RECONSTRUCTING METRIC TREES FROM ORDER INFORMATION ON TRIPLES IS NP COMPLETE

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Abstract. We show that reconstructing a tree from order information on triples is NP-hard. This is in contrast to the case for ultra-metrics and for subtree information on quadruples which are both known to allow polynomial time reconstruction.

1. Introduction

This paper deals with the computational complexity of finding a tree compatible with known combinatorial data. This type of problem arises among other places in trying to reconstruct a phylogenetic tree from partial information. We might for example know about all triples of species which two are most closely related and which two least. Equivalently we might know for each pair of species which of the others are more closely related to the first and which to the second. To get decision problems these types of data are abstracted to triples structures and midpoints structures respectively. We then address two problems. We show that it is computationally difficult (NP-complete and NP-hard respectively) to either determine whether the known data is compatible with any tree structure or given that it is compatible with some tree structure to find such a tree. A computationally equivalent formulation of a tree structure is the information about each quadruple of species of the tree structure underlying just these four.

Triples and midpoints structures are computationally closely related: There is a polynomial time bijection between them which preserves the set of compatible tree geometries. Thus we will focus entirely on midpoints structures but get the same results for both. There is a polynomial time algorithm to test whether a given tree geometry is compatible with a given midpoints structure so that determining whether there exists a tree compatible with a given structure is in NP.

Some related questions have been previously studied. It is known that if the metric tree is required to have some point in the tree equidistant from every element (an ultra-metric) the computational problems can both be solved in polynomial time. A less natural restriction than an ultra-metric is that every edge of the tree into which the set is embedded contain the midpoint between some pair of elements. Call this tree geometry the midpoints tree. This idea arises from the fact that any edge containing the midpoint between two of the elements can be quickly identified from the above types of data and hence so can the midpoints tree. There is a metric on the midpoints tree studied in where it is referred to as the order distance. Unfortunately even in cases where the given midpoints structure has a realization with the midpoints tree geometry the order distance need not be such a realization since it need not be compatible with the original
midpoints structure. In the ultra-metric situation the midpoints tree is the unique minimal
tree geometry realizing any given midpoints structure which has an ultra-metric realization.
Warnow has asked whether the midpoints tree is the unique minimal tree geometry realizing
any given realizable midpoints structure in general; see also Theorem 2 of [1]. The examples
in this paper provide counterexamples for the above.

In section 2 we introduce notation. Section 3 states the results. In section 4 we give the
main construction: a standard NP-complete problem (3-SAT) is encoded by a midpoints
structure. In sections 5 and 6 we show the equivalence of the satisfiability of any case of
3-SAT with the realizability of the midpoints structure which encodes that case.

2. Notation

For notational convenience only generic structures (for which none of the distances are
equal) will be considered. The results still hold for arbitrary data since every realizable
triples or midpoints structure without ties can be realized by a tree without ties.

It will be convenient throughout to fix a total order on the elements of the finite sets.

**Definition 2.1.** A **triples structure** on an ordered finite set $X$ is a set $\{<_x\}_{x \in X}$ of
relations on $X$ such that for all $x, y, z \in X$ with $x \neq y$ we have $x \not<_z x, x <_x y$ and either
$y <_z x$ or $x <_z y$ but not both.

A metric on $X$ will be a realization of a triples structure $\{<_x\}_{x \in X}$ if the distance from $x$
to $z$ is less than the distance from $y$ to $z$ whenever $x <_z y$.

Write $\binom{X}{2}$ for $\{A \subseteq X| |A| = 2\}.$

**Definition 2.2.** A **midpoints structure** on an ordered set $X$ is a map $m : \binom{X}{2} \to 2^X$
with $\{\max\{x, y\}\} = \{x, y\} \cap m\{x, y\}$ for all $x \neq y \in X$.

A metric on $X$ will be a realization of a midpoints structure $m$ if the distance from $x$
to $z$ is less than the distance from $y$ to $z$ whenever $y < x$ and $z \in m\{x, y\}$ or $x < y$ and
$z \not\in m\{x, y\}$.

There are polynomial time algorithms to translate between these two types of data:

To get a midpoints structure $m$ from a triples structure $\{<_x\}_{x \in X}$: For every $x < y$ set
$m\{x, y\} = \{z \in X|y <_z x\}$.

