Generalized Eilenberg Theorem I:
Local Varieties of Languages

Jiří Adámek\textsuperscript{1}, Stefan Milius\textsuperscript{2}, Robert S. R. Myers\textsuperscript{1} and Henning Urbat\textsuperscript{1}

\textsuperscript{1} Institut für Theoretische Informatik
Technische Universität Braunschweig, Germany
\textsuperscript{2} Lehrstuhl für Theoretische Informatik
Friedrich-Alexander-Universität Erlangen-Nürnberg, Germany

Dedicated to Manuela Sobral.

Abstract. We investigate the duality between algebraic and coalgebraic recognition of languages to derive a generalization of the local version of Eilenberg’s theorem. This theorem states that the lattice of all boolean algebras of regular languages over an alphabet $\Sigma$ closed under derivatives is isomorphic to the lattice of all pseudovarieties of $\Sigma$-generated monoids. By applying our method to different categories, we obtain three related results: one, due to Gehrke, Grigorieff and Pin, weakens boolean algebras to distributive lattices, one weakens them to join-semilattices, and the last one considers vector spaces over $\mathbb{Z}_2$.

1 Introduction

Regular languages are precisely the behaviours of finite automata. A machine-independent characterization of regularity is the starting point of algebraic automata theory (see e.g. \cite{16}): one defines recognition via preimages of monoid morphisms $f : \Sigma^* \to M$, where $M$ is a finite monoid, and every regular language is recognized in this way by its syntactic monoid. This suggests to investigate how operations on regular languages relate to operations on monoids. Recall that a pseudovariety of monoids is a class of finite monoids closed under finite products, submonoids and quotients (homomorphic images), and a variety of regular languages is a class of regular languages closed under the boolean operations (union, intersection and complement), left and right derivatives\footnote{The left and right derivatives of a language $L \subseteq \Sigma^*$ are $w^{-1}L = \{ u \in \Sigma^* : uw \in L \}$ and $Lw^{-1} = \{ u \in \Sigma^* : uw \in L \}$ for $w \in \Sigma^*$, respectively.} and preimages of monoid morphisms $\Sigma^* \to \Gamma^*$. Eilenberg’s variety theorem \cite{9}, a cornerstone of automata theory, establishes a lattice isomorphism

$$\text{varieties of regular languages} \cong \text{pseudovarieties of monoids}.$$ 

Numerous variations of this correspondence are known, e.g. weakening the closure properties in the definition of a variety \cite{15}\cite{18}, or replacing regular languages by formal power series \cite{20}. Recently Gehrke, Grigorieff and Pin \cite{11}\cite{12} proved a “local” version of Eilenberg’s theorem: for every fixed alphabet $\Sigma$, there is a lattice isomorphism between \textit{local varieties of regular languages} (sets of regular languages over $\Sigma$
closed under boolean operations and derivatives) and local pseudovarieties of monoids (sets of \( \Sigma \)-generated finite monoids closed under quotients and subdirect products). At the heart of this result lies the use of Stone duality to relate the boolean algebra of regular languages over \( \Sigma \), equipped with left and right derivatives, to the free \( \Sigma \)-generated profinite monoid.

In this paper we generalize the local Eilenberg theorem to the level of an abstract duality. Our approach starts with the observation that all concepts involved in this theorem are inherently categorical:

1. The boolean algebra \( \text{Reg}_\Sigma \) of all regular languages over \( \Sigma \) naturally carries the structure of a deterministic automaton whose transitions \( L \xrightarrow{a} a^{-1}L \) for \( a \in \Sigma \) are given by left derivation and whose final states are the languages containing the empty word. In other words, \( \text{Reg}_\Sigma \) is a coalgebra for the functor \( T_\Sigma : Q \rightarrow 2 \times Q^\Sigma \) on the category of boolean algebras, where \( 2 = \{0,1\} \) is the two-chain. The coalgebra \( \text{Reg}_\Sigma \) can be captured abstractly as the rational fixpoint \( \nu T_\Sigma \) of \( T_\Sigma \), i.e., the terminal locally finite \( T_\Sigma \)-coalgebra [14].
2. Monoids are precisely the monoid objects in the category of sets, viewed as a monoidal category w.r.t. the cartesian product.
3. The categories of boolean algebras and sets occurring in (1) and (2) are locally finite varieties of algebras (that is, their finitely generated algebras are finite), and the full subcategories of finite boolean algebras and finite sets are dually equivalent via Stone duality.

Inspired by (3), we call two locally finite varieties \( C \) and \( D \) of (possibly ordered) algebras predual if the respective full subcategories \( C_f \) and \( D_f \) of finite algebras are dually equivalent. Our aim is to prove a local Eilenberg theorem for an abstract pair of predual varieties \( C \) and \( D \), the classical case being covered by taking \( C = \text{boolean algebras} \) and \( D = \text{sets} \). In this setting deterministic automata are modeled both as coalgebras for the functor

\[
T_\Sigma : C \rightarrow C, \quad T_\Sigma Q = 2 \times Q^\Sigma,
\]

and as algebras for the functor

\[
L_\Sigma : D \rightarrow D, \quad L_\Sigma A = 1 + \bigcup_{\Sigma} A,
\]

where \( 2 \) is a two-element algebra in \( C \) and \( 1 \) is its dual finite algebra in \( D \). These functors are predual in the sense that their restrictions \( T_\Sigma : C_f \rightarrow C_f \) and \( L_\Sigma : D_f \rightarrow D_f \) to finite algebras are dual, and therefore the categories of finite \( T_\Sigma \)-coalgebras and finite \( L_\Sigma \)-algebras are dually equivalent. As a first approximation to the local Eilenberg theorem, we consider the rational fixpoint \( \nu T_\Sigma \) for \( T_\Sigma \) – which is always carried by the automaton \( \text{Reg}_\Sigma \) of regular languages – and the initial algebra \( \mu L_\Sigma \) for \( L_\Sigma \) and establish a lattice isomorphism

\[
\text{subcoalgebras of } \nu T_\Sigma \cong \text{ideal completion of the poset of finite quotient algebras of } \mu L_\Sigma.
\]

This is “almost” the desired general local Eilenberg theorem. For the classical case (\( C = \text{boolean algebras} \) and \( D = \text{sets} \)) one has \( \nu T_\Sigma = \text{Reg}_\Sigma \) and \( \mu L_\Sigma = \Sigma^* \), and the above
isomorphism states that the boolean subalgebras of $\text{Reg}_\Sigma$ closed under left derivatives correspond to sets of finite quotient algebras of $\Sigma^*$ closed under quotients and subdirect products. What is missing is the closure under right derivatives on the coalgebra side, and quotient algebras of $\Sigma^*$ which are monoids on the algebra side.

The final step is to prove that the above isomorphism restricts to one between local varieties of regular languages in $\mathcal{C}$ and local pseudovarieties of $\mathcal{D}$-monoids. Here a $\mathcal{D}$-monoid $M$ in the monoidal category $(\mathcal{D}, \otimes, \Psi)$ is an algebra $A$ in $\mathcal{D}$ equipped with a “bilinear” monoid multiplication $A \times A \to A$, which means that the maps $a \circ -$ and $- \circ a$ are $\mathcal{D}$-morphisms for all $a \in A$. For example, $\mathcal{D}$-monoids in $\mathcal{D} = \text{sets}$, posets, join-semilattices and vector spaces over $\mathbb{Z}_2$ are monoids, ordered monoids, idempotent semirings and $\mathbb{Z}_2$-algebras (in the sense of algebras over a field), respectively. In all these examples the monoidal category $(\mathcal{D}, \otimes, \Psi)$ is closed: the set $\mathcal{D}(A, B)$ of homomorphisms between two algebras $A$ and $B$ is an algebra in $\mathcal{D}$ with the pointwise algebraic structure. Our main result is the

**General Local Eilenberg Theorem.** Let $\mathcal{C}$ and $\mathcal{D}$ be predual locally finite varieties of algebras, where the algebras in $\mathcal{D}$ are possibly ordered. Suppose further that $\mathcal{D}$ is monoidal closed w.r.t. tensor product, epimorphisms in $\mathcal{D}$ are surjective, and the free algebra in $\mathcal{D}$ on one generator is dual to a two-element algebra in $\mathcal{C}$. Then there is a lattice isomorphism

$$\text{local varieties of regular languages in } \mathcal{C} \cong \text{local pseudovarieties of } \mathcal{D}\text{-monoids}.$$
Theorem. Under the assumptions of the General Local Eilenberg Theorem, the free profinite \( D \)-monoid on \( \Sigma \) is dual to the rational fixpoint \( \rho T \).

This extends the corresponding results of Gehrke, Grigorieff and Pin [11] for \( D = \text{sets} \) and \( D = \text{posets} \).

The present paper is a revised and extended version of the conference paper [2], providing full proofs and more detailed examples. In comparison to loc. cit. we work with a slightly modified categorical framework in order to simplify the presentation, see Section 6.

Related work. Our paper is inspired by the work of Gehrke, Grigorieff and Pin [11, 12] who showed that the algebraic operation of the free profinite monoid on \( \Sigma \) dualizes to the derivative operations on the boolean algebra of regular languages (and similarly for the free ordered profinite monoid on \( \Sigma \)). Previously, the duality between the boolean algebra of regular languages and the Stone space of profinite words appeared (implicitly) in work by Almeida [5] and was formulated by Pippenger [17] in terms of Stone duality.

A categorical approach to the duality theory of regular languages has been suggested by Rhodes and Steinberg [21]. They introduce the notion of a boolean bialgebra, and prove the equivalence of bialgebras and profinite semigroups. The precise connection to their work is yet to be investigated.

Another related work is Polák [18] and Reutenauer [20]. They consider what we treat as the example of join-semilattices and vector spaces, respectively, and obtain a (non-local) Eilenberg theorem for these cases. To the best of our knowledge the local version we prove does not follow from the global version, and so we believe that our result is new.

The origin of all the above work is, of course, Eilenberg’s theorem [9]. Later Reiterman [19] proved another characterization of pseudovarieties of monoids in the spirit of Birkhoff’s classical variety theorem. Reiterman’s theorem states that any pseudovariety of monoids can be characterized by profinite equations (i.e., pairs of elements of a free profinite monoid). We do not treat profinite equations in the present paper.

2 The Rational Fixpoint

The aim of this section is to recall the rational fixpoint of a functor, which provides an abstract coalgebraic view of the set of regular languages. As a prerequisite, we need a categorical notion of “finite automaton”, and so we will work with categories where enough “finite” objects exist – viz. locally finitely presentable categories [4].

Definition 2.1. (a) An object \( X \) of a category \( C \) is finitely presentable if the hom-functor \( C(X, -) : C \rightarrow \text{Set} \) is finitary (i.e., preserves filtered colimits). Let \( C_f \) denote the full subcategory of all finitely presentable objects of \( C \).

(b) \( C \) is locally finitely presentable if it is cocomplete, \( C_f \) is small up to isomorphism and every object of \( C \) is a filtered colimit of finitely presentable objects.
Example 2.2. Let $\Gamma$ be a finitary signature, that is, a set of operation symbols with finite arity.

(1) Denote by $\text{Alg} \, \Gamma$ the category of $\Gamma$-algebras and $\Gamma$-homomorphisms. A variety of algebras is a full subcategory of $\text{Alg} \, \Gamma$ closed under products, subalgebras (represented by injective homomorphisms) and homomorphic images (represented by surjective homomorphisms). Equivalently, by Birkhoff’s theorem [7], varieties of algebras are precisely the classes of algebras definable by equations of the form $s = t$, where $s$ and $t$ are $\Gamma$-terms. Every variety of algebras is locally finitely presentable [4, Corollary 3.7].

(2) Similarly, let $\text{Alg}_0 \, \Gamma$ be the category of ordered $\Gamma$-algebras. Its objects are $\Gamma$-algebras carrying a poset structure for which every $\Gamma$-operation is monotone, and its morphisms are monotone $\Gamma$-homomorphisms. A variety of ordered algebras is a full subcategory of $\text{Alg}_0 \, \Gamma$ closed under products, subalgebras and homomorphic images. Here subalgebras are represented by embeddings (injective $\Gamma$-homomorphisms that are both monotone and order-reflecting), and homomorphic images are represented by surjective $\Sigma$-homomorphisms that are monotone but not necessarily order-reflecting.

Bloom [8] proved an ordered analogue of Birkhoff’s theorem: varieties of ordered algebras are precisely the classes of ordered algebras definable by inequalities $s \leq t$ between $\Gamma$-terms. From this it is easy to see that every variety of ordered algebras is finitary monadic over the locally finitely presentable category of posets, and hence locally finitely presentable [4, Theorem and Remark 2.78].

In our applications we will work with the varieties in the table below. Observe that all these varieties are locally finite, that is, their finitely presentable objects are precisely the finite algebras.

| $\mathcal{C}$ | objects                  | morphisms                      |
|--------------|--------------------------|--------------------------------|
| Set          | sets                     | functions                      |
| BA           | boolean algebras         | boolean morphisms              |
| DL$_{01}$    | distributive lattices with 0 and 1 | lattice morphisms preserving 0 and 1 |
| JS$_{0}$     | join-semilattices with 0 | semilattice morphisms preserving 0 |
| Vect $\mathbb{Z}_2$ | vector spaces over the field $\mathbb{Z}_2$ | linear maps                   |
| Pos          | partially ordered sets   | monotone functions             |

Remark 2.3. For the rest of this paper the term “variety” refers to both varieties of algebras and varieties of ordered algebras.

Notation 2.4. Fix a locally finitely presentable category $\mathcal{C}$ and a finitary endofunctor $T : \mathcal{C} \to \mathcal{C}$.

Definition 2.5. A $T$-coalgebra is a pair $(Q, \gamma)$ of a $\mathcal{C}$-object $Q$ and a $\mathcal{C}$-morphism $\gamma : Q \to TQ$. A homomorphism

\[ h : (Q, \gamma) \to (Q', \gamma') \]
of $T$-coalgebras is a $C$-morphism $h : Q \to Q'$ with $\gamma' \cdot h = Th \cdot \gamma$. We denote by $\text{Coalg}_T$ the category of all $T$-coalgebras and their homomorphisms, and by $\text{Coalg}_{\text{fp}} T$ the full subcategory of $T$-coalgebras $(Q, \gamma)$ with finitely presentable carrier $Q$ (in the case where $C$ is a locally finite variety, these are precisely the finite coalgebras).

**Example 2.6.** Given a finite alphabet $\Sigma$ and an object $2$ in $C$, the endofunctor

$$T_\Sigma = 2 \times \text{Id}^2 = 2 \times \text{Id} \times \text{Id} \times \ldots \times \text{Id}$$

of $C$ is finitary since in any locally finitely presentable category filtered colimits commute with finite products. If $C$ is a locally finite variety and $2$ is a two-element algebra in $C$, then $T_\Sigma$-coalgebras are deterministic automata, see e.g. [22]. Indeed, by the universal property of the product, to give a coalgebra $Q \xrightarrow{\gamma} T_\Sigma Q = 2 \times Q^2$ means precisely to give an algebra $Q$ (of states), morphisms $\gamma_a : Q \to Q$ for every $a \in \Sigma$ (representing $a$-transitions) and a morphism $f : Q \to 2$ (representing final states). Here are two special cases:

(a) The usual concept of a deterministic automaton (without initial states) is captured as a coalgebra for $T_\Sigma$ where $C = \text{Set}$ and $2 = \{0, 1\}$. An important example of a $T_\Sigma$-coalgebra is the automaton $\text{Reg}_\Sigma$ of regular languages. Its states are the regular languages over $\Sigma$, its transitions are

$$\gamma_a(L) = a^{-1}L \quad \text{for all } L \in \text{Reg}_\Sigma \text{ and } a \in \Sigma,$$

and the final states are precisely the languages containing the empty word $\epsilon$.

