REGULARITY OF HYPERBOLIC MAGNETIC SCHRÖDINGER EQUATION WITH OSCILLATING COEFFICIENTS

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Abstract. This paper mainly discuss the regularity behavior of the hyperbolic magnetic Schrödinger equation with singular coefficients near the origin. We apply the techniques from the microlocal analysis to explore the upper bound of loss of regularity. Furthermore, in order to demonstrate the optimality of the result, a delicate counterexample with periodic coefficients will be constructed to show the lower bound of loss of regularity by the application of harmonic analysis and instability arguments.

1. INTRODUCTION TO HYPERBOLIC MAGNETIC SCHRÖDINGER EQUATION AND ITS REGULARITY BEHAVIOR

Mathematically speaking, the magnetic field $B$ is a solenoidal vector field whose field line either forms a closed curve or extends to infinity. In contrast, a field line of the electric field $E$ starts at a positive charge and ends at a negative charge. For instance, the earth’s magnetic field is a consequence of the movement of convection currents in the outer ferromagnetic liquid of the core. In the study of quantum mechanics, a magnetic field is produced by electric fields varying in time, spinning of the elementary particles, or moving electric charges, etc. Nowadays, electromagnetic theory is widely utilized in medical research of organs’ biomagnetism, vortex study in the superconductor which carries quantized magnetic flux, and geographical cataclysms forecasts, such as earthquakes, volcanic eruptions, geomagnetic reversal, etc. [8, 9, 10].

Let $A$ be the vector potential of $B$, which does not depend on time, $B = \nabla \times A$. Clearly, $\nabla \cdot B = \text{div} \text{rot} A = 0$. We deduce from the Maxwell’s equation ($\mu$ is the magnetic permeability) $\nabla \times E = -\mu \partial B/\partial t = 0$ that $E = -\nabla \phi$, where the scalar $\phi$ represents the electric potential. Next we choose an appropriate Lagrangian for the charged particle in the electromagnetic field ($q$ is the electric charge of the particle, and
v is its velocity, m is mass). \( L^e = m\dot{v}^2/2 - q\phi + qv \cdot A \). The canonical momentum is specified by the equation \( p = \nabla_v L^e = mv + qA \). Then we define the classical Hamiltonian by Legendre transform, \( \mathcal{H} := p \cdot v - L^e = (p - qA)^2/(2m) + q\phi \). In quantum mechanics, we replace \( p \) by \( -i\hbar \nabla \) (\( \hbar \) is the Planck constant), \( \mathcal{H} = (i\hbar \nabla + qA)^2/(2m) + q\phi \). This Hamiltonian operator phenomenologically describes a quantity of behaviors discovered in superconductors and quantum electrodynamics (QED). Ginzburg-Landau equations, Schrödinger equations, Dirac equations and the matrix Pauli operator are famous examples in these respects [11, 14, 16].

To be more specific, our mathematical model arises from the discussion of the extremum of the following variational problem \( I[u] \) in the study of quantum mechanics [13, 17, 18], \( u \in \mathcal{U} \) will be explained in Section 2,

\[
I[u] := \frac{1}{2} \int_\Omega \left( |u_t|^2 - b^2(t)|i\nabla + A(x)|u|^2 - \phi(x)|u|^2 \right) dx.
\]

Let the Lagrangian be

\[
\mathcal{L}(t,x_1, \cdots, x_n, u, \bar{u}, u_t, \bar{u}_t, \nabla u, \nabla \bar{u}) \triangleq |u_t|^2 - b^2(t)|(i\nabla + A(x))u|^2 - \phi(x)|u|^2.
\]

The Euler-Lagrangian equation for \( \mathcal{L} \) is of the form

\[
\frac{\partial \mathcal{L}}{\partial u} - \frac{\partial}{\partial t} \left( \frac{\partial \mathcal{L}}{\partial u_t} \right) - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left( \frac{\partial \mathcal{L}}{\partial u_{x_i}} \right) = 0.
\]

In fact, simple calculation leads to

\[
\frac{\partial \mathcal{L}}{\partial u} = b^2(t) \left( iA \cdot \nabla \bar{u} - A^2 \bar{u} - \phi(x) \bar{u} \right), \quad \frac{\partial \mathcal{L}}{\partial u_t} = \bar{u}_t, \quad \frac{\partial \mathcal{L}}{\partial u_{x_i}} = b^2(t) \left( - \bar{u}_x - ia_i \bar{u} \right).
\]

Consequently, one has

\[
\dot{u}_t + b^2(t)(i\nabla + A)^2u + \phi(x)u = 0.
\]

As is known, loss of regularity is an essential topic when we study the well-posedness of partial differential equations. For instance, there is no loss for Cauchy problem of the classical wave operator and Klein-Gordon operator. In other words, for a sufficiently large Sobolev index \( s \), when \( (u_0, u_1) \in H^{s+1} \times H^s \) are given, then there exists a unique solution \( u \) belonging to the following function spaces: \( C([0, \infty); H^{s+1}) \cap C^1([0, \infty); H^s) \).

For the classical wave equation with variable coefficients, [3, 12, 15] introduced a classification of regularity behaviors incurred by the singular coefficients. As for the difference of regularity for initial Cauchy data, [5] considered the typical \( p \)-evolution model in 1-Dimension. In effect, the principal operator determined the difference \( p \) when its coefficient is Log-Lipschitz continuous with respect to the time [2, 3, 7].

Let \( \Omega \subset \mathbb{R}^N \) be a bounded open set with a time-independent vector potential \( A \in (C^1(\Omega))^N \), its boundary \( \Gamma \subset C^2 \). Assume that \( \mathcal{H}(\Omega) \) is a Hilbert space. From the Hamiltonian, we can define the corresponding vector operator

\[
\mathcal{H}_A := i\nabla + A(x) ; \mathcal{H}(\Omega) \rightarrow (\mathcal{H}(\Omega))^N.
\]
The concerned function spaces will be detailed in the next section. In this work, we address the regularity behavior of the hyperbolic magnetic Schrödinger equation without influence from the electric field $E$.

\begin{equation}
\begin{cases}
    u_{tt} + b^2(t)\mathcal{A}^2 u = 0 & (t, x) \in (0, T) \times \Omega \\
    u = 0 & (t, x) \in (0, T) \times \Gamma \\
    u(0, x) = u_0(x), u_t(0, x) = u_1(x) & x \in \Omega.
\end{cases}
\end{equation}

And the time-dependent oscillating coefficient $b$ satisfies the following assumptions:

- **(Assumption I):** $C_1 \leq \inf_{t \in [0, T]} b(t) \leq \sup_{t \in [0, T]} b(t) \leq C_2$, $C_1, C_2 > 0$;

- **(Assumption II):** $b \in C^2(0, T]$, \( \left| \frac{d^k b(t)}{dt^k} \right| \leq C_3 (\nu(t)/t)^k \) uniformly for $k = 1, 2$, where $\nu \in C(0, T]$ is a positive, strictly decreasing function satisfying $\inf_{t \in [0, T]} \nu(t) \geq C_4 > 0$.

Under the above assumptions, we are ready to give the main result concerned with the regularity of the Cauchy problem (1).

**Theorem 1.1.** Define $\mu(t) = t/\nu(t)$. For the Sobolev index $s \geq 1$, if the oscillating coefficient $b$ satisfies Assumptions I and II, $(u_0, u_1) \in H^s \times H^{s-1}$, then, there exists a unique solution $u$ belonging to the following function spaces:

\[ u \in C\left([0, T]; \exp \left( c_1 \nu^{-1}(2^P/\sqrt{(i\nabla + A(x))^2}) \right) H^s \right), \]

\[ u_t \in C\left([0, T]; \exp \left( c_1 \nu^{-1}(2^P/\sqrt{(i\nabla + A(x))^2}) \right) H^{s-1} \right), \]

where $P$ is a fixed appropriate positive integer and $c_1$ is a positive constant. Moreover, $\mu^{-1}$ denotes the inverse function of $\mu$.

