Exact and numerical solutions to a Mindlin microcontinuum model

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January 10, 2019

Abstract

In this paper we consider a one-dimensional Mindlin model describing linear elastic behaviour of isotropic materials with micro-structural effects. After introducing the kinetic and the potential energy, we derive a system of equations of motion by means of the Euler-Lagrange equations. A class of exact solutions is obtained. They have a wave behaviour due to a good property of the potential energy. Numerical solutions are obtained by using a weighted essentially non-oscillatory finite difference scheme coupled by a total variation diminishing Runge-Kutta method. A comparison between exact and numerical solutions shows the robustness and the accuracy of the numerical scheme. A numerical example of solutions for an inhomogeneous material is also shown.

MSC-class: 74B - 74J05 (Primary) 74S20 (Secondary)

1 Introduction

In this paper we study a one-dimensional Mindlin model for longitudinal waves in microstructured materials. It is well known that matter is not continuous but has an internal structure. This underlying microstructure could influence profoundly the dynamic thermo-mechanical response of materials, as, for instance, when the scale of a deformation is of the order of the material microstructural heterogeneities or the characteristic length of the continuum is comparable to the short wavelength of a signal propagating through the material. An interesting example of the impact of microstructural peculiarities on the dynamic response of materials is the wave scattering due to the mismatch in mechanical impedance at the interface between different material phases in layered composites [1]. In [2] it has been experimentally observed that heterogeneities, such as, for instance, microvoids leading to localization of deformation and loss of shear strength, have effect on the dynamic strength of polycrystalline metals.

There are essentially two types of models to study microstructured materials: discrete and continuum models. In the microcontinuum theory, the macrostructure and microstructure of the continuum are usually separated. This leads to the formulation of separate balance laws for each structure. Mindlin [3] proposed a model, where the two structures are separated, and introduced two balance laws: one for the macrostructure and another for the microstructure. Moreover the Mindlin continuum model also incorporates the inelastic behaviour of the material. In the interesting Ref. [4], [5] and [6] general one-dimensional Mindlin-type microstructure models are discussed.
In this work we transform the equations of a one-dimensional Mindlin model in a particular set of hyperbolic partial differential equations, where it is clear how to impose correct boundary conditions. This point is important in the numerical simulations. Moreover we propose a numerical scheme, based on a weighted essentially non-oscillatory finite difference scheme coupled by a total variation diminishing Runge-Kutta method, which furnishes accurate results also for not smooth solutions.

This paper is organized as follows. In Section 2 we derive the one-dimensional wave equations for microstructured materials and we find important inequalities involving the physical parameters, by requiring the potential energy to be a strictly positive definite function. Section 3 is devoted to look for simple exact solutions and in Section 4 we transform the equations in a suitable system of first order partial differential equations, which is solved numerically. In Section 5 we consider the case when the physical parameters depend on the spatial coordinate. This happens when the continuum consists of different materials. Also in this case numerical solutions are obtained. Finally the conclusions are drawn in the last section.

2 Basic equations

Following an approach similar to that employed by the authors in [3], and [7], the one-dimensional wave equations for microstructured materials are derived starting from the Lagrangian

$$L = K - W,$$

where $K$ and $W$ are the kinetic and potential energy, respectively. We assume that the kinetic energy $K$ is given by

$$K = \frac{1}{2} \rho u^2 + \frac{1}{2} I_\mu \chi^2,$$

where $u$ is macroscopic displacement, and $\chi$ the microdeformation. The constant positive parameters $\rho$ and $I_\mu$ are the macroscopic density and the micro-inertia, respectively. As usually the subscript $t$ or $x$ denotes the first partial derivative with respect to the time $t$ or the space coordinate $x$. According to the theory of the elasticity, the potential energy $W$ is a function of the variables $u_x, \chi, \chi_x$.

The corresponding Euler-Lagrange equations have the general form

$$\left( \frac{\partial L}{\partial u} \right)_t + \left( \frac{\partial L}{\partial u_x} \right)_x = 0,$$

$$\left( \frac{\partial L}{\partial \chi} \right)_t + \left( \frac{\partial L}{\partial \chi_x} \right)_x \frac{\partial L}{\partial \chi} = 0.$$

We choose the potential energy $W$ given by

$$W = \frac{1}{2} \gamma u^2_x + A u_x \chi + \frac{1}{2} B \chi^2 + \frac{1}{2} C \chi_x^2,$$

where $\gamma, A, B, C$ are physical constant parameters of the model. Now the system (3,4) becomes

$$\rho u_{tt} = \gamma u_{xx} + A \chi_x,$$

$$I_\mu \chi_{tt} = C \chi_{xx} - A u_x - B \chi.$$

The case $A = 0$ is mathematically interesting, because Eq. (6) reduces to the classical wave equation for the unknown $u$, provided that $\gamma > 0$, and Eq. (7) becomes the telegrapher’s equation, where $\chi$ is the
unknown. We do not consider this special case. It is also possible to derive a single fourth-order partial differential equations from the system (6)-(7) (see also Ref. [8]). In fact Eq. (6) gives
\[ \chi_x = \frac{1}{A} [\rho u_{tt} - \gamma u_{xx}] \]
Now, if we differentiate Eq. (7) with respect to the variable \( x \) and use the previous equation, then it is a simple matter to obtain the following equation
\[ I_{\mu} \rho u_{tttt} - (\gamma I_{\mu} + \rho C) u_{txx} + \gamma C u_{xxxx} + (A^2 - B \gamma) u_{xx} + B \rho u_{tt} = 0. \]
In this paper we do not study this equation, but we consider the system (6)-(7).

