SIMULTANEOUS APPROXIMATION
OF A REAL NUMBER BY ALL CONJUGATES
OF AN ALGEBRAIC NUMBER

GUILLAUME ALAIN

Abstract. Using a method of H. Davenport and W. M. Schmidt, we show that, for each positive integer \( n \), the ratio \( 2/n \) is the optimal exponent of simultaneous approximation to real irrational numbers 1) by all conjugates of algebraic numbers of degree \( n \), and 2) by all but one conjugates of algebraic integers of degree \( n + 1 \).

1. Introduction

An outstanding problem in Diophantine approximation, motivated initially by Mahler’s and Koksma’s classifications of numbers, is to provide sharp estimates for the approximation of a real number by algebraic numbers of bounded degree. Starting with the pioneer work [Wi] of E. Wirsing in 1961, this problem has been studied by many authors and extended in several directions. A good account of this can be found in Chapter 3 of [Bu]. For our purpose, let us simply mention that, in 1969, H. Davenport and W. M. Schmidt gave estimates for the approximation by algebraic integers [DS] and that, more recently, D. Roy and M. Waldschmidt looked at simultaneous approximations by several conjugate algebraic integers [RW]. While the latter work was limited to at most one quarter of the conjugates, we consider here the problem of simultaneous approximation of a real number by all (resp. all but one) conjugates of an algebraic number (resp. algebraic integer). Upon defining the height \( H(P) \) of a polynomial \( P \in \mathbb{R}[T] \) to be the largest absolute value of its coefficients, and the height \( H(\alpha) \) of an algebraic number \( \alpha \in \mathbb{C} \) to be the height of its irreducible polynomial in \( \mathbb{Z}[T] \), our main result reads as follows.

Theorem A. Let \( \xi \in \mathbb{R}\setminus\mathbb{Q} \) and let \( n \in \mathbb{N}^* \). There exist positive constants \( c_1, c_2 \) depending only on \( \xi \) and \( n \) with the following properties.

(i) There are infinitely many algebraic numbers \( \alpha \) of degree \( n \) such that

\[
\max_{\sigma} |\xi - \sigma| \leq c_1 H(\alpha)^{-2/n}
\]

where the maximum is taken over all conjugates \( \sigma \) of \( \alpha \).

(ii) There are infinitely many algebraic integers \( \alpha \) of degree \( n + 1 \) such that

\[
\max_{\sigma \neq \alpha} |\xi - \sigma| \leq c_2 H(\alpha)^{-2/n}
\]

where the maximum is taken over all conjugates \( \sigma \) different from \( \alpha \).

1991 Mathematics Subject Classification. Primary 11J13; Secondary 11J70. Work partially supported by NSERC.
In the case $n = 2$, this improves the estimates of the corollary in Section 1 of [AR]. In fact, as we will see in the next section, the statement of part (i) is optimal up to the value of $c_1$ for each $\xi \in \mathbb{R} \setminus \mathbb{Q}$, while the statement of part (ii) is optimal up to the value of $c_2$ at least for quadratic irrational values of $\xi$. This seems to be the first instance where an optimal exponent of approximation is known for all values of the degree $n$ in this type of problem. The fact that we can control the degree of the approximations originates from an observation of Y. Bugeaud and O. Teulié in [BT].

An irrational real number $\xi$ is said to be badly approximable if there exists a constant $c > 0$ such that $|\xi - p/q| \geq cq^{-2}$ for any rational number $p/q$. This is equivalent to asking that $\xi$ has bounded partial quotients in its continued fraction expansion (see Theorem 5F in Chapter 1 of [Sc]). For these numbers, we can refine Theorem A as follows.

**Theorem B.** Let $\xi \in \mathbb{R} \setminus \mathbb{Q}$ be badly approximable and let $n \in \mathbb{N}^*$. Then there exist positive constants $c_1, \ldots, c_4$ depending only on $\xi$ and $n$ with the following properties. For each real number $X \geq 1$, there is an algebraic number $\alpha$ of degree $n$ satisfying (1) and $c_3 X \leq H(\alpha) \leq c_4 X$. There is also an algebraic integer $\alpha$ of degree $n + 1$ satisfying (2) and $c_3 X \leq H(\alpha) \leq c_4 X$.

