FROM TOPOLOGICAL RECURSION TO WAVE FUNCTIONS AND PDES
QUANTIZING HYPERELLPTIC CURVES

BERTRAN EYNARD AND ELBA GARCÍA-FAILDE

ABSTRACT. Starting from loop equations, we prove that the wave functions constructed from topo-
logical recursion on families of degree 2 spectral curves with a global involution satisfy a system of
partial differential equations, whose equations can be seen as quantizations of the original spectral
curves. The families of spectral curves can be parametrized with the so-called times, defined as
periods on second type cycles, and with the poles. These equations can be used to prove that
the WKB solution of many isomonodromic systems coincides with the topological recursion wave
function, which proves that the topological recursion wave function is annihilated by a quantum
curve. This recovers many known quantum curves for genus zero spectral curves and generalizes
this construction to hyperelliptic curves.

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1. INTRODUCTION

Topological recursion is a powerful tool, which was first discovered in the context of large size
asymptotic expansions in random matrix theory [15, 12, 10, 11] and established as an independent
universal theory around 2007 [21]. Its most important role was to unveil a common structure in
many different topics in mathematics and physics, which helped building bridges among them and
gaining general context. For instance, it has been related to fundamental structures in enumerative
geometry and integrable systems, such as intersection theory on moduli spaces of curves and
cohomological field theories.

A quantum curve is a Schrödinger operator-like non-commutative analogue of a plane curve that
annihilates the so-called wave function, which can be seen as a WKB asymptotic solution of the
corresponding differential equation. Inspired by the intuition coming from matrix models, it has
been conjectured that there exists such a quantum curve associated to a spectral curve, which is
the input of the topological recursion, and whose WKB asymptotic solution is reconstructed by the
topological recursion output.

This claim was verified in [9] for a class of spectral curves called admissible, which are basically
spectral curves whose Newton polygons have no interior point. Admissible spectral curves include
a very large number of spectral curves of genus 0. Therefore they recover many cases previously
studied in the literature in various algebro-geometric contexts. In the present work, we go beyond
admissible spectral curves and study the quantum curve problem for spectral curves with a global
involution, given by algebraic curves whose defining polynomials are of the form $y^2 = R(x)$, with $R$ a rational function on $x$. This setting includes the genus 0 spectral curves with a global involution $y \mapsto -y$, such as the well-known Airy curve, which are many less that the set of admissable curves, but it also includes all genus 1 spectral curves, i.e. all elliptic curves, and all hyperelliptic curves, which are curves of genus $\hat{g} > 1$ where $R$ is a polynomial in $x$.

Quantum curves encode enumerative invariants in an interesting way, and help building the bridge between the geometry of integrable systems and topological recursion. One of the most celebrated applications of quantum curves is in the context of knot theory, where the quantum curve of the $A$-polynomial of a knot provides a conjectural constructive generalization of the volume conjecture [13, 14, 5].

1.1. Quantum curves and topological recursion. We start by presenting the idea of quantum curves and their relation to topological recursion. Consider $P \in \mathbb{C}[x,y]$ and let

$$\mathcal{C} = \{(x,y) \in \mathbb{C}^2 | P(x,y) = 0\}$$

be a plane curve.

Consider $h > 0$ a formal parameter. A quantization of the plane curve $\mathcal{C}$, is a differential operator $\hat{P}$ of the form

$$\hat{P}(\hat{x}, \hat{y}; h) = P_0(\hat{x}, \hat{y}) + O(h),$$

where $\hat{x} = x$, $\hat{y} = h \frac{d}{dx}$. In fact, $\hat{P}$ is a normal-ordered operator valued (all the $\hat{y}$ in a monomial are placed to the right of all the $\hat{x}$) formal series (or transseries) whose leading order term $P_0(\hat{x}, \hat{y})$ recovers the polynomial equation of the original spectral curve. Actually, in general, $P_0(x,y)$ can be a reducible polynomial with $P(x,y)$ one of its factors. The operators $\hat{x}$ and $\hat{y}$ satisfy the following commutation relation which justifies the name “quantization”:

$$[\hat{y}, \hat{x}] = h.$$

One can consider a Schrödinger-type differential equation

(1) $$\hat{P}(\hat{x}, \hat{y})\psi(z, h) = 0, \text{ with } z \in \mathcal{C},$$

whose solution can be calculated via the WKB method, that is we require $\psi$ to have a formal series (resp. transseries) $h$ expansion of the form

$$\psi(z, h) = \exp \left( \sum_{m\geq 0} h^{m-1} S_m(z) \right)$$

(resp. of the form of a formal series in powers of exponentials of inverse powers of $h$ whose coefficients are formal power series in $h$). The coefficients $S_k(z)$ are determined recursively via (1). One fundamental question is if the formal solution $\psi$ can be computed directly from the original plane curve $\mathcal{C}$. The conjectural answer is provided by the topological recursion and is already established in many cases.

The input data of the topological recursion is called spectral curve. For the purpose of this paper, a spectral curve will be given as in [17], i.e. by the data $(\Sigma, x, y, \omega_{0,1}, \omega_{0,2})$, where $\Sigma$ is a compact Riemann surface, and $x$ and $y$ are meromorphic functions on $\Sigma$ such that the zeroes of $dx$ do not coincide with the zeroes of $dy$. Then $x$ and $y$ must be algebraically dependent, i.e. $P(x,y) = 0$ with $P \in \mathbb{C}[x,y]$. We consider $\omega_{0,1} := ydx$ and $\omega_{0,2}$ a symmetric bi-differential $B$ on $\Sigma^2$ with double poles along the diagonal and vanishing residues. The output of the topological recursion are symmetric meromorphic multi-differentials $\omega_{g,n}(z_1, \ldots, z_n)$ on $\Sigma^n$. In particular, for $n = 0$, the $F_g := \omega_{g,0}$ are complex numbers or expressions depending on parameters of the spectral curves. In the present work, we assume that the poles of $y$ but not of $x$ cannot be ramification points of $x$. 


The perturbative wave function $\psi(z)$ constructed from topological recursion is defined (see [21]) as

$$
\frac{1}{x(z) - x(o)} \exp \left( \sum_{g \geq 0, n \geq 1} \frac{h^{2g-2+n}}{n!} \int_o^z \cdots \int_o^z \left( \omega_{g,n}(z_1, \ldots, z_n) - \delta_{g,0} \delta_{n,2} \frac{dx(z_1)dx(z_2)}{(x(z_1) - x(z_2))^2} \right) \right),
$$

where $o \in \Sigma$ is a chosen base-point for integration. In general the quantum curve is obtained by choosing $o = \zeta$ such that $x(\zeta) = \infty$, and one may need to regularize the $(g,n) = (0,1)$ term in the limit $o \to \zeta$. We choose a regularization for $(g,n) = (0,2)$ which slightly differs from some part of the literature and produces the first factor of the expression. We emphasize that our definition transforms as a spinor $1/2$-form under change of coordinates. Actually, we will generalize the definition of the wave function by allowing integration over any divisor as in [17].

A further question is if the differential operator $\hat{P}$ can be directly constructed from the topological recursion. This has also been answered affirmatively in many cases. In this article, we actually construct an operator that we believe is a more fundamental object, which appears more naturally for curves of genus $g > 0$, and provides a PDE which also allows to reconstruct the wave function $\psi$. Our system of PDEs will also imply a quantum curve in the more classical sense considered in this section.

1.2. Generalized cycles. Let us now recall the concept of generalized cycles on a Riemann surface as in [17], which will help us introduce suitable local coordinates in the space of spectral curves. These local coordinates can be seen as deformation parameters giving rise to families of spectral curves and will play a key role when producing our system of PDEs for a large class of spectral curves.

The so-called times $t_i$, introduced in [17], can be viewed as local coordinates in the space of spectral curves. Time deformations $\partial t_i$ belong to the tangent space, which is isomorphic to the space of meromorphic differential forms on the spectral curve, and via form-cycle duality it can be identified with the space of generalized cycles on the spectral curve. In [17], generalized cycles are defined as elements of the dual of the space of meromorphic forms on $\Sigma$ such that integrating $\omega_{0,2} = B$ on them gives meromorphic 1-forms.

In practice a generating family is given by three kinds of cycles, dual to three kinds of forms:

- **1st kind cycles**: This type of cycles are usual non-contractible cycles, i.e. elements $[\gamma] \in H_1(\Sigma, \mathbb{C})$. If $\Sigma$ is compact of genus $\hat{g}$, then $\dim H_1(\Sigma, \mathbb{C}) = 2\hat{g}$.

- **2nd kind cycles**: A cycle of second type $\gamma = \gamma_p, f$ consists of a small circle $\gamma_p$ around a point $p \in \Sigma$ weighted by a function $f$ holomorphic in a neighborhood of $\gamma_p$ and meromorphic in a neighborhood of $p$, with a possible pole at $p$ (of any degree), i.e. by definition $\int_p \omega := 2\pi i \text{Res}_p f \omega$.

- **3rd kind cycles**: They are open chains $\gamma = \gamma_{q \rightarrow p}$ (paths up to homotopic deformation with fixed endpoints), whose boundaries $\partial \gamma = [p] - [q]$ are degree zero divisors.

Let $x : \Sigma \to \mathbb{C}$ be the meromorphic function that makes the spectral curve a branched cover of the Riemann sphere. A basis of functions which are meromorphic in a neighborhood of $p \in \Sigma$ is given by

$$
\{ \xi_p^k \}_{k \in \mathbb{Z}}, \text{ with } \xi_p = (x - x(p))^{1/\text{ord}_p(x)}.
$$

If $x(p) = \infty$, we set $\xi_p = x^{1/\text{ord}_p(x)}$, with $\text{ord}_p(x) < 0$. The following set of cycles generates an integer lattice in the space of second kind cycles:

$$
A_{p,k} = \gamma_p \xi_p^k, \quad p \in \Sigma, k \geq 0,
$$

$$
B_{p,k} = \frac{1}{2\pi i} \gamma_p \frac{\xi_p^{-k}}{k}, \quad p \in \Sigma, k \geq 1.
$$
Given a meromorphic 1-form $\omega$ on $\Sigma$, for every pole $p$ of $\omega$, we define for every $j \geq 0$ the $KP$ times:

\begin{equation}
(4) \quad t_{p,j} := \frac{1}{2\pi i} \int_{A_{p,j}} \omega = \frac{1}{2\pi i} \int_{\gamma_p} \xi_p^j \omega = \text{Res}_p (\xi_p)^j \omega,
\end{equation}

so that

$$
\omega \sim \sum_{j=0}^{\deg_p(\omega)} t_{p,j} \xi_p^{-j-1} d\xi_p + \text{analytic at } p.
$$

Since we assumed $\Sigma$ to be compact, the number of poles is finite. Moreover, all the times with $j \geq \deg_p(\omega)$ are vanishing. Therefore, only a finite number of times are non-zero.

1.3. **Context and outline.** One of our motivations to study the problem of quantum curves for any spectral curve with a global involution was to be able to recover the whole isomonodromic system associated to Painlevé I just from loop equations. We also aimed to give the first quantum curves for spectral curves of genus $\hat{g} > 1$ and we were especially interested to see how introducing deformations with respect to the times could give rise to systems of PDEs that we consider more natural in general.

1.3.1. **Comparison to the literature.** In [8], they generalize the techniques employed in [9] to find the quantum curves for admissible curves to apply them to the family of genus one spectral curves given by the Weierstrass equation. They find an order two differential operator that annihilates the perturbative wave function $\psi$. However, it is not a quantum curve, since it contains infinitely many $\hbar$ corrections which are not meromorphic functions of $x$. They also check the first orders of the conjectural quantum curve [16, 20, 4] for the non-perturbative wave function.

In [25], they focus on the Painlevé I spectral curve, which is a degenerate torus, and from topological recursion they get a PDE that annihilates the wave function, which is compatible with the isomonodromic system and, together with another identity coming from integrable systems, provides a quantum curve that annihilates the wave function. In [23], the first author slightly generalizes the same results to the case of any elliptic curve, that is he considers not only the degenerate case of Painlevé I, but tori where none of the two cycles are pinched. In both papers, they show that the $\hbar$ corrections from [8] can be controlled by a derivative with respect to a deformation parameter. The quantum curves still contain infinitely many $\hbar$-correction terms, but in this case, these corrections are given by the asymptotic expansion of the solution of Painlevé I around $\hbar \to 0$.

In [24], the approach is reversed: they prove that Lax pairs associated with $\hbar$-dependent Painlevé equations satisfy the topological type property of [1], which implies that one can reconstruct the $\hbar$-expansion of the isomonodromic $\tau$-function from topological recursion. Finally, in [28], they generalize this result showing that it is always possible to deform a differential equation $\partial_x \Psi(x) = L(x) \Psi(x)$, with $L(x) \in \mathfrak{sl}_2(\mathbb{C})$ by introducing a formal parameter $h$ in such a way that it satisfies the topological type property.

In the present work, we recover the PDE from [25, 23] from loop equations (which are necessary for topological recursion, but not sufficient\(^1\)) and as part of a system that we obtain because we consider a wave function where the integrals are over any divisor of degree zero. With our system, we are able to recover the whole isomonodromic system associated to Painlevé I just from loop equations. We also give an additional meaning to the deformation parameter that appears

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\(^1\)Topological recursion provides a specific solution of loop equations. General solutions of loop equations don’t necessarily satisfy the so-called projection formula, which implies that the purely holomorphic part of $\omega_{g,n}$ vanishes (in the sense of [7, Section 2.2.2]), and they are governed by a generalisation called blobbed topological recursion, which was introduced in [7].
naturally in [25, 23] for the case of elliptic curves, making use of the powerful idea of deforming with respect to the generalized cycles introduced in the previous section. The elliptic curve case is a very concrete case in which there is only one such deformation parameter, but we see that we need to consider several in the higher genus cases.

1.3.2. Outline. In Section 2 we introduce the type of curves we consider in this work and relate them to the concept of spectral curves as input of the topological recursion. We also give the link to the spectral curves in the setting of isomonodromy systems, which serves as a motivation to us. Moreover, we compute the deformation parameters of the family of elliptic curves, which recovers the Painlevé I isomonodromy system setting in the degenerate case; in particular, the so-called KP times.

In Section 3 we recall the loop equations for our specific setting and deduce some interesting consequences relating them to time deformations, which appear when considering spectral curves of genus $\hat{g} > 0$.

In Section 4 we prove our main result. From loop equations we obtain a system of partial differential equations that annihilates our wave function defined from topological recursion integrating over a general divisor. We also give the shape of this system in the particular cases of genus zero and elliptic curves. Finally, we consider our system for the particular case of a two-point divisor, which then consists of only two PDEs, with differentials with respect to two spectral variables and the deformation parameters, called times. We are able to combine the two PDEs in such a way that we eliminate one of the spectral variables.

In Section 5, we argue that if the spectral curve comes from an isomonodromic system, then the topological recursion non-perturbative wave function has to coincide with the solution of the isomonodromic system, which implies an ODE, which is the quantum curve we were looking for. As particular interesting cases, we recover the first Painlevé system and equation, and its higher analogues defined in terms of Gelfand–Dikii polynomials.

In Appendix A, we prove that the integrable kernel of any isomonodromic system satisfies the same PDE that we found for the two-point wave function constructed from topological recursion.

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2. Spectral curves with a global involution

In this article we focus on algebraic plane curves of the form

$$y^2 = P(x),$$

with $P(x) \in \mathbb{C}[x]$ an arbitrary polynomial of $x$, and we will generalize to $y^2 = R(x)$ with $R(x) \in \mathbb{C}(x)$ an arbitrary rational function of $x$.

\[\text{While the revised version of this article was produced, the first affirmative answer to the quantum curve conjecture for generic plane curves of arbitrary degrees was released [19].}\]
The degree of the polynomial is related to the genus of the curve. For example, in the case in which $P$ is a polynomial of degree $2m + 1$ or $2m + 2$ the curve has genus $\tilde{g} \leq m$, with equality if the plane curve is smooth. If the degree is odd, the curve has one point at infinity and if the degree is even, the curve has two points at infinity. If $\tilde{g} > 1$, the curve is called hyperelliptic; if $\tilde{g} = 1$ (with a distinguished point), it is called elliptic, and if $\tilde{g} = 0$, it is called rational.

2.1. Spectral curves as input of the topological recursion. The method of topological recursion associates to a spectral curve $S$ a doubly indexed family of meromorphic multi-differentials $\omega_{g,n}$ on $\Sigma^p$:

$$\text{TR: Spectral curve } S = (\Sigma, x, ydx, B) \leadsto \text{Invariants } \omega_{g,n} (F_g = \omega_{g,0}).$$

A spectral curve is the data of a Riemann surface $\Sigma$, a holomorphic projection $x : \Sigma \to \mathbb{C}P^1$ to the base $\mathbb{C}P^1$ which turns $\Sigma$ into a ramified cover of the sphere, a meromorphic 1-form $ydx$ on $\Sigma$, and $B$ a 2nd kind fundamental differential, i.e. a symmetric $1 \times 1$ form on $\Sigma \times \Sigma$ with normalized double pole on the diagonal and no other pole, behaving near the diagonal as:

$$B(z_1, z_2) = \frac{dz_1dz_2}{(z_1 - z_2)^2} + \text{holomorphic at } z_1 = z_2.$$

In case the spectral curve is of genus 0, i.e. $\Sigma = \mathbb{C}P^1$, it is known that such a $B$ is unique and is worth

$$B(z_1, z_2) = \frac{dz_1dz_2}{(z_1 - z_2)^2}.$$

If the genus $\tilde{g}$ of $\Sigma$ is $\geq 1$, $B$ is not unique since one can add any symmetric bilinear tensor product of holomorphic 1-forms. A way to find a unique one is to choose a Torelli marking, which is a choice of a symplectic basis $\{\{A_i\}_{i=1}^{\tilde{g}}, \{B_i\}_{i=1}^{\tilde{g}}\}$ of $H_1(\Sigma, \mathbb{Z})$. There exists a unique $B$ normalized on the $A$-cycles of $H_1(\Sigma, \mathbb{Z})$

$$\int_{A_i} B(z_1, \cdot) = 0.$$

Such a bi-differential has a natural construction in algebraic geometry and is called the normalized fundamental differential of the second kind on $\Sigma$. See [18] for constructing $B$ for general algebraic plane curves.

