Towards a Microscopic Derivation of the 
Phonon Boltzmann Equation

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1 Introduction

The thermal conductivity of insulating (dielectric) crystals is computed almost exclusively on the basis of the phonon Boltzmann equation. We refer to \cite{1} for a discussion more complete than possible in this contribution. On the microscopic level the starting point is the Born-Oppenheimer approximation (see \cite{2} for a modern version), which provides an effective Hamiltonian for the slow motion of the nuclei. Since their deviation from the equilibrium position is small, one is led to a wave equation with a weak nonlinearity. As already emphasized by R. Peierls in his seminal work \cite{3}, physically it is of importance to retain the structure resulting from the atomic lattice, which forces the discrete wave equation.

On the other hand, continuum wave equations with weak nonlinearity appear in the description of the waves in the upper ocean and in many other fields. This topic is referred to as weak turbulence. Again the theoretical treatment of such equations is based mostly on the phonon Boltzmann equation, see e.g. \cite{4}. In these applications one considers scales which are much larger than the atomistic scale, hence quantum effects are negligible. For dielectric crystals, on the other side, quantum effects are of importance at low temperatures. We refer to \cite{1} and discuss here only the classical discrete wave equation with a small nonlinearity.

If one considers crystals with a single nucleus per unit cell, then the displacement field is a 3-vector field over the crystal lattice $\Gamma$. The nonlinearity results from the weakly non-quadratic interaction potentials between the nuclei. As we will see, the microscopic mechanism responsible for the validity of the Boltzmann equation can be understood already in case the displacement field is declared to be scalar, the nonlinearity to be due to an on-site potential, and the lattice $\Gamma = \mathbb{Z}^3$. This is the model I will discuss in my notes.

As the title indicates there is no complete proof available for the validity of the phonon Boltzmann equation. The plan is to explain the kinetic scaling and to restate our conjecture in terms of the asymptotics of certain Feynman diagrams.

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2 Microscopic model

We consider the simple cubic crystal $\mathbb{Z}^3$. The displacement field is denoted by

$$q_x \in \mathbb{R}, \quad x \in \mathbb{Z}^3,$$

with the canonically conjugate momenta

$$p_x \in \mathbb{R}, \quad x \in \mathbb{Z}^3.$$

We use units in which the mass of the nuclei is $m = 1$. The particles interact harmonically and are subject to an on-site potential, which is divided into a quadratic part and a non-quadratic correction. Thus the Hamiltonian of the system reads

$$H = \frac{1}{2} \sum_{x \in \mathbb{Z}^3} \left( p_x^2 + \omega_0^2 q_x^2 \right) + \frac{1}{2} \sum_{x, y \in \mathbb{Z}^3} \alpha(x - y) q_x q_y + \sum_{x \in \mathbb{Z}^3} V(q_x) = H_0 + \sum_{x \in \mathbb{Z}^3} V(q_x).$$

(2.3)

The coupling constants have the properties

$$\alpha(x) = \alpha(-x),$$

$$|\alpha(x)| \leq \alpha_0 e^{-\gamma|x|}$$

(2.4)

(2.5)

for suitable $\alpha_0, \gamma > 0$, and

$$\sum_{x \in \mathbb{Z}^3} \alpha(x) = 0,$$

(2.6)

because of the invariance of the interaction between the nuclei under the translation $q_x \sim q_x + a$.

