On Infinite Families of Narrow-Sense Antiprimitive BCH Codes Admitting 3-Transitive Automorphism Groups and their Consequences

Qi Liu\textsuperscript{a}, Cunsheng Ding\textsuperscript{b}, Sihem Mesnager\textsuperscript{c}, Chunming Tang\textsuperscript{d}, Vladimir D. Tonchev\textsuperscript{e}

\textsuperscript{a}School of Mathematics and Information, China West Normal University, Nanchong, Sichuan, 637002, China
\textsuperscript{b}Department of Computer Science and Engineering, The Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong, China
\textsuperscript{c}Department of Mathematics, University of Paris VIII, 93526 Saint-Denis, University Sorbonne Paris Cité, Laboratory Analysis, Geometry and Applications (LAGA), UMR 7539, CNRS, 93430 Villetaneuse, and Telecom Paris, Polytechnic Institute of Paris, 91120 Palaiseau, France
\textsuperscript{d}School of Mathematics and Information, China West Normal University, Nanchong, Sichuan, 637002, China, and also Department of Computer Science and Engineering, The Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong, China
\textsuperscript{e}Department of Mathematical Sciences, Michigan Technological University, Houghton, MI 49931, USA

Abstract

The Bose-Chaudhuri-Hocquenghem (BCH) codes are a well-studied subclass of cyclic codes that have found numerous applications in error correction and notably in quantum information processing. They are widely used in data storage and communication systems. A subclass of attractive BCH codes is the narrow-sense BCH codes over the Galois field GF(q) with length q + 1, which are closely related to the action of the projective general linear group of degree two on the projective line. Despite its interest, not much is known about this class of BCH codes. This paper aims to study some of the codes within this class and specifically narrow-sense antiprimitive BCH codes (these codes are also linear complementary duals (LCD) codes that have interesting practical recent applications in cryptography, among other benefits). We shall use tools and combine arguments from algebraic coding theory, combinatorial designs, and group theory (group actions, representation theory of finite groups, etc.) to investigate narrow-sense antiprimitive BCH Codes and extend results from the recent literature. Notably, the dimension, the minimum distance of some q-ary BCH codes with length q + 1, and their duals are determined in this paper. The dual codes of the narrow-sense antiprimitive BCH codes derived in this paper include almost MDS codes. Furthermore, the classification of PGL(2, p^m)-invariant codes over GF(p^h) is completed. As an application of this result, the p-ranks of all incidence structures invariant under the projective general linear group PGL(2, p^m) are determined. Furthermore, infinite families of narrow-sense BCH codes admitting a 3-transitive automorphism group are obtained. Via these BCH codes, a coding-theory approach to constructing the Witt spherical geometry designs is presented. The BCH codes proposed in

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Email addresses: liuqijichushuxue@163.com (Qi Liu), cding@ust.hk (Cunsheng Ding), smesnager@univ-paris8.fr (Sihem Mesnager), tangchunmingmath@163.com (Chunming Tang), tonchev@mtu.edu (Vladimir D. Tonchev)
1. Introduction

An \([n,k]_q\) linear code \(C\) is a \(k\)-dimensional vector subspace of \(GF(q)^n\), where \(q\) is a prime power. If the linear code \(C\) has minimum distance \(d\), it is also called an \([n,k,d]_q\) code. The dual code \(C^⊥\) of a linear code \(C\) is the set of vectors orthogonal to all codewords of \(C\), i.e.,

\[
C^⊥ = \{w \in GF(q)^n : \langle c, w \rangle = 0 \text{ for all } c \in C\},
\]

where \(\langle c, w \rangle\) is the usual Euclidean inner product of \(c\) and \(w\). A cyclic code \(C\) of length \(n\) over \(GF(q)\) is a linear subspace of \(GF(q)^n\) such that \((c_0, c_1, \ldots, c_{n-1}) \in C\) implies that \((c_{n-1}, c_0, \ldots, c_{n-2}) \in C\). By definition, a cyclic code is a special linear code. Cyclic codes are widely employed in communication systems, storage devices and consumer electronics, as they have efficient encoding and decoding algorithms. If we identify an \(n\)-tuple \(c = (c_0, \ldots, c_{n-1}) \in GF(q)^n\) with the polynomial \(c(x) = \sum_{i=0}^{n-1} c_i x^i\) in the residue class ring \(GF(q)[x]/\langle x^n - 1 \rangle\), any cyclic code \(C\) of length \(n\) over \(GF(q)\) is an ideal in \(GF(q)[x]/\langle x^n - 1 \rangle\). Since the ring \(GF(q)[x]/\langle x^n - 1 \rangle\) is principal, there is an unique monic divisor \(g(x)\) of \(x^n - 1\) of the smallest degree such that \(C = \langle g(x) \rangle\). This polynomial \(g(x)\) is called the generator polynomial of \(C\) and \(h(x) := (x^n - 1)/g(x)\) is referred to as the check polynomial of \(C\). It is easily seen that the reciprocal of \(h(x)\) is the generator polynomial of the dual code \(C^⊥\).

Let \(n\) and \(q\) be coprime. Let \(\beta\) be a primitive \(n\)th root of unity in an extension field of \(GF(q)\). The Bose-Chaudhuri-Hocquenghem (BCH) code \(C_{(q,n,\delta,h)}\) with designed distance \(\delta\) consists of the set of all \(c(x) \in GF(q)[x]/\langle x^n - 1 \rangle\) such that \(c(\beta^{h+i}) = 0\) for all \(i\) in the range \(0 \leq i \leq \delta - 2\), where \(h\) is an arbitrary integer and \(\delta\) is a positive integer with \(2 \leq \delta \leq n\). By definition, the BCH code \(C_{(q,n,\delta,h)}\) has generator polynomial

\[
lcm\{M_{\beta^i}(x), M_{\beta^{i+1}}(x), \ldots, M_{\beta^{i+\delta-2}}(x)\},
\]

where \(M_{\beta^i}(x)\) denotes the minimal polynomial of \(\beta^i\) over \(GF(q)\), and \(lcm\) denotes the least common multiple of a set of polynomials.

It follows from the BCH bound on cyclic codes and the definition of BCH codes that \(\delta\) is a lower bound on the minimum distance of \(C_{(q,n,\delta,h)}\) [21]. When \(h = 1\), the corresponding BCH code \(C_{(q,n,\delta,h)}\) is said to be narrow-sense. If \(n = q^m - 1\), then \(C_{(q,n,\delta,h)}\) is referred to as a primitive BCH code. If \(n = q^{m+1}\), then \(C_{(q,n,\delta,h)}\) is called an antiprimitive BCH code.

The discovery of BCH codes by Bose and Ray-Chaudhuri [1] and independently Hocquenghem [12] has been an outstanding success in the construction of codes based on algebraic structures. An attractive feature of BCH codes is that one can infer valuable information on their minimum distances and dimensions from their design parameters \(q, n, \delta,\) and \(h\). However, determining the actual minimum distance of most BCH codes is a challenging problem (see [5]). For BCH codes \(C_{(q,n,\delta,h)}\), only the lower bound \(\delta\) on their minimum distances is known, and the actual minimum
distance is known only in special cases \([16,20,32]\). In particular, we have very limited knowledge about narrow-sense antiprimitive BCH codes \([17]\).

In this paper, we will consider the narrow-sense antiprimitive \(q\)-ary BCH codes \(C_{(q,q+1,\delta,1)}\) of length \(q+1\) with designed distance \(\delta\). Some of these codes have a relatively large automorphism group. Such codes are of interest from various points of view and have certain advantages since the number of computations needed for encoding and decoding can be considerably reduced when the automorphism group is sufficiently large \([19]\). Employing the action and representation of the finite linear group of degree two, we will show that the supports of the codewords of minimum weight belong to the finite linear group of degree two, which answers the question whether there is a coding-theory construction for the Witt spherical geometry designs in the affirmative. A short description of the spherical geometry designs given by Witt can be found in \([30]\). We will also present a complete classification of PGL\((2,p^m)\)-invariant \(p^h\)-ary codes and derive the \(p\)-ranks of PGL\((2,p^m)\)-invariant combinatorial designs. The codes treated in this paper are linear complementary dual (LCD) codes, which are important in coding theory for theoretical \([4]\) and practical reasons \([3]\) (especially, as discovered in cryptography, against side-channel attacks and fault injection attacks). This paper also generalizes and extends the results in \([10]\).

This article is organized as follows. Sec. 2 introduces fundamental notions of algebraic coding theory, combinatorial designs, and group actions. Sec. 3 considers actions and representations of the finite linear group of degree two. Sec. 4 completes the classification of PGL\((2,p^m)\)-invariant codes over GF\((p^h)\) and gives the \(p\)-rank of PGL\((2,p^m)\)-invariant \(t\)-designs. Sec. 5 investigates the parameters and automorphisms of the BCH codes studied in this paper and presents a coding-theory construction of the Witt spherical geometry designs. Finally, Sec. 6 concludes this paper and explains an important motivation of constructing a linear code supporting a known \(t\)-design.

2. Preliminaries

Throughout this paper, \(p\) is a prime and GF\((q)\) is the finite field of order \(q\), where \(q = p^m\) for some positive integer \(m\). The set of non-zero elements of GF\((q)\) is denoted by GF\((q)^*\). The main goal of this paper is to push further the investigation about the narrow-sense BCH codes for which the information on them is still thin. To achieve the objective of this paper, we need to introduce basic notions of algebraic coding theory, combinatorial designs, and group actions in this section. For additional background on these subjects, the reader is referred to \([6,7,13,21]\).

