Structure of the resource theory of quantum coherence

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Quantum coherence is an essential feature of quantum mechanics which is responsible for the departure between classical and quantum world. The recently established resource theory of quantum coherence studies possible quantum technological applications of quantum coherence, and limitations which arise if one is lacking the ability to establish superpositions. An important open problem in this context is a simple characterization for incoherent operations, constituted by all possible transformations allowed within the resource theory of coherence. In this work, we contribute to such a characterization by proving several upper bounds on the maximum number of incoherent Kraus operators in a general incoherent operation. For a single qubit, we show that the number of incoherent Kraus operators is not more than 5, and it remains an open question if this number can be reduced to 4. The presented results are also relevant for quantum thermodynamics, as we demonstrate by introducing the class of Gibbs-preserving strictly incoherent operations, and solving the corresponding mixed-state conversion problem for a single qubit.

Quantum resource theories [1, 2] provide a strong framework for studying fundamental properties of quantum systems and their applications for quantum technology. The basis of any quantum resource theory is the definition of free states and free operations. Free states are quantum states which can be prepared at no additional cost, while free operations capture those physical transformations which can be implemented without consumption of resources. Having identified these two main features, one can study the basic properties of the corresponding theory, such as possibility of state conversion, resource distillation, and quantification. An important example is the resource theory of entanglement, where free states are separable states, and free operations are local operations and classical communication [3, 4].

In the resource theory of quantum coherence [5–9], free states are identified as incoherent states

\[ \rho = \sum_j p_j |\phi_j\rangle \langle \phi_j|, \]

i.e., states which are diagonal in a fixed specified basis \{|\phi_j\rangle\}. The choice of this basis depends on the particular problem under study, and in many relevant scenarios such a basis is naturally singled out by the unavoidable decoherence [10].

The definition of free operations within the theory of coherence is not unique, and several approaches have been discussed in the literature, based on different physical (or mathematical) considerations [8]. Two important frameworks are known as incoherent [6] and strictly incoherent operations [7, 11], which will be denoted by IO and SIO, respectively. The characterizing feature of IO is the fact that they admit an incoherent Kraus decomposition, i.e., they can be written as [6]

\[ \Lambda(\rho) = \sum_j K_j \rho K_j^\dagger, \]

where each of the Kraus operators \(K_j\) cannot create coherence individually, \(K_j |n\rangle \sim |n\rangle\) for suitable integers \(n\) and \(m\). This approach is motivated by the fact that any quantum operation can be interpreted as a selective measurement in which outcome \(j\) occurs with probability \(p_j = \text{Tr}[K_j \rho K_j^\dagger]\), and the state after the measurement is given by \(K_j \rho K_j^\dagger / p_j\). An IO can then be interpreted as a measurement which cannot create coherence even if one applies post-selection on the measurement outcomes.

Strictly incoherent operations are incoherent operations with the additional property that all \(K_j\) are also incoherent [7, 11]. These operations have several desirable properties which distinguish them from the larger class IO. In particular, it has been shown that SIO is the most general class of operations which do not use coherence [11].

Other important frameworks which are discussed in the recent literature include maximally incoherent operations (MIO) [5]: this is the most general class of operations which cannot create coherence from incoherent states. It has recently been shown that this framework has maximally coherent mixed states, i.e., quantum states which are the optimal resource among all states with a given spectrum [12]. Another important class are translationally invariant operations (TIO) [13], these are quantum operations which commute with time translations \(e^{-iHt}\) induced by a given Hamiltonian \(H\). The set TIO is strictly larger than TIO for nondegenerate Hamiltonians [13]. Moreover, the class TIO has found several applications in the literature, including the resource theory of asymmetry [13–15] and quantum thermodynamics [16, 17]. Further approaches, also going beyond incoherent states, have been investigated recently in Refs. [18–23].

The quantification of coherence is another important research direction. Postulates for coherence quantifiers have been presented [5, 6, 8, 24], based on earlier approaches in entanglement theory [3, 4, 25]. Operational coherence measures include distillable coherence and coherence cost [7, 26], which quantify optimal rates for asymptotic coherence distillation and dilution via the corresponding set of free op-

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A general quantum operation, acting on a Hilbert space of dimension \( d \), admits a decomposition with at most \( d^2 \) Kraus operators [49] – this is the maximum Kraus rank. However, since (strictly) incoherent Kraus operators have a very specific structure, it is unclear if this result also transfers to IO and SIO [50]. In the following two statements, we provide upper bounds for the number of (strictly) incoherent Kraus operators for these operations. We refer to this minimal number as the (strictly) incoherent Kraus rank.

**Theorem 1** (Maximum number of incoherent Kraus operators for IO). Any incoherent operation acting on a Hilbert space of dimension \( d \) admits a decomposition with at most \( d^2 + 1 \) incoherent Kraus operators. For \( d = 2 \) and \( d = 3 \), this number can be improved to 5 and 39 respectively.

This theorem is a combination of Propositions 3, 4, and 5, which will be presented and discussed below. The corresponding bound for SIO is given in the following statement.

**Theorem 2** (Maximum number of strictly incoherent Kraus operators for SIO). Any strictly incoherent operation acting on a Hilbert space of dimension \( d \) admits a decomposition with at most \( \min\{d^2 + 1, \sum_{k=1}^{d} d!/(k-1)! \} \) strictly incoherent Kraus operators.

