Geometrize and conquer: the Klein–Gordon and Dirac equations in Doubly Special Relativity

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In this work we discuss the deformed relativistic wave equations, namely the Klein–Gordon and Dirac equations in a Doubly Special Relativity scenario. We employ what we call a geometric approach, based on the geometry of a curved momentum space, which should be seen as complementary to the more spread algebraic one. In this frame we are able to rederive well-known algebraic expressions, as well as to treat yet unresolved issues, to wit, the explicit relation between both equations, the discrete symmetries for Dirac particles, the fate of covariance, and the formal definition of a Hilbert space for the Klein–Gordon case.

I. INTRODUCTION

A quantum gravity theory (QGT), i.e. a theory encompassing quantum field theory (QFT) and general relativity (GR), has been looked for during several decades. The (probably) simplest trial, a quantum theory of gravitation where the mediation of the interaction is carried out by the graviton, a spin-2 particle, has lead to several inconveniences [1, 2], which ended up in its understanding as an effective field theory [3]. In spite of the success of the latter, there remain many subtleties, for example those related to the renormalization process and the gauge-independence of the computations [4–6].

Searching for a more fundamental theory and principles, the community has developed several theoretical frameworks, such as string theory [7–9], loop quantum gravity [10, 11], causal dynamical triangulations [12], causal set theory [13–15] and functional renormalization group [16]. In (almost) all of them a minimum length arises [17–19], which is heuristically associated with the Planck length

\[ \ell_P \approx 1.6 \times 10^{-33} \text{cm} \]

(or Planck mass

\[ M_P \approx 1.22 \times 10^{19} \text{GeV} \]

This small length (high-energy scale) is expected to somehow separate the regime where spacetime displays its classical nature from the one where it develops a quantum structure.

One can naturally guess that all of the above-mentioned theories should lead to novel scenarios with possible observable implications. However, the description of the latter is generally arduous. In order to enhance the connection with the observational side, another way of thinking has arisen not long ago: instead of considering a fundamental theory of quantum gravity, one can explore a low-energy limit of it, leading indeed to testable phenomenology [20].

A possible way to follow this bottom-up approach is through a modification of the special relativistic kinematics by introducing a high-energy scale. There are two different possibilities of doing this, depending on which is the fate of Lorentz symmetry: for high energies, one can consider that it is either violated or deformed. The former scenario is implemented in the framework of Lorentz invariance violation (LIV) [21–24], while the latter is inherent to Doubly (or Deformed) Special Relativity (DSR) theories [25]. An immediate consequence is that the relativity principle characterizing Special Relativity (SR) is lost in LIV, while simply deformed in DSR.

Both ideas have been extensively developed from an algebraic approach. Indeed, the original proposal by Snyder consisted in a deformed algebra of position and momentum operators, in an attempt to regularize ultraviolet divergences arising in QFT [26, 27]. More recently, after the introduction of the $q$-deformed Poincaré algebra through a Drinfeld–Jimbo deformation [28], several works have been devoted to understand $\kappa$-Minkowski and Snyder algebras [29–36], including also proposals on how to obtain a Dirac equation [37–41].

In this article we will employ a less developed, geometric point of view, to discuss generalizations of the Klein–Gordon and Dirac equations, what may be appreciated by readers as an efficient and intuitive alternative. Recently, a clear connection between the geometry in momentum space and noncommutative physics was shown to exist [42–44].
This idea was already latent in the original paper of Born [43], where a first proposal of curved momentum space was made and a lattice structure for spacetime was shown to arise. These considerations are englobed in the broader context of velocity or momentum dependent spacetime, the so-called Finsler and Hamilton geometries [10]. Among others, there are several works in LIV setups describing the propagation of particles with a modified dispersion relation through a Finsler spacetime [17–51]. Also in DSR scenarios there exist contributions which consider a velocity [52–54] and momentum [55–57] dependent geometries, taking into account a deformed dispersion relation.

Our construction below is greatly motivated by the results in [58], where it was shown that all the ingredients of a relativistic deformed kinematics (both a deformed dispersion relation and a composition law) can be singled out in the case of a maximally symmetric (curved) momentum space. In particular, $\kappa$-Poincaré kinematics can be obtained from a de Sitter space by identifying the deformed composition law, the deformed Lorentz transformations and the deformed dispersion relation: they correspond respectively to the translation isometries, the Lorentz isometries and the squared distance to the origin. Notice that the last two facts have been previously contemplated in Refs. [59, 60]. Also, it is important to keep in mind that different bases of $\kappa$-Poincaré [61] can be obtained from different choices of momentum coordinates in a de Sitter space [58].

In order to discuss the modified deformed relativistic wave equations in a geometrical framework, we organize our article as follows. In Sec. II we will show how one can introduce the Klein–Gordon equation in curved momentum space, considering also some formal aspects regarding the definition of the corresponding Hilbert space. Then, in Sec. III we generalize the technique to a fermionic system of spin one-half. In both cases we show that we reproduce firmly set results previously obtained within the scheme of Hopf algebras; additionally, we show that one can implement a variational principle in momentum space by defining appropriate actions. We discuss the discrete fermionic symmetries in Sec. III D. Finally, we state our conclusions in Sec. IV. Additional results regarding the geometry of the so-called $\kappa$-Poincaré’s symmetric basis are included in App. A.

