On the number of frequency hypercubes $F^n(4; 2, 2)^*$

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Abstract

A frequency $n$-cube $F^n(4; 2, 2)$ is an $n$-dimensional $4 \times \ldots \times 4$ array filled by 0s and 1s such that each line contains exactly two 1s. We classify the frequency 4-cubes $F^4(4; 2, 2)$, find a testing set of size 25 for $F^3(4; 2, 2)$, and derive an upper bound on the number of $F^n(4; 2, 2)$. Additionally, for any $n$ greater than 2, we construct an $F^n(4; 2, 2)$ that cannot be refined to a latin hypercube, while each of its sub-$F^{n-1}(4; 2, 2)$ can.

Keywords: frequency hypercube, frequency square, latin hypercube, testing set, MDS code

1 Introduction

A frequency hypercube (frequency $n$-cube) $F^n(q; \lambda_0, \ldots, \lambda_{m-1})$ is an $n$-dimensional $q \times \ldots \times q$ array, where $q = \lambda_0 + \cdots + \lambda_{m-1}$, filled by numbers $0, \ldots, m-1$ with the property that each line contains exactly $\lambda_i$ cells with symbol $i$, $i = 0, \ldots, m-1$ (a line consists of $q$ cells of the array differing in one coordinate). Frequency hypercubes generalize both frequency squares, $n = 2$, and latin hypercubes, $\lambda_0 = \cdots = \lambda_{q-1} = 1$ (alternatively, frequency hypercubes $F^{n+1}(q; 1, q-1)$ are also equivalent to latin $n$-cubes), see respectively [3], [8] and [10] for the enumeration of the objects of small size. In their turn, frequency squares and latin hypercubes are different generalizations of latin squares, which have parameters $F^2(q; 1, \ldots, 1)$ in our notation.

In the current work, we study frequency hypercubes of order $q = 4$. Latin hypercubes of order 4, corresponding to $F^n(4; 1, 1, 1, 1)$ and $F^n(4; 1, 3)$ are characterized in [6], and their structures are now rather well understood; in particular, not only the asymptotics [14], but also an effective recursive formula for

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Their number is known \[15\], see sequences A211214 and A211215 in OEIS \[2\]. Another case, \(F^n(4; 1, 1, 2)\), is solved in \[11\], where it is proved that every frequency \(n\)-cube \(F^n(4; 1, 1, 2)\) can be subdivided to \(F^n(4; 1, 1, 1, 1)\). We focus on the remaining case \(F^n(4; 2, 2)\). It should also be mentioned that all frequency hypercubes with \(q < 4\) correspond to latin hypercubes of order \(q\), which form only one isotopy class, for each \(n\) and \(q\).

The goals of our paper are, firstly, to enumerate all frequency hypercubes \(F^3(4; 2, 2)\) and \(F^4(4; 2, 2)\), and secondly, improve the upper bound on the number of frequency hypercubes \(F^n(4; 2, 2)\). The second result is based on calculations in small dimensions, where we study some more general class of objects than the frequency hypercubes, called unitrades. We use the language of coding theory as the working language in this paper and study sets of vertices of the Hamming graph, called codes. In particular, the frequency hypercubes \(F^n(4; 2, 2)\) are represented by the special codes called double-MDS-codes in the Hamming graph \(H(n, 4)\) in such a way that a frequency hypercube \(F^n(4; 2, 2)\) is the characteristic function of a double-MDS-code.

The trivial upper bound \(2^{3^n}\) on the number of frequency cubes \(F^n(4; 2, 2)\) is based on the existence of an easy-to-find testing set from \(3^n\) points such that the frequency cube is uniquely defined by its values on those points. Our new upper bound, of form \(2^{\alpha n}\), is obtained by finding an appropriate testing set of size \(\alpha n\), where \(\alpha < 3\). Note that a lower bound of form \(2^{\alpha n + o(1)}\) on the number of \(F^n(4; 2, 2)\) is immediate from the similar lower bound for \(F^n(4; 1, 1, 1, 1)\) (see \[14\]). So, the number of objects is double-exponential. In the framework of our study, the following questions are actual and remain open: What is the asymptotics of the double-logarithm of the number of \(F^n(4; 2, 2)\) (as well as \(F^n(q; \lambda_0, \ldots, \lambda_{m-1})\) for any fixed \(q > 4\), \(\lambda_0, \ldots, \lambda_{m-1}\))? What is the minimum size of a testing set for \(F^n(4; 2, 2)\) (in general, for \(F^n(q; \lambda_0, \ldots, \lambda_{m-1}), q > 3\), in particular, for \(F^3(4; 2, 2)\)?)

As follows from the definition, double-MDS-codes are equivalent to simple orthogonal arrays \(OA(4^n/2, n, 4, n-1)\), see \[4\] for the definition and notation. In \[13\], based on a lower bound on the number of double-MDS-codes, Potapov derived a lower bound on the number of \(n\)-ary balanced correlation immune (resilient) Boolean functions of order \(n/2\) (equivalently, simple orthogonal arrays \(OA(2^{n-1}, n, 2, n/2)\)). Substituting the results of our computation of the number of \(F^3(4; 2, 2)\) to the arguments in \[13\] straightforwardly improves the bound \[13\] for correlation immune functions for \(n = 8\).

