Scattering in a spatially varying mass $\mathcal{PT}$-symmetric double heterojunction

Anjana Sinha$^{(a)}$ and R. Roychoudhury$^{2,3(b)}$

$^1$ Department of Instrumentation Science, Jadavpur University - Kolkata 700 032, India

$^2$ Department of Mathematics, Visva Bharati University - Santiniketan 731 235, West Bengal, India

$^3$ Advanced Centre for Nonlinear and Complex Phenomena - 1175 Survey Park, Kolkata - 700075, India

received 12 June 2013; accepted in final form 4 September 2013
published online 27 September 2013

PACS 03.65.Nk - Scattering theory

PACS 03.65.-w - Quantum mechanics

Abstract – We observe that the reflection and transmission coefficients of a particle within a double, $\mathcal{PT}$-symmetric heterojunction with spatially varying mass and non-vanishing imaginary part near the edges, show interesting features, depending on the degree of non-Hermiticity, despite the absence of exceptional point and spectral singularity. Exact analytical solutions for the wave function for both bound and scattering states are obtained, and the reflection and transmission coefficients are plotted as a function of energy, for both left as well as right incidence. As expected, the spatial dependence on mass changes the nature of the scattering solutions within the heterojunctions, whereas the space-time ($\mathcal{PT}$) symmetry is responsible for the left-right asymmetry in the scattering coefficients. However, the non-vanishing imaginary component of the potential near the heterojunction edges gives new and interesting results, hitherto unnoticed.

Copyright © EPLA, 2013

Introduction. – Position-dependent effective mass (PDEM) formalism is extremely important in describing the electronic and transport properties of quantum wells and quantum dots, impurities in crystals, He-clusters, quantum liquids, semiconductor heterostructures, etc. [1–6]. In semiconductor heterostructures (e.g., Al$_x$Ga$_{1-x}$As), the varying doping concentration $x$ along the $z$-axis makes the mass of the charge carrier (electron or hole) dependent on its position. On the other hand, it is an established fact that non-Hermitian quantum systems with $\mathcal{PT}$ symmetry, open up a fascinating world, unknown to conventional Hermitian systems [7–9]. $\mathcal{PT}$ synthetic novel optical devices, with balanced gain and loss, have been engineered to exhibit several intriguing features unknown to Hermitian optical systems — e.g., double refraction, power oscillations, unidirectional invisibility, left-right asymmetry, non-reciprocal diffraction patterns, etc. [10–19].

In this work we study a special form of semiconductor device consisting of a thin layer of $\mathcal{PT}$-symmetric material sandwiched between two normal semiconductors, such that the effective mass of the charge carrier (electron or hole) varies with the position within the heterojunctions, but is constant outside. The mass $m(z)$ and the real part of the potential $V_R(z)$ are taken to be continuous throughout the device. The form of $V(z)$ (where $V(z) = V_R(z) + V_I(z)$) is such that $V_I(z)$ does not vanish near the heterojunctions. It is this non-vanishing imaginary component that gives rise to some interesting results. We obtain the exact analytical solutions for the bound and scattering states of a particle inside such a semiconductor device and also the reflection and transmission amplitudes, $R$ and $T$, respectively. Our primary aim here is to look for any anomaly in $|R|$ and/or $|T|$, with increase in the magnitude of the imaginary component $V_I(z)$.

The paper is organized as follows: For the sake of completeness, the PDEM Schrödinger equation is briefly introduced in the second section to show the procedure for obtaining the exact analytical solutions. In the third section, we study an explicit non-Hermitian $\mathcal{PT}$-symmetric double heterojunction, with the properties described earlier. Though we pay sufficient attention to obtaining the exact analytical solutions and bound-state energies, the main stress in this work is on the behaviour of the transmission and reflection coefficients, $|T|^2$ and $|R|^2$, respectively, with respect to the relative strengths of the real and imaginary parts of the potential, and also the side of...
entry of the particle. A series of graphs plotted in figs. 1 to 8 illustrate our observations. The fourth section is kept for conclusions and discussions.

