Emergent behaviors of continuous and discrete thermomechanical Cucker-Smale models on general digraphs

Jiu-Gang Dong
Department of Mathematics, Harbin Institute of Technology, Harbin 150001, China
jgdong@hit.edu.cn

Seung-Yeal Ha
Department of Mathematical Sciences and Research Institute of Mathematics
Seoul National University, Seoul 08826, and
Korea Institute for Advanced Study, Hoegiro 85, Seoul, 02455, Korea (Republic of)
syha@snu.ac.kr

Doheon Kim
Department of Mathematical Sciences, Seoul National University,
Seoul 08826, Korea (Republic of), dohun930728@snu.ac.kr

We present emergent dynamics of continuous and discrete thermomechanical Cucker-Smale (TCS) models equipped with temperature as an extra observable on general digraph. In previous literature, the emergent behaviors of the TCS models were mainly studied on a complete graph, or symmetric connected graphs. Under this symmetric setting, the total momentum is a conserved quantity. This determines the asymptotic velocity and temperature a priori using the initial data only. Moreover, this conservation law plays a crucial role in the flocking analysis based on the elementary $\ell_2$ energy estimates. In this paper, we consider a more general connection topology which is registered by a general digraph, and the weights between particles are given to be inversely proportional to the metric distance between them. Due to this possible symmetry breaking in communication, the total momentum is not a conserved quantity, and this lack of conservation law makes the asymptotic velocity and temperature depend on the whole history of solutions. To circumvent this lack of conservation laws, we instead employ some tools from matrix theory on the scrambling matrices and some detailed analysis on the state-transition matrices. We present two sufficient frameworks for the emergence of mono-cluster flockings on a digraph for the continuous and discrete models. Our sufficient frameworks are given in terms of system parameters and initial data.

Keywords: Digraph, emergence, energy estimate, scrambling matrices, state-transition matrices, thermomechanical Cucker-Smale particles

AMS Subject Classification: 39A11, 39A12, 34D05, 68M10

1. Introduction

Collective behaviors of many-particle systems are ubiquitous in our nature, e.g., flocking of birds, flashing of fireflies, swarming of fishes and herding of sheep,
etc [3, 4, 37, 49, 51]. Among the diverse collective behaviors, our concern lies on the flocking which denotes some concentration phenomenon in velocity, in which particles move with the same velocity asymptotically only using the simple environment information and basic rules. Motivated by the seminal contributions [46, 50] on the flocking modeling by Reynolds and Vicsek et al, several mechanical models were proposed in literature in diverse disciplines such as applied mathematics, control theory of multi-agent system and statistical physics. Among them, our main interest lies on the flocking models proposed by Cucker and Smale [18]. In this paper, we are mainly interested in the thermodynamic Cucker-Smale model which generalizes the classical Cucker-Smale model by adding internal temperature variables. More precisely, let $x_i, v_i, \theta_i$ be the position, velocity and temperature of the $i$-th Cucker-Smale particles. Then, dynamics of these macroscopic observables is governed by the Cauchy problem to the continuous thermodynamic Cucker-Smale model introduced in [35]:

$$\frac{dx_i}{dt} = v_i, \quad t > 0, \quad i = 1, 2, \cdots, N,$$

$$\frac{dv_i}{dt} = \frac{1}{N} \sum_{j=1}^{N} \chi_{ij} \phi(\|x_i - x_j\|) \left(\frac{v_j - v_i}{\theta_j} - \frac{v_i}{\theta_i}\right),$$

$$\frac{d\theta_i}{dt} = \frac{1}{N} \sum_{j=1}^{N} \chi_{ij} \zeta(\|x_i - x_j\|) \left(\frac{1}{\theta_i} - \frac{1}{\theta_j}\right),$$

subject to the initial data:

$$(x_i(0), v_i(0), \theta_i(0)) = (x_i^{in}, v_i^{in}, \theta_i^{in}), \quad i = 1, 2, \cdots, N$$

where $\|\cdot\|$ denotes the standard $\ell^2$-norm in $\mathbb{R}^d$. Here network topology ($\chi_{ij}$) is given as follows:

$$\chi_{ij} = \begin{cases} 1, & \text{if } j \text{ transmits information to } i, \\ 0, & \text{otherwise}, \end{cases}$$

Here, we assume that the directed graph corresponding to the network topology ($\chi_{ij}$) has at least one spanning tree. And we also assume $\chi_{ii} = 1$ for $i = 1, \cdots, N$, for some technical reason. The communication weights $\phi, \zeta : \mathbb{R}_+ \cup \{0\} \to \mathbb{R}_+$ appearing in the R.H.S. of (1.1) are assumed to be bounded, Lipschitz continuous positive non-increasing functions defined on the nonnegative real numbers:

$$0 < \phi(r) \leq \phi(0) =: \kappa_1, \quad 0 < \zeta(r) \leq \zeta(0) =: \kappa_2, \quad r \geq 0,$$

$$(\phi(r) - \phi(s))(r - s) \leq 0, \quad (\zeta(r) - \zeta(s))(r - s) \leq 0, \quad r, s \geq 0.$$ 

Note that if we choose all initial temperatures $\theta_i^{in}$ to have the same value, say unity, then it is easy to see that $\theta_i(t) \equiv 1, \quad i = 1, \cdots, N$ and system (1.1) reduces to the
Discrete and continuous thermomechanical Cucker-Smale models on general digraph

Cucker-Smale (CS) model \[18\] on a digraph:

\[
\begin{align*}
\frac{dx_i}{dt} &= v_i, \quad t > 0, \quad i = 1, 2, \ldots, N, \\
\frac{dv_i}{dt} &= \frac{1}{N} \sum_{j=1}^{N} \chi_{ij} \phi(\|x_i - x_j\|)(v_j - v_i).
\end{align*}
\]

The CS model has received lots of attention in literature from different perspectives, i.e., global and local flocking \([8, 18, 19, 32, 34, 36]\), collision avoiding \([1, 16]\), time-delay effect \([9, 10, 21, 24, 27, 44]\), hierarchical and rooted leadership, general digraph \([20, 38, 39, 47]\), application to flight navigation \([43]\), noisy effects \([2, 17, 23, 33]\), mean-field limit \([5, 32, 34]\), kinetic and hydrodynamic description \([6, 7, 25, 36, 45]\), and variants of C-S model \([41, 42]\), etc. (see recent survey papers \([14, 40]\) for details).

Compared to the above vast literature on the C-S model, there were very few works for the TCS model, e.g., global flocking \([31, 35]\), flocking effect by singular communication \([11]\), hydrodynamic TCS model \([29, 30]\). Next, we consider the discrete analogue of (1.1):

\[
\begin{align*}
x_i[t+1] &= x_i[t] + hv_i[t], \quad t \in \mathbb{N} \cup \{0\}, \quad i = 1, 2, \ldots, N, \\
v_i[t+1] &= v_i[t] + \frac{h}{N} \sum_{j=1}^{N} \chi_{ij} \phi(\|x_i[t] - x_j[t]\|) \left(\frac{v_j[t]}{\theta_j[t]} - \frac{v_i[t]}{\theta_i[t]}\right), \\
\theta_i[t+1] &= \theta_i[t] + \frac{h}{N} \sum_{j=1}^{N} \chi_{ij} \zeta(\|x_i[t] - x_j[t]\|) \left(\frac{1}{\theta_i[t]} - \frac{1}{\theta_j[t]}\right),
\end{align*}
\]

with the initial data

\[
(x_i[0], v_i[0], \theta_i[0]) = (x_i^{in}, v_i^{in}, \theta_i^{in}), \quad i = 1, 2, \ldots, N.
\]

Here \(h > 0\) denotes the time-step.

Before, we discuss our main results obtained in this paper, we recall the concept of mono-cluster flocking as follows. We first set \(X, V\) and \(\Theta\):

\[
\begin{align*}
X &:= (x_1, \ldots, x_N), \quad V := (v_1, \ldots, v_N), \quad \Theta := (\theta_1, \ldots, \theta_N), \\
D(X(t)) &:= \max_{1 \leq i, j \leq N} \|x_i(t) - x_j(t)\|, \quad D(V(t)) := \max_{1 \leq i, j \leq N} \|v_i(t) - v_j(t)\|, \\
D(\Theta(t)) &:= \max_{1 \leq i, j \leq N} |\theta_i(t) - \theta_j(t)|.
\end{align*}
\]

**Definition 1.1.** \([31, 39]\) Let \(C := (X, V, \Theta)\) be a time-dependent configuration on the extended state space \(\mathbb{R}^{Nd} \times \mathbb{R}^{Nd} \times \mathbb{R}^d\). Then the configuration \(C\) exhibits asymptotic mono-cluster flocking, if and only if the following conditions hold:

\[
\sup_{0 \leq t < \infty} D(X(t)) < \infty, \quad \lim_{t \to \infty} D(V(t)) = 0, \quad \lim_{t \to \infty} D(\Theta(t)) = 0.
\]
The main results of this paper are two-fold. Our first result is concerned with a sufficient framework leading to the asymptotic emergence of mono-cluster flocking. Our proof can be split into several steps. In the first step, we show that under the assumption $\min_{1 \leq i \leq N} \theta_i^{in} > 0$, temperatures are away from zero (Lemma 3.1). In the second step, under the a priori assumption on the boundedness of spatial diameter:

$$\sup_{0 \leq t < \infty} D(X(t)) < \infty, \quad (1.3)$$

we show that the temperature and velocity diameters decay exponentially (Proposition 3.1 and Proposition 3.2): for some $c > 0$ we have

$$D(\Theta(t)) = O(e^{-ct}) \quad \text{and} \quad D(V(t)) = O(e^{-ct}) \quad \text{as} \quad t \to \infty$$

In our final step, we show that if initial data $(X^{in}, V^{in}, \Theta^{in})$ satisfy

$$D(X^{in}) + D(V^{in}) + D(\Theta^{in}) \ll 1,$$

then, we can show that the a priori condition (1.3) is attained and conclude the emergence of mono-cluster flocking in the sense of Definition 1.1.

Our second main result is concerned with a sufficient framework for the mono-cluster flocking of the discrete model (1.2). Flocking analysis for the discrete model is almost parallel to the continuous one except one extra condition on the smallness of time-step:

$$0 < h \ll \min \left\{ \frac{1}{\kappa_1}, \frac{1}{\kappa_2} \right\}.$$

Other procedures are almost the same as the continuous one.

The rest of the paper is organized as follows: In Section 2, we briefly review directed graphs, scrambling matrices, state-transition matrices and reformulation of (1.1) and (1.2) in terms of coldness which is the inverse of temperature. In Section 3, we present our first result which concerns the emergence of mono-cluster flocking to the continuous model following procedure depicted as above. In Section 4, we perform a similar analysis as in Section 3 to prove our second result which concerns the discrete model. Finally, Section 5 is devoted to the brief summary and discussion of our main results and some future directions. In Appendix A, Appendix B and Appendix C, we provide the proofs of Lemma 4.2, Lemma 4.3 and Proposition 4.2, respectively.

Notation: For $s \in \mathbb{R}$, $\lfloor s \rfloor$ denotes the greatest integer not exceeding $s$. And for any matrix $A$, $\|A\|_F$ denotes the Frobenius norm. Throughout the paper, we denote particle $i$ by the $i$-th thermodynamic Cucker-Smale particle. For an $m \times n$ matrices $A = (a_{ij})$ and $B = (b_{ij})$, $A \geq B$ means that $a_{ij} \geq b_{ij}$ for all $i, j$. And $A \geq 0$ means that $a_{ij} \geq 0$ for all $i, j$. An $N \times N$ matrix $A = (a_{ij})$ is nonnegative means that all entries are nonnegative.
2. Preliminaries

In this section, we provide several elementary concepts on directed graphs, scrambling matrices, state-transition matrices and a reformulation of the TCS model using coldness variable instead of temperature.

2.1. A directed graph

A directed graph (digraph) $\mathcal{G} = (\mathcal{V}(\mathcal{G}), \mathcal{E}(\mathcal{G}))$ consists of a finite set $\mathcal{V}(\mathcal{G}) = \{1, \ldots, N\}$ of vertices (nodes), and a set $\mathcal{E}(\mathcal{G}) \subset \mathcal{V}(\mathcal{G}) \times \mathcal{V}(\mathcal{G})$ of arcs.

If $(j, i) \in \mathcal{E}(\mathcal{G})$, we say that $j$ is a neighbor of $i$, and we denote the neighbor set of the vertex $i$ by $\mathcal{N}_i := \{j : (j, i) \in \mathcal{E}(\mathcal{G})\}$. We define the adjacency matrix $\chi = (\chi_{ij})$, where

$$\chi_{ij} = \begin{cases} 1, & \text{if } j \text{ is a neighbor of } i, \\ 0, & \text{otherwise.} \end{cases}$$

A path in a digraph $\mathcal{G}$ from $i_0$ to $i_p$ is a finite sequence $i_0, i_1, \ldots, i_p$ of distinct vertices such that each successive pair of vertices is an arc of $\mathcal{G}$. The integer $p$ (the number of its arcs) is called the length of the path. If there exists a path from $i$ to $j$, then vertex $j$ is said to be reachable from vertex $i$, and we define the distance from $i$ to $j$, dist($i, j$), as the length of a shortest path from $i$ to $j$. We say that $\mathcal{G}$ has a spanning tree if we can find a vertex (called a root) such that any other vertex of $\mathcal{G}$ is reachable from it. For each root $r$ of digraph $\mathcal{G}$ with a spanning tree, we define $\max_{j \in \mathcal{V}} \text{dist}(r, j)$ as the depth of the spanning tree of $\mathcal{G}$ rooted at $r$. The smallest depth $\gamma_g$ of $\mathcal{G}$ is given by the following relation:

$$\gamma_g := \min_{r \text{ is a root}} \max_{j \in \mathcal{V}} \text{dist}(r, j).$$

Throughout the paper, we denote $\gamma_g$ by the smallest depth of the directed graph corresponding to the network topology $(\chi_{ij})$ in [1.1] and [1.2].

2.2. Scrambling matrices

Next, we review the concept of scrambling matrix and its properties. First, we introduce concepts of stochastic matrix, scrambling matrix and adjacency matrix as follows.

**Definition 2.1.** Let $A = (a_{ij})$ be a nonnegative $N \times N$ matrix.

1. $A$ is a stochastic matrix, if its row-sum is equal to unity:

$$\sum_{j=1}^{N} a_{ij} = 1, \quad 1 \leq i \leq N.$$

2. $A$ is a scrambling matrix, if for each pair of indices $i$ and $j$, there exist an index $k$ such that $a_{ik} > 0$ and $a_{jk} > 0$. 
A is an adjacency matrix of a digraph \( G \) if the following holds:

\[ a_{ij} > 0 \iff (j, i) \in E. \]

In this case, we write \( G = G(A) \).

**Remark 2.1.** Define the **ergodicity coefficient** of \( A \) as follows.

\[
\mu(A) := \min_{i,j} \sum_{k=1}^{N} \min\{a_{ik}, a_{jk}\}. \tag{2.1}
\]

Then, it is easy to see that

1. \( A \) is scrambling if and only if \( \mu(A) > 0 \).
2. For nonnegative matrices \( A \) and \( B \),

\[ A \geq B \implies \mu(A) \geq \mu(B). \tag{2.2} \]

For a matrix \( A = (a_{ij}) \), we set

\[ a := \min\{a_{ij} : a_{ij} > 0\}. \]

**Lemma 2.1.** (Proposition 1, \[20\] ) Let \( A = (a_{ij}) \) be a nonnegative \( N \times N \) matrix of which all diagonal entries are positive. Suppose that \( G(A) \) has a spanning tree with the smallest depth \( \gamma_g \). Then, we have

\[ \mu(A^{\gamma_g}) \geq a^{\gamma_g}. \]

The following result is a perturbative version of Lemma 2.1 in \[15\].

