METAPLECTIC REPRESENTATIONS OF AFFINE HECKE ALGEBRAS AND WEYL GROUPS

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ABSTRACT. Chinta and Gunnells expressed the functional equations of Weyl group multiple Dirichlet series in terms of an explicit, but rather intricate, multi-parameter linear Weyl group action on the fraction field of the group algebra \( \mathbb{C}[\Lambda] \) of some lattice \( \Lambda \). In this paper we realize the Chinta-Gunnells Weyl group action by localizing a suitable quotient of a parabolically induced affine Hecke algebra module. This gives an elementary and uniform proof that the Chinta-Gunnells formulas indeed define an action of the Weyl group. The key step in the paper is to realize the quotient affine Hecke algebra module in terms of Demazure-Lusztig type operators involving metaplectic generalizations of divided-difference operators.

1. INTRODUCTION

The purpose of this paper is to shed light on a “mysterious” Weyl group action discovered by Chinta and Gunnells [13, 14], in connection with their work on Weyl group multiple Dirichlet series. We recall now briefly the Chinta-Gunnells Weyl group action. See, e.g., [4, 5, 13, 14, 9, 12] and the nice overview article [8] for the connection to Weyl group multiple Dirichlet series.

Let \( W \) be the Weyl group of an irreducible root system \( \Phi \), with Coxeter generators \( \{ s_i \}_{i=1}^r \) corresponding to a choice of simple roots \( \{ \alpha_i \}_{i=1}^r \). Let \( P \) be the weight lattice of \( \Phi \). The Weyl group canonically acts on the fraction field \( \mathbb{C}(P) \) of the group algebra \( \mathbb{C}[P] \) by field automorphisms. Chinta and Gunnells have constructed a deformation of this action, which depends on the choice of a \( W \)-invariant quadratic form \( Q : P \to \mathbb{Z} \), a natural number \( n \), and on parameters \( v, g_0, \ldots, g_{n-1} \) satisfying

\[
g_0 = -1, \quad g_j g_{n-j} = v^{-1}, \quad j = 1, \ldots, n - 1.
\]

Let \( 0 \leq r_m(j) \leq m - 1 \) denote the remainder on dividing \( j \) by the natural number \( m \), and define \( g_j \) for arbitrary \( j \in \mathbb{Z} \) by setting \( g_j = g_{r_m(j)} \). Let \( B(\lambda, \mu) = Q(\lambda + \mu) - Q(\lambda) - Q(\mu) \) be the bilinear form associated to \( Q \), and put \( m(\alpha) = n / \gcd(n, Q(\alpha)) \). It defines a new root system \( \Phi^m := \{ m(\alpha) \alpha \}_{\alpha \in \Phi} \), which is either isomorphic to \( \Phi \) or to \( \Phi^\vee \). The weight lattice \( P^m \subseteq P \) of \( \Phi^m \) is

\[
P^m = \{ \lambda \in P \mid B(\lambda, \alpha) \equiv 0 \mod n \quad \forall \alpha \in \Phi \}.
\]
(see Lemma 2.2). Then the Chinta-Gunnells action \( \sigma_i = \sigma(s_i) \) of the simple reflection \( s_i \in W \) on \( \mathbb{C}(P) \) is given by the formula

\[
\sigma_i(f \chi^\lambda) := \frac{(s_if)x^{s_i\lambda}}{1 - vx^{m(\alpha_i)a_i}} \times \left[ x^{-r_{m(\alpha_i)}(n)Q(\alpha_i)} \alpha_i (1 - v) - vgQ(\alpha_i)B(\lambda, \alpha_i)x^{(1-m(\alpha_i))\alpha_i} (1 - x^{m(\alpha_i)a_i}) \right]
\]

for \( f \in \mathbb{C}(P^m) \) and \( \lambda \in P \).

It is non-trivial to show that the formula (1.1) defines a representation of \( W \). The main issue is to verify that the braid relations are satisfied. Although this reduces to a rank 2 computation, the calculations become rather formidable, and in [14] the details are only presented for \( A_2 \). Trying to find a natural interpretation of this representation was one of the main motivations for our work.

Chinta and Gunnells [14] employed the action (1.1) to give an explicit construction of the “local” parts of certain Weyl group multiple Dirichlet series, and to establish thus the analytic continuation and functional equations for these series. In this situation, the \( g_i \) are \( n \)-th order Gauss sums for the local field, and \( v = p^{-1} \) with \( p \) the cardinality of the residue field. Subsequently, Chinta-Offen [16] for type \( A \), and McNamara [25] in general, showed that these local parts are essentially Whittaker functions for principal series of certain \( n \)-fold “metaplectic” covers of quasi-split reductive groups. The resulting explicit expression for the Whittaker function in terms of the action (1.1) is the metaplectic generalization of the Casselman-Shalika formula. This result is in line with the fact that multiple Dirichlet series should themselves be Whittaker coefficients attached to metaplectic Eisenstein series [6, 9].

Still more recently, Chinta-Gunnells-Puskas [15] have shown that the \( W \)-action (1.1) gives rise to a Cherednik [11] type Demazure-Lusztig action of the Hecke algebra of \( W \). It leads to an expression of the metaplectic Whittaker functions in terms of metaplectic Demazure-Lusztig operators. Their work was partly motivated by Brubaker-Bump-Licata [7], who gave formulas for (nonmetaplectic) Iwahori-Whittaker functions in terms of Hecke operators and nonsymmetric Macdonald polynomials. The recent work of Patnaik-Puskas [26] uses the Chinta-Gunnells-Puskas Hecke algebra action to study metaplectic Iwahori-Whittaker functions.

In this paper we give a uniform construction of a Weyl group representation (Theorem 3.21) and an associated Hecke algebra representation (Theorem 4.2) that generalize the Chinta-Gunnells [14] and Chinta-Gunnells-Puskas [15] representations, respectively. Our construction does not involve case-by-case considerations, and it yields a representation for the generic Hecke algebra \( H(k) \), which has independent Hecke parameters for each root length in \( \Phi \). Our method also allows us to incorporate extra freedom in the definition of \( g_i \) by allowing them to depend on the root length (see Definition 3.5 of the representation parameters).
The Chinta-Gunnells and Chinta-Gunnells-Puskas representations are recovered in the equal Hecke and representation parameter case of our constructions.

Our starting point was the observation that (1.1) has many features in common with formulas obtained by the process of “Baxterization” [11]. The key idea behind this process is that the group algebra of the affine Weyl group and the affine Hecke algebra become isomorphic after a suitable localization, which allows one to relate certain representations of the two algebras. This inspired our search for a natural representation of the affine Hecke algebra whose associated localized affine Weyl group representation produces (1.1) for its W-action. Its first form can be recovered from the Chinta-Gunnells-Puskas Hecke algebra action as follows.

Note that the Chinta-Gunnells W-action (1.1) has an obvious extension to a representation of the extended affine Weyl group \( \tilde{W}^m := W \ltimes P^m \) with \( \mu \in P^m \) acting on \( \mathbb{C}(P) \) by multiplication by \( x^\mu \). Let \( \tilde{H}^m(k) \) be the associated extended affine Hecke algebra with single Hecke parameter \( k \) satisfying \( k^2 = v \). If the affine extension of the Chinta-Gunnells W-action on \( \mathbb{C}(P) \) arises from a \( \tilde{H}^m(k) \)-action on \( \mathbb{C}(P) \) by localization, then the generators \( \{ T_i \}_{i=1}^r \) of the finite Hecke algebra \( H(k) \) act on \( \mathbb{C}(P) \) by the Chinta-Gunnells-Puskas metaplectic Demazure-Lusztig operators associated to \( \sigma_i \) (cf. Proposition [11]). It follows that the underlying \( H(k) \)-representation is equivalent to the \( H(k) \)-representation on \( \mathbb{C}(P) \) defined by

\[
\pi(T_i)x^\lambda := (k - k^{-1})\nabla_i(x^\lambda) - k g_{-B(\lambda,\alpha_i)} x^{s_i\lambda}, \quad \lambda \in P,
\]

with \( \nabla_i \) the following metaplectic version of the divided-difference operator

\[
\nabla_i(x^\lambda) := \frac{x^{\lambda} - x^{s_i\lambda + r_{m(\alpha_i)} (\lambda, \alpha_i^\vee)} x^{\lambda}}{1 - x^{m(\alpha_i)\alpha_i}}.
\]

But now we want to have an a priori proof that (1.2) defines a \( H(k) \)-action on \( \mathbb{C}(P) \) and conclude from it that (1.1) defines a W-action on \( \mathbb{C}(P) \) via the localization technique.

Although the formulas (1.2) are much simpler than (1.1), a direct case-by-case check that it defines a \( H(k) \)-representation will be close to being as cumbersome as for the Chinta-Gunnells action. The key point of this paper is to circumvent the case-by-case check by proving that \( \pi \) is isomorphic to a quotient of the parabolically induced module \( \tilde{H}^m(k) \otimes_{H(k)} V_C \) for an appropriate \( H(k) \)-representation \( V_C \).

The \( H(k) \)-representation \( V_C \) is defined as follows. Let \( V = \bigoplus_{\lambda \in P} \mathbb{C}v_\lambda \) be the complex vector space with basis the weight lattice \( P \). It has a natural left \( H(k) \)-module structure reducing to the canonical \( \mathbb{C}[W] \)-module structure when \( k = 1 \) (see Lemma [3.4]). We call \( V \) the reflection representation of \( H(k) \). For each \( W \)-invariant subset \( D \subseteq P \), the subspace \( V_D := \bigoplus_{\lambda \in D} \mathbb{C}v_\lambda \) is a \( H(k) \)-submodule of \( V \). In particular, \( V_{\{0\}} \) is the trivial representation of \( H(k) \). The appropriate choice of \( W \)-invariant subset \( C \) of \( P \) in the above realization of \( \pi \) now turns out to be

\[
C := \{ \lambda \in P \mid (\lambda, \alpha^{\vee}) \leq m(\alpha) \quad \forall \alpha \in \Phi \}.
\]
Note that $C$ contains a complete set of coset representatives of $P/P^m$.

The following trivial example is instructive to get a feeling for what is going on. Suppose that $m(\alpha) = 1$ for all $\alpha \in \Phi$. Then $P^m = P$ and $\nabla_i$ is the standard divided-difference operator on $\mathbb{C}[P]$. In this case it is well known that (1.2) is equivalent to the induced module $\tilde{H}^m(k) \otimes_{H(k)} V_{\{0\}}$ by the Bernstein-Zelevinsky presentation of $\tilde{H}^m(k)$. The $W$-subset $C$ in this case is oversized, with $C \setminus \{0\}$ being the set of nonzero miniscule weights in $P$.

We now briefly discuss the content of the paper. We introduce in Section 2 the appropriate metaplectic structures on the root systems and affine Weyl and Hecke algebras. Section 3 is devoted to the metaplectic representation theory of the affine Weyl groups and generic affine Hecke algebras. We introduce the reflection representation in Subsection 3.1. Subsection 3.2 forms the heart of our approach: we introduce the analogue of (1.2) for generic Hecke and representation parameters and establish that it defines a representation of the generic affine Hecke algebra by identifying it with a quotient of the induced module $\tilde{H}^m(k) \otimes_{H(k)} V_C$ (see Theorem 3.7). In Subsection 3.3 we explain the localization technique and apply it to $\pi$ (Theorem 3.7) to obtain the generalized Chinta-Gunnells $W$-action (Theorem 3.21).

In Section 4 we form the associated metaplectic Demazure-Lusztig operators and generalize some of the results from [15] to the setting of unequal Hecke and representation parameters. We also simplify some of the proofs from that paper by using the standard symmetrizer and antisymmetrizer elements in the Hecke algebra. This allows us to define a natural class of “Whittaker functions” for generic Hecke algebras. It is natural to ask whether these more general functions arise as actual matrix coefficients for some class of representations of $p$-adic groups. This question is of particular interest since generic Hecke algebras have begun to play an increasing role in the study of the Bernstein components within the categories of smooth representations of $p$-adic groups, see, e.g., [10, 18] and references therein.

