Quantum mechanics-based signal and image denoising

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Abstract—Decomposition of digital signals and images into other basis or dictionaries than time or space domains is a very common approach in signal and image processing and analysis. Such a decomposition is commonly obtained using fixed transforms (e.g., Fourier or wavelet) or dictionaries learned from example databases or from the signal or image itself. In this work, we investigate in detail a new approach of constructing such a signal or image-dependent bases inspired by quantum mechanics tools, i.e., by considering the signal or image as a potential in the discretized Schrödinger equation. To illustrate the potential of the proposed decomposition, denoising results are reported in the case of Gaussian, Poisson and speckle noise and compared to state of the art algorithms based on wavelet shrinkage, total variation regularization or patch-wise sparse coding in learned dictionaries.

Index Terms—adaptive signal and image representation, denoising, quantum mechanics.

I. INTRODUCTION

In number of applications, processing or analyzing signals and images require the use of other representations than time or space. While the most famous transformation still remains the Fourier transform, other representations have been proposed to overcome the non-localization in time or space of the Fourier basis vectors. Among these time-frequency representations, the most used are the short time Fourier and the wavelet transforms \cite{1}, \cite{2}. Most often (see, e.g., image compression, restoration, reconstruction, denoising or compressed sensing), such transforms are associated to the concept of sparsity, i.e., their ability to concentrate most of the signal or image energy in a few coefficients. To reinforce the sparsity, overcomplete dictionaries have also been explored over the last decades, such as the wavelet frames or more recently patch-based or convolutional dictionaries learned from a set of training signals or images \cite{3}. The latter has been shown to be of particular interest in image denoising \cite{4}.

In this paper, we investigate a novel signal and image representation, through a dedicated basis extracted from the signal or image itself, using concepts from quantum mechanics. First preliminary results were published in \cite{5}. Compared to fixed basis such as Fourier, discrete cosine, wavelets, curvelets, etc, or dictionary learning that generally needs a training database, the proposed approach has the advantage of computing a transform adapted to the signal or image of interest.

Several attempts of translating quantum principles in image or signal processing applications have been proposed in the literature. One may note the seminal work in \cite{6}, or, more recently, the interest of quantum mechanics in image segmentation \cite{7} or in pulse-shaped signal analysis \cite{8}, \cite{9}. Note that a separate domain also exists on designing image processing algorithms adapted to quantum computers, but is of a different purpose \cite{10}, \cite{11}.

More related to our work, we note that there was a recent attempt to use quantum mechanics in the same context in \cite{12}, \cite{13}. Although there are similarities between the two approaches, there are also some important differences. The authors in \cite{12}, \cite{13} start from a continuous mathematical representation of the signal, and the discretization only occurs at the end of the process. The processing of a large image in these papers is done by decomposing it into lines and columns to get 1D signals, while the proposed work is applied blockwise, which we believe is more efficient for image denoising. Additionally, unlike \cite{12}, \cite{13}, our method fully takes into account the quantum localization phenomenon, a subtle effect due to quantum interference which makes the distribution of the eigenfunctions of the Schrödinger operator strongly dependent on noise, and has important effects on the denoising process. We also use the physics of the problem to identify the optimal domain of applicability of such methods.

The proposed framework reposes on the discrete version of the Schrödinger equation for a quantum particle in a potential. In our case, the potential is represented by the signal samples or the pixel values. The bases used to decompose the signal or the image are directly computed from the signal and image itself and correspond to the wave function representing the stationary solutions of the Schrödinger equation. These wave functions have interesting properties of temporal or spatial localization and of frequency dependence on the value of the potential. In particular, they present higher frequencies for low potential values, thus allowing an original signal or image decomposition.

Within the proposed framework, the frequency and localization properties of the basis can be controlled through several parameters, thus ensuring flexibility in applications such as denoising. A detailed description of the behavior of the proposed transform and denoising method with respect to the choice of these parameters is provided, allowing to gain insight about the practical consequences in signal and image processing of the quantum mechanical principles involved. Furthermore, the proposed transform embedded in a denoising algorithm shows promising results in different noise scenarios.
(additive Gaussian, Poisson or speckle noise). Finally using different signals and images, comparisons with several state of the art methods are finally performed.

The remainder of the paper is organized as follows. Section II gives the details of the adaptive transform design and its application to denoising. Results and comparisons are provided in Section III and concluding remarks are finally reported in Section IV.

II. THE METHOD

A. Adaptive transform from quantum mechanics

The main idea of the proposed method is to describe a signal or an image on a specific basis which is constructed through the resolution of a related problem of quantum mechanics: the probability of presence of a quantum particle in a potential related to the signal or image. In the following, we explain how this problem is formulated and how it is built from the signal or image.

Our formalism is based on the resolution of the Schroedinger equation of non-relativistic quantum mechanics. This equation determines the wave function \( \psi(y) \) which belongs to the Hilbert space of \( L^2 \)-integrable functions, \( y \) being e.g. a spatial coordinate. The function \( |\psi(y)|^2 \) gives the probability of presence of the particle, which implies that \( \int |\psi(y)|^2 dy = 1 \). For particles in a potential \( V(y) \), the stationary Schroedinger equation reads

\[
-\frac{\hbar^2}{2m} \nabla^2 \psi = -V(y)\psi + E\psi, \tag{1}
\]

with \( m \) the mass of the particle and \( \hbar \) the Planck constant that are parameters of the problem. In operator notation it corresponds to \( \hat{H}\psi = E\psi \) with \( \hat{H} = -\frac{\hbar^2}{2m} \nabla^2 + V \) the Hamiltonian operator. The energy \( E \) of the particle in (1) labels the solutions of the problem. Solutions of this stationary Schroedinger equation in a bounded domain correspond to a discrete set of energy levels, from a minimal energy to infinity.