**Remark 1.2.** It is clear that, the oscillating coefficient incurs loss of regularity. Actually, the above theorem also holds when we replace $2^P$ by a general sufficiently large positive number. One chooses $2^P$ since the contrast is very significant while comparing with the coefficient sequences constructed in the counterexample in Section 4. Moreover, it is worth noticing that if $\nu(t) = (\log(1/t))^\gamma$, $\gamma \in (0, 1)$, increasing $P$ extends the Sobolev spaces.

**Remark 1.3.** The function $\mu^{-1}$ is uniquely determined since $\mu$ is a strictly increasing function in $(0, T]$. Particularly, when $\nu(t) \leq C$, there is no loss of regularity. This theorem shows explicitly the so-called (at most) $\nu$-loss of regularity which arises from the singular coefficient near the origin.

**Remark 1.4.** We give some typical examples to explain the different influence from various kinds of oscillating coefficients. Suppose that $(u_0, u_1) \in H^s \times H^{s-1}$, then according to Theorem 1.1, there exists a unique solution $u$ belonging to the following function spaces $(\alpha, \beta, \gamma > 0)$:

1. $\nu(t) \sim 1$, no loss of regularity,

\[ u \in C([0, T]; H^s), \quad u_t \in C([0, T]; H^{s-1}). \]
(2) \( \nu(t) = \log(1/t), \) finite loss of regularity,
\[ u \in C([0,T]; H^{s-\alpha}), \quad u_t \in C([0,T]; H^{s-\alpha-1}). \]

(3) \( \nu(t) = \left( \log(1/t) \right)^{\gamma}, \quad \gamma \in (0,1), \) arbitrarily small loss of regularity,
\[ u \in C\left([0,T]; (\sqrt{(i\nabla + A(x))^2})^\beta \left( \log(\sqrt{(i\nabla + A(x))^2/2}) \right)^{-1} H^s \right), \]
\[ u_t \in C\left([0,T]; (\sqrt{(i\nabla + A(x))^2})^\beta \left( \log(\sqrt{(i\nabla + A(x))^2/2}) \right)^{-1} H^{s-1} \right). \]

(4) \( \nu(t) = \left( \log(1/t) \right) \left( \log[2](1/t) \right)^{\gamma_2} \cdots \left( \log[n](1/t) \right)^{\gamma_n}, \quad \gamma_i \in (0,1), \quad i = 2, \cdots, n, \) infinite loss of regularity,
\[ u \in C\left([0,T]; (\sqrt{(i\nabla + A(x))^2})^\gamma \left( \log[2](\sqrt{(i\nabla + A(x))^2/2}) \right)^{\gamma_2} \cdots \left( \log[n](\sqrt{(i\nabla + A(x))^2/2}) \right)^{\gamma_n} H^s \right), \]
\[ u_t \in C\left([0,T]; (\sqrt{(i\nabla + A(x))^2})^\gamma \left( \log[2](\sqrt{(i\nabla + A(x))^2/2}) \right)^{\gamma_2} \cdots \left( \log[n](\sqrt{(i\nabla + A(x))^2/2}) \right)^{\gamma_n} H^{s-1} \right). \]

The rest of the paper is organized as follows. Section 2 is devoted to the description of \( \mathcal{H}_A^2 \)-induced function spaces \( \mathcal{H}_0^1, \mathcal{H}^{-1} \) and introduction of \( \mathcal{H}_A^2 \)-pseudodifferential operators. And the usual compactness-uniqueness argument is applied to demonstrate a generalized Poincaré’s inequality. In Section 3, we use some powerful tools, such as symbol calculus, normal diagonalisation, etc. from micro-local analysis and instability arguments to obtain the precise \( \nu \)-loss of regularity. In Section 4, we discuss the optimality of the loss of regularity by the application of harmonic analysis and instability arguments.

2. Prerequisites: Basic Functional Spaces and Pseudodifferential Operators

2.1. \( \mathcal{H}_A^2 \)-induced Hilbert spaces. Now we give a function space induced by the vector operator \( \mathcal{H}_A \).

Definition 2.1. Let \( A \in (L^\infty(\Omega))^N \), and we define a complex function space
\[ \mathcal{H}^1(\Omega) := \{ \omega : \omega \in L^2(\Omega), \mathcal{H}_A \omega \in (L^2(\Omega))^N \}, \]
which is equipped with the norm
\[ \|\omega\|_{\mathcal{H}^1} := \sqrt{\|\omega\|_{L^2}^2 + \|\mathcal{H}_A \omega\|_{(L^2)^N}^2}, \]
where
\[ \|(\omega_1, \cdots, \omega_N)\|_{(L^2)^N} := \sqrt{\sum_{\ell=1}^N \|\omega_{\ell}\|_{L^2}^2}. \]

One defines \( \mathcal{H}_0^1 \) as the closure of \( \mathcal{D}(\Omega) \) in \( \mathcal{H}^1 \), and \( \mathcal{H}^{-1} \) as the dual space of \( \mathcal{H}_0^1 \).

Lemma 2.2. Actually, \( \mathcal{H}^1 \) is an equivalent definition of the Sobolev space \( H^1 \). Consequently, the imbeddings \( \mathcal{H}_0^1 \hookrightarrow L^2 \) and \( L^2 \hookrightarrow \mathcal{H}^{-1} \) are both dense and compact.
Proof. Indeed, utilizing the definition of norm in each space, one has

- $\mathcal{H}^1 \hookrightarrow H^1$
  \[
  \|\omega\|^2_{\mathcal{H}^1} = \|\omega\|^2_{L^2} + \|\nabla \omega\|^2_{(L^2)^N} = \|\omega\|^2_{L^2} + \|\mathcal{H}_A \omega - A \omega\|^2_{(L^2)^N} 
  \leq \|\omega\|^2_{L^2} + 2\|\mathcal{H}_A \omega\|^2_{(L^2)^N} + 2\|A \omega\|^2_{(L^2)^N} 
  \leq (1 + 2N\|A\|^2_{L^\infty})\|\omega\|^2_{L^2} + 2\|\mathcal{H}_A \omega\|^2_{(L^2)^N}.
  \]

- $H^1 \hookrightarrow \mathcal{H}^1$
  \[
  \|\omega\|^2_{H^1} = \|\omega\|^2_{L^2} + \|\mathcal{H}_A \omega\|^2_{(L^2)^N} = \|\omega\|^2_{L^2} + \|i\nabla + A(x)\omega\|^2_{(L^2)^N} 
  \leq \|\omega\|^2_{L^2} + 2\|\nabla \omega\|^2_{(L^2)^N} + 2\|A(x)\omega\|^2_{(L^2)^N} 
  \leq (1 + 2N\|A\|^2_{L^\infty})\|\omega\|^2_{L^2} + 2\|\nabla \omega\|^2_{(L^2)^N}.
  \]

To introduce the pseudodifferential operators, first, we give a series of generalized Green’s formulas for the second order operator $(i\mathcal{H}_A)^2$ on $H^2$.

