Energy condition in Rastall gravity theory

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Abstract. Modified gravity is believed to be a good candidate in explaining unknown components such as dark energy and dark matter. Recently, The Rastall gravity one of the modified gravity, provide the interesting result that dark energy may cluster. Some of the modified gravity use several parameters whose value must be constrained from experimental results or existing theories. In this context, energy condition is expected to be able to constrain this parameter. In our study, we are able to limit the parameter in the range $\lambda \leq 0$ and $\lambda \geq 1$ in the context stellar matter(where $\lambda$ is Rastall parameter). We display the results in the graph of the equation of state parameter $w$ againts Rastall parameter $\lambda$. So this results can be generally used in studying cosmology or stellar objects.

1. Introduction

The facts about accelerating the universe provide the new research for physicists. What mysterious force that contributed to the acceleration of the universe is still questionable. Various models have been proposed, including $\Lambda$CDM models which involve unknown components namely dark energy. In this case, the equation of state of dark energy is $w = p/\rho$, where $p$ and $\rho$ are pressure and density of dark energy, respectively[1]. The modified gravity is also a good candidate with the aim of explaining the acceleration of the universe. Mostly, one starts from the curvature description of gravity. The modification of gravity is obtained from modifying the Einstein-Hilbert Lagrangian with additional curvature terms, for example is $f(R)$ gravity (See [2,3] and the references therein).

On the other hand, there is also a modification of gravity is not started from the Lagrangian, but only merely from an assumption. For example, The Rastall gravity theory [4,5] which assumes that the covariant divergence of the stress-energy tensor is non-zero but proportional with curvature scalar i.e. $T^\mu_{\nu,\mu} \propto R_{\nu}$. One needs a parameter to change the sign of proportionality to an equal sign, which the parameter is called the Rastall parameter. The value of parameter must be constrained from experimental results or existing theories. In this paper, we try to find a range of parameter values that are permitted by energy conditions.

The presence of energy condition tries to capture the idea that energy should be positive [6-10]. The energy condition is often in proving various theorems such as the law of thermodynamics of black holes. The Raychaudhuri equation is used as a reference forms the energy condition. This equation represent by the positivity condition, which is given by the equation $R_{\mu\nu}k^\mu k^\nu \geq 0$, where $k^\mu$ and $R_{\mu\nu}$ are any null vector and Ricci tensor, respectively.
2. Formalism

Generally, the energy condition arises when one studies the Raychaudhuri equation given by

\[
\frac{d\theta}{d\tau} = -\frac{1}{2} \theta^2 - \sigma^2 + \omega^2 - R_{\mu\nu}k^\mu k^\nu,
\]

(1)

where \(\theta, \sigma_{\mu\nu}\) and \(\omega_{\mu\nu}\) are the expansion, shear and rotation, respectively. The equation will be satisfied if \(R_{\mu\nu}k^\mu k^\nu \geq 0\) is applied. Since the condition for attractive gravity, namely \(\frac{d\theta}{d\tau} < 0\). Also \(\sigma^2 \geq 0\) and \(\omega_{\mu\nu}\) for spatial shear tensor and hypersurface ortogonal congruences, respectively.

In the general relativity and using the inequality equation, one can rewrite condition in the form of energy momentum tensor given by

\[
\mathcal{T}_{\mu\nu} \geq 0.
\]

However, in another theory of gravity, one must be able to form their equation into like Einstein equation (effective form) with the modified stress-energy tensor. The field and divergence of energy-momentum tensor equations in Rastall gravity theory are given by

\[
R_{\mu\nu} - \frac{\lambda}{2} g_{\mu\nu} R = 8\pi G \mathcal{T}_{\mu\nu},
\]

(2)

\[
\mathcal{T}_{\mu\nu}^v = \left(1 - \frac{\lambda}{16\pi G}\right)R_{\mu\nu},
\]

(3)

where \(\lambda\) is the Rastall parameter, which \(\lambda = 1\) the usual general relativity theory is recovered. In this paper, we assume that the matter content of an ideal fluid which is contained in the energy-momentum tensor

\[
\mathcal{T}_{\mu\nu} = p g_{\mu\nu} + (\rho + p) u_\mu u_\nu,
\]

