Accelerated Distributed Laplacian Solvers via Shortcuts

Ioannis Anagnostides*, 1, Themis Gouleakis† 2, and Christoph Lenzen‡ 3

1National Technical University of Athens
2Max Planck Institute for Informatics
3CISPA Helmholtz Center for Information Security

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Abstract

In this work we refine the analysis of the distributed Laplacian solver recently established by Forster, Goranci, Liu, Peng, Sun, and Ye (FOCS ’21), via the Ghaffari-Haeupler framework (SODA ’16) of low-congestion shortcuts. Specifically, if $\epsilon > 0$ represents the error of the solver, we derive two main results:

- For any $n$-node graph $G$ with hop-diameter $D$ and treewidth bounded by $k$, we show the existence of a Laplacian solver with round complexity $O(n^{o(1)}kD \log(1/\epsilon))$ in the CONGEST model. For graphs with bounded treewidth this circumvents the notorious $\Omega(\sqrt{n})$ lower bound for "global" problems in general graphs.

- Following a recent line of work in distributed algorithms, we consider a hybrid communication model which enhances CONGEST with very limited global power in the form of the recently introduced node-capacitated clique. In this model, we show the existence of a Laplacian solver with round complexity $O(n^{o(1)} \log(1/\epsilon))$.

The unifying thread of these results is an application of accelerated distributed algorithms for a congested variant of the standard part-wise aggregation problem that we introduce. This primitive constitutes the primary building block for simulating "local" operations on low-congestion minors, and we believe that this framework could be more generally applicable.

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*email: ioannis.anagnostides@gmail.com
†email: tgouleak@mpi-inf.mpg.de
‡email: lenzen@cispa.de
1 Introduction

The Laplacian paradigm has emerged as one of the cornerstones of modern algorithmic graph theory, integrating techniques from combinatorial optimization with powerful machinery from numerical linear algebra. Originally pioneered by Spielman and Teng [ST14], who established the first nearly-linear time solvers for a (linear) Laplacian system on some underlying graph, there has been a considerable amount of interest in providing simpler and more efficient solvers in recent years [KMP14; Kel+13; KS16]. Indeed, this framework has led to some state of the art algorithms for a wide range of fundamental graph-theoretic problems; e.g., see [Mad16; Coh+17; Bra+20; Kel+14; Pen16; AMV20], and references therein. In the distributed context, a major breakthrough was recently made by Forster, Goranci, Liu, Peng, Sun, and Ye [For+20], developing a distributed algorithm which solves a Laplacian system on an $n$-node graph within $O(n^{o(1)}(\sqrt{n} + D) \log(1/\epsilon))$ rounds of the standard CONGEST model, where $D$ represents the hop-diameter of the underlying network and $\epsilon > 0$ the error of the solver. This complexity nearly matches the lower bound of $\Omega(\sqrt{n} + D)$ rounds, established via a reduction from the $s - t$ connectivity problem [Das+11].

This lower bound under the CONGEST model of distributed computing should hardly be of any surprise. Indeed, it is well-known by now that a remarkably wide range of global optimization problems, including minimum spanning tree, minimum cut, maximum flow, and single-source shortest paths, require $\tilde{\Omega}(\sqrt{n} + D)$ rounds\textsuperscript{1} [PR99; Elk04; Das+11]. In fact, the same limitation applies for any non-trivial approximation. However, these lower bounds are constructed on some pathological graph instances which typically do not occur in practice. This begs the question: Can we obtain more refined performance-guarantees based on the underlying topology of the communication network? The framework of low-congestion shortcuts, introduced by Ghaffari and Haeupler [GH16], offers a very elegant and compelling answer to this question. For example, they established the existence of $\tilde{O}(D)$-round distributed algorithms for MST and Min-Cut on planar graphs, bypassing the notorious $\Omega(\sqrt{n})$ lower bound for general graphs. We should remark that for many graphs of practical interest the diameter is typically remarkably small, e.g. $D = \text{polylog}(n)$ (as is folklore, this holds for most social networks). Thus, this would immediately imply exponential improvements—assuming planarity or near-planarity—over the generic algorithms for general graphs. In the context of the distributed Laplacian paradigm, we ask the following:

*Can we obtain a faster distributed Laplacian solver on restricted families of graphs in the CONGEST model?*

In this work, we provide an affirmative answer to this question for graphs with bounded treewidth. Moreover, another concrete way of bypassing the $\tilde{\Omega}(\sqrt{n} + D)$ lower bound, besides investigating restricted families of graphs, is by enhancing the local communication network with a limited amount of global power. Indeed, research concerning hybrid networks was recently initiated by Augustine et al. [Aug+20] in the realm of distributed algorithms, although networks combining different communication modes have found numerous applications in real-life systems, and as such, they have been well-studied in other areas of distributed computing; cf. [CGC16; Wan+10; KS18]. This leads to the following fundamental question:

*Can we obtain a faster distributed Laplacian solver under some additional but limited global power?*

\textsuperscript{1}As usual, we use the notation $\tilde{O}(\cdot)$ to suppress polylogarithmic factors of $n$. 
Specifically, we will enhance the standard CONGEST model with the recently introduced node-capacitated clique (henceforth NCC) [Aug+19]. The latter model enables all-to-all communication, but with severe capacity restrictions for every node. The integration of these models will be referred to as the HYBRID model for the rest of this work. In this context, we will show a very strong positive result for HYBRID, answering the second question previously raised.

1.1 Contributions

The primary conceptual contribution of our work is to extend the seminal Ghaffari-Haeupler framework of low-congestion shortcuts, incorporating congestion on the underlying part-wise aggregation problem (see Definition 2.1 and Section 3). As it turns out, this particular variant constitutes the primary building block for performing "local" operations on low-congestion minors, a concept introduced in [For+20]. Importantly, our proposed framework allows for a unifying treatment of different models in distributed computing by refining the performance of an algorithm based on the complexity of the congested part-wise aggregation problem. In the context of the Laplacian paradigm, we manage to refine the analysis of the solver presented in [For+20], leading to a substantial acceleration in the round complexity under structured network topologies:

Theorem 1.1 (Informal Statement). Consider an n-node graph with hop-diameter $D$ and treewidth bounded by $k$. Then, there exists a distributed Laplacian solver in the CONGEST model with round complexity $O(n^{o(1)}kD \log(1/\epsilon))$, where $\epsilon > 0$ represents the error of the solver.

We refer to Corollary 5.2 for a formal statement of our result. The treewidth is a fundamental parameter of a graph; roughly speaking, it is the minimum $k$ such that the graph $G$ can be decomposed into a "tree structure" of components with at most $k + 1$ nodes [BT97] (see Definition 2.5 for the formal description). It should be stressed that the family of graphs with bounded treewidth subsumes many other interesting classes, such as outerplanar graphs, cactus graphs, Halin graphs, etc. Treewidth plays a central role in parameterized complexity, and in particular, it is well-known that many intractable problems in the centralized model of computation become "easy" for graphs with bounded treewidth [Cyg+15]. Our result implies an analogous characterization, but in the distributed context; note that this connection was previously observed in [HIIZ16b] for problems such as MST and Min-Cut. Using standard results, our theorem implies a Laplacian solver with round complexity $O(n^{o(1)}D^2 \log(1/\epsilon))$ for planar graphs [Epp99], and more broadly, $O(n^{o(1)}gD^2 \log(1/\epsilon))$ for graphs of genus $g$ [Epp00]. Also notice that for a graph with bounded treewidth and $D = n^{o(1)}$, we show the existence of an algorithm which solves a Laplacian system within $O(n^{o(1)} \log(1/\epsilon))$ rounds. Moreover, in the HYBRID model we establish the following result:

Theorem 1.2 (Informal Statement). Consider any n-node graph. Then, there exists a distributed Laplacian solver in the HYBRID model with round complexity $O(n^{o(1)} \log(1/\epsilon))$, where $\epsilon > 0$ represents the error of the solver.

The formal statement of this theorem is deferred to Corollary 5.3. Observe that our result in HYBRID applies to arbitrary graphs, implying a remarkably fast subroutine for solving a Laplacian system. As a result, we corroborate the observation that a very limited amount of global power can lead to substantially faster algorithms for certain optimization problems, strongly supplementing a recent line of work [Aug+20; KS20; FHS20; CLP20; Göt+20].
As we previously alluded to, both of these theorems are obtained through a unifying approach, and we consider this to be an important conceptual contribution of our work. Indeed, the connecting thread is an extension of the Ghaffari-Haeupler framework of low-congestion shortcuts [GH16]. One of our key observations is that the round complexity of the distributed Laplacian solver of Forster et al. [For+20] can be refined based on the solution of what we call the congested part-wise aggregation problem, without requiring additional technical work. In the CONGEST model, we reduce the congested variant to the standard part-wise aggregation problem, leveraging the recent powerful result of Ghaffari and Haeupler [GH20]. In the HYBRID model, we employ certain communication primitives developed in [Aug+19] for dealing with congested part-wise problems. A byproduct of our results is that the framework of low-congestion shortcuts interacts particularly well with the HYBRID model, as was previously observed in [AG21].

1.2 Related Work

The cardinal reference point of our work is the recent Laplacian solver of Forster, Goranci, Liu, Peng, Sun, and Ye [For+20] with near-optimal complexity of $O(n^{o(1)}(\sqrt{n} + D) \log(1/\epsilon))$ rounds, where $\epsilon > 0$ represents the error of the solver. Specifically, they devised several new ideas and techniques to circumvent certain issues which mostly relate to the bandwidth restrictions of the CONGEST model; these building blocks, as well as the resulting Laplacian solver are revisited in our work in order to further refine the performance of the solver. We are not aware of any previous research addressing this problem in the distributed context. On the other hand, the Laplacian paradigm has attracted a considerable amount of interest in the community of parallel algorithms; most notably, we refer to [PS14; Ble+14]. It should be noted that these approaches in the PRAM model of parallel computing fail—at least without non-trivial modifications—to lead to a near-optimal solver in the distributed context [For+20].

Research concerning hybrid communication networks in distributed algorithms was recently initiated by Augustine, Hinnenthal, Kuhn, Scheideler, and Schneider [Aug+20]. Specifically, they investigated the power of a model which integrates the standard LOCAL model [Lin92] with the recently introduced node-capacitated clique (NCC) [Aug+19], focusing mostly on distance computation tasks. Several of their results were subsequently improved and strengthened in subsequent works [KS20; CLP20] under the same model of computation. In our work we consider a substantially weaker model, imposing a severe limitation on the communication over the "local edges". This particular variant has been already studied in some recent works for a variety of fundamental problems [FHS20; Göt+20].

