LOCALLY UNITARY GROUPOID CROSSED PRODUCTS

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Abstract. We define the notion of a principal $S$-bundle where $S$ is a groupoid
group bundle and show that there is a one-to-one correspondence between prin-
cipal $S$-bundles and elements of a sheaf cohomology group associated to $S$. We
also define the notion of a locally unitary action and show that the spectrum
of the crossed product is a principal $\hat{S}$-bundle. Furthermore, we prove that the
isomorphism class of the spectrum determines the exterior equivalence class of
the action and that every principal bundle can be realized as the spectrum of
some locally unitary crossed product.

Introduction

In this paper we study the connection between groupoid dynamical systems and
the spectrum of the crossed product. Our eventual goal will be to show there is a
strong link between the exterior equivalence class of a locally unitary action and
the isomorphism class of the spectrum $(A \rtimes S)\wedge$ as a principal bundle. What’s
more, there is a one-to-one correspondence between these principal bundles and
elements of a cohomology group, and this yields a complete cohomological invari-
ant for the exterior equivalence class of locally unitary actions on algebras with
Hausdorff spectrum. The primary objective of this work is to generalize the similar
theory of locally unitary group actions [8] to groupoids. However, the idea of using
local triviality conditions and cohomology classes to produce information about
the spectrum or primitive ideal space of crossed products has been implemented in
many contexts. It is applied to, what turns out to be, a groupoid setting in [11]
and is often connected to actions on algebras with continuous trace as in [7] or [9].

The structure of the paper is as follows. In Section 1 we introduce the notion
of a principal bundle associated to a groupoid group bundle. This theory mirrors
the classic theory of principal group bundles. In Section 2 we introduce some basic
results and constructions concerning groupoid crossed products. In Section 3 we
define what it means to be a unitary groupoid action and show that these actions
are trivial in the sense that the associated crossed product is given by a tensor
product. Finally, in Section 4 we describe locally unitary actions. Specifically, we
show that if $\alpha$ is a locally unitary action of the group bundle $S$ on a $C^*$-algebra
with Hausdorff spectrum $A$ then the spectrum of $A \rtimes_{\alpha} S$ is a principal $\hat{S}$-bundle and
the isomorphism class of the spectrum determines the exterior equivalence class of
the action. Furthermore, we show that every principal bundle can be realized as
the spectrum of some locally unitary crossed product.

Before we begin in earnest it should be noted that the results of this paper can be
found, in more detail and with a great deal of background material, in the author’s
thesis [2].

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1. Principal Group Bundles

In this section we will define the notion of a principal $S$-bundle associated to a groupoid group bundle $S$. The theory of principal $S$-bundles turns out to be nearly identical to the classic theory of principal group bundles. Consequently, we will only outline most of the proofs in this section. The following material is modeled off [15, Section 4.2].

Remark 1.1. A (groupoid) group bundle is a locally compact Hausdorff groupoid $S$ with identical range and source maps. Throughout this paper we will let $S$ denote a second countable, locally compact Hausdorff groupoid group bundle with abelian fibres and will denote both the range and the source map by $p$.

We begin with some definitions. Recall that we may view any continuous surjection $q : X \to Y$ as a “topological” bundle. We will denote the fibres by $X_y := q^{-1}(y)$ for all $y \in Y$.

Definition 1.2. Let $S$ be an abelian locally compact Hausdorff group bundle with bundle map $p$. Suppose $X$ is a locally compact Hausdorff bundle over $S^{(0)}$ with bundle map $q$. Furthermore, suppose there is an open cover $\mathcal{U} = \{U_i\}_{i \in I}$ of $S^{(0)}$ such that for each $i \in I$ there is a homeomorphism $\phi_i : q^{-1}(U_i) \to p^{-1}(U_i)$ with $p \circ \phi_i = q$. Finally, suppose that for all $i, j \in I$ there is a section $\gamma_{ij}$ of $S|_{U_{ij}} = p^{-1}(U_{ij})$ such that

$$\phi_i \circ \phi_j^{-1}(s) = \gamma_{ij}(p(s))s$$

for all $s \in S|_{U_{ij}}$. Such a bundle is called a principal $S$-bundle with trivialization $(\mathcal{U}, \phi, \gamma)$. The maps $\phi = \{\phi_i\}$ are referred to as trivializing maps and the sections $\gamma = \{\gamma_{ij}\}$ are referred to as transition maps.

Definition 1.3. Suppose $q : X \to S^{(0)}$ and $r : Y \to S^{(0)}$ are both principal $S$-bundles with trivializations $(\mathcal{U}, \phi, \gamma)$ and $(\mathcal{V}, \psi, \eta)$ respectively. Let $W$ be some common refinement of $\mathcal{U}$ and $\mathcal{V}$ with refining maps $\rho$ and $\sigma$ respectively. Furthermore, suppose $\Omega : X \to Y$ is a homeomorphism such that $r \circ \Omega = q$ and that for all $W_i \in W$, $\beta_i : W_i \to S$ is a section of $p$ such that for all $s \in p^{-1}(W_i)$,

$$\psi_{\sigma(i)} \circ \Omega \circ \phi_{\rho(i)}^{-1}(s) = \beta_i(p(s))s.$$

Then $(W, \Omega, \beta)$ is an $S$-bundle isomorphism of $X$ onto $Y$.

Remark 1.4. Suppose we have a principal $S$-bundle $X$ with trivializations $(\mathcal{U}, \phi, \gamma)$ and $(\mathcal{V}, \psi, \eta)$. We say that the trivializations are equivalent if the identity map forms an isomorphism as in Definition 1.3. We then say that $X$ is a principal $S$-bundle if it is equipped with a maximal, pairwise equivalent collection of trivializations called an atlas. We say that two principal $S$-bundles are isomorphic if they are isomorphic with respect to any pair of trivializations in their respective atlases. This is the usual method of dealing with locally trivial bundles and we won’t make much of a fuss about it.

Next, we would like to mimic the group case and characterize the set of all principal $S$-bundles by classes in some cohomology group. We will be using sheaf cohomology as defined and developed in [12, Section 4.1].

Proposition 1.5. Let $S$ be an abelian locally compact Hausdorff group bundle and for $U$ open in $S^{(0)}$ let $\mathcal{S}(U) = \Gamma(U, S)$ be the set of continuous sections from $U$.
into $S$. Then $S$ is an abelian sheaf and as such gives rise to a sheaf cohomology $H^n(S^{(0)}; S)$ which we shall denote by $H^n(S)$.

Proof. Straightforward arguments show that $S$ is a pre-sheaf with $\rho_{U,V} : S(U) \to S(V)$ given by restriction. Now suppose we have an open set $U \subset S^{(0)}$ and a decomposition $U = \bigcup_{i \in I} U_i$ of $U$ into open sets $U_i$. Furthermore, suppose we have $\gamma_i \in \Gamma(U_i, S)$ for all $i \in I$ and for all $i, j \in I$

$$\rho_{U_i, U_j}(\gamma_i) = \rho_{U_j, U_i}(\gamma_j).$$

Tracing through the definitions we see that each $\gamma_i$ is a continuous section on $U_i$ such that the $\gamma_i$ agree on overlaps. Therefore, we can define a continuous section $\gamma$ on $U$ in a piecewise fashion so that $\rho_{U,U_i}(\gamma) = \gamma_i$. Furthermore, it is clear that $\gamma$ is uniquely determined by the $\gamma_i$. Thus $S$ is a sheaf of groups on $S^{(0)}$ which is obviously abelian.

At this point we can build the desired correspondence between principal $S$-bundles and elements of $H^1(S)$.

Theorem 1.6. Suppose $S$ is an abelian locally compact Hausdorff group bundle. There is a one-to-one correspondence between the isomorphism classes of principal $S$-bundles and elements of the sheaf cohomology group $H^1(S)$. Given a principal bundle $X$ with trivialization $(U, \phi, \gamma)$ the cohomology class in $H^1(S)$ associated to $X$ is realized by the cocycle $\gamma$.

Proof. Because this proof is so similar to the corresponding proof for principal group bundles we will limit ourselves to sketching an outline. First, suppose $X$ is a principal $S$-bundle and pick a trivialization $(U, \phi, \gamma)$. Then $\gamma = \{\gamma_{ij}\}$ turns out to be a cocycle and as such we can use $\gamma$ to define a class $[\gamma] \in H^1(S^{(0)}; S)$. It is straightforward to show that $[\gamma]$ is independent of which trivialization we choose for $X$. Now let $Y$ be another bundle isomorphic to $X$. Suppose $(U, \phi, \gamma)$ is a trivialization for $X$, $(V, \psi, \eta)$ a trivialization for $Y$, and let $(W, \Omega, \beta)$ be an isomorphism from $X$ to $Y$. By passing to a common refinement we can assume, without loss of generality, that $U = V = W$. Then, for all $u \in U_{ij}$, simple calculations show that $\eta_{ij}(u)\beta_j(u) = \beta_i(u)\gamma_{ij}(u)$. Hence $\gamma^{-1}\eta$ is a boundary and therefore $[\gamma] = [\eta]$ in $H^1(S^{(0)}; S)$. This shows that the map $X \mapsto [\gamma]$ is a well defined function from the set of isomorphism classes of principal $S$-bundles into $H^1(S)$.

Next we are going to construct an inverse map. Suppose $c \in H^1(S^{(0)}; S)$ is realized by $\gamma \in Z^1(U, S)$ for some open cover $U$. Let $C = \prod p^{-1}(U_i)$ be the disjoint union of the $p^{-1}(U_i)$ and denote elements of $C$ by $(s, i)$ where $s \in p^{-1}(U_i)$. Define a relation on $C$ by $(s, i) \equiv (t, j)$ if and only if $p(s) = p(t) = u$ and $s = \gamma_{ij}(u)t$. Elementary calculations using the cocycle identity show that $\equiv$ is an equivalence relation. Let $X_\gamma$ be the quotient of $C$ by $\equiv$ with equivalence classes denoted by $[s, i]$ for $(s, i) \in C$ and associated quotient map $Q$. Since the map $(s, i) \mapsto p(s)$ is constant on $[s, i]$, we can factor it through $Q$ to obtain a continuous surjection $q : X_\gamma \to S^{(0)}$. As it turns out, $\equiv$ is trivial on $p^{-1}(U_i) \subset C$ and $\phi_i = Q|_{p^{-1}(U_i)} : p^{-1}(U_i) \to q^{-1}(U_i)$ is a homeomorphism. Because $X_\gamma$ is locally homeomorphic to $S$, it follows that $X_\gamma$ is locally compact Hausdorff and that we can view $X_\gamma$ as a bundle over $S^{(0)}$ with bundle map $q$. Furthermore, straightforward computations show $X_\gamma$ is a principal $S$-bundle with trivialization $(U, \phi, \gamma)$, and that the cohomology class associated to $X_\gamma$ is $[\gamma] = c$. 
We must show that our map is well defined in the sense that if we choose two different realizations of c we end up with isomorphic principal bundles. Let \( \eta = \{ \eta_j \} \) be some other cocycle which implements c on an open cover \( V \). Since \( [\eta] = [\gamma] = c \) we can pass to some common refinement of \( U \) and \( V \), say \( W \) with refining maps \( r \) and \( \rho \) respectively, and find continuous sections \( \beta_i \in \Gamma(W_i, S) \) such that

\[
\eta_{p(i)p(j)} \beta_j = \beta_i \gamma_r(i)r(j).
\]

We define \( \Omega : X_\gamma \to X_\eta \) locally by \( \Omega([s, r(i)]) = [\beta_i(p(s))s, \rho(i)] \). Some basic arguments show that \( \Omega \) is well defined and that \( (W, \Omega, \beta) \) is an isomorphism from \( X_\gamma \) onto \( X_\eta \). Thus we have constructed a well defined map \( [\gamma] \mapsto X_\gamma \) from \( H^1(S) \) into the set of isomorphism classes of principal \( S \)-bundles. Furthermore, it is clear that this map is a right inverse for \( X \mapsto [\gamma] \). Another simple argument shows that it is also a left inverse so that we have the desired correspondence.

We continue our exploration of principal \( S \)-bundles by showing that they are equivalent to a certain class of principal \( S \)-spaces. First, observe that only transitive groupoids can have transitive actions. Thus, we make the following definition for groupoid actions which are as transitive as they can be.

**Definition 1.7.** Suppose \( G \) is a locally compact Hausdorff groupoid acting on a locally compact Hausdorff space \( X \). Then we say the action is orbit transitive if \( G \cdot x = G \cdot y \) in \( X/G \) whenever \( G \cdot r(x) = G \cdot r(y) \) in \( G(0)/G \).

**Remark 1.8.** Recall that a \( G \)-space \( X \) is proper if the map \( (\gamma, x) \mapsto (\gamma \cdot x, x) \) from \( G \ltimes X \) into \( X \times X \) is proper. The action is principal if it is both free and proper.

Next we have the following construction.

**Proposition 1.9.** Suppose \( X \) is a principal \( S \)-bundle with trivialization \( (U, \phi, \gamma) \). Define the range map on \( X \) to be its bundle map \( q \). Then, for \( s \in S, x \in X \) such that \( p(s) = q(x) \in U_i \),

\[
s \cdot x = \phi_i^{-1}(s\phi_i(x))
\]

defines a continuous action of \( S \) on \( X \). Furthermore the following hold:

(a) The action of \( S \) on \( X \) is principal.

(b) For all \( i \) the map \( \phi_i \) is equivariant with respect to this action and the action of \( S \) on itself by left multiplication.

