Abstract

We establish extension theorems for separately holomorphic mappings defined on sets of the form $W \setminus M$ with values in a complex analytic space which possesses the Hartogs extension property. Here $W$ is a 2-fold cross of arbitrary complex manifolds and $M$ is a set of singularities which is locally pluripolar (resp. thin) in fibers.

Classification AMS 2000: Primary 32D15, 32D10

Keywords: Cross theorem, set of singularities, holomorphic extension, plurisubharmonic measure.

1 Introduction

Let $D \subset X$ (resp. $G \subset Y$) be an open set, $A \subset \overline{D}$ (resp. $B \subset \overline{G}$), where $X$ and $Y$ are complex manifolds $^1$, and let $M \subset ((D \cup A) \times B) \cup (A \times (G \cup B))$. The set $M_a := \{w \in G : (a, w) \in M\}$, $a \in A$, is called the vertical fiber of $M$ over $a$ (resp. the set $M_b := \{z \in D : (z, b) \in M\}$, $b \in B$, is called the horizontal fiber of $M$ over $b$). We say that $M$ possesses a certain property in fibers over $A$ (resp. $B$) if all vertical fibers $M_a$, $a \in A$, (resp. all horizontal fibers $M_b$, $b \in B$) possess this property.

The main purpose of this work is to study the following PROBLEM:

Let $X$, $Y$, $D$, $G$, $A$, and $B$ be as above, and let $Z$ be a complex analytic space $^2$. Define the cross

$$W := ((D \cup A) \times B) \cup (A \times (G \cup B)).$$

We want to determine an “optimal” open subset of $X \times Y$, denoted by $\widehat{W}$, which is characterized by the following property:

---

$^1$ In this paper complex manifolds are always assumed to be of finite dimension and countable at infinity.

$^2$ All complex analytic spaces are assumed to be reduced, irreducible, and countable at infinity.
Let $M \subset W$ be a subset which is relatively closed and locally pluripolar (resp. thin) in fibers over $A$ and $B$ ($M = \emptyset$ is allowed). Then there exists a new set of singularities $\hat{M} \subset \hat{W}$, which is, in some sense, of the same structure as $M$ and which fulfills the following property:

For every mapping $f : W \setminus M \rightarrow Z$ satisfying, in essence, the following condition:

\[
\begin{align*}
    f(a, \cdot) &\in \mathcal{C}((G \cup B) \setminus M_a, Z) \cap \mathcal{O}(G \setminus M_a, Z), & a \in A, \\
    f(\cdot, b) &\in \mathcal{C}((D \cup A) \setminus M^b, Z) \cap \mathcal{O}(D \setminus M^b, Z), & b \in B,
\end{align*}
\]

there exists an $\hat{f} \in \mathcal{O}(\hat{W} \setminus \hat{M}, Z)$ such that for “all” $(\zeta, \eta) \in W \setminus M$, $\hat{f}(z, w)$ tends to $f(\zeta, \eta)$ as $(z, w) \in \hat{W} \setminus \hat{M}$ tends, in some sense, to $(\zeta, \eta)$.

We briefly recall the very recent developments around this PROBLEM. The case when $M = \emptyset$ has been thoroughly investigated in the work [16, 17] of the first author. These articles also show that the natural “target spaces” $Z$ for obtaining a satisfactory answer to the above PROBLEM are the ones which possess the Hartogs extension property.

The case where $X$ and $Y$ are Riemann domains (over $\mathbb{C}^n$), $A \subset D$, $B \subset G$, and $Z = \mathbb{C}$ has been completed in some joint-articles of M. Jarnicki and the second author (see [9, 10, 11, 13]). Therefore, it is reasonable to conjecture that a positive solution to the PROBLEM may exist when the “target space” $Z$ possesses the Hartogs extension property.

The main purpose of this article is to verify the above conjecture in its full generality. Our proof is geometric in nature. Indeed, our method consists in using holomorphic discs, and it is based on the works in [10, 20, 16, 17]. Moreover, the novelty of this new approach is that it does not use the classical method of doubly orthogonal bases of Bergman type. It is worthy to note here that most of the previous works in the subject of separate holomorphy make use of the latter method.

Acknowledgment. The paper was written while the first author was visiting the Abdus Salam International Centre for Theoretical Physics in Trieste and the

\[\text{Acknowledgment.} \]
Korea Institute for Advanced Study in Seoul. He wishes to express his gratitude to these organizations.

2 Preliminaries and the statement of the main result

First we recall some notions developed in [17] such as systems of approach regions for an open set in a complex manifold, and the corresponding plurisubharmonic measures. These will provide the framework for an exact formulation of the PROBLEM and for our final solution.

2.1 Approach regions, local pluripolarity and plurisubharmonic measure

Definition 2.1. Let $X$ be a complex manifold and $D \subset X$ an open subset. A system of approach regions for $D$ is a collection $A = (A_\alpha(\zeta))_{\zeta \in D, \alpha \in I_\zeta}$ ( $I_\zeta \neq \emptyset$ for all $\zeta \in \partial D$) of open subsets of $D$ with the following properties:

(i) For all $\zeta \in D$, the system $(A_\alpha(\zeta))_{\alpha \in I_\zeta}$ forms a basis of open neighborhoods of $\zeta$ (i.e., for any open neighborhood $U$ of a point $\zeta \in D$, there is an $\alpha \in I_\zeta$ such that $\zeta \in A_\alpha(\zeta) \subset U$).

(ii) For all $\zeta \in \partial D$ and $\alpha \in I_\zeta$, $\zeta \in \overline{A_\alpha(\zeta)}$.

Moreover, $A$ is said to be canonical if it satisfies (i) and the following property (which is stronger than (ii)):

(ii') For every point $\zeta \in \partial D$, there is a basis of open neighborhoods $(U_\alpha)_{\alpha \in I_\zeta}$ of $\zeta$ in $X$ such that $A_\alpha(\zeta) = U_\alpha \cap D$, $\alpha \in I_\zeta$.

$A_\alpha(\zeta)$ is often called an approach region at $\zeta$.

In what follows we fix an open subset $D \subset X$ and a system of approach regions $A = (A_\alpha(\zeta))_{\zeta \in D, \alpha \in I_\zeta}$ for $D$.

For every function $u : D \rightarrow [-\infty, \infty)$, let

$$(A-)\limsup u(z) := \sup_{\alpha \in I_\zeta} \limsup_{A_\alpha(z) \ni w \rightarrow z} u(w), \quad z \in \overline{D}.$$

Therefore,

$$(A-)\limsup u(z) = \limsup_{D \ni w \rightarrow z} u(w), \quad \text{if } z \in D,$$

i.e. $(A-)\limsup u |_D$ coincides with the usual upper semicontinuous regularization of $u$ in case $u$ is locally bounded from above on $D$. 
For a set $A \subset \overline{D}$ put
\[ h_{A,D} := \sup \{ u : u \in \mathcal{PSH}(D), u \leq 1 \text{ on } D, \ A-\limsup u \leq 0 \text{ on } A \}, \]
where $\mathcal{PSH}(D)$ denotes the cone of all functions plurisubharmonic on $D$.

A set $A \subset D$ is said to be pluripolar in $D$ if there is $u \in \mathcal{PSH}(D)$ such that $u$ is not identically $-\infty$ on every connected component of $D$ and $A \subset \{ z \in D : u(z) = -\infty \}$. A set $A \subset D$ is said to be locally pluripolar in $D$ if for any $z \in A$, there is an open neighborhood $V \subset D$ of $z$ such that $A \cap V$ is pluripolar in $V$. A set $A \subset D$ is said to be non-pluripolar (resp. non-locally pluripolar) if it is not pluripolar (resp. not locally pluripolar). According to a classical result of Josefson and Bedford (see [14], [3]), if $D$ is a Riemann domain over a Stein manifold, then $A \subset D$ is locally pluripolar if and only if it is pluripolar.

**Definition 2.2.** The relative extremal function of $A$ relative to $D$ is the function $\omega(\cdot, A, D)$ defined by
\[ \omega(z,A,D) = \omega_A(z,A,D) := (A-\limsup h_{A,D})(z), \quad z \in \overline{D}. \]

Note that when $A \subset D$, Definition 2.2 coincides with the classical definition of Siciak’s relative extremal function for $z \in D$.

Next, we say that a set $A \subset \overline{D}$ is locally pluriregular at a point $a \in A$ if $\omega(a,A \cap U, D \cap U) = 0$ for all open neighborhoods $U$ of $a$, where the system of approach regions for $D \cap U$ is given by $\mathcal{A}|_{D \cap U} := \{ A_a(z) \cap U : z \in D \cap U, a \in I_z \}$. Moreover, $A$ is said to be locally pluriregular if it is locally pluriregular at all points $a \in A$. It should be noted from Definition 2.1 that if $a \in \overline{A} \cap D$, then the property of local pluriregularity of $A$ at $a$ does not depend on the system of approach regions $\mathcal{A}$, while the situation is different when $a \in \overline{A} \cap \partial D$: then the property does depend on $\mathcal{A}$.

We denote by $A^*$ the following set
\[ (A \cap \partial D) \cup \left\{ a \in \overline{A} \cap D : A \text{ is locally pluriregular at } a \right\}. \]

If $A \subset D$ is non-locally pluripolar, then a classical result of Bedford and Taylor (see [3, 4]) says that $A^*$ is locally pluriregular and $A \setminus A^*$ is locally pluripolar. Moreover, when $A \subset D$, $A^*$ is locally of type $G_\delta$, that is, for every $a \in A^*$ there is an open neighborhood $U \subset D$ of $a$ such that $A^* \cap U$ is a countable intersection of open sets.

Now we are in the position to introduce the following version of a plurisubharmonic measure.

**Definition 2.3.** For a set $A \subset \overline{D}$, let $\tilde{A} = \tilde{A}(A) := \bigcup_{P \in \mathcal{E}(A)} P$, where
\[ \mathcal{E}(A) = \mathcal{E}(A, A) := \left\{ P \subset \overline{D} : P \text{ is locally pluriregular, } \overline{P} \subset A^* \right\}. \]

\[ ^7 \text{Observe that this function depends on the system of approach regions.} \]

\[ ^8 \text{Note that } \overline{P} \subset (A \cap \partial D) \cup (A \cap D)^*. \]
The plurisubharmonic measure of $A$ relative to $D$ is the function $\tilde{\omega}(\cdot, A, D)$ defined by

$$\tilde{\omega}(z, A, D) := \omega(z, \tilde{A}, D), \quad z \in \overline{D}.$$ 

It is worthy to remark that $\tilde{\omega}(\cdot, A, D)|_D \in PSH(D)$ and $0 \leq \tilde{\omega}(z, A, D) \leq 1$, $z \in D$. Obviously, if $\tilde{A} \neq \emptyset$, then $\tilde{A}$ is locally pluriregular; in particular,

$$\tilde{\omega}(z, A, D) = 0, \quad z \in \tilde{A}.$$ (1)

An example in [1] shows that, in general, $\omega(\cdot, A, D) \neq \tilde{\omega}(\cdot, A, D)$ on $D$.

Now we compare the plurisubharmonic measure $\tilde{\omega}(\cdot, A, D)$ with Siciak’s relative extremal function $\omega(\cdot, A, D)$. For the moment, we only focus on the case where $A \subset D$.

If $A$ is an open subset of an arbitrary complex manifold $D$, then it can be shown that

$$\tilde{\omega}(z, A, D) = \omega(z, A, D), \quad z \in D.$$ 

If $A$ is a (not necessarily open) non-locally pluripolar subset of an arbitrary complex manifold $D$, then we have, by Proposition 7.1 in [17],

$$\tilde{\omega}(z, A, D) = \omega(z, A^*, D), \quad z \in D.$$ 

On the other hand, if, moreover, $D$ is a bounded open subset of $\mathbb{C}^n$, then we have (see, for example, Lemma 3.5.3 in [8]) $\omega(z, A, D) = \omega(z, A^*, D)$, $z \in D$. Consequently, under the last assumption,

$$\tilde{\omega}(z, A, D) = \omega(z, A, D), \quad z \in D.$$ 

The case where $A \subset \partial D$ has been investigated in [17, 18]. Our discussion shows that, at least in the case where $A \subset D$, the notion of the plurisubharmonic measure is a good candidate for generalizing Siciak’s relative extremal function to the manifold context in the theory of separate holomorphy.

