Harnack Inequalities for McKean-Vlasov SDEs Driven by Subordinate Brownian Motions

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Abstract

The existence and uniqueness are established for McKean-Vlasov SDEs driven by Lévy processes. By using an approximation technique and coupling by change of measures, Harnack inequalities are investigated for McKean-Vlasov SDEs driven by subordinate Brownian motions.

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1 Introduction

It is well known that solution to the linear Fokker-Planck-Kolmogorov equation (FPKE) (cf. [5]) can be constructed by the time marginal distributions of solution to Itô (distribution independent) stochastic differential equation (SDE), see e.g. [13]. This means that we can describe FPKEs by using a probabilistic approach ([2, 3, 17]). However, many important partial differential equations (PDEs) for probability measures are nonlinear, see, for instance, [5, 6, 8, 9, 10, 11, 21] and references therein. Such PDEs are also of Fokker–Planck type. Fortunately, nonlinear FPKEs are also closely connected to the so-called distribution dependent SDEs, also named McKean-Vlasov SDEs in the literature, in which the coefficients

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depend on the distribution of the solution. Barbu and Röckner [2, 3] investigated one-to-
one correspondence between nonlinear FPKEs with second-order differential operator and
McKean-Vlasov SDEs driven by Brownian motion, see also [12] for closely related results
on path dependent nonlinear FPKEs and path-distribution dependent SDEs with Brownian
noise.

Recently, Jourdain, Méleard and Woyczynski in [14] investigated McKean-Vlasov model
with multiplicative Lévy noises. For McKean-Vlasov SDEs driven by additive Lévy processes,
Y. Song [20] applied Malliavin calculus to get exponential ergodicity in the total variance
distance, while Liang, Majka and Wang used a different approach in [15] to derive exponential
ergodicity in the $L^1$-Wasserstein distance.

Let $P$ be the family of all probability measures on $\mathbb{R}^d$ equipped with the weak topology,
and $L_\zeta$ denote the distribution of a random variable $\zeta$. When a different probability measure
$\tilde{P}$ is concerned, we use $L_\zeta|_{\tilde{P}}$ to denote the law of $\zeta$ under $\tilde{P}$. In this paper, we consider the
following McKean-Vlasov SDEs driven by Lévy processes:

\begin{equation}
\begin{aligned}
dX_t &= b(t, X_t, L X_t) \, dt + \sigma(t) \, dZ_t,
\end{aligned}
\end{equation}

where $b : [0, \infty) \times \mathbb{R}^d \times P \to \mathbb{R}^d$ and $\sigma : [0, \infty) \to \mathbb{R}^d \otimes \mathbb{R}^d$ are measurable and locally
bounded, and $Z = \{Z_t\}_{t \geq 0}$ is a $d$-dimensional Lévy process with $Z_0 = 0$.

Note that $Z$ has stationary and independent increments and almost surely càdlàg (right-
continuous with finite left limits) paths $t \mapsto Z_t$. Since $Z$ is a (strong) Markov process, it is
completely characterized by the law of $Z_t$, hence by the characteristic function of $Z_t$. It is
well known that

$$
\mathbb{E}e^{i\langle \xi, Z_t \rangle} = e^{-t\psi(\xi)}, \quad t > 0, \ \xi \in \mathbb{R}^d,
$$

where the symbol (characteristic exponent) $\psi : \mathbb{R}^d \to \mathbb{C}$ is given by the Lévy–Khintchine
formula

$$
\psi(\xi) = -i\langle l, \xi \rangle + \frac{1}{2} \langle \xi, Q \xi \rangle + \int_{\mathbb{R}^d \setminus \{0\}} \left(1 - e^{i\langle \xi, x \rangle} + i\langle \xi, x \rangle \mathbf{1}_{(0,1]}(|x|)\right) \nu_Z(dx),
$$

where $l \in \mathbb{R}^d$ is the drift coefficient, $Q$ is a nonnegative semidefinite $d \times d$ matrix, and $\nu_Z$
is the Lévy measure on $\mathbb{R}^d \setminus \{0\}$ satisfying $\int_{\mathbb{R}^d \setminus \{0\}} (1 \wedge |x|^2) \nu_Z(dx) < \infty$. The Lévy triplet
$(l, Q, \nu_Z)$ uniquely determines $\psi$, hence $Z$ and the infinitesimal generator of $Z$ is of the form

\begin{equation}
\begin{aligned}
\mathcal{A} f &= \langle l, \nabla f \rangle + \frac{1}{2} \langle \nabla, Q \nabla \rangle f + \int_{\mathbb{R}^d \setminus \{0\}} \left(f(x + \cdot) - f - \langle x, \nabla f \rangle \mathbf{1}_{(0,1]}(|x|)\right) \nu_Z(dx)
\end{aligned}
\end{equation}

for $f \in C^2_b(\mathbb{R}^d)$.

The first contribution of the present paper is the existence and uniqueness of the solution
to (1.1), see Theorem 2.3 below. To this end, we shall follow the iteration argument used in
[24]; moreover, we need to bound the moment for solutions to Lévy-driven (distribution
independent) SDEs with one-sided Lipschitz continuous drift.

The dimension-free Harnack inequality, initialized in [22], has become an efficient tool
in stochastic analysis, and it can be used to study the strong Feller property, heat kernel
estimates, transportation-cost inequalities, hyperboundedness, and many more; we refer to
the monograph by F.-Y. Wang [23, Subsection 1.4.1] for an in-depth explanation of its
applications.

To establish Harnack inequality for McKean-Vlasov SDEs with jumps, we will restrict
ourselves to the special case
\[ Z_t = W_{S_t}, \]
where \( W = \{W_t\}_{t \geq 0} \) is a standard Brownian motion
on \( \mathbb{R}^d \), and \( S = \{S_t\}_{t \geq 0} \) is a subordinator independent of \( W \). Then the equation (1.1) reduces
to
\[
(1.3) \quad dX_t = b(t, X_t, \mathcal{L}X_t) \, dt + \sigma(t) \, dW_{S_t}.
\]

We will adopt absolutely continuous path to approximate the path of \( S \) as in [25, 26, 7],
and, as it turns out, this will be crucial for our study. As before (see e.g. [25, 7]), a coupling
argument and the Girsanov theorem will also be used.

Recall that a subordinator \( S = \{S_t\}_{t \geq 0} \) is a nondecreasing Lévy process on \([0, \infty)\), and
it is uniquely determined by its Laplace transform which is of the form
\[
E e^{-rS_t} = e^{-t\phi(r)}, \quad r > 0, t \geq 0.
\]
The characteristic (Laplace) exponent \( \phi : (0, \infty) \to (0, \infty) \) is a Bernstein function, i.e. a
\( C^\infty \)-function such that \( \phi \geq 0 \) and with alternating derivatives \((-1)^{n+1}\phi^{(n)} \geq 0, n \in \mathbb{N} \).
Every such \( \phi \) has a unique Lévy–Khintchine representation
\[
(1.4) \quad \phi(r) = \varrho r + \int_{(0,\infty)} (1 - e^{-rx}) \, \nu_S(dx), \quad r > 0,
\]
where \( \varrho \geq 0 \) is the drift parameter and \( \nu_S \) is a Lévy measure, that is, a Radon measure
on \((0, \infty)\) satisfying \( \int_{(0,\infty)} (1 \wedge x) \, \nu_S(dx) < \infty \). We use [19] as our standard reference for
Bernstein functions and subordinators.

The (random) time-changed process \( (W_{S_t})_{t \geq 0} \) is a rotationally invariant Lévy process
with symbol \( \phi(\cdot, \cdot^2/2) \) and is called a subordinate Brownian motion. If \( S \) is an \( \alpha \)-stable
subordinator with Bernstein function \( \phi(r) = r^\alpha \) \((0 < \alpha < 1)\), then \( (W_{S_t})_{t \geq 0} \) is the well-
known \( 2\alpha \)-stable Lévy process with discontinuous sample paths and its generator is given
by the fractional Laplacian operator \(-\frac{\alpha}{2} (-\Delta)^\alpha \). By choosing different Bernstein functions,
we can construct many other time-changed Brownian motions. Thus, subordinate Brownian
motions form a very large class of Lévy processes. Nonetheless, compared with general Lévy
processes, subordinate Brownian motions are much more tractable.

The remaining part of the paper is organized as follows. In Section 2, we investigate the
strong/weak existence and uniqueness of solutions to McKean-Vlasov SDEs driven by Lévy
processes. By using an approximation technique and coupling by change of measures, the
dimension-free Harnack inequalities are established in Section 5. Finally, the appendix con-
tains a result concerning moments for Lévy-driven (distribution independent) SDEs, which
has been used in Section 2.
2 Existence and uniqueness for McKean-Vlasov SDEs with Lévy noises

For $p \in [1, \infty)$, let

$$P_p := \left\{ \mu \in \mathcal{P} : \mu(|\cdot|^p) := \int_{\mathbb{R}^d} |x|^p \mu(dx) < \infty \right\}. $$

It is well known that $P_p$ is a Polish space under the Wasserstein distance

$$W_p(\mu_1, \mu_2) := \inf_{\pi \in \mathcal{C}(\mu_1, \mu_2)} \left( \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^p \pi(dx, dy) \right)^{1/p},$$

where $\mathcal{C}(\mu_1, \mu_2)$ is the set of all couplings for $\mu_1$ and $\mu_2$. Moreover, the topology induced by $W_p$ on $P_p$ coincides with the weak topology.