For a given initial Bloch vector \((r_x, r_y, r_z)^T\), these inequalities completely characterize the achievable region for the final Bloch vectors \((s_x, s_y, s_z)^T\). The achievable region is symmetric under rotations around the \( z \)-axis and corresponds to a cylinder with radius \((r_x^2 + r_z^2)^{1/2}\) and height \(2r_z\) with ellipsoids attached at the top and the bottom. In Fig. 1 we show the projection of the achievable region into the \( x-z \) plane for

![Figure 1. Achievable region for single-qubit SIO, IO, and MIO. Colored areas show the projection of the achievable region in the \( x-z \) plane for initial Bloch vectors \((0.5, 0, 0.5)^T\) [blue], \((-0.8, 0, -0.6)^T\) [green], and \((1, 0, 0)^T\) [red]. Note that the last two states are pure. The magenta line corresponds to the achievable region of an incoherent state with Bloch vector \((0, 0, 0.65)^T\).](image-url)
four initial states. The proof of Eqs. (3) in Appendix A makes use of our result that any single-qubit SIO can be decomposed into four strictly incoherent Kraus operators, see Theorem 2 and discussion below Proposition 4 for the general form of these operators. Alternatively, the form of the achievable region can be proven using results in [18–20]. We also note that for pure states the conversion problem has been completely solved for SIO [7] and IO [45].

As a second application for our results, we investigate strictly incoherent operations which preserve a given incoherent state \( \tau \), i.e.,

\[
\Lambda(\tau) = \tau.
\]

The motivation for this constraint originates from quantum thermodynamics. In particular, if \( \tau = e^{-\beta H} / \text{Tr}[e^{-\beta H}] \) is the Gibbs state of a system with Hamiltonian \( H \), then the condition (4) is known to hold for thermal operations [16, 17, 51]. For a non-degenerate Hamitonian \( H \) thermal operations cannot create coherence in the eigenbasis of \( H \), and conditions for state transformations under these operations and the role of coherence therein have been extensively studied in Refs. [52–54]. The most general quantum operations that fulfill Eq. (4) are known as Gibbs-preserving operations, and it has been shown that such operations can create coherence [55]. Here, we contribute to this discussion by introducing the set of Gibbs-preserving SIO, and giving a full solution for the mixed-state conversion problem on a single qubit. As an example, in Fig. 2 we show the achievable region in the \( x-z \) plane for an initial state \( \rho \) with Bloch vector \( r = (0.5, 0, 0.5)T \), and the preserved Gibbs state \( \tau \) has the Bloch vector \( t = (0, 0, -0.2)T \). Note that the achievable region is convex and symmetric under rotations around the \( z \)-axis. We refer to Appendix B for more details and further examples.

This discussion clearly demonstrates how Theorems 1 and 2 lead to deep insights on the structure of the resource theory of quantum coherence. In particular, they lead to a full solution of the state conversion problem under single-qubit SIO, IO, and MIO. These results have clear relevance within the resource theory of coherence, and also extend to other related fields, including quantum thermodynamics.

In the remainder of this letter we will present statements which are needed for proving the aforementioned theorems, and provide further detailed discussion of the results.

**Bounds from Choi-Jamiołkowski isomorphism.** We will now present bounds for IO and SIO arising from the Choi-Jamiołkowski isomorphism between a quantum operation \( \Lambda \) and the corresponding Choi state

\[
\rho_\Lambda = (\Lambda \otimes \text{id})(\Phi_d^*),
\]

where \( \Phi_d^* = d^{-1} \sum_{i,j=0}^{d-1} \ket{i,j} \bra{i,j} \) is a maximally entangled state of dimension \( d^2 \). The rank of the Choi state is the Kraus rank, which is the smallest number of (not necessarily incoherent) Kraus operators.

**Proposition 3** (Bounds originating from the Choi state). Any (strictly) incoherent operation acting on a Hilbert space of dimension \( d \) admits a decomposition with at most \( d^4 + 1 \) (strictly) incoherent Kraus operators.

**Bounds from the structure of (strictly) incoherent operations.** We will now provide improved bounds which explicitly make use of the structure of IO and SIO. For this, we will first consider incoherent operations on a single-qubit, i.e., the corresponding Hilbert space has dimension \( d = 2 \). The following proposition shows that any single-qubit IO can be decomposed into 5 incoherent Kraus operators with a simple structure.

**Proposition 4** (Incoherent operations on qubits). Any incoherent operation acting on a single qubit admits a decomposition with at most 5 incoherent Kraus operators. A canonical
choice of the operators is given by the set
\[
\{(a_1, b_1), (0, b_2), (a_3, b_3), (0, 0), (a_4, 0), (0, 0), (a_5, 0)\},
\]  
(7)

where \(a_i\) can be chosen real, while \(b_i \in \mathbb{C}\). Moreover, it holds that \(\sum_{i=1}^5 a_i^2 = \sum_{j=1}^4 |b_j|^2 = 1\) and \(a_1 b_1 + a_2 b_2 = 0\).

We refer to Appendix D for the proof. We also note that the same techniques can be applied to single-qubit SIO, in which case the number of strictly incoherent operators reduces to 4.

The corresponding general form of strictly incoherent Kraus operators can be given as
\[
\{(a_1, 0), (0, b_2), (a_3, 0), (0, 0), (a_4, 0), (0, 0)\},
\]  
where \(a_i\) are real and \(\sum_{i=1}^4 a_i^2 = \sum_{j=1}^2 |b_j|^2 = 1\). A more general bound for SIO for arbitrary dimensions will be given below.