We briefly mention another related avenue of research that we plan to investigate further. The Whittaker functions mentioned above arise from the representation of $\tilde{H}^m(k)$ via the antisymmetrizer (cf. the proof of Theorem 4.9). One may also consider the corresponding symmetric variant. These polynomials are indexed by dominant weights $\lambda \in P^+$, symmetric with respect to the Chinta-Gunnells $W$-action, and for $\lambda \in P^m$ reduce to Hall-Littlewood polynomials for the root system $\Phi$ (see Remark 4.11). In a work in progress, we investigate these further and construct analogues of Macdonald polynomials in this setting, along the lines of the standard Cherednik-Macdonald theory.

Let us conclude with remarking that the localization procedure we use in this paper is instrumental in Cherednik’s construction of quantum affine Knizhnik-Zamolodchikov equations attached to affine Hecke algebra modules. Closely related to it is the role of the localization procedure for type $A$ in the context of
integrable vertex models with \( U_q(\hat{\mathfrak{sl}}_n) \)-symmetry, in the special cases that the associated braid group action descends to an affine Hecke algebra action, in which case the localization procedure is often referred to as Baxterization (see, e.g., [11, 27] and references therein). This is exactly the context in which the metaplectic Whittaker function can be realized as a partition function, the corresponding integrable model being “metaplectic ice”, see [1, 3, 2]. It is an intriguing open question whether there is a conceptual connection with the current interpretation of the Chinta-Gunnells action through localization.

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2. THE EXTENDED AFFINE HECKE ALGEBRA

2.1. The root system. Let \( E \) be an Euclidean space with scalar product \((\cdot, \cdot)\) and norm \( \| \cdot \| \). Let \( \Phi \subset E \) be an irreducible reduced root system, and \( W \subset O(E) \) its Weyl group. The reflection in \( \alpha \in \Phi \) is denoted by \( s_\alpha \in W \), and its co-root is \( \alpha^\vee := 2\alpha/\|\alpha\|^2 \).

Fix a base \( \{\alpha_1, \ldots, \alpha_r\} \) of \( \Phi \), and write \( \Phi^+ \) for the corresponding set of positive roots. Let

\[
P := \{\lambda \in V \mid (\lambda, \alpha^\vee) \in \mathbb{Z} \quad \forall \alpha \in \Phi\} = \bigoplus_{i=1}^{r} \mathbb{Z}\varpi_i
\]

be the weight lattice of \( \Phi \) with \( \varpi_i \in E \) the fundamental weights, defined by \((\varpi_i, \alpha_j^\vee) = \delta_{i,j}\). Let

\[
Q = \mathbb{Z}\Phi = \bigoplus_{i=1}^{r} \mathbb{Z}\alpha_i
\]

be the root lattice of \( \Phi \).

2.2. The metaplectic structure. In the theory of metaplectic Whittaker functions, a new root system \( \Phi^m \) is attached to the metaplectic covering data of the reductive group over the non-archimedean local field, cf. [14, 15] and references therein. We recall in this subsection this additional metaplectic data on the root system.

Fix a \( W \)-invariant quadratic form \( Q : P \to \mathbb{Z} \) and write \( B : P \times P \to \mathbb{Z} \) for the associated symmetric bilinear pairing

\[
B(\lambda, \mu) := Q(\lambda + \mu) - Q(\lambda) - Q(\mu), \quad \lambda, \mu \in P.
\]
Then $Q(\cdot) = \text{cst} \| \cdot \|^2$ for some $\text{cst} \in \mathbb{R}^\times$, and hence $B(\lambda, \mu) = 2\text{cst}(\lambda, \mu)$ for all $\lambda, \mu \in P$. In particular, for all $\lambda \in P$ and $\alpha \in \Phi$,

\begin{equation}
\frac{B(\lambda, \alpha)}{Q(\alpha)} = (\lambda, \alpha^\vee).
\end{equation}

Let $n \in \mathbb{Z}_{>0}$ and define

$$m(\alpha) := \frac{n}{\gcd(n, Q(\alpha))} = \frac{\text{lcm}(n, Q(\alpha))}{Q(\alpha)} \quad \forall \alpha \in \Phi.$$ 

Note that $m : \Phi \to \mathbb{Z}_{>0}$ is $W$-invariant.

Set $\Phi^m := \{\alpha^m := m(\alpha)\alpha\}_{\alpha \in \Phi} \subset E$. Then $\Phi^m$ is a root system. In fact, if $m$ is constant then $\Phi^m$ is isomorphic to $\Phi$, while if $m$ is nonconstant then $\Phi^m$ is isomorphic to the co-root system $\Phi^\vee = \{\alpha^\vee\}_{\alpha \in \Phi}$ (this follows from the definition of $m(\alpha)$ and the fact that $Q(\cdot) = \text{cst} \| \cdot \|^2$). In particular, $\{\alpha_1^m, \ldots, \alpha_r^m\}$ is a base of $\Phi^m$ and $W$ is the Weyl group of $\Phi^m$.

Write $Q^m$ for the root lattice of $\Phi^m$ and $P^m$ for the weight lattice of $\Phi^m$. Since $(\alpha^m)^\vee = m(\alpha)^{-1}\alpha^\vee$ for $\alpha \in \Phi$, we have

$$Q^m = \bigoplus_{i=1}^r \mathbb{Z}\alpha_i^m, \quad P^m = \bigoplus_{i=1}^r \mathbb{Z}\varpi_i^m$$

with $\varpi_i^m := m(\alpha_i)\varpi_i$ the fundamental weights of $P^m$.

**Lemma 2.1.** a. For $\alpha \in \Phi$ and $\lambda \in P$ we have

$$B(\lambda, \alpha^m) = \text{lcm}(n, Q(\alpha))(\lambda, \alpha^\vee).$$

b. For $\alpha \in \Phi$ and $\lambda \in P$ we have

$$B(\lambda, \alpha) \equiv 0 \mod n \iff (\lambda, \alpha^\vee) \equiv 0 \mod m(\alpha).$$

**Proof.** a For $\lambda \in P$ and $\alpha \in \Phi$ we have

$$B(\lambda, \alpha^m) = m(\alpha)B(\lambda, \alpha)$$

$$= \frac{nB(\lambda, \alpha)}{\gcd(n, Q(\alpha))}$$

$$= \frac{nQ(\alpha)}{\gcd(n, Q(\alpha))} \frac{B(\lambda, \alpha)}{Q(\alpha)} = \frac{n}{\gcd(n, Q(\alpha))} \frac{\text{lcm}(n, Q(\alpha))}{\text{lcm}(n, Q(\alpha))}(\lambda, \alpha^\vee).$$

b. For $\lambda \in P$ and $\alpha \in \Phi$ we have

$$B(\lambda, \alpha) \equiv 0 \mod n \iff Q(\alpha)(\lambda, \alpha^\vee) \equiv 0 \mod n$$

$$\iff (\lambda, \alpha^\vee) \equiv 0 \mod \frac{n}{\gcd(n, Q(\alpha))}$$

$$\iff (\lambda, \alpha^\vee) \equiv 0 \mod m(\alpha). \quad \square$$
Lemma 2.2.

\[ P^m = \{ \lambda \in P \mid (\lambda, \alpha^\vee) \equiv 0 \mod m(\alpha) \ \forall \alpha \in \Phi \} \]
\[ = \{ \lambda \in P \mid B(\lambda, \alpha) \equiv 0 \mod n \ \forall \alpha \in \Phi \}. \]

Proof. The first equality follows from the fact that \((\alpha^m)^\vee = m(\alpha)^{-1}\alpha^\vee\) for \(\alpha \in \Phi\). The second equality follows immediately from part b of Lemma 2.1. \( \square \)

2.3. The extended affine Hecke algebra. In this subsection we will introduce the extended affine Hecke algebra associated to the finite root system \(\Phi^m\).

Take \(\{\alpha^m_0, \ldots, \alpha^m_r\}\) as base of \(\Phi^m\) and write \(\Phi^{m\pm}\) for the associated set of positive and negative roots. Let \(\theta^m \in \Phi^{m+}\) be the highest short root of \(\Phi^m\). The set of dominant integral weights of \(\Phi^m\) with respect to this base is denoted by \(P^{m+}\).

Consider the semi-direct product group

\[ \widetilde{W}^m := W \ltimes P^m. \]

An element in \(\widetilde{W}^m\) will be denoted by \(w\tau(\nu)\) with \(w \in W\) and \(\nu \in P^m\). It acts on \(E \oplus \mathbb{R}\) by

\[ w\tau(\nu)(v, c) := (wv, c - (v, \nu)) \]

for \((v, c) \in E \oplus \mathbb{R}\). The extended affine Weyl group preserves the affine root system

\[ \tilde{\Phi}^m := \{ (\alpha^m, t\|\alpha^m\|^2/2) \mid \alpha \in \Phi, \ t \in \mathbb{Z} \} \subset E \oplus \mathbb{R}. \]

The reflection \(s_a \in \tilde{W}^m\) associated to the affine root \(a = (\alpha^m, t\|\alpha^m\|^2/2) \in \tilde{\Phi}^m\) is \(s_a := s_{\alpha^m} \tau(t\alpha^m)\). The reflections \(s_a (a \in \Phi^m)\) generate the affine Weyl group \(W \ltimes Q^m\).

We fix \( (a^m_0, a^m_1, \ldots, a^m_r) := ((\theta^m, \|\theta^m\|^2/2), (-\alpha^m_1, 0), \ldots, (-\alpha^m_r, 0)) \) as base of the affine root system \(\tilde{\Phi}^m\). Then \(W \ltimes Q^m\) is a Coxeter group with Coxeter generators the simple reflections \(s_j := s_{\alpha^m_j} (j = 0, \ldots, r)\).

Write \(\Phi^{m+}\) and \(\Phi^{m-}\) for the associated set of positive and negative roots in \(\tilde{\Phi}^m\) respectively. Note that by our (unconventional) choice of base of \(\tilde{\Phi}^m\), which is convenient for us in order to be able to work with the most natural versions of divided difference operators, we have \(\Phi^{m+} \subset \Phi^{m-}\).

Define the length of \(w \in \tilde{W}^m\) by

\[ \ell(w) := \#(\tilde{\Phi}^{m+} \cap w^{-1}(\tilde{\Phi}^{m-})) \]

and set

\[ \Omega^m := \{ w \in \tilde{W}^m \mid \ell(w) = 0 \}. \]

Then \(\Omega^m\) is a finite abelian subgroup of \(\tilde{W}^m\) isomorphic to \(P^m/Q^m\). It permutes the elements of the base \(\{a^m_0, \ldots, a^m_r\}\) of \(\tilde{\Phi}^m\). This induces an action of \(\Omega^m\) on
the index set \( \{0, \ldots, r\} \) of the base, and \( \omega s_j \omega^{-1} = s_{\omega(j)} \) in \( \tilde{W}^m \) for \( \omega \in \Omega^m \) and \( j \in \{0, \ldots, r\} \). It results in the isomorphism

\[
\tilde{W}^m \simeq \Omega^m \ltimes (W \ltimes Q^m).
\]

Let \( k : \Phi \to \mathbb{C}^\times \) be a \( W \)-invariant function and write \( k_\alpha \) for the value of \( k \) at \( \alpha \in \Phi \). View \( k \) as a \( \tilde{W}^m \)-invariant function on \( \tilde{P}^m \) by setting

\[
k_{(\alpha_\ell, t || \alpha_\ell ||^2 / 2)} := k_\alpha.
\]

Each \( \tilde{W}^m \)-invariant function \( k \) on \( \tilde{P}^m \) is of this form. Set \( k_j := k_{a_j^m} \) for \( j = 0, \ldots, r \).

