Towards Data-driven LQR with KoopmanizingFlows*,**

Petar Bevanda* Max Beier* Shahab Heshmati-Alamdari** Stefan Sosnowski* Sandra Hirche*

* Chair of Information-oriented Control (ITR), Department of Electrical and Computer Engineering, Technical University of Munich D-80333 Munich, Germany (e-mail: [petar.bevanda, max.beier, sosnowski, hirche]@tum.de).
** Department of Electronic Systems, Aalborg University, Fredrik Bajers Vej 7K, 9220 Aalborg, Denmark (e-mail: shhe@es.aau.dk)

Abstract: We propose a novel framework for learning linear time-invariant (LTI) models for a class of continuous-time non-autonomous nonlinear dynamics based on a representation of Koopman operators. In general, the operator is infinite-dimensional but, crucially, linear. To utilize it for efficient LTI control, we learn a finite representation of the Koopman operator that is linear in controls while concurrently learning meaningful lifting coordinates. For the latter, we rely on KoopmanizingFlows - a diffeomorphism-based representation of Koopman operators. With such a learned model, we can replace the nonlinear infinite-horizon optimal control problem with quadratic costs to that of a linear quadratic regulator (LQR), facilitating efficacious optimal control for nonlinear systems. The prediction and control efficacy of the proposed method is verified on simulation examples.

Keywords: Machine learning, Koopman operators, Learning for control, Representation Learning, Neural networks, Learning Systems

1. INTRODUCTION

Inspired by the infinite-dimensional, linear Koopman operator (Koopman, 1931), there has been an increased interest in global linearizations of nonlinear dynamics in recent years. By taking a finite-dimensional nonlinear system and “lifting” it to a higher-dimensional linear operator representation, superior complexity-accuracy balance compared to conventional nonlinear modeling is possible via efficient linear techniques for prediction and control.

Although the autonomous setting is native to the Koopman operator, there are practicable extensions of the theory to controlled systems that open up the possibility of applying classical results from linear control theory to nonlinear systems (Bevanda et al., 2021c). First ideas of including control in Koopman-inspired frameworks can be found in Proctor et al. (2016), with ideas of optimal control analytically considered in Brunton et al. (2016) and a data-driven manner for reduced order optimal control in Kaiser et al. (2021). Yet, given predictive linear models for nonlinear systems, efficient optimal control is possible due to the considerably simplified linear control design compared to their nonlinear counterparts. Existing optimal control methods based on data-driven Koopman operator representations utilize linear model predictive control (MPC) (Korda and Mezić, 2018; Lian and Jones, 2020; Korda and Mezić, 2020), inspiring control applications in nonlinear flows (Arbabi and Mezić, 2017), soft robotics (Bruder et al., 2021) and autonomous vehicles (Cibulka et al., 2020). Often, however, the aforementioned data-driven approaches tend to provide only locally accurate models and operate in a receding-horizon fashion. However, an efficient solution of an infinite-horizon optimal control problem from data for nonlinear systems might be more attractive due to the possibility of solving it in closed form. To exploit Koopman operator representations advantageously through an LQR problem reformulation, it is crucial to obtain predictive models that are globally accurate, which has not previously been explored in a fully data-driven fashion.

Inspired by the prediction efficacy and theoretical properties of KoopmanizingFlows (Bevanda et al., 2021a), we propose an extension to controlled systems allowing for efficient infinite-horizon LQR. The contribution of this paper is the development of KoopmanizingFlow-LQR (KFLQR) - a principled framework for learning controlled Koopman operator dynamical models, ensuring operator-theoretic considerations are embedded in identification and control. The framework is entirely data-driven as the lifting and the controlled LTI dynamics are learned concurrently. Such a learned model allows us to cast a nonlinear infinite-horizon optimal control problem with quadratic costs to that of an LQR. To the best of our knowledge, this is the only Koopman-based framework for data-driven LQR control design. We demonstrate the performance of the proposed method on two simulation examples.

