GEOMETRY OF BOUNDED FRÉCHET MANIFOLDS

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Abstract. In this paper we develop the geometry of bounded Fréchet manifolds. We prove that a bounded Fréchet tangent bundle admits a vector bundle structure. But the second order tangent bundle $T^2M$ of a bounded Fréchet manifold $M$, becomes a vector bundle over $M$ if and only if $M$ is endowed with a linear connection. As an application, we prove the existence and uniqueness of the integral curve of a vector field on $M$.

1. Introduction

The geometry of Fréchet manifolds has received serious attention in recent years, cf. [3] for a survey. In particular, second order tangent bundles have been studied due to their applications in the study of second order ordinary differential equations that arise via geometric objects (such as autoparallel curves and parallel translation) on manifolds (see [1], [2]). However, due to intrinsic difficulties with Fréchet spaces only a certain type of manifolds was considered, namely those Fréchet manifolds which can be obtained as projective limit of Banach manifolds (PLB-manifolds). It was proved that the second order tangent bundle $T^2M$ of a PLB-manifold $M$ admits a vector bundle structure if and only if $M$ is endowed with a linear connection (see [4]).

Some of the basic issues in the theory of Fréchet spaces are mainly related with the space of continuous linear mappings. Indeed, the space of continuous linear mappings of one Fréchet space to another is not a Fréchet space in general. On the other hand, the general linear group of a Fréchet space does not admit any non-trivial topological group structure. This defect puts in question the way of defining vector bundle. Another drawback is the lack of a general solvability theory for ordinary differential equations. Because of these reasons, in the framework of Fréchet bundles an arbitrary connection is hard to handle.

As remarked, there is a way out of these difficulties for Fréchet manifolds which can be obtained as projective limit of Banach manifolds. However, there is another way to overcome aforementioned problems. Recently, in the suggestive paper [17], Müller introduced the concept of bounded Fréchet manifolds and provided an inverse function theorem in the sense of Nash and Moser in this category. Such spaces arise in geometry and physical field theory.

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and have many desirable properties. For instance, the space of all smooth sections of a fiber bundle (over closed or non-compact manifolds), which is the foremost example of infinite dimensional manifolds, has the structure of a bounded Fréchet manifold, see [17, Theorem 3.34]. As for the importance of bounded Fréchet manifolds, we refer to the paper [6], where Sard’s theorem was obtained in this category. The statement of the theorem is as follows: Let $M$ resp. $N$ be bounded Fréchet manifolds with compatible metrics $d_M$ resp. $d_N$ modelled on Fréchet spaces $E$ resp. $F$ with standard metrics. Let $f : M \to N$ be an $MC^k$- Lipschitz Fredholm map with $k > \max\{\text{Ind } f, 0\}$. Then the set of regular values of $f$ is residual in $N$.

One of the essential ideas of this setting is to replace the space of all continuous linear maps by the space $L_{d^*,d}(E,F)$ of all linear Lipschitz continuous maps. Then $L_{d^*,d}(E,F)$ is a topological group that has satisfactory properties. For example, the composition map $L_{d,g}(F,G) \times L_{d^*,d}(E,F) \to L_{d^*,g}(E,G)$ is bilinear continuous. In particular, the evaluation map $L_{g,d}(E,F) \times E \to F$ is continuous.

Our goal in this paper is to extend to bounded Fréchet manifolds the known results of Fréchet geometry. We define the tangent bundles $TM$ and $T^2M$ of a bounded Fréchet manifold $M$ modelled on a Fréchet space $F$ and prove that they too are endowed with bounded Fréchet manifold structures of the same type modelled on $F^2$ and $F^4$, respectively. In addition, we show that $TM$ admits a vector bundle structure, which allows us to define a connection on $TM$ via a connection map (cf. [19], [20]). We shall interpret linear connections as linear systems of ordinary differential equations on trivial bundles. Our main result is that $T^2M$ admits a vector bundle structure if and only if $M$ is endowed with a linear connection. Moreover, a linear connection on $M$ determines a vector bundle structure on $T^2M$ and a vector bundle isomorphism $T^2M \to TM \oplus TM$. We conclude by proving the existence and uniqueness of the integral curve of a vector field on $M$.

It turns out that bounded Fréchet manifolds have some advantages over both PLB-manifolds and infinite dimensional convenient manifolds. In the case of PLB-manifolds, the difficulty is that to construct a geometric object on manifolds we need to establish the existence of the projective limit of its Banach corresponding factors. While in the case of convenient manifolds, to construct geometrical structures on manifolds we need to define the notion of manifolds by charts but this restricts the consequences of Cartesian closedness drastically (see [13], [16]). In addition, for convenient manifolds we have two different kinds of tangent bundles (kinematic and operational) and hence we have two different types of vector fields. Another drawback is that operational vector fields do not necessarily have integral curves. On the other hand, for a given kinematic vector field integral curves may not exist locally, and if they exist they may not be unique for the same initial condition (see [13]).
2. Prerequisites

In this section we summarize all the necessary preliminary material that we need for a self contained presentation of the paper. For detailed studies on bounded Fréchet manifolds we refer to [6], [10] and [17].

