Supersymmetry and the Rotation Group

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Abstract

A model invariant under a supersymmetric extension of the rotation group \(O(3)\) is mapped, using a stereographic projection, from the spherical surface \(S_2\) to two dimensional Euclidean space. The resulting model does not have a manifest local structure.

1 Introduction

The relationship between the conformal group [1,2] in Minkowski space and the anti-de Sitter group is well established [3,4]. An analogue exists between the conformal group in Euclidean space and the rotation group. This has led to a mapping of models in an \(n + 1\) dimensional spherical space to an \(n\) dimensional Euclidean space [5,6].

It has been possible to construct models that are invariant under supersymmetric extensions of the invariance present in spaces of constant curvature (such as spherical or anti-de Sitter spaces) [7-11]. Since the generators of the supersymmetry transformation are no longer the “square root” of the generator of translations, there is no degeneracy between the Boson and Fermion masses as occurs in the supersymmetric extension of the Poincaré group.

In this paper we examine a model on the sphere \(S_2\) that is invariant under a supersymmetric extension of the group \(O(3)\) and show how a stereographic projection can be used to project it onto two dimensional Euclidean space. The resulting model is explicitly dependent on \(x^2\).
2 The Model

In three Euclidean dimensions, irreducible spinors are two dimensional Dirac spinors. If the generator of rotations is $J^a$ and of supersymmetry transformations if $Q_i$, then a suitable supersymmetry algebra is [10,11]

\[
\begin{align*}
[J^a, J^b] &= i\epsilon^{abc} J^c \quad (1a) \\
[J^a, Q_i] &= -\frac{1}{2} \tau^a_{ij} Q_j \quad (1b) \\
[Z, Q_i] &= \mp Q_i \quad (1c) \\
\{Q_i, Q_j^\dagger\} &= Z\delta_{ij} \mp 2\tau^a_{ij} J^a \quad (1d)
\end{align*}
\]

where $\tau^a$ is a Pauli spin matrix satisfying

\[
\tau^a \tau^b = \delta^{ab} + i\epsilon^{abc} \tau^c. \quad (2)
\]

Representations of this algebra are discussed in ref. [10]; it is shown there that the eigenvalues of $Z$ form an upper bound to the eigenvalues of $J^2$.

The action

\[
S = \int \frac{dA}{R^2} \left[ \left( \frac{1}{2} \Psi^\dagger (\tau \cdot L + \rho) \Psi - \Phi^* (L^2 + \rho(1 - \rho)) \Phi \\
- \frac{1}{4} F^* F \right) + \lambda_N \left( 2(1 - 2\rho) \Phi^* \Phi - (F^* \Phi + F \Phi^*) \\
- \Psi^\dagger \Psi \right)^N \right] \quad (3)
\]

is invariant under the supersymmetry transformation

\[
\begin{align*}
\delta \Phi &= \xi^\dagger \Psi \quad (4a) \\
\delta \Psi &= 2(\tau \cdot L + 1 - \rho) \Phi \xi - F \xi \quad (4b) \\
\delta F &= -2\xi^\dagger (\tau \cdot L + \rho) \Psi \quad (4c)
\end{align*}
\]

as well as

\[
\begin{align*}
\delta \Phi &= \lambda i [2(1 - \rho) \Phi - F] \quad (5a) \\
\delta \Psi &= \lambda i (1 + 2\tau \cdot L) \Psi \quad (5b) \\
\delta F &= \lambda i [-4 \left( L^2 + \rho(1 - \rho) \right) \Phi + 2\rho F] \quad (5c)
\end{align*}
\]

where $\xi$ is Grassmann spinor and $\lambda$ is a scalar (both are constants). A surface element on the sphere $\eta^2 = R^2$ is $dA$. The transformations of eqs. (4,5) are consistent with the algebra of eq. (1).

If now

\[
\Lambda_\pm = \frac{1 \pm i\tau \cdot \eta/R}{\sqrt{2}} = (\Lambda_\mp)^\dagger = (\Lambda_\pm)^{-1} \quad (6)
\]
then as
\[(\tau \cdot L + 1)\tau \cdot \eta = -\tau \cdot \eta (\tau \cdot L + 1)\]  \hspace{1cm} (7)

we have
\[\Lambda_{\pm}(\tau \cdot L + 1)\Lambda_{\mp} = \pm i \frac{\tau \cdot \eta}{R} (\tau \cdot L + 1).\]  \hspace{1cm} (8)

By eq. (8), if \(\Psi = \Lambda_{-}\Psi', \xi = \Lambda_{-}\xi'\), then the Fermionic terms in the model of eq. (3) become
\[S_{\Psi} = \int \frac{dA}{R^2} \left[ \frac{1}{2} \Psi'^{\dagger} \left( \frac{i\tau \cdot \eta}{R} (\tau \cdot L + 1) - (1 - \rho) \right) \Psi' + \lambda N \left[ -\Psi'^{\dagger} \Psi \right]^N \right].\]  \hspace{1cm} (9)

The invariances of eqs. (4,5) are similarly transformed.

