Notes on Spinors in Low Dimension

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0. Introduction. The purpose of these old notes (from 1998) is to determine the orbit structure of
the groups Spin(p, q) acting on their spinor spaces for certain values of n = p + q, in particular, the values
(p, q) = (8, 0), (9, 0), (10, 0), and (10, 1).

though it will turn out in the end that there are a few interesting things to say about the cases (p, q) = (10, 2)
and (9, 1), as well.

1. The Octonions. Let Ω denote the ring of octonions. Elements of Ω will be denoted by bold
letters, such as x, y, etc. Thus, Ω is the unique R-algebra of dimension 8 with unit 1 ∈ Ω endowed with
a positive definite inner product ⟨,⟩ satisfying ⟨xy, xy⟩ = ⟨x, x⟩⟨y, y⟩ for all x, y ∈ Ω. As usual, the norm
of an element x ∈ Ω is denoted |x| and defined as the square root of ⟨x, x⟩. Left and right multiplication
by x ∈ Ω define maps Lx, Rx : Ω → Ω that are isometries when |x| = 1.

The conjugate of x ∈ Ω, denoted x̄, is defined to be x̄ = 2⟨x, 1⟩1 − x. When a symbol is needed, the
map of conjugation will be denoted C : Ω → Ω. The identity x̄x = |x|2 holds, as well as the conjugation
identity xyx̄ = yxȳ. In particular, this implies the useful identities C Lx C = Rx and C Rx C = Lx.

The algebra Ω is not commutative or associative. However, any subalgebra of Ω that is generated
by two elements is associative. It follows that x (xy) = |x|2y and that xyx = x(yx) for all x, y ∈ Ω.
Thus, Rx Lx = Lx Rx (though, of course, Rx Ly ̸= Ly Rx in general). In particular, the expression xyx
is unambiguously defined. In addition, there are the Moufang Identities

(xy)z = x(y(z)),

z(xy) = (zy)x,

x(yz)x = (xy)(zx),

which will be useful below. (See, for example, Spinors and Calibrations, by F. Reese Harvey, for proofs.)

2. Spin(8). For x ∈ Ω, define the linear map mx : Ω ⊕ Ω → Ω ⊕ Ω by the formula

\[ m_x = \begin{bmatrix} 0 & CR_x \\ -CL_x & 0 \end{bmatrix}. \]

By the above identities, it follows that (mx)2 = −|x|2 and hence this map induces a representation on
the vector space Ω ⊕ Ω of the Clifford algebra generated by Ω with its standard quadratic form. This
Clifford algebra is known to be isomorphic to \( M_{16}(\mathbb{R}) \), the algebra of 16-by-16 matrices with real entries,
so this representation must be faithful. By dimension count, this establishes the isomorphism Cf(Ω, ⟨,⟩) = \( \text{End}_R(Ω ⊕ Ω) \).

The group Spin(8) ⊂ GLR(Ω ⊕ Ω) is defined as the subgroup generated by products of the form m_{x^t} m_y
where x, y ∈ Ω satisfy |x| = |y| = 1. Such endomorphisms preserve the splitting of Ω ⊕ Ω into the two given
summands since

\[ m_{x^t} m_y = \begin{bmatrix} -L_{x^t} L_y & 0 \\ 0 & -R_{x^t} R_y \end{bmatrix}. \]

In fact, setting x = −1 in this formula shows that endomorphisms of the form

\[ \begin{bmatrix} L_u & 0 \\ 0 & R_u \end{bmatrix}, \quad \text{with } |u| = 1 \]

lie in Spin(8). In fact, they generate Spin(8), since m_{x^t} m_y is clearly a product of two of these when
|x| = |y| = 1.
Fixing an identification $\mathbb{O} \simeq \mathbb{R}^8$, this defines an embedding $\text{Spin}(8) \subset \text{SO}(8) \times \text{SO}(8)$, and the projections onto either of the factors is a group homomorphism. Since neither of these projections is trivial, since the Lie algebra $\mathfrak{so}(8)$ is simple, and since $\text{SO}(8)$ is connected, it follows that each of these projections is a surjective homomorphism. Since $\text{Spin}(8)$ is simply connected and since the fundamental group of $\text{SO}(8)$ is $\mathbb{Z}_2$, it follows that each of these homomorphisms is a non-trivial double cover of $\text{SO}(8)$. Moreover, it follows that the subsets $\{ L_u \mid |u| = 1 \}$ and $\{ R_u \mid |u| = 1 \}$ of $\text{SO}(8)$ each suffice to generate $\text{SO}(8)$.

Let $H \subset (\text{SO}(8))^3$ be the set of triples $(g_1, g_2, g_3) \in (\text{SO}(8))^3$ for which

$$g_2(xy) = g_1(x) g_3(y)$$

for all $x, y \in \mathbb{O}$. The set $H$ is closed and is evidently closed under multiplication and inverse. Hence it is a compact Lie group.

By the third Moufang identity, $H$ contains the subset

$$\Sigma = \{ (L_u, L_u R_u, R_u) \mid |u| = 1 \}.$$

Let $K \subset H$ be the subgroup generated by $\Sigma$, and for $i = 1, 2, 3$, let $\rho_i : H \to \text{SO}(8)$ be the homomorphism that is projection onto the $i$-th factor. Since $\rho_1(K)$ contains $\{ L_u \mid |u| = 1 \}$, it follows that $\rho_1(K) = \text{SO}(8)$, so a fortiori $\rho_1(H) = \text{SO}(8)$. Similarly, $\rho_3(H) = \text{SO}(8)$.

The kernel of $\rho_1$ consists of elements $(I_8, g_2, g_3)$ that satisfy $g_2(xy) = x g_3(y)$ for all $x, y \in \mathbb{O}$. Setting $x = 1$ in this equation yields $g_2 = g_3$, so that $g_2(xy) = x g_2(y)$. Setting $y = 1$ in this equation yields $g_2(x) = x g_2(1)$, i.e., $g_2 = R_u$ for $u = g_2(1)$. Thus, the elements in the kernel of $\rho_1$ are of the form $(1, R_u, R_u)$ for some $u$ with $|u| = 1$. However, any such $u$ would, by definition, satisfy $(xy)u = x(yu)$ for all $x, y \in \mathbb{O}$, which is impossible unless $u = \pm 1$. Thus, the kernel of $\rho_1$ is $\{(I_8, \pm I_8, \pm I_8)\} \simeq \mathbb{Z}_2$, so that $\rho_1$ is a 2-to-1 homomorphism of $H$ onto $\text{SO}(8)$. Similarly, $\rho_3$ is a 2-to-1 homomorphism of $H$ onto $\text{SO}(8)$, with kernel $\{(\pm I_8, \pm I_8, I_8)\}$. Thus, $H$ is either connected and isomorphic to $\text{Spin}(8)$ or else disconnected, with two components.