We use the following conventions. We define the Minkowski metric $(\eta_{\mu\nu})$ with mostly minus signs; all other metrics will possess the same signature. Greek indices are used to label spacetime components of a tensor $(\mu, \nu, \cdots = 0, 1, 2, 3)$, while latin indices denote just spatial components $(i, j, \cdots = 1, 2, 3)$. The first latin characters $(a, b, \cdots = 0, 1, 2, 3)$ are employed for components in the local orthonormal frame given by the (inverse) vierbein $e_{\mu}{}^a$. Regarding momenta, i.e., coordinates, we use the following notation: we denote $\vec{p}^2 := p_\mu \eta^{\mu\nu} p_\nu$; the set of all spatial components of a vector $\vec{p}$ is written as $\vec{p}$ and $\vec{p}^2 := \vec{p} \cdot \vec{p}$. We use units $\hbar = c = 1$.

II. KLEIN–GORDON EQUATION

The fact that a nontrivial momentum metric is able to describe the kinematics of DSR has been thoroughly discussed in [58]. Following those lines, in this section we propose a generalization of the usual Klein–Gordon (KG) equation to the case in which the metric is momentum dependent. Let us first recall some basic aspects of the usual case. In SR, the KG operator acts on a wave function $\phi$ as

$$\left( \eta^{\mu\nu} \frac{\partial}{\partial x^\nu} \frac{\partial}{\partial x^\mu} + m^2 \right) \phi(x) = 0,$$

where $m$ is the mass of the particle and $\eta_{\mu\nu}$ is the metric in Minkowski spacetime. Since its construction relies in the principle of relativity, this equation is devised as invariant under Poincaré symmetries. In addition it is also invariant under general diffeomorphisms in space, which is clear once one recognizes the Beltrami–Laplace operator. The most general solution to Eq. (1) can be written in terms of plane-wave solutions. Transforming to Fourier space we have

$$\phi(x) = \frac{\sqrt{2}}{(2\pi)^3} \int d^4 p e^{i x \cdot p} \tilde{\phi}(p) \delta(C_M(p) - m^2),$$

where we have defined the usual Casimir,

$$C_M(p) := p^2 = p_\mu \eta^{\mu\nu} p_\nu,$$

and the Dirac delta function enforces the on-shell condition (or dispersion relation) to be satisfied. Some additional overall factors where chosen to simplify the notation later. Alternatively, the KG equation can be seen as an algebraic equation in momentum space,

$$(C_M(p) - m^2) \tilde{\phi}(p) = 0.$$
Once we introduce a nontrivial pseudo-Riemannian metric \( g_{\mu \nu}(p) \) which depends on the momentum, we are constrained to deform the Casimir in a consistent way. In [62] it was shown that, defining the Casimir as the pseudo-squared distance to the origin \((p^*)\) in momentum space, which we will denote by \( C_D(p) \), the following relation between the metric and the Casimir holds

\[
C_D(p) = f^\mu g_{\mu \nu}(p) f^\nu, \tag{5}
\]

\[
f^\mu(p) := \frac{1}{2} \frac{\partial C_D(p)}{\partial p_\mu}. \tag{6}
\]

Notice that our choice for \( C_D(p) \) can be understood in terms of Synge’s world function \( \sigma(p', p) \) [63], which is a biscalar equal to one half of the square of the (geodesic) distance between \( p' \) and \( p \). Under this interpretation, we are fixing the first argument of \( \sigma \) to be \( p^* \). Moreover, the functions \( f^\mu \) are the covariant derivatives of Synge’s function, \( \sigma^\mu \), and will be called generalized momenta because of their role in generating the Casimir in Eq. (5) (see also the following sections). Note thus that this discussion is valid for any reasonable pseudo-Riemannian metric, as has been shown for example in [64, 65].

In SR, i.e. considering a flat metric, expression (5) can be trivially shown to be satisfied by the Casimir defined in Eq. (3). Instead, in DSR, when regarding the Casimir as the squared of the distance in momentum space, \( f^\mu \) will be nontrivial functions of the momenta. This is the case also in the so-called classical basis of \( \kappa \)-Poincaré: even if in the algebraic context it is considered that the Casimir in this basis is the one of SR [64], one can easily see that the momentum metric describing this kinematics leads to a nontrivial (squared) distance in momentum space [66].

In the classical basis of \( \kappa \)-Poincaré, the distance \( C_D(p) \) turns out to be a function of \( C_M \). In an Euclidean setup, where both are positive, one can invert the relation to show that \( C_M \) may also play the role of a Casimir. In the Minkowskian case, one can appeal to a Wick rotation to establish the same conclusion. These different Casimirs then become equivalent (at least at the classical level) to a joint redefinition of all the particles’ masses. However, by regarding the Casimir as a propagator in QFT, we expect a different behaviour of the theory in the ultraviolet (UV) regime, i.e. for large momenta. One important argument that favours our choice \( (C_D) \) is that, as we will see, then the Klein–Gordon operator equals the “squared” Dirac one, without the need of introducing any extra function of the mass.

We may now proceed in a canonical way, meaning that the generalized KG equation should enforce the dispersion relation given by the generalized Casimir in Eq. (5). The appropriate definition is thus seen to be\(^2\)

\[
(f^\mu(p) g_{\mu \nu}(p) f^\nu(p) - m^2) \phi(p) = 0. \tag{7}
\]

The case of SR is simply recovered by considering a flat metric and the undeformed Casimir \( C_M \).

More formally, from a quantum mechanical perspective, expression (7) should be thought as a representation of the KG equation in momentum space. This entails considering a construction of momentum eigenstates in curved momentum space, a vision dual to DeWitt idea’s of nonrelativistic quantum mechanics in curved configuration space [67, 68]. The fact that we want to consider the relativistic case is by far nontrivial; we will offer a more detailed discussion of these issues in Sec. II C.