The proof of both results follows the method introduced by Davenport and Schmidt in [DS]. Let $\mathbb{R}[T]_{\leq n}$ denote the real vector space of polynomials of degree $\leq n$ in $\mathbb{R}[T]$, and let $\mathbb{Z}[T]_{\leq n}$ denote the subgroup of polynomials with integral coefficients in $\mathbb{R}[T]_{\leq n}$. We first provide estimates for the last minimum of certain convex bodies of $\mathbb{R}[T]_{\leq n}$ with respect to $\mathbb{Z}[T]_{\leq n}$ and then deduce the existence of polynomials of $\mathbb{Z}[T]_{\leq n}$ with specific inhomogeneous Diophantine properties. This is done in Section 3. In Section 4, we show that these polynomials have roots which fulfil the requirements of Theorem A or B.

Throughout this paper, all implied constants in the Vinogradov symbols $\gg$, $\ll$ and their conjunction $\asymp$ depend only on $\xi$ and $n$.

2. **Optimality of the exponents of approximation**

Let $\xi \in \mathbb{R} \setminus \mathbb{Q}$ and let $n \in \mathbb{N}^*$. If $n \geq 2$, the result in part (i) of Theorem A is optimal up to the value of the implied constant since, for any algebraic number $\alpha$ of degree $n$ with conjugates $\alpha_1, \ldots, \alpha_n$, the discriminant $D(\alpha)$ of $\alpha$ satisfies

$$|D(\alpha)| \leq H(\alpha)^{2(n-1)} \prod_{1 \leq i < j \leq n} |\alpha_i - \alpha_j|^2 \leq H(\alpha)^{2(n-1)} \left(2 \max_{1 \leq i \leq n} |\xi - \alpha_i| \right)^{n(n-1)}$$

Since $D(\alpha)$ is a non-zero integer, its absolute value is $\geq 1$, and thus we deduce that

$$\max_{1 \leq i \leq n} |\xi - \alpha_i| \geq \frac{1}{2} H(\alpha)^{-2/n}$$

(compare with §5 of [Wi]). If $n = 1$, the result is optimal for any badly approximable $\xi$. Note that a similar argument also shows that, for any algebraic integer $\alpha$ of degree $n + 1$ with conjugates $\alpha_1, \ldots, \alpha_{n+1}$, we have $\max_{1 \leq i \leq n} |\xi - \alpha_i| \geq (1/2) H(\alpha)^{-2/(n-1)}$. 
Similarly, the result in part (ii) of Theorem A is optimal up to the value of the implied constant when $\xi$ is a quadratic irrational number. To prove this, suppose that an algebraic integer $\alpha$ of degree $n+1$ has conjugates $\alpha_1, \ldots, \alpha_{n+1}$ distinct from $\xi$ with the first $n$ satisfying

$$\max_{1 \leq i \leq n} |\xi - \alpha_i| \leq 1$$

Let $Q(T) \in \mathbb{Z}[T]$ be the irreducible polynomial of $\xi$ over $\mathbb{Z}$. Since $\alpha$ is an algebraic integer, the product $Q(\alpha_1) \cdots Q(\alpha_{n+1})$ is a rational integer and since it is non-zero (because $\xi$ is not a conjugate of $\alpha$), we deduce that

$$1 \leq \prod_{i=1}^{n+1} |Q(\alpha_i)| .$$

For each $i = 1, \ldots, n$, we have $|Q(\alpha_i)| \ll |\xi - \alpha_i|$ since $\xi$ is a root of $Q$ and $|\xi - \alpha_i| \leq 1$. We also have $|Q(\alpha_{n+1})| \ll \max\{1, |\alpha_{n+1}|\}^2$ since $Q$ has degree 2. This gives

$$1 \ll H(\alpha)^2 \prod_{i=1}^{n} |\xi - \alpha_i|$$

and consequently $\max_{1 \leq i \leq n} |\xi - \alpha_i| \gg H(\alpha)^{-2/n}$.

**Remark 1.** It would be interesting to know if there exists as well transcendental numbers $\xi$ for which the exponent $2/n$ for $H(\alpha)$ in Theorem A part (ii) is best possible.

**Remark 2.** The case where $\xi \in \mathbb{Q}$ is not interesting as it leads to much weaker estimates. In this case, one finds that, for each algebraic number $\alpha$ of degree $n$ with $\alpha \neq \xi$, one has $\max_\mathbb{Q} |\xi - \bar{\alpha}| \gg H(\alpha)^{-1/n}$, and that, for each algebraic integer $\alpha$ of degree $n+1$ with $\alpha \neq \xi$, one has $\max_{\mathbb{Q} \neq \alpha} |\xi - \bar{\alpha}| \gg H(\alpha)^{-1/n}$.

