On a uniform estimate for the quaternionic Calabi problem

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Abstract

We establish a $C^0$ \textit{a priori} bound on the solutions of the quaternionic Calabi-Yau equation (of Monge-Ampère type) on compact HKT manifolds with a locally flat hypercomplex structure. As an intermediate step, we prove a quaternionic version of the Gauduchon theorem.

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1 Introduction and main results

1.1 HKT manifolds

We first recall some definitions and preliminaries from HKT geometry.

1 Definition. A smooth manifold $M^{4n}$ with a triple of complex structures $I, J, K$ such that $IJ = -JI = K$ will be called a \textit{hypercomplex manifold}.

2 Remark. Hypercomplex manifolds were explicitly introduced by Boyer [7].
3 Definition. A hypercomplex manifold \((M^{4n}, I, J, K)\) with a Riemannian metric \(\rho\) that is invariant under the three complex structures \((I, J, K)\) will be called a hyper-Hermitian manifold.

4 Definition. (The \(\partial J\)-operator) Consider the complex manifold \((M, I)\). Denote by \(\Omega^{p,q}_I(M)\) the space of \((p,q)\)-forms on \((M, I)\). Let \(\partial : \Omega^{p,q}_I(M) \rightarrow \Omega^{p+1,q}_I(M)\) and \(\overline{\partial} : \Omega^{p,q}_I(M) \rightarrow \Omega^{p+1,q}_I(M)\) be the usual \(\partial\) and \(\overline{\partial}\) operators on the complex manifold \((M, I)\). Define

\[
\partial J := J^{-1} \circ \overline{\partial} \circ J.
\]

Then \([17]\) (a) \(J : \Omega^{p,q}_I(M) \rightarrow \Omega^{q,p}_I(M),\) (b) \(\partial J : \Omega^{p,q}_I(M) \rightarrow \Omega^{p+1,q}_I(M)\) and (c) \(\partial \partial J = -\partial J \partial\).

Given a hyper-Hermitian manifold \((M, I, J, K, \rho)\) consider the differential form

\[
\Omega := -\omega_J + \sqrt{-1} \omega_K
\]

where \(\omega_L(A, B) := \rho(A, B \cdot L)\) for any \(L \in \mathbb{H}\) with \(L^2 = -1\) and any two vector fields \(A, B\) on \(M\). We use the following definition of an HKT-metric.

5 Definition. The metric \(\rho\) on \(M\) is called an HKT-metric if

\[
\partial \Omega = 0.
\]

6 Remark. HKT manifolds were introduced in the physical literature by Howe and Papadopoulos \([13]\). For the mathematical treatment see Grantcharov-Poon \([11]\) and Verbitsky \([17]\). The original definition of HKT-metrics in \([13]\) was different but equivalent to Definition 5 (the latter was given in \([11]\)).

7 Remark. The classical hyper-Kahler metrics (i.e. Riemannian metrics with holonomy contained in the group \(Sp(n)\)) form a subclass of HKT-metrics. It is well known that a hyper-Hermitian metric \(\rho\) is hyper-Kahler if and only if the form \(\Omega\) is closed, or equivalently \(\partial \Omega = 0\) and \(\overline{\partial} \Omega = 0\).

8 Definition. A form \(\omega \in \Omega^{2k,0}_I(M)\) that satisfies \(J \omega = \overline{\omega}\) will be called a real \((2k,0)\)-form \([17]\). The space of real \((2k,0)\)-forms will be denoted by \(\Omega^{2k,0}_I(M)\).

9 Definition. (The \(t\) isomorphism \([17]\)) Given a right vector space \(V\) over \(\mathbb{H}\), there is an \(\mathbb{R}\)-linear isomorphism \(t : \Lambda^{2,0}_{I,\mathbb{R}}(V) \rightarrow S_\mathbb{H}(V)\) from the space of the real \((2,0)\)-forms on \(V\) to the space of symmetric forms on \(V\) that are invariant with respect to the action of the group \(SU(2)\) of unit quaternions, given by \(t(\eta)(A, A) := \eta(A, A \circ J)\) \((17)\). We shall denote by the same letter the induced isomorphism \(t : \Omega^{2,0}_{I,\mathbb{R}}(M) \rightarrow S_\mathbb{H}(M)\), where the target is the space of global sections of the bundle with the fiber \(S_\mathbb{H}(T_x M)\) over a general point \(x \in M\).

10 Definition. Forms \(\eta \in \Omega^{2,0}_{I,\mathbb{R}}(M)\) satisfying \(t(\eta) > 0\) and \(t(\eta) \geq 0\) will be called strictly positive and positive respectively. We also call a form \(\Theta \in \Omega^{2n,0}_{I,\mathbb{R}}(M)\) strictly positive if it equals \(f \cdot \Omega^n\) for a strictly positive function \(f \in C^\infty(M, \mathbb{R})\). This definition does not depend on the choice of the hyper-Hermitian form \(\Omega\), because the real line bundle \(\Lambda^{2n,0}_{I,\mathbb{R}}\) is canonically oriented, and strict positivity of \(\Theta\) is equivalent to \(\Theta_x\) being a strictly positive element in the fiber over \(x\) for all \(x \in M\). We refer to \([4, 5]\) and references therein for further discussion of the notion of positivity.
11 Remark. On a hyper-Hermitian manifold the metric \( \rho \) and the form \( \Omega = -\omega_J + \sqrt{-1}\omega_K \) satisfy \( t(\Omega) = \rho \) and are therefore mutually defining.

The following is a hypercomplex analogue of the \( \partial \bar{\partial} \)-lemma on complex manifolds.

**Proposition 1.1.** (The \( \partial \bar{\partial} \)-lemma cf. [4]) A form \( \Omega \in \Omega^{2,0}(M) \) satisfies \( \partial \Omega = 0 \) if and only if it can be locally represented as \( \Omega = \partial \bar{\partial} f \) for a function \( f \in C^\infty(M, \mathbb{R}) \).

