Next-to-soft corrections for Drell-Yan and Higgs boson rapidity distributions beyond N^3LO

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We present a formalism that resums both soft-virtual (SV) and next to SV (NSV) contributions to all orders in perturbative QCD for the rapidity distribution of any colorless particle produced in hadron colliders. Using the state-of-the-art results, we determine the complete NSV contributions to third order in strong coupling constant for the rapidity distributions for Drell-Yan and also for Higgs boson in gluon fusion as well as bottom quark annihilation. Using our all order z space result, we show how the NSV contributions can be resummed in two-dimensional Mellin space.

Introduction.—Accurate measurement of observables at the Large Hadron Collider (LHC) and their precise theoretical predictions, provide an opportunity to test the Standard Model (SM) with unprecedented accuracy thereby constraining beyond the SM (BSM) scenarios. One of the cleanest observables at the LHC is Drell-Yan (DY) production [1] of on-shell vector bosons Z and W± or a pair of leptons and hence it has received enormous attention from the theory community. Measurements [2–4] of inclusive and differential rates of DY production are used as a standard candle to calibrate the detectors and also to fit the non perturbative parton distribution functions (PDF) [5–9]. Any deviation from the SM predictions can provide crucial information to BSM scenarios, such as R-parity violating supersymmetric models, models with Z′ and large extra dimension models [10, 11]. Similarly, the ongoing measurements of inclusive and differential cross sections [12, 13], along with the theoretical predictions [14] on strong and electroweak radiative corrections help us to probe the symmetry-breaking mechanism and the coupling of the Higgs boson with other SM particles. This is possible owing to the third order QCD predictions for DY production [15, 16] and Higgs boson productions in gluon fusion [14, 17, 18] and bottom quark annihilation [19, 20].

Like inclusive rates, differential ones also get large contributions from logarithms from phase space boundaries of the final state particles, thus spoiling the reliability of the fixed-order predictions. These large logarithms can be summed up to all orders in perturbation theory. In the seminal works of Sterman [21] and of Catani and Trentadue [22], resummation of leading large logs for the inclusive rates in Mellin space and also to differential x_F distribution [22] using double Mellin moments were achieved. Using factorization properties of differential cross sections and renormalization group (RG) invariance, an all order z-space formalism was also developed in [23], to study the threshold-enhanced contribution to rapidity distribution of any colorless particle. The formalism was also applied to Z and W± [24] and also to DY and Higgs production at N^3LO level [20, 25]. In [26], the same formalism [23] was used to study threshold resummation of rapidity distribution of Higgs bosons and later to DY production [27]. For different approaches and their applications, see [28–36].

Besides the threshold logarithms, contributions from subleading logarithms are also present in all the partonic channels beyond leading order in perturbation theory. These subleading logarithms demonstrate perturbative behaviour similar to those from threshold region, which allows one to study their all order structure. Such logarithms do appear in inclusive reactions and there have been remarkable progress to understand them. See, [37–49] for more details. Recently, in a series of articles [50, 51], we studied variety of inclusive reactions to understand these subleading logarithms and found a systematic way to sum them up to all orders in z as well as in Mellin N spaces. The latter provides resummed prediction in N space for subleading logarithms similar to that of threshold ones.

The differential distributions often show richer logarithmic structure due to multi-dimensional space (spanned by z_1 or N_l) making it a challenging task to understand the all order structure. In the present letter, using factorisation properties of physical observables and RG invariance, we have achieved the task of organising the subleading logarithms in a systematic fashion that is suitable for summing them up to all orders in perturbation theory both in z_1 and N_l spaces.

Theoretical framework.— In QCD improved parton model, the rapidity distribution of a colorless state F in hadron-hadron collision is given by

\[
\frac{d\sigma}{dy} = \sigma_{\text{Born}}(\tau, q^2) \sum_{a,b=q, g} \int_{x_1}^{1} \frac{dz_1}{z_1} \int_{z_2}^{1} \frac{dz_2}{z_2} f_a \left( \frac{x_1^0}{z_1}, \mu_F^2 \right) \\
\times f_b \left( \frac{x_2^0}{z_2}, \mu_F^2 \right) \Delta_{d,ab} (z_1, z_2, q^2, \mu_F^2, \mu_R^2),
\]

