Quasishuffle double bialgebras

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Abstract

Quasishuffle Hopf algebras, usually defined on a commutative monoid, can be more generally defined on any associative algebra $V$. If $V$ is a commutative and cocommutative bialgebra, the associated quasishuffle bialgebra $QSh(V)$ inherits a second coproduct $\delta$ of contraction and extraction of words, cointeracting with the deconcatenation coproduct $\Delta$, making $QSh(V)$ a double bialgebra. In order to generalize the universal property of the Hopf algebra of quasisymmetric functions $QSym$ (a particular case of quasishuffle Hopf algebra) as exposed by Aguiar, Bergeron and Sottile, we introduce the notion of double bialgebra over $V$. A bialgebra over $V$ is a bialgebra in the category of right $V$-comodules and an extra condition is required on the second coproduct for double bialgebras over $V$.

We prove that the quasishuffle bialgebra $QSh(V)$ is a double bialgebra over $V$, and that it satisfies a universal property: for any bialgebra $B$ over $V$ and for any character $\lambda$ of $B$, under a connectedness condition, there exists a unique morphism $\phi$ of bialgebras over $V$ from $B$ to $QSh(V)$ such that $\varepsilon \circ \phi = \lambda$. When $V$ is a double bialgebra over $V$, we obtain a unique morphism of double bialgebras over $V$ from $B$ to $QSh(V)$, and show that this morphism $\phi_1$ allows to obtain any morphism of bialgebra over $V$ from $B$ to $QSh(V)$ thanks to an action of a monoid of characters. This formalism is applied to a double bialgebra of $V$-decorated graphs.

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Contents

1 Bialgebras over another bialgebra
1.1 Définitions and notations ........................................ 4
1.2 Antipode .......................................................... 5
1.3 Nonunitary cases ............................................... 6
1.4 Double bialgebras over $V$ ....................................... 7

2 Quasishuffle bialgebras
2.1 Definition ...................................................... 8
2.2 Universal property of quasishuffle bialgebras ............. 10
2.3 Double bialgebra morphisms ................................ 13
2.4 Action on bialgebra morphisms ............................... 17
2.5 Applications to graphs ....................................... 19

Introduction

Quasishuffle bialgebras are Hopf algebras based on words, used in particular for the study of relations between multizetas [10,11]. They also appear in Ecalle’s mould calculus, as a symmetrical mould can be interpreted as a character on a quasishuffle bialgebra [3]. Hoffman’s construction
is based on commutative countable semigroups, but it can be extended to any associative algebra $(V, \cdot)$, not necessarily unitary [6]. The associated quasishuffle bialgebra $\text{QSh}(V)$ is, as a vector space, the tensor algebra $T(V)$. Its product is the quasishuffle product $\triangleright$, inductively defined as follows: if $x, y \in V$ and $v, w \in T(V)$,

$$1 \triangleright w = w,$$

$$v \triangleright 1 = v,$$

$$xv \triangleright yw = x(v \triangleright yw) + y(xv \triangleright w) + (x \cdot y)(v \triangleright w).$$

For example, if $x, y, z, t \in V$,

$$x \triangleright y = xy + yx + x \cdot y,$$

$$xy \triangleright z = xyz + xzy + xzy + (x \cdot z)y + x(y \cdot z),$$

$$xy \triangleright zt = xytz + xzyt + xzyt + zxt + ztxy + (x \cdot z)ty + (x \cdot z)yt + xz(y \cdot t) + zyx(y \cdot t) + (x \cdot z)(y \cdot t).$$

The coproduct $\Delta$ is the deconcatenation: if $x_1, \ldots, x_n \in V$,

$$\Delta(x_1 \ldots x_n) = \sum_{i=0}^{n} x_1 \ldots x_i \otimes x_{i+1} \ldots x_n.$$ 

When $(V, \cdot, \delta_V)$ is a commutative bialgebra, not necessarily unitary, then $\text{QSh}(V)$ inherits a second, less known coproduct $\delta$: if $x_1, \ldots, x_n \in V$,

$$\delta(v_1 \ldots v_n) = \sum_{1 \leq i_1 < \ldots < i_p \leq k} \left( \prod_{1 \leq j < i_1} v_{i_1} \right) \ldots \left( \prod_{i_p+1 \leq j \leq k} v_i^p \right) \otimes (v_{i_1}^{p+1} \ldots v_k^{p}),$$

with Sweedler’s notation for $\delta_V$ and where the symbols $\prod_{i_p+1 \leq j \leq k}$ mean that the products are taken in $(V, \cdot)$. The counit $\epsilon_\delta$ is given as follows: for any word $w$ of length $n \geq 1$,

$$\epsilon_\delta(w) = \begin{cases} \epsilon_V(w) & \text{if } n = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then $(T(V), \triangleright, \delta)$ is a bialgebra, and $(T(V), \triangleright, \Delta)$ is a bialgebra in the category of right $(T(V), \triangleright, \delta)$-comodules, which in particular implies that

$$(\Delta \otimes \text{Id}) \circ \delta = \triangleright_{1,3,24} \circ (\delta \otimes \delta) \circ \Delta,$$

where $\triangleright_{1,3,24} : T(V)^{\otimes 4} \longrightarrow T(V)^{\otimes 3}$ send $w_1 \otimes w_2 \otimes w_3 \otimes w_4$ to $w_1 \otimes w_3 \otimes w_2 \triangleright w_4$. Two particular cases will be considered all along this paper:

- $V = \mathbb{K}$, with its usual bialgebraic structure. The quasishuffle algebra $\text{QSh}(\mathbb{K})$ is isomorphic to the polynomial algebra $\mathbb{K}[X]$, with its two coproducts defined by

$$\Delta(X) = X \otimes 1 + 1 \otimes X,$$

$$\delta(X) = X \otimes X.$$ 

- $V$ is the algebra of the semigroup $(\mathbb{N}_{>0}, +)$. We recover the double Hopf algebra of quasisymmetric functions $\text{QSym}$ [3 9 12 14]. This Hopf algebra is studied in [2], where it is proved to be the terminal object in a category of combinatorial Hopf algebras: If $B$ is a graded and connected Hopf algebra and $\lambda$ is a character of $B$, then there exists a unique homogeneous Hopf algebra morphism $\phi_\lambda : B \longrightarrow \text{QSym}$ such that $\epsilon_\delta \circ \phi_\lambda = \lambda$. We proved in [4 5] that when $(B, m, \Delta, \delta)$ is a double bialgebra, such that:
– \((B, m, \Delta)\) is a graded and connected Hopf algebra,
– for any \(n \in \mathbb{N}\), \(\delta(B_n) \subseteq B_n \otimes B\),

then \(\phi_{\Delta}\) is the unique homogeneous double bialgebra morphism from \(B\) to \(\mathbb{Q}\)\text{Sym}. We also proved a similar result for \(\mathbb{K}[X]\), where the hypothesis "graded and connected" on \(B\) is replaced by the weaker hypothesis "connected".

In this paper, we generalize these results to any quasishuffle \(\mathbb{Q}\text{Sh}(V)\) associated to a commutative and cocommutative bialgebra \((V, \cdot, \delta_V)\), not necessarily unitary. We firstly show that \((T(V), \cdot, \Delta)\) is a bialgebra in the category of right \(V\)-comodules, with the coaction \(\rho\) defined by

\[
\forall v_1, \ldots, v_n \in V, \quad \rho(v_1 \ldots v_n) = v'_1 \ldots v''_n \otimes v'\prime \ldots v''\prime.
\]

Moreover, the second coproduct \(\Delta\) satisfies this compatibility with \(\rho\):

\[
(Id \otimes c) \circ (\rho \otimes Id) \circ \Delta = (\delta \otimes Id) \circ \rho,
\]

where \(c : V \otimes T(V) \rightarrow T(V) \otimes V\) is the usual flip. Equivalently, \((T(V), \cdot, \Delta)\) is a comodule over \((V, \delta^p)\otimes (T(V), \delta)\). This observation leads us to study bialgebras over \(V\), that is bialgebras in the category of right \((V, \cdot, \delta_V)\)-comodules (Definition 1.1 when \(V\) is unitary). Technical difficulties occur when \(V\) is not unitary, a case that cannot be neglected as it includes \(\mathbb{Q}\)\text{Sym}: this is the object of Definition 1.3, where we use the unitary extension \(uV\) of \(V\), which is also a bialgebra.

We define double bialgebras over \(V\) in Definition 1.4 in the unitary case and Definition 1.3 in the nonunitary case.

- When \(V = \mathbb{K}\), bialgebras over \(V\) are bialgebras \(B\) with a decomposition \(B = B_1 \otimes B_X\), where \(B_1\) is a subbialgebra and \(B_X\) is a biideal. This includes any bialgebra \(B\), taking \(B_1 = \mathbb{K}1_B\) and \(B_X\) the kernel of the counit.

- When \(V = \mathbb{K}(\mathbb{N}_{\geq 0}, +)\), bialgebras over \(V\) are \(\mathbb{N}\)-graded and connected bialgebras, that is \(\mathbb{N}\)-graded bialgebras \(B\) with \(B_0 = \mathbb{K}1_B\).

We prove that in a bialgebra \((B, m, \Delta, \rho)\) over \(V\) such that \((B, m, \Delta)\) is a Hopf algebra, then the antipode is automatically a comodule morphisms (Proposition 1.2), that is

\[
\rho \circ S = (S \otimes Id_V) \circ \rho.
\]

In the case of \(\mathbb{N}\)-graded bialgebras, this means that \(S\) is automatically homogeneous; more generally, if \(\Omega\) is a commutative semigroup and \(B\) is an \(\Omega\)-graded bialgebra and a Hopf algebra, then its antipode is automatically \(\Omega\)-homogeneous.

Let us now consider the double quasishuffle algebra \(\mathbb{Q}\text{Sh}(V) = (T(V), \cdot, \Delta, \delta)\), which is over \(V\) with the coaction \(\rho\). We obtain a generalization of Aguiar, Bergeron and Sottile’s result: Theorem 2.2 states that for any connected bialgebra \(B\) over \(V\) and for any character \(\lambda\) of \(B\), there exists a unique morphism \(\phi_{\lambda}\) from \(B\) to \(\mathbb{Q}\text{Sh}(V)\) of bialgebras over \(V\) such that \(\lambda \circ \phi_{\lambda} = \phi_{\lambda}\), given by an explicit formula implying the iterations of the reduced coproduct \(\tilde{\Delta}\) associated to the coproduct \(\Delta\) of \(B\).