For a good background of the pluripotential theory, see the books [8] or [15]. For a more detailed discussion on systems of approach regions as well as their corresponding plurisubharmonic measure, see [16].

### 2.2 Cross, separate holomorphicity, and $A$-limit.

Let $X$, $Y$ be two complex manifolds, let $D \subset X$, $G \subset Y$ be two nonempty open sets, let $A \subset \overline{D}$ and $B \subset \overline{G}$. Moreover, $D$ (resp. $G$) is equipped with a system of approach regions $A(D) = \{(A_\alpha(\zeta))_{\zeta \in \overline{D}, \alpha \in I_{\zeta}}\}$ (resp. $A(G) = \{(A_\alpha(\eta))_{\eta \in \overline{G}, \alpha \in I_{\eta}}\}$).\footnote{In fact we should have written $I_{\zeta}(D)$, resp. $I_{\eta}(G)$; but we skip $D$ and $G$ here to make the notions as simple as possible.}
We define a 2-fold cross $W$, its interior $W^o$ and its regular part $\tilde{W}$ (with respect to $\mathcal{A}(D)$ and $\mathcal{A}(G)$) as

$$W = \mathbb{X}(A,B;D,G) := ( (D \cup A) \times B ) \bigcup ( A \times (B \cup G) ),$$

$$W^o = \mathbb{X}^o(A,B;D,G) := ( A \times G ) \cup ( D \times B ),$$

$$\tilde{W} = \tilde{\mathbb{X}}(A,B;D,G) := \mathbb{X}(\tilde{A},\tilde{B};D,G).$$

Moreover, put

$$\omega(z,w) := \omega(z,A,D) + \omega(w,B,G), \quad (z,w) \in D \times G,$n

$$\tilde{\omega}(z,w) := \tilde{\omega}(z,A,D) + \tilde{\omega}(w,B,G), \quad (z,w) \in D \times G.$$n

For a 2-fold cross $W := \tilde{\mathbb{X}}(A,B;D,G)$ let

$$\tilde{W} := \tilde{\mathbb{X}}(A,B;D,G) = \{ (z,w) \in D \times G : \omega(z,w) < 1 \}.$$n

Therefore, we obtain

$$\tilde{\tilde{W}} = \tilde{\tilde{\mathbb{X}}}(\tilde{A},\tilde{B};D,G) = \{ (z,w) \in D \times G : \tilde{\omega}(z,w) < 1 \}.$$n

Let $Z$ be a complex analytic space and $M \subset W$ a subset which is relatively closed in fibers over $A$ and $B$. We say that a mapping $f : W^o \setminus M \longrightarrow Z$ is separately holomorphic and write $f \in \mathcal{O}_s(W^o \setminus M,Z)$, if, for all $a \in A$ (resp. $b \in B$) the mapping $f(a,\cdot)|_{G \setminus M_a}$ (resp. $f(\cdot,b)|_{D \setminus M^b}$) is holomorphic.

We say that a mapping $f : W \setminus M \longrightarrow Z$ is separately continuous and write $f \in \mathcal{C}_s(W \setminus M,Z)$ if, for all $a \in A$ (resp. $b \in B$) the mapping $f(a,\cdot)|_{(G \cup B) \setminus M_a}$ (resp. $f(\cdot,b)|_{(D \cup A) \setminus M^b}$) is continuous.

Let $\Omega$ be an open subset of $D \times G$. A point $(\zeta,\eta) \in \overline{D} \times \overline{G}$ is said to be an end-point of $\Omega$ with respect to $\mathcal{A} = \mathcal{A}(D) \times \mathcal{A}(G)$ if for any $(\alpha,\beta) \in I_\zeta \times I_\eta$ there exist open neighborhoods $U$ of $\zeta$ in $X$ and $V$ of $\eta$ in $Y$ such that

$$\left( U \cap \mathcal{A}_a(\zeta) \right) \times \left( V \cap \mathcal{A}_\beta(\eta) \right) \subset \Omega.$$n

The set of all end-points of $\Omega$ is denoted by $\text{End}(\Omega)$.

It follows from \textbf{1} that if $\tilde{A},\tilde{B} \neq \varnothing$, then $\tilde{W} \subset \text{End}(\tilde{\tilde{W}})$.

Let $S$ be a relatively closed subset of $\tilde{W}$ and let $(\zeta,\eta) \in \text{End}(\tilde{\tilde{W}} \setminus S)$. Then a mapping $f : \tilde{\tilde{W}} \setminus S \longrightarrow Z$ is said to admit the $\mathcal{A}$-limit $\lambda$ at $(\zeta,\eta)$, and one writes

$$\lambda = (\mathcal{A} \lim f)(\zeta,\eta), \quad [10]$$n

\text{Note that here } \mathcal{A} = \mathcal{A}(D) \times \mathcal{A}(G).
if, for all \( \alpha \in I_\zeta, \beta \in I_\eta \),

\[
\lim_{C \ni (z,w) \to (\zeta,\eta), z \in A_\alpha(\zeta), w \in A_\beta(\eta)} f(z,w) = \lambda.
\]

We conclude this introduction with a notion we need in the sequel. Let \( M \) be a topological space. A mapping \( f : M \to Z \) is said to be bounded if there exists an open neighborhood \( U \) of \( f(M) \) in \( Z \) and a holomorphic embedding \( \phi \) of \( U \) into a bounded polydisc of \( \mathbb{C}^k \) such that \( \phi(U) \) is an analytic set in this polydisc. \( f \) is said to be locally bounded along \( N \subset M \) if for every point \( z \in N \), there is an open neighborhood \( U \) of \( z \) (in \( M \)) such that \( f|_U : U \to Z \) is bounded. \( f \) is said to be locally bounded if it is so for \( N = M \). It is clear that, if \( Z = \mathbb{C} \), then the above notions of boundedness coincide with the usual ones.

### 2.3 Hartogs extension property.

We recall here the following notion (see, for example, Shiffman [29] and a result by Ivashkovich [7]). For \( 0 < r < 1 \), the Hartogs figure, denoted by \( H(r) \), is given by

\[
H(r) := \left\{ (z_1, z_2) \in E^2 : |z_1| < r \text{ or } |z_2| > 1 - r \right\},
\]

where, in this article, \( E \) always denotes the open unit disc of \( \mathbb{C} \).

**Definition 2.4.** A complex analytic space \( Z \) is said to possess the Hartogs extension property if every mapping \( f \in \mathcal{O}(H(r), Z) \) extends to a mapping \( \hat{f} \in \mathcal{O}(E^2, Z), r \in (0, 1) \).

We mention an important characterization due to Shiffman (see [29]).

**Theorem 2.5.** A complex analytic space \( Z \) possesses the Hartogs extension property if and only if for every subdomain \( D \) of any Stein manifold \( M \), every mapping \( f \in \mathcal{O}(D, Z) \) extends to a mapping \( \hat{f} \in \mathcal{O}(\hat{D}, Z) \), where \( \hat{D} \) is the envelope of holomorphy\(^{11}\) of \( D \).

In the light of this result, the natural “target spaces” \( Z \) for obtaining satisfactory answers to the PROBLEM are the complex analytic spaces satisfying the Hartogs extension property.

### 2.4 Statement of the main result

Recall that a subset \( S \) of a complex manifold \( M \) is said to be thin if for every point \( x \in M \) there are a connected neighborhood \( U = U(x) \subset M \) and a holomorphic function \( f \) on \( U \), not identically zero, such that \( U \cap S \subset f^{-1}(0) \). We are now ready to state our main result.

\(^{11}\) For the notion of the envelope of holomorphy, see, for example, [8].
Main Theorem. Let $X, Y$ be two complex manifolds, let $D \subset X$, $G \subset Y$ be two open sets, and let $A$ (resp. $B$) be a subset of $D$ (resp. $G$). $D$ (resp. $G$) is equipped with a system of approach regions $(\mathcal{A}_\alpha(\zeta))_{\zeta \in D, \alpha \in I_\zeta}$ (resp. $(\mathcal{A}_\beta(\eta))_{\eta \in G, \beta \in I_\eta}$).

Suppose in addition that $A = A^* = A^{**}$ and $B = B^* = B^{**}$ and that $\tilde{\omega}(\cdot, A, D) < 1$ on $D$ and $\tilde{\omega}(\cdot, B, G) < 1$ on $G$. Let $Z$ be a complex analytic space possessing the Hartogs extension property. Let $M$ be a relatively closed subset of $W$ with the following properties:

- $M$ is thin in fibers (resp. locally pluripolar in fibers) over $A$ and over $B$;

Then there exists a relatively closed analytic (resp. a relatively closed locally pluripolar) subset $\hat{M}$ of $\widehat{W}$, $\hat{M} \cap \tilde{W} \subset M$ and $\tilde{W} \setminus M \subset \operatorname{End}(\tilde{W} \setminus \hat{M})$, and for every mapping $f : W \setminus M \rightarrow Z$ satisfying the following conditions:

(i) $f \in \mathcal{C}_s(W \setminus M, Z) \cap \mathcal{O}_s(W^o \setminus M, Z)$;

(ii) $f$ is locally bounded along $\mathcal{X}(A \cap \partial D, B \cap \partial G; D, G) \setminus M$;

(iii) $f|_{(A \times B) \setminus M}$ is continuous at all points of $(A \cap \partial D) \times (B \cap \partial G)$,

there exists a unique mapping $\hat{f} \in \mathcal{O}(\widehat{W} \setminus \hat{M}, Z)$ which admits the $A$-limit $f(\zeta, \eta)$ at every point $(\zeta, \eta) \in \tilde{W} \setminus M$.

Although our main result has been stated only for the case of a 2-fold cross, it can be also formulated for the general case of an $N$-fold cross with $N \geq 2$. It remains an open question whether $\tilde{W}$ is the maximal extension region of $W$ for the family of mappings discussed in the Main Theorem (for a special case see [25]). Various applications of the Main Theorem will be given in Section 6 below. It is possible to obtain a generalization of the Main Theorem in the case where $M$ is not necessarily closed in $W$. Indeed, it suffices to make use of the works [12, 13] and combine them with our present method.

Before going further we say some words about the exposition of the paper. We only give the proof of the Main Theorem for the case where the set of singularities $M$ is locally pluripolar in fibers. It is therefore left to the interested reader to treat the case where $M$ is thin in fibers. On the other hand, as in any article

12 It is worthy to note that this assumption is not so restrictive since we know from Subsection 2.1 that $A \setminus A^*$ and $B \setminus B^*$ are locally pluripolar for arbitrary sets $A \subset D, B \subset G$.

13 Note that if $A \cap D = \emptyset$ and $B \cap G = \emptyset$, then this intersection is empty.

14 It follows from Subsection 2.2 that $\mathcal{X}(A \cap \partial D, B \cap \partial G; D, G) = (\{D \cup A\} \times (B \cap \partial G)) \cup ((A \cap \partial D) \times (G \cup B))$. 

8
of holomorphic extension, there are always two parts: describing the method of extension and justifying the gluing process. Since our primary aim is to make the article as compact as possible, we focus more on the way we extend the mappings than the gluing process. Throughout the paper, $Z$ always denotes a complex analytic space possessing the Hartogs extension property.

3 Auxiliary results

First we recall and prove some auxiliary results. From [10] we extract the following particular case of a general cross theorem with singularities which will be needed in the future.

**Theorem 3.1.** Let $X = \mathbb{C}^n$ and $Y = \mathbb{C}^m$, let $D \subset X$, $G \subset Y$ be two bounded domains, let $A \subset D$ and $B \subset G$ be non-pluripolar subsets. Let $M$ be a relatively closed subset of $W$ such that $M$ is pluripolar in fibers over $A$ and $B$.