We define the filling fractions as the $A$-periods of $\omega_{0,1}$:

$$\epsilon_j := \frac{1}{2\pi i} \int_{A_j} ydx, \text{ for } j = 1, \ldots, \tilde{g}.$$  

Remark 2.1. The coordinate $x : \Sigma \to \mathbb{C}P^1$ in the definition of spectral curve can be thought as a ramified covering of the sphere. We call degree of the spectral curve the number of sheets of the covering, i.e. the number of preimages of a generic point. In this article, we focus on spectral curves of degree 2 with a global involution $(x,y) \mapsto (x,-y)$.

2.2. Spectral curves from isomonodromic systems. Painlevé transcendentors have their origin in the study of special functions and of isomonodromic deformations of linear differential equations. They are solutions to certain nonlinear second-order ordinary differential equations in the complex plane with the Painlevé property, i.e. the only movable singularities are poles.

An $h$-dependent Lax pair is a pair $(L(x, t; h); R(x, t; h))$ of $2 \times 2$ matrices, whose entries are rational functions of $x$ and holomorphic in $t$ such that the system of partial differential equations

$$\begin{cases} 
  h \frac{\partial}{\partial x} \Psi(x, t) = L(x, t; h) \Psi(x, t), \\
  h \frac{\partial}{\partial t} \Psi(x, t) = R(x, t; h) \Psi(x, t)
\end{cases}$$
is compatible. We call such a system an isomonodromy system.

The compatibility condition, i.e. \( \frac{\partial}{\partial t} \Psi = \frac{\partial}{\partial x} \Psi \), is equivalent to the so-called zero-curvature equation:

\[
\hbar \frac{\partial \mathcal{L}}{\partial t} - \hbar \frac{\partial \mathcal{R}}{\partial x} + [\mathcal{L}, \mathcal{R}] = 0.
\]

In [26], Jimbo and Miwa gave a list of the Lax pairs whose compatibility conditions are equivalent to the six Painlevé equations.

Let us consider the expansion around \( \hbar = 0 \) of the first equation of the system: \( \mathcal{L}(x, t; \hbar) = \sum_{k \geq 0} \hbar^k \mathcal{L}_k(x, t) \). The associated spectral curve is given by

\[
\text{det}(y \text{Id} - \mathcal{L}_0(x, t)) = 0,
\]

which is actually a family of algebraic curves parametrized by \( t \).

2.2.1. Motivational example: The first Painlevé equation. Let us consider the first Painlevé equation with a formal small parameter \( \hbar \):

\[ P_I: \frac{\hbar^2}{2} \frac{\partial^2}{\partial t^2} U - 3U^2 = t. \]

The leading term \( u = u(t) \) of a formal power series solution \( U(t, \hbar) = u(t) + \sum_{k \geq 1} \hbar^k u_k(t) \) satisfies \( t = -3u^2 \) and determines the subleading terms recursively:

\[ u_k = c_k u^{1-5k}, c_k \in \mathbb{Q} \]

by the recursion

\[ c_0 = 1, \quad 2c_{k+1} = \frac{25k^2 - 1}{6^k} c_k - \sum_{j=1}^{k} c_j c_{k+1-j}. \]

The coefficient \( u_k(t) \) has a singularity at \( u = 0 \), i.e. \( t = 0 \). This special point is called a turning point of \( P_I \). We shall assume that \( t \neq 0 \). We denote by \( \dot{U} \) the derivative with respect to \( t \) of \( U(t, \hbar) \).

The Tau-function \( \mathcal{T}(t) \) is defined in such a way that

\[ U(t) = -\hbar^2 \frac{c_2^2}{6} \frac{\partial^2}{\partial t^2} \log \mathcal{T}. \]

The Painlevé equation ensures that \( \mathcal{T} \) is an entire function with simple zeros at the movable poles of \( U \).

The Lax pair associated to the first Painlevé equation is given by

\[
\mathcal{L}(x, t; \hbar) := \begin{pmatrix} \hbar \dot{U} & x - U \\ (x - U)(x + 2U) + \frac{\hbar^2}{2} \dot{U} & -\frac{\hbar}{2} \ddot{U} \end{pmatrix} \quad \text{and} \quad \mathcal{R}(x, t; \hbar) := \begin{pmatrix} 0 & 1 \\ x + 2U & 0 \end{pmatrix}.
\]

The leading term of \( \mathcal{L} \) in its expansion around \( \hbar = 0 \) is given by

\[ \mathcal{L}_0(x, t) = \begin{pmatrix} 0 & 1 \\ x^2 + ux - 2u^2 & x - u \end{pmatrix}. \]

The spectral curve reads

\[
\text{det}(y \text{Id} - \mathcal{L}_0(x, t)) = y^2 - (x - u)^2(x + 2u) = 0,
\]

which is actually a family of algebraic curves parametrized by \( t \). Since we have assumed \( t \neq 0 \), the two roots \( x = u \) and \( x = -2u \) of \( R(x) = (x - u)^2(x + 2u) \) are distinct, but the root \( x = u \) has multiplicity 2. These curves have genus 0 or, more precisely, constitute a family of tori with one of the cycles pinched.

In general, we want to study the family of tori given by

\[
y^2 = x^3 + tx + V,
\]
where \( R(x) = x^3 + tx + V \) has three different roots. The case \( t = -3u^2, V = 2u^3 \) recovers the particular degenerate case (9).

2.3. Parametrizations and deformation parameters of the elliptic case. In the elliptic case, we give the parametrizations and compute the coordinates because it has more structure than the genus 0 case, since we need to introduce one deformation parameter, and it also illustrates how the general case works. The degenerate case corresponds to Painlevé I, which is our prototypical example when making the connection to isomonodromy systems.

2.3.1. Degenerate case. When \( V = 2u^3 \), consider the parametrization of the spectral curve given by

\[
\begin{align*}
    x(z) &= z^2 - 2u, \\
    y(z) &= z^3 - 3uz,
\end{align*}
\]

with \( z \in \Sigma = \mathbb{CP}^1 \) and the fundamental form of the second kind on \( \Sigma \):

\[
B(z_1, z_2) = \frac{dz_1 dz_2}{(z_1 - z_2)^2}.
\]

It satisfies

\[
y^2 = x^3 + tx + V,
\]

with \( t = -3u^2, V = 2u^3 \).

This curve has one ramification point at \( z = 0 \) and one pole at \( z = \infty \).

Near \( z = \infty \), we have

\[
y \sim x^\frac{3}{2} + \frac{t}{2x^\frac{3}{2}} + \frac{V}{2x^\frac{1}{2}} - \frac{t^2}{8x^\frac{3}{2}} - \frac{tV}{4x^\frac{3}{2}} + O(x^{-\frac{9}{2}}).
\]

This implies that

\[
t_{x,1} = \frac{1}{2\pi i} \int_{A_{x,1}} y dx = \text{Res}_{z \to \infty} \frac{x^{-\frac{1}{2}}}{x^{-\frac{5}{2}}} y dx = -t,
\]

\[
\int_{B_{x,1}} y dx = \text{Res}_{z \to \infty} \frac{x^{-\frac{1}{2}}}{x^{-\frac{5}{2}}} y dx = -V,
\]

\[
t_{x,5} = \frac{1}{2\pi i} \int_{A_{x,5}} y dx = \text{Res}_{z \to \infty} \frac{x^{-\frac{5}{2}}}{x^{-\frac{7}{2}}} y dx = -2,
\]

\[
\int_{B_{x,5}} y dx = \frac{1}{5} \text{Res}_{z \to \infty} \frac{x^{-\frac{5}{2}}}{x^{-\frac{7}{2}}} y dx = \frac{tV}{10},
\]

where we use the generalized cycles \( A_{p,k} \) and \( B_{p,k} \) defined in (2) and we have considered only the values \( k = 1, 5 \) for which \( t_{x,k} \neq 0 \).

The degenerate cycle \( A \) corresponds to a small simple closed curve encircling \( z_0 = \sqrt{3u} \) and the cycle \( B \) corresponds to the chain \((-\sqrt{3u} \to \sqrt{3u})\). Therefore, we have

\[
\epsilon = \frac{1}{2\pi i} \oint_A y dx = 0, \quad I = \oint_B y dx = \frac{-8}{15} (3u)^\frac{5}{2}.
\]

The prepotential \( F_0 = \omega_{0,0} \), defined in general in \([17, 21]\), is worth

\[
F_0 = \frac{1}{2} \left( \epsilon I + \sum_k t_{x,k} \int_{B_{x,k}} y dx \right) = \frac{1}{2} \left( tV - \frac{tV}{10} \right) = \frac{2}{5} tV = \frac{-12}{5} u^5,
\]

and satisfies

\[
\frac{\partial F_0}{\partial t} = 2u^3 = V, \quad \frac{\partial^2 F_0}{\partial t^2} = \frac{\partial V}{\partial t} = -u.
\]
2.3.2. Non-degenerate case: elliptic curves. When \( V \neq 2u^3 \), we shall consider the Weierstrass parametrization of the torus of modulus \( \tau \), and with a scaling \( \nu \):

\[
\begin{cases}
  x(z) = \nu^2 \wp(z), \\
  y(z) = \frac{\nu}{2} \wp'(z),
\end{cases}
\]

with \( z \in \Sigma = \mathbb{C}/\mathbb{Z} + \tau \mathbb{Z} \) the torus of modulus \( \tau \). The fundamental form of the second kind on \( \Sigma \), normalized on the \( A \) cycle is:

\[
B(z_1, z_2) = (\wp(z_1 - z_2) + G_2(\tau))dz_1dz_2.
\]

with \( G_k(\tau) \) the \( k \)th Eisenstein series. It satisfies

\[
y^2 = x^3 + tx + V,
\]

with

\[
t = -15\nu^4G_4(\tau), \quad V = -35\nu^6G_6(\tau).
\]

Instead of parametrizing the spectral curve with \( t \) and \( V \), we shall parametrize it with \( t \) and \( \epsilon \), where

\[
\epsilon = \frac{1}{2\pi i} \int_A ydx = \frac{3\nu^5}{\pi i} \left( 2G_2(\tau)G_4(\tau) - 7G_6(\tau) \right) = -3\nu^5G_4'(\tau).
\]

We shall now write

\[
V = V(t, \epsilon).
\]

We have

\[
dV = \frac{5}{3} \pi ivd\epsilon - \frac{2}{3} \nu^2G_2(\tau)dt.
\]

This curve has 3 ramification points at \( z = \frac{1}{2}, \frac{1 + \tau}{2} \), and one pole at \( z = 0 \).

Near \( z = 0 \), we have

\[
y \sim x^{\frac{1}{2}} + \frac{t}{2x^{\frac{1}{2}}} + \frac{V}{2x^{\frac{3}{2}}} - \frac{t^2}{8x^{\frac{5}{2}}} - \frac{tV}{4x^{\frac{7}{2}}} + O(x^{-\frac{9}{2}}).
\]

This implies that

\[
t_{x,1} = \frac{1}{2\pi i} \int_{A_{x,1}} ydx = \text{Res}_{z \to 0} x^{\frac{1}{2}} ydx = -t,
\]

\[
\int_{B_{x,1}} ydx = \text{Res}_{z \to 0} x^{\frac{1}{2}} ydx = -V.
\]

\[
t_{x,5} = \frac{1}{2\pi i} \int_{A_{x,5}} ydx = \text{Res}_{z \to 0} x^{-\frac{2}{5}} ydx = \frac{tV}{10},
\]

\[
\int_{B_{x,5}} ydx = \frac{1}{5} \text{Res}_{z \to 0} x^{\frac{3}{5}} ydx = \frac{tV}{10}.
\]

We also consider

\[
I = \oint_B ydx.
\]

The prepotential \( F_0 = \omega_{0,0} \) is worth

\[
F_0 = \frac{1}{2} \left( tV - 2\frac{tV}{10} + I\epsilon \right) = 2\frac{tV}{5} + \frac{1}{2} I\epsilon,
\]

and satisfies

\[
dF_0 = V dt + I d\epsilon.
\]
Remark 2.2. In particular, for elliptic curves, we can express the independent term of the curve $V$ as a deformation of the prepotential $\omega_{0,0}$ with respect to the coefficient of the linear term $t$:

$$\frac{\partial}{\partial t} \omega_{0,0} = V.$$  (11)

This is a consequence of $\frac{\partial}{\partial t} \omega_{0,0} = \int_{B_{x,1}} \omega_{0,1}$ (which is already proved in [21, Theorem 5.1]).

In terms of the torus modulus $\tau$ and scaling $\nu$, we have:

$$t = -15\nu^4 G_4(\tau), \quad V = -35\nu^6 G_6(\tau), \quad \epsilon = -3\nu^5 G_4'(\tau), \quad I = 2\pi i \tau + \frac{4}{5}\nu t.$$  

We also have:

$$\omega_{1,0} = F_1 = \frac{1}{48} \log (4t^3 + 27V^2) + \frac{1}{4} \log \frac{2}{\nu}.$$  

3. LOOP EQUATIONS AND DEFORMATION PARAMETERS

We start by recalling the loop equations for the topological recursion applied to any spectral curve of degree 2 with a global involution. Let $y^2 = R(x)$, with $R \in \mathbb{C}(x)$. The family of curves that we consider has the global involution $z \mapsto -z$, i.e. $x(z) = x(-z)$.

Let $\omega_{0,1}(z) := y(z)dx(z)$, $\omega_{0,2}(z_1, z_2) := B(z_1, z_2)$ and $\omega_{g,n}$ for $2g - 2 + n > 0$ be defined as the topological recursion amplitudes for this initial data [21].

The loop equations for this particular case read:

**Theorem 3.1.** [6, Proposition 2.8] Let $g, n \in \mathbb{N}$. The linear loop equations read:

$$\omega_{g,n+1}(z, z_1, \ldots, z_n) + \omega_{g,n+1}(-z, z_1, \ldots, z_n) = \delta_{g,0}\delta_{n,1} \frac{dx(z)dx(z_1)}{(x(z) - x(z_1))^2}.$$  (12)

The quadratic loop equations claim that the following expression

$$\frac{1}{dx(z)^2} \left( \omega_{g-1,n+2}(z, -z, z_1, \ldots, z_n) + \sum_{\substack{g_1 + g_2 = g, \\ I_1 \cup I_2 = \{z_1, \ldots, z_n\}}} \omega_{g_1,1+|I_1|}(z, I_1)\omega_{g_2,1+|I_2|}(-z, I_2) \right)$$  (13)

is a rational function of $x(z)$ with no poles at the branch-points.

We will make use of an immediate consequence of the loop equations:

**Corollary 3.2.** For all $g, n \geq 0$,

$$P_{g,n}(x(z); z_1, \ldots, z_n) := \frac{-1}{dx(z)^2} \left( \omega_{g-1,n+2}(z, -z, z_1, \ldots, z_n) \right.$$

$$\left. + \sum_{\substack{g_1 + g_2 = g, \\ I_1 \cup I_2 = \{z_1, \ldots, z_n\}}} \omega_{g_1,1+|I_1|}(z, I_1)\omega_{g_2,1+|I_2|}(-z, I_2) \right)$$

$$\left. + \sum_{i=1}^{n} \frac{1}{dx(z_i)} \omega_{g,n}(z_1, \ldots, -z_i, \ldots, z_n) \right)$$

is a rational function of $x(z)$ that has no poles at the branch-points and no poles when $x(z) = x(z_i)$.

**Proof.** First, the expression (14) is invariant under $z \mapsto -z$ and meromorphic on $\Sigma$, hence a meromorphic function of $x(z) \in \mathbb{CP}^1$, i.e. a rational function of $x(z)$. From the loop equations, it
has no pole at branchpoints. Let us study the behavior at \( z = z_i \). The only term in (13) that contains a pole at \( z = z_i \) is

\[
\frac{1}{dx(z)^2} B(z, z_i) \omega_{g,n}(-z, z_1, \ldots, \hat{z}_i, \ldots, z_n).
\]

Remark that \( B(-z, z_i) \) has no pole at \( z = z_i \) and

\[
B(z, z_i) + B(-z, z_i) = \frac{dx(z)dx(z_i)}{(x(z) - x(z_i))^2} = d_i \left( \frac{dx(z)}{x(z) - x(z_i)} \right).
\]

Therefore we add a term without any poles to the previous one and consider the term with a pole at \( z = z_i \) to be

\[
\frac{1}{dx(z)^2} \frac{dx(z)dx(z_i)}{(x(z) - x(z_i))^2} \omega_{g,n}(-z, z_1, \ldots, \hat{z}_i, \ldots, z_n).
\]

We can write it as

\[
d_i \left( \frac{1}{x(z) - x(z_i)} \right) \left( \frac{\omega_{g,n}(-z, z_1, \ldots, \hat{z}_i, \ldots, z_n)}{dx(z)} - \omega_{g,n}(-z, z_1, \ldots, \hat{z}_i, \ldots, z_n) \frac{dx(z_i)}{dx(z_i)} \right).
\]

The sum of the first two terms does not have a pole at \( z = z_i \). Therefore subtracting the last term for all \( i = 1, \ldots, n \), we obtain an expression with no poles at \( z = z_i \). Since this expression is an even function of \( z \), there is no pole at \( z = -z_i \) either.

Recall that in general \( \omega_{0,2} = B \) can have poles only at coinciding points and the \( \omega_{g,n} \)'s with \( 2g - 2 + n > 0 \) can have poles only at ramification points. Therefore, from the corollary we see that \( P_{g,n}(x(z); z_1, \ldots, z_n) \) as a function of \( z \) can only have poles at the poles of \( \omega_{0,1} = ydx \).

