Optomechanical frequency combs

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Keywords: combs, optomechanics, nanophotonics

Abstract

We study the formation of frequency combs in a single-mode optomechanical cavity. The comb is composed of equidistant spectral lines centered at the pump laser frequency and located at different harmonics of the mechanical resonator. We investigate the classical nonlinear dynamics of such system and find analytically the onset of parametric instability resulting in the breakdown of a stationary continuous wave intracavity field into a periodic train of pulses, which in the Fourier domain gives rise to a broadband frequency comb. Different dynamical regimes, including a stationary state, frequency comb generation and chaos, and their dependence on the system parameters, are studied both analytically and numerically. Interestingly, the comb generation is found to be more robust in the poor cavity limit, where optical loss is equal or larger than the mechanical resonance frequency. Our results show that optomechanical resonators open exciting opportunities for microwave photonics as compact and robust sources of frequency combs with megahertz line spacing.

1. Introduction

Frequency combs are important elements in optics and photonics, which are widely used in optical metrology, precision spectroscopy, atomic clocks and radio-frequency photonics [1–3]. In general, optical frequency combs are generated by different mechanisms, such as electro-optical modulation of a continuous wave source, which induces many sidebands to the reference frequency, or through nonlinear effects where the frequency mixing processes generate a range of equidistant harmonics. Despite different techniques involved in the generation of frequency combs, in principle all approaches can be seen as parametric processes, where either external or self-induced time-harmonic modulation of a parameter results in the formation of a large number of frequency sidebands [4, 5].

On the other hand, recent progress in intense localization of light in chip-scale high-contrast and high-quality-factor dielectric microcavities has allowed the design and realization of ultra-compact and low-cost frequency comb sources [6]. In such systems, multiple sidebands are induced by a cascaded four-wave-mixing process in the nonlinear dielectric material. They are locked at the longitudinal modes of the optical cavity to form a stable comb. The generation of equidistant frequency comb is made possible by the formation of cavity temporal solitons, and their periodic analogue, i.e., cnoidal waves, in a regime where the dispersive behavior of the optical microring resonator is compensated with cubic nonlinearities [7–9].

In a different context, there is a growing interest in the coupling of optical and mechanical resonances in micro- and nano-structured photonic devices. Indeed, optical cavities with small mode volumes have enabled strong optical forces that can dominate thermal effects, resulting in an effective coupling between optical and mechanical degrees of freedom. These systems are generally referred to as optomechanical cavities [10–12]. The possibility of coupling optical and mechanical oscillations on a photonic chip has led to a range of applications [13–25] in quantum and classical optics including, for example, parametric wavelength conversion [18–20], reconfigurable optical filtering [21], as well as breaking reciprocity in light transmission [22–25]. An interesting aspect of cavity optomechanical systems is the presence of nonlinear effects induced by radiation pressure, which is itself proportional to the optical intensity.
When driven with a single laser beam, optomechanical cavities exhibit a large cubic nonlinearity in their stationary states where the mechanical resonator is forced into a fixed location under a constant radiation pressure. Recent progress in the fabrication of high-Q optical microcavities, on the other hand, has led to the development of optomechanical systems in which the period of the mechanical oscillations is much lower than the photon lifetime, enabling dynamical nonlinear interactions between optical and mechanical quantities. This scenario has resulted in interesting nonlinear dynamical effects, much richer than stationary-state nonlinearities. In this regard, the nonlinear dynamics of optomechanical cavities has been intensively explored in the recent literature, while various static and dynamics aspects, such as bistability [26–28], spontaneous symmetry breaking [29], self-induced oscillations [30–33], and chaos [34–36] have been investigated. In addition, parametric generation of equally spaced optical lines induced by the acoustic modes in bottle microresonators have also been explored [37]. Even though parametric self-sustained mechanical oscillations have been well studied in this context, no significant effort has been made to analyze and tailor the formation of frequency combs in optomechanical systems. Such clocks have been observed in recent experiments involving optomechanical cavities with bulk [38–40] and surface [41] acoustic modes, as well as in photonic crystal fibers [42, 43]. Given the experimental accessibility of optomechanical frequency combs, a rigorous theoretical analysis of the comb formation in the idealized single-mode system is timely.

