Perfect fluid and $F(T)$ gravity descriptions of inflationary universe and comparison with observational data

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Abstract We describe in this paper the observables of inflationary models, in particular the spectrum index of torsion scalar perturbations, the tensor-to-scalar ratio, and the running of the spectral index, in the framework of perfect fluid models and $F(T)$ gravity theories through the reconstruction methods. Then, our results on the perfect fluid and $F(T)$ gravity theories of inflation are compared with recent cosmological observations such as the Planck satellite and BICEP2 experiment. Our studies prove that the perfect fluid and $F(T)$ gravity models can reproduce the inflationary Universe consistent above all with the Planck data. We have reconstructed several models and considered others which give the best fit values compatible with the spectral index of curvature perturbations, the tensor-to-scalar ratio, and the running of the spectral index within the allowed ranges suggested by the Planck and BICEP2 results. By taking the trace-anomaly into consideration, we have shown that the reconstructed models $F(T)$ can not describe a finite de Sitter inflation without an additional constant $n$ that we related to cosmological constant.

Keywords Scalar field · Inflation · Slow-roll · E-folds

1 Introduction

The recent data taken by the BICEP2 experiment (Ade et al. 2014a) on the tensor-to-scalar ratio of the primordial density perturbations, additionally to the observations by the satellites of the Wilkinson Microwave Anisotropy Probe (WMAP) (Spergel et al. 2003; Hinshaw et al. 2013) and the Planck (Ade et al. 2014b, 2014c) have begotten many reflexion on inflation. The potential form of inflaton is related to the spectrum of the density perturbations generated during inflation (Lidsey et al. 1997). Several models of inflation have recently been constructed such as quantum cosmological perturbations for predictions and observations (Mukhanov 2013), and others to account for the Planck and BICEP2 experiment (Hazra et al. 2014). It exists some of them which are related to scalar field additionally to modified gravities models of inflation in comparison with the data analysis of the BICEP2 (Joergensen et al. 2014; Gao and Gong 2014; Bamba et al. 2014a, 2014b, 2014c; Kobayashi and Seto 2014).

Many studies with Various interesting results, have been done on the reconstruction of inflationary models in the framework of perfect fluid, $F(R)$ gravity and others modified gravity theories. Furthermore, the reconstruction of $F(R)$ gravity models from observational data has been executed in Starobinsky (1980). It has been also done in supergravity (Ferrara et al. 2014) and the related models (Chakravarty and Mohanty 2015). All these realizations have been the attempts to make modified gravity models to explain the Planck and BICEP2 results (Bamba et al. 2014a, 2014b, 2014c; Pallis 2014). Moreover, Bamba and his collaborators have recently and explicitly performed the reconstruction of scalar field theories with inflation leading to the theoretical consequences compatible with the observational data obtained from the Planck and BICEP2 in...
terms of the spectral index of the curvature fluctuations, the tensor-to-scalar ratio, and the running of the spectral index in Bamba et al. (2014b). They have also made the perfect fluid and \( F(R) \) gravity descriptions of inflation and its comparison with observational data in Bamba et al. (2014c). In this last one, they have re-expressed the observables of inflationary models, i.e., the spectral index \( n_s \) of curvature perturbations, the tensor-to-scalar ratio \( r \), and the running of the spectral index \( \alpha_s \), in terms of the quantities in perfect fluid models and \( F(R) \) gravity theories. They have investigated several \( F(R) \) gravity models, especially, a power-law model which gives the best fit values compatible with the spectral index and tensor-to-scalar ratio within the allowed ranges suggested by the Planck and BICEP2 results. Moreover, the spectral index’s features have been studied in induced gravity (Kaiser 1994) and scalar-tensor theories (Kaiser 1995). Among the inflation description attempts in the framework of \( F(T) \), it existed a very successful work related to \( T^2 \) gravity in which the authors, by taking the trace-anomaly into consideration, have explicitly shown that in \( T^2 \) teleparallel gravity, the de Sitter inflation can occur, although quasi de Sitter inflation happens in \( R^2 \) gravity (Bamba et al. 2014a). We have observed from this work that the de Sitter inflation can occur and finally the Universe can exit from it due to the de Sitter space instability solution coming from the trace-anomaly. Some months latter, it is proved that this \( T^2 \) gravity can lead to graceful exit inflation with no need to slow-roll technique (Nashed et al. 2014). Secondly, they have also generated slow-roll potential whose analysis can perform tensor-to-scalar ratio and spectral index parameters consistent with the recent Planck and BICEP2 data. Another important work which explains a scenario of inflation in the framework of teleparallel gravity precisely in the modified version \( F(T) \) gravity is proposed in Jamil et al. (2015). Their investigation gives a value of Spectral scalar index \( n_s \) which is compatible in a reasonable agreement with the WMAP7 and for pre Planck \( n_s < 0.961 \).

In parallel way to these works with interesting results on the inflationary models in the framework of perfect fluid and modified gravities, we investigated in this paper, the descriptions of inflation through perfect fluid and \( F(T) \) gravity models. We reformulate the observables of inflationary models in terms of the quantities of scalar torsion and in perfect fluid and compare the theoretical representations with recent data from Planck et BICEP2. By considering some forms of scalar torsion, we have reconstructed some \( F(T) \) inflation models whose behaviors will be analysed by taking the trace-anomaly into consideration. We here emphasize that the \( F(T) \) gravity coupling with scalar field theory have already been explored in several interesting cosmological works the density perturbation growth in teleparallel cosmology (Geng and Wu 2013) and dynamical features of scalar-torsion theories (Skugoreva et al. 2015). This present work wants to explain the importance of formulations of the observables for inflationary models in terms of the perfect fluid and \( F(T) \) gravity. It is physically motivated by the fact that in ordinary scalar field model of inflation, the spectral index, the tensor-to-scalar ratio, and the running of the spectral index are represented by using the potential \( V(\phi) \) of the scalar field. Consequently, scalar field models are consistent with the observations and by comparing the theoretical representations of these observables with observations, we can get information on the properties of the perfect fluid and \( F(T) \) gravity models to account for the observations in the early Universe. Our approach in terms of inflation can permit to find the conditions in which the perfect fluid and \( F(T) \) gravity models can be viable from the cosmological point of view. We normalize to unit the following constants \( k_B = c = \hbar = 1 \) and express the gravitational constant \( 8\pi G = \kappa^2 \equiv 8\pi/M_{Pl}^2 \) with Planck mass of \( M_{Pl} = G_N^{-1/2} \).

The plan of the manuscript is outlined as follows: In Sect. 2 we write the slow-roll parameters and express the observables for inflationary models in the first time as function of scalar torsion and in the second time in perfect fluid description. Then, in Sect. 3 we re-express these parameters and observables in framework of \( F(T) \) gravity. All ours investigations have been ended by comparison with the Planck and BICEP2 data. Finally, Sect. 4 is devoted to the conclusions.

2 Scalar torsion and perfect fluid description of the slow-roll parameters

As we have mentioned it in our introduction, the slow-roll parameters are related to inflaton namely the potential of scalar field. In this section, we express these parameters by the scalar torsion and in terms of perfect fluid.