The NCC model, which captures the global network in all hybrid models studied thus far, was introduced in [Aug+19] partly to address the unrealistic power of the congested clique (CLIQUE) [Lot+03], wherein each node can communicate concurrently and independently with all other nodes by $O(\log n)$-bit messages. In contrast, the NCC model allows communication with up to $O(\log n)$ (arbitrary) nodes per round. As a result, in the HYBRID model and under a sparse local network only $\tilde{O}(n)$ bits can be exchanged overall per round, whereas CLIQUE allows for the exchange of up to $\tilde{O}(n^2)$ (distinct) bits. As evidence for the power of CLIQUE we note that even slightly super-constant lower bounds would give new lower bounds in circuit complexity, as implied by a simulation argument in [DKO14].
2 Preliminaries

It is assumed that the communication network consists of a set of \(n\) entities with \( \{1, 2, \ldots, n\} \) being the set of their IDs, and a local communication topology given by a graph \( G \).\(^2\) We will assume that \( G \) is undirected throughout this work; as usual, we also posit that for all \( e \in E(G) \), \( w(e) \in \{1, 2, \ldots, W\} \), for some \( W = \text{poly}(n) \). At the beginning every node knows the identifiers of each node in its own neighborhood, but has no further knowledge about the topology of the graph. Communication occurs in synchronous rounds, and in every round nodes have unlimited computational power to process the information they possess. The local communication mode will be modeled with the CONGEST model, for which in each round every node can exchange an \( O(\log n)\)-bit message with each of its neighbors in \( G \) via the local edges. The global communication mode uses NCC for which in each round every node can exchange \( O(\log n)\)-bit messages with up to \( O(\log n)\) arbitrary nodes via global edges. If the capacity of some channel is exceeded, i.e., too many messages are sent to the same node, it will only receive an arbitrary (potentially adversarially selected) subset of the information based on the capacity of the network; the rest of the messages are dropped. We will measure the performance of a distributed algorithm in terms of its round complexity—the number of rounds required so that every node knows its part of the output; for randomized protocols it will suffice to reach the desired state with high probability.\(^3\)

Low-Congestion Shortcuts. A recurring scenario in distributed algorithms for global problems (e.g. MST) boils down to solving the following part-wise aggregation problem:

**Definition 2.1** (Part-Wise Aggregation Problem). Consider a graph \( G = (V, E) \) whose node set is partitioned into \( k \) parts \( P_1, \ldots, P_k \) such that each induced subgraph \( G[P_i] \) is connected. In the part-wise aggregation problem the input is a value \( x_u \) for each \( u \in V \), and the goal is to let every node learn a distributive aggregate function of the values held by the nodes in its own part.

Note that a distributive aggregate function simply refers to a binary operator satisfying the commutative and the associative law (e.g. \( \max \{\cdot, \cdot\}, + \)). For example, in the context of Boruvka’s algorithm for the MST problem, determining the minimum-weight outgoing edge for each part is an instance of a part-wise aggregation problem. The following definition is directly related to this aggregation problem:

**Definition 2.2** (Low-Congestion Shortcuts). Consider a graph \( G = (V, E) \) whose node set is partitioned into \( k \) parts \( P_1, \ldots, P_k \) such that each induced subgraph \( G[P_i] \) is connected. A collection of subgraphs \( H_1, \ldots, H_k \) is a shortcut of \( G \) with congestion \( c \) and dilation \( d \) if the following properties hold: (i) the diameter of each subgraph \( G[P_i] + H_i \) is at most \( d \), and (ii) every edge is included in at most \( c \) many of the subgraphs \( H_i \). The quantity \( Q = c + d \) will be referred to as the quality of the shortcut.

This concept was formally introduced by Ghaffari and Haeupler [GH16, Definition 1]. It should be noted that general graphs admit low-congestion shortcuts of quality \( O(D + \sqrt{n}) \), and this bound is existentially tight (up to logarithmic factors). On the other hand, one of the main

\(^2\)To ameliorate any possible confusion, we stress that we will sometimes switch between the notation \( G, \pi \) and \( G, n \) for the communication network, as it will be made explicit in each context.

\(^3\)We say that an event holds with high probability if it occurs with probability at least \( 1 - 1/n^c \) for some constant \( c > 0 \).
contributions of Ghaffari and Haeupler [GH16] was to show that planar graphs admit shortcuts of quality $\tilde{O}(D)$, in turn leading to $\tilde{O}(D)$-round algorithms for MST and Min-Cut. After this original work several extensions and improvements were established [HIZ16a; KP21; DG19; GL18; GP17; LP19; HL18]. For our purposes, we will use a very general result recently established by Ghaffari and Haeupler [GH20, Theorem 1.2]; but first, let us recall the following definition:

**Definition 2.3 (Minor Density).** The minor density $\delta(G)$ of a graph $G$ is defined as

$$\delta(G) = \max \left\{ \frac{|E'|}{|V'|} : H = (V', E') \text{ is a minor of } G \right\}.$$  

It should be noted that $\delta(G) = \tilde{\Theta}(r(G))$, where $r(G)$ is the complete-graph minor size, i.e., $r(G) = \max\{r : K_r \text{ is a minor of } G\}$ [Tho84; Tho01]. Moreover, any family of graphs $\mathcal{G}$ which is closed under taking minors (e.g. planar graphs) has a constant minor density.

**Theorem 2.4 ([GH20]).** Any $n$-node graph $G$ with diameter $D$ and minor density $\delta(G)$ admits shortcuts with dilation $O(\delta D)$ and congestion $O(\delta D \log D)$.

We remark that the (linear) dependency on the minor density is existentially optimal; see [GH20, Lemma 3.2]. In this work, we study graphs with bounded treewidth, so let us first recall the following definition:

**Definition 2.5 (Tree Decomposition and Treewidth).** A tree decomposition of a graph $G$ is a tree $T$ with tree-nodes $X_1, \ldots, X_k$, where each $X_i$ is a subset of $V(G)$ satisfying the following properties:

1. $V = \bigcup_{i=1}^k X_i$;
2. For any node $u \in V(G)$, the tree-nodes containing $u$ form a connected subtree of $T$;
3. For every edge $\{u, v\} \in E(G)$, there exists a tree-node $X_i$ which contains both $u$ and $v$.

The width $w$ of the tree decomposition is defined as $w := \max_{i \in [k]} |X_i| - 1$. Moreover, the treewidth $tw(G)$ of the graph $G$ is defined as the minimum of the width among all possible tree decompositions of $G$.

**Fact 2.6 ([GH20]).** Consider a graph $G$ with treewidth at most $k$. Then, $\delta(G) \leq k$, where $\delta(G)$ represents the minor density of $G$.

**Distributed Construction of Shortcuts.** From an algorithmic standpoint, low-congestion shortcuts are useful only if we can guarantee a "fast" distributed construction. To this end, it has been observed [HIZ16a; HHW20] that certain additional structural properties—namely tree-restriction and small block number—suffice in order to guarantee construction in only $\tilde{O}(Q)$ rounds, whenever quality-$Q$ shortcuts exist. In light of this, Ghaffari and Haeupler [GH20, Theorem 1.5] also established analogously fast distributed constructions:

**Theorem 2.7 ([GH20]).** Consider an $n$-node $m$-edge graph $G = (V, E)$ with minor density $\delta(G)$ and diameter $D$. There exists a randomized distributed algorithm which computes a shortcut of quality $\tilde{O}(\delta D)$ in $\tilde{O}(\delta D)$ rounds with high probability. In addition, the algorithm uses only $\tilde{O}(m)$ messages, where $m = |E|$.

It should be noted that there is also a deterministic distributed algorithm with a slightly worse performance-guarantee (see [GH20]).
Low-Congestion Minors. Here we introduce the concept of low-congestion minors [For+20]. We remark that, for notational consistency, we will generally endeavor to use the notation of Forster et al. [For+20].

Definition 2.8 ([For+20]). A graph $G$ is a minor of $\overline{G}$ if the following properties hold:

1. For every node $u^G \in V(G)$ there exists:
   (i) A subset of nodes of $\overline{G}$, which is termed as a super-node, $S^{G \to \overline{G}}(u^G)$, with a leader node $\ell(u^G) \in S^{G \to \overline{G}}(u^G)$;
   (ii) A connected subgraph of $\overline{G}$ on $S^{G \to \overline{G}}(u^G)$, for which we maintain a spanning tree $T^{G \to \overline{G}}(u^G)$.

2. There exists a mapping of the edges of $G$ onto edges of $\overline{G}$, or self-loops, such that for any $\{u^G, v^G\} \in E(G)$, the mapped edge $\{u, v\}$ satisfies $u \in S^{G \to \overline{G}}(u^G)$ and $v \in S^{G \to \overline{G}}(v^G)$.

Moreover, we say that this minor $G$ has congestion $\rho$, or $G$ is a $\rho$-minor, if:

1. Every node $u \in \overline{G}$ is contained in at most $\rho$ super-nodes $S^{G \to \overline{G}}(u^G)$, for some $u^G \in V(G)$;

2. Every edge of $\overline{G}$ appears as the image of an edge of $G$ or in one of the trees connecting super-nodes (i.e., $T^{G \to \overline{G}}(u^G)$ for some $u^G$) at most $\rho$ times.

Finally, we say that $G$ is $\rho$-minor distributed over $\overline{G}$ if every $u \in V(\overline{G})$ stores:

1. All $u^G \in V(G)$ for which $u \in S^{G \to \overline{G}}(u^G)$;

2. For every edge $e$ incident to $u$, (i) all the nodes $u^G$ for which $e \in T^{G \to \overline{G}}(u^G)$, and (ii) all edges $e^G$ that map to it.

Importantly, performing "local" operations based on a graph $\rho$-minor distributed into the underlying communication network boils down to a congested part-wise aggregation problem, which is addressed in detail in Section 3. We remark that the basis for Definition 2.8 was the earlier concept of a distributed cluster graph from [Gha+15].

The Laplacian Matrix. Consider a weighted undirected graph $G = (V, E, w > 0)$. The Laplacian of the graph $G$ is defined as

$$L(G)_{u,v} = \begin{cases} \sum_{\{u,z\} \in E} w_{u,z} & \text{if } u = v, \\ -w_{u,v} & \text{otherwise.} \end{cases}$$ (2)

Note that the Laplacian matrix of a graph is (i) symmetric ($L(G)^T = L(G)$), (ii) positive semi-definite ($x^T L(G) x \geq 0$), and (iii) weakly diagonally dominant ($L(G)_{u,u} \geq \sum_{v \neq u} |L(G)_{u,v}|$).
Definition 2.9 (Schur Complement). For a symmetric matrix $A \in \mathbb{R}^{n \times n}$ and a partition of $[n]$ into $\mathcal{T}$ and $S$, permute the rows and columns of $A$ such that

$$ A = \begin{bmatrix} A_{[S,S]} & A_{[S,T]} \\ A_{[T,S]} & A_{[T,T]} \end{bmatrix}. $$

Then, the Schur complement of $A$ onto $\mathcal{T}$ is defined as $\text{SC}(A, \mathcal{T}) := A_{[\mathcal{T},\mathcal{T}]} - A_{[\mathcal{T},S]}A_{[S,S]}^+A_{[S,T]}$, where $A^+$ denotes the Moore-Penrose pseudo-inverse matrix. For a graph $G$ and a subset $\mathcal{T} \subseteq V(G)$, we will write $\text{SC}(G, \mathcal{T}) := \text{SC}(\mathcal{L}(G), \mathcal{T})$.