The main assumption of this paper is the following

the potential energy \( W = W(u, \chi, \chi_x) \) is a strictly positive definite function.

This implies some inequalities involving the physical parameters. We have immediately
\[ W(u, 0, 0) = \frac{1}{2} \gamma u_x^2 > 0 \quad \forall u_x \neq 0 \Rightarrow \gamma > 0, \]
\[ W(0, \chi, 0) = \frac{1}{2} B \chi^2 > 0 \quad \forall \chi \neq 0 \Rightarrow B > 0, \]
\[ W(0, 0, \chi_x) = \frac{1}{2} C \chi_x^2 > 0 \quad \forall \chi_x \neq 0 \Rightarrow C > 0. \]
Since \( W(u, \chi, \chi_x) = W(u, \chi, 0) + W(0, 0, \chi_x) \), then, thanks to Eq. (10), \( W \) is a strictly positive definite function if and only if
\[ W(u, \chi, 0) = \frac{1}{2} \gamma u_x^2 + A u_x \chi + \frac{1}{2} B \chi^2 \]
is a strictly positive definite function. This holds if and only if the discriminant is negative. This gives the last condition
\[ \gamma B - A^2 > 0. \]

3 Explicit simple solutions

In this section we are interested to derive some explicit solutions to the one-dimensional wave equations for microstructured materials (6)-(7). To the best of our knowledge, the complete derivation of the following solutions has never been reported, although in [4, 9, 10, 7, 11] the authors carried out a dispersion analysis for the same equations.

To simplify notation, it is useful to define the parameters
\[ a_1 = \frac{\gamma}{\rho}, \quad a_2 = \frac{A}{\rho}, \quad a_3 = \frac{C}{I_{\mu}}, \quad a_4 = \frac{A}{I_{\mu}}, \quad a_5 = \frac{B}{I_{\mu}}. \]
Hence Eqs. (6)-(7) write
\[ u_{tt} = a_1 u_{xx} + a_2 \chi_x, \]
\[ \chi_{tt} = a_3 \chi_{xx} - a_4 u_x - a_5 \chi. \]
The inequalities (8)-(11) imply that
\[ a_1 > 0, \quad a_3 > 0, \quad a_5 > 0, \]
\[ 3 \]
Since the coefficients of Eq. (22) are positive and therefore Eq. (22) has four purely imaginary roots. If we denote

\[ a_2 a_4 - a_1 a_5 = \frac{A}{\rho I_\mu} \frac{A}{\rho I_\mu} - \frac{\gamma}{\rho I_\mu} \frac{B}{\rho I_\mu} = \frac{1}{\rho I_\mu} (A^2 - \gamma B) < 0, \]  

(16)

We look for solutions of Eqs. (13)-(14) of the kind

\[ u(t, x) = U_1(t) \sin(\omega x) , \quad \chi(t, x) = X_1(t) \cos(\omega x) , \]  

(17)

where \( \omega \) is a non-zero real parameter. Using (17), Eqs. (13)-(14) give the system of second order ordinary differential equations for the unknowns \( U_1 \) and \( X_1 \)

\[ U''_1(t) \sin(\omega x) = -a_1 \omega^2 U_1(t) \sin(\omega x) - a_2 \omega X_1(t) \sin(\omega x) , \]

\[ X''_1(t) \cos(\omega x) = -a_3 \omega^2 X_1(t) \cos(\omega x) - a_4 \omega U_1(t) \cos(\omega x) - a_5 X_1(t) \cos(\omega x) . \]

It is simplified immediately, and writes

\[ U''_1(t) = -a_1 \omega^2 U_1(t) - a_2 \omega X_1(t) , \]

(18)

\[ X''_1(t) = -a_3 \omega^2 X_1(t) - a_4 \omega U_1(t) - a_5 X_1(t) . \]

(19)

Since \( a_2 = A/\rho \) is a non-zero number, then Eq. (18) gives

\[ X_1(t) = \frac{-1}{a_2 \omega} [U''_1(t) + a_1 \omega^2 U_1(t)] , \]

(20)

and Eq. (19) becomes

\[ U''''_1(t) + (a_1 \omega^2 + a_3 \omega^2 + a_5) U''_1(t) + \omega^2 [a_1 \omega_1^2 + a_2 + a_4] U_1(t) = 0 . \]