(b) Analogously, consider $T_\Sigma$ as an endofunctor of $C = \text{BA}$ with $2 = \{0, 1\}$ (the two-element boolean algebra). A coalgebra for $T_\Sigma$ is a deterministic automaton with a boolean algebra structure on the state set $Q$. Moreover, the transition maps $\gamma_a : Q \to Q$ are boolean homomorphisms, and the final states (given by the inverse image of 1 under $f : Q \to 2$) form an ultrafilter. The above automaton $\text{Reg}_\Sigma$ is also a $T_\Sigma$-coalgebra in $\text{BA}$: the set of regular languages is a boolean algebra w.r.t. the usual set-theoretic operations, left derivatives preserve these operations, and the languages containing $\epsilon$ form a principal ultrafilter.

**Definition 2.7.** (a) A coalgebra is called **locally finitely presentable** if it is a filtered colimit of coalgebras with finitely presentable carrier. The full subcategory of $\text{Coalg}_T$ of all locally finitely presentable coalgebras is denoted $\text{Coalg}_{\text{fp}} T$.

(b) The **rational fixpoint** of $T$ is the filtered colimit

$$r : \varnothing T \to T(\varnothing T)$$

of all coalgebras with finitely presentable carrier, i.e., the colimit of the diagram $\text{Coalg}_{\text{fp}} T \to \text{Coalg}_T$.

The term “rational fixpoint” is justified by item (a) in the theorem below.

**Theorem 2.8** ([14]).

(a) $r$ is an isomorphism.
(b) \( \rho T \) is the terminal locally finitely presentable \( T \)-coalgebra, i.e., every locally finitely presentable \( T \)-coalgebra has a unique coalgebra homomorphism into \( \rho T \).

**Example 2.9.** The rational fixpoint of \( T_\Sigma : \text{Set} \to \text{Set} \) is the automaton \( \rho T_\Sigma = \text{Reg}_\Sigma \) of Example 2.6(a), see [3]. For any locally finitely presentable \( T_\Sigma \)-coalgebra \( (Q, \gamma) \), the unique homomorphism \( (Q, \gamma) \to \rho T_\Sigma \) maps each state \( q \in Q \) to its accepted language

\[
L_q = \{ a_1 \ldots a_n \in \Sigma^* : q \xrightarrow{a_1} q_1 \xrightarrow{a_2} q_2 \to \ldots \xrightarrow{a_n} q_n \text{ for some final state } q_n \}.
\]

This example can be generalized:

**Theorem 2.10.** Suppose that \( C \) is a locally finite variety and \( T \) lifts a finitary functor \( T_0 \) on \( \text{Set} \), that is, the following diagram (where \( U \) denotes the forgetful functor) commutes:

\[
\begin{array}{ccc}
C & \xrightarrow{T} & C \\
\downarrow U & & \downarrow U \\
\text{Set} & \xrightarrow{T_0} & \text{Set}
\end{array}
\]

Then the functor \( U : \text{Coalg} T \to \text{Coalg} T_0 \) given by

\[
Q \xrightarrow{\gamma} TQ \mapsto UQ \xrightarrow{U\gamma} UTQ = T_0UQ
\]

preserves the rational fixpoint, i.e.,

\[
U(\rho T) \cong \rho T_0.
\]

**Proof.** The functor \( U \) is finitary (since filtered colimits of \( T \)-coalgebras are formed on the level of \( C \) and hence on the level of \( \text{Set} \)) and restricts to finite coalgebras, so we have a commutative square

\[
\begin{array}{ccc}
\text{Coalg} T & \xrightarrow{U} & \text{Coalg} T_0 \\
\downarrow \iota & & \downarrow I_0 \\
\text{Coalg}_f T & \xrightarrow{\nu} & \text{Coalg}_f T_0
\end{array}
\]

where \( \iota \) and \( I_0 \) are the inclusion functors. We will prove below that \( \nu \) is cofinal, from which the claim follows:

\[
\begin{align*}
U(\rho T) &= U(\text{colim } I) & \text{def. } \rho T \\
&\cong \text{colim}(U I) & U \text{ finitary} \\
&\cong \text{colim}(I_0 \nu) & U I = I_0 \nu \text{ cofinal} \\
&\cong \text{colim}(I_0) & \nu \text{ cofinal} \\
&= \rho T_0 & \text{def. } \rho T_0
\end{align*}
\]

The cofinality of \( \nu \) amounts to proving that
(1) for every finite $T_0$-coalgebra $(Q, \gamma)$ there exists a $T_0$-coalgebra homomorphism $(Q, \gamma) \to \forall(Q', \gamma')$ for some finite $T$-coalgebra $(Q', \gamma')$, and

(2) any two such coalgebra homomorphisms are connected by a zig-zag.

Proof of (1). Let $\Phi : \text{Set} \to \mathcal{C}$ be the left adjoint of the forgetful functor $U : \mathcal{C} \to \text{Set}$, and denote the unit and counit of the adjunction by $\eta$ and $\varepsilon$, respectively. Given a finite $T_0$-coalgebra $(Q, \gamma)$ form the “free” $T$-coalgebra $\Phi Q \Phi \gamma \to \Phi T_0 Q \Phi T_0 \eta Q \cdot \Phi \gamma$.

Note that $\Phi Q$ is finite because $\mathcal{C}$ is locally finite. Then $\eta_Q : (Q, \gamma) \to V(\Phi Q, \varepsilon T \Phi Q \cdot \Phi T_0 \eta Q \cdot \Phi \gamma)$ is a coalgebra homomorphism. Indeed, the diagram below commutes by the naturality of $\eta$ and the triangle identity $U \varepsilon \cdot \eta_U = \text{id}$:

Proof of (2). Given any coalgebra homomorphism $h : (Q, \gamma) \to \forall(Q', \gamma')$ there exists a unique $D$-morphism $\overline{h} : \Phi Q \to Q'$ with $U \overline{h} \cdot \eta_Q = h$ by the universal property of $\eta$. We claim that $\overline{h}$ is a coalgebra homomorphism

Indeed, the lower square in the diagram below commutes when precomposed with $\eta_Q$, from which the equation $\gamma' \cdot \overline{h} = T \overline{h} \circ \varepsilon T \Phi Q \cdot \Phi T_0 \eta Q \cdot \Phi \gamma$ follows.

Now given two coalgebra homomorphisms $h : (Q, \gamma) \to \forall(Q', \gamma')$ and $k : (Q, \gamma) \to \forall(Q'', \gamma'')$, the desired zig-zag in $\text{Coalg}_T$ is $Q' \overline{k} \Phi Q \overline{h} Q''$. □
**Corollary 2.11.** Let $C$ be a locally finite variety with a two-element algebra $2$. Then the rational fixpoint of $T_\Sigma = 2 \times \text{Id}^\Sigma : C \to C$ is carried by the automaton $\text{Reg}_\Sigma$ of Example 2.6(a). For any locally finitely presentable $T_\Sigma$-coalgebra $(Q, \gamma)$, the unique homomorphism $(Q, \gamma) \to \rho T_\Sigma$ maps each state $q \in Q$ to its accepted language.

**Proof.** Apply Theorem 2.10 to $T = T_\Sigma$ and $T_0 = \{0, 1\} \times \text{Id}^\Sigma$. Since $\rho T_0 = \text{Reg}_\Sigma$ by Example 2.9, the claim follows. \qed

Next we will show that the locally finitely presentable $T$-coalgebras arise as a “free completion” of the coalgebras with finitely presentable carrier (Theorem 2.14 below).

**Remark 2.12.** (a) Recall that the free completion under filtered colimits of a small category $A$ is a full embedding $A \hookrightarrow \text{Ind}(A)$ such that $\text{Ind}(A)$ has filtered colimits and every functor $F : A \to B$ into a category $B$ with filtered colimits has a finitary extension $\overline{F} : \text{Ind}(A) \to B$, unique up to natural isomorphism:

\[
\begin{array}{ccc}
A & \xrightarrow{F} & \text{Ind}(A) \\
\downarrow & & \downarrow \overline{F} \\
B & & \\
\end{array}
\]

This determines $\text{Ind}(A)$ up to equivalence. If $A$ has finite colimits then $\text{Ind}(A)$ is locally finitely presentable and $(\text{Ind}(A))_f \cong A$. Conversely, every locally finitely presentable category $C$ arises in this way: $C \cong \text{Ind}(C_f)$.

(b) If $A$ is a join-semilattice then $\text{Ind}(A)$ is its ideal completion, see Remark 3.15.

**Lemma 2.13.** Let $B$ be a cocomplete category and $J : A \to B$ be a small full subcategory of finitely presentable objects closed under finite colimits. Then the unique finitary extension $J^* : \text{Ind}(A) \to B$ forms a full coreflective subcategory.

**Proof.** By the theorem in Section VI.1.8 of Johnstone [13], we know that $J^*$ is a full embedding so that $\text{Ind}(A)$ can be identified with the full subcategory of $B$ given by all filtered colimits of objects from $A$. We will show that this full subcategory is coreflective. Let $B$ be an object of $B$ and define $\overline{B}$ to be the colimit of the filtered diagram

\[
A/B \xrightarrow{\text{A}} A \xleftarrow{\text{B}} B,
\]

where the first arrow is the canonical projection functor and the second one the inclusion functor. We denote the corresponding colimit injections by

\[
in_f : A \to \overline{B} \quad \text{for every } f : A \to B \text{ in } A/B.
\]

Clearly, the objects in $A/B$ form a cocone on the above diagram and so we have a unique morphism $b : \overline{B} \to B$ such that

\[
b \cdot \text{in}_f = f \quad \text{for every } f : A \to B \text{ in } A/B.
\]

We will now prove this morphism $b$ to be couniversal. To this end, let $A$ be an object of $\text{Ind}(A)$, i.e.,

\[
A = \colim_{i \in I} A_i
\]
is a filtered colimit in $B$ of objects from $A$ with colimit injections $a_i : A_i \to A$, $i \in I$.

Given a morphism $f : A \to B$ in $B$, the morphism $f \cdot a_i : A_i \to B$ is an object of $A/B$, and for each connecting morphism $a_{i,j} : A_i \to A_j$ we have

$$(f \cdot a_j) \cdot a_{i,j} = f \cdot a_i.$$ 

Thus, $a_{i,j}$ is a morphism in $A/B$ and so

$$\text{in}_{f \cdot a_j} \cdot a_{i,j} = \text{in}_{f \cdot a_i},$$

i.e., the morphisms $\text{in}_{f \cdot a_i} : A_i \to \overline{B}$ form a cocone. So we get a unique $\overline{f} : A \to \overline{B}$ such that

$$\overline{f} \cdot a_i = \text{in}_{f \cdot a_i} \quad \text{for every } i \in I.$$ 

Now the following diagram commutes:

\[
\begin{array}{ccc}
\overline{B} & \xrightarrow{b} & B \\
\downarrow{\text{in}_{f \cdot a_i}} & & \downarrow{f} \\
A_i & \xrightarrow{a_i} & A
\end{array}
\]

Indeed, the outside and left-hand triangle commute, and so the right-hand one commutes when precomposed with every $a_i$, $i \in I$, whence that triangle commutes since the colimit injections $a_i$ form a jointly epimorphic family.

We still need to show that $\overline{f}$ is unique such that $b \cdot \overline{f} = f$. So assume that $\overline{f} : A \to \overline{B}$ is any such morphism. Fix $i \in I$. Then, since $\overline{B}$ is a filtered colimit and $A_i$ is finitely presentable, it follows that there exists some $g : A'_i \to B$ in $A/B$ and some morphism $\overline{f}' : A_i \to A'_i$ such that the square below commutes:

\[
\begin{array}{ccc}
A'_i & \xrightarrow{\text{in}_g} & \overline{B} \\
\downarrow{\overline{f}'} & & \downarrow{\overline{f}} \\
A_i & \xrightarrow{a_i} & A
\end{array}
\]

It follows that $\overline{f}'$ is a connecting morphism in $A/B$ from $f \cdot a_i$ to $g$:

$$g \cdot \overline{f}' = b \cdot \text{in}_g \cdot \overline{f}' = b \cdot \overline{f} \cdot a_i = f \cdot a_i.$$ 

Therefore we get $\text{in}_g \cdot \overline{f}' = \text{in}_{f \cdot a_i}$, so that

$$\overline{f} \cdot a_i = \text{in}_g \cdot \overline{f}' = \text{in}_{f \cdot a_i},$$

which determines $\overline{f}$ uniquely. This completes the proof. \qed

**Theorem 2.14.** $\text{Coalg}_{\text{fp}} T$ is the $\text{Ind}$-completion of $\text{Coalg}_f T$ and forms a coreflective subcategory of $\text{Coalg}_T$.
Proof. We apply the previous lemma to $A = \text{Coalg}_f T$ and $B = \text{Coalg} T$. Then $B$ is clearly cocomplete, and $A$ is closed under finite colimits (since colimits of coalgebras are constructed in the base category and finitely presentable objects are closed under finite colimits). Moreover, as shown in [1], every $T$-coalgebra with finitely presentable carrier is a finitely presentable object of $\text{Coalg} T$.

Hence Lemma 2.13 yields that $J^* : \text{Ind}(\text{Coalg}_f T) \to \text{Coalg} T$ is a full coreflective subcategory. The definition of $J^*$ is that it takes formal filtered diagrams of objects in $\text{Coalg}_f T$ and constructs their colimits. Therefore its image is precisely $\text{Coalg}_{lfp} T$, which gives the desired equivalence $\text{Coalg}_{lfp} T \cong \text{Ind}(\text{Coalg}_f T)$ and that $\text{Coalg}_{lfp} T$ is coreflective. □

3 Algebraic and Coalgebraic Recognition

We are ready to present our first take on the local Eilenberg theorem. At the heart of our approach lies the investigation of a duality for our categories of interest (e.g. Stone duality between finite boolean algebras and finite sets) and the induced algebra-coalgebra duality.

Definition 3.1. Two categories $C$ and $D$ are called predual if their full subcategories $C_f$ and $D_f$ of finitely presentable objects are dually equivalent, that is, $C_f \cong D_f^\text{op}$.

Example 3.2. The pairs of locally finite varieties listed in the table below are predual.

| $C$     | $D$   |
|---------|-------|
| BA      | Set   |
| $DL_{01}$ | Pos   |
| $JSL_0$ | $JSL_0$ |
| $\text{Vect} Z_2$ | $\text{Vect} Z_2$ |

In more detail:

(a) The categories $BA$ and $Set$ are predual via Stone duality. The equivalence $BA_f \xrightarrow{\cong} Set_f^\text{op}$ assigns to each finite boolean algebra the set of all atoms, and its associated equivalence $Set_f^\text{op} \xrightarrow{\cong} BA_f$ sends each finite set to the boolean algebra of all subsets.

(b) The categories $DL_{01}$ and $Pos$ are predual via Birkhoff duality. The equivalence $DL_{01,f} \xrightarrow{\cong} Pos_f^\text{op}$ assigns to each finite distributive lattice the subposet of all join-irreducible elements, and its associated equivalence $Pos_f^\text{op} \xrightarrow{\cong} DL_{01,f}$ sends each finite poset to the lattice of all down-closed subsets.

(c) The category $JSL_0$ is self-predual. The equivalence $JSL_{0,f} \xrightarrow{\cong} JSL_{0,f}^\text{op}$ sends each finite join-semilattice $X$ to its dual poset $X^\text{op}$.

(d) The category $\text{Vect} Z_2$ is self-predual. The equivalence $\text{Vect}_f Z_2 \xrightarrow{\cong} (\text{Vect}_f Z_2)^{\text{op}}$ sends each finite $Z_2$-vector space $X$ to the dual space $X^* = \text{hom}(X, Z_2)$ of all linear maps from $X$ to $Z_2$.