In this work, we present a systematic investigation of frequency comb formation in optomechanical cavities. As opposed to most works on cavity optomechanics, here we take a nonlinear optics approach by using a time-dependent nonlinear equation governing the intracavity optical field. The comb formation threshold is analytically investigated through a stability analysis of the stationary state solutions. The effect of different parameters on this highly nonlinear process is investigated, studying the overall accessibility of the comb generation. In particular, we show that the sideband resolution ratio plays an important role in the comb formation threshold, as well as in the shape of the comb and in the onset of chaotic behavior. Interestingly, the comb generation is found to be more robust in the bad cavity limit, despite the fact that large pump power levels are required.

2. Model

Consider a generic optomechanical cavity, as shown schematically in figure 1 (a), which involves a fixed mirror and a suspended mirror that can move under an optical force. For small mechanical displacements $x$ of the suspended mirror, to first order the optical resonance frequency can be written as $\omega(x) = \omega_0 - Gx$, where $\omega_0$ represents the rest frequency and the coefficient $G = -\partial \omega / \partial x$ describes the shift of the optical resonance per mechanical displacement [10]. Considering $a$ as the modal amplitude of the optical mode, being normalized such that $|a|^2$ represents the number of intracavity photons, the total optical energy can be written as $U = \hbar (\omega_0 - Gx)|a|^2$. For a finite displacement $x$, the optical force exerted on the mirror is $F_{\text{opt}} = -\partial U / \partial x$, which results in a simple expression for the optical force in this system: $F_{\text{opt}} = \hbar G |a|^2$. Assuming that the interaction between a single pair of coupled optical and mechanical modes, the dynamics of such system can thus be described through the coupled mode equations [10]

$$\frac{da}{dt} = \left(i(\Delta + Gx) - \frac{\kappa}{2}\right)a + \sqrt{\kappa} s_{\text{in}},$$

(1)

Figure 1. (a) A sketch of an optomechanical cavity involving a suspended mirror that can move under the act of the radiation pressure. (b) The normalized nonlinear response function of the optomechanical system. For demonstration purposes, here a relatively poor mechanical quality factor of $Q_m = 10$ is used.
\[
\frac{d^2 x}{dt^2} + \Gamma_m \frac{dx}{dt} + \Omega_m^2 x = \frac{hG}{m} |a|^2. \tag{2}
\]

In these relations, \(|s_{in}|^2\) is the input photon flux, \(\kappa = \kappa_e + \kappa_r\) represents the sum of external and internal optical losses, \(\Delta \equiv \omega_l - \omega_0\) is the detuning of the pump laser from the optical resonance frequency, and \(m, \Omega_m\) and \(\Gamma_m\), respectively, represent the effective mass, resonance frequency and decay rate of the mechanical resonator. The output photon flux \(|s_{out}|^2\) is related to the input via \(s_{out}(t) = s_{in} - \sqrt{\kappa_e} a(t)\). By neglecting optical and mechanical losses, and in the absence of the external drive, equations (1) and (2) exhibit a constant of motion, which is the total optomechanical energy \(H = \hbar(\Delta + Gx) |a|^2 + \frac{1}{2} m \Omega_m^2 x^2 + \frac{1}{2} m \dot{x}^2\). On the other hand, the intracavity photon is governed by the rate equation \(\frac{d |a|^2}{dt} = - \kappa |a|^2 + \sqrt{\kappa_e} (a^* s_{in} + s_{in}^* a)\), which, combined with the input–output relation, results in \(\frac{d |a|^2}{dt} = - \kappa_e |a|^2 + |s_{in}|^2 - |s_{out}|^2\). Interestingly, this relation, a direct manifestation of Poynting’s theorem in the coupled waveguide-cavity circuit, is independent from the mechanical resonator. Equations (1) and (2) can be casted into a single integro-differential equation where the back-action of the radiation pressure force on the optical mode can be considered as a non-instantaneous cubic nonlinearity described through

\[
\frac{da}{dt} = \left[ i \left( \Delta + \int_0^\infty \gamma(t - t') |a(t')|^2 dt' \right) - \frac{\kappa}{2} \right] a(t) + \sqrt{\kappa_e} s_{in}. \tag{3}
\]