It is important to note that the proofs of Propositions 3 and 4 are fundamentally different: while the argument leading to Proposition 3 is based on the Choi-Jamiołkowski isomorphism and Caratheodory's theorem, the proof of Proposition 4 makes explicit use of the structure of IO. In the next step, we will extend Proposition 4 to arbitrary dimension.

**Proposition 5** (IO for \(d\)-dimensional systems). Any incoherent operation for a quantum system of dimension \(d\) admits a decomposition with at most \(d(d^4 - 1)/(d - 1)\) incoherent Kraus operators.

We refer to Appendix E for the proof. For single-qubit IO Proposition 5 gives us 6 incoherent operators as an upper bound. As we have already seen in Proposition 4, this number can be reduced to 5. For qutrits we obtain 39 Kraus operators, while the bound in Proposition 3 gives 82 Kraus operators. For dimensions larger than 3 Proposition 3 always gives a better bound.

We will now see how the tools presented above can be applied to study the structure of SIO. By Proposition 3, any SIO admits a decomposition into (at most) \(d^4 + 1\) strictly incoherent Kraus operators. As we will show in the following, for small dimensions this number can be significantly reduced.

**Proposition 6** (SIO for \(d\)-dimensional systems). Any strictly incoherent operation acting on a Hilbert space of dimension \(d\) admits a decomposition with at most \(\sum_{k=1}^d (d^4 - 1)/(d - 1)!\) strictly incoherent Kraus operators.

The proof of the proposition can be found in Appendix F. Note that the bound in this proposition is below \(d^4 + 1\) for \(d \leq 5\). For larger dimensions \(d^4 + 1\) gives a better bound. For \(d = 2\) we see that any single-qubit SIO admits a decomposition into 4 strictly incoherent Kraus operators. This was already discussed below Proposition 4. As we will show in the following, this bound is tight.

**Proposition 7** (Lower bound). For a Hilbert space of dimension \(d\), there exist strictly incoherent operations which cannot be implemented with fewer than \(d^2\) Kraus operators.

The proof of the proposition can be found in Appendix G. Note that for \(d = 2\) the bounds in Propositions 6 and 7 coincide: any single-qubit SIO can be decomposed into 4 strictly incoherent Kraus operators, and some single-qubit SIO require 4 Kraus operators in their decomposition.

**Conclusions.** In this work we have studied the structure of the resource theory of quantum coherence, focusing in particular on the structure of incoherent and strictly incoherent operations. We have shown that any (strictly) incoherent operation can be written with at most \(d^4 + 1\) (strictly) incoherent Kraus operators, where \(d\) is the dimension of the Hilbert space under study. For small dimensions this number can be significantly reduced. For a single qubit any IO can be decomposed into 5 incoherent Kraus operators, while any SIO admits a decomposition into 4 strictly incoherent Kraus operators. While the latter bound is tight, the tightness of the other bounds remains an open question.

Our results assist in the systematic investigation of the resource theory of coherence due to the significant reduction of unknown parameters. We have applied our results to solve the mixed-state conversion problem for single-qubit SIO, IO, and MIO. We further introduced the class of Gibbs-preserving strictly incoherent operations and also solved the corresponding mixed-state conversion problem for a single qubit. A natural next step would be to consider single-qubit incoherent operations applied on one subsystem of a bipartite quantum state. Such multipartite scenarios have been previously studied in Refs. [38–41], and the results presented in this letter provide a strong framework for their further investigation. Another important question which is left open in this work is the relation of Gibbs-preserving strictly incoherent operations to thermal operations. We expect that further results in this direction will be presented in the near future, exploring Gibbs-preserving strictly incoherent operations in relation to recent works on quantum thermodynamics [16, 17, 52–54] and extending them to other notions of quantum coherence [8].

Finally, our results also transfer to other related concepts, including translationally invariant operations, which are relevant in the resource theory of asymmetry and quantum thermodynamics. Recalling that TIO is a subset of IO, the results presented in this letter also give bounds on decompositions of TIO into incoherent Kraus operators. Thus, numerical simulations and optimizations over all these classes now also become feasible, at least for small dimensions.

**Note added.** Upon completion of this manuscript, a related work has appeared on the arXiv [56].

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In the following, we will prove that via SIO it is possible to convert a single-qubit state $\rho$ with Bloch vector $\mathbf{r} = (r_x, r_y, r_z)^T$ into another single-qubit state $\sigma$ with Bloch vector $\mathbf{r}'$.
plane, i.e., we set composed into 4 Kraus operators of the form

\[ s^2 + s^2 \leq r_x^2 + r_y^2, \]

(A1)

\[ s^2 \leq 1 - \frac{1 - r_x^2}{r_x^2} (s^2 + s^2). \]

(A2)

Note that by Theorem 3 in Ref. [18], this also proves that these conditions are necessary and sufficient for state transformations via single-qubit IO and MIO.

In the first step, we will discuss the symmetries and main properties of the considered problem. For a given initial state \( \rho \), the achievable region (i.e., the set of all states which is achievable via SIO) is invariant under rotations around the \( z \)-axis of the Bloch sphere. This follows from the fact that such rotations are strictly incoherent unitaries. The achievable region is further symmetric under reflection at the \( x \)-\( y \) plane, since such a reflection corresponds to a \( \sigma_z \) rotation, which is SIO. Moreover, the achievable region is convex and contains the maximally mixed state \( |1/2 \rangle \).

