Dimensional analysis and Rutherford scattering

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Abstract
Dimensional analysis, and in particular the Buckingham Π theorem is widely used in fluid mechanics. In this paper we obtain an expression for the impact parameter from Buckingham’s theorem and we compare our result with Rutherford’s original discovery of the early twentieth century.

(Some figures may appear in colour only in the online journal)

1. Rutherford scattering

In his work on helium nuclei scattered by a heavy nucleus, Rutherford had to consider how close this beam of nuclei approaches to the heavy target. Obviously he was not able to see what happens at microscopic scales when a single helium atom is scattered by the effect of a nucleus of the target, so he became interested in macroscopic measurable quantities. One of these quantities is the differential cross section \(\sigma\), defined as the number of scattered particles per unit of time in a solid angle and divided by the incident intensity of the alpha particles beam [1, 2].

To compute \(\sigma\), he needed to find a relation between the impact parameter \(s\) of a single alpha particle and the scattering angle \(\Phi\) (see figure 1). \(\sigma\) depends on the impact parameter and the scattering angle as follows:

\[
\sigma = \frac{s}{\sin \Phi} \left| \frac{ds}{d\Phi} \right|.
\]  

(1)

Using the orbit equation of a particle under the effect of a central force, like the Coulomb interaction, it is possible to obtain \(s = s(\Phi)\). Following this procedure the desired relation, after some algebra, turns out to be [2]:

\[
s = \frac{kZZ' e^2}{2E} \cot \frac{\Phi}{2}.
\]  

(2)
Figure 1. Rutherford scattering scheme. An alpha particle is fired close to the target, then due to the electrostatic interaction it is scattered and changes its trajectory by an angle $\Phi$.

Here $Z$ and $Z'$ are the atomic numbers of the heavy nucleus and the incident alpha particle, respectively, $k$ is the force constant of electrostatic Coulomb interaction, $E = m_\alpha v^2/2$ is the energy of the alpha particle and $e$ is the electron charge.

2. Dimensional analysis

In 1968 the scientific community established the international system of units (IS). These comprised the so-called base units, where any possible unit that might describe any physical process could be derived from them. In physics a unit system such as the IS is required for dimensional analysis, which plays a fundamental role in many of its fields. Dimensional analysis allows one to verify that the equations describing physical phenomena are homogeneous.

As a simple example, consider the Bernoulli equation, which describes the energy conservation law for a inviscid fluid:

$$P_1 + \frac{1}{2}\rho v_1^2 + \rho gh_1 = P_2 + \frac{1}{2}\rho v_2^2 + \rho gh_2.$$  \hspace{1cm} (3)

The subindex 1 or 2 indicates that the quantity is taken in two different points inside the fluid. $P$ is the local pressure, $v$ the velocity, $\rho$ is the fluid density and $h$ is the local height of the fluid.

Using dimensional analysis it is easy to see that the dimensions of the above equation are given by $ML^{-1}T^{-2}$, where $M$ is a dimension of mass, $L$ the dimension of length and $T$ the dimension of time. This holds for both sides of (3), satisfying the requirement of dimensional homogeneity; the interesting feature of this kind of mathematical expression is that they can be rewritten in a dimensionless form by multiplying by a proper factor that cancels out any possible dimension in the equation.
Through this process some characteristic dimensionless numbers may appear, and they usually acquire some relevance depending on their physical significance. Let us consider the Navier–Stokes equations for an incompressible fluid affected by a gravitational field $\vec{g}$:

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla) \vec{u} = -\frac{1}{\rho} \nabla P + \vec{g} + \nu \nabla^2 \vec{u}. \quad (4)$$

The characteristic dimensionless numbers of this general problem can be obtained easily by transforming (4) into its dimensionless form by considering the dimensionless variables $\vec{r}' = \vec{r}/L, \; t' = tU/L, \; \vec{u}' = \vec{u}/U, \; P' = P/\rho U^2$, so that we get:

$$\frac{\partial \vec{u}'}{\partial t'} + (\vec{u}' \cdot \nabla') \vec{u}' = -\nabla' P' - Fr - 2(g) + Re^{-1} \nabla^2 \vec{u}'. \quad (5)$$

Where the desired non-dimensional parameters are the Reynolds and the Froude numbers, $Re = UL/\nu$ and $Fr = U/\sqrt{gL}$, respectively. But an alternative procedure to obtain dimensionless parameters without handling the equations of motion—which is more in the spirit of the Buckingham $\Pi$ theorem—is as follows.