Lemma 2.3. For $u, v \in H^2$, $\Gamma \in C^2$, one has

\[
\int_{\Omega} (i\mathcal{H}_A)^2u \bar{v} dx = \int_{\Omega} \frac{\partial u}{\partial \nu_{i\mathcal{H}_A}} \cdot \bar{v} d\Gamma - \int_{\Omega} i\mathcal{H}_A u \cdot i\mathcal{H}_A \bar{v} dx,
\]

\[
\int_{\Omega} (i\mathcal{H}_A)^2u \bar{v} dx - \int_{\Omega} u (i\mathcal{H}_A)^2 \bar{v} dx = \int_{\Omega} \frac{\partial u}{\partial \nu_{i\mathcal{H}_A}} \cdot \bar{v} d\Gamma - \int_{\Gamma} u \cdot \frac{\partial v}{\partial \nu_{i\mathcal{H}_A}} d\Gamma,
\]

\[
\int_{\Omega} (i\mathcal{H}_A)^2u dx = \int_{\Gamma} \frac{\partial u}{\partial \nu_{i\mathcal{H}_A}} d\Gamma - \int_{\Omega} A(x) \cdot \mathcal{H}_A u dx,
\]

where

\[
(2) \quad \frac{\partial}{\partial \nu_{i\mathcal{H}_A}} := (\nabla - iA) \cdot \nu = \frac{\partial}{\partial \nu} - iA(x) \cdot \nu,
\]

and $\nu$ is the unit outward normal.

Proof. Keep in mind the classical trace theory in [1]. If $\Omega$ is bounded and $\Gamma \in C^2$, then $\mathcal{D}(\Omega)$ is dense in $H^2$. And the trace mapping $v \mapsto \vec{\gamma} v = (\gamma_0 v, \gamma_1 v) = (v|_{\Gamma}, \frac{\partial v}{\partial \nu}|_{\Gamma})$ from
$H^2(\Omega)$ to $H^{1/2}(\Gamma) \times H^{1/2}(\Gamma)$ is linear and continuous. So we prove these identities on $\mathcal{D}(\Omega)$. On the one hand,

$$\int_{\Omega} i\mathcal{H}_A u \cdot i\mathcal{H}_A v dx = \int_{\Omega} \nabla u \cdot \nabla \bar{v} dx - \int_{\Omega} i\mathcal{A}(x) u \cdot \nabla \bar{v} dx + \int_{\Omega} i\mathcal{A}(x) u \cdot \nabla \bar{v} dx + \int_{\Omega} \mathcal{A}^T u \bar{v} dx + \int_{\Omega} \bar{v} \cdot (A \cdot \nu) d\Gamma + \langle u, i\nabla \cdot A \bar{v} \rangle_{L^2} + \int_{\Omega} \mathcal{A}^T u \bar{v} dx.$$ 

On the other hand,

$$\int_{\Omega} (i\mathcal{H}_A)^2 u \bar{v} dx = \int_{\Gamma} \frac{\partial u}{\partial \nu} \cdot \bar{v} d\Gamma - 2 \int_{\Gamma} i\mathcal{A}(x) \cdot \nu d\Gamma - \int_{\Omega} \nabla u \cdot \nabla \bar{v} dx - \langle u, i\mathcal{A} \cdot \nabla \bar{v} \rangle_{L^2} - \langle u, i\nabla \cdot A \bar{v} \rangle_{L^2} - \int_{\Omega} \mathcal{A}^T u \bar{v} dx.$$ 

Notice the definition (2), and one concludes the proof of the first identity. The second identity follows immediately when we consider the conjugate of the first identity. Finally, the third identity is the special case of $v \equiv 1$ of the first identity. □

**Remark 2.4.** When $A \equiv 0$, one has the classical Green’s formulas for Laplacian $\Delta$. From the above lemma, we know that $\mathcal{H}_A^2$ is a self-adjoint differential operator on $H^2 \cap H^1_0$. In this case, (2) becomes the usual unit outward normal derivative.

In the following, one introduces a considerably significant result, which plays a crucial part in the description of the dual of $H^1_0$.

**Lemma 2.5.** (Generalized Poincaré’s inequality) Let $\Gamma \in C^1$. Then for any $\omega \in H^1_0$, there is a constant $C(\Omega) > 0$ such that

$$\|\omega\|_{L^2} \leq C(\Omega) \|(i\nabla + A) \omega\|_{(L^2)^N}.$$ 

**Proof.** Assume that

$$\Omega \subset [x_{01}, x_{11}] \times [x_{02}, x_{12}] \times \cdots \times [x_{0N}, x_{1N}].$$ 

Let us define a semi-norm on $H^1$, i.e.

$$|\omega|_{\mathcal{H}_1} := \|(i\nabla + A) \omega\|_{(L^2)^N} = \sqrt{\sum_{j=1}^{N} \|i \frac{\partial \omega}{\partial x_j} + a_j(x) \omega\|_{L^2}^2}.$$ 

First we prove that the above semi-norm is actually a norm on $H^1_0$. In fact, let $\omega$ be a function from $H^1_0(\Omega)$ such that $|\omega|_{\mathcal{H}_1} = 0$. Then one has a system of ordinary differential equations in $\Omega$,

$$\forall j = 1, \cdots, N, \quad i \frac{\partial \omega}{\partial x_j} + a_j(x) \omega = 0.$$
Let $\nabla_j$ denote $\frac{\partial}{\partial x_j}$. Here we introduce an important matrix, i.e. the compatibility matrix $\Xi_A$,

$$
\Xi_A := \begin{pmatrix}
\xi_{11} & \xi_{12} & \cdots & \xi_{1N} \\
\xi_{21} & \xi_{22} & \cdots & \xi_{2N} \\
\vdots & \vdots & \ddots & \vdots \\
\xi_{N1} & \xi_{N2} & \cdots & \xi_{NN}
\end{pmatrix}
$$

where

$$
\xi_{jk} := \left| \begin{array}{c}
\nabla_j \\
\nabla_k \\
a_j \\
a_k
\end{array} \right|.
$$

Clearly, $\Xi_A$ is an antisymmetric matrix. In quantum mechanics, $\Xi_A \equiv 0$ stands for the case without magnetic field, i.e. $B = \text{rot}A = 0$.

Once the magnetic field exists, then $\Xi_A \neq 0$. Consequently, $\Xi_A$ serves as a test matrix for the magnetic field. If $\Xi_A \neq 0$, i.e. the magnetic field exists, then the above system has only the trivial solution $\omega = 0$. Otherwise, if $\Xi_A = 0$, then there exists a unique solution represented by

$$
\omega = C_0 \exp \left( i \int_{x_0}^{x_k} a_k(x_1, \cdots, x_{k-1}, s_k, x_{k+1}, \cdots, x_N) ds_k \right), \quad \forall k = 1, \text{or} \cdots, N,
$$

where $C_0$ is any complex constant. Indeed, for $j \neq k$, when $i \neq j$ and $i \neq k$, then one has

$$
\frac{\partial}{\partial x_i} \left( \int_{x_0}^{x_j} a_j ds_j - \int_{x_0}^{x_k} a_k ds_k \right) = 0.
$$

While for $j \neq k$, when $i$ is equal to any one of them, then we have the same conclusion. As a result, for $\forall j, k = 1, \cdots, N$,

$$
\int_{x_0}^{x_j} a_j ds_j - \int_{x_0}^{x_k} a_k ds_k = C.
$$

Immediately, one deduces that $|\omega|$ is a constant in $\Omega$. Notice the fact $\omega \in H^1_0$, then one has $\omega = 0$ in $\Omega$.

