Quadratic Suffices for Over-parametrization via Matrix Chernoff Bound

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Abstract

We improve the over-parametrization size over two beautiful results [Li and Liang’ 2018] and [Du, Zhai, Poczos and Singh’ 2019] in deep learning theory.

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1 Introduction

Over-parameterization theory for deep neural networks becomes extremely popular over the last few years. There is a long line (still growing very quickly) of work proving that (stochastic) gradient descent algorithm is able to find the global minimum if the network is wide enough \([LL18, DZPS19, AZLS18, AZLS19, DLL+19, ZCG18]\). One fundamental question for over-parameterization is, how wide should the neural network be?

Formally speaking, the existing results show that as long as the width \(m\) is at least polynomial of number of input data \(n\), then \((S)GD\)-type algorithm can work in the following sense: we first randomly pick a weight matrix to be the initialization point, update the weight matrix according to gradient direction over each iteration, and eventually find the global minimum. It is conjectured [Lee18] that \(m = \Omega(n^{poly}(\log(n/\delta)))\) is the right answer, where \(\delta\) is the failure probability. The randomness is from the random initialization and also algorithm itself, but not from data. There are other work relied on input data to be random [BG17, Tia17, ZSJ+17, Sol17, LY17, ZSD17, DLT+18, GLM18, BJW19], however over-parameterization theory does not allow that assumption and it only needs to make very mild assumption on data, e.g. separable. The breakthrough result by Li and Liang [LL18] is the first one that is able to explain why the greedy algorithm works very well in practice for ReLU neural network from over-parameterization perspective. The state-of-the-art result for one-hidden-layer neural network with ReLU activation function is due to Du, Zhai, Poczos and Singh [DZPS19]. Their beautiful result proves that \(m = \Omega(n^{6}\text{poly}(\log(n/\delta)))\) is sufficient. We improve the result [DZPS19] from two perspectives: one is the dependence on failure probability, and the other is the dependence on the number of input data. More precisely, we show that \(m = \Omega(n^{4}\text{poly}(\log(n/\delta)))\) is sufficient via a careful concentration analysis. More interestingly, when the input data have certain property, we can improve the bound to \(m = \Omega(n^{2}\text{poly}(\log(n/\delta)))\) via a more careful concentration analysis for random variables.

The study on concentration of summation of random variables dates back to Central Limit Theorem. The first modern concentration bounds were probably proposed by Bernstein [Ber24]. Chernoff bound is an extremely popular variant, which was introduced by Rubin and published by Chernoff [Che52]. Chernoff bound is a fundamental tool in Theoretical Computer Science and has been used in almost every randomized algorithm paper without even stating it. One common statement is the following: given a list of independent random variables \(x_1, \cdots, x_m \in [0, 1]\) with mean \(\mu\), then

\[
\Pr\left[ \frac{1}{m} \sum_{i=1}^{m} x_i - \mu > \epsilon \right] \leq 2 \exp(-\Omega(m\epsilon^2)).
\]

In many applications, we are not just dealing with scalar random variables. A natural generalization of the Chernoff bound appeared in the works of Rudelson [Rud09], Ahlswede-Winter [AW02], and Tropp [Tro12]. They proved that a similar concentration phenomenon is true even for matrix random variables. Given a list of independent complex Hermitian random matrices \(X_1, \cdots, X_m \in \mathbb{C}^{n \times n}\) with mean \(\mu\) and \(\|X_i\| \leq 1, \forall i \in [m]\), then

\[
\Pr\left[ \left\| \frac{1}{m} \sum_{i=1}^{m} X_i - \mu \right\| > \epsilon \right] \leq 2n \exp(-\Omega(n\epsilon^2)).
\]

For a more detailed survey and recent progress on the topic Matrix Chernoff bound, we refer readers to [Tro15, GLSS18, KS18].

In this work, we draw an interesting connection between deep learning theory and Matrix Chernoff bound: we can view the width of neural network as the number of independent random matrices.
1.1 Our Result

We start with the definition of Gram matrix, which can be found in [DZPS19].

**Definition 1.1** (Data-dependent function $H$). Given a collection of data $\{x_1, \ldots, x_n\} \subset \mathbb{R}^d$. For any vector $w \in \mathbb{R}^d$, we define symmetric matrix $H(w) \in \mathbb{R}^{n \times n}$ as follows

$$H(w)_{i,j} = x_i^T x_j 1_{w^T x_i \geq 0, w^T x_j \geq 0}, \forall (i,j) \in [n] \times [n].$$

Then we define continuous Gram matrix $H^{\text{cts}} \in \mathbb{R}^{n \times n}$ in the following sense

$$H^{\text{cts}} = \mathbb{E}_{w \sim N(0,I)}[H(w)].$$

Similarly, we define discrete Gram matrix $H^{\text{dis}} \in \mathbb{R}^{n \times n}$ in the following sense

$$H^{\text{dis}} = \frac{1}{m} \sum_{r=1}^{m} H(w_r).$$

We use $N(0,I)$ to denote Gaussian distribution. We use $\mathbb{E}_w$ to denote $\mathbb{E}_{w \sim N(0,I)}$ and $\Pr_w$ to denote $\Pr_{w \sim N(0,I)}$. We introduce some mild data-dependent assumption. Without loss of generality, we can assume that $\|x_i\|_2 \leq 1, \forall i \in [n].$

**Assumption 1.2** (Data-dependent assumption). We made the following data-dependent assumption:

1. Let $\lambda = \lambda_{\text{min}}(\mathbb{E}_w[H(w)])$ and $\lambda \in (0,1]$.
2. Let $\alpha \in [1,n]$ and $\gamma \in [0,1)$ be the parameter such that

$$\Pr_w \left( \left\| H(w) - \mathbb{E}_w[H] \right\| \leq \alpha \right) \geq 1 - \gamma.$$

3. Let $\beta \in [1,n^2]$ be the parameter such that

$$\left\| \mathbb{E}_w \left( (H(w) - \mathbb{E}_w[H]) \left( H(w) - \mathbb{E}_w[H] \right)^\top \right) \right\| \leq \beta.$$

4. Let $\theta \in [0,\sqrt{n}]$ be parameter such that

$$|x_i^T x_j| \leq \theta/\sqrt{n}, \forall i \neq j.$$

The first assumption is from [DZPS19]. For more detailed discussion about that assumption, we refer the readers to [DZPS19]. The last assumption is similar to assumption in [AZLS18, AZLS19], where they assumed that for $i \neq j$, $\|x_i - x_j\|_2 \geq \theta'$. If we think of $\|x_i\|_2 = 1, \forall i \in [n]$, then we know that $(\theta')^2 \leq 2 - 2x_i^T x_j$. It indicates $(\theta')^2 + 2\theta/\sqrt{n} \leq 2$. The second and the third assumption are motivated by Matrix Chernoff bound. The reason for introducing these Matrix Chernoff-type assumption is, the goal is to bound the spectral norm of the sums of random matrices in several parts of the proof. One way is to relax the spectral norm to the Frobenious norm, and bound each entry of the matrix, and finally union bound over all entries in the matrix. This could potentially lose a $\sqrt{n}$ factor compared to applying Matrix Chernoff bound. We feel these assumptions can indicate how the input data affect the over-parameterization size $m$ in a more clear way.

We state our result for the concentration of sums of independent random matrices:
Table 1: Summary of Convergence Result. Let $m$ denotes the width of neural network. Let $n$ denote the number of input data points. Let $\delta$ denote the failure probability.

| Reference | $m$ | $\lambda$ | $\alpha$ | $\theta$ |
|-----------|-----|----------|---------|---------|
| [DZPS19]  | $\lambda^{-4}n^6\text{poly}(\log n, 1/\delta)$ | Yes      | No      | No      |
| Theorem 1.4 | $\lambda^{-4}n^4\text{poly}(n/\delta)$ | Yes      | No      | No      |
| Theorem 1.5 | $\lambda^{-4}n^3\text{poly}(n/\delta) \cdot \alpha$ | Yes      | Yes     | No      |
| Theorem 1.6 | $\lambda^{-4}n^2\text{poly}(n/\delta) \cdot \alpha(\alpha + \theta^2)$ | Yes      | Yes     | Yes     |

**Proposition 1.3** (Informal of Theorem 5.1). Assume Part 1, 2 and 3 of Assumption 1.2. If $m = \Omega((\lambda^{-2}\beta + \lambda^{-1}\alpha)\log(n/\delta))$, then

$$\Pr_{w_1, \ldots, w_m \in \mathcal{N}(0, I)} [\|H^{\text{dis}} - H^{\text{cts}}\|_2 \leq \lambda/4] \geq 1 - \delta.$$  

Proposition 1.3 is a direct improvement compared to Lemma 3.1 in [DZPS19], which requires $m = \Omega(\lambda^{-2}n^2\log(n/\delta))$. Proposition 1.3 is better when input data points have some good properties, e.g., $\beta, \alpha = o(n^2)$. However the result in [DZPS19] always needs to pay $n^2$ factor, no matter what the input data points are.

We state our convergence result as follows:

**Theorem 1.4** (Informal of Theorem 4.6). Assume Part 1 of Assumption 1.2. Let $m$ denote the width of neural network, let $n$ denote the number of input data points. If $m = \Omega(\lambda^{-4}n^4\text{poly}(\log(n/\delta)))$, then gradient descent is able to find the global minimum from a random initialization point with probability $1 - \delta$.

Theorem 1.4 is a direct improvement compared to Theorem 4.1 in [DZPS19], which requires $m = \Omega(\lambda^{-4}n^6\text{poly}(\log(n/\delta)))$.

If we also allow Part 2 of Assumption 1.2, we can slightly improve Theorem 1.4 from $n^4$ to $n^3$,

**Theorem 1.5** (Informal of Theorem 5.5). Assume Part 1 and 2 of Assumption 1.2. If $m = \Omega(\lambda^{-4}n^3\alpha\text{poly}(\log(n/\delta)))$, then gradient descent is able to find the global minimum from a random initialization point with probability $1 - \delta$.

Except for $m$, Theorem 4.1 in [DZPS19] requires step size $\eta$ to be $\Theta(\lambda/n^2)$. Theorem 1.5 only needs step size $\eta$ to be $\Theta(\lambda/(\alpha n))$.

Further, if we also allow Part 4 of Assumption 1.2, we can slightly improve Theorem 1.4 from $n^4$ to $n^2$,

**Theorem 1.6** (Informal of Theorem 6.4). Assume Part 1, 2 and 4 of Assumption 1.2. If $m = \Omega(\lambda^{-4}n^2\alpha(\theta^2 + \alpha)\text{poly}(\log(n/\delta)))$, the gradient descent is able to find the global minimum from a random initialization point with probability $1 - \delta$.

1.2 Technical Overview

We follow the exact same optimization framework as Du, Zhai, Poczos and Singh [DZPS19]. We improve the bound on $m$ by doing a careful concentration analysis for random variables without changing the high-level optimization framework.

We briefly summarize the optimization framework here: the minimal eigenvalue $\lambda$ of $H^{\text{cts}}$, as introduced in [DZPS19], turns out to be closely related with the convergence rate. As time evolves, the weights $w$ in the network may vary; however if $w$ stay in a ball of radius $R$ that only depends on
the number of data $n$ and $\lambda$, and particularly does not depend on the number of neurons $m$, then we are still able to lower bound the minimal eigenvalue of $H(w)$. On the other hand, we want to upper bound $D$, the actual move of $w$, with high probability. It turns out $D$ is proportional to $\frac{1}{\sqrt{m}}$

We require $D < R$ in order to control the convergence rate. In this way we derive a lower bound of $m$.