When \(B\) is a double bialgebra over \(V\), we prove that there exists a unique morphism of double bialgebras over \(V\) from \(B\) to \(\mathbb{Q}\text{Sh}(V)\), that is \(\Phi_{\epsilon_S}\), with an explicit formula involving the counit of the second coproduct, the coaction and the iterations of the first coproduct (Theorem 2.4). Moreover, for any bialgebra \(B'\) over \(V\), the second coproduct \(\delta\) induces an action \(\star\) of the monoid of characters \(\text{Char}(B)\) (with the product induced by \(\delta\)) onto the set of morphisms of bialgebras over \(V\) from \(B\) to \(B'\) (Proposition 2.10). When \(B' = \mathbb{Q}\text{Sh}(V)\), we obtain that this action is simply transitive (Corollary 2.12), which gives a bijection between the set of characters of \(B\) and the set of morphisms of double bialgebras over \(V\) from \(B\) to \(\mathbb{Q}\text{Sh}(V)\). This is finally applied to the twisted bialgebra of graphs \(G\): for any \(V\), we obtain a double bialgebra \(H_V\) of
V-decorated graphs, and the unique morphism of double bialgebras over V from \( V \) to \( QSh(V) \) is a generalization of the chromatic polynomial and of the chromatic (quasi)symmetric series. Taking \( V = \mathbb{K} \) or \( \mathbb{K}(\mathbb{N}_{\geq 0}, +) \), we recover the terminal property of \( \mathbb{K}[X] \) and \( QSym \).

These results will be applied in a series of forthcoming papers on mixed graphs, hypergraphs, partitions, etc.

**Notations 0.1.** 1. We denote by \( \mathbb{K} \) a commutative field of characteristic zero. Any vector space in this field will be taken over \( \mathbb{K} \).

2. For any \( n \in \mathbb{N} \), we denote by \( [n] \) the set \( \{1, \ldots, n\} \). In particular, \( [0] = \emptyset \).

1 Bialgebras over another bialgebra

1.1 Définitions and notations

Let \((V, \cdot, \delta_V)\) be a commutative bialgebra, which we firstly assume to be unitary and counitary. Its counit is denoted by \( \varepsilon_V \) and its unit by \( 1_V \).

**Definition 1.1.** A bialgebra over \( V \) is a bialgebra in the category of right \( V \)-comodules, that is a family \( (B,m,\Delta,\rho) \) where \((B,m,\Delta)\) is a bialgebra and \( \rho : B \longrightarrow B \otimes V \) such that:

- \( \rho \) is a right coaction of \( V \) over \( B \), that is
  \[
  (\rho \otimes \text{Id}_V) \circ \rho = (\text{Id}_B \otimes \delta_V) \circ \rho,
  \]
  \[
  (\text{Id}_B \otimes \varepsilon_V) \circ \rho = \text{Id}_B.
  \]

- The unit of \( B \) is a \( V \)-comodule morphism, that is
  \[
  \rho(1_B) = 1_B \otimes 1_V.
  \]

- The product \( m \) of \( B \) is a \( V \)-comodule morphism, that is
  \[
  \rho \circ m = (m \otimes \cdot) \circ (\text{Id} \otimes c \otimes \text{Id}) \circ (\rho \otimes \rho),
  \]
  where \( c : B \otimes B \longrightarrow B \otimes B \) is the usual flip, sending \( a \otimes b \) to \( b \otimes a \).

- The counit \( \varepsilon_\Delta \) of \( B \) is a \( V \)-comodule morphism, that is
  \[
  \forall x \in B, \quad (\varepsilon_\Delta \otimes \text{Id}) \circ \rho(x) = \varepsilon_\Delta(x) 1_V.
  \]

- The coproduct \( \Delta \) of \( B \) is a \( V \)-comodule morphism, that is
  \[
  (\Delta \otimes \text{Id}) \circ \rho = m_{1,3,24} \circ (\rho \otimes \rho) \circ \Delta,
  \]
  where
  \[
  m_{1,3,24} : \begin{align*}
  B \otimes V \otimes B \otimes V &\longrightarrow B \otimes B \otimes V \\
  b_1 \otimes v_1 \otimes b_2 \otimes v_2 &\longrightarrow b_1 \otimes b_2 \otimes v_1 \cdot v_2.
  \end{align*}
  \]

Notice that the second and third items are equivalent to the fact that \( \rho \) is an algebra morphism.

**Example 1.1.** • Let \((\Omega, \ast)\) be a monoid and let \( V = \mathbb{K} \Omega \) be the associated bialgebra. Let \( B \) be a bialgebra over \( V \). For any \( a \in \Omega \), we put
\[
B_a = \{ x \in B \mid \rho(x) = x \otimes a \}.
\]
Then $B = \bigoplus_{a \in \Omega} B_a$. Indeed, if $x \in B$, we can write
\[
\rho(x) = \sum_{a \in \Omega} x_a \otimes \alpha.
\]
Then
\[
(\rho \otimes \text{Id}) \circ \rho(x) = \sum_{a \in \Omega} \rho(x_a) \otimes \alpha = (\text{Id} \otimes \delta_V) \circ \rho(x) = \sum_{a \in \Omega} x_a \otimes \alpha \otimes \alpha.
\]
Therefore, for any $\alpha \in \Omega$, $\rho(x_a) = x_a \otimes \alpha$, that is $x_a \in B_\alpha$. Moreover,
\[
x = (\text{Id} \otimes \epsilon_V) \circ \rho(x) = \sum_{a \in \Omega} x_a.
\]
The second item of Definition 1.1 is equivalent to $1_B \in B_{1_\Omega}$. The third item is equivalent to
\[
\forall \alpha, \beta \in \Omega, \quad B_\alpha B_\beta \subseteq B_{\alpha \beta}.
\]
The fourth item is equivalent to $\bigoplus_{\alpha \neq 1_\Omega} B_\alpha \subseteq \ker(\epsilon_\Delta)$. The last item is equivalent to
\[
\forall \alpha \in \Omega, \quad \Delta(B_\alpha) \subseteq \bigoplus_{\alpha' \bullet \alpha'' = \alpha} B_{\alpha'} \otimes B_{\alpha''}.
\]
In other words, a bialgebra over $V$ is an $\Omega$-graded bialgebra.

Let $V = \mathbb{K}[X]/\langle X^2 = X \rangle$, with the coproduct defined by $\Delta(X) = X \otimes X$. This bialgebra is isomorphic to the bialgebra of the monoid $(\mathbb{Z}/2\mathbb{Z}, \times)$. As $V$ has a basis $(1, X)$ of group-like elements, a bialgebra over $V$ admits a decomposition $B = B_1 \oplus B_X$, with $1_B \in B_1$, $\epsilon_\Delta(B_X) = (0)$, and
\[
B_1 B_X + B_X B_1 + B_X B_X \subseteq B_X, \quad B_1 B_1 \subseteq B_1, \\
\Delta(B_X) \subseteq B_X \otimes B_1 + B_1 \otimes B_X + B_X \otimes B_X, \quad \Delta(B_1) \subseteq B_1 \otimes B_1.
\]
In other words, a bialgebra over $V$ is a bialgebra with a decomposition $B = B_1 \oplus B_X$, such that $B_1$ is a subbialgebra and $B_X$ is a biideal. In particular, any bialgebra $(B, m, \Delta)$ is trivially a bialgebra over $V$, with $B_1 = \mathbb{K}1_B$ and $B_X = \ker(\epsilon_\Delta)$, or equivalently, for any $x \in B$,
\[
\rho(x) = \epsilon(x)1_B \otimes 1 + (x - \epsilon(x)1_B) \otimes X.
\]

Notations 1.1. We shall use the Sweedler’s notation $\rho(x) = x_0 \otimes x_1$. The five items of Definition 1.1 become
\[
(x_0)_0 \otimes (x_0)_1 \otimes x_1 = x_0 \otimes x'_1 \otimes x''_1, \quad x_0 \epsilon(x_1) = x, \\
(1_B)_0 \otimes (1_B)_1 = 1_B \otimes 1_V, \\
(xy)_0 \otimes (xy)_1 = x_0 y_0 \otimes x_1 y_1, \quad \epsilon_\Delta(x_0) x_1 = \epsilon_\Delta(x)1_V, \\
(x^{(1)})_0 \otimes (x^{(2)})_1 \otimes x_1 = (x^{(1)})_0 \otimes (x^{(2)})_0 \otimes (x^{(1)})_1 (x^{(2)})_1.
\]
1.2 Antipode

Proposition 1.2. Let \((V, m_V, \delta_V)\) be a bialgebra and let \((B, m, \Delta, \rho)\) be a bialgebra over \(V\). If \((B, m, \Delta)\) is a Hopf algebra of antipode \(S\), then \(S\) is a comodule morphism, that is

\[ \rho \circ S = (S \otimes \text{Id}_V) \circ \rho. \]

Proof. Let us give \(\text{Hom}(B, B \otimes V)\) its convolution product \(*\): for any linear maps \(f, g\) from \(B\) to \(B \otimes V\),

\[ f * g = m_{B \otimes V} \circ (f \otimes g) \circ \Delta. \]

In this convolution algebra,

\[
\begin{align*}
((S \otimes \text{Id}_V) \circ \rho) * \rho &= m_{B \otimes V} \circ (S \otimes \text{Id}_V \otimes \text{Id}_B \otimes \text{Id}_V) \circ (\rho \otimes \rho) \circ \Delta \\
&= (m \circ (S \otimes \text{Id}_B) \circ \Delta \otimes \text{Id}_V) \circ m_{1,3,24} \circ (\rho \otimes \rho) \circ \Delta \\
&= (m \circ (S \otimes \text{Id}_B) \circ \Delta \otimes \text{Id}_V) \circ (\Delta \otimes \text{Id}) \circ \rho \\
&= (m \circ (S \otimes \text{Id}_B) \circ \Delta \otimes \text{Id}_V) \circ \rho \\
&= (\iota_B \circ \varepsilon \Delta \otimes \text{Id}_V) \circ \rho \\
&= \iota_{B \otimes V} \circ \varepsilon \Delta.
\end{align*}
\]

So \((S \otimes \text{Id}_V) \circ \rho\) is a right inverse of \(\rho\) in \((\text{Hom}(B, B \otimes V), *)\).

\[
\rho * (\rho \circ S) = m_{B \otimes V} \circ (\rho \otimes \rho) \circ (\text{Id} \otimes S) \circ \Delta \\
= \rho \circ m \circ (\text{Id} \otimes S) \circ \Delta \\
= \rho \circ \iota_B \circ \varepsilon \Delta \\
= \iota_{B \otimes V} \circ \varepsilon \Delta.
\]

So \(\rho \circ S\) is a left inverse of \(\rho\) in \((\text{Hom}(B, B \otimes V), *)\). As \(*\) is associative, \((S \otimes \text{Id}_V) \circ \rho = \rho \circ S. \)

Example 1.2. Let \((\Omega, \ast)\) be a semigroup. If \(V\) is the bialgebra of \((\Omega, \ast)\), we recover that if \(B\) is an \(\Omega\)-graded bialgebra and a Hopf algebra, then, for any \(\alpha \in \Omega\),

\[ S(B_\alpha) \subseteq B_\alpha. \]

1.3 Nonunitary cases

We shall work with not necessarily unitary bialgebras \((V, \cdot, \delta_V)\). If so, we put \(uV = \mathbb{K} \oplus V\) and we give it a product and a coproduct defined as follows:

\[
\begin{align*}
\forall \lambda, \mu \in \mathbb{K}, \forall v, w \in V, \quad & (\lambda + v) \cdot (\mu + w) = \lambda \mu + \lambda w + \mu v + v \cdot w, \\
\forall \lambda \in \mathbb{K}, \forall v \in V, \quad & \delta_{uV}(\lambda + v) = \lambda 1 \otimes 1 + \delta_V(v).
\end{align*}
\]

Then \((uV, \cdot, \delta_{uV})\) is a counitary and unitary bialgebra, and \(V\) is a nonunitary subbialgebra of \(uV\).

