BOUNDARY CONTROLLABILITY FOR A COUPLED SYSTEM
OF DEGENERATE/SINGULAR PARABOLIC EQUATIONS

BRAHIM ALLAL AND ABDELKARIM HAJJAJ
Hassan First University of Settat
Faculté des Sciences et Techniques, MISI Laboratory
B.P. 577, Settat 26000, Morocco

JAWAD SALHI
Moulay Ismail University of Meknes
FST Errachidia, MAIS Laboratory, MAMCS Group
P.O. Box 509, Boutalamine 52000, Errachidia, Morocco

AMINE SBAI*
Hassan First University of Settat
Faculté des Sciences et Techniques, MISI Laboratory
B.P. 577, Settat 26000, Morocco

(Communicated by Scott W. Hansen)

Abstract. In this paper we study the boundary controllability for a system of two coupled degenerate/singular parabolic equations with a control acting on only one equation. We analyze both approximate and null boundary controllability properties. Besides, we provide an estimate on the null-control cost. The proofs are based on a detailed spectral analysis and the use of the moment method by Fattorini and Russell together with some results on biorthogonal families.

1. Introduction. This work is devoted to the study of the boundary controllability properties of the following controlled system:

\[
\begin{align*}
& y_t - (x^\alpha y_x)_x - \frac{\mu}{x^2} y = Ay, & & (t, x) \in Q := (0, T) \times (0, 1), \\
& y(t, 1) = Bv, & & t \in (0, T), \\
& y(t, 0) = 0, & & t \in (0, T), \\
& y(0, x) = y_0(x), & & x \in (0, 1),
\end{align*}
\]

(1)

where $T > 0$ is some final time, $0 \leq \alpha < 1$, $\mu \leq \mu(\alpha) := \frac{(1-\alpha)^2}{4}$, $y_0 \in H^{-1, \mu(0, 1)} \alpha$ (that will be defined later in section 2), $y = (y_1, y_2)^*$ is the state variable and $v = v(t)$ is the control function which acts on the system by means of the Dirichlet boundary condition at the point $x = 1$. Moreover, $A \in \mathcal{L}(\mathbb{R}^2)$ and $B \in \mathbb{R}^2$ are, respectively, a suitable coupling matrix and a control operator, chosen so that:

\[
\text{rank}[B|AB] = 2.
\]

(2)

2020 Mathematics Subject Classification. 93B05, 93C05, 35K65, 93B60.

Key words and phrases. Boundary controllability, coupled systems, degenerate parabolic equations, singular potentials, moment method.

* Corresponding author: Amine Sbai.
Notice that, taking $P = [B|AB]$, by performing the change of variables $\tilde{y} = P^{-1}y$, one obtains the following reformulation of (1):

$$
\begin{cases}
\tilde{y}_t - (x^α\tilde{y}_x)_x - \frac{μ}{x^2}\tilde{y} = \tilde{A}\tilde{y}, & (t, x) \in (0, T) \times (0, 1), \\
\tilde{y}(t, 1) = \tilde{B}v, & t \in (0, T), \\
\tilde{y}(t, 0) = 0, & t \in (0, T), \\
\tilde{y}(0, x) = P^{-1}y_0(x), & x \in (0, 1),
\end{cases}
$$

where

$$
\tilde{A} = \begin{pmatrix} 0 & a_1 \\ 1 & a_2 \end{pmatrix} \quad \text{and} \quad \tilde{B} = e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.
$$

Therefore, for simplicity, it will be assumed in the rest of the paper that $A$ and $B$ are given by

$$
A = \begin{pmatrix} 0 & a_1 \\ 1 & a_2 \end{pmatrix} \quad \text{and} \quad B = e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.
$$

This means that we are exerting only one control force on the system but we want to control the corresponding state $y = (y_1, y_2)$ which has two components.

We are more precisely interested in the study of system (1) under the assumption that the coupling matrix $A$ has two distinct eigenvalues. However, we emphasize that our controllability results can be also obtained in the case of a coupling matrix admitting a double eigenvalue by combining our proofs with the ideas of [6].

The starting point of the present work is the result established in [23] for the boundary controllability properties of the (uniformly) parabolic system:

$$
\begin{cases}
y_t - y_{xx} = Ay, & (t, x) \in Q, \\
y(t, 0) = Bv, & t \in (0, T), \\
y(t, 1) = 0, & t \in (0, T), \\
y(0, x) = y_0(x), & x \in (0, 1),
\end{cases}
$$

with $A$ and $B$ as defined previously. In fact, as shown in [23], there exists two different situations:

1. If the matrix $A$ in (5) has one double real eigenvalue or a couple of conjugate complex eigenvalues, (2) is a necessary and sufficient condition for the null controllability at any time.

2. If $A$ has two different real eigenvalues, an additional condition is needed for null controllability, independently of the considered vector $B$.

Since then, the boundary controllability results for various parabolic problems was investigated (see [4, 5, 6, 8, 7, 18, 26]).

However, all the previous cited papers deal with uniformly parabolic problems without degeneracies or singularities. In recent years, controllability issues for degenerate and/or singular parabolic problems by means of a locally distributed control have been investigated in many papers, see [11, 14, 20, 24, 25, 36, 37]. For related systems of coupled degenerate/singular parabolic equations we refer to [2, 28, 35].

To our best knowledge, the known boundary controllability results of degenerate and/or singular parabolic problems are in the scalar case (see for example [9, 10, 12, 13, 27, 34]).

The aim of this research is to establish general results in the case of coupled degenerate/singular parabolic equations in the spirit of [3]. To prove our results we
will use the moment method by Fattorini and Russell [21, 22], requiring a thorough spectral analysis of the model under consideration. At first, we will see that for every \( v \in L^2(0,T) \) and \( y_0 \in H^{-1,\mu}_\alpha(0,1)^2 \), system (1) admits a unique weak solution defined by transposition that satisfies
\[
y \in L^2(Q)^2 \cap C([0,T],H^{-1,\mu}_\alpha(0,1)^2).
\]
Observe that the previous regularity permits to pose the boundary controllability of the degenerate singular system (1) in the space \( H^{-1,\mu}_\alpha(0,1)^2 \).

Then, we pass to analyse both approximate and null controllability issues using a boundary control acting at \( x = 1 \). We emphasize that imposing a control that acts at the nonsingular point does not imply a simple adaptation of the previous distributed controllability results. For example, at a first glance, one may think that our boundary controllability results can be obtained directly by standard extension and localization arguments from the corresponding distributed controllability results as in the case of scalar parabolic equations. But this is not the case and the situation is quite different for non-scalar parabolic systems. Indeed, while the Kalman’s rank condition (2) is a necessary and sufficient condition for the null controllability at any time in the distributed case, it was proved in [23] that it is necessary, but not sufficient, for the boundary controllability for coupled parabolic systems.

To state the main results of this article, we recall the following notions:

**Definition 1.1.**

1. It will be said that system (1) is approximately controllable in \( H^{-1,\mu}_\alpha(0,1)^2 \) at time \( T > 0 \) if for every \( y_0, y_d \in H^{-1,\mu}_\alpha(0,1)^2 \) and any \( \varepsilon > 0 \), there exists a control function \( v \in L^2(0,T) \) such that the solution \( y \) to system (1) satisfies
\[
\|y(T,\cdot) - y_d\|_{H^{-1,\mu}_\alpha(0,1)^2} \leq \varepsilon.
\]

2. It will be said that system (1) is null controllable at time \( T > 0 \) if for every \( y_0 \in H^{-1,\mu}_\alpha(0,1)^2 \), there exists a control \( v \in L^2(0,T) \) such that the solution \( y \) to system (1) satisfies
\[
y(T,\cdot) = 0, \quad \text{in} \quad H^{-1,\mu}_\alpha(0,1)^2.
\]

Our first result for the boundary controllability of system (1) concerns the approximate controllability which is proved under the rank condition (2) together with a condition (see (34)) that is equivalent to the simplicity of the spectrum associated with system (1). We refer to Theorem 4.1 for a precise statement of this result.

In this work, we will also prove the boundary null controllability of (1) assuming the previous conditions which characterize the approximate controllability property (see Theorem 5.1 for a rigorous statement).

At this point, let us mention that the approximate controllability result can be derived as a by-product of the null controllability one. Here, we will provide a direct proof, which is interesting in itself and potentially useful in other situations such as pointwise control problem where the approximate controllability holds at any time \( T > 0 \), whereas a minimal time of control appears for the null controllability result.

Throughout this paper, we shall use the following notations. Given \( z \in \mathbb{C} \), \( \Re(z) \) and \( \Im(z) \) denote the real and imaginary parts of \( z \).

In order to get the approximate controllability result of (1), we will need the following known result on the existence of a biorthogonal family to the exponentials with suitable bounds (see [6] or [23]).
Theorem 1.2. Let $T > 0$. Suppose that \( \{A_n\}_{n \geq 1} \) is a sequence of complex numbers such that, for some $\delta, \rho > 0$, one has

\[
\begin{align*}
\Re(A_n) &\geq \delta |A_n|, \\
|A_n - A_m| &\geq \rho |n - m|, \\
\sum_{n \geq 1} \frac{1}{|A_n|} &< +\infty.
\end{align*}
\]

(6)

Then, there exists a family \( \{q_n\}_{n \geq 1} \subset L^2(0, T) \) biorthogonal to \( \{e^{-\Lambda_n t}\}_{n \geq 1} \) i.e., a family \( \{q_n\}_{n \geq 1} \) in \( L^2(0, T) \) such that

\[
\int_0^T q_n(t)e^{-\Lambda_n t} \, dt = \delta_{nm}, \quad \forall n, m \geq 1.
\]

Moreover, for every $\varepsilon > 0$, there exists $C_\varepsilon > 0$ for which

\[
\|q_n\|_{L^2(0, T)} \leq C_\varepsilon e^{\varepsilon \Re(A_n)}, \quad \forall n \geq 1.
\]

It is worth mentioning that the above Theorem can also be applied to get the null controllability result for the system (1). However, it does not permit to deduce the required estimate on the null-control cost. For this reason, to obtain the null controllability result together with an estimate of the control cost, we are going to apply the next result provided in [7].

Theorem 1.3. Let \( \{A_n\}_{n \geq 1} \) be a sequence of complex numbers fulfilling the following assumptions:

1. $A_n \neq A_m$ for all $n, m \geq 1$ with $n \neq m$;
2. $\Re(A_n) > 0$ for every $n \geq 1$;
3. for some $\delta > 0$

\[
|\Im(A_n)| \leq \delta \sqrt{\Re(A_n)} \quad \forall n \geq 1;
\]
4. \( \{A_n\}_{n \geq 1} \) is nondecreasing in modulus,

\[
|A_n| \leq |A_{n+1}| \quad \forall n \geq 1;
\]
5. \( \{A_n\}_{n \geq 1} \) satisfies the following gap condition: for some $\varrho, \varrho > 0$,

\[
\left\{ \begin{array}{l}
|A_n - A_m| \geq \varrho |n^2 - m^2| \quad \forall n, m : |n - m| \geq q, \\
\inf_{n \neq m, |n-m|<q} |A_n - A_m| > 0;
\end{array} \right.
\]

(7)

6. for some $p, s > 0$,

\[
|p\sqrt{r} - N(r)| \leq s, \quad \forall r > 0,
\]

(8)

where \( N \) is the counting function associated with the sequence \( \{A_n\}_{n \geq 1} \), that is the function defined by

\[
N(r) = \# \{ n : |A_n| \leq r \}, \quad \forall r > 0.
\]

Then, there exists $T_0 > 0$, such that for any $T \in (0, T_0)$, we can find a family \( \{q_n\}_{n \geq 1} \subset L^2(-T/2, T/2) \) biorthogonal to \( \{e^{-\Lambda_n t}\}_{n \geq 1} \) i.e., a family \( \{q_n\}_{n \geq 1} \) in \( L^2(-T/2, T/2) \) such that

\[
\int_{-T/2}^{T/2} q_n(t)e^{-\Lambda_n t} \, dt = \delta_{nm}.
\]

Moreover, there exists a positive constant $C > 0$ independent of $T$ for which

\[
\|q_n\|_{L^2(-T/2, T/2)} \leq C e^{C \sqrt{\Re(A_n)} + \frac{\varrho}{2}}, \quad \forall n \geq 1.
\]

(9)
The rest of the paper is organized as follows. In Section 2, we prove the well-posedness of the problem (1) in appropriate weighted spaces using the transposition method and recall some characterizations of the controllability. In section 3, we discuss the spectral analysis related to scalar singular operators and present a description of the spectrum associated with system (1) which will be useful for developing the moment method. Section 4 is devoted to studying the boundary approximate controllability problem for the system (1). Finally, in section 5, we prove the boundary null controllability result and establish an estimate of the control cost.

