Kinematic analysis of a generalized Cardanic joint

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Abstract. The necessity of transmitting the rotation motion between two shafts is commonly met in mechanical engineering. The most frequent solutions are represented mechanical systems known as couplings. The coupling solutions presented in literature depend on two main features: the relative position between the shafts and the variation of the transmission ratio. One of the most popular couplings is the Cardanic joint meant to transmit motion between intersecting axes. From technological point of view, to obtain the concurrence of all axes of the joints of the coupling it is a very difficult task. To overcome this aspect, it is assumed that the axes of the shafts are not intersecting and from the four joints of the Cardanic coupling, only the input joint remains a rotation one; the other three joints transform into cylindrical joints and thus allow for relative linear displacement of the elements besides rotation. In this manner, the new mechanism is a RCCC one. The paper presents the positional analysis of the mechanism applying the method of homogenous operators proposed by Hartenberg and Denavit. The analytical expressions of all displacements from the joints of the mechanism are obtained.

The aspects referring to the obscurity occurring when inverse trigonometric functions are involved in describing the displacements and the way to avoid these are discussed.

1. Introduction

The main task of a mechanism is to transmit motion and power from the driving element to the final one under well stipulated conditions [1]. The problem of transmission of rotation motion between two shafts is often met in technology [2]. When a rigorously constant transmission ration is required, the gear mechanisms are used [3], [4]. When the constant transmission ratio is not a stringent necessity, the use of linkage mechanisms [5-7] which are more reliable and economical than gears is a handy solution.

The kinematical study of a linkage mechanism supposes establishing the position of the mechanism for a given position of the driving elements [8]. This means that for stipulated position parameters of the driving elements, all displacements from the rest of kinematical pairs of the mechanisms are required. Every method from technical literature is based on the condition of closed kinematical chain from the structure of the mechanism [9], requirement imposed by the definition of a mechanism itself. The condition of closed kinematical chain is founded on the manner of coordinate transformation of a point using a set of coordinate systems attached to the elements of the chain, chosen in such way that the first frame coincides to the last one. The condition of closed kinematical chain is expressed in a manner that depends on the structure of the mechanism and the solutions depend on the mathematical apparatus involved.
2. Methods applied in spatial kinematical analysis

For plane mechanisms [10], the condition of closed kinematic chain can be expressed in vector format or using complex numbers. For spherical mechanisms [11], the closed loop condition is formulated via 3x3 type rotation matrices. The closing equation has the form:

\[
R_{1k}R_{2k} \ldots R_{n+1} = I_3
\]  

(1)

where \( R_{k+1} \) is the rotation matrix that brings the frame \((k)\) over the frame \((k+1)\) and \( I_3 \) is the unit matrix of third order. The general from of a rotation matrix which describes the rotation of \( \varphi \) angle around an axis of \( e \) vector is, according to Angeles [12]:

\[
R = ee^T + (I_3 - ee^T) \cos \varphi + E \sin \varphi
\]

(2)

where \( E \) is the anti-symmetric matrix attached to the \( e \) versor. The condition to be fulfilled by two rotation matrices in order to be identical is to have the angles and the versors of the axes, respectively, identical. From here it results the conclusion that from the nine scalar equations provided by equation (1) only three are independent. Another extremely useful instrument in expressing the spatial rotations is the quaternions algebra, [13]. For the case of spatial mechanisms the closing condition has a form identical to the relation (1) [14].

\[
T_{1k}T_{2k} \ldots T_{n+1} = I_4
\]

(3)

where \( T_{k+1} \) is the displacement matrix that superposes the \((k)\) frame over the \((k+1)\) frame and having the general form:

\[
T_{k+1}^{k} = \begin{bmatrix}
R_{k+1} & d_{k+1} \\
0 & 1
\end{bmatrix}
\]