Definition 1.3. Let \((V, \cdot, \delta_V)\) be a not necessarily unitary bialgebra and \((uV, \cdot, \delta_{uV})\) be its unitary extension. A bialgebra over \(V\) is a bialgebra \((B, m, \Delta, \rho)\) over \(uV\) such that

\[
\rho(\ker(\varepsilon \Delta)) \subseteq B \otimes V.
\]

A double bialgebra over \(V\) is a double bialgebra \((B, m, \Delta, \delta, \rho)\) over \(uV\) such that \((B, m, \Delta, \rho)\) is a bialgebra over \(V\).
Remark 1.1. If \((B, m, \Delta, \rho)\) is a bialgebra over the nonunitary bialgebra \((V, \cdot, \delta_V)\), then

\[\{ b \in B \mid \rho(b) = b \otimes 1 \} = \mathbb{K}1_B.\]

Indeed, if \(\rho(b) = b \otimes 1\), putting \(b' = b - \varepsilon_\Delta(b)1_B\), then \(b' \in \text{Ker}(\varepsilon_\Delta)\). Hence,

\[\rho(b') = \rho(b) - \varepsilon_\Delta(b)1_B \otimes 1 = (b - \varepsilon(b)1_B) \otimes 1 \in B \otimes V,\]

so \(b = \varepsilon_\Delta(b)1_B\).

In the sequel, we will mention that we work with a nonunitary bialgebra \((V, \cdot, \delta_V)\) if we want to use Definition 1.3 and not Definition 1.1 even if \((V, \cdot)\) has a unit – that will happen when we shall work with \(\mathbb{K}\).

Example 1.3. 1. If \((\Omega, \times)\) is a semigroup, then a bialgebra \((B, m, \Delta)\) over \(V\) is an \(u\Omega\)-graded bialgebra, where \(u\Omega = \{e\} \sqcup \Omega\) with the extension of the product of \(\Omega\) such that \(e\) is a unit:

\[B = \bigoplus_{\alpha \in u\Omega} B_\alpha,\]

\[B_e = \mathbb{K}1_B,\]

\[\forall \alpha, \beta \in \Omega, \quad \Delta(B_\alpha) \subseteq \sum_{\alpha' \alpha'' \in \Omega, \alpha' \alpha'' = \alpha} B_{\alpha'} \otimes B_{\alpha''} + B_\alpha \otimes B_e + B_e \otimes B_\alpha,\]

\[\forall \alpha \in \Omega, \quad \varepsilon_\Delta(B_\alpha) = (0).\]

A double bialgebra \((B, m, \Delta, \delta)\) over \(V\) is a bialgebra over \(V\) such that for any \(\alpha \in \Omega,\)

\[\delta(B_\alpha) \subseteq B_\alpha \otimes B.\]

2. If \(V = \mathbb{K}[X]\), then \(u\mathbb{K} = \mathbb{K}[X]/\langle X^2 = X \rangle\), and any double bialgebra \((B, m, \Delta, \delta)\) is a double bialgebra over \(V\) with \(B_1 = \mathbb{K}1_B\) and \(B_X = \text{Ker}(\varepsilon_\Delta)\).

1.4 Double bialgebras over \(V\)

Definition 1.4. Let \((B, m, \Delta, \delta)\) be a double bialgebra, \((V, \cdot, \delta_V)\) be a bialgebra and \(\rho : B \rightarrow B \otimes V\) be a right coaction of \(V\) over \(B\). We shall say that \((B, m, \Delta, \delta, \rho)\) is a double bialgebra over \(V\) if \((B, m, \Delta, \rho)\) is a bialgebra over \(V\) and

\[(\text{Id} \otimes c) \circ (\rho \otimes \text{Id}) \circ \delta = (\delta \otimes \text{Id}) \circ \rho : B \rightarrow B \otimes B \otimes V,\]

where \(c : V \otimes B \rightarrow B \otimes V\) is the usual flip. In other words, with Sweedler’s notation \(\delta(x) = x' \otimes x''\) for any \(x \in B,\)

\[(x')_0 \otimes x'' \otimes (x')_1 = (x_0)' \otimes (x_0)'' \otimes x_1.\]

Remark 1.2. In other words, in a double bialgebra \(B\) over \(V\), considering the left coaction \(\rho^{op}\) of \(V^{cop} = (V, \delta^V_V^{op})\) on \(B,\)

\[(\rho^{op} \otimes \text{Id}) \circ \delta = (\text{Id} \otimes \delta) \circ \rho^{op},\]

which means that \(B\) is a \((V, \delta^V_V^{op})-\,(B, \delta)\)-bicomodule, or equivalently a \(V^{cop} \otimes B\)-comodule.

\(^1\)which is of course unitary, but which we treat as a nonunitary bialgebra, as mentioned before.
2 Quasishuffle bialgebras

2.1 Definition

Let $(V, \cdot)$ be a nonunitary bialgebra. The tensor algebra $T(V)$ is given the quasishuffle product associated to $V$: For any $v_1, \ldots, v_{k+l} \in V$,

$$v_1 \cdots v_k \shuffle v_{k+1} \cdots v_{k+l} = \sum_{\sigma \in \text{QSh}(k,l)} \left( \prod_{i=1} \left( v_i \right) \right) \cdots \left( \prod_{i=\text{max}(\sigma)} v_i \right),$$

where $\text{QSh}(k,l)$ is the set of $(k,l)$-quasishuffles, that is surjections $\sigma : [k+l] \rightarrow [\text{max}(\sigma)]$ such that $\sigma(1) < \ldots < \sigma(k)$ and $\sigma(k+1) < \ldots < \sigma(k+l)$. The symbol $\prod$ means that the corresponding products are taken in $(V, \cdot)$. The coproduct $\Delta$ is given by deconcatenation: for any $v_1, \ldots, v_n \in V$,

$$\Delta(v_1 \cdots v_n) = \sum_{k=0}^{n} v_1 \cdots v_k \otimes v_{k+1} \cdots v_n.$$

A special case is given when $\cdot$ is the zero product of $V$. In this case, we obtain the shuffle product $\shuffle$ of $T(V)$.

If $(V, \cdot, \delta_V)$ is a not necessarily unitary commutative bialgebra, then $\text{QSh}(V)$ inherits a second coproduct $\delta$ making it a double bialgebra. For any $v_1, \ldots, v_k \in V$, with Sweeder’s notation $\delta_V(v) = v' \otimes v''$,

$$\delta(v_1 \cdots v_n) = \sum_{1 \leq i_1 < \ldots < i_p < k} \left( \prod_{1 \leq i \leq i_1} v_i \right) \cdots \left( \prod_{i_p+1 \leq i \leq k} v_i \right) \otimes (v''_1 \cdots v''_{i_p}) \shuffle \cdots \shuffle (v''_{i_p+1} \cdots v''_k).$$

Proposition 2.1. Let $(V, \cdot, \delta_V)$ be a nonunitary bialgebra. We define a coaction of $V$ on $\text{QSh}(V)$ by

$$\forall v_1, \ldots, v_n \in V, \quad \rho(v_1 \cdots v_n) = v'_1 \cdots v'_n \otimes v''_1 \cdots v''_n.$$

1. The quasishuffle bialgebra $\text{QSh}(V) = (T(V), \shuffle, \Delta, \rho)$ is a bialgebra over $V$ if and only if $(V, \cdot)$ is commutative.

2. The quasishuffle double bialgebra $\text{QSh}(V) = (T(V), \shuffle, \Delta, \delta, \rho)$ is a bialgebra over $V$ if and only if $(V, \cdot)$ is commutative and cocommutative.

