LOW REGULARITY WELL-POSEDNESS FOR THE YANG-MILLS SYSTEM IN 2D

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Abstract. The Cauchy problem for the Yang-Mills system in two space dimensions is treated for data with minimal regularity assumptions. In the classical case of data in $L^2$-based Sobolev spaces we have to assume that the number of derivatives is more than $3/4$ above the critical regularity with respect to scaling. For data in $L^r$-based Fourier-Lebesgue spaces this result can be improved by $1/4$ derivative in the sense of scaling as $r \to 1$.

1. Introduction

Let $G$ be the Lie group $SO(n, \mathbb{R})$ (the group of orthogonal matrices of determinant 1) or $SU(n, \mathbb{C})$ (the group of unitary matrices of determinant 1) and $g$ its Lie algebra $so(n, \mathbb{R})$ (the algebra of trace-free skew symmetric matrices) or $su(n, \mathbb{C})$ (the algebra of trace-free skew hermitian matrices) with Lie bracket $[X, Y] = XY - YX$ (the matrix commutator). For given $A_\alpha : \mathbb{R}^{1+n} \to g$ we define the curvature $F = F[A]$ by

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha + [A_\alpha, A_\beta],$$

where $\alpha, \beta \in \{0, 1, \ldots, n\}$ and $D_\alpha = \partial_\alpha + [A_\alpha, \cdot]$.

Then the Yang-Mills system is given by

$$D^a F_{a\beta} = 0$$

in Minkowski space $\mathbb{R}^{1+n} = \mathbb{R}_t \times \mathbb{R}^n_\mathbf{x}$, where $n \geq 2$, with metric $\text{diag}(-1, 1, \ldots, 1)$. Greek indices run over $\{0, 1, \ldots, n\}$, Latin indices over $\{1, \ldots, n\}$, and the usual summation convention is used. We use the notation $\partial_\mu = \frac{\partial}{\partial x_\mu}$, where we write $(x^0, x^1, \ldots, x^n) = (t, x_1, \ldots, x_n)$ and also $\partial_0 = \partial_t$.

Setting $\beta = 0$ in (2) we obtain the Gauss-law constraint

$$\partial^i F_{ij0} + [A^i, F_{j0}] = 0.$$

The total energy for Yang-Mills at time $t$ is given by

$$E(t) = \sum_{0 \leq \alpha, \beta \leq n} \int_{\mathbb{R}^n} |F_{\alpha\beta}(t, x)|^2 \, dx,$$

and is conserved for a smooth solution decaying sufficiently fast at spatial infinity. The Yang-Mills system is invariant with respect to the scaling

$$A_\lambda(t, x) = \lambda A(t, \lambda x), \quad F_\lambda(t, x) = \lambda^2 F(t, \lambda x).$$

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This implies
\[ \|A_\lambda(0,\cdot,\cdot)\|_{\dot{H}^{s,r}} = \lambda^{1+s-\frac{n}{2}} \|a_\lambda\|_{\dot{H}^{s,r}}, \]
\[ \|F_\lambda(0,\cdot,\cdot)\|_{\dot{H}^{s,r}} = \lambda^{2l-\frac{n}{2}} \|f_\lambda\|_{\dot{H}^{s,r}}. \]

Here \( \|u\|_{\dot{H}^{s,r}} := \|\langle \xi \rangle^{s} \hat{u}(\xi)\|_{L^{r}} \) where \( r \) and \( r' \) are dual exponents, and \( \dot{H}^{s,r} \) denotes the homogeneous space. Therefore the scaling critical exponent is \( s = \frac{n}{2} - 1 \) for \( A \) and \( l = \frac{n}{2} - 2 \) for \( F \).

The system is gauge invariant. Given a sufficiently smooth function \( U : \mathbb{R}^{1+n} \rightarrow \mathcal{G} \) we define the gauge transformation \( T \) by \( T A_0 = A_0' \), \( T(A_1,\ldots,A_n) = (A_1',\ldots,A_n') \), where

\[ A_\alpha \mapsto A_\alpha' = U A_\alpha U^{-1} - (\partial_{\alpha} U) U^{-1}. \]

It is well-known that if \( (A_0,\ldots,A_n) \) satisfies (1), (2) so does \( (A_0',\ldots,A_n') \).

Hence we may impose a gauge condition. We exclusively study the case \( n = 2 \) and Lorentz gauge \( \partial^\alpha A_\alpha = 0 \). Other convenient gauges are the Coulomb gauge \( \partial^\alpha A_\alpha = 0 \) and the temporal gauge \( A_0 = 0 \). Our aim is to obtain local well-posedness for data with minimal regularity.

The classical case \( n = 3 \) and \( r = 2 \) with data in standard Sobolev spaces was considered by Klainerman and Machedon [KM], who made the decisive decision that the nonlinearity satisfies a so-called null condition, which enabled them to prove global well-posedness in temporal and in Coulomb gauge in energy space. The corresponding result in Lorenz gauge, where the Yang-Mills equations can be formulated as a system of nonlinear wave equations, was shown by Selberg and Tesfahun [ST], who discovered that also in this case some of the nonlinearities have a null structure. Tesfahun [T] improved this result to data without finite energy, namely for \( (A(0), (\partial_t A)(0)) \in H^s \times H^{s-1} \) and \( (F(0), (\partial_t F)(0)) \in H^1 \times H^{1-1} \) with \( s = \frac{n}{2} + \epsilon \) and \( l = -\frac{n}{2} + \epsilon \) for any \( \epsilon > 0 \) by discovering an additional partial null structure. A further improvement was achieved by the author [P2], namely to \( (s,l) = (\frac{n}{2} + \epsilon, -\frac{n}{2} + \epsilon) \) by modifying the solution spaces appropriately. In view of a recent result by S. Hong [H1] who showed that the flow map is not \( C^2 \) if \( s - 1 > 2r \) this result is in a sense sharp. This especially shows that the scaling critical regularity cannot be achieved by the used iteration method. S. Hong [H] also proved local well-posedness of the Yang-Mills system in the Lorenz gauge for initial data in the Besov space \( B^\frac{s}{2}_2 \times B^\frac{s}{2}_2 \), which is critical with respect to scaling, if an additional angular regularity is assumed. Local well-posedness in energy space in 3D was also given by Oh [O] using a new gauge, namely the Yang-Mills heat flow. He was also able to show that this solution can be globally extended [O1]. The Cauchy problem was also treated in higher space dimensions by several authors ([KS], [KT], [KrT], [KrSt], [P1]).

As the critical case in 3D with respect to scaling is \( (s,l) = (\frac{n}{2}, -\frac{n}{2}) \), there is however still a gap, a phenomenon, which is also present in other gauges. In order to close this gap the author treated in [P2] the local well-posedness problem for the Yang-Mills system in Lorenz gauge and space dimension \( n = 3 \) in the case of data \( (A(0), (\partial_t A)(0)) \in \dot{H}^{s,r} \times \dot{H}^{s-1,r} \) and \( (F(0), (\partial_t F)(0)) \in \dot{H}^{l,r} \times \dot{H}^{l-1,r} \) in Fourier-Lebesgue spaces for \( r \neq 2 \), which coincide with the classical Sobolev spaces \( H^s \) for \( r = 2 \). The assumption is that \( s = \frac{n}{2} - \frac{\delta}{2} + \epsilon \) and \( l = \frac{n}{2} - \frac{\delta}{2} + \epsilon \), where any \( \delta > 0 \) is admissible. Thus \( s \rightarrow 2 + \delta \) and \( l \rightarrow 1 + \delta \) as \( r \rightarrow 1 \), which is almost optimal with respect to scaling.

Such an approach was used by several authors already, starting with Vargas-Vega [VV] for 1D Schrödinger equations. Grünrock showed LWP for the modified KdV equation [G], a result which was improved by Grünrock and Vega [GV].
Grünerck treated derivative nonlinear wave equations in 3+1 dimensions and obtained an almost optimal result as \( r \to 1 \) with respect to scaling. Systems of nonlinear wave equations in the 2+1 dimensional case for nonlinearities which fulfill a null condition were considered by Grigoryan-Nahmod. The latter two results are based on estimates by Foschi and Klainerman.

The present paper is a continuation of Grünerck. Here we consider the local well-posedness problem for space dimension \( n = 2 \). Our main result for the classical case of \( L^2 \)-based data (\( r = 2 \)) is local well-posedness under the assumption \( s > \frac{4}{3} \) and \( l > -\frac{1}{r} \), thus \( \frac{3}{2} \) away from the critical exponents \( s = 0 \) and \( l = -1 \) with respect to scaling. In order to reduce this gap we again consider data in Fourier-Lebesgue spaces \( \hat{H}^{s,r} \times \hat{H}^{l,r} \). We obtain \( s \to \frac{3}{2} \) and \( l \to \frac{1}{2} \) as \( r \to 1 \), which scales like \( (s,l) = (\frac{3}{2}, -\frac{1}{2}) \) for the case \( r = 2 \). Thus the gap shrinks by \( \frac{1}{2} \).

The approach in the present paper is similar to Grünerck. In Chapter 2 we rewrite the Yang-Mills equations in Lorenz gauge as a system of semilinear wave equations. We also formulate the main theorems (Theorem 2.1, Cor. 2.1 and Theorem 2.2) in Chapter 3. We recall some basic facts about our solution spaces and a general local well-posedness theorem for the Cauchy problem for systems of nonlinear wave equations with data in Fourier-Lebesgue spaces, which allows to reduce it to estimates for the nonlinearities. In Chapter 4 we give the final formulation of the system in terms of null forms as far as possible. The bi-, tri- and quadrilinear estimates sufficient for the local well-posedness result are formulated, and a general local well-posedness theorem for the Cauchy problem for systems fulfills a null condition were considered by Grigoryan-Nahmod. The latter two results are based on estimates by Foschi and Klainerman.

In Chapter 5 we prove bilinear estimates for the null forms and for general bilinear terms in generalized Bourgain-Klainerman-Machedon spaces \( H^{s,b}_r \) (and \( X^s_{k,b} \)) based on estimates by Foschi and Klainerman, Grünerck, Grigoryan-Nahmod and Grigoryan-Tanguy. In Chapter 6 we consider the case where \( r > 1 \) is close to 1 and prove the multilinear estimates formulated in Chapter 4 by reduction to the bilinear estimates of Chapter 5. In Chapter 7 we prove these estimates in the classical case \( r = 2 \) by reduction to bilinear estimates given by A.K.S. Finally in Chapter 8 we interpolate between the estimates for \( r = 2 \) and \( r = 1 \) to obtain the desired local well-posedness result in the whole range \( 1 < r \leq 2 \).

2. Main results

Expanding (2) in terms of the gauge potentials \( \{A_\alpha\} \), we obtain:

\[
\Box A_\beta = \partial_\beta \partial^\alpha A_\alpha - [\partial^\alpha A_\alpha, A_\beta] - [A^\alpha, \partial^\alpha A_\beta] - [A^\alpha, F_{\alpha \beta}].
\]

If we now impose the Lorenz gauge condition, the system (3) reduces to the nonlinear wave equation

\[
\Box A_\beta = -[A^\alpha, \partial_\alpha A_\beta] - [A^\alpha, F_{\alpha \beta}].
\]

In addition, regardless of the choice of gauge, \( F \) satisfies the wave equation

\[
\Box F_{\beta \gamma} = -[A^\alpha, \partial_\alpha F_{\beta \gamma}] - \partial^\alpha [A_\alpha, F_{\beta \gamma}] - [A^\alpha, [A_\alpha, F_{\beta \gamma}]] - 2[F_{\alpha \beta}, F_{\gamma \alpha}],
\]

where we refer to [ST], chapter 3.2.

Expanding the second and fourth terms in (5), and also imposing the Lorenz gauge, yields

\[
\Box F_{\beta \gamma} = -2[A^\alpha, \partial_\alpha F_{\beta \gamma}] + 2[\partial_\beta A^\alpha, \partial_\alpha A_\gamma] - 2[\partial_\beta A^\alpha, \partial_\alpha A_\gamma] + 2[\partial^\alpha A_\beta, \partial_\alpha A_\gamma] + 2[\partial^\alpha A_\beta, \partial_\alpha A_\gamma] - [A^\alpha, [A_\alpha, F_{\beta \gamma}]] + 2[F_{\alpha \beta}, [A^\alpha, A_\gamma]] - 2[F_{\alpha \gamma}, [A^\alpha, A_\beta]] - 2[[A^\alpha, A_\beta], [A_\alpha, A_\gamma]].
\]
Note on the other hand by expanding the last term in the right hand side of (1), we obtain
\[ \Box A_\beta = -2[A^\alpha, \partial_\alpha A_\beta] + [A^\alpha, \partial_\beta A_\alpha] - [A^\alpha, [A_\alpha, A_\beta]]. \] (7)

We want to solve the system (6)-(7) simultaneously for \( A \) and \( F \). So to pose the Cauchy problem for this system, we consider initial data for \((A, F)\) at \( t = 0:\)
\[ A(0) = a, \quad \partial_t A(0) = \dot{a}, \quad F(0) = f, \quad \partial_t F(0) = \dot{f}. \] (8)

In fact, the initial data for \( F \) can be determined from \((a, \dot{a})\) as follows:
\[
\begin{aligned}
f_{ij} &= \partial_\alpha a_{ij} - \partial_j a_i + [a_i, a_j], \\
f_{ii} &= \ddot{a}_i - \partial_i a_0 + [a_0, a_i], \\
\dot{f}_{ij} &= \partial_\alpha a_{ij} - \partial_j a_i + [a_i, a_j], \\
\dot{f}_{ii} &= \partial^\alpha f_{ij} + [a^\alpha, f_{ai}] 
\end{aligned}
\] (9)

where the first three expressions come from (1) whereas the last one comes from (2) with \( \beta = i \).

