Determination of the Influence of Reinforcement Direction of Open Thin-Walled Cylindrical Carbon Shells on Their Natural Vibrations

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Abstract. Nowadays, open cylindrical shells are widely used in structures in construction, aviation, energy, oil, and other industries. Shell structures are exposed to short-term cyclical influences in the course of operation; these influences cause forced vibrations of the structure which trigger the internal dynamic mechanisms, changing the natural vibrations of the structure, which has a significant impact on the strength characteristics of the shell. Carbon-fiber-reinforced plastics (CFRP) are used as an effective material for such structures. This is a durable and lightweight material. Due to its high cost, its application used to be efficient only in aircraft and in the space industry. Improvement of manufacturing techniques has made it possible to expand the range of CFRP applications. Due to their thinness, CFRP products are prone to vibrations. Vibrations can cause resonance condition, which can result in structural collapse.

1. Introduction

Nowadays, open cylindrical shells are widely used in structures in construction, aviation, energy, oil, and other industries. Shell structures are exposed to short-term cyclical influences in the course of operation; these influences cause forced vibrations of the structure which trigger the internal dynamic mechanisms, changing the natural vibrations of the structure, which has a significant impact on the strength characteristics of the shell. Carbon-fiber-reinforced plastics (CFRP) are used as an effective material for such structures. This is a durable and lightweight material. Due to its high cost, its application used to be efficient only in aircraft and in the space industry. Improvement of manufacturing techniques has made it possible to expand the range of CFRP applications. Due to their thinness, CFRP products are prone to vibrations. Vibrations can cause resonance condition, which can result in structural collapse.
2. Timeliness and scientific novelty

There have been numerous accidents in the history of construction. Errors in design, construction, and operation lead to deformations, structural failure, and human losses.

It should be noted that when designing such buildings, the load-bearing structures are designed taking into account large safety margins, while no analysis of vibrations or occurrence of the resonance condition is performed, which leads to tragedies. Therefore, it becomes necessary to analyze structures in terms of allowable variations of the amplitude and frequency of their vibrations and, based on this, create systems for preventing critical states of structures.

Hence, there is a need for further research on deformation, stability and strength characteristics. This line of research has not been studied in full but it is of fundamental importance.

In most cases, resolution of the issues regarding strength and stability is connected with the issues of the dynamics of shells and shell structures. Sometimes, the problem solution is connected specifically with the study of coupling of the shell vibration bending modes. Researchers have identified various vibration modes: bending and radial, torsional and longitudinal, bending and longitudinal vibrations, bending and shear modes, etc. It is also known that the lowest frequencies of shear, radial, and torsional vibrations are significantly exceeded by the lowest frequencies of bending vibrations of the shell; hence, various types of combined action of conjugate and non-conjugate vibration modes may be observed during operation of shells.

Change in the frequency characteristics of the shell oscillatory processes can lead to superposition of the natural and forced vibrations of shells. When certain natural frequency values are observed, the forced and natural vibrations combine, which results in the occurrence of resonance.

Experimental studies of shell bending vibrations demonstrate some patterns of their behavior which are not consistent with the existing research papers [1–4].

Therefore, it is necessary to determine the circumstances leading to contradictions in the mathematical model of open thin-walled carbon shell vibrations, in order to refine the mathematical model of the modern mechanics of shell vibrations. Applicability areas should be defined more precisely for various vibration theories. A new solution should be found in respect of open cylindrical shell vibrations, for various directions of fiber in carbon fabrics. Experimental studies, followed by numerical ones, should be conducted, in order to set all the parameters of fiber directions in carbon fabrics for open thin-walled cylindrical shell vibrations.

3. Problem formulation

In order to reduce the number of accidents, it is necessary to develop and experimentally validate a mathematical model describing the oscillatory behavior of an open shell for various reinforcement scenarios in case of multiple layers. Shell vibration frequencies should be analyzed within this problem, as well as their influence on the dynamic deflection shape for multilayered open thin-walled shell vibrations [5–7]. Priority will be given to studying such parameters as reinforcement directions in different layers.

4. Theoretical part

Let us consider the frequency characteristics of multilayered thin-walled shell vibrations.