Then there exists a relatively closed pluripolar set $\hat{M} \subset \hat{W}$ such that:

- $\hat{M} \cap W \cap \hat{W} \subset M$;
- for every mapping $f \in \mathcal{O}_s(W \setminus M, Z)$, there exists a unique mapping $\hat{f} \in \mathcal{O}(\hat{W} \setminus \hat{M}, Z)$ such that $\hat{f} = f$ on $(\hat{W} \cap W \setminus M)$.

**Proof.** The special case when $D$ and $G$ are pseudoconvex and $Z = \mathbb{C}$ has been proved in [10]. However, using a recent result in [16], the assumption that $D$ and $G$ are pseudoconvex can be removed. Now we treat the general case where $Z$ is a complex analytic space possessing the Hartogs extension property. Applying Theorem 3.3 below and using the hypothesis that $M$ is a relatively closed subset of $W$, we may obtain a local extension of $f$ defined on some open neighborhood of $\hat{W} \setminus M$. Finally, by applying Theorem 2.5, the desired conclusion of the theorem follows from its special case $Z = \mathbb{C}$ (see also [2]).

We also need the following version of Theorem 3.1 when $M$ is not necessarily closed in $W$.

**Theorem 3.2.** Let $X = \mathbb{C}^n$ and $Y = \mathbb{C}^m$, let $D_0 \subset D \subset X$, $G_0 \subset G \subset Y$ be four bounded domains, and let $A \subset D_0$ and $B \subset G_0$ be non-pluripolar subsets. Let $M$ be a subset of $W := \mathbb{R}(A, B; D, G)$ such that $M$ is relatively closed pluripolar in fibers over $A$ and $B$. Then there exist:

- pluripolar sets $P \subset A$, $Q \subset B$ such that the set $A_0 := A \setminus P$, $B_0 := B \setminus Q$ are locally pluriregular;
- a relatively closed pluripolar set $\hat{M} \subset \hat{W}$

such that:
\(\widehat{M} \cap \mathbb{X}(A_0, B_0; D, G) \subset M;\)

- for every mapping \(f \in \mathcal{O}(W \setminus M, Z) \cap \mathcal{O}(D_0 \times G_0, Z),\) there exists a unique mapping \(\hat{f} \in \mathcal{O}(\widehat{W} \setminus \widehat{M}, Z)\) such that \(\hat{f} = f\) on \(D_0 \times G_0.\)

**Proof.** The special case when \(D_0, D\) and \(G_0, G\) are pseudoconvex and \(Z = \mathbb{C}\) has been proved in Theorem 3.4 of [12]. However, using a recent result in [16], the assumption of pseudoconvexity can be removed. Finally, by applying Theorem 2.5, the desired conclusion of the theorem follows from its special case \(Z = \mathbb{C}.\) \(\square\)

The next result was proved by the first author in [17].

**Theorem 3.3.** We keep the hypotheses and notation of the Main Theorem. Suppose in addition that \(M = \emptyset.\) Then the conclusion of the Main Theorem holds for \(\widehat{M} = \emptyset.\)

The following result will play an important role in the sequel.

**Theorem 3.4 ([5]).** Let \(D \subset \mathbb{C}^n\) be a domain and let \(\widehat{D}\) be the envelope of holomorphy of \(D.\) Assume that \(S\) is a relatively closed pluripolar subset of \(D.\) Then there exists a relatively closed pluripolar subset \(\widehat{S}\) of \(\widehat{D}\) such that \(\widehat{S} \cap D \subset S\) and \(\widehat{D} \setminus \widehat{S}\) is the envelope of holomorphy of \(D \setminus S.\)

In this article, let mes denote the Lebesgue measure on the unit circle \(\partial E.\) Recall here the system of angular (or Stolz) approach regions for \(E\) (see, for example, [17]). Put

\[A_\alpha(\zeta) := \left\{ t \in E : \left| \arg \left( \frac{\zeta - t}{\zeta} \right) \right| < \alpha \right\}, \quad \zeta \in \partial E, \ 0 < \alpha < \frac{\pi}{2},\]

where \(\arg : \mathbb{C} \longrightarrow (-\pi, \pi]\) is as usual the argument function. \(A = (A_\alpha(\zeta))_{\zeta \in \partial E, \ 0 < \alpha < \frac{\pi}{2}}\) is referred to as the system of angular (or Stolz) approach regions for \(E.\) In this context \(A - \lim\) is also called angular limit.

For \(z \in \mathbb{C}^n\) and \(r > 0,\) let \(\Delta_z^r(r)\) denote the open polydisc centered at \(z\) with radius \(r.\) When \(n = 1,\) we will write for short \(\Delta_z(r)\) instead of \(\Delta^1_z(r).\)

Fix \(A \subset \partial E.\) For \(a \in \partial E\) and \(0 < \rho, \epsilon < 1,\) let

\[\Delta_a(\rho, \epsilon) := \Delta_a(\rho, \epsilon; A) := \left\{ z \in \Delta_a(\rho) \cap E : \omega(z, A \cap \Delta_a(\rho), \Delta_a(\rho) \cap E) < \epsilon \right\} .\]

It is worthy to remark that if \(a\) is a density point of \(A,\) then \(\Delta_a(\rho, \epsilon) \neq \emptyset.\)

The following result will be very useful.

**Proposition 3.5.** Let \(D = G = E\) and let \(A \subset \partial D\) be a measurable subset such that \(\text{mes}(A) > 0,\) and let \(B \subset G\) be an open set. Moreover, we assume that any point of \(A\) is a density point of \(A.\) Consider the cross \(W := \mathbb{X}(A, B; D, G).\) Let \(M\) be a relatively closed subset of \(W\) such that \(M_a\) is polar (resp. discrete) in...
There exists a relatively closed pluripolar subset $\hat{E}$ by the open set $E$. Theorem 3.7.

Remark 3.6. The previous proposition still holds if we replace the domain $D$ by the open set $D := \{z \in E : \omega(z, A, E) < \epsilon\}$ for some $0 < \epsilon < 1$.

The first main purpose of this section is to establish the following higher dimensional version of Theorem 3.5.

Theorem 3.7. Let $A$ be a measurable subset of $\partial E$ with $\text{mes}(A) > 0$ and let $r > 1$. Let $M$ be a relatively closed subset of $A \times \Delta^0(r)$ such that $M \cap (A \times E^n) = \emptyset$ and that $M_a := \{w \in \Delta^0(r) : (a, w) \in M\}$ is pluripolar for all $a \in A$. Then there exists a relatively closed pluripolar subset $\hat{M}$ of $\hat{X}(A, E^n; E, \Delta^0(r))$ with $\hat{M} \cap E^{n+1} = \emptyset$ and with the following additional property:

Let $f : \hat{X}(A, E^n; E, \Delta^0(r)) \setminus M \to \mathbb{C}$ be a locally bounded function such that

- for all $a \in A$, $f(a, \cdot)|_{\Delta^0(r) \setminus M_a}$ is holomorphic;
- for all $w \in E^n$, the function $f(\cdot, w)|_E$ is holomorphic and admits the angular limit $f(a, w)$ at all points $a \in A$.

Then there is a unique function $\hat{f} \in \mathcal{O}(\hat{X}(A, E^n; E, \Delta^0(r)) \setminus \hat{M}, \mathbb{C})$ which extends $f|_{E^{n+1}}$.

The second main purpose is to prove the following generalization of Theorem 3.7 where $D$ need not to be a disc.
Theorem 3.8. Let $X = \mathbb{C}^m$ and $Y = \mathbb{C}^n$, let $D \subset X$ be a bounded open set and $G = \Delta_0^r(r)$ for some $1 < r < \infty$, let $A$ be a subset of $\overline{D}$ with $A = A^*$, $\tilde{\omega}(\cdot, A, D) < 1$ on $D$, and let $B = E^n$. Moreover, $D$ is equipped with a system of approach regions $(A_\alpha(\zeta))_{\zeta \in \overline{D}, \alpha \in I(\zeta)}$ and $G$ is equipped with the canonical system of approach regions. Let $M$ be a closed subset of $W := \mathcal{X}(A, B; D, G)$ with the following properties:

- $M$ is locally pluripolar in fibers over $A$ and over $B$;
- $M \cap (A \times B) = \emptyset, M \cap (D \times B) = \emptyset$.

Then there exists a relatively closed locally pluripolar subset $\tilde{M}$ of $\tilde{W}$ with $\tilde{M} \cap (D \times B) = \emptyset$, $W \cap \tilde{W} \cap \tilde{M} \subset M$ and $(W \cap \tilde{W}) \setminus M \subset \text{End}(\tilde{W} \setminus \tilde{M})$ such that:

For every function $f : W \setminus M \to \mathbb{C}$ satisfying the following conditions:

- $f \in \mathcal{C}_s(W \setminus M, \mathbb{C}) \cap \mathcal{O}_s(W^o \setminus M, \mathbb{C})$;
- $f$ is locally bounded along $((A \cap \partial D) \times G) \setminus M$;

there exists a unique function $\hat{f} \in \mathcal{O}(\tilde{W} \setminus \tilde{M}, \mathbb{C})$ which admits the $A$-limit $f(\zeta, \eta)$ at every point $(\zeta, \eta) \in (W \cap \tilde{W}) \setminus M$.

In order to prove Theorem 3.7 and 3.8, our strategy is as follows. First, observe that Theorem 3.7 for the case $n = 1$ follows from Theorem 3.5. Next, we will show that Theorem 3.7 for a given $n$ implies Theorem 3.8 for this $n$. Finally, it suffices to show that Theorem 3.8 for $n = 1$ implies, in turn, Theorem 3.7 for arbitrary $n$.

To make the above strategy to work we will rely on the approach using holomorphic discs as it was done in [17]. For a bounded mapping $\phi \in \mathcal{O}(E, \mathbb{C}^n)$ and $\zeta \in \partial E$, $f(\zeta)$ denotes the angular limit value of $f$ at $\zeta$ if it exists. A classical theorem of Fatou says that $\text{mes} \left( \{ \zeta \in \partial E : \exists f(\zeta) \} \right) = 2\pi$.

Theorem 3.9. Let $D$ be a bounded open set in $\mathbb{C}^n$, $A \subset \overline{D}$, $z_0 \in D$ and $\epsilon > 0$. Let $\mathcal{A}$ be a system of approach regions for $D$. Suppose in addition that $A$ is locally pluriregular (relative to $\mathcal{A}$) and that $\omega(\cdot, A, D) < 1$ on $D$. Then there exist a bounded mapping $\phi \in \mathcal{O}(E, \mathbb{C}^n)$ and a measurable subset $\Gamma_0 \subset \partial E$ with the following properties:

1) Any point of $\Gamma_0$ is a density point of $\Gamma_0$, $\phi(0) = z_0$, $\phi(E) \subset \overline{D}$, $\Gamma_0 \subset \{ \zeta \in \partial E : \phi(\zeta) \in \overline{A} \}$, and

$$1 - \frac{1}{2\pi} \cdot \text{mes}(\Gamma_0) < \omega(z_0, A, D) + \epsilon.$$
2) Let \( f \in \mathcal{C}(D \cup \overline{A}, \mathbb{C}) \cap \mathcal{O}(D, \mathbb{C}) \) be such that \( f(D) \) is bounded. Then there exist a bounded function \( g \in \mathcal{O}(E, \mathbb{C}) \) such that \( g = f \circ \phi \) in a neighborhood of \( 0 \in E \) and \( g(\zeta) = (f \circ \phi)(\zeta) \) for all \( \zeta \in \Gamma_0 \). Moreover, \( g|_{\Gamma_0} \in \mathcal{C}(\Gamma_0, \mathbb{C}) \).

This theorem which has been proved in Theorem 3.8 of \([17]\) motivates the following

**Definition 3.10.** We keep the hypotheses and notation of Theorem 3.9. Then every pair \((\phi, \Gamma_0)\) satisfying the conclusions 1)–2) of this theorem is said to be an \( \epsilon \)-candidate for the triplet \((z, A, D)\).