Now $K$ is a connected subgroup of $H$ and the kernel of $\rho_1$ intersected with $K$ is either trivial or $\mathbb{Z}_2$. Moreover, the product homomorphism $\rho_1 \times \rho_3 : K \to \text{SO}(8) \times \text{SO}(8)$ maps the generator $\Sigma \subset K$ into generators of $\text{Spin}(8) \subset \text{SO}(8) \times \text{SO}(8)$. It follows that $\rho_1 \times \rho_3(K) = \text{Spin}(8)$ and hence that $\rho_1$ and $\rho_3$ must be non-trivial double covers of $\text{SO}(8)$ when restricted to $K$. In particular, it follows that $K$ must be all of $H$ and, moreover, that the homomorphism $\rho_1 \times \rho_3 : H \to \text{Spin}(8)$ must be an isomorphism. It also follows that the homomorphism $\rho_2 : H \to \text{SO}(8)$ must be a double cover of $\text{SO}(8)$ as well.

Henceforth, $H$ will be identified with $\text{Spin}(8)$ via the isomorphism $\rho_1 \times \rho_3$. Note that the center of $H$ consists of the elements $(\varepsilon_1 I_8, \varepsilon_2 I_8, \varepsilon_3 I_8)$ where $\varepsilon_i^2 = \varepsilon_1 \varepsilon_2 \varepsilon_3 = 1$ and is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

**Triality.** For $(g_1, g_2, g_3) \in H$, the identity $g_2(xy) = g_1(x) g_3(y)$ can be conjugated, giving

$$C g_2 C(xy) = g_2(\overline{y}x) = g_1(\overline{y}) g_3(x) = g_3(x) g_1(\overline{y}).$$

This implies that $(C g_3 C, C g_2 C, C g_1 C)$ also lies in $H$. Also, replacing $x$ by $z \overline{y}$ in the original formula and multiplying on the right by $g_3(\overline{y})$ shows that

$$g_2(z) g_3(\overline{y}) = g_1(z \overline{y}),$$

implying that $(g_2, g_1, C g_3 C)$ lies in $H$ as well. In fact, the two maps $\alpha, \beta : H \to H$ defined by

$$\alpha(g_1, g_2, g_3) = (C g_3 C, C g_2 C, C g_1 C), \quad \text{and} \quad \beta(g_1, g_2, g_3) = (g_2, g_1, C g_3 C)$$

are outer automorphisms (since they act nontrivially on the center of $H$) and generate a group of automorphisms isomorphic to $S_3$, the symmetric group on three letters. The automorphism $\tau = \alpha \beta$ is known as the triality automorphism.

**Notation.** To emphasize the group action, denote $\mathbb{O} \cong \mathbb{R}^8$ by $V_i$ when regarding it as a representation space of $\text{Spin}(8)$ via the representation $\rho_i$. Thus, octonion multiplication induces a $\text{Spin}(8)$-equivariant projection

$$V_1 \otimes V_3 \longrightarrow V_2.$$
In the standard notation, it is traditional to identify $V_1$ with $\mathbb{S}_-$ and $V_3$ with $\mathbb{S}_+$ and to refer to $V_2$ as the ‘vector representation’. Let $\rho'_i : \mathfrak{spin}(8) \to \mathfrak{so}(8)$ denote the corresponding Lie algebra homomorphisms, which are, in fact, isomorphisms. For simplicity of notation, for any $a \in \mathfrak{spin}(8)$, the element $\rho'_i(a) \in \mathfrak{so}(8)$ will be denoted by $a_i$ when no confusion can arise.

**Orbit structure.** Let $\text{SO}(\mathfrak{Im}\mathfrak{O}) \simeq \text{SO}(7)$ denote the subgroup of $\text{SO}(\mathfrak{O}) \simeq \text{SO}(8)$ that leaves $1 \in \mathfrak{O}$ fixed, and let $K_i \subset H$ be the preimage of $\text{SO}(\mathfrak{Im}\mathfrak{O})$ under the homomorphism $\rho_i : H \to \text{SO}(\mathfrak{O})$. Then $K_i$ is a non-trivial double cover of $\text{SO}(\mathfrak{Im}\mathfrak{O})$ and hence is isomorphic to $\text{Spin}(7)$. Note, in particular, that $K_1$ contains $(I_8, -I_8, -I_8)$ and hence $\rho_3(K_1) \subset \text{SO}(8)$ contains $-I_8$. This implies that $\rho_3 : K_1 \to \text{SO}(V_3)$ is a faithful representation of $\text{Spin}(7)$ and hence $K_1$ acts transitively on the unit sphere in $V_1$.

In particular, it follows that $\text{Spin}(8) \subset \text{SO}(V_1) \times \text{SO}(V_3)$ acts transitively on the product of the unit spheres in $V_1$ and $V_3$. Consequently, it follows that the quadratic polynomials

$$q_1(x, y) = |x|^2 \quad \text{and} \quad q_2(x, y) = |y|^2$$

generate the ring of $\text{Spin}(8)$-invariant polynomials on $\mathfrak{O} \oplus \mathfrak{O}$ and that every point of this space lies on the $\text{Spin}(8)$-orbit of a unique element of the form $(a \mathbf{1}, b \mathbf{1})$ for some pair of real numbers $a, b \geq 0$. For $ab \neq 0$, the stabilizer of such an element is the 14-dimensional simple group $G_2$, and this group acts transitively on the unit sphere in $\mathfrak{Im}\mathfrak{O}$.