Note that the Lorenz gauge condition \( \partial^\alpha A_\alpha = 0 \) and (2) with \( \beta = 0 \) impose the constraints
\[ \dot{a}_0 = \partial^\alpha a_\alpha, \quad \partial^\alpha f_{0\alpha} + [a^\alpha, f_{0\alpha}] = 0. \] (10)

Now we formulate our main theorem.

**Theorem 2.1.** Let \( 1 < r \leq 2 \), \( \epsilon > 0 \). Assume that \( s \) and \( l \) satisfy the following conditions:
\[ s = \frac{3}{2r} + \epsilon, \quad l = \frac{3}{2r} - 1 + \epsilon. \]

Given initial data \((a, \dot{a}) \in \tilde{H}^{s,r} \times \tilde{H}^{s-1,r}, (f, \dot{f}) \in \tilde{H}^{l,r} \times \tilde{H}^{l-1,r}\), there exists a time \( T > 0 \), \( T = T(||a||_{\tilde{H}^{s,r}}, ||\dot{a}||_{\tilde{H}^{s-1,r}}, ||f||_{\tilde{H}^{l,r}}, ||\dot{f}||_{\tilde{H}^{l-1,r}})\), such that the Cauchy problem (6), (7), (8) has a unique solution \( A_\mu \in X_{s,b}^{l,b}([0,T] + X_{s,b}^{l,b}([0,T]) \), \( F \in X_{s,b}^{l,b}([0,T] + X_{s,b}^{l,b}([0,T]) \). This solution has the regularity \( A_\mu \in C^0([0,T], \tilde{H}^{s,r}) \cap C^1([0,T], \tilde{H}^{s-1,r}), F \in C^0([0,T], \tilde{H}^{l,r}) \cap C^1([0,T], \tilde{H}^{l-1,r}) \).

The solution depends continuously on the data and persistence of higher regularity.

**Corollary 2.1.** Let \( s, r \) fulfill the assumptions of Theorem 2.1. Moreover assume that the initial data fulfill (4) and (17). Given any \((a, \dot{a}) \in \tilde{H}^{s,r} \times \tilde{H}^{s-1,r}\), there exists a time \( T = T(||a||_{\tilde{H}^{s,r}}, ||\dot{a}||_{\tilde{H}^{s-1,r}}, ||f||_{\tilde{H}^{l,r}}, ||\dot{f}||_{\tilde{H}^{l-1,r}})\), such that the solution \((A, F)\) of Theorem 2.1 satisfies the Yang-Mills system (1), (2) with Cauchy data \((a, \dot{a})\) and the Lorenz gauge condition \( \partial^\alpha A_\alpha = 0 \).

**Proof of the Corollary.** If \((a, \dot{a}) \in \tilde{H}^{s,r} \times \tilde{H}^{s-1,r}\), then \((f, \dot{f})\), defined by (9), fulfill \((f, \dot{f}) \in \tilde{H}^{l,r} \times \tilde{H}^{l-1,r}\), as one easily checks. Thus we may apply Theorem 2.1. The solution \((A, F)\) does not necessarily fulfill the Lorenz gauge condition and (11) of (12), i.e. \( F = F[A] \). If however the conditions (9) and (10) are assumed then these properties are satisfied and \((A, F)\) is a solution of the Yang-Mills system (1), (2) with Cauchy data \((a, \dot{a})\). This was shown in [47], Remark 2. □

Let us also formulate the result in the special case \( r = 2 \).

**Theorem 2.2.** Let \( \epsilon > 0 \). Assume that \( s \) and \( l \) satisfy the following conditions:
\[ s > \frac{3}{4}, \quad l > \frac{1}{4}, \quad s \geq l \geq s - 1, \quad 2s - l > \frac{9}{4}, \quad 4s - 2l > \frac{5}{2}, \quad 2l - s > -\frac{5}{4}. \]

Given initial data \((a, \dot{a}) \in H^s \times H^{s-1}, (f, \dot{f}) \in H^l \times H^{l-1},\), there exists a time \( T > 0 \), \( T = T(||a||_{H^s}, ||\dot{a}||_{H^{s-1}}, ||f||_{H^l}, ||\dot{f}||_{H^{l-1}})\), such that the
Cauchy problem \((\partial_t \mathcal{Q}, \mathcal{L})\) has a unique solution \(A_\mu \in X^{s,b}_+[0,T] + X^{s,b}_-[0,T]\), \(F \in X^{s,b}_+[0,T] + X^{s,b}_-[0,T]\) (these spaces are defined in Def. \((2.7)\)). Here \(b = \frac{1}{2}+\). This solution has the regularity
\[ A_\mu \in C^0([0,T], H^s) \cap C^1([0,T], H^{s-1}), \]
\(F \in C^0([0,T], H^s) \cap C^1([0,T], H^{s-1})\).

The solution depends continuously on the data and persistence of higher regularity holds.

**Corollary 2.2.** Let \(s, r\) fulfill the assumptions of Theorem 2.2. Moreover assume that the initial data fulfill (2) and (17). Given any \((a, \hat{a}) \in H^s \times H^{s-1}\), there exists a time \(T = T(\|a\|_{H^s}, \|\hat{a}\|_{H^{s-1}}, \|f\|_{H^s}, \|\hat{f}\|_{H^{s-1}})\), such that the solution \((A,F)\) of Theorem 2.2 satisfies the Yang-Mills system (17), (2) with Cauchy data \((a, \hat{a})\) and the Lorenz gauge condition \(\partial^\alpha A_\alpha = 0\).

Let us fix some notation. We denote the Fourier transform with respect to space and time by \(\hat{\cdot}\). \(\square = \partial^2_t - \Delta\) is the d’Alembert operator, \(a \pm \epsilon\) for a sufficiently small \(\epsilon > 0\), and \(\langle \cdot, \cdot \rangle := (1 + |\cdot|^2)^\frac{1}{2}\).

Let \(A^\alpha\) be the multiplier with symbol \(|\xi|^\alpha\). Similarly let \(D^\alpha\), and \(\partial^\alpha\) be the multipliers with symbols \(|\xi|^\alpha\) and \(|\tau - \xi|^\alpha\), respectively.

**Definition 2.1.** Let \(1 \leq r \leq 2\), \(s, b \in \mathbb{R}\). The wave-Sobolev spaces \(H^r_{s,b}\) are the completion of the Schwarz space \(\mathcal{S}(\mathbb{R}^{1+3})\) with norm
\[ \|u\|_{H^r_{s,b}} = \|\langle \xi \rangle^s (|\tau| - |\xi|)^b \hat{u}(\tau, \xi)\|_{L^r_{\tau,\xi}}, \]
where \(r'\) is the dual exponent to \(r\). We also define \(H^r_{s,b}[0,T]\) as the space of the restrictions of functions in \(H^r_{s,b}\) to \([0,T] \times \mathbb{R}^3\). Similarly we define \(X^r_{s,b,\pm}\) with norm
\[ \|\phi\|_{X^r_{s,b,\pm}} := \|\langle \xi \rangle^s (|\tau| \mp |\xi|)^b \hat{\phi}(\tau, \xi)\|_{L^r_{\tau,\xi}}, \]
and \(X^r_{s,b,\pm}[0,T]\).

In the case \(r = 2\) we denote \(H^2_{s,b} = H^s_{s,b}\) and similarly \(X^2_{s,b,\pm} = X^s_{s,\pm}\).

3. Preliminaries

We start by collecting some fundamental properties of the solution spaces. We rely on \((\mathcal{G})\). The spaces \(X^r_{s,b,\pm}\) with norm
\[ \|\phi\|_{X^r_{s,b,\pm}} := \|\langle \xi \rangle^s (|\tau| \mp |\xi|)^b \hat{\phi}(\tau, \xi)\|_{L^r_{\tau,\xi}}, \]
for \(1 < r < \infty\) are Banach spaces with \(\mathcal{S}\) as a dense subspace. The dual space is \(X^r_{s,-b,\pm}\), where \(\frac{1}{r} + \frac{1}{r'} = 1\). The complex interpolation space is given by
\[ (X^{r_0}_{s_0,b_0,\pm}, X^{r_1}_{s_1,b_1,\pm})_{\theta} = X^{r}_{s,b,\pm}, \]
where \(s = (1 - \theta)s_0 + \theta s_1\), \(\frac{1}{r} = \frac{\theta}{r_0} + \frac{1 - \theta}{r_1}\), \(b = (1 - \theta)b_0 + \theta b_1\). Similar properties has the space \(H^r_{s,b}\).

If \(u = u_+ + u_-\), where \(u_\pm \in X^{r}_{s,b,\pm}[0,T]\), then \(u \in C^0([0,T], \dot{H}^{s,r})\), if \(b > \frac{1}{r}\).

The "transfer principle" in the following proposition, which is well-known in the case \(r = 2\), also holds for general \(1 < r < \infty\) (cf. \((\mathcal{G})\), Prop. A.2 or \((\mathcal{G})\), Lemma 1). We denote \(\|u\|_{L^2_t(L^p_x)} := \|\hat{u}\|_{L^p_x(L^2_t)}\).

**Proposition 3.1.** Let \(1 \leq p, q \leq \infty\). Assume that \(T\) is a bilinear operator which fulfills
\[ \|T(e^{\pm i\xi D} f_1, e^{\pm i\xi D} f_2)\|_{L^p_t(L^q_x)} \lesssim \|f_1\|_{\dot{H}^{s_1,r}} \|f_2\|_{\dot{H}^{s_2,r}}, \]
for all combinations of signs \(\pm_1, \pm_2\), then for \(b > \frac{1}{r}\) the following estimate holds:
\[ \|T(u_1, u_2)\|_{L^p_t(L^q_x)} \lesssim \|u_1\|_{H^r_{s_1,b}} \|u_2\|_{H^r_{s_2,b}}. \]
The general local well-posedness theorem is the following (obvious generalization of) [G], Thm. 1.

**Theorem 3.1.** Let \( N_\pm(u,v) := N_\pm^r(u_+, u_-, v_+, v_-) \) and \( M_\pm(u,v) := M_\pm^r(u_+, u_-, v_+, v_-) \) be multilinear functions. Assume that for given \( s,l \in \mathbb{R}, 1 < r < \infty \) there exist \( b, a > \frac{1}{r} \) such that the estimates

\[
\|N_\pm(u,v)\|_{X^{r,a,b}_l} \leq \omega_1(\|u\|_{X^{r,a}_l}, \|v\|_{X^{r,a}_l})
\]

and

\[
\|M_\pm(u,v)\|_{X^{r,a,b}_l} \leq \omega_2(\|u\|_{X^{r,a}_l}, \|v\|_{X^{r,a}_l})
\]

are valid with nondecreasing functions \( \omega_1, \omega_2 \), where \( \|u\|_{X^{r,a}_l} := \|u_--\|_{X_{r,b,-}^l} + \|u_+-\|_{X_{r,b,+}^l} \). Then there exist \( T = T(\|u_0\|_{\tilde{H}^{r,a}}, \|v_0\|_{\tilde{H}^{r,b}}) > 0 \) and a unique solution \((u_+, u_-, v_+, v_-) \in X^{r,b,+}_{r,b,-}[0,T] \times X^{r,b,-}_{r,b,+}[0,T] \times X^{r,a,+}_{r,a,-}[0,T] \times X^{r,a,-}_{r,a,+}[0,T] \) of the Cauchy problem

\[
\partial_t u_\pm + iA u = N_\pm(u,v), \quad \partial_t v_\pm + iA v = M_\pm(u,v)
\]

\[
u_\pm(0) = u_{0,\pm} \in \tilde{H}^{r,a}, \quad v(0) = v_{0,\pm} \in \tilde{H}^{r,b}.
\]

This solution is persistent and the mapping data upon solution \((u_{0,+, u_{0,-}, v_{0,+}, v_{0,-}) \mapsto (u_+, u_-, v_+, v_-) \in \tilde{H}^{r,a,+} \times \tilde{H}^{r,b,+} \times \tilde{H}^{r,a,-} \times \tilde{H}^{r,b,-} \rightarrow [0,T_0] \times X^{r,a,+}_{r,a,-}[0,T_0] \times X^{r,a,-}_{r,a,+}[0,T_0] \times X^{r,b,+}_{r,b,-}[0,T_0] \times X^{r,b,-}_{r,b,+}[0,T_0] \) is locally Lipschitz continuous for any \( T_0 < T \).

4. Reformulation of the problem and null structure

The reformulation of the Yang-Mills equations and the reduction of our main theorem to nonlinear estimates is completely taken over from Tesfahun [T] (cf. also the fundamental paper by Selberg and Tesfahun [ST]).

