On Surjectivity of Invariant Differential Operators

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Abstract

By proving a topological Paley-Wiener Theorem for Riemannian symmetric spaces of non-compact type, we show that a non-zero invariant differential operator is a homeomorphism from the space of test functions onto its image and hence surjective when extended to the space of distributions.

1 Introduction

Let \( \mathscr{D}(\mathbb{R}^n) \) be the space of compactly supported smooth functions on \( \mathbb{R}^n \). Define the Euclidean Paley-Wiener space \( \mathcal{H}^R(\mathbb{C}^n) \) to be the space of holomorphic maps \( \varphi : \mathbb{C}^n \rightarrow \mathbb{C} \) such that

\[
|\varphi(\lambda)| \leq C_N e^{R|\text{Im}\lambda|}(1 + |\lambda|)^{-N}.
\]

This is topologized by the seminorms \( \|\varphi\|_N \) which are the smallest constants \( C_N \) for which the estimates hold. By the Paley-Wiener Theorem, the Euclidean Fourier transform

\[
\tilde{f}(\xi) := \int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi}dx
\]

is a linear homeomorphism \( \mathscr{D}(\mathbb{R}^n) \rightarrow \bigcup_{R>0} \mathcal{H}^R(\mathbb{C}^n) \) when the spaces are given the inductive limit topology.

Now let \( D \neq 0 \) be a constant coefficient differential operator on \( \mathbb{R}^n \). It is well-known that conjugation of such a differential operator with the Fourier transform is just multiplication by a (non-zero) polynomial, \( P_D \). This multiplication map turns out to be a homeomorphism onto its image, and consequently this holds also for \( D \). An immediate consequence of this is that the differential operator is surjective on the space of distributions and hence always admit weak solutions (for more on this see [4] Chapter VII).

In the article [1] it is stated that this result may be generalized to invariant differential operators on symmetric spaces of non-compact type. However the central arguments are either rather sketchy ([1], Lemma 8) or non-existing ([1], Theorem 7). In this article we remedy this by providing a different approach.

2 Notation

First we introduce some notation. Let \( X \) be a Riemannian symmetric space of non-compact type, i.e. a quotient \( G/K \) where \( G \) is a semisimple non-compact Lie-group with finite center and \( K \) is a maximal compact subgroup. The Lie algebra \( \mathfrak{g} \) has a Cartan decomposition \( \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \) and we pick in \( \mathfrak{p} \) a maximally abelian subalgebra \( \mathfrak{a} \). Define \( M := Z_K(\mathfrak{a}) \) and \( B := K/M \) and
let $\mathfrak{a}_c^*$ denote the complexification of the dual of $\mathfrak{a}$. Let $\Sigma \subseteq \mathfrak{a}^*$ denote the set of restricted roots w.r.t. $(g, \mathfrak{a})$ and let $\Sigma^+$ denote the set of positive roots relative to a fixed Weyl chamber. Put $\rho := \frac{1}{2} \sum_{\lambda \in \Sigma^+} \lambda$.

We let $\mathcal{D}(X)$ denote the set of test functions ($C^\infty$-functions with compact support) equipped with its usual inductive limit topology, and let $\mathcal{D}_R(X)$ denote the closed subspace of test functions whose support is contained in the closed $R$-ball $B_R(eK)$ around $eK \in X$.

The Fourier transform $\tilde{f} : \mathfrak{a}_c^* \times B \to \mathbb{C}$ of a function $f \in \mathcal{D}(X)$ is defined by (see e.g. [2], Chapter III)

$$\tilde{f}(\lambda, b) := \int_X f(x) e^{-i\lambda(x,b)} \, dx$$

where $A(gK, kM) := A(k^{-1}g)$ with $A : G \to \mathbb{R}$ being the Iwasawa projection from the $\text{NAK}$-decomposition of $G$.