**3. Spin(9).** For $(r, x) \in \mathbb{R} \oplus \mathfrak{O}$, define a $\mathbb{C}$-linear map $m_{(r, x)} : \mathbb{C} \otimes \mathfrak{O}^2 \to \mathbb{C} \otimes \mathfrak{O}^2$ by the formula

$$m_{(r, x)} = i \begin{bmatrix} rI_8 & CR_x \\ CL_x & -rI_8 \end{bmatrix}.$$ 

Since $(m_{(r, x)})^2 = -(r^2 + |x|^2)$ times the identity map, this defines a $\mathbb{C}$-linear representation on $\mathbb{C} \otimes \mathfrak{O}^2$ of the Clifford algebra generated by $\mathbb{R} \oplus \mathfrak{O}$ endowed with its direct sum inner product. Since this Clifford algebra is known to be isomorphic to $M_{16}(\mathbb{C})$, it follows, for dimension reasons, that this representation is one-to-one and onto, establishing the isomorphism $\text{Cl}(\mathbb{R} \oplus \mathfrak{O}, (,)) \cong \text{End}_\mathbb{C}(\mathbb{C} \otimes \mathfrak{O}^2)$.

As usual, $\text{Spin}(9)$ is the subgroup generated by the products of the form $m_{(r, x)}m_{(s, y)}$, where $r^2 + |x|^2 = s^2 + |y|^2 = 1$. Note that these products have real coefficients, and so actually lie in $\text{GL}_\mathbb{R}(\mathfrak{O}^2) \cong \text{GL}(16, \mathbb{R})$. In fact, these products are themselves seen to be products of the special form

$$p_{(r, x)} = m_{(-1, 0)}m_{(r, x)} = \begin{bmatrix} rI_8 & CR_x \\ -CL_x & rI_8 \end{bmatrix}, \quad \text{where } r^2 + |x|^2 = 1,$$

so these latter matrices suffice to generate $\text{Spin}(9)$. By the results of the previous section, products of an even number of the $p_{(0, u)}$ with $|u| = 1$ generate $\text{Spin}(8) \subset \text{Spin}(9)$.

Since the linear transformations of the form $p_{(r, x)}$ preserve the quadratic form

$$q(x, y) = |x|^2 + |y|^2,$$

it follows that $\text{Spin}(9)$ is a subgroup of $\text{SO}(\mathfrak{O}^2) = \text{SO}(16)$.

**The Lie algebra.** Since $\text{Spin}(9)$ contains $\text{Spin}(8)$, the containment $\mathfrak{spin}(8) \subset \mathfrak{spin}(9)$ yields the containment

$$\left\{ \begin{bmatrix} a_1 & 0 \\ 0 & a_3 \end{bmatrix} \bigg| a \in \mathfrak{spin}(8) \right\} \subset \mathfrak{spin}(9).$$

Moreover, since $\text{Spin}(9)$ contains the 8-sphere consisting of the $p_{(r, x)}$ with $r^2 + |x|^2 = 1$, its Lie algebra must contain the tangent space to this 8-sphere at $(r, x) = (1, \mathbf{0})$, i.e.,

$$\left\{ \begin{bmatrix} 0 & CR_x \\ -CL_x & 0 \end{bmatrix} \bigg| x \in \mathfrak{O} \right\} \subset \mathfrak{spin}(9).$$
By dimension count, this implies the equality

$$\text{spin}(9) = \left\{ \begin{pmatrix} a_1 & C R_x & x \\ -C L_x & a_3 & 0 \\ 0 & \end{pmatrix} \middle| x \in \mathbb{O}, \ a \in \text{spin}(8) \right\}.$$ 

Let $\rho : \text{Spin}(9) \to \text{SO}(\mathbb{R} \oplus \mathbb{O}) \simeq \text{SO}(9)$ be the homomorphism for which the induced map on Lie algebras is

$$\rho' \left( \begin{pmatrix} a_1 & C R_z \\ -C L_z & a_3 \end{pmatrix} \right) = \begin{pmatrix} 0 & 2z^* \\ -2z & a_2 \end{pmatrix},$$

where $x^* : \mathbb{O} \to \mathbb{R}$ is just $x^*(y) = \langle x, y \rangle$. (The triality constructions imply that $\rho'$ is, indeed, a Lie algebra homomorphism. Note that, when restricted to Spin(8), this becomes the homomorphism $\rho_2 : \text{Spin}(8) \to \text{SO}(\mathbb{O}) = \text{SO}(8)$.) Then $\rho$ is a double cover of $\text{SO}(9)$.

Define the squaring map $\sigma : \mathbb{O}^2 \to \mathbb{R} \oplus \mathbb{O}$ by

$$\sigma \left( \begin{pmatrix} x \\ y \end{pmatrix} \right) = \begin{pmatrix} |x|^2 - |y|^2 \\ 2x y \end{pmatrix}.$$ 

A short calculation using the Moufang Identities shows that $\sigma$ is $\rho$-equivariant, i.e., that $\sigma(gv) = \rho(g)(\sigma(v))$ for all $v \in \mathbb{O}^2$ and all $g \in \text{Spin}(9)$. This will be useful below.

*Orbit structure and stabilizer.* Each point of $\mathbb{O}^2$ lies on the Spin(8)-orbit of an element $(a \mathbf{1}, b \mathbf{1})$ for some real numbers $a, b \geq 0$. Thus, the orbits of Spin(9) on the unit sphere in $\mathbb{O}^2$ are unions of the Spin(8)-orbits of the elements $(\cos \theta \mathbf{1}, \sin \theta \mathbf{1})$. Now, calculation yields

$$p_{(\cos \phi, \sin \phi \mathbf{1})} \begin{pmatrix} \cos \theta \mathbf{1} \\ \sin \theta \mathbf{1} \end{pmatrix} = \begin{pmatrix} \cos(\theta - \phi) \mathbf{1} \\ \sin(\theta - \phi) \mathbf{1} \end{pmatrix}. $$

Since all of the elements $(\cos \theta \mathbf{1}, \sin \theta \mathbf{1})$ lie on a single Spin(9)-orbit, it follows that Spin(9) acts transitively on the unit sphere in $\mathbb{O}^2$ and, consequently, that the quadratic form $q$ generates the ring of Spin(9)-invariant polynomials on $\mathbb{O}^2$.

Since the orbit of $(\mathbf{1}, \mathbf{0}) \in \mathbb{O}^2$ is the 15-sphere and since Spin(9) is connected and simply connected, it follows that the Spin(9)-stabilizer of this element must be connected, simply connected, and of dimension 21. Since $K_1 \subset \text{Spin}(8) \subset \text{Spin}(9)$ lies in this stabilizer and has dimension 21, it follows that $K_1$ must be equal to this stabilizer.

