Gurariĭ operators are generic

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Abstract
Answering a question of Garbulińska-Węgrzyn and Kubiś, we prove that Gurariĭ operators form a dense $G_δ$-set in the space $B(G)$ of all nonexpansive operators on the Gurariĭ space $G$, endowed with the strong operator topology. This implies that universal operators on $G$ form a residual set in $B(G)$.

Keywords Gurariĭ space · Universal operator

Mathematics Subject Classification 47A05 · 47A65 · 46B04 · 46B28 · 54B52 · 54H05 · 54H15

1 Introduction

All Banach spaces considered in this paper are real and separable, all operators are linear bounded operators between separable Banach spaces. An isometric embedding (resp. $\varepsilon$-isometric embedding for some $\varepsilon > 0$) of Banach spaces is any injective operator $T : X \to Y$ such that $\|T(x)\| = \|x\|$ (resp. $(1 - \varepsilon)\|x\| < \|T(x)\| < (1 + \varepsilon)\|x\|$) for any nonzero $x \in X$. An operator $T : X \to Y$ is called nonexpansive if $\|T\| \leq 1$.

A Banach space $X$ is called Gurariĭ if for any finite-dimensional Banach spaces $A \subseteq B$ and any $\varepsilon > 0$, any isometric embedding $i : B \to X$ extends to an $\varepsilon$-isometric embedding $\tilde{f} : A \to X$. The first example of a Gurariĭ space $G$ was constructed by Gurariĭ in [4]. In [9], Lusky proved that any two Gurariĭ spaces are isometrically isomorphic, so up to an isometry there exists a unique Gurariĭ space $G$. A simple proof of the uniqueness of the Gurariĭ space was found by Kubiś and Solecki [8]. In [7], Kubiś suggested an elementary game-theoretic construction of the Gurariĭ space. It is known [3] that the Gurariĭ space is universal in the sense that it contains an isometric copy of any separable Banach space.

An operator $U : V \to W$ between Banach spaces is defined to be universal if for every operator $T : X \to Y$ with $\|T\| \leq \|U\|$, there exist linear isometric embeddings $i : X \to V$,
$j: Y \to W$ such that the diagram

$$
\begin{array}{ccc}
V & \xrightarrow{U} & W \\
\uparrow & & \uparrow j \\
X & \xrightarrow{T} & Y \\
\end{array}
$$

is commutative, that is, $U \circ i = j \circ T$.

In [1], [2] Garbulińska–Węgrzyn and Kubiś constructed a universal operator $\Omega: G \to G$ using the technique of Fraïssé limits. More precisely, they defined the notion of a Gurariǐ operator (which is an operator counterpart of the notion of a Gurariǐ space), constructed a Gurariǐ operator (as the Fraïssé limit in a suitable category) and proved that every Gurariǐ operator is universal.

An operator $G: X \to Y$ between Banach spaces is called Gurariǐ if $G$ is nonexpansive and for any $\varepsilon > 0$, any nonexpansive operator $T: A \to B$ between finite-dimensional Banach spaces, any Banach subspaces $A_0 \subseteq A$, $B_0 \subseteq B$ with $T[A_0] \subseteq B_0$, and any isometric embeddings $i_0: A_0 \subseteq X$, $j_0: B_0 \subseteq Y$ with $G \circ i_0 = j_0 \circ T | A_0$, there exist $\varepsilon$-isometric embeddings $i: A \to X$ and $j: B \to Y$ such that $i|_{A_0} = i_0$, $j|_{B_0} = j_0$ and $G \circ i = j \circ T$.

An example of a Gurariǐ operator $\Omega: G \to G$ was constructed in [1]. By [1, Theorem 3.5], any Gurariǐ operator $G: X \to Y$ is isometric to the Gurariǐ operator $\Omega: G \to G$ in the sense that there exist bijective isometries $i: X \to G$ and $j: Y \to G$ such that $\Omega \circ i = j \circ G$. Therefore, a Gurariǐ operator is unique up to an isometry (like the Gurariǐ space). By [1, Theorem 3.3], every Gurariǐ operator is universal. Therefore, the set $\mathcal{G}(G)$ of Gurariǐ operators from a Gurariǐ space $G$ to itself is a subset of the set $U(G)$ of nonexpansive universal operators from $G$ to $G$. In its turn, $U(G)$ is a subset of the space $B(G)$ of all nonexpansive operators from $G$ to $G$. The space $B(G)$ is endowed with the strong operator topology. Let us recall that the strong operator topology on $B(G)$ is the smallest topology such that for every $x \in G$ the map $\delta_x : B(G) \to G$, $\delta_x : T \mapsto T(x)$, is continuous.

In [2, Question 1] Garbulińska–Węgrzyn and Kubiś asked whether the set $\mathcal{G}(G)$ of Gurariǐ operators is residual in the space $B(G)$. Let us recall that a subset $R$ of a topological space $X$ is residual if it contains a dense $G_\delta$-subset of $X$.

In this paper we answer Question 1 of [2] affirmatively proving the following main result.

**Theorem 1** The set $\mathcal{G}(G)$ of Gurariǐ operators is dense $G_\delta$ in the space $B(G)$ of all nonexpansive operators on $G$.

Since $\mathcal{G}(G) \subseteq U(G)$, Theorem 1 implies the following corollary.

**Corollary 1** The set $U(G)$ of universal nonexpansive operators on $G$ is residual in the space $B(G)$ of all nonexpansive operators on $G$.

**Remark 1** Observe that the space $B(G)$ is a topological semigroup with respect to the operation of composition of operators. The group $\text{Iso}(G)$ of linear isometries of $G$ is Polish [6, 9.3] and hence is a $G_\delta$-subgroup of the topological semigroup $B(G)$. By [1, 3.5], any Gurariǐ operators are isometric, which implies the set $\mathcal{G}(G)$ coincides with the orbit $\{ I \circ \Omega \circ J : I, J \in \text{Iso}(G) \}$ of any Gurariǐ operator $\Omega: G \to G$ under the two-sided action of the Polish group $\text{Iso}(G) \times \text{Iso}(G)$ on $B(G)$, and this orbit is a unique dense $G_\delta$-orbit of this action. Theorem 1 also shows that Gurariǐ operators are “generic” elements of the space $B(G)$ in the Baire category sense.
2 Preliminaries

By $\mathbb{R}_+$ we denote the half-line $[0, \infty)$. Each natural number $n$ is identified with the set $\{0, \ldots, n - 1\}$. For a function $f : X \to Y$ between sets and a subset $A \subseteq X$ we denote by $f[A]$ the image $\{f(x) : x \in A\}$ of the set $A$ under the map $f$.

For a Banach space $X$, point $x \in X$ and positive real number $r$, let

$$B_X(x, r) = \{y \in X : \|x - y\| < r\}$$

be the open ball of radius $r$ with center $x$ in the Banach space $X$. If $x = 0$, then we shall write $B_X(r)$ instead of $B_X(0, r)$.