**Lemma 2.2.** Suppose that a nonnegative \( N \times N \) matrix \( A = (a_{ij}) \) is stochastic, and let \( B = (b_{ij}^1) \), \( Z = (z_{ij}^1) \) and \( W = (w_{ij}^1) \) be \( N \times d \) matrices such that

\[ W = AZ + B. \tag{2.3} \]

Then, we have

\[
\max_{i,k} \|w_i - w_k\| \leq (1 - \mu(A)) \max_{l,m} \|z_l - z_m\| + \sqrt{2}\|B\|_F,
\]

where

\[ z_i := (z_{i1}, \ldots, z_{id}), \quad b_i := (b_{i1}, \ldots, b_{id}), \quad w_i := (w_{i1}, \ldots, w_{id}), \quad i = 1, \ldots, N. \]

**Proof.** First we use a property of stochastic matrices to see

\[
\sum_{i=1}^{N} \max\{0, a_{il} - a_{kl}\} + \sum_{i=1}^{N} \min\{0, a_{il} - a_{kl}\} = \sum_{i=1}^{N} (a_{il} - a_{kl}) = 0. \tag{2.4}
\]
Then, for $1 \leq i, k \leq N$, we use (2.3), (2.4) and Cauchy-Schwarz’s inequality to find
\[
\|w_i - w_k\|^2 = \left\langle \sum_{l=1}^{N} a_{il} z_l + b_i - \sum_{l=1}^{N} a_{kl} z_l - b_k, w_i - w_k \right\rangle \\
= \sum_{l=1}^{N} (a_{il} - a_{kl}) \langle z_l, w_i - w_k \rangle + \langle b_i - b_k, w_i - w_k \rangle \\
\leq \sum_{l=1}^{N} \max \{0, a_{il} - a_{kl}\} \max_n \langle z_n, w_i - w_k \rangle + \sum_{l=1}^{N} \min \{0, a_{il} - a_{kl}\} \min_n \langle z_n, w_i - w_k \rangle + \langle b_i - b_k, w_i - w_k \rangle \\
= \sum_{l=1}^{N} \max \{0, a_{il} - a_{kl}\} \max_{n,m} \langle z_n - z_m, w_i - w_k \rangle + \langle b_i - b_k, w_i - w_k \rangle \\
\leq \sum_{l=1}^{N} \max \{0, a_{il} - a_{kl}\} \max_{n,m} \|z_n - z_m\| \|w_i - w_k\| + \|b_i - b_k\| \|w_i - w_k\|, 
\]
(2.5)

By (2.5), we have
\[
\max_{i,k} \|w_i - w_k\| \leq \max_{i,k} \sum_{l=1}^{N} \max \{0, a_{il} - a_{kl}\} \max_{n,m} \|z_n - z_m\| + \max_{i,k} \|b_i - b_k\|. 
\]

Finally, we use the following observations:
\[
\max_{i,k} \sum_{l=1}^{N} \max \{0, a_{il} - a_{kl}\} = \max_{i,k} \sum_{l=1}^{N} (a_{il} - \min \{a_{il}, a_{kl}\}) = 1 - \mu(A) 
\]
and
\[
\|b_i - b_k\|^2 \leq (\|b_i\| + \|b_k\|)^2 \leq 2\|b_i\|^2 + 2\|b_k\|^2 \leq 2\|B\|^2 
\]
for $i \neq k$ to derive the desired estimate. \(\square\)

2.3. *State-transition matrices*

Let $t_0 \in \mathbb{R}$ and $A : [t_0, \infty) \to \mathbb{R}^{N \times N}$ be an $N \times N$ matrix of continuous functions. Consider the following time-dependent linear ODE:
\[
\frac{d\xi(t)}{dt} = A(t)\xi(t), \quad t > t_0. 
\]
(2.6)

Then, the solution of (2.6) is given by
\[
\xi(t) = \Phi(t,t_0)\xi(t_0), 
\]
where $\Phi(t, t_0)$ is the state-transition matrix or fundamental matrix. We can write the state-transition matrix $\Phi(t, t_0)$ in the following form, which is known as the Peano-Baker series (see [48]):

$$\Phi(t, t_0) = I + \sum_{n=1}^{\infty} \int_{t_0}^{t} \int_{t_0}^{\tau_1} \cdots \int_{t_0}^{\tau_{n-1}} A(\tau_1)A(\tau_2)\cdots A(\tau_n) d\tau_n \cdots d\tau_2 d\tau_1,$$

where $I$ is $N \times N$ identity matrix. We conclude this subsection by introducing a technical lemma to be used later.

Let $t_0 \in \mathbb{R}$, $c \in \mathbb{R}$ and $A : [t_0, \infty) \to \mathbb{R}^{N \times N}$ be an $N \times N$ matrix of continuous functions. Then, we set $\Phi(t, t_0)$ and $\Psi(t, t_0)$ to be the state-transition matrices corresponding to the following linear ODEs, respectively:

$$\frac{d\xi(t)}{dt} = A(t)\xi(t) \quad \text{and} \quad \frac{d\xi(t)}{dt} = [A(t) + cI]\xi(t), \quad t \geq t_0. \quad (2.7)$$

Then, the following lemma yields a relation between $\Phi(t, t_0)$ and $\Psi(t, t_0)$.

**Lemma 2.3.** The following relation holds.

$$\Phi(t, t_0) = e^{-c(t-t_0)}\Psi(t, t_0), \quad \text{or} \quad \Psi(t, t_0) = e^{c(t-t_0)}\Phi(t, t_0), \quad t \geq t_0.$$

**Proof.** Let $\Phi(t, t_0)$ and $\Psi(t, t_0)$ be the state transition matrices of (2.7)\textsubscript{1} and (2.7)\textsubscript{2}, respectively. To derive desired estimate, we set

$$\tilde{\Phi}(t, t_0) := e^{-c(t-t_0)}\Psi(t, t_0). \quad (2.8)$$

and we will show that $\tilde{\Phi}(t, t_0)$ satisfies (2.7)\textsubscript{1} and the same initial data.

- (Equation): By direct estimate, we have

$$\frac{d}{dt}\tilde{\Phi}(t, t_0) = \frac{d}{dt}[e^{-c(t-t_0)}\Psi(t, t_0)]
\quad = -ce^{-c(t-t_0)}\Psi(t, t_0) + e^{-c(t-t_0)} \frac{d}{dt}\Psi(t, t_0)
\quad = -ce^{-c(t-t_0)}\Psi(t, t_0) + e^{-c(t-t_0)}[A(t) + cI]\Psi(t, t_0)
\quad = e^{-c(t-t_0)} A(t)\Psi(t, t_0) = A(t)\tilde{\Phi}(t, t_0). \quad (2.9)$$

- (Initial data): It is clear from (2.8) that

$$\tilde{\Phi}(t_0, t_0) = \Psi(t_0, t_0) = I. \quad (2.10)$$

Finally, we can see that (2.9) and (2.10) satisfies the same ODE system and initial data. By the uniqueness theory of ODEs, we have

$$\Phi(t, t_0) = \tilde{\Phi}(t, t_0). \quad \square$$
2.4. A reformulation of the TCS model

In this subsection, we introduce a "coldness" variable which is a reciprocal of the temperature. We set

$$\beta_i(t) := \frac{1}{\theta_i(t)}, \quad t > 0, \quad \beta_i^{in} := \frac{1}{\theta_i^{in}}, \quad i = 1, \cdots, N.$$  

Then $\beta$ measures the inverse temperature, i.e., coldness. Then, the corresponding Cauchy problem for the continuous and discrete models (1.1) and (1.2) are

$$\begin{align*}
\frac{dx_i}{dt} &= v_i, \quad t > 0, \quad i = 1, 2, \cdots, N, \\
\frac{dv_i}{dt} &= \frac{1}{N} \sum_{j=1}^{N} \chi_{ij} \phi(||x_i - x_j||)(\beta_j v_j - \beta_i v_i), \\
\frac{d\beta_i}{dt} &= \frac{1}{N} \sum_{j=1}^{N} \chi_{ij} \zeta(||x_i - x_j||) (\beta_j - \beta_i), \\
(x_i(0), v_i(0), \beta_i(0)) &= (x_i^{in}, v_i^{in}, \beta_i^{in}).
\end{align*}$$  

(2.11)

and

$$\begin{align*}
\begin{array}{l}
x_i[t + 1] = x_i[t] + hv_i[t], \quad t \in \mathbb{N} \cup \{0\}, \quad i = 1, 2, \cdots, N, \\
v_i[t + 1] = v_i[t] + \frac{h}{N} \sum_{j=1}^{N} \chi_{ij} \phi(||x_i[t] - x_j[t]||)(\beta_j[t] v_j[t] - \beta_i[t] v_i[t]), \\
\frac{1}{\beta_i[t + 1]} = \frac{1}{\beta_i[t]} + \frac{h}{N} \sum_{j=1}^{N} \chi_{ij} \zeta(||x_i[t] - x_j[t]||)(\beta_i[t] - \beta_j[t]), \\
(x_i[0], v_i[0], \beta_i[0]) = (x_i^{in}, v_i^{in}, \beta_i^{in}).
\end{array}
\end{align*}$$  

(2.12)

2.5. A review of previous results

In this subsection, we briefly review the known previous results on the emergent behaviors of the TCS model (1.1) with small diffusion velocities. The model (1.1) with all-to-all couplings with $\phi \equiv 1$ and $\zeta \equiv 1$ has been proposed in a recent work by Ha and Ruggeri [35]. They derived the model (1.1) from the system of gas mixture which is a coupled system of reactive Euler systems based on reasonable physical settings such as spatial homogeneity, Galilean invariance, small diffusion velocity assumption and entropy principle (see [35] for a detailed discussion). In their work, they derived an exponential flocking estimate for (1.1) as long as initial states satisfy a kind of small assumptions. Their work has been extended to several directions, e.g., nonexistence of mono-cluster flocking [31], time-delay effect [22], uniform stability and its kinetic limit [30], global well-posedness of the hydrodynamic TCS model in [29], coupling with fluids [12, 13], discrete TCS model [28], although they are all dealing with TCS ensemble over the complete graph. As far as the authors know, this is the first work dealing with the emergent dynamics of TCS ensemble other than the complete graph.
3. Emergence dynamics of the continuous model

In this section, we present an asymptotic flocking estimate for the continuous model (2.11). In the sequel, we derive our flocking estimate in the following three steps.

• First, we show the uniform boundedness of temperatures and monotonic properties of the diameter of the coldness, and using these estimates, we derive an exponential asymptotic alignment of temperatures under a priori uniform boundedness condition of spatial diameter.

• Second, we derive asymptotic alignment of velocities under a priori uniform boundedness condition of spatial diameter.

• Finally, we present a sufficient condition leading to the uniform boundedness of spatial diameter in terms of system parameters. This leads to the mono-cluster flocking estimate.

3.1. A priori temperature alignment

In this subsection, we show that the temperatures have some positive lower bound and upper bound. For convenience, we work with the system (2.11). Next, we set initial maximum and minimum coldness $\beta_{i}^{\text{in}}$ in $U$ and $\beta_{i}^{\text{in}}$ in $L > 0$ as follows:

$$\beta_{i}^{\text{in}} := \max_{1 \leq i \leq N} \beta_{i}^{\text{in}}$$

$$\beta_{i}^{\text{in}} := \min_{1 \leq i \leq N} \beta_{i}^{\text{in}}.$$

We also set

$$B := (\beta_{1}, \cdots, \beta_{N}), \quad D(B(t)) := \max_{1 \leq i, j \leq N} |\beta_{i}(t) - \beta_{j}(t)|.$$

Lemma 3.1 (Boundedness of temperatures). Let $\{(x_{i}, v_{i}, \beta_{i})\}$ be a solution to the Cauchy problem (2.11) with positive initial temperatures $\beta_{i}^{\text{in}} < \infty$. Then, we have

(i) $\beta_{i}^{\text{in}} \leq \beta_{i}(t) \leq \beta_{i}^{\text{in}}, \quad i = 1, \cdots, N, \quad 0 \leq t < \infty,$

(ii) $D(B(\cdot))$ is monotone decreasing.

Proof. (i) First, we define the maximal and minimal values for coldness as follows:

$$\beta_{M}(t) := \max_{1 \leq i \leq N} \beta_{i}(t), \quad \beta_{m}(t) := \min_{1 \leq i \leq N} \beta_{i}(t), \quad t \geq 0.$$

Then, for each $t > 0$, we choose extremal indices $1 \leq m_{t}, M_{t} \leq N$ satisfying

$$\beta_{m}(t) = \beta_{m_{t}}(t) \quad \text{and} \quad \beta_{M}(t) = \beta_{M_{t}}(t).$$
Note that for a.e. \( t > 0 \), \( \beta_m \) and \( \beta_M \) satisfy
\[
\frac{d\beta_m(t)}{dt} = \frac{\beta_m(t)^2}{N} \sum_{j=1}^{N} \chi_{m,j} \zeta(\|x_{m}(t) - x_{j}(t)\|) \left( \beta_j(t) - \beta_m(t) \right) \geq 0,
\]
\[
\frac{d\beta_M(t)}{dt} = \frac{\beta_M(t)^2}{N} \sum_{j=1}^{N} \chi_{M,j} \zeta(\|x_{M}(t) - x_{j}(t)\|) \left( \beta_j(t) - \beta_M(t) \right) \leq 0.
\]
(3.1)

Thus, minimal and maximal coldness are non-decreasing and non-increasing along the flow (2.11).

(ii) Recall the diameter for coldness:
\[
D(B(t)) = \max_{1 \leq i \leq N} \beta_i(t) - \min_{1 \leq i \leq N} \beta_i(t), \quad t \geq 0.
\]

We use the above defining relation and (3.1) to get the desired estimate:
\[
\frac{d}{dt} D(B(t)) = \frac{d}{dt} \left( \beta_M(t) - \beta_m(t) \right) \leq 0, \quad \text{a.e.} \ t > 0.
\]

Next, we study the exponential decay of \( D(B) \) using a more refined argument. First, we rewrite (2.11) as follows.
\[
\frac{d\beta_i(t)}{dt} = -\frac{1}{N} \beta_i^2 \left[ \beta_i \sum_{j=1}^{N} \chi_{ij} \zeta(\|x_i(t) - x_j(t)\|) - \sum_{j=1}^{N} \chi_{ij} \zeta(\|x_i(t) - x_j(t)\|) \beta_j \right]
\]
(3.2)

In order to rewrite (3.2) in a more compact form, we define an \( N \times N \) matrix \( L(t) \) by
\[
L(t) := D(t) - A(t),
\]
(3.3)

where the matrices \( A(t) = (a_{ij}(t)) \) and \( D(t) = \text{diag}(d_1(t), \cdots, d_N(t)) \) are defined by the following relations:
\[
a_{ij}(t) := \chi_{ij} \zeta(\|x_i(t) - x_j(t)\|) \quad \text{and} \quad d_i(t) = \sum_{j=1}^{N} \chi_{ij} \zeta(\|x_i(t) - x_j(t)\|).
\]
(3.4)

And we also define
\[
\Gamma(t) := \text{diag}(\beta_1(t), \cdots, \beta_N(t)).
\]
(3.5)

Then, we see from (3.2) that (2.11) can be written as
\[
\frac{d}{dt} B(t) = -\frac{1}{N} \Gamma(t)^2 L(t) B(t).
\]
(3.6)

Let \( \Phi(t_2, t_1) \) be the state transition matrix associated with (3.6). Then, for any given \( \delta > 0 \), we derive the solution formula for \( B \):
\[
B(m\delta) = \Phi \left( m\delta, (m-1)\delta \right) B((m-1)\delta), \quad m \in \mathbb{N}.
\]
(3.7)
Lemma 3.2. Let \( \{(x_i, v_i, \beta_i)\} \) be a solution to (2.11) satisfying a priori condition:

\[
\sup_{0 \leq t < \infty} D(X(t)) \leq x^\infty < \infty. \tag{3.8}
\]

Then the following assertions hold.

1. The ergodicity coefficient \( \mu(\Phi(m\delta, (m-1)\delta)) \) satisfies

\[
\mu(\Phi(m\delta, (m-1)\delta)) \geq C_1 \zeta(x^\infty)^{\gamma_s},
\]

where a positive constant \( C_1 \) is given by the following relation:

\[
C_1 = C_1(\delta) := e^{-\kappa_2 (\beta^n_L)^2 \delta} \cdot \frac{1}{\gamma_g!} \left( \frac{2}{\gamma_g N} \delta \right)^{\gamma_s} \beta^n_L.
\]

2. The state transition matrix \( \Phi(m\delta, (m-1)\delta) \) is stochastic.

Proof. (1) We claim

\[
\Phi(m\delta, (m-1)\delta) \geq C_1 (A^\infty)^{\gamma_s} \geq 0, \tag{3.9}
\]

where \( A^\infty = (a^\infty ij) \) is a nonnegative matrix defined by \( a^\infty ij := \chi_{ij} \zeta(x^\infty) \).