3. Buckingham $\Pi$ theorem

The general framework states that for a given physical problem with $(q_1, q_2, \ldots, q_n)$ fundamental variables involved, such that there is a functional relationship of the form $f(q_1, q_2, \ldots, q_n) = 0$, there must exist $(n - j)$ dimensionless parameters called the $\Pi$ numbers, which can be constructed from combinations of the $n$ characteristic variables of the problem [3], where $j$ is the rank of the corresponding dimensional matrix. Typically, the number of fundamental dimensions or base units required to write the $n$ variables of the problem is the same as $j$. From the set of the $n$ variables one must choose $j$ variables as the ‘repeating variables’ that must appear in all the non-dimensional $\Pi$ numbers. It is important to remark that these repeating variables must have different dimensions between each other.

Now each $\Pi_i$ dimensionless number ($i = 1, \ldots, n - j$) is formed from the product of the $j$ repeating variables and one of the $(n - j)$ remaining fundamental variables, where each one of the repeating variables must have an unknown exponent:

$$\Pi_i = V_i \prod_{k=1}^{j} V_{Rk}^{a_k}. \quad (6)$$

Here $V_i$ is the fundamental variable associated with $\Pi_i$ and it belongs to the set of the $(n - j)$ remaining variables. $V_{Rk}$ is the index over all the repeating variables, however it is important to note from (6) that the choice of the $\Pi$ parameters is not unique [3]. Finally, $a_k$ represents the corresponding exponent to be determined in such a way that all the parameters $\Pi_i$ should be non-dimensional. In this way is possible to rewrite $f(q_1, q_2, \ldots, q_n) = 0$ as:

$$F(\Pi_1, \Pi_2, \ldots, \Pi_{n-j}) = 0. \quad (7)$$

4. Application to Rutherford scattering

The best way to illustrate the power of the Buckingham $\Pi$ theorem is through an example. In this paper we choose a particular situation where dimensional analysis has not been used before, the deduction of the functional relation $s = s(\Phi)$ for Rutherford scattering.

First we select the set of fundamental variables of the problem as $(v, s, \phi, m, F)$, where $F$ is the force between the heavy nucleus and the incident alpha particle for a typical distance. In this case $j = 3$ and from the above set we choose $(v, s, F)$ to be the repeating variables. The theorem states that there are only two dimensionless parameters, the first is the scattering
angle itself since this is already a non-dimensional quantity ($\Pi_1 = \Phi$), and the second can be founded on (6), using the three repeating variables chosen previously and $V_i = m_a$:

$$\Pi_2 = m_a v^a s^b F_e^c,$$  \hfill (8)

In terms of the three fundamental dimensions required in the problem this turns out to be:

$$\Pi_2 = M_1^{1+c} L^{a+b+c} T^{-a-2c}.$$  \hfill (9)

The requirement for $\Pi_2$ to be dimensionless gives a simple linear system of equations for the unknown exponents ($a$, $b$, $c$). The solution is quite straightforward and gives $a = 2$, $b = -1$ and $c = -1$. Now, replacing these values of the exponents into (8) we get:

$$\Pi_2 = \frac{m_a v^2}{s F_e}.$$  \hfill (10)

Now note that a typical value for the electric force is given by:

$$F_e = \frac{kZZ' e^2}{s^2}.$$  \hfill (11)

Replacing this into the expression for $\Pi_2$ gives:

$$\Pi_2 = \frac{m_a v^2 s}{kZZ' e^2}. $$  \hfill (12)

Finally, using (7) it is easy to see that:

$$\Pi_2 = f(\Pi_1).$$

This can then be used to obtain an expression for $s$, as follows:

$$s = \frac{kZZ' e^2}{m_a v^2 f(\Phi)}. $$  \hfill (13)

This is in perfect accordance with (2).

5. Conclusion

Dimensional analysis (in particular the Buckingham $\Pi$ theorem) is mainly used in fluid dynamics. However, in spite of this we have shown it used simply but powerfully in a classical scattering problem, in which it was demonstrated that a correct functional form of the impact parameter can be obtained without using Newton’s laws or any other conservation principle. However, recall that the choice of parameters, even in this particular physical problem, is not unique; so one might be tempted to do the same with other particular arrangements of variables to obtain a new set of parameters for those cases where the Buckingham $\Pi$ theorem does not necessarily lead to the correct answer.

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