Next we cover the concentration techniques we use in this work. In order to bound $\|H\|$,

[DZPS19] relax it to Frobenius norm and then relax it to entry-wise L1 norm,

$$\|H\| \leq \|H\|_F \leq \|H\|_1.$$  

Then they can bound each term of $H_{i,j}$ individually via Markov inequality.

One key observation is that $\|H\|_1$ is a quite loose bound for $\|H\|_F$, in the sense that $\|H\|_1 = \|H\|_F$ holds only if $H$ contains at most 1 non-zero entry. This means we can work on the Frobenius norm directly, and we shall be able to obtain a tighter estimation. By definition of $H$, it can be written as a summation of $m$ independent matrices $A_1, A_2, \cdots, A_m \in \mathbb{R}^{n \times n}$,

$$H = \frac{1}{m} \sum_{r=1}^{m} A_r$$

In order to bound $\|H\|_F$, for each $i,j$, we regard each $H_{ij}$ as summation of $m$ independent random variables, then apply Bernstein bound to obtain experiential tail bound on the concentration of $H_{ij}$. Finally, by taking a union bound over all the $n^2$ pairs we obtain a tighter bound for $\|H\|_F$.

We shall mention that $\|H\|_F$ is also a loose upper bound of $\|H\|$, i.e., $\|H\|_F = \|H\|$ only if $H$ is a rank-1 matrix. Hence, if the condition number of $H$ is small, which may happen as a property of the data, then we may benefit from bounding $\|H\|$ directly. We achieve this by apply matrix Chernoff bound, which states the spectral norm of summation of $m$ independent matrices concentrates under certain conditions.

We shall stress that mutually independence plays a very important role in our argument. Throughout the whole paper we are dealing with summations of the form $\sum_{r=1}^{m} y_r$ where $\{y_m\}_{r=1}^{m}$ are independent random variables. Previous argument mainly applies Markov inequality, which pays a factor of $1/\delta$ around the mean for error probability $\delta$. But we can obtain much tighter concentration bound by taking advantage of independence as in Bernstein inequality and Hoeffding inequality. This allows us to improve the dependency on $\delta$ from $1/\delta$ to $\log 1/\delta$.

We also make use of matrix spectral norm to deal with summation of the form $\|\sum_{i=1}^{n} a_i x_i\|_2$ where $\{a_i\}_{i=1}^{n}$ are scalars and $\{x_i\}_{i=1}^{n}$ are vectors. Naively applying triangle inequality leads to an upper bound proportional to $\|a\|_1$, which can be as large as $\sqrt{n} \|a\|_2$. Instead, we observe that the matrix formed by $(x_1 \cdots x_n) := X$ has good singular value property, which allows us to obtain the bound $\|X\|_2 \cdot \|a\|_2$. Therefore, this bound does not rely on number of inputs explicitly.

Roadmap We provide some basic definitions and probability tools in Section 2. We define the optimization problem in Section 3. We present our quartic result in Section 4. We improve it to cubic and quadratic in Sections 5 and 6.

2 Preliminaries

2.1 Notation

We use $[n]$ to denote $\{1,2,\cdots,n\}$. We use $\phi$ to denote ReLU activation function, i.e., $\phi(x) = \max\{x,0\}$. For an event $f(x)$, we define $1_{f(x)}$ such that $1_{f(x)} = 1$ if $f(x)$ holds and $1_{f(x)} = 0$
otherwise. For a matrix $A$, we use $\|A\|$ to denote the spectral norm of $A$. We define $\|A\|_F = (\sum_i \sum_j A_{i,j}^2)^{1/2}$ and $\|A\|_1 = \sum_i \sum_j |A_{i,j}|$.

### 2.2 Probability tools

**Lemma 2.1** (Chernoff bound [Che52]). Let $X = \sum_{i=1}^n X_i$, where $X_i = 1$ with probability $p_i$ and $X_i = 0$ with probability $1 - p_i$, and all $X_i$ are independent. Let $\mu = \mathbb{E}[X] = \sum_{i=1}^n p_i$. Then

1. $\Pr[X \geq (1 + \delta)\mu] \leq \exp(-\delta^2 \mu/3)$, $\forall \delta > 0$;
2. $\Pr[X \leq (1 - \delta)\mu] \leq \exp(-\delta^2 \mu/2)$, $\forall 0 < \delta < 1$.

**Lemma 2.2** (Hoeffding bound [Hoe63]). Let $X_1, \cdots, X_n$ denote $n$ independent bounded variables in $[a_i, b_i]$. Let $X = \sum_{i=1}^n X_i$, then we have

$$\Pr[|X - \mathbb{E}[X]| \geq t] \leq 2 \exp\left(-\frac{2t^2}{\sum_{i=1}^n (b_i - a_i)^2}\right).$$

**Lemma 2.3** (Bernstein inequality [Ber24]). Let $X_1, \cdots, X_n$ be independent zero-mean random variables. Suppose that $|X_i| \leq M$ almost surely, for all $i$. Then, for all positive $t$,

$$\Pr\left[\sum_{i=1}^n X_i > t\right] \leq \exp\left(-\frac{t^2/2}{\sum_{j=1}^n \mathbb{E}[X_j^2] + Mt/3}\right).$$

**Lemma 2.4** (Anti-concentration of Gaussian distribution). Let $X \sim N(0, \sigma^2)$, that is, the probability density function of $X$ is given by $\phi(x) = \frac{1}{\sqrt{2\pi}\sigma^2} e^{-\frac{x^2}{2\sigma^2}}$. Then

$$\Pr[|X| \leq t] \in \left(\frac{2}{3} \frac{t}{\sigma}, \frac{4}{3} \frac{t}{\sigma}\right).$$

**Lemma 2.5** (Matrix Bernstein, Theorem 6.1.1 in [Tro15]). Consider a finite sequence $\{X_1, \cdots, X_m\} \subset \mathbb{R}^{n_1 \times n_2}$ of independent, random matrices with common dimension $n_1 \times n_2$. Assume that

$$\mathbb{E}[X_i] = 0, \forall i \in [m] \quad \text{and} \quad \|X_i\| \leq M, \forall i \in [m].$$

Let $Z = \sum_{i=1}^m X_i$. Let $\mathbb{V}ar[Z]$ be the matrix variance statistic of sum:

$$\mathbb{V}ar[Z] = \max \left\{ \left\| \sum_{i=1}^m \mathbb{E}[X_i^T X_i] \right\|, \left\| \sum_{i=1}^m \mathbb{E}[X_i^T X_i] \right\| \right\}. $$

Then

$$\mathbb{E}[\|Z\|] \leq (2 \mathbb{V}ar[Z] \cdot \log(n_1 + n_2))^{1/2} + M \cdot \log(n_1 + n_2)/3.$$

Furthermore, for all $t \geq 0$,

$$\Pr[\|Z\| \geq t] \leq (n_1 + n_2) \cdot \exp\left(-\frac{t^2/2}{\mathbb{V}ar[Z] + Mt/3}\right).$$
3 Problem Formulation

Our problem formulation is the same as [DZPS19]. We consider a two-layer ReLU activated neural network with $m$ neurons in the hidden layer:

$$f(W, x, a) = \frac{1}{\sqrt{m}} \sum_{r=1}^{m} a_r \phi(w_r^\top x),$$

where $x \in \mathbb{R}^d$ is the input, $w_1, \ldots, w_m \in \mathbb{R}^d$ are weight vectors in the first layer, $a_1, \ldots, a_m \in \mathbb{R}$ are weights in the second layer. For simplicity, we only optimize $W$ but not optimize $a$ and $W$ at the same time.

Recall that the ReLU function $\phi(x) = \max\{x, 0\}$. Therefore for $r \in [m]$, we have

$$\frac{f(W, x, a)}{\partial w_r} = \frac{1}{\sqrt{m}} a_r x_1 w_r^\top x_{i \geq 0}. \quad (1)$$

We apply the gradient descent to optimize the weight matrix $W$ in the following standard way,

$$W(k + 1) = W(k) - \eta \frac{\partial L(W(k))}{\partial W(k)}. \quad (2)$$

We define objective function $L$ as follows

$$L(W) = \frac{1}{2} \sum_{i=1}^{n} (y_i - f(W, x_i, a))^2.$$ We can compute the gradient of $L$ in terms of $w_r$

$$\frac{\partial L(W)}{\partial w_r} = \frac{1}{\sqrt{m}} \sum_{i=1}^{n} (f(W, x_i, a_r) - y_i) a_r x_i 1_{w_r^\top x_{i \geq 0}}. \quad (3)$$

We consider the ordinary differential equation defined by

$$\frac{dw_r(t)}{dt} = - \frac{\partial L(W)}{\partial w_r}. \quad (4)$$

At time $t$, let $u(t) = (u_1(t), \ldots, u_n(t)) \in \mathbb{R}^n$ be the prediction vector where each $u_i(t)$ is defined as

$$u_i(t) = f(W(t), a, x_i). \quad (5)$$

4 Quartic Suffices

4.1 Bounding the difference between continuous and discrete

**Lemma 4.1** (Lemma 3.1 in [DZPS19]). We define $H^{cts}, H^{dis} \in \mathbb{R}^{n \times n}$ as follows

$$H^{cts}_{i,j} = \mathbb{E}_{w \sim N(0, I)} \left[ x_i^\top x_j 1_{w^\top x_i \geq 0, w^\top x_j \geq 0} \right],$$

$$H^{dis}_{i,j} = \frac{1}{m} \sum_{r=1}^{m} \left[ x_i^\top x_j 1_{w_r^\top x_i \geq 0, w_r^\top x_j \geq 0} \right].$$
Let $\lambda = \lambda_{\min}(H^{\text{cts}})$. If $m = \Omega(\lambda^{-2} n^2 \log(n/\delta))$, we have
\[
\|H^{\text{dis}} - H^{\text{cts}}\|_F \leq \frac{\lambda}{4}, \text{ and } \lambda_{\min}(H^{\text{dis}}) \geq \frac{3}{4}\lambda.
\]
hold with probability at least $1 - \delta$.

For the completeness, we still provide a proof here.

