Towards quantum turbulence in cold atomic fermionic superfluids

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Abstract
Fermionic superfluids provide a new realization of quantum turbulence, accessible to both experiment and theory, yet relevant to phenomena from both cold atoms to nuclear astrophysics. In particular, the strongly interacting Fermi gas realized in cold-atom experiments is closely related to dilute neutron matter in neutron star crusts. Unlike the liquid superfluids $^4$He (bosons) and $^3$He (fermions), where quantum turbulence has been studied in laboratory for decades, superfluid Fermi gases stand apart for a number of reasons. They admit a rather reliable theoretical description based on density functional theory called the time-dependent superfluid local density approximation that describes both static and dynamic phenomena. Cold atom experiments demonstrate exquisite control over particle number, spin polarization, density, temperature, and interaction strength. Topological defects such as domain walls and quantized vortices, which lie at the heart of quantum turbulence, can be created and manipulated with time-dependent external potentials, and agree with the time-dependent theoretical techniques. While similar experimental and theoretical control exists for weakly interacting Bose gases, the unitary Fermi gas is strongly interacting. The resulting vortex line density is extremely high, and quantum turbulence may thus be realized in small systems where classical turbulence is suppressed. Fermi gases also permit the study of exotic superfluid phenomena such as the Larkin–Ovchinnikov–Fulde–Ferrell pairing mechanism for polarized superfluids which may give rise to 3D supersolids, and a pseudo-gap at finite temperatures that might affect the regime of classical turbulence. The dynamics associated with these phenomena has only started to be explored. Finally, superfluid mixtures have recently been realized, providing experimental access to phenomena like Andreev–Bashkin entrainment predicted decades ago. Superfluid Fermi gases thus provide a rich forum for addressing phenomena related to quantum turbulence with applications ranging from terrestrial superfluidity to astrophysical dynamics in neutron stars.

Keywords: quantum turbulence, superfluid mixtures, cold atoms, fermions, vortex dynamics

Some figures may appear in colour only in the online journal.
tangled spaghetti, will evolve in time in a rather chaotic fashion, cross and recombine in similar fashion to strands of DNA and lead to quantum turbulence. At finite temperatures, but below the critical temperature, a normal component is also present and the normal and superfluid components interact. In the literature, the term quantum turbulence also refers to the dynamics of finite-temperature superfluids where the vortices in the superfluid component interacts with a classically turbulent normal component.

Until recently studies of quantum turbulence have been performed mostly in liquid $^4$He and $^3$He [5–10], and lately also in Bose–Einstein condensates (BECs) of cold atoms [11]. Both liquid $^4$He and $^3$He are strongly interacting systems of bosons and fermions respectively, but unfortunately a microscopic theory describing their dynamics does not exists yet. On the other hand the BECs of cold atoms can be described quite accurately either by the Gross–Pitaevskii equation (GPE) [12, 13] at very low temperatures, when the fraction of the normal component is negligible, or by the theory of Zaremba, Nikuni, and Griffin (ZNG) [14], in which the dynamics and the coupling between the normal and superfluid components are accounted for. There exists also stochastic extensions of the GPEs [15, 16], in which two types of modes are introduced, slow quantum modes describes by the GP equation, and the fast modes modeled stochastically, separated by a cutoff energy. Unfortunately, these stochastic extensions appear to generate results which are dependent on the choice of the cutoff energy [17]. The BECs of cold gases are systems of weakly interacting atoms, which is the main reason why the derivation of either the GPE or the ZNG framework were possible.

There exist a number of alternative phenomenological approaches to study superfluid dynamics. As noted by Sato and Packard [18], in the case of Bose superfluids ‘two phenomenological theories explain almost all experiments’: the two-fluid hydrodynamics theories due to Tisza and Landau [19–24], and the ‘complementary view provided by Fritz London, Lars Onsager, and Richard Feynman, [that] treats the superfluid as a macroscopic quantum state.’ A shortcoming of the two-fluid hydrodynamics, which is ‘essentially thermodynamics’ [18], is the absence of the Planck’s constant and the corresponding lack of quantized vortices. Vortex quantization was later implemented ‘by hand’ when needed [24], or by using the GPE. Another remedy is the widely-used filament model [25–27] in which quantized vortices and their crossing and reconnections are introduced ad hoc using the Biot–Savart law as a recipe to construct the resulting velocity from the vortex lines.