Definition 3.3. Let $C$ and $D$ be predual categories. Two functors $T : C \to C$ and $L : D \to D$ are called predual if they restrict to functors $T_f : C_f \to C_f$ and $L_f : D_f \to D_f$.
\( \mathcal{D}_f \) and these restrictions are dual, i.e., the following diagram commutes up to natural isomorphism:

\[
\begin{array}{ccc}
\mathcal{D}^{op} & \xrightarrow{L_f^{op}} & \mathcal{D}^{op} \\
\downarrow & & \downarrow \\
\mathcal{C}_f & \xrightarrow{T_f} & \mathcal{C}_f
\end{array}
\]

Assumptions 3.4. For the remainder of this paper we fix the following data:

(a) \( \mathcal{C} \) is a locally finite variety of algebras and \( \mathcal{D} \) is a locally finite variety of algebras or ordered algebras.
(b) \( \mathcal{C} \) and \( \mathcal{D} \) are predual with equivalence functors

\[
\begin{array}{l}
(\sim) : \mathcal{C}_f \xrightarrow{\sim} \mathcal{D}_f^{op} \\
(\sim) : \mathcal{D}_f^{op} \xrightarrow{\sim} \mathcal{C}_f
\end{array}
\]

(c) \( T : \mathcal{C} \to \mathcal{C} \) and \( L : \mathcal{D} \to \mathcal{D} \) are predual finitary functors. Moreover, \( T \) preserves monomorphisms and intersections and \( L \) preserves epimorphisms.

Example 3.5. The endofunctor \( T_{\Sigma} = 2 \times \text{Id}_{\Sigma} \) of \( \mathcal{C} \) (see Example 2.6) has the predual endofunctor

\[
L_{\Sigma} = 1 + \bigsqcup_{\Sigma} \text{Id}
\]

of \( \mathcal{D} \), where \( 1 = \hat{2} \). Clearly both functors are finitary, \( T_{\Sigma} \) preserves monomorphisms and intersections and \( L_{\Sigma} \) preserves epimorphisms.

Definition 3.6. An \( L \)-algebra \((A, \alpha)\) consists of a \( \mathcal{D} \)-object \( A \) and a \( \mathcal{D} \)-morphism \( \alpha : LA \to A \). A homomorphism

\[
h : (A, \alpha) \to (A', \alpha')
\]

of \( L \)-algebras is a \( \mathcal{D} \)-morphism \( h : A \to A' \) with \( h \cdot \alpha = \alpha' \cdot Lh \). We denote by \( \text{Alg} L \) the category of \( L \)-algebras, and by \( \text{Alg}_f L \) the full subcategory of finite \( L \)-algebras, that is, algebras \((A, \alpha)\) with finite carrier \( A \).

Example 3.7. Algebras for the functor \( L_{\Sigma} \) of Example 3.5 correspond to deterministic automata in \( \mathcal{D} \) with an initial state but without final states. Indeed, by the universal property of the coproduct an \( L_{\Sigma} \)-algebra \( L_{\Sigma} A = 1 + \bigsqcup_{\Sigma} A \xrightarrow{\alpha} A \) is determined by a morphism \( i : 1 \to A \) (specifying an initial state) and morphisms \( \alpha_a : A \to A \) for each \( a \in \Sigma \) (specifying the \( a \)-transitions). Here are two special cases:

(a) If \( \mathcal{C} = \text{BA} \) and \( \mathcal{D} = \text{Set} \) then \( 1 = \hat{2} \) is the one-element set since the two-element boolean algebra \( \hat{2} \) has one atom, so \( L_{\Sigma} \)-algebras are precisely the (classical) deterministic automata with an initial state but without final states.
(b) Similarly, if \( \mathcal{C} = \text{DL}_{01} \) and \( \mathcal{D} = \text{Pos} \) then \( 1 \) is the one-element poset, and \( L_{\Sigma} \)-algebras correspond to ordered deterministic automata with an initial state but without final states.
**Remark 3.8.** The categories $\text{Coalg}_T$ and $\text{Alg}_L$ are dually equivalent. Indeed, the equivalence functor $\tilde{\phi} : C \xrightarrow{\sim} D^\text{op}$ lifts to an equivalence $\text{Coalg}_T \xrightarrow{\sim} (\text{Alg}_L)^\text{op}$ given by

$$\begin{align*}
(Q \xrightarrow{\gamma} TQ) &\mapsto (LQ = TQ \xrightarrow{\tilde{\gamma}} Q).
\end{align*}$$

**Notation 3.9.**

1. By a subcoalgebra of a $T$-coalgebra $(Q, \gamma)$ is meant one represented by a homomorphism $m : (Q', \gamma') \rightarrow (Q, \gamma)$ with $m$ monic in $C$. Subcoalgebras are ordered as usual: $m \leq m'$ iff $m$ factorizes through $m'$ in $\text{Coalg}_T$. We denote by $\text{Sub}(gT)$ the poset of all subcoalgebras of $gT$, and by $\text{Sub}_f(gT)$ the subposet of all finite subcoalgebras of $gT$.

2. Likewise, a quotient algebra of an $L$-algebra is one represented by an epimorphism in $D$. Quotient algebras are ordered by $e \leq e'$ iff $e$ factorizes through $e'$ in $\text{Alg}_L$. For the initial $L$-algebra $\mu_L$, which exists because $L$ is finitary, we have the posets $\text{Quo}_f(\mu_L) \subseteq \text{Quo}(\mu_L)$ of all (finite) quotient algebras of $\mu_L$.

**Remark 3.10.** (a) Since $L$ preserves epimorphisms, $\text{Alg}_L$ has a factorization system consisting of homomorphisms carried by epimorphisms and strong monomorphisms in $D$, respectively. Indeed, given any homomorphism $h : (A, \alpha) \rightarrow (B, \beta)$ of $L$-algebras take its (epi, strong mono)-factorization $h = m \cdot e$ in $D$ and then use that $L$ preserves epis and diagonalization to obtain an algebra structure on the domain of $m$ such that $m$ and $e$ are $L$-algebra homomorphisms:

Moreover, given a commutative square of $L$-algebra homomorphisms, where $e$ is an epimorphism and $m$ is a strong monomorphism in $D$, the unique diagonal $d$ is easily seen to be a homomorphism of $L$-algebras.

(b) Dually, since $T$ preserves monomorphisms, $\text{Coalg}_T$ has a factorization system of homomorphisms carried by strong epimorphisms and monomorphisms in $C$.

Using factorizations, locally finitely presentable coalgebras can be described in terms of subcoalgebras.

**Proposition 3.11.** (a) A $T$-coalgebra is locally finitely presentable iff it is locally finite, i.e., every state is contained in a some finite subcoalgebra.
(b) Every subcoalgebra of a locally finite coalgebra is locally finite.

Proof. (a) Let \((Q, \gamma)\) be a locally finitely presentable coalgebra, i.e., it arises as a filtered \((Q, \gamma_i) \xrightarrow{c_i} (Q, \gamma) (i \in I)\) of finite coalgebras \((Q, \gamma_i)\). Since filtered colimits of \(T\)-coalgebras are formed on the level of \(C\) and hence on the level of \(\text{Set}\), the maps \(c_i\) are jointly surjective. It follows that every state \(q \in Q\) is contained in \(c_i[Q_i]\) for some \(i\), and hence in the subcoalgebra of \((Q, \gamma)\) obtained by factorizing \(c_i\) as in Remark 3.10(b).

Conversely, suppose that every state is contained in some finite subcoalgebra of \((Q, \gamma)\). Then the filtered cocone \((Q, \gamma_i) \rightarrow (Q, \gamma) (i \in I)\) of all finite subcoalgebras of \((Q, \gamma)\) is jointly surjective. This implies that \(Q_i \rightarrow Q (i \in I)\) is a filtered colimit in \(\text{Set}\) and hence also in \(C\) and \(\text{Coalg} \ T\).

(b) Let \(Q\) be a subcoalgebra of a locally finite coalgebra \(Q'\). Then every state \(q \in Q\) is contained in some finite subcoalgebra \(Q''\) of \(Q'\). Since \(T\) preserves intersections, \(Q'' \cap Q\) is a finite subcoalgebra of \(Q\) containing \(q\), so \(Q\) is locally finite. \(\square\)

Example 3.12. A coalgebra for the functor \(T_{\Sigma}Q = 2 \times Q^{\Sigma}\) on \(C\) is locally finitely presentable iff from every state only finitely many states are reachable by transitions.

Proposition 3.13. Sub\((\mathcal{g}T)\) and Quo\((\mu L)\) are complete lattices, and Sub\(_f\)(\(\mathcal{g}T\)) and Quo\(_f\)(\(\mu L\)) are join-subsemilattices.

Proof. Given a family of subcoalgebras \(m_i : (Q_i, \gamma_i) \rightarrow \mathcal{g}T (i \in I)\) it is easy to see that their join in \(\text{Sub}(\mathcal{g}T)\) is the subcoalgebra \(m : (Q, \gamma) \rightarrow \mathcal{g}T\) obtained by factorizing the homomorphism \([m_i] : \bigsqcup_i (Q_i, \gamma_i) \rightarrow \mathcal{g}T\) as in Remark 3.10(b).

\[ \begin{array}{ccc} \bigsqcup_i (Q_i, \gamma_i) & \xrightarrow{[m_i]} & \mathcal{g}T \\ & \searrow^{m} & \downarrow^{\gamma} \\ & (Q, \gamma) & \end{array} \]

If \(I\) and all \((Q_i, \gamma_i)\) are finite, then \((Q, \gamma)\) is clearly also finite. This proves that \(\text{Sub}(\mathcal{g}T)\) is a complete lattice and \(\text{Sub}_f(\mathcal{g}T)\) is a join-subsemilattice. The corresponding statements about Quo\((\mu L)\) and Quo\(_f\)(\(\mu L\)) are shown by a dual argument. \(\square\)

Proposition 3.14. The semilattices \(\text{Sub}_f(\mathcal{g}T)\) and Quo\(_f\)(\(\mu L\)) are isomorphic. The isomorphism is given by \((m : (Q, \gamma) \rightarrow \mathcal{g}T) \mapsto (e : \mu L \rightarrow (\tilde{Q}, \tilde{\gamma}))\), where \(e\) is the unique \(L\)-algebra homomorphism defined by the initiality of \(\mu L\).

Proof. Consider any finite \(T\)-coalgebra \((Q, \gamma)\) and its dual finite \(L\)-algebra \((\tilde{Q}, \tilde{\gamma})\), see Remark 3.8. Since \(\mathcal{g}T\) is the terminal locally finite \(T\)-coalgebra and \(\mu L\) is the initial \(L\)-algebra, there are unique homomorphisms \((Q, \gamma) \xrightarrow{m} \mathcal{g}T\) and \(\mu L \xrightarrow{\epsilon} (\tilde{Q}, \tilde{\gamma})\).
We will prove that

\[ m \text{ is monic (in } C) \iff e \text{ is epic (in } D) \]

from which the claim immediately follows. Assume first that \( m \) is monic in \( C \), and let \( e = e_2 \cdot e_1 \) be the factorization of \( e \) as in Remark 3.10(a):

Since \( e_2 \) is injective, the algebra \( (A, \alpha) \) is finite. Moreover, the strong \( D \)-monomorphism \( e_2 \) is also strongly monic in \( D_f \) because the full embedding \( D_f \rightarrow D \) preserves epis (since \( D_f \) is closed under finite colimits). Hence the dual morphism \( e_2 \) is strongly epic in \( C_f \). Since \( \bar{T} \) is the terminal locally finite \( T \)-coalgebra and \( (\bar{A}, \bar{\alpha}) \) is finite, there exists a unique coalgebra homomorphism \( f \) making the triangle below commute:

By assumption \( m \) is monic, so \( e_2 \) is monic in \( C \) and hence in \( C_f \). But \( e_2 \) is also strongly epic in \( C_f \), and hence an isomorphism (both in \( C_f \) and \( C \)). It follows that \( e_2 \) is an isomorphism (both in \( D_f \) and \( D \)), so \( e = e_2 \cdot e_1 \) is epic in \( D \).

The converse direction is proved by a symmetric argument. \( \square \)

**Remark 3.15.** Recall that the **ideal completion** \( \text{Ideal}(A) \) of a join-semilattice \( A \) is the complete lattice of all ideals (= join-closed downsets) of \( A \) ordered by inclusion. Up to isomorphism \( \text{Ideal}(A) \) is characterized as a complete lattice containing \( A \) such that:

1. every element of \( \text{Ideal}(A) \) is a directed join of elements of \( A \), and
2. the elements of \( A \) are compact in \( \text{Ideal}(A) \): if \( x \in A \) lies under a directed join of elements \( y_i \in \text{Ideal}(A) \), then \( x \leq y_i \) for some \( i \).

**Theorem 3.16.** \( \text{Sub}(\bar{T}) \) is the ideal completion of \( \text{Quo}_f(\mu L) \).

**Proof.** Since \( \text{Sub}_f(\bar{T}) \cong \text{Quo}_f(\mu L) \) by Proposition 3.14, it suffices to prove that \( \text{Sub}(\bar{T}) \) (which forms a complete lattice by Proposition 3.13) is the ideal completion of \( \text{Sub}_f(\mu L) \). To this end we verify the properties (1) and (2) of Remark 3.15.

1. We need to prove that every subcoalgebra \( m : (Q, \gamma) \rightarrow \bar{T} \) is a directed join of subcoalgebras in \( \text{Sub}_f(\bar{T}) \). The coalgebra \( (Q, \gamma) \) is locally finite, being a subcoalgebra of the locally finite coalgebra \( \bar{T} \) (see Proposition 3.11), and hence a filtered colimit
$c_i : (Q_i, \gamma_i) \to (Q, \gamma) \quad (i \in I)$ of coalgebras in $\text{Coalg}_T$. Factorize each $c_i$ as in Remark 3.10(b):

\[
\begin{array}{ccc}
(Q_i, \gamma_i) & \xrightarrow{c_i} & (Q, \gamma) \\
\downarrow{e_i} & & \downarrow{m} \\
(Q', \gamma'_i) & & (Q', \gamma') \\
\end{array}
\]

Then $n_i = m \cdot m_i$ ($i \in I$) is a directed set in $\text{Sub}_f(\varrho T)$. Since $(c_i)$ is colimit cocone, the morphism $[c_i] : \bigsqcup_{i \in I} Q_i \to Q$ is a strong epimorphism in $\mathcal{C}$. This implies that $\bigcup m_i = \text{id}_Q$, hence $\bigcup n_i = m$.

(2) We show that every finite subcoalgebra $m : (Q, \gamma) \to \varrho T$ is compact in $\text{Sub}_f(\varrho T)$. So let $n = \bigcup_{i \in I} n_i$ be a directed union of subcoalgebras $n_i : (Q, \gamma_i) \to \varrho T$ in $\text{Sub}_f(\varrho T)$. Then in $\mathcal{C}$ we also have a directed union $n = \bigcup n_i$ because coproducts in $\text{Coalg}_T$ are formed on the level of $\mathcal{C}$. Indeed, $\bigcup n_i$ is formed from $[n_i]$ by an image factorization, see the proof of Proposition 3.13 and recall from Remark 3.10 that image factorizations in $\text{Coalg}_T$ are the liftings of (strong epi, mono)-factorizations in $\mathcal{C}$.

Since $Q$ is finite, from $m \subseteq n$ follows that $m \subseteq n_i$ for some $i \in I$. That is, there exists a morphism $f : Q \to Q_i$ in $\mathcal{C}$ with $m = n_i \cdot f$. It remains to verify that $f$ is a $T$-coalgebra homomorphism:

\[
\begin{array}{ccc}
Q & \xrightarrow{\gamma} & TQ \\
\downarrow{f} & & \downarrow{Tf} \\
Q_i & \xrightarrow{\gamma_i} & TQ_i \\
\downarrow{n_i} & & \downarrow{Tn_i} \\
\varrho T & \xrightarrow{T} & T(\varrho T) \\
\end{array}
\]

Since the lower square and the outer square commute, it follows that the upper square commutes when composed with $Tn_i$. By assumption $T$ preserves monomorphisms, so we can conclude that upper square commutes.

Example 3.17. Let $T_\Sigma : \text{BA} \to \text{BA}$ and $L_\Sigma : \text{Set} \to \text{Set}$ as in Example 3.5. The rational fixpoint of $T_\Sigma$ is the boolean algebra $\text{Reg}_\Sigma$ with transitions $L \xrightarrow{a^{-1}} L$, see Corollary 2.11 and the initial algebra of $L_\Sigma$ is the automaton $\Sigma^*$ with initial state $\varepsilon$ and transitions $w \xrightarrow{a} wa$ for $a \in \Sigma$. Hence the previous theorem gives a one-to-one correspondence between

(i) sets of regular languages over $\Sigma$ closed under boolean operations and left derivates, and

(ii) ideals of $\text{Quo}_f(\Sigma^*)$, i.e., sets of quotient automata of $\Sigma^*$ closed under quotients and joins.

