Properties and Problems Related to the Smarandache Type Functions

Sebastian Martin Ruiz, Avda. De Regla, 43, Chipiona 11550 (Cadiz), Spain
M. Perez, Rehoboth, Box 141, NM 87301, USA

Abstract:
In this paper we present the definitions and some properties of several Smarandache type functions that are involved in many proposed solved and unsolved problems and conjectures in number theory and recreational mathematics. Examples are also provided. Interesting solved problems related to them are attached as addenda to this article.

1. Introduction:
The most known Smarandache Function, which became a classical function in the number theory, is the following:

\[ S: \mathbb{N}^* \to \mathbb{N}^*, \quad S(1) = 1, \quad S(n) \text{ is the smallest integer such that } S(n)! \text{ is divisible by } n. \]

For example: \( S(6) = 3 \), because \( 3! = 1 \cdot 2 \cdot 3 = 6 \), which is divisible by 6, and 3 is the smallest number with this property, i.e. 2! is not divisible by 6.

\( S(8) = 4 \), \( S(11) = 11 \).
This function has been very much studied in the last decade and interesting properties have been found related to it.

2. Properties:

2.1. Max \( \{ p, p \text{ prime and } p \text{ divides } n \} \leq S(n) \leq n \) for any positive integer \( n \).

2.2. If \( n = (p_1^{s_1}) \cdot (p_2^{s_2}) \cdot \ldots \cdot (p_k^{s_k}) \), where \( p_1, p_2, \ldots, p_k \) are distinct primes, then \( S(n) = \max \{ S(p_i^{s_i}) \} \leq \max \{ p_i \cdot s_i \} \).

2.3. Characterization of a prime number:
Let \( p \) be an integer \( > 4 \). Then: \( p \) is prime if and only if \( S(p) = p \).

Proof:
Let \( p \) be prime \( > 4 \), and suppose \( S(p) = m < p \), then \( m! \) is not be divisible by \( p \), therefore \( S(p) = p \).
Now, let \( S(p) = p \) and \( p \neq 4 \); suppose \( p \) is not prime, whence there exist two integers \( s \) and \( t \), with \( s \leq t < p \), such that \( p = s \cdot t \), but then \( S(p) \leq t \neq p \) because \( t! \) is divisible by \( s \) and by \( t \) in the same time (i.e. \( t! \) is divisible by \( p \)). Contradiction.

2.4 An exact formula to calculate the number of primes less than or equal to \( x \) (L. Seagull):
If \( x \) is an integer \( \geq 4 \), then the number of prime numbers \( \leq x \) is:
\[ \Pi(x) = -1 + \sum_{k=2}^{\lfloor x \rfloor} S(k) \]

where \( S(k) \) is the classical Smarandache Function, and \( \lfloor a \rfloor \)
means the interior integer part of \( a \) (the smallest integer greater than or equal to \( a \)).

**Proof:**
Knowing the Smarandache Function has the property that if \( p > 4 \) then \( S(p) = p \) if only if \( p \) is prime, and \( S(k) \leq k \) for any \( k \), and \( S(4) = 4 \) (the only exception from the first rule),
we easily find an exact formula for the number of primes less than or equal to \( x \).

**4. Conjectures:**
4.1. The diophantine equation \( S(n) = S(n+1) \) has no solutions. (L. Tutescu)
4.2. The diophantine equation \( S(n) + S(n+1) = S(n+2) \) has infinitely many solutions. (I. M. Radu)

**5. More Smarandache Type Functions** have been also considered and studied, such as:

5.1. *Smarandache Double Factorial Function.* \( Sdf(n) \) is the smallest integer such that \( Sdf(n)!! \) is divisible by \( n \), where the double factorial
\[ m!! = 1 \cdot 3 \cdot 5 \cdots m, \text{ if } m \text{ is odd;} \]
and \( m!! = 2 \cdot 4 \cdot 6 \cdots m, \text{ if } m \text{ is even.} \)
For example:
\[
\begin{array}{cccccccccccccccc}
 n & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\
SDF(n) & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 4 & 9 & 10 & 11 & 6 & 13 & 14 & 5 & 6 \\
\end{array}
\]

