Sunspot Equilibrium in General Quitting Games

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Abstract

We prove that general quitting games, which are quitting games in which each player may have more than one continue action, admit a sunspot \( \varepsilon \)-equilibrium, for every \( \varepsilon > 0 \). To this end we show that the equilibrium set of strategic-form games can be uniformly approximated by a smooth manifold, and develop a new fixed-point theorem for manifolds.

1 Introduction

One of the central open questions in game theory to date is whether every multiplayer stochastic games admits a uniform equilibrium payoff. Mertens and Neyman (1981) proved that two-player zero-sum stochastic games admit a uniform value, Vieille (2000a, 2000b) proved that two-player nonzero-sum stochastic games admit a uniform equilibrium payoff, and Solan (1999) proved that three-player absorbing games admit a uniform equilibrium payoff. Solan and Vieille (2001) presented the class of quitting games, and showed that a certain class of these games admit a uniform equilibrium payoff. Further results on this class of games were proven by Simon (2012) and Solan and Solan (2017).

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While the existence of a uniform equilibrium payoff in general stochastic games is still an open problem, the existence of an extensive-form correlated equilibrium payoff in multiplayer stochastic game was proven by Solan and Vieille (2002). Recall that an extensive-form correlated equilibrium payoff is a uniform equilibrium payoff in an extended game, which includes a correlation device that sends at every stage a private signal to each player, where the signal can depend on past signals sent to all players. Solan and Vohra (2001, 2002) proved that every absorbing game admits a normal-form correlated equilibrium payoff, which is a uniform equilibrium payoff in an extended game that includes a correlation device that sends one private signal to each player at the outset of the game.

Recently Solan and Solan (2017) proved that every quitting game admits a sunspot equilibrium payoff, which is an equilibrium payoff in an extended game that includes a correlation device, which sends at every stage a public signal that is uniformly distributed on $[0,1]$ and independent of past signals and play.

In this paper we extend the result of Solan and Solan (2017) to a more general class of absorbing games, namely, the class of general quitting games. A general quitting game is a quitting game in which each player has a single quitting action and possibly several continue actions.

In addition to proving that a sunspot equilibrium payoff exists in the class of general quitting games, the paper has several additional contributions, which are needed in the proof of the main result.

• We show that the equilibrium set can be uniformly approximated by smooth manifolds, a property that allows us to use topological results that require manifolds to be smooth.

• We develop a new fixed point result for manifolds.

• We develop a new technique to studying multiplayer absorbing games, which reduces an absorbing game into a collection of quitting games.

• As noted by Solan, Solan, and Solan (2018), our results imply that if at least two players have at least two continue actions, then the general quitting game admits a uniform equilibrium payoff.

The paper is organized as follows. The model and the main result appear in Section 2. The proof for the case in which one player has two continue
actions and all other players have one continue action appears in Section 3. Section 4 presents the results in topology that we need in the main proofs, and Section 5 shows that the equilibrium set can be uniformly approximated by a smooth manifold. In Section 6 we provide the proof of the main result. Section 7 discusses extensions of our main result to other classes of absorbing games.

2 The Model and Main Results

2.1 General Quitting Games

Definition 2.1 A general quitting game is a vector \( \Gamma = (I, (A^c_i)_{i \in I}, u) \) where

- \( I \) is a finite set of players.
- \( A^c_i \) is a finite nonempty set of continue actions, for each player \( i \in I \). The set of all actions of player \( i \) is \( A_i := A^c_i \cup \{Q_i\} \), where \( Q_i \) is interpreted as a quitting action. The set of all action profiles is \( A = \times_{i \in I} A_i \).
- \( u : A \rightarrow [0, 1]^I \) is a payoff function.

The game proceeds as follows. At every stage \( t \in \mathbb{N} \), each player \( i \in I \) chooses an action \( a^t_i \in A_i \). Let \( a^t = (a^t_i)_{i \in I} \) be the action profile chosen at stage \( t \). We denote by \( t_* \) the first stage in which a quitting action is played; that is,

\[
t_* := \min\{t \in \mathbb{N} : a^t_i = Q_i \text{ for some } i \in I\},
\]

with the convention that the minimum of an empty set is \( \infty \). The stage payoff at stage \( t \) is \( r(a^{\min\{t,t_*\}}) \). Thus, the play in effect terminates at stage \( t_* \), and the payoff after stage \( t_* \) is equal to the payoff at stage \( t_* \).

General quitting games are an extension of the class of quitting games, in which each player has a single continue action.

Definition 2.2 A general quitting game \( \Gamma = (I, (A^c_i)_{i \in I}, u) \) is a quitting game if \( |A^c_i| = 1 \) for every player \( i \in I \).

Quitting games were first studied by Flesch, Thuijsman, and Vrieze (1997), who studied a specific three-player quitting game and identified the set of its
uniform equilibrium payoffs, and were formally defined by Solan and Vieille (2001).

A (behavior) strategy of player $i$ is a function $\sigma_i: (\cup_{t=0}^{\infty} A^t) \to \Delta(A_i)$. A strategy profile $\sigma = (\sigma_i)_{i \in I}$, one for each player. Every strategy profile $\sigma$ induces a probability distribution over the set of plays $A^\infty$. Denote by $E_\sigma$ the corresponding expectation operator and by

$$\gamma(\sigma) := E_\sigma \left[ \lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} u(\min\{t, t^*\}) \right]$$

the (undiscounted) payoff under strategy profile $\sigma$. Note that the way a strategy is defined after the termination stage $t^*$ does not affect the payoff.

The set of mixed action profiles is $X := \times_{i \in I} \Delta(A_i)$. For every mixed action profile $x \in X$ denote by $p(x) := 1 - \prod_{i \in I} x_i(A^c_i)$ the per-stage probability of absorption under $x$. The mixed action profile $x$ is absorbing if $p(x) > 0$, and nonabsorbing if $p(x) = 0$. For every absorbing mixed action profile $x \in X$ denote by $u(x)$ the expected absorbing payoff under $x$:

$$u(x) := \frac{\sum_{a \in A} \left( \prod_{i \in I} x_i(a_i) \right) u(a)}{p(x)}.$$

**Definition 2.3** Let $(I, (A_i^c)_{i \in I}, u)$ be a general quitting game and let $\varepsilon > 0$. A strategy profile $\sigma = (\sigma_i)_{i \in I}$ is an $\varepsilon$-equilibrium if for every player $i \in I$ and every strategy $\sigma'_i$ of player $i$,

$$\gamma_i(\sigma) \geq \gamma_i(\sigma'_i, \sigma_{-i}) - \varepsilon.$$

The equilibrium concept that we study in this paper is undiscounted equilibrium. By arguments similar to those of Solan and Vieille (2001), our results apply to the stronger notion of uniform equilibrium.

Solan (1999) proved that every three-player absorbing game admits an $\varepsilon$-equilibrium, for every $\varepsilon > 0$. To date it is not known whether this result extends to absorbing games with more than three players; for partial results, see Solan and Vieille (2001), Simon (2012), and Solan and Solan (2017).

**2.2 Sunspot Equilibrium**

We enrich the game by introducing a public correlation device: at the beginning of every stage $t \in \mathbb{N}$ the players observe a public signal $y^t \in [0, 1]$ that is drawn by the uniform distribution, independently of past signals and play.
A strategy of player $i$ in the game with public correlation device is a sequence of measurable functions $\xi_i = (\xi_i^t)_{t \in \mathbb{N}}$, where $\xi_i^t : [0, 1]^t \times A^{t-1} \to \Delta(A_i)$. The interpretation of $\xi_i^t$ is that if the play was not absorbed before stage $t$, then at stage $t$ player $i$ plays the mixed action $\xi_i^t(y^1, a^1, y^2, a^2, \cdots, a^{t-1}, y^t)$.

Every strategy profile $\xi = (\xi_i)_{i \in I}$ induces a probability distribution over the set of plays in the game with public correlation device, with a corresponding expectation operator that is denoted by $E_{\xi}$. Denote by $\gamma(\xi) := E_{\xi} \left[ \lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} u(a_{\min\{t, t_i\}}) \right]$ the expected payoff under strategy profile $\xi$. An $\varepsilon$-equilibrium in the extended game is called a sunspot $\varepsilon$-equilibrium of the original game.

**Definition 2.4** A strategy profile $\xi$ is a sunspot $\varepsilon$-equilibrium if it is an $\varepsilon$-equilibrium in the game with public correlation device, that is, if for every $i \in I$ and every strategy $\xi'_i$ of player $i$ we have

$$\gamma_i(\xi) \geq \gamma_i(\xi'_i, \xi_{-i}) - \varepsilon.$$ 

Solan and Solan (2017) proved that every quitting game admits a sunspot $\varepsilon$-equilibrium, for every $\varepsilon > 0$. Our main result concerns the extension of this result to general quitting games.

**Theorem 2.5** Every general quitting game admits a sunspot $\varepsilon$-equilibrium, for every $\varepsilon > 0$.

### 2.3 Linear Complementarity Problems

Our study of sunspot $\varepsilon$-equilibrium uses the concept of linear complementarity problems, which we introduce in this section.

Let $r^1, r^2, \cdots, r^n$ be $n$ vectors in $\mathbb{R}^n$, and let $q \in \mathbb{R}^n$. Denote

$$\mathcal{R} := \{ x \in \mathbb{R}^n : x_i \geq r^i_i, \ \forall i \in \{1, 2, \ldots, n\} \}. $$

The linear complementarity problem $\text{LCP}(((r^i)_{i=1}^n, q))$ is the following problem that consists of linear equalities and inequalities:

Find $w \in \mathcal{R}$, and $z = (z_0, z_1, \cdots, z_n) \in \Delta(\{0, 1, \cdots, n\})$,
such that $w = z_0 q + \sum_{i=1}^{n} z_i r^i$,

$$z_i = 0 \text{ or } w_i = r^i_i, \ \forall i \in \{1, 2, \ldots, n\}. $$
The last condition in the problem (1) is the complementarity condition. We note that for \( q \in \mathbb{R} \) there is always at least one solution to the problem (1), namely, \( z = (1, 0, \cdots, 0) \) and \( w = q \).

It will be convenient to present the vectors \( r^1, r^2, \cdots, r^n \) as a matrix \( R \) whose \( i \)'th column is \( r^i \), for \( i \in \{1, 2, \ldots, n\} \). We will then denote the linear complementarity problem \( \text{LCP}((r^i_{i=1}^n), q) \) by \( \text{LCP}(R, q) \).

**Definition 2.6** A matrix \( R \) is a \( Q \)-matrix if for every \( q \in \mathbb{R}^n \) the linear complementarity problem \( \text{LCP}(R, q) \) has a solution.

In this paper we will be interested in approximate solutions to linear complementarity problems, which we define now.

**Definition 2.7** Let \( \varepsilon > 0 \). The vector \((w, (z_i)_{i=0}^n) \in \mathbb{R}^n \times \Delta(\{0, 1, \cdots, n\})\) is an \( \varepsilon \)-approximate solution of the problem \( \text{LCP}(R, q) \) if the following conditions hold:

- \( w_i \geq r^i_i - \varepsilon \) for every \( i \in \{1, \ldots, n\} \).
- \( w_i \leq r^i_i + \varepsilon \) or \( z_i = 0 \), for every \( i \in \{1, \ldots, n\} \).
- \( \|w - z_0q - Rz\|_\infty < \varepsilon \).

If \((R_\varepsilon, q_\varepsilon)_{\varepsilon > 0}\) is a sequence of \( n \times n \) matrices and vectors in \( \mathbb{R}^n \) that converge to a limit \((R, q)\), then any limit of \( \varepsilon \)-approximate solutions of the problems \( \text{LCP}(R_\varepsilon, q_\varepsilon) \) as \( \varepsilon \) goes to 0 is a solution of the problem \( \text{LCP}(R, q) \). Consequently, if the problem \( \text{LCP}(R, q) \) has no solution, then any nearby problem does not have an approximate solution. This observation is summarized by the following lemma.