By construction, Eq. (7) is well-defined under the action of diffeomorphisms in momentum space (leading to different basis bases of the Hopf algebra structure), since the Casimir has been defined as a distance. Moreover, Eq. (7) satisfies a deformed Lorentz invariance. Indeed, in the context of (classical, i.e. not quantum) DSR, there exists a notion, albeit deformed, of Lorentz transformations [25]. The action of the latter is such that the Casimir is left invariant and the metric transforms appropriately as a \((2,0)\)-tensor [58]. Once we assume that the field transforms as a scalar under diffeomorphisms \( p \to p' \), i.e.

\[
\phi'(p') = \phi(p), \tag{8}
\]

the action of the quantum generators of Lorentz invariance in momentum space can be seen to be given by simple multiplication with their classical expression as given in [58].

Contrary to what happens in SR, where the Lorentz generators \( J_{\mu \nu} \) are linear functions of \( p \), in DSR the Lorentz transformations are generally nonlinear. However, the equation

\[
C_D(p) = C_D(p') \tag{9}
\]

holds if \( p \) and \( p' \) are connected through a deformed Lorentz transformation. This shows that our proposal for the KG equation is invariant under Lorentz transformations which are the isometries of the metric defining the kinematics of DSR.

\(^2\) From now on we drop the tilde over \( \phi \), since the basic wave function will be defined in momentum space.
A. Klein–Gordon equation in the symmetric basis of $\kappa$-Poincaré

As an example of the above-derived equations, we will write the KG equation in the symmetric basis of $\kappa$-Poincaré. Using the formalism developed in [58] one can obtain all the relevant geometric quantities; more details on these computations can be found in App. A. Then, the distance in momentum space can be readily obtained by employing Eq. (5) in conjunction with \( (A6) \):

\[
C_D^{(S)}(p) = \Lambda^2 \text{arccosh}^2 \left( \cosh \left( \frac{p_0}{\Lambda} \right) - \frac{\vec{p}^2}{2\Lambda^2} \right).
\]

One can compare this result with the proposal in [37], which was obtained in an algebraic context. From their definition of Casimir, the associated KG equation is found to be

\[
\left( C_A^{(S)}(p) - m^2 \right) \phi(p) := \left( \left( 2\Lambda \sinh \left( \frac{p_0}{2\Lambda} \right) \right)^2 - \vec{p}^2 - m^2 \right) \phi(p).
\]

Evidently, Eq. (11) differs from our proposal (10) of constructing the KG equation from the squared distance in momentum space. However, similar to the situation discussed in the previous subsection, one can indeed see that the following relation between them exists:

\[
C_D^{(S)}(p) = \Lambda^2 \text{arccosh}^2 \left( 1 + \frac{C_A(p)}{2\Lambda^2} \right).
\]

As commented previously, this difference in the definition of the Casimir should be appreciable in the UV regime after the application of a quantization procedure. Additionally, as will be seen in Sec. III C, our proposal provides a direct link between Dirac and KG equations.

B. An action for scalar fields

Assuming a principle of stationary action, the KG equation (7) can be obtained from the action

\[
S_{\text{KG}} := \int d^4 p \sqrt{-g} \phi^*(p) \left( C_D(p) - m^2 \right) \phi(p).
\]

The factor $\sqrt{-g}$ guarantees invariance under a change of momentum basis. This action may serve as a departing point to construct a free quantum field theory, as has been proposed in the work by Gol’fand [69, 70], and refined by Mir-Kasimov [71–73] and others [74, 75]. Indeed, one can first define the theory in a Riemannian momentum space, where the action is positive defined, and afterwards employ an analogue of the Wick-rotation to consider pseudo-Riemannian spaces [70].

Although this theory would share many features with the proposal of Gol’fand, there is a fundamental difference: we propose the Casimir as a distance in momentum space, whereas in [70] just $p^2$ in the Beltrami chart [70] is employed (which does not correspond to the squared distance in that coordinates). Notice that, as commented previously, even if we would work with the same coordinates, the models are expected to differ in the UV sector because of the different choice of the Casimir. Moreover, as we will see, our choice of Casimir is the only one allowing to make a direct identification between the KG and Dirac equations.

A subtle point arises when one tries to introduce interactions. Many proposals of QFT in $\kappa$-Minkowski and Snyder space have been done in the literature [35, 77–85], most of them relying on the definition of an appropriate Moyal–Groenewold product (also called star-product) [86–89]. The latter encodes the nontrivial addition of momenta and, generally speaking, can be built in a case by case analysis. In our case, the adoption of the translation operators [58] seems better suited. We will leave a concrete analysis to a future publication.

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3 In a Casimir, superscripts denote the choice of coordinates, while subscripts refer to the chosen Casimir. For example, the subscript $A$ will label the Casimir considered in the algebraic approach [37], while subscript $D$ refers to the squared distance in momentum space, cf. [58].

4 Actually, the proposal in [72] can be “squared” but just to obtain one component of the KG operator in [74]: the latter proposes a separation of the field into two components, one which vanishes on the mass shell and one that doesn’t.
C. On the definition of a Hilbert space

