MATING, PAPER FOLDING, AND AN ENDOMORPHISM OF $\mathbb{P} \mathbb{C}^2$

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ABSTRACT. We are studying topological properties of the Julia set of the map $F(z, p) = \left(\left(\frac{2z}{p+1} - 1\right)^2, \left(\frac{p+1}{p+2}\right)^2\right)$ of the complex projective plane $\mathbb{P} \mathbb{C}^2$ to itself. We show a relation of this rational function with an uncountable family of “paper folding” plane filling curves.

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1. Introduction

We study topology of the Julia set of the map

\[ F(z, p) = \left( \frac{2z}{p+1} - 1, \frac{p-1}{p+1} \right) \]

of the projective plane \( \mathbb{P} \mathbb{C}^2 \) to itself. In particular, we show an interesting connection between this map and an uncountable class of plane-filling curves coming from folding a strip of paper. One of the goals of the paper is to show how techniques of iterated monodromy groups can be used to obtain interesting topological properties of dynamical systems. All our results are proved using algebraic computations with self-similar groups.

Let us start from the end of the article, and describe the plane-filling curves. Take a long narrow strip of paper and fold it in two. Then repeat the procedure several times. Note that each time you have a choice of two directions of folding. Then unfold it so that you get right angles at the creases. You will get something like it is shown on Figure 1.

Let us record the way we folded the paper as a sequences of letters \( L, R \), standing for “left” and “right”, respectively.

Fold now two equal strips of paper in the same way (described by the same sequence of letters \( L, R \)), and rotate one with respect to the other by \( 180^\circ \). Put them down so that their endpoints touch, see Figure 2.

You will get a closed curve \( \gamma \), bounding a connected maze of square rooms. See two examples of such mazes on Figure 3. The rooms are shaded black. Two red dots mark the endpoints of the strips of paper, the green dots mark their midpoints (i.e., the creases of the first folding).

The marked points are vertices of a square (this easily follows from the construction). Let us choose an infinite sequence \( w = X_1X_2 \ldots \) of letters \( L, R \). Let us draw rescaled closed curves \( \gamma_{w_n} \) corresponding to finite sequences \( w_n = X_1X_2 \ldots X_n \) of instructions in such a way that the marked points stay at the vertices of a fixed square \( Q \). Let us parametrize the curves \( \gamma_{w_n} \) uniformly (proportionally to the arclength) by \( t \in [0, 1] \), so that \( \gamma_{w_n}[0,1/2] \) and \( \gamma_{w_n}[1/2,1] \) are copies of the folded paper strip. Then the vertices of \( Q \) are \( \gamma_{w_n}(0), \gamma_{w_n}(1/2) \) (the endpoints of the strips), and \( \gamma_{w_n}(1/4), \gamma_{w_n}(3/4) \) (their midpoints).

It follows from the description of the folding procedure that the maze \( \gamma_{w_{n+1}} \) is obtained from the maze \( \gamma_{w_n} \) by replacing each wall by a corner (so that the old
wall is the hypotenuse and the new walls are legs of an isosceles right triangle). We get then a sequence of curves converging uniformly to some limit curve $\gamma_{x_1, x_2, \ldots}$.

The best known and studied example is the Heighway dragon curve, which corresponds to a constant sequence $w = \text{LLL} \cdots$ (or $\text{RRR} \cdots$). It was defined for the first time by NASA physicists J. Heighway, B. Banks, and W. Harter, and popularized by M. Gardner in Scientific American. It is also called sometimes the “Jurassic Park Fractal”, as the curves $\gamma_{n} |_{[0,1/2]}$ appear at the beginning of each chapter of “Jurassic Park” by M. Crichton. The closed version $\gamma_w$ is called sometimes the twin-dragon curve. See the images of these curves on Figure 4. In [7, p. 190] a relation of the twin dragon curve to numeration systems on complex numbers is discussed.

Consider the group $H$ of transformations of the plane generated by rotations by $180^\circ$ around the vertices of the square $Q$ (recall that it is the square whose vertices are the endpoints and the midpoints of the strips of paper). The fundamental
domain of the group $H$ is the square $Q'$ of twice bigger area such that vertices of $Q$ are midpoints of the sides of $Q'$. The quotient $\mathbb{R}^2/H$ of the plane by the action of $H$ is homeomorphic to the sphere, and can be realized as the pillowcase obtained from the square $Q'$ by folding its corners over the sides or the square $Q$.

If we take the curve $\gamma_{w_n}$, then its image $\gamma_{w_n}/H$ on the pillowcase $\mathbb{R}^2/H$ is a nice Eulerian path tracing a square grid on the pillow, see Figure 5. The figure shows how pieces of the curve $\gamma_{w_n}$ that are outside the fundamental domain $Q'$ are moved inside by elements of $H$ (actually, just by the generators).

For every infinite sequence $w = X_1X_2 \ldots$ the image $\gamma_w/H$ of the curve $\gamma_w$ is a curve passing through every point of the sphere $\mathbb{R}^2/H$.

It follows directly from the construction, that the curves $\gamma_{X_1X_2 \ldots X_n}|_{I}$ for $I = [0, 1/4], [1/4, 1/2], [1/2, 3/4], [3/4, 1]$ are similar (with the similarity coefficient $1/\sqrt{2}$) to the curves $\gamma_{X_2X_3 \ldots X_n}|_{[0,1/2]}$ and $\gamma_{X_3X_4 \ldots X_n}|_{[1/2,1]}$. The partition of $\gamma_{X_1X_2 \ldots X_n}$ into the defined above sub-curves correspond to splitting the original strip of paper in two (and “forgetting” about the first folding). Moreover, the similarities $\gamma_{X_1X_2 \ldots X_n}|_{[0,1/4]} \to \gamma_{X_2X_3 \ldots X_n}|_{[0,1/2]},$ $\gamma_{X_1X_2 \ldots X_n}|_{[1/4, 1/2]} \to \gamma_{X_2X_3 \ldots X_n}|_{[1/2,1]}$, 

![Figure 4. Dragon and Twin Dragon curves](image)

![Figure 5. Curve on the pillowcase $\mathbb{R}^2/H$](image)
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$\gamma_{X_1X_2...X_n}|_{[1/2,3/4]} \rightarrow \gamma_{X_1X_2...X_n}|_{[0,1/2]}$, $\gamma_{X_1X_2...X_n}|_{[3/4,1]} \rightarrow \gamma_{X_2X_3...X_n}|_{[1/2,1]}$ are restrictions of a branched self-covering $S_{X_1}: \mathbb{R}^2/H \rightarrow \mathbb{R}^2/H$. See Figure 6 for a description of $S_{X_1}$. In the limit $S_{X_1}$ induces a piece-wise similarity map from $\gamma_{X_1X_2...}$ to $\gamma_{X_2X_3...}$.