**Lemma 2.8** Let \( R \) be an \( n \times n \) matrix that is not a \( Q \)-matrix. Let \( q \in \mathbb{R}^n \) satisfy that the problem \( \text{LCP}(R, q) \) has no solution. There is \( \varepsilon = \varepsilon(R, q) > 0 \) such that the problem \( \text{LCP}(R', q') \) has no \( \varepsilon \)-approximate solution\(^1\) for every \((R', q') \in \mathcal{M}_{n,n} \times \mathbb{R}^n \) that satisfy \( \|R' - R\|_\infty \leq \varepsilon \) and \( \|q' - q\|_\infty \leq \varepsilon \).

\(^1\)The set of \( n \times n \) matrices is denoted \( \mathcal{M}_{n,n} \).
2.4 Quitting Games and Linear Complementarity Problems

When the game $\Gamma$ is a quitting game, we denote the unique continue action of player $i$ by $C_i$. Denote by $\vec{C} = (C_i)_{i \in I}$ the action profile under which all players continue. For every player $i \in I$ denote by $C_{-i} = (C_j)_{j \neq i}$ the action profile in which all players except player $i$ continue.

Denote by $R(\Gamma)$ the $(|I| \times |I|)$-matrix whose $i$’th column is $u(Q_i, C_{-i})$, that is, the payoff vector when player $i$ quits and all other players continue. Solan and Solan (2017) related linear complementarity problems to equilibria and sunspot equilibria.

Theorem 2.9 (Solan and Solan, 2017) Let $\Gamma = (I, \{C_i, Q_i\}_{i \in I}, u)$ be a quitting game.

- If the matrix $R(\Gamma)$ is not a $Q$-matrix, then there is an absorbing stationary strategy profile $x$ such that for every $\varepsilon > 0$ the stationary strategy $x$, supplemented with threats of punishment, is a Nash $\varepsilon$-equilibrium of $\Gamma$.

- If the matrix $R(\Gamma)$ is a $Q$-matrix, then for every $\varepsilon > 0$ the game $\Gamma$ admits a sunspot $\varepsilon$-equilibrium $\xi$ in which, after every finite history, at most one player $i$ plays the action $Q_i$, and the probability by which this player plays the action $Q_i$ is at most $\varepsilon$. Moreover, $\gamma_i(\xi) \geq u(Q_i, C_{-i})$ and, if under $\xi$ at least two players quit with positive probability, then $\gamma_i(\xi) \geq \min_{j \neq i} u(Q_j, C_{-j})$.

When $\Gamma$ is a quitting game such that the matrix $R(\Gamma)$ is not a $Q$-matrix, then the limit of $\lambda$-discounted stationary equilibria as the discount factor goes to 0 is either the nonabsorbing entry or absorbing with a positive probability that is bounded away from 0. This is the content of the next result, whose proof follows from Section 3.2 in Solan and Solan (2017) and Lemma 2.8.

Lemma 2.10 Let $\Gamma$ be a quitting game such that the matrix $R = R(\Gamma)$ is not a $Q$-matrix. For every $\lambda \in (0, 1]$ let $x^\lambda$ be a $\lambda$-discounted stationary equilibrium of the game $\Gamma$, such that the limit $x := \lim_{\lambda \to 0} x^\lambda$ exists. Then either $x = C$, or $\sum_{i \in I} x_i \geq \varepsilon(R, q)$, where $q$ is any vector such that the problem LCP($R, q$) has no solution and $\varepsilon(R, q)$ is given by Lemma 2.8.
3 The Proof for the Case $|A^c_i| = 2$ and $|A^c_j| = 1$ for Every $i \neq 1$

In this section we provide a proof for the special case in which Player 1 has two continue actions and each other player has a single continue action. We furthermore assume that the game is positive and recursive; that is, $u_i(a) = 0$ for every player $i \in I$ and every action profile $a \in \times_{j \in I} A^c_j$ (recall that by assumption payoffs are between 0 and 1). This case exhibits some important aspects of the proof of the general case, and will help us explain the need for the new tools that we develop in the sequel. The extension from positive recursive quitting games to general quitting games will not require new tools.

One interesting aspect of this case is that it uses Browder’s Theorem, instead of a more sophisticated fixed point theorem that we will need for the general case. The authors are not aware of another application of Browder’s Theorem in dynamic games.

**Theorem 3.1 (Browder, 1960)** Let $X \subseteq \mathbb{R}^n$ be a convex and open set, Let $K \subseteq X$ be compact, and let $F : [0, 1] \times X \to K$ be a continuous function. Define $C_F := \{(t, x) \in [0, 1] \times X : x = f(t, x)\}$ be the set of fixed points of $f$. There is a connected component $T$ of $C_F$ such that $T \cap \{(0) \times X\} \neq \emptyset$ and $T \cap \{(1) \times X\} \neq \emptyset$.

**Step 1:** Definition of auxiliary quitting games.

We will show that, for a given $\delta > 0$, the game $\Gamma$ admits a sunspot $\delta$-equilibrium. The proof will exhibit an interplay between the general quitting game $\Gamma = (I, (A^c_i)_{i \in I}, u)$ and a family of auxiliary quitting games, which we define now.

Denote the two continue actions of Player 1 by $A^c_1 = \{C^1_1, C^2_1\}$. For every $\alpha \in [0, 1]$ and every vector $q \in \mathbb{R}^I$ define a quitting game $\Gamma^\alpha(q)$ over the set of players $I$, whose payoff function is denoted by $u^\alpha$, as follows:

- The nonabsorbing payoff is $u^\alpha(C) := q$.
- For every action profile $a_{-1} \in \times_{i \neq 1}\{C_i, Q_i\}$ we have $u^\alpha(Q_1, a_{-1}) := u(Q_1, a_{-1})$.
- For every action profile $a_{-1} \in (\times_{i \neq 1}\{C_i, Q_i\})\{C_{-1}\}$ we have $u^\alpha(C_1, a_{-1}) := \alpha u(C^1_1, a_{-1}) + (1 - \alpha) u(C^2_1, a_{-1})$. 

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The game $\Gamma^\alpha(q)$ is a restricted version of the original game $\Gamma$: whenever Player 1 continues in the game $\Gamma^\alpha(q)$ it is as if he played the mixed action $\alpha C_1^1 + (1 - \alpha) C_1^2$ in the game $\Gamma$. We note that every mixed action profile $\hat{x}$ in the game $\Gamma^\alpha(q)$ naturally induces a strategy profile $x$ in the game $\Gamma$ as follows:

$$x_i = \begin{cases} \hat{x}_i & \text{if } i \neq 1, \\ \hat{x}_1(C_1)\alpha C_1^1 + x_1(C_1)(1 - \alpha) C_1^2 + x_1(Q_1)Q_1 & \text{if } i = 1. \end{cases}$$  \hspace{1cm} (2)$$

With the game $\Gamma^\alpha(q)$ we associate the $(|I| \times |I|)$-matrix $R^\alpha$ whose $i$'th column is equal to $u_i^\alpha(Q_i, C_{-i})$. Note that this matrix is independent of $q$.

**Step 2:** There is $\alpha_0 \in [0, 1]$ for which $R^{\alpha_0}$ is a $Q$-matrix.

Let $q_0 \notin R^1_+$ be arbitrary and let $\varepsilon > 0$ be sufficiently small. By Theorem 2.9, the game $\Gamma^{\alpha_0}(q_0)$ admits a sunspot $\varepsilon$-equilibrium $\hat{\xi}$ in which the players mainly continue, and at every stage $t$ at most one of the players, denoted $i^*$, quits, and does so with probability at most $\varepsilon$.

To show that the game $\Gamma$ admits a sunspot $\delta$-equilibrium, we will transform the correlated strategy profile $\hat{\xi}$ in the game $\Gamma^{\alpha_0}(q)$ into a correlated strategy profile $\xi$ in the game $\Gamma$. The correlated strategy profile $\xi$ is defined as follows:

- For every $i \neq 1$ we have $\xi_i = \hat{\xi}_i$.
- Whenever Player 1 continues under $\hat{\xi}$, he plays $\alpha_0 C_1^1 + (1 - \alpha_0) C_1^2$ under $\xi$; whenever Player 1 quits under $\hat{\xi}$, he also quits under $\xi$.

The reader may verify that the incentive constraints of each player $i \neq 1$ are the same under $\hat{\xi}$ and under $\xi$. Regarding Player 1, since the probability by which players quit is at most $\varepsilon$, and since Player 1 plays the mixed action $\alpha_0 C_1^1 + (1 - \alpha_0) C_1^2$ until the game terminates, by conducting statistical tests players $I \setminus \{1\}$ can identify if Player 1 does not play the actions $C_1^1$ and $C_1^2$ with frequencies that are close to $\alpha_0$ and $1 - \alpha_0$, and punish Player 1 at his min-max level if he is found deviating. Since the game is positive and recursive, such a deviation cannot be profitable for Player 1. Consequently, provided $\varepsilon$ is sufficiently small, this strategy is a $\delta$-equilibrium.

**Step 3:** There is $\alpha_0 \in [0, 1]$ for which $R^{\alpha_0}$ is not a $Q$-matrix.

The conditions in Steps 2 and 3 are not mutually exclusive. Thus, there are games for which two different types of sunspot $\delta$-equilibria exist.
Let $q_0 \in \mathbb{R}^I$ be such that the linear complementarity problem $\text{LCP}(R^\alpha, q_0)$ has no solution. By Lemma 2.10 it follows that in the game $\Gamma^\alpha(q_0)$ all accumulation points of discounted equilibria as the discount factor goes to 0 are bounded away from $\bar{C}$.

**Step 3a:** Applying Browder’s Theorem.

For every $\lambda \in (0, 1]$, every $\alpha \in [0, 1]$, and every stationary strategy profile $\hat{x} \in [0, 1]^I$ in the game $\Gamma^\alpha(q_0)$ denote by $\gamma^{\alpha, \lambda}(\hat{x})$ the $\lambda$-discounted payoff under $x$ in the auxiliary game $\Gamma^\alpha(q_0)$. Denote by $M^\lambda(\alpha) \subseteq [0, 1]^I$ the set of $\lambda$-discounted stationary equilibria of the game $\Gamma^\alpha(q_0)$:

$$M^\lambda(\alpha) := \{ \hat{x} \in [0, 1]^I : \hat{x} \text{ is a } \lambda\text{-discounted stationary equilibrium of } \Gamma^\alpha(q_0) \}.$$  

Denote by $M^\lambda \subseteq [0, 1] \times [0, 1]^I$ the graph of the function $M^\lambda(\cdot)$.

Browder’s Theorem implies that there exists a connected component of $M^\lambda$ that intersects both $\{0\} \times [0, 1]^I$ and $\{1\} \times [0, 1]^I$. Because the set $M^\lambda$ is semialgebraic, this in turn implies that there is a continuous path $y^\lambda$ in $M^\lambda$ that intersections both $\{0\} \times [0, 1]^I$ and $\{1\} \times [0, 1]^I$; that is, there exists a continuous function $y^\lambda : [0, 1] \rightarrow M^\lambda$ such that $y^\lambda(0) \in \{0\} \times [0, 1]^I$ and $y^\lambda(1) \in \{1\} \times [0, 1]^I$.

**Step 3b:** Constructing a continuous path of equilibria.

Denote by $Y \subseteq [0, 1]^I$ the set of all accumulation points of sequences in $(y^\lambda)_{\lambda \in (0, 1]}$ as $\lambda$ goes to 0; that is, $Y$ is the set of all limits $\lim_{k \rightarrow \infty} (\alpha^k, y^{\lambda^k}(\alpha^k))$ such that $\lim_{k \rightarrow \infty} \lambda^k = 0$ and the two limits $\lim_{k \rightarrow \infty} \alpha^k$ and $\lim_{k \rightarrow \infty} y(\alpha^k)$ exist. The set $Y$ is closed and connected. Moreover, it is a subset of the set $Z$ of all accumulation points of limits of discounted equilibria as the discount factor goes to 0, which is semialgebraic. It follows that there is a continuous path in $Z$ that intersects both $\{0\} \times X$ and $\{1\} \times X$. This in turn implies that there is a continuous function $s : [0, 1] \rightarrow [0, 1]$ satisfying $s(0) = 0$ and $s(1) = 1$, and a continuous function $\hat{x} : [0, 1] \rightarrow [0, 1]^I$, such that $\hat{x}(\alpha)$ is an accumulation point of discounted equilibria in the game $\Gamma^{s(\alpha)}(q_0)$, for every $\alpha \in [0, 1]$. Lemma 2.10 together with the choice of $q_0$, imply that the mixed

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3To apply Browder’s Theorem we need to show that the set $M^\lambda$ is the set of fixed points of some continuous function $f : [0, 1] \times [0, 1]^I \rightarrow [0, 1]^I$. Such a function can be constructed using the function devised in Nash (1950) to prove the existence of equilibrium in strategic form games, by observing that a $\lambda$-discounted stationary equilibrium is a fixed point of the Shapley operator. Browder’s Theorem is applied to $K = [0, 1]^I$ and $X = [-\varepsilon, \varepsilon]^I$. 