**Theory.** – Within the heterojunctions, $a_1 < z < a_2$, where the particle mass varies with position, the Hermitian kinetic energy term $T_{EM}$ is given by [4,20]

$$T_{EM} = \frac{1}{2} \left( m^0 \frac{\partial}{\partial z} + m \frac{\partial}{\partial z} \right)^2 + \frac{\gamma^2}{2} \left( \frac{\partial}{\partial z} \right)^2$$

(1)

where $p = -i\hbar \frac{\partial}{\partial z}$ is the momentum operator. The ambiguity parameters $\alpha, \beta, \gamma$ obey the von Roos constraint [4,20]

$$\alpha + \beta + \gamma = -1 \quad (2)$$

but have no unique definition. For simplicity of calculations, we shall work in units $\hbar = c = 1$, and use prime to denote differentiation with respect to $z$. For continuity conditions at the abrupt interfaces, well-behaved ground-state energy, and the best fit to experimental results [21], we shall restrict the ambiguity parameters to satisfy the BenDaniel-Duke choice, $viz, \alpha = \gamma = 0, \beta = -1$ [22]. Hence, in the intermediate layer $a_1 < z < a_2$, the kinetic energy term in eq. (1) reduces to

$$T = \frac{1}{2} p \left( \frac{1}{m} \right) p$$

(3)

so that the Hamiltonian for the particle with PDEM becomes [23]

$$H = -\frac{1}{2m(z)} \frac{d^2}{dz^2} + \left( \frac{1}{2m(z)} \right) \frac{d}{dz} + V_R(z) + iV_I(z), \quad (4)$$

whereas, outside the well, $z < a_1$ and $z > a_2$, the particle obeys the conventional Schrödinger equation

$$\left\{ -\frac{1}{2m_{1,2}(z)} \frac{d^2}{dz^2} + V_{01,02}(z) \right\} \psi(z) = E\psi(z)$$

(5)

having plane-wave solutions. In case we consider a wave incident from the left, the solutions in the two regions are

$$\psi_R(z) = Te^{ik_1z}, \quad -\infty < z < a_1,$$

$$\psi_L(z) = e^{ik_2z}, \quad a_2 < z < \infty,$$

(6)

where $R$ and $T$ denote the reflection and transmission amplitudes, and

$$k_{1,2} = \sqrt{2m_{1,2}(E - V_{01,02})}.$$

(7)

The important point worth remembering here is that for PDEM systems, the solutions $\psi(z)$ obey modified boundary conditions [22,24] —the functions $\psi(z)$ and $\frac{d\psi(z)}{dz}$ are continuous at each heterojunction $a_1$ and $a_2$. These are used to calculate $R, T$ and the bound-state energies. Using the transformations [25]

$$\psi_m(z) = \left(2m(z)\right)^{1/4} \phi(\rho),$$

$$\rho = \int \sqrt{2m(z)}dz,$$

(8)

the Schrödinger equation (4) for PDEM in the region $a_1 < z < a_2$, reduces to one for constant mass $m$,

$$-\frac{\partial^2 \phi}{\partial \rho^2} + \left\{ \tilde{V}(\rho) - E \right\} \phi = 0$$

(9)

with

$$\tilde{V}(\rho) = V(z) + \frac{7}{32} \frac{m'^2}{m^3} - \frac{m''}{8m^2}.$$  

(10)

Evidently, eq. (9) can be solved analytically for some particular cases of $V(z)$ and $m(z)$ only. In ref. [21], we had given the exact analytical solutions for one such case, with vanishing imaginary part near the heterojunctions, which shows the phenomenon of spontaneous $PT$-symmetry breaking, and admits a spectral singularity. In this work, we shall study a second case, where the most important contribution of the imaginary part of the potential is near the heterojunctions. Thus, this present study is distinctively different from the work done in ref. [21].

**Explicit model: $PT$-symmetric diffused potential well with PDEM.** – We assume the real part of the intermediate layer to be a diffused quantum well, with the following ansatz for the potential $V(z)$ and mass $m(z)$:

$$V(z) = \begin{cases} \mu_1 \frac{1}{1+z^2} + i\frac{\mu_2}{\sqrt{1+z^2}}, & |z| < a_0, \\ \mu_1 \frac{1}{1+a_0^2} = V_0, & |z| > a_0; \end{cases}$$

(11)

$$m(z) = \begin{cases} \frac{g^2}{2(1+z^2)}, & |z| < a_0, \\ \frac{g^2}{2(1+a_0^2)} = m_0, & |z| > a_0. \end{cases}$$

(12)

Here $\mu_1, \mu_2, g$ are some constant parameters.