Due to these properties we can significantly simplify our further analysis by focusing on the positive part of the \( x \)-\( z \) plane, i.e., we set \( r_x = s_x = 0 \), and all the other coordinates are nonnegative. Our statement then reduces to the following inequalities:

\[ s^2 \leq r_x^2, \]

(A3)

\[ s^2 \leq 1 - \frac{1 - r_x^2}{r_x^2} s^2. \]

(A4)

In a next step, we note that any single-qubit SIO can be decomposed into 4 Kraus operators of the form

\[ K_1 = \begin{pmatrix} a_1 & 0 \\ 0 & b_1 \end{pmatrix}, \quad K_2 = \begin{pmatrix} 0 & b_2 \\ a_2 & 0 \end{pmatrix}, \]

\[ K_3 = \begin{pmatrix} a_3 & 0 \\ 0 & 0 \end{pmatrix}, \quad K_4 = \begin{pmatrix} 0 & b_3 \\ 0 & 0 \end{pmatrix}. \]

(A5)

In particular, any SIO is completely determined by the vectors \( a = (a_1, a_2, a_3)^T \) and \( b = (b_1, b_2, b_3)^T \). The normalization condition \( \Sigma_{a_1} K_i K_i^T = 1 \) implies that \( |a| = |b| = 1 \). Moreover, \( a \) can be chosen real. The proof for this form of single-qubit SIOs follows the same line of reasoning as the proof of Proposition 4, see also Appendix D for more details. By using the explicit form of the Kraus operators (A5), the Bloch coordinates \( s_x \) and \( s_z \) of the final state take the form

\[ s_x = r_x (a_1 \text{Re} \{b_1\} + a_2 \text{Re} \{b_2\}), \]

(A6a)

\[ s_z = 1 - a_2^2 (1 + r_z) - |b|^2 (1 - r_z), \]

(A6b)

where \( \text{Re} \{b_i\} = (b_i + b_i^\dagger)/2 \) denotes the real part of the complex number \( b_i \).

We will now focus on the boundary of the achievable region in the positive part of the \( x \)-\( z \) plane. In particular, we will show that this boundary can be attained with real vectors \( a \) and \( b \) with \( a_3 = b_3 = 0 \). For this, let \( (s_x, s_z) \) be a point at the boundary. Assume now that the corresponding vectors \( a \) and \( b \) do not have the aforementioned properties. Then, we can introduce the following normalized real vectors

\[ a' = ([a_1^2 + a_2^2]^{1/2}, |a_2|, 0)^T, \]

(A7)

\[ b' = ([b_1^2 + |b_2|^2]^{1/2}, |b_3|, 0)^T. \]

(A8)

The vectors \( a' \) and \( b' \) give rise to a state with Bloch coordinates \( s_x' \) and \( s_z' \), as given in Eqs. (A6a) and (A6b). It is now straightforward to check that \( s_x' = s_z \), and \( s_x' \geq s_z \). However, we assumed that \( (s_x, s_z) \) belongs to the boundary. Since the achievable region is convex, it must be that \( s_x' = s_z \). This proves that the boundary can be attained with real vectors \( a \) and \( b \) with \( a_3 = b_3 = 0 \).

In the final step, we introduce the parametrization

\[ a_1 = \cos \frac{\theta - \phi}{2}, \quad a_2 = \sin \frac{\theta - \phi}{2}, \]

(A9)

\[ b_1 = \sin \frac{\theta + \phi}{2}, \quad b_2 = \cos \frac{\theta + \phi}{2}. \]

(A10)

Using these parameters in Eqs. (A6a) and (A6b) we obtain

\[ s_x = r_x \sin \theta, \]

(A11)

\[ s_z = r_z \sin \theta \sin \phi + \cos \theta \cos \phi. \]

(A12)

Recall that due to the symmetry of the problem we are considering only nonnegative values of \( r_x \) and \( s_z \). Thus, without loss of generality, the parameter \( \theta \) can be chosen from the range \( 0 \leq \theta \leq \pi/2 \).

From Eq. (A11) we can immediately verify Eq. (A3), and it remains to prove Eq. (A4). For this, we will evaluate the maximal value for \( s_z \) achievable for given values of \( r_x \), \( r_z \), and \( s_z \). The maximum of \( s_z \) is determined by \( ds_z/d\phi = 0 \). Taking the aforementioned parameter regions into account, we can solve this problem explicitly for \( \phi \), with the solution

\[ \phi = \arctan \left[ r_z \tan \theta \right]. \]

(A13)

Using this solution in Eq. (A12) we obtain the explicit form for the maximal \( s_z \):

\[ s_z = (\cos^2 \theta + r_z^2 \sin^2 \theta)^{1/2}. \]

(A14)

In summary, we have shown that the points \( (r_x \sin \theta, [\cos^2 \theta + r_z^2 \sin^2 \theta]^{1/2}) \) with parameter \( \theta \in [0, \pi/2] \) are located at the boundary of the achievable region on the positive \( x \)-\( z \) plane. Note that all these points fulfill Eq. (A4) with equality. Moreover, this result completely characterizes the full achievable region due to convexity, rotational symmetry around the \( z \)-axis, and reflection symmetry at the \( x \)-\( y \) plane. This completes the proof. We note that the statement can also be proven in an alternative way, by using results in section III of Ref. [19].