For an operator $T : X \to Y$ between Banach spaces, let

$$\|T\| = \inf \{r \in \mathbb{R}_+ : T[B_X(1)] \subseteq B_Y(r)\} \quad \text{and} \quad \|T^\leftarrow\| = \inf \{\|r \in \mathbb{R}_+ : T[X] \cap B_Y(1) \subseteq T[B_X(r)] \cup \{\infty\}\}.$$ 

The (finite or infinite) number $\|T^\leftarrow\|$ is called the openness norm of $T$. Observe that an operator $T : X \to Y$ between Banach spaces has finite openness norm $\|T^\leftarrow\|$ if and only if the operator $T : X \to T[X]$ is open if and only if $T[X]$ is a closed subspace of $Y$. In particular, for any operator $T : X \to Y$ from a finite-dimensional Banach space $X$, the openness norm $\|T^\leftarrow\|$ is finite. For a bijective operator $T : X \to Y$ the openness norm $\|T^\leftarrow\|$ is equal to the norm $\|T^{-1}\|$ of the inverse operator $T^{-1} : Y \to X$.

For a non-negative integer $n$ and a real number $p \in [1, \infty)$, let $\ell_p^n$ be the Banach space $\mathbb{R}^n$ endowed with the $\ell_p$-norm

$$\|x\| = \left(\sum_{i \in n} |x_i|^p\right)^{\frac{1}{p}} \text{ for any } x = (x_i)_{i \in n} \in \mathbb{R}^n.$$ 

A Banach space is Hilbert if its norm is generated by an inner product $\langle \cdot, \cdot \rangle$ in the sense that $\|x\| = \sqrt{\langle x, x \rangle}$ for all $x \in X$. In particular, for every $n \in \mathbb{N}$ the Banach space $\ell_p^n$ is Hilbert: its norm is generated by the inner product

$$\langle x, y \rangle = \sum_{i \in n} x_i y_i.$$ 

It is well-known that the inner product $\langle \cdot, \cdot \rangle$ of a Hilbert space can be uniquely recovered from the norm by the formula

$$\langle x, y \rangle = \frac{1}{4} \left(\|x + y\|^2 - \|x - y\|^2\right).$$ 

For a subset $A \subseteq X$ of a Hilbert space $X$ we denote by

$$A^\perp = \{x \in X : \forall a \in A \ \langle x, a \rangle = 0\}$$

the orthogonal complement of $A$ in $X$.

For an operator $T : X \to Y$ between Banach spaces, $\ker(T)$ stands for the kernel $T^{-1}(0)$ of $T$.

**Lemma 1** Let $T : X \to Y$ be an operator from a Hilbert space $X$ to a Banach space $Y$ and $K = \ker(T)$ be the kernel of $T$. Then $T[B_X(1)] = T[B_X(1) \cap K^\perp]$.

**Proof** The inclusion $T[B_X(1) \cap K^\perp] \subseteq T[B_X(1)]$ is trivial. To prove that $T[B_X(1)] \subseteq T[B_X(1) \cap K^\perp]$, fix any element $y \in T[B_X(1)]$ and find $x \in B_X(1)$ with $T(x) = y$. Let $x'$ be the orthogonal projection of $x$ onto the subspace $K$. Then $x - x' \in K^\perp$ and
Given any nonzero embedding. If

Lemma 2 Let $X$ be a Hilbert space of finite dimension $n > 0$, and $T : X \to Y$ be an operator to a Banach space $Y$. Then

$$\|T\| \leq \sqrt{n} \max_{1 \leq i \leq n} \|T(e_i)\|$$

for any orthonormal basis $e_1, \ldots, e_n$ in $X$.

Proof Consider the isomorphism $I : \ell^n \to X$ defined by $I : (x_1, \ldots, x_n) \mapsto \sum_{i=1}^n x_i e_i$. It is easy to see that $\|I\| = 1$. Using Cauchy–Bunyakovsky–Schwartz inequality, one can show that $\|I^{-1}\| = \sqrt{n}$. It follows that

$$\|T\| = \|T \circ I \circ I^{-1}\| \leq \|T \circ I\| \cdot \|I^{-1}\| = \max_{1 \leq i \leq n} \|T(e_i)\| \cdot \sqrt{n}. \quad \Box$$

The Banach–Mazur distance between two isomorphic Banach spaces $X, Y$ is defined as

$$d_{BM}(X, Y) := \inf\{\|I\| \cdot \|I^{-1}\| : I \text{ is an isomorphism between } X \text{ and } Y\}.$$ 

The following proposition is a classical result of John [5] (see also [10, Theorem 2.3.2]).

Proposition 1 [John] For any $n$-dimensional Banach space $X$, we have $d_{BM}(X, \ell^n) \leq \sqrt{n}$.

Lemma 3 Let $\epsilon, \delta$ be positive real numbers. Let $i, j : X \to Y$ be two operators between Banach spaces. If $\|i - j\| \leq \delta$ and $i$ is an $\epsilon$-isometric embedding, then $j$ is an $(\epsilon + \delta)$-isometric embedding.

Proof Given any nonzero $x \in X$, observe that

$$\|j(x)\| \leq \|i(x)\| + \|i(x) - j(x)\| < (1 + \epsilon)\|x\| + \|i - j\| \cdot \|x\|$$

$$\leq (1 + \epsilon)\|x\| + \delta \|x\| = (1 + \epsilon + \delta)\|x\|$$

and

$$\|j(x)\| \geq \|i(x)\| - \|i(x) - j(x)\| > (1 - \epsilon)\|x\| - \delta \|x\| = (1 - \epsilon - \delta)\|x\|,$$

which means that $j$ is an $(\epsilon + \delta)$-isometric embedding. \quad \Box

Lemma 4 If $G : \mathbb{G} \to \mathbb{G}$ is a Gurari˘ı operator, then $G[B_G(1)] = B_G(1)$ and hence $\|G\| = 1 = \|G^\rightarrow\|$.

Proof The nonexpansive property of $G$ ensures that $G[B_G(1)] \subseteq B_G(1)$. To show that $B_G(1) \subseteq G[B_G(1)]$, fix any element $y \in B_G(1)$. If $\|y\| = 0$, then $y = 0 = G(0) \in G[B_G(1)]$. So, we assume that $\|y\| > 0$.

Let $X = Y = \mathbb{R}$ and $T : X \to Y$ be the identity operator. Let $X_0 = \{0\} \subset X$, $Y_0 = Y$, $i_0 : X_0 \to \{0\} \subset \mathbb{G}$ be the trivial isometric embedding and $j : Y_0 \to \mathbb{G}$ be the isometric embedding defined by $j(t) = \frac{ty}{\|y\|}$. Since $\|y\| < 1$, there exists $\epsilon > 0$ such that $(1 + \epsilon)\|y\| < 1$. Since the operator $G$ is Gurari˘ı, there exists an $\epsilon$-isometric embedding $i : X \to \mathbb{G}$ such that $G \circ i = j \circ T$. Then the element $x = i(\|y\|)$ has $G(x) = G \circ i(\|y\|) = j \circ T(\|y\|) = j(\|y\|) = y$ and $\|x\| = \|i(\|y\|)\| < (1 + \epsilon)\|y\| < 1$. Therefore, $y = G(x) \in G[B_G(1)]$ and hence $B_G(1) = G[B_G(1)]$. The latter equality implies that $\|G\| = 1 = \|G^\rightarrow\|$. \quad \Box
In the proof of Theorem 1 we shall apply the following homogeneity property of the Gurari˘ı space, proved by Kubiš and Solecki in [8, 1.1].