We denote by \( F, d \) a Fréchet space whose topology is defined by a complete translation-invariant metric \( d \). We define \( \| f \|_d = d(f, 0) \) for \( f \in F \) and write \( L.f \) instead of \( L(f) \) when \( L \) is a linear map between Fréchet spaces. A metric with absolutely convex balls will be called a standard metric. Note that every Fréchet space admits a standard metric which defines its topology: If \( \alpha_n \) is an arbitrary sequence of positive real numbers converging to zero and if \( \rho_n \) is any sequence of continuous semi-norms defining the topology of \( F \). Then

\[
d_{\alpha, \rho}(e, f) := \sup_{n \in \mathbb{N}} \alpha_n \frac{\rho_n(e - f)}{1 + \rho_n(e - f)}
\]

is a metric on \( F \) with the desired properties.

As mentioned in the Introduction, we replace the space of all linear continuous maps between Fréchet spaces by the space of all linear Lipschitz continuous maps. Let \( (E, g) \) be another Fréchet space and let \( \mathcal{L}_{g,d}(E, F) \) be the set of all globally linear Lipschitz continuous maps, i.e. linear maps \( L : E \to F \) such that

\[
\| L \|_{g,d} := \sup_{x \in E \setminus \{0\}} \frac{\| Lx \|_d}{\| x \|_g} < \infty.
\]

We abbreviate \( \mathcal{L}_g(E) := \mathcal{L}_{g,g}(E, E) \) and write \( \| L \|_g = \| L \|_{g,g} \) for \( L \in \mathcal{L}_g(E) \). If \( d \) is a standard metric, then

\[
D_{g,d} : \mathcal{L}_{g,d}(E, F) \times \mathcal{L}_{g,d}(E, F) \longrightarrow [0, \infty), \quad (L, H) \mapsto \| L - H \|_{g,d}
\]

is a translational-invariant metric on \( \mathcal{L}_{d,g}(E, F) \) turning it into an Abelian topological group (see [10] Remark 1.9). The latter is not a topological vector space in general, but a locally convex vector group with absolutely convex balls. We shall always equip Fréchet spaces with standard metrics and define the topology on \( \mathcal{L}_{d,g}(E, F) \) by the metric \( D_{g,d} \). The vector groups \( \mathcal{L}_{g,d}^{(i+1)}(F, E) := (F, \mathcal{L}_{g,d}^i(F, E)) \) are defined by induction.

Let \( E, F \) be Fréchet spaces, \( U \) an open subset of \( E \), and \( P : U \to F \) a continuous map. Let \( CL(E, F) \) be the space of all continuous linear maps from \( E \) to \( F \) topologized by the compact-open topology. We say \( P \) is differentiable at the point \( p \in U \) if there exists a linear map \( dP(p) : E \to F \) with \( dP(p)h = \lim_{t \to 0} \frac{P(p+th) - P(p)}{t} \), for all \( h \in E \). If \( P \) is differentiable at all points \( p \in U \), if \( dP(p) : U \to CL(E, F) \) is continuous for all \( p \in U \) and if the induced map \( P' : U \times E \to F, (u, h) \mapsto dP(u)h \) is continuous in the product topology, then we say
that $P$ is Keller-differentiable. We define $P^{(k+1)}: U \times E^{k+1} \to F$ inductively by

$$P^{(k+1)}(u, f_1, \ldots, f_{k+1}) = \lim_{t \to 0} \frac{P^{(k)}(u + tf_{k+1})(f_1, \ldots, f_{k}) - P^{(k)}(u)(f_1, \ldots, f_{k})}{t}.$$  

If $P$ is Keller-differentiable, $dP(p) \in \mathcal{L}_{d,g}(E,F)$ for all $p \in U$, and the induced map $dP(p) : U \to \mathcal{L}_{d,g}(E,F)$ is continuous, then $P$ is called b-differentiable. We say $P$ is $MC^0$ and write $P^0 = P$ if it is continuous. We say $P$ is an $MC^1$ and write $P^{(1)} = P'$ if it is b-differentiable. Let $\mathcal{L}_{d,g}(E,F)_0$ be the connected component of $\mathcal{L}_{d,g}(E,F)$ containing the zero map. If $P$ is b-differentiable and if $V \subseteq U$ is a connected open neighborhood of $x_0 \in U$, then $P'(V)$ is connected and hence contained in the connected component $P'(x_0) + \mathcal{L}_{d,g}(E,F)_0$ of $P'(x_0)$ in $\mathcal{L}_{d,g}(E,F)$. Thus, $P' \mid_V - P'(x_0) : V \to \mathcal{L}_{d,g}(E,F)_0$ is again a map between subsets of Fréchet spaces. This enables a recursive definition: If $P$ is $MC^1$ and $V$ can be chosen for each $x_0 \in U$ such that $P' \mid_V - P'(x_0) : V \to \mathcal{L}_{d,g}(E,F)_0$ is $MC^{k-1}$, then $P$ is called an $MC^k$-map. We make a piecewise definition of $P^{(k)}$ by $P^{(k)} \mid_V := (P' \mid_V - P'(x_0))^{(k-1)}$ for $x_0$ and $V$ as before. The map $P$ is $MC^\infty$ if it is $MC^k$ for all $k \in \mathbb{N}_0$. We shall denote by $D, D^2$ the first and the second differential, respectively.