Define an equivalent norm in $H^1$, i.e.

$$
||\omega||_{H^1} := ||\omega||_{L^2} + |\omega|_{H^1}.
$$

We prove the inequality by contradiction. If there does not exist any such constant $C(\Omega)$ such that, $\forall \omega \in H^1_0$,

$$
||\omega||_{L^2} \leq C(\Omega)|\omega|_{H^1},
$$

then one can find a sequence $\{\omega_m\}_m$ from $H^1_0$ such that

$$
1/m ||w_m||_{L^2} > |w_m|_{H^1}.
$$

Let

$$
v_m \triangleq w_m/||w_m||_{H^1},
$$

then one defines a sequence $\{v_m\}_m$ from $H^1_0$ such that

$$
(3) \quad ||v_m||_{H^1}^2 = 1,
$$
Since \( \Omega \) is bounded and open, \( \Gamma \in C^1 \), then the canonic injection from \( H^1(\Omega) \) to \( L^2(\Omega) \) is compact. From (3), one can extract a subsequence \( \{v_\mu\}_\mu \) from the sequence \( \{v_m\}_m \) such that

\[
v_\mu \to v \quad \text{in } L^2(\Omega).
\]

From (4), one has

\[
\forall j = 1, \cdots, N, \quad i \frac{\partial v_\mu}{\partial x_j} + a_j(x)v_\mu \to 0 \quad \text{in } L^2(\Omega).
\]

Since \( H^1_0 \) is complete, then one obtains

\[
v_\mu \to v \quad \text{in } H^1_0,
\]

with

\[
\forall j = 1, \cdots, N, \quad i \frac{\partial v}{\partial x_j} + a_j(x)v = 0.
\]

Accordingly, one has

\[
v \equiv 0.
\]

This is impossible since

\[
\|v\|_{H^1} = \lim_{\mu \to \infty} \|v_\mu\|_{H^1} = 1.
\]

This concludes the proof. \( \square \)

### 2.2. \( H^2_A \)-induced pseudodifferential operators.

According to Lemma 2.5, we introduce an equivalent norm in \( H^1_0 \),

\[
\|u\|_{H^1_0} := \|H_A u\|_{(L^2)^N}, \quad \forall u \in H^1_0.
\]

Lemma 2.3 indicates

\[
(H^2_A u, v)_{L^2} = (u, v)_{H^1_0}, \quad \forall u \in H^1_0 \quad \text{such that } H^2_A u \in L^2, \quad \forall v \in H^1_0.
\]

As is discussed, the imbeddings \( H^1_0 \hookrightarrow L^2 \) and \( L^2 \hookrightarrow H^{-1} \) are both dense and compact. Consequently, \( H^1_0 \hookrightarrow H^{-1} \) is dense and compact. As a result, it is reasonable to introduce the duality mapping

\[
H^2_A : H^1_0 \to H^{-1}
\]

defined by

\[
(H^2_A u, v)_{H^{-1}} := (u, v)_{H^1_0}, \quad \forall u, v \in H^1_0.
\]

By Riesz-Frèchet representation theorem, it holds that \( H^2_A \) is an isometric isomorphism of \( H^1_0 \) onto \( H^{-1} \). This indicates, \( \mathcal{D}(\Omega) \) is also dense in \( H^{-1} \). Denoting the compact imbedding

\[
I : H^1_0 \to H^{-1}.
\]

Then we define a linear and compact mapping

\[
S \triangleq (H^2_A)^{-1} \circ I : H^1_0 \to H^1_0.
\]

Furthermore, \( S \) is positive and self-adjoint. Indeed, for \( \forall u, v \in H^1_0 \), on the one hand,

\[
(Su, v)_{H^1_0} = ((H^2_A)^{-1}u, v)_{H^1_0} = (u, v)_{L^2}.
\]
On the other hand,

\[(u, Sv)_{\mathcal{H}_0^1} = (u, (\mathcal{H}_0^1)^{-1}v)_{\mathcal{H}_0^1} = (u, v)_{L^2}.\]

Hence,

\[(Su, v)_{\mathcal{H}_0^1} = (u, Sv)_{\mathcal{H}_0^1}.\]

Applying the spectral theorem in \[17\] to the compact, self-adjoint and positive linear operator \(S\), we conclude that the spectrum for \(\mathcal{H}_0^2\) on \(\mathcal{H}_0^1\) is discrete. Here we denote as \(\Lambda := \{\lambda_k^2\}_k\). And the point spectrum satisfies

\[0 < \lambda_1^2 \leq \lambda_2^2 \leq \lambda_3^2 \leq \cdots \to \infty,\]

with finite multiplicity. In addition, there exists an orthogonal system of complex-valued eigenfunctions \(\{\phi_{\lambda}(x)\}_{\lambda \in \Lambda}\) in \(\mathcal{H}_0^1\), and for each \(\lambda^2 \in \Lambda\),

\[\|\phi_{\lambda}\|_{L^2} = 1.\]

More importantly, \(\{\phi_{\lambda}(x)\}_{\lambda \in \Lambda}\) is dense in \(\mathcal{H}_0^1\). Hereafter, we denote by \(Z\) the finite combinations of eigenfunctions \(\phi_{\lambda}\). Obviously, \(Z\) is dense in \(\mathcal{H}_0^1\).

**Remark 2.6.** \(\{\phi_{\lambda}(x)\}_{\lambda^2 \in \Lambda}\) has orthogonality in both \(L^2\) and \(\mathcal{H}^{-1}\). Indeed, for \(k \neq l\),

\[0 = (\phi_{\lambda_k}, \phi_{\lambda_l})_{\mathcal{H}_0^1} = (\mathcal{H}_0^2 \phi_{\lambda_k}, \phi_{\lambda_l})_{\mathcal{H}_0^1} = \lambda_k^2 (\phi_{\lambda_k}, \phi_{\lambda_l})_{L^2}.\]

While by using the isometric property of \(\mathcal{H}_0^2\) on \(\mathcal{H}_0^1\), one has

\[(\phi_{\lambda_k}, \phi_{\lambda_l})_{\mathcal{H}^{-1}} = ((\mathcal{H}_0^2)^{-1} \phi_{\lambda_k}, (\mathcal{H}_0^2)^{-1} \phi_{\lambda_l})_{\mathcal{H}_0^1} = \lambda_k^{-2} \lambda_l^{-2} (\phi_{\lambda_k}, \phi_{\lambda_l})_{\mathcal{H}_0^1} = 0.\]

**Remark 2.7.** Moreover, \(\{\phi_{\lambda}(x)\}_{\lambda^2 \in \Lambda}\) is also dense in both \(L^2\) and \(\mathcal{H}^{-1}\) since the density of the imbeddings

\[\mathcal{H}_0^1 \hookrightarrow L^2 \hookrightarrow \mathcal{H}^{-1}.\]

**Remark 2.8.** Let \(\Omega = (0, \pi)\), for the Dirichlet operator

\[(i \frac{d}{dx} - 1)^2 : \mathcal{H}_0^1 \to \mathcal{H}^{-1},\]

it is easy to calculate that \(\{1, 2^2, 3^2, \cdots, N^2, \cdots\}\) is the set of eigenvalues which are bounded away from 0. And the associated orthonormal basis (in the sense of \(L^2\)-norm) in \(\mathcal{H}_0^1\) is

\[\{\sqrt{2/\pi} \sin(x)e^{-ix}, \sqrt{2/\pi} \sin(2x)e^{-ix}, \sqrt{2/\pi} \sin(3x)e^{-ix}, \cdots, \sqrt{2/\pi} \sin(Nx)e^{-ix}, \cdots\}.\]

With the above notations, one can define the generalized Fourier transform for \(f \in \mathcal{H}^{-1}\) as follows:

\[(5) \quad \hat{f}(\lambda) := \langle f, \phi_{\lambda} \rangle_{\mathcal{H}^{-1}, \mathcal{H}_0^1}.\]

And the corresponding Fourier series, a unique orthogonal expansion in \(\mathcal{H}^{-1}\), is of the form

\[(6) \quad f(x) = \sum_{\lambda^2 \in \Lambda} \hat{f}(\lambda) \phi_{\lambda}(x),\]
with the RHS converging in $\mathcal{H}^{-1}$. Indeed, for $\forall f \in \mathcal{H}^{-1}$, there is a unique $u_f \in \mathcal{H}_0^1$ such that $(f, v)_{\mathcal{H}^{-1}, \mathcal{H}_0^1} = (u_f, v)_{\mathcal{H}_0^1}$ for $\forall v \in \mathcal{H}_0^1$. Then

$$\sum_{\lambda \in \Lambda} |\hat{f}(\lambda)|^2 \|\phi_{\lambda}\|_{\mathcal{H}^{-1}}^2 = \sum_{\lambda \in \Lambda} |(u_f, \phi_{\lambda})_{\mathcal{H}_0^1}|^2 \|\phi_{\lambda}\|_{\mathcal{H}^{-1}}^2 = \sum_{\lambda \in \Lambda} \lambda^2 |(u_f, \phi_{\lambda})_{L^2}|^2 \|\phi_{\lambda}\|_{L^2}^2 < \infty.$$ 

At the moment, we are ready to introduce the pseudodifferential operators induced by $\mathcal{H}_0^1$.