2.1 Slow-roll parameters

The action of teleparallel gravity coupled with the model of scalar field \( \phi \) is given by Geng and Wu (2013)

\[
S = \int \left( \frac{T}{2\kappa^2} - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) \right) e^a dx, \tag{1}
\]

where \( T \) is the scalar torsion and \( e \) the determinant of tetrad \( e^a_{\mu} \). The slow-roll parameters, \( \varepsilon \), \( \eta \) and \( \xi \) are defined by Bamba et al. (2014a)

\[
\varepsilon \equiv \frac{1}{2\kappa^2} \left( \frac{V'(\phi)}{V(\phi)} \right)^2, \quad \eta \equiv \frac{1}{\kappa^2} \frac{V''(\phi)}{V(\phi)}, \quad \xi \equiv \frac{1}{\kappa^4} \frac{V'(\phi)V'''(\phi)}{(V(\phi))^2}. \tag{2}
\]
Here and for the rest, the prime means the derivative with respect to argument such as \( V' (\phi) \equiv \partial V (\phi) / \partial \phi \), and others. For the scalar field models, the spectral index \( n_s \), of curvature perturbations, the tensor-to-scalar ratio \( r \) of the density perturbations and the running of spectral index \( \alpha_s \) are expressed as Bamba et al. (2014a)

\[
n_s - 1 \sim -6\epsilon + 2\eta, \quad r = 16\epsilon,
\]

\[
\alpha_s \equiv \frac{dn_s}{d \ln k} \sim 16\epsilon\eta - 24\xi^2 - 2\xi^2.
\]

(3)

The variation of action (1) with respect to the tetrad \( e_a^\mu \) gives (Geng and Wu 2013; Skugoreva et al. 2015)

\[
\frac{1}{\kappa^2} \left[ \frac{1}{4} \delta^\rho_\nu T + T^\sigma_{\nu \phi} S_\sigma^{\rho \nu} + e^{-1} e^\sigma_{ \beta} \partial_a \left( e_a^\sigma S^\sigma_{\nu \mu} \right) \right] = \frac{1}{2} \left[ \frac{\delta^\phi_\nu}{1} \frac{1}{2} g^{\rho \nu} \partial_\mu \phi_\nu + V (\phi) \right] - \frac{1}{2} \delta^\phi_\nu g^{\rho \nu} \partial_\mu \phi_\nu
\]

\[- \frac{1}{2} \delta^\nu_\nu g^{\rho \nu} \partial_\mu \phi_\nu \phi_\nu \].

(4)

We consider an Universe described by the following flat Friedmann-Lemaître-Roberson-Walker FRW metric:

\[
ds^2 = -dt^2 + a^2 (t) \sum_{i=1,2,3} \left( dx_i \right)^2.
\]

(5)

Here, \( a (t) \) is the scale factor and \( H \equiv \dot{a} / a \) is the Hubble parameter. From (5), one obtains the torsion scalar in function of \( H \) by \( T = -6H^2 \). The point (,) means here the derivative with respect to time as \( \partial / \partial t \).

We can now find out the expressions of the slow-roll parameters as function of torsion scalar. Indeed, by using the relations (4) and (5), one gets the field equation given by:

\[
\frac{T}{2\kappa^2} = -\frac{1}{2} \dot{\phi}^2 - V (\phi),
\]

(6)

\[
\frac{\dot{T}}{12H\kappa^2} = \frac{1}{2} \ddot{\phi}.
\]

(7)

We define the scalar field \( \phi \) by a new scalar field \( \varphi \) such as \( \phi = \varphi (\varphi) \) and we identify \( \varphi \) to e-folds number \( N \). If one defines \( N \) by scale factor as \( N \equiv \ln (a / a_0) \), we obtain \( \dot{N} = \varphi' = H \). Equations (6) and (5) become

\[
\frac{T (N)}{2\kappa^2} = \frac{1}{12} w(\varphi) T (N) + V (\varphi (\varphi)),
\]

(8)

\[
\frac{T' (N)}{12\kappa^2} = -\frac{1}{12} w(\varphi) T (N),
\]

(9)

with \( w(\varphi) \equiv (d\varphi / d\varphi)^2 \). So, if we consider the torsion scalar as function of \( N \), by combining (8) and (9), one expresses \( w(\varphi) \) and \( V (\phi) \equiv V (\varphi (\varphi)) \) by \( N \) as

\[
w(\varphi) = \left[ -\frac{T' (N)}{\kappa^2 T (N)} \right]_{\varphi \equiv \varphi}.
\]

(10)

\[
V (\varphi) = -\frac{1}{12\kappa^2} \left[ 6T (N) + T' (N) \right]_{\varphi \equiv \varphi}.
\]

(11)

We find \( T = T (N) \) and \( \varphi = N \) as solution for field equation \( \phi \) or \( \varphi \) and Einstein equations because of equivalence between Teleparallel and General Relativity. We also find that \( T' (N) > 0 \) because \( w(\varphi) > 0 \) and \( T (N) < 0 \). Thus, we can now express in terms of \( T (N) \) all the slow-roll parameters and the observables by making using the relations (10) and (11) in their basic expressions defined in (2) and (3) respectively. One gets

\[
e = -\frac{1}{2T' (N)} \left( \frac{6T' (N) + T''(N)}{6T (N) + T' (N)} \right)^2 - 1,
\]

(12)

\[
\eta = -\left[ 6 (T' (N))^3 + T''(N)T' (N) \right]^2
\]

\[+ 6T'' (N) T' (N) \left( 6T (N) + T' (N) \right) + 2T'' (N) T' (N) \left( 6T (N) + T' (N) \right)]
\]

\[+ \left[ 2 (6T' (N))^2 T (N) + (T' (N))^3 \right].
\]

\[
\xi^2 = \frac{T (N)}{2T' (N)} \frac{6T' (N) + T''(N)}{2T' (N)(6T (N) + T' (N))^2} \left[ 7T'' (N)
\]

\[+ 3T'' (N) + \frac{2T'' (N) T (N)}{T' (N)} + 6T'' (N) T (N)
\]

\[- \frac{4T'' (N) T'' (N) (N)}{(T' (N))^2} + \frac{2(T'' (N))^3 T (N)}{(T' (N))^3}
\]

\[- \frac{6(T'' (N))^2 T (N)}{(T' (N))^2} - \frac{(T'' (N))^2}{(T' (N))^2}\right],
\]

(14)

\[
n_s = 1 + \frac{3T (N) (6T' (N) + T''(N))^2}{T' (N)(6T (N) + T' (N))^2}
\]

\[+ \left[-6T' (N)^3 - T' (N)^2 T'' (N) + T (N) T'' (N)^2
\]

\[-2T (N) T' (N) (3T'' (N) + T (N)^3) \right] \left[ T' (N)^2 (6T (N) + T' (N)) \right]^{-1},
\]

(15)

\[
r = -\frac{8T (N)(6T' (N) + T''(N))^2}{T' (N)(6T (N) + T' (N))^2},
\]

(16)

\[
\alpha_s = \left[ T (N) (6T' (N) + T''(N))(T (N))^4 \left( 144T' (N)^2
\right.
\]

\[+ 5T'' (N)^2 + T' (N)(41T'' (N) - 3T (N)^3))
\]

\[+ 12T (N)^2 T' (N)(-4T'' (N)^3)
\]

\[+ T' (N) T'' (N)(9T'' (N) + 8T (N)^3)
\]

\[+ T' (N)^2 (51T'' (N) + 9T (N)^3 - 2T (N)^4))
\]

\[+ 2T (N) T' (N)^2 (216T' (N))^3 + 6T'' (N)^3
\]

\[+ 3T' (N) T'' (N)(117T' (N) - 2T (N)^3))
\]

\[+ T' (N)^2 (150T'' (N) - 3T (N)^3 + T (N)^4))
\]

\]

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\[
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\[ -72T(N)^3(T''(N)^3 - T'(N)T''(N)(3T''(N) + 2T^{(3)}(N)) + T^{(4)}(N)) \]
\[ / (T'(N)^4(6T(N) + T'(N))^4). \]  