(4)

where \( R_{k+1} \) has the same significance as in equation (1), \( d_{k+1} \) is the vector that moves the origin \( O_k \) over \( O_{k+1} \) origin and \( I_4 \) is the unit matrix of fourth order. It is easily noticed that the relation (3) supposes the identity between the rotation matrices and displacement vectors of the matrices from both members of equation. Thus, a system of six scalar equations actually results. The kinematic study of spatial chains was substantially simplified by Hartenberg and Denavit [15] who proved that for a kinematic chain with lower cylindrical pairs, the number of necessary parameters needed for stipulation of the relative position of two elements may be reduced from six to four by the suitable selection of the position and orientation of the axes of the frames attached to the chains’ elements. To complete this operation, it is required that the axes of the cylindrical pairs should be chosen as \( Ox \) axes of the frames and each of the \( Ox \) axis should be along the common normal of two neighboring axes, as in Figure 1, [15]. With the axes chosen in this manner, the \((k)\) coordinate system can be superposed over the \((k+1)\) frame by a roto-translation of parameters \( \theta_k \) and \( s_k \) followed by a roto-translation of parameters \( \alpha_{k+1} \) and \( a_{k+1} \).

\[
T_{k+1}^{k} = \begin{bmatrix}
R_{k+1} & d_{k+1} \\
0 & 1
\end{bmatrix}
\]

(5)

The two displacement vectors are expressed under the form:

\[
\begin{align*}
\{ d_{k+1}^{k} \} &= \begin{bmatrix} 0 \\ s_k \end{bmatrix} \\
\{ d_{k+1}^{k} \} &= \begin{bmatrix} a_{k+1} \\ 0 \end{bmatrix}
\end{align*}
\]

(6)
In order to obtain the rotation matrices from relation (1), this relation is applied for \( \mathbf{e} = \mathbf{k} \), \( \phi = \theta_k \) and \( \mathbf{e} = \mathbf{i} \), \( \phi = \alpha_{k,k+1} \) respectively, and it results:

\[
\mathbf{R}^z_{k,k+1} = \begin{bmatrix}
\cos \theta & -\sin \theta & 0 \\
\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

and:

\[
\mathbf{R}^x_{k,k+1} = \begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \alpha_{k,k+1} & -\sin \alpha_{k,k+1} \\
0 & \sin \alpha_{k,k+1} & \cos \alpha_{k,k+1}
\end{bmatrix}.
\]

The following notations are introduced for a simpler writing manner:

\[\sin \alpha = S\alpha\]
\[\cos \alpha = C\alpha\]

The \( \mathbf{T}_{k,k+1} \) operator has the matrix:

\[
\mathbf{T}_{k,k+1} = \begin{bmatrix}
C\theta_k - S\theta_k & 0 & 0 \\
S\theta_k & C\theta_k & 0 \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
1 & 0 & 0 & a_{k,k+1} \\
0 & C\alpha_{k,k+1} & -S\alpha_{k,k+1} & 0 \\
0 & S\alpha_{k,k+1} & C\alpha_{k,k+1} & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

Another especially useful tool in kinematical analysis of spatial mechanisms is the dual numbers concept introduced by Ball and Clifford [16]. Similarly to complex numbers use, they propose the employment of numbers having the form:

\[\hat{x} = x + \epsilon_0\]

acknowledged as dual numbers where \( \epsilon \) is the imaginary unit defined by the Phillips equality [17]:

\[\epsilon^2 = 0\]
Based on the Kotelnikov transfer principle [16] that states: all relations deduced within spherical trigonometry remain valid if the real numbers are replaced by dual numbers, any spatial mechanism may be studied using a spherical mechanism. Founded on this remark, Yang [18] analyses the spatial mechanisms with five elements employing matrices of 3x3 type with dual numbers. It is also Yang who studies spatial mechanisms applying dual quaternions.

3. The kinematical analysis of the proposed mechanism

Next, the positional analysis of generalized Cardanic mechanism is presented. The mechanism of the Cardanic transmission presented in Figure 2 is a spherical one.