Appendix B: Gibbs-preserving SIO

In the following, we consider single-qubit SIO which preserve a given incoherent state \( \tau \), i.e.,

\[ \Lambda(\tau) = \tau, \]

(B1)

where the Bloch vector corresponding to \( \tau \) is given by \( t = (0, 0, \tau) \).

Before presenting the solution for this problem, we note that any single-qubit incoherent state

\[ \tau = p|0\rangle\langle 0| + (1 - p)|1\rangle\langle 1| \]

(B2)
The initial state has the Bloch vector \( r = (0.5, 0, 0.5)^T \) [blue dot], and the corresponding Bloch vector \( t \) is shown as a green dot.

The equality leads to constraints on the range of the parameter \( t_z \). In particular, \( a_2 \) can take any real value compatible with

\[
0 \leq a_2^2 \leq \min \left \{ 1, \frac{1 - t_z}{1 + t_z} \right \}.
\]  

\( a_2 \) can take any value between \(-1 \) and 1. Using these results in Eq. (A6b), we obtain

\[
s_z = r_z - 2a_2^2r_z,
\]

and it follows that \( s_z \) can take any value between \(-|r_z|\) and \(|r_z|\). It remains to show that \( s_z \) can take any value between \(-|r_z|\) and \(|r_z|\) for any given value of \( s_r \). This can be seen by choosing the parameters \( a_i \) and \( b_i \) as follows,

\[
\begin{align*}
a_1 &= b_1 = \sqrt{1 - a_2^2}, \\
a_2 &= b_2 = \sqrt{\frac{r_z - s_z}{2r_z}}, \\
a_3 &= b_3 = 0.
\end{align*}
\]

For this choice of parameters Eq. (A6a) gives us \( s_y = r_y \). Note that a larger value for \( s_y \) cannot be achieved via SIO in general, see also the discussion in Appendix A. By symmetry, this completes the proof of Eq. (B6). In general (i.e., without restricting to the \( x-z \) plane), the achievable region for unital single-qubit SIOs is given by

\[
s_x^2 + s_y^2 \leq r_x^2 + r_y^2, \quad s_z^2 \leq r_z^2.
\]

In the right part of Fig. 3 we show this region for the initial state with the Bloch vector \( r = (0.5, 0, 0.5)^T \) [dot].
The initial state [blue dot] is the same as in Fig. 3, and the corresponding Bloch vector $s$.

In the next step we will focus on another special case, namely $t_z = 1$. From Eq. (B4) we immediately see that $a_2 = 0$, and $|b_1|$ can take any value between 0 and 1. From Eq. (A6b) we further obtain

$$s_z = 1 - |b_1|^2(1-r_z),$$

which means that $s_z$ can take any value in the range $r_z \leq s_z \leq 1$. We now use Eq. (A6a), noting that for $a_2 = 0$ and fixed value of $|b_1|$ the maximal value of $s_z$ is given as $s_z = |r_z b_1|$. Together with Eq. (B13), we finally obtain

$$s_z = |r_z| \sqrt{\frac{1 - s_z}{1 - r_z}},$$

for the maximal value of $s_z$. By symmetry, the achievable region in the $x$-$z$ plane is thus determined by the inequalities

$$\frac{s_z^2}{r_z^2} \leq 1 - s_z, \quad r_z \leq s_z \leq 1.$$  

In general (i.e., without restricting to the $x$-$z$ plane) the achievable region is given by

$$\frac{s_z^2 + r_z^2}{r_z^2 + r_y^2} \leq 1 - s_z, \quad r_z \leq s_z \leq 1.$$  

In the left part of Fig. 4 we show this region for the initial state with the Bloch vector $r = (0.5, 0, 0.5)^T$.

We will now consider another special case, namely $t_z = -1$. In this situation we see from Eq. (B4) that $|b_1| = 1$, and $a_z^2$ can take any value between 0 and 1. From Eq. (A6b) we get

$$s_z = r_z - a_z^2(1 + r_z),$$

which implies that $s_z$ can take any value between $-1$ and $r_z$. Moreover, for $|b_1| = 1$ and a fixed value of $|a_z|$ the maximal value of $s_z$ is given as $s_z = |r_z(1 - a_z^2)^{1/2}|$, which can be seen directly from Eq. (A6a). Together with Eq. (B19) we arrive at the following result for the maximal value of $s_z$:

$$s_z = |r_z| \sqrt{\frac{1 + s_z}{1 + r_z}}.$$  

By symmetry, this proves that the achievable region in the $x$-$z$ plane is determined by

$$\frac{s_z^2}{r_z^2} \leq \frac{1 + s_z}{1 + r_z}, \quad -1 \leq s_z \leq r_z.$$  

In general, i.e., without restricting to the $x$-$z$ plane, the achievable region is given by

$$\frac{s_z^2 + r_z^2}{r_z^2 + r_y^2} \leq \frac{1 + s_z}{1 + r_z}, \quad -1 \leq s_z \leq r_z.$$  

In the right part of Fig. 4 we show this region for the initial state with the Bloch vector $r = (0.5, 0, 0.5)^T$.