Proof. 1. Let us assume that $\text{QSh}(V)$ is a double bialgebra over $V$ with this coaction $\rho$. For any $v, w \in V$,

$$\rho(v \shuffle w) = \rho(vw + vw + v \cdot w)$$

$$= v'w' \otimes v'' \cdot w'' + v'w' \otimes v''' \cdot w'' + v'' \cdot w' \otimes v'' \cdot w''',$$

$$(\shuffle m) \circ (\rho \otimes \rho)(v \otimes w) = v' \shuffle w' \otimes v'' \cdot w''$$

$$= (v'w' + w'v' + v' \otimes w') \otimes v'' \cdot w''.$$

As $\shuffle$ is comodule morphism, we obtain that for any $v, w \in V$,

$$w' \otimes v' \otimes v'' \cdot w'' = w' \otimes v' \otimes v'' \cdot w''.$$

Applying $\epsilon_V \otimes \epsilon_V \otimes \text{Id}_V$, this gives $v \cdot w = w \cdot v$, so $\cdot$ is commutative.
Let us now assume that \( \cdot \) is commutative. The compatibilities of the unit and of the counit with the coaction \( \rho \) are obvious. Let \( v_1, \ldots, v_{k+l} \in V \) and let \( \sigma \in \text{QSh}(k, l) \).

\[
\rho \left( \prod_{i \in \sigma^{-1}(1)} v_i \right) \cdots \left( \prod_{i \in \sigma^{-1}(\max(\sigma))} v_i \right) \\
= \left( \prod_{i \in \sigma^{-1}(1)} v'_i \right) \cdots \left( \prod_{i \in \sigma^{-1}(\max(\sigma))} v'_i \right) \otimes \left( \prod_{i \in \sigma^{-1}(1)} v''_i \right) \cdots \left( \prod_{i \in \sigma^{-1}(\max(\sigma))} v''_i \right)
\]

as \((V, \cdot)\) is commutative. Summing over all \( \sigma \):

\[
\rho(v_1 \cdots v_k \uplus v_{k+1} \cdots v_{k+l}) = \sum_{\sigma \in \text{QSh}(k, l)} \left( \prod_{i \in \sigma^{-1}(1)} v'_i \right) \cdots \left( \prod_{i \in \sigma^{-1}(\max(\sigma))} v'_i \right) \otimes v''_1 \cdots v''_n
\]

\[
= (v'_1 \cdots v'_k \uplus v'_{k+1} \cdots v'_{k+l}) \otimes (v''_1 \cdots v''_k) \cdot (v''_{k+1} \cdots v''_{k+l})
\]

\[
= \rho(v_1 \cdots v_k) \rho(v_{k+1} \cdots v_{k+l}).
\]

Let \( v_1, \ldots, v_k \in V \). If \( 0 \leq i \leq k \),

\[
m_{1,3,24} \circ (\rho \otimes \rho)(v_1 \cdots v_i \otimes v_{i+1} \cdots v_k) = v'_1 \cdots v'_i \otimes v'_{i+1} \cdots v''_n \otimes v''_1 \cdots v''_k.
\]

Summing over all \( i \),

\[
m_{1,3,24} \circ (\rho \otimes \rho) \circ \Delta(v_1 \cdots v_k) = \left( \sum_{i=0}^{k} v'_1 \cdots v'_i \otimes v'_{i+1} \cdots v'_k \right) \otimes v''_1 \cdots v''_k
\]

\[
= (\Delta \otimes \text{Id}) \circ \rho(v_1 \cdots v_k).
\]

2. Let us assume that \( \text{QSh}(V) \) is a double bialgebra over \( V \). By the first part of this proof, \( V \) is commutative. For any \( v \in V \),

\[
(\text{Id} \otimes \delta_V) \circ \delta_V(v) = (\delta_V \otimes \text{Id}) \circ \delta_V(v)
\]

\[
= (\delta \otimes \text{Id}) \circ \rho(v)
\]

\[
= (\text{Id} \otimes c) \circ (\rho \otimes \text{Id}) \circ \delta(v)
\]

\[
= (\text{Id} \otimes c) \circ (\delta \otimes \text{Id}) \circ \delta(v)
\]

\[
= (\text{Id} \otimes \delta_V^\text{op}) \circ \delta_V(v).
\]

Applying \( c_V \otimes \text{Id} \otimes \text{Id} \), we obtain that \( \delta_V^\text{op} = \delta_V \), so \((V, \delta_V)\) is cocommutative.

Let us assume that \((V, \delta_V)\) is commutative and cocommutative. It is proved in [6] that \( \text{QSh}(V) \) is a double bialgebra. By the first item, \( \text{QSh}(V) \) is a bialgebra over \( V \). For any \( v_1, \ldots, v_n \in V \),

\[
(\delta \otimes \text{Id}) \circ \rho(v_1 \cdots v_k)
\]

\[
= \sum_{1 \leq i_1 < \cdots < i_p \leq k} \left( \prod_{i \in i_1} v'_i \right) \cdots \left( \prod_{i \in i_{p+1} \leq i \leq k} v'_i \right) \otimes (v''_1 \cdots v''_{i_1} \uplus \cdots \uplus (v''_{i_{p+1}} \cdots v''_{i_p}) \otimes v''_{i_1} \cdots v''_k,
\]

9
whereas

$$(\text{Id} \otimes c) \circ (\rho \otimes \text{Id}) \circ \delta(v_1 \ldots v_k)$$

$$= \sum_{1 \leq i_1 < \ldots < i_p < k} \left( \prod_{1 \leq i \leq i_1} v_i' \right) \ldots \left( \prod_{i_p+1 \leq i \leq k} v_i'' \right) \otimes (v_1'' \ldots v_{i_1}'' ) \oplus \ldots \oplus (v_{i_p+1}'' \ldots v_k'')$$

$$\otimes \left( \prod_{1 \leq i \leq i_1} v_i' \right) \ldots \left( \prod_{i_p+1 \leq i \leq k} v_i'' \right)$$

$$= \sum_{1 \leq i_1 < \ldots < i_p < k} \left( \prod_{1 \leq i \leq i_1} q_i' \right) \ldots \left( \prod_{i_p+1 \leq i \leq k} q_i'' \right) \otimes (v_1'' \ldots v_{i_1}'' ) \oplus \ldots \oplus (v_{i_p+1}'' \ldots v_k'') \otimes v_1'' \ldots v_k'',$n as $(V, \cdot)$ is commutative. By the cocommutativity of $\delta_V$,

$$(\delta \otimes \text{Id}) \circ \rho(v_1 \ldots v_k) = (\text{Id} \otimes c) \circ (\rho \otimes \text{Id}) \circ \delta(v_1 \ldots v_k),$$

so $(T(V), \oplus, \Delta, \delta, \rho)$ is a double bialgebra over $V$. \hfill \Box

### 2.2 Universal property of quasishuffle bialgebras

Let us recall the definition of connectivity for bialgebras:

**Notations 2.1.**

1. Let $(B, m, \Delta)$ be a bialgebra, of unit $1_B$ and of counit $\varepsilon$. For any $x \in \text{Ker}(\varepsilon)$, we put

$$\tilde{\Delta}(x) = \Delta(x) - x \otimes 1 - 1 \otimes x.$$  

Then $\tilde{\Delta}$ is a coassociative coproduct on $\text{Ker}(\varepsilon)$. Its iterations will be denoted by $\tilde{\Delta}^{(n)} : \text{Ker}(\varepsilon) \to \text{Ker}(\varepsilon) \otimes (n+1)$, inductively defined by

$$\tilde{\Delta}^{(n)} = \begin{cases} 
\text{Id}_{\text{Ker}(\varepsilon)} & \text{if } n = 0, \\
(\tilde{\Delta}^{(n-1)} \otimes \text{Id}) \circ \tilde{\Delta} & \text{otherwise.}
\end{cases}$$

2. Recall that $(B, m, \Delta)$ is connected if

$$\text{Ker}(\varepsilon) = \bigcup_{n=0}^{\infty} \text{Ker}(\tilde{\Delta}^{(n)}).$$

3. If $(B, m, \Delta)$ is a connected bialgebra, for any $n \geq 0$ we put

$$B_{\leq n} = K1_B \oplus \text{Ker}(\tilde{\Delta}^{(n)}).$$

As $B$ is a connected, this is a filtration of $B$, known as the coradical filtration $[1, 15]$. Moreover, for any $n \geq 1$, because of the coassociativity of $\Delta$,

$$\text{Ker}(\varepsilon) \subseteq B_{\leq n-1}.$$  

In the case of bialgebras over a bialgebra $(V, \cdot, \delta_V)$, the connectedness is sometimes automatic:
Proposition 2.2. Let \((V, \cdot, \Delta)\) be a nonunitary bialgebra. For any \(n \geq 1\), we put
\[ V^n = \text{Vect}(v_1 \cdots v_n, v_1, \ldots, v_n \in V). \]

If \(\bigcap_{n \geq 1} V^n = (0)\), then any bialgebra over \(V\) is a connected bialgebra.

Proof. Let \((B, m, \Delta, \rho)\) be a bialgebra over \(V\) and let \(x \in \text{Ker}(\varepsilon_{\Delta})\). We put
\[ \rho(x) = \sum_{i=1}^{p} x_i \otimes v_i. \]

Let us denote by \(W\) the vector space generated by the elements \(v_i\). By definition, this is a finite-dimensional vector space and \(\rho(x) \in B \otimes W\). As \(W\) is finite-dimensional, the decreasing sequence of vector spaces \((W \cap V^n)_{n \geq 1}\) is stationary, so there exists \(N \geq 1\) such that if \(n \geq N\), \(W \cap V^n = W \cap V^N\). Therefore
\[ W \cap V^{-N} = W \cap \bigcap_{n \geq 1} V^n = (0). \]

Moreover,
\[ \left. m_{1,3,\ldots,2N-1,24\ldots,2N} \circ \rho^{\otimes N} \circ \tilde{\Delta}^{(N-1)}(x) \right|_{eB^{\otimes N} @ V^{-N}} = \left. (\tilde{\Delta}^{(N-1)} \otimes \text{Id}) \circ \rho(x) \right|_{eB^{\otimes N} @ W}. \]

As \(V^{-N} \cap W = (0)\), \((\tilde{\Delta}^{(N-1)} \otimes \text{Id}) \circ \rho(x) = 0\). Then
\[ (\text{Id}^{\otimes N} \otimes \varepsilon_V) \circ (\tilde{\Delta}^{(N-1)} \otimes \text{Id}) \circ \rho(x) = \tilde{\Delta}^{(N-1)}(x) = 0. \]

So \((B, m, \Delta)\) is connected. \(\square\)

Example 2.1. 1. If \((V, \cdot, \delta_V)\) is the bialgebra of the semigroup \((\mathbb{N}_{>0}, +)\), then \(\bigcap_{n \geq 1} V^{-n} = (0)\).

We recover the classical result that any \(\mathbb{N}\)-graded bialgebra \(B\) such that \(B_0 = \mathbb{K} 1_B\) is connected. This also works for algebras of semigroups \(\mathbb{N}^n \setminus \{0\}\), for example.

2. This does not hold if \(V\) is unitary, as then \(V^{-n} = V\) for any \(n \in \mathbb{N}\).

Theorem 2.3. Let \(V\) be a nonunitary, commutative bialgebra and let \((B, m, \Delta, \rho)\) be a connected bialgebra over \(V\). For any character \(\lambda\) of \(B\), there exists a unique morphism \(\phi\) from \((B, m, \Delta, \rho)\) to \((T(V), \cdot, \Delta, \rho)\) of bialgebras over \(V\) such that \(\varepsilon_{\delta} \circ \phi = \lambda\). Moreover, for any \(x \in \text{Ker}(\varepsilon_{\Delta})\),
\[ \phi(x) = \sum_{n=1}^{\infty} \left. (\lambda \otimes \text{Id}) \circ \rho \right|_{eV^{\otimes n}} \circ \tilde{\Delta}^{(n-1)}(x). \]  

(1)

Proof. Let us first prove that for any \(\lambda \in V^*\), such that \(\lambda(1_B) = 1\), there exists a unique coalgebra morphism \(\phi : (B, \Delta, \rho) \rightarrow (T(V), \Delta, \rho)\) of coalgebras over \(V\) such that \(\varepsilon_{\delta} \circ \phi = \lambda\).