![Figure 6. Pillowcase folding](image)

Each of the curves $\gamma_{X_1X_2...X_n}$ divides the sphere into two parts: it separates squares of different color in a checkerboard coloring of the pillow $\mathbb{R}^2/\mathbb{Z}^2$. The squares of one color are connected to each other by the corridors of the maze in a tree-like fashion. It is reasonable to assume that in the limit the curve $\gamma_{X_1X_2...}$ goes around a dendrite that is a limit of the sequence of the trees bounded by $\gamma_{X_1X_2...X_n}$. See, for example [10 II.7], where relation between plane-filling curves and “rivers” they bound is explored.

The map $S_{X_1}$ induces a degree two branched covering of the dendrite bounded by $\gamma_{X_1X_2...X_n}$ by the dendrite bounded by $\gamma_{X_1X_2...}$.

The case of the twin-dragon curve was analyzed in detail by J. Milnor in [11]. He showed that the dendrites into which the curve $\gamma_{RRR...}/H$ separates the sphere can be naturally identified with the Julia set of the polynomial $f(z) = z^2 + c$, where $c \approx -0.228 + 1.115 i$ is a root of the polynomial $c^3 + 2c^2 + 2c + 2$. Moreover, the curve $\gamma_{RRR...}/H$ goes around the dendrite in the same way as the Caratheodory loop goes around the Julia set, so that the sphere $\mathbb{R}^2/H$ is the mating of the polynomial $z^2 + c$ with itself: it is obtained by gluing two Julia sets along the Caratheodory loops so that one loop is a complex conjugate of the other.

The general paper folding curves $\gamma_w$ were studied in [5] where a connection with numeration systems on complex numbers was described (and closely related to the “pillowcase folding” maps $S_L, S_R: \mathbb{R}^2/H \rightarrow \mathbb{R}^2/H$).

In our paper we show a relation of the sphere-filling curves $\gamma_w$ with dynamics of the map

$$F(z,p) = \left( \left( \frac{2z}{p+1} - 1 \right)^2, \left( \frac{p-1}{p+1} \right)^2 \right)$$

defined on the projective plane $\mathbb{P}C^2$ to itself. Note that on the second coordinate $F$ is the rational function $f : p \mapsto \left( \frac{p-1}{p+1} \right)^2$, whereas on the first coordinate iterations of $F$ are compositions of polynomials $h_p : z \mapsto \left( \frac{2z}{p+1} - 1 \right)^2$, where $p$ runs through a forward orbit of iterations of $f$. Intersections of the Julia set of $F$ with planes $p = p_0$ are then Julia sets of the non-autonomous iterations $C \xrightarrow{h_{p_0}} C \xrightarrow{h_{p_1}} C \xrightarrow{h_{p_2}} \cdots$, where $h_p$.
where \( p_{n+1} = \left( \frac{p_n - 1}{p_n + 1} \right)^2 \). These Julia sets, which we will denote \( J(p_0) \), are dendrites, see Figure \( \text{\ref{fig:julia}} \) if \( p_0 \) belongs to the Julia set of \( f \).

We show that the dendrites \( J(p_0) \) are precisely the dendrites around which the loops \( \gamma_w \) go. Moreover, the pillow \( \mathbb{R}^2/\mathbb{Z}^2 \) can be obtained by gluing two copies of \( J(p_0) \) along the Caratheodory loop going around \( J(p_0) \) (one loop is glued to the other using reflection with respect to a diameter). The curve \( \gamma_w \) is the image of the Caratheodory loop. Moreover, the construction is dynamical: the map \( h_{p_0} \) agrees with the double coverings \( \gamma_{X_1 X_2 \ldots} \rightarrow \gamma_{X_2 X_3 \ldots} \).

The paper is organized as follows. We start with a short reminder of the main notions and techniques of self-similar groups. Section 3 is devoted to the study of dynamics of the endomorphism \( F : \mathbb{PC}^2 \rightarrow \mathbb{PC}^2 \). We start with computation of the iterated monodromy group of \( F \) in Theorem 3.3. It is a self-similar group acting on a degree 4 rooted tree. A computationally more convenient group is an index two extension of \( \text{IMG}(F) \), which is defined in Subsection 3.2. We denote it \( \mathcal{G} \), and it is the iterated monodromy group of the quotient of the dynamical system \( F : \mathbb{PC}^2 \rightarrow \mathbb{PC}^2 \) by complex conjugation.

The group \( \text{IMG}(F) \) contains a natural subgroup \( \mathcal{G} \) corresponding to non-autonomous iterations \( h_p \) of polynomials on the first coordinate. It is not transitive on the levels of the rooted tree, and we get an uncountable family of quotients of \( \mathcal{G} \) coming from restricting of \( \mathcal{G} \) to invariant binary subtrees. A similar family of groups was studied in \[14, 18, 20, 16\].

The graphs of the action of \( \mathcal{G} \) on the levels of binary sub-trees are approximations of the Julia sets \( J(p) \) of the non-autonomous iterations of \( h_p \) (i.e., the corresponding slices of the Julia set of \( F \)). They are trees, in accordance with the fact that \( J(p) \) are dendrites. We study these trees in \[3.4\] using recursive description of the generators of the self-similar group \( \mathcal{G} \). In particular, we describe inductive algorithms of constructing these graphs, see Corollaries \[3.11 \text{ and } 3.12\].

In the next subsection we study external angles to the Julia sets \( J(p) \), i.e., the Caratheodory loops around them. The bundle of Caratheodory loops is the limit space of the subgroup \( \mathcal{R} \) of \( \text{IMG}(F) \) generated by the loops not intersecting the Julia sets \( J(p) \). We derive from the structure of the group \( \mathcal{R} \) how the bundle of Caratheodory loops is glued together from a Cantor set of circles, and which external angles land on the points of the line \( z = p \). In particular, we show that for a countable set of parameters \( p \) (equal to the backward orbit of the unique real fixed point of \( f(p) = \left( \frac{1 - p}{1 + p} \right)^2 \) ) there are two external rays to \( J(p) \) landing on \( (p, p) \), and that in all the other cases such ray is unique.