Solutions of (1) form a basis of the Hilbert space to which the wavefunctions belong. This space is infinite-dimensional for continuous values of the position \( y \) in (1). However, we are interested in signal or image processing applications, where the space is discretized in a finite number of points. Specifically, we assume that the potential \( V \) is represented by the value of signal sample or image pixel \( x \). In the case of a discretized problem, the operators become finite matrices and the resolution of (1) is equivalent to diagonalizing the Hamiltonian matrix. In (1), the Laplacian operator should be replaced by its discrete version, following the standard numerical definitions of the gradient operator, developed in the following for a 2D image \( x \in \mathbb{R}^{N\times N} \):

\[
(\partial_x x)(i,j) = x(i+1,j) - x(i,j) \quad \text{if } i < N
\]

\[
(\partial_y x)(i,j) = x(i,j+1) - x(i,j) \quad \text{if } j < N
\]

where \( \partial_x \) and \( \partial_y \) are associated to the horizontal and vertical gradients. The boundary conditions correspond simply to a zero padding of the image.

The resolution of (1) is thus equivalent to finding eigenvectors and eigenvalues of the discretized Hamiltonian matrix \( H \in \mathbb{R}^{N^2\times N^2} \) defined assuming that the signal or the image \( x \) is the potential:

\[
H(i,j) = \begin{cases} x(i,j) + 4\frac{\hbar^2}{2m} & \text{for } i = j, \\ -\frac{\hbar^2}{2m} & \text{for } i = j \pm 1, \\ -\frac{\hbar^2}{2m} & \text{for } i = j \pm N, \\ 0 & \text{otherwise.} \end{cases} \tag{2}
\]

As the boundary conditions correspond to zero padding of the image, a few individual coefficients of the matrix \( H \) follow specific rules. Indeed, \( H(i,j) = x(i,j) + 2\frac{\hbar^2}{2m} \) for \( i = j \) and \( i \in \{1, N, N^2-N+1, N^2\} \), \( H(i,j) = x(i,j) + 3\frac{\hbar^2}{2m} \) for \( i = j \) and \( i \in \{2, 3, ..., N-1, N^2-N+2, N^2-N+3, ..., N^2\} \), \( H(i,j) = x(i,j) + 4\frac{\hbar^2}{2m} \) for \( i = j \) and other than the previous set with \( i \mod N \in \{0, 1\} \) and \( H(i,i+1) = H(i+1,i) = 0 \) for any \( i \) multiple of \( N \) apart from \( N^2 \). As an example, for an image of size \( 4 \times 4 \) (i.e. \( N = 4 \)), the corresponding Hamiltonian matrix will be of size \( 16 \times 16 \) (explicitly shown in Appendix A).

The set of eigenvectors gives a basis of the Hilbert space, with each eigenvector associated to an energy \( E \), which is the corresponding eigenvalue of the Hamiltonian operator. The \( N^2 \) eigenvectors, denoted by \( \psi_i \in \mathbb{R}^{N^2\times 1} \) are the main tool for the proposed adaptive transform in this work. Indeed, our method consists in projecting the signal or image on this particular basis and use the energy associated to each eigenfunction as a parameter on which we perform the thresholding of these coefficients.

B. Properties of the adaptive transform

The basis vectors are ranked from the lowest to the highest corresponding eigenvalue, associated with the energy \( E \). In the same way as the Fourier or wavelet basis, these basis vectors \( \psi_i \) are oscillating functions, but with a local frequency depending on the local value of \( \sqrt{2m(E-V)}/\hbar \). This local frequency thus depends on the difference between the energy \( E \) and potential \( V \), such that for a given energy, the higher frequency oscillations are associated with the lower values of the potential. In this way, the basis of eigenvectors of (1) naturally describes with different frequencies the different parts of the signal or image, in contrast to e.g. the Fourier or wavelet bases. The precise relation between the local frequency of the eigenvectors and the value of the signal or image pixel is governed by the parameter \( \hbar^2/2m \). In the physical problem of quantum mechanics, this quantity is linked to Planck’s constant and the particle mass, but in our framework it is a free parameter. It should be chosen with care, as extreme values are clearly inadequate. Indeed, as the problem is discretized there is a maximal frequency in the problem, linked to the inverse of the discretization step. If \( \hbar^2/2m \) is very small, the local frequencies \( \sqrt{2m(E-V)}/\hbar \) become very high even for low values of the energy, the maximal energy becomes very low, and the basis does not explore properly high values of the signal or pixels of the image. On the other side for very large values of \( \hbar^2/2m \), the local frequencies become smaller and smaller at fixed energy, the maximal energy becomes larger and larger, and eventually when \( \hbar^2/2m \) tends to infinity most vectors of the adaptive basis are so high above the signal or
image pixel that they do not discriminate between low and high values, becoming closer and closer to the standard Fourier basis vectors. Therefore it is crucial to tune the free parameter $\hbar^2/2m$ in the right way.