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NOTATION

Vectors/matrices are denoted with lower/upper case bold symbols \( \mathbf{x}/X \). Symbols \( \mathbb{N}/\mathbb{R}/\mathbb{C} \) denote sets of natural/real/complex numbers, while \( \mathbb{N}_0 \) denotes all natural numbers with zero, and \( \mathbb{R}_{+,0}/\mathbb{R}_{+,0} \) all positive reals with/without zero. Function spaces with a specific integrability/regularity order are denoted as \( L^p/C \) with the order (class) specified in their exponent. The Jacobian matrix of vector-valued map \( \psi \) evaluated at \( x \) is denoted as \( \mathbf{J}_\psi(x) \). The \( L^2 \)-norm on a set \( X \) is denoted as \( \| \cdot \|_{\mathbb{P},X} \). Writing \( \circ \) denotes the Hadamard product, \( exp \) pointwise exponential and \( \circ \) function composition. Underlined matrices \( \underline{X} \) represent ones in the immediate (original) state dimension.

2. PROBLEM STATEMENT

2.1 Modeling Assumptions

Consider an unknown, continuous-time nonlinear dynamical system of the following form

\[
\dot{x} = a(x) + Bu =: f(x,u)
\]

(1)

with continuous states and controls on compact sets \( x \in X \subset \mathbb{R}^d \) and \( u \in U \subset \mathbb{R}^m \), respectively. The autonomous dynamics are smooth such that \( a \in C^2(X) \), with control assumed to be entering the dynamics linearly via \( B \in \mathbb{R}^{d \times m} \).

Assumption 1. We assume the sole fixed point of \( f(x,0) \) for (1) is hyperbolic and contained in \( X \).

Assumption 1 is quite general in practice as it admits dynamical systems representing motion, e.g., human reaching movements (Khansari-Zadeh and Billard, 2011) or physical systems such as a neutrally buoyant underwater vehicle (Fossen, 2011).

2.2 Koopman Operator Theory

The solutions of forward-complete continuous-time dynamics (Bittracher et al., 2015) are fully described by the flow map of \( a(x) := f(x,0) \)

\[
x(t_0) \equiv x_0, \quad F^t(x_0) := x_0 + \int_{t_0}^{t_0+t} a(x(\tau))d\tau,
\]

(2)

which has a unique solution on \([0, +\infty[\) from the initial condition \( x \) at \( t = 0 \) (Angeli and Sontag, 1999). Due to the hyperbolicity of the isolated attractor, this holds for (1). The above flow map induces the associated Koopman operator semigroup as defined in the following.

Definition 1. The semigroup \( \{K^t\}_{t \in \mathbb{R}_{+,0}} : C(X) \to C(X) \) of Koopman operators for the autonomous flow of (2) acts on a scalar observable function \( h \in C(X) \) on the state space \( X \) through \( K^t_a h = h \circ F^t \).

Simply put, the operator applied to an observable function \( h \) at time \( t_0 \) advances it along the flow (2) as follows \( K^t_a h(x(t_0)) = h(x(t_0+t)) \). By applying it component-wise to the “state-observer” function \( h(x) = x \), it is identical to the flow map defined in (2). Critically, every \( K^t_a \) is a linear \(^1\) operator. With a well-defined Koopman operator semigroup, we introduce its infinitesimal generator.

\[
\mathcal{G}_{K_a} h = \lim_{t \to 0^+} \frac{K^t_a h - h}{t} = \frac{d}{dt} h, \quad (3)
\]

is the infinitesimal generator of the semigroup of Koopman operators \( \{K^t\}_{t \in \mathbb{R}_{+,0}} \).

Crucially, the Koopman operator formalism allows one to decompose dynamics into linearly evolving coordinates, which naturally arise through the eigenfunctions of evolution operator \( \mathcal{G}_{K_a} \). These eigenfunctions are formally defined as follows.

Definition 3. An observable \( \phi \in C(X) \) is an eigenfunction of \( \mathcal{G}_{K_a} \) if it satisfies \( \mathcal{G}_{K_a} \phi = \lambda \phi \), for an eigenvalue \( \lambda \in \mathbb{C} \). The span of eigenfunctions \( \phi \) of \( \mathcal{G}_{K_a} \) is denoted by \( \Phi \).