(c) The action of \( S \) on \( X \) is orbit transitive.

**Proof.** Elementary calculations show that the action is well defined on overlaps, respects the groupoid operations, and is continuous.

Part (a): Suppose \( s \cdot x = x \) for \( s \in S \) and \( x \in X \). Then for some \( i \) we have \( \phi_i^{-1}(s\phi_i(x)) = x \) so that \( s\phi_i(x) = \phi_i(x) \). It follows that \( s \in S(0) \) and that the action is free. Now suppose \( \{x_l\} \) and \( \{s_l\} \) are nets in \( X \) and \( S \), respectively, so that \( x_l \to x \) and \( s_l \cdot x_l \to y \). We can pass to a subnet and assume that \( p(s) = q(x) = q(y) \in U_i \) and \( p(s_l) = q(x_l) \in U_i \) for all \( l \). In this case \( s_l\phi_i(x_l) \to \phi_i(y) \) and, combining this with the fact that \( \phi_i(x_l) \to \phi_i(x) \), we have \( s_l \to \phi_i(y)\phi_i(x)^{-1} \). It follows quickly that the action of \( S \) on \( X \) is proper, and therefore principal.

Part (b): Suppose \( s \in S \) and \( x \in X \) such that \( p(s) = q(x) = u \in U_i \). Then

\[
\phi_i(s \cdot x) = \phi_i(\phi_i^{-1}(s\phi_i(x))) = s\phi_i(x).
\]

Part (c): Suppose \( x, y \in X \) such that \( q(x) = q(y) \in U_i \) and let \( s = \phi_i(y)\phi_i(x)^{-1} \). Then we are done since \( s \cdot x = \phi_i^{-1}(\phi_i(y)\phi_i(x)^{-1}\phi_i(x)) = y \).
This next proposition shows that we can view principal $S$-bundles as particularly nice $S$-spaces. It is often useful think of principal $S$-bundles in this manner.

**Theorem 1.10.** Suppose $S$ is an abelian locally compact Hausdorff group bundle and $X$ is a locally compact Hausdorff space. Then $X$ is a principal $S$-bundle if and only if $X$ is a principal, orbit transitive, $S$-space such that the range map on $X$ has local sections.

**Proof.** If $X$ is a principal $S$-bundle then let $(U, \phi, \gamma)$ be a trivialization of $X$ and let $S$ act on $X$ as in Proposition 1.9. On $U_i$ define $\sigma_i : U_i \to X$ by $\sigma_i(u) = \phi_i^{-1}(u)$. Now suppose $\Omega : S \to X$ is a continuous equivariant map. By passing to a subnet, relabel, and find $t$ such that $s_i \to t$. However, using the continuity of the action, this implies $s_i \cdot \sigma_i(p(s_i)) \to t \cdot \sigma_i(p(s))$. Using the fact that $X$ is Hausdorff and the action is free, we have $s = t$. It follows that $\psi_i$ is a homeomorphism and we define the trivializing maps to be $\phi_i = \psi_i^{-1}$. Next, we need to compute the transition functions. Suppose $s \in p^{-1}(U_{ij})$. Then

$$\phi_i \circ \phi_j^{-1}(s) = \psi_i^{-1} \circ \psi_j(s) = \psi_i^{-1}(s \cdot \sigma_j(p(s))) = \psi_i^{-1}(\gamma_{ij}(p(s))s \cdot \sigma_i(p(s))) = \gamma_{ij}(p(s))s$$

where $\gamma_{ij}(u)$ is the unique element of $S$ such that $\gamma_{ij}(u) \cdot \sigma_i(u) = \sigma_j(u)$. We know $\gamma_{ij}(u)$ is guaranteed to exist because the action is orbit transitive and that $\gamma_{ij}(u)$ is unique because the action is free. It is simple enough to show that $\gamma_{ij}$ is continuous. Hence $X$ is a principal $S$-bundle with trivialization $(U, \phi, \gamma)$. \qed

This next proposition is nice because it frees our idea of principal bundle isomorphism from the hassle of having to keep track of local trivializations. It is also mildly remarkable that $\Omega$ is not required to be a homeomorphism, or even a bijection.

**Proposition 1.11.** Suppose $X$ and $Y$ are principal $S$-bundles. Then $X$ and $Y$ are isomorphic if and only if there exists a continuous map $\Omega : X \to Y$ which is $S$-equivariant with respect to the actions of $S$ on $X$ and $Y$.

**Proof.** Let $X$ and $Y$ be as above with bundle maps $q$ and $r$, and trivializations $(U, \phi, \gamma)$ and $(V, \psi, \eta)$, respectively. Elementary calculations show that if $(W, \Omega, \beta)$ is a principal bundle isomorphism from $X$ to $Y$ then $\Omega$ is equivariant.

Now suppose $\Omega : X \to Y$ is a continuous equivariant map. By passing to a common refinement we may assume without loss of generality that $U = V$. Given $U_i$ let $\Omega_i = \psi_i \circ \Omega \circ \phi_i^{-1}$. Since each of its component maps preserves fibres, $\Omega_i$ does as well, and therefore $\Omega_i|_{S_u}$ maps $S_u$ into $S_u$ for $u \in U_i$. If $s, t \in S_u$ then

$$\Omega_i(st) = \psi_i \circ \Omega(s \cdot \phi_i^{-1}(t)) = \psi_i(s \cdot \Omega(\phi_i^{-1}(t))) = s\Omega_i(t).$$
Observe that given an abelian group homomorphism \( h : H \to H \) such that \( h(st) = sh(t) \) for all \( s, t \in H \) we have \( h(s) = h(es) = h(e)s \). Hence, \( h \) is actually just left multiplication by \( h(e) \). Applying this to the current situation we find that \( \Omega_i|_{S_u} \) is left multiplication by \( \Omega_i(u) \) on \( S_u \). Define \( \beta_i \) on \( U_i \) by \( \beta_i(u) = \Omega_i(u) \). The function \( \beta_i \) is a section of \( S \) on \( U_i \) which is continuous because \( \Omega_i \) is continuous. Since \( \Omega_i \) is defined by left multiplication against \( \beta_i \), it follows immediately that \( \Omega_i \) has a continuous inverse given by left multiplication against \( \beta_i^{-1} \). Thus \( \Omega_i \) is a homeomorphism. It is straightforward to show that this implies that \( \Omega_i \) is a homeomorphism. Furthermore, we know that for \( s \in q^{-1}(U_i) \)

\[
\psi_i \circ \Omega \circ \phi_i^{-1}(s) = \Omega_i(s) = \beta_i(p(s))s.
\]

Hence \((U, \Omega, \beta)\) is a principal bundle isomorphism of \( X \) onto \( Y \). \( \square \)

**Remark 1.12.** It is philosophically important to see that the theory of principal \( S \)-bundles is an extension of the classical theory of principal group bundles. Suppose \( H \) is an abelian locally compact Hausdorff group and \( X \) and \( Y \) are locally compact Hausdorff spaces. Let \( S = Y \times H \) be the trivial group bundle. Then it is not difficult to show that \( X \) is a principal \( H \)-bundle over \( Y \) if and only if \( X \) is a principal \( S \)-bundle. What’s more, \( H^0(S) \cong H^0(Y; H) \) and under this identification \( X \) generates the same cohomology class when viewed as either a principal \( S \)-bundle or a principal \( H \)-bundle.

### 1.1. Locally \( \sigma \)-trivial Spaces

As observed in Remark 1.12 principal “group bundle bundle” theory is a natural extension of classical principal group bundle theory. The real question is if there are principal \( S \)-bundles which are not generated by principal \( H \)-bundles. Fortunately, principal \( S \)-bundles are also an extension of the notion of \( \sigma \)-trivial spaces as defined in [11] and there are nontrivial examples given there. First, however, we define what it means to be \( \sigma \)-trivial.

**Definition 1.13.** Suppose the abelian locally compact Hausdorff group \( H \) acts on the locally compact Hausdorff space \( X \) and the stabilizers vary continuously in \( H \) with respect to the Fell topology. We shall say that \( X \) is a locally \( \sigma \)-trivial space if \( X/H \) is Hausdorff and if every \( x \in X \) has a \( H \)-invariant neighborhood \( U \) such that there exists a homeomorphism \( \phi : U \to (U/H \times H)/ \cong \) where

\[
(H \cdot x, s) \cong (H \cdot y, t) \text{ if and only if } H \cdot x = H \cdot y \text{ and } st^{-1} \in H_x.
\]

Furthermore we require that

(a) If \( x \in U \) then \( \phi(x) = [H \cdot x, s] \) for some \( s \in H \) and,

(b) If \( x \in U, s \in H \) and \( \phi(x) = [H \cdot x, t] \) then \( \phi(s \cdot x) = [H \cdot x, st] \).

Our goal will be to construct an abelian group bundle \( S \) associated to \( G \) and \( X \). Because this construction is unimportant for what follows, we will omit some of the details.

**Proposition 1.14.** Suppose the abelian locally compact Hausdorff group \( H \) acts on the locally compact Hausdorff space \( X \). Furthermore, suppose the stabilizers vary continuously in \( H \) and that \( X/H \) is Hausdorff. Define \( S_{(X,H)} := (X/H \times H)/ \cong \), often denoted \( S \), where \( \cong \) is as in Definition 1.13. Then \( S \) is an abelian locally compact Hausdorff group bundle with a Haar system whose unit space can be identified with \( X/H \). The bundle map is given by \( p([H \cdot x, s]) = H \cdot x \) and the operations are

\[
[H \cdot x, s][H \cdot x, t] := [H \cdot x, st], \quad \text{and} \quad [H \cdot x, s]^{-1} := [H \cdot x, s^{-1}].
\]
The fibre $S_{H,x}$ over $H \cdot x$ is (isomorphic to) $H/H_x$.

**Proof.** Define $S$ as in the statement of the proposition. A relatively simple argument shows that $S$ is locally compact Hausdorff and that the quotient map $Q : X/H \times H \to S$ is open. It is similarly straightforward to show that, given the operations above, $S$ is algebraically a group bundle over $X/H$ with bundle map $p$ given by $p([H \cdot x, s]) = H \cdot x$. We now assert that the operations are continuous and that $p$ is open. This can be proved by using the fact that $Q$ is open. It then follows from [13] Lemma 1.3 that $S$ has a Haar system. Given $H \cdot x \in X/H$ we have $S_{H,x} = \{[H \cdot x, s] : s \in H\}$. We can define a continuous surjective homomorphism $\phi : H \to S_{H,x}$ by $\phi(s) = [H \cdot x, s]$. Straightforward arguments show that $\phi$ is open, and hence it is clear from the definition of $\cong$ that $\phi$ factors to an isomorphism of $H/H_x$ with $S_{H,x}$. Since $H/H_x$ is clearly abelian, this proves that $S$ has abelian fibres and we are done. \( \square \)

The reason we went through all of this rigmarole is that given a locally $\sigma$-trivial system $(H, X)$ we would like to show that $X$ is a principal $S_{(H,X)}$-bundle.

**Proposition 1.15.** Suppose $H$ is an abelian locally compact Hausdorff group acting on a locally compact Hausdorff space $X$ with continuously varying stabilizers such that $X/H$ is Hausdorff. If $X$ is locally $\sigma$-trivial then $X$ is a principal $S_{(H,X)}$-bundle.

**Proof.** Let $q : X \to X/H$ be the quotient map. We know from Definition [13] that if $x \in X$ then there is an $H$-invariant neighborhood $U$ that is homeomorphic to $U/H \times H \cong$. If we let $V = U/H$ then $V$ is an open neighborhood of $H \cdot x$ and $q^{-1}(V) = U$. Let $p$ be the bundle map for $S$ and observe that

$$p^{-1}(V) = \{[H \cdot x, s] \in S : H \cdot x \in V\} = U/H \times H \cong.$$  

Thus we have a homeomorphism $\phi_V : q^{-1}(V) \to p^{-1}(V)$. Find one of these neighborhoods for every $x \in X$ and use them to build an open cover $\mathcal{V}$ of $X/H$. For $V_i$ in this open cover let $\phi_i = \phi_{V_i}$. Given $\gamma_{ij} : U_{ij} \to S$ by $\gamma_{ij}(H \cdot x) = \phi_i \circ \phi_j^{-1}([H \cdot x, e])$ where $e$ is the unit in $H$. It is clear that $\gamma_{ij}$ is a continuous section on $V_{ij}$. Since $\gamma_{ij}$ is a section, we can find a function $\tilde{\gamma}_{ij}$ from $V_{ij}$ into $H$ such that $\gamma_{ij}(H \cdot x) = [H \cdot x, \tilde{\gamma}_{ij}(H \cdot x)]$. Suppose $[H \cdot x, s] \in p^{-1}(V_{ij})$. Then, using the equivariance condition of Definition [13], we have

$$\phi_i \circ \phi_j^{-1}([H \cdot x, s]) = \phi_i(s) \circ \phi_j^{-1}([H \cdot x, e]) = [H \cdot x, \tilde{\gamma}_{ij}(H \cdot x)s] = \gamma_{ij}(H \cdot x)[H \cdot x, s].$$

Thus $X$ is a principal $S$-bundle with trivialization $(\mathcal{V}, \phi, \gamma)$. \( \square \)

**Remark 1.16.** The upshot of Proposition 1.15 is that every nontrivial example of a locally $\sigma$-trivial space given in [11] is an example of a nontrivial principal $S$-bundle. This shows that such bundles do exist, and it then follows from Theorem 1.6 that there are group bundles with nontrivial cohomology.