This correspondence is refined in the following section.
4 The Local Eilenberg Theorem

In this section we establish our main result, the generalized local Eilenberg theorem. We continue to work under the Assumptions 3.4 – that is, a locally finite variety of algebras \( \mathcal{C} \) and a predual locally finite variety of (possibly ordered) algebras \( \mathcal{D} \) are given – and restrict our attention to deterministic automata, modeled as coalgebras and algebras for the predual functors

\[
T_{\Sigma} = 2 \times \text{Id}^\Sigma : \mathcal{C} \to \mathcal{C} \quad \text{and} \quad L_{\Sigma} = 1 + \bigcup_\Sigma \text{Id} : \mathcal{D} \to \mathcal{D},
\]

respectively. Here \( 2 = \{0, 1\} \) is a fixed two-element algebra in \( \mathcal{C} \) and \( 1 = \overline{2} \) is its dual algebra in \( \mathcal{D} \).

The crucial step towards Eilenberg’s theorem is to prove that the isomorphism

\[
\text{Sub}_f(\varrho T_{\Sigma}) \cong \text{Quo}_f(\mu L_{\Sigma})
\]

of Proposition 3.14 restricts to one between the finite subcoalgebras of \( \varrho T_{\Sigma} \) closed under right derivatives and the finite quotient algebras of \( \mu L_{\Sigma} \) whose transitions are induced by a monoid structure. To this end we will characterize right derivatives and monoids from a categorical perspective and show that they are dual concepts (Sections 4.1 and 4.2). The general local Eilenberg theorem is proved in Section 4.3.

4.1 Right derivatives

The closure of the regular languages under right derivatives is usually proved via the following automata construction: suppose a deterministic \( \Sigma \)-automaton in \( \text{Set} \) (with states \( Q \) and final states \( F \subseteq Q \)) accepts a language \( L \subseteq \Sigma^* \). Then given \( w \in \Sigma^* \) replace the set of final states by

\[
F' = \{ q \in Q : q \xrightarrow{w} q' \text{ for some } q' \in F \}.
\]

The resulting automaton accepts the right derivative \( Lw^{-1} = \{ u \in \Sigma^* : uw \in L \} \) of \( L \). This construction generalizes to arbitrary \( T_{\Sigma} \)-coalgebras:

**Notation 4.1.** \( T_{\Sigma} \)-coalgebras \( Q \xrightarrow{\gamma} 2 \times Q^\Sigma \) are represented as triples

\[
(Q, \gamma_a : Q \to Q, f : Q \to 2),
\]

see Example 2.6. For each \( T_{\Sigma} \)-coalgebra \( Q = (Q, \gamma_a, f) \) and \( w \in \Sigma^* \) we put

\[
Q_w := (Q, \gamma_a, f \cdot \gamma_w)
\]

where, as usual, \( \gamma_w = \gamma_{a_n} \cdots \gamma_{a_1} \) for \( w = a_1 \cdots a_n \).
Remark 4.2. A $C$-morphism $h : Q \to Q'$ is a $T_S$-coalgebra homomorphism $h : (Q, \gamma_a, f) \to (Q', \gamma'_a, f')$ iff the following diagram commutes for all $a \in \Sigma$:

\[
\begin{array}{ccc}
Q \xrightarrow{\gamma_a} Q \\
\downarrow h & \Rightarrow & \downarrow h \\
Q' \xrightarrow{\gamma'_a} Q'
\end{array}
\]

In this case also the square below commutes, which implies that $h$ is a $T_S$-coalgebra homomorphism $h : Q_w \to Q'_w$ for all $w \in \Sigma^*$.

\[
\begin{array}{ccc}
Q \xrightarrow{\gamma_w} Q \\
\downarrow h & \Rightarrow & \downarrow h \\
Q' \xrightarrow{\gamma'_w} Q'
\end{array}
\]

Proposition 4.3. A subcoalgebra $Q$ of $gT_S$ is closed under right derivatives (i.e., $L \in Q$ implies $Lw^{-1} \in Q$ for each $w \in \Sigma^*$) iff there exists a coalgebra homomorphism from $Q_w$ to $Q$ for each $w \in \Sigma^*$.

Proof. A subcoalgebra $Q$ of $gT_S$ is up to isomorphism a set of regular languages over $\Sigma$ carrying a $C$-algebraic structure and closed under left derivatives, see Corollary 2.11. The ordering of subcoalgebras is induced by inclusion of sets, i.e., $Q \subseteq Q'$ in $\text{Sub}(gT_S)$ iff $Q \subseteq Q'$.

Suppose that a coalgebra homomorphism $\alpha_w : Q_w \to Q$ exists for each $w \in \Sigma^*$. The languages accepted by $Q_w$ are precisely $\{Lw^{-1} : L \in Q\}$ because we have moved the final states backwards along all $w$-paths. The morphism $\alpha_w$ assigns to each state of $Q_w$ its accepted language, so $\{Lw^{-1} : L \in Q\} \subseteq Q$. Hence $Q$ is closed under right derivatives.

Conversely suppose that $Q \hookrightarrow gT_S$ is closed under right derivatives. Clearly $Q_w$ is locally finite for each $w \in \Sigma^*$, so there are unique homomorphisms $Q_w \to gT_S$ and $\bigsqcup Q_w \to gT_S$. Factorize them as in Remark 3.10 (b):

\[
Q_w \to \tilde{Q}_w \Rightarrow gT_S \\
\bigsqcup_{w \in \Sigma^*} Q_w \to \bigsqcup \Rightarrow gT_S
\]

It is now easy to see (using the factorization system) that

\[
\tilde{Q} = \bigvee \{ \tilde{Q}_w : w \in \Sigma^* \}
\]

in $\text{Sub}(gT_S)$. Since $Q$ is closed under right derivatives we have $\tilde{Q}_w = \{Lw^{-1} : L \in Q\} \subseteq \tilde{Q}$ and hence $\bigsqcup \tilde{Q} \subseteq \tilde{Q}$. Since $Q = Q_\varepsilon$ the reverse inclusion holds, so $Q = \tilde{Q}$. Therefore we obtain a coalgebra homomorphism $Q_w \xrightarrow{m_w} \bigsqcup_{w \in \Sigma^*} Q_w \to \tilde{Q} = Q$.  

Lemma 4.4. Every finite subcoalgebra of $gT_S$ is contained in a finite subcoalgebra of $gT_S$ closed under right derivatives.
Proof. Let $Q \hookrightarrow \eta T_{\Sigma}$ be a finite subcoalgebra of $\eta T_{\Sigma}$. Since $\mathcal{C}(Q, Q)$ is finite, there exists a finite set $W \subseteq \Sigma^*$ of words such that for every $u \in \Sigma^*$ there exists $w \in W$ with $Q_u = Q_w$. Then the coalgebra $\bigsqcup_{w \in W} Q_w$ is finite because the coproduct is constructed on the level of $\mathcal{C}$, and $\mathcal{C}_f$ is closed under finite colimits as $\mathcal{C}$ is locally finite. Factorize the unique homomorphism $\bigsqcup_{w \in W} Q_w \rightarrow \eta T_{\Sigma}$ as in Remark 3.10(b): 

$$Q = Q \varepsilon \rightarrow \bigsqcup_{w \in W} Q_w \xrightarrow{\varepsilon} Q' \xrightarrow{m} \eta T_{\Sigma}$$

The coalgebra $Q'$ is clearly also finite. Moreover, if $w$ is chosen such that $Q = Q_w$, the above diagram yields $Q \hookrightarrow Q'$. It remains to show that $Q'$ is closed under right derivatives. First observe that, for each $w \in \Sigma^*$, one has 

$$(\bigsqcup_{w' \in W} Q_{w'})_w = \bigsqcup_{w' \in W} (Q_{w'})_w = \bigsqcup_{w' \in W} Q_{w'w'}$$

because $(-)_w$ commutes with coproducts and $f \circ \gamma_{w'} \circ \gamma_w = f \circ \gamma_{w'w'}$. Consider the diagram of coalgebra homomorphisms 

$$\bigsqcup_{w' \in W} Q_{w'w'} = (\bigsqcup_{w' \in W} Q_{w'})_w \xrightarrow{e} Q'_w \xrightarrow{\alpha_w} Q' \xleftarrow{\beta} \eta T_{\Sigma}$$

The strong epic $e$ and mono $m$ were defined above, $\beta$ is the final morphism, the topmost $e$ exists by Remark 4.2, and $h$ exists because each $Q_{w'w'}$ equals some $Q_v$ with $v \in W$. Hence the square commutes by finality, and diagonal fill-in gives a coalgebra homomorphism $\alpha_w : Q'_w \rightarrow Q'$. Since $w$ was an arbitrary word we deduce from Proposition 4.3 that $Q'$ is closed under right derivatives. \hfill \Box

Corollary 4.5. Let $Q \hookrightarrow Q' \hookrightarrow \eta T_{\Sigma}$ be subcoalgebras where $Q$ is finite and $Q'$ is closed under right derivatives. Then $Q$ is contained in a finite subcoalgebra of $Q'$ closed under right derivatives.

Proof. By the previous lemma there exists a finite subcoalgebra $Q'' \hookrightarrow \eta T_{\Sigma}$ that contains $Q$ and is closed under right derivatives. Since $T_{\Sigma}$ preserves intersections, $Q'' \cap Q'$ is a subcoalgebra of $Q'$ with the desired properties.

Notation 4.6. $\text{Sub}^r(\eta T_{\Sigma})$ and $\text{Sub}^f(\eta T_{\Sigma})$ are the posets of (finite) subcoalgebras of $\eta T_{\Sigma}$ closed under right derivatives.

Proposition 4.7. $\text{Sub}^r(\eta T_{\Sigma})$ is the ideal completion of $\text{Sub}^f(\eta T_{\Sigma})$. 

Generalized Eilenberg Theorem I: Local Varieties of Languages 19
Proof. Since $T_{\Sigma}$ preserves intersections, $\text{Sub}^r(\varrho T_{\Sigma})$ forms a complete lattice whose meet is set-theoretic intersection. It remains to check the conditions (1) and (2) of Remark 3.15. For (2), just note that directed joins are directed unions of subcoalgebras, so that every finite subcoalgebra closed under right derivatives is compact. For (1) we use that every subcoalgebra $Q \hookrightarrow \varrho T_{\Sigma}$ is locally finite, see Proposition 3.11, and hence arises as the directed union of its finite subcoalgebras. But by Corollary 4.5 the poset of all finite subcoalgebras of $Q$ contains the ones closed under right derivatives as a final subposet, so $Q$ is also the directed union of its finite subcoalgebras closed under right derivatives. \hfill \Box

4.2 $\mathcal{D}$-Monoids

By Proposition 4.3 closure under right derivatives of a subcoalgebra $Q \hookrightarrow \varrho T_{\Sigma}$ is characterized by the existence of $T_{\Sigma}$-coalgebra homomorphisms $Q_w \to Q$. In this section we investigate the dual $L_{\Sigma}$-algebra homomorphisms $\mathcal{Q} \to \mathcal{Q}_w$ and show that they define a monoid structure on $\mathcal{Q}$. This requires the following additional assumptions on the variety $\mathcal{D}$:

Assumptions 4.8. For the rest of Section 4 we assume that

(a) epimorphisms in $\mathcal{D}$ are surjective;
(b) for any two algebras $A$ and $B$ in $\mathcal{D}$, the set $[A, B]$ of homomorphisms from $A$ to $B$ is an algebra in $\mathcal{D}$ with the pointwise algebraic structure, i.e., a subalgebra of the power $B^A = \prod_{x \in A} B$;
(c) $1 = \mathcal{Q}_1$ is a free $\mathcal{D}$-algebra on the one-element set 1, that is,

\[ 1 \cong \Psi 1 \]

for the left adjoint $\Psi : \text{Set} \to \mathcal{D}$ to the forgetful functor $\mathcal{D} \to \text{Set}$.

Remark 4.9. Here is a more categorical view of Assumption 4.8(b). Given algebras $A$, $B$ and $C$ in $\mathcal{D}$, a bimorphism is a function $f : A \times B \to C$ such that $f(a, -) : B \to C$ and $f(-, b) : A \to C$ are $\mathcal{D}$-morphisms for all $a \in A$ and $b \in B$. A tensor product of $A$ and $B$ is a universal bimorphism $t : A \times B \to A \otimes B$, i.e., for every bimorphism $f : A \times B \to C$ there is a unique $\mathcal{D}$-morphism $f'$ making the diagram below commute.

\[ \begin{array}{ccc}
A \times B & \xrightarrow{t} & A \otimes B \\
\downarrow f & & \downarrow f' \\
C & \downarrow \end{array} \]

The tensor product exists in every variety $\mathcal{D}$ and turns it into a symmetric monoidal category $(\mathcal{D}, \otimes, \mathcal{Q}_1)$, see [6]. Assumption 4.8(b) then states precisely that $\mathcal{D}$ is monoidal closed.

Example 4.10. The varieties $\mathcal{D} = \text{Set}, \text{Pos}, \text{JSL}_0, \text{Vect}_{\mathbb{Z}_2}$ of Example 3.2 meet the Assumptions 4.8.
Remark 4.11. (1) For all algebras $A$ and $B$ in $D$ and $x \in A$ let $\text{ev}_x$ be the composite

$$[A, B] \ni f \mapsto B^A \xrightarrow{\pi_x} B.$$ 

The morphism $\text{ev}_x$ is evaluation at $x$, i.e.,

$$\text{ev}_x(f) = f(x) \quad \text{for all } f \in [A, B].$$

(2) For any two $D$-morphisms $f : B \to B'$ and $g : A \to A'$, the maps

$$c_f : [A, B] \to [A, B'] \quad \text{and} \quad c'_g : [A, B] \to [A', B]$$

given by composition with $f$ and $g$, respectively, are $D$-morphisms.

The assumption that $D$ is monoidal closed gives rise to inductive definition and proof principles that we shall use extensively.

Definition 4.12 (Inductive Extension Principle). Let $(g_i : A \to B)_{i \in I}$ be a set-indexed family of morphisms between two fixed $D$-objects $A$ and $B$. Its inductive extension is the family $(g_x : A \to B)_{x \in \Psi I}$ defined as follows:

1. Extend the function $g : I \to [A, B], i \mapsto g_i$, to a $D$-morphism $\Psi I \to [A, B]$

2. Put $g_x := \Psi I \to [A, B]$ for all $x \in \Psi I$.

Lemma 4.13. (a) In Definition 4.12 we have $g_{\eta i} = g_i$ for all $i \in I$.

(b) For all sets $I$ and $D$-objects $A$, the family $(\text{ev}_x : [\Psi I, A] \to A)_{x \in \Psi I}$ is the inductive extension of $(\text{ev}_{\eta i} : [\Psi I, A] \to A)_{i \in I}$.