5.2. *Smarandache-Kurepa Function.*
For \( p \) prime, \( SK(p) \) is the smallest integer such that \( !SK(p) \) is divisible by \( p \), where \( !(m) = 0! + 1! + 2! + \ldots + (m-1)! \)
For example:
\[
\begin{array}{cccccccccccccccc}
p & 2 & 3 & 7 & 11 & 17 & 19 & 23 & 31 & 37 & 41 & 61 & 71 & 73 & 89 \\
SK(p) & 2 & 4 & 6 & 6 & 5 & 7 & 7 & 12 & 22 & 16 & 55 & 54 & 42 & 24 \\
\end{array}
\]

5.3. *Smarandache-Wagstaff Function.*
For \( p \) prime, \( SW(p) \) is the smallest integer such that \( W(SW(p)) \) is divisible by \( p \), where \( W(m) = 1! + 2! + \ldots + (m)! \)
For example:
\[
\begin{array}{cccccccccccccccc}
p & 3 & 11 & 17 & 23 & 29 & 37 & 41 & 43 & 53 & 67 & 73 & 79 & 97 \\
SW(p) & 2 & 4 & 5 & 12 & 19 & 24 & 32 & 19 & 20 & 20 & 7 & 57 & 6 \\
\end{array}
\]

5.4. *Smarandache Ceil Functions of k-th Order.*
\( Sk(n) \) is the smallest integer for which \( n \) divides \( Sk(n)^k \).
For example, for \( k=2 \), we have:
\[
\begin{array}{cccccccccccccccc}
n & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\
S2(n) & 1 & 2 & 3 & 2 & 5 & 6 & 7 & 4 & 3 & 10 & 11 & 6 & 13 & 14 & 15 & 4 \\
\end{array}
\]
5.5. **Pseudo-Smarandache Function:**

\( Z(n) \) is the smallest integer such that \( 1 + 2 + \ldots + Z(n) \) is divisible by \( n \).

For example:

| \( n \) | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| \( Z(n) \) | 1 | 3 | 2 | 3 | 4 | 3 | 6 |

5.6. **Smarandache Near-To-Primordial Function:**

\( \text{SNTP}(n) \) is the smallest prime such that either \( p\# - 1 \), \( p\# \), or \( p\# + 1 \) is divisible by \( n \), where \( p\# \), of a prime number \( p \), is the product of all primes less than or equal to \( p \).

For example:

| \( n \) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | \ldots | 59 | \ldots |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| \( \text{SNTP}(n) \) | 2 | 2 | 2 | 5 | 3 | 3 | 3 | 5 | ? | 5 | 11 | \ldots | 13 | \ldots |

6. **Other Smarandache Type Functions** also studied in the last years are:

6.1. Let \( f: \mathbb{Z} \rightarrow \mathbb{Z} \) be a strictly increasing function and \( x \) an element in \( \mathbb{R} \). Then:

a) **Inferior Smarandache \( f \)-Part** of \( x \), \( ISf(x) \), is the smallest \( k \) such that \( f(k) \leq x < f(k+1) \).

b) **Superior Smarandache \( f \)-Part** of \( x \), \( SSf(x) \) is the smallest \( k \) such that \( f(k) < x \leq f(k+1) \).

Particular cases:

a) Inferior S-Prime Part:

For any positive real number \( n \) one defines \( ISp(n) \) as the largest prime number less than or equal to \( n \).

The first values of this function are:

\( 2,3,5,7,7,7,11,11,13,13,13,17,17,17,17,19,19,19,23,23 \).

b) Superior S-Prime Part:

For any positive real number \( n \) one defines \( SSp(n) \) as the smallest prime number greater than or equal to \( n \).

The first values of this function are:

\( 2,2,2,3,5,7,7,11,11,11,11,13,13,13,17,17,17,17,17,19,19,23,23,23 \).

c) Inferior S-Square Part:

For any positive real number \( n \) one defines \( ISs(n) \) as the largest square less than or equal to \( n \).