The material is organized as follows. Section \[2\] sets up the basic notations and definitions. Section \[3\] gives a straightforward algorithm to classify double-MDS-codes and unitrades. The results obtained by implementation of the algorithm are shown in tables. Section \[4\] introduces the splittability property of double-MDS-codes. The main result of this paper is established in Section \[5\]. We give a testing set of size 25 for \(F^3(4; 2, 2)\) and derive an upper bound on the number of \(F^n(4; 2, 2)\).
2 Basic notations and definitions

2.1 Double-codes and related objects

Let $\Sigma = \{0, 1, \ldots, q-1\}$ and $\Sigma^n$ be the set of words of length $n$ over the alphabet $\Sigma$. Subsets of $\Sigma^n$ are called codes of length $n$. We set $q = 4$ in this paper, while the definitions are valid for arbitrary $q$. The Hamming graph $H(n, q)$ is a graph on the vertex set $\Sigma^n$, where two vertices are adjacent if they differ in one coordinate. We call a set of $q$ pairwise adjacent vertices in $H(n, q)$ a line (essentially, a line is a maximal clique in $H(n, q)$). A (distance-2) MDS code is a subset of $\Sigma^n$ intersecting every line in exactly one element (we consider only the minimum-distance-2 MDS codes, while in coding theory this concept is more general). A set $S \subset \Sigma^n$ is called a double-MDS-code if each line of $\Sigma^n$ contains exactly two elements from $S$. A set $S \subset \Sigma^n$ is called a unitrade if each line of $\Sigma^n$ contains an even number of elements from $S$. If each line of $\Sigma^n$ contains zero or two elements from $S$, then $S$ is called a double-code. A double-code is called splittable if it can be represented as a union of two independent sets (if the double-code is a double-MDS-code, then these two sets are MDS codes). The characteristic function $\chi_S : \Sigma^n \rightarrow \{0, 1\}$ of a double-MDS-code is a frequency $n$-cube $F^n(q; q-2, 2)$.

For example, the picture below illustrates an incomplete collection of unitrades in $H(2, 4)$; the first four of them (and the complement of the 5th) are double-codes, while only the 3rd one and the 4th one are double-MDS-codes, both splittable.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{unitrades.png}
\caption{Illustration of unitrades in $H(2, 4)$.}
\end{figure}

Remark 1. The concept “unitrade” was introduced in [12] in the context of study of MDS codes. The reader can observe a difference between our definition and that in [12]. The reason is that the definition of a unitrade is based on the properties of the symmetric difference of two objects of the main class under the study. Our main class is double-MDS-codes, and the symmetric difference can have 0, 2, or 4 elements in a line; this property is taken as the definition of a unitrade. In [12], the main class is MDS codes, and a unitrade is allowed to have only 0 or 2 elements in a line, as the symmetric difference of any two MDS codes.

Given a set $M \subset \Sigma^n$, a layer of $M$ consists of all elements of $M$ that have some fixed value $a$ in a given coordinate $i$ (referred to as the direction of the layer): $\{(c_0, \ldots, c_n) \in C \mid c_i = a\}$. We identify a layer with the corresponding subset of $\Sigma^{n-1}$ by ignoring the fixed coordinate. Note that any layer inherits the property of the set to be an MDS code, double-MDS-code, double-code, or unitrade.
2.2 Isotopy and equivalence

An isotopy in $\Sigma^n$ is an $n$-tuple of permutations $\theta_i : \Sigma \to \Sigma$, $i \in \{1, \ldots, n\}$. For an isotopy $\bar{\theta} = (\theta_1, \ldots, \theta_n)$ and a set $S \subset \Sigma^n$, we put

$$\bar{\theta}S = \{(\theta_1 x_1, \ldots, \theta_n x_n) \mid (x_1, \ldots, x_n) \in S\}.$$ 

Two sets $S_1 \subset \Sigma^n$ and $S_2 \subset \Sigma^n$ are isotopic if there exists an isotopy $\bar{\theta}$ such that $\bar{\theta}S_1 = S_2$. An autotopy of a set $S \subset \Sigma^n$ is an isotopy $\bar{\theta}$ such that $\bar{\theta}S = S$.

Two sets $S_1 \subset \Sigma^n$ and $S_2 \subset \Sigma^n$ are equivalent if there exists an isotopy $\bar{\theta}$ and a permutation $\sigma$ of $n$ coordinates such that $\sigma \bar{\theta}S_1 = S_2$, where

$$\sigma S = \{(x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)}) \mid (x_1, \ldots, x_n) \in S\}.$$ 

If $\sigma \bar{\theta}S = S$ holds for some set $S \subset \Sigma^n$, isotopy $\bar{\theta}$ and coordinate permutation $\sigma$, then the pair $(\sigma, \bar{\theta})$ is called an automorphism of $S$. The group of all automorphisms (autotopies) of a set $S$ with the composition as the group operation is denoted $\text{Aut}(S)$ (respectively $\text{Atop}(S)$).