Here, \(\gamma(\tau)\) represents a temporal response function, which is found to be

\[
\gamma(\tau) = \frac{hG^2}{m \sqrt{\Omega_m^2 - \Gamma_m^2 / 4}} \exp \left( \frac{\Gamma_m}{\Omega_m} \sin \left( \sqrt{\Omega_m^2 - \Gamma_m^2 / 4} / \tau \right) \right). \tag{4}
\]

For a high-Q mechanical resonator, i.e., \(\Omega_m/\Gamma_m \gg 1\), this expression can be simplified to \(\gamma(\tau) = \frac{hG^2}{m \Omega_m} \exp (-\Gamma_m^2/2) \sin (\Omega_m \tau)\). This oscillatory-damping response function is shown in figure 1(b). It is worth noting the close analogy between equation (4) and the Raman response function of optical materials, which has been widely used to describe the delayed nonlinear response of silica fibers (see, e.g., [44]), for example. In fact, the coupled mechanical resonator model can describe the molecular and lattice vibrations of materials interacting with light [45]. The optomechanical cavity system can thus be regarded as an ideal platform for nonlinear Raman or Brillouin effects, where the optomechanical interactions are reduced down to a pair of optical and mechanical modes. In the same way, as we discuss in this work, notions from nonlinear optics can be applied to optomechanical cavities and provide a useful playground for technological breakthroughs.

3. Comb formation threshold

Under continuous wave laser excitation, the stationary state intracavity field \(a(t) = \bar{a}\) is governed by

\[
\left( i \Delta + \gamma_0 |\bar{a}|^2 \right) \bar{a} + \sqrt{\kappa_e} s_{in} = 0, \tag{5}
\]

where, \(\gamma_0 = \int_0^\infty \gamma(\tau) d\tau = \hbar G^2/m \Omega_m^2\) represents the steady state cubic nonlinearity coefficient. As discussed in the following section, such steady state solution is stable as long as the drive laser power is below certain threshold. The steady state equation (5) and its fixed point solutions have been largely investigated in earlier works [26, 27]. In particular, equation (5) is known to exhibit bistability for negative detuning and at critically large power levels.

In order to find the threshold for the formation of the first self-oscillation harmonics, we consider a weak modulation of the pump photons, \(a(t) = \bar{a} + \alpha(t)\). Assuming \(|\alpha| \ll |\bar{a}|\), the pump photon amplitude \(\bar{a}\) is governed by equation (5) while the small signal \(\alpha(t)\) follows:

\[
\frac{d \alpha}{dt} = \left( i \Delta + \gamma_0 |\bar{a}|^2 - \frac{\kappa}{2} \right) \alpha(t) + i \int_{-\infty}^t \gamma(t - t') (|\bar{a}|^2 \alpha(t') + \bar{a}^2 \alpha^*(t')) dt'. \tag{6}
\]

To find the instability threshold, we assume the ansatz \(\alpha = \alpha_+ e^{\kappa t} + \alpha_- e^{-\kappa t}\), which results in

\[
\alpha_+ = \left( i \Delta + \gamma_0 |\bar{a}|^2 - \frac{\kappa}{2} \right) \alpha_+ + i \tilde{\gamma}(s) |\bar{a}|^2 \alpha_+ + i \tilde{\gamma}(s) \bar{a}^2 \alpha_-, \tag{7.a}
\]

\[
\alpha_- = - \left( i \Delta + \gamma_0 |\bar{a}|^2 - \frac{\kappa}{2} \right) \alpha_- - i \tilde{\gamma}(s) |\bar{a}|^2 \alpha_- - i \tilde{\gamma}(s) \bar{a}^2 \alpha_+, \tag{7.b}
\]

