self-coupling \(\lambda(\mu)\) becomes negative at the high-scale \(\Lambda_1\). Therefore, if the renormalization scale becomes larger than the instability scale as \(\mu \approx (R + \langle \delta \phi^2 \rangle_{\text{ren}})^{1/2} > \Lambda_1\), the quartic term \(\lambda(\mu) \phi^4/4\) becomes negative and destabilizes the standard model Higgs effective potential in curved spacetime.

Although the standard Higgs mass is the electroweak scale \(\mathcal{O}(M_{\text{EW}})\), the Higgs field can phenomenologically acquire various effective masses from various couplings in the cosmological background. The non-minimal coupling \(\xi(\mu)\) provides an extra contribution to the Higgs mass via the curvature \(R\). Furthermore, if there are classical and coherent scalar fields to couple the Higgs field with \(\lambda_{S}S\), the dynamical mass of the Higgs field can be generated by \(\lambda_{S}S^2\) where \(S\) is the classical scalar field like the inflaton. In present paper, we consider the curvature mass \(\Lambda(\mu)\) and the dynamical mass \(\lambda_{S}S^2\).

The magnitude relation of the effective mass \(m^2_{\text{eff}} \approx \xi(\mu) R + \lambda_{S}S^2\) and the renormalized Higgs vacuum fluctuations \(\langle \delta \phi^2 \rangle_{\text{ren}}\) govern the destabilization of the Higgs effective potential in the curved background. If the Higgs effective potential is destabilized by the vacuum field fluctuations, the Higgs effective potential becomes negative, i.e. \(\partial V_{\text{eff}}(\phi)/\partial \phi \lesssim 0\), and therefore, the classical and coherent Higgs field \(\phi(t)\) on the entire Universe rolls down to the Planck-scale true vacuum. In Section III, Section IV and Section V, we have considered the vacuum field fluctuations in the adiabatic, the non-adiabatic background or the varying scalar field background. By using the obtained results, we can obtain the stabilization or destabilization condition of the Higgs effective potential in the cosmological background.

In \(\xi(\mu) R \gg \lambda_{S}S^2\), the renormalized vacuum fluctuations of the Higgs field are summarize as
\[
\langle \delta \phi^2 \rangle_{\text{ren}} \begin{cases} 
\approx R/288 \pi^2 & \langle \xi(\mu) \gtrsim \mathcal{O}(10^{-1}) \rangle \\
\gtrsim \mathcal{O}(R) & \langle \xi(\mu) \lesssim \mathcal{O}(10^{-1}) \rangle
\end{cases}
\]
where Eq. (126) express the vacuum fluctuations in the adiabatic and the non-adiabatic background. In the de-Sitter spacetime as \(R = 12H^2\), the renormalized Higgs vacuum fluctuations can be given by
\[
\langle \delta \phi^2 \rangle_{\text{ren}} \approx \begin{cases} 
H^2/24 \pi^2 & \langle \xi(\mu) \gtrsim \mathcal{O}(10^{-1}) \rangle \\
H^2/32 \pi^2 \xi(\mu) & \langle \xi(\mu) \lesssim \mathcal{O}(10^{-1}) \rangle
\end{cases}
\]
\[1\] The instability scale \(\Lambda_1\) can be approximately determined by the value of the Higgs boson mass and the top quark mass. The current measurements of the Higgs boson mass \(m_H = 125.09 \pm 0.21\,(\text{stat})\pm 0.11\,(\text{syst})\) GeV [1–4] and the top quark mass \(m_t = 172.44 \pm 0.13\,(\text{stat})\pm 0.47\,(\text{syst})\) GeV [5] show the instability scale to be \(\Lambda_1 \approx 10^{11}\) GeV [70] although this instability scale \(\Lambda_1\) depends on the gauge (see [71–75] for the detail discussions).
where the above equations are valid during the inflation. However, after inflation, the non-minimal curvature term \( \xi (\mu) R \) can generate the enormous Higgs vacuum fluctuations via tachyonic resonance as \( \langle \delta \phi^2 \rangle_{\text{ren}} \gg O(R) \) where the non-minimal curvature term \( \xi (\mu) \) is relatively large. If we assume the simple \( m_S^2 S^2 \) chaotic inflation model, we can numerically obtain the constraint of the tachyonic resonance not to generate the large Higgs vacuum fluctuations as \( |\xi (\mu)| \lesssim O(10) \) (see Ref.[25, 28–32] for the detailed discussions).