where \(\sigma_{\text{Born}}(\mu_F^2) = \sigma_{\text{Born}}(x_1^0, x_2^0, q^2, \mu_F^2)\) is the born cross section and \(\mu_R\) is the ultraviolet (UV) renormalization scale. The scaling variables \(x_l^0 (l = 1, 2)\) are defined through hadronic rapidity: \(y = \frac{1}{2} \ln (p_2 q/p_1 q) = \frac{1}{2} \ln \left( \frac{x_1^0}{x_2^0} \right)\) and \(\tau = q^2/S = x_1^0 x_2^0\). Here \(q\) denotes the momentum of colorless state F and \(S = (p_1 + p_2)^2\) is the hadronic center of mass energy, with \(p_l (l = 1, 2)\) the momenta.
of incoming hadrons. For \( F \) being state of a pair of leptons \( \sigma^e = d\sigma^q(\tau, q^2, y)/dq^2 \), i.e., its invariant mass distribution, whereas for the Higgs production in gluon fusion or in bottom quark annihilation \( \sigma^c = \sigma^b(\tau, q^2, y) \) respectively. The PDFs \( f_c(x_i, \mu_F^2) \) of colliding partons \( c = q, \bar{q}, g, b \) with momentum fractions \( x_l (l = 1, 2) \) are renormalized at the factorization scale \( \mu_F \). The partonic coefficient functions (CFs), \( \Delta_{d,a,b} \), are perturbatively calculated in QCD in powers of strong coupling constant \( a_s(\mu_F^2) = g^2(\mu_F^2)/16\pi^2 \) and are functions of the scaling variables \( z_l = x_l/\tau_l (l = 1, 2) \). They are obtained from the partonic processes through mass factorization. The UV finite partonic processes contain soft and collinear divergences associated with the soft gluons and collinear partons, beyond leading order in perturbation theory, which can be removed by summing over degenerate final states and by mass factorization. In this letter we restrict ourselves to partonic CFs of only quark-antiquark initiated processes for DY, gluon-gluon and bottom-bottom initiated processes for Higgs productions. We call them diagonal CFs (dCFs) \( \Delta_{d,a,b} \). These dCFs comprise of contributions from \( \delta(1-z_1) \) and \( D_{ij}(z_l) \equiv \ln(z_l) \) (namely SV) and the coefficients regular in \( z_l \). The leading contributions of the latter near the threshold region \( z_1 = 1 \) contain terms of the form \( D_{ij}(z_l) \ln^k(1-z_j) \) and \( \delta(1-z_1) \ln^k(1-z_j) \) with \((l, j = 1, 2), (i, k = 0, 1, \ldots) \). We call them next to soft-virtual (NSV) contributions. In the Mellin \( N_l \) space, these terms are of the form of \( \ln^k N_l \) with \((j, l = 1, 2), (k = 0, 1, \ldots) \). The dominant SV contribution has been studied in the earlier works of one of the authors in [23]. In the following we discuss the NSV contributions of the dCFs in \( z_l \) as well as in \( N_l \) space.

**Fixed Order Formalism.**— Using RG invariance and factorization properties of differential dCF [23], the threshold-enhanced SV and NSV terms of dCF, denoted by \( \Delta_{d,c}^{SV+NSV} \), is found to exponentiate as

\[
\Delta_{d,c}^{SV+NSV} = C \exp \left( \Psi_{d}^{c}(q^2, \mu_F^2, \mu_F^2, z_1, z_2, \epsilon) \right) \bigg|_{\epsilon = 0}, \tag{2}
\]

where the function \( \Psi_d^c \) is computed in perturbative QCD in \( 4 + \epsilon \) space-time dimensions and \( z_1 = 1 - z_1 \) and \( z_2 = 1 - z_2 \) are the shifted scaling variables. It is shown in Eq.(9) of [23] that the UV and IR finite function \( \Psi_d^c \) can be decomposed in terms of form factor \( F^c \), soft distribution \( \Phi_{d}^{c} \) and the diagonal Altarelli-Parisi kernels \( \Gamma_{cc} \). The soft distribution discussed in [23] using K+G type Sudakov differential equations, accounts for the soft enhancements associated with the real emissions in the production channel and is universal in nature. This universality ensures that \( \Phi_d^c \) is only sensitive to the initial legs and is blind to the hard process under study. In this letter we find that the K+G equation admits solution that can account for next-to-soft contributions as well:

\[
\Phi_d^{c} = \sum_{i = 1}^{\infty} \hat{a}_i \left( \frac{q^2 \ln \tau_2}{\mu^2} \right) \frac{i^n}{n!} \left( \frac{\epsilon}{4 \pi \tau_2} \right) \exp \left( \frac{i e}{4 \pi \tau_2} \Phi_d^{c,i}(\epsilon) \right) + \frac{i e}{4 \pi \tau_2} \Phi_d^{c}(\tau_2, \epsilon) + \frac{i e}{4 \pi \tau_2} \Phi_d^{c}(\tau_1, \epsilon), \tag{3}
\]

where \( \delta = \exp \left( \frac{i e}{2} \gamma_E \ln(4\pi) \right) \) with \( \gamma_E \) being the Euler-Mascheroni constant. The first term within the parenthesis accounts for the soft contributions and remaining two terms correspond to next-to-soft contributions. The soft part of the solution was proposed along with the predictions for Higgs production and DY in [23] till third order, without \( \delta(\tau_1) \delta(\tau_2) \) terms. Later on [20, 25] gives the complete result for SV. Through mass factorisation the divergent part of the NSV solution cancels against the collinear singularities from AP kernels and the finite part contributes to dCFs. The coefficients \( \varphi_d^{c,i} \) depend on \( \tau_1 \) and \( \epsilon \) in such a way that the NSV part is RG invariant provided we sum the series to all orders. In addition, we find that the logarithmic structure of \( \Phi_d^c \) and consequently their predictions remain unaltered under the simultaneous transformation of the exponent in first parenthesis and the \( z_l \)-dependence in \( \varphi_d^{c,i}(\tau_l, \epsilon) \). The AP kernels satisfy,

\[
\mu_F^2 \frac{d}{d\mu_F^2} \ln \Gamma_{cc}(\mu_F^2, \tau_1) = \frac{1}{2} P^c(a_s(\mu_F^2), \tau_1) + \delta P^c, \tag{4}
\]

where

\[
P^c(a_s, \tau_1) = 2 \left( A^c(a_s) + B^c(a_s) \delta(\tau_1) + L^c(a_s, \tau_1) \right), \tag{5}
\]

with \( A^c \) and \( B^c \) being the cusp and collinear anomalous dimensions, \( L^c(a_s, \tau_1) \equiv C^c(a_s) \ln(\tau_1) + D^c(a_s) \) and the \( \delta P^c \) denote NSV and beyond the NSV terms respectively. We drop \( \delta P^c \) throughout. The NSV improved solution \( \Phi_d^c \) results in an integral representation of the finite function \( \Psi_d^c \) which embeds the all order information of the mass-factorised differential distribution.

\[
\Psi_d^c = \delta(\tau_1) \left( \frac{q^2 \ln \tau_2}{\mu^2} \frac{d}{dq^2} P^c(a_s(q_1^2), \tau_2) + Q_{d}^{c}(a_s(q_2^2), \tau_2) + \frac{1}{4} \left( \frac{1}{\tau_1} \right)^2 \left( P^c(a_s(q_1^2), \tau_2) + 2 L^c(a_s(q_1^2), \tau_2) \right) \right) + \frac{1}{2} \delta(\tau_1) \delta(\tau_2) \ln \left( q_{d,a}^{c}(a_s(\mu_F^2)) \right) + \tau_1 \leftrightarrow \tau_2, \tag{6}
\]

where \( P^c(a_s, \tau_1) = P^c(a_s, \tau_1) - 2 B^c(a_s) \delta(\tau_1), q_1^2 = q^2 (1 - z_1) \) and \( q_2^2 = q^2 z_1 \tau_2 \). The subscript + indicates standard
plus distribution. The function $\mathcal{Q}_d^f$ in (6) is given as
\[
\mathcal{Q}_d^f(a_s, \tau_l) = \frac{2}{\tau_l} D_d^f(a_s) + 2\mathcal{P}_d^f(a_s, \tau_l). \tag{7}
\]
The splitting function $P^c$ and the SV coefficient $D_d^c$ are known to third order [26] in QCD. Here $\mathcal{P}_d^f$ constitutes the finite part of $\phi_{d,c}^{(i)}$ in (3) and is parametrized in the following way,
\[
\phi_{d,c}^{f}\left(a_s(\lambda^2), \tau_l\right) = \sum_{i=1}^{\infty} \sum_{k=0}^{\infty} \hat{a}_s \left(\lambda^2\right)^{i \hat{2}} S_{\epsilon}^{i,k}(\tau_l) \ln^k \tau_l,
\]
\[
= \sum_{i=1}^{\infty} \sum_{k=0}^{\infty} a_i^c(\lambda^2) \phi_{d,c}^{(i,k)} \ln^k \tau_l. \tag{8}
\]

