Generalized Landau-Pollak Uncertainty Relation

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Abstract

The Landau-Pollak uncertainty relation treats a pair of rank one projection valued measures and imposes a restriction on their probability distributions. It gives a nontrivial bound for summation of their maximum values. We give a generalization of this bound (weak version of the Landau-Pollak uncertainty relation). Our generalization covers a pair of positive operator valued measures. A nontrivial but slightly weak inequality that can treat an arbitrary number of positive operator valued measures is also presented. A possible application to the problem of separability criterion is also suggested.

1 Introduction

The uncertainty relation represents one of the most fundamental aspects of the quantum theory. It is not only interesting by itself but also plays crucial roles in the various fields. The uncertainty relation imposes a restriction on probability distributions of measurement outcomes with respect to two (or more) noncommutative observables. In addition to the most famous Robertson-type uncertainty relation [1], there are a several types of the relation, such as entropic type [2, 3, 4, 5, 6] and Landau-Pollak type [2, 8]. The difference among these uncertainty relations lies on the quantities measuring the randomness (unsharpness) of the probability distributions. In the Landau-Pollak type, the randomness of a probability distribution \( \{p_i\} \) is characterized by its maximum value \( \max_i p_i \). The Landau-Pollak uncertainty relation shows that the maximum values with
respect to a pair of noncommutative observables must satisfy a nontrivial inequality. From the inequality, a weaker but simpler nontrivial inequality for the summation of the maximum values is obtained. According to the weak version of Landau-Pollak uncertainty relation, the probabilities of two noncommutative observables cannot have 1 as their maximum values simultaneously. In spite of the simplicity of the theorem for a pair of observables, no nontrivial generalization to three or more observables has been known yet. More precisely, the Landau-Pollak uncertainty relation has been able to treat only pairs of rank one projection valued measures so far. In this paper, we try to generalize the weak version of the relation to cover an arbitrary number of positive operator valued measures. That is, our generalization is two-fold. It enables us to treat general types of observables, and gets rid of the restriction on the numbers of observables. This paper is organized as follows. In the next section we give our main results. We first show a lemma that plays key roles in our generalization. We give a theorem for a pair of general observables that enables us to derive the weak version of the Landau-Pollak uncertainty relation. Another theorem for an arbitrary numbers of general observables is also presented. It relates the bound with pairwise noncommutativity. Although the theorem for an arbitrary number of general observables is not trivial, it is not strong enough to derive the original weak version of the Landau-Pollak uncertainty relation for a pair of observables. In section 3 we show a several examples and a possible application of our theorem to separability criterion in qudit system.

2 Main Results

A POVM $A$ (on a discrete space) is a family of positive operators $\{A_i\}$ satisfying $\sum_i A_i = 1$. Although, for simplicity, the measures are restricted to the discrete case, it is straightforward to generalize our theorems to the general (continuous) measurable space. POVMs give the most general description of the observables. An important subclass of the observables is a class of projection valued measures (PVMs). A PVM is a family of projection operators whose summation equals the identity operator. Suppose we have a POVM: $A = \{A_i\}$. For any state $\rho$, $\langle A_i \rangle_{\rho} := \text{tr}(\rho A_i)$ means the probability of getting an outcome $i$ when we measure the observable $A_i$ in the state $\rho$.

In its original version [7] interpreted by Maassen and Uffink[2,4], the Landau-Pollak uncertainty relation treats the simplest projection valued measures (PVMs) $P := \{P_i\}$ and $Q = \{Q_j\}$, where $P_i$’s and $Q_j$’s are rank one projection operators as $P_i = \{|i\rangle\langle i|\}$ and $Q_j = \{|j\rangle\langle j|\}$. They derived an inequality:

$$\text{Arccos} \langle P_i \rangle_{\rho} + \text{Arccos} \langle Q_j \rangle_{\rho} \geq \text{Arccos} |\langle i|j \rangle|.$$  

By maximizing $\langle P_i \rangle_{\rho} + \langle Q_j \rangle_{\rho}$ under this bound, one can obtain a weaker but simpler inequality:

$$\langle P_i \rangle_{\rho} + \langle Q_j \rangle_{\rho} \leq 1 + |\langle i|j \rangle|. \quad (1)$$

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To relate it with generalized entropic quantities, Vicente and Sánchez-Ruiz introduced a quantity \(M_\infty(P : \rho) = \max_i \langle P_i \rangle_\rho\) and showed an inequality for rank one PVMs \(P\) and \(Q\),

\[M_\infty(P : \rho) + M_\infty(Q : \rho) \leq 1 + \max_{i,j} |\langle i | j \rangle|.

