On the universality of the Carter and McLenaghan formula

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received 23 September 2008; accepted in final form 20 March 2009

PACS 04.20.Cv - Fundamental problems and general formalism
PACS 04.62.+v - Quantum fields in curved spacetime
PACS 11.30.-j - Symmetry and conservation laws

Abstract – It is shown that the formula of the isometry generators of the spinor representation given by Carter and McLenaghan is universal in the sense that this holds for any representation, in either local frames or even natural ones. Starting with the observation that point-dependent spin matrices in natural frames can be defined for any tensor representation, the covariant form of the isometry generators is written down, showing that this is just the Carter and McLenaghan formula in natural frames.

Introduction. – In general relativity, the Noether theorem leads to general conservation laws associated to the relativistic and gauge covariance [1,2]. In the particular case of isometries associated to Killing vectors, the physical fields minimally coupled to gravity take over this symmetry transforming according to appropriate representations of the isometry group. In the case of the scalar vector or tensor fields these representations are completely defined by the well-known rules of the general coordinate transformations, since the isometries are in fact particular automorphisms. However, the theory of spinor fields is formulated in orthogonal local frames where the basis-generators of the spinor representation were discovered by Carter and McLenaghan [3].

Some time ago we proposed a theory of external symmetry in the context of the gauge covariant theories in local orthogonal frames. We introduced there new transformations which combine isometries and gauge transformations such that the tetrad fields should remain invariant [4]. In this way, we obtained the external symmetry group which is nothing other than the universal covering group of the isometry one. Moreover, we derived the general form of the basis-generators of the representations of this group showing that these are given by a formula which is similar to that of Carter and McLenaghan.

In the present letter, we would like to show that this formula holds even in natural frames. To this aim we define the spin matrices of the vector and tensor representations in natural frames which are point-dependent and give rise to the spin terms of the covariant form of our basis-generators. The conclusion is that the Carter and McLenaghan formula is universal.

We start in the second section with a short introduction to the tetrad gauge covariant theories in which we defined our theory of external symmetry briefly presented in the next section. The spin matrices in natural frames are introduced in section three where the basis-generators of the vector and tensor representations of the external symmetry group are obtained.

Tetrad gauge covariance. – Let us consider the pseudo-Riemannian spacetime \((M, g)\) and a local chart (or natural frame) of coordinates \(x^\mu\) (labeled by natural indices, \(\mu, \nu, \ldots = 0, 1, 2, 3\)). Given a gauge, we denote by \(e^\mu\) the tetrad fields that define the local frames and by \(\hat{e}^\mu\) those of the corresponding coframes. These have the usual duality, \(e^\mu \, \hat{e}^\lambda = \delta^\mu_\lambda\), \(e^\mu \, e^\mu = \delta^\mu_\mu\), and orthonormalization, \(e_\mu \cdot e_\nu = \eta_{\mu\nu} \), \(\hat{e}_\mu \cdot \hat{e}_\nu = \hat{\eta}^{\mu\nu} \), properties. The metric tensor \(g_{\mu\nu} = \eta_{\alpha\beta} e^\alpha_\mu e^\beta_\nu\) raises or lowers the natural indices, while for the local indices (\(\hat{\mu}, \hat{\nu}, \ldots = 0, 1, 2, 3\)) we have to use the flat Minkowski metric \(\eta = \text{diag}(1, -1, -1, -1)\). This metric remains invariant under the transformations of its gauge group, \(O(1, 3)\). This has as subgroup the Lorentz group, \(L^1\), of the transformations \(\Lambda[A(\omega)]\) corresponding to the transformations \(A(\omega) \in SL(2, \mathbb{C})\) through the canonical homomorphism. In the standard covariant parametrization, with the real parameters \(\omega^{\hat{\alpha}\hat{\beta}} = -\omega^{\hat{\beta}\hat{\alpha}}\), the \(SL(2, \mathbb{C})\) transformation reads

\[
A(\omega) = e^{-\frac{i}{2} \omega^{\hat{\alpha}\hat{\beta}} S_{\hat{\alpha}\hat{\beta}}},
\]

where \(S_{\hat{\alpha}\hat{\beta}}\) are the covariant basis-generators of the \(sl(2, \mathbb{C})\) Lie algebra. In fact, these are the principal spin operators that generate the spin terms of other operators.
of the quantum theory. For small values of $\omega^{\alpha\beta}$, the matrix elements of the transformations $\Lambda$ in the local basis can be expanded as $\Lambda^{\alpha'}_{\alpha}[A(\omega)] = \delta^{\alpha'}_{\alpha} + \omega^{\alpha'}_{\mu} \delta^{\mu}_{\alpha} + \cdots$.