In the scale \( \mu \approx (R + \langle \delta \phi^2 \rangle_{\text{ren}})^{1/2} > \Lambda_I \), the Higgs self-coupling \( \lambda (\mu) \) becomes negative \(^2\) and the destabilization of the effective Higgs potential can be determined by the following relation \( \xi (\mu) R < |\lambda (\mu)| \langle \delta \phi^2 \rangle_{\text{ren}} \) where we assume \( \lambda (\mu) \approx -0.01 \). In the de-Sitter spacetime, we can obtain the condition of the non-minimal coupling as \( \xi (\mu) \lesssim O(10^{-3}) \) not to destabilize the effective Higgs potential \(^3\). In other cosmological background, we can expect the same constraint of the non-minimal coupling to be \( \xi (\mu) \lesssim O(10^{-3}) \). If \( \xi (\mu) \) does not satisfy the condition, the effective Higgs potential \( V_{\text{eff}} (\phi) \) is destabilized, the coherent Higgs field \( \phi (t) \) goes out to the negative Planck-energy vacuum and leads to the catastrophic collapse of the Universe.

In \( \xi (\mu) R \ll \lambda_{\phi S} S^2 \), the renormalized vacuum fluctuations of the Higgs field can be given by

\[
\langle \delta \phi^2 \rangle_{\text{ren}} \approx \left\{ \begin{array}{ll}
M^2 (\phi) / 48 \pi^2 & (\lambda_{\phi S} S^2 \lesssim \lambda \phi^2) \\
\frac{m_S^2}{48 \pi^2} & (\lambda_{\phi S} S^2 \gtrsim \lambda \phi^2)
\end{array} \right.
\]

(128)

where the above equations are valid for the adiabatic background. In the non-adiabatic case, the vacuum fluctuations of the Higgs field become generally larger than the adiabatic vacuum fluctuations. As well-known facts, in the parametric resonance or the tachyonic resonance during preheating stage, the vacuum field fluctuations exponentially grow as \( \langle \delta \phi^2 \rangle_{\text{ren}} \gg m_S^2 / 48 \pi^2 \) where the complicated numerical analysis must be required. If we assume the simple \( m_S^2 S^2 \) chaotic inflation model, we can numerically obtain the restriction of the parametric Higgs resonance not to generate the large vacuum fluctuations as \( \lambda_{\phi S} \lesssim O(10^{-8}) \) (see Ref.[25, 28–32] for the detailed discussions).

In the scale \( \mu \approx (R + \langle \delta \phi^2 \rangle_{\text{ren}})^{1/2} > \Lambda_I \), the destabilization of the effective Higgs potential \( V_{\text{eff}} (\phi) \) can be determined by \( \lambda_{\phi S} S^2 < |\lambda (\mu)| \langle \delta \phi^2 \rangle_{\text{ren}} \). For instance, if we assume the simple \( m_S^2 S^2 \) chaotic inflation where \( S \) is the inflaton, the coherent inflaton field \( S \) has the Planck-scale field value \( S \approx M_{\text{Pl}} \approx 10^{19} \) GeV and the stabilization condition during inflation can be given by \( \lambda_{\phi S} \gtrsim O(10^{-13}) \). Therefore, if the inflaton field \( S \) weakly couples the Higgs field, the destabilization of the Higgs potential can not happen during inflation. However, after inflation, the parametric resonance or the tachyonic resonance via the coherent oscillation of \( S \) can generate the enormous Higgs vacuum fluctuations with the relatively large \( \lambda_{\phi S} \).

On the other hand, the vacuum field fluctuations of the Higgs field expressed as \( \langle \delta \phi^2 \rangle_{\text{ren}} \) can cause directly the vacuum transition to the Planck-scale true vacuum \([13–25]\). This situation is essentially different from the phenomenon discussed previously. If the inhomogeneous and local Higgs fields overcome the hill of the effective potential, the localized Higgs fields classically go out to the negative Planck-scale vacuum and catastrophic Anti-de Sitter (AdS) domains are formed. Although not all AdS domains threaten the existence of the Universe \([22, 24]\), which highly depends on the evolution of the AdS domains (for the details see Ref.[24, 36]), some AdS domains expand eating other regions of the electroweak vacuum, and consume the entire Universe. Therefore, the existence of AdS domains in our Universe is catastrophic, and so let us consider the conditions not to generate the AdS domains or bubbles in the curved background.