Existence. Let \(\phi : B \rightarrow \text{QSh}(V)\) defined by \((1)\) and by \(\phi(1_B) = 1\). By connectivity of \(B\), \((1)\) makes perfectly sense. Let us prove that \(\phi\) is a coalgebra morphism. As \(\phi(1_B) = 1\), it is enough to prove that for any \(x \in \text{Ker}(\varepsilon_{\Delta})\), \(\Delta \circ \phi(x) = (\phi \otimes \phi) \circ \tilde{\Delta}(x)\). We shall use Sweedler’s notation
\( \tilde{\Delta}^{(n-1)}(x) = x^{(1)} \otimes \ldots \otimes x^{(n)}. \)

\[ \tilde{\Delta} \circ \phi(x) \]

\[ = \sum_{n=1}^{\infty} \lambda \left( x^{(1)}_0 \right) \ldots \lambda \left( x^{(n)}_0 \right) \tilde{\Delta} \left( x^{(1)}_1 \ldots x^{(n)}_1 \right) \]

\[ = \sum_{n=1}^{\infty} \sum_{i=1}^{n-1} \lambda \left( x^{(1)}_0 \right) \ldots \lambda \left( x^{(n)}_0 \right) x^{(1)}_1 \ldots x^{(i+1)}_1 \ldots x^{(n)}_1 \]

\[ = \sum_{i,j \geq 1} \lambda \left( x^{(1)}_0 \right) \ldots \lambda \left( x^{(i)}_0 \right) \lambda \left( x^{(j)}_0 \right) \ldots \lambda \left( x^{(j)}_1 \right) x^{(1)}_1 \ldots x^{(i)}_1 \otimes x^{(1)}_1 \ldots x^{(j)}_1 \]

\[ = (\phi \otimes \phi) \left( x^{(1)} \otimes x^{(2)} \right) \]

\[ = (\phi \otimes \phi) \circ \tilde{\Delta}(x). \]

Let us prove that \( \epsilon_\delta \circ \phi = \lambda \). If \( x = 1_B \), then \( \epsilon_\delta \circ \phi(1_B) = \epsilon_\delta(1) = 1 = \lambda(1_B) \). If \( x \in \text{Ker}(\varepsilon_\Delta) \), as \( \epsilon_\delta(V^\otimes n) = (0) \) for any \( n \geq 2 \),

\[ \epsilon_\delta \circ \phi(x) = \epsilon_\delta \circ (\lambda \otimes \text{Id}) \circ \rho \circ \tilde{\Delta}(0)(x) + 0 = \lambda(\text{Id} \otimes \epsilon_\delta) \circ \rho(x) = \lambda(x). \]

Let us prove that \( \phi \) is a comodule morphism. If \( x = 1_B \), then

\[ \rho \circ \phi(1_B) = 1 \otimes 1 = (\phi \otimes \text{Id})(1_B \otimes 1) = (\phi \otimes \text{Id}) \circ \rho(1_B). \]

Let us assume that \( x \in \text{Ker}(\varepsilon_\Delta) \).

\[ (\phi \otimes \text{Id}) \circ \rho(x) = \phi(x_0) \otimes x_1 \]

\[ = \sum_{n=1}^{\infty} \lambda \left( x^{(1)}_0 \right) \ldots \lambda \left( x^{(n)}_0 \right) x^{(1)}_0 \ldots x^{(n)}_0 \otimes x^{(1)}_1 \ldots x^{(n)}_1 \]

\[ = \sum_{n=1}^{\infty} \lambda \left( x^{(1)}_0 \right) \ldots \lambda \left( x^{(n)}_0 \right) x^{(1)}_1 \ldots x^{(1)}_1 \otimes x^{(1)}_1 \ldots x^{(n)}_1 \]

\[ = \sum_{n=1}^{\infty} \lambda \left( x^{(1)}_0 \right) \ldots \lambda \left( x^{(n)}_0 \right) \rho \left( x^{(1)}_1 \ldots x^{(n)}_1 \right) \]

\[ = \rho \circ \phi(x). \]

Uniqueness. Let \( \psi : (B, \Delta, \rho) \rightarrow (T(V), \Delta, \rho) \) such that \( \epsilon_\delta \circ \psi = \lambda \). As 1 is the unique group-like element of \( \text{QSh}(V) \), necessarily \( \psi(1_B) = 1 = \phi(1_B) \). It is now enough to prove that \( \psi(x) = \phi(x) \) for any \( x \in \text{Ker}(\varepsilon_\Delta) \). We assume that \( x \in B_{\leq n} \) and we proceed by induction on \( n \). If \( n = 0 \), there is nothing to prove. Let us assume that \( n \geq 1 \). As \( \tilde{\Delta}(x) \in B_{\leq n-1}^{\otimes 2} \), by the induction hypothesis,

\[ \tilde{\Delta} \circ \psi(x) = (\psi \otimes \psi) \circ \tilde{\Delta}(x) = (\phi \otimes \phi) \circ \tilde{\Delta}(x) = \tilde{\Delta} \circ \phi(x), \]

so \( \psi(x) - \phi(x) \in \text{Ker}(\tilde{\Delta}) = V \). We put \( \psi(x) - \phi(x) = v \in V \). Then

\[ v = (\epsilon_V \otimes \text{Id}) \circ \delta_V(v) \]

\[ = (\epsilon_\delta \otimes \text{Id}) \circ \rho(v) \]

\[ = (\epsilon_\delta \otimes \text{Id}) \circ \rho \circ \phi(x) - (\epsilon_\delta \otimes \text{Id}) \circ \rho \circ \psi(x) \]

\[ = (\epsilon_\delta \otimes \text{Id}) \circ (\phi \otimes \text{Id})(x) - (\epsilon_\delta \otimes \text{Id}) \circ (\psi \otimes \text{Id})(x) \]

\[ = (\lambda \otimes \text{Id})(x) - (\lambda \otimes \text{Id})(x) \]

\[ = 0. \]
So $\psi(x) = \phi(x)$.

Let us now consider a character $\lambda$. As $\lambda(1_B) = 1$, we already proved that there exists a unique coalgebra morphism $\phi : (B, \Delta, \rho) \rightarrow (T(V), \Delta, \rho)$ such that $\epsilon_\delta \circ \phi = \lambda$. Let us prove that it is an algebra morphism. We consider the two morphisms $\phi_1 = \psi \circ (\phi \otimes \phi)$ and $\phi_2 : \phi \circ m$, both from $B \otimes B$ to $\text{QSh}(V)$. As $\phi$, $\psi$ and $m$ are both comodule and coalgebra morphisms, $\phi_1$ and $\phi_2$ are comodule and coalgebra morphisms. Moreover, $B \otimes B$ is connected and, as $\epsilon_\delta$ is a character of $(T(V), \psi)$ and $\lambda$ is a character of $(B, m)$,

$$
\epsilon_\delta \circ \psi \circ (\phi \otimes \phi) = (\epsilon_\delta \otimes \epsilon_\delta) \circ (\phi \otimes \phi) = \lambda \otimes \lambda = \lambda \otimes m = \epsilon_\delta \circ \phi \circ m.
$$

So $\epsilon_\delta \circ \phi_1 = \epsilon_\delta \circ \phi_2$. By the uniqueness part, $\phi_1 = \phi_2$. \hfill \Box

**Lemma 2.4.** 1. The double bialgebras $\text{QSh}(\mathbb{K}) = (T(\mathbb{K}), \psi, \Delta)$ and $(\mathbb{K}[X], m, \Delta, \delta)$ are isomorphic, through the map

$$
H : \begin{cases}
\text{QSh}(\mathbb{K}) & \rightarrow \mathbb{K}[X] \\
\lambda_1 \ldots \lambda_n & \rightarrow \lambda_1 \ldots \lambda_n H_n(X),
\end{cases}
$$

where $H_n$ is the $n$-th Hilbert polynomial

$$
H_n(X) = \frac{X(X - 1)\ldots(X - n + 1)}{n!}.
$$

2. Let $V$ be a nonunitary, commutative and cocommutative bialgebra. The following map is a morphism of double bialgebras:

$$
H_V : \begin{cases}
\text{QSh}(V) & \rightarrow \mathbb{K}[X] \\
v_1 \ldots v_n & \rightarrow \epsilon_V(v_1)\ldots\epsilon(v_n)H_n(X).
\end{cases}
$$

**Proof.** 1. In order to simplify the reading of the proof, the element $1 \in \mathbb{K} \subseteq \text{QSh}(\mathbb{K})$ is denoted by $x$. We apply Theorem 2.3 with $B = \mathbb{K}[X]$, with its usual product $m$ and coproducts $\Delta$ and $\delta$, with the character $\epsilon_\delta$ of $\mathbb{K}[X]$, which sends any polynomial $P$ on $P(1)$. Let us denote by $\phi$ the following morphism. Then $\phi(X) = \epsilon_\delta(X)x = x$. By multiplicativity, for any $n \geq 1$,

$$
\phi(X^n) = x^{\epsilon_\delta n} = n!x^n + \text{a linear span of } x^k \text{ with } k < n.
$$

By triangularity, $\phi$ is an isomorphism. Let us denote by $H$ the inverse isomorphism, and let us prove that $H(x^n) = H_n(X)$ for any $n$ by induction on $n$. This obvious if $n = 0$ or 1. Let us assume that $n \geq 2$. Let us prove that for any $0 \leq k \leq n - 1$, $H(x^n)(k) = 0$ by induction on $k$. As $\epsilon_\Delta \circ H = \epsilon_\Delta,

$$
H(x^n)(0) = \epsilon_\Delta \circ H(x^n) = \epsilon_\Delta(x^n) = 0.
$$

If $k \geq 1$, as $H$ is a coalgebra morphism,

$$
H(x^n)(k) = H(x^n)(k - 1 + 1) = \Delta \circ H(x^n)(k - 1, 1) = (H \otimes H) \circ \Delta(x^n)(k - 1, k) = \sum_{l=0}^{n} H(x^l)(k - 1)H(x^{n-l})(1) = H(x^n)(k - 1) + \sum_{l=1}^{n-1} H_l(k - 1)H_{n-l}(1) + H(x^n)(1) = H(x^n)(1),
$$

13
by the induction hypotheses on \(k\) and \(n\). As \(\epsilon_\delta \circ \phi = \epsilon_\delta\), we obtain that \(\epsilon_\delta \circ H = \epsilon_\delta\),
\[
H(x^n)(1) = \epsilon_\delta \circ H(x^n) = \epsilon_\delta(x^n) = 0.
\]
Therefore, \(H(x^n)\) is a multiple of \(X(X-1)\ldots (X-n+1)\). By the triangularity of \(\phi\), we obtain that
\[
H(x^n) = \frac{X^n}{n!} + \text{terms of degree < } n.
\]
Consequently, \(H(x^n) = H_n(X)\).