Unlike BECs, the unitary Fermi gas (UFG) or a Fermi gas near unitarity—a gas of cold fermionic atoms with interactions tuned at or near a Feshbach resonance—is strongly interacting, and exhibits a large pairing gap of the order of the Fermi energy, a short coherence length of the order of the average interparticle separation, and a high critical velocity of the order of the Fermi velocity [28–33]. The UFG shares many similarities with the dilute neutron matter, which is present in the skin of neutron-rich nuclei, and in the crust of neutron stars where neutron-rich nuclei form a Coulomb lattice immersed in a neutron superfluid. This motivated the initial independent microscopic studies of the UFG by nuclear theorists [28, 34, 35]. A system of fermions with zero-range interaction and infinite scattering length should have a ground state energy and properties determined by their only existing dimensional scale, namely their density, similarly to a free Fermi gas. Theoretical quantum Monte Carlo (QMC) studies of the Fermi gas near unitarity have achieved percent-level accuracy [36–39] and agree with experiment [40] at an accuracy better than 1% for the energy per particle, and a few percent for the magnitude of the pairing gap [41, 42].

A microscopic theoretical framework capable of describing quantum turbulence in fermionic superfluids and implementable in realistic calculations has become possible only recently. Two factors played an crucial role: (i) the development and validation against experiment of an appropriate microscopic framework for the structure and dynamics of fermionic superfluids, and (ii) the implementation of this framework using sophisticated numerical algorithms that fully utilize the advanced capabilities of modern leadership class computers, such as Titan.

QMC algorithms cannot be used to describe turbulent dynamics and the only available theoretical candidate with a

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5 The Many-Body Challenge Problem (MBX) formulated by G F Bertsch in 1999. See also [34, 35].

6 https://olcf.ornl.gov/computing-resources/titan-cray-sk7/. The ensemble of all 18,688 GPUs on Titan delivers about 90% of the computing power, while the rest of 299,088 CPUs provide the rest of about 10%. A single GPU provides the same number of floating point operations as approximately 134 CPUs.
microscopic underpinning is time-dependent density functional theory (TDDFT) [43–46] extended to superfluid systems. Such an extension of TDDFT performed in the spirit of the Kohn and Sham [44] is known as the time-dependent superfluid local density approximation (TDSLDA) [47, 48]. This is in principle an exact approach [49], in the same sense as the initial formulation of DFT by Hohenberg and Kohn [43]. The superfluid local density approximation (SLDA) and the Schrödinger equation description of a given system should be indistinguishable at the one-body level. One can prove mathematically that an energy density functional (EDF) for a given system exists [43], but unfortunately no mathematical recipe has been ever devised for constructing this EDF. In the case of the UFG, however, the structure of the functional can be fixed with high accuracy by rather general requirements: dimensional arguments require that it can depend only on the fermion mass $m$, Planck’s constant $\hbar$, the number density $n(\vec{r})$, the kinetic energy density $\tau(\vec{r})$, the anomalous density $\nu(\vec{r})$, and the current density $\vec{J}(\vec{r})$. The inclusion of the anomalous density is required in order to be able to disentangle the normal from the superfluid phases, and the presence of currents is necessary to put in evidence matter flow. These must be combined in an EDFs that preserves Galilean covariance and appropriate symmetries (parity, translation, rotation, gauge, etc.). As in the implementation by Kohn and Sham [44] of the underlying DFT [44], the kinetic energy density largely accounts for gradient effects, and additional corrections appear to be rather small [39, 50, 51] in case of the UFGs. The resulting (unregulated and unrenormalized) EDF for an unpolarized UFG reads:

$$\varepsilon(\vec{r}) = \frac{\hbar^2}{m} \left[ \frac{\tau(\vec{r})}{2} + \gamma \frac{\nu(\vec{r})^2}{n^{1/3}}(\vec{r}) + \beta \frac{3(3\pi^2)^{2/3}n^{2/3}}{5} \frac{\vec{J}(\vec{r})^2}{n(\vec{r})} \right] - (\alpha - 1) \frac{\vec{J}(\vec{r})^2}{2n(\vec{r})},$$

(1a)

where

$$n(\vec{r}) = \sum_{0 < E < E_c} |v_k(\vec{r})|^2,$$

(1b)

$$\tau(\vec{r}) = \sum_{0 < E < E_c} |v_k(\vec{r})|^2,$$

(1c)

$$\nu(\vec{r}) = \sum_{0 < E < E_c} u_k(\vec{r})^2,$$

(1d)

$$\vec{J}(\vec{r}) = 2 \text{Im} \sum_{0 < E < E_c} v_k^*(\vec{r}) \vec{\nabla} v_k(\vec{r}),$$