**Definition 2.9.** Let $A \in (C^1(\Omega))^N$. Assume that the complex-valued functional $F \in C(\mathbb{R}_+)$ is polynomially bounded. One defines a generalized linear pseudodifferential operator on $\mathcal{H}_0^1$ as follows:

$$(7) \quad F\left(\sqrt{\mathcal{H}_0^1}\right) u(x) := \sum_{\lambda \in \Lambda} F(\lambda) \hat{u}(\lambda) \phi_{\lambda}(x).$$

The sequence $\{F(\lambda)\}_{\lambda \in \Lambda}$ is referred to as the symbol of $F\left(\sqrt{\mathcal{H}_0^1}\right)$.

**Remark 2.10.** As a matter of fact, when $f \in \mathcal{H}_0^1$, then

$$\|f\|_{\mathcal{H}_0^1}^2 = \sum_{\lambda \in \Lambda} |\hat{f}(\lambda)|^2 (\phi_{\lambda}, \phi_{\lambda})_{\mathcal{H}_0^1} = \sum_{\lambda \in \Lambda} \lambda^2 |\hat{f}(\lambda)|^2 < \infty,$n

$$\|f\|_{L^2}^2 = \sum_{\lambda \in \Lambda} |\hat{f}(\lambda)|^2 (\phi_{\lambda}, \phi_{\lambda})_{L^2} = \sum_{\lambda \in \Lambda} |\hat{f}(\lambda)|^2 < \infty,$n

$$\|f\|_{\mathcal{H}^{-1}}^2 = \sum_{\lambda \in \Lambda} |\hat{f}(\lambda)|^2 (\phi_{\lambda}, \phi_{\lambda})_{\mathcal{H}^{-1}} = \sum_{\lambda \in \Lambda} 1/\lambda^2 |\hat{f}(\lambda)|^2 < \infty.$$ 

3. **Techniques of microlocal analysis and proof of Theorem 1.1**

3.1. **Useful auxiliary tools from microlocal analysis.**

**Definition 3.1.** The extended phase space is divided into the following low frequency zone $Z_{low}(M)$, pseudodifferential zone $Z_{pd}(P, M)$ and $p$-evolution type zone $Z_{pe}(P, M)$: $(M$ and $P$ will be given later)

- $Z_{low}(M) := \{(t, \lambda) \in [0, T] \times \{\lambda_1 \leq \lambda \leq M\}\};$
- $Z_{pd}(P, M) := \{(t, \lambda) \in [0, T] \times \{\lambda \geq M\} : t\lambda \leq 2^P \nu(t)\};$
- $Z_{pe}(P, M) := \{(t, \lambda) \in [0, T] \times \{\lambda \geq M\} : t\lambda \geq 2^P \nu(t)\}.$

**Definition 3.2.** The following classes of symbols are defined in the $p$-evolution type zone for large frequencies: ($\ell \in \mathbb{N}, (m_1, m_2) \in \mathbb{R} \times \mathbb{R}$)

$$S^{\ell\{m_1, m_2\}} := \left\{ a \in C^\infty([0, T]; C^\infty(\lambda \geq M)) : |D_\lambda^k D_\lambda^\alpha a(t, \lambda)| \leq C_{k, \alpha} \lambda^{m_1-\alpha} (\nu(t)/t)^{m_2+k} \right\}$$

for all $k, \alpha \in \mathbb{N}, \ k \leq \ell, \ (t, \lambda) \in Z_{pe}(P, M)$.

**Definition 3.3.** We define micro-energy in each zone, denoted uniformly as $V(t, \lambda) = (V_1, V_2)^T$. 

10
In the low frequency zone $Z_{\text{low}}(M)$,

$$V(t, \lambda) := (\hat{u}, D_\lambda \hat{u})^T;$$

In the pseudo-differential zone $Z_{\text{pd}}(P, M)$,

$$V(t, \lambda) := (\lambda \hat{u}, D_\lambda \hat{u})^T;$$

In the p-evolution type zone $Z_{\text{pe}}(P, M)$,

$$V(t, \lambda) := (\lambda b(t) \hat{u}, D_\lambda \hat{u})^T.$$

**Definition 3.4.** $t_{\lambda}$ is defined as the solution of $t \lambda = 2^P \nu(t)$, and correspondingly, $t_{\lambda} \lambda = 2^P \nu(t_{\lambda})$ is called the separating line in the time-higher frequency part.

**Lemma 3.5.** According to Definition 3.2 and 3.4, in $Z_{\text{pe}}(P, M)$ we have the following symbol calculus properties,

- $(\lambda b(t))^{-2} \in S^2\{-2, 0\}$;
- $S^r_m \{m_1, m_2\} \subset S^r_m \{m_1 + k, m_2 - k\}$ for all $k \in \mathbb{N}$;
- If $a \in S^r_m \{m_1, m_2\}$ and $b \in S^r_m \{k_1, k_2\}$, then $ab \in S^r_m \{m_1 + k_1, m_2 + k_2\}$;
- If $a \in S^r_m \{m_1, m_2\}$, then $D^a_\lambda a \in S^r_m \{m_1 + \alpha, m_2\}$;
- If $a(t, \lambda) \in S^r_m \{-1, 2\}$, then for $(t, \lambda) \in Z_{\text{pe}}(P, M)$, $\left| \int_{t_{\lambda}}^t \nu(\tau, \lambda) d\tau \right| \lesssim \nu(t_{\lambda})$.

**Proof.** We only prove the last statement. Actually,

$$\left| \int_{t_{\lambda}}^t \nu(\tau, \lambda) d\tau \right| \lesssim \int_{t_{\lambda}}^t \nu^2(\tau)/(\nu^2(\lambda) d\tau \lesssim \nu^2(t_{\lambda})/(t_{\lambda}) \lesssim \nu(t_{\lambda}).$$

\[\square\]

3.2. **Estimates in $Z_{\text{low}}(M)$ and $Z_{\text{pd}}(P, M)$**. For the hyperbolic magnetic Schrödinger equation (1), the treatments in $Z_{\text{low}}(M)$ and $Z_{\text{pd}}(P, M)$ are essentially the same.