We make the following assumptions on the Lévy measure $\nu_Z$ of $Z$ and the coefficient $b$:

There exists some $\theta \geq 1$ such that

(H1) $\int_{|x| \geq 1} |x|^\theta \nu_Z(dx) < \infty$;

(H2) (Continuity) For every $t \geq 0$, $b(t, \cdot, \cdot)$ is continuous on $\mathbb{R}^d \times \mathcal{P}_\theta$;

(H3) (Monotonicity) There exist locally bounded functions $\kappa_1 : [0, \infty) \to \mathbb{R}$ and $\kappa_2 : [0, \infty) \to [0, \infty)$ such that

$$2\langle b(t, x, \mu) - b(t, y, \nu), x - y \rangle \leq \kappa_1(t)|x - y|^2 + \kappa_2(t)W_\theta(\mu, \nu)|x - y|, \quad t \geq 0, x, y \in \mathbb{R}^d, \mu, \nu \in \mathcal{P}_\theta;$$

(H4) (Growth) There exists a locally bounded function $\Theta : [0, \infty) \to [0, \infty)$ such that

$$|b(t, 0, \mu)| \leq \Theta(t) \left\{ 1 + (\mu(|\cdot|^\theta))^{1/\theta} \right\}, \quad t \geq 0, \mu \in \mathcal{P}_\theta.$$

Remark 2.1. It is well known that (H1) is equivalent to $\mathbb{E}|Z_t|^\theta < \infty$ for some (or, equivalently, all) $t > 0$, cf. [18, Theorem 25.3].

Definition 2.2. A càdlàg adapted process $(X_t)_{t \geq 0}$ on $\mathbb{R}^d$ is called a (strong) solution of (1.1), if

$$\mathbb{E} \int_0^t |b(s, X_s, \mathcal{L}_{X_s})|ds < \infty, \quad t \geq 0,$$

and $\mathbb{P}$-a.s.

$$X_t = X_0 + \int_0^t b(s, X_s, \mathcal{L}_{X_s})ds + \int_0^t \sigma(s)dZ_s, \quad t \geq 0.$$

We say that (1.1) has strong (or pathwise) existence and uniqueness in $\mathcal{P}_\theta$, if for any $\mathcal{F}_0$-measurable random variable $X_0$ with $\mathcal{L}_{X_0} \in \mathcal{P}_\theta$, the equation has a unique solution $(X_t)_{t \geq 0}$ satisfying $\mathbb{E}|X_t|^\theta < \infty$ for all $t > 0$.  

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(2) A couple $(\tilde{X}_t, \tilde{Z}_t)_{t \geq 0}$ is called a weak solution to (1.1), if $\tilde{Z} = (\tilde{Z})_{t \geq 0}$ is a Lévy process having the same symbol as $Z$ with respect to a complete filtered probability space $(\Omega, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$, and $(\tilde{X}_t)_{t \geq 0}$ satisfies

$$d\tilde{X}_t = b(t, \tilde{X}_t, \mathcal{L}_{\tilde{X}_t}) \, dt + \sigma(t) \, d\tilde{Z}_t.$$ \hfill (2.1)

Moreover, for any $s \geq 0$ and $s, t \geq s$,

$$|\mathcal{L}_{\tilde{X}}| = |\mathcal{L}_{\tilde{X}_s}| = \mathcal{L}_{\tilde{X}_s}.$$ \hfill (H1)

(3) (1.1) is said to have weak uniqueness in $\mathcal{P}_\theta$, if any two weak solutions of the equation with common initial distribution in $\mathcal{P}_\theta$ are equal in law. Precisely, if $(X_t, Z_t)_{t \geq 0}$ with respect to $(\Omega, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ and $(\tilde{X}_t, \tilde{Z}_t)_{t \geq 0}$ with respect to $(\Omega, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ are weak solutions of (1.1), then $\mathcal{L}_{X_0} = \mathcal{L}_{\tilde{X}_0}$ implies $\mathcal{L}_{X_t} = \mathcal{L}_{\tilde{X}_t}$ for all $t > 0$. (1.1) is said to have strong/weak well-posedness in $\mathcal{P}_\theta$ if it has strong/weak existence and uniqueness in $\mathcal{P}_\theta$.

**Theorem 2.3.** Assume (H1)-(H4). Then the following assertions hold.

1. The equation (1.1) has strong well-posedness in $\mathcal{P}_\theta$.
2. The equation (1.1) has weak well-posedness in $\mathcal{P}_\theta$.

We will prove Theorem 2.3 by the argument used in [24]. For fixed $s \geq 0$ and $\mathcal{F}_s$-measurable $\mathbb{R}^d$-valued random variable $X_{s,s}$ with $\mathbb{E}|X_{s,s}|^\theta < \infty$, set

$$X^{(0)}_{s,t} = X_{s,s}, \quad \mu^{(0)}_{s,t} = \mathcal{L}_{X^{(0)}_{s,t}}, \quad t \geq s.$$ \hfill (2.2)

For $n \in \mathbb{N}$, let $(X^{(n)}_{s,t})_{t \geq s}$ solve the classical (distribution independent) SDE

$$dX^{(n)}_{s,t} = b(t, X^{(n)}_{s,t}, \mu^{(n-1)}_{s,t}) \, dt + \sigma(t) \, dZ_t, \quad t \geq s,$$ \hfill (1.1)

with $X^{(n)}_{s,s} = X_{s,s}$, where $\mu^{(n-1)}_{s,t} := \mathcal{L}_{X^{(n-1)}_{s,t}}$.

**Lemma 2.4.** Assume (H1)-(H4). Then for every $n \in \mathbb{N}$, the SDE (2.1) has a unique strong solution $X^{(n)}_{s,t}$ with

$$\mathbb{E} \sup_{t \in [s,T]} |X^{(n)}_{s,t}|^\theta < \infty, \quad T > s, n \in \mathbb{N}. \hfill (2.2)$$

Moreover, for any $T > 0$, there exists $t_0 > 0$ such that for all $s \in [0,T]$ and $X_{s,s} \in L^\theta(\Omega \rightarrow \mathbb{R}^d, \mathcal{F}_s)$,

$$\mathbb{E} \sup_{t \in [s,s+t_0]} |X^{(n+1)}_{s,t} - X^{(n)}_{s,t}|^\theta \leq 2^\theta \, e^{-n} \mathbb{E} \sup_{t \in [s,s+t_0]} |X^{(1)}_{s,t}|^\theta, \quad n \in \mathbb{N}. \hfill (2.2)$$

**Proof.** We only need to apply Proposition 4.1 in the appendix and use the argument in [24, proof of Lemma 2.3] to obtain the desired assertions. Here we omit the details to save space. \hfill □
Proof of Theorem 2.3. (1) First, we prove the existence of strong solution in $\mathcal{P}_\theta$. For simplicity, we only consider $s = 0$ and denote $X_{0,t} = X_t$, $t \geq 0$.

For $t > 0$, let $\mathcal{D}_t$ be the family of all $\mathbb{R}^d$-valued càdlàg functions on $[0, t]$ equipped with the uniform norm. Since $\mathcal{D}_t$ is a Banach space, so is $L^\theta(\Omega; \mathcal{D}_t)$. Let $(X_t)_{t \in [0, t_0]}$ be the unique limit of $(X_{t_0}^{(n)})_{t \in [0, t_0]}$ in Lemma 2.4. Then $(X_t)_{t \in [0, t_0]}$ is an adapted càdlàg process and satisfies

(2.3) \[ \lim_{n \to \infty} \sup_{t \in [0, t_0]} \mathbb{W}_\theta(\mu_t^{(n)}, \mathcal{L}_t^\theta) \leq \lim_{n \to \infty} \mathbb{E} \sup_{t \in [0, t_0]} |X_t^{(n)} - X_t|^\theta = 0. \]

Reformulate (2.1) as

$$X_t^{(n)} = X_0 + \int_0^t b(s, X_s^{(n)}, \mu_s^{(n-1)}) \, ds + \int_0^t \sigma(s) \, dZ_s.$$ 

Now (2.3), (H2), the local boundedness of $b$, and the dominated convergence theorem imply that $\mathbb{P}$-a.s.

$$X_t = X_0 + \int_0^t b(s, X_s, \mathcal{L}_s) \, ds + \int_0^t \sigma(s) \, dZ_s, \quad t \in [0, t_0].$$

Moreover, (2.2) and (2.3) lead to $\mathbb{E} \sup_{s \in [0, t_0]} |X_s|^\theta < \infty$. Therefore, $(X_t)_{t \in [0, t_0]}$ solves (1.1) up to time $t_0$. The same assertion holds for $(X_s, t \in [s, (s + t_0) \wedge T]$ and $s \in [0, T]$. By solving the equation piecewise in time, and using the arbitrariness of $T > 0$, we conclude that (1.1) has a unique strong solution $(X_t)_{t \geq 0}$ with

$$\mathbb{E} \sup_{s \in [0, t]} |X_s|^\theta < \infty, \quad t \geq 0.$$

Next, we prove strong uniqueness in $\mathcal{P}_\theta$. Let $X_t$ and $Y_t$ be two solutions to (1.1) with $X_0 = Y_0$ and $\mathbb{E}|X_0|^\theta + \mathbb{E}|Y_0|^\theta < \infty$, $t \geq 0$. It follows from (H3) that

$$d|X_t - Y_t|^2 \leq \kappa_1(t)|X_t - Y_t|^2 \, dt + \kappa_2(t)\mathbb{W}_\theta(\mathcal{L}_X, \mathcal{L}_Y)|X_t - Y_t| \, dt.$$ 