The upper limit on the sum over $k$ is controlled by the dimensionally regularised Feynman integrals that contribute to order $a_i^c$. The constant $g_{d,0}^c$ in (6) results from finite part of the virtual contributions and pure $\delta(\tau_l)$ terms of $\Phi^{c}_d$. The exponent $\Psi^{c}_d$ that captures both SV as well as NSV terms to all orders in perturbation theory is one of the main results of this letter.

**Matching with the Inclusive.**— The SV function $\phi_{d,c}^f$ can be determined at every order in perturbation theory using fixed order predictions of $\Delta_{d,c}$. Alternatively, we can determine $\phi_{d,c}^f$ from corresponding inclusive cross sections using the relation [23]:
\[
\int_0^1 dx_1 \int_0^1 dx_2 \frac{x_1^0 x_2^0}{(x_1 x_2)^{N-1}} \frac{d\sigma^c}{dy} = \int_0^1 d\tau \Gamma^{N-1} \sigma^c, \tag{9}
\]
where $\sigma^c$ is the inclusive cross section. This relation in the large $N$ limit gives
\[
\sum_{i=1}^{\infty} \hat{a}_s^i \left(\frac{g_s^c}{\mu^2}\right)^{i \hat{2}} S_{\epsilon}^{i,k} \left[ t_i^1(\epsilon) \phi_{d,c}^{(i,k)}(\epsilon) - t_i^2(\epsilon) \phi_{d,c}^{(i)}(\epsilon) \right] + \sum_{k=0}^{\infty} \left[ t_{3,k}^i(\epsilon) \phi_{d,c}^{(i,k)}(\epsilon) - t_{4,k}^i(\epsilon) \phi_{d,c}^{(i)}(\epsilon) \right] = 0. \tag{10}
\]

Here we keep $\ln^k N$ as well as $O(1/N)$ terms for the determination of the SV and NSV coefficients. The constants $\phi_{d,c}^{(i)}$ and $\phi_{d,c}^{(i,k)}$ are the inclusive counterparts to the SV and NSV coefficients respectively which are known to third order in QCD for DY ($c = g$), for Higgs production in gluon fusion ($c = g$) and in bottom quark annihilation ($c = b$) (for NSV see [50]). The coefficients are
\[
t_i^1 = \frac{i e(2 - i \epsilon)}{4 N \epsilon} \Gamma^2 \left(1 + i \frac{\epsilon}{2}\right), \quad t_i^2 = \frac{i e(1 - i \epsilon)}{2 N \epsilon} \Gamma(1 + i \epsilon),
\]
\[
t_{3,k}^i = \Gamma \left(1 + i \frac{\epsilon}{2}\right) \frac{\partial^k}{\partial \alpha^k} \left(\frac{\Gamma(1 + \alpha)}{\Gamma(\alpha + i \epsilon / 2)}\right)_{\alpha = i \epsilon},
\]
\[
t_{4}^{i,k} = \frac{\partial^k}{\partial \alpha^k} \left(\frac{\Gamma(1 + \alpha)}{\Gamma(\alpha + i \epsilon / 2)}\right)_{\alpha = i \epsilon}. \tag{11}
\]

**All order prediction.**— In [20, 23, 25], we studied the predictive power of SV part of $\Psi^{c}_d$ to dCFs to all orders using lower order results. Here, in particular, we predict NSV terms of the form $\delta(\tau_l) \ln^k \tau_l$, $n + 1 \leq k \leq 2n - 1$ and $D_i(z_i) \ln^k \tau_l$ for $i, k = 0, 1, \cdots, n; i + k < 2n - 1$ at every order $a_i^n$ provided $\Psi^{c}_d$ is known to order $a_i^{n-1}$. From $\Psi^{c}_d$, $c = q, b, g$ determined from second order inclusive results [50], we obtain for the first time the results for the third order NSV contributions to dCFs, $\Delta_{d,c}$, for $c = q, b$ and also for $c = g$ [52]. Further, the knowledge of third order results [50] for inclusive reactions and using (10) we have determined the NSV coefficients $\phi_{d,c}^{(i,k)}$ and dCFs to third order. They will be presented towards the end in concise form.

