Pseudo-Finsleroid spatial-anisotropic relativistic space

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Abstract

The pseudo-Finsleroid relativistic metric was constructed upon assuming that the involved vector field $b_i$ is time-like. In the present paper it is shown that the metric admits just the alternative counterpart in which the field is space-like. The entailed pseudo-Finsleroid-spatial framework is systematically described. We face on various remarkable properties, including the constant curvature of the associated indicatrix, the explicit Hamiltonian function, transparent presentations for the angle and scalar product. The spray coefficients are found to be of a rather simple structure. The Berwald case is attractively realized. Interesting conformal properties are stemming.

Keywords: Finsler metrics, relativistic spaces, spray coefficients.
1. Introduction: the pseudo-Finsleroid metric background

The Finslerian methods propose various interesting tools for addressing the anisotropy to the geometry (see [1-7]). Presence of local spatial anisotropy in the relativistic space-time is typical of many pictures appeared in physical applications. Can a preferred spatial vector field, to be denoted as \( b_i(x) \), be geometrically distinguished, in the sense that the vector \( b_i \) can make traces on dependence of the fundamental metric function on local directions? The Finsleroid tools (developed in [8-12]) can provide us with an adequate method to develop such a subject.

Namely, the simplest standpoint to start from is the stipulation that the Finsler space is specified in accordance with the condition that the squared Finslerian metric function \( F^2(x, y) \) possesses the functional dependence

\[
F^2(x, y) = \Phi \left(g(x), b_i(x), a_{ij}(x), y\right), \tag{1.1}
\]

where \( g(x) \) is a scalar, \( a_{ij}(x) \) is a pseudo-Riemannian metric tensor, and \( y \) denote tangent vectors supported by points \( x \) of the underlining manifold. Next, we require that the entailed Finslerian metric tensor \( g_{ij}(x, y) \) be of the time-space signature:

\[
\text{sign}(g_{ij}) = (+ - - \ldots). \tag{1.2}
\]

The pseudo-Riemannian norm \( ||b|| \) of the involved vector \( b_i \) should be negative. Also, when \( ||b|| = -1 \), the associated indicatrix should be a space of constant curvature. Finally, sufficient regularity properties are implied. The respective anisotropic space \( \mathcal{AR}_{g, c} \) defined in (2.16) meets hopefully all these requirements. The space admits the norm \( ||b|| \) to be \( ||b|| = -c \) with \( c \in (0, 1] \). The time-like and space-like sectors are separated according to the sign of the involved quadratic form \( B(x, y) \). The scalar

\[
h(x) = \sqrt{1 + \epsilon g^2(x) \frac{4}{4}}, \tag{1.3}
\]

where \( \epsilon = 1 \) in the time-like sector and \( \epsilon = -1 \) in the space-like sector, plays an important role in the theory developed below.

In Section 2, the \( \mathcal{AR} \)-space as well as the unimodular \( \mathcal{UAR} \)-case which uses \( c = 1 \) are rigorously introduced, basic notions are explained, and various key properties are set forth. The Finslerian metric tensor constructed from the function \( F^2 \) given by (2.13) does inherit from the tensor \( a_{ij}(x) \) the time-space signature (2.4), so that (1.2) is valid. Elucidating the structure of the respective Cartan tensor and indicatrix curvature tensor of the \( \mathcal{AR} \)-space results in the remarkable special types (2.48) and (2.50). The induced geometry in the tangent spaces of the \( \mathcal{AR} \)-space is of the conformally flat type. When \( c = 1 \), from (2.43) we can conclude that \( X = 1/N \), which reduces (2.46) to the simple representation

\[
F^2C_hC^h = -\frac{N^2 g^2 \epsilon}{4}, \tag{1.4}
\]

which right-hand part is independent of vectors \( y \) and, therefore, the constancy of the indicatrix curvature is realized (as manifested by the formulas (2.55)-(2.57)). At any \( c \in (0, 1] \), the spray coefficients \( G^i \) can be represented in the clear explicit form (2.61). The Berwald case thereof is obviously determined by the conditions \( \nabla_j b_i = 0 \) and \( g = \text{const} \), which reduce \( G^i \) to the associated pseudo-Riemannian coefficients \( a^i_{nm}(x)y^n y^m \) (where \( a^i_{nm}(x) \) are the Christoffel symbols given rise to by the associated pseudo-Riemannian metric tensor \( a_{ij}(x) \)).
The \( \mathcal{UAR} \)-space angle \( \alpha_{\{x\}}(y_1, y_2) \) formed by two vectors in symmetric way is proposed. The angle is supported by a point \( x \in M \) of the base manifold \( M \) (in just the same sense as in the pseudo-Riemannian geometry) and is independent of any vector element \( y \) of support. This gives rise to the notion of the scalar product of the vectors \( (y_1, y_2) \). The \( \mathcal{UAR} \)-space coordinates are indicated in terms of which the metric line element \( ds \) can be represented in the transparent forms, extending the spherical-coordinate representations handling in the Riemannian geometry. This \( ds \) (see (2.100)) shows that the angle is of the “canonical” geometrical meaning: \( \alpha_{\{x\}}(y_1, y_2) \) is the length of the geodesic curve on the indicatrix, the curve joining the points at which the entered vectors \( (y_1, y_2) \) intersect the indicatrix. The success has been predetermined by the observation that the squared AR-pseudo-Finsleroid metric function \( F^2(x, y) \) can conveniently be represented by the formula

\[
F^2(x, y) = B(x, y)e^{-g(x)\chi(x,y)} \tag{1.5}
\]

such that in the \( \mathcal{UAR} \)-space the scalar \( \chi \) possesses the lucid geometrical meaning of the azimuthal angle measured from the direction assigned by the input vector \( b \). Indeed, fixing a point \( x \) in the background manifold \( M \), a vector \( y^i \in T_xM \) is parallel to \( b^i(x) \) if and only if the angle-representation of the \( y^i \) corresponds to \( \chi = 0 \). The property is also obvious from the derived formula (2.76), namely \( \alpha_{\{x\}}(y, b) = \chi \). The arisen functions \( Sh \chi, Ch \chi, Sin \chi, Cos \chi \) can be interpreted as the required extensions of the trigonometric functions to the \( \mathcal{UAR} \)-space.

In Section 3, we confine the attention to the future-time-like sector (the respective function \( F^2 \) introduced by (3.11)-(3.12) is of the type proposed earlier in the kinematic-study paper [11]). In Section 4, the space-like sector is attentively evaluated. The amazing uniformity between structures of the representations of various involved tensors can be traced, despite the form of the fundamental metric function differs drastically in these sectors (compare between (3.11)-(3.12) and (4.10)-(4.14)).

In Section 5, we fix the tangent space and write down the respective components of the covariant vector and metric tensor relative to the orthonormal frame of the input pseudo-Riemannian metric tensor \( a_{ij} \), choosing the \((N - 1)\)-th component of the tangent vector to be parallel to the 1-form \( b \). The formulas obtained in this way are convenient in various evaluations.

In Section 6, the possibility of obtaining the \( \mathcal{AR} \)-space Hamiltonian function \( H \) in a convenient explicit form is indicated. When \( c = 1 \), the function \( H^2 \) derived is totally similar to the initial function \( F^2 \). In contrast to the later function \( F^2 \) which depends primarily on the contravariant vectors \( y = \{y^i\} \), the \( H^2 \) is the function of the co-vectors \( \hat{y} = \{\hat{y}_i\} \). The relationship between the variables \( y = \{y^i\} \) and \( \hat{y} = \{\hat{y}_i\} \) is assigned by the simple formula (2.37). The knowledge of the Hamiltonian function makes it possible to formulate the Hamilton-Jacobi equation which is of important physical meaning.

In Appendix A, various calculations with the help of the \( \mathcal{UAR} \)-space coordinates are systematically presented.

In Appendix B we indicate a straightforward and convenient way to obtain the \( \mathcal{AR} \)-space spray coefficients.

In Appendix C, we observe the remarkable conformal property of the \( \mathcal{UAR} \)-space that the explicit and simple form can be proposed for the respective conformal multiplier, which proves to be expressed through the fundamental metric function, according to (C.2). At the same time, the \( \mathcal{UAR} \)-space is not conformal to any pseudo-Riemannian space. Instead, the \( \mathcal{UAR} \)-space proves to be metrically isomorphic to the factor-pseudo-
Riemannian space $\mathcal{G}$ in which the metric tensor reads

$$t_{mn}(x, \zeta) = p(x, \zeta)a_{mn}(x),$$

where $\zeta \in \mathcal{G}$ and the factor is

$$p(x, \zeta) = \frac{1}{h^2(x)}|S^2(x, \zeta)|^{(1-h(x))/h(x)},$$

with $S^2(x, \zeta) = a_{ij}(x)\zeta^i\zeta^j$ standing for the pseudo-Riemannian metric in the space $\mathcal{G}$. The isomorphism is the diffeomorphism, being initiated by the explicit and simple formulas (C.1)-(C.4). There appears the relationship

$$|F^2(x, y)|^h(x) = |S^2(x, \zeta)|,$$

which shows that at each point $x$ the Finslerian indicatrix (defined by $|F^2(x, y)| = 1$) is isomorphic to the pseudo-Euclidean sphere $S_x(\mathcal{G})$ (defined by $|S^2(x, \zeta)| = 1$) in the space $\mathcal{G}$. The curvature of the pseudo-sphere $S_x(\mathcal{G})$ in the space $\mathcal{G}$ is not, however, of the unit value, for the metric tensor (1.6) in the space $\mathcal{G}$ differs from the pseudo-Riemannian tensor by the factor $p$. If we construct on $S_x(\mathcal{G})$ the curvature tensor, we readily arrive at the representation (C.29) which says us that in the $\mathcal{UAR}$-space the indicatrix is of the constant curvature

$$\mathcal{R}_{\mathcal{UAR}}\text{-pseudo-Finsleroid indicatrix} = -\epsilon h^2.$$ (1.9)

This value agrees with the conclusions (2.55)-(2.57) obtainable in the traditional manner when the known tensors of Finsler geometry are systematically evaluated. Also, the attentive consideration leads to the remarkable equality

$$\alpha = \frac{1}{h}\alpha_{\text{pseudo-Riemannian}}$$ (1.10)

(see (C.31)), where the right-hand part belongs to the sense of the ordinary pseudo-Riemannian geometry and operates in the space $\mathcal{G}$. The equality (1.10) is a direct implication of (1.6)-(1.8).

To come to the framework with changing the curvature of the pseudo-Finsleroid indicatrix along the background manifold of points $x$, we must allow for a dependence $g = g(x)$ of the involved pseudo-Finsleroid charge $g$ of $x$, avoiding the stipulation that the $g$ be a constant. In this direction, the idea of involutive dependence (set forth in the previous work [12]), which permits for the scalar $g(x)$ to vary just in the direction assigned by the input preferred vector field $b_i(x)$, seems to be attractive to follow.

### 2. Basic notions of the $\mathcal{A}\mathcal{R}$-space

Let $M$ be an $N$-dimensional $C^\infty$ differentiable manifold, $T_xM$ denote the tangent space to $M$ at a point $x \in M$, and $y \in T_xM$ mean tangent vectors. Suppose we are given on $M$ a pseudo-Riemannian metric $S = S(x, y)$. Denote by $\mathcal{R}_N = (M, S)$ the obtained $N$-dimensional pseudo-Riemannian space. Let us also assume that the manifold $M$ admits a non-vanishing space-like 1-form $b = b(x, y)$ of the length $c(x)$, so that

$$||b|| = -c.$$ (2.1)

We shall assume

$$0 < c(x) \leq 1.$$ (2.2)
With respect to natural local coordinates in the space $R_N$ we have the local representations

$$
b = b_i(x)y^i, \quad S = a_{ij}(x)y^i y^j, \quad a^{ij}(x)b_i(x)b_j(x) = -c^2(x),$$

(2.3)

with a pseudo-Riemannian metric tensor $a_{ij}(x)$, so that the time-space signature

$$\text{sign}(a_{ij}) = (+ - \ldots)$$

(2.4)

takes place. The reciprocity $a^{in}a_{nj} = \delta^i_j$ is implied, where $\delta^i_j$ stands for the Kronecker symbol. The covariant index of the vector $b_i$ will be raised by means of the tensorial rule $b^i = a^{ij}b_j$, which inverse reads $b_i = a_{ij}b^j$. We also introduce the tensor

$$r_{ij}(x) := a_{ij}(x) + b_i(x)b_j(x)$$

(2.5)

and construct the quadratic form

$$\gamma = r_{ij}y^i y^j,$$

(2.6)

in terms of which we introduce the variable

$$q = \sqrt{|\gamma|},$$

(2.7)

obtaining

$$S = \gamma - b^2.$$

(2.8)

We also introduce on the background manifold $M$ a scalar field $g = g(x)$. The consideration will be structured by signs of the quadratic form

$$B(x, y) := \gamma - gbq - b^2 \equiv S - gbq.$$  

(2.9)

Namely, to generalize the pseudo-Riemannian geometry in a desired pseudo-Finsleroid Finslerian way, we adapt the consideration to the following decomposition of the tangent bundle $TM$:

$$TM = T_g^+ \cup \Sigma_g^+ \cup R_g^+ \cup \mathcal{R}_g \cup \mathcal{R}_g^{-0} \cup \mathcal{R}_g^- \cup \Sigma_g^- \cup T_g^-,$$

(2.10)

which sectors relate to the cases where the tangent vectors $y \in TM$ are, respectively, time-like, upper-cone isotropic, space-like, lower-cone isotropic, or past-like. The upperscripts "+" and "−" stand for "future" and "past", respectively. We take the sums

$$T_g^+ = T_g^{+r} \cup T_g^{+l}, \quad \Sigma_g^+ = \Sigma_g^{+r} \cup \Sigma_g^{+l}, \quad \Sigma_g^- = \Sigma_g^{-r} \cup \Sigma_g^{-l}, \quad T_g^- = T_g^{-r} \cup T_g^{-l},$$

(2.11)

$$R_g^+ = R_g^{+r} \cup R_g^{+l}, \quad \mathcal{R}_g = \mathcal{R}_g^r \cup \mathcal{R}_g^l, \quad R_g^- = R_g^{-r} \cup R_g^{-l},$$

(2.12)

where $r$ means in the direction of the vector $b^i$ (= to the right), and $l$ means opposite to the direction of $b^i$ (= to the left).

We propose

**Definition.** The squared AR-pseudo-Finsleroid metric function $F^2(x, y)$ is given by the formula

$$F^2(x, y) := B(x, y)J^2(x, y)$$

(2.13)

with the function $J$ taken from (3.12) and (4.11).
The positive (not absolute) homogeneity holds: $F^2(x, \lambda y) = \lambda^2 F^2(x, y)$ for any $\lambda > 0$ and all admissible $(x, y)$.

We introduce the indicator $\epsilon$:

$$\epsilon = 1, \text{ if vector } y \text{ is time-like, \ and } \epsilon = -1, \text{ if vector } y \text{ is space-like.} \quad (2.14)$$

We can obtain from the function $F^2$ the distinguished Finslerian tensors, and first of all the covariant tangent vector $\hat{y} = \{y_i\}$, the Finslerian metric tensor $\{g_{ij}\}$ together with the contravariant tensor $\{g^{ij}\}$ defined by the reciprocity conditions $g_{ij}g^{jk} = \delta^k_i$, and the angular metric tensor $\{h_{ij}\}$, by making use of the conventional Finslerian rules in succession:

$$y_i := \frac{1}{2} \frac{\partial F^2}{\partial y^i}, \quad g_{ij} := \frac{1}{2} \frac{\partial^2 F^2}{\partial y^i \partial y^j} = \frac{\partial y_i}{\partial y^j}, \quad h_{ij} := g_{ij} - \frac{1}{F^2} y_i y_j. \quad (2.15)$$

Under these conditions, we introduce

**Definition.** The arisen space

$$\mathcal{AR}_{g;c} := \{\mathcal{R}_N; TM; y \in TM; b_i(x); g(x); F^2(x, y); g_{ij}(x, y)\} \quad (2.16)$$

is called the $\mathcal{AR}$-pseudo-Finsleroid space.

**Definition.** The space $\mathcal{R}_N = (M, S)$ entering the above definition is called the associated pseudo-Riemannian space.

**Definition.** Within any tangent space $T_xM$, the function $F^2(x, y)$ produces the $\mathcal{AR}$-pseudo-Finsleroid indicatrix:

$$\mathcal{IAR}_{g;c}(x) := \{y \in \mathcal{IAR}_{g;c}(x) : y \in T_xM, F^2(x, y) = 1\}, \text{ time-like } \{y\}; \quad (2.17)$$

$$\mathcal{IAR}_{g;c}(x) := \{y \in \mathcal{IAR}_{g;c}(x) : y \in T_xM, F^2(x, y) = 0\}, \text{ isotropic } \{y\}; \quad (2.18)$$

$$\mathcal{IAR}_{g;c}(x) := \{y \in \mathcal{IAR}_{g;c}(x) : y \in T_xM, F^2(x, y) = -1\}, \text{ space-like } \{y\}. \quad (2.19)$$

**Definition.** $\mathcal{IAR}_{g;c}(x) \subset T_xM$ is the boundary of the $\mathcal{AR}$-pseudo-Finsleroid $\mathcal{PAR}_{g;c}(x) \subset T_xM$.

**Definition.** The scalar $g(x)$ is called the $\mathcal{AR}$-pseudo-Finsleroid charge. The 1-form $b = b_i(x) y^i$ is called the $\mathcal{AR}$-pseudo-Finsleroid-axis 1-form.