Proof of claim (3.9): We use (3.3), (3.4) and (3.5) to estimate the coefficient matrix for (3.6) as follows.

\[
-\frac{1}{N} \Gamma(t)^2 L(t) = \frac{1}{N} \Gamma(t)^2 (A(t) - D(t)) \geq \frac{(\beta^n_L)^2}{N} A^\infty - \kappa_2 (\beta^n_L)^2 I.
\]

Now, we decompose the coefficient matrix \( -\frac{1}{N} \Gamma(t)^2 L(t) \) into the sum of the following two matrices:

\[
-\frac{1}{N} \Gamma(t)^2 L(t) = \left( -\frac{1}{N} \Gamma(t)^2 L(t) + \kappa_2 (\beta^n_L)^2 I \right) - \kappa_2 (\beta^n_L)^2 I. \tag{3.10}
\]

The terms in the parenthesis of (3.10) can also be estimated as follows:

\[
-\frac{1}{N} \Gamma(t)^2 L(t) + \kappa_2 (\beta^n_L)^2 I \geq \frac{(\beta^n_L)^2}{N} A^\infty \geq 0. \tag{3.11}
\]

On the other hand, for any \( 0 \leq t_1 < t_2 < \infty \), let \( \Phi(t_2, t_1) \) and \( \Psi(t_2, t_1) \) be the state-transition matrices of \( -\frac{1}{N} \Gamma(t)^2 L(t) \) and \( -\frac{1}{N} \Gamma(t)^2 L(t) + \kappa_2 (\beta^n_L)^2 I \) on \( [t_1, t_2] \), respectively. Then it follows from Lemma 2.3 that

\[
\Phi(t_2, t_1) = e^{-\kappa_2 (\beta^n_L)^2 (t_2 - t_1)} \Psi(t_2, t_1). \tag{3.12}
\]
In (3.11), we can use the Peano-Baker series to obtain
\[
\Psi(t_2, t_1) = I + \sum_{n=1}^{\infty} \int_{t_1}^{t_2} \int_{t_1}^{\tau_1} \cdots \int_{t_1}^{\tau_{n-1}} \left( - \frac{1}{N} \Gamma(\tau_1)^2 L(\tau_1) + \kappa_2 (\beta_L^n)^2 I \right) \cdots \\
\left( - \frac{1}{N} \Gamma(\tau_n)^2 L(\tau_n) + \kappa_2 (\beta_L^n)^2 I \right) d\tau_n \cdots d\tau_1
\geq I + \sum_{n=1}^{\infty} \int_{t_1}^{t_2} \int_{t_1}^{\tau_1} \cdots \int_{t_1}^{\tau_{n-1}} \left( \frac{\beta_L^n}{N} A^\infty \right)^n d\tau_n \cdots d\tau_1
\]
\[
= I + \sum_{n=1}^{\infty} \frac{1}{n!} (t_2 - t_1)^n \left( \frac{\beta_L^n}{N} A^\infty \right)^n
\]
\[
= \exp \left( (t_2 - t_1) \left( \frac{\beta_L^n}{N} A^\infty \right) \right).
\]
(3.13)

For a fixed \( m \in \mathbb{N} \), we combine (3.12) and (3.13) and put \( t_1 = (m-1)\delta \), \( t_2 = m\delta \) to obtain
\[
\Phi(m\delta, (m-1)\delta) \geq e^{-\kappa_2 (\beta_L^n)^2 \delta} \exp \left[ \delta \left( \frac{\beta_L^n}{N} A^\infty \right) \right]
\]
\[
= e^{-\kappa_2 (\beta_L^n)^2 \delta} \left[ I + \sum_{n=1}^{\infty} \frac{1}{n!} \left( \delta \left( \frac{\beta_L^n}{N} A^\infty \right) \right)^n \right]
\]
\[
\geq e^{-\kappa_2 (\beta_L^n)^2 \delta} \cdot \frac{1}{\gamma_g} \left( \delta \left( \frac{\beta_L^n}{N} A^\infty \right) \right) \gamma_g (A^\infty)^{\gamma_g}
\]
\[
= C_1 (A^\infty)^{\gamma_g} \geq 0,
\]
(3.14)

which proves the claim (3.9).

(3.14) and (2.2) yield
\[
\mu \left( \Phi(m\delta, (m-1)\delta) \right) \geq C_1 \mu((A^\infty)^{\gamma_g}) \geq C_1 \zeta(x^\infty)^{\gamma_g},
\]
where \( \mu \) is the ergodicity coefficient defined in (2.1), and the last inequality is due to Lemma 2.1.

(2) By (3.14), \( \Phi(m\delta, (m-1)\delta) \) is nonnegative, so it remains to show that each of its rows sums to 1. Note that the constant state \( \xi(t) := [\xi_1(t), \cdots, \xi_N(t)]^T \equiv [1, \cdots, 1]^T \) is a solution to (3.6), i.e.
\[
\frac{d}{dt} \xi(t) = - \frac{1}{N} \Gamma(t)^2 L(t) \xi(t).
\]
Hence, it satisfies
\[
[1, \cdots, 1]^T = \Phi(m\delta, (m-1)\delta)[1, \cdots, 1]^T.
\]
This implies that \( \Phi(m\delta, (m-1)\delta) \) is stochastic.
\( \square \)
Next, we are ready to present a priori temperature alignment based on Lemma 3.2.

**Proposition 3.1.** Let \( \{(x_i, v_i, \beta_i)\} \) be a solution to (2.11) satisfying a priori condition (3.8). Then, we have the exponential decay of \( D(B(t)) \): For any given \( \delta > 0 \), we have

\[
D(B(t)) \leq \left( 1 - C_1 \zeta(x^\infty)^{\gamma} \right)^{\frac{t}{\delta}} D(B(0)), \quad t \geq 0,
\]

where \( C_1 = C_1(\delta) \) is the constant defined in Lemma 3.2.

**Proof.** Since \( \Phi\left(m\delta, (m - 1)\delta\right) \) is stochastic (Lemma 3.2), we can combine (3.7), Lemma 2.2 and Lemma 3.2 to obtain

\[
D(B(m\delta)) \leq \left( 1 - \mu \Phi\left(m\delta, (m - 1)\delta\right) \right) D(B((m - 1)\delta)) \leq \cdots \leq (1 - C_1 \zeta(x^\infty)^{\gamma})^m D(B(0)), \quad m \in \mathbb{N}.
\]

So for any real \( t = \delta p \geq 0 \), we use (3.15) and Lemma 3.1 to get

\[
D(B(t)) = D(B(\delta p)) \leq (1 - C_1 \zeta(x^\infty)^{\gamma})^{\frac{p}{\delta}} D(B(0)) = (1 - C_1 \zeta(x^\infty)^{\gamma})^{\frac{t}{\delta}} D(B(0)).
\]

3.2. **A priori velocity alignment**

In this subsection, we provide velocity alignment estimate under the a priori assumption (3.8). For notational simplicity, we introduce the following notation:

\[
u_i(t) := \frac{v_i(t)}{\theta_i(t)} = \beta_i(t)v_i(t), \quad i = 1, \ldots, N, \quad \text{and} \quad R_u(t) := \max_{1 \leq i \leq N} \|u_i(t)\|, \quad t \geq 0.
\]

To derive the velocity alignment, we use a bootstrapping argument, i.e., first we derive a uniform boundedness of velocity diameter, and then using the differential inequalities for velocity diameter, we improve our rough boundedness to the exponential decay of the velocity diameter. As a first step, we prove the boundedness of velocities.

**Lemma 3.3 (Boundedness of velocities).** Let \( \{(x_i, v_i, \beta_i)\} \) be a solution to (2.11) satisfying a priori condition (3.8). Then, velocities of the particles are uniformly bounded: for any given \( \delta > 0 \) we have

\[
\|v_i(t)\| \leq \frac{1}{\beta_i^\mu} R_u(0) \exp\left( \frac{\kappa_2 \beta_i^\mu D(B(0))}{C_1 \zeta(x^\infty)^{\gamma}} \right) =: R_v^c = R_v^c(x^\infty, \delta), \quad i = 1, \ldots, N, \quad t \geq 0,
\]

where \( C_1 = C_1(\delta) \) is the constant defined in Lemma 3.2.

**Proof.** For the desired estimate, it suffices to derive an estimate:

\[
R_u(t) \leq R_u(0) \exp\left( \frac{\kappa_2 \beta_i^\mu D(B(0))}{C_1 \zeta(x^\infty)^{\gamma}} \right), \quad t \geq 0.
\]
First, we derive a differential inequality of $R_u$. For each $i = 1, \ldots, N$, we have
\[
\frac{d}{dt} \| u_i \|^2 = \frac{d}{dt}(\beta_i^2 \| v_i \|^2) = 2\beta_i \| v_i \|^2 \frac{d\beta_i}{dt} + 2\beta_i^2 \left< v_i, \frac{dv_i}{dt} \right>, \quad t > 0. \tag{3.16}
\]
We estimate the two terms of the right-hand side of (3.16). We have
\[
2\beta_i \| v_i \|^2 \frac{d\beta_i}{dt} = \frac{2\beta_i \| v_i \|^2}{N} \sum_{j=1}^{N} \chi_{ij} \zeta(\| x_i - x_j \|) \beta_j^2 (\beta_j - \beta_i) \\
= \frac{2\beta_i \| v_i \|^2}{N} \sum_{j=1}^{N} \chi_{ij} \zeta(\| x_i - x_j \|) (\beta_j - \beta_i) \tag{3.17}
\]
and
\[
2\beta_i^2 \left< v_i, \frac{dv_i}{dt} \right> = 2\beta_i^2 \left< v_i, \frac{1}{N} \sum_{j=1}^{N} \chi_{ij} \phi(\| x_i - x_j \|) (\beta_j v_j - \beta_i v_i) \right> \\
= 2\beta_i \left< u_i, \frac{1}{N} \sum_{j=1}^{N} \chi_{ij} \phi(\| x_i - x_j \|) (u_j - u_i) \right> \\
= \frac{2\beta_i}{N} \sum_{j=1}^{N} \chi_{ij} \phi(\| x_i - x_j \|)(u_j - u_i) \tag{3.18}
\]

We combine (3.16), (3.17), and (3.18) to obtain
\[
\frac{d}{dt} \| u_i \|^2 \leq 2\kappa_2 \beta_i \mathcal{D}(\mathcal{B}) \| u_i \|^2 + \frac{2\beta_i}{N} \| u_i \| \sum_{j=1}^{N} \chi_{ij} \phi(\| x_i - x_j \|)(\| u_j \| - \| u_i \|), \quad t > 0.
\]

For each $t > 0$, we take a maximal index $i_t$ satisfying $R_u(t) = \| u_{i_t}(t) \|$. Then we have
\[
\frac{d}{dt} R_u(t)^2 \leq 2\kappa_2 \beta_{i_t} \mathcal{D}(\mathcal{B}) R_u(t)^2 + \frac{2\beta_{i_t} R_u(t)}{N} \sum_{j=1}^{N} \chi_{i_t j} \phi(\| x_{i_t} - x_j \|)(\| u_j \| - R_u(t)) \\
\leq 2\kappa_2 \beta_{i_t} \mathcal{D}(\mathcal{B}) R_u(t)^2 \leq 2\kappa_2 \beta_{i_t}^{in} \mathcal{D}(\mathcal{B}(t)) R_u(t)^2, \quad \text{a.e. } t > 0.
\]

Hence
\[
2R_u(t) \frac{d}{dt} R_u(t) \leq 2\kappa_2 \beta_{i_t}^{in} \mathcal{D}(\mathcal{B}(t)) R_u(t)^2, \quad \text{a.e. } t > 0.
\]
If \( R_u(t) > 0 \), then we can divide the above inequality by \( 2R_u(t) \). If \( R_u(t) = 0 \), then \( R_u \) attains a global minimum at \( t \), so \( \frac{d}{dt}R_u(t) = 0 \). Hence we have the following differential inequality:

\[
\frac{d}{dt}R_u(t) \leq \kappa_2 \beta_i^\infty D(B(t))R_u(t), \text{ a.e. } t > 0.
\]

Now we apply Gronwall’s inequality and Proposition 3.1 to obtain

\[
R_u(t) \leq R_u(0) \exp \left( \kappa_2 \beta_i^\infty \int_0^t D(B(s))ds \right) \leq R_u(0) \exp \left( \kappa_2 \beta_i^\infty D(B(0)) \sum_{n=0}^{\infty} (1 - C_1 \zeta(x^\infty) \gamma_s)^n \right) \leq R_u(0) \exp \left( \frac{\kappa_2 \beta_i^\infty D(B(0))}{C_1 \zeta(x^\infty) \gamma_s} \right).
\]

Next, we derive a differential inequality for the velocity diameter.

**Lemma 3.4 (Differential inequality of \( D(V(t)) \)).** Let \( \{(x_i, v_i, \beta_i)\} \) be a solution to (2.11) satisfying a priori condition (3.8). Then, for any given \( \delta > 0 \) we have

\[
\frac{d}{dt}D(V(t)) \leq 2\kappa_1 R_V^\infty D(B(t)), \text{ a.e. } t > 0,
\]

where \( R_V^\infty = R_V^\infty(x^\infty, \delta) \) is the constant defined in Lemma 3.3.

**Proof.** For a given \( t \), let \( i \) and \( j \) be indices satisfying the relation:

\[
D(V) = \|v_i - v_j\|.
\]

Then, we have

\[
\frac{1}{2} \frac{d}{dt} \|v_i - v_j\|^2 = \left\langle v_i - v_j, \frac{dv_i}{dt} - \frac{dv_j}{dt} \right\rangle = \left\langle v_i - v_j, \frac{1}{N} \sum_{k=1}^{N} \chi_{ik}\phi_{ik}(\beta_k v_k - \beta_i v_i) \right\rangle \left\langle v_j - v_i, \frac{1}{N} \sum_{k=1}^{N} \chi_{jk}\phi_{jk}(\beta_k v_k - \beta_j v_j) \right\rangle =: \mathcal{I}_{11} + \mathcal{I}_{12},
\]

where we wrote \( \phi_{ij} := \phi(\|x_i - x_j\|) \), \( i, j = 1, 2, \cdots, N \) for notational convenience.
Below, we estimate the terms $I_{11}$, $i = 1, 2$ one by one.

• (Estimate of $I_{11}$): We use $\phi_{ik} \leq \phi(0) = \kappa_1$ to find
  
  $$I_{11} = \frac{1}{N} \sum_{k=1}^{N} \chi_{ik} \phi_{ik} \langle v_i - v_j, \beta_k v_k - \beta_i v_i \rangle + \frac{1}{N} \sum_{k=1}^{N} \chi_{ik} \phi_{ik} \langle v_i - v_j, \beta_k v_k - \beta_i v_i \rangle$$

  $$\leq \frac{1}{N} \sum_{k=1}^{N} \chi_{ik} \phi_{ik} \langle v_i - v_j, (\beta_k - \beta_i) v_k \rangle + 0$$

  $$\leq \kappa_1 \mathcal{D}(B)\|v_i - v_j\| \left( \frac{1}{N} \sum_{k=1}^{N} \|v_k\| \right).$$

  The first inequality followed from

  $$\langle v_k - v_i, v_i - v_j \rangle = \frac{\|v_k - v_j\|^2 - \|v_k - v_i\|^2 - \|v_i - v_j\|^2}{2}$$

  $$\leq \frac{\|v_i - v_j\|^2 - 0 - \|v_i - v_j\|^2}{2} = 0.$$ 

• (Estimate of $I_{12}$):

  $$I_{12} = \frac{1}{N} \sum_{k=1}^{N} \chi_{jk} \phi_{jk} \langle v_j - v_i, \beta_k v_k - \beta_j v_j \rangle + \frac{1}{N} \sum_{k=1}^{N} \chi_{jk} \phi_{jk} \langle v_j - v_i, \beta_j v_k - \beta_j v_j \rangle$$

  $$\leq \frac{1}{N} \sum_{k=1}^{N} \chi_{jk} \phi_{jk} \langle v_j - v_i, (\beta_k - \beta_j) v_k \rangle + 0$$

  $$\leq \kappa_1 \mathcal{D}(B)\|v_i - v_j\| \left( \frac{1}{N} \sum_{k=1}^{N} \|v_k\| \right).$$

  The first inequality followed from

  $$\langle v_k - v_j, v_j - v_i \rangle = \frac{\|v_k - v_i\|^2 - \|v_k - v_j\|^2 - \|v_j - v_i\|^2}{2}$$

  $$\leq \frac{\|v_j - v_i\|^2 - 0 - \|v_j - v_i\|^2}{2} = 0.$$ 

  Now, we combine estimates for $I_{11}$ and $I_{12}$ in (3.19) and Lemma 3.3 to obtain

  $$I_{11} + I_{12} \leq 2\kappa_1 R_\psi^c \mathcal{D}(B)\|v_i - v_j\|.$$ 

  Since $\mathcal{D}(V) = \|v_i - v_j\|$, we have

  $$\mathcal{D}(V(t)) \frac{d}{dt} \mathcal{D}(V(t)) \leq 2\kappa_1 R_\psi^c \mathcal{D}(B)\mathcal{D}(V(t)), \quad \text{a.e. } t > 0.$$ 

  If $\mathcal{D}(V(t)) > 0$, then we can divide the above inequality by $\mathcal{D}(V(t))$. If $\mathcal{D}(V(t)) = 0$, then $\mathcal{D}(V)$ attains a global minimum at $t$, so $\frac{d}{dt} \mathcal{D}(V(t)) = 0$. Hence we have the following differential inequality:

  $$\frac{d}{dt} \mathcal{D}(V(t)) \leq 2\kappa_1 R_\psi^c \mathcal{D}(B(t)), \quad \text{a.e. } t > 0.$$ 

where the matrices \( \tilde{N} \) are defined by

\[
\tilde{N} := \frac{1}{N} \sum_{j=1}^{N} \chi_{ij} \phi(\|x_i - x_j\|) v_j.
\]

In order to express (3.20) in matrix form, we define an \( N \times N \) matrix \( \tilde{L}(t) \) by

\[
\tilde{L}(t) := \tilde{D}(t) - \tilde{A}(t),
\]

where the matrices \( \tilde{A}(t) = (\tilde{a}_{ij}(t)) \) and \( \tilde{D}(t) = \text{diag}(\tilde{d}_1(t), \ldots, \tilde{d}_N(t)) \) are defined by the following relations:

\[
\tilde{a}_{ij}(t) := \chi_{ij} \phi(\|x_i(t) - x_j(t)\|) \quad \text{and} \quad \tilde{d}_i(t) = \sum_{j=1}^{N} \chi_{ij} \phi(\|x_i(t) - x_j(t)\|).
\]

On the other hand, recall that

\[
\Gamma(t) := \text{diag}(\beta_1(t), \ldots, \beta_N(t)).
\]

Thus, (3.20) can be rewritten as

\[
\frac{d}{dt} V(t) = -\frac{1}{N} \Gamma(t) \tilde{L}(t) V(t) + \frac{1}{N} \Lambda(t),
\]

where the \( N \times d \) matrix \( \Lambda(t) := (\lambda^{ik}_j(t))_{1 \leq i \leq N, 1 \leq k \leq d} \) is defined by

\[
\lambda^{ik}_j(t) := \sum_{j=1}^{N} \chi_{ij} \phi(\|x_i(t) - x_j(t)\|)(\beta_j(t) - \beta_i(t)) u^k_j(t), \quad i = 1, \ldots, N, \quad k = 1, \ldots, d, \quad t \geq 0.
\]

We also define the \( N \times N \) matrix \( B(t) = (b_{ij}(t)) \) by

\[
b_{ij}(t) = \chi_{ij} \phi(\|x_i(t) - x_j(t)\|)(\beta_j(t) - \beta_i(t)), \quad i, j = 1, \ldots, N.
\]

Then we have

\[
\Lambda(t) = B(t)V(t).
\]

Next, we perform the same analysis as in Lemma 3.2. Let \( \hat{\Phi}(t_2, t_1) \) be the state transition matrix associated with the homogeneous part of (3.21). Then, for any given \( \delta > 0 \), we derive the solution formula for \( V \):

\[
V(m\delta) = \hat{\Phi}(m\delta, (m - 1)\delta)V((m - 1)\delta) + \frac{1}{N} \int_{(m-1)\delta}^{m\delta} \hat{\Phi}(m\delta, s)\Lambda(s)ds, \quad m \in \mathbb{N}.
\]

In next lemma, we study properties of the matrix \( \hat{\Phi}(m\delta, (m - 1)\delta) \).