To get a triples structure $\{<_x\}_{x \in X}$ from a midpoints structure $m$: For every $y < x \in X$
and $z \in X$ if $z \in m\{x, y\}$ set $x <_z y$ and otherwise set $y <_z x$.

**Lemma 2.3.** These algorithms are inverses to each other and a metric is a realization of a
given midpoints structure if and only if it is a realization of the associated triples structure.

If $S \subseteq X$ write $S^1 = S$ and $S^c = X\backslash S$. Think of $S \subseteq X$ as an edge of a tree with leaves
labeled by $X$ for which the edge splits $X$ into the sets $S^1$ and $S^c$.

**Definition 2.4.** If $S, T \subseteq X$ define $[S, T] = \{U \subseteq X\}$ for some $e, f, g \in \{c, 1\}, S^c \subseteq U^g \subseteq
T^f$, and define $[S, T] = [S, T] \backslash \{T\}$. If $U \in [S, T]$ then write $S : U : T$.

Thus $[S, T]$ is all edges which appear on the path between the edges $S$ and $T$ in some
tree. If $t := \sum_{e \in E} t(e)x_e \in \mathbb{R}^E$ and $W \subseteq E$ then write $t_W := \|W^t\|_1 = \Sigma_{e \in W} t(e)$. A
metric tree will be represented as an element $t \in \mathbb{R}_+^{2^X}$ with $t(S) = t(S^c)$ being the
length of the edge splitting the leaves into $S^1$ and $S^c$. Write $R_1 : R_2 : \ldots : R_r$ if $R_i : R_j : R_k$ for every $1 \leq i < j < k \leq r$. Note that if any two of $R_1 : R_2 : R_3$, $R_1 : R_2 : R_4$, $R_1 : R_3 : R_4$ and $R_2 : R_3 : R_4$ hold then all four hold.

A structure will be called realizable if there is a tree metric which is a realization of it. More precisely,

**Definition 2.5.** A midpoints structure $m$ on $X$ is called **realizable with realization** $t \in \mathbb{R}^{2^X}$ and **tree structure** $\{S \subseteq X | t(S) > 0\}$ if

1. if $S \subseteq X$ then $t(S) = t(S^c) \geq 0$,
2. if $S, T \subseteq X$ with $S^c \cap T^f \neq \emptyset$ for every $e, f \in \{1, c\}$ then $t(S)t(T) = 0$ and
3. if $x \neq x' \in X$ then $t_{\{(x), m(x,x')\}} > t_{\{(x'), m(x,x')\}}$.

The first condition ensures that the length $t(S)$ of the edge $S$ is the same as that of $S^c$ (which is the same edge) and is nonnegative. The second condition ensures that the nonzero edges form a tree. The third condition ensures that the edge $m(x,x')$ contains in its interior the midpoint of the path from $x$ to $x'$ and hence that if $x < x'$ then $z$ is closer to $x'$ than it is to $x$ if and only if $z \in m(x,x')$.

3. **Theorem**

**Theorem 3.1.** The question: Is a given midpoints structure (or triples structure) realizable? is NP complete.

**Proof.** We will encode 3-SAT and then apply 5.1 and 6.1.

**Corollary 3.2.** Determining a compatible metric tree structure given a realizable midpoints structure (or triples structure) is NP hard.

4. **Encoding**

We will encode a case of 3-satisfiability in conjunctive normal form with $V$ variables and $C$ length 3 or clauses. The function $I$ names the variables appearing in a particular clause, while $\sigma$ indicates whether each variable appears with a not. We assume that all three variables appearing in any given clause are distinct.

If $a \leq b \in \mathbb{Z}$ write $[a, b]$ for $\{a, a+1, \ldots, b\}$.

**Definition 4.1.** A case of 3-SAT is a quadruple $(V, C, \nu, \sigma)$ with $V, C \in \mathbb{N}$, $\nu : [1, C] \times [0, 2] \rightarrow [1, V]$ and $\sigma : [1, C] \times [0, 2] \rightarrow \{-1, +1\}$ with $\nu(c,0) < \nu(c,1) < \nu(c,2)$ for every $c \in [1, C]$. The case $P = (V, C, \nu, \sigma)$ is said to be **satisfiable by** $h$ if $h : [1, V] \rightarrow \{-1, +1\}$ and for each $c \in [1, C]$ there is some $a \in [0, 2]$ with $h(\nu(c,a)) = \sigma(c,a)$.