**Lemma 5** Let $X$ be a finite-dimensional linear subspace of the Gurari˘ı space $G$. For every $\varepsilon > 0$ and every $\varepsilon$-isometric embedding $f : X \to G$ there exists a bijective isometry $I : G \to G$ such that $\|f - I\|_X < \varepsilon$.

### 3 Characterizing Gurari˘ı operators

In this section we shall present several characterizations of Gurari˘ı operators. The following characterization was proved in [1].

**Theorem 2** An operator $G : G \to G$ is Gurari˘ı if and only if it satisfies the following condition:

\[(GA)\] for any $\varepsilon > 0$, any nonexpansive operator $T : X \to Y$ between finite-dimensional Banach spaces, any Banach subspaces $X_0 \subseteq X$, $Y_0 \subseteq Y$ with $T[X_0] \subseteq Y_0$, and any isometric embeddings $i_0 : X_0 \to G$, $j_0 : Y_0 \to G$ with $G \circ i_0 = j_0 \circ T|X_0$, there exist $\varepsilon$-isometric embeddings $i : X \to G$, $j : Y \to G$ such that

$$\|i|_{X_0} - i_0\| < \varepsilon, \quad \|j|_{Y_0} - j_0\| < \varepsilon, \quad \text{and} \quad \|G \circ i - j \circ T\| < \varepsilon.$$  

Next, we show that the isometric embeddings $i_0, j_0$ in the above characterization can be replaced by $\delta$-isometric embeddings for a sufficiently small $\delta$. For non-negative real numbers $\varepsilon, n, m, t$ let $\delta(\varepsilon, n, m, t)$ be the largest number $\delta$ such that

$$\delta(1 + (1 + \delta)(n + 1) \max\{1, t\sqrt{nm}\}) \leq \varepsilon.$$  

**Theorem 3** A nonexpansive operator $G : G \to G$ is Gurari˘ı if and only if it satisfies the condition:

\[(GB)\] for any $\varepsilon \in (0, 1]$, any nonexpansive operator $T : X \to Y$ between finite-dimensional Banach spaces, any Banach subspaces $X_0 \subseteq X$, $Y_0 \subseteq Y$ with $T_0|X_0 \subseteq Y_0$, $T_0 = T|X_0 : X_0 \to Y_0$, and any $\delta$-isometric embeddings $i_0 : X_0 \to G$, $j_0 : Y_0 \to G$ with

$$\|G \circ i_0 - j_0 \circ T_0\| < \delta := \delta(\varepsilon, \dim(X_0), \dim(Y_0), \|T_0\|)$$

there exist $\varepsilon$-isometric embeddings $i : X \to G$, $j : Y \to G$ such that

$$\|i|_{X_0} - i_0\| < \varepsilon, \quad \|j|_{Y_0} - j_0\| < \varepsilon, \quad \text{and} \quad \|G \circ i - j \circ T\| < \varepsilon.$$  

**Proof** The “if” part follows immediately from Theorem 2 and the trivial implication (GB)$\Rightarrow$(GA).

To prove the “only if” part, we need to show that every Gurari˘ı operator $G : G \to G$ has the property (GB). Fix any $\varepsilon \in (0, 1]$, any nonexpansive operator $T : X \to Y$ between finite-dimensional Banach spaces, any Banach subspaces $X_0 \subseteq X$ and $Y_0 \subseteq Y$ with $T[X_0] \subseteq Y_0$, and any $\delta$-isometric embeddings $i_0 : X_0 \to G$, $j_0 : Y_0 \to G$ such that $\|G \circ i_0 - j_0 \circ T_0\| < \delta$ where $\delta = \delta(\varepsilon, \dim(X_0), \dim(Y_0), \|T_0\|)$ and $T_0 = T|X_0 : X_0 \to Y_0$.

Let $n = \dim X_0$ and $m = \dim Y_0$. By Proposition 1, there exist isomorphisms $I : X_0 \to \ell^2_n$ and $J : Y_0 \to \ell^2_m$ such that $\|I\| \cdot \|I^{-1}\| \leq \sqrt{n}$ and $\|J\| \cdot \|J^{-1}\| \leq \sqrt{m}$.

Fix a basis $e_1, \ldots, e_k, e_{k+1}, \ldots, e_n$ in $X_0$ and a basis $e'_1, \ldots, e'_{n-k}, e'_{n-k+1}, \ldots, e'_m$ in $Y_0$ such that:

$\square$ Springer
Let \( K = \mathcal{I}[\ker(T_0)] \subseteq \ell^2_2 \) and \( K^\perp \) be the orthogonal complement of \( K \) in the Hilbert space \( \ell^2_2 \). The choice of the basis \( e_1, \ldots, e_n \) guarantees that \( I(e_1), \ldots, I(e_k) \) and \( I(e_{k+1}), \ldots, I(e_n) \) are orthonormal bases for the spaces \( K \) and \( K^\perp \), respectively. Observe that the operator \( T_\perp = T \circ I^{-1} |_{K^\perp} : K^\perp \to T[X_0] \) is an isomorphism. By Lemma 1,
\[
T_\perp[B_K \perp (1)] = T \circ I^{-1}[B_K \perp (1)] = T \circ I^{-1}[B_{\ell^2_2} \perp (1)].
\]
Also \( I[B_{X_0} (1)] \subseteq B_{\ell^2_2} (\| I \|) \) and hence \( B_{X_0} (r) \subseteq I^{-1}[B_{\ell^2_2} (\| I \| \| r \|)] \) for every \( r \in \mathbb{R}_+ \). Then
\[
\{ r \in \mathbb{R}_+ : B_{T[X_0] (1)} \subseteq T[B_{X_0} (r)] \} \subseteq \{ r \in \mathbb{R}_+ : B_{T[X_0] (1)} \subseteq T \circ I^{-1}[B_{\ell^2_2} (\| I \| \| r \|)] \} = \{ r \in \mathbb{R}_+ : B_{T[X_0] (1)} \subseteq T_\perp[B_K \perp (\| I \| \| r \|)] \}
\]
and thus
\[
\| T_\perp^{-1} \| = \inf \{ r \in \mathbb{R}_+ : B_{T[X_0] (1)} \subseteq T[B_{X_0} (r)] \} \geq \inf \{ r \in \mathbb{R}_+ : B_{T[X_0] (1)} \subseteq T_\perp[B_K \perp (\| I \| \| r \|)] \} = \| I \|^{-1} \inf \{ \| I \| \| r \| : B_{T[X_0] (1)} \subseteq T_\perp[B_K \perp (\| I \| \| r \|)] \} = \| I \|^{-1} \| T_\perp^{-1} \|.
\]
Therefore,
\[
\| T_\perp^{-1} \| \leq \| T_0^{-1} \| \cdot \| I \|.
\]
By condition (2), for any \( l \in \{ 1, \ldots, k \} \) we have \( e_l \in \ker(T_0) \) and hence
\[
\| G(i_0(e_l)) \| = \| G \circ i_0(e_l) - j_0 \circ T_0(e_l) \| \leq \| G \circ i_0 - j_0 \circ T_0 \| \cdot \| e_l \| < \delta \cdot \| I \|^{-1} \| I(e_l) \| = \delta \cdot \| I \|^{-1} \|.
\]
Lemma 4 implies that
\[
G[i_0(e_l) + B_G(\delta \| I^{-1} \|)] = G(i_0(e_l)) + B_G(\delta \| I^{-1} \|) \ni 0.
\]
Then there exist \( y_l \in i_0(e_l) + B_G(\delta \| I^{-1} \|) \) such that \( y_l \in \ker G \).
Consider the operator \( i'_0 : X_0 \to \mathbb{G} \) such that \( i'_0(e_l) = y_l \) for all \( l \in \{ 1, \ldots, k \} \) and \( i'_0(e_l) = i_0(e_l) \) for all \( l \in \{ k + 1, \ldots, n \} \).

