ANDREWS-GORDON TYPE SERIES FOR THE LEVEL 5 AND 7
STANDARD MODULES OF THE AFFINE LIE ALGEBRA $A_2^{(2)}$

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Abstract. We give Andrews-Gordon type series for the principal characters
of the level 5 and 7 standard modules of the affine Lie algebra $A_2^{(2)}$. We also
give conjectural series for some level 2 modules of $A_{13}^{(2)}$.

1. Introduction

In this paper, we use the $q$-Pochhammer symbol: for $n \in \mathbb{N}$ and $m \in \mathbb{N} \cup \{\infty\}$,
$$
(x; q)_{\infty} := \prod_{i \geq 0} (1 - xq^i), \quad (x; q)_n := \prod_{i=0}^{n-1} (1 - xq^i), \quad (a_1; q)_m := (a_1; q)_m \cdots (a_k; q)_m.
$$

1.1. The Andrews-Gordon identities. The Rogers-Ramanujan identities

$$
\sum_{n \geq 0} \frac{q^{n^2}}{(q; q)_n} = \frac{1}{(q, q^4; q^5)_{\infty}}, \quad \sum_{n \geq 0} \frac{q^{n^2 + n}}{(q; q)_n} = \frac{1}{(q^2, q^3; q^5)_{\infty}}
$$

was one of the motivations for inventing the vertex operators [15, §14] in the theory
of affine Lie algebras (see [22]). It started from Lepowsky-Milne’s observation [23]:

$$
\chi_{A_1^{(1)}}(2\Lambda_0 + \Lambda_1) = \frac{1}{(q, q^4; q^5)_{\infty}}, \quad \chi_{A_1^{(1)}}(3\Lambda_0) = \frac{1}{(q^2, q^3; q^5)_{\infty}}.
$$

Here, $\chi_A(\lambda)$ (called the principal character) stands for the principally specialized
character of the vacuum space $\Omega(V(\lambda))$ [11, §7] for the integrable highest weight
module (a.k.a. the standard module) $V(\lambda)$ associated with a dominant integral
weight $\lambda \in P^+$ of the affine Lie algebra $g(A)$. We obey the numbering of vertices
of the affine Dynkin diagram $A$ in [15, §4] and duplicate $A_1^{(1)}$, $A_2^{(2)}$, $A_{odd}^{(2)}$ as Figure 1.

The level of $\sum_{i \in I} d_i \Lambda_i \in P^+$ is given by $\sum_{i \in I} \tilde{a}_i d_i$, where the colabel $\tilde{a}_i$ is
the number written on the vertex $\alpha_i$ in the figure. We can expand $\chi_A(\lambda)$ into an explicit infinite product via Lepowsky’s numerator formula (see [4]).

After the success of vertex operator theoretic proofs of the Rogers-Ramanujan
identities [25, 26, 27], it has been expected that, for each $A$ and $\lambda$, there should exist
“Rogers-Ramanujan type identities” whose infinite products are given by $\chi_A(\lambda)$.

The Andrews-Gordon identities (Theorem 1.1) can be seen as an instance of this
expectation because of an existence of a vertex operator theoretic proof for it [24].

Note that the infinite product (the right hand side) in Theorem 1.1 is equal to

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\[ A_1^{(1)} \sim \frac{1}{\alpha_0} \quad A_2^{(2)} \sim \frac{2}{\alpha_0} \quad A_2^{(2)} = \frac{1}{\alpha_1} - \frac{2}{\alpha_2} - \frac{3}{\alpha_3} - \cdots - \frac{2}{\alpha_{r-1}} \sim \frac{2}{\alpha_r} \]

**Figure 1.** The affine Dynkin diagrams \( A_1^{(1)}, A_2^{(2)}, A_2^{(2)}_{\text{odd}} \).

\( \chi_{A_1^{(1)}}((2k-i)\Lambda_0 + (i-1)\Lambda_1) \). This is the case of level \( 2k-1 \) and the Rogers-Ramanujan identities \( [1] \) are the cases when \( k = 2 \) and \( i = 2, 1 \). An even level analog is known as the Andrews-Bressoud identities (see [31, §2.2]).

**Theorem 1.1** ([2]). Let \( 1 \leq i \leq k \). Putting \( N_j = n_j + \cdots + n_{k-1} \), we have

\[
\sum_{n_1, \ldots, n_{k-1} \geq 0} q^{N_i^2 + \cdots + N_{k-1}^2 + N_i + \cdots + N_{k-1}} (q;q)_{n_1} \cdots (q;q)_{n_{k-1}} = \frac{(q^4, q^{2k+1-i}, q^{2k+1}; q^{2k+1})_{\infty}}{(q;q)_{\infty}}.
\]

As do the Rogers-Ramanujan identities, Theorem 1.1 has an interpretation as a partition theorem [12]. A partition theorem is a statement of the form “For any \( n \geq 0 \), partitions \( (\lambda_1, \ldots, \lambda_t) \) of \( n \) with condition \( C \) are equinumerous to partitions of \( n \) with condition \( D \).” Theorem 1.1 is equivalent to the partition theorem, where

- \( C: 1 \leq \forall j \leq \ell - k + 1, \lambda_j - \lambda_{j+k-1} \geq 2 \) and \( \{|1 \leq j \leq \ell | \lambda_j = 1\} < i \),
- \( D: 1 \leq \forall j \leq \ell, \lambda_j \neq 0, \pm i \pmod{2k+1} \).