Two frequency $n$-cubes $\chi S_1$ and $\chi S_2$ are isotopic (equivalent) if $S_1$ is isotopic (equivalent) to $S_2$ or $\Sigma^n \setminus S_2$. So, we see that the number of equivalence classes of frequency $n$-cubes $F^n(4; 2, 2)$ is in general smaller than that of double-MDS-codes in $H(n, 4)$: a double-MDS-code can be nonequivalent to its complement, but the corresponding two frequency $n$-cubes are equivalent by the definition. An autotopy of a frequency $n$-cube $\chi S$ is an isotopy that sends the corresponding double-MDS-code $S$ to itself or to its complement $\Sigma^n \setminus S$. The automorphisms of a frequency $n$-cube are defined similarly. So, the set of autotopies (automorphisms) of a frequency $n$-cube $F^n(4; 2, 2)$ either coincides with the set of autotopies (automorphisms) of the corresponding double-MDS-code, or is twice larger than it.

2.3 Testing sets

A subset $T$ of $\Sigma^n$ is a testing set for double-MDS-codes (or, equivalently, for any other class of subsets of $\Sigma^n$) if $C_1 \cap T \neq C_2 \cap T$ for any two different double-MDS-codes in $H(n, 4)$.

Tab. 1: A double-MDS-code (right table) is reconstructed by its intersection with the testing set (left table)

For example, $\{0, 1, 2\}^n$ (see, e.g., Table 1) is a testing set for double-MDS-codes in $H(n, 4)$: its size is $3^n$, and it is not difficult to show that it is minimum for $n = 1, 2$. However, we can find a smaller testing set for $n = 3$; the minimum size of a testing set in the three-dimensional case is still unknown.
3 Classification

Using a computational approach, we classify double-MDS-codes in $H(3, 4)$ and $H(4, 4)$ and unitrades in $H(3, 4)$. The algorithm is rather straightforward. Below, we describe it for double-MDS-codes, while for unitrades it is absolutely the same (just replace “double-MDS-code” by “unitrade” everywhere). Before we give the details of the algorithm, we say few words about recognizing the equivalence. A usual way to work with the equivalence of codes is to represent them by graphs in such a way that two codes are equivalent if and only if the corresponding graphs are isomorphic, see [5, §3.3.2]. A standard software that helps to recognize the graph isomorphism is nauty&traces [9]; it is realized as a package that can be used in c or c++ programs. With this package, for a graph one can compute its canonically-labelled version, such that two graphs are isomorphic if and only if the corresponding canonically-labelled graphs are equal to each other. The same procedure computes the automorphism group of the graph, which can be used for the numerical validation of the results, see Section 3.2.

3.1 Algorithm

Assume that, as a result of the past classification, we have that the number of double-MDS-codes of length $n - 1$ is $N_{n-1}$ and they are partitioned into $\alpha$ equivalence classes.

We consider a double-MDS-code of length $n$ as the union of 4 of its layers in the last direction, essentially, 4 double-MDS-codes of length $n - 1$. As the fourth layer is uniquely reconstructed from the first three ones, we can run through $N_{n-1}^3$ triples of double-MDS-codes of length $n - 1$ and check if they can be completed by the fourth layer to form a double-MDS-code of length $n$. In such a way, we construct all double-MDS-codes of length $n$. However, simple calculations show that the number $N_{n-1}^3$, $n = 4$, is too large to manage all these possibilities. A standard way to make the search faster is, at each step, to choose only nonequivalent cases (this procedure is often called isomorph rejection).

At step 1, we choose the first layer from a nonequivalent double-MDS-codes of length $n - 1$ (representatives of the equivalence classes).

At step 2, for each choice of the first layer, we choose the second layer from all $N_{n-1}$ double-MDS-codes of length $n - 1$. So, we process $\alpha \cdot N_{n-1}$ cases at this step. Among all possible unions of the first layer and the second layer, called semi-codes, we choose only nonequivalent $\alpha'$ representatives, where $\alpha'$ is the number of the equivalence classes of semi-codes.

At step 3, for each of $\alpha'$ representatives of semi-codes, we choose the third layer from $N_{n-1}$ different double-MDS-codes of length $n - 1$. So, we process $\alpha' \cdot N_{n-1}$ cases at this step. Some of the resulting cases cannot be completed to a double-MDS-code of length $n$ (for example, if there are already three codewords in some line of the last direction); for the remaining cases, the completion is unique, and we use isomorph rejection to keep only nonequivalent representatives of the resulting double-MDS-codes of length $n$. 
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3.2 Validation

We can check the results of the classification using a general double-counting approach, described in [5, §10.2]. We know that the number of semi-codes is $N_{n-1}^2$, and at step 2 we found representatives of their $a'$ equivalence classes. The number of continuations to a double-MDS-code of length $n$ is the same for all semi-codes from the same, say $i$th, equivalence class. We denote this number by $R_i$ and remark that it was found, for each $i$, at step 3. Hence, the total number $N_n$ of double-MDS-codes of length $n$ can be found as

$$N_n = \sum_{i=1}^{a'} M_i R_i,$$

(1)

where $M_i$ is the number of semi-codes in the $i$th equivalence class (it can be easily counted from the automorphism group of the corresponding representative). The number $N_n$ can be alternatively found as

$$N_n = \sum_{j} T_j, \quad T_j = \frac{4^n \times n!}{|\text{Aut}(C_j)|},$$

(2)

where $T_j$ is the size of the $j$th equivalence class of double-MDS-codes of length $n$ and $C_j$ is its representative (which is kept after the final isomorph rejection at step 3). Coincidence of the values of $N_n$ obtained from (1) and (2) certifies that our algorithm gives correct numbers $R_i$ and, in particular, it did not miss anything.

