On dynamic optimization sensitivity analysis

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Abstract. This article describes a linear penalty method for getting sensitive information, i.e. the effect of variation on the constraint to the objective function. The dynamic optimization problem discussed here is a problem with state constraints. At first, the problem is formulated to unconstrained dynamic optimization problem by adding the constraints to the objective function as linear penalty terms. Then, the mathematical formulas to evaluate the change in the objective function and the optimal solution are derived if small perturbation introduced in the constraints. Lastly, the formulas are applied to some numerical examples. Comparison the result of numerical simulations to the numerical solutions obtained from reliable software, i.e. MlSER 3.3, shows that the method in general is quite effective. The differences between two solutions are very small and insignificant.

1. Introduction
Dynamic optimization problems also known as optimal control problems arise naturally in many daily life applications. In general, the problem is to optimize (minimize or maximize) some predetermined objective function by choosing the best possible controls subject to dynamical systems which govern the change of the systems. This problem is usually, except for simple cases (linear or quadratic), not tractable analytically. Therefore, numerical methods are the only necessary way to solve the problem.

Furthermore, the problem becomes more complex if there are several constraints such as control constraints, mixed state-control constraints and purely state constraints involved. One of the simplest ways in numerical methods to deal with the constraints is to convert them as additional penalty terms in the objective function. If at a particular time, states and controls the constraints are satisfied, then these penalty terms will have a zero value. Otherwise, they will have a very big or small value which is in contradiction with the aims to optimize the objective function. Some authors, for examples in [1], [2], [3], have successfully applied this method to solve the constrained optimal control problem but they did not exploit the sensitivity information further. To the author knowledge, there are not many articles discussed the use of penalty functions to obtain sensitivity information, except Fiacco [4] in nonlinear programming to name but a few.

However, there are some other methods used in the existing literatures to analyze the sensitivity of dynamic optimization, such as [5], [6], [7] and [8]. SQP-methods in [5] proposed by Busken were able to solve and give additional information about sensitivity. Evers in [6] evaluated the change in the objective function due to small variations in initial conditions, terminal conditions and parameters of the dynamical system. Mallanowski [7] proposed the application of the classical implicit function theorem to investigate the sensitivity of the system, whereas Pfeiffer [8] used relaxation and decomposition techniques to obtain such information.
In this article, we utilize linear penalty functions to convert mixed state-control constrained problem to unconstrained problem. Afterwards, the Pontryagin Minimum Principle [9] is used to derive new formulas in the form of integral to measure the change in the objective function because of small changes on the right hand side of constraints (see Theorem 1). This integral can be evaluated numerically after solving a two point boundary value problem using Sweep method (see Theorem 2). In summary, using the proposed method we can easier determine how big the changes in the objective function, states and controls from the reference optimal solution without solving the problem directly.

2. The linear penalty function

In this section, we will discuss how to use linear penalty function to obtain information from some constrained dynamic optimization problems, namely the effect of small perturbation on the constraint to a predetermined objective function.

2.1. The problem formulation

At first, we state the problem as the followings. We would like to minimize the objective function, i.e.

$$\min_{u \in \mathbb{F}} I(u, \alpha) = \int_{0}^{T_f} F(x, u, t) \, dt + \psi(x(T))$$

subject to initial value

$$\frac{dx}{dt} = f(x, u, t), \quad x \in \mathbb{R}^n, \quad \forall \, t \in (0, T]$$

$$x(0) = x_0, \quad x_0 \in \mathbb{R}^n$$

and state-constraint

$$\Gamma = \{ \, u \in \mathbb{R}^m \mid h(x, u, t, \alpha) \equiv g(x, u, t) - \alpha \leq 0 \, \}, \quad \forall \, t \in (0, T] .$$

In this context, $x$ is a vector of states and $u$ is a control vector and $t$ is time. The constants $T_f > 0$ and $\alpha \in \mathbb{R}^p$ represents respectively a final time horizon and a small perturbation on the constraint. The functions such as $F: \mathbb{R}^{n+m+1} \rightarrow \mathbb{R}^n$, $\psi: \mathbb{R}^n \rightarrow \mathbb{R}$, $f: \mathbb{R}^{n+m+1} \rightarrow \mathbb{R}^n$ and $h: \mathbb{R}^{n+m+1+p} \rightarrow \mathbb{R}^p$ are known and second order continuously differentiable functions with respect to all the arguments. If $\alpha = 0$ then the problem become constrained dynamic optimization problem without perturbation. We consider here the problem of $\alpha \neq 0$ whose quantities are quite small.

