An upper bound for the number of perfect matchings in graphs

Shmuel Friedland∗

Department of Mathematics, Statistics, and Computer Science,
University of Illinois at Chicago
Chicago, Illinois 60607-7045, USA
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Abstract

We give an upper bound on the number of perfect matchings in an undirected simple graph \( G \) with an even number of vertices, in terms of the degrees of all the vertices in \( G \). This bound is sharp if \( G \) is a union of complete bipartite graphs. This bound is a generalization of the upper bound on the number of perfect matchings in bipartite graphs on \( n + n \) vertices given by the Bregman-Minc inequality for the permanents of \((0, 1)\) matrices.

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1 Introduction

Let \( G = (V, E) \) be an undirected simple graph with the set of vertices \( V \) and edges \( E \). For a vertex \( v \in V \) denote by \( \deg v \) the degree of the vertex \( v \). Assume that \( |V| \) is even. Denote by \( \text{perfmat} G \) the number of perfect matching in \( G \). Our main result states that

\[
\text{perfmat} G \leq \prod_{v \in V} ((\deg v)!)^{\frac{1}{2 \deg v}}, \tag{1.1}
\]

We assume here that \( 0^\frac{1}{2} = 0 \). This result is sharp if \( G \) is a disjoint union of complete bipartite graphs. For bipartite graphs the above inequality follows from the Bregman-Minc inequality for the permanents of \((0, 1)\) matrices, conjectured by Minc [4] and proved by Bregman [2]. In fact, the inequality (1.1) is the analog of the Bregman-Minc inequality for the hafnians of \((0, 1)\) symmetric of even order with zero diagonal. Our proof follows closely the proof of the Bregman-Minc inequality given by Schrijver [6].

2 Permanents and Hafnians

If \( G \) is a bipartite graph on \( n + n \) vertices then \( \text{perfmat} G = \text{perm} B(G) \), where \( B(G) = [b_{ij}] \in \{0, 1\}^{n \times n} \) is the incidence matrix of the bipartite graph \( G \). Thus \( V = V_1 \cup V_2 \) and \( E \subset V_1 \times V_2 \), where \( V_i = \{v_{1,i}, \ldots, v_{n,i}\} \) for \( i = 1, 2 \). Then \( b_{ij} = 1 \) if and only if \((v_{i,1}, v_{j,2}) \in E \). Recall that the permanent of \( B \in \mathbb{R}^{n \times n} \) is given by \( \text{perm} B = \sum_{\sigma \in S_n} \prod_{i=1}^{n} b_{i \sigma(i)} \), where \( S_n \) is the symmetric group of all permutations \( \sigma : \langle n \rangle \to \langle n \rangle \).

Vice versa, given any \((0, 1)\) matrix \( B = [a_{ij}] \in \{0, 1\}^{n \times n} \), then \( B \) is the incidence matrix of the induced \( G(B) = (V_1 \cup V_2, E) \). Denote by \( \langle n \rangle := \{1, \ldots, n\} \), \( m + \langle n \rangle := \{m + 1, \ldots, m + n\} \).

∗Visiting Professor, Fall 2007 - Winter 2008, Berlin Mathematical School, Berlin, Germany
n} for any two positive integers m, n. It is convenient to identify V₁ = ⟨n⟩, V₂ = n + ⟨n⟩. Then rᵢ := ∑ₙᵢ=₁ bᵢᵢ is the i-th degree of i ∈ ⟨n⟩. The celebrated Bregman-Minc inequality, conjectured by Minc [1] and proved by Bregman [2], states
\[ \text{perm} B \leq \prod_{i=1}^{n} (rᵢ) \frac{1}{rᵢ}. \] (2.1)

A simple proof Bregman-Minc inequality is given [6]. Furthermore the above inequality is generalized to nonnegative matrices. See [1, 5] for additional proofs of (2.1).