**Claim 1** \( \| i'_0 - i_0 \| < \delta n \) and \( i'_0 \) is a \( \delta(n+1) \)-isometric embedding.

**Proof** By Lemma 2,
\[
\| i'_0 - i_0 \| = \| (i'_0 - i_0) \circ I^{-1} \circ I \| \leq \| i'_0 - i_0 \| \circ I^{-1} \| \cdot \| I \| \leq \| I \| \sqrt{n} \max_{1 \leq i \leq n} \| (i'_0 - i_0) \circ I^{-1}(e_i) \| = \| I \| \sqrt{n} \max_{1 \leq i \leq n} \| i'_0(e_i) - i_0(e_i) \| = \| I \| \sqrt{n} \max_{1 \leq i \leq k} \| y_l - i_0(e_i) \| < \| I \| \sqrt{n} \delta \| I^{-1} \| \leq \delta n.
\]
Since \( i_0 \) is a \( \delta \)-isometric embedding and \( \| i_0 - i'_0 \| < \delta n \), the operator \( i'_0 : X_0 \to \mathbb{G} \) is a \( \delta(n+1) \)-isometric embedding by Lemma 3. \( \square \)
Let \( j'_0 : Y_0 \to \mathbb{C} \) be a unique operator such that \( j'_0(e'_l) = G \circ i'_0(e_{l+k}) = G \circ i_0(e_{l+k}) \) for all \( l \in \{1, \ldots, n-k\} \) and \( j'_0(e'_l) = j_0(e'_l) \) for all \( l \in \{n-k+1, \ldots, m\} \). Observe that for every \( l \in \{1, \ldots, k\} \) we have

\[
j'_0 \circ T_0(e_l) = j'_0(0) = 0 = G(y_l) = G \circ i'_0(e_l),
\]

and for every \( l \in \{k+1, \ldots, n\} \) we have

\[
j'_0 \circ T_0(e_l) = j'_0(e'_{l-k}) = G \circ i'_0(e_{l-k}) = G \circ i'_0(e_l).
\]

Therefore, \( j'_0 \circ T_0(e_l) = G \circ i'_0(e_l) \) for every \( l \in \{1, \ldots, n\} \). Taking into account that \( e_1, \ldots, e_n \) is a basis of the space \( X_0 \), we conclude that

\[
j'_0 \circ T_0 = G \circ i'_0.
\]  \( (1) \)

**Claim 2** \( \|j'_0 - j_0\| \leq \delta(n+1)\sqrt{nm}\|T_0^-\| \) and \( j'_0 \) is a \( \delta(1 + (n + 1)\sqrt{nm}\|T_0^-\|) \)-isometric embedding.

**Proof** Since \( j'_0 \circ T_0 = G \circ i'_0 \), we have

\[
\| (j'_0 - j_0) \circ T_0 \| = \| G \circ i'_0 - G \circ i_0 \| \leq \| G \circ i'_0 - G \circ i_0 \| + \| G \circ i_0 - j_0 \circ T_0 \| \\
< \| G \| \cdot \| i'_0 - i_0 \| + \delta \leq \delta n + \delta = \delta(n+1)
\]

and

\[
\| (j'_0 - j_0) \rvert_{T[X_0]} \| = \| (j'_0 - j_0) \circ T_{\perp} \circ T_{\perp}^{-1} \| \\
= \| (j'_0 - j_0) \circ T_0 \circ I^{-1} \circ T_{\perp}^{-1} \| \leq \| (j'_0 - j_0) \circ T_0 \circ I^{-1} \| \cdot \| T_{\perp}^{-1} \| \\
\leq \delta(n+1)\| I^{-1} \| \cdot \| T_{\perp}^{-1} \| \cdot \| I \| \leq \delta(n+1)\sqrt{n}\| T_{0}^- \|.
\]

Given any element \( z \in Y_0 \), write \( J(z) \) as \( J(z) = J(x) + J(y) \) for some \( x \in T[X_0] \) and \( y \in Y_0 \) such that \( J(y) \in (J[T[X_0]])^\perp \). Since the vectors \( J(x) \) and \( J(y) \) are orthogonal in the Hilbert space \( e_2^m \), the Pitagoras Theorem ensures that \( \| J(x) \| \leq \| J(z) \| \). Taking into account that \( y \) belongs to the linear hull of the vectors \( e'_{n-k+1}, \ldots, e'_m \) and \( j'_0(e'_l) = j_0(e'_l) \) for all \( l \in \{n-k+1, \ldots, m\} \), we conclude that \( j'_0(y) = j_0(y) \). Then

\[
\| j'_0(z) - j_0(z) \| = \| j'_0(x+y) - j_0(x+y) \| = \| j'_0(x) - j_0(x) \| \\
= \| (j'_0 - j_0) \circ J^{-1} \circ J(x) \| \leq \| (j'_0 - j_0) \rvert_{T[X_0]} \| \cdot \| J^{-1} \| \cdot \| J(x) \| \\
\leq \| (j'_0 - j_0) \rvert_{T[X_0]} \| \cdot \| J^{-1} \| \cdot \| J(z) \| \\
\leq \delta(n+1)\sqrt{n}\| T_{0}^- \| \cdot \| J^{-1} \| \cdot \| J \| \cdot \| z \| \leq \delta(n+1)\sqrt{n}\| T_{0}^- \| \sqrt{m} \cdot \| z \|.
\]

and hence

\[
\| j'_0 - j_0 \| \leq \delta(n+1)\sqrt{nm}\| T_0^- \|.
\]

Since \( j_0 \) is a \( \delta \)-isometric embedding, the operator \( j'_0 \) is a \( \delta(1 + (n + 1)\sqrt{nm}\|T_0^-\|) \)-isometric embedding by Lemma 3.