**Definition 2.3.** The extended affine Hecke algebra \( \tilde{H}^m(k) \) is the unital associative algebra over \( \mathbb{C} \) generated by \( T_0, \ldots, T_r \) and the finite group \( \Omega^m \), with defining relations

a. \( (T_j - k_j)(T_j + k_j^{-1}) = 0 \) for \( j = 0, \ldots, r \).

b. For \( 0 \leq i \neq j \leq r \) the braid relation \( T_i T_j T_i \cdots = T_j T_i T_j \cdots \) (\( m_{ij} \) factors on each side, with \( m_{ij} \) the order of \( s_i s_j \) in \( \tilde{W}^m \)).

c. \( \omega T_j = T_{\omega(j)} \omega \) for \( \omega \in \Omega^m \) and \( j = 0, \ldots, r \).

For \( w = \omega s_{i_1} \cdots s_{i_r} \in \tilde{W}^m \) (\( \omega \in \Omega^m \) and \( 0 \leq i_j \leq r \)) a reduced expression of \( w \in \tilde{W}^m \) (i.e. \( \ell = \ell(w) \)), set

\[
T_w := \omega T_{i_1} \cdots T_{i_r} \in \tilde{H}^m(k).
\]

The \( T_w \) (\( w \in \tilde{W}^m \)) are well defined and form a linear basis of \( \tilde{H}^m(k) \).

The finite Hecke algebra \( H(k) \) is the subalgebra of \( \tilde{H}^m(k) \) generated by \( T_i \) (\( 1 \leq i \leq r \)). The elements \( T_w \) (\( w \in W \)) form a linear basis of \( H(k) \).

We recall the Bernstein-Zelevinsky presentation of \( \tilde{H}^m(k) \) (see [22]). Let \( \tilde{H}^m(k)^\times \) be the group of invertible elements in \( \tilde{H}^m(k) \). There exists a unique group homomorphism \( P^m \to \tilde{H}^m(k)^\times \), which we denote by \( \nu \mapsto Y^\nu \), such that

\[
Y^\nu = T_{\tau(-\nu)} \quad \forall \nu \in P^m^+.
\]

The subalgebra \( \mathbb{C}Y[P^m] := \text{span}\{Y^\nu\}_{\nu \in P^m} \) of \( \tilde{H}^m(k) \) is isomorphic to the group algebra \( \mathbb{C}[P^m] = \bigoplus_{\nu \in P^m} \mathbb{C}x^\nu \) via the map \( Y^\nu \mapsto x^\nu \) for \( \nu \in P^m \), and the multiplication map defines a linear isomorphism

\[
H(k) \otimes \mathbb{C}Y[P^m] \simeq \tilde{H}^m(k).
\]

The cross relations between the two subalgebras \( H(k) \) and \( \mathbb{C}Y[P^m] \) in \( \tilde{H}^m(k) \) are captured by

\[
T_i Y^\nu = (k_i - k_i^{-1}) \left( \frac{Y^{s_i \nu} - Y^\nu}{Y^{a_i^m} - 1} \right) + Y^{s_i \nu} T_i
\]

for \( i = 1, \ldots, r \) and \( \nu \in P^m \).
For later purposes it is convenient to write the cross relation (2.3) as
\[
T_i Y^\nu = (k_i - k_i^{-1}) \tilde{\nabla}_i^m (Y^\nu) + Y^{s_i \nu} T_i
\]
with \( \tilde{\nabla}_i^m : C_Y [P^m] \to C_Y [P^m] \) the divided difference operator, which is defined as the linear operator satisfying
\[
\tilde{\nabla}_i^m (Y^\nu) := \frac{Y^\nu - Y^{s_i \nu}}{1 - \alpha_i^m}
\]
for \( \nu \in P^m \).

3. Metaplectic representations

3.1. The reflection representation of \( H(k) \). Set
\[
V := \bigoplus_{\lambda \in P} C v_\lambda.
\]
It inherits a left \( W \)-action by the linear extension of the canonical action of \( W \) on \( P \). For a \( W \)-invariant subset \( D \subset P \) we write
\[
V_D := \bigoplus_{\lambda \in D} C v_\lambda
\]
for the corresponding \( W \)-submodule of \( V \). Then \( V = \bigoplus_{\lambda \in P^+} V_{O_{\lambda}} \) with \( O_{\lambda} = W \lambda \) the \( W \)-orbit of \( \lambda \) in \( P \) and \( P^+ \subset P \) the cone of dominant weights of \( \Phi \) with respect to the base \( \{ \alpha_1, \ldots, \alpha_r \} \). In this subsection we deform the \( W \)-action on \( V_D \) and \( V \) to a \( H(k) \)-action.

Fix \( \lambda \in P^+ \). The stabilizer subgroup
\[
W_\lambda := \{ w \in W \mid w \lambda = \lambda \}
\]
is a standard parabolic subgroup of \( W \). It is generated by the simple reflections \( s_i \) \((i \in I_{\lambda})\), with \( I_{\lambda} \) the index subset
\[
I_{\lambda} := \{ i \in \{1, \ldots, r\} \mid s_i \lambda = \lambda \}.
\]
Note that \( V_{O_{\lambda}} \cong C[\lambda] \otimes_{C[\lambda]} C \) as \( W \)-modules, with \( C \) regarded as the trivial \( W \lambda \)-module. This description leads to a natural Hecke deformation of the \( W \)-action on \( V_{O_{\lambda}} \) as follows.

Let \( W^\lambda \) be the minimal coset representatives of \( W/W_\lambda \), which can be characterized by
\[
W^\lambda = \{ w \in W \mid \ell(ws_i) = \ell(w) + 1 \quad \forall i \in I_{\lambda} \}.
\]
For \( \lambda \in P^+ \) let \( H_\lambda(k) \subset H(k) \) be the subalgebra generated by the \( T_i \) \((i \in I_{\lambda})\). Write \( C_{\lambda} \) for the trivial one-dimensional \( H_\lambda(k) \)-module defined by \( T_i \mapsto k_i \) \((i \in I_{\lambda})\). Consider the linear isomorphism
\[
\phi_\lambda : H(k) \otimes_{H_\lambda(k)} C_{\lambda} \xrightarrow{\sim} V_{O_{\lambda}}
\]
defined by \( \phi_\lambda(T_w \otimes_{H_\lambda(k)} 1) = v_{w \lambda} \) for \( w \in W^\lambda \). Transporting the canonical \( H(k) \)-module structure on \( H(k) \otimes_{H_\lambda(k)} C_{\lambda} \) to \( V_{O_{\lambda}} \) through the linear isomorphism \( \phi_\lambda \)
Since Case (3):

\[ \ell > \ell \]

Hence Case (1):

\[ \ell > \ell \]

\[ \ell > \ell \]

It is now easy to conclude the proof of the lemma:

**Case (1):** \( \ell(s_iw) = \ell(w) + 1 \) and \( s_iw \in W^\lambda \). Then \( \ell(s_iw) = \ell(w) + 1 \) implies \( w^{-1} \alpha_i \in \Phi^+ \), hence \( (\mu, \alpha_i^\vee) = (\lambda, w^{-1} \alpha_i^\vee) \geq 0 \). The assumption \( s_iw \in W^\lambda \) implies \( s_iw \lambda \neq w\lambda \), in particular \( (\mu, \alpha_i^\vee) \neq 0 \). Hence \( (\mu, \alpha_i^\vee) > 0 \).

**Case (2):** \( \ell(s_iw) = \ell(w) + 1 \) and \( s_iw \not\in W^\lambda \). Then \( s_iw = ws_j \) for some \( j \in I_\lambda \) by [17, Lem. 3.2]. Hence \( s_iw \lambda = w\lambda \) and consequently \( (\mu, \alpha_i^\vee) = 0 \).

**Case (3):** \( \ell(s_iw) = \ell(w) - 1 \). Then \( (\mu, \alpha_i^\vee) = (\lambda, w^{-1} \alpha_i^\vee) \leq 0 \) since \( w^{-1} \alpha_i \in \Phi^- \). If \( s_iw \lambda = w\lambda \) then \( s_iw \) would be a representative of \( wW_\lambda \) of small length than \( w \), which is absurd. Hence \( s_iw \lambda \neq w\lambda \), and consequently \( (\mu, \alpha_i^\vee) < 0 \). If \( s_iw \not\in W^\lambda \) then the minimal length representative \( w' \in W^\lambda \) of the coset \( s_iwW_\lambda \) has length strictly smaller than \( \ell(s_iw) = \ell(w) - 1 \). But then \( wW_\lambda \) contains an element of length strictly smaller than \( \ell(w) \), which is absurd. Hence \( s_iw \in W^\lambda \).

It is now easy to conclude the proof of the lemma:

**Case (1):** \( \ell(s_iw) = \ell(w) + 1 \) and \( s_iw \in W^\lambda \). Then

\[ T_i v_\mu = \phi_\lambda(T_i T_w \otimes H_\lambda(k) 1) = \phi_\lambda(T_{s_iw} \otimes H_\lambda(k) 1) = v_{s_i \mu}. \]

**Case (2):** \( \ell(s_iw) = \ell(w) + 1 \) and \( s_iw \not\in W^\lambda \). Let \( j \in I_\lambda \) such that \( s_iw = ws_j \).

Note that \( \alpha_j \in W^\alpha \), hence \( k_i = k_j \), and that \( \ell(w s_j) = \ell(w) + 1 \), so that \( T_i T_{s_iw} = T_{s_iw} = T_{ws_j} = T_w T_j \).

Then

\[ T_i v_\mu = \phi_\lambda(T_i T_w \otimes H_\lambda(k) 1) = \phi_\lambda(T_w T_j \otimes H_\lambda(k) 1) = k_i v_\mu. \]

**Case (3):** \( \ell(s_iw) = \ell(w) - 1 \). Using \( T_i^2 = (k_i - k_i^{-1}) T_i + 1 \) we get

\[ T_i T_{s_iw} = (k_i - k_i^{-1}) T_w + T_{s_iw}. \]

in \( H(k) \), and hence

\[ T_i v_\mu = (k_i - k_i^{-1}) v_\mu + v_{s_i \mu} \]

since \( s_iw \in W^\lambda \). \( \square \)
3.2. The metaplectic affine Hecke algebra representation. For \( s \in \mathbb{Z}_{>0} \) and \( t \in \mathbb{Z} \) let \( r_s(t) \in \{0, \ldots, s-1\} \) be the remainder of \( t \) modulo \( s \). Define \( q, r : P \to P \) by
\[
q(\lambda) := \lambda - r(\lambda),
\]
\[
r(\lambda) := \sum_{i=1}^{r} r_{m(\alpha_i)}((\lambda, \alpha_i^\vee)) x_i.
\]

**Lemma 3.2.** \( q(P) \subseteq P^m \).

**Proof.** For \( i = 1, \ldots, r \) and \( \lambda \in P \) we have
\[
(q(\lambda), \alpha_i^{m\nu}) = m(\alpha_i)^{-1}(q(\lambda), \alpha_i^\vee)
\]
\[
= m(\alpha_i)^{-1}((\lambda, \alpha_i^\vee) - r_{m(\alpha_i)}((\lambda, \alpha_i^\vee))) \in \mathbb{Z}.
\]

Let \( \mathbb{C}[P] = \text{span}\{x^\lambda\}_{\lambda \in P} \) be the group algebra of the weight lattice \( P \) and \( \mathbb{C}[P^m] \) the group algebra of \( P^m \). The Weyl group acts naturally on \( \mathbb{C}[P] \) and \( \mathbb{C}[P^m] \) by algebra automorphisms.

Under the canonical identification \( \mathbb{C}[P^m] \cong \mathbb{C}[Y] \), the divided difference operator \( \nabla_i \), featuring in the Bernstein-Zelevinsky cross relations (2.4) of the extended affine Hecke algebra \( \tilde{H}(k) \) becomes the linear operator \( \nabla_i : \mathbb{C}[P^m] \to \mathbb{C}[P^m] \) satisfying
\[
\nabla_i(x^\nu) := \frac{x^\nu - x^\lambda}{1 - x^{\alpha_i^\vee}} = \left(1 - x^{-(\nu, \alpha_i^{m\nu})} \alpha_i^m\right) x^\nu, \quad \nu \in P^m
\]
for \( i = 1, \ldots, r \).

