Nucleon Decay with Domain-Wall Fermions

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We report on our on-going project to calculate the nucleon decay matrix elements with domain-wall fermions. Operator mixing is discussed employing a non-perturbative renormalization. Bare matrix elements of all the possible decay modes induced by the dimension-six operators are calculated with the direct method, which are compared with the indirect calculation using chiral perturbation theory.

1. INTRODUCTION

There are two methods to calculate nucleon decay matrix elements using lattice QCD. One is the direct calculation of the matrix elements dealing with three- and various two-point functions \cite{1,2,3}. The other is the indirect method where one relies on chiral perturbation theory whose low energy constants are calculated on the lattice \cite{4,5,1,2,3}. The former method is preferable but requires huge computation, typically 10 times larger than the latter to achieve similar statistical accuracy.

In a previous work \cite{3}, we made use of good chiral symmetry of the domain-wall fermion with the DBW2 gauge action \cite{6}. The proton decay matrix element of the pion final state was calculated with both methods. The results are consistent with each other within the relatively large error. There we assumed the mixing of operators arising from the explicit chiral symmetry breaking is negligible. Here we reconsider this issue by calculating the mixing with a non-perturbative renormalization (NPR). Also the matrix elements of the proton decay in case of the non-degenerate quark mass in the pseudoscalar state are calculated allowing further discussion on the effectiveness of the chiral perturbation theory.

All of our calculations are done on the quenched DBW2 gauge configurations with size $N^3_r \times N^\tau = 16^3 \times 32$. The domain-wall height is $M = 1.8$, the fifth dimension size is $L_s = 16$ for the NPR and 12 for the matrix element.

2. OPERATOR MIXING

The nucleon decay is induced by the dimension-six operator which consists of one lepton and three quarks. The three quark operator, with which we calculate matrix element with initial nucleon and final pseudoscalar state, in general takes a form as

$$O^{a}_{uds} = \epsilon^{ijk}(u^T \overline{C} \Gamma d^j)\Gamma^s s^k.$$ (1)

The superscript $a$ specifies a combination of the gamma matrix $(\Gamma \Gamma^\prime)$ listed in Table 1. Operators are classified by the ordinary parity ($P$) and by the parity of switching ($S$) $u$ and $d$ quarks. The $S$ is a symmetry of the (lattice) QCD Lagrangian when quark masses are degenerate. The $u$, $d$, $s$ do not necessarily mean the real flavor, but simply stand for the different flavors. The other arrangements of flavors, such like $O^{a}_{usd}$ and $O^{a}_{das}$ are always expressed in terms of $O^{a}_{uds}$ by Fierz rearrangement and/or by $S$. Operators with two flavors like $O^{a}_{uda}$ follows the same argument. Lattice operators are renormalized as

$$O^{a}_{ren} = Z^{ab}O^{b}_{latt}.$$ (2)

For example, for operators in $P^+$, $S^-$ sector, which will not mix the operators in the other sectors, $a$ and $b$ in eq. 2 may be $SS$, $PP$, $AA$. The
same 3×3 matrix Z applies to the \( P^- \), \( S^- \) sector: \( SP, PS, −AV \), too.

For the NPR with RI-MOM scheme \( 7,8 \) we calculate the Green’s function of the operator with the three quark external states in the Landau gauge,

\[ G^a(x_0, x_1, x_2, x_3) = \langle \mathcal{O}^a_{ads}(x_0)\bar{u}(x_1)\bar{d}(x_2)s(x_3) \rangle. \]  

We set \( x_0 = 0 \). Fourier transformation is performed on the other legs with the same momentum \( p \), which are then amputated to obtain the vertex function,

\[ \Lambda^a(p^2) = \text{F.T.} \, G^a(0, x_1, x_2, x_3)|_{\text{Amp.}}. \]  

Writing the tensor indices explicitly, the renormalization condition of the RI-MOM scheme leads,

\[ P^a_{ijk} \alpha \beta \delta \gamma \cdot Z_q^{-3/2} Z^{be}_{bc} \Lambda^c_{ijk} \alpha \beta \gamma \delta = \delta^{ab}, \]  

where \( Z_q \) is the quark wave function renormalization \( i, j, k \) are color indices, \( \alpha, \beta \) and \( \gamma, \delta \) are Dirac indices associated with \( \Gamma, \Gamma' \) respectively. The renormalization condition should be applied where the pion pole effect \( 8 \) is absent and the lattice artifact is small : \( \Lambda_{QCD} \ll |p| \ll 1/a \). The projection matrix \( P^a \), for example onto \( SS \), is \( P^{SS} = \epsilon^{ijk} (C^{-1})^\beta \alpha \delta \gamma / (6 \cdot 4 \cdot 4) \). We define the matrix \( M \) as,

\[ M^{ab} = P^a_{ijk} \alpha \beta \delta \gamma \cdot \Lambda^b_{ijk} \alpha \beta \gamma \delta, \]  

which will give \( Z_q^{3/2} (Z^{-1})^{ab} \).