In the time-like sectors the quadratic form (2.9) reads merely

$$B = q^2 - gbq - b^2 > 0,$$

and, therefore, is of the positive discriminant

$$D_B = 4h^2 > 0, \quad \epsilon = 1,$$

with $h = \sqrt{1 + (q^2/4)}$. Alternatively, in the space-like sectors, the formula (2.7) yields $\gamma = -q^2$ and the $B$ of (2.9) takes on the form

$$B = -(q^2 + gbq + b^2) < 0,$$
which discriminant is negative:

\[ D_{(B)} = -4h^2 < 0, \quad \epsilon = -1, \]

where this time \( h = \sqrt{1 - (g^2/4)} \). This distinction is the reason why the form of the pseudo-Finsleroid function \( F^2 \) is essentially different (see (3.11)-(3.12) and (4.10)-(4.13)) in the time-like and space-like sectors.

In the limit \( g \to 0 \), the definition (2.9) degenerates to the quadratic form of the input pseudo-Riemannian metric tensor \( a_{ij}(x) \):

\[ B|_{g=0} = a_{ij}(x)y^iy^j \equiv S(x,y). \]

We have also

\[ J(x,y)|_{g=0} = 1, \quad F^2(x,y)|_{g=0} = S(x,y), \quad (2.20) \]

and

\[ a_{ij}(x) = g_{ij}(x,y)|_{g=0}. \quad (2.21) \]

**Definition.** The space \( \mathcal{AR}_{g,c} \) is unimodular, if \( c = 1 \):

\[ \mathcal{UAR}_g := \mathcal{AR}_{g,c=1}. \quad (2.22) \]

The derivatives

\[ \frac{\partial J}{\partial b} = -\frac{g}{2} \frac{q}{B} J, \quad \frac{\partial J}{\partial q} = \frac{g}{2} \frac{b}{B} J \quad (2.23) \]

can readily be obtained from (3.12) and (4.11).

In many cases it is convenient to use the variables

\[ u_i := a_{ij}y^j, \quad v^i := y^i + bb^i, \quad v_m := u_m + bb_m = r_{mn}y^n \equiv a_{mn}v^n. \quad (2.24) \]

The identities

\[ u_i v^i = v_i y^i = \epsilon q^2, \quad v_i b^i = v^i b_i = (1 - c^2)b_i, \quad r_{ij} b^j = (1 - c^2)b_i, \quad (2.25) \]

\[ v_i v^i = \epsilon q^2 + (1 - c^2)b^2, \quad \frac{\partial b}{\partial y^i} = b_i, \quad \frac{\partial q}{\partial y^i} = \frac{v_i}{q}, \quad \frac{\partial (q/b)}{\partial y^i} = \frac{q}{b^2}e_i \quad (2.26) \]

can readily be verified, where

\[ e_i = -b_i + \epsilon \frac{b}{q^2} v_i \quad (2.27) \]

is the vector showing the property \( e_i y^i = 0 \). Particularly, the vector enters the equality

\[ \frac{\partial (J^2)}{\partial y^k} = \frac{gg}{B} J^2 e_k. \quad (2.28) \]

By performing required direct calculation, we find the representations

\[ y_i = (v_i - (b + gq)b_i)J^2 \quad (2.29) \]

and

\[ g_{ij} = \left[ a_{ij} - \frac{g}{B} \left( -q(b + gq)b_i b_j + q(b_i v_j + b_j v_i) - \epsilon b \frac{v_i v_j}{q} \right) \right] J^2, \quad (2.30) \]
together with the reciprocal (contravariant) components
\[ g^{ij} = \begin{bmatrix} a^{ij} + \frac{g}{B} \left( -bq b^i b^j + q(b^i v^j + b^j v^i) - \epsilon(b + gc^2 q) \frac{v^i v^j}{\nu} \right) \end{bmatrix} \frac{1}{J^2}. \] (2.31)

The determinant of the metric tensor (2.30) can straightforwardly be evaluated, yielding
\[ \text{det}(g_{ij}) = \frac{\nu}{q} J^{2N} \text{det}(a_{ij}), \] (2.32)
where
\[ \nu = q - \epsilon(1 - c^2)gb. \] (2.33)

It is useful to note that
\[ b(b + gc^2 q) = -B + \epsilon q \nu, \] (2.34)
\[ g_{ij} b^j = \begin{bmatrix} b_i - \frac{g}{B} \left( q(b + gc^2 q)b_i - \frac{1}{q} (c^2 q^2 + \epsilon(1 - c^2)b^2) v_i \right) \end{bmatrix} J^2, \] (2.35)
and
\[ g_{ij} v^j = \frac{\nu}{q} \left[ (\epsilon q^2 - b^2) v_i - \epsilon g q^3 b_i \right] \frac{1}{B} J^2. \] (2.36)

In terms of the set \( \{ b_i, u_i = a_{ij} y^j \} \), we obtain the alternative representations
\[ y_i = (u_i - g q b_i) J^2 \] (2.37)
and
\[ g_{ij} = \begin{bmatrix} a_{ij} + \frac{g}{B} \left( (gq^2 - \frac{b(q^2 - \epsilon b^2)}{q}) b_i b_j + \frac{b}{q} u_i u_j - \frac{q^2 - \epsilon b^2}{q} (b_i u_j + b_j u_i) \right) \end{bmatrix} J^2. \] (2.38)
We get also
\[ g^{ij} = \begin{bmatrix} a^{ij} + \epsilon \frac{q}{\nu} (bb^i b^j + b^i y^j + b^j y^i) - \epsilon \frac{g}{B \nu} (b + gc^2 q) y^i y^j \end{bmatrix} \frac{1}{J^2}. \] (2.39)

With the help of (2.37) we can transform (2.38) to
\[ g_{ij} = \begin{bmatrix} a_{ij} + \epsilon \frac{q}{q} \left( -(b + gq) b_i b_j + \frac{1}{B} b \frac{1}{J^2} \frac{1}{J^2} y_i y_j - \frac{1}{J^2} (b_i y_j + b_j y_i) \right) \end{bmatrix} J^2. \] (2.40)

To raise the index, it is convenient to apply the rules
\[ g^{ij} b_j = \frac{1}{F^2} \left[ (B + gbq) b^i - \epsilon \frac{g}{\nu} (c^2 B + b(b + gc^2 q)) v^i \right] \] and, for any co-vector \( t_j \), from (2.31) we obtain
\[ g^{ij} t_j = \begin{bmatrix} Ba^{ij} t_j + gq(yt) b^i + \frac{\epsilon g}{\nu} \left( B(bt) - (b + gc^2 q)(yt) \right) v^i \end{bmatrix} \frac{1}{F^2}, \]
where \((yt) = y^j t_j\) and \((bt) = b^j t_j\).

From the determinant value (2.32) we can explicate the vector

\[
C_i = \frac{\partial \ln \left( \sqrt{\det(g_{mn})} \right)}{\partial y^i},
\]

(2.41)

obtaining

\[
C_i = g \frac{1}{2B X} \frac{q}{\epsilon} e_i,
\]

(2.42)

where

\[
\frac{1}{X} = N + \epsilon \frac{(1 - \epsilon^2)B}{q\nu}.
\]

(2.43)

Also,

\[
C^i = g \frac{1}{2F^2 X} \frac{q}{\nu} \left( -b^i + \epsilon \frac{1}{q\nu} (b + gc^2q)\psi^i \right),
\]

(2.44)

or alternatively,

\[
C^i = \epsilon g \frac{1}{2F^2 X} \frac{1}{\nu} \left( -Bb^i + (b + gc^2q)y^i \right).
\]

(2.45)

We can evaluate the contraction

\[
C^h C_h = -\epsilon g^2 \frac{1}{4F^2X^2} \left( N + 1 - \frac{1}{X} \right).
\]

(2.46)

The Cartan tensor

\[
C_{ijk} := \frac{1}{2} \frac{\partial g_{ij}}{\partial y^k},
\]

(2.47)

when evaluated with the help of the components \(g_{ij}\) given by (2.30), is representable in the form

\[
C_{ijk} = X \left[ C_i h_{jk} + C_j h_{ik} + C_k h_{ij} - \left( N + 1 - \frac{1}{X} \right) \frac{1}{C_h C_h} C_i C_j C_k \right].
\]

(2.48)

The curvature of indicatrix is described by the tensor

\[
\hat{R}^i_{mn} := C^{j}_{h m} C^{h}_{i n} - C^{j}_{h n} C^{h}_{i m}.
\]

(2.49)

Inserting here (2.48) results in

\[
\hat{R}^i_{mn} g^m = -(C_h C^h) X^2 (h_{im} h_{jn} - h_{in} h_{jm})
\]

\[
+ X^2 \left( N - \frac{1}{X} \right) \left( C_i C_m h_{jn} - C_i C_n h_{jm} + C_j C_n h_{im} - C_j C_m h_{in} \right).
\]

(2.50)

Contracting this tensor yields

\[
\hat{R}^{i}{}_{jmn} g^n = -(C_h C^h) X^2 (N - 2) h_{im} + X^2 \left( N - \frac{1}{X} \right) \left( (N - 3) C_i C_m + C_j C^j h_{im} \right)
\]

(2.51)

and
\[ \hat{R}_{ijmn}g^{jn}g^{im} = -(C_hC^h)X^2(N - 2)(N - 1) + X^2 \left( N - \frac{1}{X} \right) (C_hC^h)(2N - 4). \] (2.52)

From these formulas it can readily be concluded that the tangent pseudo-Riemannian space is conformally flat. For instance, taking the dimensions \( N > 3 \), we can evoke the conformal Weyl tensor \( W_{ijmn} \) and use the representations (2.50)-(2.52). We straightforwardly obtain \( W_{ijmn} = 0 \).

In the unimodular space \( \mathcal{UA}R_g \) defined by (2.22), the equality \( c = 1 \) entails \( X = 1/N \) (see (2.43)), so that the representation (2.48) reduces to

\[ C_{ijk} = \frac{1}{N} \left( h_{ij}C_k + h_{ik}C_j + h_{jk}C_i - \frac{1}{C_hC^h}C_iC_jC_k \right) \] (2.53)

with (1.2) being valid. The curvature tensor representation (2.50) reduces to merely

\[ F^2\hat{R}_{ijmn} = \frac{e^g}{4}(h_{im}h_{jn} - h_{in}h_{jm}), \] (2.54)

which entails the following formulas to characterize the value \( \mathcal{R} \) of the indicatrix curvature:

\[ \mathcal{R}_{\mathcal{UA}R}\text{-pseudo-Finsleroid indicatrix} = -\left( 1 + \frac{1}{4}g^2 \right) \leq -1, \text{ when } \epsilon = 1, \] (2.55)

and

\[ \mathcal{R}_{\mathcal{UA}R}\text{-pseudo-Finsleroid indicatrix} = \left( 1 - \frac{1}{4}g^2 \right) \leq 1, \text{ when } \epsilon = -1. \] (2.56)

We have

\[ \mathcal{R}_{\mathcal{UA}R}\text{-pseudo-Finsleroid indicatrix} \xrightarrow{g \to 0} \mathcal{R}_{\text{pseudo-Euclidean sphere}}. \] (2.57)

The \( \mathcal{UA}R\text{-pseudo-Finsleroid indicatrix of the time-like case is a space of constant negative curvature} \) (according to (2.55)). The positive curvature value (2.56) is obtained in the space-like case.

With the expression \( g^{ij}t_j \) indicated before (2.41), it follows that

\[ F^2(b + gc^2q)g^{ij}t_j = \left[ Ba^{ij}t_j + gq(yt)b^i \right](b + gc^2q) \]

\[ + \frac{e^g}{\nu} \left( B(bt) - (b + gc^2q)(yt) \right)(b + gc^2q)v^i. \]

Since

\[ \frac{2F^2X}{g}C^i + qb^i = \frac{\epsilon}{\nu}(b + gc^2q)v^i \]

(see (2.44)), we can write

\[ F^2(b + gc^2q)g^{ij}t_j = Ba^{ij}t_j(b + gc^2q) + (bt)gqb^i \]

\[ + 2\left( B(bt) - (b + gc^2q)(yt) \right)F^2XC^i. \]

Use here

\[ gBb^i = g(b + gc^2q)y^i - \frac{2F^2X}{\epsilon}C^i \]
(see (2.45)) and the equality \( B = -b(b + gc^2q) + cq \nu \) (see (2.34)) to get

\[
F^2 g^{ij} t_j = B a^{ij} t_j + (bt) gq y^i - 2bF^2 X(bt)C^i - 2(yt) F^2 X C^i.
\]

In this way we arrive at the expansion

\[
F^2 g^{ij} t_j = B \left( a^{ij} t_j + \frac{1}{c^2} (bt) b^i \right) + 2 \frac{q}{gc^2} F^2 X (bt) C^i
\]

\[
- 2 \left( \frac{1}{c^2} b(bt) + (yt) \right) F^2 X C^i - \frac{1}{c^2} b(bt) y^i.
\]

(2.58)

We use the pseudo-Riemannian covariant derivative

\[
\nabla_i b_j := \partial_i b_j - b_k a^k_{ij},
\]

(2.59)

where

\[
a^k_{ij} := \frac{1}{2} b^{kn} (\partial_j a_{ni} + \partial_i a_{nj} - \partial_n a_{ji})
\]

(2.60)

are the Christoffel symbols given rise to by the associated pseudo-Riemannian metric \( \mathcal{S} \).

Attentive direct calculations (see Appendix B) of the induced spray coefficients \( G^i = \gamma^i_{nm} y^n y^m \), where \( \gamma^i_{nm} \) denote the associated Finslerian Christoffel symbols, can be used to arrive at the following result.

**PROPOSITION.** In the \( \mathcal{A} \mathcal{R} \)-space the spray coefficients \( G^i \) can explicitly be written in the form

\[
G^i = -\varepsilon g^i_j \left( y^i y^h \nabla_j b_h - gq b_j f_j \right) v^i + gq f^i + E^i + a^i_{nm} y^n y^m,
\]

(2.61)

with the notation \( v^i = y^i + bb^i \) and

\[
f^i = f^i_{\eta} y^\eta, \quad f^i_{\eta} = a^i_{\kappa} f_{\kappa \eta}, \quad f_{\eta \kappa} = \nabla_{\eta} b_{\kappa} - \nabla_{\kappa} b_{\eta} \equiv \frac{\partial b_{\eta}}{\partial x^m} - \frac{\partial b_{\eta}}{\partial x^m}.
\]

(2.62)

The coefficients \( E^i \) involve the gradients \( g_h = \partial g / \partial x^h \) and can be taken as

\[
E^i = g^{ih} \frac{\partial g_h}{\partial g} (yy) - \frac{1}{2} \tilde{M} F^2 g^h g^{ih},
\]

(2.63)

where \( X \) is the function given in (2.43), \( (yy) = y^h g_h \), and the function \( \tilde{M} \) is defined by the equality

\[
\frac{\partial F^2}{\partial g} = \tilde{M} F^2.
\]

(2.64)

Applying the formula (2.58) to the case \( t_i = g_i \) yields

\[
F^2 g_h g^{ih} = B \left( g^i + \frac{1}{c^2} (bg) b^i \right) + 2 \frac{q}{gc^2} F^2 X (bg) C^i
\]

\[
- 2 \left( \frac{1}{c^2} b(bg) + (yg) \right) F^2 X C^i - \frac{1}{c^2} b(bg) y^i.
\]

(2.65)
where \((bg) = b^h g_h\).

In (2.61), the difference \(G^i - a^i_{nm} y^n y^m\) involves the crucial terms linear in the covariant derivative \(\nabla_j b_h\). When \(g = \text{const}\), we have \(E^i = 0\) and the metric function \(F\) does not enter the right-hand side of (2.61), the only trace of the function \(F\) in the spray coefficients (2.61) being the occurrence of the pseudo-Finsleroid charge \(g\) in the right-hand side.

Now we perform differentiation with respect to the pseudo-Finsleroid parameter \(g\), obtaining
\[
\frac{\partial h}{\partial g} = \epsilon \frac{1}{4} G, \quad \frac{\partial G}{\partial g} = \frac{1}{h^3}.
\]

From (4.12) and (4.13) we find
\[
\frac{\partial f}{\partial g} = -\frac{1}{2h} + \frac{b}{-B} \left( \frac{1}{4} G q + \frac{1}{2h} b \right), \quad \epsilon = -1,
\]
and
\[
\frac{\partial F^2}{\partial g} = -b q J^2 - \frac{1}{h^3} f F^2 + G \left[ \frac{1}{2h} - \frac{b}{-B} \left( \frac{1}{4} G q + \frac{1}{2h} b \right) \right] F^2 = \bar{M} F^2, \quad \epsilon = -1,
\]
where
\[
\bar{M} = -\frac{b q}{B} - \frac{1}{h^3} f + \frac{1}{2} G \left( q^2 + \frac{1}{2} gbq \right), \quad \epsilon = -1,
\]
or
\[
\bar{M} = -\frac{1}{h^3} f + \frac{1}{2} G q^2 + \frac{1}{-B h^2} b q, \quad \epsilon = -1.
\]

We find the vector \(\bar{M}_i = \partial \bar{M} / \partial y^i\):
\[
\bar{M}_i = \frac{4q^2}{g B} X C_i, \quad \epsilon = -1.
\]

Differentiating (2.64) with respect to \(y^i\) and using (2.69) just yield
\[
\frac{\partial y_i}{\partial g} = \bar{M} y_i + \frac{2q^2}{g} J^2 X C_i, \quad \epsilon = -1.
\]

Inserting this result in (2.63) yields
\[
E^i = \bar{M} (yg) y^i + F^2 \frac{2q^2}{g B} (yg) X C^i - \frac{1}{2} \bar{M} F^2 g_h g^{ih}, \quad \epsilon = -1.
\]

Similar expressions can be derived in the case \(\epsilon = 1\) (see (3.12)-(3.13)).