### 3. Construction of Polynomials

Throughout this section, we fix an irrational real number $\xi \in \mathbb{R} \setminus \mathbb{Q}$ and a positive integer $n \geq 1$. For each integer $q \geq 1$, we denote by $\mathcal{C}(q)$ the convex body of $\mathbb{R}[T]_{\leq n}$ which consists of all polynomials $P \in \mathbb{R}[T]_{\leq n}$ satisfying

$$|P^{[k]}(\xi)| \leq q^{2k-n} \quad (0 \leq k \leq n)$$

where $P^{[k]}(\xi) = P^{(k)}(\xi)/k!$ denotes the $k$-th divided derivative of $P$ at $\xi$ (the coefficient of $(T - \xi)^k$ in the Taylor expansion of $P$ at $\xi$). We first prove:

**Proposition 3.1.** Let $q$ be the denominator of a convergent of $\xi$. Then the last minimum of $\mathcal{C}(q)$ with respect to the lattice $\mathbb{Z}[T]_{\leq n}$ is $\leq 2^n$, and its first minimum is $\geq \left(\frac{2^n}{(n+1)!}\right)^{-1}$. Moreover, the convex body $2^n \mathcal{C}(q)$ contains a basis of $\mathbb{Z}[T]_{\leq n}$ over $\mathbb{Z}$.

**Proof.** Put $L_1 = qT - p$ where $p/q$ denotes a convergent of $\xi$ with denominator $q$. If $q > 1$, we also define $L_0 = q_0T - p_0$ where $p_0/q_0$ is the previous convergent of $\xi$ (in reduced form).
If \( q = 1 \), we simply take \( L_0 = 1 \). The theory of continued fractions tells us that these linear forms satisfy

\[
|L_i(\xi)| \leq q^{-1} \quad \text{and} \quad |L_i'(\xi)| \leq q
\]

for \( i = 0, 1 \), and moreover that their determinant (or Wronskian) is \( \pm 1 \) (see §4 in Chapter I of [Sc]). The latter fact means that \( \{L_0, L_1\} \) spans \( \mathbb{Z}[T]_{\leq 1} \) over \( \mathbb{Z} \). Therefore the products

\[
P_j = L_0^n L_1^{n-j} \quad (0 \leq j \leq n)
\]

span \( \mathbb{Z}[T]_{\leq n} \) over \( \mathbb{Z} \) and, since the rank of \( \mathbb{Z}[T]_{\leq n} \) is \( n + 1 \), they form in fact a basis of \( \mathbb{Z}[T]_{\leq n} \) over \( \mathbb{Z} \). Using (3), we also find that

\[
|P_j^{[k]}(\xi)| \leq \binom{n}{k} q^{2k-n} \leq 2^n q^{2k-n} \quad (0 \leq j, k \leq n).
\]

Thus \( \{P_0, \ldots, P_n\} \) is a basis of \( \mathbb{Z}[T]_{\leq n} \) contained in \( 2^n C(q) \). This proves the last assertion of the proposition as well as the fact that the last minimum of \( C(q) \) is \( \leq 2^n \).

Identify \( \mathbb{R}[T]_{\leq n} \) with \( \mathbb{R}^{n+1} \) under the map which sends a polynomial \( a_0 + a_1 T + \ldots + a_n T^n \) to the point \( (a_0, a_1, \ldots, a_n) \). Then the linear map \( \theta : \mathbb{R}[T]_{\leq n} \to \mathbb{R}^{n+1} \) given by \( \theta(P) = (P(\xi), P^{[1]}(\xi), \ldots, P^{[n]}(\xi)) \) has determinant 1 and so \( C(q) \) has volume \( \prod_{k=0}^n (2q^{2k-n}) = 2^{n+1} \).

Since the lattice \( \mathbb{Z}[T]_{\leq n} \) has co-volume 1 (it is identified with \( \mathbb{Z}^{n+1} \)), Minkowski’s second convex body theorem shows that the successive minima \( \lambda_1, \ldots, \lambda_{n+1} \) of \( C(q) \) with respect to \( \mathbb{Z}[T]_{\leq n} \) satisfy \(( (n+1)! )^{-1} \leq \lambda_1 \cdots \lambda_{n+1} \leq 1 \). Since \( \lambda_2 \leq \ldots \leq \lambda_{n+1} \leq 2^n \), this implies that \( \lambda_1 \geq \left( 2^{n^2 (n+1)!} \right)^{-1} \).