![Figure 2. Cardanic transmission](image-url)

The structure of the mechanism contains four rotation pairs whose axes should be theoretically concurrent in a point. This requirement is extremely restrictive and therefore, from technological point of view, special tasks must be fulfilled. To avoid these complications, the kinematical pairs of the mechanism (excluding the driving pair) are transformed into cylindrical pairs. Thus, the resulting mechanism is a $RCCC$ spatial one, with the degree of mobility 1, and which doesn’t require the concurrence of the axes of two pairs. For the mechanism obtained in the above manner, applying a series of particular values for the constructive parameters, the positional analysis made by way of the Hartenberg-Denavit method is presented next. The general case of the analysis of the $RCCC$ mechanism was completed by Yang who applies as manner of study the dual quaternions and by Fischer who studies the kinematics of the mechanisms using matrices with dual elements. The generalized Cardanic mechanism presented in Figure 2 consists in four elements: the driving element 1, the intermediary coupling element 2 that has perpendicular (without being a strict requirement) but not intersecting arms, the final element 3 and the ground, 4. The elements 1 and 3 are identical as shape but they distinguish by the joint they have with the ground. Thus, while the element 1 and the ground 4 complete a rotation pair, $R$, the driven element 3 and the ground 1 make a cylindrical pair, $C$. The intermediate element 2 creates cylindrical pairs with the driven element and with the driving element respectively, and for this reasons the mechanism’s name - $RCCC$, is, from structural point of view, justified. For the kinematical analysis of the mechanisms, the Hartenberg-Denavit methodology is applied. To this purpose, the $O_z$ axes of the pairs are chosen with the orientation presented in Figure 2. Once the $O_z$ axes are stipulated, the $O_{\alpha_{k,k+1}}$ axes are chosen as to superpose the current axis $O_{z_k}$ aver the next $O_{z_{k+1}}$ axis by a screw motion with the $\alpha_{k,k+1}$ rotation in trigonometric sense, as shown in Figure 2. The motions of the mechanism will be described by the displacements from the three cylindrical pairs (three rotations $\theta_2$, $\theta_3$ and $\theta_4$ together to the translations $s_2$, $s_3$ and $s_4$) expressed as functions of the position of the driving element $\theta_1$. The constructive characteristics of the mechanism will be represented by:
\[ a_{12} = a_{23} = a_{34} = \pi/2 \quad (13) \]

and

\[ a_{12} = a_{34} = 0. \quad (14) \]

Figure 3. Generalized Cardanic mechanism

\[ Z(\theta_1, s_1)X(a_{12}, a_{12})Z(\theta_2, s_2)X(a_{23}, a_{23})Z(\theta_3, s_3)X(\alpha_{34}, \alpha_{34})Z(\theta_4, s_4)X(\alpha_{41}, \alpha_{41}) = I_4 \quad (15) \]

where \( Z(\theta, s) \) and \( X(\alpha, a) \) are the displacements with respect to \( z \) axis and \( x \) axis respectively, applying the equations (7) and (8) and \( I_4 \) is the unit matrix of fourth order. It is obvious that by replacing the equations (1) and (2), the concrete form of the matrix equation (3) will be considerably simplified. Though, to solve this equation is not an easy task. McCarthy recommends that the equation (15) should be written under the form:

\[ Z(\theta_1, s_1)X(a_{12}, a_{12})...Z(\theta_4, s_4)X(\alpha_{41}, \alpha_{41})^{-1} \quad (16) \]

without stipulating the point where the two terms from the initial equation must be separated. In the present case, the McCarthy method was applied by writing the equation (16) in the following manner:

\[ Z(\theta_1, s_1)X(a_{12}, a_{12})Z(\theta_2, s_2)X(a_{23}, a_{23}) = [Z(\theta_3, s_3)X(\alpha_{34}, \alpha_{34})Z(\theta_4, s_4)X(\alpha_{41}, \alpha_{41})]^{-1} \quad (17) \]
where $O_4$ is the zero matrix of $4 \times 4$ type. The matrix equation (15) assumes the annulations of the rotation matrix and of the displacement vector from the matrix on the left member of equation (15). The displacement vector doesn’t create problems as it means the simultaneous cancellation of the first three elements from the last column of the left member matrix, but concerning the rotation matrix, it is characterized by 9 elements (the $3 \times 3$ type matrix from upper left).