(4)

where \(u_\mu\) is the 4-velocity and \(g_{\mu\nu}\) is the component of metric tensor. The quantities \(p\) and \(\rho\) represent the pressure and the density of the fluid. We can calculate the effective forms in the context of ideal fluid, by defining \(p\) and \(\rho\) are effective pressure and density, respectively. The new effective quantities \(p\) and \(\rho\) are related to \(p\) and \(\rho\) by

\[
p = \frac{1}{2} \left(\frac{\lambda - 1}{2\lambda - 1}\right)\rho + \frac{1}{2} \left(\frac{\lambda + 1}{2\lambda - 1}\right) p,
\]

(5)

\[
\rho = \frac{1}{2} \left(\frac{3\lambda - 1}{2\lambda - 1}\right)\rho + \frac{1}{2} \left(\frac{3\lambda - 3}{2\lambda - 1}\right) p.
\]

(6)

It is worth noting that, in the above notation, the field and divergence of energy-momentum tensor equations can be written by

\[
R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G \mathcal{F}_{\mu\nu},
\]

(7)

\[
\mathcal{F}_{\mu\nu} = 0,
\]

(8)

where effective energy-momentum tensor \(\mathcal{F}_{\mu\nu}\) is given by

\[
\mathcal{F}_{\mu\nu} = p g_{\mu\nu} + (\rho + p) u_\mu u_\nu.
\]

(9)

We can obtain the energy condition as follows

\[
\rho + p \geq 0, \text{for NEC}
\]

\[
\rho \geq 0 \land \rho + p \geq 0, \text{for WEC}
\]

\[
\rho + 3p \geq 0 \land \rho + p \geq 0, \text{for SEC}
\]

\[
\rho \geq 0 \land \rho \pm p \geq 0, \text{for DEC}
\]
Here we use the symbols NEC, WEC, SEC and DEC for the null, weak, strong and dominant energy conditions, respectively. We can also manipulate the inequalities above becomes the inequalities between parameter of state \( w = \frac{p}{\rho} \) and parameter of Rastall \( \lambda \).

\[
\begin{align*}
\frac{1}{2} \left( \frac{3\lambda - 1}{2\lambda - 1} \right) + \frac{1}{2} \left( \frac{3\lambda - 3}{2\lambda - 1} \right) w & \geq -1, \text{ for NEC} \\
\frac{1}{2} \left( \frac{3\lambda - 1}{2\lambda - 1} \right) + \frac{1}{2} \left( \frac{3\lambda - 3}{2\lambda - 1} \right) w & \geq 0, w \geq -1, \text{ for WEC} \\
\frac{1}{2} \left( \frac{3\lambda - 1}{2\lambda - 1} \right) + \frac{1}{2} \left( \frac{3\lambda - 3}{2\lambda - 1} \right) w & \geq 0, w \geq -1, \text{ for SEC} \\
\frac{1}{2} \left( \frac{3\lambda - 1}{2\lambda - 1} \right) + \frac{1}{2} \left( \frac{3\lambda - 3}{2\lambda - 1} \right) w & \geq 0, w \geq -1, \text{ for DEC}
\end{align*}
\]

3. Results and Discussion

3.1 Null Energy Condition

It should be noted that \( \rho + p \) does not depend on the Rastall parameter or it can be written as \( \rho + p = \rho + p \). Therefore we can’t use null energy condition because this condition can’t restrict the Rastall parameter \( \lambda \).

![Figure 1](image-url)  

**Figure 1.** Plot of the weak energy condition in parameter equation of state \( w \) versus parameter of Rastall \( \lambda \)

3.2 Weak Energy Condition (WEC)

Plot of the weak energy condition in equation of state parameter \( w \) against parameter of Rastall \( \lambda \) is shown on the Figure 1. The blue area indicates the region allowed by weak energy condition and outside the blue area indicates the region prohibited by weak energy condition. If we compare with ultra-relativistic matter equation of state (for example matter in the very early universe or radiation) which equation of state parameter is one-third, all of the Rastall parameter except \( \lambda = \frac{1}{2} \) will be used in the calculation because they do not violate the weak energy condition. On the other hand, if our calculations use the equation of state of non-relativistic matter (for example cold-dust) which equation of state parameter is zero, Rastall parameter that are permitted are \( \lambda \leq \frac{1}{3} \) and \( \lambda > \frac{1}{2} \). As for dark energy equation of state which equation of state parameter is zero, Rastall parameter that are permitted are
\( \lambda > \frac{1}{2} \). When we deal with stellar matter, we always expect that the equation of state parameter must be greater than zero, because we expect the pressure and energy density to always be positive. So that in this case, it is better to choose \( \lambda < \frac{1}{2} \) and \( \lambda \geq 1 \) because the Rastall parameters in this range are always in blue area (the area allowed by WEC).