12 Definition. A function \( u \in C^\infty(M, \mathbb{R}) \) on a hypercomplex manifold \((M, I, J, K)\) for which \( \partial \bar{\partial} J u \) is (strictly) positive will be called (strictly) plurisubharmonic.

13 Definition. On any hypercomplex manifold there exists a unique connection \( \nabla \) with zero torsion that satisfies \( \nabla I = \nabla J = \nabla K = 0 \). It will be called the Obata connection [14] of the hypercomplex manifold.

For a quadratic form \( Q \) on a right vector space \( V \) over the quaternions we shall denote by \( \langle Q \rangle_u \) its average over the action of the group \( SU(2) \) of unit quaternions, that is

\[
\langle Q \rangle_u(x) := \int_{SU(2)} Q(x \circ L) d\nu(L),
\]

for the Haar probability measure \( \nu \) on \( SU(2) \). We shall denote by \( Q_+ \) the projection \( Q_+ := Q - \langle Q \rangle_u \) of \( Q \).

14 Lemma. For a quadratic form \( Q \) on a right vector space \( V \) over the quaternions

\[
\langle Q \rangle_u(x) = \frac{1}{4} (Q(x) + Q(x \circ I) + Q(x \circ J) + Q(x \circ K)).
\]

The lemma follows by noting that both sides of the equality are \( SU(2) \)-invariant on one hand, and both their averages over \( SU(2) \) equal \( \langle Q \rangle_u(x) \) on the other.

15 Definition. For a quaternion \( q \in \mathbb{H} \) written in the standard form

\[
q = t + x \cdot i + y \cdot j + z \cdot k
\]

define the Dirac-Cauchy-Riemann operator \( \partial_{\bar{q}} \) on an \( \mathbb{H} \)-valued function \( F \) by

\[
\partial_{\bar{q}} F = \frac{\partial}{\partial t} F + \frac{\partial}{\partial x} F + \frac{\partial}{\partial y} F + \frac{\partial}{\partial z} F,
\]

and its quaternionic conjugate \( \partial_q \) by

\[
\partial_q F = \overline{\partial_{\bar{q}} F} = \frac{\partial}{\partial t} F + \frac{\partial}{\partial x} F + \frac{\partial}{\partial y} F + \frac{\partial}{\partial z} F.
\]

Note that the quaternionic Hessian \( Hess_\mathbb{H}(u) := \langle \frac{\partial^2 u}{\partial q_i \partial q_j} \rangle \) is the average of the quadratic form \( D^2 u \) over \( SU(2) \), as a short computation shows. As for different \( i \) and \( j \) the operators \( \partial_{\bar{q}_i} \) and \( \partial_{\bar{q}_j} \) commute, this matrix is hyper-Hermitian, namely satisfies the following definition.
16 Definition. We call a quaternionic matrix \( A = (a_{ij})_{i,j=1}^n \) hyper-Hermitian if \( a_{ji} = \overline{a_{ij}} \) for the quaternionic conjugation \( \mathbb{H} \ni q \mapsto \overline{q} \in \mathbb{H} \).

We shall also use a version of a determinant defined for hyper-Hermitian quaternionic matrices referring for further details, properties, and references to [1]. Considering \( \mathbb{H} \) as an \( \mathbb{R} \)-linear vector space of dimension 4 we have the embedding \( 0 \to \text{Mat}(n, \mathbb{H}) \to \text{Mat}(4n, \mathbb{R}) \) of matrix \( \mathbb{R} \)-algebras. Denote by \( \mathcal{H}_n \) the image of the subspace of hyper-Hermitian matrices.

**Proposition 1.2.** There exists a polynomial \( P \) on \( \mathcal{H}_n \) such that \( P^4 = \det |_{\mathcal{H}_n} \) and \( P(\text{Id}) = 1 \). Moreover \( P \) is uniquely determined by these properties, is homogenous of degree \( n \) and has integer coefficients.

17 Definition. For a hyper-Hermitian matrix \( A \) the Moore determinant \( \det(A) := P(\mathbb{R}A) \in \mathbb{R} \) for \( \mathbb{R}A \) the matrix in \( \mathcal{H}_n \) corresponding to \( A \).

**Theorem 1.** The Moore determinant restricts to the usual determinant on the \( \mathbb{R} \)-subspace of complex Hermitian matrices.

On flat HKT manifolds, the Moore determinant of \( \text{Hess}_{\mathbb{H}}(u) \) can be naturally identified (up to a positive multiplicative constant) with \( (\partial \partial^J u)^n \), see [4, Corollary 4.6].

1.2 The quaternionic Calabi-Yau equation

A quaternionic version of the classical Calabi problem was introduced by M. Verbitsky and the first named author in [5]. It says that if \( (M, I, J, K, \Omega) \) is a compact HKT-manifold and \( f \in C^\infty(M, \mathbb{R}) \) is a smooth real valued function, then there exists a constant \( A > 0 \) such that the equation

\[
(\Omega + \partial \partial^J \phi)^n = Ae^f \Omega^n
\]

has a solution \( \phi \in C^\infty(M, \mathbb{R}) \). Analogously to the Kähler case, if such a \( \phi \) exists then \( \Omega + \partial \partial^J \phi \) is an HKT form. This equation is non-linear and elliptic of second order. The uniqueness of a solution is shown in [5] and the existence of solutions was conjectured. Notice that recently M. Verbitsky [18] has found a geometric interpretation of this equation.

In [5] it was also shown that a solution of the above equation satisfies a \( C^0 \) a priori estimate provided that the holonomy of the Obata connection of \( (M, I, J, K) \) is contained in the subgroup \( SL_n(\mathbb{H}) \subset GL_n(\mathbb{H}) \). The main result of this paper is to show the \( C^0 \) estimate under different assumptions on the manifold, namely for the case of locally flat hypercomplex structure (equivalently, whenever the Obata connection is flat).