**Remark 1.17.** In [11] a $\sigma$-trivial space is said to be locally liftable if given a continuous section $c : U \to S$ then there exists $V \subset U$ and a continuous map $\tilde{c} : V \to H$ such that $c(H \cdot x) = [H \cdot x, \tilde{c}(H \cdot x)]$. Locally $\sigma$-trivial bundles are defined to be locally $\sigma$-trivial spaces which are also locally liftable. The reason for this extra requirement has to do with finding a cohomological invariant for $X$. Let $\mathcal{T}$ be the sheaf defined for $U \subset X/H$ by $\mathcal{T}(U) = C(U, H)$ and $\mathcal{R}$ be the subsheaf of $\mathcal{T}$ where
\( \mathcal{R}(U) \) is the subset of \( T(U) \) such that \( f(H \cdot x) \in H_x \) for all \( x \). Then sheaf cohomological considerations will show that we can construct a quotient sheaf \( T/\mathcal{R} \) and an associated cohomology \( H^n(X/H; T/\mathcal{R}) \). Given a \( \sigma \)-trivial space one would like to use the transition maps \( \gamma_{ij} \), as defined in the proof of Proposition \[13,15\] to construct an element of \( H^1(X/H; T/\mathcal{R}) \). The problem is that while \( \gamma_{ij} \) is a continuous section of \( U_{ij} \) into \( S \) the associated map \( \tilde{\gamma}_{ij} : U_{ij} \to H \) may not be continuous. If \( \tilde{\gamma}_{ij} \) is not continuous then it doesn’t define an element of \( T(U_{ij}) \) and we cannot construct the appropriate cohomology element. However, if \( X \) is required to be locally liftable then, by passing to a smaller open set, we can guarantee that \( \tilde{\gamma}_{ij} \) is a continuous function. As such it defines an element of \( T(U_{ij}) \) and hence a cohomology element in \( H^1(X/H; T/\mathcal{R}) \). In fact, it is shown in \[11\] that this construction leads to a one-to-one correspondence between locally \( \sigma \)-trivial bundles with a fixed orbit space \( X/H \) and \( H^1(X/H; T/\mathcal{R}) \).

This is an artificial restriction in our setting. The \( \gamma_{ij} \) can always be used to define an element of \( H^1(S(X,H)) \), regardless of whether \( \sigma \) is locally liftable or not. It is comforting to observe the following, however. Let \( S \) be the sheaf of local sections of \( S \) so that \( H^n(S) = H^n(X/H; S) \) by definition. It is straightforward to show that if \( \sigma \) is locally liftable then we get a short exact sequence of sheaves

\[
0 \to \mathcal{R} \to T \to S \to 0
\]

and that \( H^n(X/H; S) \) is naturally isomorphic to \( H^n(X/H, T/\mathcal{R}) \). Furthermore, once one sorts out all of the various constructions, it is clear that the different cohomological invariants of a \( \sigma \)-trivial bundle are identified under this isomorphism.

2. **Groupoid Crossed Products**

For the rest of the paper we will assume that \( G \), or \( S \), is a second countable locally compact Hausdorff groupoid, or group bundle, with a Haar system. We will let \( A \) be a separable \( C_0(G^{(0)}) \)-algebra and \( \mathcal{A} \) be its associated upper-semicontinuous bundle (usc-bundle). In general we will use the \( C_0(X) \)-algebra notation from \[13\] Appendix C] or \[4\] Section 3.1] and refer readers to these sources as references.

Next, let \((A,G,\alpha)\) be a groupoid dynamical system as defined in \[6\] or \[2\] Section 3.2]. We will assume that the reader is familiar with groupoid dynamical systems, although we take the time here to establish some of the basics. First, recall that \( \alpha \) is given by a collection of isomorphisms \( \{\alpha_s\}_{\gamma \in G} \) such that \( \alpha_\gamma : A(s(\gamma)) \to A(r(\gamma)) \), \( \alpha_{s\eta} = \alpha_s \circ \alpha_\eta \) whenever \( \gamma \) and \( \eta \) are composable, and \( \alpha : a = \alpha_s(a) \) defines a continuous action of \( G \) on \( A \). We then form the convolution algebra \( \Gamma_c(G, r^*\mathcal{A}) \) given the operations

\[
f \ast g(\gamma) = \int_G f(\eta) \alpha_\eta(g(\eta^{-1}\gamma)) d\lambda^\gamma(\eta) \quad \text{and} \quad f^*(\gamma) = \alpha_r(g(\gamma^{-1}))^*.
\]

Next, suppose we have a representation \((U,X \ast \mathfrak{H},\mu)\) of \( G \) and a \( C_0(G^{(0)}) \)-linear representation \( \pi \) of \( A \) on \( L^2(X \ast \mathfrak{H}, \mu) \). It follows from some fairly heavy representation theory \[6\] Section 7], \[2\] Section 3.3] that there exists a collection of representations \( \pi_u : A(u) \to B(\mathcal{H}_u) \) such that \( \pi(a) \) is equal to the direct integral

\[
\int_{G(0)} \pi_u(a(u)) d\mu(u) \quad \text{for all} \quad a \in A.
\]

This decomposition is an essential aspect of \( C_0(G^{(0)}) \)-linear representations of \( A \) and will be used in Section \[8\] Next, recall that we can use the quasi-invariant measure \( \mu \) to induce a measure \( \nu = \int_{G(0)} \lambda^\nu d\mu(u) \)
on $G$. We say that $(\pi, U, X \ast S, \mu)$ is a covariant representation if the relation

\[ U_\gamma \pi_{\alpha(\gamma)}(a) = \pi_{\alpha(\gamma)}(a) U_\gamma \text{ for all } a \in A(\alpha(\gamma)) \]

holds $\mu$-almost everywhere on $G$. We can then form the integrated representation $\pi \times U$ of $\Gamma_c(G, r^* A)$ on $L^2(X \ast S, \mu)$ by

\[ \pi \times U(f) h(u) = \int_G \pi_u(f(\gamma)) U_\gamma h(s(\gamma)) \Delta(\gamma)^{-\frac{1}{2}} d\lambda^u(\gamma) \]

where $\Delta$ is the modular function associated to $\mu$. It is then either a definition or a theorem that the crossed product $A \times G$ is the completion of $\Gamma_c(G, r^* A)$ with respect to the universal norm arising from the integrated covariant representations.

Remark 2.1. This construction mirrors the construction of groupoid $C^*$-algebras. In the groupoid case the integrated form of a unitary representation is given on $C_c(G)$ by

\[ U(f) h(u) = \int_G f(\gamma) U_\gamma h(s(\gamma)) \Delta(\gamma)^{-\frac{1}{2}} d\lambda^u(\gamma) \]

and $C^*(G)$ is the completion of $C_c(G)$ with respect to the universal norm determined by these representations.

With the exception of this section we will generally be interested only in actions of group bundles on $C^*$-algebras. As such, most of our crossed products will have extra structure. Recall from [3] Proposition 1.2] that if $S$ is a group bundle and $(A, S, \alpha)$ is a dynamical system then $A \times S$ is a $C_0(S^{(0)})$-algebra and that restriction from $\Gamma_c(S, p^* A)$ to $C_c(S, A(u))$ factors to an isomorphism of the fibre $A \times S(u)$ with the group crossed product $A(u) \times S_u$. In fact, this identification is even more robust. Recall that if $A$ is a $C_0(X)$-algebra and $U \subset X$ is open then we define $A(U)$ to be the section algebra $\Gamma_0(U, A)$ [3] Section 1].

Proposition 2.2. Suppose $(A, S, \alpha)$ is a groupoid dynamical system, $S$ is a group bundle, and that $U$ is an open subset of $S^{(0)}$. Then $A \times_{\alpha} S(U)$ and $A(U) \times_{\alpha} S|_U$ are isomorphic as $C_0(U)$-algebras.

Proof. Let $C = S^{(0)} \setminus U$ and recall that $A \times S$ is a $C_0(S^{(0)})$-algebra. It follows from some general $C_0(X)$-algebra theory that $A \times S(U)$ is isomorphic to the ideal

\[ I_C = \overline{\text{span}} \{ \phi : \phi \in C_0(S^{(0)}), f \in \Gamma_c(S, p^* A), \phi(C) = 0 \} \]

via the inclusion map $\iota_1 : A \times S(U) \to A \times S$ where we view both spaces as section algebras of their associated bundle. Because $S$ acts trivially on its unit space, $U$ is $S$-invariant and it follows from [3] Theorem 3.3] that the inclusion map $\iota_2 : \Gamma_c(S|_U, p^* A) \to \Gamma_c(S, p^* A)$ extends to an isomorphism of $A(U) \times S|_U$ with $\text{Ex}(U)$ where $\text{Ex}(U)$ is the closure of $\Gamma_c(S|_U, p^* A)$ in $A \times S$. A standard approximation argument shows that $I_C = \text{Ex}(U)$. Thus $\iota_2^{-1} \circ \iota_1$ yields the desired isomorphism. The fact that it is $C_0(U)$-linear follows from a computation. \hfill $\square$

Remark 2.3. The identification made in Proposition 2.2 extends to the spectrum. Recall from [15] Proposition C.5 that $(A \times S)^\wedge$ can be identified as a set with the disjoint union $\bigsqcup_{u \in S^{(0)}} (A(u) \times S_u)^\wedge$ and that there is a continuous map $q : (A \times S)^\wedge \to S^{(0)}$ such that $q(\pi) = u$ if and only if $\pi$ factors to a representation of $A(u) \times S_u$. Suppose $U \subset S^{(0)}$ is open. Then each of $q^{-1}(U)$, $(A \times S(U))^\wedge$ and $(A(U) \times S|_U)^\wedge$ can be identified setwise with $\bigsqcup_{u \in U} (A(u) \times S_u)^\wedge$. It is a matter of
sorting out definitions and applying Proposition 2.2 to see that all three algebras induce the same topology on the disjoint union.

Groupoid dynamical systems are fairly common and in particular arise naturally from groupoid actions on spaces, which we shall demonstrate after proving the following

Lemma 2.4. Suppose $X$, $Y$ and $Z$ are locally compact Hausdorff spaces and that $\sigma : Y \to X$ and $\tau : Z \to X$ are continuous surjections. Let $Z \ast Y = \{(z, y) \in Z \times Y : \tau(z) = \sigma(y)\}$. Then the map $\iota : C_0(Z \ast Y) \to \tau^*(C_0(Y))$ such that $\iota(f)(z)(y) = f(z, y)$ is an isomorphism. Furthermore, $\iota(f)$ is compactly supported if $f$ is and $\iota$ preserves convergence with respect to the inductive limit topology.

Remark 2.5. Recall from [15] Example C.4 that if $\sigma : Y \to X$ is a continuous surjection then $C_0(Y)$ has a $C_0(X)$-algebra structure. Furthermore, the fibre $C_0(Y)(x)$ is isomorphic to $C_0(\sigma^{-1}(x))$ via restriction.

Proof. Let $\mathcal{C}$ be the usc-bundle associated to $C_0(Y)$ as a $C_0(X)$-algebra. Define $\iota : C_c(Z \ast Y) \to \Gamma_c(\mathcal{C}, \tau^*\mathcal{C})$ by $\iota(f)(z)(y) = f(z, y)$. It is straightforward to show that $\iota(f)$ is a compactly supported section of $\tau^*\mathcal{C}$. We need to see that $\iota(f)$ is continuous. Start by demonstrating this in a simpler case. Suppose $g \in C_c(Z, Z \ast Y)$ and define $g \otimes h(z, y) = g(z)h(y)$ for all $(z, y) \in Z \ast Y$. Suppose $z_i \to z$. Since $h \in C_c(Y)$, we can view $h$ as a continuous section of $\mathcal{C}$ with $h(x) = h|_{\sigma^{-1}(x)}$ for all $x \in X$ and therefore $h(\tau(z_i)) \to h(\tau(z))$ in $\mathcal{C}$, since $\tau$ is continuous. It follows quickly from the fact that scalar multiplication is continuous that $\iota(f)$ is a continuous section.

Now suppose we have $f \in C_c(Z \ast Y)$. Since $Z \ast Y$ is closed in $Z \times Y$, we can extend $f$ to the product space and then find $g^i \in C_c(Z)$ and $h_i \in C_c(Y)$ such that $k_i = \sum_j g^i_j \otimes h^i_j \to f$ uniformly. Let $z_i \to z$ and observe that $\tau(z_i) \to \tau(z)$. We will show $\iota(f)(z_i) \to \iota(f)(z)$ using [15] Proposition C.20. Let $\epsilon > 0$ and choose $I$ such that $\|k_I - f\|_\infty < \epsilon$. Since sums of continuous functions are continuous, $\iota(k_I)(z_i) \to \iota(k_I)(z)$ by the previous paragraph. Furthermore, given $w \in Z$ we have
\[
\|\iota(k_I)(w) - \iota(f)(w)\|_\infty = \sup_{y \in \sigma^{-1}(\tau(w))} |k_I(w, y) - f(w, y)| \leq \|k_I - f\|_\infty < \epsilon.
\]
Since this is true for all $z_i$ and $z$, we are done.

An elementary calculation now shows that $\iota$ is isometric and it follows from an application of [15] Proposition C.24 that ran $\iota$ is dense in $\Gamma_0(\mathcal{C}, \tau^*\mathcal{C})$. Since $\iota : C_c(Z \ast Y) \to \tau^*C_0(Y)$ is an isometry mapping onto a dense set we can extend $\iota$ to an isomorphism $\iota : C_0(Z \ast Y) \to \tau^*C_0(Y)$. It is now straightforward to see that $\iota$ has the desired form on all of $C_0(Z \ast Y)$ and that $\iota$ preserves convergence with respect to the inductive limit topology. 