For use in the next two sections, it will be useful to understand the orbits of Spin(9) acting on $\mathbb{O}^2 \oplus \mathbb{O}^2$ and to understand the ring of Spin(9)-invariant polynomials on this vector space of real dimension 32. The first observation is that the generic orbit has codimension 4. This can be seen as follows: Since Spin(9) acts transitively on the unit sphere in $\mathbb{O}^2$, every element lies on the Spin(9) orbit of an element of the form

$$\begin{pmatrix} \begin{pmatrix} a \mathbf{1} \\ 0 \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \end{pmatrix},$$

where $a \geq 0$. Assuming $a > 0$, the stabilizer in Spin(9) of this first component is $K_1 \simeq \text{Spin}(7)$ and this acts transitively on the unit sphere in the second $\mathbb{O}$-summand of $\mathbb{O}^2$, so that an element of the above form lies on the orbit of an element of the form

$$\begin{pmatrix} \begin{pmatrix} a \mathbf{1} \\ 0 \end{pmatrix}, \begin{pmatrix} x \\ b \mathbf{1} \end{pmatrix} \end{pmatrix},$$

where $b \geq 0$. Assuming $b > 0$, the stabilizer in $K_1$ of $\mathbf{1}$ in this second $\mathbb{O}$-summand is $G_2$, which acts transitively on the unit sphere in $\text{Im} \mathbb{O}$ in the first $\mathbb{O}$-summand. This implies that an element of the above form lies on the orbit of an element of the form

$$z = \begin{pmatrix} \begin{pmatrix} a \mathbf{1} \\ 0 \end{pmatrix}, \begin{pmatrix} c \mathbf{1} + du \mathbf{1} \\ b \mathbf{1} \end{pmatrix} \end{pmatrix},$$
for some \( c, d \geq 0 \) and \( u \in \text{Im} \Omega \) some fixed unit imaginary octonion. Thus, the generic Spin(9)-orbit has codimension at most 4. It is still possible that two elements of the above form with distinct values of \( a, b, c, d > 0 \) might lie on the same Spin(9)-orbit, but this will be ruled out directly.

To see that these latter elements lie on distinct Spin(9)-orbits, it will be sufficient to construct Spin(9)-invariant polynomials on \( \mathbb{O}^2 \oplus \mathbb{O}^2 \) that separate these elements. To do so, write the typical element of \( \mathbb{O}^2 \oplus \mathbb{O}^2 \) in the form

\[
(v_1, v_2) = \left( \frac{x_1}{y_1}, \frac{x_2}{y_2} \right),
\]

and first consider the three quadratic polynomials

\[
q_{2,0} = |x_1|^2 + |y_1|^2, \\
q_{1,1} = x_1 \cdot x_2 + y_1 \cdot y_2, \\
q_{0,2} = |x_2|^2 + |y_2|^2.
\]

These polynomials are manifestly Spin(9)-invariant and satisfy

\[
q_{2,0}(z) = a^2, \quad q_{1,1}(z) = ac, \quad q_{0,2}(z) = b^2 + c^2 + d^2.
\]

Evidently, these polynomials span the vector space of Spin(9)-invariant quadratic polynomials on \( \mathbb{O}^2 \oplus \mathbb{O}^2 \).

Since Spin(9) contains \(-1\) times the identity, there are no Spin(9)-invariant cubic polynomials. A representation-theoretic argument shows that the Spin(9)-invariant quartic polynomials on \( \mathbb{O}^2 \oplus \mathbb{O}^2 \) form a vector space of dimension 7. Six of these are accounted for by quadratic polynomials in \( q_{2,0}, q_{1,1}, \) and \( q_{0,2} \), while a seventh can be constructed as follows. Define

\[
q_{2,2} = \sigma(v_1) \cdot \sigma(v_2) = (|x_1|^2 - |y_1|^2) (|x_2|^2 - |y_2|^2) + 4 (x_1 y_1) \cdot (x_2 y_2).
\]

Using the Spin(9)-equivariance of the squaring map \( \sigma \), it follows that \( q_{2,2} \) is indeed invariant under Spin(9). Note that

\[
q_{2,2}(z) = a^2 (c^2 + d^2 - b^2),
\]

so that knowledge of \( (q_{2,0}(z), q_{1,1}(z), q_{0,2}(z), q_{2,2}(z)) \) suffices to recover \( a, b, c, d > 0 \) when these numbers are all non-zero. It now follows that the simultaneous level sets of these four polynomials are exactly the Spin(9)-orbits on \( \mathbb{O}^2 \oplus \mathbb{O}^2 \). (It seems likely that these polynomials generate the ring of Spin(9)-invariant polynomials on \( \mathbb{O}^2 \oplus \mathbb{O}^2 \), but such a result will not be needed, so this problem will not be discussed further.)

4. Spin(10). Rather than construct the Clifford representation for an inner product on a vector space of dimension 10, it is convenient to use the fact that Spin(10) already appears as a subgroup of \( \text{Cl}(\mathbb{R} \oplus \mathbb{O}, \langle , \rangle) = \text{End}_\mathbb{C}(\mathbb{C} \otimes \mathbb{O}^2) \). In fact, by the discussion in the last section, Spin(10) is the connected subgroup of this latter algebra whose Lie algebra is

\[
\text{spin}(10) = \left\{ \left( \begin{array}{cc} a_1 + ir I_8 & C R_x + i C R_y \\ -C L_x + i C L_y & a_3 - ir I_8 \end{array} \right) \bigg| r \in \mathbb{R}, \ x, y \in \mathbb{O}, \ a \in \text{spin}(8) \right\}.
\]

Note that \( \text{spin}(10) \) appears as a subspace of \( \text{su}(16) \), so that Spin(10) acts on \( \mathbb{C}^{16} = \mathbb{C} \otimes \mathbb{O}^2 \) preserving the complex structure and the quadratic form

\[
q = q_{2,0} + q_{0,2} = |x_1|^2 + |y_1|^2 + |x_2|^2 + |y_2|^2,
\]

where, now, the typical element of \( \mathbb{C} \otimes \mathbb{O}^2 \) will be written as

\[
z = \left( \frac{x_1 + i x_2}{y_1 + i y_2} \right).
\]
Note that, because there are no connected Lie groups that lie properly between Spin(9) and Spin(10), it follows that Spin(10) is generated by Spin(9) and the circle subgroup

\[ T = \left\{ \begin{pmatrix} e^{ir} I_8 & 0 \\ 0 & e^{-ir} I_8 \end{pmatrix} \right\} \quad r \in \mathbb{R}/2\pi\mathbb{Z}, \]

which lies in Spin(10), but does not lie in Spin(9). In particular, a polynomial on \( C \otimes \mathbb{O}^2 \) is Spin(10)-invariant if and only if it is both Spin(9)-invariant and \( T \)-invariant.

