Critical exponents of the three-dimensional classical plane rotator model on the sc lattice from a high temperature series analysis

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High temperature series expansions of the spin-spin correlation function for the plane rotator (or XY) model on the sc lattice are extended by three terms through order $\beta^{17}$. Tables of the expansion coefficients are reported for the correlation function spherical moments of order $l = 0, 1, 2$. Our analysis of the series leads to fairly accurate estimates of the critical parameters.

In three dimensions the two-component vector model is the simplest spin model in the universality class of the superfluid $^{4}$He transition of $^{4}$He, and of the ferromagnetic transition of magnets with an easy magnetization plane[1]. No high temperature (HT) series studies of this model have appeared in the last two decades in spite of remarkable experimental measurements of the critical parameters in superfluid $^{4}$He and intense theoretical activity in Renormalization Group calculations and by direct Monte Carlo simulations.

In particular we should mention that the critical index $\nu$, which describes the leading singularity of the superfluid fraction in $^{4}$He near the superfluid transition temperature, has been measured with high precision in a long series of experiments by G. Ahlers and his collaborators[2]. As stressed by Ahlers, the superfluid fraction is the most accurately known singular parameter at a critical point, and correspondingly $\nu$ is the most accurately known critical index. The most recent experiments yield the value $\nu = 0.6705 \pm 0.0006$.

Unfortunately the critical exponent $\gamma$ cannot be measured in liquid $^{4}$He and, as far as magnetic systems are concerned, no precise measurements exist either for $\gamma$ or for $\nu$. A review of static critical properties of $^{4}$He can be found in Ref.[3] and a general discussion of the interpretation of the measurements on $^{4}$He in connection with the problem of confluent singularities is given in Ref.[4].

The Hamiltonian of the three-dimensional plane rotator (or XY) model is

$$H\{s\} = -\sum_x \sum_{\mu=1,3} s(x) \cdot s(x + e_\mu). \tag{1}$$

Here $s(x)$ is a two-component classical spin of unit length associated to the site with position vector $x = n_1 e_1 + n_2 e_2 + n_3 e_3 = (n_1, n_2, n_3)$ of a 3-dimensional simple cubic lattice and $e_1$, $e_2$, $e_3$ are the elementary lattice vectors. The sum over $x$ extends to all lattice sites.

It has been rigorously proved that the model exhibits a ferromagnetic phase transition [5].

We present here series which extend by three terms, to order $\beta^{17}$, the series of Ref.[6]. They have been computed by a FORTRAN code which iteratively solves the Schwinger-Dyson equations for the correlation functions[7].

We have tabulated the HTE coefficients of the two-point correlation function $C(x; \beta) = \langle s(0) \cdot s(x) \rangle \tag{2}$ for all inequivalent sites $x$ for which the expansion is non trivial to order $\beta^{17}$.

We have analyzed the series for the spherical moments of the correlation function $m^{(l)}(\beta)$ defined as follows:
temperature than do the susceptibility series approximants, nota bly behaved, though, like the analogous s.a.w. and Ising series, unbiase d approximants at first glance give a lower critical 

\[ \gamma \]

value, so that we can only estimate 

\[ \beta \]

0

studied lattice is still only 64

3

only recently after the invention of algorithms with reduced critical slowing down \[ [12] \]. The largest accurately 

\[ \nu \]

definition.

Let us now briefly review previous high temperature series analyses, restricting our review to the sc lattice results.

In conclusion our estimates of \( \gamma \) are fairly precise and, as shown later, in good agreement with the Renormalization Group (RG) results. However our estimates of \( \nu \) cannot yet compete either with the precision of the experimental data nor with the RG or Monte Carlo determinations, although they are perfectly compatible with both. This is probably due to the slow convergence of \( m^{(2)}(\beta) \), and as noted above, has already been observed in the study of high order expansions for the SAW and Ising models \[ [3] \]. Longer series are then required in order to make a more accurate analysis possible and in particular to account properly for the confluent singularities.