For the anharmonic on-site potential we set

$$V(u) = \sqrt{\varepsilon} \frac{1}{3} \lambda u^3 + \varepsilon \left( \frac{\lambda^2}{18} \omega_0^2 \right) u^4, \quad u \in \mathbb{R}.$$

(2.7)

$\varepsilon$ is the dimensionless scale parameter, eventually $\varepsilon \to 0$. The quartic piece is added so to make sure that $H \geq 0$. In the limit $\varepsilon \to 0$ its contribution will vanish and for simplicity of notation we will omit it from the outset. Then the equations of motion are

$$\frac{d}{dt} q_x(t) = p_x(t),$$

$$\frac{d}{dt} p_x(t) = -\sum_{y \in \mathbb{Z}^3} \alpha(y - x) q_y(t) - \omega_0^2 q_x(t) - \sqrt{\varepsilon} \lambda q_x(t)^2, \quad x \in \mathbb{Z}^3.$$

(2.8)

We will consider only finite energy solutions. In particular, it is assumed that $|p_x| \to 0$, $|q_x| \to 0$ sufficiently fast as $|x| \to \infty$. In fact, later on there will be the need to impose random initial data, which again are assumed to be supported on
finite energy configurations. In the kinetic limit the average energy will diverge as \( \varepsilon^{-3} \).

It is convenient to work in Fourier space. For \( f : \mathbb{Z}^3 \to \mathbb{R} \) we define
\[
\hat{f}(k) = \sum_{x \in \mathbb{Z}^3} e^{-i2\pi k \cdot x} f_x ,
\]

\( k \in \mathbb{T}^3 = [-\frac{1}{2}, \frac{1}{2}]^3 \), with inverse
\[
f_x = \int_{\mathbb{T}^3} dk e^{i2\pi k \cdot x} \hat{f}(k) ,
\]

\( dk \) the 3-dimensional Lebesgue measure. The dispersion relation for the harmonic part \( H_0 \) is
\[
\omega(k) = (\omega_0^2 + \hat{\alpha}(k))^{1/2} \geq \omega_0 > 0 ,
\]
since \( \hat{\alpha}(k) > 0 \) for \( k \neq 0 \) because of the mechanical stability of the harmonic lattice with vanishing on-site potential.

In Fourier space the equations of motion read
\[
\frac{\partial}{\partial t} \tilde{q}(k, t) = \hat{p}(k, t) ,
\]
\[
\frac{\partial}{\partial t} \tilde{p}(k, t) = -\omega(k)^2 \tilde{q}(k, t) - \sqrt{\varepsilon \lambda} \int_{\mathbb{T}^6} dk_1 dk_2 \delta(k - k_1 - k_2) \tilde{q}(k_1, t) \tilde{q}(k_2, t) ,
\]

with \( k \in \mathbb{T}^3 \). Here \( \delta \) is the \( \delta \)-function on the unit torus, to say, \( \delta(k') \) carries a point mass whenever \( k' \in \mathbb{Z}^3 \).

It will be convenient to concatenate \( q_x \) and \( p_x \) into a single complex-valued field. We set
\[
a(k) = \frac{1}{\sqrt{2}} \left( \sqrt{\omega} \tilde{q}(k) + i \frac{1}{\sqrt{\omega}} \tilde{p}(k) \right) ,
\]

with the inverse
\[
\tilde{q}(k) = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{\omega}} (a(k) + a(-k)^*) , \quad \tilde{p}(k) = \frac{1}{\sqrt{2}} i \sqrt{\omega} \left( -a(k) + a(-k)^* \right) .
\]

To have a concise notation, we introduce
\[
a(k, +) = a(k)^* , \quad a(k, -) = a(k) .
\]

Then the \( a \)-field evolves as
\[
\frac{\partial}{\partial t} a(k, \sigma, t) = i \sigma \omega(k) a(k, \sigma, t) + i \sigma \sqrt{\varepsilon \lambda} \sum_{\sigma_1, \sigma_2 = \pm 1} \int_{\mathbb{T}^6} dk_1 dk_2 \left( 8 \omega(k) \omega(k_1) \omega(k_2) \right)^{-1/2} \delta(-\sigma k + \sigma_1 k_1 + \sigma_2 k_2) a(k_1, \sigma_1, t) a(k_2, \sigma_2, t) .
\]

(2.16)
3 Kinetic limit and Boltzmann equation

The kinetic limit deals with a special class of initial probability measures. Their displacement field has a support of linear size $\varepsilon^{-1}$ and average energy of order $\varepsilon^{-3}$. More specifically, these probability measures have the property of being locally Gaussian and almost stationary under the dynamics. Because of the assumed slow variation in space the covariance of such probability measures changes only slowly, i.e. on the scale $\varepsilon^{-1}$, in time.