To apply a linear penalty method we incorporate the constraint as penalty terms in the objective function. Therefore, the above problem become as follows.

$$\min_{u \in \mathbb{F}} J(u, \alpha, \theta) = I(u, \alpha) + \int_{0}^{T_f} \theta^T h^+(x, u, t, \alpha) \, dt$$

subject to initial value

$$\frac{dx}{dt} = f(x, u, t), \quad x \in \mathbb{R}^n, \quad \forall \, t \in (0, T]$$

$$x(0) = x_0, \quad x_0 \in \mathbb{R}^n$$

Vectors $\theta = (\theta_1, \theta_2, \ldots, \theta_p)^T$ are penalty factor whose elements are larger than 0 ( $\theta_i > 0$, $\forall \, i = 1, 2, \ldots, p$ ) and function of time $t$. The vector function $h^+ = (h_1^+, h_2^+, \ldots, h_p^+)$ is defined as

$$h_i^+ = \begin{cases} 0, & \text{if } h_i \leq 0 \\ h_i, & \text{if } h_i > 0 \end{cases}$$

for $i = 1, 2, \ldots, p$. Alternatively, $h_i^+$ can be defined as $h_i^+ = \max\{0, h_i\}$. Therefore, if at particular time, states and controls the constraints are violated ($h_i > 0$) then the objective function will be penalized by $h_i$. On the other hand, the term $h_i^+$ will add a value of zero to the objective function if the constraints are satisfied at some time, states and controls.
2.2. The sensitivity analysis
In this part, we state two main theorems which measure how big the influences of some small perturbation in the constraints on the objective function. As the functions $h^+ = (h^+_1, h^+_2, \ldots, h^+_p)$ are not differentiable, we need to replace them with the smooth version as in [10], i.e.

$$
\begin{align*}
  h^+_h &= \begin{cases} 
    0 & \text{if } h \leq 0 \\
    2\epsilon^{-1}h^2 - \epsilon^{-2}h^3 & \text{if } 0 < h \leq \epsilon \\
    \epsilon & \text{if } h > \epsilon 
  \end{cases}
\end{align*}
$$

where $\epsilon$ is a very small constant.

**Theorem 2.1.** Let $(x, u)$ and $(x + \delta x, u + \delta u)$ be respectively the unique solution of the problem for $\alpha = 0$ and $\alpha \neq 0$. If $\omega = \max\{|\delta x|, |\delta u|\}$, then the change on the objective function due a small perturbation in the constraints can be calculated by one of the follows.

$$
\Delta J = \int_0^T H(x + \delta x, u + \delta u, t, \lambda, \theta, \alpha) - H(x + \delta x, u, t, \lambda, \theta, \alpha) \, dt + O(\omega^2)
$$

or

$$
\Delta J = \int_0^T H(x + \delta x, u + \delta u, t, \lambda, \theta, 0) - H(x + \delta x, u, t, \lambda, \theta, 0) \, dt + O(\omega^2),
$$

where $H$ is a Hamiltonian function defined as

$$
H(x, u, t, \lambda, \theta, \alpha) = F(x, u, t) + \lambda^T f(x, u, t) + \theta^T h^+_\chi(x, u, t, \alpha).
$$

**Proof of Theorem 2.1** We will prove the theorem by evaluating

$$
\Delta J = \Delta F + \Delta \psi + \Delta h^+_\chi
$$

where

$$
\Delta F = \int_0^T F(x + \delta x, u + \delta u, t) - F(x, u, t) \, dt
$$

and

$$
\Delta \psi = \psi(x + \delta x)(T_f) - \psi(x)(T_f)
$$

and

$$
\Delta h^+_\chi = \sum_{i=1}^p \int_0^T \theta_i[(h^+_\chi)_{ij}(x + \delta x, u + \delta u, t, \alpha_i) - (h^+_\chi)_{ij}(x, u, t, 0)] \, dt.
$$

At first, we reformulate $\Delta F$ as the following.

$$
\Delta F = \int_0^T F(x + \delta x, u + \delta u, t) - F(x + \delta x, u, t) + F(x + \delta x, u, t) - F(x, u, t) \, dt.
$$

