New perturbation theory of low-dimensional quantum liquids II: operator description of Virasoro algebras in integrable systems

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We show that the recently developed pseudoparticle operator algebra which generates the low-energy Hamiltonian eigenstates of multicomponent integrable systems also provides a natural operator representation for the the Virasoro algebras associated with the conformal-invariant character of the low-energy spectrum of the these models. Studying explicitly the Hubbard chain in a non-zero chemical potential and external magnetic field, we establish that the pseudoparticle perturbation theory provides a correct starting point for the construction of a suitable critical-point Hamiltonian. We derive explicit expressions in terms of pseudoparticle operators for the generators of the Virasoro algebras and the energy-momentum tensor, describe the conformal-invariant character of the critical point from the point of view of the response to curvature of the two-dimensional space-time, and discuss the relation to Kac-Moody algebras and dynamical separation.

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I. INTRODUCTION

In this paper we continue our study of a new perturbation theory for low-dimensional quantum liquids by showing that the pseudoparticle operator algebra and perturbation theory introduced in the preceding paper (Ref. [1], henceforth I) leads, in a natural way, to the correct operator description of the critical-point physics of multicomponent integrable quantum liquids [2,3,4,5,6].

Here by “critical point” we mean that regime in which conformal invariance holds in the continuum limit. Among the specific physical quantities of interest at the critical point are the velocities of the gapless modes, matrix elements of various currents coupled to experimental probes, and correlation functions for charge, spin, and other observable operators.

Considerable previous work has established that conformal invariance provides particularly powerful insight into massless field theories in two space-time dimensions [7,8,9,10] and hence into the critical-point physics of integrable quantum liquids. In particular, at the critical point these theories are characterized by the “conformal anomaly,” \( c \), which can be extracted from either [8,10]

(i) The finite-size-scaling behavior of the ground-state energy; or

(ii) The stress tensor-tensor correlation function associated with the response to curvature of the two-dimensional space.

On the one hand, it was shown from the analysis of finite-size-scaling results of these systems [4,5] that each gapless excitation corresponds to one Virasoro algebra with conformal anomaly \( c = 1 \) and that the complete critical theory is given as a direct product of the Virasoro algebras.

On the other hand, method (ii) has not been applied to multicomponent models solvable by Bethe ansatz (BA), since conformal-field theory refers to Lorentz-invariant systems only. In the multicomponent quantum liquids there are several different “light velocities” and it is not immediately clear how we can take all these velocities to be simultaneously equal to one. One of our tasks in this article will be to clarify this particular point.
For definiteness, as in I we focus our discussion on the Hubbard chain in a non-zero chemical potential and an external magnetic field, which corresponds to a two component quantum liquid. Further, again as in I, we limit our study to the $U(1) \otimes U(1)$ sector of parameter space where the low-energy physics is dominated by lowest-weight states (LWS) of both the $\eta$ spin and spin algebras [11]. Although our explicit calculations are for the Hubbard model, our general results and approach apply to multicomponent integrable systems [2,3,4,5,6].

The critical-point energy spectrum of the Hubbard chain in a magnetic field has been obtained previously by combining the BA solution, finite-size studies, and conformal-field theory [5,12]. Further, the asymptotic behavior of correlation functions and the corresponding critical exponents in the case of multicomponent-integrable quantum liquids have also been obtained [5].

Our goal in this article is to extend these results – which have dealt with energies and expectation values only – to a full operator representation of the critical-point theory. In particular, we will show that the perturbative character of the pseudoparticle basis studied in I permits full, explicit evaluation of the properties of the critical point and in particular allows the operator description of the Virasoro algebras of integrable quantum liquids. This permits us to construct, and to calculate using simple operator forms, the stress tensor and generators of the Virasoro algebras [6] and to characterize the highest-weight states (HWS) [6,8,9,10] of these algebras; such states correspond to the “primary fields” [9,10].

Before outlining our approach in this paper, let us recall briefly the crucial results from I. As in the Landau Fermi-liquid approach, we use the interacting ground state as the reference state [13,14]. Importantly, in the pseudoparticle basis, this reference state has the form of a simple Slater determinant and further the pseudoparticle interaction $f$ functions are, in general, non-singular. This allows the introduction of a well-defined pseudoparticle perturbation theory. Since the one-particle spectral function is fully incoherent – i.e., $Z_F = 0$ – the system is not a Fermi liquid but has what we have termed a “Landau-liquid” character [15,16,17,18,19,20].
In the pseudoparticle basis, the low-energy physics is fully determined by two-pseudoparticle (zero-momentum) forward scattering. Near the pseudo-Fermi points we can classify the few types of pseudoparticle scattering processes which are left over. In order to describe the critical-theory spectrum we construct an effective Hamiltonian containing only the relevant two-pseudoparticle scattering terms of the full Hamiltonian (see Eq. (1) below). This is done by linearizing the pseudoparticle bands of the Hamiltonian in the pseudoparticle basis and considering only the values of the two-pseudoparticle interaction at the pseudo-Fermi points. Hence, the critical-point Hamiltonian is constructed directly in the pseudoparticle basis, in contrast to the usual procedure of deriving the critical-point Hamiltonian in the electronic basis and normal-ordered relative to the electronic non-interacting ground state \[21\].

The remainder of the paper is organized into five additional sections. In Sec. II we construct the critical-point Hamiltonian and show that the second-order pseudoparticle perturbation theory leads to the energy spectrum of conformal-field theory. In Sec. III we describe the conformal-invariant character of the multicomponent integrable quantum liquids from the point of view of the response to the curvature of the two-dimensional space-time. Further, we use the pseudoparticle operator basis to show that at the critical point the energy-momentum tensor operator decouples into two (or \(\nu\), for the general multicomponent case) new tensors which act on orthogonal Hilbert subspaces. Each gapless excitation branch corresponds to an independent Minkowski space (each with common space and time but a different “light” velocity). The Lorentz-invariance in each of these spaces is associated with independent Virasoro algebras and two (or \(\nu\)) related affine-Lie algebras (Kac-Moody algebras) \[6,9,10\]. We also introduce the pseudoparticle field theory, which describes the quantum liquid at low energy and small momentum. In Sec. IV we write in operator form the generators of the Virasoro algebras. The treatment of the associated affine-Lie algebras and of the dynamical separation is presented in Sec. V. Finally, Sec. VI contains the discussion and concluding remarks.
II. CRITICAL-POINT EFFECTIVE HAMILTONIAN

In this section we use the perturbative character of the pseudoparticle basis to derive a critical-point Hamiltonian. This is constructed from the Hamiltonian in the pseudoparticle basis normal-ordered with respect to the ground state of the many-particle problem. As shown in I, the normal-ordered Hamiltonian has an infinite number of terms and is given by

\[ : \hat{H} := \sum_{i=1}^{\infty} \hat{H}^{(i)} , \]  

(1)

where the first- and second-order pseudoparticle scattering terms read

\[ \hat{H}^{(1)} = \sum_{q,\alpha} \epsilon_\alpha(q) : \hat{N}_\alpha(q) : , \]  

(2)

and

\[ \hat{H}^{(2)} = \frac{1}{N_a} \sum_{q,\alpha,q',\alpha'} \frac{1}{2} f_{\alpha\alpha'}(q,q') : \hat{N}_\alpha(q) :: \hat{N}_{\alpha'}(q') : , \]  

(3)

respectively. Here \( \epsilon_\alpha(q) \) are the \( \alpha \) pseudoparticle bands \[16,18\] and the four (or, in the general case, \( \nu \times \nu \)) “Landau” \( f \) functions have the universal form

\[ f_{\alpha\alpha'}(q,q') = 2\pi v_\alpha(q) \Phi_{\alpha\alpha'}(q,q') + 2\pi v_{\alpha'}(q') \Phi_{\alpha\alpha'}(q',q) + \sum_{j=\pm1} \sum_{\alpha''=c,s} 2\pi v_{\alpha''} \Phi_{\alpha''\alpha}(jq_Fa'', q) \Phi_{\alpha''\alpha'}(jq_Fa'', q') , \]  

(4)

where the pseudoparticle group velocities are given by

\[ v_\alpha(q) = \frac{d\epsilon_\alpha(q)}{dq} ; \quad v_\alpha \equiv v_\alpha(q_Fa) , \]  

(5)

and the velocities \( v_\alpha \) play a determining role at the critical point, representing the “light” velocities which appear in the conformal-invariant expressions \[1,5,6,16\]. \( \Phi_{\alpha\alpha'}(q,q') \) is the two-pseudoparticle forward-scattering phase shift (see Eqs. (20) – (23) below and Eqs. (23) – (26) of Ref. \[18\]). The form of the normal-ordered Hamiltonian (1) – (3) is universal for the class of integrable multicomponent quantum liquids \[1\].