**Proposition 2.1** Let G = (V₁ ∪ V₂, E) be a bipartite graph with #V₁ = #V₂. Then (2.1) holds. If G is a union of complete bipartite graphs then equality holds in (2.1).

**Proof** Assume that #V₁ = #V₂ = n. Clearly,
\[ \text{perfmat}_G = \text{perm} B(G) = \text{perm} B(G)^\top = \sqrt{\text{perm} B(G)} \sqrt{\text{perm} B(G)^\top}. \]

Note that the i-th row sum of B(G)^\top is the degree of the vertex n + i ∈ V₂. Apply the Bregman-Minc inequality to perm B(G) and perm B(G)^\top to deduce (1.1).

Assume that G is the complete bipartite graph Kᵣᵣ on r + r vertices. Then B(Kᵣᵣ) = Jᵣ = {1}ᵣ×r. So perfmat Kᵣᵣ = r!. Hence equality holds in (1.1). Assume that G is a (disjoint) union of G₁, . . . , Gᵢ. Since perfmat G = ∏ᵢ=1 perfmat Gᵢ, we deduce (1.1) is sharp if each Gᵢ is a complete bipartite graph.

Let A(G) ∈ {0, 1}ᵐ×ᵐ be the adjacency matrix of an undirected simple graph G on m vertices. Note that A(G) is a symmetric matrix with zero diagonal. Vice versa, any symmetric (0, 1) matrix with zero diagonal induces an indirected simple graph G(A) = (V, E) on m vertices. Identify V with ⟨m⟩. Then rᵢ, the i-th row sum of A, is the degree of the vertex i ∈ ⟨m⟩.

Let K₂n be the complete graph on 2n vertices, and denote by M(K₂n) the set of all perfect matches in K₂n. Then α ∈ M(K₂n) can be represented as α = {(i₁, j₁), (i₂, j₂), . . . , (iₙ, jₙ)} with iₖ < jₖ for k ∈ ⟨n⟩. It is convenient to view (iₖ, jₖ) as an edge in K₂n. We can view α as an involution in S₂n with no fixed points. So for l ∈ ⟨2n⟩ α(l) is second vertex corresponding to l in the perfect match given by α. Vice versa, any fixed point free involution of (2n) induces a perfect match α ∈ M(K₂n). Denote by Sₘ the space of m × m real symmetric matrices. Assume that A = [aᵢⱼ] ∈ S₂n. Then the hafnian of A is defined as
\[ \text{hafn} A := \sum_{\alpha = \{(i₁, j₁),(i₂, j₂), . . . , (iₙ, jₙ)\} \in \mathcal{M}(K₂n)} \prod_{k=1}^{n} a_{iₖ, jₖ}. \] (2.2)

Note that hafn A does not depend on the diagonal entries of A. Let i ≠ j ∈ ⟨2n⟩. Denote by A(i, j) ∈ S₂n₋₂ the symmetric matrix obtained from A by deleting the i, j rows and columns of A. The following proposition is straightforward, and is known as the expansion of the hafnian by the row, (column), i.

**Proposition 2.2** Let A ∈ S₂n. Then for each i ∈ ⟨2n⟩
\[ \text{hafn} A = \sum_{j \in ⟨2n⟩ \setminus \{i⟩} aᵢⱼ \text{hafn} A(i, j) \] (2.3)

It is clear that perfmat G = hafn A(G) for any G = ⟨⟨2n⟩⟩, E). Then (1.1) is equivalent to the inequality
\[ \text{hafn} A \leq \prod_{i=1}^{2n} (rᵢ) \frac{1}{rᵢ}, \text{ for all } A \in \{0, 1\}^{⟨2n⟩×⟨2n⟩} \cap S₂n,₀ \] (2.4)

Our proof of the above inequality follows the proof of the Bregman-Minc inequality given by A. Schrijver [6].
3 Preliminaries