Here, \(\tilde{\gamma}(s) = \hbar G^2/[m(s^2 + s \Gamma_m + \Omega_m^2)]\) is the Laplace transform of the retarded response function \(\gamma(\tau)\). Relations (7) result in a quartic algebraic equation for the exponent \(s^2\):
\[ g = \frac{-W}{W} < - W < W \]

Figure 2. (a) Normalized parametric instability growth rate \( R(s)/\Gamma_m \) versus intracavity pump photon number \( |\ell|^2 \) for different frequency detunings, \( \Delta = -\Omega_m, 0, \Omega_m \). (b) The instability region (shaded area) versus laser detuning according to the linear stability analysis. Here, we consider a microtoroid cavity with \( \kappa/2\pi = 2\nu_c/2\pi = 2\nu_c/2\pi = 200 \text{ MHz}, \Omega_m/2\pi = 50 \text{ MHz}, \Gamma_m/2\pi = 50 \text{ kHz}, G/2\pi = 6 \text{ GHz nm}^{-1} \), and \( m = 6 \text{ ng} \).

\[ (s^2 + \Gamma_m s + \Omega_m^2)(s^2 + \kappa s + \Delta^2 + \kappa^2/4) + 2\Delta h G^2/m = 0, \]  

where \( \Delta = \Delta + \gamma_m |\ell|^2 \) represents the modified frequency detuning and \( G = G_m \) is the enhanced optomechanical frequency shift per displacement. The stability of the system can be investigated by evaluating the complex roots of equation (8). The emergence of roots with positive real parts marks the onset of instabilities and the magnitude of unstable roots can be regarded as an instability gain. The onset of instability can be predicted without a direct calculation of the roots by using the Routh–Hurwitz criteria, as typically done in control systems [46]. The Routh–Hurwitz stability criterion for a generic quartic polynomial characteristic equation \( s^4 + c_1 s^3 + c_2 s^2 + c_3 s + c_4 = 0 \) can be expressed as \( c_1 > 0, c_3 > 0, c_4 > 0 \) and \( c_2 c_3 c_4 > c_1^3 + c_2^2 c_4 \).

On the other hand, equation (8) can be expanded in the form of a standard fourth-order equation with
\[ c_1 = \kappa + \Omega_m, c_2 = \kappa \Omega_m = \Omega_m + (\Delta^2 + \kappa^2/4), c_3 = \kappa \Omega_m^2 + \Gamma_m (\Delta^2 + \kappa^2/4) \] and \[ c_4 = \Omega_m^2 (\Delta^2 + \kappa^2/4) + 2\Omega_m \Delta (\Delta - \Delta), \] where we have used \( \hbar G^2/m = \Omega_m/\gamma_m |\ell|^2 = \Omega_m^2 (\Delta - \Delta) \). Given that for the optomechanical system \( c_1 \) and \( c_2 \) are always positive, the necessary and sufficient conditions for the stability of such system reduce to the positivity of the following expressions, which are real-coefficient polynomials, in terms of the optomechanically-induced static frequency shift \( \gamma_m |\ell|^2 \):

\[ 3(\gamma_m |\ell|^2)^2 + 4\Delta (\gamma_m |\ell|^2) + \Delta^2 + \kappa^2/4 > 0, \] \[ (\gamma_m |\ell|^2)^4 + (\gamma_m |\ell|^2)^4 + (\gamma_m |\ell|^2)^2 + d_4 > 0. \]