Theorem 3.9 says that there always exist \( \epsilon \)-candidates for all triplets \((z, A, D)\).

The following result reduces the Main Theorem to local situations.

**Proposition 3.11.** We keep the hypotheses and notation of the Main Theorem.

1) Suppose in addition that the following property holds:

Let \( A_0 \) (resp. \( B_0 \)) be a subset of \( \overline{D} \) (resp. \( \overline{G} \)) such that \( A_0 \) and \( B_0 \) are locally pluriregular and that \( \overline{A}_0 \subset A \) and \( \overline{B}_0 \subset B \) and that \( \overline{A}_0, \overline{B}_0 \) are compact. Then there exists a relatively closed pluripolar subset \( \hat{M} \) of \( \hat{\mathcal{K}}(A_0, B_0; D, G) \) such that \( \mathcal{K}(A_0, B_0; D, G) \setminus M \subset \text{End}(\hat{\mathcal{K}}(A_0, B_0; D, G) \setminus \hat{M}) \) and that for every mapping \( f : W \to Z \) which satisfies conditions (i)–(iii) of the Main Theorem, there exists a mapping \( \hat{f} \) defined and holomorphic on \( \hat{\mathcal{K}}(A_0, B_0; D, G) \setminus \hat{M} \) which admits \( A \)-limit \( f(\zeta, \eta) \) at all points \((\zeta, \eta) \in \mathcal{K}(A_0, B_0; D, G) \setminus M\).

Then the conclusion of the Main Theorem holds.

2) Suppose that the property of Part 1) is satisfied for all \( B_0 \in \mathcal{F} \), where \( \mathcal{F} \subset \mathcal{E}(B) \) such that \( \hat{B} = \bigcup_{P \in \mathcal{F}} P \). Then the conclusion of the Main Theorem holds. Here we recall that

\[
\mathcal{E}(B) = \mathcal{E}(B, A) := \{ P \subset \overline{G} : P \text{ is locally pluriregular, } \overline{P} \subset B \}\.
\]

This result permits to pass from the relative extremal functions \( \omega(\cdot, A_0, D) \), where \( A_0 \in \mathcal{E}(A) \), to the plurisubharmonic measure \( \widetilde{\omega}(\cdot, A, D) \).

**Proof.** The case where \( M = \emptyset \) is treated by the first author in \([17]\) where he starts from Theorem 8.2 therein in order to prove the main theorem in Section 9 of that article. This method also works in the present context making the obviously necessary changes.

In the light of Part 2) of Proposition 3.11, Theorem 3.8 is reduced to prove the following result.

**Theorem 3.12.** Let \( X = \mathbb{C}^m \) and \( Y = \mathbb{C}^n \), let \( D \subset X \) be a bounded open set and \( G = \Delta_0^0(r) \) for some \( r > 1 \), let \( A \) be a subset of \( \overline{D} \) such that \( A = A^* \).

\(^{15}\) Note here that by Part 1), \((f \circ \phi)(\zeta)\) exists for all \( \zeta \in \Gamma_0 \).
\[ \tilde{\omega}(<1, A, D) < 1 \text{ on } D, \] and let \( B = E^n \). Moreover, \( D \) is equipped with a system of approach regions \( (A_\alpha(\zeta), \zeta \in \mathcal{A}, \alpha \in I_\zeta) \) and \( G \) is equipped with the canonical system of approach regions. Let \( A_0 \) be a subset of \( D \) such that \( A_0 \) is locally pluriregular and that \( \overline{A}_0 \subset A \). Put \( W_0 := X(A_0, B; D, G) \). Let \( M \) be a relatively closed subset of \( W \) with the following properties:

- \( M \) is locally pluripolar in fibers over \( A \);
- \( M \cap ((A \cup D) \times B) = \emptyset \).

Then there exists a relatively closed locally pluripolar subset \( \hat{M} \) of \( \hat{W}_0 \) with \( \hat{M} \cap (D \times B) = \emptyset \) and \( (W \cap W_0) \setminus M \subset \text{End}(\hat{W}_0 \setminus \hat{M}) \) such that:

- for every function \( f : W \setminus M \to \mathbb{C} \) satisfying the following conditions:
  - \( f \in \mathcal{O}_s(W \setminus M, \mathbb{C}) \cap \mathcal{O}_s(W^0 \setminus M, \mathbb{C}) \);
  - \( f \) is locally bounded along \( ((A \cap \partial D) \times G) \setminus M \);

there exists a unique function \( \hat{f} \in \mathcal{O}(\hat{W}_0 \setminus \hat{M}, \mathbb{C}) \) which admits the \( \mathcal{A} \)-limit \( f(\zeta, \eta) \) at every point \( (\zeta, \eta) \in (W \cap W_0) \setminus M \).

Proof. First we will find a subset \( \mathcal{M} \subset \hat{W}_0 \) such that

- \( \mathcal{M} \cap (D \times B) = \emptyset \);
- for all \( z \in D \), the vertical fibers \( \mathcal{M}_z := \{ w \in G : (z, w) \in \mathcal{M} \} \) are relatively closed pluripolar in \( (\hat{W}_0)_z := \{ w \in G : (z, w) \in \hat{W}_0 \} \);
- \( f|_{D \times B} \) extends to \( \hat{f} \) which is well-defined on \( \hat{W}_0 \setminus \mathcal{M} \) and which satisfies \( \hat{f}(z, \cdot) \in \mathcal{O}(\hat{W}_0 \setminus \mathcal{M}_z, \mathbb{C}) \) for all \( z \in D \), where

\[
(\hat{W}_0 \setminus \mathcal{M})_z := \{ w \in G : (z, w) \in \hat{W}_0 \setminus \mathcal{M} \}.
\]

To this end fix a \( z_0 \in D \). We want to construct the vertical fiber \( \mathcal{M}_{z_0} \). Take an arbitrary \( \epsilon > 0 \) such that

\[
\omega(z_0, A_0, D) + \epsilon < 1.
\]

By Theorem [3.9] and Definition [3.10] there is an \( \epsilon \)-candidate \( (\phi, \Gamma) \) for \( (z_0, A_0, D) \). By shrinking \( \Gamma \) using Lusin’s theorem, we may assume without loss of generality that \( \phi|_{\Gamma} \) is continuous. Moreover, using the hypotheses on \( M \) and on \( f \), we see that the function \( f_\phi \), defined by

\[
f_\phi(t, w) := f(\phi(t), w), \quad (t, w) \in X(\Gamma, B; E, G) \setminus M_\phi,
\]

satisfies the hypotheses of Theorem [3.7] where

\[
M_\phi := \{ (t, w) \in \Gamma \times G : (\phi(t), w) \in M \}.
\]
By this theorem, let \( \hat{M}_\phi \) be the relatively closed pluripolar subset of \( \hat{\mathcal{X}}(\Gamma, B; E, G) \) with \( \hat{M}_\phi \cap (E \times B) = \emptyset \) and let \( \hat{f}_\phi \in \mathcal{O}(\hat{\mathcal{X}}(\Gamma, B; E, G) \setminus \hat{M}_\phi, \mathbb{C}) \) be such that

\[
(A - \lim \hat{f}_\phi)(t, w) = f_\phi(t, w), \quad (t, w) \in \hat{\mathcal{X}}(\Gamma, B; E, G) \setminus M_\phi.
\]

Using the above discussion we will define \( \mathcal{M}_{z_0} \) and the desired extension function \( \hat{f}(z_0, \cdot) \) on \( (\hat{W}_0 \setminus \mathcal{M})_{z_0} \) as follows:

fix a point \( (z_0, w_0) \in \hat{W} \) and an \( \epsilon > 0 \) such that

\[
\omega(z_0, A_0, D) + \omega(w_0, B, G) + \epsilon < 1
\]

and there exists an \( \epsilon \)-candidate \( (\phi, \Gamma) \) for \( (z_0, A, D) \) with \( (0, w_0) \in \hat{\mathcal{X}}(\Gamma, B; E, G) \setminus \hat{M}_\phi \). Then the value of \( \hat{f} \) at \( (z_0, w_0) \) is, by our definition, given as

\[
\hat{f}(z_0, w_0) := \hat{f}_\phi(0, w_0),
\]

where \( \hat{f}_\phi \) is defined in \((2) - (3)\). On the other hand, put

\[
M_{z_0} := \{ w \in G : \forall \phi \text{ as above}, (0, w) \in \hat{M}_\phi \}.
\]

Using Lemma 4.5 in \[17\] it can be checked that \( \hat{f}(z_0, \cdot) \) is well-defined on \( (\hat{W}_0 \setminus \mathcal{M})_{z_0} \). Moreover, it is easy to see that \( \mathcal{M} \cap (D \times B) = \emptyset \) and that all vertical fibers \( \mathcal{M}_z \) with \( z \in D \) are relatively closed pluripolar. This completes the construction of \( \mathcal{M} \subset \hat{W}_0 \) and of \( \hat{f} \) on \( \hat{W}_0 \setminus \mathcal{M} \).

For all \( 0 < \delta < \frac{1}{2} \) let

\[
A_\delta := \{ z \in D : \omega(z, A_0, D) < \delta \} \quad \text{and} \quad G_\delta := \{ w \in G : \omega(w, B, G) < 1 - \delta \}.
\]

We are able to define a new function \( \tilde{f}_\delta \) on \( \mathcal{X}(A_\delta, B; D, G_\delta) \setminus \mathcal{M} \) as follows

\[
\tilde{f}_\delta(z, w) := \begin{cases} 
\hat{f}(z, w) & (z, w) \in (A_\delta \times G_\delta) \setminus \mathcal{M}, \\
\tilde{f}(z, w) & (z, w) \in D \times B.
\end{cases}
\]

Using the hypotheses on \( f \) and the previous paragraph, we see that

\[
\tilde{f}_\delta \in \mathcal{O}(\mathcal{X}(A_\delta, B; D, G_\delta) \setminus \mathcal{M}, \mathbb{C}).
\]

Observe that \( A_\delta \) is an open set in \( D \), all vertical fibers \( \mathcal{M}_z \) with \( z \in D \) are relatively closed pluripolar and all horizontal fibers \( \mathcal{M}^w \) with \( w \in B \) are empty, and \( \tilde{f}_\delta|_{D \times B} \) is holomorphic. Consequently, \( \tilde{f}_\delta \) satisfies the hypotheses of Theorem 3.2 for \( D_0 := D, G_0 := B, A := A_\delta, B := B, D := D, G := G_\delta \), where the left sides of the above assignments are the notation of Theorem 3.2. Applying this
Remark 3.6 to the function \( f \) where \( \text{mes} \) denotes the linear measure on \( \partial E \).

Now fix a sequence \( (\delta_r) \) to show that these extended functions. On the other hand, it follows from (5) that theorem yields a relatively closed pluripolar subset \( \hat{\mathcal{A}} \) to \( (\delta_r) \). Fix a point \( a \).

Proof. \( \text{Theorem } 3.7 \) extends holomorphically to \( \hat{\mathcal{A}} \). Let \( \hat{W}_0 \) denote the set of all \( (\delta_r) \) such that every function \( \hat{\mathcal{W}} \) and a relatively closed pluripolar set \( \hat{\mathcal{A}} \) exists with the above properties. Note that \( \hat{\mathcal{A}} \) may be taken as singular with respect to all these extended functions. On the other hand, it follows from [3] that

\[
\hat{W}_0 = \hat{\mathcal{X}}(A_0 \times G_0) = \bigcup_{0 < \delta < 1} A_0 \times G_0.
\]

Now fix a sequence \( (\delta_k) \) such that \( 0 < \delta_k < 1 \) and \( \delta_k \setminus 0^+ \). Then, using the last equality, we may glue \( \hat{\mathcal{A}} \) together in order to obtain a relatively closed pluripolar subset \( \hat{\mathcal{M}} \) of \( \hat{\mathcal{W}}_0 \) and an extension function \( \hat{f} \in \hat{\mathcal{O}}(\hat{W}_0 \setminus \hat{\mathcal{M}}, \mathbb{C}) \) with the desired properties of the theorem.