A bounded Fréchet manifold is a Hausdorff second countable topological space with an atlas of coordinate charts taking their values in Fréchet spaces such that the coordinate transition functions are all $MC^\infty$-maps.

We will need to consider the space of all globally Lipschitz continuous $k$-multilinear maps. Let $B = \prod_{i=1}^{i=k} F_i$ be the topological product of any finite number $k$ of Fréchet spaces $(F_1, d_1), \ldots, (F_k, d_k)$. For $x = (x_1, \ldots, x_k) \in B$ and $y = (y_1, \ldots, y_k) \in B$, we define the maximum metric $d_{\max}$ as follows: $d_{\max}(x,y) = \max_{1 \leq i \leq k} d_i(x_i, y_i)$. We shall always use this metric on $B$. Let $(F_1, d_1), \ldots, (F_k, d_k)$ and $(F, d)$ be Fréchet spaces. The space of all globally Lipschitz continuous $k$-multilinear maps is the space of all $k$-multilinear maps $L : F_1 \times \ldots \times F_k \to F$ such that for all $f_i \in F_i \setminus \{0\}$, $1 \leq i \leq k$,

$$\| L \|_{d_1, \ldots, d_k, d} := \sup_{f_i \in F_i \setminus \{0\}} \frac{\| L(f_1, \ldots, f_k) \|_d}{\| f_1 \|_{d_1} \cdots \| f_k \|_{d_k}} < \infty.$$  

This space is denoted by $\mathcal{L}_{d_1, \ldots, d_k, d}(F_1, \ldots, F_k; F)$. We define on the latter space a metric

$$D_{d_1, \ldots, d_k, d}(L, H) = \| L - H \|_{d_1, \ldots, d_k, d}$$

which makes it into an Abelian topological group.

Throughout the paper, we suppose that $d_1, \ldots, d_k, d$ are fixed metrics and we will not write them when they appear as indices in the notations to make the notations more readable.

Convention. The terms bounded Fréchet tangent bundle and bounded Fréchet second order tangent bundle are too long, so we remove “bounded Fréchet” from the terms.
3. Constructions of $TM$ and $T^2M$

3.1. **Tangent bundle.** Let $M$ be a bounded Fréchet manifold modelled on a Fréchet space $F$. Let $\mathcal{MC}_p(M)$ be the set of all $MC^\infty$- mappings $f : \mathbb{R} \to M$ that send zero to $p \in M$. We define on $\mathcal{MC}_p(M)$ an equivalence relation $\sim$ as follows: Let $\Phi = \{(U_\alpha, \varphi_\alpha)\}_{\alpha \in \mathcal{A}}$ be a compatible atlas for $M$, $(p \in U_\alpha, \varphi_\alpha)$ an admissible chart, and $f, g \in \mathcal{MC}_p(M)$. Let $r$ be a fixed natural number. We say that $f$ and $g$ are equivalent and write $f \sim g$ if they satisfy the following:

$$(\varphi_\alpha \circ f)'(0) = (\varphi_\alpha \circ g)'(0), \ldots, (\varphi_\alpha \circ f)^r(0) = (\varphi_\alpha \circ g)^r(0),$$

where the orders of the derivatives run between 1 and $r$. It follows from the chain rule for $MC^k$-maps (see [10] Lemma B.1 (f)) that the equivalency at a point $p$ is well defined. The equivalence class containing a mapping $f \in \mathcal{MC}_p(M)$ is called the $r$-jet of $f$ at $p$ and is denoted by $j^r_p f$.

Let $TM$ be the set of all 1-jets of $M$ and let $\pi_M : TM \to M$ be a natural projection. The fiber $\pi_M^{-1}(p)$ is the tangent space $T_pM$. The space $T_pM$ has the structure of a Fréchet space which is isomorphic to $F$ by means of the mapping $\varphi_\alpha \circ \pi_M : T_pF \to F$ given by $j^1_p f \mapsto \varphi_\alpha(p)$. It is easily verified that this structure of $T_pM$ is independent of the choice of the chart $(U_\alpha, \varphi_\alpha)$. Then $TM$ is the disjoint union of the tangent spaces $T_pM$ and is called the tangent bundle over $M$. Let $h : M \to N$ be an $MC^k$-map of manifolds. The tangent map $Th : TM \to TN$ is defined by $Th(j^1_p(f)) = j^1_{h\circ p}(h \circ f)$.

The following lemma is fundamental for constructing trivializing atlases and vector bundle structures for $TM$ and $T^2M$.

**Lemma 3.1.**

(i): Let $h : M \to N$ and $g : N \to K$ be $MC^k$-maps of manifolds. Then $T(h \circ g) = Tg \circ Th$.