### 3.1. Variational formulas of the topological recursion

We recall one of the most important properties of the topological recursion: we know how the output data behaves when we vary the family of spectral curves to which we apply the topological recursion with respect to some parameters. This result already appeared in the original paper where topological recursion was introduced [21, Theorem 5.1] and was revisited in terms of generalised cycles in [17, Theorem 3.2]. A detailed proof of it with applications in the context of combinatorics can be found in [3, Section 3.2].

**Theorem 3.3.** Let \((\Sigma, x_t, y_tdx_t, B_{\Sigma})\) be a holomorphic family of spectral curves, where \( B_{\Sigma} \) is the normalised bidifferential of the second kind on \( \Sigma \), depending on a parameter \( t \in \mathbb{C} \). Let \( \omega_{g,n}^t \) be the topological recursion differentials associated to that family of spectral curves. For the base topologies, we have \( \omega_{0,1}^t = y_tdx_t \) and \( \omega_{0,2}^t = B_{\Sigma} \). Assume there exists a generalised cycle \( \gamma \) on \( \Sigma \) whose support does not contain the zeroes of \( dx \) and such that

\[
\partial_t \omega_{0,1}^t(z) = \int_\gamma B(z, \cdot),
\]

where the derivatives are taken with \( x \) fixed. Then,

\[
\partial_t \omega_{g,n}^t(z_1, \ldots, z_n) = \int_\gamma \omega_{g,n+1}^t(z_1, \ldots, z_n, \cdot),
\]

where the derivatives are taken at \( x_i = x(z_i) \) fixed.

We remark that the equation (16) for \((0,2)\) follows from the formula for \( \omega_{0,3} \) in terms of \( B_{\Sigma} \) [21, Theorem 4.1] and the Rauch variational formula (see [3, Lemma 3.13] and references therein).
3.2. Relation to time deformation for elliptic curves. Now we restrict ourselves to curves described by polynomials of the form

$$y^2 = x^3 + tx + V.$$  

In this case, $\omega_{0,1} = ydx$ can only have poles at $x(z) = \infty$.

The topological recursion amplitudes for $2g - 2 + n \geq 0$ are analytic away from branchpoints; in particular they are analytic at $\infty$, with the following behavior near $z = \infty$:

$$\omega_{g,n}(z, z_1, \ldots, z_n) = O(z^{-2}).$$

In our case, this implies the following behavior at $x(z) = \infty$:

$$\frac{\omega_{g,n+1}(z, z_1, \ldots, z_n)}{dx(z)} = O(x(z)^{-\frac{n}{2}}), \text{ for } 2g - 2 + n \geq 0.$$  

Since the only pole can come from terms that contain $\omega_{0,1} = ydx$, $P_{g,n}$ has the following behavior at $x(z) \to \infty$:

$$P_{g,n}(x(z); z_1, \ldots, z_n) = 2y(x)\frac{dx(z)\omega_{g,n+1}(z, z_1, \ldots, z_n)}{dx(z)^2} + O(x(z)^{-1}),$$  

i.e.

$$P_{g,n}(x(z); z_1, \ldots, z_n) = 2(y(z)O(x(z)^{-\frac{n}{2}}) + O(x(z)^{-1})) = O(x(z)^0), \text{ for } 2g - 2 + n \geq 0,$$

where the last behavior comes from the fact that in the elliptic curve case, we have $y \sim x^\frac{3}{2}$.

We have seen that $P_{g,n}$ is a polynomial of degree 0, that is independent of $z$, and can be written:

$$P_{g,n}(x(z); z_1, \ldots, z_n) = 2\lim_{z \to \infty} x(z)\frac{\omega_{g,n+1}(z, z_1, \ldots, z_n)}{dx(z)}.$$

**Corollary 3.4.** For $(g, n) \neq (0,0), (0,1)$:

$$P_{g,n}(x(z); z_1, \ldots, z_n) = -\oint_{\mathcal{B}_{x,1}} \omega_{g,n+1}(z, z_1, \ldots, z_n) = \frac{\partial}{\partial t} \omega_{g,n+1}(z_1, \ldots, z_n),$$

with $\mathcal{B}_{x,1}$ the second kind cycle given by $\frac{1}{2\pi i}C_{\infty}\sqrt{x(z)}$, where $C_{\infty}$ denotes a small contour around $\infty$.

Moreover

$$P_{0,0}(x(z)) = y(z)^2 = x^3 + tx + V = x^3 + tx + \frac{\partial}{\partial t} \omega_{0,0},$$

$$P_{0,1}(x(z); z_1) = 2y(z)\frac{dx(z)B(z, z_1)}{dx(z)} - d_1\left(\frac{y(z) + y(z_1)}{x(z) - x(z_1)}\right) = \frac{\partial}{\partial t} \omega_{0,1}(z_1).$$

**Proof.** For $(g, n) = (0,0)$, we obtain $P_{0,0}(x(z)) = y(z)^2$ from (14), which together with $V = \frac{\partial}{\partial t} \omega_{0,0}$ from (11) gives the expression for this case.

The expression for $(g, n) = (0,1)$ is a direct computation using (14):

$$P_{0,1}(x(z); z_1) = \frac{y(z)}{dx(z)}(B(z, z_1) - B(-z, z_1)) - d_1\left(\frac{y(z_1)}{x(z) - x(z_1)}\right).$$

In order to get the second equality in (21), observe from the first equality that also $P_{0,1}$ has the following behavior at $x(z) \to \infty$:

$$P_{0,1}(x(z); z_1) = 2\frac{y(z)B(z, z_1)}{dx(z)} + O(x(z)^{-1}) = 2y(z)O(x(z)^{\frac{3}{2}}) + O(x(z)^{-1}) = O(x(z)^0).$$
For \((g, n) \neq (0, 0)\), since \(P_{g,n}\) is constant with respect to \(z\), we can write
\[
 P(x(z); z_1, \ldots, z_n) = 2 \lim_{x(z) \to \infty} \frac{x(z)^{\frac{3}{2}} \omega_{g,n+1}(z, z_1, \ldots, z_n)}{dx(z)}
\]
\[
= -\text{Res}_{z=0} \sqrt{x(z)} \omega_{g,n+1}(z, z_1, \ldots, z_n)
\]
\[
= - \oint_{B_{x,1}} \omega_{g,n+1}(z, z_1, \ldots, z_n).
\]

Moreover, since \(t = -t_{x,1}\), we have
\[
\frac{\partial}{\partial t} \omega_{0,1} = -\frac{\partial}{\partial t_{x,1}} \omega_{0,1} = -\oint_{B_{x,1}} \omega_{0,2},
\]
and since \(B = \omega_{0,2}\) is the Bergman kernel normalized on the \(A\)-cycles, Theorem 3.3 implies for any topology that
\[
\oint_{B_{x,1}} \omega_{g,n+1}(z, z_1, \ldots, z_n) = \frac{\partial}{\partial t_{x,1}} \omega_{g,n}(z_1, \ldots, z_n) = -\frac{\partial}{\partial t} \omega_{g,n}(z_1, \ldots, z_n).
\]

\[\square\]

3.3. Generalization to any plane curve with a global involution. Our goal is to find the relation between \(P_{g,n}\) from loop equations and an operator depending on time deformations acting on the topological recursion amplitudes. In this section, we generalize the relation that we have just found in the Painlevé I case to all plane curves with a global involution.

Consider algebraic curves of the form
\[
y^2 = R(x),
\]
with \(R(x) \in \mathbb{C}(x)\) an arbitrary rational function.

This is parametrized by a pair of meromorphic functions \(x, y\) on a Riemann surface \(\Sigma\). The ramified covering given by \(x: \Sigma \to \mathbb{C}P^1\) is a double cover. Depending on the parity of the behavior of \(y\) as \(x \to \infty\), \(x\) has either one double pole (order \(d = -2\)) or 2 simple poles (order \(d = -1\)). Let us denote \(\sigma\) the global involution which sends \((x, y) \mapsto (x, -y)\).

Let \(\{\lambda_i\}_{i=1}^N\) be the set of zeroes of the denominator of \(R(x)\). The 1-form \(\omega_{0,1} = ydx\) can have a pole over \(x = \infty\) and poles over \(\{\lambda_i\}_{i=1}^N\). We call \(\zeta_i\) the poles of \(\omega_{0,1}\) of respective degrees given by \(m_i\).

Let us define \(d_i := \text{ord}_{\zeta_i}(x)\). If \(\zeta_i\) is not a pole of \(x\), we assume that it is not a ramification point, hence \(d_i = 1\). If \(\zeta_i\) is a pole of \(x\), then \(d_i\) can be either \(-2\) or \(-1\) as we commented, depending on \(\zeta_i\) being a ramification point or not. Notice that if \(\zeta_i\) is a pole, so is \(\sigma(\zeta_i)\), and thus poles come in pairs. Moreover, we can only have \(\sigma(\zeta_i) = \zeta_i\), if \(\zeta_i\) is a double pole of \(x\), which corresponds to the case \(d_i = -2\). Near \(\zeta_i\) we use the local variable
\[
\xi_i = (x - x(\zeta_i))^{\frac{1}{d_i}},
\]
where we define \(x(\zeta_i) = 0\), if \(\zeta_i\) is a pole of \(x\).

We write the Laurent expansion:
\[
ydx \sim \sum_{j=0}^{m_i} t_{\zeta,1,j} \xi_i^{-1-j} d\xi_i + \text{analytic at } \zeta_i.
\]
This defines the local KP times [17]
\[
t_{\zeta,1,j} = \text{Res}_{\xi_i} \xi_i^j ydx = \frac{1}{2\pi i} \oint_{A_{\zeta,1,j}} ydx.
\]
Notice that for poles for which \( \sigma(\zeta_i) \neq \zeta_i \), we have

\[
\tag{27} t_{\sigma(\zeta),j} = -t_{\zeta,j}.
\]

Again, loop equations imply that the \( P_{g,n}(x; z_1, \ldots, z_n) \) defined in (14) has no pole at coinciding points or at branchpoints. It must be a rational function of \( x \), whose poles can be only at the poles of \( ydx \), i.e. at the poles of \( R(x) \) and possibly at \( x = \infty \).

For every second type cycle \( B_{p,k} \), \( k \geq 1 \), we define the operator \( \hat{\partial}_{B_{p,k}} \) as

\[
\hat{\partial}_{B_{p,k}} \omega_{g,n}(z_1, \ldots, z_n) := \int_{B_{p,k}} \omega_{g,n+1}(\cdot, z_1, \ldots, z_n) = \text{Res}_{x=x(p)} \frac{\zeta - \zeta}{k} \omega_{g,n+1}(\cdot, z_1, \ldots, z_n).
\]

**Proposition 3.5.** Let \( \zeta_i \) be the poles of \( \omega_{0,1} \) of respective degrees given by \( m_i \). The operator

\[
\tag{28} L(x) := \sum_{i, \varepsilon(\zeta_i) = \infty} \sum_{j=1}^{m_i} t_{\zeta_i,j} \sum_{0 \leq k \leq \frac{1}{d_i} - 2} x^k (-j - k - 2) \hat{\partial}_{B_{\zeta_i,j+d_i(k+2)}}
\]

\[
+ \sum_{i, \varepsilon(\zeta_i) \neq \infty} \sum_{j=0}^{m_i} t_{\zeta_i,j} \sum_{k=0}^{j} (x - x(\zeta_i))^{-(k+1)} (j + 1 - k) \hat{\partial}_{B_{\zeta_i,j+1-k}}
\]

gives

\[
\tag{29} P_{g,n}(x; z_1, \ldots, z_n) = L(x) \omega_{g,n}(z_1, \ldots, z_n).
\]

**Proof.** Let us write the Cauchy formula for \( x = x(z) \)

\[
P_{g,n}(x; z_1, \ldots, z_n) = \text{Res}_{x' = x} \frac{dx'}{x' - x} P_{g,n}(x'; z_1, \ldots, z_n)
\]

\[
= \frac{1}{2} \text{Res}_{z' = z} \frac{dx'}{z' - x} P_{g,n}(x(z'); z_1, \ldots, z_n)
\]

\[
+ \frac{1}{2} \text{Res}_{z' = \sigma(z)} \frac{dx'}{z' - x} P_{g,n}(x(z'); z_1, \ldots, z_n)
\]

\[
= \frac{1}{2} \sum_{i} \text{Res}_{z' = \zeta_i} \frac{dx'}{x - z'} P_{g,n}(x(z'); z_1, \ldots, z_n)
\]

\[
= -\frac{1}{2} \sum_{i, \varepsilon(\zeta_i) = \infty} \sum_{k \geq 0} x(z)^k \text{Res}_{z' = \zeta_i} x(z')^{-(k+1)} dx(z') P_{g,n}(x'; z_1, \ldots, z_n)
\]

\[
+ \frac{1}{2} \sum_{i, \varepsilon(\zeta_i) \neq \infty} \sum_{k \geq 0} \xi_i(z)^{-(k+1)} \text{Res}_{z' = \zeta_i} \xi_i(z')^k dx(z') P_{g,n}(x'; z_1, \ldots, z_n).
\]

Since the behavior at the poles is given by the terms containing \( \omega_{0,1} = ydx \) in (14), near any of the poles \( \zeta_i \) we have

\[
\tag{30} P_{g,n}(x(z); z_1, \ldots, z_n) \sim \frac{2y(z)}{dx(z)} \omega_{g,n+1}(z, z_1, \ldots, z_n) + O(\xi_i^{-2(d_i+1)}).
\]
First consider the poles over \( x = \infty \), with \( d_i = -1 \) or \( d_i = -2 \), which contribute to \( P_{g,n} \) as

\[
- \sum_{i \in \{\zeta_i\}} \sum_{k \geq 0} x(z)^k \operatorname{Res}_{z' \to \zeta_i} x(z')^{-(k+1)} y(z') \omega_{g,n+1}(z', z_1, \ldots, z_n)
\]

\[
= - \sum_{i \in \{\zeta_i\}} \sum_{k \geq 0} t_{i,j}^{(1)} \sum_{j = 0}^{m_i} x(z)^k \operatorname{Res}_{z' \to \zeta_i} x(z')^{-(k+1)} \xi_i(z')^{j-1} \frac{1}{d_i \xi_i(z')} \omega_{g,n+1}(z', z_1, \ldots, z_n)
\]

\[
= \sum_{i \in \{\zeta_i\}} \sum_{j = 0}^{m_i} t_{i,j}^{(2)} \sum_{k \geq 0} x(z)^k \operatorname{Res}_{z' \to \zeta_i} x(z')^{-(k+1)} \omega_{g,n+1}(z', z_1, \ldots, z_n)
\]

\[
= - \sum_{i \in \{\zeta_i\}} \sum_{j = 0}^{m_i} t_{i,j}^{(2)} \sum_{k \geq 0} x(z)^k \left( k + 2 + \frac{j}{d_i} \right) \int_{B_{i,j+k}} \omega_{g,n+1}(z', z_1, \ldots, z_n)
\]

The finite poles contribute as

\[
\frac{1}{2} \sum_{i \in \{\zeta_i\} \neq \infty} \sum_{k \geq 0} \xi_i(z)^{-(k+1)} \operatorname{Res}_{z' \to \zeta_i} \xi_i(z')^k d(x(z')) P_{g,n}(x'; z_1, \ldots, z_n)
\]

\[
= \sum_{i \in \{\zeta_i\} \neq \infty} \sum_{k \geq 0} \xi_i(z)^{-(k+1)} \operatorname{Res}_{z' \to \zeta_i} \xi_i(z')^k y(z') \omega_{g,n+1}(z', z_1, \ldots, z_n)
\]

\[
= \sum_{i \in \{\zeta_i\} \neq \infty} \sum_{j = 0}^{m_i} t_{i,j}^{(1)} \sum_{k \geq 0} \xi_i(z)^{-(k+1)} \operatorname{Res}_{z' \to \zeta_i} \xi_i(z')^{j-1} \omega_{g,n+1}(z', z_1, \ldots, z_n)
\]

\[
= \sum_{i \in \{\zeta_i\} \neq \infty} \sum_{j = 0}^{m_i} t_{i,j}^{(2)} \sum_{k \geq 0} \xi_i(z)^{-(k+1)} (j + 1 - k) \int_{B_{i,j+k-1}} \omega_{g,n+1}(z', z_1, \ldots, z_n)
\]

\[
= \sum_{i \in \{\zeta_i\} \neq \infty} \sum_{j = 0}^{m_i} t_{i,j}^{(2)} \sum_{k \geq 0} \xi_i(z)^{-(k+1)} (j + 1 - k) \partial_{B_{i,j+k-1}} \omega_{g,n}(z_1, \ldots, z_n).
\]

\( \square \)

Let us now rewrite the operator \( L(x) \) as a differential operator with respect to the moduli of the spectral curve, i.e. the times appearing in the pole structure of \( \omega_{0,1} \) and the poles \( \Lambda = \{ \lambda_l \}_{l=1}^N \) of \( R(x) \).