**Lemma 3.5.** Let \( \{x_i, v_i, \beta_i\} \) be a solution to (2.11) satisfying a priori condition (3.8). Then the following assertions hold.
(1) The ergodicity coefficient \( \mu(\tilde{\Phi}(m\delta, (m-1)\delta)) \) satisfies
\[
\mu(\tilde{\Phi}(m\delta, (m-1)\delta)) \geq C_2 \phi(x^\infty)^\gamma_g,
\]
where
\[
C_2 = C_2(\delta) := e^{-\kappa_1 \beta_1^\infty \delta} \cdot \frac{1}{\gamma_g} \left( \frac{\beta_1^\infty}{N} \right)^{\gamma_g}.
\]

(2) The state transition matrix \( \tilde{\Phi}(m\delta, (m-1)\delta) \) is stochastic.

Proof. (1) Note that
\[
-\frac{1}{N} \Gamma(t) \tilde{L}(t) = -\frac{1}{N} \Gamma(t)(\tilde{A}(t) - \tilde{D}(t)) \geq \frac{\beta_1^\infty}{N} \tilde{A}^\infty - \kappa_1 \beta_1^\infty I,
\]
where \( \tilde{A}^\infty = (\tilde{a}^\infty_{ij}) \) is a nonnegative matrix whose entries are defined by
\[
\tilde{a}^\infty_{ij} := \chi_{ij} \phi(x^\infty).
\]
Motivated by (3.24), we again decompose the matrix \( -\frac{1}{N} \Gamma(t) \tilde{L}(t) \) into a sum of two matrices:
\[
-\frac{1}{N} \Gamma(t) \tilde{L}(t) = \left( -\frac{1}{N} \Gamma(t) \tilde{L}(t) + \kappa_1 \beta_1^\infty I \right) - \kappa_1 \beta_1^\infty I.
\]
Again, we use (3.24) to obtain
\[
-\frac{1}{N} \Gamma(t) \tilde{L}(t) + \kappa_1 \beta_1^\infty I \geq \frac{\beta_1^\infty}{N} \tilde{A}^\infty \geq 0.
\]
For any \( 0 \leq t_1 < t_2 < \infty \), let \( \tilde{\Phi}(t_2, t_1) \) and \( \tilde{\Psi}(t_2, t_1) \) be the state-transition matrices of \( -\frac{1}{N} \Gamma(t) \tilde{L}(t) \) and \( -\frac{1}{N} \Gamma(t) \tilde{L}(t) + \kappa_1 \beta_1^\infty I \) on \([t_1, t_2] \), respectively. Then, it follows from Lemma 2.3 that we have
\[
\tilde{\Phi}(t_2, t_1) = e^{-\kappa_1 \beta_1^\infty (t_2 - t_1)} \tilde{\Psi}(t_2, t_1).
\]
And from (3.24), we can use the Peano-Baker series and obtain the following:
\[
\tilde{\Psi}(t_2, t_1) = I + \sum_{n=1}^{\infty} \int_{t_1}^{t_2} \int_{t_1}^{\tau_1} \cdots \int_{t_1}^{\tau_{n-1}} \left( -\frac{1}{N} \Gamma(\tau_1) \tilde{L}(\tau_1) + \kappa_1 \beta_1^\infty I \right) \cdots \\
\left( -\frac{1}{N} \Gamma(\tau_n) \tilde{L}(\tau_n) + \kappa_1 \beta_1^\infty I \right) \cdot d\tau_n \cdots d\tau_1 \\
\geq I + \sum_{n=1}^{\infty} \int_{t_1}^{t_2} \int_{t_1}^{\tau_1} \cdots \int_{t_1}^{\tau_{n-1}} \left( \frac{\beta_1^\infty}{N} \tilde{A}^\infty \right)^n \cdot d\tau_n \cdots d\tau_1 \\
= I + \sum_{n=1}^{\infty} \frac{1}{n!} (t_2 - t_1)^n \left( \frac{\beta_1^\infty}{N} \tilde{A}^\infty \right)^n = \exp \left( (t_2 - t_1) \frac{\beta_1^\infty}{N} \tilde{A}^\infty \right).
\]
Now, we fix $m \in \mathbb{N}$, and combine (3.26) and (3.27), and put $t_1 = (m-1)\delta$, $t_2 = m\delta$

to obtain

$$
\hat{\Phi}(m\delta, (m-1)\delta) \geq e^{-\kappa_1 t^{1/2} \delta} \exp \left[ \frac{\delta^{\beta_{1n}}}{N} \tilde{A}^\infty \right] = e^{-\kappa_1 t^{1/2} \delta} \left[ I + \sum_{n=1}^{\infty} \frac{1}{n!} \left( \frac{\delta^{\beta_{1n}}}{N} \tilde{A}^\infty \right)^n \right] \\
\geq e^{-\kappa_1 t^{1/2} \delta} \cdot \frac{1}{\gamma_{g}^{1}} (\delta^{\beta_{1n}} \tilde{A}^\infty)^{\gamma_{g}} = C_2(\tilde{A}^\infty)^{\gamma_{g}} \geq 0.
$$

(3.28)

Hence, we use (2.2) and Lemma 2.1 to get

$$
\mu(\hat{\Phi}(m\delta, (m-1)\delta)) \geq C_2\mu(\tilde{A}^\infty)^{\gamma_{g}} \geq C_2 \phi(x^\infty)^{\gamma_{g}}.
$$

(3.29)

(2) Note that by (3.28) $\hat{\Phi}(m\delta, (m-1)\delta)$ is nonnegative and the constant state

$\xi(t) := [\xi_1(t), \cdots, \xi_N(t)]^T \equiv [1, \cdots, 1]^T$ is a solution to the corresponding homogeneous part of (3.21):

$$
[1, \cdots, 1]^T = \hat{\Phi}(m\delta, (m-1)\delta)[1, \cdots, 1]^T,
$$

which implies that $\hat{\Phi}(m\delta, (m-1)\delta)$ is stochastic.

Next, we are ready to provide the exponential decay estimate of $\mathcal{D}(V)$ in the following proposition.

**Proposition 3.2 (Exponential decay of $\mathcal{D}(V(t))$).** Let $\{(x_i, v_i, \beta_i)\}$ be a solution to (2.11) satisfying a priori condition (3.8). Then, we have the exponential decay of $\mathcal{D}(V(t))$: For any given $\delta > 0$, we have

$$
\mathcal{D}(V(t)) \leq \left( 1 - C_2 \phi(x^\infty)^{\gamma_{g}} \right)^{\frac{1}{\delta} \left[ \mathcal{D}(V(0)) + 2\delta \kappa_1 R_{f} \mathcal{D}(\mathcal{B}(0)) \right]} \left( 1 - C_1 \phi(x^\infty)^{\gamma_{g}} \right)^{\frac{1}{\delta} \left[ \max\{1 - C_1 \phi(x^\infty)^{\gamma_{g}}, 1 - C_2 \phi(x^\infty)^{\gamma_{g}}\} \right]^{\frac{1}{\delta} - 1},
$$

where $C_1 = C_1(\delta)$ and $C_2 = C_2(\delta)$ are the constants defined in Lemmas 3.2 and 3.3, respectively.

**Proof.** We combine (3.23), Lemma 2.2 and (3.28) to obtain

$$
\mathcal{D}(V(m\delta)) \leq \left( 1 - \mu(\hat{\Phi}(m\delta, (m-1)\delta)) \right) \mathcal{D}(V((m-1)\delta)) + \frac{\sqrt{2}}{N} \left\| \int_{(m-1)\delta}^{m\delta} \hat{\Phi}(m\delta, s) \Lambda(s) ds \right\|_F \\
\leq \left( 1 - C_2 \phi(x^\infty)^{\gamma_{g}} \right) \mathcal{D}(V((m-1)\delta)) + \frac{\sqrt{2}}{N} \left\| \int_{(m-1)\delta}^{m\delta} \hat{\Phi}(m\delta, s) \right\|_F \|\Lambda(s)\|_F ds \\
\leq \left( 1 - C_2 \phi(x^\infty)^{\gamma_{g}} \right) \mathcal{D}(V((m-1)\delta)) + \frac{\sqrt{2}}{N} \int_{(m-1)\delta}^{m\delta} \sqrt{N} \|\Lambda(s)\|_F ds.
$$

(3.30)
The last inequality followed from the fact that $\Phi(m\delta, s)$ is a stochastic matrix with $N$ rows. We also use Proposition 3.1, Lemma 3.3 and (3.22) to get

$$
\|A(s)\|_F \leq \|B(s)\|_F \|V(s)\|_F \leq N\kappa_1 \mathcal{D}(B(s)) \cdot \sqrt{N} \max_{1 \leq i \leq N} \|v_i(s)\|
$$

$$
\leq N\kappa_1 (1 - C_1 \zeta(x^\infty)^\gamma \phi)^{\frac{1}{2}} \mathcal{D}(B(0)) \cdot \sqrt{N} R_v
$$

$$
\leq N \sqrt{N\kappa_1} R_v \mathcal{D}(B(0)) (1 - C_1 \zeta(x^\infty)^\gamma \phi)^{m-1}, \quad s \in [(m-1)\delta, m\delta].
$$

(3.31)

Finally, we combine (3.30) and (3.31) and get the following relation: for $m \in \mathbb{N}$,

$$
\mathcal{D}(V(m\delta)) \leq (1 - C_2 \phi(x^\infty)^\gamma \phi) \mathcal{D}(V((m-1)\delta)) + \sqrt{2} N\kappa_1 R_v \mathcal{D}(B(0)) \delta (1 - C_1 \zeta(x^\infty)^\gamma \phi)^{m-1}
$$

$$
=: (1 - C_2 \phi(x^\infty)^\gamma \phi) \mathcal{D}(V((m-1)\delta)) + C_3 (1 - C_1 \zeta(x^\infty)^\gamma \phi)^{m-1}.
$$

(3.32)

We divide both sides of (3.32) by $(1 - C_2 \phi(x^\infty)^\gamma \phi)^m$ to obtain

$$
\frac{\mathcal{D}(V(m\delta))}{(1 - C_2 \phi(x^\infty)^\gamma \phi)^m} \leq \frac{\mathcal{D}(V((m-1)\delta))}{(1 - C_2 \phi(x^\infty)^\gamma \phi)^{m-1}} + \frac{C_3}{1 - C_2 \phi(x^\infty)^\gamma \phi} \left[ \frac{1 - C_1 \zeta(x^\infty)^\gamma \phi}{1 - C_2 \phi(x^\infty)^\gamma \phi} \right]^{m-1}, \quad m \in \mathbb{N}.
$$

This and inductive arguments yield

$$
\frac{\mathcal{D}(V(m\delta))}{(1 - C_2 \phi(x^\infty)^\gamma \phi)^m} \leq \mathcal{D}(V(0)) + \frac{C_3}{1 - C_2 \phi(x^\infty)^\gamma \phi} \sum_{n=0}^{m-1} \left[ \frac{1 - C_1 \zeta(x^\infty)^\gamma \phi}{1 - C_2 \phi(x^\infty)^\gamma \phi} \right]^n, \quad m \in \mathbb{N}.
$$

Thus for any $m \in \mathbb{N}$ we have

$$
\mathcal{D}(V(m\delta)) \leq (1 - C_2 \phi(x^\infty)^\gamma \phi)^m \mathcal{D}(V(0))
$$

$$
+ C_3 \sum_{n=0}^{m-1} [1 - C_1 \zeta(x^\infty)^\gamma \phi]^n [1 - C_2 \phi(x^\infty)^\gamma \phi]^{m-n-1}
$$

$$
\leq (1 - C_2 \phi(x^\infty)^\gamma \phi)^m \mathcal{D}(V(0))
$$

$$
+ C_3 m \max\{1 - C_1 \zeta(x^\infty)^\gamma \phi, 1 - C_2 \phi(x^\infty)^\gamma \phi\}^{m-1}.
$$

(3.33)
3.3. Emergence of mono-cluster flocking

In this subsection, we present a mono-cluster flocking estimate. Note that in Proposition 3.1 and Proposition 3.2, we have alignment estimates for temperatures and velocities under the following a priori condition:

\[
\sup_{0 \leq t < \infty} \mathcal{D}(X(t)) \leq x^\infty < \infty.
\]

In the sequel, we will look for sufficient condition to guarantee the above a priori condition in terms of initial data and system parameters. Roughly speaking, our sufficient conditions can be stated as follows. If the initial position, velocity, and temperature of the particles are close enough, i.e., their corresponding diameters are sufficiently small, then spatial diameter will stay as bounded, hence temperature and velocity alignments emerge exponentially fast. For positive constants \(x^\infty > 0\) and \(\delta > 0\), we recall constants defined before:

\[
C_1 = C_1(\delta) := e^{-\kappa_2 \beta \beta_\infty^m} \cdot \frac{1}{\gamma_g^4} \left( \delta \frac{\beta \beta_\infty^m}{N} \right)^\gamma, \\
C_2 = C_2(\delta) := e^{-\kappa_1 \beta \beta_\infty^m} \cdot \frac{1}{\gamma_g^4} \left( \delta \frac{\beta \beta_\infty^m}{N} \right)^\gamma, \\
R_V^\delta = R_V^\delta(x^\infty, \delta) := \frac{1}{\beta \beta_\infty^m} R_a(0) \exp \left( \kappa_2 \beta \beta_\infty^m \mathcal{D}(B(0)) \right) C_1 \zeta(x^\infty)^{\gamma}, \\
\]

**Theorem 3.1.** Suppose that for a given positive constants \(x^\infty > 0\) and \(\delta > 0\), the
Discrete and continuous thermomechanical Cucker-Smale models on general digraph

initial data \((X^m, V^m, B^m)\) satisfy the following relation:

\[
\mathcal{D}(X(0)) + \frac{\mathcal{D}(V(0))\delta}{C_2 \phi(x^{\infty})^γ_γ} + \frac{\sqrt{2}N_κ_1R_κ V_m \mathcal{D}(B(0))\delta^2}{\min\{C_1ζ(x^{\infty})^γ_γ, C_2 \phi(x^{\infty})^γ_γ\}^2} + \frac{2κ_1 R_κ V_m \mathcal{D}(B(0))\delta^2}{C_1ζ(x^{\infty})^γ_γ} \leq x^{\infty}.
\]

(3.34)

Then, we have

(i) \(\sup_{0 \leq t < \infty} \mathcal{D}(X(t)) \leq x^{\infty}, \quad \mathcal{D}(B(t)) \leq (1 - C_1ζ(x^{\infty})^γ_γ)^\frac{1}{1+\delta} \mathcal{D}(B(0))\),

(ii) \(\mathcal{D}(V(t)) \leq (1 - C_2 \phi(x^{\infty})^γ_γ)^\frac{1}{1+\delta} \mathcal{D}(V(0)) + 2δκ_1 Rκ V_m \mathcal{D}(B(0))(1 - C_1ζ(x^{\infty})^γ_γ)^\frac{1}{1+\delta} + \sqrt{2}Nκ_1 Rκ V_m \mathcal{D}(B(0))\delta^2 \left[\max\{1 - C_1ζ(x^{\infty})^γ_γ, 1 - C_2 \phi(x^{\infty})^γ_γ\}\right]^{\frac{1}{1+\delta}}\).