(ii): If $h : M \to N$ is an $MC^k$-diffeomorphism, then $Th : TM \to TN$ is a bijection and $(Th)^{-1} = T(h^{-1})$.

(iii): Let $h : U \subset E \to V \subset F$ be a diffeomorphism of open sets of Fréchet spaces. The tangent map $Th : U \times F \to V \times E$ is a local vector bundle isomorphism.

(iv): If $h : U \subset E \to V \subset F$ is an $MC^{k-1}$-diffeomorphism of open sets of Fréchet spaces, then $Th$ is an $MC^{k-1}$-diffeomorphism.

**Proof.**

(i): $g \circ h$ is $MC^k$ ([10] Lemma B.1 (f)). Furthermore,

$$T(g \circ h)(j^1_p f) = j^1_{(g \circ h)(p)}(g \circ h \circ f) = Tg(j^1_{h(p)})(h \circ f) = (Tg \circ Th)(j^1_p f).$$
(ii): By the previous part and the definition of the tangent map, $Th \circ Th^{-1} = Tid_{TN}$ while $Th^{-1} \circ Th = Tid_{TM}$.

(iii): $Th$ is a local vector bundle morphism. Since $h$ is a diffeomorphism it follows that $(Th)^{-1} = T(h^{-1})$ is a local vector bundle morphism, thus $Th$ is a vector bundle isomorphism.

(iv): Let $C$ be a curve passing through $u \in U$ such that $DC(0) \cdot 1 = e$ for a given $e \in F$. Define the map $\lambda(t) : \mathbb{R} \to E$ by $\lambda(t) = u + et$ which is tangent to $C$ at $t = 0$. Define $\lambda : U \times F \to TU$ by $\lambda(u, e) = j^1_u(\eta(t))$. We have $(Th \circ \lambda)(u, e) = Th \cdot j^1_u(\eta(t)) = j^1_{h(u)}(h \circ \eta(t))$. Also we have $(\lambda \circ h')(u, e) = \lambda(h(u), D h(u) \cdot e) = j^1_{h(u)}(h(u) + (D h(u), e)t)$. These are equal because the curves $t \mapsto h(u + et)$ and $t \mapsto h(u) + (D h(u), e)t$ are tangent at 0 by the definition of the derivative and the previous parts. Therefore, $Th \circ \lambda = \lambda \circ h'$, which means $\lambda$ identifies $U \times E$ with $TU$. Correspondingly, we can identify $h'$ with $Th$, so the results of earlier parts imply statement (iv).

\[ \Box \]

**Proposition 3.1.** Let $\pi_M : TM \to M$ be a tangent bundle. Then the atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in \mathcal{A}}$ gives rise to a trivializing atlas $\{(\pi^{-1}_M(U_\alpha), T\varphi_\alpha)\}_{\alpha \in \mathcal{A}}$ on $TM$, with

$$ T\varphi_\alpha : \pi^{-1}_M(U_\alpha) \to \varphi_\alpha(U_\alpha) \times F, \quad j^1_p(f) \mapsto (\varphi_\alpha(p), (\varphi_\alpha \circ f)'(0)); \ f \in \mathcal{MC}_p(M). $$

This makes $TM$ into a bounded Fréchet manifold modelled on $F \times F$.

**Proof.** It follows from Lemma 3.1 \[ \Box \]

We will apply the definition of differentiable vector bundles due to Neeb [18]. We will need the following notion of differentiability which is the adaption of the differentiability given for Keller $C^k$-maps in [18, Definition II.3.1].

**Definition 3.1.** Let $M$ be an $MC^k$, $(k \geq 1)$ Fréchet manifold, and Diff$(M)$ the group of diffeomorphisms of $M$. Further, let $N$ be an $MC^k$ Fréchet manifold. Although, in general, Diff$(M)$ has no natural Lie group structure, a map $\varphi : N \to Diff(M)$ is said to be $MC^k$, if the following map is of class $MC^k$:

$$ \hat{\varphi} : N \times M \to M \times M, \quad (n, x) \mapsto (\varphi(n)(x), \varphi^{-1}(n)(x)) $$

**Definition 3.2.** Let $M$ be an $MC^k$-Fréchet manifold modeled on a Fréchet space $F$, $k \geq 1$, and $E$ another Fréchet space. A $MC^r$-vector bundle of type $E$ over $M$ is a triple $(\Pi, V, E)$, consisting of a $MC^r$-manifold $V$, a $C^r$-map $\Pi : V \to M$ and a Fréchet space $E$, with the following properties:
(VB.1): \( \forall m \in M, \) the fiber \( V_m := \Pi^{-1}(m) \) is a Fréchet space isomorphic to \( E \).

( VB.2): Each \( m \in M \) has an open neighborhood \( U \) for which there exists a diffeomorphism

\[
\phi_U : \Pi^{-1}(U) \to U \times E
\]

with \( \phi_U = (\Pi|_U, \psi_U) \), where \( \psi_U : \Pi^{-1}(U) \to E \) is linear on each \( V_m, m \in U \).