3.3 Results of the classification

The results of the classification by the algorithm in Section 3.1 are reflected in following theorems and Tables 2, 3, and 4. The database with representatives of equivalence classes can be found in [1].

**Theorem 3.1.** There are 51678 double-MDS-codes of length 3, divided into 10 equivalence classes or 26 isotopy classes. There are 51678 frequency $F^3(4; 2, 2)$ cubes, divided into 10 equivalence classes or 26 isotopy classes.

**Theorem 3.2.** There are 55720396530 double-MDS-codes of length 4, divided into 8895 equivalence classes or 192214 isotopy classes. There are 55720396530 frequency $F^4(4; 2, 2)$ hypercubes, divided into 7203 equivalence classes or 154078 isotopy classes.

**Theorem 3.3.** There are $2^{2^7}$ unitrades in $H(3, 4)$, divided into 2528 equivalence classes. Representatives of 312 of them are equivalent to their complements.

As we will see below (Lemma 5.2), the total number of unitrades in $H(n, 4)$ is $2^{3^n}$ for any $n$. 

### 3.4 Splittability of double-MDS-codes in \( H(4, 4) \)

From the definitions, we can get the following relationship between the splittability of a double-MDS-code and the splittability of each of its layers. If a double-MDS-code is splittable, then each layer of this code is splittable. If there exists a non-splittable layer, then the entire code is non-splittable. But if all layers are splittable, it does not necessarily mean that the code itself is splittable.

As a result of the classification, we found that all double-MDS-codes in \( H(4, 4) \), except for two equivalence classes, are splittable if their layers are splittable. In the diagram below, the black vertices induce a cycle of length 9.
| $|\text{Aut}(C)|$ | $N$ | $N'$ | $N''$ | $N^*$ | $6 \cdot 8$ | $10$ | $10$ | $10$ | $3$ |
|---|---|---|---|---|---|---|---|---|---|
| $24 \cdot 2048$ | 1 | 1 | 1 | 1 | $4' \cdot 8$ | 10 | 10 | 10 | 2 |
| $4' \cdot 512$ | 1 | 1 | 1 | 1 | $3 \cdot 8$ | 1 | 1 | | |
| $6 \cdot 256$ | 1 | 1 | 1 | 1 | $2' \cdot 8$ | 58 | 58 | 57 | 2 |
| $2' \cdot 256$ | 1 | 1 | 1 | 1 | $2'' \cdot 8$ | 7 | 7 | 7 | 1 |
| $24 \cdot 128$ | 1 | 1 | 1 | 1 | $1 \cdot 8$ | 71 | 65 | 63 | |
| $8 \cdot 128$ | 2 | 2 | 2 | 2 | $12' \cdot 4$ | 1 | 1 | 0 | |
| $6 \cdot 128$ | 1 | 1 | 1 | 1 | $6' \cdot 4$ | 3 | 3 | 3 | |
| $4' \cdot 128$ | 1 | 1 | 1 | 1 | $4'' \cdot 4$ | 18 | 16 | 16 | |
| $2' \cdot 128$ | 2 | 2 | 2 | 2 | $3' \cdot 4$ | 1 | 1 | 0 | |
| $2'' \cdot 128$ | 1 | 1 | 1 | 1 | $2'' \cdot 4$ | 173 | 169 | 169 | |
| $6 \cdot 64$ | 1 | 1 | 1 | 1 | $2'''' \cdot 4$ | 6 | 6 | 6 | |
| $4' \cdot 64$ | 2 | 2 | 2 | 2 | $1 \cdot 4$ | 220 | 212 | 207 | |
| $2' \cdot 64$ | 3 | 3 | 3 | 2 | $24' \cdot 2$ | 1 | 1 | 1 | |
| $1 \cdot 64$ | 1 | 1 | 1 | | $6' \cdot 2$ | 12 | 10 | 10 | |
| $8 \cdot 32$ | 1 | 1 | 1 | 1 | $4'' \cdot 2$ | 38 | 30 | 30 | |
| $6 \cdot 32$ | 1 | 1 | 1 | 1 | $3' \cdot 2$ | 7 | 3 | 1 | |
| $4' \cdot 32$ | 3 | 3 | 3 | | $2' \cdot 2$ | 297 | 279 | 273 | |
| $2' \cdot 32$ | 2 | 2 | 2 | 2 | $2'' \cdot 2$ | 46 | 40 | 36 | |
| $1 \cdot 32$ | 7 | 7 | 7 | | $1 \cdot 2$ | 1057 | 917 | 875 | |
| $24 \cdot 16$ | 2 | 2 | 2 | 2 | $8' \cdot 1$ | 4 | 0 | 0 | |
| $6 \cdot 16$ | 4 | 4 | 4 | 3 | $6' \cdot 1$ | 21 | 19 | 19 | |
| $4' \cdot 16$ | 8 | 8 | 8 | 7 | $4'' \cdot 1$ | 16 | 0 | 0 | |
| $3 \cdot 16$ | 1 | 1 | 1 | 0 | $4' \cdot 1$ | 12 | 0 | 0 | |
| $2' \cdot 16$ | 27 | 27 | 25 | 6 | $3' \cdot 1$ | 16 | 10 | 6 | |
| $2'' \cdot 16$ | 4 | 4 | 4 | 2 | $2' \cdot 1$ | 728 | 506 | 506 | |
| $1 \cdot 16$ | 18 | 16 | 16 | 15 | $2'' \cdot 1$ | 78 | 0 | 0 | |
| $24 \cdot 8$ | 2 | 2 | 2 | 2 | $1 \cdot 1$ | 5862 | 3018 | 2781 | |
| $8 \cdot 8$ | 2 | 2 | 2 | 2 | total: | 8895 | 5511 | 5200 | 50 | |