**Invariant polynomials.** Among the quadratic polynomials that are Spin(9)-invariant, only the multiples of \( q = q_{2,0} + q_{0,2} \) are also \( T \)-invariant. Thus, \( q \) spans the space of Spin(10)-invariant quadratic forms on \( C \otimes \mathbb{O}^2 \). In particular, this implies that the action of Spin(10) on \( C \otimes \mathbb{O}^2 \) is irreducible (even as a real vector space).

Among the quartic polynomials that are Spin(9)-invariant, a short calculation shows that only linear combinations of \( q^2 \) and

\[
p = \frac{1}{2} \left( q_{2,0} + q_{0,2} - 2 q_{1,1} \right)
= |x_1|^2 |x_2|^2 + |y_1|^2 |y_2|^2 - (x_1 \cdot x_2 + y_1 \cdot y_2)^2 + 2 (x_1 y_1) \cdot (x_2 y_2)
= |x_1 \wedge x_2|^2 + |y_1 \wedge y_2|^2 - 2 (x_1 \cdot x_2) (y_1 \cdot y_2) + 2 (x_1 y_1) \cdot (x_2 y_2).
\]

are invariant under the action of \( T \). Thus, it follows that \( q^2 \) and \( p \) span the space of Spin(10)-invariant quartics. (Note the interesting feature that, in the latter expression for \( p \), only the final term makes use of octonion multiplication operations.)

**Orbits and stabilizers.** Let \( M \subset C \otimes \mathbb{O}^2 \) be the Spin(10)-orbit of \( z_0 = (1 + i 0, 0 + i 0) \). The tangent space to \( M \) at \( z_0 \) is the set of vectors of the form

\[
\left( \begin{array}{cc}
a_1 + ir & CR_x + i CR_y \\
-C L_x + i C L_y & a_3 - ir I_8
\end{array} \right) \begin{pmatrix} 1 + i 0 \\ 0 + i 0 \end{pmatrix} = \begin{pmatrix} a_1 + i r1 \\ -\bar{y} + i \bar{y} \end{pmatrix},
\]

and the Lie algebra of the Spin(10)-stabilizer of \( z_0 \) is defined by the equations \( a_1 1 = r = x = y = 0 \). Thus, the identity component of the stabilizer of \( z_0 \) is \( K_1 \cong \text{Spin}(7) \) and the full stabilizer must lie in the normalizer of \( K_1 \) in Spin(10). Evidently, the normalizer of \( K_1 \) in Spin(10) is \( K_1 \cdot T \). Since only the identity in the subgroup \( T \) stabilizes \( z_0 \), the full stabilizer of \( z_0 \) is \( K_1 \). Thus, \( M \) is diffeomorphic to Spin(10)/Spin(7), which is a smooth manifold of dimension \( 45 - 21 = 24 \) that is 2-connected, i.e., \( \pi_0(M) = \pi_1(M) = \pi_2(M) = 0 \).

The normal space to \( M \) at \( z_0 \) is the orthogonal direct sum of the line \( \mathbb{R} z_0 \) (which is normal to the unit sphere in \( C \otimes \mathbb{O}^2 \)) and the subspace of dimension 7

\[ N_{z_0} = \left\{ \begin{pmatrix} 0 + i x \\ 0 + i 0 \end{pmatrix} \right\} \quad x \in \text{Im} \mathbb{O}. \]

The stabilizer \( K_1 \) acts as SO(7) on this subspace. In particular, it acts transitively on the unit sphere in \( N_{z_0} \), and hence it acts transitively on the space of geodesics in the unit 31-sphere that meet \( M \) orthogonally at \( z_0 \). Since \( M \) is itself a Spin(10)-orbit, it follows that Spin(10) acts transitively on the normal tube of any radius about \( M \) in the unit 31-sphere. Since, for generic radii, these normal tubes are hypersurfaces, it follows that the generic Spin(10)-orbit in the 31-sphere must be a hypersurface of dimension 30. Using the fact that such a hypersurface is an \( S^6 \)-bundle over \( M \), the long exact sequence in homotopy implies that these hypersurface orbits are also 2-connected, which implies that the Spin(10)-stabilizer of any point on such a hypersurface must be both connected and simply connected.

Now, the full group Spin(10) must act transitively on the space of geodesics in the unit 31-sphere that meet \( M \) orthogonally at any point while every point of the unit 31-sphere lies on some geodesic that meets \( M \) orthogonally. Thus, fixing some \( u \in \text{Im} \mathbb{O} \) with \( |u| = 1 \), it follows that every element of the 31-sphere lies on the Spin(10)-orbit of an element of the form

\[ z_0 = \begin{pmatrix} \cos \theta + i \sin \theta u \\ 0 + i 0 \end{pmatrix}. \]
Note that \( p(z_\theta) = \cos^2 \theta \sin^2 \theta = \frac{1}{4} \sin^2(2\theta) \), so it follows that for \( 0 \leq \theta \leq \pi/4 \), the elements \( z_\theta \) lie on distinct orbits, and that \( 0 \leq p \leq 1/4 \), with the endpoints of this interval being the only critical values of \( p \). While \( M = p^{-1}(0) \) is one critical orbit, the other critical orbit is \( M^* = p^{-1}(\frac{1}{2}) \) and consists of the points of the 31-sphere that are at geodesic distance \( \pi/4 \) from \( M \). It follows from this that \( M^* \) is also connected and is a single orbit of \( \text{Spin}(10) \). In particular, the simultaneous level sets of \( q \) and \( p \) are exactly the \( \text{Spin}(10) \)-orbits in \( \mathbb{C} \otimes \mathbb{O}^2 \).

For \( 0 < \theta < \pi/4 \), the nearest point on \( M \) to \( z_\theta \) is \( z_0 \), so the \( \text{Spin}(10) \)-stabilizer of \( z_\theta \) is a subgroup of \( K_1 \) that has already been seen to be both connected and simply connected. Also, the orbit of \( z_\theta \) is a 6-plane bundle over \( M \). By dimension count, this stabilizer must be of dimension 15 and must contain the stabilizer in \( K_1 \) of \( 1 \) and \( u \), which is \( \text{Spin}(6) \). Thus, the stabilizer of such a \( z_\theta \) is exactly \( \text{Spin}(6) \simeq \text{SU}(4) \).

