Mixing and Decay Constants of Pseudoscalar Mesons: Octet-Singlet vs. Quark Flavor Basis

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Although \(\eta - \eta'\) mixing is qualitatively well understood as a consequence of the \(U(1)_A\) anomaly in QCD together with a broken \(SU(3)_F\) flavor symmetry, until recently the values of decay and mixing parameters of the \(\eta\) and \(\eta'\) were only approximately known, e.g. values for the octet-singlet mixing angle between \(-20^\circ\) and \(-10^\circ\) could be found in the literature. New experimental data, especially for the reactions \(\gamma\gamma \rightarrow \eta, \eta'\) and \(B \rightarrow \eta' K\), together with new theoretical results from higher order corrections in chiral perturbation theory stimulated a phenomenological re-analysis of this subject, which led to a coherent qualitative and quantitative picture of \(\eta - \eta'\) mixing and even of \(\eta - \eta' - \eta_c\) mixing.

1. \(\eta - \eta'\) Mixing Schemes

A crucial observation of our analysis [1] is the fact that for a proper treatment of the mixing one clearly has to distinguish between matrix elements of \(\eta, \eta'\) states with local currents (e.g. weak decay constants) and overall state mixing. While in the former the \(SU(3)_F\) symmetry breaking effects, \((2m_u/(m_u + m_d) \approx 26)\) turn out to be essential, in the latter the gluon anomaly plays the important role [2]. Correspondingly, one may think of two possible choices of appropriate basis states as a starting point for the description of \(\eta - \eta'\) mixing, namely the quark flavor basis (which becomes exact in the limit \(m_s \rightarrow \infty\)) and the octet-singlet basis (which becomes exact for \(m_u = m_d = m_s\), respectively.

In order to define these bases properly, it is useful to consider a Fock state decomposition of the mesonic states in the parton picture. One then defines the quark flavor basis through

\[
\left. \eta \right\rangle = \cos \phi \left. \eta_q \right\rangle - \sin \phi \left. \eta_u \right\rangle, \\
\left. \eta' \right\rangle = \sin \phi \left. \eta_q \right\rangle + \cos \phi \left. \eta_u \right\rangle
\]

(1)

with \(\phi\) being the mixing-angle and

\[
\left. \eta_q \right\rangle := \frac{\Psi_q |u\bar{u} + d\bar{d}|}{\sqrt{2}} + \frac{\Psi^q_s |gg\rangle}{\sqrt{3}} + \ldots
\]

\[
\left. \eta_u \right\rangle := \frac{\Psi_u |d\bar{u} + s\bar{s}|}{\sqrt{2}} + \frac{\Psi^u_s |gg\rangle}{\sqrt{3}} + \ldots
\]

(2)

Here \(\Psi^q_i\) denote (light-cone) wave functions of the corresponding parton states. The effect of higher Fock states \((|gg\rangle + \ldots)\) is twofold: First, they are necessary for the correct normalization, \(\langle \eta_i | \eta_j \rangle = \delta_{ij}\). Secondly, they reflect the mixing (e.g. through the twist-4 \(|gg\rangle\) component which is present due to the anomaly).

Analogously, in the octet-singlet basis, one obtains

\[
\left. \eta \right\rangle = \cos \theta \left. \eta_8 \right\rangle - \sin \theta \left. \eta_1 \right\rangle, \\
\left. \eta' \right\rangle = \sin \theta \left. \eta_8 \right\rangle + \cos \theta \left. \eta_1 \right\rangle
\]

(3)

with the usual pseudoscalar octet-singlet mixing angle \(\theta = \phi - \arctan \sqrt{2}\). However, the flavor decomposition in the Fock state expansion looks now more complicated due to the broken \(SU(3)_F\) symmetry

\[
\left. \eta_8 \right\rangle := \frac{(\Psi_q |u\bar{u} + d\bar{d}| - 2\Psi_s |s\bar{s}|)}{\sqrt{6}} + \frac{(\Psi^q_s - \sqrt{2} \Psi^u_s |gg\rangle)}{\sqrt{3}} + \ldots
\]

\[
\left. \eta_1 \right\rangle := \frac{(\Psi_q |u\bar{u} + d\bar{d}| + \Psi_s |s\bar{s}|)}{\sqrt{3}} + \frac{(\sqrt{2} \Psi^q_s + \Psi^u_s |gg\rangle)}{\sqrt{3}} + \ldots
\]