Due to Assumption 1, the Koopman operator generator has a pure point spectrum for the dynamics (1) (Mauroy and Mezić, 2016). Therefore, for each observable \( h \), there exists a sequence \( v_j(h) \in C \) of mode weights, such that the following decomposition completely describes its dynamics as

\[
\dot{h} = \mathcal{G}_{K_a} h = \mathcal{G}_{K_a} \left( \sum_{j=1}^{\infty} v_j(h) \phi_j \right) = \sum_{j=1}^{\infty} v_j(h) \lambda_j \phi_j. \quad (4)
\]

With a slight abuse of operator notation, we can write the decomposition (4) compactly as

\[
\dot{h} = \mathcal{V}_h \mathcal{G}_{K_a} \Phi + \mathcal{V}_h \mathbf{B} u, \quad (5)
\]

holds for all systems of type (1). This is easily checked by considering an additional linear operator \( \mathcal{B} \) in the following operator-based description of (5):

\[
\dot{h} = \mathcal{V}_h \mathcal{G}_{K_a} \Phi + \mathcal{V}_h \mathbf{B} u, \quad (6)
\]

The autonomous part of (6) satisfies (4) and the additive control part remains linear as in (1).

Remark 1. If the actuation entry in (1) was nonlinear, assuming an LTI controlled system in the lifted model becomes only locally accurate (Bevanda et al., 2021c) but still can suffice for short-horizon prediction and control (Korda and Mezić, 2018; Lian and Jones, 2020).

Clearly, the infinite-dimensional model as in (6) is not helpful for a practical representation. Thus, as finding meaningful finite-dimensional models for (6) is not even analytically possible in general with perfect system knowledge, we seek a finite-dimensional representation from data. Note that, due to the linearity of input entry, it need not be lifted but projected onto the LTI system of the autonomous dynamics.

\(^1\) Consider \( h_1, h_2 \in C(\mathbb{X}) \) and \( \beta \in \mathbb{C} \). Then, using Definition 1, \( K^t (\beta h_1 + h_2) = (\beta h_1 + h_2) \circ F^t = \beta h_1 \circ F^t + h_2 \circ F^t = \beta \gamma^t h_1 + \gamma^t h_2 \).
Assumption 2. A data-set of $N$ state-input-output tuples $\mathbb{D}_N = \{ \mathbf{x}^{(i)}, \mathbf{u}^{(i)}, \hat{\mathbf{x}}^{(i)} \}_{i=1}^N$ for the system (1) is available.

The above measurements are commonly assumed to be at disposal. If not directly accessible, the time-derivative of the state can be approximated through finite differences for practical applications. Based on the data from Assumption 2, we consider the problem of learning a finite-dimensional Koopman generator model for the forced system (1) by solving the following optimization problem

$$
\min_{A,B,C,\psi(\cdot)} \sum_{i=1}^N \left\| \hat{\mathbf{x}}^{(i)} - C \left( A\psi(\mathbf{x}^{(i)}) + B\mathbf{u}^{(i)} \right) \right\|_2^2 \quad \text{(7a)}
$$

reconstruction

$$
+ \| \mathbf{x}^{(i)} - C\psi(\mathbf{x}^{(i)}) \|_2^2
$$

subject to: $\psi \in \Phi$ \quad \text{(Koopman-invariance)}

with $\psi = [\psi_1, \ldots, \psi_D]^T$, $A \in \mathbb{R}^{D \times D}$, $B \in \mathbb{R}^{D \times m}$ and $C \in \mathbb{R}^{d \times D}$ providing a finite-dimensional representation in terms of a state-space model

$$
z_0 = \psi(x_0), \quad \dot{z} = A\dot{z} + B\mathbf{u}, \quad \mathbf{x} = C\mathbf{z}. \quad \text{(8a)-(8b)}
$$

This model trades the nonlinearity of a $d$-dimensional ODE (1) for a nonlinear “lift” (8a) of the initial condition $x_0$ to higher dimensional ($D \gg d$) Koopman-invariant coordinates (8b) such that the original state can be linearly reconstructed via (8c). Moreover, (7) learns an arbitrary amount of Koopman-invariant features directly instead of only finding ones that lie in a heuristically predetermined dictionary of functions.