Assuming now that $(M,g)$ is orientable and time-orientable, we can consider $G(\eta) = L_+^4$ as the gauge group of the Minkowski metric $g$. This is the structure group of the principal fiber bundle whose basis is $M$. The group Spin($\eta) = SL(2,C)$ is the universal covering group of $G(\eta)$ and represents the structure group of the spin fiber bundle [5–7]. In general, a matter field $\psi(\rho) : M \to V(\rho)$ is locally defined over $M$ with values in the vector space $V(\rho)$ of a representation $\rho$, generally reducible, of the group Spin($\eta$). The covariant derivatives of the field $\psi(\rho)$,

$$D^{(\rho)}_{\alpha} = e^{\alpha}_{\mu} D_{\mu}^{(\rho)} = \tilde{D}_{\alpha} + \frac{i}{2} \left( S_{\beta}^{\alpha} \tilde{\Gamma}_{\alpha\beta}^{\gamma} \right) \xi^{\gamma}(x)$$

depend on the connection coefficients in local frames, \( \tilde{\Gamma}_{\alpha\beta}^{\gamma} = \tilde{e}_{\mu}^{\alpha} \tilde{e}_{\nu}^{\beta} \left( \tilde{\epsilon}_{\alpha\beta}^{\gamma} - \tilde{\epsilon}_{\beta\alpha}^{\gamma} \right) \), which assure the covariance of the whole theory under tetrad gauge transformations produced by automorphisms $A$ of the spin fiber bundle. This is the general framework of the theories involving fields with half integer spin which cannot be treated in natural frames.

**External symmetries in local frames.** – A special difficulty in local frames is that the theory is no longer covariant under isometries, since these can change the tetrad fields that carry natural indices. For this reason, we proposed a theory of external symmetry in which each isometry transformation is coupled to a gauge one that is able to correct the position of the local frames such that the whole transformation should preserve not only the metric but the tetrad gauge too [4]. Thus, for any isometry transformation $x \to x' = \phi_{\xi}(x) = x + \xi^{a} k_{a} + \cdots$, depending on the parameters $\xi^{a}$ (\( a, b, \ldots = 1, 2, \ldots N \)) of the isometry group $I(M)$, one must perform the gauge transformation $A_{\xi}$ defined as

$$A^{(\xi)}_{\alpha} = \hat{\delta}^{\alpha}_{\mu} \phi_{\xi}(x) \frac{\partial \phi^{\alpha}_{\nu}(x)}{\partial x^{\mu}} e^{\nu}_{\beta}(x)$$

with the supplementary condition $A_{\xi=0}(x) = 1 \in SL(2,C)$. Then the transformation laws of our fields are

$$e(x) \to e'(x') = e(\phi_{\xi}(x))$$

$$A_{\xi}(x, \phi_{\xi}) : \hat{e}(x) \to \hat{e}'(x') = e(\frac{\partial \phi^{\alpha}_{\nu}(x)}{\partial x^{\mu}} e^{\nu}_{\beta}(x))$$

$$\psi(\rho)(x) \to \psi'(\rho)(x') = \rho[A_{\xi}(x)] \psi(\rho)(x).$$

We have shown that the pairs $(A_{\xi}, \phi_{\xi})$ constitute a well-defined Lie group that we called the external symmetry group, $S(M)$, pointing out that this is just the universal covering group of $I(M)$ [4]. For small values of $\xi^{a}$, the $SL(2,C)$ parameters of $A_{\xi}(x) = A_{\xi}[\omega_{\xi}(x)]$ can be expanded as $\omega^{\alpha\beta}_{\xi}(x) = \dot{\xi}^{a} \Omega^{\alpha\beta}_{a}(x) + \cdots$, in terms of the functions

$$\Omega^{\alpha\beta}_{a} = \frac{\partial \omega^{\alpha\beta}_{\xi}}{\partial \xi^{a}}|_{\xi=0} = (e^{\alpha}_{\mu} k_{a}, \nu + e^{\nu}_{\mu} k_{a}, \alpha e^{\alpha}_{\mu} \eta^{\beta\gamma},$$

which depend on the Killing vectors $k_{a} = \partial_{\xi^{a}} \phi_{\xi}(\xi=0)$ associated to $\xi^{a}$.