Proof. We will use continuity argument. For this, we define the set \(S\) as follows:

\[
S := \left\{T > 0 : \mathcal{D}(X(t)) < x^{\infty}, \quad t \in [0, T]\right\}.
\]

Then, by (3.34), the set \(S\) is nonempty. Now, we claim that \(\sup S = \infty\). Suppose not, i.e. \(T^* := \sup S < \infty\). Then, we have

\[
\mathcal{D}(X(T^*)) = x^{\infty}.
\]

It follows from (2.11) that we have

\[
\|x_i(T^*) - x_j(T^*)\| \leq \|x_i(0) - x_j(0)\| + \int_0^{T^*} \|v_i(s) - v_j(s)\|ds
\]

\[
\leq \mathcal{D}(X(0)) + \int_0^{T^*} \mathcal{D}(V(s))ds
\]

\[
\leq \mathcal{D}(X(0)) + \mathcal{D}(V(0)) \int_0^{\infty} (1 - C_2 \phi(x^{\infty})^γ_γ)^\frac{1}{1+\delta} ds
\]

\[
+ 2δκ_1 Rκ V_m \mathcal{D}(B(0)) \int_0^{\infty} (1 - C_1ζ(x^{\infty})^γ_γ)^\frac{1}{1+\delta} ds
\]

\[
+ \sqrt{2}Nκ_1 Rκ V_m \mathcal{D}(B(0))\delta^2 \sum_{n=0}^{\infty} (1 - C_1ζ(x^{\infty})^γ_γ)^n
\]

\[
+ 2κ_1 Rκ V_m \mathcal{D}(B(0))\delta^2 \sum_{n=0}^{\infty} (1 - C_2 \phi(x^{\infty})^γ_γ)^n
\]

\[
+ \sqrt{2}Nκ_1 Rκ V_m \mathcal{D}(B(0))\delta^2 \sum_{n=1}^{\infty} n \left[\max\{1 - C_1ζ(x^{\infty})^γ_γ, 1 - C_2 \phi(x^{\infty})^γ_γ\}\right]^{n-1}
\]

\[
= \mathcal{D}(X(0)) + \frac{\mathcal{D}(V(0))\delta}{C_2 \phi(x^{\infty})^γ_γ} + \frac{\sqrt{2}Nκ_1 Rκ V_m \mathcal{D}(B(0))\delta^2}{\min\{C_1ζ(x^{\infty})^γ_γ, C_2 \phi(x^{\infty})^γ_γ\}^2} + \frac{2κ_1 Rκ V_m \mathcal{D}(B(0))\delta^2}{C_1ζ(x^{\infty})^γ_γ}
\]

\[
\leq x^{\infty}.
\]
This implies $T^* \in S$, which is a contradiction. Therefore we have $\sup S = \infty$, i.e. (i) holds. (ii) and (iii) follow from (i) by Propositions 3.1 and 3.2.

4. Emergent dynamics of the discrete model

In this section, we present an asymptotic flocking estimate for the discrete model (2.12). We perform our flocking estimate in the following three steps, which are mostly parallel to those of the continuous model.

4.1. A priori temperature alignment

In this subsection, we will derive a priori asymptotic alignment in temperature under the a priori assumption on the uniform boundedness of spatial diameter. For the convenience of presentation, we deal with the system (2.12). In the sequel, the order of presentation will be exactly parallel to that of the continuous model.

Lemma 4.1 (Boundedness of temperatures). Let $\{(x_i[t], v_i[t], \beta_i[t])\}$ be a solution to the system (2.12) with initial data $\{(x_i^{\text{in}}, v_i^{\text{in}}, \beta_i^{\text{in}})\}$. Suppose that the time-step satisfies

$$0 < h \leq \frac{1}{\kappa_2(\beta_U^{\text{in}})^2}. \quad (4.1)$$

Then, the following assertions hold:

(i) $\beta_i^{\text{in}} \leq \beta_i[t] \leq \beta_i^{\text{in}}, \quad i = 1, \cdots, N, \quad t \in \mathbb{N} \cup \{0\},$

(ii) $\mathcal{D}(\mathcal{B}[t])$ is monotone decreasing.

Proof. We define the maximal and minimal values for coldness as

$$\beta_M[t] := \max_{1 \leq i \leq N} \beta_i[t], \quad \beta_m[t] := \min_{1 \leq i \leq N} \beta_i[t], \quad t \in \mathbb{N} \cup \{0\}.$$  \hspace{1cm} (4.2)

For each $t \in \mathbb{N} \cup \{0\}$, we choose extremal indices $1 \leq m_t, M_t \leq N$ satisfying

$$\beta_m[t] = \beta_{m_t}[t] \quad \text{and} \quad \beta_M[t] = \beta_{M_t}[t].$$

We claim the following relation:

$$\beta_L^{\text{in}} \leq \beta_m[t-1] \leq \beta_m[t] \leq \beta_M[t] \leq \beta_M[t-1] \leq \beta_U^{\text{in}}, \quad t \in \mathbb{N}, \quad (4.2)$$

Proof of claim (4.2): We will use the proof by induction.

• (Initial step): The base case $t = 1$ can be shown in almost the same way as in the following inductive step.
• (Inductive step): Suppose that the relation (4.2) holds for \( t \geq 1 \). Then for \( t + 1 \), we have
\[
\frac{1}{\beta_{m[t + 1]}} - \frac{1}{\beta_{m[t]}} = \frac{1}{\beta_{m[t + 1]}} - \frac{1}{\beta_{m[t]}} = \frac{1}{\beta_{m[t + 1]}} + \frac{h}{N} \sum_{j=1}^{N} \chi_{m[t + 1]} \zeta(||x_{m[t + 1]} - x_{j}[t]||)(\beta_{m[t + 1]} - \beta_{j[t]}) - \frac{1}{\beta_{m[t]}} \\
\leq \frac{1}{\beta_{m[t + 1]}} + \frac{h}{N} \sum_{j=1}^{N} \chi_{m[t + 1]} \zeta(||x_{m[t + 1]} - x_{j}[t]||)(\beta_{m[t + 1]} - \beta_{m[t]}) - \frac{1}{\beta_{m[t]}} \\
= \left(\beta_{m[t + 1]} - \beta_{m[t]}\right) \left(- \frac{1}{\beta_{m[t]} \beta_{m[t]}} + \frac{h}{N} \sum_{j=1}^{N} \chi_{m[t + 1]} \zeta(||x_{m[t + 1]} - x_{j}[t]||)\right) \\
\leq \left(\beta_{m[t + 1]} - \beta_{m[t]}\right) \left(- \frac{1}{(\beta_{m[t]} \beta_{m[t]})^2} + h\kappa_{2}\right) \leq 0.
\]

The second and the last inequalities followed from the fact that
\[
\beta_{i[t]} - \beta_{m[t]} \geq 0, \quad i = 1, \cdots, N \Rightarrow \beta_{m[t + 1]} - \beta_{m[t]} \geq 0.
\]

Hence we have
\[
\beta_{m[t + 1]} \geq \beta_{m[t]}.
\]

On the other hand, we have
\[
\frac{1}{\beta_{M[t + 1]}} - \frac{1}{\beta_{M[t]}} = \frac{1}{\beta_{M[t + 1]}} - \frac{1}{\beta_{M[t]}} = \frac{1}{\beta_{M[t + 1]}} + \frac{h}{N} \sum_{j=1}^{N} \chi_{M[t + 1]} \zeta(||x_{M[t + 1]} - x_{j}[t]||)(\beta_{M[t + 1]} - \beta_{j[t]}) - \frac{1}{\beta_{M[t]}} \\
\geq \frac{1}{\beta_{M[t + 1]}} + \frac{h}{N} \sum_{j=1}^{N} \chi_{M[t + 1]} \zeta(||x_{M[t + 1]} - x_{j}[t]||)(\beta_{M[t + 1]} - \beta_{M[t]}) - \frac{1}{\beta_{M[t]}} \\
= \left(\beta_{M[t + 1]} - \beta_{M[t]}\right) \left(- \frac{1}{\beta_{M[t]} \beta_{M[t]}} + \frac{h}{N} \sum_{j=1}^{N} \chi_{M[t + 1]} \zeta(||x_{M[t + 1]} - x_{j}[t]||)\right) \\
\geq \left(\beta_{M[t + 1]} - \beta_{M[t]}\right) \left(- \frac{1}{(\beta_{M[t]} \beta_{M[t]})^2} + h\kappa_{2}\right) \geq 0.
\]

The second and the last inequalities followed from the fact that
\[
\beta_{i[t]} - \beta_{M[t]} \leq 0, \quad i = 1, \cdots, N \Rightarrow \beta_{M[t + 1]} - \beta_{M[t]} \leq 0.
\]

Therefore, we have
\[
\beta_{M[t + 1]} \leq \beta_{M[t]}.
\]

Finally, we combine (4.3) and (4.4) to derive the estimate (4.2). Hence (4.2) also holds for \( t + 1 \), and the induction is complete.
Next, note that (2.12) is equivalent to the following relation:

$$\beta_i[t+1] = \beta_i[t] + \frac{h}{N} \beta_i[t] \beta_j[t+1] \sum_{j=1}^{N} \chi_{ij} \zeta(\|x_i[t] - x_j[t]\|) \left( \beta_j[t] - \beta_i[t] \right), \quad t \in \mathbb{N} \cup \{0\}. \tag{4.5}$$

In fact, we can rewrite (4.5) in vector form. For this, we define an $N \times N$ matrix $L[t]$:

$$L[t] := D[t] - A[t],$$

where the matrices $A[t] = (a_{ij}[t])$ and $D[t] = \text{diag}(d_1[t], \cdots, d_N[t])$ are defined by the following relations:

$$a_{ij}[t] := \chi_{ij} \zeta(\|x_i[t] - x_j[t]\|) \quad \text{and} \quad d_i[t] = \sum_{j=1}^{N} \chi_{ij} \zeta(\|x_i[t] - x_j[t]\|)$$

We also define

$$\Gamma[t] := \text{diag}(\beta_1[t], \cdots, \beta_N[t]).$$

Then we can rewrite (4.5) as follows.

$$B[t+1] = \left( I - \frac{h}{N} \Gamma[t] \Gamma[t+1] L[t] \right) B[t], \quad t \in \mathbb{N} \cup \{0\}. \tag{4.6}$$

**Proposition 4.1.** Suppose that time-step and initial data satisfy (2.1), and let $\{(x_i, v_i, \beta_i)\}$ be a solution to system (2.12) satisfying a priori condition:

$$\sup_{t \in \mathbb{N} \cup \{0\}} \mathcal{D}(X[t]) \leq x^\infty < \infty. \tag{4.7}$$

Then, we have the exponential decay of $\mathcal{D}(B[t])$: for any given integer $n_0 \geq \gamma_g$ we have

$$\mathcal{D}(B[t]) \leq (1 - D_1(x^\infty)^{\gamma_g})^{n_0} \mathcal{D}(B[0]), \quad t \in \mathbb{N} \cup \{0\},$$

where the positive constant $D_1$ is given as follows.

$$D_1 = D_1(n_0) := \binom{n_0}{\gamma_g} (1 - h \kappa_2 (\beta^{in}_U)^2)^{n_0 - \gamma_g} \left( \frac{h(\beta^{in}_U)^2}{N} \right)^{\gamma_g}.$$ 

**Proof.** First, note that

$$-\frac{1}{N} \Gamma[t] \Gamma[t+1] L[t] = \frac{1}{N} \Gamma[t] \Gamma[t+1] (A[t] - D[t]) \geq \frac{(\beta^{in}_U)^2}{N} A^\infty - \kappa_2 (\beta^{in}_U)^2 I,$$

where $A^\infty = (a_{ij}^\infty)$ is a nonnegative matrix defined by

$$a_{ij}^\infty := \chi_{ij} \zeta(x^\infty).$$

Then, the terms inside the parenthesis of (4.6) can be estimated as follows.

$$I - \frac{h}{N} \Gamma[t] \Gamma[t+1] L[t] \geq (1 - h \kappa_2 (\beta^{in}_U)^2) I + \frac{h(\beta^{in}_U)^2}{N} A^\infty \geq 0, \quad t \in \mathbb{N} \cup \{0\}. \tag{4.8}$$
For any $t_1,t_2 \in \mathbb{N} \cup \{0\}$ with $t_2 - t_1 \geq \gamma_g$, define the matrix $\Phi[t_2, t_1]$ as follows:

$$
\Phi[t_2, t_1] := \left( I - \frac{h}{N} \Gamma[t_2 - 1] \Gamma[t_2 - 1] L[t_2 - 1] \right) \left( I - \frac{h}{N} \Gamma[t_2 - 2] \Gamma[t_2 - 1] L[t_2 - 1] \right) \cdots \\
\times \left( I - \frac{h}{N} \Gamma[t_1] \Gamma[t_1 + 1] L[t_1] \right).
$$

Then it follows from (4.8) that

$$
\Phi[t_2, t_1] \geq \left( 1 - h \kappa_2 (\beta_L^{in})^2 I + \frac{h (\beta_L^{in})^2}{N} A^\infty \right)^{t_2 - t_1}
\geq \left( \sum_{n=0}^{t_2 - t_1} \binom{t_2 - t_1}{n} (1 - h \kappa_2 (\beta_L^{in})^2)^{t_2 - t_1 - n} \left( \frac{h (\beta_L^{in})^2}{N} A^\infty \right)^n \right) \left( 1 - h \kappa_2 (\beta_L^{in})^2 \right)^{t_2 - t_1 - \gamma_g} \left( \frac{h (\beta_L^{in})^2}{N} A^\infty \right)^{\gamma_g}.
$$

Now, we fix $m \in \mathbb{N}$, and put $t_1 = (m - 1) n_0$, $t_2 = mn_0$ in (4.9) to obtain

$$
\Phi[mn_0, (m - 1)n_0] \geq \left( n_0 \gamma_g \right) \left( 1 - h \kappa_2 (\beta_L^{in})^2 \right)^{n_0 - \gamma_g} \left( \frac{h (\beta_L^{in})^2}{N} A^\infty \right)^{\gamma_g} = D_1 (A^\infty)^{\gamma_g} \geq 0.
$$

Therefore, we have

$$
\mu \left( \Phi[mn_0, (m - 1)n_0] \right) \geq D_1 \mu((A^\infty)^{\gamma_g}) \geq D_1 \zeta (x^\infty)^{\gamma_g},
$$

where in the last inequality, we used (2.2) and Lemma 2.1.

The nonnegative matrix $\Phi[mn_0, (m - 1)n_0]$ is actually stochastic, because we have the following for each $(m - 1)n_0 \leq t < mn_0$:

$$[1, \ldots, 1]^T = \left( I - \frac{h}{N} \Gamma[t] \Gamma[t + 1] L[t] \right) [1, \ldots, 1]^T.
$$

Now, it follows from the relation:

$$
B[mn_0] = \Phi[mn_0, (m - 1)n_0] B[(m - 1)n_0]
$$

that we can use Lemma 2.2 with $B = 0$ and (4.10) to obtain

$$
\mathcal{D}(B[mn_0]) \leq \left( 1 - \mu \left( \Phi[mn_0, (m - 1)n_0] \right) \right) \mathcal{D}(B[(m - 1)n_0]) \leq (1 - D_1 \zeta (x^\infty)^{\gamma_g}) \mathcal{D}(B[(m - 1)n_0]), \quad m \in \mathbb{N}.
$$

By induction, we have

$$
\mathcal{D}(B[mn_0]) \leq (1 - D_1 \zeta (x^\infty)^{\gamma_g})^m \mathcal{D}(B[0]), \quad m \in \mathbb{N}.
$$

So for any $t \in \mathbb{N} \cup \{0\}$, we have the following:

$$
\mathcal{D}(B[t]) \leq \mathcal{D} \left( B \left[ n_0, \frac{t}{mn_0} \right] \right) \leq (1 - D_1 \zeta (x^\infty)^{\gamma_g})^{\left\lfloor \frac{t}{\gamma_g} \right\rfloor} \mathcal{D}(B[0]).
$$

The first inequality was due to Lemma 4.1.
4.2. *A priori* velocity alignment

In this subsection, we will derive asymptotic velocity alignment under the *a priori* assumption on the uniform boundedness of spatial diameters. For the convenience of presentation, we set

\[ u_i[t] := v_i[t], \quad \theta_i[t] = \beta_i[t] v_i[t], \quad i = 1, \cdots, N, \text{ and} \]

\[ R_u[t] := \max_{1 \leq i \leq N} \|u_i[t]\|, \quad t \in \mathbb{N} \cup \{0\}. \]

As a first step, we study the boundedness of velocities.

**Lemma 4.2 (Boundedness of velocities).** Suppose that the time-step and initial data satisfy

\[ 0 < h \leq \min \left\{ \frac{1}{\kappa_2 (\beta^{in}_U)^2}, \frac{\beta^{in}_L}{2\kappa_1 (\beta^{in}_U)^2} \right\}, \]

and let \( \{x_i, v_i, \beta_i\} \) be a solution to system (2.12) satisfying a priori condition (4.7). Then, velocities of the particles are bounded: for any given integer \( n_0 \geq \gamma_g \) we have

\[ \|v_i[t]\| \leq \frac{1}{\beta^{in}_L} R_u[0] \exp \left[ \frac{hn_0 \kappa_2 \beta^{in}_U D(B[0])}{D_1 \zeta(x^\infty)^\gamma_g} \right] =: R^d_V = R^d_V(x^\infty, n_0), \quad t \in \mathbb{N} \cup \{0\}, \]

where \( D_1 \) is the constant defined in Proposition 4.1.

**Proof.** Since the proof is rather lengthy, we leave its proof in Appendix A.

Our next job is to introduce an inequality for \( D(V) \) which will be used later.

**Lemma 4.3.** Suppose that the time-step and initial data satisfy

\[ 0 < h \leq \min \left\{ \frac{1}{\kappa_2 (\beta^{in}_U)^2}, \frac{\beta^{in}_L}{2\kappa_1 (\beta^{in}_U)^2} \right\}, \]

and let \( \{x_i, v_i, \beta_i\} \) be a solution to system (2.12) satisfying a priori condition (4.7). Then, for any given integer \( n_0 \geq \gamma_g \) we have

\[ D(V[t+1]) \leq D(V[t]) + 2h\kappa_1 R^d_V D(B[t]), \quad t \in \mathbb{N} \cup \{0\}, \]

where \( R^d_V = R^d_V(x^\infty, n_0) \) is the constant defined in Lemma 4.2.