In what follows, we give a generalization of the weak version of the Landau-Pollak uncertainty relation (1). The following lemma plays essential roles in the generalization.

**Lemma 1** Let \(P_1, P_2, \ldots, P_m\) be projection operators on a Hilbert space \(\mathcal{H}\), and let \(|\psi\rangle\) be some unit vector in \(\mathcal{H}\) satisfying \(P_j |\psi\rangle \neq 0\) for all \(j\). Let \(\Lambda(|\psi\rangle)\) denote the maximal eigenvalue of the positive semidefinite \(m \times m\) matrix \(G\) given by

\[G_{ij} := \frac{\langle \psi | P_i P_j | \psi \rangle}{\| P_i |\psi\| \cdot \| P_j |\psi\|}.

Then

\[\sum_{i=1}^{m} \langle \psi | P_i |\psi\rangle \leq \Lambda(|\psi\rangle)

holds.

**Proof:**

To show the positive semidefiniteness of the matrix \(G\), we define a normalized vector \(|\psi_i\rangle\) for each \(i\) as

\[|\psi_i\rangle := \frac{P_i |\psi\rangle}{\| P_i |\psi\|}.

Since the matrix \(G\) can be written as \(G_{ij} = \langle \psi_i | \psi_j \rangle\), for all \(x_1, x_2, \ldots, x_m \in \mathbb{C}\),

\[\sum_{i,j=1}^{m} x_i x_j = \| \sum_{i=1}^{m} x_i |\psi_i\rangle \|^2 \geq 0\]

holds. The matrix \(G\) is thus obviously positive semidefinite. Thanks to this semidefiniteness, we have for all \(x_1, x_2, \ldots, x_m \in \mathbb{C}\),

\[\sum_{i,j=1}^{m} x_i G_{ij} x_j \leq \Lambda(|\psi\rangle) \sum_{i=1}^{m} |x_i|^2.

Hence

\[\left( \sum_{i=1}^{m} \langle \psi | P_i |\psi\rangle \right)^2 = \langle \psi | \sum_{i=1}^{m} P_i |\psi\rangle^2 \leq \langle \psi | \left( \sum_{i=1}^{m} P_i \right)^2 |\psi\rangle = \sum_{i,j=1}^{m} \langle \psi | P_i P_j |\psi\rangle \]

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\[
\sum_{i,j=1}^{m} \|P_i|\psi\rangle\| \cdot G_{ij} \cdot \|P_j|\psi\rangle\|
\leq \Lambda(|\psi\rangle) \sum_{i=1}^{m} \|P_i|\psi\rangle\|^2
\]
\[
= \Lambda(|\psi\rangle) \sum_{i=1}^{m} \langle \psi | P_i | \psi \rangle
\]
holds, where the inequality in the second line follows from a relation, \(|\langle \psi | A | \psi \rangle|^2 \leq \langle \psi | A^2 | \psi \rangle\) that holds for an arbitrary self-adjoint operator \(A = A^*\). Q.E.D.

The estimation of the maximal eigenvalue \(\Lambda(|\psi\rangle)\) of the above lemma gives a generalization of the weak version of the Landau-Pollak uncertainty relation. In case \(m = 2\), the exact diagonalization of \(G\) can be easily performed. Due to this fact, we can generalize the uncertainty relation for a pair of general observables. To obtain the theorem for a pair of POVMs, one needs an extension technique to represent positive operators as projection operators in an enlarged space. To clarify the argument, we first begin with a generalization to a pair of PVMs that is obtained by a direct application of Lemma 1, and next show a theorem for a pair of general observables.

**Theorem 2** Let \(P_1\) and \(P_2\) be projection operators on a Hilbert space \(\mathcal{H}\). For any state \(\rho\),

\[
\langle P_1 \rangle_{\rho} + \langle P_2 \rangle_{\rho} \leq 1 + \|P_1 P_2\|
\]

holds, where \(\|\cdot\|\) represents the operator norm defined as \(\|A\| := \sup_{|\psi\rangle \neq 0} \|A|\psi\rangle\| / \|\psi\rangle\|\).