Proof. (a) For all $x \in A$ we have:

$$g_{\eta i}(x) = \text{ev}_x(g_{\eta i}) \quad \text{def. ev}$$
$$= \text{ev}_x(\Psi I(\eta i)) \quad \text{def. } g_{\eta i}$$
$$= \text{ev}_x(g_i) \quad \text{def. } \Psi I$$
$$= g_i(x) \quad \text{def. ev}$$

(b) Extend the function $g : I \to [[\Psi I, A], A], i \mapsto \text{ev}_{\eta i}$, to a $D$-morphism $\Psi I \to [[\Psi I, A], A]$. 

$$I \xrightarrow{\eta} \Psi I \xrightarrow{\Psi I} [[\Psi I, A], A]$$
We need to show that the extended family consists of \( g_x = \text{ev}_x \) for all \( x \in \Psi I \). To this end, we first prove the equation

\[
\text{ev}_f \cdot \overline{\gamma} = f \quad \text{for all} \quad f : \Psi I \rightarrow A.
\]

It suffices to prove \( \text{ev}_f \cdot \overline{\gamma} = f \), and indeed we have for all \( i \in I \):

\[
\begin{align*}
\text{ev}_f(\overline{\gamma}(\eta_i)) &= \text{ev}_f(g_i) & \text{def. } \overline{\gamma} \\
&= \text{ev}_f(\text{ev}_{\eta_i}) & \text{def. } g \\
&= \text{ev}_{\eta_i}(f) & \text{def. } \text{ev} \\
&= f(\eta_i) & \text{def. } \text{ev}
\end{align*}
\]

Therefore, for all \( x \in \Psi I \) and \( f : \Psi I \rightarrow A \),

\[
\begin{align*}
g_x(f) &= \text{ev}_f(g_x) & \text{def. } \text{ev} \\
&= \text{ev}_f(\overline{\gamma}(x)) & \text{def. } g_x \\
&= f(x) & \text{eqn. above} \\
&= \text{ev}_x(f) & \text{def. } \text{ev}
\end{align*}
\]

so \( g_x = \text{ev}_x \) as claimed.

\[\Box\]

**Lemma 4.14 (Inductive Proof Principle).** Let \( f, f', h, h' \) and \( g_i, g'_i \) (\( i \in I \)) be \( D \)-morphisms as in the diagram \((*)\) below:

\[
\begin{array}{c}
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{f'} & & \downarrow{g_i} \\
B' & \xrightarrow{h'} & C'
\end{array}
& \quad & \\
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{f'} & & \downarrow{g_i} \\
B' & \xrightarrow{h'} & C'
\end{array}
\end{array}
\]

If \((*)\) commutes for all \( i \in I \), then \((**\)) commutes for all \( x \in \Psi I \).

**Proof.** Suppose that \( h \cdot g_i \cdot f = h' \cdot g'_i \cdot f' \) for all \( i \in I \). We first prove the equation

\[
c'_f \cdot c_h \cdot \overline{\gamma} = c'_{f'} \cdot c_{h'} \cdot \overline{g'}
\]

where \( c_h, c_{h'}, c'_f \) and \( c'_{f'} \) are the \( D \)-morphisms of Remark 4.11(b) and \( \overline{\gamma}, \overline{g'} : \Psi I \rightarrow [B, C] \) are as in Definition 4.12. Indeed, we have for all \( i \in I \):

\[
\begin{align*}
c'_f(c_h(\overline{\gamma}(\eta_i))) &= c'_f(c_h(g_i)) \quad & \text{def. } \overline{\gamma} \\
&= h \cdot g_i \cdot f \quad & \text{def. } c_h, c'_f \\
&= h' \cdot g'_i \cdot f' \quad & \text{assumption} \\
&= c'_{f'}(c_{h'}(\overline{g'}(\eta_i))) \quad & \text{compute backwards}
\end{align*}
\]
We conclude that, for all \( x \in \Psi I \),

\[
    h \cdot g_x \cdot f = c'_f(c_h(g_x)) = c'_f(c_h(\overline{f}(x))) = c'_f(c_{h'}(\overline{g'}(x))) = h' \cdot g'_x \cdot f' \quad \text{compute backwards}
\]

**Definition 4.15.** A \( \mathcal{D} \)-monoid \((A, \circ, i)\) consists of an algebra \( A \) in \( \mathcal{D} \), a bimorphism \( \circ : A \times A \to A \) and an element \( i : 1 \to A \) subject to the usual monoid axioms, i.e., the multiplication \( \circ \) is associative and \( i \) is its unit. A morphism of \( \mathcal{D} \)-monoids

\[
h : (A, \circ, i) \to (A', \circ', i')
\]

is a \( \mathcal{D} \)-morphism \( h : A \to A' \) that is also a monoid morphism, i.e., it preserves the multiplication and the unit. We denote by \( \text{Mon}(\mathcal{D}) \) the category of \( \mathcal{D} \)-monoids, and by \( \text{Mon}_f(\mathcal{D}) \) the full subcategory of all finite \( \mathcal{D} \)-monoids.

**Remark 4.16.** \( \mathcal{D} \)-monoids are precisely the monoid objects in the monoidal category \((\mathcal{D}, \otimes, \Psi 1)\), see Remark 4.9.

**Example 4.17.** \( \mathcal{D} \)-monoids in \( \mathcal{D} = \text{Set}, \text{Pos}, \text{JSL}_0, \text{Vect} \mathbb{Z}_2 \) correspond to monoids, ordered monoids, idempotent semirings and \( \mathbb{Z}_2 \)-algebras, respectively.

**Lemma 4.18.** (a) \( \text{Mon}(\mathcal{D}) \) is Set complete and \( \text{Mon}_f(\mathcal{D}) \) is finitely complete, with limits formed on the level of \( \mathcal{D} \).

(b) The (epi, strong mono)-factorization system of \( \mathcal{D} \) lifts to \( \text{Mon}(\mathcal{D}) \).

**Proof.** (a) Let \( V : \text{Mon}(\mathcal{D}) \to \mathcal{D} \) denote the forgetful functor. Given a diagram \( D : S \to \text{Mon}(\mathcal{D}) \) where \( D_s = (A_s, \circ_s, i_s) \) for \( s \in S \), form the limit cone \((L \xrightarrow{p_s} V D_s)_{s \in S}\) in \( \mathcal{D} \) (which is also a limit cone in Set as limits in \( \mathcal{D} \) are formed on the level of underlying sets). Since

\[
    (L \times L \xrightarrow{p_s \times p_s} V D_s \times V D_s \xrightarrow{\circ_s} V D_s)_{s \in S} \quad \text{and} \quad (1 \xrightarrow{i_s} V D_s)_{s \in S}
\]

are also cones (in Set) over \( V D \), there are unique mediating maps \( \circ : L \times L \to L \) and \( i : 1 \to L \). It is now easy to verify that \((L, \circ, i)\) is \( \mathcal{D} \)-monoid and that \((L, \circ, i) \to D_s)_{s \in S}\) forms a limit cone over \( D \) in \( \text{Mon}(\mathcal{D}) \). This proves the completeness of \( \text{Mon}(\mathcal{D}) \). The proof that \( \text{Mon}_f(\mathcal{D}) \) is finitely complete is identical – just start with a finite diagram \( D \) and use that \( \mathcal{D}_f \) is closed under finite limits.

(b) Let \( f : (M, \bullet, i) \to (M', \bullet', i') \) be a morphism of \( \mathcal{D} \)-monoids and \( M \xrightarrow{e} M_0 \xrightarrow{m} M' \) be its (epi, strong mono)-factorization in \( \mathcal{D} \). Then \( m \) is injective and \( e \) is surjective by Assumption 4.8(a). Consequently, there exists a unique monoid structure \( \bullet \) and \( i_0 \) on \( M_0 \) for which \( e \) and \( m \) are monoid morphisms. All that needs proving is that for every element \( x \in M_0 \) the function \( x \bullet - \) is an endomorphism of \( M_0 \) in \( \mathcal{D} \) (and similarly for \( - \bullet x \)). Let \( d' : M' \to M' \) be the \( \mathcal{D} \)-morphism

\[
d'(x) = x' \bullet - \quad \text{for } x' = m(x).
\]
Since $e$ is surjective, we have $y \in M$ with $e(y) = x$ and we denote by $d : M \to M$ the $D$-morphism $d = y \cdot -$. Then $f \cdot d = d' \cdot f$ because, for all $z \in M$,

$$f \cdot d(z) = f(y \cdot z)$$

$$= f(y) \cdot' f(z)$$

$$= x' \cdot' f(z)$$

$$= d'(f(z))$$

$$= d' \cdot f(z).$$

The unique diagonal fill in

\[
\begin{array}{ccc}
M & \xrightarrow{e} & M_0 \\
\downarrow d & \searrow & \downarrow m \\
M' & \searrow & M'
\end{array}
\]

yields a $D$-morphism $d_0$ with $d_0 = x \star -$. Indeed, given $p \in M_0$, choose $q \in M$ such that $p = e(q)$. Then:

$$d_0(p) = d_0 \cdot e(q) = e \cdot d(q) = e(y \cdot q) = e(y) \star e(q) = x \star p.$$  

Our next goal is to show that the free monoid $(\Sigma^*, \emptyset, \varepsilon)$ on $\Sigma$ in Set extends to a free $D$-monoid $(\Psi \Sigma^*, \bullet, \eta \varepsilon)$ on $\Sigma$.

**Definition 4.19.** For every word $w \in \Sigma^*$ the endomaps $w \cdot -$ and $- \cdot w$ of $\Sigma^*$ yield unique $D$-endomorphisms $l_w$ and $r_w$ of $\Psi \Sigma^*$ making the squares below commute:

\[
\begin{array}{ccc}
\Sigma^* & \xrightarrow{w} & \Sigma^* \\
\eta & \downarrow & \eta \\
\Psi \Sigma^* & \xrightarrow{l_w} & \Psi \Sigma^*
\end{array}
\quad
\begin{array}{ccc}
\Sigma^* & \xrightarrow{-w} & \Sigma^* \\
\eta & \downarrow & \eta \\
\Psi \Sigma^* & \xrightarrow{r_w} & \Psi \Sigma^*
\end{array}
\]

Let $l_x, r_x : \Psi \Sigma^* \to \Psi \Sigma^*$ ($x \in \Psi \Sigma^*$) be the arrows obtained by inductively extending the families $(l_w)_{w \in \Sigma^*}$ and $(r_w)_{w \in \Sigma^*}$, respectively.

**Lemma 4.20.** For all $x, y \in \Psi \Sigma^*$ the following equations hold:

(a) $r_x \cdot l_y = l_y \cdot r_x$

(b) $r_y(x) = l_x(y)$

**Proof.** (a) Consider the diagrams below:

\[
\begin{array}{ccc}
\Psi \Sigma^* & \xrightarrow{r_w} & \Psi \Sigma^* \\
\downarrow l_x & & \downarrow l_y \\
\Psi \Sigma^* & \xrightarrow{r_w} & \Psi \Sigma^*
\end{array}
\quad
\begin{array}{ccc}
\Psi \Sigma^* & \xrightarrow{r_x} & \Psi \Sigma^* \\
\downarrow l_w & & \downarrow l_y \\
\Psi \Sigma^* & \xrightarrow{r_x} & \Psi \Sigma^*
\end{array}
\quad
\begin{array}{ccc}
\Psi \Sigma^* & \xrightarrow{r_y} & \Psi \Sigma^* \\
\downarrow l_x & & \downarrow l_y \\
\Psi \Sigma^* & \xrightarrow{r_y} & \Psi \Sigma^*
\end{array}
\]
The left square commutes for all \( v, w \in \Sigma^* \): indeed, for all \( u \in \Sigma^* \) we have
\[
l_v(r_w(\eta u)) = \eta(vuw) = r_w(l_v(\eta u))
\]
by the definition of \( l_v \) and \( r_w \). By induction it follows that the middle square commutes for all \( v \in \Sigma^* \) and \( x \in \Psi \Sigma^* \). Using induction again we conclude that the right square commutes for all \( x, y \in \Psi \Sigma^* \).

(b) We first prove that the following diagram commutes for all \( y \in \Psi \Sigma^* \) (where \( \tilde{\mathcal{T}} \) is defined as shown in Definition 4.12(1)):

\[
\begin{array}{ccc}
\Psi \Sigma^* & \xrightarrow{\tilde{\mathcal{T}}} & [\Psi \Sigma^*, \Psi \Sigma^*] \\
\downarrow & & \downarrow \\
\Psi \Sigma^* & \xrightarrow{\mathcal{E}_y} & \Psi \Sigma^*
\end{array}
\]

By Lemma 4.13(b) and the induction principle it suffices to prove that it commutes for \( y = \eta w \) where \( w \in \Sigma^* \). In fact, we have for all \( v \in \Sigma^* \):
\[
ev_{\eta w}(\tilde{\mathcal{T}}(\eta v)) = \ev_{\eta w}(l_v) \quad \text{def. } \tilde{\mathcal{T}}
= \mathcal{E}_v(\eta w) \quad \text{def. } \mathcal{E}_v
= \eta(vw) \quad \text{def. } l_v
= r_w(\eta v) \quad \text{def. } r_w
= r_{\eta w}(\eta v) \quad \text{Lemma 4.13(a)}
\]

and therefore \( \ev_{\eta w} \circ \tilde{\mathcal{T}} = r_{\eta w} \). It follows that, for all \( x, y \in \Psi \Sigma^* \):
\[
r_y(x) = \ev_y(\tilde{\mathcal{T}}(x)) \quad \text{diagram above}
= \ev_y(l_x) \quad \text{def. } l_x
= l_x(y) \quad \text{def. } \mathcal{E}_y
\]

Definition 4.21. We define a multiplication on \( \Psi \Sigma^* \) as follows:
\[
x \bullet y := r_y(x) = l_x(y) \quad \text{for all } x, y \in \Psi \Sigma^*.
\]

Proposition 4.22. \((\Psi \Sigma^*, \bullet, \eta \varepsilon)\) is the free \( \mathcal{D} \)-monoid on \( \Sigma \): for any \( \mathcal{D} \)-monoid \((A, \circ, i)\) and any function \( f : \Sigma \to A \), there is a unique extension to a \( \mathcal{D} \)-monoid morphism \( \mathcal{F} : \Psi \Sigma^* \to A \).

\[
\begin{array}{ccc}
\Sigma^+ & \xrightarrow{\eta} & \Psi \Sigma^* \\
\downarrow & \mathcal{F} & \downarrow \\
\Sigma & \xrightarrow{f} & A
\end{array}
\]

Proof. We first show that \((\Psi \Sigma^*, \bullet, \eta \varepsilon)\) is a \( \mathcal{D} \)-monoid. Indeed:
(1) \(\bullet\) is associative: for all \(x, y, z \in \Psi \Sigma^*\) we have

\[ x \bullet (y \bullet z) = l_x(r_z(y)) = r_x(l_x(y)) = (x \bullet y) \bullet z \]

using the definition of \(\bullet\) and Lemma 4.20(a).

(2) \(\eta \varepsilon\) is the neutral element: for all \(x \in \Psi \Sigma^*\) we have

\[ x \bullet \eta \varepsilon = r_x \varepsilon(x) = r_x(x) = \text{id}(x) = x \]

and symmetrically \(\eta \varepsilon \bullet x = x\).

(3) \(\bullet\) is a \(D\)-bimorphism since, for all \(x \in \Psi \Sigma^*\), the functions \(l_x = x \bullet -\) and \(r_x = - \bullet x\) are \(D\)-morphisms.

It remains to verify the universal property. Given \(f\) as in the diagram above, one can first extend \(f\) to a monoid morphism \(f' : \Sigma^* \to A\) (using that \(\Sigma^*\) is the free monoid on \(\Sigma\)) and then extend \(f'\) to a \(D\)-morphism \(\overline{f} : \Psi \Sigma^* \to A\) with \(\overline{f} \cdot \eta = f'\) (by the universal property of \(\eta\)). We only need to verify that \(\overline{f}\) is a monoid morphism. Firstly, \(\overline{f}\) preserves the unit:

\[ \overline{f}(\eta \varepsilon) = f'(\varepsilon) \quad \text{def. } \overline{f} \]

\[ = i \quad f' \text{ monoid morphism} \]

To prove that \(\overline{f}\) also preserves the multiplication, consider the squares below where \(l'_x, r'_w : A \to A\) are the \(D\)-morphisms \(l'_x = f(x) \circ -\) and \(r'_w = - \circ f\eta\).

\[ \Psi \Sigma^* \xrightarrow{r_w} \Psi \Sigma^* \quad \Psi \Sigma^* \xrightarrow{l_x} \Psi \Sigma^* \]

\[ A \xrightarrow{r'_w} A \quad A \xrightarrow{l'_x} A \]

The left square commutes for all \(w \in \Sigma^*\) because, for all \(v \in \Sigma^*\),

\[ \overline{f}(r_w(\eta v)) = \overline{f}(\eta(vw)) \quad \text{def. } r_w \]

\[ = f'(vw) \quad \text{def. } \overline{f} \]

\[ = f'(v) \circ f'(w) \quad f' \text{ monoid morphism} \]

\[ = \overline{f}(\eta v) \circ \overline{f}(\eta w) \quad \text{def. } \overline{f} \]

\[ = r'_w(\overline{f}(\eta w)) \quad \text{def. } r'_w \]

Then also the right square commutes for all \(x \in \Psi \Sigma^*\) because, for all \(w \in \Sigma^*\),

\[ \overline{f}(l_x(\eta w)) = \overline{f}(r_w(x)) \quad \text{Lemma 4.20(b)} \]

\[ = r'_w(\overline{f}(x)) \quad \text{left square} \]

\[ = \overline{f}(x) \circ \overline{f}(\eta w) \quad \text{def. } r'_w \]

\[ = l'_x(\overline{f}(\eta w)) \quad \text{def. } l'_x \]
We conclude that, for all \(x, y \in \Psi \Sigma^*\),
\[
\overline{f}(x \cdot y) = \overline{f}(l_x(y)) = l'_x(\overline{f}(y)) = \overline{f}(x) \circ \overline{f}(y)
\]
def. \(\cdot\)
\[
\text{right square}
\]
def. \(l'_x\)

**Example 4.23.** (a) For \(D = \text{Set or Pos}\) we have \(\Psi \Sigma^* = \Sigma^*\) (discretely ordered in the case \(D = \text{Pos}\)). The monoid multiplication is concatenation of words, and the unit is \(\varepsilon\).