In particular, the stabilizer of any point of the 31-sphere not on \( M \) or \( M^* \) must be a conjugate of \( \text{SU}(4) \).

Now, the tangent space to \( M^* \) at \( z_{\pi/4} \) is the set of vectors of the form

\[
\left( \begin{array}{cccc}
  a_1 + ir I_8 & C R_x + i C R_y & 1 + i u & \frac{(a_1 1 - r u) + i (a_1 u + r 1)}{-(x + y u) + i (y - xu)} \\
  -C L_x + i C L_y & a_3 - ir I_8 & 0 + i 0 & \end{array} \right).
\]

Thus, the Lie algebra of the stabilizer \( G \) of \( z_{\pi/4} \) is defined by the relations \( a_1 1 - r u = a_1 u + r 1 = y - xu = 0 \). (Remember that \( u^2 = -1 \).) It follows that \( a_1 \in \mathfrak{so}(8) \) must belong to the stabilizer of the 2-plane spanned by \( \{1, u\} \), so that \( a_1 \) lies in \( \mathfrak{so}(2) \oplus \mathfrak{so}(6) \). Conversely, if \( a_1 \) lies in this subspace, then there exists a unique \( r \in \mathbb{R} \) so that \( a_1 1 - r u = a_1 u + r 1 = 0 \). From the matrix representation, it is clear that the maximal torus in \( \mathfrak{so}(2) \oplus \mathfrak{so}(6) \) (which has rank 4) is a maximal torus in the full stabilizer algebra, which has dimension 24. The root pattern is evident from the matrix representation, implying that the stabilizer algebra is isomorphic to \( \mathfrak{su}(5) \).

Now \( M^* \) has dimension 21 and is the base of a fibration whose total space is one of the hypersurface orbits and whose fiber is a 9-sphere. The 2-connectivity of the hypersurface orbits implies, by the long exact sequence in homotopy, that \( M^* \) is also 2-connected, which implies that \( M^* = \text{Spin}(10)/G \) where \( G \) is both connected and simply connected. Since its Lie algebra is \( \mathfrak{su}(5) \), it follows that \( G \) is isomorphic to \( \text{SU}(5) \).

5. \( \text{Spin}(10, 1) \). To construct the spinor representation of \( \text{Spin}(10, 1) \), it will be easiest to construct the Lie algebra representation by extending the Lie algebra representation of \( \text{Spin}(10) \) that was constructed in §3. It is convenient to identify \( \mathbb{C} \otimes \mathbb{O}^2 \) with \( \mathbb{O}^4 \) explicitly via the identification

\[
z = \left( \begin{array}{c} x_1 + i x_2 \\ y_1 + i y_2 \end{array} \right) = \left( \begin{array}{c} x_1 \\ y_1 \\ x_2 \\ y_2 \end{array} \right).
\]

Via this identification, \( \mathfrak{spin}(10) \) becomes the subspace

\[
\mathfrak{spin}(10) = \left\{ \begin{array}{cccc}
  a_1 & C R_x & -r I_8 & -C R_y \\
  -C L_x & a_3 & -C L_y & r I_8 \\
  r I_8 & C R_y & a_1 & C R_x \\
  C L_y & -r I_8 & -C L_x & a_3 \\
\end{array} \right\} \quad r \in \mathbb{R}, \quad x, y \in \mathbb{O}, \quad a \in \mathfrak{so}(8) \right\}.
\]

Consider the one-parameter subgroup \( R \subset \text{SL}_2(\mathbb{O}^4) \) defined by

\[
R = \left\{ \left( \begin{array}{cc}
  t I_{16} & 0 \\
  0 & t^{-1} I_{16} \end{array} \right) \right\} \quad t \in \mathbb{R}^+ \right\}.
\]

It has a Lie algebra \( r \subset \mathfrak{sl}(\mathbb{O}^4) \). Evidently, the the subspace \( [\mathfrak{spin}(10), r] \) consists of matrices of the form

\[
\left( \begin{array}{cccc}
  0 & 0 & r I_8 & C R_y \\
  0 & 0 & C L_y & -r I_8 \\
  r I_8 & C R_y & 0 & 0 \\
  C L_y & -r I_8 & 0 & 0 \end{array} \right), \quad r \in \mathbb{R}, \quad y \in \mathbb{O}.
\]
Let $\mathfrak{g} = \mathfrak{spin}(10) \oplus \mathfrak{r} \oplus [\mathfrak{spin}(10), \mathfrak{r}]$. Explicitly,

$$
\mathfrak{g} = \begin{cases}
\begin{pmatrix}
    a_1 + x I_8 & C R_x & y I_8 & C R_y \\
    -C L_x & a_3 + x I_8 & C L_y & -y I_8 \\
    z I_8 & C R_z & a_1 - x I_8 & C R_x \\
    C L_z & -z I_8 & -C L_x & a_3 - x I_8
\end{pmatrix} & \text{for } x, y, z \in \mathbb{R}, \\
    x, y, z \in \mathbb{O}, \\
    a \in \mathfrak{spin}(8)
\end{cases}.
$$

Computation using the Moufang Identities shows that $\mathfrak{g}$ is closed under Lie bracket and hence is a Lie algebra of (real) dimension 55 that contains $\mathfrak{spin}(10)$. The induced representation of $\mathfrak{spin}(10)$ on $\mathfrak{g}/\mathfrak{spin}(10)$ evidently restricts to $\mathfrak{spin}(9)$ to preserve the splitting corresponding to the sum $\mathfrak{r} \oplus [\mathfrak{spin}(10), \mathfrak{r}] \cong \mathbb{R} \oplus \mathbb{R}^9$ and acts as the standard (irreducible) representation on the $\mathbb{R}^3$ summand. It follows that $\mathfrak{spin}(10)$ must act via its standard (irreducible, ten dimensional) representation on $\mathfrak{g}/\mathfrak{spin}(10)$. Since the trace of the square of a non-zero element in the subspace $\mathfrak{r} \oplus [\mathfrak{spin}(10), \mathfrak{r}]$ is positive, $\mathfrak{g}$ is semisimple of non-compact type. It follows that $\mathfrak{g}$ is isomorphic to $\mathfrak{so}(10,1)$ and hence is the Lie algebra of a representation of $\mathfrak{spin}(10,1)$. This representation must be faithful since it is faithful on the maximal compact subgroup $\mathfrak{spin}(10)$.

