Constant dissipation rate is optimal for thermodynamic protocols: experimental implementation of Landauer erasure through thermodynamic length

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In this work we explore the use of thermodynamic length to improve the performance of experimental protocols. In particular, we implement Landauer erasure on a driven electron level in a semiconductor quantum dot, and compare the standard protocol in which the energy is increased linearly in time with the one coming from geometric optimisation. The latter is obtained by choosing a suitable metric structure, whose geodesics correspond to optimal finite-time thermodynamic protocols in the slow driving regime. We show experimentally that geodesic drivings minimise dissipation for slow protocols, with a bigger improvement as one approaches perfect erasure. Moreover, the geometric approach also leads to smaller dissipation even when the time of the protocol becomes comparable with the equilibration timescale of the system, i.e., away from the slow driving regime. Our results also illustrate, in a single-electron device, a fundamental principle of thermodynamic geometry: optimal finite-time thermodynamic protocols are those with constant dissipation rate along the process.

I. INTRODUCTION

Landauer erasure represents one of the most paradigmatic protocols in stochastic and quantum thermodynamics. Its relevance is not only historical, as it was the first case in which a strong argument for the physicality of information was made, but also conceptual, as it shows how logical irreversibility inevitably leads to dissipation, and practical, as it imposes a fundamental bound on the minimal heat released by an operating computer with finite memory. In particular, Landauer’s limit establishes that the minimal amount of heat dissipated in order to erase a bit is bounded by [1]:

$$\Delta Q \geq -k_B T \Delta S$$

where $T$ is the temperature of the bath and $\Delta S$ is the difference in entropy between the final and the initial state, which turns out to be negative for erasing protocols.

Equality in Eq. (1) corresponds to an ideal isothermal process. This can only be realised in infinite time, which makes Landauer’s limit de facto unattainable in practice. Nevertheless, it is a crucial task to minimise dissipation (i.e. $\Delta Q$) in information-processing devices, and much experimental effort has been devoted to approach the Landauer’s limit [2]. Experimental demonstrations of (almost-perfect) Landauer erasure have been reported in different platforms, including colloidal particles [3–6], nanomagnets [7–9], superconducting flux logic cells [10], underdamped micromechanical oscillators [11, 12] and optomechanical systems [13] (see also related works in quantum systems such as nuclear magnetic resonance setups [14] and ion traps [15]).

Despite how well studied this problem is, in all the experimental works above the driving chosen in order to induce the erasure is linear in time. We show here that this is suboptimal, which is in agreement with previous theoretical works [16–24]. In particular, we study how to exploit the concept of thermodynamic length [19, 25–34] to devise better erasing protocols in finite time. This quantity arises from the expansion of the entropy production for protocols that are performed in a long, but finite time. In this regime, optimal protocols are governed by the principle of constant dissipation rate, meaning that the optimal protocol is the one that allocates the dissipation in the most uniform way [33, 35–37]. The corresponding thermodynamic metric also gives a prescription to find minimally dissipating drivings: in fact, the geodesics associated to this metric realise optimal protocols in the slow driving regime [19, 25–27, 31].

We experimentally demonstrate how this geometric approach can be exploited to minimise dissipation in a Landauer erasure protocol. Our device is based on a semiconductor quantum dot which allows for the manipulation of discrete energy levels, see Fig. 1. We study both the regime of slow driving, for which we demonstrate the expected improvement, and the fast driving regime. For the latter, which is in principle outside of the realms of application of thermodynamic length, we still observe substantial reductions in dissipation compared to the linear drive. Finally, we show that the improvements become more and more relevant the closer one gets to complete erasure.

These results can be regarded as the experimental proof of principle for the relevance of thermodynamic length in devising optimal finite-time protocols. As it was argued theoretically in [19, 31, 33–37], thermodynamic length offers a flexible and powerful tool for minimising dissipation. It is particularly interesting to see
that even for a problem as well explored as the one of Landauer erasure it is possible to find an improvement over present experimental protocols.

II. EXPERIMENTAL SET-UP

The experiment is performed using the same device as in [38], shown in Figure 1 (a). Three quantum dots (QDs) are formed by polytype engineering in an InAs nanowire [39–42]. The occupancy \( n \in \{0, 1\} \) of a spin-degenerate energy level \( E \) in the leftmost QD (QD1) encodes the bit of information to be erased in the experiment. We drive the energy level with the plunger gate voltage \( V_{g1} \) which has a lever arm \( \alpha = 1.6 \times 10^4 \, k_B T / V \). The rightmost QD is voltage biased with \( V_b = 0.5 \, mV \) and tuned so that the current \( I_d \) is sensitive to changes in the QD1 occupancy, giving a real-time probe of \( n \) [43, 44].