**Lemma 3.6.** Let \( \zeta_l^{(1)} \) and \( \zeta_l^{(2)} \) be the two preimages of \( \lambda_l \) by \( x \), which are poles of order \( m_l \) of \( \omega_{0,1} \). We assume that the filling fractions \( \epsilon_i \) are independent of the poles \( \lambda_l \), i.e.

\[
\frac{\partial}{\partial \lambda_l} \epsilon_i = \int_{A_i} \frac{\partial}{\partial \lambda_l} \omega_{0,1} = 0.
\]

Then we can decompose the operator derivative with respect to \( \lambda_l \) as

\[
\frac{\partial}{\partial \lambda_l} \omega_{g,n}(z_1, \ldots, z_n) = \sum_{i=1,2} \sum_{j=0}^{m_i} (j + 1) t_{i,j}^{(1)} \partial_{B_{i,j+1}} \omega_{g,n}(z_1, \ldots, z_n),
\]

for all \( g, n \geq 0 \).
Proof. Let \( \zeta \) be one of the two preimages of \( \lambda_l \) by \( x \) and \( \xi_l = x - \lambda_l \) be the local variable around it. Then, for \( z \to \zeta \), we have

\[
\frac{\partial}{\partial \lambda_l} \omega_{0,1}(z) \sim \sum_{j=0}^{m_l}(j+1)t_{\zeta,j}(\xi_l(z))^{-j-2}d\xi_l + \text{ analytic.}
\]

On the other hand, we have

\[
\partial\mathcal{B}_{\zeta,j}\omega_{0,1}(z) = \int_{\mathcal{B}_{\zeta,j}} \omega_{0,2}(\cdot, z) = \text{Res}_{z' = \lambda_l} \frac{\omega_{0,2}(z', z)}{j(\xi_p(z'))^j}
\]

\[
= -\text{Res}_{z' = \lambda_l} \omega_{0,2}(z', z) + \text{ analytic} = \frac{d\xi_l(z)}{(\xi_l(z))^{j+1}} + \text{ analytic.}
\]

The last equality follows from the fact that the fundamental differential of the second kind \( \omega_{0,2} \) only has a double pole along the diagonal and the next-to-last equality follows from the decomposition

\[
\oint_{\gamma_{\lambda_l}} \frac{\omega_{0,2}(\cdot, z)}{j(\xi_p(\cdot))^j} = \oint_{\gamma_z} \frac{\omega_{0,2}(\cdot, z)}{j(\xi_p(\cdot))^j} - \oint_{\gamma_z} \frac{\omega_{0,2}(\cdot, z)}{j(\xi_p(\cdot))^j} = \text{ analytic} - \oint_{\gamma_z} \frac{\omega_{0,2}(\cdot, z)}{j(\xi_p(\cdot))^j},
\]

where \( \gamma_z \) and \( \gamma_{\lambda_l} \) are small counterclockwise contours around \( z \) and \( \lambda_l \), and \( \gamma_{z,\lambda_l} \) is a counterclockwise contour that surrounds the segment from \( z \) to \( \lambda_l \), which gives rise to an analytic term.

Therefore,

\[
\frac{\partial}{\partial \lambda_l} \omega_{0,1}(z) = -\sum_{i=1,2} \sum_{j=0}^{m_l}(j+1)t_{\zeta,(i)} \partial\mathcal{B}_{\zeta,(i),j+1}\omega_{0,1}(z)
\]

is a holomorphic 1-form with vanishing \( A \)-periods and thus it vanishes.

This yields the equality for \( \omega_{0,1} \). Since we chose \( \omega_{0,2} \) to be the Bergman kernel with vanishing \( A \)-periods, we obtain the equality for the rest of the \( \omega_{g,n} \) from Theorem 3.3.

\[\square\]

**Corollary 3.7.** Let \( \zeta \in x^{-1}(x) \) and \( \zeta \in x^{-1}(\lambda_l) \) be poles of \( \omega_{0,1} \) of orders \( m_x \) and \( m_l \), \( l = 1, \ldots, N \), respectively. Let \( d_x := \text{ord}_x(x) \). The operator (28) is equal to the differential operator \( L(x) = L_x(x) + L_A(x) \), with

\[
L_x(x) = \sum_{j=1,2}^{m_x} t_{\zeta,x,j} \left( \sum_{k=0}^{j-2} \frac{x^{-k}}{d_x} \right) \frac{\partial}{\partial \zeta}\left( \frac{x^{-j}}{d_x} \right)
\]

and

\[
L_A(x) = \sum_{l=1}^{N} \left( \frac{1}{x - \lambda_l} \frac{\partial}{\partial \lambda_l} + \sum_{j=1}^{m_l-1} t_{\zeta,(i),j} \sum_{k=1}^{j} (x - \lambda_l)^{-(k+1)}(j+1-k) \frac{\partial}{\partial \zeta}\right).
\]

**Proof.** From the variational formulas of topological recursion (see Theorem 3.3), if \( \zeta \) is a ramified pole of \( \omega_{0,1} \) of order \( m \), i.e. \( x(\zeta) = \infty \) and \( \text{ord}_x(x) = -2 \), then we have

\[
\frac{\partial}{\partial t_{\zeta,j}} \omega_{g,n}(z_1, \ldots, z_n) = \partial\mathcal{B}_{\zeta,j} \omega_{g,n}(z_1, \ldots, z_n), \text{ for } j = 1, \ldots, m.
\]

On the other hand, if \( \zeta \) is an unramified pole of \( \omega_{0,1} \) of order \( m \), either \( x^{-1}(\infty) = \{\zeta, \sigma(\zeta)\} \) and \( \text{ord}_x(x) = -1 \) or \( x^{-1}(\lambda_l) = \{\zeta, \sigma(\zeta)\} \) and \( \text{ord}_x(x) = 1 \), then the coefficients of the expansion of \( \omega_{0,1} \) around \( \zeta = \zeta^+ \) and the other preimage \( \sigma(\zeta) = \zeta^- \) are not independent. Therefore, the variational formulas include residues at the two preimages:

\[
\frac{\partial}{\partial t_{\zeta,j}} \omega_{g,n}(z_1, \ldots, z_n) = \partial\mathcal{B}_{\zeta,j} \omega_{g,n}(z_1, \ldots, z_n) - \partial\mathcal{B}_{\sigma(\zeta),j} \omega_{g,n}(z_1, \ldots, z_n), \text{ for } j = 1, \ldots, m.
\]
The variational formulas allow to rewrite all the operators \( \partial_{\zeta,j} \) in (28), for \( j = 1, \ldots, m \), in terms of derivatives with respect to the times \( t_{\zeta,j} \) appearing as coefficients of the polar part of \( \omega_{0,1} \). This is enough to rewrite the first line of (28) as \( L_x(x) \) in (33).

For \( L_\Lambda(x) \), we make use of (32) to express the remaining operators \( \partial_{\zeta,m+1} \) in terms of allowed times \( t_{\zeta,j}, j = 1, \ldots, m \), and derivatives with respect to the poles \( \lambda_t \) of \( R(x) \):

\[
(m_l + 1) t_{\zeta_l,m_l} \xi_{\zeta_l}^{-1} (\partial_{\zeta_l,m_l+1} - \partial_{\zeta_l(m_l),m_l+1}) = \xi_{\zeta_l}^{-1} \left( \frac{\partial}{\partial \lambda_t} - \sum_{j=0}^{m_l-1} (j+1) t_{\zeta_l,j} \frac{\partial}{\partial t_{\zeta_l,j+1}} \right).
\]

The second type of terms of the RHS of (35) cancels the terms with \( k = 0 \) from second line of (28), yielding (34).

We have just found a differential operator \( L(x) \) in the times and the poles of \( R(x) \), whose coefficients are rational functions of \( x \), with poles at \( x = \infty \) or \( x = x(\zeta_i) \), i.e. the same poles as \( R(x) \), with at most the same degrees.

**Example 3.1.** In the elliptic case of curves of the form \( y^2 = x^3 + tx + V \) we have only one pole, at \( \zeta_i = \infty \), of degree \( m_i = 5 \), with \( d_i = -2 \). The only non-vanishing times are \( t_{\infty,5} = -2 \) and \( t_{\infty,1} = -t \), and thus only the terms with \( j = 5 \) and \( k = 0 \) contribute:

\[
L(x) = \sum_{j=1,5}^{m_l} t_{\infty,j} \sum_{0 \leq k \leq 2 + (j-1)/2} x^k (j/2 - k + 2) \frac{\partial}{\partial t_{\infty,j-2k+2}}
\]

\[
= t_{\infty,5} (5/2 - 0 - 2) \frac{\partial}{\partial t_{\infty,5-2k+2}}
\]

\[
= -2 \left( \frac{5}{2} - 2 \right) \frac{\partial}{\partial t_{\infty,5-2k+2}}
\]

\[
= -2 \left( \frac{5}{2} - 2 \right) \frac{\partial}{\partial t_{\infty,1}}
\]

\[
= \frac{\partial}{\partial t_{\infty,1}}.
\]

**Example 3.2.** In the Airy case, \( y^2 = x \), we have only one pole, at \( \zeta_i = \infty \), of degree \( m_i = 3 \), with \( d_i = -2 \). The sum is empty and

\[
L(x) = 0.
\]

**Remark 3.3.** More generally, the admissible curves considered in [9] are those for which

\[
L(x) = 0.
\]

4. **PDEs quantizing any hyperelliptic curve**

For \( r \geq 1 \), let \( D = \sum_{i=1}^r \alpha_i [p_i] \) be a divisor on \( \Sigma \), with \( p_i \in \Sigma \). We call \( \sum \alpha_i \) the *degree* of the divisor and denote \( \text{Div}_0(\Sigma) \) the set of divisors of degree 0. For \( D \in \text{Div}_0(\Sigma) \), we define the integration of a 1-form \( \rho(z) \) on \( \Sigma \) as

\[
\int_D \rho(z) := \sum_i \alpha_i \int_o^{p_i} \rho(z),
\]

where \( o \in \Sigma \) is an arbitrary base point. This integral is well defined locally, meaning that it is independent of the base point \( o \) because the degree of the divisor is zero, however it depends on a choice of homotopy class from \( o \) to \( p_i \).
For \((g, n) \neq (0, 2)\) consider the functions of \(D\), defined locally:

\[
F_{g, 0}(D) := F_g = \omega_{g, 0},
\]

\[
F_{g, n}(D) := \int_D \cdots \int_D \omega_{g, n}(z_1, \ldots, z_n),
\]

\[
F'_{g, n}(z; D) := \frac{1}{dx(z)} \int_D \cdots \int_D \omega_{g, n}(z, z_2, \ldots, z_n),
\]

\[
F''_{g, n}(z, \bar{z}; D) := \frac{1}{dx(z)dx(\bar{z})} \int_D \cdots \int_D \omega_{g, n}(z, \bar{z}, z_3, \ldots, z_n).
\]

Recall that

\[
B(z_1, z_2) := d_1d_2 \log \left( E(z_1, z_2) \sqrt{dx(z_1)dx(z_2)} \right),
\]

with \(E(z_1, z_2)\) being the prime form, which is defined in [21], and satisfies that it vanishes only if \(z_1 = z_2\) with a simple zero and has no pole.

For \((g, n) = (0, 2)\) define:

\[
F_{0, 2}(D) := 2 \sum_{i < j} \alpha_i \alpha_j \log \left( E(p_i, p_j) \sqrt{dx(p_i)dx(p_j)} \right),
\]

\[
F'_{0, 2}(z; D) := \frac{1}{dx(z)}d_z \left( \sum_{i=1}^r \alpha_i \log \left( E(z, p_i) \sqrt{dx(z)dx(p_i)} \right) \right),
\]

\[
F''_{0, 2}(z, \bar{z}; D) := \frac{B(z, \bar{z})}{dx(z)dx(\bar{z})}.
\]

Since the \(\omega_{g, n}\) are symmetric, we have the following relations for \((g, n) \neq (0, 2)\):

\[
\frac{d}{dx_i} F_{g, n}(D) = n\alpha_i F'_{g, n}(p_i; D),
\]

\[
\left( \frac{d}{dx_i} \right)^2 F_{g, n}(D) = n(n - 1)\alpha_i^2 F''_{g, n}(p_i, p_i; D) + n\alpha_i \left( \frac{d}{dx} F'_{g, n}(\bar{p}; D) \right)_{\bar{p}=p_i},
\]

where \(x_i := x(p_i)\), \(\bar{x} := x(\bar{p})\), and \(\frac{d}{dx}\) acts on meromorphic functions by taking exterior derivative and dividing by \(dx\), which amounts to derive an analytic expansion of the meromorphic function with respect to a local variable \(x\).

For \((g, n) = (0, 2)\) we have

\[
\frac{d}{dx_i} F_{0, 2}(D) = 2\alpha_i \lim_{{z \to p_i}} \left( F'_{0, 2}(z; D) - \alpha_i \frac{d}{dx(z)} \log \left( E(z, p_i) \sqrt{dx(z)dx(p_i)} \right) \right)
\]

\[
= 2\alpha_i \sum_{j \neq i} \alpha_j \frac{d}{dx(p_i)} \log \left( E(p_i, p_j) \sqrt{dx(p_i)dx(p_j)} \right).
\]

Remark 4.1. Integrating the first part of Theorem 3.1 over a divisor \(D\) of degree 0, we obtain:

\[
F'_{0, 2}(z; D) + F'_{0, 2}(-z; D) = \sum_{i=1}^r \frac{\alpha_i}{x(z) - x(p_i)}.
\]
Lemma 4.1. Let \( \zeta \) be a pole of \( \omega_{0,1} \) and \( B_{\zeta,k} \) the \( k \)th 2nd type cycle around \( \zeta \), with \( k \geq 1 \). Consider \( t_{\zeta,k} \) the corresponding KP time. Then,

\[
\int_D \int_D \int_{B_{\zeta,k}} \omega_{0,3}(z, z_1, z_2) = \int_D \int_D \frac{\partial}{\partial t_{\zeta,k}} \omega_{0,2}(z_1, z_2) = \frac{\partial}{\partial t_{\zeta,k}} F_{0,2}(D).
\]

Proof.

\[
\int_D \int_D \left( B(z_1, z_2) - \frac{dx(z_1)dx(z_2)}{(z_1 - x(z_2))^2} \right) + 2 \sum_{i<j} \alpha_i \alpha_j \log (x_i - x_j) = 2 \sum_{i<j} \alpha_i \alpha_j \log \left( \frac{E(p_i, p_j) \sqrt{dx(z_2)}}{x_i - x_j} \right) + 2 \sum_{i<j} \alpha_i \alpha_j \log (x_i - x_j) + \sum_i \alpha_i^2 \log \frac{dx_i}{dx_i} = F_{0,2}(D).
\]

Taking the derivative with respect to \( t_{\zeta,k} \) of the first line gives the left hand side of (44) because we are taking this derivative at fixed \( x \).

Consider \( dE(z, p_1) := \frac{d}{dz} \log \left( E(z, p_1) \sqrt{dx(z)dx(p_1)} \right) \) and observe that

\[
\lim_{z \to p_1} \frac{dE(z, p_1)}{dz} - \frac{1}{x(z) - x(p_1)} = 0.
\]

With this notation one can rewrite (38) as

\[
F'_{0,2}(z; D) = \sum_{i=1}^{r} \alpha_i \frac{dE(z, p_i)}{dz},
\]

and \( \frac{d}{dx_i} F_{0,2}(D) = 2 \alpha_i \sum_{j \neq i} \alpha_j \frac{dE(p_i, p_j)}{dx_i} \).

We define

\[
S_0(D) := \int_D ydx = F_{0,1}(D), \quad S_1(D) := \log \prod_{i<j} \left( E(p_i, p_j) \sqrt{dx(p_i)dx(p_j)} \right)^{\alpha_i \alpha_j} = \frac{F_{0,2}(D)}{2},
\]

\[
S_m(D) := \sum_{2g-2+n=m-1} \frac{F_{g,n}(D)}{n!},
\]

and

\[
\psi = \psi(D, h) := \exp(S(D, h)), \quad S(D, h) := \sum_{m=0}^{\infty} h^{m-1} S_m(D).
\]

Theorem 4.2. Let \( F := \sum_{g \geq 0} h^{2g-2} F_g \). For every \( k = 1, \ldots, r \), assuming \( \alpha_k^2 = 1 \), we obtain

\[
\hbar^2 \left( \frac{d^2}{dx_k^2} - \sum_{i \neq k} \frac{\alpha_i}{x_k - x_i} \frac{d}{dx_k} - L(x_k) - L(x_k).F + \sum_{i \neq j} \frac{\alpha_i \alpha_j}{(x_k - x_i)(x_k - x_j)} \right) \psi = R(x_k) \psi.
\]

Proof. We will give the proof of the claim for the case \( k = 1 \), but it works exactly the same for every \( k \).

Let us first consider the generic situation with \( (g, n) \neq (0, 0), (0, 1), (0, 2), (1, 0) \). Using (14) and (29) for \( n = 0 \), we can write \( P_{g,0}(x) \), with \( x = x(z) \), as

\[
F'_{g-1,2}(z, z; D) + \sum_{g_1 + g_2 = g} \frac{dF'_{g_1,1}(z; D)F'_{g_2,1}(z; D)}{dx_1} = L(x).F_{g,0}(D) = L(x).F_g.
\]
Setting \( z = p_1, x_1 = x(p_1) \) and using (40) and (41), we obtain

\[
\frac{1}{2\alpha_1^2} \left( \frac{d}{dx_1} \right)^2 F_{g-1,2}(D) - \frac{1}{\alpha_1} \left( \frac{d}{dx(p)} F'_{g-1,2}(\tilde{p}; D) \right)_{\tilde{p}=p_1} + \sum_{g_1+g_2=g} \frac{1}{\alpha_1^2} \frac{dF_{g_1,1}(D)}{dx_1} \frac{dF_{g_2,1}(D)}{dx_1}
= L(x_1).F_g(D).
\]

For \( n > 0 \), we integrate (14) \( n \) times over \( D \) and use (29) to get

\[
F''_{g-1,n+2}(z, z; D) + \sum_{g_1+g_2=g \atop n_1+n_2=n} \binom{n}{n_1} F'_{g_1,n_1+1}(z; D)F'_{g_2,n_2+1}(z; D) +
- 2nF'_{0,2}(-z; D)F''_{g,n}(z; D) - n \sum_{i=1}^r \alpha_i \frac{F'_{g,n}(p_i; D) - F'_{g,n}(z; D)}{x(z) - x(p_i)} = L(x).F_{g,n}(D),
\]

where the sum of the second term is taken over \( g_i, n_i \geq 0 \) and “no \((0,2)\)” means we exclude the cases with \((g, n) = (0, 1)\) for \( i = 1, 2 \), for which we have used that

\[
(F'_{0,2}(z; D) - F'_{0,2}(-z; D))F''_{g,n}(z; D) = -2F'_{0,2}(-z; D)F''_{g,n}(z; D) + \sum_{i=1}^r \alpha_i \frac{F'_{g,n}(z; D)}{x(z) - x(p_i)},
\]

which follows from (43).