The coefficients \(G^i\) determine the geodesic equation
\[
\frac{d^2 x^i}{ds^2} + G^i \left( x, \frac{dx}{ds} \right) = 0,
\]
where \(ds = \sqrt{|F^2(x, dx)|}\).

Also, it is possible to transfer to the present theory the concept of angle that was introduced in the previous work [8] (dealt with the case when the involved vector field
$b_i(x)$ is time-like). Indeed, whenever $c = 1$ (whence $\|b\| = -1$), any two nonzero tangent vectors $y_1, y_2 \in T_x M$ of a fixed tangent space form the $\mathcal{UAR}$-angle

$$\alpha_{\{x\}}(y_1, y_2) := \frac{1}{h} \arccosh \frac{h^2 \langle y_1, y_2 \rangle_{\{x\}} - A(x, y_1)A(x, y_2)}{|B(x, y_1)| |B(x, y_2)|},$$

(2.71)
in the future-time-like sector, and

$$\alpha_{\{x\}}(y_1, y_2) := \frac{1}{h} \arccos \frac{A(x, y_1)A(x, y_2) - h^2 \langle y_1, y_2 \rangle_{\{x\}}}{|B(x, y_1)| |B(x, y_2)|},$$

(2.72)
in the space-like sector, where $\langle y_1, y_2 \rangle_{\{x\}} = r_{ij}(x)y_i^1y_j^2$ and $A = b + (1/2)gq$.

At equal vectors, the zero-value

$$\alpha_{\{x\}}(y_1, y_2) = 0$$

(2.73)
takes place. The zero-degree homogeneity

$$\alpha_{\{x\}}(y_1, ky_2) = \alpha_{\{x\}}(ky_1, y_2) = \alpha_{\{x\}}(y_1, y_2), \quad k > 0, \forall y_1, y_2,$$

(2.74)
holds. The proposal obviously exhibits the symmetry:

$$\alpha_{\{x\}}(y_1, y_2) = \alpha_{\{x\}}(y_2, y_1).$$

(2.75)

The angle $\alpha_{\{x\}}(y_1, y_2)$ is supported by a point $x \in M$ of the base manifold $M$ (in just the same sense as in the pseudo-Riemannian geometry), and is independent of any vector element $y$ of support.

If, fixing a point $x$, we consider the angle $\alpha_{\{x\}}(y, b)$ formed by a vector $y \in T_x M$ with the input characteristic vector $b'(x)$, taking into account that the nullification $q = 0$ and the equality $A = -1$ take place at $y = b$ whenever the $\mathcal{UAR}$-pseudo-Finsleroid space is used, from (2.72) we get in the space-like sector the respective value to be

$$\alpha_{\{x\}}(y, b) := \frac{1}{h} \arccos \frac{A(x, y)}{|B(x, y)|}.$$  

(2.76)

In the Euclidean and Riemannian geometries, an important role is played by the spherical coordinates. Their use enables to conveniently represent vectors, evaluate squares and volumes, study curvature of surfaces, in many cases simplify consideration and solve rigorously equations, and also introduce and use various trigonometric functions. Similar coordinates are available in the pseudo-Euclidean geometry and applications founded upon such the geometry. In the context of the present theory, such coordinates can meaningfully be extended. The key observation thereto is the appropriate choice of the angle $\chi$ as follows:

$$\chi = \frac{1}{h} J,$$

(2.77)
in agreement with

$$J = e^{-\frac{1}{2}g\chi}$$

(2.78)
(see (4.11) and formulas below (3.12)).

Accordingly, by fixing the tangent space according to Section 5, and choosing the four-dimensional case $N = 4$, the $\mathcal{UAR}$-space coordinates $\{z^p\}$ are given by

$$z^0 = \sqrt{|F^2|}, \quad z^1 = \eta, \quad z^2 = \phi, \quad z^3 = \chi,$$

(2.79)
where $\phi$ is the polar angle in the $\mathbb{R}^1 \times \mathbb{R}^2$-plane, $\chi$ plays the role of the pseudo-Finsleroid azimuthal angle measured from the direction of the input vector $b^i$, and $\eta$ serves to measure the time-components. The indices $p, q, ...$ will be specified over the range $0,1,2,3$.

For the vectors $\{R^p\}$ (see (5.4)-(5.7)) it proves possible to construct the representation

$$R^p = R^p(g; z^q)$$

which possesses the invariance property

$$F^2(g; R^p(g; z^q)) = (z^0)^2 \epsilon. \quad (2.80)$$

We shall transform the Finslerian metric tensor $g_{pq}(g; R)$ given by the components (5.17)-(5.20) to such coordinates, obtaining the tensor

$$A_{rs}(g; z) := g_{pq}(g; R) \frac{\partial R^p}{\partial z^r} \frac{\partial R^q}{\partial z^s} \quad (2.81)$$

which is of the diagonal structure.

In the future-time-like sector, so that $\epsilon = 1$, we have $F > 0$ and can take $z^0 = F$. The following representations are appropriate:

$$R^0 = F \cosh \eta \operatorname{Ch} \chi,$$

$$R^1 = F \sinh \eta \operatorname{Ch} \chi \cos \phi, \quad R^2 = F \sinh \eta \operatorname{Ch} \chi \sin \phi, \quad R^3 = F \operatorname{Sh} \chi; \quad (2.83)$$

with

$$\operatorname{Ch} \chi = \frac{1}{J \hbar} \cosh f, \quad \operatorname{Sh} \chi = \frac{1}{J} \left( \sinh f - \frac{G}{2} \cosh f \right), \quad (2.84)$$

entailing

$$q = F \operatorname{Ch} \chi, \quad b + \frac{1}{2} g q = \frac{F}{J} \sinh f. \quad (2.85)$$

In the space-like sector, when $\epsilon = -1$ is used, we introduce the function

$$K = \sqrt{|F^2|} > 0, \quad \text{so that} \quad z^0 = K, \quad (2.86)$$

and set forth the representations

$$R^0 = K \sinh \eta \operatorname{Sin} \chi,$$

$$R^1 = K \cosh \eta \operatorname{Sin} \chi \cos \phi, \quad R^2 = K \cosh \eta \operatorname{Sin} \chi \sin \phi, \quad R^3 = K \operatorname{Cos} \chi; \quad (2.87)$$

with

$$\operatorname{Sin} \chi = \frac{1}{J \hbar} \sin f, \quad \operatorname{Cos} \chi = \frac{1}{J} \left( \cos f - \frac{G}{2} \sin f \right), \quad (2.88)$$

which entails

$$q = K \operatorname{Sin} \chi, \quad b + \frac{1}{2} g q = \frac{K}{J} \cos f. \quad (2.89)$$

The arisen functions $\operatorname{Sh} \chi$, $\operatorname{Ch} \chi$, $\operatorname{Sin} \chi$, $\operatorname{Cos} \chi$ can be interpreted as the required extensions of the trigonometric functions to the $\mathcal{UAR}$-space.
When these representations are applied to the \textit{UAR\angle} (2.71)-(2.72), the following result is obtained:

$$\alpha_{\{x\}}(y_1, y_2) = \frac{1}{h} \arccosh \tau_{12} \quad (2.92)$$

with

$$\tau_{12} = \cosh(f_2 - f_1) + \Omega_{12} \cosh f_1 \cosh f_2 \quad (2.93)$$

and

$$\Omega_{12} = \cosh(\eta_2 - \eta_1) - 1 + \left(1 - \cos(\phi_2 - \phi_1)\right) \sinh \eta_1 \sinh \eta_2 \quad (2.94)$$
in the future-time-like sector; alternatively,

$$\alpha_{\{x\}}(y_1, y_2) = \frac{1}{h} \arccos \tau_{12} \quad (2.95)$$

with

$$\tau_{12} = \cos(f_2 - f_1) + \Omega_{12} \sin f_1 \sin f_2 \quad (2.96)$$

and

$$\Omega_{12} = \cosh(\eta_2 - \eta_1) - 1 - \left(1 - \cos(\phi_2 - \phi_1)\right) \cosh \eta_1 \cosh \eta_2 \quad (2.97)$$
in the space-like sector.

From (2.82) we get the diagonal tensor, so that \(A_{01} = A_{02} = A_{03} = A_{12} = A_{13} = A_{23} = 0\), and the squared linear element \(ds^2 = \epsilon_{\alpha\beta} dz^\alpha dz^\beta\) which reads

$$\begin{align*}
(ds)^2 &= (dz^0)^2 - (z^0)^2 \left[(d\chi)^2 + \frac{1}{h^2} \cosh^2 f \left(\sinh^2 \eta (d\phi)^2 + (d\eta)^2\right)\right] \quad (2.98)
\end{align*}$$
in the future-time-like sector, and

$$\begin{align*}
(ds)^2 &= (dz^0)^2 + (z^0)^2 \left[(d\chi)^2 - \frac{1}{h^2} \sin^2 f \left((d\eta)^2 - \cosh^2 \eta (d\phi)^2\right)\right] \quad (2.99)
\end{align*}$$
in the space-like sector.

If we consider the infinitesimal version of the representations (2.92)-(2.97), putting \(\eta_2 - \eta_1 = d\eta, \chi_2 - \chi_1 = d\chi, \) and \(\phi_2 - \phi_1 = d\phi, \) then from (2.98) and (2.99) we can directly conclude that

$$\begin{align*}
(ds)^2 &= (dz^0)^2 - (z^0)^2 (d\alpha)^2, \quad (2.100)
\end{align*}$$

where \(d\alpha\) is the value issued from (2.92) and (2.95). The formula (2.100) is remarkable because tells us that, with the angle \(\alpha\) defined by (2.71)-(2.72), \(d\alpha\) is the arc-length on the indicatrix; for along the indicatrix we have \(z^0 = 1.\)

With the angle (2.71)-(2.72), we can naturally propose the \textit{UAR\angle-scalar product}

$$\langle y_1, y_2 \rangle_{\{x\}} := F(x, y_1) F(x, y_2) \cosh \left(\alpha_{\{x\}}(y_1, y_2)\right), \quad (2.101)$$
in the future-time-like sector, and

$$\langle y_1, y_2 \rangle_{\{x\}} := K(x, y_1) K(x, y_2) \cos \left(\alpha_{\{x\}}(y_1, y_2)\right), \quad (2.102)$$
in the space-like sector, where \(K = \sqrt{|F^2|} > 0\). At equal vectors, the reduction

$$\langle y, y \rangle_{\{x\}} = F^2(x, y) \quad (2.103)$$
takes place, that is, the two-vector scalar product proposed reduces exactly to the squared U.A.R-Finsler metric function. The homogeneity

\[ \langle y_1, ky_2 \rangle_x = k \langle y_1, y_2 \rangle_x, \quad \langle ky_1, y_2 \rangle_x = k \langle y_1, y_2 \rangle_x, \quad k > 0, \ \forall y_1, y_2, \quad (2.104) \]

holds. The proposal obviously exhibits the symmetry:

\[ \langle y_1, y_2 \rangle_x = \langle y_2, y_1 \rangle_x, \quad (2.105) \]

3. Future-time-like sector of the space \( \mathcal{A} \mathcal{R}_{g; c} \)

Assuming

\[ -\infty < g(x) < \infty, \quad (3.1) \]

we construct the scalar

\[ h(x) = \sqrt{1 + \frac{1}{4} (g(x))^2}. \quad (3.2) \]

In the future-time-like sector

\[ y \in T^+_g, \quad T^+_g = T^+_g \cup T^+_l \quad (3.3) \]

with

\[ T^+_g = \left\{ y \in T^+_g : y \in T_x M, b \geq 0, q > -g_- b \right\} \quad (3.4) \]

and

\[ T^+_l = \left\{ y \in T^+_l : y \in T_x M, b \leq 0, q > -g_+ b \right\}, \quad (3.5) \]

where

\[ g_+ = -\frac{1}{2} g + h, \quad g_- = -\frac{1}{2} g - h, \]

we have

\[ r_{ij} y^i y^j > 0 \quad (3.6) \]

and can write the variable (2.7) merely as

\[ q = \sqrt{r_{ij}(x) y^i y^j}. \quad (3.7) \]

The quadratic form (2.9) reads now

\[ B = q^2 - gbq - b^2 = (b - g_- q)(g_+ q - b) > 0. \quad (3.8) \]

Notice that

\[ b - g_- q > 0, \quad g_+ q - b > 0 \quad (3.9) \]

throughout the sector (3.3). The form \( B \) can also conveniently be represented as

\[ B = h^2 q^2 - \left( b + \frac{1}{2} gq \right)^2. \quad (3.10) \]

Throughout the sector (3.3), we take the metric function (2.13) as follows:

\[ F(x, y) := \sqrt{B(x, y)} J(x, y) \equiv (b - g_- q)^{G_+/2} (g_+ q - b)^{-G_- / 2} \quad (3.11) \]
with
\[ J(x, y) = \left( \frac{b - g - q}{g + q - b} \right)^{-G/4}, \]  
(3.12)
where \( G = g/h, \ G_+ = g_+/h, \) and \( G_- = g_-/h. \)

Introducing the function
\[ f = \frac{1}{2} \ln \left( \frac{b - g - q}{g + q - b} \right), \]
we obtain the representation
\[ J(x, y) = e^{-\frac{1}{2} G(x)f(x, y)} \]
of the type (4.11). We can write
\[ \cosh f = \frac{h q}{\sqrt{B}}, \quad \sinh f = \frac{b + \frac{1}{2} g q}{\sqrt{B}}. \]

With the function
\[ L = q - \frac{1}{2} g b, \]
we can write the identity (3.10) in the alternative form:
\[ B = L^2 - h^2 b^2. \]

The derivatives
\[ \frac{\partial f}{\partial g} = \frac{q}{B} \left( \frac{1}{4} G b - \frac{1}{2h} g \right), \quad \frac{\partial J^2}{\partial g} = -\frac{1}{h^3} f J^2 - G \frac{q}{B} \left( \frac{1}{4} G b - \frac{1}{2h} g \right) J^2, \]
and
\[ \frac{\partial J}{\partial b} = -\frac{g}{2B} J, \quad \frac{\partial J}{\partial q} = \frac{b}{2B} J \]
(3.13)
can readily be obtained from (3.12).

In many cases it is convenient to use the variables
\[ u_i := a_{ij} y^j, \quad v^i := y^i + b b^i, \quad v_m := u_m + b b_m = r_{mn} y^n \equiv a_{mn} v^n. \]
(3.14)
The identities
\[ u_i v^i = v_i y^i = q^2, \quad v_i b^i = v^i b_i = (1 - c^2) b, \quad r_{ij} b^j = (1 - c^2) b_i, \]
(3.15)
\[ v_i v^i = q^2 + (1 - c^2) b^2, \quad \frac{\partial b}{\partial y^i} = b_i, \quad \frac{\partial q}{\partial y^i} = \frac{v_i}{q}, \quad \frac{\partial (q/b)}{\partial y^i} = \frac{q}{b^2} e_i \]
(3.16)
can readily be verified, where
\[ e_i = -b_i + \frac{b}{q^2} v_i \]
(3.17)
is the vector showing the property \( e_i y^i = 0. \) Particularly, the vector enters the equality
\[ \frac{\partial \left( \frac{F^2}{B} \right)}{\partial y^k} = \frac{g q F^2}{B} e_k. \]
(3.18)
By performing required direct calculation, we find the representations
\[
y_i = \left( v_i - (b + gq)b_i \right) \frac{F^2}{B},
\]
(3.19)

\[
g_{ij} = \left[ a_{ij} - \frac{g}{B} \left( -q(b + gq)b_i b_j + q(b_i v_j + b_j v_i) - \frac{b v_i v_j}{q} \right) \right] \frac{F^2}{B},
\]
(3.20)

and
\[
g^{ij} = \left[ a^{ij} + \frac{g}{B} \left( -b q \delta_i^j + q(b^i v^j + b^j v^i) - (b + g c^2 q) \frac{v^i v^j}{\nu} \right) \right] \frac{B}{F^2}.
\]
(3.21)

The determinant of the metric tensor is
\[
det(g_{ij}) = \nu \frac{F^2}{B}^N \det(a_{ij}),
\]
(3.22)

where
\[
\nu = q - (1 - c^2)gb.
\]
(3.23)

Contracting shows that
\[
g_{ij} b^j = \left[ b_i - \frac{g}{B} \left( q(b + gq)c^2 b_i - q c^2 v_i + q b(1 - c^2)b_i - \frac{b}{q}(1 - c^2) b v_i \right) \right] \frac{F^2}{B},
\]
or
\[
g_{ij} b^j = \left[ b_i - \frac{g}{B} \left( q(b + gc^2 q)b_i - \frac{1}{q}(c^2 q^2 + (1 - c^2)b^2)v_i \right) \right] \frac{F^2}{B},
\]
(3.24)

and
\[
g_{ij} v^j = \left[ v_i - (b + gq)b_i + bb_i - \frac{g}{B} \left( q(b + gc^2 q)b_i - \frac{1}{q}(c^2 q^2 + (1 - c^2)b^2)v_i \right) \right] \frac{F^2}{B},
\]
or
\[
g_{ij} v^j = \nu \frac{F^2}{B^2} \left[ (q^2 - b^2)v_i - g q^3 b_i \right].
\]
(3.25)

It is of help to note that
\[
b(b + gc^2 q) = -B + q \nu.
\]
(3.26)

To verify that the tensor (3.21) is reciprocal to (3.20), we are to demonstrate that
\[
g^{ij} g_{ij} - \delta^n_i = 0. \]

We have
\[
g^{ij} g_{ij} - \delta^n_i = -\frac{g}{B} \left( -q(b + gq)b_i b^n + q(b_i v^n + b^n v_i) - \frac{b v_i v^n}{q} \right)
\]
\[
+ \frac{g q}{B} (v^n - b b^n) \left[ b_i - \frac{g}{B} \left( q(b + gc^2 q)b_i - \frac{1}{q}(c^2 q^2 + (1 - c^2)b^2)v_i \right) \right]
\]
\[
+ \frac{g}{B^2 q} \left( q v b^n - (b + gc^2 q)v^n \right) \left[ (q^2 - b^2)v_i - g q^3 b_i \right].
\]
Simplifying yields
\[ g^{nj}g_{ij} - \delta^n_i = -\frac{g}{B} \left( q(b_i v^n + b^n v_i) - b_i v^n \right) \]

\[ + \frac{gq}{B} v^n \left[ b_i - \frac{g}{B} \left( q(b + gc^2 q)b_i - \frac{1}{q}(c^2 q^2 + (1 - c^2)b^2)v_i \right) \right] - \frac{gq}{B} b^n \frac{1}{gq} (c^2 q^2 + (1 - c^2)b^2)v_i \]

\[ + \frac{g}{B^2 q} qv^n (q^2 - b^2)v_i - \frac{g}{B^2 q} (b + gc^2 q)v^n \left[ (q^2 - b^2)v_i - gq^2 b_i \right], \]

so the rest is
\[ g^{nj}g_{ij} - \delta^n_i = \frac{g}{B} b^n \left[ v_i + \frac{gq}{B} \frac{1}{q} (c^2 q^2 + (1 - c^2)b^2)v_i \right] - \frac{gq}{B} (b + gc^2 q)v^n (q^2 - b^2)v_i = 0. \]

Thus, the reciprocity is valid.