5
The ground state associated with a canonical ensemble of $(\eta_z, S_z)$ values (here $\eta_z$ and $S_z$ are the eigenvalues of the diagonal generator of the $\eta$ spin and spin $SU(2)$ algebras, respectively – see Eqs. (9) and (10) of paper I) has the form 

$$|0; \eta_z, S_z\rangle = \prod_{\alpha=c,s} \prod_{q=q_{F\alpha}} [b_{q\alpha}^\dagger |V\rangle],$$

(6)

where the operator $b_{q\alpha}^\dagger$ creates one $\alpha$ pseudoparticle of pseudomomentum $q$, $|V\rangle$ is the $n = 0$ electronic and pseudoparticle vacuum and $q_{F\alpha}$ are the pseudo-Fermi points. When the number of $\alpha$ pseudoparticles, $N_\alpha$, is odd (even) and the quantum numbers \[ I^\alpha \] are integers (half integers) these are symmetric and given by 

$$q_{F\alpha}^{(+)} = -q_{F\alpha}^{(-)} = \frac{\pi}{N_a} [N_\alpha - 1],$$

(7)

whereas when $N_\alpha$ is odd (even) and $I^\alpha_j$ are half integers (integers) we have that 

$$q_{F\alpha}^{(+)} = \frac{\pi}{N_a} N_\alpha, \quad -q_{F\alpha}^{(-)} = \frac{\pi}{N_a} [N_\alpha - 2],$$

(8)

or

$$q_{F\alpha}^{(+)} = \frac{\pi}{N_a} [N_\alpha - 2], \quad -q_{F\alpha}^{(-)} = \frac{\pi}{N_a} N_\alpha .$$

(9)

(Similar expressions are obtained for the pseudo-Brioullin zones limits $q^{(\pm)}_{\alpha}$ if we replace in Eqs. (12) – (14) $N_\alpha$ by $N^*_\alpha$.) In many expressions the quantities (7) – (9) are replaced by the leading order term, $q_{F\alpha}^{(\pm)} \approx \pm q_{F\alpha}$, where $q_{F\alpha}$ is given by 

$$q_{F\alpha} = \frac{\pi N_\alpha}{N_a} .$$

(10)

As discussed in detail in I, in the $U(1)$ $\otimes$ $U(1)$ sector of parameter space and for energy scales smaller than the gaps of the non-LWS multiplets and LWS II \[ [22,23,24], \] the Hilbert space of the Hamiltonian in (1) – (3) coincides with the full Hilbert space of the quantum problem \[ [1], \] so that it provides a complete description of the low-energy physics and can be used as starting point for the construction of a critical-point Hamiltonian.
This construction proceeds by linearizing the pseudoparticle bands $\epsilon_\alpha(q)$ of the above Hamiltonian around the pseudo-Fermi points and keeping only the two-pseudoparticle interaction term (3). In addition, one replaces the full $f$ function expressions (4) by the corresponding values at the pseudo-Fermi points. As shown previously, the forms of these $f$ functions imply that at the critical point the two-pseudoparticle phase shifts play a crucial role. Measuring the pseudomomentum from the pseudo-Fermi points adds the index $\iota = \text{sgn}(q)1 = \pm 1$, which defines the right ($\iota = 1$) and left ($\iota = -1$) movers, to the pseudoparticle operators, which also depend on the pseudomomentum $q$ and the color $\alpha$. The pseudoparticle operators $b^\dagger_{qa} (b_{qa})$ become $b^\dagger_{\kappa\alpha} (b_{\kappa\alpha})$ and the pseudoparticle number operators $\hat{N}_\alpha(q) = b^\dagger_{qa} b_{qa}$ become

$$\hat{N}_\alpha,\iota(\kappa) = b^\dagger_{\kappa\alpha} b_{\kappa\alpha},$$

where the new pseudomomentum $\kappa$ is such that

$$\kappa = q - q^{(+)}_F \alpha ; \quad -q^{(+)}_F \alpha < \kappa < (q^{(+)}_\alpha - q^{(+)}_F \alpha), \quad \iota = 1,$$

$$\kappa = q - q^{(-)}_F \alpha ; \quad (q^{(-)}_\alpha - q^{(-)}_F \alpha) < \kappa < -q^{(-)}_F \alpha, \quad \iota = -1,$$

and the pseudo-Fermi points $q^{(\pm)}_F \alpha$ are given by Eqs. (7) – (9). The number of $\alpha, \iota$ pseudoparticle operator is given by $\hat{N}_{\alpha,\iota} = \sum_\kappa \hat{N}_{\alpha,\iota}(\kappa)$.

In normal order relative to the ground state (6), the operators (11) are thus given by

$$: \hat{N}_{\alpha,\iota}(\kappa) : = \hat{N}_{\alpha,\iota}(\kappa) - \Theta(-\kappa) .$$

The critical-point Hamiltonian reads

$$: \hat{H} : = \sum_{\kappa,\alpha,\iota} \kappa \nu \nu : \hat{N}_{\alpha,\iota}(\kappa) : + \frac{1}{\hat{N}_\alpha} \sum_{\kappa,\kappa',\alpha,\alpha'} \sum_{\iota} \frac{1}{2} [f^1_{\alpha\alpha'} : \hat{N}_{\alpha,\iota}(\kappa) :: \hat{N}_{\alpha',\iota}(\kappa') :$$

$$+ f^{-1}_{\alpha\alpha'} : \hat{N}_{\alpha,\iota}(\kappa) :: \hat{N}_{\alpha',-\iota}(\kappa') :] ,$$

where the two (or $\nu$) “light velocities” $\nu_\alpha$ are given in Eq. (5) and

$$f^1_{\alpha\alpha'} = f_{\alpha\alpha'}(q^{(\pm)}_F \alpha, q^{(\pm)}_F \alpha'), \quad f^{-1}_{\alpha\alpha'} = f_{\alpha\alpha'}(q^{(\pm)}_F \alpha, q^{(\mp)}_F \alpha'),$$

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and the pseudo-Fermi points $q^{(\pm)}_F \alpha$ are given by Eqs. (7) – (9). The number of $\alpha, \iota$ pseudoparticle operator is given by $\hat{N}_{\alpha,\iota} = \sum_\kappa \hat{N}_{\alpha,\iota}(\kappa)$.
are two values of the $f$ functions. $f_{aa'}^{1}$ ($f_{aa'}^{-1}$) refers to forward-scattering of two right-moving or two left-moving pseudoparticles (one right-moving and one left-moving pseudoparticles) close to the corresponding pseudo-Fermi points.

To clarify the scale-invariant character of the critical-point Hamiltonian (14) it is useful to express the $f$ functions $f_{aa'}^{±1}$ (15) as

$$f_{aa'}^{±1} = 2\pi \sum_{α''} 2v_α G_{aa'}^{±1}(α, α'),$$

where the two-pseudoparticle interaction functions $G_{aa'}^{1}(α', α'')$ and $G_{aa'}^{-1}(α', α'')$ are given explicitly by

$$2G_{aa'}^{1}(α', α'') = -δ_{α,α'}δ_{α,α''} + \frac{1}{2} [ξ_0^{α}ξ_0^{α''} + ξ_1^{α}ξ_1^{α''}],$$

(17)

and

$$2G_{aa'}^{-1}(α', α'') = \frac{1}{2} [ξ_0^{α}ξ_0^{α''} - ξ_1^{α}ξ_1^{α''}],$$

(18)

respectively, and the dimensionless parameters $ξ_j^{α}$ (with $j = 0, 1$) are simple combinations of the two-pseudoparticle phase shifts at the pseudo-Fermi points

$$ξ_j^{α} = δ_{α,α'} + Φ_{aa'}(q_{Fα}, q_{Fα'}) + (-1)^j Φ_{aa'}(q_{Fα}, -q_{Fα'}),$$

(19)

where the phase shifts $Φ_{aa'}(q, q')$ are given by

$$Φ_{cc}(q, q') = Φ_{cc}(\frac{\sin K_0(q)}{u}, \frac{\sin K_0(q')}{u}),$$

(20)

$$Φ_{cs}(q, q') = Φ_{cs}(\frac{\sin K_0(q)}{u}, S_0(q')),$$

(21)

$$Φ_{sc}(q, q') = Φ_{sc}(S_0(q), \frac{\sin K_0(q')}{u}),$$

(22)

and

$$Φ_{ss}(q, q') = Φ_{ss}(S_0(q), S_0(q')),$$

(23)
and \( K_0(q) \) and \( S_0(q) \) are the ground-state solution of Eqs. (A1) and (A2) of paper I. The phase shifts \( \bar{\Phi}_{cc}(x, x') \), \( \bar{\Phi}_{cs}(x, y') \), \( \bar{\Phi}_{sc}(y, x') \), and \( \bar{\Phi}_{ss}(y, y') \) are the solutions of the four coupled integral equations (32) – (35) of Ref. \[15\]. For a study of the relation of these phase shifts to two-electron matrix elements see Ref. \[14\].

Inserting (18) into the RHS of Eq. (14) leads to

\[
: \hat{H} : = \sum_{\alpha, \iota = \pm 1} v_{\alpha} \{ \iota \sum_{\kappa} \hat{N}_{\alpha, \iota}(\kappa) :
+ \frac{2\pi}{N_{a}} \sum_{\kappa, \kappa'} \sum_{\alpha', \alpha''}[G_{\alpha}^{1}(\alpha', \alpha'') : \hat{N}_{\alpha', \iota}(\kappa) :: \hat{N}_{\alpha'', -\iota}(\kappa') :]
+ G_{\alpha}^{-1}(\alpha', \alpha'') : \hat{N}_{\alpha', -\iota}(\kappa) :: \hat{N}_{\alpha'', \iota}(\kappa') :}\right],
\]

which is precisely the critical-point Hamiltonian presented in our earlier work \[6\]. It includes two-pseudoparticle (momentum \( k = 0 \)) forward-scattering only.

Note that the Hamiltonian (24) can be written as \( : \hat{H} : = \sum_{\alpha} : \hat{H}_{\alpha} : \). Remarkably, despite the interaction terms, each term \( : \hat{H}_{\alpha} : \) in the Hamiltonian is scale invariant, the only scale being the “light velocity” \( v_{\alpha} \). In the renormalization group terms, this means that we are working at the interacting fixed point \[6\]. Since we have considered the relevant two-pseudoparticle interactions only, it follows from the pseudoparticle perturbation theory presented in I that the critical-point spectrum corresponds, exclusively, to the second-order energy expansion in the density of excited pseudoparticles.