2. Preliminary results.

2.1. Functional framework. In the study of degenerate/singular problems, it is by now classical that of great importance is the following generalized Hardy inequality (see, for example, [36] or [17, Lemma 5.3.1]): for all $\alpha \in [0, 2)$,

$$
\frac{(1 - \alpha)^2}{4} \int_0^1 x^{\alpha - 2} z^2 \, dx \leq \int_0^1 x^{\alpha} z_x^2 \, dx, \quad \forall z \in C_c^\infty(0, 1).
$$

(10)

For any $\mu \leq \mu(\alpha) = \frac{(1 - \alpha)^2}{4}$, we introduce the functional space associated to degenerate/singular problems:

$$
H^{1,\mu}_\alpha(0, 1) := \left\{ z \in L^2(0, 1) \cap H^1_{loc}((0, 1)) \mid \int_0^1 (x^{\alpha} z_x^2 - \frac{\mu}{x^{2-\alpha}} z^2) \, dx < +\infty \right\}
$$

$$
H^{1,\mu}_{\alpha,0}(0, 1) := \left\{ z \in H^{1,\mu}_\alpha(0, 1) \mid z(0) = z(1) = 0 \right\}.
$$

Further, we define $H^{-1,\mu}_{\alpha,0}(0, 1)$ the dual space of $H^{1,\mu}_{\alpha,0}(0, 1)$ with respect to the pivot space $L^2(0, 1)$, endowed with the natural norm

$$
\|f\|_{H^{-1,\mu}_{\alpha,0}(0, 1)} := \sup_{\|g\|_{H^{1,\mu}_{\alpha,0}(0, 1)} = 1} \langle f, g \rangle_{H^{1,\mu}_{\alpha,0}(0, 1), H^{-1,\mu}_{\alpha,0}(0, 1)}.
$$

We also define

$$
H^{1,\mu}_{\alpha,0}(0, 1) := \left\{ z \in H^{1,\mu}_\alpha(0, 1) \cap H^2_{loc}((0, 1)) \mid (x^{\alpha} z_x)_x + \frac{\mu}{x^{2-\alpha}} z \in L^2(0, 1) \right\}.
$$

Notice besides that, as $C_c^\infty(0, 1)$ is dense both in $L^2(0, 1)$ and in $H^{1,\mu}_{\alpha,0}(0, 1)$, $H^{1,\mu}_{\alpha,0}(0, 1)$ is dense in $H^{2}_{loc}(0, 1)$.

In what follows, for simplicity, we will always denote by $\langle \cdot, \cdot \rangle$ the standard scalar product of either $L^2(0, 1)$ or $L^2(0, 1)^2$, by $\langle \cdot, \cdot \rangle_{X',X}$ the duality pairing between the Hilbert space $X$ and its dual $X'$. On the other hand, we will use $\| \cdot \|_\alpha$ (resp. $\| \cdot \|^{-1,\mu}_\alpha$) for denoting the norm of $H^{1,\mu}_{\alpha,0}(0, 1)^2$ (resp. $H^{-1,\mu}_{\alpha,0}(0, 1)^2$).

2.2. Well-posedness. Now, we are ready to give some results related to the existence, uniqueness and continuous dependence with respect to the data of the degenerate/singular problem (1). To this aim, let us consider the following nonhomogeneous adjoint problem:

$$
\begin{cases}
-\varphi_t - (x^{\alpha} \varphi_x)_x - \frac{\mu}{x^{2-\alpha}} \varphi = A^* \varphi + g, & \text{in } Q, \\
\varphi(t, 0) = \varphi(t, 1) = 0, & \text{in } (0, T), \\
\varphi(T, x) = \varphi_0, & \text{in } (0, 1),
\end{cases}
$$

(11)

where $A$ is given in (4) and $\varphi_0$ and $g$ are functions in appropriate spaces.

Let us start with a first result on existence and uniqueness of strict solutions to system (11). One has (see [1] or [10, Definition 4.1]):
Proposition 1. Assume that \( \varphi_0 \in H^{1,\mu}_{\alpha,0}(0,1)^2 \) and \( g \in L^2(Q)^2 \). Then, system (11) admits a unique solution

\[
\varphi \in W := C^0([0,T]; H^{1,\mu}_{\alpha,0}(0,1)^2) \cap H^1(0,T; L^2(0,1)^2) \\
\cap L^2(0,T; H^{1,\mu}_{\alpha,0}(0,1)^2) \cap H^{1,\mu}_{\alpha,0}(0,1)^2
\]

such that

\[
\|\varphi\|_{C^0([0,T]; H^{1,\mu}_{\alpha,0}(0,1)^2)} + \|\varphi\|_{H^1(0,T; L^2(0,1)^2)} + \|\varphi\|_{L^2(0,T; H^{1,\mu}_{\alpha,0}(0,1)^2)} \\
\leq C\left(\|\varphi_0\|_{\alpha} + \|g\|_{L^2(Q)^2}\right),
\]

for some positive constant \( C \).

In view of Proposition 1, the following definition makes sense:

Definition 2.1. Let \( y_0 \in H^{-1,\mu}_{\alpha}(0,1)^2 \) and \( v \in L^2(0,T) \) be given. It will be said that \( y \in L^2(Q)^2 \) is a solution by transposition to (1) if, for each \( g \in L^2(Q)^2 \), the following identity holds

\[
\int_Q y \cdot g \, dx \, dt = \langle y_0, \varphi(0,\cdot) \rangle_{H^{-1,\mu}_{\alpha,0}, H^{1,\mu}_{\alpha,0}} - \int_0^T B^* \varphi_x(t,1) v(t) \, dt,
\]

where \( \varphi \in W \) is the solution of (11) associated to \( g \) and \( \varphi_0 = 0 \).

With this definition we can state the result of existence and uniqueness of solution by transposition to system (1).

Proposition 2. Assume that \( y_0 \in H^{-1,\mu}_{\alpha}(0,1)^2 \) and \( v \in L^2(0,T) \). Then, system (1) admits a unique solution by transposition \( y \) that satisfies

\[
\begin{cases}
  y \in L^2(Q)^2 \cap C^0([0,T], H^{-1,\mu}_{\alpha}(0,1)^2), \\
  y_t \in L^2(0,T; (H^{2,\mu}_{\alpha}(0,1)^2 \cap H^{1,\mu}_{\alpha,0}(0,1)^2)',) \\
  y_t - (x^n y_x)_x - \frac{\mu}{x^n} y = Ay \quad \text{in} \quad L^2(0,T; (H^{2,\mu}_{\alpha}(0,1)^2 \cap H^{1,\mu}_{\alpha,0}(0,1)^2)'), \\
  y(0,\cdot) = y_0 \quad \text{in} \quad H^{-1,\mu}_{\alpha}(0,1)^2
\end{cases}
\]

and

\[
\|y\|_{L^2(Q)^2} + \|y\|_{C^0(H^{-1,\mu}_{\alpha})} + \|y_t\|_{L^2((H^{2,\mu}_{\alpha}(0,1)^2 \cap H^{1,\mu}_{\alpha,0}(0,1)^2)'),} \\
\leq C\left(\|v\|_{L^2(0,T)} + \|y_0\|_{\alpha}^{-1,\mu}\right),
\]

for a constant \( C = C(T) > 0 \).

Proof. Let \( y_0 \in H^{-1,\mu}_{\alpha}(0,1)^2 \), \( v \in L^2(0,T) \) and consider the following functional \( T : L^2(Q)^2 \to \mathbb{R} \) given by

\[
T(g) = \langle y_0, \varphi(0,\cdot) \rangle_{H^{-1,\mu}_{\alpha,0}, H^{1,\mu}_{\alpha,0}} - \int_0^T B^* \varphi_x(t,1) v(t) \, dt,
\]

where \( \varphi \in W \) is the solution of the adjoint system (11) associated to \( g \in L^2(Q)^2 \) and \( \varphi_0 = 0 \). From (12), we can deduce the existence of a positive constant \( C \) such that

\[
|T(g)| \leq C\left(\|v\|_{L^2(0,T)} + \|y_0\|_{\alpha}^{-1,\mu}\right)\|g\|_{L^2(Q)^2},
\]

for all \( g \in L^2(Q)^2 \). We infer that \( T \) is bounded. Hence, by Riesz-Fréchet representation theorem, there exists a unique \( y \in L^2(Q)^2 \) satisfying (13), i.e., a solution by transposition of (1) in the sense of Definition 2.1. It is also clear that this solution satisfies the equality \( y_t - (x^n y_x)_x - \frac{\mu}{x^n} y = Ay \) in \( \mathcal{D}'(Q)^2 \) and the estimate

\[
\|y\|_{L^2(Q)^2} = \|T\| \leq C\left(\|v\|_{L^2(0,T)} + \|y_0\|_{\alpha}^{-1,\mu}\right).
\]
Next, we are going to prove that the solution $y$ of system (1) is more regular. To be precise, let us show that $(x^\alpha y_x)_x + \frac{\mu}{x^{2-\alpha}} y \in L^2((0,T); (H^{1,\mu}_{\alpha,0}(0,1))^2 \cap H^{1,\mu}_{\alpha,0}(0,1))^2)$ and
\[ \|(x^\alpha y_x)_x + \frac{\mu}{x^{2-\alpha}} y\|_{L^2} \leq C \left( \|v\|_{L^2} + \|y_0\|_{\alpha,1} \right). \] (16)

To this end, let us consider two sequences $\{y^m_0\}_{m \geq 1} \subset H^{1,\mu}_{\alpha,0}(0,1)^2$ and $\{v^m\}_{m \geq 1} \subset H^{1,\mu}_{\alpha,0}(0,1, T)$ such that
\[ y^m_0 \to y_0 \text{ in } H^{-1,\mu}(0,1)^2 \text{ and } v^m \to v \text{ in } L^2(0,T). \]

Now, the strategy consists in transforming our original system (1) (as done for instance in [10] in the context of a scalar degenerate/singular parabolic equation) into a problem with homogeneous boundary conditions and a source term. To this end, let us introduce the following function:
\[ \forall x \in [0,1], \quad p(x) := x^{\alpha/2} \quad \text{where} \quad q^\alpha := \frac{1 - \alpha}{2} + \frac{\sqrt{\mu(\alpha)} - \mu}{2}. \]

Formally, if $y^m$ is the solution of (1) associated to $y^m_0$ and $v^m$, then the function defined by
\[ \tilde{y}^m(t,x) = y^m(t,x) - Bp(x)v^m(t), \]
is solution of
\[ \begin{cases}
\tilde{y}^m = A\tilde{y}^m + f^m(t,x), & (t,x) \in Q, \\
\tilde{y}^m(0,x) = \tilde{y}^m_0(x), & x \in (0,1), \\
\end{cases} \] (17)
where $f^m(t,x) = p(x)v^m(t)AB - p(x)v^m(t)B \in L^2(Q)^2$. With the previous regularity assumptions on the data, we can apply Proposition 1, to deduce that system (17) has a unique strict solution
\[ \tilde{y}^m \in C^0([0,T]; H^{1,\mu}_{\alpha,0}(0,1)^2) \cap H^1(0,T; L^2(0,1)^2) \cap L^2(0,T; H^{2,\mu}_{\alpha,0}(0,1)^2). \]

By setting
\[ \tilde{v}^m(t,x) := Bp(x)v^m(t), \]
we observe that $\tilde{v}^m$ satisfies
\[ \tilde{v}^m \in C^0([0,T]; H^{1,\mu}_{\alpha,0}(0,1)^2) \cap H^1(0,T; L^2(0,1)^2) \cap L^2(0,T; H^{2,\mu}_{\alpha,0}(0,1)^2). \]

Therefore, the problem (1) for $v^m$ and $y^m_0$ has a unique solution
\[ y^m \in C^0([0,T]; H^{1,\mu}_{\alpha,0}(0,1)^2) \cap H^1(0,T; L^2(0,1)^2) \cap L^2(0,T; H^{2,\mu}_{\alpha,0}(0,1)^2) \]
which satisfies
\[ \iint_Q y^m \cdot \varphi \, dx \, dt = \{y^m_0, \varphi(0,x)\}_{H^{1,\mu}_{\alpha,0}} + \int_0^T B^* \varphi(t,1)v^m(t) \, dt, \quad \forall m \geq 1, \]
for all $g \in L^2(Q)$, where $\varphi$ is the solution of the system (11) associated to $g$ and $\varphi_0 = 0$. The previous identity and (13) also provide:
\[ \begin{cases}
\|y^m\|_{L^2(Q)^2} \leq C \left( \|v\|_{L^2} + \|y_0\|_{\alpha,1,1} \right) \\
y^m \to y \text{ in } L^2(Q)^2 \text{ and } (x^\alpha y^m_x)_x + \frac{\mu}{x^{2-\alpha}} y^m \to (x^\alpha y_x)_x + \frac{\mu}{x^{2-\alpha}} y \in D'(Q)^2. \end{cases} \] (18)
On the other hand, integrations by parts that may be justified as explained in a detailed manner in [15], lead to
\[
\int_Q \left( (x^\alpha y^m)_x + \frac{\mu}{x^{2-\alpha}} y^m \right) \cdot \psi \, dt \\
= \int_Q y^m \cdot \left( (x^\alpha \psi)_x + \frac{\mu}{x^{2-\alpha}} \psi \right) \, dx - \int_0^T B^* \psi_x(t,1) v^m(t) \, dt,
\]
for every \( \psi \in L^2(0,T; H^{2,\mu}_\alpha(0,1)^2 \cap H^{1,\mu}_{\alpha,0}(0,1)^2) \). From this equality we deduce that the sequence \( \{(x^\alpha y^m)_x + \frac{\mu}{x^{2-\alpha}} y^m \}_{m \geq 1} \) is bounded in \( L^2(0,T; (H^{2,\mu}_\alpha(0,1)^2 \cap H^{1,\mu}_{\alpha,0}(0,1)^2)^\prime) \). This property together with (18) implies that \( (x^\alpha y_x)_x + \frac{\mu}{x^{2-\alpha}} y \in L^2(0,T; (H^{2,\mu}_\alpha(0,1)^2 \cap H^{1,\mu}_{\alpha,0}(0,1)^2)^\prime) \) and satisfies the estimate (16).