**Proof.** Since the proof is lengthy, we leave its proof in Appendix B.

**Proposition 4.2 (Exponential decay of the velocity diameter).** Suppose that the time-step and initial data satisfy

\[ 0 < h \leq \min \left\{ \frac{1}{\kappa_2 (\beta^{in}_U)^2}, \frac{\beta^{in}_L}{2\kappa_1 (\beta^{in}_U)^2} \right\}, \]
and let \( \{(x_i, v_i, \beta_i)\} \) be a solution to system (2.12) satisfying a priori condition (4.7). Then we have the exponential decay of \( \mathcal{D}(V[t]) \): for any given integer \( n_0 \geq \gamma_g \) we have
\[
\mathcal{D}(V[t]) \leq (1 - D_2 \phi(x^{\infty})^{\gamma_g})^{\frac{1}{\gamma_g}} \mathcal{D}(V[0]) + 2hn_0 \kappa_1 R_V \alpha \left(1 - D_1 \zeta(x^{\infty})^{\gamma_g}\right)^{\frac{1}{\gamma_g}} \mathcal{D} \left( \mathcal{B}[0] \right)
+ \sqrt{2hn_0 N \kappa_1 R_V \mathcal{D} \left( \mathcal{B}[0] \right)} \left[ \frac{1}{n_0} \right] \left[ \max \{1 - D_1 \zeta(x^{\infty})^{\gamma_g}, 1 - D_2 \phi(x^{\infty})^{\gamma_g}\} \right]^{\frac{1}{\gamma_g} - 1},
\]
where \( D_1 \) is the constant defined in Proposition 4.1 and
\[
D_2 = D_2(n_0) := \left(\frac{n_0}{\gamma_g}\right) (1 - h \kappa_1 \beta_U^{\alpha})^{n_0 - \gamma_g} \left(\frac{h \beta_U^{\gamma_g}}{N}\right)^{\gamma_g}.
\]

**Proof.** We leave its proof in Appendix C.

### 4.3. Emergence of mono-cluster flocking

In this subsection, we derive a mono-cluster flocking estimate by verifying the a priori assumption (4.7) by imposing some conditions on system parameters and initial data. More precisely, our second main result can be summarized as follows. We set
\[
D_1 = D_1(n_0) := \left(\frac{n_0}{\gamma_g}\right) (1 - h \kappa_2 \beta_U^{\gamma_g})^{n_0 - \gamma_g} \left(\frac{h \beta_U^{\gamma_g}}{N}\right)^{\gamma_g},
D_2 = D_2(n_0) := \left(\frac{n_0}{\gamma_g}\right) (1 - h \kappa_1 \beta_U^{\gamma_g})^{n_0 - \gamma_g} \left(\frac{h \beta_U^{\gamma_g}}{N}\right)^{\gamma_g},
R_V = R_V(\alpha, n_0) := \frac{1}{\beta_U^{\gamma_g}} R_u[0] \exp \left[ \frac{hn_0 \kappa_2 \beta_U^{\gamma_g} \mathcal{D} \left( \mathcal{B}[0] \right)}{D_1 \zeta(x^{\infty})^{\gamma_g}} \right].
\]

**Theorem 4.1.** Let a real number \( \alpha > 0 \) and an integer \( n_0 \geq \gamma_g \) be given, and suppose that the time-step and initial data satisfy
\[
0 < h \leq \min \left\{ \frac{1}{\kappa_2 \beta_U^{\gamma_g}}, \frac{\beta_U^{\gamma_g}}{2 \kappa_1 \beta_U^{\gamma_g}} \right\},
\]
\[
\mathcal{D}(X[0]) + \frac{hn_0 \mathcal{D}(V[0])}{D_2 \phi(x^{\infty})^{\gamma_g}} + \sqrt{2hn_0 \kappa_1 R_V} \left[ \min \{D_1 \zeta(x^{\infty})^{\gamma_g}, D_2 \phi(x^{\infty})^{\gamma_g}\} \right]^{\frac{1}{\gamma_g}} \mathcal{D} \left( \mathcal{B}[0] \right) \leq x^{\infty},
\]
and let \( \{(x_i, v_i, \beta_i)\} \) be a solution to system (2.12). Then we have
\[
(i) \quad \sup_{t \in [0, \infty)} \mathcal{D}(X[t]) \leq x^{\infty}, \quad \mathcal{D}(\mathcal{B}[t]) \leq (1 - D_1 \zeta(x^{\infty})^{\gamma_g})^{\frac{1}{\gamma_g}} \mathcal{D}(\mathcal{B}[0]),
(ii) \quad \mathcal{D}(V[t]) \leq (1 - D_2 \phi(x^{\infty})^{\gamma_g})^{\frac{1}{\gamma_g}} \mathcal{D}(V[0]) + 2hn_0 \kappa_1 R_V \left(1 - D_1 \zeta(x^{\infty})^{\gamma_g}\right)^{\frac{1}{\gamma_g}} \mathcal{D} \left( \mathcal{B}[0] \right)
+ \sqrt{2hn_0 N \kappa_1 R_V} \mathcal{D} \left( \mathcal{B}[0] \right) \left[ \frac{1}{n_0} \right] \left[ \max \{1 - D_1 \zeta(x^{\infty})^{\gamma_g}, 1 - D_2 \phi(x^{\infty})^{\gamma_g}\} \right]^{\frac{1}{\gamma_g} - 1}.
\]
Proof. We claim:

$$\mathcal{D}(X[t]) \leq x^\infty$$ for $t \in \mathbb{N} \cup \{0\}$.

We will prove this by induction on $t$.

- Initial step: For the case $t = 0$, it is clear from (4.11).

- Induction step: Suppose that the claim holds for $0 \leq t \leq m$. Then, we have

$$\|x_i[m+1] - x_j[m+1]\|$$
$$\leq \|x_i[0] - x_j[0]\| + h \sum_{n=0}^{m} \|v_i[n] - v_j[n]\|$$
$$\leq \mathcal{D}(X[0]) + h \sum_{n=0}^{m} \mathcal{D}(V[n])$$
$$\leq \mathcal{D}(X[0]) + h \sum_{n=0}^{m} \sum_{n=0}^{\infty} (1 - D_2 \phi(x^\infty)^{\gamma_\phi})^{\frac{n}{n_0}}$$
$$+ 2h^2n_0 \kappa_1 R_V^d \mathcal{D}(B[0]) \sum_{n=0}^{\infty} (1 - D_1 \zeta(x^\infty)^{\gamma_\zeta})^{\frac{n}{n_0}}$$
$$+ \sqrt{2h^2n_0^2 \kappa_1 R_V^d \mathcal{D}(B[0])} \sum_{n=0}^{\infty} n [\max\{1 - D_1 \zeta(x^\infty)^{\gamma_\zeta}, 1 - D_2 \phi(x^\infty)^{\gamma_\phi}\}]^{\frac{n}{n_0} - 1}$$
$$= \mathcal{D}(X[0]) + h n_0 \mathcal{D}(V[0]) \sum_{n=0}^{\infty} (1 - D_2 \phi(x^\infty)^{\gamma_\phi})^n$$
$$+ 2h^2n_0^2 \kappa_1 R_V^d \mathcal{D}(B[0]) \sum_{n=0}^{\infty} (1 - D_1 \zeta(x^\infty)^{\gamma_\zeta})^n$$
$$+ \sqrt{2h^2n_0^2 \kappa_1 R_V^d \mathcal{D}(B[0])} \sum_{n=0}^{\infty} n [\max\{1 - D_1 \zeta(x^\infty)^{\gamma_\zeta}, 1 - D_2 \phi(x^\infty)^{\gamma_\phi}\}]^{n-1}$$
$$= \mathcal{D}(X[0]) + h n_0 \mathcal{D}(V[0]) \sum_{n=1}^{\infty} \frac{\sqrt{2h^2n_0^2 \kappa_1 R_V^d \mathcal{D}(B[0])}}{D_2 \phi(x^\infty)^{\gamma_\phi}} \left[\min\{D_1 \zeta(x^\infty)^{\gamma_\zeta}, D_2 \phi(x^\infty)^{\gamma_\phi}\}\right]^{n-1}$$
$$\leq x^\infty.$$

Therefore, the claim holds for $t = m + 1$, and the induction is complete. So a priori condition (i) does hold, and the alignment estimates (ii) and (iii) follow from Proposition 4.1 and Proposition 4.2 respectively.

Remark 4.1. Fix a real number $\delta > 0$ and take $n_0 = \left\lfloor \frac{\delta}{h}\right\rfloor$. Then as $h \to 0$, the left-hand side of (4.11) approaches that of (3.34).
5. Conclusion

In this paper, we presented a mono-cluster flocking estimate for a thermodynamic Cucker-Smale model. As aforementioned in Introduction, most flocking models in literature deal with mechanical models, i.e., position and momentum are macroscopic observables. Thus, internal structures of particles, e.g., spin, temperature, vibration, etc., are often ignored in the modeling. Recently, Ha and Ruggeri introduced thermodynamic particle models which are consistent with thermodynamics and the Cucker-Smale model for the isothermal case. They derived the generalized Cucker-Smale model with temperature from the gas mixture models. Thus, it inherits the entropy principle as gas mixture system does. In a previous series of works on the emergent dynamics on the TCS model, most flocking analysis has been done mostly for the complete networks. Thus, interaction between network structure and system dynamics are completely decoupled. In this work, we presented exponential flocking estimates for the continuous and discrete TCS model with small diffusion velocities. Our proposed frameworks are formulated in terms of system parameters and initial data. Of course, there are many issues which have not been addressed in this paper. For example, we have only dealt with mono-cluster flocking. However, as noticed in the Cucker-Smale model, depending on the initial data and nature of communication weight (short range or long range), we might have multi-cluster flockings. Moreover, we do not have a detailed information on the spatial structure of resulting asymptotic flocking states. We leave these interesting issues for a future work.
Appendix A. Proof of Lemma 4.2

For the proof, it suffices to show the upper bound of $R_u[t]$: \[ R_u[t] \leq R_u[0] \exp \left[ \frac{h\kappa_2\beta_{in}^U D(B[0])}{D_1\zeta(x_0)^\gamma_9} \right], \quad t \in \mathbb{N} \cup \{0\}. \]

For each $t \in \mathbb{N} \cup \{0\}$, we choose an extremal index $1 \leq M_i \leq N$ satisfying the relation:

\[ \|u_{M_i}[t]\| = R_u[t]. \]

For each $i = 1, \ldots, N$, we have

\[
\|u_i[t + 1]\|^2 - \|u_i[t]\|^2 = \beta_i[t + 1]^2\|v_i[t + 1]\|^2 - \beta_i[t]^2\|v_i[t]\|^2 \\
= \|v_i[t]\|^2(\beta_i[t + 1]^2 - \beta_i[t]^2) + \beta_i[t + 1]^2(\|v_i[t + 1]\|^2 - \|v_i[t]\|^2) \\
=: I_{21} + I_{22}, \quad t \in \mathbb{N} \cup \{0\}.
\]

Below, we estimate the terms $I_{2i}, i = 1, 2$ one by one.

- (Estimate of $I_{21}$): By direct calculation, we have

\[
I_{21} = \|v_i[t]\|^2(\beta_i[t + 1] + \beta_i[t])(\beta_i[t + 1] - \beta_i[t]) \\
= \frac{h}{N}\|v_i[t]\|^2(\beta_i[t + 1] + \beta_i[t])\beta_i[t + 1]\|u_i[t]\|^2 \sum_{j=1}^N \chi_{ij}\zeta(\|x_i[t] - x_j[t]\|)(\beta_j[t] - \beta_i[t]) \\
= \frac{h}{N} \left( 1 + \frac{\beta_i[t + 1]}{\beta_i[t]} \right) \beta_i[t + 1]\|u_i[t]\|^2 \sum_{j=1}^N \chi_{ij}\zeta(\|x_i[t] - x_j[t]\|)(\beta_j[t] - \beta_i[t]) \\
\leq \left( 1 + \frac{\beta_i[t + 1]}{\beta_i[t]} \right) h\kappa_2\beta_{in}^U D(B[t])\|u_i[t]\|^2 \\
= \left( 1 + \frac{\beta_i[t + 1]}{\beta_i[t]} \right) h\kappa_2\beta_{in}^U D(B[t])\|u_i[t]\|^2 \\
\leq \left( 1 + \frac{\beta_i[t + 1]}{\beta_i[t]} \right) h\kappa_2\beta_{in}^U D(B[t])\|u_i[t]\|^2 \tag{A.2}
\]

- (Estimate of $I_{22}$): We set

\[
P := \frac{h}{N} \sum_{j=1}^N \chi_{ij}\phi(\|x_i[t] - x_j[t]\|) \quad \text{and} \quad Q := \frac{h}{N} \sum_{j=1}^N \chi_{ij}\phi(\|x_i[t] - x_j[t]\|)\|u_j[t]\|.
\]

Then, we use the estimate

\[
P \leq h\kappa_1 \leq \frac{1}{\beta_{in}^U} \leq \frac{1}{\beta_i[t]} \tag{A.3}
\]
to obtain

\[ T_{22} = \beta_i[t + 1]^2 \left( \| v_i[t] + \frac{h}{N} \sum_{j=1}^{N} \chi_{ij} \phi(\| x_i[t] - x_j[t] \|) (\beta_j[t] v_j[t] - \beta_i[t] v_i[t]) \|^2 - \| v_i[t] \|^2 \right) \]

\[ = \beta_i[t + 1]^2 \left( 2 \left\langle v_i[t], \frac{h}{N} \sum_{j=1}^{N} \chi_{ij} \phi(\| x_i[t] - x_j[t] \|) (\beta_j[t] v_j[t] - \beta_i[t] v_i[t]) \right\rangle \right. \]

\[ + \left. \left\| \frac{h}{N} \sum_{j=1}^{N} \chi_{ij} \phi(\| x_i[t] - x_j[t] \|) \left( \beta_i[t] v_i[t] - \chi_{ij} \phi(\| x_i[t] - x_j[t] \|) u_j[t] - Pu_i[t] \right) \right\| \right)^2 \]

\[ = \beta_i[t + 1]^2 \left( 2 \left\langle u_i[t], \frac{h}{N} \sum_{j=1}^{N} \chi_{ij} \phi(\| x_i[t] - x_j[t] \|) u_j[t] - Pu_i[t] \right\rangle \right. \]

\[ + \left. \left\| \frac{h}{N} \sum_{j=1}^{N} \chi_{ij} \phi(\| x_i[t] - x_j[t] \|) u_j[t] - Pu_i[t] \right\|^2 \right)^2 \]

\[ \leq \beta_i[t + 1]^2 \left( P^2 - \frac{2P}{\beta_i[t]} \| u_i[t] \|^2 + \left( \frac{h}{N} \sum_{j=1}^{N} \chi_{ij} \phi(\| x_i[t] - x_j[t] \|) u_j[t] \right)^2 \right) \]

\[ + \left( \frac{1}{\beta_i[t]} - P \right) \frac{2h}{N} \sum_{j=1}^{N} \chi_{ij} \phi(\| x_i[t] - x_j[t] \|) \langle u_i[t], u_j[t] \rangle \]

\[ = \beta_i[t + 1]^2 \left( Q - P \| u_i[t] \| \right) \left( Q - \left( P - \frac{2}{\beta_i[t]} \| u_i[t] \| \right) \right) \]

\[ =: \mathcal{F}_1(Q). \]

Note that \( \mathcal{F}_1 \) is a convex function in \( Q \), and we have

\[ 0 \leq Q \leq P \| u_{M_i}[t] \|. \]

Thus, we have

\[ \mathcal{F}_1(Q) \leq \max\{ \mathcal{F}_1(0), \mathcal{F}_1(P \| u_{M_i}[t] \|) \}. \]

By (A.3), we have

\[ \mathcal{F}_1(0) \leq 0 \leq \mathcal{F}_1(P \| u_{M_i}[t] \|). \]
Hence, we have

\[ I_{22} \leq \beta_i[t+1]^2 \left( P \| u_M[t] \| - P \| u_i[t] \| \right) \left( P \| u_M[t] \| - \left( P - \frac{2}{\beta_i[t]} \right) \| u_i[t] \| \right) \]

\[ = P \beta_i[t+1]^2 \left( \| u_M[t] \| - \| u_i[t] \| \right) \left( \left( \| u_M[t] \| - \| u_i[t] \| \right) P + \frac{2}{\beta_i[t]} \| u_i[t] \| \right) \]

\[ \leq h\kappa_1(\beta_U^m)^2 \left( \| u_M[t] \| - \| u_i[t] \| \right) \left( \| u_M[t] \| - \| u_i[t] \| \right) h\kappa_1 + \frac{2}{\beta_L^m} \| u_i[t] \| \]

\[ \leq h\kappa_1(\beta_U^m)^2 \left( \| u_M[t] \| - \| u_i[t] \| \right) \left( h\kappa_1 \| u_M[t] \| + \frac{2}{\beta_L^m} \| u_i[t] \| \right). \]

\[ (A.4) \]