The standard null forms are given by

\[
\begin{align*}
Q_u(u,v) &= \partial_\alpha u \partial^\alpha v = -\partial_\alpha u \partial_\beta v + \partial_\beta u \partial_\alpha v, \\
Q_{u\beta}(u,v) &= \partial_\alpha u \partial_\beta v - \partial_\beta u \partial_\alpha v.
\end{align*}
\]

For \( g \)-valued \( u, v \), define a commutator version of null forms by

\[
\begin{align*}
Q_u(u,v) &= [\partial_\alpha u, \partial^\alpha v] = Q_u(u,v) - Q_v(v,u), \\
Q_{u\beta}(u,v) &= [\partial_\alpha u, \partial_\beta v] - [\partial_\beta u, \partial_\alpha v] = Q_{u\beta}(u,v) + Q_{v\beta}(v,u).
\end{align*}
\]

Note the identity

\[
[\partial_\alpha u, \partial_\beta u] = \frac{1}{2} ([\partial_\alpha u, \partial_\beta v] - [\partial_\beta u, \partial_\alpha v]) = \frac{1}{2} Q_{u\beta}[u,v].
\]

Define

\[
Q[u,v] = Q_{12}[R_1u_2 - R_2u_1, \phi] - Q_{01}[R^1u_0, v],
\]

where \( R_\alpha = \Lambda^{-1} \partial_\alpha \) are the Riesz transforms.

We split the spatial part \( A = (A_1, A_2, A_3) \) of the potential into divergence-free and curl-free parts and a smoother part:

\[
A = A^{df} + A^{cf} + \Lambda^{-2}A,
\]

where

\[
A^{df} = (\partial_2(\partial_1 A_2 - \partial_2 A_1), -\partial_1(\partial_1 A_2 - \partial_2 A_1))
\]

\[
A^{cf} = -\Lambda^{-2} \partial_1(\partial_2 A - \partial_1 A_1).
\]
4.1. Terms of the form $[A^\alpha, \partial_\alpha \phi]$ and $[\partial_\alpha A^\alpha, \partial_\alpha \phi]$. In the Lorenz gauge, terms of the form $[A^\alpha, \partial_\alpha \phi]$, where $A_\alpha, \phi \in S$ with values in $\mathfrak{g}$, can be shown to be a sum of bilinear null forms and a smoother bilinear part whereas the term $[\partial_\alpha A^\alpha, \partial_\alpha \phi]$ is a null form.

Lemma 4.1. In the Lorenz gauge, we have the identities

$$[A^\alpha, \partial_\alpha \phi] = \Omega \{ [\Lambda^{-1} A, \phi] + [\Lambda^{-2} A^\alpha, \partial_\alpha \phi] \},$$

(16)

$$[\partial_\alpha A^\alpha, \partial_\alpha \phi] = Q_{0\alpha} [A^\alpha, \phi].$$

(17)

Proof. To show (16) we modify the proof in [ST], Lemma 1 whereas (17) is proved in the same paper (see identity (2.7) therein).

Using (15) we write

$$A^\alpha \partial_\alpha \phi = (-A_0 \partial_t \phi + A^{cf} \cdot \nabla \phi) + A^{df} \cdot \nabla \phi + \Lambda^{-2} A \cdot \nabla \phi.$$

Let us first consider the first term in the parentheses. We use the Lorenz gauge, $\partial_t A_0 = \nabla \cdot A$, to write

$$A^{cf} \cdot \nabla \phi = -\Lambda^{-2} \partial^i (\partial_i A_0) \partial_t \phi = -\partial_t (\Lambda^{-1} R^i A_0) \partial_t \phi.$$

We can also write

$$A_0 \partial_t \phi = -\Lambda^{-2} \partial_t \partial^\alpha A_0 \partial_t \phi + \Lambda^{-2} A_0 \partial_t \phi$$

$$= -\partial_t (\Lambda^{-1} R^i A_0) \partial_t \phi + \Lambda^{-2} A_0 \partial_t \phi.$$

Combining the above identities, we get

$$-A_0 \partial_t \phi + A^{cf} \cdot \nabla \phi = Q_{0\alpha} (\Lambda^{-1} R^i A_0, \phi) - \Lambda^{-2} A_0 \partial_t \phi.$$

Next, we consider the second term. We have

$$A^{df} \cdot \nabla \phi = \Lambda^{-2} (\partial_2 (\partial_1 A_2 - \partial_2 A_1) \partial_1 \phi - \partial_1 (\partial_1 A_2 - \partial_2 A_1) \partial_2 \phi)$$

$$= -\Lambda^{-2} Q_{12} (\partial_1 A_2 - \partial_2 A_1, \phi)$$

$$= -Q_{12} (\Lambda^{-1} (R^i A_2 - R_2 A_1), \phi).$$

Thus, we have shown

$$A^\alpha \partial_\alpha \phi = -Q_{12} (\Lambda^{-1} (R^i A_2 - R_2 A_1), \phi) + Q_{0\alpha} (\Lambda^{-1} R^i A_0, \phi) + \Lambda^{-2} A^\alpha \partial_\alpha \phi.$$

(18)

Similarly, modifying the above argument one can show

$$\partial_\alpha \phi A^\alpha = -Q_{12} (\phi, \Lambda^{-1} (R^i A_2 - R_2 A_1)) + Q_{0\alpha} (\phi, \Lambda^{-1} R^i A_0) + \partial_\alpha \phi \Lambda^{-2} A^\alpha.$$

(19)

Subtracting (18) and (19) yields (16). \qed

4.2. Terms of the form $[A^\alpha, \partial_\beta A_\alpha]$. In the Lorenz gauge, this term can be written as a sum of bilinear null form terms, bilinear terms which are smoother, a bilinear term which contains only $F$ and higher order terms in $(A, F)$.

Lemma 4.2. In the Lorenz gauge, we have the identity

$$[A^\alpha, \partial_\beta A_\alpha] = \sum_{i=1}^{4} \Gamma'_i (A, \partial A, F, \partial F),$$

where $\Gamma'_i$ are the symmetric tensors that are used in the proof of the lemma.
where
\[
\begin{align*}
\Gamma^{1}_{\beta}(A, \partial A, F, \partial F) &= -[A_0, \partial_\beta A_0] + [\Lambda^{-1} R_j (\partial_t A_0), \Lambda^{-1} R^j \partial_\beta A_0], \\
\Gamma^{2}_{\beta}(A, \partial A, F, \partial F) &= -\left\{Q_{12}[\Lambda^{-1} R^n A_n, \Lambda^{-1} (R^1 \partial_\beta A^2 - R_2 \partial_\beta A_1)]
\right. \\
&
\left. + Q_{12}[\Lambda^{-1} R^n \partial_\beta A_n, \Lambda^{-1} (R^1 A_2 - R_2 A_1)]\right\}, \\
\Gamma^{3}_{\beta}(A, \partial A, F, \partial F) &= [\Lambda^{-2} \partial_j F_{12}, \Lambda^{-2} \partial_\beta \partial^j F_{12}] \\
&- [\Lambda^{-2} \partial_j F_{12}, \Lambda^{-2} \partial_\beta \partial^j [A_1, A_2]] \\
&- [\Lambda^{-2} \partial_j [A_1, A_2], \Lambda^{-2} \partial_\beta \partial^j F_{12}] \\
&+ [\Lambda^{-2} \partial_j [A_1, A_2], \Lambda^{-2} \partial_\beta \partial^j [A_1, A_2]], \\
\Gamma^{4}_{\beta}(A, \partial A, F, \partial F) &= [A^{df} + A^{df}, \Lambda^{-2} \partial_\beta A] + [\Lambda^{-2} A, \partial_\beta A].
\end{align*}
\] (20)

Thus, \(\Gamma^{2}_{\beta}\) is a combination of the commutator version \(Q\)-type null forms. The term \(\Gamma^{3}_{\beta}\) is also a null form (of non-\(Q\)-type) as shown below.

4.3. The system \(6\)–\(9\) in terms of the null forms. In view of Lemma 4.1 the first, second and third bilinear terms in \(6\) are null forms up to some smoother bilinear terms. By the identity \(8\), the fourth and fifth terms are identical to \(2Q_0[A_{\beta}, A_\gamma]\) and \(Q_{12}[A^0, A_\alpha]\), respectively.

By Lemma 4.2 the first term in \(7\) is a null form up to some smoother bilinear terms. By Lemma 4.2 the second term in \(7\) is a sum of bilinear null form terms, bilinear terms which are smoother, a bilinear term which contains only \(F\) and higher order terms in \((A, F)\).

Thus the system \(6\), \(7\) in Lorenz gauge can be written in the following form
\[
\begin{align*}
\Box A_\beta &= M_\beta(A, \partial_t A, F, \partial_t F), \\
\Box F_{\beta\gamma} &= N_{\beta\gamma}(A, \partial_t A, F, \partial_t F),
\end{align*}
\] (21)

where
\[
M_\beta(A, \partial_t A, F, \partial_t F) = -2Q[A^{-1} A, A_\beta] + \sum_{i=1}^{4} \Gamma^{i}_{\beta}(A, \partial A, F, \partial F) - 2[\Lambda^{-2} A^0, \partial_\alpha A_\beta] \\
- [A^0, [A_\alpha, A_\beta]],
\]
\[
N_{12}(A, \partial_t A, F, \partial_t F) = -2Q[A^{-1} A, F_{12}] + 2Q[A^{-1} \partial_2 A, A_1] - 2Q[A^{-1} \partial_1 A, A_2] \\
+ 2Q_0[A_2, A_1] + Q_{12}[A^0, A_\alpha] - 2[\Lambda^{-2} A^0, \partial_\alpha F_{12}] \\
+ 2[\Lambda^{-2} \partial_2 A^0, \partial_\alpha A_1] - 2[\Lambda^{-2} \partial_1 A^0, \partial_\alpha A_2] \\
- [A^0, [A_\alpha, F_{12}]] + 2[F_{\alpha 1}, [A^0, A_2]] - 2[F_{\alpha 0}, [A^0, A_1]] \\
- 2[[A_\alpha, A_1], [A_\alpha, A_2]],
\]
\[
N_{00}(A, \partial_t A, F, \partial_t F) = -2Q[A^{-1} A, F_{00}] + 2Q[A^{-1} \partial_0 A, A_0] - 2Q_{0[j}[A^j, A_1] \\
+ 2Q_0[A_0, A_1] + Q_{00}[A^0, A_\alpha] - 2[\Lambda^{-2} A^0, \partial_\alpha F_{00}] \\
+ 2[\Lambda^{-2} \partial_0 A^0, \partial_\alpha A_0] - [A^0, [A_\alpha, F_{00}]] + 2[F_{00}, [A^0, A_1]] \\
- 2[F_{00}, [A^0, A_0]] - 2[[A_\alpha, A_0], [A_\alpha, A_1]],
\]

In a standard way we rewrite the system \(21\) as a first order (in \(t\)) system. Defining \(A_\pm = \frac{1}{2}(A \pm (iA)^{-1}\partial_t A)\), \(F_\pm = \frac{1}{2}(F \pm (iA)^{-1}\partial_t F)\), so that \(A = \)
$A_+ + A_-, \partial_+ A = i\Lambda(A_+ - A_-), F = F_+ + F_-, \partial_+ F = i\Lambda(F_+ - F_-)$ the system transforms to

\begin{align}
(i\partial_t + \Lambda)A_+^\beta &= -(2\Lambda)^{-1}M_\beta(A, \partial_+ A, F, \partial_+ F), \quad (22) \\
(i\partial_t + \Lambda)F_+^{\beta\gamma} &= -(2\Lambda)^{-1}N^{\beta\gamma}(A, \partial_+ A, F, \partial_+ F). \quad (23)
\end{align}

The initial data transform to

\[ A_{\pm}(0) = \frac{1}{2}(a \pm (i\Lambda)^{-1}\dot{a}) \in \hat{H}^{s,r}, \quad F_{\pm}(0) = \frac{1}{2}(f \pm (i\Lambda)^{-1}\dot{f}) \in \hat{H}^{s,r}. \]

Now, looking at the terms in $M_\beta$ and $N^{\beta\gamma}$ and noting the fact that the Riesz transforms $R_k$ are bounded in the spaces involved, the estimates in Theorem 3.4 reduce to proving:

1. the estimates for the null forms $Q_{12}, Q_0$ and $Q \in \{Q_{0l}, Q_{12}\}$:

\[
\left\|Q[A^{-1}A,A]\right\|_{H^s_{1+b-1+}} \lesssim \|A\|_{X^{s,b}_1} \|A\|_{X^{s,b}_2}, \quad (24)
\]

\[
\left\|Q_{12}[A^{-1}A,\Lambda^{-1}\partial A]\right\|_{H^s_{1+b-1+}} \lesssim \|A\|_{X^{s,b}_1} \|\Lambda^{-1}\partial A\|_{X^{s,b}_2}, \quad (25)
\]

\[
\left\|Q[A^{-1}A,F]\right\|_{H^s_{1+b-1+}} \lesssim \|A\|_{X^{s,b}_1} \|F\|_{X^{s,b}_2}, \quad (26)
\]

\[
\left\|Q[A,A]\right\|_{H^s_{1+b-1+}} \lesssim \|A\|_{X^{s,b}_1} \|A\|_{X^{s,b}_2}, \quad (27)
\]

\[
\left\|Q_0[A,A]\right\|_{H^s_{1+b-1+}} \lesssim \|A\|_{X^{s,b}_1} \|A\|_{X^{s,b}_2}, \quad (28)
\]

the following estimate for $\Gamma^1$ and other bilinear terms

\[
\left\|\Gamma^1(A, \partial A)\right\|_{H^s_{1+b-1+}} \lesssim \|A\|_{X^{s,b}_1} \|A\|_{X^{s,b}_2}, \quad (29)
\]

\[
\left\|\Pi(A, \Lambda^{-2}\partial A)\right\|_{H^s_{1+b-1+}} \lesssim \|A\|_{X^{s,b}_1} \|A\|_{X^{s,b}_2}, \quad (30)
\]

\[
\left\|\Pi(\Lambda^{-2}A, \partial A)\right\|_{H^s_{1+b-1+}} \lesssim \|A\|_{X^{s,b}_1} \|A\|_{X^{s,b}_2}, \quad (31)
\]

\[
\left\|\Pi(\Lambda^{-1}F, \Lambda^{-2}\partial F)\right\|_{H^s_{1+b-1+}} \lesssim \|F\|_{X^{s,b}_1} \|F\|_{X^{s,b}_2}, \quad (32)
\]

\[
\left\|\Pi(\Lambda^{-2}A, \partial F)\right\|_{H^s_{1+b-1+}} \lesssim \|A\|_{X^{s,b}_1} \|F\|_{X^{s,b}_2}, \quad (33)
\]

\[
\left\|\Pi(\Lambda^{-1}A, \partial A)\right\|_{H^s_{1+b-1+}} \lesssim \|A\|_{X^{s,b}_1} \|A\|_{X^{s,b}_2}, \quad (34)
\]

and

2. the following trilinear and quadrilinear estimates:

\[
\left\|\Pi(\Lambda^{-1}F, \Lambda^{-2}\partial(AA))\right\|_{H^s_{1+b-1+}} \lesssim \|F\|_{X^{s,b}_1} \|A\|_{X^{s,b}_2} \|A\|_{X^{s,b}_2}, \quad (35)
\]

\[
\left\|\Pi(\Lambda^{-1}\partial F, \Lambda^{-1}(AA))\right\|_{H^s_{1+b-1+}} \lesssim \|F\|_{X^{s,b}_1} \|A\|_{X^{s,b}_2} \|A\|_{X^{s,b}_2}, \quad (36)
\]

\[
\left\|\Pi(\Lambda^{-1}(AA), \Lambda^{-2}\partial(AA))\right\|_{H^s_{1+b-1+}} \lesssim \|A\|_{X^{s,b}_1} \|A\|_{X^{s,b}_2} \|A\|_{X^{s,b}_2} \|A\|_{X^{s,b}_2}, \quad (37)
\]

\[
\left\|\Pi(A, A, A)\right\|_{H^s_{1+b-1+}} \lesssim \|A\|_{X^{s,b}_1} \|A\|_{X^{s,b}_2} \|A\|_{X^{s,b}_2}, \quad (38)
\]

\[
\left\|\Pi(A, A, F)\right\|_{H^s_{1+b-1+}} \lesssim \|A\|_{X^{s,b}_1} \|A\|_{X^{s,b}_2} \|F\|_{X^{s,b}_2}, \quad (39)
\]

\[
\left\|\Pi(A, A, A)\right\|_{H^s_{1+b-1+}} \lesssim \|A\|_{X^{s,b}_1} \|A\|_{X^{s,b}_2} \|A\|_{X^{s,b}_2}, \quad (40)
\]

$\Pi(\cdot \cdot \cdot)$ denotes a multilinear operator in its arguments and $\|u\|_{X^{s,b}_{b+}} := \|u_+\|_{X^{s,b}_{b+}} + \|u_\pm\|_{X^{s,b}_{b+}}$.

The matrix commutator null forms are linear combinations of the ordinary ones, in view of (22). Since the matrix structure plays no role in the estimates under consideration, we reduce (24)–(28) to estimates of the ordinary null forms for $C$-valued functions $u$ and $v$ (as in (11)).
Next we consider the term $\Gamma^1_1$ and want to show that it is a null form. In fact the detection of this null structure was the main progress of his paper over Selberg-Tesfahun [ST].

We may ignore its matrix form and treat

$$\Gamma^1_k(A_0, \vartheta_k A_0) = -A_0(\partial_k A_0) + \Lambda^{-1} R_j(\partial_k A_0) \Lambda^{-1} R^j \partial_i(\partial_k A_0)$$

for $k = 1, 2, 3$ and

$$\Gamma^0(A_0, \partial^t A_1) = -A_0(\partial^t A_0) + \Lambda^{-1} R_j(\partial^t A_0) \Lambda^{-1} R^j \partial_i(\partial^t A_1),$$

where we used the Lorenz gauge $\partial_0 A_0 = \partial^t A_1$ in the last line in order to eliminate one time derivative. Thus we have to consider

$$\Gamma^1(u, v) = -uv + \Lambda^{-1} R_j(\partial_t u) \Lambda^{-1} R^j(\partial_t v),$$

where $u = A_0$ and $v = \partial^t A_1$ or $v = \partial_0 A_0$.

The proof of the following lemma was essentially given by Tesfahun [T].

**Lemma 4.3.** Let $q_{12}(u, v) := Q_{12}(D^{-1}u, D^{-1}v)$, $q_0(u, v) := Q_0(D^{-1}u, D^{-1}v)$. The following estimate holds:

$$\Gamma^1(u, v) q_{12}(u, v) + q_0(u, v) + (\Lambda^{-2} u) v + u(\Lambda^{-2} v). \tag{41}$$

Here $u \leq v$ means $|\hat{u}| \leq |\hat{v}|$.

**Proof.** $\Gamma^1(u, v)$ has the symbol

$$p(\xi, \tau, \eta, \lambda) = -1 + \frac{\langle \xi, \eta \rangle \tau \lambda}{\langle \xi \rangle^2 \langle \eta \rangle^2} = \left( -1 + \frac{\langle \xi, \eta \rangle_1}{\langle \xi \rangle^2 \langle \eta \rangle^2} \right) + \frac{(\tau \lambda - \langle \xi, \eta \rangle_1)(\xi, \eta)_1}{\langle \xi \rangle^2 \langle \eta \rangle^2} = I + II$$

Now we estimate

$$|I| = \left| \frac{\langle \xi \rangle^2 \langle \eta \rangle^2 \cos^2 \angle(\xi, \eta)}{\langle \xi \rangle^2 \langle \eta \rangle^2} - 1 \right| \leq \left| \frac{\langle \xi \rangle^2 \langle \eta \rangle^2 \cos^2 \angle(\xi, \eta)}{\langle \xi \rangle^2 \langle \eta \rangle^2} - 1 \right| + \left| \frac{\langle \xi \rangle^2 \langle \eta \rangle^2 - \langle \xi \rangle^2 \langle \eta \rangle^2}{\langle \xi \rangle^2 \langle \eta \rangle^2} \right|$$

$$= \sin^2 \angle(\xi, \eta) + \left| \frac{\langle \xi \rangle^2 \langle \eta \rangle^2 - \langle \xi \rangle^2 \langle \eta \rangle^2}{\langle \xi \rangle^2 \langle \eta \rangle^2} \right|,$$

where $\angle(\xi, \eta)$ denotes the angle between $\xi$ and $\eta$. We have

$$\left| \frac{\langle \xi \rangle^2 \langle \eta \rangle^2 - \langle \xi \rangle^2 \langle \eta \rangle^2}{\langle \xi \rangle^2 \langle \eta \rangle^2} \right| = \left| \frac{\langle \xi \rangle^2 + \langle \eta \rangle^2 + 1}{\langle \xi \rangle^2 \langle \eta \rangle^2} \right| \leq \frac{1}{\langle \xi \rangle^2} + \frac{1}{\langle \eta \rangle^2}$$

and

$$\sin \angle(\xi, \eta) = \frac{|\xi \eta_2 - \xi_2 \eta|}{|\xi| |\eta|}.$$

Thus the operator belonging to the symbol $I$ is controlled by $q_{12}(u, v) + (\Lambda^{-2} u) v + u(\Lambda^{-2} v)$. Moreover

$$|II| \leq \frac{|\tau \lambda - \langle \xi, \eta \rangle_1|}{\langle \xi \rangle \langle \eta \rangle} \leq |q_0(\xi, \eta)|.$$

Thus we obtain (41). \qed
5. Bilinear estimates

The proof of the following bilinear estimates relies on estimates given by Foschi and Klainerman [FK]. We first treat the case \( r > 1 \), but close to 1.

**Lemma 5.1.** Assume \( 0 \leq \alpha_1, \alpha_2, \alpha_1 + \alpha_2 \geq \frac{1}{r} \) and \( b > \frac{1}{r} \). The following estimate applies

\[
\|q_{12}(u, v)\|_{H^r_{t, \vec{\xi}}} \lesssim \|u\|_{X^{\alpha_1, b, \pm_1}_t} \|v\|_{X^{\alpha_2, b, \pm_2}_t}.
\]

**Proof.** Because we use inhomogeneous norms it is obviously possible to assume \( \alpha_1 + \alpha_2 = \frac{1}{r} \). Moreover, by interpolation we may reduce to the case \( \alpha_1 = \frac{1}{r} \), \( \alpha_2 = 0 \).

The left hand side of the claimed estimate equals

\[
\|\mathcal{F}(q_{12}(u, v))\|_{L^{r'}_{t, \vec{\xi}}} = \| \int q_{12}(\tau, \xi - \eta) \hat{u}(\lambda, \eta) \hat{v}(\tau - \lambda, \xi - \eta) d\lambda d\eta \|_{L^{r'}_{t, \vec{\xi}}}.
\]

Let now \( u(t, x) = e^{\pm i D_{u_0}^{\pm_1}(x)} \), \( v(t, x) = e^{\pm i D_{v_0}^{\pm_2}(x)} \), so that

\[
\hat{u}(\tau, \xi) = c\delta(\tau \mp_1 |\xi|) \hat{u}_0^{\pm_1}(\xi), \quad \hat{v}(\tau, \xi) = c\delta(\tau \mp_2 |\xi|) \hat{v}_0^{\pm_2}(\xi).
\]

This implies

\[
\|\mathcal{F}(q_{12}(u, v))\|_{L^{r'}_{t, \vec{\xi}}} = c^2 \| \int q_{12}(\eta, \xi - \eta) \hat{u}_0^{\pm_1}(\eta) \hat{v}_0^{\pm_2}(\xi - \eta) \delta(\lambda \mp_1 |\eta|) \delta(\tau - \lambda \mp_2 |\xi - \eta|) d\lambda d\eta \|_{L^{r'}_{t, \vec{\xi}}}
\]

\[
= c^2 \| \int q_{12}(\eta, \xi - \eta) \hat{u}_0^{\pm_1}(\eta) \hat{v}_0^{\pm_2}(\xi - \eta) \delta(\tau \mp_1 |\eta| \mp_2 |\xi - \eta|) d\eta \|_{L^{r'}_{t, \vec{\xi}}}
\]

By symmetry we only have to consider the elliptic case \( \pm_1 = \pm_2 = + \) and the hyperbolic case \( \pm_1 = +, \pm_2 = - \).

**Elliptic case.** We obtain by [FK], Lemma 13.2:

\[
|q_{12}(\eta, \xi - \eta)| \leq \frac{|\eta|}{|\xi - \eta|} \lesssim \frac{|\xi (\eta) + |\xi - \eta| - |\xi|}{|\eta|^\pm |\xi - \eta|^\pm}.
\]

By Hölder’s inequality we obtain

\[
\|\mathcal{F}(q_{12}(u, v))\|_{L^{r'}_{t, \vec{\xi}}} \lesssim \| \| \| \delta(\tau - |\eta| - |\xi - \eta|) \hat{u}_0^{\pm_1}(\eta) \hat{v}_0^{\pm_2}(\xi - \eta) d\eta \|_{L^{r'}_{t, \vec{\xi}}}
\]

\[
\lesssim \sup_{\tau, \xi} I \| D_{\tau}^{\pm_1} u_0^{\pm_1} \|_{L^{r'}_{t, \vec{\xi}}} \| v_0^{\pm_2} \|_{L^{r'}_{t, \vec{\xi}}},
\]

where

\[
I = |\xi|^{\frac{1}{r}} |\tau| - |\xi|^{\frac{1}{r}} \left( \int \delta(\tau - |\eta| - |\xi - \eta|) |\eta|^{-\frac{1}{r}} |\xi - \eta|^{-B} d\eta \right)^{\frac{1}{r}}.
\]

We want to prove \( \sup_{\tau, \xi} I \lesssim 1 \). By [FK], Lemma 4.3 we obtain

\[
\int \delta(\tau - |\eta| - |\xi - \eta|) |\eta|^{-\frac{1}{r}} |\xi - \eta|^{-\frac{1}{r}} d\eta \sim \tau^A |\tau| - |\xi|^B,
\]

where \( A = \max(1 + \frac{2}{r}, \frac{3}{2}) - 1 - r = -\frac{3}{r} \) and \( B = 1 - \max(1 + \frac{2}{r}, \frac{3}{2}) = -\frac{5}{r} \). Using \( |\xi| \leq |\tau| \) this implies

\[
I \lesssim |\xi|^{\frac{1}{r}} |\tau| - |\xi|^{\frac{1}{r}} \tau^A |\tau| - |\xi|^{-\frac{1}{r}} \lesssim 1.
\]
Hyperbolic case. We start with the following bound (cf. [FK], Lemma 13.2):

\[ |q_{12}(\eta, \xi - \eta)| \leq \frac{|\eta_1(\xi - \eta_2 - \eta_2(\xi - \eta)|}{|\eta_1| |\xi - \eta|} \lesssim \frac{1/2(|\xi| - |\eta| - |\xi - \eta|)\dot{\xi}}{|\eta_1|^{1/2}|\xi - \eta|^{1/2}}, \]

so that similarly as in the elliptic case we have to estimate

\[ I = |\xi|^{1/2}|\tau| - |\xi|^{1/2} \left( \int \delta(\tau - |\eta| + |\xi - \eta|) |\eta|^{-1 - \frac{5}{2}}|\xi - \eta|^{-\frac{3}{2}} \, d\eta \right)^{1/2}. \]