**Example 4.2.** The case $(x \lor \overline{y} \lor z) \land (w \lor x \lor y)$ is encoded with $V = 4$, $C = 2$, $\nu(1,1) = 2$, $\sigma(1,1) = 1$, ...

$$\{\nu(c,a)\} = \begin{pmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \end{pmatrix} \text{ and } \{\sigma(c,a)\} = \begin{pmatrix} +1 & -1 & +1 \\ +1 & +1 & +1 \end{pmatrix}.$$ and is satisfiable by 11 choices of signs one of which is $h(1) = -h(2) = -h(3) = h(4) = 1.$
The idea will be to construct a single midpoints structure for each case of 3-SAT. This will be accomplished by using two constructions and combining them (one copy of the first and many of the second) using the following definition.

**Definition 4.3.** Combining Midpoints Structures:
If \( m : \left( \frac{V}{2} \right) \to 2^X \) and \( n : \left( \frac{V}{2} \right) \to 2^Y \) are midpoints structures on ordered sets \( X \) and \( Y \) respectively and \( f : Y \to X \), then \( m \cup f n \) is a midpoints structure on \( X \cup Y \) (with \( X \) and \( Y \) ordered as before and \( x < y \) for every \( x \in X \) and \( y \in Y \)) defined by

\[
\begin{align*}
(m \cup f n)\{x, x'\} &= m\{x, x'\} \cup f^{-1}m\{x, x'\} & \text{if } x, x' \in X \\
(m \cup f n)\{y\} &= \{y\} & \text{if } x \in X, y \in Y \\
(m \cup f n)\{y, y'\} &= n\{y, y'\} & \text{if } y, y' \in Y
\end{align*}
\]

If \( m \cup f n \) is realizable any realization restricts to realizations of \( m \) and \( n \) but if \( m \) and \( n \) are both realizable \( m \cup f n \) might or might not be. The idea for realizing \( m \cup f n \) when possible is to choose the distances for \( m \) to be much smaller than those for \( n \) and put the tree for \( m \) into the middle of the tree for \( n \) with leaves \( Y \) attached to leaves \( X \) according to the map \( f \).

**Example 4.4.** If \( X = \{a, b, c\} \), \( Y = \{A, B, C, D\} \), \( f(A) = a \), \( f(B) = b \), \( f(C) = c \), \( f(D) = c \) and the first two figures represent tree metrics realizing \( m \) and \( n \) respectively then the third figure represents a tree metric realizing \( m \cup f n \):

For the remainder of this paper fix a case \( P = (V, C, \nu, \sigma) \) of 3-SAT. We will construct in several steps a midpoints structure \( m_P \) associated to \( P \). The construction will be polynomial size and require polynomial time in the size of \( P \) and \( m_P \) will be realizable iff \( P \) is satisfiable.

4.5. **Variable Structure.** We start with a midpoints structure \( m_0 \) with two elements for each variable and four extra end elements. For each clause a copy of a second midpoints structure \( m \) will be attached to \( m_0 \) using definition 4.3.

The set of elements for \( m_0 \) is \( X_0 = \{x_{v,s}\} | v \in [0, V + 1], s \in \{-1, 1\} \) making a total of \( 2V + 4 \) elements. Order these elements with \( x_{v',s'} < x_{v,s} \) if \( v' < v \) or if \( v' = v \) and \( s' < s \). If \( x_{v',s'} < x_{v,s} \in X_0 \) then set

\[
m_0\{x_{v',s'}, x_{v,s}\} := \begin{cases} \{x_{u,r} \in X_0 | u \geq v, r \in \{1, -1\}\} & \text{if } 2v' + s' = 2v + s \text{ and } v' < v \\
\{x_{v,s}\} & \text{if } 2v' + s' < 2v + s
\end{cases}
\]


Note that \( m_0 \) is realizable with its midpoints tree geometry. For instance the vector \( t \in \mathbb{R}^{2X_0} \) with \( t_0(x_u,v) = 1 \) and \( t(S) = 10^{v+\frac{1}{2}} \) except that \( t_0(x_0,-1) = 0 \) and with \( t(S) = 0 \) for all other \( S \subseteq X_0 \). is a realization of \( m_0 \).