Let $\mathcal{H}^R(\mathfrak{a}_c^* \times B)$ denote the space of smooth functions $\psi : \mathfrak{a}_c^* \times B \to \mathbb{C}$ which are holomorphic on $\mathfrak{a}_c^*$ and which satisfy the following growth condition

$$\forall N \in \mathbb{Z}_{>0} \exists C_N \forall \lambda, b : |\psi(\lambda, b)| \leq C_N e^{R|\text{Im}\lambda|(1 + |\lambda|)^{-N}}. \quad (1)$$

Furthermore, we denote by $\mathcal{H}^R(\mathfrak{a}_c^* \times B)_W$ the subset of functions satisfying the following Weyl invariance for each $s \in W$:

$$\forall x \in X \forall \lambda \in \mathfrak{a}_c^* : \int_B e^{(is - \lambda)b} A(x,b) \psi(s \cdot \lambda, b) \, db = \int_B e^{(i\lambda + b)x} \psi(\lambda, b) \, db. \quad (2)$$

3 A Topological Paley-Wiener Theorem

First we need a topological Paley-Wiener Theorem and in order to do so, we have to topologize the space $\mathcal{H}^R(\mathfrak{a}_c^* \times B)_W$. For this, introduce the space $\mathcal{H}^R(\mathfrak{a}_c^*, L^2(B))$ to be the space consisting of holomorphic maps $\psi : \mathfrak{a}_c^* \to L^2(B)$ satisfying

$$||\psi(\lambda)||_{L^2(B)} \leq C_N e^{R|\text{Im}\lambda|(1 + |\lambda|)^{-N}}$$

for all $N$. We define $||\psi||_N$ to be the smallest such constant $C_N$, and we topologize $\mathcal{H}^R(\mathfrak{a}_c^*, L^2(B))$ by this family of seminorms. This turns it into a Fréchet space. The Weyl invariance (2) still makes sense in this generalized setting, and thus we define $\mathcal{H}^R(\mathfrak{a}_c^*, L^2(B))_W$ to be the subset of Weyl invariant elements. This is a closed subspace and hence a Fréchet space.

We have an obvious inclusion $\mathcal{H}^R(\mathfrak{a}_c^* \times B)_W \to \mathcal{H}^R(\mathfrak{a}_c^*, L^2(B))_W$ and this inclusion turns out to be surjective:

**Lemma 3.1.** It holds that $\mathcal{H}^R(\mathfrak{a}_c^*, L^2(B))_W = \mathcal{H}^R(\mathfrak{a}_c^* \times B)_W$ as vector spaces.

**Proof.** For $\psi \in \mathcal{H}^R(\mathfrak{a}_c^*, L^2(B))_W$ define

$$f(x) := \int_{\mathfrak{a}_c^* \times B} \psi(\lambda, b) e^{(i\lambda + b)x} \, d\lambda = \int_{\mathfrak{a}_c^*} \langle \psi(\lambda), \lambda \rangle \, d\lambda.$$

Obviously, $f$ is a smooth function. By examining the proof of bijectivity of $F : \mathcal{D}(X) \to \mathcal{H}(\mathfrak{a}_c^* \times B)_W$ (e.g. in [2], p. 278–280), it is seen that $f$ is supported in the closed $R$-ball $B_R(eK)$ and that $\tilde{f} - \tilde{\psi} = 0$ almost everywhere, and thus $f$ is a smooth representative of $\psi$. Furthermore, $\tilde{f}$ satisfies the stronger growth condition (1) and hence $f \in \mathcal{H}^R(\mathfrak{a}_c^* \times B)_W$. \(\Box\)

Now $\mathcal{H}^R(\mathfrak{a}_c^* \times B)_W$ inherits the topology from $\mathcal{H}^R(\mathfrak{a}_c^*, L^2(B))_W$ (given by the seminorms $|| \cdot ||_N$), and hence it becomes a Fréchet space. Furthermore we define

$$\mathcal{H}(\mathfrak{a}_c^* \times B)_W := \bigcup_{R \in \mathbb{Z}_{>0}} \mathcal{H}^R(\mathfrak{a}_c^* \times B)_W$$

and give it the inductive limit topology.
Theorem 3.2 (Topological Paley-Wiener). The Fourier transform $\mathcal{F} : \mathcal{D}(X) \rightarrow \mathcal{H}(a^*_C \times B)_W$ is a linear homeomorphism. Furthermore $\tilde{f} \in \mathcal{H}^R(a^*_C \times B)_W$ if and only if $f \in \mathcal{D}(X)$.

Proof. The bijectivity of $\mathcal{F}$ as well as the last claim is stated and proved in [?] Theorem III.5.1.