Let us assume then that the initial data for (2.16) are random and specified by a Gaussian probability measure on phase space. It is assumed to have mean

$$\langle a(k, \sigma) \rangle^G_{\varepsilon} = 0 \quad (3.1)$$

and for the covariance we set

$$\langle a(k, \sigma)a(k', \sigma) \rangle^G_{\varepsilon} = 0 , \quad (3.2)$$

$$W_{\varepsilon}(y, k) = \varepsilon^3 \int_{(T/\varepsilon)^3} d\eta e^{i2\pi y \cdot \eta} \langle a(k - \varepsilon\eta/2, +)a(k + \varepsilon\eta/2, -) \rangle^G_{\varepsilon}, \quad (3.3)$$

$y \in (\varepsilon \mathbb{Z})^3$, which defines the Wigner function rescaled to the lattice $(\varepsilon \mathbb{Z})^3$. Local stationarity is ensured by the condition

$$\lim_{\varepsilon \to 0} W_{\varepsilon}([r]_{\varepsilon}, k) = W^0(r, k) , \quad (3.4)$$

where $[r]_{\varepsilon}$ denotes integer part modulo $\varepsilon$. Note that $W_{\varepsilon}$ is normalized as

$$\sum_{y \in (\varepsilon \mathbb{Z})^3} \int_{T^3} dk W_{\varepsilon}(y, k) = \int_{T^3} dk \langle a(k, +)a(k, -) \rangle^G_{\varepsilon} . \quad (3.5)$$

The condition that the limit in (3.4) exists thus implies that the average phonon number increases as $\varepsilon^{-3}$, equivalently the average total energy increases as

$$\langle \int_{T^3} d^3k \omega(k)a(k, +)a(k, -) \rangle^G_{\varepsilon} = \langle H_0 \rangle^G_{\varepsilon} = O(\varepsilon^{-3}) . \quad (3.6)$$

Let $\langle \cdot \rangle_t$ be the time-evolved measure at time $t$. Its rescaled Wigner function is

$$W_{\varepsilon}(y, k, t) = \varepsilon^3 \int_{(T/\varepsilon)^3} d\eta e^{i2\pi y \cdot \eta} \langle a(k - \varepsilon\eta/2, +)a(k + \varepsilon\eta/2, -) \rangle_{t/\varepsilon} . \quad (3.7)$$

Kinetic theory claims that

$$\lim_{\varepsilon \to 0} W_{\varepsilon}([r]_{\varepsilon}, k, t) = W(r, k, t) , \quad (3.8)$$
where \( W(r, k, t) \) is the solution of the phonon Boltzmann equation

\[
\frac{\partial}{\partial t} W(r, k, t) + \frac{1}{2\pi} \nabla \omega(k) \cdot \nabla_r W(r, k, t) = \frac{\pi}{2} \lambda^2 \sum_{\sigma_1, \sigma_2 = \pm 1} \int_T dk_1 dk_2 (\omega(k) \omega(k_1) \omega(k_2))^{-1} \delta(\omega(k) + \sigma_1 \omega(k_1) + \sigma_2 \omega(k_2)) \delta(k + \sigma_1 k_1 + \sigma_2 k_2) (W(r, k, t) W(r, k_1, t) + \sigma_1 W(r, k, t) W(r, k_2, t) + \sigma_2 W(r, k, t) W(r, k, t))
\]

(3.9)

to be solved with the initial condition \( W(r, k, 0) = W^0(r, k) \).

