Diagram groups are totally orderable

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Abstract

In this paper, we introduce the concept of the independence graph of a directed 2-complex. We show that the class of diagram groups is closed under graph products over independence graphs of rooted 2-trees. This allows us to show that a diagram group containing all countable diagram groups is a semi-direct product of a partially commutative group and R. Thompson’s group $F$. As a result, we prove that all diagram groups are totally orderable.

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1 Introduction

This paper is based on [13], where we define many of the concepts used below.

We say that $G$ is left-orderable (right-orderable) whenever there exists a total order $<$ on $G$ invariant under left (right) multiplication, that is, $a < b$ implies $ca < cb$ ($ac < bc$) for any $a, b, c \in G$. It is easy to see that a group is left-orderable if and only if it is right-orderable. The group $G$ is orderable whenever there exists a total order $<$ on $G$ such that $a < b$ implies $ca < cb$ and $ac < bc$ for any $a, b, c \in G$. One of the main results of this paper is that all diagram groups are totally orderable. This answers in positive [10, Problem 17.6]. When this paper was almost completed, we found out from B. Wiest that he had recently proved that diagram groups are left-orderable (which is weaker than being orderable). First he proved that diagram groups are embeddable into a certain braid group on infinitely many strings, then he proved that this braid

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group is left-orderable [19]. Note that this very interesting approach is completely independent from ours.

We use standard notation for conjugation and commutators. If \( a, b \) are elements of a group \( G \), then \( a^b = b^{-1}ab \), \([a, b] = a^{-1}a^b = a^{-1}b^{-1}ab \). If \( A, B \) are subgroups of \( G \), then \([A, B] \) is the subgroup of \( G \) generated by all commutators of the form \([a, b] \), where \( a \in A, b \in B \).

A group \( G \) is called partially commutative (right angled Artin group or graph group) if it has a group presentation such that all defining relations have the form \([a, b] = 1 \), where \( a, b \) are generators.

We will use some properties of the R. Thompson group \( F \) which is the diagram group of the Dunce hat \( H_0 = \langle x | x^2 = x \rangle \) (see [12, 13]). Recall [5] that \( F \) is isomorphic to the group of all increasing continuous piecewise linear functions from the interval \([0, 1]\) onto itself such that all singularities (breakpoints of the derivative) occur only at finitely many dyadic rational points (points of the form \( m/2^n \)) and all slopes are integer powers of 2. The group operation is the composition of functions (we shall write function symbols to the right of the argument).

In [13], we proved that the diagram group \( G_1 = D(H_1, x) \) of the directed 2-complex \( H_1 = \langle x | x^2 = x, x = x \rangle \) is universal, that is, it contains all countable diagram groups. Notice that \( H_1 \) is obtained from the Dunce hat by adding one 2-cell of the form \( x = x \).

Throughout the paper, we shall always assume that when we add a 2-cell, we add its inverse as well.

In this paper, we first prove, using diagram products of groups [11], that if we add 2-cells of the form \( e = e \) to a directed 2-complex \( K \), then the diagram groups of the new complex are semi-direct products of partially commutative groups and the diagram groups of the initial complex. Moreover, the partially commutative group can be explicitly described as a diagram group of some directed 2-complex. In order to do that, we use the concept of a rooted 2-tree from [13] and introduce the concept of the independence graph of a directed 2-complex. We show that the class of diagram groups is closed under graph products over independence graphs of rooted 2-trees (in particular, it is closed under countable direct products). We also use natural representations of diagram groups and groupoids by homeomorphisms of intervals of the real line using transition functions as in [10]. As a result, we obtain a complete description of the structure of \( G_1 \), and prove that \( G_1 \) is orderable. That, in turn, implies that all diagram groups are totally orderable because the orderability is a local property.

### 2 Diagram products of groups and graph products of diagram groups

Let us recall the definition of the diagram product of groups (we translate the definition from [11] into the language of directed 2-complexes). Let \( K \) be a directed 2-complex with the set of edges \( E \) and the set of positive 2-cells \( E^+ \). Let \( G_E = \{ G_e \mid e \in E \} \) be a collection of groups. Finally, let \( w \) be a nonempty 1-path in \( K \). Then the diagram product \( D(G_E; K, w) \) of \( G_E \) over \( K \) with base \( w \) is the fundamental group with base \( w \) of the following 2-complex \( C(G_E) \) of groups [1] described below:

- the underlying 2-complex is the 2-skeleton of the Squier complex \( Sq(K) \),
- the vertex group \( G(s) \) assigned to a vertex \( s = e_1 \cdots e_n \) (\( e_i \in E \)) is the direct product \( G_{e_1} \times \cdots \times G_{e_n} \),

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• the edge group assigned to the edge \( x = (p, f, q) \) is the direct product \( G(p) \times G(q) \),

• the embeddings \( \iota_x \) of the edge group \( G(p) \times G(q) \) into the vertex groups \( G(p \mathbin{[} f \mathbin{]} \mathbin{q}) = G(p) \times G([f]) \times G(q) \) and \( \tau_x \) of \( G(p) \times G(q) \) into \( G(p[ f ] q) = G(p) \times G([f]) \times G(q) \) are coordinate-wise,

• the group assigned to every 2-cell in \( \text{Sq}(K) \) is trivial.

The procedure of calculating the diagram product can be described as follows. First we choose a spanning forest \( T \) in \( \text{Sq}(K) \). Then every edge \( x \) of \( \text{Sq}(K, w) \) defines an element \( \bar{x} \) of the fundamental group \( \pi_1(Sq(K), w) \cong D(K, w) \). Namely, this is the element represented by the loop \( p_{(x)}^{-1}xp_{r(x)} \), where \( p_v \) is the geodesic path from \( v \) to \( w \) in \( T \). According to formulas (4), (5) of [11], the diagram product \( D(\mathcal{G}_E; K, w) \) is isomorphic to the group

\[
\left( \ast \ G(v) \ast D(K, w) \right) / \mathcal{N},
\]

where the free product of the groups \( G(v) \) is taken over all vertices of \( \text{Sq}(K, w) \) and \( \mathcal{N} \) is the normal closure of the following defining relations:

\[
\iota_x(g)^2 = \tau_x(g) \text{ for every positive edge } x \text{ of } \text{Sq}(K, w), \quad g \in G(x).
\]

Theorem 4 of [11] shows that this algebraic construction corresponds to the following topological construction provided all groups in \( \mathcal{G}_E \) are diagram groups.

The directed 2-complex on Figure 1 will be called a switch. It has 4 vertices, edges \( a, b, c, s \) and one positive 2-cell \( a = bsc \).

![Diagram of a switch](image)

Figure 1.

Let \( K \) be a directed 2-complex with the set of edges \( E \). For every \( e \in E \), let \( K_e \) be a directed 2-complex with a distinguished 1-path \( p_e \), where \( G_e \cong D(K_e, p_e) \). For each edge \( e \in E \) we attach a switch of the form \( e = b_e s_e c_e \) to \( K \). Then attach \( K_e \) by subdividing \( s_e \) into \( |p_e| \) edges and gluing \( p_e \) with \( s_e \). As a result, we obtain a new directed 2-complex \( \bar{K} \). Note that \( G_e \) is also isomorphic to the diagram group \( D(\bar{K}_e, e) \), where \( \bar{K}_e \) is \( K_e \) with the switch added. This follows from [13], Theorem 4.1, part 1 and Corollary 3.4.

One can easily check that this description coincides with the construction in [11, Theorem 4] if \( K = K_P \) for some semigroup presentation \( P \). The only difference is that in [11] we used (in the new terminology) switches of the form \( a = bsb \) instead of \( a = bsc \). This difference is insignificant and the proof of [11, Theorem 4] can be easily generalized to arbitrary directed 2-complexes. We also need to describe explicitly the isomorphism, which was presented in the proof of that theorem. In order to do this, we need to introduce some notation.

Let \( v \) be a 1-path in \( K \) that contains an edge \( z \). One can write \( v = v' z v'' \) for some 1-paths \( v', v'' \). Then the vertex group \( G(v) \) in the complex \( C(\mathcal{G}_E) \) of groups equals the direct product \( G(v') \times G_z \times G(v'') \). The direct factor \( G_z \) in this direct product projects onto a subgroup in the diagram product. This subgroup will be denoted by \( G(v', z, v'') \). According to (1), all these subgroups together with the image of diagram group \( D(\bar{K}, w) \) generate the diagram product.

For any vertex \( v \) of \( \text{Sq}(K, w) \), we let \( P_v \) be the \((v, w)\)-diagram over \( K \) corresponding to the geodesic path \( p_v \) in \( T \) from \( v \) to \( w \). The next theorem immediately follows from [11, Theorem 4] (more precisely, from the straightforward generalization of this theorem to directed 2-complexes) and its proof.
Theorem 2.1. For every nonempty 1-path \(w\) in \(K\), the diagram product \(\mathcal{D}(G_E;K,w)\) is isomorphic to the diagram group \(\mathcal{D}(\mathcal{K},w)\). The isomorphism takes an element \(g\) of the subgroup \(G(v',z,v'')\) to the diagram \(P^{-1}_v(\varepsilon(v') + \Delta_g + \varepsilon(v''))P_v\), where \(v = v'zv''\), \(\Delta_g\) is a \((z,z)\)-diagram that represents the element of \(\mathcal{D}(\mathcal{K}_z,z)\) corresponding to \(g\).

This theorem gives an explicit representation by a diagram from the group \(\mathcal{D}(\mathcal{K},w)\) for any element of the diagram product \(\mathcal{D}(G_E;K,w)\). Formula (1) says that the diagram product is generated by the images of the diagram group \(\mathcal{D}(K,w)\) and the vertex groups \(G(v)\). The elements of \(\mathcal{D}(K,w)\) are already \((w,w)\)-diagrams over \(\mathcal{K}\). The vertex group \(G(v)\) is a direct product of the groups of the form \(G(v',z,v'')\) so we know the \((w,w)\)-diagrams that represent elements of \(G(v)\).

In this paper, we consider diagram products of groups in the case when the underlying directed 2-complex \(K\) is a rooted 2-tree. Recall [13, Theorem 7.2] that a rooted 2-tree with root \(w\) is a directed 2-complex \(K\) satisfying the following conditions:

**T1** For any vertex \(o\) of \(K\), there exist a 1-path from \(i(w)\) to \(\tau(w)\) containing \(o\).

**T2** Every two 1-paths in \(K\) with the same endpoints are (directly) homotopic in \(K\).

**T3** The diagram group \(\mathcal{D}(K,w)\) is trivial.

We are going to show now that if \(K\) is a rooted 2-tree then any diagram product of groups over \(K\) is a graph product of these groups.

Let \(K\) be a directed 2-complex with a distinguished 1-path \(w\). Two (different) edges \(e_1, e_2\) of \(K\) will be called independent provided there exists a 1-path in \(K\) homotopic to \(w\) that contains both \(e_1\) and \(e_2\). Consider the following independence graph \(\Gamma(K,w)\). Its vertices are all edges of \(K\) that belong to 1-paths homotopic to \(w\). Two vertices are adjacent in the graph if and only if they represent independent edges.