Further Notation. Consider two positive semi-definite matrices $A, B \in \mathbb{R}^{n \times n}$. For a vector $x \in \mathbb{R}^n$, we define $||x||_A := \sqrt{x^T A x}$ (Mahalanobis norm). We will write $A \succeq B$ if $\exp(-e)A \preceq B \preceq \exp(e)A$, where $A \preceq B$ if and only if the matrix $B - A$ is positive semi-definite. For an edge $e = \{u, v\}$, we will let $b_e := 1_u - 1_v$, where $1_u \in \mathbb{R}^n$ represents the characteristic vector of a node $u$. In this context, for a graph $G$ with resistances $r_e$, we define the leverage scores as $\text{lev}_G(e) := r_e^{-1} b_e^T \mathcal{L}(G) b_e$. Note that $0 \leq \text{lev}_G(e) \leq 1$.

3. The Congested Part-Wise Aggregation Problem

A fundamental communication primitive employed in the distributed Laplacian solver of Forster et al. [For+20] consists of performing aggregation operations with respect to a graph $\rho$-minor distributed into the underlying communication network $\mathcal{G}$. In the context of the low-congestion shortcuts framework, and in particular Definition 2.1, we view this problem as a congested variant of the part-wise aggregation problem: There exists a collection of $k$ parts $P_1, \ldots, P_k$ such that each induced subgraph $G[P_i]$ is connected, but unlike the standard part-wise aggregation problem of Definition 2.1, we assume that every node can be included in up to $\rho$ parts. One of our important observations is that the performance of the distributed Laplacian solver of Forster et al. [For+20] drastically depends on the round-complexity of this problem. In particular, in order to present a unifying analysis for both models of interest, as well as for future applications and extensions, we shall analyze the distributed Laplacian solver under the following abstract hypothesis:

Assumption 3.1. Consider a model of computation which incorporates CONGEST. Moreover, let $G = (V, E)$ be an $n$-node graph $\rho$-minor distributed into an $\pi$-node communication network $\mathcal{G}$. Then, we can solve with high probability the $\rho$-congested part-wise aggregation problem in $Q = Q(\rho)$ rounds; that is, all the nodes in $S^{G \to \mathcal{G}}(u^G)$ can (simultaneously for all $u^G \in V(G)$) compute a distributive aggregate function on $O(\log \pi)$-bit inputs. For the function $Q(\rho) \geq \rho$ we posit that the dependence on $\rho$ is polynomial, implying that $Q(\rho_1, \rho_2) \leq c Q(\rho_2)$ for some universal constant $c$.

Naturally, $Q$ may also depend on certain parameters of the network (e.g. the hop diameter), but for notational simplicity we only make explicit the dependence on the congestion parameter $\rho$. Returning to Assumption 3.1, we should note that for general graphs we can solve the $\rho$-congested part-wise aggregation problem in $O(\rho \sqrt{n} + D)$ rounds of CONGEST. Indeed, for any part $P_i$ for which $|P_i| \leq \sqrt{n}$ we let $H_i = \emptyset$; otherwise, we let $H_i = E(T)$, where $T$ represents a BFS tree over the entire network $\mathcal{G}$. Now every part $P_i$ for which $|P_i| \leq \sqrt{n}$ can perform the aggregation using the corresponding trees of Definition 2.8; the constraint for the congestion implies that $\rho \sqrt{n}$ rounds will suffice. Afterwards, observe that the parts that remain are at most $\rho \sqrt{n}$, due
to node-congestion barrier. This implies that they can perform the aggregation via a standard pipelining technique on the BFS tree, which requires $O(\rho \sqrt{n} + D)$ rounds. In the sequel, we will show that the congested part-wise aggregation problem can be solved substantially faster if we restrict our attention to special families of graphs, or if we allow some limited amount of global power. We remark that no attempt was made to optimize the dependence of $Q$ on $\rho$ since (as long as the dependence stays polynomial) it does not substantially affect the complexity of the Laplacian solver.

3.1 The CONGEST Model

The purpose of this subsection is to establish the following communication primitive in the CONGEST model:

Lemma 3.2. Let $G = (V,E)$ be an $n$-node graph $\rho$-minor distributed into an $\pi$-node communication network $\overline{G} = (\overline{V}, \overline{E})$ of diameter at most $D$, such that $\text{tw}(\overline{G}) \leq k$. Then, we can perform with high probability the following operations in the CONGEST model, simultaneously for all $u^G \in V(G)$, within $\tilde{O}(\rho^3 k D)$ rounds:

1. Every leader $\ell(u^G)$ sends an $O(\log \pi)$-bit message to all the nodes in $S^{G\to\overline{G}}(u^G)$;
2. All the nodes in $S^{G\to\overline{G}}(u^G)$ compute a distributive aggregate function on $O(\log \pi)$-bit inputs.

This lemma will be established based on a reduction to the standard part-wise aggregation problem of Definition 2.1 by a suitable augmentation of the graph. Before we proceed with this construction, let us note that we will use $\rho(u)$ to indicate the number of times a node $u \in V(\overline{G})$ is included in some super-node $S^{G\to\overline{G}}(u^G)$, for some $u^G \in V(G)$. With this notation, it follows that $\rho = \max_u \rho(u)$.

Now consider a graph $\overline{G}_\rho$ constructed from $\overline{G}$ as follows: Every node $u \in V(\overline{G})$ with congestion $k := \rho(u)$ will be replaced with the set of nodes $\{u_1, u_2, \ldots, u_k\}$; moreover, every edge $\{u,v\} \in E(\overline{G})$ in $\overline{G}$ shall induce the set of edges $\{u_i,v_j\} \in E(\overline{G}_\rho)$ for all $(i,j) \in [\rho(u)] \times [\rho(v)]$ (see Figure 1). In the graph $\overline{G}$ every node $u$ possesses $\rho(u)$ many labels, which correspond to the aggregation parts node $u$ is included in. Our reduction will ensure that every node in $\overline{G}_\rho$ will be included in a single part by assigning a different label to each element in the node-set.

This reduction will only be useful if we can guarantee (i) a small overhead in the quality of the shortcuts, and (ii) efficient construction/simulation on graph $\overline{G}_\rho$. This is shown in the following propositions, where we assume that $\pi \geq 2$ to avoid trivialities.

Claim 3.3. $D(\overline{G}_\rho) \leq D(\overline{G}) + 1$.

Proof. Consider two nodes $u,v \in V(\overline{G})$ with distance $d(u,v)$. Then, it is easy to see that for all $(i,j) \in [\rho(u)] \times [\rho(v)]$, $d(u_i,v_j) = d(u,v)$. Moreover, for $i,j \in [\rho(u)]$ it follows that $d(u_i,u_j) \leq 2 \leq D(\overline{G}) + 1$, where we used the fact that $\overline{G}$ is connected.  

Lemma 3.4. If the treewidth of $\overline{G}$ is $\text{tw}(\overline{G})$, then $\text{tw}(\overline{G}_\rho) \leq \rho \text{tw}(\overline{G}) + \rho$.

Proof. Consider a tree decomposition of $\overline{G}$ into tree-nodes $\{X_j\}_{j=1}^k$, such that the corresponding width satisfies $w = \text{tw}(\overline{G})$. We will show that there exists a tree decomposition on the graph $\overline{G}_\rho$
with width at most $\rho(w + 1)$, which in turn will imply that $tw(\hat{G}_p) \leq \rho(w + 1) = \rho(tw(G) + 1)$. Indeed, consider the following sets:

$$\hat{X}_j := \{u_i : u \in X_j, i \in [\rho(u)]\},$$

for all $j \in [k]$. Observe that by construction, $|\hat{X}_j| = \sum_{u \in X_j} \rho(u) \leq \rho|X_j|$. Moreover, we will show that the collection of sets $\{\hat{X}_j\}_{j=1}^k$ is indeed a tree decomposition (recall Definition 2.5). In particular, since $V(\hat{G}) \subseteq \bigcup X_j$, it directly follows that $V(\hat{G}_p) \subseteq \bigcup \hat{X}_j$. Moreover, consider two sets $\hat{X}_j, \hat{X}_\ell$, both containing some node $u_i \in V(\hat{G}_p)$ for $i \in [\rho(u)]$. Then, we know that all the tree-nodes in the (unique) path between $X_j$ and $X_\ell$ based on the original tree decomposition include $u$, implying that all the tree nodes in the path between $\hat{X}_j$ and $\hat{X}_\ell$ also contain $u_i$. Finally, we know that for every edge $\{u, v\} \in E(\hat{G})$ there exists a subset $X_j$ such that $u, v \in X_j$. Thus, we can infer that for every edge in $E(\hat{G}_p)$ there is a subset $\hat{X}_j$ which includes both incident endpoints. As a result, we have constructed a tree decomposition in $\hat{G}_p$ with width $\max_{j \in [k]} |\hat{X}_j| - 1 \leq \rho(w + 1) - 1$, completing the proof. \hfill \Box

**Lemma 3.5.** Any distributed algorithm which requires $Q$ rounds of communication on graph $\hat{G}_p$ under the CONGEST model can be simulated in at most $\rho^2 Q$ rounds of CONGEST on graph $G$.

**Proof.** Let us consider one round of communication in $\hat{G}_p$. In particular, note that any two nodes $u, v \in \hat{G}$ can simulate the message exchange between all the nodes $u_i, v_j \in V(\hat{G}_p)$, where $i \in [\rho(u)]$ and $j \in [\rho(v)]$, in at most $\rho(u) \times \rho(v) \leq \rho^2$ rounds of CONGEST in the underlying communication network $G$. \hfill \Box

**Proof of Lemma 3.2.** It suffices to establish the lemma for the aggregation operation, as the broadcast operation can be performed by reversing the message pattern of an aggregation operation in time. To this end, we will first leverage Theorem 2.7 for the graph $\hat{G}_p$. In particular, note that it suffices to solve the standard part-wise aggregation problem (Definition 2.1) in the graph $\hat{G}_p$; indeed, our reduction ensures the connectivity of the induced parts. Moreover, Lemma 3.4, Claim 3.3, and Theorem 2.4 imply that the graph $\hat{G}_p$ admits low-congestion shortcuts of quality $O(\rho k D)$ (recall...
Fact 2.6). In particular, we can compute via Theorem 2.7 shortcuts of quality $\tilde{O}(\rho kD)$ within $\tilde{O}(\rho kD)$ rounds of communication on $G_\rho$, with high probability. By Lemma 3.5 this construction will be concluded after $\tilde{O}(\rho kD)$ rounds of communication in the underlying network $\overline{G}$.