2. The counit \(\epsilon_V : V \to \mathbb{K}\) is a bialgebra morphism. By functoriality, we obtain a double bialgebra morphism from \(\text{QSh}(V)\) to \(\text{QSh}(\mathbb{K})\). Composing with the isomorphism of the preceding item, we obtain \(H_V\).

As any bialgebra is trivially a bialgebra over \(\mathbb{K}\), we immediately obtain:

**Corollary 2.5.** Let \((B, m, \Delta)\) be a connected bialgebra and let \(\lambda\) be a character of \(B\). There exists a unique bialgebra morphism \(\phi : (B, m, \Delta) \to (\mathbb{K}[X], m, \Delta)\) such that for any \(x \in B\), \(\phi(x)(1) = \lambda(x)\). For any \(x \in \text{Ker}(\epsilon_\Delta)\),
\[
\phi(x) = \sum_{n=1}^{\infty} \lambda^{\otimes n} \circ \tilde{\Delta}^{(n-1)}(x)H_n(X).
\]

When \(V\) is the bialgebra of the semigroup \((\mathbb{N}_{>0}, +)\), we recover Aguiar, Bergeron and Sottile’s result \([2]\), with Proposition 2.2.

**Corollary 2.6.** Let \((B, m, \Delta)\) be a graded bialgebra with \(B_0 = \mathbb{K}1_B\) and let \(\lambda\) be a character of \(B\). There exists a unique bialgebra morphism \(\phi : (B, m, \Delta) \to (\text{QSym}, \shuffle, \Delta)\) such that \(\epsilon_\delta \circ \phi = \lambda\).

### 2.3 Double bialgebra morphisms

**Theorem 2.7.** Let \(V\) be a nonunitary, commutative and cocommutative bialgebra, and let \((B, m, \Delta, \delta, \rho)\) be a connected double bialgebra over \(V\). There exists a unique morphism \(\phi\) from \((B, m, \Delta, \delta, \rho)\) to \((T(V), \Delta, \delta, \rho)\) of double bialgebras over \(V\). For any \(x \in \text{Ker}(\epsilon_\Delta)\),
\[
\phi(x) = \sum_{n=1}^{\infty} ((\epsilon_\delta \otimes \text{Id}) \circ \rho)^\otimes n \circ \tilde{\Delta}^{(n-1)}(x).
\]

**Proof.** Uniqueness: such a morphism is a morphism \(\phi\) from \((B, m, \Delta, \rho)\) to \((B, m, \Delta, \rho)\) with \(\epsilon_\delta \circ \phi = \epsilon_\delta\). By Theorem 2.3, it is unique.

Existence: let \(\phi : (B, m, \Delta, \rho) \to (B, m, \Delta, \rho)\) be the (unique) morphism such that \(\epsilon_\delta \circ \phi = \epsilon_\delta\). Let us prove that for any \(x \in B_{\leq n}\), \(\delta \circ \phi(x) = (\phi \otimes \phi) \circ \delta(x)\) by induction on \(n\). If \(n = 0\), we can assume that \(x = 1_B\). Then
\[
\delta \circ \phi(1_B) = (\phi \otimes \phi) \circ \delta(1_B) = 1 \otimes 1.
\]

Let us assume the result at all ranks \(< n\), with \(n \geq 2\). Let \(x \in \text{Ker}(\epsilon_\Delta)\). As \((\epsilon_\Delta \otimes \text{Id}) \circ \delta(x) = \epsilon_\Delta(x)1, \delta(x) \in \text{Ker}(\epsilon_\Delta) \otimes B\).
\[
(\Delta \otimes \text{Id}) \circ \delta \circ \phi(x) = m_{1,3,24} \circ (\delta \otimes \delta) \circ \tilde{\Delta} \circ \phi(x)
\]
\[
= m_{1,3,24} \circ (\delta \otimes \delta) \circ (\phi \otimes \phi) \circ \tilde{\Delta}(x)
\]
\[
= m_{1,3,24} \circ (\phi \otimes \phi) \circ m_{1,3,24} \circ (\delta \otimes \delta) \circ \tilde{\Delta}(x)
\]
\[
= (\phi \otimes \phi) \circ m_{1,3,24} \circ (\delta \otimes \delta) \circ \tilde{\Delta}(x)
\]
\[
= (\phi \otimes \phi) \circ (\phi \otimes \phi) \circ \tilde{\Delta}(x).
\]
We used the induction hypothesis on the both sides of the tensors appearing in $\tilde{\Delta}(x)$ for the third equality. We deduce that $\delta \circ \phi(x) - (\phi \otimes \phi) \circ \delta(x) \in \text{Ker}(\Delta \otimes \text{Id}) = V \otimes T(V)$. Moreover,

$$(\text{Id} \otimes c) \circ (\rho \otimes \text{Id}) \circ \delta \circ \phi(x) = (\delta \otimes \text{Id}) \circ \rho \circ \phi(x)$$

and

$$(\text{Id} \otimes c) \circ \rho \circ (\delta \otimes \text{Id}) \circ \delta(x) = (\text{Id} \otimes c) \circ (\phi \otimes \phi) \circ (\rho \otimes \text{Id}) \circ \delta(x)$$

Moreover,

$$(\text{Id} \otimes c) \circ \rho \circ (\delta \otimes \text{Id}) \circ \phi(x) = (\delta \circ \phi(x) - (\phi \otimes \phi) \circ \delta) \otimes \text{Id}) \circ \rho(x).$$

Putting $y = \delta \circ \phi(x) - (\phi \otimes \phi) \circ \delta(x) \in V \otimes T(V)$, we proved that

$$(\text{Id} \otimes c) \circ \rho \circ \text{Id}(y) = ((\delta \circ \phi - (\phi \otimes \phi) \circ \delta) \otimes \text{Id}) \circ \rho(x).$$

As $y \in V \otimes T(V)$,

$$\rho \otimes \text{Id}(y) = \delta V \otimes \text{Id}(y).$$

Consequently,

$$(\epsilon_3 \otimes \text{Id} \otimes \text{Id}) \circ \rho \circ \text{Id}(y) = (\epsilon V \otimes \text{Id} \otimes \text{Id}) \circ \delta V \otimes \text{Id}(y) = y.$$

Moreover,

$$(\epsilon_3 \otimes \text{Id} \otimes \text{Id}) \circ \rho \circ \text{Id}(y) = (\epsilon_3 \otimes \text{Id} \otimes \text{Id}) \circ (\delta \circ \phi \otimes \text{Id}) \circ \rho(x)$$

and

$$(\epsilon_3 \otimes \text{Id} \otimes \text{Id}) \circ (\delta \circ \phi \otimes \text{Id}) \circ \rho(x) = (\delta \circ \phi \otimes \text{Id}) \circ \rho(x)$$

Moreover,

$$\epsilon_{\delta} \otimes \text{Id} \otimes \text{Id} \circ \rho \circ \text{Id}(y) = (\epsilon V \otimes \text{Id} \otimes \text{Id}) \circ \delta V \otimes \text{Id}(y) = y.$$

Hence, $y = 0$, so $\delta \circ \phi(x) = (\phi \otimes \phi) \circ \delta(x)$.

Applying to $V = \mathbb{K}$ or $V = \mathbb{K}(> 0, +)$:

**Corollary 2.8.** 1. Let $(B, m, \Delta)$ be a connected double bialgebra. There exists a unique double bialgebra morphism $\phi$ from $(B, m, \Delta, \delta)$ to $(\mathbb{K}[X], m, \Delta, \delta)$. For any $x \in \text{Ker}(\varepsilon_\Delta)$,

$$\phi(x) = \sum_{n=1}^{\infty} \epsilon_{\delta}^n \circ \tilde{\Delta}^{(n-1)}(x)H_n(X).$$

2. Let $(B, m, \Delta)$ be a graded, connected double bialgebra, such that for any $n \in \mathbb{N}$,

$$\delta(B_n) \subseteq B_n \otimes B.$$

There exists a unique homogeneous double bialgebra morphism $\phi$ from $(B, m, \Delta, \delta)$ to $(\text{QSym}, \circ, \Delta, \delta)$. For any $x \in \text{Ker}(\varepsilon_\Delta)$,

$$\phi(x) = \sum_{n=1}^{\infty} \sum_{k_1, \ldots, k_n \geq 1} \epsilon_{\delta}^n \circ (\pi_{k_1} \otimes \ldots \otimes \pi_{k_n}) \circ \tilde{\Delta}^{(n-1)}(x)(k_1, \ldots, k_n).$$

3. Let $\Omega$ be a commutative monoid and let $(B, m, \Delta)$ be a connected $\Omega$-graded double bialgebra, connected as a coalgebra, such that for any $\alpha \in \Omega$,

$$\delta(B_\alpha) \subseteq B_\alpha \otimes B.$$

There exists a unique homogeneous double bialgebra morphism $\phi$ from $(B, m, \Delta, \delta)$ to $(\text{QSh}(\mathbb{K}\Omega))$. For any $x \in \text{Ker}(\varepsilon_\Delta)$,

$$\phi(x) = \sum_{n=1}^{\infty} \sum_{\alpha_1, \ldots, \alpha_n \in \Omega} \epsilon_{\delta}^n \circ (\pi_{\alpha_1} \otimes \ldots \otimes \pi_{\alpha_n}) \circ \tilde{\Delta}^{(n-1)}(x)(\alpha_1, \ldots, \alpha_n).$$
We can give a generalization of Hoffman’s isomorphism between the shuffle and the quasi-shuffle algebras [10, 11].

**Theorem 2.9.** Let \((V, \cdot)\) be a nonunitary, commutative algebra. The following map is a Hopf algebra isomorphism:

\[
\Theta_V : \begin{cases}
(T(V), \shuffle, \Delta) & \longrightarrow (T(V), \shuffle, \Delta) \\
\Theta_V : w & \longrightarrow \frac{1}{\ell(w_1)! \cdots \ell(w_k)!} |w_1| \cdots |w_k|,
\end{cases}
\]

where for any word \(w\), \(|w|\) is the product in \(V\) of its letters and \(\ell(w)\) its length.