The last of eqs. (4) defines the operator-valued representation $(A_{\xi}, \phi_{\xi}) \to T^{(\rho)}_{\xi}$ of the group $S(M)$ whose transformations,

$$(T^{(\rho)}_{\xi}(\psi(x))[\phi_{\xi}(x)] = \rho[A_{\xi}(x)] \psi(\rho)(x)),$$

leave the field equation invariant. Their basis-generators [4],

$$X^{a}_{\rho} = i \partial_{\xi} - T^{(\rho)}_{\xi}|_{\xi=0} = -ik_{a} \partial_{\mu} + \frac{1}{2} \Omega^{\alpha\beta}_{a} \rho(\hat{\Gamma}^{\alpha\beta}_{a}),$$

commute with the operator of the field equation and satisfy the commutation rules $[X^{a}_{\rho}, X^{b}_{\rho}] = i\epsilon^{abc} X^{c}_{\rho}$ determined by the structure constants, $\epsilon_{abc}$, of the $(s(M) \sim i(M))$ algebras. Equation (7) can be put in the covariant form

$$X^{a}_{\rho} = -ik_{a} D^{a}_{\rho} + \frac{1}{2} k_{a,\mu,\nu} e^{\mu}_{a} e^{\nu}_{b} \rho(\hat{\Gamma}^{ab}_{a}),$$

which represents the generalization to any representation $\rho$ of the famous formula given by Carter and McLenaghan for the spinor representation $\rho_{\text{spin}} = (\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ [3].

We specify that our theory of isometries can be formulated in the most general terms of Kosmann’s Lie derivatives which are defined for any vector field that is not necessarily Killing [8,9]. Thus, for example, the functions (5) are just those arising in the spin terms of these Lie derivatives, but the condition $\Omega^{\alpha\beta}_{a} = -\hat{\Omega}^{\alpha\beta}_{a}$ is fulfilled only when $k_{a}$ are Killing vectors [4]. In other respects, it is obvious that when the isometries are absent we remain only with the general conservation laws due to the relativistic and gauge covariance [1,2,7].

**Isometries in natural frames.** – Whenever there are no spinors, the matter fields are vectors or tensors of different ranks and the whole theory is independent of the tetrad fields dealing with the natural frames only. Let us see what happens with our theory of external symmetry in natural frames focussing on tensor fields. Any tensor field, $\Theta$, transforms under isometries as $\Theta \to \tilde{\Theta} = T^{\Theta}_{\xi} \Theta$, according to a tensor representation of the group $S(M)$ defined by the well-known rule in natural frames

$$\left[ \frac{\partial \phi^{\alpha}_{\xi}(x) \partial \phi^{\beta}_{\xi}(x)}{\partial x^{\mu}} \cdots \right] (T^{\Theta}_{\alpha\beta})_{\mu\nu\ldots}(\phi_{\xi}(x)) = \Theta_{\mu\nu\ldots}(x).$$

Hereby, one derives the basis-generators of the tensor representation, $X^{a}_{\Theta} = i \partial_{\xi} T^{\Theta}_{\xi}|_{\xi=0}$, whose action reads

$$(X^{a}_{\Theta})_{\alpha\beta\ldots} = -ik_{a} \Theta_{\alpha\beta\ldots} + k_{a,\alpha} \Theta_{\alpha\beta\ldots} + k_{a,\nu} \Theta_{\alpha\nu\beta\ldots} + k_{a,\alpha} \Theta_{\alpha\beta\nu\ldots}.\)$$