In the final step we consider the remaining case $-1 < t_z < 1$, $t_z \neq 0$. By solving Eq. (B4) for $|b_1|^2$ and inserting it in Eq. (A6b) we obtain

$$s_z = r_z + 2 \left(\frac{t_z - r_z}{1 - t_z}\right) a_z^2.$$  

Using this result together with Eq. (B5) we can determine the
range of \( s_z \), given by

\[
s_z \in \begin{cases} \frac{t_z - r_z}{1 - t_z}, & \text{for } t_z \geq r_z, \\ \frac{r_z - t_z}{1 - t_z}, & \text{for } t_z < r_z. 
\end{cases}
\]

We will now determine all possible values of \( s_z \) for a given value of \( s_x \). Note that due to Eqs. (B4) and (B25) a fixed value of \( s_x \) immediately fixes the values of \(|a_2|\) and \(|b_1|\) as follows:

\[
|a_2| = \sqrt{\frac{(s_z - r_z)(1 - t_z)}{2(t_z - r_z)}}, \tag{B27a}
\]

\[
|b_1| = \sqrt{1 - \frac{(1 + t_z)(s_z - r_z)}{2(t_z - r_z)}}. \tag{B27b}
\]

In the next step, we use the explicit expression for \( s_x \) in Eq. (A6a). Recalling that the vectors \( a = (a_1, a_2, a_3) \) and \( b = (b_1, b_2, b_3) \) are normalized, it is straightforward to see that for fixed values of \(|a_2|\) and \(|b_1|\) the maximal value for \( s_z \) is given by

\[
s_z = \frac{r_z}{|a_1|} \left[ \frac{1}{1 - a_2^2} + \frac{|a_1^2|}{1 - |b_1|^2} \right]. \tag{B28}
\]

By using Eqs. (B27) it is straightforward to express \( s_x \) as a function of \( r_x, r_z, s_z, \) and \( t_z \). We do not display the final expression here. Recalling that the achievable region is symmetric under rotation around the \( z \)-axis, it is straightforward to obtain the full achievable region from Eq. (B28). In the left part of Fig. 3 we show the achievable region for the initial state with the Bloch vector \( r = (0.5, 0, 0.5)^T \), and the state \( \tau \) has the Bloch vector \( t = (0, 0, 0.7)^T \).

Appendix C: Proof of Proposition 3

We will present the proof for IO. The proof for SIO follows the same lines of reasoning. In the first step we define the operator

\[ M = (K \otimes I)\Phi^\dagger_\sigma(K^\dagger \otimes I), \tag{C1} \]

where \( K \) is an arbitrary incoherent operator. The operators \( M \) are Hermitian operators of dimension \( d^4 \), i.e., the number of real parameters is \( d^4 \). For any incoherent operation \( \Lambda \), the corresponding Choi state \( \rho_\Lambda \) belongs to the convex hull of the operators \( M \). This follows directly from the definition of an incoherent operation in Eq. (2). By Caratheodory’s theorem, any Choi state can thus be written as a convex combination of at most \( n = d^4 + 1 \) operators of the form (C1). That is, there exist \( n \) incoherent operators \( N_j \) and a probability distribution \( p_j \) such that

\[ \rho_\Lambda = \sum_{j=1}^{n} p_j (N_j \otimes I)\Phi^\dagger_\sigma(N^\dagger_j \otimes I). \tag{C2} \]

This, together with the Choi-Jamiołkowski isomorphism implies that the incoherent operation \( \Lambda \) can be written as

\[ \Lambda(\sigma) = \sum_{j=1}^{n} L_j \sigma L^\dagger_j \tag{C3} \]

with \( n \) incoherent Kraus operators defined as \( L_j = \sqrt{p_j}N_j \).

The fact that \( L_j \) are indeed Kraus operators, i.e., fulfill the completeness relation \( \sum_{j=1}^{n} L^\dagger_j L_j = I \) can be checked directly from Eq. (C2), recalling that \( Tr_1(\rho_\Lambda) = \|d\). 

Appendix D: Proof of Proposition 4

The main ingredient in the proof is the fact that two sets of Kraus operators \( \{K_j\}_{j=1}^{n} \) and \( \{L_i\}_{i=1}^{k} \) give rise to the same quantum operation if and only if \([49]\)

\[ L_i = \sum_{j=1}^{n} U_{ij} K_j \tag{D1} \]

where \( U_{ij} \) are the elements of a unitary \( U \in U(\max(n,k)) \). That is to say, the two sets of Kraus operators are connected by an isometry. The smallest number of Kraus operators achievable for a given quantum operation is the Kraus rank, which is identical with the rank of the Choi state.

Before we proceed, we note that any incoherent operation on a finite dimensional Hilbert space admits a finite set of incoherent Kraus operators, as was proven in Proposition 3. For a single qubit, the incoherence condition \( K_j|m\rangle \sim |n\rangle \) implies that every incoherent Kraus operator belongs to one of the following four types:

\[ K^I = \begin{pmatrix} * & 0 \\ 0 & 0 \end{pmatrix}, \quad K^{II} = \begin{pmatrix} * & 0 \\ 0 & 0 \end{pmatrix}, \tag{D2} \]

\[ K^{III} = \begin{pmatrix} 0 & 0 \\ 0 & * \end{pmatrix}, \quad K^{IV} = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}. \tag{D3} \]

where \( * \) denotes an arbitrary complex number.