**Proof:**

Since an arbitrary state can be decomposed into a mixture of pure states, it suffices to prove the inequality only for the case that the state \(\rho\) is pure and is written with a normalized vector \(|\psi\rangle\) as \(\rho = |\psi\rangle \langle \psi |\). If either \(P_1 |\psi\rangle\) or \(P_2 |\psi\rangle\) equals 0, the statement is trivially true. Thus we may assume that both \(P_1 |\psi\rangle \neq 0\) and \(P_2 |\psi\rangle \neq 0\) hold. We apply lemma 1 with \(m = 2\). The 2 \times 2 matrix \(G\) can be written in this case as,

\[
G := \begin{pmatrix}
1 & \langle \psi_1 | \psi_2 \rangle \\
\langle \psi_2 | \psi_1 \rangle & 1
\end{pmatrix},
\]

where \(|\psi_k\rangle := \frac{P_k |\psi\rangle}{\|P_k |\psi\rangle\|}\) for \(k = 1, 2\). Its maximal eigenvalue is easily obtained as \(\Lambda(|\psi\rangle) = 1 + |\langle \psi_1 | \psi_2 \rangle|\). It can be bounded as \(\Lambda(|\psi\rangle) = 1 + |\langle \psi | P_1 P_2 | \psi \rangle| \leq 1 + \|P_1 P_2\|\). It ends the proof. Q.E.D.

To accomplish a generalization of the weak version of the Landau-Pollak uncertainty relation to a pair of positive operator valued measures, we need an extension technique as discussed below.
Theorem 3 Let $A$ and $B$ be positive operators that satisfy $A \leq 1$ and $B \leq 1$ on a Hilbert space $\mathcal{H}$. For an arbitrary state $\rho$,

$$\langle A \rangle_\rho + \langle B \rangle_\rho \leq 1 + \|A^{1/2}B^{1/2}\|$$

holds.

Proof:

Also in this case it suffices to prove the inequality only for the case that the state $\rho$ is pure. We hence write $\rho$ as $|\Omega\rangle \langle \Omega|$ with a unit vector $|\Omega\rangle$. If either $A|\Omega\rangle$ or $B|\Omega\rangle$ is vanishing, the statement is trivially true. We thus may assume that neither of them are vanishing. An extension of the Hilbert space enables us to represent the positive operators $A$ and $B$ as projection operators as follows. Let us consider an enlarged Hilbert space $\mathcal{K} := \mathcal{H} \oplus \mathcal{H} \oplus \mathcal{H}$ and operators $P_1$ and $P_2$ as,

$$P_1 := \begin{pmatrix} A & \sqrt{A(1 - A)} & 0 \\ \sqrt{A(1 - A)} & 1 - A & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$P_2 := \begin{pmatrix} B & 0 & \sqrt{B(1 - B)} \\ 0 & 0 & 0 \\ \sqrt{B(1 - B)} & 0 & 1 - B \end{pmatrix},$$

where we have used the commutativity between $A$ and $1 - A$ (or, $B$ and $1 - B$) to guarantee the well-definedness of $\sqrt{A(1 - A)}$ (or, $\sqrt{B(1 - B)}$). It can be easily checked that $P_1$ and $P_2$ are indeed projection operators in the enlarged space $\mathcal{K}$. We define a unit vector $|\psi\rangle := |\Omega\rangle \oplus 0 \oplus 0$ and apply lemma 1 to them. Since $\langle \Omega|A|\Omega\rangle = \langle \psi|P_1|\psi\rangle$ and $\langle \Omega|B|\Omega\rangle = \langle \psi|P_2|\psi\rangle$ hold, we obtain,

$$\langle \Omega|A|\Omega\rangle + \langle \Omega|B|\Omega\rangle \leq 1 + |\langle \psi_1|\psi_2\rangle|,$$

where $|\psi_k\rangle := \frac{P_k|\psi\rangle}{\|P_k|\psi\rangle\|}$ (for $k = 1, 2$) is a unit vector in $\mathcal{K}$. By using $P_1|\psi\rangle = A|\Omega\rangle \oplus \sqrt{A(1 - A)}|\Omega\rangle \oplus 0$ and $P_2|\psi\rangle = B|\Omega\rangle \oplus 0 \oplus \sqrt{B(1 - B)}|\Omega\rangle$, we can calculate $|\langle \psi_1|\psi_2\rangle|$ as,

$$|\langle \psi_1|\psi_2\rangle| = \frac{|\langle \Omega|AB|\Omega\rangle|}{\|A^{1/2}|\Omega\rangle\| \cdot \|B^{1/2}|\Omega\rangle\|}.$$