(1e)

and where the sums are taken up to a specified energy cutoff $E_c$ in the quasiparticle energy $E_k$, see below. This EDF is written for simplicity in its unregulated form, and the full theory requires regularization to properly deal with the kinetic and anomalous densities which diverge as functions of $E_c$. The regularization and renormalization process is well established and described in [47, 52]. Terms in this functional have very simple physical meaning. The first represents kinetic energy contribution, while the second is due to the pairing interaction—these two terms must enter in a specific combination in order to assure the theory can be regularized. The third term represents the normal interaction energy, and the last term restores Galilean covariance of the functional. The dimensionless constants $\alpha$, $\beta$, and $\gamma$ are determined from independent QMC calculations of the uniform UFGs [28, 37, 41, 53]. The functional has also been extended to spin-polarized case. (For an extensive review, see [47]). In the stationary case, the quasi-particle wave-functions satisfy the Bogoliubov-de Gennes like equations

$$\left( \begin{array}{c} \hbar(\vec{r}) - \mu \\ \Delta(\vec{r}) \\ \Delta^*(\vec{r}) - (h^*(\vec{r}) - \mu) \end{array} \right) \left( \begin{array}{c} u_k(\vec{r}) \\ v_k(\vec{r}) \\ \bar{v}_k(\vec{r}) \end{array} \right) = E_k \left( \begin{array}{c} u_k(\vec{r}) \\ v_k(\vec{r}) \\ \bar{v}_k(\vec{r}) \end{array} \right),$$

(2)

where

$$h(\vec{r}) = \frac{\delta \varepsilon(\vec{r})}{\delta n(\vec{r})} + V_{\text{ext}}(\vec{r}), \quad \Delta(\vec{r}) = \frac{\delta \varepsilon(\vec{r})}{\delta \nu^*(\vec{r})},$$

(3)

and where $V_{\text{ext}}(\vec{r})$ is an arbitrary external potential and $\mu$ is the chemical potential.

The parameters of the SLDA fixed by homogeneous systems have been further validated against solutions of the Schrödinger equation for inhomogeneous systems, and found to agree on the few percent level or better with QMC results for almost 100 systems [54–56] including both polarized and unpolarized systems, as well as superfluid and normal systems [47]. This step is necessary to demonstrate that the Schrödinger equation and SLDA for the UFG deliver identical descriptions of various quantum systems as required by the general DFT theorem [43, 44].

In the case of time-dependent problems, the external potential could become time-dependent (an external stirrer) and the equations for the quasi-particle wave-functions for a spin unpolarized system become

$$i\hbar \left( \begin{array}{c} \dot{u}_k(\vec{r}, t) \\ \dot{v}_k(\vec{r}, t) \end{array} \right) = \left( \begin{array}{c} \hbar(\vec{r}, t) - \mu \\ \Delta(\vec{r}, t) \\ \Delta^*(\vec{r}, t) - (h^*(\vec{r}, t) - \mu) \end{array} \right) \left( \begin{array}{c} u_k(\vec{r}, t) \\ v_k(\vec{r}, t) \end{array} \right).$$

The structure of these TDSLDA equations and the static SLDA equations (2) illustrates the numerical complexity of the problem: one must solve them for all quasi-particle states up to cutoff energy $E_c$. The number of such equations is in principle infinite and they must be accurately discretized, but even after discretization one ends up with the order of $N_v N_s N_s$ quasi-particle wave-functions depending on $N_v N_s N_s$ spatial lattice points, where $N_v$, $N_s$, and $N_s$ are the number of spatial lattice points in each spatial direction. For typical problems studied so far, the number of quasi-particle wave-functions ranges from several tens of thousands to a fraction of a million, leading correspondingly to twice the number of nonlinear coupled complex time-dependent three-dimensional partial differential equations. The solution of these coupled equations requires leadership class computers with hardware accelerators such as graphics processing units (GPU). In practical applications (due to the numerical complexity and memory requirements) thousands of GPU are required, and presently the supercomputer Titan at Oak Ridge National Laboratory (the largest supercomputer currently in US) is one of the few than can handle this type of calculations (see footnote$^6$).
The dynamics of quantized vortices, their crossing, and recombination, requires a full solution in 3D of the TDSLDA equations for a large number of time-steps (hundreds of thousands to millions). This makes the use of leadership class computers an inextricable part of the solution of these kind of problems. Both the development of a validated theoretical microscopic framework and the availability of supercomputers were crucial developments in making the study of fermionic superfluid possible.