**Proof.** For every fixed pair $(i, j)$, $H^{\text{dis}}_{i,j}$ is an average of independent random variables, i.e.
\[
H^{\text{dis}}_{i,j} = \frac{1}{m} \sum_{r=1}^{m} x_i^T x_j 1_{w_r^T x_i \geq 0, w_r^T x_j \geq 0}.
\]

Then the expectation of $H^{\text{dis}}_{i,j}$ is
\[
\mathbb{E}[H^{\text{dis}}_{i,j}] = \frac{1}{m} \sum_{r=1}^{m} \mathbb{E}_{w_r \sim N(0, I_d)} [x_i^T x_j 1_{w_r^T x_i \geq 0, w_r^T x_j \geq 0}]
\]
\[
= \mathbb{E}_{w \sim N(0, I_d)} [x_i^T x_j 1_{w^T x_i \geq 0, w^T x_j \geq 0}]
\]
\[
= H^{\text{cts}}_{i,j}.
\]

For $r \in [m]$, let $z_r = \frac{1}{m} x_i^T x_j 1_{w_r^T x_i \geq 0, w_r^T x_j \geq 0}$. Then $z_r$ is a random function of $w_r$, hence $\{z_r\}_{r \in [m]}$ are mutually independent. Moreover, $-\frac{1}{m} \leq z_r \leq \frac{1}{m}$. So by Hoeffding inequality (Lemma 2.2) we have for all $t > 0$,
\[
\Pr \left[ |H^{\text{dis}}_{i,j} - H^{\text{cts}}_{i,j}| \geq t \right] \leq 2 \exp \left( - \frac{2t^2}{4/m} \right) = 2 \exp(-mt^2/2)
\]

Setting $t = \left( \frac{1}{m} 2 \log(2n^2/\delta) \right)^{1/2}$, we can apply union bound on all pairs $(i, j)$ to get with probability at least $1 - \delta$, for all $i, j \in [n]$,
\[
|H^{\text{dis}}_{i,j} - H^{\text{cts}}_{i,j}| \leq \left( \frac{2}{m} \log(2n^2/\delta) \right)^{1/2} \leq 4 \left( \frac{\log(n/\delta)}{m} \right)^{1/2}.
\]

Thus we have
\[
\|H^{\text{dis}} - H^{\text{cts}}\|^2 \leq \|H^{\text{dis}} - H^{\text{cts}}\|_F^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} |H^{\text{dis}}_{i,j} - H^{\text{cts}}_{i,j}|^2 \leq \frac{1}{m} 16n^2 \log(n/\delta).
\]

Hence if $m = \Omega(\lambda^{-2} n^2 \log(n/\delta))$ we have the desired result.

We define the event
\[
A_{i,r} = \left\{ \exists u : \|u - \tilde{w}_r\|_2 \leq R, 1_{x_i^T u \geq 0} \neq 1_{x_i^T u \geq 0} \right\}.
\]

Note this event happens if and only if $|\tilde{w}_r^T x_i| < R$. Recall that $\tilde{w}_r \sim N(0, I)$. By anti-concentration inequality of Gaussian (Lemma 2.4), we have
\[
\Pr[A_{i,r}] = \Pr_{z \sim N(0, 1)} [\|z\| < R] \leq \frac{2R}{\sqrt{2\pi}}.
\]
4.2 Bounding changes of $H$ when $w$ is in a small ball

We improve the Lemma 3.2 in [DZPS19] from the two perspective: one is the probability, and the other is upper bound on spectral norm.

**Lemma 4.2** (perturbed $w$). Let $R \in (0, 1)$. If $\tilde{w}_1, \ldots, \tilde{w}_m$ are i.i.d. generated $\mathcal{N}(0, I)$. For any set of weight vectors $w_1, \ldots, w_m \in \mathbb{R}^d$ that satisfy for any $r \in [m]$, $\|\tilde{w}_r - w_r\|_2 \leq R$, then the $H : \mathbb{R}^{m \times d} \rightarrow \mathbb{R}^{n \times n}$ defined

$$H(w)_{i,j} = \frac{1}{m} x_i^\top x_j \sum_{r=1}^m 1_{w_r^\top x_i \geq 0, w_r^\top x_j \geq 0}.$$ 

Then we have

$$\|H(w) - H(\tilde{w})\|_F < 2nR,$$

holds with probability at least $1 - n^2 \cdot \exp(-mR/10)$.

**Proof.** The random variable we care is

$$\sum_{i=1}^n \sum_{j=1}^n |H(\tilde{w})_{i,j} - H(w)_{i,j}|^2 = \frac{1}{m^2} \sum_{i=1}^n \sum_{j=1}^n \left| x_i^\top x_j \sum_{r=1}^m (1_{\tilde{w}_r^\top x_i \geq 0, \tilde{w}_r^\top x_j \geq 0} - 1_{w_r^\top x_i \geq 0, w_r^\top x_j \geq 0}) \right|^2 \leq \frac{1}{m^2} \sum_{i=1}^n \sum_{j=1}^n \left( \sum_{r=1}^m (1_{\tilde{w}_r^\top x_i \geq 0, \tilde{w}_r^\top x_j \geq 0} - 1_{w_r^\top x_i \geq 0, w_r^\top x_j \geq 0}) \right)^2 = \frac{1}{m^2} \sum_{i=1}^n \sum_{j=1}^n \left( \sum_{r=1}^m s_{r,i,j} \right)^2,$$

where the last step follows from for each $r, i, j$, we define

$$s_{r,i,j} := 1_{\tilde{w}_r^\top x_i \geq 0, \tilde{w}_r^\top x_j \geq 0} - 1_{w_r^\top x_i \geq 0, w_r^\top x_j \geq 0}.$$

We consider $i, j$ are fixed. We simplify $s_{r,i,j}$ to $s_r$.

Then $s_r$ is a random variable that only depends on $\tilde{w}_r$. Since $\{\tilde{w}_r\}_{r=1}^m$ are independent, $\{s_r\}_{r=1}^m$ are also mutually independent.

If $\neg A_{i,r}$ and $\neg A_{j,r}$ happen, then

$$\left| 1_{\tilde{w}_r^\top x_i \geq 0, \tilde{w}_r^\top x_j \geq 0} - 1_{w_r^\top x_i \geq 0, w_r^\top x_j \geq 0} \right| = 0.$$

If $A_{i,r}$ or $A_{j,r}$ happen, then

$$\left| 1_{\tilde{w}_r^\top x_i \geq 0, \tilde{w}_r^\top x_j \geq 0} - 1_{w_r^\top x_i \geq 0, w_r^\top x_j \geq 0} \right| \leq 1.$$

So we have

$$\mathbb{E}_{\tilde{w}_r} [s_r] \leq \mathbb{E}_{\tilde{w}_r} [1_{A_{i,r} \vee A_{j,r}}] \leq \Pr[A_{i,r}] + \Pr[A_{j,r}] \leq \frac{4R}{\sqrt{2\pi}} \leq 2R,$$

and

$$\mathbb{E}_{\tilde{w}_r} \left[ (s_r - \mathbb{E}[s_r])^2 \right] = \mathbb{E}_{\tilde{w}_r} [s_r^2] - \mathbb{E}[s_r]^2 \leq \mathbb{E}_{\tilde{w}_r} [s_r^2] \leq \mathbb{E}_{\tilde{w}_r} \left[ (1_{A_{i,r} \vee A_{j,r}})^2 \right] \leq \frac{4R}{\sqrt{2\pi}} \leq 2R.$$
Table 2: Table of Parameters for the \( m = \tilde{\Omega}(n^4) \) result in Section 4. \( \text{Nt.} \) stands for notations.

| \( \text{Nt.} \) | Choice | Place | Comment |
|----------------|--------|-------|---------|
| \( \lambda \) | \( \lambda_{\min}(H^{\text{cls}}) \) | Assumption 1.2 | Data-dependent |
| \( R \) | \( \lambda/n \) | Eq. (11) | Maximal allowed movement of weight |
| \( D_{\text{cts}} \) | \( \sqrt{m\|y-n(0)\|_2^2} / \sqrt{m\lambda} \) | Lemma 4.4 | Actual moving distance of weight, continuous case |
| \( D \) | \( \sqrt{m\|y-n(0)\|_2^2} / \sqrt{m\lambda} \) | Lemma 4.7 | Actual moving distance of weight, discrete case |
| \( \eta \) | \( \lambda/n^2 \) | Eq. (11) | Step size of gradient descent |
| \( m \) | \( \lambda^{-2}n^2 \log(n/\delta) \) | Lemma 4.1 | Bounding discrete and continuous |
| \( m \) | \( \lambda^{-4}n^4 \log^3(n/\delta) \) | Lemma 4.5 and Claim 4.8 | \( D < R \) and \( \|y-u(0)\|_2^2 = O(n) \) |

We also have \( |s_r| \leq 1 \). So we can apply Bernstein inequality (Lemma 2.3) to get for all \( t > 0 \),

\[
\Pr \left[ \sum_{r=1}^{m} s_r \geq 2mR + mt \right] \leq \Pr \left[ \sum_{r=1}^{m} (s_r - \mathbb{E}[s_r]) \geq mt \right] \leq \exp \left( -\frac{m^2t^2/2}{2mR + mt/3} \right).
\]

Choosing \( t = R \), we get

\[
\Pr \left[ \sum_{r=1}^{m} s_r \geq 3mR \right] \leq \exp \left( -\frac{m^2R^2/2}{2mR + mR/3} \right) \leq \exp \left( -mR/10 \right).
\]

Thus, we can have

\[
\Pr \left[ \frac{1}{m} \sum_{r=1}^{m} s_r \geq 2R \right] \leq \exp(-mR/10).
\]

Therefore, we complete the proof.

4.3 Loss is decreasing while weights are not changing much

For simplicity of notation, we provide the following definition.

**Definition 4.3.** For any \( s \in [0,t] \), we define matrix \( H(s) \in \mathbb{R}^{n \times n} \) as follows

\[
H(s)_{i,j} = \frac{1}{m} \sum_{r=1}^{m} x_i^\top x_j 1_{w_r(s)^\top x_i \geq 0, w_r(s)^\top x_j \geq 0}.
\]

With \( H \) defined, it becomes more convenient to write the dynamics of predictions. For each
\(i \in [n]\), we have
\[
\frac{d}{dt} u_i(t) = \sum_{r=1}^{m} \left\langle \frac{\partial f(W(t), a, x_i)}{\partial w_r(t)}, \frac{dw_r(t)}{dt} \right\rangle \\
= \sum_{r=1}^{m} \left\langle \frac{\partial f(W(t), a, x_i)}{\partial w_r(t)}, -\frac{\partial L(w(t), a)}{\partial w_r(t)} \right\rangle \\
= \sum_{r=1}^{m} \left\langle \frac{\partial f(W(t), a, x_i)}{\partial w_r(t)}, -\frac{1}{\sqrt{m}} \sum_{i=1}^{n} (f(W, x_i, a_r) - y_i)a_r x_i 1_{w_r x_i \geq 0} \right\rangle \\
= \sum_{j=1}^{n} (y_j - u_j(t)) \left\langle \frac{\partial f(W(t), a, x_i)}{\partial w_r(t)}, \frac{\partial f(W(t), a, x_j)}{\partial w_r(t)} \right\rangle \\
= \sum_{j=1}^{n} (y_j - u_j(t)) H(t)_{i,j}
\]

where the first step follows from (5) and the chain rule of derivatives, the second step uses (3), the third step uses (4), the fourth step uses (1) and (5), and the last step uses the definition of the matrix \(H\).

Hence, we have the following compact expression for \(\frac{d}{dt} u(t)\) as a whole vector:
\[
\frac{d}{dt} u(t) = H(t) \cdot (y - u(t)).
\]

**Lemma 4.4** (Lemma 3.3 in [DZPS19]). Suppose for \(0 \leq s \leq t\), \(\lambda_{\min}(H(w(s))) \geq \lambda/2\). Let \(D_{cts}\) be defined as
\[
D_{cts} := \frac{\sqrt{n}||y - u(0)||_2}{\sqrt{m\lambda}}.
\]

Then we have
\[
||y - u(t)||_2^2 \leq \exp(-\lambda t) \cdot ||y - u(0)||_2^2,
\]

and
\[
||w_r(t) - w_r(0)||_2 \leq D_{cts}.
\]

For the completeness, we still provide a proof.