Let us now briefly review previous high temperature series analyses, restricting our review to the sc lattice results.

Bowers and Joyce \[ [10] \], computed series to order \( \beta^8 \) and gave the following estimates: \( \beta_c = 0.4530 \pm 0.0016 \), and \( \gamma = 1.312 \pm 0.006 \).

In Ref. \[ [13] \] the series were extended to order \( \beta^{11} \). The estimated inverse critical temperature was \( \beta_c = 0.4539 \pm 0.0013 \) and the corresponding estimates for \( \gamma \) and \( \nu \) were \( \gamma = 1.32 \pm 0.05 \) and \( \nu = 0.675 \pm 0.015 \). A comparison with our results shows that our central values for \( \beta_c \) and \( \gamma \) are significantly lower and that the precision in our estimates has improved by a factor two.

Reliable Monte Carlo simulations with good statistical accuracy, on reasonably sized lattices, have become possible only recently after the invention of algorithms with reduced critical slowing down \[ [12] \]. The largest accurately studied lattice is still only 64

3

sites large (present practical limits seem to be around 100

3

sites), which means that a very accurate treatment of finite size effects is required and that the estimate of systematic errors is very delicate. The oldest analysis is due to Li and Teitel \[ [13] \] who performed a Metropolis simulation ( supplemented by over-relaxation method ) on lattices up to 16

3

sites. A finite size scaling analysis of their data yields \( \beta_c = 0.4533 \pm 0.0006 \)
and \( \nu = 0.67 \pm 0.02 \). (The model actually simulated is a clock model with 512 states.)

More recently Hasenbusch and Meyer [14] used the Wolff single cluster algorithm on lattices up to \( 96^3 \) sites. From a fit of the data to \( \chi \propto (\beta_c - \beta)^{-\gamma} \), they found \( \beta_c = 0.45421 \pm 0.00008 \) and \( \gamma = 1.327 \pm 0.008 \). A recent update [15] of this study using the Wolff single cluster algorithm on lattices up to \( 64^3 \) sites gave \( \beta_c = 0.45420 \pm 0.00002, \nu = 0.664 \pm 0.006 \) and \( \gamma = 1.324 \pm 0.001 \).

W. Janke [16] also used the Wolff single cluster algorithm on lattices up to \( 48^3 \) sites. From a study of the fourth order cumulant he obtained \( \beta_c = 0.4542 \pm 0.0001 \) and \( \nu = 0.670 \pm 0.002 \). Fitting data to the formula \( \chi \propto (\beta_c - \beta)^{-\gamma} \) he obtained \( \beta_c = 0.45408 \pm 0.00008 \), and \( \gamma = 1.316 \pm 0.005 \). Repeating his fit with fixed \( \beta_c = 0.4542 \) the value of \( \gamma \) increases to \( \gamma = 1.323 \pm 0.002 \).

The previous computations should also be compared to the estimates by the Renormalization Group applied to an \( O(2) \) symmetric field theory model.

Sixth order perturbation expansion in three dimensions by Baker, Nickel and Meiron [19], gave \( \beta_c \pm 0.003 \). Subsequently, taking into account the large order behavior of the perturbation series coefficients, Le Guillou and Zinn Justin [20] refined these estimates and obtained \( \gamma = 1.316 \pm 0.0025 \) and \( \nu = 0.6695 \pm 0.001 \).

Performing the computation [21] by the Wilson-Fisher \( \epsilon = 4 - d \) expansion Borel resummed to order \( \epsilon^3 \), Le Guillou and Zinn Justin subsequently obtained the following estimates: \( \gamma = 1.315 \pm 0.007 \) and \( \nu = 0.671 \pm 0.005 \).

It thus appears that the RG results for \( \gamma \) are slightly smaller than the old HT and some of the new Monte Carlo estimates, but perfectly compatible with the results of our analysis, while our estimate of \( \nu \) is compatible with, but less accurate than, the most recent RG results.