The middle QD is kept in Coulomb blockade, reducing the system to the one shown in Figure 1 (b): a discrete QD1 energy level coupled to a fermionic reservoir at temperature \( T = 100 \, mK \) (set by the cryostat temperature). Electrons tunnel between them with the rates \( \Gamma_{in} = 2 \Gamma_0 (1 + b E) f(E) \) and \( \Gamma_{out} = \Gamma_0 (1 + b E) (1 - f(E)) \) where \( \Gamma_0 = 39 \, Hz \) and \( b = 0.0036/k_B T \) were determined using a feedback protocol [45], and \( f(E) = 1/(1 + e^{E/k_B T}) \) is the Fermi-Dirac distribution. The average occupation at equilibrium for each energy is given by \( n_{eq}(E) = 1/(1 + \frac{1}{2} e^{E/k_B T}) \), corresponding to the thermal state for a system with a degeneracy 2 in the \( n = 1 \) state.

FIG. 1. a) Scanning electron microscope image of the nanowire device. Embedded in the nanowire are three QDs, each aligned to one of the plunger gates \( V_{g1}, V_{g2} \) or \( V_{gd} \). Contacts separate the device into one part with two QDs and one part with a single QD. The coupler couples the two systems together, allowing the current \( I_d \) through the lone QD to provide a measure of the charge state of the other system. Here, the QD involved in the experiment is marked in blue (close to the plunger gate with \( V_{g1} \)) while the quantum dot marked in red is tuned into Coulomb blockade. The sensor quantum dot is marked in purple and the tunnel barriers are coloured orange. b) The energy diagram for the protocol.

III. ERASURE PROTOCOL

The erasure protocol is realised as follows: first, the system is allowed to thermalise in contact with the reservoir bath while keeping its energy at \( E_0 = \log_2 k_B T \) corresponding to a \( 50\% - 50\% \) occupation condition, see Fig. 1 (b). Then, while still keeping it in contact with the bath, we ramp up the energy of the dot until we reach \( E_1 := E_0 + E_A \), where \( E_A \) defines the driving amplitude. When \( E_A \gg k_B T \), we have \( n_{eq}(E_1) = 0 \), i.e., the dot is unoccupied with probability close to 1. As the last step, the energy is quenched back to \( E_0 \), so that the system is effectively erased.

We measure the heat \( \Delta Q \) by monitoring the electron transitions: whenever the dot is occupied and an electron tunnels out, the energy at that time gets transferred to the reservoir where it dissipates and adds to \( \Delta Q \); similarly, if an electron tunnels into the dot that energy is the reservoir where it dissipates and adds to \( \Delta Q \). Moreover, since the quench is instantaneous, it does not contribute to the heat production, as the state of the system is unaffected by it. Repeating the protocol many times one can then compute the average heat \( \langle \Delta Q \rangle \) simply by adding up the resulting heat for each round and dividing by the number of rounds. Alternatively, the same result can also be obtained from the average occupation \( \langle n(t) \rangle \) thanks to the equality:

\[
\langle \Delta Q \rangle = - \int_0^\tau dt \, \langle \dot{n}(t) \rangle \, E(t),
\]

where \( \tau \) is the total time of the protocol, while \( E(t) \) is the drive used to interpolate between \( E_0 \) and \( E_1 \). The most usual choice is to take it to be a linear drive \( E(t) := E_0 + E_A \cdot t / \tau \), but in principle \( E(t) \) could be any function satisfying \( E(0) = E_0 \) and \( E(\tau) = E_1 \). In fact, it turns out that the linear protocol is suboptimal.