Using Taylor series, we expand

$$
F(x + \delta x, u, t) - F(x, u, t) = \sum_{i=1}^n \frac{\partial F}{\partial x_i}(x, u, t) \delta x_i + O(\omega^2).
$$

Moreover, from the Pontryagin Minimum Principle we have

$$
\frac{d\lambda_i}{dt} = -\sum_{j=1}^n \lambda_j \frac{\partial f_i}{\partial x_j} - \sum_{k=1}^p \theta_k \frac{\partial (h^+_\chi)_k}{\partial x_i}.
$$

Taking the derivative with respect to $t$ of the last equation and expanding $d(\delta x_i)/dt$ as

$$
\frac{d(\delta x_i)}{dt} = \delta f_i = f_i(x + \delta x, u + \delta u, t) - f_i(x, u, t) + \sum_{j=1}^n \frac{\partial f_i}{\partial x_j} \delta x_j + O(\omega^2),
$$

we get
\[
\sum_{i=1}^{n} \frac{\partial F}{\partial x_i} \delta x_i = - \frac{d}{dt} \sum_{i=1}^{n} \lambda_i \delta x_i + \sum_{i=1}^{n} \lambda_i \left( f_i(x + \delta x, u + \delta u, t) - f_i(x, u, t) \right) - \sum_{i=1}^{n} \sum_{k=1}^{p} \theta_k \frac{\partial (h^+_k)}{\partial x_i} + O(\omega^2).
\]

Therefore, in the final form
\[
\Delta F = \int_{0}^{T_f} F(x + \delta x, u + \delta u, t) - F(x + \delta x, u, t) - \frac{d}{dt} \sum_{i=1}^{n} \lambda_i \delta x_i + \sum_{i=1}^{n} \lambda_i \left( f_i(x + \delta x, u + \delta u, t) - f_i(x, u, t) \right) - \sum_{i=1}^{n} \sum_{k=1}^{p} \theta_k \frac{\partial (h^+_k)}{\partial x_i} + O(\omega^2).
\]

Secondly, we expand \( \Delta \psi \) according to Taylor series as
\[
\Delta \psi = \psi(x + \delta x) (T_f) - \psi(x) (T_f) = \sum_{i=1}^{n} \frac{\partial \psi}{\partial x_i} (x_i (T_f)) \delta x_i + O(\omega^2).
\]

Analogously, we can write \( \Delta h^+_k \) as
\[
\Delta h^+_k = \sum_{k=1}^{p} \int_{0}^{T_f} \theta_k \left( h^+_k (x + \delta x, u + \delta u, t, \alpha_k) - h^+_k (x + \delta x, u, t, \alpha_k) \right) dt + \sum_{i=1}^{n} \frac{\partial h^+_k}{\partial x_i} \delta x_i + O(\omega^2).
\]

Alternatively,
\[
\Delta h^+_k = \sum_{k=1}^{p} \int_{0}^{T_f} \theta_k \left( h^+_k (x + \delta x, u + \delta u, t, \alpha_k) - h^+_k (x + \delta x, u, t, 0) \right) dt + \sum_{i=1}^{n} \frac{\partial h^+_k}{\partial x_i} \delta x_i + O(\omega^2).
\]

Summing up all the final form of \( \Delta F, \Delta \psi, \Delta h^+_k \), we come to
\[
\Delta f = \int_{0}^{T_f} H(x + \delta x, u + \delta u, t, \lambda, \theta, \alpha) - H(x + \delta x, u, t, \lambda, \theta, \alpha) dt + O(\omega^2)
\]

or alternatively
\[
\Delta f = \int_{0}^{T_f} H(x + \delta x, u + \delta u, t, \lambda, \theta, \alpha) - H(x + \delta x, u, t, \lambda, \theta, 0) dt + O(\omega^2).
\]

The next theorem presents a method for evaluating \( \delta x \) and \( \delta u \).