Letting \( z = p_1, x_i = x(p_i) \), dividing by \( n! \) and using (40) and (41), we obtain

\[
\frac{1}{(n+2)!\alpha_1^2} \left( \frac{d}{dx_1} \right)^2 F_{g-1,n+2}(D) - \frac{1}{(n+1)!\alpha_1} \left( \frac{d}{dx(p)} F'_{g-1,n+2}(\tilde{p}; D) \right)_{\tilde{p}=p_1} +
\sum_{g_1+g_2=g \atop n_1+n_2=n} \frac{1}{(n_1+1)!(n_2+1)!\alpha_1^2} \frac{dF_{g_1,n_1+1}(D)}{dx_1} \frac{dF_{g_2,n_2+1}(D)}{dx_1}
- \frac{1}{n!} \sum_{i=2}^r \frac{1}{x(p_1) - x(p_i)} \left( \frac{dF_{g,n}(D)}{dx_i} - \frac{\alpha_i}{\alpha_1} \frac{dF_{g,n}(D)}{dx_1} \right) + \frac{\alpha_1}{(n-1)!} \left( \frac{dF'_{g,n}(\tilde{p}; D)}{dx(\tilde{p})} \right)_{\tilde{p}=p_1}
- \frac{2}{n!\alpha_1} \frac{dF_{g,n}(D)}{dx_1} F'_{0,2}(-p_1; D) = L(x_1) \frac{F_{g,n}(D)}{n!}.
\]
Using (43) again, we obtain the following expression for the left hand side:

\[
\begin{align*}
\frac{1}{(n+2)!\alpha_1^2} \left( \frac{d}{dx_1} \right)^2 F_{g-1,n+2}(D) - \frac{1}{(n+1)!\alpha_1} \left( \frac{d}{dx(\bar{p})} F'_{g-1,n+2}(\bar{p}; D) \right)_{\bar{p}=p_1} + \\
\sum_{g_1+g_2=g \atop n_1+n_2=n} \frac{1}{(n_1+1)!(n_2+1)!\alpha_1^2} \frac{dF_{g_1,n_1+1}(D)}{dx_1} \frac{dF_{g_2,n_2+1}(D)}{dx_1} + \\
\frac{\alpha_1}{(n-1)!} \left( \frac{dF'_{g,n}(\bar{p}; D)}{dx(\bar{p})} \right)_{\bar{p}=p_1} - \frac{1}{n!} \sum_{i=2}^r \frac{1}{x(p_1) - x(p_i)} \left( \frac{dF_{g,n}(D)}{dx_i} + \frac{\alpha_i}{\alpha_1} \frac{dF_{g,n}(D)}{dx_1} \right)_{\bar{p}=p_1} - \\
\frac{2}{n!\alpha_1} \left( \frac{dF_{g,n}(D)}{dx_1} \right)^2 F_{g-1,n+2}(D) - \frac{1}{(n+1)!\alpha_1} \left( \frac{d}{dx(\bar{p})} F'_{g-1,n+2}(\bar{p}; D) \right)_{\bar{p}=p_1} + \\
\frac{\alpha_1}{(n-1)!} \left( \frac{dF'_{g,n}(\bar{p}; D)}{dx(\bar{p})} \right)_{\bar{p}=p_1} + \sum_{g_1+g_2=g \atop n_1+n_2=n} \frac{1}{(n_1+1)!(n_2+1)!\alpha_1^2} \frac{dF_{g_1,n_1+1}(D)}{dx_1} \frac{dF_{g_2,n_2+1}(D)}{dx_1} + \\
\frac{1}{n!} \sum_{i=2}^r \frac{1}{x(p_1) - x(p_i)} \left( \frac{dF_{g,n}(D)}{dx_i} + \frac{\alpha_i}{\alpha_1} \frac{dF_{g,n}(D)}{dx_1} \right),
\end{align*}
\]

where for the last equality we have used (42) and (46).

For all \( \ell \geq 3 \), we sum this expression for all \( g \geq 0, n \geq 1 \) such that \( 2g - 2 + n = \ell - 2 \) and the corresponding \( P_{g,0}(x(z_1)) \) for \( n = 0 \) from (50) for all \( g \geq 0 \) such that \( 2g - 2 = \ell - 2 \) to obtain:

\[
\sum_{\substack{2g+n=\ell \\ g \geq 0, n \geq 0}} \frac{1}{(n+2)!\alpha_1^2} \left( \frac{d}{dx_1} \right)^2 F_{g-1,n+2}(D) \\
- \frac{1}{(n+1)!\alpha_1} \left( \frac{d}{dx(\bar{p})} F'_{g-1,n+2}(\bar{p}; D) \right)_{\bar{p}=p_1} + \frac{\alpha_1}{(n-1)!} \left( \frac{dF'_{g,n}(\bar{p}; D)}{dx(\bar{p})} \right)_{\bar{p}=p_1} \\
- \frac{1}{n!} \sum_{i=2}^r \frac{1}{x(p_1) - x(p_i)} \left( \frac{dF_{g,n}(D)}{dx_i} + \frac{\alpha_i}{\alpha_1} \frac{dF_{g,n}(D)}{dx_1} \right) \\
+ \frac{1}{\alpha_1^2} \sum_{\ell_1+\ell_2=\ell} \left( \sum_{g_1 \geq 0, n_1 \geq 1} \frac{dF_{g_1,n_1}(D)}{dx_1} \sum_{g_2 \geq 0, n_2 \geq 1} \frac{dF_{g_2,n_2}(D)}{dx_1} \right)_{\ell_1+\ell_2=\ell} = \sum_{\substack{2g+n=\ell \\ g \geq 0, n \geq 0}} L(x_1) \frac{F_{g,n}}{n!},
\]

In this sum over topologies, assuming that \( \frac{1}{\alpha_1} = \alpha_1 \), all the first terms of the second line cancel against all the second terms except the ones with \( n = 1 \), which complete the terms of the first line, using (41) for the special case \( n = 1 \) in which only the second term of the right hand side survives.

Therefore, for \( \ell \geq 3 \), we have proved

\[
\left( \frac{d}{dx_1} \right)^2 S_{\ell-1} + \frac{1}{\alpha_1^2} \sum_{\ell_1+\ell_2=\ell} \frac{d}{dx_1} S_{\ell_1} \frac{d}{dx_1} S_{\ell_2} - \sum_{i=2}^r \frac{dS_{\ell-1}}{dx_i} + \frac{\alpha_1}{\alpha_1^2} \frac{dS_{\ell-1}}{dx_1} \]

\[
= \sum_{\substack{2g+n=\ell \\ g \geq 0, n \geq 0}} L(x_1) \frac{F_{g,n}}{n!},
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Thus for  For

\[ (52) \quad [h^\ell] \left[ h^2 \left( \frac{d}{dx_1} \right)^2 S + \frac{1}{\alpha_1^2} \frac{d}{dx_1} S \frac{d}{dx_1} S - \sum_{i=2}^{r} \frac{dS_i}{dx_1} + \frac{\alpha_i}{\alpha_1} \frac{dS_i}{dx_1} \right) \right] = [h^\ell] \left[ h^2 (L(x_1).S + L(x_1).F) + R(x_1) \right]. \]

Let us finally consider the special cases before the assumption \(\alpha_1^2 = 1\):

- For \((g, n) = (0, 0)\) we get

\[ (53) \quad \frac{1}{\alpha_1^2} \left( \frac{dF_{0,1}(D)}{dx_1} \right)^2 = R(x_1), \quad \text{i.e.} \quad \frac{1}{\alpha_1^2} \left( \frac{d}{dx_1} S_0 \right)^2 = R(x_1), \quad \text{or} \]

\[ [h^0] \left[ h^2 \left( \frac{d}{dx_1} \right)^2 S + \frac{1}{\alpha_1^2} \frac{d}{dx_1} S \frac{d}{dx_1} S - \sum_{i=2}^{r} \frac{dS_i}{dx_1} + \frac{\alpha_i}{\alpha_1} \frac{dS_i}{dx_1} - L(x_1).S - L(x_1).F \right) - R(x_1) \right] = 0. \]

- For \((g, n) = (0, 1)\) we get

\[ 2F'_{0,1}(z; D)F'_{0,2}(z; D) - \sum_{i=1}^{r} \alpha_i \frac{F'_{0,1}(p_i; D) + F'_{0,1}(z; D)}{x(z) - x(p_i)} = L(x).F_{0,1}(D). \]

Thus

\[ 2F'_{0,1}(z; D)F'_{0,2}(z; D) - 2\alpha_1 \frac{F'_{0,1}(z; D)}{x(z) - x(p_1)} - \alpha_1 \frac{F'_{0,1}(p_1; D) - F'_{0,1}(z; D)}{x(z) - x(p_1)} \]

\[ - \sum_{i=2}^{r} \frac{F'_{0,1}(p_i; D) - F'_{0,1}(z; D)}{x(z) - x(p_i)} = L(x).F_{0,1}(D). \]

At \(z = p_1\) this gives

\[ 2F'_{0,1}(p_1; D) \left( F'_{0,2}(z; D) - \alpha_1 \frac{1}{x(z) - x(p_1)} \right) \]

\[ + \alpha_1 \left( \frac{dF'_{0,1}(z; D)}{dx} \right)_{z=p_1} \]

\[ - \sum_{i=2}^{r} \frac{1}{x(p_i) - x(p_1)} \left( \frac{dF_{0,1}(D)}{dx_i} + \frac{\alpha_i}{\alpha_1} \frac{dF_{0,1}(D)}{dx_i} \right) = L(x_1).F_{0,1}(D), \]

and equivalently:

\[ \left( \frac{d}{dx_1} \right)^2 F_{0,1}(D) + \frac{1}{\alpha_1^2} \frac{d}{dx_1} F_{0,1}(D) \frac{d}{dx_1} F_{0,2}(D) - \sum_{i=2}^{r} \frac{dF_{0,1}(D)}{dx_i} + \frac{\alpha_i}{\alpha_1} \frac{dF_{0,1}(D)}{dx_i} \right] = L(x_1).F_{0,1}(D), \]

\[ \left( \frac{d}{dx_1} \right)^2 S_0 + \frac{2}{\alpha_1^2} \frac{d}{dx_1} S_0 \frac{d}{dx_1} S_1 - \sum_{i=2}^{r} \frac{dS_i}{dx_1} + \frac{\alpha_i}{\alpha_1} \frac{dS_i}{dx_1} \right] = L(x_1).S_0, \]

\[ [h] \left[ h^2 \left( \frac{d}{dx_1} \right)^2 S + \frac{1}{\alpha_1^2} \frac{d}{dx_1} S \frac{d}{dx_1} S - \sum_{i=2}^{r} \frac{dS_i}{dx_1} + \frac{\alpha_i}{\alpha_1} \frac{dS_i}{dx_1} - L(x_1).S - L(x_1).F \right) - R(x_1) \right] = 0. \]
For \((g, n) = (0, 2)\) we first rewrite (14):

\[
P_{0,2}(x(z), z_1, z_2) - 2y(z)\omega_{0,3}(z, z_1, z_2) = \frac{-B(z, z_1)B(-z, z_2) - B(-z, z_1)B(z, z_2)}{(dx)^2} + d_1 \frac{B(z_2, -z_1)}{(x - x_1)dx_1} + d_2 \frac{B(z_1, -z_2)}{(x - x_2)dx_2} = 2\frac{B(z, z_1)B(z, z_2)}{(dx)^2} + d_1 \frac{1}{x - x_1} \left(\frac{B(z_2, -z_1) - B(z_2, z_2)}{dx_1} - \frac{B(z_1, -z_2)}{dx_2}\right) + d_2 \frac{1}{x - x_2} \left(\frac{B(z_1, z_2)}{dx_1} + \frac{B(z_1, z_2)}{dx_2}\right) + d_1 d_2 \frac{1}{x - x_1} \frac{1}{x - x_2}.
\]

Now we integrate twice over \(D\):

\[
\int_D \int_D P_{0,2}(x(z), z_1, z_2) - 2F'_{0,1}(z; D)F'_{0,3}(z; D) = 2(F'_{0,2}(z; D))^2 - 2 \sum_{i=1}^r \frac{1}{x - x_i} F_{0,2}(z; D) - \sum_{i=1}^r \frac{2\alpha_i}{x - x_i} \sum_{j \neq i} \alpha_j \left(\frac{dE(p_i, p_j)}{dx_i} - \frac{1}{x_i - x_j}\right).
\]

We introduce \(\hat{F}'_{0,2}(z; D) = F'_{0,2}(z; D) - \alpha_1 \frac{dE(z, p_1)}{dx_1}\) and obtain:

\[
\frac{2\alpha_1^2}{x - x_i} \frac{dE(z, p_1)}{dx} - 2 \sum_{i \neq j} \frac{\alpha_i}{x - x_i} \hat{F}'_{0,2}(z; D) - 2 \sum_{i \neq j} \frac{\alpha_1 \alpha_i}{x - x_i} \frac{dE(z, p_1)}{dx} - \frac{1}{x - x_i} \frac{d}{dx_i} F_{0,2}(D) - \sum_{i \neq j} \frac{1}{x - x_i} \frac{d}{dx_i} F_{0,2}(D) + 2 \sum_{i \neq j} \frac{\alpha_i \alpha_j}{(x - x_i)(x_i - x_j)} + 2 \sum_{j \neq i} \frac{\alpha_1 \alpha_j}{(x - x_1)(x_1 - x_j)}.
\]

Observe that

\[
\lim_{z \to p_1} \left(- \sum_{i \neq 1} \frac{\alpha_1 \alpha_i}{x - x_i} \frac{dE(z, p_1)}{dx} + \sum_{j \neq 1} \frac{\alpha_1 \alpha_j}{(x - x_1)(x_1 - x_j)}\right) = \sum_{i \neq 1} \frac{\alpha_1 \alpha_i}{(x - x_i)^2}.
\]

Using this, at \(z = p_1\) we obtain

\[
\frac{1}{2\alpha_1^2} \left(\frac{dF_{0,2}(D)}{dx_1}\right)^2 + \left(\frac{d}{dx_1}\right)^2 F_{0,2}(D) - \sum_{i \neq 1} \frac{1}{x - x_i} \left(\frac{dF_{0,2}(D)}{dx_i} + \frac{\alpha_i}{\alpha_1} \frac{dF_{0,2}(D)}{dx_1}\right) + 2\alpha_1^2 S(p_1) + 2 \sum_{i \neq j, i \neq 1} \frac{\alpha_i \alpha_j}{(x - x_i)(x_i - x_j)},
\]

where we have called \(S(p_1)\) the limit

\[
\lim_{z \to p_1} \frac{dE(z, p_1)}{dx(z)} \left(\frac{dE(z, p_1)}{dx(z)} - \frac{1}{x(z) - x(p_1)}\right).
\]

Using (29) in this special case, together with Lemma 4.1, we obtain

\[
\int_D \int_D F_{0,2}(x; z_1, z_2) = \int_D \int_D L(x) \omega_{0,2}(z_1, z_2) = L(x) F_{0,2}(D).
\]
For the first term of (55), we thus obtain:

\begin{equation}
L(x_1). \frac{F_{0,2}(D)}{2} = \frac{1}{3 \alpha_1^2} \frac{d}{dx_1} F_{0,1}(D) \frac{d}{dx_1} F_{0,3}(D) + \frac{1}{4 \alpha_1^2} \left( \frac{dF_{0,2}(D)}{dx_1} \right)^2 + \frac{1}{2} \left( \frac{d}{dx_1} \right)^2 F_{0,2}(D) + \alpha_1^2 S(p_1) - \frac{1}{2} \sum_{i \neq 1} \frac{1}{x_1 - x_i} \left( \frac{dF_{0,2}(D)}{dx_i} \frac{\alpha_i}{\alpha_1} \frac{dF_{0,2}(D)}{dx_1} \right) + \sum_{i \neq j, i \neq 1} \frac{\alpha_i \alpha_j}{(x_1 - x_i)(x_i - x_j)}.
\end{equation}

\[ \text{Remark}\]

- For \((g, n) = (1, 0)\) we get

\[ -B(z, -z) \frac{d}{dx(z)^2} + 2F_{0,1}(z; D)F_{1,1}(z; D) = L(x_1).F_{1,0}(D). \]

At \(z = p_1\) this gives

\[ -B(p_1, -p_1) \frac{d^2}{dx_1^2} + 2 \frac{1}{\alpha_1^2} \frac{dF_{0,1}(D)}{dx_1} \frac{dF_{1,1}(D)}{dx_1} = L(x_1).F_{1,0}(D). \]

Using that \(B(p_1, -p_1) = S(p_1)\) and summing the expressions for \((0, 2)\) and \((1, 0)\), we obtain:

\begin{equation}
\left( \frac{d}{dx_1} \right)^2 S_1 + \frac{1}{\alpha_1^2} \left( \frac{d}{dx_1} S_1 \right)^2 + \frac{2}{\alpha_1^2} \frac{d}{dx_1} S_0 \frac{d}{dx_1} S_2 - \sum_{i = 2}^r \frac{dS_i}{dx_i} + \frac{\alpha_i}{\alpha_1} \frac{dS_i}{dx_1} + \frac{\alpha_i \alpha_j}{(x_1 - x_i)(x_i - x_j)} = L(x_1).S_1 + L(x_1).S_1,
\end{equation}

that is

\begin{equation}
\left[ \hbar^2 \left( \frac{d}{dx_1} \right)^2 S + \frac{1}{\alpha_1^2} \frac{d}{dx_1} S \frac{d}{dx_1} \right] - \sum_{i = 2}^r \frac{dS_i}{dx_i} + \frac{\alpha_i}{\alpha_1} \frac{dS_i}{dx_1} + (\alpha_1^2 - 1)S(p_1) + (*) \right) \right]
= \left[ \hbar^2 \left( L(x_1).S + L(x_1).F \right) + R(x_1) \right],
\end{equation}

with

\begin{equation}
(*) = \sum_{i \neq j, i \neq 1} \frac{\alpha_i \alpha_j}{(x_1 - x_i)(x_i - x_j)}.
\end{equation}

From the assumption \(\alpha_1 = \frac{1}{\alpha_1}\) and summing over all topologies, we get the claim. \(\square\)

\[ \text{Remark 4.2.}\]

Very often in the literature a different convention is used to regularize the \((0, 2)\) term of the wave function:

\[ \tilde{\psi}(D, t, h) := \exp \left( \tilde{S}_1(D, t) + \sum_{m \geq 0, m \neq 1} \hbar^{m-1} S_m(D, t) \right), \]

where \(\tilde{S}_1(D, t) := \frac{1}{2} \int_D \int_D \left( B(z_1, z_2) - \frac{dx(z_1)dx(z_2)}{(x(z_1) - x(z_2))^2} \right) \). Using (45), we obtain that the relation to our wave function is the following

\[ \psi(D, t, h) = \tilde{\psi}(D, t, h) \cdot \prod_{i < j} (x_i - x_j)^{\alpha_0} \].