An alternative protocol can be designed as follows. First, it should be noticed that in the limit of \( (\Gamma_0 \tau) \gg 1 \), Eq. (2) can be brought to the form (Appendix A):

\[
\langle \Delta Q \rangle = -k_B T \Delta S + k_B T \int_0^\tau dt \, g(t) \dot{E}(t)^2
\]
up to corrections of order $(\Gamma_0 \tau)^{-2}$ and regardless of the particular choice of the protocol. The quantity $g(t)$ is called thermodynamic metric: it is always positive and depends smoothly on the drive $E(t)$. For reasons of space, we refer for the particular expression of the metric to Appendix A. The integral in Eq. (3) is a standard quantity in differential geometry, usually called energy functional. The name comes from the analogy with the action of a particle moving with velocity $\dot{E}(t)$ and variable mass $g(t)$. Interestingly, thanks to the form of the dissipation in Eq. (3), we can automatically construct minimally dissipating drives simply by solving the geodesic equation for $E(t)$. Further details are provided in Appendix A and in [19, 33].

The corresponding trajectory are shown in Fig. 2 for two driving amplitudes. Compared with the linear drive, the geodesic one allocates more time to ramp up the energy when the QD is occupied with larger chance (at low $E$) and becomes steeper towards the end of the protocol. This can be intuitively understood as follows: since the dissipation is linear in $\langle n(t) \rangle$ while $\langle n(t) \rangle$ decreases exponentially with $E(t)$, it is better to allocate more time at the beginning, when the variation $\langle n(t) \rangle$ is big, and to reserve little time to the final jump in the energy, because exponentially small amount of the tunneling events take place at large $E$. Notice that this reasoning is justified by the fact that for slow driving $\langle n(t) \rangle \simeq n_{eq}(E(t))$. Still, we show that this intuition is also relevant for drives where $\Gamma_0 \tau \simeq 3$ (which we dub fast driving regime).

The reasoning above intuitively captures a characteristic of geodesic drives: it can be proven that the entropy production rate, i.e., the integrand in Eq. (3), is constant along optimal protocols [33, 35–37]. This effect is exemplified in Fig. 3, where we plot the heat production rate both in the fast and in the slow driving regime ($\Gamma_0 \tau \simeq 40$ for the latter), comparing the behaviour of a linear drive with the one of the geodesic. We see that for the non-optimized drive, the heat production peaks at the beginning, while decreasing towards zero at the end of the protocol. For geodesic drives instead, the heat is produced more uniformly along the protocol\(^1\).

IV. COMPARISON BETWEEN LINEAR AND GEODESIC DRIVE

In this section we compare the performance of the geodesic protocol with the usual choice of a linear drive. The data are presented in Fig. 4.

The two plots on the left represent the quality of erasure as a function of the driving amplitude. This quantity is measured by the percentage of residual population in the dot at the end of the drive or, equivalently, with the population probability $p(n = 0)$. On the right of Fig. 4, we also plot the heat produced as a function of the driving amplitude. The continuous lines are the theoretical predictions, while the dotted lines correspond to the experimental data. The grey dotted line corresponds to the ideal case, that is the maximal erasure for each $E_A$ on the left, and the Landauer’s limit $k_B T \Delta S$ on the right.

\(1\) The fact that the entropy production rate is not perfectly constant along the trajectory arises from finite time effects. We numerically verified that increasing $\Gamma_0 \tau$ makes the heat production closer to a constant value.
for small amplitudes the geodesic does not depart much from the linear drive, see Fig. 2. For larger amplitudes, we can appreciate the strength of geodesic protocols. In the slow driving regime (top figures), we observe that the geodesic drive dissipates less (right top Fig. 4) for a similar quality of erasure (left top Fig. 4). In fact, the dissipation grows linearly as the amplitude $E_A$ increases for the linear protocol (i.e., as the quality of the erasure increases), whereas it tends to a constant for the geodesic drive. This makes geodesic drives more and more relevant when one wants to reach higher erasing quality. Indeed, the geodesic drive stays much closer to the Landauer’s limit of $k_B T \Delta S$. These results demonstrate the reduction in dissipation when erasing a qubit in the slow driving regime theoretically predicted in previous works [19].

Interestingly, as we depart from the slow driving regime (bottom Fig. 4), we observe a trade-off: on the one hand, the fast linear drive achieves a higher quality of erasure than the geodesic protocol. This happens because for such a short protocol duration, the system does not have enough time to respond to the steep ramp at the end of the geodesic drive making that energy range effectively lost in the erasure attempt. On the other hand, the dissipation produced by the geodesic protocol saturates, as expected from the theory (see Appendix A), so one can achieve the same erasing precision as the one given by the linear drive at the same dissipation just by increasing the amplitude.

