P-TH POWERS IN MOD P COHOMOLOGY OF FIBERS

by Luc Menichi

Abstract. — Let $F \hookrightarrow E \to B$ be a fibration whose base space $B$ is a finite simply-connected CW-complex of dimension $\leq p$ and whose total space $E$ is a path-connected CW-complex of dimension $\leq p - 1$. If $\alpha \in H^+(F; \mathbb{F}_p)$ then $\alpha^p = 0$.

We work over the prime field $\mathbb{F}_p$ with $p$ an odd or even prime. The homology and cohomology of spaces are considered with coefficients in $\mathbb{F}_p$.

In [1], Anick proved using algebraic models:

Theorem [1, 9.1]. — Let $r$ be a non-negative integer. Let $B$ be a simply-connected space with a finite type homology concentrated in degrees $i \in [r + 1, rp]$. Then all $p$-th powers vanish in $H^+(\Omega B)$.

This result was suggested by McGibbon and Wilkerson [6, p. 699]. The aim of this note is to give two different generalisations of Anick Theorem: Theorem A and Theorem C below.

Luc Menichi, Université d’Angers, Faculté des Sciences, 2 Boulevard Lavoisier, 49045 Angers, FRANCE, Telephone: 02-41-73-50-25, Fax: 02-41-73-54-54
E-mail: Luc.Menichi@univ-angers.fr

2000 Mathematics Subject Classification. — 57T35, 55T20, 55R20.

Key words and phrases. — Eilenberg-Moore spectral sequence, fiber space, loop space, free loop space, $p$-th powers.

Topologie/Topology.
The first one, whose proof is inspired by a result of Lannes and Schwartz [4, Prop 0.6], uses the (vertical) Steenrod operations in the Eilenberg-Moore spectral sequence:

**Theorem A.** — Let $r$ and $k$ be two non-negative integers. Consider a fiber product of spaces

\[
\begin{array}{c}
E \times_B X \\
\downarrow \pi \\
X \\
\end{array}
\]

where

- $\pi$ is a Serre fibration and
- $H^*(E)$, $H^*(X)$ and $H^*(B)$ are of finite type.

If $B$ is simply-connected with homology concentrated in degrees $i \in [r + 1, rp^k]$ and the product space $E \times X$ is path connected with homology $H^*(E \times_B X)$ concentrated in degrees $i \in [r, rp^k - 1]$, then all $p^k$-th powers vanish in $H^+(E \times_B X)$.

**Proof.** — We suppose that $p$ is an odd prime. The case $p = 2$ is similar. Let $\mathcal{A}$ denote the mod $p$ Steenrod algebra. The degree of an element $\alpha$ is denoted $|\alpha|$. Recall from [8, 11, 10], that the Eilenberg-Moore spectral sequence is a strongly convergent second quadrant cohomological spectral sequence of $\mathcal{A}$-modules

\[
E_2^{s,*} \cong \text{Tor}^{H^*(B)}_{H^*(E), H^*(X)}(\mathcal{A}) \Rightarrow H^*(E \times_B X).
\]

More precisely, there exists a convergent filtration of $\mathcal{A}$-modules on $H^*(E \times_B X)$:

\[
H^*(E \times_B X) \supset F_s \supset F_{s-1} \supset F_1 \supset F_0 \supset F_{-1} = \{0\}
\]

such that $\Sigma^{-s} F_s/F_{s-1} \cong E_{\infty}^{-s,*}$, $s \geq 0$. Here $\Sigma^{-s}$ denotes the $s$-th desuspension of an $\mathcal{A}$-module.

Let $\alpha \in F_s$ such that the class $[\alpha] \in F_s/F_{s-1}$ is non zero. We want to prove that $\alpha^{p^k} = 0$. As an $\mathcal{A}$-module, $\text{Tor}^{H^*(B)}_{H^*(E), H^*(X)}(\mathcal{A})$ is the $s$-th homology group of a complex of $\mathcal{A}$-modules, namely the Bar construction, whose $s$-th term is $H^*(E) \otimes H^+(B)^{\otimes s} \otimes H^*(X)$.