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4.1. **PDE for Airy curve.** In this particular case, we had $P_{g,n} = 0$.

Therefore, in this case we obtain the following system of PDEs:

$$
\hbar^2 \left( \frac{d^2}{dx_k^2} - \sum_{i \neq k} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_k} + \sum_{i \neq j, j \neq k} \frac{\alpha_i \alpha_j}{(x_k - x_i)(x_i - x_j)} \right) \psi(x) = \hbar \psi(x),
$$

for every $k = 1, \ldots, r$.

**Example 4.3.** Considering the divisor $D = [z_1] - [z_2]$, sending $z_2 \to \infty$ and regularizing the $(0,1)$ factor of the wave function, we recover the Airy quantum curve from our PDE for $x = x_1$:

$$
\left( \frac{d^2}{dx^2} - x \right) \psi = 0.
$$

4.2. **PDE for Painlevé case.** In this case we have $P_{g,n} = \frac{\partial}{\partial t} \omega_{g,n}(z_1, \ldots, z_n)$.

Therefore, we obtain the following PDE:

$$
\hbar^2 \left( \frac{d^2}{dx_k^2} - \sum_{i \neq k} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_k} \right) \psi(D) = (x_1^3 + tx_k + V) \psi = (x_k^3 + tx_k + \frac{\partial}{\partial t} \omega_{0,0}) \psi,
$$

for every $k = 1, \ldots, r$, where $(\ast)$ is given by (63).

4.3. **Reduced equation.** Consider a divisor $D = [z] - [z']$ with 2 points, and call $x = x(z), x' = x(z')$. The equation we have obtained is a PDE: it involves both $d/dx$ and $d/dx'$, as well as partial derivatives with respect to times when $L(x) \neq 0$. Let us show here that it is possible to eliminate $d/dx'$ and arrive to an equation involving only $d/dx$, as well as possibly times derivatives.

Define

$$
\tilde{\psi}(z, z') := (x - x') \psi([z] - [z'], t, h) e^F.
$$

Define the differential operators

$$
\mathcal{D} := \hbar^2 \frac{d^2}{dx^2} - \hbar^2 L(x) - R(x),
$$

$$
\mathcal{D}' := \hbar^2 \frac{d^2}{dx'^2} - \hbar^2 L(x') - R(x').
$$

Equation (48) is equivalent to

$$
\mathcal{D} \tilde{\psi} = \frac{\hbar^2}{x - x'} \left( \frac{d}{dx} + \frac{d}{dx'} \right) \tilde{\psi} = -\mathcal{D}' \tilde{\psi}.
$$

In particular this implies

$$
\hbar^2 \frac{d}{dx'} \tilde{\psi} = -\hbar^2 \frac{d}{dx} \tilde{\psi} + (x - x') \mathcal{D} \tilde{\psi},
$$

and applying $d/dx'$ again we find

$$
\hbar^2 \frac{d^2}{dx'^2} \tilde{\psi} = -\mathcal{D} \tilde{\psi} - \hbar^2 \frac{d}{dx} \frac{d}{dx'} \tilde{\psi} + (x - x') \mathcal{D} \frac{d}{dx'} \tilde{\psi}
$$

$$
= \hbar^2 \frac{d^2}{dx^2} \tilde{\psi} - 2\mathcal{D} \tilde{\psi} - (x - x') \left( \frac{d}{dx} \mathcal{D} + \mathcal{D} \frac{d}{dx} - \hbar^{-2} \mathcal{D}(x - x') \mathcal{D} \right) \tilde{\psi}.
$$
Therefore
\[
0 = (\mathcal{D} + \mathcal{D}') \tilde{\psi} \\
= \mathcal{D} \tilde{\psi} + \left( h^2 \frac{d^2}{dx^2} - R(x') - h^2 L(x') \right) \tilde{\psi} \\
= \left( h^2 \frac{d^2}{dx^2} - \mathcal{D} - R(x') - h^2 L(x') - (x - x') \left( \frac{d}{dx} \mathcal{D} + \mathcal{D} \frac{d}{dx} - h^{-2} \mathcal{D}(x - x') \mathcal{D} \right) \right) \tilde{\psi} \\
= \left( R(x) - R(x') + h^2 (L(x) - L(x')) - (x - x') \left( \frac{d}{dx} \mathcal{D} + \mathcal{D} \frac{d}{dx} - h^{-2} (x - x') \mathcal{D} \right) \right) \tilde{\psi} \\
= \left( R(x) - R(x') + h^2 (L(x) - L(x')) - (x - x') \left( \frac{dR(x)}{dx} + h^2 \frac{dL(x)}{dx} - h^{-2} (x - x') \mathcal{D} \right) \right) \tilde{\psi},
\]
and thus
\[
\frac{R(x) - R(x')}{x - x'} \tilde{\psi} + h^2 \frac{L(x) - L(x')}{x - x'} \tilde{\psi} = \left( \frac{dR(x)}{dx} + h^2 \frac{dL(x)}{dx} - h^{-2} (x - x') \mathcal{D} \right) \tilde{\psi}.
\]
Finally,
\[
\mathcal{D}^2 \tilde{\psi} = \frac{h^2}{x - x'} \left( \frac{R(x) - R(x')}{x - x'} + h^2 \frac{L(x) - L(x')}{x - x'} - \frac{dR(x)}{dx} - h^2 \frac{dL(x)}{dx} \right) \tilde{\psi}.
\]
This equation is a PDE, with rational coefficients \( \in \mathbb{C}(x) \), involving \( d/dx \) and \( \partial/\partial t \)'s but no \( d/dx' \) anymore.

Notice that the right hand side is of order \( O(h^2) \) in the limit \( h \to 0 \), and \( \mathcal{D} \to \tilde{\psi}^2 - R(x) \), where \( \tilde{\psi} = h \hat{\psi}/dx \).

5. Quantum curves

The goal now is to prove that \( \psi(D) \) obeys an isomonodromic system of differential equations, and in particular this implies the existence of a quantum curve \( \hat{P}(\hat{x}, \hat{y}, h) \) that annihilates \( \psi(D) \). To this purpose, we first prove that \( \psi([z] - [z']) \) coincides with the integrable kernel of an isomonodromic system. The way to prove it generalizes the method of [2], i.e. first proving that the ratio of \( \psi \) and the integrable kernel has to be a formal series of the form \( 1 + O(z^{L-1}) \) and then showing that the only solution of equations (48) which has that behavior implies that the ratio must be 1.

5.1. Painlevé I (genus 0 case). We shall prove that \( \psi([z] - [z']) \) coincides with the integrable kernel associated to the Painlevé I kernel.

Consider a solution of the Painlevé system (48), written as
\[
\Psi(x) = \begin{pmatrix} A(x) & B(x) \\ \hat{A}(x) & \hat{B}(x) \end{pmatrix}, \quad \det \Psi(x) = 1,
\]
i.e. \( \Psi(x) \) satisfying (48):
\[
\left( h \frac{\partial}{\partial x} - \mathcal{L}(x, t; h) \right) \Psi(x) = 0, \quad \left( h \frac{\partial}{\partial t} - \mathcal{R}(x, t; h) \right) \Psi(x) = 0.
\]
Define $A(x), \tilde{A}(x), B(x), \tilde{B}(x)$ as WKB $h$-formal series solutions, with leading orders

\[
A(x) \sim \frac{i}{\sqrt{2\pi}} e^{h^{-1}\int_0^y y'dx} (1 + O(h)),
\]

\[
B(x) \sim \frac{i}{\sqrt{2\pi}} e^{-h^{-1}\int_0^y y'dx} (1 + O(h)),
\]

\[
\tilde{A}(x) \sim \frac{i}{\sqrt{2}} e^{h^{-1}\int_0^y y'dx} (1 + O(h)),
\]

\[
\tilde{B}(x) \sim -\frac{i}{\sqrt{2}} e^{-h^{-1}\int_0^y y'dx} (1 + O(h)),
\]

and with each coefficient of higher powers of $h$ in $(1 + O(h))$ being a polynomial of $1/z$ that tends to 0 as $z \to \infty$. The choice of normalisation constants for the WKB solutions $C_A = C_B = \sqrt{2}$ is made such that $\det \Psi = -2C_AC_B = 1$ and $C_A = C_B$. The integrable kernel is defined as (a WKB formal series of $h$):

\[
K(x, x') := \frac{A(x)\tilde{B}(x') - \tilde{A}(x)B(x')}{x - x'}.
\]

From the isomonodromic system (8), one can verify that this kernel obeys the same equation (48) as $\psi([z] - [z'])$. In fact, they are equal:

**Theorem 5.1.** We have, as formal WKB power series of $h$

\[
\psi([z] - [z'], t, h) = \frac{A(x)\tilde{B}(x') - \tilde{A}(x)B(x')}{x - x'},
\]

where $x = x(z)$ and $x' = x(z')$.

**Proof.** Define the ratio

\[
H(z, z') := \frac{(x - x')\psi([z] - [z'], t, h)}{A(x)\tilde{B}(x') - \tilde{A}(x)B(x')}.
\]

It is a formal series of $h$ whose coefficients are rational functions of $z$ and $z'$. The leading orders show that

\[
H(z, z') = 1 + O(h).
\]

More explicitly, we have

\[
H(z, z') = \frac{(z^2 - z'^2)\sqrt{\frac{\pi}{2\pi}} \sum h^k c_k \left(\frac{1}{z}, \frac{1}{z'}\right)}{
\frac{1}{2} \sqrt{\frac{\pi}{2\pi}} \left(\sum h^k \alpha_k \left(\frac{1}{z}\right)\right) \left(\sum h^k \beta_k \left(\frac{1}{z'}\right)\right) + 
\frac{1}{2} \sqrt{\frac{\pi}{2\pi}} \left(\sum h^k \tilde{\alpha}_k \left(\frac{1}{z}\right)\right) \left(\sum h^k \tilde{\beta}_k \left(\frac{1}{z'}\right)\right)}
\]

\[
= \frac{\sum h^k c_k \left(\frac{1}{z}, \frac{1}{z'}\right)}{
\frac{1}{2} \left(\sum h^k \alpha_k \left(\frac{1}{z}\right)\right) \left(\sum h^k \tilde{\beta}_k \left(\frac{1}{z'}\right)\right) + 
\frac{1}{2} \left(\sum h^k \tilde{\alpha}_k \left(\frac{1}{z}\right)\right) \left(\sum h^k \beta_k \left(\frac{1}{z'}\right)\right)}/\left(\frac{1}{z} + \frac{1}{z'}\right).
\]

Since we are in the hyperelliptic case, we have that $\beta_k \left(\frac{1}{z}\right) = \alpha_k \left(\frac{-1}{z}\right)$ and $\tilde{\beta}_k \left(\frac{1}{z}\right) = \tilde{\alpha}_k \left(\frac{-1}{z}\right)$, which implies that the first factor of the denominator is divisible by $\frac{1}{z} + \frac{1}{z'}$. Hence the series at the denominator is invertible.

Therefore, at each order of $h$, the coefficient is a polynomial of $1/z, 1/z'$, which tends to 0 at $z, z' \to \infty$:

\[
H(z, z') - 1 \in \frac{1}{z z'} C[[z^{-1}, z'^{-1}]][[h]].
\]
We shall prove that \( H = 1 \) by following the method of [2]. Let us assume that \( H \neq 1 \), and write
\[
H(z, z') = 1 + H_M(z)z'^{-M} + O(z'^{-M-1}),
\]
where \( M \geq 1 \) is the smallest possible power of \( z' \) whose coefficient \( H_M \) would be \( \neq 0 \) as a formal series of \( h \). Let \( \tilde{K}(x, x') := A(x)\tilde{B}(x') - \tilde{A}(x)B(x') \). Using equation (68) for \( \tilde{\psi}(z, z') = H(z, z')\tilde{K}(x, x')e^F \), together with the fact that \( \overline{K}(x, x') := \tilde{K}(x, x')e^F \) satisfies the same equation (68) (which is proved in Appendix A in general), we obtain the following equation for \( H = H(z, z') \):
\[
\begin{align*}
\frac{d^2 H}{dx'^2} + 2\frac{d \ln B(x')}{dx'} \frac{dH}{dx'} + 2\frac{d \ln \tilde{A}(x)/A(x) - \tilde{B}(x')/B(x')}{dx'} \frac{dH}{dx'} \\
- \tilde{K}(x, x')(HL(x'), \tilde{K}(x, x') - H\tilde{K}(x, x')L(x').F) = \frac{1}{x' - x} \left( \frac{d}{dx} + \frac{d}{dx'} \right) H,
\end{align*}
\]
whose leading power of \( z' \) comes only from the second term and is
\[
2\gamma z' H_M(z)z'^{-M-2} + O(z'^{-M-2}) = 2H_M(z)z'^{-M-1} + O(z'^{-M-2}) = 0,
\]
implying that \( H_M = 0 \), and thus contradicting the hypothesis that \( H \neq 1 \). This proves the theorem.

As a corollary this implies that
\[
\lim_{z' \to \infty} \frac{(x(z) - x(z'))\psi([z] - [z'], t, \hbar)}{B(x(z'))} = A(x(z)).
\]
In other words
\[
\frac{i}{2\sqrt{\pi}} e^{\hbar^{-1}\int_0^z \omega_0 dz} e^{\sum_{g,n \neq (0,1)} \frac{h^{2g-2+n}}{2n!} \int_{y_1}^z \int_{y_2}^z \omega_{g,n}} = A(x(z)).
\]
In the Painlevé system, the function \( A(x) \) is annihilated by the quantum curve
\[
\frac{d^2}{dx^2} - \left( x - U \right)^2 \left( x + 2U \right) + \frac{h^2}{2} \frac{d}{dx} \left( x - U \right) + \frac{h^2}{4} \frac{d^2}{dx^2} \left( x - U \right) + \frac{\hbar}{x - U} \frac{d}{dx} - \frac{\hbar}{x - U} \frac{d^2}{dx^2} = 0.
\]

5.2. General genus 0 case. The same argument applies to any isomonodromic system thanks to the fact that both the 2-point wave function \( \tilde{\psi} = (x - x')\psi e^F \) constructed from topological recursion and the integrable kernel \( \overline{K}(x, x') = \tilde{K}(x, x')e^F \) satisfy the same PDE (68). The first claim is the main result of the article and the second claim was proved in Appendix A in general, since we could not find this result in the literature for a general isomonodromic system. Consider that we have a Lax system of the type
\[
\begin{align*}
\hbar \frac{\partial}{\partial x} \Psi(x) &= \mathcal{L}(x; \hbar) \Psi(x), \\
\hbar \frac{\partial}{\partial t_k} \Psi(x) &= \mathcal{R}_k(x; \hbar) \Psi(x).
\end{align*}
\]
whose spectral curve in the limit $h \to 0$ is a genus zero curve of the form $\det(y-\mathcal{L}_0(x)) = y^2 - R(x) = 0$, and which has a WKB formal power series solution of the form
\[
A(x) \sim \frac{i}{\sqrt{2dx/dz}} e^{h^{-1} \int_0^x ydx} (1 + O(h)), \\
B(x) \sim \frac{i}{\sqrt{2dx/dz}} e^{-h^{-1} \int_0^x ydx} (1 + O(h)), \\
\tilde{A}(x) \sim i \frac{1}{2} \frac{dx/dz}{e^{h^{-1} \int_0^x ydx}} (1 + O(h)), \\
\tilde{B}(x) \sim -i \frac{1}{2} \frac{dx/dz}{e^{-h^{-1} \int_0^x ydx}} (1 + O(h)),
\]
where each coefficient of higher powers of $h$ in $(1 + O(h))$ is a rational function of $z$ that tends to 0 at poles $z \to \zeta$, where we have chosen a pole $x(\zeta) = \infty$. This pole has degree $-d = 1$ or $-d = 2$, and $\xi = x^{1/d}$ is a local coordinate near the pole. We emphasize that for all genus 0 spectral curves where $y^2 = P_{\text{odd}}(x)$ with $P_{\text{odd}}(x)$ an odd polynomial of $x$, such systems are explicitly known as Gelfand–Dikii systems \cite{GelfandDikii} described in Section 5.4 below, and for more general spectral curves (genus 0, possibly with non-empty Newton polygon), it was proved in \cite{Its:1993} that a $2 \times 2$ autonomous system $\mathcal{L}_0$ always admits an $h$-deformation $\mathcal{L}$ with this property.

We define the following formal series of $h$:
\[
H(z, z') := \frac{(x(z)-x(z'))\psi([z]-[z'])}{A(x)B(x') - A(x)B(x')} = 1 + O(h).
\]
We have proved that the integrable kernel $\tilde{K}(x, x')$ associated to the Lax system satisfies equation (68) (see Theorem A.1), and thus $H$ satisfies the PDE (76).