4.6. Example: For example 4.2 above \( X_0 = \{ x_0, -1 \} \) as an ordered set, \( m_0 \{ x_0, -1 \} = \{ x_2, -1 \} \), \( m_0 \{ x_3, -1 \} = \{ x_4, -1 \} \), ... and the figure represents the realization \( t \) given above.

4.7. Encoding Signs: By lemma 5.4 below every realization of \( m_0 \) must have every edge in the image of \( m_0 \) with a positive length and hence every realization must have a contraction to the midpoints tree. Thus every realization involves pulling apart some (or none) of the \( V + 2 \) degenerate vertices in this tree. For each degree four vertex there are 4 ways to do this.

The idea of the construction will be to encode a choice of \( h : [1, V] \rightarrow \{ 1, -1 \} \) by the choice of this splitting with the second configuration corresponding to \( h(v) = 1 \) and the third corresponding to \( h(v) = -1 \). The other two configurations will correspond to \( h(v) \) being undetermined. We will add on a copy of the midpoints structure \( m \) given below for each clause so that a splitting of the branches arises in a realization of the final midpoints structure if and only if the associated choice of \( h : [1, V] \rightarrow \{ 1, -1 \} \) satisfies the given case of 3-SAT. If a value \( h(v) \) is undetermined it must satisfy the case for both values.

4.8. Clause Structures. In this section we give the main construction. This is a midpoints structure \( m \) with 48 elements which has the property that after being added to \( m_0 \) there are at least three minimal sets of edges which can appear as a tree structure realizing the result. This means that there is no realization with the midpoints tree geometry since by lemma 5.4 this geometry is a contraction of every geometry realizing a midpoints structure. This makes the sum of \( m \) and \( m_0 \) a counterexample to Warnow’s question. In this case there are three other edges so that every tree structure containing one of these three along with the
edges containing midpoints is a realization of \( m \). We will use a copy of \( m \) for each clause \( c \) of the case \( P \) of 3-SAT and denote it by \( m_c \). These will be combined with \( m_0 \) from above using definition 4.3 to get the final midpoints structure \( m_P \). The clause is encoded by the choice of map with which to combine \( m_c = m \) with \( m_0 \). The three special edges mentioned above will be positioned by the combining map so as to correspond to the three variable sign choices 4.7 which will satisfy the \( c \)th triple term or clause. The midpoints structure \( m \) splits into 4 isomorphic substructures each with 12 elements (indexed by \( p \) below) with \( m \) applied to two elements with different \( p \) indices being simply the singleton with larger \( p \) so that in any realization the different 12 element subtrees will have successively much longer leaf lengths. Each 12 element midpoints substructure is simply a total order with a realization by its midpoints geometry which is simply a star with the ordering on leaf lengths given by the order below 4.9. The other 12 element substructures differ only by a relabeling. The arrangement of the elements of these 12 element substructures given by the combining maps to the underlying variable midpoints structure \( m_0 \) is shown in the figure after example 4.11. The subtlety in \( m \) arises from the fact that some of the midpoints between two elements of a 12 element substructure are not singletons but rather doubletons involving an element from an adjacent substructure. These give 16 more edges than in the star in the midpoints geometry of \( m \) so that the midpoints geometry has one degree 32 vertex adjacent to 16 leaves and 16 degree 3 vertices each adjacent to two more leaves. To get a counterexample to theorem 2 of [4] choose \( P \) to be any non-satisfiable case of 3-SAT and consider \( m_P \).

The midpoints structure \( m \) is the central construction of this paper and is given explicitly below.

**Definition 4.9.** \( Y = [0, 3] \times [0, 3] \times [-1, 1] \). Define an involution \( \mu \) on \( Y \) by \( \mu(p, q, e) = ((p+e), (q+e), -e) \) where \( 0 \leq (r)_4 \leq 3 \) is the reduction of \( r \) modulo 4. Every midpoint will be either a singleton or a doubleton \( \{y, \mu y\} \). Consider the total ordering on \([0, 3] \times [-1, 1]\) with \((0, 0) < (1, 0) < (0, 1) < (1, -1) < (2, 0) < (1, 1) < (2, -1) < (3, 0) < (3, 1) < (0, -1) < (2, 1) < (3, -1)\) Totally order \( Y \) by setting \((p', q', e') < (p, q, e)\) if \( p' < p \) or if \( p' = p \) and \((q', e') < (q, e)\) above. Define \( m \) so that for every \((p', q', e') < (p, q, e) \in Y\) we have \( m\{(p', q', e'), (p, q, e)\} = \{(p, q, e)\}\) if \( p' < p \) or if \( [p' = p \text{ and } (q', e') < (q, 0)] \) above or if \( [p' = p, q' = q, e' = 0 \text{ and } e = 1] \) and otherwise \( m\{(p', q', e'), (p, q, e)\} = \{(p, q, e), \mu(p, q, e)\}\).

