Two-pion contribution to the muon magnetic moment

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The two-pion contribution to hadronic vacuum polarization can be extracted from \( \tau \) decay data when isospin violating and radiative corrections are taken into account. When the dominant corrections are applied to the photon-inclusive decay \( \tau^- \to \nu_\tau \pi^- \pi^0[\gamma] \), one obtains a shift \( \Delta a_\mu = (-12.0 \pm 2.6) \times 10^{-10} \) for the anomalous magnetic moment of the muon. The shift appears to be too small to reconcile the determinations of hadronic vacuum polarization from existing \( \tau \) and \( e^+e^- \) data. The reliability of electromagnetic corrections in the photon-inclusive \( \tau \) decay is examined.

1 Status of \( g_\mu - 2 \)

The magnetic moment of a particle with spin \( \vec{S}_p \),

\[
\vec{\mu}_p = g_p \frac{e \hbar}{2m_p c} \vec{S}_p ,
\]

is expressed in terms of its gyromagnetic factor \( g_p \). With polarized muons in a properly tuned storage ring [1], the spin precession frequency is directly proportional to the anomalous magnetic moment \( a_\mu = (g_\mu - 2)/2 \). On the basis of the most recent experimental result from Brookhaven [2], the present world average is

\[
a_\mu^{\exp} = (11659203 \pm 8) \times 10^{-10} .
\]

The standard model prediction for \( a_\mu \) consists of three parts:

\[
a_\mu^{\text{SM}} = a_\mu^{\text{QED}} + a_\mu^{\text{weak}} + a_\mu^{\text{had}} .
\]

The first two contributions are known with very high accuracy [3]:

\[
\begin{align*}
a_\mu^{\text{QED}} &= (11658470.6 \pm 0.3) \times 10^{-10} \\
a_\mu^{\text{weak}} &= (15.2 \pm 0.1) \times 10^{-10} .
\end{align*}
\]

\(^1\)Work supported in part by RTN, EC-Contract No. HPRN-CT-2002-00311 (EURIDICE).
To set the stage, we may compare the experimental result with the combined electroweak contribution,

$$a^\text{exp}_\mu - a^\text{QED+weak}_\mu = (717 \pm 8) \times 10^{-10},$$  

(5)
a difference of 90 standard deviations.

Although the primary motivation of the Brookhaven experiment was to test the weak contribution $a^\text{weak}_\mu$, the main issue at present for a meaningful comparison between theory and experiment is to understand the hadronic contribution $a^\text{had}_\mu$:

$$a^\text{had}_\mu = a^\text{had,LO}_\mu + a^\text{had,HO}_\mu + a^\text{had,LBL}_\mu. \quad (6)$$

By far the most important hadronic contribution is due to vacuum polarization at lowest order in $\alpha$ (shown in Fig. 1):

$$a^\text{had,LO}_\mu = a^\text{hvp}_\mu. \quad (7)$$

The most recent value for the higher-order hadronic contribution [4] is displayed in Eq. (6) (all values for $a^\mu$ are given in units of $10^{-10}$ from now on). The hadronic light-by-light contribution $a^\text{had,LBL}_\mu$ still carries a large theoretical uncertainty but at least the sign is now established [5, 6].

Most of the hadronic vacuum polarization contribution to $a_\mu$ originates at rather low energies (about 70 % for $t \leq 0.8$ GeV$^2$). Therefore, nonperturbative methods are needed together with experimental input. The most recent evaluation, using either only $e^+e^-$ data or including $\tau$ decay data (for the two- and four-pion channels), finds [7]

$$a^\text{hvp}_\mu = \begin{cases} 
684.7 \pm 7.0 & [e^+e^-] \\
701.9 \pm 6.2 & [\tau]
\end{cases}. \quad (8)$$

Comparison of the total standard model contribution with the experimental result (2) leads to [7]

$$a^\text{exp}_\mu - a^\text{SM}_\mu = \begin{cases} 
33.9 \pm 11.2 & (3.0 \ \sigma) \ [e^+e^-] \\
16.7 \pm 10.7 & (1.6 \ \sigma) \ [\tau]
\end{cases}. \quad (9)$$
The two determinations of $a_{\mu}^{\text{hvp}}$ in Eq. (8) differ by more than four standard deviations [7]. This discrepancy could be of experimental origin or the theoretical analysis might be inaccurate or incomplete. It is the main purpose of this talk to discuss whether the discrepancy could be due to an underestimate of isospin violation.

2 Hadronic vacuum polarization and isospin violation

The contribution of hadronic vacuum polarization at $O(\alpha^2)$ to $a_\mu$ (Fig. 1) is given by [8]

$$a_{\mu}^{\text{hvp}} = \int_{\Lambda M_\pi^2}^{\infty} dt K(t) \sigma_0(e^+e^- \rightarrow \text{hadrons})(t)$$  \hspace{1cm} (10)

with a smooth kernel $K(t)$ concentrated at low energies. I discuss here only the two-pion contribution that accounts for 73 % of $a_{\mu}^{\text{hvp}}$.