Let us recall the concept of a graph product of groups. Let \(\Gamma\) be a graph. Consider a family of groups \(\{G_v, v \in V\}\), where \(V\) is the set of vertices of \(\Gamma\). The graph product of groups \(\{G_v, v \in V\}\) over the graph \(\Gamma\) is the factor-group of the free product \(*_{v \in V} G_v\) over the normal subgroup generated by all commutators \([g,h]\) for all \(g \in G_s, h \in G_t\), where \(s, t\) are adjacent vertices in \(\Gamma\).

Clearly, if \(\Gamma\) is a complete graph, then the graph product coincides with the direct product, and if \(\Gamma\) is the graph with no edges, then the graph product coincides with the free product. Partially commutative groups are precisely the graph products of free (cyclic) groups.

**Lemma 2.2.** Let \(K\) be a directed 2-complex and let \(w\) be its nonempty 1-path. Suppose that \(K\) satisfies conditions T2 and T3. Let \(E\) be the set of edges of \(K\), and \(\mathcal{G}_E = \{G_e \mid e \in E\}\) be a family of groups. Then the diagram product of the family of groups \(\{G_e \mid e \in E\}\) over \(K\) with base \(w\) is isomorphic to the graph product of the same family of groups over the independence graph \(\Gamma(K,w)\).

**Proof.** Let \(S = \text{Sq}(K,w)\). The vertices of \(S\) are all 1-paths in \(K\) that are homotopic to \(w\). Let \(v = z_1 \cdots z_m\) \((z_i \in E)\) be a vertex of \(S\). In the 2-complex of groups \(C(G_E)\), the vertex group \(G(v)\) is the direct product \(G_{z_1} \times \cdots \times G_{z_m}\). Since the diagram group \(\mathcal{D}(K,w)\) is trivial, the diagram product is generated by the images of all the vertex groups \(G(v), v \in S\) according to (1).
Let $z$ be any vertex of $\Gamma(K, w)$. This means that $z$ belongs to some 1-path $v$ homotopic to $w$, that is, $v = v'zv''$ for some 1-paths $v', v''$. We are going to show that the image of the group $G(v', z, v'')$ in the diagram product depends only on $z$.

Suppose that $v'zv''_1$, $v'zv''_2$ are 1-paths in $K$ homotopic to $w$. The paths $v'_1$, $v''_2$ have the same endpoints so they are homotopic. Then there exists a $(v'_1, v'_2)$-diagram $\Delta'$ over $K$. Analogously, there exists a $(v''_1, v''_2)$-diagram $\Delta''$ over $K$. Each of these diagrams can be represented as a concatenation of atomic diagrams. Thus in order to prove that the groups $G(v'_1, z, v''_1)$, $G(v'_2, z, v''_2)$ are identified in the diagram product, it suffices to consider the case when one of the diagrams $\Delta'$ or $\Delta''$ is trivial and the other one is an atomic diagram. These cases are symmetric, so without loss of generality let $\Delta'$ be trivial and let $\Delta''$ be the atomic diagram corresponding to a positive edge $x = (p, f, q)$ in $\text{Sq}(K, w)$. The edge $z$ is contained in $p$, that is, $p = p'zp''$. We need to compare the groups $G(p', z, p''[f]q)$ and $G(p', z, p''[f]q)$. Notice that the edge $x$ represents the trivial element of the fundamental group $\pi_1(\text{Sq}(K), w)$ since the group is trivial itself. So the element $\bar{x}$ in formula (2) is trivial. The edge group $G(x) = G(p) \times G(q)$ embeds naturally into the vertex groups $G(p[f]q) = G(p) \times G([f] \times G(q)$ and $G(p[f]q) = G(p) \times G([f] \times G(q)$. These embeddings have been denoted by $\iota_x$ and $\tau_x$. Formula (2) now says that $\iota_x(g) = \tau_x(g)$ for any $g \in G(x)$. Thus the images of $G(p)$ in both vertex groups are identified in the diagram product. In particular, the direct factors of the form $G_z$ are also identified. But these are the groups $G(p', z, p''[f]q)$ and $G(p', z, p''[f]q)$. So they coincide in the diagram product.

Thus the images of groups $G_z$, $z \in \mathbf{E}$, generate the diagram product. The only defining relations we impose on the free product of these groups, are commutativity relations involving elements of different factors of the direct products $G(v)$, $v \in \text{Sq}(K, w)$. If $z_1$, $z_2$ are independent edges and $u_0z_1u_1z_2u_2$ is a 1-path homotopic to $w$, then the groups $G(z_1)$ and $G(z_2)$ pairwise commute in the diagram product because they are the factors of the direct product $G(u_0) \times G(z_1) \times G(u_1) \times G(z_2) \times G(u_2)$. These are the only defining relations we have. So the diagram product is the free product of the groups $G_z$, where $z$ occurs in a 1-path homotopic to $w$, factored by the relations of commutativity of the form $[G_{z_1}, G_{z_2}] = 1$ for each pair of independent edges $z_1$, $z_2$. This completes the proof.

**Remark 2.3.** In case the groups $G_z$ are diagram groups, we are interested in finding an explicit form of their elements in the diagram product. Suppose that $g \in G_z$ is represented by a $(z, z)$-diagram $\Delta_z$ over $\hat{K}_z$. For any 1-paths $v'$, $v''$ in $K$, where $v'zv''$ is homotopic to $w$ in $K$, we can consider the diagram of the form

$$P^{-1}(\varepsilon(v') + \Delta_g + \varepsilon(v''))P,$$

where $P$ is a $(v, w)$-diagram over $K$. If $P = P_v$ from the statement of Theorem 2.1, then $g \in G(v', z, v'')$ is represented in the diagram product by (3). However, if the diagram group $D(K, w)$ is trivial, then, replacing $P_v$ by $P$ leads to an equivalent diagram. So in this case (3) gives us the desired representation of an element $g \in G_z$. Since the groups $G(v', z, v'')$ are independent of $v'$, $v''$, the diagram we get also depends on $z$ and $g \in G_z$ only.

Let us call a graph $\Gamma$ appropriate if the class of diagram groups is closed under graph products over $\Gamma$. If every two 1-paths in $K$ with the same endpoints are homotopic and the diagram group $D(K, w)$ is trivial, then Lemma 2.2 shows that the independence graph $\Gamma(K, w)$ is appropriate. Clearly, the class of appropriate graphs is closed under taking full subgraphs. (Indeed, we can assign a trivial group to each vertex not in the subgraph.) On the other hand, [11, Theorem 30] shows that an appropriate graph cannot contain a cycle of odd length $\geq 5$ as a full subgraph.
Lemmas 2.2 and [13, Lemma 7.1] provide us with a large class of appropriate graphs.

**Theorem 2.4.** Let $K$ be a rooted 2-tree with the root $w$. Then the independence graph $\Gamma(K, w)$ is appropriate.

It is possible to prove that the converse of Theorem 2.4 also holds: every appropriate graph is the independence graph of some rooted 2-tree. We shall include the proof in our future paper.

We know [10, 11] that the class of diagram groups is closed under finite direct products and countable direct powers. Now we have a stronger result.

**Theorem 2.5.** The class of diagram groups is closed under countable direct products.

**Proof.** Direct products are graph products over complete graphs. Thus by Theorem 2.4, it suffices to construct a rooted 2-tree whose independence graph contains a countable complete subgraph. It is not difficult to understand that this is true for any “sufficiently branching” infinite rooted 2-tree.

In particular, let us start with the edge $e_0$ and then for each $n \geq 0$ add a 2-cell of the form $e_{2n} = e_{2n+1}e_{2n+2}$. In the resulting rooted 2-tree, the edges $e_1, e_3, e_5, \ldots$ with odd subscripts are obviously pairwise independent. $\square$

### 3 Expansions of directed 2-complexes

Let $K$ be a directed 2-complex and let $\nu$ be a function from the set of edges of $K$ to a set of (possibly infinite) cardinal numbers. For any edge $e$ of $K$, let us add $\nu(e)$ positive 2-cells of the form $e = e$. The new directed 2-complex $K_\nu$ is called an expansion of $K$. The new cells of the form $e = e$ are called leaves. In particular, if $K = H_0$ is the Dunce hat and $\nu$ is the function defined by $\nu(x) = 1$, then $K_\nu = H_1$ and $D(K_\nu, x)$ is the universal group $G_1$.

Expansions arise naturally when we consider directed 2-complexes $K$ with redundant 2-cells. A 2-cell $f$ is called redundant if $\lceil f \rceil$ is homotopic to $\lfloor f \rfloor$ in the complex $K \setminus \{f\}$ obtained from $K$ by removing $f^{\pm 1}$. In this case, by [13, Theorem 4.1, part 1], we can add a new edge $e$ with $\iota(e) = \iota(f)$, $\tau(e) = \tau(f)$ and a new 2-cell of the form $\lceil f \rceil = e$. The diagram groups will not change. After that, by [13, Theorem 4.1, part 2], we can replace the 2-cell $f$ by the 2-cell $f'$ of the form $e = e$ without changing the diagram groups. Thus adding redundant 2-cells to a directed 2-complex is essentially equivalent to taking expansions.

Complexes with redundant 2-cells are needed when we use complete directed 2-complexes in order to compute diagram groups. Recall that we have quite powerful technical tools to compute diagram groups of complete directed complexes [13, Section 6]. If we want to compute a diagram group of a directed 2-complex, which is not necessarily complete, then one way to do this is to embed $K$ into a bigger complex $K'$, where $K'$ is complete. This can be done using a kind of the Knuth – Bendix completion procedure, so $K'$ is obtained from $K$ by adding some redundant 2-cells.

Notice that if $K'$ is obtained from $K$ by adding redundant 2-cells, then for every nonempty 1-path $p$ of $K$, the diagram group $H = D(K, p)$ is a retract of $G = D(K', p)$ (see [13, Lemma 4.1, part 1]) and the retraction can be described explicitly. Thus if we know $H$, we can compute $G$.

The main goal of this section is to give a description of the diagram groups of the expansion $K_\nu$ as semi-direct products of partially commutative groups and the diagram groups of $K$.

Let $p$ be a nonempty 1-path in $K$. Given a $(p, p)$-diagram $\Delta$ over $K_\nu$, we can collapse all cells of $\Delta$ that correspond to 2-cells of the form $e = e$ from $K_\nu \setminus K$. (Collapsing a cell, means identifying its top and bottom path so the cell becomes an edge.) The resulting diagram over
\( \mathcal{K} \) is denoted by \( \theta(\Delta) \). Clearly, \( \theta \) is a retraction from \( \mathcal{D}(\mathcal{K}_\nu, p) \) to \( \mathcal{D}(\mathcal{K}, p) \) because all diagrams over \( \mathcal{K} \) are fixed by \( \theta \).

Therefore, the group \( \mathcal{D}(\mathcal{K}_\nu, p) \) is a semi-direct product of the kernel \( \mathcal{A} \) of \( \phi \) and \( G = \mathcal{D}(\mathcal{K}, p) \). We are going to prove that \( \mathcal{A} \) is a partially commutative group that can be presented as a diagram group of a certain directed 2-complex associated with \( \mathcal{K}_\nu \).

Let \( p \) be any 1-path in \( \mathcal{K} \). Recall [13] that a universal 2-cover of \( \mathcal{K} \) with base \( p \) is a directed 2-complex \( \mathcal{M} \) over \( \mathcal{K} \) with a labelling map \( \phi \) which satisfy the following properties:

**U1** \( \mathcal{M} \) is a rooted 2-tree with root \( \bar{p} \), where \( \phi(\bar{p}) = p \);

**U2** for any 1-path \( \bar{q} \) in \( \mathcal{M} \) from \( \iota(\bar{p}) \) to \( \tau(\bar{p}) \), the local map of \( \mathcal{M} \) at \( \bar{q} \) is bijective.

