A Characterization of Metric Projection in CAT(0) Spaces

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Abstract. In this paper, we present a characterization of metric projection in CAT(0) spaces by using the concept of quasilinearization. Furthermore, some basic properties of metric projection are investigated.

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1. Introduction

A metric space \((X, d)\) is a CAT(0) space if it is geodesically connected and if every geodesic triangle in \(X\) is at least as thin as its comparison triangle in the Euclidean plane. For other equivalent definitions and basic properties, we refer the reader to standard texts such as [1]. Complete CAT(0) spaces are often called Hadamard spaces. Let \(x, y \in X\). We write \(\lambda x \oplus (1 - \lambda)y\) for the unique point \(z\) in the geodesic segment joining from \(x\) to \(y\) such that
\[d(z, x) = (1 - \lambda)d(x, y)\quad \text{and}\quad d(z, y) = \lambda d(x, y)\]
for all \(x, y, z \in X\) and \(\lambda \in [0, 1]\).

We also denote by \([x, y]\) the geodesic segment joining from \(x\) to \(y\), that is, \([x, y] = \{\lambda x \oplus (1 - \lambda)y : \lambda \in [0, 1]\}\). A subset \(C\) of a CAT(0) space is convex if \([x, y] \subseteq C\) for all \(x, y \in C\).

Berg and Nikolaev in [2] have introduced the concept of quasilinearization. Let us formally denote a pair \((a, b)\) \(\in X \times X\) by \(\overrightarrow{ab}\) and call it a vector. Then quasilinearization is the map \(\langle \cdot, \cdot \rangle : (X \times X) \times (X \times X) \to \mathbb{R}\) defined by
\[
\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \frac{1}{2} \left(d^2(a, d) + d^2(b, c) - d^2(a, c) - d^2(b, d)\right), \quad (a, b, c, d \in X).
\]

We say that \(X\) satisfies the Cauchy-Schwarz inequality if
\[
\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle \leq d(a, b)d(c, d)
\]
for all \(a, b, c, d \in X\). It known [2, Corollary 3] that a geodesically connected metric space is CAT(0) space if and only if it satisfies the Cauchy-Schwarz inequality.

We need the following lemma in the sequel.

Lemma 1.1. [4, Lemma 2.5] A geodesic space \(X\) is a CAT(0) space if and only if the following inequality
\[
d^2(\lambda x \oplus (1 - \lambda)y, z) \leq \lambda d^2(x, z) + (1 - \lambda)d^2(y, z) - \lambda(1 - \lambda)d^2(x, y)
\]
is satisfied for all \(x, y, z \in X\) and \(\lambda \in [0, 1]\).
2. Main results

Let $C$ be a nonempty complete convex subset of a CAT(0) space $X$. It is known [1, Proposition 2.4] that for any $x \in X$ there exists a unique point $x_0 \in C$ such that
\[ d(x, x_0) = \min_{y \in C} d(x, y). \]
The mapping $P_C : X \to C$ defined by $P_C x = x_0$ is called the metric projection from $X$ onto $C$.

We need the following useful lemma to prove our main result.

Lemma 2.1. (For a general case see [3, Lemma 4.1.1]) Let $X$ be a CAT(0) space, $x, y \in X$, $\lambda \in [0, 1]$ and $z = \lambda x \oplus (1 - \lambda)y$. Then,
\[ \langle z, z \rangle \leq \lambda \langle x, z \rangle \]
for all $y \in X$.

Proof. Using (1.1) and (1.3), we have
\[ 2(\langle z, z \rangle - \lambda \langle x, z \rangle) = d_2(x, w) + d_2(y, z) - d_2(y, w) \]
\[ = -\lambda(d_2(x, w) + d_2(y, z) - d_2(x, z) - d_2(y, w)) \]
\[ \leq \lambda d_2(x, w) + (1 - \lambda)d_2(y, w) - \lambda(1 - \lambda)d_2(x, y) + d_2(y, z) \]
\[ = (1 - \lambda)d_2(x, z) + \lambda d_2(x, y) - \lambda(1 - \lambda)d_2(x, y) \]
\[ = \lambda^2(1 - \lambda)d_2(x, y) + \lambda(1 - \lambda)^2d_2(x, y) - \lambda(1 - \lambda)d_2(x, y) \]
\[ = 0, \]
which is the desired inequality. \qed

Theorem 2.2. Let $C$ be a nonempty convex subset of a CAT(0) space $X$, $x \in X$ and $u \in C$. Then $u = P_C x$ if and only if
\[ \langle x, u \rangle \geq 0 \] (2.2)
for all $y \in C$.