(21)

Eq. (21) is a linear homogeneous fourth-order ordinary differential equation with constant coefficients and then it can be solved easily. The characteristic equation is

\[ \lambda^4 + (a_1 \omega^2 + a_3 \omega^2 + a_5) \lambda^2 + \omega^2 [a_1 a_3 \omega^2 + (a_1 a_5 - a_2 a_4)] = 0 , \]

(22)

where \( \lambda \) is an eigenvalue. The discriminant of the biquadratic equation (22) is

\[ \Delta = \left( a_1 \omega^2 + a_3 \omega^2 + a_5 \right)^2 - 4 \omega^2 [a_1 a_3 \omega^2 + (a_1 a_5 - a_2 a_4)] \]

\[ = \left[ (a_3 - a_1) \omega^2 + a_5 \right]^2 + 4 a_2 a_4 \omega^2 . \]

Since \( a_2, a_4 \) are positive, then also \( \Delta \) is positive. Taking into account (15) and (16), it is evident that all the coefficients of Eq. (22) are positive and therefore Eq. (22) has four purely imaginary roots. If we denote by \( \pm i \xi \) and \( \pm i \eta \) the four roots, then the general solution of Eq. (21) writes

\[ U_1(t) = k_1 \cos(\xi t) + k_2 \sin(\xi t) + k_3 \cos(\eta t) + k_4 \sin(\eta t) , \]

(23)

where \( k_i (i = 1, 2, 3, 4) \) are arbitrary real numbers. Eq. (20) gives the solution \( X_1(t) \) easily

\[ X_1(t) = \frac{\xi^2 - a_1 \omega^2}{a_2 \omega} [k_1 \cos(\xi t) + k_2 \sin(\xi t)] + \frac{\eta^2 - a_1 \omega^2}{a_2 \omega} [k_3 \cos(\eta t) + k_4 \sin(\eta t)] . \]

(24)

Another set of exact solutions can be obtained, looking for solutions of Eqs. (13)-(14) of the kind

\[ u(t, x) = U_2(t) \cos(\omega x) , \quad \chi(t, x) = X_2(t) \sin(\omega x) . \]

(25)
Using (25), Eqs. (13)-(14) give the system of second order ordinary differential equations for the unknowns $U_2$ and $X_2$

\[ U_2''(t) \cos(\omega x) = -a_1 \omega^2 U_2(t) \cos(\omega x) + a_2 \omega X_2(t) \cos(\omega x), \]
\[ X_2''(t) \sin(\omega x) = -a_3 \omega^2 X_2(t) \sin(\omega x) + a_4 \omega U_2(t) \sin(\omega x) - a_5 X_2(t) \sin(\omega x). \]

It is equivalent to the system

\[ U_2''(t) = -a_1 \omega^2 U_2(t) + a_2 \omega X_2(t), \]
\[ X_2''(t) = -a_3 \omega^2 X_2(t) + a_4 \omega U_2(t) - a_5 X_2(t), \]

that is similar to the system (18)-(19). So we can use the same procedure to derive a fourth order ordinary differential equation. Since the characteristic equation coincides with Eq. (22), we do not give further details on the solutions of this equation.

We remark that Eqs. (13)-(14) are linear and homogeneous; so any linear (finite or, under suitable conditions, numerable) combination of solutions is also a solution.

4 Numerical solutions and numerical tests

The numerical treatment of Eqs. (13)-(14) requires some transformations of variables in order to derive a suitable system of first order partial differential equations. As the equations are of hyperbolic type, this step is of fundamental importance to achieve good numerical solutions, and to take into account boundary conditions, correctly.

Firstly we introduce the new variables $\alpha$ and $\beta$ defined by

\[ \alpha = u_t - \sqrt{a_1} u_x, \quad \beta = \chi_t - \sqrt{a_3} \chi_x. \quad (26) \]

Now Eq. (13) becomes

\[ \frac{\partial}{\partial t} (\alpha + \sqrt{a_1} u_x) - \sqrt{a_1} \frac{\partial}{\partial x} (u_t - \alpha) + a_2 \chi_x = 0 \quad \Leftrightarrow \quad \alpha_t + \sqrt{a_1} \alpha_x - a_2 \chi_x = 0, \]

and Eq. (14) is

\[ \frac{\partial}{\partial t} (\beta + \sqrt{a_3} \chi_x) - \sqrt{a_3} \frac{\partial}{\partial x} (\chi_t - \beta) + a_4 u_x + a_5 \chi = 0 \quad \Leftrightarrow \quad \beta_t + \sqrt{a_3} \beta_x + a_4 u_x + a_5 \chi = 0. \]