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strategy profiles \((\hat{x}(\alpha))_{\alpha \in [0,1]}\) are absorbing, that is, \(\sum_{i \in I} \hat{x}_i(\alpha) > 0\) for every \(\alpha \in [0,1]\).

Recall that \(x(\alpha)\) is the strategy profile in the game \(\Gamma\) induced by \(\hat{x}(\alpha)\). We next show that for every \(\alpha \in [0,1]\), only Player 1 may have profitable deviations from \(x(\alpha)\), and such a profitable deviation can only be to play some continue action with probability 1.

**Step 3c:** For every \(\alpha \in [0,1]\), when the players follow the stationary strategy profile \(x(\alpha)\) no player \(i \neq 1\) can profit by deviating from \(x_i(\alpha)\). Moreover, Player 1 cannot profit by deviating to \(Q_1\).

Fix \(i \neq 1\) and \(\alpha \in [0,1]\). If \(\hat{x}_j(\alpha) = 0\) for every \(j \neq i\), then necessarily \(\hat{x}_i(\alpha) > 0\). Since the game is positive and recursive, player \(i\) cannot profit by deviating from \(x_i(\alpha)\). Otherwise, \(\sum_{j \neq i} \hat{x}_j(\alpha) > 0\), and therefore standard continuity arguments imply that

\[
\gamma(Q_i, x_{-i}(\alpha)) = \lim_{\lambda \to 0} \gamma^{\alpha,\lambda}(Q_i, \hat{x}_{-1}(\alpha)) \leq \lim_{\lambda \to 0} \gamma^{\alpha,\lambda}(\hat{x}(\alpha)) = \gamma(x(\alpha)),
\]

and

\[
\gamma(C_i, x_{-i}(\alpha)) = \lim_{\lambda \to 0} \gamma^{\alpha,\lambda}(C_i, \hat{x}_{-1}(\alpha)) \leq \lim_{\lambda \to 0} \gamma^{\alpha,\lambda}(\hat{x}(\alpha)) = \gamma(x(\alpha)).
\]

It follows that player \(i\) cannot profit by deviating. A similar argument shows that Player 1 cannot profit by deviating from \(x_1(\alpha)\) to \(Q_1\).

Our goal now is to show that there is \(\alpha \in [0,1]\) such that Player 1 cannot profit by deviating from \(x(\alpha)\) to either \(C^1_1\) or \(C^2_1\). Such a strategy profile \(x(\alpha)\) is a 0-equilibrium of the original game \(\Gamma\).

**Step 3d:** The case that there is \(\alpha_0 \in [0,1]\) such that \(\sum_{i \neq 1} \hat{x}_i(\alpha_0) = 0\).

Assume first that there is \(\alpha_0 \in [0,1]\) such that \(\sum_{i \neq 1} \hat{x}_i(\alpha_0) = 0\). Then \(\hat{x}_1(\alpha_0) > 0\). Since the game is recursive and positive, and all players except of Player 1 continue, Player 1 cannot profit by deviating from \(x(\alpha)\) to either \(C^1_1\) or \(C^2_1\).

**Step 3e:** The case that \(\sum_{i \neq 1} \hat{x}_i(\alpha) > 0\) for every \(\alpha \in [0,1]\).

Assume now that \(\sum_{i \neq 1} \hat{x}_i(\alpha) > 0\) for every \(\alpha \in [0,1]\). Denote by \(u^1_1(\alpha) := u_1(C^1_1, x_{-1}(\alpha))\) (resp. \(u^2_1(\alpha) := u_1(C^2_1, x_{-1}(\alpha))\)) the payoff of Player 1 if he plays the stationary strategy \(C^1_1\) (resp. \(C^2_1\)) while all other players follow
the stationary strategy profile \(x(\alpha)\). These quantities are well defined because \(\sum_{i \neq 1} \hat{x}_i(\alpha) > 0\) for every \(\alpha \in [0, 1]\). Since the function \(\alpha \mapsto \hat{x}(\alpha)\) is continuous, the functions \(\alpha \mapsto u_1^1(\alpha)\) and \(\alpha \mapsto u_1^2(\alpha)\) are continuous.

By Eq. (2), for \(\alpha = 1\), the strategy \(x_1(1)\) assigns probability 0 to the action \(C_1^2\). Similarly, for \(\alpha = 0\), the strategy \(x_1(0)\) assigns probability 0 to the action \(C_1^1\). Consequently, if \(u_1^1(1) \geq u_1^2(1)\), then Player 1 cannot profit by deviating from \(x_1(1)\) to \(C_1^2\), and therefore the stationary strategy profile \(x(1)\) is a 0-equilibrium. Similarly, if \(u_1^1(0) \leq u_1^2(0)\), then Player 1 cannot profit by deviating from \(x_1(0)\) to \(C_1^1\), and therefore the stationary strategy profile \(x(0)\) is a 0-equilibrium.

We can thus assume that \(u_1^1(1) < u_1^2(1)\) while \(u_1^1(0) > u_1^2(0)\). The continuity of the functions \(u_1^1\) and \(u_1^2\) imply that there is \(\alpha_0 \in (0, 1)\) such that \(u_1^1(\alpha_0) = u_1^2(\alpha_0)\). But then both \(C_1^1\) and \(C_1^2\) yield the same payoff against \(x_{-1}(\alpha_0)\), and therefore Player 1 cannot profit by deviating from \(x_1(\alpha_0)\) to either \(C_1^1\) or \(C_1^0\). In particular, the stationary strategy profile \(x(\alpha_0)\) is a 0-equilibrium.

We now discuss the adaptation of this proof to the general case, still assuming the game is positive and recursive. For every vector \(\alpha = (\alpha_i)_{i \in I} \in \times_{i \in I} \Delta(A_i^c)\) and for every vector \(q \in \mathbb{R}^I\) we will define an auxiliary game \(\Gamma^\alpha(q)\) with the interpretation that whenever player \(i\) continues in \(\Gamma^\alpha(q)\), it is as if he chose the mixed action \(\alpha_i\) in the original game \(\Gamma\). If there is \(\alpha_0 \in \times_{i \in I} \Delta(A_i^c)\) such that the matrix \(R^{\alpha_0}\) is a \(Q\)-matrix, then as above, the game \(\Gamma^{\alpha_0}(q)\) admits a sunspot \(\varepsilon\)-equilibrium (that is independent of \(q\)), which can be turned into a sunspot \(\varepsilon\)-equilibrium of the original game \(\Gamma\).

We therefore restrict attention to the case in which there is \(\alpha_0 \in \times_{i \in I} \Delta(A_i^c)\) such that the matrix \(R^{\alpha_0}\) is not a \(Q\)-matrix. Denote by \(M(\alpha, q)\) the set of all accumulation points of \(\lambda\)-discounted equilibria as \(\lambda\) goes to 0 of the auxiliary game \(\Gamma^\alpha(q)\), and by \(M(\alpha) := \cup_{q \in \mathbb{R}^I} M(\alpha, q)\). By Browder’s Theorem one can prove that the set-valued function \(\alpha \mapsto M(\alpha)\) has a connected component, whose boundary, when projected to \(\times_{i \in I} \Delta(A_i^c)\), coincides with the boundary of \(\times_{i \in I} \Delta(A_i^c)\). In the proof above, to show that a stationary equilibrium exists we used the Mean Value Theorem. In the general case we need to use a fixed point theorem applied to the set-valued function \(M\). In Section 4 we will develop such a theorem. Our proof utilizes the theory of intersection index, which requires the graph of \(M\) to be a smooth manifold. By Kohlberg and Mertens (1986), given the set of players and the sets of actions of the players of a strategic form game, the equilibrium set is homeomorphic to
the set of games, which is a Euclidean space. This set, however, is not a smooth manifold. In Section 5 we will prove that the equilibrium set can be uniformly approximated by smooth manifolds, a property that will suffice for our purposes.

When the game is not positive and recursive, the minmax value of some player in \( \Gamma \) may be higher than his minmax value in the auxiliary game \( \Gamma^\alpha \). When this happens, the sunspot \( \varepsilon \)-equilibrium is constructed by Solan and Solan (2017) for the matrix \( R^\alpha \) may not be converted into a sunspot \( \varepsilon \)-equilibrium of \( \Gamma \). To handle this issue, we will assume in Step 2 that the matrix \( R^\alpha \) is a \( Q \)-matrix for every \( \alpha \in \times_{i \in I} \Delta(A_i^\varepsilon) \) (which is the complement of the condition in Step 3) and use a result that extends Laraki (2010) to show that there is \( \alpha \) for which (a) the matrix \( R^\alpha \) is a \( Q \)-matrix and (b) in the sunspot \( \varepsilon \)-equilibrium constructed by Solan and Solan (2017) for the matrix \( R^\alpha \) all players receive at least their minmax value in the original game \( \Gamma \).

4 Topological Foundation

In this section we present the results from topology that we need in the paper. We refer to Guillemin and Pollack (2010) for the relevant background on manifolds, including the definition of transversality, oriented manifolds, and the intersection index. One should bear in mind that Guillemin and Pollack (2010) often consider the case of closed manifolds without boundary, while in our case some manifolds have boundary. Nevertheless, our assumptions will ensure that the results still hold, with the same proofs, when the manifolds have boundary.

All manifolds in this paper are oriented. In this section we use simplexes and products of simplexes, which are not smooth manifolds in the usual definition since their boundary is not a manifold. One way to handle such manifolds is as manifolds with corners, see, e.g., Joyce (2010). This issue will not arise in our results; the only place where we do care about the boundary being a manifold is in Theorem 4.1, in which we will deal with it specifically.

Given \( \varepsilon > 0 \) and a function \( f : X \to Y \), where \( Y \) is a metric space with metric \( \rho_Y \), the function \( f_\varepsilon : X \to Y \) is an \( \varepsilon \)-perturbation of \( f \) if \( \rho_Y(f(x), f_\varepsilon(x)) < \varepsilon \) for every \( x \in X \). The basic result in topology that we need is a variation of Browder’s Theorem.

**Theorem 4.1** Let \( X \) be a compact \( k \)-dimensional manifold with boundary. Let \( U \) be an \( n \)-dimensional connected open boundaryless manifold. Let \( M \subseteq X \times U \) be a closed subset, and let \( f : X \to U \) be a continuous function. Then there exists a continuous function \( g : X \to U \) such that for every \( x \in X \), the function \( g \) is \( \varepsilon \)-close to \( f \) and \( g(x) \) is in \( M \).
$U \times X$ be an $n$-dimensional boundaryless manifold that satisfies $M \cap (U \times \partial X) = \emptyset$. Let $N$ be an $l$-dimensional compact manifold with boundary. Let $y : N \times X \rightarrow U$ be a continuous function such that for every $\alpha \in N$ the function $y(\alpha, \cdot) : X \rightarrow U$ is homotopic to a constant function.

Consider the function $\tilde{y} : N \times X \rightarrow U \times X$ defined by

$$\tilde{y}(\alpha, x) = (y(\alpha, x), x).$$

Let $\pi : U \times X \rightarrow U$ be the projection and denote $d := \deg(\pi_{|M})$.