Our purpose is to show that the operators $X^{a}_{\Theta}$ can be written in a form that is equivalent to eq. (8) of Carter and McLenaghan.
In order to accomplish this we start with the vector representation \( \rho_\nu = \left( \frac{1}{2}, -\frac{1}{2} \right) \) of the \( SL(2, \mathbb{C}) \) group, generated by the spin matrices \( \rho_\nu(S^{\lambda \delta}) \) that have the well-known matrix elements

\[
[\rho_\nu(S^{\lambda \delta})]\tilde{\mu}_\delta = i(\eta^{\lambda \mu} \delta_\nu - \eta^{\lambda \mu} \delta_\nu)
\]

in local bases. Furthermore, we define the point-dependent spin matrices in natural frames whose matrix elements in the natural basis read

\[
(\tilde{S}^{\mu \nu})_{\tau} = e^\mu_\alpha e^\nu_\beta e^\tau_\gamma [\rho_\nu(S^{\lambda \delta})]_\lambda \delta = i(g^{\mu \sigma} \delta_\nu - g^{\mu \sigma} \delta_\nu).
\]

In what follows, we assume that these matrices represent the spin operators of the vector representation in natural frames. We observe that these are the basis-generators of the groups \( G[g(x)] \sim G(\eta) \) which leave the metric tensor \( g(x) \) invariant in each point \( x \). Since the representations of these groups are point-wise equivalent with those of \( G(\eta) \), one can show that in each point \( x \) the basis-generators \( S^{\mu \nu}(x) \) satisfy the standard commutation rules of the vector representation \( \rho_\nu \) (but with \( g(x) \) instead of \( \eta \)).

In general, the spin matrices of a tensor \( \Theta \) of any rank \( n \), are the basis-generators of the representation \( \rho_n = \rho_1 \otimes \rho_2 \otimes \rho_3 \otimes \cdots \otimes \rho_n \) which read \( \rho_n(\tilde{S}) = \tilde{S}^1 \otimes \tilde{I}^2 \otimes I^3 \otimes \tilde{S}^4 \otimes \cdots + \tilde{I} \). Using these spin matrices, a straightforward calculation shows that eq. (8) can be rewritten in natural frames as

\[
X_\alpha^\mu = -ik_\alpha^\mu \nabla_\mu + \frac{1}{2} k_{\alpha \mu \nu} \rho_\nu(\tilde{S}^{\mu \nu}),
\]

where \( \nabla_\mu \) are the usual covariant derivatives. It is not difficult to verify that the action of these operators is just that given by eq. (10), which means that \( X_\alpha^\mu \) are the basis-generators of a tensor representation of rank \( n \) of the group \( S(M) \). Thus, it is clear that eq. (13) represents the generalization to natural frames of the Carter and McLenaghan formula (8). We stress that this result is not trivial, since it cannot be seen as a simple basis transformation as in the usual tensor theory of the linear algebra.

Concluding remarks. – The principal conclusion here is that the Carter and McLenaghan formula is universal since it holds not only in local frames but in natural frames too. In local frames, this formula can be written in the covariant form (8) or as in eq. (7), where the orbital part (i.e., the first term) is completely separated from the spin term (the second one). It is worth pointing out that eq. (7) cannot be rewritten in natural frames even though it might be useful for analyzing the operators’ structure in the relativistic quantum mechanics. This is because the connection in local frames cannot be expressed in terms of spin matrices \( \tilde{S} \). Therefore, in natural frames we are forced to use only the genuine Carter and McLenaghan formula (13).

This formula exhibits explicitly the point-dependent spin matrices of the vector or tensor representations which may provide us with information about the spin content of the various fields involved in theory. However, some results could be surprising. Thus, for example, if a second order rank tensor has the form \( G_{\mu \nu} = g_{\mu \nu} F \), where \( F \) is a scalar field then we calculate \( \rho_2(\tilde{S}^{\alpha \beta})G = 0 \) which indicates that the tensor \( G \) is spinless despite the fact that it is of second rank. Consequently, the metric tensor is also spinless as it seems to be natural since this is covariantly constant and invariant under isometries (such that \( X_\alpha^2 g = 0 \)). Another example is presented in ref. [10] where we define the photon polarization in de Sitter manifolds without considering local frames.

Finally we note that, despite our partial results, the problem of the spin in general relativity remains open as long as we do not have an effective theory similar to the Wigner theory of the induced representations of the Poincaré group.

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We are grateful to C. Crucean and N. Nicolaevici for interesting and useful discussions on closely related subjects.

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