(b) For \(D = \text{JS}L_0\) we have \(\Psi \Sigma^* = \mathcal{P}_\omega \Sigma^*\), the semilattice of all finite languages over \(\Sigma\) w.r.t. union. The monoid multiplication is concatenation of languages, and the unit is the language \(\{\varepsilon\}\).

(c) For \(D = \text{Vect} \mathbb{Z}_2\) we have \(\Psi \Sigma^* = \mathcal{P}\Sigma^*\), the vector space of all finite languages over \(\Sigma\) where vector addition is symmetric difference. The monoid multiplication is the \(\mathbb{Z}_2\)-weighted concatenation \(L \odot L'\) of languages, (i.e., \(L \odot L'\) consists of all words \(w\) having an odd number of factorizations \(w = uu'\) with \(u \in L\) and \(u' \in L'\)), and the unit is again \(\{\varepsilon\}\).

**Definition 4.24.** (a) A \(\Sigma\)-generated \(D\)-monoid is a quotient of \(\Psi \Sigma^*\), i.e., a \(D\)-monoid morphism \(e : \Psi \Sigma^* \to A\) with \(e\) epic in \(D\). A morphism between two \(\Sigma\)-generated \(D\)-monoids \(e : \Psi \Sigma^* \to A\) and \(e' : \Psi \Sigma^* \to A'\) is a generator-preserving \(D\)-monoid morphism \(f : A \to A'\), i.e., \(f \cdot e = e'\).

(b) We denote by \(\Sigma\text{-Mon}(D)\) the poset of all \(\Sigma\)-generated \(D\)-monoids under the usual quotient ordering (see Notation 4.9), and by \(\Sigma\text{-Mon}_f(D)\) the subposet of all \(\Sigma\)-generated finite \(D\)-monoids.

**Remark 4.25.** The poset \(\Sigma\text{-Mon}(D)\) is a complete lattice – the join of a family \(e_i : \Psi \Sigma^* \to A_i\) of \(\Sigma\)-generated \(D\)-monoids is their subdirect product \(S\), obtained by forming their product in \(\text{Mon}(D)\) and the (strong epi, mono)-factorization of the morphism \((e_i)\):

\[
\begin{CD}
\Psi \Sigma^* @>{e_i}>> A_i \\
S @>>{\pi_i}>> \prod A_i
\end{CD}
\]

Indeed, this follows from the fact that \(\text{Mon}(D)\) is complete and inherits the factorization system of \(D\), see Lemma 4.18 Analogously, since \(\text{Mon}_f(D)\) is finitely complete, \(\Sigma\text{-Mon}_f(D)\) is a join-semilattice, in fact a join-subsemilattice of \(\Sigma\text{-Mon}(D)\).

\(\Sigma\)-generated monoids are closely related to algebras for the functor \(L_\Sigma\).

**Notation 4.26.** In analogy to Notation 4.1 we represent \(L_\Sigma\)-algebras \(1 + \prod \Sigma A \xrightarrow{\alpha} A\) as triples
\[
A = (A, \alpha_i : A \to A, i : 1 \to A).
\]
For any \(L_\Sigma\)-algebra \(A = (A, \alpha_a, i)\) and \(w \in \Sigma^*\) we put
\[
A_w := (A, \alpha_a, \alpha_w \cdot i)
\]
where \(\alpha_w = \alpha_a \cdots \alpha_{a_j}\) for \(w = a_1 \cdots a_n \in \Sigma^*\).
**Remark 4.27.** Dually to Remark 4.2, a \( \mathcal{D} \)-morphism \( h : A \to A' \) is an \( L_\Sigma \)-algebra homomorphism \( h : (A, \alpha_a, i) \to (A', \alpha'_{a'}, i') \) iff the following diagram commutes for all \( a \in \Sigma \):

\[
\begin{array}{ccc}
1 & \xrightarrow{i} & A \\
\downarrow & & \downarrow h \\
A' & \xrightarrow{\alpha'_{a'}} & A'
\end{array}
\]

In this case also the square below commutes, which implies that \( h \) is \( L_\Sigma \)-coalgebra homomorphism \( h : A_w \to A'_w \) for all \( w \in \Sigma^* \).

\[
\begin{array}{ccc}
A & \xrightarrow{\alpha_w} & A \\
\downarrow h & & \downarrow h \\
A' & \xrightarrow{\alpha'_{w}} & A'
\end{array}
\]

**Remark 4.28.** Every \( \Sigma \)-generated \( \mathcal{D} \)-monoid \( e : \Psi \Sigma^* \to (A, \circ, i) \) induces an \( L_\Sigma \)-algebra \( \overline{A} = (A, \alpha_a, \overline{i}) \) where \( \alpha_a = - \circ e(\eta a) : A \to A \) and \( \overline{i} : 1 \to 1 \) is the free extension of \( i : 1 \to A \). In particular, for \( e = \text{id} \) we obtain

\[
\Psi \Sigma^* = (\Psi \Sigma^*, r_a, \overline{\eta}).
\]

Clearly every generator-preserving morphism \( f : A \to B \) of \( \Sigma \)-generated \( \mathcal{D} \)-monoids is also an \( L_\Sigma \)-algebra homomorphism \( f : \overline{A} \to \overline{B} \).

**Proposition 4.29.** \( \overline{\Psi \Sigma^*} = \mu L_\Sigma \).

**Proof.** Consider the functor \( L_\Sigma^0 = 1 + \text{id}^\Sigma \) on Set. Since \( L_\Sigma^0 \) and \( L_\Sigma \) are finitary, their initial algebras arise as colimits of the respective initial chains:

\[
\varnothing \to 1 \to 1 + \bigsqcup \Sigma 1 \to \ldots \quad 0 \to 1 \to 1 + \bigsqcup \Sigma 1 \to \ldots
\]

Now \( \Psi \) preserves colimits (being a left adjoint) and \( 1 = \Psi 1 \), so \( \Psi \) maps the initial chain of \( L_\Sigma^0 \) to the initial chain of \( L_\Sigma \) and preserves the colimit of the left chain, which implies that \( \Psi (\mu L_\Sigma^0) = \mu L_\Sigma \). Since \( \mu L_\Sigma = (\Sigma^*, \cdot, a, \varepsilon) \) we have \( \mu L_\Sigma = (\Psi \Sigma^*, \Psi (\cdot, a), \overline{\eta}) \). Moreover \( \Psi (\cdot, a) = r_a \) by the definition of \( r_a \), so

\[
\mu L_\Sigma = (\Psi \Sigma^*, r_a, \overline{\eta}) = \overline{\Psi \Sigma^*}.
\]

**Proposition 4.30.** \( \Sigma \text{-Mon}(\mathcal{D}) \) is a sublattice of \( \text{Quo}(\mu L_\Sigma) \), and \( \Sigma \text{-Mon}_f(\mathcal{D}) \) is a sub-semilattice of \( \text{Quo}_f(\mu L_\Sigma) \).

**Proof.** Remark 4.28 and Proposition 4.29 show that every (finite) \( \Sigma \)-generated \( \mathcal{D} \)-monoid \( e : \Psi \Sigma^* \to A \) induces a (finite) quotient algebra \( e : \mu L_\Sigma \to \overline{A} \) of \( \mu L_\Sigma \) carried by the same morphism \( e \). We show that the map \( \overline{(-)} : \Sigma \text{-Mon}(\mathcal{D}) \to \text{Quo}(\mu L_\Sigma) \) and its restriction \( \overline{(-)} : \Sigma \text{-Mon}_f(\mathcal{D}) \to \text{Quo}_f(\mu L_\Sigma) \) are order-embeddings. Clearly \( \overline{(-)} \) is
injective and monotone. To show that \( \tilde{\pi} \) it is order-reflecting, consider a commutative diagram as below, where \( e \) and \( e' \) are \( D \)-monoid morphisms and \( f : \tilde{A} \rightarrow \tilde{A}' \) is an \( L_\Sigma \)-algebra homomorphism.

\[
\begin{array}{ccc}
\mu L_\Sigma = \Psi \Sigma^* \\
A & \xrightarrow{e} & \tilde{A}' \\
\downarrow f & & \downarrow e' \\
A & \xrightarrow{e} & \tilde{A}'
\end{array}
\]

We need to show that \( f \) is a monoid morphism. Let \( x', y' \in A \) and choose \( x, y \in \Psi \Sigma^* \) with \( e(x) = x' \) and \( e(y) = y' \), using that \( e \) is surjective by Assumption 4.8(a). Then

\[
f(x' \circ y') = f(e(x \bullet y)) = e'(x \bullet y) = e'x \circ e'y = f e(x) \circ f e(y) = f x' \circ f y'
\]

and moreover \( f \) preserves the unit because \( f \) is an \( L_\Sigma \)-algebra homomorphism. \( \square \)

Hence \( \Sigma \)-Mon(\( D \)) is isomorphic to a sublattice of Quo(\( \mu L_\Sigma \)). The following proposition characterizes its elements in terms of \( L_\Sigma \)-algebra homomorphisms.

**Proposition 4.31.** A quotient algebra \( e : \mu L_\Sigma \rightarrow (A, \alpha_a, i) \) of \( \mu L_\Sigma \) is induced by a \( \Sigma \)-generated \( D \)-monoid if and only if there exists an \( L_\Sigma \)-algebra homomorphism from \( A \) to \( A_w \) for each \( w \in \Sigma^* \).

**Proof.** \((\Rightarrow)\) Suppose that \( e : \mu L_\Sigma \rightarrow (A, \alpha_a, i) \) is induced by some \( \Sigma \)-generated \( D \)-monoid \( e : \Psi \Sigma^* \rightarrow (A, \circ, i') \), that is, \( \alpha_a = \circ e(\eta a) \) and \( i = \eta i' \). For each \( w \in \Sigma^* \) consider the \( D \)-morphism

\[
\beta_w = e(\eta w) \circ : A \rightarrow A.
\]

We claim that \( \beta_w \) is an \( L_\Sigma \)-algebra homomorphism

\[
\beta_w : (A, \alpha_a, i) \rightarrow (A, \alpha_a, \alpha_w \cdot i),
\]

that is, a homomorphism \( \beta_w : A \rightarrow A_w \). In fact, we have for each \( x \in A \) and \( a \in \Sigma \):

\[
\beta_w(\alpha_a(x)) = e(\eta w) \circ (x \circ e(\eta a)) \quad \text{def. } \beta_w, \alpha_a
\]

\[
= (e(\eta w) \circ x) \circ e(\eta a) \quad \text{associativity}
\]

\[
= \alpha_a(\beta_w(x)) \quad \text{def. } \beta_w, \alpha_a
\]
so $\beta_w \cdot \alpha_a = \alpha_a \circ \beta_w$. To prove preservation of initial states (i.e., $\beta_w \cdot i = \alpha_w \cdot i$) it suffices to prove $\beta_w \cdot i \cdot \eta_1 = \alpha_w \cdot i \circ \eta_1$ where $\eta_1 : 1 \to \Psi 1 = 1$ is the unit. We compute:

$$\begin{align*}
\beta_w \cdot i \cdot \eta_1 (*)(\ast) &= \beta_w(i') \\
&= e(\eta w) \circ i' \\
&= e(\eta w) \\
&= e(\eta \varepsilon \cdot \eta w) \\
&= e(\varepsilon)(\eta \varepsilon) \\
&= \alpha_w(e(\varepsilon)) \\
&= \alpha (i'(\eta_1 (*))) \\
&= \alpha_w \cdot i \circ \eta_1 (*)
\end{align*}$$

Hence $\beta_w : A \to A_w$ is an $L_{\Sigma^*}$-algebra homomorphism as claimed.

($\Leftarrow$) Suppose that an $L_{\Sigma^*}$-algebra homomorphism $\beta_w : A \to A_w$ is given for each $w \in \Sigma^*$, and let $(\beta_x : A \to A)_{x \in \Psi \Sigma^*}$ and $(\alpha_x : A \to A)_{x \in \Psi \Sigma^*}$ be the inductive extensions of the families $(\beta_w : A \to A)_{w \in \Sigma^*}$ and $(\alpha_w : A \to A)_{w \in \Sigma^*}$ of $D$-morphisms.

(1) For all $x \in \Psi \Sigma^*$, let $A_x := (A, \alpha_a, \alpha_x \cdot i)$. We claim that $\beta_x : A \to A_x$ is an $L_{\Sigma^*}$-algebra homomorphism, which means that the following squares commute:

$$\begin{array}{c}
A \xrightarrow{\beta_x} A \\
\downarrow \alpha_x \quad \quad \quad \quad \quad \quad \quad \downarrow \alpha_x \\
A \xrightarrow{\beta_x} A
\end{array}$$

Indeed, they clearly commute if $x = \eta (w)$ for some $w \in \Sigma^*$ because $\beta_{\eta w} = \beta_w : A \to A_w$ is an $L_{\Sigma^*}$-algebra homomorphism, and therefore they commute for all $x$ by the induction principle (Lemma 4.14).