Next, the problem will be cast in the general distributed algorithm scheduling (DAS) framework of Ghaffari [Gha15]. Specifically, we have to execute in parallel $n$ distributed algorithms $A_1, A_2, \ldots, A_n$, where each algorithm corresponds to an aggregation operation on a super-node $S_{G} \rightarrow \overline{G}(u_G)$, for all $u_G \in V(G)$. Observe that if we execute every algorithm $A_i$ separately we would require dilation $= \tilde{O}(\rho kD)$ rounds in $\hat{G}$, while if we execute all the algorithms simultaneously an edge could have congestion at most congestion $= \tilde{O}(\rho kD)$, where the congestion here is measured with respect to the graph $\hat{G}_\rho$. As a result, we can employ the random delays technique of Leighton, Maggs, and Rao [LMR94] to execute all the algorithms simultaneously in $O(\text{congestion} + \text{dilation} \cdot \log(\rho \pi))$ rounds in the graph $\hat{G}_\rho$; e.g., see [Gha15, Theorem 1.1].

Thus, we know from Lemma 3.5 that the overall communication in the actual network $\overline{G}$ will incur a multiplicative overhead of at most $\rho^{-2}$. It should be noted that this assumes shared randomness, but Ghaffari has devised a pre-processing step with only a logarithmic overhead that removes this assumption [Gha15, Theorem 1.2]. This completes the proof of this lemma.

3.2 The NCC Model

Next, it will be established that the $\rho$-congested part-wise aggregation problem admits a very efficient solution in the NCC model, as implied by the following lemma:

Lemma 3.6. Let $G = (V, E)$ be an $n$-node graph $\rho$-minor distributed into an $n$-node communication network $\overline{G} = (\overline{V}, \overline{E})$. Then, we can perform with high probability the following operations in the NCC model, simultaneously for all $u_G \in V(\overline{G})$, within $O(\rho^2 + \rho \log \pi)$ rounds:

1. Every leader $\ell(u_G)$ sends an $O(\log \pi)$-bit message to all the nodes in $S_{G} \rightarrow \overline{G}(u_G)$;
2. All the nodes in $S_{G} \rightarrow \overline{G}(u_G)$ compute a distributive aggregate function on $O(\log \pi)$-bit inputs.

This lemma essentially follows from the non-trivial machinery developed in [Aug+19]. More specifically, we will state some of their communication primitives, which in turn will imply Lemma 3.2 after an appropriate translation to our setting.

The Aggregation Problem. In the aggregation problem, as defined by Augustine et al. [Aug+19], we are given a distributive function and a set of aggregation groups $\{C_1, \ldots, C_N\}$, with $C_i \subseteq V(G)$ for all $i$, and every aggregation group being associated to some target node $t_i \in C_i$. If every node holds some input value, the goal is to let all the target nodes learn the aggregate value with respect to their aggregation group. Observe that in this setting a node may be part of multiple groups, and in particular, we let $\ell$ be an upper bound on the groups a given node may be included in (local load). If $L = \sum_{i=1}^{N} |C_i|$ represents the global load of the aggregation problem, Augustine et al. [Aug+19, Theorem 2.3] established the following result:

Lemma 3.7 ([Aug+19]). There exists an aggregation algorithm which solves the aggregation problem in $O(L/\pi + \ell/\log \pi + \log \pi)$ rounds of NCC with high probability.
The Multicast Problem. Consider a set of multicast groups \( \{C_1, \ldots, C_N\} \), with \( C_i \subseteq V(\bar{G}) \) and \( N \) sources such that \( s_i \in C_i \). We assume that every node serves as a source in at most one multicast group, but as before it may be part of multiple groups. If we represent the local and global load as \( \ell \) and \( L \) respectively, defined as previously, and we let \( C \) be the congestion of the multicast trees, i.e., the maximum number of trees that share the same butterfly node, Augustine et al. [Aug+19, Theorem 2.4] established the following:

**Lemma 3.8** ([Aug+19]). There exists an algorithm which determines a set of multicast trees in \( O\left(\frac{L}{\pi} + \frac{\ell}{\log \pi + \log \bar{\pi}}\right) \) rounds of NCC with high probability, such that the congestion of the resulting multicast trees is \( O\left(\frac{L}{\pi} + \log \bar{\pi}\right) \).

Having constructed the multicast trees, the goal is to let every source \( s_i \) deliver a message to every node in \( C_i \). This procedure can be efficiently implemented through the multicast algorithm of Augustine et al. [Aug+19, Theorem 2.5]:

**Lemma 3.9.** There exists an algorithm which solves the multicast problem in \( O(C + \frac{\ell}{\log \pi + \log \bar{\pi}}) \) rounds of NCC with high probability, where \( C \) is the congestion of the multicast trees.

**Proof of Lemma 3.6.** We will explain how the previous lemmas can be used to show Lemma 3.7. As before, allow us to restrict our attention to the aggregation operation. In particular, in the context of the aggregation problem we described, assume that the target node coincides with the leader of every group. Given that \( G \) is a \( \rho \)-minor of \( \bar{G} \), we can infer that the local load satisfies \( \ell \leq \rho \), while the global load satisfies \( L \leq \rho \bar{\pi} \). As a result, Lemma 3.7 guarantees that \( O(\log \bar{\pi} + \rho) \) rounds of NCC suffice so that the leader (or equivalently the target node) learns the aggregate value with respect to the corresponding super-node—with high probability.

The next step is to disseminate this value to all the nodes within the super-node. To this end, we will proceed in \( \rho \) rounds, so that in every round a leader acts as a source to a separate super-node. Again, given that \( G \) is a \( \rho \)-minor of \( \bar{G} \) we can conclude that a leader is a source in at most \( \rho \) super-nodes. In every iteration, we shall determine a set of multicast trees in \( O(\rho + \log \bar{\pi}) \) rounds with congestion \( O(\rho + \log \bar{\pi}) \), with high probability (Lemma 3.8). Afterwards, it suffices to solve (in each iteration) a multicast problem, which—having constructed the multicast trees—can be solved after \( O(\rho + \log \bar{\pi}) \) rounds (Lemma 3.9). This leads to the desired conclusion. \( \square \)

### 4 The Laplacian Building Blocks

To keep the exposition reasonably self-contained, in this section we review the basic ingredients of the distributed Laplacian solver developed in [For+20]. In particular, the guarantees established in [For+20] will be extended under Assumption 3.1. Then, we will combine these pieces in Section 5 to establish our main results.

#### 4.1 Ultra-Sparsification

As is standard in the Laplacian paradigm, we will require a preconditioner in the form of an ultra-sparsifier. In particular, the following lemma is established in Appendix A.2, and it is a direct refinement of [For+20, Lemma 4.9]:
**Lemma 4.1 (Ultra-Sparsification).** Consider an $n$-node $m$-edge graph $G$ which is $\rho$-minor distributed into an $\pi$-node communication network $\mathcal{T}$ for which Assumption 3.1 holds for some $Q = Q(\rho)$. Then, UltraSparseify$(G, k)$ takes as input a parameter $k$ and returns after $O(n^{o(1)} Q(\rho))$ rounds a graph $H$ such that

1. $H$ is a subgraph of $G$;
2. $H$ has $n - 1 + m2^{O((\log n \log \log n))} / k$ edges;
3. $\mathcal{L}(G) \preceq \mathcal{L}(H) \preceq k \mathcal{L}(G)$.

Moreover, the algorithm returns $\hat{G}, Z_1, Z_2, C$ such that

1. $\hat{G}$ $1$-minor distributes into $H$ such that $\hat{G} = \text{SC}(H, C)$, with $|C| = m2^{O((\log n \log \log n))} / k$;
2. The operators $Z_1$ and $Z_2$ can be evaluated in $O(Q(\rho) \log n)$ rounds, and are such that

\[
\mathcal{L}(H)^\dagger = Z_1^T \begin{bmatrix} Z_2 & 0 \\ 0 & \mathcal{L}(\hat{G})^\dagger \end{bmatrix} Z_1. \tag{5}
\]

Let us briefly review the pieces required for this lemma. First, we need the distributed implementation of the low-stretch spanning tree algorithm of Alon, Karp, Peleg, and West [Alo+95], which is due to Ghaffari et al. [Gha+15]. Then, this spanning tree is augmented with off-tree edges based on the sampling procedure of Koutis et al. [KMP10], leading to a graph with a spectral-approximation guarantee with respect to the original graph. Finally, the parallel elimination procedure of Blelloch et al. [Ble+14] is used in order to perform a series of contractions, leading to a subset with size analogous to the number of off-tree edges. We revisit these steps in Appendix A.2.

### 4.2 Sparsified Cholesky

The next building block is the sparsified Cholesky algorithm of Kyng et al. [Kyn+16], which manages to effectively eliminate in every iteration a non-negligible fraction of the nodes. In the distributed context, we state the following lemma, which is a direct refinement of [For+20, Lemma 4.10].

**Lemma 4.2 (Sparsified Cholesky).** Let $G$ be an $n$-node graph $\rho$-minor distributed into a communication network $\mathcal{T}$ for which Assumption 3.1 holds for some $Q = Q(\rho)$. Then, for a given parameter $d$ and error $\epsilon$, the algorithm Eliminate$(G, d, \epsilon)$ runs in $O(Q(\rho) (\log n / \epsilon^c)^d)$ rounds, where $c$ represents some universal constant, and returns a subset $T \subset V(G)$ and access to operators $Z_1$ and $Z_2$ such that

1. $|T| \leq (49/50)^d |V(G)|$;
2. The operators $Z_1, Z_1^T, Z_2$ can be applied to vectors in $O(Q(\rho) (\log n / \epsilon^c)^d)$ rounds;
3. 

\[
(1 - \epsilon)^d \mathcal{L}(G)^\dagger \preceq Z_1^T \begin{bmatrix} Z_2 & 0 \\ 0 & \text{SC}(G, T)^\dagger \end{bmatrix} Z_1 \preceq (1 + \epsilon)^d \mathcal{L}(G)^\dagger. \tag{6}
\]
This lemma is established based on a distributed implementation of the *sparsified Cholesky* algorithm of Kyng et al. [Kyn+16]. In particular, the Cholesky decomposition essentially reduces solving a Laplacian to inverting (i) any sub-matrix of the Laplacian induced on a set $S$, and (ii) the Schur complement on $V \setminus S$. Thus, Kyng et al. [Kyn+16] initially develop a procedure for identifying an "almost independent" subset of nodes $F$ (more precisely, a *strongly* diagonally dominant subset) for which inverting the Laplacian restricted on $F$ can be done efficiently through preconditioning (e.g. via the *Jacobi method*), while $F$ also contains at least a constant fraction of the nodes. Next, a combinatorial view of the Schur complement based on a certain family of random walks (see [Dur+19]) is employed in order to construct a spectral sparsifier of the Schur complement on $T = V \setminus F$. This process is then repeated for $d$ iterations, leading to Lemma 4.2. Several technical challenges that arise are discussed in Appendix A.3. Next, the main idea is to recurse on the set of terminals $T$. However, in our context this requires maintaining the invariance that the underlying subgraph is cast as a minor (with a reasonable congestion) of $\overline{G}$. This is ensured in the following subsection.