We will now show that there always exists a set of incoherent Kraus operators \( \{L_i\} \) where each of the four types occurs at most twice, i.e., the total number of incoherent Kraus operators \( L_i \) is at most eight. For this, assume that the incoherent Kraus representation \( \{K_j\}_{j=1}^{n} \) contains at least three nonzero Kraus operators of the first type, i.e., \( K_1, K_2, K_3 \in K^I \). Note that these three Kraus operators are then linearly dependent: there is a nontrivial choice of numbers \( z_i \in \mathbb{C} \) such that \( \sum_{i=1}^{3} z_i K_i = 0 \). Without loss of any generality, we can assume that the vector \( (z_1, z_2, z_3)^T \) is normalized, i.e., \( \sum_{i=1}^{3} |z_i|^2 = 1 \). In the next step we introduce a \( U \in U(n) \) as

\[ U = V \otimes I_{n-3}, \tag{D4} \]

where \( I_{n-3} \) is the identity operator acting on dimension \( n-3 \) and \( V \in U(3) \) unitary defined such that its first row corresponds to \( (z_1, z_2, z_3) \). With this definition, it is straightforward to check that the Kraus operators \( L_i = \sum_j U_{ij} K_j \) are all incoherent, and moreover \( L_1 = 0 \). These arguments are not limited
to Kraus operators of the first type $K^I$, but can be applied in the same way for all types given above. Applying this procedure repeatedly, we see that any incoherent single-qubit quantum operation can be written with at most two incoherent Kraus operators of every type, i.e., with at most 8 incoherent Kraus operators in total.

In the next step, we show that the number of incoherent Kraus operators can be further reduced to 6. For this, note that without any loss of generality, the respective sets are given by

$$K^I = \left\{ \begin{pmatrix} * & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & * \end{pmatrix} \right\}, \quad K^{II} = \left\{ \begin{pmatrix} * & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & * \end{pmatrix} \right\},$$

(D5)

$$K^{III} = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}, \quad K^{IV} = \left\{ \begin{pmatrix} * & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & * \end{pmatrix} \right\}.$$  (D6)

Following similar arguments as above, the joint set $K^I \cup K^{II}$ can be reduced to three operators, and the same is true for $K^{III} \cup K^{IV}$. This proves that any incoherent single-qubit operation can be written with 6 Kraus operators of the form

$$\left\{ \begin{pmatrix} * & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}.$$  (D7)

We will now complete the proof, showing that this set of 6 Kraus operators can be reduced to 5 Kraus operators of the form (7). For achieving this, we will rearrange the aforementioned Kraus operators, focusing in particular on the 3 operators

$$K_1 = \begin{pmatrix} 0 & 0 \\ a_1 & b_1 \end{pmatrix}, \quad K_2 = \begin{pmatrix} a_2 & 0 \\ 0 & b_2 \end{pmatrix}, \quad K_3 = \begin{pmatrix} 0 & 0 \\ a_3 & 0 \end{pmatrix},$$  (D8)

where $a_1$ and $b_1$ are complex numbers. The missing 3 Kraus operators will be denoted by $K_4$, $K_5$, and $K_6$. Their elements will not be important in the following discussion, but we note that they have the form

$$K_4 = \begin{pmatrix} * & * \\ 0 & 0 \end{pmatrix}, \quad K_5 = \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}, \quad K_6 = \begin{pmatrix} * & 0 \\ 0 & 0 \end{pmatrix},$$  (D9)

where $*$ denotes some complex number.

Consider now the following $3 \times 3$ unitary matrix

$$U = \begin{pmatrix} m b_1 |a_3|^2 & 0 \\ m |a_1|^2 + |a_3|^2 & b_2 \end{pmatrix}_{a_3 b_2} - \frac{ma_3^* b_1 a_1}{-na_3 b_1} - \frac{na_1 b_2}{-na_1 b_2},$$  (D10)

where the parameters $l$, $m$, and $n$ are nonnegative and chosen as

$$l^2 = \frac{1}{|a_1|^2 + |a_3|^2}, \quad m^2 = \frac{1}{(|a_1|^2 + |a_3|^2) (|a_3|^2 |b_1|^2 + |b_2|^2) + |a_1 b_2|^2}}, \quad n^2 = \frac{1}{|a_3|^2 |b_1|^2 + |b_2|^2) + |a_1 b_2|^2}.$$  

We now introduce a new Kraus decomposition $\{L_i\}$ as

$$L_i = \sum_{j=1}^{3} U_{i,j} K_j \quad \text{for } 1 \leq i \leq 3, \quad K_i \quad \text{for } 4 \leq i \leq 6,$$  (D11)

where $U_{i,j}$ are elements of the unitary matrix in Eq. (D10). It can now be verified by inspection that the operators $L_1$, $L_2$, and $L_3$ have the form

$$L_1 = \begin{pmatrix} 0 & 0 \\ 0 & * \end{pmatrix}, \quad L_2 = \begin{pmatrix} * & 0 \\ 0 & 0 \end{pmatrix}, \quad L_3 = \begin{pmatrix} * & 0 \\ 0 & 0 \end{pmatrix},$$  (D12)

where $*$ denote some complex numbers which can be written in terms of the parameters $a_i$ and $b_i$, but the explicit form is not important for the following discussion.

Recall now that the remaining Kraus operators have the same form as in Eq. (D9), i.e.,

$$L_4 = \begin{pmatrix} * & * \\ 0 & 0 \end{pmatrix}, \quad L_5 = \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}, \quad L_6 = \begin{pmatrix} * & 0 \\ 0 & 0 \end{pmatrix},$$  (D13)

and in particular the operator $L_6$ has the same form as $L_3$. Thus, the set $L_5 \cup L_6$ can be reduced to one Kraus operator. This proves that any incoherent operation on a single qubit can be written with at most 5 Kraus operators as given in Eq. (7).