Tab. 3: The table reflects the number of nonequivalent double-MDS-codes in $H(n, 4)$, $n = 4$, with the given number of automorphisms. In the first column, the number of automorphisms of a double-MDS-code $C$ is represented in the form $P \cdot T$, where $T = |\text{Aut}(C)|$ and $P$ is the order of the group $\text{Aut}(C)/\text{Atop}(C)$ acting on the four coordinates. The accents reflect the type of this group: groups of type $2'$ and $4'$ contain a transposition; groups of type $2''$ and $4''$ consist of the identity permutation and involutions without fixed points; group of type $4''$ corresponds to the cyclic group of order 4; all other groups of permutations of the four coordinates are defined by their orders up to conjugancy. The second column contains the number $N$ of nonequivalent double-MDS-codes with the corresponding restrictions on the automorphism set. The third column contains the number $N'$ of such codes that are equivalent to their complement. The fourth column contains the number $N''$ of such codes that are isotopic to their complement. $N^*$ in the fifth column is the number of splittable double-MDS-codes. Using these data, the following numbers can be found for each row. The number of isotopy classes of double-MDS-codes is $N \cdot n!/P$. The number of nonequivalent frequency $n$-cubes $F^n(4; 2, 2)$ is $(N + N')/2$; and $(N - N')/2$ of them have $P \cdot T$ automorphisms, while the remaining $N'$ have twice more (including the automorphisms that change the value of the function). The number of non-isotopic frequency $n$-cubes $F^n(4; 2, 2)$ is $(N + N') \cdot n!/2P$; and $(N + N' - 2N'') \cdot n!/2P$ of them have $T$ autotopies, while the remaining $N'' \cdot n!/P$ have twice more.
4 Non-splittable double-MDS-codes with splittable layers

| Aut(C) | N | N' | N'' | N* | 2 · 8 | 1 · 8 | 2 · 64 | 1 · 64 | 6 · 32 | 1 · 32 | 2 · 32 | 1 · 32 | 2 · 16 | 1 · 16 | total |
|--------|---|----|-----|----|------|------|-------|-------|------|-------|------|-------|------|-------|------|
| 6 · 256 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 · 64 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 6 · 32 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 · 32 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 · 16 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

Tab. 4: The table reflects the number of nonequivalent double-MDS-codes in $H(n, 4), n = 3$, with the given number of automorphisms. The notation and calculation of the derived values are the same as those in Table 3.

In the next section, we generalize found examples by constructing, for every length $n \geq 3$, a non-splittable double-MDS-code whose layers are all splittable.

4 Non-splittable double-MDS-codes with splittable layers

The theorem proved in this section is important for us to understand the inter-relation between the classes of double-MDS-codes and splittable double-MDS-codes. A double-MDS-code is defined by a local property; a code is a double-MDS-code if and only if each of its layers in each direction is a double-MDS-code. As we will see below, the similar is not true for splittable double-MDS-codes.

We first introduce some notations. Two adjacent vertices of the Hamming graph differ in one coordinate, whose number is called the direction of the corresponding edge. Edges of the same direction are called parallel. We color all edges of $H(n, 4)$ by three colors depending on the values of the two words in the coordinate in which they are different. If the values are either 0 and 1 or 2 and 3, then the color is 1; if the values are either 0 and 2 or 1 and 3, the color is 2; for 0, 3 or 1, 2, the color is 3. If a double-MDS-code is not splittable, then its induced subgraph of the Hamming graph is not bipartite. In this case, there is an odd-length cycle in this subgraph.

**Lemma 4.1.** Any odd-length cycle in $H(n, 4)$ contains three parallel edges of three different colors.

**Proof.** If a cycle in $H(n, 4)$ contains edges of at most two colors of some direction, then the number of the edges of this direction in the cycle is even. For example, if a cycle only has edges of color 2 and 3 in direction $i$, then each edge of these direction changes the value of the $i$th coordinate between the sets $\{0, 1\}$ and $\{2, 3\}$. Walking along all of the edges of the cycle, we have an even number of such changes because finally we return to the starting vertex.