As is well-known, the formal construction and interpretation of the KG theory poses several problems. One is able to build a conserved current in SR, \( J^{\text{KG}}_\mu := i\psi^*(x) \partial_\mu \psi(x) \), where \( \partial_\mu = \partial_x - \partial_{\dot{x}} \), and even extend its zero component to define a sesquilinear form

\[
(\phi, \psi)_{\text{KG}} := i \int d^3 x [\partial_0 \phi^*(x) \psi(x) - \phi^*(x) \partial_0 \psi(x)],
\]

which is conserved in time when \( \psi \) and \( \phi \) are solutions of the KG equation and sufficiently well-behaved. However, an issue exists, inasmuch as (14) is not positive definite, leading to a failure in its interpretation as a density and in the definition of a scalar product. The way around it, whose generalization enables the construction of QFT in curved spaces, has been to understand that Eq. (14) indeed corresponds to a scalar product if one restricts to positive-energy modes [90]. This assertion can be immediately seen if one writes a solution \( \phi \) of the KG equation in Fourier space; employing the delta function in (2) to perform the integral in \( p_0 \) we obtain

\[
\phi(x) = \frac{1}{\sqrt{2(2\pi)^3}} \int \frac{d^3 p}{\omega_p} \left( e^{i(t\omega_p + \vec{p} \cdot \vec{x})} \phi_+(\vec{p}) + e^{-i(t\omega_p + \vec{p} \cdot \vec{x})} \phi_-(\vec{p}) \right),
\]

where \( \omega_p := \sqrt{\vec{p}^2 + m^2} \) is the positive solution one obtains for \( p_0 \) by enforcing the dispersion relation, while the two distinct components \( \phi_{\pm} := \phi(\pm \omega_p, \vec{p}) \) arise because of the disconnected support of the delta function in (2). Additionally, in (15) one can recognize the Lorentz invariant measure. Using this expansion one may recast the scalar product as

\[
(\phi, \psi)_{\text{KG}} = \int \frac{d^3 p}{\omega_p} \left[ \phi^*_+ (p) \psi_+(p) - \phi^*_+(p) \psi_-(p) \right],
\]

from which the positive and negative part of the scalar product are evident.

A generalization of Eq. (15) has been developed in [84, 91]. In particular, Ref. [84] includes a term which is proportional to the square root of the product \( f^\mu(p) g_{\mu\nu}(p) f^\nu(p) \), which is nothing but our Casimir. Therefore, this term can be easily reabsorbed in a constant.

The fact that an extension of (16) may be important also in the construction of a QFT in curved momentum space has been acknowledged in [91]. In order to generalize expression (16) in our setup, we need to introduce two changes. First, we rewrite it in the space of four-momenta, including the measure factor and a Dirac delta function that guarantees the satisfaction of Eq. (7). A further step is to notice that one can introduce a time-like vector \( t_\mu = (1, 0, 0, 0) \). Summing these elements and given sufficiently well-behaved functions \( \phi, \psi \in L^2(\text{d}S_4) \), i.e. the space of square-integrable complex-valued functions over \( \text{d}S_4 \), we can define a positive Hermitian form

\[
(\phi, \psi)_{\text{CD}} := 2 \int d^4 p \sqrt{-g} \delta \left( C_D - m^2 \right) \Theta(f^\mu t_\nu) \phi^*(p) \psi(p),
\]

where \( \Theta(\cdot) \) is the Heaviside function. It is important to notice that this scalar product is clearly invariant under diffeomorphisms in momentum space. An alternative expression can be obtained using the Dirac delta to integrate over \( p_0 \); one then obtains

\[
(\phi, \psi)_{\text{CD}} = \sum_i \int \frac{d^3 p}{|f^\nu t_\nu|} \sqrt{-g} \Theta(f^\mu t_\nu) \phi^*(p) \psi(p) \Big|_{p_0=\tilde{p}_0^{(i)}},
\]

where \( \tilde{p}_0^{(i)} \) denotes the several solutions to the Casimir equation \( C_D(\tilde{p}_0^{(i)}, \vec{p}) - m^2 = 0 \). Expression (18) evidently resembles the first term in the RHS of (16) and suggests to proceed in the following formal way: one should consider smooth functions with local support on \( \text{d}S_4 \) and define a projected subspace \( \mathcal{S} \) by multiplying by \( \Theta(f^\mu t_\nu) \). This is the analogous of choosing the subspace of solutions with positive energy in the standard flat case. Then the Cauchy completion of \( \mathcal{S} \) with the scalar product (17) will define a Hilbert space, from which one may try to implement a quantization.

At this point two comments are in order. The description of \( \kappa \)-Poincaré kinematics customarily involves the introduction of a constant temporal vector\(^5\), privileging thus the temporal component over the space coordinates.

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\(^5\) Strictly speaking, \( \kappa \)-Poincaré introduces an object which is not a vector under momentum diffeomorphisms. We use the name “constant temporal vector” because it is widely employed in this context.
since only the boosts are modified but not the rotations \([61]\). This means that invariance under diffeomorphisms will be restricted to a subset respecting the aforementioned splitting. Notice also that the on-shell condition defines a foliation of spacetime with spacelike hyper-surfaces, whose normal vectors are the generalized momenta \(f^\mu\) (which are timelike). This implies that, even if in such case \(f^\mu\) would not transform as a vector under momentum diffeomorphisms, the term \(\Theta(f^\nu t^\nu)\) in Eq. \((17)\) would indeed be invariant in the aforementioned restricted sense.

Additionally, as discussed in several papers (see \([84]\) and references therein), the kinematics of \(\kappa\)-Poincaré can be described in just one-half of de Sitter momentum space. Moreover, this half is not closed under the action of Lorentz transformations. This restriction is not necessary in our current discussion, but may be implemented in future discussions involving interactions.