In the recent preprint [3] the above conjecture was solved by the first named author under the additional assumption that \( M \) admits a hyperKähler metric compatible with the underlying hypercomplex structure.

A different quaternionic version of the Monge-Ampère equation on the flat quaternionic space \( \mathbb{H}^n \) was introduced earlier by the first named author [2] (based also on [1]) where he proved the solvability of the Dirichlet problem under appropriate assumptions.
1.3 A-priori bounds

In this paper we show uniform a priori bounds on the solution $\phi$ of the quaternionic Calabi-Yau equation on a locally flat HKT manifold. That is, we show that there exists a constant $C$ depending only on $(M, I, J, K, \Omega)$ and $f$ such that the solution $\phi$ normalized e.g. by $\max_M \phi = 0$ satisfies

$$||\phi||_{L^\infty} \leq C.$$ 

Such bounds were shown in [5] under the condition that there exists a holomorphic $(2n, 0)$-form on $M$ (with respect to the complex structure $I$) that is nowhere vanishing. We show these bounds in the case of general locally flat HKT manifolds.

More specifically we prove the following.

**Theorem 2.** Let $(M, I, J, K, \Omega)$ be a compact HKT manifold such that the Obata connection is flat. Let $\phi \in C^\infty(M, \mathbb{R})$ be a solution of the equation

$$(\Omega + \partial \partial_J \phi)^n = f \Omega^n; \quad f > 0$$

with the normalization condition $\max_M \phi = 0$. Then

$$||\phi||_{L^\infty} \leq C,$$

for a constant $C$ depending only on $(M, I, J, K, \Omega)$ and $||f||_{L^\infty}$.

**18 Remark.** Let us describe an example of an HKT manifold satisfying the assumptions of Theorem 2 which does not satisfy the assumptions of the uniform estimate obtained in [5]. This example was kindly mentioned to us by the anonymous referee; it appears explicitly in [11, Section 4.3].

Let us consider the space $\mathbb{H}^n \setminus \{0\}$ with the standard flat hypercomplex structure. Consider the HKT metric corresponding to the form $\Omega = \partial \partial_J (\ln r)$ where $r$ is the Euclidean distance to the origin; the condition of positivity can be easily checked by a direct computation.

Let us fix a real number $q > 0$, $q \neq 1$. The multiplication by $q$ acts on $\mathbb{H}^n \setminus \{0\}$; this action preserves the hypercomplex structure and the HKT metric. Hence it induces an HKT structure on the quotient manifold $(\mathbb{H}^n \setminus \{0\})/\langle q \rangle$ where $\langle q \rangle$ denotes the group generated by the multiplication by $q$. This quotient is a compact HKT manifold; it is called the quaternionic Hopf manifold. It is well known (and not hard to check) that this manifold, equipped with the complex structure $I$, has no non-zero holomorphic $(2n, 0)$-forms. In particular the uniform estimate of [5] does not apply to it.

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2 Proof

Here we present the proof of Theorem 2 assuming several important propositions that we prove later. The method of the proof is heavily based on the paper [6] of Z. Blocki. We use the generic notation \textit{const} to denote any positive constant that depends only on \( n, M, I, J, K, \Omega \).

**Proof. (Theorem 2)**

**Step 1:** the \( L^1 \) bound.

As proven in [5] \( \Omega + \partial\partial_J \phi > 0 \) in this case, or in other words it is an HKT form for a certain HKT metric. Whence by Proposition 2.3 that we prove in Section 2.2 below, with the normalization \( \max_M \phi = 0 \), the norm \( ||\phi||_{L^1} \) is bounded by a constant depending on \( M, I, J, K, \Omega \).

**Step 2:** The Taylor expansion of the potential of the metric.

We first formulate and prove the exact statements that allow us to make our calculations uniform in an appropriate sense, and then make the calculations.

**19 Lemma.** Given an HKT-manifold \((M, I, J, K, \Omega)\) there exist \( r > 0 \) and \( a > 0 \) such that \( \forall z_0 \in M \exists g \in C^\infty(B(z_0, 2r)) \) such that

1. \( g < 0 \) on \( B(z_0, 2r) \),
2. \( \Omega = \partial\partial_J g \) on \( B(z_0, 2r) \),
3. \( \inf_{B(z_0, 2r) \setminus B(z_0, r)} g \geq \inf_{B(z_0, r)} g + a \),
4. \( \inf_{B(z_0, r)} g = g(z_0) \),
5. \( ||g||_{C^{10}(B(z_0, 2r))} \leq \text{const.} \)

**Proof. (Lemma 19)**

1. **Claim 1. Suitable atlas:** There exists a finite open cover \( \{U_i\}_{i \in I} \) of \( M \) such that

   1. \( U_i \subset V_i \) for open sets \( V_i \) isomorphic as hypercomplex manifolds to open subsets in \( \mathbb{H}^n \),
   2. there exist functions \( g_i \in C^\infty(V_i, \mathbb{R}) \) such that \( \Omega = \partial\partial_J g_i \) on \( V_i \),
   3. \( ||g_i||_{C^{20}(U_i)} < \text{const} \),
   4. there exists \( r_0 > 0 \) such that for any point \( z_0 \in M \) there exists \( i_0 \in I \) satisfying \( B(z_0, 2r_0) \subset U_{i_0} \).

   This claim is an easy consequence of Proposition 1.15 of [4].

2. **Claim 2. Uniform lower bound on the quaternionic Hessian:** There exists a positive constant \( \epsilon \) such that \( \text{Hess}_{\mathbb{H}}(g_i) > \epsilon I \) on \( U_i \) for all \( i \in I \).