A more subtle property of the basis consists in its localization properties. Indeed, it is known in quantum mechanics that a disordered potential localizes the wavefunctions in one and two dimensions. Due to destructive interference, the different wave functions are exponentially localized at different positions of the potential, an effect known as Anderson localization, which earned the Nobel prize in 1977 to its discoverer [14]. If the signal or image are not smooth, which certainly arises in the case of a noisy signal or image, we expect the vectors of the basis to be localized, with a localization length which will be smaller and smaller for increasing noise power.

The level of localization is measured by computing the inverse participation ratio (IPR) of the wave functions, mathematically defined as:

$$IPR = \frac{\sum_{i=1}^{N^2} |\psi_i|^2}{\sum_{i=1}^{N^2} |\psi_i|^4}$$

where $N^2$ is the dimension of the Hilbert space. For a vector uniformly spread over $P$ indices and zero elsewhere, this quantity is exactly $P$. For an exponentially localized vector such as the wavefunctions in a disordered potential, it is proportional to the localization length. This phenomenon should be taken into account when producing the adaptative basis.

C. Application to the denoising problem

The significant difficulties for signal or image denoising are to sharpen the edges without blurring and preserve the image textures without generating artifacts. The most common denoising strategies are based on three primary steps. To distinguish the useful information and the noise the noisy signal or image is projected onto a dictionary. This is then accompanied by a hard or soft thresholding process in the transformed space. Finally, the revised coefficients are back projected to the time or space domain, so that the denoised signal or image could be retrieved. We will apply the same procedure using the adaptative basis defined by the eigenvectors $\psi_i$ obtained by solving the Schrödinger equation (1).

The basic assumptions is that the noise is more present in high frequency components of the signal or image, corresponding to eigenvectors associated with large energy eigenvalues. The thresholding will therefore be performed in energy, leaving out the components of the signal or image on high energy eigenvectors. The fact that our basis has frequencies which vary depending on the position should be an asset, especially for signal or image dependent noise (e.g. Poisson noise). In the following, we will show that it is indeed the case in some examples of signals and images with various types of noise.

The denoising process unfolds as follows; for a noisy signal or image denoted by $x$, the denoised signal or image is rebuilt through:

$$\hat{x} = \sum_{i=1}^{N^2} \alpha_i \psi_i \tau_i,$$

with

$$\tau_i = \begin{cases} 1 - \frac{1}{\rho} & \text{for } i \leq s, \\ 1 - \frac{i-s}{\rho} > 0, & \text{for } i > s \text{ and } 1 - \frac{i-s}{\rho} > 0, \\ 0 & \text{otherwise.} \end{cases}$$

where $\alpha_i$ are the coefficients representing the signal or image $x$ in the proposed adaptive basis. $s$ and $\rho$ are two hyperparam-
eters, used to define the thresholding function for the proposed denoising algorithm.

In order to use this procedure, we will need to specify which values of the parameter $\hbar^2/2m$ should be selected. As we will see, there is a relatively large range of values where the algorithm is efficient, meaning that it can be set to a specific value independent of the signal or image on which the algorithm is used.

The localization property of the basis vectors should also be taken into account. A given noisy signal or image introduced in (1) will give rise to localized wave functions with a localization length which will depend on the noise level. To modify this characteristic of the basis, and to construct the Hamiltonian matrix, we use a smoothed adaptation of the noisy signal or image, computed by a simple convolution with a Gaussian kernel whose standard deviation is denoted by $\sigma$. In our framework, this standard deviation $\sigma$ is an additional free parameter. If $\sigma$ is chosen too large, then the noisy signal or image becomes so smooth that many characteristics needed for the adaptative basis will be lost. On the opposite, if $\sigma$ is too small the basis vector will remain strongly localized. To balance both sides we need to tune the parameter $\sigma$ to get the best achievable outcome.

D. Algorithm description

Denoising a signal or an image using the proposed method requires the computation of eigenvalues and eigenvectors of the discretized Hamiltonian matrix (2) for appropriate values of the parameters $\hbar^2/2m$ and $\sigma$, project the signal or image on this basis, threshold the coefficients by an appropriate threshold in energy, and reconstruct from this a denoised signal or image. These steps are summarized in Algorithm 1 and Fig. 1.

**Algorithm 1:** Denoising algorithm using the proposed adaptive transform.

**Input:** $x$, $\frac{\hbar^2}{2m}$, $s$, $\rho$, $\sigma$

1. Compute a smooth version of $x$ by Gaussian filtering
2. Construct the Hamiltonian matrix $H$ based on the smoothed version of $x$ using (2)
3. Calculate the eigenvectors $\psi_i$ of $H$
4. Compute the coefficients $\alpha_i$ by projecting $x$ onto the basis formed by $\psi_i$
5. Threshold the coefficients $\alpha_i$ and recover the denoised signal or image following (5) and (4)

**Output:** $\hat{x}$

For very large signals and images, where the size of the matrix (2) becomes too large for practical simulations, we implement a modified version of the algorithm where the matrix (2) is diagonalized for subparts of the signal or image independently, and then a complete signal or image is reconstructed:

- Use algorithm 1 for each sub-block.
- Reconstruct the denoised signal or image by integrating each denoised sub-block.