In terms of the set \( \{ b_i, u_i = a_{ij} y^j \} \), we obtain the alternative representations
\[ y_i = (u_i - gqb_i) \frac{F^2}{B} \] (3.27)

and
\[ g_{ij} = \left[ a_{ij} + \frac{g}{B} \left( (gq^2 - \frac{b(q^2 - b^2)}{q})b_i b_j + \frac{b}{q} u_i u_j - \frac{q^2 - b^2}{q} (b_i u_j + b_j u_i) \right) \right] \frac{F^2}{B}. \] (3.28)

We get also
\[ g^{ij} = \left[ a^{ij} + \frac{g}{\nu} (bb' b^i + b'y^j + b_i y^j) - \frac{g}{B
u} (b + gc^2 q)y^i y^j \right] \frac{B}{F^2}. \] (3.29)

With the help of (3.27) we can transform (3.28) to the representation
\[ g_{ij} = \left[ a_{ij} + \frac{g}{q} \left( -(b + gq) b_i b_j + \frac{1}{B} \frac{B}{B^2} B \frac{B}{F^2} y_i y_j - \frac{B}{F^2} (b_i y_j + b_j y_i) \right) \right] \frac{F^2}{B}. \] (3.30)

From the determinant value (3.22) we can explicate the vector
\[ A_i = F \frac{\partial \ln \left( \sqrt{|\det(g_{mn})|} \right)}{\partial y^i}, \] (3.31)

obtaining
\[ A_i = \frac{1}{2} F (1 - c^2) \frac{g}{\nu} e_i + \frac{N}{F} y_i - \frac{N F}{2B} \left( 2v_i - 2bb_i - gqb_i - gb \frac{1}{q} v_i \right), \]

or
\[ A_i = g \frac{F}{2B X} \frac{q}{\nu} e_i, \] (3.32)

where
\[ \frac{1}{X} = N + \frac{(1 - c^2)B}{q \nu}. \] (3.33)
Also,

\[ A^i = g \frac{1}{2F} \frac{q}{X} \left( -b^i + \frac{b}{q^2} v^i + \frac{g}{q^2 \nu} v^i (c^2 q^2 + (1 - c^2) b^2) \right). \]

Simplifying yields

\[ A^i = g \frac{1}{2F} \frac{q}{X} \left( -b^i + \frac{1}{q \nu} (b + gc^2 q) v^i \right), \quad (3.34) \]

or alternatively,

\[ A^i = g \frac{1}{2F} \frac{1}{X \nu} \left( -b^i + (b + gc^2 q) y^i \right). \quad (3.35) \]

We obtain the contraction

\[ A^h A_h = -g^2 \frac{1}{4} \frac{1}{X^2} \left( N + 1 - \frac{1}{X} \right). \quad (3.36) \]

Differentiating (3.30) yields

\[
\frac{\partial g_{ij}}{\partial y^k} - \frac{2}{F} X A_k g_{ij} = g \frac{e_k b_i b_j}{q} \frac{F^2}{B} - \frac{g}{q} \left[ e_k l_i l_j + 2 \left( \frac{1}{F} l_k - \frac{1}{F} X A_k \right) b_l l_j \right] + \frac{g b^1}{q} \frac{1}{F} (g_{ik} l_j + g_{jk} l_i) - \frac{g}{q} (g_{ik} b_j + g_{jk} b_i) 
\]

\[ -\frac{g}{q} \left( \frac{2b^1}{F} l_i l_j - l_j b_i - l_i b_j \right) 2X A_k + \frac{g}{q b} e_k (g_{j} b_i + y_i b_j) + \frac{g}{q b} y_j b_k b_i + \frac{g}{q b} y_i b_k b_j. \]

Applying here the equalities

\[ b_i = bl_i \frac{1}{F} - \frac{q^2}{B} e_i \quad (3.37) \]

(see (3.19)) and

\[ b_i b_j = b^2 l_i l_j \frac{1}{F^2} - b q^2 \frac{1}{B} \frac{1}{F} (e_i l_j + e_j l_i) + q^4 \frac{1}{B^2} e_i e_j, \quad (3.38) \]

we obtain

\[
\frac{\partial g_{ij}}{\partial y^k} - \frac{2}{F} X A_k g_{ij} = g \frac{e_k b_i b_j}{q} \frac{F^2}{B} - \frac{g}{q} e_k l_i l_j 
\]

\[ + \frac{g b^1}{q} \frac{1}{F} (h_{ik} l_j + h_{jk} l_i) - \frac{g}{q} \left( g_{ik} b_j + g_{jk} b_i \right) - \frac{2g b^1}{q} \frac{1}{F} X A_k l_i l_j + \frac{2g}{q} \left( l_j b_i + l_i b_j \right) X A_k 
\]

\[ + \frac{g}{q b} e_k \left( y_j b_i + y_i b_j \right) + \frac{g}{q b} y_j \left( b_l k \frac{1}{F} - \frac{q^2}{B} e_k \right) b_i + \frac{g}{q b} y_i \left( b_l k \frac{1}{F} - \frac{q^2}{B} e_k \right) b_j \]
\[
= -\frac{g}{q} e_k l_j l_j - 2 \frac{g}{q} b \frac{1}{F} X A_k l_j l_j + \frac{g}{q} e_k \left( b^2 l_j l_j \frac{1}{F^2} - b q^2 \frac{1}{B} F (e_i l_j + e_j l_i) + q^4 \frac{1}{B^2} e_i e_j \right) \frac{F^2}{B}
\]

\[
+ \frac{g}{q} b \frac{1}{F} (h_{ik} l_j + h_{jk} l_i) - \frac{g}{q} \left( h_{ik} b_j + h_{jk} b_i \right) - \frac{g}{q} b \frac{1}{B} e_k \left( y_j b_i + y_i b_j \right)
\]

\[
= -\frac{g}{q} e_k l_j l_j - 2 \frac{g}{q} b \frac{1}{F} X A_k l_j l_j + \frac{g}{q} e_k \left( b^2 l_j l_j \frac{1}{F^2} - b q^2 \frac{1}{B} F (e_i l_j + e_j l_i) + q^4 \frac{1}{B^2} e_i e_j \right) \frac{F^2}{B}
\]

\[
+ \frac{g}{q} b \frac{1}{F} (h_{ik} l_j + h_{jk} l_i) - \frac{g}{q} \left( h_{ik} b_j + h_{jk} b_i \right) - \frac{g}{q} b \frac{1}{B} e_k \left( y_j b_i + y_i b_j \right)
\]

so that

\[
= \frac{\partial g_{ij}}{\partial y^k} - \frac{2}{F} X A_k g_{ij} = gq^3 e_k e_i e_j - \frac{g}{q} e_k l_j l_j + gq \frac{1}{B} \left( h_{ik} e_j + h_{jk} e_i \right).
\]

Therefore, the associated Cartan tensor

\[
A_{ijk} := \frac{F}{2} \frac{\partial g_{ij}}{\partial y^k}
\]

is representable in the form

\[
A_{ijk} = X \left[ A_i h_{jk} + A_j h_{ik} + A_k h_{ij} - \left( N + 1 - \frac{1}{X} \right) \frac{1}{A_i A_j A_k} \right].
\]

It is useful to verify that the contraction

\[
A_i = g^{jk} A_{ijk}
\]

when obtained from (3.41) is identical to the representation (3.32).

In various processes of evaluation, it is useful to apply the formulas

\[
\frac{\partial B}{\partial y^k} = 2 B \frac{F^2}{F^2} y_k - 2 B \frac{F}{F} X A_k, \quad \frac{\partial \left( \frac{F^2}{B} \right)}{\partial y^k} = 2 F \frac{B}{B} X A_k,
\]

and

\[
F b_n = b_l - \frac{2q}{g} X A_n, \quad F \frac{1}{q} v_n = q l_n + \frac{B - q^2 2}{B} X A_n,
\]
together with

\[
\frac{\partial F}{\partial y^n} - \frac{2(B - q^2) X A_n}{gb^2} = -\frac{2q^2(2b + qg)}{gF B} X A_k, \quad (3.45)
\]

which can be verified by simple straightforward calculation. It follows that

\[
\frac{\partial(X A_k)}{\partial y^n} = -\frac{1}{F}X A_n + \frac{g b}{2F q} h_{kn} - \frac{2}{gbqF}(B - q^2)X^2 A_k A_n. \quad (3.46)
\]

4. The space-like sector of the space \( \mathcal{A} \mathcal{R}_{g;c} \)

Instead of (3.1) and (3.2), we now take

\[-2 < g(x) < 2 \quad (4.1)\]

and

\[h(x) = \sqrt{1 - \frac{1}{4}(g(x))^2}. \quad (4.2)\]

In the space-like sector

\[y \in \mathcal{R}_g^+, \quad \mathcal{R}_g^+ = \mathcal{R}_g^{+r} \cup \mathcal{R}_g^{+l} \quad (4.3)\]

with

\[\mathcal{R}_g^{+r} = \left( y \in \mathcal{R}_g^{+r} : y \in T_x M, b \geq 0, q < -g_b \right) \quad (4.4)\]

and

\[\mathcal{R}_g^{+l} = \left( y \in \mathcal{R}_g^{+l} : y \in T_x M, b \leq 0, q < -g_b \right), \quad (4.5)\]

where

\[g_+ = -\frac{1}{2}g + h, \quad g_- = -\frac{1}{2}g - h, \quad (4.6)\]

we have

\[r_{ij} y^i y^j < 0 \quad (4.6)\]

and can write the variable (2.7) merely as

\[q = \sqrt{-r_{ij}(x)y^i y^j}. \quad (4.7)\]

The quadratic form (2.9) takes on the form

\[B = -(q^2 + gbq + b^2) < 0, \quad (4.8)\]

which is of the negative discriminant

\[D_{(B)} = -4h^2 < 0, \quad (4.9)\]

where \(h\) is given by (4.2).

In the space-like region (4.3), the squared metric function \(F^2(x, y)\) is given by the formulas

\[F^2(x, y) = B(x, y) J^2(x, y) \quad (4.10)\]
and
\[ J(x, y) = e^{-\frac{1}{2}G(x)f(x,y)} \] (4.11)
with
\[ f = -\arctan \frac{G}{2} + \arctan \frac{L}{hb} = \arctan \frac{hq}{b + \frac{1}{2}gq}, \quad \text{if} \ b \leq 0, \] (4.12)
and
\[ f = -\pi - \arctan \frac{G}{2} + \arctan \frac{L}{hb} = -\pi + \arctan \frac{hq}{b + \frac{1}{2}gq}, \quad \text{if} \ b \geq 0, \] (4.13)
where \( G = g/h \) and
\[ L = q + \frac{q}{2}b. \] (4.14)

We have
\[ f = 0, \quad \text{if} \ q = 0 \quad \text{and} \quad b < 0; \] (4.15)
\[ f = -\pi, \quad \text{if} \ q = 0 \quad \text{and} \quad b > 0. \] (4.16)

Sometimes it is convenient to use also the function
\[ A = b + \frac{q}{2}q. \] (4.17)

The identities
\[ L^2 + h^2b^2 = -B, \quad h^2q^2 + A^2 = -B \] (4.18)
are valid.

The derivatives
\[ \frac{\partial J}{\partial b} = -\frac{g}{2B}qJ, \quad \frac{\partial J}{\partial q} = \frac{g}{2B}bJ \] (4.19)
can readily be obtained from (4.11)-(4.13).

It is useful to compare (4.10)-(4.13) with (3.11) and (3.12).

Again, we use the variables
\[ u_i := a_{ij}y^j, \quad v^i := y^i + bb^i, \quad v_m := u_m + bb_m = r_{mn}y^n \equiv a_{mn}v^n, \] (4.20)
and apply the identities
\[ u_iv^i = v_iy^i = -q^2, \quad v_ib^i = v^ib_i = (1 - c^2)b, \quad r_{ij}b^j = (1 - c^2)b_i, \] (4.21)
\[ v_iy^i = -q^2 + (1 - c^2)b^2, \quad \frac{\partial b}{\partial y^i} = b_i, \quad \frac{\partial q}{\partial y^i} = -\frac{v_i}{q}, \quad \frac{\partial (q/b)}{\partial y^i} = \frac{q}{b^2}e_i, \] (4.22)
where
\[ e_i = -b_i - \frac{b}{q^2}v_i \] (4.23)
is the vector showing the property \( e_iy^i = 0 \). Particularly, the vector enters the equality
\[ \frac{\partial (J^2)}{\partial y^k} = \frac{qq}{B}J^2e_k. \] (4.24)
By performing required direct calculation, we find the representations

\[ y_i = \left( v_i - (b + gq)b_i \right) J^2, \quad (4.25) \]

\[ g_{ij} = \left[ a_{ij} - \frac{g}{B} \left( -q(b + gq)b_ib_j + q(b_i v_j + b_j v_i) + b \frac{v_i v_j}{q} \right) \right] J^2, \quad (4.26) \]

and

\[ g^{ij} = \left[ a^{ij} + \frac{g}{B} \left( -bq b^i b^j + q(b^i v^j + b^j v^i) + (b + gc^2 q) \frac{v^i v^j}{q} \right) \right] \frac{1}{J^2}. \quad (4.27) \]

The determinant of the metric tensor is

\[ \det(g_{ij}) = \frac{\nu}{q} \left( J^2 \right)^N \det(a_{ij}), \quad (4.28) \]

where

\[ \nu = q + (1 - c^2)gb. \quad (4.29) \]

Contracting shows that

\[ g_{ij} b^j = \left[ b_i - \frac{g}{B} \left( q(b + gq)c^2 b_i - qc^2 v_i + qb(1 - c^2)b_i + b \frac{b}{q}(1 - c^2)b v_i \right) \right] J^2, \]

or

\[ g_{ij} b^j = \left[ b_i - \frac{g}{B} \left( q(b + gc^2 q)b_i - \frac{1}{q}(c^2 q^2 - (1 - c^2)b^2) v_i \right) \right] J^2, \quad (4.30) \]

and

\[ g_{ij} v^j = \left[ v_i - (b + gq)b_i + bb_i - b \frac{g}{B} \left( q(b + gc^2 q)b_i - \frac{1}{q}(c^2 q^2 - (1 - c^2)b^2) v_i \right) \right] J^2, \]

or

\[ g_{ij} v^j = \frac{\nu}{q} \left[ -(q^2 + b^2) v_i + q g^3 b_i \right] \frac{1}{B} J^2. \quad (4.31) \]

It is useful to note that

\[ b(b + gc^2 q) = -B - q\nu. \quad (4.32) \]

Let us verify that the tensor \((4.27)\) is reciprocal to \((4.26)\):

\[ g^{bij} g_{ij} - \delta^i \left[ \cdot \right] = -\frac{g}{B} \left( -q(b + gq)b_i b^j + q(b_i v^j + b^j v_i) + b \frac{v_i v^j}{q} \right) \]

\[ + \frac{gq}{B} (v^j - b b^j) \left[ b_i - \frac{g}{B} \left( q(b + gc^2 q)b_i - \frac{1}{q}(c^2 q^2 - (1 - c^2)b^2) v_i \right) \right] \]

\[ + \frac{g}{B^2 q} \left( q vb^j + (b + gc^2 q)v^j \right) \left[ -(q^2 + b^2) v_i + q g^3 b_i \right]. \]

Simplifying yields
\[ g^{nj}g_{ij} - \delta^n_i = -\frac{g}{B} \left( q(b_i v^n + b^n v_i) + b_i v^n v_i \right) \]
\[ + \frac{g q}{B} v^n \left[ b_i - \frac{g}{B} \left( q(b + gc^2 q)b_i - \frac{1}{q} (c^2 q^2 - (1 - c^2)b^2) v_i \right) \right] - \frac{g g q}{B^2 q} \left( c^2 q^2 - (1 - c^2)b^2 \right) v_i \]
\[- \frac{g}{B^2 q} q v b^m (q^2 + b^2) v_i + \frac{g}{B^2 q} (b + gc^2 q) v^n \left[ -(q^2 + b^2) v_i + g q b_i \right], \]
so the rest is
\[ g^{nj}g_{ij} - \delta^n_i = -\frac{g}{B} b_i v^n + \frac{g g q}{B} v^n \frac{1}{q} (c^2 q^2 - (1 - c^2)b^2) v_i - \frac{g}{B^2 q} (b + gc^2 q) v^n (q^2 + b^2) v_i = 0. \]
Thus, the reciprocity is valid.