2.1. Linear codes and combinatorial \(t\)-designs

An \([n,k,d]\) linear code \(C\) over GF\((q)\) is a linear subspace of GF\((q)^n\) with dimension \(k\) and minimum (Hamming) distance \(d\). An \([n,k,n-k+1]\) linear code is called a maximum distance separable (MDS) code. An \([n,k,n-k]\) linear code is said to be almost maximum distance separable (almost MDS, for short). Given a linear code \(C\) of length \(n\) over GF\((q)\), its (Euclidean) dual code is denoted by \(C^\perp\). The code \(C^\perp\) is defined by

\[ C^\perp = \{ (b_0,b_1, \ldots, b_{n-1}) \in \text{GF}(q)^n : \langle c, b \rangle := \sum_{i=0}^{n-1} c_i b_i = 0, \quad \forall (c_0, c_1, \ldots, c_{n-1}) \in C \}. \]

A linear complementary dual code (abbreviated LCD) is defined as a linear code \(C\) whose dual code \(C^\perp\) satisfies \(C \cap C^\perp = \{0\}\). Let \(v\) be a positive integer, \(a = (a_0, \ldots, a_{v-1}) \in \text{GF}(q)^*^v\) and \(C\)
be a \([v,k]_q\) linear code. Let \(a \cdot C\) denote the linear code \(\{(a_0c_0, \ldots, a_{v-1}c_{v-1}) : (c_0, \ldots, c_{v-1}) \in C\}\). It is a simple matter to check that

\[
(a \cdot C)^\perp = a^{-1} \cdot C^\perp,
\]

where \(a^{-1} = (a_0^{-1}, \ldots, a_{v-1}^{-1})\). Let \(C\) be a \([v,k,d]\) linear code over \(GF(q)\). Let \(P\) be the set of coordinate positions of codewords of \(C\) and let \(Sym(P)\) be the symmetric group acting on \(P\). An element \(c\) of \(C\) could be written as \(c = (c_x)_{x \in P}\). The permutation group \(PAut(C)\) of \(C\) is the subgroup of \(Sym(P)\) which leaves the code globally invariant. More precisely, it is the subgroup of those \(g\) satisfying

\[
g(c_x)_{x \in P} = (c_{g^{-1}x})_{x \in P} \in C \text{ for all } (c_x)_{x \in P} \in C.
\]

The monomial automorphism group \(MAut(C)\) of \(C\) is the subgroup of \((GF(q)^*)^n \rtimes Sym(P)\) which leaves the code globally invariant. More precisely, it is the subgroup of those \((((a_x)_{x \in P}; g)\) satisfying

\[
((a_x)_{x \in P}; g) (c_x)_{x \in P} = (a_gc_{g^{-1}x})_{x \in P} \in C \text{ for all } (c_x)_{x \in P} \in C.
\]

Let Gal\((GF(q))\) denote the Galois group of \(GF(q)\) over its prime field. The automorphism group \(Aut(C)\) of \(C\) is the subgroup of \((GF(q)^*)^n \rtimes (Sym(P) \times Gal(GF(q)))\) which maps \(C\) onto itself. More precisely, it is the subgroup of those \((((a_x)_{x \in P}; g; \gamma)\) satisfying

\[
((a_x)_{x \in P}; g; \gamma) (c_x)_{x \in P} = (a_{\gamma}c_{g^{-1}x})_{x \in P} \in C \text{ for all } (c_x)_{x \in P} \in C.
\]

We say that \(Aut(C)\) is \(t\)-homogeneous (respectively, \(t\)-transitive) if for every pair of \(t\)-element sets of coordinates (respectively, \(t\)-element ordered sets of coordinates), there is an element \(((a_x)_{x \in P}; g; \gamma)\) of the automorphism group \(Aut(C)\) such that its permutation part \(g\) sends the first set to the second set.

Let \(P\) be a set of \(v \geq 1\) elements, and let \(B\) be a set of \(k\)-subsets of \(P\), where \(k\) is a positive integer with \(1 \leq k \leq v\). Let \(t\) be a positive integer with \(t \leq k\). The pair \(\mathbb{D} = (P, B)\) is called a \(t-(v,k,\lambda)\) design, or simply \(t\)-design, if every \(t\)-subset of \(P\) is contained in exactly \(\lambda\) elements of \(B\). The elements of \(P\) are called points, and those of \(B\) are referred to as blocks. A \(t-(v,k,\lambda)\) design is referred to as a Steiner system if \(t \geq 2\) and \(\lambda = 1\), and is denoted by \(S(t,k,v)\).

The interplay between coding theory and \(t\)-designs started many years ago. Let \(C\) be a \([v,k,d]\) linear code over \(GF(q)\). Let \(A_i := A_i(C)\) denote the number of codewords with Hamming weight \(i\) in \(C\), where \(0 \leq i \leq v\). For each \(k\) with \(A_k \neq 0\), let \(B_k(C)\) denote the set of the supports of all codewords with Hamming weight \(k\) in \(C\), where the coordinates of codewords are indexed by \(\{p_1, \ldots, p_v\}\). Let \(\mathcal{P} = \{p_1, \ldots, p_v\}\). The pair \((\mathcal{P}, B_k(C))\) may be a \(t-(v,k,\lambda)\) design for some positive integer \(\lambda\), which is called a design supported by the code, or shortly the support design of the code, and is denoted by \(\mathbb{D}_k(C)\). In such a case, we say that the code \(C\) holds or supports a \(t-(v,k,\lambda)\) design. If \(C\) has a \(t\)-homogeneous or \(t\)-transitive automorphism group, the codewords of any weight \(i \geq t\) of \(C\) hold a \(t\)-design [7, Theorem 4.30].

The incidence matrix \(A = (a_{i,j})\) of a \(t\)-design \(\mathbb{D}\) is a \((0,1)\)-matrix with rows indexed by the blocks, and columns indexed by the points of \(\mathbb{D}\), where \(a_{i,j} = 1\) if the \(j\)th point belongs to the \(i\)th block, and \(a_{i,j} = 0\) otherwise. If \(p\) is a prime, the \(p\)-rank of \(\mathbb{D}\) (or rank\(_p\)\(\mathbb{D}\)) is defined as the rank of its incidence matrix \(A\) over \(GF(p)\): \(\text{rank}_p \mathbb{D} = \text{rank}_p A\). Equivalently, the \(p\)-rank of a design is
the dimension of the linear $p$-ary code spanned by the rows of its $(0, 1)$-incidence matrix. The $p$-rank of incidence structures, i.e., the dimension of the corresponding codes, can be used to classify incidence structures of certain types. For example, the 2-rank and 3-rank of Steiner triple and quadruple systems were intensively studied and employed for counting and classifying Steiner triple and quadruple systems [14], [15], [24], [28], [29], [34], [35], [36], [37].

2.2. Constructions of $t$-designs from group actions

If $X$ is a set (usually, some kind of geometric space), the “symmetries" of $X$ are often captured by the action of a group $G$ on $X$.

**Definition 1.** Given a set $X$, and a group $G$, a left action of $G$ on $X$ (for short, an action of $G$ on $X$) is a mapping $\phi : G \times X \rightarrow X$, such that

1. For all $g, h \in G$ and all $x \in X$,
   \[
   \phi(g, \phi(h, x)) = \phi(gh, x);
   \]
2. For all $x \in X$, $\phi(1, x) = x$,

   where $1 \in G$ is the identity element of $G$.

We also call this data a (left) $G$-set $X$ or say that “$G$ acts on $X$" (on the left). To alleviate the notation, we usually write $g(x)$ or even $gx$ for $\phi(g, x)$. Given an action $\phi : G \times X \rightarrow X$, for every $g \in G$, we have a permutation $\phi_g$ over $X$ defined by $\phi_g(x) = g \cdot x$, for all $x \in X$. Then, the map $g \mapsto \phi_g$ is a group homomorphism from $G$ to the symmetric group $\text{Sym}(X)$ of $X$. With a slight abuse of notation, this group homomorphism $G \rightarrow \text{Sym}(X)$ is also denoted $\phi$.

Recall that a finite group $G$ acting on a set $X$ is said to be $t$-transitive if for every pair of ordered $k$-tuples of distinct points $(x_1, \ldots, x_t)$ and $(x'_1, \ldots, x'_t)$ there exists an element $g \in G$ such that $gx_i = x'_i$, $1 \leq i \leq t$. In particular, a transitive group is 1-transitive. Let $t\binom{k}{t}$ be the set of subsets of $X$ consisting of $k$ elements. A group action of $G$ on $X$ induces an action of $G$ on the set $t\binom{k}{t}$ for each $1 \leq k \leq |X|$ and given by $(g, B) \mapsto gB := \{gx : x \in B\}$. The group $G$ is said to act $t$-homogenously on $X$ if $G$ acts transitively on $t\binom{k}{t}$.

We recall a well-known general fact (see, e.g. [2, Proposition 4.6]), that for a $t$-homogeneous group $G$ on a finite set $X$ with $|X| = v$ and a subset $B$ of $X$ with $|B| = k > t$, the pair $(X, \text{Orb}_B)$ is a $t$-$(v, k, \lambda)$ design, where $\text{Orb}_B$ is the orbit of $B$ under the action of $G$ on $t\binom{k}{t}$, $\lambda = \frac{v}{t} \frac{(\binom{t}{k}) |G|}{(\binom{k}{t}) |\text{Stab}_B|}$ and $\text{Stab}_B$ is the stabilizer of $B$ for this action. For some recent works on $t$-designs from group actions, we refer the reader to [25, 31].