   Indeed, since \( g_i \) is strictly plurisubharmonic on \( V_i \) we have \( \text{Hess}_{\mathbb{H}}(g_i) > 0 \) and as \( U_i \subset V_i \), there exist constants \( \epsilon_i \) such that \( \text{Hess}_{\mathbb{H}}(g_i) > \epsilon_i I \) on \( U_i \). Take \( \epsilon \leq \min_{i \in I} \epsilon_i \).

   Given a point \( z_0 \in M \), we shall now construct the function \( g \) and choose \( r > 0, a > 0 \). Consider \((U, \tilde{g}) = (U_{i_0}, g_{i_0})\) with \( B(z_0, 2r_0) \subset U_{i_0} \).

   The Taylor expansion of \( \tilde{g} \) about \( z_0 \) (in flat coordinates on \( V_i \supset U_i \)) gives

   \[
   \tilde{g}(z_0 + h) = \tilde{g}(z_0) + d_{z_0} \tilde{g}(h) + D^2_{z_0} \tilde{g}(h) + \Theta(h),
   \]

   where \( \Theta(h) \) represents the remainder term.
where $\Theta(h) = o(|h|^2)$. Now we split the quadratic form $D^2_{z_0} \tilde{g}(h)$ into its invariant and complementary parts with respect to the action of $SU(2)$:

$$D^2_{z_0} \tilde{g}(h) = \langle D^2_{z_0} \tilde{g} \rangle_u(h) + \langle D^2_{z_0} \tilde{g} \rangle_u(h) = Hess_{\mathbb{H}}(\tilde{g})_{z_0}(h) + \langle D^2_{z_0} \tilde{g} \rangle_u(h).$$

Define $g$ by

$$g(z_0 + h) := Hess_{\mathbb{H}}(g)_{z_0}(h) + \Theta(h).$$

Claim 3. Equality of Hessians: We have the pointwise equality $Hess_{\mathbb{H}}(g) \equiv Hess_{\mathbb{H}}(\tilde{g})$ on $U$ (and therefore $\Omega = \partial \partial_f g$ on $U$).

Proof. (Claim 3) It is sufficient to prove that the function $p(h) := \tilde{g} - g = g(z_0) + d_{z_0} g(h) + \langle D^2_{z_0} g \rangle_u(h)$ satisfies $Hess_{\mathbb{H}}(p) \equiv 0$ on $U$. The quaternionic Hessian of the affine function $g(z_0) + d_{z_0} g(h)$ certainly vanishes. Therefore it is enough to prove that $Hess_{\mathbb{H}}(\langle D^2_{z_0} g \rangle_u(h)) \equiv 0$. And indeed

$$Hess_{\mathbb{H}}(\langle D^2_{z_0} g \rangle_u(h)) = \langle D^2(D^2_{z_0} g - \langle D^2_{z_0} g \rangle_u) \rangle_u = \langle D^2 g - \langle D^2_{z_0} g \rangle_u \rangle_u = \langle D^2_{z_0} g(u) - \langle D^2_{z_0} g \rangle_u \rangle_u = \langle D^2 g \rangle_u - \langle \langle D^2 g \rangle_u \rangle_u = \langle D^2 g \rangle_u - \langle D^2 g \rangle_u = 0.$$ 

The second equality follows since for a quadratic form $Q$ we have $D^2 Q = Q$. \hfill \Box

Note that by construction $D^2_{z_0} g = Hess_{\mathbb{H}}(g)_{z_0} = Hess_{\mathbb{H}}(\tilde{g})_{z_0} > \epsilon I$ where $\epsilon$ is provided by Claim 2. We shall choose $r < r_0$ small enough so that $D^2_{(z_0 + h)} g \geq \epsilon \frac{1}{2} I$ for all $h \leq 2r$. Consider the Taylor expansion of $D^2_{(z_0 + h)} g$ in $h$ about $h = 0$:

$$D^2_{(z_0 + h)} g = D^2_{z_0} g + \Theta_1(h).$$

The function $\Theta_1$ satisfies $|\Theta_1| < \kappa |h| I$ for a constant $\kappa$ depending only on $||\tilde{g}||_{C^1(U)}$. Therefore, by Claim 1, $\kappa$ depends on $M, I, J, K, \Omega$ only. Consequently

$$D^2_{(z_0 + h)} g \geq \epsilon I - \kappa |h| I = (\epsilon - \kappa |h|) I.$$ 

Hence for $|h| < \frac{\epsilon}{\kappa}$ we have $D^2_{(z_0 + h)} g \geq \frac{\epsilon}{2} I$. Therefore we can choose $r = \min\{r_0, \frac{\epsilon}{2\kappa}\}/4$. Note that $r = const$. By the choice of $r$, the function $g$ is convex with $D^2 g \geq \frac{\epsilon}{2} I$ in $B(z_0, 2r)$ with minimum in $z_0$, thus satisfying condition 4.

By a straightforward computation we have then the estimate $g(z_0 + h) > const \cdot \frac{\epsilon r^2}{2}$. Hence $a = const \cdot \frac{\epsilon r^2}{2} = const$ can be chosen to satisfy condition 3. Condition 2 is satisfied by Claim 3. The function $g$ can be modified by adding a constant to satisfy condition 1. Condition 5 is satisfied by property 3 of Claim 1. \hfill \Box

Step 3: Wrapping up.