2.3 Perfect fluid models reconstructions

We investigate here the linear form of \( T(N) \) given by:

\[ T(N) = D_0 N + D_1, \]  

with \( D_0 > 0 \) and \( D_1 < 0 \) constants. The physical motivation of this choice is that, according to exponential inflation of slow-roll, the scale factor is given by \( a = \hat{a} \exp(H_{int} t) \) where \( \hat{a} \) is a constant (Bamba et al. 2014a). \( H_{int} \) is the Hubble parameter at the inflationary stage and it is approximately constant i.e. it weakly depends of time. To express this weak time dependence of \( H \) and consequently of scalar torsion \( T \) we use the form in (24) where the e-folds number plays the role of time. In this case, if \( D_1 / D_0 \ll N \), the time dependence of scalar torsion during inflation is negligible. By using the relations (18), (19), one obtains:

\[ \rho(N) = -\frac{1}{2\kappa^2}(D_0 N + D_1), \]  

\[ P(N) = \frac{1}{6\kappa^2}(3N + 1)D_0 + 3D_1. \]  

By eliminating \( N \) between these last equations, we find the following relation

\[ P(N) = \frac{D_0}{6\kappa^2} - \rho(N), \]  

which, added to the general equation of state (20), gives

\[ f(\rho) = \frac{D_0}{6\kappa^2}. \]  

From equations in (25), we can deduce the state equation of perfect fluid according to the linear form of scalar torsion as

\[ \omega(N) = \frac{P(N)}{\rho(N)} = -1 + \frac{f(\rho)}{\rho(N)} = \frac{(3N + 1)D_0 + 3D_1}{3(D_0 N + D_1)}. \]  

2.3.2 The exponential form

The second example of perfect fluid model studied in this work is whose scalar torsion is given by the following exponential function of \( N \)

\[ T(N) = D_2 N e^{\beta N} + D_3, \]  

with \( D_2 > 0 \), \( D_3 < 0 \) and \( \beta > 0 \) constants. The physical reason of the choice of this form is the following. During power law inflation, the scale factor is given by \( a = \hat{a} t^p \) with \( \hat{a} \) constant. Then, the scalar torsion during inflation is so \( T = -6(\hat{p}/t)^2 \) and becomes \( T = -6\hat{p}^2 \exp(-2N/\hat{p}) \). This last form is equivalent to (29) if we make \( D_2 = -6\hat{p}^2 \), \( \beta = -2/\hat{p} \) and \( D_3 = 0 \). Such an exponential form can reproduce the power-low inflation.
By introducing relation (29) in the gravitational equations (18) and (19), one gets
\[ \rho(N) = -\frac{1}{2\kappa^2}(D_2 N e^{\beta N} + D_3), \]
\[ P(N) = -\frac{1}{6\kappa^2}[(3 + \beta)D_2 N e^{\beta N} + 3D_3]. \]  
(30)
Elimination of \( N \) between these equations gives
\[ P(N) = -\left(1 + \frac{\beta}{3}\right)\rho(N) - \frac{\beta D_3}{6\kappa^2}. \]  
(31)
From (20), one obtains:
\[ f(\rho) = \frac{\rho(\rho(N))}{\rho(N)} = -\frac{\beta D_3}{6\kappa^2}. \]
(32)
Then, we find the parameter of state of perfect fluid model corresponding to the exponential form
\[ \omega(N) = \frac{(3 + \beta)D_2 N e^{\beta N} + 3D_3}{3(D_2 N e^{\beta N} + D_3)}. \]  
(33)
2.3.3 Another form
We consider here a model studied in Mukhanov (2013) whose state parameter is given by
\[ \omega(N) = -1 + \frac{\beta}{1 + N^{\gamma}}, \]
(34)
with \( \beta \) and \( \gamma \) free parameters. By solving the system of equations formed by (18), (19) and (34), we obtain
\[ T = \tilde{T} \exp \left[ \frac{-3\tilde{\beta}(1 + N)^{\gamma - \tilde{\gamma}}}{1 - \tilde{\gamma}} \right] \]  
(35)
where \( \tilde{T} \) is constant.

We can now specify the expressions of some observables as example the spectral index and the tensor-to-scalar ratio by introducing (35) in (15) and (16)
\[ n_s = \frac{1}{3(-2(1 + N)^{\gamma} + \beta)\gamma}(1 + N)^{1 - \gamma}\left[-9(1 + N)^2\beta^3 \right. \]
\[ \left. + 3\beta^2(1 + N)^{1 + \gamma}(13 + 13N - \gamma) \right. \]
\[ - 2(1 + N)^{3\gamma} \beta(24(1 + N)^2 - \gamma + \gamma^2) \]
\[ \left. + 2(1 + N)^{3\gamma}(-6 + 6N - (4 + \gamma)\gamma + 6N(2 + \gamma)) \right]| \]
(36)
\[ r = \left[8(1 + N)^{-2 - \gamma}\beta(3(1 + N)\beta \right. \]
\[ \left. + (1 + N)^{\gamma}(-6 - 6N + \gamma) \right]^\gamma \]
\[ \times \left[3(-2(1 + N)^{\gamma} + \beta)\right]^{-1}. \]  
(37)
We deduce that for the appropriate value of parameters \( \{\beta, \gamma\} \), we can get the values of observables \( n_s \) and \( r \).

2.3.4 Comparison with the observations
Suppose that the time variation of \( f(\rho) \) and \( \rho \) during inflation is sufficiently small and that inflation is almost exponential as \( \omega(N) \equiv P(N)/\rho(N) = -1 + f(\rho)/\rho(N) \approx -1 \), i.e., \( |f(\rho)/\rho(N)| \ll 1 \), one gets from expressions obtained in Sect. 2.2 after writing the slow-roll parameters and the observables in term of perfect fluid, the following approximative relations:
\[ n_s \approx 1 - 6\frac{f(\rho)}{\rho(N)}, \quad r \approx 24\frac{f(\rho)}{\rho(N)}. \]  
(38)
We present here the recent observations on spectral index \( n_s \), the tensor-to-scalar ratio \( r \) and the running of spectral index \( \alpha_s \). The recent data of Planck satellite (Ade et al. 2014a) suggested \( n_s = 0.9603 \pm 0.0073 \) (68 % CL), \( r < 0.11 \) (95 % CL), and \( \alpha_s = -0.0134 \pm 0.0090 \) (68 % CL) [Planck and WMAP (Spergel et al. 2003; Hinshaw et al. 2013)], whose negative sign is at 1.5\( \sigma \). The BICEP2 experiment (Ade et al. 2014a) implies \( r = 0.20^{+0.07}_{-0.05} \) (68 % CL). It is mentioned that discussions exist on how to subtract the foreground, for example in Ade et al. (2015) and Kamionskowski and Kovetz (2014). Recently, progress appear also in Colley and Gott (2015) to ensure the BICEP2 declarations. It has been also remarked that the representation of \( \alpha_s \) is also given in Bassett et al. (2006).