Prior to the proof of Theorem 3.7 for all \( n \) we make some preparation. Under the hypotheses and notation of Theorem 3.7 we establish the following

**Proposition 3.13.** Let \( A \) be a measurable subset of \( \partial E \) with \( \text{mes}(A) > 0 \). Then, for every density point \( a_0 \in A \) and every \( r' \in (1, r) \), there exist \( 0 < \rho = \rho_{r'}, \epsilon = \epsilon_{r'}, \rho_{r'}, \epsilon_{r'} < 1 \) and a relatively closed pluripolar set \( T = T_{r'} \subset \Delta_{a_0}(\rho, \epsilon) \times \Delta_0^0(r') \) with \( T \cap (\Delta_{a_0}(\rho, \epsilon) \times E^n) = \emptyset \) such that every function \( f \) satisfying the hypotheses of Theorem 3.7 extends holomorphically to \( (\Delta_{a_0}(\rho, \epsilon) \times \Delta_0^0(r')) \setminus T \).

In other words, this proposition says that some local extensions are possible.

**Proof.** Fix a point \( a_0 \) of \( A \) and let \( r_0' \) be the supremum of all \( r' \in (0, r) \) such that \( \rho_{r'}, \epsilon_{r'}, \rho_{r'}, \epsilon_{r'} \), and \( T_{r'} \) exist with the above properties. Note that \( 1 \leq r_0' \leq r \). It suffices to show that \( r_0' = r \).

Suppose that \( r_0' < r \). Fix \( r_0' < r'' < r \) and choose \( r'' \in (0, r_0') \) such that \( \sqrt[r]{r''^m} > r_0' \). Let \( \rho := \rho_{r''}, \epsilon := \epsilon_{r''}, \) and \( S := T_{r''} \).

Write \( w = (w', w_n) \in \mathbb{C}^n = \mathbb{C}^{n-1} \times \mathbb{C} \). Let \( C \) denote the set of all \( (a, b') \in (A \cap \Delta_{a_0}(\rho)) \times \Delta_0^{n-1}(r'') \) such that the fiber \( M_{(a, b', \cdot)} \) is polar.

\[
F := \left\{ w' \in \Delta_0^{n-1}(r') : \text{mes}\left( \{ z \in A : (z, w') \notin C \} \right) > 0 \right\},
\]

where \( \text{mes} \) denotes the linear measure on \( \partial E \). For every \( w' \in \Delta_0^{n-1}(r') \setminus F \), we have that \( \text{mes}(A \setminus C_{w'}) = 0 \), where \( C_{w'} := \{ z \in A : (z, w') \in C \} \). Applying Remark 3.6 to the function \( f(\cdot, w', \cdot) \) restricted to the cross

\[
Y_{w'} := \mathbb{X}(C_{w'}, \Delta_0(r'); \Delta_{a_0}(\rho, \epsilon), \Delta_0(r)) \setminus (M \cup S)
\]
we conclude that there exists a closed pluripolar set $T_{w'} \subset \widehat{Y}_{w'}$ such that $T_{w'} \cap \Delta_{a_0}(\rho, \epsilon) \times \Delta_0(r') \subset S$ and every function $f$ satisfying the hypotheses of Theorem 3.12 extends holomorphically to a function $\hat{f}_{w'}$ defined on $\widehat{Y}_{w'} \setminus T_{w'}$.

Next, we will argue as in the proof of Theorem 3.12. More precisely, for all $0 < \delta < \frac{1}{2}$ let

$A_\delta := \{(z, w') \in \Delta_{a_0}(\rho, \epsilon) \times \Delta_0^{n-1}(r') : \omega((z, w'), C, \Delta_{a_0}(\rho, \epsilon) \times \Delta_0^{n-1}(r')) < \delta\},$

$G_\delta := \{w_n \in \Delta_0(r) : \omega(w_n, \Delta_0(r'), \Delta_0(r)) < 1 - \delta\}.$

We define a set $\mathcal{M}$ as follows:

$\mathcal{M}(z, w') := \{(T_{w'})_n := \{w_n \in \Delta_0(r) : (z, w_n) \in T_{w'}, \Delta_0(r)\} : w'_n \in \Delta_0^{n-1}(r') \setminus F, \ w' \in F.$

Moreover, we define a new function $\bar{f}_\delta$ on

$\mathbb{X}\left(A_\delta, \Delta_0(r'); \Delta_{a_0}(\rho, \epsilon) \times \Delta_0^{n-1}(r'), G_\delta\right) \setminus \mathcal{M}$

as follows

$\bar{f}_\delta(z, w) := \begin{cases} \hat{f}(z, w) & (z, w) \in (A_\delta \times G_\delta) \setminus \mathcal{M}, \\ f(z, w) & (z, w) \in (\Delta_{a_0}(\rho, \epsilon) \times \Delta_0^{n-1}(r') \times \Delta_0(r')) \setminus \mathcal{M}. \end{cases}$

Using the hypotheses on $f$ and the previous paragraph, we see that

$\bar{f}_\delta \in \mathcal{O}_s\left(\mathbb{X}\left(A_\delta, \Delta_0(r'); \Delta_{a_0}(\rho, \epsilon) \times \Delta_0^{n-1}(r'), G_\delta\right) \setminus \mathcal{M}, \mathbb{C}\right).$

Arguing as in the proof of Theorem 3.12 we can show that there exists a relatively closed pluripolar subset $T_n$ of $\widehat{Y}_n$ such that every function $f$ as in the hypothesis extends holomorphically to a function $f_n$ defined on $\widehat{Y}_n \setminus T_n$, where

$Y_n := \mathbb{X}(C, \Delta_0(r'); \Delta_{a_0}(\rho, \epsilon) \times \Delta_0^{n-1}(r'), \Delta_0(r)).$

In order to calculate $\widehat{Y}_n$ we need the following lemma.

**Lemma 3.14.** For $(z, w') \in \Delta_{a_0}(\rho, \epsilon) \times \Delta_0^{n-1}(r')$,

$\omega((z, w'), C, \Delta_{a_0}(\rho, \epsilon) \times \Delta_0^{n-1}(r')) = \max\left\{\frac{1}{\epsilon} \cdot \omega(z, A \cap \Delta_{a_0}(\rho), \Delta_{a_0}(\rho)), \omega(w', \Delta_0^{n-1}(r'), \Delta_0^{n-1}(r'))\right\}.$
Proof. Using Proposition 5.2 in [17] we may assume without loss of generality that \( \epsilon = 1 \). Observe that the \((2n-1)\)-dimensional Lebesgue measure of

\[
\left( (A \cap \Delta_{a_0}(\rho)) \times \Delta_0^{n-1}(r') \right) \setminus C
\]

is zero and that the set \((A \cap \Delta_{a_0}(\rho)) \times \Delta_0^{n-1}(r')\) is living on the boundary of the smooth hypersurface \( \partial E \times \Delta_0^{n-1}(r') \) in \( \mathbb{C}^n \). Consequently, in the desired equality we may suppose that \( C = (A \cap \Delta_{a_0}(\rho)) \times \Delta_0^{n-1}(r') \). Then the equality follows easily from the product property for the extremal function.

We come back to the proof of the proposition. Using the above lemma, we get

\[
\hat{V}_n = \{(z, w', w_n) \in \Delta_{a_0}(\rho, \epsilon) \times \Delta_0^{n-1}(r') \times \Delta_0(r) : \omega((z, w'), C, \Delta_{a_0}(\rho, \epsilon) \times \Delta_0^{n-1}(r')) + \omega(w_n, \Delta_0(r'), \Delta_0(r)) < 1 \} = \{(z, w', w_n) \in \Delta_{a_0}(\rho) \times \Delta_0^{n-1}(r') \times \Delta_0(r) : \omega((z, w'), (A \cap \Delta_{a_0}(\rho)) \times \Delta_0^{n-1}(r'), \Delta_{a_0}(\rho, \epsilon) \times \Delta_0^{n-1}(r')) + \omega(w_n, \Delta_0(r'), \Delta_0(r)) < 1 \}
\]

\[
= \{(z, w', w_n) \in \Delta_{a_0}(\rho) \times \Delta_0^{n-1}(r') \times \Delta_0(r) : \max \{ \frac{1}{\epsilon} \cdot \omega(z, A \cap \Delta_{a_0}(\rho), \Delta_{a_0}(\rho)) , \omega(w', \Delta_0^{n-1}(r'), \Delta_0^{n-1}(r')) \} + \omega(w_n, \Delta_0(r'), \Delta_0(r)) < 1 \}
\]

Since \( r'' < r \), we may find an \( \rho_n \in (0, \rho] \) such that every function \( f \) in the hypotheses of Theorem 3.7 extends holomorphically to a function \( \hat{f}_n \) defined on \( \Delta_{a_0}(\rho_n) \times \Delta_0^{n-1}(r') \times \Delta_0(r'') \setminus T_n \). We may assume that \( T_n \) is singular with respect to the family \( \{ \hat{f}_n : f \text{ as in the hypotheses of Theorem 3.7} \} \).

Repeating the above argument for the coordinates \( w_\nu, \nu = 1, \ldots, n - 1 \), and gluing the obtained sets, we find an \( \rho_0 \in (0, \rho], \epsilon_0 \in (0, \epsilon] \) and a relatively closed pluripolar set \( T_0 := \bigcup_{j=1}^n T_j \) such that every function \( f \) as in the hypotheses of Theorem 3.7 extends holomorphically to a function \( \hat{f}_0 := \bigcup_{j=1}^n \hat{f}_j \) holomorphic in \( \Delta_{a_0}(r_0, \epsilon_0) \times \Omega \setminus T_0 \), where

\[
\Omega := \bigcup_{j=1}^n \Delta_0^{n-1}(r') \times \Delta_0(r'') \times \Delta_0^{n-j}(r').
\]

Let \( \hat{\Omega} \) denote the envelope of holomorphy of \( \Omega \). Applying Theorem 3.4, we find a relatively closed pluripolar subset \( T \) of \( \Delta_{a_0}(\rho_0, \epsilon_0) \times \hat{\Omega} \) such that every function \( f \) as in the hypotheses of Theorem 3.7 extends to a function \( \hat{f} \) holomorphic on \( (\Delta_{a_0}(\rho_0, \epsilon_0) \times \hat{\Omega}) \setminus T \). Let \( r''' := \sqrt[n]{r''^{n-1}r''} \). Observe that \( \Delta_0(r''') \subset \hat{\Omega} \). Recall that \( r''' > r''_0 \). We may assume that \( T \) is singular with respect to the family \( \{ \hat{f} : f \text{ as in the hypotheses of Theorem 3.7} \} \). Hence, the proof is finished. \( \square \)
Now we are in the position to show that Theorem 3.8 for \( n = 1 \) implies Theorem 3.7.

**Proof of Theorem 3.7.** Suppose without loss of generality that all points of \( A \) are density points of \( A \). Using a classical exhaustion argument it suffices to prove the following

**Assertion.** For every compact set \( A_0 \subset A \) and every \( r' \in (1, r) \), there exist \( 0 < \rho = \rho' < 1 \) and a relatively closed pluripolar set \( T = T' \subset \hat{X}(A_0, E^n; E, \Delta_0^n(r')) \) such that every function \( f \) satisfying the hypotheses of Theorem 3.7 extends holomorphically to \( \hat{X}(A_0, E^n; E, \Delta_0^n(r')) \setminus T \).