**Lemma 3.6.** For all $(t, \lambda) \in Z_{\text{low}}(M)$, we have the following energy estimate:

$$\left| \begin{pmatrix} \hat{u}(t, \lambda) \\ D_\lambda \hat{u}(t, \lambda) \end{pmatrix} \right| \lesssim |\hat{u}_0(\lambda)| + |\hat{u}_1(\lambda)|.$$  

**Proof.** Apply the partial Fourier transform, and we have the following equation

$$D^2_\lambda \hat{u}(t, \lambda) - \lambda^2 \hat{b}^2(t) \hat{u}(t, \lambda) = 0.$$  

Next, we study the system of first order

$$D_\lambda V = \mathcal{A}(t, \lambda)V := \begin{pmatrix} 0 \\ \lambda^2 \hat{b}^2(t) \end{pmatrix} V.$$  

In fact, the method of successive approximation enables us to construct the fundamental solution of the system

$$D_\lambda \mathcal{E}(t, s, \lambda) = \mathcal{A}(t, \lambda)\mathcal{E}(t, s, \lambda), \quad \mathcal{E}(s, s, \lambda) = I.$$  

More precisely, $\mathcal{E}(t, s, \lambda)$ is given in the form of matrizing representation:

$$\mathcal{E}(t, s, \lambda) = I + \sum_{k=1}^{\infty} i^k \int_s^t \mathcal{A}(t_1, \lambda) \int_s^{t_1} \mathcal{A}(t_2, \lambda) \cdots \int_s^{t_{k-1}} \mathcal{A}(t_k, \lambda) dt_k \cdots dt_1.$$  

Actually, we have
Lemma 3.7. For $k \in \mathbb{N}_+$, it holds
\[
\left\| \mathcal{A}(t_1, \lambda) \int_s^{t_1} \mathcal{A}(t_2, \lambda) \cdots \int_s^{t_{k-1}} \mathcal{A}(t_k, \lambda) dt_k \cdots dt_1 \right\| \leq 1/k! \left( \int_s^t \| \mathcal{A}(r, \lambda)\| dr \right)^k.
\]
Indeed,
\[
\int_s^t \| \mathcal{A}(t_1, \lambda) \| \int_s^{t_1} \| \mathcal{A}(t_2, \lambda)\| dt_2 dt_1
= \int_s^t \frac{\partial}{\partial t_1} \left( \int_s^{t_1} \| \mathcal{A}(t_2, \lambda)\| dt_2 \right) \left( \int_s^{t_1} \| \mathcal{A}(t_2, \lambda)\| dt_2 \right) dt_1
= 1/2 \int_s^t \frac{\partial}{\partial t_1} \left( \int_s^{t_1} \| \mathcal{A}(t_2, \lambda)\| dt_2 \right)^2 dt_1 = 1/2 \left( \int_s^t \| \mathcal{A}(r, \lambda)\| dr \right)^2.
\]
By induction method, the statement follows immediately. Consequently, by applying the definition of the zone $Z_{low}(M)$, we have
\[
\| \mathcal{E}(t, s, \lambda)\| \leq \exp \left( \int_s^t \| \mathcal{A}(r, \lambda)\| dr \right) \leq \exp \left( \int_0^T C(M) ds \right) \leq C(M, T).
\]
And this estimate leads to the final conclusion. \hfill \Box

Remark 3.8. The constant $M$ is chosen in order to separate large frequencies from lower frequencies. Actually, we can choose any large constant as $M$. As a matter of fact, the small frequencies play an insignificant role in the discussion of loss of regularity, and it is reasonable to neglect the influence from the electric field $E$.

Now we sketch the discussion in $Z_{pd}(P, M)$.

Lemma 3.9. For all $(t, \lambda) \in Z_{pd}(P, M)$, we have the following energy estimate with a positive constant $c_1$ depending upon $P$ and $M$:
\[
\left| \left( \begin{array}{c} \lambda \hat{u}(t, \lambda) \\ D_t \hat{u}(t, \lambda) \end{array} \right) \right| \lesssim \exp (c_1 \nu(t, \lambda)) (\lambda |\hat{u}_0(\lambda)| + |\hat{u}_1(\lambda)|).
\]

Proof. We study the system of first order
\[
D_t V = \mathcal{B}(t, \lambda) V := \left( \begin{array}{cc} 0 & \lambda \\ \lambda b^2(t) & 0 \end{array} \right) V.
\]
Similarly, we construct the fundamental solution of the system
\[
D_t \mathcal{E}(t, s, \lambda) = \mathcal{B}(t, \lambda) \mathcal{E}(t, s, \lambda), \quad \mathcal{E}(s, s, \lambda) = I.
\]
More precisely, $\mathcal{E}(t, s, \lambda)$ is given in the form of matrifiant representation
\[
\mathcal{E}(t, s, \lambda) = I + \sum_{k=1}^{\infty} i^{k} \int_s^t \mathcal{B}(t_1, \lambda) \int_s^{t_1} \mathcal{B}(t_2, \lambda) \cdots \int_s^{t_{k-1}} \mathcal{B}(t_k, \lambda) dt_k \cdots dt_1.
\]
By the induction method as in the proof of Lemma 3.7, one has
\[
\| \mathcal{E}(t, s, \lambda)\| \leq \exp \left( \int_s^t \| \mathcal{B}(r, \lambda)\| dr \right) \leq \exp \left( \int_0^T c_1 \lambda ds \right) \leq \exp (c_1 \nu(t, \lambda)).
\]
The final inequality holds when we take account of the definition of $t_\lambda$. \hfill \Box
3.3. Estimates in $Z_{pe}(P,M)$.

**Lemma 3.10.** For all $(t, \lambda) \in Z_{pe}(P,M)$, we have the following energy estimate with a positive constant $c_1$ depending upon $P, M$:

$$
(17) \quad \left| \begin{pmatrix} \lambda b(t) \hat{u}(t, \lambda) \\ D_t \hat{u}(t, \lambda) \end{pmatrix} \right| \lesssim \exp\left( c_1 \nu(t, \lambda) \right) \left( \lambda b(t) |\hat{u}(t, \lambda)| + |D_t \hat{u}(t, \lambda)| \right).
$$

**Proof.** The whole process is based on the application of two steps of diagonalisation procedure and the construction of fundamental solution. Taking account of the definition of micro-energy in this zone, we study the following first order system:

$$
(18) \quad D_t V = \begin{pmatrix} 0 & \lambda b(t) \\ \lambda b(t) & 0 \end{pmatrix} V + \begin{pmatrix} D_t b(t)/b(t) & 0 \\ 0 & 0 \end{pmatrix} V.
$$

**Step 1: First step of diagonalisation**

Choose the diagonalizer:

$$
(19) \quad \mathcal{M} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.
$$

It is evident that $\mathcal{M}^{-1}$ exists. Through the transform $V = \mathcal{M} V_0$, we obtain

$$
0 = D_t V_0 - \mathcal{D} V_0 + B_1 V_0 + B_2 V_0
$$

$$
:= D_t V_0 - \begin{pmatrix} \lambda b(t) & 0 \\ 0 & -\lambda b(t) \end{pmatrix} V_0
$$

$$
- \frac{1}{2} \begin{pmatrix} D_t b(t)/b(t) & -D_t b(t)/b(t) \\ -D_t b(t)/b(t) & D_t b(t)/b(t) \end{pmatrix} V_0,
$$

where $\mathcal{D} \in S^2\{1,0\}$ and $B = B_1 + B_2 \in S^1\{0,1\}$.

**Step 2: Second step of diagonalisation**

To carry out this step of diagonalisation, or the so-called normal form diagonalisation, we follow the procedure of asymptotic theory of differential equations. Namely, we construct an invertible matrix $N_1(t, \xi) := I + N^{(1)}(t, \lambda)$. Define $N^{(0)} := I, B^{(0)} := B, F^{(0)} := \text{diag}(B^{(0)})$,

$$
N^{(1)}_{\alpha\beta} := B^{(0)}_{\alpha\beta} / (\tau_q - \tau_r), q \neq r; \quad N^{(1)}_{qq} := 0, \quad \tau_k = (-1)^{k+1} \lambda b(t), k = 1, 2;
$$

$$
B^{(1)} := (D_t - \mathcal{D} + B)(I + N^{(1)}) - (I + N^{(1)})(D_t - \mathcal{D} + F^{(0)}).
$$

According to the properties of the symbol calculus, $N^{(1)} \in S^1\{-1,1\}$ and $F^{(0)} \in S^1\{0,1\}$. As for $B^{(1)}$, we obtain the following relation:

$$
(20) \quad B^{(1)} = B + [N^{(1)}, \mathcal{D}] - F^{(0)} + D_t N^{(1)} + B N^{(1)} - N^{(1)} F^{(0)}.
$$
The construction principle implies that the sum of the first three terms vanishes, hence $B^{(1)} \in S^0\{-1, 2\}$. Finally, let

$$R_1 := N_1^{-1}B^{(1)} = N_1^{-1}\left((D_t - \mathcal{D} + B)(I + N^{(1)}) - (I + N^{(1)})(D_t - \mathcal{D} + F^{(0)})\right).$$

By virtue of the calculus of generalized symbols, this definition means $R_1 \in S^0\{-1, 2\}$. In addition, due to the definition of $Z_{pe}(P, M)$, then $N^{(1)} \in S^1\{-1, 1\}$ indicates $|N^{(1)}_{qr}| \leq C/2^P$. Consequently, an appropriate integer $P$ assures that $\|N_1 - I\| < 1/2$ in $Z_{pe}(P, M)$, which implies the invertibility of $N_1$. As a result, we have the following system after the second step of diagonalisation:

$$\tag{22} (D_t - \mathcal{D} + F^{(0)} + R_1)V_1 = 0,$$

where $R_1 \in S^0\{-1, 2\}$.