For any $\varepsilon > 0$, it is easy to see that

$$d(|X_t - Y_t|^2 + \varepsilon)_{\theta^2} = \frac{\theta}{2} (|X_t - Y_t|^2 + \varepsilon)^{\frac{\theta-2}{2}} d|X_t - Y_t|^2$$

(2.4) \[ \leq \frac{\theta}{2} \kappa_1(t)(|X_t - Y_t|^2 + \varepsilon)^{\frac{\theta-2}{2}} |X_t - Y_t|^2 \, dt \]

$$+ \frac{\theta}{2} \kappa_2(t)\mathbb{W}_\theta(\mathcal{L}_X, \mathcal{L}_Y)(|X_t - Y_t|^2 + \varepsilon)^{\frac{\theta-1}{2}} \, dt.$$ 

Using the following inequality

(2.5) \[ yz^{\rho-1} \leq \frac{1}{\rho} y^\rho + \frac{\rho - 1}{\rho} z^\rho, \quad y, z \geq 0, \rho \geq 1 \]

with $\rho = \theta$, we get

$$d(|X_t - Y_t|^2 + \varepsilon)^{\theta^2} \leq (|X_0 - Y_0|^2 + \varepsilon)^{\theta^2} + \int_0^t \frac{\theta}{2} \kappa_1(s)(|X_s - Y_s|^2 + \varepsilon)^{\frac{\theta-2}{2}} |X_s - Y_s|^2 \, ds$$

$$+ \frac{\theta}{2} \kappa_2(s)\mathbb{W}_\theta(\mathcal{L}_X, \mathcal{L}_Y)(|X_s - Y_s|^2 + \varepsilon)^{\frac{\theta-1}{2}} \, ds.$$
Since it follows from (1) that (2.7) has a strong well-posedness in $P$

the weak uniqueness of (2.8) implies

jump processes (cf. [4, Theorem 1]), it also satisfies weak uniqueness. Noting that

has a unique strong solution. According to Yamada–Watanabe’s theory for SDEs driven by

Moreover,

By (H1)-(H4) and Proposition 4.1 below, the following SDE

has a unique strong solution. According to Yamada–Watanabe’s theory for SDEs driven by

jump processes (cf. [4, Theorem 1]), it also satisfies weak uniqueness. Noting that

the weak uniqueness of (2.8) implies

So, (2.8) can be rewritten as

Since it follows from (1) that (2.7) has a strong well-posedness in $P$, we know that $\bar{X} = \tilde{X}$. Therefore, (2.9) implies $\mathcal{L}_{\bar{X}_t}|_P = \mathcal{L}_{X_t}|_P$ for all $t \geq 0$, as required.

For $\mu_0 \in P$, let $X_t(\mu_0)$ be the solution to (1.1) with $\mathcal{L}_{X_0} = \mu_0$. Let $P_t^*\mu_0$ be the distribution of $X_t(\mu_0)$.

**Proposition 2.5.** Assume (H1)-(H4). For any $\mu_0, \nu_0 \in P$,

\begin{equation}
\mathbb{W}_\theta(P_t^*\mu_0, P_t^*\nu_0) \leq \exp \left[ \frac{1}{2} \int_0^t \{\kappa_1(s) + \kappa_2(s)\} \, ds \right] \mathbb{W}_\theta(\mu_0, \nu_0), \quad t \geq 0.
\end{equation}
Proof. It follows from (2.6) and Gronwall’s inequality that

\[ E|X_t - Y_t|^\theta \leq E|X_0 - Y_0|^\theta \exp \left[ \frac{\theta}{2} \int_0^t \{\kappa_1(s) + \kappa_2(s)\} \, ds \right]. \]

For any \( \mu_0, \nu_0 \in \mathcal{P}_\theta \), we can take \( \mathcal{P}_0 \)-measurable random variables \( X_0 \) and \( Y_0 \) such that \( \mathcal{L}X_0 = \mu_0, \mathcal{L}Y_0 = \nu_0 \) and \( \mathbb{W}_\theta(\mu_0, \nu_0)^\theta = E|X_0 - Y_0|^\theta \). Combining this with \( \mathbb{W}_\theta(P_t^{\mu_0}, P_t^{\nu_0})^\theta \leq E|X_t(\mu_0) - Y_t(\nu_0)|^\theta \), we obtain the desired assertion. \( \square \)

3 Harnack inequalities

In this section, we study the Harnack inequality for (1.3). In this case, the Lévy noise \((Z_t)_{t \geq 0}\) is given by subordinate Brownian motion \((W_{S_t})_{t \geq 0}\), where \( W = \{W_t\}_{t \geq 0} \) is a standard Brownian motion on \( \mathbb{R}^d \), and \( S = \{S_t\}_{t \geq 0} \) is an independent subordinator with Bernstein function (Laplace exponent) \( \phi \) given by (1.4). Since the Lévy measure of \( Z_t = W_{S_t} \) is

\[ \nu_Z(dx) = \int_{[0, \infty)} (2\pi s)^{-d/2} e^{-|x|^2/(2s)} \nu_S(ds) \, dx, \]

where \( \nu_S \) is the Lévy measure of subordinator \( S \), it is not hard to verify that (H1) is equivalent to

(H1') \( \int_{(1, \infty)} x^{\theta/2} \nu_S(dx) < \infty. \)

Remark 3.1. We list here some typical examples for Bernstein function \( \phi \) satisfying (H1').

- (Stable subordinators) Let \( \phi(r) = r^\alpha \) with drift \( \varrho = 0 \) and Lévy measure \( \nu_S(dx) = \frac{\alpha}{\Gamma(1-\alpha)} x^{1-\alpha} \, dx \), where \( 1/2 < \alpha < 1 \). Then (H1') holds if \( 1 \leq \theta < 2\alpha \);

- (Relativistic stable subordinators) Let \( \phi(r) = (r + m^{1/\alpha})^\alpha - m \) with drift \( \varrho = 0 \) and Lévy measure \( \nu_S(dx) = \frac{\alpha}{\Gamma(1-\alpha)} e^{-m^{1/\alpha} x} x^{1-\alpha} \, dx \), where \( 0 < \alpha < 1 \) and \( m > 0 \). Then (H1') holds for all \( \theta \geq 1 \);

- (Gamma subordinators) Let \( \phi(r) = \log(1 + r/a) \) with drift \( \varrho = 0 \) and Lévy measure \( \nu_S(dx) = x^{-1} e^{-a x} \, dx \), where \( a > 0 \). Then (H1') holds for all \( \theta \geq 1 \);

- Let \( \phi(r) = r \log(1 + a/r) \) with drift \( \varrho = 0 \) and Lévy measure \( \nu_S(dx) = x^{-2}(1 - e^{-a x}(1 + a x)) \, dx \), where \( a > 0 \). Then (H1') holds if \( 1 \leq \theta < 2 \);

- Let \( \phi(r) = re^r \int_1^\infty e^{-ry} y^{-n} \, dy \) with drift \( \varrho = 0 \) and Lévy measure \( \nu_S(dx) = n(1 + x)^{-n-1} \, dx \), where \( n \in \mathbb{N} \). Then (H1') holds if \( 1 \leq \theta < 2n \).

We refer to [19, Chapter 16] for an extensive list of such Bernstein functions.

Moreover, we need the following assumption on \( \sigma \):
(H5) For any \( t \geq 0, \sigma(t) \) is invertible and there exists a non-decreasing function \( \lambda : [0, \infty) \to [0, \infty) \) such that
\[
\|\sigma(t)^{-1}\| \leq \lambda(t), \quad t \geq 0.
\]

For \( t > 0 \), let
\[
K_1(t) := \exp \left[-\int_0^t \kappa_1(r) \, dr \right],
\]
and
\[
K(t, \theta) := \frac{1}{2} \int_0^t \exp \left[\frac{\theta}{2} \{\kappa_1(s) + \kappa_2(s)\} - \frac{1}{2} \int_0^s \kappa_1(r) \, dr \right] \kappa_2(s) \, ds,
\]
where \( \kappa_1 \) and \( \kappa_2 \) are from (H3).

Under (H1') and (H2)-(H5), it follows from Theorem 2.3 that for \( \mu_0 \in \mathcal{P}_\theta \), equation (1.3) with \( \mathcal{L}_{X_0} = \mu_0 \) has a unique solution \( X_t(\mu_0) \). Define
\[
P_t f(\mu_0) := \mathbb{E} f(X_t(\mu_0)), \quad t \geq 0, f \in \mathcal{B}_0(\mathbb{R}^d).
\]

Note that, in general, \((P_t)_{t \geq 0}\) is not a semigroup, see [24].

The main result in this section is the following theorem.