In Section 4 we define matings of the non-autonomous iterations \( h_p \). Namely, for every \( p \) in the Julia set of \( f(p) = \left( \frac{1 - p}{1 + p} \right)^2 \) consider the corresponding slice \( J(p) \) of the Julia set of \( F \), and the Caratheodory loop \( \gamma \) around it. Take then another copy of \( J(p) \), and glue them together along the Caratheodory loops so that one loop is a mirror image of the other with respect to the diameter containing an external angle landing on \( (p, p) \). Note that in the case when \( p \) belongs to the backward orbit of the real fixed point of \( f \), there are two such rays, and we have therefore two possible choices for the mating.

We define the mating and study it in purely algebraic terms. We construct an “amalgam” of the group \( \mathcal{G} \) with itself, generating a group \( \hat{\mathcal{G}} \) by two copies of
Then the inclusion of the two copies of $G$ induce semi-conjugacies of the limit dynamical systems. We show that these semi-conjugacies realize the matings as described in the previous paragraph.

We study then the limit dynamical system of $\hat{G}$, i.e., the obtained matings. We show that the group $\hat{G}$ contains a virtually abelian subgroup $H$ such that the inclusion $H \hookrightarrow \hat{G}$ induces a conjugacy of the limit dynamical systems. Then we show that the limit space of $\hat{G}$ is a direct product of the Cantor set $\{L, R\}^\infty$ with the sphere $\mathbb{C}/H$, where $H$ is the group of affine transformations of the form $z \mapsto \pm z + a + ib$ for $a, b \in \mathbb{Z}$, i.e., the pillowcase described above. The limit dynamical system acts as the binary one-sided shift on the Cantor set and as multiplication by $1+i$ or $1-i$ on the pillowcase $\mathbb{C}/H$ (the choice of the coefficient $1 \pm i$ depends on the first letter of the corresponding element of the one-sided shift).

It follows that the constructed matings are Lattès examples, though non-autonomous, as we can choose one of two multiplications.

Since $G < \hat{G}$ the graphs of action of $G$ on the levels of the tree are sub-graphs of the graphs of action of $\hat{G}$. The graphs of action of $\hat{G}$ are square grids on the pillowcases. We show that the graphs of action of the two copies of $G$ partition the edges of the grid into two disjoint sub-trees, see Figure 23. These partitions of a square grid into two subtrees converge to the decomposition of the pillowcase $\mathbb{C}/H$ into two dendrites.

In Section 5 we relate the obtained results about the mating with the paper folding curves. Namely, we show that the curve separating the two subtrees of the square grid on $\mathbb{C}/H$ coincides with the paper-folding curves $\gamma_v$ described in this introduction, and that in the limit the curves $\gamma_v$ converge to the image of the Caratheodory loop in the mating.

The last section describes structure of the slices of the Julia set of $F$ that correspond to the values of $p$ belonging to the boundaries of the Fatou components of $f(p) = \left(\frac{p+1}{p-1}\right)^2$. We show that they are obtained by “flattening” the boundaries of the Fatou components of the polynomials $16z^2(1-z)^2$ and $(2z^2 - 4z + 1)^2$. In other terminology, they are obtained by (rotated) tuning of these polynomials by the polynomial $z^2 - 2$. As usual, the proof is carried out using just algebraic computations with the iterated monodromy groups.

2. Self-similar groups

We present here, in a very condensed form, the main definitions and results of the theory of self-similar and iterated monodromy groups. For a more detailed account, see [13, 15, 17, 19].

2.1. Covering bisets.

**Definition 2.1.** Let $G$ be a group. A $G$-biset is a set $\mathfrak{M}$ together with commuting left and right actions of $G$ on $\mathfrak{M}$. In other words, we have two maps $G \times \mathfrak{M} \to \mathfrak{M} : (g, x) \mapsto g \cdot x$ and $\mathfrak{M} \times G \to \mathfrak{M} : (x, g) \mapsto x \cdot g$ satisfying $1 \cdot x = x \cdot 1 = x$ for all $x \in \mathfrak{M}$, and:

$$g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x, \quad (x \cdot g_1) \cdot g_2 = x \cdot (g_1 g_2) \quad (g_1 \cdot x) \cdot g_2 = g_1 \cdot (x \cdot g_2)$$

for all $g_1, g_2 \in G, x \in \mathfrak{M}$. 

We say that \( \mathcal{M} \) is a \textit{covering biset} if the right action of \( G \) on \( \mathcal{M} \) is free, i.e., if \( x \cdot g = x \) implies \( g = 1 \). We also assume then that the number of right orbits is finite.

Sometimes we also consider \( G_1 - G_2 \) bisets \( \mathcal{M} \), which are sets with commuting left \( G_1 \)-action and right \( G_2 \)-action.

\textbf{Definition 2.2.} The isomorphism class of a pair \((G, \mathcal{M})\), where \( G \) is a group and \( \mathcal{M} \) is a covering \( G \)-biset is called a \textit{self-similar group}. Here two pairs \((G_1, \mathcal{M}_1)\) and \((G_2, \mathcal{M}_2)\) are \textit{isomorphic} (the corresponding self-similar groups are called \textit{equivalent}) if there exists an isomorphism \( \phi : G_1 \to G_2 \) and a bijection \( f : \mathcal{M}_1 \to \mathcal{M}_2 \) such that \( f(g_1 \cdot x \cdot g_2) = \phi(g_1) \cdot f(x) \cdot \phi(g_2) \) for all \( g_1, g_2 \in G_1 \) and \( x \in \mathcal{M}_1 \).

Let \( \mathcal{M}_1, \mathcal{M}_2 \) be \( G \)-biset. Then their \textit{tensor product}, denoted \( \mathcal{M}_1 \otimes \mathcal{M}_2 \), is the quotient of \( \mathcal{M}_1 \times \mathcal{M}_2 \) by the equivalence relation \( x_1 \otimes g \cdot x_2 = x_1 \cdot g \otimes x_2 \), with the actions \( g_1 \cdot (x_1 \otimes x_2) \cdot g_2 = (g_1 \cdot x_1) \otimes (x_2 \cdot g_2) \). It is easy to show that tensor product of two covering bisets is a covering biset. It is also easy to see that \((\mathcal{M}_1 \otimes \mathcal{M}_2) \otimes \mathcal{M}_3\) is naturally isomorphic to \( \mathcal{M}_1 \otimes (\mathcal{M}_2 \otimes \mathcal{M}_3) \).