Proof. Let $\langle x, u \rangle \geq 0$ for all $y \in C$. If $d(x, u) = 0$, then the assertion is clear. Otherwise, we have
\[ \langle x, u \rangle - \langle x, x \rangle = \langle x, u \rangle \geq 0. \]
This together with Cauchy-Schwarz inequality implies that
\[ d_2(x, u) = \langle x, u \rangle \leq \langle x, x \rangle \leq d(x, y). \]
That is, $d(x, u) \leq d(x, y)$ for all $y \in C$ and so $u = P_C x$.

Conversely, let $u = P_C x$. Since $C$ is convex, then $z = \lambda y \oplus (1 - \lambda)u \in C$ for all $y \in C$ and $\lambda \in (0, 1)$. Thus, $d(x, u) \leq d(x, z)$. Using (1.1) we have
\[ \langle x, u \rangle \geq \frac{1}{2}d_2(x, z) - \frac{1}{2}d_2(x, u) \geq 0. \] (2.3)
On the other hand, by using Lemma 2.1, we have $\langle x, x \rangle \leq \lambda \langle x, y \rangle$. This together with (2.3) implies that
\[ \langle x, u \rangle \geq 0. \]
Since the function $d(\cdot, x) : X \to \mathbb{R}$ is continuous for all $x \in X$, letting $\lambda \to 0^+$, we have $\langle x, u \rangle \geq 0$. This completes the proof. \qed
Theorem 2.3. Let $C$ be a nonempty subset of a CAT(0) space $X$ and $x \in X$. Then $P_C x \subset \partial C$, where $P_C x = \{ z \in C : d(x, z) = \inf_{y \in C} d(x, y) \}$ and $\partial C$ is the boundary of $C$.

Proof. Let $u \in P_C x$ and $u \not\in \partial C$. Then there exists an $\varepsilon > 0$ such that $B(u, \varepsilon) \subset C$, where $B(u, \varepsilon)$ denotes the open ball with center $u$ and radius $\varepsilon$. For each $n \geq 1$, let $z_n = 1/n x \oplus (1 - 1/n) u$. We know that $d(z_n, u) = \frac{1}{n} d(x, u)$. Hence, for sufficiently large $N \geq 1$, $d(z_N, u) < \varepsilon$. Thus $z_N \in \partial C$. On the other hand, $d(z_N, x) = \left(1 - \frac{1}{N}\right) d(x, u) < d(x, C)$, which contradicts the fact that $u \in P_C x$. Therefore, $u \in \partial C$. \hfill \Box

A self-mapping $T$ of $C \subset X$ is said to be

(i) nonexpansive if $d(Tx, Ty) \leq d(x, y)$,
(ii) firmly nonexpansive if $\langle x - Ty, x - Ty \rangle \geq d^2(Tx, Ty)$,
(iii) monotone if $\langle x - Ty, x - Ty \rangle \geq 0$,

for all $x, y \in C$. It is clear that every firmly nonexpansive mapping is monotone. Also, it follows from Cauchy-Schwarz inequality that every firmly nonexpansive mapping is nonexpansive.

Proposition 2.4. Let $C$ be a nonempty closed convex subset of a Hadamard space $X$. Then, the metric projection $P_C : X \to C \subset X$ is firmly nonexpansive and so it is monotone and nonexpansive.

Proof. Let $x, y \in X$. Since $P_C x, P_C y \in C$, it follows from Theorem 2.2 that $\langle x P_C x, P_C x P_C y \rangle \geq 0$ and $\langle y P_C y, P_C y P_C x \rangle \geq 0$.

Therefore,

$$\langle x P_C x, P_C x P_C y \rangle = \langle x P_C x, P_C x P_C y \rangle + \langle P_C x P_C y, P_C x P_C y \rangle + \langle y P_C y, P_C y P_C x \rangle$$
$$\geq \langle P_C x P_C y, P_C x P_C y \rangle$$
$$= d^2(P_C x, P_C y),$$

which completes the proof. \hfill \Box

REFERENCES

[1] M. Bridson and A. Haefliger, Metric Spaces of Nonpositive Curvature, Springer-Verlag, Berlin, Heidelberg, New York, 1999.
[2] I.D. Berg and I.G. Nikolaev, Quasilinearization and curvature of Alexandrov spaces, Geom. Dedicata 133 (2008) 195–218.
[3] H. Dehghan, Fixed Point Approximations, Scholar’s Press, Saarbrücken, 2013.
[4] S. Dhompongsa, B. Panyanak, On $\Delta$-convergence theorems in CAT(0) spaces, Comput. Math. Appl. 56 (2008) 2572-2579.