**Theorem 3.2.** Assume (H1') and (H2)-(H5).

1. For any \( \mu_0, \nu_0 \in \mathcal{P}_\theta, \) \( T > 0, \) and \( f \in \mathcal{B}_0(\mathbb{R}^d) \) with \( f \geq 1, \)
\[
P_T \log f(\nu_0)
\]
\[
\leq \log P_T f(\mu_0) + \lambda(T)^2 \left\{ \mathbb{W}_2(\mu_0, \nu_0)^2 + K(T, \theta)^2 \mathbb{W}_\theta(\mu_0, \nu_0)^2 \right\} \mathbb{E} \left( \int_0^T K_1(s) \, dS_s \right)^{-1}.
\]

2. For any \( p > 1, \mu_0, \nu_0 \in \mathcal{P}_\theta, \) \( \mathcal{F}_0 \)-measurable random variables \( X_0, Y_0 \) with \( \mathcal{L}_{X_0} = \mu_0, \mathcal{L}_{Y_0} = \nu_0, \) \( T > 0, \) and non-negative \( f \in \mathcal{B}_0(\mathbb{R}^d), \)
\[
(P_T f(\nu_0))^p \leq P_T f^p(\mu_0)
\]
\[
\times \left( \mathbb{E} \exp \left[ \frac{p \lambda(T)^2}{(p-1)^2} \{X_0 - Y_0|^2 + K(T, \theta)^2 \mathbb{W}_\theta(\mu_0, \nu_0)^2 \} \left( \int_0^T K_1(s) \, dS_s \right)^{-1} \right] \right)^{p-1}.
\]

For \( \mu_0, \nu_0 \in \mathcal{P}_\theta \) and \( t > 0, \) let \( \mu_t := P_t^* \mu_0 \) and \( \nu_t := P_t^* \nu_0. \) The following corollary is a direct consequence of Theorem 3.2, see [23, Theorem 1.4.2].

**Corollary 3.3.** Assume (H1') and (H2)-(H5). Let \( \mu_0, \nu_0 \in \mathcal{P}_{\theta \lor 2} \) and \( T > 0. \) If \( \mathbb{E} S_T^{-1} < \infty, \) then \( \mu_T \) and \( \nu_T \) are equivalent. Furthermore, the following assertions hold.

1. It holds that
\[
\int_{\mathbb{R}^d} \log \left( \frac{d\nu_T}{d\mu_T} \right) \, d\nu_T \leq \left\{ \mathbb{W}_2(\mu_0, \nu_0)^2 + K(T, \theta)^2 \mathbb{W}_\theta(\mu_0, \nu_0)^2 \right\} \mathbb{E} \left( \int_0^T K_1(s) \, dS_s \right)^{-1}.
\]
(2) For any \( p > 1 \) and \( \mathcal{F}_0 \)-measurable random variables \( X_0, Y_0 \) with \( \mathcal{L}_{X_0} = \mu_0, \mathcal{L}_{Y_0} = \nu_0 \),

\[
\int_{\mathbb{R}^d} \left( \frac{d\nu_T}{d\mu_T} \right)^{1/(p-1)} d\nu_T 
\leq \mathbb{E} \exp \left[ \frac{p}{(p-1)^2} \left\{ |X_0 - Y_0|^2 + K(T, \theta)^2\|W|^2(\mu_0, \nu_0) \right\} \left( \int_0^T K_1(s) \, d\ell_s \right)^{-1} \right].
\]

\[3.1\] Harnack inequalities under deterministic time-change

Let \( \ell : [0, \infty) \to [0, \infty) \) be a sample path of subordinator \( S \), which is a non-decreasing and càdlàg function with \( \ell(0) = 0 \). For \( \mu_0 \in \mathcal{P}_\theta \), let \( X_t(\mu_0) \) be the solution to (1.1) with \( \mathcal{L}_{X_0} = \mu_0 \). By (H2) and (H3), \( b(t, \cdot, X_t(\mu_0)) \) is continuous and satisfies the one-sided Lipschitz condition

\[2\langle b(t, x, \mathcal{L}_{X_t(\mu_0)}) - b(t, y, \mathcal{L}_{X_t(\mu_0)}), x - y \rangle \leq \kappa_1(t)|x - y|^2, \quad t \geq 0, x, y \in \mathbb{R}^d.
\]

Thus, for any \( \mu_0 \in \mathcal{P}_\theta \), the following SDE has a unique non-explosive solution with \( \mathcal{L}_{X_0} = \mu_0 \):

\[3.1\]

\[dX_t^\ell = b(t, X_t^\ell, \mathcal{L}_{X_t(\mu_0)}) \, dt + \sigma(t) \, dW_t.
\]

We denote the solution by \( X_t^\ell(\mu_0) \). The associated Markov operator is defined by

\[3.2\]

\[P_t^\ell f(\mu_0) := \mathbb{E} f \left( X_t^\ell(\mu_0) \right), \quad t \geq 0, f \in \mathcal{B}_b(\mathbb{R}^d), \mu_0 \in \mathcal{P}_\theta.
\]

**Proposition 3.4.** Assume (H1') and (H2)-(H5).

(1) For any \( \mu_0, \nu_0 \in \mathcal{P}_\theta, T > 0, \) and \( f \in \mathcal{B}_b(\mathbb{R}^d) \) with \( f \geq 1 \), it holds

\[P_T^\ell \log f(\nu_0) \leq \log P_T^\ell f(\mu_0) + \lambda(T)^2 \left\{ \|W_2(\mu_0, \nu_0)\|^2 + K(T, \theta)^2\|W|\|2(\mu_0, \nu_0)\|^2 \right\} \left( \int_0^T K_1(s) \, d\ell_s \right)^{-1}.
\]

(2) For any \( p > 1, \mu_0, \nu_0 \in \mathcal{P}_\theta, \) \( \mathcal{F}_0 \)-measurable random variables \( X_0, Y_0 \) with \( \mathcal{L}_{X_0} = \mu_0, \mathcal{L}_{Y_0} = \nu_0, T > 0, \) and non-negative \( f \in \mathcal{B}_b(\mathbb{R}^d) \), we have

\[(P_T^\ell f(\nu_0))^p \leq P_T^\ell f^p(\mu_0) \cdot \left( \mathbb{E} \exp \left[ \frac{p\lambda(T)^2}{(p-1)^2} |X_0 - Y_0|^2 \left( \int_0^T K_1(s) \, d\ell_s \right)^{-1} \right] \right)^{p-1} \times \exp \left[ \frac{p\lambda(T)^2}{p-1} K(T, \theta)^2\|W|\|2(\mu_0, \nu_0)\|^2 \left( \int_0^T K_1(s) \, d\ell_s \right)^{-1} \right].
\]

Following the line of [25, 26, 7], for \( \varepsilon \in (0, 1) \), consider the following regularization of \( \ell \):

\[\ell_t^\varepsilon := \frac{1}{\varepsilon} \int_t^{t+\varepsilon} \ell_s \, ds + \varepsilon t = \int_0^1 \ell_{\varepsilon s+t} \, ds + \varepsilon t, \quad t \geq 0.
\]
It is clear that, for each $\varepsilon \in (0, 1)$, the function $\ell^\varepsilon$ is absolutely continuous, strictly increasing and satisfies for any $t \geq 0$

$$\ell^\varepsilon_t \downarrow \ell_t \quad \text{as} \quad \varepsilon \downarrow 0.$$  

(3.3)

For $\mu_0 \in \mathcal{P}_\theta$, let $X^\varepsilon_t(\mu_0)$ be the solution to the following SDE with $\mathcal{L}X^\varepsilon_0 = \mu_0$:

$$dX^\varepsilon_t = b(t, X^\varepsilon_t, \mathcal{L}X^\varepsilon_t(\mu_0)) dt + \sigma(t) dW^\varepsilon_{t-t_0^\varepsilon}.$$  

Define the associated Markov operator $P^\varepsilon_t$ by (3.2) with $\ell$ replaced by $\ell^\varepsilon$.

**Lemma 3.5.** Fix $\varepsilon \in (0, 1)$ and assume $(H1')$ and $(H2)$-$(H5)$. Then the assertions in Proposition in 3.4 hold with $\ell$ replaced by $\ell^\varepsilon$.

**Proof.** Fix $T > 0$. Take $\mathcal{F}_0$-measurable random variables $X_0, Y_0$ with $\mathcal{L}X_0 = \mu_0, \mathcal{L}Y_0 = \nu_0$. Let $Y_t$ solve the SDE

$$dY_t = b(t, Y_t, \mathcal{L}X_t(\nu_0)) dt + \xi(t) \mathbf{1}_{[0, \tau)}(t) \frac{X^\varepsilon_t(\mu_0) - Y_t}{|X^\varepsilon_t(\mu_0) - Y_t|} d\ell^\varepsilon_t + \sigma(t) dW^\varepsilon_{t-t_0^\varepsilon}$$  

with $\mathcal{L}Y_0 = \nu_0$, where

$$\tau := T \wedge \inf\{t \geq 0; X^\varepsilon_t(\mu_0) = Y_t\}$$  

and

$$\xi(t) := \{X_0 - Y_0| + K(t, \theta)\mathbb{W}_\theta(\mu_0, \nu_0)\} \frac{\sqrt{K_1(t)}}{\int_0^T K_1(s) d\ell^\varepsilon_s}.$$  

It is clear that $(X^\varepsilon_t, Y_t)$ is well defined for $t < \tau$. By $(H3)$, it follows that for $t < \tau$

$$d|X^\varepsilon_t(\mu_0) - Y_t| \leq \frac{1}{2} \kappa_1(t)|X^\varepsilon_t(\mu_0) - Y_t| dt + \frac{1}{2} \kappa_2(t)\mathbb{W}_\theta(\mu_t, \nu_t) dt - \xi(t) d\ell^\varepsilon_t.$$  