By [13, Remark 8.2], property U2 can be replaced by the following property:

**U2’** for any 1-path \( \bar{r} \) in \( \mathcal{M} \) and for any 2-cell \( f \) of \( \mathcal{K} \) such that \( [f] = \phi(\bar{r}) \), there is exactly one 2-cell \( \bar{f} \) of \( \mathcal{M} \) labelled by \( f \) with top path \( \bar{r} \).

Theorem [13, Theorem 8.1] shows that for every directed 2-complex \( \mathcal{K} \) and a 1-path \( p \) in \( \mathcal{K} \), the universal 2-cover \( \bar{\mathcal{K}}_p \) exists and is unique up to an isomorphism that preserves the labels.

Throughout this section, we fix a directed 2-complex \( \mathcal{K}_\nu \), a 1-path \( p \) in it, the expansion \( \mathcal{K}_\nu \), the retraction \( \theta \), the kernel \( \mathcal{A} \) of \( \theta \), the universal 2-cover \( \bar{\mathcal{K}}_p \) with the root \( \bar{p} \), and the labelling map \( \phi \).

Let \( \bar{\nu} \) be the function on the edges of \( \bar{\mathcal{K}}_p \) that sends every edge \( \bar{e} \) to \( \nu(\phi(\bar{e})) \). The corresponding expansion of \( \bar{\mathcal{K}}_p \) is denoted by \( \bar{\mathcal{K}}_{\nu,p} \). There exists a natural labelling morphism \( \bar{\phi} \) from \( \bar{\mathcal{K}}_{\nu,p} \) to \( \mathcal{K}_\nu \) that acts on \( \bar{\mathcal{K}}_p \) as \( \phi \) and acts as a one-to-one correspondence between the set of leaves attached to each edge \( \bar{e} \) of \( \bar{\mathcal{K}}_p \) and the set of leaves attached to the edge \( e = \phi(\bar{e}) \) of \( \mathcal{K} \) (these two sets of leaves have the same cardinality by definition). Thus \( \bar{\mathcal{K}}_{\nu,p} \) can be considered as a directed 2-complex over \( \mathcal{K}_\nu \).

**Theorem 3.1.** The kernel \( \mathcal{A} \) of the natural homomorphism \( \theta \) from \( \mathcal{D}(\mathcal{K}_\nu, p) \) to \( \mathcal{D}(\mathcal{K}, p) \) is isomorphic to the diagram group \( \mathcal{D}(\bar{\mathcal{K}}_{\nu,p}, \bar{p}) \).

**Proof.** Let \( \phi_p: \mathcal{D}(\bar{\mathcal{K}}_{\nu,p}, \bar{p}) \rightarrow \mathcal{D}(\mathcal{K}, p) \) and \( \bar{\phi}_p: \mathcal{D}(\bar{\mathcal{K}}_{\nu,p}, \bar{p}) \rightarrow \mathcal{D}(\mathcal{K}_\nu, p) \) be the group homomorphisms induced by the maps \( \phi \) and \( \bar{\phi} \) (see [13, Section 5]). Let \( \bar{\theta}: \mathcal{D}(\bar{\mathcal{K}}_{\nu,p}, \bar{p}) \rightarrow \mathcal{D}(\bar{\mathcal{K}}_p, \bar{p}) \) be the homomorphism that collapses the leaves of this expansion. Then the diagram

\[
\begin{array}{ccc}
\mathcal{D}(\bar{\mathcal{K}}_{\nu,p}, \bar{p}) & \xrightarrow{\phi_p} & \mathcal{D}(\mathcal{K}, p) \\
\downarrow \phi & & \downarrow \theta \\
\mathcal{D}(\bar{\mathcal{K}}_p, \bar{p}) & \xrightarrow{\bar{\phi}_p} & \mathcal{D}(\mathcal{K}_\nu, p)
\end{array}
\]

is clearly commutative. The group \( \mathcal{D}(\bar{\mathcal{K}}_p, \bar{p}) \) is trivial because \( \bar{\mathcal{K}}_p \) is a rooted 2-tree and satisfies property T3 by [13, Theorem 7.2]. Hence the image of \( \bar{\phi}_p \) is contained in \( \mathcal{A} \).

Let us show that \( \phi_p \) is injective. Consider any reduced \((\bar{p}, \bar{p})\)-diagram \( \Delta \) over \( \bar{\mathcal{K}}_{\nu,p} \). It suffices to prove that the diagram \( \Delta = \bar{\phi}_p(\bar{\Delta}) \) has no dipoles. By contradiction, suppose that two cells \( \pi_1, \pi_2 \) of \( \Delta \) form a dipole. Then their labels are \( f, f^{-1} \) for some \( f \) and the cells have the form \( u = v, v = u \), respectively, for some \( u, v \). Let \( q' \) be a path in \( \Delta \) that connects \( \iota(\Delta) \) and \( \iota(\pi_1) = \iota(\pi_2) \). Also let \( q'' \) be a path in \( \Delta \) that connects \( \tau(\pi_1) = \tau(\pi_2) \) and \( \tau(\Delta) \). By \( r \) we denote the path \([\pi_1] = [\pi_2] \) in \( \Delta \). The diagram \( \bar{\Delta} \) has the corresponding path \( \bar{q}' \bar{r} \bar{q}'' \) from \( \iota(\bar{\Delta}) \)
to \(\tau(\hat{\Delta})\). The cells \(\pi_1, \pi_2\) have natural preimages \(\pi_1, \pi_2\). We claim that these cells form a dipole in \(\Delta\).

If \(\pi_1, \pi_2\) are leaves of the form \(e = e\) then their labels are \(f\) and \(f^{-1}\). Hence \(\pi_1, \pi_2\) are cells of the form \(\check{e} = \check{e}\) with labels \(\check{f}, \check{f}^{-1}\). Now suppose that the labels of \(\pi_1, \pi_2\) belong to \(K\). The atomic diagrams \(\varepsilon(q') + \varepsilon(q')^{-1}\) and \(\varepsilon(q') + \pi_2 + \varepsilon(q'')\) coincide so the corresponding atomic 2-paths (with top 1-path \(q'rq''\)) also coincide. The atomic diagrams \(\varepsilon(q') + \varepsilon(q')^{-1} + \varepsilon(q'')\) and \(\varepsilon(q') + \pi_2 + \varepsilon(q'')\) have the same top path \(q'rq''\). They are diagrams over \(K_\nu\) and they have the same image under \(\phi\). So they must also coincide according to U2. Thus \(\Delta\) has a dipole, so it is not reduced, a contradiction.

It remains to show that \(\check{\phi}_p\) is surjective. Suppose that \(\Delta\) is a diagram over \(K\) that can be reduced to the trivial diagram \(\varepsilon(p)\). Let \(m\) be the number of dipoles one needs to cancel when reducing \(\Delta\) to \(\varepsilon(p)\). We prove by induction on \(m\) that there exists a \((\check{p}, \check{p})\)-diagram \(\hat{\Delta}\) over \(K_\nu\) such that \(\check{\phi}_p\) takes \(\hat{\Delta}\) to \(\Delta\). If \(m = 0\), then we have nothing to prove. Let \(m \geq 1\). Take the dipole in \(\Delta\) that is cancelled on the first step. Suppose it is formed by a cell of the form \(u = v\) and a cell of the form \(v = u\). The diagram \(\Delta\) can be cut into two parts by a path with label of the form \(q'rq''\). Let \(\Delta'\) be the result of cancelling the dipole in \(\Delta\). Then \(\Delta'\) can be also cut into two parts by a path labelled by \(q'rq''\). In order to reduce this diagram, we have to cancel \(m - 1\) pairs of dipoles. So we can apply the inductive assumption and find a diagram \(\Delta'\) that maps to \(\Delta'\) under \(\check{\phi}_p\). Consider the cut of \(\Delta'\) by the path labelled by \(q'rq''\). Notice that there is an atomic diagram \(\varepsilon(q') + (v = u) + \varepsilon(q'')\) over \(K\) whose top path is \(q'rq''\). So we also have the atomic diagram \(\varepsilon(q') + (v = \check{u}) + \varepsilon(q'')\) over \(K_\nu\) with top 1-path \(q'\check{v}q''\) (because of U2). Now we can replace the subpath \(\check{u}\) in \(\Delta'\) by the dipole formed by the pair of mirror cells of the form \(\check{u} = \check{v}\) and \(\check{v} = \check{u}\). The resulting diagram denoted by \(\hat{\Delta}\) clearly maps to \(\Delta\) under \(\check{\phi}_p\).

Now let \(\Delta_\nu\) be an arbitrary diagram in \(A\). By \(\Delta\) we denote its image under \(\theta\) (that is, the result of collapsing all leaves). Thus \(\Delta\) represents a trivial element of the group \(D(K, p)\) and so it can be reduced to the trivial diagram. From the previous paragraph, we know that there exists a \((\check{p}, \check{p})\)-diagram \(\hat{\Delta}\) that is taken to \(\Delta\) by \(\phi_p\). Passing from \(\Delta_\nu\) to \(\Delta\), we can collapse one cell of the form \(e = e\) at a time.

Suppose that \(\Delta\) is a diagram over \(K_\nu\) obtained from a diagram \(\hat{\Delta}\) over \(\overline{K}_{\nu, p}\) by applying \(\check{\phi}_p\). Also let \(\Delta\) be the result of collapsing one cell of the form \(e = e\) in a diagram \(\Delta_\nu\) over \(K_\nu\). Then it suffices to show that some diagram \(\Delta\) over \(\overline{K}_{\nu, p}\) is mapped to \(\Delta_\nu\) under \(\check{\phi}_p\) and is mapped to \(\Delta\) under \(\theta\). This is obvious: indeed, we just find an edge \(\check{e}\) in \(\Delta\) that maps to \(e\), the edge that remains after the cell \(e = e\) (with some label \(f\)) is collapsed. If we insert the cell \(\check{e} = \check{e}\) with label \(\check{f}\) into \(\Delta\) instead of the edge \(\check{e}\), then we get exactly what we need.

Thus the homomorphism \(\check{\phi}_p\) is in fact an isomorphism between \(D(\overline{K}_{\nu, p}, \check{p})\) and the subgroup \(A = \text{Ker} \theta \in D(K_\nu, p)\).

Now we can use the representation of \(A\) as a diagram group in order to complete the description of \(A\). First we show that \(A\) can be represented as a diagram product. In order to do this, we need to modify the directed 2-complex \(\overline{K}_{\nu, p}\) preserving its diagram groups. Let \(\check{e}\) be an edge of \(\overline{K}_{\nu, p}\). If \(e = \phi(\check{e})\in K\) is the label of this edge, then there are \(\nu(e)\) leaves of the form \(\check{e} = \check{e}\) in \(\overline{K}_{\nu, p}\). We replace these leaves by a “switch” (see Section 2), that is, a 2-cell of the form \(\check{e} = bsc\), and \(\nu(e)\) positive cells of the form \(s = s\). (Here \(b, s, c\) depend of \(\check{e}\).) The label of a 2-cell \(s = s\) will be the same as the label of the leaf it replaces. Applying this operation to all edges of \(\overline{K}_{\nu, p}\) at once, we get a new directed 2-complex \(\hat{K}_{\nu, p}\).