Now fix a compact set \( A_0 \subset A \) and an \( r' \in (1, r) \). Applying Proposition 3.13 to all points of \( A_0 \) and using the compactness of \( A_0 \), we may find \( k \) points \( a_1, \ldots, a_k \subset A \) and \( 2k \) numbers \( 0 < \rho_1, \epsilon_1, \ldots, \rho_k, \epsilon_k < 1 \) and a relatively closed pluripolar set \( T' \subset \Omega \times \Delta_0^n(r') \) with \( \Omega := \bigcup_{j=1}^k \Delta_{a_j}(\rho_j, \epsilon_j) \) such that

- \( A_0 \subset \bigcup_{j=1}^k \Delta_{a_j}(\rho_j) \)
- every function \( f \) satisfying the hypotheses of Theorem 3.7 extends holomorphically to \( \Omega \times \Delta_0^n(r') \setminus T' \).

We are able to define a new function \( \tilde{f} \) on \( \hat{X}(\Omega, E^n; E, \Delta_0^n(r')) \setminus T' \) as follows

\[
\tilde{f}_{\delta}(z, w) := \begin{cases} 
\hat{f}(z, w) & (z, w) \in (\Omega \times \Delta_0^n(r')) \setminus T', \\
f(z, w) & (z, w) \in (E \times E^n).
\end{cases}
\]  

(7)

Using the hypotheses on \( f \) and the previous argument, we see that

\[ \tilde{f}_{\delta} \in \mathcal{G}(\hat{X}(\Omega, E^n; E, \Delta_0^n(r')) \setminus T', \mathbb{C}) \].

Consequently, \( \tilde{f} \) satisfies the hypotheses of Theorem 3.2. Applying this theorem yields a relatively closed pluripolar subset \( T \) of \( \hat{X}(\Omega, E^n; E, \Delta_0^n(r')) \) with \( T \cap (E \times E^n) = \emptyset \) and a function \( \hat{f} \in \mathcal{G}(\hat{X}(\Omega, E^n; E, \Delta_0^n(r')) \setminus T, \mathbb{C}) \) such that \( \hat{f} = f \) on \( E \times E^n \). Using the above-listed properties of \( a_1, \ldots, a_k \), we see that

\[ \hat{X}(A_0, E^n; E, \Delta_0^n(r')) \subset \hat{X}(\Omega, E^n; E, \Delta_0^n(r')) \].

This proves the above assertion, and thereby completes the theorem. \( \square \)

4 Using holomorphic discs

In this section we combine Poletsky’s theory of discs [26, 27], Rosay’s Theorem on holomorphic discs [28] and Theorem 3.1.
Let us recall some facts from Poletsky’s theory of discs. For a complex manifold $M$, let $\mathcal{O}(E, M)$ denote the set of all holomorphic mappings $\phi : E \to M$ which extend holomorphically to a neighborhood of $E$. Such a mapping $\phi$ is called a holomorphic disc on $M$. Moreover, for a subset $A$ of $M$, let

$$1_A(z) := \begin{cases} 1, & z \in A, \\ 0, & z \in M \setminus A. \end{cases}$$

In the work [28] Rosay proved the following result.

**Theorem 4.1.** Let $u$ be an upper semicontinuous function on a complex manifold $M$. Then the Poisson functional of $u$ defined by

$$\mathcal{P}[u](z) := \inf \left\{ \frac{1}{2\pi} \int_{0}^{2\pi} u(\phi(e^{i\theta}))d\theta : \phi \in \mathcal{O}(E, M), \phi(0) = z \right\},$$

is plurisubharmonic on $M$.

This implies the following important consequence (see, for example, Proposition 3.4 in [16]).

**Corollary 4.2.** Let $M$ be a complex manifold equipped with the canonical system of approach regions and $A$ a nonempty open subset of $M$. Then $\omega(z, A, M) = \mathcal{P}[1_{M \setminus A}](z), z \in M$.

The main result of this section is

**Theorem 4.3.** Let $X, Y$ be two complex manifolds, let $D \subset X, G \subset Y$ be two connected open sets, let $A \subset D, B \subset G$ be two non-empty open subsets. Let $M$ be a relatively closed subset of $W := X(A, B; D, G)$ such that $M$ is locally pluripolar in fibers over $A$ and over $B$. Then there exists a relatively closed locally pluripolar subset $\widehat{M}$ of $\widehat{W}$ such that $\widehat{M} \cap W \subset M$ and that for every mapping $f \in \mathcal{O}_*(W \setminus M, Z)$, there exists a unique mapping $\hat{f} \in \mathcal{O}(\widehat{W} \setminus \widehat{M}, Z)$ such that $\hat{f} = f$ on $W \setminus M$.

**Proof.** First shall prove the following weaker version of Theorem 4.3.

**Assertion.** For every $(z_0, w_0) \in \widehat{W}$, there are a connected open neighborhood $U \times V$ of $(z_0, w_0)$ in $\widehat{W}$ and a relatively closed locally pluripolar subset $S$ of $U \times V$ and a mapping $\hat{f} \in \mathcal{O}((U \times V) \setminus S, Z)$ with $U \cap A \neq \emptyset \neq V \cap B$ such that $\hat{f} = f$ on $W \setminus (M \cup S)$.

Taking for granted this assertion, the theorem follows immediately from a routine gluing process. Now we shall present the proof of the assertion. Applying
Theorem 4.1 and Corollary 4.2, we may find \( \phi \in \mathcal{O}(E, D) \) and \( \psi \in \mathcal{O}(E, G) \) such that

\[
\phi(0) = z_0, \quad \psi(0) = w_0, \quad \frac{1}{2\pi} \left( \int_0^{2\pi} 1_{D \setminus A}(\phi(e^{i\theta}))d\theta + \int_0^{2\pi} 1_{G \setminus B}(\psi(e^{i\theta}))d\theta \right) < 1.
\]

(8)

Recall some ideas in the work of Rosay [28, p. 166]. First we construct some kind of Hartogs figure \( H \) with gaps which contains the image by a holomorphic embedding of a neighborhood \( \{ \phi(t) : t \in \mathbb{E} \} \) in \( D \) such that \( H \) is contained in a complex manifold \( D' \) spreading over \( \mathbb{C}^2 \times D \). Composing with the projection \( \Pi : \mathbb{C}^2 \times D \to D \), we get a natural map \( \tilde{\Pi} : D' \to D \). Let \( U' \) be a Stein neighborhood of \( H \) in \( D' \). There exists a proper holomorphic embedding \( \tau \) of \( U' \) into \( \mathbb{C}^\mu \) (for some \( \mu \)), and \( \tau(U') \) has a neighborhood \( \tilde{U} \) in \( \mathbb{C}^\mu \) with a holomorphic retract \( r \) from \( \tilde{U} \) onto \( \tau(U') \). Consequently, setting \( \Phi := \tilde{\Pi} \circ \tau^{-1} \circ r \) we obtain a surjective mapping \( \Phi \in \mathcal{O}(\tilde{U}, U) \), where \( U \) is a connected open neighborhood of \( \{ \phi(t) : t \in \mathbb{E} \} \) in \( D \). Analogously, we may find a connected open neighborhood \( V \) of \( \{ \psi(t) : t \in \mathbb{E} \} \) in \( G \), an open subset \( \tilde{V} \) in \( \mathbb{C}^\nu \) (for some \( \nu \)) and a surjective mapping \( \Psi \in \mathcal{O}(\tilde{V}, V) \).

Next, consider the cross

\[
\mathcal{W} := \mathbb{X}(\Phi^{-1}(A), \Psi^{-1}(B); \tilde{U}, \tilde{V})
\]

and the set

\[
\mathcal{M} := \left\{ (x, y) \in \tilde{U} \times \tilde{V} : (\Phi(x), \Psi(y)) \in M \right\}.
\]

Since \( \Phi \) and \( \Psi \) are surjective, it is clear that \( \mathcal{M} \) is a relatively closed subset of \( \mathcal{W} \) and that \( \mathcal{M} \) is locally pluripolar in fibers over \( \Phi^{-1}(A) \) and \( \Psi^{-1}(B) \). Now consider the mapping \( F : \mathcal{W} \setminus \mathcal{M} \to Z \) defined by

\[
F(x, y) := f(\Phi(x), \Psi(y)), \quad (x, y) \in \mathcal{W} \setminus \mathcal{M}.
\]

Using the hypotheses of the theorem, we are able to apply Theorem 3.1 to \( F \). Consequently, we obtain a relatively closed locally pluripolar subset \( \tilde{\mathcal{M}} \) of \( \tilde{\mathcal{W}} \) and a mapping \( \tilde{F} \in \mathcal{O}(\tilde{\mathcal{W}} \setminus \tilde{\mathcal{M}}, Z) \) such that \( \tilde{\mathcal{W}} \cap \tilde{\mathcal{M}} \subset \mathcal{M} \) and \( \tilde{F} = F \) on \( \tilde{\mathcal{W}} \setminus \tilde{\mathcal{M}} \).

Let \( \mathcal{C} \) be the set of critical points of \( (x, y) \mapsto (\Phi(x), \Psi(y)) \). This is a proper analytic subset of \( \tilde{U} \times \tilde{V} \) since \( \Phi \) and \( \Psi \) are surjective (using Sard Theorem). Now define the set

\[
\mathcal{C} := \left\{ (\Phi(x), \Psi(y)) : (x, y) \in \tilde{C} \right\}.
\]

It is not difficult to show that \( \mathcal{C} \) is relatively closed and is contained in a proper analytic subset of \( U \times V \).

Using the above formula for \( F \) we see that

\[
F(x, y) = F(x', y'), \quad \forall (x, y), (x', y') \in \mathcal{W} \setminus \mathcal{M} : \Phi(x) = \Phi(x'), \; \Psi(y) = \Psi(y').
\]
Consequently, using the Uniqueness Principle we can show that
\[ \hat{F}(x, y) = \hat{F}(x', y'), \quad \forall (x, y), (x', y') \in \hat{W} \setminus \hat{M} : \Phi(x) = \Phi(x'), \; \Psi(y) = \Psi(y'). \]

Therefore, we can define the mapping
\[ \hat{f}(z, w) := \hat{F}(\Phi^{-1}(z), \Psi^{-1}(w)), \quad (z, w) \in (\Phi, \Psi)(\hat{W} \setminus \hat{M}) \setminus S. \]

Since \( \Phi \) and \( \Psi \) look like fibrations outside \( C \), \( \hat{f} \) is holomorphic. Now letting
\[ S := C \cup (\Phi, \Psi)(\hat{M}), \]
we see that \( S \) is locally pluripolar in \( \mathcal{U} \times \mathcal{V} \). Using \([8]\) we may choose a connected open neighborhood \( U \times V \) of \( (z_0, w_0) \) in \( (\mathcal{U} \times \mathcal{V}) \cap \hat{W} \) with the required properties of the assertion. This completes the proof. \( \square \)

The following result is an immediate consequence of the above theorem.

**Corollary 4.4.** Theorem 3.8 still holds in the following general settings: \( Y \) is an arbitrary complex manifold and \( G \subset Y \) is an arbitrary domain and \( B \subset G \) is an arbitrary open subset.

**Proof.** Observe that the proof of Theorem 3.8 still works if in the original hypothesis of the latter theorem we only change the following: \( B \) is a polydisc, that is, \( B \) is not necessarily centered at the center of the polydisc \( G \). The second step will be to require that \( B \) is (only!) an open subset of the polydisc \( G \). For this step we should apply Theorem 1.3 in order to obtain local extensions. Then by a routine patching process, we may obtain the global extension from the local ones. The last step will be to require simply that \( B \) is an open set of an arbitrary domain \( G \). In fact, this step is reduced to the second one by using parameterized families of holomorphic discs (see Lemma 3.2 in [16]). \( \square \)

**Corollary 4.5.** Theorem 3.8 still holds in the following general settings: \( Y \) is an arbitrary complex manifold, \( G \subset Y \) is an arbitrary domain, \( B \subset G \) is an arbitrary open subset, \( M \) is not necessarily relatively closed in \( W \) but we require instead that \( A \subset D \).