Figure 1 shows the position-dependent effective mass $m(z)$ and the potential $V(z)$ in the entire semiconductor device, as a function of $z$, for a suitable set of parameter values, $viz, g = 1.5, \mu_1 = 4, \mu_2 = 3$: (a) $a_0 = 2.5$; (b) $a_0 = 3.2$. While the imaginary part of the potential $V_I(z)$ may appear to be the same as in ref. [21] for a very thin intermediate layer, the difference between the two models becomes more prominent as the thickness of the intermediate layer increases —while $V_I(z)$ tapers off as one goes farther away from the centre in ref. [21], it reaches a constant value in the model considered in this work.

For the spatial mass dependence given by eq. (12), eq. (8) transforms the coordinate $z$ to

$$\rho = g \sinh^{-1} z,$$

(13)

so that, after some straightforward algebra, $\tilde{V}(\rho)$ in eq. (10) reduces to

$$\tilde{V}(\rho) = \frac{1}{4g^2} - \left( \mu_1 - \frac{1}{g^2} \right) \coth^2 \rho + i\mu_2 g^2 \tanh \rho.$$  

(14)
Thus, eq. (9) may be written as
\[
\frac{d^2\phi}{d\rho^2} + \left\{ \kappa^2 + s \left( s + 1 \right) \text{sech}^2 \rho - 2i\lambda \tanh \rho \right\} \phi = 0, \quad (15)
\]
where
\[
\kappa^2 = Eg^2 - \frac{1}{4}, \quad \rho = \frac{\theta}{g}
\]
and the new parameters \( s \) and \( \lambda \) are expressed in terms of the constants \( \mu_1, \mu_2 \) and \( g \), as
\[
\lambda = \frac{1}{2}\mu_2g^2, \quad s = -\frac{1}{2} \pm g\sqrt{\mu_1}.
\]
One can check from eq. (14) that the diffused potential well (eq. (11)) within the intermediate layer \(|z| < a_0\), with spatially varying mass \( m(z) \), reduces to the standard PT-symmetric Rosen-Morse II potential for constant mass [26,27]. Now, the standard PT-symmetric Rosen-Morse II potential has the following unique characteristics: i) absence of quasi-parity; ii) only real energy due to the absence of spontaneous breakdown of PT symmetry; iii) switching of the bound-state energies from negative to positive values, with increase in the magnitude of the non-Hermiticity parameter.

Additionally, the soliton solution of the corresponding non-linear Schrödinger equation is always unstable [27], unlike that in the Scarf II case.

These points of difference from the PT-symmetric Scarf II model studied in ref. [21] provide the main motivation for studying this particular case.

Our aim in this work is basically two-fold: i) to find the bound states of the system, check for exceptional points, and observe the effect of \( \mu_2 \), if any; ii) to see the effect of \( \mu_2 \) on the behaviour of the reflection and transmission coefficients.

To obtain the solution of (15), we introduce a new variable
\[
y = \frac{1 - i \tanh \rho}{2},
\]
and write the solution as
\[
\phi = y^{5/2}(1 - y)^{\nu/2} \chi(y).
\]
After some straightforward algebra, eq. (15) reduces to the hypergeometric equation [28]
\[
y(1 - y)\frac{d^2\chi}{dy^2} + \left\{ \delta + 1 - (\delta + \nu + 2) y \right\} \frac{d\chi}{dy} - \left\{ \frac{\delta + \nu + 1}{2} \right\}^2 - \mu_1g^2 \chi = 0, \quad (20)
\]
where \( \delta \) and \( \nu \) are determined from the expressions
\[
\delta^2 + \kappa^2 - 2i\lambda = 0, \quad \nu^2 + \kappa^2 + 2i\lambda = 0. \quad (21)
\]

Now, (20) has complete solution [28]
\[
\chi = P \binom{2F1}{a, b; c; y} + Qy^{1-c} \binom{2F1}{1 + a - c, 1 + b - c, 2 - c; y}, \quad (22)
\]
where \( P \) and \( Q \) are constants, and the parameters \( a, b \) and \( c \) are as defined below
\[
a = \frac{\delta + \nu + 1}{2} + g\sqrt{\mu_1},
b = \frac{\delta + \nu + 1}{2} - g\sqrt{\mu_1},
c = \delta + 1. \quad (23)
\]
It is known from the literature [26] that for bound states of the standard PT-symmetric Rosen-Morse II potential, \( \text{Re}(\delta) > 0 \) and \( \text{Re}(\delta) < 0 \) are two mutually exclusive cases. Additionally, the regularity of the solution demands \( \text{Re}(\nu) > 0 \). These conditions follow from the asymptotic form of the Rosen-Morse II wave functions. However, since the Rosen-Morse II solutions are restricted to the central region only in the present model, these conditions are no longer relevant. This allows for a wider variety of solutions, although the qualitative picture remains unchanged, as we shall see in fig. 2.
After some straightforward algebra, the final solution to the PDEM Schrödinger equation within the potential well $|z| < a_0$, reduces to