The dynamic deflection can be written in the following form [8–11]:

\[
w(x, y, t) = e^{-k \cdot n \cdot y} \cdot (A(t) \sin \frac{ny}{R} + B(t) \cos \frac{ny}{R}) \cdot \sin \frac{\pi x}{l}
\]

The parentheses are removed:

\[
w(x, y, t) = e^{-k \cdot n \cdot y} \cdot (A(t) \sin \frac{ny}{R} \cdot \sin \frac{\pi x}{l} + B(t) \cos \frac{ny}{R} \cdot \sin \frac{\pi x}{l})
\]
Let us calculate the second partial derivative with respect to $x$ of the deflection function $w(x, y, t)$:

$$\frac{\partial w(x, y, t)}{\partial x} = e^{-\frac{kny}{R}} * (\frac{\pi}{l} A(t) \sin \frac{ny}{R} * \cos \frac{\pi x}{l} + \frac{\pi}{l} B(t) \cos \frac{ny}{R} * \cos \frac{\pi x}{l})$$

$$\frac{\partial^2 w(x, y, t)}{\partial x^2} = e^{-\frac{kny}{R}} * ((-\frac{\pi^2}{l^2}) A(t) \sin \frac{ny}{R} * \sin \frac{\pi x}{l} - (\frac{\pi^2}{l^2}) B(t) \cos \frac{ny}{R} * \sin \frac{\pi x}{l})$$

As a result, we arrive at the following differential equation:

$$\frac{1}{E} \nabla^4 \Phi = e^{-\frac{kny}{R}} * \left( \frac{n^4}{l^4} \left[ A(t) \sin \frac{ny}{R} * \sin \frac{\pi x}{l} + B(t) \cos \frac{ny}{R} * \sin \frac{\pi x}{l} \right] \right)$$

Next, we find a partial solution of the inhomogeneous equation obtained. We seek the solution as follows:

$$\Phi_{\text{I.H.}}(x, y, t) = e^{-\frac{kny}{R}} * (\Phi_1(t) \sin \frac{ny}{R} * \sin \frac{\pi x}{l} + \Phi_2(t) \cos \frac{ny}{R} * \sin \frac{\pi x}{l})$$

where $\Phi_1(t)$, $\Phi_2(t)$ are some functions of time.

$$\frac{\partial^4 \Phi_{\text{I.H.}}(x, y, t)}{\partial x^4} = e^{-\frac{kny}{R}} * \left( \frac{n^4}{l^4} \left[ \Phi_1(t) \sin \frac{ny}{R} * \sin \frac{\pi x}{l} + \Phi_2(t) \cos \frac{ny}{R} * \sin \frac{\pi x}{l} \right] \right)$$

$$\frac{\partial^4 \Phi_{\text{I.H.}}(x, y, t)}{\partial y^4} = e^{-\frac{kny}{R}} * \left( \frac{n^4}{l^4} \left[ \Phi_1(t) \sin \frac{ny}{R} * \sin \frac{\pi x}{l} + \Phi_2(t) \cos \frac{ny}{R} * \sin \frac{\pi x}{l} \right] \right)$$

Thus, we obtain the following equation:

$$\left[ \frac{\pi^2}{l^2} + 2 \frac{n^2}{lR} \right] e^{-\frac{kny}{R}} (\Phi_1(t) \sin \frac{ny}{R} * \sin \frac{\pi x}{l} + \Phi_2(t) \cos \frac{ny}{R} * \sin \frac{\pi x}{l}) = \frac{E}{R} \left( \frac{n^4}{l^4} \left[ \Phi_1(t) \sin \frac{ny}{R} * \sin \frac{\pi x}{l} + \Phi_2(t) \cos \frac{ny}{R} * \sin \frac{\pi x}{l} \right] \right)$$

The dependences follow from the equation obtained:

$$\left[ \frac{\pi^2}{l^2} + 2 \frac{n^2}{lR} \right] e^{-\frac{kny}{R}} \Phi_1(t) = \frac{E}{R} \left( \frac{n^4}{l^4} \right) A(t)$$

$$\left[ \frac{\pi^2}{l^2} + 2 \frac{n^2}{lR} \right] e^{-\frac{kny}{R}} \Phi_2(t) = \frac{E}{R} \left( \frac{n^4}{l^4} \right) B(t)$$

A few simple transformations yield the following:
\[ \Phi_1(t) = \frac{E}{R^2} \left( \frac{n^2}{l^2} \right) \sin \left( \frac{n}{l} y \right) \frac{1}{\sqrt{\left( \frac{n}{l} \right)^2 + \left( \frac{m}{R} \right)^2}} \left( e^{-\frac{k \pi n y}{R}} - e^{-\frac{k \pi m y}{R}} \right) A(t) , \]
\[ \Phi_2(t) = \frac{E}{R^2} \left( \frac{n^2}{l^2} \right) \sin \left( \frac{n}{l} y \right) \frac{1}{\sqrt{\left( \frac{n}{l} \right)^2 + \left( \frac{m}{R} \right)^2}} \left( e^{-\frac{k \pi n y}{R}} - e^{-\frac{k \pi m y}{R}} \right) B(t) , \]

We will use the obtained result later.