**Lemma 3.3.** For \( i = 1, \ldots, r \) there exists a unique linear map
\[
\nabla_i : \mathbb{C}[P] \to \mathbb{C}[P]
\]
satisfying
\[
(3.1) \quad \nabla_i(x^\lambda) := \frac{1 - x^{-(q(\lambda), \alpha_i^{m\nu})} \alpha_i^m}{1 - x^{\alpha_i^{m\nu}}} x^\lambda
\]
for \( \lambda \in P \). Furthermore,
\[
\nabla_i|_{\mathbb{C}[P^m]} = \nabla_i^m.
\]

**Proof.** Note that \( \nabla_i : \mathbb{C}[P] \to \mathbb{C}[P] \) is a well defined linear operator by the previous lemma. In fact,
\[
(3.2) \quad \nabla_i(x^\lambda) := \begin{cases} 
-x^{\lambda-\alpha_i^{m\nu}} - \cdots - x^{\lambda-(q(\lambda), \alpha_i^{m\nu})} \alpha_i^m, & \text{if } (q(\lambda), \alpha_i^{m\nu}) > 0, \\
0 & \text{if } (q(\lambda), \alpha_i^{m\nu}) = 0, \\
x^{\lambda} + x^{\lambda+\alpha_i^{m\nu}} + \cdots + x^{\lambda-(1+(q(\lambda), \alpha_i^{m\nu}))} \alpha_i^m, & \text{if } (q(\lambda), \alpha_i^{m\nu}) < 0.
\end{cases}
\]
The second statement follows from the observation that
\[
P^m = \{ \lambda \in P \mid q(\lambda) = \lambda \}. \quad \square
Remark 3.4. Note that the action of $\nabla_i$ can alternatively be described by
\[
\nabla_i(x^\lambda) = \frac{x^\lambda - x^{s_i\lambda+(r(\lambda),\alpha_i^m)\alpha_i^m}}{1 - x^{\alpha_i^m}}, \quad \lambda \in P.
\]
Write $\Phi^m = \Phi^m_{sh} \cup \Phi^m_{lg}$ for the division of $\Phi^m$ into short and long roots, with the convention $\Phi^m = \Phi^m_{lg}$ if all roots have the same length. Write
\[
\text{size} : \Phi^m \to \{sh, lg\}
\]
for the function on $\Phi^m$ satisfying $\text{size}(\alpha) = \text{sh}$ iff $\alpha \in \Phi^m_{sh}$. Write $k_{sh}$ and $k_{lg}$ for the value of $k$ on $\Phi^m_{sh}$ and $\Phi^m_{lg}$ respectively.

**Definition 3.5** (Representation parameters). Let $g_j(x) \in \mathbb{C}^\times$ for $j \in \mathbb{Z}$ and $x \in \{sh, lg\}$ be parameters satisfying the following conditions:

- a. $g_j(x) = -1$ if $j \in n\mathbb{Z}$,
- b. $g_j(x) = g_{r(j)}(x)$,
- c. $g_j(x)g_{n-j}(x) = k_x^{-2}$ if $j \in \mathbb{Z} \setminus n\mathbb{Z}$.

Remark 3.6. The special case where $g_j(x) = g_j$, i.e., the parameters do not depend on root length, was considered in [14, 13] and motivated the generalization above.

In the applications considered in those papers, the $g_i$ were taken to be Gauss sums.

Write $\overline{\lambda}$ for the class of $\lambda \in P$ in the finite abelian quotient group $P/P^m$. By Lemma 2.2
\[
p_i(\overline{\lambda}) := -k_ig_{-B(\lambda,\alpha_i)}(\text{size}(\alpha_i^m))
\]
is a well defined function $p_i : P/P^m \to \mathbb{C}^\times$ for $i = 1, \ldots, r$. Note that $p_i(\overline{\lambda}) = k_i$ if $m(\alpha_i) \mid (\lambda, \alpha_i^\vee)$ by Lemma 2.1(a). The following theorem is the main result of this subsection.

**Theorem 3.7.** The formulas
\[
p(T_i)x^\lambda := (k_i - k_i^{-1})\nabla_i(x^\lambda) + p_i(\overline{\lambda})x^{s_i\lambda},
p(Y^\mu)x^\lambda := x^{\lambda+\nu}
\]
for $\lambda \in P$, $i = 1, \ldots, r$ and $\nu \in P^m$ turn $\mathbb{C}[P]$ into a left $\tilde{H}^m(k)$-module.

Remark 3.8. (i) Note that $\mathbb{C}[P^m] \subseteq \mathbb{C}[P]$ is a $\tilde{H}^m(k)$-submodule with respect to the action (3.3). The action on $\mathbb{C}[P^m]$ simplifies to
\[
p(T_i)x^\mu = (k_i - k_i^{-1})\nabla_i^m(x^\mu) + k_i x^{s_i\mu},
p(Y^\mu)x^\nu = x^{\mu+\nu}
\]
for $i = 1, \ldots, r$ and $\mu, \nu \in P^m$. It follows that $\mathbb{C}[P^m] \simeq \tilde{H}^m(k) \otimes_{H(k)} \mathbb{C}_0$ as $\tilde{H}^m(k)$-modules. In particular for $m \equiv 1$ (which happens for instance when $n = 1$), the representation $\pi$ itself is isomorphic to $\tilde{H}^m(k) \otimes_{H(k)} \mathbb{C}_0$. 

Let $\Lambda$ be a lattice in $E$ satisfying $Q \subseteq \Lambda \subseteq P$. Note that $\Lambda$ is automatically $W$-invariant. The lattice $\Lambda_0 := \Lambda \cap P_m$ then satisfies $Q^m \subseteq \Lambda_0 \subseteq P^m$, and

$$\Lambda_0 = \{ \lambda \in \Lambda \mid B(\lambda, \alpha) \equiv 0 \mod n \ \forall \alpha \in \Phi \}$$

by Lemma 2.2. Furthermore, $\mathbb{C}[\Lambda] \subseteq \mathbb{C}[P]$ is a $\tilde{H}^m(k, \Lambda_0)$-submodule for the action (3.4), with $\tilde{H}^m(k, \Lambda_0)$ the subalgebra of $\tilde{H}^m(k)$ generated by $H(k)$ and $C_Y[\Lambda_0] := \text{span}\{Y^\nu\}_{\nu \in \Lambda_0}$. We write $\pi_\Lambda : \tilde{H}^m(k, \Lambda_0) \to \text{End}(\mathbb{C}[\Lambda])$ for the corresponding representation map.

The remainder of this subsection is devoted to the proof of Theorem 3.7. The strategy is to realize the $\tilde{H}^m(k)$-module $(\pi, \mathbb{C}[P])$ as a quotient of the induced $\tilde{H}^m(k)$-module

$$N_C := \tilde{H}^m(k) \otimes_{H(k)} V_C$$

for an appropriate choice of $W$-invariant subset $0 \in C \subseteq P$. Note that the subrepresentation $N_{\{0\}}$ is isomorphic to $\mathbb{C}[P]^m$ viewed as module over $\tilde{H}^m(k)$ by Remark 3.5(ii).

The elements

$$Y^\nu \otimes_{H(k)} v_\lambda \quad (\nu \in P^m, \lambda \in C)$$

form a linear basis of $N_C$ and, by the Bernstein-Zelevinsky commutation relations (2.4), the $\tilde{H}^m(k)$-action on $N_C$ is explicitly given by

$$(3.5) \quad T_i(Y^\nu \otimes_{H(k)} v_\lambda) = (k_i - k_i^{-1}) \tilde{\nabla}^m_i (Y^\nu) \otimes_{H(k)} v_\lambda + Y^{s_\nu} \otimes_{H(k)} T_i v_\lambda,$$

$$Y^\mu (Y^\nu \otimes_{H(k)} v_\lambda) = Y^{\mu + \nu} \otimes_{H(k)} v_\lambda$$

for $\lambda \in C$, $i = 1, \ldots, r$ and $\mu, \nu \in P^m$.

We now continue the analysis of the proof of Theorem 3.7 by viewing the group algebra $\mathbb{C}[P] := \text{span}\{x^\lambda\}_{\lambda \in P}$ as a free left $C_Y[P^m]$-module by

$$Y^\nu \cdot x^\lambda := x^{\lambda + \nu}, \quad \nu \in P^m, \lambda \in P.$$  

This $C_Y[P^m]$-module structure on $\mathbb{C}[P]$ coincides with the $C_Y[P^m]$-structure that will arise from the desired $\tilde{H}^m(k)$-action (3.3) by restriction.

Let $C \subseteq P$ be a $W$-invariant subset and let

$$\mathbf{c} : C \to \mathbb{C}^\times$$

be a (for the moment arbitrary) non-vanishing complex-valued function on $C$.

**Definition 3.9.** We write

$$\psi^C_C : N_C \to \mathbb{C}[P]$$

for the morphism of $C_Y[P^m]$-modules satisfying

$$\psi^C_C(Y^\nu \otimes_{H(k)} v_\lambda) := \mathbf{c}(\lambda)^{-1} x^{\lambda + \nu}, \quad \lambda \in C, \nu \in P^m.$$  

We fix from now on the $W$-invariant subset $C \subseteq P$ to be

$$(3.6) \quad C := \{ \lambda \in P \mid |(\lambda, \alpha^\vee)| \leq m(\alpha) \ \forall \alpha \in \Phi \}.$$
Lemma 3.10. $\psi_C^\varepsilon : N_C \to \mathbb{C}[P]$ is an epimorphism of $\mathbb{C}_Y[P^m]$-modules.

Proof. We need to show that $\psi_C^\varepsilon$ is surjective. Consider the action of $\tilde{W}^m = W \ltimes P^m$ on $P$ and $E$ by reflections and translations. Since $C$ is $W$-invariant it suffices to show that each $\tilde{W}^m$-orbit in $P$ intersects $C$. We show the stronger statement that each $W \ltimes Q^m$-orbit in $P$ intersects $C \cap P^+$ in exactly one point.

Write

$$E^+ := \{v \in E \mid (v, \alpha) \geq 0 \quad \forall \alpha \in \Phi^+\}$$

for the closure of the fundamental Weyl chamber of $E$ with respect to $\Phi^+$. By [19, §4.3] and the fact that $\theta^{m\nu} \in \Phi^{m\nu+}$ is the highest root of $\Phi^{m\nu}$, each $W \ltimes Q^m$-orbit in $E$ intersects the fundamental alcove

$$A_o := \{v \in E^+ \mid (v, \theta^{m\nu}) \leq 1\}$$

in exactly one point. Hence each $W \ltimes Q^m$-orbit in $P$ intersects $A_o \cap P$ in exactly one point. Now note that

$$A_o \cap P = \{\lambda \in P^+ \mid (\lambda, \theta^{m\nu}) \leq 1\} = \{\lambda \in P^+ \mid (\lambda, \alpha^{m\nu}) \leq 1 \quad \forall \alpha \in \Phi^+\} = \{\lambda \in P^+ \mid (\lambda, \alpha^\nu) \leq m(\alpha) \quad \forall \alpha \in \Phi^+\} = C \cap P^+. \quad \square$$

The map $\psi_C^\varepsilon$ gives rise to an isomorphism

$$(3.7) \quad \tilde{\psi}_C^\varepsilon : N_C/\ker(\psi_C^\varepsilon) \sim \mathbb{C}[P]$$

of $\mathbb{C}_Y[P^m]$-modules by Lemma 3.10. We now fine-tune the normalizing factor $c$ such that the kernel $\ker(\psi_C^\varepsilon) \subseteq N_C$ is in fact a $\tilde{H}^m(k)$-submodule of $N_C$. We start with deriving some elementary properties of the metaplectic divided difference operators $\nabla_i^\nu (i = 1, \ldots, r)$.

