Triviality of GHZ operators of higher spin

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Abstract

We prove that local observables of the set of GHZ operators for particles of spin higher than 1/2 reduce to direct sums of the spin 1/2 operators $\sigma_x$, $\sigma_y$ and, therefore, no new contradictions with local realism arise by considering them.

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1 Introduction

The GHZ theorem \cite{1} provides a powerful test of quantum non-locality, which can be confirmed or refuted by the outcome of just one single experiment \cite{2}. Formulated for three spin 1/2 particles \cite{2} \cite{3}, the argument is based on the anti-commutative nature of the 2x2 spin operators $\sigma_x, \sigma_y$. The values of the three mutually commuting observables

$$\sigma^a_x \otimes \sigma^b_y \otimes \sigma^c_y \equiv \sigma^a_x \sigma^b_y \sigma^c_y, \quad \sigma^a_y \sigma^b_x \sigma^c_y,$$

and their product, $-$ $\sigma^a_x \sigma^b_x \sigma^c_x$, cannot be obtained, consistently, by making local assignments to each of the individual spin operators, $m_I^x, m_I^y = \pm 1, I = a, b, c$. This is not a contradiction of Quantum Mechanics: the state $|\psi\rangle = \frac{1}{\sqrt{2}} (|\uparrow\uparrow\uparrow\rangle - |\downarrow\down\down\rangle)$, for instance, is one of the common eigenstates of the four operators, with eigenvalues $\lambda_1 = \lambda_2 = \lambda_3 = 1, \lambda_4 = -1$, respectively. $|\psi\rangle$ is a highly correlated (entangled) state of the three parties which has no defined value for $\sigma^I_x, \sigma^I_y$.

In this note we address the question of how to generalize the argument to particles of higher spin and find that there are no non-trivial extensions other than direct sums of operators that can be brought into the form $\sigma_x, \sigma_y$ by means of local unitarity transformations. (For odd dimensional Hilbert spaces the direct sum is completed by a one-dimensional submatrix, i.e., a c-number in the diagonal). We give a proof for the cases of spin 1 and 3/2. Similar problems have been addressed in \cite{4}.

Let us look for observables $A, B$ such that $AB = \omega BA$ (their hermiticity implies that $\omega$ is at most a phase): this is a necessary condition for the commutator relations $[A^a_i A^b_i A^c_i, B^a_i B^b_i B^c_i] = etc... = 0$ to hold. As we shall see, all interesting cases correspond to $\omega = -1$. Without loss of generality, $A$ can always be taken diagonal, $A = \text{diag}(\lambda_1, \lambda_2)$, for the simplest case $s=1/2$. The above condition reads

$$AB - \omega BA = \begin{pmatrix} (1 - \omega)\lambda_1 b_{11} & (\lambda_1 - \omega \lambda_2) b_{12} \\ (\lambda_2 - \omega \lambda_1) b_{12}^* & (1 - \omega) \lambda_2 b_{22} \end{pmatrix} = 0.$$  

If $\omega \neq 1$, a solution with non-vanishing off-diagonal elements is allowed if $\omega^2 = 1$, i.e., $\omega = -1$ This leads to

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & b \\ b^* & 0 \end{pmatrix},$$

which can always be transformed to $\sigma_x$ and $\sigma_y$, by rotations and adequate normalization. These are the operators of the example \cite{1}. For spin 1/2 the set of GHZ operators are in this sense unique.

2 Spin one

For higher spins the proof proceeds along the same lines. We find one case of interest, with $\omega = -1$,

$$A = \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & b & c \\ b^* & 0 & 0 \\ c^* & 0 & 0 \end{pmatrix}.$$
In the basis where $B$ is diagonal $A$ and $B$ read

$$A = -\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad B = \sqrt{|b|^2 + |c|^2} \begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

which proves the assertion in the case of spin one, as a rotation around $x$ brings $B$ into the form $0 \oplus \sigma_y$, while $A$ is left as $1 \oplus \sigma_x$, up to normalizations.

### 3 Spin 3/2

For spin 3/2, in addition to cases that reduce straightforwardly to those of lower spins, we find:

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad B = \sqrt{|a|^2 + |b|^2 + |c|^2} \begin{pmatrix} 0 & a & b & c \\ a^* & 0 & 0 & 0 \\ b^* & 0 & 0 & 0 \\ c^* & 0 & 0 & 0 \end{pmatrix}$$

In the basis where $B$ is diagonal $A$ and $B$ read

$$A = -\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad B = \sqrt{|a|^2 + |b|^2 + |c|^2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

which is again diagonal in two, 2x2, blocks.

The last case corresponds to

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & a & 0 & b \\ a^* & 0 & c & 0 \\ 0 & c^* & 0 & d \\ b^* & 0 & d^* & 0 \end{pmatrix}$$

The following list of unitary transformations bring these matrices to the desired form:

a) With

$$F = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$F^\dagger = F = F^{-1}$, we find

$$A' = FAF = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad B' = FBF = \begin{pmatrix} B^\dagger \mathcal{B} \end{pmatrix},$$

where

$$\mathcal{B} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$
b) A unitary transformation of the form $U = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}$ leaves $A'$ invariant and allows to diagonalize $B$

$$A'' = A', \quad B'' = U B' U^\dagger = \begin{pmatrix} U_1 B U_2^\dagger \end{pmatrix} \begin{pmatrix} m & 0 \\ 0 & n \end{pmatrix}. \quad (11)$$

We have used the result that the generic matrix $B$ can be brought to a diagonal form with two unitary matrices $U_1, U_2$.

c) Finally, acting with $F$ again,

$$A''' = A, \quad B''' = \begin{pmatrix} 0 & m \\ m^* & 0 \\ 0 & n \end{pmatrix} \begin{pmatrix} n^* & 0 \end{pmatrix}, \quad (12)$$

which completes the proof.

4 Conclusions

We conclude that the equation $AB = \omega BA$ is very restrictive on $\omega$ and on the possible forms of $A$ and $B$; as the Hilbert space dimension increases, with increasing spin, all its solutions for $\omega \neq 1$ have $\omega = -1$ and are essentially direct sums of the two-dimensional $\sigma_x$ and $\sigma_y$. In this sense there are no solutions that could, in principle, enrich the possibilities opened by the GHZ theorem.

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