Next, we use the Bubnov–Galerkin method. As a result, the following equations are obtained:

\[ \int_0^{\frac{l}{2\pi}} \int_0^{\frac{b}{2\pi}} \int_{x_0}^{x_0+\varepsilon} \int_{y_0}^{y_0+\varepsilon} \frac{D}{h} \nabla^4 w - \frac{1}{R} \frac{\partial^2 \Phi}{\partial x^2} + \rho \frac{\partial^2 w}{\partial t^2} \cdot \left( e^{-\frac{k \pi n y}{R}} - e^{-\frac{k \pi m y}{R}} \right) \cos \left( \frac{n}{l} x \right) \sin \left( \frac{m}{R} y \right) \frac{dx \, d\theta}{\pi} = 0 ; \]
\[ \int_0^{\frac{l}{2\pi}} \int_0^{\frac{b}{2\pi}} \int_{x_0}^{x_0+\varepsilon} \int_{y_0}^{y_0+\varepsilon} \frac{M}{h} \frac{\partial^2 w(x_0, y_0, t)}{\partial t^2} \cdot \left( e^{-\frac{k \pi n y}{R}} - e^{-\frac{k \pi m y}{R}} \right) \sin \left( \frac{n}{l} x \right) \sin \left( \frac{m}{R} y \right) \frac{dx \, d\theta}{\pi} = 0 ; \]
\[ \int_0^{\frac{l}{2\pi}} \int_0^{\frac{b}{2\pi}} \int_{x_0}^{x_0+\varepsilon} \int_{y_0}^{y_0+\varepsilon} \frac{M}{h} \frac{\partial^2 w(x_0, y_0, t)}{\partial t^2} \cdot \left( e^{-\frac{k \pi n y}{R}} - e^{-\frac{k \pi m y}{R}} \right) \cos \left( \frac{n}{l} x \right) \frac{dx \, d\theta}{\pi} = 0 ; \]
\[ \int_0^{\frac{l}{2\pi}} \int_0^{\frac{b}{2\pi}} \int_{x_0}^{x_0+\varepsilon} \int_{y_0}^{y_0+\varepsilon} \frac{M}{h} \frac{\partial^2 w(x_0, y_0, t)}{\partial t^2} \cdot \left( e^{-\frac{k \pi n y}{R}} - e^{-\frac{k \pi m y}{R}} \right) \sin \left( \frac{n}{l} x \right) \frac{dx \, d\theta}{\pi} = 0 ; \]

Let us apply the deflection and stress function expressions

\[ w(x, y, t) = \frac{A(t)}{R} \sin \frac{n y}{R} \sin \frac{n x}{l} + B(t) \cos \frac{n y}{R} \sin \frac{n x}{l} , \]
\[ \Phi(x, y, t) = \frac{A(t)}{R} \sin \frac{n y}{R} \sin \frac{n x}{l} + 2 \Phi_2(t) \cos \frac{n y}{R} \sin \frac{n x}{l} + \frac{k y^2}{2} , \]

\[ \Phi_2(t) \] and \[ \Phi_1(t) \] have been obtained earlier. They should be applied after integration. Let \( \frac{k y^2}{2} \) be equal to zero.