$$\psi_{in}(z) = (2m)^{1/4} e^{\sqrt{2}y} |1 - y|^{\nu/2} \left\{ P_2 F_1(a, b, c; y) + Qy^{1-c} 2^c F_1(1 + a - c, 1 + b - c, 2 - c; y) \right\}.$$  

(24)

Outside the well ($|z| > a_0$), the scattering solutions are given by eq. (6), with $k_1 = k_2 = k$ (say), viz,

$$\psi_L(z) = e^{ikz} + Re^{-ikz}, \quad -\infty < z < a_1,$nineq1$$

$$\psi_R(z) = T e^{ikz}, \quad a_2 < z < \infty,$nineq2$$

while the bound states are given by

$$\psi_L^{(b)}(z) = A_1 e^{k_1 z}, \quad -\infty < z < a_1,$nineq3$$

$$\psi_R^{(b)}(z) = A_2 e^{-k_2 z}, \quad a_2 < z < \infty,$nineq4$$

while $k_b = 2m_0 \sqrt{V_0 - E}$.

To obtain the solution in the entire region, $A_1$, $A_2$, $P$, $Q$, $R$ and $T$ are determined by applying the modified boundary conditions at each heterojunction $\pm a_0$, and the properties of the hypergeometric functions $2F_1(a, b, c; y)$ [28], and taking the help of Mathematica. Analogous to our previous studies on non-Hermitian [21] and Hermitian [29] models, the scattering solution (plotted for the set of parameter values $g = 1.5$, $\mu_1 = 4$, $\mu_2 = 0.3$, $a_0 = 2.5$, for $E = 44$) depicts the change in the nature of the otherwise plane-wave solutions due to the dependence of mass on the position of the particle in the intermediate region. For the above parameter values, bound-state energies are obtained at: ground state $E_0 = -8.82$, first excited state $E_1 = -2.57$, second excited state $E_2 = -0.9$, etc. The corresponding wave functions are plotted in fig. 2, for $Re \delta > 0$, $Re \nu > 0$. The inset curve on the top right shows the scattering state solution, and that on the bottom left shows the fist excited state for $Re \delta < 0$, $Re \nu < 0$. Positive and negative values of $Re(\delta)$ and $Re(\nu)$ leave the qualitative nature of the wave functions unchanged.

We observe a very interesting phenomenon — the standard $PT$-symmetric Rosen-Morse II potential for constant mass (I) and the diffused quantum well with PDEM sandwiched between two heterojunctions (II), share some similar features: i) the bound-state energy is always real, hence there is no spontaneous breakdown of $PT$ symmetry; ii) the bound-state energy switches from a negative to a positive value occurs at a lower value of $\mu_5$, for particle entering the device from left or right.

As $\mu_5$ increases, the behaviour of $|R_L|$, $|R_R|$ and $|T|$ changes abruptly. For low values of $\mu_2$, for particle entering the device from left or right, $|T|$ increases with increasing energy, finally reaching unity — total transmission. This observation is similar to that given in our earlier study [21]. However, as $\mu_2$ increases, $|T|$ first decreases,
reaches a minimum, and then increases to reach a saturation value. Once again, the pattern is identical for left and right incidence. This peculiar behaviour is shown in the 3D plot of fig. 5. This abrupt change of behaviour occurs at a particular value of $\mu_2$, and the trend continues for all values of $\mu_2$ greater than this value.