Let \( \mathcal{M} \) be a covering \( G \)-biset. Consider the disjoint union \( \mathcal{M}^* = \bigcup_{n \geq 0} \mathcal{M} \otimes^n \) of the tensor powers \( \mathcal{M} \otimes^n, n \geq 0 \), where \( \mathcal{M} \otimes^0 \) is the group \( G \) with the natural left and right actions on itself. Denote by \( T_{\mathcal{M}} \) the set of orbits of the right action of \( G \) on \( \mathcal{M}^* \). Then \( G \) acts on \( T_{\mathcal{M}} \) from the left. The set \( T_{\mathcal{M}} \) has a natural structure of a rooted tree, where the right orbit of \( x_1 \otimes x_2 \otimes \cdots \otimes x_n \otimes x_{n+1} \) is connected by an edge to the right orbit of \( x_1 \otimes x_2 \otimes \cdots \otimes x_n \otimes x_{n+1} \) for \( x_i \in \mathcal{M} \). The left action of \( G \) on \( T_{\mathcal{M}} \) is an action by automorphisms of the rooted tree (the root is the unique right orbit of the action of \( G \) on itself).

The action of \( G \) on \( T_{\mathcal{M}} \) is not faithful in general. Let \( N \) be the kernel of the action. Denote then by \( \mathcal{M}/N \) the set of right \( N \)-orbits. It is easy to see that \( h \cdot x \) and \( x \) belong to one right \( N \)-orbit for every \( x \in \mathcal{M} \) and \( h \in N \), and that the left and right actions of \( G \) on \( \mathcal{M} \) descend to left and right actions of \( G/N \) on \( \mathcal{M}/N \). We call the self-similar group \((G/N, \mathcal{M}/N)\) the \textit{faithful quotient} of the self-similar group \((G, \mathcal{M})\).

2.2. \textbf{Wreath recursions and virtual endomorphisms.} Let \((G, \mathcal{M})\) be a self-similar group. We say that \( X = \{ x_1, x_2, \ldots, x_d \} \subset \mathcal{M} \) is a \textit{basis} if it is a transversal of right \( G \)-orbits. In other words, it is such a subset of \( \mathcal{M} \) that for every \( x \in \mathcal{M} \) there exists a unique \( x_i \in X \) such that \( x = x_i \cdot g \) for some \( g \in G \). Note that \( g \) is also uniquely determined by \( x \), since the right action is free.

For every \( n \) the set \( X^n = \{ a_1 \otimes a_2 \otimes \cdots \otimes a_n : a_i \in X \} \) is a basis of \( \mathcal{M} \otimes^n \). Consequently, \( X^* = \bigcup_{n \geq 0} X^n \) is a basis of \( \mathcal{M}^* \). Here \( X^0 \) consists of a single \textit{empty word} identified with the identity element of \( G = \mathcal{M} \otimes^0 \). We will usually write \( a_1 \otimes a_2 \otimes \cdots \otimes a_n \) just as a word \( a_1 a_2 \ldots a_n \). We get a natural bijection between the set \( T_{\mathcal{M}} \) of right orbits of \( \mathcal{M}^* \) and the set \( X^* \) of finite words over \( X \). The vertex adjacency on the tree \( T_{\mathcal{M}} \) corresponds to a similar adjacency of elements of \( X^* \): a word \( v \in X^* \) is adjacent to the words of the form \( x v x \) for \( x \in X \). In other words, we consider \( X^* \) as the right Cayley graph of the free monoid generated by \( X \). The action of \( G \) on \( T_{\mathcal{M}} \) is transformed hence to an action of \( G \) on \( X^* \). We call this action the \textit{self-similar action} associated with the basis \( X \) of the biset \( \mathcal{M} \).

Let \( g \in G \) and \( x \in X \), then there exist unique \( y \in X \) and \( h \in G \) such that \( g \cdot x = y \cdot h \). The induced map \( x \mapsto y \) is a permutation coinciding with the
restriction of the action of \( g \) to the first level \( X \subset X^* \) of the rooted tree \( X^* \). Let us denote this permutation by \( \sigma_g \in \text{Symm}(X) \). Denote also \( h = g|_x \). We get a map

\[
\Phi : g \mapsto \sigma_g(g|_x)_{x \in X}
\]

from \( G \) to the semidirect product \( \text{Symm}(X) \ltimes G^X \). The semi-direct product is called the (permutational) wreath product \( X \). Let us call it the wreath recursion

\[
\text{basis endomorphism in the following way. Let}
\]

\[
T \in G \text{ (where}
\]

\[
\phi
\]

\[
\text{the vertex}
\]

\[
x
\]

\[
\text{translations}
\]

\[
h \text{ hence to}
\]

\[
x\]

\[
\text{orbits is transitive, every element}
\]

\[
\pi
\]

\[
\text{structure of a covering}
\]

\[
\text{Iterated monodromy groups.}
\]

\[
\text{Let} \ M_1, M_0 \ 
\]

\[
\text{be path connected and locally}
\]

\[
\text{path connected topological spaces or orbispaces. A topological correspondence is a pair of maps}
\]

\[
f, \iota : M_1 \rightarrow M_0,
\]

\[
\text{where}
\]

\[
f \text{ is a finite degree covering map and}
\]

\[
\iota \text{ is a continuous map.}
\]

\[
\text{Choose a basepoint} \ t \in M_0. \text{ Let} \ \mathcal{M} \ 
\]

\[
\text{be the set of pairs} \ (z, \ell), \text{ where}
\]

\[
z \in f^{-1}(t), \text{ and}
\]

\[
\ell \text{ is homotopy class of a path in} \ M_0 \text{ from} \ t \text{ to} \ \iota(z). \text{ The set} \ \mathcal{M} \ 
\]

\[
\text{has a natural structure of a covering} \ \pi_1(M_0, t) \text{-biset. Namely, for a loop} \ \gamma \in \pi_1(M_0, t), \text{ denote by}
\]

\[
(z, \ell) \cdot \gamma \text{ the element} \ (z, \ell \gamma). \text{ Here and in the sequel we compose paths as functions: in a product} \ \ell \gamma \text{ the path} \ \gamma \text{ is passed first, then}
\]

\[
\ell. \text{ Denote by} \ \gamma \cdot (z, \ell) \text{ the element} \]
(y, \iota(\gamma)\ellt), \text{ where } \gamma_z \text{ is the unique lift of } \gamma \text{ by } f \text{ that starts at } z, \text{ and } y \text{ is the end of } \gamma_z.\\