For a cycle of odd length, there is an odd number of edges of some direction. Hence, the edges of these directions are colored by three different colors.

**Theorem 4.2.** For any $n$ larger than 2, there is an unsplittable double-MDS-code in $H(n, 4)$ whose all layers in all directions are splittable.
Proof. The idea of the construction is based on Lemma 4.1. We will construct an unsplittable double-MDS-code, with an induced cycle of odd length $2n + 1$, such that the edges of the induced graph satisfy the following:

(i) the edges of any direction from 1 to $n - 1$ are colored by only two colors;
(ii) for any two edges of direction $n$ and colors 1 and 3, the distance between them is $n - 1$ (i.e., two words from different edges differ in each coordinate from 1 to $n - 1$).

Property (i) guarantees that every layer in the $n$th direction is splittable by Lemma 4.1. On the other hand, Property (ii) guarantees that every layer in any other direction is splittable, again by Lemma 4.1.

To construct such a code, we divide the vertex set of $H(n, 4)$ into $2^n - 1$ sectors $V_a$, $a \in \{0, 2\}^{n-1} \times \{0\}$, where $V_a = \{a + b \mid b \in \{0, 1\}^{n-1} \times \{0, 1, 2, 3\}\}$. Define three $\{0, 1\}$-valued functions $\alpha$, $\beta$, $\gamma$ on $\{0, 1, 2, 3\}$:

| $x$   | 0 | 1 | 2 | 3 |
|-------|---|---|---|---|
| $\alpha(x)$ | 1 | 1 | 0 | 0 |
| $\beta(x)$  | 1 | 0 | 1 | 0 |
| $\gamma(x)$ | 0 | 1 | 1 | 0 |

On $\{0, 1, 2, 3\}^n$, define the following function:

$$f(x) = \begin{cases} 
  x_1 + \ldots + x_{n-1} + \alpha(x_n) \mod 2 & \text{if } x \in V_{(0, \ldots, 0, 0)}, \\
  x_1 + \ldots + x_{n-1} + \beta(x_n) + 1 \mod 2 & \text{if } x \in V_{(2, \ldots, 2, 0, \ldots, 0, 0)} \ (n - 2 \text{ sectors}), \\
  x_1 + \ldots + x_{n-1} + \gamma(x_n) \mod 2 & \text{if } x \in V_{(2, \ldots, 2, 0)}, \\
  x_1 + \ldots + x_{n-1} + \beta(x_n) \mod 2 & \text{otherwise}.
\end{cases}$$

Denote by $M$ the set of ones of $f$. It is straightforward to see that $f$ is an $F^n(4; 2, 2)$ frequency hypercube, and so $M$ is a double-MDS-code. Any edge of $H(n, 4)$ of color 1 and direction from 1 to $n - 1$ is in one sector, and the values of $f$ on the two vertices of the edge are different. Hence, such an edge does not lie in $M$, and (i) is satisfied. Further, any edge in $M$ of color 1 and direction $n$ lies in the sector $V_{(0, \ldots, 0, 0)}$, while any edge in $M$ of color 3 and direction $n$ lies in $V_{(2, \ldots, 2, 0)}$. Hence, (ii) is also satisfied. As was mentioned above, properties (i) and (ii) imply that any layer of $M$ is splittable. It remains to find a cycle of odd length

- $(0, \ldots, 0, 0), (0, \ldots, 0, 1),$
- $(2, 0, \ldots, 0, 1), (2, 2, 0, \ldots, 0, 1), \ldots, (2, \ldots, 2, 0, 1), \ (n - 2 \text{ vertices}),$
- $(2, \ldots, 2, 1), (2, \ldots, 2, 2),$
- $(0, 2, \ldots, 2, 2), (0, 2, \ldots, 2, 0), (0, 0, 2, \ldots, 2, 0), \ldots, (0, \ldots, 0, 2, 0), \ (n - 1 \text{ vertices})$

(for convenience, we group vertices according to the four cases in the definition of $f$). The cycle has 3 edges of direction $n$ (underlined) and two edges of each other direction. □
5  An upper bound on the number of double-MDS-codes

The number of double-MDS-codes of length $n$ has a trivial upper bound $2^{3n}$ because every such code can be reconstructed from only 3 of 4 layers in any direction. Thus, the size of the minimum testing set is no more than $3^n$. For $n = 3$, this value is 27.

For a subset $P$ of $\Sigma^n$, we denote by $X_P$ the column-vector $(x_0, \ldots, x_{4^n-1})^T = (x(0,\ldots,0,0), x(0,\ldots,0,1), \ldots, x(3,\ldots,3,3))^T$, over the Galois field $GF(2)$ of order 2, where $x(t_1,\ldots,t_n) = 1$ if $(t_1,\ldots,t_n) \in P$ and $x(t_1,\ldots,t_n) = 0$ otherwise (the elements $(t_1,\ldots,t_n)$ of $\Sigma^n$ in the indices are listed in the lexicographic order).