The two-pion contribution can also be extracted from $\tau$ decay. In the isospin limit,

$$\sigma_0(e^+e^- \rightarrow \pi^+\pi^-)(t) = h(t) \frac{d\Gamma(\tau^- \rightarrow \pi^0\pi^-\nu_\tau)}{dt}$$  \hspace{1cm} (11)

with a known kinematic function $h(t)$. At the level of accuracy needed to match the present experimental precision, a systematic account of isospin violation including electromagnetic corrections is required. I report here on a recent analysis of those corrections [9, 10].

We expect the size of isospin violating corrections to lie somewhere between

$$\frac{M_{\pi^+}\pi^- - M_{\pi^0}\pi^-}{M_\rho^2} = 2 \times 10^{-3} \quad \text{and} \quad \frac{M_{\pi^+}\pi^- - M_{\pi^0}\pi^-}{M_\pi^2} = 0.067.$$  \hspace{1cm} (12)

In a first step one integrates out all heavy fields with masses $> m_\tau$. This generates an electroweak short-distance correction factor for semihadronic $\tau$ decays [11] $S_{\text{EW}} = 1.0194$ (in the $\overline{MS}$ scheme). The second and final step is then to calculate the isospin violating corrections in the theory with light fields only.

The CVC relation (11) gets modified in the presence of isospin violation:

$$\sigma_0(t) = h(t) \frac{d\Gamma(\tau^- \rightarrow \pi^0\pi^-\nu_\tau)}{dt} \frac{R_{IB}(t)}{S_{\text{EW}}},$$  \hspace{1cm} (12)

with an isospin breaking correction function

$$R_{IB}(t) = \frac{1}{G_{EM}(t)} \frac{\beta_{\pi^+\pi^-}(t)}{\beta_{\pi^0\pi^-}(t)} \left| \frac{F_V(t)}{f_+(t)} \right|^2.$$

\hspace{1cm} (13)
The phase space correction factor \cite{12}

\[
\frac{\beta^3_{\pi^0\pi^+}(t)}{\beta^3_{\pi^+\pi^-}(t)} = 1 + \frac{3(M^2_{\pi^0} - M^2_{\pi^+})}{t - 4M^2_{\pi}} + O[(M^2_{\pi^0} - M^2_{\pi^+})^2]
\]

is especially important near threshold. The ratio of form factors \(F_V(t)\) in \(e^+e^-\) annihilation, \(f_+(t)\) in \(\tau\) decay) is mainly sensitive to \(\rho - \omega\) mixing and (to a lesser extent) to the width difference \(\Gamma_{\rho^+} - \Gamma_{\rho^0}\) \cite{7, 10}.

The main task is the calculation of radiative corrections that go into the function \(G_{EM}(t)\). As usual, the radiative corrections consist of two parts and only the sum is infrared finite and well defined: the exclusive rate with one-loop corrections \cite{9} and the radiative rate. I concentrate here on the calculation of the radiative rate \cite{10}, i.e., \(\Gamma(\tau^- \rightarrow \pi^0\pi^-\nu_\tau\gamma)\) under ALEPH conditions \cite{13}, with photons of all energies included.

To describe this decay, we have used a gauge invariant chiral resonance model with the following features \cite{10}:

- Low’s theorem (leading and subleading terms) is manifestly satisfied in terms of an explicit representation for the pion form factor \(f_+(t)\) \cite{14}.
- The amplitude exhibits the correct low-energy behaviour to \(O(p^4)\).
- The low-energy amplitude is extended into the resonance region using the standard chiral resonance Lagrangian \cite{15}. The implicit assumption is that \(\rho\) and (to a lesser extent) \(a_1\) exchange, which contribute already at \(O(p^4)\), are the dominant mechanisms at all accessible energies.

For photon energies \(E_\gamma < 100\) MeV (in the \(\tau\) rest frame), the rate is dominated by bremsstrahlung (leading Low approximation). However, under ALEPH conditions with all photons included, the bremsstrahlung approximation is not sufficient. The infrared finite sum of loop-corrected and radiative rate translates into the function \(G_{EM}(t)\) shown in Fig. 2.

We are now ready to calculate the total shift in \(a_{hvp}^\mu\) due to isospin violation in the two-pion channel:

\[
\Delta a_{hvp}^\mu = \int_{4M^2_{\pi}}^{t_{max}} dtK(t) \left[ h(t) \frac{d\Gamma_{\pi\pi[\gamma]}}{dt} \right] \times \left( \frac{R_{IB}(t)}{S_{EW}} - 1 \right).
\]

In Table 1 the various contributions and the total shifts are displayed for two values of \(t_{max}\): electroweak short-distance correction \((S_{EW})\), threshold correction \((KIN)\), radiative corrections \((EM)\) and form factor ratio \((FF)\). Obviously, the low-energy region below 1 GeV^2 is dominating.