Hence, system (13)-(14) is equivalent to the set of four partial differential equations

\[ u_t = \sqrt{a_1} u_x + \alpha, \quad (27) \]
\[ \chi_t = \sqrt{a_3} \chi_x + \beta, \quad (28) \]
\[ \alpha_t = -\sqrt{a_1} \alpha_x + a_2 \chi_x, \quad (29) \]
\[ \beta_t = -\sqrt{a_3} \beta_x - a_4 u_x - a_5 \chi. \quad (30) \]

At this step it is necessary the change of variables

\[ v = \alpha + c_1 \chi, \quad w = \beta + c_2 u \]
where \(c_1\) and \(c_2\) are real constants to be determined. The new system writes

\[
\frac{\partial u}{\partial t} = \sqrt{a_1} \frac{\partial u}{\partial x} + v - c_1 \chi,
\]
\[
\frac{\partial \chi}{\partial t} = \sqrt{a_3} \frac{\partial \chi}{\partial x} + w - c_2 u,
\]
\[
\frac{\partial v}{\partial t} = c_1 \frac{\partial \chi}{\partial t} - \sqrt{a_1} \left( \frac{\partial v}{\partial x} - c_1 \frac{\partial \chi}{\partial x} \right) + a_2 \frac{\partial \chi}{\partial x},
\]
\[
\frac{\partial w}{\partial t} = c_2 \frac{\partial u}{\partial t} - \sqrt{a_3} \left( \frac{\partial w}{\partial x} - c_2 \frac{\partial u}{\partial x} \right) - a_4 \frac{\partial u}{\partial x} - a_5 \chi,
\]

that is

\[
\frac{\partial u}{\partial t} = \sqrt{a_1} \frac{\partial u}{\partial x} + v - c_1 \chi,
\]
\[
\frac{\partial \chi}{\partial t} = \sqrt{a_3} \frac{\partial \chi}{\partial x} + w - c_2 u,
\]
\[
\frac{\partial v}{\partial t} = c_1 \left[ \sqrt{a_3} \frac{\partial \chi}{\partial x} + w - c_2 u \right] - \sqrt{a_1} \left( \frac{\partial v}{\partial x} - c_1 \frac{\partial \chi}{\partial x} \right) + a_2 \frac{\partial \chi}{\partial x},
\]
\[
\frac{\partial w}{\partial t} = c_2 \left[ \sqrt{a_1} \frac{\partial u}{\partial x} + v - c_1 \chi \right] - \sqrt{a_3} \left( \frac{\partial w}{\partial x} - c_2 \frac{\partial u}{\partial x} \right) - a_4 \frac{\partial u}{\partial x} - a_5 \chi.
\]

Now, if we choose

\[
c_1 = \frac{-a_2}{\sqrt{a_1} + \sqrt{a_3}}\quad \text{and} \quad c_2 = \frac{a_4}{\sqrt{a_1} + \sqrt{a_3}},
\]

the partial derivative of the unknowns \(u\) and \(\chi\) disappear in the last two equations. The final result is

\[
\frac{\partial u}{\partial t} = \sqrt{a_1} \frac{\partial u}{\partial x} - c_1 \chi + v,
\]
\[
\frac{\partial \chi}{\partial t} = \sqrt{a_3} \frac{\partial \chi}{\partial x} - c_2 u + w,
\]
\[
\frac{\partial v}{\partial t} = - \sqrt{a_1} \frac{\partial v}{\partial x} - c_1 c_2 u + c_1 w,
\]
\[
\frac{\partial w}{\partial t} = - \sqrt{a_3} \frac{\partial w}{\partial x} - (c_1 c_2 + a_5) \chi + c_2 v.
\]

Eqs. (31)-(34) is a system of partial differential equations of hyperbolic type, and the new unknowns \(u, \chi, v, w\) are the Riemann invariants. These equations are similar to advection equations with source terms. This makes simple a proper imposition of the initial and boundary conditions. For instance, if \(x \in [0, L]\), then we must assign the initial conditions for all the variables, and the boundary conditions

\[
u(t, L^+), \quad \chi(t, L^+), \quad v(t, 0^-), \quad w(t, 0^-).
\]

In order to find numerical solutions of Eqs. (31)-(34), we use a Weighted Essentially Non-Oscillatory (WENO) finite difference scheme [12], [13]. The main advantage of WENO schemes is their capability to achieve arbitrarily high-order accuracy in regions where the solution are smooth, while maintaining stable, non-oscillatory and sharp discontinuity transitions. In particular the schemes are suitable for hyperbolic partial differential equations admitting solutions containing both strong discontinuities and complex smooth features. WENO schemes approximate spatial partial derivatives by means of suitable
finite differences, so the original system of partial differential equations is replaced by a set of ordinary
differential equations, which are solved by means of a Runge-Kutta method. In this paper we employ
a third-order Total Variation Diminishing (TVD) Runge-Kutta method [13]. These methods guarantee
that the total variation of the solution does not increase, so that no new extrema are generated.