A basis of } \mathfrak{M} \text{ is any set } \{(z_1, \ell_1), (z_2, \ell_2), \ldots, (z_d, \ell_d)\}, \text{ where } d = \deg f, \text{ and } f^{-1}(t) = \{z_1, z_2, \ldots, z_d\}.\\

The faithful quotient of the self-similar group } (\pi_1(M_0, t), \mathfrak{M}) \text{ is called the iterated monodromy group of the correspondence } f, \iota : M_1 \to M_0.\\

The virtual endomorphism associated with the biset } \mathfrak{M} \text{ is equal to } \iota_* : \pi_1(M_1) \to \pi_1(M_0), \text{ where } \pi_1(M_1) \text{ is identified with a subgroup finite index in } \pi_1(M_0) \text{ by the isomorphism } f_* \text{. It is well defined up to compositions with inner automorphisms of } \pi_1(M_0), \text{ as any virtual endomorphism associated with a self-similar group.}\\

If } f : M \to M \text{ is a branched self-covering, then we may transform it into a topological correspondence by removing from } M \text{ the closure } P \text{ of the union of the forward orbits of branch points of } M. \text{ If } P \text{ is not big enough, in particular, if it does not disconnect } M, \text{ then we can consider the topological correspondence } f, \iota : M_1 \to M_0, \text{ where } M_0 = M \setminus P, \text{ and } M_1 = f^{-1}(M_0), \text{ and } \iota : M_1 \to M_0 \text{ is the identical embedding. This is done, for example, if } f \text{ is a post-critically finite rational function, or a post-critically finite endomorphism of } \mathbb{P}^1(\mathbb{C}) \text{ (which means that } P \text{ is a union of a finite number of varieties). In these cases we represent the elements of } \mathfrak{M} \text{ just as paths } \ell = \iota(\ell), \text{ since their endpoints are uniquely determined.}\\

2.4. Contracting self-similar groups. Let } \mathfrak{M} \text{ be a covering } G\text{-biset, and let } X \text{ be a basis of } \mathfrak{M}. \text{ For every } v \in X^* \text{ and } g \in G \text{ denote by } g|_v \text{ the unique element such that } g \cdot v = u \cdot g|_v \text{ for some } u \in X^*. \text{ We call it the section of } g \text{ in } v.\\

**Definition 2.3.** The self-similar group } (G, \mathfrak{M}) \text{ (with a chosen basis } X \text{ of } \mathfrak{M}) \text{ is said to be contracting if there exists a finite set } \mathcal{N} \subset G \text{ such that for every } g \in G \text{ there exists } n \text{ such that } g|_v \in \mathcal{N} \text{ for every } v \in X^k \text{ such that } k \geq n. \text{ The smallest set } \mathcal{N} \text{ satisfying the above condition is called the nucleus of the group.}\\

It is proved in [13, Corollary 2.11.7] that the property of a biset to be hyperbolic does not depend on the choice of a basis } X. \text{ The nucleus, however, depends on } X.\\

Since a biset } \mathfrak{M} \text{ and a basis } X \text{ is uniquely determined, up to an isomorphism, by the associated wreath recursion, we call a wreath recursion } G \to \text{Symm}(X) \ltimes G^X \text{ contracting if the corresponding self-similar group is contracting. Sometimes we say that a } G\text{-biset } \mathfrak{M} \text{ is hyperbolic if the self-similar group } (G, \mathfrak{M}) \text{ is contracting. It is easy to see that if } (G, \mathfrak{M}) \text{ is contracting, then its faithful quotient is also contracting.}\\

The nucleus } \mathcal{N} \text{ satisfies the property that } g|_x \in \mathcal{N} \text{ for all } g \in \mathcal{N} \text{ and } x \in X. \text{ We will often represent } \mathcal{N} \text{ as an automaton using its Moore diagram. It is the oriented graph with the set of vertices } \mathcal{N} \text{ in which for every } g \in \mathcal{N} \text{ and } x \in X \text{ we have an arrow from } g \text{ to } g|_x \text{ labeled by } x|y, \text{ where } y \text{ is the image of } x \text{ under the action of } g \text{ on the first level of the tree } X^*, \text{ i.e., we have } g \cdot x = y \cdot g|_x.\\

Let } (G, \mathfrak{M}) \text{ be a contracting self-similar group, and let } X \subset \mathfrak{M} \text{ be a basis. Consider the space } X^{-\omega} \text{ of left-infinite sequences } \ldots x_2x_1 \text{ of elements of } X. \text{ We say that } \ldots x_2x_1 \ldots y_2y_1 \in X^{-\omega} \text{ are asymptotically equivalent if there exists a sequence } g_n \in G \text{ taking a finite number of values such that } g_n(x_n \ldots x_1) = y_n \ldots y_1 \text{ (with respect to the action of } G \text{ on } X^*). \text{ The quotient of the topological space } X^{-\omega} \text{ by the asymptotic equivalence relation is called the limit space of the group } (G, \mathfrak{M}), \text{ and is denoted } J_G. \text{ The shift } \ldots x_2x_1 \mapsto \ldots x_3x_2 \text{ agrees with the asymptotic equivalence.
relation, so that it induces a continuous map \( s : \mathcal{J}_G \to \mathcal{J}_G \). We call the pair \((\mathcal{J}_G, s)\) the \textit{limit dynamical system} of the self-similar group.

One can show, see [13, Theorem 3.6.3], that two sequences \( \ldots x_2 x_1 \) and \( \ldots y_2 y_1 \) are asymptotically equivalent if and only if there exists an oriented path \( \ldots e_2 e_1 \) of arrows in the Moore diagram of the nucleus such that \( e_n \) is labeled by \( x_n | y_n \).