III. Results

This section regroups results showing the interest of the proposed approach in signal and image denoising and analyze the optimal choice of parameters. Subsection III-A illustrates the localization property of the wave functions in the presence of noise, and consequently the interest of pre-smoothing with parameter $\sigma$ embedded in the denoising algorithm. Subsection III-B elaborates the dependence of the proposed denoising method on the choice of the hyperparameters $\hbar^2/2m$ and $\sigma$. Subsection III-C compares the denoising results obtained with the proposed approach to several state of the art methods. Finally, the section ends with an example of real medical application in subsection III-D, showing the ability of the proposed method to denoise real world (dental) cone beam computed tomography (CBCT) images.

Fig. 2. (a) Synthetic signal and (b) cropped version of Lena used to illustrate the localization property of the wave functions.

Fig. 3. IPR corresponding to the wave function calculated from the signal in Fig. 2(a) degraded by an additive Gaussian noise for several SNR. The size of the signal was 512.

A. Localization of wave functions and interest of Gaussian smoothing

The localization property of the wave functions and the impact of Gaussian smoothing are analyzed through two
examples, i.e., a synthetic signal and a cropped version of the standard Lena image shown in Fig. 2.

Fig. 3 shows the average IPR of solutions of (1) obtained from the signal in Fig. 2(a) degraded by an additive Gaussian noise with different signal to noise ratios (SNR). The results displayed clearly highlight the increasing localization of the wave functions when the SNR decreases. This is consistent with the theory. Indeed, as explained in the previous section, due to the phenomenon of Anderson localization the wave functions $\psi_i$ are exponentially localized in the presence of noise. In addition, they become more and more localized when noise increases. This property can be measured by the IPR defined in (3), which measures the number of significant components of a vector.

Fig. 4(b), Fig. 5(b-c) and Fig. 6(b-c) show examples of wave functions calculated from a noisy signal and image, where the same noisy image corrupted by additive Gaussian noise is considered in Fig. 5(a) and Fig. 6(a). The Fig. 5 and Fig. 6 in fact show two different wave functions for the same example. From these examples, one can observe that the wave functions are completely localized in a specific location and present a fast decrease due to the destructive interference. On the contrary, in the case where the same wave functions are calculated from low-pass filtered versions of the noisy signal and image (i.e. a smoothed version of the potential), they are shown to delocalize and spread over the whole available space using high frequencies for regions with low potential and low frequencies for high potential value regions, as illustrated in
The localization of the wave functions in the presence of noise has an important impact on the proposed signal or image representation and furthermore on the efficiency of the denoising process. To illustrate this claim, Fig. 7 shows a denoising result with and without the use of the low pass Gaussian filter prior to the computation of the wave functions. In this example, the cropped version of Lena in Fig. 7(a) was degraded by a Poisson noise resulting into a SNR of 15 dB. The denoised images in Fig. 7(c,d) were obtained using the algorithm detailed in Algo. 1. However, while the result in Fig. 7(c) exploits the image decomposition through localized wave functions computed directly from the noisy image, the result in Fig. 7(d) was obtained by filtering the noisy image by a low pass Gaussian filter before using (1), in order to delocalize the wave functions. The interest of such delocalization can be visually appreciated in this example and allows a peak SNR (PSNR) gain of more than 3 dB. In the following, we will always use a pre-smoothed signal or image in (1), and the parameter $\sigma$ of the smoothing will be studied as one of the parameters of the algorithm. To avoid confusion, let us insist on the fact that although the smoothening of the noisy signal or image before using (1) may look like a denoising process, it is only a tool to construct the appropriate adaptative basis, which will be used in a second step to denoise the original noisy signal or image.
B. Influence of hyperparameters on the efficiency of the algorithm

In this subsection, we provide a detailed discussion about the influence of the hyperparameters on the proposed adaptive bases. As mentioned above, the parameter $\hbar^2/2m$ specifies how the local frequencies of the vectors of the basis vary with the amplitude of the signal or image pixel value.

To illustrate this relationship, the effect of $\hbar^2/2m$ on local frequencies is shown in Fig. 8 for three distinct values of this parameter. For each case, three wave functions (number 25, 70 and 100) computed from the synthetic signal in Fig. 2(a) are displayed. For low values of $\hbar^2/2m$ (i.e., 0.08 for the results in Fig. 8(a)), one may remark that the wave functions are oscillating at very high frequencies, even for higher values of the potential (i.e., of the signal). The presence of a maximal oscillation period due to the discretization of the signal implies that in this limit the high values of the signal are not taken properly into account. For very high values of $\hbar^2/2m$ (15 for the results in Fig. 8(c)), most of the wave functions are at an energy well above the potential values, and they discriminate less and less between the regions with different potential height. In this limit, wave functions behave very similarly to cosine functions with increasing frequencies, thus reducing the interest of the proposed bases that becomes very similar to the Fourier transform. At intermediate values of $\hbar^2/2m$ (1 for the results in Fig. 8(b)), wave functions explore the different regions but with clearly different oscillation frequencies, i.e., wave vectors have significantly larger frequencies or short wavelengths for the low potential valued regions as opposed to high potential regions.