**Resummation.**— Near the hadronic threshold region, $z_l \to 1$, the PDFs often become large (due to their small momentum fractions) which allows the threshold contributions from CFs to dominate at every order in $a_s$. Hence, truncated perturbative predictions become unreliable. In Mellin space, these dominant ones show up as order one terms of the form $a_s \beta_0 \ln N_1 N_2$ in the large $N_1$ region at every order. Thanks to all order integral representation for $\Psi^{c}_d$ in (6) and RG equation of $a_s$, we can resum these terms to all orders. Defining double Mellin moment of any arbitrary function $F(z_1, z_2)$ by $F_N = \int_0^1 dz_1 z_1^{N-1} \int_0^1 dz_2 z_2^{N-1} F(z_1, z_2)$, we obtain $\Delta^{c}_{d,N} = \tilde{g}_{d,0} \exp(\Psi^{c}_{d,N})$, which can be expanded in terms of $a_s$: $\Delta^{c}_{d,N} = \sum_{i=0}^{\infty} a_s^i (\mu^2) \Delta^{c,(i)}_{d,N}$. The resummed result for $\Psi^{c}_{d,N}$ takes the following form:
\[
\Psi^{c}_{d,N} = \left( g_{d,0}^i (\omega) + \frac{1}{N_1} \overline{g}_{d,i,1}(\omega) \right) \ln N_1
\]
\[
+ \sum_{i=0}^{\infty} a_s^i \begin{array}{l} \frac{1}{2} \overline{g}_{d,i,2}(\omega) + \frac{1}{N_1} \overline{g}_{d,i,2}(\omega) \\
+ \frac{1}{N_1} \sum_{i=0}^{\infty} a_s^i h_{d,i}^c(\omega, N_1) + (N_1 \leftrightarrow N_2), \end{array} \tag{12}
\]
where
\[
h_{d,0}^c(\omega, N_1) = h_{d,00}^c(\omega) + h_{d,01}^c(\omega) \ln N_1,
\]
\[
h_{d,i}^c(\omega, N_1) = \sum_{k=0}^{\infty} h_{d,i,k}^c(\omega) \ln^k N_1, \tag{13}
\]
where $\omega = a_s^c \beta_0 \ln N_1 N_2$. The SV resummation coefficients, which comprises of $\tilde{g}_{d,0}^c$ and $\overline{g}_{d,1}^c$, are greatly discussed in [26, 53, 54] and so from here onwards we focus on the NSV resummation coefficients namely $\overline{g}_{d,i}$ and $\overline{g}_{d,i}$. In $\tilde{N}$ space, the use of resummed $a_s$ allows us to organise the series in such a way that $\omega$ is treated as order one at every order in $a_s$. The coefficient $\overline{g}_{d,i}$ is found to be zero. The coefficients $\overline{g}_{d,i+2}$ are controlled by the universal cusp anomalous dimension $A^c$, while $h_{d,i}^c$ by the NSV coefficients $\phi_{d,c}$ as well as by $C^c, D^c$ from $P^c(a_s, \tau_l)$. 
The resummation coefficients \( \tilde{g}_d^{(i)}, g_d^{(i)}(\omega), \tilde{\sigma}_d^{(i)}(\omega) \) and \( h_d^{(i)}(\omega) \) encode the entire all order information in a systematic fashion through leading, next-to-leading, \( \cdots \), SV and NSV logarithms present in the \( \Psi_d^{(i)} \). For instance, the knowledge of second order resummation coefficients, \( \{ \tilde{g}_{d,0}, g_{d,1}, \tilde{g}_{d,2}, g_{d,2}, h_{d,2} \}, \) is sufficient to predict the \( \ln^{(2)} N_{c} \) of \( \Delta_{d,N}^{c(i)} \) for \( i > 2 \) to all orders. We present Table II towards the end which demonstrate this feature for \( (\ln^2 N_{f}/N_{f}) \) terms. In summary, we study the all order logarithmic structure of the NSV terms in \( \tilde{N} \) space and the resummation coefficients till 4-loop are provided in the Supplementary Material [55].