in the above written equations (with \( k = \text{const} \)), but first let us calculate all the partial derivatives required:

\[ \frac{\partial^4 w(x, y, t)}{\partial x^4} = \frac{e^{-\frac{k \pi n y}{R}}}{R^4} \left( \frac{n^4}{l^4} \right) \sin \left( \frac{n y}{R} \sin \frac{n x}{l} \right) + B(t) \cos \frac{n y}{R} \sin \frac{n x}{l} , \]
\[ \frac{\partial^4 w(x, y, t)}{\partial y^4} = \frac{e^{-\frac{k \pi n y}{R}}}{R^4} \left( \frac{n^4}{l^4} \right) \sin \left( \frac{n y}{R} \sin \frac{n x}{l} \right) + B(t) \cos \frac{n y}{R} \sin \frac{n x}{l} , \]
\[ \frac{\partial^4 w(x, y, t)}{\partial x^2 \partial y^2} = \frac{e^{-\frac{k \pi n y}{R}}}{R^4} \left( \frac{n^4}{l^4} \right) \sin \left( \frac{n y}{R} \sin \frac{n x}{l} \right) + B(t) \cos \frac{n y}{R} \sin \frac{n x}{l} , \]
\[ \psi^4_w = \frac{\partial^4 w(x,y,t)}{\partial x^4} + 2 \frac{\partial^4 w(x,y,t)}{\partial x^2 \partial y^2} + \frac{\partial^4 w(x,y,t)}{\partial y^4} \]

Thus, we will have the following:

\[
\frac{D}{R} \int_0^l \int_0^{b \pi R} \left\{ \left( \frac{\pi}{l} \right)^2 + \left( \frac{n}{R} \right)^2 \right\} \left[ \frac{-k \pi n R^2}{l} \sin \left( \frac{ny}{R} \right) + B(t) \cos \left( \frac{ny}{R} \right) \sin \left( \frac{\pi x}{l} \right) \right] \, dx \, dy + \\
+ \frac{1}{R} \int_0^l \int_0^{b \pi R} \left\{ \frac{-k \pi n R^2}{l} \sin \left( \frac{ny}{R} \right) \sin \left( \frac{\pi x}{l} \right) + \Phi_2(t) \cos \left( \frac{ny}{R} \right) \sin \left( \frac{\pi x}{l} \right) \right\} \, dx \, dy + \\
+ \rho \int_0^l \int_0^{b \pi R} \left\{ \frac{-k \pi n R^2}{l} \sin \left( \frac{ny}{R} \right) \sin \left( \frac{\pi x}{l} \right) + B(t) \cos \left( \frac{ny}{R} \right) \sin \left( \frac{\pi x}{l} \right) \right\} \, dx \, dy + \\
+ \frac{M}{4h \pi^2} \int_0^{x_0+\varepsilon} \int_0^{y_0+\varepsilon} \left\{ \frac{-k \pi n R^2}{l} \sin \left( \frac{ny}{R} \right) + B(t) \cos \left( \frac{ny}{R} \right) \sin \left( \frac{\pi x}{l} \right) \right\} \, dx \, dy
\]

Next, let us proceed to calculate the limits:

\[
\lim_{\varepsilon \to 0} \left( \frac{M}{4h \pi^2} \cdot \frac{4lR}{n \pi n} \sin \left( \frac{\pi}{l} x_0 \right) \sin \left( \frac{\pi}{l} \varepsilon \right) \right) = \\
= \frac{MlR}{h \pi n} \sin \left( \frac{\pi}{l} x_0 \right) \cos \left( \frac{\pi n y_0}{R} \right) \lim_{\varepsilon \to 0} \left( \frac{\sin \left( \frac{\pi}{l} \varepsilon \right) \sin \left( \frac{\pi}{l} \varepsilon \right)}{\varepsilon^2} \right) = \\
= \frac{MlR}{h \pi n} \sin \left( \frac{\pi}{l} x_0 \right) \cos \left( \frac{\pi n y_0}{R} \right) \sin \left( \frac{\pi n y_0}{R} \right)
\]

\[
\lim_{\varepsilon \to 0} \left( \frac{M}{4h \pi^2} \cdot \frac{4lR}{n \pi n} \sin \left( \frac{\pi}{l} x_0 \right) \sin \left( \frac{\pi}{l} \varepsilon \right) \right) = \\
= \frac{MlR}{h \pi n} \sin \left( \frac{\pi}{l} x_0 \right) \sin \left( \frac{\pi n y_0}{R} \right) \lim_{\varepsilon \to 0} \left( \frac{\sin \left( \frac{\pi}{l} \varepsilon \right) \sin \left( \frac{\pi}{l} \varepsilon \right)}{\varepsilon^2} \right) = \\
= \frac{MlR}{h \pi n} \sin \left( \frac{\pi}{l} x_0 \right) \sin \left( \frac{\pi n y_0}{R} \right) \sin \left( \frac{\pi n y_0}{R} \right)
\]