The construction of polynomials given by the next proposition uses only the last assertion of Proposition 3.1.

**Proposition 3.2.** Let \( q \) be the denominator of a convergent of \( \xi \). There exist an irreducible polynomial \( P(T) \in \mathbb{Z}[T] \) of degree \( n \) and an irreducible monic polynomial \( Q(T) \in \mathbb{Z}[T] \) of degree \( n + 1 \) satisfying

\[
c_5 q^{2k-n} \leq \left| P^{[k]}(\xi) \right|, \left| Q^{[k]}(\xi) \right| \leq 3c_5 q^{2k-n} \quad (0 \leq k \leq n)
\]

where \( c_5 = (n + 1)2^{n+1} \).

Note that such polynomials have height \( \asymp q^n \).

**Proof.** The last assertion of Proposition 3.1 tells us the existence of a basis \( \{P_0, \ldots, P_n\} \) of \( \mathbb{Z}[T]_{\leq n} \) satisfying

\[
|P_j^{[k]}(\xi)| \leq 2^n q^{2k-n} \quad (0 \leq j, k \leq n).
\]

Since \( \{P_0, \ldots, P_n\} \) is a basis of \( \mathbb{Z}[T]_{\leq n} \) over \( \mathbb{Z} \), we can write \( T^n + 2 = \sum_{j=0}^n b_j P_j(T) \) for some \( b_0, \ldots, b_n \in \mathbb{Z} \). Consider the polynomial

\[
R(T) = 2c_5 \sum_{k=0}^n q^{2k-n} (T - \xi)^k
\]
where \( c_5 = (n + 1)2^{n+1} \). Since \( \{P_0, \ldots, P_n\} \) is also a basis of \( \mathbb{R}[T]_{\leq n} \) over \( \mathbb{R} \), we can also write \( R(T) = \sum_{j=0}^{n} \theta_j P_j(T) \) for some \( \theta_0, \ldots, \theta_n \in \mathbb{R} \). Choose integers \( a_0, \ldots, a_n \) such that \( a_j \equiv b_j \mod 4 \) and \( |a_j - \theta_j| \leq 2 \) for \( j = 0, \ldots, n \), and define \( P(T) = \sum_{j=0}^{n} a_j P_j(T) \).

By construction \( P(T) \) belongs to \( \mathbb{Z}[T]_{\leq n} \) and is congruent to \( T^n + 2 \) modulo 4. Thus it is a polynomial of degree \( n \) over \( \mathbb{Q} \) and it is irreducible by virtue of Eisenstein’s criterion (for the prime 2). Since \( P(T) - R(T) = \sum_{j=0}^{n} (a_j - \theta_j) P_j(T) \), we deduce from (4) that

\[
|P^{[k]}(\xi) - R^{[k]}(\xi)| \leq \sum_{j=0}^{n} |a_j - \theta_j| \left| P_j^{[k]}(\xi) \right| \leq c_5 q^{2k-n} \quad (0 \leq k \leq n).
\]

Since \( R^{[k]}(\xi) = 2c_5 q^{2k-n} \), it follows that \( c_5 q^{2k-n} \leq |P^{[k]}(\xi)| \leq 3c_5 q^{2k-n} \) for \( k = 0, \ldots, n \), as required.

The construction of \( Q(T) \) is similar. Write

\[
T^{n+1} + 2 = T^{n+1} + \sum_{j=0}^{n} b'_j P_j(T) \quad \text{and} \quad (T - \xi)^{n+1} + R(T) = T^{n+1} + \sum_{j=0}^{n} \theta'_j P_j(T),
\]

with \( b'_0, \ldots, b'_n \in \mathbb{Z} \) and \( \theta'_0, \ldots, \theta'_n \in \mathbb{R} \), and choose integers \( a'_0, \ldots, a'_n \) such that \( a'_j \equiv b'_j \mod 4 \) and \( |a'_j - \theta'_j| \leq 2 \) for \( j = 0, \ldots, n \). Then the polynomial

\[
Q(T) = T^{n+1} + \sum_{j=0}^{n} a'_j P_j(T) \in \mathbb{Z}[T]
\]

is irreducible (by virtue of Eisenstein’s criterion for 2), monic of degree \( n + 1 \), and it satisfies also \( |Q^{[k]}(\xi) - R^{[k]}(\xi)| \leq c_5 q^{2k-n} \) for \( k = 0, \ldots, n \).

\[\square\]

4. Proof of Theorems A and B

In this section, we prove the main theorems A and B of the introduction by combining Proposition 3.2 with the following result.