Now we consider $\mathcal{F} : \mathcal{D}_R(X) \rightarrow \mathcal{H}^R(a^*_C \times B)_W$ for a given $R$. For $f \in \mathcal{D}_R(X)$ it is straightforward to check the inequality for each $N$:

$$\|\mathcal{F} f\|_N \leq C \sup_{\lambda \in a^*_C, b \in B} e^{-R|\text{Im}\lambda|} \int_{B_R(e^K)} |Df(x)||e^{(-i\lambda + \rho)A(x,b)}|dx$$

where $D$ is the invariant differential operator (of order $2N$) on $X$ corresponding to the invariant polynomial $(1 + |\lambda|^2)^N$ (as in \(\Box\)) and where $C$ is a constant depending on $N$ and $R$. Since $x \in B_R(e^K)$ we have by [?] p. 476 eq. (13) that $|A(x,b)| \leq R$ and hence we see that

$$e^{-R|\text{Im}\lambda|} |e^{(-i\lambda + \rho)A(x,b)}| = e^{(\text{Im}\lambda + \rho)A(x,b) - R|\text{Im}\lambda|} \leq e^{R|\rho|}.$$ 

Hence we get $\|\mathcal{F} f\|_N \leq C\|f\|_{2N}$, where $\| \cdot \|_{2N}$ is one of the standard seminorms on $\mathcal{D}_R(X)$, i.e. $\mathcal{F} : \mathcal{D}_R(X) \rightarrow \mathcal{H}^R(a^*_C \times B)_W$ is continuous.

Thus the Fourier transform is a homeomorphism $\mathcal{D}_R(X) \xrightarrow{\sim} \mathcal{H}^R(a^*_C \times B)_W$ since these spaces are Fréchet. Hence it is also a homeomorphism when defined on $\mathcal{D}(X)$.

4 Consequences of the Paley-Wiener Theorem

Now, let $D$ be a non-zero invariant differential operator. There exists a $W$-invariant polynomial $P_D \neq 0$ on $a^*_C$ (this is a consequence of [3] Theorem II.4.6 and Lemma II.5.14 where we identify $\mathbb{D}(A)$ with $W$-invariant polynomials on $a$) such that the following diagram commutes

\[
\begin{array}{ccc}
\mathcal{D}_R(X) & \xrightarrow{\sim} & \mathcal{H}^R(a^*_C \times B)_W \\
D \downarrow & & \downarrow M_{P_D} \\
\mathcal{D}_R(X) & \xrightarrow{\sim} & \mathcal{H}^R(a^*_C \times B)_W
\end{array}
\]

where $M_{P_D}$ is multiplication by $P_D$. The first goal is to show that $M_{P_D}$ and hence $D$ are linear homeomorphisms onto their images. Injectivity of $M_{P_D}$ is clear by holomorphicity since $P_D \neq 0$.

Another payoff of considering $\mathcal{H}^R(a^*_C, L^2(B))$ rather than $\mathcal{H}^R(a^*_C \times B)$, is that it admits a nice description as a tensor product. First, however, note that the spaces we defined in Section 1 actually make sense for any finite-dimensional inner product space $V$ and its complexification $V_C$. Thus we can define $\mathcal{D}(V)$, $\mathcal{D}_R(V)$, $\mathcal{H}^R(V_C)$ and so on, as well as a Euclidean Fourier transform which will be a homeomorphism $\mathcal{F} : \mathcal{D}(V) \xrightarrow{\sim} \mathcal{H}(V_C)$. In the following we will take $V = a^*$ with the Killing form as inner product.

Let $\mathcal{H}^R(a^*_C) \hat{\otimes} L^2(B)$ denote the completion of the algebraic tensor product in either the projective or injective topology (they are both equal since $\mathcal{H}^R(a^*_C) \cong \mathcal{D}(a^*)$ is nuclear). Then

Lemma 4.1. There exists a natural linear homeomorphism $\mathcal{H}^R(a^*_C) \hat{\otimes} L^2(B) \xrightarrow{\sim} \mathcal{H}^R(a^*_C, L^2(B))$.