### 4.3 Minor Schur Complement

This subsection introduces a subroutine that will be invoked after the ELIMINATE algorithm in order to return a low-congestion minor based on the set of terminals $T$ returned by ELIMINATE; while doing so, the algorithm will incur a small overhead in the spectral guarantee, and a limited growth in the number of nodes with respect to $T$; this increase will be eventually negligible due to the selection of parameter $d$ in ELIMINATE. In particular, the following lemma is a refinement of [For+20, Theorem 3].

**Lemma 4.3.** Let $G$ be an $n$-node graph $\rho$-minor distributed into an $\overline{n}$-node communication network for which Assumption 3.1 holds for some $Q = Q(\rho)$. Then, for an error parameter $0 < \epsilon < 0.1$ and a subset $T$ of nodes, the algorithm $\text{ApproxSC}$ returns with high probability a graph $H$ as a $\rho$-minor distribution into $\overline{G}$ such that

1. $T \subseteq V(H)$;
2. $H$ has $O(|T| \log^2 n/\epsilon^2)$ edges;
3. $\text{SC}(H, T) \approx \epsilon \text{SC}(G, T)$.

This algorithm requires $O(\log^{10} n/\epsilon^3)$ calls to a distributed Laplacian solver to accuracy $1/\text{poly}(n)$ on graphs that $2\rho$-minor distribute into $\overline{G}$, and an overhead of $O(Q(\rho) \log^{10} \pi/\epsilon^3)$ rounds.

This result builds upon the work of Li and Schild [LS18], which (roughly speaking) established that randomly contracting an edge with probability equal to its leverage score (and otherwise deleting) would suffice. In the distributed context, Forster et al. [For+20] devise a parallelized implementation of this scheme based on the localization of electrical flows [SRS18]. More precisely, they manage to identify a non-negligible subset of edges—which they refer to as *steady edges*—with small mutual (electrical) "correlation", allowing for independent (and hence highly parallelized) contractions/deletions within this set. This approach employs the recursive and sketching-based method of random projections due to Spielman and Srivastava [SS08], similarly to [LS18], in order to estimate quantities such as leverage scores and electrical correlation. These steps are carefully reviewed in Appendix A.4.
4.4 Schur Complement Chain

Finally, let us introduce the concept of a Schur complement chain, and explain how they can be employed in order to produce a Laplacian solver.

**Definition 4.4.** For an \(n\)-node graph \(G, \{(G_i, Z_{i,1}, Z_{i,2}, T_i)\}_{i=1}^t\) is a \((\gamma, \epsilon)\)-Schur complement chain if the following conditions hold:

1. \(G_1 = G\);
2. \(T_i \subseteq V(G_{i+1}) \subseteq V(G_i)\) and \(\text{SC}(G_i, T_i) \approx \epsilon \text{SC}(G_{i+1}, T_i)\);
3. \(|V(G_{i+1})| \leq |V(G_i)|/\gamma\) for \(i < t\), and \(|V(G_t)| \leq \gamma\).
4. \((1 - \epsilon) L(G_i)^\dagger \preceq Z_{i,1}^T \begin{bmatrix} Z_{i,2}^T & 0 \\ 0 & \text{SC}(G_i, T_i)^\dagger \end{bmatrix} Z_{i,1} \preceq (1 + \epsilon) L(G_i)^\dagger.\) (7)

In particular, note that in Section 5 a Schur complement chain will be developed through Lemma 4.1, Lemma 4.2, and Lemma 4.3. The following lemma implies a solution to the Laplacian system based on a suitable Schur complement chain.

**Lemma 4.5 ([For+20]).** Consider an \(n\)-node communication network for which Assumption 3.1 holds for some \(Q = Q(\rho)\), and let \(\{(G_i, Z_{i,1}, Z_{i,2}, T_i)\}_{i=1}^t\) be a \((\gamma, \epsilon)\)-Schur complement chain for an \(n\)-node graph \(G\) for some \(\gamma \geq 2\) and \(\epsilon \leq 1/(C \log n)\), for a sufficiently large constant \(C\), such that for all \(i:\)

1. \(G_i\) \(\rho\)-minor distributes into \(\overline{G}\);
2. The linear operators \(Z_{i,1}\) and \(Z_{i,2}\) can be evaluated in at most \(O(\overline{n}^{(1)}Q(\rho))\) rounds.

Then, for any given vector \(b\), there is an algorithm which computes a vector \(x\) in \(O(\rho \overline{n}^{(1)}Q(\rho))\) rounds such that

\[||x - L(G)^\dagger b||_{L(G)} \leq \epsilon \log n ||b||_{L(G)^\dagger}.\]  (8)

5 The Laplacian Solver

In this section we combine the building blocks previously developed to establish our main result. In particular, the distributed Laplacian solver of Forster et al. [For+20] is given in Algorithm 1, and the guarantee of this algorithm is shown in Theorem 5.1.

**Theorem 5.1.** Consider a weighted \(n\)-node graph \(G\) for which Assumption 3.1 holds for some \(Q(\rho) = O(\rho^c Q)\), where \(c\) is a universal constant and \(Q\) some parameter. Then, for any vector \(b \in \mathbb{R}^n\) stored on its nodes and a sufficiently small error parameter \(\epsilon > 0\), \text{Solver}(G, \epsilon) returns after \(O(n^{(1)}Q \log(1/\epsilon))\) rounds a vector \(x\) distributed on its nodes such that

\[||x - L(G)^\dagger b||_{L(G)} \leq \epsilon ||b||_{L(G)}.\]  (9)
Algorithm 1: Distributed Laplacian Solver [For+20]: Solver($G, \epsilon$)

Input: An undirected weighted graph $G$;

$G' := \text{SpectralSparsefy}(G)$;

$(G_1, Z_{1,1}, Z_{1,2}, T_1, G_2) := \text{UltraSparsefy}(G', k)$;  
// Lemma 4.1

$(G_i, Z_{i,1}, Z_{i,2}, T_i)$ for $i=2$ := \text{BuildChain}($G_2, d, \epsilon, k$);  
// Lemma 4.5

Solve $L(G)x = b$ via Chebyshev preconditioning;

Procedure \text{BuildChain}($G, d, \epsilon, k$);

if $|V(G)| \leq k$ then

return $\emptyset$;  
// Lemma 4.2

$(Z_1, Z_2, C) := \text{Eliminate}(G, d, \epsilon)$;  
// Lemma 4.3

$H := \text{ApproxSC}(G, C, \epsilon)$;

return $(G, Z_1, Z_2, C) \cup \text{BuildChain}(H, d, \epsilon, k)$;

The proof of this theorem is included in Appendix A.5. We note that a guarantee with respect to the $L(G)^{\dagger}$-norm—as in Lemma 4.5—can be translated to a guarantee in the $L(G)$-norm. This incurs only a logarithmic multiplicative overhead, since it is assumed that the weights are polynomially bounded and the dependence on $1/\epsilon$ is logarithmic. Thus, the overhead is subsumed by the factor $\pi^{(1)}$; see [Vis12, pp. 19–20]. Finally, based on this theorem we derive two main corollaries, which are implied directly by Lemma 3.2 and Lemma 3.6:

Corollary 5.2. Consider a weighted $n$-node graph $G$ of diameter $D$ and treewidth $k$. Then, for any vector $b \in \mathbb{R}^n$ stored on its nodes and a sufficiently small error parameter $\epsilon > 0$, Solver($G, \epsilon$) returns after $O(n^{\pi^{(1)}}kD \log(1/\epsilon))$ rounds of CONGEST a vector $x$ distributed on its nodes such that

$$||x - L(G)^{\yhead}b||_{L(G)} \leq \epsilon ||b||_{L(G)}.$$  
(10)

Corollary 5.3. Consider a weighted $n$-node graph $G$. Then, for any vector $b \in \mathbb{R}^n$ stored on its nodes and a sufficiently small error parameter $\epsilon > 0$, Solver($G, \epsilon$) returns after $O(n^{\pi^{(1)}} \log(1/\epsilon))$ rounds of HYBRID a vector $x$ distributed on its nodes such that

$$||x - L(G)^{\yhead}b||_{L(G)} \leq \epsilon ||b||_{L(G)}.$$  
(11)

6 Concluding Remarks

Solving a Laplacian system on an underlying graph has emerged as a central component in many state of the art algorithms for graph-theoretic problems, and as such, understanding the fundamental limitations of this problem constitutes a major and active area of research. In this context, our work has revealed that we can obtain substantially accelerated distributed Laplacian solvers if we either restrict our attention to more "structured" topologies, or if we have access to a limited amount of global power, considerations which are both concretely motivated in practice. This characterization was obtained through a unifying analysis based on an extension of the Ghaffari-Haeupler framework of low-congestion shortcuts, and we believe that this "congested" extension may lead to further research in the future. In terms of applications, Forster et al. [For+20]
showed that the Laplacian paradigm can offer sublinear and exact distributed algorithms for many important network optimization problems (such as max-flow), an objective which previously appeared elusive. In light of this, we are optimistic that our refinements will be used as building blocks towards improved distributed algorithms in the future.

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A Omitted Proofs

In this section we elaborate on the Laplacian building blocks, establishing the claims made in Section 4, and eventually leading to the proof of the main result, namely Theorem 5.1. We stress that the results appearing in the appendix were essentially established in [For+20], so the purpose of this appendix is to translate them into our more general setting.

A.1 Useful Routines

First, we present several useful routines that can be performed efficiently under Assumption 3.1.

Corollary A.1 (Matrix-Vector Products). Consider a matrix $A$ with non-zeros supported on the edges of an $n$-node graph $G$ which is $\rho$-minor distributed over a communication network $\overline{G}$ for which Assumption 3.1 holds for some $Q = Q(\rho)$, with values stored in the endpoints of the corresponding edges, and a vector $v$.
\( x \in \mathbb{R}^n \) stored on the nodes \( \ell(u^G) \) for \( u^G \in V(G) \). Then, we can compute the vector \( Ax \in \mathbb{R}^n \) stored on the leader nodes \( \ell(u^G) \) for all \( u^G \in V(G) \) after \( O(Q(\rho)) \) rounds with high probability.