2.3. PGL(2, $q$) and Witt spherical geometry designs

The projective linear group $\text{PGL}(2, q)$ of degree two is defined as the group of invertible $2 \times 2$ matrices with entries in $\text{GF}(q)$, modulo the scalar matrices $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$, where $a \in \text{GF}(q)^{*}$. 
Here the following convention for the action of \( \text{PGL}(2, q) \) on the projective line \( \text{PG}(1, q) \) is used. A matrix \( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{PGL}(2, q) \) acts on \( \text{PG}(1, q) \) by

\[
(x_0 : x_1) \mapsto \begin{bmatrix} a & b \\ c & d \end{bmatrix} (x_0 : x_1) = (ax_0 + bx_1 : cx_0 + dx_1),
\]

or, via the usual identification of \( GF(q) \) with the usual conventions of defining elements in \( GF(q) \),

\[
\text{PGL}(2, q) \text{ acts \ on \ } \text{PG}(1, q) \text{, by linear fractional transformation}
\]

\[
x \mapsto \frac{ax + b}{cx + d},
\]

with the usual conventions of defining \( \frac{a(-d/c)+b}{c(-d/c)+d} = \infty \) and \( \frac{a\infty + b}{c\infty + d} = a/c \).

This is an action on the left, i.e., for \( g_1, g_2 \in \text{PGL}(2, q) \) and \( x \in \text{PG}(1, q) \) the following holds: \( g_1(g_2(x)) = (g_1g_2)(x) \). The action of \( \text{PGL}(2, q) \) on \( \text{PG}(1, q) \) defined in (3) is sharply 3-transitive, i.e., for any distinct \( a, b, c \in GF(q) \) there is \( g \in \text{PGL}(2, q) \) taking \( \infty \) to \( a \), \( 0 \) to \( b \), and \( 1 \) to \( c \). In fact, \( g \) is uniquely determined and it equals

\[
g = \begin{bmatrix} a(b-c) & b(c-a) \\ b-c & c-a \end{bmatrix}.
\]

Thus, \( \text{PGL}(2, q) \) is in one-to-one correspondence with the set of ordered triples \( (a, b, c) \) of distinct elements in \( GF(q) \) and

\[
|\text{PGL}(2, q)| = (q + 1)q(q - 1).
\]

Put \( B = \text{PG}(1, q) \) and \( \text{Orb}_B = \{ gB : g \in \text{PGL}(2, q^m) \} \). Note that \( B \) is a subset of \( \text{PG}(1, q^m) \). Define \( \mathbb{D} = (B, \text{Orb}_B) \). Since \( \text{PGL}(2, q^m) \) acts 3-transitively on \( \text{PG}(1, q^m) \), \( \mathbb{D} \) is a \( 3-(q^m + 1, q + 1, \lambda) \) design for some \( \lambda \). Since \( \text{PGL}(2, q^m) \) is sharply 3-transitive on \( \text{PG}(1, q^m) \) and \( \text{PGL}(2, q) \) is sharply 3-transitive on \( B \), \( \text{PGL}(2, q) \) is the setwise stabiliser of \( B \). Consequently, \( \lambda = 1 \) and \( \mathbb{D} = (\text{PG}(1, q^m), \text{Orb}_B) \) is a Steiner system \( S(3, q + 1, q^m + 1) \). These Steiner systems were constructed by Witt [30], and are called Witt spherical geometry designs. A coding-theoretic construction of the Witt spherical geometry design \( S(3, q + 1, q^m + 1) \) was given in [9] for \( q = 3 \) and in [10, 28] for \( q = 4 \). Whether there exists an infinite family of linear codes holding the Witt spherical geometry design \( S(3, q + 1, q^m + 1) \) for \( q \geq 5 \) being a prime power has been an open problem. This paper will settle this open problem by presenting an infinite family of BCH codes holding the Witt spherical geometry design \( S(3, q + 1, q^m + 1) \) for any prime power \( q \) and positive integer \( m \geq 2 \).

2.4. The cyclicity-defining sets and trace representations of cyclic codes

Given a linear code \( C \) of length \( n \) and dimension \( k \) over \( GF(r) \), we define a linear code \( GF(r^h) \otimes C \) over \( GF(r^h) \) by

\[
GF(r^h) \otimes C = \left\{ \sum_{i=1}^{k} a_i c_i : (a_1, a_2, \ldots, a_k) \in GF(r^h)^k \right\},
\]

where \( \{c_1, c_2, \ldots, c_k\} \) is a basis of \( C \) over \( GF(r) \). This code is independent of the choice of the basis \( \{c_1, c_2, \ldots, c_k\} \) of \( C \), is called the lifted code of \( C \) to \( GF(r^h) \). Clearly, \( GF(r^h) \otimes C \) and \( C \) have
the same length, dimension and minimum distance, but different weight distributions. A trivial verification shows that if \((c_0, \ldots, c_{n-1}) \in \mathbb{GF}(r^h) \otimes C\), then \((c_0', \ldots, c_{n-1}') \in \mathbb{GF}(r^h) \otimes C\).

Let \(n\) be a positive integer with \(\gcd(n, r) = 1\). The order \(\text{ord}_n(r)\) of \(r\) modulo \(n\) is the smallest positive integer \(h\) such that \(r^h \equiv 1 \pmod{n}\). Let \(\mathbb{Z}_n\) denote the ring of residue classes of integers modulo \(n\). The \(r\)-cyclotomic coset of \(e \in \mathbb{Z}_n\) is the set \([e]_{(r,n)} = \{r^i e \pmod{n} : 0 \leq i \leq \text{ord}_n(r) - 1\}\), where \(x \pmod{n}\) denotes the unique integer \(\ell\) such that \(0 \leq \ell \leq n - 1\) and \(x \equiv \ell \pmod{n}\). Then any two \(r\)-cyclotomic cosets are either equal or disjoint. A subset \(E\) of \(\mathbb{Z}_n\) is called \(r\)-invariant if the set \(\{re \pmod{n} : e \in E\}\) equals \(E\), that is, \(E\) is the union of some \(r\)-cyclotomic cosets. A subset \(\tilde{E} = \{e_1, \ldots, e_t\}\) of an \(r\)-invariant set \(E\) is called a complete set of representatives of \(r\)-cyclotomic cosets of \(E\) if \([e_1]_{(r,n)}, \ldots, [e_t]_{(r,n)}\) are pairwise distinct and \(E = \bigcup_{i=1}^t [e_i]_{(r,n)}\).

Let \(\gamma\) be a primitive \(n\)-th root of unity in \(\mathbb{GF}(r^h)\), where \(h = \text{ord}_n(r)\). It is known \([13]\) that any \(r\)-ary cyclic code of length \(n\) with \(\gcd(n, r) = 1\) has a simple description by means of the trace function. The trace function \(\text{Tr}_{q^f/q} : \mathbb{GF}(q^h) \to \mathbb{GF}(q)\) is defined as:

\[
\text{Tr}_{q^f/q}(x) := \sum_{i=0}^{h-1} x^i = x + x^q + x^{q^2} + \cdots + x^{q^{h-1}}.
\]

The trace function from \(\mathbb{F}_{q^h}\) to its prime subfield is called the absolute trace function.

**Theorem 2.** \([13]\) Let \(C\) be an \([n,k]_r\) cyclic code with \(\gcd(n, r) = 1\) and \(\gamma\) be a primitive \(n\)-th root of unity in \(\mathbb{GF}(r^h)\), where \(h = \text{ord}_n(r)\). Then there exists a unique \(r\)-invariant set \(E \subseteq \mathbb{Z}_n\) such that

\[
C = \left\{ \left( \sum_{i=1}^t \text{Tr}_{r^{h_i}/r} \left( a_i \gamma^{f_{ij}} \right) \right)^{n-1} : a_i \in \mathbb{GF}(r^{h_i}) \right\},
\]

where \(\{e_1, \ldots, e_t\}\) is any complete set of representatives of \(r\)-cyclotomic cosets of \(E\) and \(h_i = \left| [e_i]_{(r,n)} \right|\). Moreover, \(k = |E| = \sum_{i=1}^t h_i\).

Theorem 2 states that there is a one-to-one correspondence between cyclic linear codes over \(\mathbb{GF}(r)\) with length \(n\) and \(r\)-invariant subsets of \(\mathbb{Z}_n\) with respect to a fixed \(n\)-th root of unity \(\gamma\). We will call the set \(E\) in Theorem 2 the cyclicity-defining set of \(C\) with respect to \(\gamma\).

The following corollary is an immediate consequence of Theorem 2.

**Corollary 3.** \([13]\) Let \(n\) be a positive integer such that \(\gcd(n, r) = 1\). Let \(C\) be an \([n,k]_r\) cyclic code with cyclicity-defining set \(E\) and \(\mathbb{GF}(r^f) \otimes C\) be the lifted code of \(C\) to \(\mathbb{GF}(r^f)\). Then \(\mathbb{GF}(r^f) \otimes C\) is an \([n,k]_r\) cyclic code defined by the cyclicity-defining set \(E\) of \(C\). In particular,

\[
\mathbb{GF}(r^h) \otimes C = \left\{ \left( \sum_{e \in E} a_e \gamma^{fe} \right)^{n-1} : a_e \in \mathbb{GF}(r^h) \right\},
\]

where \(h = \text{ord}_n(r)\) and \(\gamma\) is a primitive \(n\)-th root of unity in \(\mathbb{GF}(r^h)\).