Lemma 3.11. Let $i \in \{1, \ldots, r\}$.

(i) For $\lambda \in P$ and $\nu \in P^m$ we have

$$x^{\lambda} \nabla_i^\nu (x^\nu) = \nabla_i (x^{\lambda+\nu}) - \nabla_i (x^{\lambda}) x^{s_i \nu}.$$

(ii) For $\lambda \in P$ and $\nu \in P^m$ we have

$$x^{\lambda} \nabla_i^\nu (x^\nu) = \begin{cases} \nabla_i (x^{\lambda+\nu}) - x^{s_i \nu} & \text{if } -m(\alpha_i) \leq (\lambda, \alpha^\nu_i) < 0, \\ \nabla_i (x^{\lambda+\nu}) & \text{if } 0 \leq (\lambda, \alpha^\nu_i) < m(\alpha_i), \\ \nabla_i (x^{\lambda+\nu}) + x^{s_i (\lambda+\nu)} & \text{if } (\lambda, \alpha^\nu_i) = m(\alpha_i). \end{cases}$$

Proof. (i) This follows by a direct computation.

(ii) Note that

$$(q(\lambda), \alpha^m_i) = \begin{cases} -1 & \text{if } -m(\alpha_i) \leq (\lambda, \alpha^\nu_i) < 0, \\ 0 & \text{if } 0 \leq (\lambda, \alpha^\nu_i) < m(\alpha_i), \\ 1 & \text{if } (\lambda, \alpha^\nu_i) = m(\alpha_i), \end{cases}$$

as desired.
hence
\[
\nabla_i(x^\lambda) = \begin{cases} 
  x^\lambda & \text{if } -m(\alpha_i) \leq (\lambda, \alpha_i^\vee) < 0, \\
  0 & \text{if } 0 \leq (\lambda, \alpha_i^\vee) < m(\alpha_i), \\
  -x^{\lambda - \alpha_i^m} = -x^{s_i \lambda} & \text{if } (\lambda, \alpha_i^\vee) = m(\alpha_i).
\end{cases}
\]

Now use (i).

The following lemma will play an important role in finding the proper choice of normalizing factor \(c\).

**Lemma 3.12.** For \(\nu \in P^m\), \(\lambda \in C\) and \(i = 1, \ldots, r\) we have
\[
\psi_C^c(T_i Y^\nu \otimes_{H(k)} v_\lambda) = c(\lambda)^{-1}((k_i - k_i^{-1})x^\lambda \nabla_i(x^{\lambda + \nu}) + d_i(\lambda)x^{s_i(\lambda + \nu)})
\]
with \(d_i : C \to \mathbb{C}^r\) given by
\[
d_i(\lambda) := \begin{cases} 
  c(\lambda)/c(s_i \lambda) & \text{if } -m(\alpha_i) \leq (\lambda, \alpha_i^\vee) < 0, \\
  k_i c(\lambda)/c(s_i \lambda) & \text{if } (\lambda, \alpha_i^\vee) = 0, \\
  c(\lambda)/c(s_i \lambda) & \text{if } 0 < (\lambda, \alpha_i^\vee) < m(\alpha_i), \\
  k_i - k_i^{-1} + c(\lambda)/c(s_i \lambda) & \text{if } (\lambda, \alpha_i^\vee) = m(\alpha_i).
\end{cases}
\]

**Proof.** By a direct computation using (3.5), we have
\[
\psi_C^c(T_i Y^\nu \otimes_{H(k)} v_\lambda) = c(\lambda)^{-1}(k_i - k_i^{-1})x^\lambda \nabla_i(x^{\lambda + \nu}) + x^{s_i \nu} \psi_C(1 \otimes_{H(k)} T_i v_\lambda)
\]
for \(\lambda \in C\) and \(\nu \in P^m\). We now consider four cases.

**Case 1:** \(-m(\alpha_i) \leq (\lambda, \alpha_i^\vee) < 0.
Then
\[
\psi_C^c(1 \otimes_{H(k)} T_i v_\lambda) = c(\lambda)^{-1}(k_i - k_i^{-1})x^\lambda + c(s_i \lambda)^{-1}x^{s_i \lambda}.
\]
Substituting into (3.9) and using Lemma 3.11 we get the desired formula
\[
\psi_C^c(T_i Y^\nu \otimes_{H(k)} v_\lambda) = c(\lambda)^{-1}(k_i - k_i^{-1})x^\lambda \nabla_i(x^{\lambda + \nu}) + \frac{c(\lambda)}{c(s_i \lambda)}x^{s_i(\lambda + \nu)}.
\]

**Case 2:** \((\lambda, \alpha_i^\vee) = 0.
Now we have
\[
\psi_C^c(1 \otimes_{H(k)} T_i v_\lambda) = c(\lambda)^{-1}k_i x^\lambda = c(s_i \lambda)^{-1}k_i x^{s_i \lambda}.
\]
Substituting into (3.9) and using Lemma 3.11 we now get the desired formula
\[
\psi_C^c(T_i Y^\nu \otimes_{H(k)} v_\lambda) = c(\lambda)^{-1}(k_i - k_i^{-1})x^\lambda \nabla_i(x^{\lambda + \nu}) + k_i \frac{c(\lambda)}{c(s_i \lambda)}x^{s_i(\lambda + \nu)}.
\]

**Case 3:** \(0 < (\lambda, \alpha_i^\vee) < m(\alpha_i).
Then
\[
\psi_C^c(1 \otimes_{H(k)} T_i v_\lambda) = c(s_i \lambda)^{-1}x^{s_i \lambda}.
\]
Substitution into \((3.9)\) and using Lemma \([3.11]\) gives the desired formula
\[
\psi_C^e(T_i Y^\nu \otimes_{H(k)} v_\lambda) = c(\lambda)^{-1}\left( (k_i - k_i^{-1}) \nabla_i (x^{\lambda+\nu}) + \frac{c(\lambda)}{c(s_i \lambda)} x^{s_i (\lambda+\nu)} \right).
\]

**Case 4:** \((\lambda, \alpha_i^\vee) = m(\alpha_i)\).

In this case
\[
\psi_C^e(1 \otimes_{H(k)} T_i v_\lambda) = c(s_i \lambda)^{-1} x^{s_i \lambda},
\]
hence substitution into \((3.9)\) and using Lemma \([3.11]\) gives
\[
\psi_C^e(T_i Y^\nu \otimes_{H_0(k)} v_\lambda) = c(\lambda)^{-1}\left( (k_i - k_i^{-1}) \nabla_i (x^{\lambda+\nu}) + \left( k_i - k_i^{-1} + \frac{c(\lambda)}{c(s_i \lambda)} \right) x^{s_i (\lambda+\nu)} \right),
\]
as desired. \(\square\)

We now continue with the proof of Theorem \(3.7\). Define parameters \(h_j(x) \in \mathbb{C}^\times\) for \(j \in \mathbb{Z}\) and \(x \in \{sh, lg\}\) by
\[
\begin{align*}
    h_j(x) &:= k_x & \text{if } j \in n\mathbb{Z}_{<0}, \\
h_j(x) &:= -k_x^{-1} g_j(x)^{-1} & \text{if } j \in \mathbb{Z}_{<0} \setminus n\mathbb{Z}_{<0}, \\
h_j(x) &:= 1 & \text{if } j \in \mathbb{Z}_{\geq 0}.
\end{align*}
\]
Then \(h_j(x) = h_{-n+j}(x)\) if \(j \in \mathbb{Z}_{<0}\), and \(h_j(x)h_{-sn-j}(x) = 1\) for \(j \in \mathbb{Z}_{<0} \setminus n\mathbb{Z}_{<0}\) and \(s \in \mathbb{Z}_{\geq 0}\) such that \(-sn < j < 0\).

Choose \(c : C \to \mathbb{C}^\times\) by
\[
(3.10) \quad c(\lambda) := \prod_{\alpha \in \Phi^+} h_{Q(\alpha)(\lambda, \alpha^\vee)}(\text{size}(\alpha^m)), \quad \lambda \in C.
\]
Using
\[
(3.11) \quad \frac{c(\lambda)}{c(s_i \lambda)} = \frac{h_{Q(\alpha_i)(\lambda, \alpha_i^\vee)}(\text{size}(\alpha_i^m))}{h_{-Q(\alpha_i)(\lambda, \alpha_i^\vee)}(\text{size}(\alpha_i^m))}.
\]
\([2.1], \quad [3.8]\) and Lemma \([2.1]\), one verifies that for \(i = 1, \ldots, r\) and \(\lambda \in C\),
\[
\begin{align*}
d_i(\lambda) &= \begin{cases} h_{-Q(\alpha_i)(\lambda, \alpha_i^\vee)}(\text{size}(\alpha_i^m))^{-1} & \text{if } m(\alpha_i) \not\mid (\lambda, \alpha^\vee), \\
k_i & \text{if } m(\alpha_i) \mid (\lambda, \alpha^\vee). \end{cases}
\end{align*}
\]
Rewriting in terms of the representation parameters \(g_j(x)\) and using Lemma \([2.1]\) we get
\[
(3.12) \quad d_i(\lambda) = p_i(\lambda)
\]
for \(i = 1, \ldots, r\) and \(\lambda \in C\), with \(p_i(\lambda)\) given by \((3.3)\).

Now let \(S_i : \mathbb{C}[P] \to \mathbb{C}[P]\) be the linear map defined by
\[
S_i(x^\lambda) := (k_i - k_i^{-1}) \nabla_i (x^\lambda) + p_i(\lambda) x^{s_i \lambda}, \quad \lambda \in P,
\]
then Lemma \([3.12]\) and \((3.12)\) show that for \(i = 1, \ldots, r\) and \(\lambda \in C, \ \nu \in P^m\),
\[
(3.13) \quad S_i(\psi_C^e(Y^\nu \otimes_{H(k)} v_\lambda)) = \psi_C^e(T_i Y^\nu \otimes_{H(k)} v_\lambda).
\]
Hence the kernel of the epimorphism $\psi_C^m : N_C \to \mathbb{C}[P]$ is a $\tilde{H}^m(k)$-submodule. By (3.13) it follows that the $\tilde{H}^m(k)$-module structure on $\mathbb{C}[P]$, inherited from the quotient $\tilde{H}^m(k)$-module $N_C/\ker(\psi_C^m)$ by the $\mathbb{C}_Y[P^m]$-module isomorphism $\tilde{\psi}_C^m$ (see (3.7)), is explicitly given by (3.3). This completes the proof of Theorem 3.7.

In subsequent sections, we will work with some conjugations of $\pi$, so the following Lemma will be useful.

**Lemma 3.13.** Let $\Lambda$ be a lattice in $E$ satisfying $Q \subseteq \Lambda \subseteq P$. Let $h \in \tilde{H}^m(k, \Lambda_0)$ and $\mu \in P$. Then $x^{-\mu} \pi(h)x^{\mu}$ preserves $\mathbb{C}[\Lambda]$.

**Proof.** Since $\pi(Y^\nu)$ for $\nu \in \Lambda_0$ commutes with multiplication by $x^\mu$, we need only check that $x^{-\mu} \pi(T_i)x^{\mu}$ preserves $\mathbb{C}[\Lambda]$ for $1 \leq i \leq r$. Let $\lambda \in \Lambda$. By Theorem 3.7 we have

$$x^{-\mu} \pi(T_i)(x^{\lambda+\mu}) = x^{-\mu}(k_i - k_i^{-1}) \nabla_i(x^{\lambda+\mu}) + x^{-\mu} p_i(\lambda + \mu)x^{s_i(\lambda+\mu)}.$$

We have

$$x^{-\mu+s_i(\lambda+\mu)} = x^{\lambda-(\lambda+\mu, \alpha_i)^\vee} \alpha_i \in \mathbb{C}[\Lambda],$$

since $(\lambda + \mu, \alpha_i)^\vee \alpha_i \in Q$. For the other term, by (3.1) and Lemma 3.2, we have

$$\nabla_i(x^{\lambda+\mu}) = x^{\lambda+\mu} g,$$

where $g \in \mathbb{C}[Q^m]$. So now $x^{-\mu} \nabla_i(x^{\lambda+\mu}) \in \mathbb{C}[\Lambda]$.

\[\square\]

3.3. **The metaplectic Weyl group representation.** For $f = \sum c_i Y^\nu \in \mathbb{C}[P^m]$ write $f(Y) := \sum c_i Y^\nu \in \tilde{H}^m(k)$. Let $\tilde{H}_{loc}^m(k)$ be algebra obtained by localizing the extended affine Hecke algebra $\tilde{H}^m(k)$ at $\mathbb{C}_Y[P^m]^\times$.