### III. DIRAC EQUATION

Now that we have illustrated our ideas on how the geometrical approach works in the scalar case, a generalization to Dirac particles of spin \(1/2\) seems natural. Our proposal in DSR for the Dirac equation, in momentum space and as an algebraic equation, is given by

\[
(\gamma^\mu f_\mu(p) - m) \psi(p) = 0, \tag{19}
\]

where the functions \(f_\mu\) are obtained from the covariant functions in Eq. \((6)\) by a contraction with the inverse metric,

\[
f_\mu(p) := g^{\mu\nu}(p)f^\nu(p). \tag{20}
\]

The curved-momentum-space gamma matrices (with greek indices and underlined) are defined in terms of the usual Dirac matrices \(\gamma^a\) (with latin indices, with \(a = 0, 1, 2, 3\)), the latter providing a spin-1/2 finite-dimensional representation of the \(\text{SO}(3,1)\) group. More concretely, we consider the tetrad \(e^\nu_a(p)\) (or nonholonomic connection) in momentum space, from which the metric can be built

\[
g^{\mu\nu}(p) =: e^\mu_a(p)\eta^{ab}e^\nu_b(p), \tag{21}
\]

so that the gamma matrices can be written as

\[
\gamma^\mu := \gamma^a e^\mu_a(p). \tag{22}
\]

As a consequence of the definition \((22)\), the gamma matrices share several properties with the gamma matrices in curved configuration space \([92]\). For example, it is immediate to see that they satisfy a Clifford algebra using the momentum-space metric, i.e.

\[
\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}(p)1, \tag{23}
\]

where the operator \(\{\cdot, \cdot\}\) is the anti-commutator. Another similarity with the curved configuration space is that, if we multiply Eq. \((19)\) by the operator \((\gamma^\nu f_\nu(p) + m)\), we reobtain the KG equation derived in the previous section, cf. Eq. \((7)\). If instead of considering the Casimir as the squared of the distance in momentum space one chooses any other function of it, this fundamental relationship will not hold. This is for example the case of the Klein–Gordon and Dirac equations obtained in the algebraic approach \([89]\), a detailed comparison will be performed in Sec. \([11]\).

We close this general discussion by noting that one can derive the modified Dirac equation from an action principle in momentum space. Indeed, one can define the Dirac adjoint \(\bar{\psi} := \psi^\dagger \gamma^0\) and the following action,

\[
S_{\text{Dirac}} := \int d^4 p \sqrt{-g} \bar{\psi}(-p) \left(\gamma^\mu f_\mu(p) - m\right) \psi(p), \tag{24}
\]

from which the Dirac equation follows when looking for stationary configurations.

#### A. Deformed Lorentz invariance of the Dirac equation

A proof of the fact that the proposed Dirac equation is Lorentz invariant can be done in a way analogous to the one followed in curved configuration space. We will assume that the spinor \(\psi(p)\) transforms linearly (with a matrix \(S\)) under a Lorentz transformation \(p \to p'\); by requiring Lorentz invariance, we will then see that \(S\) corresponds to a fermionic (Dirac) representation of the Lorentz group. We start by applying \(S\) to Eq. \((19)\), so that we obtain

\[
(S\gamma^\mu S^{-1} f_\mu(p) - m) S\psi(p) = (S\gamma^a S^{-1} e^{\mu_a}(p')f_\mu'(p') - m) \psi'(p') = 0, \tag{25}
\]
where we have used appropriate transformations for the vielbein and the functions $f^\mu$, cf. the definition (6) (all the quantities in the new system of coordinates are denoted with a prime).

Then, since the proposed diffeomorphism is an isometry of the metric, the vielbein satisfies

$$e^{\alpha}_{\, a}(p') = e^{\alpha}_{\, b}(p') \Lambda^b_{\, a}(p'),$$

where $\Lambda^b_{\, a}(p')$ is a (local) Lorentz transformation that may depend on the point $p'$. Note that this transformation is not the deformed Lorentz transformation obtained from the isometries of the metric. Considering (26) and the symmetry of $f$ is not the deformed Lorentz transformation obtained from the isometries of the metric. Considering (26) and the symmetry of $f$ (cf. the definition (6)), we obtain

$$\left(S \gamma^a S^{-1} \Lambda^b_{\, a}(p') e^a_{\, \mu}(p') f_{\mu}(p') - m\right) \psi'(p') = 0.$$  \hspace{1cm} (27)

Of course, local Lorentz transformations near the identity can be expanded in terms of antisymmetric parameters $\epsilon_{ab} := \eta_{ac} \epsilon^c_b$ as customarily,

$$\Lambda^b_{\, a}(p') = \delta^b_{\, a} + \epsilon^b_{\, a}(p') + \cdots.$$  \hspace{1cm} (28)

Thus, we can also expand the matrix $S$ in Eq. (25) for transformations around the identity; this allows us to determine the infinitesimal form of $S$ as a function of the Lorentz coefficients $\epsilon_{ab}(p')$:

$$S = 1 - \frac{i}{4} \sigma^{ab} \epsilon_{ab}(p') + \cdots,$$  \hspace{1cm} (29)

$$\sigma^{ab} := \frac{i}{2} \left[ \gamma^a, \gamma^b \right].$$  \hspace{1cm} (30)

B. The Dirac equation in simple coordinates

To show the previously-described formalism in action, let us fix the coordinates and consider a particular example of the exact deformed Dirac equation, i.e. keeping all orders in the deformation parameter $\Lambda$. In [93] a really simple basis of $\kappa$-Poincaré kinematics was found, in which the composition law is linear in momenta

$$(p \oplus q)_{\mu} = p_{\mu} + (1 - p_0/\Lambda) q_0.$$  \hspace{1cm} (31)

The momentum metric with these isometries can be defined by the following inverse metric and inverse tetrad [43]:

$$g_{\mu\nu}(p) := \eta_{\mu\nu} (1 - p_0/\Lambda)^2, \quad p_0 < \Lambda,$$  \hspace{1cm} (32)

$$e^a_{\, \mu}(p) := \delta^a_{\, \mu} (1 - p_0/\Lambda).$$  \hspace{1cm} (33)