Using Lemma 2.4 we can show that left translation yields a groupoid dynamical system from any $G$-space.

Proposition 2.6. Suppose a locally compact Hausdorff groupoid $G$ acts on a locally compact Hausdorff space $X$. Then $C_0(X)$ is a $C_0(G^{(0)})$-algebra and there is an action of $G$ on $C_0(X)$ given by $\text{lt}_\gamma : C_0(r_X^{-1}(s(\gamma))) \to C_0(r_X^{-1}(r(\gamma)))$ where
\[
\text{lt}_\gamma(f)(y) = f(\gamma^{-1} \cdot y)
\]
for all $f \in C_0(r_X^{-1}(s(\gamma)))$ and $y \in r_X^{-1}(r(\gamma))$. 

\[ (4) \]
Proof. The $C_0(G^{(0)})$-algebra structure on $C_0(X)$ arises from the range map on $X$ as in Remark [2.3]. Let $C$ be the usc-bundle associated to $C_0(X)$. Define $\lambda_{\gamma}(\iota) : C_0(r_X(s(\gamma)\iota)) \to C_0(r_X(r(\gamma)\iota))$ as in the statement of the proposition. It is straightforward to show that each $\lambda_{\gamma}(\iota)$ is an isomorphism and that the groupoid operations are preserved. The only difficult part is proving that the action is continuous. Suppose $f_i \to f$ in $C$ and $\gamma_i \to \gamma$ in $G$ such that $u_i = p(f_i) = s(\gamma_i)$ for all $i$ and $u = p(f) = s(\gamma)$. Observe from Lemma [2.3] that there is an isomorphism $\iota : C_0(G \ast X) \to r^*C_0(X)$ and find $g \in C_0(G \ast X)$ such that $\iota(g)(u) = f$. Define $\tilde{g} \in C_0(G \ast X)$ by $\tilde{g}(\gamma, x) = g(\gamma, \gamma^{-1} \cdot x)$. A simple calculation shows that $\iota(\tilde{g})(\eta) = \lambda_{\gamma}(\iota(\tilde{g})(\eta))$ for all $\eta \in G$. In particular, since $\iota(\tilde{g})$ is a continuous section, we must have $\lambda_{\gamma}(\iota(\tilde{g})(\gamma)) \to \lambda_{\gamma}(\iota(\tilde{g})(\gamma))$. Finally, observe that

$$\| \lambda_{\gamma_i}(\iota(\tilde{g}(\gamma_i))) - \lambda_{\gamma_i}(f_i) \| = \| \iota(\tilde{g})(\gamma_i) - f_i \| \to 0$$

so that by [15, Proposition C.20] we have $\lambda_{\gamma_i}(f_i) \to \lambda_{\gamma}(f)$.

Remark 2.7. It is not particularly difficult to use Lemma [2.3] to show that if $X$ is a $G$-space and $G \ltimes X$ is the transformation groupoid then $C_0(X) \rtimes_H G$ is naturally isomorphic to the transformation groupoid $C^\ast$-algebra $C^\ast(G \ltimes X)$. However, since this fact has little bearing on our current discussion, the proof has been omitted.

Remark 2.8. There is a converse to Proposition [2.6] for abelian algebras. Given a groupoid dynamical system $(C_0(X), G, \alpha)$ it follows from [4, Proposition 1.1] that there is an action of $G$ on $C_0(X)^\wedge = X$. It is straightforward to show, once one sorts out all the definitions, that $\alpha$ is given by left translation with respect to this action.

Example 2.9. If $X$ is a principal $S$-space then we showed in Proposition [1.3] that there is a continuous action of $S$ on $X$. Hence there is an associated crossed product $C_0(X) \rtimes_H S$ which will be essential in the latter half of Section [4].

3. Unitary Actions

In this section we will discuss what it means for a groupoid to act trivially. The main goal will be to show that if the action is trivial then the crossed product reduces to a balanced tensor product. As with group crossed products, trivial actions are going to be defined by unitaries.

Definition 3.1. Suppose $S$ is a locally compact Hausdorff groupoid group bundle and $A$ is a $C_0(S^{(0)})$-algebra. Then a unitary action of $S$ on $A$ is defined to be a collection $\{u_s\}_{s \in S}$ such that

1. $u_s \in UM(A(p(s)))$ for all $s \in S$,
2. $u_{st} = u_su_t$ whenever $p(s) = p(t)$, and
3. $s \cdot a := u_{s}a$ defines a continuous action of $S$ on $A$.

The triple $(A, S, u)$ is called a unitary dynamical system.

Remark 3.2. If $u$ is a unitary action of $S$ on $A$ then the restriction of $u$ to $S_v$ for $v \in S^{(0)}$ gives a unitary action of $S_v$ on $A(v)$ in the sense of [15, Definition 2.70]. Thus, Definition 3.1 is really just a “bundled” version of the notion of a unitary action of a group on a $C^\ast$-algebra.

As with groupoid dynamical systems there is an “unbundled” definition.
Proposition 3.3. Suppose \( (A, S, u) \) is a unitary dynamical system. Then there is an element \( u \in UM(p^*A) \) such that \( u(s) = u_s \) for all \( s \in S \). Conversely, if we have \( u \in UM(p^*A) \) then there are elements \( u_s \in UM(A(p(s))) \) for all \( s \in S \) and if \( u_{st} = u_s u_t \) whenever \( p(s) = p(t) \) then \( \{u_s\} \) defines a unitary action of \( S \) on \( A \).

Proof. Suppose \( (A, S, u) \) is a unitary action and \( f \in p^*A \). We need to show that

\[
\begin{align*}
h(s) &:= u_s f(s), \\
g(s) &:= u_s^* f(s)
\end{align*}
\]

define elements of \( p^*A \). The continuity of \( h \) is obvious from condition (c) of Definition 3.1. Suppose \( s_i \to s \) and \( a_i \to a \) such that \( a_i \in A(p(s_i)) \) for all \( i \). First, observe that condition (b) of Definition 3.1 guarantees that \( u_{s^{-1}} = u_s^{-1} = u_s^* \) for all \( s \in S \). Therefore

\[
u_{s^{-1}} a_i = u_s a_i \to u_s^* a = u_{s^{-1}} a.
\]

It follows immediately that \( g \) is continuous as well. Furthermore, both \( h \) and \( g \) must vanish at infinity because \( f \) does. Thus \( h, g \in p^*A \). Hence [5, Lemma 2] implies that there is a multiplier \( u \) such that \( u(f)(s) = u_s f(s) \) for all \( s \in S \). Since each \( u_s \) is a unitary, it is clear that \( u \) must be a unitary.

Next, suppose we are given \( u \in UM(p^*A) \). Then, via [5, Lemma 2], we know there exists multipliers \( u_s \) such that \( u_s(f(s)) = u(f)(s) \). However, since \( u \) is a unitary, each \( u_s \) must be as well. It is now straightforward, using [13, Proposition C.20], to show that \( \{u_s\} \) defines a unitary action of \( S \) on \( A \). \( \square \)

Given a unitary dynamical system we can form an associated groupoid dynamical system in the obvious way by using the adjoint map.

Proposition 3.4. Suppose \( (A, S, u) \) is a unitary dynamical system. Then the collection \( \{\text{Ad } u_s\}_{s \in S} \) defines a groupoid action of \( S \) on \( A \). We say that such an action is unitary or unitarily implemented.

Proof. Given a unitary action let \( u \) be the corresponding element of \( UM(p^*A) \) guaranteed by Proposition 3.3. Then define \( \text{Ad } u : p^*A \to p^*A \) by \( \text{Ad } u(f) = uf u^* \). Clearly \( \text{Ad } u \) is a \( C_0(S^{(0)}) \)-linear automorphism of \( p^*(A) \) and it is straightforward to use [6, Lemma 4.3] to show that \( \text{Ad } u \) yields the desired dynamical system. \( \square \)

At this point we need to make a brief detour through the notion of equivalent actions. The following construction will play the role of isomorphism.

Definition 3.5. Suppose \( G \) is a locally compact Hausdorff groupoid and \( A \) is a \( C_0(G^{(0)}) \)-algebra. Furthermore, suppose \( \alpha \) and \( \beta \) are actions of \( G \) on \( A \). Then we say that \( \alpha \) and \( \beta \) are exterior equivalent if there is a collection \( \{u_\gamma\}_{\gamma \in G} \) such that

\[
\begin{align*}
(a) & \ u_\gamma \in UM(A(r(\gamma))) \text{ for all } \gamma \in G, \\
(b) & \ u_{\gamma \eta} = u_\gamma u_\eta \text{ for all } \gamma, \eta \in G \text{ such that } s(\gamma) = r(\eta), \\
(c) & \ \text{the map } (\gamma, a) \mapsto u_\gamma a \text{ is jointly continuous on the set } \{ (\gamma, a) \in G \times A : r(\gamma) = q(a) \}, \text{ and} \\
(d) & \ \beta_\gamma = \text{Ad } u_\gamma \circ \alpha_\gamma \text{ for all } \gamma \in G.
\end{align*}
\]

As before, we present an alternate definition which removes the bundle theory.

Proposition 3.6. Suppose \( \alpha \) and \( \beta \) are exterior equivalent actions of the locally compact groupoid \( G \) on the \( C_0(G^{(0)}) \)-algebra \( A \) with the collection \( \{u_\gamma\} \) implementing the equivalence. Then there is an element \( u \in UM(r^*A) \) such that \( u(f)(\gamma) = u_\gamma f(\gamma) \) for all \( f \in r^*A \) and \( \gamma \in G \).
Conversely, if \( u \in UM(r^*A) \) then there are \( u_\gamma \in UM(A(r(\gamma))) \) for all \( \gamma \in G \). If \( u_{\gamma \eta} = u_\gamma u_{\eta} \) whenever \( s(\gamma) = r(\eta) \) and \( \beta_\gamma = \text{Ad} u_\gamma \circ \alpha_\gamma \) for all \( \gamma \in G \) then \( \alpha \) and \( \beta \) are exterior equivalent.

**Proof.** This is demonstrated in almost exactly the same way as Proposition 3.3 and the proof is omitted for brevity.

The most important fact about exterior equivalent actions is the following

**Proposition 3.7.** Suppose \((A, G, \alpha)\) and \((A, G, \beta)\) are exterior equivalent groupoid dynamical systems with the equivalence implemented by \(\{u_\gamma\}\). Then the map \( \phi : \Gamma_c(G, r^*A) \to \Gamma_c(G, r^*A) \) defined by

\[
\phi(f)(\gamma) = f(\gamma)u_\gamma^*
\]

for all \( \gamma \in G \) extends to an isomorphism from \( A \times_\alpha G \) onto \( A \times_\beta G \).

**Proof.** Let \((A, G, \alpha)\) and \((A, G, \beta)\) be exterior equivalent dynamical systems with the equivalence implemented by \(\{u_\gamma\}\). Use Proposition 3.4 to find \( u \in UM(r^*A) \) such that \( uf(\gamma) = u_\gamma f(\gamma) \) for all \( \gamma \). Given \( f \in \Gamma_c(G, r^*A) \) view \( f \) as an element of the pull back \( r^*A \) and define \( \phi(f) = fu^* \). It is clear that \( \phi : \Gamma_c(G, r^*A) \to \Gamma_c(G, r^*A) \) is given by (5). Some lengthy, but unenlightening, computations show that \( \phi \) is a \( * \)-homomorphism with respect to the actions arising from \( \alpha \) in its domain and from \( \beta \) on its range. Since \( \|\phi(f)(\gamma)\| = \|f(\gamma)\| \) for all \( \gamma \in G \), it follows quickly that \( \phi \) is continuous with respect to the inductive limit topology. Therefore the Disintegration Theorem [6, Theorem 7.8,7.12] implies that \( \phi \) extends to a \( * \)-homomorphism from \( A \times_\alpha G \) into \( A \times_\beta G \). We can define an inverse \( \psi \) for \( \phi \) on \( \Gamma_c(G, r^*A) \) by \( \psi(f)(\gamma) = f(\gamma)u_\gamma \) in an identical fashion.

Moving on, our statement that unitary actions are “trivial” dynamical systems will be supported by the next lemma. However, let us first introduce an action which is as simple as possible.

**Example 3.8.** Suppose \( S \) is a locally compact group bundle and \( A \) is a \( C_0(S^{(0)}) \)-algebra. It is easy to show that the collection of identity maps \( \text{id}_x : A(p(s)) \to A(p(s)) \) defines an action of \( S \) on \( A \). Observe that group bundles are the only groupoids which can act trivially in this manner.

**Lemma 3.9.** If \((A, S, \alpha)\) is a unitary dynamical system then it is exterior equivalent to the trivial system \((A, S, \text{id})\).

**Proof.** Suppose \( \alpha \) is implemented by the unitaries \( \{u_\gamma\} \). Then an elementary calculation shows that the \( \{u_\gamma\} \) also implement an equivalence between \( \text{id} \) and \( \alpha \).

**Remark 3.10.** The curious reader may wonder why we have only defined unitary actions for a special class of groupoids. Unitary actions should always be equivalent to the trivial action and, as stated in Example 3.8, the trivial action only makes sense for group bundles. Thus, it only makes sense to define unitary actions for group bundles.