In the subcase \(|\eta| + |\xi - \eta| \leq 2|\xi|\) we apply [FK], Prop. 4.5 and obtain

\[ \int \delta(\tau - |\eta| + |\xi - \eta|) |\eta|^{-1 - \frac{5}{2}}|\xi - \eta|^{-\frac{3}{2}} \, d\eta \sim |\xi|^4 |\xi|^{-|\tau|}. \]

where in the subcase \(0 \leq \tau \leq |\xi|\) we obtain \(A = \max(1, 1, \frac{3}{2}) - 1 - r = \frac{1}{2} - r\) and \(B = 1 - \max(1, 1, \frac{3}{2}) = -\frac{1}{2},\) which implies

\[ I \lesssim |\xi|^{1/2}|\tau| - |\xi|^{1/2}|\xi|^{-1} \lesssim |\xi|^{-\frac{1}{2}} \lesssim 1. \]

Similarly in the subcase \(-|\xi| \leq \tau \leq 0\) we obtain \(A = \max(1 + \frac{3}{2}, \frac{3}{2}) - 1 - r = -\frac{3}{2}\), \(B = 1 - \max(1 + \frac{3}{2}, \frac{3}{2}) = -\frac{1}{2},\) which implies

\[ I \sim |\xi|^{1/2}|\tau| - |\xi|^{1/2}|\xi|^{-\frac{5}{2}}|\tau| - |\xi|^{-\frac{1}{2}} = 1. \]

In the subcase \(|\eta| + |\xi - \eta| \geq 2|\xi|\) we obtain by [FK], Lemma 4.4:

\[ \int \delta(\tau - |\eta| + |\xi - \eta|) |\eta|^{-1 - \frac{5}{2}}|\xi - \eta|^{-\frac{3}{2}} \, d\eta \sim |\xi|^4 |\xi|^{-|\tau|} \lesssim |\xi|^4 |\xi|^{-|\tau|} \sim |\xi|^4 |\xi|^{-|\tau|}. \]

We remark that in fact the lower limit of the integral can be chosen as 2 by inspection of the proof in [FK]. The integral converges, because \(|\tau| \leq |\xi|\) and \(r > 1.\) This implies the bound

\[ I \lesssim |\xi|^{1/2}|\tau| - |\xi|^{1/2}|\xi|^{-1} \lesssim 1. \]

Summarizing we obtain

\[ \|q_{12}(u, v)\|_{H_{6, 0}^0} \lesssim \|D^{1/2}u_0^{1,1} \|_{L^{r'}} \|v_0^{1/2} \|_{L^{r'}}. \]

By the transfer principle Prop. 5.1 we obtain the claimed result. \(\square\)

In a similar manner we can also estimate the nullform \(q_{02}(u, v).\)

**Lemma 5.2.** Assume \(0 \leq \alpha_1, \alpha_2, \alpha_1 + \alpha_2 \geq \frac{1}{2}\) and \(b > \frac{1}{2}.\) The following estimate applies

\[ \|q_{02}(u, v)\|_{H_{6, 0}^0} \lesssim \|u\|_{X_{\alpha_1, b, \pm 1}^{1,1}} \|v\|_{X_{\alpha_2, b, \pm 2}^{1,1}}. \]

**Proof.** Again we may reduce to the case \(\alpha_1 = \frac{1}{2}\) and \(\alpha_2 = 0.\) Arguing as in the proof of Lemma 5.1 we use in the elliptic case the estimate (cf. [FK], Lemma 13.2):

\[ |q_{02}(\eta, \xi - \eta)| \lesssim \frac{(|\eta| + |\xi - \eta| - |\xi|)\dot{\xi}}{\min(|\eta|, |\xi - \eta|, |\xi - \eta|^{1/2})}. \]

In the case \(|\eta| \leq |\xi - \eta|\) we obtain

\[ I = |\tau| - |\xi|^{1/2} \left( \int \delta(\tau - |\eta| + |\xi - \eta|) |\eta|^{-1 - \frac{5}{2}} \, d\eta \right)^{1/2} \sim |\tau| - |\xi|^{1/2}|\tau|^{1/2}|\tau| - |\xi|^{1/2} = 1, \]

where in the subcase \(|\eta| + |\xi - \eta| \leq 2|\xi|\) we apply [FK], Prop. 4.5 and obtain

\[ \int \delta(\tau - |\eta| + |\xi - \eta|) |\eta|^{-1 - \frac{5}{2}}|\xi - \eta|^{-\frac{3}{2}} \, d\eta \sim |\xi|^4 |\xi|^{-|\tau|}. \]
Lemma 5.3. Assume $|\eta| \geq |\xi - \eta|$ we obtain

$$I = \|\tau - |\xi|| \left( \int \delta(\tau - |\eta| - |\xi - \eta|) |\eta|^{-1} |\xi - \eta|^{-r} d\eta \right)^{\frac{1}{r}}$$

$$\sim \|\tau - |\xi|| |\tau - |\xi||^{\frac{1}{r}} |\tau - |\xi||^{\min(1, -\frac{1}{r})},$$

where $A = \max(1, \frac{3}{2}, \frac{3}{2}) - 1 - \frac{r}{2} = 0$ and $B = 1 - \max(1, \frac{3}{2}, \frac{3}{2}) - \frac{r}{2} = -\frac{r}{2}$.

In the case $|\eta| \geq |\xi - \eta|$ we obtain

$$I \lesssim \|\tau - |\xi|| \frac{1}{r} |\tau - |\xi||^{\frac{1}{r}} |\tau - |\xi||^{\frac{1}{r}} \lesssim 1.$$ 

In the hyperbolic case we obtain by [FK], Lemma 13.2:

$$|q_0(\eta, \xi - \eta)| \lesssim |\eta| \left( |\xi| - |\eta| - |\eta - \xi| \right)^{\frac{1}{r}}$$

and argue exactly as in the proof of Lemma 7. The proof is completed as before.

We also need the same result for $q_0(u, v)$.

**Lemma 5.3.** Assume $0 \leq \alpha_1, \alpha_2$ , $\alpha_1 + \alpha_2 \geq \frac{1}{r}$ and $b > \frac{1}{r}$. The following estimate applies

$$\|q_0(u, v)\|_{H^b} \lesssim \|u\|_{X^\alpha_{1+b, 1+b}} \|v\|_{X^\alpha_{2+b, 1+b}}.$$ 

**Proof.** As before we reduce to the case $\alpha_1 = \frac{1}{r}$ and $\alpha_2 = 0$. We use in the elliptic case the estimate (cf. [FK], Lemma 13.2):

$$|q_0(\eta, \xi - \eta)| \lesssim \frac{|\eta| + |\xi - \eta| - |\xi - \eta|}{\min(|\eta|, |\xi - \eta|)}.$$ 

In the case $|\eta| \leq |\xi - \eta|$ to have to estimate

$$I = \|\tau - |\xi|| \left( \int \delta(\tau - |\eta| - |\xi - \eta|) |\eta|^{-1-r} d\eta \right)^{\frac{1}{r}}$$

$$\sim \|\tau - |\xi|| |\tau - |\xi||^{\frac{1}{r}} |\tau - |\xi||^{\min(1, -\frac{1}{r})},$$

because $A = \max(1 + r, \frac{3}{2}) - 1 - r = 0$ and $B = 1 - \max(1 + r, \frac{3}{2}) = -r$.

In the case $|\eta| \geq |\xi - \eta|$ we obtain

$$I \lesssim \|\tau - |\xi|| \frac{1}{r} |\tau - |\xi||^{\frac{1}{r}} |\tau - |\xi||^{\frac{1}{r}} \lesssim 1,$$

because $A = \max(1, r, \frac{3}{2}) - 1 - r = \frac{1}{2} - r$, $B = 1 - \frac{3}{2} = -\frac{1}{2}$ and $|\xi| \leq |\tau|$.

In the hyperbolic case we obtain by [FK], Lemma 13.2:

$$|q_0(\eta, \xi - \eta)| \lesssim \frac{|\xi| - |\eta| - |\eta - \xi||}{|\eta| |\xi - \eta|}.$$ 

In the subcase $|\eta| + |\xi - \eta| \leq 2|\xi|$ we apply [FK], Prop. 4.5 and obtain

$$\int \delta(\tau - |\eta| + |\xi - \eta|) |\eta|^{-1-r} |\xi - \eta|^{-r} d\eta \sim |\xi|^A |\tau - |\xi||^{B}$$

where in the subcase $0 \leq r \leq |\xi| : A = \max(r, \frac{3}{2}) - 1 - 2r = \frac{1}{2} - 2r$, $B = 1 - \max(r, \frac{3}{2}) = -\frac{1}{2}$, so that

$$I \lesssim |\xi| |\tau - |\xi|| |\xi|^{r-2} |\tau - |\xi||^{\frac{1}{r}} \lesssim 1,$$
whereas in the subcase \(-|\xi| \leq \tau \leq 0\) we obtain \(A = \max(r, \frac{2}{3}) - 1 - 2r = -r\), 
\(B = 1 - \max(1 + r, \frac{4}{3}) = -r\), so that \(I \sim 1\).

In the subcase \(|\eta| + |\xi - \eta| \geq 2|\xi|\) we obtain by [FK], Lemma 4.4:

\[
\int \delta(\tau - |\eta| + |\xi - \eta|)|\eta|^{-1-r}|\xi - \eta|^{-r} d\eta
\]

\[
\sim ||\tau| - |\xi|| - |\xi||^{-\frac{2}{3}}||\tau| + |\xi||^{-\frac{2}{3}}|r|^{-\frac{2}{3}}|\xi|^{-\frac{1}{2}} \lesssim |\xi|^{-\frac{1}{2}}|r|^{-\frac{1}{2}} \lesssim 1.
\]

The integral converges, because \(|\tau| \leq |\xi|\). This implies the bound

\[I \lesssim |\xi|||\tau| - |\xi|| + |\xi|^{-\frac{1}{2}}|r| + |\xi|^{-\frac{1}{2}}|\xi|^{-\frac{1}{2}} \lesssim |\xi|^{-\frac{1}{2}}|r|^{-\frac{1}{2}} \lesssim 1.\]

The proof is completed as the proof of Lemma 7.1.

\[\square\]

**Lemma 5.4.** Let \(1 < r \leq 2\). Assume \(\alpha_1, \alpha_2 \geq 0\), \(\alpha_1 + \alpha_2 > \frac{3}{2} r\), \(b_1, b_2 > \frac{1}{2} r\), \(b_1 + b_2 > \frac{3}{2} r\). Then the following estimate applies:

\[\|uv\|_{H^\alpha_{b_0}} \lesssim \|u\|_{X_{\alpha_1, b_1, z_1}} \|v\|_{X_{\alpha_2, b_2, z_2}}.\]

**Proof.** This follows from [GT], Prop. 3.1 by summation over the dyadic parts.

\[\square\]

**Lemma 5.5.** If \(\alpha_1, \alpha_2, b_1, b_2 \geq 0\), \(\alpha_1 + \alpha_2 > \frac{2}{3} r\) and \(b_1 + b_2 > \frac{1}{r}\) the following estimate applies:

\[\|uv\|_{H^\alpha_{b_0}} \lesssim \|u\|_{H^\alpha_{\alpha_1, b_1}} \|v\|_{H^\alpha_{\alpha_2, b_2}}.\]

**Proof.** We may assume \(\alpha_1 = \frac{1}{3} + \frac{1}{2} + , \alpha_2 = 0\), \(b_1 = \frac{1}{3} + \), \(b_2 = 0\) (or similarly \(b_1 = 0\), \(b_2 = \frac{1}{3} + \)). By Young’s and Hölder’s inequalities we obtain

\[
\|uv\|_{H^\alpha_{b_0}} = \|\tilde{u}\|_{L^\infty_{\xi}} \|\tilde{v}\|_{L^\infty_{\xi}} \lesssim \|\tilde{u}\|_{L^\infty_{\xi}} \|\tilde{v}\|_{L^\infty_{\xi}}
\]

\[
\lesssim \|\xi\|^{-\frac{1}{2}}\|\tau - |\xi||^{-\frac{1}{2}}\|\xi\|^{\frac{1}{2}} \|\xi\|^{-\frac{1}{2}} \|\xi\|^{\frac{1}{2}} \|\tilde{u}\|_{L^\infty_{\xi}} \|\tilde{v}\|_{L^\infty_{\xi}}
\]

\[
\lesssim \|u\|_{H^\alpha_{\frac{1}{3} + \frac{1}{2} +}} \|v\|_{H^\alpha_{b_0}}.
\]

\[\square\]

**Lemma 5.6.** Let \(1 < r \leq 2\), \(0 \leq \alpha_1, \alpha_2\) and \(\alpha_1 + \alpha_2 \geq \frac{1}{r} + b\), \(b > \frac{1}{r}\). Then the following estimate applies:

\[\|uv\|_{H^\alpha_{b_0}} \lesssim \|u\|_{X_{\alpha_1, b_1, z_1}} \|v\|_{X_{\alpha_2, b_2, z_2}}.\]