The probability of the vacuum field fluctuations can

\[^2\text{In the situation } (R + \langle \delta \phi^2 \rangle_{\text{ren}})^{1/2} < \Lambda_I, \text{ the Higgs self-coupling term } \lambda (\mu) \phi^4/4 \text{ becomes positive unless } \phi > \Lambda_I. \text{ Therefore, the homogeneous Higgs field } \phi (t) \text{ can not classically roll down into the Planck-scale true vacuum. However, the large vacuum fluctuations of the Higgs field can generate AdS domains or bubbles as shown in Eq. (135).}\]

\[^3\text{During inflation, the curvature mass-term } \xi (\mu) R \text{ can stabilize the effective Higgs potential and suppress AdS domains or bubbles. Therefore, the catastrophic phenomena can be avoided if the relatively large non-minimal Higgs-gravity coupling } \xi (\mu) \text{ is introduced. However, after inflation, } \xi (\mu) R \text{ via the non-minimal coupling drops rapidly, sometimes become negative and lead to the exponential growth of the coherent Higgs field } \phi (t) \text{ at the end of the inflation, or the large Higgs vacuum fluctuations via the tachyonic resonance (see Ref.[26] for the detailed discussion). Therefore, the non-minimal coupling } \xi (\mu) \text{ can not prevent the catastrophic phenomena in the inflationary Universe.}\]

\[^4\text{In the } m_S^2 S^2 \text{ chaotic inflation where } m_S \approx H \approx 10^{13} \text{ GeV, the vacuum fluctuations of the Higgs field come from de-Sitter background and slow varying inflaton background as follows:}\]

\[
\langle \delta \phi^2 \rangle_{\text{ren}} \approx \frac{3 H^4}{8 \pi^2 \lambda_{\phi S} S^2} \approx \frac{m_S^2}{48 \pi^2}
\]

(129)

where we ignore the curvature mass term \( \xi (\mu) 12 H^2 \).
be expressed as the Gaussian distribution function

\[ P(\phi, \langle \delta \phi^2 \rangle_{\text{ren}}) = \frac{1}{\sqrt{2\pi \langle \delta \phi^2 \rangle_{\text{ren}}}} \exp \left( -\frac{\phi^2}{2 \langle \delta \phi^2 \rangle_{\text{ren}}} \right). \]  

\begin{equation} \tag{130} \end{equation}

By using Eq. (130), the probability not to produce AdS domains or bubbles can be given by

\[ P(\phi < \phi_{\text{max}}) = 1 - \text{erf} \left( \frac{\phi_{\text{max}}}{\sqrt{2 \langle \delta \phi^2 \rangle_{\text{ren}}}} \right), \]  

\begin{equation} \tag{131} \end{equation}

where we define \( \phi_{\text{max}} \) as the effective Higgs potential of Eq. (124) takes its maximal value \(^5\). Therefore, the probability that the localized Higgs fields roll down into the true vacuum can be given by

\[ P(\phi > \phi_{\text{max}}) = \frac{\sqrt{2 \langle \delta \phi^2 \rangle_{\text{ren}}}}{\pi \phi_{\text{max}}} \exp \left( -\frac{\phi^2}{2 \langle \delta \phi^2 \rangle_{\text{ren}}} \right). \]  

\begin{equation} \tag{132} \end{equation}

The vacuum decay probability of the Universe can be expressed as

\[ e^{3N_{\text{hor}}} P(\phi > \phi_{\text{max}}) < 1, \]  

\begin{equation} \tag{134} \end{equation}

where \( e^{3N_{\text{hor}}} \) corresponds to the physical volume of our Universe at the end of the inflation and we can take the e-folding number \( N_{\text{hor}} \approx N_{\text{CMB}} \approx 60 \). By substituting Eq. (134) into Eq. (133), we can obtain the following relation of the electroweak vacuum stability

\[ \frac{\langle \delta \phi^2 \rangle_{\text{ren}}}{\phi_{\text{max}}^2} < \frac{1}{6N_{\text{hor}}}. \]  

\begin{equation} \tag{135} \end{equation}

The above condition can be determined by the effective Higgs potential of Eq. (124) and the Higgs vacuum fluctuations of Eq. (126), Eq. (127) and Eq. (128). In the inflationary Universe, we can obtain the restriction of the non-minimal coupling \( \xi(\mu) \gtrsim O(10^{-2}) \) and the Higgs-inflaton coupling \( \lambda_{\phi S} \gtrsim O(10^{-12}) \) not to generate the unwanted AdS domains or bubbles. Therefore, if the relatively large non-minimal Higgs-gravity coupling or the Higgs-inflaton coupling are introduced, the Higgs metastability vacuum can be safe during the inflation. Therefore, in other cosmological background, these situations remain unchanged, namely the large coupling of \( \xi \) or \( \lambda_{\phi S} \) can stabilize the effective Higgs potential and suppress the catastrophic AdS domains or bubbles.