We combine (A.1), (A.2), and (A.4) to obtain

\[ \| u_i[t+1] \|^2 - \| u_i[t] \|^2 \leq \left( 2 + h\kappa_2\beta_U^m D(B[t]) \right) h\kappa_2\beta_U^m D(B[t]) \| u_i[t] \|^2 + h\kappa_1(\beta_U^m)^2 \left( \| u_M[t] \| - \| u_i[t] \| \right) \left( h\kappa_1 \| u_M[t] \| + \frac{2}{\beta_L^m} \| u_i[t] \| \right). \]

\[ (A.5) \]

Now, we take \( i = M_{t+1} \) in (A.5) to get

\[ \| u_{M_{t+1}}[t+1] \|^2 - \| u_M[t] \|^2 \]

\[ = \| u_{M_{t+1}}[t+1] \|^2 - \| u_M[t] \|^2 + \| u_{M_{t+1}}[t+1] \|^2 - \| u_{M_{t+1}}[t] \|^2 \]

\[ \leq \| u_{M_{t+1}}[t+1] \|^2 - \| u_M[t] \|^2 + \left( 2 + h\kappa_2\beta_U^m D(B[t]) \right) h\kappa_2\beta_U^m D(B[t]) \| u_{M_{t+1}}[t] \|^2 \]

\[ + h\kappa_1(\beta_U^m)^2 \left( \| u_M[t] \| - \| u_{M_{t+1}}[t] \| \right) \left( h\kappa_1 \| u_M[t] \| + \frac{2}{\beta_L^m} \| u_{M_{t+1}}[t] \| \right) \]

\[ = \left( 2 + h\kappa_2\beta_U^m D(B[t]) \right) h\kappa_2\beta_U^m D(B[t]) \| u_{M_{t+1}}[t] \|^2 \]

\[ - \left( \| u_M[t] \| - \| u_{M_{t+1}}[t] \| \right) \left( 1 - h\kappa_1(\beta_U^m)^2 \right) \| u_M[t] \| + \frac{2h\kappa_1(\beta_U^m)^2}{\beta_L^m} \| u_{M_{t+1}}[t] \| \]

\[ \leq \left( 2 + h\kappa_2\beta_U^m D(B[t]) \right) h\kappa_2\beta_U^m D(B[t]) \| u_{M_{t+1}}[t] \|^2 \]

\[ \leq \left( 2 + h\kappa_2\beta_U^m D(B[t]) \right) h\kappa_2\beta_U^m D(B[t]) \| u_M[t] \|^2. \]

This yields

\[ \| u_{M_{t+1}}[t+1] \|^2 \leq \left( 1 + h\kappa_2\beta_U^m D(B[t]) \right) \| u_M[t] \|^2. \]

Hence, we have

\[ R_u[t+1] \leq \left( 1 + h\kappa_2\beta_U^m D(B[t]) \right) R_u[t], \quad t \in \mathbb{N} \cup \{0\}. \]
Now, we apply Proposition 4.1 to obtain

\[
R_u[t] \leq R_u[0] \prod_{n=0}^{t-1} \left( 1 + h\kappa_2 \beta_U^{in} D(B[n]) \right)
\]

\[
= R_u[0] \exp \left[ \sum_{n=0}^{t-1} \log \left( 1 + h\kappa_2 \beta_U^{in} D(B[n]) \right) \right]
\]

\[
\leq R_u[0] \exp \left[ \sum_{n=0}^{t-1} \left( h\kappa_2 \beta_U^{in} D(B[n]) \right) \right]
\]

\[
\leq R_u[0] \exp \left[ h\kappa_2 \beta_U^{in} D(B[0]) \sum_{n=0}^{\infty} (1 - D_1 \zeta(x^\infty)^{\gamma_s}) \frac{1}{\pi_n} \right]
\]

\[
= R_u[0] \exp \left[ h\kappa_2 \beta_U^{in} D(B[0]) \sum_{n=0}^{\infty} \frac{(n+1)n_0 - 1}{\pi_n} \sum_{k=nn_0} (1 - D_1 \zeta(x^\infty)^{\gamma_s}) \right]
\]

\[
= R_u[0] \exp \left[ h\kappa_2 \beta_U^{in} D(B[0]) \sum_{n=0}^{\infty} n_0 (1 - D_1 \zeta(x^\infty)^{\gamma_s}) \right]
\]

\[
= R_u[0] \exp \left[ \frac{h\kappa_2 \beta_U^{in} D(B[0])}{D_1 \zeta(x^\infty)^{\gamma_s}} \right].
\]

This completes the proof of Lemma 4.2.

**Appendix B. Proof of Lemma 4.3**

For each \( t \in \mathbb{N} \cup \{0\} \), we choose extremal indices \( 1 \leq i_t, j_t \leq N \) satisfying the relation:

\[
D(V[t]) = \|v_{i_t}[t] - v_{j_t}[t]\|.
\]

For each \( i = 1, \ldots, N \), we have

\[
\|v_i[t + 1] - v_j[t + 1]\|^2 - \|v_i[t] - v_j[t]\|^2
\]

\[
= \left\| v_i[t] + h \sum_{k=1}^{N} \chi_{ik} \phi_k[t] \left( \beta_k[t] v_k[t] - \beta_i[t] v_i[t] \right) - v_j[t] - h \sum_{k=1}^{N} \chi_{jk} \phi_k[t] \left( \beta_k[t] v_k[t] - \beta_j[t] v_j[t] \right) \right\|^2
\]

\[
= \| (1 - P)(v_i[t] - v_j[t]) + X + Y \|^2 - \|v_i[t] - v_j[t]\|^2,
\]
where the quantities $P, X$ and $Y$ are defined as follows:

$$
P := \frac{h}{N} \sum_{k=1}^{N} \chi_{ik} \phi_{ik}[t] \beta_i[t] + \frac{h}{N} \sum_{k=1}^{N} \chi_{jk} \phi_{jk}[t] \beta_j[t],
$$

$$
X := \frac{h}{N} \sum_{k=1}^{N} \chi_{ik} \phi_{ik}[t] \left( \beta_i[t] v_k[t] - \beta_i[t] v_j[t] \right) - \frac{h}{N} \sum_{k=1}^{N} \chi_{jk} \phi_{jk}[t] \left( \beta_j[t] v_k[t] - \beta_j[t] v_j[t] \right),
$$

$$
Y := \frac{h}{N} \sum_{k=1}^{N} \chi_{ik} \phi_{ik}[t] \left( \beta_i[t] v_k[t] - \beta_i[t] v_i[t] \right) - \frac{h}{N} \sum_{k=1}^{N} \chi_{jk} \phi_{jk}[t] \left( \beta_k[t] v_k[t] - \beta_j[t] v_i[t] \right).
$$

We write $\phi_{ij}[t] := \phi(\|v_i[t] - x_j[t]\|), i, j = 1, 2, \ldots, N$ for convenience.

Next, we rewrite (B.1) as follows:

$$
\|v_i[t] + 1 - v_j[t + 1]\|^2 - \|v_i[t] - v_j[t]\|^2 = \left( \|1 - P\|v_i[t] - v_j[t]\| + X \right)^2 - \|v_i[t] - v_j[t]\|^2
$$

$$
+ \left( \|1 - P\|v_i[t] - v_j[t]\| + X \right)^2 - \|(1 - P)(v_i[t] - v_j[t]) + X\|^2
$$

$$
=: \mathcal{I}_{31} + \mathcal{I}_{32}. \quad (B.2)
$$

Below, we estimate the terms $\mathcal{I}_{3i}, i = 1, 2$ one by one.

- (Estimate of $\mathcal{I}_{31}$): Note that

$$
P \leq 2h\kappa_1 \beta_{ij}^{\text{ini}} \leq 1. \quad (B.3)
$$

So we have

$$
\mathcal{I}_{31} = (P^2 - 2P)\|v_i[t] - v_j[t]\|^2 + 2(1 - P)(v_i[t] - v_j[t], X) + \|X\|^2
$$

$$
\leq (P^2 - 2P)\|v_i[t] - v_j[t]\|^2 + 2(1 - P)(v_i[t] - v_j[t])\|X\| + \|X\|^2
$$

$$
= \left( \|X\| - P\|v_i[t] - v_j[t]\| \right) \left( \|X\| + (2 - P)\|v_i[t] - v_j[t]\| \right)
$$

$$
=: \mathcal{F}_2(\|X\|).
$$

Note that $\mathcal{F}_2$ is a convex function, and we have

$$
0 \leq \|X\| \leq \frac{h}{N} \sum_{k=1}^{N} \chi_{ik} \phi_{ik}[t] \beta_i[t] v_k[t] - v_j[t] \| + \frac{h}{N} \sum_{k=1}^{N} \chi_{jk} \phi_{jk}[t] \beta_j[t] v_k[t] - v_i[t] \|
$$

$$
\leq P\|v_i[t] - v_j[t]\|.
$$

Thus, we have

$$
\mathcal{F}_2(\|X\|) \leq \max \left\{ \mathcal{F}_2(0), \mathcal{F}_2(P\|v_i[t] - v_j[t]\|) \right\}.
$$

On the other hand, it follows from (B.3) that we have

$$
\mathcal{F}_2(0) \leq 0 \leq \mathcal{F}_2(P\|v_i[t] - v_j[t]\|).
$$
Hence, we have
\[
I_{31} \leq \mathcal{F}_2(P\|v_i[t] - v_j[t]\|)
\]
\[
= \left( P\|v_i[t] - v_j[t]\| - P\|v_i[t] - v_j[t]\| \right) \left( P\|v_i[t] - v_j[t]\| + (2 - P)\|v_i[t] - v_j[t]\| \right)
\]
\[
= \left( \|v_i[t] - v_j[t]\| - \|v_i[t] - v_j[t]\| \right) \left( P^2\|v_i[t] - v_j[t]\| + (2P - P^2)\|v_i[t] - v_j[t]\| \right)
\]
\[
\leq \left( \|v_i[t] - v_j[t]\| - \|v_i[t] - v_j[t]\| \right) \left( \|v_i[t] - v_j[t]\| + \|v_i[t] - v_j[t]\| \right)
\]
\[
= \|v_i[t] - v_j[t]\|^2 - \|v_i[t] - v_j[t]\|^2
\]
\[
= D(V[t])^2 - \|v_i[t] - v_j[t]\|^2.
\]
(B.4)

- (Estimate of $I_{32}$): In this case, note that

\[
\|Y\| \leq \frac{h}{N} \sum_{k=1}^{N} \chi_{jk} \phi_{jk}[t] |\beta_k[t] - \beta_i[t]| v_k[t] || + \frac{h}{N} \sum_{k=1}^{N} \chi_{jk} \phi_{jk}[t] |\beta_k[t] - \beta_j[t]| v_k[t] ||
\]
\[
\leq 2h\kappa_1 R_V D(B[t])
\]

and

\[
\|(1 - P)(v_i[t] - v_j[t]) + X\| \leq D(V[t]),
\]

by (B.4). Hence, we have

\[
I_{32} = 2\|(1 - P)(v_i[t] - v_j[t]) + X, Y\| \|^2
\]
\[
\leq 2\|(1 - P)(v_i[t] - v_j[t]) + X\|\|Y\| \|^2
\]
\[
\leq 2D(V[t]) \cdot 2h\kappa_1 R_V^4 D(B[t]) + \left( 2h\kappa_1 R_V^4 D(B[t]) \right)^2 \]
\[
= \left( D(V[t]) + 2h\kappa_1 R_V^4 D(B[t]) \right)^2 - D(V[t])^2.
\]

(B.5)

Now, it follows from (B.2), (B.4) and (B.5) that we have

\[
\|v_i[t+1] - v_j[t+1]\|^2 - \|v_i[t] - v_j[t]\|^2
\]
\[
\leq \left( D(V[t]) + 2h\kappa_1 R_V^4 D(B[t]) \right)^2 - \|v_i[t] - v_j[t]\|^2,
\]
i.e.

\[
\|v_i[t+1] - v_j[t+1]\|^2 \leq \left( D(V[t]) + 2h\kappa_1 R_V^4 D(B[t]) \right)^2.
\]

(B.6)

We take \((i, j) = (i_{t+1, j_{t+1}})\) in (B.6) to obtain

\[
D(V[t+1]) \leq \left( D(V[t]) + 2h\kappa_1 R_V^4 D(B[t]) \right)^2,
\]
i.e.

\[
D(V[t+1]) \leq D(V[t]) + 2h\kappa_1 R_V^4 D(B[t]).
\]

This completes the proof of Lemma 4.3.
Appendix C. Proof of Proposition 4.2

We can rewrite (2.12) in a more concise form. We define an $N \times N$ matrix $\tilde{L}[t]$ by

$$\tilde{L}[t] := \tilde{D}[t] - \tilde{A}[t],$$

where the matrices $\tilde{A}[t] = (\tilde{a}_{ij}[t])$ and $\tilde{D}[t] = \text{diag}(\tilde{d}_1[t], \ldots, \tilde{d}_N[t])$ are defined by the following relations:

$$\tilde{a}_{ij}[t] := \chi_{ij}(||x_i[t] - x_j[t]||) \quad \text{and} \quad \tilde{d}_i[t] = \sum_{j=1}^{N} \chi_{ij}(||x_i[t] - x_j[t]||).$$

Recall that $\Gamma[t] := \text{diag}(\beta_1[t], \ldots, \beta_N[t])$. Then we can rewrite (2.12) as

$$V[t + 1] = \left[ I - \frac{h}{N} \Gamma[t] \tilde{L}[t] \right] V[t] + \frac{h}{N} \Lambda[t], \quad (C.1)$$

where the $N \times d$ matrix $\Lambda[t] := (\lambda_i^k[t])_{1 \leq i \leq N, 1 \leq k \leq d}$ is defined by

$$\lambda_i^k[t] := \sum_{j=1}^{N} \chi_{ij}(||x_i[t] - x_j[t]||)(\beta_j[t] - \beta_i[t])u_j^k[t], \quad i = 1, \ldots, N, \quad k = 1, \ldots, d, \quad t \in \mathbb{N} \cup \{0\}.$$ 

Lastly, we define the $N \times N$ matrix $B[t] = (b_{ij}[t])$ by

$$b_{ij}[t] = \chi_{ij}(||x_i[t] - x_j[t]||)(\beta_j[t] - \beta_i[t]), \quad i, j = 1, \ldots, N.$$ 

Then we have $\Lambda[t] = B[t]V[t]$.

Note that

$$-\frac{1}{N} \Gamma[t] \tilde{L}[t] = \frac{1}{N} \Gamma[t] (\tilde{A}[t] - \tilde{D}[t]) \geq \frac{\beta_{1n}^n}{N} \tilde{A}^\infty - \kappa_1 \beta_{1n}^n I,$$

where $\tilde{A}^\infty = (\tilde{a}_{ij}^\infty)$ is a nonnegative matrix defined by

$$\tilde{a}_{ij}^\infty := \chi_{ij}(x^\infty).$$

Hence we have

$$I - \frac{h}{N} \Gamma[t] \tilde{L}[t] \geq (1 - h\kappa_1 \beta_{1n}^n)I + \frac{h\beta_{1n}^n}{N} \tilde{A}^\infty \geq 0, \quad t \in \mathbb{N} \cup \{0\}. \quad (C.2)$$

For any $t_1, t_2 \in \mathbb{N} \cup \{0\}$ with $t_2 - t_1 \geq \gamma_g$, define the matrix $\tilde{\Phi}[t_2, t_1]$ as follows:

$$\tilde{\Phi}[t_2, t_1] := \left( I - \frac{h}{N} \Gamma[t_2 - 1] \tilde{L}[t_2 - 1] \right) \left( I - \frac{h}{N} \Gamma[t_2 - 2] \tilde{L}[t_2 - 2] \right) \cdots \left( I - \frac{h}{N} \Gamma[t_1] \tilde{L}[t_1] \right).$$
It follows from (C.2) that
\[
\Phi[t_2, t_1] \geq \left((1 - h \kappa_1 \beta_{U}^{(n)}) I + \frac{h \beta_{U}^{(n)}}{N} \tilde{A} \right)^{t_2 - t_1} \\
= \sum_{n=0}^{t_2 - t_1} \binom{t_2 - t_1}{n} (1 - h \kappa_1 \beta_{U}^{(n)})^{t_2 - t_1 - n} \left( \frac{h \beta_{U}^{(n)}}{N} \tilde{A} \right)^n \geq \left( \frac{t_2 - t_1}{\gamma_g} \right) (1 - h \kappa_1 \beta_{U}^{(n)})^{t_2 - t_1 - \gamma_g} \left( \frac{h \beta_{U}^{(n)}}{N} \tilde{A} \right)^{\gamma_g}.
\]

(C.3)

Now, we fix \( m \in \mathbb{N} \) and put \( t_1 = (m - 1)n_0, t_2 = mn_0 \) in (C.3) to obtain
\[
\Phi[mn_0, (m - 1)n_0] \geq \left( \frac{n_0}{\gamma_g} \right) (1 - h \kappa_1 \beta_{U}^{(n)})^{n_0 - \gamma_g} \left( \frac{h \beta_{U}^{(n)}}{N} \tilde{A} \right)^{\gamma_g} = D_2(\tilde{A})^{\gamma_g} \geq 0.
\]

Then, we use Lemma 2.1 to derive
\[
\mu(\Phi[mn_0, (m - 1)n_0]) \geq D_2 \mu((\tilde{A})^{\gamma_g}) \geq D_2 \phi(x^{\infty})^{\gamma_g}.
\]

(C.4)

Note that for each \((m - 1)n_0 \leq t < mn_0:\)
\[
[1, \ldots, 1]^T = (I - \frac{h}{N} \Gamma[t] \tilde{L}[t])[1, \ldots, 1]^T.
\]

Thus, the nonnegative matrix \( \Phi[mn_0, (m - 1)n_0] \) is actually stochastic.