**Proof.** We may assume \(\alpha_1 = \frac{1}{3} + b\), \(\alpha_2 = 0\). We apply the “hyperbolic Leibniz rule” (cf. [AFS], p. 128):

\[||\tau| - |\xi|| \lesssim ||\rho| - |\eta|| + ||\tau - \rho|| - |\xi - \eta|| + b_+ (\xi, \eta),\]

where

\[b_+ (\xi, \eta) = |\eta| + |\xi - \eta| - |\xi| , \quad b_- (\xi, \eta) = |\xi| - |\eta|| - |\xi - \eta||.\]

Let us first consider the term \(b_\pm (\xi, \eta)\) in (42). Decomposing as before \(uv = u_+ v_+ + u_- v_+ + u_- v_- + u_- v_-\), where \(u_\pm (t) = e^{\pm itD} f, v_\pm (t) = e^{\pm itD} g\), we use

\[
\tilde{u}_\pm (\tau, \xi) = c\delta (\tau = |\xi|) \hat{f} (\xi) , \quad \tilde{v}_\pm (\tau, \xi) = c\delta (\tau = |\xi|) \hat{g} (\xi)
\]

and have to estimate

\[
\| \int b_\pm (\xi, \eta) \delta (\tau - |\eta| \mp |\xi - \eta|) \hat{f} (\xi) \hat{g} (\xi - \eta) d\eta \|_{L^\infty_{\xi}}
\]

\[
= \| \int |\tau| - |\xi| ||b_\mp (\tau - |\eta| \mp |\xi - \eta|) \hat{f} (\xi) \hat{g} (\xi - \eta) d\eta \|_{L^\infty_{\xi}}
\]

\[
\lesssim \sup_{\tau, \xi} I \| \tilde{f} (\tau, \xi) \hat{g} (\xi - \eta) \|_{L^\infty_{\xi}}.
\]
Here we used Hölder’s inequality, where

\[ I = ||r|| - |\xi||^b \left( \int \delta(\tau - |\eta|) ||r - |\xi||^{-1} d\eta \right)^\frac{1}{b}. \]

In order to obtain \( I \lesssim 1 \) we first consider the elliptic case \( \pm_1 = \pm_2 = \pm \) and use [FK], Prop. 4.3. Thus

\[ I \sim ||r|| - |\xi||^b \left( ||r - |\xi||^b \right)^\frac{1}{b} = ||r - |\xi||^b ||r - |\xi||^{-b} = 1 \]

with \( A = \max(1 + br, \frac{3}{2}) - (1 + br) = 0 \) and \( B = 1 - \max(1 + br, \frac{3}{2}) = -br \).

Next we consider the hyperbolic case \( \pm_1 = +, \pm_2 = - \).

First we assume \(|\eta| + |\xi - \eta| \leq 2|\xi|\) and use [FK], Prop. 4.5 which gives

\[ \int \delta(\tau - |\eta|) ||r - |\xi||^{-1} d\eta \sim |||4||_4 ||r - |\xi||^{-b} \]

where \( A = \frac{3}{2} - (1 + br) = \frac{1}{2} - br \), \( B = 1 - \frac{3}{2} = -\frac{1}{2} \), if \( 0 \leq \tau \leq |\xi| \), so that

\[ I \sim ||r|| - |\xi||^b \left( ||r - |\xi||^b \right)^\frac{1}{b} ||r - |\xi||^{-b} \lesssim 1. \]

If \(|\xi| \leq \tau \leq 0\) we obtain \( A = \max(1 + br, \frac{3}{2}) - (1 + br) = 0 \), \( B = 1 - \max(1 + br, 2) = -br \), which implies \( I \lesssim 1 \).

Next we assume \(|\eta| + |\xi - \eta| \geq 2|\xi|\) , use [FK], Lemma 4.4 and obtain

\[ I \sim ||r|| - |\xi||^b \left( \int \delta(\tau - |\eta|) ||r - |\xi||^{-1} d\eta \right)^\frac{1}{b} \]

\[ \sim ||r|| - |\xi||^b \left( ||r - |\xi|| + |\xi||^{-b} \right)^\frac{1}{b} = \int_2^\infty \left( |r - |\xi|| + |\xi||^{-b} \right)^\frac{1}{b} \]

This integral converges, because \( \tau \leq |\xi| \) and \( b > \frac{1}{2} \). This implies

\[ I \lesssim ||r|| - |\xi||^b \left( ||r|| + |\xi|^{-b} \right)^\frac{1}{b} \lesssim 1. \]

using \( |r| \leq |\xi| \).

By the transfer principle we obtain

\[ ||B^b_{\pm}(u, v)||_{X^0_b} \lesssim ||u||_{X^{2, b, \pm_1}_{\pm_0}} ||v||_{X^{2, b, \pm_2}_{\pm_0}}. \]

Here \( B^b_{\pm} \) denotes the operator with Fourier symbol \( b_{\pm} \).

Consider now the term \( ||r - |\xi|| \) (or similarly \( ||r - |\xi|| \) in (42)). We have to prove

\[ ||UD^b_v||_{H^0_b} \lesssim ||u||_{X^{2, b, \pm_1}_0} ||v||_{X^{2, b, \pm_2}_0}, \]

which is implied by

\[ ||uv||_{H^0_b} \lesssim ||u||_{X^{2, b, \pm_1}_0} ||v||_{X^{2, b, \pm_2}_0}. \]

This results from Lemma 5.5 because \( \alpha_1 + \alpha_2 \geq \frac{1}{r} + b > \frac{2}{r} \), which completes the proof.

6. Proof of (24) - (40) in the case \( r = 1+ \):

Proof. The estimates are proven by the results of chapter 5.

Assumption: \( s > 1 + \frac{1}{b} \), \( l \geq \frac{1}{b} \), \( s - 1 \leq l \leq s \), \( 2s - l > 1 + \frac{1}{b} \), \( 2l - s + 2 > \frac{3}{b} \) and \( b = \frac{1}{b} + \).

Proof of (24) and (25): This reduces to

\[ ||q(u, v)||_{H^{s-1, 0}} \lesssim ||u||_{X^{s, b}_{-1, 0}} ||v||_{X^{s, b}_{-1, 0}}. \]
By the fractional Leibniz rule this results from Lemma 5.1 or Lemma 5.2 for $s > 1/r$.

**Proof of (26):** This reduces to
\[ \left\| q(u, v) \right\|_{H^s_{-1, 0}} \lesssim \left\| u \right\|_{X^{s, b}_{r, a}} \left\| v \right\|_{X^{s, b}_{r, a}}. \]

Let us first consider the case $|\xi - \eta| \leq 1$. It suffices to show
\[ \left\| Q(u, v) \right\|_{H^s_{-1, 0}} \lesssim \left\| u \right\|_{X^{s, b}_{r, a}} \left\| v \right\|_{X^{s, b}_{r, a}} \]
for any $N \in \mathbb{N}$. By Lemma 5.3 we obtain easily:
\[ \left\| Q(u, v) \right\|_{H^s_{-1, 0}} \lesssim \left\| \Lambda u \Lambda v \right\|_{H^s_{-1, 0}} \lesssim \left\| \Lambda u \right\|_{X^{s, b}_{r, a}} \left\| \Lambda v \right\|_{X^{s, b}_{r, a}} \lesssim \left\| u \right\|_{X^{s, b}_{r, a}} \left\| v \right\|_{X^{s, b}_{r, a}} \]
for sufficiently large $N$.

From now on we assume $|\xi - \eta| \geq 1$.

In the elliptic case we use the estimate (cf. [FK], Lemma 13.2):
\[ |q(\eta, \xi - \eta)| \lesssim \left( \frac{1}{|\eta|} + \frac{|\xi - \eta| - |\xi|}{\min(|\eta|, |\xi - \eta|)} \right)^{s/2}. \]

We argue as in Lemma 5.1 and Lemma 5.2. In the subcase $|\xi - \eta| \lesssim |\eta|$ we estimate for $l \geq 1 \geq b$ and $s = \frac{1}{2} + \frac{r}{2}$:
\[ I = |\tau| - |\xi|^{\frac{1}{2}} \left( \int |\xi - \eta|^{-s} d|\eta| \right)^{\frac{1}{2}} \sim |\tau| - |\xi|^{\frac{1}{2}} - |\xi - \eta|^{\frac{s}{2}} - 1, \]
where by [FK], Prop. 4.3 we obtain $A = \max(s, 1/r) - sr = 0$, $B = 1 - \max(s, 1/r) = -1/2$, which implies
\[ \left\| q(u, v) \right\|_{H^s_{-1, 0}} \leq \left\| q(u, v) \right\|_{H^s_{-1, 0}} \leq \left\| u \right\|_{X^{s, b}_{r, a}} \left\| v \right\|_{X^{s, b}_{r, a}} \leq \left\| u \right\|_{X^{s, b}_{r, a}} \left\| v \right\|_{X^{s, b}_{r, a}}. \]

If $l \leq 1$ we apply the fractional Leibniz and reduce to Lemma 5.1 or Lemma 5.2 using $s > 1/r$.

In the subcase $|\eta| \ll |\xi - \eta| \sim |\xi|$ we assume $s = \frac{1}{2}$ and obtain similarly
\[ I = |\xi|^{l-1} |\tau| - |\xi|^{\frac{1}{2}} \left( \int |\xi - \eta|^{-s} d|\eta| \right)^{\frac{1}{2}} \sim |\tau| - |\xi|^{\frac{1}{2}} - |\xi - \eta|^{(1-l)r} - 1, \]
where $A = \max((s + \frac{1}{2})r, 1/r) - (s + \frac{1}{2})r = 0$, $B = 1 - \max((s + \frac{1}{2})r, 1/r) = -1/2$. This implies the claimed estimate.

In the hyperbolic case we use (cf. [FK], Lemma 13.2):
\[ |q(\eta, \xi - \eta)| \lesssim \frac{|\xi|^{\frac{s}{2}} (|\xi| - |\eta| - |\xi - \eta|)^{\frac{s}{2}}}{(|\eta| - |\xi - \eta|)^{\frac{s}{2}}}. \]

Now by an elementary calculation (cf. [AFS1]) we obtain
\[ (|\xi| - |\eta| - |\xi - \eta|)^{\frac{s}{2}} \lesssim |\tau| - |\xi|^{\frac{s}{2}} + |\lambda + |\eta||^{\frac{s}{2}} + |\lambda - |\tau||^{\frac{s}{2}}. \]

Thus we reduce to the estimates
\[ \left\| uv \right\|_{H^{s}_{-1, 0}} \lesssim \left\| u \right\|_{X^{s, b}_{r, a}} \left\| v \right\|_{X^{s, b}_{r, a}}, \]
\[ \left\| uv \right\|_{H^{s}_{-1, 0}} \lesssim \left\| u \right\|_{X^{s, b}_{r, a}} \left\| v \right\|_{X^{s, b}_{r, a}}, \]
\[ \left\| uv \right\|_{H^{s}_{-1, 0}} \lesssim \left\| u \right\|_{X^{s, b}_{r, a}} \left\| v \right\|_{X^{s, b}_{r, a}}. \]

For $l \geq 1/2$, $s \geq 1 + 1/r$, the first estimate follows from Lemma 5.6 by the fractional Leibniz rule, because $s + \frac{1}{2} \geq 1 + \frac{1}{2r} > \frac{1}{r} + b$ for $b = \frac{1}{2} + r$. The other two estimates
follow from Lemma 5.5. The proof of (20) is complete.

**Proof of (27) and (28):** Concerning (27) we have to show
\[ \|q(u, v)\|_{H_{-1,0}^s} \lesssim \|u\|_{X_{-1,0}^s} \|v\|_{X_{-1,0}^s}. \]
This is implied by Lemma 5.1 or Lemma 5.2 if \( l \leq 1 \) and \( s > 1 + \frac{1}{2l} \) or more generally \( l \leq s \) and \( 2s - 2 - (l - 1) > \frac{1}{2} \Leftrightarrow 2s - l > 1 + \frac{1}{2} \). Similarly (28) follows from Lemma 5.3.

**Proof of (29):** By Lemma 4.3 we have to prove
\[ \|q_{12}(u, Av)\|_{X_{-1,0}^s} \lesssim \|u\|_{X_{-1,0}^s} \|v\|_{X_{-1,0}^s}, \]
which by the fractional Leibniz rule results from Lemma 5.4. Moreover we need
\[ \|g_0(u, Av)\|_{X_{-1,0}^s} \lesssim \|u\|_{X_{-1,0}^s} \|v\|_{X_{-1,0}^s}, \]
which is given by Lemma 5.3. Finally
\[ \|\Lambda^{-2} u\|_{X_{-1,0}^s} + \|a(\Lambda^{-2} v)\|_{X_{-1,0}^s} \lesssim \|u\|_{X_{-1,0}^s} \|v\|_{X_{-1,0}^s} \]
by Lemma 5.5 for \( s > \frac{1}{2} \), which is fulfilled.

**Proof of (30) and (31):** The estimates result from Lemma 5.5 if \( s \geq 1 \).

**Proof of (32):** We reduce to
\[ \|u v\|_{H_{-1,0}^s} \lesssim \|u\|_{X_{-1}^{s+1}} \|v\|_{X_{-1,0}^s}. \]
By the fractional Leibniz rule this is implied by Lemma 5.4 if \( l \geq s - 1 \) and \( 2l + 1 - (s - 1) = 2l - s + 2 > \frac{3}{2} \). This is one of our assumptions. It is fulfilled for \( l = \frac{1}{2} \) and \( s = 1 + \frac{3}{2} \).