3. LEARNING A CONTROLLED DIFFEOMORPHISM-BASED LTI MODEL

To preface the learning approach, we formally define the notion of LTI-coordinates - ones that evolve linearly under the dynamics.

Definition 4. Consider the system (1), a matrix $A \in \mathbb{R}^{D \times D}$ and a finite set of features $\psi := [\psi_1, \ldots, \psi_D]^T$ with $\psi_i(\mathbf{x}) \in C^1(\mathbb{X})$ on a compact set $\mathbb{X}$. If this feature set solves the following linear partial differential equation (PDE)

$$
J_\psi(x)\mathbf{a}(x) = A\psi(x), \quad \text{(9)}
$$

the features are admissible Koopman-invariant coordinates satisfying (7b).

3.1 Structured Lifting Construction from a Latent Space

As in KoopmanizingFlows (Bevanda et al., 2021a), we use a diffeomorphic relation to a linear latent model to obtain solutions for (9), providing us with Koopman-invariant lifting coordinates that fulfill (7b). The KoopmanizingFlows framework relies on the fact that lifting a linear system to a monomial basis is invariant to the dynamics, meaning the monomial coordinates still form an LTI system whose spectral properties are determined by the original linear system itself. To exploit the former for nonlinear system modeling, one is tasked with morphing

![Fig. 1. Construction diagram for learning model for the autonomous part of (8a)-(8c) with the construction pathway in bold and the maps to be learned in magenta. The sets $\mathbb{X}, \mathbb{Y}, \mathbb{Z}$ correspond to the immediate state-space, latent unit-box space and lifted linear model space, respectively; with corresponding tangent spaces denotes as $T_\mathbb{X}, T_\mathbb{Y}, T_\mathbb{Z}$.](image-url)

the nonlinear system to a linear one by a smooth change of coordinates.

Definition 5. Vector fields $\dot{\mathbf{x}} = \mathbf{a}(\mathbf{x})$ and $\dot{\mathbf{y}} = \mathbf{t}(\mathbf{y})$ are diffeomorphic, or smoothly equivalent, if there exists a diffeomorphism $d : \mathbb{R}^d \rightarrow \mathbb{R}^d$ such that $\mathbf{a}(\mathbf{x}) = J_{\hat{\mathbf{a}}}^{-1}(\mathbf{x})t(d(\mathbf{x}))$.

We first formally define monomial coordinates based on the latent vector $d(\mathbf{x}) = \mathbf{y} = [y_1, \ldots, y_d]^T$ through $y^\alpha = y_1^{\alpha_1}y_2^{\alpha_2}\cdots y_d^{\alpha_d}$, where $\alpha \in \mathbb{N}_0^d$ is a multi-index. Then, we obtain a lifted coordinate vector by concatenating all monomials $y^\alpha$ up to order $\|\alpha\|_1 = a_1 + \cdots + a_d \leq p$ in a lexicographical ordering in a vector $\mathbf{y}^{[\mathcal{P}]}$. Due to the construction of this vector, it inherits the linear dynamical system description from $\mathbf{y} = d(\mathbf{x})$ with the dynamics of $\mathbf{y}^{[\mathcal{P}]}$ linearly dependent on $\mathcal{A}$ (Bevanda et al., 2021a, Lemma 6). This is formalized in the following proposition.

Proposition 1. (Bevanda et al. 2021a). Assume the linear system $\dot{\mathbf{y}} = \mathcal{A}\mathbf{y}$ is smoothly equivalent to the autonomous part of (1) via a diffeomorphism $d$ such that $\mathbf{y} = d(\mathbf{x})$. Then, the lifted features $\psi = \mathbf{d}^{[\mathcal{P}]}$ satisfy (7b), i.e., $\psi(\mathbf{x}) = \mathbf{d}^{[\mathcal{P}]}(\mathbf{x}) = \psi^{[\mathcal{P}]}$ are Koopman-invariant coordinates and define a latent linear system

$$
z_0 = d^{[\mathcal{P}]}(x_0), \quad \dot{z} = \mathcal{A}\mathbf{d}^{[\mathcal{P}]}(\mathbf{z}). \quad \text{(10a)-(10b)}
$$

Due the above result, learning Koopman-invariant features reduces to learning a diffeomorphism, which allows us to replace the constraint (7b) by the condition

$$
\mathbf{a}(\mathbf{x}) = J_{\hat{\mathbf{a}}}^{-1}(\mathbf{x})\mathcal{A}\mathbf{d}(\mathbf{x}), \quad \text{(11)}
$$

which ensures the smooth equivalence between $\mathbf{a}$ and $\mathcal{A}$.