**Lemma 3.2.** The diagram groups \(D(\overline{K}_{\nu, p}, \check{p})\) and \(D(\hat{K}_{\nu, p}, \check{p})\) are isomorphic.
Proof. Let \( \psi \) be the morphism from \( \K_{\nu,p} \) to \( \K_{\nu,p} \) that sends the leave \( \tilde{e} = \tilde{e} \) with label \( f \) to the 2-path \((1, \tilde{e} = bsc, 1) \circ (b, s = s, c) \circ (1, \tilde{e} = bsc, 1)^{-1} \), where the 2-cell \( s = s \) has the same label \( f \). This morphism induces a homomorphism \( \psi : \mathcal{D}(\K_{\nu,p}, \tilde{p}) \) to \( \mathcal{D}(\K_{\nu,p}, \tilde{p}) \) (see [13, Section 5]). We shall show that this homomorphism is an isomorphism.

Let us consider a reduced \((\tilde{p}, \tilde{p})\)-diagram \( \Delta \) over \( \K_{\nu,p} \). Suppose that there is an edge of \( \Delta \) labelled by \( s \). Let us consider a reduced \((\tilde{p}, \tilde{p})\)-diagram \( \Delta \) over \( \K_{\nu,p} \). Consider the maximal subdiagram \( \Xi \) of \( \Delta \) that contains this edge and is a product of \( (s,s) \)-cells. In particular, \( \Xi \) is an \( (s,s) \)-diagram. The top path of \( \Xi \) is not contained in the top path of \( \Delta \) because \( [\Delta] \) has no edges labelled by \( s \). So there exists a cell \( \pi_1 \) whose bottom path contains \( [\Xi] \). This cell must be a switch, that is, a \((\tilde{e}, bsc)\)-cell. Analogously, we can find a \((bsc, \tilde{e})\)-cell \( \pi_2 \) in \( \Delta \) whose top path contains \( [\Xi] \). We can find some paths \( q_1, q_2 \) from \( \iota(\Delta) \) to \( \iota(\pi_1), \iota(\pi_2) \), respectively, such that there is a subdiagram \( \Delta' \) with the top path of the form \( q_1r_1 \) and the bottom path of the form \( q_2r_2 \). Here \( r_1 \) is the first edge of \( [\pi_1] \) and \( r_2 \) is the first edge of \( [\pi_2] \). This implies that \( r_1 \) and \( r_2 \) must coincide in \( \Delta' \) because they are labelled by \( b \). Indeed, there are no 2-cells whose top or bottom path ends with the edge \( b \). Therefore, the first edge of \( [\pi_1] \) coincides with the first edge of \( [\pi_2] \).

A similar argument implies that the third edge of \( [\pi_1] \) coincides with the third edge of \( [\pi_2] \).

This shows that any edge labelled by \( s \) in \( \Delta \) is contained in a \((\tilde{e}, \tilde{e})\)-diagram, which is the product of a \((\tilde{e}, bsc)\)-cell, some \((s,s)\)-cells, and a \((bsc, \tilde{e})\)-cell. Since \( \Delta \) is reduced, all labels of the \((s,s)\)-cells, read top to bottom, form a freely irreducible word. Now we can replace our \((\tilde{e}, \tilde{e})\)-subdiagram by a product of \((\tilde{e}, \tilde{e})\)-cells with the same labels. This operation can be done simultaneously for all edges in \( \Delta \) labelled by \( s \), where \( s \) is involved in some switch of the form \( \tilde{e} = bsc \). The result is obviously a preimage of \( \Delta \) under \( \psi \). It is easy to see that the operation we just described defines the inverse of \( \psi \), hence \( \psi^* \) is an isomorphism. \( \Box \)

It follows from Section 2 that the diagram group \( \mathcal{D}(\K_{\nu,p}, \tilde{p}) \) is the diagram product over \( \K_{\nu,p} \) of the family of groups \( \{G_\xi\} \), where \( \xi \) runs over all edges of \( \K_{\nu,p} \). Each group \( G_\xi \) is the diagram group of the directed 2-complex \( \mathcal{F}_\nu(e) \) with \( \nu(e) \) positive 2-cells of the form \( e = e \) and base \( e \).

It is clear that the connected component \( \text{Sq}(\mathcal{F}_\nu(e), e) \) of the Squier complex is a wedge of \( \nu(e) \) circles. So \( G_\xi \) is the free group of rank \( \nu(e) \). Now we are going to prove that \( A \) is a partially commutative group.

**Theorem 3.3.** 1. For every edge \( \tilde{e} \) of \( \K_{\nu,p} \), let \( G_\xi \) be the free group of rank \( \nu(e) \), where \( e = (\phi(\tilde{e})) \). Then the kernel \( A \) of the natural retraction \( \theta \) from \( \mathcal{D}(\K_{\nu,p}, \tilde{p}) \) to \( \mathcal{D}(\K_{\nu,p}) \) is a graph product of the free groups \( G_\xi \) over the independence graph of \( \K_{\nu,p} \). Thus \( A \) has a partially commutative presentation, where the generators are arbitrary symbols of the form \( a(\tilde{e}, i) \) with \( \tilde{e} \) an edge of \( \K_{\nu,p} \), \( 1 \leq i \leq \nu(e) \), and the defining relations are of the form \( [a(\tilde{e}_1, i_1), a(\tilde{e}_2, i_2)] = 1 \), with \( \tilde{e}_1, \tilde{e}_2 \) independent edges of \( \K_{\nu,p} \).

2. As an element of the diagram group \( \mathcal{D}(\K_{\nu,p}, \tilde{p}) \), the symbol \( a(\tilde{e}, i) \) is represented by a diagram of the form \( \Delta^{-1}\Psi\Delta \) over \( \K_{\nu,p} \), where \( \Delta = \phi(\tilde{\Delta}) \) for some diagram \( \tilde{\Delta} \) over \( \K_{\nu,p} \) with \( [\tilde{\Delta}] = \tilde{p} \) and \( [\Delta] = q'q'' \) for some \( q', q'' \), and \( \Psi \) is the atomic diagram \( \epsilon(q') + f_i + \epsilon(q'') \), where \( q' = \phi(q') \), \( q'' = \phi(q'') \), and \( f_i \) is the \( i \)-th leaf attached to \( e = \phi(\tilde{e}) \). This diagram is independent on the choice of \( \tilde{\Delta} \).

3. For every diagram \( \Gamma \in \mathcal{D}(\K_{\nu,p}) \), and every element \( a(\tilde{e}, i) \) represented by the diagram \( \Delta^{-1}\Psi\Delta \) as in part 2, the element \( \Gamma^{-1}a(\tilde{e}, i)\Gamma \) is equal to the generator \( a(\tilde{e}, i) = \Delta^{-1}\Psi\Delta \), where \( \Delta \) is the lift of the diagram \( \Delta \) over \( \K \) to the universal 2-cover \( \K_{\nu,p} \). The edge \( \tilde{e} \) is the image of the occurrence of \( e = \phi(\tilde{e}) \) in \( q'q'' \) after we lift the diagram into \( \K_{\nu,p} \). 9
Proof. 1. We have already shown that $\mathcal{A}$ is isomorphic to the diagram product over $\tilde{K}_p$ of the family of free groups $\{G_e\}$ of rank $\nu(e)$. Since $\tilde{K}_p$ is a rooted 2-tree (by property U1), it remains to use Lemma 2.2.

2. The first statement follows from properties T1 and T2, and Remark 2.3.

3. The fact that the lift of $\Delta \Gamma$ exists follows from [13, Lemma 8.3]. The fact that $\bar{\Delta}^{-1}\Psi \Delta$ is a generator of $\mathcal{A}$ and the statement about $\bar{e}$ follows from Remark 2.3.

This completes the description of the diagram group $\mathcal{D}(K_\nu, p)$ as a semi-direct product of $\mathcal{A}$ and $\mathcal{D}(K, p)$.

4 Representations of diagram groups by homeomorphisms

For every two natural numbers $m, n$, let $H(m \to n)$ be the set of all increasing homeomorphisms from the interval $[0, m]$ onto the interval $[0, n]$ of the real line. The union $H = \bigcup_{m,n} H(m \to n)$ is a groupoid where the objects are all intervals $[0, m]$ and morphisms are the functions from $H(m \to n)$. The local groups of this groupoid are the groups $H(m \to m)$.

We are also going to consider the subgroupoid $PLF$ of $H$ whose morphisms are piecewise linear homeomorphisms with finitely many singular points (breakpoints of the derivative). The set of such piecewise linear homeomorphisms between an interval $[0, m]$ and the interval $[0, n]$ will be denoted by $PLF(m \to n)$.

Another useful subgroupoid of $H$ is the groupoid $PLF_2$ whose morphisms are functions from $PLF$ whose singularities occur at dyadic rational points only and whose slopes are integer powers of 2. It is well known that the local groups of this groupoid are isomorphic to R. Thompson’s group $F$ [5].

In this section, we consider natural representations of diagram groupoids of directed 2-complexes in the groupoids $H, PLF, PLF_2$.

Let $K$ be any directed 2-complex, and $\Gamma$ be one of the groupoids $H, PLF, or PLF_2$. To each 2-cell $f$ of $K$ we assign a homeomorphism $T_f$ in $\Gamma(m \to n)$, where $m$ is the length of $[f]$ and $n$ is the length of $[f]$. It is assumed that $T_{f^{-1}} = T_f^{-1}$ for every 2-cell $f$. We call $T_f$ a transition function of $f$. The assignment of a transition function $T_f$ for each 2-cell $f$ is called a transition scheme on $K$ (over $\Gamma$).

We can extend every transition scheme $T$ to the set of all 2-paths in $K$. For an atomic 2-path $\delta = (u, f, v)$, where $u, v$ are 1-paths, $f$ is a 2-cell, we define a function $T_{\delta}$ from $[0, m]$ onto $[0, n]$, where $m = |u|, n = |u|, v|$ as follows:

\[
(t)T_{\delta} = \begin{cases} 
  t, & 0 \leq t \leq |u| \\
  (t-|u|)T_f, & |u| \leq t \leq m - |v| \\
  t + n - m, & m - |v| \leq t \leq m 
\end{cases}
\]

If $\delta = \delta_1 \circ \cdots \circ \delta_r$ is a 2-path decomposed into a product of atomic 2-paths, then by definition $T_{\delta}$ is the composition $T_{\delta_1} \cdots T_{\delta_r}$ (it is clear that the composition is defined). This function is called a transition function of $\delta$. It is easy to see that this function belongs to $\Gamma(m \to n)$.

Let $\delta_1, \delta_2$ be independent atomic 2-paths. Without loss of generality, they can be written as $\delta_1 = (u, f_1, v, f_2)w, \delta_2 = (u, f_1, v, f_2, w)$ for some 2-cells $f_1, f_2$ and 1-paths $u, v, w$. Let $\delta'_1 = (u, f_1, v, f_2)w, \delta'_2 = (u, f_1, v, f_2, w)$. Then the 2-paths $\delta_1 \circ \delta'_2$ and $\delta_2 \circ \delta'_1$ are isotopic, they correspond to the same diagram $\varepsilon(u) + f_1 + \varepsilon(v) + f_2 + \varepsilon(w)$. It follows easily from the definitions that the transition functions of both $\delta_1 \circ \delta'_2$ and $\delta_2 \circ \delta'_1$ are equal to the continuous function that is affine with slope 1 on the intervals that correspond to $u, v, w$, acts as $T_{f_1}$ on
the interval of length $|f_1|$, and acts as $T_{f_2}$ on the interval of length $|f_2|$. This implies that 2-paths with the same diagrams have the same transition function, so any diagram $\Delta$ over $K$ has a unique transition function (depending only on the transition scheme $T$). This function has a nice geometric description.