Thus, by (2.10), we obtain that

$$\sqrt{K_1(t)} |X^\varepsilon_t(\mu_0) - Y_t|$$  

$$\leq |X_0 - Y_0| + \frac{1}{2} \int_0^t \sqrt{K_1(s)} \kappa_2(s)\mathbb{W}_\theta(\mu_s, \nu_s) ds - \int_0^t \sqrt{K_1(s)} \xi(s) d\ell^\varepsilon_s$$  

$$\leq |X_0 - Y_0| + K(t, \theta)\mathbb{W}_\theta(\mu_0, \nu_0) - \int_0^t \sqrt{K_1(s)} \xi(s) d\ell^\varepsilon_s$$  

$$= \left\{|X_0 - Y_0| + K(t, \theta)\mathbb{W}_\theta(\mu_0, \nu_0)\right\} \left\{1 - \frac{\int_0^t K_1(s) d\ell^\varepsilon_s}{\int_0^T K_1(s) d\ell^\varepsilon_s}\right\}$$  

for all $t < \tau$. If $\tau(\omega) > T$ for some $\omega \in \Omega$, we can take $t = T$ in the above inequality to get

$$0 < \sqrt{K_1(T)} |X^\varepsilon_T(\mu_0, \omega) - Y_T(\omega)| \leq 0,$$  

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Thus, the distribution of \( Y_t \) for \( t \in [\tau, T] \), then \( Y_t \) solves (3.4) for \( t \in [\tau, T] \). In particular, \( X^{\tau\varepsilon}_T(\mu_0) = Y_T \).

Denote by \( \gamma^{\varepsilon} : [\ell^{\varepsilon}_0, \infty) \to [0, \infty) \) the inverse function of \( \ell^{\varepsilon} \). Then \( \ell^{\varepsilon}_{t_0} = t \) for \( t \geq \ell^{\varepsilon}_0 \), \( \gamma^{\varepsilon}_{t_0} = t \) for \( t \geq 0 \), and \( t \mapsto \gamma^{\varepsilon}_t \) is absolutely continuous and strictly increasing. Let

\[
\widetilde{W}_t := \int_0^t \Psi(r) \, dr + W_t \quad \text{and} \quad M_t := -\int_0^t \langle \Psi(r), dW_r \rangle, \quad t \geq 0,
\]

where

\[
\Psi(r) := \sigma_{\gamma^{\varepsilon}_{t_0}+\varepsilon}^{-1} \Phi(\gamma^{\varepsilon}_{t_0}+\varepsilon) \quad \text{and} \quad \Phi(r) := \xi(r) \mathbb{1}_{[0,\tau)}(r) \frac{X^{\tau\varepsilon}_T(\mu_0) - Y_r}{X^{\tau\varepsilon}_T(\mu_0) - Y_{\tau}}.
\]

By (H5) and the elementary inequality that \((a+b)^2 \leq 2a^2 + 2b^2\) for \( a, b \geq 0 \), the compensator of the martingale \( M_t \) satisfies, for \( t \geq 0 \),

\[
\langle M \rangle_t = \int_0^t |\Psi(r)|^2 \, dr \leq \int_0^T |\sigma^{-1}_s \Phi(s)|^2 \, d\ell^{\varepsilon}_s \\
\leq \int_0^T \lambda(s)^2 \Phi(s)^2 \, d\ell^{\varepsilon}_s \leq \lambda(T)^2 \int_0^T \xi(s)^2 \, d\ell^{\varepsilon}_s = \lambda(T)^2 \left( \int_0^T K_1(s) \, d\ell^{\varepsilon}_s \right)^{-1} \\
\leq 2\lambda(T)^2 \left( |X_0 - Y_0|^2 + K(T, \theta)^2 \mathbb{W}_\theta(\mu_0, \nu_0)^2 \right) \left( \int_0^T K_1(s) \, d\ell^{\varepsilon}_s \right)^{-1}.
\]

By Novikov’s criterion, we have \( \mathbb{E}[R|\mathcal{F}_0] = 1 \), where

\[
R := \exp \left[ M_{\ell^{\varepsilon}_{t_0}} - \frac{1}{2} \langle M \rangle_{\ell^{\varepsilon}_{t_0}} \right].
\]

According to Girsanov’s theorem, \( (\widetilde{W}_t)_{0 \leq t \leq \ell^{\varepsilon}_T(\tau) - \ell^{\varepsilon}_0} \) is a d-dimensional Brownian motion under the new probability measure \( \mathbb{R}P(\cdot|\mathcal{F}_0) \). Rewrite (3.4) as

\[
dY_t = b(t, Y_t, \mathcal{L}_{\mathcal{X}_t}(\nu_0)) \, dt + \sigma(t) \, d\widetilde{W}_t^{\varepsilon}_{t_0}.
\]

Thus, the distribution of \( (Y_t)_{0 \leq t \leq T} \) under \( \mathbb{R}P(\cdot|\mathcal{F}_0) \) coincides with that of \( (X^{\tau\varepsilon}_t(\nu_0))_{0 \leq t \leq T} \) under \( \mathbb{P}(\cdot|\mathcal{F}_0) \); in particular, it holds that for any \( f \in \mathcal{B}(\mathbb{R}^d) \),

\[
\mathbb{E}_{\mathbb{R}P(\cdot|\mathcal{F}_0)} f(X_T^{\tau\varepsilon}(\nu_0)) = \mathbb{E}_{\mathbb{R}P(\cdot|\mathcal{F}_0)} f(Y_T) = \mathbb{E}_{\mathbb{P}(\cdot|\mathcal{F}_0)} [R f(Y_T)] = \mathbb{E}_{\mathbb{P}(\cdot|\mathcal{F}_0)} [R f(X_T^{\tau\varepsilon}(\mu_0))].
\]

By (3.6), the Young inequality (cf. [23, p. 24]), and the observation that

\[
\log R = -\int_0^{\ell^{\varepsilon}_{t_0}} \langle \Psi(r), dW_r \rangle - \frac{1}{2} \int_0^{\ell^{\varepsilon}_{t_0}} |\Psi(r)|^2 \, dr \\
= -\int_0^{\ell^{\varepsilon}_{t_0}} \langle \Psi(r), d\widetilde{W}_r \rangle + \frac{1}{2} \langle M \rangle_{\ell^{\varepsilon}_{t_0}},
\]

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we get that, for any positive \( f \in \mathcal{B}_b(\mathbb{R}^d) \),
\[
E_{P(\cdot, \mathcal{F}_0)} \log f(X^e_T(\nu_0)) = E_{P(\cdot, \mathcal{F}_0)} \left[ R \log f(X^e_T(\mu_0)) \right]
\leq \log E_{P(\cdot, \mathcal{F}_0)} f(X^e_T(\mu_0)) + E_{P(\cdot, \mathcal{F}_0)} [R \log R]
= \log E_{P(\cdot, \mathcal{F}_0)} f(X^e_T(\mu_0)) + E_{R^p(\cdot, \mathcal{F}_0)} \log R
= \log E_{P(\cdot, \mathcal{F}_0)} f(X^e_T(\mu_0)) + \frac{1}{2} E_{R^p(\cdot, \mathcal{F}_0)} \langle M \rangle \epsilon_T^e - \epsilon_0^e.
\]
Combining this with the Jensen inequality and (3.5), we obtain
\[
P_T^e \log f(\nu_0) = E \{ E_{P(\cdot, \mathcal{F}_0)} \log f(X^e_T(\nu_0)) \}
\leq \log E \{ E_{P(\cdot, \mathcal{F}_0)} f(X^e_T(\mu_0)) \} + \frac{1}{2} E \{ E_{P(\cdot, \mathcal{F}_0)} [R \langle M \rangle \epsilon_T^e - \epsilon_0^e] \}
\leq \log P_T^e f(\mu_0) + \lambda(T)^2 \left\{ E |X_0 - Y_0|^2 + K(T, \theta)^2 \mathbb{W}_0(\mu_0, \nu_0)^2 \right\} \left( \int_0^T K_1(s) \, d\epsilon_s^e \right)^{-1}.
\]
Taking infimum over \( \mathcal{L}_{X_0} = \mu_0, \mathcal{L}_{Y_0} = \nu_0 \), we derive the log-Harnack inequality.
Next, we prove the power-Harnack inequality. For any \( p > 1 \) and positive \( f \in \mathcal{B}_b(\mathbb{R}^d) \), we find with (3.6) and the Hölder inequality that
\[
E_{P(\cdot, \mathcal{F}_0)} f(X^e_T(\nu_0)) = E_{P(\cdot, \mathcal{F}_0)} \left[ R f(X^e_T)(\mu_0) \right]
\leq \left( E_{P(\cdot, \mathcal{F}_0)} f^p(X^e_T)(\mu_0) \right)^{1/p} \cdot \left( E_{P(\cdot, \mathcal{F}_0)} [R^{p/(p-1)}] \right)^{(p-1)/p}.
\]
Since by (3.5)
\[
R^{p/(p-1)} = \exp \left[ \frac{p}{p-1} M_{\epsilon_T^e - \epsilon_0^e} - \frac{p}{2(p-1)} \langle M \rangle \epsilon_T^e - \epsilon_0^e \right],
\]
we know that
\[
E_{P(\cdot, \mathcal{F}_0)} [R^{p/(p-1)}] \leq \exp \left[ \frac{p \lambda(T)^2}{(p-1)^2} \left\{ |X_0 - Y_0|^2 + K(T, \theta)^2 \mathbb{W}_0(\mu_0, \nu_0)^2 \right\} \left( \int_0^T K_1(s) \, d\epsilon_s^e \right)^{-1} \right].
\]
Inserting this estimate into (3.7), we obtain
\[
P_T^e f(\nu_0) = E \{ E_{P(\cdot, \mathcal{F}_0)} f(X^e_T(\nu_0)) \}\]
\[
\begin{align*}
&\leq \mathbb{E} \left\{ \left( \mathbb{E}^{\mathbb{P}(\mathcal{F}_0)} f^p(X^\varepsilon_T(\mu_0)) \right)^{1/p} \exp \left[ \frac{\lambda(T)^2}{p - 1} |X_0 - Y_0|^2 \left( \int_0^T K_1(s) \, dl_s^\varepsilon \right)^{-1} \right] \right\} \\
&\times \exp \left[ \frac{\lambda(T)^2}{p - 1} K(T, \theta)^2 \mathbb{W}^{(\mu_0, \nu_0)}^2 \left( \int_0^T K_1(s) \, dl_s^\varepsilon \right)^{-1} \right].
\end{align*}
\]