The second hyperparameter studied in this section that has a strong impact on the proposed denoising algorithm is the cut-off frequency of the Gaussian low pass filter used to smooth the noisy signal or images before computing the wave functions through (1), as explained in the preceding
subsection. In particular, this cut off frequency is fixed through the choice of the standard deviation $\sigma$ of the Gaussian filter. At last, in order to denoise the signal or image one has to threshold the coefficients of the signal or image on the adapta-
Fig. 11. (a) Sample A, (b,c) influence of the hyperparameters $\hbar^2/2m$ and $\sigma$ on proposed method performed on the sample A corrupted with Poisson noise corresponding to a SNR of 15 dB, (d) sample B, (e,f) influence of the hyperparameters $\hbar^2/2m$ and $\sigma$ on proposed method performed on the sample B corrupted with Poisson noise corresponding to a SNR of 15 dB, (g) sample C, (h,i) influence of the hyperparameters $\hbar^2/2m$ and $\sigma$ on proposed method performed on the sample C corrupted with Poisson noise corresponding to a SNR of 15 dB. The hyperparameters $s$ and $\rho$ have been manually tuned up to their best possible values for each set of experiment.

tive basis; this process uses two thresholding hyperparameters $s$ and $\rho$ defined in (5), which define respectively the threshold value and the abruptness of the cut off. In particular, the parameter $s$ corresponds to the threshold in energy of the wave functions taken into account in the expansion (4) to reconstruct the signal or image. Fig. 9 illustrates the variation of the SNR while reconstructing the denoised result corresponding to the signal in Fig. 2(a) (contaminated by additive Gaussian noise of 15 dB) for changing values of the hyperparameter $s$. For $\sigma^2 = 0$, the reconstructed signal has a SNR worse or similar to the original noisy one, indicating once more the importance of the smoothing process before calculating the wave functions through (1). For nonzero values of $\sigma^2$, there is a relatively small range of optimal $s$ values, where the SNR is much better than in the original noisy signal. Of course this threshold value should eventually depend on the level of noise. The adaptative transform makes the filtering of high frequencies stronger at high values of the potential, but the overall level of filtering should still depend on the noise properties.

Numerical experiments on the synthetic signal and on the cropped Lena in Fig. 2 were carried out to analyze the impact of $\hbar^2/2m$ and $\sigma$ on the denoising quality and subsequently...
Fig. 12. Influence of the hyperparameters $\hbar^2/2m$ and $\sigma$ on proposed decomposition carried out on the 2D system Fig. 2(b) in presence of (a,b) Poisson noise, (c,d) Gaussian noise and (e,f) speckle noise corresponding to a SNR of 15 dB respectively. The hyperparameters $s$ and $\rho$ have been manually tuned up to their best possible values for each set of experiment.

to adjust these parameters to their best values for assessment of the efficiency of the algorithm. Two additional synthetic signals were also generated as shown in Fig. 11(d)(g) (Fig. 11(a) corresponds to the same synthetic signal used previously)
Fig. 13. Signal and images used to compare the proposed denoising method to existing algorithms: (a) Synthetic signal, (b) Synthetic image, (c) Elaine, (d) Lena, (e) Still image, (f) Moon.