Suppose that $\Delta$ is a $(u,v)$-diagram over $K$, where $m = |u|, n = |v|$. Each edge of $\Delta$ can be identified with a closed unit interval so each point on the edge has a coordinate from $[0,1]$. Then every 1-path of length $r$ in $\Delta$ is naturally identified with the interval $[0,r]$ so that each point on this path has a coordinate between 0 and $r$. For every cell $\pi$ of $\Delta$ labelled by $f$, we connect each point with coordinate $t$ on the top path of $\pi$ with the point on the bottom path of $\pi$ that has the coordinate $(t)T_{f}$ on it. All these connecting lines can be chosen disjoint since $T_{f}$ is strictly increasing. If we draw the connecting lines for all the cells of $\Delta$, then we have a disjoint “vertical” family of curves (a lamination) each of which consists of finitely many connecting lines. These curves will be called the transition lines of $\Delta$ (some of them may consist of just one point). Now it is easy to describe the function $T_{\Delta}$: for every point $x$ with coordinate $t \in [0,m]$ on the top path of $\Delta$, the value of $(t)T_{\Delta}$ is the coordinate of the end point of the transition line starting at $x$.

Clearly, the transition function of a dipole is the identity. Therefore, equivalent diagrams have the same transition functions. This implies that given a transition scheme $T$ on $K$ over a groupoid $\Gamma$ (which is equal to either $H$ or PLF or PLF$_2$), we have an induced functor $\psi_T$ from the diagram groupoid $D(K)$ into the groupoid $\Gamma$: to any diagram $\Delta$, the functor $\psi_T$ assigns the transition function $T_{\Delta}$ of $\Delta$ which is an element of $\Gamma(m \rightarrow n)$, where $m = |[\Delta]|, n = |[\Delta]|$.

This functor maps every diagram group $D(K,w)$ into the local group $\Gamma([w] \rightarrow |w|)$. Thus for every diagram group, we have a large collection of representations by homeomorphisms of intervals of the real line.

Some directed 2-complexes have transition schemes which induce faithful representations of diagram groups. This is true, for example, in the case of the Dunce hat (see below). Notice that since the group $\text{PLF}(m \rightarrow m)$ is totally orderable [2], every diagram group that has enough representations in $\text{PLF}(m \rightarrow m)$ to separate all elements, is totally orderable as well. However, there are directed 2-complexes $K$ such that the intersection of kernels of all representations induced by transition schemes of $K$ is not trivial. This was mentioned in [10, Section 17] without a proof. Now we are going to give an example.

**Example 4.1.** Let $K$ be the directed 2-complex $(x | x^2 = x^2)$ with one vertex, one edge, and one positive 2-cell $f$. Let $A, B$ be the $(x^5, x^5)$-diagrams over $K$ shown on Figure 2:

![Figure 2](image)

Thus $A$ corresponds to the 2-path $(x,f,x^2) \circ (x^2,f^{-1},x) \circ (x^3,f,1) \circ (x^2,f,x) \circ (x,f^{-1},x^2)$ whereas $B$ corresponds to the 2-path $(1,f,x^3)$. The cells of $A$ are enumerated according to their appearance in the 2-path. Let $T_f$ be any strictly increasing continuous function from $[0,2]$ onto itself. First we show that the transition function $T_A$ from $[0,5]$ onto itself is identical on
[0, 2]. If \( t \in [0, 1] \), then \((t)T_A = t\). So let \( t \in [1, 2] \). Suppose that \((t - 1)T_f \leq 1\). In this case the transition line of the point with coordinate \( t \) on the top of \( A \) will consist of two paths that go through the first and the fifth cell of \( A \). These paths are mirror images of each other. So \((t)T_A = 1 + (t - 1)T_fT_f^{-1} = t\) in this case. Now suppose that \((t - 1)T_f > 1\). This means that the part of the transition line of \( t \) inside the first cell of \( A \) ends on some point on the top path of the second cell of \( A \) and its coordinate there will be some number \( s = (t - 1)T_f - 1 \leq 1 \). The portion of the transition line inside the second cell of \( A \) will connect the point with coordinate \( s \) on the top to the point with coordinate \((s)T_f^{-1}\) on the bottom. Since \( T_f \) is increasing, \((s)T_f^{-1} \leq (1)T_f^{-1} < t - 1 \leq 1\) so the line goes through the common boundary of the second and the fourth cell. Therefore, it also consists of two parts that are mirror images of each other, and so \((t)T_A = t\).

We proved that \( T_A \) is identical on \([0, 2]\). It is obvious that \( T_B \) is identical on \([2, 5]\). Thus \( T_A \) and \( T_B \) commute. Hence the transition function of the commutator \( C = [A, B] \) is identical on the whole interval \([0, 5]\). The diagram \( C \) has 12 cells and it is clearly reduced. So \( C \) is an example of a diagram that is contained in the kernel of any homomorphism induced by a transition scheme.

Notice that the diagram group \( G \) of the complex \( \langle x \mid x^2 = x^2 \rangle \) with base \( x^5 \) is a partially commutative group given by \( \langle a, b, c, d \mid [a, b] = [b, c] = [c, d] = 1 \rangle \). The component \( \text{Sq}(K, x^5) \) has only one vertex \( x^5 \) and four positive edges \( a = (x, f, x^2), b = (x^3, f, 1), c = (1, f, x^3), d = (x^2, f, x) \). There are three 2-cells (squares \( \varepsilon(x) + f + f + \varepsilon(1), \varepsilon(1) + f + \varepsilon(x) + f + \varepsilon(1), \varepsilon(1) + f + f + \varepsilon(x) \)) which correspond to the three commutativity relations.

It is obvious that the groups \( H(n \to n) \) (resp., \( \text{PLF}(n \to n) \)) are isomorphic to each other for all \( n > 0 \). A more traditional notation for the groups \( H(1 \to 1) \) and \( \text{PLF}(1 \to 1) \) is \( \text{Homeo}_+[0, 1] \) and \( \text{PLF}_+[0, 1] \), respectively. Example 4.1 shows that homomorphisms into \( \text{Homeo}_+[0, 1] \) induced by transition schemes do not always separate elements of a diagram group. Thus it is natural to ask the following question (a similar question was mentioned in [10]).

**Problem 4.2.** Is it true that any diagram group is residually \( \text{Homeo}_+[0, 1] \) or even residually \( \text{PLF}_+[0, 1] \)?

Notice that we also do not know if any diagram group is residually \( F \).

Consider the Dunce hat \( \mathcal{H}_0 = \langle x \mid x = x^2 \rangle \). In this case every transition scheme consists of one function \( h: [0, 1] \to [0, 2] \). Let \( h \) be the function \( t \mapsto 2t \), and \( T \) be the corresponding transition scheme. We shall show, in particular, that the induced homomorphism of the R. Thompson group \( F = D(\mathcal{H}_0, x) \) into \( \text{PLF}_2 \) is the well known representation of \( F \) by piecewise linear transformations of the unit interval [5].

**Lemma 4.3.** The homomorphism \( \psi_T \) is injective on \( D(\mathcal{H}_0, x) \).

**Proof.** Consider the following two diagrams over \( \mathcal{H}_0 \):

![Figure 3](image-url)
It was mentioned in [11, Section 1] that these diagrams generate the group \( D(H_0, x) \).

The straightforward computation gives \( (1/2)T_{x_0}T_{x_1} = (1/4)T_{x_1} = 1/8 \) and \( (1/2)T_{x_1}T_{x_0} = (1/2)T_{x_0} = 1/4 \). So the transition functions \( T_{x_0} \) and \( T_{x_1} \) do not commute. This means that the image of \( D(H_0, x) \) under \( \psi_T \) is non-Abelian. It is well known, however, that all proper homomorphic images of the group \( F \) are Abelian [5]. Thus \( \psi_T \) is injective on \( D(H_0, x) \). \(\Box\)

**Lemma 4.4.** The functor \( \psi_T \) is an isomorphism from \( D(H_0) \) onto \( \text{PLF}_2 \).

**Proof.** Let \( h \) be a function from \( \text{PLF}_2(m \to n) \). We need to show that there exists a unique reduced \( (x^m, x^n) \)-diagram \( \Delta \) over \( H_0 \) such that the transition function \( T_\Delta \) coincides with \( h \).

We begin with proving the uniqueness. Let \( \Delta_1, \Delta_2 \) have the same transition function \( h \). Since each equivalence class of diagrams has exactly one reduced representative [13, Theorem 2.5], it suffices to prove that \( \Delta_1 \) and \( \Delta_2 \) are equivalent. Both diagrams are \( (x^m, x^n) \)-diagrams so we can consider the \( (x^a, x^n) \)-diagram \( \Delta_1^{-1}\Delta_2 \). The transition function of it is the identity. Hence the same is true for the transition function of \( \Theta = \Delta_1^{-1}\Delta_1 \Delta_2 \Delta \), where \( \Delta \) is any \( (x^m, x^n) \)-diagram over \( H_0 \). By Lemma 4.3, the diagram \( \Theta \) represents the trivial element of \( D(H_0, x) \) so it is equivalent to \( \varepsilon(x) \). It follows immediately that \( \Delta_1, \Delta_2 \) are equivalent.

Now let us prove the existence. Given a function \( h \) from \( \text{PLF}_2(m \to n) \), let \( a_0 = 0, a_1, \ldots, a_k, a_{k+1} = m \) be the increasing sequence that contains all the singularity points of \( h \) \( (k \geq 0) \). Let \( b_i = (a_i)h \) for all \( 0 \leq i \leq k + 1 \). By \( d \) we denote a smallest natural number such that all the numbers \( 2^da_i, 2^db_i \) are integers \( (0 \leq i \leq k + 1) \). Consider the function \( g \) from \([0, 2^dm]\) onto \([0, 2^dn]\) given by \( (t)g = 2^d \cdot (2^dt)h \). All singularities of \( g \) occur at integer points and all slopes are the same as the corresponding slopes of \( h \). The map \( g \) takes \([2^da_i, 2^da_{i+1}]\) onto \([2^db_i, 2^nd_{i+1}]\) for all \( 0 \leq i \leq k \). On each of these intervals, \( g \) has some slope of the form \( 2^c \), where \( c \) is an integer.

For any \( j \geq 0 \), let \( \Delta_j \) be an \( (x, x^2) \)-diagram over \( H_0 \) that can be defined by induction as follows. We let \( \Delta_0 = \varepsilon(x) \) and \( \Delta_{j+1} = \pi \circ (\Delta_j + \Delta_j) \) for all \( j \geq 0 \). Here \( \pi \) is the \((x, x^2)\)-diagram of one cell. The transition function of \( \Delta_j \) is obviously linear. Now for any number \( r \geq 1 \), we may take the sum of \( r \) copies of \( \Delta_j \). We will denote this sum by \( r \cdot \Delta_j \). It corresponds to the linear function from \([0, r]\) onto \([0, 2^dr]\).