(4)

\[\text{Of course, to construct the wave functions of all Fock states explicitly, one has to solve the QCD bound state problem.}\]
Only in the flavor symmetry limit one would have trivial relations between the wave functions: 
\[ \Psi_q = \Psi_s = \Psi_8 = \Psi_1, \quad \Psi_{q'} = \sqrt{2}\Psi_s = \sqrt{2}\Psi_5^{1}\sqrt{3}, \]
\[ \Psi_{8} = 0, \text{ etc.} \] Only in this case one would recover the usually anticipated form of octet and singlet states \(|n_k\rangle \rightarrow \Psi_s|\bar{u}u + d\bar{d} - 2ss\rangle/\sqrt{6} + \ldots \) and \(|n_1\rangle \rightarrow \Psi_1|\bar{u}u + d\bar{d} + s\bar{s}\rangle/\sqrt{3} + \Psi_5^{1}|gg\rangle + \ldots \)

Note that in higher Fock states with increasing number of partons the effect of \(SU(3)_F\) symmetry breaking is washed out (e.g. the ratio of constituent quark masses is only \(2\tilde{m}_u/(\tilde{m}_u + \tilde{m}_d) \approx 5/3\)), and thus the octet-singlet basis is still useful for low-energy expansions of QCD like e.g. chiral perturbation theory (ChPT). However, weak decay constants only probe the short-distance properties of the valence Fock states and are thus rather sensitive to \(SU(3)_F\) breaking effects. To see this in more detail, let us define the decay constant\(^2\) as (\(f_\pi = 131 \text{ MeV}\))

\[ \langle 0| J^\mu_{5}\rangle |\pi\rangle \equiv i f_\pi p_\mu \] (5)

with \(P = n, n'\); \(i = q, s \ (i = 8, 1)\), and the relevant flavor combinations of axial-vector currents denoted as \(J^\mu_{5}\). Using Eqs. (6) one obtains

\[
\begin{pmatrix}
    f_8^q & f_8^s \\
    f_8^{q'} & f_8^{s'}
\end{pmatrix}
= 
\begin{pmatrix}
    f_q \cos \phi & -f_s \sin \phi \\
    f_q \sin \phi & f_s \cos \phi
\end{pmatrix}
= 
U(\phi) \text{ diag}[f_q, f_s] \tag{6}
\]

with \(f_q(f_s)\) related to the wave function \(\Psi_q(\Psi_s)\) at the origin\(^3\) and with \(U\) being a usual rotation

\(^2\)We stress that occasionally used decay constants “\(f_\pi, f_{\pi'}\)” are ill-defined quantities.

\(^3\) The decay constants are calculated from the Fock state decomposition as follows (for concreteness we chose the \(|n_k\) state as an example)

\[ \langle 0| J^\mu_{5}|n_k\rangle \rangle \equiv \sum_n \int d^4x \int d^3k_\perp \frac{\Psi_s(x, k_\perp)}{16\pi^3} \langle 0| J^\mu_{5}|8\rangle \langle 8|n_k\rangle \]

\[ = \sum_{\alpha} \frac{\delta_{\alpha 8}}{\sqrt{N_\alpha}} \int d^4x \int d^3k_\perp \frac{\Psi_s(x, k_\perp)}{16\pi^3} \langle \gamma_\mu \gamma_5 \gamma_{\alpha}|8\rangle \]

\[ = \frac{i}{2}\sqrt{2}N_8 \int d^4x \int d^3k_\perp \frac{\Psi_s(x, k_\perp)}{16\pi^3} \langle \gamma_\mu |8\rangle \]

Here \(x\) denotes the usual (light-cone +) momentum fraction of the quark and \(k_\perp\) its transverse momentum. Note that only the leading quark-antiquark Fock state contributes to the decay constant, i.e. Eq. (6) is exact.

matrix identical to the one of the state mixing\(^4\).