Choose $z_0 \in M$ at which the function $\phi$ attains its minimum: $\phi(z_0) = \min_M \phi$. Lemma 19 provides us then with an appropriate $r, a > 0$ and $g \in C^\infty(B(z_0, 2r))$. Then the function $u = \phi + g$ in the domain $D = B(z_0, 2r)$ satisfies the conditions of Lemma 20 below. Indeed $\partial \partial_f u = \Omega + \partial \partial_f (\phi + g) > 0$ whence $u$ is plurisubharmonic and $u$ is negative on $D$ as both $\phi$ and $g$ are. The set $\{u < \inf_D u + a\}$ is contained in $B(z_0, r) \subset B(z_0, 2r)$ and hence is relatively compact.
Lemma \[20\] gives us

$$\|\phi\|_{L^\infty(M)} - \|g\|_{L^\infty(D)} \leq \|u\|_{L^\infty(D)} \leq a + \text{const} \cdot (r/a)^4 n \|u\|_{L^1(D)} \|f\|^4_{L^\infty(D)}.$$ 

Moreover $\|u\|_{L^1(D)} \leq \|\phi\|_{L^1(M)} + \|g\|_{L^\infty(D)} \leq \text{const}$ by Step 1 and property 5 of $g$ and $f > \text{const} \cdot f_0$ by property 5 of $g$. Hence

$$\|\phi\|_{L^\infty(M)} \leq \text{const} + \text{const} \cdot \|f\|^4_{L^\infty(D)},$$

which finishes the proof. \qed

2.1 The ABP inequality

We would like to use the following lemma in the proof.

20 Lemma. Let $D \subset \mathbb{H}^n$ be a bounded domain. Let $u$ be a negative $C^2$ strictly plurisubharmonic function in $D$ and $a > 0$ a constant such that the set $\{u < \inf_D u + a\}$ is relatively compact in $D$. Denote by $f$ the Moore determinant $f = \det(\frac{\partial^2 u}{\partial q_i \partial q_j})$. Then

$$\|u\|_{L^\infty(D)} \leq a + \text{const} \cdot \frac{\text{diam}(D)^{4n}}{a^{4n}} \|u\|_{L^1(D)} \|f\|^4_{L^\infty(D)}.$$ 

This lemma is a formal consequence of the following proposition:

Proposition 2.1. Let $D \subset \mathbb{H}^n$ be a bounded domain. Let $u \in C^2(D) \cap C(\overline{D})$ be a non-positive strictly plurisubharmonic function in $D$, that vanishes on $\partial D$. Denote by $f$ the Moore determinant $f = \det(\frac{\partial^2 u}{\partial q_i \partial q_j})$. Then

$$\|u\|_{L^\infty(D)} \leq \text{const} \cdot \text{diam}(D) \|f\|^{1/n}_{L^4(D)}.$$ 

We will first show how the lemma follows from the proposition and then prove the latter.

Proof. (Lemma \[20\]) Define $v := u - \inf_D u - a$. Set $D' = \{v < 0\}$. Then $D' \Subset D$ by assumption. By Proposition \[2.1\]

$$a = \|v\|_{L^\infty(D')} \leq \text{const} \cdot \text{diam}(D') \|f\|^{1/n}_{L^4(D')} \leq \text{const} \cdot \text{diam}(D')(\text{Vol}(D'))^{1/4n} \|f\|^{1/n}_{L^\infty(D')}.$$ 

On the other hand

$$\text{Vol}(D') \leq \frac{\|u\|_{L^1(D')}}{\inf_D u + a} = \frac{\|u\|_{L^1(D)}}{\|u\|_{L^\infty(D)} - a}.$$ 

The lemma now follows by direct substitution. \qed

Proof. (Proposition \[2.1\]) By the Alexandrov-Bakelman-Pucci inequality (Lemma 9.2 in [10]) we have

$$\|u\|_{L^\infty(D)} \leq \text{const} \cdot \text{diam}(D) \left(\int_{\Gamma} \det D^2 u\right)^{1/4n}, \quad (2.1)$$

where

$$\Gamma := \{x \in D| u(x) + (Du(x), y - x) \leq u(y) \forall y \in D\} \subset \{D^2 u \geq 0\}.$$
Using quaternionic linear algebra (cf. [1] and the references therein) if \( w^1, \ldots, w^n \) are a hyper-Hermitian orthonormal \( \mathbb{H} \)-base of eigenvectors of \( \frac{\partial^2 u}{\partial ar{q}_i \partial q_j} \) then \( w^1, \ldots, w^n, w^1 \circ I, \ldots, w^n \circ I, w^1 \circ J, \ldots, w^n \circ J, w^1 \circ K, \ldots, w^n \circ K \) form an orthonormal basis in \( \mathbb{R}^{4n} \cong \mathbb{H}^n \) and at a point where \( D^2 u \geq 0 \) we obtain

\[
\det(\frac{\partial^2 u}{\partial ar{q}_i \partial q_j}) = \prod_{l=1}^{n} \frac{n!}{\prod_{i,j=1}^{n} w^l_i \partial^2 u(w^l_i) + D^2 u(w^l_i \circ I) + D^2 u(w^l_i \circ J) + D^2 u(w^l_i \circ K)}
\]

\[
\geq \text{const} \prod_{l=1}^{n} \sqrt{(D^2 u(w^l_i))(D^2 u(w^l_i \circ I))(D^2 u(w^l_i \circ J))(D^2 u(w^l_i \circ K))} \geq \text{const} \sqrt{\det D^2 u}.
\]

The last inequality follows from the fact that the determinant of a real nonnegative symmetric matrix does not exceed the product of its diagonal entries. Thus at a point where \( D^2 u \geq 0 \) we have

\[
\det D^2 u \leq \text{const} \cdot \det(\frac{\partial^2 u}{\partial ar{q}_i \partial q_j})^4.
\]

Plugging this into (2.1) yields the proposition.

2.2 The \( L^1 \)-bound

Let \( (M^{4n}, I, J, K) \) be a compact hypercomplex manifold. Let us fix \( \Omega \in \Lambda^{2,0}_{I,\mathbb{R}}(M) \) which is real and strictly positive (we do not assume that \( \Omega \) is HKT, i.e. \( \partial \Omega \) may not vanish). The following result is a version of the Gauduchon theorem [9]; the proof follows closely the lines of his arguments.

