A plastic model based on the bounding surface formulation for cyclic behavior of soil

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Abstract. Cyclic behavior of saturated soil under cyclic loading, such as earthquake, is one significant cause of the strain accumulation which can lead to the foundation destruction. This paper presents the relationship between hardening rules in the multi-surface plasticity and hardening function in the bounding surface plasticity, because both of them need to calculate the plastic modulus at every current stress state, applying the newly developed hardening rule into the hardening function to calculate the variable plastic modulus, this new idea adds the adversity of plasticity model, and a new bounding surface is established. The simulation result of undrained triaxial test verified the reliability of the new bounding surface model.

1. Introduction
Under the function of cyclic loading such as the seismic, flow liquefaction of soil and deformation caused by cyclic mobility are still the main phenomenon relating to significant foundation destruction. Cyclic mobility is characterized by the progressive accumulation of shear deformation under cyclic loading [1]. Generally, there are three categories of explorations about cyclic mobility, including experimental studies [2, 3, 4], numerical analysis [5-7], constitutive models for post-liquefaction deformations. In terms of the constitutive model, there are all kinds of models formed based on different theories, like multi-surface models [8-10], single-surface models [11], bounding surface models [12, 13], generalized plastic models [14, 15]. Among them, multi-surface model and bounding surface model are always the main exploration objects because of their flexibility of simulating the deformation mechanism.

The multi-surface model is an extension of the nonlinear kinematic hardening model, it imposes that there are infinite loading surfaces except for the first yield surface when material enters into the process of plastic deformation, the plastic modulus vary between different loading surfaces according to the certain hardening rule so as to correspond to the practical behavior of material under loading. In
this kind of model, different stress point in the stress path locates on different loading surface, so all stress points in will have their own plastic modulus. As for the bounding surface model, there is just one surface called bounding surface which is regarded as a bounding surface material state, created by the manufacturing process or the past loading history relating to pre-consolidation stress. Based on the bounding surface concept, the plastic modulus corresponds to physical stress point on the stress path is determined through some mapping rule which is used to get the stress point on bounding surface and certain hardening rule.

Comparing with the bounding surface model, multi-surface model has been developed more progress because of the diversity of hardening rule and choices of internal variables. Considering that these two models can be applied to tackle the same plastic modulus of the same stress state under identical stress path, this paper introduces a kind of multi-surface model and a $J_2$ hardening rule using in the multi-surface model, then adjust the model into $p-q$ axes system to coincide with the system of bounding surface model. Secondly, the plastic modulus of one certain stress state was calculated by using two models to get the relationship between two functions which is the expression of this plastic modulus, thus, the function of hardening coefficient in the bounding model can be optimized by the new development of the multi-surface model. Finally, the new bounding model with adjusted hardening parameter will be verified by using it into saturated soil under cyclic loading.

2. Cyclic behavior mechanism

Understanding the main mechanism of cyclic behavior is very important to replicate the process in drained or undrained triaxial experiments and to know the effective stress path. Cyclic behavior can occur in a wide range of soils and site condition. It is characterized by progressive accumulation of shear deformations and progressive reduction in effective stress under cyclic loading, eventually, result in large permanent displacements. The main pattern of this behavior is shown in Figure 1 [5].

![Figure 1. Main pattern of cyclic mobility](image)

The type of cyclic loading applied above is same as seismic loading. The pattern showed the soil behavior in two cycles of loading at a state of low confining stress. Because of low confining stress, there exists a short phase of soil contraction at first. During the phase 1-2, the shear stress has no change with the increase of shear strain because of the movement of soil particles in pores. With the decrease of pore rate, the stress path surpassed PT surface [16], the shear dilatation occurred in phase 2-3, this phenomenon is led by rolling between particles when the density becomes high, as well as the amount of particle contact becomes more so that the effective stress becomes higher compared with before. Phase 3-4 indicates there is few of elastic deformation in the process of unloading with dramatic degradation of effective stress, then the stress path come to the shear extension area, the situation is similar to the shear compaction except that stress path does not surpass the PT surface.
Phase 7-9 is similar to phase 1-3. As the variation of shear strain shown above, the shear strain has been accumulating in the process of cyclic loading.

3. Constitutive model

In this paper, the constitutive model is based on the bounding surface plastic model proposed by Yannis F. Dafalias [17]. The concept of bounding surface is different from that of yield surface and loading surface. The function of bounding surface is related to the original proconsolidation stress state, the shape of bounding surface is obtained by cutting the roscoe surface and hvorslev surface which is in the $p - q - v$ space by a plane of certain $v$ whose value is equal to that of the preconsolidation stress state. Because a shape-hardening function is used in the process of calculating plastic modulus corresponding to current stress point, and a certain hardening rule is also applied to describe the variation of loading surface to trace the plastic modulus during the phase of plastic deformation in the multi-surface plastic model, attention is placed on the new development of shape-hardening function in bounding surface plastic model after a brief description of bounding surface model.