**Proof.** Recall we can write the dynamics of predictions as
\[
\frac{d}{dt} u(t) = H(t) \cdot (y - u(t)).
\]

We can calculate the loss function dynamics
\[
\frac{d}{dt} ||y - u(t)||_2^2 = -2(y - u(t))^T \cdot H(t) \cdot (y - u(t)) \\
\leq -\lambda ||y - u(t)||_2^2.
\]

Thus we have \(\frac{d}{dt}(\exp(\lambda t)||y - u(t)||_2^2) \leq 0\) and \(\exp(\lambda t)||y - u(t)||_2^2\) is a decreasing function with respect to \(t\).
Using this fact we can bound the loss
\[ \| y - u(t) \|_2^2 \leq \exp(-\lambda t) \| y - u(0) \|_2^2. \] (7)

Now, we can bound the gradient norm. Recall for \( 0 \leq s \leq t \),
\[
\left\| \frac{d}{ds} w_r(s) \right\|_2 = \left\| \sum_{i=1}^{n} (y_i - u_i) \frac{1}{\sqrt{m}} a_r x_i \cdot 1_{w_r(s) \geq 0} \right\|_2 \\
\leq \frac{1}{\sqrt{m}} \sum_{i=1}^{n} |y_i - u_i(s)| \\
\leq \frac{\sqrt{n}}{\sqrt{m}} \| y - u(s) \|_2 \\
\leq \frac{\sqrt{n}}{\sqrt{m}} \exp(-\lambda s) \| y - u(0) \|_2.
\] (8)

where the first step follows from (3), the second step follows from triangle inequality and \( a_r = \pm 1 \) for \( r \in [m] \) and \( \| x_i \|_2 = 1 \) for \( i \in [n] \), the third step follows from Cauchy-Schwartz inequality, and the last step follows from (7).

Integrating the gradient, we can bound the distance from the initialization
\[
\| w_r(t) - w_r(0) \|_2 \leq \int_0^t \left\| \frac{d}{ds} w_r(s) \right\|_2 ds \\
\leq \frac{\sqrt{n}}{\sqrt{m}} \| y - u(0) \|_2.
\]

\[ \square \]

**Lemma 4.5** (Lemma 3.4 in [DZPS19]). If \( D_{cts} < R \). then for all \( t \geq 0 \), \( \lambda_{\text{min}}(H(t)) \geq \frac{1}{2} \lambda \). Moreover, for all \( r \in [m] \),
\[
\| w_r(t) - w_r(0) \| \leq D_{cts},
\]
and
\[
\| y - u(t) \|_2^2 \leq \exp(-\lambda t) \cdot \| y - u(0) \|_2^2.
\]

For the completeness, we still provide a proof.

**Proof.** Assume the conclusion does not hold at time \( t \). We argue there must be some \( s \leq t \) so that \( \lambda_{\text{min}}(H(s)) < \frac{1}{2} \lambda \).

If \( \lambda_{\text{min}}(H(t)) < \frac{1}{2} \lambda \), then we can simply take \( s = t \).

Otherwise since the conclusion does not hold, there exists \( r \) so that
\[
\| w_r(t) - w_r(0) \| \geq D_{cts} \quad \text{or} \quad \| y - u(t) \|_2^2 > \exp(-\lambda t) \| y - u(0) \|_2^2.
\]

Then by Lemma 4.4, there exists \( s \leq t \) such that
\[
\lambda_{\text{min}}(H(s)) < \frac{1}{2} \lambda.
\]
By Lemma 4.2, there exists $t_0 > 0$ defined as 
\[
t_0 = \inf \left\{ t > 0 : \max_{r \in [m]} \|w_r(t) - w_r(0)\|_2^2 \geq R \right\}.
\]
Thus at time $t_0$, there exists $r \in [m]$ satisfying $\|w_r(t_0) - w_r(0)\|_2^2 = R$. By Lemma 4.2,
\[
\lambda_{\min}(H(t')) \geq \frac{1}{2} \lambda, \forall t' \leq t_0.
\]
However, by Lemma 4.4, this implies 
\[
\|w_r(t_0) - w_r(0)\|_2 \leq D_{cts} < R,
\]
which is a contradiction.

### 4.4 Convergence

**Theorem 4.6.** Recall that $\lambda = \lambda_{\min}(H^{cts}) > 0$. Let $m = \Omega(\lambda^{-4}n^4\log(n/\delta))$, we i.i.d. initialize $w_r \in \mathcal{N}(0, I)$, $a_r$ sampled from $\{-1, +1\}$ uniformly at random for $r \in [m]$, and we set the step size $\eta = O(\lambda/n^2)$ then with probability at least $1 - \delta$ over the random initialization we have for $k = 0, 1, 2, \ldots$

\[
\|u(k) - y\|_2^2 \leq (1 - \eta \lambda/2)^k \cdot \|u(0) - y\|_2^2.
\]

**(9)**

**Correctness** We prove Theorem 4.6 by induction. The base case is $i = 0$ and it is trivially true. Assume for $i = 0, \cdots, k$ we have proved (9) to be true. We want to show (9) holds for $i = k + 1$.

From the induction hypothesis, we have the following Lemma stating that the weights should not change too much.

**Lemma 4.7** (Corollary 4.1 in [DZPS19]). If (9) holds for $i = 0, \cdots, k$, then we have for all $r \in [m]$

\[
\|w_r(k+1) - w_r(0)\|_2 \leq \frac{4\sqrt{n}\|y - u(0)\|_2}{\sqrt{m} \lambda} := D
\]

For the completeness, we still provide the proof

**Proof.** We use the norm of gradient to bound this distance,

\[
\|w_r(k + 1) - w_r(0)\|_2 \leq \eta \sum_{i=0}^{k} \left\| \frac{\partial L(W(i))}{\partial w_r(i)} \right\|_2 \\
\leq \eta \sum_{i=0}^{k} \frac{\sqrt{n}\|y - u(i)\|_2}{\sqrt{m}} \\
\leq \eta \sum_{i=0}^{k} \frac{\sqrt{n}(1 - \eta \lambda/2)^{i/2}}{\sqrt{m}} \|y - u(0)\|_2 \\
\leq \eta \sum_{i=0}^{\infty} \frac{\sqrt{n}(1 - \eta \lambda/2)^{i/2}}{\sqrt{m}} \|y - u(0)\|_2 \\
= \frac{4\sqrt{n}\|y - u(0)\|_2}{\sqrt{m} \lambda},
\]

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where the first step follows from (2), the second step follows from (8), the third step follows from the induction hypothesis, the fourth step relaxes the summation to an infinite summation, and the last step follows from $\sum_{i=0}^{\infty}(1 - \eta\lambda/2)^{i/2} = \frac{2}{\eta\lambda}$.

Thus, we complete the proof. \qed

Next, we calculate the different of predictions between two consecutive iterations, analogue to $\frac{du_{i}(t)}{dt}$ term in Lemma 4.4. For each $i \in [n]$, we have

\[
    u_i(k + 1) - u_i(k) = \frac{1}{\sqrt{m}} \sum_{r = 1}^{m} a_r \cdot \left( \phi(w_r(k + 1)\top x_i) - \phi(w_r(k)\top x_i) \right) \\
    = \frac{1}{\sqrt{m}} \sum_{r = 1}^{m} a_r \cdot \left( \phi \left( \left( w_r(k) - \eta \frac{\partial L(W(k))}{\partial w_r(k)} \right)\top x_i \right) - \phi(w_r(k)\top x_i) \right).
\]

Here we divide the right hand side into two parts. $v_{1,i}$ represents the terms that the pattern does not change and $v_{2,i}$ represents the term that pattern may changes. For each $i \in [n]$, we define $v_{1,i}$ and $v_{2,i}$ as follows

\[
    v_{1,i} := \frac{1}{\sqrt{m}} \sum_{r \in S_i} a_r \cdot \left( \phi \left( \left( w_r(k) - \eta \frac{\partial L(W(k))}{\partial w_r(k)} \right)\top x_i \right) - \phi(w_r(k)\top x_i) \right), \\
    v_{2,i} := \frac{1}{\sqrt{m}} \sum_{r \in S_i} a_r \cdot \left( \phi \left( \left( w_r(k) - \eta \frac{\partial L(W(k))}{\partial w_r(k)} \right)\top x_i \right) - \phi(w_r(k)\top x_i) \right).
\]

Thus, we can rewrite $u(k + 1) - u(k) \in \mathbb{R}^n$ in the following sense

\[
    u(k + 1) - u(k) = v_1 + v_2.
\]

In order to analyze $v_1 \in \mathbb{R}^n$, we provide definition of $H$ and $H^\perp \in \mathbb{R}^{n\times n}$ first,

\[
    H(k)_{i,j} = \frac{1}{m} \sum_{r = 1}^{m} x_i\top x_j 1_{w_r(k)\top x_i \geq 0, w_r(k)\top x_j \geq 0}, \\
    H(k)_{i,j}^\perp = \frac{1}{m} \sum_{r \in S_i} x_i\top x_j 1_{w_r(k)\top x_i \geq 0, w_r(k)\top x_j \geq 0}.
\]

Then, we can rewrite $v_{1,i} \in \mathbb{R}$

\[
    v_{1,i} = -\frac{\eta}{m} \sum_{j = 1}^{n} x_i\top x_j (u_j - y_j) \sum_{r \in S_i} 1_{w_r(k)\top x_i \geq 0, w_r(k)\top x_j \geq 0} \\
    = -\eta \sum_{j = 1}^{n} (u_j - y_j) (H_{i,j}(k) - H_{i,j}^\perp(k)),
\]

which means vector $v_1 \in \mathbb{R}^n$ can be written as

\[
    v_1 = \eta(y - u(k))\top (H(k) - H^\perp(k)). \tag{10}
\]

We are ready to prove the induction hypothesis. We can rewrite $\|y - u(k + 1)\|_2^2$ as follows:

\[
    \|y - u(k + 1)\|_2^2 = \|y - u(k) - (u(k + 1) - u(k))\|_2^2 \\
    = \|y - u(k)\|_2^2 - 2(y - u(k))\top (u(k + 1) - u(k)) + \|u(k + 1) - u(k)\|_2^2.
\]
We can rewrite the second term in the above Equation in the following sense,
\[
(y - u(k))\top (u(k+1) - u(k)) \\
= (y - u(k))\top (v_1 + v_2) \\
= (y - u(k))\top v_1 + (y - u(k))\top v_2 \\
= \eta(y - u(k))\top H(k)(y - u(k)) - \eta(y - u(k))\top H(k)\perp(y - u(k)) + (y - u(k))\top v_2,
\]
where the third step follows from Eq. (10).

We define
\[
C_1 = -2\eta(y - u(k))\top H(k)(y - u(k)), \\
C_2 = 2\eta(y - u(k))\top H(k)\perp(y - u(k)), \\
C_3 = -2(y - u(k))\top v_2, \\
C_4 = \|u(k+1) - u(k)\|_2^2.
\]

Thus, we have
\[
\|y - u(k+1)\|_2^2 = \|y - u(k)\|_2^2 + C_1 + C_2 + C_3 + C_4 \\
\leq \|y - u(k)\|_2^2 (1 - \eta\lambda + 8\eta nR + 8\eta nR + \eta^2 n^2),
\]
where the last step follows from Claim 4.9, 4.10, 4.11 and 4.12, which we will prove given later.