The element $\Sigma^{-s} [\alpha] \in E_{\infty}^{-s,*}$ is represented by a cycle of the form $e[b_1] \cdots [b_s]x$, where $e \in H^*(E)$, $(b_i)_{1 \leq i \leq s} \in H^+(B)$ and $x \in H^*(X)$. So $rs \leq |\alpha| \leq (rp^k - 1)(s + 1)$.

Case 1: $|e| + |x| \geq r$. Then $|\alpha| \geq r(s + 1)$. Therefore, by a degree argument, the element $\alpha^{p^k}$ of $F_s$ is zero.
Case 2: $e = x = 1$. Since the Cartan formula applies, $\Sigma^{-s}[\alpha^p] = P^{[\alpha]/2} \Sigma^{-s}[\alpha]$ is represented by the element of the Bar construction,

$$\sum_{i_1 + \cdots + i_s = [\alpha]} [P^{i_1} b_1] \cdots [P^{i_s} b_s].$$

So $\Sigma^{-s}[\alpha^p]$ is zero for degree reasons. Therefore $\alpha^p \in F_{s-1}$ which is concentrated in degrees $\leq (rp^k - 1)s$, thus $\alpha^p = 0$.

Let $B$ be a space. The free loop space on $B$, denoted $BS^1$, is the set of continuous (unpointed) maps from the circle $S^1$ to $B$. It can be defined as a fibre product:

$$\begin{array}{ccc}
BS^1 & \rightarrow & B[0,1] \\
\downarrow \scriptstyle{ev} & & \downarrow \scriptstyle{\tau} \\
B & \rightarrow & B \times B \\
\Delta & \rightarrow & B \times B
\end{array}$$

In this particular case, we can improve Theorem A.

**Theorem B.** — Let $r$ and $k$ be two non-negative integers. If $B$ is a simply-connected space with finite type homology concentrated in degrees $i \in [r+1, rp^k]$ then all $p^k$-th powers vanish in $H^+(BS^1)$.

**Proof.** — The Eilenberg-Moore spectral sequence for the previous fibre product satisfies

$$E^2_{r,s} = HH_s(H^*(B)) \Rightarrow H^*(BS^1).$$

Here $HH_*$ denotes the Hochschild homology. As an $\mathcal{A}$-module, $HH_*(H^*(B))$ is the $s$-th homology group of a complex of $\mathcal{A}$-modules, namely the Hochschild complex, whose $s$-th term is $H^*(B) \otimes H^+(B)^{\otimes s}$.

The same arguments as in the proof of Theorem A allow us to conclude except in Case 2 for $s = 1$. If $\alpha \in F_1 \subset H^*(BS^1)$, we can only affirm that $\alpha^p \in F_0$. The evaluation map $ev : BS^1 \rightarrow B$ admits a section $\sigma$. So $H^*(ev) : H^*(B) \hookrightarrow (BS^1)$ admits $H^*(\sigma)$ as retract. The edge homomorphism

$$H^*(B) = E^0_{2,*} \rightarrow E^3_{3,*} \rightarrow E^0_{\infty,*} = F_0 \subset H^*(BS^1)$$

correspond to $H^*(ev)$. Since $\alpha^p \in F_0 = H^*(B)$, $\alpha^p = [H^*(\sigma)(\alpha)]^{p^k}$. For degree reason, all $p^k$-th powers are zero in $H^+(B)$. So $\alpha^p = 0$. □

In [3], Félix, Halperin and Thomas give a slightly more complicated proof of Anick Theorem. Their proof uses the vertical and horizontal Steenrod operations in the Serre spectral sequence:
Theorem [3, 2.9(i)]. — Let \( r \) and \( k \) be two non-negative integers. If \( B \) is a simply-connected space with a finite type homology concentrated in degrees \( i \in [r + 1, rp^k] \) then all \( p^k \)-th powers vanish in \( H^*(\Omega B) \).