1.2. The main theorems. In this paper (see also [9, Conjecture 1.1]), “Andrews-Gordon type series” stands for an infinite sum of the form

\[
\sum_{i_1, \ldots, i_s \geq 0} \frac{(-1)^{\sum_{\ell=1}^s L_{\ell i}} q^{\sum_{\ell=1}^s a_{\ell i}(i_{\ell})^2 + \sum_{1 \leq j < k \leq s} a_{jk}i_jk + \sum_{\ell=1}^s B_{\ell i} k}}{(q^{C_{i_1}}; q^{C_{i_2}})_{i_1} \cdots (q^{C_{i_s}}; q^{C_{i_s}})_{i_s}},
\]

for some \( (L_{\ell i})_{i \geq 1}, (B_{\ell i})_{i \geq 1} \in \mathbb{Z}^s, (C_{i})_{i \geq 1} \in \mathbb{N}^s \) and \( (a_{jk})_{1 \leq j < k \leq s} \in \mathbb{Z}^{(s+1)/2} \). Usually, we would like to impose some non-degeneracy conditions such as \( a_{jk} > 0 \) for all \( 1 \leq j < k \leq s \) in order not to make the problem trivial. In the rest, we put

\[
[x;q]_m = (x, q/x, q^2/x, \ldots, q^{m-1}/x)_m, \quad [a_1, \ldots, a_k; q]_m = [a_1; q]_m \cdots [a_k; q]_m
\]

for \( m \in \mathbb{N} \cup \{\infty\} \). The purpose of this paper is to give the following.

**Theorem 1.2.** Concerning level 5 standard modules for \( A_2^{(2)} \), we have

\[
\sum_{i,j,k \geq 0} (-1)^k q^{(i_{\ell}^2 + 8(i_{\ell} + 2j + 2k + 4j + i + 5j + k)} (q;q)_i (q^2;q^2)_j (q^2;q^2)_k = \frac{1}{[q^8, q^4, q^3, q^2; q^{16}]_{\infty}} (\chi_{A_2^{(2)}}(5\Lambda_0)),
\]

\[
\sum_{i,j,k \geq 0} (-1)^k q^{(i_{\ell}^2 + 8(i_{\ell} + 2j + 2k + 4j + i + 7j + 3k)} (q;q)_i (q^2;q^2)_j (q^2;q^2)_k = \frac{1}{[q^8, q^4, q^6, q^2; q^{16}]_{\infty}} (\chi_{A_2^{(2)}}(\Lambda_0 + 2\Lambda_1)),
\]

**Theorem 1.3.** Concerning level 7 standard modules for \( A_2^{(2)} \), we have

\[
\sum_{i,j,k \geq 0} q^{(i_{\ell}^2 + 8(i_{\ell} + 10i_{\ell} + 2j + 2k + 8j + i + 4j + 5k)} (q;q) (q^2;q^2)_j (q^2;q^2)_k = \frac{1}{[q, q^4, q^6, q^8, q^9; q^{20}]_{\infty}} (\chi_{A_2^{(2)}}(5\Lambda_0 + \Lambda_1)),
\]

\[
\sum_{i,j,k \geq 0} q^{(i_{\ell}^2 + 8(i_{\ell} + 10i_{\ell} + 2j + 2k + 8j + i + 8j + 9k)} (q;q) (q^2;q^2)_j (q^2;q^2)_k = \frac{1}{[q, q^4, q^5, q^6, q^8, q^9; q^{20}]_{\infty}} (\chi_{A_2^{(2)}}(\Lambda_0 + 3\Lambda_1)).
\]
Theorem 1.4. Concerning level 7 standard modules for $A_2^{(2)}$, we have

\[
\sum_{i,j,k,l \geq 0} (-1)^k q^{(i+j+2i+2j+k+4\ell)} (q;q)_i(q^2;q^2)_j(q^4;q^4)_k(q^4;q^4)_l
= \frac{1}{[q,q^2,q^4,q^5,q^7;q^{20}]_{\infty}} = \chi_{A_2^{(2)}}(5\Lambda_0),
\]

\[
\sum_{i,j,k,l \geq 0} (-1)^k q^{(i+j+2i+2j+k+4\ell)} (q;q)_i(q^2;q^2)_j(q^4;q^4)_k(q^4;q^4)_l
= \frac{1}{[q,q^2,q^4,q^5,q^7,q^8;q^{20}]_{\infty}} = \chi_{A_2^{(2)}}(3\Lambda_0 + 2\Lambda_1).
\]

Note that the level 5 module $V(3\Lambda_0 + \Lambda_1)$ is missing in Theorem 1.2, but

\[
\chi_{A_2^{(2)}}(3\Lambda_0 + \Lambda_1) = \frac{1}{[q,q^3,q^5,q^7;q^{16}]_{\infty}} = \frac{1}{(q;q^2)^{\infty}},
\]

can be written as a form of Andrews-Gordon type series (e.g. put $x = q$ in [22] (B)).

1.3. Comments on the proofs. The following steps are common in proving that a multim is equal to an infinite product.

(S1) Reduce the multim to a single sum.

(S2) Search for lists of identities which deduce the desired result.

The step (S1) for Theorem 1.2, 1.3, 1.4 uses a similar technique to the proof of Kanade-Russell’s conjectures of modulo 12 [38], I_5, I_6] by Bringmann et.al. [5] §4.10. For (S2), we employ Slater’s list [33] (see also Remark 1.1).

Theorem 1.5. (33) (39) = (83), (38) = (86), (99), (94))

\[
\sum_{n \geq 0} q^{2n^2} (q;q)_{2n} = \chi_{A_2^{(2)}}(5\Lambda_0), \quad \sum_{n \geq 0} q^{2n^2+2n} (q;q)_{2n+1} = \chi_{A_2^{(2)}}(\Lambda_0 + 2\Lambda_1),
\]

\[
\sum_{n \geq 0} q^{n^2+n} (q;q)_{2n} = \chi_{A_2^{(2)}}(7\Lambda_0), \quad \sum_{n \geq 0} q^{n^2+n} (q;q)_{2n+1} = \chi_{A_2^{(2)}}(3\Lambda_0 + 2\Lambda_1).
\]

1.4. Toward $A_2^{(2)}$ analog of the Andrews-Gordon identities. Let us recall the previous studies for $A_2^{(2)}$. For the level 2 case, the principal characters are obtained by inflating $q$ to $q^2$ from the infinite products in [1]. For the level 3 (resp. level 4) case, the vacuum spaces are studied in [6] (resp. in [29]), which resulted in conjectural partition theorems (see [33] Theorem 5.2, Theorem 5.3, Conjecture 5.5, Conjecture 5.6, Conjecture 5.7) that were later proved in [3] [7] [33] (resp. [34]). The Andrews-Gordon type series are known as [21] Corollary 18) (resp. [34]), which are duplicated in Theorem 1.2, 1.3, 1.4 resp. Remark 1.1. For the level 3 case, see also [47].