We want to show that crossed products of unitary dynamical systems are tensor products. However, we are working with fibred objects so we need to use a tensor product which respects the bundle structure on the algebras. It is assumed that the reader is familiar with the basics of balanced tensor products between \( C_0(X) \)-algebras, although [10] and [12, Appendix B] will serve as references.
Remark 3.11. For completeness, let us recall that if $A$ and $B$ are $\mathcal{C}_0(X)$-algebras then the balancing ideal $I_X$ is the ideal in $A \otimes_{\max} B$ generated by
\[ \{ f \cdot a \otimes b - a \otimes f \cdot b : f \in \mathcal{C}_0(X), a \in A, b \in B \}. \]
The balanced tensor product $A \otimes_{\mathcal{C}_0(X)} B$ is defined to be the quotient $A \otimes_{\max} B/I_X$. Our tensor products will generally be maximal tensor products. However, most of the time we will be working with nuclear $C^*$-algebras so that we will not have to make this distinction.

We now prove the main theorem concerning unitary actions, which is that they have trivial crossed products. Because we are working with universal completions, the proof is surprisingly technical.

**Theorem 3.12.** Suppose $(A,S,\alpha)$ is a unitary dynamical system with $\alpha$ implemented by $u$. Then there is a $\mathcal{C}_0(S(0))$-linear isomorphism $\phi : C^*(S) \otimes_{\mathcal{C}_0(S(0))} A \to A \rtimes \alpha S$ which is characterized for $a \in A$ and $f \in \mathcal{C}_c(S)$ by
\[ \phi(f \otimes a)(s) = f(s)a(p(s))u_s^* \]
(6)

**Remark 3.13.** Since $\phi$ is $\mathcal{C}_0(S(0))$-linear it factors to isomorphisms $\phi_u : C^*(S_u) \otimes A(u) \to A(u) \times S_u$. It is not difficult to check that these are the usual isomorphisms that arise from unitary actions [15 Lemma 2.73].

**Proof.** First, let $\beta$ be a Haar system for $S$ and consider the trivial action $\text{id}$ of $S$ on $A$. Given $a \in A$ and $f \in \mathcal{C}_c(S)$ define $\iota(f \otimes a)(s) := f(s)a(p(s))$. It follows easily that $\iota(f \otimes a) \in \Gamma_c(S,p^*A)$. Extend $\iota$ to the algebraic tensor product $\mathcal{C}_c(S) \otimes A$ by linearity so that $\iota : \mathcal{C}_c(S) \otimes A \to \Gamma_c(S,p^*A)$. Observe that $\iota$ is dense with respect to the inductive limit topology by [3 Proposition 1.3]. Simple calculations on elementary tensors show that $\iota$ is a *-homomorphism. Now we check that $\iota$ is bounded. Suppose $(\pi,U,S(0)*\mathcal{F},\mu)$ is a covariant representation of $(A,S,\text{id})$. Then $U$ is a groupoid representation of $S$ and we can form the integrated representation as in Remark 2.11 which we also denote by $U$. Let the collection $\{\pi_u\}_{u \in S(0)}$ be a decomposition of $\pi$ as in Section 2. Since $(\pi,U)$ is covariant, we must have, for all $a \in A$ and almost every $s \in S$,
\[ \pi_p(s)(a(p(s)))U_s = U_s\pi_p(s)(a(p(s))). \]
However, we can now compute for $f \in \mathcal{C}_c(S)$ and $h \in \mathcal{L}^2(S(0)*\mathcal{F},\mu)$ that
\[ (\pi(a)U(f))h(u) = \int_S \pi_u(a(u))f(s)U(s)h(u)\Delta(s)^{-\frac{1}{2}}d\beta^n(s) = (U(f)\pi(a))(h(u)). \]
We can extend this by continuity to all $f \in C^*(S)$ and conclude that $\pi$ and $U$ are commuting representations of $A$ and $C^*(S)$. It follows from [12 Theorem B.27] that there exists a representation $U \otimes \pi$ on $C^*(S) \otimes A$ such that $U \otimes \pi(f \otimes a) = U(f)\pi(a)$. Given $f \in \mathcal{C}_c(S)$ and $a \in A$ we check that
\[ \pi \rtimes U(\iota(f \otimes a))h(u) = \int_S \pi_u(f(s)a(u))U_s h(u)\Delta(s)^{-\frac{1}{2}}d\beta^n(s) \]
\[ = \pi_u(a(u)) \int_S f(s)U_s h(u)\Delta(s)^{-\frac{1}{2}}d\beta^n(s) \]
\[ = \pi(a)U(f)h(u) = U \otimes \pi(f \otimes a)h(u). \]
Using linearity, we conclude that $\pi \rtimes U(\iota(\xi)) = U \otimes \pi(\xi)$ for all $\xi \in \mathcal{C}_c(S) \otimes A$. Thus, given $\xi \in \mathcal{C}_c(S) \otimes A$, $\|\pi \rtimes U(\iota(\xi))\| = \|U \otimes \pi(\xi)\| \leq \|\xi\|$. Since this is
true for all covariant representations, it follows that \( \iota \) is bounded and extends to a homomorphism on \( C^*(S) \otimes A \). Furthermore, since the range of \( \iota \) is dense, it must be surjective. What’s more, given \( \phi \in C_0(S^{(0)}) \), \( f \in C_c(S) \) and \( a \in A \) we have

\[
\iota(\phi \cdot f \otimes a)(s) = \phi(p(s))f(s)a(p(s)) = \iota(f \otimes \phi \cdot a)(s).
\]

It follows by continuity and linearity that \( \iota \) factors through the balancing ideal and induces a surjective homomorphism \( \hat{\iota} : C^*(S) \otimes_{C_0(S^{(0)})} A \to A \rtimes S \).

We would like to show that \( \hat{\iota} \) is isometric. Suppose \( R \) is a faithful representation of \( C^*(S) \otimes_{C_0(S^{(0)})} A \) and let \( R^\ast \) be its lift to \( C^*(S) \otimes A \). It follows [12, Corollary B.22] that there are commuting representations \( \pi \) and \( U \) of \( A \) and \( C^*(S) \) such that \( R^\ast = U \otimes \pi \). Furthermore, since \( U \otimes \pi \) contains the balancing ideal, a quick computation shows that \( U(\phi \cdot f)\pi(a) = U(f)\pi(\phi \cdot a) \) for all \( \phi \in C_0(S^{(0)}) \), \( f \in C^*(S) \), and \( a \in A \). Now, without loss of generality, we can use the Disintegration Theorem to assume that \( U \) is the integrated form of some groupoid representation \((U, S^{(0)} \rtimes \mathfrak{g}, \mu)\). Furthermore we have, for all \( \phi \in C_0(S^{(0)}) \), \( a \in A \), and \( f \in C_c(S) \)

\[
\pi(\phi \cdot a)U(f)h(u) = \int_S \phi(u)f(s)U_s\pi(a)h(u)\Delta(s)^{-\frac{1}{2}}d\mu(s) = \phi(u)U(f)\pi(a)h(u).
\]

Since \( U \) is nondegenerate, this implies that \( \pi \in C_0(S^{(0)}) \)-linear. Suppose the collection \( \{\pi_u\} \) is a decomposition of \( \pi \) and let \( \nu \) be the measure on \( S \) induced by \( \mu \). All we need to do to prove that \( (\pi, U) \) is a covariant representation of \((A, S, \text{id})\) is verify the covariance relation. Let \( \{a_i\} \) be a countable dense subset in \( A \) and \( e_l \) a special orthogonal fundamental sequence for \( S^{(0)} \rtimes \mathfrak{g} \) [13, Remark F.7]. Since \( \pi \) and \( U \) commute, we have for all \( i, l, k \) and \( f \in C_c(S) \)

\[
0 = (\pi(a_i)U(f)e_l, e_k) = (U(f)\pi(a_i)e_l, e_k)
\]

\[
= \int_S (f(s)\pi_p(a_i(p(s)))U_se_l(p(s)), e_k(p(s)))\Delta(s)^{-\frac{1}{2}}d\nu(s)
\]

\[
- \int_S (f(s)U_s\pi_p(a_i(p(s)))e_l(p(s)), e_k(p(s)))\Delta(s)^{-\frac{1}{2}}d\nu(s)
\]

\[
= \int_S (f(s)(\pi_p(a_i(p(s)))U_s - U_s\pi_p(a_i(p(s))))e_l(p(s)), e_k(p(s)))\Delta(s)^{-\frac{1}{2}}d\nu(s).
\]

This holds for all \( f \in C_c(S) \) so that we may conclude for each \( i, l \) and \( k \) there exists a \( \nu \)-null set \( N_{i,l,k} \) such that

\[
(\pi_p(a_i(p(s)))U_s - U_s\pi_p(a_i(p(s))))e_l(p(s)), e_k(p(s))) = 0
\]

for all \( s \not\in N_{i,l,k} \). However, if we let \( N = \bigcup_{i,l,k} N_{i,l,k} \) then \( N \) is still a \( \nu \)-null set and for each \( s \not\in N \) \[9\] holds for all \( i, l \) and \( k \). Since \( \{e_l(p(s))\} \) is a basis (plus zero vectors) for each \( p(s) \), this implies that for \( s \not\in N \) we have \( \pi_p(a_i(p(s)))U_s = U_s\pi_p(a_i(p(s))) \) for all \( i \). Because \( \{a_i\} \) is dense in \( A \), this holds for all \( a \in A \). Thus \( (\pi, U) \) is a covariant representation of \((A, S, \text{id})\). Furthermore, we can reuse the computation in \[5\] to conclude that \( \pi \rtimes U \circ \iota = \pi \rtimes U \). Given \( \xi \in C^*(S) \otimes A \) let \( \iota' \) be its image in \( C^*(S) \otimes_{C_0(S^{(0)})} A \). We then have

\[\|\xi'\| = \|R(\xi')\| = \|U \otimes \pi(\xi)\| = \|\pi \rtimes U(\iota(\xi))\| \leq \|\iota(\xi)\| = \|\iota'(\xi')\|\]

It follows that \( \iota \) is isometric and is therefore an isomorphism.

To finish the proof, observe that because of Proposition \[3.7\] and Lemma \[3.9\] the map \( \psi : A \rtimes_{\text{id}} S \to A \rtimes_{\alpha} S \) given by \( \psi(f)(s) = f(s)u^*_s \) is an isomorphism. Thus
Proposition 4.3. Assume separability, every fibre is elementary and isomorphic to the compacts. As a bundle, C-algebra, it follows quickly that \( \phi \) is \( \mathcal{C}_0(S^{(0)}) \)-linear and has the correct form. \( \square \)

4. Locally Unitary Actions

Now that we have developed the theory of unitary actions we can modify Definition 3.1 and introduce a new concept. The basic idea is that we weaken the continuity condition and see what kind of structure we have left.

Definition 4.1. Suppose \( S \) is a group bundle and \( A \) is a \( \mathcal{C}_0(S^{(0)}) \)-algebra. A dynamical system \( (A, S, \alpha) \) is said to be locally unitary if there is an open cover \( \{U_i\}_{i \in I} \) of \( S^{(0)} \) such that \( (A(U_i), S|_{U_i}, \alpha|_{S|_{U_i}}) \) is unitarily implemented for all \( i \in I \).

Our goal will be to analyze the exterior equivalence classes of abelian locally unitary actions on \( C^* \)-algebras with Hausdorff spectrum. In particular, the rest of the \( C^* \)-algebras in this section will have Hausdorff spectrum and we will view them as \( \mathcal{C}_0(\hat{A}) \)-algebras in the usual fashion.

Remark 4.2. If \( A \) has Hausdorff spectrum then \( A \) is naturally a \( \mathcal{C}_0(\hat{A}) \)-algebra. The fibres are given by \( A(\pi) = A/\ker \pi \) for all \( \pi \in \hat{A} \). In particular, since we are assuming separability, every fibre is elementary and isomorphic to the compacts. As such each fibre has a unique faithful irreducible representation (up to equivalence).

With this assumption we obtain a very nice identification of the spectrum for unitary crossed products.

Proposition 4.3. Suppose \( S \) is an abelian locally compact Hausdorff group bundle with a Haar system, that \( A \) is a \( C^* \)-algebra with Hausdorff spectrum \( S^{(0)} \) and that \( (A, S, \alpha) \) is a unitary dynamical system. Let \( \{u_s\} \) be the unitaries implementing \( \alpha \) and for all \( v \in S^{(0)} \) let \( \pi_v \) be the unique irreducible representation of \( A(v) \). Define, for \( \omega \in \hat{S} \),

\[
\omega \pi_{\hat{p}(\omega)}(u)(s) := \omega(s) \pi_{\hat{p}(\omega)}(u_s).
\]

Then the map \( \phi: \hat{S} \to (A \rtimes_\alpha S)^\wedge \) given by \( \phi(\omega) = \pi_{\hat{p}(\omega)} \rtimes \omega \pi_{\hat{p}(\omega)}(u) \) is a bundle homeomorphism.