Recall that \( P = (V, C, \nu, \sigma) \) is an instance of 3-SAT. For each \( c \in [1, C] \) define \( m_c \) to be the midpoints structure on \( X_c = \{x_{c,p,q,e} | p, q \in [0, 3], e \in [-1, 1]\} \) isomorphic to \( m \) above. Extend the total order and the involution \( \mu \) from \( Y \) to \( X_c \). (Explicitly take \( \phi : X_c \to Y \) to be \( \phi x_{c,p,q,e} = (p, q, e) \) and set \( m_c\{x, x'\} = \phi^{-1} m\{\phi x, \phi x'\}, x < x' \text{ if } \phi x < \phi x' \) and \( \mu x = \phi^{-1} \mu \phi x \) for every \( x, x' \in X_c \).) The following maps will be used to combine the \( m_c \) with \( m_0 \) using 4.5. Define \( X_{\leq 0} := X_0 \) and \( X_{\leq c} := X_{\leq (c-1)} \cup X_c \) for every \( c \in [1, C] \) and
$X := X_P := X_{\leq C}$. Define $f : X \to X_0$ by setting $f|_{X_0}$ to be the identity and

\[
\begin{align*}
    f_{x_{c,p},0,0} & := x_{\nu(c,0),-\sigma(c,0)} \\
    f_{x_{c,p},1,0} & := x_{\nu(c,1),-\sigma(c,1)} \\
    f_{x_{c,p},2,0} & := x_{\nu(c,2),-\sigma(c,2)} \\
    f_{x_{c,p},3,0} & := x_{0,1} \\
    f_{x_{c,p},0,-1} & := f_{x_{c,p},3,1} := x_{\nu(c,0),\sigma(c,0)} \\
    f_{x_{c,p},1,-1} & := f_{x_{c,p},0,1} := x_{\nu(c,1),\sigma(c,1)} \\
    f_{x_{c,p},2,-1} & := f_{x_{c,p},1,1} := x_{\nu(c,2),\sigma(c,2)} \\
    f_{x_{c,p},3,-1} & := f_{x_{c,p},2,1} := x_{V+1,1}
\end{align*}
\]

Note that $f$ is invariant under the involution $\mu : X \to X$. We combine sequentially to get the final construction of the midpoints structure $m_P$.

**Definition 4.10.** If $P = (V, C, \nu, \sigma)$ is a case of 3-SAT then take $m_{\leq 0} := m_0$ from above and for every $c \in [1, C]$ take $m_c$ and $f$ from above and define $m_{\leq c} := m_{\leq (c-1)} \cup f|_{X_c} m_c$ a midpoints structure on $X_{\leq c}$. Finally take $m_P := m_{\leq C}$ the midpoints structure on $X$.

**Example 4.11.** For the example 4.2 above we have $C = 2$, $X_1 = \{x_{1,0,0,-1}, x_{1,0,0,0}, \ldots, x_{1,3,3,1}\}$ and $X = X_0 \cup X_1 \cup X_2$ with 108 elements. The map $f : X \to X_0$ has for instance $f^{-1}(x_{0,1}) = \{x_{0,1}, x_{1,0,3,0}, x_{1,1,3,0}, x_{1,2,3,0}, x_{1,3,3,0}, x_{2,0,3,0}, x_{2,1,3,0}, x_{2,2,3,0}, x_{2,3,3,0}\}$, $f^{-1}(x_{1,1}) = \{x_{1,1}, x_{2,0,0,-1}, x_{2,0,3,1}, x_{2,1,0,-1}, x_{2,1,3,1}, x_{2,2,0,-1}, x_{2,2,3,1}, x_{2,3,0,-1}, x_{2,3,3,1}\}$, and $f^{-1}(x_{1,-1}) = \{x_{1,-1}, x_{2,0,0,0}, x_{2,1,0,0}, x_{2,2,0,0}, x_{2,3,0,0}\}$. Below is part of a realization of $m_P$ based on the solution $h$ from example 4.2. Only 26 of the 108 elements are shown (the 12 in $X_0$, 13 of those from $X_1$ and one from $X_2$). The lengths given are those of $t_h$ given in 6.1. The adjustment $t'$ to $t$ given in 6.1 is the difference between the labeled leaf lengths and the nearest multiple of $10^{12}$ (the position in the numbers below marked with a semicolon); $t'$ does not adjust the interior edges.
5. Realizable implies satisfiable