The proof of this corollary follows the one by Forster et al. [For+20, Corollary 4.4], but nonetheless we state it here for completeness.

**Proof of Corollary A.1.** The first step is to use Assumption 3.1 in order to disseminate the coordinates of vector \( x \) to the corresponding super-nodes after \( Q(\rho) \) rounds; that is, for every \( u^G \in V(G) \) the leader \( \ell(u^G) \) passes to \( S^G \rightarrow G(u^G) \) the corresponding coordinate. Then, every node performs locally all the multiplications for its corresponding indices, and after \( \rho \) rounds the node can deliver this information to the corresponding super-node. Observe that this is possible because \( A \) is supported on edges of \( G \), and Definition 2.8 imposes an edge-congestion bound. Finally, we invoke again Assumption 3.1 to sum all of the values of each super-node to the leader node, which coincides with the output requirement of the claim.

Another important corollary of Assumption 3.1 is that we can simulate the spectral sparsification algorithm of Koutis (henceforth SpectralSparify) on \( G \) [Kou14]:

**Corollary A.2 (Spectral Sparsification).** Consider an \( n \)-node graph \( G \) that \( \rho \)-minor distributes into an \( n \)-node communication network \( \mathcal{T} \) for which Assumption 3.1 holds for some \( Q = Q(\rho) \). Then, for any \( 0 < \epsilon < 0.1 \) we can implement the SpectralSparify algorithm of Koutis for \( G \) after \( O(Q(\rho) \log^2 n/e^2) \) rounds, which returns with high probability a graph \( \tilde{G} \) distributed as a \( \rho \)-minor into \( \mathcal{T} \) such that

- \( \mathcal{L}(G) \approx_{\epsilon} \mathcal{L}(\tilde{G}) \) (Spectral approximation);
- \( \tilde{G} \) is a reweighted subgraph of \( G \) with \( O(n \log^6 n/e^2) \) edges in expectation.

The proof of this corollary is fairly simple (see [For+20, Corollary 4.4]), but we give a sketch for completeness.

**Proof of Corollary A.2.** The SpectralSparify algorithm of Koutis iteratively uses the spanner scheme of Baswana and Sen [BS07]. The latter algorithm gradually grows clusters, and in particular, in each round clusters are sampled at random—a "leader" node determines whether the cluster is included in the sample, and then forwards the information to the rest of the cluster. Then, nodes compare the weights of their incident edges to decide whether they will join some cluster, and which incident edges will be added to the spanner. As a result, all the operations of the Baswana-Sun algorithm can be performed via the routine of Assumption 3.1, and the claim follows.

**Composition of Minors.** We also state the extensions of [For+20, Lemma 4.6] and [For+20, Corollary 4.7], which are related to the composition of \( \rho \)-minors.

**Lemma A.3 (Composing Minors).** Consider a graph \( G_2 \) which is \( \rho_2 \)-minor distributed into a communication network \( \mathcal{T} \) for which Assumption 3.1 holds for some \( Q = Q(\rho) \), and a graph \( G_1 \) which is \( \rho_1 \)-minor distributed into \( G_2 \). Then, we can compute with high probability and after \( \tilde{O}(Q(\rho_1 \rho_2)) \) rounds a \( (\rho_1 \times \rho_2) \)-minor distribution of \( G_1 \) into \( G \).

**Corollary A.4 (Parallel Contraction).** Consider a graph \( G \) which is \( \rho \)-minor distributed into a communication network \( \mathcal{T} \) for which Assumption 3.1 holds for some \( Q = Q(\rho) \). If \( F \) represents a subset of the edges of graph \( G \), we can obtain with high probability a \( \rho \)-minor distribution of \( G/F \) into \( \mathcal{T} \) in \( \tilde{O}(Q(\rho)) \) rounds.

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Recall that the notation $G/F$ implies the graph obtained from $G$ after contracting all the edges in the set $F \subseteq E(G)$.

### A.2 Ultra-Sparsification: Proof of Lemma 4.1

The first ingredient required for Lemma 4.1 is a distributed version of the celebrated Alon-Karp-Peleg-West (AKPW) low-stretch spanning tree construction [Alo+95], which is due to Ghaffari, Karrenbauer, Kuhn, Lenzen, and Patt-Shamir [Gha+15]. We commence by stating their definition of a distributed $N$-node cluster graph, which incidentally was the basis for Definition 2.8.

**Definition A.5** (Distributed Cluster Graph, [Gha+15]). A distributed $N$-node cluster graph is a 5-tuple $G = (V, E, L, T, \psi)$ satisfying the following properties:

1. $V = \{S_1, \ldots, S_N\}$ forms a partition of the node set into $N$ clusters;
2. $E$ represents a multi-set of (weighted) edges;
3. $L$ is the set of leaders such that every cluster $S_i$ has exactly one leader $\ell_i \in L$. The ID of the leader node will also serve as the ID of the cluster, while it is assumed that nodes know the ID of their leader, as well as the size of their cluster;
4. $T = \{T_1, \ldots, T_N\}$ is a set of cluster trees such that each cluster tree $T_i = (S_i, E_i)$ is a (rooted) spanning tree of the induced subgraph $G[S_i]$ of $G$, with root the leader of the cluster $\ell_i \in S_i$ (observe that this implies that the subgraph induced by each cluster $S_i$ is connected);
5. $\psi: E \mapsto E$ is a bijective function that maps every edge $\{S_i, S_j\} \in E$ to some edge $\{u_i, u_j\} \in E$ connecting the corresponding clusters; i.e., it holds that $u_i \in S_i$ and $u_j \in S_j$. It is assumed that the two nodes $u_i$ and $u_j$ know that the edge $\{u_i, u_j\}$ is used to connect their respective clusters, as well as its weight.

Having introduced the concept of a distributed cluster graph, we state the following lemma, which is a direct corollary of the communication primitives we previously described.

**Lemma A.6.** Let $G = (V, E)$ be an $n$-node graph $\rho$-minor distributed into an $\pi$-node communication network $\overline{G} = (\overline{V}, \overline{E})$ for which Assumption 3.1 holds for some $Q = Q(\rho)$. If $G = (V, E, L, T, \psi)$ is a distributed cluster graph for $G$, the following operations can be performed in $\tilde{O}(Q(\rho))$ rounds:

1. The leader $\ell_i$ of each cluster $S_i$ broadcasts an $O(\log \pi)$-bit message to every node in $S_i$;
2. Computing distributive aggregate functions on $O(\log \pi)$-bit inputs simultaneously for all clusters, assuming the tree $T_i$ is known.

**Proof.** The definition of a distributed $N$-node cluster graph (Definition A.5) implies that $G$ is 1-minor distributed over $G$, and in turn $\rho$-minor distributed into $\overline{G}$. Note that the induced distributed mapping can be obtained using $\tilde{O}(Q(\rho))$ rounds of communication, by virtue of Lemma A.3. Thus, Assumption 3.1 leads to the desired claim.

As a result, it follows that the SplitGraph algorithm in [Gha+15] can be simulated on a graph $G$ which is $\rho$-minor distributed into $\overline{G}$ after $O(n^{\omega(1)}Q(\rho))$ communication rounds—under Assumption 3.1. In particular, this observation directly gives a distributed construction of a low-stretch spanning tree:
Lemma A.7 ([For+20; Gha+15]). Consider an $n$-node $m$-edge graph $G$ which is $\rho$-minor distributed into an $\overline{n}$-node communication network $\overline{G}$ for which Assumption 3.1 holds for some $Q = Q(\rho)$. Then, we can construct a spanning tree $T$ of $G$ after $O(n^{\omega(1)}Q(\rho))$ rounds such that the nodes know upper bounds on the corresponding stretches that sum to at most $m2^{O(\sqrt{\log n \log \log n})}$.

Importantly, it turns out that the guarantee of Lemma A.7 suffices to sample edges by stretch, as implied by the following lemma:

Lemma A.8 ([KMP10]). Consider an $n$-node graph $G$ and a tree $T$ such that the nodes know upper bounds on the corresponding stretches that sum up to $\alpha$. Then, for any parameter $k$ there is a sampling procedure, implementable locally, that gives a graph $H$ which satisfies with high probability the following:

1. $\mathcal{L}(G) \preceq \mathcal{L}(H) \preceq kL(G)$;
2. $H$ contains the edges of $T$ and $O(\alpha \log n / k)$ additional edges.

The final step for establishing Lemma 4.1 uses the parallel elimination procedure of Blelloch et al. [Ble+14], which requires a logarithmic number of rounds under the PRAM model of computation. Thus, we can show the following lemma:

Lemma A.9. Consider an $n$-node graph $H$ which is $\rho$-minor distributed into an $\overline{n}$-node communication network $\overline{G}$ for which Assumption 3.1 holds for some $Q = Q(\rho)$. Moreover, let $T$ be a spanning tree of $H$ and $W$ be the set of off-tree edges of $H$ with respect to $T$. Then, there is an algorithm which runs in $O(Q(\rho) \log n)$ rounds and returns a graph $\hat{G}$, 1-embeddable into $H$, satisfying the following:

1. $\hat{G}$ contains $O(|W|)$ nodes and edges;
2. There are operators $Z_1$ and $Z_2$, which can be evaluated in $O(Q(\rho) \log n)$ rounds, such that

$$\mathcal{L}(H)^+ = Z_1^T \begin{bmatrix} Z_2 & 0 \\ 0 & \mathcal{L}(\hat{G})^+ \end{bmatrix} Z_1.$$  \hspace{1cm} (12)

With these pieces in place, Lemma 4.1 follows directly from Lemma A.7, Lemma A.8, and Lemma A.9.

A.3 Sparsified Cholesky: Proof of Lemma 4.2

The proof of Lemma 4.2 mainly relies on a distributed implementation of the Schur Complement Chain (SCC) construction of Kyng et al. [Kyn+16]. In particular, the first step is to formalize a notion of almost-independence:

Definition A.10. A matrix $M$ is $\alpha$-diagonally dominant (henceforth $\alpha$-DD) if

$$M_{i,j} \geq (1 + \alpha) \sum_{j \neq i} |M_{i,j}|, \hspace{1cm} \forall i.$$  \hspace{1cm} (13)

Moreover, an index set $F$ is $\alpha$-DD if $M_{[F,F]}$ is $\alpha$-DD.
An important observation is that computing the inverse $M^{-1}[F,F]$ for an $\alpha$-DD set can be efficiently performed using a preconditioned gradient descent method. In this context, Kyng et al. [Kyn+16] give a simple sampling algorithm for finding "large" $\alpha$-DD sets given a Laplacian matrix. More precisely, their algorithm initially selects a random subset of nodes, and then it filters out these which do not meet the condition of Definition A.10. This leads to the following result:

**Lemma A.11.** Let $G$ be an $n$-node graph $\rho$-minor distributed into a communication network $\mathcal{G}$ for which Assumption 3.1 holds for some $Q = Q(\rho)$. Then, if $L$ is the Laplacian matrix of $G$ and $\alpha \geq 0$ some parameter, there is an algorithm which computes an $\alpha$-DD subset $F$ of $L$ of size at least $n/(8(1+\alpha))$ in $O(Q(\rho) \log n)$ rounds with high probability.