At small momentum and low energy the only relevant term in the Hamiltonian (14), (24) is the non-interacting pseudoparticle term. This explains the Landau-liquid character of the problem in the pseudoparticle basis. In contrast, in the case of excitations with large momentum and low energy or excitations involving small changes in the numbers of \( \alpha \) type pseudoparticles both the non-interacting and the two-pseudoparticle terms of (14), (24) must be included.

From the adiabatic continuity principle studied in Refs. \[17,19\], which refers to small momentum and low energy excitations only, it follows that, in the limit of zero onsite electronic interaction \( U \), the non-interacting term of the critical point Hamiltonian becomes the corre-
sponding non-interacting critical-point Hamiltonian *in terms of electron operators*. Further, in this limit the colors $\alpha$ become the spin projections $\sigma(\alpha)$ and the velocities $v_\alpha$ the Fermi velocities $v_{F\sigma(\alpha)}$ \[^{19}\]. Note, however, that this does not hold true when we consider low-energy excitations of large momentum or involving changes in the numbers of pseudoparticles. In this case the interaction term of the Hamiltonian (14), (24) becomes relevant. Importantly, in the present model and in multicomponent systems, that pseudoparticle interaction term does not necessarily vanish in the $U \rightarrow 0$ limit \[^{1}\]. This is related to the exotic properties of the electron - pseudoparticle canonical transformation \[^{22,23}\].

At the critical-point, the ground-state fluctuations of the finite system correspond, in the infinite system, to low-energy excitations. In the case of the fluctuations associated with the response to curvature of the space-time, which are related to Lorentz invariance, the relevant critical-point excitations have small momentum and low energy. This corresponds to the usual definition of the critical point.

On the other hand, however, the scale-invariant character of the Hamiltonian (24) suggests that considering *all* the low-energy excitations of the infinite system, including eigenstates with large momentum and/or with different numbers $N_\alpha$ of $\alpha$ pseudoparticles, should lead to the same energy spectrum as the finite-size studies of the finite system \[^{5}\]. Our second-order pseudoparticle perturbation theory, which is equivalent to the use of the Hamiltonian (14), (24), confirms this suggestion.

Therefore, we can define the critical point in a broader way by considering the low-energy Hilbert space associated with the conformal-invariant character of the quantum liquid \[^{5,6}\]. In this more general case, we need to consider the Hilbert space spanned by *all* Hamiltonian eigenstates of low energy. This Hilbert space can be divided into three Hilbert subspaces which we call (A), (B), and (C) \[^{6}\]. The corresponding BA Hamiltonian eigenstates were used in Refs. \[^{4,5,12}\] to derive the conformal spectrum of the Hubbard chain. Here we intend to characterize these excitations in terms of the operators which generate them from the ground state (6), which in the pseudoparticle basis can be rewritten in terms of right and left $\alpha$ pseudoparticles as

\[^{19}\]
\[ |0; \eta_z, S_z \rangle = \prod_{\alpha, \iota} \prod_{-\kappa > 0} b^\dagger_{\kappa \alpha \iota} |V \rangle. \quad (25) \]

The three types of eigenstates that span the entire low-energy Hilbert space include:

(A) Eigenstates associated with small changes \( \delta N_\alpha \) in the numbers of \( \alpha \) pseudoparticles \( N_\alpha \). These states correspond to an “adiabatic” change from a given point of the \( U(1) \otimes U(1) \) sector of parameter space to a neighboring point of the same space. Since these excitations involve changes in the values of the numbers \( N_\alpha \), they imply a change in the values of the pseudo-Fermi points \( q_{F \alpha}^{(\pm)} \) \((7) - (9)\). This can change the integer or half-integer character of the quantum numbers \( I_\j \) which, as discussed in detail in I, determine the nature of the BA eigenstate. We assume that the density of pseudoparticles “added” or “removed”, \( \delta n_\alpha = \frac{\delta N_\alpha}{N_a} \), is small. Note that the number of \( \alpha \) pseudoparticles is a good quantum number. In addition, what is added, removed, or spin-flipped are electrons, not pseudoparticles \([1]\).

These low-energy states in the Hilbert space are in fact ground states of the form \((25)\) of neighboring parameter-space points. The density \( \delta n_\alpha \) can be expressed in terms of the changes in the pseudo-Fermi points as follows \(2\pi \delta n_\alpha = \delta q_{F \alpha}^{(+)} - \delta q_{F \alpha}^{(-)} \[1\]. When \( \delta n_\alpha > 0 \) these states are of the form

\[
\langle (A) \rangle = \prod_\alpha \left[ \prod_{\kappa=0}^{\delta q_{F \alpha}^{(+)}} b^\dagger_{\kappa \alpha 1} \prod_{\kappa=0}^{\delta q_{F \alpha}^{(-)}} b_{\kappa \alpha -1} \right] |0; \eta_z, S_z \rangle, \quad \delta n_\alpha > 0, \quad (26)
\]

whereas for \( \delta n_\alpha < 0 \) they are given by

\[
\langle (A) \rangle = \prod_\alpha \left[ \prod_{\kappa=0}^{\delta q_{F \alpha}^{(+)}} b_{\kappa \alpha 1} \prod_{\kappa=0}^{\delta q_{F \alpha}^{(-)}} b_{\kappa \alpha -1} \right] |0; \eta_z, S_z \rangle, \quad \delta n_\alpha < 0. \quad (27)
\]

(B) One-pair and multipair eigenstates involving the transfer of a small density of \( \alpha \)-pseudoparticles from the pseudo-Fermi points \( q = q_{F \alpha}^{(\pm)} \) to \( q = q_{F \alpha}^{(\mp)} \pm \frac{2\pi}{N_a} \). We introduce the number \( D_\alpha \), such that \( 2D_\alpha = \delta N_\alpha^- = [\delta N_{\alpha, \iota=1} - \delta N_{\alpha, \iota=-1}] \), which is related to the number of \( \alpha \)-pseudoparticles transferred. These eigenstates have large momentum and “transform” \( \alpha, \iota \) pseudoparticles into \( \alpha, -\iota \) pseudoparticles. When \( D_\alpha > 0 \) and for small but finite densities of pseudoparticles transferred these states can be generated from the ground state \((25)\) as

\[
\langle (B) \rangle = \prod_\alpha \left[ \prod_{\kappa=0}^{2\pi D_\alpha/N_a} b^\dagger_{\kappa \alpha 1} b_{\kappa \alpha -1} \right] |0; \eta_z, S_z \rangle, \quad D_\alpha > 0, \quad (28)
\]
whereas if $D_\alpha < 0$ and for small finite densities they are of the form

$$|(B)\rangle = \prod_\alpha \left[ \prod_{\kappa=0}^N b_{\kappa\alpha}^{\dagger} b_{\kappa\alpha} \right] |0; \eta_z, S_z\rangle, \quad D_\alpha < 0. \quad (29)$$

The states (B) have large momentum

$$k = \sum_\alpha D_\alpha \left[ q_{F \alpha}^{(+)} + \frac{2\pi}{N_a} - q_{F \alpha}^{(-)} \right] = \sum_\alpha D_\alpha 2q_{F \alpha}, \quad (30)$$

where $q_{F \alpha}$ is given in Eq. (10).

(C) One-pair and multipair eigenstates involving $\alpha$ pseudoparticle-pseudohole processes around the same point of the pseudo-Fermi surface $q_{F \alpha}^{(\pm)}$, the numbers of $\alpha$, $\iota$ pseudoparticle-pseudohole processes of this kind being denoted by

$$N_{ph}^{\alpha \iota} = \iota \frac{N_a}{2\pi} \left[ \sum_p \kappa_{p,\alpha,\iota} - \sum_h \kappa_{h,\alpha,\iota} \right], \quad (31)$$

and the density of excited pseudoparticles $\frac{N_{ph}^{\alpha \iota}}{N_a}$ being small. Here $\kappa_{p,\alpha,\iota}$ ($\kappa_{h,\alpha,\iota}$) defines the pseudomomentum values of the pseudoparticles (pseudoholes). These states are of the form

$$|(C)\rangle = L_{-N_{ph}^{\alpha \iota}}^{\alpha \iota} |0; \eta_z, S_z\rangle, \quad (32)$$

where the explicit form of the generator $L_{-N_{ph}^{\alpha \iota}}^{\alpha \iota}$ will be discussed below (see Sec. IV, Eq. (56)); as the notation suggests, we shall in fact see that it is related to generators of the Virasoro algebra associated with the theory.

The eigenstates (A) and (B) are pseudoparticle collective excitations which include pseudo-Fermi points displacements. While the states (A) connect two ground states of neighboring canonical ensembles because the displacements of the two kinds of $\iota$ pseudoparticles have opposite sign, the displacements associated with the excitations (B) have the same sign and lead to the finite excitation momentum (30). In addition, when in the case of the eigenstates (A) the quantum numbers $I_j^\alpha$ change from integers (half integers) to half integers (integers), these excitations involve a global pseudomomentum shift which affects all pseudoparticles of the corresponding $\alpha$ branch. This leads to a so-called orthogonal catastrophe \[25\] and explains the fully incoherent – i.e., $Z_F = 0$ – nature of the one-electron spectral function \[19\].
The pure zero-momentum forward-scattering character of the Hamiltonian (14), (24) implies that the numbers of right and left \( \alpha \) pseudoparticles obey independent conservation laws \[21\], i.e. the four (or \( 2\nu \)) numbers \( N_{\alpha,i} \) of \( \alpha, i \) pseudoparticles are good quantum numbers. From the perspective of Lorentz invariance, this means that the \( i = 1 \) and \( i = -1 \) problems are independent and the states (26), (27) and (28), (29) are seen as \( i \) “ground states” of different canonical ensembles. Therefore, the fluctuations which determine the Lorentz invariance involve only the excitations (C), whereas the states (A) and (B) represent different choices of starting ground states. In the language of conformal-field theory this means that (A) and (B) are HWS of the Virasoro algebras, whereas the excitations (C) correspond to the towers \[9\]. We will discuss this in detail below. One consequence of this fact is that instead of the ground state (24) we can use as starting states for the excitations \(|(C)\rangle\) both states of form \(|(A)\rangle\) and \(|(B)\rangle\).