For any \( 0 \leq i \leq k \), if \( d_i \geq 0 \), then we let \( r_i = 2^n(a_{i+1} - a_i) \). If \( d_i < 0 \), then we let \( r_i = 2^n(b_{i+1} - b_i) \). By definition, the diagram \( \Xi \) is the sum of diagrams \( \Xi_i \) \( (0 \leq i \leq k) \), where \( \Xi_i = r_i \cdot \Delta_{d_i} \) if \( d_i \geq 0 \) and \( \Xi_i = (r_i \cdot \Delta_{-d_i})^{-1} \) if \( d_i < 0 \). The transition function of \( \Xi \) is exactly \( g \). Now by \( \Delta \) we will denote the reduced form of the diagram \( (m \cdot \Delta_d) \circ \Xi \circ (n \cdot \Delta_d)^{-1} \), whose transition function is \( h \). \(\Box\)

**Remark 4.5.** One can also consider representations of diagram groups by differentiable functions. For example, consider subgroupoid of \( H \) that consists of \( C^\infty\)-homeomorphisms. It is easy to see that if every function in a transition scheme of a directed 2-complex \( K \) is \( C^\infty \) and has derivative 1 at the ends of its domain then the induced representation maps the diagram groupoid into groupoid of \( C^\infty\)-homeomorphisms. In particular, one can construct many faithful \( C^\infty\)-representations of the R. Thompson group \( F \). The existence of such representations was first proven in [9].

**Remark 4.6.** Notice that one can consider transition schemes of directed 2-complexes over other groupoids. The groupoid should only have an operation of tensor product [15]. The groupoid \( H \) has a natural tensor product: if \( f \in H(k \to l), g \in H(m \to n) \) then the tensor product of \( f \) and \( g \) is the function from \( H(k + m \to l + m) \) which acts on \([0, k] \) as \( f \) and on \([k, k + m] \) as \( g(x - k) + l \). Another example of a groupoid with a tensor product is, of course, the
diagram groupoid of any directed 2-complex with one vertex. In that case the tensor product is the sum of diagrams.

Instead of $\mathbf{H}$, one can consider the groupoid of all homeomorphisms between rectangles of the form $[0, m] \times [0, 1]$ on the plane $\mathbb{R}^2$ ($m \in \mathbb{N}$), that map isometrically $\{0\} \times [0, 1]$ to $\{0\} \times [0, 1]$ and $\{m\} \times [0, 1]$ to $\{n\} \times [0, 1]$. This groupoid also has a natural operation of tensor product. In that case a transition scheme $T$ of a directed 2-complex $\mathcal{K}$ associates with every 2-cell $f$ a homeomorphism from $[0, m] \times [0, 1]$ onto $[0, n] \times [0, 1]$, where $m = \lfloor |f| \rfloor$, $n = \lceil |f| \rceil$. We can prove that the functors induced by these transition schemes separate all elements of any diagram groupoid.

5 The structure of $\mathcal{G}_1$

The complex $\mathcal{H}_1$ is an expansion of the Dunce hat $\mathcal{H}_0 = \langle x \mid x^2 = x \rangle$ with $\nu(x) = 1$. Hence we can apply the results of Section 3. In this section, we are going to give a precise description of the group $\mathcal{G}_1 = \mathcal{D}(\mathcal{H}_1, x)$.

As in Section 3, let $\theta$ be the (natural) retraction from $\mathcal{G}_1 = \mathcal{D}(\mathcal{H}_1, x)$ to $F \cong \mathcal{D}(\mathcal{H}_0, x)$, the group $\mathcal{A}$ be the kernel of this retraction, $\mathcal{K} = \tilde{\mathcal{K}}_x$ be the universal 2-cover of $\mathcal{K} = \mathcal{H}_0$ with base $x$. Our first goal is to give a nice description of $\tilde{\mathcal{K}}$. We are going to construct a certain directed 2-complex $\mathcal{M}$ over $\mathcal{H}_0$, and then show that it satisfies properties U1 and U2' for $\mathcal{K} = \mathcal{H}_0$. Notice that we do not need to describe the labelling morphism $\phi$ because $\mathcal{H}_0$ has only one edge and only one positive 2-cell.

The set of vertices of $\mathcal{M}$ is the set of all dyadic rational numbers in the unit interval $[0; 1]$. The complex $\mathcal{M}$ is a union of complexes $\mathcal{M}_i$ ($i \geq 0$). All these complexes have the same set of vertices. (Warning: this is not a natural filtration of $\mathcal{M}$.)

The complex $\mathcal{M}_0$ is defined by the following process. We start with an edge $\tilde{x}$ that goes from 0 to 1. This is the edge of level 0. Suppose that all edges of level $r \geq 0$ have been constructed. To each edge $e$ of level $r$ we assign two new edges, $e'$ and $e''$. If $e$ goes from $\alpha$ to $\beta$, where $0 \leq \alpha < \beta \leq 1$ are dyadic rational numbers, then $e'$ goes from $\alpha$ to $(\alpha + \beta)/2$ and $e''$ goes from $(\alpha + \beta)/2$ to $\beta$. We also attach a positive 2-cell $f$ of the form $e = e'e''$. By definition, we say that the 2-cell $f$ and the edges $e', e''$ have level $r + 1$.

This inductive process creates $2^r$ edges for each level $r \geq 0$ and the same number of positive 2-cells that have level $r + 1$. This gives us a complex $\mathcal{M}_0$ that consists of all edges and 2-cells constructed during this process. We can draw a part of $\mathcal{M}_0$ in the following picture.

![Diagram](image)

Figure 4.

It shows all 2-cells and all edges of level $\leq 3$. It is convenient to view $\mathcal{M}_0$ as a plane complex. By definition, we set the height of all edges and 2-cells of $\mathcal{M}_0$ to 0. For any $h \geq 1$, assuming that $\mathcal{M}_{h-1}$ is already constructed, we consider all pairs of consecutive edges $e', e''$ in $\mathcal{M}_{h-1}$ (this means that $e'e''$ is a 1-path). For each of these pairs, if $\mathcal{M}_{h-1}$ contains no 2-cells of the
form $e = e'e''$, where $e$ is an edge, we add a new edge $e$ and a new 2-cell of the form $e = e'e''$. We set the height of all edges and 2-cells added in this way as $h$. The new directed 2-complex is denoted by $\mathcal{M}_h$.

By definition, $\mathcal{M} = \bigcup_{h \geq 0} \mathcal{M}_h$ is a rooted 2-tree with root $\tilde{x}$ so property U1 holds.

Let $E$ be the set of edges of $\mathcal{M}$. It is easy to see that for each edge $e \in E$, there is a unique 2-cell of the form $e = e'e''$, where $e', e'' \in E$. This cell depends on $e$ only so we denote it by $\pi_e$. Thus given an edge $e$, we have two functions $e \mapsto e'$ and $e \mapsto e''$ from $E$ to itself. Moreover, for any pair of consecutive edges of $\mathcal{M}$ we have a unique edge $e$ in $\mathcal{M}$ such that these consecutive edges are exactly $e'$, $e''$. These properties can also be used to define $\mathcal{M}$ axiomatically. They immediately imply that $\mathcal{M}$ satisfies U2'. So $\mathcal{M}$ is the universal 2-cover of the Dunce hat $\mathcal{H}_0$ with base $x$ according to [13, Remark 8.2]. Notice also that one can regard the edges of $\mathcal{M}$ as elements of the one-generated free Cantor algebra (see [14]).

From now on, we will denote the universal 2-cover of the Dunce hat with base $x$ by $\tilde{\mathcal{K}}$. As in the Section 3, $\mathcal{K} = \tilde{\mathcal{K}}_{\nu, x}$ is an expansion of $\tilde{\mathcal{K}}$ obtained by adding a leaf to each edge of $\tilde{\mathcal{K}}$. Theorem 3.1 implies:

**Corollary 5.1.** The kernel $\mathcal{A}$ of the retraction $\theta$ of $G_1$ onto $F$ is the diagram group of the directed complex $\mathcal{K}$ with base $\tilde{x}$.

Theorem 3.3 gives the following partially commutative presentation of $\mathcal{A}$. We will denote the generators of $\mathcal{A}$ by $a(z)$ instead of $a(z, i)$ as in Theorem 3.3 because in this case $i = 1$ always. By definition of $\mathcal{K}$, the endpoints of every edge $z$ in $\mathcal{K}$ are dyadic rational numbers with $0 \leq \iota(e) < \tau(e) \leq 1$, and for every two dyadic numbers $0 \leq \lambda < \mu \leq 1$, there is an edge connecting $\lambda$ with $\mu$ (in fact there are countably many such edges of different heights). Thus every edge in $E$ corresponds to an open interval $(\iota(e), \tau(e))$ on the real line. We claim that (by Theorem 3.3) symbols $a(z_1), a(z_2)$ commute if and only if the corresponding intervals do not intersect. Indeed, if $z_1$ and $z_2$ occur in the same 1-path $q$ in $\mathcal{K}$ from $0 = \iota(\tilde{x})$ to $1 = \tau(\tilde{x})$, then we can assume without loss of generality that $z_1$ is to the left of $z_2$ in that path. This means $\tau(z_1) \leq \iota(z_2)$ and the intervals do not intersect. Conversely, if the two open intervals do not intersect, then without loss of generality $\tau(z_1) \leq \iota(z_2)$. Then there exists a path in $\mathcal{K}$ (of length at most 5) that connects vertices 0 and 1 and contains $z_1$ and $z_2$, so the symbols $a(z_1), a(z_2)$ commute.

So we have the following structural description of the subgroup $\mathcal{A}$ of $G_1$.

**Theorem 5.2.** For every open subinterval $P$ of $(0, 1)$ with dyadic rational endpoints let $Z(P)$ be a countable set of symbols. Then $\mathcal{A}$ is generated by the union of all $Z(P)$ subject to commutativity relations: two symbols commute if and only in the corresponding intervals do not intersect.

Since $G_1$ is a semi-direct product of $\mathcal{A}$ and R. Thompson’s group $F$, it remains to describe the action of $F$ on $\mathcal{A}$. We know that the generators of $\mathcal{A}$ are in one-to-one correspondence with edges of $\mathcal{K}$. Now to every edge of $\mathcal{K}$, we assign a piecewise linear function. In order to do that, we will use transition functions defined in the previous section. We shall always assume that the transition scheme $T$ on $\mathcal{H}_0$ consists of linear functions.

We need some useful technical lemma. Recall that $\phi$ denotes the labelling function from $\mathcal{K}$ to $\mathcal{H}_0$. Applying $\phi$ to a diagram, means that we replace all its edge labels by $x$. (One can forget about the labels of cells because they can be easily recovered.) We also recall that vertices of $\mathcal{K}$ are dyadic rational numbers from $[0, 1]$.
Lemma 5.3. 1. Suppose that $\Delta$ is a diagram over $\mathcal{H}_0$ that has cells of the form $x^2 = x$ only. Then the transition function $T_\Delta$ is linear on each edge of the top path of $\Delta$.

2. Let $\tilde{\Delta}$ be a $(\tilde{q}, \tilde{x})$-diagram over $\widetilde{K}$. Suppose that $\tilde{o}$ is a vertex on $\tilde{q}$. Let $\Delta = \phi(\tilde{\Delta})$ and let $o$ be the image of $\tilde{o}$ in $\Delta$. Then $o$ is connected by a transition line in $\Delta$ to the point on $[\Delta]$ that has coordinate $\tilde{o}$.