One of the first illustrations of the power of the TDSLDA was to show for the first time in literature how various type of quantized vortices can be generated dynamically by stirring a cloud, and how these vortices propagate, cross, recombine, and eventually disappear from the system by interacting with the boundaries \[57\]. It was also demonstrated that the superfluid system can become normal at relatively large stirring velocities, but also, somewhat unexpectedly, that a system stirred at a velocity nominally larger than the speed of sound can remain superfluid. This behavior is different than in superfluid \(^{3}\)He or \(^{4}\)He, and results from the high compressibility of the UFG, where the speed of sound can locally increase and stabilize the superfluid phase.

Solutions of the time-dependent Schrödinger equation for many-fermion systems are hard to come by. There are a few cases where analytical solutions can be obtained \[58-60\] and some have been compared to the corresponding solutions of the similar TDSEDA systems \[61\], corresponding to excitations of the Anderson–Higgs modes and related excitations that await experimental confirmation. The Anderson–Higgs mode is rather spectacular and has some similarities with a plasmon mode in the sense that it has a finite excitation energy for zero momentum. The Anderson–Higgs mode is however not a number density oscillation, the number density remains constant in a homogeneous system and only the amplitude of the pairing gap undergoes large slow anharmonic oscillations in time.

There are a number of non-trivial non-equilibrium phenomena studied experimentally and these results can be confronted with the TDSLDA predictions. One such experiment addresses the excitation of so called quantum shock-waves \[62\]. In a shock-wave, properties of the system vary drastically across the shock-wave front, and have distinct and almost time-independent values on either side of the shock-wave front. Classically shock-waves owe their existence to dissipative effects and their interplay with dispersive effects \[63\]. Dissipation, however, is strongly inhibited in quantum systems at low temperatures and the nature of quantum shock-waves is not obvious \[64\]. The TDSLDA is able to describe rather accurately the existence and the shape of quantum shock-waves observed in experiments, without the need to introduce any dissipative effects \[65\]. The shock-wave front propagates at supersonic speeds, as expected. Moreover, the TDSLDA also predicts the excitation of domain walls in the wake of the shock-wave front, with speeds lower than the speed of sound. These domain walls are topological excitations of the pairing field, which has a negative but almost constant value on one side of the domain wall and positive value on the other (defined up to an obvious gauge transformation). The accuracy of the experiments on quantum shock-waves performed so far is insufficient to put in evidence these domain walls, but at a hydrodynamic level, these experiments are well described by a GPE-like theory tuned to the UFG equation of state \[66\]. In such a GPE-like description one cannot distinguish between density and pairing order parameters and such an approach will also fail to describe the pair-breaking excitations and, moreover, the Anderson–Higgs mode as well, when the density is constant, but the pairing field oscillates. In a GPE approach, the number density cannot be disentangled from the superfluid order parameter.

Another strong validation of the correctness of the TDSLDA was the unambiguous and correct identification of the nature of the excitation observed by the MIT group, identified by the authors as a ‘heavy soliton’ \[69\]. Using the TDSLDA framework (right side of figure 2), we demonstrated how an initially formed domain wall would evolve rapidly into a vortex ring \[68\] in the elongated axially symmetric traps described in the initial paper \[69\]. Vortices in the UFG have a small core and are difficult to image in experiment. For this reason the MIT group devised an ingenious imaging procedure to visualization the excitations. This procedure significantly altered the size of the excitations, originally leading to the incorrect identification of these objects as thick heavy solitons, instead of the thin vortices and vortex rings that were actually produced. Using TDDFTs techniques we theoretically simulated this experimental visualization protocol, demonstrating that vortex rings expand to a size consistent with the thick objects observed in the experiment. The MIT group subsequently refined their imaging procedure, developing a new tomographic imaging technique capable of isolating portions of the excitations, thereby confirming that the imprinted domain wall indeed evolves into a vortex ring, and ultimately a vortex segment when axial symmetry is broken \[70\]. In this subsequent paper it was revealed that the axial symmetry of the trap was spoiled by gravity. TDSLDA simulation of these asymmetric traps confirmed the evolution of the vortex ring into a vortex segment as the ring collides with the trap boundaries \[71\]. Similar processes were first demonstrated in dynamical simulations of the UFG \[57\]. Thus, all of the qualitative aspects of these dynamic experiments have been reproduced by the TDSLDA framework.