Then, with a direct application of Eq. (5), we obtain the Casimir to be

$$C_D^{\text{simp}}(p) = \Lambda^2 \arccosh^2 \left( 1 + \frac{p^2}{2 \Lambda^2 (1 - p_0/\Lambda)} \right),$$  \hspace{1cm} (34)

such that the Dirac equation in this basis reads

$$\Lambda \sqrt{C_D^{\text{simp}}(p)} \left[ 2 \Lambda \gamma^a \delta^\mu_{\, \mu} p_\mu - \gamma^0 (p_0^2 + \vec{p}^2) - 2 \gamma^i p_i p_0 \right] \psi(p) = m \psi(p).$$  \hspace{1cm} (35)

In spite of the rather complex aspect of this equation, the solutions can be found in a straightforward way. Analogue to what happens in SR, this modified Dirac equation allows four different solutions, namely

$$\psi_1 = \begin{pmatrix} 1 \\ 0 \\ \frac{p_1}{p_0 + \Lambda \left( e^{-m/\Lambda} - 1 \right)} \\ \frac{p_1}{p_0 + \Lambda \left( e^{-m/\Lambda} - 1 \right)} \end{pmatrix}, \quad \psi_2 = \begin{pmatrix} 0 \\ 1 \\ \frac{-p_1 - ip_2}{p_0 + \Lambda \left( e^{-m/\Lambda} - 1 \right)} \\ \frac{-p_1 - ip_2}{p_0 + \Lambda \left( e^{-m/\Lambda} - 1 \right)} \end{pmatrix}, \quad \psi_3 = \begin{pmatrix} -\frac{p_1}{p_0 + \Lambda \left( e^{-m/\Lambda} - 1 \right)} \\ \frac{p_0 + \Lambda \left( e^{-m/\Lambda} - 1 \right)}{p_0 + \Lambda \left( e^{-m/\Lambda} - 1 \right)} \\ 0 \\ 1 \end{pmatrix}, \quad \psi_4 = \begin{pmatrix} -\frac{p_1 - ip_2}{p_0 + \Lambda \left( e^{-m/\Lambda} - 1 \right)} \\ \frac{p_0 + \Lambda \left( e^{-m/\Lambda} - 1 \right)}{p_0 + \Lambda \left( e^{-m/\Lambda} - 1 \right)} \\ 0 \\ 1 \end{pmatrix}.$$  \hspace{1cm} (36)
where the zeroth-component of the momentum can be obtained from the dispersion relation implied by Eq. (34):

\[
p_0 = \Lambda \left( 1 - \cosh \left( \frac{m}{\Lambda} \right) \left( 1 \pm \sqrt{1 - \text{sech}^2 \left( \frac{m}{\Lambda} \right) \left( 1 - \frac{p^2}{\Lambda^2} \right)} \right) \right).
\] (37)

These expressions explicitly display how the deformation occurs in this fermionic case. Indeed, one can see by expanding the two previous equations in \( \Lambda \) that the deformation is proportional to \( m/\Lambda \). This is related to the explicit form of the Casimir in these coordinates, which leads to the energy-momentum relation of SR for massless particles. Thus, if we employ the Casimir [43], hadronic showers [20] (but not gamma ray bursts) may be used to set lower limits on \( \Lambda \). This situation could be different for different basis of \( \kappa \)-Poincaré, i.e. different momentum coordinates.

**C. Dirac equation in the symmetric \( \kappa \)-Poincaré basis**

In this part of the work we will compare our results with other proposals in the literature. With that aim, we consider the symmetric \( \kappa \)-Poincaré basis (see details in App. A); using our construction in Sec. III we obtain the following Dirac operator:

\[
D^{(S)}_D := \sqrt{\frac{C^{(S)}_D(p)}{\Lambda^2}} \left[ 2\Lambda e^{-\frac{p_0}{2\Lambda}} \gamma^i p_i + \gamma^0 \left( 2\Lambda^2 \sinh \left( \frac{p_0}{\Lambda} \right) - p^2 \right) \right].
\] (38)

By construction, the Dirac equation (35) in the simple basis can be mapped to this one by a change of coordinates in momentum space.

Now we can consider the Dirac equation proposed in [39], which was obtained by considering the standard real form of the quantum anti-de Sitter algebra, \( SO_q(3,2) \). More precisely they have introduced a finite-dimensional representation of \( SO_q(3,2) \) and consistently modified the coproduct. In our language, it should consist in employing the symmetric \( \kappa \)-Poincaré basis with the Casimir given by expression (10). It is important to emphasize that this is not simply a change of coordinates, but a different choice of Casimir. When inserting Eqs. (10) and (A5) into (19), we find the Dirac operator

\[
D^{(S)}_A := \gamma^0 \left( \Lambda \sinh \left( \frac{p_0}{\Lambda} \right) - \frac{p^2}{2\Lambda} \right) + e^{-p_0/2\Lambda} p_i \gamma^i,
\] (39)

which is indeed the same expression as that obtained in [39]. Notice that from the definitions of the Casimirs of Eqs. (10) and (11), one can see that Eqs. (38) and (39) are related. Basically,

\[
D^{(S)}_A = \gamma^\mu g_{\mu\nu}(p) \frac{\partial C^{(S)}_A}{\partial p_\nu} = \gamma^\mu g_{\mu\nu}(p) \frac{\partial C^{(S)}_D}{\partial p_\nu} = D^{(S)}_D \frac{\partial C^{(S)}_A}{\partial C^{(S)}_D}.
\] (40)

It is important to note that the “square” of the Dirac operator in Eq. (39) does not lead to the KG expression (11). Indeed, as discussed in [39], they are related by

\[
\left( D^{(S)}_A \right)^2 = C^{(S)}_A(p) \left( 1 + \frac{C^{(S)}_A(p)}{4\Lambda^2} \right).
\] (41)

Instead, our proposal (38) is such that the KG equation (10) is directly obtained when squaring the Dirac one.