**Theorem 2.2** Let \((x, u)\) be the optimal solution of the problem for \(\alpha = 0\). If \(\frac{\partial^2 H}{\partial u^2} > 0\) then \((x + \delta x, u + \delta u)\) that is the approximation to optimal solution of the problem for \(\alpha \neq 0\) can be evaluated by solving the following system of equations.

\[
\begin{bmatrix}
\frac{d(\delta x)}{dt} \\
\frac{d(\delta \lambda)}{dt}
\end{bmatrix} = \begin{bmatrix}
A & B \\
-C & -D_T
\end{bmatrix} \begin{bmatrix}
\delta x \\
\delta \lambda
\end{bmatrix} + \begin{bmatrix}
S_x \\
S_\lambda
\end{bmatrix}
\]

\[
\begin{bmatrix}
\delta u \\
\delta \theta
\end{bmatrix} = -E \begin{bmatrix}
\frac{\partial^2 H}{\partial u \partial x} & \frac{\partial f^T}{\partial u} \\
\frac{\partial h}{\partial x} & 0
\end{bmatrix} \begin{bmatrix}
\delta x \\
\delta \lambda
\end{bmatrix} - \begin{bmatrix}
0 \\
I
\end{bmatrix} \alpha.
\]
Here,

\[
A = \frac{\partial f}{\partial x} - \left[ \frac{\partial f}{\partial u} 0 \right] E \left[ \frac{\partial^2 H}{\partial u \partial x} \right],
\]

\[
B = \left[ \frac{\partial f}{\partial u} 0 \right] E \left[ \frac{\partial f^T}{\partial x} \right],
\]

\[
C = \frac{\partial^2 H}{\partial x^2} + \left[ - \frac{\partial^2 H}{\partial u \partial x} - \frac{\partial h^T}{\partial x} \right] E \left[ \frac{\partial^2 H}{\partial u \partial x} \right],
\]

\[
S_x = \left[ \frac{\partial f}{\partial u} 0 \right] E \left[ 0 \right] \alpha,
\]

\[
S_\lambda = \left[ - \frac{\partial^2 H}{\partial u \partial x} - \frac{\partial h^T}{\partial x} \right] E \left[ 0 \right] \alpha,
\]

\[
E = \begin{cases}
\left[ \frac{\partial^2 H^{-1}}{\partial u^2} 0 \\
0 0
\right] & \text{if constraint } h \text{ is inactive at } t \\
\left[ \frac{\partial^2 H}{\partial u^2} \frac{\partial h^T}{\partial u} \\
\frac{\partial h}{\partial u} 0
\right] & \text{if constraint } h \text{ is active at } t
\end{cases}
\]

**Proof of Theorem 2.2** The Pontryagin Minimum Principle provides the necessary conditions for the optimality. It stated that if \((x, u)\) is an optimal solution then

a. the partial derivative of Hamiltonian \(H\) with respect to control variables \(u\), evaluated at the optimal solution should be zero.

\[
\frac{\partial H(x, u, t)}{\partial u} = 0.
\]

Using the variational process, the variation of this equation is

\[
\frac{\partial^2 H}{\partial u^2} \delta u + \frac{\partial^2 H}{\partial u \partial x} \delta x + \frac{\partial^2 H}{\partial u \partial \lambda} \delta \lambda + \frac{\partial^2 H}{\partial u \partial \theta} \delta \theta = 0.
\]

Because

\[
\frac{\partial^2 H}{\partial u \partial \lambda} = \frac{\partial^2 H}{\partial \lambda \partial u} = \frac{\partial f^T}{\partial u}
\]

and

\[
\frac{\partial^2 H}{\partial u \partial \theta} = \frac{\partial^2 H}{\partial \theta \partial u} = \frac{\partial h^T}{\partial u},
\]

then

\[
\frac{\partial^2 H}{\partial u^2} \delta u + \frac{\partial^2 H}{\partial u \partial x} \delta x + \frac{\partial f^T}{\partial u} \delta \lambda + \frac{\partial h^T}{\partial u} \delta \theta = 0.
\]

b. the derivative of the adjoint variable \(\lambda\) with respect to time \(t\), equals to negative of the partial derivative of \(H\) with respect to state variable \(x\), i.e.

\[
\frac{d\lambda^T(t)}{dt} = -\frac{\partial H(x, u, t)}{\partial x}.
\]