In the octet-singlet basis one obtains on the other hand

\[
\begin{pmatrix}
    f_8^q & f_8^s \\
    f_8^{q'} & f_8^{s'}
\end{pmatrix}
= 
\begin{pmatrix}
    f_q \cos \theta_8 & -f_s \sin \theta_8 \\
    f_q \sin \theta_8 & f_s \cos \theta_8
\end{pmatrix}
= 
U(\theta) \text{ diag}[f_q, f_s] \tag{8}
\]

where we introduced the parametrization of \(\theta_8 = \phi - \arctan \frac{\sqrt{2}f_q}{f_s} \quad f_8^2 = \frac{f_q^2 + f_s^2}{3}, \quad f_8^3 = \frac{2f_q^2 + f_s^2}{3} \tag{9}\)

Note that the decay constants do not simply follow the state mixing in the octet-singlet basis; – only in the \(SU(3)_F\) symmetry limit one has \(\theta_8 \rightarrow \theta \leftarrow \theta_1\). Especially the matrix elements of octet/singlet currents with the opposite states do not vanish, \(\langle 0| J^\mu_{5}|n_s\rangle = i p_\mu \sin(\theta - \theta_1) f_1\) and \(\langle 0| J^\mu_{5}|n_1\rangle = i p_\mu \sin(\theta_1 - \theta) f_1\).

The difference between \(\theta_8\) and \(\theta_1\) following from Eq. (8) is analogous to the one derived within ChPT\(^5\).

2. Masses and Decay Constants

The important relation that connects short-distance properties, i.e. decay constants, with long-distance phenomena, i.e. mass-mixing, is provided by the divergences of axial-vector currents including the anomaly \((i = u, d, s, c, \ldots)\)

\[ \partial^\mu \bar{q}_i \gamma_\mu \gamma_5 q_i = 2m_i j_5^i + \frac{\alpha_s}{4\pi} G \bar{G}, \tag{10} \]

with \(j_5^i = \bar{q}_i \gamma_\gamma q_i\). Taking matrix elements \(\langle 0| \ldots |P\rangle\) (for instance \(\langle 0| \partial^\mu j^\mu_{5}\rangle \rangle = M^2 f_8^2\)) and using the definition of the decay constants\(^6\), the mass matrix in the quark flavor basis is fixed to have the following structure

\[ U(\phi) \text{ diag}[M^2 q, M^2 s] \]

\[ \equiv \left( \begin{array}{cc}
    m^2_{qq} + 2a^2 & \sqrt{2}ya^2 \\
    \sqrt{2}ya^2 & m^2_{ss} + y^2a^2
\end{array} \right) \tag{11} \]

with

\[ m^2_{qq} = 2m_q \langle 0| j^\mu_{5}|q\rangle / f_q \simeq M^2_q; \]

\[ m^2_{ss} = 2m_s \langle 0| j^\mu_{5}|s\rangle / f_s \simeq M^2_s - M^2_q \tag{12} \]

\(^4\)We like to emphasize that Eq. (6) is not to be read as \(|\eta\rangle = \cos \theta_8 |n_q\rangle - \sin \theta_8 |n_1\rangle\) etc., i.e. Eq. (9) still holds.
and
\[
\alpha^2 = \frac{1}{\sqrt{2}f_q} \langle 0|\frac{\alpha_s}{4\pi} G\bar{G}|\eta_9\rangle, \quad y = \frac{f_q}{f_s} \tag{13}
\]

The mass matrix in the octet-singlet basis can simply be obtained from \(11\) by a rotation about the ideal mixing angle. Solving for \(\phi, y, a^2\) and using \(f_q \simeq f_\pi, f_s \simeq \sqrt{2}f_K^2 - f_\pi^2\), one obtains the “theoretical” values quoted in Table 1.