The free streaming part is an immediate consequence of the evolution of \( W \) as generated by \( H_0 \). The strength of the cubic nonlinearity was assumed to be of order \( \sqrt{\varepsilon} \), which results in an effect of order 1 on the kinetic time scale. The specific form of the collision operator will be explained in the following section. It can be brought into a more familiar form by performing the sum over \( \sigma_1, \sigma_2 \). Then the collision operator has two terms. The first one describes the merging of two phonons with wave number \( k \) and \( k_1 \) into a phonon with wave number \( k_2 = k + k_1 \), while the second term describes the splitting of a phonon with wave number \( k \) into two phonons with wave numbers \( k_1 \) and \( k_2, k = k_1 + k_2 \). In such a collision process energy is conserved and wave number is conserved modulo an integer vector.

In (4.7) the summand with \( \sigma_1 = 1 = \sigma_2 \) vanishes trivially. However it could be the case that the condition for energy conservation,

\[
\omega(k) + \omega(k') - \omega(k + k') = 0,
\]

(3.10)

has also no solution. If so, the collision operator vanishes. In fact, for nearest neighbor coupling only, \( \alpha(0) = 6, \alpha(e) = -1 \) for \( |e| = 1, \alpha(x) = 0 \) otherwise, it can be shown that (3.10) has no solution whenever \( \omega_0 > 0 \). To have a non-zero collision term we have to require

\[
\int dk \int dk' \delta(\omega(k) + \omega(k') - \omega(k + k')) > 0,
\]

(3.11)

which is an implicit condition on the couplings \( \alpha(x) \). A general condition to ensure (3.11) is not known. A simple example where (3.11) can be checked by hand is

\[
\omega(k) = \omega_0 + \sum_{\alpha=1}^3 (1 - \cos(2\pi k^\alpha)), \quad k = (k^1, k^2, k^3).
\]

(3.12)

It corresponds to suitable nearest and next nearest neighbor couplings.

There is a second more technical condition which requires that

\[
\sup_k \int dk' \delta(\omega(k) + \omega(k') - \omega(k + k')) = c_0 < \infty.
\]

(3.13)

It holds for the dispersion relation (3.12). This uniform bound allows for a simple proof that the Boltzmann equation has a unique solution for short times provided \( W^0(r, k) \) is bounded.
4 Feynman diagrams

Denoting by \( \langle \cdot \rangle \) the average with respect to the measure at time \( t \) (in microscopic units), the starting point of the time-dependent perturbation series is the identity

\[
\langle \prod_{j=1}^{n} a(k_j, \sigma_j) \rangle_t = \exp \left[ it \left( \sum_{j=1}^{n} \sigma_j \omega(k_j) \right) \right] \langle \prod_{j=1}^{n} a(k_j, \sigma_j) \rangle^G
\]

\[
+i \sqrt{\varepsilon} \int_{0}^{t} ds \exp \left[ i(t-s) \left( \sum_{j=1}^{n} \sigma_j \omega(k_j) \right) \right]
\]

\[
\left( \sum_{\ell=1}^{n} \sum_{\sigma', \sigma'' = \pm 1} \sigma_{\ell} \int_{\mathbb{R}^6} dk' dk'' \phi(k_{\ell}, k', k'') \delta(-\sigma_{\ell} k_{\ell} + \sigma' k' + \sigma'' k'') \right)
\]

\[
\langle \left( \prod_{j=1}^{n} a(k_j, \sigma_j) \right) a(k', \sigma') a(k'', \sigma'') \rangle_s . \tag{4.1}
\]

Here

\[
\phi(k, k', k'') = \lambda(8 \omega(k) \omega(k') \omega(k''))^{-1/2} \tag{4.2}
\]

One starts with \( n = 2 \) and \( \sigma_1 = 1, \sigma_2 = -1 \). Then on the right hand side of (4.1) there is the product of three \( a \)'s. One resubstitutes (4.1) with \( n = 3 \), etc. Thereby one generates an infinite series, in which only the average over the initial Gaussian measure \( \langle \cdot \rangle^G \) appears.