Proof. Let \( (A, S, \alpha) \) and \( u \) be as above. It follows from Theorem 3.12 that the map \( \psi: C^*(S) \otimes_{\mathcal{C}_0(S^{(0)})} A \to A \rtimes S \) characterized by \( \psi(a \otimes f)(s) = f(s)au_s^* \) is an isomorphism. Therefore, there is a homomorphism \( \phi_1: (C^*(S) \otimes_{\mathcal{C}_0(S^{(0)})} A)^\wedge \to (A \rtimes S)^\wedge \) such that \( \psi_1(R) = R \circ \psi^{-1} \). Next, recall that we identify the dual group bundle \( \hat{S} \) with \( C^*(S)^\wedge \) [1]. Define \( \hat{S} \times_{S^{(0)}} \hat{A} := \{(\omega, \pi_{\hat{p}(\omega)}) \in \hat{S} \times \hat{A} : \omega \in \hat{S}\} \). Since \( C^*(S) \) is an abelian \( C^* \)-algebra, and is therefore GCR and nuclear, it follows from [10] Lemma 1.1 that \( \phi_2: \hat{S} \times_{S^{(0)}} \hat{A} \to (C^*(S) \otimes_{\mathcal{C}_0(S^{(0)})} A)^\wedge \) given by \( \phi_2(\omega, \pi) = \omega \otimes_\sigma \pi \) is a homeomorphism. Recall that if \( \pi \) is a representation on \( \mathcal{H} \) then \( \omega \otimes_\sigma \pi \) is the representation on \( \mathbb{C} \otimes \mathcal{H} \), which we will of course identify with \( \mathcal{H} \), characterized by \( \omega \otimes_\sigma \pi(f \otimes a) = \omega(f)\pi(a) \). Moving on, since \( \hat{A} = S^{(0)} \) we can define another homeomorphism \( \phi_3: \hat{S} \to \hat{S} \times_{S^{(0)}} \hat{A} \) by \( \phi_3(\omega) = (\omega, \pi_{\hat{p}(\omega)}) \). Let \( \phi = \phi_1 \circ \phi_2 \circ \phi_3 \) and observe that \( \phi: \hat{S} \to (A \rtimes S)^\wedge \) is a homeomorphism. Furthermore given \( \omega \in \hat{S} \) we have \( \phi(\omega) = \omega \otimes_\sigma \pi_{\hat{p}(\omega)} \circ \psi^{-1} \).

Now fix \( x \in S^{(0)} \), \( \omega \in \hat{S}_x \) and define the map \( U: S_x \to U(\mathcal{H}) \) by \( U_x = \omega(s)\pi_x(u_s) \). Since \( u \) and \( \omega \) are continuous, it follows quickly that \( U \) is a unitary
representation of $S_x$. Furthermore, we can compute for $a \in A(x)$ and $s \in S_x$ that

$$U_s \pi_x(a) = \omega(s) \pi_x(u_s a) = \omega(s) \pi_x(u_s \alpha_s U_s) = \pi_x(\alpha_s(a)) U_s.$$ 

Thus $(\pi_x, U)$ is a covariant representation of $(A(x), S_x, \alpha)$. As such we can form the integrated representation $\pi_x \rtimes U$. Recall that $A \rtimes S$ is a $C_0(S^{(0)})$-algebra and that the restriction map $\rho$ factors to an isomorphism between $A \rtimes S(x)$ and $A(x) \rtimes S_x$. Using the restriction map to view $\pi_x \rtimes U$ as a representation of $A \rtimes S$ we claim that $\pi_x \rtimes U = \phi(\omega)$. It will suffice to show that given an elementary tensor $f \otimes a$ then $\pi_x \rtimes U(\psi(f \otimes a)) = \omega \otimes_{\sigma} \pi_x(f \otimes a)$. We compute, observing that the modular function is one since $S$ is abelian,

$$\pi_x \rtimes U(\psi(f \otimes a)) = \int_S \pi_x(f(s) a(x) u_s^*) \omega(s) \pi_x(u_s) h d\beta^x(s) = \int_S f(s) \omega(s) d\beta^x(s) \pi_x(a(x)) h = \omega(f) \pi_x(a) h = (\omega \otimes_{\sigma} \pi_x)(f \otimes a)h.$$

Thus $\phi(\omega) = \pi_x \rtimes U$ and, since $U$ is just an abbreviated notation for $\omega \pi_x(u)$, we are done.

\[ \square \]

4.1. Characterization. We saw in Proposition 4.3 that the spectrum of the unitary crossed product was homeomorphic to $\hat{S}$. If the action were locally unitary then it is interesting to ask if the spectrum is locally homeomorphic to $\hat{S}$ and if it is, in fact, a principal $\hat{S}$-bundle. The answer is given in the following

**Theorem 4.4.** Suppose $S$ is an abelian group bundle with a Haar system, that $A$ is a $C^*$-algebra with Hausdorff spectrum $S^{(0)}$ and that $(A, S, \alpha)$ is a locally unitary dynamical system. Let $u^i$ implement $\alpha$ on $S|_{U_i}$ where $\{U_i\}$ is an open cover of $S^{(0)}$ and let $q : (A \rtimes_{\alpha} S)^\wedge \rightarrow S^{(0)}$ be the bundle map. Then for each $i$ the map $\psi_i : \hat{p}^{-1}(U_i) \rightarrow q^{-1}(U_i)$ such that

$$\psi_i(\omega) = \pi_{\hat{p}(\omega)} \rtimes \omega \pi_{\hat{p}(\omega)}(u^i)$$

is a homeomorphism and the map $\gamma_{ij}$ such that

$$(11) \quad \gamma_{ij}(p(s))(s) = \pi_{p(s)}((u^i_j)^* u_j^i)$$

defines a continuous section of $\hat{S}$. Furthermore, these maps make $(A \rtimes S)^\wedge$ into a principal $\hat{S}$-bundle with trivialization $(U, \psi^{-1}, \gamma)$.

**Proof.** Let $(A, S, \alpha)$ be as in the statement of the theorem. Let $\{u^i\}$ implement $\alpha$ on $S|_{U_i}$ where $U_i$ is an element of some open cover $U$. Given an open set $U \in U$ we identify each of $(A(U) \rtimes S|_U)^\wedge$, $(A \rtimes S(U))^\wedge$, and $q^{-1}(U)$ with the disjoint union $\bigsqcup_{x \in U} (A(x) \rtimes S_x)^\wedge$ as in Remark 2.3. In a similar fashion we identify each of $(C^*(S(U))^\wedge)$, $C^*(S|_U)^\wedge$ and $\hat{p}^{-1}(U)$ with the disjoint union $\bigsqcup_{x \in U} \hat{S}_x$.

Now, fix $U_i \in U$. By assumption $\alpha|_{S|_{U_i}}$, denoted $\alpha$ whenever possible, is unitarily implemented by $\{u^i\}$ and as such Proposition 4.3 implies that the map $\psi_i : (S|_{U_i})^\wedge \rightarrow (A(U_i) \rtimes S|_{U_i})^\wedge$ defined via (10) is a homeomorphism. However, under the identifications made in the previous paragraph, we can view $\psi_i$ as a map from $\hat{p}^{-1}(U_i)$ onto $q^{-1}(U_i)$. We define the trivializing maps on $(A \rtimes S)^\wedge$ to be $\phi_i = \psi_i^{-1}$. What’s more, since $(A \rtimes S)^\wedge$ is locally homeomorphic to a locally compact Hausdorff space, we can conclude that $(A \rtimes S)^\wedge$ is locally compact Hausdorff.
Next, suppose $U_i, U_j \in \mathcal{U}$ and for each $x \in U_{ij}$ let $\pi_x$ be the (unique) irreducible representation of $A(x)$. On $A(x) \rtimes S_x$ both $u_i$ and $u_j$ implement $\alpha$ so that we compute, for $s \in S_x$ and $a \in A(x)$,

$$\pi_x((u_i^*)^*u_j^*) = \pi_x((u_i^*)^*u_j^*) = \pi_x(\alpha_s^{-1}(\alpha_s(a))(u_i^*)^*u_j^*) = \pi_x(a)\pi_x((u_i^*)^*u_j^*).$$

Since $\pi_x$ is irreducible, it follows [12] Lemma A.1 that $\gamma_{ij}(x)(s) := \pi_x((u_i^*)^*u_j^*)$ is a scalar. Since $u_i^*$ and $u_j^*$ are unitaries, $\gamma_{ij}(x(s))$ must be a unitary as well and therefore has modulus one. Some simple computations show that $\gamma_{ij}(x)$ is a continuous homomorphism so that $\gamma_{ij}(x) \in \hat{S}_x$. Thus $\gamma_{ij}$ is a section of $\hat{S}$ on $U_{ij}$ and we compute for $\omega \in \hat{p}^{-1}(U_{ij})$

$$\phi_i \circ \phi_j^{-1}(\omega) = \psi_i^{-1} \circ \psi_j(\omega) = \psi_i(\pi_{\hat{p}(\omega)} \rtimes \omega \pi_{\hat{p}(\omega)}(u_j^*)),$$

Given $s \in S_{\hat{p}(\omega)}$ we have

$$\pi_{\hat{p}(\omega)}((u_i^*)^*u_j^*) = \pi_{\hat{p}(\omega)}((u_i^*)^*u_j^*) = \gamma_{ij}(\hat{p}(\omega))(s)\pi_{\hat{p}(\omega)}(u_i^*).$$

Applying (13) to (12) we obtain

$$\phi_i \circ \phi_j^{-1}(\omega) = \psi_i^{-1}(\pi_{\hat{p}(\omega)} \rtimes (\omega \gamma_{ij}(\hat{p}(\omega))\pi_{\hat{p}(\omega)}(u_i^*)) = \omega \gamma_{ij}(\hat{p}(\omega))$$

This shows that the $\gamma_{ij}$ are transition functions for the $\phi_i$. It follows quickly that $\gamma_{ij}$ is continuous and that the trivialization $(\mathcal{U}, \phi, \gamma)$ makes $(A \times S)^\wedge$ into a principal $\hat{S}$-bundle.

Of course this is little more than a curiosity unless we can use the principal bundle structure to tell us something about the action $\alpha$. Fortunately, we can do just that.

**Theorem 4.5.** Suppose $S$ is an abelian group bundle with a Haar system and that $A$ has Hausdorff spectrum $S^{(0)}$. Two locally unitary actions $(A, S, \alpha)$ and $(A, S, \beta)$ are exterior equivalent if and only if $(A \rtimes_\alpha S)^\wedge$ and $(A \rtimes_\beta S)^\wedge$ are isomorphic as $\hat{S}$-bundles.

**Remark 4.6.** Before we begin the proof of Theorem 4.5 let us give a quick application to place it into context. In [4] we identified the spectrum of certain groupoid crossed products $A \rtimes G$ as a quotient of the spectrum of the stabilizer crossed product $A \rtimes S$. Coupled with Theorem 4.5 this says that if the action of $S$ on $A$ is locally unitary then the spectrum of the global crossed product $A \rtimes G$ is a quotient of a principal $\hat{S}$-bundle and as such has a cohomological invariant.

**Proof.** Suppose $\alpha$ and $\beta$ are equivalent locally unitary actions and the equivalence is implemented by the collection $\{u_s\}$. It follows from Proposition 3.7 that the map $\phi : A \rtimes_\alpha S \to A \rtimes_\beta S$ defined for $f \in \Gamma_c(S, p^*A)$ by $\phi(f)(s) = f(s)u_s^*$ is an isomorphism. As such it induces a homeomorphism $\Phi : (A \rtimes_\beta S)^\wedge \to (A \rtimes_\alpha S)^\wedge$ via the map $\Phi(\pi) = \pi \circ \phi$.

Next, let us establish some notation. Since $\alpha$ and $\beta$ are both locally trivial we may as well pass to some common refinement and assume that there exists an open cover $\mathcal{U}$ of $S^{(0)}$ such that on $S|_{U_i}$ the unitary actions $v^i$ and $w^i$ implement $\alpha$ and $\beta$, respectively. Let $\phi_i$ and $\psi_i$ be the trivializing maps induced by $v^i$ and $w^i$, respectively. Furthermore, given $x \in S^{(0)}$ let $\pi_x$ be the (unique) irreducible representation of $A(x)$ associated to $x$. Now fix $U_i \in \mathcal{U}$ and $x \in U_i$. In order to
We conclude from (14) that is a continuous section so that (\(U\) refinement of the local trivializing cover for \(w\) unitary as well. Let \(\beta\) therefore \(\beta_i(x)(s) := \pi_x(u^*_s w_s v^*_s)\) commutes with \(\pi_x(A(x))\). Since \(\pi_x\) is irreducible, this implies that \(\beta_i(x)(s)\) must be a scalar. As before, it is straightforward to show that \(\beta_i(x)\) is a continuous \(\mathbb{T}\)-valued homomorphism and hence \(\beta_i\) is a section of \(\hat{S}\) on \(U_i\). Given \(\omega \in \hat{S}_x\) we then compute for \(f \in \Gamma_e(S, p^*A)\)

\[
\pi_x \times (\omega \pi_x(w)) (\phi(f)) = \int_{S} \pi_x(f(s)) \omega(s) \pi_x(u^*_s w_s) d\beta^x (x) = \pi_x \times (\omega \beta_i(x) \pi_x(v))(f).
\]

We conclude from (14) that

\[
\phi_i \circ \Phi \circ \psi_i^{-1}(\omega) = \phi_i(\pi_x \times (\omega \pi_x(w)) \circ \phi) = \phi_i(\pi_x \times (\omega \beta_i(x) \pi_x(v)) = \omega \beta_i(x).
\]

Therefore \(\beta_i\) implements \(\Phi\) on trivializations. It is straightforward to show that \(\beta_i\) is a continuous section so that \((\mathcal{U}, \Phi, \beta)\) is an \(\hat{S}\)-bundle isomorphism.

