Moon in a puddle and the four-vertex theorem

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Abstract. We present a proof of the moon in a puddle theorem, and use its key lemma to prove a generalization of the four-vertex theorem.
INTRODUCTION. The theorem about the Moon in a puddle provides the simplest meaningful example of a local-to-global theorem which is mainly what differential geometry is about. Yet, the theorem is surprisingly not well-known. This paper aims to redress this omission by calling attention to the result and applying it to a well-known theorem.

MOON IN A PUDDLE. The following question was initially asked by Abram Fet and solved by Vladimir Ionin and German Pestov [10].

**Theorem 1.** Assume \( \gamma \) is a simple closed smooth regular plane curve with curvature bounded in absolute value by 1. Then the region surrounded by \( \gamma \) contains a unit disc.

We present the proof from our textbook [12] which is a slight improvement of the original proof. Both proofs work under the weaker assumption that the signed curvature is at most one, assuming that the sign is chosen suitably. A more general statement for a barrier-type bound on the curvature was given by Anders Aamand, Mikkel Abrahamsen, and Mikkel Thorup [1]. There are other proofs. One is based on the curve-shortening flow; it is given by Konstantin Pankrashkin [8]. Another one uses cut locus; it is sketched by Victor Toponogov [13, Problem 1.7.19]; see also [11].

Let us mention that an analogous statement for surfaces does not hold — there is a solid body \( V \) in the Euclidean space bounded by a smooth surface whose principal curvatures are bounded in absolute value by 1 such that \( V \) does not contain a unit ball; moreover one can assume that \( V \) is homeomorphic to the 3-ball. Such an example can be obtained by inflating a nontrivial contractible 2-complex in \( \mathbb{R}^3 \) (Bing’s house constructed in [3] would do the job). This problem is discussed by Abram Fet and Vladimir Lagunov [5, 6]; see also [12].

A path \( \gamma : [0, 1] \to \mathbb{R}^2 \) such that \( \gamma(0) = \gamma(1) \) will be called a loop; the point \( \gamma(0) \) is called the base of the loop. A loop is smooth, regular, and simple if it is smooth and regular in \([0, 1]\), and injective in the open interval \((0, 1)\).

Let us use the term circline as a shorthand for a circle or line. Note that the osculating circline of a smooth regular curve is defined at each of its points — there is no need to assume that the curvature does not vanish.

Suppose that \( \gamma \) is a closed simple smooth plane loop. We say that a circline \( \sigma \) supports \( \gamma \) at a point \( p \) if the point \( \gamma(p) \) lies on both \( \sigma \) and \( \gamma \), and the circline \( \sigma \) lies in one of the closed regions that \( \gamma \) cuts from the plane. If furthermore this region is bounded, then we say that \( \sigma \) supports \( \gamma \) from inside. Otherwise, we say that \( \sigma \) supports \( \gamma \) from the outside.

**Key lemma.** Assume \( \gamma \) is a simple smooth regular plane loop. Then at one point of \( \gamma \) (distinct from its base), its osculating circle \( \sigma \) supports \( \gamma \) from inside.

Spherical and hyperbolic versions of this lemma were given in [9, Lemma 8.2] and [2, Proposition 7.1] respectively.

**Proof of the theorem modulo the key lemma.** Since \( \gamma \) has absolute curvature of at most 1, each osculating circle has radius of at least 1.

According to the key lemma, one of the osculating circles \( \sigma \) supports \( \gamma \) from inside. In this case, \( \sigma \) lies inside \( \gamma \), whence the result.

**Proof of the key lemma.** Denote by \( F \) the closed region surrounded by \( \gamma \). Arguing by contradiction, assume that the osculating circle at each point \( p \neq p_0 \) on \( \gamma \) does not lie in \( F \). Given such a point \( p \), let us consider the maximal circle \( \sigma \) that lies entirely in \( F \) and is tangent to \( \gamma \) at \( p \). The circle \( \sigma \) will be called the incircle of \( F \) at \( p \).
Note that the curvature of the incircle $\sigma$ has to be strictly larger than the curvature of $\gamma$ at $p$, hence there is a neighborhood of $p$ in $\gamma$ that intersects $\sigma$ only at $p$. Further note that the circle $\sigma$ has to touch $\gamma$ at another point at least; otherwise, we could increase $\sigma$ slightly while keeping it inside $F$.

Choose a point $p_1 \neq p_0$ on $\gamma$, and let $\sigma_1$ be the incircle at $p_1$. Choose an arc $\gamma_1$ of $\gamma$ from $p_1$ to a first point $q_1$ on $\sigma_1$. Denote by $\hat{\sigma}_1$ and $\check{\sigma}_1$ the two arcs of $\sigma_1$ from $p_1$ to $q_1$ such that the cyclic concatenation of $\hat{\sigma}_1$ and $\gamma_1$ surrounds $\check{\sigma}_1$.