Let

\[ r = \frac{1}{1 + \delta(n+1)\max\{1, \sqrt{nm}\|T_0^-\|\}} \]

\( D_X \) be the convex hull of the set \( (i'_0)^{-1}[B_G(1)] \cup B_X(r) \) in \( X \) and \( D_Y \) be the convex hull of the set \( (j'_0)^{-1}[B_G(1)] \cup B_Y(r) \) in \( Y \).
Claim 3 \( B_X(r) \subseteq D_X \subseteq B_X \left( \frac{1}{1-\delta(n+1)} \right) \) and \( D_X \cap X_0 = (i'_0)^{-1}[B_G(1)] \).

**Proof** The inclusion \( B_X(r) \subseteq D_X \) follows from the definition of \( D_X \). To see that \( D_X \subseteq B_X \left( \frac{1}{1-\delta(n+1)} \right) \), take any \( x \in X \). By Claim 1, \( i'_0 \) is a \( \delta(n+1) \)-isometric embedding and hence \( (1-\delta(n+1))\|x\| \leq \|i'_0(x)\| < 1 \) and \( \|x\| < \frac{1}{1-\delta(n+1)} \). Then
\[
D_X \subseteq \text{conv} \left( (i'_0)^{-1}[B_G(1)] \right) \subseteq B_X \left( \frac{1}{1-\delta(n+1)} \right).
\]

By Claim 1, \( i'_0[B_X(r)] \subseteq B_G(r\|i'_0\|) \subseteq B_G(r(1+\delta(n+1))) \subseteq B_G(1) \) and \( i'_0[(i'_0)^{-1}[B_G(1)]] \subseteq B_G(1) \), which implies that \( i'_0[D_X \cap X_0] \subseteq B_G(1) \) and \( D_X \cap X_0 = (i'_0)^{-1}[B_G(1)] \) by the injectivity of \( i'_0 \).

By analogy, we can apply Claim 2 and prove

Claim 4 \( B_Y(r) \subseteq D_Y \subseteq B_Y \left( \frac{1}{1-\delta(1+(n+1))\sqrt{n\|i'_0\|}} \right) \) and \( D_Y \cap Y_0 = (j'_0)^{-1}[B_G(1)] \).

Claim 5 \( T[D_X] \subseteq D_Y \).

**Proof** Observe that for every \( x \in (i'_0)^{-1}[B_G(1)] \), the equality \( j'_0 \circ T_0 = G \circ i'_0 \) proved in (1) and the inequality \( \|G\| \leq 1 \) imply
\[
j'_0 \circ T_0(x) = G \circ i'_0(x) \in G[B_G(1)] \subseteq B_G(1)
\]
and hence \( T(x) = T_0(x) \in (j'_0)^{-1}[B_G(1)] \subseteq D_Y \). On the other hand, for every \( x \in B_X(r) \) the inequality \( \|T\| \leq 1 \) implies \( T(x) \in T[B_X(r)] \subseteq B_Y(r) \subseteq D_Y \). Then
\[
T[D_X] = T \left[ \text{conv} \left( (i'_0)^{-1}[B_G(1)] \cup B_X(r) \right) \right]
= \text{conv} \left( T[(i'_0)^{-1}[B_G(1)] \cup B_X(r)] \right) \subseteq \text{conv}(D_Y) = D_Y.
\]

Let \( X' \) be the space \( X \) endowed with the norm \( \| \cdot \|_{X'} \) defined by
\[
\|x\|_{X'} = \inf \{ t \in \mathbb{R}_+ : x \in t \cdot D_X \},
\]
and let \( Y' \) be the space \( Y \) endowed with the norm \( \| \cdot \|_{Y'} \)
\[
\|y\|_{Y'} = \inf \{ t \in \mathbb{R}_+ : y \in t \cdot D_Y \}.
\]

It follows that \( B_{X'}(1) = D_X \) and \( B_{Y'}(1) = D_Y \). Let \( \text{Id}_{X,X'} : X \to X' \), \( \text{Id}_{X',X} : X' \to X \), \( \text{Id}_{Y,Y'} : Y \to Y' \), and \( \text{Id}_{Y',Y} : Y' \to Y \) be the identity operators.

In the Banach spaces \( X' \) and \( Y' \) consider the subspaces \( X'_0 = \text{Id}_{X,X'}[X_0] \) and \( Y'_0 = \text{Id}_{Y,Y'}[Y_0] \). Let \( \text{Id}_{X_0,X'_0} : X_0 \to X'_0 \), \( \text{Id}_{X'_0,X_0} : X'_0 \to X_0 \), \( \text{Id}_{Y_0,Y'_0} : Y_0 \to Y'_0 \), and \( \text{Id}_{Y'_0,Y_0} : Y'_0 \to Y_0 \) be the identity operators.

Claims 3 and 4 imply that \( i'_0 \circ \text{Id}_{X'_0,X_0} : X'_0 \to G \) and \( j'_0 \circ \text{Id}_{Y'_0,Y_0} : Y'_0 \to G \) are isometric embeddings, and Claim 5 implies that the operator \( T' = \text{Id}_{Y,Y'} \circ T \circ \text{Id}_{X',X} : X' \to Y' \) is nonexpansive. Let \( T'_0 = T'|_{X'_0} \). Since \( G \circ i'_0 \circ \text{Id}_{X'_0,X_0} = j'_0 \circ T'_0 \), we can apply the definition of the Gurari˘ı space and find \( \delta \)-isometric embeddings \( i' : X' \to G \) and \( j' : Y' \to G \) such that
\[
\|i'|_{X'_0} - i'_0 \circ \text{Id}_{X'_0,X_0}\| < \delta, \quad \|j'|_{Y'_0} - j'_0 \circ \text{Id}_{Y'_0,Y_0}\| < \delta, \quad \|G \circ i' - j \circ T'\| < \delta.
\]

Consider the operators \( i = i' \circ \text{Id}_{X,X'} : X \to G \) and \( j = j' \circ \text{Id}_{Y,Y'} : Y \to G \). It remains to prove that \( i, j \) are \( \varepsilon \)-isometric embeddings and
\[
\max\{\|i\|_{X_0} - i_0\|, \|j\|_{Y_0} - j_0\|, \|G \circ i - j \circ T\|\} < \varepsilon.
\]
To see that \( i \) is an \( \epsilon \)-isometric embedding, take any nonzero \( x \in X \) and applying Claim 1, conclude that

\[
\|i(x)\|_G = \|i' \circ \text{Id}_{X,V}(x)\|_G \leq \|i'\| \cdot \|\text{Id}_{X,V}(x)\|_V < (1 + \delta)\frac{1}{2}\|x\|_X
\]

\[
= (1 + \delta) \cdot (1 + \delta(n + 1) \max\{1, \sqrt{nm}\} \|T_0\|) \|x\|_X \leq (1 + \epsilon)\|x\|_X,
\]

by the choice of \( \delta = \delta(\epsilon, n, m, \|T_0\|) \). On the other hand the inclusion \( D_X \subseteq B_X(1 - \delta(n + 1)) \)

proven in Claim 3 implies

\[
\|i(x)\|_G = \|i' \circ \text{Id}_{X,V}(x)\|_G > (1 - \delta)\|\text{Id}_{X,V}(x)\|_V
\]

\[
\geq (1 - \delta)(1 - \delta(n + 1)) \|x\|_X \geq (1 - \epsilon)\|x\|_X
\]

by the choice of \( \delta = \delta(\epsilon, n, m, \|T_0\|) \). This means that \( i : X \to G \) is an \( \epsilon \)-isometric embedding.