Consider now \( X^{-\omega} \times G \), where \( G \) is discrete. We write elements of the space \( X^{-\omega} \times G \) as \( \ldots x_2 x_1 \cdot g \) for \( x_i \in X \) and \( g \in G \). Two sequences \( \ldots x_2 x_1 \cdot g \) and \( \ldots y_2 y_1 \cdot h \) are asymptotically equivalent if there exists a sequence \( g_n \in G \) taking a finite set of values such that \( g_n \cdot x_n \ldots x_2 x_1 \cdot g = y_n \ldots y_2 y_1 \cdot h \) in \( \mathcal{M}^\otimes n \) for all \( n \).

One can show that \( \ldots x_2 x_1 \cdot g \) and \( \ldots y_2 y_1 \cdot h \) are asymptotically equivalent if and only if there exists an oriented path \( \ldots e_2 e_1 \) in the Moore diagram of the nucleus such that \( e_n \) is labeled by \( x_n | y_n \) for every \( n \) and the last vertex of the path is \( h g^{-1} \).

The quotient of \( X^{-\omega} \times G \) by the asymptotic equivalence relation is called the \textit{limit G-space} and is denoted \( \mathcal{X}_G \). The group \( G \) acts naturally on \( X^{-\omega} \times G \) by right multiplication. This action agrees with the asymptotic equivalence relation, so that it induces a right action of \( G \) on \( \mathcal{X}_G \) by homeomorphisms.

For every \( x \in \mathcal{M} \) and \( \ldots x_2 x_1 \cdot g \in X^{-\omega} \times G \) the asymptotic equivalence class of \( \ldots x_2 x_1 \cdot g \otimes x = \ldots x_2 x_1 y \cdot h \), where \( h \in G \) and \( x \in X \) are such that \( g \cdot x = y \cdot h \), is uniquely determined by \( x \) and the asymptotic equivalence class of \( \ldots x_2 x_1 \cdot g \). It follows that we get a well defined continuous map \( \xi \mapsto \xi \otimes x \) of \( \mathcal{X}_G \) to itself.

The biset structure of \( \mathcal{M} \) agrees with the maps \( \xi \mapsto \xi \otimes x \) on \( \mathcal{X}_G \), so that \( (\xi \cdot g_1 \otimes x) g_2 = \xi \otimes (g_1 \cdot x \cdot g_2) \) for all \( \xi \in \mathcal{X}_G, g_1, g_2 \in G, x \in \mathcal{M} \). Moreover, we have the following rigidity theorem, see [13, Theorem 3.4.13].

**Theorem 2.1.** Let \( \mathcal{M} \) be a hyperbolic \( G \)-biset. Let \( \mathcal{X} \) be a metric space such that \( G \) acts on \( \mathcal{X} \) co-compactly and properly by isometries from the right. Suppose that for every \( x \in \mathcal{M} \) we have a continuous strictly contracting map \( \xi \mapsto \xi \otimes x \) such that \( (\xi \cdot g_1 \otimes x) g_2 = \xi \otimes (g_1 \cdot x \cdot g_2) \) for all \( \xi \in \mathcal{X}, g_1, g_2 \in G, x \in \mathcal{M} \).

Then there exists a homeomorphism \( \Phi : \mathcal{X} \to \mathcal{X}_G \) such that \( \Phi(\xi \cdot g) = \Phi(\xi) \cdot g \) and \( \Phi(\xi \otimes x) = \Phi(\xi) \otimes x \) for all \( \xi \in \mathcal{X}, g \in G, x \in \mathcal{M} \).

**2.5. Contracting correspondences.** Let \( f, \iota : M_1 \to M_0 \) be a topological correspondence. Its \textit{limit space} \( M_\infty \) is subspace of all sequences \( (x_1, x_2, \ldots) \in M_1^\infty \) such that \( f(x_n) = \iota(x_{n+1}) \). For example, if \( \iota \) is an identical embedding, then \( M_\infty \) is the intersection of the domains of all iterations of the partial map \( f \).

The shift \( (x_1, x_2, \ldots) \mapsto (x_2, x_3, \ldots) \) is a continuous self-map on \( M_\infty \), which we will denote \( f_\infty \).

**Definition 2.4.** Let \( f, \iota : M_1 \to M_0 \) be a topological correspondence. We say that it is \textit{contracting} if \( M_0 \) is a compact length metric space (i.e., there is a notion of length of arcs such that distance between two points is the infimum of lengths of arcs connecting them), and \( \iota \) is contracting with respect to the length metric on \( M_0 \) and the lift of the length metric from \( M_0 \) to \( M_1 \) by \( f \).

In particular, if \( f \) is expanding, and \( \iota \) is an identical embedding, then the correspondence \( f, \iota : M_1 \to M_0 \) is contracting.

One can show that the iterated monodromy group of a contracting topological correspondence is a contracting self-similar group, see [13].

**Theorem 2.2.** The limit dynamical system of a contracting topological correspondence \( \mathcal{F} \) is topologically conjugate to the limit dynamical system of the iterated monodromy group of \( \mathcal{F} \).
The correspondence between the contracting self-similar groups and expanding self-coverings is functorial in a precise way, see [15]. For example, any embedding of self-similar contracting groups (preserving self-similarity) induces a semi-conjugacy of their limit dynamical systems.

3. The Julia set of an endomorphism of $\mathbb{P}C^2$

3.1. The endomorphism and its iterated monodromy group. Consider the following map on $\mathbb{C}^2$:

$$F(z, p) = \left( \left( \frac{2z}{p+1} - 1 \right)^2, \left( \frac{p - 1}{p+1} \right)^2 \right).$$

It can be extended to an endomorphism of $\mathbb{P}C^2$ given in homogeneous coordinates by the formula

$$F[z : p : u] = [(2z - p - u)^2 : (p - u)^2 : (p + u)^2].$$

Note that this map has no points of indeterminacy, since $(2z - p - u)^2 = (p - u)^2 = (p + u)^2 = 0$ implies $p = u = z = 0$. The Jacobian of the map $F$ is

$$\begin{vmatrix}
4(2z - p - u) & 0 & 0 \\
-2(2z - p - u)^2 & 2(p - u) & 2(p + u) \\
-2(2z - p - u) & -2(p - u) & 2(p + u)
\end{vmatrix} = 32(2z - p - u)(p - u)(p + u),$$

hence the critical locus consists of three lines $2z - p - u = 0$, $p = u$, and $p + u = 0$. Their orbits under the action of $F$ are

$$\{2z - p - u = 0\} \mapsto \{z = 0\} \mapsto \{z = p\} \mapsto \{z = u\},$$

$$\{p = -u\} \mapsto \{u = 0\} \mapsto \{p = u\} \mapsto \{p = 0\} \mapsto \{p = u\}.$$

We see that the post-critical set of $F$ is the union of the six lines $z = 0$, $z = u$, $z = p$, $p = 0$, $p = u$, and $u = 0$. (Or, in affine coordinates, $z = 0$, $z = 1$, $z = p$, $p = 0$, $p = 1$, and the line at infinity.)