For the level 5 and 7 cases, the infinite products in Theorem 1.2, Theorem 1.3, Theorem 1.4 appear in [14] Theorem 4, Theorem 3, [14] Theorem 1) and [8] Theorem 1.6) and [14] Theorem 2] respectively. Those partition theorems look quite different from those for the aforementioned level 3 and 4 cases [6] [29]. It is natural to investigate partition theorems via the vertex operators (other than [14] [8]) that are related with our identities (like Proposition 7.3). As far as we know, almost nothing is known on the level 6 case except [28] (1.3)–(1.6)] (see also [32]).
1.5. Toward $A_{ord}$ level 2 analog of the Andrews-Gordon identities. Recently, related to the expectation mentioned in §4.1 some (conjectural) Andrews-Gordon type series for other affine Dynkin diagrams were found. A famous example is Kurşungöz’s reformulation [20, Conjecture 6.1] of Kanade-Russell conjectures of modulo 9 [18], which are regarded as the level 3 identities of $D_{4}^{(3)}$ (see §5.2).

Another actively studied levels and Lie types are level 2 of $A_4$, level 2 analog of the Andrews-Gordon identities. Recall the Euler’s identities

When $\ell = 2$, Andrews-Gordon type series for $\chi_{A_{4}^{(2)}}(0 A + A_1), \chi_{A_{4}^{(2)}}(A_3)$ are found as in [21, (21), (22)]. As shown by [16], they are related with the classical Göllnitz partition theorems [31, Theorem 2.42, Theorem 2.43]. When $\ell = 4$, Bringmann et.al. [5, §3, §4.1, §4.2] proved Andrews-Gordon type series for $\chi_{A_{4}^{(2)}}(A_0 + A_1), \chi_{A_{4}^{(2)}}(A_3), \chi_{A_{4}^{(2)}}(A_5)$ which were conjectured in [19, (3.2), (3.4), (3.6)] together with interpretations as partition theorems. See also [30].

Noting the principal characters for level 4 modules of $A_{4}^{(2)}$ coincide with some of those for level 2 of $A_{11}^{(2)}$ (see Remark 3.1), in Conjecture 6.1, Conjecture 6.2 we give conjectural Andrews-Gordon type series for

$$
\chi_{A_{3}^{(2)}}(A_0 + A_1), \chi_{A_{3}^{(2)}}(A_3), \chi_{A_{3}^{(2)}}(A_5), \chi_{A_{3}^{(2)}}(A_7).
$$

It would be interesting if one can find a pattern in Andrews-Gordon type series obtained so far. For example, some of them for level 4 and 5 modules of $A_{2}^{(2)}$ (resp. level 2 of $A_{11}^{(2)}$ and $A_{13}^{(2)}$) share a recursive structure as in Remark 3.1 and Remark 6.3 that we can also observe in the original (Theorem 1.1): namely, the Andrews-Gordon series for $\chi_{A_{4}^{(1)}}((2(k-1) - i)A_0 + (i - 1)A_1)$ is obtained by deleting $n_{k-1}$ (or substituting $n_{k-1} = 0$) in that for $\chi_{A_{4}^{(1)}}((2k - i)A_0 + (i - 1)A_1)$.

We hope this paper contributes to the expectation mentioned in §4.1 which has been shared in the community since Lepowsky-Milne’s observation.

Organization of the paper. The paper is organized as follows. In [2] we show some auxiliary summation formulas via Euler’s identities, the $q$-binomial theorem and the $q$-WZ method. Then, we prove Theorem 1.2. Theorem 1.3. Theorem 1.3 in [3, 4, 6] respectively. In [8] (resp. [17]), we discuss some Andrews-Gordon type series related to level 2 (resp. level 3) modules of $A_{11}^{(2)}$ (resp. $A_{13}^{(2)}$).

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2. Preparations

Recall the Euler’s identities and the $q$-binomial theorem [13, (II.1),(II.2),(II.3)].

$$
\sum_{n \geq 0} \frac{x^n}{(q; q)_n} = (A) \frac{1}{(x; q)_\infty}, \quad \sum_{n \geq 0} q^{\frac{n(n+1)}{2}} x^n = (B) \frac{(-x; q)_\infty}{(q; q)_n}, \quad \sum_{n \geq 0} \frac{(a; q)_n}{(q; q)_n} x^n = (C) \frac{(-a; q)_\infty}{(x; q)_\infty}
$$
Lemma 2.1. For any $M \in \mathbb{Z}_{\geq 0}$, we have

\[
(1) \sum_{i,j \geq 0 \atop i+j=M} \frac{q^i}{(q^2;q^2)_i(q^2;q^2)_j} = \frac{1}{(q;q)_M},
\]

\[
(2) \sum_{i,j \geq 0 \atop i+j=M} (-1)^j \frac{q^{2(j)+i}}{(q^2;q^2)_i(q^2;q^2)_j} = \frac{(q;q^2)_M}{(q^2;q^2)_M},
\]

\[
(3) \sum_{i,j \geq 0 \atop i+j+2k=M} (-1)^j \frac{q^{ij+j+k}}{(q;q)_i(q;q)_j(q^2;q^2)_k} = \frac{(-q;q)_{[M/2]}}{(q;q)_{[M/2]}},
\]

\[
(4) \sum_{i,j \geq 0 \atop i+j=M} (-1)^j \frac{q^{3(i)+ij+2j}}{(q;q)_i(q;q)_j(q^2;q^2)_M} = \frac{(q;q)_{2M}}{(q^2;q^2)_M^2}.
\]