Starting from any point of the \( U(1) \otimes U(1) \) sector of parameter space we can arrive to any other point of the same sector by performing a set of adiabatic excitations of type (A). On the other hand, the excitations (B) and (C) span the low-energy Hilbert space associated with each point of parameter space (i.e. each canonical ensemble). The Hamiltonian eigenstates (A)-(C) span the critical-point Hilbert space. These states are both eigenstates of the full pseudoparticle Hamiltonian (1) and of the critical-point Hamiltonian (14), (24). Note, however, that the critical-point Hamiltonian leads to the correct energy spectrum up to second order in the density of excited pseudoparticles only. This is because it does not contain the higher-order terms of the full Hamiltonian (1) – according to the pseudoparticle perturbation theory discussed in Sec. IV of paper I the truncated scattering-second-order Hamiltonian leads to the correct second-order energy in the density of excited pseudoparticles.

Since the quantum-liquid Hamiltonian can be expressed exclusively in terms of the pseudomomentum distribution operators (11) and (13), we can calculate the energy spectrum from the eigenvalue equations of these operators for the states \(|(A)\rangle\), \(|(B)\rangle\), and \(|(C)\rangle\) defined by Eqs. (26), (27), (28), (29), and (32). These equations are of the form

13
\[ \hat{N}_{\alpha,\iota}(\kappa)|(A)\rangle = \Theta(-\delta q_{\alpha}^{(-)} + \kappa)|(A)\rangle \quad \iota = -1, \]
\[ \hat{N}_{\alpha,\iota}(\kappa)|(B)\rangle = \Theta\left( \frac{2\pi}{N_a} D_{\alpha} - \kappa \right)|(B)\rangle \quad \iota = 1 \]
\[ = \Theta\left( -\frac{2\pi}{N_a} D_{\alpha} + \kappa \right)|(B)\rangle \quad \iota = -1, \]  \tag{33}

and
\[ \hat{N}_{\alpha,\iota}(\kappa)|(C)\rangle = \Theta(-\iota \kappa) + \frac{2\pi}{N_a} \left( \sum_{\alpha} \delta(\kappa - \kappa_{p,\alpha,1}) - \sum_{h} \delta(\kappa - \kappa_{h,\alpha,1}) \right)|\langle C)\rangle. \]  \tag{34}

In order to calculate the critical-point energy spectrum we consider an eigenstate \(|\Psi\rangle\) involving all three types of excitations (A), (B), and (C). The corresponding eigenvalue equation is
\[ \hat{N}_{\alpha,\iota}(\kappa)|\Psi\rangle = \Theta(-\iota \kappa) + \frac{2\pi}{N_a} \left( \sum_{\alpha} \delta(\kappa - \kappa_{p,\alpha,1}) - \sum_{h} \delta(\kappa - \kappa_{h,\alpha,1}) \right)|\langle C)\rangle. \]  \tag{35}

By combining the critical-point-Hamiltonian expression (14), (24) with the eigenvalue Eq. (36) we can calculate the corresponding excitation energy. Expanding to second order in the density of excited pseudoparticles we find in units of \(2\pi/N_a\) (and in the thermodynamic limit) \[\Delta E = \langle \hat{H} \rangle = \sum_{\alpha, \iota = \pm 1} v_\alpha [h_\alpha^\iota + N_{p,\alpha}^{\iota}] \]  \tag{37}
where
\[ h_\alpha^\iota \equiv \frac{1}{2} \left[ \sum_{\alpha'} \xi_{\alpha\alpha'}^1 D_{\alpha'} + \iota \sum_{\alpha'} \xi_{\alpha\alpha'}^0 (\delta N_{\alpha'/2})^2 \right]. \]  \tag{38}

Here the \(2 \times 2\) (or \(\nu \times \nu\) coefficients \(\xi_{\alpha\alpha'}^j\) (19) are the entries of the dressed-charge matrix \(Z\) when \(j = 1\) and the entries of the matrix \([Z^T]^{-1}\) when \(j = 0\). (Our definition of the dressed-charge matrix is the transpose of that used in Refs. [5].)
We can now relate the low-energy spectrum we have derived for our critical point Hamiltonian to the form of the energy spectrum obtained from finite-size scaling studies: a direct comparison shows that our expression (37) is precisely of the same form as that derived from finite-size studies, with $h_\alpha$ in (38) being the conformal dimensions of the primary fields. The perturbative character of the pseudoparticle basis then implies that it should provide the correct energy spectrum up to second order in the density of excited pseudoparticles.

We emphasize that the excitation energy (37) of the state $|\Psi\rangle$ is decoupled into two types of terms: the terms containing the parameters $h_\alpha$ are generated by the excitations (A) and (B), whereas the terms involving the numbers $N_{ph}^{\alpha\iota}$ arise from the excitations (C). This implies that the HWS of the Virasoro algebras correspond to the pseudoparticle eigenstates of types (A) and (B), whereas the low-momentum excitations (C) correspond to the towers, as we have mentioned previously. Therefore, only the excitations (A) and (B) refer to primary fields.

At the critical point, which corresponds to second-order energy spectrum (37), the multi-pair eigenstates of the Hilbert sub-space (C) can be described as a direct product of one-pair eigenstates, as in a non-interacting system. This is because at low energy and small momentum the two-pseudoparticle terms of the Hamiltonian (14), (24) are irrelevant because they give zero when acting onto the eigenstates (C) (32). It follows that the energy (37) is additive in the numbers $N_{ph}^{\alpha\iota}$, i.e., the $\alpha$ branches are fully decoupled in the case of the excitations belonging to the space (C). On the other hand, the $\iota = 1$ and $\iota = -1$ excitations also refer to orthogonal Hilbert subspaces. Therefore, (C) decouples into a set of four (or $2 \times \nu$) smaller Hilbert subspaces corresponding to each of the $\alpha, \iota$ branches. Due to the orthogonality of these $\alpha, \iota$ spaces, we can uniquely define the projections of the Hamiltonian and momentum operators in each of them.
III. LORENZ INVARIANCE, THE RESPONSE TO CURVATURE OF SPACE-TIME, AND CONFORMAL ANOMALIES

The response of a system with fixed values of the conserved quantum numbers $N_{\alpha,\iota}$ to the curvature of space-time can be determined by the ground-state fluctuations of the finite system [8] which correspond in the infinite system to the low-momentum, low-energy excitations of type (C), the restriction to fixed $N_{\alpha,\iota}$ eliminating the (A) and (B) excitations. Importantly, the two-pseudoparticle terms of the Hamiltonian (14), (24) give zero when acting onto the states (C), so that for these excitations the critical-point Hamiltonian reduces to a non-interacting pseudoparticle problem

\[ : \hat{H}_0 := \sum_{\kappa, \alpha, \iota} \iota \kappa v_{\alpha} : \hat{N}_{\alpha,\iota}(\kappa) : , \]  

(39)

which can be written as \[ : \hat{H}_0 := \sum_{\alpha} : \hat{H}_0^\alpha : . \] This non-interacting form provides a tremendous simplification, enabling us to establish using straightforward calculations several important results concerning Lorenz invariance and conformal anomalies. Before presenting these calculations, let us make some remarks concerning the nature of the states (C) in terms of both pseudoparticles and electrons.

In a later section we shall prove (see Eq. (56) below) that the excitations (C) are generated by acting on the ground state with products of one-pair $\alpha, \iota$ pseudoparticle operators. These generators creating the states (C) are interaction dependent mixtures of the electronic operators, and their explicit forms will be studied elsewhere [22, 23]. This interaction dependence is reflected in the fact that the colors $\alpha$ associated with the conserved pseudoparticle quantum numbers $N_{\alpha}$ cannot be identified with usual quantum numbers such as charge, spin, or spin projection [22, 23]. However, $N_{\alpha}$ is a conserved quantum number. Similarly, that the non-interacting pseudoparticle Hamiltonian (39) describes interacting electrons is indicated by the $U$ dependence of the velocities $v_{\alpha}$ of Eq. (5) [16].

The adiabatic principle of Ref. [19] implies that the limit $U \to 0$ of the Hamiltonian (38) describes correctly the non-interacting electrons in the sector of lowest symmetry $U(1) \otimes U(1)$,
which corresponds to $U = 0$, $0 < n < 1$, and $n^\uparrow > n^\downarrow$. In this case, the colors $c$ and $s$ become the spin projections $\uparrow$ and $\downarrow$, respectively \[13,22,23\]. The properties of that transformation also imply that the limits $\eta_z \to 0$ or (and) $S_z \to 0$ of the energy spectrum (37) provide the correct values for the corresponding higher symmetry sectors of parameter space.

The non-interacting character of the Hamiltonian (39) in the pseudoparticle basis follows from the symmetries of the problem and is the result of the Lorentz invariance which holds in each of the Hilbert subspaces $\alpha, \iota$ involving states of type (C): we find below that at the critical point the energy-momentum tensor decouples in four (or $2 \times \nu$) new tensors which act on orthogonal Hilbert subspaces. Each gapless excitation branch corresponds to independent Minkowski spaces, each with common space and time but different “light” $\alpha$ velocities. The Lorentz-invariance in each of the four (or $2 \times \nu$) $\alpha, \iota$ subspaces of (C) excitations determines the complete critical theory of the quantum liquids, which is simply given by the direct product of two (or $\nu$) Virasoro algebras (each including the $\iota = 1$ and $\iota = -1$ components), in full agreement with the finite-size results of Refs. \[4,5\]. In connection to these algebras there are two (or $\nu$) affine-Lie algebras (Kac-Moody algebras) \[6,9,10\].