Combining the identity \( y = (x^\alpha y_x)_x + \frac{\mu}{x^{2-\alpha}} y + A y \) and the regularity property for \( (x^\alpha y_x)_x + \frac{\mu}{x^{2-\alpha}} y \), we also see that \( y \in L^2(0,T; (H^{2,\mu}_\alpha(0,1)^2 \cap H^{1,\mu}_{\alpha,0}(0,1)^2)^\prime) \) and
\[
\| y_t \|_{L^2((H^{2,\mu}_\alpha(0,1)^2 \cap H^{1,\mu}_{\alpha,0}(0,1)^2)^\prime)} \leq C (\| v \|_{L^2(0,T)} + \| y_0 \|_{H^{-1,\mu}_\alpha} ),
\]
for some positive constant \( C \). Therefore \( y \in C([0,T]; X^2) \), where \( X \) is the interpolation space \( X = [L^2(0,1), (H^{2,\mu}_\alpha(0,1)^2 \cap H^{1,\mu}_{\alpha,0}(0,1)^2)]_{1/2} = H^{-1,\mu}_\alpha(0,1) \) (see [32, Proposition 2.1, p. 22]). In conclusion, we get
\[
\| y \|_{C(H^{-1,\mu}_\alpha)} \leq C (\| v \|_{L^2(0,T)} + \| y_0 \|_{H^{-1,\mu}_\alpha}).
\]
Finally, one can easily check that \( y(0,\cdot) = y_0 \in H^{-1,\mu}_\alpha(0,1)^2 \). This ends the proof.

\[\square\]

2.3. Duality. Let us consider the adjoint system of (1) given by:
\[
\begin{cases}
-\varphi_t - (x^\alpha \varphi)_x - \frac{\mu}{x^{2-\alpha}} \varphi = A^* \varphi, & \text{in } Q, \\
\varphi(t,0) = \varphi(t,1) = 0, & t \in (0,T), \\
\varphi(T,x) = \varphi_0, & \text{in } (0,1),
\end{cases}
\]
where \( \varphi_0 \in H^{1,\mu}_{\alpha,0}(0,1)^2 \). In the sequel, the solution to (19) will be called the adjoint state associated to \( \varphi_0 \). The controllability of system (1) can be characterized in terms of appropriate properties of the solutions to (19). In order to provide these characterizations, we use the following result which relates the solutions of systems (1) and (19). One has:

**Proposition 3.** Let \( y_0 \in H^{-1,\mu}_\alpha(0,1)^2, v \in L^2(0,T) \) and \( \varphi_0 \in H^{1,\mu}_{\alpha,0}(0,1)^2 \) be given. Let \( y \) be the state associated to \( y_0 \) and \( v \) and let \( \varphi \) be the adjoint state associated to \( \varphi_0 \). Then:
\[
\int_0^T B^* (x^\alpha \varphi_x)(t,1) v(t) \, dt = \langle y_0, \varphi(0,\cdot) \rangle_{H^{1,\mu}_{\alpha,0} \cdot H^{-1,\mu}_\alpha} - \langle y(T), \varphi_0 \rangle_{H^{-1,\mu}_\alpha \cdot H^{1,\mu}_{\alpha,0}}.
\]

This result is a straightforward consequence of the properties of \( y \) stated in Proposition 2.

One important result that will be useful for treating the approximate controllability of the system (1) is the following characterization in terms of the unique continuation property for the corresponding adjoint system (19). More precisely, we have:
Theorem 2.2. Let us consider $T > 0$. Then, system (1) is approximately controllable at time $T$ if and only if for all initial condition $\varphi_0 \in H^1(0, 1)^2$ the solution to system (19) satisfies the unique continuation property
\[ B^*(x^a \varphi_x)(\cdot, 1) = 0 \quad \text{on} \quad (0, T) \Rightarrow \varphi_0 = 0 \quad \text{in} \quad (0, 1) \quad \text{(i.e.,} \quad \varphi = 0 \quad \text{in} \quad Q) . \]

This result is well known. For a proof see, for instance [23], [16] and [39].

3. Spectral analysis. In order to transform the question of null controllability into a moment problem, we need to study the eigenvalue problem of the degenerate/singular operator associated to system (1). To this end, we first recall the spectral properties of scalar degenerate/singular operators.

3.1. Scalar degenerate/singular operators. In this section, we discuss some preliminary results related to the eigenvalue problem associated to the degenerate/singular operator $y \mapsto -(x^a y_x) - \frac{\mu}{x^2} y$, i.e., the nontrivial solutions $(\lambda, \Phi)$ of
\[
\begin{aligned}
\left\{ \begin{array}{l}
-(x^a \Phi')'(x) - \frac{\mu}{x^2} \Phi(x) = \lambda \Phi(x), \quad x \in (0, 1), \\
\Phi(0) = \Phi(1) = 0,
\end{array} \right.
\end{aligned}
\]
that will be essential for our purposes. For this reason, we first recall some results concerning the Bessel functions that will be useful in the rest of the paper (see [38], for more details).

For a real number $\nu \in \mathbb{R}_+$, we denote by $J_\nu$ the Bessel function of the first kind of order $\nu$ defined by the following Taylor series expansion around $x = 0$:
\[ J_\nu(x) = \sum_{m \geq 0} \frac{(-1)^m}{m! \Gamma(1 + \nu + m)} \left( \frac{x}{2} \right)^{2m+\nu} , \]
where $\Gamma(.)$ is the Gamma function.

We recall that the Bessel function $J_\nu$ satisfies the following differential equation
\[ x^2 y''(x) + xy'(x) + (x^2 - \nu^2)y(x) = 0, \quad x \in (0, +\infty) . \]
Moreover, the function $J_\nu$ has an infinite number of real zeros which are simple with the possible exception of $x = 0$ (see [31, 19]). We denote by $(j_{\nu,n})_{n \geq 1}$ the strictly increasing sequence of the positive zeros of $J_\nu$:
\[ 0 < j_{\nu,1} < j_{\nu,2} < \cdots < j_{\nu,n} < \cdots \]
and we recall that
\[ j_{\nu,n} \to +\infty \quad \text{as} \quad n \to +\infty \]
and the following bounds on the zeros $j_{\nu,n}$, which are provided in [33]:
\[ \forall \nu \in \left[ 0, \frac{1}{2} \right] , \forall n \geq 1 , \quad (n + \nu - \frac{1}{4})\pi \leq j_{\nu,n} \leq (n + \nu - \frac{1}{8})\pi . \quad (22) \]
\[ \forall \nu \geq \frac{1}{2} , \forall n \geq 1 , \quad (n + \nu - \frac{1}{4})\pi \leq j_{\nu,n} \leq (n + \nu - \frac{1}{8})\pi . \quad (23) \]

In our analysis, we will need the following classical result (see [30, Proposition 7.8]):

Lemma 3.1. Let $j_{\nu,n}, n \geq 1$ be the positive zeros of the Bessel function $J_\nu$. Then, the following hold:
Lemma 3.2. Let (21) be the associated normalized (in $\delta$, where, 1. Let $L$.

Proof. 1. For all $n, m \in \mathbb{N}^*$, we have the following result on the expression of the eigenvalues and eigenfunctions related to problem (21) that have been computed in [10]:

$$\lambda_{\alpha, \mu, n} = \left(\frac{2-\alpha}{2}\right)^2 (\nu_{\alpha, \mu, n})^2 \quad \forall n \geq 1,$$  \hspace{1cm} (24)

and the associated normalized (in $L^2(0, 1)$) eigenfunctions are

$$\Phi_{\alpha, \mu, n}(x) = \frac{\sqrt{2-\alpha}}{|J_{\nu_{\alpha, \mu, n}}(J_{\nu_{\alpha, \mu, n}})|^{\frac{1-n}{2}}} J_{\nu_{\alpha, \mu, n}} \left(J_{\nu_{\alpha, \mu, n}} x \frac{2\alpha}{\pi}\right), \quad n \geq 1.$$  \hspace{1cm} (25)

Moreover, the family $(\Phi_{\alpha, \mu, n})_{n \geq 1}$ forms an orthonormal basis of $L^2(0, 1)$.

Now, we pass to prove the following result which will be used later.

Lemma 3.2. Let $(\lambda_{\alpha, \mu, k})_{k \geq 1}$ be the sequence of eigenvalues of the spectral problem (21). Then, the following properties hold:

1. For all $n, m \in \mathbb{N}^*$, there is a constant $\rho > 0$ such that $(\lambda_{\alpha, \mu, k})_{k \geq 1}$ satisfies the following gap condition: there is a constant $\rho > 0$ such that

$$|\lambda_{\alpha, \mu, n} - \lambda_{\alpha, \mu, m}| \geq \rho |n^2 - m^2|, \quad \forall n, m \geq 1.$$  \hspace{1cm} (26)

2. The series $\sum_{n \geq 1} \frac{1}{\lambda_{\alpha, \mu, n}}$ is convergent.

Proof. 1. Let $n, m \in \mathbb{N}^*$ with $n \geq m$. We have

$$\lambda_{\alpha, \mu, n} - \lambda_{\alpha, \mu, m} = \left(\frac{2-\alpha}{2}\right)^2 (\nu_{\alpha, \mu, n} - \nu_{\alpha, \mu, m})$$

$$= \left(\frac{2-\alpha}{2}\right)^2 (\nu_{\alpha, \mu, n} - \nu_{\alpha, \mu, m})(\nu_{\alpha, \mu, n} + \nu_{\alpha, \mu, m})$$

$$= \left(\frac{2-\alpha}{2}\right)^2 \left((\nu_{\alpha, \mu, n} - \nu_{\alpha, \mu, m})(\nu_{\alpha, \mu, n+1} - \nu_{\alpha, \mu, m}) + \cdots + (\nu_{\alpha, \mu, m+1} - \nu_{\alpha, \mu, m})(\nu_{\alpha, \mu, m}-\nu_{\alpha, \mu, m})\right)$$

$$\times (\nu_{\alpha, \mu, n} + \nu_{\alpha, \mu, m}).$$

We can now distinguish the two different cases $\nu(\alpha, \mu) \in \left[0, \frac{1}{2}\right]$ and $\nu(\alpha, \mu) \geq \frac{1}{2}$ depending on the parameter $\mu$. 

2. The series $\sum_{n \geq 1} \frac{1}{\lambda_{\alpha, \mu, n}}$ is convergent.
• if $\nu(\alpha, \mu) \in \left[0, \frac{1}{2}\right]$ (i.e. $\mu \in \left(\frac{\alpha}{16}(3\alpha - 4), \mu(\alpha)\right)$), by virtue of Lemma 3.1 we immediately have that
  \[ j_{\nu(\alpha, \mu), n} - j_{\nu(\alpha, \mu), n-1} \geq j_{\nu(\alpha, \mu), 2} - j_{\nu(\alpha, \mu), 1}, \quad \forall n \geq 2. \]
  Therefore,
  \[ \lambda_{\alpha, \mu, n} - \lambda_{\alpha, \mu, m} \geq (n - m)(j_{\nu(\alpha, \mu), 2} - j_{\nu(\alpha, \mu), 1})(j_{\nu(\alpha, \mu), n} + j_{\nu(\alpha, \mu), m}). \]
  By using (22), the last inequality becomes:
  \[ \lambda_{\alpha, \mu, n} - \lambda_{\alpha, \mu, m} \geq \frac{7}{8}\pi^2 \left(\frac{2 - \alpha}{2}\right)^2 (n - m)(n + m + \nu(\alpha, \mu) - \frac{1}{2}). \quad (27) \]
  Moreover, we have
  \[ (n + m + \nu(\alpha, \mu) - \frac{1}{2}) > \frac{n + m}{2}, \]
  and thus, that there exists $\rho = \frac{7}{4\pi}\pi^2(2 - \alpha)^2$ such that
  \[ \lambda_{\alpha, \mu, n} - \lambda_{\alpha, \mu, m} \geq \rho(n^2 - m^2). \]
  
  • Let us now see the case $\nu(\alpha, \mu) \geq \frac{1}{2}$ (i.e. $\mu \leq \frac{\alpha}{16}(3\alpha - 4)$). Here we use the fact that the sequence $(j_{\nu(\alpha, \mu), n+1} - j_{\nu(\alpha, \mu), n})$ is nonincreasing and converges to $\pi$. This ensures that
  \[ j_{\nu(\alpha, \mu), n+1} - j_{\nu(\alpha, \mu), n} \geq \pi, \quad \forall n \geq 1. \]
  Therefore:
  \[ \lambda_{\alpha, \mu, n} - \lambda_{\alpha, \mu, m} \geq \left(\frac{2 - \alpha}{2}\right)^2 \pi(n - m)(j_{\nu(\alpha, \mu), n} + j_{\nu(\alpha, \mu), m}). \]
  Owing to (23), we also have
  \[ j_{\nu(\alpha, \mu), n} + j_{\nu(\alpha, \mu), m} \geq (n + m + \nu(\alpha, \mu) - \frac{1}{4})\pi \geq \pi(n + m). \]
  Combining the above last two estimates, the thesis follows with $\rho = \left(\frac{2 - \alpha}{2}\right)^2 \pi^2$.
  Thus, in every case there holds
  \[ \lambda_{\alpha, \mu, n} - \lambda_{\alpha, \mu, m} \geq \rho(n^2 - m^2). \]
  In both cases, after reversing the roles of $n$ and $m$, one has
  \[ \lambda_{\alpha, \mu, m} - \lambda_{\alpha, \mu, n} \geq \rho(m^2 - n^2). \]
  Hence,
  \[ |\lambda_{\alpha, \mu, n} - \lambda_{\alpha, \mu, m}| \geq \rho|n^2 - m^2|, \quad \forall n, m \geq 1, \]
  for a constant $\rho > 0$.