**Proposition 2.2.** There exists a unique, up to a positive multiplicative constant, form \( \Theta \in \Lambda^{2n,0}_{I,\mathbb{R}} \) which is strictly positive pointwise (in particular non-vanishing) and

\[
\partial \partial_J(\Omega^{n-1} \wedge \Theta) = 0.
\]

**Proof.** Let us fix a positive non-vanishing form \( \Theta_0 \in \Lambda^{2n,0}_{I,\mathbb{R}} \) (e.g. \( \Omega^n \)). Let us consider the operator \( A \) on real functions on \( M \) given by

\[
Af := \frac{\partial \partial_J f \wedge \Omega^{n-1} \wedge \Theta_0}{\Omega^n \wedge \Theta_0}.
\]

Clearly \( A \) is elliptic and \( A(1) = 0 \).

Define a positive definite scalar product on functions by

\[
< f, g > = \int_M f g \cdot \Omega^n \wedge \Theta_0.
\]

Let us compute the conjugate operator \( A^* \) with respect to this product:

\[
< Af, g > = \int (\partial \partial_J f) \cdot g \wedge \Omega^{n-1} \wedge \Theta_0 = \int f \cdot \partial \partial_J (g \cdot \Omega^{n-1} \wedge \Theta_0) = < f, \frac{\partial \partial_J (g \cdot \Omega^{n-1} \wedge \Theta_0)}{\Omega^n \wedge \Theta_0} > .
\]

Hence

\[
A^* g = \frac{\partial \partial_J (g \cdot \Omega^{n-1} \wedge \Theta_0)}{\Omega^n \wedge \Theta_0}.
\]
Since the operators $A$ and $A^*$ have the same symbol, their indices vanish. By the maximum principle the kernel of $A$ consists only of constant functions. Hence the kernel of $A^*$ is one dimensional. Let us denote by $G$ a generator of the kernel of $A^*$. If we show that $G$ never vanishes then $\Theta := \pm G \cdot \Theta_0$ will satisfy the proposition.

21 Lemma.

$$<G, 1> \neq 0.$$  

Proof. Assume the contrary. Then $1 \in (\text{Ker} A^*)^\perp = \text{Im} A$. Then there exists a function $f$ such that $Af = 1$. Let $x_0$ be a point of maximum of $f$. Then since $A(1) = 0$ we get$$1 = Af(x_0) \leq 0.$$ 

This is a contradiction. The lemma is proved. \qed

Let us now show that $G$ cannot change sign. Let $\phi > 0$ be any positive function. Put $\Omega_\phi := \phi \cdot \Omega$. Define the operator $A_\phi$ exactly as $A$ but with $\Omega_\phi$ instead of $\Omega$:

$$A_\phi f = \frac{\partial \theta J f \wedge \Omega_\phi^{n-1} \wedge \bar{\Theta}_0}{\Omega_\phi^n \wedge \bar{\Theta}_0} = \phi^{-1} \cdot Af.$$  

Let us compute the adjoint $A_\phi^*$ with respect to the pairing $<\cdot, \cdot>_\phi$ which is defined by$$<f, g>_\phi = \int_M f g \cdot \Omega_\phi^n \wedge \bar{\Theta}_0.$$  

Similarly to (2.5), we have$$A_\phi^* g = \frac{\partial \theta J (g \cdot \Omega_\phi^{n-1} \wedge \bar{\Theta}_0)}{\Omega_\phi^n \wedge \bar{\Theta}_0}.$$  

By assumption $A^* G = 0$, namely $\partial \theta J (G \cdot \Omega^{n-1} \wedge \bar{\Theta}_0) = 0$. Trivially$$0 = \partial \theta J (\phi^{1-n} G \cdot \Omega_\phi^{n-1} \wedge \bar{\Theta}_0) = A_\phi^* (\phi^{1-n} \cdot G) \cdot (\Omega_\phi^n \wedge \bar{\Theta}_0).$$ 

Hence $\phi^{1-n} G \in \text{Ker} A_\phi^*$. 

Let us apply Lemma [21] to $\phi^{1-n} \cdot G$ instead of $G$ and to $<\cdot, \cdot>_\phi$ instead of $<\cdot, \cdot>$. We have$$0 \neq <\phi^{1-n} \cdot G, 1>_\phi = \int \phi^{1-n} G \cdot \Omega_\phi^n \wedge \bar{\Theta}_0 = \int \phi \cdot G \cdot \Omega^n \wedge \bar{\Theta}_0.$$  

Thus we have shown that the function $G$ is such that for any $\phi > 0$$$
\int \phi G \cdot (\Omega^n \wedge \bar{\Theta}_0) \neq 0.$$

This implies easily that $G$ cannot change its sign.

It remains to show that $G$ cannot vanish at any point. This immediately follows from the following general lemma.
22 Lemma ([9], Lemma 2). Any non-negative solution of a linear elliptic differential equation of second order with $C^\infty$-smooth real coefficients is either strictly positive or vanishes identically.

Thus Proposition 2.2 is proved.

23 Lemma. Let $(M^{4n}, I, J, K)$ be a compact hypercomplex manifold with an HKT-form $\Omega$. Let $\Theta \in \Lambda_{I,\mathbb{R}}^{2n,0}$ be a positive form as in Proposition 2.2, i.e. $\partial \bar{\partial}(\Omega^{n-1} \wedge \bar{\Theta}) = 0$. Let

$$A \phi = \frac{\partial \bar{\partial} \phi \wedge \Omega^{n-1} \wedge \bar{\Theta}}{\Omega^n \wedge \Theta}.$$ 

Then the operator $A$ admits a non-negative Green function $G(x, y) \geq 0$, namely

$$- \int G(x, y) \cdot A \phi(y) \cdot (\Omega^n \wedge \bar{\Theta}) = \phi(x) - \frac{1}{\text{vol}(M)} \int \phi \cdot (\Omega^n \wedge \bar{\Theta})$$

for any function $\phi$ and point $x$ (here $\text{vol}(M) = \int \Omega^n \wedge \bar{\Theta}$).