**D. Discrete symmetries**

It is well-known that, in Minkowski space, the Dirac equation possesses invariance not only under continuous Lorentz transformations but also under discrete ones: parity and time reversal, which connect with the improper and
These results are in contrast with the findings in [94], where the action of their modified Dirac equation. A PCT theorem can be proved in more or less general terms [95, 96]. In this sense, the situation can be understood as an analog situation in curved spacetime: not every geometry would admit the definition of such symmetries [95], even if we require the assumption

\[ \kappa \]

that the time direction is privileged in the question. However, we can study the behaviour of the free part of Dirac’s equation. To prove the invariance under conjugation symmetry, since we have not introduced a coupling with electromagnetic fields, we cannot fully tackle the rotations remain undeformed [61]. Alternatively, as discussed in [97], one can describe the kinematics of also recover the Cartesian SR result in the large \( \Lambda \) limit, it is then always possible to recast the gradient of the Casimir (\( f^\mu \)) as

\[ f^\mu(p) = p^\mu \bar{f}_1\left( \frac{p_\alpha n^\alpha}{\Lambda}, \frac{p^2}{\Lambda^2} \right) + n^\mu \Lambda \bar{f}_2\left( \frac{p_\alpha n^\alpha}{\Lambda}, \frac{p^2}{\Lambda^2} \right), \]

where \( \bar{f}_1 \) and \( \bar{f}_2 \) are dimensionless functions satisfying the following properties

\[ \bar{f}_1(0, 0) = 1, \quad \lim_{\Lambda \to \infty} \Lambda \bar{f}_2\left( \frac{p_\alpha n^\alpha}{\Lambda}, \frac{p^2}{\Lambda^2} \right) = 0. \]

Recall that these definitions of time reversal and charge conjugation are valid in Dirac’s basis of the gamma matrices.

Appearing in tensorial expressions, \( n^\mu \) can be simply understood as a shorthand to introduce a deformation of usual Lorentz invariance, i.e. a special direction in spacetime.
Notice that, in order to simplify the notation, we are employing the Minkowski metric to raise and lower indices of \( p_\mu \) and \( n^\mu \).

We can also write the most general form of the momentum tetrad that respects rotational invariance:

\[
e^{\mu} a(p) = \delta_\alpha^\mu \bar{e}_i \left( \frac{p_\alpha n^\alpha}{\Lambda}, \frac{p^2}{\Lambda^2} \right) + \delta_\alpha^\mu \bar{e}_2 \left( \frac{p_\alpha n^\alpha}{\Lambda}, \frac{p^2}{\Lambda^2} \right) + n^\mu p_\alpha \bar{e}_3 \left( \frac{p_\alpha n^\alpha}{\Lambda}, \frac{p^2}{\Lambda^2} \right)
\]

\[+ \frac{p^\mu p_\alpha}{\Lambda^2} \bar{e}_4 \left( \frac{p_\alpha n^\alpha}{\Lambda}, \frac{p^2}{\Lambda^2} \right) + n^\mu n_\alpha \bar{e}_5 \left( \frac{p_\alpha n^\alpha}{\Lambda}, \frac{p^2}{\Lambda^2} \right),
\]

(53)

Since in the limit \( \Lambda \to \infty \) we want to recover the canonical tetrad in SR \( (e^{\mu} a \to \delta_\alpha^\mu) \), the coefficients \( \bar{e}_i \) will satisfy conditions similar to those in (52). Obviously, relationships between the coefficients \( \bar{f}_i \) and \( \bar{e}_i \) must exist if Eq. (5) is to be satisfied. Analogously, the contraction between the tetrad and \( f^\mu \) may be written as

\[
e^{\mu} a(p) f_\mu(p) = p_\alpha \bar{a}_1 \left( \frac{p_\alpha n^\alpha}{\Lambda}, \frac{p^2}{\Lambda^2} \right) + n_\alpha \Lambda \bar{a}_2 \left( \frac{p_\alpha n^\alpha}{\Lambda}, \frac{p^2}{\Lambda^2} \right),
\]

(54)

where \( \bar{a}_1 \) and \( \bar{a}_2 \) are simple functions of \( \bar{f}_i \) and \( \bar{e}_i \).

From Eq. (54) we can directly see that the condition imposed by Eq. (49) is satisfied. Additionally, for Eq. (50) to be accomplished, we need to replace \( \Lambda \to -\Lambda \) under the action of \( C \). This can be understood in the following way: taking into account that when applying the symmetry \( C \) we are changing the sign of the energy, it is quite natural to think that a cutoff scale should change in the same way. Given that such a role is played by the deformation parameter \( \Lambda \), we should also do the replacement \( \Lambda \to -\Lambda \). In doing so, it is important to mention that, since the momentum scalar of curvature is proportional to \( \Lambda^{-2} \), the change \( \Lambda \to -\Lambda \) corresponds to an automorphism in the considered space (de Sitter).

This is not the first time that symmetries involving an action on the deformation parameters have been employed. Indeed, there are several examples in the quantum groups’ literature proposing that, by introducing a change of sign in the deformation parameter, one can obtain automorphisms of the Hopf algebra \([98, 99]\). This has also been considered in the context of QFT in DSR, as done in \([74]\) by introducing a pair of scalar fields (linked through the \( \Lambda \to -\Lambda \) replacement) and more recently in \([35]\).