To keep the presentation transparent, let me assume that \( \langle \cdot \rangle^G \) is a translation invariant Gaussian measure with

\[
\langle a(k, \pm) \rangle^G = 0 , \quad \langle a(k, \sigma) a(k', \sigma') \rangle^G = 0 , \quad \langle a(k, +) a(k', -) \rangle^G = \delta(k - k') W(k) . \tag{4.3}
\]

Then the measure at time \( t \) is again translation invariant. Kinetic scaling now merely amounts to considering the long times \( t/\varepsilon \). The Wigner function at that time is then represented by the infinite series

\[
\langle a(q, -) a(p, +) \rangle_{t/\varepsilon} = \delta(q - p) \left( W(q) + \sum_{n=1}^{\infty} W_n^\varepsilon(q, t) \right) . \tag{4.4}
\]

The infinite sum is only formal. Taking naively the absolute value at iteration \( 2n \) one finds that

\[
|W_n^\varepsilon(q, t)| \leq \varepsilon^n (t/\varepsilon)^{2n} ((2n)!)^{-1} (2n)! c^{2n} ((2n + 2)!/2^{n+1} (n + 1)!) . \tag{4.5}
\]

Here \( \varepsilon^n = (\sqrt{\varepsilon})^{2n} \), \( (t/\varepsilon)^{2n}/(2n)! \) comes from the time integration, \( (2n)! \) from the sum over \( \ell \) in (4.1), \( c^{2n} \) from the \( k \)-integrations and the initial \( W(k) \), and the factor \( (2n + 2)!/2^{n+1} (n + 1)! \) from the Gaussian pairings in the initial measure. Thus even at fixed \( \varepsilon \) there are too many terms in the sum.
Since no better estimate is available at present, we concentrate on the structure of a single summand \( W_n^\varepsilon(q, t) \). \( \delta(q - p) W_n^\varepsilon(q, t) \) is a sum of integrals. The summation comes from

- the sum over \( \sigma', \sigma'' \) in (4.1)
- the sum over \( \ell \) in (4.1)
- the sum over all pairings resulting from the average with respect to the initial Gaussian measure \( \langle \cdot \rangle^G \).

Since each single integral has a rather complicated structure, it is convenient to visualize them as Feynman diagrams.

A Feynman diagram is a graph with labels. Let us first explain the graph. The graph consists of two binary trees. It is convenient to draw them on a “backbone” consisting of \( 2n + 2 \) equidistant horizontal level lines which are labelled from 0 (bottom) to \( 2n + 1 \) (top). The two roots of the tree are two vertical bonds from line \( 2n + 1 \) to level line \( 2n \). At level \( m \) there is exactly one branch point with two branches in either tree. Thus there are exactly \( 2n \) branch points. At level 0 there are then \( 2n + 2 \) branches. They are connected according to the pairing rule, see Figure below.

In the Feynman graph each bond is oriented with arrows pointing either up (\( \sigma = +1 \)) or down (\( \sigma = -1 \)). The left root is down while the right root is up. If there is no branching the orientation is inherited from the upper level. At a pairing the orientation must be maintained. Thus at level 0 a branch with an up arrow can be paired only with a branch with a down arrow, see (4.3). Every internal line in the graph must terminate at either end by a branch point. Every such internal line admits precisely two orientations.
Next we insert the labels. The level lines 0 to $2n + 1$ are labelled by times $0 < t_1 \ldots < t_{2n} < t$. The left root carries the label $q$ while the right root carries the label $p$. Each internal line is labelled with a wave number $k$.

To each Feynman diagram one associates an integral through the following steps.

(i) The time integration is over the simplex $0 \leq t_1 \ldots \leq t_{2n} \leq t$ as $dt_1 \ldots dt_{2n}$.
(ii) The wave number integration is over all internal lines as $\int dk_1 \ldots \int dk_n$, where $\kappa = 3n - 1$ is the number of internal lines.
(iii) One sums over all orientations of the internal lines. The integrand is a product of three factors.