**Proposition 5.1.** If $P$ is a case of 3-SAT and $m_P$ is realizable then $P$ is satisfiable.

Recall definition 2.4

**Lemma 5.2.** If $t$ is a realization of some midpoints structure on $X$ and $S, T, U \subseteq X$ with $t(T)t(S)t(U) \neq 0$ and $T \in [S, U]$ (that is $S : T : U$ then $t(S, U) = t(S, T) + 2t(T) + t(T, U)$.

Proof. This follows geometrically from the fact that $t(S, U)$ is twice the distance in the tree between the ends of the edges $S$ and $U$ which are closest to each other, and $T \in [S, U]$ with $t(T) > 0$ if $T$ is an edge on the unique geodesic in the tree $t$ between $S$ and $U$. □

**Lemma 5.3.** If $t$ is a realization of some midpoints structure on $X$ and $U_0 \cup U_1 \cup U_2 \cup U_3 = X$ is a partition of $X$ with $t(U_j) > 0$ for every $i$ then $t(U_j \cup U_3) > 0$ for at most one $j \in [0, 2]$. Further, $t(U_j \cup U_k) = 2t(U_j \cup U_k)$ so the sum $\tau = \Sigma_{j \in [0, 3]}(-1)^j t(U_j, U_{(j+1)}) = 4t(U_0 \cup U_3) - 4t(U_2 \cup U_3)$ and hence is positive if $t(U_0 \cup U_3) > 0$ (so $j = 0$) and negative if $t(U_2 \cup U_3) > 0$ (so $j = 2$). If $j = 1$ this sum is 0.

Proof. For all $e, f \in \{1, c\}$ we have $(U_j \cup U_3)^e \cap (U_k \cup U_3)^f \in \{U_i\}_{i \in [0, 3]}$ and since each $U_i$ is nonempty we get that $t(U_j \cup U_3)t(U_k \cup U_3) = 0$ by definition 2.3. For the second statement if $T \in (U_j, U_k)$ with $t(T) > 0$ then using definition 2.4 and replacing $T$ with $T^c$ if necessary we get $U_j \subseteq T \subseteq U_k^c$. Since we are taking the open interval $T \neq U_j$ and $T \neq U_k^c$. Choose $U_i \notin \{U_j, U_k\}$ with $T \cap U_i \neq \emptyset$. By definition 2.5 there is some choice of signs $e, f \in \{1, c\}$ with $T^c \cap U_i^f = \emptyset$ so $U_i \subseteq T$ and hence $T = U_i \cup U_j$. □

**Lemma 5.4.** For any realization $t$ of a midpoints structure $m$ on $X$ and any $x \neq y \in X$ we have $t(m\{x, y\}) > 0$, so the edge containing any midpoint must have positive length and hence every tree geometry for which there is a realization of $m$ contains the midpoints geometry as a contraction.

Proof. By (3) in the definition of realization 2.5 we have $t_{[x, m\{x, y\}]} > t_{[y, m\{x, y\}]}$ and similarly, switching the roles of $x$ and $y$ we get $t_{[y, m\{x, y\}]} > t_{[x, m\{x, y\}]}$. Adding these and canceling gives $4t(m\{x, y\}) > 0$. □
Proof of Proposition: First we will define a choice of signs \( h_t : [V] \to \{1, -1\} \) given a realization \( t \) of \( m_P \). We will then check that \( h_t \) satisfies \( P \).