Proof. To prove part 1, we proceed by induction on the number of cells in $\Delta$. If $\Delta$ has no cells, then the transition function of it is identical. Suppose that $\Delta$ has cells. In this case there is a cell $\pi$ in $\Delta$ such that $[\pi]$ is contained in $[\Delta]$. Let $\Delta'$ be a subdiagram in $\Delta$ that is obtained from $\Delta$ by deleting $\pi$ together with $[\pi]$. If the edge $e$ on the top of $\Delta$ is not contained in $[\pi]$, then the transition functions of $\Delta$ and $\Delta'$ coincide on the unit interval $e$. Hence the result follows by the inductive assumption. Now suppose that the edge $e$ is contained in $[\pi]$. Since the transition function of $\pi$ is linear, the top of $\pi$ maps $e$ linearly to a half of the edge $e' = [\pi]$. The transition function of $\Delta'$ is linear on $e'$ by the inductive assumption. It remains to compose two linear functions.

To prove part 2, we can first assume that $\tilde{\Delta}$ is reduced. By the properties of $\widetilde{K}$, the diagram $\Delta$ is also reduced. So according to [11, Example 2], it can be decomposed as $\Delta_+ \circ \Delta_-$, where $\Delta_+$ has only cells labelled by $x = x^2$ and $\Delta_-$ has only cells labelled by $x^2 = x$. In addition, all vertices of $\Delta$ belong to the path $r = [\Delta_+] = [\Delta_-]$. The transition line of $o$ is thus contained in $\Delta_-$.

Since $\phi(\tilde{\Delta}) = \Delta$, the diagram $\Delta$ has an induced decomposition $\Delta = \Delta_+ \circ \Delta_-$ and the point $\tilde{o}$ belongs to the path $\tilde{r}$ that cuts $\tilde{\Delta}$ into the two subdiagrams. For any cell of $\Delta_-$, its top path has length 2 and its bottom path has length 1. Since the bottom of $\Delta_-$ is $\tilde{x}$, it is easy to see by induction on the number of cells that $\Delta'$ is a diagram over the directed 2-complex $\mathcal{M}_0$. One can lift the transition line of $o$ in $\Delta_-$ into $\Delta$. We just need to show that this line connects $\tilde{o}$ with the point on $\tilde{x}$ that has the coordinate $\tilde{o}$. This is obvious if $\Delta_-$ has no cells. Let us proceed by induction on the number of cells in $\Delta$. Let $\pi$ be a cell in $\Delta_-$ such that $[\pi]$ is contained in $[\Delta_-]$. Let $\tilde{\Delta}'$ be a subdiagram in $\Delta_-$ obtained by deleting $\pi$ together with $[\pi]$.

If $\tilde{o}$ is not an inner point of $[\pi]$, then $\tilde{o}$ is contained in $\Delta'$. Hence the inductive assumption can be applied to $\tilde{\Delta}'$.

Now let $\tilde{o}$ be an inner point of $[\pi]$. In this case $[\pi]$ is a path of length 2 of the form $\tilde{e}_1 \tilde{e}_2$, where $\tilde{e}_1$ connects some vertex $\tilde{o}_1$ with $\tilde{o}$ and $\tilde{e}_2$ connects $\tilde{o}$ with some vertex $\tilde{o}_2$. The inductive assumption applied to $\tilde{\Delta}'$ allows us to conclude that the points $\tilde{o}_j$ are connected by their transition lines in $\tilde{\Delta}'$ to the points on $\tilde{x} = [\Delta'] = [\Delta]$ with coordinates $\tilde{o}_j$ ($j = 1, 2$). By definition of the complex $\mathcal{M}_0$, one has $\tilde{o} = (\tilde{o}_1 + \tilde{o}_2)/2$. The transition line of $\tilde{o}$ connects it first to the midpoint of the edge $[\pi]$. Applying part 1 to $\tilde{\Delta}'$, we see that the transition function of $\tilde{\Delta}'$ has to be linear on the edge $[\pi]$. So the midpoint of this edge will be taken to the midpoint of the interval between the images of $\tilde{o}_1$ and $\tilde{o}_2$ on $\tilde{x}$. By the inductive assumption, the latter points have coordinates $\tilde{o}_1$, $\tilde{o}_2$. Thus the midpoint is $\tilde{o}$, as desired.

Let $\tilde{e}$ be a boundary of $\tilde{K}$. First we choose any 1-paths $\tilde{q}', \tilde{q}''$ in $\tilde{K}$ such that the 1-path $\tilde{q}' \tilde{e} \tilde{q}''$ connects 0 and 1. By Properties T2 and T3, there exists a unique reduced $(\tilde{q}' \tilde{e} \tilde{q}'', \tilde{x})$-diagram over $\tilde{K}$. If we apply $\phi$ to it, then we get a diagram $\Delta$ over $\mathcal{H}_0$. The top path of $\Delta$ is decomposed as $q' \tilde{e} q''$ in a natural way, that is, $e$ is the image of $\tilde{e}$. Let $e$ be the $k$th edge of $[\Delta]$ and let $n = |q' \tilde{e} q''|$. The points with coordinates $k - 1$ and $k$ (the endpoints of $e$) are taken to some dyadic rationals $\lambda$ and $\mu$ via the transition function of $\Delta$. It follows from Lemma 5.3 that these numbers are exactly the endpoints of $\tilde{e}$ in $\tilde{K}$. Consider the function $g: [0, 1] \rightarrow [0, 1]$ defined by
Suppose that we change the paths $\tilde{q}^\prime$, $\tilde{q}^\prime\prime$. Then $\Delta$ also changes to some diagram, which will be equivalent to the product of some diagram $\Psi = \Psi_1 + \varepsilon(x) + \Psi_2$ and $\Delta$, where $\Psi_1$, $\Psi_2$ are spherical. The transition function of $\Psi$ takes $[0, k-1]$ and $[k, n]$ to themselves and it is identical on $[k-1, k]$. So the restriction of $T_\Delta \Psi$ on $[k-1, k]$ will coincide to the restriction of $T_\Delta$ on $[k-1, k]$. Thus we can say that to any edge $\tilde{e}$ of $\tilde{K}$ we assign a unique function $g_\tilde{e}$ from $[0, 1]$ to itself. By Lemma 4.4, this is a piecewise linear homeomorphism from $[0, 1]$ to $[\lambda, \mu]$ with dyadic $\lambda$, $\mu$, finitely many dyadic singularities and all slopes of the form $2^k$, $k \in \mathbb{Z}$.

Let us denote by $\Phi$ the set of all such functions from $[0, 1]$ to itself. It is easy to see that $\Phi$ is a semigroup under composition. We are going to show that for any $h \in \Phi$, there exists a unique edge $\tilde{e}$ of $\tilde{K}$ such that $h = g_\tilde{e}$.

**Lemma 5.4.** There is a natural one-to-one correspondence between edges of $\tilde{K}$ and continuous piecewise linear functions from $[0,1]$ onto a subinterval in $[0,1]$ with dyadic rational endpoints. All these functions have singularities at finitely many dyadic rational points only and all slopes of these functions are integer powers of 2.

**Proof.** Let $h \in \Phi$. We can write $(0)h = l/2^d$, $(1)h = m/2^d$ for some integers $l$, $m$, $d$. Let us consider the function

$$(t)h = \begin{cases} 
  t/2^d, & 0 \leq t \leq l \\
  (t-l)h, & l \leq t \leq l+1 \\
  1 + (t-n)/2^d, & l+1 \leq t \leq n
\end{cases}$$

from $[0, n]$ onto $[0,1]$, where $n = 2^d + l + 1 - m$. It clearly belongs to $\text{PLF}_2(n \rightarrow 1)$. By Lemma 4.4, there exists a reduced $(x^n, x)$-diagram $\Delta$ over $\mathcal{H}_0$ that has $h$ as its transition function. Using [13, Lemma 8.3], we can lift $\Delta$ to $\tilde{K}$. This gives us a $(\tilde{q}, \tilde{x})$-diagram $\tilde{\Delta}$. Since $m \leq 2^d$, one has $|\tilde{q}| = n \geq l + 1$. So let $\tilde{e}$ be the $(l+1)$th edge of $\tilde{q}$. Then $g_\tilde{e} = h$. Indeed, if we replace all edges of $\tilde{\Delta}$ by $x$, we get the diagram $\Delta$ by the definition of the lift. Its transition function is $\tilde{h}$, the restriction of this function to the unit interval $[l, l+1]$ is the function $h$ with argument shifted by $l$.

It remains to prove the uniqueness. Suppose that $\tilde{e}$, $\tilde{e}'$ are two edges of $\tilde{K}$ such that $g_\tilde{e} = g_\tilde{e}' = h$. By Lemma 5.3, both edges have the same endpoints $(0)h$, $(1)h$. So they are homotopic in $\tilde{K}$ by property T2. One can choose a reduced $(\tilde{e}', \tilde{e})$-diagram $\tilde{\Gamma}$ over $\tilde{K}$. Let us use any 1-path $\tilde{q}$ in $\tilde{K}$ from 0 to 1 that contains $\tilde{e}$ and some $(\tilde{q}, \tilde{x})$-diagram $\tilde{\Delta}$ over $\tilde{K}$. Then one can take the diagram $\tilde{\Delta}' = (\tilde{\Psi}_1 + \tilde{\Gamma} + \tilde{\Psi}_2)\tilde{\Delta}$ over $\tilde{K}$ whose top path contains $\tilde{e}'$, where $\tilde{\Psi}_1$, $\tilde{\Psi}_2$ have no cells. Now we replace all edge labels in $\tilde{\Delta}$, $\tilde{\Delta}'$, $\tilde{\Gamma}$, $\tilde{\Psi}_j$ by $x$ and get diagrams $\Delta$, $\Delta'$, $\Gamma$, $\Psi_j$ ($j = 1, 2$), respectively. Notice that $\Gamma$ is reduced because so was $\tilde{\Gamma}$. The transition function of $\Delta'$ is the composition of the transition function of $\Psi_1 + \Gamma + \Psi_2$ and $\Delta$. If we restrict these functions on the corresponding unit interval, then we see that $g_\tilde{e}'$ will be a composition of $g$ and $g_\tilde{e}$, where $g$ is the transition function of $\Gamma$. Thus $g$ must be identical so $\Gamma$ has no cells by Lemma 4.3. But this means that $\tilde{e}$, $\tilde{e}'$ coincide. \hfill \Box

Now it is natural to change the notation for the generators of $A$. We replace the symbol $a(\tilde{e})$ by the symbol $\alpha_{h_1}$, where $h = g_\tilde{e}$. So $A$ is generated by the symbols of the form $\alpha_{h_i}$, where $h$ runs over $\Phi$. According to Theorem 5.2, two generators $\alpha_{h_1}$, $\alpha_{h_2}$ commute if and only if the images of $h_1$ and $h_2$ have disjoint interiors.