Let $p_2$ be the midpoint of $\gamma_1$, and $\sigma_2$ be the incircle at $p_2$.

Note that $\sigma_2$ cannot intersect $\check{\sigma}_1$. Otherwise, if $s$ is a point of the intersection, then $\sigma_2$ must have two more common points with $\check{\sigma}_1$, say $x$ and $y$ — one for each arc of $\sigma_2$ from $p_2$ to $s$. Therefore $\sigma_1 = \sigma_2$ since these two circles have three common points: $s$, $x$, and $y$. On the other hand, by construction, $p_2 \in \sigma_2$ and $p_2 \notin \sigma_1$ — a contradiction.

Recall that $\sigma_2$ has to touch $\gamma$ at another point. From above it follows that it cannot touch $\gamma \setminus \gamma_1$, and therefore we can choose an arc $\gamma_2$ in $\gamma_1$ that runs from $p_2$ to a first point $q_2$ on $\sigma_2$. Since $p_2$ is the midpoint of $\gamma_1$, we have that

$$\text{length } \gamma_2 < \frac{1}{2} \cdot \text{length } \gamma_1.$$  

Repeating this construction recursively, we obtain an infinite sequence of arcs $\gamma_1 \supset \gamma_2 \supset \ldots$; by (\ast), we also get that

$$\text{length } \gamma_n \to 0 \quad \text{as} \quad n \to \infty.$$  

Therefore the intersection $\gamma_1 \cap \gamma_2 \cap \ldots$ contains a single point; denote it by $p_\infty$.

Let $\sigma_\infty$ be the incircle at $p_\infty$; it has to touch $\gamma$ at another point, say $q_\infty$. The same argument as above shows that $q_\infty \in \gamma_n$ for any $n$. It follows that $q_\infty = p_\infty$ — a contradiction. \hfill \blacksquare

**Exercise.** Assume that a closed smooth regular curve (possibly with self-intersections) $\gamma$ lies in a figure $F$ bounded by a closed simple plane curve. Suppose that $R$ is the maximal radius of a disc contained in $F$. Show that the absolute curvature of $\gamma$ is at least $\frac{1}{R}$ at some parameter value.

**FOUR-VERTEX THEOREM.** Recall that a vertex of a smooth regular curve is defined as a critical point of its signed curvature; in particular, any local minimum (or maximum) of the signed curvature is a vertex. For example, every point of a circle is a vertex.

The classical four-vertex theorem says that any closed smooth regular plane curve without self-intersections has at least four vertices. It has many different proofs and
generalizations. A very transparent proof was given by Robert Osserman [7]; his paper contains a short account of the history of the theorem.

Note that if an osculating circline $\sigma$ at a point $p$ supports $\gamma$, then $p$ is a vertex. The latter can be checked by direct computation, but it also follows from the Tait–Kneser spiral theorem [4]. It states that the osculating circlines of a curve with monotonic curvature are disjoint and nested; in particular, none of these circlines can support the curve. Therefore the following theorem is indeed a generalization of the four-vertex theorem:

**Theorem 2.** Any smooth regular simple plane curve is supported by its osculating circlines at 4 distinct points; two from inside and two from outside.

**Proof.** According to the key lemma, there is a point $p \in \gamma$ such that its osculating circle supports $\gamma$ from inside.

The curve $\gamma$ can be considered as a loop with $p$ as its base. Therefore the key lemma implies the existence of another point $q$ with the same property.

This shows the existence of two osculating circles that support $\gamma$ from inside; it remains to show the existence of two osculating circles that support $\gamma$ from outside.

Let us apply to $\gamma$ an inversion with respect to a circle whose center lies inside $\gamma$. Then the obtained curve $\gamma_1$ also has two osculating circles that support $\gamma_1$ from inside.

Note that these osculating circlines are inverses of the osculating circlines of $\gamma$. Indeed, the osculating circline at a point $x$ can be defined as the unique circline that has second order of contact with $\gamma$ at $x$. It remains to note that inversion, being a local diffeomorphism away from the center of inversion, does not change the order of contact between curves.

Note that the region lying inside $\gamma$ is mapped to the region outside $\gamma_1$ and the other way around. Therefore these two new circlines correspond to the osculating circlines supporting $\gamma$ from outside.

**Advanced exercise.** Suppose $\gamma$ is a closed simple smooth regular plane curve and $\sigma$ is a circle. Assume $\gamma$ crosses $\sigma$ at the points $p_1, \ldots, p_{2\cdot n}$ and these points appear in the same cycle order on $\gamma$ and on $\sigma$. Show that $\gamma$ has at least $2\cdot n$ vertices.

The order of the intersection points is important. An example with only 4 vertices and arbitrarily many intersection points can be guessed from the diagram on the right.

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