In terms of the set \( \{ b_i, u_i = a_{ij} y^j \} \), we obtain from (4.25) and (4.26) the alternative representations
\[ y_i = (u_i - g q b_i) J^2 \] (4.33)
and
\[ g_{ij} = \left[ a_{ij} + \frac{g}{B} \left( (g q^2 - b(q^2 + b^2)) b_i b_j - \frac{b}{q} u_i u_j - \frac{b^2}{q} (b_i u_j + b_j u_i) \right) \right] J^2. \] (4.34)
We get also from (4.27):
\[ g^{ij} = \left[ a^{ij} - \frac{g}{\nu} (b b^i b^j + b^i y^j + b^j y^i) + \frac{g}{B \nu} (b + gc^2 q) y^i y^j \right] \frac{1}{J^2}. \] (4.35)
With the help of (4.33) we can transform (4.34) to
\[ g_{ij} = \left[ a_{ij} + \frac{g}{q} \left( (b + g q) b_i b_j - \frac{1}{B} b \frac{1}{J^2} \frac{1}{J^2} u_i u_j + \frac{1}{J^2} (b_i y_j + b_j y_i) \right) \right] J^2. \] (4.36)

From the determinant value (4.28) we can explicate the vector
\[ C_i = \frac{\partial \ln \left( \sqrt{\det(g_{mn})} \right)}{\partial y^i}, \] (4.37)
obtaining
\[ C_i = -\frac{1}{2} (1 - c^2) \frac{g}{\nu} e_i + \frac{1}{F^2} N \frac{N y_i}{2B} \left( 2v_i - 2bb_i - g q b_i + gb \frac{1}{q} v_i \right), \]
or
\[ C_i = g \frac{1}{2B X} c_i, \] (4.38)
where
\[ \frac{1}{X} = N - \frac{(1 - c^2) B}{q \nu}. \] (4.39)
Also,
\[ C^i = g \frac{1}{2F^2 X} \left( -b^i - \frac{b^i}{q^2} v^i - \frac{g}{q^2 \nu} v^i (c^2 q^2 - (1 - c^2)b^2) \right). \]
Simplifying yields
\[ C^i = g \frac{1}{2F^2 X} \left( -b^i - \frac{1}{q\nu} (b + gc^2 q) v^i \right), \] (4.40)
or alternatively,
\[ C^i = g \frac{1}{2F^2} \frac{1}{X \nu} \left( B b^i - (b + gc^2 q) y^i \right). \] (4.41)

We obtain the contraction
\[ F^2 C^h C_h = \frac{g^2}{4} \frac{1}{X^2} \left( N + 1 - \frac{1}{X} \right). \] (4.42)

Differentiating (4.36) yields
\[ \frac{\partial g_{ij}}{\partial y^k} - 2 X C_k g_{ij} = - \frac{g}{q} e_k b_i b_j J^2 \]
\[ + \frac{g}{q} \left[ e_k \frac{1}{F^2} y_i y_j - \left( 2 \frac{1}{F^2} y_k - X C_k \right) b_i \frac{1}{F^2} y_i y_j \right] - \frac{g}{q} b_i \frac{1}{F^2} (g_{ik} y_j + g_{jk} y_i) + \frac{g}{q} (g_{ik} b_j + g_{jk} b_i) \]
\[ + \frac{g}{q} \left( 2 b_i \frac{1}{F^2} y_i y_j - (y_j b_i + y_i b_j) \right) 2 X C_k - \frac{g}{q} b_i y_j b_i - \frac{g}{q} b_i y_j b_i. \]

Applying here the equalities
\[ b_i = b y_i \frac{1}{F^4} + \frac{q^2}{B} e_i \] (4.43)
(see (4.25)) and
\[ b_i b_j = b^2 y_i y_j \frac{1}{F^4} + b q^2 \frac{1}{B F^2} (e_i y_j + e_j y_i) + q^4 \left( \frac{1}{B^2} e_i e_j, \right) \] (4.44)
we obtain
\[ \frac{\partial g_{ij}}{\partial y^k} - 2 X C_k g_{ij} = - \frac{g}{q} e_k b_i b_j J^2 + \frac{g}{q} e_i \frac{1}{F^2} y_i y_j \]
\[ - \frac{g}{q} b_i \frac{1}{F^2} (h_{ik} y_j + h_{jk} y_i) + \frac{g}{q} \left( g_{ik} b_j + g_{jk} b_i \right) + 2 \frac{g}{q} b_i \frac{1}{F^2} X C_k y_i y_j - 2 \frac{g}{q} (y_j b_i + y_i b_j) X C_k \]
\[ - \frac{g}{q b} e_k \left( y_j b_i + y_i b_j \right) \]
\[ - \frac{g}{q b} y_j \left( b y_k \frac{1}{F^2} + \frac{q^2}{B} e_k \right) b_i - \frac{g}{q b} y_i \left( b y_k \frac{1}{F^2} + \frac{q^2}{B} e_k \right) b_j \]
\[ = \frac{g}{q} e_k \frac{1}{F^2} y_i y_j + 2 \frac{g}{q} b \frac{1}{F^2} X C_k y_i y_j - \frac{g}{q} e_k \left( b^2 y_i y_j \frac{1}{F^4} + b q^2 \frac{1}{B F^2} (e_i y_j + e_j y_i) + q^4 \frac{1}{B^2} e_i e_j \right) J^2 \]
\[
-g_{ij} \frac{1}{F^2} (h_{ik} y_j + h_{jk} y_i) + \frac{g}{q} \left( h_{ik} b_j + h_{jk} b_i \right) + \frac{g}{q} \frac{1}{B} \epsilon_k \left( y_j b_i + y_i b_j \right)
\]

\[
= \frac{g}{q} \epsilon_k \frac{1}{F^2} y_i y_j + 2 \frac{g}{q} \frac{1}{F^2} X C_k y_i y_j - \frac{g}{q} \epsilon_k \left( b^2 y_i y_j \frac{1}{F^4} + b q \frac{1}{B} \frac{1}{F^2} (\epsilon_i y_j + \epsilon_j y_i) + q^4 \frac{1}{B^2} \epsilon_i \epsilon_j \right) J^2
\]

\[
- \frac{g}{q} \frac{1}{F^2} (h_{ik} y_j + h_{jk} y_i) + \frac{g}{q} \left( h_{ik} (b y_j \frac{1}{F^2} + q^2 \frac{1}{B} \epsilon_i) + h_{jk} (b y_i \frac{1}{F^2} + q^2 \frac{1}{B} \epsilon_i) \right)
\]

\[
+ \frac{g}{q} \frac{1}{B} \epsilon_k \left( y_j (b y_i \frac{1}{F^2} + q^2 \frac{1}{B} \epsilon_i) + y_i (b y_j \frac{1}{F^2} + q^2 \frac{1}{B} \epsilon_i) \right),
\]

so that

\[
\frac{\partial g_{ij}}{\partial y^k} - 2 X C_k g_{ij} = -g q^3 \frac{F^2}{B^3} \epsilon_k \epsilon_i \epsilon_j - \frac{g q}{B} \frac{1}{F^2} \epsilon_k y_i y_j + g q \frac{1}{B} \left( h_{ik} \epsilon_j + h_{jk} \epsilon_i \right). \tag{4.45}
\]

This shows that the associated Cartan tensor is representable in the form (2.48). We have applied the equalities

\[
\frac{\partial B}{\partial y^k} = \frac{2 B}{F^2} y_k - 2 B X C_k, \quad \frac{\partial (J^2)}{\partial y^k} = \frac{2}{B} X F^2 C_k. \tag{4.46}
\]

### 5. Fixing the tangent space

Let us introduce the orthonormal frame \( h^p_i(x) \) of the input pseudo-Riemannian metric tensor \( a_{ij}(x) \):

\[
a_{ij} = e_{pq} h^p_i h^q_j, \tag{5.1}
\]

where \( \{e_{pq}\} \) is the pseudo-Euclidean diagonal:

\[
e_{pq} = \text{diagonal}(+ - ... -); \tag{5.2}
\]

the indices \( p, q, ... \) will be specified on the range 0, 1, ..., \( N - 1 \). Denote by \( h^i_p \) the reciprocal frame, so that

\[
h^i_p h^p_i = \delta^i_j. \tag{5.3}
\]

At any fixed point \( x \), we can represent the tangent vectors \( y \) by their frame-components:

\[
R^p = h^p_i y^i, \tag{5.4}
\]

and consider the respective metric tensor components

\[
g_{pq} = h^p_i h^q_j g_{ij} \tag{5.5}
\]

of the Finslerian metric tensor \( g_{ij} \). We obtain the decomposition

\[
R^p = \{R^a, R^{N-1}\}, \tag{5.6}
\]
often denoting
\[ R^{N-1} = z. \quad (5.7) \]

The indices \( a, b, \ldots \) will be specified on the range \( 1, \ldots, N-1 \).

It is convenient to specify the frame such that the \((N-1)\)-th component \( h_i^{N-1}(x) \) is collinear to the input vector field \( b_i(x) \). Under these conditions the input 1-form \( b \) reads
\[ b = cz \quad (5.8) \]

and we have
\[ b^p = \{0, 0, \ldots, -c\}, \quad b_p = \{0, 0, \ldots, c\}, \quad (5.9) \]

together with
\[ q^2 = \epsilon \left[ e_{ad} R^a R^d - (1 - c^2)(R^{N-1})^2 \right]. \quad (5.10) \]

Notice the inequality
\[ \epsilon \left[ e_{ad} R^a R^d - (1 - c^2)(R^{N-1})^2 \right] > 0, \quad (5.11) \]

so that \( q \) is a nowhere vanishing variable in both the time-like and space-like sectors.

From (2.37) it follows that the covariant components \( R_p = h_p y_i \) are equal to
\[ R_a = e_{ab} R^b J^2, \quad R_{N-1} = -\frac{1}{c} (b + gc^2 q) J^2. \quad (5.12) \]

In the four-dimensional relativistic space-time,
\[ N = 4, \quad R^p = \{R^0, R^1, R^2, R^3\}, \quad (5.13) \]

it is convenient to relabel the coordinates \( R^p \) as follows:
\[ R^0 = t, \quad R^1 = x, \quad R^2 = y, \quad R^3 = z. \quad (5.14) \]

We get
\[ q = \sqrt{\epsilon \left( t^2 - x^2 - y^2 - (1 - c^2) z^2 \right)} \quad (5.15) \]

and
\[ B = \epsilon q^2 - c^2 z^2 - gcq z \equiv t^2 - x^2 - y^2 - z^2 - gcq. \quad (5.16) \]

In terms of such coordinates, the metric tensor components \( g_{pq} \) are obtained from the tensorial components \( g_{ij} \) given in (2.38). The result reads
\[ g_{00} = \left(1 + \epsilon \frac{gc}{Bq} z t^2 \right) J^2, \quad g_{01} = -\epsilon \frac{gc}{Bq} z x t J^2, \quad g_{02} = -\epsilon \frac{gc}{Bq} z y t J^2, \quad (5.17) \]
\[ g_{03} = -\frac{gc}{Bq} E t J^2, \quad g_{11} = \left(-1 + \epsilon \frac{gc}{Bq} z x^2 \right) J^2, \quad g_{22} = \left(-1 + \epsilon \frac{gc}{Bq} z y^2 \right) J^2, \quad (5.18) \]
\[ g_{12} = \epsilon \frac{gc}{Bq} z x y J^2, \quad g_{13} = \frac{gc}{Bq} E x J^2, \quad g_{23} = \frac{gc}{Bq} E y J^2, \quad (5.19) \]
\[ g_{33} = \left[-1 + \frac{gc}{Bq} \left((gq^3 - b(q^2 - \epsilon b^2)) c + \epsilon z^3 + 2(q^2 - \epsilon b^2) z \right) \right] J^2. \quad (5.20) \]
Here, $E = \epsilon z^2 + (q^2 - eb^2)$; $\epsilon = 1$ in the future-time-like sector $y \in T^+_g$ and $\epsilon = -1$ in the space-like sector $y \in R^+_g$. From (2.39) we can obtain the contravariant components

$$g^{00} = \left(1 - \epsilon \frac{g}{B^2} \beta t^2\right) \frac{1}{J^2}, \quad g^{01} = -\epsilon \frac{g}{B^2} \beta x t \frac{1}{J^2}, \quad g^{02} = -\epsilon \frac{g}{B^2} \beta y t \frac{1}{J^2}, \quad (5.21)$$

$$g^{03} = -\epsilon \frac{g}{B^2} (Bc + \beta z) t \frac{1}{J^2}, \quad g^{11} = \left(1 - \epsilon \frac{g}{B^2} \beta t^2\right) \frac{1}{J^2}, \quad g^{22} = \left(1 - \epsilon \frac{g}{B^2} \beta t^2\right) \frac{1}{J^2}, \quad (5.22)$$

$$g^{12} = -\epsilon \frac{g}{B^2} \beta y x \frac{1}{J^2}, \quad g^{13} = -\epsilon \frac{g}{B^2} (Bc + \beta z) x \frac{1}{J^2}, \quad g^{23} = -\epsilon \frac{g}{B^2} (Bc + \beta z) y \frac{1}{J^2}, \quad (5.23)$$

$$g^{33} = \left[1 + \epsilon \frac{g}{B^2} (c^2 - 2) cz - \epsilon \frac{g}{B^2} \beta z^2\right] \frac{1}{J^2}, \quad (5.24)$$

where $\beta = b + gc^2 q$.

At an arbitrary dimension $N \geq 2$, from (2.38) we obtain the respective components

$$g_{ab} = \left[e_{ab} + \epsilon g \frac{e_{ad} R^d e_{be} R^e b}{Bq}\right] J^2, \quad g_{N-1,a} = \frac{g}{Bq} \left(-ebz - (q^2 - eb^2)c\right) e_{ab} R^b J^2, \quad (5.25)$$

$$g_{N-1,N-1} = \left[1 + \frac{g}{Bq} \left((gq^3 - b(q^2 - eb^2))c^2 + ebz^2 + 2(q^2 - eb^2)b\right)\right] J^2. \quad (5.26)$$

We may alternatively obtain them through the rule $g_{pq} = \partial R_q / \partial R^p$. From (2.39) it follows that

$$g^{ab} = \left[e^{ab} - \epsilon \frac{g}{B^2} (b + gc^2 q) R^a R^b\right] \frac{1}{J^2}, \quad g^{N-1,a} = -\epsilon \frac{g}{B^2} \left[cB + (b + gc^2 q)z\right] R^a \frac{1}{J^2}, \quad (5.27)$$

$$g^{N-1,N-1} = \left[1 + \epsilon \frac{g}{B^2} \left((bc^2 - 2b)B - (b + gc^2 q)z\right)\right] \frac{1}{J^2}. \quad (5.28)$$

6. Hamiltonian function

It proves possible to obtain explicitly the Hamiltonian function $H$ of the $AR$-space. To this end we introduce the co-form $\hat{b} := y_i b^i$. From (2.29) (the formula (2.37) can also be used) we obtain

$$\hat{b} = (b + qg c^2) J^2. \quad (6.1)$$

Using the tensor $r^{ij} = a^{ij} + b^i b^j$ (see (2.5)), it is natural to introduce the co-counterpart

$$\hat{\gamma} = y_i y_j r^{ij} \quad (6.2)$$
to the scalar (2.6). Inserting here (2.37) yields
\[ \hat{\gamma} = \left[ \gamma - 2(1 - c^2)gbq - c^2(1 - c^2)g^2q^2 \right] J^4 \]  \hspace{1cm} (6.3)

(the formulas (2.25)-(2.26) have been used).

It is appropriate to use the co-counterpart
\[ \hat{q} = \sqrt{\vert \hat{\gamma} \vert} \]  \hspace{1cm} (6.4)
to (2.7) and rewrite (6.3) as follows:
\[ \hat{q}^2 = \left[ q^2 - 2(1 - c^2)\epsilon gbq - c^2(1 - c^2)\epsilon g^2 q^2 \right] J^4. \]  \hspace{1cm} (6.5)

On resolving the equation set (6.1)-(6.5) to get the functions \( \hat{b} = b(\hat{b}, \hat{q}) \) and \( q = q(\hat{b}, \hat{q}) \), we can insert the functions in the \( F^2 \) to obtain the squared Hamiltonian function \( H^2 \) through the equality
\[ H^2 = F^2 \]  \hspace{1cm} (6.6)

(we apply the method presented in Section 7.1 of [1]). This leads to the Hamiltonian function representation
\[ H^2(x, \hat{y}) = \hat{\Phi} \left( g(x), b^i(x), a^{ij}(x), \hat{y} \right), \]  \hspace{1cm} (6.7)

which is the co-counterpart to (1.1).