Interestingly, the TDSLDA predicts the crossing and reconnection of vortex lines in the intermediate stage between the vortex ring and the vortex line as demonstrated in figure 3. This prediction still awaits experimental verification. The latest MIT paper \[67\] (see figure 2) suggests the existence of exotic structures in the intermediate stage (between domain wall and vortex line state), a combination of a vortex ring and one or more vortex lines, called a \(\Phi\)-soliton \[72\]. The existence of a \(\Phi\)-soliton needs further experimental investigation as no dynamical mechanism for their generation, from a domain wall to a \(\Phi\)-soliton state, has yet been demonstrated in any simulation. Quantitative validation of the TDSLDA is still required, and will need accurate experimental measurements with improved imaging capable of resolving such states in systems small enough to accurately simulate (~1000 fermions on present computers).
Attempts were made to simulate the MIT experiments using the same GPE tuned to the UFG equation of state [66] that explained the shock-wave experiment [62]. While these kind of simulations were able to reproduce some aspects of the vortex dynamics, all of our attempts to generate a stable vortex ring out of a domain wall using only a GPE treatment failed. The main reason seems to be that the GPEs lacks any mechanism other than phonon emission for dissipating excitations, apart from the fact that there is no way to distinguish between the normal density and anomalous density in such an approach. This lack of dissipation is related to the inability of the pure GPE to crystalize vortex lattices. We are not aware of any successful result to stabilize vortex rings in the GPEs without ad hoc mechanisms, introduced to remove energy from the system [73]. It is possible that dynamics in fermionic superfluids may admit a simpler description in terms of GPE-like hydrodynamics, but direct comparison with the TDSLDA demonstrates that additional dissipation is required [74], and any such theory needs a careful and thorough validation against the TDSLDA and experiment, which has yet to be performed.

Interestingly, both of these effects seem to be properly characterized by the TDSLDA treatment, even though it formally conserves energy. In particular, the TDSLDA allows for pair-breaking effects, and having the fermionic degrees of freedom allows the theory to dissipate excess energy along the vortices such that vortex lattices crystalize [57] and the vortex rings stabilizes [68, 71]. In a GPE simulation of a recent experiment on the Josephson effect in fermionic condensates across the Bardeen–Cooper–Schrieffer (BCS)-BECs region the

Figure 2. Evolution of a phase imprinted domain wall in a harmonically trapped unitary Fermi gas. The images show the density along a central slice as a function of time from top to bottom. The left panel shows the evolution of the experiment [67] demonstrating the evolution of the initial domain wall into vortex rings and finally solitonic vortex segments. The right images show the numerical TDSLDA evolution of a similar (but smaller) system in an axially symmetric geometry from [68] demonstrating the evolution of the initial domain wall into a vortex ring which oscillates. The TDSLDA results appear to extrapolate correctly to larger systems. (Images reproduced with permission.)
authors also failed to reproduce the generation of vortices at high values of the current across the junction in the neighborhood of the Feshbach resonance [75], which points to the need to simulate such an experiment within TDSLDA.

The availability of a reliable theory to address fine details and make quite accurate predictions is not available for liquid helium, and together with the ability of experimenters to visualize in detail the spatial vortex configurations and their time evolution [67, 76] makes fermionic cold-atom systems quite unique. One can now also mix cold Bose and Fermi systems [77–79] and can hopefully study in the future in detail both theoretically and experimentally their entrainment, a phenomenon predicted a long time ago by Andreev and Bashkin [80, 81], and so far not yet put in evidence in any other system. The Andreev–Bashkin entrainment can be relevant to the neutron star physics [82–86], where the two superfluids, the superfluid neutrons and the superconducting protons and their respective normal components, and the electrons in a normal phase, and their currents and vortices can entrain each other and where experiments or direct observations cannot be performed. Moreover, as in any other superfluid in rotation, where a massive number of quantized vortices are formed, which also have to exists in a background of a Coulomb lattice of nuclei immersed in a neutron superfluid, the pinning and unpinning of vortices occur, the vortices deform, and during their evolution they can cross and recombine, and quantum turbulence emerges [87].