Defining unit vectors $|\phi_1\rangle = \frac{A^{1/2}|\Omega\rangle}{\|A^{1/2}|\Omega\rangle\|}$ and $|\phi_2\rangle = \frac{B^{1/2}|\Omega\rangle}{\|B^{1/2}|\Omega\rangle\|}$, we obtain,

$$\frac{|\langle \Omega|AB|\Omega\rangle|}{\|A^{1/2}|\Omega\rangle\| \cdot \|B^{1/2}|\Omega\rangle\|} = |\langle \phi_1|A^{1/2}B^{1/2}|\phi_2\rangle| \leq \|A^{1/2}B^{1/2}\|.$$

It ends the proof. Q.E.D.

Note that we did not use theorem 2 to prove theorem 3. Theorem 2 actually is regarded as a corollary of theorem 3. Although the following statement is obvious, it is mentioned for clarifying the relation between our theorem and the original version 1.
**Corollary 4** Let \( A = \{ A_i \} \) and \( B = \{ B_j \} \) be POVMs on a Hilbert space \( \mathcal{H} \). For an arbitrary state \( \rho \) and \( i, j \),
\[
\langle A_i \rangle_{\rho} + \langle B_j \rangle_{\rho} \leq 1 + \| A_i^{1/2} B_j^{1/2} \|
\]
holds.

Application of rank one PVMs \( A := \{ P_i \} = \{|i\rangle \langle i|\} \) and \( B = \{ Q_j \} = \{|j\rangle \langle j|\} \) to the above corollary immediately recovers the weak version of Landau-Pollak uncertainty relation (1).

The lemma 1 enables us to obtain a nontrivial bound also for an arbitrary number of POVMs. Although it is hard to obtain eigenvalues of the matrix \( G \) for a general \( m \), we can estimate the maximal value in terms of pairwise noncommutativity such as \( \| P_i P_j \| (i \neq j) \). We obtain the following theorem.

**Theorem 5** Let us consider a Hilbert space \( \mathcal{H} \) and a family of \( m \) positive operators \( \{ A_j \}_{j=1}^m \) satisfying \( A_j \leq 1 \). For any state \( \rho \),
\[
\sum_{i=1}^m \langle A_i \rangle_{\rho} \leq 1 + \left( \sum_{i \neq j} \| A_i^{1/2} A_j^{1/2} \|^2 \right)^{1/2},
\]
holds.

**Proof:**

It suffices to prove only the case that the state \( \rho \) is pure and is written as \( \rho = |\Omega\rangle \langle \Omega| \) with a unit vector \( |\Omega\rangle \in \mathcal{H} \). We may assume that \( A_i |\Omega\rangle \neq 0 \) holds for all \( i \). In fact, \( \langle A_i \rangle \) is zero for \( i \) with \( A_i |\Omega\rangle = 0 \) and does not contribute to the left hand side of the inequality (2). To make use of Lemma 1 also in this case, we first employ the extension technique as in the previous theorem. Let us consider an enlarged Hilbert space \( \mathcal{K} = \mathcal{H} \oplus \mathcal{H} \oplus \cdots \oplus \mathcal{H} \) (\( m + 1 \) times). An operator \( R \) on \( \mathcal{K} \) can be represented in a matrix form as,
\[
R = \begin{pmatrix}
R_{00} & R_{01} & R_{02} & \cdots & R_{0m} \\
R_{10} & R_{11} & R_{12} & \cdots & R_{1m} \\
R_{20} & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
R_{m0} & \cdots & \cdots & \cdots & R_{mm} 
\end{pmatrix}.
\]

That is, \( R \) acts on a vector \( |\Phi\rangle = |\Phi_0\rangle \oplus |\Phi_1\rangle \oplus \cdots \oplus |\Phi_m\rangle \) as,
\[
R|\Phi\rangle = \sum_{j_0=0}^m R_{0j_0}|\Phi_{j_0}\rangle \oplus \sum_{j_1=0}^m R_{1j_1}|\Phi_{j_1}\rangle \oplus \cdots \oplus \sum_{j_m=0}^m R_{mj_m}|\Phi_{j_m}\rangle.
\]