Let $\mathbb{C}(P^m)$ the quotient field of $\mathbb{C}[P^m]$ and write $\mathbb{C}_Y(P^m) \subseteq \tilde{H}^m_{loc}(k)$ for the subalgebra consisting of the elements $f(Y) = g(Y)/h(Y)$ with $g \in \mathbb{C}[P^m]$ and $h \in \mathbb{C}[P^m]^\times$. Then $\mathbb{C}_Y(P^m) \simeq \mathbb{C}(P^m)$ as fields by $Y^\nu \leftrightarrow x^\nu$ ($\nu \in P^m$), and

$$\tilde{H}^m_{loc}(k) \simeq H(k) \otimes \mathbb{C}_Y(P^m)$$

as vector spaces by the multiplication map. The defining relations of $\tilde{H}^m_{loc}(k)$ with respect to the decomposition (3.14) are captured by the extended cross relations

$$T_i f(Y) = (s_i f)(Y) T_i + (k_i - k_i^{-1}) \left( \frac{(s_i f)(Y) - f(Y)}{Y^{\alpha_i^\vee} - 1} \right)$$

for $i \in \{1, \ldots, r\}$ and $f \in \mathbb{C}(P^m)$, where we use the extension of the $W$-action on $\mathbb{C}[P^m]$ to $\mathbb{C}(P^m)$ by field automorphisms.

If the multiplicity function $k$ is identically equal to one then $\tilde{H}^m_{loc}(k)$ is isomorphic to the semi-direct product algebra

$$W \ltimes \mathbb{C}(P^m) := \mathbb{C}[W] \otimes \mathbb{C}(P^m)$$

with algebra structure given by $(v \otimes f)(w \otimes g) := vw \otimes (w^{-1} f) g$ for $v, w \in W$ and $f, g \in \mathbb{C}(P^m)$. We write $g w$ for the element $(1 \otimes g)(w \otimes 1) = w \otimes w^{-1} g$ in $W \ltimes \mathbb{C}(P^m)$ if no confusion can arise.
Define for $\alpha \in \Phi$ the $c$-functions $c_\alpha = \phi^m_\alpha \in \mathbb{C}(Q^m)$ by

\begin{equation}
(3.15) \quad c_\alpha := \frac{1 - k^2_\alpha x^m}{1 - x^m}.
\end{equation}

We write $c_i := c_{\alpha_i}$ ($i = 1, \ldots, r$) for the $c$-functions at the simple roots. Note that $w(c_\alpha) = c_{\alpha w}$ for $w \in W$ and $\alpha \in \Phi$.

By [20] we have the following result.

**Theorem 3.14.** There exists a unique algebra isomorphism

\begin{equation}
(3.16) \quad \varphi : W \ltimes \mathbb{C}(P^m) \rightarrow \tilde{H}^m_{\text{loc}}(k)
\end{equation}

given by $\varphi(f) = f(Y)$ for $f \in \mathbb{C}(P^m)$ and

\begin{equation}
(3.17) \quad \varphi(s_i) := \frac{k_i}{c_i(Y)} T_i + 1 - \frac{k^2_i}{c_i(Y)}
\end{equation}

for $i = 1, \ldots, r$.

The $\varphi(s_i)$ are the so-called normalized $Y$-intertwiners of the extended affine Hecke algebra $\tilde{H}^m(k)$ (see [20] and, e.g., [11, §3.3.3]). They play an instrumental role in the representation theory of $\tilde{H}^m(k)$.

Note that for $i = 1, \ldots, r$ we have

$$\varphi^{-1}(T_i) = k_i + k_i^{-1} c_i(s_i - 1)$$

in $W \ltimes \mathbb{C}(P^m)$, which are the Demazure-Lusztig operators [22].

**Remark 3.15.** The localization isomorphism (3.16) extends to the double affine Hecke algebra, see [11, §3.3.3]. Its most natural form involves the normalized $X$-intertwiners, obtained from the intertwiners $\varphi(s_i)$ by applying the (Fourier) automorphism of the double affine Hecke algebra, as well as an additional normalized $X$-intertwiner naturally attached to the simple reflection $s_0 \in \tilde{W}^m$.

**Definition 3.16.** Let $(\rho, M)$ be a left $\tilde{H}^m(k)$-module. Write $(\rho_{\text{loc}}, M_{\text{loc}})$ for the associated localized $W \ltimes \mathbb{C}(P^m)$-module

$$M_{\text{loc}} := \tilde{H}^m_{\text{loc}}(k) \otimes_{\tilde{H}^m(k)} M$$

with representation map $\rho_{\text{loc}} : W \ltimes \mathbb{C}(P^m) \rightarrow \text{End}(M_{\text{loc}})$ defined by

$$\rho_{\text{loc}}(X)(h \otimes \tilde{H}^m(k) m) := (\varphi(X)h) \otimes \tilde{H}^m(k) m$$

for $X \in W \ltimes \mathbb{C}(P^m)$, $h \in \tilde{H}^m_{\text{loc}}(k)$ and $m \in M$.

Note that $M_{\text{loc}} \cong \mathbb{C}_Y(P^m) \otimes_{\mathbb{C}_Y[P^m]} M$ as vector spaces with the isomorphism mapping $f(Y)h \otimes \tilde{H}^m(k) m$ to $f(Y) \otimes f_Y[P^m] \rho(h)m$ for $f \in \mathbb{C}(P^m)$, $h \in H(k)$ and $m \in M$ (the map is well defined by the Bernstein-Zelevinsky presentation of $\tilde{H}^m_{\text{loc}}(k)$).
Remark 3.17. Identifying $M$ as subspace of $M_{\text{loc}}$ by the linear embedding $M \hookrightarrow M_{\text{loc}}$, $m \mapsto 1 \otimes \tilde{H}^m(k)$, we have

$$\rho_{\text{loc}}(\varphi^{-1}(h))m = \rho(h)m, \quad h \in \tilde{H}^m(k), \ m \in M.$$  

Remark 3.18. A Bethe integrable system with extended affine Hecke algebra symmetry is a $\tilde{H}^m(k)$-module $V$ endowed with the integrable structure obtained from the action of the associated $X$-intertwiners on $\mathbb{C}(P^m) \otimes V$. The integrable structure is thus encoded by solutions of (braid version of) generalized quantum Yang-Baxter equations with spectral parameter. In the literature on integrable systems one sometimes says that the integrable structure arises from Baxterizing the affine Hecke algebra module structure on the quantum state space. See e.g. [27] for an example involving the Heisenberg XXZ spin-$\frac{1}{2}$ chain.

The intertwiners are also instrumental in the construction of the quantum affine KZ equations, see, e.g., [11, §1.3.2].

Recall the metaplectic affine Hecke algebra representation $(\pi, \mathbb{C}[P])$ from Theorem 3.7. In the following proposition we explicitly describe $(\pi_{\text{loc}}, \mathbb{C}[P]_{\text{loc}})$.

**Proposition 3.19.** (i) $\mathbb{C}(P) = \bigoplus_{x \in P/P^m} \mathbb{C}(P^m)x^\lambda$.

(ii) $\mathbb{C}[P]_{\text{loc}} \simeq \mathbb{C}(P)$ as vector spaces by

$$f(Y)h \otimes \tilde{H}^m(k) g \mapsto f(\pi(h)g), \quad f \in \mathbb{C}(P^m), \ g \in \mathbb{C}[P], \ h \in H(k).$$

(iii) The $\pi_{\text{loc}}$-action of $W \ltimes \mathbb{C}(P^m)$ on $\mathbb{C}(P)$ (identifying $\mathbb{C}[P]_{\text{loc}}$ with $\mathbb{C}(P)$ using the linear isomorphism from (ii)) is explicitly given by

$$\pi_{\text{loc}}(s_i)(fx^\lambda) = (s_i f) \left( \left( \frac{(1-k_i^2)x^{-q(\lambda), \alpha_i^m)}{1-k_i^2x^{\alpha_i^m}} \right)x^{s_i\lambda} + \left( \frac{k_i p_i(\lambda)(1-x^{-q(\lambda)})}{1-k_i^2x^{\alpha_i^m}} \right)x^{s_i\lambda} \right),$$

$$\pi_{\text{loc}}(g)(fx^\lambda) = gfx^\lambda$$

for $f, g \in \mathbb{C}(P^m)$, $\lambda \in P$ and $i = 1, \ldots, r$ (recall that $p_i(\lambda)$ is given by [33]).

**Proof.** (i) Let $G$ be the group of characters of the finite abelian group $P/P^m$. It acts by field automorphisms on $\mathbb{C}(P)$ by

$$\chi \cdot x^\lambda := \chi(\lambda)x^\lambda, \quad \lambda \in P, \ \chi \in G.$$  

Decomposing $\mathbb{C}(P)$ in $G$-isotypical components yields

$$\mathbb{C}(P) = \bigoplus_{\chi \in P/P^m} \mathbb{C}(P)^Gx^\lambda,$$

with $\mathbb{C}(P)^G$ the subfield of $G$-invariant elements in $\mathbb{C}(P)$. It remains to show that $\mathbb{C}(P)^G = \mathbb{C}(P^m)$, for which it suffices to show that $\mathbb{C}[P]^G = \mathbb{C}[P^m]$. The latter follows from the fact that

$$\text{pr}(x^\lambda) = \delta_{\lambda,0}x^\lambda, \quad \lambda \in P.$$
for the projection map \( \text{pr} : \mathbb{C}[P] \to \mathbb{C}[P]^G \) defined by

\[
\text{pr}(f) := \frac{1}{\# G} \sum_{\chi \in G} \chi \cdot f, \quad f \in \mathbb{C}[P].
\]

(ii) We have

\[
\mathbb{C}[P]_{\text{loc}} = \hat{H}^m_{\text{loc}}(k) \otimes \widetilde{H}^m(k) \mathbb{C}[P] \\
\cong \mathbb{C}_Y(P^m) \otimes \mathbb{C}_Y(P^m) \mathbb{C}[P] \cong \mathbb{C}(P)
\]

with the last isomorphism mapping \( f(Y) \otimes \mathbb{C}_Y(P^m) \to fg \) for \( f \in \mathbb{C}(P^m) \) and \( g \in \mathbb{C}[P] \). This is well defined and an isomorphism due to the second formula of (3.4) and due to part (i) of the proposition. The result now immediately follows.