2. This point follows easily from (22). Indeed, $\forall \nu(\alpha, \mu) \in \left(0, \frac{1}{2}\right]$, $\forall n \geq 1$
  \[ \left(n - \frac{1}{4}\right)\pi \leq j_{\nu(\alpha, \mu), n}. \]
  Thus
  \[ \sum_{n \geq 1} \frac{1}{\lambda_{\alpha, \mu, n}} \leq \frac{1}{\pi^2} \left(\frac{2 - \alpha}{2}\right)^2 \sum_{n \geq 1} \frac{1}{(n - 1)^2} \leq \frac{4}{\pi^2} \left(\frac{2 - \alpha}{2}\right)^2 \sum_{n \geq 1} \frac{1}{n^2} < +\infty. \]

$\square$
3.2. Vectorial degenerate/singular operators. Let $A$ be given by (4) and consider the degenerate/singular vectorial operator

$$L : D(L) \subset L^2((0,1)^2) \rightarrow L^2((0,1)^2)$$

$$y \mapsto -(x^3 y_x)_x - \frac{\mu}{x^{2-\alpha}} y - Ay,$$

with domain $D(L) = H^{2,\mu}(0,1)^2 \cap H^{1,\mu}_0(0,1)^2$ and also its adjoint $L^*$.

In the sequel, we derive some properties of the eigenvalues and eigenfunctions of the operators $L$ and $L^*$ which will be useful for developing the moment method. Let us first analyze the spectrum of the operators $L$ and $L^*$:

**Proposition 5.** Let us consider the operator $L$ given by (28) and its adjoint $L^*$. Then,

1. The spectra of $L$ and $L^*$ are given by $\sigma(L) = \sigma(L^*) = \{\lambda^{(1)}_{\alpha,\mu,n}, \lambda^{(2)}_{\alpha,\mu,n}\}_{n \geq 1}$ with

$$\lambda^{(1)}_{\alpha,\mu,n} = \lambda_{\alpha,\mu,n} - \alpha_1, \quad \lambda^{(2)}_{\alpha,\mu,n} = \lambda_{\alpha,\mu,n} - \alpha_2, \quad \forall n \geq 1,$$

where $\alpha_1$ and $\alpha_2$ are the eigenvalues of the matrix $A$ defined by :

- **Case 1:** $a_2^2 + 4a_1 > 0$,

$$\alpha_1 = \frac{1}{2} \left( a_2 - \sqrt{a_2^2 + 4a_1} \right) \quad \text{and} \quad \alpha_2 = \frac{1}{2} \left( a_2 + \sqrt{a_2^2 + 4a_1} \right).$$

- **Case 2:** $a_2^2 + 4a_1 < 0$,

$$\alpha_1 = \frac{1}{2} \left( a_2 + i\sqrt{-(a_2^2 + 4a_1)} \right) \quad \text{and} \quad \alpha_2 = \frac{1}{2} \left( a_2 - i\sqrt{-(a_2^2 + 4a_1)} \right).$$

2. For each $n \geq 1$, the corresponding eigenfunctions of $L$ (resp., $L^*$) associated to $\lambda^{(1)}_{\alpha,\mu,n}$ and $\lambda^{(2)}_{\alpha,\mu,n}$ are respectively given by

$$\psi^{(1)}_n = U_1 \Phi_{\alpha,\mu,n}, \quad \psi^{(2)}_n = U_2 \Phi_{\alpha,\mu,n},$$

with

$$U_1 = \frac{1}{\alpha_1 - \alpha_2} \begin{pmatrix} -\alpha_2 \\ 1 \end{pmatrix} \quad \text{and} \quad U_2 = \frac{1}{\alpha_2 - \alpha_1} \begin{pmatrix} -\alpha_1 \\ 1 \end{pmatrix}$$

(resp.,

$$\Psi^{(1)}_n = V_1 \Phi_{\alpha,\mu,n}, \quad \Psi^{(2)}_n = V_2 \Phi_{\alpha,\mu,n},$$

with

$$V_1 = \begin{pmatrix} 1 \\ \alpha_1 \end{pmatrix} \quad \text{and} \quad V_2 = \begin{pmatrix} 1 \\ \alpha_2 \end{pmatrix}.$$

**Proof.** Using the fact that the function $\Phi_{\alpha,\mu,n}$ is the eigenfunction of the Dirichlet degenerate/singular operator $-\partial_x(x^\alpha \partial_x) - \frac{\mu}{x^{2-\alpha}}$, associated to the eigenvalue $\lambda_{\alpha,\mu,n}$, one can see that the eigenvalues of the operator $L$ correspond to the eigenvalues of the matrices

$$\lambda_{\alpha,\mu,n} Id - A, \quad \forall n \geq 1,$$

($Id \in L(\mathbb{C}^2)$ is the identity matrix) and the associated eigenfunctions of $L$ are given under the form $\psi_n(\cdot) = z_n \Phi_{\alpha,\mu,n}(\cdot)$, where $z_n \in \mathbb{C}^2$ is the associated eigenvector of the matrix $\lambda_{\alpha,\mu,n} Id - A$.

Taking into account the expression of the characteristic polynomial of $\lambda_{\alpha,\mu,n} Id - A$:

$$P(z) = z^2 - z(2\lambda_{\alpha,\mu,n} - a_2) + \lambda_{\alpha,\mu,n} (\lambda_{\alpha,\mu,n} - a_2) - a_1, \quad n \geq 1,$$
Proposition 6. Assume that the following condition holds
\[ \lambda_{\alpha_\mu,n} - \lambda_{\alpha_\mu,l} \neq \alpha_1 - \alpha_2, \quad \forall n, l \in \mathbb{N}^*, \quad \text{with} \quad n \neq l. \] (34)

Let us introduce the sequence \( \{\Lambda_{\alpha_\mu,n}\}_{n \geq 1} \), where
\[ \{\Lambda_{\alpha_\mu,n}\}_{n \geq 1} := \{\lambda_{\alpha_\mu,n} + \alpha_2, \lambda_{\alpha_\mu,n} + \alpha_2\}_{n \geq 1}. \] (35)

Then, the sequence given by (35) can be rearranged into a sequence of complex numbers of increasing modulus satisfying the assumptions of Theorem 1.3.

Proof. We distinguish between two cases depending on the spectrum of the matrix \( A \).

Case 1. \( A \) has two real eigenvalues \( \alpha_1 \) and \( \alpha_2 \), chosen such that \( \alpha_1 < \alpha_2 \).

Let us introduce the sequence \( \{\Lambda_{\alpha_\mu,n}\}_{n \geq 1} \), where
\[ \{\Lambda_{\alpha_\mu,n} : n \geq 1\} := \{\lambda_{\alpha_\mu,n} + \alpha_2, \lambda_{\alpha_\mu,n} + \alpha_2\}_{n \geq 1}. \]

The hypothesis 1) holds true if and only if the condition (34) is satisfied. In addition, the hypotheses 2) and 3) are obviously satisfied by definition.

Let us now show the hypothesis 4). Since \( \alpha_2 - \alpha_1 > 0 \), observe that \( \lambda_{\alpha_\mu,n} + \alpha_2 \) and \( \lambda_{\alpha_\mu,n} + \alpha_2 \) are increasing sequences satisfying
\[ 0 < \lambda_{\alpha_\mu,n} + \alpha_2 < \lambda_{\alpha_\mu,n} + \alpha_2, \quad \forall n \geq 1. \]

Thus, we deduce that the sequence \( \{\Lambda_{\alpha_\mu,n}\}_{n \geq 1} \) can be rearranged into a positive increasing sequence.

Let us move to prove hypothesis 5). For this purpose, we are going to give an explicit rearrangement of the sequence \( \{\lambda_{\alpha_\mu,n} + \alpha_2, \lambda_{\alpha_\mu,n} + \alpha_2\}_{n \geq 1} \). Firstly, observe that there exists an integer \( n_0 \geq 1 \) and a constant \( C > 0 \) such that
\[ \lambda_{\alpha_\mu,n-1}^{(1)} < \lambda_{\alpha_\mu,n}^{(2)} < \lambda_{\alpha_\mu,n}^{(1)} < \lambda_{\alpha_\mu,n+1}^{(2)} < \cdots, \quad \forall n \geq n_0, \text{ and} \]
\[ \min_{n \geq n_0} \{\lambda_{\alpha_\mu,n}^{(2)} - \lambda_{\alpha_\mu,n}^{(1)}, \lambda_{\alpha_\mu,n}^{(1)} - \lambda_{\alpha_\mu,n}^{(2)}\} > C. \] (36)

Indeed, using (26), one has
\[ \lambda_{\alpha_\mu,n}^{(2)} - \lambda_{\alpha_\mu,n-1}^{(1)} = \lambda_{\alpha_\mu,n} - \lambda_{\alpha_\mu,n-1} + \alpha_1 - \alpha_2 \]
\[ \geq \rho(2n - 1) + \alpha_1 - \alpha_2 \rightarrow_{n \to +\infty} +\infty. \] (37)

From (37) and the fact that \( \lambda_{\alpha_\mu,n}^{(1)} - \lambda_{\alpha_\mu,n}^{(2)} = \alpha_2 - \alpha_1 > 0 \), we can conclude (36). Therefore, if \( 1 \leq n \leq 2n_0 - 2 \), we define \( \Lambda_{\alpha_\mu,n} \) such that
\[ \{\Lambda_{\alpha_\mu,n}\}_{1 \leq n \leq 2n_0 - 2} = \{\lambda_{\alpha_\mu,n}^{(1)} + \alpha_2\}_{1 \leq n \leq n_0 - 1} \cup \{\lambda_{\alpha_\mu,n}^{(2)} + \alpha_2\}_{1 \leq n \leq n_0 - 1} \quad \text{and} \quad \Lambda_{\alpha_\mu,n} < \Lambda_{\alpha_\mu,n+1} \quad \forall n : 1 \leq n \leq 2n_0 - 3. \]
Moreover, from (214) one has:

\[ \Lambda_{\alpha,\mu,2n-1} = \lambda_{\alpha,\mu,n}^{(2)} + \alpha_2 \quad \text{and} \quad \Lambda_{\alpha,\mu,2n} = \lambda_{\alpha,\mu,n}^{(1)} + \alpha_2, \forall n \geq n_0. \] (38)

Since the elements of the sequence \( \{\Lambda_{\alpha,\mu,n}\}_{n \geq 1} \) are pairwise different and from (36), one has:

\[ \inf_{n,m \geq 1; n \neq m} |\lambda_{\alpha,\mu,n}^{(1)} - \lambda_{\alpha,\mu,m}^{(2)}| > 0. \] (39)

Hence, thanks to (39), the sequence \( \{\Lambda_{\alpha,\mu,n}\}_{n \geq 1} \) satisfies the second inequality in (7) for every \( q \geq 1 \).

Our next task will be to prove the first inequality of (7) for appropriate \( q > 0 \) and \( \varrho > 0 \). To this aim, as it has been remarked in [26], it is enough to prove the existence of \( q > 0 \) and \( \tilde{\varrho} > 0 \) such that

\[ |\Lambda_n - \Lambda_m| \geq \tilde{\varrho}|n^2 - m^2| \quad \forall n, m \geq q, |n - m| \geq q. \] (40)

We divide the proof of (40) into two steps.

1. Observe that, if \( n, m \in \mathbb{N}^* \) are such that \( n, m \geq n_0 \) and \( |n - m| \geq n_0 \), then by (38) and using (26) we have

\[
|\Lambda_{\alpha,\mu,2n} - \Lambda_{\alpha,\mu,2m}| = |\lambda_{\alpha,\mu,n} - \lambda_{\alpha,\mu,m}| \geq \rho|n^2 - m^2| = \frac{\rho}{4}|(2n)^2 - (2m)^2|
\]

and

\[
|\Lambda_{\alpha,\mu,2n-1} - \Lambda_{\alpha,\mu,2m-1}| = |\lambda_{\alpha,\mu,n} - \lambda_{\alpha,\mu,m}| \geq \rho|n^2 - m^2|
\]

\[
\geq \frac{\rho}{4}|(2n - 1)^2 - (2m - 1)^2|.
\]

We obtain thus the proof of (40) for \( q = n_0 \) and \( \tilde{\varrho} = \frac{\rho}{4} \).