**Proposition 4.1.** Let \( \xi \in \mathbb{R} \), let \( n \in \mathbb{N}^{\ast} \), let \( \delta > 0 \) and let \( \mathcal{P} \) be a subset of \( \mathbb{Z}[T] \). Suppose that the elements of \( \mathcal{P} \) are either polynomials of degree \( n \) or monic polynomials of degree \( n + 1 \). Then the following conditions are equivalent:

(i) There exists a constant \( c_6 > 0 \) such that \( |P^{[k]}(\xi)| \leq c_6 H(P)^{1-(n-k)\delta} \) for each \( P \in \mathcal{P} \) and each \( k = 0, 1, \ldots, n \).

(ii) There exists a constant \( c_7 > 0 \) such that \( |\xi - \alpha| \leq c_7 H(P)^{\delta} \) for each \( P \in \mathcal{P} \) and for \( n \) of the roots \( \alpha \) of \( P \), counting multiplicity.

**Proof.** Fix \( P \in \mathcal{P} \) and write it in the form

\[
P(T) = a_0(T - \alpha_1) \cdots (T - \alpha_m)
\]
where \( m = \deg P \) and \( \alpha_1, \ldots, \alpha_m \) are the roots of \( P \) ordered so that \( |\xi - \alpha_1| \leq \ldots \leq |\xi - \alpha_m| \).

We put \( \varepsilon = H(P)^{-\delta} \) and consider the polynomial

\[
R(T) = P(\varepsilon T + \xi) = a_0 \varepsilon^m \prod_{k=1}^{m} (T + \varepsilon^{-1}(\xi - \alpha_k)).
\]

The height of \( R \) is

\[
H(R) = \max_{0 \leq k \leq m} |R^{[k]}(0)| = \max_{0 \leq k \leq m} |P^{[k]}(\xi)| \varepsilon^k,
\]

and its Mahler measure is

\[
M(R) = |a_0| \varepsilon^m \prod_{k=1}^{m} \max \left\{ 1, \varepsilon^{-1}|\xi - \alpha_k| \right\} = |a_0| \prod_{k=1}^{m} \max \{ \varepsilon, |\xi - \alpha_k| \}.
\]

For convenience, we also define

\[
L = \begin{cases} 
|a_0| & \text{if } m = n \\
\max \{ \varepsilon, |\xi - \alpha_m| \} & \text{if } m = n + 1
\end{cases}
\]

so that the formula for \( M(R) \) becomes

\[
M(R) = L \prod_{k=1}^{n} \max \{ \varepsilon, |\xi - \alpha_k| \}
\]

(recall that \( a_0 = 1 \) when \( m = n + 1 \)). Our argument below is based on the standard inequalities relating these notions of heights, namely

\[
M(R) \leq (m + 1)H(R) \quad \text{and} \quad H(R) \leq 2^m M(R).
\]

If condition (ii) holds, we find that \( M(R) \leq c_0 \varepsilon^n L \). We also have \( L \ll H(P) \) since \( |a_0| \leq H(P) \) and since \( |\xi - \alpha| \ll \max \{1, |\alpha| \} \ll H(P) \) for any root \( \alpha \) of \( P \). Then, for each \( k = 0, \ldots, n \), we obtain

\[
|P^{[k]}(\xi)| \ll \varepsilon^{-k}H(R) \ll \varepsilon^{-k}M(R) \ll \varepsilon^{n-k}H(P)
\]

which shows that condition (i) holds.

Conversely assume that condition (i) holds. In this case we find that \( H(R) \leq c_0 \varepsilon^n H(P) \).

We claim that \( H(P) \ll L \). If we take this for granted, we deduce that

\[
L \varepsilon^{n-1}|\xi - \alpha_n| \leq M(R) \ll H(R) \ll \varepsilon^n L
\]

which implies that condition (ii) holds.