From (38), we can see that when the condition \( f(\rho)/\rho(N) = 6.65 \times 10^{-3} \) is realized at the inflationary era, we find \( (n_s, r, \alpha_s) = (0.960, 0.160, -3.98 \times 10^{-4}) \). In the case of linear form of \( T \) from (24), if \( D_1/D_0 \gg N \) and \( -D_0/(3D_1) = 6.65 \times 10^{-3} \), the change of value of \( \omega(N) \) is consider to be negligible, and the above condition can be met at the inflationary stage. In the second time, for the exponential form with the condition \( \beta = 2.0 \times 10^{-4} \) and then \( \beta N \ll 1 \), and that \(-1/3[(1 + D_3/(D_2, \beta))] \approx 6.65 \times 10^{-3} \), \( n_s \) can be seen as a constant and above condition can be satisfied during inflation. As consequence, the conclusion is that perfect fluid can lead to Planck results with \( r = O(0.1) \), which value is compatible with BICEP2 experiment.

Thus, we mention here some concrete perfect fluid models (Astashenok et al. 2012). For model of \( P(N) = -\rho(N) + f(\rho) \) with \( f(\rho) = f_s \sin(\rho(N)/\tilde{\rho}) \) where \( f_s \) is constant and \( \tilde{\rho} \), the fiducial value of \( \rho \) known to produce the scenario of Pseudo-Rip (Frampton et al. 2011), if \( \rho(N)/\tilde{\rho} \ll 1 \) and \( f(\rho)/\rho(N) \approx f/\tilde{\rho} = 6.65 \times 10^{-3} \) behaves almost constant and the above condition can be satisfied. Then, we have examined another model of \( P(N) = -\rho(N) + f(\rho) \), where \( f(\rho) = (\rho(n))^{\tau} \) with \( \tau \neq 0 \) constant. By using (18), one gets \( f(\rho)/\rho(N) \approx (\rho(n)/\kappa^2)^{\tau-1} \), where the constant \( \rho(n) \) means the scalar torsion at the slow-roll inflation regime. So, similarly to the first example, when
3 Description of the slow-roll in \( F(T) \) gravity

We describe in this section, the slow-roll parameters in the framework of \( F(T) \) gravity.

3.1 Observables of inflationary models in terms of the quantities in \( F(T) \) gravity theories

The action in context of \( F(T) \) gravity theories is defined by

\[
S_{F(T)} = \int e \left( \frac{F(T)}{2\kappa^2} + \mathcal{L}_{\text{matter}} \right) dx^4.
\]  

(39)

The equation of motion is determined by Salako et al. (2013)

\[
\begin{align*}
S_{\rho} & \equiv \mathcal{S}_\rho \equiv \rho \nabla_\alpha T f_{\alpha \beta} + \left[ e^{-1} e^a e_a \partial_\alpha \left( e e_a e_\alpha S_\rho \right) \right] \\
+ T_\rho \nabla_\nu S_\rho \nabla^\nu f_T + \frac{1}{4} \delta_\rho^\beta f &= 4\pi T_\rho^\beta,
\end{align*}
\]  

(40)

where \( T_\rho^\beta \) represents energy momentum tensor of matter. By using the metric FLRW in (5), the modified Friedmann equations take the forms

\[
\frac{1}{2} F(T) - T F'(T) + \kappa^2 p_{\text{matter}} = 0,
\]  

(41)

\[
\frac{1}{2} F(T) + (6H^2 + 2H) F'(T) - 24H^2 \dot{H} F''(T) - \kappa^2 \rho_{\text{matter}} = 0.
\]  

(42)

\( \rho_{\text{matter}} \) and \( p_{\text{matter}} \) are respectively energy density and the pressure of matter. The prime of \( F(T) \) means it derivative with respect to scalar torsion \( T \) as example \( F'(T) = \frac{dF(T)}{dT} \), \( F''(T) = \frac{d^2 F(T)}{dT^2} \), etc. In vacuum, (41) gives

\[
T = \frac{F(T)}{2F'(T)}.
\]  

(43)

With the fact that the scalar tension \( T \) is function of \( N \), by introducing the relation (43) in (18) and (21), one gets \( \rho(N) \) and \( f(\rho) \) in the contest of \( F(T) \) gravity. All the slow-roll parameters and the observables can be expressed as function of \( F(T) \).

3.2 Reconstruction of \( F(T) \) gravity models

We use the method of reconstruction as it is developed in Nojiri and Odintsov (2006) and Vasquez et al. (2013). We define the e-folds number as \( N \equiv -\ln(\alpha/a_0) \) with \( a_0 \) the scale factor at present time \( t_0 \), and the redshift becomes \( z \equiv a_0/\alpha - 1 \). Consequently, we have \( N = -\ln(1 + z) \). We write the scalar torsion in terms of \( N \) through the function \( D(N) \) as

\[
T(N) = D(N) = D(-\ln(1 + z)).
\]  

(44)

By using the relation (44) and the continuity equation \( \dot{\rho} + 3H(1 + \omega) \rho = 0 \), (42) can be rewritten as

\[
0 = -3F(T) + \left[ 6D(N) + D'(N) \right] F'(T) + 2D(N)D'(N) F''(T)
\]  

\[
+ 6 \sum \rho_{\text{matter}} \omega_i a_0^{-3(1+\omega_i)} \times \exp[-3(1 + \omega_i) N(T)].
\]  

(45)

where the last term is equal to the total pressure of matter \( p_{\text{matter}} \). Here, the matters are supposed to be fluid labeled by \( i \) with the same equation of state \( \omega_i \equiv \rho_{\text{matter}} / p_{\text{matter}} \), where \( \rho_{\text{matter}} \) and \( p_{\text{matter}} \) are energy density and the pressure of \( i \)th fluid and \( \rho_{\text{matter}} \) a constant. It follows that for any form of scalar tensor i.e. the function \( D(N) \), the differential equation (45) can be resolved and the corresponding gravitational action \( F(T) \) (the model) which reproduces the expansion history described by scalar torsion via the Hubble parameter \( H \). Moreover, for particular \( F(T) \) model, the function \( D(N) \) can also be obtained. Then, with the expressions of observables of inflationary models described in Sect. 2.1, we can get the corresponding predictions on the inflation for this particular model \( F(T) \).

3.2.1 The linear form

We use the linear form of scalar tensor \( T \) of the relation (24). This prove that the e-folds number can also be expressed in terms of the scalar torsion as \( N(T) = (T - D_1) / D_0 \). The Friedmann equation (45), in the vacuum, becomes second order differential equation in \( T \) labeled by

\[
-3F(T) + (6T + D_0) F'(T) + 2T D_0 F''(T) = 0,
\]  

(46)

whose resolution gives

\[
F(T) = C_1 \sqrt{-T} + C_2 \sqrt{-T} \times \left( \frac{2e^{\frac{3}{2}T_0} \sqrt{-T}}{\sqrt{-T} - 2\sqrt{\frac{3\pi}{2} \text{Erf} \left( \frac{\sqrt{-T}}{\sqrt{D_0}} \right)}} \right).
\]  

(47)

\( C_1 \) and \( C_2 \) constants of integration whereas \( \text{Erf}(z) \) is the Gauss integral distribution given by \( \text{Erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt \). We have consequently reconstructed the gravitational Lagrangian density which can generate the required expansion for any given expansion history \( T \) or for any given e-folds number \( N \). To putting out the observables of inflationary
models corresponding to the expansion history form considered, one puts the relation (24) in those defined by (15), (16) and (17) and gets
\[ n_s = 1 + \frac{6D_0(12D_1 + D_0(-1 + 12N))}{(D_0 + 6D_1 + 6D_0N)^2}, \]
\[ r = \frac{-288D_0(D_1 + D_0N)}{(D_0 + 6D_1 + 6D_0N)^2}, \]
\[ \alpha_s = -\frac{864D_0^2(D_1 + D_0N)[3D_1 + D_0(-1 + 3N)]}{(D_0 + 6D_1 + 6D_0N)^4}. \]

The inflationary phase must last enough to account for initial conditions problems as example the so called problem of horizon and flatness. The value of e-folds number at the end of inflation must be \( N_e \gtrsim 50 \). The slow-roll parameters should be smaller than unity during inflation; whereas they became larger than or equal to unity at the end of inflation, \( N = N_e \). As examples, if \((N, D_0, D_1) = (50.0, 0.850, -95.0)\) and \((60.0, 0.950, -115)\), one gets from the previous relations (48), (49), and (50) the following value of the observables \((n_s, r, \alpha_s) = (0.967, 0.130, -5.32 \times 10^{-4})\) and \((0.967, 0.131, -5.45 \times 10^{-4})\) respectively. Thus, the Planck results for \( n_s \) with \( r = \mathcal{O}(0.1) \) can be reached.