2. Let \( n, m \in \mathbb{N}^* \) such that \( n, m \geq n_0 \). From (38), by denoting \( \bar{n} = 2n \) and \( \bar{m} = 2m - 1 \) and using again (26), we readily see that

\[
|\Lambda_{\alpha,\mu,\bar{n}} - \Lambda_{\alpha,\mu,\bar{m}}| = |\lambda_{\alpha,\mu,n}^{(1)} - \lambda_{\alpha,\mu,m}^{(2)}|
\]

\[
= |\lambda_{\alpha,\mu,n} - \lambda_{\alpha,\mu,m} + (\alpha_2 - \alpha_1)|
\]

\[
\geq \rho|n^2 - m^2| - (\alpha_2 - \alpha_1)
\]

\[
= \frac{\rho}{4}|\bar{n}^2 - (\bar{m} + 1)^2| - (\alpha_2 - \alpha_1)
\]

\[
= \frac{\rho}{4}|\bar{n}^2 - \bar{m}^2 - 2\bar{m} - 1| - (\alpha_2 - \alpha_1).
\]

Now, observe that if \( \bar{n} < \bar{m} \), we have

\[
|\Lambda_{\alpha,\mu,\bar{n}} - \Lambda_{\alpha,\mu,\bar{m}}| \geq \frac{\rho}{4}\left(\bar{m}^2 - \bar{n}^2\right)\left(1 - \frac{4(\alpha_2 - \alpha_1)}{\rho(\bar{m}^2 - \bar{n}^2)}\right).
\]

Let us take an integer \( q_0 \geq \max\{2n_0 - 1, \frac{4(\alpha_2 - \alpha_1)}{\rho}\} \). Then, \( \forall \bar{m}, \bar{n} \geq q_0 \) with \( |\bar{m} - \bar{n}| \geq q_0 \), one has

\[
|\Lambda_{\alpha,\mu,\bar{n}} - \Lambda_{\alpha,\mu,\bar{m}}| \geq \frac{\rho}{4}\left(\bar{m}^2 - \bar{n}^2\right)\left(1 - \frac{4(\alpha_2 - \alpha_1)}{\rho(\bar{m} + \bar{n})q_0}\right)
\]

\[
\geq \frac{\rho}{4}\left(\bar{m}^2 - \bar{n}^2\right)\left(1 - \frac{2(\alpha_2 - \alpha_1)}{\rho q_0}\right)
\]

\[
\geq \frac{\rho}{8}(\bar{m}^2 - \bar{n}^2).
\]
On the other hand, if \( \tilde{n} > \tilde{m} \), we have

\[
|\Lambda_{\alpha,\mu,\tilde{n}} - \Lambda_{\alpha,\mu,\tilde{m}}| \geq \frac{\rho}{4} (\tilde{n}^2 - \tilde{m}^2) \left( 1 - \frac{4(\alpha_2 - \alpha_1)}{\rho} + \frac{1}{2\tilde{m}q_1} \right).
\]

Let us work with an integer \( q_1 \) given by

\[
q_1 \geq \max\{2n_0 - 1, \frac{4(\alpha_2 - \alpha_1)}{\rho} + 4\}.
\]

Thus, if \( \tilde{n}, \tilde{m} \in \mathbb{N}^* \) are such that \( \tilde{n}, \tilde{m} \geq q_1 \) and \( |\tilde{n} - \tilde{m}| \geq q_1 \), then one has

\[
|\Lambda_{\alpha,\mu,\tilde{n}} - \Lambda_{\alpha,\mu,\tilde{m}}| \geq \frac{\rho}{4} (\tilde{n}^2 - \tilde{m}^2) \left( 1 - \frac{1}{q_1} \left( \frac{2(\alpha_2 - \alpha_1)}{\rho} + 2 \right) \right)
\]

Hence, choosing \( q = \max\{q_0, q_1\} \), (40) follows immediately for \( \tilde{q} = \frac{q}{8} \).

In conclusion, we have proved the existence of a number \( q \geq 1 \) such that (40) holds.

Let us now show the hypothesis 6). From the definition of \( \{\Lambda_{\alpha,\mu,n}\}_{n \geq 1} \), for any \( r > 0 \), we can write:

\[
N(r) = \#\{k : \lambda_{\alpha,\mu,k} + \alpha_2 - \alpha_1 \leq r\} + \#\{k : \lambda_{\mu,k} \leq r\}
= \#A_1(r) + \#A_2(r) = n_1 + n_2.
\]

where \( A_i = \{k : \lambda_{\alpha,\mu,k} + \alpha_2 - \alpha_1 \leq r\} \) and \( n_i = \#A_i, \text{ i=1,2} \). Our purpose is to prove suitable estimates for \( n_1 \) and \( n_2 \).

From the definition of \( A_2 \) and \( n_2 \), we have that \( n_2 \) is a natural number which is characterized by \( \lambda_{\alpha,\mu,n_2} \leq r \) and \( \lambda_{\alpha,\mu,n_2+1} > r \). We distinguish two cases depending on the value of \( \nu_{\mu}. \) Let us start by the case \( \nu(\alpha,\mu) \leq \frac{1}{2} \). From the inequality \( \lambda_{\alpha,\mu,n_2} \leq r \) and by (22), we have \( (n_2 + \frac{\nu(\alpha,\mu)}{2})^2 \leq r \) so that

\[
n_2 \leq \frac{\sqrt{r}}{\pi} - \frac{\nu(\alpha,\mu)}{2} + \frac{1}{4}.
\]  

(41)

On the other hand, from the inequality \( \lambda_{\alpha,\mu,n_2+1} > r \), we get

\[
n_2 > \frac{\sqrt{r}}{\pi} - \frac{\nu(\alpha,\mu)}{4} - \frac{7}{8}.
\]

Summarizing, \( n_2 \) is a nonnegative integer such that

\[
\frac{\sqrt{r}}{\pi} - \frac{\nu(\alpha,\mu)}{4} - \frac{7}{8} < n_2 \leq \frac{\sqrt{r}}{\pi} - \frac{\nu(\alpha,\mu)}{2} + \frac{1}{4}, \quad \forall r > 0.
\]

(42)

Next we are going to estimate \( n_1 \). Using arguments similar to the ones used above, we can see that

\[
\lambda_{\alpha,\mu,n_1} + \alpha_2 - \alpha_1 \leq r
\]

and

\[
\lambda_{\alpha,\mu,n_1+1} + \alpha_2 - \alpha_1 > r
\]

imply that

\[
\frac{\sqrt{r} + \alpha_1 - \alpha_2}{\pi} - \frac{\nu(\alpha,\mu)}{4} - \frac{7}{8} < n_1 \leq \frac{\sqrt{r} + \alpha_1 - \alpha_2}{\pi} - \frac{\nu(\alpha,\mu)}{2} + \frac{1}{4}.
\]

(43)
Then, using the fact that $\sqrt{a} - \sqrt{b} \leq \sqrt{a - b}$ and $\sqrt{a - b} \leq \sqrt{a}$ provided $a \geq b > 0$, one gets
\[
\frac{\sqrt{r}}{\pi} - \frac{\sqrt{\alpha_2 - \alpha_1}}{\pi} - \frac{\nu(\alpha, \mu)}{2} - \frac{7}{8} < n_1 \leq \frac{\sqrt{r}}{\pi} - \frac{\nu(\alpha, \mu)}{2} + \frac{1}{4}, \quad \forall r > 0.
\] (43)
Recall that $N(r) = n_1 + n_2$. Thus, combining (42) and (43), it follows that for $\nu(\alpha, \mu) \leq \frac{1}{2}$:
\[
\frac{2\sqrt{r}}{\pi} - \frac{\sqrt{\alpha_2 - \alpha_1}}{\pi} - \frac{\nu(\alpha, \mu)}{2} - \frac{7}{4} < N(r) \leq \frac{2\sqrt{r}}{\pi} - \frac{\nu(\alpha, \mu)}{2} + \frac{1}{2}, \quad \forall r > 0,
\] and deduce (8) with
\[
p = \frac{2}{\pi} \quad \text{and} \quad s = \max\{\frac{\sqrt{\alpha_2 - \alpha_1}}{\pi} + \frac{\nu(\alpha, \mu)}{2} + \frac{7}{4}, -\nu(\alpha, \mu) + \frac{1}{2}\} = \frac{\sqrt{\alpha_2 - \alpha_1}}{\pi} + \frac{\nu(\alpha, \mu)}{2} + \frac{7}{4}.
\]
The case $\nu_\mu \geq \frac{1}{2}$ can be treated in a similar way, but, instead of working with the bounds (22), we will use (23) to obtain
\[
\frac{\sqrt{r}}{\pi} - \frac{\nu(\alpha, \mu)}{2} - \frac{3}{4} < n_2 \leq \frac{\sqrt{r}}{\pi} - \frac{\nu(\alpha, \mu)}{4} + \frac{1}{8}, \quad \forall r > 0,
\] (44)
and
\[
\frac{\sqrt{r}}{\pi} - \frac{\sqrt{\alpha_2 - \alpha_1}}{\pi} - \frac{\nu(\alpha, \mu)}{2} - \frac{3}{4} < n_1 \leq \frac{\sqrt{r}}{\pi} - \frac{\nu(\alpha, \mu)}{4} + \frac{1}{8}, \quad \forall r > 0.
\] (45)
From the inequalities (44) and (45), we obtain that:
\[
\frac{2\sqrt{r}}{\pi} - \frac{\sqrt{\alpha_2 - \alpha_1}}{\pi} - \frac{\nu(\alpha, \mu)}{2} - \frac{3}{2} < N(r) \leq \frac{2\sqrt{r}}{\pi} - \frac{\nu(\alpha, \mu)}{2} + \frac{1}{4}, \quad \forall r > 0,
\] and again deduce (8) with
\[
p = \frac{2}{\pi} \quad \text{and} \quad s = \max\{\frac{\sqrt{\alpha_2 - \alpha_1}}{\pi} + \frac{\nu(\alpha, \mu)}{2} + \frac{3}{2}, -\nu(\alpha, \mu) + \frac{1}{4}\} = \frac{\sqrt{\alpha_2 - \alpha_1}}{\pi} + \frac{\nu(\alpha, \mu)}{2} + \frac{3}{2}.
\]
We thus obtain the last hypothesis 6) of Theorem 1.3. This ends the proof in this case.

**Case 2. A has two complex eigenvalues $\alpha_1$ and $\alpha_2$.**

In this case $a_2^2 + 4a_1 < 0$,
\[
\alpha_1 = \frac{a_2}{2} + i\beta, \quad \text{and} \quad \alpha_2 = \frac{a_2}{2} - i\beta,
\]
where $\beta := \frac{1}{2} \sqrt{-(a_2^2 + 4a_1)}$.

Now, we consider the complex sequence $\{\Lambda_{\alpha,\mu,n}\}_{n \geq 1}$, with
\[
\Lambda_{\alpha,\mu,2n-1} = \lambda_{\alpha,\mu,n}^{(2)} + \alpha_1 = \lambda_{\alpha,\mu,n}, \quad \forall n \geq 1,
\]
\[
\Lambda_{\alpha,\mu,2n} = \lambda_{\alpha,\mu,n}^{(1)} + \alpha_2 = \lambda_{\alpha,\mu,n} - 2i\beta, \quad \forall n \geq 1.
\] (46)
Let us check if the hypotheses in Theorem 1.3 hold true for $\{\Lambda_{\alpha,\mu,n}\}_{n \geq 1}$.

First, it is clearly that the sequence $\{\Lambda_{\alpha,\mu,n}\}_{n \geq 1}$ always satisfies the hypothesis 1). Furthermore, the hypothesis 2) follows directly from the fact that $\Re(\Lambda_{\alpha,\mu,2n}) = \Re(\Lambda_{\alpha,\mu,2n-1}) = \lambda_{\alpha,\mu,n} > 0$.

The hypothesis 3) is clearly fulfilled. Indeed, one can find $\delta > 0$ (which depends on $\beta$) such that
\[
|\Im(\Lambda_{\alpha,\mu,2n})| = 2\beta \leq \delta \sqrt{\Re(\Lambda_{\alpha,\mu,2n})}
\]
and
\[
|3(Λ_{α,μ,2n-1})| = 0 \leq \delta \sqrt[3]{|R(Λ_{α,μ,2n-1})|}.
\]

Let us now prove hypothesis 4). To this end, it suffices to prove that there exists an integer \( \tilde{n}_0 \geq 1 \) such that for all \( n \geq \tilde{n}_0 \), \( |Λ_{α,μ,2n}| \leq |Λ_{α,μ,2n+1}| \). Using (26), we have
\[
|Λ_{α,μ,2n+1}|^2 - |Λ_{α,μ,2n}|^2 = λ_{α,μ,n+1}^2 - λ_{α,μ,n}^2 - 4β^2
\geq (λ_{α,μ,n+1} - λ_{α,μ,n})^2 - 4β^2
\geq ρ^2((n + 1)^2 - n^2 - 4β^2
= ρ^2(2n + 1)^2 - 4β^2,
\]
which implies that
\[
\lim_{n \to +∞} \bigl( |Λ_{α,μ,2n+1}|^2 - |Λ_{α,μ,2n}|^2 \bigr) = +∞.
\]
Therefore, there exists \( \tilde{n}_0 \geq 1 \) such that \( \{Λ_{α,μ,n}\}_{n \geq 2\tilde{n}_0} \) is nondecreasing in modulus. This shows hypothesis 4).