**Proof of (33):** The estimate reduces to
\[ \|u v\|_{H_{-1,0}^s} \lesssim \|u\|_{X_{-1,0}^s} \|v\|_{X_{-1,0}^s}. \]
If \( l \geq 1 \) this easily follows from Lemma 5.5. Assume from now on \( 0 \leq l < 1 \). The result is by duality equivalent to
\[ \|u w\|_{H_{l,0}^{s+1}} \lesssim \|u\|_{H_{l+1,0}^{s+1}} \|w\|_{H_{l,0}^{s+1}}, \]
which by the fractional Leibniz rule reduces to the estimates
\[ \|u w\|_{H_{l,0}^{s+1}} \lesssim \|u\|_{H_{l+1,0}^{s+1}} \|w\|_{H_{l,0}^{s+1}} \]
and
\[ \|u w\|_{H_{l,0}^{s+1}} \lesssim \|u\|_{H_{l+1,0}^{s+1}} \|w\|_{H_{l,0}^{s+1}}. \]
Now we obtain
\[ \|u w\|_{H_{l,0}^{s+1}} = \|\hat{u} \hat{w}\|_{L_{x,t}^{s+1}} \lesssim \|\hat{u}\|_{L_{x,t}^{s+1}} \|\hat{w}\|_{L_{x,t}^{s+1}} \]
\[ \lesssim \|\xi\|^{s+1} \langle |\tau| - |\xi| \rangle^{-1} \|L_{\xi:t}^{s+1}\| \langle |\tau| - |\xi| \rangle^{s+1} \|\hat{u}\|_{L_{\xi:t}^{s+1}} \|\hat{w}\|_{L_{\xi:t}^{s+1}}, \]
if \( s > \frac{3}{2} - 1 \), which is fulfilled.

**Proof of (34):** We reduce to
\[ \|u v\|_{H_{-l,0}^s} \lesssim \|u\|_{X_{-l+1,0}^s} \|v\|_{X_{-l,0}^s}, \]
which results from Lemma 5.5 if \( l \leq 1 \) and \( s \geq 1 \) or \( l \geq 2 \) and \( 2s - l > \frac{3}{2} - 1 \) and moreover \( s \geq l \).

**Proof of (35):** The estimate
\[ \|u w\|_{H_{-l,0}^s} \lesssim \|u\|_{X_{-l+1,0}^s} \|v\|_{X_{-l,0}^s} \|w\|_{X_{-l,0}^s} \]
reduces by the fractional Leibniz rule to the estimates
\[ \|u w\|_{H_{-l,0}^s} \lesssim \|u\|_{X_{-l+2,0}^s} \|v\|_{X_{-l,0}^s} \|w\|_{X_{-l,0}^s} \]
and
\[ \|uvw\|_{X_{s,0}^{2}} \lesssim \|u\|_{X_{l+1,b}^{s}} \|v\|_{X_{l+2,b}^{s}} \|w\|_{X_{l,b}^{s}}. \]

Now we obtain by Lemma 5.5:
\[ \|uvw\|_{X_{s,0}^{5}} \lesssim \|u\|_{X_{s+2,b}^{l}} \|vw\|_{X_{s-1,b}^{2}}. \]
\[ \lesssim \|u\|_{X_{s+2,b}^{l}} \|v\|_{X_{s-1,b}^{2}} \|w\|_{X_{s,b}^{l}}. \]

The last estimate results from Lemma 5.5 because \(2s-(s-2+2) = s+l-\frac{2}{p}+2 \geq 1 + \frac{2}{p} + \frac{2}{p} - \frac{2}{p} + 2 = 3 - \frac{1}{p} > \frac{2}{p} \). Moreover in exactly the same way we obtain
\[ \|uvw\|_{X_{s,0}^{5}} \lesssim \|u\|_{X_{l+1,b}^{s}} \|vw\|_{X_{l-1,b}^{5}}. \]
\[ \lesssim \|u\|_{X_{l+1,b}^{s}} \|v\|_{X_{l,b}^{s}} \|w\|_{X_{l,b}^{s}}. \]

**Proof of (36):** We apply Lemma 5.5 which implies
\[ \|u\Lambda^{-1}(vw)\|_{X_{l+1,b}^{5}} \lesssim \|u\|_{X_{l,b}^{5}} \|\Lambda^{-1}(vw)\|_{X_{l-1,b}^{5}} \lesssim \|u\|_{X_{l,b}^{5}} \|v\|_{X_{l,b}^{5}} \|w\|_{X_{l,b}^{5}}, \]
because \(2s-(\frac{2}{p} - l + s - 2) > 1 + \frac{2}{p} - \frac{2}{p} - \frac{2}{p} + 2 = 3 - \frac{1}{p} > \frac{2}{p} \) and \(l \geq s - 1 \).

**Proof of (38):** By Lemma 5.3, we obtain
\[ \|uvw\|_{H_{s+1,b}^{l}} \lesssim \|u\|_{X_{l,b}^{s}} \|vw\|_{X_{l-1,b}^{5}} \lesssim \|u\|_{X_{l,b}^{s}} \|v\|_{X_{l,b}^{s}} \|w\|_{X_{l,b}^{s}}, \]
because \(2s - \frac{2}{p} + 1 > 2 + \frac{1}{p} - \frac{2}{p} + \frac{1}{p} > \frac{2}{p} \).

**Proof of (39):** For \(l \leq 1\) we obtain by Lemma 5.5
\[ \|uvw\|_{H_{s}^{l}} \lesssim \|uvw\|_{H_{s}^{l}} \lesssim \|u\|_{X_{l,b}^{s}} \|v\|_{X_{l,b}^{s}} \|w\|_{X_{l,b}^{s}} \]

For the last estimate we applied Lemma 5.4, where we used \(2s - (\frac{2}{p} - l) > 2 + \frac{1}{p} - \frac{2}{p} = 2 - \frac{2}{p} > \frac{2}{p} \).

If \(l \geq 1\) we use the fractional Leibniz rule which reduces the claimed estimate to
\[ \|uvw\|_{H_{s}^{l}} \lesssim \|u\|_{H_{l,b}^{s}} \|vw\|_{H_{l-1,b}^{s}} \lesssim \|u\|_{H_{l,b}^{s}} \|v\|_{H_{l,b}^{s}} \|w\|_{H_{l,b}^{s}} \]
by applying Lemma 5.5 twice, where we used \(2s - (\frac{2}{p} - 1) > \frac{2}{p} \). Moreover
\[ \|uvw\|_{H_{s}^{l}} \lesssim \|u\|_{H_{l,b}^{s}} \|vw\|_{H_{l-1,b}^{s}} \lesssim \|u\|_{H_{l,b}^{s}} \|v\|_{H_{l-1,b}^{s}} \|w\|_{H_{l,b}^{s}} \]
by Lemma 5.5 using \(2s - (l - 1) - (\frac{2}{p} - l) > \frac{2}{p} \).

**Proof of (40):** We obtain
\[ \|\Lambda^{-1}(vw)wz\|_{H_{s-1,b}^{l}} \lesssim \|\Lambda^{-1}(vw)wz\|_{H_{s-1,b}^{l}} \]
\[ \lesssim \|u\|_{X_{l,b}^{s}} \|v\|_{X_{l,b}^{s}} \|wz\|_{X_{l,b}^{s}} \]

For the first step we applied Lemma 5.5 using \(s - (\frac{2}{p} - 1) + \frac{2}{p} - 2 - (s - 1) = \frac{2}{p} \).
For the last estimate we apply Lemma 5.6 using \(2s - (\frac{2}{p} - 3) > 2 + \frac{1}{p} - \frac{2}{p} + 3 = 5 - \frac{2}{p} > \frac{2}{p} \) and also Lemma 5.5 using \(2s - s + \frac{2}{p} - 1 > 1 + \frac{1}{p} + \frac{2}{p} - 1 = \frac{2}{p} \).

**Proof of (40):** For \(l \leq 1\) we use Lemma 5.5 and Lemma 5.6:
\[ \|uvwz\|_{H_{s}^{l}} \lesssim \|uvwz\|_{H_{s}^{l}} \lesssim \|uvwz\|_{H_{s}^{l}} \]
\[ \lesssim \|u\|_{X_{l,b}^{s}} \|v\|_{X_{l,b}^{s}} \|wz\|_{X_{l,b}^{s}}, \]

where we used \(2s - \frac{1}{p} + b > 2 + \frac{1}{p} + b > \frac{2}{p} \) for \(b = \frac{1}{p} \). For \(l > 1\) we use the fractional Leibniz rule which reduces the claimed estimate to
\[ \|uvwz\|_{H_{s}^{l}} \lesssim \|uvwz\|_{H_{s}^{l}} \]
\[ \lesssim \|u\|_{X_{l,b}^{s}} \|v\|_{X_{l,b}^{s}} \|wz\|_{X_{l,b}^{s}}, \]
provided \(2s - (l - 1) - \frac{1}{r} > \frac{2}{r}\) ⇔ \(2s - l > \frac{3}{r} - 1\) and \(2s - \frac{1}{r} > 2 > \frac{1}{r} + b\) for \(b = \frac{1}{r} + \).

\[\square\]

### 7. Proof of (24) - (40) in the case \(r = 2\).

We recall the null forms given by

\[
Q_0(u, v) = -\partial_t u \partial_t v + \partial^\alpha u \partial^\alpha v
\]
\[
Q_0(u, v) = \partial_t u \partial_t v - \partial^\alpha u \partial^\alpha v
\]
\[
Q_{12}(u, v) = \partial_t u \partial_2 v - \partial_2 u \partial_t v
\]

and the substitution

\[
u = u_+ + u_-, \quad \partial_t u = i\Lambda(u_+ - u_-), \quad v = v_+ + v_-, \quad \partial_t v = i\Lambda(v_+ - v_-).
\]

We consider the Fourier symbols

\[
q_0(\xi, \eta) = (\xi)\eta - (\xi, \eta) = |\xi||\eta|(1 - \frac{\xi, \eta}{|\xi||\eta|}) + (\xi)\eta - |\xi||\eta|
\]
\[
q_{01}(\xi, \eta) = -(\xi)\eta + \xi_1(\eta) = |\xi||\eta|(\xi_1 - \frac{\eta_1}{|\eta|}) + \xi_1(\eta) - |\eta| - (\xi) - |\xi|\eta_1
\]
\[
q_{12}(\xi, \eta) = -2\xi_1\eta_2 + 2\xi_2\eta_1.
\]

Then the Fourier symbols of \(Q_0(u, v)\), \(Q_{01}(u, v)\) and \(Q_{12}(u, v)\) are linear combinations of \(q_0(\pm i\xi, \pm 2\eta)\), \(q_{01}(\pm i\xi, \pm 2\eta)\) and \(q_{12}(\pm i\xi, \pm 2\eta)\), respectively.

The following simple observation can be found e.g. in [ST].

### Lemma 7.1. The Fourier symbols satisfy

\[
|q_0(\pm i\xi, \pm 2\eta)| \lesssim |\xi||\eta|\lesssim(\pm 1, \pm 2\eta)^2 + \frac{1}{\min(|\xi|, |\eta|)}
\]
\[
|q_{01}(\pm i\xi, \pm 2\eta)| \lesssim |\xi||\eta|\lesssim(\pm 1, \pm 2\eta)^2 + \frac{|\xi|}{|\eta|} + \frac{|\eta|}{|\xi|}
\]
\[
|q_{12}(\pm i\xi, \pm 2\eta)| \lesssim |\xi||\eta|\lesssim(\pm 1, \pm 2\eta)^2
\]

The estimate for the angle in the following Lemma was proven in [AFS1], Lemma 5:

### Lemma 7.2. Let \(\alpha, \beta, \gamma \in [0, \frac{1}{2}]\), \(\tau, \lambda \in \mathbb{R}\), \(\xi, \eta \in \mathbb{R}^2\), \(\xi, \eta \neq 0\). Then the following estimate applies for all signs \(\pm_1, \pm_2\):

\[
\angle(\pm_1, \pm_2) \lesssim \left(\frac{|\tau + \lambda - |\xi + \eta||}{\min(|\xi|, |\eta|)}\right)^\alpha + \left(\frac{|\xi, \eta + \tau|\frac{|\xi|}{\min(|\xi|, |\eta|)}\right)^\beta + \left(\frac{|\xi, \eta + \lambda - \eta|\frac{|\eta|}{\min(|\xi|, |\eta|)}\right)^\gamma.
\]

The following bilinear estimates for wave-Sobolev spaces were proven in [AFS], Lemma 7.
Corollary 7.1. Let \( s_0, s_1, s_2 \in \mathbb{R} \), \( b_0, b_1, b_2 \geq 0 \). Assume that
\[
\begin{align*}
b_0 + b_1 + b_2 &> \frac{1}{2} \\
s_0 + s_1 + s_2 &> \frac{3}{2} - (b_0 + b_1 + b_2) \\
s_0 + s_1 + s_2 &> 1 - \min_{i \neq j}(b_i + b_j) \\
s_0 + s_1 + s_2 &> \frac{1}{2} - \min_i b_i \\
s_0 + s_1 + s_2 &> 1 - \min(b_0 + s_1 + s_2, s_0 + b_1 + s_2, s_0 + s_1 + b_2) \\
s_0 + s_1 + s_2 &\geq \frac{3}{4} \\
\min_{i \neq j}(s_i + s_j) &\geq 0,
\end{align*}
\]
where the last two inequalities are not both equalities. Then the following estimate applies:
\[
\|uv\|_{H^{-r_0} - b_0} \lesssim \|u\|_{H^{s_1} - b_1} \|v\|_{H^{s_2} - b_2}.
\]
If \( b_0 < 0 \), this remains true provided we additionally assume \( b_0 + b_1 > 0 \), \( b_0 + b_2 > 0 \) and \( s_1 + s_2 > -b_0 \).