Indeed, the algorithm of Kyng et al. [Kyn+16] determines an $\alpha$-DD subset of size $n/(8(1+\alpha))$, while the round-complexity guarantee follows similarly to the proof of [For+20, Lemma 6.7]. Here we should note that the global aggregation steps required in the distributed implementation of [For+20, Lemma 6.7] can be trivially performed in $O(Q(\rho))$ rounds.

The next step is to construct an operator that approximates $L^{-1}[F,F]$ where $F$ is an $\alpha$-DD set, and can be efficiently applied to vectors. This is ensured by the following lemma:

**Lemma A.12 ([For+20]).** Let $G$ be a graph $\rho$-minor distributed into an $n$-node communication network $\mathcal{G}$ for which Assumption 3.1 holds for some $Q = Q(\rho)$. Moreover, let $L$ be the Laplacian matrix associated with $G$, and $F$ be a subset of $V(G)$ such that $L^{-1}[F,F]$ is $\alpha$-DD for some $\alpha \geq 4$. Then, for any vector $b$ stored on the leaders of the super-nodes, there is an algorithm which returns in $O(Q(\rho) \log(1/\epsilon))$ rounds the vector $Zb$ stored on the same nodes, where $Z$ is a linear operator such that

$$L^{-1}[F,F] \preceq Z^{-1} \preceq L^{-1}[F,F] + \epsilon \cdot SC(L,F),$$

for any sufficiently small $\epsilon > 0$.

Again, this lemma follows from the guarantee in [Kyn+16] regarding the Jacobi procedure, as well as by directly adapting the distributed implementation in [For+20] using Corollary A.1.

**Approximating the Schur Complement.** Moreover, $\alpha$-DD sets will be useful in the approximation of the Schur complement on the complementary subset of nodes. First, let us recall a combinatorial view of the Schur complement as a Laplacian matrix with weights estimated by certain random walks:

**Lemma A.13 ([Dur+19]).** Let $G$ be an $n$-node weighted graph and a subset of nodes $T$. Moreover, consider parameters $0 < \epsilon < 1$ and $\mu = O(\log n/\epsilon^2)$. If $H$ is an initially empty graph, repeat for every edge $\{u,v\} \in E(G)$ and for $\mu$ iterations the following procedure:

1. Simulate a random walk starting from $u$ until it first hits $T$ at some node $t_1$;
2. Simulate a random walk starting from $v$ until it first hits $T$ at some node $t_2$;
3. Combine these two walks to get a walk $t_1 = u_0, \ldots, u_\ell = t_2$, where $\ell$ is the length of the combined walk.
4. Add the edge \( \{t_1, t_2\} \) to \( H \) with weight

\[
\frac{1}{\mu \sum_{i=0}^{\ell - 1} 1/w_{u_i, u_{i+1}}}. \tag{15}
\]

Then, the resulting graph \( H \) satisfies \( \mathcal{L}(H) \approx \epsilon \text{SC}(G,T) \) with high probability.

It should be noted that the random walks in the lemma are implied in the usual sense, wherein a step from a node is taken with probability proportional to the edge-weights of the incident edges. In the sequel, we will compute an \( \alpha \)-DD set \( F \) via Lemma A.11, and then the goal will be to approximate the Schur complement on the set \( T = V \setminus F \). Importantly, given that \( F \) is \( \alpha \)-DD, we can guarantee that the random walks required in Lemma A.13 will be short in expectation. Nonetheless, a challenge that arises in the distributed context—and in particular under the CONGEST model—is that the expected congestion of an edge may by prohibitively large. This issue will be resolved by incorporating new nodes to the terminals whenever they exceed some threshold of congestion. At the same time, however, we also have to limit the node-congestion since \( G \) is minor distributed into \( \mathcal{G} \), and we can only deal with limited congestion. This will be addressed by invoking the spectral sparsification algorithm, ensuring that the average degree, and subsequently the congestion, remains limited.

Before we proceed with the algorithm that approximates the Schur complement, we note that we can implement the random walks of Lemma A.13 in \( \tilde{O}(Q(\rho)) \) rounds, under Assumption 3.1, as was explained in [For+20].

**Lemma A.14 ([For+20]).** Let \( G \) be an \( n \)-node graph \( \rho \)-minor distributed into an \( \mathcal{G} \)-node communication network \( \mathcal{G} \) for which Assumption 3.1 holds for some \( Q = Q(\rho) \). Moreover, let \( F \) be an \( \alpha \)-DD set, \( T = V \setminus F \) the set of terminals, \( \epsilon \in (0,1) \) some error parameter, and \( \gamma \geq 1 \) the congestion parameter. Then, the algorithm \textsc{RandomWalkSchur} runs in \( O(\alpha^{-1}\gamma Q(\rho) \log^2 n/e^2) \) rounds, and returns a graph \( H \) along with its \( (\alpha^{-1}\gamma \log \nu \rho) \)-minor distribution into \( \mathcal{G} \) such that

\[
\mathcal{L}(H) \approx \epsilon \text{SC}(G,\hat{T}), \tag{16}
\]

with high probability, where \( \hat{T} \supseteq T \) has size at most \( n - |F| + O(\alpha^{-1}m\epsilon^{-2} \log^2 n/\gamma) \).

**Proof.** Let us briefly describe the \textsc{RandomWalkSchur} algorithm. First, we compute the expected congestion of the family of random walks \( W \) predicted by Lemma A.13 with respect to the set of terminals \( T \). This is done by propagating the congestion to neighbors for \( O(\alpha^{-1}\log n) \) steps. Then, we create a new set \( \hat{T} \) which includes \( T \) as a subset, as well as all the nodes which exceeded the congestion threshold of \( \gamma \), based on the estimation procedure of the previous step. Note that the congestion of a node with respect to \( W \) is simply the number of times this particular node participates in some random walk of \( W \). By construction, it follows that the size of \( \hat{T} \) is \( n - |F| \) along with all the nodes that exceeded the congestion threshold of \( \gamma \). However, since \( F \) is an \( \alpha \)-DD set it follows that the length of a random walk is \( O(\alpha^{-1} \log n) \) with high probability, while for every edge we simulate \( \mu = O(\log n/e^2) \) random walks (this is related to the concentration of the corresponding random variables, as implied by Lemma A.13), in turn implying that the total congestion generated by these random walks is \( O(\alpha^{-1}m\epsilon^{-2} \log^2 n) \). As a result, only \( O(\alpha^{-1}m\epsilon^{-2} \log^2 n/\gamma) \) nodes can have congestion more than \( \gamma \), verifying the assertion regarding the size of \( \hat{T} \). Next, the algorithm implements the random walks of Lemma A.13, but with respect
to the augmented set of terminals $\hat{T}$. A Chernoff bound argument assures us that all nodes in $V \setminus \hat{T}$ will have congestion $O(\gamma)$ with high probability.

In terms of the distributed implementation, estimating the congestion can be implemented in $O(n^{-1}Q(\rho)\log^2 n/\epsilon^2)$ rounds; this follows since every walk has length $O(n^{-1}\log n)$ with high probability, and we execute $\mu = O(\log n/\epsilon^2)$ iterations for every edge. Also note that a single step in the congestion-estimation procedure can be implemented in $O(n^{-1}\gamma Q(\rho)\log^2 n/\epsilon^2)$ rounds with high probability; this uses the aforementioned guarantee for the congestion. The final step is to minor-distribute the graph $H$ with weights as dictated by Lemma A.13. This is done by assigning to the terminals the leaders of all intermediate (non-terminal) nodes. The congestion guarantee ensures that the resulting mapping is an $O(n^{-1}\gamma\log n\rho)$-minor distribution into $\overline{G}$.

Proof of Lemma 4.2. The Eliminate algorithm proceeds in $d$ rounds, initializing $M^{(0)}$ to be an $\epsilon$-spectral sparsifier of $L(G)$ (recall Corollary A.2). In every round $i \geq 1$, (i) we compute an $\alpha$-DD set $F_i$ with $\alpha := 4$, (ii) we employ Lemma A.12 in order to have access to an operator that approximates $M^{(i-1)}_{i[F,F]}$, and (iii) we compute an $\epsilon$-spectral sparsifier $M^{(i)}$ of the Schur complement $SC(M^{(i-1)},\hat{T}_i)$ approximated via Lemma A.14; here, $\hat{T}_i = \hat{T}_{i-1} - F_i + U_i$, where $U_i$ represents the set of extra nodes added to ensure low congestion. In particular, Lemma A.14 is invoked with congestion parameter $\gamma := 1000Cn^{-1}\log^2 n/\epsilon^2$, where $C$ is a sufficiently large constant. The sparsification algorithm of Koutis Corollary A.2 tells us that the number of edges will be $m = (n\log^6 n/\epsilon^2)$, in turn implying that the number of nodes drops by at least a multiplicative factor of $49/50$.

In terms of the distributed implementation, notice that due to the selection of the parameters the approximation of the Schur complement (Lemma A.14) can be performed in $O(Q(\rho)\log^{10} n/\epsilon^6)$ rounds. Next, the spectral sparsification step can be implemented in $O(Q(\rho')\log^7 n/\epsilon^2)$, where $\rho' = n^{-1}\gamma\rho = O(\log^2 n/\epsilon^2)\rho$. Thus, by virtue of Assumption 3.1 we can infer that $Q(\rho') = O(\log^c n/\epsilon^2)Q(\rho)$, where $c$ is some constant. Thus, after $d$ iterations the cost of these operations is bounded by $O(Q(\rho')\log^c n/\epsilon^2)^d)$, where $c$ is some constant. Finally, the error guarantee follows directly from the previous lemmas (Lemma A.12, Lemma A.13 and Lemma A.14), and arguing about the accumulation of the error.

\[ \square \]

A.4 Minor Schur Complement: Proof of Lemma 4.3

We commence this subsection by introducing the notion of steady edges, which are in a sense edges which are mutually "uncorrelated":

Definition A.15 ([For+20]). A stochastic subset of edges $Z \subseteq E$ is called $(\alpha, \delta)$-steady with respect to an $m$-edge graph $H$ if

1. $E_z \left[ \sum_{e \in Z} r_e^{-1} b_e b_e^T \right] \leq \alpha L(H)$;

2. For all $e \in Z$ we have $\sum_{e \neq f \in Z} \frac{|b_e^T L(H) b_f|}{\sqrt{\epsilon_e \epsilon_f}} \leq \delta$;

3. For all $e \in Z$ it holds that
\[ r_e^{-1} b_e^T \mathcal{L}(H)^\dagger \begin{bmatrix} \text{SC}(H, T) & 0 \\ 0 & 0 \end{bmatrix} \mathcal{L}(H)^\dagger b_e \leq \frac{32|T|}{m}. \]  

(17)

In words, the first constraint ensures that no edge will be selected in the steady set with too high of a probability, the second corresponds to the localization constraint, circumscribing the (mutual) correlation of edges within the set, while the final constraint imposes a bound on the variance, and will be used in the martingale analysis (recall Freedman’s inequality). It should be stressed that the existence of such objects follows from the localization of electrical flows [SRS18].