The cold Fermi gases studied in the neighborhood of the Feshbach resonance have several ‘unfair’ advantages over other physical systems. One can easily manipulate the magnitude and even the sign of the interaction and thus be able to confront theory in experiment in a much wider range of parameters than otherwise one can envision. The number of particles, their density, the temperature, the shape of the confining potentials, a multitude of external probes and stirrers can be rather easily experimentally realized and theoretically simulated. Since the interaction is strong in the Fermi gases, the coherence length is comparable to the average interparticle separation, and the gases are dilute, the vortex-line separation can becomes comparable to the inter-particle separation for modest rotation rates. For example in TDSLDA simulations presented in [57, 71], the average separation between vortices (forming the vortex) lattice was about 3 times average inter-particle distance. (See also top panel of figure 4).

In figure 4 we show an example of how one can generate a large number of quantized vortices in a rather small cloud with only 1410 fermions, together with a domain wall, which subsequently evolve, cross and recombine, leading to manifestly non-equilibrium probability distribution function (PDF) of the particle velocities [71]. We start with the ground state of a cloud cut in half with a knife-edge potential and subsequently we then stir the system with two circulating laser beams parallel to the long axis of the trap. Once a vortex lattice is generated, we imprint a π phase shift between the the two halves of the cloud. Just before removing the knife-edge, we introduce a slight tilt to speed the formation of a vortex tangle. After the knife-edge is removed, the vortex lines twist, cross, and reconnect, and vortices untie in a manner qualitatively in agreement with a recent theoretical analysis of vortex knots [88]. From the velocity PDF one sees a clear departure from the equilibrium Gaussian behavior as the tangle evolves — a hallmark of quantum turbulence [5, 89]. Eventually the system relaxes to a vortex lattice and equilibrates in $v_\parallel$. Somewhat similar velocity PDF are seen in the theoretical studies of dilute Bose gases [89] and in the phenomenological filament model of the crossing-recombination vortex line dynamics [90].

A question of crucial importance in nuclear astrophysics is the pinning mechanism of quantized vortices in the neutron star crust. It was conjectured almost four decades ago by Anderson and Itoh [91] that a neutron star glitch, a sudden

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**Figure 3.** Vortex reconnections transforming a vortex ring into a vortex line. An initial vortex ring, touches the boundaries and coverts into two vortex lines. The first reconnection occurs closer to one edge of the cloud and ends with formation of a new vortex ring (top row). The new vortex ring moves to other side of the cloud. It is deformed and undergoes asymmetric deformation into two vortex lines, one that is absorbed by the boundary and one that survives (bottom line).
spin-up of the rotational frequency, is caused by a cata-

crophic unpinning of a huge number of quantized vortices in
the neutron superfluid from the Coulomb lattice formed by the
nuclei immersed in this fluid. It has been a matter of debate
ever since whether a quantized vortex is pinned to nuclei
forming the lattice or interstitially, therefore whether a vortex
is attracted or repelled by a nucleus. The interaction of a
quantized vortex with a nucleus immersed in a neutron
superfluid is an extremely difficult problem to solve theore-
tically and a direct observation is impossible, and over four
decades theorists produced contradictory predictions. At the
same time it is known that the properties of dilute neutron
matter are very similar to the properties of a UFG. By applying
the TDSLDA and using an accurate nuclear EDF it was shown
recently [92] that a vortex and a nucleus repel always, by
studying the dynamics of such a system and extracting the

Figure 4. Generation of quantum turbulence with a phase imprint of the vortex lattice [71]. In the left column consecutive frames show a vortex lattice with the knife edge dividing the cloud (top frame) and the decay of turbulent motion. In the right column we show the corresponding PDFs for longitudinal $v_\parallel$ and transversal $v_\perp$ components of collective velocity (in arbitrary units). Dotted lines show the Gaussian best fit to the data.
force between a vortex line and a nucleus. The interaction is rather complex, the nucleus and the vortex line deform during their evolution. The situation illustrated in figure 5, apart from the presence of a vortex, is identical to the experimental setups realized in mixtures of Fermi and Bose systems already [77, 78], where spatially the Bose system is of smaller size than the Fermi system. This kind of experiments can be easily performed in the presence of one or even many vortices in the presence of one or even many ‘impurities,’ and the motion of such impurities can be also controlled. Performing and studying such systems will definitely shed a lot of light on the vortex-pinning mechanism and the dynamics in neutron star crust.