Generally speaking, the adiabatic vacuum fluctuations of the Higgs field satisfying the condition of Eq. (65) have little effect on the dynamics of the Higgs field since they are accompanied by large effective masses. However, if there are large coherent scalar fields \( S \) which satisfy the following relations \( \lambda m_S^2/4\pi^2 \gtrsim \Lambda_I^2 \) and \( \lambda n_S^2/4\pi^2 \gtrsim \lambda_{\phi S} S^2 \), the adiabatic Higgs vacuum fluctuations as \( \langle \delta \phi^2 \rangle_{\text{ren}} \gtrsim m_S^2/4\pi^2 \) can destabilize the effective Higgs potential and generate the AdS domains or bubbles. This situation generally occurs at the end of the inflation. If we assume the instability scale as \( \Lambda_I \approx 10^{11} \text{ GeV} \) and the Higgs self-coupling \( \lambda \approx 0.01 \), we can obtain the constraint of the mass of such scalar fields as \( m_S \lesssim 10^{13} \text{ GeV} \). Moreover, the non-adiabatic vacuum fluctuations of the Higgs field lead to the unwanted phenomena and cause the catastrophic collapse of the Universe. In conclusion, the metastable electroweak vacuum is highly unstable on the cosmological background even if the Higgs field has the non-minimal Higgs-gravity coupling or the Higgs-scalar coupling, and suggest that the new physics should stabilize the Higgs potential.

**VII. CONCLUSION AND DISCUSSION**

In this paper, we have investigated the electroweak vacuum instability in the adiabatic or non-adiabatic cosmological background. In general cosmological background, the vacuum fluctuations \( \langle \delta \phi^2 \rangle \) of the Higgs field can glow to the cosmological scale and destabilize the effective Higgs potential, or generate the catastrophic AdS domains or bubbles. These undesirable phenomena induce serious problems for the Higgs metastability Universe. We have obtained the exact renormalized vacuum fluctuations \( \langle \delta \phi^2 \rangle_{\text{ren}} \) in the adiabatic, the non-adiabatic background, and the slow varying scalar background by using the adiabatic expansion or the adiabatic regularization methods. The non-adiabatic vacuum fluctuations of the Higgs field generally destabilize the effective Higgs potential, or generate the AdS domains or bubbles. On the other hand, the adiabatic Higgs vacuum fluctuations does not cause the catastrophic phenomena of the Higgs vacuum. However, if there are large background scalar fields, the adiabatic vacuum fluctuations of the Higgs field can destabilize the effective Higgs potential and give the upper bound of the scalar mass as \( m_S \lesssim 10^{13} \text{ GeV} \) where we set the instability scale \( \Lambda_I \approx 10^{11} \text{ GeV} \).
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Appendix: Adiabatic (WKB) expansion method

In this appendix we provide a detailed description of the adiabatic (WKB) expansion method following the literature [38, 58, 76, 77]. In order to give the renormalized vacuum field fluctuations we must solve Eq. (68) with the suitable in-vacuum as follows

\[ n'_k = \frac{\Omega'_k}{\Omega_k} \Re z_k, \quad z'_k = \frac{\Omega'_k}{\Omega_k} \left( n_k + \frac{1}{2} \right) - 2 i \Omega_k z_k. \]  \hspace{1cm} (A.1)

For simplicity we assume \( z_k = u_k + i v_k \), i.e \( u_k = \Re z_k \) and \( v_k = \Im z_k \). By using these relations we can rewrite Eq. (A.1) as follows

\[ n'_k = \frac{\Omega'_k}{\Omega_k} u_k, \]  \hspace{1cm} (A.2)
\[ u'_k = \frac{\Omega'_k}{\Omega_k} \left( n_k + \frac{1}{2} \right) + 2 \Omega_k v_k, \]  \hspace{1cm} (A.3)
\[ v'_k = -2 \Omega_k u_k. \]  \hspace{1cm} (A.4)