We consider two simple explicit solutions and make a comparison with the corresponding numerical solu-
tions obtained using WENO scheme. The spatial domain of the solutions is the interval \((0,1)\). We assume
periodic boundary conditions in the simulations. The numerical physical parameters, with arbitrary units,
used in the simulations are

\[
\rho = 1, \quad I_\mu = 1, \quad \gamma = 0.99, \quad A = -0.01, \quad B = 10, \quad C = 1,
\]

which are of the same order of the parameters employed in Ref. [9]. We point out that the physical
parameters are not of same order. This implies that, in general, the solutions are not very smooth. We
use different partitions of the interval \((0,1)\) and we denote by \(N\) the number of the cells. So the spatial
step \(\Delta x\) is given by \(1/N\), because the length of the interval is equal to one. We measure the difference
between exact and numerical solutions by means of the formula

\[
err(u) = \max_{t,x} |u(t,x) - \hat{u}(t,x)| / (1 + |u(t,x)|),
\]

where \(\hat{u}(t,x)\) denotes the numerical solution for the unknown \(u\) at time \(t\) and position \(x\); of course, only
the grid points, in time and space, are used to evaluate the maximum. An analogous formula is employed
for the unknown \(\chi\).

**Numerical test A**

The parameter \(\omega\) of first solution (case A) of type (17) is equal to \(2\pi\), and \(k_i = 1\) with \((i = 1,2,3,4)\). In
our simulations, we choose \([0,10]\) as interval for time integration. The Table 1 shows the errors between
exact and numerical solutions in the case A.

| \(N\)  | 128   | 256   | 512   | 1024  | 2048  |
|-------|-------|-------|-------|-------|-------|
| \(err(u)\) | \(3.663 \times 10^{-5}\) | \(4.471 \times 10^{-6}\) | \(5.561 \times 10^{-7}\) | \(6.947 \times 10^{-8}\) | \(8.684 \times 10^{-9}\) |
| \(err(\chi)\) | \(1.749 \times 10^{-4}\) | \(1.259 \times 10^{-5}\) | \(1.132 \times 10^{-6}\) | \(1.184 \times 10^{-7}\) | \(1.479 \times 10^{-8}\) |

**Numerical test B**

For the second test (case B), we use the sum of two solutions of kind (17) by choosing two values \((2\pi\)
and \(4\pi\)) for the parameter \(\omega\). The integration constants \(k_i\) and the domain parameters are the same as
in the case A. The Table 2 shows the errors between exact and numerical solutions in the case B.

| \(N\)  | 128   | 256   | 512   | 1024  | 2048  |
|-------|-------|-------|-------|-------|-------|
| \(err(u)\) | \(1.151 \times 10^{-3}\) | \(1.252 \times 10^{-4}\) | \(1.482 \times 10^{-5}\) | \(1.822 \times 10^{-6}\) | \(2.268 \times 10^{-7}\) |
| \(err(\chi)\) | \(6.183 \times 10^{-2}\) | \(8.600 \times 10^{-3}\) | \(1.136 \times 10^{-4}\) | \(1.428 \times 10^{-5}\) | \(1.814 \times 10^{-6}\) |
Remark.
The numerical simulations show the robustness and the accuracy of the method both for fine and coarse meshes. The differences in the errors between test A and B depend on the smoothness of the solutions; in the test B the maximum absolute value of the partial derivatives with respect to the coordinate $x$ is greater than in the first case. Moreover the different order of magnitude of the physical parameters introduces a stiffness in the set of partial differential equations.

5 Spatial depending parameters

When the physical parameters $\rho$, $I_\mu$, $\gamma$, $A$, $B$ and $C$ are not constant, but they are differentiable functions of the variable $x$, and we assume valid the definitions of the kinetic energy (2) and the potential energy
then the Euler-Lagrange equations write
\[ \rho u_{tt} = \gamma u_{xx} + \frac{d\gamma}{dx} u_x + A \chi_x + \frac{dA}{dx} \chi, \]
(36)
\[ I_\mu \chi_{tt} = C \chi_{xx} + \frac{dC}{dx} \chi_x - A u_x - B \chi. \]
(37)

If we define
\[ a_1(x) = \frac{\gamma}{\rho}, \quad b_1(x) = \frac{1}{\rho} \frac{d\gamma}{dx}, \quad a_2(x) = \frac{A}{\rho}, \quad b_2(x) = \frac{1}{\rho} \frac{dA}{dx}, \]
\[ a_3(x) = \frac{C}{I_\mu}, \quad b_3(x) = \frac{1}{I_\mu} \frac{dC}{dx}, \quad a_4(x) = \frac{A}{I_\mu}, \quad a_5(x) = \frac{B}{I_\mu}, \]
then the equations for the unknowns \( u \) and \( \chi \) becomes
\[ u_{tt} = a_1(x) u_{xx} + b_1(x) u_x + a_2(x) \chi_x + b_2(x) \chi, \]
(38)
\[ \chi_{tt} = a_3(x) \chi_{xx} + b_3(x) \chi_x - a_4(x) u_x - a_5(x) \chi. \]
(39)