The knowledge of \( H^2 \) makes it possible to formulate the \( \mathcal{A}\mathcal{R} \)-Hamilton-Jacobi equation
\[ H^2 \left( x, \frac{\partial S}{\partial x^i} \right) = m^2, \]  \hspace{1cm} (6.8)

where \( S = S(x) \) is the characteristic function and \( m \) is the rest mass of the particle.

When \( c = 1 \), from (6.1) and (6.5) it ensues that
\[ \hat{b} = (b + gq)J^2, \quad \hat{q} = qJ^2, \]  \hspace{1cm} (6.9)

which entails
\[ \frac{\hat{q}}{b} = \frac{q}{b + gq}, \quad \frac{q}{b} = \frac{\hat{q}}{b - gb}. \]  \hspace{1cm} (6.10)

Introducing the quadratic form
\[ \hat{B} = \epsilon \hat{q}^2 + gb\hat{q} - \hat{b}^2, \]  \hspace{1cm} (6.11)

we get the equality
\[ \hat{B} = BJ^4, \]  \hspace{1cm} (6.12)

where \( B \) is the initial quadratic form (2.9). When the co-vectors \( \hat{y} \) are relatable to the vectors \( y \) of the future-time-like sector, so that \( \epsilon = 1 \), from (6.9) it follows that the function \( H^2(x, \hat{y}) \) is obtained from (3.11)-(3.12) to read
\[ H^2(x, \hat{y}) = \hat{B}(x, \hat{y})J^2(x, \hat{y}) \equiv (\hat{b} + g\hat{q})^{-G_+} (-g_{-\hat{q}} - \hat{b})^{G_+}, \quad \epsilon = 1, \]  \hspace{1cm} (6.13)

and
\[ \hat{J}(x, \hat{y}) = \left( \frac{\hat{b} + g\hat{q}}{-g_{-\hat{q}} - \hat{b}} \right)^{G/4} \equiv \frac{1}{J(x, y)}, \quad \epsilon = 1, \]  \hspace{1cm} (6.14)
with \( G = g/h, \ G_+ = g_+/h, \) and \( G_- = g_-/h; \) the quantities \( g_+ \) and \( g_- \) have been indicated below (4.5). The quadratic form

\[
\hat{B} = (\hat{b} + g_+ \hat{q})(-g_- \hat{q} - \hat{b}) , \quad \epsilon = 1,
\]

substitutes now the initial \( B. \)

In the space-like sector, from (4.10)-(4.13) and (6.9)-(6.12) we obtain

\[
H^2(x, \hat{y}) = \hat{B}(x, \hat{y}) \hat{J}^2(x, \hat{y}), \quad \epsilon = -1,
\]

and

\[
\hat{J}(x, \hat{y}) = e^{\frac{1}{2} G(x) f(x, \hat{y})} \equiv \frac{1}{J(x, y)}, \quad \epsilon = -1,
\]

with

\[
\hat{f} = \arctan \frac{G}{2} + \arctan \frac{L}{hb}, \quad \text{if} \quad \hat{b} \geq 0,
\]

and

\[
\hat{f} = \pi + \arctan \frac{G}{2} + \arctan \frac{\hat{L}}{hb}, \quad \text{if} \quad \hat{b} \leq 0,
\]

where

\[
\hat{L} = \hat{q} - \frac{g}{2} \hat{b}.
\]

We have taken into account the identities

\[
\arctan \frac{\hat{L}}{hb} - \arctan \frac{L}{hb} = - \arctan \frac{gh}{1 - \frac{g^2}{2}},
\]

and

\[
\arctan \frac{G}{2} - \arctan \frac{gh}{1 - \frac{g^2}{2}} = - \arctan \frac{G}{2}.
\]

**Appendix A: Applying the \( \mathcal{UAR} \)-space coordinates**

Below the coordinates \( z^p \) defined in (2.79) are used.

We start with examining the future-time-like sector. We have \( F > 0 \) and can take \( z^0 = F. \) We introduce the representations

\[
R^0 = F \cosh \eta \mathrm{Ch} \chi, \quad (A.1)
\]

\[
R^1 = F \sinh \eta \mathrm{Ch} \chi \cos \phi, \quad R^2 = F \sinh \eta \mathrm{Ch} \chi \sin \phi, \quad R^3 = F \mathrm{Sh} \chi, \quad (A.2)
\]

obtaining

\[
q = F \mathrm{Ch} \chi, \quad (A.3)
\]

with

\[
\mathrm{Ch} \chi = \frac{1}{Jh} \cosh f, \quad \mathrm{Sh} \chi = \frac{1}{J} \left( \sinh f - \frac{G}{2} \cosh f \right). \quad (A.4)
\]
We also introduce the function
\[ \text{Sh}^* \chi = \frac{1}{J} \left( \sinh f + \frac{G}{2} \cosh f \right). \] (A.5)

From this it follows
\[ \text{Sh}' \chi = \text{Ch} \chi, \quad \text{Ch}' \chi = \text{Sh}^* \chi, \] (A.6)
where the prime ' means differentiation with respect to the angle \( \chi \). The angle is taken as follows:
\[ \chi = \frac{1}{\hbar} f, \quad J = e^{-\frac{1}{2}g \chi} \] (A.7)
(see the formulas below (3.12)).

The identities
\[ \text{Sh}^* \chi = \text{Sh} \chi + g \text{Ch} \chi \] (A.8)
and
\[ \text{Sh}^* \chi \text{Sh}^* \chi - g \text{Sh}^* \chi \text{Ch} \chi - \text{Ch}^2 \chi = - \frac{1}{J^2} \] (A.9)
are valid.

Evaluating the partial derivatives yields
\[ \frac{\partial R^0}{\partial z^0} = \frac{1}{F} R^p, \] (A.10)
\[ \frac{\partial R^0}{\partial z^1} = F \sinh \eta \text{Ch} \chi, \quad \frac{\partial R^0}{\partial z^2} = 0, \quad \frac{\partial R^0}{\partial z^3} = F \cosh \eta \text{Sh}^* \chi, \] (A.11)
\[ \frac{\partial R^1}{\partial z^1} = F \cosh \eta \text{Ch} \chi \cos \phi, \quad \frac{\partial R^1}{\partial z^2} = -F \sinh \eta \text{Ch} \chi \sin \phi, \] (A.12)
\[ \frac{\partial R^1}{\partial z^3} = F \sinh \eta \text{Sh}^* \chi \cos \phi, \quad \frac{\partial R^2}{\partial z^2} = F \sinh \eta \text{Sh}^* \chi \sin \phi, \] (A.13)
\[ \frac{\partial R^2}{\partial z^1} = F \cosh \eta \text{Ch} \chi \sin \phi, \quad \frac{\partial R^2}{\partial z^2} = F \sinh \eta \text{Ch} \chi \cos \phi, \] (A.14)
\[ \frac{\partial R^3}{\partial z^1} = 0, \quad \frac{\partial R^3}{\partial z^2} = 0, \quad \frac{\partial R^3}{\partial z^3} = F \text{Ch} \chi. \] (A.15)

The derivatives fulfill the identities
\[ R^1 \frac{\partial R^1}{\partial z^2} + R^2 \frac{\partial R^2}{\partial z^2} \equiv 0, \quad \frac{\partial R^1}{\partial z^2} \frac{\partial R^1}{\partial z^3} + \frac{\partial R^2}{\partial z^2} \frac{\partial R^2}{\partial z^3} \equiv 0. \] (A.16)

Let us apply the transformation (A.1)-(A.3) to the \( UAR \)-angle (2.71). This yields
\[ \alpha_{(3)}(y_1, y_2) = \frac{1}{\hbar} \arccosh \tau_{12}, \] (A.17)
where
\[ \tau_{12} = \cosh f_1 \cosh f_2 Z_{12} - \sinh f_1 \sinh f_2 \] (A.18)
with
\[ Z_{12} = \cosh \eta_1 \cosh \eta_2 - \sinh \eta_1 \sinh \eta_2 \cos(\phi_2 - \phi_1). \] (A.19)

We can write also
\[ \tau_{12} = \cosh(f_2 - f_1) + \cosh f_1 \cosh f_2 \left( \cosh(\eta_2 - \eta_1) - 1 + Y \sinh \eta_1 \sinh \eta_2 \right), \] (A.20)

where \( Y = 1 - \cos(\phi_2 - \phi_1) \).

Let us use the Finslerian metric tensor components \( g_{pq}(g; R) \) indicated in (5.17)-(5.20) to obtain the tensor \( A_{rs}(g; z) \) introduced by the transformation (2.82). We get the diagonal tensor:
\[ A_{00} = 1, \quad A_{33} = -(z^0)^2, \] (A.21)

\[ A_{11} = -(z^0)^2 \frac{1}{h^2} \cosh^2 f, \quad A_{22} = -(z^0)^2 \sinh^2 \eta \frac{1}{h^2} \cosh^2 f, \] (A.22)

so that the respective squared linear element is
\[ (ds)^2 = (dz^0)^2 - (z^0)^2 \left[ (d\chi)^2 + \frac{1}{h^2} \cosh^2 f \left( \sinh^2 \eta (d\phi)^2 + (d\eta)^2 \right) \right]. \] (A.23)

Let us verify the involved coefficients. We have
\[
A_{11} = \left( \frac{\partial R^0}{\partial z^1} g_{00} + 2 \frac{\partial R^1}{\partial z^1} g_{10} + 2 \frac{\partial R^2}{\partial z^1} g_{20} \right) \frac{\partial R^0}{\partial z^1} + \frac{\partial R^1}{\partial z^1} \frac{\partial R^1}{\partial z^1} g_{11} + 2 \frac{\partial R^1}{\partial z^1} \frac{\partial R^2}{\partial z^1} g_{12} + \frac{\partial R^2}{\partial z^1} \frac{\partial R^2}{\partial z^1} g_{22},
\]

so that
\[
\frac{1}{F^2 F^2} A_{11} = \sinh^2 \eta \text{Ch}^2 \chi \left( 1 + \frac{g}{Bq} z t^2 \right) + \cosh^2 \eta \text{Ch}^2 \chi \cos^2 \phi \left( -1 + \frac{g}{Bq} z x^2 \right) - 2 \frac{g}{Bq} z x t \sinh \eta \text{Ch} \chi \cosh \eta \text{Ch} \chi \cos \phi - 2 \frac{g}{Bq} z y t \sinh \eta \text{Ch} \chi \cosh \eta \text{Ch} \chi \sin \phi + 2 \cosh^2 \eta \text{Ch}^2 \chi \cos \phi \sin \frac{g}{Bq} z x y + \cosh^2 \eta \text{Ch}^2 \chi \sin^2 \phi \left( -1 + \frac{g}{Bq} z y^2 \right)
\]
\[
= \sinh^2 \eta \text{Ch}^2 \chi \left( 1 + \frac{g}{Bq} z t^2 \right) - 2 \frac{g}{Bq} z t F \sinh^2 \eta \text{Ch}^2 \chi \cosh \eta \text{Ch} \chi + \cosh^2 \eta \text{Ch}^2 \chi \left( -1 + \frac{g}{Bq} z F^2 \sinh^2 \eta \text{Ch}^2 \chi \right).
\]

By reducing we arrive at
\[
\frac{1}{F^2 F^2} A_{11} = - \text{Ch}^2 \chi.
\]

Therefore,
\[ A_{11} = -F^2 \frac{1}{h^2} \cosh^2 f. \] (A.24)
The similar chain:

\[ A_{22} = \frac{\partial R^1 \partial R^1}{\partial z^2} g_{11} + 2 \frac{\partial R^1 \partial R^2}{\partial z^2} g_{12} + \frac{\partial R^2 \partial R^2}{\partial z^2} g_{22} \]

and

\[ \frac{1}{J^2 F^2} A_{22} = \sinh^2 \eta \, \text{Ch}^2 \chi \, \sin^2 \phi \left( -1 + \frac{g}{Bq} z x^2 \right) \]

\[-2 \sinh^2 \eta \, \text{Ch}^2 \chi \, \cos \phi \sin \phi \frac{g}{Bq} z x y + \sinh^2 \eta \, \text{Ch}^2 \chi \, \cos^2 \phi \left( -1 + \frac{g}{Bq} z y^2 \right)\]

leads to

\[ \frac{1}{J^2 F^2} A_{22} = - \sinh^2 \eta \, \text{Ch}^2 \chi, \]

or

\[ A_{22} = - F^2 \sinh^2 \eta \frac{1}{h^2} \cosh^2 f. \quad (A.25) \]

Next, we consider the component

\[ A_{33} = \left( \frac{\partial R^0}{\partial z^3} g_{00} + 2 \frac{\partial R^1}{\partial z^3} g_{10} + 2 \frac{\partial R^2}{\partial z^3} g_{20} + 2 \frac{\partial R^3}{\partial z^3} g_{30} \right) \frac{\partial R^0}{\partial z^3} + \frac{\partial R^1}{\partial z^3} \frac{\partial R^1}{\partial z^3} g_{11} + 2 \frac{\partial R^1}{\partial z^3} \frac{\partial R^2}{\partial z^3} g_{12} + \frac{\partial R^1}{\partial z^3} \frac{\partial R^3}{\partial z^3} g_{13} \]

\[ + 2 \frac{\partial R^2}{\partial z^3} \frac{\partial R^2}{\partial z^3} g_{22} + 2 \frac{\partial R^2}{\partial z^3} \frac{\partial R^3}{\partial z^3} g_{23} + \frac{\partial R^3}{\partial z^3} \frac{\partial R^3}{\partial z^3} g_{33}, \]

getting

\[ \frac{1}{J^2 F^2} A_{33} = \cosh^2 \eta \, \text{Sh}^* \chi \, \text{Sh}^* \chi \left( 1 + \frac{g}{Bq} z t^2 \right) \]

\[-2 \frac{g}{Bq} z x t \cosh \eta \, \text{Sh}^* \chi \, \text{Sh}^* \chi \sinh \eta \cos \phi - 2 \frac{g}{Bq} z y t \cosh \eta \, \text{Sh}^* \chi \, \text{Sh}^* \chi \sinh \eta \sin \phi \]

\[-2 \cosh \eta \, \text{Sh}^* \chi \, \text{Ch} \chi \frac{g}{Bq} \left( z^2 + (q^2 - b^2) \right) t + \sinh^2 \eta \, \text{Sh}^* \chi \, \text{Sh}^* \chi \cos^2 \phi \left( -1 + \frac{g}{Bq} z x^2 \right) \]

\[+ 2 \sinh^2 \eta \, \text{Sh}^* \chi \, \text{Sh}^* \chi \cos \phi \sin \phi \frac{g z x y}{Bq} + 2 \sinh \eta \, \text{Ch} \chi \, \text{Sh}^* \chi \cos \phi \frac{g x}{Bq} \left( z^2 + (q^2 - b^2) \right) \]

\[+ \sinh^2 \eta \, \text{Sh}^* \chi \, \text{Sh}^* \chi \sin^2 \phi \left( -1 + \frac{g z y^2}{Bq} \right) + 2 \sinh \eta \, \text{Sh}^* \chi \, \text{Ch} \chi \sin \phi \frac{g y}{Bq} \left( z^2 + (q^2 - b^2) \right) \]

\[+ \text{Ch}^2 \chi \left[ -1 + \frac{g}{Bq} \left( (g q^3 - b(q^2 - b^2)) + z^3 + 2(q^2 - b^2) z \right) \right]. \]
Due simplifying yields

\[
\frac{1}{J^2 F^2} A_{33} = \cosh^2 \eta \cdot \text{Sh}^* \chi \cdot \text{Sh}^* \chi \left( 1 + \frac{g}{Bq} z t^2 \right)
\]

\[-2 \frac{g}{Bq} F z t \cosh \eta \sinh^2 \eta \cdot \text{Sh}^* \chi \cdot \text{Sh}^* \chi \cdot \text{Ch} \cdot \chi - 2 \cosh \eta \cdot \text{Sh}^* \chi \cdot \text{Ch} \cdot \chi \cdot \frac{g}{Bq} q^2 t \]

\[-\sinh^2 \eta \cdot \text{Sh}^* \chi \cdot \text{Sh}^* \chi + \sinh^2 \eta \cdot \text{Sh}^* \chi \cdot \text{Sh}^* \chi \cdot \frac{g}{Bq} z F^2 \sinh^2 \text{Ch} \cdot \text{Ch} \cdot \chi \]

\[+ 2 \sinh \eta \cdot \text{Ch} \cdot \chi \cdot \text{Sh}^* \chi \cdot \frac{g}{Bq} q^2 F \sinh \eta \cdot \text{Ch} \cdot \chi + \text{Ch}^2 \chi \left[ -1 + \frac{g}{Bq} q^2 (b + gq) \right] \]

\[= \text{Sh}^* \chi \cdot \text{Sh}^* \chi \left( 1 + \frac{g}{Bq} F^2 \cdot \text{Ch} \cdot \text{Ch} \cdot \chi \right) - 2 \text{Sh}^* \chi \cdot \text{Ch} \cdot \chi \cdot \frac{g q}{B} F \cdot \text{Ch} \cdot \chi - \text{Ch}^2 \chi \left[ 1 - \frac{g q}{B} (b + gq) \right].\]

In this way we obtain simply

\[
\frac{1}{J^2 F^2} A_{33} = \text{Sh}^* \chi \cdot \text{Sh}^* \chi + \text{Sh}^* \chi \cdot \text{Sh}^* \chi \cdot \frac{g b q}{B} - 2 \text{Sh}^* \chi \cdot \text{Ch} \cdot \chi \cdot \frac{g q}{B} F - \text{Ch}^2 \chi + \text{Ch} \cdot \chi \cdot \frac{g q}{B} \frac{1}{F} (b + gq)
\]

\[= \text{Sh}^* \chi \cdot \text{Sh}^* \chi - g \cdot \text{Sh}^* \chi \cdot \text{Ch} \cdot \chi - \text{Ch}^2 \chi = -\frac{1}{J^2} \]

(the identity (A.9) has been used), so that

\[A_{33} = -F^2. \quad \text{(A.26)}\]

Thus, the representations (A.21) and (A.22) are correct.