Proof. Let $G_0$ be a Green function bounded from below (as in Appendix A). However we can add to $G_0$ a large constant. Indeed

$$\int A \phi(y) \cdot (\Omega^n \wedge \bar{\Theta}) = \int \partial \bar{\partial} \phi \wedge \Omega^{n-1} \wedge \bar{\Theta} = \int \phi \cdot \partial \bar{\partial}(\Omega^{n-1} \wedge \bar{\Theta}) = 0.$$

Proposition 2.3. Assume that $\Omega + \partial \bar{\partial} \phi > 0$ and $\max_M \phi = 0$. Then $||\phi||_{L^1}$ is bounded by a constant depending on $(M, I, J, K, \Omega)$ only.

Proof. Let us choose positive $\Theta$ as in Proposition 2.2. Let $x \in M$ be a point of maximum of $\phi$. Denote by $A$ the operator as in Lemma 23 and let $G \geq 0$ be its Green function. We have by assumption

$$-A \phi = -\frac{\partial \bar{\partial} \phi \wedge \Omega^{n-1} \wedge \bar{\Theta}}{\Omega^n \wedge \Theta} \leq 1.$$ 

We have

$$||\phi||_{L^1} = 0 \int \phi \cdot (\Omega^n \wedge \bar{\Theta}) = - \int G(x, y) A \phi(y) \cdot (\Omega^n \wedge \bar{\Theta}) \leq \int G(x, y) \cdot (\Omega^n \wedge \Theta) \leq \text{const}(M, \Omega).$$
A Green function bounded from below

We adapt several proofs from [16] together with a classical local result to prove that for any elliptic second-order differential operator $A$ on a compact manifold $M$ there exists a Green function $G(x,y)$ that satisfies the properties below. This result seemed plausible to experts, however we have not found a proof in the literature and hence choose to fill this gap here.

Consider the Hilbert space $L^2(M,\mathbb{R})$ where the inner product on smooth functions is given by $\langle f,g \rangle := \int_M f(x)g(x)dV(x)$ for a normalized smooth measure $dV$. Denote by $U: L^2(M,\mathbb{R}) \to L^2(M,\mathbb{R})$ the operator of projection onto the orthogonal complement $J_0$ of the kernel $K \subset C^\infty(M)$ of $A$. Then the properties are:

0. $\int_M G(x,y)A\phi(y)dV(y) = U\phi(x)$, for any $\phi \in C^\infty(M)$.
1. $G(x,y)$ is smooth outside the diagonal $\Delta \subset M \times M$.
2. $G(x,y) \geq -D_1$ (a.e.) on $M \times M$ for a constant $D_1 > 0$.
3. For any fixed $x \in M$, $\|G(x,\cdot)\|_{L^1(M)} < D_2$ for a constant $D_2 > 0$.

Note that these properties are what is left to show to establish our desired a priori $L^1$-bound. Indeed in our case $U\phi = \phi - \int_M \phi dV$, for $dV$ the normalized smooth measure $dV = \Omega^n / \int_M \Omega^n$. We now construct an operator $A^1: C^\infty(M,\mathbb{R}) \to C^\infty(M,\mathbb{R})$ whose Schwartz kernel (c.f. [12] Section 5.2) satisfies property (0). We then show that it also satisfies the other properties.

First consider the spaces $K = \text{Ker}A \subset C^\infty(M,\mathbb{R})$ and $C = \text{Ker}A^* \subset C^\infty(M,\mathbb{R})$. These have additive close complements $I = \text{Image}A$ and $J = \text{Image}A^*$ where $A^*$ is the adjoint differential operator to $A$ with respect to $dV$. For each Sobolev completion $L^2_s$ of $C^\infty(M,\mathbb{R})$ denote by $I_s$, $J_s$ the corresponding completions of subspaces. These are the images of the corresponding operators $A_{s+2}: L^2_{s+2} \to L^2_s$ and $(A^*)_{s+2}: L^2_{s+2} \to L^2_s$. For what follows choose an $L^2$-orthonormal basis $\{f_1,\ldots,f_k\} \subset C^\infty(M)$ of the kernel $K$ of $A$, and an $L^2$-orthonormal basis $\{g_1,\ldots,g_l\} \subset C^\infty(M)$ of the kernel $C$ of $A^*$. Denote by $P$ the operator $C^\infty(M) \to C^\infty(M)$ given by $f \mapsto \Sigma_j (f,f_j) f_j$. Its image is $K$ and it is a smoothing operator - its Schwartz kernel is $K_P(x,y) = \Sigma_j f_j(x)f_j(y) \in C^\infty(M \times M)$. Denote by $P_s$ its extension $L^2_s \to C^\infty(M)$.

Similarly define $Q$ with image $C$, with Schwartz kernel $K_Q(x,y) = \Sigma_j g_j(x)g_j(y) \in C^\infty(M \times M)$ and $Q_s$ its extension to $L^2_s$.

For every $s$, we have $L^2_s = K \oplus J_s$ and $L^2_s = C \oplus I_s$ as Banach spaces. Denote by $U_s$ and $V_s$ the projections complementary to $P_s$ and to $Q_s$ - namely $U_s = Id - P_s$ and $V_s = Id - Q_s$. Noting that $A_{s}|_{I_s}: J_s \to I_{s-2}$ is a bijective continuous map of Hilbert spaces and has as such, by the Banach inverse mapping theorem, a continuous inverse $D_{s-2}: I_{s-2} \to J_s$, we construct a continuous linear map

$$(A^1)_{s-2}: L^2_{s-2} \to L^2_s$$

as the composition

$$L^2_{s-2} \xrightarrow{V_{s-2}} I_{s-2} \xrightarrow{D_{s-2}} J_s \xrightarrow{I_s} L^2_s.$$
It is easy to check that these diagrams for different $s$ are commuting with the embeddings $L^2_{s'} \to L^2_s$ for $s' > s$ and therefore define a continuous linear operator

$$A^i : C^\infty(M) \to C^\infty(M).$$

We note that as $(A^i)^s(A^i)_s = U_s$ for all $s$, we have $A^iA = U$ on the $C^\infty$ level - i.e. property (0).