(iv) There is a product over all branch points. At each branchpoint there is a root, say wave vector $k_1$ and orientation $\sigma_1$, and there are two branches, say wave vectors $k_2, k_3$ and orientations $\sigma_2, \sigma_3$. Then each branch point carries the weight

$$\delta(-\sigma_1 k_1 + \sigma_2 k_2 + \sigma_3 k_3)\sigma_1 \phi(k_1, k_2, k_3).$$

If one regards the wave vector $k$ as a current with orientation $\sigma$, then (4.6) expresses Kirchhoff’s rule for conservation of the current.

(v) By construction each bond carries a time difference $t_{m+1} - t_m$, a wave vector $k$, and an orientation $\sigma$. Then to this bond one associates the phase factor

$$\exp[i(t_{m+1} - t_m)\sigma \omega(k)/\varepsilon].$$

The second factor is the product of such phase factors over all bonds.

(vi) The third factor of the integrand is given by

$$\prod_{j=1}^{n+1} W(k_j),$$

where $k_1, \ldots, k_{n+1}$ are the wave numbers of the bonds between level 0 and level 1.

(vii) Finally there is the prefactor $(-1)^n \varepsilon^{-n}$.

To illustrate these rules we give an example for $n = 2$, see Figure above. The associated integral is given by, more transparently keeping the $\delta$-functions from the pairings,

$$\varepsilon^{-2} \int_{0 \leq t_1 \leq \ldots \leq t_4 \leq t} dt_1 \ldots dt_4 \int_{\mathbb{T}^{24}} dk_1 \ldots dk_8 \delta(q + k_1 - k_2)\delta(k_2 + k_3 + k_4)\delta(-p - k_5 - k_6)\delta(-k_1 + k_7 - k_8) \phi(q, k_1, k_2)\phi(k_2, k_3, k_4)\phi(p, k_5, k_6)\phi(k_1, k_7, k_8) \delta(k_7 - k_6)W(k_7)\delta(k_8 - k_3)W(k_8)\delta(k_4 - k_5)W(k_4) \exp \left[ \{i(t_1 - t_4)(-\omega_q + \omega_p) + i(t_4 - t_3)(\omega_1 - \omega_2 + \omega_p) + i(t_3 - t_2)(\omega_1 + \omega_3 + \omega_4 + \omega_p) + i(t_2 - t_1)(\omega_1 + \omega_3 + \omega_4 - \omega_5 - \omega_6) + it_1(\omega_7 - \omega_8 + \omega_3 + \omega_4 - \omega_5 - \omega_6) \}/\varepsilon \right]$$

(4.9)
with \( \omega_q = \omega(p), \omega_p = \omega(p), \omega_j = \omega(k_j). \)

\( \delta(q - p)W_n^\varepsilon(q, t) \) is the sum over all Feynman diagrams with \( 2n + 2 \) levels and thus is a sum of oscillatory integrals. In the limit \( \varepsilon \to 0 \) only a few leading terms survive while all remainders vanish. E.g., the Feynman diagram above is subleading. In fact, the conjecture of kinetic theory can be stated rather concisely:

**Kinetic Conjecture:** In a leading Feynman diagram the Kirchhoff rule never forces an identification of the form \( \delta(k_j) \) with some wave vector \( k_j \). In addition, the sum of the \( 2(n - m + 1) \) phases from the bonds between level lines \( 2m \) and \( 2m + 1 \) vanishes for every choice of internal wave numbers. This cancellation must hold for \( m = 0, \ldots, n. \)

Since we assumed the initial data to be spatially homogeneous, the phonon Boltzmann equation (3.10) simplifies to

\[
\frac{\partial}{\partial t} W(k, t) = 4\pi \lambda^2 \sum_{\sigma_1, \sigma_2 = \pm 1} \int_{\mathbb{T}^6} dk_1 dk_2 \phi(k, k_1, k_2)^2 \delta(\omega(k) + \sigma_1 \omega(k_1) + \sigma_2 \omega(k_2)) \delta(k + \sigma_1 k_1 + \sigma_2 k_2)(W(k_1, t)W(k_2, t) + 2\sigma_2 W(k, t)W(k_1, t)),
\]