### 3.2.2 The exponential form

Here, we use the exponential form of the scalar torsion in (29). In this case, the relation between \( N \) and the scalar torsion is expressed by \( D_2e^{\beta N} = T - D_3 \). Then, the second order differential equation obtained from the Friedmann equation (45) in the vacuum according to this form of scalar torsion is given by
\[ -3F(T) + (6T + \beta T - \beta D_3)F'(T) + 2T(\beta T - \beta D_3)F''(T) = 0. \]

Its resolution gives
\[ F(T) = C_1\sqrt{-T} - 2C_2(-D_3 - T)^{-3/\beta} \left( 1 + \frac{T}{D_3} \right)^{3/\beta} \]
\[ \times 2F_1\left[ -\frac{1}{2}, \frac{3}{2}, \frac{1}{2}, \frac{-T}{D_3} \right]. \]

with \( C_1 \) and \( C_2 \) the constants of integration and \( 2F_1(x_1, x_2, x_3; y) \), is the hypergeometric function where \( x_i (i = 1, \ldots, 3) \) are constants and \( y \) the variable. The corresponding observables of inflationary models for the exponential expansion history are given by
\[ n_s = 1 + \beta(6 + \beta) \left[ -6D_3^2 + e^{N\beta}D_2\left[ 2D_3\beta + e^{N\beta}D_2(6 + \beta) \right] \right] \]
\[ \times \left[ 6D_3 + e^{N\beta}D_2(6 + \beta) \right]^{-2}, \]
\[ r = \frac{-8e^{N\beta}D_2(e^{N\beta}D_2 + D_3)\beta(6 + \beta)^2}{(6D_3 + e^{N\beta}D_2(6 + \beta))^2}, \]
\[ \alpha_s = \left[ e^{N\beta}D_2(e^{N\beta}D_2 + D_3)\beta^2(6 + \beta) \right]^2 \times \left[ 6e^{N\beta}D_2(6 + \beta) \right]^{-4}. \]

We deduce from these that for \((N, D_2, D_3) = (50.0, 1.10, -10)\) and \((60.0, 1.20, -15.0)\), one gets \((n_s, r, \alpha_s) = (0.9627, 6.89 \times 10^{-2}, -6.4 \times 10^{-4})\) and \((0.9652, 5.83 \times 10^{-2}, -5.23 \times 10^{-4})\) respectively.

We recall here that our investigation tries to explain the inflation scenario in the framework of \( F(T) \) gravity coupled with scalar field as it was already done in a lot of works and also through another gravity (Bamba et al. 2014a; Jamil et al. 2015). The obtained models in (47) and (52) represent the geometrical part of the initial model described by the action in (39). The reconstruction approach followed to obtain these models which consists to neglect all matter fields, can also be justified by the fact that in the inflationary era the inflation is driven by inflaton field \( \phi \) (Jamil et al. 2015). Furthermore, for appropriate choice of the parameters of these models, they can lead to the very interesting models studied in Jamil et al. (2015).

But the question which is actually to be answered is to know if these reconstructed models described an eternal or finite inflation. In others words, we are called to examine if the Universe can exit from the inflation in the framework of our present \( F(T) \) inflation description. According to literature, the exist from occurs frequently in the de Sitter inflation. For the moment, the only best and the most successful inflationary model \( F(T) \) which have perfectly described the exist in inflation is the \( T^2 \) model studied by Bamba et al. (2014a). They have clearly demonstrated that this model can begot the de Sitter inflation from which the Universe can exist due to instability of the de Sitter solution coming from the trace anomaly. In order to verify if the reconstructed models (47) and (52) can make end the inflation, we follow the approach developed by Bamba et al. (2014a). Before all investigations, we generalize these models by making \( F(T) = G(T) + n \) where \( n \) is a constant and which can be related to the cosmological constant \((n = -\Lambda)\). By introducing this new form of Lagrangian density in the both differential equations (46), (51), one obtains respectively after solution two generalized models given by
\[ G_1(T) = C_1\sqrt{-T} + C_2\sqrt{-T} \]
\[ \times \left( -\frac{2e^{N\beta}}{\sqrt{T}} - \frac{2\sqrt{3\pi}\text{Erf}(\sqrt{T})}{\sqrt{D_0}} \right) - n, \]
\[ r = \frac{288D_0(D_1 + D_0N)}{(D_0 + 6D_1 + 6D_0N)^2}, \]
\[ \alpha_s = \left[ e^{N\beta}D_2(e^{N\beta}D_2 + D_3)\beta^2(6 + \beta) \right]^2 \times \left[ 6e^{N\beta}D_2(6 + \beta) \right]^{-4}. \]
where \( c_1, c_2, v_1 \) and \( v_2 \) are integration constants. For \( n = 0 \) the reconstructed models in (47) and (52) will be recovered respectively by \( G_1(T) \) and \( G_2(T) \). So \( G_1(T) \) and \( G_2(T) \) are extensions of the models in (47) and (52) and they will be considered now as density Lagrangian, in others words as \( F(T) \) models, in order to study the end form inflation through them and easily deduce the cases of the models (47) and (52).

### 3.2.3 Exist from inflation description with the first generalized model \( G_1(T) \)

During the inflationary era, the cosmological evolution is a nearly de Sitter evolution and can exit by considering the quantum effect such as the trace anomaly. Here, our attempts consist to find out at what conditions the latest models can generate the unstable de Sitter solution. By putting the model \( G_1(T) \) of (56) in the first modified Friedmann equation (41) in vacuum namely \( F(T) - 2T F'(T) = 0 \), one obtains

\[
n + 2e^{-\frac{18}{16}H^2}c_2 = 0.
\]  

(58)

If \( n \) is considered as a cosmology constant \( n = -2\Lambda < 0 \) and \( c_2 > 0 \), we can get the de Sitter solution at inflationary stage which depends on the cosmology constant as

\[
H_s = \left[ \frac{D_0}{18} \ln \left( \frac{-2c_2}{n} \right) \right]^{1/2}.
\]  

(59)

We can see directly that if \( n = 0 \) which is equivalent to the case of the model (47), \( H_s \) diverges and then the de Sitter solution will not be reached. The corresponding scalar factor also diverges at any time and the Universe will be in permanence in inflation. However, the scale factor at de Sitter inflationary stage in (59) can be determined from \( H_s = \dot{a}(t)/a(t) \) as

\[
a(t) = a_0 \exp \left\{ \frac{D_0}{18} \ln \left( \frac{-2c_2}{n} \right) \right\}^{1/2} t
\]  