Also in this case, we introduce new variables in order to make the system of partial differential equations suitable for a numerical integration. It is possible to prove (see Appendix A) that Eqs. 36–39 are equivalent to the four partial differential equations
\[ u_t = \sqrt{a_1(x)} u_x - \phi_1(x) u - \phi_2(x) \chi + v, \]
(40)
\[ \chi_t = \sqrt{a_3(x)} \chi_x - \phi_3(x) u - \phi_4(x) \chi + w, \]
(41)
\[ v_t = -\sqrt{a_1(x)} v_x + \phi_1(x) u + \phi_2(x) \chi + \phi_1(x) v + \phi_2(x) w, \]
(42)
\[ w_t = -\sqrt{a_3(x)} w_x + \phi_3(x) u + \phi_4(x) \chi + \phi_3(x) v + \phi_4(x) w, \]
(43)

where
\[ \phi_1(x) = \sqrt{a_1(x)} \frac{d\phi_1}{dx} - [\phi_1(x)]^2 - \phi_2(x) \phi_3(x), \]
\[ \phi_2(x) = \sqrt{a_1(x)} \frac{d\phi_2}{dx} + b_2(x) - \phi_1(x) \phi_2(x) - \phi_2(x) \phi_4(x), \]
\[ \phi_3(x) = \sqrt{a_3(x)} \frac{d\phi_3}{dx} - \phi_3(x) \phi_1(x) - \phi_4(x) \phi_3(x), \]
\[ \phi_4(x) = \sqrt{a_3(x)} \frac{d\phi_4}{dx} - a_5(x) - \phi_3(x) \phi_2(x) - [\phi_4(x)]^2, \]
\[ \varphi_1(x) = \frac{-1}{2} \rho(x) \frac{d}{dx} \left[ \rho(x) \sqrt{a_1(x)} \right], \quad \varphi_2(x) = \frac{-a_2(x)}{\sqrt{a_1(x)} + \sqrt{a_3(x)}}, \]
\[ \varphi_3(x) = \frac{a_4(x)}{\sqrt{a_1(x)} + \sqrt{a_3(x)}}, \quad \varphi_4(x) = \frac{-1}{2I_\mu(x)} \frac{d}{dx} \left[ \sqrt{a_3(x)} I_\mu(x) \right]. \]

We show a simple numerical example, where the numerical scheme is the same of the cases described in the previous section. The spatial domain is the interval \([0, 1]\). The assume that
\[ \gamma(x) = \gamma^* (1 + \psi(x)), \quad A(x) = A^* (1 + \psi(x)), \quad B(x) = B^* (1 + \psi(x)) \]
\[ \rho(x) = \rho^* (1 + \psi(x)), \quad C(x) = C^* (1 + \psi(x)), \quad I_\mu(x) = I_\mu^* (1 + \psi(x)), \]
with
\[ \rho^* = 1, \quad I_{\mu}^* = 1, \quad \gamma^* = 0.99, \quad A^* = -0.01, \quad B^* = 10, \quad C^* = 1, \]
and (see, Figure 3)
\[ \psi(x) = h \left[ \frac{1}{1 + \exp(400 (0.5 - x))} - \frac{1}{1 + \exp(400 (0.7 - x))} \right], \]
where \( h \) is a parameter. In our simulations we have chosen \( h = 0.1 \) and \( h = 1 \). We point out that the function \( \psi \) is not smooth. The model simulates an approximation of a continuum, which consists of two different materials. Following Ref. [5] we assume that the continuum is at rest at the initial time, that is
\[ u(0,x) = 0, \quad \chi(0,x) = 0, \quad v(0,x) = 0, \quad w(0,x) = 0. \]
The boundary conditions must be simulated an excitation of the strain at \( x = 0 \) for a short time period; so we must assign \( u_x(t,0^-) \). This is possible, by observing that only Eq. (40) contains the term \( u_x \). Now, if \( u_x(t,0^-) = \varepsilon(t) \), then Eq. (40) gives
\[ v(t,0^-) = u_1(t,0) - \sqrt{a_1(0)} \varepsilon(t) + \varphi_1(0) u(t,0) + \varphi_2(0) \chi(t,0), \quad (44) \]
which is the boundary condition for the unknown \( v \) at time \( t \). The right hand side of (44) can be evaluated, since it is known at the previous time step. The partial derivative of \( u \) with respect to the time is approximated by means of a simple finite difference. The other boundary conditions (35) are assumed null at every time. So, when we find the approximated solution at each time step by means of numerical integration. We have chosen
\[ \varepsilon(t) = \begin{cases} 1 + \cos(\pi (1 - 50t)) / 2 & \text{if } 0 \leq t \leq 0.04 \\ 0 & \text{otherwise} \end{cases} \]
in the simulation.
We have found the solutions (see Figure 3 and 5) for \( h = 0.1 \) and \( h = 1 \), by using 1024 grid points in the spatial interval \([0, 1]\), and we verify a nice accuracy by changing the number of grid points. In the figure...
the partial derivative of $u(t, x)$ with respect to $x$ is shown at different times. We plot $u_x(t, x) + \kappa t$, where $\kappa$ is a positive constant, instead of $u_x(t, x)$. The fictitious shift was introduced only to make clear the figure. The reflected waves, which arise from the inhomogeneity, are evident.