The map $F$ is a particular case of a general class of post-critically finite skew-product maps related to the Teichmüller theory of post-critically finite branched self-coverings of the sphere (Thurston maps). See [2], where the map $F$ was (somewhat implicitly) constructed, and [3, 4] where other different classes of similar examples are studied.

Denote by $J_2$ the Julia set of $F$, i.e., the set of points without neighborhoods on which the sequence $F^n$ is normal. Denote by $J_1$ the support of the measure of maximal entropy of $F$, which coincides with the attractor of backward iterations of $F$. Both sets are completely $F$-invariant and we have $J_1 \subset J_2$, see more in [6].

**Proposition 3.1.** The limit dynamical system of the iterated monodromy group of $F$ is topologically conjugate with the action of $F$ on $J_1$.

**Proof.** By [13, Theorem 5.5.3] (see also Theorem 2.2 in our paper), it is enough to construct an orbifold metric on a neighborhood of $J_1$ with respect to which $F$ is expanding. Let $U$ be an open subset of $\mathbb{C} \setminus \{0, 1\}$ containing the Julia set of the rational function $f(p) = \left( \frac{p-1}{p+1} \right)^2$ and such that $f^{-1}(U) \subset U$. Consider the inverse image $W$ of $U$ in $\mathbb{P}C^2$ under the projection map $(z, p) \mapsto p$ of $\mathbb{C}^2 \subset \mathbb{P}C^2$ onto $\mathbb{C}$. Note that the lines $p = 0$ and $p = 1$ are disjoint from $W$, hence the intersection points of the lines $z = p$, $z = 0$, $z = 1$, $p = 0$ and $p = 1$ do not belong to $W$. 

Consider the orbifold with the underlying space $W$, where the lines $z = 0$, $z = 1$ and $z = p$ are singular with the isotropy groups of order 2 uniformized in an atlas of the orbifold as rotations by $180^\circ$ in the $z$-planes (that are projected to the identity map on the $p$-plane). The function $F$ can be realized as a covering $F : W_1 \to W$ of a sub-orbispace $W_1$ of $W$, where $W_1$ is the orbifold with the underlying space $F^{-1}(W)$ and singular lines $z = 0$, $z = 1$, $z = p$, $z = p + 1$ of order two (also uniformized by a rotation in the $z$-planes).

The orbifold $W$ is a locally trivial bundle over the set $U$ with hyperbolic fibers (as the fundamental group of every fiber is free product of three copies of the group of order two). It follows from Proposition 3.2.2 and Theorem 3.2.15 of [8] that the orbifold $W$ is Kobayashi hyperbolic (i.e., that its universal covering is Kobayashi hyperbolic). Consequently, the embedding of orbifolds $\iota : W_1 \to W$ is contracting with respect to the Kobayashi metrics on $W_1$ and $W$, while the map $F : W_1 \to W$ is a local isometry. Theorem 2.2 finishes the proof.

An important property of the map $F$, greatly facilitating its study is a skew-product structure: the second coordinate $\left(\frac{p-1}{p+1}\right)^2$ of $F(z,p)$ depends only on $p$. See Figure 7 for the Julia set of $f(p) = \left(\frac{p-1}{p+1}\right)^2$ together with marked post-critical points 0 and 1.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure7}
\caption{Julia set of $\left(\frac{p-1}{p+1}\right)^2$}
\end{figure}

On the first coordinate we have a quadratic polynomial $h_p(z) = (2z/(p+1) - 1)^2$ in $z$, depending on the parameter $p$. This makes it possible, in particular, to draw the intersections of the Julia set of $F$ with the $z$-lines $p = p_0$. See Figures 8 and 9 where some slices of the Julia set of $F$ are shown. We will denote by $J_1(q)$ the intersection of the Julia set $J_1$ of $F$ with the line $p = q$.

The rational function $\left(\frac{p-1}{p+1}\right)^2$ has three fixed points

$q_0 \approx -0.6478 + 1.7214i, \quad q_0^* \approx -0.6478 - 1.7214i, \quad q_1 \approx 0.2956$. 
The corresponding polynomials $h_q(z) = \left(\frac{2z^q - 1}{\sqrt{1} + 1}\right)^2$, for $q \in \{q_0, q_0, q_1\}$, are post-critically finite, with the dynamics
\[
0 \mapsto 1 \mapsto q \mapsto q
\]
on the post-critical set. These polynomials and their iterated monodromy groups were studied in [2].

Consider the polynomial for $q_0 \approx -0.6478 + 1.7214i$. Its iterated monodromy group is generated by
\[
\begin{align*}
\alpha &= \sigma, \\
\beta &= (1, \alpha), \\
\gamma &= (\gamma, \beta),
\end{align*}
\]
where $\alpha$, $\beta$ and $\gamma$ are loops around 0, 1 and $p$, respectively (see a general formula for iterated monodromy groups of quadratic polynomials in [3]).

**Figure 8.** Slices of the Julia set $J_1$ of $F$
We can interpret the complement $M$ of the post-critical set of $F$ as the configuration space of pairs of complex numbers $(z, p)$ which are different from $\infty, 0, 1$ and from each other. We can identify the loops $\alpha, \beta, \gamma$ with the loops in the configuration space $M$ coming from $p$ staying and $z$ traveling along the loops $\alpha, \beta, \gamma$ in the line $p = q_0$.