Proof. By (A), we get (1) as follows.

\[
\sum_{i,j \geq 0} \frac{q^i}{(q^2;q^2)_i(q^2;q^2)_j} x^{i+j} = \frac{1}{(x;q^2)_\infty} \frac{1}{(xq^2;q^2)_\infty} = \frac{1}{(x;q)_\infty} = \sum_{M \geq 0} \frac{x^M}{(q;q)_M}
\]

Similarly, we get (2) by (A), (B), (C) as follows.

\[
\sum_{i,j \geq 0} \frac{x^i}{(q^2;q^2)_i(q^2;q^2)_j} (-1)^j \frac{q^{2(j)+i}x^j}{(q;q)_i(q;q)_j(q^2;q^2)_k} = \frac{1}{(x;q^2)_\infty} \frac{1}{(x^2;q^2)_\infty} = \sum_{M \geq 0} \frac{(q;q^2)_M}{(q^2;q^2)_M} x^M.
\]

To prove (3), we calculate the generating series of both sides by (A), (B).

\[
\sum_{i,j \geq 0} (-1)^j \frac{q^{ij+j+k}}{(q;q)_i(q;q)_j(q^2;q^2)_k} x^{i+j+2k} = \left( \sum_{j \geq 0} \frac{(-xq)^j}{(q;q)_j} \sum_{i \geq 0} \frac{(xq^j)^i}{(q;q)_i} \right) \cdot \sum_{k \geq 0} \frac{(x^2q^k)^k}{(q^2;q^2)_k}
\]

\[
= \left( \sum_{j \geq 0} \frac{(-xq)^j}{(q;q)_j(xq^j;q^2)_\infty} \right) \cdot \frac{1}{(x;q^2)_\infty} \frac{1}{(x^2;q^2)_\infty} \sum_{j \geq 0} \frac{(xq)^j}{(q;q)_j} (-xq)^j.
\]

Thus, the generating series of the left hand side is equal to

\[
\frac{1}{(x;q^2)_\infty} \frac{1}{(x^2;q^2)_\infty} \frac{(-x^2q;q)^\infty}{(-xq;q)^\infty}
\]

by (C). We easily see that it is equal to the generating series of the right hand side:

\[
\sum_{M \geq 0} \frac{(-q;q)_{[M/2]}}{(q;q)_{[M/2]}} x^M = \sum_{N \geq 0} \frac{(-q;q)_N}{(q;q)_N} x^{2N+1} (1+x) = (1+x) \frac{(-x^2q;q)_\infty}{(x^2;q^2)_\infty}.
\]

In the proof of (4), we promise $1/(q;q)_n = 0$ if $n < 0$. For $M, j \geq 0$, we let

\[
f_{M,j} := (-1)^j \frac{q^{3(i)+(M-j)(j+2j)}}{(q;q)_M(q;q)_{M-j}(q;q)_j(q;q)_{2M}}
\]

so that it suffices to prove (by the q-WZ method \textbf{[17]}) $\sum_{j \geq 0} f_{M,j} = 1$ (note that $f_{M,j} = 0$ when $j > M$). The $q$-Zeilberger algorithm helps us finding an expression

\[
g_{M,j} = (-1)^j \frac{(1-q^{M+1-j}-q^{2M+2-j})q^{3(i)+(M-j)(j+2j)}}{(1+q^{M+1})(1-q^{2M+1})(q;q)_{M-j+1}(q;q)_{j-1}(q;q)_{2M}}
\]

for which we can verify that $g_{M,j} \neq 0 \Rightarrow 0 \leq j-1 \leq M$ and $f_{M+1,j} - f_{M,j} = g_{M,j+1} - g_{M,j}$ for any $M, j \geq 0$. This implies $\sum_{j \geq 0} f_{M+1,j} - \sum_{j \geq 0} f_{M,j} = 0$ for any $M \geq 0$. Now we only need to see $\sum_{j \geq 0} f_{0,j} = 1$, which is obvious. \qed
Hence, by Lemma 2.1 (2), we see (3)

Thus, we see (2) = (3)

for \( a = 0, 1 \). We rewrite the inner sum on \( j \) and \( k \) as

\[
\sum_{j,k \geq 0} (-1)^k q^{8(j)+2(2j)+4jk+(5+2a)j+(1+2a)k} (q^2)^j (q^2)^k
\]

By Lemma 2.1 (4) (with \( q \) replaced by \( q^2 \)), we see (2) is reduced to

\[
\sum_{i,M \geq 0} (-1)^M \frac{(q^2)^{2M}}{(q; q)^M} \sum_{j,k \geq 0} (-1)^{j+k} q^{6(j)+2j+4j+2(2a+1)} + (2a+1) M (q^2)^j (q^2)^k
\]

With (B), the inner sum on \( i \) is rewritten as

\[
\sum_{i \geq 0} q^{(i)}(q^2)^{2M+1} (q; q)^M = (-q^2)^{M+1} (q; q)^M = (-q; q)^{2M+1} (q; q)^M = 1 \frac{(q; q)^M}{(q^2; q^2)^M}
\]

Hence, by Lemma 2.1 (4), we see (3) \( (q; q^2)_{\infty} \) is equal to

\[
\sum_{M \geq 0} (-1)^M q^{M + 2aM} (q^2)^M = \sum_{m,n \geq 0} (-1)^m q^{2m + 4n + 2mn + (1+2a)m + (2+2a)n} (q^2)^m (q^2)^n
\]