Another degree of freedom to be exploited in future studies of quantum turbulence is the spin polarization of the Fermi systems. Clogston [93] and Chandrasekhar [94] noted the normal phase competes with the BCS superfluidity phase in the case of spin polarized Fermi systems. It was predicted a long time ago that the s-wave pairing mechanism is modified by the emergence of the so called Larkin–Ovchinnikov (LO) [95] and Fulde–Ferrell (FF) [96] phase (LOFF or Fulde–Ferrell–Larkin–Ovchinnikov (FFL0)), when the two fermions in the Cooper pair couple to a non-zero total momentum and the pairing gap oscillates in space. The equation of state of the polarized UFG has been extracted independently experimentally [97] and within the SLDA [98, 99], where it was also shown that the LOFF phase exists at unitarity for a wide range of spin polarization. The studies of the structure and dynamics of vortices, and particularly the possibility and features of the quantum turbulence, in this regime in the UFG still awaits its implementation. It was shown recently the the Clogston–Chandrasekhar critical polarization for fermionic superfluidity is enhanced in Fermi–Bose systems [100], making such systems even more promising to study in non-equilibrium.

Cold Fermi gases around the Feshbach resonance have another peculiar feature. Systems with fewer than about a million atoms can have quantum turbulence below the critical temperature. However, this number is too small to sustain large-scale flow and support classical turbulence [101]. In this respect these systems are different from traditional liquid $^4$He experiments at finite temperatures, where both normal and superfluid systems could become turbulent. Classical turbulence in 3D systems is achieved for values of the Reynolds number of the order of $10^{15}$ or higher [1, 2]. In the UFG quantized vortices, and therefore quantum turbulence, can exist in clouds with as little as 500–1000 fermions and flow velocities $v \approx 0.7v_F$, see supplemental material in [57]. Interestingly, due to the compressibility of these gasses, this velocity can be larger than the average Landau critical velocity $v_c \approx 0.4v_F$. The largest cold atomic clouds created so far in the laboratory have about $10^6$ atoms and a Reynolds number $Re \approx 600$.The Reynolds number scales with particle number roughly as $Re = n m v_L /\eta \sim N^{2/3}$ at constant atom number density $n$, where $m$ is the mass of the atom, $L$ the characteristic size of the system, and $\eta$ the shear viscosity. Increasing the flow velocity to values much higher than the Fermi velocity or critical velocity will render the system normal, with properties close those of a free Fermi gas.
Conclusion

Superfluid Fermi gases provide a new forum in which to study quantum turbulence. As with cold atomic BECs, one has exquisite experimental control with the ability to excite and observe topological defects such as domain walls and vortices that lie at the heart of quantum turbulence. Unlike BECs, the UFG is strongly interacting and demonstrates an extremely high vortex line density. Consequently, turbulence can be realized in a new regime: namely in small systems where classical turbulence is suppressed. To date, small numbers of topological defects have been formed and imaged in cold Fermi gases and there appears to be no impediment to realizing turbulence. Current imaging processes, however, are generally destructive and require expansion which can impede their interpretation. Some questions concerning phenomena related to quantum turbulence may therefore require improved non-destructive imaging techniques, perhaps using multi-component systems as has been done with BECs [102] or using some sort of tracer as done with He [9, 103–105]. Particularly useful would be the development of imaging techniques that allow for the extraction of the velocity distribution in a turbulent system.

Although Fermi gases can realize turbulence in small systems with less than 1000 fermions, these are still beyond the reach of direct \textit{ab initio} quantum techniques like QMC. Validated dynamical models will therefore be required. In weakly interacting systems, GPE-based theories and ZNG-like extensions work well, and similar techniques may apply for Fermi gases, but the most accurate approach that remains tractable for strongly interacting systems appears to be superfluid DFT such as the SLDA which has been validated at the percent level for static systems and qualitatively validated against dynamical experiments. Precision measurements of quantitative dynamics in relatively small systems (≤500 particles) is now required to quantitatively benchmark the dynamical component of these theories. The extension of TDSLDA to include fluctuations will also be needed. Above the critical temperature, in the pseudogap regime, where both dimers and atoms coexist, a new kinetic or hydrodynamic approach would be desirable as well. Benchmarking these theories is critical for other branches of physics. In particular, weakly interacting systems, GPE-based theories and ZNG-like systems with less than 1000 fermions, these are still beyond the reach of \textit{ab initio} quantum techniques like QMC.

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