By analogy we can prove that \( j \) is an \( \epsilon \)-isometric embedding.

The inclusion \( B_X(r) \subseteq D_X \) implies \( \|\text{Id}_{X,Y}r\| \leq \|\text{Id}_{X,V}\| \leq \frac{1}{r} \) and hence

\[
\|i \, x_0 - i_0\| = \|i' \circ \text{Id}_{X,Y}r - i_0' \circ \text{Id}_{X,Y}r\| \leq \|i' - i_0'\| \cdot \|\text{Id}_{X,Y}r\| < \frac{\delta}{r} \leq \epsilon.
\]

By analogy we can prove that \( \|j \, y_0 - j_0\| < \epsilon \). Finally, observe that

\[
\|G \circ i - j \circ T\| = \|G \circ i' \circ \text{Id}_{X,V} - j' \circ \text{Id}_{V,Y} \circ T\|
\]

\[
= \|G \circ i' \circ \text{Id}_{X,V} - j' \circ T' \circ \text{Id}_{X,V}\|
\]

\[
\leq \|G \circ i' - j' \circ T'\| \cdot \|\text{Id}_{X,V}\| < \delta \cdot \|\text{Id}_{X,V}\| \leq \frac{\delta}{r} \leq \epsilon.
\]

\[\square\]

Next, we show that the spaces \( X, Y \), operator \( T \), subspaces \( X_0, Y_0 \) and \( \delta \)-isometries \( i_0, j_0 \)
in Theorem 3 can be chosen from a suitable countable family of standard objects, defined below.

A standard linear space is the linear space \( \mathbb{R}^n \) for some \( n \in \omega \). For a standard linear space \( X = \mathbb{R}^n \) by \( E_X = \{x \in [0, 1]^n : |x^{-1}(1)| = 1\} \) we denote the standard unit basis in \( X = \mathbb{R}^n \).

A linear subspace \( X_0 \) of a standard linear space \( X \) is called standard if \( X_0 \) coincides with the linear hull of some subset \( E \subseteq E_X \). It is clear that each standard linear space \( X \) has exactly \( 2^{\dim(X)} \) standard linear subspaces.

An operator \( T : X \to Y \) between linear spaces is called standard if there exist numbers \( k, n, m \) such that \( X = \mathbb{R}^n, Y = \mathbb{R}^m, 0 \leq n - k \leq m \) and \( T \) assigns to each \( x \in \mathbb{R}^n \) the vector \( y = T(x) \) defined by

\[
y(i) = \begin{cases} x(i + k) & \text{if } i < n - k \\ 0 & \text{otherwise.} \end{cases}
\]

In this case \( \ker(T) = \mathbb{R}^k \times \{0\}^{n-k} \) and \( T[X] = \mathbb{R}^{n-k} \times \{0\}^{m-n+k} \).

A norm \( \| \cdot \| \) on a standard linear space \( X = \mathbb{R}^n \) is called rational if its closed unit ball \( \{x \in X : \|x\| \leq 1\} \) coincides with the convex hull of some finite set \( F \subseteq \mathbb{Q}^n \subseteq \mathbb{R}^n \). It is clear that the family of rational norms on \( \mathbb{R}^n \) is countable.

A rational Banach space is a standard linear space endowed with a rational norm.

For a countable dense set \( D \) in the Gurari˘ı space \( G \), let \( \mathcal{F}_D \) be the family of all 9-tuples \( (\epsilon, X, Y, T, X_0, Y_0, T_0, i_0, j_0) \) consisting of

\[\bullet \text{ a rational number } \epsilon \in (0, 1);\]
• rational Banach spaces $X, Y$;
• a nonexpansive standard operator $T : X \to Y$;
• standard linear subspaces $X_0 \subseteq X$ and $Y_0 \subseteq Y$ such that $T[X_0] \subseteq Y_0$;
• the operator $T_0 = T|_{X_0} : X_0 \to Y_0$;
• $\delta(\varepsilon, \dim(X_0), \dim(Y_0), \|T_0^{-}\|)-$isometric embeddings $i_0 : X_0 \to \mathbb{G}$ and $j_0 : Y_0 \to \mathbb{G}$ such that $i_0(E_X \cap X_0) \cup j_0(E_Y \cap Y_0) \subseteq D$.

It is easy to see that the family $\mathcal{F}_D$ is countable.

**Theorem 4** Let $D$ be a countable dense subset of the Gurariǐ space $\mathbb{G}$. A nonexpansive operator $G : \mathbb{G} \to \mathbb{G}$ is Gurariǐ if and only if it satisfies the condition:

(GC) for any $9$-tuple $(\varepsilon, X, Y, T, X_0, Y_0, T_0, i_0, j_0) \in \mathcal{F}_D$ with

\[
\|G \circ i_0 - j_0 \circ T|_{X_0}\| < \delta(\varepsilon, \dim(X_0), \dim(Y_0), \|T_0^{-}\|),
\]

there exist $\varepsilon$-isometric embeddings $i : X \to \mathbb{G}$, $j : Y \to \mathbb{G}$ such that

\[
\|i|_{X_0} - i_0\| < \varepsilon, \quad \|j|_{Y_0} - j_0\| < \varepsilon, \quad \text{and} \quad \|G \circ i - j \circ T\| < \varepsilon.
\]

**Proof** The “only if” part follows immediately from the “only if” part of Theorem 3. To prove the “if” part, assume that the condition (GC) is satisfied. By Theorem 3, the Gurariǐ property of the operator $G$ will follow as soon as we show that $G$ satisfies the condition (GB). To this end, fix any $\varepsilon \in (0, 1]$, any nonexpansive operator $T : X \to Y$ between finite-dimensional Banach spaces, any linear subspaces $X_0 \subseteq X$ and $Y_0 \subseteq Y$ with $T[X_0] \subseteq Y_0$, and $\delta$-isometric embeddings $i_0 : X_0 \to \mathbb{G}$ and $j_0 : Y_0 \to \mathbb{G}$ such that $\|G \circ i_0 - j_0 \circ T_0\| < \delta := \delta(\varepsilon, \dim(X_0), \dim(Y_0), \|T_0^{-}\|)$, where $T_0 = T|_{X_0} : X_0 \to Y_0$.