By making the substitution

\[z^0 = e^\sigma, \quad \text{(A.27)}\]

from (A.23) we get

\[(ds)^2 = e^{2\sigma} \left[ (d\sigma)^2 - (d\chi)^2 - \frac{1}{h^2} \cosh^2 f (\sinh^2 \eta (d\phi)^2 + (d\eta)^2) \right]. \quad \text{(A.28)}\]

Using here new coordinates

\[\rho = e^{h\sigma} \cosh(h\chi), \quad \tau = e^{h\sigma} \sinh(h\chi) \quad \text{(A.29)}\]

just shows the property that the space under study is \textit{conformally flat}:

\[(ds)^2 = \mathcal{K} \left( (ds)^2 \right)_{\text{pseudo-Euclidean}}, \quad \text{(A.30)}\]

where

\[\left( (ds)^2 \right)_{\text{pseudo-Euclidean}} = (d\rho)^2 - (d\tau)^2 - \rho^2 \left( \sinh^2 \eta (d\phi)^2 + (d\eta)^2 \right) \quad \text{(A.31)}\]
If we apply here (A.27) and remind that $z^0 = F$, we observe that the conformal multiplier $\mathcal{K}$ can be expressed through the metric function, namely we get

$$\mathcal{K} = \frac{1}{h^2} (F^2)^{1-h}.$$ (A.33)

Turning to the *space-like sector*, so that $\epsilon = -1$, we introduce the function

$$K = \sqrt{|F^2|} > 0, \quad \text{so that} \quad z^0 = K,$$ (A.34)

and set forth the representations

$$R^0 = K \sinh \eta \sin \chi,$$ (A.35)

$$R^1 = K \cosh \eta \sin \chi \cos \phi, \quad R^2 = K \cosh \eta \sin \chi \sin \phi, \quad R^3 = K \cos \chi,$$ (A.36)

which entails

$$q = K \sin \chi,$$ (A.37)

with

$$\sin \chi = \frac{1}{Jh} \sin f, \quad \cos \chi = \frac{1}{J} \left( \cos f - \frac{G}{2} \sin f \right).$$ (A.38)

We need also the function

$$\cos^* \chi = \frac{1}{J} \left( \cos f + \frac{G}{2} \sin f \right).$$ (A.39)

The derivatives

$$\sin^' \chi = \cos^* \chi, \quad \cos^' \chi = -\sin \chi$$ (A.40)

can readily be verified. Again, we take the angle $\chi$ according to (A.7) (notice (4.11)).

There arise the identities

$$\cos^* \chi = \cos \chi + g \sin \chi, \quad \sin^2 \chi + g \sin \chi \cos \chi + \cos^2 \chi = \frac{1}{J^2},$$ (A.41)

and also

$$-\sin^2 \chi + g \sin \chi \cos^* \chi - \cos^* \chi \cos^* \chi = -\frac{1}{J^2}.$$ (A.42)

From (A.35)-(A.36) we calculate the partial derivatives, obtaining

$$\frac{\partial R^0}{\partial z^0} = \frac{1}{K} R^0, \quad \frac{\partial R^3}{\partial z^1} = \frac{\partial R^3}{\partial z^2} = 0, \quad \frac{\partial R^3}{\partial z^3} = -K \sin \chi,$$ (A.43)

$$\frac{\partial R^0}{\partial z^1} = K \cosh \eta \sin \chi, \quad \frac{\partial R^0}{\partial z^2} = 0, \quad \frac{\partial R^0}{\partial z^3} = K \sinh \eta \cos^* \chi,$$ (A.44)

$$\frac{\partial R^1}{\partial z^1} = K \sinh \eta \sin \chi \cos \phi, \quad \frac{\partial R^1}{\partial z^2} = -K \cosh \eta \sin \chi \sin \phi,$$ (A.45)
\[
\frac{\partial R^1}{\partial z^3} = K \cosh \eta \cos^* \chi \cos \phi, \quad \frac{\partial R^2}{\partial z^3} = K \cosh \eta \cos^* \chi \sin \phi, \quad (A.46)
\]

\[
\frac{\partial R^2}{\partial z^1} = K \sinh \eta \sin \chi \sin \phi, \quad \frac{\partial R^2}{\partial z^2} = K \cosh \eta \sin \chi \cos \phi. \quad (A.47)
\]

Using the substitution (A.35)-(A.37) in the \textit{UAR}-angle (2.71) yields

\[
\alpha \{x \} (y_1, y_2) = \frac{1}{h} \arccosh \tau_{12}, \quad (A.48)
\]

where

\[
\tau_{12} = \sin f_1 \sin f_2 Z_{12} + \cos f_1 \cos f_2 \quad (A.49)
\]

and

\[
Z_{12} = \cosh \eta_1 \cosh \eta_2 \cos (\phi_2 - \phi_1) - \sinh \eta_1 \sinh \eta_2. \quad (A.50)
\]

Whence we have

\[
\tau_{12} = \cos (f_2 - f_1) + \sin f_1 \sin f_2 \left( \cosh (\eta_2 - \eta_1) - 1 - P \cosh \eta_1 \cosh \eta_2 \right), \quad (A.51)
\]

where \( P = 1 - \cos (\phi_2 - \phi_1) \).

Again, we evaluate the transform \( A_{rs}(g; z) \) according to (2.82), now taken the components (5.17)-(5.20) with the value \( \epsilon = -1 \). We get the diagonal tensor: \( A_{01} = A_{02} = A_{03} = A_{12} = A_{13} = A_{23} = 0 \) and

\[
A_{00} = -1, \quad A_{33} = -(z^0)^2, \quad (A.52)
\]

\[
A_{11} = (z^0)^2 \frac{1}{h^2} \sin^2 f, \quad A_{22} = -(z^0)^2 \cosh^2 \eta \frac{1}{h^2} \sin^2 f, \quad (A.53)
\]

so that the object \( ds^2 = -A_{rs}dz^rdz^s \) takes on the form

\[
(d\eta)^2 - (d\chi)^2 \left[ \frac{1}{h^2} \sin^2 f \left( (d\eta)^2 - \cosh^2 \eta (d\phi)^2 \right) - (d\chi)^2 \right] + (dz^0)^2. \quad (A.54)
\]

Let us check the validity of the key components. We get

\[
\frac{1}{f^2 K^2} A_{11} = \cosh^2 \eta \sin^2 \chi \left( 1 - \frac{g}{Bq} z^t t^2 \right) + \sinh^2 \eta \sin^2 \chi \cos^2 \phi \left( -1 - \frac{g}{Bq} z^x x^2 \right)
\]

\[
+ 2 \frac{g}{Bq} z^t \sinh \eta \sin^2 \chi \cosh \eta \cos \phi + 2 \frac{g}{Bq} z^t y t \ sinh \eta \sin^2 \chi \cosh \eta \sin \phi
\]

\[
- 2 \sinh^2 \eta \sin^2 \chi \cos \phi \sin \phi \frac{g}{Bq} z^x y + \sinh^2 \eta \sin^2 \chi \sin^2 \phi \left( -1 - \frac{g}{Bq} y^2 \right)
\]

\[
= \cosh^2 \eta \sin^2 \chi \left( 1 - \frac{g}{Bq} z^t t^2 \right) + 2 \frac{g}{Bq} z^t K \cosh^2 \eta \sin^2 \chi \sinh \eta \sin \chi
\]
\[ + \sinh^2 \eta \sin^2 \chi \left( -1 - \frac{g}{Bq} z K^2 \cosh^2 \eta \sin^2 \chi \right), \]

so that
\[ \frac{1}{J^2 K^2} A_{11} = \sin^2 \chi, \]
or
\[ A_{11} = K^2 \frac{1}{h^2} \sin^2 f. \]

After that, we consider the component \( A_{22} \), getting
\[ \frac{1}{J^2 K^2} A_{22} = \cosh^2 \eta \sin^2 \chi \sin^2 \phi \left( -1 - \frac{g}{Bq} z x^2 \right) \]

\[ + 2 \cosh^2 \eta \sin^2 \chi \cos \phi \sin \phi \frac{g}{Bq} z xy + \cosh^2 \eta \sin^2 \chi \cos^2 \phi \left( -1 - \frac{g}{Bq} z y^2 \right). \]

On reducing, we are left with
\[ \frac{1}{J^2 K^2} A_{22} = - \cosh^2 \eta \sin^2 \chi, \]
which results in
\[ A_{22} = -K^2 \cosh^2 \eta \frac{1}{h^2} \sin^2 f. \]

Finally, we evaluate the component \( A_{33} \), obtaining
\[ \frac{1}{J^2 K^2} A_{33} = \sinh^2 \eta \cos^* \chi \cos^* \chi \left( 1 - \frac{g}{Bq} z t^2 \right) \]

\[ + 2 \frac{g}{Bq} z x t \cosh \eta \cos^* \chi \cos^* \chi \sinh \eta \cos \phi + 2 \frac{g}{Bq} z y t \cosh \eta \cos^* \chi \cos^* \chi \sinh \eta \sin \phi \]

\[ + 2 \sinh \eta \cos^* \chi \sin \chi \frac{g}{Bq} q^2 t + \cosh^2 \eta \cos^* \chi \cos^* \chi \cos^2 \phi \left( -1 - \frac{g}{Bq} z x^2 \right) \]

\[ - 2 \cosh^2 \eta \cos^* \chi \cos^* \chi \cos \phi \sin \phi \frac{g}{Bq} z xy \]

\[ - 2 \cosh \eta \sin \chi \cos^* \chi \cos \phi \frac{g q}{B} x + \cosh^2 \eta \cos^* \chi \cos^* \chi \sin^2 \phi \left( -1 - \frac{g z y^2}{Bq} \right) \]

\[ - 2 \cosh \eta \cos^* \chi \sin \chi \frac{g}{Bq} q^2 y + \sin^2 \chi \left[ -1 + \frac{q}{Bq} q^2 (b + qg) \right]. \]

Canceling similar terms leads to
\[ \frac{1}{J^2 K^2} A_{33} = \sinh^2 \eta \cos^* \chi \cos^* \chi \left( 1 - \frac{g}{Bq} z t^2 \right) \]
\[ +2 \frac{g}{Bq} z^t K \cosh^2 \eta \cos^* \chi \cos^* \chi \sin \eta \sin \chi + 2 \sin \eta \cos^* \chi \sin \frac{g}{Bq} q^2 t \]

\[ - \cosh^2 \eta \cos^* \chi \cos^* \chi - \cosh^2 \eta \cos^* \chi \cos^* \chi \frac{g}{Bq} z^t K^2 \cosh^2 \eta \sin \chi \sin \chi \]

\[ -2 \cosh \eta \cos^* \chi \sin \chi \frac{g}{Bq} q^2 K \cosh \eta \sin \chi + \sin^2 \chi \left[ -1 + \frac{g}{Bq} q^2 (b + gq) \right] \]

\[ = - \cos^* \chi \cos^* \chi \left( 1 + \frac{g}{Bq} z^t K^2 \sin \chi \sin \chi \right) \]

\[ -2 \frac{g}{Bq} q^2 K \cos^* \chi \sin^2 \chi + \sin^2 \chi \left[ -1 + \frac{g}{Bq} q^2 (b + gq) \right] . \]

We are left with

\[ \frac{1}{J^2 K^2} A_{33} = - \cos^* \chi \cos^* \chi - \cos^* \chi \cos^* \chi \frac{gbq}{K} \]

\[ -2 \cos^* \chi \sin \chi \frac{g}{B} q^2 - \sin^2 \chi + \sin \chi \frac{g}{B} q^2 q^2 (b + gq) \]

\[ = - \sin^2 \chi + g \sin \chi \cos^* \chi - \cos^* \chi \cos^* \chi = - \frac{1}{J^2} \]

(we have used the identity (A.42)), so that

\[ A_{33} = - K^2 . \]

Thus all the coefficients in the representation (A.54) of \((ds)^2\) are valid.

Performing the choice \(z^0 = e^\sigma\) in (A.54) yields

\[ (ds)^2 = - e^{2\sigma} \left[ \frac{1}{h^2} \sin^2 f \left( (d\eta)^2 - \cosh^2 \eta (d\phi)^2 \right) - (d\chi)^2 - (d\sigma)^2 \right] . \]

The subsequent use of the coordinates

\[ \rho = e^{h\sigma} \sin(h\chi), \quad \tau = e^{h\sigma} \cos(h\chi) \]

leads to the conclusion that the metric is \textit{conformally flat}

\[ (ds)^2 = \kappa \left( (ds)^2 \right) \text{ pseudo-Euclidean} , \]
where
\[
\left. \left( (ds)^2 \right) \right|_{\text{pseudo-Euclidean}} = -\rho^2 \left( (d\eta)^2 - \cosh^2 \eta (d\phi)^2 \right) + (d\rho)^2 + (d\tau)^2
\] (A.61)

and
\[
\kappa = \frac{1}{h^2} \rho^2 + \tau^2 \right)^{(1-h)/h},
\] (A.62)

which can be written as
\[
\kappa = \frac{1}{h^2} \left( K^2 \right)^{1-h}.
\] (A.63)

**Appendix B: Evaluation of spray coefficients of the AR-space**

Let us represent the spray coefficients \( G^i = \gamma^i_{kj} y^k y^j \) in the way
\[
G^i = g^{ik} G_k
\] (B.1)
such that
\[
G_k = y^m \frac{\partial y_k}{\partial x^m} - \frac{1}{2} \frac{\partial F^2}{\partial x^k}.
\] (B.2)

Using the notation \( b_{j,k} = \partial b_j / \partial x^k \) and \( s_i = y^k b_{k,i} \), we shall apply the equalities
\[
\frac{\partial b}{\partial x^i} = s_i, \quad \frac{\partial q}{\partial x^i} = \epsilon b_i s_i + \Delta,
\] (B.3)

and their implications
\[
\frac{\partial J^2}{\partial x^k} = - \frac{g}{B} J^2 \left( q - \frac{b^2}{q} \right) s_k + \frac{\partial J^2}{\partial g} \frac{\partial g}{\partial x^k} + \Delta
\] (B.4)

(see (2.23)),
\[
\frac{\partial B}{\partial x^k} = - \frac{1}{q} \left( q^2 + \epsilon b^2 \right) s_k - b q \frac{\partial g}{\partial x^k} + \Delta
\] (B.5)

(see (2.9)), and
\[
\frac{\partial F^2}{\partial x^k} = -2 g q J^2 s_k + \frac{\partial F^2}{\partial g} \frac{\partial g}{\partial x^k} + \Delta
\] (B.6)

(see (2.13)), where \( \Delta \) symbolizes the summary of the terms which involve partial derivatives of the input pseudo-Riemannian metric tensor \( a_{ij} \) with respect to the coordinate variables \( x^k \).

Taking into account the representation (2.37), from (B.2) we get
\[
G_k = -g \left( \epsilon \frac{b}{q} (ys) b_k + q b_{km} y^m \right) J^2 - y_k \frac{g}{B} \left( q - \frac{b^2}{q} \right) (ys) + g q J^2 s_k
\]
\[
+ \frac{\partial y_k}{\partial g} (ys) - \frac{1}{2} \frac{\partial F^2}{\partial g} \frac{\partial g}{\partial x^k} + \Delta,
\] (B.7)

where
\[
(ys) = \frac{\partial g}{\partial x^i} y^i.
\] (B.8)
To raise the index, it is convenient to apply the rules

\[ g^{ij}b_j = \frac{1}{F^2} \left[ (B + g bq)b^i - \frac{eg}{\nu} \left( c^2 B + b(b + gc^2 q) \right) v^i \right] \]

and, for any co-vector \( t_j \), from (2.31) we obtain

\[ g^{ij}t_j = \left[ Ba^{ij}t_j + gq(yt)b^i + \frac{eg}{\nu} \left( B(bt) - (b + gc^2 q)(yt) \right) v^i \right] \frac{1}{F^2}, \]

where \( (yt) = y^i t_j \) and \( (bt) = b^i t_j \). In this way we obtain

\[ G^i = -g \frac{eb}{qB} (ys) \left[ (B + gbq)b^i - \frac{eg}{\nu} \left( c^2 B + b(b + gc^2 q) \right) v^i \right] \]

\[ -g \frac{q}{B} \left[ Ba^{ij}b^j_i h + gq(ys)b^i \right] - g \frac{q}{B} \frac{eg}{\nu} \left( B(b^i b^j_i h^h) - (b + gc^2 q)(ys) \right) v^i \]

\[ -v^i \frac{g}{B} \left( q - \frac{b^2}{q} \right) (ys) + eb^i \frac{g}{B} \left( q - \frac{b^2}{q} \right) (ys) \]

\[ + \frac{gq}{B} \left[ Ba^{ij}s^j + gq(ys)b^i + \frac{eg}{\nu} \left( B(bs) - (b + gc^2 q)(ys) \right) v^i \right] + E^i + a^n_{nm} y^n y^m, \]

or

\[ G^i = g \frac{b}{qB} (ys) \frac{g}{\nu} \left( c^2 B + b(b + gc^2 q) \right) v^i \]

\[ -gqa^{ij}b^j_i h - g \frac{q}{B} \frac{eg}{\nu} \left( B(b^i b^j_i h^h) - (b + gc^2 q)(ys) \right) v^i - v^i \frac{eg}{B} + gbq \frac{(ys)}{q} \]

\[ + \frac{gq}{B} \left[ Ba^{ij}s^j + \frac{eg}{\nu} \left( B(bs) - (b + gc^2 q)(ys) \right) v^i \right] + E^i + a^n_{nm} y^n y^m. \]

Due simplifying yields

\[ G^i = g \frac{b}{q} (ys) \frac{g}{\nu} \left( c^2 - 1 \right) v^i - gqa^{ij}b^j_i h - gq \frac{eg}{\nu} \left( b^i b^j_i h^h \right) v^i - v^i \frac{eg}{q} (ys) \]

\[ + gq \left[ a^{ij} s^j + \frac{eg}{\nu} (bs) v^i \right] + E^i + a^n_{nm} y^n y^m, \]

which can be written as

\[ G^i = -\frac{g}{\nu} (ys) v^i + gq \left[ a^{ij} s^j + \frac{eg}{\nu} (bs) v^i \right] - gq \left[ a^{ij} b^j_i h^h + \frac{eg}{\nu} (b^i b^j_i h^h) v^i \right] \]
This method results in the representations (2.61)-(2.63).