It should be noticed that if one allows for a small extra time at the end of the protocol in which the system thermalizes at a fixed energy, the difference in the quality of erasure between the linear and the geodesic drive would disappear. On the other hand, since the biggest contribution to the dissipation comes from the initial part of the protocol, if one can allow for this additional time, this would make the geodesic drive preferable because it would give the same erasure quality at lower dissipation. This intuition is made precise in Appendix B via numerical simulations of the process. In this way, there is a trade-off between the precision of erasure and time at optimal dissipation, or between dissipation and quality of erasure for a fixed time.

V. CONCLUSIONS

The present work shows the relevance of thermodynamic length in the design of minimally dissipating experimental protocols. In particular, by considering the erasure of information in a quantum dot we showed that even in such a well studied protocol a simple application of our method decreases the amount of dissipation released during the driving. This comes at the cost of a small decrease in the quality of the erasure, which can arguably be recovered by allowing a small transient at the end of the transformation, or by moving to bigger driving amplitudes (thanks to the saturation of the dissipated heat shown on the right of Fig 4). Moreover, we showed that even if the thermodynamic length in principle should only apply to the slow driving limit, it improves also on relatively fast protocols, proving the wide applicability of this approach.

This universality comes from an underlying physical principle: the dissipation rate in optimal protocols should be constant along the trajectory [33, 35–37]. In this sense, what the geodesic drive does is allocating the heat production in a more uniform way compared with the one arising from the naive choice of a linear drive. This fact is of key importance in the interpretation of the shape of the geodesic protocols and provides an intuitive method to develop optimal drives.

Beyond the minimisation of average dissipation, the geodesic drives considered here can also become useful for the minimisation of work and heat fluctuations [29, 46], for probabilistic work extraction [47, 48], and for increasing the efficiency of thermal machines [49–52]. Future works include the implementation of optimal protocols in the fast driving regime [20–22, 24, 53] and observing effects arising from quantum coherence in erasure processes [23, 54, 55].

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where in the second line we exploited the fact that the trace condition for $\rho$ to $Z$ state to complete the thermal state.

Appendix A: Irreversibility and thermodynamic length

Heat can be split in a path independent contribution, proportional to the difference in entropy at the endpoints, and a path dependent term, that goes under the name of dissipation. In this appendix we show how one can derive further details.

1. General theory

Consider a generic quantum state $\rho(t)$ undergoing an open system dynamics, to which we can associate a driven system Hamiltonian $H(t)$. The corresponding thermal state at each moment is denoted by $\rho_{eq}(t) := \frac{e^{-\frac{H(t)}{k_B T}}}{Z(t)}$, where $Z(t) := Tr \left[ e^{-\frac{H(t)}{k_B T}} \right]$ is called partition function. We assume that for each $t$ the dynamics tries to equilibrate the state to $\rho_{eq}(t)$. Now, simply by using the functional form of the thermal distribution, we can rewrite the average heat as:

$$\langle \Delta Q \rangle = -\int_0^T dt \ Tr \left[ \dot{\rho}(t) H(t) \right] = k_B T \int_0^T dt \ Tr \left[ \dot{\rho}(t) \log e^{-\frac{H(t)}{k_B T}} \right] =$$

$$= k_B T \int_0^T dt \ Tr \left[ \dot{\rho}(t) \log \rho_{eq}(t) \right], \quad (A1)$$

where in the second line we exploited the fact that the trace condition for $\rho(t)$ implies that $Tr \left[ \dot{\rho}(t) \right] = 0$, so we can complete the thermal state.

With the hindsight of Clausius’ inequality, we add and subtract the total derivative of the entropy to Eq. (A2),
where $\Delta n$.

In this regime, we can perturbatively solve the expression of duration of the protocol. This explains the positivity and the symmetry.

the excited state.

that in the limit $\tau \rightarrow \infty$ the population is exactly given by the one of the figures presented here to the one in the main text). We further comment on the differences between this simplified experiment, but we chose this expression to obtain analytical results. It can be proven by perturbative methods, or through numerical simulations, that the results are not very sensitive to this change (as it can be noticed by comparing the figures presented here to the one in the main text). We further comment on the differences between this simplified version and the realistic model in the next subsection for what regards the metric and the Christoffel symbol.