IV. CONCLUSIONS

We have interpreted the relativistic quantum equations, namely the Klein–Gordon and Dirac equations, in a geometric language of curved momentum space, following the ideas of \([58, 60]\). Previous works on the topics were mainly restricted to an algebraic approach. In our discussion we have found several features that are worth comment.

First of all, our geometric derivations are shown to be equivalent to the previous algebraic ones \([54, 39]\), in the sense that we can rederive them with an appropriate choice of the Casimir. The fact that one can introduce a spinorial representation in two different ways, i.e. by modifying the coproduct of the quantum algebra \( SO_q(3, 2) \) to include a finite representation or, equivalently, by using the tetrad formalism, seems rather striking. In order to prove this, the correct identification of the functions \( f^\mu \) as generalized momenta turned out to be crucial.

Second, the selection of the Casimir as the distance in momentum space establishes a direct connection between the Dirac and Klein–Gordon operators, being one the “square” of the other. This is not the case in other trials, where a nontrivial function should be introduced to link them \([29]\). This is related to the fact that, in principle, one is entitled to choose as Casimir any function of the distance. Our choice can be thus seen as a minimal choice, a characteristic which has been already used as guiding principle several times in the past.

Third, we have also discussed the implementation of discrete symmetries in curved momentum space for the fermionic case. We have found that parity, time-reversal and charge conjugation (at the free level) are all preserved in our formalism. This was not the case in previous studies \([94]\), partially because the authors have not realized the need of trading the deformation parameter \( \Lambda \to -\Lambda \) under charge conjugation. Additionally, we have identified the conditions imposed on the vierbeins and the generalized momentum functions, cf. Eqs. (49) and (51), in order for the discrete symmetries to be valid. A deeper interpretation of these expressions and their role in more general geometries is still to be built. In any case, we can affirm that PCT is satisfied in this model (at the free level), in accordance with the fundamental role that it plays in standard QFT.

Fourth, we have made a first attempt into the identification of the relevant Hilbert space in a quantization process. Further developments in this direction are currently being pursued.

This work admits several generalizations. The most important deals with the incorporation of interactions. In this scenario one possibility would be to consider gauge theories, which were recently studied both in the quantum \( \kappa \)-Minkowski (including Becchi–Rouet–Stora–Tyutin symmetries) \([100, 101]\) and classical curved-momentum-space level
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Appendix A: The symmetric basis of $\kappa$-Poincaré

In [37, 39] the authors considered the symmetric basis of $\kappa$-Poincaré, where the deformed composition law reads

\[(p \oplus q)_0 = p_0 + q_0, \quad (p \oplus q)_i = p_i e^{\kappa_0/2\Lambda} + q_i e^{-\kappa_0/2\Lambda}.\]  

(A1)

As discussed in the introduction, in [58] it was realized that the kinematics of $\kappa$-Poincaré can be obtained from the geometrical ingredients of a curved momentum space. In particular, given a de Sitter metric, the isometries associated to quasi-translations can be identified with the deformed composition law $\oplus$, which should satisfy

\[g_{\mu\nu} (p \oplus q) = \frac{\partial (p \oplus q)_\mu}{\partial q_\nu} g_{\rho\sigma} (q) \frac{\partial (p \oplus q)_\nu}{\partial q_\rho}.\]  

(A2)

When taking the limit $q \to 0$ in the previous equation we obtain

\[g_{\mu\nu} (p) = \frac{\partial (p \oplus q)_\mu}{\partial q_\nu} \eta_{\rho\sigma} \frac{\partial (p \oplus q)_\nu}{\partial q_\rho} \bigg|_{q \to 0}.\]  

(A3)

Therefore, once the composition law is known, one can easily associate a tetrad to the momentum space metric, the inverse of the former given by [58]

\[e^a_\mu(p) = \frac{\partial (p \oplus q)_\mu}{\partial q_\alpha} \bigg|_{q \to 0}.\]  

(A4)

Hence, inserting Eq. (A1) into Eq. (A4), we find a tetrad for the momentum space of the symmetric basis of $\kappa$-Poincaré

\[e^0_0(p) = 1, \quad e^0_i(p) = 0, \quad e^i_0(p) = \frac{p_i}{2\Lambda}, \quad e^j_i(p) = \delta^j_i e^{-\kappa_0/2\Lambda},\]  

(A5)

where $i, j = 1, 2, 3$. From here one can easily obtain the associated metric in momentum space by using [21],

\[g_{00}(p) = 1, \quad g_{0i}(p) = g_{i0}(p) = \frac{p_i}{2\Lambda}, \quad g_{ij}(p) = -\delta^j_i e^{-\kappa_0/2\Lambda} + \frac{p_i p_j}{4\Lambda^2}.\]  

(A6)

Resorting to Eq. (69) one can find the Casimir defined as the squared of the distance to the origin in momentum space, which is

\[C_D^{(S)}(p) = \Lambda^2 \arccosh^2 \left( \cosh \left( \frac{p_0}{\Lambda} \right) - \frac{\hat{p}^2}{2\Lambda^2} \right).\]  

(A7)

The use of this Casimir and the tetrad (A5) in Eq. (19) leads to the following Dirac equation

\[\frac{\sqrt{C_D^{(S)}(p)}}{2\Lambda \sinh \sqrt{C_D^{(S)}(p)}} \left[ 2\Lambda e^{-\frac{\kappa_0}{2\Lambda}} \gamma^0 p_i + \gamma^0 \left( 2\Lambda^2 \sinh \left( \frac{p_0}{\Lambda} \right) - \hat{p}^2 \right) \right] \psi = m \psi.\]  

(A8)

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