Then for every $\varepsilon > 0$ there is an $\varepsilon$-perturbation $\tilde{y}_\varepsilon$ of $\tilde{y}$ such that

(a) $\tilde{y}_\varepsilon$ is transversal to $M$, and

(b) the manifold $M' := (\tilde{y}_\varepsilon)^{-1}(M) \subseteq N \times X$ satisfies that its boundary is contained in $\partial N \times X$. Moreover, the projection $M' \rightarrow N$ has degree $d$.

To allow game theorists to properly interpret the data of Theorem 4.1, we explain its relation to games. Suppose that the set $I$ of players and the action sets of the players $(A_i)_{i \in I}$ are fixed. The compact manifold with boundary $X$ will be the set of mixed action profiles in binary games, namely, $X = (\times_{i \in I} \Delta(A_i))^I$. The connected open boundaryless manifold $U$ will be the set of possible payoff functions in binary games, namely, $U = \mathbb{R}^{(\times_{i \in I} A_i) \times I}$. The manifold $M \subseteq U \times X$ will be a smooth manifold that uniformly approximates the equilibrium set. Let $N$ be some parameter space which is a compact manifold with boundary, for example, a finite product of simplexes, and let $y : N \rightarrow U$ be some function that assigns a game to each parameter. In the statement of Theorem 4.1, the domain of the function $y$ is not $N$ but $N \times X$, but to understand the theorem we ignore this point. Extend $y$ to a function $\tilde{y} : N \times X \rightarrow U \times X$ by setting $\tilde{y}(\alpha, x) = (y(\alpha), x)$. Theorem 4.1 roughly states that $\tilde{y}^{-1}(M)$ is a manifold, that its boundary, when projected to $N$, contains the boundary of the parameter set $N$. In other words, it roughly says that the equilibrium set restricted to games in the range of $y$ is a manifold whose boundary covers the boundary of the parameter set.

To prove Theorem 4.1 we will need a couple of observations, which follow from the definition of the intersection index.

**Lemma 4.2** Let $X$, $Y$, and $Z$ be three manifolds with boundary, let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be smooth functions, and let $M \subseteq Z$ be a boundaryless manifold (see Figure 1). Assume that
X is compact,
g is transversal to M,
M is disjoint of \(g \circ f(\partial X)\), and
dim M + dim X = dim Z.

Then the intersection index of \(g \circ f\) and M is equal to the intersection index of f and \(g^{-1}(M)\)

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
M & \xrightarrow{g} & Z
\end{array}
\]

Figure 1: The data of Lemma 4.2

Proof. For every \(\varepsilon > 0\) there is a smooth \(\varepsilon\)-perturbation \(f_\varepsilon : X \to Y\) of f such that

• \(f_\varepsilon\) is transversal to \(g^{-1}(M)\),

• \(f_\varepsilon\) is homotopic to f through the homotopy function \(H : [0, 1] \times X \to Y\) with the condition that every points moves at most \(\varepsilon\) along the homotopy; that is, \(\rho(H(0, x), H(t, x)) < \varepsilon\) for every \(t \in [0, 1]\) and every \(x \in X\), where \(\rho\) is the metric on \(Y\), and

• \(H([0, 1] \times \partial X) \cap g^{-1}(M) = \emptyset\).

Since \(H([t] \times \partial X) \cap g^{-1}(M) = \emptyset\) for every \(t \in [0, 1]\), and since \(X\) is compact, it follows that the homotopy \(H\) preserves the intersection index (see, for example, Guillemin and Pollack (2010, page 108)), and therefore we may assume that f is transversal to \(g^{-1}(M)\).

It follows from the definitions of the intersection index and of the inverse image that the intersection index of \(g \circ f\) and M is the sum of the orientations of \((g \circ f)^{-1}(M)\). For the same reason, the intersection index of f and \(g^{-1}(M)\) is the sum of the orientations of \(f^{-1}(g^{-1}(M))\). Since the inverse image of a manifold is functorial, we get the desired result.

Let \(f : X \to Y\) be a smooth function between manifolds. A point \(y \in Y\) is a regular value of f if for every \(x \in f^{-1}(y)\) the differential of f at x, denoted \(df_x\), is onto the tangent bundle at y, denoted \(T_y(Y)\).
Lemma 4.3 Let $X$, $Y$, and $Z$ be three manifolds with boundary such that $X$ and $Z$ are compact with dimension $n$, and $Y$ has dimension $n + k$. Let $g : Y \to Z$ be smooth. Assume that $X \subseteq Y$ and $\partial X \subseteq \partial Y = g^{-1}(\partial Z)$. Then the degree of $g$ restricted to $X$ is equal to the intersection index of $X$ and $g^{-1}(Z)$, for some $z \in Z$ which is a regular value of $g$.

Proof. We will apply Lemma 4.2 to $X$, $Y$, $Z$, $g$, $f$ that is the inclusion from $Y$ to $Z$, and $X$ is a regular value $z$ of $g$ thought of as a 0-dimensional manifold with positive orientation. By Sard’s Lemma such a regular value exists. By Lemma 4.2 the intersection index of $g|_X$ and the point $z$, which is the degree of $g$, is equal to the intersection index of the inclusion and $g^{-1}(z)$, as desired.

Proof of Theorem 4.1 We will denote the set $X$ without its boundary by $X^\circ := X \setminus \partial X$. Fix $\varepsilon > 0$. By the Transversality Theorem (see, e.g., Guillemin and Pollack (2010, Theorem 70)) there is an $\varepsilon$-perturbation $\tilde{y}_\varepsilon$ of $\tilde{y}$ that satisfies the following two conditions:

- $\tilde{y}_\varepsilon$ is transversal to $M$, so that Part (a) holds, and
- $\tilde{y}_\varepsilon$ is homotopic to $\tilde{y}$ through the homotopy function $H : [0, 1] \times N \times X \to X \times U$ with the condition that every points moves at most $\varepsilon$ along the homotopy.

Since $X$ is compact, provided $\varepsilon$ is sufficiently small, along the homotopy $H$ we have $H([0, 1] \times N \times \partial X) \cap M = \emptyset$. This implies that the first claim in Part (b) holds. Indeed, since $M$ is boundaryless, we have $\partial(\tilde{y}_\varepsilon)^{-1}(M)$ is contained in $\partial(N \times X) = (\partial N \times X) \cup (N \times \partial X)$. Since $(\text{image}([0, 1] \times N \times \partial X) \cap M = \emptyset$, we deduce that $\partial(\tilde{y}_\varepsilon)^{-1}(M)$ is contained in $\partial N \times X$.

From now on we fix an arbitrary element $\alpha \in N$. By the construction of $\tilde{y}_\varepsilon$, the function $\tilde{y}_\varepsilon(\alpha, \cdot)$ is homotopic to $\tilde{y}(\alpha, \cdot)$, which is homotopic to $\text{const} \times Id : X \to U \times X$. Moreover, by assumption, along the homotopy $\partial X$ is disjoint of $M$. Since homotopy preserves the intersection index, the intersection index of $\tilde{y}_\varepsilon(\alpha, \cdot)$ and $M$ is $d$. 

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We will now apply Lemma 4.3 with the following data:

| Lemma 4.3 | Our proof |
|------------|-----------|
| X          | $M' = (\tilde{y}_\epsilon)^{-1}(M)$, |
| Y          | $N \times X^\circ$, |
| Z          | $N$, |
| $g : X \to Y$ | $\pi : N \times X^\circ \to N$. |

We verify that the conditions of the lemma hold for these data, and therefore we will deduce that the degree of the projection from $N \times X$ to $N$, restricted to $M'$, is equal to the intersection index of $\{\alpha\} \times X^\circ$ and $(\tilde{y}_\epsilon)^{-1}(M)$.

- The set $M'$ is the inverse image of a closed set under a smooth function, hence it is closed. Since $N$ and $X$ are compact, the manifold $M'$ is compact.

- By definition, the manifold $N$ is compact.

- We argue that $\dim(M') = \dim(N)$. Indeed, by definition, $\text{codim}(M') = \dim(N \times X) - \dim(M')$, and therefore, since $\tilde{y}_\epsilon$ is transversal,

  $\dim(N \times X) - \dim(M') = \text{codim}(M')$

  $= \text{codim}(M)$

  $= \dim(U \times X) - \dim(M)$

  $= n + k - n = k = \dim(X)$.

  It follows that $\dim(M') = \dim(N)$, as desired.

- We argue that $M' \subseteq N \times X^\circ$. Indeed, $M' \subseteq N \times X$ and $M \cap (N \times \partial X) = \emptyset$.

- We note that $\pi^{-1}(\partial N) = \partial(N \times X^\circ) = \partial N \times X^\circ$.

Recall that $\alpha$ is an arbitrary element in $N$. We now apply Lemma 4.2 with the following data:

| Lemma 4.2 | Our proof |
|------------|-----------|
| X          | $\{\alpha\} \times X$ |
| Y          | $N \times X$ |
| Z          | $U \times X$ |
| $f : X \to Y$ | inclusion : $\{\alpha\} \times X \to N \times X$ |
| $g : Y \to Z$ | $\tilde{y}_\epsilon : N \times X \to U \times X$ |
| $M$        | $M$. |
We note that since \( N \times \partial X \) is disjoint of \( M \), we also have that \( g \circ f(\partial(\{\alpha\} \times X)) \) is disjoint of \( M \). We leave to the reader the verification that the other conditions of Lemma 4.2 hold. We deduce from Lemma 4.2 that the intersection index of \( \{\alpha\} \times X^0 \) and \( (\tilde{y}_e)^{-1}(M) \) is equal to the intersection index of \( \tilde{y}_e(\alpha, \cdot) \) and \( M \), which is equal to \( d \). The result follows.

The following result is a fixed point theorem for manifolds.

**Theorem 4.4** Let \( \Delta \) be the \( d \)-dimensional convex compact set and let \( M \) be a \( d \)-dimensional compact manifold with boundary. Let \( f : M \rightarrow \Delta \) be a smooth function that satisfies the following conditions:

1. \( \partial M \subseteq f^{-1}(\partial \Delta) \).
2. The degree of \( f \) is not zero: \( \deg(f) \neq 0 \).

Let \( g : M \rightarrow \Delta \) be a continuous function. Then there is \( x \in M \) such that \( f(x) = g(x) \).

**Proof.**

**Step 1:** We can assume that \( \Delta \) has a smooth boundary, that the image of \( g \) does not intersect \( \partial \Delta \), and that \( f \) is transversal to \( \partial \Delta \).

Assume to the contrary that Theorem 4.4 holds whenever \( \Delta \) has a smooth boundary and \( \text{image}(g) \cap \partial \Delta = \emptyset \), but does not hold without these restrictions. Let \( g_0 : M \rightarrow \Delta \) be a continuous function for which \( \text{image}(g_0) \cap \partial \Delta \neq \emptyset \). Fix \( \varepsilon \in (0, 1) \), and let \( \Delta'_\varepsilon \subseteq \Delta \subseteq \Delta \) be two convex compact subsets of \( \Delta \) whose boundary is smooth, whose Hausdorff distance from \( \Delta \) is smaller than \( \varepsilon \), and such that \( \Delta'_\varepsilon \) and \( \partial \Delta'_\varepsilon \) are disjoint.

For every \( x \in M \) let \( g(x) \) be the point in \( \Delta'_\varepsilon \) closest to \( g_0(x) \). Then the image of the function \( g_\varepsilon \) does not intersect \( \partial \Delta'_\varepsilon \). Let \( f_\varepsilon : M \rightarrow \Delta \) be an \( \varepsilon \)-perturbation of \( f \) that is transversal to \( M \) and coincides with \( f \) on \( \partial M \). Let \( M' := f_\varepsilon^{-1}(\Delta'_\varepsilon) \), and apply the theorem to \( \Delta'_\varepsilon, M', f_\varepsilon \), and \( g_\varepsilon \). It follows that there exists \( x_\varepsilon \in M \) such that \( f_\varepsilon(x_\varepsilon) = g_\varepsilon(x_\varepsilon) \). Since the manifold \( M \) is compact, the sequence \( (x_\varepsilon)_{\varepsilon > 0} \) has an accumulation point \( x \in M \) as \( \varepsilon \) goes to 0, which, by continuity, satisfies \( f(x) = g(x) \).