Recall that $x \log x$ is a strict convex function on $\mathbb{R}_+ = [0, \infty)$, where $0 \log 0 = 0$. Hence

$$\frac{\sum_{j=1}^r t_j}{r} \log \frac{\sum_{j=1}^r t_j}{r} \leq \frac{1}{r} \sum_{j=1}^r t_j \log t_j, \text{ for } t_1, \ldots, t_r \in \mathbb{R}_+. \tag{3.1}$$

Clearly, the above inequality is equivalent to the inequality

$$\left(\sum_{j=1}^r t_j\right)\prod_{j=1}^r t_j^{\frac{1}{r}} \leq \prod_{j=1}^r t_j^{\frac{1}{r}}, \text{ for } t_1, \ldots, t_r \in \mathbb{R}_+. \tag{3.2}$$

Here $0^0 = 1$.

**Lemma 3.1** Let $A = [a_{ij}] \in \{0, 1\}^{(2n) \times (2n)} \cap S_{2n,0}$. Then for each $i \in \langle 2n \rangle$\)

$$(\text{hafn } A)_{hafn A} A \leq \prod_{j:a_{ij}=1} (\text{hafn } A(i,j))_{hafn A(i,j)} \cdot \tag{3.3}$$

**Proof** Let $t_j = \text{hafn } A(i,j)$ for $a_{ij} = 1$. Use (2.3) and (3.2) to deduce (3.3). ■

To prove our main result we need the following two lemmas.

**Lemma 3.2** The sequence $(k!\frac{1}{1})_{k=1,\ldots}^{\infty}$ is an increasing sequence.

**Proof** Clearly, the inequality $(k!\frac{1}{1})_{k=1,\ldots}^{\infty} < ((k+1)!\frac{1}{1})_{k=1,\ldots}^{\infty}$ is equivalent to the inequality $(k!\frac{1}{1})_{k=1,\ldots}^{\infty} < ((k+1)!\frac{1}{1})_{k=1,\ldots}^{\infty}$, which is in turn equivalent to $k! < (k+1)!$, which is obvious. ■

**Lemma 3.3** For an integer $r \geq 3$ the following inequality holds.

$$((r-1)!\frac{1}{r-2})_{r=1,\ldots}^{\infty} < \frac{2}{r-2} \tag{3.4}$$

**Proof** Raise the both sides of (3.1) to the power $r(r-1)(r-2)$ to deduce that (3.4) is equivalent to the inequality

$$(r-1)!^{(r-2)}((r-2)!)^{r-1} < ((r-1)!)^{2r(r-2)}. \tag{3.4}$$

Use the identities

$$r! = r(r-1)!, \quad (r-1)! = (r-1)(r-2)!, \quad 2r(r-2) = (r-1)(r-2) + r(r-1) - 2, \quad r(r-1) - 2 = (r+1)(r-2)$$

to deduce that the above inequality is equivalent to

$$(r-1)!^{(r-2)}((r-2)!)^{r-1} < (r-1)^{(r+1)(r-2)}. \tag{3.4}$$

Take the logarithm of the above inequality, divide it by $(r-2)$ deduce that (3.4) is equivalent to the inequality

$$(r-1) \log r + \frac{2}{r-2} \log(r-2)! - (r+1) \log(r-1) < 0.\tag{3.4}$$

This inequality is equivalent to

$$s_r := (r-1) \log \frac{r}{r-1} + 2(\frac{1}{r-2} \log(r-2)! - \log(r-1)) < 0 \text{ for } r \geq 3. \tag{3.5}$$

Clearly

$$(r-1) \log \frac{r}{r-1} = (r-1) \log(1 + \frac{1}{r-1}) < (r-1) \frac{1}{r-1} = 1.$$
Hence (3.5) holds if
\[
\frac{1}{r-2} \log (r-2)! - \log (r-1) < -\frac{1}{2}.
\] (3.6)

Recall the Stirling’s formula [3, pp. 52]
\[
\log k! = \frac{1}{2} \log (2\pi k) + k \log k - k + \frac{\theta_k}{12k} \text{ for some } \theta_k \in (0,1).
\] (3.7)