### 3.3 Strong Energy Condition (SEC)

Plot of the strong energy condition in equation of state parameter \( w \) against parameter of Rastall \( \lambda \) is shown on the Figure 2. As with WEC, for ultra-relativistic matter equation of state, all of the Rastall parameter except \( \lambda = \frac{1}{2} \) are allowed. Otherwise, if we use non-relativistic matter equation of state, Rastall parameter that are permitted are \( \lambda \geq \frac{2}{3} \) and \( \lambda < \frac{1}{2} \) and for dark energy, Rastall parameter that are permitted are \( \lambda < \frac{1}{2} \). Whereas when we deal with stellar matter, it is better to choose \( \lambda > \frac{1}{2} \) and \( \lambda \leq 0 \) because the Rastall parameters in this range are always in blue area (the area allowed by SEC).

![Figure 2](image.png)

**Figure 2.** Plot of the strong energy condition in parameter equation of state \( w \) versus parameter of Rastall \( \lambda \)

### 3.4 Dominant Energy Condition (DEC)

Plot of the dominant energy condition in equation of state parameter \( w \) against parameter of Rastall \( \lambda \) is shown on the Figure 3. As with WEC and SEC, for ultra-relativistic matter equation of state, all of the Rastall parameter except \( \lambda = \frac{1}{2} \) are allowed. Otherwise, if we use non-relativistic matter equation of state, Rastall parameter that are permitted are \( \lambda \leq 0 \) and \( \lambda > \frac{1}{2} \) and for dark energy, Rastall parameter that are permitted are \( \lambda > \frac{1}{2} \). Whereas when we deal with stellar matter, it is better to choose \( \lambda \geq 1 \) and \( \lambda \leq 0 \) because the Rastall parameters in this range are always in blue area (the area allowed by SEC).
In this paper, we study energy condition to restrict the Rastall parameter as summarized in Table 1. From the table it is known that for the stellar matter, the region parameters are allowed from the third energy condition is \( \lambda \geq 1 \) and \( \lambda \leq 0 \).

### Table 1. Several energy condition in the context Rastall gravity

| Ultra-relativistic matter (e.g. radiation) | Non-relativistic matter (e.g. cold-dust) | Dark energy | Stellar matter |
|-------------------------------------------|------------------------------------------|-------------|----------------|
| \( \lambda \in \text{Reals} \land \lambda \neq \frac{1}{2} \) | \( \lambda \in \text{Reals} \land \lambda \neq \frac{1}{2} \) | \( \lambda \in \text{Reals} \land \lambda \neq \frac{1}{2} \) | \( \lambda \in \text{Reals} \land \lambda \neq \frac{1}{2} \) |
| NEC \( \lambda \leq \frac{1}{3} \land \lambda > \frac{1}{2} \) | \( \lambda > \frac{1}{2} \) | \( \lambda < \frac{1}{2} \) | \( \lambda < \frac{1}{2} \land \lambda \geq 1 \) |
| WEC \( \lambda \leq \frac{1}{3} \land \lambda > \frac{1}{2} \) | \( \lambda > \frac{1}{2} \) | \( \lambda < \frac{1}{2} \) | \( \lambda > \frac{1}{2} \land \lambda \leq 0 \) |
| SEC \( \lambda \geq \frac{2}{3} \land \lambda < \frac{1}{2} \) | \( \lambda < \frac{1}{2} \) | \( \lambda > \frac{1}{2} \) | \( \lambda \geq 1 \land \lambda \leq 0 \) |
| DEC \( \lambda \leq 0 \land \lambda > \frac{1}{2} \) | \( \lambda > \frac{1}{2} \) | \( \lambda < \frac{1}{2} \) | \( \lambda > \frac{1}{2} \land \lambda \leq 0 \) |

#### 4. Conclusion
We have studied the energy condition to restricts the Rastall parameter. The region parameters for the stellar matter are allowed from the third energy condition is \( \lambda \geq 1 \) and \( \lambda \leq 0 \). On the other hand, for ultra-relativistic matter, energy condition is not able to limit the parameters which all of the Rastall parameter except \( \lambda = \frac{1}{2} \). In contrast to stellar matter and ultra-relativistic matter, none of the parameters remain from limiting energy condition for dark energy and non-relativistic matter.
Acknowledgments
We are supported by the UI’s PITTA grant No. 622/UN2.R3.1/HKP.05.00/2017.

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