Corollary 7.2. If \( b_0, b_1, b_2 \geq 0 \), \( b_0 + b_1 + b_2 > \frac{1}{2} \) and \( \min_{i \neq j}(s_i + s_j) \geq 0 \) the assumption \( s_0 + s_1 + s_2 > 1 \) is sufficient.

Corollary 7.3. If \( b_0 \geq 0 \), \( b_1, b_2 > \frac{1}{2} \) the following assumptions are sufficient:
\[
s_0 + s_1 + s_2 > 1 - (b_0 + s_1 + s_2), s_0 + s_1 + s_2 \geq \frac{3}{4}, \min_{i \neq j}(s_i + s_j) \geq 0,
\]
where the last two inequalities are not both equalities.

Now we are ready to prove the inequalities (24) - (40) in the case \( r = 2 \).

Assumption: \( s > \frac{3}{4}, l > -\frac{1}{4}, s \geq l \geq s - 1, 2s - l > \frac{5}{4}, 4s - l > 3, 3s - 2l > \frac{3}{2}, 2l - s > -\frac{5}{2} \).

Proof of (24) and (25): We have to prove
\[
\|Q(u, v)\|_{H^{-l-\frac{1}{2}+2\epsilon}} \lesssim \|u\|_{X^{s+\frac{1}{2}+\epsilon}} \|v\|_{X^{s'+\frac{1}{2}+\epsilon}}.
\]
By Lemma 7.1 and Lemma 7.2 we may reduce to the following estimates:
\[
\begin{align*}
\|uv\|_{H^{-l-\frac{1}{2}+\epsilon}} &\lesssim \|u\|_{H^{s+\frac{1}{2}+\epsilon}} \|v\|_{H^{s'+\frac{1}{2}+\epsilon}} \\
\|uv\|_{H^{-l-\frac{1}{2}+\epsilon}} &\lesssim \|u\|_{H^{s+\frac{1}{2}+\epsilon}} \|v\|_{H^{s'+\frac{1}{2}+\epsilon}} \\
\|uv\|_{H^{-l-\frac{1}{2}+\epsilon}} &\lesssim \|u\|_{H^{s+\frac{1}{2}+\epsilon}} \|v\|_{H^{s'+\frac{1}{2}+\epsilon}} \\
\|uv\|_{H^{-l-\frac{1}{2}+\epsilon}} &\lesssim \|u\|_{H^{s+\frac{1}{2}+\epsilon}} \|v\|_{H^{s'+\frac{1}{2}+\epsilon}} \\
\|uv\|_{H^{-l-\frac{1}{2}+\epsilon}} &\lesssim \|u\|_{H^{s+\frac{1}{2}+\epsilon}} \|v\|_{H^{s'+\frac{1}{2}+\epsilon}}.
\end{align*}
\]
By Cor. 7.1 these estimates are fulfilled for a sufficiently small \( \epsilon > 0 \), if \( s > \frac{1}{2} \). In the case \( Q = Q_0 \) we additionally have to show
\[
\begin{align*}
\|uv\|_{H^{-l-\frac{1}{2}+2\epsilon}} &\lesssim \|u\|_{H^{s+\frac{1}{2}+\epsilon}} \|v\|_{H^{s'+\frac{1}{2}+\epsilon}} \\
\|uv\|_{H^{-l-\frac{1}{2}+2\epsilon}} &\lesssim \|u\|_{H^{s+\frac{1}{2}+\epsilon}} \|v\|_{H^{s'+\frac{1}{2}+\epsilon}},
\end{align*}
\]
which are also fulfilled by Cor. 7.1.

Proof of (26): We need
\[
\|Q(u, v)\|_{H^{-l-\frac{1}{2}+2\epsilon}} \lesssim \|u\|_{X^{s+\frac{1}{2}+\epsilon}} \|v\|_{X^{s'+\frac{1}{2}+\epsilon}}.
\]
Using Lemma 7.1 and Lemma 7.2 we reduce to six estimates as above. A typical one is
\[ \|uv\|_{H^{s-1.5,0}} \lesssim \|u\|_{H^{s+\frac{1}{4},0}} \|v\|_{H^{s+\frac{1}{4},0}}, \]
which by Cor. 7.1 is fulfilled if \( s > \frac{1}{2} \) and \( s > l - \frac{3}{4} \). The other estimates may be handled similarly.

**Proof of (27) and (28):** We have to prove
\[ \|Q(u, v)\|_{H^{s-1.5,0}} \lesssim \|u\|_{H^{s+\frac{1}{4},0}} \|v\|_{H^{s+\frac{1}{4},0}}. \]
This reduces to the following estimates by Lemma 7.1 and Lemma 7.2
\[ \|uv\|_{H^{s-1.5,0}} \lesssim \|u\|_{H^{s+\frac{1}{4},0}} \|v\|_{H^{s+\frac{1}{4},0}}, \]
\[ \|uv\|_{H^{s-1.5,0}} \lesssim \|u\|_{H^{s+\frac{1}{4},0}} \|v\|_{H^{s+\frac{1}{4},0}}, \]
\[ \|uv\|_{H^{s-1.5,0}} \lesssim \|u\|_{H^{s+\frac{1}{4},0}} \|v\|_{H^{s+\frac{1}{4},0}}. \]
For the first estimate we use Cor. 7.2. This requires the conditions \( s \leq l \) and
\[ (s - 1) + (s - \frac{1}{2}) > 0 \Leftrightarrow s > \frac{3}{4}, \]
\[ (1 - l) + (s - 1) + (s - \frac{1}{2}) \geq 2s - 2l > \frac{5}{4}, \]
\[ (1 - l) + (s - 1) + (s - \frac{1}{2}) > 1 - ((s - 1) + (s - \frac{1}{2})) \Leftrightarrow 4s - l > 3, \]
\[ (1 - l) + (s - 1) \geq 0 \Leftrightarrow s \geq l. \]
The second estimate is by duality equivalent to
\[ \|uv\|_{H^{s,0}} \lesssim \|u\|_{H^{s+\frac{1}{4},0}} \|v\|_{H^{s+\frac{1}{4},0}}, \]
which by Cor. 7.2 requires \( 2s - l > \frac{5}{4} \) and moreover
\[ 2s - l - \frac{1}{2} > 1 - ((s - 1) + (1 - l)) \Leftrightarrow 3s - 2l > 1. \]
The third estimate is by duality equivalent to
\[ \|uv\|_{H^{s,0}} \lesssim \|u\|_{H^{s+\frac{1}{4},0}} \|v\|_{H^{s+\frac{1}{4},0}}, \]
which moreover requires
\[ 2s - l - \frac{1}{2} > 1 - ((s - 1) + (1 - l)) \Leftrightarrow 3s - 2l > \frac{3}{2}. \]
In the case of \( Q = Q_0 \) and \( Q = Q_0 \) we also need
\[ \|uv\|_{H^{s-1.5,0}} \lesssim \|u\|_{H^{s+\frac{1}{4},0}} \|v\|_{H^{s+\frac{1}{4},0}}, \]
which by Cor. 7.1 is fulfilled for \( s \geq l \) and
\[ (s + 1) + (s - 1) + (1 - l) > 0 \Leftrightarrow 2s - l > 0. \]

**Proof of (29):** We have to prove
\[ \Gamma^1(u, v)_{H^{s-1.5,0}} \lesssim \|u\|_{H^{s+\frac{1}{4},0}} \|v\|_{H^{s+\frac{1}{4},0}}. \]
Now we use Lemma 1.3:
\[ \Gamma^1(u, v) \leq Q_{12}(D^{-1}u, D^{-1}v) + Q_0(D^{-1}u, D^{-1}v) + (\Lambda^{-2}u)v + u(\Lambda^{-1}v). \]
Thus we need
\[ \|Q(u, v)\|_{H^{s-1.5,0}} \lesssim \|u\|_{H^{s+\frac{1}{4},0}} \|v\|_{H^{s+\frac{1}{4},0}}. \]
which is the same estimate, which was treated already in (21) and (25). Moreover we need the following estimates:
\[\|uv\|_{H^{s+1}} \lesssim \|u\|_{H^{s+1}} \|v\|_{H^{s+1}},\]
\[\|uv\|_{H^{s+1}} \lesssim \|u\|_{H^{s+1}} \|v\|_{H^{s+1}},\]
which easily follow from Cor. 7.1.

**Proof of (30), (31) and (33):** These estimates result from Cor. 7.1.

**Proof of (32):** This reduces to
\[\|uv\|_{H^{s+1}} \lesssim \|u\|_{H^{s+1}} \|v\|_{H^{s+1}}.\]
By Cor. 7.2 this requires \(1 - s + l + 1 > \frac{1}{4}\) \(\iff\) \(2l - s > -\frac{3}{2}\) and \(1 - s + l + 1 + l > 1 - (\frac{1}{4} + 2l + 1) \iff 4l - s > -\frac{3}{2}\) and \(l \geq s - 1\), which hold by our assumptions.

**Proof of (34):** This reduces to
\[\|uv\|_{H^{s+1}} \lesssim \|u\|_{H^{s+1}} \|v\|_{H^{s+1}},\]
which requires by Cor. 7.2 \(1 - l + s + 1 - l - 1 > \frac{1}{4}\) \(\iff\) \(2s - l > -\frac{1}{2}\) and \(1 - l + s + 1 - l - 1 > 1 - (\frac{1}{4} + 2s) \iff 4s - l > -\frac{1}{2}\).

**Proof of (35):** The desired estimate follows from Cor. 7.1 as follows:
\[\|u\Lambda^{-1}(vw)\|_{H^{s+1}} \lesssim \|u\|_{H^{s+1}} \|\Lambda^{-1}(vw)\|_{H^{s+1}},\]
where we used \(l > -\frac{1}{2}\) and \(s > \frac{1}{2}\).

**Proof of (36):** By Cor. 7.1 we obtain
\[\|u\Lambda^{-1}(vw)\|_{H^{s+1}} \lesssim \|u\|_{H^{s+1}} \|\Lambda^{-1}(vw)\|_{H^{s+1}},\]
because \(l \geq s - 1\), \(l > -\frac{1}{2}\) and \(s > \frac{1}{2}\).

**Proof of (38) and (39):** By Cor. 7.1 we obtain for \(s > \frac{1}{2}\):
\[\|uv\|_{H^{s+1}} \lesssim \|u\|_{H^{s+1}} \|v\|_{H^{s+1}} \lesssim \|u\|_{H^{s+1}} \|v\|_{H^{s+1}} \|w\|_{H^{s+1}}.

**Proof of (37):** Using Prop. 7.1 we obtain for \(s > \frac{1}{2}\):
\[\|\Lambda^{-1}(wv)\Lambda^{-1}(wz)\|_{H^{s+1}} \lesssim \|\Lambda^{-1}(wv)\|_{H^{s+1}} \|\Lambda^{-1}(wz)\|_{H^{s+1}} \lesssim \|u\|_{H^{s+1}} \|v\|_{H^{s+1}} \|w\|_{H^{s+1}}.

**Proof of (40):** We obtain
\[\|uvwz\|_{H^{s+1}} \lesssim \|uvwz\|_{H^{s+1}} \lesssim \|uvwz\|_{H^{s+1}} \lesssim \|uvwz\|_{H^{s+1}} \lesssim \|uvwz\|_{H^{s+1}} \lesssim \|uvwz\|_{H^{s+1}} \lesssim \|uvwz\|_{H^{s+1}} \lesssim \|uvwz\|_{H^{s+1}} \lesssim \|uvwz\|_{H^{s+1}} \lesssim \|uvwz\|_{H^{s+1}} \lesssim \|uvwz\|_{H^{s+1}} \lesssim \|uvwz\|_{H^{s+1}},\]
by Prop. 7.1 where we used \(1 - l + 2s - \frac{1}{2} > 1 \iff 2s - l > -\frac{3}{2}\), \(l \geq s - \frac{3}{4}\) and \(s > \frac{1}{4}\).

8. **Proof of Theorem 3.1**

In section 6 we proved the estimates (21) - (40) for
\[r = 1 +, s = \frac{3}{2} + \epsilon, l = \frac{1}{2} + \epsilon\]
for any \(\epsilon > 0\) and \(b = \frac{1}{2}\) and in section 7 for
\[r = 2, s = \frac{3}{4} + \epsilon, l = -\frac{1}{4} + \epsilon\]
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for any $\epsilon > 0$ and $b = \frac{1}{2^+}$. By multilinear interpolation this implies the validity of these estimates for

$$1 < r \leq 2, \quad s = \frac{3}{2r} + \epsilon, \quad l = \frac{3}{2r} - 1 + \epsilon, \quad b = \frac{1}{r}$$

for any $\epsilon > 0$. An application of Theorem 2.1 implies Theorem 3.1.

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