**Proof.** Arguing as in the proof of Corollary 4.4, we only need to prove Theorem 3.8 when \( A \subset D \) and \( M \) is not necessarily relatively closed in \( W \). But this follows readily from Theorem 3.1. \( \square \)

### 5 Proof of the Main Theorem

First we will prove the following local version of the Main Theorem.
Theorem 5.1. We keep the hypotheses and notation of the Main Theorem. Let $A_0$ (resp. $B_0$) be a subset of $\overline{D}$ (resp. $\overline{G}$) such that $A_0$ and $B_0$ are locally pluriregular and that $\overline{A_0} \subset A$ and $\overline{B_0} \subset B$. Then for every $(a,b) \in A_0 \times B_0$, there exist an open neighborhood $U$ of $a$ in $X$, an open neighborhood $V$ of $b$ in $Y$, and a relatively closed locally pluripolar subset $\hat{M} = \hat{M}_{(a,b)}$ of $\widehat{X}(A_0 \cap U, B_0 \cap V; D, G \cap V)$ such that:

- $X(A_0 \cap U, B_0 \cap V; D, G \cap V) \setminus M \subset \text{End}(\widehat{X}(A_0 \cap U, B_0 \cap V; D, G \cap V) \setminus \hat{M})$;
- $X(A_0 \cap U, B_0 \cap V; D, G \cap V) \cap \hat{M} \subset M$;
- for every mapping $f : W \setminus M \to Z$ satisfying conditions (v)–(iii) of the Main Theorem, there exists a unique mapping $\hat{f} \in \mathcal{O}(\widehat{X}(A_0 \cap U, B_0 \cap V; D, G \cap V) \setminus \hat{M}, Z)$ such that $\hat{f}$ admits the $\mathcal{A}$-limit $f(\zeta, \eta)$ at every point $(\zeta, \eta) \in X(A_0 \cap U, B_0 \cap V; D, G \cap V) \setminus M$.

Proof. There are two cases to consider.

Case $(a,b) \notin D \times G$.

Invoking the hypothesis on $M$ we see there exist an open neighborhood $U$ of $a$ in $X$ and an open neighborhood $V$ of $b$ in $Y$ such that

$$X(A \cap U, B \cap V; D \cap U, G \cap V) \cap M = \emptyset.$$ 

Moreover, we may assume without loss of generality that $U$ and $V$ are biholomorphic to some bounded Euclidean domains. Using this we are able to apply Theorem 3.3 to $f$ restricted to $X(A \cap U, B \cap V; D \cap U, G \cap V)$. Consequently, we obtain a mapping $\hat{f}_0 \in \mathcal{O}(\widehat{X}(A \cap U, B \cap V; D \cap U, G \cap V), Z)$ which extends $f$. For $0 < \delta < 1$ let

$$A_\delta := \{ z \in U \cap D : \omega(z, A_0 \cap U, D \cap U) < \delta \},$$
$$V_\delta := \{ w \in V \cap G : \omega(w, B_0 \cap V, G \cap V) < 1 - \delta \},$$
$$B_\delta := V_{1-\delta}.$$ 

Now let $0 < \delta < \frac{1}{2}$. Then by the above discussion $\hat{f}_0$ is holomorphic on $A_\delta \times V_\delta$. On the other hand, by the hypotheses $f(\cdot, w)$ is holomorphic on $D \setminus M^w$ for all $w \in B \cap V$. Therefore, we are in the position to apply Corollary 4.4 to the following mapping $f_\delta : X(A_\delta, B \cap V; D, V_\delta) \setminus M \to Z$ given by

$$f_\delta(z, w) := \begin{cases} 
\hat{f}_0(z, w), & (z, w) \in A_\delta \times V_\delta, \\
f(z, w), & (z, w) \in (D \times (B \cap V)) \setminus M.
\end{cases}$$

Consequently, we obtain a relatively closed locally pluripolar subset $\hat{M}_\delta$ of

$$\hat{X}(A_\delta, B_0 \cap V; D, V_\delta)$$

23
and a mapping
\[ \hat{f}_\delta \in \mathcal{O}\left( \hat{\mathcal{X}}(A_\delta, B_\delta \cap V; D, V_\delta) \setminus \hat{M}_\delta, Z \right) \]
which extends \( f \). Since \( \omega(\cdot, A_\delta, D) \leq \omega(\cdot, A_0 \cap U, D) \) on \( D \), it follows that
\[ \hat{\mathcal{X}}(A_0 \cap U, B_0 \cap V; D, V_\delta) \subset \hat{\mathcal{X}}(A_\delta, B_0 \cap V; D, V_\delta). \]

On the other hand, by Proposition 3.5 in [17],
\[ \hat{\mathcal{X}}(A_0 \cap U, B_0 \cap V; D, V_\delta) \nrightarrow \hat{\mathcal{X}}(A_0 \cap U, B_0 \cap V; D, G \cap V) \text{ as } \delta \searrow 0. \]

Now fix a sequence \((\delta_k)_{k=1}^\infty\) such that \( 0 < \delta_k < 1 \) and \( \delta_k \searrow 0^+ \). Therefore, using the last equality, we may glue \((\hat{M}_{\delta_k})_{k=1}^\infty\) (take again the smallest singular sets) together in order to obtain a relatively closed pluripolar subset \( \hat{M} \) of \( \hat{\mathcal{X}}(A_0 \cap U, B_0 \cap V; D, G \cap V) \) and an extension mapping \( \hat{f} \) holomorphic on \( \hat{\mathcal{X}}(A_0 \cap U, B_0 \cap V; D, G \cap V) \setminus \hat{M} \) with the desired properties of the theorem.  

**Case** \((a, b) \in D \times G)\.

Choose an open neighborhood \( U \subset D \) of \( a \) (resp. \( V \subset G \) of \( b \)) which is biholomorphic to a bounded Euclidean domain. Using the hypotheses, we are able to apply Theorem 3.11 to \( f|_{\mathcal{X}(A \cap U, B \cap V; U, V)} \). Consequently, we obtain a relatively closed locally pluripolar subset \( \hat{M}_0 \) of \( \hat{\mathcal{X}}(A_0 \cap U, B_0 \cap V; U, V) \) and a mapping \( \hat{f}_0 \in \mathcal{O}(\hat{\mathcal{X}}(A_0 \cap U, B_0 \cap V; U, V) \setminus \hat{M}_0, Z) \) which extends \( f \). The remaining part of the proof follows along the same lines as those given in the previous case. The only difference is that we will apply Corollary 4.5 instead of Corollary 4.4. \( \square \)

Finally, we arrive at the

**Proof of the Main Theorem.** By Proposition 3.11 we only need to check the condition stated in that proposition. In the sequel we are under the hypotheses and notation introduced in that condition. The proof will be divided into two steps.

**Step 1.** Under the hypothesis and notation of Part 1) of Proposition 3.11, let \( b_0 \in \overline{B}_0 \). Then there exists an open set neighborhood \( V \) of \( b_0 \) in \( Y \) and a relatively closed locally pluripolar subset \( \hat{M} \) of \( \hat{\mathcal{X}}(A_0, B_0 \cap V; D, V \cap G) \) such that:

- \( \mathcal{X}(A_0, B_0 \cap V; D, G \cap V) \setminus M \subset \text{End} \left( \hat{\mathcal{X}}(A_0, B_0 \cap V; D, G \cap V) \setminus \hat{M} \right); \)
- \( \mathcal{X}(A_0, B_0 \cap V; D, G \cap V) \cap \hat{M} \subset M; \)
- for every mapping \( f : W \setminus M \rightarrow Z \) satisfying conditions (i)–(iii) of the Main Theorem, there exists a unique mapping \( \hat{f} \in \mathcal{O}(\hat{\mathcal{X}}(A_0, B_0 \cap V; D, G \cap V) \setminus \hat{M}, Z) \) such that \( \hat{f} \) admits the \( A \)-limit \( f(\zeta, \eta) \) at every point \( (\zeta, \eta) \in \mathcal{X}(A_0, B_0 \cap V; D, G \cap V) \setminus M. \)
Since $\overline{A}_0$ is compact, we may apply Theorem 5.1 to all pairs $(a, b_0)$, $a \in \overline{A}_0$. Consequently, we may find a finite number of points $a_1, \ldots, a_M$, their respective open neighborhoods $U_1, \ldots, U_M$ in $X$ and an open neighborhood $V$ of $b_0$ in $Y$ with the following properties:

- $\overline{A}_0 \subset \bigcup_{j=1}^M U_j$;
- there exist a relatively closed locally pluripolar subset $\hat{M}_j$ of $\hat{X}(A_0 \cap U_j, B_0 \cap V; D, G \cap V)$ and a mapping $\hat{f}_j \in \mathcal{O}(\hat{X}(A_0 \cap U_j, B_0 \cap V; D, G \cap V) \setminus \hat{M}_j, Z)$ which admits the $\mathcal{A}$-limit $f(\zeta, \eta)$ at every point $(\zeta, \eta) \in \hat{X}(A_0 \cap U_j, B_0 \cap V; D, G \cap V) \setminus M$.

For any $0 < \delta < \frac{1}{2}$ put

$$B_\delta := \{w \in G \cap V : \omega(w, B_0 \cap V, G \cap V) < \delta\}, \quad V_\delta := B_{1-\delta},$$

$$A_{j,\delta} := \{z \in D \cap U_j : \omega(z, A_0 \cap U_j, D) < \delta\}, \quad j = 1, \ldots, M;$$

$$D_{j,\delta} := A_{j,1-\delta}, \quad A_{\delta} := \bigcup_{j=1}^M A_{j,\delta}, \quad D_{\delta} := \bigcup_{j=1}^M D_{j,\delta}.$$

Observe that $A_{\delta}$ and $D_{\delta}$ are open subsets of $D$, and $B_{\delta}$ and $V_{\delta}$ are open subsets of $V$. Moreover, $D_{\delta} \nearrow D$, $V_{\delta} \nearrow V$ and

$$\omega(\cdot, A_0, D_{\delta}) \searrow \omega(\cdot, A_0, D) \quad \text{and} \quad \omega(\cdot, B_0 \cap V, V_{\delta}) \searrow \omega(\cdot, B_0, V) \quad (9)$$

as $\delta \searrow 0$. Using the above constructions, we may glue the sets $\hat{M}_j$ (resp. the mappings $(\hat{f}_j)_{j=1}^M$) together in order to obtain a relatively closed locally pluripolar subset $M_\delta$ of $X(A_\delta, B_\delta; D_\delta, V_\delta)$ and a mapping $\hat{f}_\delta \in \mathcal{O}(X(A_\delta, B_\delta; D_\delta, V_\delta) \setminus M_\delta, Z)$. Applying Theorem 4.3 to $\hat{f}_\delta$, we get a relatively closed locally pluripolar subset $\hat{M}_\delta$ of $\hat{X}(A_\delta, B_\delta; D_\delta, V_\delta)$ and a mapping $\hat{f}_\delta \in \mathcal{O}(\hat{X}(A_\delta, B_\delta; D_\delta, V_\delta) \setminus \hat{M}_\delta, Z)$. Now using (9), we may glue the mappings $(\hat{f}_\delta)_{0<\delta<1}$ together in order to obtain a relatively closed locally pluripolar subset $\hat{M}$ of $X(A_0, B_0 \cap V; D, V)$ and a mapping $\hat{f} \in \mathcal{O}(\hat{X}(A_0, B_0 \cap V; D, V) \setminus \hat{M}, Z)$. Hence, Step 1 is complete.

**Step 2. End of the proof.**

Note that $\overline{B}_0$ is compact. Applying step 1 we may now proceed in exactly the same way as we did in step 1 starting from Theorem 5.1. Consequently, Step 2 follows. Details are left to the interested reader.

Hence the condition in Proposition 3.11 are verified and, therefore, the proof of the Main Theorem is complete.
6 Applications

In [17] the first author gives various applications of the Main Theorem for the case where $M = \emptyset$ using three different systems of approach regions. These are the canonical one, the system of angular approach regions and the system of conical approach regions. We only give here some applications of the system of conical approach regions. The reader may try to treat the first two cases, that is, to translate Theorem 10.2 and 10.3 of [17] into the context of the Main Theorem.