Theorem 1. The minimizers $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathbf{d}(\cdot)$ of the optimization problem

$$
\min_{\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathbf{d}(\cdot)} \sum_{i=1}^N \left\| \hat{\mathbf{x}}^{(i)} - C(\mathcal{A}^i\mathcal{A})\mathbf{d}^{[\mathcal{P}]}(\mathbf{x}^{(i)}) + B\mathbf{u}^{(i)} \right\|_2^2 + \| \mathbf{x}^{(i)} - C\mathbf{d}^{[\mathcal{P}]}(\mathbf{x}^{(i)}) \|_2^2 \quad \text{(12a)}
$$

subject to: $\mathbf{a}(\mathbf{x}) = J_{\hat{\mathbf{a}}}^{-1}(\mathbf{x})\mathcal{A}\mathbf{d}(\mathbf{x}) \quad \text{(12b)}$

define a solution

$$
\psi = \mathbf{d}^{[\mathcal{P}]} \quad \mathcal{A} = \mathcal{A}^i\mathcal{A}, \quad \mathcal{B} = \mathcal{B}, \quad \mathcal{C} = \mathcal{C} \quad \text{(13)}
$$
for the optimization problem (7) and thereby define a model of the form (8).

**Proof.** For the system class (1) satisfying Assumption 1, there exists a smooth equivalence to a linear system without a loss of generality (Lan and Mezić, 2013, Theorem 2.3, Corollary 2.1) as the Koopman-invariance is decoupled from control influence per definition in (6). By following Proposition 1, we can replace the Koopman-invariance condition (7b) with the constraint (12b). Thus, (13) represents a solution for (7) - defining a model of the form (8).

The overview of the construction of LTI coordinates can be found in Figure 1. The unit-box bounded latent space \( \mathcal{Y}_0 \) from Fig. 1 is very beneficial for numerical stability.

**Remark 2.** Notably, we are able to learn the autonomous and control part of (1) concurrently. Works such as Korda and Mezić (2020) might need separate learning approaches to ease the use of standard training algorithms for neural networks, we relax the optimization problem (12) by considering (12b) as a soft constraint through an additional cost. This can limit the ability of exploring the state-space from a small amount of initial conditions or become unsafe for unstable systems.

### 3.2 Relaxing the Smooth Equivalence through Costs

To ease the use of standard training algorithms for neural networks, we relax the optimization problem (12) by considering (12b) as a soft constraint through an additional summand in the cost (12a). This results in the unconstrained optimization problem

\[
\min_{A, B, C, d(t)} \sum_{i=1}^{N} \| \dot{x}^{(i)} - C(A_{p}[A]d^{p}(x^{(i)}) + Bu^{(i)} \|_2^2 + \| x^{(i)} - C d^{p}(x^{(i)}) \|_2^2 + \mathcal{L}_{SE}(x^{(i)}, \dot{x}^{(i)}) \]

where the cost

\[
\mathcal{L}_{SE}(x^{(i)}, \dot{x}^{(i)}) = \| \dot{x} - J_a^{-1}(x) A_d(x) \|_2^2
\]

replaces the constraint (12b). The summands in (14) could also be individually weighted by scalar multipliers to additionally penalize the soft constraints. This allows one to, e.g., selectively bias the learning procedure to prioritize prediction or reconstruction, based on the loss terms defined in (7a).

In order to finally solve (14), one needs to ensure the function approximator used for learning \( d \) is guaranteed to be a diffeomorphism. For this, we utilize coupling flow invertible neural networks (CF-INN) (Kobyzev et al., 2021) which demonstrated their utility for Koopman operator demonstrations Bevanda et al. (2021b,a).