Taking variation of this equation, we have

\[
\frac{d}{dt} (\delta \lambda) = -\frac{\partial^2 H}{\partial x^2} \delta x - \frac{\partial^2 H}{\partial x \partial u} \delta u - \frac{\partial^2 H}{\partial x \partial \lambda} \delta \lambda - \frac{\partial^2 H}{\partial x \partial \theta} \delta \theta.
\]
Due to the fact that
\[
\frac{\partial^2 H}{\partial x \partial \lambda} = \frac{\partial^2 H}{\partial \lambda \partial x} = \frac{\partial f^T}{\partial x}
\]
and
\[
\frac{\partial^2 H}{\partial x \partial \theta} = \frac{\partial^2 H}{\partial \theta \partial x} = \frac{\partial h^T}{\partial x},
\]
then
\[
\frac{d(\delta \lambda)}{dt} = -\frac{\partial^2 H}{\partial x^2} \delta x - \frac{\partial^2 H}{\partial x \partial u} \delta u - \frac{\partial f^T}{\partial x} \delta \lambda - \frac{\partial h^T}{\partial x} \delta \theta.
\]

c. the adjoint variable \( \lambda \) evaluated at the final time \( T_f \) is the same as the derivative of the terminal cost in the objective function \( (\psi) \) with respect to state variable \( x \) at the final time \( T_f \), i.e.
\[
\lambda^T(T_f) = \frac{\partial \psi(x(T_f))}{\partial x}.
\]

This terminal condition can provide the terminal condition for \( \delta \lambda \) as the following.
\[
\delta \lambda(T_f) = \frac{\partial^2 \psi(x(T_f))}{\partial x^2}.
\]

Additionally, the differential equation for \( \delta x \) can be obtained from the differential equation of the state variable \( x \), that is
\[
\frac{d(\delta x)}{dt} = \frac{\partial f}{\partial x} \delta x + \frac{\partial f}{\partial u} \delta u
\]
with the initial value \( \delta x(0) = 0 \). The above differential equation for \( \delta x \) and \( \delta \lambda \) can be combined as
\[
\begin{bmatrix}
\frac{d(\delta x)}{dt} \\
\frac{d(\delta \lambda)}{dt}
\end{bmatrix}
= -\begin{bmatrix}
\frac{\partial f}{\partial x} & 0 \\
\frac{\partial^2 H}{\partial x^2} & \frac{\partial f^T}{\partial x} \\
\frac{\partial^2 H}{\partial x \partial u} & \frac{\partial^2 H}{\partial u \partial \lambda}
\end{bmatrix}
\begin{bmatrix}
\delta x \\
\delta \lambda
\end{bmatrix}
+ \begin{bmatrix}
\frac{\partial f}{\partial u} \\
\frac{\partial^2 H}{\partial x \partial u} \\
\frac{\partial h^T}{\partial x}
\end{bmatrix}
\begin{bmatrix}
0 \\
\delta \lambda
\end{bmatrix}.
\]

If at time \( t \) the constraint \( h < 0 \) (inactive) then \( h^+ = 0 \) so that \( \theta^T h^+ = 0 \) regardless of the value of \( \theta \). Therefore the variation of \( \delta \theta = 0 \). This result simplifies the former equations (part a.) and leads to
\[
\delta u = -\frac{\partial^2 H^{-1}}{\partial u^2} \left( \frac{\partial^2 H}{\partial u \partial x} \delta x + \frac{\partial f^T}{\partial u} \delta \lambda \right).
\]

Therefore, we finally obtain
\[
\begin{bmatrix}
\delta u \\
\delta \theta
\end{bmatrix}
= \left[ \frac{\partial^2 H^{-1}}{\partial u^2} \right]
\begin{bmatrix}
\delta x \\
\delta \lambda
\end{bmatrix}
- \left[ \begin{bmatrix}
\frac{\partial^2 H}{\partial u \partial x} \\
\frac{\partial^2 H}{\partial u \partial \lambda}
\end{bmatrix}
\right]
\begin{bmatrix}
0 \\
\delta \lambda
\end{bmatrix},
\]
provided that the constraint is inactive at time \( t \).