Hence, the two code \(\mathbb{GF}(r^f) \otimes C\) and \(C\) have the same cyclicity-defining sets. Let \(n\) be a positive integer with \(\gcd(n, r) = 1\) and \(h = \text{ord}_n(r)\). Let \(U_n\) be the cyclic multiplicative group of all \(n\)-th roots
of unity in \(GF(q^2)\). By polynomial interpolation, every function \(f\) from \(U_n\) to \(GF(r)\) has a unique univariate polynomial expansion of the form

\[
f(u) = \sum_{i=0}^{n-1} a_i u^i,
\]

where \(a_j \in GF(r^h), u \in U_n\). As a direct result of Theorem 2, we have the following conclusion concerning cyclicity-defining sets of cyclic codes.

**Corollary 4.** [10] Let \(n\) be a positive integer with \(gcd(n, r) = 1\), \(h = \text{ord}_r(\gamma)\) and \(\gamma\) a primitive \(n\)-th root of unity in \(GF(r^h)\). Let \(C\) be an \([n,k]\_r\) cyclic code with cyclicity-defining set \(E\). Let \(f(u) = \sum_{i=0}^{n-1} a_i u^i \in GF(r^h)[u]\). If \((f(\gamma^j))_{j=0}^{n-1} \in C\) and \(a_i \neq 0\), then \(i \in E\).

3. **Group actions and representations of \(GL(2,q)\) and \(PGL(2,q)\)**

This section considers actions and representations of the projective general linear group \(PGL(2,q)\). These results will play an important role in Sections 4 and 5.

3.1. **Stabilizers of certain subsets of the projective line \(PG(1,q^2)\)**

Let \(U_{q+1}\) be the subgroup of \(GF(q^2)\) consisting of elements whose norm to \(GF(q)\) is 1. By Hilbert Theorem 90, we may describe the elements in \(U_{q+1}\) in terms of the elements of the projective line \(PG(1,q)\) as

\[
U_{q+1} = \left( \begin{array}{cc} u_0 & 1 \\ 1 & u_0 \end{array} \right) \text{PG}(1,q) = \left\{ \frac{u_0 x + 1}{x + u_0} : x \in \text{PG}(1,q) \right\},
\]

with \(u_0 \in U_{q+1} \setminus \{\pm 1\}\) and the convention that the quotient is \(u_0\) for \(x = \infty\). Recall that the setwise stabilizer of \(\text{PG}(1,q)\) under the action of \(PGL(2,q^2)\) on \(\text{PG}(1,q^2)\) is \(PGL(2,q)\). Then the two setwise stabilizer groups \(\text{Stab}_{U_{q+1}}\) and \(\text{Stab}_{PG(1,q)} = PGL(2,q)\) are related by

\[
\text{Stab}_{U_{q+1}} = \left( \begin{array}{cc} u_0 & 1 \\ 1 & u_0 \end{array} \right) \text{PGL}(2,q) \left( \begin{array}{cc} u_0 & 1 \\ 1 & u_0 \end{array} \right)^{-1}.
\]

Let \(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PGL(2,q)\). A standard computation gives

\[
\left( \begin{array}{cc} u_0 & 1 \\ 1 & u_0 \end{array} \right) \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \left( \begin{array}{cc} u_0 & 1 \\ 1 & u_0 \end{array} \right)^{-1} = \frac{u_0'}{(u_0'^2 - 1)} \left( \frac{u_0'}{c} \frac{c'}{d} \right),
\]

where \(u_0' = \sqrt{-1}u_0, c = u_0'^{-1}(cu_0^2 + (a - d)u_0 - b), d = u_0'^{-1}(du_0^2 + (b - c)u_0 - a)\) and \(\sqrt{1} \in GF(q^2)\).

Then the following commutative diagram is obtained:

\[
\begin{array}{ccc}
GF(q) \cup \{\infty\} & \xrightarrow{\frac{u_0 x + 1}{x + u_0}} & U_{q+1} \\
\xrightarrow{\frac{ax+b}{cx+d}} & \downarrow \xrightarrow{\frac{u_0' x + 1}{x + u_0'}} \\
GF(q) \cup \{\infty\} & \xrightarrow{\frac{u_0 x + 1}{x + u_0}} & U_{q+1}
\end{array}
\]
Write $A = \left\{ \left( \frac{d^q}{c}, \frac{c^q}{d} \right) : c, d \in \text{GF}(q^2), c^{q+1} \neq d^{q+1} \right\}$. Combining (6) and (7) yields $\text{Stab}_{U_{q+1}} \subseteq A$. On the other hand, it is clear that $A \subseteq \text{Stab}_{U_{q+1}}$ by Hilbert Theorem 90. We thus deduce that $\text{Stab}_{U_{q+1}} = A$. By the commutativity of the diagram in (8), we see that the action of $\text{Stab}_{U_{q+1}}$ on $U_{q+1}$ is equivalent to the action of $\text{PGL}(2, q)$ on $\text{PG}(1, q)$.

A summary of the above discussion implies the following.

**Proposition 5.** The setwise stabilizer $\text{Stab}_{U_{q+1}}$ of $U_{q+1}$ can be expressed as

\[
\text{Stab}_{U_{q+1}} = \left\{ \left( \frac{d^q}{c}, \frac{c^q}{d} \right) \in \text{PGL}(2, q^2) : c^{q+1} \neq d^{q+1} \right\}.
\]

Furthermore, the action of $\text{Stab}_{U_{q+1}}$ on $U_{q+1}$ is equivalent to the action of $\text{PGL}(2, q)$ on $\text{PG}(1, q)$. Hence, $\text{Stab}_{U_{q+1}}$ is sharply 3-transitive.

### 3.2. A modular representation of $\text{GL}(2, q)$

Representation theory of finite groups is a potent tool for constructing linear codes invariant under a given group $G$. Basic material related to the representation theory of groups can be found in [18].

**Definition 6.** A representation of a group $G$ is a pair $(V, \rho)$ where $V$ is a vector space over a field $\mathbb{F}$ and $\rho$ is a map from $G \times V$ to $V$ such that

1. $\rho$ is a group action (the action is associative and $\rho(1, v) = v$ for all $v \in V$), and
2. the map $V \to V$ defined by $v \mapsto \rho(g, v)$ is linear for all $g \in G$.

The dimension of $V$ over $\mathbb{F}$ is called the degree of the representation. A representation is called an ordinary representation if $\text{char}(\mathbb{F}) \nmid |G|$, and it is called a modular representation if $\text{char}(\mathbb{F}) | |G|$.

A representation is nothing but a linear action of $G$ on $V$. We can also think of a group representation $(V, \rho)$ of $G$ as a group homomorphism from $G$ to $\text{GL}(V)$, where $\text{GL}(V)$ denotes the group of invertible linear transformations from $V$ to itself. If $V$ has a basis $v_1, \ldots, v_n$ then we can identify $\text{GL}(V)$ with the more familiar group $\text{GL}_n$ of invertible $n \times n$ matrices. Informally speaking, a representation of a group $G$ is a way of writing the group elements as square matrices of the same size, which is multiplicative and assigns $1 \in G$ the identity matrix.

Consider the set

\[
\mathcal{P}(\delta, q) := \left\{ \text{Tr}_{q^2/q} \left( \sum_{i=1}^{\delta} a_i u^i \right) \in \text{GF}(q^2)[u]/\langle u^{q+1} - 1 \rangle : a_i \in \text{GF}(q^2) \right\}.
\]

For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \in \text{GL}(2, q^2)$ and $f \in \mathcal{P}(\delta, q)$ we set

\[
(A \circ f)(u) := (cu + d)^{(q+1)(\delta-1)} f \left( \frac{au+b}{cu+d} \right).
\]

The following result on binomial coefficients will be used in evaluating $A \circ f$.

**Lemma 7.** Let $\delta$ be a power of a prime $p$ and let $e$ be a positive integer with $e \leq \delta - 1$. Let $s$ be an integer with $e \leq s \leq \delta - 1$. Then the binomial coefficient $\binom{\delta-1+e}{s}$ is divisible by $p$. 9
Proof. Expanding the power \((1 + x)^{\delta-1+e} \in \text{GF}(p)[x]\) by the Binomial Theorem, we have
\[
(1 + x)^{\delta-1+e} = \sum_{i=0}^{\delta-1+e} \binom{\delta-1+e}{i} x^i.
\]
Since \(\delta\) is a power of \(p\), we get
\[
(1 + x)^{\delta-1+e} \equiv (1 + x)^{\delta(1+x)^{e-1}} \equiv (1+x^\delta)(1+x)^{e-1} \pmod{p}.
\]
Then, we have
\[
\sum_{i=0}^{\delta-1+e} \binom{\delta-1+e}{i} x^i \equiv (1+x^\delta)(1+x)^{e-1} \pmod{p}.
\] (11)
By comparing the coefficients of \(x^i\) \((\leq s \leq \delta - 1)\) on both sides we see that \(\binom{\delta-1+e}{s} \equiv 0 \pmod{p}\), which completes the proof of the lemma.

The following lemma shows that for \(A = \begin{pmatrix} d^q & c^q \\ c & d \end{pmatrix}^{-1} \in \text{GL}(2, q^2)\), \(f \mapsto A \circ f\) defines a GF\((q)\)-linear transformation over \(\mathcal{P}(\delta, q)\) in some cases.