**Appendix C: Conformal transformation in the UAR-space**

It proves possible to indicate the explicit transformation

\[ \zeta^m = \zeta^m(x, y) \] (C.1)

to fulfill the conformal claim. Indeed, let us take

\[ \zeta^m = \left[h v^m - (b + \frac{1}{2} g q) b^m\right] J \frac{1}{\zeta h}, \quad \zeta = \frac{1}{h} \left(|F^2|\right)^{(1-h)/2}, \] (C.2)

where \( v^m = y^m + b b^m \) and both the values \( \epsilon = 1 \) or \( \epsilon = -1 \) are admissible. Evaluating the derivatives

\[ \zeta^m_n := \frac{\partial \zeta^m}{\partial y^n} \] (C.3)

yields

\[ \zeta^m_n = E^m_n + \frac{1}{N} \zeta^m C_n - \frac{1}{\zeta} \zeta_n \zeta^m, \] (C.4)

where

\[ E^m_n = \left[h(\delta^m_n + b_n b^m) - \left(b_n + \epsilon \frac{1}{2q} g q b_n\right) b^m\right] J \frac{1}{\zeta h} \] (C.5)

and

\[ \zeta_n = \frac{\partial \zeta}{\partial y^n} = (1 - h)y_n \frac{1}{F^2} \zeta, \] (C.6)

and we have used the equality

\[ \frac{\partial J}{\partial y^n} = \frac{1}{N} C_n \] (C.7)

(which is a direct implication of the formulas (2.32) and (2.41)). It is useful to note that

\[ E^m_n y^n = \zeta^m, \quad E^m_n b^n = b^m J \frac{1}{\zeta h}. \] (C.8)

Let us find the transform

\[ s^{ij} := g^{mn} \zeta_m^i \zeta_n^j \] (C.9)

of the initial Finslerian metric tensor \( g^{mn} \). To this end we apply the representation (2.39) of \( g^{mn} \) (with \( c = 1 \) which entails \( \nu = q \)) and note the vanishing

\[ -\frac{\zeta^i \zeta^j}{N^2} C_m C^m - \frac{(1 - h)^2}{F^2} \zeta^i \zeta^j + \frac{1 - h}{F^2} (\zeta^i y^m E^j_m + \zeta^j y^m E^i_m) = 0, \]

getting

\[ s^{ij} = \frac{1}{N} (\zeta^i C^m E^j_m + \zeta^j C^m E^i_m) \]

+ \[ a^{mn} E^i_m E^j_n + \frac{g}{q} b^m b^n E^i_m E^j_n + \frac{g}{q} (b^m y^n + y^m b^n) E^i_m E^j_n - \frac{g}{B q} (b + g q) y^n y^m E^i_m E^j_n \] \( \frac{1}{J^2} \).
Simple calculation shows that

\[
a^{mn} E^i_m E^j_n = h a^{in} \left[ h (\delta^i_n + b_n b^i) + \left( b_n + \frac{1}{2q} g v_n \right) b^i \right] \frac{J^2}{x^2 h^2} - h \left( h + 1 \right) b^i + \frac{1}{x^2} E^i_j,\]

so that

\[
a^{mn} E^i_m E^j_n = h [ h (a^{ij} + b^i b^j) + \left( b^i + \frac{1}{2q} g v^i \right) b^j ] \frac{J^2}{x^2 h^2} - [ (h + 1) + \frac{1}{2q} g b ] b^j b^i \frac{J^2}{x^2 h^2} + \frac{1}{2q} g \zeta_j J b^i,\]

In this way we get

\[
h^2 s^{ij} = \epsilon \frac{g}{q F^2} (b + g q) \zeta^i \zeta^j h^2 + \epsilon \frac{g}{2q} (\zeta^i b^j + \zeta^j b^i) J \frac{1}{x} h
\]

\[\quad + h^2 a^{ij} \frac{1}{x^2} + h^2 b^i b^j \frac{1}{x^2} + \frac{1}{2q} g b^i h v^i \frac{1}{x^2} - \left[ 1 + \frac{1}{2q} g b \right] b^j b^i \frac{1}{x^2} + b^i \epsilon \frac{1}{2q} g \zeta_j J \frac{1}{x} h\]

\[\quad + \epsilon \frac{g}{q} b^i b^j \frac{1}{x^2} - \epsilon \frac{g}{q} (b^i \zeta^j + \zeta^i b^j) J \frac{1}{x} h - \epsilon \frac{g}{B q} (b + g q) \zeta^i \zeta^j \frac{1}{J^2 h^2},\]

or

\[
h^2 s^{ij} = \epsilon \frac{g}{2q} (\zeta^i b^j + \zeta^j b^i) J \frac{1}{x} h + h^2 a^{ij} \frac{1}{x^2} + h^2 b^i b^j \frac{1}{x^2} + \frac{1}{2q} g b^i h v^i \frac{1}{x^2}
\]

\[\quad - \left[ 1 + \frac{1}{2q} g b \right] b^j b^i \frac{1}{x^2} + b^i \epsilon \frac{1}{2q} g \zeta_j J \frac{1}{x} h + \frac{g}{q} b^i b^j \frac{1}{x^2} - \epsilon \frac{g}{q} (b^i \zeta^j + \zeta^i b^j) J \frac{1}{x} h.\]

Inserting here the equality

\[
h v^i = x h \frac{1}{J} \zeta^i - \left( b + \frac{1}{2} g q \right) b^i
\]

(which ensues from (C.2)), we eventually find

\[
s^{ij} = \frac{1}{x^2} a^{ij}. \tag{C.10}\]

From (C.4), (C.10), and (2.32) it follows that

\[
\det(\zeta_m^i) = (J/x)^N > 0. \tag{C.11}\]

The metric tensor transformation (C.9) can be inverted to read

\[
g_{mn} = x^2 a_{ij} \zeta_m^i \zeta_n^j. \tag{C.12}\]
The functions (C.2) possess the property of homogeneity of degree $h$:

$$\zeta_i(x, ky) = k^h \zeta_i(x, y), \quad k > 0, \forall y,$$

which entails the identity

$$y^i \zeta_n = h \zeta_i.$$  \hfill (C.14)

Therefore, from (C.12) and the equality $g_{mn}y^m y^n = F^2$ we can infer that

$$F^2 = \kappa^2 S^2(x, \zeta),$$

where

$$S^2(x, \zeta) = a_{ij}(x) \zeta^i \zeta^j.$$ \hfill (C.16)

The equality (C.12) motivates introducing the tensor

$$t_{mn}(x, \zeta) = \kappa^2 a_{mn}.$$ \hfill (C.17)

From (C.2) and (C.15) we can obtain the remarkable equality

$$|F^2(x, y)|^{h(x)} = |S^2(x, \zeta)|.$$ \hfill (C.18)

So, if we introduce the scalar

$$p(x, \zeta) = \kappa^2,$$

from (C.2) we get the representation

$$p(x, \zeta) = \frac{1}{\kappa^2(x)} |S^2(x, \zeta)|^{(1-h(x))/h(x)}$$

which enables us to read the tensor (C.17) as follows:

$$t_{mn}(x, \zeta) = p(x, \zeta) a_{mn}(x).$$ \hfill (C.21)

This line of reasoning leads to the definition

$$\mathcal{C}_g := \{ \mathcal{R}_N; TM; \zeta \in TM; g(x); t_{mn}(x, \zeta) \}$$

which we call the *factor-pseudo-Riemannian space*. The transformation (C.1)-(C.4) assigns the map

$$Z : \mathcal{UAR}_g \rightarrow \mathcal{C}_g,$$ \hfill (C.23)

which sends vectors to vectors

$$y \xrightarrow{Z} \zeta$$ \hfill (C.24)

(at each point $x \in M$) and replaces the Finslerian metric tensor $\{g_{mn}(x, y)\}$ by the factored tensor (C.21):

$$\{g_{mn}\} \xrightarrow{Z} \{t_{mn}\}.$$ \hfill (C.25)

Obviously, the map is the diffeomorphism.

Thus we are entitled to formulate the following proposition.

**PROPOSITION.** The *UAR*-space is metrically isomorphic to the factor-pseudo-Riemannian space:

$$g_{mn}(x, y) dy^m dy^n = t_{mn}(x, \zeta) d\zeta^m d\zeta^n.$$ \hfill (C.26)
Since \( v^i = q = 0 \) at \( y^i = b^i \), from (C.2) it follows that
\[
\zeta^i(x, b) = b^i(x)
\] (C.27)

(the equality \( K^2(x, b) = 1 \) has been into account; examine (4.10)-(4.14)). Therefore, we can formulate the following.

**PROPOSITION.** The involved preferred vector field \( b^i(x) \) is the proper element of the \( \mathcal{UR\!AR} \)-space isomorphism:
\[
\{b^i(x)\} \xrightarrow{\mathcal{Z}} \{b^i(x)\}.
\] (C.28)

The transformation (C.1) is trivial in the pseudo-Riemannian limit:
\[
\zeta^i \bigg|_{g=0} = y^i.
\]

Let us examine the indicatrix curvature. At each point \( x \), the invention (C.23)-(C.25) maps isomorphically the Finslerian indicatrix (defined by \( |F^2(x, y)| = 1 \)) onto the pseudo-Euclidean sphere \( \mathcal{S}_x(C_g) \) (defined by \( |S^2(x, \zeta)| = 1 \)) in space \( C_g \). The factor \( p \), having been proposed by (C.20), equals \( 1/h^2(x) \) on this \( \mathcal{S}_x(C_g) \) and thereby influences the form of the metric tensor induced on \( \mathcal{S}_x(C_g) \). Namely, let some coordinates \( m^a \) be introduced on \( \mathcal{S}_x(C_g) \), so that the tensor is of the type \( i_{ab} = \epsilon_{ab}(x, m^c) \). For instance, in the time-like sector we can take \( m^a = \zeta^a/\epsilon^0, \epsilon = 1 \). Let \( L^i \) denote the unit vectors, so that \( \epsilon L^i L^j t_{ij} = 1 \). We use the coordinates to parameterize the vectors, obtain the projection factors \( L^i_a = \partial L^i/\partial m^a \), and construct the tensor \( i_{ab} = t_{ij} L^i_a L^j_b \). In view of the factored structure (C.20)-(C.21) of the tensor \( t_{ij} \), we get the equality
\[
i_{ab} = \frac{1}{h^2} \hat{i}_{ab},
\]
where \( \hat{i}_{ab} = a_{ij} L^i_a L^j_b \) is the indicatrix metric tensor that is obtainable in the pseudo-Riemannian geometry proper. Since \( h \) is independent of \( \zeta \), the associated Christoffel symbols
\[
i^e_{a b} = \frac{1}{2} \left( \frac{\partial i_{e a}}{\partial m^b} + \frac{\partial i_{e b}}{\partial m^a} - \frac{\partial i_{a b}}{\partial m^e} \right), \quad \hat{i}^e_{a b} = \frac{1}{2} \left( \frac{\partial \hat{i}_{e a}}{\partial m^b} + \frac{\partial \hat{i}_{e b}}{\partial m^a} - \frac{\partial \hat{i}_{a b}}{\partial m^e} \right)
\]
are equivalent:
\[
i^e_{a b} = \hat{i}^e_{a b}.
\]
Therefore, the indicatrix curvature tensor
\[
I^e_{bd} = \frac{\partial i^e_{a b}}{\partial m^d} - \frac{\partial i^e_{a d}}{\partial m^b} + i^e_{a b} i_{c d} - i^e_{a d} i_{c b}
\]
is identical to the tensor \( \hat{I}^e_{bd} \) constructible by the same rule from the tensor \( \hat{i}_{ab} \), that is, \( I^e_{bd} = \hat{I}^e_{bd} \). Let us now consider the tensors \( I_{acbd} = i_{ce} I^e_{a bd} \) and \( \hat{I}_{acbd} = \hat{i}_{ce} \hat{I}^e_{a bd} \). Since \( \hat{I}_{acbd} = -\epsilon (\hat{i}_{cb} \hat{i}_{ad} - \hat{i}_{cd} \hat{i}_{ab}) \) is the ordinary case characteristic of the pseudo-Riemannian geometry (which reflects the fact that the indicatrix constant curvature is of the unit type), we get \( I_{acbd} = -\epsilon (i_{cb} i_{ad} - i_{cd} i_{ab})/h^2 \), which in turn entails
\[
I_{acbd} = -\epsilon h^2 (i_{cb} i_{ad} - i_{cd} i_{ab}),
\] (C.29)
whence
\[ R_{\text{URR-pseudo-Finsleroid indicatrix}} = -\epsilon h^2. \] (C.30)

Amazingly, the angle (2.71)-(2.72) may be written as
\[ \alpha(x)(y_1, y_2) = \frac{1}{h} \alpha_{\text{pseudo-Riemannian}}, \] (C.31)
where
\[ \alpha_{\text{pseudo-Riemannian}} = \arccosh \frac{a_{mn}(x)\zeta^n(x, y_1)\zeta^n(x, y_2)}{\sqrt{S^2(\zeta(x, y_1))} \sqrt{S^2(\zeta(x, y_2))}}, \quad \epsilon = 1, \] (C.32)
and
\[ \alpha_{\text{pseudo-Riemannian}} = \arccos \frac{-a_{mn}(x)\zeta^n(x, y_1)\zeta^n(x, y_2)}{\sqrt{|S^2(\zeta(x, y_1))|} \sqrt{|S^2(\zeta(x, y_2))|}}, \quad \epsilon = -1, \] (C.33)
(insert here (C.2) and make comparison with (2.71)-(2.73)). The conformal factor \( p \) does not enter the right-hand parts of (C.32)-(C.33). The angle (C.32)-(C.33) belongs to the sense of the ordinary pseudo-Riemannian geometry and operates in the factor-pseudo-Riemannian space (C.22).

If we insert (C.2) in (C.16), we obtain the useful equality
\[ \left( \frac{J}{\alpha h} \right)^2 = \frac{S^2(x, \zeta)}{B(x, y)}. \] (C.34)

Also, from (C.2) it follows that
\[ \zeta^i b_i = \left( b + \frac{1}{2} \eta g \right) \frac{J}{\alpha h}, \] (C.35)
and
\[ \zeta^i + (\zeta^n b_n) b^i = \frac{J}{\alpha h} h v^i. \] (C.36)

According to (2.86) and (2.91), we have
\[ \chi = \frac{1}{h} \arcsinh \frac{\zeta^i b_i(x)}{\sqrt{S^2(x, \zeta)}}, \quad \epsilon = 1; \quad \chi = \frac{1}{h} \arccos \frac{\zeta^i b_i(x)}{\sqrt{|S^2(x, \zeta)|}}, \quad \epsilon = -1, \] (C.37)
and from (2.84) and (2.89) we get
\[ b = F \text{Sh} \chi, \quad \epsilon = 1; \quad b = K \text{Cos} \chi, \quad \epsilon = -1. \] (C.38)

From (C.19)-(C.20) and (2.78) it follows that
\[ h \alpha = |S^2(x, \zeta)|^{(1-\zeta)/(2h)}, \quad \frac{1}{J} = e^{\frac{1}{2} g \chi}. \] (C.39)

The indicated formulas allow us to write down the explicit form of the inverse to (C.1) and (C.2), namely we find
\[ y^i = y^i(x, \zeta) \] (C.40)
with
\[ y^i = -b b^i + \frac{1}{h} \left( \zeta^i + (\zeta^n b_n) b^i \right) \frac{h \alpha}{J}. \] (C.41)
(examine (C.36) and remind that \( v^i = y^i + b_i^i \)).

It is possible to find straightforwardly the coefficients

\[ y_j^i := \frac{\partial y^i}{\partial \zeta_j}. \tag{C.42} \]

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