Proof. Letting $\mathcal{D}_R(a^*, L^2(B))$ denote the space of smooth $L^2(B)$-valued functions on $a^*$ with support in the $R$-ball, we define a Fourier transform $\mathcal{F} : \mathcal{D}_R(a^*, L^2(B)) \rightarrow \mathcal{H}^R(a^*_C, L^2(B))$

by

$$\mathcal{F}f)(\lambda) := \int_{a^*} f(x)e^{-i(x,\lambda)}dx$$
(L^2(B)-valued integration). For all \( v \in L^2(B) \) it holds that \( \langle Ff(\lambda), v \rangle = F(\langle f, v \rangle)(\lambda) \). Since the function \( x \mapsto (f(x), v) \) is an element of \( \mathcal{D}_R(a^*) \) we see that \( \lambda \mapsto \langle Ff(\lambda), v \rangle \) is holomorphic, i.e. \( Ff \) is weakly holomorphic, hence holomorphic (as it takes values in a Hilbert space). Furthermore we note that \( F \) has an inverse:

\[
(F^{-1} \psi)(x) = \int_{a^*} \psi(\lambda) e^{i(x, \lambda)} d\lambda.
\]

Continuity of \( F \) is easily checked and since the spaces in question are both Fréchet, \( F \) is a linear homeomorphism. The lemma now follows from the fact that \( D \) is injective and continuous. We just need to show that it’s open. For this, let \( \lambda \in X \) and \( u \in H(X) \) be an invariant differential operator, then

\[
\mathcal{D}_R(a^*, L^2(B)) \cong \mathcal{D}_R(a^*) \otimes L^2(B)
\]

(which is a consequence of [5] Theorem 44.1) and that \( \mathcal{D}_R(a^*) \cong H^R(a^*_C) \) by the Euclidean Fourier transform.

Now returning to the commuting diagram \([3]\) we see that under the identification of \( H^R(a^*_C, L^2(B)) \) with \( H^R(a^*_C) \otimes L^2(B) \), the map \( M_{PD} : H^R(a^*_C, L^2(B)) \rightarrow H^R(a^*_C, L^2(B)) \) is replaced by \( M_{PD} \otimes \text{id}_{L^2(B)} \). And from the Euclidean theory we know that \( M_{PD} : H^R(a^*_C) \rightarrow H^R(a^*_C) \) is a homeomorphism onto its image, hence the same holds for \( M_{PD} \otimes \text{id}_{L^2(B)} \) (cf. [5] Proposition 43.7). By restriction to \( H^R(a^*_C \times B)_W \) we get:

**Lemma 4.2.** The multiplication map \( M_{PD} : H^R(a^*_C \times B)_W \rightarrow H^R(a^*_C \times B)_W \) is a homeomorphism onto its image.

Finally, we need to transfer this to the inductive limit. For this we need the following lemma which is an immediate generalization of [2] Lemma III.5.13 to functions with Hilbert space values:

**Lemma 4.3.** Assume \( P : a^*_C \rightarrow \mathbb{C} \) is a nonzero polynomial and that \( \psi : a^*_C \rightarrow L^2(B) \) is a holomorphic function such that \( P\psi \in H^R(a^*_C, L^2(B)) \), then \( \psi \in H^R(a^*_C, L^2(B)) \).

Restricting to \( H^R(a^*_C, L^2(B))_W = H^R(a^*_C \times B)_W \) and referring to the commutative diagram \([3]\) we get that if \( f \in \mathcal{D}(X) \) is such that \( Df \in \mathcal{D}(X) \), then \( f \in \mathcal{D}(X) \).

We now arrive at our main theorem:

**Theorem 4.4.** Let \( D \neq 0 \) be an invariant differential operator, then \( D : \mathcal{D}(X) \rightarrow \mathcal{D}(X) \) is a homeomorphism onto its image.

**Proof.** Since \( D : \mathcal{D}(X) \rightarrow \mathcal{D}(X) \) is a homeomorphism onto its image, it is clear that \( D \) is injective and continuous. We just need to show that it’s open. For this, let \( U \subseteq \mathcal{D}(X) \) be open, i.e. \( U \cap D_R(X) \) is open for all \( R \). We need to show that \( D(U) \cap \mathcal{D}(X) \) is open for all \( R \). Claim: \( D(U) \cap \mathcal{D}(X) = D(U \cap \mathcal{D}(X)) \). The inclusion “\( \supseteq \)" is clear, since \( D \) decreases support. The inclusion “\( \subseteq \)" is a consequence of Lemma 4.3. But the right hand side \( D(U \cap \mathcal{D}(X)) \) is open in \( \mathcal{D}(X) \), and thus the result follows.

Let \( \mathcal{D}'(X) \) denote the set of distributions over \( X \), i.e. the dual of \( \mathcal{D}(X) \). A simple Hahn-Banach argument (see e.g. [4] p. 236) yields

**Corollary 4.5.** \( D : \mathcal{D}'(X) \rightarrow \mathcal{D}'(X) \) is surjective.

**References**

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