It remains to use the Hölder inequality to get the desired power-Harnack inequality. \qed

The following two assumptions will be used:

(A1) $\sigma$ is piecewise constant, i.e. there exists a sequence $\{t_n\}_{n \geq 0}$ with $t_0 = 0$ and $t_n \uparrow \infty$ such that

\[
\sigma(t) = \sum_{n=1}^{\infty} 1_{[t_{n-1}, t_n)}(t) \sigma(t_{n-1});
\]

(A2) For every $t > 0$, there exists $C_t > 0$ depending only on $t$ such that

\[
|b(s, x, \mu) - b(s, y, \mu)| \leq C_t |x - y|, \quad 0 \leq s \leq t, \ x, y \in \mathbb{R}^d, \ \mu \in \mathcal{P}_\theta.
\]

Lemma 3.6. Assume (H1') and (H2)-(H5). If (A1) and (A2) hold, then for all $t \geq 0$ and $\mu_0 \in \mathcal{P}_\theta$,

\[
\lim_{\varepsilon \downarrow 0} X^\varepsilon_T(\mu_0) = X^T_T(\mu_0) \quad \mathbb{P}\text{-a.s.}
\]

Proof. It is not hard to obtain from (A1) and (A2) that, for all $t \geq 0$ and $\varepsilon \in (0, 1)$,

\[
|X^\varepsilon_T(\mu_0) - X^\varepsilon_T(\mu_0)| \leq C_t \int_0^t |X^\varepsilon_s(\mu_0) - X^\varepsilon_s(\mu_0)| \, ds + g(\varepsilon, t),
\]

where

\[
g(\varepsilon, t) := \sup_{s \in [0, t]} \|\sigma(s)\| \cdot \left( |W^\varepsilon_{t-\varepsilon_0} - W_{t-\varepsilon_0}| + 2 \sum_{n : t_n < t} |W^\varepsilon_{t_n-\varepsilon_0} - W^\varepsilon_{t_n}| \right).
\]

Using Gronwall’s inequality, we get

\[
|X^\varepsilon_T(\mu_0) - X^\varepsilon_T(\mu_0)| \leq g(\varepsilon, t) + C_t \int_0^t g(\varepsilon, s) e^{C_t (t-s)} \, ds.
\]

Due to (3.3), for all $s \geq 0$, $\lim_{\varepsilon \downarrow 0} g(\varepsilon, s) = 0$ a.s. It remains to use the dominated convergence theorem to finish the proof. \qed

Proof of Proposition 3.4. Fix $T > 0$. By a standard approximation argument, we may and do assume that $f \in C_b(\mathbb{R}^d)$.

Step 1: Assume (A1) and (A2). Since $\ell_t$ is of bounded variation, it is not hard to verify from (3.3) that

\[
\lim_{\varepsilon \downarrow 0} \int_0^T K_1(s) \, dl_s^\varepsilon = \int_0^T K_1(s) \, dl_s.
\]
Letting $\varepsilon \downarrow 0$ in Lemma 3.5, and using Lemma 3.6, we get the desired inequalities.

**Step 2:** Assume (A2). Clearly, we can pick a sequence of $\mathbb{R}^d \otimes \mathbb{R}^d$-valued functions \{\(\sigma_n : n \in \mathbb{N}\) on $[0, \infty)$ such that each $\sigma_n$ is piecewise constant, $\| (\sigma_n(t))^{-1} \| \leq \lambda(t)$ for all $n \in \mathbb{N}$ and $t \in [0, T]$, and $\sigma_n \rightarrow \sigma$ in $L^2([0, T]; d\ell)$ as $n \rightarrow \infty$. Let $X_t^{\ell,n}$ solve (3.1) with $\sigma$ replaced by $\sigma_n$ and $X_0^{\ell,n} = X_0^{\ell}$, and denote by $P_T^{\ell,n}$ the associated Markov operator. By Step 1, the statement of Proposition 3.4 holds with $P_T^\ell$ replaced by $P_T^{\ell,n}$. It suffice to prove that

\[
\lim_{n \rightarrow \infty} P_T^{\ell,n} f = P_T^\ell f, \quad f \in C_b(\mathbb{R}^d).
\]

It follows from (A2) that

\[
|X_t^{\ell,n} - X_t^{\ell}| \leq C_t \int_0^t |X_s^{\ell,n} - X_s^{\ell}| \, ds + \left| \int_0^t \{\sigma_n(s) - \sigma(s)\} \, dW_s \right|.
\]

Noting that

\[
\lim_{n \rightarrow \infty} \mathbb{E} \left| \int_0^t \{\sigma_n(s) - \sigma(s)\} \, dW_s \right|^2 = \lim_{n \rightarrow \infty} \int_0^t \|\sigma_n(s) - \sigma(s)\|^2_{HS} \, d\ell_s = 0,
\]

we have (up to a subsequence) a.s. $\lim_{n \rightarrow \infty} \int_0^t \{\sigma_n(s) - \sigma(s)\} \, dW_s = 0$. Then as in the proof of Lemma 3.6, we find that for all $t \geq 0$,

\[
\lim_{n \rightarrow \infty} X_t^{\ell,n} = X_t^{\ell} \text{ a.s.,}
\]

which implies (3.8).

**Step 3:** For the general case, we shall make use of the approximation argument in [25, part (c) of proof of Theorem 2.1] (see also [7]). Let

\[
\tilde{b}(t, x, \mu_t) := b(t, x, \mu_t) - \frac{1}{2} \kappa_1(t) x, \quad t \geq 0, \quad x \in \mathbb{R}^d.
\]

By (H3), it is easy to see that the mapping $\text{id} - \varepsilon \tilde{b}(t, \cdot, \mu_t) : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is injective for any $\varepsilon > 0$ and $t \geq 0$. For $\varepsilon > 0$ and $t > 0$, let

\[
\tilde{b}^{(\varepsilon)}(t, x, \mu_t) := \frac{1}{\varepsilon} \left[ \left( \text{id} - \varepsilon \tilde{b}(t, \cdot, \mu_t) \right)^{-1} (x) - x \right], \quad x \in \mathbb{R}^d.
\]

Then for any $\varepsilon > 0$ and $t > 0$, $\tilde{b}^{(\varepsilon)}(t, \cdot, \mu_t)$ is dissipative and satisfies (A2) with $b$ replaced by $\tilde{b}^{(\varepsilon)}$, $|\tilde{b}^{(\varepsilon)}(t, \cdot, \mu_t)| \leq |\tilde{b}(t, \cdot, \mu_t)|$ and $\lim_{\varepsilon \downarrow 0} \tilde{b}^{(\varepsilon)}(t, \cdot, \mu_t) = \tilde{b}(t, \cdot, \mu_t)$. Let $b^{(\varepsilon)}(t, x, \mu_t) := \tilde{b}^{(\varepsilon)}(t, x, \mu_t) + \frac{1}{2} \kappa_1(t) x$. Then $b^{(\varepsilon)}(t, \cdot, \mu_t)$ also satisfies (A2) with $b$ replaced by $b^{(\varepsilon)}$ and

\[
2(b^{(\varepsilon)}(t, x, \mu_t) - b^{(\varepsilon)}(t, y, \mu_t), x - y) \leq \kappa_1(t)|x - y|^2.
\]