Fig. 1 shows the resulting \( M^{ab} \) at a quark mass \( m_f = 0.025 \) for the domain-wall fermion. 51 configurations have been analyzed \( 9 \). The renormalization factor of \( SS, SS \) and \( PP, PP \) are identical within the error except for the vicinity of zero momentum, the others are zero within the two standard deviation level. Absence of the off-diagonal element and coincidence of the \( SS, SS \) and \( PP, PP \) element are expected in the case of exact chiral symmetry. We have examined four different masses \( m_f = 0.025, 0.04, 0.055, 0.07 \), where mass dependences are hardly seen in all range of momentum except near zero.

As a result, the nucleon decay operator \( \mathcal{O}_{R/L}^B = \epsilon^{ijk} (u^T C P_{R/L} d^j) P_{L/s} [S \pm P_s - P] \) is renormalized multiplicatively without any mixing at practical level for this parameter of the domain-wall fermion. In the next section we show the results of the bare matrix elements with \( L_s = 12 \). The residual mass \( m_{res} = 1.6 \text{ MeV} \) at \( L_s = 12 \) is still comparable to \( 0.7 \text{ MeV} \) at \( L_s = 16 \). Hence the effect of \( L_s \) on the renormalization factor is expected to be absent in this case, too, which needs to be checked by further performing simulation with \( L_s = 12 \).

3. MATRIX ELEMENTS

We follow the ref. \( 2 \) to calculate the nucleon decay matrix elements by the direct method. Our results for the decay to a meson with degenerate quark mass are already shown in \( 3 \), where we have used 100 gauge configurations. We extend our calculation to the non-degenerate quark mass case to treat the decay to kaon. Our masses are \( m_{f1}, m_{f2} = 0.02, 0.04, 0.06, 0.08 \) with \( m_{f1} \leq m_{f2} \), where \( m_{f1} \) is for the \( u, d \) quarks, \( m_{f2} \) for the \( u, d \) or \( s \) quark. The other parameters are the

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**Table 1**

Operator classification by parity and switching symmetry.

| \( \mathcal{P}^+ \) | \( S^- \) | \( S^+ \) |
|-------------------|-------|-------|
| \( SS \)          | \( PP \) | \( AA \) |
| \( PP \)          | \( AV \) | \( VA \) |

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**Figure 1.** Elements of \( M \) for \( m_f = 0.025 \) as a function of momentum squared.
Figure 2. Summary $W_0$ for both direct and indirect calculations with unrenormalized operators.

same as in [3]. Result of the relevant form factor $W_0$ of the matrix element for various decay modes are shown in Fig. 2. We are assuming the SU(2) symmetry for the $u$ and $d$ quarks. There are other possible matrix elements, but they can be calculated with the matrix elements in the figure when the SU(2) symmetry is intact.

The indirect method uses the chiral perturbation theory, which relates the low energy parameters in the chiral Lagrangian to the matrix elements. JLQCD extended the calculation by Claudson et. al [10] to all possible decay modes [2]. The low energy parameters are mass ratio of nucleon and baryon with strangeness $S = -1$, $m_N/m_B$, pion decay constant $f$, $D$ and $F$ parameterizing the semileptonic decay of baryons, nucleon decay parameter $\alpha$ and $\beta$. These parameters are in principle calculable on the lattice. As we are using the chiral fermion, observables calculated on the lattice should obey the chiral perturbation whose parameters are calculated on the lattice. However we use the experimental values except for $\alpha$ and $\beta$. The reasons are that we know the results of $f_\pi$ [6] and $g_A$ [11] ($= D + F$) are anyway consistent with the experiment, and that we have no estimate of $m_B$ and individual $D$ and $F$. Using the $\alpha$ and $\beta$ in our previous study [3] and $D = 0.8, F = 0.47$ [12], $f = 0.131$ GeV, $m_N/m_B = 0.817$, the matrix elements are calculated and shown in Fig. 2 The lattice cutoff calculated by $\rho$ mass input, $a^{-1} = 1.31$ GeV is used to make dimensionless value.

The indirect method gives quite good estimate of the matrix element. Consistency of the two method within the error is in contrast to the result of JLQCD [2] with Wilson fermion. However, this could be caused by larger statistical error in our calculation. We need more statistics to be able to make more reliable statements.

4. SUMMARY

We have investigated the mixing of three-quark nucleon decay operators with domain-wall fermions by NPR. The mixing is practically negligible at the parameter set similar to that where the matrix element are calculated. We have extended our nucleon decay matrix element to those for the final kaon state matrix elements. The direct and indirect methods give consistent results within our statistical error. Performing NPR at the parameter of matrix element calculation together with the matching calculation will give us the matrix element in MS.

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