Since every convex compact set with smooth boundary is diffeomorphic to a ball, we will assume from now on that \( \Delta \) is a \( d \)-dimensional unit ball.

**Step 2:** We can assume that \( g \) is smooth.
Suppose that the theorem holds whenever the function $g$ is smooth, and let $g_0$ be an arbitrary continuous function. To show that the result holds for $g_0$, we will consider its convolution with a sequence of smooth bump functions that converge to a Dirac function.

Embed $M$ in a Euclidean space $R^m$, for $m$ sufficiently large. Denote by $\rho$ the restriction of the standard metric on $R^m$ to $M$. The metric and the orientation define a maximal form $\omega$ on $M$.

Let $\mu : R \to R$ be the smooth function defined by

$$\mu(z) := \begin{cases} \exp(-1/z^2) & z \geq 0, \\ 0 & z < 0. \end{cases}$$

and let $\kappa : R^m \to R$ be the smooth function defined by

$$\kappa(y) := \prod_{i=1}^{m} \mu(y_i + 1)\mu(1 - y_i), \quad \forall y \in R^m.$$  

The function $\kappa$ vanishes outside a ball of radius 1 around the origin. For every $\varepsilon > 0$ define a function $\kappa_{\varepsilon} : R^m \to R$ by $\kappa_{\varepsilon}(y) := \kappa(y)\varepsilon$, for every $y \in R^m$. The function $\kappa_{\varepsilon}$ is smooth and vanishes outside a ball of radius $\varepsilon$ around the origin. Finally, for every $\varepsilon > 0$ define the convolution $g_{\varepsilon} : M_\varepsilon \to R$, whose domain is $M_\varepsilon := \{ x \in R^m : \rho(x, M) < \varepsilon \}$, the $\varepsilon$-neighborhood of $M$, by

$$g_{\varepsilon}(x) := \frac{\int_M g(x)\kappa_{\varepsilon}(x-p)\omega}{\int_M \kappa_{\varepsilon}(x-p)\omega}.$$  

The function $g_{\varepsilon}|_M$ satisfies the conditions of the theorem and is smooth. Since the result holds for smooth functions, for every $\varepsilon > 0$ there is a point $x_\varepsilon \in M$ that satisfies $g_{\varepsilon}(x_\varepsilon) = f(x_\varepsilon)$. Since $M$ is compact the function $g$ is uniformly continuous, and therefore the pointwise convergence of the functions $(g_{\varepsilon})_{\varepsilon>0}$ to $g$ is uniform. Since $f$ is continuous as well it follows that any accumulation point $x$ of the sequence $(x_\varepsilon)_{\varepsilon>0}$ as $\varepsilon$ goes to 0 satisfies $g(x) = f(x)$, as desired.

**Step 3:** Deriving a contradiction.

Assume to the contrary that the theorem does not hold, and $f(x) \neq g(x)$ for every $x \in M$. Let $h : M \to \partial \Delta$ be the function that is defined by

$$h(x) := \frac{f(x) - g(x)}{\|f(x) - g(x)\|_2}.$$  

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Since there is no $x$ such that $f(x) = g(x)$, the function $h$ is well defined. The function $h|_{\partial M}$ is homotopic to $f|_{\partial M}$, by the homotopy

$$h_t(x) := \frac{f(x) - tg(x)}{\|f(x) - tg(x)\|_2}, \quad \forall t \in [0, 1].$$

Note that the $h_t$ is well defined for every $t \in [0, 1]$: for every $x \in \partial M$ we have $f(x) \in \partial \Delta$, hence $\|f(x)\|_2 = 1$ while $\|tg(x)\|_2 < 1$. In particular, the denominator of $h_t$ does not vanish.

It follows that the degree of the restricted function $h|_{\partial M}$ is equal to the degree of the restricted function $f|_{\partial M}$. By the definition of the orientation of the boundary of $M$, the degree of $f$ is equal to the degree of $f|_{\partial M}$. Thus, the degree of $h|_{\partial M}$ is nonzero.

Now, the function $h|_{\partial M}$ can be extended to a continuous function $h : M \to \Delta$, and hence by Guillemin and Pollack (2010, page 108, first proposition) it follows that the intersection index of $h|_{\partial M}$ with a point $y \in \partial \Delta$ is 0. By Lemma 4.3 it follows that the degree of $h|_{\partial M}$ is 0, a contradiction. \(\blacksquare\)

5 Approximating the Equilibrium Set by a Smooth Manifold

In this section we consider strategic-form games with a fixed set of players and fixed sets of actions for each player. Kohlberg and Mertens (1986) showed that the equilibrium set when one varies the payoff function is homeomorphic to the set of games. The goal of this section is to show that the equilibrium set can be uniformly approximated by a smooth manifold.

**Definition 5.1** A strategic game form is a pair $(I, A)$ where $I$ is a finite set of players and $A = \times_{i \in I} A_i$ is the Cartesian product of finite action sets for the players.

A payoff function for player $i$ in the strategic game form $(I, A)$ is a function $u_i : A \to \mathbb{R}$, and a payoff function is a collection $u = (u_i)_{i \in I}$ of payoff functions for the players. Consequently, the set of all payoff functions is equivalent to $\mathbb{R}^{A \times I}$. A triplet $(I, A, u)$ where $u$ is a payoff function for the strategic game form $(I, A)$ define a game.

A strategy for player $i$ is a probability distribution $x_i \in \Delta(A_i)$, and a strategy profile is a collection $x = (x_i)_{i \in I}$ of strategies for the players.
follows that the set of all strategy profiles, denoted $X$, is equivalent to $\times_{i \in I} \Delta(A_i) \subset \mathbb{R}^{\cup_{i \in I} A_i}$. A payoff function $u_i$ for player $i$ is extended to a function from $X$ to $\mathbb{R}$ in a multilinear fashion.

A strategy profile $x \in X$ is an equilibrium in the game $(I, A, u)$ if $u_i(x) \geq u_i(a_i, x_{-i})$ for every player $i \in I$ and every action $a_i \in A_i$. When the strategic game form is fixed, the equilibrium set is the collection of all pairs of a payoff function and equilibrium in the game induced by this payoff function.

**Definition 5.2** Let $(I, A)$ be a strategic game form. The equilibrium set of $(I, A)$ is the set

$$M := \{(u, x) \in \mathbb{R}^A \times X : x \text{ is a Nash equilibrium of } u\} \subset \mathbb{R}^A \times \mathbb{R}^{\cup_{i \in I} A_i}.$$ 

As mentioned above, Kohlberg and Mertens (1986) proved that the set $M$ is homeomorphic to the set of games, namely, to $\mathbb{R}^{A \times I}$. An important concept that we will need is that of $O_n$-equilibria, which we define now.

**Definition 5.3** Let $(I, A)$ be a strategic game form, let $u : A \to \mathbb{R}^I$ be a payoff function, and let $n > 0$. The strategy profile $x$ is an $O_n$-equilibrium of the game $(I, A, u)$ if for every player $i \in I$ and every action $a_i \in A_i$,

$$x_i(a_i) = \frac{\exp(nu_i(x, a_i))}{\sum_{a_i' \in A_i} \exp(nu_i(x, a_i'))}.$$  \hspace{1cm} (3)

Standard continuity arguments show that a limit of $O_n$ equilibria as $n$ goes to infinity is a Nash equilibrium. This observation is stated in the following lemma for future reference.

**Lemma 5.4** Let $(u[k])_{k=1}^{\infty}$ be a sequence of real numbers that go to infinity. Let $(u[k])_{k=1}^{\infty}$ be a sequence of positive payoff functions and let $(x[k])_{k=1}^{\infty}$ be a sequence of strategy profiles such that $x[k]$ is an $O_{u[k]}$-equilibrium in the game $(I, A, u[k])$. If the two limits $u := \lim_{k \to \infty} u[k]$ and $x := \lim_{k \to \infty} x[k]$ exist, then the strategy profile $x$ is a Nash equilibrium in the game $u$.

**Proof.** Fix a player $i \in I$ and two actions $a_i, \hat{a}_i \in A_i$. We will prove that if $u_i(a_i, x_{-i}) > u_i(\hat{a}_i, x_{-i})$ then $x_i(\hat{a}_i) = 0$. Since $u_i(a_i, x_{-i}) > u_i(\hat{a}_i, x_{-i})$ it follows that there is $\delta > 0$ such that for every $k$ sufficiently large,

$$u_i[k](a_i, x_{-i}) > u_i[k](\hat{a}_i, x_{-i}) + \delta.$$  \hspace{1cm} (4)

\textsuperscript{4}When writing $\cup_{i \in I} A_i$ we implicitly assume that the action sets of the players are disjoint.
Since \( x^{[k]} \) is an \( O_n[u] \)-equilibrium in the game \((I, A, u^{[k]})\), we have by Eq. (3)

\[
x_i^{[k]}(\tilde{a}_i) = \frac{\exp(nu_i^{[k]}(x_{-i}^{[k]}, \hat{a}_i))}{\sum_{a'_i \in A_i} \exp(nu_i(x_{-i}^{[k]}, a'_i))}.
\]

Eq. (4) implies that \( \lim_{k \to \infty} x_i^{[k]}(\tilde{a}_i) = 0 \), as desired. ■

For every real number \( n \) denote the set of all \( O_n \)-equilibrium by

\[
M_n := \{(u, x): x \text{ is an } O_n \text{-equilibrium in } u\}.
\]

We will show that \( M_n \) is a manifold, and that as \( n \) goes to infinity, the manifold \( M_n \) converges uniformly to the equilibrium set \( M \).

**Theorem 5.5** The set \( M_n \) is an \((A \times I)\)-dimensional manifold.

To prove Theorem 5.5 we need to study a certain function that will be used in the definition of the immersion between \( \mathbb{R}^{A \times I} \) and \( M_n \). The keen reader will identify the origin of this function and the proof of Theorem 5.5 in the work of Kohlberg and Mertens (1986).

**Lemma 5.6** For every \( n > 0 \) define the function \( g^{(n)}: \mathbb{R}^d \to \mathbb{R}^d \) by

\[
g^{(n)}_i(x) = x_i + \frac{\exp(nx_i)}{\sum_{j=1}^{d} \exp(nx_j)}, \quad \forall i \in \{1, 2, \cdots, d\}.
\]

The function \( g^{(n)} \) is one-to-one, onto, and an immersion.

**Proof.**

**Step 1:** The function \( g \) is an immersion.

An \( n \times n \) matrix \( A \) is strictly diagonal dominant if (a) its diagonal entries are positive, (b) its off-diagonal entries are negative, and (c) the sum of elements in each row is positive. Note that every strictly diagonal dominant matrix is invertible.

We first argue that the Jacobian matrix of \( g \) is a strictly diagonal dominant matrix at all points. Indeed, simple algebraic calculations show that for every \( i \in \{1, 2, \cdots, d\} \),

\[
\frac{\partial g_i}{\partial x_i}(x) = 1 + n \exp(nx_i) \left( \sum_{j \neq i} \exp(nx_j) \right) > 0, \quad (5)
\]

\[
\frac{\partial g_i}{\partial x_j}(x) = -n \frac{\exp(n(x_i + x_j))}{\left( \sum_{j=1}^{d} \exp(nx_j) \right)^2} < 0, \quad \forall j \neq i. \quad (6)
\]
In particular, Conditions (a) and (b) hold for the Jacobian matrix of \( g \) at every point \( x \). We also have

\[
\sum_{i=1}^{d} g_i(x) = 1 + \sum_{i=1}^{d} x_i,
\]

and therefore

\[
\sum_{i=1}^{d} \frac{\partial g_i}{\partial x_j}(x) = 1 > 0, \quad \forall j.
\]

So that Condition (c) holds as well, and the Jacobian matrix is strictly diagonal dominant at all points. It follows that \( g \) is an immersion.