On the other hand, it follows from (C.1) that we have
\[
V[mn_0] = \Phi[mn_0, (m - 1)n_0]V[(m - 1)n_0] \\
+ \sum_{n=0}^{n_0 - 1} \Phi[mn_0, (m - 1)n_0 + n + 1] \left( \frac{h}{N} \Lambda[(m - 1)n_0 + n] \right),
\]

with the convention \( \Phi[mn_0, mn_0] := I. \)

Now, we use Lemma 2.2 and (C.4) to obtain
\[
\mathcal{D}(V[mn_0]) \leq \left(1 - \mu(\Phi[mn_0, (m - 1)n_0])\right) \mathcal{D}(V[(m - 1)n_0]) \\
+ \frac{\sqrt{h}}{N} \left\| \sum_{n=0}^{n_0 - 1} \Phi[mn_0, (m - 1)n_0 + n + 1] \Lambda[(m - 1)n_0 + n] \right\|_F \\
\leq \left(1 - D_2 \phi(x^{\infty})^{\gamma_g}\right) \mathcal{D}(V[(m - 1)n_0]) \\
+ \frac{\sqrt{h}}{N} \sum_{n=0}^{n_0 - 1} \left\| \Phi[mn_0, (m - 1)n_0 + n + 1] \Lambda[(m - 1)n_0 + n] \right\|_F \\
\leq \left(1 - D_2 \phi(x^{\infty})^{\gamma_g}\right) \mathcal{D}(V[(m - 1)n_0]) + \frac{\sqrt{h}}{N} \sum_{n=0}^{n_0 - 1} \sqrt{N} \left\| \Lambda[(m - 1)n_0 + n] \right\|_F.
\]

(C.5)
The last inequality follows from the fact that $\Phi[mn_0, (m-1)n_0+n+1]$ is a stochastic matrix with $N$ rows. We use Proposition 4.1 and Lemma 4.2 to derive

\[
\|\Lambda[(m-1)n_0 + n]\|_F \\
\leq \|B[(m-1)n_0 + n]\|_F \|V[(m-1)n_0 + n]\|_F \\
\leq N\kappa_1 D(B[(m-1)n_0 + n]) \cdot \sqrt{N} \max_{1 \leq i \leq N} \|v_i[(m-1)n_0 + n]\|_F \\
\leq N\kappa_1 (1 - D_1 \zeta(x^{\infty})^{\gamma_0})^{m-1} D(B[0]) \cdot \sqrt{N} R_V, 0 \leq n \leq n_0 - 1.
\] (C.6)

Next, we combine (C.5) and (C.6) to get the following relation:

\[
D(V[mn_0]) \leq (1 - D_2 \phi(x^{\infty})^{\gamma_0}) D(V[(m-1)n_0]) \\
+ \sqrt{2hn_0} N\kappa_1 R_V^d D(B[0]) (1 - D_1 \zeta(x^{\infty})^{\gamma_0})^{m-1} \\
= : (1 - D_2 \phi(x^{\infty})^{\gamma_0}) D(V[(m-1)n_0]) + D_3 (1 - D_1 \zeta(x^{\infty})^{\gamma_0})^{m-1}, m \in \mathbb{N}.
\] (C.7)

We divide both sides of (C.7) by $(1 - D_2 \phi(x^{\infty})^{\gamma_0})^m$ to obtain

\[
\frac{D(V[mn_0])}{(1 - D_2 \phi(x^{\infty})^{\gamma_0})^m} \leq \frac{D(V[(m-1)n_0])}{(1 - D_2 \phi(x^{\infty})^{\gamma_0})^{m-1}} + \frac{D_3}{1 - D_2 \phi(x^{\infty})^{\gamma_0}} \left[ \frac{1 - D_1 \zeta(x^{\infty})^{\gamma_0}}{1 - D_2 \phi(x^{\infty})^{\gamma_0}} \right]^{m-1}, m \in \mathbb{N}.
\]

This yields

\[
\frac{D(V[mn_0])}{(1 - D_2 \phi(x^{\infty})^{\gamma_0})^m} \leq D(V[0]) + \frac{D_3}{1 - D_2 \phi(x^{\infty})^{\gamma_0}} \sum_{n=0}^{m-1} \left[ \frac{1 - D_1 \zeta(x^{\infty})^{\gamma_0}}{1 - D_2 \phi(x^{\infty})^{\gamma_0}} \right]^n, m \in \mathbb{N}.
\]

Thus for any $m \in \mathbb{N}$ we have

\[
D(V[mn_0]) \\
\leq (1 - D_2 \phi(x^{\infty})^{\gamma_0})^m D(V[0]) + D_3 \sum_{n=0}^{m-1} [1 - D_1 \zeta(x^{\infty})^{\gamma_0}]^n [1 - D_2 \phi(x^{\infty})^{\gamma_0}]^{m-n-1} \\
\leq (1 - D_2 \phi(x^{\infty})^{\gamma_0})^m D(V[0]) + D_3 m \left[ \max\{1 - D_1 \zeta(x^{\infty})^{\gamma_0}, 1 - D_2 \phi(x^{\infty})^{\gamma_0}\} \right]^{m-1}.
\] (C.8)
So for any $t \in \mathbb{N} \cup \{0\}$, we combine Lemma 4.3, Proposition 4.1, and (C.8) to deduce

$$D(V[t]) \leq D \left( V \left[ n_0 \left\lfloor \frac{t}{n_0} \right\rfloor \right] \right) + \sum_{n=n_0(\frac{t}{n_0})}^{t-1} 2h\kappa_1 R_V^d D(B[n])$$

$$\leq D \left( V \left[ n_0 \left\lfloor \frac{t}{n_0} \right\rfloor \right] \right) + 2h \left( t - n_0 \left\lfloor \frac{t}{n_0} \right\rfloor \right) \kappa_1 R_V^d (1 - D_1 \zeta(x^\infty \gamma_8) \frac{t}{n_0}) D(B[0])$$

$$\leq D \left( V \left[ n_0 \left\lfloor \frac{t}{n_0} \right\rfloor \right] \right) + 2hn_0\kappa_1 R_V^d (1 - D_1 \zeta(x^\infty \gamma_8) \frac{t}{n_0}) D(B[0])$$

$$\leq (1 - D_2 \phi(x^\infty \gamma_8) \frac{t}{n_0}) D(V[0]) + D_3 \left\lfloor \frac{t}{n_0} \right\rfloor \left[ \max \{1 - D_1 \zeta(x^\infty \gamma_8), 1 - D_2 \phi(x^\infty \gamma_8) \} \right] \frac{t}{n_0} |\frac{t}{n_0}|^{-1}$$

$$+ 2hn_0\kappa_1 R_V^d (1 - D_1 \zeta(x^\infty \gamma_8) \frac{t}{n_0}) D(B[0])$$

$$= (1 - D_2 \phi(x^\infty \gamma_8) \frac{t}{n_0}) D(V[0]) + 2hn_0\kappa_1 R_V^d (1 - D_1 \zeta(x^\infty \gamma_8) \frac{t}{n_0}) D(B[0])$$

$$+ \sqrt{2hn_0N\kappa_1 R_V^d} D(B[0]) \left\lfloor \frac{t}{n_0} \right\rfloor \left[ \max \{1 - D_1 \zeta(x^\infty \gamma_8), 1 - D_2 \phi(x^\infty \gamma_8) \} \right] \frac{t}{n_0} |\frac{t}{n_0}|^{-1}.$$
Acknowledgment The work of S.-Y. Ha is supported by the Samsung Science and Technology Foundation under Project Number SSTF-BA1401-03. The work of J.-G. Dong was supported in part by NSFC grant 11671109.

References

1. S. Ahn, H. Choi, S.-Y. Ha and H. Lee, On the collision avoiding initial-configurations to the Cucker-Smale type flocking models, *Comm. Math. Sci.* 10 (2012) 625-643.
2. S. Ahn and S.-Y. Ha, Stochastic flocking dynamics of the Cucker-Smale model with multiplicative white noises, *J. Math. Phys.* 51 (2010) 103301.
3. M. Ballerini, N. Cabibbo, R. Candelier, A. Cavagna, E. Cisbani, I. Giardina, V. Lecomte, A. Orlandi, G. Parisi, A. Procaccini, M. Viale and V. Zdravkovic, Interaction ruling animal collective behavior depends on topological rather than metric distance: evidence from a field study, *Proc. Natl. Acad. Sci. USA*, 105 (2008) 1232-1237.
4. N. Bellomo and S.-Y. Ha, A quest toward a mathematical theory of the dynamics of swarms, *Math. Mod. Meth. Appl. Sci.* 27 (2017) 745-770.
5. F. Bolley, J. A. Canizo and J. A. Carrillo, Stochastic mean-field limit: non-Lipschitz forces and swarming, *Math. Mod. Meth. Appl. Sci.* 21 (2011), 2179-2210.
6. J. A. Canizo, J. A. Carrillo and J. Rosado, A well-posedness theory in measures for some kinetic models of collective motion, *Math. Mod. Meth. Appl. Sci.* 21 (2011) 515-539.
7. J. A. Carrillo, M. Fornasier, J. Rosado and G. Toscani, Asymptotic flocking dynamics for the kinetic Cucker-Smale model, *SIAM J. Math. Anal.* 42 (2010) 218-236.
8. J. Cho, S.-Y. Ha, F. Huang, C. Jin and D. Ko, Emergence of bi-cluster flocking for the Cucker-Smale model, *Math. Mod. Meth. Appl. Sci.* 26 (2016) 1191-1218.
9. Y.-P. Choi and J. Haskovec, Cucker-Smale model with normalized communication weights and time delay, *Kinetic and Related Models*, 10 (2017), 1011-1033.
10. Y.-P. Choi and Z. Li, Emergent behavior of Cucker-Smale flocking particles with time-lags, Submitted.
11. Y.-P. Choi, S.-Y. Ha, and J. Kim, Propagation of regularity and finite-time collisions for the thermomechanical Cucker-Smale model with a singular communication, To appear in Networks and Heterogeneous Media.
12. Y.-P. Choi, S.-Y. Ha, J. Jung and J. Kim, Global dynamics of the thermodynamic Cucker-Smale ensemble immersed in incompressible viscous fluids, Submitted.
13. Y.-P. Choi, S.-Y. Ha, J. Jung and J. Kim, On the coupling of kinetic thermomechanical Cucker-Smale equation and compressible viscous fluid system, Submitted.
14. Y.-P. Choi, S.-Y. Ha and Z. Li, Emergent dynamics of the Cucker-Smale flocking model and its variants, In N. Bellomo, P. Degond, and E. Tadmor (Eds.), Active Particles Vol.I - Theory, Models, Applications(tentative title), Series: Modeling and Simulation in Science and Technology, Birkhauser-Springer.
15. F. Cucker and J.-G. Dong, On flocks influenced by closest neighbors, *Math. Mod. Meth. Appl. Sci.* 26 (2016) 2685-2708.
16. F. Cucker and J.-G. Dong, Avoiding collisions in flocks, *IEEE Trans. Automat. Control* 55 (2010) 1238-1243.
17. F. Cucker and E. Mordecki, Flocking in noisy environments, *J. Math. Pure Appl.* 89 (2008) 278-296.
18. F. Cucker and S. Smale, Emergent behavior in flocks, *IEEE Trans. Automat. Control* 52 (2007) 852-862.
19. F. Cucker and S. Smale, On the mathematics of emergence, *Japan. J. Math.* 2 (2007)
Discrete and continuous thermomechanical Cucker-Smale models on general digraph

197-227.

20. J.-G. Dong and L. Qiu, Flocking of the Cucker-Smale model on general digraphs, IEEE Trans. Automat. Control 62 (2017) 5234-5239.

21. J.-G. Dong, S.-Y. Ha and D. Kim, Interplay of time-delay and velocity alignment in the Cucker-Smale model on a general digraph, Submitted.

22. J.-G. Dong, S.-Y. Ha, D. Kim and J. Kim, Time-delay effect on the flocking in an ensemble of thermomechanical Cucker-Smale particles, J. Differential Equations (2018) https://doi.org/10.1016/j.jde.2018.08.034.

23. R. Duan, M. Fornasier and G. Toscani, A kinetic flocking model with diffusion, Commun. Math. Phys. 300 (2010) 95-145.

24. R. Erban, J. Haskovec and Y. Sun, On Cucker-Smale model with noise and delay, SIAM. J. Appl. Math. 76 (2016) 1535-1557.

25. M. Fornasier, J. Haskovec and G. Toscani, Fluid dynamic description of flocking via Povzner-Boltzmann equation, Phys. D 240 (2011) 21-31.

26. S.-Y. Ha, D. Ko and Y. Zhang, Critical coupling strength of the Cucker-Smale model for flocking, Math. Mod. Meth. Appl. Sci. 27 (2017) 1051-1087.

27. Y. Liu and J. Wu, Flocking and asymptotic velocity of the Cucker-Smale model with processing delay, J. Math. Anal. Appl. 415 (2014) 53-61.

28. S.-Y. Ha, D. Kim and Z. Li, Emergent flocking dynamics of the discrete thermodynamic Cucker-Smale model, Submitted.

29. S.-Y. Ha, J. Kim, C. Min, T. Ruggeri and X. Zhang, A global existence of classical solutions to the hydrodynamic Cucker-Smale model in presence of a temperature field, Submitted.

30. S.-Y. Ha, J. Kim, C. Min, T. Ruggeri and X. Zhang, Uniform stability of approximate thermodynamic Cucker-Smale model, Submitted.

31. S.-Y. Ha, J. Kim and T. Ruggeri, Emergent behaviors of thermodynamic Cucker-Smale particles, To appear in SIAM. J. Math. Anal.

32. S.-Y. Ha, J. Kim and X. Zhang, Uniform stability of the Cucker-Smale model and its application to the mean-field limit, Submitted.

33. S.-Y. Ha, K. Lee and D. Levy, Emergence of time-asymptotic flocking in a stochastic Cucker-Smale system, Commun. Math. Sci. 7 (2009) 453-469.

34. S.-Y. Ha and J.-G. Liu, A simple proof of Cucker-Smale flocking dynamics and mean field limit, Commun. Math. Sci. 7 (2009) 297-325.

35. S.-Y. Ha and T. Ruggeri, Emergent dynamics of a thermodynamically consistent particle model, Arch. Ratson. Mech. Anal. 223 (2017) 1397-1425.

36. S.-Y. Ha and E. Tadmor, From particle to kinetic and hydrodynamic description of flocking, Kinetic Relat. Models 1 (2008) 415-435.

37. N. E. Leonard, D. A. Paley, F. Lekien, R. Sepulchre, D. M. Fratantoni and R. E. Davis, Collective motion, sensor networks and ocean sampling, Proc. IEEE 95 (2007) 48-74.

38. Z. Li and S.-Y. Ha, On the Cucker-Smale flocking with alternating leaders, Quart. Appl. Math. 73 (2015) 693-709.

39. Z. Li and X. Xue, Cucker-Smale flocking under rooted leadership with fixed and switching topologies, SIAM J. Appl. Math. 70 (2010) 3156-3174.

40. S. Motsch and E. Tadmor, Heterophilious dynamics: Enhanced Consensus, SIAM Review 56 (2014) 577-621.

41. S. Motsch and E. Tadmor, A new model for self-organized dynamics and its flocking behavior, J. Statist. Phys. 144 (2011) 923-947.

42. J. Park, H. Kim and S.-Y. Ha, Cucker-Smale flocking with inter-particle bonding forces, IEEE Trans. Automat. Control 55 (2010) 2617-2623.
43. L. Perea, P. Elosegui and G. Gómez, Extension of the Cucker-Smale control law to space flight formation, *J. of Guidance, Control and Dynamics* 32 (2009) 527-537.
44. C. Pignotti and I. R. Vallejo, Flocking estimates for the Cucker-Smale model with time lag and hierarchical leadership, [arXiv:1707.09244] (2017).
45. D. Poyato and J. Soler, Euler-type equations and commutators in singular and hyperbolic limits of kinetic Cucker-Smale models, *Math. Mod. Meth. Appl. Sci.* 6 (2017) 1089-1152.
46. C. W. Reynolds, Flocks, herds, and schools: A distributed behavioral model, *Comput. Graph.*, 21 (1987) 25–34.
47. J. Shen, Cucker-Smale flocking under hierarchical leadership, *SIAM J. Appl. Math.* 68 (2007), 694-719.
48. E.D. Sontag, Mathematical Control Theory, 2nd edn. Texts in Applied Mathematics, vol. 6 Springer-Verlag 1998.
49. J. Toner and Y. Tu, Flocks, herds, and Schools: A quantitative theory of flocking, *Physical Review E.* 58 (1998) 4828-4858.
50. T. Vicsek, A. Czirók, E. Ben-Jacob, I. Cohen and O. Schochet, Novel type of phase transition in a system of self-driven particles, *Phys. Rev. Lett.* 75 (1995) 1226-1229.
51. T. Vicsek and A. Zefeiris, Collective motion, *Phys. Rep.* 517 (2012) 71-140.