**Choice of \( \eta \) and \( R \).** Next, we want to choose \( \eta \) and \( R \) such that
\[
(1 - \eta\lambda + 8\eta nR + 8\eta nR + \eta^2 n^2) \leq (1 - \eta\lambda/2).
\]

If we set \( \eta = \frac{\lambda}{4n^2} \) and \( R = \frac{\lambda}{64n} \), we have
\[
8\eta nR + 8\eta nR = 16\eta nR \leq \eta\lambda/4, \quad \text{and} \quad \eta^2 n^2 \leq \eta\lambda/4.
\]
This implies
\[
\|y - u(k+1)\|_2^2 \leq \|y - u(k)\|_2^2 \cdot (1 - \eta\lambda/2)
\]
holds with probability at least \( 1 - 2n \exp(-mR) \).

**Over-parameterization size, lower bound on \( m \).** We require
\[
D = \frac{4\sqrt{n}\|y - u(0)\|_2}{\sqrt{m\lambda}} < R = \frac{\lambda}{64n},
\]
and
\[
2n \exp(-mR) \leq \delta.
\]
By Claim 4.8, it is sufficient to choose \( m = \Omega(\lambda^{-1} n^4 \log(m/\delta) \log^2(n/\delta)) \).
4.5 Technical claims

Claim 4.8. For $0 < \delta < 1$, with probability at least $1 - \delta$,

$$\|y - u(0)\|_2^2 = O(n \log(m/\delta) \log^2(n/\delta)).$$

Proof.

$$\|y - u(0)\|_2^2 = \sum_{i=1}^n (y_i - f(W(0), a, x_i))^2$$

$$= \sum_{i=1}^n \left( y_i - \frac{1}{\sqrt{m}} \sum_{r=1}^m a_r \phi(w_r^\top x_i) \right)^2$$

$$= \sum_{i=1}^n y_i^2 - 2 \sum_{i=1}^n \frac{y_i}{\sqrt{m}} \sum_{r=1}^m a_r \phi(w_r^\top x_i) + \sum_{i=1}^n \frac{1}{m} \sum_{r=1}^m a_r \phi(w_r^\top x_i)^2.$$

Fix $r \in [m]$ and $i \in [n]$. Since $w_r \sim N(0, I)$ and $\|x_i\|_2 = 1$, $w_r^\top x_i$ follows distribution $N(0, 1)$. From concentration of Gaussian distribution, we have

$$\Pr[w_r^\top x_i \geq \sqrt{2 \log(2mn/\delta)}] \leq \frac{\delta}{2mn}.$$

Let $E_1$ be the even that for all $r \in [m]$ and $i \in [n]$ we have

$$\phi(w_r^\top x_i) \leq \sqrt{2 \log(2mn/\delta)}.$$

Then by union bound, $\Pr[E_1] \geq 1 - \frac{\delta}{2}$.

Fix $i \in [n]$. For every $r \in [m]$, we define random variable $z_{i,r}$ as

$$z_{i,r} := \frac{1}{\sqrt{m}} a_r \cdot \phi(w_r^\top x_i) \cdot 1_{w_r^\top x_i \leq \sqrt{2 \log(2mn/\delta)}}.$$

Then $z_{i,r}$ only depends on $a_r \in \{-1, 1\}$ and $w_r \sim N(0, I)$. Notice that $E_{a_r, w_r}[z_{i,r}] = 0$, and $|z_{i,r}| \leq \sqrt{2 \log(2mn/\delta)}$. Moreover,

$$E_{a_r, w_r} \left[ z_{i,r}^2 \right] = \frac{1}{m} a_r^2 \phi^2(w_r^\top x_i) 1_{w_r^\top x_i \leq \sqrt{2 \log(2mn/\delta)}}$$

$$= \frac{1}{m} a_r^2 \Pr[w_r^\top x_i \leq \sqrt{2 \log(2mn/\delta)}]$$

$$\leq \frac{1}{m} \cdot E_{w_r} \left[ \phi^2(w_r^\top x_i) 1_{w_r^\top x_i \leq \sqrt{2 \log(2mn/\delta)}} \right]$$

$$= \frac{1}{m},$$

where the second step uses independence between $a_r$ and $w_r$, the third step uses $a_r \in \{-1, 1\}$ and $\phi(t) = \max\{t, 0\}$, and the last step follows from $w_r^\top x_i \sim N(0, 1)$.

Now we are ready to apply Bernstein inequality (Lemma 2.3) to get for all $t > 0$,  

$$\Pr \left[ \sum_{r=1}^m z_{i,r} > t \right] \leq \exp \left( -\frac{t^2/2}{m \cdot \frac{1}{m} + \sqrt{2 \log(2mn/\delta)} \cdot t/3} \right).$$
Setting $t = \sqrt{2\log(2mn/\delta) \cdot \log(4n/\delta)}$, we have with probability at least $1 - \frac{\delta}{4n}$,
\[
\sum_{i=1}^{m} z_{i,r} \leq \sqrt{2\log(2mn/\delta) \cdot \log(4n/\delta)}.
\]

Notice that we can also apply Bernstein inequality (Lemma 2.3) on $-z_{i,r}$ to get
\[
\Pr\left[ \sum_{i=1}^{m} z_{i,r} < -t \right] \leq \exp\left( -\frac{t^2/2}{m \cdot \frac{1}{m} + \sqrt{2\log(2mn/\delta) \cdot t/3}} \right).
\]

Let $E_2$ be the event that for all $i \in [n]$,
\[
\left| \sum_{i=1}^{m} z_{i,r} \right| \leq \sqrt{2\log(2mn/\delta) \cdot \log(4n/\delta)}.
\]

By applying union bound on all $i \in [n]$, we have $\Pr[E_2] \geq 1 - \frac{\delta}{2}$.

If both $E_1$ and $E_2$ happen, we have
\[
\|y - u(0)\|_2^2 = \sum_{i=1}^{n} y_i^2 - 2 \sum_{i=1}^{n} y_i \sqrt{m} \sum_{r=1}^{m} a_r \phi(w_r^\top x_i) + \sum_{i=1}^{n} \frac{1}{m} \left( \sum_{r=1}^{m} a_r \phi(w_r^\top x_i) \right)^2
\]
\[
= \sum_{i=1}^{n} y_i^2 - 2 \sum_{i=1}^{n} y_i \sum_{r=1}^{m} z_{i,r} + \sum_{i=1}^{n} \left( \sum_{r=1}^{m} z_{i,r} \right)^2
\]
\[
\leq \sum_{i=1}^{n} y_i^2 + 2 \sum_{i=1}^{n} |y_i| \sqrt{2\log(2mn/\delta) \cdot \log(4n/\delta)} + \sum_{i=1}^{n} \left( \sqrt{2\log(2mn/\delta) \cdot \log(4n/\delta)} \right)^2
\]
\[
= O(n \log(m/\delta) \log^2(n/\delta)),
\]

where the second step uses $E_1$, the third step uses $E_2$, and the last step follows from $|y_i| = O(1), \forall i \in [n]$.

By union bound, this will happen with probability at least $1 - \delta$.

Claim 4.9. Let $C_1 = -2\eta(y - u(k))^\top H(k)(y - u(k))$. We have
\[
C_1 \leq -\|y - u(k)\|_2^2 \cdot \eta \lambda.
\]

Proof. By Lemma 4.2 and our choice of $R < \frac{\lambda}{8n}$, We have $\|H(0) - H(k)\|_F \leq 2n \cdot \frac{\lambda}{8n} = \frac{\lambda}{4}$. Recall that $\lambda = \lambda_{\min}(H(0))$. Therefore
\[
\lambda_{\min}(H(k)) \geq \lambda_{\min}(H(0)) - \|H(0) - H(k)\| \geq \lambda/2.
\]

Then we have
\[
(y - u(k))^\top H(k)(y - u(k)) \geq \|y - u(k)\|_2^2 \cdot \lambda/2.
\]

Thus, we complete the proof.

Claim 4.10. Let $C_2 = 2\eta(y - u(k))^\top H(k)^\perp(y - u(k))$. We have
\[
C_2 \leq \|y - u(k)\|_2^2 \cdot 8\eta m R.
\]

holds with probability $1 - n \exp(-mR)$.
Proof. Note that
\[ C_2 \leq 2\eta\|y - u(k)\|_2^2\|H(k)^\perp\|. \]

It suffices to upper bound \(\|H(k)^\perp\|\). Since \(\|\cdot\|_F \leq \|\cdot\|_\infty\), then it suffices to upper bound \(\|\cdot\|_F\).

For each \(i \in [n]\), we define \(y_i\) as follows
\[ y_i = \sum_{r \in S_i}^m 1_r. \]

Then we have
\[
\|H(k)^\perp\|_F^2 = \sum_{i=1}^n \sum_{j=1}^n (H(k)_{i,j})^2 \\
= \sum_{i=1}^n \sum_{j=1}^n \left( \frac{1}{m} \sum_{r \in S_i} x_i^\top x_j 1_{w_r(k)^\top x_i \geq 0, w_r(k)^\top x_j \geq 0} \right)^2 \\
= \sum_{i=1}^n \sum_{j=1}^n \left( \frac{1}{m} \sum_{r = 1}^m x_i^\top x_j 1_{w_r(k)^\top x_i \geq 0, w_r(k)^\top x_j \geq 0} \cdot 1_{r \in S_i} \right)^2 \\
\leq \frac{1}{m^2} \sum_{i=1}^n \sum_{j=1}^n \left( \sum_{r = 1}^m 1_{r \in S_i} \cdot 1_{w_r(k)^\top x_i \geq 0, w_r(k)^\top x_j \geq 0} \cdot 1_{r \in S_i} \right)^2 \\
= \frac{n}{m^2} \sum_{i=1}^n \left( \sum_{r = 1}^m 1_{r \in S_i} \right)^2 \\
= \frac{n}{m^2} \sum_{i=1}^n \sum_{r = 1}^m 1_{r \in S_i}. 
\]

Fix \(i \in [n]\). The plan is to use Bernstein inequality to upper bound \(y_i\) with high probability. First by Eq. (6) we have
\[ E[1_{r \in S_i}] \leq R. \]

We also have
\[
E\left[ (1_{r \in S_i} - E[1_{r \in S_i}])^2 \right] = E[1_{r \in S_i}^2] - E[1_{r \in S_i}]^2 \\
\leq E[1_{r \in S_i}]^2 \\
\leq R.
\]

Finally we have \(1_{r \in S_i} - E[1_{r \in S_i}]^2 \leq 1\).

Notice that \(\{1_{r \in S_i}\}_{r=1}^m\) are mutually independent, since \(1_{r \in S_i}\) only depends on \(w_r(0)\). Hence from Bernstein inequality (Lemma 2.3) we have for all \(t > 0\),
\[
\Pr[y_i > m \cdot R + t] \leq \exp\left(-\frac{t^2/2}{m \cdot R + t/3}\right).
\]

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By setting \( t = 3mR \), we have
\[
\Pr[y_i > 4mR] \leq \exp(-mR).
\]
Hence by union bound, with probability at least \( 1 - n \exp(-mR) \),
\[
\|H(k)\|^2_F \leq \frac{n}{m^2} \cdot n \cdot (4mR)^2 = 16n^2R^2.
\]
Putting all together we have
\[
\|H(k)\|^2 \leq \|H(k)\|^2_F \leq 4nR
\]
with probability at least \( 1 - n \exp(-mR) \).