On the other hand, in the case that at time \( t \) the constraint is active \( (h = 0) \) we have
\[
\frac{\partial h}{\partial u} \delta u = -\frac{\partial h}{\partial x} \delta x + \alpha.
\]

Therefore, it can be rewritten as
\[
\begin{bmatrix}
\delta u \\
\delta \theta
\end{bmatrix}
= \left[ \frac{\partial^2 H^{-1}}{\partial u^2} \frac{\partial h^T}{\partial u} \right]
\begin{bmatrix}
\delta x \\
\delta \lambda
\end{bmatrix}
- \left[ \begin{bmatrix}
\frac{\partial^2 H}{\partial u \partial x} \\
\frac{\partial^2 H}{\partial u \partial \lambda}
\end{bmatrix}
\right]
\begin{bmatrix}
\delta \lambda
\end{bmatrix} - \left[ \begin{bmatrix}
0 \\
\alpha
\end{bmatrix}
\right].
\]

In short, the equations for both cases of constraints (either active or inactive) can be compactly written as
\[
\begin{bmatrix}
\delta u \\
\delta \theta
\end{bmatrix} = -E \left( \begin{bmatrix}
\frac{\partial^2 H}{\partial u \partial x} & \frac{\partial f^T}{\partial u} \\
\frac{\partial h}{\partial x} & 0
\end{bmatrix} \begin{bmatrix}
\delta x \\
\delta \lambda
\end{bmatrix} - \begin{bmatrix}
0 \\
I
\end{bmatrix} \alpha \right),
\]

where

\[
E = \begin{cases}
\frac{\partial^2 H^{-1}}{\partial u^2} & \text{if constraint } h \text{ is inactive at } t \\
\frac{\partial^2 H}{\partial u^2} \frac{\partial h^T}{\partial u}^{-1} & \text{if constraint } h \text{ is active at } t
\end{cases}
\]

Here, the matrix \(E\) always exists as long as matrix \(\frac{\partial^2 H}{\partial u^2}\) positive definite, namely \(\frac{\partial^2 H}{\partial u^2} > 0\),

Rearranging all of the above results, it results the following formula

\[
\begin{bmatrix}
\frac{d(\delta x)}{dt} \\
\frac{d(\delta \lambda)}{dt}
\end{bmatrix} = \begin{bmatrix}
A & B \\
-C & -D^T
\end{bmatrix} \begin{bmatrix}
\delta x \\
\delta \lambda
\end{bmatrix} + \begin{bmatrix}
S_x \\
S_\lambda
\end{bmatrix},
\]

where

\[
A = \frac{\partial f}{\partial x} - \left[ \frac{\partial f}{\partial u} \right] E \begin{bmatrix}
\frac{\partial^2 H}{\partial u \partial x} \\
\frac{\partial h}{\partial x}
\end{bmatrix},
\]

\[
B = \left[ \frac{\partial f}{\partial u} \right] E \begin{bmatrix}
\frac{\partial f^T}{\partial u} \\
0
\end{bmatrix},
\]

\[
C = \frac{\partial^2 H}{\partial x^2} + \left[ - \frac{\partial^2 H}{\partial u \partial x} - \frac{\partial h^T}{\partial x} \right] E \begin{bmatrix}
\frac{\partial^2 H}{\partial u \partial x} \\
\frac{\partial h}{\partial x}
\end{bmatrix},
\]

\[
S_x = \left[ \frac{\partial f}{\partial u} \right] E \begin{bmatrix}
0 \\
I
\end{bmatrix} \alpha,
\]

\[
S_\lambda = \left[ - \frac{\partial^2 H}{\partial u \partial x} - \frac{\partial h^T}{\partial x} \right] E \begin{bmatrix}
0 \\
I
\end{bmatrix} \alpha.
\]

To solve the system of differential equations mentioned in Theorem 2, we can modify Sweep Method [11], generally used for solving unconstrained linear quadratic dynamic optimization problems. Assume \(\delta \lambda(t) = P(t) \delta x(t) + Q(t)\) such that at the final time

\[
\delta \lambda(T_f) = P(T_f) \delta x(T_f) + Q(T_f).
\]

From the Pontryagin Minimum Principles, we know that

\[
\delta \lambda(T_f) = \frac{\partial^2 \psi(T_f)}{\partial x^2} \delta x(T_f).
\]

Therefore, we have

\[
P(T_f) = -\frac{\partial^2 \psi(T_f)}{\partial x^2} \text{ and } Q(T_f) = 0.
\]

Substitution of \(\delta \lambda\) from (2.5) to the first row of (2.4) yields

\[
\frac{d(\delta x)}{dt} = (A - BP) \delta x + (S_x - BQ), \text{ with } \delta x(0) = 0
\]
Integrating forward in time this initial value problem, we get \( \delta x \). To obtain \( \delta \lambda \) we proceed as follows. Substituting (5) and its derivative into the second part of (4), we will have

\[
\frac{dP}{dt} \delta x + P \frac{d(\delta x)}{dt} + Q = -C \delta x - AT(P \delta x + Q) + S_2.
\]