**Proof.** We first prove this result when \((V, \cdot, \delta_V)\) is a commutative, cocommutative, counitary bialgebra, of counit \(\epsilon_V\). First, observe that \((T(V), \shuffle, \Delta, \rho)\) is a bialgebra over \((V, \cdot, \delta_V)\) and that the following map is a character of \((T(V), \shuffle)\): for any word \(w = x_1 \cdots x_k\),

\[
\lambda(w) = \frac{1}{k!} \epsilon_V(x_1) \cdots \epsilon_V(x_k).
\]

By the universal property of the quasishuffle algebra, there exists a unique Hopf algebra morphism \(\Theta_V : (T(V), \shuffle, \Delta) \longrightarrow (T(V), \shuffle, \Delta)\) such that \(\epsilon \circ \Theta_V = \lambda\). For any word \(w = v_1 \cdots v_k\),

\[
(\lambda \otimes \text{Id}) \circ \rho(w) = \lambda(v'_1 \cdots v'_k) \, v''_1 \cdots v''_k
\]

\[
= \frac{1}{k!} \epsilon_V(v'_1) \cdots \epsilon_V(v'_k) \, v''_1 \cdots v''_k
\]

\[
= \frac{1}{k!} v'_1 \cdots v'_k
\]

\[
= \frac{1}{\ell(w)!} |w|.
\]

Hence,

\[
\Theta_V(w) = \sum_{k=1}^{\infty} \frac{((\lambda \otimes \text{Id}) \circ \rho)^{\otimes k} \circ \tilde{\Delta}^{(k-1)}(w)}{\ell(w_1)! \cdots \ell(w_k)!} |w_1| \cdots |w_k|.
\]

Let us now consider an commutative algebra \((V, \cdot)\). Let \((S(V), m, \Delta)\) be the symmetric algebra generated by \(V\), with its usual product and coproduct. Applying the first item to \((S(V), \cdot)\), we obtain a Hopf algebra morphism \(\Theta_{S(V)} : (T(S(V)), \shuffle, \Delta) \longrightarrow (T(S(V)), \shuffle, \Delta)\). By restriction, we obtain a Hopf algebra morphism \(\Theta_{S_+(V)} : (T(S_+(V)), \shuffle, \Delta) \longrightarrow (T(S_+(V)), \shuffle, \Delta)\). The canonical algebra morphism \(\pi : S_+(V) \longrightarrow V\), sending \(v_1 \cdots v_k\) to \(v_1 \cdots v_k\) (which exists as \(V\) is commutative), induces a surjective morphism \(\varpi : T(S_+(V)) \longrightarrow T(V)\), which is obviously a Hopf algebra morphism from \((T(S_+(V)), \shuffle, \Delta)\) to \((T(V), \shuffle, \Delta)\) and from \((T(S_+(V)), \shuffle, \Delta)\) to \((T(V), \shuffle, \Delta)\). Moreover, the following square is commutative:

\[
\begin{array}{ccc}
(T(S_+(V)), \shuffle, \Delta) & \xrightarrow{\Theta_{S_+(V)}} & (T(S_+(V)), \shuffle, \Delta) \\
\varpi \downarrow & & \varpi \downarrow \\
(T(V), \shuffle, \Delta) & \xrightarrow{\Theta_V} & (T(V), \shuffle, \Delta)
\end{array}
\]

As the vertical arrows are surjective Hopf algebra morphisms and the top horizontal arrow is also a Hopf algebra morphism, the bottom horizontal arrow is also a Hopf algebra morphism. For any word \(w\), \(\Theta_V(w) - w\) is a linear span of words of length \(< \ell(w)\). By a triangularity argument, \(\Theta_V\) is bijective.
2.4 Action on bialgebra morphisms

We here fix a bialgebra \((V, \cdot, \delta_V)\), nonunitary, commutative and cocommutative.

**Notations** 2.2.

1. Let \((B, m, \Delta)\) and \((B', m', \Delta')\) be bialgebras. We denote by \(M_{B \to B'}\) the set of bialgebra morphisms from \((B, m, \Delta)\) to \((B', m', \Delta')\).

2. Let \((B, m, \Delta, \rho)\) and \((B', m', \Delta', \rho')\) be bialgebras over \(V\). We denote by \(M_{B \to B'}^\rho\) the set of morphisms of bialgebra over \(V\) from \(B\) to \(B'\), that is to say morphisms, both of bialgebras and of comodules over \(V\).

**Proposition 2.10.** Let \((B, m, \Delta, \delta, \rho)\) be a double bialgebra over \(V\) and \((B', m', \Delta', \rho')\) be a bialgebra over \(V\). The following map is a right action of the monoid of characters \((\text{Char}(B), \ast)\) attached to \((B, m, \delta)\) on \(M_{B \to B'}^\rho\):

\[
\phi \mapsto \lambda = (\phi \otimes \lambda) \circ \delta.
\]

**Proof.** Let \((\phi, \lambda) \in M_{B \to B'}^\rho \times \text{Char}(B)\). Let us prove that \(\psi = (\phi \otimes \lambda) \circ \delta\) is a bialgebra morphism. As \(\phi, \lambda\) and \(\delta\) are algebra morphisms, by composition \(\psi\) is an algebra morphism.

\[
\Delta' \circ \psi = \Delta' \circ (\phi \otimes \lambda) \circ \delta
\]

\[
= (\phi \otimes \phi) \circ \Delta \circ (\text{Id} \otimes \lambda) \circ \delta
\]

\[
= (\phi \otimes \phi \otimes \lambda) \circ (\Delta \otimes \text{Id}) \circ \delta
\]

\[
= (\phi \otimes \phi \otimes \lambda) \circ m_{1,3,24} \circ (\delta \otimes \delta) \circ \Delta
\]

\[
= (\phi \otimes \phi \otimes \lambda \otimes \lambda) \circ (\text{Id} \otimes c \otimes \text{Id}) \circ (\delta \otimes \delta) \circ \Delta
\]

\[
= (\phi \otimes \lambda \otimes \phi \otimes \lambda) \circ (\delta \otimes \delta) \circ \Delta
\]

\[
= (\psi \otimes \psi) \circ \Delta.
\]

We used that \(\lambda\) is a character for the fifth equality. Moreover,

\[
\varepsilon_{\Delta}' \circ \Psi = (\varepsilon_{\Delta}' \otimes \lambda) \circ \delta = \lambda \circ \eta \circ \varepsilon_{\Delta} = \varepsilon_{\Delta},
\]

as \(\lambda(1_B) = 1\) so \(\lambda \circ \eta = \text{Id}_\mathbb{K}\). So \(\psi \in M_{B \to B'}\). Let us now prove that \(\psi\) is a comodule morphism. As \(\rho' \circ \phi = (\phi \otimes \text{Id}) \circ \rho\),

\[
\rho' \circ \psi = \rho' \circ (\phi \otimes \lambda) \circ \delta
\]

\[
= (\phi \otimes \text{Id} \otimes \lambda) \circ (\rho \otimes \text{Id}) \circ \delta
\]

\[
= (\phi \otimes \text{Id} \otimes \lambda) \circ (\text{Id} \otimes c) \circ (\delta \otimes \text{Id}) \circ \rho
\]

\[
= (\phi \otimes \lambda \otimes \text{Id}) \circ (\delta \otimes \text{Id}) \circ \rho
\]

\[
= (\psi \otimes \text{Id}) \circ \rho.
\]

So \(\psi \in M_{B \to B'}\).

Let \(\phi \in M_{B \to B'}^\rho\), \(\lambda, \mu \in \text{Char}(B)\).

\[
(\phi \otimes \lambda) \otimes \mu = (\phi \otimes \lambda \otimes \mu) \circ (\delta \otimes \text{Id}) \circ \delta
\]

\[
= (\phi \otimes \lambda \otimes \mu) \circ (\text{Id} \otimes \delta) \circ \delta
\]

\[
= (\phi \otimes \lambda \ast \mu) \circ \delta
\]

\[
= (\phi \otimes \lambda \ast \mu).
\]

Moreover,

\[
\phi \otimes \varepsilon_{\delta} = (\phi \otimes \varepsilon_{\delta}) \circ \delta = \phi.
\]

Therefore, \(\otimes\) is an action.
Moreover, any bialgebra morphism is compatible with these actions:

**Proposition 2.11.** Let \((B, m, \Delta, \delta, \rho)\) be a double bialgebra over \(V\) and \(B'\) and \(B''\) be bialgebras over \(V\). For any morphisms \(\phi : B \to B'\) and \(\psi : B' \to B''\) of bialgebras over \(V\), for any character \(\lambda\) of \(B\),

\[
(\psi \circ \phi) \leftrightarrow \lambda = \psi \circ (\phi \leftrightarrow \lambda).
\]

**Proof.** Indeed,

\[
(\psi \circ \phi) \leftrightarrow \lambda = ((\psi \circ \phi) \otimes \lambda) \circ \delta = \psi \circ (\phi \otimes \lambda) \circ \delta = \psi \circ (\phi \leftrightarrow \lambda).
\]

**Corollary 2.12.** Let \((B, m, \Delta, \delta, \rho)\) be a connected double bialgebra over \(V\) and let \(B'\) be the double bialgebra \((T(V), \shuffle, \Delta, \delta, \rho)\). Let us denote by \(\phi_1 : B \to B'\) the unique morphism of double bialgebras of Theorem 2.7. The following maps are bijections, inverse one from the other:

\[
\theta : \left\{ \begin{array}{c}
\text{Char}(B) \\
\lambda \\
\phi_1 \leftrightarrow \lambda,
\end{array} \right. \quad \theta' : \left\{ \begin{array}{c}
M_{B \to T(V)}^0 \\
\phi \\
\epsilon_\delta \circ \phi.
\end{array} \right.
\]

**Proof.** Let \(\phi \in M_{B \to T(V)}^0\). We put \(\phi' = \theta \circ \theta'\) and \(\lambda = \epsilon_\delta \circ \phi\). Then

\[
\epsilon_\delta \circ \phi' = \epsilon_\delta \circ (\phi_1 \leftrightarrow \lambda) = (\epsilon_\delta \circ \phi_1) \star \lambda = \epsilon_\delta \star \lambda = \lambda = \epsilon_\delta \circ \phi.
\]

By the uniqueness in Theorem 2.7, \(\phi = \phi'\).