The first values of this function are:

\( 0,1,1,4,4,4,4,9,9,9,9,9,9,16,16,16,16,16,16,16,16,25,25 \).

b) Superior S-Square Part:

For any positive real number \( n \) one defines \( SSs(n) \) as the smallest square greater than or equal to \( n \).

The first values of this function are:

\( 0,1,4,4,4,9,9,9,16,16,16,16,16,16,16,25,25,25,25,25,25,25,25,25,25,36 \).

d) Inferior S-Cubic Part:

For any positive real number \( n \) one defines \( ISc(n) \) as the largest cube less than or equal to \( n \).
The first values of this function are:
0,1,1,1,1,1,1,1,8,8,8,8,8,8,8,8,8,8,8,8,8,8,8,8,27,27,27,27.
e) Superior S-Cube Part:
For any positive real number n one defines SSs(n) as the smallest
cube greater than or equal to n.
The first values of this function are:
0,1,8,8,8,8,8,8,27,27,27,27,27,27,27,27,27,27,27,27,27,27,27,27,27,27.
f) Inferior S-Factorial Part:
For any positive real number n one defines ISf(n) as the largest
factorial less than or equal to n.
The first values of this function are:
1,2,2,2,2,6,6,6,6,6,6,6,6,6,6,6,24,24,24,24,24,24,24,24.
g) Superior S-Factorial Part:
For any positive real number n one defines SSf(n) as the smallest
factorial greater than or equal to n.
The first values of this function are:
1,2,6,6,6,24,24,24,24,24,24,24,24,24,24,24,24,24,24,24,120.
Remark 1: This is a generalization of the inferior/superior integer part of a number.

6.2. Let f: Z → Z be a strictly increasing function and x an element in R. Then:
Fractional Smarandache f-Part of x, FSf(x) = x - ISf(x), where ISf(x) is the Inferior
Smarandache f-Part of x defined above.
Particular cases:
a) Fractional S-Prime Part:
FSp(x) = x - ISp(x),
where ISp(x) is the Inferior S-Prime Part defined above.
Example: FSp(12.501) = 12.501 - 11 = 1.501.
b) Fractional S-Square Part:
FSs(x) = x - ISs(x),
where ISs(x) is the Inferior S-Square Part defined above.
Example: FSs(12.501) = 12.501 - 9 = 3.501.
c) Fractional S-Cubic Part:
FSc(x) = x - ISC(x),
where ISC(x) is the Inferior S-Cubic Part defined above.
Example: FSc(12.501) = 12.501 - 8 = 4.501.
d) Fractional S-Factorial Part:
FSf(x) = x - ISf(x),
where ISf(x) is the Inferior S-Factorial Part defined above.
Example: FSf(12.501) = 12.501 - 6 = 6.501.

Remark 2.1: This is a generalization of the fractional part of a number.
Remark 2.2: In a similar way one defines:
- the Inferior Fractional Smarandache f-Part:
IFSf(x) = x - ISf(x) = FSf(x);
- and the Superior Fractional Smarandache f-Part:
SFSf(x) = SSf(x) - x;
for example: Superior Fractional S-Cubic Part of 12.501
6.3. Let \( g: A \mapsto A \) be a strictly increasing function, and let "~" be a given internal law on \( A \). Then we say that \( f: A \mapsto A \) is \textit{smarandachely complementary with respect to the function} \( g \) \textit{and the internal law} "~" if:

\[ f(x) \text{ is the smallest } k \text{ such that there exists a } z \text{ in } A \text{ so that} \]
\[ x \sim k = g(z). \]