**Step 2:** The function \( g \) is onto.

To prove that \( g \) is onto we will show that its image is both open and closed. Since the Jacobian matrix of \( g \) at every point \( x \) is invertible, by the Open Mapping Theorem the image of \( g \) is an open set. To show that the image of \( g \) is closed, note that \( \| x - g(x) \|_2 \leq 1 \) for every \( x \in \mathbb{R}^d \), and consider a sequence \( (y^k)_{k\in \mathbb{N}} \) of points in the image of \( g \) that converges to a point \( y \). For each \( k \in \mathbb{N} \) let \( x^k \in \mathbb{R}^d \) satisfy \( y^k = g(x^k) \). Since \( \| x^k - y^k \|_2 \leq 1 \), and since the sequence \( (y^k)_{k\in \mathbb{N}} \) converges, it follows that there is a subsequence \( (x^{k_l})_{l\in \mathbb{N}} \) that converges to a limit \( x \). Since the function \( g \) is continuous, \( g(x) = y \), so that \( y \) is in the image of \( g \).

**Step 3:** The function \( g \) is one-to-one.

We argue that any function whose Jacobian matrix is strictly diagonal dominant is one-to-one. Indeed, let \( f \) be such a function, assume w.l.o.g. that \( f(\bar{0}) = \bar{0} \), and fix \( x \neq \bar{0} \). We will show that \( f(x) \neq \bar{0} \). We have

\[
f(x) = f(0) + \int_{t=0}^{1} df_{tx} \cdot x dt = \left( \int_{t=0}^{1} df_{tx} dt \right) \cdot x.
\]

The matrix \( \int_{t=0}^{1} df_{tx} dt \), as an integral of strictly diagonal dominant matrices, is strictly diagonal dominant, hence invertible. In particular, \( \left( \int_{t=0}^{1} df_{tx} dt \right) \cdot x \neq \bar{0} \). □

**Proof of Theorem 5.5.** Kohlberg and Mertens (1986) provided an equivalent representation to games. Let \( u : A \to \mathbb{R}^I \) be a payoff function.
For every $i \in I$ define two functions $\tilde{u}_i : A \to \mathbb{R}$ and $\overline{u}_i : A_i \to \mathbb{R}$ by
\[
\overline{u}_i(a_i) := \frac{1}{|A_i|} \sum_{a_{-i} \in A_{-i}} u_i(a_i, a_{-i}), \quad (7)
\]
\[
\tilde{u}_i(a) := u_i(a) - \overline{u}_i(a_i). \quad (8)
\]
We denote this representation by $u = \langle \tilde{u}, \overline{u} \rangle$.

Fix $n > 0$ and define a function $z_n : M_n \to \bigcup_{i \in I} A_i$ by
\[
z_{n,i,a_i}(u, x) := u_i(a_i, x_{-i}) + \frac{\exp(nu_i(a_i, x_{-i}))}{\sum_{j \in I} \exp(nu_j(a_j, x_{-j}))}, \quad \forall i \in I, a_i \in A_i.
\]
Define now a function $\varphi_n : M_n \to \mathbb{R}^{A \times I}$ by
\[
\varphi_n(u, x) := \langle \tilde{u}, z_n(u, x) \rangle. \quad (9)
\]
Lemma 5.6 implies that the function $\varphi_n$ is one-to-one, onto, and an immersion. The result follows.

We now prove that the inverse of $g^{(n)}$ converges uniformly as $n$ goes to infinity, and we provide an explicit form to the limit function, which is nothing but the function defined by Kohlberg and Mertens (1986).

**Lemma 5.7** For every $n > 0$ let $h^{(n)} : \mathbb{R}^d \to \mathbb{R}^d$ be the inverse of $g^{(n)}$. Let $h : \mathbb{R}^d \to \mathbb{R}^d$ be the function defined by
\[
h_i(y) := \min\{y_i, \alpha^*\}, \quad \forall i = 1, 2, \cdots, d,
\]
where $\alpha^* := \max \left\{ \alpha \in \mathbb{R} : \sum_{i=1}^{d} (y_i - \alpha)_+ = 1 \right\}$. Then the sequence of functions $(h^{(n)})_{n>0}$ converges uniformly to the function $h$.

**Proof.** Fix $\varepsilon > 0$, and let $n > 0$ be sufficiently large so that $\varepsilon > 1/(1 + \exp(\varepsilon n))$. Fix $y \in \mathbb{R}^d$ and define $x := h(y)$ and $x^{(n)} := h^{(n)}(y)$. Assume w.l.o.g. that $y_1 \leq y_2 \leq \cdots \leq y_d$. By the definition of $g^{(n)}$ we have $x_1^{(n)} \leq x_2^{(n)} \leq \cdots \leq x_d^{(n)}$. By the definition of $h$ we have $x_1 \leq x_2 \leq \cdots \leq x_d$. Since
\[
\sum_{i=1}^{d} (y_i - \alpha^*)_+ = 1 = \sum_{i=1}^{d} (y_i - x_i^{(n)}) = \sum_{i=1}^{d} (y_i - x_i^{(n)})_+,
\]

and since $x_1^{(n)} \leq x_2^{(n)} \leq \cdots \leq x_d^{(n)}$, it follows that $x_d^{(n)} \geq \alpha^* = x_d$.

Denote 
\[ \alpha_i := y_i - x_i \geq 0, \]
and 
\[ \alpha_i^{(n)} := y_i^{(n)} - x_i^{(n)} \geq 0. \]
We now claim that $\alpha_i^{(n)} < \alpha_i + \varepsilon$. Indeed, assume to the contrary that for some $i$ we have $\alpha_i^{(n)} \geq \alpha_i + \varepsilon$. Then in particular
\[ x_i^{(n)} = y_i - \alpha_i^{(n)} \leq y_i - \alpha_i - \varepsilon = x_i - \varepsilon \leq x_d - \varepsilon \leq x_d^{(n)} - \varepsilon. \]
Therefore, and by the definition of $g^{(n)}$,
\[ \varepsilon \leq \alpha_i^{(n)} = \frac{\exp(nx_i^{(n)})}{\sum_{j=1}^{d} \exp(nx_j^{(n)})} \leq \frac{\exp(nx_i^{(n)})}{\exp(nx_i^{(n)} + nx_d^{(n)})} = \frac{1}{1 + \exp(n(x_d^{(n)} - x_i^{(n)}))} \leq \frac{1}{1 + \exp(\varepsilon n)}; \]
a contradiction to the choice of $n$. Since $\sum_{i=1}^{d} \alpha_i^{(n)} = 1 = \sum_{i=1}^{d} \alpha_i$, we deduce that for every $i \in \{1, 2, \ldots, d\}$ we have
\[ \alpha_i - d\varepsilon \leq \alpha_i^{(n)} \leq \alpha_i + \varepsilon, \]
which implies that $\|h^{(n)}(y) - h(y)\|_{\infty} \leq d\varepsilon$, and the desired result follows. \[ \square \]

Kohlberg and Mertens (1986) proved that the following function $\varphi : M \to \mathbb{R}^{A \times I}$ is a homeomorphism:
\[ \varphi(u, x) := \langle \tilde{u}, z(u, x) \rangle, \quad \forall (u, x) \in M, \]
where notations follow the proof of Theorem 5.5 and
\[ z_{i, a_i}(u, x) := u_i(a_i, x_{-i}) + x_i(a_i), \quad \forall i \in I, a_i \in A_i. \]
As a conclusion of Lemma 5.7 we deduce that the manifolds $(M_n)_{n>0}$ converge to the equilibrium set $M$ in a strong sense.

**Theorem 5.8** For every $\varepsilon > 0$ there is $N = N(\varepsilon) > 0$ such that for every $n \geq N$ we have
\[ \|\varphi^{-1}(y) - (\varphi_n)^{-1}(y)\|_2 \leq \varepsilon, \quad \forall y \in \mathbb{R}^{A \times I}. \]
6 Proof of the Main Result

In this section we prove Theorem 2.5 which states that every generalized quitting games admits a sunspot $\varepsilon$-equilibrium, for every $\varepsilon > 0$. Fix then a generalized quitting game $\Gamma = (I, (A^c_i)_{i \in I}, u)$. For every $i \in I$ denote by $\bar{v}_i$ player $i$’s uniform minmax value in the game $\Gamma$. We start by a technical lemma that extends Laraki (2010) to multiplayer general quitting games.

**Lemma 6.1** For every vector $\alpha_{-i} \in \times_{j \neq i} \Delta(A^c_j)$ either $u_i(Q_i, \alpha_{-i}) \geq \bar{v}_i$, or there exists $\alpha_i \in \Delta(A^c_i)$ such that the following two conditions hold:

1. $u_i(\alpha_i, \alpha_{-i}) \geq \bar{v}_i$, and
2. $u_i(\alpha_i, \alpha_{-ij}, Q_j) \geq \bar{v}_i$ for every player $j \neq i$.

**Proof.** Assume by contradiction that the claim does not hold. Then there exists $\alpha_{-i} \in \times_{j \neq i} \Delta(A^c_j)$ such that $r^i(Q_i, \alpha_{-i}) < \bar{v}_i$ and for every $\alpha_i \in \Delta(A^c_i)$ at least one of the following conditions hold:

1. $u_i(\alpha_i, \alpha_{-i}) < \bar{v}_i$.
2. $u_i(\alpha_i, \alpha_{-ij}, Q_j) < \bar{v}_i$ for some player $j \neq i$.

We will show that in the original game $\Gamma$ players $I \setminus \{i\}$ can lower player $i$’s payoff below $\bar{v}_i$, a contradiction.

Fix $\varepsilon > 0$. Let $\hat{\Gamma}^\varepsilon$ be the following auxiliary two-player zero-sum absorbing game that is derived from the game $\Gamma$, and in which Player I corresponds to player $i$ in $\Gamma$, and Player II corresponds to all other players in $\Gamma$:

- Player I is similar to player $i$ in the game $\Gamma$.

- The set of actions of Player II is $\{\emptyset\} \cup (I \setminus \{i\})$, with the following interpretation: the action $\emptyset$ corresponds to having all players $I \setminus \{i\}$ play the mixed action profile $\alpha_{-i}$ in the game $\Gamma$, and the action $j \in I \setminus \{i\}$ corresponds to having player $j$ play the mixed action $\varepsilon Q_j + (1 - \varepsilon)\alpha_j$ in the game $\Gamma$ while all players $k \in I \setminus \{i, j\}$ play the mixed action $\alpha_k$.

- Transitions and payoffs are the ones derived from the above interpretation.
Denote by $\hat{\gamma}(\cdot)$ the payoff function in the game $\hat{\Gamma}$. The game $\hat{\Gamma}$ is a zero-sum stochastic game, hence by Mertens and Neyman (1981) it admits a uniform value $\hat{v}$. By standard arguments, $\hat{v}^0 := \lim_{\varepsilon \to 0} \hat{v}$ exists.

The strategy set of Player I in the game $\hat{\Gamma}$ coincides with the strategy set of player $i$ in the game $\Gamma$. Every strategy $\sigma_{II}$ in the game $\hat{\Gamma}$ naturally defines a strategy profile $\sigma_{-i} = (\sigma_j)_{j \neq i}$ in the game $\Gamma$ as follows: after the finite history $h^t$, the strategy $\sigma_j$ stops with probability $\varepsilon \sigma_{II}(j \mid h^t)$, and plays the mixed action $\alpha_j$ otherwise. We can thus treat any strategy pair $(\sigma_I, \sigma_{II})$ in the game $\hat{\Gamma}$ also as a strategy profile in the game $\Gamma$.

We claim that for every strategy pair $(\sigma_I, \sigma_{II})$ in the game $\hat{\Gamma}$ we have

$$\|\hat{\gamma}(\sigma_I, \sigma_{II}) - \gamma(\sigma_I, \sigma_{II})\|_\infty < \varepsilon.$$  \hfill (10)

Indeed, the difference in payoffs between the two games arises only because in the game $\hat{\Gamma}$ joint quittings cannot occur, while in the game $\Gamma$ joint quittings may occur. Note, though, that in the game $\hat{\Gamma}$ the probability that a player quits is bounded from above by $\varepsilon$. Consequently, the probability of a joint quitting in the game $\hat{\Gamma}$ after any finite history is smaller than $\varepsilon$ times the probability of a single quitting after that history. In particular, Eq. (10) holds.