Eliminate \( \frac{d}{dt}(\delta x) \) in the above equation with equation (8) results

\[
\frac{dp}{dt} = PBP - PA - ATP - C, \quad (9)
\]
\[
\frac{dq}{dt} = (PB - AT)Q + (S_2 - PSx). \quad (10)
\]

Integrating backward in time equations (9) and (10) with initial value from (7), we finally obtain \( \delta \lambda \). After having \( \delta x \) and \( \delta \lambda \), we can calculate \( \delta u \) from equation (4) such that \( (x + \delta x, u + \delta u) \) can be constructed from the optimal solution of problem with no perturbation, i.e. \((x, u)\).

3. Numerical simulations

This section gives a numerical example to show the application of Theorem 1 and Theorem 2. The accuracy of the result of the simulation done using MATLAB can be seen by comparison to the result produced by the reliable dynamic optimization solver software MISER 3.3 [10].

We modify an example from [3] by introducing a small perturbation \( \alpha = 0.001 \) in the constraint. Then the problem is

\[
\min \left\{ \int_0^1 (x^2 + u^2 - 2u) \ dt + 0.5 \ (x(1))^2 \right\}
\]

subject to

\[
\frac{dx}{dt} = u, \ x(0) = 0
\]

and state-constrained

\[
h(x, u, t, \alpha) = -(x^2 + u^2 - t^2 - 1) - 0.001 \leq 0.
\]

Because \((x, u, t)\) satisfying \(h(x, u, t, 0) \leq 0\) will also satisfy \(h(x, u, t, 0.001) \leq 0\) then the feasible region of \(h(x, u, t, 0) \leq 0\) will be a subset of the feasible region of \(h(x, u, t, 0.001) \leq 0\). Hence, we use (1) of Theorem 1 to evaluate \(\Delta J\), i.e.

\[
\Delta J = \int_0^1 H(x + \delta x, u + \delta u, t, \lambda, \theta, 0.001) - H(x + \delta x, u, t, \lambda, \theta, 0.001) \ dt + O(\alpha^2),
\]

where \(\delta x\) and \(\delta u\) are from Theorem 2. The result of the computation is \(\Delta J = -6.3698.10^{-4}\) which is quite close to the result from MISER 3.3, i.e. \(-6.1723.10^{-4}\). The comparison of the optimal solution from computation and MISER 3.3 are shown in the figure 1 and figure 2.

If the perturbation is changed to \(\alpha = -0.001\), then the constraint becomes

\[
h(x, u, t, \alpha) = -(x^2 + u^2 - t^2 - 1) + 0.001 \leq 0
\]

Instead of using equation (1) from Theorem 1, we need to use equation (2) of Theorem 1. This is due to the fact that the feasible solution of \(h(x, u, t, 0) \leq 0\) is a superset of the feasible region \(h(x, u, t, -0.001) \leq 0\). Using

\[
\Delta J = \int_0^1 H(x + \delta x, u + \delta u, t, \lambda, \theta, -0.001) - H(x + \delta x, u, t, \lambda, \theta, 0) \ dt + O(\alpha^2)
\]

we get \(\Delta J = 6.3727.10^{-4}\) which is little bit different to the result from MISER 3.3, i.e. \(6.5745.10^{-4}\). The figure 3 and figure 4 compare the optimal solution obtained from applying Theorem 1 and 2 to the results from MISER 3.3.

4. Conclusions

This article discusses a method to evaluate directly the effect of small perturbation on the constraint to the value of the objective function. By applying the Theorem 1, Theorem 2 and Sweep Method described in section 3, we do not need to solve the perturbed optimal control problem from the beginning. Instead,
we just need to solve the two boundary value problems which are much easier. The accuracy of the method is good that can be seen from the comparison to the result from Miser 3.3.

Figure 1. Optimal control comparison for $\alpha = 0.001$.

Figure 2. Optimal state comparison for $\alpha = 0.001$. 
Figure 3. Optimal control comparison for $\alpha = -0.001$.

Figure 4. Optimal state comparison for $\alpha = -0.001$.

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