Moreover the coefficients of $H$ are analytic functions of $z'$, which tend to 0 at $z' \to \zeta$. Let us write $H(z, z') = 1 + O(x^{1/d})$. The subleading coefficient of $H = 1 + H_M(z)x^{M/d} + O(x^{M/d-1})$ must satisfy $y(z')H_M(z)x^{M/d-1} = O(x^{M/d-2})$, and therefore $H = 1$.

This implies that $\psi([z]-[z'])$ coincides with the integrable kernel
\[
\psi([z]-[z']) = \frac{A(x(z))\tilde{B}(x(z')) - \tilde{A}(x(z))B(x(z'))}{x(z) - x(z')}.
\]

Then taking the limit $z' \to \zeta$ this implies that
\[
\lim_{z' \to \zeta} \frac{(x(z)-x(z'))\psi([z]-[z'])}{B(x(z'))} = A(x(z)).
\]
Knowing that $A(x), \tilde{A}(x)$ satisfy an isomonodromic system with first equation
\[
\frac{\partial}{\partial x} \begin{pmatrix} A(x) \\ \tilde{A}(x) \end{pmatrix} = \mathcal{L}(x, t, h) \begin{pmatrix} A(x) \\ \tilde{A}(x) \end{pmatrix},
\]
with
\[
\mathcal{L}(x, t, h) = \begin{pmatrix} \alpha(x, t, h) & \beta(x, t, h) \\ \gamma(x, t, h) & \delta(x, t, h) \end{pmatrix},
\]
where $\alpha, \beta, \gamma, \delta$ are rational functions of $x$, with coefficients being formal power series of $h$, we get the quantum curve annihilating $A(x)$:
\[
\hat{y}^2 - (\alpha + \delta)\hat{y} + (\alpha\delta - \beta\gamma) - h\left(\frac{d\alpha/dx - \alpha\frac{d\beta/dx}{\beta}}{\beta} + \frac{d\beta/dx}{\beta}\hat{y}\right).
\]
The transseries linear combination introduced in \((88)\) solution, which we denote as follows

\[
\psi((\epsilon_j \rightarrow \epsilon_j + n_j)_j; [z] - [z']) = e^{\pi i} \psi((\epsilon_j)_j; [z] - [z']) \psi((\epsilon_1, \ldots, \epsilon_j \rightarrow \epsilon_j + 1, \ldots, \epsilon_j); [z] - [z']).
\]

The combination \(\hat{\psi}\) hence remains invariant \([17]\) if we modify the integration homotopy class from \(z'\) to \(z\) by adding 1st kind \(B\)-cycles, and it is, order by order as a transseries of \(h\), a function of \(z\) and \(z'\) on the spectral curve. Shifts by other kinds of cycles of the spectral curve were already
trivial for \( \psi \), in the sense that they result in the multiplication by simple factors, which one expects from a solution of an isomonodromic system.

From there, the same argument as for the genus zero case applies. Assume that there is an isomonodromic system

\[\begin{align*}
\hbar \frac{\partial}{\partial z} \Psi(x) &= \mathcal{L}(x; \hbar) \Psi(x), \\
\hbar \frac{\partial}{\partial t_k} \Psi(x) &= \mathcal{R}_k(x; \hbar) \Psi(x).
\end{align*}\]

(91)

whose associated spectral curve is our spectral curve, with

\[\Psi(x) = \begin{pmatrix} A(x) & B(x) \\ \hat{A}(x) & \hat{B}(x) \end{pmatrix}\]

a formal transseries solution. Then the formal transseries

\[H(z, z') = \frac{(x(z) - x(z'))\hat{\psi}([z] - [z'])}{A(x)\hat{B}(x') - \hat{A}(x(z))B(x')} = 1 + O(\hbar)\]

(92)

satisfies the PDE (76), and is such that \( H(z, z') = 1 + O(x'^{1/d}) \). The subleading order of \( H = 1 + H_M(z)x^{M/d} + O(x'^{M/d - 1}) \) must satisfy \( y(z')H_M(z)x^{M/d - 1} = O(x'^{M/d - 2}) \), and therefore \( H = 1 \). This implies that

\[\hat{\psi}([z] - [z']) = \frac{A(x(z))\hat{B}(x(z')) - \hat{A}(x(z))B(x(z'))}{x(z) - x(z')}\]

(93)

This also implies that

\[\lim_{z' \to \zeta} \frac{(x(z) - x(z'))\hat{\psi}([z] - [z'])}{B(x(z'))} = A(x(z)).\]

Since \( A(x), \hat{A}(x) \) satisfy the isomonodromic system

\[\frac{\hbar}{x} \frac{\partial}{\partial x} \begin{pmatrix} A(x) \\ \hat{A}(x) \end{pmatrix} = \mathcal{L}(x) \begin{pmatrix} A(x) \\ \hat{A}(x) \end{pmatrix},\]

where

\[\mathcal{L}(x, t, \hbar) = \begin{pmatrix} \alpha(x, t, \hbar) & \beta(x, t, \hbar) \\ \gamma(x, t, \hbar) & \delta(x, t, \hbar) \end{pmatrix},\]

we find the quantum curve annihilating \( A(x) \):

\[\hat{y}^2 - (\alpha(x) + \delta(x))\hat{y} + (\alpha(x)\delta(x) - \beta(x)\gamma(x)) - \hbar \left( \frac{\alpha'(x) - \alpha(x)}{\beta(x)} + \frac{\beta'(x)}{\beta(x)} \right) = 0.\]

(95)

Its classical part \( \hbar \to 0 \) is indeed the equation

\[\det(y\text{Id} - \mathcal{L}(x, t, 0)) = 0.\]

(96)

5.4. **Examples: Gelfand–Dikii systems.** These systems generalize the Painlevé I equation; they appear in the enumeration of maps in the large size limit \([22]\). For these Gelfand–Dikii systems, the proof that \( \psi([z] - [z']) \) coincides with the integrable kernel (which then implies the quantum curve) can be found in \([22, \text{Chapter 5}]\), by another method. Here let us provide another proof with our current method.

The Gelfand–Dikii polynomials are defined as differential polynomials of a function \( U(t) \), by the recursion

\[R_0(U) = 2, \quad \frac{\partial}{\partial t} R_{k+1}(U) = -2U \frac{\partial R_k(U)}{\partial t} - R_k(U) \frac{\partial U}{\partial t} + \frac{\hbar}{4} \frac{\partial^2 R_k(U)}{\partial t^2}.\]

(97)
At each step the integration constant is chosen so that $R_k(U)$ is homogeneous in powers of $U$ and \(\partial^2 \partial t^2\). The first few are given by

\[
\begin{align*}
R_0 &= 2, \\
R_1 &= -2U, \\
R_2 &= 3U^2 - \frac{h^2}{2} \dddot{U}, \\
R_3 &= -5U^3 + \frac{5h^2}{2} U \dddot{U} + \frac{5h^2}{4} \dddot{U}^2 - \frac{h^4}{8} \partial^4 U.
\end{align*}
\]

(98)

Let \(m \geq 1\) be an integer, and let \(\tilde{t}_0, \tilde{t}_1, \tilde{t}_2, \ldots, \tilde{t}_m\) be a set of “times”. Let \(U(t; \tilde{t}_0, \tilde{t}_1, \tilde{t}_2, \ldots, \tilde{t}_m)\) be a solution of the following non-linear ODE:

\[
\sum_{j=0}^{m} \tilde{t}_j R_{j+1}(U) = t.
\]

(99)

Notice that, formally, \(t = -2\tilde{t}_{-1}\). For \(m = 1\), the equation (99) is the Painlevé I equation. The case \(m = 2\) is called the Lee–Yang equation. The case \(m = 0\) is simply \(U(t) = \frac{-t}{2\theta_0}\).

Consider the Lax pair (adopting the normalizations of [22]) given by

\[
\mathcal{R}(x, t, h) = \begin{pmatrix} 0 & 1 \\ x + 2U(t) & 0 \end{pmatrix}
\]

and

\[
\mathcal{L}(x, t, h) = \sum_{j=0}^{m} \tilde{t}_j \mathcal{L}_j(x, t, h),
\]

where

\[
\beta_j(x, t) = \frac{1}{2} \sum_{k=0}^{j} x^{j-k} R_k(U), \quad \alpha_j(x, t) = -\frac{h}{2} \partial_t \beta_j(x, t), \quad \gamma_j(x, t) = (x + 2U) \beta_j(x, t) + h \frac{\partial}{\partial t} \alpha_j(x, t).
\]

One can easily verify that the Gelfand–Dikii polynomials are such that the zero curvature equation is satisfied

\[
h \partial_t \mathcal{L}(x, t, h) + h \frac{\partial}{\partial x} \mathcal{R}(x, t, h) = [\mathcal{R}(x, t, h), \mathcal{L}(x, t, h)].
\]

(102)

The differential equation (99) admits a formal power series solution \(U(t, h)\) with only even powers of \(h\):

\[
U(t, h) = \sum_k h^{2k} u_k(t),
\]

(103)

whose first term \(u(t) = u_0(t)\) satisfies an algebraic equation

\[
\sum_{j=0}^{m} \frac{(2j + 1)!}{j!(j + 1)!} \tilde{t}_j (-u/2)^{j+1} = -\frac{1}{4} t.
\]

(104)

In the Painlevé I case, \(m = 1, \tilde{t}_k = \delta_{k,1}\), we recover \(t = -3u^2\).

The spectral curve, in the limit \(h \to 0\):

\[
\det(y - \mathcal{L}(x, t, 0)) = 0
\]

(104)
is always a genus 0 curve. It admits the rational parametrization

\[
\left\{ \begin{array}{l}
x(z) = z^2 - 2u(t) \\
y(z) = \sum_{j=0}^{m} \tilde{t}_j \left( z^{2j+1}(1 - 2u(t)/z^2)j! + \frac{1}{z^{2j-2k+1}} \right),
\end{array} \right.
\]

where \((\cdot)_+\) means the positive part in the Laurent series expansion near \(z = \infty\), i.e.

\[
y(z) = \sum_{j=0}^{m} \tilde{t}_j \sum_{k=0}^{j} (-u)^k \frac{(2j + 1)!!}{(2j - 2k + 1)!!} z^{2j-2k+1}.
\]

In the Painlevé I case, \(\tilde{t}_j = \delta_{j,1}\), we recover \(y(z) = z^3 - 3uz\).

In the case \(m = 0\), with \(t_0 = 1\), we recover the Airy system

\[
\mathcal{L}(x, t, \hbar) = \begin{pmatrix} 0 & 1 \\ x - t & 0 \end{pmatrix}
\]

with spectral curve \(y^2 = x - t\).

Let \(\Psi(x, t, \hbar)\) as follows

\[
\Psi(x, t, \hbar) = \left( \begin{array}{c} A(x) \\ B(x) \\ A(x) \\ B(x) \end{array} \right)
\]

be a WKB \(\hbar\) formal series solution of

\[
\hbar \frac{\partial}{\partial x} \Psi(x, t, \hbar) = -\mathcal{L}(x, t, \hbar) \Psi(x, t, \hbar), \quad \hbar \frac{\partial}{\partial t} \Psi(x, t, \hbar) = \mathcal{R}(x, t, \hbar) \Psi(x, t, \hbar).
\]

Our previous results show that the formal series \(\psi([z] - [x], \tilde{t}, \hbar)\) coincide with

\[
A(x, t, \hbar) = \frac{1}{\sqrt{2\pi}} e^{\int_0^{\tilde{t}_0} ydx} e^{\sum_{g,n}(g,n)^{(0,1),(0,2)}} \frac{\hbar}{\hbar^{2g-2+n}} \int_{x_1}^{x_2} \cdots \int_{x_{g-n}}^{x_n} \omega_{g,n},
\]

and is annihilated by the quantum curve

\[
\hat{y}^2 - (\alpha(x) + \delta(x))\hat{y} + (\alpha(x)\delta(x) - \beta(x)\gamma(x)) - \hbar \left( \alpha'(x) - \alpha(x) \frac{\beta'(x)}{\beta(x)} + \delta(x) \right).
\]

**APPENDIX A. PDE FOR ANY ISOMONODROMIC SYSTEM**

The goal of this appendix is to show that the integrable kernel \(\overline{K}(x, x') := \overline{K}(x, x') e^F\) of an isomonodromic system satisfies the same PDE (68) that we obtained for the 2-point wave function built from topological recursion. Here \(F = \log T\), with \(T\) the tau function of the isomonodromic system.

Consider a solution of a \(2 \times 2\) isomonodromic system written as:

\[
\Psi(x) = \left( \begin{array}{c} A(x) \\ B(x) \\ A(x) \\ B(x) \end{array} \right), \quad \det \Psi(x) = 1,
\]

i.e. \(\Psi(x)\) satisfying the (compatible) system of equations:

\[
\left\{ \begin{array}{l}
\hbar \frac{\partial}{\partial x} \Psi = \mathcal{L}(x; \hbar) \Psi, \\
\hbar \frac{\partial}{\partial t} \Psi = \mathcal{R}_k(x; \hbar) \Psi.
\end{array} \right.
\]

The equations with respect to the isomonodromic times \(t_k\) can be seen as isomonodromic deformations of the first equation.

Consider the deformed spectral curve

\[
P(x, y; \hbar) = \det(y \text{Id} - \mathcal{L}(x; \hbar)) = y^2 + R(x) + \sum_{m \geq 1} \hbar^m P_m(x, y) = P_0(x, y) + O(\hbar).
\]

We will make use of the following technical assumption:
Assumption A.1. Let $\mathcal{N}$ be the Newton polygon associated to $P_0(x, y) = y^2 - R(x)$ and let $\hat{\mathcal{N}}$ be its interior. For $m \geq 1$ we only allow $P_m(x, y) = \sum_{i,j} P_{i,j} x^i y^j$ whose only non-zero coefficients $P_{i,j} \neq 0$ are such that $(i + 1, j + 1) \in \hat{\mathcal{N}}$, which in our case only allows for $j = 1$, so $P_m(x, y) = P_m(x)$.

Remark A.2. This assumption is to ensure that $P$ has the same Casimirs as $P_0$. It is always possible to transform a deformed spectral curve into one satisfying this assumption, by redefining what we mean by Casimirs of $P_0$, but here we assume we already have this shape for simplicity.

The associated classical spectral curve $\Sigma = \{(x, y) \mid y^2 = R(x)\}$ is presented as a two-sheeted covering of the Riemann sphere $x : \Sigma \to \mathbb{P}^1$.

We assume that the entries of the matrix $L$ are rational functions of $x$:

$$L(x; h) = \sum_{i,j=0}^{m} \frac{L_{ij}^{(l)}}{(x - \lambda_j)_{j+1}} - \sum_{j=1}^{m_{\infty}} L_{j-1}^{(\infty)} x^{j-1}.$$  

We often omit the dependence on the variables $h$ and even $x$ for simplicity. The solutions $\Psi(x)$ of the first linear differential equation of (110) have essential singularities at $x = \lambda_i$ and $x = \infty$. Let us define $\sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. We know that around each pole of $L(x)$ the function $\Psi(x)$ admits an asymptotic expansion of the form

$$\hat{\Psi}(x)e^{h^{-1}\sigma_3 T(x)},$$

where $\hat{\Psi}(x)$ is regular at the poles $P \in \{\infty, \lambda_1, \ldots, \lambda_l\}$ of $L(x)$ and

$$T(x) \sim \frac{\tilde{t}_{P,0}}{x} \log \xi_P + \sum_{j=1}^{m_P} \frac{\tilde{t}_{P,j}}{\xi_P^j},$$

where $\xi_P = x^{d_\infty}$, with $d_\infty$ equal to $-1$ (unramified) or $-2$ (ramified), for $P = \infty$, and $\xi_P = x - \lambda_i$, for $P = \lambda_i$.

The isomonodromic deformation parameters $t_k = T_{\alpha,j}$ of (110), using the multi-index notation $k = (\alpha, j)$, include:

- $\alpha = \infty$: $T_{\infty,j} := \tilde{t}_{\infty,j} = t_{\zeta^+,j}$, for $j = 1, \ldots, m_{\infty}$.
  - If $\infty$ is an unramified critical value of $x$, i.e. $d_{\infty} = -1$ and $x^{-1}(\infty) = \{\zeta^+, \zeta^-\}$, then $t_{\zeta^+,j} = -t_{\zeta^-,j}$.
  - If $\infty$ is a ramified critical value of $x$, i.e. $d_{\infty} = -2$ and $\zeta^+ = \zeta^-$.

- $\alpha = i = 1, \ldots, N$:
  - For $j = 1, \ldots, m_i$, $T_{i,j} := \tilde{t}_{\lambda_i,j} = t_{\zeta^+,j} = -t_{\zeta^-,j}$.
  - For $j = -1$, $T_{-1,j} := \lambda_i$.

We will also use this multi-index notation for the matrices $R_k = R_{\alpha,j}$ of the system (110).

Remark A.3. From our Assumption A.1, we have that all $t_i$ are independent of $h$ and coincide with the moduli of the classical spectral curve $y^2 = R(x)$.

Remark A.4. Since $\text{Tr} \ L = 0$, we have

$$L^2 = -\det L \cdot \text{Id}.$$  

Let $\mathbb{K}(x, x') := \Psi^{-1}(x')\Psi(x)$. Observe that $\tilde{K}(x, x') = A(x)\tilde{B}(x') - \tilde{A}(x)B(x') = \text{Tr} \left( \mathbb{K} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right).$

We called $\zeta_i$ the poles of $ydx$. Recall the operator defined by:

$$D := h^2 \frac{\partial^2}{\partial x^2} - h^2 L(x) - R(x),$$
where

\begin{equation}
L(x) := \sum_{i,x(\zeta_i) = x} \sum_{j=1}^{m_i} t_{\zeta_i,j} \sum_{0 \leq k \leq \frac{j}{d_i} - 2} x^k \left( -\frac{j}{d_i} - k - 2 \right) \partial_{K_{\zeta_i,j+d_i(k+2)}}
\end{equation}

\begin{equation}
\quad + \sum_{i,x(\zeta_i) \neq x} \sum_{j=0}^{m_i} t_{\zeta_i,j} \sum_{k=0}^{j} (x - x(\zeta_i))^{-(k+1)}(j + 1 - k) \partial_{K_{\zeta_i,j+1-k}},
\end{equation}

with

\begin{equation}
\partial_{K_{p,k}} \omega_{g,n}(z_1, \ldots, z_n) := \int_{B_{p,k}} \omega_{g,n+1}(\cdot, z_1, \ldots, z_n) = \text{Res}_{x=p} \frac{\partial^k}{\partial x^k} \omega_{g,n+1}(\cdot, z_1, \ldots, z_n),
\end{equation}

for every second type cycle $B_{p,k}$, $k \geq 1$, which we introduced in (3).