Let \(\lambda \in \text{Char}(B)\) and let \(\lambda' = \theta' \circ \theta(\lambda)\). Then

\[
\lambda' = \epsilon_\delta \circ (\phi_1 \leftrightarrow \lambda) = (\epsilon_\delta \circ \phi_1 \otimes \lambda) \circ \delta = (\epsilon_\delta \otimes \lambda) \circ \delta = \epsilon_\delta \star \lambda = \lambda.
\]

So \(\theta\) and \(\theta'\) are bijections, inverse one from the other.

**Corollary 2.13.**

1. Let \((B, m, \Delta, \delta)\) be a connected double bialgebra. Let us denote by \(\phi_1\) the unique morphism of double bialgebras from \(B\) to \(\mathbb{K}[X]\) of Theorem 2.7. The following maps are bijections, inverse one from the other:

\[
\theta : \left\{ \begin{array}{c}
\text{Char}(B) \\
\lambda \\
\phi_1 \leftrightarrow \lambda,
\end{array} \right. \quad \theta' : \left\{ \begin{array}{c}
M_{B \to \mathbb{K}[X]}^0 \\
\phi \\
\epsilon_\delta \circ \phi.
\end{array} \right.
\]

2. Let \((B, m, \Delta, \delta)\) be a connected, graded double bialgebra such that for any \(n \in \mathbb{N}\),

\[
\delta(B_n) \subseteq B_n \otimes B.
\]

Let us denote by \(\phi_1\) the unique homogeneous morphism of double bialgebras from \(B\) to \(\text{QSym}\) of Theorem 2.7. We denote by \(M_{B \to \text{QSym}}^0\) the set of bialgebra morphisms from \((B, m, \Delta)\) to \((\text{QSym}, \shuffle, \Delta)\) which are homogeneous of degree 0. The following maps are bijections, inverse one from the other:

\[
\theta : \left\{ \begin{array}{c}
\text{Char}(B) \\
\lambda \\
\phi_1 \leftrightarrow \lambda,
\end{array} \right. \quad \theta' : \left\{ \begin{array}{c}
M_{B \to \text{QSym}}^0 \\
\phi \\
\epsilon_\delta \circ \phi.
\end{array} \right.
\]

3. Let \(\Omega\) be a commutative monoid and let \((B, m, \Delta, \delta)\) be a connected, \(\Omega\)-graded double bialgebra, connected as a coalgebra, such that for any \(\alpha \in \Omega\),

\[
\delta(B_\alpha) \subseteq B_\alpha \otimes B.
\]

Let us denote by \(\phi_1\) the unique homogeneous morphism of double bialgebras from \(B\) to \(\text{QSh}(\mathbb{K}\Omega)\) of Theorem 2.7. We denote by \(M_{B \to \text{QSh}(\mathbb{K}\Omega)}^0\) the set of bialgebra morphisms from \((B, m, \Delta)\) to \(\text{QSh}(\mathbb{K}\Omega)\) which are homogeneous of degree the unit of \(\Omega\). The following maps are bijections, inverse one from the other:

\[
\theta : \left\{ \begin{array}{c}
\text{Char}(B) \\
\lambda \\
\phi_1 \leftrightarrow \lambda,
\end{array} \right. \quad \theta' : \left\{ \begin{array}{c}
M_{B \to \text{QSh}(\mathbb{K}\Omega)}^0 \\
\phi \\
\epsilon_\delta \circ \phi.
\end{array} \right.
\]
2.5 Applications to graphs

We postpone the detailed construction of the double bialgebras of $V$-decorated graphs to a forthcoming paper [2]. For any nonunital commutative bialgebra $(V, \cdot, \delta_V)$, we obtain a double bialgebra over $V$ of $V$-decorated graphs $\mathcal{H}_V[G]$, generated by graphs $G$ which any vertex $v$ is decorated by an element $d_G(v)$, with conditions of linearity in each vertex. For example, if $v_1, v_2, v_3, v_4 \in V$ and $\lambda_2, \lambda_3 \in \mathbb{K}$, if $w_1 = v_1 + \lambda_2 v_2$ and $w_2 = v_3 + \lambda_4 v_4$,

$$I^{v_3}_{v_1} = I^{v_3}_{w_1} + \lambda_4 I^{v_3}_{v_2} + \lambda_2 I^{v_3}_{w_2} + \lambda_2 \lambda_4 I^{v_3}_{v_4}.$$

The product is given by the disjoint union of graphs, the decorations being untouched. For any graph $G$, for any $X \subseteq V(G)$, we denote by $G|_X$ the graph defined by

$$G|_X = X, \quad E(G|_X) = \{(x, y) \in E(G) \mid x, y \in X\}.$$

Then

$$\Delta(G) = \sum_{V(G)=A \cup B} G|_A \otimes G|_B,$$

the decorations being untouched. For any equivalence relation $\sim$ on $V(G)$:

- $G/\sim$ is the graph defined by

$$V(G/\sim) = V(G)/\sim, \quad E(G/\sim) = \{(\overline{x}, \overline{y}) \mid \{x, y\} \in E(G), \overline{z} \neq \overline{y}\},$$

where for any $z \in V(G)$, $z$ is its class in $V(G)/\sim$.

- $G |\sim$ is the graph defined by

$$V(G |\sim) = V(G), \quad E(G |\sim) = \{(x, y) \in E(G) \mid x \sim y\}.$$

- We shall say that $\sim \in \mathcal{E}[G]$ if for any equivalence class $X$ of $\sim$, $G|_X$ is connected.

With these notations, the second coproduct $\delta$ is given by

$$\delta(G) = \sum_{\sim \in \mathcal{E}[G]} G/\sim \otimes G |\sim.$$

Any vertex $w \in V(G/\sim) = V(G)/\sim$ is decorated by

$$\prod_{v \in w} d_G(v)^\prime,$$

where the symbol $\prod$ means that the product is taken in $V$ (recall that any vertex of $V(G/\sim)$ is a subset of $V(G)$). Any vertex $v \in V(G |\sim) = V(G)$ is decorated by $d_G(v)^\prime$. We use Sweedler’s notation $\delta_V(v) = v^\prime \otimes v^\prime\prime$, and it is implicit that in the expression of $\delta(G)$, everything is developed by multilinearity in the vertices. For example, if $v_1, v_2, v_3 \in V$,

$$\Delta(I^{v_3}_{v_1}) = I^{v_3}_{v_1} \otimes 1 + 1 \otimes I^{v_3}_{v_1} + I^{v_3}_{v_1} \otimes v_3 + I^{v_3}_{v_1} \otimes v_1 + v_1 \otimes v_3 \otimes v_2 + v_1 \otimes v_3 \otimes v_1 \otimes v_2 \otimes v_3,
\delta(I^{v_3}_{v_1}) = I^{v_3}_{v_1} \otimes v_3 \otimes v_2 \otimes v_3 + v_1 \otimes v_2 \otimes v_3 \otimes I^{v_3}_{v_1} + I^{v_3}_{v_1} \otimes v_2 \otimes v_3 \otimes v_2 \otimes v_3 \otimes v_3 + I^{v_3}_{v_1} \otimes v_2 \otimes v_3 \otimes v_3 \otimes v_2 \otimes v_3.$$

For any $V$-decorated graph,

$$\epsilon_\delta(G) = \begin{cases} \prod_{v \in V(G)} \epsilon_V(d_G(v)) \text{ if } E(G) = \emptyset, \\ 0 \text{ otherwise.} \end{cases}$$

19
Proposition 2.14. For any graph $G$, we denote by $\mathcal{C}(G)$ the set of packed valid colorations of $G$, that is to say surjective maps $c : V[G] \to [\max(f)]$ such that for any $\{x, y\} \in E(g)$, $c(x) \neq c(y)$. We denote by $\Phi$ the unique morphism of double bialgebras over $\mathcal{V}$ from $\mathcal{H}_V[G]$ to $\mathcal{QSh}(V)$. For any $V$-decorated graph $G$,

$$\Phi_1(G) = \sum_{c \in \mathcal{C}(G)} \left( \prod_{c(x) = 1} d_V(x), \ldots, \prod_{c(x) = \max(c)} d_V(x) \right),$$

where for any vertex $x \in V(G)$, $d_V(x) \in V$ is its decoration.

Proof. Let $G$ be a $V$-decorated graph. For any vertex $i$ of $G$, we denote by $v_i \in V$ the decoration of $i$. The number of vertices of $G$ is denoted by $n$.

$$\Phi_1(G) = \sum_{k=1}^n \epsilon_\delta(g_{|I_1}) \ldots \epsilon_\delta(g_{|I_k}) \left( \prod_{i \in I_1} v_i, \ldots, \prod_{i \in I_k} v_i \right)$$

$$= \sum_{k=1}^n \epsilon_\delta(g_{|c-1(1)}) \ldots \epsilon_\delta(g_{|c-1(k)}) \left( \prod_{c(x) = 1} d_V(x), \ldots, \prod_{c(x) = k} d_V(x) \right)$$

as for any surjective map $c : V[G] \to [\max(f)]$,

$$\epsilon_\delta(g_{|c-1(1)}) \ldots \epsilon_\delta(g_{|c-1(k)}) = \begin{cases} 1 & \text{if } c \in \mathcal{C}(G), \\ 0 & \text{otherwise}. \end{cases} \quad \square$$

Example 2.2. For any $v_1, v_2, v_3 \in V$,

$$\Phi_1(\vec{v}_1^2) = v_1 v_2 + v_2 v_1,$$

$$\Phi_1(\vec{v}_2^3) = v_1 v_2 v_3 + v_1 v_3 v_2 + v_2 v_1 v_3 + v_2 v_3 v_1 + v_3 v_1 v_2 + v_3 v_2 v_1 + (v_1 \cdot v_3) v_2 + v_2 (v_1 \cdot v_3),$$

$$\Phi_1(\vec{v}_3^3) = v_1 v_2 v_3 + v_1 v_3 v_2 + v_2 v_1 v_3 + v_2 v_3 v_1 + v_3 v_1 v_2 + v_3 v_2 v_1.$$

If $V = \mathbb{K}$, we obtain the double bialgebra morphism $\phi_{\text{chr}} : \mathbb{H}[G] \to \mathbb{K}[X]$, sending any graph on its chromatic polynomial. If $V$ is the algebra of the semigroup $(> 0, +)$, we obtain the morphism $\Phi_{\text{chr}} : \mathcal{H}_V[G] \to \mathcal{QSym}$, sending any graphs which vertices are decorated by positive integers to its chromatic (quasi)symmetric function $[13]$

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