Let us see how the dynamics of $\langle n(t) \rangle$ is affected by a change in the duration $\tau$ of the protocol. First, it is apparent that in the limit $\tau \rightarrow \infty$ the driving appears frozen to the system, so the population is exactly given by the one of the excited state $\langle n(t) \rangle = n_{eq}(E(t))$.

Define the thermalisation timescale $\tau_{eq} := \Gamma_0^{-1}$. When we talk about slow driving we always mean that $\tau_{eq}/\tau \ll 1$. In this regime, we can perturbatively solve the expression of $\langle n(t) \rangle$. We rewrite the population as:

$\langle n(t) \rangle = n_{eq}(E(t)) + \Delta n(t), \quad \text{(A8)}$

where $\Delta n(t)$ is of order $O(\tau_{eq}/\tau)$. Plugging this ansatz into the rate equation we obtain:

$\frac{d}{dt} n_{eq}(E(t)) = -\Gamma_0 \Delta n(t) + O\left(\frac{\tau_{eq}}{\tau}\right) \Rightarrow \Delta n(t) = \frac{\tau_{eq}}{2} \frac{\dot{E}(t)}{1 + \cosh(E(t) - \log 2)} \quad \text{(A10)}$
FIG. 5. On the left, representation of the geodesic for a qubit, whose analytical expression is given in Eq. (A16). On the right, average heat produced in a protocol to cancel a qubit as a function of the driving amplitude. Here we only represent the first order contribution to the dissipation (i.e., Eq. (A19) and Eq. (A21)) so the effect appears to be more stark than the one presented in the main text. Still, one can obtain the same behaviour by extending the duration of the experiment.

where we kept only the thermal state in the right hand side, as every differentiation increases one order in \( \tau_{eq}/\tau \).

Then, the dissipation takes the form:

\[
\langle \Delta Q \rangle + k_B T \Delta S = k_B T \int_0^\tau dt \left( -\partial_{\rho(t)} S(\rho(t)||n_{eq}(E(t))) \right) = k_B T \tau_{eq} \int_0^\tau dt \frac{\dot{E}(t)^2}{2 + 2 \cosh(E(t) - \log 2)} + O \left( \frac{(\tau_{eq})^2}{\tau} \right),
\]

where we omitted the lengthy but straightforward calculations. In this case the metric is given by:

\[
g(t) = \frac{1}{2 + 2 \cosh(E(t) - \log 2)},
\]

where we dropped the indices, since there is only one parameter. The corresponding Christoffel symbol can be computed as:

\[
\Gamma(t) = \frac{1}{g(t)} \frac{dg(t)}{dE(t)} = -\frac{1}{2} \tanh \left( \frac{E(t) - \log 2}{2} \right),
\]

and the geodesic equation is given by:

\[
\ddot{E}(t) + \Gamma(t) \dot{E}(t)^2 = 0.
\]

This can be analytically solved to give a closed form for the optimal protocol:

\[
E_g(t) = 2 \log \left( \sqrt{2} \cot \left( A + B \frac{t}{\tau} \right) \right),
\]

where \( A \) and \( B \) can be chosen to fix the initial and final energy \( E_0, E_1 \). In particular, choosing \( E_0 = \log 2 \), and \( E_1 = E_0 + E_A \), the two constants take the value \( A = \pi/4 \) and \( B = -\frac{1}{2} \arcsin(\tanh E_A^2) \).

There is an interesting property that geodesics satisfy in general: they keep the entropy production rate constant. This means that the integrand in Eq. (A12) does not depend on time, as it can be checked by direct calculation:

\[
\frac{\dot{E}_g(t)^2}{2 + 2 \cosh(E_g(t) - \log 2)} = \frac{4 B^2}{\tau^2} = \frac{1}{\tau^2} \left( \arcsin \tanh \frac{E_A}{2} \right)^2,
\]

hence the total dissipation can be easily computed to be:

\[
\langle \Delta Q \rangle + k_B T \Delta S = k_B T \tau_{eq} \int_0^\tau dt \left( \arcsin \tanh \frac{E_A}{2} \right)^2 + O \left( \frac{(\tau_{eq})^2}{\tau} \right) = k_B T \tau_{eq} \left( \arcsin \tanh \frac{E_A}{2} \right)^2 + O \left( \frac{(\tau_{eq})^2}{\tau} \right).\]
This should be compared with the behaviour for a linear drive, for which one obtains:

$$\langle \Delta Q \rangle + k_B T \Delta S = k_B T \frac{T_{eq}}{\tau} \int_0^\tau \frac{dt}{\tau} \frac{E_A^2}{2 + 2 \cosh(E_A \frac{t}{\tau})} + \mathcal{O}\left(\left(\frac{T_{eq}}{\tau}\right)^2\right) = k_B T \frac{T_{eq}}{\tau} \left(\frac{E_A}{2} \tanh\left(\frac{E}{2}\right)\right) + \mathcal{O}\left(\left(\frac{T_{eq}}{\tau}\right)^2\right).$$