Here, we introduce a single formal adiabatic parameter \( T \) and a rescaling time variable \( \tau \equiv \eta/T \). The adiabatic (WKB) condition of Eq. (30) can be restated by

\[ \frac{d}{d \eta} \left( \eta/T \right) = \frac{1}{T} \frac{d}{d \tau} \Omega \left( \tau \right), \]  \hspace{1cm} (A.5)

where \( T \to \infty \). By using this procedure we can rewrite Eq. (A.2), Eq. (A.3) and Eq. (A.4) as follows

\[ \frac{1}{T} n'_k = \frac{\Omega'_k}{\Omega_k} u_k, \]  \hspace{1cm} (A.6)
\[ \frac{1}{T} u'_k = \frac{\Omega'_k}{\Omega_k} \left( n_k + \frac{1}{2} \right) + 2 \Omega_k v_k, \]  \hspace{1cm} (A.7)
\[ \frac{1}{T} v'_k = -2 \Omega_k u_k. \]  \hspace{1cm} (A.8)

Next we expand \( n_k, u_k \) and \( v_k \) in inverse powers of \( T \) as

\[ n_k = n_k^{(0)} + \frac{1}{T} n_k^{(1)} + \frac{1}{T^2} n_k^{(2)} + \cdots, \]  \hspace{1cm} (A.9)
\[ u_k = u_k^{(0)} + \frac{1}{T} u_k^{(1)} + \frac{1}{T^2} u_k^{(2)} + \cdots, \]  \hspace{1cm} (A.10)
\[ v_k = v_k^{(0)} + \frac{1}{T} v_k^{(1)} + \frac{1}{T^2} v_k^{(2)} + \cdots, \]  \hspace{1cm} (A.11)

where superscripts \((i)\) express the adiabatic order and the zeroth order expressions are given by

\[ n_k^{(0)} = \text{const}, \quad u_k^{(0)} = 0, \quad v_k^{(0)} = 0, \]  \hspace{1cm} (A.12)

where we solve Eq. (A.6), Eq. (A.7) and Eq. (A.8) with an iterative procedure. The above integration constant can be determined by the initial conditions for \( n_k (\eta_0) \), and \( z_k (\eta_0) \) which corresponds to the choice of the in-vacuum. For the conformal vacuum \( n_k (\eta_0) = z_k (\eta_0) = 0 \), the zeroth-order adiabatic number density \( n_k^{(0)} \) is zero. For the first adiabatic order, we can obtain the following expression

\[ n_k^{(1)} = 0, \quad u_k^{(1)} = 0, \quad v_k^{(1)} = -\frac{\Omega'_k}{2 \Omega_k^{(1)}} \left( n_k^{(0)} + \frac{1}{2} \right), \]  \hspace{1cm} (A.13)

where the odd-order adiabatic number density is zero. Next we can obtain the second order adiabatic expressions as follows

\[ n_k^{(2)} = \frac{1}{16} \frac{\Omega_k'}{\Omega_k^{(2)}}, \quad u_k^{(2)} = \frac{1}{8} \frac{\Omega_k''}{\Omega_k^{(3)}} - \frac{1}{4} \frac{\Omega_k'}{\Omega_k^{(2)}}, \quad v_k^{(2)} = 0, \]  \hspace{1cm} (A.14)

In the same way, the third order adiabatic expressions can be given by

\[ n_k^{(3)} = 0, \quad u_k^{(3)} = 0, \]  \hspace{1cm} (A.15)
\[ v_k^{(3)} = \frac{1}{16 \Omega_k^{(3)}} \left( \Omega_k'' - \frac{7}{2} \Omega_k' \Omega_k' + 15 \Omega_k^{(3)} \right), \]  \hspace{1cm} (A.16)

Finally, the forth order adiabatic expressions are given by

\[ n_k^{(4)} = -\frac{\Omega_k'}{32 \Omega_k^{(4)}} + \frac{\Omega_k''}{64 \Omega_k^{(5)}} + \frac{5 \Omega_k'}{32 \Omega_k^{(4)}} - \frac{45 \Omega_k^{(4)}}{256 \Omega_k^{(5)}}, \]  \hspace{1cm} (A.17)
\[ u_k^{(4)} = -\frac{\Omega_k'''}{32 \Omega_k^{(4)}} + \frac{11 \Omega_k' \Omega_k''}{64 \Omega_k^{(5)}} - \frac{115 \Omega_k'}{32 \Omega_k^{(4)}} + \frac{7 \Omega_k'}{32 \Omega_k^{(4)}} + \frac{45 \Omega_k^4}{32 \Omega_k^{(5)}}, \]  \hspace{1cm} (A.18)
\[ v_k^{(4)} = 0. \]  \hspace{1cm} (A.19)
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