3.1 Bounding surface plastic model

Bounding surface plastic model was studied in $J-I$ stress invariant space by Yannis F. Dafalias., in order to make it convenient to combine hardening function of bounding surface model with the hardening rule in the multi-surface model, the $J-I$ stress invariant space was replaced by $q-p$ stress invariant space. The basic parts of calculating the plastic modulus of stress state point on stress path include: equation of bounding surface, mapping rule, shape-hardening function.

3.1.1 Bounding surface. According to the bounding surface proposed by Yannis F. Dafalias, general expression of bounding surface in $q-p$ stress invariant space is as follows:

$$F(p_b, q_b, \alpha, e^p) = 0$$  \hspace{1cm} (1)

Where: $(p_b, q_b)$ are stress point corresponding to current stress point $(p, q)$ which is on the loading stress path; $\alpha$ is lode angle, its change will not be taken into account, $e^p$ is the plastic void ratio standing for $p_0$ which is used to measure the preconsolidation history.

The bounding surface is a smooth surface consisting of two parts $B-C, C-D$ which are shown in Figure 2. Curve $CD$ is one ellipse, curve $BC$ is one hyperbola with continuous tangents at the connecting points $B$ and $C$. The initial stress state within curve $BC$ is corresponding to heavily overconsolidated state comparing with $p_0$ state, the initial stress state within the curve of $CD$ is corresponding to lightly overconsolidated state and normally consolidated state. Here, curve $AB$ in reference [18] was not selected because there is almost no cohesion force for cohesionless soils, not to mention soil. CSL means the critical state line. The situation that is studied is just triaxial compression because the situation of triaxial extension is the same.
Figure 2. Schematic illustration of bounding surface and radial mapping rule in q-p space

Formulations of these three curves are presented as follows:

For curve \(CD\):

\[
F = 3(p_b - p_0)
\left(p_b + \frac{R-2}{R}p_0\right)
+ \left(\frac{(R-1)^2}{3}\right)\left(\frac{q_b}{N}\right)^2 = 0
\]  
(2)

For curve \(BC\):

\[
F = 9(p_b - \frac{p_0}{R})^2 - \left(\frac{q_b}{\sqrt{3N}}\right)
\left(\frac{q_b}{\sqrt{3N}} - \frac{3p_0}{R}
\left(1 + \frac{2RA}{N}\right)\right)
\]  
(3)

The parameters including \(N, A, R\) in these two functions all depend on lode angle \(\alpha\), the specific meaning and methods of calculating these three parameters can be found in Ref [4].

3.1.2 Mapping rule. The mapping rule applied in this paper is radial mapping rule and general plastic formulation is as follows:

\[
\sigma_{ijb} = b(\sigma_{ij} - \alpha_{ij}) + \alpha_{ij}
\]  
(4)

Where \(\sigma_{ij}\) and \(\sigma_{ijb}\) are current stress state and corresponding stress state on the bounding surface, \(\alpha_{ij}\) is the projection center, \(b\) is the similarity ratio.

When equation 4 is applied in \(p-q\) space, the specific expression is as follows:

\[
p_b = b(p - p_c) + p_c
\]  
(5)

\[
q_b = bq
\]  
(6)

\[
\alpha_b = \alpha
\]  
(7)

Where \(p, q, \alpha\) and \(p_b, q_b, \alpha_b\) are current stress state and corresponding stress state on bounding surface, \(p_c\) is the projection center, \(b\) is the similarity ratio.

In equation 5, parameter \(p_c\) can be obtained by:

\[
p_c = cp_0
\]  
(8)

Where \(p_0\) is associated with preconsolidation history and coefficient of \(c\) is constant or variable.
3.1.3 Shape-hardening function $\overline{H}$. After the introduction above, the plastic modulus $K_p$ of stress point $(p_b, q_b)$ on bounding surface corresponding to current stress state $(p, q)$ can be calculated. In order to obtain the plastic modulus $\overline{K}_p$ of current stress point $(p, q)$, the shape-hardening function is needed to establish the relationship between $K_p$ and $\overline{K}_p$, the expression of the general form is as follows:

$$K_p = \overline{K}_p \left( \delta, \sigma_{ij}, q_n \right)$$  \hspace{1cm} (9)

Where the value of $\delta$ is the distance between stress point $\sigma_{ij}$ and $\sigma_{ij_b}$, $q_n$ is a set of tensorial internal variables.