For realizing complex diffeomorphisms, CF-INN successively compose simpler diffeomorphisms called **coupling layers** \( \hat{d}_i \) using the fact that diffeomorphic maps are closed under composition, so that \( y = \hat{d}_i(x) = \hat{d}_{i+1} \circ \ldots \circ \hat{d}_1(x) \). Each coupling layer \( \hat{d}_i \) is defined to couple a disjoint partition of the input \( x = [x_a, x_b]^\top \) with two subspaces \( x_a \in \mathbb{R}^{d-n} \) and \( x_b \in \mathbb{R}^n \) where \( n \in \mathbb{N} \) and \( d \geq 2 \), in a manner that ensures bijectivity. This can be realized via affine coupling flows (ACF), which have coupling layers

\[
\hat{d}_i(x^{(i)}) = \left[ \begin{array}{c} x_b^{(i)}(t) + t_i(x_a^{(i)}) \end{array} \right] \]

with scaling functions \( s_i : \mathbb{R}^n \rightarrow \mathbb{R}^{N-n} \) and translation functions \( t_i : \mathbb{R}^n \rightarrow \mathbb{R}^{N-n} \) that can be chosen freely. The parameters of the diffeomorphic learner consist of the weights and biases in the neural networks of the scaling and translation functions concatenated in parameters \( w = [w_{s_1}, w_{t_1}, \ldots, w_{s_k}, w_{t_k}]^\top \).

**Theorem 2.** Consider diffeomorphisms \( d = \hat{d}_k \circ \ldots \circ \hat{d}_1(x) \) parameterized through coupling layers (16), which are defined using continuously differentiable functions \( s_i, t_i \). Then, by construction, every optimization problem (14) yields a candidate solution for (12), resulting in a model of the form (8).

**Proof.** Following (Bevanda et al., 2021a, Appendix - Lemma 11), the composition of coupling layers defined using continuously differentiable functions \( s_i, t_i \) is guaranteed to be a diffeomorphism. Therefore, any \( (A, d) \)-pair fulfills (10) by construction, providing the necessary hypothesis to approximately fulfill (12b) by minimizing (15).

**Remark 3.** Less formally, minimizing loss (15) allows the smooth equivalence defined through \( (A, d) \) to approximately correspond to the autonomous dynamics of (1). Thus, when the loss contribution of (15) vanishes, the sole error source in the resulting system (8) is due to a finite truncation of (6).

The result of Theorem 2 allows to efficacious obtain approximate solutions to the optimization problem (7), as it transforms a practically intractable problem (7) into an easily implementable deep learning problem (14).

### 4. DATA-DRIVEN LQR FOR NONLINEAR SYSTEMS

**Assumption 3.** The pairs \( (A, C) \) and \( (A, B) \) are observable and stabilizable, respectively.

The above assumption is common and can be numerically verified for the learned model.

#### 4.1 Linear Quadratic Optimal Control

The optimal control problem for (1) with quadratic costs is defined as follows

\[
\min_u \int_{t_0}^{\infty} \left( x(t)^\top Q x(t) + u(t)^\top R u(t) \right) \ dt
\]

s.t. \( \dot{x} = f(x, u), \quad x_0 = x(t_0) \)

where \( Q \succeq 0 \) and \( R > 0 \) are user-defined parameters. Using the solution of (14), this optimal control problem can be approximated by the following LQR problem

\[
\min_u \int_{t_0}^{\infty} \left( z(t)^\top C^\top Q C z(t) + u(t)^\top R u(t) \right) dt
\]

s.t. \( \dot{z} = Az + Bu, \quad z_0 = \psi(x_0) \)

by expressing dynamics via the state-space model (8). Due to the linear relation of immediate to lifted-state costs through \( C^\top Q C \) the resulting optimal control problem
(18) aims at minimizing an equivalent cost functional. This allows for balancing control-energy with aggressiveness in physically relevant coordinates. Its solution delivers a linear feedback gain $K \in \mathbb{R}^{m \times D}$ that defines a nonlinear optimal policy in original coordinates $u = -K \psi(x)$.