The coefficients of the latter equation are defined as \( d_1 = 4\Delta, \)
\[ d_2 = \Gamma_m \kappa + 6\Delta^2 + \Gamma_m^2 - 6\Omega_m^2 + \kappa^2/2 - 2\Omega_m \Gamma_m^2/\kappa - 2\Omega_m^2 \kappa/\Gamma_m, \]
\[ d_3 = \Delta^2 (2\Gamma_m \kappa + 4\Delta^2 + 2\Gamma_m^2 - 8\Omega_m^2 + \kappa^2/2 - 2\Omega_m \Gamma_m^2/\kappa - 2\Omega_m^2 \kappa/\Gamma_m) \]
and \( d_4 = \Delta^2 (\Delta^2 + \Gamma_m^2 - 2\Omega_m^2 + \kappa^2/2 - 2\Omega_m \Gamma_m^2 + \Gamma_m \kappa^2/2 + \Omega_m \Gamma_m \kappa^2 + \kappa^2/2 + \kappa^2/4 + \Omega_m^2 \kappa^2/4 + (\Omega_m^2 + \kappa^2/4)^2) \). The two conditions in equation (9) can be simply tested for a given set of parameters and at a particular pump bias. On the other hand, one can directly find an instability region where the first polynomial becomes negative. This happens when the discriminant of this quadratic equation (9.a) is negative, which requires \( \Delta < -\sqrt{3} \kappa/2 \).

Interestingly, this latter condition is the same as the bistability criterion. According to equation (9.a), once this condition is satisfied, an unstable root appears for a pump photon ranging between two critical points of \( |\ell|^2 = (-4\Delta \pm \sqrt{4\Delta^2 - 3\kappa^2})/(6\gamma_m) \), which are the same as the bistability turning points [29].

Even though the Routh–Hurwitz analysis determines the instability range, it does not provide further information about the exact value of the roots. In the following, we directly derive the roots of equation (8) through numerical calculation of the corresponding eigenvalue problem. Figure 2(a) shows the real part of the unstable root, which represents the instability growth rate, versus the intracavity pump photon number \( |\ell|^2 \) for different frequency detunings \( \Delta = -\Omega_m, 0, \Omega_m \). In this example, we have considered realistic parameters associated with a microtoroid cavity, \( \kappa/2\pi = 2\nu_c/2\pi = 2\nu_c/2\pi = 200 \text{ MHz}, \Omega_m/2\pi = 50 \text{ MHz}, \Gamma_m/2\pi = 50 \text{ kHz}, G/2\pi = 6 \text{ GHz nm}^{-1} \), and \( m = 6 \text{ ng} \). As this figure indicates, the instability threshold occurs at a much lower photon intensity for a blue-detuned system. To better illustrate this, we have plotted the intracavity photon range associated with the presence of unstable roots (shaded area) for different laser detunings in figure 2(b). This figure clearly shows a lower threshold power required for bringing the optomechanical system to instability in the blue-detuned regime. Notice that the footprint of bistability of the intracavity photon number \( |\ell|^2 \) is reflected in this figure for the negative laser detuning range of \( \Delta < -\sqrt{3} \kappa/2 \). It is worth noting that while predicting the onset of self-oscillations, the linear instability analysis does not
represents the mechanical quality factor and \( v = 6 \) in the frequency domain. Interestingly, the nonlinear term appears as a strong \( a_k v^2 \) and the coupling strength of this nonlinear process is proportional to \( v \). Signal associated with multiple harmonics at laser at different powers. By increasing the input photon flux is assumed to be \((a) \) for all \( n \) and \( (e) \) for \( n \), the input laser power, a periodic train of short pulses emerges in the time domain response of the cavity, which corresponds to a frequency comb with tens of harmonics. In order to better understand the dynamics of the formation of such a frequency comb, we expand the intracavity frequency combs. By further increasing the input laser power, a periodic train of short pulses emerges in the time domain response of the cavity, which corresponds to a frequency comb with tens of harmonics. In order to provide any information about higher order harmonics and the shape of the frequency comb forming in larger input powers, as discussed in the following.