Choose a basis $E_X = \{e_1, \ldots, e_{\dim(X)}\}$ in the space $X$ such that

• $\{e_1, \ldots, e_{\dim(\ker(T_0))}\}$ is a basis for the linear space $\ker(T_0)$;
• $\{e_1, \ldots, e_{\dim(\ker(T))}\}$ is a basis for the linear space $\ker(T)$;
• $\{e_1, \ldots, e_{\dim(\ker(T_0))}\} \cup \{e_{\dim(\ker(T))} + 1, \ldots, e_{\dim(\ker(T)) + \dim(X) - \dim(\ker(T_0))}\}$ is a basis for the space $X$;
• $\{e_1, \ldots, e_{\dim(T^{-1}[Y_0])}\}$ is a basis for the linear space $T^{-1}[Y_0]$.

Let $E_Y = \{e'_1, \ldots, e'_{\dim(Y)}\}$ be a basis for the linear space $Y$ such that $e'_i = T(e_{\dim(\ker(T)) + i})$ for all $i \in \{1, \ldots, \dim(X) - \dim(\ker(T))\}$. The bases $E_X$ and $E_Y$ allow us to identify the spaces $X, Y$ with the standard linear spaces $\mathbb{R}^{\dim(X)}$ and $\mathbb{R}^{\dim(Y)}$. The choice of the bases $E_X, E_Y$ ensures that the operator $T : X \to Y$ is standard and the subspaces $X_0 \subseteq X$ and $Y_0 \subseteq Y$ are standard.

By the compactness of the unit spheres in the Banach spaces $X_0, Y_0$, there exists $\delta' < \delta$ such that the $\delta'$-isometric embeddings $i_0 : X_0 \to \mathbb{G}$ and $j_0 : Y_0 \to \mathbb{G}$ are $\delta'$-isometric embeddings. Since $\|G \circ i_0 - j_0 \circ T_0\| < \delta$, we can additionally assume that $\|G \circ i_0 - j_0 \circ T_0\| < \delta'$.

By the continuity of the function $\delta(\cdot, \dim(X_0), \dim(Y_0), \cdot)$, there exists a positive real number $\nu$ such that

\[
\delta' + 2\nu \leq \delta(\varepsilon', \dim(X_0), \dim(Y_0), t)
\]

for any $\varepsilon', t \in \mathbb{R}$ with

\[
\frac{\varepsilon - 2\nu}{1 + \nu} \leq \varepsilon' \leq \varepsilon \quad \text{and} \quad \|T_0^{-}\| \leq t \leq (1 + \nu)^2\|T_0^{-}\|.
\]

By the density of $D$ in $\mathbb{G}$, the $\delta'$-isometric embeddings $i_0 : X_0 \to \mathbb{G}$ and $j_0 : Y_0 \to \mathbb{G}$ can be approximated by $\delta'$-isometric embeddings $i''_0 : X_0 \to \mathbb{G}$ and $j''_0 : Y_0 \to \mathbb{G}$ such that
Gurarienko operators are generic

\( i_0''(E_X \cap X_0) \cup j_0''(E_Y \cap Y_0) \subseteq D, \max\{\|i''_0 - i_0\|, \|j''_0 - j_0\|\} < \nu, \text{ and } \|G \circ i''_0 - j''_0 \circ T_0\| < \delta'.\)

The norms \( \|\cdot\|_X \) and \( \|\cdot\|_Y \) on the standard linear spaces \( X, Y \) can be approximated by rational norms \( \|\cdot\|_{X'} : X \to [0, \infty) \) and \( \|\cdot\|_{Y'} : Y \to [0, \infty) \) such that for every nonzero elements \( x \in X \) and \( y \in Y \) the following conditions are satisfied:

(a) \( \|x\|_X \leq \|\pi X\|_{X'} < (1 + \nu)\|x\|_X \) and \( \frac{1}{1+\nu}\|y\|_Y \leq \|y\|_{Y'} \leq \|y\|_Y; \)

(b) \( (1 - \delta')\|x\|_{X'} < \|i''_0(x)\|_{G} < (1 + \delta')\|x\|_{X'} \) and \( (1 - \delta')\|y\|_{Y'} < \|j''_0(y)\|_{G} < (1 + \delta')\|y\|_{Y'}. \)

Let \( X' \) (resp. \( Y' \)) be the rational Banach space \( X \) (resp. \( Y \)) endowed with the rational norm \( \|\cdot\|_{X'} \) (resp. \( \|\cdot\|_{Y'} \)). The condition (a) implies

\[ B_{X'}(1) \subseteq B_X(1 + \nu) \text{ and } B_{Y'}(1) \subseteq B_Y(1 + \nu). \]  

(2)

Let

\[ \text{Id}_{X,X'} : X \to X', \quad \text{Id}_{X',X} : X' \to X, \quad \text{Id}_{Y,Y'} : Y \to Y', \quad \text{and } \text{Id}_{Y',Y} : Y' \to Y \]

be the identity operators. Let \( X'_0 = \text{Id}_{X,X'}[X_0], \ Y'_0 = \text{Id}_{Y,Y'}[Y_0], \) and

\[ \text{Id}_{X_0,X_0} : X_0 \to X'_0, \quad \text{Id}_{X_0,X_0} : X'_0 \to X_0, \quad \text{Id}_{Y_0,Y_0} : Y_0 \to Y'_0 \]

be the identity operators. Let \( i'_0 = i''_0 \circ \text{Id}_{X_0,X_0} \) and \( j'_0 = j''_0 \circ \text{Id}_{Y_0,Y_0}. \)

The condition (a) implies that \( \max\{|\|\text{Id}_{X,W}_X\|, \|\text{Id}_{Y,U}_Y\|\| \leq 1 \) and hence the operator \( T' = \text{Id}_{Y,Y'} \circ T \circ \text{Id}_{X',X} : X' \to Y' \) is nonexpansive. Let \( T'_0 = T'_{|X'_0}. \) The inclusions (2) imply

\[ \|T'_0\| \leq \|(T'_0)_{|X'_0}\| \leq (1 + \nu)^2\|T_0\|. \]

Choose any rational \( \varepsilon' \) such that \( \frac{\varepsilon - 2\nu}{1 + \nu} \leq \varepsilon' \leq \frac{\varepsilon - \nu}{1 + \nu}. \) The choice of \( \nu \) ensures that

\[ \delta' + 2\nu \leq \delta(\varepsilon', \dim(X_0), \dim(Y_0), \|(T'_0)_{|X'_0}\|). \]