Suppose that \((\mathcal{U}, \Phi, \beta)\) is an \(\hat{S}\)-bundle isomorphism of \((A \rtimes_{\alpha} S)\) onto \((A \rtimes_{\beta} S)\). Let \(w^i\) and \(v^i\) implement \(\alpha\) and \(\beta\), respectively. Notice that \(\mathcal{U}\) must be a common refinement of the local trivializing cover for \(\alpha\) and \(\beta\) so that we may as well assume \(w^i\) and \(v^i\) are defined on \(\mathcal{U}\). Fix \(U_i \in \mathcal{U}\) and \(x \in U_i\). For each \(s \in S_x\) we define a unitary \(u_s \in UM(A(x))\) by

\[
u_s := \beta_i(x)(s) w_s^i (v_s^i)^*.
\]

We need to show that (15) doesn’t depend on the choice of \(U_i\). So suppose \(x \in U_j\) as well. Let \(\gamma_{ij}\) and \(\eta_{ij}\) be the transition maps for \((A \rtimes_{\alpha} S)^{\wedge}\) and \((A \rtimes_{\beta} S)^{\wedge}\), respectively. It follows from general principal bundle nonsense that \(\beta_i \gamma_{ij} = \eta_{ij} \beta_i\).

We use this fact to compute

\[
\begin{align*}
\beta_i(x)(s) \pi_x (w^*_i (v^*_i)^*) &= \beta_i(x)(s) \pi_x (w^*_i (v^*_i)^*) = \beta_i(x)(s) \gamma_{ij}(x)(s) \pi_x (w^*_i (v^*_i)^*) \\
&= \beta_j(x)(s) \eta_{ij}(x)(s) \pi_x (w^*_i (v^*_i)^*) \\
&= \beta_j(x)(s) \pi_x (w^*_i (w^*_i)^*) (v^*_i)^* = \beta_j(x)(s) \pi_x (w^*_i (v^*_i)^*).
\end{align*}
\]

Since \(\hat{A}\) has Hausdorff spectrum, \(A(x)\) is simple and therefore (16) implies

\[
\beta_i(x)(s) w_s^i (v_s^i)^* = \beta_j(x)(s) w_s^j (v_s^j)^*.
\]

Thus \(u_s\) is well defined. We now show that the \(u_s\) implement an equivalence between \(\alpha\) and \(\beta\). The second condition is the result of a simple computation. The continuity condition is straightforward to prove using the fact that the actions \(v\) and \(w\) are continuous, as well as the fact that \(\beta_i\) is a continuous section. The last condition follows from the calculation

\[
\pi(\text{Ad } u_s \alpha_s(a)) = \beta_i(x)(s) \pi(w_v v_* v_s \alpha v^*_s v_* w_s^* w_s)^* = \pi(w_v w_s^*) = \pi(\beta_s(a)).
\]

Hence \(\{u_s\}\) implements an equivalence between \(\alpha\) and \(\beta\). \(\square\)

Of course, this leads to the following
Corollary 4.7. A locally unitary action of an abelian group bundle $S$ on a $C^*$-algebra $A$ with Hausdorff spectrum $S^{(0)}$ is determined, up to exterior equivalence, by the associated cohomological invariant of $(A \rtimes S)^\wedge$ as a principal $\hat{S}$-bundle. Furthermore, this cohomology class is an invariant for the isomorphism class of $A \rtimes S$.

4.2. Existence. The final piece of the puzzle will be to prove that locally unitary actions are about as abundant as they can be. In other words, we will show that every principal bundle can be obtained through a locally unitary action.

Theorem 4.8. Suppose $S$ is an abelian group bundle with a Haar system and $q : X \to S^{(0)}$ is a principal $S$-bundle. Then $C_0(X) \rtimes S$ has Hausdorff spectrum $S^{(0)}$ and the dual action of $\hat{S}$ on $C_0(X) \rtimes S$ defined for $\omega \in \hat{S}_u$ and $f \in C_c(S_u \times X_u)$ by

$$\hat{r}_\omega(f)(s,x) = \omega(s)f(s,x)$$

is locally unitary. Furthermore, $((C_0(X) \rtimes S) \rtimes \hat{S})^\wedge$ and $X$ are isomorphic $S$-bundles.

We begin by proving the following

Proposition 4.9. Suppose $S$ is an abelian group bundle with a Haar system and that $X$ is a principal $S$-bundle. Then $C_0(X) \rtimes S$ has Hausdorff spectrum $S^{(0)}$.

Proof. We know from [4, Proposition 1.2] that $C_0(X) \rtimes S$ is a $C_0(S^{(0)})$-algebra with fibres $C_0(X_u) \rtimes S_u$. Hence there is a continuous surjection $r$ of $(C_0(X) \rtimes S)^\wedge$ onto $S^{(0)}$. Furthermore, we may identify $r^{-1}(u)$ with $(C_0(X_u) \rtimes S_u)^\wedge$ in the usual fashion. Next, let $\phi : X_u \to S_u$ be the restriction of one of the trivializing maps to $X_u$. Since $\phi$ is a homeomorphism, we can pull back the group structure from $S_u$ to $X_u$ and turn $\phi$ into a group isomorphism. Furthermore, it follows from Proposition 1.9 that $\phi$ is equivariant with respect to the action of $S_u$ on $X_u$. Therefore if we identify $X_u$ with $S_u$ then the action of $S_u$ on $X_u$ becomes the action of $S_u$ on itself by translation. In other words, $C_0(X_u) \rtimes S_u$ is isomorphic to $C_0(S_u) \rtimes S_u$. We know from the Stone-von Neumann Theorem [14] that $C_0(S_u) \rtimes S_u$ is isomorphic to the compact operators on some separable Hilbert space. Hence $C_0(S_u) \rtimes S_u$ and therefore $C_0(X) \rtimes S(u)$, has a unique irreducible representation. It follows that the map $r$ is injective.

All that remains is to show that $r$ is open, or equivalently, closed. Suppose $C$ is a closed subset of $(C_0(X) \rtimes S)^\wedge$. Then there is some ideal $I$ such that $C = \{\pi \in (C_0(X) \rtimes S)^\wedge : I \subset \ker \pi\}$. Let $D = \{u \in S^{(0)} : I \subset I_u\}$ where the ideal $I_u$ in $C_0(X) \rtimes S$ is given by

$$I_u = \overline{\text{span}}\{\phi \cdot f : \phi \in C_0(S^{(0)}), f \in C_0(X) \rtimes S, \phi(u) = 0\}.$$ 

It is straightforward to show that $D = r(C)$. We will limit ourselves to proving that $D$ is closed. Suppose $u_i \to u$ in $S^{(0)}$ and $u_i \in D$ for all $i$. Then, since $I \subset I_{u_i}$ for all $i$, we have $f(u_i) = 0$ for all $i \in I$. However, $f$ is continuous when viewed as a function on $S^{(0)}$ so that $f(u) = 0$. Thus $f \in I_u$ and $u \in D$.}

Next, we show that there is a dual action of $\hat{S}$ on $C_0(X) \rtimes S$ induced by left translation. Since it isn’t much harder, we actually prove this result in greater generality. Unfortunately, we can’t just jump right in. Verifying the continuity condition will take work. In particular, we have to deal with the topology on theusc-bundle associated to $A \rtimes S$. 
Lemma 4.10. Suppose \((A, S, \alpha)\) is a dynamical system and that \(S\) is an abelian group bundle. Let \(\mathcal{A}\) be the usc-bundle associated to \(A\), define
\[
\hat{S} \ast S = \{ (\omega, s) \in \hat{S} \times S : \hat{p}(\omega) = p(s) \},
\]
and let \(p : \hat{S} \ast S \to S^{(0)}\) be given by \(p(\omega, s) = p(s)\). Then there is a map \(\iota : \Gamma_c(\hat{S} \ast S, \p^*\mathcal{A}) \to \p^*(A \times S)\) such that \(\iota(f)(\omega)(s) = f(\omega, s)\). Furthermore, \(\iota\) is continuous with respect to the inductive limit topology and the range of \(\iota\) is dense.

Proof. The only difficult part is showing that \(\iota(f)\) is continuous as a function into \(\mathcal{E}\) where \(\mathcal{E}\) is the usc-bundle associated to \(A \times S\). We start out with a simpler function. Suppose \(g \in C_c(\hat{S}), h \in C_c(S)\) and \(a \in A\). Define \(g \otimes_h a\) on \(\hat{S} \ast S\) by \(g \otimes_h a(\omega, s) = g(\omega) h(s) a(p(s))\). It is clear that \(g \otimes_h a \in \Gamma_c(\hat{S} \ast S, \p^*\mathcal{A})\).

Furthermore, if we view \(h \otimes a\) as an element of \(\Gamma_c(S, \p^*\mathcal{A})\) then \(\iota(g \otimes_h a)(\omega) = g(\omega)(h \otimes a)(\hat{p}(\omega))\) where \((h \otimes a)(\hat{p}(\omega))\) is just the restriction of \(h \otimes a\) to \(S_{\hat{p}(\omega)}\). Since \(h \otimes a\) defines a continuous section of \(\mathcal{E}\), it is easy to see that \(\iota(g \otimes_h a)\) is a continuous function from \(\hat{S}\) into \(\mathcal{E}\). Thus \(\iota(g \otimes_h a) \in \Gamma_c(\hat{S}, \p^*\mathcal{E})\).

We now show \(\iota\) preserves convergence with respect to the inductive limit topology. Suppose \(f_i \to f\) uniformly in \(\Gamma_c(\hat{S} \ast S, \p^*\mathcal{A})\) and that eventually the supports are contained in some fixed compact set \(K\). Clearly the supports of \(\iota(f)\) are eventually contained in the projection of \(K\) to \(\hat{S}\). Fix \(\epsilon > 0\) and let \(M\) be an upper bound for \(\{ \beta^n(L)\}\) where \(L\) is the projection of \(K\) to \(S\). Then eventually \(\| f_i - f \|_\infty < \epsilon/M\).

Therefore for large \(i\) we have, given \(\omega \in \hat{S}_a\) and making use of the fact that \(S_a\) is abelian so the \(I\)-norm on \(C_c(S_a, \{a\}(u))\) only has one term,
\[
\| \iota(f_i)(\omega) - \iota(f)(\omega) \| \leq \| \iota(f_i)(\omega) - \iota(f)(\omega) \|_I \\
= \int_{\hat{S}} \| f_i(\omega, s) - f(\omega, s) \| d\beta^n(s) \\
\leq \| f_i - f \|_\infty M < \epsilon
\]
Thus \(\iota(f_i) \to \iota(f)\) uniformly and hence with respect to the inductive limit topology.

Now suppose \(f \in \Gamma_c(\hat{S} \ast S, \p^*\mathcal{A})\) and that \(\omega_i \to \omega\) in \(\hat{S}\). Fix \(\epsilon > 0\) and let \(U\) and \(V\) be relatively compact neighborhoods of the projection of \(\text{supp} \ f\) to \(\hat{S}\) and \(S\), respectively. Since sums of elementary tensors are dense [3 Proposition 1.3] we may find \(\{g_i\}_{i=1}^N \in C_c(\hat{S} \ast S)\) and \(\{a_i\}_{i=1}^N \in A\) such that \(\| f - \sum_i g_i \otimes a_i \|_\infty < \epsilon/2\).

For each \(1 \leq i \leq N\) extend \(g_i\) to all of \(C_c(\hat{S} \times S)\) and choose \(h_i^j \in C_c(\hat{S})\) and \(k_i^j \in C_c(S)\) such that \(\| g_i - \sum_j h_i^j \otimes k_i^j \|_\infty < \epsilon/(2N\|a_i\|)\). It then follows from some simple computations that
\[
\left\| f - \sum_{i=1}^N \sum_j h_i^j \otimes k_i^j \otimes a_i \right\|_\infty < \epsilon/2 + \sum_{i=1}^N \|a_i\| \left\| g_i - \sum_j h_i^j \otimes k_i^j \right\|_\infty < \epsilon.
\]
Furthermore, we can multiply the \(h_i^j\) and \(k_i^j\) by functions which vanish off \(U\) and \(V\), respectively, so that \(\text{supp} \ h_i^j \otimes k_i^j \otimes a \subset \overline{U} \times \overline{V}\). This construction shows that sums of elements of the form \(h \otimes k \otimes a\) for \(h \in C_c(\hat{S}), k \in C_c(S)\), and \(a \in A\) are dense in \(\Gamma_c(\hat{S} \ast S, \p^*\mathcal{A})\) with respect to the inductive limit topology.

At last we can show that \(\iota(f)\) is continuous for \(f \in \Gamma_c(\hat{S} \ast S, \p^*\mathcal{A})\). Let \(g_i = \sum_k h_i^k \otimes k_i^k \otimes a_i^k\) be a sequence converging to \(f\) in the inductive limit topology as above. Since sums of continuous functions are continuous, \(\iota(g_i)(\omega_j) \to \iota(g_i)(\omega)\) for
all \( i \) and it now follows from a straightforward application of \cite{15} Proposition C.20 that \( \iota(f)(\omega_i) \to \iota(f)(\omega) \). Thus \( \iota(f) \) is a continuous section. Showing that \( \iota \) has dense range now follows from \cite{15} Proposition C.24 after a brief argument. \( \square \)

The following corollary isn’t necessary to build the dual action, but it will be needed in the proof of Theorem \ref{4.8} so we include it here.

**Corollary 4.11.** Suppose \( S \) is an abelian group bundle with a Haar system and \( q : X \to S^{(0)} \) is a principal \( S \)-bundle. Define
\[
\hat{S} \ast S \ast X := \{ (\omega, s, x) \in \hat{S} \times S \times X : \hat{p}(\omega) = p(s) = q(x) \}
\]
Then there is a map \( \iota : C_c(\hat{S} \ast S \ast X) \to \hat{p}^*(C_0(X) \rtimes S) \) such that
\[
\iota(f)(\omega)(s)(x) = f(\omega, s, x).
\]
Furthermore, \( \iota \) is continuous with respect to the inductive limit topology and the range of \( \iota \) is dense.