Figure 4: The case $h = 0.1$. The function $u(t, x)$ at time $t = 0.8$ (left) and the function $u_x(t, x) + \kappa t$ (right)

Figure 5: The case $h = 1$. The function $u(t, x)$ at time $t = 0.8$ (left) and the function $u_x(t, x) + \kappa t$ (right)

6 Conclusions

In this paper a class of exact solutions of a one dimensional Mindlin model to describe linear elastic behaviour is obtained. The main assumption of this paper concerns the potential energy, which is assumed strictly positive definite. Due to this property the solutions of the model have a wave behaviour. We find exact solutions to test the accuracy of the proposed numerical method. It is based on a weighted essentially non-oscillatory finite difference scheme, coupled by a total variation diminishing Runge-Kutta method. The results obtained with the numerical methods were matched to the exact analytical solutions, and they agree very well. In this way we also verify the robustness and the accuracy of numerical scheme
both for fine and coarse meshes. Also in the case when the physical parameters are not constant and smooth, the numerical scheme seems to give accurate solutions to the equations.

Acknowledgments

The first author was partially supported by the Italian FIR project "Innovative techniques in computational mechanics based on high continuity interpolation for the integrated design of advanced structures" (Principal Investigator: Massimo Cuomo).

Appendix A

Let us consider Eqs. (38)-(39). Firstly we define the new unknowns

$$\alpha = u_t - \sqrt{a_1(x)} u_x \quad \text{and} \quad \beta = \chi_t - \sqrt{a_3(x)} \chi_x.$$ 

So Eqs. (38)-(39) are equivalent to the system

$$u_t = \sqrt{a_1(x)} u_x + \alpha, \quad (45)$$
$$\chi_t = \sqrt{a_3(x)} \chi_x + \beta, \quad (46)$$
$$\alpha_t = - \sqrt{a_1(x)} \alpha_x + \left[ b_1(x) - \frac{1}{2} \frac{da_1(x)}{dx} \right] u_x + a_2(x) \chi_x + b_2(x) \chi, \quad (47)$$
$$\beta_t = - \sqrt{a_3(x)} \beta_x + \left[ b_3(x) - \frac{1}{2} \frac{da_3(x)}{dx} \right] \chi_x - a_4(x) u_x - a_5(x) \chi. \quad (48)$$

Since

$$b_1(x) - \frac{1}{2} \frac{da_1(x)}{dx} = \frac{1}{\rho(x)} \frac{d}{dx} [\rho(x) a_1(x)] - \frac{1}{2} \frac{da_1(x)}{dx} = \frac{a_1(x)}{\rho(x)} \frac{dp(x)}{dx} + \frac{1}{2} \frac{da_1(x)}{dx},$$
$$b_3(x) - \frac{1}{2} \frac{da_3(x)}{dx} = \frac{1}{I_\mu(x)} \frac{d}{dx} [I_\mu(x) a_3(x)] - \frac{1}{2} \frac{da_3(x)}{dx} = \frac{a_3(x)}{I_\mu(x)} \frac{dI_\mu(x)}{dx} + \frac{1}{2} \frac{da_3(x)}{dx},$$

then Eqs. (45)-(48) write

$$u_t = \sqrt{a_1(x)} u_x + \alpha,$$
$$\chi_t = \sqrt{a_3(x)} \chi_x + \beta,$$
$$\alpha_t = - \sqrt{a_1(x)} \alpha_x + \left[ \frac{a_1(x)}{\rho(x)} \frac{dp(x)}{dx} + \frac{1}{2} \frac{da_1(x)}{dx} \right] u_x + a_2(x) \chi_x + b_2(x) \chi,$$
$$\beta_t = - \sqrt{a_3(x)} \beta_x + \left[ \frac{a_3(x)}{I_\mu(x)} \frac{dI_\mu(x)}{dx} + \frac{1}{2} \frac{da_3(x)}{dx} \right] \chi_x - a_4(x) u_x - a_5(x) \chi.$$