Turning to the calculational details, we apply to our pseudoparticle Hamiltonian (39) the standard continuum limit techniques: in the limit of small momentum (long wavelength) and low energy we can ignore the discrete character of the lattice and map the problem into a continuum field theory whose fields are labeled by the colors $\alpha$. Let us then introduce the pseudoparticle fields $\psi^\dagger_{\alpha \iota}$ and $\psi_{\alpha \iota}$. The Hamiltonian density which corresponds to the Hamiltonian (39) reads

$$\hat{\mathcal{H}} = \sum_{\alpha, \iota} \hat{\mathcal{H}}^\alpha_{\iota},$$

where the individual terms $\hat{\mathcal{H}}^\alpha_{\iota}$

$$\hat{\mathcal{H}}^\alpha_{\iota} = -i \nu_{\alpha}[\psi^\dagger_{\alpha \iota} \frac{\partial}{\partial x} \psi_{\alpha \iota}],$$

refer to orthogonal Hilbert subspaces of (C) and, therefore, correspond to four (or $2 \times \nu$) independent field theories. The total Lagrangian density is given by
\[ \hat{\mathcal{L}} = \sum_{\alpha, \iota} \psi_{\alpha \iota}^\dagger \left( \frac{\partial}{\partial t} + i v_\alpha \frac{\partial}{\partial x} \right) \psi_{\alpha \iota}. \] (42)

It is convenient to introduce the light-cone combinations

\[ x_\alpha^\iota = \frac{1}{2} \left\{ t - i \frac{x}{v_\alpha} \right\}. \] (43)

By construction, the pseudoparticle field associated with each orthogonal Hilbert space of colors \( \alpha \) and \( \iota \) is such that

\[ \psi_{\alpha \iota} = \psi_{\alpha \iota}(x_1^\alpha, x_{-1}^\alpha). \] (44)

Then the Lagrangian density (42) can be written as

\[ \hat{\mathcal{L}} = \sum_{\alpha, \iota} i \psi_{\alpha \iota}^\dagger \partial_\iota^\alpha \psi_{\alpha \iota}, \] (45)

where

\[ \partial_\iota^\alpha = \frac{\partial}{\partial x_\iota^\alpha}. \] (46)

The classical equations of motions are

\[ \partial_\iota^\alpha \psi_{\alpha \iota} = 0, \] (47)

which implies that

\[ \psi_{\alpha \iota} = \psi_{\alpha \iota}(x_{-1}^\alpha). \] (48)

From Eqs. (40) – (48) it is straightforward to find that the energy-momentum tensor is given by

\[ \hat{T} = \sum_{\alpha, \iota} \hat{T}_{\iota}^\alpha, \] (49)

where each component tensor \( \hat{T}_{\iota}^\alpha \) reads

\[ \hat{T}_{\iota}^\alpha = \frac{i}{2} : \psi_{\alpha-\iota}^\dagger \partial_\iota^\alpha \psi_{\alpha-\iota} :. \] (50)
and corresponds to one of the independent field theories which refer to orthogonal Hilbert subspaces of (C).

From the equal-time commutation relation

$$\{\psi_{\alpha}(x,t), \psi_{\alpha'}^\dagger(x',t')\} = \delta_{\alpha,\alpha'}\delta_{\iota,\iota'}\delta(x-x'),$$  (51)

one finds the two-point correlation functions

$$\langle 0; \eta_z, S_z | \psi_{\alpha}(x_{\iota}) \psi_{\alpha'}^\dagger(y_{\iota'}) | 0; \eta_z, S_z \rangle = -\frac{i}{\pi} \frac{\delta_{\alpha,\alpha'}\delta_{\iota,\iota'}}{(x_{\iota} - y_{\iota} + i\epsilon)},$$  (52)

Let us evaluate the leading singularity term of operator product expansion associated with the tensor-tensor correlation function \[3\]. This leading term measures the response of the quantum problem to the curvature of the two-dimensional space-time. In one-component conformal-invariant problems this term introduces the conformal anomaly \(c\) \[8\]. In the case of multicomponent systems, the problem has not previously been studied. However, the present pseudoparticle operator basis makes this problem trivial because of the remarkable decoupling (49). A straightforward calculation using Wick’s theorem shows that

$$\langle 0; \eta_z, S_z | \hat{T}_{\alpha}^\alpha(x_{\iota}) \hat{T}_{\alpha'}^{\alpha'}(y_{\iota'}) | 0; \eta_z, S_z \rangle = c_{\alpha\iota} \frac{\delta_{\alpha,\alpha'}\delta_{\iota,\iota'}}{[2(x_{\iota} - y_{\iota} + i\epsilon)^4]}, \quad c_{\alpha\iota} = 1,$$  (53)

where, following the notation of our previous work \[3\], we have used units of \(1/(4\pi^2)\).

On the other hand, since the \(\alpha, \iota\) Hilbert sub spaces are orthogonal, each tensor (50) corresponds to an independent Minkowski space with “light” velocity \(v_{\alpha}\). Therefore, we can in each of these independent spaces consider \(v_{\alpha} = 1\) in Eq. (43), which leads to \[3\]

$$\langle 0; \eta_z, S_z | \hat{T}_{\iota}^\alpha(x_{\iota}) \hat{T}_{\iota'}^{\alpha'}(y_{\iota'}) | 0; \eta_z, S_z \rangle = c_{\alpha\iota} \frac{\delta_{\alpha,\alpha'}\delta_{\iota,\iota'}}{[2(x_{\iota} - y_{\iota} + i\epsilon)^4]}, \quad c_{\alpha\iota} = 1,$$  (54)

where \(x_{\iota} = \frac{1}{2}(t - \iota x)\).

This confirms the finite-size result of Refs. \[3,4\] that the present multicomponent systems correspond to a direct sum of conformal field theories, each possessing a central charge equal to one. In the present basis the study of the operator algebra of these Virasoro algebras becomes a simple problem, as we show in the following section.
IV. OPERATOR REPRESENTATION OF THE VIRASORO ALGEBRAS

Each pair of tensors $\hat{T}_e^{\alpha,\iota}(x_i)$ is associated with one Virasoro algebra and the corresponding set of generators $L_j^{\alpha,\iota}$ [8,9,10]. The $\hat{T}_e^{\alpha,\iota}(x_i)$ can be expressed in terms of these generators as follows

$$\hat{T}_e^{\alpha,\iota}(x_i) = \frac{2\pi}{N_a^2} \sum_j L_j^{\alpha,\iota} e^{-2\pi ijx_i/N_a}.$$  \hspace{1cm} (55)

The “tower” of states associated with the terms $N_{\text{ph}}^{\alpha'i}$ (recall the definition of this quantity in (31)) in the energy spectrum (37) can be constructed by the action of the generators of the Virasoro algebras on the HWS [8,9,10] of these algebras. On the other hand, we have shown that these “tower” terms are associated with the multipair eigenstates (C). It follows that the generators $L_j^{\alpha,\iota}$ of the RHS of Eq. (55) annihilate ($j > 0$) and create ($j < 0$) $\alpha,\iota$ - pseudoparticle-pseudohole pairs (C) [8]. Thus, rather than referring to electrons, the generators of the Virasoro algebra $\alpha,\iota$ refer to the $\alpha,\iota$ - pseudoparticles. For example, for $j < 0$ these generators are given by

$$L_j^{\alpha,\iota} = \prod_{\kappa,\alpha,\iota,\kappa',\alpha',\iota'} [b_{\kappa,\alpha,\iota,\alpha',\kappa',\iota'}^\dagger b_{\kappa,\alpha,\iota,\alpha',\kappa',\iota'}],$$  \hspace{1cm} (56)

where the product contains $|j| = N_{\text{ph}}^{\alpha'i}$ factors with $|j| > 0$ giving the number of pseudoparticle-pseudohole processes (see Eq. (31)) and the pseudomomentum values $\kappa_{\alpha,\iota,\iota'}$ and $\kappa_{\alpha,\iota,\iota'}$ are the pseudomomentum values associated with these processes and are the same as in the RHS of Eq. (31).

The $j = 0$ generators $L_0^{\alpha,\iota}$, which are not of the form (56), play a particularly important role because they define the critical-point Hamiltonian and momentum operator [9]. In the present case they have the universal form

$$L_0^{\alpha,\iota} = \alpha \sum_{\kappa} \kappa : \hat{N}_{\alpha,\iota}(\kappa) :$$

$$+ \frac{2\pi}{N_a} \sum_{\kappa,\alpha',\iota',\alpha'',\iota''} [F^{\dagger}_{\alpha,\iota}(\alpha',\alpha'',\iota') : \hat{N}_{\alpha',\iota'}(\kappa) :: \hat{N}_{\alpha'',\iota''}(\kappa') :] + F^{-1}_{\alpha,\iota}(\alpha',\alpha'',\iota') : \hat{N}_{\alpha',\iota'}(\kappa) :: \hat{N}_{\alpha'',\iota''}(\kappa') :],$$  \hspace{1cm} (57)
where

\[ F_{\alpha,\iota}^{\pm 1}(\alpha', \alpha'', \iota') = \frac{1}{2} G_{\alpha}^{\pm 1}(\alpha', \alpha'') + \frac{i\iota'}{4 \xi_{\alpha\alpha'}^{0} \xi_{\alpha\alpha''}} , \] (58)