I have only listed our error estimate for the form factor ratio because the uncertainty associated with some electromagnetic low-energy constants appearing in \(G_{EM}(t)\)
is much smaller [10]. The short-distance and kinematic corrections are model independent (although there could be higher-order corrections in $S_{\text{EW}}$). The form factor ratio is dominated by $\rho - \omega$ mixing taken directly from the most recent experimental analysis [16].

The total isospin violating correction goes in the right direction towards reconciling the $\tau$ with the $e^+e^-$ data but the shift seems to be too small in absolute magnitude.

**Table 1 Contributions to $\Delta a^\mu_{\text{hvp}}$ from various sources of isospin violation (in units of $10^{-10}$) for two different values of $t_{\text{max}}$ (in units of GeV$^2$; $t_{\text{max}} \leq m_\tau^2$).**

| $t_{\text{max}}$ (GeV$^2$) | $S_{\text{EW}}$ | KIN | EM | FF   | total       |
|-----------------------------|-----------------|-----|----|------|-------------|
| 1                           | -9.5            | -7.5| -1.1| 6.1±2.6 | -11.9±2.6  |
| 3                           | -9.7            | -7.5| -1.0| 6.1±2.6 | -12.0±2.6  |
While the signs and magnitudes of the shifts caused by the short-distance correction, the kinematic threshold effect and the form factor ratio are well understood, the small electromagnetic shift deserves further discussion.

A first observation is that loop and radiative contributions tend to interfere destructively (for reasonable infrared cutoffs because only the sum is infrared finite). A more instructive exercise is the comparison with the radiative corrections for the inclusive rate $\Gamma(\tau^{-} \rightarrow d\bar{u}\nu_{\tau}\gamma)$ at the quark level, calculated some time ago by Braaten and Li [17]. Translating their result into a shift in $a_{\mu}$, one finds

$$\Delta a_{\mu}^{EM,\text{quark}} = 1.6 \ ,$$

of opposite sign but similar magnitude as in Table 1. There is no fundamental reason why the inclusive result at the quark level should agree with the exclusive two-pion result. Nevertheless, performing the radiative corrections for the two-pion mode in the leading Low approximation for the radiative mode (independent of any resonance contributions to the decay amplitude), one obtains accidentally exactly the same value:

$$\Delta a_{\mu}^{EM,\text{Low}} = 1.6 \ .$$

Note that the shift in the inclusive case and in the bremsstrahlung approximation for the exclusive channel goes in the “wrong” direction increasing the discrepancy with the $e^{+}e^{-}$ result.

What is then the origin of the “correct” sign of the shift $\Delta a_{\mu}^{EM}$ in Table 1? It turns out that the difference to the bremsstrahlung value is completely due to the subleading terms in the Low expansion for the radiative amplitude proportional to the derivative $df_{+}(t)/dt$ of the pion form factor [10]. In other words, the full curve in Fig. 2 could not be distinguished from the curve based only on the first two terms in the Low expansion. To the extent that the pion form factor is known experimentally, the shift $\Delta a_{\mu}^{EM}$ is therefore model independent and certainly independent of details of the resonance exchange model for the radiative decay $\tau^{-} \rightarrow \pi^{0}\pi^{-}\nu_{\tau}\gamma$. The result depends only on the (shape of the) pion form factor.

3 Conclusions

The total shift of $a_{\mu}^{hvp}$ due to isospin violation and radiative corrections in the two-pion channel,

$$\Delta a_{\mu}^{hvp} = -12.0 \pm 2.6 \ ,$$

agrees well with a similar more data-oriented analysis of Davier et al. [7]. Performing the shift for $a_{\mu}^{hvp}$ extracted from the two-pion decay of the $\tau$, one arrives at the results [7] already displayed in Eq. (9):

$$a_{\mu}^{\exp} - a_{\mu}^{\text{SM}} = \begin{cases} 33.9 \pm 11.2 & (3.0 \ \sigma) \ [e^{+}e^{-}] \\ 16.7 \pm 10.7 & (1.6 \ \sigma) \ [\tau] \end{cases} \ .$$
Before drawing any far-reaching conclusions about possible evidence for new physics, the reason for the discrepancy between the two determinations of $a_{\mu}^{\text{BVP}}$ must be understood. One possible uncertainty has to do with the experimental procedure of applying radiative corrections to the $e^+e^-$ data where the corrections are much bigger than in the $\tau$ decay. If both the $e^+e^-$ result and the raw $\tau$ data were correct isospin violation would have to be more than twice as big as calculated [10]. In view of the discussion presented here, especially on the natural size of electromagnetic corrections in the $\tau$ decay to two pions, I consider such a drastic underestimate very unlikely.

For the time being and pending clarification of the discrepancy between $e^+e^-$ and $\tau$-based extractions of $a_{\mu}^{\text{BVP}}$, the standard model prediction for the anomalous magnetic moment of the muon is in good shape.

Acknowledgments

I wish to congratulate G. Martinská, J. Urbán, S. Vokál and their team for the excellent organization of Hadron Structure '02. I also thank V. Cirigliano and H. Neufeld for a very pleasant collaboration.

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