Claim 4.11. Let \( C_3 = -2(y - u(k))^\top v_2 \). Then we have
\[
C_3 \leq \|y - u(k)\|_2^2 \cdot 8\eta R.
\]
with probability at least \( 1 - n \exp(-mR) \).

Proof. We have
\[
\text{LHS} \leq 2\|y - u(k)\|_2 \cdot \|v_2\|_2.
\]
We can upper bound \( \|v_2\|_2 \) in the following sense
\[
\|v_2\|_2^2 \leq \sum_{i=1}^n \left( \frac{\eta}{\sqrt{m}} \sum_{r \in S_i} \left| \frac{\partial L(W(k))}{\partial w_r(k)} \right|^\top x_i \right)^2
\]
\[
= \frac{\eta^2}{m} \sum_{i=1}^n \left( \sum_{r=1}^m 1_{r \in S_i} \left| \frac{\partial L(W(k))}{\partial w_r(k)} \right|^\top x_i \right)^2
\]
\[
\leq \frac{\eta^2}{m} \cdot \max_{r \in [m]} \left| \frac{\partial L(W(k))}{\partial w_r(k)} \right|^2 \cdot \sum_{i=1}^n \left( \sum_{r=1}^m 1_{r \in S_i} \right)^2
\]
\[
\leq \frac{\eta^2}{m} \cdot \left( \frac{\sqrt{n}}{\sqrt{m}} \|u(k) - y\|_2 \right)^2 \cdot \sum_{i=1}^n \left( \sum_{r=1}^m 1_{r \in S_i} \right)^2
\]
\[
\leq \frac{\eta^2}{m} \cdot \left( \frac{\sqrt{n}}{\sqrt{m}} \|u(k) - y\|_2 \right)^2 \cdot (4mR)^2
\]
\[
= 16n^2R^2 \eta^2 \|u(k) - y\|_2^2,
\]
where the first step follows from definition of \( v_2 \), the fourth step follows from \( \max_{r \in [m]} \left| \frac{\partial L(W(k))}{\partial w_r(k)} \right| \leq \frac{\sqrt{n}}{\sqrt{m}} \cdot \|u(k) - y\|_2 \), the fifth step follows from \( \sum_{r=1}^m 1_{r \in S_i} \leq 4mR \) with probability at least \( 1 - \exp(-mR) \).

Claim 4.12. Let \( C_4 = \|u(k + 1) - u(k)\|_2^2 \). Then we have
\[
C_4 \leq \eta^2 n^2 \|y - u(k)\|_2^2.
\]
Proof. We have
\[ \text{LHS} \leq \eta^2 \sum_{i=1}^{n} \frac{1}{m} \left( \sum_{r=1}^{m} \left\| \frac{\partial L(W(k))}{\partial w_r(k)} \right\|_2 \right)^2 \]
\[ \leq \eta^2 n^2 \| y - u(k) \|_2^2. \]
\[ \square \]

5 Cubic Suffices

We prove a more general version of Lemma 4.1 in this section.

**Theorem 5.1** (Data-dependent version, bounding the difference between discrete and continuous). Let \( H^{\text{cts}} \) and \( H^{\text{dis}} \) be defined as Definition 1.1. Let \( \lambda, \alpha, \beta \) be satisfied Assumption 1.2. If
\[ m = \Omega((\lambda^{-2}\beta + \lambda^{-1}\alpha) \log(n/\delta)), \]
we have
\[ \| H^{\text{dis}} - H^{\text{cts}} \|_2 \leq \lambda/4, \text{ and } \lambda_{\min}(H^{\text{dis}}) \geq \frac{3}{4} \lambda \]
holds with probability at least \( 1 - \exp(-\Omega(\log(n/\delta))) \).

**Proof.** Recall the definition, we know
\[ H^{\text{cts}} = \mathbb{E}_w[H(w)], \text{ and } H^{\text{dis}} = \frac{1}{m} \sum_{r=1}^{m} H(w_r). \]
We define matrix \( Y_r = H(w_r) - \mathbb{E}_w[H(w)] \). We know that, \( Y_r \) are all independent,
\[ \mathbb{E}[Y_r] = 0, \quad \| Y_r \| \leq \alpha, \quad \left\| \sum_{r=1}^{m} \mathbb{E}[Y_r Y_r^\top] \right\| \leq m \beta. \]
Let \( Y = \sum_{r=1}^{m} Y_r \). We apply Matrix Bernstein inequality (Lemma 2.5) with \( t = \sqrt{m \beta \log(n/\delta)} + \alpha \log(n/\delta) \),
\[ \Pr[\| Y \| \geq t] \leq 2n \exp \left( -\frac{t^2/2}{m \beta + \alpha t/3} \right) \]
\[ \leq 2n \exp(-\log(n/\delta)) \]
\[ \leq \exp(-\Omega(\log(n/\delta))). \]
Thus, we have
\[ \Pr \left[ \left\| \frac{1}{m} \sum_{r=1}^{m} Y_r \right\| \geq \frac{1}{m} \left( \sqrt{m \beta \log(n/\delta)} + \alpha \log(n/\delta) \right) \right] \leq \exp(-\Omega(\log(n/\delta))). \]
In order to guarantee that \( \frac{1}{m}(\sqrt{m \beta \log(n/\delta)} + \alpha \log(n/\delta)) \leq \lambda \), we need
\[ \sqrt{m} \geq \lambda^{-1} \sqrt{\beta \log(n/\delta)} \]
Table 3: Table of Parameters for the $m = \tilde{\Omega}(n^3)$ result in Section 5. Nt. stands for notations.

| Nt.      | Choice                  | Place                      | Comment                                      |
|----------|-------------------------|----------------------------|----------------------------------------------|
| $\lambda$ | $\lambda_{\min}(H_{\text{cts}})$ | Part 1 of Assumption 1.2  | Data-dependent                               |
| $\alpha$ | Absolute                | Part 2 of Assumption 1.2  | Data-dependent                               |
| $\beta$  | Variance                | Part 3 of Assumption 1.2  | Data-dependent                               |
| $R$      | $\min\{(H_{\text{cts}})\}$ | Eq. (11)                  | Maximal allowed movement of weight           |
| $D_{\text{cts}}$ | $\sqrt{\alpha\|y-u(0)\|^2/m\lambda}$ | Lemma 5.2                  | Actual moving distance, continuous case      |
| $D$      | $\sqrt{4\alpha\|y-u(0)\|^2/m\lambda}$ | Theorem 5.5               | Actual moving distance, discrete case        |
| $\eta$   | $\max\{\lambda/(\alpha n), \}^2$ | Eq. (11)                  | Step size of gradient descent                |
| $m$      | $(\lambda^{-2}\beta + \lambda^{-1}\alpha)\log(n/\delta)$ | Theorem 5.1               | Bounding discrete and continuous             |
| $m$      | $\lambda^{-1}\alpha n^3 \log^2 (n/\delta)$ | Lemma 4.5 and Claim 4.8     | $D < R$ and $\|y - u(0)\|^2_2 = O(n)$       |

when the first term is the dominated one; we need

$$m \geq \lambda^{-1}\alpha \log(n/\delta).$$

Overall, we need

$$m \geq \Omega((\lambda^{-2}\beta + \lambda^{-1}\alpha) \log(n/\delta)).$$

Thus, we complete the proof.

**Lemma 5.2** (Stronger version of Lemma 3.3 in [DZPS19]). Let Part 4 in Assumption 1.2 hold. Let $D_{\text{cts}} = \sqrt{\alpha\|y-u(0)\|^2/m\lambda}$. Suppose for $0 \leq s \leq t$, $\lambda_{\min}(H(s)) \geq \lambda/2$. Then we have

$$\|y - u(t)\|^2_2 \leq \exp(-\lambda t) \cdot \|y - u(0)\|^2_2,$$

and

$$\|w_r(t) - w_r(0)\|^2_2 \leq D_{\text{cts}}.$$

**Proof.** Recall we can write the dynamics of predictions as

$$\frac{d}{dt} u(t) = H(t) \cdot (y - u(t)).$$

We can calculate the loss function dynamics

$$\frac{d}{dt}\|y - u(t)\|^2_2 = -2(y - u(t))^T \cdot H(t) \cdot (y - u(t)) \leq -\lambda\|y - u(t)\|^2_2.$$

Thus we have $\frac{d}{dt}(\exp(\lambda t)\|y - u(t)\|^2_2) \leq 0$ and $\exp(\lambda t)\|y - u(t)\|^2_2$ is a decreasing function with respect to $t$.

Using this fact we can bound the loss

$$\|y - u(t)\|^2_2 \leq \exp(-\lambda t)\|y - u(0)\|^2_2.$$

Therefore, $u(t) \to y$ exponentially fast.
Now, we can bound the gradient norm. Recall for $0 \leq s \leq t$,

$$\left\| \frac{d}{ds} w_r(s) \right\|_2^2 = \left\| \sum_{i=1}^{n} (y_i - u_i) \frac{1}{\sqrt{m}} a_r x_i \cdot 1_{w_r(s) \cdot x_i \geq 0} \right\|_2^2.$$

Define matrix $X_r \in \mathbb{R}^{d \times n}$ by setting the $i$-th column to be $1_{w_r(s) \cdot x_i \geq 0} \cdot x_i$, then $X_r^T X_r = H(w_r(s))$, where $H(\cdot)$ is the matrix defined in Definition 1.1. Then we have $\|X_r^T X_r\|_2 \leq \alpha$ by Part 4 in Assumption 1.2, which leads to $\|X_r\|_2 \leq \sqrt{\alpha}$. So we have

$$\left\| \frac{d}{ds} w_r(s) \right\|_2^2 = \frac{1}{\sqrt{m}} \|X_r (y - u(s))\|_2^2 \leq \frac{\sqrt{\alpha}}{\sqrt{m}} ||y - u(s)||_2 \leq \frac{\sqrt{\alpha}}{\sqrt{m}} \exp(-\lambda s) \|y - u(0)\|_2^2.$$

Integrating the gradient, we can bound the distance from the initialization

$$\|w_r(t) - w_r(0)\|_2 \leq \int_0^t \left\| \frac{d}{ds} w_r(s) \right\|_2^2 ds \leq \frac{\sqrt{\alpha}}{\sqrt{m\lambda}} \|y - u(0)\|_2^2.$$ 

\[\square\]

### 5.1 Technical claims

**Claim 5.3.** Let $C_3 = -2(y - u(k))^\top v_2$. Then we have

$$C_3 \leq \|y - u(k)\|_2^2 \cdot 8\eta(\alpha n)^{1/2} R.$$

with probability at least $1 - n \exp(-mR)$.