Let $X_t^{\ell, (\varepsilon)}(\mu_0)$ solve the SDE (3.1) with $b$ replaced by $b^{(\varepsilon)}$ and $X_0^{\ell, (\varepsilon)} = X_0^{\ell}$. Denote by $P_T^{\ell, (\varepsilon)}$ the associated Markov operator. Due to the second part of the proof, Proposition 3.4 holds with $P_T^\ell$ replaced by $P_T^{\ell, (\varepsilon)}$. Then we only need to show that

\[
\lim_{\varepsilon \downarrow 0} P_T^{\ell, (\varepsilon)} f = P_T^\ell f, \quad f \in C_b(\mathbb{R}^d).
\]
To this end, we obtain from (3.9) and (2.5) with \( \rho = 2 \) that
\[
\begin{aligned}
d|X_t^{\ell,(e)} - X_t^\ell|^2 &= 2\langle X_t^{\ell,(e)} - X_t^\ell, b(e)(t, X_t^{\ell,(e)}, \mu_t) - b(e)(t, X_t^\ell, \mu_t) \rangle dt \\
+ 2\langle X_t^{\ell,(e)} - X_t^\ell, b(e)(t, X_t^\ell, \mu_t) - b(t, X_t^\ell, \mu_t) \rangle dt \\
&\leq \kappa_1(t)|X_t^{\ell,(e)} - X_t^\ell|^2 dt + 2|X_t^{\ell,(e)} - X_t^\ell| \cdot |b(e)(t, X_t^\ell, \mu_t) - b(t, X_t^\ell, \mu_t)| dt \\
&\leq (\kappa_1(t) + 1)^+|X_t^{\ell,(e)} - X_t^\ell|^2 dt + |b(e)(t, X_t^\ell, \mu_t) - b(t, X_t^\ell, \mu_t)|^2 dt.
\end{aligned}
\]
This yields that
\[
|X_t^{\ell,(e)} - X_t^\ell|^2 \leq \int_0^t (\kappa_1(s) + 1)^+|X_s^{\ell,(e)} - X_s^\ell|^2 ds + \int_0^t |b(e)(s, X_s^\ell, \mu_s) - b(s, X_s^\ell, \mu_s)|^2 ds.
\]
Combining this with Gronwall’s inequality, we obtain
\[
|X_t^{\ell,(e)} - X_t^\ell|^2 \leq \exp \left[ \int_0^t (\kappa_1(s) + 1)^+ ds \right] \cdot \int_0^t |b(e)(s, X_s^\ell, \mu_s) - b(s, X_s^\ell, \mu_s)|^2 ds.
\]
By (H2) and (H4), letting \( \epsilon \downarrow 0 \) and using the dominated convergence theorem, we get
\[
\lim_{\epsilon \downarrow 0} X_t^{\ell,(\epsilon)} = X_t^\ell \text{ for all } t \geq 0.
\]
In particular, (3.10) holds. The proof is now finished. \( \square \)

### 3.2 Proof of Theorem 3.2

**Proof of Theorem 3.2.** Since the processes \( W \) and \( S \) are independent, it holds that
\[
P_T f(\cdot) = \mathbb{E} \left[ P_T^f(\cdot) |_{\mathcal{F} = S} \right], \quad T > 0, f \in \mathcal{B}(\mathbb{R}^d).
\]
Combining the estimates in Proposition 3.4 with the Jensen inequality and the Hölder inequality, we obtain the desired Harnack type inequalities. \( \square \)

### 4 Appendix

The following result should be known, but we could not find a reference and so we include a proof for the sake of completeness.

**Proposition 4.1.** Assume that \( b : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d \) is measurable and continuous in the space variable \( x \in \mathbb{R}^d \) and \( \sigma : [0, \infty) \to \mathbb{R}^d \otimes \mathbb{R}^d \) is measurable and locally bounded. Let \( (Z_t)_{t \geq 0} \) be a Lévy process with Lévy measure \( \nu_Z \) satisfying \( \int_{|x| \geq 1} |x|^{\theta} \nu_Z(dx) < \infty \) for some \( \theta \geq 1 \). If there exists a locally bounded function \( \kappa : [0, \infty) \to \mathbb{R} \) such that
\[
2\langle b(t, x) - b(t, y), x - y \rangle \leq \kappa(t)|x - y|^2, \quad x, y \in \mathbb{R}^d, t \geq 0,
\]
and \( b(t, 0) \) is locally bounded in the time variable \( t \geq 0 \), then the SDE
\[
\begin{aligned}
\text{d}X_t &= b(t, X_t) \text{d}t + \sigma(t) \text{d}Z_t
\end{aligned}
\]
starting from \( \mathcal{F}_0 \)-measurable initial value \( X_0 \) with \( \mathcal{L}_X \in \mathcal{P}_\theta \) has a unique strong solution satisfying
\[
\mathbb{E} \sup_{s \in [0,t]} |X_s|^\theta < \infty \quad \text{for all } t > 0.
\]
Proof. Under our assumptions, it is well known that the SDE has a unique (strong) solution. It remains to prove that the moments are finite. Denote by \((l, Q, \nu_z)\) the Lévy triplet of \((Z_t)_{t \geq 0}\). By the Lévy-Itô decomposition (see e.g. [1, Theorem 2.4.16]),

\[
Z_t = lt + \sqrt{Q}W_t + \int_0^t \int_{|x| \geq 1} x \, N(ds, dx) + \int_0^t \int_{|x| < 1} x \tilde{N}(ds, dx),
\]

where \(W = (W_t)_{t \geq 0}\) is a \(d\)-dimensional (standard) Brownian motion, \(N\) is a Poisson random measure with intensity \(\nu_z(dx)ds\) and independent of \(W\), and \(\tilde{N}\) is the associated compensated Poisson random measure. By Itô’s formula (cf. [1, Theorem 4.4.7]),

\[
d|X_t|^2 = 2\langle X_{t-}, b(t, X_{t-}) + \sigma(t)l \rangle \, dt + 2\langle X_{t-}, \sigma(t)\sqrt{Q} \, dW_t \rangle + \|\sigma(t)\sqrt{Q}\|_{HS}^2 \, dt
\]

\[
+ \int_{|x| \geq 1} (|X_{t-} + \sigma(t)x|^2 - |X_{t-}|^2) \, N(dt, dx)
\]

\[
+ \int_{|x| < 1} (|X_{t-} + \sigma(t)x|^2 - |X_{t-}|^2) \tilde{N}(dt, dx)
\]

\[
+ \int_{|x| < 1} (|X_{t-} + \sigma(t)x|^2 - |X_{t-}|^2 - 2\langle X_{t-}, \sigma(t)x \rangle) \nu_z(dx) \, dt.
\]

Set \(p := \theta/2 \geq 1/2\). Applying Itô’s formula again, we obtain

\[
d(1 + |X_t|^2)^p = 2p(p - 1)(1 + |X_{t-}|^2)^{p-2}|(\sigma(t)\sqrt{Q})^* X_{t-}|^2 \, dt
\]

\[
+ p(1 + |X_{t-}|^2)^{p-1} \left(2\langle X_{t-}, b(t, X_{t-}) + \sigma(t)l \rangle + \|\sigma(t)\sqrt{Q}\|_{HS}^2 + \int_{|x| < 1} |\sigma(t)x|^2 \nu_z(dx) \right) \, dt
\]

\[
+ 2p(1 + |X_{t-}|^2)^{p-1}\langle X_{t-}, \sigma(t)\sqrt{Q} \, dW_t \rangle
\]

\[
+ \int_{|x| \geq 1} J_1(x, t, p) \, N(dt, dx) + \int_{|x| < 1} J_1(x, t, p) \tilde{N}(dt, dx) + \int_{|x| < 1} J_2(x, t, p) \nu_z(dx) \, dt,
\]

where

\[
J_1(x, t, p) := (1 + |X_{t-} + \sigma(t)x|^2)^p - (1 + |X_{t-}|^2)^p,
\]

and

\[
J_2(x, t, p) := (1 + |X_{t-} + \sigma(t)x|^2)^p - (1 + |X_{t-}|^2)^p - p(1 + |X_{t-}|^2)^{p-1}(|X_{t-} + \sigma(t)x|^2 - |X_{t-}|^2).
\]

Since \(\sigma(t)\) and \(b(t, 0)\) are locally bounded in \(t \geq 0\), it follows from (4.1) that we may find out a nondecreasing function \(H_1 : [0, \infty) \to (0, \infty)\) such that

\[
\max \left\{ (1 + |X_{t-}|^2)^{-1}|(\sigma(t)\sqrt{Q})^* X_{t-}|^2,
\right. \]

\[
2\langle X_t, b(t, X_{t-}) + \sigma(t)l \rangle + \|\sigma(t)\sqrt{Q}\|_{HS}^2 + \int_{|x| < 1} |\sigma(t)x|^2 \nu_z(dx) \right\} \leq H_1(t)(1 + |X_{t-}|^2).
\]

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Then, we get

\[(1 + |X_s|^2)^p \leq (1 + |X_0|^2)^p + p(2p - 1)H_1(s) \int_0^s (1 + |X_r|^2)^p \, dr\]
\[+ 2p \int_0^s (1 + |X_r|^2)^{p-1} (X_r, \sigma(r) \sqrt{Q} \, dW_r) + \int_0^s \int_{|x| \geq 1} J_1(x, r, p) N(dr, dx)\]
\[+ \int_0^s \int_{|x| < 1} J_1(x, r, p) \tilde{N}(dr, dx) + \int_0^s \int_{|x| < 1} J_2(x, r, p) \nu_z(dx)dr\]
\[=: (1 + |X_0|^2)^p + \sum_{i=1}^5 I_i(s, p).\]

Let \(\tau_n := \inf\{t \geq 0 : |X_t| \geq n\}\) for \(n \in \mathbb{N}\). Then we have

\[(4.2) \quad \mathbb{E} \sup_{s \in [0, t \wedge \tau_n]} (1 + |X_s|^2)^p \leq \mathbb{E}(1 + |X_0|^2)^p + \sum_{i=1}^5 \mathbb{E} \sup_{s \in [0, t \wedge \tau_n]} |I_i(s, p)|.\]