Alternatively, the mixing parameters can be determined from phenomenology without using the \(SU(3)_F\) relations for \(m_\eta^2\) and \(f_\eta^2\). The mixing angle \(\phi\) can be determined by considering appropriate ratios of decay widths/cross sections, in which only the \(\eta_9\) or \(\eta_8\) component is probed, respectively. The analysis of several independent decay and scattering processes performed in \(10\) leads to \(\phi = 39.3^\circ \pm 1.0^\circ\). It is to be stressed that the so-obtained values for the mixing angle \(\phi\) (or equivalently for \(\theta = \phi - \arctan \sqrt{2}\)) are all consistent with each other with a small experimental uncertainty and agree with the “theoretical” ones within 10%.

With this value of the mixing angle the decay constants \(f_q\) and \(f_s\) can be estimated from the \(\eta, \eta' \rightarrow \gamma \gamma\) decay widths:
\[
\Gamma[\eta \rightarrow \gamma \gamma] = \frac{9\alpha^2 M_\eta^2}{16\pi^3} \left( \frac{C_q \cos \phi}{f_q} - \frac{C_s \sin \phi}{f_s} \right)^2,
\]
\[
\Gamma[\eta' \rightarrow \gamma \gamma] = \frac{9\alpha^2 M_{\eta'}^2}{16\pi^3} \left( \frac{C_q \sin \phi}{f_q} + \frac{C_s \cos \phi}{f_s} \right)^2
\]
\tag{14}

where \(C_q = 5/9\sqrt{2}\) and \(C_s = 1/9\) are the proper charge factors. Combined with the additional information from the structure of the mass matrix, one obtains \(f_q = (1.07 \pm 0.02) f_\pi\) and \(f_s = (1.34 \pm 0.06) f_\pi\) (see also Table 1). Note that the corresponding difference between \(\theta_8, \theta, \theta_1\) (although formally a higher order \(SU(3)_F\) breaking effect) is enormous!

A prominent example which illustrates the difference between the conventional approach with \(\theta_8 = \theta = \theta_1\) and the present one is given by the \(J/\psi \rightarrow P\gamma\) decays. Following \(14\) the decay rates are proportional to the matrix elements \(\langle 0|\pi G\bar{G}|P\rangle|^2\) which can be calculated using Eqs. \(11,14\) and \(m_c \simeq m_d \simeq 0\), leading to
\[
\frac{\Gamma[J/\psi \rightarrow \eta' \gamma]}{\Gamma[J/\psi \rightarrow \eta \gamma]} = \frac{\tan^2 \phi}{M_{\eta'}^2/M_\eta^2} \left( \frac{k_\eta'}{k_\eta} \right)^3 \approx \cot^2 \theta_8 \left( \frac{k_\eta}{k_\eta'} \right)^3
\]
\tag{15}

from which one obtains by comparison with the experimental value \(\theta = 39.0^\circ \pm 1.6^\circ\) (or \(\theta = -15.7^\circ \pm 1.6^\circ\)) and \(\theta_8 = -22.0^\circ \pm 1.2^\circ\).

Direct information on the decay constants \(f_\eta\) can also be obtained from the analysis of the form factors for \(\gamma^* \gamma \rightarrow P\) at large photon virtualities, which are dominated by the valence Fock states in \(24\). Using the modified hard-scattering approach (see \(6,8\) and references therein), again, the phenomenological parameter set in Table 1 leads to a perfect description of the experimental data \(14,15\).