### TABLE I

**QUANTITATIVE DENOISING RESULTS.**

| Data         | Gaussian Noise (15dB) | Poisson Noise (15dB) | Speckle Noise (15dB) |
|--------------|-----------------------|----------------------|----------------------|
|              | Method                | SNR (dB) | PSNR (dB) | SSIM | SNR (dB) | PSNR (dB) | SSIM | SNR (dB) | PSNR (dB) | SSIM |
| Synthetic Signal | Wavelet hard       | 18.84     | 25.21      | NA    | 17.21     | 24.64      | NA    | 17.36     | 24.13      | NA    |
|              | Wavelet soft        | 18.53     | 24.21      | NA    | 17.79     | 23.62      | NA    | 17.02     | 22.70      | NA    |
|              | VST                  | NA        | NA         | NA    | NA        | NA         | NA    | NA        | NA         | NA    |
|              | TV                   | 16.20     | 23.01      | NA    | 15.94     | 23.45      | NA    | 15.92     | 22.92      | NA    |
|              | DL                   | NA        | NA         | NA    | NA        | NA         | NA    | NA        | NA         | NA    |
|              | Proposed             | 22.21     | 27.50      | NA    | 22.51     | 27.63      | NA    | 20.75     | 26.86      | NA    |
| Synthetic Image     | Wavelet hard       | 15.01     | 24.46      | 0.61  | 15.01     | 25.68      | 0.69  | 15.01     | 25.34      | 0.76  |
|              | Wavelet soft        | 15.71     | 25.05      | 0.64  | 15.61     | 26.20      | 0.70  | 15.49     | 25.80      | 0.77  |
|              | VST                  | NA        | NA         | NA    | 15.09     | 25.83      | 0.69  | 15.06     | 25.58      | 0.76  |
|              | TV                   | 15.74     | 25.07      | 0.64  | 15.62     | 26.23      | 0.71  | 15.53     | 25.78      | 0.77  |
|              | DL                   | 17.35     | 26.15      | 0.71  | 17.14     | 27.22      | 0.75  | 17.21     | 27.48      | 0.80  |
|              | Proposed             | 23.42     | 31.78      | 0.89  | 23.92     | 32.78      | 0.92  | 25.32     | 33.50      | 0.95  |
| Elaine       | Wavelet hard       | 20.52     | 27.02      | 0.52  | 20.08     | 28.17      | 0.49  | 19.75     | 25.99      | 0.48  |
|              | Wavelet soft        | 21.99     | 27.69      | 0.53  | 21.67     | 28.59      | 0.51  | 21.31     | 26.61      | 0.50  |
|              | VST                  | NA        | NA         | NA    | 21.71     | 28.64      | 0.53  | 22.51     | 27.81      | 0.56  |
|              | TV                   | 23.67     | 29.63      | 0.62  | 22.03     | 28.84      | 0.55  | 23.06     | 27.61      | 0.59  |
|              | DL                   | 24.97     | 29.92      | 0.68  | 23.96     | 29.84      | 0.62  | 22.99     | 27.58      | 0.58  |
|              | Proposed             | 24.70     | 29.87      | 0.68  | 23.89     | 29.03      | 0.65  | 25.32     | 28.32      | 0.64  |
| Lena         | Wavelet hard       | 20.84     | 28.17      | 0.72  | 20.01     | 28.89      | 0.68  | 19.22     | 27.49      | 0.66  |
|              | Wavelet soft        | 21.23     | 28.12      | 0.71  | 20.75     | 28.54      | 0.67  | 20.29     | 27.31      | 0.66  |
|              | VST                  | NA        | NA         | NA    | 20.82     | 29.50      | 0.73  | 21.24     | 28.55      | 0.69  |
|              | TV                   | 21.95     | 29.32      | 0.70  | 21.34     | 29.58      | 0.68  | 21.83     | 28.71      | 0.72  |
|              | DL                   | 23.14     | 30.02      | 0.77  | 21.89     | 29.61      | 0.71  | 20.35     | 27.24      | 0.71  |
|              | Proposed             | 23.01     | 29.89      | 0.78  | 22.86     | 29.95      | 0.77  | 23.21     | 30.10      | 0.78  |
| Still Image     | Wavelet hard       | 16.89     | 23.75      | 0.56  | 16.99     | 23.84      | 0.57  | 16.82     | 23.17      | 0.37  |
|              | Wavelet soft        | 17.24     | 23.64      | 0.67  | 17.27     | 23.63      | 0.68  | 17.05     | 23.04      | 0.67  |
|              | VST                  | NA        | NA         | NA    | 19.37     | 25.36      | 0.76  | 19.01     | 25.01      | 0.76  |
|              | TV                   | 18.75     | 24.74      | 0.75  | 18.70     | 24.71      | 0.75  | 18.35     | 24.34      | 0.74  |
|              | DL                   | 20.96     | 26.97      | 0.78  | 19.87     | 27.16      | 0.71  | 19.09     | 25.08      | 0.68  |
|              | Proposed             | 21.39     | 28.07      | 0.69  | 21.93     | 28.31      | 0.79  | 21.83     | 28.29      | 0.82  |
| Moon         | Wavelet hard       | 20.08     | 28.89      | 0.41  | 19.45     | 28.48      | 0.43  | 18.89     | 27.03      | 0.42  |
|              | Wavelet soft        | 20.79     | 29.44      | 0.45  | 21.14     | 29.51      | 0.77  | 19.47     | 27.36      | 0.75  |
|              | VST                  | NA        | NA         | NA    | 22.08     | 30.17      | 0.85  | 20.90     | 28.79      | 0.84  |
|              | TV                   | 21.35     | 30.22      | 0.50  | 23.51     | 31.41      | 0.86  | 21.29     | 29.18      | 0.80  |
|              | DL                   | 22.48     | 30.84      | 0.55  | 22.29     | 30.40      | 0.85  | 19.45     | 27.59      | 0.74  |
|              | Proposed             | 24.81     | 33.11      | 0.73  | 24.65     | 33.34      | 0.86  | 23.48     | 31.55      | 0.89  |
with the objective of analyzing the behavior of the hyperparameters for different signal configurations. Three different types of noise were considered: Poisson, additive Gaussian and multiplicative speckle noise. In all cases, the level of noise was adjusted to correspond to a SNR of 15 dB.

Fig. 10 shows the quality of the denoising results for the initial synthetic signal, in terms of SNR, versus the value of the hyperparameters $h^2/2m$ and $\sigma^2$ for different types of noise: Poisson noise, Gaussian noise and speckle noise. Several observations can be made from these results. As expected, an optimal value arises in each case. However, a small variation in the choice of the hyperparameters around this optimal values only slightly influences the quality of the denoising. Moreover, the optimal values are only slightly dependent on the nature of the noise. This means that for this type of signal the hyperparameters could be fixed beforehand at a fixed value which can be chosen independently of the type of noise present.

Next, the dependence of $h^2/2m$ and $\sigma$ hyperparameters on the shape of the signals is analyzed. For this purpose, three different synthetic signals were generated, as shown in Fig. 11(a)(d)(g), further normalized to 1 and corrupted by Poisson noise. From the results in Fig. 11(b)-c(e)-f(b-i), it can be clearly observed that the quality of the denoising does depend on the shape of the signals, which can be expected given the nature of the adaptative basis used by the proposed approach. However, the denoising process is efficient for a fairly large interval around the optimal values. As there is a big overlap in the acceptable range of values of the hyperparameters for various signal shape, again this means that the hyperparameters could be fixed beforehand at a fixed value which can be chosen independently of the type of noise present.