Hence
\[
\frac{\log(r-2)!}{r-2} < \frac{\log 2\pi(r-2)}{2(r-2)} + \log(r-2) - 1 + \frac{1}{12(r-2)^2}.
\]

Thus
\[
\frac{1}{r-2} \log (r-2)! - \log (r-1) < \frac{\log 2\pi(r-2)}{2(r-2)} + \frac{r-2}{r-1} + \frac{1}{12(r-2)^2} - 1.
\]

Since \(e^x\) is convex, it follows that \(1 + x \leq e^x\). Hence
\[
\frac{1}{r-2} \log (r-2)! - \log (r-1) < \frac{\log 2\pi(r-2)}{2(r-2)} - \frac{1}{r-1} + \frac{1}{12(r-2)^2} - 1.
\]

Note that 
\[\frac{1}{r-2} + \frac{1}{12(r-2)^2} < 0\] for \(r \geq 3\). Therefore
\[
\frac{1}{r-2} \log (r-2)! - \log (r-1) < \frac{\log 2\pi(r-2)}{2(r-2)} - 1.
\] (3.8)

Observe next that that the function \(\frac{\log 2\pi x}{2x}\) is decreasing for \(x > \frac{e}{2\pi}\). Hence the right-hand side of (3.8) is a decreasing sequence for \(r = 3, \ldots\). Since \(\frac{\log 2\pi \cdot 3}{2 \cdot 3} = 0.4894\), it follows that the right-hand side of (3.8) is less than \(-0.51\) for \(r \geq 5\). Therefore (3.5) holds for \(r \geq 5\). Since
\[
s_3 = \log \frac{9}{16} < 0, \quad s_4 = \log \frac{128}{243} < 0
\]
we deduce the lemma.

The arguments of the Proof of Lemma 3.3 yield that \(s_r, r = 3, \ldots\), converges to \(-1\). We checked the values of this sequence for \(r = 3, \ldots, 100\), and we found that this sequence decreases in this range. We conjecture that the sequence \(s_r, r = 3, \ldots\) decreases.

4 Proof of generalized Bregman-Minc inequality

**Theorem 4.1** Let \(G = (V,E)\) be undirected simple graph on an even number of vertices. Then the inequality (1.1) holds.

**Proof** We prove (2.4). We use the induction on \(n\). For \(n = 1\) (2.4) is trivial. Assume that theorem holds for \(n = m - 1\). Let \(n = m\). It is enough to assume that \(\text{hafn} \ A > 0\).

In particular each \(r_i \geq 1\). If \(r_i = 1\) for some \(i\), then by expanding \(\text{hafn} \ A\) by the row \(i\), using the induction hypothesis and Lemma 3.2 we deduce easily the theorem in this case.

Hence we assume that \(r_i \geq 2\) for each \(i \in \langle 2n \rangle\). Let \(G = G(A) = (\langle 2n \rangle, E)\) be the graph induced by \(A\). Then \(\text{hafn} \ A > 0\) is the number of perfect matchings in \(G\). Denote by \(\mathcal{M} := \mathcal{M}(G) \subset \mathcal{M}(K_{2n})\) the set of all perfect matchings in \(G\). Then \(#\mathcal{M} = \text{hafn} \ A\). We now follow the arguments in the proof of the Bregman-Minc theorem given in [4] with the corresponding modifications.
We now explain each step of the proof.

1. Trivial.

2. Use (3.3).

3. The number of factors of \( r_i \) is equal to \( \text{hafn} A \) on both sides, while the number of factors \( \text{hafn} A(i, j) \) equals to the number of \( \alpha \in \mathcal{M} \) such that \( \alpha(i) = j \).