Finally we use (B) to rewrite the inner sum on \( m \) as

\[
\sum_{m \geq 0} (-1)^m q^{2m + (1+2a+2n)m} (q^2)^m = (q^{1+2a+2n}; q^2)^{\infty} = \frac{(q; q^2)^{\infty}}{(q^2; q^2)_{n+a}}
\]

Thus, we see (2) = \( \frac{(q; q^2)^{\infty}}{(q^2; q^2)_{\infty}} \) is equal to

\[
\sum_{n \geq 0} q^{4n+2(2a)n} (q; q^2)_{n+a} (q^2; q^2) = \sum_{n \geq 0} q^{2n^2 + 2an} (q; q^2)_{2n+a}
\]

which proves Theorem 1.2 in virtue of Theorem 1.3.
Remark 3.1. In [34, Theorem 2.2] the following identities are shown:

\[
\sum_{i, k \geq 0} (-1)^k q^{i(\frac{1}{2})+2j(\frac{1}{2})+2ik+i+k} (q; q)_i(q^2; q^2)_k = \frac{1}{[q^2, q^2, q^4; q^{14}]_{\infty}} (\chi_{A_2^{(2)}}(4A_0) = \chi_{A_2^{(2)}}(\Lambda_3)),
\]

(4)

\[
\sum_{i, k \geq 0} (-1)^k q^{i(\frac{1}{2})+2j(\frac{1}{2})+2ik+i+3k} (q; q)_i(q^2; q^2)_k = \frac{1}{[q, q^4, q^6; q^{14}]_{\infty}} (\chi_{A_2^{(2)}}(2A_0 + A_1) = \chi_{A_2^{(2)}}(A_0 + A_1)),
\]

(5)

\[
\sum_{i, k \geq 0} (-1)^k q^{i(\frac{1}{2})+2j(\frac{1}{2})+2ik+2i+3k} (q; q)_i(q^2; q^2)_k = \frac{1}{[q^2, q^2, q^6; q^{14}]_{\infty}} (\chi_{A_2^{(2)}}(2A_1) = \chi_{A_2^{(2)}}(\Lambda_3)).
\]

(6)

We remark that [11] and [13] coincide with double sums obtained by taking the “j = 0 part” of the triple sums in Theorem 1.3.

4. Proof of Theorem 1.3

Note that the Andrews-Gordon type series is of the form

\[
\sum_{i, j, k \geq 0} q^{i(\frac{1}{2})+8j(\frac{1}{2})+2i+2jk+8j+8k+(4+4\epsilon)+2(5+4\epsilon)k} (q; q)_i(q^2; q^2)_j(q^2; q^2)_k
\]

(7)

for \( a = 0, 1 \). We rewrite the inner sum on \( j \) and \( k \) in (7) as

\[
\sum_{j, k \geq 0} q^{8j(\frac{1}{2})+10j(\frac{1}{2})+8j+2i+4\epsilon+4j+2(5+4\epsilon)k} (q^2; q^2)_j(q^2; q^2)_k = \sum_{M \geq 0} \sum_{j+k=M} q^{2j(\frac{1}{2})+8j(\frac{1}{2})+4\epsilon+2i+4\epsilon} (q^2; q^2)_j(q^2; q^2)_k.
\]

By Lemma 2.1 (2) (with \( q \) replaced by \(-q\)), we see (7) is reduced to

\[
\sum_{i, M \geq 0} q^{(i+2M)+2jM+4M^2+4aM} (q; q)_i(q^2; q^2)_M.
\]

(8)

With (B), the inner sum on \( i \) is rewritten as

\[
\sum_{i \geq 0} q^{i(\frac{1}{2})+2M+M} (q; q)_i = (q; q)_{2M} = (q; q)_{2M} = \frac{1}{(q; q)_{2M}}.
\]

By (1), we see \((q; q^2)_{\infty}\) is reduced to

\[
\sum_{M \geq 0} q^{4M^2+4aM} (q; q^2)_M = \frac{1}{(q; q^2)_{\infty}}.
\]

This is equal to \((q; q^2)_{\infty} \cdot \chi_{A_2^{(2)}}(5 - 4a)A_0 + (1 + 2a)A_1)\) and proves Theorem 1.3.

Remark 4.1. In Slater’s list [33] (79) = (98), (96), there are identities whose infinite products matches those in Theorem 1.3 (but we do not need them).

\[
\sum_{n \geq 0} q^{n^2} (q; q)_{2n} = \chi_{A_2^{(2)}}(5A_0 + A_1), \quad \sum_{n \geq 0} q^{n^2+2n} (q; q)_{2n+1} = \chi_{A_2^{(2)}}(A_0 + 3A_1).
\]
5. Proof of Theorem 1.4