In refs. [5,6] it is shown that under the change of variable from \(x^a\) to \(\eta^a\) in \((n + 1)\) dimensions
\[\eta^a = R h^a + 2 \frac{x^a - h^a x^2 / R}{1 - 2 x \cdot h / R + x^2 / R^2}\]  \hspace{1cm} (10)

the plane \(x \cdot h = 0\) is mapped onto the sphere \(\eta^2 = R^2\). (The vector \(h^a\) is a unit vector in the direction of \(\eta^{n+1}\).) Since on this plane \(\eta^{n+1} = R \left( \frac{1 - x^2 / R^2}{1 + x^2 / R^2} \right)\) and \(\eta' = \frac{2 x^i}{1 + x^2 / R^2} (i = 1 \ldots n)\), it follows that if
\[\kappa = 1 + \eta^{n+1} / R = \frac{2}{1 + x^2 / R^2}\]  \hspace{1cm} (11)

then
\[d^{a}x = \kappa^{-n} dA.\]  \hspace{1cm} (12)

The fields \(\Phi\) and \(\Psi'\) map onto fields \(\phi\) and \(\psi\) so that [12,13]
\[
\Phi = \kappa^{-n/2} \phi \hspace{1cm} (13a) \\
\Psi' = \kappa^{(1-n)/2} (\kappa/2)^{1/2} (1 - h \cdot \alpha \cdot x / R) \psi \hspace{1cm} (13b) \\
\equiv \kappa^{(1-n)/2} U \psi
\]

where \(\alpha^a\) is a matrix in \(n + 1\) dimensional space satisfying the Clifford algebra
\[
\{ \alpha^a, \alpha^b \} = 2 \delta^{ab}. \]  \hspace{1cm} (14)

If now
\[\gamma^{ab} = \frac{1}{4} [\alpha^a, \alpha^b] \]  \hspace{1cm} (15)

and
\[L^{ab} = \eta^a \partial^b - \eta^b \partial^a\]  \hspace{1cm} (16)
then in three dimensions where $\alpha^a = \tau^a$ (by eq. (2)), $\gamma^{ab} = \frac{1}{2} \epsilon^{abc} \tau^c$ and $L^{ab} = i \epsilon^{abc} L^c$ so that

$$\tau \cdot L = -\gamma^{ab} L_{ab}. \quad (17)$$

It can then be shown that [12]

$$\int dA \Phi^* \left( \frac{L^{ab} L_{ab} - \frac{1}{2} n (n - 2)}{2 R^2} \right) \Phi = \int d^n x \phi^* \partial^2 \phi \quad (18)$$

and [13]

$$\int dA \Psi^\dagger \left( \frac{\alpha \cdot \eta (\gamma^{ab} L_{ab} - n/2)}{R^2} \right) \Psi = \int d^n x \psi^\dagger \alpha \cdot \partial \psi. \quad (19)$$

When in $n = 2$ dimensions, eqs. (12,13,17,18,19) show that the model of eqs. (3,9) becomes in two dimensional Euclidean space

$$S = \int d^2 x \left[ \left( -\frac{1}{2} \psi^\dagger \left( \frac{\partial}{R} \tau \cdot \partial + \frac{\kappa (1 - \rho)}{R^2} \right) \right) - 2 \phi^* \left( \partial^2 + \frac{\kappa^2 \rho (1 - \rho)}{2 R^2} \right) \phi \right. \right.$$

$$\left. \left. - \frac{\kappa^2}{4 R^2} f^* f \right) + \frac{\lambda N \kappa^2}{R^2} \left( 2 (1 - 2 \rho) \phi^* \phi - (f^* \phi + f \phi^*) - \kappa^{-1} \psi^\dagger \psi \right) \right]^N \quad (20)$$

where $F = \kappa^{1-n/2} f$ (as in eq. (13a)). Since $RU \tau \cdot hU^{-1} = \tau \cdot \eta$, it follows from eq. (13) that eq. (4) becomes

$$\delta \phi = \kappa^{-1} \xi^i \psi \quad (21a)$$

$$\delta f = 2 i \kappa^{-2} R \xi^i \frac{\partial}{\partial x^i} \psi + 2 \kappa^{-1} (1 - \rho) \xi^i \psi \quad (21b)$$

$$\delta \psi = -2 \left( i \kappa^{-1} R \xi^i \frac{\partial}{\partial x^i} + \rho \right) \phi \xi - f \xi. \quad (21c)$$

Here $\xi = \kappa^{(1-n)/2} U \zeta$ as in eq. (13b). Both the model of eq. (20) and the symmetry of eq. (21) are non local because of the contribution of $\kappa$ defined in eq. (11).

### 3 Discussion

The model of eq. (20) is unusual, having explicit dependence on $x^2$. However, it does possess the supersymmetry of eq. (21) as a consequence of the original model of eq. (1) being invariant under the transformations of eqs. (4,5). Since this supersymmetry is not the “square root” of a generator of translations, there is no degeneracy between the Boson and Fermion masses.

We are examining if a more realistic model which incorporates supersymmetry in this fashion can be devised. Radiative corrections to the model of eq. (1) are also being considered.
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