Thus, define $\mathfrak{spin}(10,1)$ to be the (connected) subgroup of $\mathfrak{SL}_2(\mathbb{O}^4)$ that is generated by $\mathfrak{spin}(10)$ and the subgroup $\mathfrak{r}$. Its Lie algebra $\mathfrak{g}$ will henceforth be written as $\mathfrak{spin}(10,1)$.

**Invariant Polynomials and Orbits.** Consider the $\mathfrak{spin}(10)$-invariant polynomial

$$
p = |x_1|^2 |x_2|^2 + |y_1|^2 |y_2|^2 - (x_1 \cdot x_2 + y_1 \cdot y_2)^2 + 2 (x_1 y_1) \cdot (x_2 y_2),
$$

Evidently, $p$ is invariant under $\mathfrak{r}$ and is therefore invariant under $\mathfrak{spin}(10,1)$. In particular, it follows that the orbits of $\mathfrak{spin}(10,1)$ on $\mathbb{O}^4 \cong \mathbb{R}^{32}$ must lie in the level sets of $p$.

Also from the previous section, it is known that every element of $\mathbb{O}^4$ lies on the $\mathfrak{spin}(10)$-orbit of exactly one of the elements

$$
z_{a,b} = \begin{pmatrix}
    a & 1 \\
    0 & b & u \\
    0
\end{pmatrix}
$$

where $0 \leq b \leq a$.

and where $u \in \text{Im}\mathbb{O}$ is a fixed unit imaginary octonion. However, all of the elements of the form

$$
\begin{pmatrix}
    at & 1 \\
    0 \\
    (b/t) & u \\
    0
\end{pmatrix}
$$

(where $0 < t$)

lie on the same $\mathfrak{r}$-orbit and, hence, on the same $\mathfrak{spin}(10,1)$-orbit. Since $p(z_{a,b}) = a^2 b^2$, it now follows that each of the nonzero level sets of $p$ is a single $\mathfrak{spin}(10,1)$-orbit while the zero level set is the union of the origin and a single $\mathfrak{spin}(10,1)$-orbit, say, the orbit of $z_{1,0}$. Moreover, it follows that $p$ generates the ring of $\mathfrak{spin}(10,1)$-invariant polynomials on $\mathbb{O}^4$.

**Stabilizers.** Multiplication by positive scalars acts transitively on the non-zero level sets of $p$, so they are all diffeomorphic. In fact, each such level set is contractible to the $\mathfrak{spin}(10)$-invariant locus where $q$ reaches its minimum on this level set and this is a manifold of dimension 21 that is diffeomorphic to $M^*$. In particular, it follows that each of the non-zero level sets of $p$ is 2-connected, so that the stabilizer in $\mathfrak{spin}(10,1)$ of a point on such a level set must be connected and simply connected.

If $z \in \mathbb{O}^4$ has $p(z) \neq 0$, then the $\mathfrak{spin}(10,1)$-orbit of $z$ has dimension 31 and so its stabilizer in $\mathfrak{spin}(10,1)$ must be of dimension $55-31 = 24$. Moreover all of these stabilizers must be conjugate in $\mathfrak{spin}(10,1)$. Since the $\mathfrak{spin}(10)$-stabilizer of the point $z_{1,1}$ is already known to be $\text{SU}(5)$, which has dimension 24, it follows that this must be the $\mathfrak{spin}(10,1)$-stabilizer as well.

The $\mathfrak{spin}(10,1)$-orbit consisting of nonzero vectors in the zero locus of $p$ is just the deleted cone on $M$, and so has dimension 25. Since it is contractible to $M$, it is 2-connected, so that the stabilizer in $\mathfrak{spin}(10,1)$
of a point in this orbit must be connected and simply connected and of dimension 55−25 = 30. In fact, the Lie algebra of this stabilizer is just

\[
\begin{pmatrix}
    a_1 & 0 & y I_8 & C R_x \\
    0 & a_3 & C L_y & -y I_8 \\
    0 & 0 & a_1 & 0 \\
    0 & 0 & 0 & a_3
\end{pmatrix}
\begin{pmatrix}
y \\
\end{pmatrix}
\begin{pmatrix}
y \in \mathbb{R}, \\
y \in \mathbb{O}, \\
a \in \mathfrak{t}_1
\end{pmatrix}
\]

where \( \mathfrak{t}_1 \) is the Lie algebra of \( K_1 \subset \text{Spin}(8) \). Thus, the stabilizer is a semi-direct product of \( \text{Spin}(7) \) with a copy of \( \mathbb{R}^9 \).

Consider the squaring map \( \sigma : \mathbb{O}^4 \to \mathbb{R}^{2+1} \oplus \mathbb{O} = \mathbb{R}^{10+1} \) that takes spinors for \( \text{Spin}(10, 1) \) to vectors. This map \( \sigma \) is defined as follows:

\[
\sigma \left( \begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{pmatrix} \right) = \begin{pmatrix} |x_1|^2 + |y_1|^2 \\ 2 (x_1 \cdot x_2 - y_1 \cdot y_2) \\ x_2^2 + |y_2|^2 \\ 2 (x_1 y_2 + x_2 y_1) \end{pmatrix}.
\]

Define the inner product on vectors in \( \mathbb{R}^{2+1} \oplus \mathbb{O} \) by the rule

\[
\left( \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ x \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ y \end{pmatrix} \right) = -2(a_1 b_3 + a_3 b_1) + a_2 b_2 + x \cdot y
\]

and let \( \text{SO}(10, 1) \) denote the subgroup of \( \text{SL}(\mathbb{R}^{2+1} \oplus \mathbb{O}) \) that preserves this inner product. This group still has two components of course, but only the identity component \( \text{SO}^+ (10, 1) \) will be of interest here. Let \( \rho : \text{Spin}(10, 1) \to \text{SO}^+ (10, 1) \) be the homomorphism whose induced map on Lie algebras is given by the isomorphism

\[
\rho' \left( \begin{pmatrix} a_1 + x I_8 & C R_x & y I_8 & C R_y \\ -C L_x & a_3 + x I_8 & C L_y & -y I_8 \\ z I_8 & C R_z & a_1 - x I_8 & C R_x \\ C L_z & -z I_8 & -C L_x & a_3 - x I_8 \end{pmatrix} \right) = \begin{pmatrix} 2x & y & 0 & \mathbf{y}^* \\ 2z & 0 & 2y & 2 \mathbf{y}^* \\ 0 & z & -2x & \mathbf{z}^* \\ 2 \mathbf{z} & -2 \mathbf{z} & 2 \mathbf{y} & a_2 \end{pmatrix}.
\]

With these definitions, the squaring map \( \sigma \) is seen to have the equivariance \( \sigma(g z) = \rho(g) \sigma(z) \) for all \( g \) in \( \text{Spin}(10, 1) \) and all \( z \in \mathbb{O}^4 \).