Note that the Andrews-Gordon type series is of the form
\[
\sum_{i,j,k,\ell \geq 0} (-1)^{k+1+(1-a)(j+k)} \frac{q^\left(\frac{j}{2}\right) + 2\left(\frac{k}{2}\right) + 8\left(\frac{j}{2}\right) + 4(i+j+k+2\ell) + (1+a)(1+2(j+k+2\ell))}{(q; q)_i(q^2; q^2)_j(q^2; q^2)_k(q^2; q^2)_\ell}
\]
(9)
for \(a = 0, 1\) (swapping \(j\) and \(k\) from the expression in Theorem 1.4 when \(a = 0\)). We rewrite the inner sum on \(j, k\) and \(\ell\) in (9) as
\[
\sum_{j,k,\ell \geq 0} (-1)^{k+1+(1-a)(j+k)} q^\left(\frac{j}{2}\right) + 2\left(\frac{k}{2}\right) + 8\left(\frac{j}{2}\right) + 4(j+k+2\ell) + (1+a)(1+2(j+k+2\ell))
\]
(10)
\[
q^j q^k (q^2; q^2)_j (q^2; q^2)_k (q^2; q^2)_\ell
\]
For (B), (12) is reduced to
\[
\sum_{i,M \geq 0} (-1)^{(1-a)M} \frac{q^\left(\frac{j}{2}\right) + 2\left(\frac{k}{2}\right) + 8\left(\frac{j}{2}\right) + 4(i+j+k+2\ell) + (1+a)(1+2(j+k+2\ell))}{(q; q)_i(q^2; q^2)_j(q^2; q^2)_k(q^2; q^2)_\ell}
\]
(11)
By Lemma 2.1 (3) (with \(q\) replaced by \(q^2\)), we see (10) is reduced to
\[
\sum_{i,M \geq 0} (-1)^{(1-a)M} \frac{q^\left(\frac{j}{2}\right) + 2\left(\frac{k}{2}\right) + 8\left(\frac{j}{2}\right) + 4(i+j+k+2\ell) + (1+a)(1+2(j+k+2\ell))}{(q; q)_i(q^2; q^2)_j(q^2; q^2)_k(q^2; q^2)_\ell}
\]
(12)
With (B), the inner sum on \(i\) is rewritten as
\[
\sum_{i \geq 0} q^{\left(\frac{j}{2}\right) + (1+M)i}_i = (-q^{1+M}; q)_\infty = (-q; q)_\infty = \frac{1}{(-q; q)_\infty}.
\]
Hence, (10)-(9, 2, 12) is equal to
\[
\sum_{M \geq 0} (-1)^{(1-a)M} q^{\left(\frac{j}{2}\right) + 2(k+2+) + 8(\frac{j}{2}) + 4(i+j+k+2\ell) + (1+a)(1+2(j+k+2\ell))}_{\ell}\]
(11)
Further, by Lemma 2.1 (1) (with \(q\) replaced by \(-q\)), we see (11) is equal to
\[
\sum_{m,n \geq 0} (-1)^{1-(a)(m+n)+n} q^{\left(\frac{m+n}{2}\right) + (1+a)(m+n)+n}_{(q^2; q^2)_m(q^2; q^2)_n}.
\]
(12)
Finally, by (B), (12) is reduced to
\[
\left\{
\begin{align*}
\sum_{n \geq 0} & q^{2(\frac{n}{2})+2n}_{(q^2; q^2)_n} \sum_{m \geq 0} (-1)^{m} q^{2(\frac{n}{2})+2m+n}_{(q^2; q^2)_m} = \sum_{n \geq 0} q^{2(\frac{n}{2})+2n}_{(q^2; q^2)_n} \frac{(q)q^{2n+1}_\infty}{q^2} = (a = 0), \\
\sum_{m \geq 0} & q^{2(\frac{m}{2})+2m}_{(q^2; q^2)_m} \sum_{n \geq 0} (-1)^{n} q^{2(\frac{m}{2})+2m+n}_{(q^2; q^2)_n} = \sum_{m \geq 0} q^{2(\frac{m}{2})+2m}_{(q^2; q^2)_m} \frac{(q)q^{2m+3}_\infty}{q^2} = (a = 1).
\end{align*}
\right.
\]
In each case, we see (9) = (12)/(q^2)_\infty is equal to
\[
\sum_{s \geq 0} q^{2s+2s}_{(q^2; q^2)_s} (q^2)_\infty = \sum_{s \geq 0} q^{2s+2s}_{(q^2; q^2)_s} (q; q)_{2s+a} = \sum_{s \geq 0} q^{2s+2s}_{(q^2; q^2)_s} (q; q)_{2s+a}.
\]
This proves Theorem 1.4 in virtue of Theorem 1.5.
6. Conjectures for level 2 modules of $A_{13}^{(2)}$

The level 2 principal characters of $A_{13}^{(2)}$ are

$$\chi_{A_{13}^{(2)}}((\delta_{0} + \delta_{1})\Lambda_{0} + \Lambda_{i}) = \frac{(q^{16}; q^{16})_{\infty} [q^{2i}; q^{16}]_{\infty}}{(q^{2}; q^{2})_{\infty} [q^{i}; q^{16}]_{\infty}}$$

where $0 \leq i \leq 7$ and $\delta$ is the Kronecker delta.

We give conjectural Andrews-Gordon type series for $\chi_{A_{13}^{(2)}}(\Lambda_{0} + \Lambda_{1})$ and $\chi_{A_{13}^{(2)}}(\Lambda_{2n+1})$, where $n = 1, 2, 3$. Note that $\chi_{A_{13}^{(2)}}(2\Lambda_{0}) = \chi_{A_{13}^{(2)}}(2\Lambda_{1})$ and $\chi_{A_{13}^{(2)}}(\Lambda_{2n})$, where $n = 1, 2, 3$, are obtained by inflating $q$ to $q^2$ from infinite products with smaller period.

**Conjecture 6.1.** We have $F_{1}(2, 2, 2) = \chi_{A_{13}^{(2)}}(\Lambda_{3})$, $F_{1}(4, 2, 6) = \chi_{A_{13}^{(2)}}(\Lambda_{5})$ and $F_{1}(6, 4, 6) = \chi_{A_{13}^{(2)}}(\Lambda_{7})$, where

$$F_{1}(a, b, c) := \sum_{i, j, k \geq 0} (-1)^{k} \frac{q^{4(\frac{3}{2}) + 2(\frac{1}{2}) + 4i} + 2j + 4k + a + b + j + c}{(q; q)_{i}(q^{2}; q^{2})_{j}(q^{4}; q^{4})_{k}}.$$

**Conjecture 6.2.** We have $F_{2}(1, 3, 12) = \chi_{A_{13}^{(2)}}(\Lambda_{0} + \Lambda_{1})$, $F_{2}(1, 1, 8) = \chi_{A_{13}^{(2)}}(\Lambda_{3})$ and $F_{2}(3, 3, 16) = \chi_{A_{13}^{(2)}}(\Lambda_{7})$, where

$$F_{2}(a, b, c) := \sum_{i, j, k \geq 0} (-1)^{k} \frac{q^{4(\frac{3}{2}) + 2(\frac{1}{2}) + 4i} + 2j + 4k + a + b + j + c}{(q; q)_{i}(q^{2}; q^{2})_{j}(q^{4}; q^{4})_{k}}. \quad (13)$$