Now we are ready to describe the action of $F$ on $A$. We can think about the $\alpha_{h_i}$’s as of elements in $G_1$. The group $F$, as a subgroup of $G_1$, acts on $A$ by conjugation. Recall that $F$ is...
Thus three factors. The first of them is $\Delta$−diagram $\epsilon$ function of $\Gamma$, which is $g$ of this form represents a generator of $\mathcal{A}$. To find which one, we must take the first factor and lift it into $\tilde{K}$ according to Lemma [13, Lemma 8.3]. The top edge of the $(x,x)$-cell will be mapped onto some edge $\tilde{e}$ of $\tilde{K}$. This will tell us what will be the generator of $\mathcal{A}$ we want to find. In our case lift of the first factor is a product of $\Gamma^{-1}$ and $\Delta^{-1}$. So the transition function $g_{\tilde{e}}$ of $\tilde{e}$ is the product of the transition function that corresponds to $\Delta$, which is $T_{\tilde{e}} = h$, and the transition function of $\Gamma$, which is $g$. So the composition of the corresponding functions is $hg$, as desired. Thus $g^{-1}\alpha_{hg} = \alpha_{hg}$.

Now we finally have the following structural description of $\mathcal{G}_1$.

**Theorem 5.6.** The universal diagram group $\mathcal{G}_1$ is isomorphic to the semi-direct product of the group $\mathcal{A}$ and the R. Thompson’s group $F$. Here $\mathcal{A}$ is a partially commutative group generated by symbols of the form $\alpha_h$, where $h$ runs over $\Phi$, the set of all continuous piecewise linear functions from $[0,1]$ into $[0,1]$ with finitely many singularity points, where $0f$, $(1)f$ and all the singularity points of $h$ are dyadic rational and all slopes are of the form $2^k$, $k \in \mathbb{Z}$. Two symbols $\alpha_{h_1}$ and $\alpha_{h_2}$ commute if and only if the interiors of the images of $h_1$ and $h_2$ are disjoint. The group $F$, as the group of piecewise linear functions, is a subset in $\Phi$. Its action on $\mathcal{A}$ is defined as follows: $g^{-1}\alpha_{hg} = \alpha_{hg}$.

### 6 Orderability

Now we are finally in a position to prove the main result of the paper:

**Theorem 6.1.** Any diagram group of any directed 2-complex is totally orderable.

**Proof.** It is well known that the property of a group to be orderable is local [16]. This means that the group is orderable if and only if all its finitely generated subgroups are orderable. If we have finitely many diagrams as elements of some diagram group, then we can find a finite subcomplex in the original directed 2-complex such that all these diagrams will be diagrams over a finite directed 2-complex, that is, they will be elements of some countable diagram group $H$. Since $\mathcal{G}_1$ is universal [13, Theorem 5.6], the group $H$ embeds into $\mathcal{G}_1$. If we can prove that $\mathcal{G}_1$ is orderable, then $H$ will be also orderable. So any diagram group will be orderable.
By Theorem 5.6, $G_1$ is a semi-direct product of $A$ and $F$. The group $A$ is partially commutative and thus every total order on the set of generators of $A$ extends to a total order of $A$ by [7]. The group $F$ is also orderable [2]. It is easy to see that to show that $G_1$ is orderable, it suffices to show that the action of $F$ on $A$ respects a total order on $A$.

There exists a well known natural total order on the set $\Phi$ [2]. Let $h_1, h_2 \in \Phi$. Let $c$ be the least upper bound of the set $\{ t \mid h_1 equals h_2 on [0, t] \}$. If $h_1 \neq h_2$, then $c < 1$. Both functions have the same value at $c$ and they are affine on a small right neighbourhood of $c$. So they have different right derivatives at $c$. Denoting these derivatives by $k_1$ and $k_2$, respectively, we set $h_1 < h_2$ whenever $k_1 < k_2$ and $h_2 < h_1$ whenever $k_2 < k_1$. Since $F$ is a subset in $\Phi$, this induces also the order on $F$. This order on $F$ is stable under multiplication from both sides [2]. We also need to notice that $h_1 < h_2$ implies $h_1g < h_2g$ whenever $h_1, h_2 \in \Phi, g \in F$. This is obvious because all functions from $\Phi$ are increasing.

We can introduce a total order on the set of generators of $A$ in the following natural way: $\alpha_{h_1} < \alpha_{h_2}$ if and only if $h_1 < h_2$ in $\Phi$. This order induces a total order on $A$ as described in [7]. Let

$$A = A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n \supseteq \cdots \quad (5)$$

be the lower central series of $A$: $A_{n+1} = [A_n, A]$ for all $n \geq 1$. It is known [7] that the intersection of all the $A_n$’s is trivial for any partially commutative group. Following [7], we define basic commutators of weight $n \geq 1$ by induction and also introduce a total ordering on the set of basic commutators. By definition, basic commutators of weight 1 are the elements $\alpha_h$ ($h \in \Phi$). We already have a total order on that set. Now let $n > 1$ and suppose that we have a definition of basic commutators of weight $< n$ and also a total order $<$ on them. Basic commutators of weight $n$ will be bigger than basic commutators of weight $< n$ with respect to the order we define. Any basic commutator of weight $n > 1$ is a commutator of the form $[c', c'']$, where

- $c', c''$ are basic commutators of weight $< n$ and the sum of their weights is $n - 1$,
- $[c', c'']$ is not equal to 1 in $A$.
- $c' > c''$ in the order we have,
- if $c' = [c_1, c_2]$ for some basic commutators $c_1, c_2$, then $c'' \geq c_2$.

Basic commutators of weight $n$ are ordered lexicographically: $[c_1, c_2] < [c_3, c_4]$ if and only if either $c_1 < c_2$, or $c_1 = c_2$ and $c_3 < c_4$.

By [7], basic commutators of weight $n$ freely generate the Abelian group $A_n/A_{n+1}$. These groups can be totally ordered in a standard way. Say, if $Y$ is the basis of a free Abelian group totally ordered with $<$, then any nontrivial element $g$ of the group generated by $Y$ is a (uniquely defined) product of powers of elements of $Y$ with non-zero exponents: $g = y_1^{k_1} \cdots y_m^{k_m}$, where $y_1 < \cdots < y_m$. By definition, $g$ is positive whenever the exponent on $y_m$ is positive. By [7, Theorem 3.1], there exists a unique total order on $A$ such that the canonical projections $\pi_n: A_n \to A_n/A_{n+1}$ are increasing. To find out whether an element $g \in A, g \neq 1$, is positive, it suffices to find the lowest $n$ such that $g \notin A_{n+1}$. Since $g \in A_n$, one can project $g$ onto a nontrivial element $\pi_n(g) \in A_n/A_{n+1}$. One has $g > 1$ if and only if $\pi_n(g) > 1$ in the corresponding group.

Notice that the word problem is decidable in all partially commutative groups (see, for example, [6]) so one can effectively decide if an element of $A$ is positive.

It remains to prove that this order $<$ on $A$ is stable under the action of $F$. We know that any element $g \in F$ induces a permutation on the set of generators $\{ \alpha_h \mid h \in \Phi \}$ and preserves...
the order on this set. Since \([a, b]^{g} = [a^{g}, b^{g}]\) for any commutator, the action by \(g\) also preserves
the set of basic commutators and the order on that set. This implies that the order on \(A\) is
preserved as well.

We would like to add a few more applications of the results of the preceding sections. First
we have the following surprising result.

**Theorem 6.2.** Every diagram group is residually countable.

*Proof.* The proof of [13, Lemma 5.3] shows that any diagram group embeds into the diagram
group \(\mathcal{D}(\mathcal{H}_{\alpha}, x)\), where \(\mathcal{H}_{\alpha}\) is the expansion of the Dunce hat with \(\nu(x) = \alpha\) (here \(\alpha\) is a cardinal
number). Any diagram over \(\mathcal{H}_{\alpha}\) involves only finitely many leaf cells. So for any reduced
nontrivial diagram \(\Delta\) over \(\mathcal{H}_{\alpha}\), we have a homomorphism from \(\mathcal{D}(\mathcal{H}_{\alpha}, x)\) to the countable group
\(\mathcal{G}_{n} = \mathcal{D}(\mathcal{H}_{n}, x)\) (here \(n\) is the number of leaf cells in \(\Delta\)) that collapses all leaves not involved in
\(\Delta\) to edges. This homomorphism takes \(\Delta\) to itself, so it separates \(\Delta\) from 1.

We know that there are simple non-trivial diagram groups: the derived subgroup of the
group \(F\) is an example (see [11, Theorem 26]). Other examples include the derived subgroups
of all generalized Thompson groups \(F_{n}\) (see [4] or [13] for definitions). It is known [3] that all
generalized Thompson groups are embeddable into each other. In particular, all of them are
subgroups in \(F\). Here is our result.

**Theorem 6.3.** Any simple subgroup of a diagram group embeds into R. Thompson’s group \(F\).

*Proof.* Let \(G\) be a simple subgroup of a diagram group. By Theorem 6.2, \(G\) is countable.
Therefore it is embeddable into \(\mathcal{G}_{1}\) by [13, Theorem 5.6]. Since \(\mathcal{G}_{1}\) is an extension of \(A\) by \(F\),
the group \(G\) either embeds into \(A\) or it embeds into \(F\). The first option is impossible because
every partially commutative group is residually nilpotent [7]. Hence \(G\) is a subgroup of \(F\).

It is natural to ask the following question.

**Problem 6.4.** Is it true that any simple subgroup of \(F\) is isomorphic to the derived subgroup
of \(F_{n}\) for some \(n\)?

One more corollary deals with groups representable by diagrams that do not contain non-
Abelian free subgroups. We know that \(F\) is one of such groups [5]. The following question is
still open.

**Problem 6.5.** Is it true that every group representable by diagrams and containing no non-
Abelian free subgroups embeds into \(F\)?

However, the following result is a step in this direction.

**Theorem 6.6.** Let \(G\) be a group representable by diagrams that does not contain non-Abelian
free subgroups. Then \(G\) is an extension of some free Abelian group \(A\) by a subgroup \(B\) of the
group \(F\).

This theorem will follow from the next lemma.

**Lemma 6.7.** Every subgroup of a partially commutative group is either Abelian or contains a
free non-Abelian subgroup.
Proof. Indeed, it suffices to prove the statement for subgroups of finitely generated partially commutative groups. By [6], every finitely generated partially commutative group has a finite index subgroup embeddable into a Coxeter group.

By [17], every finitely generated subgroup of a Coxeter group is either virtually Abelian or contains a finite index subgroup with a non-Abelian free factor. Having a free non-Abelian factor implies having a free non-Abelian subgroup.

Since partially commutative groups are orderable, they satisfy the centralizer property [16], that is \([a^n, b] = 1\) implies \([a, b] = 1\) for every pair of elements \(a, b\). This immediately implies that every virtually Abelian subgroup of a partially commutative group is Abelian.

Now it is easy to complete the proof of Theorem 6.6.

Proof. Let \(H\) be a subgroup of a diagram group. Suppose that all free subgroups of \(G\) are Abelian. As above, we can regard \(G\) as a subgroup of \(G_\alpha = \mathcal{D}(\mathcal{H}_\alpha, x)\) for some cardinal \(\alpha\). By Theorem 3.3, \(G_\alpha\) is an extension of a partially commutative group \(A_\alpha\) by \(F\). Then \(G\) is an extension of \(G \cap A_\alpha\) by a subgroup of \(F\). By Lemma 6.7, \(G \cap A_\alpha\) is Abelian. It remains to apply [11, Theorem 16] that says that every Abelian subgroup of a diagram group is free Abelian.

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