(60)

with \( a_0 \) constant. We do not know if the reconstructed de Sitter solution is attractor or not i.e. if its inflation will be eternal or not. The problem will be solved by introducing the quantum effect such as the trace anomaly and then studying the stability of the obtained de Sitter solution for its small perturbation which can lead to the exit from inflation. Indeed, the contribution of trace anomaly in the framework of \( F(T) \) theory with the action defined in (39), combines the Friedmann equations (41) and (42) in the vacuum in the following equation:

\[
-\frac{2}{k^2} \left( F(T) - 3\dot{H}F'(T) + 2TF'(T) - 6\dot{H}TF''(T) \right) - \left( \frac{2}{3} \ddot{b} + \ddot{b}' \right) \left( \frac{d^2}{dt^2} + 3H \frac{d}{dt} \right) (12H^2 + 6\dot{H}) + 24\ddot{b}'(H^4 + 2\dot{H}) = 0,
\]  

(61)

where \( \ddot{b}, \ddot{b}' \) and \( \ddot{b}'' \) are the parameters come from the trace anomaly expression. The parameters \( \ddot{b} \) and \( \ddot{b}' \) are related to the \( N \) scalars, \( N_{1/2} \) spinors, \( N_1 \) vector fields, \( N_2 = 0 \) or 1 gravitons and \( N_{1D} \) higher-derivative conformal scalars and \( \ddot{b}'' \) can be an arbitrary coefficient. For full informations on these parameters and about the full demonstration of (61), the reader is referred to Bamba et al. (2014a). In the de Sitter space where the Hubble rate is constant, \( H = H_0 \), the relation (61) becomes

\[
-\frac{2}{k^2} \left( F(T) - 12H_0^2F'(T) \right) + 24\ddot{b}'H_0^2 = 0.
\]  

(62)

By putting the model \( G_1(T) \) of (56) in (62), one obtains

\[
24\ddot{b}'H_0^2 - 2n - 4e^{-\frac{18}{16}H_0^2}c_2 = 0.
\]  

(63)

Our main goal consists to verify if (63) can be solved. The first approach that we follow here consists to find the conditions on the parameters of this equation in order to obtain the de Sitter solution \( H_0 \). To do so, we pose

\[
f(x) = 24\ddot{b}'x^4 \kappa^2 - 2n - 4e^{-\frac{18}{16}H_0^2}c_2,
\]  

(64)

with \( x = H_0 \). At the inflationary stage, we can have \( x \in [0, +\infty] \) and the first derivative of the function \( f \) with respect to \( x \) can be put in the following form

\[
f'(x) = x \left( 96\ddot{b}' \kappa^2 x^2 + 72e^{-\frac{18}{16}H_0^2}c_2 \right).
\]  

(65)

For ordinary matter, \( \ddot{b}' < 0 \), and from the precedent hypotheses we have \( x > 0 \) and \( D_0 > 0 \). If also \( c_2 < 0 \), we can have \( f'(x) < 0 \) which begets \( f(0, +\infty) = ] -\infty, -2n - 4c_2] \). Equation (63) is reduced to \( f(x) = 0 \) and have solution if \( 0 \in ] -\infty, -2n - 4c_2] \). This can be possible if \( -2n - 4c_2 > 0 \). This last condition is true if for example, we take \( n = -2\Lambda \) because we have already constrained \( c_2 \) to be negative. Consequently, the de Sitter solution can be found. In the second approach, we have investigated the numerical solution of (63). We pose

\[
f_1(x) = 24\ddot{b}'x^4 \kappa^2,
\]  

(66)

\[
f_2(x) = +2n + 4e^{-\frac{18}{16}H_0^2}c_2,
\]  

(67)
and then plot these two functions and we obtain Fig. 1.

The intersection of the curves reflecting the evolution of the two functions \( f_1 \) and \( f_2 \) for \( x = H_0 \) in \([0, 10^9]\) shows that the de Sitter inflation can be realized and the de Sitter solution can be in \([0, 10^9]\). The corresponding inflation duration can be enough and only the instability of this de Sitter solution can lead to the end of the inflationary stage. Indeed, by following the procedure proposed in Vilenkin (1985), we introduce the \( G_1(T) \) model in trace anomaly resulting equation (61) and obtain

\[
0 = \frac{2}{\kappa^2} \left( -n - \frac{2e^{-\frac{18}{70}H^2c_2}}{D_0} - 9\tilde{H}c_2 \right) \nonumber
\]
\[
- \left( \frac{2}{3} \tilde{b} + \tilde{b}'' \right) (6\tilde{H} + 42\tilde{H}^2H + 24\tilde{H}^2 + 72H^2\tilde{H}) \nonumber
\]
\[
+ 24\tilde{b}'(H^4 + H^2\tilde{H}). \tag{68} \nonumber
\]

We then consider the small perturbation from the de Sitter solution as

\[
H = H_i(1 - \delta(t)), \quad \delta \ll 1. \tag{69} \nonumber
\]

Here, \( H_i \) stays for the constant Hubble parameter at the inflationary stage for the case that the small perturbation vanishes \( (\delta(t)) \). It comes from relation (69) that if \( \delta(t) > 0 \), \( H < H_i \) and \( H \) will not diverge contrary to the case where \( \delta(t) < 0 \). So, we assume \( \delta(t) > 0 \) and substitute relation (69) into (68). After linear perturbation studying, we have the following perturbation equation:

\[
0 = 6H_iD_0\kappa^2 \left( \frac{2}{3} \tilde{b} + \tilde{b}'' \right) \tilde{H} + 42H_i^2D_0\kappa^2 \left( \frac{2}{3} \tilde{b} + \tilde{b}'' \right) \tilde{H} \nonumber
\]
\[
- \left[ 72H_i^3D_0\kappa^2 \left( \frac{2}{3} \tilde{b} + \tilde{b}'' \right) \right] \tilde{H} \nonumber
\]
\[
- 24\tilde{b}'D_0\kappa^2H_i^3 - 36c_2H_ie^{-\frac{18}{70}H^2c_2} \tilde{H} \nonumber
\]
\[
- (96\tilde{b}'D_0\kappa^2H_i^4 + 144c_2e^{-\frac{18}{70}H^2c_2}H_i^2)\delta. \tag{70} \nonumber
\]

By considering the form of \( \delta \) as \( \delta = \exp(\tau t) \) where \( \tau \) is a constant, it results that if \( \tau > 0 \) the amplitude of \( \delta \) increases as the cosmic time passes, so the de Sitter solution becomes unstable, consequently the Universe can exit from the inflationary stage. By making using this form \( \delta = \exp(\tau t) \) in perturbation equation (70) one gets

\[
\tau^3 + 7H_i\tau^2 + \left( -12H_i^2 + \frac{4\tilde{b}'D_0\kappa^2H_i^2 + 6c_2e^{-\frac{18}{70}H^2c_2}}{D_0\kappa^2} \right) \tau \nonumber
\]
\[
- \left( 16\tilde{b}'D_0\kappa^2H_i^3 + 24c_2e^{-\frac{18}{70}H^2c_2}H_i \right) = 0. \tag{71} \nonumber
\]

As long as at the initial stage of inflation, the Hubble rate \( H \) varies slowly so that \( \dot{H} \ll \ddot{H}H \) and by neglecting the term \( \tau^3 \) one gets a second order equation on \( \tau \) whose resolution gives

\[
\tau = \frac{1}{P} \left( A \pm \frac{1}{2} \sqrt{R + S} \right), \tag{72} \nonumber
\]

where

\[
P = 7D_0H_i(2\tilde{b} + 3\tilde{b}'')\kappa^2, \tag{73} \nonumber
\]
\[
A = -9c_2e^{-\frac{18}{70}H^2c_2} + 30D_0H_i^2(-2\tilde{b} + \tilde{b}' - 3\tilde{b}'')\kappa^2, \tag{74} \nonumber
\]
\[
R = 672D_0H_i\kappa^2(2\tilde{b} + 3\tilde{b}'')3c_2e^{-\frac{18}{70}H^2c_2}H_i + 2D_0H_i^2\kappa^2\tilde{b}'', \tag{75} \nonumber
\]
\[
S = 36[3c_2e^{-\frac{18}{70}H^2c_2} + 2D_0H_i^2(-2\tilde{b} + \tilde{b}' - 3\tilde{b}'')\kappa^2]^2. \tag{76} \nonumber
\]