4. Apply the induction hypothesis to each \( \text{hafn} A(i, j) \). Note that since the edge \((i, \alpha(i))\) appears in the perfect matching \( \alpha \in \mathcal{M} \), it follows that \( \text{hafn} A(i, \alpha(i)) \geq 1 \). Hence if \( j \in (2n) \setminus \{i, \alpha(i)\} \) and \( r_j = 2 \) we must have that \( a_{ij} + a_{\alpha(i)j} \leq 1 \).

5. Change the order of multiplication.

6. Fix \( \alpha \in \mathcal{M} \) and \( j \in (2n) \). Then \( j \) is matched with \( \alpha(j) \). Consider all other \( n - 1 \) edges \((i, \alpha(i))\) in \( \alpha \). \( j \) is connected to \( r_j - 1 \) vertices in \( (2n) \setminus \{j, \alpha(j)\} \). Assume there are \( s \) triangles formed by \( j \) and the \( s \) edges out of \( n - 1 \) edges in \( \alpha \setminus \{j, \alpha(j)\} \). Then \( j \) is connected to \( t = r_j - 1 - 2s \) edges vertices \( i \in (2n) \setminus \{j, \alpha(j)\} \) such that \( j \) is not connected to \( \alpha(i) \). Hence there are \( 2n - 2 - (2t + 2s) \) vertices \( k \in (2n) \setminus \{j, \alpha(j)\} \) such that \( j \) is not connected to \( k \) and \( \alpha(k) \). Therefore, for this \( \alpha \) and \( j \) we have the following terms in (5):

\[
(\text{hafn} A)^{(1)} 2n \text{hafn} A = \prod_{i=1}^{2n} (\text{hafn} A)^{(2)} \prod_{i=1}^{2n} (r_i \text{hafn} A \prod_{j,a_{ij}=1} (\text{hafn} A(i, j))^{(3)} \prod_{\alpha \in \mathcal{M}} ((\prod_{i=1}^{2n} r_i (\prod_{i=1}^{2n} \text{hafn} A(i, \alpha(i))))
\]

\[
\leq \prod_{\alpha \in \mathcal{M}} ((\prod_{i=1}^{2n} r_i) \prod_{j=1}^{2n} ((r_j!)^{(4)} \prod_{i=1}^{2n} (j) \prod_{j, a_{ij}, a_{\alpha(i)j}=1 (r_j-1)!}^{(4.1)}) \prod_{i=1}^{2n} (j)))^{(5)} = \prod_{\alpha \in \mathcal{M}} ((\prod_{i=1}^{2n} r_i) \prod_{j=1}^{2n} ((r_j!)^{(5)} \prod_{i=1}^{2n} (j)))^{(5)}
\]

\[
\leq \prod_{\alpha \in \mathcal{M}} ((\prod_{i=1}^{2n} r_i) \prod_{j=1}^{2n} ((r_j!)^{(6)} \prod_{i=1}^{2n} (j)))^{(6)} = \prod_{\alpha \in \mathcal{M}} ((\prod_{i=1}^{2n} r_i) \prod_{j=1}^{2n} ((r_j!)^{(7)} \prod_{i=1}^{2n} (j)))^{(7)} 2n \text{hafn} A.
\]
In the last step we used the equality $r_j - 1 = 2s + t$. Assume first that $r_j > 2$. Use Lemma 3.3 to deduce that (4.1) increases in $t$. Hence the maximum value of (4.1) is achieved when $s = 0$ and $t = r_j - 1$. Then (4.1) is equal to

$$\left(\frac{r_j!}{r_j} \right)^{2n-2r_j} \frac{2^{(r_j-1)}}{2^{r_j-1} \cdot (r_j-1)!}.$$ 

If $r_j = 2$ then, as we explained above, $s = 0$. Hence (4.1) is also equal to the above expression. Hence (6) holds.

7. Trivial.
8. Trivial.

Thus

$$(\text{hafn } A)^{2n} \leq \left( \prod_{i=1}^{2n} \left( \frac{1}{r_i!} \right)^{1/2} \right)^{2n} \text{hafn } A.$$ 

This establishes (2.4).

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