\( \xi_{\alpha\alpha'}^{j} \) is given in Eq. (19) and the remaining parameters of the RHS of Eqs. (57) and (58) are the same as in Eq. (24). The conformal- and scale-invariant character of each \( \alpha \) term of the Hamiltonian (24) allows us to take \( v_{\alpha} = 1 \) in Eq. (57). The critical-point Hamiltonian (24) can be expressed in terms of the generators (57) as

\[ \hat{\mathcal{H}} := \sum_{\alpha,\iota} v_{\alpha} L_{0}^{\alpha,\iota} , \] (59)

whereas the momentum simply reads

\[ \hat{P} := \sum_{\alpha,\iota} \iota L_{0}^{\alpha,\iota} + \sum_{\kappa,\alpha,\iota} \iota q_{F\alpha} \hat{N}_{\alpha,\iota}(\kappa) : = \sum_{\kappa,\alpha,\iota} [\kappa + \iota q_{F\alpha}] \hat{N}_{\alpha,\iota}(\kappa) : + \frac{2\pi}{N_{a}} \sum_{\kappa,\kappa',\alpha,\iota} \sum_{\lambda,\lambda'} \frac{t}{2} [\hat{N}_{\alpha,\iota}(\kappa) :: \hat{N}_{\alpha,-\iota}(\kappa') : + \hat{N}_{\alpha,\iota}(\kappa) :: \hat{N}_{\alpha,-\iota}(\kappa') :] . \] (60)

We emphasize that the absence of the interaction-dependent parameters \( \xi_{\alpha\alpha'}^{j} \) (19) in the quantity \( \sum_{\alpha,\iota} \iota L_{0}^{\alpha,\iota} \) of the RHS of Eq. (60) follows from the fact that the matrix of entries \( \xi_{\alpha\alpha'}^{0} \) is the inverse of the transposition of the matrix of entries \( \xi_{\alpha\alpha'}^{1} \). Notice that the two-pseudoparticle interaction terms of the normal-ordered critical-point momentum operator (60) are not present in the original momentum expression (see Eq. (22) of I). This is because the latter expression refers to the Hilbert space associated with a canonical ensemble characterized by constant \( (\eta_{z}, S_{z}) \) eigenvalues. In turn, the two-pseudoparticle interaction terms of the normal-ordered critical-point momentum operator (60) account for the changes in the pseudo-Fermi points (7) – (9) due to changes in the eigenvalues \( \eta_{z} \) and \( S_{z} \): these interaction terms give zero when acting on states with the same values of \( \eta_{z} \) and \( S_{z} \) as the reference ground state. We stress that both the Hamiltonian expression (59) and the momentum expression (60) are valid only at the critical point. Therefore, they lead to the correct excitation energy and momentum, respectively, to second order in the density of excited pseudoparticles only.
The HWS of the Virasoro algebras, \( |h^\alpha_\iota\rangle \), are eigenstates of the generators (57) satisfying
\[
L^\alpha_\iota_0 |h^\alpha_\iota\rangle = h^\alpha_\iota |h^\alpha_\iota\rangle.
\]
(61)
The \( \alpha, s \) towers are constructed by applying the generators (56) to the HWS of the Virasoro algebras, \( i.e., \)
\[
L^\alpha_\iota_{-N^\text{ph}} |h^\alpha_\iota\rangle = (h^\alpha_\iota + N^\text{ph}_\iota) |h^\alpha_\iota\rangle,
\]
(62)
where the states \( |h^\alpha_\iota + N^\text{ph}_\iota\rangle \) are also eigenstates of the operators \( L^\alpha_\iota_0 \)
\[
L^\alpha_\iota_0 |h^\alpha_\iota + N^\text{ph}_\iota\rangle = (h^\alpha_\iota + N^\text{ph}_\iota) |h^\alpha_\iota + N^\text{ph}_\iota\rangle.
\]
(63)
From Eqs. (59) and (60) we find that the energy spectrum associated with all \( \alpha, \iota \) conformal families is given by Eq. (37) and that the corresponding momentum is
\[
P = \sum_{\alpha, \iota} \iota [h^\alpha_\iota + N^\text{ph}_\iota] + \sum_{\alpha} D_\alpha 2q_F\alpha.
\]
(64)
In the present class of quantum liquids the HWS of the Virasoro algebras are the Hamiltonian eigenstates (A) and (B) of form (26), (27) and (28), (29), respectively. There is a one-to-one correspondence between the primary-fields operators and the HWS \( \frac{3}{2}, \frac{11}{4}, \frac{15}{2} \). Therefore, the primary-field operators correspond to excitations which connect adiabatically ground states (25) of neighboring parameter-space points (excitations (A)); and large-momentum, low-energy pseudoparticle-pseudohole excitations (excitations (B)). In contrast to the states (C), both the excitations (A) and (B) change the conserved pseudoparticle numbers \( N_{\alpha, \iota} \). The simple form of the generators of the states (A) and (B) (see Eqs. (26), (27) and (28), (29), respectively) reveals that the corresponding primary-field operators have simple expressions in the pseudoparticle basis.

The interaction terms of the critical-point Hamiltonian (14), (24) are controlled by two-pseudoparticle forward scattering through the corresponding pseudoparticle phase shifts. These appear in the excitation energies of the HWS of the Virasoro algebras through the conformal dimensions \( h^\alpha_\iota \) (38) and, therefore, the interaction dependence of the critical
exponents associated with these HWS, as for example the exponents of the oscillating terms in the correlation functions \[5\], is determined by the pseudoparticle interactions \[6,17\].

On the other hand, the energy spectrum of the small-momentum and low-energy excitations (C), which determines the response of the energy-momentum tensor to the curvature of the two-dimensional space-time, does not include, at the critical point, the interaction-dependent two-pseudoparticle phase shifts. It follows that \(c_\alpha = 1\) in Eq. (54). This is because the excitations (C) involve only the pseudoparticle non-interacting Hamiltonian (39) and correspond to the towers (62) whose spectrum is given, exclusively, in terms of the integer numbers (31).

V. AFFINE LIE ALGEBRAS AND DYNAMICAL SEPARATION

For each of the two (or \(\nu\)) above Virasoro algebras, there is associated an affine-Lie algebra \[4\]. These affine-Lie algebras \[26\] are often in the literature \[3,8,10\] called Kac-Moody algebras \[27,28\], although the Kac-Moody algebras are in fact more general and also include the hyperbolic algebras which are not affine-Lie algebras \[26\].

The affine-Lie algebras are generated by considering the conserved quantities related to the original particles of the theory, \(N_{\gamma(\alpha)} = N_\alpha\), where \(N_\alpha\) is the eigenvalue of the \(\alpha\)-pseudoparticle number operator introduced in I, \(\hat{N}_\alpha \equiv \sum_q \hat{N}_\alpha(q) = \sum_q b_{q,\alpha}^\dagger b_{q,\alpha}\), and \(\gamma(\alpha)\) is associated with the particle conserving number, \(N_{\gamma(\alpha)}\), which equals the number of \(\alpha\) pseudoparticles. For the Hubbard chain (see I) \(N_c = N = N_\uparrow + N_\downarrow\) and \(N_s = N_\downarrow\) and, therefore, \(\gamma(c) = \rho\) (charge) and \(\gamma(s) = \downarrow\) (down spin). Since \(N_{\gamma(\alpha)} = N_\alpha\) and \(\hat{N}_{\gamma(\alpha)} = \hat{N}_\alpha\), it follows that the particle-number operators \(\hat{N}_{\gamma(\alpha)}\) can be written in the pseudoparticle basis as

\[
\hat{N}_{\gamma(\alpha)} \equiv \sum_q \hat{N}_\alpha(q) = \sum_q b_{q,\alpha}^\dagger b_{q,\alpha}.
\]

We can associate to each conserved quantity in (65) a \(\gamma(\alpha)\) current. These two (or \(\nu\)) \(\gamma(\alpha)\) currents are the diagonal generators of the corresponding affine-Lie algebras \[4\]. Using the notation of the previous sections, we introduce the following currents
\[ \hat{J}^{\gamma(a)}_0 = \sum \hat{N}_{\alpha,\ell}, \]
\[ \hat{J}^{\gamma(a)}_1 = \sum \ell \hat{N}_{\alpha,\ell}, \]

which can be combined in the holomorphic and anti-holomorphic components, \( \hat{J}^{\gamma(a)}_\ell = \hat{J}^{\gamma(a)}_0 + \ell \hat{J}^{\gamma(a)}_1 \), such that
\[ \hat{J}^{\gamma(a)}_\ell = 2 \hat{N}_{\alpha,\ell}. \]

(We denote \( \hat{J}^{\gamma(a)}_\ell \) with \( \ell = +1 \) by \( \hat{J}^{\gamma(a)}_+ \), to distinguish it from the current \( \hat{J}^{\gamma(a)}_1 \) of Eq. (66).)

Since the numbers \( N_{\alpha,\ell} \) are good quantum numbers, at the critical point right (left) pseudoparticles are made out of right (left) electrons only. This implies that the particle currents \( \hat{J}^{\gamma(a)}_\ell \) (67) and the pseudoparticle currents \( \hat{J}^{\alpha}_\ell \) are equal. However, the associated particle current densities, \( \hat{J}^{\gamma(a)}_\ell(x) \), and pseudoparticle current densities, \( \hat{J}^{\alpha}_\ell(x) \), are different objects. This can be shown by considering the Fourier transform of the current densities \( \hat{J}^{\gamma(a)}_\ell(x) \) and \( \hat{J}^{\alpha}_\ell(x) \), which we call \( \hat{J}^{\gamma(a)}_\ell(k) \) and \( \hat{J}^{\alpha}_\ell(k) \), respectively. As we show in Eqs. (68) and (69) below, these two small-momentum currents are not equal.