Similarly, if one draws the 3D plots for $|R_L|^2$ and $|R_R|^2$, with respect to energy and $\mu_2$, as shown in fig. 6 and fig. 7, respectively, once again there is an abrupt change in their behaviour at and beyond some critical value of $\mu_2$. The qualitative behaviour of $|T|^2$, $|R_L|^2$ and $|R_R|^2$ for large values of $\mu_2$, is shown in fig. 8 (for $\mu_2 = 3$). This is in sharp contrast to their behaviour at low values of $\mu_2$, as shown in fig. 4. However, the interesting point to note here is that the scattering coefficients remain finite everywhere, so the system does not exhibit spectral singularity. Thus, in spite of the absence of spectral singularity, the non-Hermiticity parameter $\mu_2$ plays a crucial role in the behaviour of the scattering amplitudes, similar to its role in deciding the sign of bound state (negative or positive). These are the most important findings of the present study. In this respect, the present work gives results quite different from ref. [21]. In the limit $\mu_2 \to 0$, we get back the Hermitian result: $|R_L| = |R_R| = |R|$, and $|T|^2 + |R|^2 = 1$.

Conclusions and discussions. – To conclude, in the present work we obtain the exact analytical solutions for the bound and scattering states of a particle (charge carrier) when a thin layer of non-Hermitian $\mathcal{PT}$ material with spatially varying mass is sandwiched between two layers of a normal semiconductor, thus forming a double heterojunction. The intermediate layer described by a diffused quantum well with spatially varying mass, reduces to the $\mathcal{PT}$-symmetric Rosen-Morse II potential with constant mass, after a suitable transformation. Despite the absence of spontaneous breakdown of $\mathcal{PT}$ symmetry and spectral singularity (ss), this particular model displays some unique characteristics, possibly due to the non-vanishing imaginary part of the potential near the heterojunctions. This is in stark contrast to the spontaneous breakdown of $\mathcal{PT}$ symmetry and ss observed in our earlier work [21].

The series of graphs plotted in the paper show the potential and mass functions in fig. 1, and the first three bound-state solutions in the semiconductor device in fig. 2, for $\text{Re}\ \delta > 0$, $\text{Re}\ \nu > 0$. Additionally, in fig. 2, the inset graph on the top right shows a typical exact analytical scattering state solution, while the inset graph on the bottom left shows the first excited state for negative values of $\text{Re}\ \delta$ and $\text{Re}\ \nu$. The truncation of the imaginary potential at the bounding walls at $\pm a_0$ allows for all possible combinations of $\delta, \nu$. While the effect of the PDEM is to
change the nature of the otherwise plane-wave scattering solutions in the intermediate layer, bound-state solutions show kinks at the heterojunctions. Numerical calculations show that the bound-state energy switches from a negative to a positive value with the increase in the value of the non-Hermiticity parameter $\mu_2$, similar to the standard $\mathcal{PT}$-symmetric Rosen-Morse II potential for constant mass, but quite unlike the model considered in ref. [21]. The relative strengths of $\mu_1, \mu_2$, where this switching occurs for the ground-state energy, are shown in the contour plot of fig. 3. The behaviour of the scattering amplitudes, as shown in figs. 4 to 8, gives credence to the important role played by $\mu_2$. For low values of $\mu_2$, the nature of the reflection and transmission coefficients, as shown in fig. 4, is analogous to the observation in our previous study [21]. However, as $\mu_2$ increases beyond a certain value, the qualitative picture of these coefficients changes abruptly. Nevertheless, this cannot be called a spectral singularity (ss), since contrary to the blowing up of these coefficients viz, $|T|^2, |R_L|^2$ and $|R_R|^2$, at ss [21,30,32], they remain finite. This observation is totally different from that observed in our earlier study done in ref. [21]. The behaviour of $|R|^2$ and $|T|^2$ depends on whether the particle is entering the device from the left or from the right — this non-Hermitian system too possesses left-right asymmetry, despite the particle having PDEM in the region within the heterojunctions.

For the particle entering the semiconductor device from the absorptive side ($V_I(z) < 0$), reflection is normal ($|R_L| < 1$), while for the particle entering the device from the emissive side ($V_I(z) > 0$), reflection is anomalous ($|R_R| > 1$). At the same time $|T|^2 + |R|^2 \neq 1$. In a fairly recent work it has been shown that for the $\mathcal{PT}$-symmetric Scarf II potential, in a particular regime, $|T|^2 + |R_L||R_R| = 1$ [33]. However, it did not consider spatially varying mass, nor an abrupt heterojunction. In our present study of a $\mathcal{PT}$-symmetric heterojunction in the form of a diffused quantum well with PDEM, this conjecture is not valid.