Assume that \( t \in \mathbb{R}^{2X} \) is a realization of \( m_P \). For convenience we give new names to some of the elements \( A \subseteq X \) in the image of \( m_P \) and their complements. Write

\[
\begin{align*}
\{c,p,q,-\}^\mu := & \{x_{c,p,q,e}\}^\mu := m_P \{x_{c,p,q,e}, x_{v,s}\}, \\
A_{v,s} := & f^{-1}\{x_{v,s}\}, \\
A_{>v} := & (A_{<v-1})^c := f^{-1}\{x_{v',s}' > v, s \in \{1, -1\}\} = m_P \{x_{v,1}, x_{v+1,-1}\}.
\end{align*}
\]

For every \( v \) and \( s \) the set \( \{A_{v,s}, A_{v,-s}, A_{>v}, A_{<v}\} \) partitions \( X \).

Now by lemma 5.3 we get that for every midpoint above \( t(m_P\{x,y\}) > 0 \). Thus we can apply lemma 5.3 to each of the (ordered) partitions above. Write \( \tau_{v,s} = t(A_{v,s}, A_{v,-s}) - t(A_{v,-s}, A_{>v}) + t(A_{>v}, A_{<v}) - t(A_{<v}, A_{v,s}) \) which by \( 5.3 \) will have \( \tau_{v,s} > 0 \) (and \( \tau_{v,-s} = 0 \)) if and only if \( t(A_{v,s} \cup A_{<v}) > 0 \). Since replacing \( s \) with \( -s \) simply switches the order of the first two elements of the partition, \( 5.3 \) also gives that \( \tau_{v,-s} > 0 \) (and \( \tau_{v,s} = 0 \)) if \( t(A_{v,s} \cup A_{<v}) > 0 \). In particular these situations are mutually exclusive.

Definition 5.5. Define \( h_t(v) = s \) if \( \tau_{v,s} > 0 \) and choose \( h_t(v) \) arbitrarily if \( \tau_{v,1} = \tau_{v,-1} \leq 0 \).

We will show that \( h_t \) satisfies \( P \). Fix a clause \( c \in [1, C] \). For each fixed \( p \in [0, 3] \) order lexicographically the 12 elements \( \{x_{c,p,q,e}\} q \in [0, 3], e \in [-1, 1] \} \subseteq X \) (so \( x_{c,p,0,0} < x_{c,p,0,1} < x_{c,p,1,0} < x_{c,p,1,1} < x_{c,p,2,0} < x_{c,p,2,1} < x_{c,p,3,0} < x_{c,p,3,1} \) and consider the 12 inequalities \( 0 < t([x], m\{x,y\}) - t([y], m\{x,y\}) \) (from part (3) of 2.5) obtained by taking \( y < x \) to be adjacent in the above order (11 cases) or else \( x = x_{c,p,3,1} \) the last element and \( y = x_{c,0,0} \) the first. Add the 48 inequalities obtained by taking these 12 inequalities for all choices of \( p \in [0, 3] \) to obtain:

\[
0 < \sum_{p,q \in [0,3]} (t([x_{c,p,q,0}], [x_{c,p,q,-1}]^\mu) - t([x_{c,p,q,-1}], [x_{c,p,q,-1}]^\mu) + t([x_{c,p,q,1}], [x_{c,p,q,1}]^\mu) - t([x_{c,p,q+1,1}], [x_{c,p,q+1,1}]^\mu))
\]

For every \( c \in [1, C] \), \( p \in [0, 3] \), and \( q \in [0, 3] \) for the first line, \( a \in [0, 2] \) for the second and \( b \in [0, 1] \) for the fourth we have:

\[
\begin{align*}
\{x_{c,p,q,1}\} & : \{x_{c,p,q,1}\}^\mu : \{x_{c,p,q+1,1}\}^\mu : \{x_{c,p,q+1,1}\}, \\
\{x_{c,p,a,0}\} : & A_{\nu(c,a), -\sigma(c,a)} : A_{\nu(c,a), 0} : \{x_{c,p,a,1}\}, \\
\{x_{c,p,0,0}\} & : A_{\nu(c,0), -\sigma(c,0)} : A_{\nu(c,0), 0} : \{x_{c,p,0,1}\}, \\
\{x_{c,p,0,1}\} & : A_{\nu(c,0), -\sigma(c,0)} : A_{\nu(c,0), 0} : \{x_{c,p,0,1}\}, \\
\{x_{c,p,2,0}\} & : A_{\nu(c,2), -\sigma(c,2)} : A_{\nu(c,2), 0} : \{x_{c,p,2,1}\}, \\
\{x_{c,p,2,1}\} & : A_{\nu(c,2), -\sigma(c,2)} : A_{\nu(c,2), 0} : \{x_{c,p,2,1}\}, \\
\{x_{c,p,3,0}\} & : A_{\nu(c,3), -\sigma(c,3)} : A_{\nu(c,3), 0} : \{x_{c,p,3,1}\}, \\
\{x_{c,p,3,1}\} & : A_{\nu(c,3), -\sigma(c,3)} : A_{\nu(c,3), 0} : \{x_{c,p,3,1}\}, \\
\{x_{c,p,3,2}\} & : A_{\nu(c,3), -\sigma(c,3)} : A_{\nu(c,3), 0} : \{x_{c,p,3,1}\}, \\
\{x_{c,p,3,3}\} & : A_{\nu(c,3), -\sigma(c,3)} : A_{\nu(c,3), 0} : \{x_{c,p,3,1}\}, \\
\end{align*}
\]