We then call \( U \) a trivializing subset of \( M \) and \( \phi_U \) a bundle chart. If \( \phi_U \) and \( \phi_V \) are two bundle charts and \( U \cap V \neq \emptyset \), then we obtain a diffeomorphism

\[
\phi_U \circ \phi_V^{-1} : \phi_V(U \cap V) \times E \to \phi_U(U \cap V) \times E
\]

of the form \( (x, v) \mapsto (x, \psi_{UV}(x)v) \). This leads to a map \( \psi_{UV} : U \cap V \to \text{GL}(E) \) for which it does not make sense to speak about smoothness because \( \text{GL}(E) \) is not a Lie group. Nevertheless, \( \psi_{UV} \) is of class \( C^r \) in the sense (Definition 3.1) that the map

\[
\widehat{\psi}_{UV} : (U \cap V) \times E \to (U \cap V) \times E
\]

\[
(x, v) \mapsto (\psi_{UV}(x)v, \psi_{UV}(x)^{-1}v) = (\psi_{UV}(x)v, \psi_{VU}(x)v)
\]

is of class \( C^r \). Here, \( \text{GL}(E) \) is the general linear group of \( E \).

**Theorem 3.1.** Let \( M \) be an \( MC^k \)-Fréchet manifold modeled on a Fréchet space \( F \), \( k \geq 1 \). \( TM \) admits a vector bundle structure over \( M \) with fiber of type \( F \).

**Proof.** Consider the above atlas of \( M \) and its corresponding trivializing atlas for \( TM \). Let \( \pi_1, \pi_2 \) be the projections to the first and the second factors, respectively. For all \( \alpha \in A \) we have \( \pi_1 \circ T \varphi_\alpha = \pi_M \), therefore \( TM \) is a fiber bundle. Suppose \( U_\alpha \cap U_\beta \neq \emptyset \), then by Lemma 3.1 (iii) the overlap map

\[
T \varphi_\alpha \circ T \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \times F \to \varphi_\alpha(U_\alpha \cap U_\beta) \times F
\]

is a local vector bundle isomorphism. Let \( \Theta_{\alpha \beta} = T \varphi_\alpha \circ T \varphi_\beta^{-1} \) be the transition map. The following map

\[
\widehat{\Theta}_{\alpha \beta} : \varphi_\beta(U \cap V) \times E \to \varphi_\alpha(U \cap V) \times E
\]

\[
(x, v) \mapsto (\Theta_{\alpha \beta}(x)v, \Theta_{\alpha \beta}(x)^{-1}v) = (\Theta_{\alpha \beta}(x)v, \Theta_{\beta \alpha}(x)v)
\]

is an \( MC^{k-1} \) morphism. Thus, \( TM \) is an \( MC^{k-1} \) vector bundle over \( M \). \( \square \)
3.2. Second order tangent bundle. Now that $TM$ is a manifold we can define second order tangents: Assume $r = 2$ in the equivalence relation (1). Let $T^2_pM$ be the set of all 2-jets at $p$ and let $T^2M = \bigcup_{p \in M} T^2_pM$. Let $\Pi_{TM} : T^2M \to M$ be a natural projection defined by $\Pi_{TM}(j^2_p(f)) = p$. If we topologize $T^2M$ in a natural way, then $T^2M$ is called the second order tangent bundle over $M$.

By virtue of Lemma 3.1, we have a trivializing atlas $\{((\Pi^{-1}_{TM}(\pi^{-1}_M(U_\alpha))), \tilde{\phi}_\alpha)\}_{\alpha \in A}$ for $T^2M$ with

$$\tilde{\phi}_\alpha : \Pi^{-1}_{TM}(\pi^{-1}_M(U_\alpha)) \to \varphi_\alpha(U_\alpha) \times F, \quad j^2_p(f) \mapsto (\varphi_\alpha(p), (\varphi_\alpha \circ f)^{\text{th}}(0)); \quad f \in MC_p(M).$$

$T^2_pM$ can be identified with $F \times F$ under the isomorphism:

$$\Psi : T^2_pM \to F \times F, \quad j^2_p(f) \mapsto ((\varphi_\alpha \circ f)'(0), (\varphi_\alpha \circ f)^{\text{th}}(0)),$$

but fails to be a vector bundle over $M$ because the trivializing isomorphism does not respect the linear structure of the fibers. The submersion $\pi_{12} : T^2M \to TM$ defined by $\pi_{12}(j^2_p(f)) = j^1_p(f)$ is a vector bundle. Let $\pi_2 : T(TM) \to TM$ be an ordinary tangent bundle over $TM$. The space $T^2M$ coincides with

$$\{Y \in T(TM) \mid \pi_2(Y) = T\pi_M(Y)\},$$

and can be identified with a submanifold of $T(TM)$, see [15, Proposition 3.2, p. 372 ]. The bundle $T(TM)$ is a fiber bundle over $M$ with the projection $\pi^2 = \pi_M \circ T\pi_M$. The restriction $\pi^2 \mid_{T^2M} : T^2M \to M$ is again a fiber bundle.