From Corollary 3.7, the operator $L(x)$ can be re-written in terms of derivatives with respect to the isomonodromic parameters: $t_{\zeta_i,k}$, $k = 1, \ldots, m_i$, and $\lambda_l$, for $l = 1, \ldots, N$.

**Theorem A.1.** The operator

\begin{equation}
\mathcal{O} := \mathcal{D} - \hbar^2 L(x)F - \frac{\hbar^2}{x - x'} \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial x'} \right)
\end{equation}

annihilates the integrable kernel:

\begin{equation}
\mathcal{O} \mathcal{K}(x, x') = 0.
\end{equation}

The differential equation (116) is equivalent to

\begin{equation}
\left( \mathcal{D} - \frac{\hbar^2}{x - x'} \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial x'} \right) \right) (\mathcal{K}(x, x') e^F) = 0.
\end{equation}

The proof of this theorem will follow from the three following lemmas.

**Lemma A.2.** Let $\mathcal{T}$ be the tau function of the isomonodromic system and $F := \log \mathcal{T}$. Then,

\begin{equation}
\frac{1}{2} \text{Tr} \mathcal{L}(x; \hbar)^2 = - \det \mathcal{L}(x; \hbar) = \hbar^{-2} R(x) + L.F.
\end{equation}

**Proof.** It was proved by Jimbo–Miwa–Ueno [27] that there exists a tau function $\mathcal{T}(\{t_k\})$ of the isomonodromic times $t_k = T_{n,j}$ such that

\begin{equation}
\frac{\partial}{\partial t_k} \log \mathcal{T} = - \text{Res}_{x=P} \text{Tr} \left( \frac{\partial}{\partial t_k} T(x) \sigma_3 \Psi^{-1}(x) \frac{\partial}{\partial x} \Psi(x) \right),
\end{equation}

where $P = \infty$, if $\alpha = \infty$, and $P = \lambda_l$, if $\alpha = l$, for $l = 1, \ldots, N$. Let $y_s$ be the singular part of $y$. We have

\begin{equation}
\frac{\partial}{\partial x} \Psi = \frac{\partial}{\partial x} \hat{\Psi} e^{h^{-1} \sigma_3 T} + h^{-1} \hat{\Psi} \sigma_3 y_s e^{h^{-1} \sigma_3 T},
\end{equation}

which implies

\begin{equation}
\mathcal{L} = \frac{\partial}{\partial x} \Psi \cdot \Psi^{-1} = \frac{\partial}{\partial x} \hat{\Psi} \cdot \hat{\Psi}^{-1} + h^{-1} \hat{\Psi} \sigma_3 y_s \hat{\Psi}^{-1}.
\end{equation}

Therefore

\begin{equation}
\text{Tr} \mathcal{L}^2 = \text{Tr} \left( \frac{\partial}{\partial x} \hat{\Psi} \cdot \hat{\Psi}^{-1} \right)^2 + 2h^{-2} y_s^2 + 2 \text{Tr} \left( y_s \sigma_3 \hat{\Psi}^{-1} \frac{\partial}{\partial x} \hat{\Psi} \right).
\end{equation}

For every $P \in \{\infty, \lambda_1, \ldots, \lambda_l\}$, $\zeta_i \in x^{-1}(P)$, we have

\begin{equation}
\frac{\partial}{\partial t_{\zeta_i,k}} T(x) = - \frac{\xi_{P,k}}{k},
\end{equation}

where

\begin{equation}
\xi_{P,k} := \int_{B_{p,k}} \omega_{g,n+1}(\cdot, z_1, \ldots, z_n) = \text{Res}_{x=p} \frac{\partial^k}{\partial x^k} \omega_{g,n+1}(\cdot, z_1, \ldots, z_n),
\end{equation}

for every second type cycle $B_{p,k}$, $k \geq 1$. The proof of the above equation follows from the properties of the tau function and the isomonodromic parameters.
for \( k = 1, \ldots, m_i \). Around \( \zeta_l \in x^{-1}(\lambda_l) \), we have

\[
(121) \quad \frac{\partial}{\partial \lambda_l} T(x) = - \sum_{j=0}^{m_l} -t_{\zeta_l,j} \xi_{\lambda_l}^{-j-1},
\]

which behaves as \(-y(z)\) around \( z = \zeta_l \). Substituting (119) in (118), we get

\[
\frac{\partial}{\partial t_k} F = -\text{Res}_{x=P} \frac{1}{2y_s(x)} \frac{\partial}{\partial t_k} T(x) \left( \text{Tr} L(x)^2 - \text{Tr} \left( \frac{\partial}{\partial x} \hat{\Psi}(x) \cdot \hat{\Psi}(x)^{-1} \right)^2 - 2h^{-2} y_s(x)^2 \right).
\]

Since \( \hat{\Psi}(x) \) is analytic at \( x = P \), we have \( \frac{\partial}{\partial x} \hat{\Psi}(x) \cdot \hat{\Psi}(x)^{-1} = O(\xi_P^2) \) for \( x \to P \). Thus the middle term does not contribute to the residue:

\[
\frac{\partial}{\partial t_k} F = -\text{Res}_{x=P} \frac{1}{2y_s(x)} \frac{\partial}{\partial t_k} T(x) (\text{Tr} L(x)^2 - 2h^{-2} R(x)).
\]

Using the behaviors (120) and (121), we get that \( \text{Tr} L(x)^2 = 2(L(x) \cdot F + h^{-2} R(x)) \). \( \square \)

**Lemma A.3.** Let \( \zeta_\infty \in x^{-1}(x) \) and \( \zeta_l \in x^{-1}(\lambda_l) \) be poles of \( \omega_{0,1} \) of orders \( m_\infty \) and \( m_l \), \( l = 1, \ldots, N \), respectively. Let \( d_x := \text{ord}_{\zeta_x}(x) \). We have

\[
(122) \quad \mathcal{O} \mathbb{K}(x, x') = \left( \Psi(x')^{-1} h^2 \left( \frac{\partial}{\partial x} L(x; h) \right) - \frac{\mathcal{L}(x; h) - \mathcal{L}(x'; h)}{x - x'} - \mathcal{R}(x, x') \right) \Psi(x') \mathbb{K}(x, x'),
\]

with

\[
\mathcal{R}(x, x') := \sum_{j=1-2d_x}^{m_\infty} t_{\zeta_x,j} \sum_{s=1}^{\xi_x} \xi_{\lambda_l}^{-j-2d_x} \left( \frac{\partial}{\partial x} \mathcal{R}_{\zeta_x, s}(x) - \mathcal{R}_{\zeta_x, s}(x') \right)
\]

\[
+ \sum_{l=1}^{N} \left( \xi_{\lambda_l}^{-1}(\mathcal{R}_{l,1}(x) - \mathcal{R}_{l,1}(x')) + \sum_{s=1}^{m_l-1} \sum_{j=s}^{m_l} t_{\zeta_l,j} \xi_{\lambda_l}^{-j-s} \mathcal{R}_{l, s}(x) - \mathcal{R}_{l, s}(x') \right).
\]

**Proof.** On the one hand, we have

\[
(123) \quad \frac{\partial}{\partial x} \mathbb{K}(x, x') = \Psi^{-1}(x') \frac{\partial}{\partial x} \Psi(x) = \Psi^{-1}(x') L(x; h) \Psi(x) = \Psi^{-1}(x') L(x; h) \Psi(x') \mathbb{K}(x, x');
\]

\[
(124) \quad \frac{\partial}{\partial x} \mathbb{K}(x, x') = -\Psi^{-1}(x') \frac{\partial}{\partial x} \Psi(x') \Psi^{-1}(x') \Psi(x) = -\Psi^{-1}(x') \mathcal{L}(x'; h) \Psi(x') \mathbb{K}(x, x');
\]

\[
(125) \quad \frac{\partial}{\partial x} \mathbb{K}(x, x') = \Psi^{-1}(x') \left( \frac{\partial}{\partial x} L(x; h) + \mathcal{L}(x; h)^2 \right) \Psi(x') \mathbb{K}(x, x').
\]

On the other hand,

\[
(126) \quad \frac{\partial}{\partial t_k} \mathbb{K}(x, x') = \Psi^{-1}(x') \frac{\partial}{\partial t_k} \Psi(x) + \frac{\partial}{\partial t_k} \Psi^{-1}(x') \Psi(x) = \Psi^{-1}(x') (\mathcal{R}_k(x; h) - \mathcal{R}_k(x'; h)) \Psi(x).
\]

Using (123), (124), (125) and (113), we can rewrite

\[
(D + L(x) \cdot F) \mathbb{K}(x, x') = -h^2 \frac{\partial}{\partial x} \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial x'} \right) \mathbb{K}(x, x') =
\]

\[
\left( \Psi(x')^{-1} h^2 \left( \frac{\partial}{\partial x} L(x; h) - \mathcal{L}(x; h) - \frac{\mathcal{L}(x; h) - \mathcal{L}(x'; h)}{x - x'} \right) \right) \Psi(x')
\]

\[
- \left( h^2 L(x) + R(x) + h^2 L(x) \cdot F \right) \mathbb{K}.
\]

Making use of (117), we obtain

\[
(127) \quad \mathcal{O} \mathbb{K} = \left( \Psi(x')^{-1} h^2 \left( \frac{\partial}{\partial x} L(x; h) - \frac{\mathcal{L}(x; h) - \mathcal{L}(x'; h)}{x - x'} \right) \Psi(x') - h^2 L(x) \right) \mathbb{K}.
\]
Using the shape of the operator $L(x)$ from Corollary 3.7 in terms of derivatives with respect to the isomonodromic parameters and the action on $\mathbb{K}$ by these derivatives given by (126), we deduce

$$L(x)\mathbb{K} = \Psi^{-1}(x')\mathcal{R}(x, x')\Psi(x')\mathbb{K}.$$ 

Let us define $T(x) := \int y(x)dx$ and $\sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. We can write

$$\mathbb{K}(x, x') = e^{\frac{-h}{x}x^T(x')}\hat{\mathbb{K}}(x, x')e^{\frac{-h}{x}\sigma_3 T(x)}$$

with $\hat{\mathbb{K}}(x, x')$ analytic when $x' \to \zeta_i$, where the $\zeta_i$ are the poles of $ydx$. Let us use the notation $\sigma := \frac{h}{x}\sigma_3$.

**Lemma A.4.** The expression

$$e^{\sigma T(x')} (O\hat{\mathbb{K}}(x, x'))^\sigma \hat{\mathbb{K}}(x, x')^{-1} e^{-\sigma T(x')}$$

is a rational function of $x$ and $x'$ with no poles at $x' \to \infty$ or $x' \to \lambda_i$.

**Proof.** The idea now is to rewrite the operator acting on $\hat{\mathbb{K}} = \hat{\mathbb{K}}(x, x')$, instead of on $\mathbb{K} = \mathbb{K}(x, x')$, and analyze the poles of the equation, as a function of $x'$. We have:

$$\frac{\partial}{\partial x}\mathbb{K} = e^{-\sigma T(x')}\left(\frac{\partial}{\partial x} \hat{\mathbb{K}} + \hat{\mathbb{K}} \sigma y(x)\right)e^{\sigma T(x)}.$$

$$\frac{\partial}{\partial x}\mathbb{K} \cdot \mathbb{K}^{-1} = e^{-\sigma T(x')}\left(\frac{\partial}{\partial x} \hat{\mathbb{K}} \mathbb{K}^{-1} + y(x)\hat{\mathbb{K}} \sigma \mathbb{K}^{-1}\right)e^{\sigma T(x')}.$$

$$\frac{\partial^2}{\partial^2 x}\mathbb{K} \cdot \mathbb{K}^{-1} = e^{-\sigma T(x')}\left(\frac{\partial^2}{\partial^2 x} \hat{\mathbb{K}} + 2y(x)\frac{\partial}{\partial x} \hat{\mathbb{K}} \sigma + \frac{\partial}{\partial x} y(x)\hat{\mathbb{K}} \sigma + h^{-2}y(x)^2\hat{\mathbb{K}}\right)\mathbb{K}^{-1}e^{\sigma T(x')}.$$

$$L(x)\mathbb{K} \cdot \mathbb{K}^{-1} = e^{-\sigma T(x')} (-L(x)T(x')\sigma L(x)T(x)\hat{\mathbb{K}} \mathbb{K}^{-1} + L(x)\hat{\mathbb{K}} \mathbb{K}^{-1})e^{\sigma T(x')}.$$ 

Therefore we can write

$$e^{\sigma T(x')} (O\hat{\mathbb{K}}(x, x'))^\sigma \hat{\mathbb{K}}(x, x')^{-1} e^{-\sigma T(x')} = \frac{y(x')\sigma}{x-x'} + L(x)T(x')\sigma + F(x, x'),$$

with $F(x, x')$ a rational function of $x$ and $x'$ with no poles at $x' \to \infty$ or $x' \to \lambda_i$. Let $z'$ be one of the preimages of $x'$: $x' = x(z')$.

**Behavior at $x' \to \infty$:** We consider the behavior of $L(x)T(x')$ at $z' \to \zeta_x$, with $\zeta_x \in x^{-1}(\infty)$.

Here we call $d = -d_x$ and $m = m_x$ to simplify the notations, and use $\xi = x^{-\frac{1}{d}}, \xi' = x'^{-\frac{1}{d}}$ as local coordinates. The only terms of $L(x)$ that may bring poles at $z' \to \zeta_x$ when applied to $T(x')$ read

$$L_{\infty}(x) = \sum_{j=1+2d}^m t_{\xi_x, j} \sum_{k=0}^{\frac{j-1}{d} - 2} \xi^{-dk} \left(\frac{j}{d} - k - 2\right) \frac{\partial}{\partial t_{\xi_x, j} - d(k+2)}$$

We have

$$\frac{s}{d} \frac{\partial}{\partial t_{\xi_x, s}} T(x') = -\frac{1}{d} \xi^{l-s}.$$
Therefore, around $z' = \zeta_\infty$, we get

$$L_\infty(x).T(x') = -\frac{1}{d} \sum_{j=1+2d}^{m} t_{\zeta x,j} \sum_{k=0}^{j-1-2d} \xi^{-dk} \xi^{-j+(k+2)d},$$

which is indeed singular when $z' \rightarrow \zeta_\infty$. On the other hand, we consider the local behavior of the first term of the RHS of (129) around $z' = \zeta_\infty$

$$\frac{y(x')}{x-x'} = -\frac{y(x')}{x'-x} = \frac{1}{d} \sum_{j=0}^{m} t_{\zeta x,j} \sum_{k=0}^{j-1} \xi^{-j+(k+2)d} \xi^{-dk},$$

whose singular terms around $z' = \zeta_\infty$ cancel exactly (130).

- Behavior at $x' \rightarrow \lambda_l$: We consider the behavior of $L(x).T(x')$ at $z' \rightarrow \zeta_l$, with $\zeta_l \in \lambda_l^{-1}(\lambda_l)$.

The only terms of $L_\lambda$ that may bring poles at $z' \rightarrow \zeta_l$ applied to $y(x')dx'$ read

$$\sum_{s=1}^{m_l+1} \sum_{j=s-1}^{m_l} t_{\zeta_l,j}(x-\lambda_l)^{-j+s-2} \partial_{\zeta_l,s} \omega_{0,1}(z') = \sum_{s=1}^{m_l+1} \sum_{j=s-1}^{m_l} t_{\zeta_l,j}(x-\lambda_l)^{-j+s-2} \int_{B_{\zeta_l,s}} \omega_{0,2}(\cdot, z')$$

where we denoted $x_1 = x(p_1)$ and $\omega_{0,2} = B$ is the Bergman kernel. Developing the last sum and dropping the terms which are regular at $p_1 \rightarrow \zeta_l$, we obtain

$$\text{Res}_{p_1=\zeta_l} \frac{y(p_1)}{x_1-x} B(p_1, z'),$$

which is indeed irregular when $z' \rightarrow \zeta_l$. Observe that this is the local behavior around $z' = \zeta_l$ of the derivative of the second term of the RHS of (129) with respect to $x'$. Now taking the derivative of the first term as well, we get

$$\frac{d}{dx'} \frac{y(x')}{x-x'} = \text{Res}_{p_1=\zeta_l} \frac{y(p_1)}{x_1-x} B(z', p_1).$$

Finally, adding (131) and (132), we obtain an expression which is regular at $z' \rightarrow \zeta_l$:

$$\frac{1}{2\pi i} \int_{\gamma} \frac{y(\cdot)}{x-\lambda_l} B(z', \cdot),$$

where $\gamma$ is a contour around $z'$ and $\zeta_l$.

Thus, (129) is a rational function of $x$ and $x'$ with no poles at $x' \rightarrow \infty$ or $x' \rightarrow \lambda_l$. \hfill \square

**Proof of Theorem A.1.** Using the shape of Lemma A.3, we deduce that $\mathcal{O} \mathbb{K}(x, x') = 0$ when $x' \rightarrow x$ and hence also the expression given by (128) vanishes when $x' \rightarrow x$. In Lemma A.4, we have shown that (128) has no poles as a rational function of $x'$. Thus it must be constant, and since it vanishes at $x' \rightarrow x$, it must be zero, which implies $\mathcal{O} \mathbb{K}(x, x') = 0$. \hfill \square

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