(iii) For \( f, g \in \mathbb{C}(P^m) \) and \( \lambda \in P \) we have

\[
\pi_{\text{loc}}(g)(f x^\lambda) = \pi_{\text{loc}}(g)(f(Y) \otimes \hat{H}^m(k) x^\lambda) \\
= (gf)(Y) \otimes \hat{H}^m(k) x^\lambda = gf x^\lambda = \pi_{\text{loc}}(gf)x^\lambda,
\]

this establishes the second formula. For the first formula it then suffices to prove that

\[
(3.18) \quad \pi_{\text{loc}}(s_i)(x^\lambda) = \frac{(1 - k_i^2)x^{-\alpha_i^m} - (\chi(\lambda), \alpha_i^m)\alpha_i^m}{1 - k_i^2 x^\alpha_i^m} x^{\lambda_i} + \frac{k_i \pi_i(\lambda)(1 - x^{\alpha_i^m})}{1 - k_i^2 x^\alpha_i^m} x^{s_i \lambda}
\]

for \( i = 1, \ldots, r \) and \( \lambda \in P \). By the first formula of (3.4) we have

\[
\pi_{\text{loc}}(s_i)x^\lambda = \frac{k_i}{c_i} \pi(T_i)x^\lambda + \left(1 - \frac{k_i^2}{c_i}\right)x^\lambda \\
= \frac{k_i}{c_i} \left((k_i - k_i^{-1}) \frac{x^\lambda - x^{\lambda - (\chi(\lambda), \alpha_i^m)\alpha_i^m}}{1 - x^{\alpha_i^m}} + \pi_i(\lambda)x^{s_i \lambda}\right) + \left(1 - \frac{k_i^2}{c_i}\right)x^\lambda.
\]

Substituting the definition of the \( c \)-function \( c_i \) (see (3.15)) gives

\[
\pi_{\text{loc}}(s_i)x^\lambda = \frac{(k_i^2 - 1)(x^\lambda - x^{\lambda - (\chi(\lambda), \alpha_i^m)\alpha_i^m}) + k_i \pi_i(\lambda)(1 - x^{\alpha_i^m})x^{s_i \lambda} + (1 - k_i^2)x^\lambda}{1 - k_i^2 x^\alpha_i^m}.
\]

Simplifying the expression gives (3.18). \( \Box \)

Remark 3.20. Since \( q(0) = 0 \) and \( \pi_i(0) = k_i \), we have \( \pi_{\text{loc}}(s_i)1 = 1 \). Hence \( \mathbb{C}(P^m) \) is a \( \pi_{\text{loc}} \)-submodule of \( \mathbb{C}(P) \) with the \( W \times \mathbb{C}(P^m) \)-action reducing to the standard one,

\[
\pi_{\text{loc}}(s_i)f = s_i f, \quad \pi_{\text{loc}}(g)f := gf
\]

for \( i = 1, \ldots, r \) and \( f, g \in \mathbb{C}(P^m) \).

Recall the definition of the representation parameters \( g_j(x) \) \((j \in \mathbb{Z}, x \in \{sh, lg\})\), see Definition 3.3. We conjugate the \( \pi_{\text{loc}} \)-action by a certain factor, in order to line it up with the Weyl group action of Chinta-Gunnells [13, 14].
**Theorem 3.21** (Metaplectic Weyl group representation). The following formulas turn \( \mathbb{C}(P) \) into a left \( W \ltimes \mathbb{C}(P^m) \)-module,

\[
\sigma(s_i)(f x^\lambda) := \frac{(1 - k_i^2 x^{(q(-\lambda),\alpha_i)\alpha_i^m})}{(1 - k_i^2 x^{\alpha_i^m})} (s_i f) x^\lambda
\]

\[
+ k_i^2 q_{(\alpha_i)-B(\lambda,\alpha_i)}(\text{size}(\alpha_i^m)) \frac{(1 - x^{-\alpha_i^m})}{(1 - k_i^2 x^{\alpha_i^m})} (s_i f) x^{\alpha_i + s_i \lambda},
\]

(3.19)

\[
\sigma(g) f x^\lambda := g f x^\lambda
\]

for \( f, g \in \mathbb{C}(P^m), \lambda \in P \) and \( i = 1, \ldots, r \).

**Proof.** Write \( \rho := \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha \) and \( \rho^m := \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha^m \) for the half sum of positive roots of \( \Phi \) and \( \Phi^m \) respectively. Then \( s_i(\rho) = \rho - \alpha_i \) and \( s_i(\rho^m) = \rho^m - \alpha_i^m \), in particular \( \rho = \sum_{i=1}^r \omega_i \in P \) and

\[
\rho^m = \sum_{i=1}^r \omega_i^m = \sum_{i=1}^r m(\alpha_i) \omega_i \in \mathbb{C}(P).
\]

Consider now the \( W \ltimes \mathbb{C}(P^m) \) on \( \mathbb{C}(P) \) defined by

(3.20) \[ \sigma(X)f := x^{p^m - \rho} \pi_{loc}(X)(x^{p^m - \rho} f), \quad X \in W \ltimes \mathbb{C}(P^m), \quad f \in \mathbb{C}(P). \]

Then \( \sigma(g) f = g f \) for \( g \in \mathbb{C}(P^m) \) and \( f \in \mathbb{C}(P) \), and

(3.21) \[
\sigma(s_i) x^\lambda = \frac{(1 - k_i^2 x^{-(q(\lambda + \rho^m - \rho),\alpha_i)\alpha_i^m})}{1 - k_i^2 x^{\alpha_i^m}} x^\lambda
\]

\[
- \frac{1}{1 - k_i^2 x^{\alpha_i^m}} \frac{1 - k_i^2 x^{\alpha_i^m}}{1 - k_i^2 x^{\alpha_i^m}} (1 - x^{-\alpha_i^m}) x^{\alpha_i + s_i \lambda}.
\]

Note that

\[
\rho(\lambda + \rho^m - \rho) = \rho^m - \rho - \rho(-\lambda)
\]

since \( r_s(t + s - 1) = s - 1 - r_s(-t) \) for \( s \in \mathbb{Z}_{>0} \) and \( t \in \mathbb{Z} \), hence

\[
q(\lambda + \rho^m - \rho) = -q(-\lambda).
\]

Furthermore,

\[
-k_i p(\lambda - \rho) = k_i^2 q_{(\alpha_i)-B(\lambda,\alpha_i)}(\text{size}(\alpha_i^m))
\]

for \( \lambda \in P \) since \( -B(\lambda - \rho, \alpha_i) = Q(\alpha_i) - B(\lambda, \alpha_i) \). Substituting these two formulas in (3.21) gives the desired result. \[ \square \]

As in Remark 3.8(ii), fix a lattice \( \Lambda \subseteq E \) satisfying \( Q \subseteq \Lambda \subseteq P \) and set \( \Lambda_0 := \Lambda \cap P^m \). Then \( Q^m \subseteq \Lambda_0 \subseteq P^m \) and recall that \( \Lambda_0 \) can alternatively be described as

\[
\Lambda_0 = \{ \lambda \in \Lambda \mid B(\lambda, \alpha) \equiv 0 \ \text{mod } n \ \forall \alpha \in \Phi \},
\]

which places us directly in the context of [14]. Note that \( \Lambda \) and \( \Lambda_0 \) are automatically \( W \)-stable. In particular the subalgebra of \( W \ltimes \mathbb{C}(P^m) \) generated by \( W \) and \( \mathbb{C}(\Lambda_0) \) is isomorphic to the semi-direct product algebra \( W \ltimes \mathbb{C}(\Lambda_0) \).
Let $\mathbb{C}(\Lambda_0)$ and $\mathbb{C}(\Lambda)$ be the subfields of $\mathbb{C}(P)$ generated by $x^\nu$ ($\nu \in \Lambda_0$) and $x^\lambda$ ($\lambda \in \Lambda$) respectively. Similarly to Proposition 3.19(i) we have the decomposition

$$\mathbb{C}(\Lambda) = \bigoplus_{\lambda \in \Lambda/\Lambda_0} \mathbb{C}(\Lambda_0) x^\lambda.$$ 

Then $\mathbb{C}(\Lambda) \subseteq \mathbb{C}(P)$ is a $W \rtimes C(\Lambda_0)$-submodule with respect to the action $\sigma$. Writing

$$\sigma_\Lambda : W \rtimes C(\Lambda_0) \to \text{End}(\mathbb{C}(\Lambda))$$

for the resulting representation map, we get

**Corollary 3.22.** In the setup as above, the representation map $\sigma_\Lambda$ is explicitly given by

$$\sigma_\Lambda(s_i)(f x^\lambda) := \frac{(1 - k_i^2 x^{q(\alpha_i) - \beta(\lambda, \alpha_i) - B(\lambda, \alpha_i) (\text{size}(\alpha_i))} (1 - k_i^2 x^{\alpha_i}))}{(1 - k_i^2 x^{\alpha_i})} (s_i f) x^{\alpha_i + s_i \lambda},$$

with $f, g \in \mathbb{C}(\Lambda_0)$, $\lambda \in \Lambda$ and $i = 1, \ldots, r$.

**Remark 3.23.** Consider the special case that $k : \Phi^m \to \mathbb{C}^\times$ is constant and the representation parameters $g_j(x)$ satisfy $g_j(s h) = g_j(l g)$ for all $j \in \mathbb{Z}$. We call this the equal Hecke and representation parameter case. Then $\sigma_\Lambda$ is exactly the Chinta-Gunnells [13, 14] Weyl group action. This is immediately apparent by comparing (3.22) with [15, (7)] (the parameter $v$ in [15] corresponds to $k^2$). Note that our technique gives an independent and uniform proof that the formulas of Chinta-Gunnells do indeed give an action of the Weyl group.

**Remark 3.24.** Note that $\sigma_\Lambda$ reduces at $n = 1$ to the standard $W$-action. However, it is in fact not the standard action on $\mathbb{C}(P^m)$, due to the fact that we have conjugated $\pi_{\text{loc}}$ by $x^{\rho^m}$ (compare with Remark 3.20).

Set $\Phi(w) := \Phi^+ \cap w^{-1} \Phi^-$ ($w \in W$) and let $w_0 \in W$ be the longest Weyl group element.

**Definition 3.25.** For $\lambda \in P^+$ define $\widetilde{\mathcal{W}}_\lambda \in \mathbb{C}(P)$ by

$$\widetilde{\mathcal{W}}_\lambda := \left( \prod_{\alpha \in \Phi^+} c_\alpha \right) \sum_{w \in W} (-1)^{f(w)} \left( \prod_{\alpha \in \Phi(w^{-1})} x^{\alpha^m} \right) \sigma(w)(x^{w_0 \lambda}).$$

In the equal Hecke and parameter case, McNamara’s [25, Thm. 15.2] metaplectic Casselman-Shalika formula relates $\widetilde{\mathcal{W}}_\lambda$ to the spherical Whittaker function of metaplectic covers of unramified reductive groups over local fields, see also [15, Thm. 16]. It is a natural open problem what the corresponding representation
theoretic interpretation is of $\tilde{W}_\lambda$ in the unequal Hecke and/or representation parameter case.

In the following section we will obtain in Theorem 4.9 an expression of $\tilde{W}_\lambda$ in terms of metaplectic analogues of Demazure-Lusztig operators, generalizing [15, Thm. 16].

4. Metaplectic Demazure-Lusztig operators

In the previous section we used the localization isomorphism $\varphi: W \ltimes C(P_m) \xrightarrow{\sim} \tilde{H}^m_{\text{loc}}(k)$ to obtain the metaplectic Weyl group representation $\sigma$ from the metaplectic affine Hecke algebra representation $\pi$. In this section we use the localization isomorphism to turn the metaplectic Weyl group representation $\sigma$ into a localized affine Hecke algebra representation involving metaplectic Demazure-Lusztig type operators. This leads to a generalization of some of the results in [15, §3] to unequal Hecke and representation parameters, and simplifies some of the proofs in [15, §3].

Define the algebra map

$$\tau: \tilde{H}^m_{\text{loc}}(k) \to \text{End}(\mathbb{C}(P))$$

by $\tau := \sigma \circ \varphi^{-1}$.

**Proposition 4.1.** For $h \in \tilde{H}^m(k)$ and $g \in C[P]$, we have

$$\tau(h)(g) = x^{\rho^m} \pi(h)x^{\rho - \rho^m}g.$$  

In particular, the restriction of $\tau$ to $\tilde{H}^m(k)$ preserves $\mathbb{C}[P]$, and the restriction of $\tau$ to $\tilde{H}^m(k, \Lambda_0)$ preserves $\mathbb{C}[\Lambda]$.