The specific expression of the relationship between $K_p$ and $\overline{K}_p$ is shown below:

$$K_p = \overline{K}_p + \overline{H} \frac{\delta}{(r-\delta)} = \overline{K}_p + \overline{H} \left( \frac{b}{b-1} - s \right)^{-1}$$  \hspace{1cm} (10)

$$r = \frac{b}{(b-1)}$$  \hspace{1cm} (11)

Where the $r$ means the distance of $\alpha_{ij}$ from $\bar{\sigma}_{ij}$.

3.2 Hardening rule of multi-surface plastic model

The multi-surface is another kind of theory that was developed to describe the plastic strain accumulation in the process of cyclic loading. Comparing with only one surface of bounding surface theory, multi-surface theory can trace the variation of loading surfaces more accurately, there exists an infinite amount of loading surfaces. The multi-surface plastic used in this paper was proposed by Jean H. Prevost [18]. In this paper, the method of calculating parameters including $a, \mu$ will not be introduced, this method can be found in Ref. [18].The functions of yield surface and loading surfaces are expressed as follows:

$$f = \frac{3}{2} (s - p \beta): (s - p \beta) - m^2 p^2 = 0$$  \hspace{1cm} (12)

Where $s$ is deviatoric stress tensor; $p$ is effective mean normal stress which is selected as internal variation; $\alpha$ is kinematic deviatoric tensor defining the coordinates of loading surface center in deviatoric stress subspace; $m$ is the material parameter.

In order to apply it into bounding surface theory introduced above, this function of yield surface and loading surface expressed in $(q, p)$ stress space, the function is as follows:

$$f = (q - p \beta)^2 - m^2 p^2 = 0$$  \hspace{1cm} (13)

Where $\beta = (\beta_1 - \beta_2) = \frac{3\beta_1}{2}$.

The flow rule of plastic shear strain is selected as an associative flow which is the same as that of the bounding surface. The hardening rule used in this paper is deviatoric kinematic hardening rule which means this hardening just take the change of loading surface on $\pi$ plane into account. The expression of hardening rule is as follows:

$$p \beta = a \mu$$  \hspace{1cm} (14)
Where $\mu$ is deviatoric tensor defining the direction of translation; $a$ is the amount of translation.

According to the consistency condition applied in kinematic hardening rule, the equation can be obtained as follows:

$$\frac{\partial f}{\partial \sigma_{ij}} (d\sigma_{ij} - db_{ij}) = 0 \quad (15)$$

Where $\frac{\partial f}{\partial \sigma_{ij}}$ is a tensor which symbolizes the outer normal to yield surface and loading surfaces, and this tensor consists of deviatoric and dilatational components. $db_{ij}$ means the back stress which indicates the movement of loading surface center, it is also a tensor consists of deviatoric and dilatational components.

After applying the equation.15 into $(p, q)$ stress space, the relationship between plastic modulus and current stress state can be obtained as follows:

$$H d\lambda - \frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij} = 0 \quad (16)$$

Where $H$ is the plastic modulus of current loading surface, $d\lambda$ means the value of plastic strain increment and the expression of equation.16 is due to the equation $H d\lambda = \frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij}$

Because there is no dilatational component in the tensor of $b_{ij}$, so the final form of equation.16 is as follows:

$$H d\lambda = \frac{\partial f}{\partial q} \frac{\partial q}{\partial \sigma_{ij}} p d\beta \quad (17)$$

Considering the condition of $\sigma_2 = \sigma_3$ of triaxial test that is studied about, the equation.17 can be transformed into:

$$H d\lambda = \pm \frac{3}{2} p^2 q (\alpha p - q) d\beta \quad (18)$$

Where the “+” and “−” indicates the shear compaction and shear extension respectively.

3.3 Modification of hardening function $\tilde{H}$

According to two plastic models which were adapted in $(q, p)$ stress space, the parameter $K_p$ and $H$ has the same mechanical meaning, as well as, the value of $K_p$ and value of $H$ is equal for the same current stress state on the stress path, so a relationship between bounding surface plastic model and multi-surface plastic model can be established as follows:

$$\frac{3}{2} p^2 q (\beta p - q) \frac{d\beta}{d\lambda} = K_p + \tilde{H} \left( \frac{b}{b-1} - s \right)^{-1} \quad (19)$$

Where the “+” was selected to correspond to the situation of shear compaction in bounding surface theory.