**Lemma 8.** Let \(q = p^m\) and \(\delta\) be a power of \(p\). Let \(A = \begin{pmatrix} d^q & c^q \\ c & d \end{pmatrix}^{-1} \in \text{GL}(2, q^2)\) and \(f \in \mathcal{P}(\delta, q)\). Then \(A \circ f \in \mathcal{P}(\delta, q)\).

*Proof.* In order to prove this lemma it suffices to prove the following claim: for any integer \(e\) with \(1 \leq e \leq \delta - 1\) and \(\beta \in \text{GF}(q^2)\), we have \(A \circ g \in \mathcal{P}(\delta, q)\), where \(g = \text{Tr}_{q^2/q}(\beta u^e)\). By the definition of the operator \('\circ'\), we have
\[
(A \circ g)(u) = (cu + d)^{(q+1)(\delta-1)} \text{Tr}_{q^2/q}(\beta \left(\frac{d^q u + c^q}{cu + d}\right)^e) = (c^q u^{-1} + d^q)^{\delta-1} (cu + d)^{\delta-1} \text{Tr}_{q^2/q}(\beta \left(\frac{d^q u + c^q}{cu + d}\right)^e) = \text{Tr}_{q^2/q}(\beta u^e (cu + d)^{\delta-1} (c^q u^{-1} + d^q)^{\delta-1+e}).
\] (12)
Using the Binomial Theorem, we get
\[
u^e (cu + d)^{\delta-1-e} = \sum_{i=0}^{\delta-1-e} \binom{\delta-1-e}{i} a_i u^{i+e}
\]
and
\[
(c^q u^{-1} + d^q)^{\delta-1+e} = \sum_{j=0}^{\delta-1+e} \binom{\delta-1+e}{j} b_j u^{-j},
\]
where \(a_i = c^i d^{\delta-1-e-i}\) and \(b_j = c^j d^q (\delta-1+e-j)\). By (12) we have
\[
(A \circ g)(u) = \text{Tr}_{q^2/q} \left( \beta \sum_{0 \leq i \leq \delta-1-e} \sum_{0 \leq j \leq \delta-1+e} \binom{\delta-1-e}{i} \binom{\delta-1+e}{j} a_i b_j u^{i-j+e} \right).
\] (13)
Note that $0 \leq i \leq \delta - 1 - e$ and $0 \leq j \leq \delta - 1 + e$ imply that $- (\delta - 1) \leq i - j + e \leq \delta - 1$. Hence there are $c_i \in \text{GF}(q^2)$ $(1 \leq t \leq \delta - 1)$ such that

$$(A \circ g)(u) = \text{Tr}_{q^2/q} \left( \beta \sum_{i=1}^{\delta-1} c_i u^i \right) + \text{Tr}_{q^2/q}(\beta W),$$

(14)

where $W = \sum_{t=0}^{\delta-1-t} \beta_{t+e} \beta_{t+e-1} a_{t+e} b_{t+e}$. By Lemma 7 we deduce that $(p^{-1+e}) \equiv \beta \pmod{p}$. It follows that $W = 0$, which implies that $A \circ g \in \mathcal{P}(\delta, q)$. This completes the proof.

We consider the lifting of $\text{Stab}_{U_{q+1}}$ in $\text{GL}(2, q^2)$

$$\overline{\text{Stab}}_{U_{q+1}} = \left\{ \begin{pmatrix} d^q & c^q \\ c & d \end{pmatrix} : c, d \in \text{GF}(q^2), c^{q+1} \neq d^{q+1} \right\}.$$  

(15)

The following lemma implies that the action $\circ'$ defined in (10) gives a representation of $\overline{\text{Stab}}_{U_{q+1}}$ on the linear space $\mathcal{P}(\delta, q)$. It comes straightforwardly from Lemma 8 and the definition of the action $\circ'$.

**Lemma 9.** Let $q = p^m$ and $\delta$ be a power of $p$. Let $A_1, A_2 \in \overline{\text{Stab}}_{U_{q+1}}$ and $f_1, f_2 \in \mathcal{P}(\delta, q)$, and denote by $E$ the $2 \times 2$ identity matrix. Then the following hold:

1. $A_1 \circ f_1 \in \mathcal{P}(\delta, q)$,
2. $E \circ f_1 = f_1$,
3. $(A_1 A_2) \circ f_1 = A_1 \circ (A_2 \circ f_1)$,
4. $A_1 \circ (af_1 + bf_2) = aA_1 \circ f_1 + bA_2 \circ f_2$ for all $a, b \in \text{GF}(q)$.

By (6), we have $\overline{\text{Stab}}_{U_{q+1}} = \left( \begin{pmatrix} u_0 & 1 \\ 1 & u_0 \end{pmatrix} \right)^{-1} \text{GL}(2, q) \left( \begin{pmatrix} u_0 & 1 \\ 1 & u_0 \end{pmatrix} \right)$. Hence, the representation of $\overline{\text{Stab}}_{U_{q+1}}$ gives rise to a representation of $\text{GL}(2, q)$ on $\mathcal{P}(\delta, q)$.

**4. PGL(2, q)-invariant codes**

The main objective of this section is to classify all linear codes over $\text{GF}(p^h)$ of length $p^m + 1$ that are invariant under $\text{PGL}(2, p^m)$. As an immediate application, we derive the $p$-rank of the incidence matrices of $t$-$(p^m + 1, k, \lambda)$ designs that are invariant under $\text{PGL}(2, p^m)$.

Let $C$ be a $[p^m + 1, k]_{\rho, \lambda}$ linear code. We can regard $U_{p^m + 1}$ as the set of the coordinate positions of $C$ and write the codeword of $C$ as $(c_u)_{u \in U_{p^m + 1}}$. Then the set of coordinate positions of $C$ could be endowed with the action of $\text{Stab}_{U_{p^m + 1}}$. According to Proposition 5, we only need to find all linear codes over $\text{GF}(p^h)$ of length $p^m + 1$ which are invariant under $\text{Stab}_{U_{p^m + 1}}$.

The following lemma gives the polynomial expansion of the linear fractional transformation $\frac{u - c^q}{-cu + 1}$, where $c \in \text{GF}(q^2)^* \setminus U_{q+1}$.

**Lemma 10.** Let $c \in \text{GF}(q^2)^* \setminus U_{q+1}$. Then for any $u \in U_{q+1}$, the following holds:

$$\frac{u - c^q}{-cu + 1} = \sum_{i=1}^{q} c^{i-1} u^i.$$
Proof. An easy computation shows that
\[
\sum_{i=1}^{q} c^{i-1} u^i = \frac{1 - (cu)^q}{1 - cu} u \\
= \frac{u - c^q u^q + 1}{u - c^q (u^q + 1)},
\]
which completes the proof.

The following lemma expresses the coefficients of the polynomial expansion of a function \( f \) over \( U_{q+1} \) in terms of the sums over \( U_{q+1} \) of the product function of \( f \) and the power functions \( u^j \). It comes directly using the same argument as in the case that \( q \) is a power of 2.[10, Lemma].

Lemma 11. Let \( f \) be a function from \( U_{q+1} \) to \( GF(q^{2h}) \) with \( h \geq 1 \). Let \( \sum_{i=0}^{q} a_i u^i \) be the polynomial expansion of \( f \), where \( a_i \in GF(q^{2h}) \). Then \( a_i = \sum_{u \in U_{q+1}} f(u) u^{-i} \), where \( 0 \leq i \leq q \).

Proof. Applying Lemma 11 to the function \( f(u) = (gu)^e \), we obtain
\[
a_1 = \sum_{u \in U_{q+1}} (gu)^e u^{-1} = \sum_{u \in U_{q+1}} u^e (g^{-1} u)^{-1} = \sum_{u \in U_{q+1}} u^e \frac{u^{-1}}{u^{-e} / u^e + 1} = \frac{e}{c} \sum_{u \in U_{q+1}} u^e \sum_{j=1}^{q} u^j / c^q j^{-1}
\]

Employing Lemma 11 on \( (gu)^e \) again, we have
\[
a_0 = \sum_{u \in U_{q+1}} (gu)^e = \sum_{u \in U_{q+1}} u^e = 0.
\]

This completes the proof.

Now we are ready to prove the main result of this section.
Theorem 13. Let $q = p^m$ with $m \geq 1$ and $p$ being a prime. If $C$ is a linear code over $\text{GF}(p^h)$ of length $q + 1$ that is invariant under the permutation action of $\text{PGL}(2, q)$, then $C$ must be one of the following:

(I) the zero code $C_0 = \{(0,0,\ldots,0)\};$ or 

(II) the whole space $\text{GF}(p^h)^{q+1}$, which is the dual of $C_0$; or 

(III) the repetition code $C_1 = \{(c,c,\ldots,c) : c \in \text{GF}(p^h)\}$ of dimension 1; or 

(IV) the code $C_1^\perp$, given by 

$$C_1^\perp = \left\{(c_0,\ldots,c_q) \in \text{GF}(p^h)^{q+1} : c_0 + \cdots + c_q = 0\right\}.$$ 

Proof. It is evident that the four trivial $p^h$-ary linear codes $C_0, C_0^\perp, C_1$ and $C_1^\perp$ of length $q + 1$ are invariant under $\text{PGL}(2, q)$.

Let $C$ be a $p^h$-ary linear code of length $q + 1$ which is invariant under $\text{PGL}(2, q)$, which amounts to saying that $C$ is invariant under $\text{Stab}_{U_{q+1}}$ by Proposition 5. Observe that the translation $\pi(u) = u_0u$ belongs to $\text{Stab}_{U_{q+1}}$, where $u_0 \in U_{q+1}$. This clearly forces $C$ to be a cyclic code. Let $E$ be the cyclicity-defining set of $C$. We consider the following four cases for $E$.