\[
\begin{bmatrix}
C\theta_1 C\theta_2 - C\theta_3 C\theta_4 & S\theta_1 C\theta_2 + C\theta_3 S\theta_4 a & S\theta_1 S\theta_2 + S\theta_3 S\theta_4 a & -a_23 - a_4 S\theta_2 a - a_4 C\theta_3 C\theta_4 - a_4 S\theta_3 \\
S\theta_1 - S\theta_3 C\theta_4 & -C\theta_1 + S\theta_3 S\theta_4 a & -S\theta_1 S\theta_4 S\theta_4 a + C\theta_3 C\theta_4 & -a_23 S\theta_2 a - a_4 C\theta_3 C\theta_4 + a_4 S\theta_3 \\
C\theta_1 S\theta_2 - S\theta_4 & S\theta_1 S\theta_2 - C\theta_4 C\theta_4 a & -C\theta_2 + C\theta_4 S\theta_4 a & s_1 C\theta_2 - a_4 S\theta_4 a - s_3
\end{bmatrix} = O_4
\]

(18)

From the nine equations obtained, only three are independent. A rigorous solving assumes separation of the symmetrical parts from the two matrices:

\[
R^i = \frac{R + R^T}{2}
\]

(19)

which permits finding the angle of rotation and the disjoining of the asymmetrical parts:

\[
R^a = \frac{R - R^T}{2}
\]

(20)

which permits finding the versor of the axis of rotation.

For the current case, a different technique will be used consisting in the convenient selections of the three equations from which the angles $\theta_2$, $\theta_3$ and $\theta_4$ should be found. The elements from the positions (3,1) and (3,2) are equaled to zero and the following system results:

\[
\begin{align*}
C\theta_1 S\theta_2 - S\theta_4 &= 0 \\
S\theta_1 S\theta_2 - C\theta_4 C\theta_4 a &= 0
\end{align*}
\]

(21)

that allows for the $\theta_4$ to be found.

\[
\theta_4 = atan\{C\theta_4 a / S\theta_1 \}
\]

(22)

The elements from the positions (3,2) and (3,3) are equaled to zero:

\[
\begin{align*}
S\theta_1 S\theta_2 - C\theta_4 C\theta_4 a &= 0 \\
-C\theta_2 + C\theta_4 S\theta_4 a &= 0
\end{align*}
\]

(23)

and it is found:

\[
\theta_2 = atan\{tan\theta_2 / (S\theta_4 C\theta_1)\}
\]

(24)

The elements from the positions (1,1) and (2,1) are zero:

\[
\begin{align*}
C\theta_1 C\theta_2 - C\theta_3 C\theta_4 &= 0 \\
S\theta_1 - S\theta_3 C\theta_4 &= 0
\end{align*}
\]

(25)

and it results:

\[
\theta_3 = atan\{tan\theta_3 / C\theta_2 \}
\]

(26)

The displacements from the cylindrical pairs are obtained straightforward:
\[s_3 = s_1C\theta_2 - a_{41}S\theta_4\]
\[s_4 = \frac{a_{41}C\theta_3C\theta_4 - (a_{22} + s_1S\theta_2)}{S\theta_3}\]
\[s_2 = s_4C\theta_3 - a_{41}S\theta_3C\theta_4\]

(27)

Next it will be shown that the employment of the function \(\text{atan}(x)\) in the expressions of the rotations (22), (24) and (26) may generate errors in the description of the kinematics of the mechanism. Instead of function \(\text{atan}(y/x)\), the use of the function \(\text{angle}(x,y)\), existing in any calculus utilitarian, is proposed; this function returns the value of the angle from the domain \([0, 2\pi]\) made by the vector radius of the point of coordinates \((x, y)\) with the positive half-axis \(Ox\). The expressions of the rotation angles from the cylindrical pairs, as dependencies only of the angle of the driving element \(\theta_1\), are presented next:

\[\theta_2 = \text{angle}(S\alpha_{41}S\theta_1, C\alpha_{41})\]
\[\theta_3 = \text{angle}(S\alpha_{41}C\theta_1, \sqrt{S^2\theta_1 + C^2\alpha_{41}C^2\theta_1})\]
\[\theta_4 = \text{angle}(S\theta_1, C\alpha_{41}C\theta_1)\]

(28)

4. Discussions

The variation of the linear displacement from the pair between the driven element and the ground is presented in Figure 4, with continuous line, found by means of relations (28), with broken line, found by using the relations (22), (24) and (26). One can observe that, the two plots coincide only for the first quarter of rotation of the driving element and differ from the rest. At a first sight it should seem that it is only a vertical translation, on portions, of the graph with continuous line.