4. Connection

In this section we define connections by using Vilms [20] point of view for connections on infinite dimensional vector bundles. Also, we show that each linear connection corresponds in a bijective way to an ordinary differential equation analogous to the case of Banach manifolds (see [19]).

Henceforth, we keep the formalism of Section 3 for tangent bundles and second order tangent bundles.

Definition 4.1. A strengthened connection map for $TM$ is a map $\mathcal{K} : T(TM) \to TM$, which is fully determined by its local form:

$$\mathcal{K}_\alpha := \varphi_\alpha \circ \mathcal{K} \circ (\tilde{\phi}_\alpha)^{-1},$$

$$\varphi_\alpha(U_\alpha) \times F \times F \times F \to \varphi_\alpha(U_\alpha) \times F, \quad \mathcal{K}_\alpha = (f, g, h, k) = (f, k + \tau_\alpha(f, g)h),$$

for a family of mappings

$$\tau_\alpha : \varphi_\alpha(U_\alpha) \times F \to \mathcal{L}_d(F)^\times.$$
Here $\mathcal{L}_d(F)^\times$ is a subset of $\mathcal{L}_d(F)$ consists of invertible mappings. The mapping $\tau_\alpha$ is $MC^{k-1}$ in the sense that the map

$$\hat{\tau}_\alpha : (\varphi_\alpha(U_\alpha) \times F) \times F \to F \times F, \quad (x, y, h) \mapsto (\tau_\alpha(x, y)(h), \tau_\alpha^{-1}(x, y)(h))$$

is $MC^{k-1}$, see Definition 3.1. It follows of course that $\mathbb{K}$ is of class $MC^{k-1}$.

**Remark 4.1.** In the case of Banach manifolds, it is not required that the maps $\tau_\alpha$ be invertible. However, we require that they be invertible to compensate for the fact that $\mathcal{L}_F(F)$ is not a Fréchet manifold.

A connection on $M$ is a connection map on the tangent bundle $\pi_M : TM \to M$. A connection $\mathcal{K}$ is linear if and only if it is linear on the fibers of the tangent map. Locally $T\pi$ is the map $U_\alpha \times F \times F \times F \to U_\alpha \times F$ defined by $T\pi(f, \xi, h, \gamma) = (f, h)$, hence locally its fibers are the spaces $\{f\} \times F \times \{h\} \times F$. Therefore, $\mathcal{K}$ is linear on these fibers if and only if the maps $(g, k) \mapsto k + \tau_\alpha(f, g)h$ are linear, and this means that the mappings $\tau_\alpha$ need to be linear with respect to the second variable.

Assume that the connection $\mathcal{K}$ is linear and $f \in U_\alpha$. By the canonical isomorphism of Lemma 2.3 to $\tau_\alpha(f, .) \in \mathcal{L}(F, \mathcal{L}(F, F)) \cong \mathcal{L}(F \times F; F)$ is associated the unique local Christoffel symbol $\Gamma_\alpha(p) : \varphi_\alpha(U_\alpha) \to \mathcal{L}(F \times F; F)$ satisfying $\Gamma_\alpha(p)(g, h) = \tau_\alpha(p, g, h)h$. Christoffel symbols satisfy the following compatibility condition (cf. [2]):

$$\tau_\alpha(\nabla \Theta_{\alpha\beta}(f)(g), \nabla \Theta_{\alpha\beta}(f)(h))(\Theta_{\alpha\beta}(f)) + (\nabla^2 \Theta_{\alpha\beta}(f)(h))(g) = \nabla \Theta_{\alpha\beta}(f)(\tau_{\beta}(g, h)(f)) \quad (1)$$

for all $(f, g, h) \in \varphi_\alpha(U_\alpha \cap U_\beta) \times F \times F$. Here, $\Theta_{\alpha\beta} := \varphi_\alpha \circ \varphi_\beta^{-1}$.

**Theorem 4.1.** Every linear connection on $M$ induces a vector bundle structure on $\pi_2 |_{T^2 M} : T^2 M \to M$ and gives rise to an isomorphism of this vector bundle with the vector bundle $TM \oplus TM$.

**Proof.** If we have a connection, then the connection map $\mathcal{K} : T(TM) \to M$ is defined. The following map

$$\pi_2 \oplus \mathcal{K} \oplus T\pi_M : T(TM) \to TM \oplus TM \oplus TM \quad (2)$$

is a diffeomorphism (see [3]). The diffeomorphism determines a unique vector bundle structure for $T(TM)$ over $M$. Let $(U_\alpha, \varphi_\alpha)$ be a chart of $M$. The induced chart $\{(\pi_2^{-1}(U_\alpha), T\varphi_\alpha)\}$ in $TM$ takes a vector bundle structure by means of the Diffeomorphism [2]. Let $\iota : TM \oplus TM \to TM \oplus TM \oplus TM$ be the natural isomorphism. $T^2 M$ is a submanifold of $T(TM)$ consisting of tangent vectors $\mathcal{Y}$ such that $\pi_2(\mathcal{Y}) = T\pi_M(\mathcal{Y})$. Therefore, the inclusion $\iota$ is the isomorphism onto $(\pi_2 \oplus \mathcal{K} \oplus T\pi_M)(T^2 M)$, thus