In the distributed setting, the following result will be established:

**Lemma A.16 ([For+20]).** Let \( G \) be an \( n \)-node m-edge graph \( \rho \)-minor distributed into \( \overline{G} \) for which Assumption 3.1 holds for some \( Q = Q(\rho) \). For a constant \( \delta \in (0, 1) \) and a subset of terminals \( T \subseteq V(G) \), there exists an algorithm which has access to a distributed Laplacian solver, and returns with high probability a set of at least \( \delta m / (2000C \log^2 m) \) edges in expectation which is \( (\delta / (1000C \log^2 m), \delta) \)-steady, where \( C \) is a sufficiently large constant. This algorithm requires \( O(\log^2 n) \) calls to a distributed Laplacian solver to \( 1 / \text{poly}(n) \) accuracy on graphs that \( 2\rho \)-minor distribute into \( \overline{G} \), and \( O(Q(\rho) \log^2 n) \) communication rounds.

The first step towards establishing this lemma is to approximate the correlation of edges within some arbitrary set:

**Lemma A.17 ([For+20]).** Let \( G \) be an \( n \)-node graph with resistances \( r_e, \rho \)-minor distributed into a communication network \( \overline{G} \) for which Assumption 3.1 holds for some \( Q = Q(\rho) \). Then, there is an algorithm, with access to a distributed Laplacian solver, which for any subset \( W \subseteq E(G) \) and any edge \( e \in W \) returns with high probability the quantity

\[ \sum_{e \neq f \in W} \frac{|b_e^T \mathcal{L}(G)^\dagger b_e|}{\sqrt{r_e} \sqrt{r_f}} \]  

(18)

to within a factor of 2. This algorithm requires \( O(\log^2 n) \) calls to a distributed Laplacian solver on graphs that \( \rho \)-minor distribute into \( \overline{G} \) to accuracy \( 1 / \text{poly}(n) \), and an additional \( O(Q(\rho) \log^2 n) \) communication rounds.

The proof of this lemma follows directly from [For+20, Lemma 5.13], and leverages the \( \ell_1 \)-sketch of Indyk [Ind06]. Similarly, a sketch can be employed to estimate the effect of each edge on the Schur complement:

**Lemma A.18 ([For+20]).** Let \( G \) be an \( n \)-node with resistances \( r_e, \rho \)-minor distributed into a communication network \( \overline{G} \) for which Assumption 3.1 holds for some \( Q = Q(\rho) \). Then, for a subset \( T \subseteq V(G) \), there exists an algorithm which returns with high probability an estimate of

\[ r_e^{-1} b_e^T \mathcal{L}(G)^\dagger \begin{bmatrix} \text{SC}(G, T) & 0 \\ 0 & 0 \end{bmatrix} \mathcal{L}(G)^\dagger b_e \]  

(19)

to within a factor of 2. This algorithm requires \( O(\log n) \) calls to a distributed Laplacian solver to accuracy \( 1 / \text{poly}(n) \) on graphs that \( 2\rho \)-minor distribute into \( \overline{G} \), and \( O(Q(\rho) \log n) \) communication rounds.
As a result, Lemma A.16 is established based on the algorithm \texttt{FindSteady} in [For+20], with the round complexity guarantee following directly from Lemma A.17 and Lemma A.18. The next ingredient is a pre-processing step which ensures that all the edges have leverage scores bounded away from 0 and 1.

**Lemma A.19** ([LS18]). Let \( G \) be an \( n \)-node graph \( \rho \)-minor distributed into a communication network \( \overline{G} \) for which Assumption 3.1 holds for some \( Q = Q(\rho) \). If 1.1-approximate leverage scores \( \tilde{\text{lev}}_G(e) \) for the edges in \( G \) are known, then there exists a process which returns after \( \tilde{O}(Q(\rho)) \) rounds a graph \( H \) such that

1. \( H \) is electrically equivalent to \( G \);
2. \( H \) is \( 2\rho \)-minor distributed into \( \overline{G} \);
3. All the leverage scores of edges in \( H \) are between \( \left[ \frac{3}{16}, \frac{13}{16} \right] \).

Moreover, there exists a procedure which takes as input \( G \), and returns in \( O(Q(\rho)) \) rounds a graph resulting from collapsing paths and parallel edges, and removing non-terminal leaves, along with a \( \rho \)-minor distribution into \( \overline{G} \).

The distributed implementation of this lemma is fairly simple, and relies on Lemma A.3. We will also use the following lemma, which is based on the random projection scheme of Spielman and Srivastava [SS08]:

**Lemma A.20** ([For+20]). Let \( G \) be an \( n \)-node graph \( \rho \)-minor distributed into a communication network \( \overline{G} \) for which Assumption 3.1 holds for some \( Q = Q(\rho) \). Then, there is an algorithm with access to a distributed Laplacian solver which for all edges \( e \in E(G) \) approximates the leverage score \( \text{lev}_G(e) \) to within a factor of \( 1 + \delta \) with high probability. This algorithm requires \( O(\log n / \delta^2) \) calls to a distributed Laplacian solver on graphs which \( \rho \)-minor distribute into \( \overline{G} \) to accuracy \( 1 / \text{poly}(n) \), as well as \( O(Q(\rho) \log n / \delta^2) \) communication rounds.

The proof of this lemma follows directly from [For+20, Lemma 5.4], and uses Achliopta’s variant of the Johnson-Lindenstrauss lemma [Ach03]. With these pieces at hand, we are ready to describe the algorithm for computing a minor Schur complement. In particular, at each iteration we first determine a set of steady edges via Lemma A.16. Then, we estimate the leverage scores via the random projection scheme of Lemma A.20, and each edge in the steady set is contracted (independently) with probability given by its (approximate) leverage scores; otherwise, the edge is deleted (for this we will use Corollary A.4). We also employ Lemma A.19 in every iteration in order to ensure that leverage scores are bounded away from 0 and 1. This process is repeated as long as the number of edges exceeds a threshold, leading to the algorithm \textsc{ApproxSC} in [For+20].

The next theorem was shown in [For+20] using matrix martingale analysis:

**Lemma A.21** ([For+20]). The algorithm \textsc{ApproxSC} takes as input a graph \( G \) with a set of terminals \( T \) and an error parameter \( \epsilon \), and returns with high probability a graph \( H \) satisfying \( |E(H)| = O(|T| \log^2 n / \epsilon^2) \) and \( \text{SC}(H, T) \approx \epsilon \text{SC}(G, T) \).

**Proof of Lemma 4.3.** First, the algorithm only performs deletions and contractions, implying that it indeed returns a minor. Moreover, the correctness follows directly from Lemma A.21. To bound the requirements of the algorithm note that \textsc{ApproxSC} executes \( O(\log m / \alpha) \) iterations, where \( \alpha := \delta / (1000C \log^2 m) = O(\epsilon / \log^4 m) \), with high probability. In each iteration the dominant cost in
where we used that

\[ |V(G)\| \leq |V(G_1)|2^{O((\sqrt{\log n \log \log n})/k)} \]

this follows since we have sparsified the graph in the first step. Thus, for \( k = 2^{(\log \pi)^{2/3}} \) we can infer that \( |V(G_2)| \leq |V(G_1)|/k^{1-o(1)} \). Next, with regards to the Schur complement chain, Lemma 4.2 and Lemma 4.3 imply that \( |V(G_{i+1})| \leq |V(G_i)|O(0.98^d \log^2 n/e^2) \). Hence, setting \( d = 2^{(\log \pi)^2} \) and \( \epsilon = 1/(\log \pi)^2 \) gives us that \( |V(G_{i+1})| \leq |V(G_i)|2^{-\Theta((\log \log \pi)^2)} \).

As a result, BUILDCHAIN returns a \( (2^{\Theta((\log \pi)^2)}, \epsilon) \)-Schur complement chain, which in turn implies that this chain has length \( O(\log \pi/(\log \log \pi)^2) \). Thus, Lemma 4.5 implies that we can use this chain to produce a solution in \( O(\rho \pi^{o(1)} Q(\rho)) \) rounds, where \( \rho \) represents the maximum congestion of a graph along the chain; it will be establish that \( \rho = O(\pi^{o(1)}) \).

Let \( f(n, \rho) \) represent the number of rounds required by SOLVER on a graph with \( n \) nodes which \( \rho \)-minor distributes into \( \overline{G} \), and \( g(n, \rho) \) the number of rounds required by BUILDCHAIN with input an \( n \)-node graph which \( \rho \)-minor distributes into \( \overline{G} \). Then, if we ignore lower order terms, it follows that

\[ f(n, \rho) = O(\rho \pi^{o(1)} Q(\rho)) + g(n/k^{1-o(1)}, \rho), \]

where we used that \( |V(G_2)| \leq |V(G_1)|/k^{1-o(1)} \). Moreover, we have that

\[ g(n, \rho) = O((\log^e n/e^c)(\log \log \pi)^2 Q(\rho)) + f(n, 2\rho)O(\log^{10} n/e^3) + g(n/2^{\Theta((\log \log \pi)^2)}, \rho) \]

\[ = O(\pi^{o(1)} Q(\rho)) + O(\text{polylog}(\pi))f(n, 2\rho) + g(n/2^{\Theta((\log \log \pi)^2)}, \rho), \]

where we used that \( |V(G_{i+1})| \leq |V(G_i)|2^{-\Theta((\log \log \pi)^2)} \), and we ignored lower order terms. As a result, the overall increase in congestion is \( 2^{O(\log \pi/(\log \log \pi)^2)} = O(\pi^{o(1)}) \). That is, all the graphs constructed \( O(\pi^{o(1)}) \)-minor distribute into \( \overline{G} \). Finally, the theorem follows since by Assumption 3.1 the dependence of \( Q(\rho) \) on \( \rho \) is polynomial. \( \square \)