**Step 3: Estimate of the fundamental solution**

We apply the transform $V_0 = N_1V_1$ and consider the system

$$\tag{23} (D_t - \mathcal{D} + F^{(0)} + R_1)V_1 = 0.$$

The fundamental solution for this equation is $\mathcal{E} = \mathcal{E}_1\mathcal{H}$, where $\mathcal{E}_1(t, s, \lambda)$ has the following form,

$$\begin{align*}
\mathcal{E}_1(t, s, \lambda)^{(11)} &= \exp \left(i \int_s^t \lambda b(\tau)d\tau + 1/2 \int_s^t b'(\tau)/b(\tau)d\tau \right), \\
\mathcal{E}_1(t, s, \lambda)^{(22)} &= \exp \left(-i \int_s^t \lambda b(\tau)d\tau + 1/2 \int_s^t b'(\tau)/b(\tau)d\tau \right), \\
\mathcal{E}_1(t, s, \lambda)^{(12)} &= \mathcal{E}_1(t, s, \lambda)^{(21)} = 0.
\end{align*}$$

Furthermore, $\mathcal{H}(t, s, \lambda)$ satisfies

$$\tag{25} D_t\mathcal{H} + \mathcal{E}_1(s, t, \lambda)R_1(t, \lambda)\mathcal{E}_1(t, s, \lambda)\mathcal{H} = 0, \quad \mathcal{H}(s, s, \lambda) = I.$$

Since

$$\tag{26} \|\mathcal{E}_1(t, s, \lambda)\| \leq C,$$

for all $s, t \in [t_\lambda, T]$, then by applying the same estimation procedure as in $Z_{pd}(P, M)$, we have

$$\tag{27} \|\mathcal{H}(t, t_\lambda, \lambda)\| \leq \exp \left(\int_{t_\lambda}^t \|\mathcal{R}_1(\tau, t_\lambda, \lambda)\|d\tau \right) \leq \exp(c_1\nu(t_\lambda)).$$

Therefore, we conclude the estimate for the fundamental solution:

$$\tag{28} \|\mathcal{E}(t, t_\lambda, \lambda)\| = \|\mathcal{E}_1\mathcal{H}\| \lesssim \exp(c_1\nu(t_\lambda)).$$

Using the invertibility of $\mathcal{H}$, $N_1$, and the two transforms

$$\tag{29} V_1(t, \lambda) = \mathcal{E}(t, t_\lambda, \lambda)V_1(t_\lambda, \lambda), \quad V_0 = N_1V_1, V = \mathcal{H}V_0.$$
we transform $V_1(t, \lambda)$ back to the original micro-energy $V(t, \lambda)$ and obtain
\begin{equation}
\|V(t, \lambda)\| \lesssim \exp(c_1 \nu(t_\lambda))\|V(t_\lambda, \lambda)\|.
\end{equation}
\[\square\]

Combining the estimates in Lemma 3.6, 3.9 and 3.10, one gets the following energy estimate in $[0, T] \times \{\lambda \geq M\}$ with a positive constant $c_1$:
\begin{equation}
\left| \left( \begin{array}{c}
\lambda \hat{u}(t, \lambda) \\
D_t \hat{u}(t, \lambda)
\end{array} \right) \right| \lesssim \exp(c_1 \nu(t_\lambda)) \left( |\lambda \hat{u}_0(\lambda)| + |\hat{u}_1(\lambda)| \right).
\end{equation}

Taking into account the definition of $t_\lambda$, generalized pseudodifferential operators and Plancherel theorem, we arrive at the statement of Theorem 1.1 immediately.

4. Optimality of the loss of regularity

In this section, we discuss the optimality of our estimates in Theorem 1.1. The method of instability argument to be used was developed in [4] to show that a Log-type loss really appears for hyperbolic Cauchy problems. Now we further develop this idea to demonstrate that the precise $\nu$-loss of derivatives really appears for the magnetic Schrödinger equation. Let us consider the Cauchy problem in $[0, T] \times \Omega$, $(\Omega = (0, 2\pi), A(x) = a(x))$
\begin{equation}
\partial^2_t u + b^2(t)(i\partial_x + a(x))^2 u = 0, \quad u(0, x) = u_0(x), \quad \partial_t u(0, x) = u_1(x),
\end{equation}
with $2\pi$-periodic initial Cauchy data $u_0, u_1$.

**Definition 4.1.** For a $2\pi$-periodic solution $u = u(t, x)$ in the $x$ variable, we introduce the homogeneous energy
\begin{equation}
\hat{E}_s(u)(t) := \|u(t, \cdot)\|_{H^s(\Omega)}^2 + \|\partial_t u(t, \cdot)\|_{H^{s-1}(\Omega)}^2, \quad s \in \mathbb{R},
\end{equation}
where $H^s(\Omega)$ denotes the homogeneous Sobolev space of index $s$.

First we introduce some useful auxiliary functions and sequences.

**Definition 4.2.** For a sufficiently small $\varepsilon > 0$, we define
\begin{equation}
w_\varepsilon(t) := \sin t \exp(2\varepsilon \int_0^t \psi(\tau) \sin^2 \tau d\tau),
\end{equation}
\begin{equation}
a_\varepsilon(t) := 1 - 4\varepsilon \psi(t) \sin(2t) - 2\varepsilon \psi'(t) \sin^2 t - 4\varepsilon^2 \psi^2(t) \sin^4 t,
\end{equation}
where the real non-negative $C^\infty$ function $\psi$ is $2\pi$-periodic on $\mathbb{R}$ and identically 0 in a neighborhood of 0. Furthermore, it satisfies
\begin{equation}
\int_0^{2\pi} \psi(\tau) \sin^2(\tau) d\tau = \pi.
\end{equation}

It is easy to verify the following fact.

**Lemma 4.3.** According to Definition 4.2, $a_\varepsilon \in C^\infty(\mathbb{R})$ and $w_\varepsilon \in C^\infty(\mathbb{R})$. Particularly, $w_\varepsilon$ is the unique solution of following differential equation with initial Cauchy data
\begin{equation}
w_\varepsilon''(t) + a_\varepsilon(t)w_\varepsilon(t) = 0, \quad w_\varepsilon(0) = 0, \quad w_\varepsilon'(0) = 1.
\end{equation}
Definition 4.4. We define a sequence of oscillating intervals $\{I_k\}_k$ by
\[
I_k := [t_k - \rho_k/2, t_k + \rho_k/2],
\]
and a zero sequence $\{t_k\}_k$ satisfying
\[
2^p \nu(t_k)t_k^{-1} = \lambda_k,
\]
for each $k \in \mathbb{N}$. Furthermore, define
\[
\{\rho_k\}_k := \left\{2^{-p+2\pi t_k[\nu(t_k)]/\nu(t_k)}\right\}_k,
\]
where $p \in \mathbb{N}$ is chosen such that $2^p-1 \varepsilon \pi > c_1 + 1$.