**Conjecture 6.3.** We have $F_{3}(1, 5, 1, 12) = \chi_{A_{13}^{(2)}}(\Lambda_{5})$, where

$$F_{3}(a, b, c, d) := \sum_{i, j, k, t \geq 0} (-1)^{k} \frac{q^{4(\frac{3}{2}) + 2(\frac{1}{2}) + 16i} + 2j + 4k + a + b + j + c}{(q; q)_{i}(q^{2}; q^{2})_{j}(q^{4}; q^{4})_{k}(q^{4}; q^{4})_{t}}.$$

**Remark 6.4.** One can prove $F_{2}(a, b, c) = F_{3}(a, 2a + 1, b, c)$ for $a, b, c \geq 0$ by rewriting the inner sum on $i$ in $(13)$ as

$$\sum_{i \geq 0} \frac{q^{(\frac{3}{2}) + 2j + 4k + a}i}{(q; q)_{i}} = \sum_{i \geq 0} \sum_{s, t \geq 0} \frac{q^{(\frac{3}{2}) + 2j + 4k + a}i + (\frac{3}{2}) + 2j + 4k + a + b + j + c}{(q; q)_{s}(q^{2}; q^{2})_{t}} = \sum_{s, t \geq 0} \frac{q^{(\frac{3}{2}) + 2j + 4k + a}i + 2st + 2j + 4k(s + 2t) + a + (2a + 1)t}{(q; q)_{s}(q^{2}; q^{2})_{t}}.$$

Here, the first equality follows from

$$\sum_{i \geq 0} \frac{q^{(\frac{3}{2})}i}{(q; q)_{i}(q^{2}; q^{2})_{j}} = \frac{1}{(q; q)_{M}} \text{ for } M \geq 0,$$

which is proved similarly to Lemma 2.7 (using (A) and (B)).

Hence, if Conjecture 6.2 is true, we have $F_{3}(1, 3, 3, 12) = \chi_{A_{13}^{(2)}}(\Lambda_{0} + \Lambda_{1})$, $F_{3}(1, 3, 1, 8) = \chi_{A_{13}^{(2)}}(\Lambda_{3})$, $F_{3}(3, 7, 3, 16) = \chi_{A_{13}^{(2)}}(\Lambda_{7})$ and Conjecture 6.3 gives the “missing” case.

**Remark 6.5.** The double sums (11) and (13) coincide with those obtained by taking the “$k = 0$ part” of the triple sums $F_{2}(1, 1, 8)$ and $F_{2}(1, 3, 12)$ in Conjecture 6.2.
7. Notes on Capparelli’s identities

We fix the conditions (C1) and (C2) on a partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ to recall Capparelli’s partition theorems (Theorem 7.1). See also [14, 15].

(C1) $1 \leq \forall j \leq \ell - 1$, $\lambda_j - \lambda_{j+1} \geq 2$,
(C2) $1 \leq \forall j \leq \ell - 1$, $\lambda_j - \lambda_{j+1} \leq 3 \implies \lambda_j + \lambda_{j+1} \equiv 0 \pmod{3}$

**Theorem 7.1** ([3, 7, 15]). Let $a = 1, 2$. For any $n \geq 0$, partitions $\lambda$ of $n$ with condition $C_a$ are equinumerous to those with condition $D_a$, where

$C_a$: (C1) and (C2) and $1 \leq \forall j \leq \ell, \lambda_i \neq a$,

$D_a$: $1 \leq \forall j \leq \ell, \lambda_j \neq \pm a \pmod{6}$, and $\lambda_1, \ldots, \lambda_{\ell(\lambda)}$ are distinct.

In [21, Theorem 10, Theorem 11], Kurşungöz showed we fix the conditions $C_a$ and $f_a(x, q) := \sum_{\lambda \in C_a} x^{t(\lambda)} q^{\lambda}$ for $a = 1, 2$. Combining Theorem 7.1 and (14), (15) with (12), we have

$\sum_{i,j \geq 0} q^{4(i^2)+12(j^2)+6ij+2i+6j} (q; q)_i (q^3; q^2)_j x^{i+j+2}$,

$\sum_{i,j \geq 0} q^{4(i^2)+12(j^2)+6ij+3i+9j} (q; q)_i (q^3; q^2)_j x^{i+j+2} (1 + xq^{1+2i+3j})$

where $C_a$ denote the set of partitions with the condition $C_a$ and $f_a(x, q) := \sum_{\lambda \in C_a} x^{t(\lambda)} q^{\lambda}$.

Combining Theorem 7.1 and (12), (14, 15) with $x = 1$, Kurşungöz got the following identities.

**Theorem 7.2** ([21 Corollary 18]). Concerning the level 3 modules of $A_2^{(2)}$, we have

$\sum_{i,j \geq 0} q^{2i^2+6ij+6j^2} (q; q)_i (q^3; q^3)_j = (-q^2, -q^3, -q^4, -q^6; q^6)_{\infty} \left( = \frac{1}{[q^2, q^3, q^{12}]_{\infty}} = \chi_{A_2^{(2)}(3A_0)} \right)$,

$\sum_{i,j \geq 0} q^{2i^2+6ij+6j^2+i+3j} (1 + q^{2i+3j+1}) (q; q)_i (q^3; q^3)_j = (-q, -q^3, -q^5, -q^6; q^6)_{\infty}$

$\left( = \frac{[q^2, q^{12}]_{\infty}}{[q, q^3, q^5, q^{12}]_{\infty}} = \chi_{A_2^{(2)}(A_0 + A_1)} \right)$.