(4.10)

where we used the symmetry with respect to \( (k_1, \sigma_1) \) and \( (k_2, \sigma_2) \). To (4.10) we associate the Boltzmann hierarchy

\[
\frac{\partial}{\partial t} f_n = \mathcal{C}_{n,n+1} f_{n+1}, \quad n = 1, 2, \ldots,
\]

(4.11)

acting on the symmetric functions \( f_n(k_1, \ldots, k_n) \) with

\[
\mathcal{C}_{n,n+1} f_{n+1}(k_1, \ldots, k_n) = 4\pi \lambda^2 \sum_{\ell = 1}^n \sum_{\sigma', \sigma'' = \pm 1} \int_{\mathbb{T}^6} dk' dk'' \phi(k_\ell, k', k'')^2 \delta(\omega(k_\ell) + \sigma' \omega(k') + \sigma'' \omega(k'')) \delta(k_\ell + \sigma' k' + \sigma'' k'') [f_{n+1}(k_1, \ldots, k', k_{\ell+1}, \ldots, k_n, k')] + 2\sigma'' f(k_1, \ldots, k, k)\]

(4.12)

Under the condition (3.13) and provided \( \|W\|_\infty < \infty \), the hierarchy (4.11) has a unique solution for short times. In case

\[
f_n(k_1, \ldots, k_n, 0) = \prod_{j=1}^n W(k_j),
\]

(4.13)

the factorization is maintained in time and each factor agrees with the solution of the Boltzmann equation (4.10). From (4.11) one easily constructs the perturbative solution to (4.10) with the result

\[
W(k, t) = W(k) + \sum_{n=1}^\infty \frac{1}{n!} t^n (\mathcal{C}_{1,2} \ldots \mathcal{C}_{n,n+1} W^{\otimes n+1})(k)
\]

\[
= W(k) + \sum_{n=1}^\infty W_n(k, t).
\]

(4.14)
The series in (4.14) converges for $t$ sufficiently small.

For $n = 1, 2$ the oscillating integrals can be handled by direct inspection with the expected results $\lim_{\epsilon \to 0} W_1^\epsilon(k, t) = W_1(k, t)$, $\lim_{\epsilon \to 0} W_2^\epsilon(k, t) = W_2(k, t)$. If the leading terms are as claimed in the Kinetic Conjecture, then they agree with the series (4.14). The complete argument is a somewhat tricky counting of diagrams, which would lead us too far astray. Thus the most immediate project is to establish that all subleading diagrams vanish in the limit $\epsilon \to 0$. This would be a step further when compared to the investigations [5], [6].

Of course a complete proof must deal with the uniform convergence of the series in (4.14).

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References

[1] H. Spohn, The Phonon Boltzmann Equation, Properties and Link to Weakly Anharmonic Lattice Dynamics, preprint.

[2] S. Teufel, Adiabatic Perturbation Theory in Quantum Dynamics, Lecture Notes in Mathematics 1821, Springer-Verlag, Berlin, Heidelberg, New York 2003.

[3] R.E. Peierls, Zur kinetischen Theorie der Wärmeleitung in Kristallen, Annalen Physik 3, 1055–1101 (1929).

[4] V.E. Zakharov, V.S. L’vov, and G. Falkovich, Kolmogorov Spectra of Turbulence: I Wave Turbulence. Springer, Berlin 1992.

[5] D. Benedetto, F. Castella, R. Esposito, and M. Pulvirenti, Some considerations on the derivation of the nonlinear quantum Boltzmann equation, J. Stat. Phys. 116, 381–410 (2004).

[6] L. Erdős, M. Salmhofer, and H.T. Yau, On the quantum Boltzmann equation, J. Stat. Phys. 116, 367–380 (2004).