With these definitions, the polynomial \( p \) has the expression

\[
p(z) = -\frac{1}{4} \sigma(z) \cdot \sigma(z),
\]

from which its invariance is immediate. Moreover, it follows from this that the squaring map carries the orbits of \( \text{Spin}(10, 1) \) to the orbits of \( \text{SO}^+ (10, 1) \) and that the image of \( \sigma \) is the union of the origin, the forward light cone, and the future-directed timelike vectors.

6. \( \text{Spin}(10, 2) \). It might be tempting to conjecture that \( \text{Spin}(10, 1) \) could be defined directly as the stabilizer of \( p \). However, this is not the case, as the stabilizer of \( p \) is larger. One can see this directly by looking at the alternative expression

\[
p = |x_1 \wedge x_2|^2 + |y_1 \wedge y_2|^2 - 2 (x_1 \cdot x_2) (y_1 \cdot y_2) + 2 (x_1 y_1) \cdot (x_2 y_2),
\]

which makes it evident that \( p \) is invariant under the 6-dimensional Lie group

\[
G = \left\{ \begin{pmatrix} a I_8 & 0 & b I_8 & 0 \\ 0 & a' I_8 & 0 & b' I_8 \\ c I_8 & 0 & d I_8 & 0 \\ 0 & c' I_8 & 0 & d' I_8 \end{pmatrix} \middle| ad-bc = \pm 1 \right\}.
\]
Since $G$ does not lie in $\text{Spin}(10, 1)$, the invariance group of $p$ must be properly larger than $\text{Spin}(10, 1)$.

In particular, consider the $G$-subgroup $\mathbb{R}' \simeq \mathbb{R}^+$ consisting of matrices of the form

\[
\begin{pmatrix}
  t I_8 & 0 & 0 & 0 \\
  0 & t^{-1} I_8 & 0 & 0 \\
  0 & 0 & t^{-1} I_8 & 0 \\
  0 & 0 & 0 & t I_8
\end{pmatrix}
\]

where $t > 0$,

which is not a subgroup of $\text{Spin}(10, 1)$. Let $\mathfrak{r}'$ denote its Lie algebra. Calculation shows that

\[
\mathfrak{r}' \oplus [\mathfrak{spin}(10, 1), \mathfrak{r}'] = \left\{ \begin{pmatrix}
  w I_8 & C R_w & u I_8 & 0 \\
  C L_w & -w I_8 & 0 & u I_8 \\
  v I_8 & 0 & -w I_8 & -C R_w \\
  0 & v I_8 & -C L_w & w I_8
\end{pmatrix} \middle| u, v, w \in \mathbb{R}, \quad w \in \mathbb{O} \right\}
\]

and that the sum $\mathfrak{spin}(10, 1) \oplus \mathfrak{r}' \oplus [\mathfrak{spin}(10, 1), \mathfrak{r}']$ is closed under Lie bracket. Thus, this defines a Lie algebra of dimension 66 that lies in the stabilizer of $p$.

The details of further analysis will be omitted, but by using arguments similar to those used in previous sections, one sees that this algebra is isomorphic to $\mathfrak{so}(10, 2)$ and that the connected Lie subgroup of $\text{GL}_R(\mathbb{O}^4)$ whose Lie algebra is this one is simply connected, so that it this group is $\text{Spin}(10, 2)$. Henceforth, this algebra will be denoted $\text{spin}(10, 2)$. Thus,

\[
\text{spin}(10, 2) = \left\{ \begin{pmatrix}
  a_1 + x I_8 & C R_w & y I_8 & C R_y \\
  C L_x & a_3 + w I_8 & C L_y & u I_8 \\
  z I_8 & C R_z & a_1 - x I_8 & -C R_x \\
  C L_z & v I_8 & -C L_w & a_3 - w I_8
\end{pmatrix} \middle| u, v, w, x, y, z \in \mathbb{R}, \quad w, x, y, z \in \mathbb{O}, \quad a \in \mathfrak{spin}(8) \right\}.
\]

Moreover, representation theoretic methods show that the only connected proper subgroup of $\text{SL}_R(\mathbb{O}^4)$ that properly contains $\text{Spin}(10, 2)$ is $\text{Sp}(16, \mathbb{R})$, the symplectic group preserving the symplectic 2-form $\Omega$ defined by

\[
\Omega = dx_1 \wedge dx_2 + dy_1 \wedge dy_2.
\]

Of course, $\text{Sp}(16, \mathbb{R})$ does not stabilize any nonzero polynomials. It follows that $\text{Spin}(10, 2)$ is the identity component of the stabilizer of $p$, and hence that the stabilizer of $p$ must lie in the normalizer of $\text{Spin}(10, 2)$ in $\text{GL}_R(\mathbb{O}^4)$. However, this normalizer is just $\mathbb{R}^+ \cdot I_{32} \times \text{Spin}(10, 2)$ and the only element in $\mathbb{R}^+ \cdot I_{32}$ that stabilizes $p$ is the identity element. It follows that $\text{Spin}(10, 2)$ is the stabilizer of $p$.

7. $\text{Spin}(9, 1)$. As a final note, inspection reveals that the subalgebra

\[
\text{spin}(9, 1) = \left\{ \begin{pmatrix}
  a_1 + x I_8 & C R_w \\
  C L_x & a_3 - x I_8
\end{pmatrix} \middle| x \in \mathbb{R}, \quad w, x \in \mathbb{O}, \quad a \in \mathfrak{spin}(8) \right\} \subset \mathfrak{sl}(16, \mathbb{R}),
\]

which contains $\mathfrak{spin}(9)$, is actually the Lie algebra of a faithful representation of $\text{Spin}(9, 1)$ on $\mathbb{R}^{16} \simeq \mathbb{O}^2$. This action of $\text{Spin}(9, 1)$ has the interesting feature that it has only two orbits: The origin and the set of all non-zero vectors. This follows because the compact group $\text{Spin}(9) \subset \text{Spin}(9, 1)$ already acts transitively on the unit spheres, but the larger group does not even preserve the quadratic form.