Following Ref. [8], the dressed-charge matrix [5] is a representation of the small-momentum current \( \hat{J}^{\gamma(a)}_\ell(k) \) in the reduced \( \ell \) Hilbert space of vanishing energy and momentum. This \( \ell \) space is spanned by the ground state (25) and two single-pair \( \alpha, \ell \)-pseudoparticle eigenstates (C) which are constructed from that ground state by transferring the pseudoparticle at \( \kappa = 0 \) to \( \kappa = \ell \frac{2\pi}{N_a} \). We call these excited eigenstates \( |\alpha, \ell; \eta_z, S_z\rangle \).

In Appendix A we use the methods introduced in Ref. [19] to show that at the smallest momentum \( k = \ell' \frac{2\pi}{N_a} \) (which vanishes as \( N_a \to \infty \))
\[ \langle \alpha, \ell; \eta_z, S_z | \hat{J}^{\alpha'}_{\ell'}(k) |0; \eta_z, S_z\rangle = \delta_{\ell,\ell'} \delta_{\alpha,\alpha'}, \]
(68)

whereas,
\[ \langle \alpha, \ell; \eta_z, S_z | \hat{J}^{\gamma(a')}_\ell(k) |0; \eta_z, S_z\rangle = \delta_{\ell,\ell'} \delta_{\alpha,\alpha'}. \]
(69)

Comparison of Eqs. (68) and (69) confirms that although \( \hat{J}^{\gamma(a)}_\ell = \hat{J}^{\alpha}_\ell \) we have that \( \hat{J}^{\gamma(a)}_\ell(k) \neq \hat{J}^{\alpha}_\ell(k) \), and thus \( \hat{J}^{\gamma(a)}_\ell(x) \neq \hat{J}^{\alpha}_\ell(x) \).
From our previous discussions, it is clear that in the phases of lowest symmetry and at low energies the excitations of the system are pseudoparticle-pseudohole excitations (B) and (C). Thus, the transport carriers of the quantum liquid are the $\alpha$ pseudoparticles [19]. These pseudoparticles couple to external fields, whose couplings determine the exotic instabilities observed in quasi-one-dimensional synthetic metals [20]. The pure zero-momentum forward-scattering character of the Hamiltonian (1) implies that for these energy scales the $\alpha$ pseudoparticle currents are non-dissipative and contribute to the coherent part of the $\gamma(\alpha)$ and other conductivities only [3,17,19]. (See the Lehmann representation of the general $\vartheta$ conductivity spectrum in Eq. (A10) of Appendix A.)

By introducing a magnetic flux through a ring (periodic boundary conditions), it is possible to calculate the charge stiffness associated with these pseudoparticles. This is done by taking the second derivative of the energy with respect to the flux [29]. On the other hand, the charge stiffness can be shown to be proportional to the weight of the $\delta$-peak which constitutes the coherent part of the charge conductivity spectrum [17,19,30,31]. (This is given in the general case (charge, spin, spin projection) by Eq. (A10) of Appendix A.) The stiffness determines the transport masses of the pseudoparticles. These masses are defined in Eq. (59) of Ref. [19] and are proportional to the inverse of the pseudoparticle elementary currents [17,19].

Consider the real part of the frequency-dependent conductivity, $\sigma^\vartheta(\omega)$, associated with the current $\hat{J}^\vartheta_1$, Eq. (A10) of Appendix A. (In the case of the Hubbard chain, for example, $\vartheta$ can be charge $\vartheta = \rho$, spin $\vartheta = \sigma_z$, and spin projection $\vartheta = \sigma_\varphi$.) In that Appendix we use the method used in Ref. [19] for charge $\vartheta = \rho$ and spin $\vartheta = \sigma_z$ to evaluate the expression for the general $\vartheta$ stiffness in terms of the pseudoparticle transport masses. The result is

$$2\pi D^\vartheta = \sum_{\alpha'} \frac{q_{F\alpha'}}{m_{\alpha'}^{\vartheta}},$$

(70)

where the transport masses are given by

$$m_{\alpha}^{\vartheta} \equiv \frac{q_{F\alpha}}{[g_{\alpha}^{\vartheta} \sum_{\alpha'} \sum_{\alpha''} g_{\alpha''}^{\vartheta} v_{\alpha'} \xi_{1\alpha'}^1 \xi_{1\alpha''}^1]},$$

(71)
The latter equation represents the \( \vartheta \)-transport mass for a \( \alpha \), \( \iota \) pseudoparticle of pseudomomentum \( \kappa = 0 \) \( (q = \iota q_{F\alpha}) \) \[19\]. In the RHS of Eqs. (70) and (71) the velocity \( v_{\alpha} \) and pseudomomentum \( q_{F\alpha} \) are given in Eqs. (5) and (10), respectively. The interaction-dependent quantities \( \xi_{\alpha\alpha'}^1 \) are the elements of the dressed-charge matrix (19) which are the two-particle matrix elements (69) (when \( \iota = \iota' \)). The integer numbers \( g_{\alpha}^\vartheta \) are model dependent and are the coefficients of the general equation (A3) of Appendix A. They are given in that Appendix for the case of the Hubbard chain.

The properties of the \( \gamma(\alpha) \) currents, which are the diagonal generators of the affine-Lie algebras associated with the Virasoro algebras studied in Sec. IV, lead to the concept of dynamical separation \[6\]. Indeed, consider \( \vartheta = \gamma(\alpha) \) in the above expressions. In Appendix A we find

\[
2\pi D^{\gamma(\alpha)} = \frac{q_{F\alpha}}{m_{\gamma(\alpha)}} ,
\]

where

\[
m_{\gamma(\alpha)} = \frac{q_{F\alpha}}{\left[\sum_{\alpha'} v_{\alpha'} (\xi_{\alpha'\alpha})^2\right]} ,
\]

and

\[
m_{\gamma(\alpha)} = \infty ,
\]

for \( \alpha \neq \alpha' \). Therefore, the stiffness of the \( \gamma(\alpha) \) conductivity spectrum involves the transport mass (and elementary current) of the corresponding \( \alpha \) pseudoparticle only. This holds true for the two (or \( \nu \)) \( \gamma(\alpha) \) stiffnesses only. It follows that in the present \( U(1) \otimes U(1) \) sector the peaks of the charge and down-spin conductivity spectra are determined, in the Hubbard chain, by elementary currents of \( c \) and \( s \) pseudoparticles only, respectively.

This does not hold true, however, for the non-\( \gamma(\alpha) \) conductivities whose stiffness expressions involve a superposition of several elementary currents of different \( \alpha \) pseudoparticle branches. This is the case of the spin and up-spin conductivities of the Hubbard chain.

The concept of dynamical separation introduced in \[6\] associates the \( \alpha \) pseudoparticles with the \( \gamma(\alpha) \) conductivities. It means that from the point of view of the two \( \gamma(\alpha) \) diagonal
generators of the affine-Lie algebras each \( \alpha \) branch of pseudoparticles can be identified with the conserved quantity \( \gamma(\alpha) \). However, it will be shown elsewhere \(^{22,23}\) that from the point of view of the \( \alpha, \iota \) generators which transform the ground state (25) onto the small-momentum and low-energy one-pair \( \alpha, \iota \) pseudoparticle excitations the colors \( \alpha \) are not the quantum numbers \( \gamma(\alpha) \). This follows from the properties of the canonical transformation which at low energy connects the electronic and pseudoparticle basis.

VI. CONCLUDING REMARKS

Our results in the preceding sections establish that the pseudoparticle basis and perturbation theory developed in I can be applied to study the critical-point physics of multicomponent integrable systems. Consistent with earlier work based on the finite-size corrections \(^{14,15}\), we have found that the complete critical theory of these systems is determined by a direct product of Virasoro algebras. We were able to write the expressions for the generators of these algebras and other operators explicitly, \( i.e. \) we have introduced the operator-Virasoro algebras of integrable quantum liquids. Taken together, the results of I and the present paper clarify considerably several points concerning the non-Fermi liquid physics in one-dimensional quantum liquids.

First, they have revealed that the correct critical-point Hamiltonian of integrable systems refers to the pseudoparticle basis.

Second, they have shown that despite the many-pseudoparticle \( (k = 0 \) forward-scattering) interaction terms in the critical-point Hamiltonian, the theory is nonetheless conformal and scale invariant, \( i.e. \) the pseudoparticle operator basis permits a perturbative expansion around an interacting electron fixed point.

Third, the Landau-liquid character of the \( U(1) \otimes U(1) \) sectors of the Hubbard chain \(^{15,16,17,18,19,20}\) follows from the fact that at low energy and at small momentum the Hamiltonian pseudoparticle interaction terms are irrelevant. Moreover, the validity of the Landau-liquid expansion of Refs. \(^{16,18,19}\) was shown to follow from the perturbative char-
acter of the quantum problem in the pseudoparticle operator basis.

Fourth, our results reveal that the present kind of many-body problem defined in a discrete lattice can be mapped into a set of non-interacting quantum-field theories at the critical point. However, despite the correspondence between the pseudoparticle colors $\alpha$ and the particle quantum numbers $\gamma(\alpha)$ in terms of the $\alpha$ elementary currents and $\gamma(\alpha)$ conductivities (dynamical separation), we will show elsewhere $^{22,23}$ that, from the point of view of the generators of the low-energy and small-momentum excitations, the pseudoparticle colors $\alpha$ and the particle quantum numbers $\gamma(\alpha)$ are not equivalent. This means that the quantum numbers $\alpha$ which label the above set of independent and non-interacting quantum-field theories are not usual quantum numbers such as spin projection, charge, and spin. This also implies the non-perturbative character of the usual electronic basis, where the above quantum-field theories are more difficult to describe.

Finally, it has been argued that non-Fermi liquid behavior occurs in some sectors of parameter space of two-dimensional interacting quantum liquids $^{25,32,33,34}$. If this proves true, some of the pseudoparticle concepts and techniques developed in the present one-dimensional context may prove useful in that case as well.