From the completeness condition $\sum_{i=1}^{5} K_i^\dagger K_i = \mathbb{1}$, it is straightforward to see that $\sum_{i=1}^{5} |a_i|^2 = \sum_{j=1}^{2} |b_j|^2 = 1$ and $a_1^* b_1 + a_2^* b_2 = 0$. Finally, the numbers $a_j = |a_j| e^{i \theta_j}$ can be chosen real by multiplying each of the Kraus operators with the corresponding phase $e^{-i \theta_j}$.

### Appendix E: Proof of Proposition 5

The proof follows similar ideas as the proof of Proposition 4. We start with the fact that every incoherent operation admits a finite decomposition into incoherent Kraus operators, see Proposition 3. Now we classify the incoherent Kraus operators according to their types, i.e., the location of their nonzero elements. In general, there are $d^d$ different types. For each type, at most $d$ incoherent Kraus operators can be linearly independent. Following the arguments in the proof of Proposition 4, this implies that any incoherent operation admits a decomposition with at most $d^{d+1}$ incoherent Kraus operators.

To reduce this number further, let $C_d$ denote the number of incoherent Kraus operators such that the first $k - 1$ columns have all entries zero, and the $k$-th column has one nonzero entry. In particular, $C_1$ denotes the number of Kraus operators which have one nonzero element in the first column, and possibly some nonzero elements in the other columns. Moreover, $C_d$ denotes the number of Kraus operators which have one nonzero entry in the last column, and all other elements are zero.

As we will show in the following, every incoherent operation admits an incoherent Kraus decomposition such that

$$C_k \leq d^{d-k+1}$$  (E1)

for all $k \in [1, d]$. For proving the statement, note that $d^{d-k+1}$ is exactly the number of different shapes for incoherent Kraus operators which have zero entries in the first $k - 1$ columns. Assume now that some Kraus decomposition $\{K_i\}$ has $C_k > d^{d-k+1}$. Then, there must be two Kraus operators $K_1$ and $K_2$ which have the same shape and all entries in the
first $k - 1$ columns are zero. Then – similar as in the proof of Proposition 4 – we can introduce a new incoherent Kraus decomposition $\{L_i\}$ such that $L_1$ and $L_2$ are linear combinations of $K_1$ and $K_2$. Moreover, we can choose the new Kraus operators such that $L_i = K_i$ for $i > 2$, and all entries in the $k$-th column of $L_1$ are zero. This proves the existence of an incoherent Kraus decomposition fulfilling Eq. (E1).

To complete the proof of the proposition we evaluate the sum over all $C_k$, giving rise to

$$
\sum_{k=1}^{d} C_k \leq \sum_{n=1}^{d} d^n = \frac{d(d^d - 1)}{d - 1}.
$$

(E2)

Appendix F: Proof of Proposition 6

A strictly incoherent Kraus operator can have at most one nonzero element in each row and column. Similar as in the proof of Proposition 5, we now introduce the number $C_k$, which counts the number of strictly incoherent Kraus operators with the first $k - 1$ columns having all entries zero, and the $k$-th column having one nonzero entry. We will now show that every SIO admits a strictly incoherent Kraus decomposition such that

$$
C_k \leq \frac{d!}{(k - 1)!}.
$$

(F1)

For this, note that $d!/(k - 1)!$ is the number of different shapes of strictly incoherent Kraus operators which have zero entries in the first $k - 1$ columns. Assume now that some strictly incoherent Kraus decomposition $\{K_i\}$ has $C_k > d!/(k - 1)!$. Then, there must be two Kraus operators $K_1$ and $K_2$ which have the same shape and all entries in the first $k - 1$ columns are zero. We can then introduce a new Kraus decomposition $\{L_i\}$ such that $L_1$ and $L_2$ are linear combinations of $K_1$ and $K_2$. Moreover, we can choose the new Kraus operators such that $L_i = K_i$ for $i > 2$, and all elements in the $k$-th column of $L_1$ are zero. This proves the existence of a strictly incoherent Kraus decomposition which fulfills Eq. (F1). The proof of the proposition is complete by noticing that the sum $\sum_{k=1}^{d} C_k$ is an upper bound on the total number of strictly incoherent Kraus operators, and thus

$$
\sum_{k=1}^{d} C_k \leq \sum_{k=1}^{d} \frac{d!}{(k - 1)!}.
$$

(F2)

Appendix G: Proof of Proposition 7

Consider a quantum operation defined via the Kraus operators

$$
K_{i,j} = \frac{1}{\sqrt{d}} |\text{mod}(j + i, d)\rangle \langle j|
$$

(G1)

with $0 \leq i, j \leq d - 1$. We will now prove that the strictly incoherent operation $\Lambda(\rho) = \sum_{i,j} K_{i,j} \rho K_{i,j}^\dagger$ cannot be implemented with fewer than $d^2$ Kraus operators. For this, it is enough to show that the corresponding Choi state $\rho_\Lambda$ has rank $d^2$. This can be seen by applying the operators $K_{i,j} \otimes \mathbb{1}$ to the maximally entangled state vector $|\Phi_+^d\rangle = \sum_{k=0}^{d-1} |k,k\rangle / \sqrt{d}$:

$$
(K_{i,j} \otimes \mathbb{1}) |\Phi_+^d\rangle = \frac{1}{d} |\text{mod}(j + i, d)\rangle |j\rangle.
$$

(G2)

The proof is complete by observing that all these $d^2$ (unnormalized) states are linearly independent.