(2) We prove the equation

$$e(x \bullet y) = \alpha_y (e(x))$$

for all $x, y \in \Psi \Sigma^*$, where $\bullet$ is the multiplication of the free $D$-monoid $\Psi \Sigma^* = \mu L_{\Sigma^*}$. Observe first that the following diagram commutes for all $y \in \Psi \Sigma^*$:

$$\begin{array}{c}
\Psi \Sigma^* \xrightarrow{r_y} \Psi \Sigma^* \\
\downarrow e \quad \quad \quad \quad \quad \quad \quad \quad \downarrow e \\
A \xrightarrow{\alpha_y} A
\end{array}$$

In fact, it commutes if $y = \eta w$ for some $w \in \Sigma^*$ because $e$ is an $L_{\Sigma^*}$-algebra homomorphism, so it commutes for all $y$ by induction. Therefore

$$e(x \bullet y) = e(r_y(x)) = \alpha_y (e(x)).$$
(3) We prove the equation
\[ e(x \cdot y) = \beta_x(e(y)) \quad \text{for all } x, y \in \Psi \Sigma^*. \]

First note that \( l_x : \mu L \Sigma \to (\mu L \Sigma)_x \) is an \( L \Sigma \)-algebra homomorphism: we have
\[ l_x \cdot r_a = r_a \cdot l_x \text{ by Lemma } 4.20(a) \]
and \( l_x \cdot \eta e = r_x \cdot \eta e \) because
\[ l_x \cdot \eta e \cdot \eta_1(*) = l_x(\eta e) = x \cdot \eta e = \eta e \cdot x = r_x(\eta e) = r_x \cdot \eta e \cdot \eta_1(*). \]

Since also \( \beta_x : A \to A_x \) is an \( L \Sigma \)-algebra homomorphism by (1), the following diagram commutes by initiality of \( \mu \hat{L} \Sigma \):

\[
\begin{array}{ccc}
\mu L \Sigma & \xrightarrow{l_x} & (\mu L \Sigma)_x \\
\downarrow e & & \downarrow e \\
A & \xrightarrow{\beta_x} & A_x
\end{array}
\]

Therefore
\[ e(x \cdot y) = e(l_x(y)) = \beta_x(e(y)). \]

(4) We define the desired monoid structure on \( A \). The unit is \( i \cdot \eta_1(*) \in A \), and the multiplication is given as follows: for all \( x', y' \in A \), choose \( x, y \in \Psi \Sigma^* \) with \( x' = e(x) \) and \( y' = e(y) \) (using that \( e \) is surjective by Assumption 4.8(a)) and put
\[ x' \circ y' := e(x \cdot y). \]

We need to prove that \( x' \circ y' \) is well-defined, i.e., independent of the choice of \( x \) and \( y \). In fact, by (2) above, \( x' \circ y' \) is independent of the choice of \( x \) and moreover \( - \circ y' = \alpha_y \). Analogously (3) states that \( x' \circ y' \) is independent of the choice of \( y \) and that \( x' \circ - = \beta_x \). It follows that \( \circ : A \times A \to A \) is a well-defined \( D \)-bimorphism, and by definition we have \( e(x \cdot y) = e(x) \circ e(y) \) for all \( x, y \in \Psi \Sigma^* \). The associativity and unit laws for \( \circ \) hold in \( A \) because they hold in in \( \Psi \Sigma^* \) and \( e \) is surjective, concluding the proof that \( (A, \circ, i \cdot \eta_1(*)) \) is a \( D \)-monoid. Moreover, \( e \) is clearly a monoid morphism
\[ e : \Psi \Sigma^* \to (A, \circ, i \cdot \eta_1(*)), \]

and the quotient algebra of \( \mu L \Sigma \) it induces is precisely \( (A, \alpha, i) \). For the latter we need to show \( - \circ e(\eta a) = \alpha_a \) for all \( a \in \Sigma \). Given \( x' \in A \), we choose \( x \in \Psi \Sigma^* \) with \( e(x) = x' \) and compute
\[ x' \circ e(\eta a) = e(x \cdot \eta a) = e(r_a(x)) = \alpha_a(e(x)) = \alpha_a(x'), \]

using the definitions of \( \circ \) and \( \cdot \) and the fact the \( e \) is an \( L \Sigma \)-algebra homomorphism. □
4.3 The Local Eilenberg Theorem

We can now put our (co-)algebraic characterizations of right derivatives and monoids together to prove the general local Eilenberg theorem. The key result is

Proposition 4.32. The semilattices $\text{Sub}_f(qT_{\Sigma})$ and $\Sigma$-Mon$_f(D)$ are isomorphic.

Proof. We show that the isomorphism $\text{Sub}_f(qT_{\Sigma}) \cong \text{Quo}_f(qT_{\Sigma})$ of Proposition 3.14 restricts to an isomorphism $\text{Sub}_f(qT_{\Sigma}) \cong \Sigma$-Mon$_f(D)$. Indeed, a finite subcoalgebra $(Q, \gamma_a, f) \hookrightarrow qT_{\Sigma}$ is closed under right derivatives if and only if a $T_{\Sigma}$-coalgebra homomorphism $Q_w \rightarrow Q$ exists for each $w \in \Sigma^*$ (see Proposition 4.3). By Remark 3.8 the $L_{\Sigma}$-algebra dual to $(Q, \gamma_a, f)$ is $\hat{Q} = (\hat{Q}, \hat{\gamma}_a, \hat{f})$, and the one dual to $Q_w = (Q, \gamma_a, f \cdot \gamma_w)$ is $\hat{Q}_w = (\hat{Q}, \hat{\gamma}_a, \hat{\gamma}_w \cdot \hat{f}) = \hat{Q}_{w^r}$, where $w^r$ is the reversed word of $w$. Indeed, for $w = a_1 \ldots a_n$ we have $\gamma_w = \gamma_{a_1} \ldots \gamma_{a_n}$. Hence by duality an $L_{\Sigma}$-algebra homomorphism $\hat{Q} \rightarrow \hat{Q}_{w^r}$ exists for each $w \in \Sigma^*$, which by Proposition 4.30 and 4.31 means precisely that $\hat{Q} \in \Sigma$-Mon$_f(D)$. \qed

Definition 4.33. A local variety of regular languages in $C$ is a subcoalgebra of $qT_{\Sigma}$ closed under right derivatives.

Example 4.34. (a) A local variety of regular languages is a set of regular languages over $\Sigma$ closed under the boolean operations and derivatives and containing $\emptyset$. This is the case $C = BA$.  

(b) A local lattice variety of regular languages is a set of regular languages over $\Sigma$ closed under union, intersection and derivatives and containing $\emptyset$ and $\Sigma^*$. This is the case $C = DL_{01}$.  

(c) A local semilattice variety of regular languages is a set of regular languages over $\Sigma$ closed under union and derivatives and containing $\emptyset$. This is the case $C = JSL_{01}$.  

(d) A local linear variety of regular languages is a set of regular languages over $\Sigma$ closed under symmetric difference and derivatives and containing $\emptyset$. This is the case $C = Vect Z_2$.

Definition 4.35. A local pseudovariety of $D$-monoids is a set of finite $\Sigma$-generated $D$-monoids closed under subdirect products and quotients, i.e., an ideal in the join-semilattice $\Sigma$-Mon$_f(D)$.

This leads to the main result of this paper. For convenience, we recall all assumptions used so far in the statement of the theorem.

Theorem 4.36 (General Local Eilenberg Theorem). Let $C$ and $D$ be predual locally finite varieties of algebras, where the algebras in $D$ are possibly ordered. Suppose further that $D$ is monoidal closed w.r.t. tensor product, epimorphisms in $D$ are surjective, and the free algebra in $D$ on one generator is dual to a two-element algebra in $C$. Then there is a lattice isomorphism

$$\text{local varieties of regular languages in } C \cong \text{local pseudovarieties of } D\text{-monoids}.$$
Proof. By Proposition 4.32 we have a semilattice isomorphism

$$\text{Sub}_f(gT_{\Sigma}) \cong \Sigma\text{-Mon}_f(D)$$

Taking ideal completions on both sides yields a complete lattice isomorphism

$$\text{Ideal}(\text{Sub}_f(gT_{\Sigma})) \cong \text{Ideal}(\Sigma\text{-Mon}_f(D))$$

The ideals of the join-semilattice $\Sigma\text{-Mon}_f(D)$ are by definition precisely the pseudovarieties of $D$-monoids. Moreover, Proposition 4.7 yields

$$\text{Ideal}(\text{Sub}_f(gT_{\Sigma})) \cong \text{Sub}_f(gT_{\Sigma})$$

and the elements of $\text{Sub}_f(gT_{\Sigma})$ are by definition precisely the local varieties of regular languages in $C$. \qed

Corollary 4.37. By instantiating Theorem 4.36 to the categories of Example 3.2 we obtain the following lattice isomorphisms:

| $C$     | $D$     | local varieties of regular languages $\cong$ local pseudovarieties of . . . |
|---------|---------|--------------------------------------------------------------------------------|
| BA      | Set     | local varieties $\cong$ monoids                                                   |
| DL_{01} | Pos     | local lattice varieties $\cong$ ordered monoids                                   |
| JSL_{0} | JSL_{0} | local semilattice varieties $\cong$ idempotent semirings                          |
| Vect $\mathbb{Z}_2$ | Vect $\mathbb{Z}_2$ | local linear varieties $\cong$ $\mathbb{Z}_2$-algebras |

The first two isomorphisms were proved in [11, 12], the last two are new local Eilenberg correspondences.

5 Profinite Monoids

In [11, 12] Gehrke, Grigorieff and Pin demonstrated that the boolean algebra $\text{Reg}_{\Sigma}$, equipped with left and right derivatives, is dual to the free $\Sigma$-generated profinite monoid. In this section we generalize this result to our categorical setting (Assumptions 3.4). For this purpose we will introduce below a category $\hat{D}$ that is dually equivalent (rather than just predual) to $C$, and arises as a “profinite” completion of $D_f$.

Definition 5.1. (a) Dually to Definition 2.1, an object $X$ of a category $B$ is called co-finitely presentable if the hom-functor $B(\cdot, X) : B \to \text{Set}^{op}$ is cofinitary, i.e., preserves cofiltered limits. The full subcategory of all co-finitely presentable objects is denoted by $B_{\text{cofp}}$. The category $B$ is locally co-finitely presentable if $B_{\text{cofp}}$ is essentially small, $B$ is complete and every object arises as a cofiltered limit of co-finitely presentable objects.

(b) The dual of Ind is denoted by Pro: the free completion under cofiltered limits of a small category $A$ is a full embedding $A \to \text{Pro}A$ such that $\text{Pro}A$ has cofiltered limits and every functor $F : A \to B$ into a category $B$ with cofiltered colimits has an essentially unique cofinitary colimit $\hat{F} : \text{Pro}A \to B$.
Note that \((\text{Pro } \mathcal{A})^{\text{op}} \cong \text{Ind}(\mathcal{A}^{\text{op}})\). If \(\mathcal{A}\) has finite limits then \(\text{Pro } \mathcal{A}\) is locally cofinitely presentable and \((\text{Pro } \mathcal{A})_{\text{cfp}}^{\text{op}} \cong \mathcal{A}\). Conversely, for every locally cofinitely presentable category \(\mathcal{B}\) one has \(\mathcal{B} \cong \text{Pro}(\mathcal{B}_{\text{cfp}})\).

**Notation 5.2.** Let \(\hat{\mathcal{D}}\) denote the free completion of \(\mathcal{D}\) under cofiltered limits, i.e.,

\[
\hat{\mathcal{D}} = \text{Pro}(\mathcal{D}_{\text{cfp}}).
\]

**Remark 5.3.** \(\hat{\mathcal{D}}\) is dually equivalent to \(\mathcal{C}\) since

\[
\mathcal{C} = \text{Ind}(\mathcal{C}_f) \cong \text{Ind}(\mathcal{D}_f^{\text{op}}) \cong \text{Pro}(\mathcal{D}_f)^{\text{op}} = \hat{\mathcal{D}}^{\text{op}}.
\]

Moreover, the equivalence functors

\[
\begin{array}{ccc}
\mathcal{C} \cong & \text{Ind}(\mathcal{C}_f) & \cong \text{Ind}(\mathcal{D}_f^{\text{op}}) \\
\text{Ind}(\mathcal{C}_f) \cong & \text{Pro}(\mathcal{D}_f)^{\text{op}} & \cong \hat{\mathcal{D}}^{\text{op}} \\
\end{array}
\]

extend essentially uniquely to equivalences – also called \(\sim\) and \(\cong\) – between \(\mathcal{C}\) and \(\hat{\mathcal{D}}^{\text{op}}\):

\[
\begin{array}{ccc}
\mathcal{C} = \text{Ind}(\mathcal{C}_f) & \overset{\sim}{\cong} & \hat{\mathcal{D}}^{\text{op}} \\
\text{Ind}(\mathcal{C}_f) & \overset{\cong}{\cong} & \text{Pro}(\mathcal{D}_f)^{\text{op}} = \hat{\mathcal{D}}^{\text{op}} \\
\end{array}
\]

**Example 5.4.** For the predual varieties \(\mathcal{C}\) and \(\mathcal{D}\) of Example 3.2 we have the following categories \(\hat{\mathcal{D}}\):

| \(\mathcal{C}\) | \(\mathcal{D}\) | \(\hat{\mathcal{D}}\) |
|----------------|----------------|----------------|
| BA | Set | Stone |
| DL_{01} | Pos | Priest |
| JSL_{0} | JSL_{0} | JSL_{0} in Stone |
| Vect \(\mathbb{Z}_2\) | Vect \(\mathbb{Z}_2\) | Vect \(\mathbb{Z}_2\) in Stone |

In more detail:

(a) For \(\mathcal{C} = \text{BA}\) and \(\mathcal{D} = \text{Set}\), we have the classical Stone duality: \(\hat{\mathcal{D}}\) is the category of Stone spaces (i.e., compact Hausdorff spaces with a base of clopen sets) and continuous maps. The equivalence functor \(\sim: \text{Stone}^{\text{op}} \to \text{BA}\) assigns to each Stone space the boolean algebra of clopen sets, and its associated equivalence \(\cong: \text{BA} \to \text{Stone}^{\text{op}}\) assigns to each boolean algebra the Stone space of all ultrafilters.

(b) For the category \(\mathcal{C} = \text{DL}_{01}\) and \(\mathcal{D} = \text{Pos}\) we have the classical Priestly duality: \(\hat{\mathcal{D}}\) is the category of Priestley spaces (i.e., ordered Stone spaces such that given \(x \nleq y\) there is a clopen set containing \(x\) but not \(y\)) and continuous monotone maps. The equivalence functor \(\sim: \text{Priest}^{\text{op}} \to \text{DL}_{01}\) assigns to each Priestley space the lattice of all clopen upsets, and its associated equivalence \(\cong: \text{DL}_{01} \to \text{Priest}^{\text{op}}\) assigns to each distributive lattice the Priestley space of all prime filters.
(c) For \( C = D = JSL_0 \) the dual category \( \hat{D} \) is the category of join-semilattices in Stone. Similarly, for \( C = D = V e c t \mathbb{Z}_2 \) the dual category \( \hat{D} \) is the category of \( \mathbb{Z}_2 \)-vector spaces in Stone, see [13].

**Definition 5.5.** We denote by \( \hat{L} : \hat{D} \to \hat{D} \) the dual of the functor \( T : C \to C \), i.e., the essentially unique functor for which the following diagram commutes up to natural isomorphism:

\[
\begin{array}{ccc}
\hat{D}^{op} & \rightarrow & \hat{D}^{op} \\
\downarrow \cong & & \downarrow \cong \\
C & \xrightarrow{T} & C
\end{array}
\]

**Remark 5.6.** In analogy to Remark 3.8, the categories \( Coalg T \) and \( Alg \hat{L} \) are dually equivalent: the equivalence \( \cong : C \xrightarrow{\cong} D^{op} \) lifts to an equivalence \( Coalg T \xrightarrow{\cong} (Alg \hat{L})^{op} \) given by

\[
(Q \xrightarrow{\cong} TQ) \mapsto (\hat{L},Q = \hat{T}Q \xrightarrow{\cong} \hat{Q}).
\]

**Example 5.7.** The dual of the endofunctor \( T_{\Sigma}Q = \Sigma \times Q^\Sigma \) of \( C \), see Example 2.6, is the endofunctor of \( \hat{D} \)

\[
\hat{L}_{\Sigma}Z = 1 + \coprod_{\Sigma} Z
\]

where \( 1 = \hat{2} \). In \( \hat{D} = \text{Stone} \) the object \( 1 \) is the one-element space. Hence, by the universal property of the coproduct, an \( \hat{L}_{\Sigma} \)-algebra \( \hat{L}_{\Sigma}Z = 1 + \coprod_{\Sigma} Z \to Z \) is a deterministic \( \Sigma \)-automaton (without final states) in Stone, given by a Stone space \( Z \) of states, continuous transition functions \( \alpha_a : Z \to Z \) for \( a \in \Sigma \), and an initial state \( 1 \to Z \). Analogously for the other dualities of Example 5.4.

**Notation 5.8.** The category of all \( \hat{L} \)-algebras with a cofinitely presentable carrier (shortly \( cfp \)-algebras) is denoted by \( Alg_{cfp} \hat{L} \).

**Remark 5.9.** Note that \( Alg_{cfp} \hat{L} \cong Alg_{f} L \) because the restrictions of \( \hat{L} \) and \( L \) to \( \hat{D}_{cfp} \cong D_f \) are naturally isomorphic.

**Definition 5.10.** Dually to Definition 2.7 an \( \hat{L} \)-algebra is called **locally cofinitely presentable** if it is a cofiltered limit of \( cfp \)-algebras.

**Remark 5.11.** The category of all locally cofinitely presentable algebras is equivalent to \( Pro(Alg_{cfp} \hat{L}) \). This is the dual of Theorem 2.14. The initial object \( \tau \hat{L} \) is what one can call the **dual of the rational fixpoint**. By the dual of Definition 2.7 one can construct \( \tau \hat{L} \) as the limit of all algebras in \( Alg \hat{L} \) with cofinitely presentable carrier.