![Figure 4. The translation of the driven element calculated in two ways](image)

If the graphs from Figure 4 are parts of the same continuous function, translated vertically on intervals, their derivatives must coincide. In Figure 5 there are presented the derivatives of the two functions from Figure 4. It can be observed that the plots are identical only for the portion on which the graphs from Figure 4 coincide. This fact confirms the statement that the use of the function \(\text{atan}(x)\) generates errors in the kinematical analysis of the mechanism. To clarify this aspect, one must remark that the expressions of linear displacements from the cylindrical pairs depend on the rotation angles \(\theta_2\), \(\theta_3\) and \(\theta_4\) as arguments of the trigonometric functions \(\cos(x)\) and \(\sin(x)\).
The following simple functions are considered next in order to explain the effect of applying the $\text{atan}(x)$ function to the above mentioned ones:

\[
\begin{align*}
    f(\phi) &= \sin(\text{angle}(\cos(\phi),\sin(\phi))) \\
    f'(\phi) &= \sin(\text{atan}(\sin(\phi)/\cos(\phi)))
\end{align*}
\]  

(30)

Theoretically, these expressions should represent the same function or at most one of them could be vertically translated on portions but in the end their derivatives should be equal.

A similar situation arises for the $\cos(x)$ function. It can be concluded that, applying the trigonometric functions sine and cosine to the functions $\text{atan}(y/x)$ and $\text{angle}(x,y)$ different results are obtained. Finally, using the relations (28), the rotation velocities and displacement velocities from cylindrical pairs were found and there are shown in Figure 8 and Figure 9 respectively. Both the rotation velocity and the translation velocity from the pair between the final element and the ground
are the result of the superposition of a periodic component over a constant component, the frequency of the periodic component being twice the frequency of the rotation of the driving element. The law of angular velocity variation is identical to the one of a Cardanic mechanism [2].

Figure 8. The rotation velocities from cylindrical pairs

Concerning the motions from the intermediate pairs, from Figure 8 it possibly will say that the rotations from these pairs are the same only delayed with a quarter of period and of opposite signs. However the plots of the translation velocities disagree with this hypothesis. For a true such hypothesis, the plot $s_3 = s_3(s_2)$ should be a symmetrical shape with respect to coordinate axes. The graph from Figure 10 confirms that the motions from the two pairs present different characteristics.

Figure 9. The linear velocities from cylindrical pairs

Figure 10. The trajectory of the center of the coupling element
5. Conclusions

The paper presents the kinematical analysis of the spatial mechanism obtained from a Cardanic joint, which is a spherical mechanism, by transforming the rotation pairs into cylindrical pairs. The new mechanism presents the advantage that transmits the rotation motion identically to the spherical Cardanic mechanism. But, compared to it, the condition of intersection of the axes of all pairs, which is difficult to ensure in applications, must not be strictly obeyed.

After a short review of the methods applied in spatial analysis, the method of homogenous operators proposed by Hartenberg and Denavit is chosen as technique for the analysis of the mechanism. The analytical expressions for all the displacements from the pairs of the mechanism as functions of the rotation angle of the driving element are resulting from the analysis.

In the final part of the work, the errors produced by the use of multiform function $atan(x)$ are highlighted and necessity of replacing this function with the uniform function $angle(x, y)$ is discussed.

The variations of the rotation and translation velocities from all the pairs of the mechanism are presented and the remark that in the final pair the motion is helicoidally periodic with the period half of the period of the driving element is made.

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