Let us define operators \( P_k \) for \( k = 1, \ldots m \) by their matrix elements as,
\[
(P_k)_{ij} = \begin{cases}
A_k & (i = j = 0) \\
\sqrt{A_k(1 - A_k)} & (i = k, j = 0) \\
\sqrt{A_k(1 - A_k)} & (i = 0, j = k) \\
0 & \text{(otherwise)}
\end{cases}
\]
It is easy to check that they are projection operators in $K$. For a unit vector $|\psi\rangle := |\Omega\rangle \oplus 0 \oplus 0 \oplus \cdots \oplus 0$, $\langle \psi | P_k | \psi \rangle = \langle \Omega | A_k | \Omega \rangle$ holds, hence we can apply Lemma 1 to obtain a bound for $\sum_k \langle \Omega | A_k | \Omega \rangle$. Definition of unit vectors $|\psi_k\rangle := P_k |\psi\rangle / \| P_k |\psi\rangle \|$ $(k = 1, 2, \ldots, m)$ allows an expression, $G_{ij} = \langle \psi_i | | \psi_j \rangle$. Since each eigenvalue $\lambda_t (t = 1, 2, \ldots, m)$ can be obtained by the diagonalization of this matrix, there exists a unitary matrix $u$ satisfying for each $t$,

$$\lambda_t = \sum_{ij} u^*_t \langle \psi_i | \psi_j \rangle u_{ij}$$

$$= \sum_i u^*_t u_{i1} + \sum_{i \neq j} u^*_t u_{ij} \langle \psi_i | \psi_j \rangle,$$

whose right hand side can be bounded by the unitarity of $u$ and the Cauchy-Schwarz inequality as,

$$\lambda_t \leq 1 + \left( \sum_{i \neq j} |u^*_t u_{i1}|^2 \right)^{1/2} \left( \sum_{i \neq j} |\langle \psi_i | \psi_j \rangle|^2 \right)^{1/2}$$

$$\leq 1 + \left( \sum_{ij} |\langle \psi_i | \psi_j \rangle|^2 \right)^{1/2} \left( \sum_{i \neq j} |\langle \psi_i | \psi_j \rangle|^2 \right)^{1/2}$$

$$\leq 1 + \left( \sum_{i \neq j} |\langle \psi_i | \psi_j \rangle|^2 \right)^{1/2}.$$

Since the above inequality holds for all $t$, $\Lambda(|\psi\rangle) \leq 1 + \left( \sum_{i \neq j} |\langle \psi_i | \psi_j \rangle|^2 \right)^{1/2}$ holds. As in the proof of theorem 3, $(\langle \psi_i | \psi_j \rangle)$ can be bounded as,

$$|\langle \psi_i | \psi_j \rangle| = \frac{|\langle \Omega | A_i A_j | \Omega \rangle|}{\| A_i^{1/2} | \Omega \rangle \cdot \| A_j^{1/2} | \Omega \rangle \|} \leq \| A_i^{1/2} A_j^{1/2} \|.$$

It ends the proof. Q.E.D.

As discussed in the next section, this bound is not strong enough although it is nontrivial.

The following corollary is obvious.

**Corollary 6** Let $A^{(j)} = \{ A^{(j)}_{s_j} \}_s$ be a POVM for each $j = 1, 2, \ldots, m$ on a Hilbert space $\mathcal{H}$. For an arbitrary state $\rho$ and arbitrary $s_1, s_2, \ldots, s_m$,

$$\sum_{j=1}^m \langle A^{(j)}_{s_j} | \rho \rangle \leq 1 + \left( \sum_{i \neq j} \| A^{(i)}_{s_i} \|^{1/2} \| A^{(j)}_{s_j} \|^{1/2} \right)^{1/2}$$

holds.
3 Discussions

In this section, we discuss a several examples and suggest a possible application to the problem of separability criterion.

First let us consider the relationship between theorem 3 (or corollary 4) and theorem 5 (or corollary 6). Since in theorem 5 the number of observables is arbitrary, one may put it as \( m = 2 \). Then we obtain, for positive operators \( A \) and \( B \),

\[
\langle A \rangle_\rho + \langle B \rangle_\rho \leq 1 + \sqrt{2} \| A^{1/2} B^{1/2} \|,
\]

which is slightly worse than theorem 3.

Next let us consider the most trivial case that \( \| A^{1/2} A^{1/2} \| = 0 \) holds for all \( i \neq j \). For instance, if \( A = \{ A_i \} \) itself forms a PVM, the condition \( \| A^{1/2} A^{1/2} \| = 0 \) for all \( i \neq j \) follows. It gives the trivial bound,

\[
\sum_i \langle A_i \rangle_\rho \leq 1.
\]

Although also in the case \( A = \{ A_i \} \) forms a POVM this trivial bound \( \sum_i \langle A_i \rangle \leq 1 \) should hold, unfortunately our inequality is not strong enough to show it.