To prove the claim, we observe that

\[
H(P) = H(P(T + \xi)) = \max_{0 \leq k \leq m} |P^{[k]}(\xi)|.
\]

By hypothesis, we have \( |P^{[k]}(\xi)| \leq c_0 H(P)^{1-\delta} \) for \( k = 0, \ldots, n - 1 \) and we also have \( |P^{[m]}(\xi)| = 1 \) if \( m = n + 1 \). Finally, we have \( |P^{[n]}(\xi)| = |a_0| \) if \( m = n \), and \( |P^{[n]}(\xi)| = |\sum_{k=1}^{m}(\xi - \alpha_k)| \leq m|\xi - \alpha_m| \) if \( m = n + 1 \), showing that \( |P^{[n]}(\xi)| \ll L \). All this implies that

\[
H(P) \ll \max \{1, L\}.
\]

Since \( L \geq \varepsilon = H(P)^{-\delta} \), this in turn implies that \( H(P) \ll L \). \( \square \)
**Proof of the theorems.** Let $\xi \in \mathbb{R} \setminus \mathbb{Q}$ and $n \in \mathbb{N}^*$. We simply prove Part (ii) of Theorems A and B since the proof of Part (i) is similar and slightly easier.

For each denominator $q$ of a convergent of $\xi$, Proposition 3.2 shows the existence of an irreducible monic polynomial $Q \in \mathbb{Z}[T]$ of degree $n + 1$ satisfying $H(Q) \approx q^n$ and

$$|Q^k(\xi)| \leq c_6 H(Q)^{(2k-n)/n} = c_6 H(Q)^{1-(n-k)(2/n)} , \quad (0 \leq k \leq n)$$

for some constant $c_6 = c_6(\xi, n)$. The family $\mathcal{P}$ of these polynomials satisfies the condition (i) of Proposition 4.1 for the choice $\delta = 2/n$, and so it satisfies also the condition (ii) of the same proposition for the same value of $\delta$ and for some constant $c_7$. For each $Q \in \mathcal{P}$, choose a root $\alpha$ of $Q$ for which $|\xi - \alpha|$ is maximal. Since $Q$ is irreducible, this root $\alpha$ is an algebraic integer of degree $n + 1$ and height $H(\alpha) = H(Q)$ whose conjugates $\overline{\alpha}$ over $\mathbb{Q}$ are the $n + 1$ distinct roots of $Q$. Therefore, we get $\max_{\alpha \neq \alpha} |\xi - \overline{\alpha}| \leq c_7 H(\alpha)^{-2/n}$. This proves Part (ii) of Theorem A since we find infinitely many such numbers $\alpha$ by varying $Q$.

If $\xi$ is badly approximable, the ratios of the denominators of consecutive convergents of $\xi$ are bounded. Thus, for each $X \geq 1$, there exists such a denominator $q$ with $q \approx X^{1/n}$, and so there exists a polynomial $Q \in \mathcal{P}$ with $H(Q) \approx X$. Consequently, the root $\alpha$ of $Q$ that we chose above satisfies $H(\alpha) \approx X$ and this proves Part (ii) of Theorem B. $\square$

**Acknowledgments.** The author thanks his MSc thesis supervisor Damien Roy for suggesting this problem and for his help in writing the present paper.

**References**

[AR] B. Arbour and D. Roy, A Gel’fond type criterion in degree two, *Acta Arith.*, 111 (2004), 97-103.
[Bu] Y. Bugeaud, *Approximation by algebraic numbers*, Cambridge Tracts 160, Cambridge Univ. Press, 2004
[BT] Y. Bugeaud and O. Teulié, Approximation d’un nombre réel par des nombres algébriques de degré donné, *Acta Arith.* 93 (2000), 77-86.
[DS] H. Davenport and W. M. Schmidt, Approximation to real numbers by quadratic irrationals, *Acta Arith.* 15 (1969), 393-416.
[RW] D. Roy and M. Waldschmidt, Diophantine approximation by conjugate algebraic integers, *Compositio Math.* 140 (2004), 595-612.
[Sc] W. M. Schmidt, *Diophantine approximation*, Lecture Notes in Math. 785, Springer-Verlag, 1980.
[Wi] E. Wirsing, Approximation mit algebraischen Zahlen beschränkten Grades, *J. reine angew. Math.* 206 (1961), 67-77.

DÉPARTEMENT DE MATHEMATIQUES, UNIVERSITÉ D’OTTAWA, 585 KING EDWARD, OTTAWA, ONTARIO K1N 6N5, CANADA
E-mail address: gyomalin@gmail.com