Then the 9-tuple \( (\varepsilon', X', Y', T', X'_0, Y'_0, T'_0, i'_0, j'_0) \) belongs to the family \( \mathcal{F}_D. \) Since

\[ \|G \circ i'_0 - j'_0 \circ T_0\| \leq \|G \circ i''_0 - j''_0 \circ T_0\| \cdot \|\text{Id}_{X_0,X_0}\| \leq \|G \circ i''_0 - j''_0 \circ T_0\|
\]

\[ \leq \|G \circ i''_0 - G \circ i_0 + \|G \circ i_0 - j_0 \circ T_0\| + \|j_0 \circ T_0 - j''_0 \circ T_0\|
\]

\[ \leq \|G \| \cdot \|i''_0 - i_0\| + \delta' + \|j_0 - j''_0\| \cdot \|T_0\|
\]

\[ \leq \nu + \delta' + v \leq \delta(\varepsilon', \dim(X_0), \dim(Y_0), \|(T'_0)_{|X'_0}\|). \]

By the condition (GC), there exist \( \varepsilon'-\)isometric embeddings \( i' : X' \to \mathbb{G} \) and \( j' : Y' \to G \) such that

\[ \max\{|i'|_{X'_0} - i'_0\|, \|j'|_{Y'_0} - j'_0\|, \|G \circ i' - j' \circ T'\|\| < \varepsilon'. \]

The inequality \( \varepsilon'(1 + \nu) \leq \varepsilon \) and the condition (a) imply that the maps \( i = i' \circ \text{Id}_{X,X'} \) and \( j = j' \circ \text{Id}_{Y,Y'} \) are \( \varepsilon \)-isometric embeddings. It remains to show that

\[ \max\{|i|_{X_0} - i_0\|, \|j|_{Y_0} - j_0\|, \|G \circ i - j \circ T\|\| < \varepsilon. \]

Observe that

\[ \|i\|_{X_0} - i_0\| = \|i' \circ \text{Id}_{X_0,X_0} - i'_0\| + \|i''_0 - i_0\|
\]

\[ \leq \|i'\|_{X'_0} - i'_0 \circ \text{Id}_{X_0,X_0} \cdot \|\text{Id}_{X_0,X_0}\| + \|i''_0 - i_0\| < \varepsilon'(1 + \nu) + \nu \leq \varepsilon. \]

By analogy we can prove that \( \|j\|_{Y_0} - j_0\| < \varepsilon. \) Finally,
\[ \|G \circ i - j \circ T\| = \|G \circ i' \circ \text{Id}_{X,X'} - j' \circ T' \circ \text{Id}_{X,X'}\| \leq \|G \circ i' - j' \circ T'\| \cdot \|\text{Id}_{X,X'}\| \leq \varepsilon'(1 + \nu) < \varepsilon. \]

\[ \square \]

4 Proof of Theorem 1

First we show that the set \( G(\mathbb{G}) \) of Gurari˘ı operators is dense in the space \( B(\mathbb{G}) \).

Fix any nonexpansive operator \( T \in B(\mathbb{G}) \) and any neighborhood \( O_T \) of \( T \) in \( B(\mathbb{G}) \). By the definition of the strong operator topology, there exist \( \varepsilon > 0 \) and a finite-dimensional subspace \( E \subseteq \mathbb{G} \) such that \( \{S \in B(\mathbb{G}) : \|(S - T)|_E\| \leq 2\varepsilon\} \subseteq O_T \). Consider the finite-dimensional Banach space \( F = T[E] \subseteq \mathbb{G} \). Let \( \Omega : \mathbb{G} \to \mathbb{G} \) be the Gurari˘ı operator, constructed in [1]. By [1, 3.3], the operator \( \Omega \) is universal. So, there exist isometric embeddings \( i : E \to \mathbb{G} \) and \( j : F \to \mathbb{G} \) such that \( \Omega \circ i = j \circ T|_E \). By Lemma 5, for the isometric embeddings \( i : E \to \mathbb{G} \) and \( j : F \to \mathbb{G} \) there exist bijective isometries \( I, J \in \text{Iso}(\mathbb{G}) \) such that \( \|i - I|_E\| < \varepsilon \) and \( \|j - J\| < \varepsilon \). Then \( S = J^{-1} \circ \Omega \circ I \) is a Gurari˘ı operator such that for every \( x \in E \) we have

\[ \|S(x) - T(x)\| = \|J^{-1} \circ \Omega \circ I(x) - T(x)\| \leq \|J^{-1} \circ \Omega \circ i(x) - T(x)\| + \|J^{-1} \circ \Omega \circ i(x) - (I - j) \circ T(x)\| \leq \|J^{-1}\| \cdot \|\Omega\| \cdot \|I(x) - i(x)\| + \|J^{-1} \circ j \circ T(x) - T(x)\| \leq \|I|_E - i\| \cdot \|x\| + \|j - J\| \cdot \|T\| \cdot \|x\| \leq \varepsilon \|x\| + \varepsilon \|x\| = 2\varepsilon \|x\| \]

and hence \( \|(S - T)|_E\| \leq 2\varepsilon \). The choice of \( \varepsilon \) ensures that \( S \in O_T \cap G(\mathbb{G}) \), witnessing that the subspace \( G(\mathbb{G}) \) is dense in \( B(\mathbb{G}) \).

Now we shall prove that the dense set \( G(\mathbb{G}) \) is of type \( G_\delta \) in \( B(\mathbb{G}) \). Fix any countable dense set \( D \) in the Gurari˘ı space \( \mathbb{G} \). For every 9-tuple

\[ t = (\varepsilon, X, Y, T, X_0, Y_0, T_0, i_0, j_0) \in \mathcal{F}_D \]

and the number \( \delta_t = \delta(\varepsilon, \text{dim}(X_0), \text{dim}(Y_0), \|T_0^\text{inc}\|) \), consider the set

\[ \mathcal{F}_t = \{G \in B(\mathbb{G}) : \|G \circ i_0 - j_0 \circ T_0\| \geq \delta_t\} \]

and the set \( \mathcal{U}_t \) of all operators \( G \in B(\mathbb{G}) \) such that \( \|G \circ i_0 - j_0 \circ T_0\| < \delta_t \) and there exist \( \varepsilon \)-isometric embeddings \( i : X \to \mathbb{G} \) and \( j : Y \to \mathbb{G} \) such that

\[ \max\{\|i\|_{X_0} - i_0\|, \|j\|_{Y_0} - j_0\|, \|G \circ i - j \circ T\|\} < \varepsilon. \]

It is easy to see that the set \( \mathcal{F}_t \) is closed in \( B(\mathbb{G}) \) and the set \( \mathcal{U}_t \) is open in \( B(\mathbb{G}) \). Then their union \( (\mathcal{F}_t \cup \mathcal{U}_t) \) is a \( G_\delta \)-set in \( B(\mathbb{G}) \). Theorem 4 ensures that \( G(\mathbb{G}) = \bigcap_{t \in \mathcal{F}_D} (\mathcal{F}_t \cup \mathcal{U}_t) \) and hence \( G(\mathbb{G}) \) is a dense \( G_\delta \)-set in \( B(\mathbb{G}) \).

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