As an example to show the non-triviality of our theorem, we consider a \( D \)-dimensional Hilbert space and its mutually unbiased basis (MUB). A MUB \[9\] is a family of bases \( \{ B^{(i)} \} \), where \( B^{(j)} = \{ |1 : j\rangle, |2 : j\rangle, \ldots, |D : j\rangle \} \) is an orthonormalized basis. They satisfy, for \( i \neq j \),

\[
|\langle s : j | t : i \rangle| = \frac{1}{\sqrt{D}}.
\]

It is a longstanding problem to ask how many bases are compatible with respect to this condition. It is known that its upper bound is \( D + 1 \) for \( D \)-dimensional Hilbert space, and for \( D = p^r \) (\( p \) is a prime) this upper bound is actually attained \[9\] \[10\]. We assume that we treat a Hilbert space that attains this upper bound \( D + 1 \). Let us take a vector \( |s_i : i\rangle \) from each basis \( B^{(i)} \), and put \( P_i := |s_i : i\rangle \langle s_i : i| \). According to our theorem, we obtain,

\[
\sum_{i=1}^{D+1} \langle P_i \rangle_\rho \leq 1 + \sqrt{D + 1}.
\]

(3)

Let us compare it with the bound that is trivially obtained by combining the inequality for a pair of observables. According to theorem 3 for each pair of \( i \) and \( j \) (\( i \neq j \)),

\[
\langle P_i \rangle_\rho + \langle P_j \rangle_\rho \leq 1 + \sqrt{\frac{1}{D}}
\]

holds. Trivially combining them into an inequality gives,

\[
\sum_i \langle P_i \rangle_\rho \leq \frac{D + 1}{2} + \frac{1}{2} \left( \sqrt{D} + \frac{1}{\sqrt{D}} \right).
\]

One can easily see that our inequality (3) gives better bound for \( D \geq 3 \), and the discrepancy between our upper bound and the bound obtained by the trivial combination becomes larger for larger dimension \( D \). An analogous discussion
has been done for the entropic uncertainty relations, and the nontrivial bound for MUB has been obtained \cite{11,12}.

Finally we make a comment on a possible application of our generalized Landau-Pollak uncertainty relation in the quantum information. The problem of finding a criterion to distinguish between separable states and entangled states is one of the important problems. Recently, criteria based upon the various uncertainty relations \cite{13,14,15,16,8} have been proposed. Among them, Vicente and Sánchez-Ruiz \cite{8} employed the Landau-Pollak uncertainty relation. While they examined the strength of their criterion for bipartite qubit system, they proposed a criterion for general bipartite (D-dimensional) qudit system ((57) in \cite{8}). As they suggested there, a generalization of Landau-Pollak uncertainty relation to arbitrary numbers of observables enables us to propose another criterion for \(D = p^r\) (\(p: \text{prime}\)) as follows. Let us consider a MUB \(\{\mathcal{B}^{(j)}\}\) on \(\mathcal{H}_A\), where \(\mathcal{B}^{(j)} = \{|1 : j\rangle, |2 : j\rangle, \ldots, |D : j\rangle\}\) is an orthonormalized basis satisfying, for \(i \neq j\), \(|\langle s : j|t : i\rangle| = \frac{1}{\sqrt{D}}\). One can define a self-adjoint operator \(A^{(j)}\) as \(A^{(j)} := \sum_s f(s)|s : j\rangle\langle s : j|\) for a suitable function \(f\). Its copy on \(\mathcal{H}_B\) is also defined and written as, \(B^{(j)}\). We write the maximum probability of the outcome with respect to an observable \(A\) and a state \(\rho\) as \(M_\infty(A : \rho)\).

For separable states, they satisfy,

\[
\sum_i M_\infty(A^{(i)} \otimes B^{(i)} : \rho^{\text{sep}}) \leq 1 + \sqrt{D + 1}.
\]

It is a natural extension of the separability criterion suggested in \cite{8}. The detailed formulation and analysis will be appeared elsewhere.

In conclusion, we showed a generalization of the weak version of the Landau-Pollak uncertainty relation. Our generalized inequality gives a nontrivial bound for an arbitrary number of positive operator valued measures. A possible application to the problem of separability criterion was also suggested.

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