By Laraki (2010) it follows that $\hat{v}^0 < \overline{v}_i$. Since for $\varepsilon$ small, the strategy set of Player II in the auxiliary game $\hat{\Gamma}$ is “smaller” than the product of the strategy spaces of players $I \setminus \{i\}$ in the game $\Gamma$, it follows that $\overline{v}_i \leq \lim\limits_{\varepsilon \to 0} \hat{v}$. This is a contradiction. $\blacksquare$

We now turn to prove Theorem 2.5. For each $i \in I$ denote an element $\alpha^{[i]} \in \Delta(A^i_i)$ by $\alpha_1^{[i]} 1^{i} + \cdots + \alpha_{k_i}^{[i]} C_{k_i}^i$, where $k_i := |A^i_i|$ is the number of continue actions of player $i$ and $A^i_i = \{C_1^i, \ldots, C_{k_i}^i\}$.

**Step 1:** Definition of a family of auxiliary quitting games.

For every $\alpha = (\alpha^{[i]}, \ldots, \alpha^{[I]}) \in \times_{i \in I} \Delta(A^i_i)$ and every $q \in \mathbb{R}^I$ let $\Gamma^\alpha(q)$ be the quitting game that is based on $\Gamma$ and is defined as follows:

- Whenever player $i$ continues, it is as if he plays each continue action $C_k^i$ in $\Gamma$ with probability $\alpha_k^{[i]}$, for $1 \leq k \leq k_i$.
- The nonabsorbing payoff is $q_0$. 

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Formally, the payoff function in the game $\Gamma^\alpha(q)$, denoted $u^\alpha(\cdot)$, is defined as follows. For every subset $J \subseteq I$ denote $A^*_J := (\times_{i \in J} A_i) \setminus (\times_{i \in J} A_i^c)$; this is the set of action profiles of players $J$ in which at least one player quits. Then

$$u^\alpha(\tilde{C}) := q,$$

$$u^\alpha(\tilde{C}_J, \tilde{Q}_{I\setminus J}) := \sum_{a^J \in A^*_J} \left( \prod_{i \in J} \alpha^{i}(a_i) \right) u(a^J, \tilde{Q}_{I\setminus J}), \quad \emptyset \subseteq J \subset I.$$

For every $\alpha \in \times_{i \in I} \Delta(A_i^c)$ denote $R^\alpha = R(\Gamma^\alpha(q))$, for any $q \in R^I$.

**Step 2:** The matrix $R^\alpha$ is a $Q$-matrix for every $\alpha \in \times_{i \in I} \Delta(A_i^c)$.

We will define a set-valued function $F : (\times_{i \in I} \Delta(A_i^c)) \to (\times_{i \in I} \Delta(A_i^c))$, and show that it satisfies the conditions of Kakutani’s Fixed Point Theorem. We then show that for every fixed point $\alpha$ of $F$, the sunspot $\varepsilon$-equilibrium of the auxiliary game $\Gamma^\alpha$ can be converted into a sunspot $\varepsilon$-equilibrium of the original game $\Gamma$.

For every vector $\alpha = (\alpha_i)_{i \in I} \in \times_{i \in I} \Delta(A_i^c)$ and every player $i \in I$ define $F_i(\alpha) \subseteq \Delta(A_i^c)$ as follows:

$$F_i(\alpha) := \begin{cases} \Delta(A_i^c), & \text{if } u_i(Q_i, \alpha_{-i}) \geq \overline{v}_i, \\ \{ \alpha'_i \in \Delta(A_i^c) : u_i(\alpha'_i, \alpha_{-i}) \geq \overline{v}_i \text{ and } u_i(\alpha'_i, \alpha_{-i}, j) \geq \overline{v}_i, \forall j \neq i \}, & \text{otherwise.} \end{cases}$$

We note that $F_i(\alpha)$ is independent of $\alpha_i$, and by Lemma [6.1] it has nonempty values. The reader can verify that $F$ has convex values, and by standard continuity arguments as well as from the fact that each player has a single quitting action, it has a closed graph. By Kakutani’s Fixed Point Theorem, there is $\alpha \in \times_{i \in I} \Delta(A_i^c)$ such that $\alpha \in F(\alpha)$. Since the matrix $R^\alpha$ is a $Q$-matrix, by Solan and Solan (2017) the game $\Gamma^\alpha(q)$ admits a sunspot $\varepsilon$-equilibrium $\xi$, which is independent of $q$, and in which at most one player quits with positive probability at every stage, and does so with probability smaller than $\varepsilon$. The definition of $F$ together with Theorem [2.9] yield that $\gamma_i^\alpha(\xi) \geq \overline{v}_i$ for every player $i \in I$. Consequently, by adding statistical tests to ensure that each player $i$ plays the mixed action $\alpha_i$ whenever $\xi$ indicates that he should continue, we deduce that the original game $\Gamma$ has a sunspot $\varepsilon$-equilibrium.

We note that if there are a sequence $(\alpha_i^{[n]})_{n \in \mathbb{N}}$ of vectors in $\times_{i \in I} \Delta(A_i^c)$ and a sequence $(x^{[n]})_{n \in \mathbb{N}}$ of stationary strategies in $[0, 1]^I$ such that $\lim_{n \to \infty} p(x^{[n]}) = \varepsilon$.
0 and for every \( n \in \mathbb{N} \), the stationary strategy profile \( x^{[n]} \) is (a) absorbing and (b) a 0-equilibrium of the game \( \Gamma^{[n]} \), then the matrix \( R^\alpha \) is a Q matrix for every accumulation point \( \alpha \) of the sequence \( (\alpha^{[n]})_{n \in \mathbb{N}} \).

Consequently we can assume that there is \( \varepsilon_0 > 0 \) such that for every \( \varepsilon < \varepsilon_0 \) and every \( \alpha \in \times_{i \in I} \Delta(A^c_i) \) the game \( \Gamma^\alpha \) does not admit a stationary 0-equilibrium \( x \) that satisfies \( p(x) < \varepsilon_0 \).

**Step 3:** There is \( \alpha_0 \in \times_{i \in I} \Delta(A^c_i) \) such that the matrix \( R^{\alpha_0} \) is not a Q-matrix.

Since the matrix \( R^{\alpha_0} \) is not a Q-matrix, there is \( q_0 \in \mathbb{R}^I \) such that the linear complementarity problem \( \text{LCP}(R^{\alpha_0}, q_0) \) has no solution.

Fix \( \lambda \in (0, 1] \), \( n > 0 \), and \( \varepsilon < \frac{1}{1+\exp(n)} \).

**Step 3a:** Applying Theorem 4.1.

A binary game is a strategic-form game in which every player has two actions. Denote the set of mixed action profiles in an \( I \)-player binary game by \( Z := [0, 1]^I \); this is a compact manifold.

Denote \( N := \times_{i \in I} \Delta(A^c_i) \).

Note that \( N \) is a compact manifold with boundary. The set \( N \times Z \) is equivalent to the set of mixed action profiles \( X \) in the original game \( \Gamma \). Indeed, for every pair \((\alpha, z) \in N \times Z\), where \( \alpha = (\alpha^{[i]})_{i \in I} \) and \( z = (z_i)_{i \in I} \), corresponds the mixed action profile \( x = x(\alpha, z) \in \times_{i \in I} \Delta(A^c_i) \) under which \( z_i \) is the probability that player \( i \) chooses the action \( Q_i \) and the product \( 1 - z_i )^{\alpha^{[i]}} \) determines the probability that player \( i \) uses each of his continue actions. Formally,

\[
\begin{align*}
x_i(Q_i) &:= z_i, \quad (11) \\
x_i(C^k_i) &:= (1 - z_i)\alpha^{[i]}_k, \quad \forall k \in \{1, 2, \ldots, k_i\}. \quad (12)
\end{align*}
\]

Let \( U := \mathbb{R}^{2|I|\times |I|} \) be the set of payoff functions for binary \(|I|\)-player games. The set \( U \) is a connected open boundaryless manifold. Denote by \( M \subset \mathbb{R}^{2|I|\times |I|} \times [0, 1]|I| \) the equilibrium set of binary games, and by \( M_n \) the smooth manifold of \( O_n \)-equilibria of binary games. Let \( \pi : U \times Z \to U \) be the projection. We can choose the orientation of \( M_n \) and \( U \) in such a way that the degree of \( \pi|_{M_n} \) is 1.

Let \( y_\lambda : N \times Z \to U \) be the continuous function that is defined by

\[
y_\lambda(\alpha, z, a) := \begin{cases} 
\lambda q_0 + (1 - \lambda)\gamma(\alpha(\alpha, z)), & \text{if } a = \vec{C}, \\
u(\alpha_J, Q_{-J}), & \text{if } a = (\vec{C}_J, Q_{-J}).
\end{cases} \quad (13)
\]
This is the payoff function of the binary strategic-form game that is derived from the game \( \Gamma^0(q_0) \), assuming players discount their payoffs and the continuation strategy profile is \( x(\alpha, z) \). Since \( U \) is convex, the function \( y_\lambda(\alpha, \cdot) : Z \rightarrow U \) is homotopic to a constant function, for every \( \alpha \in N \).

Let \( y_1 : N \times Z \rightarrow U \) be the function that is defined on the right-hand side of Eq. (13) with \( \lambda = 1 \). For every fixed \( \delta > 0 \), on the region

\[
X_\delta^* := \{ (\alpha, z) \in N \times Z : \sum_{i \in I} z_i \geq \delta \}
\]

the functions \( (y_\lambda)_{\lambda \in (0,1]} \) converge uniformly to \( y_1 \) as \( \lambda \) goes to 0.

For every \( \lambda \in (0,1] \) let \( \tilde{y}_\lambda : N \times Z \rightarrow U \times Z \) be the function defined by

\[
\tilde{y}_\lambda(\alpha, z) := (y_\lambda(\alpha, z), z), \quad \forall \alpha \in N, \forall z \in Z.
\]

By Theorem 4.1 applied to \( X = Z, U, M = M_n, N \), and \( y = y_\lambda \), there is an \( \varepsilon \)-perturbation \( \tilde{y}_{\lambda,n,e} \) of \( \tilde{y}_\lambda \) that is transversal to \( M_n \) and such that the set \( M_{\lambda,n,e} := (\tilde{y}_{\lambda,n,e})^{-1}(M_n) \subseteq N \times Z \) is a \( (\sum_{i \in I}(|A_i| - 1)) \)-dimensional manifold whose boundary is contained in \( \partial N \times Z \).

**Step 3b:** Dividing the manifold \( M_{\lambda,n,e} \) to absorbing and nonabsorbing points.

As we noted above, for every \( \alpha \in N \) the game \( \Gamma^\alpha(q_0) \) does not admit a stationary equilibrium that is absorbing with probability in \((0, \varepsilon_0)\).

Since the sequence of functions \( (\tilde{y}_\lambda)_{\lambda > 0} \) converges uniformly to \( \tilde{y}_1 \) on the region \( X_{\varepsilon_0}^* \) in which the probability of absorption is at least \( \varepsilon_0 \), there is \( \lambda_0(\varepsilon_0) > 0 \) such that the intersection of the image of \( \tilde{y}_\lambda \) and \( M \) is disjoint of \( U \times (B(\overline{C}, \frac{6\varepsilon_0}{7}) \setminus B(\overline{C}, \frac{5\varepsilon_0}{7})) \), for every \( \lambda \in (0, \lambda_0(\varepsilon_0)) \).

By Theorem 5.8 the manifolds \((M_n)_{n \in \mathbf{N}} \) converge uniformly to \( M \) on every compact set of games, hence there exists \( n_0 = n_0(\varepsilon_0) \in \mathbf{N} \) such that the intersection of the image of \( \tilde{y}_1 \) and \( M_n \) is disjoint of \( U \times (B(\overline{C}, \frac{5\varepsilon_0}{7}) \setminus B(\overline{C}, \frac{2\varepsilon_0}{7})) \), for every \( n \geq n_0(\varepsilon_0) \) and every \( \lambda \in (0, \lambda_0(\varepsilon_0)) \).