C. Efficiency of the denoising process

This section presents denoising results on a synthetic signal, a synthetic image and four standard testing images of size $512 \times 512$ and $320 \times 320$ pixels, shown in Fig. 13.

Denoising is an extensively explored research field that prevents an exhaustive comparison of the proposed approach to all the existing methods. Moreover, we remind that the most important contribution herein is to investigate a novel way of decomposing signals or images, which is not meant to outperform all the denoising algorithms in any scenario. Four algorithms from the literature were used for comparison purpose: i) wavelet denoising based on hard and soft thresholding, (d) wavelet soft thresholding, (e) total variation regularization, (f) dictionary learning and (g) proposed method. The proposed adaptive transform was computed with the hyperparameter $h^2/2m = 0.23$, $\sigma^2 = 7.5$ and the hyperparameters $s$ and $\rho$ have been manually tuned up to their best possible values.
thresholding of detail coefficients [1], [2], ii) the variance stabilization transform (VST) relevant for data dependent noise models [15], iii) an optimization-based approach using the total variation (TV) semi-norm to regularize the solution [16], [17], and iv) a dictionary learning (DL) method exploiting sparse and redundant representations over learned patch-based dictionaries [18]. Note that for all the methods and for all the simulation scenarios, their hyperparameters were manually
tuned to obtain optimal denoising results in the sense of the quantitative measurements employed. We used the Matlab implementations available in the Numerical tours website [19].

Three quantitative measurements were used to evaluate the denoised images: the signal to noise ratio (SNR), the peak signal to noise ratios (PSNR) and the structure similarity (SSIM) [20]. All the quantitative results are regrouped in Table I where the best and the second best values have been highlighted by red and blue colors respectively for each dataset. Note that VST is only used for data-dependent noise and VST and DL were only tested for images, as initially suggested by the seminal papers. Illustrative results for Still image (Fig. 13 (e)) corrupted by Gaussian noise, Moon image (Fig. 13(f)) with Poisson noise and Lena image (Fig. 13 (d)) with speckle noise are shown respectively in Fig. 14, 15 and 16. All these results allow us to draw some conclusions. First, one may remark that in almost all the cases, regardless of the noise nature and the image, the proposed method is one of the two best ones. This proves its adaptability to different scenarios which can be considered a strong point in number of practical applications. Second, we may remark that for the synthetic signal and image, our method outperforms all the others. The main reason is that the synthetic signal and image were generated to promote the main characteristic of the proposed decomposition, that keeps preferentially higher frequencies for low gray levels and lower frequencies for high gray levels. For such images or signals, the proposed method is very efficient. On the contrary, TV and DL, for example, fail in these cases because of the non piece-wise constant nature of the synthetic data. Finally, we remark that the proposed denoising algorithm provides competitive results compared to DL that learns the redundant dictionary from a database of clean images. Of course the proposed method does not need such a database.

D. Application to CBCT dental image denoising

This section illustrates the ability of the proposed method to denoise real medical images. In particular, the application considered in this work for illustration purpose is CBCT dental imaging. CBCT is a medical imaging modality that allows tooth visualization with low radiation doses, and is thus suitable for dental applications. However, the low radiation prevents the current scanners to provide images with high SNR. In [21], the quality of CBCT dental image within phantom and in vivo data were evaluated. Fig. 17 shows a noisy image resulting from that study, as well as the denoised images with the proposed approach. The region of interest in this image is the dark region in the middle of the tooth, that represents the canal root. The results displayed show that the method has some practical applications in this field.

IV. Conclusions

We investigated in this paper an original approach of constructing an adaptive transform in the context of signal and image processing based on the resolution of a quantum mechanical problem. More precisely, the signal or image is used as the potential in a quantum problem, the resolution of which gives as eigenvectors the proposed adaptive basis. The basis vectors automatically use a different range of frequencies to explore low potential valued regions compared to the regions corresponding to the high potential values. Therefore, thresholding the coefficients of the signal or image expanded in this basis will process differently high and low values of the signal or image. This framework has been illustrated through denoising applications on different signals and images in presence of Gaussian, Poisson and speckle noise. We have performed a detailed investigation of the impact of the hyperparameters. We have also presented a quantitative comparison of the denoising efficiency of the proposed adaptive method compared to state-of-the-art methods on synthetic signals and standard images. The results of our investigation show that our method has interesting potential to denoise signals and images, especially for Poisson and speckle noise to which it is well adapted; indeed, as a vector in the adaptative basis naturally uses higher frequencies for low values of the signal compared to low values, the thresholding process keeps more frequencies for low values than for high values. Our results show that our denoising procedure outperforms standard methods in specific cases, and ranks among the best methods in most cases. Our study of the hyperparameters shows that they cannot be chosen at random, but that the range of optimality is large enough to allow to set them beforehand independently of the signal or image, although the choice may be modified according to the type of noise present in the application.