We have \( c_2 < 0, \tilde{b} > 0, \tilde{b}' < 0 \) and \( \tilde{b}'' \) as an arbitrary constant so can be chosen as \( \tilde{b}'' > 0 \). So \( A \) and \( P \) can be positive and \( R \) negative; \( S \) is automatically positive and so \( P + S \) can be positive. It results that \( \tau > 0 \) can be found and \( \delta \) can grow and the Hubble parameter decreases so the de Sitter solution becomes unstable. Otherwise, the consistent choice of these previous parameters can also show at what condition the stability of the de Sitter solution, power of an eternal inflation, can occur. It is well known that at inflationary stage, the Hubble parameter represented here by our de Sitter solution \( H_i \) is great, so for \( D_0 > 0 \), it follows \( D_0H_i^2 > e^{-\frac{18}{70}H^2c_2} > 0 \).
According to the fact that \( c_2 < 0 \) and \( \bar{b}'' > 0 \) are arbitrary free constant parameters, they can be chosen in order to satisfy \( | -2\bar{b} + \bar{b}' - 3\bar{b}'' | \geq \frac{3}{10}|c_2| \) and as result, \( A \) can be negative because \( -2\bar{b} + \bar{b}' - 3\bar{b}'' < 0 \). The others quantities can maintain their sign respectively namely \( P \) positive, \( R \) negative, \( S \) and \( R + S \) can be still positive. We conclude from (72) that with the sign (−), the control parameter \( \tau \) can be negative and then \( \delta \) decreases in time leading to stability of the de Sitter solution so that the Universe can not exit from the inflationary stage.

### 3.2.4 Exist from inflation description with the second generalized model \( G_2(T) \)

We proceed as the same manner as in the previous subsection on the first generalized model. So we plug the second generalized model \( G_2(T) \) in (57) in the first Friedmann equation in vacuum and obtain

\[
n + 2(-D_3 - 6H^2)^{-3/\beta}v_2 = 0. \tag{77}
\]

If \( n = 0 \), this last equation can’t have finite solution because \( \beta > 0 \). For \( v_2 > 0 \), we extract the possible de Sitter solution as

\[
H_i = \frac{\sqrt{6}}{6}(-D_3 - \left(-\frac{n}{2v_2}\right)^{-\beta/3})^{1/2}. \tag{78}
\]

To guarantee the exit from inflation with the model, we put it in the trace anomaly contribution equation (61), and it results

\[
\frac{1}{\kappa^2}[24\bar{b}'H^4\kappa^2 - 2n - 4(-D_3 - 6H^2)^{-3/\beta}v_2] = 0. \tag{79}
\]

By making

\[
d(x) = 24\bar{b}'x^4\kappa^2 - 2n - 4(-D_3 - 6x^2)^{-3/\beta}v_2, \tag{80}
\]

the parallel studying as the latter on the first model shows in the case of the second generalized model \( G_2(T) \) prove that a de Sitter solution can be met if \( v_2 > 0 \) and the following conditions are satisfied:

\[
\beta = \frac{3}{2k - 1} \quad \text{with} \quad k = 1, 2, 3, \ldots,
\]

in order to have a positive function derivative, \( (81) \)

and

\[
D_3 < -\left(-\frac{n}{2v_2}\right)^{-\beta/3} \quad \text{to have 0 in interval image.} \tag{82}
\]

Posing

\[
d_1(x) = 24\bar{b}'x^4\kappa^2, \tag{83}
\]

we illustrate again through the intersection of the curves reflecting the evolution of the two functions \( d_1 \) and \( d_2 \) in Fig. 2 that the de Sitter solution under the trace anomaly can be met with the second generalized model.

In order to find out the instability of the de Sitter solution, source of the exit from inflation, we plug the relation (57) in (61) and we obtain

\[
0 = \frac{2}{\kappa^2} \left[ -n + \frac{2v_2}{\beta}(-D_3 + 6H^2)^{(1+\frac{2}{\beta})} \right] \times [(D_3 - 6H^2)\beta + 9\dot{H}] \nonumber
\]

\[
- \left(\frac{2}{3}\bar{b} + \bar{b}''\right)(6\ddot{H} + 42\dot{H}H + 24H^2 + 72H^2\dot{H}) + 24\bar{b}'(H^4 + H^2\dot{H}). \tag{85}
\]

The associated perturbation equation is

\[
0 = 6H_i \left(\frac{2}{3}\bar{b} + \bar{b}''\right)\delta + 42H_i^2 \left(\frac{2}{3}\bar{b} + \bar{b}''\right)\delta \nonumber
\]

\[
+ \left[-24\bar{b}'H_i^3 + 72H_i^3 \left(\frac{2}{3}\bar{b} + \bar{b}''\right) \right] \delta \nonumber
\]

\[
- \frac{36v_2}{\kappa^2\beta}(-D_3 + 6H_i^2)^{-2k}H_i \delta \nonumber
\]

\[
+ \left[-96\bar{b}'H_i^4 + 48\frac{v_2}{\kappa^2}(-D_3 + 6H_i^2)^{-2k} \right] \delta, \tag{86}
\]

where we have used the relation (81). We consider again the form \( \delta = \exp \sigma t \) with \( \sigma \) a constant supposed to be positive in order to ensure the growth with the cosmic time of the amplitude of \( \delta \) which can lead to the instability of the de Sitter
solution. We introduce this expression in (86) and obtain
\[
0 = \sigma^3 + 7H_1 \sigma^2 + 12H_2^2 \sigma 
\]
\[
+ \left[ -16 \tilde{b}' H_1^2 \kappa^2 \beta + 8 \beta v_2 (-D_3 + 6 H_1^2)^{-2k} \right] \sigma 
\]
\[
+ \left[ 16 \beta v_2 (-D_3 + 6 H_1^2)^{-2k} \right] \sigma 
\]
\[
\times \left[ \beta \kappa^2 H_1 \left( \frac{2 \tilde{b} + \tilde{b}''}{3} \right) \right]^{1/2} \] (87)

Neglecting the term \( \sigma^3 \), the resolution of the obtained second order equation on \( \sigma \) gives
\[
\sigma = \frac{1}{L} (M \pm \sqrt{3P + Q}), \] (88)

with
\[
L = \frac{(-D_3 + 6 H_1^2)^{-2k}}{7H_1 (2 \tilde{b} + 3 \tilde{b}'') \beta \kappa^2}, \] (89)
\[
M = 9 v_2 - 6 H_1 (-D_3 + 6 H_1^2)^{2k} (2 \tilde{b} - \tilde{b}' + 3 \tilde{b}'') \beta \kappa^2, \] (90)
\[
P = 56 \beta \kappa^2 (-D_3 + 6 H_1^2)^{2k} \left( 2 \tilde{b} + 3 \tilde{b}'' \right) \left( v_2 (2k - \beta) \right) + 2 H_1^4 (-D_3 + 6 H_1^2)^{2k} \beta \kappa^2 \] (91)
\[
Q = 3 \left[ 3 v_2 - 2 H_1^4 (-D_3 + 6 H_1^2)^{2k} (2 \tilde{b} - \tilde{b}' + 3 \tilde{b}'') \beta \kappa^2 \right]^{1/2}, \] (92)

From the previous analysis \((2 \tilde{b} - \tilde{b}' + 3 \tilde{b}'')\) can be positive, and from relation (81) \( k > 0 \), one can conclude that \( L > 0 \). From Fig. 2, the constant \( v_2 \) must be positive and very small in order to generate the de Sitter solution so it can be chosen in aim to have \( M < 0 \), so \( P \) can be negative because we have \( \tilde{b}' < 0 \) and \( v_2 (2k - \beta) > 0 \) and very small according to (81). \( Q \) is positive so the sum \( P + Q \) can be positive and \( \sigma > 0 \) can be reached by considering the solution (88) with sign (+). We also emphasize here that for a great positive value of free parameter \( \tilde{b}'', \) \( M \) can be negative as it previously said. The consideration of the sign (-) in (88) leads to \( \sigma \) negative and the exit can not occur.