**Proof.** The formula follows from (3.20), Proposition 3.19(ii) and Remark 3.17, and then the statements about restrictions follow from Theorem 3.7 and Lemma 3.13. □

**Proposition 4.2.** We have

$$\tau(T_i)(fx^\lambda) = k_ifx^\lambda + k_i^{-1}c_i(s_i)(fx^\lambda) - fx^\lambda,$$

$$\tau(g(Y))(fx^\lambda) = gfx^\lambda$$

for $f, g \in \mathbb{C}(P^m)$, $\lambda \in P$ and $i = 1, \ldots, r$.

**Proof.** This is immediate from the fact that $\varphi^{-1}(T_i) = k_i + k_i^{-1}c_i(s_i - 1)$ and $\varphi^{-1}(g(Y)) = g$ for $i = 1, \ldots, r$ and $g \in \mathbb{C}(P^m)$. □

Define the linear operator

$$T_i := -k_i\tau(Y^{\rho^m}T_i^{-1}Y^{-\rho^m}) \in \text{End}(\mathbb{C}(P)).$$

**Definition 4.3.** We call $T_i \in \text{End}(\mathbb{C}(P))$ ($i = 1, \ldots, r$) the metaplectic Demazure-Lusztig operators.
By a direct computation,
\begin{equation}
\mathcal{T}_i(f) = (1 - k_i^2 x^{\alpha_i^m}) \left( \frac{f - x^{\alpha_i^m} \sigma(s_i)f}{1 - x^{\alpha_i^m}} \right) - f, \quad f \in \mathbb{C}(P).
\end{equation}

They restrict to well-defined linear operators on $\mathbb{C}([\Lambda])$ for any lattice $\Lambda$ in $V$ satisfying $Q \subseteq \Lambda \subseteq P$, in which case they reduce for the equal Hecke and representation parameter case to the Demazure-Lusztig operators $[15, (11)]$.

**Lemma 4.4.** The metaplectic Demazure-Lusztig operator $\mathcal{T}_i$ stabilizes $\mathbb{C}[P]$ and $\mathbb{C}([\Lambda])$ for $i = 1, \ldots, r$.

**Proof.** Follows from (4.1), Proposition 4.1, and Lemma 3.13.

The realization (4.1) of the $\mathcal{T}_i$’s through the $\tilde{H}^{m}_{loc}(k)$-representation $\tau$ directly imply that the metaplectic Demazure-Lusztig operators $\mathcal{T}_i$ ($i = 1, \ldots, r$) satisfy the braid relations of $W$ and the quadratic Hecke relations
\begin{equation}
\mathcal{T}_i^2 = (k_i^2 - 1) \mathcal{T}_i + k_i^2, \quad i = 1, \ldots, r
\end{equation}
(this in particular provides an alternative and uniform proof of the braid relations and quadratic Hecke relations of the metaplectic Demazure-Lusztig operators in [15], see [15 Prop. 5(ii)] and formula (13) in [15 Prop. 7]). For $w = s_{i_1} \cdots s_{i_r} \in W$ a reduced expression we write $\mathcal{T}_w := \mathcal{T}_{i_1} \cdots \mathcal{T}_{i_r} \in \text{End}(\mathbb{C}(P))$.

**Remark 4.5.** Using that $\sigma(s_i)f = x^{\rho^m} \pi_{loc}(s_i)(x^{\rho^m} - f)$ we have
\begin{equation}
\tau(Y^{\rho^m-p} T_i Y^{\rho^m-p}) f = k_i f + k_i^{-1} c_i(\pi_{loc}(s_i)f - f) = \pi_{loc}(\varphi^{-1}(T_i)) f
\end{equation}
for $f \in \mathbb{C}(P)$. Hence
\begin{equation}
\mathcal{T}_i = -k_i \pi_{loc}(\varphi^{-1}(Y^{\rho} T_i^{-1} Y^{-\rho})).
\end{equation}

**Remark 4.6.** Let $\Lambda$ be a lattice in $E$ satisfying $Q \subseteq \Lambda \subseteq P$. The localization isomorphism $\varphi$ restricts to an isomorphism of algebras
\[
\varphi_{\Lambda} : W \ltimes \mathbb{C}([\Lambda_0]) \overset{\sim}{\longrightarrow} \tilde{H}^{m}_{loc}(k, \Lambda_0),
\]
with $\tilde{H}^{m}_{loc}(k, \Lambda_0)$ the subalgebra of $\tilde{H}^{m}_{loc}(k)$ generated by $H(k)$ and $\mathbb{C}([\Lambda_0]) := \{g(Y)/h(Y) \mid g \in \mathbb{C}([\Lambda_0]), h \in \mathbb{C}([\Lambda_0])^\times \}$. The algebra map
\[
\tau_{\Lambda} : \tilde{H}^{m}_{loc}(k, \Lambda_0) \rightarrow \text{End}(\mathbb{C}([\Lambda]))
\]
defined by $\tau_{\Lambda} := \sigma_{\Lambda} \circ \varphi^{-1}_{\Lambda}$ then satisfies
\[
\tau_{\Lambda}(T_i)(fx^\lambda) = k_i f x^\lambda + k_i^{-1} c_i(\sigma_{\Lambda}(s_i)(fx^\lambda) - fx^\lambda),
\]
\[
\tau_{\Lambda}(g(Y))(fx^\lambda) = g f x^\lambda
\]
for $f, g \in \mathbb{C}([\Lambda_0)$, $\lambda \in \Lambda$ and $i = 1, \ldots, r$, where $\Lambda_0 := \Lambda \cap P^m$. Note that
\[
\tau_{\Lambda}(X) = \tau(X)|_{\mathbb{C}([\Lambda])}, \quad X \in \tilde{H}^{m}_{loc}(k, \Lambda_0).
\]
The metaplectic Demazure-Lusztig operators $T_i$ then restrict to the following linear operators on $\mathbb{C}(\Lambda)$,
\[
T_i|_{\mathbb{C}(\Lambda)} = -k_it_\Lambda(Ad_{Y_{\rho_m}}(T_i^{-1})),
\]
where $Ad_{Y_{\rho_m}} \in \text{Aut}(\widetilde{H}_m^{\text{loc}}(k, \Lambda_0))$ is the restriction of the inner automorphism $X \mapsto Y_{\rho_m}XY_{-\rho_m}$ of $\widetilde{H}_m^{\text{loc}}(k)$ to the subalgebra $\widetilde{H}_m^{\text{loc}}(k, \Lambda_0)$.

We now use these results to generalize results from [15, §3] to the case of unequal Hecke and representation parameters. We first analyze certain symmetrizer and antisymmetrizer elements in $\widetilde{H}_m^{\text{loc}}(k)$. We then use the metaplectic Weyl group representation $\sigma$ to obtain generalizations of the formula [15, Thm. 16] for the metaplectic Whittaker function.

Recall from Section 2.3 that $k : \Phi \to \mathbb{C}^\times$ is a $W$-invariant function and $k_j := k_{a_j}$ for $j = 0, \ldots, r$. For $w = s_{i_1} \cdots s_{i_m} \in W$ a reduced word ($1 \leq i_j \leq r$), we define
\[
k_w := \prod_{j=1}^{m} k_{i_j}.
\]
Note that, in the special case that $k$ is a constant function (the equal Hecke algebra parameters case), we have $k_w = k^\ell(w)$. Also let
\[
W(k^{\pm 2}) := \sum_{w \in W} k_w^\pm 2.
\]

Define the symmetrizer $1_+ \in H(k)$ and antisymmetrizer $1_- \in H(k)$ by
\[
1_+ := \sum_{w \in W} k_w T_w, \quad 1_- := \sum_{w \in W} (-1)^{\ell(w)} k_w^{-1} T_w.
\]

It is well known (see e.g., [20, 1.19.1] and [11]) that the symmetrizer $1_+$ and antisymmetrizer $1_-$ satisfy the following properties.

**Proposition 4.7.** We have the following identities in $H(k)$:
\[
T_i 1_\pm = \pm k_i 1_\pm = 1_\pm T_i, \\
1_\pm^2 = W(k^{\pm 2}) 1_\pm
\]
for $i = 1, \ldots, r$.

The equations $T_i 1_\pm = \pm k_i 1_\pm$ for $i = 1, \ldots, r$ characterize $1_\pm$ as an element in $H(k)$ up to a multiplicative constant. It follows from this observation that
\[
1_+ = k_{w_0}^2 \sum_{w \in W} k_{w_0}^{-1} T_{w_1}^{-1}, \quad 1_- = k_{w_0}^{-2} \sum_{w \in W} (-1)^{\ell(w)} k_w T_{w_1}^{-1}.
\]

The multiplicative constant is determined by comparing the coefficient of $T_{w_0}$ in the linear expansion in terms of the basis $\{T_w\}_{w \in W}$ of $H(k)$.

Recall the definition (3.15) of the $c$-functions $c_\alpha$ ($\alpha \in \Phi$).
Proposition 4.8. We have the following identities in $W \ltimes \mathbb{C}(P^m)$:

$$
\varphi^{-1}(1_+) = \left( \sum_{w \in W} w \right) \prod_{\alpha \in \Phi^+} c_{\alpha},
$$

$$
\varphi^{-1}(1_-) = k_{w_0}^{-2} \left( \prod_{\alpha \in \Phi^+} c_{\alpha} \right) \sum_{w \in W} (-1)^{\ell(w)} w.
$$

Proof. See [24, (5.5.14)].

We now obtain the following main result of this section.

Theorem 4.9. We have the following identity of operators in $\text{End}(\mathbb{C}(P))$:

$$
\sum_{w \in W} T_w = \left( \prod_{\alpha \in \Phi^+} c_{\alpha} \right) x^{\rho^m} \left( \sum_{w \in W} (-1)^{\ell(w)} \sigma(w) \right) x^{-\rho^m}
$$

$$
= \left( \prod_{\alpha \in \Phi^+} c_{\alpha} \right) \sum_{w \in W} (-1)^{\ell(w)} \left( \prod_{\alpha \in \Phi(w^{-1})} x^{\alpha^m} \right) \sigma(w).
$$

In particular, for $\lambda \in P^+$ we have

$$
\widetilde{W}_\lambda = \sum_{w \in W} T_w (x^{w_0 \lambda}).
$$

Proof. By (4.1) and (4.10) we have

$$
\sum_{w \in W} T_w = \sum_{w \in W} (-1)^{\ell(w)} k_w \tau \left( Y^{\rho^m} T_{w^{-1}} Y^{-\rho^m} \right)
$$

$$
= k_{w_0}^2 \tau \left( Y^{\rho^m} 1_- Y^{-\rho^m} \right).
$$

The first formula now follows directly using $\tau = \sigma \circ \varphi^{-1}$ and the previous proposition. The second formula follows from the observation that

$$
\rho^m - w \rho^m = \sum_{\alpha \in \Phi(w^{-1})} \alpha^m
$$

for $w \in W$.

Corollary 4.10. Let $\Lambda \subset E$ be a lattice satisfying $Q \subseteq \Lambda \subseteq P$. Then $\widetilde{W}_\lambda \in \mathbb{C}[\Lambda]$ for $\lambda \in \Lambda^+ := P^+ \cap \Lambda$.

Proof. This follows from Lemma 4.4 and the previous theorem.

Remark 4.11. Note that the symmetric variant $\tau(1_+)(x^\lambda)$ of $\widetilde{W}_\lambda$ for $\lambda \in \Lambda^+$ may also be of interest. These are polynomials (again by Lemma 4.4), symmetric with respect to the Chinta-Gunnells $W$-action $\sigma$ (by Proposition 4.8(a)), which reduce for $m \equiv 1$ to Hall-Littlewood polynomials [23, §10] (by e.g., Remark 3.20).
Remark 4.12. In the equal Hecke and parameter case, \( \tilde{W}_\lambda \) admits the interpretation as a metaplectic Whittaker function attached to a metaplectic cover of a reductive group over a nonarchimedean local field, see [15, Thm. 16]. It is a natural open problem what the corresponding representation theoretic interpretation is of \( \tilde{W}_\lambda \) in the unequal Hecke and/or representation parameter case.

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