As it is shown in Fig. 5, where we plotted the total heat in the two cases as a function of the amplitude, for the geodesic drive the dissipation saturates to $\pi^2/4$, while for the linear drive it linearly diverges. This behaviour mirrors the one shown in Fig. 4.

3. Quantum dot in contact with a bath: realistic model

In the experiment the rate equation is given by:

$$\langle \dot{n}(t) \rangle = \Gamma(0)(1 + bE(t))(1 + f(E(t)))(n_{eq}(E(t)) - \langle n(t) \rangle).$$

Reproducing the considerations of the previous section, we can postulate the ansatz $\langle n(t) \rangle = n_{eq}(E(t)) + \Delta n(t)$, where in this case the perturbation is given by:

$$\Delta n(t) = \frac{T_{eq}}{2} \frac{\dot{E}(t)}{(1 + bE(t))(1 + f(E(t)))(1 + \cosh(E(t) - \log 2))}.$$

Then, Eq. (A12) becomes in this context

$$\langle \Delta Q \rangle + k_B T \Delta S = k_B T T_{eq} \int_0^\tau \frac{dt}{\tau} \frac{\dot{E}(t)^2}{(1 + bE(t))(1 + f(E(t)))(2 + 2 \cosh(E(t) - \log 2))} + \mathcal{O}\left(\left(\frac{T_{eq}}{\tau}\right)^2\right),$$

so that the metric is given by:

$$g(t) = \frac{1}{(1 + bE(t))(1 + f(E(t)))(2 + 2 \cosh(E(t) - \log 2))},$$

while the Christoffel symbol reads:

$$\Gamma(t) = \frac{1}{2} \left(-\frac{b}{bE(t) + 1} + \frac{1}{e^{-E(t)} + 1} + \frac{6}{e^{E(t)} + 2} - 2\right).$$

Despite the lengthy expressions of these quantities, the effective difference between this model and the one in the previous section are quite small.

Appendix B: Geodesic erasure with a transient

In the main text it is argued that allowing the system to equilibrate at the end of the protocol would make the geodesic drive optimal both in terms of minimal dissipation and erasure quality even for fast drivings. We present here numerical evidences that this is in fact the case.

To this end, we consider a transient time $\tau_{trans}$ of twice $\Gamma(0)(1 + bE_1)(1 + f(E_1))$ (the prefactor in Eq. (A22)), which in our experimental set-up corresponds to $\sim 0.05$ s. There are two ways of allocating the total time $\tau + \tau_{trans}$: either the full time is used for the drive, or the drive takes place in time $\tau$, while for the remaining $\tau_{trans}$ the system thermalises at fixed energy. These two possible choices are shown in Fig. 6: the dashed lines correspond to the first choice, while the solid lines correspond to the latter.

The solid lines show how the extra thermalisation makes the erasure quality of the geodesic and linear drive comparable (the difference is of order $\sim 0.1\%$) while the dissipation for the geodesic is always lower in the first case. On the other hand, when considering also the red dashed line (the one corresponding to a linear drive which takes $\tau + \tau_{trans}$) there is a trade-off: for lower amplitudes it dissipates less than the geodesic, but at the cost of a lower erasure quality; for higher amplitudes, instead, the geodesic starts dissipating less, whereas the erasure quality saturates to a similar value. This shows that even if the geodesic drive is in principle designed only for slowly driven systems, it can be minimally modified to give drives that are optimal also in the fast driving regime.
FIG. 6. Numerical simulations showing how allowing time for an extra transient can improve the quality of erasure of the geodesic protocol, while still making it preferable to the linear drive in respect to the amount of dissipated heat. The solid lines correspond to the drivings in which the system is driven for time $\tau$ and then it is let thermalise for an extra time $\tau_{\text{trans}}$, while the dashed lines correspond to allocating time $\tau + \tau_{\text{trans}}$ to the drive.