Let $D \subset \mathbb{C}^n$ be a domain and $A \subset \partial D$. We suppose in addition that $D$ is locally $C^2$ smooth on $A$ (i.e. for any $\zeta \in A$, there exist an open neighborhood $U = U_\zeta$ of $\zeta$ in $\mathbb{C}^n$ and a real function $\rho = \rho_\zeta \in C^2(U)$ such that $D \cap U = \{ z \in U : \rho(z) < 0 \}$ and $d\rho(\zeta) \neq 0$). We define the system of conical approach regions supported on $A$: $A = (A_\alpha(\zeta))_{\zeta \in \partial D, \alpha \in I_\zeta}$ as follows:

- If $\zeta \in \partial D \setminus A$, then $(A_\alpha(\zeta))_{\alpha \in I_\zeta}$ coincide with the canonical approach regions.

- If $\zeta \in A$, then
  \[
  A_\alpha(\zeta) := \{ z \in D : |z - \zeta| < \alpha \cdot \text{dist}(z, T_\zeta) \},
  \]
  where $I_\zeta := (1, \infty)$ and $\text{dist}(z, T_\zeta)$ denotes the Euclidean distance from the point $z$ to the to the tangent hyperplane $T_\zeta$ of $\partial D$ at $\zeta$.

We can also generalize the previous construction to a global situation:

$X$ is an arbitrary complex manifold, $D \subset X$ is an open set and $A \subset \partial D$ is a subset with the property that $D$ is locally $C^2$ smooth on $A$.

Let $X$ be an arbitrary complex manifold and $D \subset X$ an open subset. We say that a set $A \subset \partial D$ is locally contained in a generating manifold if there exist an (at most countable) index set $J \neq \emptyset$, a family of open subsets $(U_j)_{j \in J}$ of $X$ and a family of generating manifolds $16 (\mathcal{M}_j)_{j \in J}$ such that $A \cap U_j \subset \mathcal{M}_j$, $j \in J$, and that $A \subset \bigcup_{j \in J} U_j$. The dimensions of $\mathcal{M}_j$ may vary according to $j \in J$.

Suppose that $A \subset \partial D$ is locally contained in a generating manifold. Then we say that $A$ is of positive size if under the above notation $\sum_{j \in J} \text{mes}(\mathcal{M}_j, A \cap U_j) > 0$, where $\text{mes}(\mathcal{M}_j)$ denotes the Lebesgue measure on $\mathcal{M}_j$. A point $a \in A$ is said to be a density point of $A$ if it is a density point of $A \cap U_j$ on $\mathcal{M}_j$ for some $j \in J$. Denote by $A$ the set of density points of $A$.

Suppose now that, in addition, $A \subset \partial D$ is of positive size. We equip $D$ with the system of conical approach regions supported on $A$. Using the work of B. Coupet (see Théorème 2 in [6]), one can show that\footnote{A differentiable submanifold $\mathcal{M}$ of a complex manifold $X$ is said to be a generating manifold if for all $\zeta \in \mathcal{M}$, every complex vector subspace of $T_\zeta X$ containing $T_\zeta \mathcal{M}$ coincides with $T_\zeta X$.} $A$ is locally pluriregular at all density points of $A$ and $A' \subset A$. Consequently, it follows from Definition 2.3 that

\[
\bar{\omega}(z, A, D) \leq \omega(z, A', D), \quad z \in D.
\]
This estimate, combined with the Main Theorem, implies the following result.

**Theorem 6.1.** Let $X, Y$ be two complex manifolds, let $D \subset X$, $G \subset Y$ be two connected open sets, and let $A$ (resp. $B$) be a subset of $\partial D$ (resp. $\partial G$). $D$ (resp. $G$) is equipped with a system of conical approach regions $(A_\alpha(\zeta))_{\zeta \in \overline{D}, \alpha \in I_\zeta}$ (resp. $(A_\beta(\eta))_{\eta \in \overline{G}, \beta \in I_\eta}$) supported on $A$ (resp. on $B$). Suppose in addition that $A$ and $B$ are of positive size. Define

$$W' := X(A', B; D, G),$$
$$\hat{W}' := \left\{ (z, w) \in D \times G : \omega(z, A', D) + \omega(w, B', G) < 1 \right\},$$

where $A'$ (resp. $B'$) is the set of density points of $A$ (resp. $B$). Let $M$ be a relatively closed subset of $W$ with the following properties:

- $M$ is thin in fibers (resp. locally pluripolar in fibers) over $A$ and over $B$;
- $M \cap (A \times B) = \emptyset$.

Then there exists a relatively closed analytic (resp. a relatively closed locally pluripolar) subset $\hat{M}$ of $\hat{W}'$ such that for every mapping $f : W \setminus M \to Z$ satisfying the following conditions:

(i) $f \in \mathcal{C}_s(W \setminus M, Z) \cap \mathcal{C}_s(W^o \setminus M, Z)$;
(ii) $f$ is locally bounded along $X(A, B; D, G) \setminus M$;
(iii) $f|_{(A \times B)}$ is continuous,

there exists a unique mapping $\hat{f} \in \mathcal{C}(\hat{W}' \setminus \hat{M}, Z)$ which admits the $A$-limit $f(\zeta, \eta)$ at every point $(\zeta, \eta) \in \hat{W}' \setminus \hat{M}$.

The second application is a very general mixed cross theorem.

**Theorem 6.2.** Let $X, Y$ be two complex manifolds, let $D \subset X$, $G \subset Y$ be connected open sets, let $A$ be a subset of $\partial D$, and let $B$ be a subset of $G$. $D$ is equipped with the system of conical approach regions $(A_\alpha(\zeta))_{\zeta \in \overline{D}, \alpha \in I_\zeta}$ supported on $A$ and $G$ is equipped with the canonical system of approach regions $(A_\beta(\eta))_{\eta \in \overline{G}, \beta \in I_\eta}$. Suppose in addition that $A \subset \partial D$ is of positive size and that $B = B^* \neq \emptyset$. Define

$$W' := X(A', B; D, G),$$
$$\hat{W}' := \left\{ (z, w) \in D \times G : \omega(z, A', D) + \omega(w, B, G) < 1 \right\},$$

where $A'$ is the set of density points of $A$. Let $M$ be a relatively subset of $W$ with the following properties:
• $M$ is thin in fibers (resp. locally pluripolar in fibers) over $A$ and over $B$;

• $M \cap (A \times B) = \emptyset$.

Then there exists a relatively closed analytic (resp. a relatively closed locally pluripolar) subset $\widehat{M}$ of $\widehat{W}'$ such that $W' \setminus M \subset \text{End}(\widehat{W}' \setminus \widehat{M})$ and that for every mapping $f : W \setminus M \rightarrow Z$ satisfying the following conditions:

(i) $f \in \mathcal{C}_s(W \setminus M, Z) \cap \mathcal{O}_s(W^o \setminus M, Z)$;

(ii) $f$ is locally bounded along $(A \times G) \setminus M$,

there exists a unique mapping $\hat{f} \in \mathcal{O}(\widehat{W}' \setminus \widehat{M}, Z)$ which admits the $A$-limit $f(\zeta, \eta)$ at every point $(\zeta, \eta) \in W' \setminus M$.

References

[1] O. Alehyane et J. M. Hecart, Propriété de stabilité de la fonction extrémale relative, Potential Anal., 21, (2004), no. 4, 363–373.

[2] O. Alehyane et A. Zeriahi, Une nouvelle version du théorème d’extension de Hartogs pour les applications séparément holomorphes entre espaces analytiques, Ann. Polon. Math., 76, (2001), 245–278.

[3] E. Bedford, The operator $(dd^c)^n$ on complex spaces, Semin. P. Lelong - H. Skoda, Analyse, Années 1980/81, Lect. Notes Math., 919, (1982), 294–323.

[4] E. Bedford, B. A. Taylor, A new capacity for plurisubharmonic functions, Acta Math., 149, (1982), 1–40.

[5] E. M. Chirka, The extension of pluripolar singularity sets, Proc. Steklov Inst. Math., 200 (1993), 369–373.

[6] B. Coupet, Construction de disques analytiques et régularité de fonctions holomorphes aubord, Math. Z. 209 (1992), no. 2, 179–204.

[7] S. M. Ivashkovich, The Hartogs phenomenon for holomorphically convex Kähler manifolds, Math. USSR-Izv., 29, (1997), 225–232.

[8] M. Jarnicki and P. Pflug, Extension of Holomorphic Functions, de Gruyter Expositions in Mathematics 34, Walter de Gruyter, 2000.

[9] ——, An extension theorem for separately holomorphic functions with analytic singularities, Ann. Pol. Math., 80, (2003), 143–161.
[10] ——, An extension theorem for separately holomorphic functions with pluripolar singularities, Trans. Amer. Math. Soc., 355, No. 3, (2003), 1251–1267.

[11] ——, An extension theorem for separately meromorphic functions with pluripolar singularities, Kyushu J. Math., 57, (2003), No. 2, 291–302.

[12] ——, A remark on separate holomorphy, Studia Math., 174, (2006), no. 3, 309–317.

[13] ——, A general cross theorem with singularities, Analysis (Munich), 27 (2007), no. 2-3, 181–212.

[14] B. Josefson, On the equivalence between polar and globally polar sets for plurisubharmonic functions on $\mathbb{C}^n$, Ark. Mat., 16, (1978), 109–115.

[15] M. Klimek, Pluripotential theory, London Mathematical society monographs, Oxford Univ. Press., 6, (1991).

[16] V.-A. Nguyên, A general version of the Hartogs extension theorem for separately holomorphic mappings between complex analytic spaces, Ann. Scuola Norm. Sup. Pisa Cl. Sci., (2005), serie V, Vol. IV(2), 219–254.

[17] ——, A unified approach to the theory of separately holomorphic mappings, Ann. Scuola Norm. Sup. Pisa Cl. Sci., (2008), serie V, Vol. VII(2), 181–240.

[18] ——, Conical plurisubharmonic measure and new cross theorems, arXiv:math.CV.0901.

[19] ——, Recent developments in the theory of separately holomorphic mappings, arXiv:math.CV.0901.1991.

[20] V.-A. Nguyên and P. Pflug, Boundary cross theorem in dimension 1 with singularities, Indiana Univ. Math. J. (to appear).

[21] P. Pflug, Extension of separately holomorphic functions—a survey 1899–2001, Ann. Polon. Math., 80, (2003), 21–36.

[22] P. Pflug and V.-A. Nguyên, A boundary cross theorem for separately holomorphic functions, Ann. Polon. Math., 84, (2004), 237–271.

[23] ——, Boundary cross theorem in dimension 1, Ann. Polon. Math., 90(2), (2007), 149-192.

[24] ——, Generalization of a theorem of Gonchar, Ark. Mat., 45, (2007), 105–122.
[25] —, Envelope of holomorphy for boundary cross sets, Arch. Math. (Basel), (2007), 89, 326–338.

[26] E. A. Poletsky, Plurisubharmonic functions as solutions of variational problems, Several complex variables and complex geometry, Proc. Summer Res. Inst., Santa Cruz/CA (USA) 1989, Proc. Symp. Pure Math. 52, Part 1, (1991), 163–171.

[27] —, Holomorphic currents, Indiana Univ. Math. J., 42, No.1, (1993), 85–144.

[28] J. P. Rosay, Poletsky theory of disks on holomorphic manifolds, Indiana Univ. Math. J., 52, No.1, (2003), 157–169.

[29] B. Shiffman, Extension of holomorphic maps into Hermitian manifolds, Math. Ann., 194, (1971), 249–258.

V.-A. Nguyêñ, School of Mathematics, Korea Institute for Advanced Study, 207-43Cheongryangni-2dong, Dongdaemun-gu, Seoul 130-722, Korea.

vietanh@kias.re.kr

Peter Pflug, Carl von Ossietzky Universität Oldenburg, Institut für Mathematik, Postfach 2503, D–26111, Oldenburg, Germany.
pflug@mathematik.uni-oldenburg.de