Now let’s introduce the new unknowns $v$ and $w$ to replace $\alpha$ and $\beta$, by means of the relationship

$$v = \alpha + \varphi_1(x) u + \varphi_2(x) \chi, \quad w = \beta + \varphi_3(x) u + \varphi_4(x) \chi,$$
where the functions $\varphi_i$ must be chosen opportune. It is a simple matter to verify that the new equations are

$$
\begin{align*}
  u_t &= \sqrt{a_1(x)} u_x - \varphi_1(x) u - \varphi_2(x) \chi + v, \\
  \chi_t &= \sqrt{a_3(x)} \chi_x - \varphi_3(x) u - \varphi_4(x) \chi + w, \\
  v_t &= \left[ \sqrt{a_1(x)} \varphi_1(x) + \frac{a_1(x)}{\rho(x)} \frac{d\rho}{dx} + \frac{1}{2} \frac{da_1(x)}{dx} + \varphi_1(x) \sqrt{a_1(x)} \right] u_x \\
  &+ \left[ \sqrt{a_1(x)} \varphi_2(x) + a_2(x) + \varphi_2(x) \sqrt{a_3(x)} \right] \chi_x - \sqrt{a_1(x)} v_x \\
  &+ \left[ \sqrt{a_1(x)} \frac{d\varphi_1}{dx} - [\varphi_1(x)]^2 - \varphi_2(x) \varphi_3(x) \right] u \\
  &+ \left[ \sqrt{a_1(x)} \frac{d\varphi_2}{dx} + b_2(x) - \varphi_1(x) \varphi_2(x) - \varphi_2(x) \varphi_4(x) \right] \chi + \varphi_1(x) v + \varphi_2(x) w, \\
  w_t &= \left[ \sqrt{a_3(x)} \varphi_3(x) - a_4(x) + \varphi_3(x) \sqrt{a_1(x)} \right] u_x \\
  &+ \left[ \sqrt{a_3(x)} \varphi_4(x) + \frac{a_3(x)}{I_{\mu}(x)} \frac{dI_{\mu}}{dx} + \frac{1}{2} \frac{da_3(x)}{dx} + \varphi_4(x) \sqrt{a_3(x)} \right] \chi_x - \sqrt{a_3(x)} w_x \\
  &+ \left[ \sqrt{a_3(x)} \frac{d\varphi_3}{dx} - \varphi_3(x) \varphi_1(x) - \varphi_4(x) \varphi_3(x) \right] u \\
  &+ \left[ \sqrt{a_3(x)} \frac{d\varphi_4}{dx} - a_5(x) - \varphi_3(x) \varphi_2(x) - [\varphi_4(x)]^2 \right] \chi + \varphi_3(x) v + \varphi_4(x) w.
\end{align*}
$$

We choose the functions $\varphi_i$ to make null the coefficients of $u_x$ and $\chi_x$ in the third and fourth equation. We derive immediately

$$
\begin{align*}
  \varphi_1(x) &= -\frac{1}{2} \sqrt{a_1(x)} \left[ \frac{a_1(x)}{\rho(x)} \frac{d\rho}{dx} + \frac{1}{2} \frac{da_1(x)}{dx} \right] = -\frac{1}{2} \frac{d}{dx} \left[ \rho(x) \sqrt{a_1(x)} \right], \\
  \varphi_2(x) &= -\frac{a_2(x)}{\sqrt{a_1(x)} + \sqrt{a_3(x)}}, \\
  \varphi_3(x) &= -\frac{a_4(x)}{\sqrt{a_1(x)} + \sqrt{a_3(x)}}, \\
  \varphi_4(x) &= -\frac{1}{2} \frac{a_2(x)}{\sqrt{a_3(x)}} \frac{dI_{\mu}}{dx} + \frac{1}{2} \frac{da_3(x)}{dx} + \frac{1}{2} \frac{da_3(x)}{dx} + \frac{1}{2} \frac{dI_{\mu}(x)}{dx} \frac{\sqrt{a_3(x)}}{I_{\mu}(x)} \frac{dI_{\mu}}{dx}.
\end{align*}
$$

Therefore the set of equations reduces to

$$
\begin{align*}
  u_t &= \sqrt{a_1(x)} u_x - \varphi_1(x) u - \varphi_2(x) \chi + v, \\
  \chi_t &= \sqrt{a_3(x)} \chi_x - \varphi_3(x) u - \varphi_4(x) \chi + w, \\
  v_t &= -\sqrt{a_1(x)} v_x + \left[ \sqrt{a_1(x)} \frac{d\varphi_1}{dx} - [\varphi_1(x)]^2 - \varphi_2(x) \varphi_3(x) \right] u \\
  &+ \left[ \sqrt{a_1(x)} \frac{d\varphi_2}{dx} + b_2(x) - \varphi_1(x) \varphi_2(x) - \varphi_2(x) \varphi_4(x) \right] \chi + \varphi_1(x) v + \varphi_2(x) w, \\
  w_t &= -\sqrt{a_3(x)} w_x + \left[ \sqrt{a_3(x)} \frac{d\varphi_3}{dx} - \varphi_3(x) \varphi_1(x) - \varphi_4(x) \varphi_3(x) \right] u \\
  &+ \left[ \sqrt{a_3(x)} \frac{d\varphi_4}{dx} - a_5(x) - \varphi_3(x) \varphi_2(x) - [\varphi_4(x)]^2 \right] \chi + \varphi_3(x) v + \varphi_4(x) w.
\end{align*}
$$

\[ 13 \]
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