$$\iota^{-1} \circ (\pi_2 \oplus \mathcal{K} \oplus T\pi_M)(T^2 M) = \pi_2 \oplus \mathcal{K}(T^2 M).$$
Hence the diffeomorphism
\[ \pi_2 \oplus K : T^2M \to TM \oplus TM \]
gives the structure of a vector bundle to \( T^2M \). Since \( T^2M \) is isomorphic to \( TM \oplus TM \), it can be considered as a vector bundle with group structure \( \text{Aut}(F \times F) \).

The proof of the following theorem is the same as the usual proof given for Banach manifolds (see [4, Theorem 2.4]). We just give literally the scheme of proof.

**Theorem 4.2.** If \( T^2M \) admits a vector bundle structure isomorphic to \( TM \oplus TM \), then there exists a linear connection on \( M \).

**Proof.** Let \( \{(\Pi^{-1}(U_\alpha), \Omega_\alpha)\}_{\alpha \in \mathcal{A}} \) be a trivializing atlas of \( T^2M \). By hypothesis \( \Omega_{\alpha,p} = \Omega_{\alpha,p}^1 \times \Omega_{\alpha,p}^2 \), where \( \Omega_{\alpha,p}^i : \pi_{M}^{-1}(p) \to F \) \((i=1,2)\). Let \((U, \Omega)\) be an arbitrary chart such that \( U \subseteq U_\alpha \). Define \( \Omega_\alpha = \Omega \circ (\Omega_{\alpha,p}^1 \circ (D_x \Omega)^{-1}) \). Then define the mappings as follows:
\[ \tau_\alpha(u, u)(y) = \Omega_{\alpha,p}^2(j_f^2) - (\Omega_\alpha \circ f)^{(0)}, \quad y \in \Omega_\alpha(U_\alpha), \]
where \( f \) is the representative of the vector \( u \). The remaining values of \( \tau_\alpha(y) \) on elements of the form \((u, v)\) with \( u \neq v \) are automatically defined if we demand \( \tau_\alpha(y) \) to be symmetric and bilinear. They satisfy the compatibility condition since the trivializations \( \{(\Pi^{-1}(U_\alpha), \Omega_\alpha)\}_{\alpha \in \mathcal{A}} \) coincide on all common areas of their domains, hence give rise to a linear connection on \( M \).

\[ \square \]

5. **Vector fields on \( TM \)**

Having introduced the tangent bundle over a manifold \( M \), we now consider sections of these bundles. A vector field on \( M \) is a section \( \xi : M \to TM \) of its tangent bundle, i.e. \( \pi_M \circ \xi = \text{id}_M \). For a vector field \( \xi \) and a chart \( U \subset M \xrightarrow{\varphi} \varphi(U) \subset F \), the principle part \( \xi_\varphi : \varphi(U) \to F \) of \( \xi \) is defined by \( \xi_\varphi(\varphi(p)) = \text{pr}_2 \circ T\varphi(\xi_p) \). Let \( I \) be an open interval in \( \mathbb{R} \) and let \( \ell : I \to M \) be a curve passing through \( p_0 \). If \( \xi \) is a vector field on \( M \) and if \( \xi_\varphi \) denotes the principle part of its local representative in a chart \( \varphi \), then \( \ell(t) \) is called an integral curve of \( \xi \) when \((\varphi \circ \ell)'(t) = \xi_\varphi(\varphi \circ \ell(t)) \) for each \( t \), where \( \varphi \circ \ell \) is the local representative of the curve \( \ell \). Note that if the base manifold \( M \) is a Fréchet space \( F \) with differential structure induced by the chart \((F, \text{id}_F)\), then the above condition reduces to: \( \ell'(t) = D\ell(t)(1_{\mathbb{R}}) \). That is, our definition is a natural generalization of the notion of derivative on a manifold \( M \).