If $E = \emptyset$, then $C = C_0$. 

If $E = \{0\}$, then $C = C_1$. 

If $\{0\} \subseteq E$, then there exists an $e \in E \setminus \{0\}$. Applying Corollary 3 the lifted code $\text{GF}(q^{2h}) \otimes C$ to $\text{GF}(q^{2h})$ is the cyclic code over $\text{GF}(q^{2h})$ with respect to the cyclicity-defining set $E$. We see at once that $\text{GF}(q^{2h}) \otimes C$ also stays invariant under $\text{Stab}_{U_{q+1}}$ from the definition of lifting of a cyclic code. Combining Corollary 3 with Proposition 5 we obtain $((gu^e)_{u \in U_{q+1}} \in \text{GF}(q^{2h}) \otimes C$, where 

$$g = \begin{pmatrix} 1 & -c^q \\ -c & 1 \end{pmatrix}^{-1}$$

and $c \in \text{GF}(q^2)^* \setminus U_{q+1}$. Applying Corollary 4 and Lemma 12 we can assert that $1 \in E$. Thus $(gu)_{u \in U_{q+1}} \in \text{GF}(q^{2h}) \otimes C$. Combining Corollary 4 and Lemma 12 we deduce $E = \{0,1,\ldots,q\}$. We thus get $C = \text{GF}(p^h)^{q+1}$.

If $E \neq \emptyset$ and $0 \not\in E$, then there exists an $e \in E \setminus \{0\}$. An analysis similar to that in the proof of the case of $\{0\} \subseteq E$ shows that $E = \{1,\ldots,q\}$ and $C = C_1^\perp$. This completes the proof. 

For any set $A$ and a positive integer $k$, recall that $\binom{A}{k}$ denotes the set of all $k$-subsets of $A$. The following result is an important consequence of Theorem 13.

Theorem 14. Let $p$ be a prime. Let $\mathcal{B} \subseteq (\text{PG}(1,p^m))$ such that $m \geq 1$, $1 \leq k \leq p^m$ and $\mathcal{B}$ is invariant under the action of $\text{PGL}(2, p^m)$. Then the incidence structure $\mathbb{D} = (\text{PG}(1,p^m), \mathcal{B})$ has $p$-rank $p^m$ or $p^m + 1$ depending on whether $k$ is a multiple of $p$ or not.

Proof. Since $\mathcal{B}$ is invariant under the action of $\text{PGL}(2, p^m)$, then so is the code $C_\mathcal{D}(\mathbb{D})$ of $\mathbb{D}$. It then follows from Theorem 13 that $C_\mathcal{D}(\mathbb{D}) = C_1^\perp$ or $C_\mathcal{D}(\mathbb{D}) = \text{GF}(p)^{p^m+1}$. The desired conclusion then follows.
5. Narrow-sense $q$-ary BCH codes of length $q+1$

In this section, we shall determine parameters and automorphisms of some narrow-sense $q$-ary BCH codes of length $q+1$ and present a coding-theory construction of the Witt spherical geometry designs.

For a positive integer $\ell \leq q+1$, define a $2\delta \times \ell$ matrix $M_{\delta,\ell}$ by

$$
\begin{bmatrix}
{u_1}^{-(\delta-1)} & {u_2}^{-(\delta-1)} & \cdots & {u_\ell}^{-(\delta-1)} \\
{u_1}^{-(\delta-1)} & {u_2}^{-(\delta-1)} & \cdots & {u_\ell}^{-(\delta-1)} \\
{u_1}^{-2} & {u_2}^{-2} & \cdots & {u_\ell}^{-2} \\
{u_1}^{-1} & {u_2}^{-1} & \cdots & {u_\ell}^{-1} \\
{u_1}^{+1} & {u_2}^{+1} & \cdots & {u_\ell}^{+1} \\
{u_1}^{+2} & {u_2}^{+2} & \cdots & {u_\ell}^{+2} \\
\vdots & \vdots & \cdots & \vdots \\
{u_1}^{+(\delta-1)} & {u_2}^{+(\delta-1)} & \cdots & {u_\ell}^{+(\delta-1)}
\end{bmatrix},
$$

where $u_1, \ldots, u_\ell \in U_{q+1}$. For all integers $r_1$ and $r_2$ with $-(\delta-1) \leq r_1 < r_2 \leq (\delta-1)$, let $M_{\delta,\ell}[r_1, r_2]$ denote the submatrix of $M_{\delta,\ell}$ obtained by choosing the row $\{u_1', u_2', \ldots, u_\ell'\}$, where $r_1 \leq i \leq r_2$. By adopting similar arguments as in [26, Lemma 29], we deduce the following result.

**Lemma 15.** Let $M_{\delta,\ell}$ be the matrix given by (16) with $\{u_1, \ldots, u_\ell\} \in (U_{q+1})^\ell$. Consider the system of homogeneous linear equations defined by

$$
M_{\delta,\ell}(x_1, \ldots, x_\ell)^T = 0.
$$

Then (17) has a nonzero solution $(x_1, \ldots, x_\ell)$ in $\text{GF}(q)^\ell$ if and only if $\text{rank}(M_{\delta,\ell}) < \ell$, where $\text{rank}(M_{\delta,\ell})$ denotes the rank of the matrix $M_{\delta,\ell}$.

The $\ell$th elementary symmetric polynomial $\sigma_\ell$ over a set $\{u_1, \ldots, u_n\}$ is a specific sum of products without permutation of repetitions and defined by

$$
\sigma_\ell(u_1, \ldots, u_n) = \sum_{1 \leq i_1 < \cdots < i_\ell \leq n} u_{i_1} \cdots u_{i_\ell}.
$$

The following lemma shows a general equation for a generalized Vandermonde determinant with one deleted row in terms of the elementary symmetric polynomial [11, p. 466].

**Lemma 16.** For each $\ell$ with $0 \leq \ell \leq n$, it holds that

$$
\begin{vmatrix}
1 & 1 & \cdots & 1 & 1 \\
u_1 & u_2 & \cdots & u_{n-1} & u_n \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
u_1^{\ell-1} & u_2^{\ell-1} & \cdots & u_{n-1}^{\ell-1} & u_n^{\ell-1} \\
u_1^{\ell+1} & u_2^{\ell+1} & \cdots & u_{n-1}^{\ell+1} & u_n^{\ell+1} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
u_1^\ell & u_2^\ell & \cdots & u_{n-1}^\ell & u_n^\ell
\end{vmatrix}
= \left(\prod_{1 \leq j < i \leq n} (u_i - u_j)\right) \sigma_{n-\ell}(u_1, \ldots, u_n).
$$

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The following lemma will be used to determine the codewords with weight $\delta + 1$ in $C(\delta^m, \delta^{m+1}, \delta, 1)$.

**Lemma 17.** Let $\delta$ be a power of a prime $p$ and $q = \delta^m$. Let $u_0 \in U_{q+1} \setminus \{1, -1\}$. Set

$$u_c = \frac{c + u_0}{c + u_0}, \quad u_\infty = 1,$$

$$a_c = (c + u_0)^{(q+1)(\delta-1)}, \quad a_\infty = 1,$$

where $c \in \text{GF}(\delta)$. Then $u_c \in U_{q+1}, a_c \in \text{GF}(q)^*$ and it holds that

$$\sum_{c \in \text{GF}(\delta) \cup \{\infty\}} a_c u_c^e = 0,$$

where $e \in \{1, 2, \ldots, \delta - 1\}$.

**Proof.** It is straightforward to check that $u_c \in U_{q+1}$ and $a_c \in \text{GF}(q)^*$. We compute

$$\sum_{c \in \text{GF}(\delta)} a_c u_c^e = \sum_{c \in \text{GF}(\delta)} \left( \frac{c + u_0}{c + u_0} \right)^{\delta - 1 + e} \left( \frac{c + u_0}{c + u_0} \right)^{\delta - 1 - e} + u_0^{2e} = \sum_{c \in \text{GF}(\delta)} \sum_{i=0}^{\delta - 1 + e} \sum_{j=0}^{\delta - 1 - e} \left( \frac{c + u_0}{c + u_0} \right)^{\delta - 1 + e} \left( \frac{c + u_0}{c + u_0} \right)^{\delta - 1 - e} u_0^{2e+i-j} c^{i+j} + u_0^{-2e}. $$

Together with the fact $\sum_{c \in \text{GF}(\delta)} c^{i+j} = \begin{cases} 0, & (\delta - 1) | (i + j) \\ -1, & (\delta - 1) \not| (i + j) \end{cases}$, we have

$$\sum_{c \in \text{GF}(\delta)} a_c u_c^e = -\sum_{j=0}^{\delta - 1 - e} \left( \frac{c + u_0}{c + u_0} \right)^{\delta - 1 + e} \left( \frac{c + u_0}{c + u_0} \right)^{\delta - 1 - e} u_0^{2e+i-j} + u_0^{-2e} - u_0^{-2e} - 1 = -\sum_{j=0}^{\delta - 1 - e} \left( \frac{c + u_0}{c + u_0} \right)^{\delta - 1 + e} \left( \frac{c + u_0}{c + u_0} \right)^{\delta - 1 - e} u_0^{2e+\delta-1-2j} - 1.$$

By Lemma 7, we have

$$\sum_{c \in \text{GF}(\delta)} a_c u_c^e = -1.$$

This completes the proof of the lemma.