We recursively define the matrix $A$ of size $4n \times 4n-1$ over $GF(2)$ as follows:

$$A = \begin{pmatrix} A_1 \\ \vdots \\ A_n \end{pmatrix}, \quad A_i = I_{4^{i-1}} \otimes (1,1,1,1) \otimes I_{4^{n-i}}, \quad 1 \leq i \leq n.$$

Here and below, the symbol $\otimes$ denotes the Kronecker product of matrices, and $I_m$ is the $m \times m$ identity matrix. The submatrices $A_i$ of $A$ have sizes $4^{n-1} \times 4^n$.

**Lemma 5.1.** A subset $D$ of $\Sigma^n$ is a unitrade if and only if $AX_D = 0$, where $A$ is the matrix defined above.

**Proof.** The claim is straightforward from the definition of unitrade. Indeed, each equation from the system $AX_D = 0$ sums the values of $X_D$ over the four elements of one line; this sum equals 0 if and only if $X$ has an even number of elements in that line (the rows of the submatrix $A_i$ correspond to the lines of direction $i$). For example, for a unitrade of length 2, the definition of a unitrade says that

$$\sum_{i=0}^{3} x_{(i,j)} = 0, \quad j = 0,1,2,3,$$

and

$$\sum_{j=0}^{3} x_{(i,j)} = 0, \quad i = 0,1,2,3,$$

which is the same as $AX_D = 0$, where $A = \begin{pmatrix} I_4 \otimes (1,1,1,1) \\ (1,1,1,1) \otimes I_4 \end{pmatrix}$.

**Lemma 5.2.** The number of unitrades in $\Sigma^n$ is $2^{3n}$. In particular, the rank of the matrix $A$ is $4^n - 3^n$.

**Proof.** We firstly note that each element of $\Sigma^n$ with at $m$, $m > 0$, zeros belongs to a line where each of the other three elements has $m-1$ zeros. It follows that the elements of a unitrade with a zero component are uniquely reconstructed from its elements without zero components. That is, a unitrade is uniquely defined by its intersection with $\{1,2,3\}^n$. So, the number of unitrades is not larger than the number $2^{3^n}$ of subsets of $\{1,2,3\}^n$.

It remains to find $3^n$ linearly independent solutions of $AX_D = 0$. For every $a$ from $\{1,2,3\}^n$, let the set $D_a$ consist of all $2^n$ elements of $\Sigma^n$ obtained from $a$ by replacing some (maybe all, maybe none) of components by zeros. It is easy
to see that $D_a$ is a unitrade. Moreover, all $X_{D_a}$, $a \in \{1, 2, 3\}^n$, are linearly independent because each element of $\{1, 2, 3\}^n$ belongs to a unique $D_a$. \qed

The following proposition allows to construct testing sets for double-MDS-codes.

**Proposition 5.3.** Suppose that there exists unitrade $D$ in $H(n, 4)$ such that no two double-MDS-codes $C_1$, $C_2$ satisfy $\emptyset \neq C_1 \triangle C_2 \subset D$. Then for double-MDS-codes of length $n$, there is a testing set $T \subset \Sigma^n \setminus D$ of size $3^n - k_D$, where $2^{k_D}$ is the number of unitrade subsets of $D$, $k_D \geq 1$.

**Proof.** Denote $\overline{D} = \Sigma^n \setminus D$ and $N = |\overline{D}|$. Let $\overline{D} = \{u_0, u_1, \ldots, u_{N-1}\}$. Note that $\overline{D}$ is also a unitrade. By $B$, we denote the $N \times 4^n$ matrix with the rows $X_{\{u_0\}}, \ldots, X_{\{u_{N-1}\}}$. The solution space of $\begin{pmatrix} A \\ B \end{pmatrix}X = 0$ over GF(2) is $\{X_P \mid P \subset D, \ P \text{ is a unitrade}\}$. The dimension of this space is $k_D$, because $2^{k_D} = |\{P \mid P \subset D, \ P \text{ is a unitrade}\}|$ by the definition of $k_D$. Hence, the rank of the matrix $\begin{pmatrix} A \\ B \end{pmatrix}$ is $4^n - k_D$. The rank of $A$ is $4^n - 3^n$ by Lemma 5.2. So, we can find $3^n - k_D$ rows of $B$, with numbers $i_1, \ldots, i_{3^n - k_D}$, forming a $(3^n - k_D) \times 4^n$ submatrix $B'$ such that the rank of $\begin{pmatrix} A \\ B' \end{pmatrix}$ is also $4^n - k_D$. Each row of $B'$ has 1 in exactly one position, corresponding to some $u_i$. We state that the subset $T = \{u_i \mid i = i_1, \ldots, i_{3^n - k_D}\}$ of $\overline{D}$ is a testing set for double-MDS-codes. Indeed, if two double-MDS-codes $C_1$ and $C_2$ meet $C_1 \cap T = C_2 \cap T$, then $C = C_1 \triangle C_2$ does not intersect with $T$. Hence, the vector $X_C$ is in the kernel of $B'$, i.e., $B'X_C = 0$. Moreover, $C$ is a unitrade because it is the symmetric difference of unitrades. Therefore, $X_C$ is also in the kernel of $A$, i.e., $AX_C = 0$.