**Proposition 5.1.** Let \( U \subseteq F \) be open and let \( \xi : U \to F \) be \( MC^k, k \geq 1 \). Then for \( p_0 \in U \), there is an integral curve \( \ell : I \to F \) at \( p_0 \). Furthermore, any two such curves are equal on the intersection of their domains.
Proof. Since $\xi$ is $MC^k$, it is bounded, say by $R$. Let $L$ be a positive real number. Pick a positive real number $r$ such that $B_r(p_0) \subseteq U$ and $\| \xi(p) \|_d \leq L$ for all $p \in B_r(p_0)$. Let $m = \min\{1/R, r/L\}$ and let $t_0$ be a real number. We shall show that there is a unique $MC^1$-curve $\ell(t), t \in [t_0 - m, t_0 + m]$ whose image lies in $B_r(p_0)$ and that satisfies
\[
\ell'(t) = \xi(\ell(t)), \quad \ell(t_0) = p_0. \tag{1}
\]
The conditions $\ell'(t) = \xi(\ell(t)), \ell(t_0) = p_0$ are equivalent to the integral equation
\[
\ell(t) = p_0 + \int_{t_0}^{t} \xi(\ell(u))du \tag{2}
\]
Now define $\ell_n(t)$ by induction
\[
\ell_0(t) = p_0, \quad \ell_{n+1}(t) = p_0 + \int_{t_0}^{t} \xi(\ell_n(u))du.
\]
The estimation on the size of integral (see [10, Lemma 1.10]) yields $\ell_n(t) \in B_r(p_0)$ for all $n$ and $t \in [t_0 - m, t_0 + m]$. Furthermore,
\[
\| \ell_{n+1}(t) - \ell_n(t) \|_d \leq \frac{LR^n}{(n+1)!} | t - t_0 |^{n+1}.
\]
To see this, assume that
\[
\| \ell_n(t) - \ell_{n-1}(t) \|_d \leq \frac{LR^{n-1}}{n!} | t - t_0 |^{n}.
\]
Then we estimate as follows: (again assuming that $t \geq t_0$ for simplicity)
\[
\| \ell_{n+1}(t) - \ell_n(t) \|_d = \left\| \int_{t_0}^{t} \xi(\ell_n(u)) - \xi(\ell_{n-1}(u))du \right\|_d
\leq \int_{t_0}^{t} R\| \xi(\ell_n(u)) - \xi(\ell_{n-1}(u)) \|_d du
\leq R \int_{t_0}^{t} \frac{LR^{n-1}}{n!}(u - t_0)^n du
= \frac{LR^n}{(n+1)!}(u - t_0)^{n+1}.
\]
Thus, since
\[
\frac{LR^n}{(n+1)!}(u - t_0)^{n+1} \leq \frac{LR^n}{(n+1)!}m^{n+1}
\]
and the series with these quantities as terms is convergent, we see, writing
\[
\| \ell_{n+p} - \ell_n \|_d
\]
as a telescoping sum, that the functions $\ell_n$ form a uniformly Cauchy sequence and hence converge uniformly to a continuous curve $\ell(t)$ satisfying (2). Since $\ell(t)$ is continuous, the
integral equation in fact shows that it is $MC^1$. This proves existence. Now let $j(t)$ be another solution. By a similar induction argument as above, we find that

$$\|\ell_n(t) - j(t)\|_d \leq \frac{LR^n}{(n+1)!} |t - t_0|^{n+1}. $$

Therefore, letting $n \to \infty$ gives $\ell(t) = j(t)$. 

\[ \square \]

**Corollary 5.1.** Suppose the hypotheses of the previous proposition hold. Let $I_t(p_0)$ be the solution of $\ell'(t) = \xi(\ell(t))$, $\ell(t_0) = p_0$. Then there is an open neighborhood $U_0$ of $p_0$ and a positive real number $\alpha$ such that for every $q \in U_0$ there exists a unique integral curve $\ell(t) = I_t(q)$ satisfying $\ell(0) = q$ and $\ell'(t) = \xi(\ell(t))$ for all $t \in (-\alpha, \alpha)$.

**Proof.** Suppose $U_0 = B_{r/2}(p_0)$ and $\alpha = \min\{1/R, r/2L\}$. Fix an arbitrary point $q_0$ in $U_0$. Then $B_{r/2}(q_0) \subset B_{r}(p_0)$, thereby $\|\xi(z)\|_d < L$ for all $z \in B_{r/2}(q_0)$. By Proposition 5.1 with $p_0$ replaced by $q$, $r$ by $r/2$, and $t_0$ by $0$, for all $t \in (-\alpha, \alpha)$ there exists a unique integral curve $\ell(t)$ such that $\ell(0) = q$. 

\[ \square \]

The proof of the following theorem is the same as the usual proof given for Banach manifolds (see [14, Theorem 2.1]).

**Theorem 5.1.** Let $\xi : M \to TM$ be a vector field. Then there exits an integral curve for $\xi$ at $p \in M$. Furthermore, any two such curves are equal on the intersection of their domains.

**Proof.** The existence follows from Proposition 5.1 by means of local representation. But that is not applicable for the proof of uniqueness since these curves may lie in different charts. Let $\rho_i(t) : I_i \to M$ (i=1,2) be two integral curves. Let $I = I_1 \cap I_2$ and $J = \{t \in I \mid \rho_1(t) = \rho_2(t)\}$. $J$ is closed since $M$ is Hausdorff. From Proposition 5.1, $J$ contains some neighborhood of 0. Now define $\delta_1(u) = \rho_1(u + t)$ and $\delta_2(u) = \rho_2(u + t)$ for $t \in J$. They are integral curves with initial conditions $\rho_1(t)$ and $\rho_2(t)$, respectively. By Proposition 5.1 they coincide on a some neighborhood of 0. Therefore, $J$ contains an open neighborhood of $t$, so $J$ is open. Since $I$ is connected it follows that $J = I$. 

\[ \square \]

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