4. Optomechanical frequency comb generation

Figure 3 shows the time and frequency response of the optomechanical system driven with constant intensity laser at different powers. By increasing the input photon flux, the stationary state solution breaks into a periodic signal associated with multiple harmonics at \( k \Omega_m \), \( k = 0, \pm 1, \pm 2, \ldots \), in the frequency domain. Interestingly, the comb lines exhibit a triangular shape, which is analogous to the solitary hyperbolic secant profile in Kerr frequency combs. By further increasing the input laser power, a periodic train of short pulses emerges in the time domain response of the cavity, which corresponds to a frequency comb with tens of harmonics. In order to better understand the dynamics of the formation of such a frequency comb, we expand the intracavity field as an infinite sum of multiple periodic harmonics

\[
a(t) = \sum_{k=-\infty}^{\infty} a_k e^{-i k \omega_m t},
\]

which results into a set of coupled nonlinear difference equations

\[
\left( i (\Delta + k \Omega_m) - \frac{\kappa}{2} \right) a_k + i \gamma_0 \sum_{k'} \frac{1}{1 - k'^2 + i k' / Q_m} \sum_{k''} a_{k-k'} a_{k+k''} a_{k''} + \sqrt{\kappa} s_0 \delta_{k,0} = 0.
\]

Here, \( Q_m = \Omega_m / \Gamma_m \) represents the mechanical quality factor and \( \delta_{k,0} \) is the Kronecker delta, equal to unity for \( k = 0 \) and zero otherwise. Interestingly, these coupled harmonic equations are similar to the modal rate equations describing cascaded four-wave-mixing processes in the microresonator Kerr frequency combs \cite{47}. According to this relation, any \( k \)th harmonic \( a_k \) interacts with each triple of \( a_{i}, a_{i}^{*}, a_{i} \), for \( i, j, k \) being integers satisfying the selection rule \( k = u - v + w \), and the coupling strength of this nonlinear process is proportional to \((1 - (w - v)^2 + i(w - v) / Q_m)^{-1} \). For \( w = v \), the nonlinear term reduces to \( |a_k|^2 a_k \), which represents Kerr-type self-phase-modulation. On the other hand, for \( w - v = \pm 1 \), the nonlinear term appears as a strong dissipative coupling of \( a_k \) to \( a_{k-1} a_{k+1} a_{k} \) and \( a_{k+1} a_{k-1} a_{k} \) for all \( v \). Given that \( Q_m \gg 1 \), for \( |w - v| > 1 \), the nonlinear coupling rate can be approximated with \((1 - (w - v)^2)^{-1} \), which rapidly drops for large index mismatches.

An important parameter in optomechanical cavities is the so-called sideband resolution ratio, which is defined as the ratio of the mechanical resonance frequency to the linewidth of the optical mode, i.e., \( \Omega_m / \kappa \). This parameter determines whether the optical mode effectively interacts with one or both sidebands of the mechanical mode. The former happens for \( \Omega_m / \kappa \ll 1 \) and it is typically known as the resolved sideband regime, while the latter scenario happens for an optical linewidth comparable or larger than the mechanical resonance frequency and it is called the unresolved sideband regime. In addition, the sideband resolution is a measure of dynamical interactions between optical and mechanical degrees of freedom, since it determines the
The ratio of the mechanical oscillation period to the photon lifetime. Hereafter, we interchangeably use the terminologies of good and bad cavity regimes for the resolved and unresolved sideband scenarios, respectively.

As shown in figure 2(b), the frequency detuning associated with the minimum instability threshold is slightly upshifted with respect to $\Delta = +\Omega_m$ which is associated with the parametric amplification point. This is indeed due to the fact that the optomechanical system of figure 2 operates in the unresolved sideband regime. In such a system, in order to excite only the upper mechanical sideband, we need to increase the detuning such that the overlap between the optical mode linewidth and the lower mechanical sideband is minimized. On the other hand, when operating in the resolved sideband regime, the frequency detuning associated with the minimum instability threshold is expected to be located at the point $\Delta = -\Omega_m$. This behavior is clearly shown in figure 4, which depicts the frequency detuning for minimum instability threshold $D_{th, min}$ as a function of the sideband resolution ratio $k_W/\Omega_m$.