For curve $CD$: $K_p = \frac{1 + e_m}{\lambda - k} \left( \left( 1 - \frac{p_l}{p_0} \right) + \frac{p_l}{p_0} \right) \cdot 9 \left[ 2bp + (1 - 2b)p_c + \frac{2p_0}{R} \right] \cdot \left\{ 3 \left[ 2bp + (1 - 2b)p_c \right] + \frac{(R - 1)^2 h^2 q^2}{3N^2} \right\} \quad (20)$
For curve BC: \( R_p = \frac{1+e_{in}}{\lambda-k} \left( \left( 1 - \frac{p_l}{p_o} \right) + \frac{p_l}{p_o} \right) \cdot 54 \left[ bp + (1-b)p_c - \frac{p_o}{R} \right] \cdot \left\{ 18 \left[ bp + (1-b)p_c - \frac{p_o}{R} \right] \cdot \left[ bp + (1-b)p_c \right] + \frac{1}{\sqrt{3N}} \left[ \frac{3p_o(2N+RA)}{R} - \frac{2bq}{\sqrt{3N}} \right] bq \right\} \) (21)

Where \( e_{in} \) is the initial void ratio on \( e - \ln p \) plot for reference preconsolidation history, \( \lambda, k \) are consolidation slope and typical swelling slope respectively, \( p_l \) is a turning point where the relationship between \( p \) and \( e \) changes from logarithmic to linear.

After merging same terms of \( R_p \) and left part of equation 19, polynomials of \( p \) and \( q \) can be obtained in terms of \( g(p, q, b, \beta, c, R, N, A) \) because the formulation is complex and \( R, N, A \) are determined by \( \alpha \) so the new hardening function \( \bar{H} \) can be expressed as follows:

\[ g(p, q, b, \beta, c, \alpha) \cdot \left( \frac{b}{b-1} - s \right) = \bar{H} \] (22)

As it shows in the equation 22, the value of \( \bar{H} \) is determined by current stress state and \( \alpha \), this is corresponding to parameter \( z \) and \( h \) in original \( \bar{H} \); \( b, c \) is corresponding to certain mapping rule; \( \beta \) is a new parameter for \( \bar{H} \) which is used to increase the accuracy of plastic modulus.

4. Performance of new hardening function

In this part, the undrained triaxial tests were conducted to verify the reliability of the new bounding surface plasticity, the confining pressure is 100 kPa, void ratio is 0.668 and the test is controlled by certain stress CSR=0.225, this experiment condition was used to simulate the phenomenon of floe liquefaction, the comparison of simulation result and test in laboratory is shown in Figure 3.

![Figure 3. Comparison of simulation result and laboratory for flow liquefaction](image)

Figure 3 shows the new bounding surface can generally simulate the flow liquefaction phenomenon, although there is out of sync at the beginning of cycle loading. After several cycles of loading and unloading, the curve of laboratory and simulation are basically identical in the process of intensive decrease of effective mean stress ranging from 0kPa to 40kPa. The new bounding surface plasticity can simulate the decrease of effective mean stress range from 40kPa to 100kPa.

Another undrained triaxial test under symmetric stress-controlled cyclic shear loading was conducted to verify the performance of new hardening function for cyclic mobility, the confining pressure is 80kPa, the stress time history is shown in Figure 4.
The relationship between shear strain $\gamma$ and time is shown Figure 5.

According to the Figure 5, this new bounding surface plasticity can simulate the accumulation of shear strain after liquefaction effectively, reflecting the intensive increase of plastic strain.

The image of stress path and relation curve of shear stress $q$ and shear strain $\gamma$ were plotted in Figure 6 and Figure 7 respectively.

In Figure 6, the variation of effective mean stress was reflected explicitly before the pre-liquefaction reached. After the pre-liquefaction, during the process of cyclic mobility, the similarity of increase and decrease of effective mean stress was simulated by this new
bounding surface plasticity. In Figure 7, the increase of strain is corresponding to the increase in Figure 5, the increase is lightly before the pre-liquefaction reached, but the increase becomes more intensive during cyclic mobility in a similar way. All these phenomena can be reflected by using this new hardening function

5. Conclusion
In this paper, a new research idea was proposed to develop the bounding surface plasticity by aiming at the hardening function. The multi-surface plasticity is developed from the bounding surface theory, both of them focus on the variation of plastic modulus to assure every physical value of plastic modulus at every certain stress state. Considering the development of hardening rules and the wide application of multi-surface theory, these hardening rules are applied into the hardening function in the bounding surface theory. By applying the new hardening function into the undrained triaxial test, the performance is verified effective.

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