\[
\lim_{\varepsilon \to 0} \left( \frac{M}{4h \pi^2} \cdot \frac{4lR}{n \pi} \sin \left( \frac{\pi}{l} x_0 \right) \sin \left( \frac{\pi}{l} \varepsilon \right) \right) = \\
= \frac{MlR}{h \pi} \sin \left( \frac{\pi}{l} x_0 \right) \lim_{\varepsilon \to 0} \left( \frac{\sin \left( \frac{\pi}{l} \varepsilon \right)}{\varepsilon} \right) = \\
= \frac{MlR}{h \pi} \sin \left( \frac{\pi}{l} x_0 \right) \left( \frac{\sin \left( \frac{\pi}{l} \varepsilon \right)}{\varepsilon} \right) = \frac{MlR}{h \pi} \sin \left( \frac{\pi}{l} x_0 \right).
\]
As a result, we obtain the following differential equation system:

\[
\begin{align*}
\omega^2 A(t) + \ddot{A}(t) + \frac{2M}{\rho \pi l Rh} [\kappa_1 \dot{A}(t) + \kappa_2 \dot{B}(t)] &= 0, \\
\omega^2 B(t) + \ddot{B}(t) + \frac{2M}{\rho \pi l Rh} [\kappa_2 \dot{A}(t) + \kappa_4 \dot{B}(t)] &= 0, \\
\frac{M}{\rho \pi l Rh} [\kappa_3 \dot{A}(t) + \kappa_5 \dot{B}(t)] &= 0.
\end{align*}
\]

Simple transformations of the above obtained equation yield the second equation:

\[
2.12 \left(\frac{\lambda}{\omega}\right)^4 - \left(2.08 \frac{p^2}{\omega^2} + 2.04\right) \left(\frac{\lambda}{\omega}\right)^2 + 2 \frac{p^2}{\omega^2} = 0,
\]

where

\[
\frac{p^2}{\omega^2} = \frac{\frac{1}{\rho} \left[ \frac{D}{h} \left( \frac{\pi}{T} \right)^2 + \frac{E}{R^2} \right]}{1 + \frac{\frac{1}{\rho} \left[ \frac{D}{h} \left( \frac{\pi}{T} \right)^2 + \frac{E}{R^2} \left( \frac{\pi}{T} \right)^2 \right]}{\frac{DR^2}{hE} \left( \frac{\pi}{T} \right)^4 + 1}}.
\]

Figure 1. Theoretical Values of Vibration Frequencies.

5. Practical relevance, results obtained in experimental studies

An experiment was designed in order to validate the experimental results. An experiment program was developed in the course of experimental studies, shells made of carbon fabric TWILL 2/2 3K-1000-240 \( \rho = 240 \, \text{g/m}^2 \) with different directions of reinforcement. As a result of these studies, a new, refined mathematical model was obtained for open shell vibrations (for different directions of fiber in carbon fabrics) which is associated with the experimental studies.

Solving the problem of development of a mathematical model for calculating vibrations of a reinforced open shell will make it possible to proceed with the development of software which forms a
constituent part of integrated safety systems (systems for online monitoring of structural health of load-bearing structures) of building structures, aircraft, etc.

This study corresponds to the world level of research [12–14], since the results obtained in the course of generation of experimental data will contribute to the reduction in the manufacturing cost of airframes and water vehicle hulls, building structures, buildings, and structures [15]. These products will have high competitive advantages in the Russian and world markets.

Figure 2. Design of Experiments.

Figure 3. Experimental Studies of CFRP Shell Vibrations.
Figure 4. Relationship Between Vibration Frequency and Number of Half-Waves
1 — Vibration frequency when the carbon fabric fibers are directed across the shell; 2 — Theoretical data for vibration frequencies when the carbon fabric fibers are directed across the shell; 3 — Vibration frequency when the carbon fabric fibers are directed at an angle of 45 degrees across the shell; 4 — Vibration frequency when the carbon fabric fibers are directed along the shell;

Experimental results are close to the theoretical ones, which indicates reliability of the studies conducted. Vibration frequency increases with changes in the shell reinforcement direction.

6. Conclusions (summary)
Initially, the direction of open thin-walled CFRP shell reinforcement has a substantial influence on the frequency characteristics of the oscillatory process; however, this influence decreases with the increase in the number of half-waves. The mathematical model of composite shell vibrations for different directions has been refined and validated in the course of the studies.

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