\[ \Delta_{d,q}^{NSV,(3)} = C_{F}^{2} \left\{ L_{1}^{2} \left[ (1632 + 32 \zeta_{4}) - 160 \zeta_{3} - 160 \zeta_{0} \right] + L_{1}^{2} \left[ \zeta_{3} \right] + \left( 32 \zeta_{4} + 160 \zeta_{3} + 160 \zeta_{0} \right) \right\} + \Delta_{d,q}^{NSV,(3)} \]

**Results.** We present the third order NSV results for dCFs, \( \Delta_{d,c} \), with \( e = q, b \), corresponding to DY process and for bottom quark induced Higgs production after expanding them as \( \Delta_{d,c} = \sum_{i=0}^{\infty} \nu_{d}^{(i)}(\Delta_{d,c}^{SV,(i)} + \Delta_{d,c}^{NSV,(i)} + \cdots) \). We have set \( \mu^{2} = \mu_{F}^{2} = q^{2} \) and express the results in terms of \( SU(N_{c}) \) Casimirs, namely \( C_{F} = (N_{c}^{2} - 1)/2N_{c} \) and \( C_{A} = N_{c} \) and \( n_{f} \), the number of active quark flavours.
Here, \( L_{z_1} = \ln(\bar{\tau}_1) \), \( \delta(\bar{\tau}_2) = \delta(\bar{\tau}_2) \), \( \bar{\tau}_j = \left( \frac{\ln(\bar{\tau}_j)}{\ln(\bar{\tau}_2)} \right) \) and 
\( \zeta_2 = 1.6449 \cdots \) and \( \zeta_3 = 1.20205 \cdots \). Complete third order results for the Higgs production in gluon fusion are already known [52, 56], however we can not confirm our results, which is given in [55], with them as they are not publicly available. For the DY, we have found that our third order prediction is in complete agreement with the [56] for terms of the type \( D_i(z_l) \ln^j(\bar{\tau}_m) \), \( i, j \geq 0, l, m = 1, 2 \). The remaining \( \delta(\bar{\tau}_m) \ln^j(\bar{\tau}_m) \) terms in DY and the complete NSV predictions for Higgs production in bottom quark annihilation channel at third order are new. Using results up to third order, we can predict three highest NSV logarithms to all order. Here, the results at fourth order for \( \ln^j(\bar{\tau}_m) \), \( j = 7, 6, 5 \) are presented.

This way, we can predict most of the leading NSV terms to all orders in \( a_s \). In fact, the resummation in \( \bar{N} \) space organises SV and NSV threshold logarithms to all orders and the resulting resummation coefficients are controlled by anomalous dimensions as well as \( \varphi^f_{d,c} \), known to a specific order. The knowledge of these coefficients to specific order in \( a_s \) is sufficient to predict the infinite tower of SV and NSV logarithms to a specific accuracy. We summarise our findings in Table I. The results for dCFs and NSV contributions are provided in Supplemental Material [55].

**Table I:** The all order predictions for NSV logarithms in \( \Delta_{c,(2)}^{d,N} \) for a given set of resummation coefficients

| GIVEN | PREDICTIONS |
|-------|-------------|
| Resummation Coefficients | \( \Delta_{c,(2)}^{d,N} \) | \( \Delta_{c,(3)}^{d,N} \) | \( \Delta_{c,(4)}^{d,N} \) |
| \( \tilde{g}_{d_0,0}, \tilde{g}^{c}_{d_1,1}, \tilde{g}^{c}_{d_2,2}, \bar{g}^{c}_{d_0,1}, \tilde{g}^{c}_{d_3,3}, \bar{g}^{c}_{d_0,2} \) | \( \ln^3 N_i \) | \( \ln^3 N_i \) | \( \ln^{(2k-3)} N_i \) |
| \( \tilde{g}^{c}_{d_0,n-1}, \tilde{g}^{c}_{d_0,n+1}, \tilde{g}^{c}_{d_n,0} \) | \( \ln^{4 N_i} \) | \( \ln^{(2k-2)} N_i \) | \( \ln^{(2k-n)} N_i \) |

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