**Proof.** We have

$$\text{LHS} \leq 2\|y - u(k)\|_2 \cdot \|v_2\|_2.$$
We can upper bound $\|v_2\|_2$ in the following sense:

$$\|v_2\|_2^2 \leq \sum_{i=1}^{n} \left( \frac{\eta}{\sqrt{m}} \sum_{r \in S_i} \left( \frac{\partial L(W(k))}{\partial w_r(k)} \right)^\top x_i \right)^2$$

$$= \frac{\eta^2}{m} \sum_{i=1}^{n} \left( \sum_{r=1}^{m} 1_{r \in S_i} \left( \frac{\partial L(W(k))}{\partial w_r(k)} \right)^\top x_i \right)^2$$

$$\leq \frac{\eta^2}{m} \cdot \max_{r \in [m]} \left| \frac{\partial L(W(k))}{\partial w_r(k)} \right|^2 \cdot \sum_{i=1}^{n} \left( \sum_{r=1}^{m} 1_{r \in S_i} \right)^2$$

$$\leq \frac{\eta^2}{m} \cdot \left( \frac{\sqrt{\alpha}}{\sqrt{m}} \|u(k) - y\|_2 \right)^2 \cdot \sum_{i=1}^{n} \left( \sum_{r=1}^{m} 1_{r \in S_i} \right)^2$$

$$\leq \frac{\eta^2}{m} \cdot \left( \frac{\sqrt{\alpha}}{\sqrt{m}} \|u(k) - y\|_2 \right)^2 \cdot (4mR)^2$$

$$= 16\alpha n R^2 \eta^2 \|u(k) - y\|_2^2,$$

where the first step follows from definition of $v_2$, the fourth step follows from (12) and

$$\max_{r \in [m]} \left| \frac{\partial L(W(k))}{\partial w_r(k)} \right| = \max_{r \in [m]} \left| \frac{\partial w_r(k)}{\partial k} \right|$$

$$\leq \frac{\sqrt{\alpha}}{\sqrt{m}} \|y - u(k)\|_2,$$

the fifth step follows from $\sum_{r=1}^{m} 1_{r \in S_i} \leq 4mR$ with probability at least $1 - \exp(-mR)$.

\[ \square \]

**Claim 5.4.** Let $C_4 = \|u(k + 1) - u(k)\|_2^2$. Then we have

$$C_4 \leq \eta^2 \alpha n \|y - u(k)\|_2^2.$$

**Proof.** We have

$$\text{LHS} \leq \eta^2 \frac{1}{m} \sum_{i=1}^{n} \left( \sum_{r=1}^{m} \left| \frac{\partial L(W(k))}{\partial w_r(k)} \right| \right)^2$$

$$\leq \eta^2 \frac{1}{m} \sum_{i=1}^{n} \left( \sum_{r=1}^{m} \frac{\sqrt{\alpha}}{\sqrt{m}} \|u(k) - y\|_2 \right)^2$$

$$\leq \eta^2 \alpha n \|y - u(k)\|_2^2.$$  

\[ \square \]

**5.2 Main result**

**Theorem 5.5.** Recall that $\lambda = \lambda_{\min}(H^{\text{cts}}) > 0$. Let $m = \Omega(\lambda^{-4}n^3\alpha \log^3(n/\delta))$, i.i.d. initialize $w_r \in \mathcal{N}(0, I)$, $a_r$ sampled from $\{-1, +1\}$ uniformly at random for $r \in [m]$, and we set the step size $\eta = O(\lambda/(\alpha n))$ then with probability at least $1 - \delta$ over the random initialization we have for $k = 0, 1, 2, \cdots$

$$\|u(k) - y\|_2^2 \leq (1 - \eta \lambda/2)^k \cdot \|u(0) - y\|_2^2.$$  

(13)
Proof. This proof, similar to the proof of Theorem 4.6, is again by induction. (13) trivially holds when $k = 0$, which is the base case.

If (13) holds for $k' = 0, \ldots, k$, then we claim that for all $r \in [m]$

$$
\|w_r(k + 1) - w_r(0)\|_2 \leq \frac{4\sqrt{\alpha}\|y - u(0)\|_2}{\sqrt{m}\lambda} := D
$$

(14)

To see this, we use the norm of gradient to bound this distance,

$$
\|w_r(k + 1) - w_r(0)\|_2 \leq \eta \sum_{k'=0}^k \left\| \frac{\partial L(W(k'))}{\partial w_r(k')} \right\|_2
$$

$$
\leq \eta \sum_{k'=0}^k \frac{\sqrt{\alpha}\|y - u(k')\|_2}{\sqrt{m}}
$$

$$
\leq \eta \sum_{k'=0}^k \frac{\sqrt{\alpha}(1 - \eta \lambda/2)^{k'/2}}{\sqrt{m}}\|y - u(0)\|_2
$$

$$
\leq \eta \sum_{k'=0}^\infty \frac{\sqrt{\alpha}(1 - \eta \lambda/2)^{k'/2}}{\sqrt{m}}\|y - u(0)\|_2
$$

$$
= \frac{4\sqrt{\alpha}\|y - u(0)\|_2}{\sqrt{m}\lambda},
$$

where the first step follows from (2), the second step follows from (12), the third step follows from the induction hypothesis, the fourth step relaxes the summation to an infinite summation, and the last step follows from $\sum_{k'=0}^\infty (1 - \eta \lambda/2)^{k'/2} = \frac{2}{\eta\lambda}$.

Then from Claim 5.4, it is sufficient to choose $\eta = \frac{\lambda}{4\alpha n}$ so that (13) holds for $k' = k + 1$. This completes the induction step.

Over-parameterization size, lower bound on $m$.

We require

$$
D = \frac{4\sqrt{\alpha}\|y - u(0)\|_2}{\sqrt{m}\lambda} < R = \frac{\lambda}{64n},
$$

and

$$
2n \exp(-mR) \leq \delta.
$$

This implies that

$$
m = \Omega(\lambda^{-4}n^2\alpha\|y - u(0)\|_2^2)
$$

$$
= \Omega(\lambda^{-4}n^2\alpha\log(m/\delta) \log^2(n/\delta)),
$$

where the last step follows from Claim 4.8.

\[ \square \]

6 Quadratic Suffices

Lemma 6.1 (perturbed $w$). Let $R \in (0,1)$. Let Assumption 4 in 1.2 hold, i.e. for all $i \neq j$, $|x_i^T x_j| \leq \theta/\sqrt{n}$. If $\tilde{w}_1, \cdots, \tilde{w}_m$ are i.i.d. generated $\mathcal{N}(0, I)$. For any set of weight vectors $w_1, \cdots, w_m \in \mathbb{R}^d$.
that satisfy for any \( r \in [m] \), \( \| \bar{w}_r - w_r \|_2 \leq R \), then the \( H : \mathbb{R}^{m \times d} \to \mathbb{R}^{n \times n} \) defined

\[
H(w)_{i,j} = \frac{1}{m} x_i^\top x_j \sum_{r=1}^m 1_{w_r^\top x_i \geq 0, w_r^\top x_j \geq 0}.
\]

Then we have

\[
\|H(w) - H(\bar{w})\|_F < 2 \left( n(1 + \theta^2) \right)^{1/2} R,
\]

holds with probability at least \( 1 - n^2 \cdot \exp(-mR/10) \).

**Proof.** The random variable we care is

\[
\sum_{i=1}^n \sum_{j=1}^n |H(\bar{w})_{i,j} - H(w)_{i,j}|^2 = \frac{1}{m^2} \sum_{i=1}^n \sum_{j=1}^n |x_i^\top x_j \sum_{r=1}^m (1_{\tilde{w}_r^\top x_i \geq 0, \tilde{w}_r^\top x_j \geq 0} - 1_{w_r^\top x_i \geq 0, w_r^\top x_j \geq 0})|^2
\]

\[= B_1 + B_2,
\]

where \( B_1, B_2 \) are defined as

\[
B_1 = \frac{1}{m^2} \sum_{i=1}^n \sum_{j=1}^n |(1_{\tilde{w}_r^\top x_i \geq 0} - 1_{w_r^\top x_i \geq 0})|^2,
\]

\[
B_2 = \frac{1}{m^2} \sum_{i=1}^n \sum_{j \in [n] \setminus \{i\}} |x_i^\top x_j \sum_{r=1}^m (1_{\tilde{w}_r^\top x_i \geq 0, \tilde{w}_r^\top x_j \geq 0} - 1_{w_r^\top x_i \geq 0, w_r^\top x_j \geq 0})|^2.
\]

We can further bound \( B_2 \) as

\[
B_2 \leq \frac{1}{m^2} \sum_{i=1}^n \sum_{j \in [n] \setminus \{i\}} \frac{\theta^2}{n} \left| \sum_{r=1}^m (1_{\tilde{w}_r^\top x_i \geq 0, \tilde{w}_r^\top x_j \geq 0} - 1_{w_r^\top x_i \geq 0, w_r^\top x_j \geq 0}) \right|^2
\]

\[= \frac{\theta^2}{nm^2} \sum_{i=1}^n \sum_{j \in [n] \setminus \{i\}} \left| \sum_{r=1}^m (1_{\tilde{w}_r^\top x_i \geq 0, \tilde{w}_r^\top x_j \geq 0} - 1_{w_r^\top x_i \geq 0, w_r^\top x_j \geq 0}) \right|^2.
\]

For each \( r, i, j \), we define

\[
s_{r,i,j} := 1_{\tilde{w}_r^\top x_i \geq 0, \tilde{w}_r^\top x_j \geq 0} - 1_{w_r^\top x_i \geq 0, w_r^\top x_j \geq 0}.
\]

Then we can rewrite \( B_1 \) and \( B_2 \) as

\[
B_1 = \frac{1}{m^2} \sum_{i=1}^n \left( \sum_{r=1}^m s_{r,i,i} \right)^2,
\]

\[
B_2 = \frac{\theta^2}{nm^2} \sum_{i=1}^n \sum_{j \in [n] \setminus \{i\}} \left( \sum_{r=1}^m s_{r,i,j} \right)^2.
\]

Therefore it is sufficient to bound \( \sum_{r=1}^m s_{r,i,j} \) simultaneously for all pair \( i, j \). Using same technique in the proof of Theorem 4.2, we have

\[
\Pr \left[ \frac{1}{m} \sum_{r=1}^m s_{r,i,j} \geq 2R \right] \leq \exp(-mR/10).
\]
Table 4: Table of Parameters for the \( m = \tilde{\Omega}(n^2) \) result in Section 6. \( \text{Nt.} \) stands for notations.

| \( \text{Nt.} \) | \( \text{Choice} \) | \( \text{Place} \) | \( \text{Comment} \) |
|----------------|----------------|----------------|----------------|
| \( \lambda \) | := \( \lambda_{\min}(H^{cte}) \) | Part 1 of Assumption 1.2 | Data-dependent |
| \( \alpha \) | Absolute | Part 2 of Assumption 1.2 | Data-dependent |
| \( \beta \) | Variance | Part 3 of Assumption 1.2 | Data-dependent |
| \( \theta \) | Inner product | Part 4 of Assumption 1.2 | Data-dependent |
| \( R \) | \( \frac{\lambda}{\sqrt{n}} \cdot \min\left\{ \frac{1}{\sqrt{\alpha n}}, \frac{1}{\sqrt{1+\theta^2}} \right\} \) | Eq. (15) | Maximal allowed movement of weight |
| \( D \) | \( \frac{4\sqrt{\alpha \|y-u(0)\|^2}}{\sqrt{m\lambda}} \) | Theorem 6.4 | Actual moving distance, discrete case |
| \( \eta \) | \( \frac{\lambda}{(\alpha n)} \) | Eq. (11) | Step size of gradient descent |
| \( m \) | \( \lambda^{-2}\beta + \lambda^{-1}\alpha \log(n/\delta) \) | Theorem 5.1 | Bounding discrete and continuous |
| \( m \) | \( \lambda^{-4}\alpha(\alpha + \theta^2)n^2\log^3(n/\delta) \) | Lemma 4.5 and Claim 4.8 | \( D < R \) and \( \|y - u(0)\|^2_2 = \tilde{O}(n) \) |