Since artificial $\mathcal{PT}$-symmetric optical structures are now a reality, as are semiconductor devices with position-dependent effective mass, it is anticipated that the observations made in this work will provide some significant insight in the study of electron transport in semiconductor heterostructures.

***

One of the authors (AS) acknowledges financial assistance from the Department of Science and Technology, Government of India, through its grant SR/WOS-A/PS-11/2012. The authors thank the unknown referee for some invaluable comments. Thanks are also due to B. Roy and B. Midya for some interesting discussions.

REFERENCES
[1] GELLER M. R. and KOHN W., Phys. Rev. Lett., 70 (1993) 3103.
[2] YOUNG K., Phys. Rev. B, 39 (1989) 13434.
[3] BASTARD G., Phys. Rev. B, 24 (1981) 5693.
[4] HARRISON P., Quantum Wells, Wires and Dots, 2nd edition (Wiley-Interscience) 2005.
[5] BASTARD G., Wave Mechanics Applied to Semiconductor Heterostructures (Les Editions de Physique, Les Ulis, France) 1988.
[6] LÉVY-LEBLOND J.-M., Phys. Rev. A, 52 (1995) 1845.
[7] BENDER C. M., Rep. Prog. Phys., 70 (2007) 947.
[8] MOSTAFAZADEH A., Int. J. Geom. Methods Mod. Phys., 7 (2010) 1191.
[9] For numerous works on the topic, see the $\mathcal{PT}$-symmetry-meetings’ homepage http://gemma.ujf.cas.cz/~znojil/conf.
[10] LONGHI S., Phys. Rev. A, 81 (2010) 022102.
[11] MUSSILIMANI Z. H., MAKRIS K. G., EL-GANAINY R. and CHRISTODOULIDES D. N., Phys. Rev. Lett., 100 (2008) 030402; J. Phys. A, 41 (2008) 244019.
[12] GUO A. et al., Phys. Rev. Lett., 103 (2009) 093902.
[13] RÜTER C. E. et al., Nat. Phys., 6 (2010) 192.
[14] MAKRIS K. G. et al., Phys. Rev. Lett., 100 (2008) 103904; Phys. Rev. A, 81 (2010) 063807.
[15] RAMEZANI H. et al., Phys. Rev. A, 82 (2010) 043803.
[16] LONGHI S., Phys. Rev. Lett., 103 (2009) 123601; Phys. Rev. A, 82 (2010) 031801(R).
[17] LIN Z. et al., Phys. Rev. Lett., 106 (2011) 213901.
[18] CHUNG Y. D., LI GE and DOUGLAS STONE A., Phys. Rev. Lett., 106 (2011) 093902.
[19] SCHOMERUS H., Phys. Rev. Lett., 104 (2013) 233601.
[20] VON ROOS O., Phys. Rev. B, 27 (1983) 7547.
[21] SINHA A., J. Phys. A: Math. Theor., 45 (2012) 185305 and references therein.
[22] BENDANIEL D. J. and DUKE C. B., Phys. Rev. Lett., 152 (1966) 683.
[23] PLASTINO A. R., RIGO A., CASAS M., GARCÍAS F. and PLASTINO A., Phys. Rev. A, 60 (1999) 4318.
[24] CONLEY J. W., DUKE C. B., MAHAN G. D. and TIEMANN J. J., Phys. Rev., 150 (1966) 466.
[25] ROY B. and ROY P., J. Phys. A: Math. Gen., 35 (2002) 3961.
[26] LÉVAI G. and MAGYARI E., J. Phys. A: Math. Theor., 42 (2009) 195302.
[27] MIDYA B. and ROYCHOWDHURY R., Phys. Rev. A, 87 (2013) 045803.
[28] ABRAMOWITZ I. and STEGUN I. A., Handbook of Mathematical Functions (Dover, New York) 1970.
[29] SINHA A., Eur. Phys. Lett., 96 (2011) 20008.
[30] AHMED Z., Phys. Lett. A, 324 (2004) 152; Phys. Rev. A, 64 (2001) 042716; J. Phys. A: Math. Theor., 45 (2012) 032004.
[31] CANNATA F., DEDONDER J. P. and VENTURA A., Ann. Phys. (N.Y.), 322 (2007) 397.
[32] MOSTAFAZADEH A., Phys. Rev. A, 80 (2009) 032711.
[33] AHMED Z., Phys. Lett. A, 377 (2013) 957.