Particular cases:

a) S-Square Complementary Function:
\[ f: \mathbb{N} \mapsto \mathbb{N}, \ f(x) = \text{the smallest } k \text{ such that } x^k \text{ is a perfect square.} \]
\[ \text{The first values of this function are:} \]
\[ 1, 2, 3, 1, 5, 6, 7, 2, 1, 10, 11, 3, 14, 15, 1, 17, 2, 19, 5, 21, 22, 23, 6, 1, 26, 3, 7. \]

b) S-Cubic Complementary Function:
\[ f: \mathbb{N} \mapsto \mathbb{N}, \ f(x) = \text{the smallest } k \text{ such that } x^k \text{ is a perfect cube.} \]
\[ \text{The first values of this function are:} \]
\[ 1, 4, 9, 2, 25, 36, 49, 1, 3, 100, 121, 18, 169, 196, 225, 4, 289, 12, 361, 50. \]

More generally:

c) S-m-power Complementary Function:
\[ f: \mathbb{N} \mapsto \mathbb{N}, \ f(x) = \text{the smallest } k \text{ such that } x^k \text{ is a perfect } m \text{-power.} \]
d) S-Prime Complementary Function:
\[ f: \mathbb{N} \mapsto \mathbb{N}, \ f(x) = \text{the smallest } k \text{ such that } x+k \text{ is a prime.} \]
\[ \text{The first values of this function are:} \]
\[ 1, 0, 0, 1, 0, 3, 2, 1, 0, 1, 0, 3, 2, 1, 0, 1, 0, 3, 2, 1, 0, 5, 4, 3, 2, 1, 0, 1, 0, 5. \]

6.4. \textit{S-Multiplicative Function:}

\[ \text{is a function } f: \mathbb{N}^* \mapsto \mathbb{N}^* \text{ such that for any } (a, b) = 1, \ f(a \cdot b) = \max \{ f(a), f(b) \}; \]
[i.e. it reflects the main property of the Smarandache function].

The following functions are obviously S-multiplicative:
a) The constant function \( f: \mathbb{N}^* \mapsto \mathbb{N}^* \), \( f(n) = 1. \)
b) The Smarandache function \( S: \mathbb{N}^* \mapsto \mathbb{N}, \ S(n) = \max\{ \ p! : n \}. \)

Certainly, many properties of multiplicative functions can be translated for S-multiplicative functions.

7. \textbf{Functional Smarandache Iterations:}

7.1. \textit{Functional Smarandache Iteration of First Kind:}

Let \( f: A \mapsto A \) be a function, such that \( f(x) \leq x \text{ for all } x, \) and
\[ \min \{ f(x), \ x \in A \} \geq m_0 \neq -\infty. \]

Let \( f \) have \( p \geq 1 \) fix points: \( m_0 \leq x_1 < x_2 < \ldots < x_p. \)
[The point \( x \) is called fix if \( f(x) = x. \) Then:
\[ \text{SI}_1 f (x) = \text{the smallest number of iterations } k \text{ such that} \]
\[ f(f(\ldots f(x)\ldots))) = \text{constant}. \]
iterated k times

Example:
Let $n > 1$ be an integer, and $d(n)$ be the number of positive divisors of $n$, $d: \mathbb{N} \rightarrow \mathbb{N}$. Then $SI_{1d}(n)$ is the smallest number of iterations $k$ such that $d(d(...d(n)...)) = 2$; because $d(n) < n$ for $n > 2$, and the fix points of the function $d$ are 1 and 2. Thus $SI_{1d}(6) = 3$, because $d(d(d(6))) = d(d(4)) = d(3) = 2 = \text{constant}$. $SI_{1d}(5) = 1$, because $d(5) = 2$.

7.2. Functional Smarandache Iteration of Second Kind:
Let $g: A \rightarrow A$ be a function, such that $g(x) > x$ for all $x$, and let $b > x$. Then:
$SI_{2g}(x, b) =$ the smallest number of iterations $k$ such that $g(g(...g(x)...)) \geq b$. because $g(n) > n$ for $n > 1$. Thus $SI_{2g}(4, 11) = 3$, because $g(g(g(4))) = g(g(7)) = g(8) = 15 \geq 11$.

7.3. Functional Smarandache Iteration of Third Kind:
Let $h: A \rightarrow A$ be a function, such that $h(x) < x$ for all $x$, and let $b < x$. Then:
$SI_{3h}(x, b) =$ the smallest number of iterations $k$ such that $h(h(...h(x)...)) \leq b$. Example:
Let $n$ be an integer and $gd(n)$ be the greatest divisor of $n$, less than $n$, $gd: \mathbb{N}^* \rightarrow \mathbb{N}^*$. Then $gd(n) < n$ for $n > 1$.