Let us now check if the hypothesis 5) holds true. To this aim, we choose to arrange the sequence \( \{Λ_{α,μ,n}\}_{n \geq 1} \) defined in (46) as follows:
\[
\{Λ_{α,μ,n}\}_{1\leq n \leq 2\tilde{n}_0 - 2} = \{λ_{α,μ,n}^{(1)} + α_2\}_{1\leq n \leq \tilde{n}_0 - 1} \cup \{λ_{α,μ,n}^{(2)} + α_2\}_{1\leq n \leq \tilde{n}_0 - 1} \quad \text{and} \quad |Λ_{α,μ,n}| < |Λ_{α,μ,n+1}| \quad \forall n : 1 \leq n \leq 2\tilde{n}_0 - 3.
\]
Moreover, from (2\( \tilde{n}_0 - 1 \))-th term, we set:
\[
Λ_{α,μ,2n-1} = λ_{α,μ,n}^{(2)} + α_2 \quad \text{and} \quad Λ_{α,μ,2n} = λ_{α,μ,n}^{(1)} + α_2, \quad \forall n \geq n_0.
\]
First, observe that the second property is actually satisfied for any \( q \). Our next objective will be to prove the first inequality in (7). Arguing as done in the real case, by Lemma 3.2, there exists \( ρ > 0 \) such that
\[
|Λ_{α,μ,2n} - Λ_{α,μ,2m}| \geq \frac{ρ}{4}|(2n|^2 - (2m|^2|, \quad \forall n, m \geq n_0
\]
and
\[
|Λ_{α,μ,2n-1} - Λ_{α,μ,2m-1}| \geq \frac{ρ}{4}|(2n - 1|^2 - (2m - 1|^2|, \quad \forall n, m \geq n_0.
\]
Moreover, denoting \( \tilde{n} = 2n \) and \( \tilde{m} = 2m - 1 \), one can prove that there exists \( q \geq \max\{4, 2\tilde{n}_0 - 1\} \) such that \( \forall \tilde{n}, \tilde{m} \geq q \) with \( |\tilde{n} - \tilde{m}| \geq q \), we have
\[
|Λ_{α,μ,\tilde{n}} - Λ_{α,μ,\tilde{m}}|^2 = |Λ_{α,μ,2n} - Λ_{α,μ,2m-1}|^2 \geq \left(\frac{ρ}{8}|\tilde{n}^2 - \tilde{m}^2|\right)^2.
\]
Indeed, by (26), for \( \tilde{n}, \tilde{m} \geq 2\tilde{n}_0 - 1 \) we have
\[
|Λ_{α,μ,\tilde{n}} - Λ_{α,μ,\tilde{m}}|^2 = |Λ_{α,μ,2n} - Λ_{α,μ,2m-1}|^2
\geq |λ_{α,μ,n} - λ_{α,μ,m}|^2 + 4β^2
\geq \left(\frac{ρ}{4}|\tilde{n}^2 - \tilde{m}^2|\right)^2 = \left(\frac{ρ}{4}|\tilde{n}^2 - \tilde{m}^2 - 2\tilde{m} - 1|\right)^2.
\]
Next, if $|\tilde{n} - \tilde{m}| \geq 4$, simple computation gives
\[
|\tilde{n}^2 - \tilde{m}^2 - 2\tilde{m} - 1| = |(\tilde{n}^2 - \tilde{m}^2)(1 - \frac{2\tilde{m} + 1}{\tilde{n}^2 - \tilde{m}^2})| \\
\geq \frac{1}{2}|\tilde{n}^2 - \tilde{m}^2|.
\]
Hence, the conclusion follows by working with $q$ given by
\[
q \geq \max\{4, 2\tilde{n}_0 - 1\}.
\]
Finally, proceeding as in the real case, it is not difficult to obtain some suitable
parameters $p$ and $s$ for which the inequality (8) holds. This finishes the proof of
Proposition 6.

We will finish this section giving a result on the set of eigenfunctions of the
operators $L$ and $L^*$. It reads as follows:

**Proposition 7.** Let us consider the sequences
\[
B = \{\psi_n^{(1)}, \psi_n^{(2)} : n \geq 1\} \quad \text{and} \quad B^* = \{\Psi_n^{(1)}, \Psi_n^{(2)} : n \geq 1\}. \tag{48}
\]
Then,
1. $B$ and $B^*$ are biorthogonal families in $L^2(0, 1)^2$.
2. $B$ and $B^*$ are complete sequences in $L^2(0, 1)^2$.
3. The sequences $B$ and $B^*$ are biorthogonal Riesz bases of $L^2(0, 1)^2$.
4. The sequence $B^*$ is a basis of $H_{\alpha,0}^1(0, 1)^2$ and $B$ is its biorthogonal basis in
   $H_{\alpha,0}^{-1,\pi}(0, 1)^2$.

**Proof.** From the expressions of $\psi_n^{(i)}$ and $\Psi_n^{(i)}$, we can write
\[
\psi_n^{(i)} = U_i \Phi_{\alpha,\mu,n} \quad \text{and} \quad \Psi_n^{(i)} = V_i \Phi_{\alpha,\mu,n}, \quad i = 1, 2, \quad n \geq 1,
\]
where $U_i, V_i \in \mathbb{R}^2$ and $\Phi_{\alpha,\mu,n}$ is given in (25).

1. It is not difficult to check that $\{U_i\}_{i=1,2}$ and $\{V_i\}_{i=1,2}$ are biorthogonal families
   of $\mathbb{R}^2$. Moreover, since $(\Phi_{\alpha,\mu,n})_{n \geq 1}$ is an orthonormal basis for $L^2(0, 1)$, we
   readily deduce
\[
\langle \psi_n^{(i)}, \Psi_{k}^{(j)} \rangle = (U_i)^{tr} V_j \langle \Phi_{\alpha,\mu,n}, \Phi_{\alpha,\mu,k} \rangle = \delta_{ij}\delta_{nk}, \quad \forall n, k \geq 1, \quad i, j = 1, 2.
\]
   This proves the claim.

2. We will use [29, Lemma 1.44]. For this purpose, let us consider $f = (f_1, f_2)^{tr} \in
   L^2(0, 1)^2$ such that
\[
\langle f, \psi_n^{(i)} \rangle = 0, \quad \forall n \geq 1, \quad i = 1, 2.
\]
   If we denote $f_{i,n}$ ($i = 1, 2$) the corresponding Fourier coefficients of the function
   $f_i \in L^2(0, 1)$ with respect to the basis $(\Phi_{\alpha,\mu,n})_{n \geq 1}$, then the previous
   equality can be written as
\[
(f_{1,n}, f_{2,n})[U_1|U_2] = 0_{\mathbb{R}^2}, \quad \forall n \geq 1.
\]
   Using the fact that $det[U_1|U_2] \neq 0$, we deduce $f_{1,n} = f_{2,n} = 0$, for all $n \geq 1$.
   This implies that $f_1 = f_2 = 0$ (since $(\Phi_{\alpha,\mu,n})_{n \geq 1}$ is an orthonormal basis in
   $L^2(0, 1)$) and, therefore, $f = 0$ which proves the completeness of $B$. A similar
   argument can be used for $B^*$ and the conclusion follows immediately.
3. By [29, Theorem 7.13], we know that \( \{ \psi_n^{(1)}, \psi_n^{(2)} \}_{n \geq 1} \) is a Riesz basis for \( L^2(0,1)^2 \) if and only if \( \{ \psi_n^{(1)}, \psi_n^{(2)} \}_{n \geq 1} \) is a complete Bessel sequence and possesses a biorthogonal system that is also a complete Bessel sequence. Using the previous properties 1) and 2), we only have to prove that the sequence \( \{ \psi_n^{(1)}, \psi_n^{(2)} \}_{n \geq 1} \) and \( \{ \Psi_n^{(1)}, \Psi_n^{(2)} \}_{n \geq 1} \) are Bessel sequences. This amounts to prove that the series

\[
S_1(f) = \sum_{n \geq 1} [(f, \psi_n^{(1)})^2 + (f, \psi_n^{(2)})^2] \quad \text{and}
\]

\[
S_2(f) = \sum_{n \geq 1} [(f, \Psi_n^{(1)})^2 + (f, \Psi_n^{(2)})^2]
\]

converge for any \( f = (f_1, f_2)^T \in L^2(0,1)^2 \).

From the definition of the functions \( \psi_n^{(i)} \) and \( \Psi_n^{(i)} \), it is easy to see that there exists some constant \( C > 0 \) such that

\[
S_1(f) \leq C \sum_{n \geq 1} (|f_{1,n}|^2 + |f_{2,n}|^2) \quad \text{and}
\]

\[
S_2(f) \leq C \sum_{n \geq 1} (|f_{1,n}|^2 + |f_{2,n}|^2).
\]

Recall that \( f_{i,n} \) is the Fourier coefficient of the function \( f_i \in L^2(0,1) \) with respect to \( \Phi_{\alpha, \mu, i} \). Accordingly, the series \( S_1(f) \) and \( S_2(f) \) converge since \( (\Phi_{\alpha, \mu, i})_{n \geq 1} \) is an orthonormal basis for \( L^2(0,1) \). We obtain thus the proof of desired result.

4. For showing item 4) we make use of [29, Theorem 5.12]. First, by taking \( L^2(0,1) \) as a pivot space, one has

\[
H_{\alpha,0}^{1,\mu}(0,1) \subset L^2(0,1) \subset (H_{\alpha,0}^{1,\mu}(0,1))' = H_{\alpha}^{-1,\mu}(0,1).
\]

Furthermore, observe that \( B^* \subset H_{\alpha,0}^{1,\mu}(0,1)^2 \) and is complete in this space since it is in \( L^2(0,1)^2 \). On the other hand, by the definition of the duality pairing, we have

\[
\langle \psi_n^{(i)}, \Psi_k^{(j)} \rangle_{H_{\alpha,0}^{-1,\mu},H_{\alpha,0}^{1,\mu}} = \langle \psi_n^{(i)}, \Psi_k^{(j)} \rangle = \delta_{ij} \delta_{nk}, \quad \forall n, k \geq 1, \quad i, j = 1, 2.
\]

Thus, \( B \subset H_{\alpha}^{-1,\mu}(0,1)^2 \) and is biorthogonal to \( B^* \), which also yields that \( B^* \) is minimal in \( H_{\alpha,0}^{1,\mu}(0,1)^2 \) thanks to [29, Lemma 5.4]. To conclude the proof, it remains to prove that for any \( f = (f_1, f_2) \in H_{\alpha,0}^{1,\mu}(0,1)^2 \), the series

\[
S(f) = \sum_{n \geq 1} [(\psi_n^{(1)}, f)_{H_{\alpha,0}^{-1,\mu},H_{\alpha,0}^{1,\mu}} \Psi_n^{(1)} + (\psi_n^{(2)}, f)_{H_{\alpha,0}^{-1,\mu},H_{\alpha,0}^{1,\mu}} \Psi_n^{(2)}]
\]

converges in \( H_{\alpha,0}^{1,\mu}(0,1)^2 \).

Using again the definitions of \( \psi_n^{(i)} \) and \( \Psi_n^{(i)} \), one can prove that

\[
\langle \psi_n^{(1)}, f \rangle_{H_{\alpha,0}^{-1,\mu},H_{\alpha,0}^{1,\mu}} \Psi_n^{(1)} = \frac{1}{\alpha_1 - \alpha_2} \left( -\alpha_2 f_{1,n} + f_{2,n} \right) \Phi_{\alpha, \mu, n}
\]

and

\[
\langle \psi_n^{(2)}, f \rangle_{H_{\alpha,0}^{-1,\mu},H_{\alpha,0}^{1,\mu}} \Psi_n^{(2)} = \frac{1}{\alpha_1 - \alpha_2} \left( -\alpha_1 \alpha_2 f_{1,n} + \alpha_1 f_{2,n} \right) \Phi_{\alpha, \mu, n}
\]

with \( \alpha_1, \alpha_2 > 0 \) and \( \Phi_{\alpha, \mu, n} \) defined in (5.43).
and
\[
\langle \psi_{n}^{(2)}, f \rangle_{H_{\alpha,0}^{-\mu},H_{\alpha,0}^{1}} = \frac{1}{\alpha_2 - \alpha_1} \left( -\alpha_1 f_{1,n} + f_{2,n} \right) \Phi_{\alpha,\mu,n}
\]
where \( f_{i,n} \) is the Fourier coefficient of the function \( f_i \in H_{\alpha,0}^{1}(0,1), i = 1, 2 \).

But, we know that the series \( \sum_{n \geq 1} f_{i,n} \Phi_{\alpha,\mu,n}, i = 1, 2 \) converges in \( H_{\alpha,0}^{1}(0,1) \) since \( (\Phi_{\alpha,\mu,n})_{n \geq 1} \) is an orthogonal basis for \( H_{\alpha,0}^{1}(0,1) \) and \( f_1, f_2 \in H_{\alpha,0}^{1}(0,1) \).