The computational time of the eigenvectors of the Hamiltonian operator is the major drawback of this method, which

Fig. 17. (a) Clean CBCT dental image, (b) Noisy CBCT dental image, (c) CBCT dental image after denoising considering the hyperparameter $h^2/2m = 0.5$, $\sigma^2 = 20$, $\rho = 1$ and the hyperparameter $s$ has been manually tuned up to their best possible values.
can be tackled by more refined algorithms. It should be also noted that in many applications the computational efficiency of the algorithm, while important, is less crucial than the efficiency to denoise the signal or image considered. As a future perspective of this study, it would be very interesting to extend this framework to other reconstruction applications available in the literature, such as deconvolution, super-resolution or compressed sensing.

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APPENDIX A

EXPLICIT FORM OF HAMILTONIAN MATRIX

Following equation (1),
\[ H = -\frac{\hbar^2}{2m} \nabla^2 + x. \]

Note that:

* The potential \( V \) is represented by \( x \) (the signal or image).
* If \( x \) is a signal of size \( N \), then the size of \( H \) is \( N \times N \).
* If \( x \) is an image of size \( N \times N \), it is transformed into a vector (in the lexicographical order) of size \( N^2 \) and \( H \) is a \( N^2 \times N^2 \) matrix.
* In both cases (\( x \) is a signal or an image), \( x \) is considered in a vector form.

For a 1D signal, on obtain:

Numerical derivatives of \( \psi \): \( (D\psi)_i = \psi(i + 1) - \psi(i) \)

Numerical Laplacian of \( \psi \): \( (\nabla^2\psi)_i = \psi(i + 1) - 2\psi(i) + \psi(i - 1) \)

Thus, \( (H\psi)_i = -\frac{\hbar^2}{2m}(\psi(i+1) - 2\psi(i) + \psi(i-1)) + x(i)\psi(i) \)

\( (H\psi)_i = \left( x(i) + 2\frac{\hbar^2}{2m}\right)\psi(i) - \frac{\hbar^2}{2m}(\psi(i+1) - \frac{\hbar^2}{2m}\psi(i-1)) \)

\( i = 1, 2, 3, ..., N \)

where

\[ H(i, j) = \begin{cases} 
  x(i) + 2\frac{\hbar^2}{2m}, & \text{for } j = i, \\
  -\frac{\hbar^2}{2m}, & \text{for } j = i \pm 1, \\
  0, & \text{otherwise.}
\end{cases} \]

For a 2D image \( \mathbf{\psi} \in \mathbb{R}^{N \times N} \) the methodology is similar. A discretized Hamiltonian matrix \( H \in \mathbb{R}^{N^2 \times N^2} \) is constructed using (2), where zero padding is used to handle the boundary effects. Precisely if \( N = 4 \), \textit{i.e.} for an image of size \( 4 \times 4 \) the discretized Hamiltonian becomes of size \( 16 \times 16 \). The first and last six columns and rows of the Hamiltonian matrix are explicitly given in Tables II and III accordingly.

### TABLE II

| 1     | 2        | 3     | 4     | 5     | 6     |
|-------|----------|-------|-------|-------|-------|
| \( x(1, 1) + 2\frac{\hbar^2}{2m} \) | -\frac{\hbar^2}{2m} | 0     | 0     | -\frac{\hbar^2}{2m} | 0     |
| \( \frac{\hbar^2}{2m} \) | \( x(2, 2) + 3\frac{\hbar^2}{2m} \) | -\frac{\hbar^2}{2m} | 0     | -\frac{\hbar^2}{2m} | 0     |
| 0     | -\frac{\hbar^2}{2m} | \( x(3, 3) + 3\frac{\hbar^2}{2m} \) | -\frac{\hbar^2}{2m} | 0     | 0     |
| 0     | 0        | -\frac{\hbar^2}{2m} | \( x(4, 4) + 2\frac{\hbar^2}{2m} \) | 0     | 0     |
| \( \frac{\hbar^2}{2m} \) | 0        | 0     | 0     | \( x(5, 5) + 3\frac{\hbar^2}{2m} \) | -\frac{\hbar^2}{2m} |
| 0     | \( \frac{\hbar^2}{2m} \) | 0     | 0     | -\frac{\hbar^2}{2m} | \( x(6, 6) + 4\frac{\hbar^2}{2m} \) |

### TABLE III

| 11    | 12        | 13     | 14     | 15     | 16     |
|-------|-----------|--------|--------|--------|--------|
| \( x(11, 11) + 4\frac{\hbar^2}{2m} \) | -\frac{\hbar^2}{2m} | 0     | 0     | -\frac{\hbar^2}{2m} | 0     |
| \( \frac{\hbar^2}{2m} \) | \( x(12, 12) + 3\frac{\hbar^2}{2m} \) | 0     | 0     | 0     | -\frac{\hbar^2}{2m} |
| 0     | 0         | \( x(13, 13) + 2\frac{\hbar^2}{2m} \) | -\frac{\hbar^2}{2m} | 0     | 0     |
| \( \frac{\hbar^2}{2m} \) | 0         | 0     | \( x(14, 14) + 3\frac{\hbar^2}{2m} \) | -\frac{\hbar^2}{2m} | 0     |
| \( \frac{\hbar^2}{2m} \) | 0         | 0     | -\frac{\hbar^2}{2m} | \( x(15, 15) + 3\frac{\hbar^2}{2m} \) | -\frac{\hbar^2}{2m} |
| 0     | \( \frac{\hbar^2}{2m} \) | 0     | 0     | -\frac{\hbar^2}{2m} | \( x(16, 16) + 2\frac{\hbar^2}{2m} \) |
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