$SI_{3gd}(60, 3) = 4$, because $gd(gd(gd(60))) = gd(gd(gd(30))) = gd(gd(15)) = gd(5) = 1 \leq 3$.

References:
[1] Ashbacher, C., "A Note on the Smarandache Near-To-Primordial Function", <Smarandache Notions Journal>, Vol. 7, No. 1-2-3, 46-49, 1996.
[2] Ashbacher, C., "Some Properties of the Smarandache-Kurepa and Smarandache-Wagstaff Functions", <Mathematics and Informatics Quarterly>, Vol. 7, No. 3, pp. 114-116, 1997.
[3] Begay, A., "Smarandache Ceiling Functions", in <Bulletin of Pure and Applied Sciences>, India, Vol. 16E, No. 2, 227-229, 1997.
[4] Castillo, Jose, "Other Smarandache Type Functions", http://www.gallup.unm.edu/~smarandache/funct2.txt
[5] Dumitrescu, C., Seleacu, V., "Some notions and questions in number theory", Erhus Univ. Press, Glendale, 1994. http://www.gallup.unm.edu/~smarandache/snaqint.txt
[6] Ibstedt, H., "Smarandache Iterations of First and Second Kinds", <Abstracts of Papers Presented to the American Mathematical Society>, Vol. 17, No. 4, Issue 106, 680, 1996.
[7] Kashihara, K., "Comments and Topics on Smarandache Notions and Problems", Erhus Univ. Press, Vail, USA, 1996. http://www.gallup.unm.edu/~smarandache/Kashihara.pdf
[8] Mudge, Mike, "The Smarandache Near-To-Primordial (S.N.T.P.) Function", <Smarandache Notions Journal>, Vol. 7, No. 1-2-3, 45, 1996.
[9] Popescu, Marcela, Nicolescu, Mariana, "About the Smarandache Complementary Cubic Function", <Smarandache Notions Journal>, Vol. 7, no. 1-2-3, 54-62, 1996.
[10] Popescu, Marcela, Seleacu, Vasile, "About the Smarandache Complementary Prime Function", <Smarandache Notions Journal>, Vol. 7, No. 1-2-3, 12-22, 1996.
[11] Ruiz, Sebastian Martin, “Applications of Smarandache Function, Prime and Coprime Functions”, American Research Press, Rehoboth, 2002; http://www.gallup.unm.edu/~smarandache/SMRuiz-eBook.pdf
[12] Seagull, L., "The smarandache Function and the number of primes up to x", <Mathematical Spectrum>, University of Shielfield, Vol. 28, No. 3, 53, 1995/6. http://www.gallup.unm.edu/~smarandache/FORMULA.TXT
[13] Smarandache, F., “A Function in Number Theory”, Analele Univ. Timisoara, XVIII, fasc. 1, 79-88, 1980; http://www.gallup.unm.edu/~smarandache/SFBook1.pdf
[14] Smarandache, Florentin, "Only Problems, not Solutions!", Xiquan Publishing House, Phoenix-Chicago, 1990, 1991, 1993; http://www.gallup.unm.edu/~smarandache/OPNS.pdf
[15] "The Florentin Smarandache papers" Special Collection, Arizona State University, Hayden Library, Tempe, Box 871006, AZ 85287-1006, USA;
[16] Tabirca, Sabin, "About S-Multiplicative Functions", <Octogon>, Brasov, Vol. 7, No. 1, 169-170, 1999.
[17] Weisstein, Eric W., “CRC Concise Encyclopedia of Mathematics”, CRC Press, Boca Raton, 1998.

Other e-books on Smarandache Functions and Sequences can be downloaded from http://www.gallup.unm.edu/~smarandache/eBooks-otherformats.htm and research papers from http://www.gallup.unm.edu/~smarandache/math.htm.