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TABLE I. HTE coefficients of the nearest neighbor correlation $C(0, x)$ with $x = (1, 0, 0)$.

| order | coefficient |
|-------|-------------|
| 1     | 0.50000000000000000000000000000000 |
| 3     | 0.43750000000000000000000000000000 |
| 5     | 1.01041666666666666666666666666667 |
| 7     | 2.49169218750000000000000000000000 |
| 9     | 7.48240598958333333333333333333333 |
| 11    | 24.729247938469328703703703703704 |
| 13    | 86.7042412409706721230158730158730 |
| 15    | 317.8007506891941898083560681217 |
| 17    | 1205.066024541349586174488161069 |

TABLE II. HTE coefficients of the susceptibility $m^{(0)}$.

| order | coefficient |
|-------|-------------|
| 0     | 1.00000000000000000000000000000000 |
| 1     | 3.00000000000000000000000000000000 |
| 2     | 7.50000000000000000000000000000000 |
| 3     | 18.37500000000000000000000000000000 |
| 4     | 43.50000000000000000000000000000000 |
| 5     | 102.33750000000000000000000000000000 |
| 6     | 237.054875000000000000000000000000 |
| 7     | 546.94628906250000000000000000000000 |
| 8     | 1252.040821250000000000000000000000 |
| 9     | 2858.817529296875000000000000000000 |
| 10    | 6496.151407877604166666666666666666 |
| 11    | 14735.3746124891493055555555555555 |
| 12    | 33314.7537746853298611111111111111 |
| 13    | 75222.256639208112441964285714286 |
| 14    | 16944.488235923221330915178571434 |
| 15    | 381306.311343971793613736591641865 |
| 16    | 856543.263379992410619422872230489 |
| 17    | 1922537.91945074856684251367029620 |
### TABLE III. HTE coefficients of the first correlation moment $m^{(1)}$

| order | coefficient |
|-------|-------------|
| 0     | 0.00000000000000000000000000000000     |
| 1     | 3.00000000000000000000000000000000     |
| 2     | 11.4852813742385702928101323452582    |
| 3     | 35.391916429113710288472410676167     |
| 4     | 100.3914579738235211177404733391      |
| 5     | 270.169140885332810622174619548742    |
| 6     | 703.928165009702962171567107355945    |
| 7     | 1789.196531376491796388959830865      |
| 8     | 4468.327891804604696625866305929854    |
| 9     | 11000.8726669685811734428842857616    |
| 10    | 26788.05694784612641692323831814      |
| 11    | 64627.3429637161982839763208977200    |
| 12    | 154749.8127323977592584196634614      |
| 13    | 368132.797893714045088109726930470    |
| 14    | 870977.871997489895140365839695762    |
| 15    | 2050710.75491296800902977572988208    |
| 16    | 4808405.28831745065018387682551317    |
| 17    | 11232734.4966907585972846903939187    |

### TABLE IV. HTE coefficients of the second correlation moment $m^{(2)}$

| order | coefficient |
|-------|-------------|
| 0     | 0.00000000000000000000000000000000     |
| 1     | 3.00000000000000000000000000000000     |
| 2     | 18.00000000000000000000000000000000    |
| 3     | 72.37500000000000000000000000000000    |
| 4     | 247.50000000000000000000000000000000   |
| 5     | 770.59375000000000000000000000000000  |
| 6     | 2261.343750000000000000000000000000  |
| 7     | 6360.665039625000000000000000000000  |
| 8     | 17343.77343750000000000000000000000 |
| 9     | 46158.42104492187500000000000000000 |
| 10    | 120515.31930338541666666666666666   |
| 11    | 309746.425031873914930555555555556  |
| 12    | 785831.29642740885416666666666666  |
| 13    | 197180.9920570928431919642857143    |
| 14    | 4901417.59164962163047185019841270   |
| 15    | 12084656.317853839434553784567212    |
| 16    | 29584235.76402013523082377438823     |
| 17    | 71970593.8709586784015817546900548   |