Apart from the instability threshold, there is a major difference between the frequency combs formed in the bad and good cavity regimes. This behavior is reflected in both the time and frequency response of the system as shown in figure 5, which compares the two scenarios $\Omega_m/\kappa = 0.1$ (a), (b) and $\Omega_m/\kappa = 10$ (c), (d). In the unresolved sideband regime, ultra-short high-intensity pulses are formed in the time domain, which correspond to a broad frequency comb exhibiting a triangular shape. In the resolved sideband scenario, on the other hand, the temporal response is oscillatory, while the comb shape is distorted. This behavior can be understood from the fact that in the resolved sideband case, only the fundamental harmonic falls within the bandwidth of the optical mode, while in the unresolved regime, the cavity linewidth covers a broad range of harmonics and can thus shape the comb.
For input powers exceeding a certain level, the optomechanical system enters a chaotic regime. Therefore, depending on the parameters involved, the frequency comb can be formed over a certain pump power range. In order to show this transition, we have investigated the general behavior of the system in a parameter map of the input photon flux $|s_m|^2$ and its frequency detuning $\Delta$. The results are shown in figures 6(a) and (b) for two different cases, unresolved ($\Omega_m/\kappa = 0.1$) and resolved ($\Omega_m/\kappa = 10$) sideband regimes, respectively. Here, three different regimes of stationary state, frequency comb and chaos are, respectively, shown with circles, rectangles and triangles. As expected, in the unresolved sideband system, the comb formation starts around $\Delta \approx 2.5\Omega_m$ at intermediate power levels. In addition, in this case, the frequency combs persist at very high powers without any signature of chaos. In fact, in the bad cavity regime, large optical losses filter out undesired frequency components that in principle initiate chaos. Quite interestingly, similar behavior is observed in Kerr frequency combs [48]. In the resolved sideband system, on the other hand, the comb initiates at $\Delta = \Omega_m$ and at much lower pump powers. However, for an increasing power, the optomechanical system enters a chaotic regime. This transition is accompanied by a sequence of period doubling bifurcations that results in the formation of intermediate frequency components between the mechanical sidebands.

5. Conclusion

In conclusion, in this paper we have studied the effect of large optical nonlinearities induced by radiation pressure in optomechanical cavities. In particular, the formation of an optical frequency comb was investigated. The comb formation threshold was derived analytically and the effect of different parameters was investigated. In particular, we showed that frequency comb generation does not require resolved sideband operation, which is typically a demanding condition in optomechanical micro- and nano-cavities. In contrast, combs formed in the unresolved sideband regime were found to be less prone to chaos.

Our results suggest that optomechanical cavities, due to their strong nonlinearities, are suitable candidates for chip-scale nonlinear devices for different applications. In particular, optomechanically-induced frequency combs offer frequency spacings of tens of megahertz, which is much narrower than what is usually achieved using nonlinear optical microresonators. It is worth mentioning that the frequency spacing in microring Kerr frequency combs is governed by their free spectral range and generating combs with megahertz frequency spacings generally requires large ring resonators with tens of centimeters of diameters. Optomechanical frequency combs can be therefore of potential use as compact sources in RF and microwave photonics, given that the comb formation process in optomechanical cavities is all-optical and it does not require bulky electro-optic devices. Moreover, optomechanical frequency combs can also be utilized in the optical domain, offering a better resolution for applications in spectroscopy due to their narrower line spacing compared to optical microcombs. The usefulness of optomechanical combs for broad applications in optics, however, is tied to an octave expansion of the comb frequency range [49]. In principle, such an ultra-wideband optomechanical comb can be
generated through a multimode optomechanical cavity where a sequence of optical longitudinal modes are strongly coupled to mechanical modes, generating intermediate harmonics at mechanical sidebands. Another potential direction for optomechanical combs is in sensing applications. In principle, the relatively small sideband spacing in the frequency comb allows for accurate baseband electronic detection of any changes in the linewidth and spacing of the sidebands, which instead allows for sensitive measurements.

The theoretical investigation presented here, explores some of the fundamental concepts of optomechanical comb formation based on a minimal coupled mode model. This work can provide a basis for future theoretical and experimental investigations on optomechanical frequency combs.

Acknowledgments
The authors would like to thank Dr H Taheri for helpful discussions. This work was partially supported by the Office of Naval Research with grant No. N00014-15-1-2685.

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