**Theorem 18.** Let $\delta$ be a power of a prime $p$ and $q = \delta^m$ with $m \geq 2$. Then the narrow-sense antiprimitive BCH code $C_{(q,q+1,\delta,1)}$ has parameters $[q + 1, q - 2\delta + 3, \delta + 1]_q$.

**Proof.** Note that the generator polynomial of the narrow-sense antiprimitive BCH code $C_{(q,q+1,\delta,1)}$ is $\prod_{i=1}^{\delta - 1} (x^2 - (\beta^i + \beta^{-i})x + 1)$, where $\beta$ is a primitive $(q + 1)$-th root of unity. The desired conclusion on the dimension of the code follows.

We now determine the minimum distance of the code $C_{(q,q+1,\delta,1)}$. According to the BCH bound, the minimum distance $d$ of the code $C_{(q,q+1,\delta,1)}$ satisfies $d \geq \delta$. In order to prove $d \geq (\delta + 1)$ it suffices to prove the following claim: there is no codeword of weight $\delta$ in $C_{(q,q+1,\delta,1)}$. Define

$$H = \begin{bmatrix}
1 & \beta^{-1}(\delta-1) & \beta^{-2}(\delta-1) & \beta^{-3}(\delta-1) & \ldots & \beta^{-q}(\delta-1) \\
\vdots & \vdots & \vdots & \vdots & \ldots & \vdots \\
1 & \beta^{1-1} & \beta^{-2-1} & \beta^{-3-1} & \ldots & \beta^{-q+1} \\
1 & \beta^{1+1} & \beta^{2+1} & \beta^{3+1} & \ldots & \beta^{q+1} \\
\vdots & \vdots & \vdots & \vdots & \ldots & \vdots \\
1 & \beta^{1+1}(\delta-1) & \beta^{2+1}(\delta-1) & \beta^{3+1}(\delta-1) & \ldots & \beta^{q+1}(\delta-1)
\end{bmatrix}. \quad (19)$$
It is easily seen that $H$ is a parity-check matrix of $C_{(q,q+1,\delta,1)}$, i.e.,

$$C_{(q,q+1,\delta,1)} = \{ c \in \text{GF}(q)^{q+1} : cH^T = 0 \}.$$  

Suppose that there is a codeword of weight $\delta$. Then there exist $\{u_1, \cdots, u_\delta\} \in \binom{U_{q+1}}{\delta}$ and $(x_1, \cdots, x_\delta) \in (\text{GF}(q)^*)^\delta$ such that $M_{\delta,\delta}(x_1, \cdots, x_\delta)^T = 0$. Applying Lemma [17], we find that $\text{rank}(M_{\delta,\delta}) < \delta$. Then the determinant of the square matrix $M_{\delta,\delta}[-(\delta - 1) + i, 1 + i]$ equals zero for any $i \in \{0, 1, \ldots, \delta - 2\}$. By Lemma [16] we have

$$\sigma_\ell(u_1, \ldots, u_\delta) = 0,$$

where $1 \leq \ell \leq \delta - 1$. By Vieta’s formula, we obtain

$$\prod_{i=1}^\delta (u - u_i) = u^\delta + (-1)^\delta \prod_{i=1}^\delta u_i,$$

where $u$ is an indeterminate. Substituting $u$ in both sides of the above equation by $u_i$ and $u_j$ ($1 \leq i < j \leq \delta$), respectively, we find that $u_i^\delta = u_j^\delta$: a contradiction. We thus deduce that $d \geq (\delta + 1)$.

Let $u_0$ be a fixed element in $U_{q+1} \setminus \{+1, -1\}$. Write $u(x) = \frac{x+u_0^q}{x+a_0}$ and $a(x) = (x + u_0)^{(q+1)(\delta-1)}$, where $x \in \text{GF}(\delta)$, and $u(\infty) = 1, a(\infty) = 1$. Set $c = (c_u)_{u \in U_{q+1}}$ where

$$c_u = \begin{cases} a(x), & \text{if } u = u(x), \\ 0, & \text{otherwise}, \end{cases}$$

where $x \in \text{GF}(\delta) \cup \{\infty\}$. By Lemma [17] $c \in C_{(q,q+1,\delta,1)}$ and $\text{wt}(c) = \delta + 1$. Thus, $d = \delta + 1$.

The following theorem shows that the dual $C_{(\delta^m,\delta^m+1,\delta,1)}$ of $C_{(\delta^m,\delta^m+1,\delta,1)}$ is an almost maximum distance separable code (almost MDS code), where $\delta \geq 3$ is a prime power.

**Theorem 19.** Let $\delta$ be a power of a prime $p$ and $q = \delta^m$ with $\delta \geq 3, m \geq 2$. Then the dual $C_{(q,q+1,\delta,1)}^{\perp}$ of the narrow-sense antiprimitive BCH code $C_{(q,q+1,\delta,1)}$ has parameters $[q + 1, 2\delta - 2, q - 2\delta + 3]_q$.

**Proof.** By Delsarte’s Theorem, any nonzero codeword of the code $C_{(q,q+1,\delta,1)}^{\perp}$ can be written as $c = (f(u))_{u \in U_{q+1}}$, where $f(u) = \text{Tr}_{q^2/q} \left( \sum_{i=1}^{\delta-1} a_i u^i \right)$ and $a_i \in \text{GF}(q^2)$. Rewrite $f(u)$ in the form

$$f(u) = u^{-(\delta-1)} \left( a_{\delta-1} u^{2(\delta-1)} + a_{\delta-2} u^{2\delta-3} + \cdots + a_{1}^q \right).$$

It follows that there are at most $2(\delta - 1)$ values of $u \in U_{q+1}$ such that $f(u) = 0$. That is, the minimum distance $d^{\perp}$ of $C_{(q,q+1,\delta,1)}^{\perp}$ satisfies $d^{\perp} \geq q - 2\delta + 3$. On the other hand, we have $d^{\perp} \leq q - 2\delta + 4$ by the Singleton bound. Suppose that $d^{\perp} = q - 2\delta + 4$. Then $C_{(q,q+1,\delta,1)}$ is an MDS code and so is $C_{(q,q+1,\delta,1)}$: a contradiction to the minimum distance of $C_{(q,q+1,\delta,1)}$ given in Theorem [18] This completes the proof.
Let $\text{GF}(q)^{U_{q+1}}$ denote the vector space consisting of all elements $(c_u)_{u \in U_{q+1}}$, where $c_u \in \text{GF}(q)$. The action of the semidirect product $(\text{GF}(q^*)^{U_{q+1}} \rtimes \text{Stab}_{U_{q+1}})$ on $\text{GF}(q)^{U_{q+1}}$ is defined by

$$(a_u)_{u \in U_{q+1}} \rtimes g \equiv (a_u)_{u \in U_{q+1}} = (a_u c_{g^{-1} u})_{u \in U_{q+1}}.$$ 

Thus the multiplication in $\text{GF}(q^*)^{U_{q+1}} \rtimes \text{Stab}_{U_{q+1}}$ is given by

$$(a_u)_{u \in U_{q+1}} \rtimes g_1 \cdot (b_v)_{v \in U_{q+1}} \rtimes g_2 = (a_u)_{u \in U_{q+1}} \rtimes (g_1 g_2),$$

where $c_u = a_u b_{g_1^{-1} u}$.

Note that $C_{\perp,q+1,\delta,1} = \{(f(u))_{u \in U_{q+1}} : f \in \mathcal{P}(\delta,q)\}$. Then the results on the group representation in Lemma 9 translate immediately into corresponding results on the monomial isomorphisms of $C_{\perp,q+1,\delta,1}$.

**Theorem 20.** Let $\delta$ be a power of a prime $p$ and $q = \delta^m$ with $m \geq 2$. Define a subgroup of $(\text{GF}(q^*)^{U_{q+1}} \rtimes \text{Stab}_{U_{q+1}}$ by

$$G_{\delta}^+ = \left\{ \left( \left( (cu + d)^{(q+1)(\delta-1)} \right)_{u \in U_{q+1}} \cdot \left( \begin{array}{cc} d^q & c^q \\ c & d \end{array} \right)^{-1} \right) : c,d \in \text{GF}(q^2), c^{q+1} \neq d^{q+1} \right\}.$$ 

Then $G_{\delta}^+$ is a subgroup of the monomial automorphism group $\text{MAut}(C_{\perp,q+1,\delta,1})$. In particular, the automorphism group of $C_{\perp,q+1,\delta,1}$ is 3-transitive.

The following theorem is an immediate consequence of Theorem 20.

**Theorem 21.** Let $\delta$ be a power of a prime $p$ and $q = \delta^m$ with $m \geq 2$. Define a subgroup of $(\text{GF}(q^*)^{U_{q+1}} \rtimes \text{Stab}_{U_{q+1}}$ by

$$G_{\delta} = \left\{ \left( \left( (cu + d)^{-(q+1)(\delta-1)} \right)_{u \in U_{q+1}} \cdot \left( \begin{array}{cc} d^q & c^q \\ c & d \end{array} \right)^{-1} \right) : c,d \in \text{GF}(q^2), c^{q+1} \neq d^{q+1} \right\}.$$ 

Then $G_{\delta}$ is a subgroup of the monomial automorphism group $\text{MAut}(C_{\perp,q+1,\delta,1})$. In particular, the automorphism group of $C_{\perp,q+1,\delta,1}$ is 3-transitive.