We now construct a pseudodifferential parametrix $B$ for $A$ and then show that $B$ and $A^i$ differ by a smoothing operator.

24 Construction. It is well-known (c.f. [16, Proposition 3.4]) that for the elliptic differential operator $A$ there exists a pseudo-differential operator $B$ on $M$ such that $AB = Id$ and $BA = Id$ are smoothing pseudo-differential operators. Such an operator $B$ is called a (pseudo-differential) parametrix for $A$. For the convenience of the reader and for use in establishing properties (2) and (3) we recall the construction of this operator. Cover the manifold $M$ by small balls $B(x_i, \varepsilon)$ with respect to an auxiliary Riemannian metric. Consider the restrictions $A_i$ of $A$ to the space of functions compactly supported in $B(x_i, 2\varepsilon)$. Then each operator $A_i$ has a pseudo-differential parametrix $B_i$. Then

$$B = \Sigma_i \Psi_i B_i \Phi_i$$

is the required parametrix [16, Proposition 3.4]. Here $\Phi_i, \Psi_i$ are the operators of multiplication by $\phi_i, \psi_i$ respectively for $\{\phi_i\}_i$ forming a partition of unity subordinate to the covering $\{B(x_i, \varepsilon)\}_i$ and $\psi_i$ is a positive smooth function supported in $B(x_i, 1.5\varepsilon)$ and identically equal to 1 on $B(x_i, \varepsilon)$.

25 Lemma. The uniformly elliptic operator $L = A_i$ on the ball $\Omega = B(x_i, 2\varepsilon)$ has a pseudodifferential parametrix $B_i$ whose Schwartz kernel $G_i(x, y)$ satisfies

$$G(x, x + z) = G_0(x, x + z) + o(|z|^{2-n})$$

uniformly for $|z| < 1$, $x \in K \subset \Omega$ as $z \to 0$, where

$$G_0(x, y) = \frac{1}{(n-2)\omega_n} \det(L_2(x))^{-1/2} \langle L_2(x)(x - y), (x - y) \rangle^{2-n},$$

$$Lu = -\Sigma_{i,j}(L_2)_{ij} \partial_i \partial_j u + \Sigma b_k \partial_k u + cu,$$

for a uniformly positive definite matrix $(L_2(x))$ and $\omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}$.

This can be shown in several ways - one is by [15, Chapter III, Theorem 3.3] together with an explicit calculation (up to a smooth function) of the leading term as a Fourier transform of a locally integrable homogenous function.

By Lemma [25] the functions $G_i$ are all bounded from below on $B(x_i, 1.5\varepsilon) \times B(x_i, 1.5\varepsilon)$ since they are positive near the diagonal and smooth off the diagonal. Now the Schwarz kernel of $B$ is

$$K'(x, y) = \Sigma_i \psi_i(x) G_i(x, y) \phi_i(y).$$

We would like to provide lower bounds on $K'(x, y)$ and upper bounds on $||K'(x, \cdot)||_{L^1(M)}$.

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The sign in the definition of $L$ appears already in the case of the constant coefficient Laplace operator.
26 Lemma. Since all $G_1(x, y)$ are bounded from below, we have $K'(x, y) \geq -D'_i$ for $D'_i > 0$ on $M \times M$. Moreover since for all $G_1(x, y)$ we have $\| G_1(x, \cdot) \|_{L^1(B(x, \varepsilon))} < D'_{2,i}$ we have $\| K'(x, \cdot) \|_{L^1(M)} < D'_2 = \Sigma_i D'_{2,i}$.

We now compare $A^t$ and $B$. We follow [16] Theorem 3.6 to show the following.

Proposition A.1.

$$A^t = B + T$$

(A.2)

where $T$ has a Schwartz kernel $K_T \in C^\infty(M \times M)$.

Indeed we have $AB = Id + R$, for a smoothing operator $R$. Multiplying on the left by $A^t$ we have $UB = A^t + A^t R$, that is $A^t = B - PB - A^t R$, where $P$ is the projection operator onto $K = Ker A$.

Note first that $A^t R$ is smoothing as $R$ is smoothing and $A^t$ is continuous as an operator $C^\infty(M, \mathbb{R}) \to C^\infty(M, \mathbb{R})$. Indeed smoothing operators can be described on one hand as those operators $C^\infty(M, \mathbb{R}) \to D(M, \mathbb{K})$ that have smooth kernels, and on the other as those operators that can be extended to linear maps $D(M, \mathbb{R}) \to C^\infty(M, \mathbb{R})$ continuous in the weak topology on every bounded subset of $D(M, \mathbb{K})$. We refer to [3] (23.11.1) for the equivalence of these descriptions. From the second description, the statement is immediate.

Note also that $PB$ is smoothing since $B$ is a pseudo-differential operator and $P$ is smoothing. Hence $A^t = B + T$ for $T = -PB - A^t R$ - a smoothing operator.

27 Wrapping up. Now note that as $A^t = B + T$, we have $G(x, y) = K'(x, y) + K_T(x, y)$, yet $K'(x, y) \geq -D'_1$ on $M \times M$ and $K_T(x, y) \geq -D''_1$ as $K_T \in C^\infty(M \times M)$. Hence $G(x, y) \geq -D_1$ for $D_1 = D'_1 + D''_1$ and property (1) is established. Property (0) follows by definition and property (2) follows from $\| K'(x, \cdot) \|_{L^1(M)} < D'_2$ and $\| K_T(x, \cdot) \|_{L^1(M)} \leq E = \| K_T \|_{L^\infty(M \times M)} < \infty$ for $D_2 = E + D'_2$.

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