Remark 4.5. Here we only consider the case $\lim_{t \to 0} \nu(t) = +\infty$. It is easy to check that the sequences $\{t_k\}_k, \{\rho_k\}_k, t_k$ tend to $0$. Such choice of $\rho_k$ guarantees that $I_k$ is contained in $(0, T]$. Furthermore, $\lambda_k \rho_k/(4\pi) \in \mathbb{N}^+$. With these auxiliary functions and sequences, the optimality argument can be expressed as the following theorem.

Theorem 4.6. For the Cauchy problem (1.1), there exists
- a sequence of coefficients $\{b^2_k(t)\}_k$ satisfying all assumptions of Theorem 1.1 with constants independent of $k$;
- a sequence of initial Cauchy data $\{(u^0_k(x), u^1_k(x))\}_k \in H^s(\Omega) \times H^{s-1}(\Omega)$; such that the sequence of corresponding solutions $\{u^k(t, x)\}_k$ satisfies
\[
\sup_k \mathbb{E}_1(u^k)(0) \leq C(\varepsilon),
\]
\[
\sup_k \mathbb{E}_1(\exp(-c_1(\varepsilon) \nu(\mu^{-1}(2^p/(\sqrt{(i \nabla + A(x))^2}))))u^k)(t) = +\infty, \text{ for any } t \in (0, T],
\]
where $C(\varepsilon)$ and $c_1(\varepsilon)$ depend on the sufficiently small positive constant $\varepsilon$.

Proof. We divide our proof into three steps.

Step 1: Construction of a sequence of oscillating coefficients

For each $k \in \mathbb{N}$, we define the oscillating coefficient $b^2_k(t)$ as
\[
b^2_k(t) := \begin{cases} 
1, & t \in [0, T] \setminus I_k; \\
\alpha \varepsilon (\lambda_k^{-1}(t - t_k)), & t \in I_k.
\end{cases}
\]

Remark 4.7. The above definition indicates, on the one hand, $b^2_k \in C^\infty(\mathbb{R})$ since $\alpha \varepsilon$ is identically equal to $1$ in a neighborhood of $I_k$. On the other hand,
\[
0 < b^2_0 \leq \inf_{t \in [0, T]} b^2_k(t) \leq \sup_{t \in [0, T]} b^2_k(t) \leq b^2_1 < \infty,
\]
where the positive constants $b_0$ and $b_1$ are independent of $k$ when we choose an appropriate $\varepsilon > 0$. Simple calculations show that the coefficient $b^2_k$ satisfies all assumptions of Theorem 1.1 in the interval $I_k$. While in $[0, T] \setminus I_k$, it is trivial.
**Step 2: Construction of auxiliary functions**

Next we study the family of Cauchy problems in $[t_k - \rho_k/2, t_k + \rho_k/2] \times \Omega$,

$$\partial_t^2 u^k + \partial_k^2(t)(i\partial_x + a(x))^2 u^k = 0, \quad u^k(t_k, x) = 0, \quad \partial_t u^k(t_k, x) = u_1^k(x).$$

Let the initial Cauchy data be

$$u_1^k(x) = \phi_{\lambda_k}(x)$$

and apply the coordinate transform

$$s = \lambda_k(t - t_k).$$

At the same time, define

$$v^k(s, x) := u^k(t(s), x),$$

then for $s \in [-\lambda_k\rho_k/2, \lambda_k\rho_k/2]$, we get

$$\partial_s^2 v^k + \lambda_k^{-2}\alpha(s)(i\partial_x + a(x))^2 v^k = 0, \quad v^k(0, x) = 0, \quad \partial_s v^k(0, x) = u_1^k(x)/\lambda_k.$$

As a matter of fact, we have a unique solution for (49) in the form of

$$v^k(s, x) = \lambda_k^{-1}\phi_{\lambda_k}(x)w(s).$$

Transforming back to $u^k(t, x)$, we arrive at

$$u^k(t, x) = \lambda_k^{-1}\phi_{\lambda_k}(x)w(\lambda_k(t - t_k))$$

in $I_k$. Further calculations lead to

$$u^k(t_k - \rho_k/2, x) = 0, \quad \partial_t u^k(t_k - \rho_k/2, x) = \phi_{\lambda_k}(x)\exp(-\varepsilon\rho_k\lambda_k/2),$$

$$u^k(t_k + \rho_k/2, x) = 0, \quad \partial_t u^k(t_k + \rho_k/2, x) = \phi_{\lambda_k}(x)\exp(\varepsilon\rho_k\lambda_k/2).$$

**Step 3: Existence of $\nu$-loss of regularity**

Now we introduce an energy conservation law in the sense of pseudo-differential operators.

**Lemma 4.8.** For the Cauchy problem in $(t, x) \in \mathbb{R} \times \Omega$,

$$\partial_t^2 u + (i\partial_x + a(x))^2 u = 0, \quad u(t_0, x) = 0, \quad \partial_t u(t_0, x) = \phi_{\lambda}(x),$$

then the energy conservation law holds, that is,

$$\dot{E}_s(u)(t) = E_s(u)(t_0).$$

**Proof.** In effect, we have the following explicit representation of the unique solution by virtue of separation of variables:

$$u(t, x) = \sin(\lambda(t - t_0))\phi_{\lambda}(x)/\lambda.$$
By applying the definition of homogeneous Sobolev spaces \( \dot{H}^s(\Omega) \), \( s \in \mathbb{R} \), we calculate the homogeneous energy for the solution \( u \). It holds that

\[
\dot{E}_s(u)(t) = \| u(t, \cdot) \|_{\dot{H}^s(\Omega)}^2 + \| \partial_t u(t, \cdot) \|_{\dot{H}^{s-1}(\Omega)}^2
\]

\[
= \sum_{\lambda^2 \in \Lambda} |\hat{u}(t, \lambda)|^2 \lambda^{2s} + \sum_{\lambda^2 \in \Lambda} |\partial_t \hat{u}(t, \lambda)|^2 \lambda^{2(s-1)}
\]

\[
= \lambda^{2(s-1)}(\sin^2(\lambda(t - t_0)) + \cos^2(\lambda(t - t_0)))
\]

\[
= \lambda^{2(s-1)} = \dot{E}_s(u)(t_0).
\]

Therefore,

\[
\dot{E}_1(u^k)(t) = \exp(-\varepsilon \rho_k \lambda_k), \quad \text{for} \quad t \in [0, t_k - \rho_k/2];
\]

\[
\dot{E}_1(u^k)(t) = \exp(\varepsilon \rho_k \lambda_k), \quad \text{for} \quad t \in [t_k + \rho_k/2, T].
\]

It is evident that (41) follows directly from (57). While for \( t \in [t_k + \rho_k/2, T] \), we have

\[
\dot{E}_1(\exp(-c_1 \nu(\mu^{-1}(2^P/\sqrt{(i\nabla + A(x))^2}))u^k))(t)
\]

\[
= \dot{E}_1(\exp(-c_1 \nu(\mu^{-1}(2^P/\lambda_k))))u^k)(t)
\]

\[
= \exp(-2c_1 \nu(\mu^{-1}(2^P/\lambda_k)))\dot{E}_1(u^k)(t)
\]

\[
= \exp(-2c_1 \nu(\mu^{-1}(2^P/\lambda_k))) + \varepsilon \rho_k \lambda_k
\]

\[
= \exp(-2c_1 \nu(t_k) + \varepsilon \rho_k \lambda_k).
\]

Taking into account the choice of \( \rho_k \) and \( t_k \), we have (42). This concludes our proof. \( \square \)

**Remark 4.9.** Periodic functions are very useful tools in the construction of coefficients for instability arguments. This kind of techniques is frequently used in the discussion of Floquet theory etc. [6, 7, 12, 15].

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X. Lu and X. Lv  

Regularity of hyperbolic magnetic Schrödinger equation

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