**Example 5.12.** (a) For \( C = BA \) and \( \hat{D} = \text{Stone} \), we have

\[
\tau \hat{L}_{\Sigma} = \text{ultrafilters of regular languages}.
\]
(b) Analogously, for $C = DL_{01}$ and $\hat{D} = Priest$, we have 
\[ \tau L^\Sigma = \text{prime filters of regular languages.} \]

**Definition 5.13.** We denote by $F : D \to \hat{D}$ the essentially unique finitary functor for which 
\[ \xymatrix{ D & \ar[l]_{D_f} \ar[r]^{\text{Ind}(D_f)} & Pro(D_f) = \hat{D} \ar[l]_F } \]
commutes, and by $U : \hat{D} \to D$ the essentially unique cofinitary functor for which 
\[ \xymatrix{ \hat{D} & \ar[l]_{D_f} \ar[r]^{\text{Pro}(D_f)} & \text{Ind}(D_f) = D \ar[l]_U } \]
commutes.

**Lemma 5.14.** The functors $F$ and $U$ are well-defined and $F$ is a left adjoint to $U$.

*Proof.* $F$ is well-defined because $\hat{D}$ has filtered colimits: being locally cofinitely presentable it is cocomplete. Analogously for $U$. Furthermore, $F$ and $U$ form an adjunction: every object $A \in D$ is a filtered colimit 
\[ A = \colim_{i \in I} A_i \text{ with } A_i \in D_f. \]
This implies, since $FA_i = A_i$, that 
\[ FA = \colim_{i \in I} A_i \text{ in } \hat{D}. \]
Analogously, every object $B \in \hat{D}$ is a cofiltered limit 
\[ B = \lim_{j \in J} B_j \text{ with } B_j \in D_f. \]
This implies, since $UB_j = B_j$, that, 
\[ UB = \lim_{j \in J} B_j \text{ in } D. \]
Consequently we have the desired natural isomorphism 
\[ \hat{D}(FA, B) \cong \lim_{i \in I} \lim_{j \in J} D_f(A_i, B_j) \cong D(A, UB). \]

**Example 5.15.** 1. For $C = BA$, $D = Set$ and $\hat{D} = Stone$, the functor $F : Set \to Stone$ is the Stone-Čech compactification and $U : Stone \to Set$ is the forgetful functor.
2. For $C = DL_{01}$, $D = Pos$ and $\hat{D} = Priest$, the functor $F : Pos \to Priest$ constructs the free Priestley space on a poset and $U : Priest \to Pos$ is the forgetful functor.
Definition 5.16. Let \( \hat{U} : \text{Pro}(\text{Alg}_{cfp} \hat{L}) \to \text{Alg} \hat{L} \) denote the essentially unique cofinitary functor that makes the triangle below commute:

\[
\begin{array}{ccc}
\text{Alg}_{cfp} \hat{L} & \cong & \text{Alg} \hat{L} \\
\downarrow \hat{U} & & \downarrow \hat{U} \\
\text{Pro}(\text{Alg}_{cfp} \hat{L}) & \rightarrow & \text{Alg} \hat{L}
\end{array}
\]

Example 5.17. For \( T_\Sigma Q = 2 \times Q_\Sigma : \text{BA} \to \text{BA} \) we have \( \hat{L}_\Sigma Z = 1 + \coprod \Sigma Z : \text{Stone} \to \text{Stone} \) and \( \hat{L}_\Sigma A = 1 + \coprod \Sigma A : \text{Set} \to \text{Set} \). The objects of \( \text{Pro}(\text{Alg}_{cfp} \hat{L}_\Sigma) \) are the locally cofinitely presentable \( \hat{L}_\Sigma \)-algebras, and the functor \( \hat{U} : \text{Pro}(\text{Alg}_{cfp} \hat{L}_\Sigma) \to \text{Alg} \hat{L}_\Sigma \) simply forgets the topology on the carrier of an \( \hat{L}_\Sigma \)-algebra.

Proposition 5.18. \( \hat{U} \) is a right adjoint.

Proof. We have a commutative square

\[
\begin{array}{ccc}
\text{Pro}(\text{Alg}_{cfp} \hat{L}) & \xrightarrow{\hat{U}} & \text{Alg} \hat{L} \\
\downarrow & & \downarrow \\
\text{Pro} \hat{D}_{cfp} = \hat{D} & \xrightarrow{\hat{U}} & \hat{D}
\end{array}
\]

where the vertical functors are the obvious forgetful functors. Recall that limits in \( \text{Alg} \hat{L} \) are formed on the level of \( \hat{D} \), analogously for limits in \( \text{Pro}(\text{Alg}_{cfp} \hat{L}) \). Since \( U \) preserves limits by Lemma 5.14, we conclude that so does \( \hat{U} \). By the Special Adjoint Functor Theorem, \( \hat{U} \) is a right adjoint: the category \( \text{Pro}(\text{Alg}_{cfp} \hat{L}) \) is complete, \( \text{Alg}_{cfp} \hat{L} \) is its (essentially small) cogenerator, and \( \text{Pro}(\text{Alg}_{cfp} \hat{L}) \) is wellpowered. Indeed, it is dual to a locally finitely presentable category which is co-wellpowered by [4, Theorem 1.58].

Remark 5.19. It follows that the left adjoint \( \hat{F} \) of \( \hat{U} \) maps the initial \( L \)-algebra to the initial locally cofinitely presentable \( \hat{L} \)-algebra: \( \hat{F}(\mu L) = \tau \hat{L} \). One can prove that \( \hat{F} \) assigns to every \( L \)-algebra \( \alpha : LA \to A \) the limit of the diagram of all its quotients in \( \text{Alg}_f \hat{L} = \text{Alg}_{cfp} \hat{L} \). Thus, we see that \( \tau \hat{L} \) can be constructed as the limit (taken in \( \text{Alg} \hat{L} \)) of all finite quotient \( \Sigma \)-algebras of \( \mu L \). This construction generalizes a similar one given by Gehrke [10].

Remark 5.20. Under the Assumptions 4.8 of the previous section we obtain a generalization of the result of Gehrke, Grigorieff and Pin [11, 12] that \( \text{Reg} \Sigma \) endowed with boolean operations and derivatives is dual to the free profinite monoid on \( \Sigma \). By Proposition 4.32 and Lemma 4.18 the finite \( \Sigma \)-generated \( D \)-monoids form a cofinal subposet of \( \text{Quo}_f(\mu L_\Sigma) \). Thus, the corresponding diagrams have the same limit in \( \text{Alg} \hat{L}_\Sigma \). Hence by the previous remark \( \tau \hat{L}_\Sigma \) is also the limit of the directed diagram of all finite \( \Sigma \)-generated \( D \)-monoids. Since \( \hat{U} \) preserves this limit and the forgetful functor \( \text{Mon} \hat{D} \) creates limits (see Lemma 4.18) it follows that \( \hat{U}(\tau \hat{L}_\Sigma) \) carries the structure of a \( D \)-monoid and it is then easy to see that it is the free profinite \( D \)-monoid on \( \Sigma \): for every
$\mathcal{D}$-monoid morphism $e : \Psi \Sigma^* \to M$ into a finite $\mathcal{D}$-monoid $M$, there exists a unique $L\Sigma$-algebra homomorphism $\tilde{\pi} : \tau L\Sigma \to M$ such that $\tilde{U}\tilde{\pi}$ is a $\mathcal{D}$-monoid morphism and the diagram below commutes:

\[ \begin{array}{c}
\Psi \Sigma^* \\
\downarrow \tilde{\eta}
\end{array} \xymatrix{
\hat{\Psi}^* \mu L\Sigma \\
\hat{\Psi} \tau L\Sigma \\
\hat{U} \hat{F} M
\} \end{array} \]

Here $\tilde{\eta}$ is the unit of the adjunction $\hat{F} \dashv \hat{U}$. In summary:

**Theorem 5.21.** Under the assumptions of the General Local Eilenberg Theorem, $\tau \hat{L}\Sigma$ is the free profinite $\mathcal{D}$-monoid on $\Sigma$.

### 6 A Categorical Framework

Although we have assumed $\mathcal{C}$ and $\mathcal{D}$ to be locally finite varieties throughout this paper, our methodology was purely categorical (rather than algebraic) in spirit. In fact, all our results and their proofs can be adapted to the following categorical setting:

1. $\mathcal{C}$ and $\mathcal{D}$ are predual categories, i.e., the categories $\mathcal{C}_f$ and $\mathcal{D}_f$ of finitely presentable objects are dually equivalent.
2. $\mathcal{C}$ has the following additional properties:
   (a) $\mathcal{C}$ is locally finitely presentable.
   (b) $\mathcal{C}_f$ is closed under strong epimorphisms.
   (c) $\mathcal{C}$ is concrete, i.e., a faithful functor $\cdot c : \mathcal{C} \to \text{Set}$ is given.
   (d) $\cdot c$ is a finitary right adjoint and maps finitely presentable objects to finite sets.
   (e) $\cdot c$ is amnestic, i.e., every subset $B \subseteq A|_c$, where $A$ is an object of $\mathcal{C}$, is carried by at most one subobject of $A$ in $\mathcal{C}$.
3. $\mathcal{D}$ has the following additional properties:
   (a) $\mathcal{D}$ is locally finitely presentable.
   (b) $\mathcal{D}_f$ is closed under strong monomorphisms and finite products.
   (c) $\mathcal{D}$ is concrete, i.e., a faithful functor $\cdot d : \mathcal{D} \to \text{Set}$ is given.
   (d) $\cdot d$ is a finitary right adjoint and maps finitely presentable objects to finite sets.
   (e) $\cdot d$ preserves epimorphisms.
   (f) For any two objects $A$ and $B$ of $\mathcal{D}$, there exists an embedding $[A, B] \xrightarrow{e_{A,B}} B|_A$ given by hom-sets, i.e.,
      i. $[A, B]$ has the underlying set $\mathcal{D}(A, B)$,
      ii. $e_{A,B}$ is a monomorphism,
      iii. $[e_{A,B}]$ takes $f : A \to B$ to its underlying function $|f| \in |B| |_A$, and
      iv. whenever a $\mathcal{D}$-morphism $h : X \to [A, B]$ factorizes through $e_{A,B}$ in $\text{Set}$, it factorizes through $e_{A,B}$ in $\mathcal{D}$.
4. The $\mathcal{D}$-object $1 = \Psi 1$, where $\Psi : \text{Set} \to \mathcal{D}$ is the left adjoint of $\cdot d$, is dual to an object $2$ of $\mathcal{C}$ with two-element underlying set.
5. $T: C \to C$ and $L: D \to D$ are finitary predual functors, i.e., they restrict to functors $T_f: C_f \to C_f$ and $L_f: D_f \to D_f$ and these restrictions are dual. Moreover, $T$ preserves monomorphisms and preimages, and $L$ preserves epimorphisms. In Section 4 one works with the functors $T = T_\Sigma = 2 \times \text{Id}_\Sigma$ on $C$ and $L = L_\Sigma = 1 + \bigsqcup \Sigma \text{Id}$ on $D$.

7 Conclusions and Future Work

Inspired by recent work of Gehrke, Grigorieff and Pin [11, 12] we have proved a generalized local Eilenberg theorem, parametric in a pair of dual categories $C$ and $\hat{D}$ and a type of coalgebras $T: C \to C$. By instantiating our framework to deterministic automata, i.e., the functor $T_\Sigma = 2 \times \text{Id}_\Sigma$ on $C = \text{BA, DL}_0$, $\text{JSL}_0$ and $\text{Vect Z}_2$, we derived the local Eilenberg theorems for (ordered) monoids as in [11], as well as two new local Eilenberg theorems for idempotent semirings and $\text{Z}_2$-algebras.

There remain a number of open points for further work. Firstly, our general approach should be extended to the ordinary (non-local) version of Eilenberg’s theorem. Secondly, for different functors $T$ on the categories we have considered our approach should provide the means to relate varieties of rational behaviours of $T$ with varieties of appropriate algebras. In this way, we hope to obtain Eilenberg theorems for systems such as Mealy and Moore automata, but also weighted or probabilistic automata – ideally, such results would be proved uniformly for a certain class of functors.

Another very interesting aspect we have not treated in this paper are profinite equations and syntactic presentations of varieties (of $D$-monoids or regular languages, resp.) as in the work of Gehrke, Grigorieff and Pin [11]. An important role in studying profinite equations will be played by the $\hat{L}$-algebra $\tau \hat{L}$, the dual of the rational fixpoint, that we identified as the free profinite $D$-monoid. A profinite equation is then a pair of elements of $\tau \hat{L}$. We intend to investigate this in future work.

References

[1] Adámek, J., Porst, H.E.: On tree coalgebras and coalgebras presentations. Theoret. Comput. Sci. 311, 257–283 (2004)
[2] Adámek, J., Milius, S., Myers, R., Urbat, H.: Generalized eilenberg theorem i: Local varieties of languages. In: Proc. Seventeenth International Conference on Foundations of Software Science and Computation Structures (FoSSaCS’14). Lecture Notes Comput. Sci. (ARCoSS), vol. 8412, pp. 366–380 (2014)
[3] Adámek, J., Milius, S., Velebil, J.: Iterative algebras at work. Math. Structures Comput. Sci. 16(6), 1085–1131 (2006)
[4] Adámek, J., Rosický, J.: Locally presentable and accessible categories. Cambridge University Press (1994)
[5] Almeida, J.: Finite semigroups and universal algebra. World Scientific Publishing, River Edge (1994)
[6] Banaschewski, B., Nelson, E.: Tensor products and bimorphisms. Canad. Math. Bull. 19, 384–401 (1976)
[7] Birkhoff, G.: On the structure of abstract algebras. Proc. Cambridge Phil. Soc. 31 pp. 433–454 (1935)
[8] Bloom, S.L.: Varieties of ordered algebras. J. Comput. Syst. Sci. 13(2), 200–212 (1976)
[9] Eilenberg, S.: Automata, languages and machines, vol. B. Academic Press [Harcourt Brace Jovanovich Publishers, New York (1976)
[10] Gehrke, M.: Stone duality, topological algebra and recognition (2013), available at
http://hal.archives-ouvertes.fr/hal-00859717
[11] Gehrke, M., Grigorieff, S., Pin, J.É.: Duality and equational theory of regular languages. In: Proc. ICALP 2008, Part II. Lecture Notes Comput. Sci., vol. 5126, pp. 246–257. Springer (2008)
[12] Gehrke, M., Grigorieff, S., Pin, J.É.: A topological approach to recognition. In: Proc. ICALP 2010, Part II. Lecture Notes Comput. Sci., vol. 6199, pp. 151–162. Springer (2010)
[13] Johnstone, P.T.: Stone Spaces, Cambridge studies in advanced mathematics, vol. 3. Cambridge University Press (1982)
[14] Milius, S.: A sound and complete calculus for finite stream circuits. In: Proc. 25th Annual Symposium on Logic in Computer Science (LICS’10), pp. 449–458. IEEE Computer Society (2010)
[15] Pin, J.: A variety theorem without complementation. Russian Mathematics (Izvestija vu-zov.Matematika) 39, 80–90 (1995)
[16] Pin, J.É.: Mathematical foundations of automata theory (2013), available at
http://www.liafa.jussieu.fr/~jep/PDF/MPRI/MPRI.pdf
[17] Pippenger, N.: Regular languages and Stone duality. Theory Comput. Syst. 30(2), 121–134 (1997)
[18] Polák, L.: Syntactic semiring of a language. In: Sgall, J., Pultr, A., Kolman, P. (eds.) Proc. International Symposium on Mathematical Foundations of Computer Science (MFCS), vol. 2136, pp. 611–620. Springer (2001)
[19] Reiterman, J.: The Birkhoff theorem for finite algebras. Algebra Universalis 14(1), 1–10 (1982)
[20] Reutenauer, C.: Séries formelles at algèbres syntactiques. J. Algebra 66, 448–483 (1980)
[21] Rhodes, J., Steinberg, B.: The Q-theory of Finite Semigroups. Springer Publishing Company, Incorporated, 1st edn. (2008)
[22] Rutten, J.J.M.M.: Universal coalgebra: A theory of systems. Theor. Comput. Sci. 249(1), 3–80 (Oct 2000)