This result generalizes in:

Theorem C (Compare with [7, 10.8]). — Let \( r \) and \( k \) be two non-negative integers. Let \( F \overset{j}{\to} E \overset{\pi}{\to} B \) be a Serre fibration with \( E \) path connected. If \( B \) is a simply-connected space with finite type homology concentrated in degrees \( i \in [r + 1, rp^k] \) then, for any \( \alpha \in H^*(F) \), \( \alpha p^k \in \text{Im} H^*(j) \).

Proof. — The proof follows the lines of [3, 2.9]. Since \( H^{\leq r}(B) = 0 \), \( \alpha \in E_{2,r}^{0,*} \) survives till \( E_{r+1}^{0,*} \). Therefore by a Theorem of Araki [2] and Vázquez [12] (See also [5, Prop 2.5 Case 2]), \( \alpha p^k \in E_{2,r}^{0,*} \) survives till \( E_{rp^k+1}^{0,*} \). Since \( H^{> rp^k}(B) = 0 \),
\[
E_{rp^k+1}^{0,*} = E_{\infty}^{0,*} = \text{Im} H^*(j).
\]

In order to see that the hypothesis in Félix-Halperin-Thomas Theorem (and in Theorem B) cannot be improved, consider \( B = \Sigma \mathbb{C}P^k \), the suspension of the \( p^k \)-dimension complex projective space.

Observe also that in Theorem C, \( \alpha p^k \) is not zero in general. Indeed, take \( \pi \) to be the fibration associated to the suspension of the Hopf map from \( S^{2p^k-1} \) to \( \mathbb{C}P^k \) [5, Remark 9.9].

Finally, remark that the following question of McGibbon and Wilkerson remains unsolved.

Question [3, p. 699] (See also [5, section 9. Question 3])

Let \( B \) be a finite simply-connected CW-complex and \( p \) a prime large enough. Do all the Steenrod operations act trivially on \( H^*(\Omega B) \)?

I wish to thank Professor Katsuhiko Kuribayashi, for his precious help with the Eilenberg-Moore spectral sequence.

References

[1] D. J. Anick – Hopf algebras up to homotopy, J. Amer. Math. Soc. 2 (1989), no. 3, p. 417–453.
[2] S. Araki – Steenrod reduced powers in the spectral sequences associated with a fibering. I, II, Mem. Fac. Sci. Kyushu Univ. Ser. A 11 (1957), p. 15–64, 81–97.
[3] Y. Félix, S. Halperin & J.-C. Thomas – The Serre spectral sequence of a multiplicative fibration, Prépublications, Université d’Angers, 1998, http://sciences.univ-angers.fr/math/preprint.html.

[4] J. Lannes & L. Schwartz – A propos de conjectures de Serre et Sullivan, Invent. Math. 83 (1986), p. 593–603.

[5] C. A. McGibbon – Phantom maps, Handbook of Algebraic Topology (I. M. James, ed.), North-Holland, Amsterdam, 1995, p. 1209–1257.

[6] C. A. McGibbon & C. W. Wilkerson – Loop spaces of finite complexes at large primes, Proc. Amer. Math. Soc. 96 (1986), p. 698–702.

[7] L. Menichi – On the cohomology algebra of a fiber, Preprint Hopf Topology Archive, 1999.

[8] D. L. Rector – Steenrod operations in the Eilenberg-Moore spectral sequence, Comment. Math. Helv. 45 (1970), p. 540–552.

[9] J. Sawka – Odd primary Steenrod operations in first-quadrant spectral sequences, Trans. Am. Math. Soc. 273 (1982), p. 737–752.

[10] L. Schwartz – Unstable modules and unstable algebras. Applications to homotopy theory, Lectures Notes of the Grenoble Summer School, http://www-fourier.ujf-grenoble.fr, 1997.

[11] L. Smith – On the Künneth theorem. I: The Eilenberg-Moore spectral sequence, Math. Z. 116 (1970), p. 94–140.

[12] R. Vázquez García – Note on Steenrod squares in the spectral sequence of a fibre space, Bol. Soc. Math Mexicana 2 (1957), p. 1–8.