The following theorem presents a coding-theoretic construction of the Witt spherical geometry designs. This theorem shows that the supports of the codewords of minimum Hamming weight in the BCH code $C_{\perp,q+1,\delta,1}$ yield a Witt spherical geometry design.

**Theorem 22.** Let $\delta$ be a power of a prime $p$ and $q = \delta^m$ with $m \geq 2$. Then the incidence structure $(U_{q+1}, \mathcal{B}_{\delta+1}(C_{\perp,q+1,\delta,1}))$ is isomorphic to the Witt spherical geometry design with parameters $3-(\delta^m + 1, \delta + 1, 1)$.

**Proof.** By Theorems 18 and 21, $(U_{q+1}, \mathcal{B}_{\delta+1}(C_{\perp,q+1,\delta,1}))$ is a $3-(\delta^m + 1, \delta + 1, \lambda)$, where $\lambda$ is a positive integer. Let $u_0$ be a fixed generator element of $U_{q+1}$. Then we have $\lambda = |B_{u_0}|$, where $B_{u_0} = \{ B \in \mathcal{B}_{\delta+1}(C_{\perp,q+1,\delta,1}) : \{1, u_0, u_0^2 \} \subseteq B \}$. Let $B = \{ u_1, \ldots, u_{\delta+1} \} \in B_{u_0}$ with $u_1 = 1, u_2 = u_0, u_3 = u_0^2$. Recall that

$$C_{\perp,q+1,\delta,1} = \{ c \in \text{GF}(q)^{q+1} : cH^T = 0 \},$$
where \( H \) is given by (19). By Lemma 15, the rank of the matrix \( M_{\delta, \delta+1} \) defined in (16) is less than \( \delta + 1 \). So the determinant of the square matrix \( M_{\delta, \delta+1}[-(\delta - 1) + i, 2 + i] \) equals zero, where \( 0 \leq i \leq \delta - 3 \). By Lemma 16, we have

\[
\sigma_{\ell}(u_1, \ldots, u_{\delta+1}) = 0,
\]

where \( 2 \leq \ell \leq \delta - 1 \). By Vieta’s formula, we obtain

\[
\prod_{i=1}^{\delta+1}(u - u_i) = u^{\delta+1} + au^\delta + bu + c,
\]

where \( u \) is an indeterminate and \((a, b, c) \in GF(q^2)^3\). Substituting \( u \) in both sides of the above equation by \( u_i \) (\( 1 \leq i \leq 3 \)), we get

\[
\begin{cases}
  a + b + c = -1 \\
  u_0^0a + u_0b + c = -u_0^{\delta+1} \\
  u_0^2a + u_0^2b + c = -u_0^{2(\delta+1)}
\end{cases}
\]

Note that the coefficient matrix of the above system of equations is nonsingular. Thus the system has a unique solution of \((a, b, c) \in GF(q^2)^3\). Substituting \( u \) in both sides of the above equation by \( u_i \) (\( 1 \leq i \leq 3 \)), we get

\[
\begin{cases}
  a + b + c = -1 \\
  u_0^0a + u_0b + c = -u_0^{\delta+1} \\
  u_0^2a + u_0^2b + c = -u_0^{2(\delta+1)}
\end{cases}
\]

Let \( g_0 = \begin{pmatrix} u_0 & 1 \\ 0 & u_0 \end{pmatrix} \). By Lemma 17, \( \frac{1}{u_0}g_0PG(1, \delta) \in B_{\delta+1}(C(q, q+1, \delta, 1)) \). It follows that \( g_0PG(1, \delta) \in B_{\delta+1}(C(q, q+1, \delta, 1)) \) from Theorem 21. Since \( \lambda = 1 \), by (5), we conclude that

\[
B_{\delta+1}(C(q, q+1, \delta, 1)) = \text{Stab}_{U_{q+1}}(g_0PG(1, \delta)) = \{g_0gPG(1, \delta) : g \in PGL(2, q)\} = g_0\text{Orb}_{PG(1, \delta)},
\]

where \( \text{Orb}_{PG(1, \delta)} = \{gPG(1, \delta) : g \in PGL(2, q)\} \). Therefore, the map from \( PGL(2, q) \) to \( U_{q+1} \) given by \( x \rightarrow g_0x \) gives rise to an isomorphism between the two incidence structures \((PGL(2, q), \text{Orb}_{PG(1, \delta)})\) and \((U_{q+1}, B_{\delta+1}(C(q, q+1, \delta, 1)))\). The desired conclusion then follows from the definition of the Witt spherical geometry design.

Combining Theorems 21 and 22 with Theorem 14 yields the following result.

**Theorem 23.** Let \( \delta \) be a power of a prime \( p \) and \( m \) an integer with \( m \geq 2 \). Then the \( p \)-rank of the Witt spherical geometry design with parameters \( S(3, \delta + 1, \delta^m + 1) \) is \( \delta^m + 1 \).

**Example 24.** Let \( q = 25 \). Then the narrow-sense BCH code \( C_{(q, q+1, 5, 1)} \) has parameters \([26, 18, 6]_{25} \).
and weight enumerator

\[
1 + 3120z^6 + 1053000z^8 \\
+ 52478400z^9 + 2246164440z^{10} \\
+ 76730209920z^{11} + 2313008100000z^{12} \\
+ 5973754888000z^{13} + 1331420089708800z^{14} \\
+ 2556301945153920z^{15} + 421789956437369520z^{16} \\
+ 595468120248610000z^{17} + 71456174963080050000z^{18} \\
+ 72208345183198710720z^{19} + 606550099565746236960z^{20} \\
+ 41592006827232472278720z^{21} + 226865491784954611290000z^{22} \\
+ 946916835276318186384000z^{23} + 2840750505828957328830600z^{24} \\
+ 5454240971191597731710304z^{25} + 5034683973407628695013720z^{26},
\]

and \((U_{q+1}, B_6 (C_{q,q+1,8,1}))\) is a 3-(26, 6, 1) design.

The dual code \(C_{q,q+1,5,1}^\perp\) has parameters \([26, 8, 18]_{25}\) and weight enumerator

\[
1 + 1645800z^{18} + 4180800z^{19} + 70265520z^{20} + 426192000z^{21} \\
+ 23933520000z^{22} + 9911491200z^{23} + 29801335200z^{24} \\
+ 57185869104z^{25} + 527935590000z^{26},
\]

and \((U_{q+1}, B_{18} (C_{q,q+1,8,1}^\perp))\) is a 3-(26, 18, 21522) design.

6. Summary and concluding remarks

We have provided a detailed discussion of the interplay among the narrow-sense antiprimitive BCH codes, group actions and group representations concerning PGL(2, q), and combinatorial 3-designs in this paper. The main contributions of this paper are the following:

- Infinite families of narrow-sense antiprimitive BCH codes admitting a 3-transitive automorphism group were proposed in Theorem 21. The dimensions and the minimum distances of these codes and their duals were also determined in Theorems 18 and 19. Using Delsarte’s Theorem, it is shown in Theorem 19 that the dual codes of the narrow-sense antiprimitive BCH codes derived in this paper are almost MDS.

- A coding-theory construction of the Witt spherical geometry design \(S(3, \delta + 1, \delta^m + 1)\) was presented in Theorem 22.

- A complete classification of PGL(2, \(p^m\))-invariant \(p^h\)-ary linear codes was established in Theorem 13.

- The \(p\)-ranks of incidence structures that are invariant under the action of PGL(2, \(p^m\)) were derived in Theorem 14. In particular, the \(p\)-rank of the Witt spherical geometry design \(S(3, \delta + 1, \delta^m + 1)\) was determined in Theorem 23.
The results of this paper generalize and extend the work in [10]. It would be interesting to determine structures and parameters for more 3-designs held in $C(δ^m, δ^m + 1, δ, 1)$ and $C(δ^m, δ, 1)$, where $δ$ is a prime power. It would be valuable to determine the full automorphism groups of the BCH codes introduced in this paper.

Finally, we would explain an important motivation for constructing a linear code over a finite field to support a known $t$-design constructed with an algebraic, combinatorial, or group-theoretic approach. Such an investigation may not be interesting in combinatorics, as the known $t$-design was already discovered earlier. However, this would be very interesting in coding theory and enhances coding theory, as the newly discovered linear code supporting the known $t$-design should have special properties compared with general linear codes. It is known that the dual code of such a code admits majority-logic decoding [22, 23, 27]. By definition, $t$-designs have a certain level of symmetry. The larger the strength $t$ of a $t$-design is, the higher the level of symmetry the $t$-design has. For instance, all the known linear codes supporting a 4-design or a 5-design have special properties [7, 26]. While the Witt spherical geometry design $S(3, δ + 1, δ^m + 1)$ was discovered 80 years ago, the effort of constructing a linear code supporting this design carried out in this paper has led to the discovery of the codes $C(δ^m, δ^m + 1, δ, 1)$ which are PGL($2, δ^m$)-invariant and a complete classification of PGL($2, p^m$)-invariant $p^m$-ary linear codes. In addition, an infinite family of almost MDS codes $C(δ^m, δ^m + 1, δ, 1)$ were obtained in this project. The results obtained in this paper enhance coding theory and strengthen the interplay between coding theory and design theory. In addition to the Witt spherical geometry design $S(3, δ + 1, δ^m + 1)$, more 3-designs with new parameters are supported by the codewords of other weights in $C(δ^m, δ^m + 1, δ, 1)$. Hence, this paper does have contributions to the theory of combinatorial designs. Furthermore, the codes presented in this paper also have applications in cryptography (secret sharing [33] and authentication codes [8]).

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