Using lemma 5.2 and the above betweenness relations to substitute into the previous inequality gives (after some cancellation):

\[
0 < - \sum_{p,q \in [0,3]} (2t([x_{c,p,q,1}], [x_{c,p,q,1}]^\mu) + t([x_{c,p,q+1,1}], [x_{c,p,q+1,1}]^\mu))
\]
Thus there is some $a \in [0, 2]$ for which $\tau_{\nu(c,a), \sigma(c,a)} > 0$ and hence $h_t(\nu(c,a)) = \sigma(c, a)$.

6. Satisfiable implies realizable

**Proposition 6.1.** If $P$ is a satisfiable case of 3-SAT then the midpoints structure $m_P$ on $X_P$ is realizable.

**Proof.** Assume that $h : [1, V] \rightarrow \{-1, 1\}$ satisfies $P$. Construct $t_h \in \mathbb{R}^{2^X}$ realizing $m_P$ by starting with $t \in \mathbb{R}^{2^X}$ which almost realizes $m_P$ and then perturbing the values on the leaf edges by $t'$ to get $t_h$. Start with

\[ t(A_{>v} \cup A_{v, h(v)}) = 6 \quad \text{for all } v \in [1, V], \]

\[ t(A_{>v}) = 10^V \quad \text{for all } v \in [0, V], \]

\[ t(A_{v,s}) = 10^V + 15 \frac{1}{2} \quad \text{for all } v \in [0, V + 1] \text{ and } s \in \{-1, 1\}, \]

\[ t(\{c, p, q, e\}^x) = 10^2V + 4c + (p + q + e) \quad \text{for all } c \in [1, C], p, q \in [0, 3] \text{ and } e \in \{-1, 1\}, \]

\[ t(\{c, p, q\}) = 10^2V + 4c + 4p + 2 \times 10^2V + 4c + (p + q)^2 \quad \text{for all } c \in [1, C], p, q \in [0, 3] \text{ and } e \in [-1, 1] \]

and $t(S) = 0$ for all other $S \subseteq X$. Note that the only edges of $t$ (and $t_h$) not containing midpoints are $A_{0, -1}$ and those in the first line above with length 6. The vector $t$ fails to realize $m_P$ only for the midpoints of lexicographically adjacent pairs of elements $x_{c, p, q, e}$ of $X$ with the same $c$ and $p$ coordinates. This is corrected by the slight perturbation below. Denote the number of agreements by $n_h(c) = \{|a \in [0, 2]| h(\nu(c, a)) = \sigma(c, a)\}$.

Correction:

\[ t'(\{c, p, 0, -1\}) = 0, \]

\[ t'(\{c, p, q, 0\}) = t'(\{c, p, q, -1\} - u_{x_{c, p, q, -1}, x_{c, p, q, 0}} + n_h(c), \]

\[ t'(\{c, p, q, 1\}) = t'(\{c, p, q, 0\} + u_{x_{c, p, q, 0}, x_{c, p, q, 1}} + n_h(c), \]

\[ t'(\{c, p, q, -1\}) = t'(\{c, p, q, -1\} + n_h(c) \text{ if } q \neq 0 \]

and $t'(S) = 0$ for all other $S \subseteq X$. Finally set $t_h(S) = t(S) + t'(S)$ for every $S \subseteq X$.

It is now straightforward to check that $t_h$ is a realization of $m_P$.

\[ \square \]

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