We shall estimate these terms separately. First,

\[\mathbb{E} \sup_{s \in [0, t \wedge \tau_n]} I_1(s, p) \leq p(2p - 1)H_1(t)\mathbb{E} \int_0^{t \wedge \tau_n} (1 + |X_r|^2)^p \, dr\]
\[\leq p(2p - 1)H_1(t) \int_0^t \mathbb{E} \sup_{r \in [0, r \wedge \tau_n]} (1 + |X_r|^2)^p \, dr.\]

By the Burkholder-Davis-Gundy inequality, there exist a constant \(c_1 > 0\) and a nondecreasing function \(H_2 : [0, \infty) \to (0, \infty)\) such that

\[\mathbb{E} \sup_{s \in [0, t \wedge \tau_n]} |I_2(s, p)| \leq 2c_1p\mathbb{E} \left( \int_0^{t \wedge \tau_n} (1 + |X_r|^2)^{2p-2} |\sigma(r)\sqrt{Q}|^{2p} |X_r|^2 \, dr \right)^{1/2}\]
\[\leq 2c_1pH_2(t)\mathbb{E} \left( \int_0^{t \wedge \tau_n} (1 + |X_r|^2)^{2p-1} \, dr \right)^{1/2}\]
\[\leq 2c_1p\sqrt{t}H_2(t)\mathbb{E} \sup_{s \in [0, t \wedge \tau_n]} (1 + |X_s|^2)^{p-1/2}.\]

Applying the following inequality (recall \(p \geq 1/2\))

\[(4.3) \quad yz^{p-1/2} \leq \frac{[3(2p - 1)]^{2p-1}}{(2p)^{2p}} y^{2p} + \frac{1}{3} z^p, \quad y, z \geq 0,\]

it holds that

\[\mathbb{E} \sup_{s \in [0, t \wedge \tau_n]} |I_2(s, p)| \leq [3(2p - 1)]^{2p-1} [c_1\sqrt{t}H_2(t)]^{2p} + \frac{1}{3} \mathbb{E} \sup_{s \in [0, t \wedge \tau_n]} (1 + |X_s|^2)^p.\]
By the Burkholder-Davis-Gundy inequality, cf. Novikov [16, Theorem 1.1 (a)], there exists H such that

\[ \sup_{s \in [0,t \wedge \tau_n]} |I_3(s, p)| = \sup_{s \in [0,t \wedge \tau_n]} \left| \sum_{r \in [0,s], |\Delta Z_r| \geq 1} J_1(\Delta Z_r, r, p) \right| \leq E \left( \sum_{r \in [0,t \wedge \tau_n], |\Delta Z_r| \geq 1} |J_1(\Delta Z_r, r, p)| \right) \]

\[ = E \int_0^{t \wedge \tau_n} \int_{|x| \geq 1} |J_1(x, r, p)| N(dr, dx) \]

\[ = E \int_0^{t \wedge \tau_n} \int_{|x| \geq 1} |J_1(x, r, p)| \nu_Z(dx)dr. \]

Since there exist \( c_2 = c_2(p) > 0 \) and nondecreasing function \( H_3 : [0, \infty) \to (0, \infty) \) such that

\[ |J_1(x, r, p)| \leq (1 + |X_r + \sigma(r)x|^2)^p + (1 + |X_r|^2)^p \leq c_2(1 + |X_r|^2)^p + c_2H_3(r)|x|^{2p}, \]

we know that

\[ E \sup_{s \in [0,t \wedge \tau_n]} |I_3(s, p)| \leq c_2 \int_{|x| \geq 1} \nu_Z(dx) \cdot E \int_0^{t \wedge \tau_n} (1 + |X_r|^2)^p dr \]

\[ + c_2 \int_0^{t \wedge \tau_n} H_3(r) dr \cdot \int_{|x| \geq 1} |x|^{2p} \nu_Z(dx) \]

\[ \leq c_2 \int_{|x| \geq 1} \nu_Z(dx) \cdot \int_0^t E \sup_{s \in [0,t \wedge \tau_n]} (1 + |X_s|^2)^p dr \]

\[ + c_2 tH_3(t) \int_{|x| \geq 1} |x|^{2p} \nu_Z(dx). \]

By the Burkholder-Davis-Gundy inequality, cf. Novikov [16, Theorem 1.1 (a)], there exists \( c_3 > 0 \) such that

\[ E \sup_{s \in [0,t \wedge \tau_n]} |I_4(s, p)| \leq c_3 E \left( \int_0^{t \wedge \tau_n} \int_{|x| < 1} |J_1(x, r, p)|^2 \nu_Z(dx)dr \right)^{1/2}. \]

It is easy to verify that there exists a nondecreasing function \( H_4 : [0, \infty) \to [1, \infty) \) such that

\[ H_4(r)^{-1} \leq \frac{1 + |X_r + \sigma(r)x|^2}{1 + |X_r|^2} \leq H_4(r), \quad |x| < 1, r \geq 0. \]

Combining this with the following elementary inequality

\[ |y^p - z^p| \leq p(y^{p-1} + z^{p-1})|y - z|, \quad y, z \geq 0, \]

one has

\[ |J_1(x, r, p)| \leq p \left[ (1 + |X_r + \sigma(r)x|^2)^p - 1 + (1 + |X_r|^2)^p - |X_r|^2 \right]. \]
\[
\leq p \left( H_4(r)^{|p-1|} + 1 \right) (1 + |X_r-|^2)^{p-1} \cdot |X_r+ + \sigma(r)x|^2 - |X_r-|^2).
\]

For \( |x| < 1 \), since it holds for some nondecreasing function \( H_5 : [0, \infty) \to (0, \infty) \) that
\[
|X_r+ + \sigma(r)x|^2 - |X_r-|^2 = 2\langle X_r+, \sigma(r)x \rangle + |\sigma(r)x|^2
\]
\[
\leq 2|X_r-||\sigma(r)x| + |\sigma(r)x|^2
\]
\[
\leq H_5(r)(1 + |X_r-|^2)^{1/2}|x|,
\]
we obtain
\[
|J_1(x, r, p)| \leq p \left( H_4(r)^{|p-1|} + 1 \right) H_5(r)(1 + |X_r-|^2)^{p-1/2}|x|.
\]

This yields that
\[
\mathbb{E} \sup_{s \in [0, t \wedge \tau_n]} |I_4(s, p)|
\leq c_3p \left( H_4(t)^{|p-1|} + 1 \right) H_5(t) \sqrt{t} \left( \int_{|x| < 1} |x|^2 \nu_Z(dx) \right)^{1/2} \mathbb{E} \sup_{s \in [0, t \wedge \tau_n]} (1 + |X_s|^2)^{p-1/2}
\]
\[
\leq \left[ 3(2p - 1) \right]^{2p-1} \left[ 2^{-1}c_3 \left( H_4(t)^{|p-1|} + 1 \right) H_5(t) \sqrt{t} \left( \int_{|x| < 1} |x|^2 \nu_Z(dx) \right)^{1/2} \right]^{2p}
\]
\[
+ \frac{1}{3} \mathbb{E} \sup_{s \in [0, t \wedge \tau_n]} (1 + |X_s|^2)^{p},
\]

where in the last inequality we have used (4.3). By the inequality
\[
|y^p - z^p - pz^{p-1}(y - z)| \leq \frac{p|p-1|}{2} (y^{p-2} + z^{p-2}) (y - z)^2, \quad y, z \geq 0,
\]
(4.4) and (4.5), we get that for \( |x| < 1 \),
\[
|J_2(x, r, p)|
\leq \frac{p|p-1|}{2} \left[ (1 + |X_r- + \sigma(r)x|^2)^{p-2} + (1 + |X_r-|^2)^{p-2} \right] (|X_r- + \sigma(r)x|^2 - |X_r-|^2)
\]
\[
\leq \frac{p|p-1|}{2} \left( H_4(r)^{|p-2|} + 1 \right) H_5(r)^2 (1 + |X_r-|^2)^{p-1}|x|^2.
\]

This implies that
\[
\mathbb{E} \sup_{s \in [0, t \wedge \tau_n]} |I_5(s, p)| \leq \mathbb{E} \int_0^{t \wedge \tau_n} \int_{|x| < 1} |J_2(x, r, p)| \nu_Z(dx) dr
\]
\[
\leq \frac{p|p-1|}{2} \left( H_4(t)^{|p-2|} + 1 \right) H_5(t)^2 \int_{|x| < 1} |x|^2 \nu_Z(dx) \cdot \int_0^t \mathbb{E} \sup_{s \in [0, r \wedge \tau_n]} (1 + |X_s|^2)^p dr.
\]

Substituting the above estimates into (4.2), we conclude that there exist \( C = C(p) > 0 \) and nondecreasing function \( \Phi : [0, \infty) \to (0, \infty) \) such that
\[
\mathbb{E} \sup_{s \in [0, t \wedge \tau_n]} (1 + |X_s|^2)^p \leq 3\mathbb{E}(1 + |X_0|^2)^p + \Phi(t)C + \Phi(t)C \int_0^t \mathbb{E} \sup_{s \in [0, r \wedge \tau_n]} (1 + |X_s|^2)^p dr.
\]
By Gronwall’s inequality and letting $n \to \infty$, we obtain that for all $t > 0$

$$
\mathbb{E} \sup_{s \in [0,t]} (1 + |X_s|^2)^p \leq [3 \mathbb{E}(1 + |X_0|^2)^p + \Phi(t)^C] \cdot \exp \left[ t \Phi(t)^C \right] < \infty,
$$

which completes the proof. \qed

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