Note that the left hand side of the latter is not an Andrews-Gordon type series in our sense (see [4, 5]. The purpose of this section is to prove

$\sum_{i,j,k \geq 0} q^{5(i^2)+5(j^2)+12(k^2)+3ij+6ik+6jk+(3-a)i+(2+a)j+6k} (q^2; q^2)_i (q^3; q^3)_j (q^3; q^3)_k = \chi_{A_2^{(2)}((5-2a)A_0 + (a-1)A_1)}$

for $a = 1, 2$. This follows from substituting $x = 1$ to Theorem 7.3.

**Theorem 7.3.** For $a = 1, 2$, we have

$f_a(x, q) = \sum_{i,j,k \geq 0} q^{5(i^2)+5(j^2)+12(k^2)+3ij+6ik+6jk+(3-a)i+(2+a)j+6k} (q^2; q^2)_i (q^3; q^3)_j (q^3; q^3)_k x^{i+j+2k}$.

**Proof.** Since the set of partitions with the conditions (C1), (C2) is a linked partition ideal (see [1, §8]), one can derive a $q$-difference equation algorithmically

$F(x) = (1 + xq^3)F(xq^3) + x(q^3a + q^3 + xq^6)F(xq^6) + x^2q^3(1 - xq^6)F(xq^6)$,

(16)
where \( F(x) := f_a(x, q) \). Putting \( F(x) =: \sum_{M \geq 0} f_M x^M \), by \([10]\) we have
\[
(1 - q^{3M}) f_M = q^{3M-3} (q^{3M+a} + q^{3M-a} + q^3) f_{M-1} + q^{6M-6}(1 + q^{3M-3}) f_{M-2} - q^{9M-12} f_{M-3}
\]
for all \( M \in \mathbb{Z} \) (we consider \( f_M = 0 \) for \( M < 0 \)). Putting \( g_M := q^{-3(M/2)} f_M \), we have
\[
(1 - q^{3M}) g_M = (q^{3M+a} + q^{3M-a} + q^3) g_{M-1} + (q^3 + q^{3M}) g_{M-2} - q^6 g_{M-3}.
\]
Putting \( G(x) := \sum_{M \geq 0} g_M x^M \), we have
\[
(1 - xq^3)(1 - x^2q^3) G(x) = (1 + xq^{3-a})(1 + xq^{3+a}) G(xq^3),
\]
and hence
\[
G(x) = \frac{(-xq^{3-a} - xq^{3+a}, q^3_\infty)}{(xq^3; q^3_\infty)(x^2q^3; q^3_\infty)} = \frac{(-xq^{3-a} - xq^{2+a}, q^2_\infty)}{(x^2q^3; q^3_\infty)},
\]
where the latter equality is because \( a = 1, 2 \). By (A) and (B) (in \([2]\)) we have
\[
G(x) = \sum_{i,j,k \geq 0} \frac{q^{2(j)+(3-a)i} q^{2(j)+(2+a)j} q^{3k}}{(q^2; q^2)_i (q^2; q^2)_j (q^3; q^3)_k} x^{i+j+2k}.
\]
Finally, since \( f_M := q^{3(M/2)} g_M \) we have
\[
F(x) = \sum_{i,j,k \geq 0} \frac{q^{4(j)+(3-a)i} q^{2(j)+(2+a)j} q^{3k}}{(q^2; q^2)_i (q^2; q^2)_j (q^3; q^3)_k} q^{3(i+j+2k)} x^{i+j+2k},
\]
which is precisely Theorem \([7,3]\).

\( \square \)

**Remark 7.4.** In place of \([17]\), if we write
\[
G(x) = \frac{(-xq^{2}; q^3_\infty)}{(x^2q^3; q^3_\infty)} \quad \text{for } a = 1 \quad \text{and } \quad G(x) = (1 + xq) \frac{(-xq^{3}; q^3_\infty)}{(x^2q^3; q^3_\infty)} \quad \text{for } a = 2,
\]
then we get the double sum expression \([14]\) and an alternative one to \([15]\)
\[
f_2(x, q) = \sum_{i,j \geq 0} \frac{q^{4(j)+(3-a)i} q^{2(j)+(2+a)j} q^{6(k)} q^{6(i+j+2k)}}{(q^2; q^2)_i (q^3; q^3)_j (q^6; q^6)_k} x^{i+j+2k} (1 + xq^{1+3i+6j}).
\]

**Remark 7.5.** We can reprove Theorem \([7,1]\) using the equation \([16]\). If we put
\[
G(x) = \sum_{M \geq 0} g_M x^M := F(x)(x; q^3)_\infty, \quad h_M := g_M/(q^3; q^3)_M \quad \text{and } \quad H(x) := \sum_{M \geq 0} h_M x^M,
\]
by a similar argument to the proof of Proposition \([7,3]\), we get
\[
(1 - x)(1 - xq^3) H(x) = (1 + xq^{3-a})(1 + xq^{3+a}) H(xq^6)
\]
and \( H(x) = (-xq^{3-a} - xq^{3+a}, q^6)_\infty/(x; q^3)_\infty \). Again, by similar arguments (using (A) and (B)), we see
\[
g_M = \sum_{i+j+k=M} \frac{1}{(q^3; q^3)_i (q^6; q^6)_j (q^6; q^6)_k} q^{6(j)+(3-a)i} q^{6(j)+(3+a)k},
\]
Since \( F(x) = G(x)(x; q^3)_\infty \), by Appell’s Comparison Theorem \([10]\) page 101 we get
\[
F(1) = (q^3; q^3)_\infty \lim_{M \to \infty} g_M = (q^3; q^3)_\infty \sum_{j,k \geq 0} \frac{q^{6(j)+(3-a)i} q^{6(j)+(3+a)k}}{(q^6; q^6)_j (q^6; q^6)_k} = (-q^3; q^3)_\infty (-q^{3-a}, -q^{3+a}; q^6)_\infty,
\]
which proves Theorem \([7,1]\)
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