Since \( \tilde{y}_{\lambda,n,e} \) is an \( \varepsilon \)-perturbation of \( \tilde{y}_\lambda \), it follows that the intersection of the image of \( \tilde{y}_{\lambda,n,e} \) and \( M_n \) is disjoint of \( U \times (B(\overline{C}, \frac{4\varepsilon_0}{7}) \setminus B(\overline{C}, \frac{3\varepsilon_0}{7})) \), for every \( n \geq n_0(\varepsilon_0) \), every \( \lambda \leq \lambda_0(\varepsilon_0) \), and every \( \varepsilon \leq \frac{3\varepsilon_0}{7} \).

We can therefore divide \( M_{\lambda,n,e} \) into two disjoint parts: the points in \( N \times B(\overline{C}, \frac{3\varepsilon_0}{7}) \) and the points in \( N \times (Z \setminus B(\overline{C}, \frac{4\varepsilon_0}{7})) \), denoted respectively \( M_{\lambda,n,e}^{in} \) and \( M_{\lambda,n,e}^{out} \).

By the choice of \( q_0 \), the set \( \tilde{y}_1(\{\alpha_0\}) \times M_n \) is disjoint of \( B(\overline{C}, \frac{4\varepsilon_0}{7}) \), for every \( n \geq n_0 \). By Theorem 4.1 the projection \( f : N \times Z \rightarrow N \), restricted
to $M_{\lambda,n,\varepsilon}$, has degree 1. When restricted to $M_{\lambda,n,\varepsilon}^{in}$, the projection $f$ is not onto and therefore it has degree 0. It follows that the projection $f|_{M_{\lambda,n,\varepsilon}^{out}}$ has degree 1.

**Step 3c:** Applying Theorem 4.4

In an $O_n$-equilibrium of a binary game whose payoffs are in the interval $[0, c]$, each action is played with probability at least $\frac{1}{1 + \exp(nc)}$. It follows that for every $(\alpha, z) \in M_{\lambda,n,\varepsilon}^{out}$ we have $z_{i}^{\lambda,n,\varepsilon} \geq \frac{1}{1 + \exp(n)} - \varepsilon$, which is positive by the choice of $\varepsilon$. For every $n > 0$ and every $\varepsilon \geq 0$ denote

$$Z_{n,\varepsilon} := \left\{ z \in Z : z_{i} \geq \frac{1}{1 + \exp(n)} - \varepsilon, \ \forall i \in I \right\}.$$ 

The set $Z_{n,\varepsilon}$ is nonempty and compact, and, as mentioned above, it satisfies $M_{\lambda,n,\varepsilon}^{out} \subseteq N \times Z_{n,\varepsilon}$. For every $z \in Z_{n,\varepsilon}$ and every $i \in I$ we have $z_{i} > 0$, hence the absorbing payoff $u_{i}(C_{i}^{k}, x_{-i}(\alpha, z))$ is well defined for every $k \in \{1, 2, \ldots, k_{i}\}$.

Define a continuous function $g^{[n]} : N \times Z_{n,\varepsilon} \to N$ by

$$g_{i}^{[n]}(\alpha, z) := \sum_{k=1}^{k_{i}} \exp\left( nu_{i}(C_{i}^{k}, x_{-i}(\alpha, z)) \right) C_{i}^{k}.$$ 

We would like to apply Theorem 4.4 with $\Delta = N$, $M = M_{\lambda,n,\varepsilon}^{out}$, $f : N \times Z \to Z$ the natural projection, and $g = g^{[n]}$. We need to verify that $\partial M_{\lambda,n,\varepsilon}^{out} \subseteq f^{-1}(\partial N)$. Since $\tilde{y}_{\lambda,n,\varepsilon}$ is transversal to $M_{\lambda,n,\varepsilon}$, it follows that

$$\partial M_{\lambda,n,\varepsilon}^{out} \subseteq \partial M_{\lambda,n,\varepsilon} \subseteq \partial(\text{domain}(\tilde{y}_{\lambda,n,\varepsilon})) = \partial(N \times Z) = (\partial N \times Z) \cup (N \times \partial Z).$$ 

Since every $O_n$-equilibrium is completely mixed, $\partial M_{\lambda,n,\varepsilon}^{out}$ is disjoint of $N \times \partial Z$, hence $\partial M_{\lambda,n,\varepsilon}^{out} \subseteq \partial N \times Z$, so that indeed $\partial M_{\lambda,n,\varepsilon}^{out} \subseteq f^{-1}(\partial N)$. By Theorem 4.4 we obtain the existence of a point $(\alpha_{\lambda,n,\varepsilon}, z_{\lambda,n,\varepsilon}) \in M_{\lambda,n,\varepsilon}^{out}$ that satisfies

$$g^{[n]}(\alpha_{\lambda,n,\varepsilon}, z_{\lambda,n,\varepsilon}) = \alpha_{\lambda,n,\varepsilon}.$$ 

The fact that $(\alpha_{\lambda,n,\varepsilon}, z_{\lambda,n,\varepsilon}) \in M_{\lambda,n,\varepsilon}^{out}$ has two implications:

- Under the strategy profile $x(\alpha_{\lambda,n,\varepsilon}, z_{\lambda,n,\varepsilon})$ the per-stage probability of absorption is bounded away from 0: $\sum_{i \in I} z_{\lambda,n,\varepsilon,i} \geq \varepsilon_{0}$.
- $\tilde{y}_{\lambda,n,\varepsilon}(\alpha_{\lambda,n,\varepsilon}, z_{\lambda,n,\varepsilon}) \in M_{\lambda,n,\varepsilon}^{out}$. 

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Step 3d: Taking limits.

We let \( \varepsilon \) go to 0, then \( \lambda \) go to 0, and finally \( n \) go to infinity. Since the set \( N \times \mathbb{Z} \) is compact, for every fixed \( n > 0 \) and \( \lambda \in (0, 1] \), the sequence \( (\alpha_{\lambda,n,\varepsilon}, z_{\lambda,n,\varepsilon})_{\varepsilon > 0} \) has an accumulation point \( (\alpha_{\lambda,n}, z_{\lambda,n}) \in N \times \mathbb{Z}_{n,0} \) as \( \varepsilon \) goes to 0. By continuity this accumulation points satisfies the following properties:

\[
\begin{align*}
g^{[n]}(\alpha_{\lambda,n}, z_{\lambda,n}) &= \alpha_{\lambda,n}, \\
\sum_{i \in I} z_{\lambda,n,i} &\geq \varepsilon_0, \\
\tilde{g}_\lambda(\alpha_{\lambda,n}, z_{\lambda,n}) &\in M_n.
\end{align*}
\]

For every fixed \( n > 0 \) consider an accumulation point of the sequence \( x(\alpha_{\lambda,n}, z_{\lambda,n})_{\lambda \in (0, 1]} \) as \( \lambda \) goes to 0, denoted \( (\alpha_n, z_n) \). Since \( y^\lambda \) converges uniformly to \( y_1 \) on \( X_{\varepsilon_0}^* \), we deduce that

(D.1) \( g^{[n]}(\alpha_n, z_n) = \alpha_n \).

(D.2) The strategy profile \( z_n \) is absorbing: \( \sum_{i \in I} z_{n,i} \geq \varepsilon_0 \).

(D.3) The strategy profile \( z_n \) is an \( O_n \)-equilibrium in the binary game \( y_1(\alpha_n, z_n) \).

Consider now an accumulation point \( (\alpha, z) \) of the sequence \( (\alpha_n, z_n)_{n \in \mathbb{N}} \) as \( n \) goes to infinity. We will show that the strategy profile \( x(\alpha, z) \) is a Nash equilibrium in the game \( \Gamma \). By continuity this strategy profile is absorbing: \( \sum_{i \in I} z_i \geq \varepsilon_0 \). By Lemma [5.4] the stationary strategy profile \( z \) is a 0-equilibrium in the binary game \( y_1(\alpha, z) \).

We now show that \( x(\alpha, z) \) is a stationary 0-equilibrium in the game \( \Gamma \) as well. Fix a player \( i \in I \) such that whatever he plays, the play is absorbed; that is, \( \sum_{j \neq i} z_j > 0 \). We will show that player \( i \) is indifferent among all actions in the support of \( \alpha^{[i]} \). Indeed, fix two continue actions \( a_i, a'_i \in A^c_i \) of player \( i \). If \( u_i(a_i, x_{-i}(\alpha, z)) < u_i(a'_i, x_{-i}(\alpha, z)) \), then there is \( \eta > 0 \) such that \( u_i(a_i, x_{-i}(\alpha, z)) < u_i(a'_i, x_{-i}(\alpha, z)) - \eta \). Consequently, for every \( n \) sufficiently large we have

\[
u_i(a_i, x_{-i}(\alpha_n, z_n)) < u_i(a'_i, x_{-i}(\alpha_n, z_n)) - \eta.
\]

By the definition of \( g^{[n]} \) and by (D.1), this implies

\[
\lim_{n \to \infty} \frac{\alpha^{[n]}_{n}(a_i)}{\alpha^{[n]}_{n}(a'_i)} = \lim_{n \to \infty} \frac{\exp(nu_i(a_i, x_{-i}(\alpha_n, z_n)))}{\exp(nu_i(a'_i, x_{-i}(\alpha_n, z_n)))} = 0.
\]

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In particular, under the mixed action $x_i(\alpha, z)$ the action $a_i$ is selected with probability 0.

Since $z_n$ is an $O_n$-equilibrium of the binary game, if $z_i > 0$, then player $i$ is indifferent between continuing and quitting.

It is left to consider the case that player $i$ is the sole player who quits with positive probability: $\sum_{j \neq i} z_j = 0$. It is standard to show that the stationary strategy profile $x(\alpha_n, z_n)$ is an $\varepsilon$-equilibrium, provided $n$ is sufficiently large.

## 7 Extensions

In this paper we proved the existence of a sunspot $\varepsilon$-equilibrium in the class of general quitting games. A natural question is whether our techniques can be applied to more general classes of games. These include (a) games with more than one nonabsorbing state and (b) games in which the absorption structure is not rectangular.

Regarding extension (a), our techniques can be used to prove the existence of a sunspot $\varepsilon$-equilibrium in positive recursive stochastic games with two nonabsorbing states, in which in each state the absorption structure is rectangular.

Regarding extension (b), we believe that our techniques apply whenever each player has a quitting action, that is, an action that leads to absorption, whatever the other players play. For example, in general positive recursive absorbing games we have the following sufficient condition for the existence of a sunspot equilibrium payoff.

**Definition 7.1** Let $i \in I$ be a player and let $x \in X$ be a nonabsorbing mixed action profile, that is, $p(x) = 0$. An action $a_i \in A_i$ is an absorbing action given $x$ if $p(a_i, x_{-i}) > 0$. The set of players who have absorbing actions given the mixed action profile $x$ is denoted by $I(x)$. If player $i$ has an absorbing action $a_i$ given $x$, we say that the action $a_i$ is a best absorbing action given $x$ if it yields to player $i$ the maximal payoff among all absorbing actions: $p(a_i, x_{-i}) > 0$ and $u_i(a_i, x_{-i}) \geq u_i(a'_i, x_{-i})$ for every absorbing action $a'_i$ of player $i$ given $x$.

If the game is positive and recursive, and if there is a nonabsorbing mixed action profile $x$ such that $I(x) = \emptyset$, then the stationary strategy $x$ is a stationary Nash equilibrium. When $I(x) \neq \emptyset$ for every nonabsorbing mixed action profile $x \in X$, one can prove the following result.
Theorem 7.2 Let $\Gamma = (I, (A_i)_{i \in I}, p, u)$ be a positive recursive absorbing game and let $x \in X$ be a nonabsorbing mixed action profile such that $I(x) = \{1, 2, \ldots, k\}$. For each player $i \in I(x)$ let $\hat{a}_i \in A_i$ be a best absorbing action of player $i$ given $x$. Denote by $R$ the $(k \times k)$-matrix whose $i$'th row is the vector $u(\hat{a}_i, x_{-i})$, restricted to its first $k$ coordinates. If the matrix $R$ is a $Q$-matrix, then for every $\varepsilon > 0$ the game $\Gamma$ admits a sunspot $\varepsilon$-equilibrium.

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