3. \(\eta - \eta' - \eta_c\) Mixing

Since the derivation of the pseudoscalar mass matrix via Eq. \(13\) does not have to make use of flavor symmetry, it can be generalized to \(\eta - \eta' - \eta_c\) mixing in a straight forward manner \(14\), leading to a similar mass matrix as in Eq. \(13\)
\[
\left( \begin{array}{ccc}
 m_{\eta 0}^2 + 2a^2 & \sqrt{2}ya^2 & \sqrt{2}za^2 \\
 \sqrt{2}ya^2 & m_{s 0}^2 + y^2a^2 & yza^2 \\
 \sqrt{2}za^2 & yza^2 & m_{c 0}^2 + z^2a^2
\end{array} \right)
\]
\tag{16}

Of course the mixing between light and heavy pseudoscalars is suppressed by the heavy masses, i.e. \(a^2/m_{\eta_c}^2\) may be treated as a small parameter, leading to \(m_{\eta_c}^2 \approx M_{\eta_c}^2\). The second new parameter is also unambiguously fixed \(z = f_q/f_c \approx f_q/f_{J/\psi} = 0.35\).

From the phenomenological point of view, namely from the rather large branching ratio for \(B \rightarrow K\eta'\) reported by CLEO \(11\), one is mostly interested in the matrix elements of \(\eta, \eta'\) with the charm axial-vector current \(\langle 0|\gamma_{\mu}\gamma_5 c|P\rangle\)
Table 1
Theoretical and phenomenological values of mixing parameters (for details, see [1]).

|                  | $f_{\eta}/f_{\pi}$ | $f_{\eta}/f_{\pi}$ | $\phi$ | $y$ | $a^2$ [GeV$^2$] | $f_{\phi}/f_{\pi}$ | $f_{1}/f_{\pi}$ | $\theta$ | $\theta_{\phi}$ | $\theta_{1}$ |
|------------------|---------------------|---------------------|--------|----|----------------|---------------------|-----------------|---------|----------------|----------|
| theory           | 1.00                | 1.41                | 42.4$^o$ | 0.78 | 0.281          | 1.28                | 1.15            | -12.3$^o$ | -21.0$^o$ | -2.7$^o$ |
| phenom.          | 1.07                | 1.34                | 39.3$^o$ | 0.81 | 0.265          | 1.26                | 1.17            | -15.4$^o$ | -21.2$^o$ | -9.2$^o$ |

$t f_{\pi} p_{\mu}$. From the diagonalization of the mass matrix one obtains the following values

$$
\begin{align*}
 f_{\eta}^c &= -f_c \theta_c \sin \theta_8 = (-2.4 \pm 0.2) \text{ MeV}, \\
 f_{\eta}'^c &= f_c \theta_c \cos \theta_8 = (-6.3 \pm 0.6) \text{ MeV}
\end{align*}
$$

(17)

where we have defined the mixing angle $\theta_c = -z \sqrt{2 + y^2/a^2/M^2_{\eta'}} \approx -1.0^o$, which is reasonably small and in accord with Refs. [2] and, in particular, with the independent bounds found from the analysis of the $\eta \gamma$ and $\eta' \gamma$ transition form factors [1]. Obviously, the intrinsic charm in $\eta'$ cannot induce a dominant contribution to the $B \to K \eta'$ decays (via $b \to s c \bar{c}$), contrary to what is assumed occasionally [13,14].

An immediate test of the parameter values is provided by a similar ratio of $J/\Psi$ decay widths as in Eq. (11). Most interestingly, via Eq. (11), the intrinsic charm picture (i.e. $J/\Psi \to c \bar{c} \gamma$, $c \bar{c} \to \eta'$) and the gluon picture of ref. [1] turn out to be equivalent with the result [1]

$$
\frac{\Gamma[J/\Psi \to \eta' \gamma]}{\Gamma[J/\Psi \to \eta \gamma]} = \theta_c^2 \cos \theta_8^2 \left( \frac{k_{\eta'}}{k_{\eta}} \right)^3 = \left( \frac{|G|^2}{\sqrt{2 f_{\eta} M_{\eta'}}} \right)^2 \left( \frac{k_{\eta'}}{k_{\eta}} \right)^3
$$

(18)

The values of $\theta_c$ and $\theta_8$ found in our approach perfectly reproduce the experimental value for this ratio [1].

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