By applying union bound on all \( i, j \) pairs, we get with probability at least \( 1 - \exp(-mR/10) \),

\[
\|H(w) - H(\tilde{w})\|_F^2 \leq B_1 + B_2 \leq 4nR^2(1 + \theta)^2.
\]

which is precisely what we need. \( \square \)

Claim 6.2. Assume \( R \leq \frac{\lambda}{64\sqrt{n}} \cdot \frac{1}{\sqrt{1+\theta^2}} \). Let \( C_1 = -2\eta(y - u(k))^\top H(k)(y - u(k)) \). We have

\[
C_1 \leq -\|y - u(k)\|_2^2 \cdot \eta \lambda
\]

Proof. By Lemma 6.1 and our choice of \( R \leq \frac{\lambda}{64\sqrt{n}} \cdot \frac{1}{\sqrt{1+\theta^2}} \), we have

\[
\|H(0) - H(k)\|_F \leq 2\left( n(1 + \theta^2) \right)^{1/2} \cdot \frac{\lambda}{64\sqrt{n}} \cdot \frac{1}{\sqrt{1+\theta^2}} \leq \frac{\lambda}{4}.
\]

Recall that \( \lambda = \lambda_{\min}(H(0)) \). Therefore

\[
\lambda_{\min}(H(k)) \geq \lambda_{\min}(H(0)) - \|H(0) - H(k)\| \geq \lambda/2.
\]

Then we have

\[
(y - u(k))^\top H(k)(y - u(k)) \geq \|y - u(k)\|_2^2 \cdot \lambda/2.
\]

Thus, we complete the proof. \( \square \)

Claim 6.3. Let \( C_2 = 2\eta(y - u(k))^\top H(k)^\perp(y - u(k)) \). We have

\[
C_2 \leq \|y - u(k)\|_2^2 \cdot 8\eta R \left( n(1 + \theta^2) \right)^{1/2}.
\]

holds with probability \( 1 - n \exp(-mR) \).

Proof. Note that

\[
C_2 \leq 2\eta \cdot \|y - u(k)\|_2^2 \cdot \|H(k)^\perp\|.
\]

It suffices to upper bound \( \|H(k)^\perp\| \). Since \( \| \cdot \| \leq \| \cdot \|_F \), then it suffices to upper bound \( \| \cdot \|_F \).
For each $i \in [n]$, we define $y_i$ as follows

$$y_i = \sum_{r=1}^{m} 1_{r \in S_i}.$$ 

Then we have

$$\|H(k)^\perp\|_F^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} (H(k)_{ij})^2$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \left( \frac{1}{m} \sum_{r \in S_i} x_i^\top x_j 1_{w_r(k)^\top x_i \geq 0, w_r(k)^\top x_j \geq 0} \right)^2$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \left( \frac{1}{m} \sum_{r=1}^{m} x_i^\top x_j 1_{w_r(k)^\top x_i \geq 0, w_r(k)^\top x_j \geq 0} \cdot 1_{r \in S_i} \right)^2$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \left( \frac{1}{m^2} \sum_{r=1}^{m} x_i^\top x_j \cdot 1_{w_r(k)^\top x_i \geq 0, w_r(k)^\top x_j \geq 0} \cdot 1_{r \in S_i} \right)^2$$

$$= B_1 + B_2,$$

where $B_1$ and $B_2$ are defined as:

$$B_1 := \sum_{i=1}^{n} \frac{1}{m^2} \left( \sum_{r=1}^{m} 1_{w_r(k)^\top x_i \geq 0} \cdot 1_{r \in S_i} \right)^2,$$

$$B_2 := \sum_{i=1}^{n} \sum_{j \in [n] \setminus \{i\}} \frac{x_i^\top x_j}{m^2} \left( \sum_{r=1}^{m} 1_{w_r(k)^\top x_i \geq 0, w_r(k)^\top x_j \geq 0} \cdot 1_{r \in S_i} \right)^2.$$

We bound $B_1$ and $B_2$ separately.

We first bound $B_1$.

$$B_1 = \sum_{i=1}^{n} \frac{1}{m^2} \left( \sum_{r=1}^{m} 1_{w_r(k)^\top x_i \geq 0} \cdot 1_{r \in S_i} \right)^2$$

$$\leq \frac{1}{m^2} \sum_{i=1}^{n} \left( \sum_{r=1}^{m} 1_{r \in S_i} \right)^2$$

$$= \frac{1}{m^2} \sum_{i=1}^{n} y_i^2.$$

Fix $i \in [n]$. The plan is to use Bernstein inequality to upper bound $y_i$ with high probability. First by Eq. (6) we have

$$\mathbb{E} \left[ 1_{r \in S_i} \right] \leq R.$$

We also have

$$\mathbb{E} \left[ \left( 1_{r \in S_i} - \mathbb{E} \left[ 1_{r \in S_i} \right] \right)^2 \right] = \mathbb{E} \left[ 1_{r \in S_i}^2 \right] - \mathbb{E} \left[ 1_{r \in S_i} \right]^2$$

$$\leq \mathbb{E} \left[ 1_{r \in S_i}^2 \right] \leq R.$$
Finally we have $|1_r \in \mathcal{S}_i - \mathbb{E}[1_r \in \mathcal{S}_i]| \leq 1$.

Notice that $\{1_r \in \mathcal{S}_i\}_{r=1}^m$ are mutually independent, since $1_r \in \mathcal{S}_i$ only depends on $w_r(0)$. Hence from Bernstein inequality (Lemma 2.3) we have for all $t > 0$,

$$\Pr[y_i > m \cdot R + t] \leq \exp\left(\frac{-t^2/2}{m \cdot R + t/3}\right).$$

By setting $t = 3mR$, we have

$$\Pr[y_i > 4mR] \leq \exp(-mR).$$

Hence by union bound, with probability at least $1 - n \exp(-mR)$, for all $i \in [n]$,

$$y_i \leq 4mR.$$

If this happens, we have

$$B_1 \leq 16nR^2.$$

Next we bound $B_2$. We have

$$B_2 = \sum_{i=1}^{n} \sum_{j \in [n] \setminus \{i\}} \frac{|x_i^\top x_j|^2}{m^2} \left( \sum_{r=1}^{m} 1_{w_r(k)^\top x_i \geq 0, w_r(k)^\top x_j \geq 0} \cdot 1_{r \in \mathcal{S}_i} \right)^2$$

$$\leq \frac{1}{m^2} \sum_{i=1}^{n} \left( \sum_{j \in [n] \setminus \{i\}} (x_i^\top x_j)^4 \right)^{1/2} \left( \sum_{j \in [n] \setminus \{i\}} \left( \sum_{r=1}^{m} 1_{w_r(k)^\top x_i \geq 0, w_r(k)^\top x_j \geq 0} \cdot 1_{r \in \mathcal{S}_i} \right)^4 \right)^{1/2}$$

$$\leq \frac{1}{m^2} \sum_{i=1}^{n} \left( \sum_{j \in [n] \setminus \{i\}} (x_i^\top x_j)^4 \right)^{1/2} \left( \sum_{j \in [n] \setminus \{i\}} y_i^4 \right)^{1/2}$$

$$= \sqrt{n-1} \sum_{i=1}^{n} \left( \sum_{j \in [n] \setminus \{i\}} (x_i^\top x_j)^4 \right)^{1/2} y_i^2$$

$$\leq 16R^2 \sqrt{n} \sum_{i=1}^{n} \left( \sum_{j \in [n] \setminus \{i\}} (x_i^\top x_j)^4 \right)^{1/2},$$

where the last step happens when $y_i \leq 4mR$ for all $i \in [n]$.

Now, using the assumption $x_i^\top x_j \leq \frac{\theta}{\sqrt{n}}$ (Part 4 of Assumption 1.2), we have

$$B_2 \leq 16nR^2 \theta^2.$$\[\]

Putting things together, we have with probability at least $1 - n \exp(-mR)$,

$$\|H(k)^\perp\|_F^2 \leq B_1 + B_2$$

$$\leq 16nR^2(1 + \theta^2).$$

This gives us $\|H(k)^\perp\| \leq 4R \left(n(1 + \theta^2)\right)^{1/2}$, which is precisely what we need. \[\]
6.1 Main result

**Theorem 6.4.** Let $\lambda, \alpha, \beta, \theta$ be defined as Assumption 1.2. Let

$$m = \Omega\left(\lambda^{-4}n^2\alpha \max\{1 + \theta^2, \alpha\} \log^3(n/\delta)\right).$$

We i.i.d. initialize $w_r \in \mathcal{N}(0, I)$, $a_r$ sampled from $\{-1, +1\}$ uniformly at random for $r \in [m]$, and we set the step size $\eta = O(\lambda/(\alpha n))$ then with probability at least $1 - \delta$ over the random initialization we have for $k = 0, 1, 2, \cdots$

$$\|u(k) - y\|_2^2 \leq (1 - \eta\lambda/2)^k \cdot \|u(0) - y\|_2^2.$$

**Proof. Choice of $\eta$ and $R$.** We want to choose $\eta$ and $R$ such that

$$(1 - \eta\lambda + 8\eta R \left(n(1 + \theta^2)\right)^{1/2} + 8\eta(\alpha n)^{1/2}R + \eta^2 \alpha n) \leq (1 - \eta\lambda/2). \quad (15)$$

Now, if we set $\eta = \frac{\lambda}{4an}$ and $R = \frac{\lambda}{64\sqrt{n}} \cdot \min\left\{\frac{1}{\sqrt{1+\theta^2}}, \frac{1}{\sqrt{\alpha}}\right\}$, we have

$$8\eta R \left(n(1 + \theta^2)\right)^{1/2} + 8\eta(\alpha n)^{1/2}R \leq \frac{1}{4}\eta\lambda,$$

and $\eta^2 n^2 \leq \frac{1}{4}\eta\lambda$. This gives us

$$\|y - u(k + 1)\|_2^2 \leq \|y - u(k)\|_2^2 (1 - \eta\lambda/2)$$

with probability at least $1 - 2n \exp(-mR)$.

**Over-parameterization size, lower bound on $m$.** By same analysis as in the proof of Theorem 5.5, we still have

$$\|w_r(k + 1) - w_r(0)\|_2 \leq \frac{4\sqrt{\alpha}\|y - u(0)\|_2}{\sqrt{m}\lambda} := D$$

We require

$$D = \frac{4\sqrt{\alpha}\|y - u(0)\|_2}{\sqrt{m}\lambda} < R = \frac{\lambda}{64\sqrt{n}} \cdot \min\left\{\frac{1}{\sqrt{1+\theta^2}}, \frac{1}{\sqrt{\alpha}}\right\}$$

and

$$2n \exp(-mR) \leq \delta.$$

This implies that

$$m = \Omega(\lambda^{-4}n\alpha\|y - u(0)\|_2^2 \max\{1 + \theta^2, \alpha\})$$

$$= \Omega(\lambda^{-4}n^2\alpha \cdot \max\{1 + \theta^2, \alpha\} \cdot \log(m/\delta) \log^2(n/\delta)),$$

where the last step follows from Claim 4.8. \qed

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