This implies that, the series
\[
\sum_{n \geq 1} \langle \psi_{n}^{(1)}, f \rangle_{H_{\alpha,0}^{-\mu},H_{\alpha,0}^{1}} \Psi_{n}^{(1)} \quad \text{and} \quad \sum_{n \geq 1} \langle \psi_{n}^{(2)}, f \rangle_{H_{\alpha,0}^{-\mu},H_{\alpha,0}^{1}} \Psi_{n}^{(2)}
\]
converge in \( H_{\alpha,0}^{1}(0,1)^2 \) and assure the convergence of \( S(f) \) in \( H_{\alpha,0}^{1}(0,1)^2 \).

This concludes the proof of the result.

\[\Box\]

4. Boundary approximate controllability. We will devote this section to proving the approximate controllability at time \( T > 0 \) of system (1). To this aim, we are going to use Theorem 1.2.

First of all, using Lemma 3.2 and similar techniques as in Proposition 6, one can prove that the following result holds.

**Proposition 8.** Assume that condition (34) holds. Then, the family defined in (35) satisfies (6).

Now, we are ready to state our first main result on approximate controllability. One has:

**Theorem 4.1.** Let \( \mu \leq \mu(\alpha) \) and let us denote by \( \alpha_1 \) and \( \alpha_2 \) the eigenvalues of the matrix \( A \). Then, system (1) is approximately controllable in \( H_{\alpha,0}^{-\mu}(0,1)^2 \) at time \( T > 0 \) if and only if condition (34) is satisfied.

**Proof.** As said in section 2, in order to prove this theorem we will follow a duality approach leading us to study the unique continuation property for the adjoint system.

**Necessary condition:** By contradiction, let us assume that condition (34) does not hold, i.e., that there is \( n_0, l_0 \in \mathbb{N}^* \) with \( n_0 \neq l_0 \) such that
\[
\lambda_{\alpha,\mu,n_0}^{(1)} = \lambda_{\alpha,\mu,l_0}^{(2)} := \lambda.
\]

Let us see that the unique continuation property for the adjoint system (19) is no longer valid. Indeed, let us take \( \varphi_0 = a \Psi_{n_0}^{(1)} + b \Psi_{l_0}^{(2)} \in H_{\alpha,0}^{1}(0,1)^2 \), with \( a, b \in \mathbb{R} \) to be determined. In this case, it is not difficult to see that the corresponding solution to the adjoint problem (19) is given by
\[
\varphi(t, x) = (a \Psi_{n_0}^{(1)}(x) + b \Psi_{l_0}^{(2)}(x)) e^{-\lambda(T-t)}, \quad \forall (t, x) \in Q.
\]

By recalling the definition of \( \Psi_{n}^{(i)} \), we have that
\[
B^*(x^{\alpha} \varphi_x)(t, 1) = B^*(a \Psi_{n_0}^{(1)}(1) + b \Psi_{l_0}^{(2)}(1)) e^{-\lambda(T-t)}
\]
\[
= \left( a B^* V_1(x^{\alpha} (\Phi_{\alpha,\mu,n_0})(x))(1) + b B^* V_2(x^{\alpha} (\Phi_{\alpha,\mu,l_0})(x))(1) \right) e^{-\lambda(T-t)}
\]
\[
= \left( a (x^{\alpha} (\Phi_{\alpha,\mu,n_0})(x))(1) + b (x^{\alpha} (\Phi_{\alpha,\mu,l_0})(x))(1) \right) e^{-\lambda(T-t)}.
\]
Then, choosing

\[ a = (x^\alpha (\Phi_{\alpha,\mu,1})_x)(1) \quad \text{and} \quad b = -(x^\alpha (\Phi_{\alpha,\mu,0})_x)(1), \]

we get that \( B^* (x^\alpha \varphi_x)(t,1) = 0 \) but \( \varphi_0 \neq 0 \), which proves that the unique continuation property for the adjoint system (19) fails to be true. This ends the proof of the necessary part.

**Sufficient condition:** Let us now assume that the condition (34) holds and prove that the unique continuation property for the solutions of the adjoint system (19) holds.

Let us consider \( \varphi_0 \in H^1_{\alpha,0} (0,1)^2 \) and suppose that the corresponding solution \( \varphi \) of the adjoint problem (19) satisfies

\[ B^* (x^\alpha \varphi_x)(t,1) = 0, \quad \forall t \in (0,T). \quad (49) \]

From Proposition 7, we know that \( B^* \) is a basis for \( H^1_{\alpha,0} (0,1)^2 \) and thus \( \varphi_0 \in H^1_{\alpha,0} (0,1)^2 \) can be written as

\[ \varphi_0 = \sum_{n \geq 1} (b_n \Psi_n^{(1)} + c_n \Psi_n^{(2)}), \]

where

\[ b_n = \langle \psi_n^{(1)}, \varphi_0 \rangle_{H^{-1,\mu}_{\alpha,\nu}} H_{\alpha,\nu}^{1,\mu} \quad \text{and} \quad c_n = \langle \psi_n^{(2)}, \varphi_0 \rangle_{H^{-1,\mu}_{\alpha,\nu}} H_{\alpha,\nu}^{1,\mu}, \quad \text{for any} \quad n \geq 1. \]

Using Proposition 5, the corresponding solution \( \varphi \) of system (19) associated to \( \varphi_0 \) is given by

\[ \varphi(t, \cdot) = \sum_{n \geq 1} \left( b_n \Psi_n^{(1)} e^{-\lambda_n^{(1)}(\alpha,\mu,n)(T-t)} + c_n \Psi_n^{(2)} e^{-\lambda_n^{(2)}(\alpha,\mu,n)(T-t)} \right), \quad \forall t \in (0,T). \]

On the other hand, direct computations show that

\[ (x^\alpha (\Phi_{\alpha,\mu,\nu,n})_x)(1) = \frac{(2-\alpha)^2}{2} J_{\nu(\alpha,\mu),n} J_{\nu(\alpha,\mu),n} J_{\nu(\alpha,\mu),n}. \]

Hence,

\[ 0 = B^* (x^\alpha \varphi_x)(T-t,1) \]

\[ = \sum_{n \geq 1} B^* (b_n (x^\alpha \psi_n^{(1)})(1) e^{-\lambda_n^{(1)}(\alpha,\mu,n)T} + c_n (x^\alpha \psi_n^{(2)})(1) e^{-\lambda_n^{(2)}(\alpha,\mu,n)T}) \]

\[ = (x^\alpha (\Phi_{\alpha,\mu,\nu,n})_x)(1) (b_n B^* V_1 e^{-\lambda_n^{(1)}(\alpha,\mu,n)T} + c_n B^* V_2 e^{-\lambda_n^{(2)}(\alpha,\mu,n)T}) \]

\[ = -\frac{(2-\alpha)^2}{2} \sum_{n \geq 1} J_{\nu(\alpha,\mu),n} J_{\nu(\alpha,\mu),n} J_{\nu(\alpha,\mu),n} (b_n e^{-\lambda_n^{(1)}(\alpha,\mu,n)T} + c_n e^{-\lambda_n^{(2)}(\alpha,\mu,n)T}) \]

\[ = -\frac{(2-\alpha)^2}{2} \sum_{n \geq 1} J_{\nu(\alpha,\mu),n} J_{\nu(\alpha,\mu),n} J_{\nu(\alpha,\mu),n} e^{\alpha T} (b_n e^{-\lambda_n^{(1)}(\alpha,\mu,n)+(\alpha+2)T} + c_n e^{-\lambda_n^{(2)}(\alpha,\mu,n)+(\alpha+2)T}). \]

From Proposition 8, we can apply Theorem 1.2 in order to deduce the existence of a biorthogonal family \( \{ q_n^{(1)}, q_n^{(2)} \}_{n \geq 1} \) to \( \{ e^{-(\lambda_n^{(1)}(\alpha,\mu,n)+(\alpha+2)t)}, e^{-(\lambda_n^{(2)}(\alpha,\mu,n)+(\alpha+2)t)} \}_{n \geq 1} \) in \( L^2 (0,T) \).
Thus, the previous identity, in particular, implies
\[
\begin{cases}
\int_0^T B^*(x^n \varphi_x)(T-t,1)e^{-\alpha t} q_n^{(1)}(t) \, dt = -\frac{(2-\alpha)^3}{2} \frac{J_{\nu(\alpha,\mu)}(J_{\nu(\alpha,\mu)};x_n)}{|J_{\nu(\alpha,\mu)}(J_{\nu(\alpha,\mu)};x_n)|} J_{\nu(\alpha,\mu)} x_n b_n \\
= 0, \quad \forall n \geq 1
\end{cases}
\]

Thus \( b_n = c_n = 0 \) for any \( n \geq 1 \). In conclusion, \( \varphi_0 = 0 \). This proves the continuation property for the solutions to the adjoint problem (19) and, thanks to Theorem 2.2, the approximate controllability of system (1) at any positive time \( T \) holds.

5. Boundary null controllability. In this section, we will address the main achievement of this work which is the boundary null controllability result of system (1), providing an estimate of the control cost as a function of \( T \). In this sense, one has:

**Theorem 5.1.** Let \( \mu \leq \mu(\alpha) \) and let us denote by \( \alpha_1 \) and \( \alpha_2 \) the eigenvalues of \( A \). Then, system (1) is null controllable in \( H^{1,\mu}_0(0,1)^2 \) at time \( T > 0 \) if and only if condition (34) is satisfied.

Moreover, for every \( T > 0 \) and \( y_0 \in H^{1,\mu}_0(0,1)^2 \) there exists a null control \( v \in L^2(0,T) \) for system (1) which, in addition, satisfies
\[
\|v\|_{L^2(0,T)} \leq C e^{CT + \frac{2}{T}} \|y_0\|_{H^{1,\mu}_0},
\]
where \( C > 0 \) does not depend on \( T \).

**Proof.** To prove Theorem 5.1, we transform the controllability problem into a moment problem. Using Proposition 3, we deduce that the control \( v \in L^2(0,T) \) drives the solution of (1) to zero at time \( T \) if and only if \( v \in L^2(0,T) \) fulfills
\[
\int_0^T B^*(x^n \varphi_x)(t,1) v(t) \, dt = \langle y_0, \varphi(0,\cdot) \rangle_{H^{1,\mu}_0, H^{1,\mu}_0}, \quad \forall \varphi_0 \in H^{1,\mu}_0(0,1)^2
\]

where \( \varphi \in C^0([0,T]; H^{1,\mu}_0(0,1)^2) \cap L^2(0,T; H^{2,\mu}(0,1)^2 \cap H^{1,\mu}_0(0,1)^2) \) is the solution of the adjoint system (19) associated to \( \varphi_0 \).

Using Proposition 5, the corresponding solution \( \varphi \) of system (19) associated to \( \varphi_0 \) is given by
\[
\varphi(t,x) = \sum_{k \geq 1} \left( \langle \psi_k^{(1)}, \varphi_0 \rangle_{H^{-1,\mu}_0, H^{1,\mu}_0} \Psi_k^{(1)} e^{-\lambda_{\nu_{k},\mu}(t-t)} + \langle \psi_k^{(2)}, \varphi_0 \rangle_{H^{-1,\mu}_0, H^{1,\mu}_0} \Psi_k^{(2)} e^{-\lambda_{\nu_{k},\mu}(t-t)} \right).
\]

From Proposition 7, we have that \( B^* \) is a basis for \( H^{1,\mu}_0(0,1)^2 \). In particular, we also deduce that \( \varphi(t,x) = \Psi_n^{(i)}(x)e^{-\lambda_{\nu_{k},\mu}(t-t)} \) is the solution of system (19) corresponding to \( \varphi_0 = \Psi_n^{(i)} \in H^{1,\mu}_0(0,1)^2 \). Therefore, we can deduce that the identity (51) is equivalent to
\[
\int_0^T B^*(x^n \Psi_n^{(i)})(1) v(t) e^{-\lambda_{\nu_{k},\mu}(t-t)} \, dt = e^{-\lambda_{\nu_{k},\mu}(t-t)} y_0, \quad \forall n \geq 1, \, i = 1, 2.
\]
Taking into account the expressions of $\Psi_n^{(i)}$ (see (33)), we infer that $v \in L^2(0, T)$ is a null control for system (1) associated to $y_0$ if and only if

$$\frac{(2 - \alpha)^2}{2|J'_\nu(\nu, \mu, \alpha, \beta)|} \int_0^T v(t)e^{-\lambda_n^{(i)}(T-t)}dt = e^{-\lambda_n^{(i)}(T-T_0)}\langle y_0, \Psi_n^{(i)} \rangle_{H^{-1,\nu,\mu,0}_n}, \forall n \geq 1, \ i = 1, 2$$

and equivalently,

$$\int_0^T v(t)e^{-\lambda_n^{(i)}(T-t)}dt = C_{\nu(\alpha, \mu), n}^{(i)} \Psi_n^{(i)} \lambda_n^{(i)}(n) \geq 1, \ i = 1, 2, \ (52)$$

where $C_{\nu(\alpha, \mu), n}^{(i)}$ is given by

$$C_{\nu(\alpha, \mu), n}^{(i)} = \frac{2|J'_\nu(\nu, \mu, \alpha, \beta)|e^{-\lambda_n^{(i)}(T)}}{(2 - \alpha)^2|J'_\nu(\nu, \mu, \alpha, \beta)|} e^{-\lambda_n^{(i)}(T-T_0)}\langle y_0, \Psi_n^{(i)} \rangle_{H^{-1,\nu,\mu,0}_n}, \forall n \geq 1, \ i = 1, 2.$$  

Performing the change of variable $s = T/2 - t$ in (52), the controllability problem reduces then to the following moment problem: Given $y_0 \in H^{-1,\nu}(0, 1)^2$, find $v \in L^2(0, T)$ such that $u(s) = v(T/2 - s)e^{\alpha_2 s} \in L^2(-T/2, T/2)$ satisfies

$$\int_{-T/2}^{T/2} u(s)e^{-\lambda_n^{(i)}(T_n^{(i)} + \alpha_2 s)}ds = \tilde{C}_{\nu(\alpha, \mu), n}^{(i)} \forall n \geq 1, \ i = 1, 2, \ (53)$$

with

$$\tilde{C}_{\nu(\alpha, \mu), n}^{(i)} = e^{-\lambda_n^{(i)}(T_n^{(i)} + \alpha_2 T/2)} C_{\nu(\alpha, \mu), n}^{(i)}.$$  

At this stage, the strategy to solve the moment problem (53) is to use the concept of biorthogonal family. In fact, Proposition 6 and Theorem 1.3 guarantee the existence of $T_0 > 0$, such that for any $T \in (0, T_0)$, there exists a biorthogonal family $\{q_n^{(1)}, q_n^{(2)}\}_{n \geq 1}$ to $\{e^{-\lambda_n^{(i)}(T_n^{(i)} + \alpha_2 T/2)} \}_{n \geq 1}$ in $L^2(-T/2, T/2)$ which also satisfies

$$\|q_n^{(i)}\|_{L^2(-T/2, T/2)} \leq C e^{\lambda_n^{(i)}(T_n^{(i)} + \alpha_2 T/2)} \forall n \geq 1, \ i = 1, 2 \ (55)$$

for some positive constant $C$ independent of $T$.