We conclude that $\begin{pmatrix} A \\ B' \end{pmatrix}X_C = 0$, and hence $\begin{pmatrix} A \\ B \end{pmatrix}X_C = 0$ (indeed, the row spaces of the matrices $\begin{pmatrix} A \\ B' \end{pmatrix}$ and $\begin{pmatrix} A \\ B \end{pmatrix}$ coincide). The last equality means that $C$ is in $\{P \mid P \subset D, \ P \text{ is a unitrade}\}$. By the hypothesis of the proposition, $C$ is an empty set. It follows that $C_1 = C_2$ and $T$ is a testing set. \qed

The straightforward check, utilizing the classification in Theorems 3.1 and 3.3 shows that there are two nonequivalent unitrades in $H(n, 4)$, of cardinalities 32 and 38, that satisfy the hypothesis of the proposition with $k_D = 2$, and there are no unitrades with $k_D = 3$. The two examples are shown at the following diagram where white and black bullets indicate unitrade subsets $D'$ and $D''$, so $2^{k_D} = |\{\emptyset, D', D'', D\}|$. 

![Diagram showing unitrade subsets](image-url)
The second unitrade has the following complement,

where removing any white vertex results in a testing set of size 25.

This result is generalized by the following lemma, which is a special case of [7, Proposition 26] (for completeness, we give a proof).

**Lemma 5.4.** Let $T \subset \Sigma^3$ be a testing set for double-MDS-codes of length 3. Then the Cartesian product $T^l \subset \Sigma^{3l}$ of the testing set $T$ is a testing set for double-MDS-codes of length $3l$.

**Proof.** We prove the claim by induction. Suppose that for two double-MDS-codes $C_1$ and $C_2$ we have $C_1 \mid_{T^l} = C_2 \mid_{T^l}$. Then, by the induction hypothesis, for any $c \in T$, $C_1 \mid_{T^l \times \{c\}} = C_2 \mid_{T^l \times \{c\}}$ implies $C_1 \mid_{\Sigma^{3(l-1)} \times \{c\}} = C_2 \mid_{\Sigma^{3(l-1)} \times \{c\}}$. Hence, for any $\omega \in \Sigma^{3(l-1)}$, we have $C_1 \mid_{\{\omega\} \times T^l} = C_2 \mid_{\{\omega\} \times T^l}$. The set $\{\omega\} \times T$ is testing on $\{\omega\} \times \Sigma^3$. Then $C_1 \mid_{\{\omega\} \times \Sigma^3} = C_2 \mid_{\{\omega\} \times \Sigma^3}$ for any $\omega \in \Sigma^{3(l-1)}$.

Substituting the testing set of size 25 found above to Lemma 5.4 we get a testing set of size $25^l$ for double-MDS-codes in $H(3l, 4)$.

**Corollary 5.5.** The number of double-MDS-codes in $\Sigma^n$, where $n$ is divisible by 3, is at most $2^{3n^n}$, where $\beta = 25^{1/2} < 3$.

With a slightly worth bound, the result is generalized to an arbitrary $n \geq 3$.

**Theorem 5.6.** (upper bound) The number of $N_n$ double-MDS-codes in $\Sigma^n$, $n \geq 3$, is at most $2^{2^n}$, where $\alpha < 2.955$. Moreover, for $n$ sufficiently large, we have $N_n < 2^{2.925^n}$.

**Proof.** By Corollary 5.5 for $n = 3l$, $l = 1, 2, \ldots$, the size of a testing set is no more than $25^l = \alpha_n^n$, where

$$\alpha_n = \alpha_{3l} \equiv \beta, \quad \beta = 25^{1/2} < 2.925.$$  

For $n = 3l + 1$, a double-MDS-code can be considered as the union of 4 layers, 4 double-MDS-codes of length $3l$. Moreover, each of them is uniquely determined by the other three layers. Therefore, the minimum size of a testing set is no more than $25^l \cdot 3 = \alpha_n^n$, where

$$\alpha_n = \alpha_{3l+1} = (25^l \cdot 3)^{1/3} = \beta \cdot \exp \frac{\ln 3 - \ln \beta}{n}.$$
Similarly, for \( n = 3l + 2 \) the size of a testing set is bounded by \( 25^l \cdot 3 \cdot 3 = \alpha_n^l \)
where

\[
\alpha_n = \alpha_{3l+2} = (25^l \cdot 3^2)^{\frac{1}{n}} = \beta \cdot \exp^{\frac{2(\ln 3 - \ln \beta)}{n}}.
\]

Since the sequences \( \alpha_{3l} \), \( \alpha_{3l+1} \), \( \alpha_{3l+2} \) are non-increasing, the first claim of the theorem holds with

\[
\alpha = \max\{\alpha_3, \alpha_4, \alpha_5\} = \max\{25^\frac{1}{3}, 75^\frac{1}{4}, 225^\frac{1}{5}\} = 225^\frac{1}{5} < 2.955.
\]

Finally, we see that \( \alpha_n \to \beta \), which proves the second claim. \( \square \)

**Remark 2.** It is worth to check if the same approach works for more general class of objects, e.g. for double-codes. Direct computations show that the set of symmetric differences of double-codes in \( H(3, 4) \) is the set of all unitrades, and a testing set of size less than 27 cannot be found in the way described. So, the best upper bound on the number of double-codes remains trivial, i.e., \( 2^{3n} \).

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