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In this Appendix we use the methods used in Ref. [19] for \( \vartheta = \rho \) and \( \vartheta = \sigma_z \), to derive the current representations of Eqs. (68) and (69) and the stiffness expressions (70) – (74) for general particle quantum numbers \( \vartheta \). Although we consider mainly the case of the Hubbard chain, our results and expressions are valid for multicomponent integrable quantum liquids.

Equation (68) follows from the fact that \( \hat{J}_i^{\vartheta}(k)|0; \eta_z, S_z\rangle = |\alpha, \nu; \eta_z, S_z\rangle \).

Let us consider the \( \vartheta - \vartheta \) correlation function, \( \chi^{\vartheta}(k, \omega) \). (In the case of the Hubbard chain \( \vartheta \) can denote charge (\( \vartheta = \rho \)), spin (\( \vartheta = \sigma_z \)), and spin projection (\( \vartheta = \sigma \)).

At the smallest momentum \( k = \frac{i2\pi}{N_a} \) (which vanishes as \( N_a \to \infty \) and vanishing frequency \( \omega \), the only excited eigenstates are the single-pair pseudoparticle states \( |\alpha, \nu; \eta_z, S_z\rangle \) of Eq. (69). Therefore, the use of a Lehmann representation for \( \chi^{\vartheta}(k, \omega) \) gives when \( k = \frac{i2\pi}{N_a} \) and \( \omega \) is vanishing small [19]

\[
\chi^{\vartheta}(k, \omega) = -\frac{N_a}{\pi} \sum_{\alpha} \langle \langle \alpha, \nu; \eta_z, S_z | \hat{J}_i^{\vartheta}(k)|0; \eta_z, S_z\rangle \rangle^2 \frac{k^2 v_\alpha}{(kv_\alpha)^2 - (\omega + i\eta)^2}. \tag{A1}
\]

In order to derive the matrix elements \( \langle \alpha, \nu; \eta_z, S_z | \hat{J}_i^{\vartheta}(k)|0; \eta_z, S_z\rangle \) in (A1) we evaluate \( \chi^{\vartheta}(k, \omega) \) by a second method. Comparison of the obtained expression with (A1) and the use of a boundary condition leads then to Eq. (69).

Let us couple the pseudoparticles to a weak external \( \vartheta \) probe of long wavelength and low frequency. According to the fluctuation-dissipation theorem, the correlation function \( \chi^{\vartheta}(k, \omega) \) equals the response function

\[
\chi^{\vartheta}(k, \omega) = \frac{\delta\langle \hat{N}_\vartheta(k, \omega) \rangle}{\varrho^\vartheta(k, \omega)}, \tag{A2}
\]

where \( \hat{N}_\vartheta \) is the \( \vartheta \) particle conserving number, \( e^\vartheta \varrho^\vartheta(k, \omega) \) is the external potential, and \( e^\vartheta \) is the corresponding elementary constant. For instance, \( e^\rho = -e \) is the electronic charge, \( e^{\sigma_z} = 1/2 \) is the electronic spin, and \( e^\sigma = \sigma 1/2 = \pm 1/2 \) are the electronic spin projections.

The particle number \( N_\vartheta \) can be written in terms of the numbers \( N_{\gamma(\alpha)} = N_\alpha \) as follows
\[ N_\vartheta = \sum_\alpha g_\vartheta^\alpha N_\alpha , \quad \text{(A3)} \]

where \( g_\vartheta^\alpha \) are model-dependent integers. In the case of the Hubbard chain, where \( \gamma(c) = \rho \) and \( \gamma(s) = \downarrow \), they read \( g_\vartheta^\rho = 1 \), \( g_\vartheta^s = 0 \), \( g_\vartheta^+ = 1 \), \( g_\vartheta^- = -1 \), \( g_\vartheta^\sigma = 1 \), and \( g_\vartheta^\sigma = -2 \). The following result holds true for all multicomponent models:

\[ g_\vartheta^{\gamma(\alpha)\alpha'} = \delta_{\alpha,\alpha'} . \quad \text{(A4)} \]

As discussed in Ref. [19], the quantity

\[ C_\vartheta^\alpha = e^{\vartheta} g_\vartheta^\alpha , \quad \text{(A5)} \]

is the coupling of the \( \alpha \) pseudoparticles to the \( \vartheta \) probe.

The suitable inhomogeneous fluctuation \( \delta \langle \hat{N}_\vartheta(k,\omega) \rangle \) to be used in Eq. (A2) is of the form [19]

\[ \delta \langle \hat{N}_\vartheta(k,\omega) \rangle = \frac{N_a}{2\pi} \sum_\alpha \int_{q_{\vartheta}}^{q_{\vartheta}^{(+)}} \delta N_\alpha(q; k, \omega) g_\vartheta^\alpha , \quad \text{(A6)} \]

where the two inhomogeneous pseudomomentum deviations \( \delta N_\alpha(q; k, \omega) \) (with \( \alpha = c, s \)) are determined by the following system of two (or \( \nu \)) kinetic equations

\[ (kv_\alpha(q) - \omega) \delta N_\alpha(q; k, \omega) + (\text{sgn}(q))k\delta(\hat{q}F_{\alpha} - |q|) \sum_{\alpha'} \frac{1}{2\pi} \int_{q_{\alpha'}}^{q_{\alpha'}^{(+)}} dq' \delta N_{\alpha'}(q'; k, \omega) f_{\alpha\alpha'}(q, q') + g_\vartheta^{\alpha} V^\vartheta(k, \omega) = 0 , \quad \alpha = c, s . \quad \text{(A7)} \]

In order to evaluate the response function (A2) we have to solve the coupled kinetic equations (A7). The form of the general solutions for these equations is

\[ \delta N_\alpha(q; k, \omega) = -\frac{1}{N_a} \left[ \delta(\hat{q}F_{\alpha} - |q|) \frac{\text{sgn}(q)k}{kv_\alpha(q) - \omega - i\eta} X_\alpha^\vartheta(q; k, \omega) \right] , \quad \text{(A8)} \]

where the imaginary term, \( i\eta \), conforms to the usual adiabatic boundary conditions. The equations which determine the functions \( X_\alpha^\vartheta(q; k, \omega) \) are obtained by replacing the expressions (A8) in the kinetic equations. Following Eqs. (A2), (A6), and (A8) the response function is given by
$$\chi^{\theta}(k, \omega) = -\frac{N_a}{2\pi V^\theta(k, \omega)} \sum_{j=\pm 1} \sum_{\alpha} \frac{k}{k v_{\alpha} - j(\omega + i\eta)} g_{\alpha}^\theta X_{\alpha}^\theta(j q_{F\alpha}; k, \omega). \quad (A9)$$

Evaluation of the functions $X_{\alpha}^\theta(j q_{F\alpha}; k, \omega)$ leads to expression (A1) with the amplitudes $|\langle \alpha, \iota; \eta_z, S_z | \hat{J}^\theta_i(k) |0; \eta_z, S_z \rangle|$ expressed in terms of the $j = 1$ phase-shift combinations (19). Finally, the use of a suitable boundary conditions provides the relative phases of the matrix elements with the result

$$\langle \alpha, \iota; \eta_z, S_z | \hat{J}^\theta_i(k) |0; \eta_z, S_z \rangle = \sum_{\alpha'} g_{\alpha'}^\theta \langle \alpha, \iota; \eta_z, S_z | \hat{J}^\gamma_i(\alpha') (k) |0; \eta_z, S_z \rangle, \quad (A10)$$

where $\langle \alpha, \iota; \eta_z, S_z | \hat{J}^\gamma_i(\alpha') (k) |0; \eta_z, S_z \rangle$ is given by Eq. (69) with $\iota' = \iota$. (It is obvious that (69) vanishes when $\iota' \neq \iota$.)

In order to calculate the $\theta$ stiffnesses (70)–(74) we use the correlation-function expression (A1) with the matrix elements given by Eq. (69). The Lehmann representation of the $\theta$ conductivity spectrum reads

$$Re \sigma^\theta(\omega) = 2\pi (e^\theta)^2 \left[ D^\theta \delta(\omega) + \frac{1}{N_a} \sum_{i \neq 0} |\langle i | \hat{J}^\theta_i |0; \eta_z, S_z \rangle|^2 \delta \left( \omega_{i,0}^2 - \omega^2 \right) \right], \quad (A11)$$

where the $i$ summation refers to both the “dissipative” LWS II and corresponding non-LWS multiplet eigenstates of zero momentum, which we denote by $|i\rangle$. In the lowest-symmetry sectors of parameter space they have finite excitation energy $\omega_{i,0} = E_i - E_0$ ($E_0$ denotes the ground-state energy). $D^\theta$ is the $\theta$ stiffness. In order to derive $D^\theta$, we use the following standard relation of the $\theta$ conductivity spectrum (A11) to the response function $\chi^{(\theta)}(k, \omega)$

$$Re \sigma^\theta(\omega) = Re \left[ \lim_{k \to 0} i(e^\theta)^2 \omega \frac{\chi^{(\theta)}(k, \omega)}{k^2 N_a} \right]. \quad (A12)$$

In the present lattice model this relation is only valid for $\omega \to 0$ and at $\omega = 0$. Therefore, it provides the coherent part of (A11). The use of (A1) and (69) in the RHS of Eq. (A12) leads to

$$Re \sigma^\theta(\omega) = 2\pi (e^\theta)^2 D^\theta \delta(\omega), \quad (A13)$$
with $D^{\rho}$ given by Eqs. (70) – (71). The $D^{\gamma(\alpha)}$ expression (72) – (74) follows from Eqs. (70) – (71) and (A4).

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