**Proof.** Let \( \mathcal{C} \) be the usc-bundle associated to \( C_0(X) \) as a \( C_0(S^{(0)}) \)-algebra. Consider the map \( \iota_1 : \Gamma_c(\hat{S} \ast S, p^*\mathcal{C}) \to \hat{p}^*(C_0(X) \rtimes S) \) given by \( \iota_1(f)(\omega)(s)(x) := f(\omega, s, x) \). It follows from Lemma \ref{4.10} that this map is continuous with respect to the inductive limit topology and its range is dense in \( \hat{p}^*(C_0(X) \rtimes S) \). Now consider the map \( \iota_2 : C_c(\hat{S} \ast S \ast X) \to \Gamma_c(\hat{S} \ast S, p^*\mathcal{C}) \) given by \( \iota_2(f)(\omega, s)(x) = f(\omega, s, x) \). It follows from Lemma \ref{2.4} that \( \iota_2 \) is surjective and preserves the inductive limit topology. Thus the map \( \iota = \iota_2 \circ \iota_1 \) has the correct form and all the right properties. \( \square \)

Now we can finally tackle the dual action construction. This will provide the last tool we need to demonstrate Theorem \ref{4.8}

**Proposition 4.12.** Suppose \((A, S, \alpha)\) is a dynamical system and that \( S \) is an abelian group bundle with a Haar system. Then for each \( \omega \in \hat{S} \) there is an automorphism \( \hat{\alpha}_\omega \) on \( A \times S(\hat{p}(\omega)) \) defined for \( f \in C_c(S_{\hat{p}(\omega)}, A(\hat{p}(\omega))) \) by
\[
\hat{\alpha}_\omega(f)(s) = \omega(s)f(s).
\]
With this action \((A \times S, \hat{S}, \hat{\alpha})\) is a dynamical system.

**Proof.** Since everything else is straightforward, we will limit ourselves to demonstrating the continuity of the action. Let \( E \) be the bundle associated to \( A \times S \) and suppose \( \omega_i \to \omega \) in \( \hat{S} \) and \( f_i \to f \) in \( E \) such that \( f_i \in A \times S(\hat{p}(\omega_i)) \) for all \( i \). Now choose \( g \in \hat{p}^*(A \times S) \) such that \( g(\omega) = f \). It follows from Lemma \ref{4.10} that we can choose \( h \in \Gamma_c(\hat{S} \ast S, p^*A) \) such that \( \|\iota(h) - g\|_\infty < \epsilon/2 \). Define \( \alpha(h)(\omega, s) = \omega(s)h(\omega, s) \). It is clear that \( \alpha(h) \in \Gamma_c(\hat{S} \ast S, p^*A) \). It is also easy to see that \( \iota(\alpha(h))(\omega) = \alpha_\omega(\iota(h)(\omega)) \). Thus \( \|\iota(\alpha(h))(\omega) - \alpha_\omega(f)\| = \|\alpha_\omega(\iota(h)(\omega)) - g(\omega)\| < \epsilon/2 < \epsilon \). Next, since \( g(\omega_i) \to g(\omega) = f \) and \( f_i \to f \) we have \( \|g(\omega_i) - f_i\| \to 0 \). Therefore, eventually, we have
\[
\|\iota(\alpha(h))(\omega_i) - \alpha_\omega(f_i)\| \leq \|\alpha_\omega(\iota(h)(\omega_i) - g(\omega_i))\| + \|\alpha_\omega(g(\omega_i) - f_i)\| \leq \epsilon/2 + \|g(\omega_i) - f_i\| < \epsilon.
\]
It follows from \cite{15} Proposition C.20 that \( \alpha_\omega(f_i) \to \alpha_\omega(f) \). \( \square \)

**Remark 4.13.** The action from Proposition \ref{4.12} is a generalization of the usual Takai dual action for abelian groups.
We are now ready to prove our existence theorem.

**Proof of Theorem 4.8.** We have shown in Proposition 4.13 that \( C_0(X) \times S \) has Hausdorff spectrum \( S^{(0)} \). Furthermore, we showed in Proposition 4.12 that there is an action of \( \hat{S} \) which, if we view \( C_c(S_u \times X_u) \) as sitting densely inside \( C_c(S_u, C_0(X_u)) \), is given by

\[
\hat{r}_\omega(f)(s,x) = \omega(s)f(s,x)
\]

for \( f \in C_c(S_u \times X_u) \). We need to show that \( \hat{r} \) is locally unitary. Let \( U \) be a trivializing cover of \( X \) and let \( \phi_i \) be the local trivializations. Fix \( U_i \in U \). Then for all \( \omega \in \hat{S}_w \), and \( f \in C_c(S_w \times X_w) \) define

\[
(18) \quad u_\omega f(s,x) := \omega(\phi_i(x))f(s,x).
\]

Simple calculations show that \( u \) is a homomorphism on \( S_w \). Next we will show that \( u \) is adjointable. Equip \( C_0(\mathbb{X}_w) \times S_w \) with its usual inner product as a Hilbert module. For all \( f, g \in C_c(S_w \times X_w) \) we have

\[
\langle u_\omega f, g \rangle(s,x) = \int_S \omega(\phi_i(t^{-1} \cdot x))f(t^{-1}, t^{-1} \cdot x)g(t^{-1}s, t^{-1} \cdot x)d\beta^w(t) = \langle f, u_\omega^- g \rangle(s,x).
\]

This shows that \( u_\omega \) is adjointable on \( C_c(S_w \times X_w) \) and we can also observe that

\[
\|u_\omega f\|^2 = \|\langle u_\omega f, u_\omega f \rangle\| = \|\langle f, u_\omega^- u_\omega f \rangle\| = \|\langle f, f \rangle\| = \|f\|^2.
\]

Thus \( u_\omega \) is isometric on \( C_c(S_w \times X_w) \) and as such it can be extended to a unitary multiplier on \( C_0(\mathbb{X}_w) \times S_w \). All that remains for the collection \( \{u_\omega\} \) to define a unitary action of \( \hat{p}^{-1}(U_i) \) on \( C_0(X) \times S(U_i) \) is continuity.

Let \( E \) be the bundle associated to \( C_0(X) \times S \) and fix \( \epsilon > 0 \). Suppose \( \omega_j \to \omega \) in \( \hat{p}^{-1}(U_i) \) and \( f_j \to f \) in \( E|U_i \) such that \( f_j \in C_0(X) \times S(\hat{p}(\omega_j)) \) for all \( j \). Choose \( g \in \hat{p}^*(C_0(X) \times S) \) such that \( g(\omega) = f \). Using Corollary 4.11 we can find a continuous, compactly supported function \( \psi \) on \( \hat{S} \cap \hat{S} \times X \) such that \( \|\epsilon(\psi) - \epsilon\|_\infty < \epsilon/2 \). Consider the open set \( O = \hat{p}^{-1}(U_i) \ast \hat{p}^{-1}(U_i) \ast q^{-1}(U_i) \) in \( \hat{S} \ast \hat{S} \times X \). We define a new function \( k \in C(O) \) by

\[
k(\chi, s, x) = \psi(p(s))\chi(\phi_i(x))h(\chi, s, x)
\]

where \( \psi \in C_c(U_i) \) is some function which is one on a neighborhood of \( \hat{p}(\omega) \). Now \( k \) is clearly compactly supported with \( \text{supp} \ k \subseteq O \). Therefore we can, and do, extend \( k \) by zero to all of \( \mathbb{S} \ast \mathbb{S} \times X \). Next we observe the following facts. First, \( \iota(k)(\omega) = u_\omega^{-}k(\omega) \), and eventually \( \iota(k)(\omega_j) = u_{\omega_j}^{-}\iota(k)(\omega_j) \). Second, that

\[
\|\iota(k)(\omega) - u_\omega f\| = \|u_\omega \iota(k)(\omega) - g(\omega)\| = \|\iota(h)(\omega) - g(\omega)\| < \epsilon/2.
\]

Third, \( f_i \to f \) and \( g(\omega_i) \to g(\omega) = f \) so that \( \|f_i - g(\omega_i)\| \to 0 \). Thus, eventually, we have

\[
\|\iota(k)(\omega_i) - u_{\omega_i}f_i\| \leq \|u_{\omega_i}(\iota(h)(\omega_i) - g(\omega_i))\| + \|u_{\omega_i}(g(\omega_i) - f_i)\|
\]

\[
< \epsilon/2 + \|g(\omega_i) - f_i\| < \epsilon.
\]

Finally, we observe that \( \iota(k)(\omega_i) \to \iota(k)(\omega) \) since \( \iota(k) \) is a continuous section. It follows from [15, Proposition C.20] that \( u_\omega f_i \to u_\omega f \) and hence \( \{u_\omega\} \) defines a unitary action of \( \hat{p}^{-1}(U_i) \) on \( C_0(X) \times S(U_i) \). What’s more, the calculation

\[
u_\omega f u_\omega^*(s,x) = \overline{\omega(\phi_i(x))} u_\omega f^*(s^{-1}, s^{-1} \cdot x)
\]
shows that \( u \) implements \( \hat{\rho}^{-1}(U_1) \). Since we performed this construction for an arbitrary element of the cover \( U \), it follows that \( \hat{\rho} \) is locally unitary.

Consider \( Y = ((C_0(X) \times S) \times \hat{S})^\wedge \). Now, \( Y \) is a principal \( S \)-bundle and, in light of duality [4, Theorem 16], a principal \( S \)-bundle as well. We would like to show that \( Y \) is isomorphic to \( X \). Using Theorem 1.6 it suffices to show that \( X \) and \( Y \) have the same cohomological invariant. Let \( \gamma_{ij} \) be the transition functions for \( X \) with respect to the trivializing maps \( \phi_i \). Let \( \eta_{ij} \) be the transition functions for \( Y \) and recall from Theorem 1.3 that we have \( \eta_{ij}(v)(\omega) = \pi_{\nu}(u_v^*)^{-1} u_{v^*} \) where \( \pi_{\nu} \) is the unique irreducible representation of \( A(v) \) and \( u_v^*, u^*_{v^*} \) are the unitaries constructed above. It follows from general principal bundle nonsense that \( \phi_i(x) = \gamma_{ij}(x)\phi_j(x) \) for all \( x \). We now compute for \( f \in C_c(S \times X \times V) \)

\[
((u_v^*)^* u_{v^*} f)(s, x) = \omega^{-1}(\phi_i(x)) \omega(\phi_j(x)) f(s, x) = \omega(\gamma_{ij}(v) \phi_j(x)) \omega(\phi_j(x)) f(s, x)
\]

where \( \gamma_{ij}(v) \) denotes the image of \( \gamma_{ij}(v) \) in the double dual. Therefore, since \( \pi_{\nu} \) is faithful, \( \eta_{ij}(v)(\omega) = \gamma_{ij}(v)(\omega) \). Thus, once we identify \( S \) with \( \hat{S} \), the cohomological invariants of \( X \) and \( Y \) are identical. \( \square \)

Example 4.14. Theorem 1.3 says that any principal \( S \)-bundle \( X \) gives rise to a locally unitary action of \( \hat{S} \) on \( C_0(X) \times S \) and in particular this holds for locally \( \sigma \)-trivial bundles. Thus any example in [4] yields a locally unitary action.

Remark 4.15. It is worth describing, at least briefly, how this material generalizes [8]. Suppose \( H \) is an abelian group and \( A \) has Hausdorff spectrum \( X \). If \( \alpha \) is an action of \( H \) on \( A \) then, as in Example 1.4, we can form the transformation groupoid \( H \times X \) and there is an action \( \beta \) of \( H \times X \) on \( A \). Furthermore, we have \( A \rtimes_{\alpha} H \cong A \rtimes_{\beta} (H \times X) \). Without getting into the details, \( \alpha \) is locally unitary according [8] if, for each \( \pi \in X \), there is an open neighborhood \( U \) of \( \pi \) and a strictly continuous map \( u: H \to M(A) \) such that for each \( \rho \in U \), \( \hat{u} \circ u \) is a representation of \( H \) on \( H_\pi \) which implements \( \alpha \). In particular this implies that \( \rho \circ \alpha^{-1} = \hat{u} \circ u \circ \rho \circ \alpha^{-1} \) is equivalent to \( \rho \). Thus the action of \( H \) on \( X \) induced by \( \alpha \) is trivial and the transformation groupoid \( H \times X \) is the trivial group bundle. What’s more, since \( A \) has Hausdorff spectrum, it is not hard to show that \( u_s(x) \) implements \( \beta(s, x) \) on \( A(x) \) and that \( \beta \) is unitarily implemented by \( \{ u_s(x) \} \) on \( H \times X \). Thus \( \beta \) is a locally unitary action of \( H \times S \) on \( A \). Now, the dual of \( H \times X \) is \( \hat{H} \times X \) and we have, according to Theorem 1.3, that \( (A \rtimes_{\alpha} H)^\wedge \cong (A \rtimes_{\beta} (H \times X))^\wedge \) is a principal \( \hat{H} \times X \) bundle. However, it follows from Remark 1.2 that this implies \( (A \rtimes_{\alpha} H)^\wedge \) is a principal \( \hat{H} \)-bundle. From here it is straightforward to see how the results of this section are related to those in [8].

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