For $T < T_0$, a solution to the moment problem (53) is then given for every $t \in (0, T)$ by

$$u(t) = \sum_{n \geq 1} (\tilde{C}_{\nu(\alpha, \mu), n}^{(i)} q_n^{(1)}(t) + \tilde{C}_{\nu(\alpha, \mu), n}^{(2)} q_n^{(2)}(t)).$$

Thus

$$v(t) = \sum_{n \geq 1} (\tilde{C}_{\nu(\alpha, \mu), n}^{(1)} q_n^{(1)}(T/2 - t) + \tilde{C}_{\nu(\alpha, \mu), n}^{(2)} q_n^{(2)}(T/2 - t)) e^{-\alpha_2(T/2 - t)}. \ (56)$$

The only remaining point is to prove that $v \in L^2(0, T)$ and to estimate its norm with respect to $T$ and $y_0$. This can be achieved thanks to the estimate (55). Indeed, from the expression of $\Psi_n^{(i)}$ and $\lambda_n^{(i)}$, we can easily deduce the existence of a constant $C_1 > 0$ such that for $i = 1, 2 :$

$$\|\Psi_n^{(i)}\|_\mu \leq C_1 \sqrt{\lambda_n^{(i)}}, \forall n \geq 1.$$

From (54), it is easy to see that there exists a new constants $C$ not depending on $n$ and $T$ such that

$$|\tilde{C}_{\nu(\alpha, \mu), n}^{(i)}| \leq C e^{-\lambda_n^{(i)}(T_n^{(i)} + \alpha_2 T/2)} \|y_0\|_{H^{-1,\nu,\mu}_n} \forall n \geq 1, \ i = 1, 2. \ (57)$$
Coming back to the expression (56) of the null control $v$, taking into account the definition of $\lambda_{\alpha,\mu,n}^{(i)}$ and using the estimates (55) and (57), we get

$$\|v\|_{L^2(0,T)} \leq C e^{C T} \|y_0\|_{H^{-1,\mu}} \sum_{n \geq 1} e^{-\lambda_{\alpha,\mu,n} T/2} e^{C \sqrt{\lambda_{\alpha,\mu,n} T/2}}. \tag{58}$$

Moreover, Young’s inequality gives

$$C \sqrt{\lambda_{\alpha,\mu,n}} \leq \frac{\lambda_{\alpha,\mu,n} T}{4} + \frac{C^2 T}{4}$$

for every $n \geq 1$ and $T > 0$, so that

$$\|v\|_{L^2(0,T)} \leq C e^{C T} \|y_0\|_{H^{-1,\mu}} \sum_{n \geq 1} e^{-\lambda_{\alpha,\mu,n} T/4}.$$

On the other hand, by (22) and (23), it can be easily checked that there exists a constant $C > 0$ such that

$$C n^2 \leq \lambda_{\alpha,\mu,n} = j_{\rho(\alpha,\mu)}^2, \forall n \geq 1.$$

Finally, for every $T < T_0$, we then have

$$\|v\|_{L^2(0,T)} \leq C e^{C T + \frac{C}{T}} \|y_0\|_{H^{-1,\mu}} \sum_{n \geq 1} e^{-C n^2 T}$$

$$\leq C e^{C T + \frac{C}{T}} \|y_0\|_{H^{-1,\mu}} \int_0^\infty e^{-C T s^2} ds$$

$$= C e^{C T + \frac{C}{T}} \|y_0\|_{H^{-1,\mu}} \sqrt{\frac{\pi}{T}}$$

$$\leq C e^{C T + \frac{C}{T}} \|y_0\|_{H^{-1,\mu}},$$

where $C$ is independent of $T$. This inequality shows that $v \in L^2(0,T)$ and yields the desired estimate on the null control in the case where $T < T_0$. The case $T \geq T_0$ is actually reduced to the previous one. Indeed, any continuation by zero of a control on $(0,T_0/2)$ is a control on $(0,T)$ and the estimate follows from the decrease of the cost with respect to the time. This completes the proof of Theorem 5.1. \hfill \Box

**REFERENCES**

[1] B. Allal, G. Fragnelli and J. Salhi, Controllability for degenerate/singular parabolic systems involving memory terms, submitted.

[2] B. Allal, A. Hajjaj, L. Maniar and J. Salhi, Null controllability for singular cascade systems of $n$-coupled degenerate parabolic equations by one control force, *Evol. Equ. Control Theory*, 10 (2021), 545–573.

[3] B. Allal, J. Salhi and A. Sbai, Boundary controllability for a coupled system of parabolic equations with singular potentials, in revision, 2021.

[4] F. Ammar-Khodja, A. Benabdallah, M. González-Burgos and L. de Teresa, New phenomena for the null controllability of parabolic systems: Minimal time and geometrical dependence, *J. Math. Anal. Appl.*, 444 (2016), 1071–1113.

[5] F. Ammar-Khodja, A. Benabdallah, M. González-Burgos and L. de Teresa, Minimal time for the null controllability of parabolic systems: The effect of the condensation index of complex sequences, *J. Funct. Anal.*, 267 (2014), 2077–2151.

[6] F. Ammar-Khodja, A. Benabdallah, M. González-Burgos and L. de Teresa, The Kalman condition for the boundary controllability of coupled parabolic systems. Bounds on biorthogonal families to complex matrix exponentials, *J. Math. Pures Appl.*, 96 (2011), 555–590.
[7] A. Benabdallah, F. Boyer, M. González-Burgos and G. Olive, Sharp estimates of the one-dimensional boundary control law for parabolic systems and application to the N-dimensional boundary null controllability in cylindrical domains, *SIAM J. Control Optim.*, 52 (2014), 2970–3001.

[8] A. Benabdallah, F. Boyer and M. Morancey, A block moment method to handle spectral condensation phenomenon in parabolic control problems, *Ann. H. Lebesgue*, 3 (2020), 717–793.

[9] U. Biccari, Boundary controllability for a one-dimensional heat equation with a singular inverse-square potential, *Math. Control Relat. Fields*, 9 (2019), 191–219.

[10] U. Biccari, V. Hernández-Santamaría and J. Vancostenoble, Existence and Cost of Boundary Controls for a Degenerate/Singular Parabolic Equation, Mathematical Control & Related Fields, 2021.

[11] U. Biccari and E. Zuazua, Null controllability for a heat equation with a singular inverse-square potential involving the distance to the boundary function, *J. Differential Equations*, 261 (2016), 2809–2853.

[12] P. Cannarsa, P. Martinez and J. Vancostenoble, The cost of controlling weakly degenerate parabolic equations by boundary controls, *Math. Control Relat. Fields*, 7 (2017), 171–211.

[13] P. Cannarsa, P. Martinez and J. Vancostenoble, The cost of controlling strongly degenerate parabolic equations, *ESAIM Control Optim. Calc. Var.*, 26 (2020), Paper No. 2, 50 pp.

[14] C. Cazacu, Controllability of the heat equation with an inverse-square potential localized on the boundary, *SIAM J. Control Optim.*, 52 (2014), 2055–2089.

[15] C. Cazacu, Schrödinger operators with boundary singularities: Hardy inequality, Pohozaev identity and controllability results, *J. Funct. Anal.*, 263 (2012), 3741–3783.

[16] J.-M. Coron, *Control and Nonlinearity*, Mathematical Surveys and Monographs, 136, American Mathematical Society, Providence, RI, 2007.

[17] E. B. Davies, *Spectral Theory And Differential Operators*, Cambridge Studies in Advanced Mathematics, vol. 42, Cambridge University Press, Cambridge, 1995.

[18] M. Duprez, Controllability of a 2 x 2 parabolic system by one force with space-dependent coupling term of order one, *ESAIM Control Optim. Calc. Var.*, 23 (2017), 1473–1498.

[19] A. Elbert, Some recent results on the zeros of Bessel functions and orthogonal polynomials, *J. Comput. Appl. Math.*, 133 (2001), 65–83.

[20] S. Ervedoza, Control and stabilization properties for a singular heat equation with an inverse-square potential, *Comm. Partial Differential Equations*, 33 (2008), 1996–2019.

[21] H. O. Fattorini and D. L. Russell, Uniform bounds on biorthogonal functions for real exponentials with an application to the control theory of parabolic equations, *Quart. Appl. Math.*, 32 (1974/75), 45–69.

[22] H. O. Fattorini and D. L. Russell, Exact controllability theorems for linear parabolic equations in one space dimension, *Arch. Ration. Mech. Anal.*, 43 (1971), 272–292.

[23] E. Fernández-Cara, M. González-Burgos and L. de Teresa, Boundary controllability of parabolic coupled equations, *J. Funct. Anal.*, 259 (2010), 1720–1758.

[24] G. Fragnelli, Interior degenerate/singular parabolic equations in nondivergence form: well-posedness and Carleman estimates, *J. Differential Equations*, 260 (2016), 1314–1371.

[25] G. Fragnelli and D. Mugnai, Carleman estimates for singular parabolic equations with interior degeneracy and non smooth coefficients, *Adv. Nonlinear Anal.*, 6 (2017), 61–84.

[26] M. González-Burgos and G. R. Sousa-Neto, Boundary controllability of a one-dimensional phase-field system with one control force, *J. Differential Equations*, 269 (2020), 4286–4331.

[27] M. Gueye, Exact boundary controllability of 1-D parabolic and hyperbolic degenerate equations, *SIAM J. Control Optim.*, 52 (2014), 2037–2054.

[28] A. Hajjaj, L. Maniar and J. Salhi, Carleman estimates and null controllability of degenerate/singular parabolic systems, *Electron. J. Differential Equations*, 2016 (2016), Paper No. 292, 25 pp.

[29] C. Heil, *A Basis Theory Primer, Expanded Edition, Applied and Numerical Harmonic Analysis*, Birkhäuser/Springer, New York, 2011.

[30] V. Komornik and P. Loreti, *Fourier Series in Control Theory*, Springer, Berlin, 2005.

[31] N. N. Lebedev, *Special Functions and their Applications*, Dover Publications, New York, 1972.

[32] J.-L. Lions and E. Magenes, *Problèmes aux Limites non Homogènes et Applications*, Vol. 3. (French) Travaux et Recherches Mathématiques, No. 20. Dunod, Paris, 1970.
[33] L. Lorch and M. E. Muldoon, Monotonic sequences related to zeros of Bessel functions, *Numer. Algorithms*, 49 (2008), 221–233.

[34] P. Martinez and J. Vancostenoble, The cost of boundary controllability for a parabolic equation with inverse square potential, *Evol. Equ. Control Theory*, 8 (2019), 397–422.

[35] J. Salhi, Null controllability for a singular coupled system of degenerate parabolic equations in nondivergence form, *Electron. J. Qual. Theory Differ. Equ.*, (2018), Paper No. 31, 28 pp.

[36] J. Vancostenoble, Improved Hardy-Poincaré inequalities and sharp Carleman estimates for degenerate/singular parabolic problems, *Discrete Contin. Dyn. Syst. Ser. S*, 4 (2011), 761–790.

[37] J. Vancostenoble and E. Zuazua, Null controllability for the heat equation with singular inverse-square potentials, *J. Funct. Anal.*, 254 (2008), 1864–1902.

[38] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, second edition, Cambridge University Press, Cambridge, 1944.

[39] J. Zabczyk, *Mathematical Control Theory: An Introduction*, Birkhäuser, Boston, 1995.

Received for publication April 2021; early access October 2021.

E-mail address: b.allal@uhp.ac.ma
E-mail address: abdelkarim.hajjaj@uhp.ac.ma
E-mail address: sj.salhi@gmail.com
E-mail address: a.sbai@uhp.ac.ma