5. EVALUATION

For all examples, ACF with 7 coupling layers are used to learn the diffeomorphisms. The neural networks for the scaling and translation functions in each of the affine coupling layers have 3 hidden layers, with 120 neurons, each with a smooth Exponential Linear Unit (ELU) as the activation function. The dimension of the lifting coordinates is $D = 65$ ($\beta = 10$). As the optimizer, the ADAM (Kingma and Ba, 2015) was employed. For Example 2 both components were learned jointly from forced data, while for Example 1 the autonomous model and the input matrix were learned separately, with forced and unforced trajectories from the same initial conditions.

5.1 Open Loop Prediction

Example 1. Consider the following stable dynamical system containing essential characteristic hydrodynamic damping of underwater vehicles (Fossen, 2011)

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_1 - x_2 - x_2|x_2| + u \end{bmatrix}.$$ (19)

We gather trajectory data from 50 uniformly spaced initial conditions, starting from the edge of $[-2.5, 2.5]^2$ for 5s with sampling time $dt = 0.025s$, while control inputs are generated from an amplitude modulated pseudo-random bit sequences (APRBS) signal with amplitude from $-1$ to 1 and hold time from 0.025s to 0.1s, resulting in 200 datapoints each. The neural network is trained for 10 000 epochs with a full-batch gradient descent. For prediction we discretized state-space matrices $A_d, B_d$ from (8) for a specified $dt = 0.025$ using the well known closed-form expressions $A_d = e^{A dt}$ and $B_d = A^{-1}(A_d - I)B$. We compare 2s KF-prediction with 200 trajectories generated from our training examples, but with randomly chosen initial conditions from the set $[-2.5, 2.5]^2$. The reduction in root mean square error (RMSE) compared to a Taylor linearization amounts to 68%. By examining Figure 2 that describes the 5s open-loop prediction of Example 1, the accuracy of long-term prediction is evident as the linear predictor predicts an open loop trajectory from the edge of the trained state-space.

5.2 Control

Example 2. Here we consider control of the following system

$$\dot{\hat{x}} = \begin{bmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \end{bmatrix} = \begin{bmatrix} 5x_2(x_2^2 + x_2|x_2|) + t(x_1, x_2) \end{bmatrix} + u.$$ (20)

with velocity-related nonlinear damping $v(x_2) = 5(x_2^3 + x_2|x_2|)$ as well as complex damping effects $t(x_1, x_2) = 10x_2\sin(5x_1)\cos(2x_1)$ dependent on the position.

For this example, 1024 uniformly spaced initial conditions are sampled from the set $[-1, 1]^2$ for 0.5s ($dt = 0.005s$) resulting in 100 datapoints each while controlled using an APRBS signal with amplitude from $-5$ to $5$ and hold time 0.01s to 0.1s. The neural network is trained for 20 000 epochs with a batch size of 2000.

To evaluate the control performance, 50 uniformly sampled trajectories close to the edges of the state-space box are sampled, evaluating the cost (17a) along 10s long trajectories.

Table 1 summarizes decrease in accumulated cost compared to an exact first order Taylor linearization around the equilibrium. The Koopman operator generator model provides overall better performance and drastically reducing the required control effort for Example 2 as it is able to exploit the nonlinearity of the system’s dynamics.
Table 1. Reduction of accumulated cost for KF-LQR for Example 2 w.r.t. Jacobian linearization LQR. $J$ and $J_u$ denote total and control costs, respectively.

| KF-LQR | mean $J$ var $J_u$ mean $J_u$ |
|--------|------------------|------------------|
| $Q = 10I$, $R = 1$ | 14.96% 97.77% 89.66% |

6. CONCLUSION

In this paper, we present a novel framework for learning Koopman generator models for controlled systems by constructing controlled LTI prediction models for a class of nonlinear dynamics. The presented results demonstrate the utility and transferrability of Koopman operator theory to an optimal control problem with quadratic subject to nonlinear dynamics that results in an LQR problem. Future work considerations include rigorously addressing controllability and observability, as well as extending the model capacity to that of systems with state-dependent actuation.

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