As conclusion, we need to stress however, that our previous analysis have permitted to realize an exist de Sitter inflation in this \( F(T) \) description of inflation and require the consideration of the cosmological constant which have allowed us to constrain the free parameters of ours models in some conditions.

### 3.3 Power-law model of \( F(T) \) gravity

In this part of section, we choose one concrete model to describe the observables of inflationary models. The concerning model is the power-law model of \( F(T) \) gravity given by \( F(T) = \mu (-T)^\mu \) where \( \mu \) are \( n \) constants. By introducing this expression in the Friedmann equation (45) in the vacuum, we get a first order differential equation in \( D(N) \) which solution is:
\[
D(N) = T(N) = D_4 \exp \left( -\frac{3N}{n} \right), \] (93)

with \( D_4 \) a negative constant. This reconstructed form is equivalent to exponential form in (29) with the following conditions \( D_2 = D_4, D_3 = 0 \) and \( \beta = -3/n \). This solution is valid for \( n \neq 0 \). The slow-roll parameters with respect to (93) are written as \( \epsilon = 3/(2n), \eta = 3/n, \) and \( \xi^2 = 9/n^2 \). Since these parameters are constants, if \( n \gg 1 \), on gets \( \epsilon \ll 1, \eta \ll 1 \) and \( \xi^2 \ll 1 \) during inflation. These results prove that the conditions of slow-rolling are realized with the reconstructed form of scalar torsion in (93) and then prove the inflation actually occurs. But the inflation stops when the slow-roll parameters become of order \( \sim 1 \) (Odintsov and Oikonomou 2015) or if \( M_{\text{max}}(\epsilon, |\eta|) = 1 \) (Nashed et al. 2014). This exist can be met if we take \( n = 3 \) because it gives \( M_{\text{max}}(\epsilon, |\eta|) = M_{\text{max}}(\frac{1}{3}, 1) = 1 \) and the expression of the associated scalar torsion is \( T(N) = D_4 \exp(-N) \) where \( D_4 \) is constant and \( N \) is a e-folds number. In other hand, if \( n \gg 1 \), namely if \( n \) is very larger then unit, it follows \( \exp(-\frac{3N}{n}) \sim 1 \) and the scalar torsion behaves like constant \( T(N) \sim D_4 \). This corresponds to a de Sitter solution or in another words de Sitter inflation. If we take the constant \( D_4 \) as \( D_4 = -2A \), we can get \( T(N) = T \sim -2A \) which corresponds to a vacuum Universe with the state parameter \( \omega \sim -1 \) and so represents the key of the inflationary Universe. Such inflationary models produces a de Sitter expansion case of the Universe and there are completely invariant under space and time translations so that they are incapable of ending the inflation epoch. Consequently, inflation occurs and the Universe can not exit from it.

Moreover, it is was developed in \( F(T) \) that the graceful exit inflation without slow-roll approximation in the quadratic \( T^2 \) theory (Nashed et al. 2014). But it was recently performed the activity on the so-called singular inflation in the \( f(R) \) where exist occurs due to the type IV singularity at the end of slow-roll inflation (Odintsov and Oikonomou 2015). In such descriptions, there exist two alternative descriptions for the slow-roll parameters, one that takes into account the potential (Liddle et al. 1994), that is the case of our present work, and the other one takes into account the Hubble rate only namely the so-called Hubble slow-roll parameters which were explored for example by (Odintsov and Oikonomou 2015). In the singularity studying, the Hubble parameter was considered as polynomial function of cosmic time as \( H = n_1 + n_2 (t - t_0)^\alpha \) and the IV singularity occurs if \( t = t_0 \) and \( \alpha > 1 \). In our present slow parameters description within perfect fluid and \( F(T) \) theory, the first slow parame-
\[
\epsilon = \frac{3}{2} \rho(N) f(\rho) \left[ \frac{f'(\rho) - 2}{2\rho(N) - f(\rho)} \right]^2,
\]

(94)

with \( N \) a e-folds number used as scalar field, \( \rho(N) \) and \( f(\rho) \) the perfect fluid quantities which can be determined in the context of \( F(T) \) by using (18), (21) and (43) as

\[
\rho(N) = -\frac{F(T)}{4\kappa^2 F'(T)},
\]

(95)

\[
f(\rho) = \frac{T'(N) [ (F'(T))^2 - F''(T)F(T) ]}{12\kappa^2 F'(T)}.
\]

(96)

We recall here that in these equations, the prime means the derivative with respect to argument. The two last equations hold because \( \rho \) and \( T \) are function of \( N \) so the \( F(T) \) must be a function of scalar field \( N \). Consequently, the slow parameter in (94) will be function only of \( N \) and according to the fact that \( dN/dt = H \), one can find out the possibility of the graceful exit inflation. So it will be necessary to couple the scalar field to \( F(T) \) in such investigation. This means that any graceful exit can not occur if we consider only \( F(T) \) and without taking into consideration a scalar field.

4 Conclusion

We have described in this paper the observables of inflationary models. These observables are related to the slow-roll parameters which explicitly depend from the potential of inflaton. This justifies the choose of the scalar field in the description of the observables. We have found out the expressions of these observables as function of scalar torsion. This description have been done in the framework of scalar field model coupled with the Teleparallel gravity. We have chased our investigations on the perfect fluid description of slow-roll parameters. After expressing the slow-roll parameters and the observables in terms of quantities of perfect fluid, we have reconstructed two perfect fluid models according to the linear and exponential forms of scalar tensor and gives additional model. The have compared the results with observation. We have found that perfect fluid can lead to the Planck results with \( r = O(0.1) \).

We have ended our study on the \( F(T) \) gravity description of the slow-roll parameters. We have also reconstructed two models as we have done in the case of perfect fluid. By choosing appropriate values for the free parameters of the models, one gets results on the observables especially \( n_s \) and \( r \) which are compatible with the Planck data on these observables.

We have studied the exit from inflation by generalizing these reconstructed models with a constant \( n \) which we have related to a constant cosmology. With these new obtained models and by making using the trace anomaly contribution as it is done by some authors mentioned in the manuscript, we have demonstrated that the de Sitter inflation can be realized if \( n \) is necessary no null. Ours investigations have found out that the de Sitter inflation can occur and finally the Universe can exit from it due to the de Sitter space instability solution coming from the trace anomaly. The last studied model in this work, the Power-law model of \( F(T) \) gravity, has permitted to confirm the realization of the conditions of slow-rolling in the framework of \( F(T) \) gravity.

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