Let $S$ and $P$ be the elements of the fundamental group of $M$ uniquely determined by the following relations (see [18] for details)

\[
\begin{align*}
P\alpha P^{-1} &= \beta \alpha^{-1}, \\
S\alpha S^{-1} &= \alpha \gamma \alpha^{-1} \alpha^{-1}, \\
P\beta P^{-1} &= \beta \alpha \alpha^{-1} \beta^{-1}, \\
S\beta S^{-1} &= \beta, \\
P\gamma P^{-1} &= \gamma, \\
S\gamma S^{-1} &= \alpha \gamma \alpha^{-1}.
\end{align*}
\]

Denote also $T = \gamma S^{-1} P^{-1} \beta \alpha$. We have then:

\[
\begin{align*}
T\alpha T^{-1} &= \alpha, \\
T\beta T^{-1} &= \gamma \beta \gamma^{-1}, \\
T\gamma T^{-1} &= \gamma \beta \gamma^{-1} \gamma^{-1}.
\end{align*}
\]

**Proposition 3.2.** The virtual endomorphism associated with $F$ is given by

\[
\phi(\alpha^2) = 1, \quad \phi(\beta) = 1, \quad \phi(\gamma) = \gamma, \\
\phi(\alpha^{-1} \beta \alpha) = \alpha, \quad \phi(\alpha^{-1} \gamma \alpha) = \beta, \\
\phi(S^2) = \beta \alpha \gamma S^{-1} P^{-1}, \quad \phi(T) = P, \quad \phi(P^2) = 1.
\]

**Proof.** The elements of the fundamental group of $M$ are uniquely determined by their action by conjugation on the normal subgroup generated by $\alpha, \beta$ and $\gamma$. Let us use this fact (following [2] and [18]) to compute the virtual endomorphism of $\pi_1(M)$ associated with the partial self-covering $F$.

The domain of the restriction of the virtual endomorphism onto $\langle \alpha, \beta, \gamma \rangle$ (associated with the first coordinate of the recursion (1)–(3)) is generated by $\alpha^2, \beta, \gamma, \alpha^{-1} \beta \alpha$. 

---

**Figure 9.** Slices of the Julia set $J_2$ of $F$. 

[Diagram of Slices of the Julia set]
and $\alpha^{-1}\gamma\alpha$. The action of the virtual endomorphism is given by
\[
\phi(\alpha^2) = 1, \quad \phi(\beta) = 1, \quad \phi(\gamma) = \gamma,
\]
\[
\phi(\alpha^{-1}\beta\alpha) = \alpha, \quad \phi(\alpha^{-1}\gamma\alpha) = \beta.
\]

The domain of the induced virtual endomorphism is generated by the above generators of the domain of $\phi$ and the automorphisms $S^2$, $P^2$ and $T = \gamma S^{-1}P^{-1}\beta\alpha$.

Denote $\tau = \gamma^{-1}\alpha^{-1}\beta^{-1}$. A direct computation shows that $\tau$ commutes with $S$ and with $P$. We also have
\[
\tau^{-1}S^{-1}P^{-1}\alpha PS\tau = \beta\gamma\beta^{-1}\gamma^{-1}\alpha\gamma\beta\gamma^{-1}\beta,
\]
and
\[
\tau^{-1}S^{-1}P^{-1}\beta PS\tau = \beta\gamma\beta^{-1}\beta^{-1},
\]
and
\[
\tau^{-1}S^{-1}P^{-1}\gamma PS\tau = \beta\gamma\beta^{-1}.
\]

We have
\[
\phi(S^2)\alpha\phi(S^{-2}) = \phi(S^2\alpha^{-1}\beta\alpha S^{-2}) = \alpha\gamma^{-2}\beta(\alpha\gamma^{-2})\beta(\alpha\gamma^{-2}) = \beta\gamma\beta^{-1}\gamma^{-1}\alpha\beta^{-1},
\]
\[
\phi(S^2)\beta\phi(S^{-2}) = \phi(S^2\alpha^{-1}\gamma\alpha S^{-2}) = \beta(\alpha\gamma^{-2}\gamma\alpha(\alpha\gamma^{-2})) = \beta\gamma\beta^{-1} \gamma^{-1},
\]
and
\[
\phi(S^2)\gamma\phi(S^{-2}) = \phi(S^2\gamma S^{-2}) = \phi(\alpha\gamma\alpha\gamma^{-1}\alpha^{-1}) = \beta\gamma^{-1}.
\]

We get that
\[
\phi(S^2) = \tau^{-1}S^{-1}P^{-1}.
\]

We have
\[
\phi(T)\alpha\phi(T^{-1}) = \phi(T\alpha^{-1}\beta\alpha T^{-1}) = \phi(\alpha^{-1}\gamma\beta\gamma^{-1}\alpha^{-1}) = \beta\alpha^{-1},
\]
\[
\phi(T)\beta\phi(T^{-1}) = \phi(T\alpha^{-1}\gamma\alpha T^{-1}) = \phi(\alpha^{-1}\gamma\beta\gamma^{-1}\gamma^{-1}\alpha) = \beta\alpha^{-1} \beta^{-1},
\]
and
\[
\phi(T)\gamma\phi(T^{-1}) = \phi(T\gamma T^{-1}) = \phi(\gamma\beta\gamma^{-1} \gamma^{-1}) = \gamma,
\]
which implies that $\phi(T) = P$.

It remains to compute $\phi(P^2)$.
\[
\phi(P^2)\alpha\phi(P^{-2}) = \phi(P^2\alpha^{-1}\beta\alpha P^{-2}) = \phi(\beta\alpha\beta^{-1}\beta^{-1}) = \alpha,
\]
\[
\phi(P^2)\beta\phi(P^{-2}) = \phi(P^2\alpha^{-1}\gamma\alpha P^{-2}) = \phi(\beta\alpha\beta^{-1}\alpha^{-1} \beta^{-1} \gamma\beta\alpha\beta^{-1}\alpha^{-1} \beta^{-1}) = \beta,
\]
which implies that $\phi(P^2) = 1$. \hfill \square

**Theorem 3.3.** The iterated monodromy group IMG $(F)$ is generated by the wreath recursion
\[
\alpha = \sigma(\beta, \beta^{-1}, \beta\alpha, \alpha^{-1}\beta^{-1}),
\]
\[
\beta = (1, \beta\alpha^{-1}, \alpha, 1),
\]
\[
\gamma = (\gamma, \beta, \gamma, \beta),
\]
\[
P = \pi,
\]
\[
S = \sigma\pi(P^{-1}\tau^{-1}, P^{-1}, S^{-1}\tau^{-1}, S^{-1}).
\]
where $\sigma = (12)(34)$, $\pi = (13)(24)$, and $\tau = \gamma^{-1}\alpha^{-1}\beta^{-1}$.

Note that it follows from the recursions that the elements $\alpha, \beta, \gamma, P$ of the iterated monodromy group are involutions, hence the wreath recursion can be written as

\[
\begin{align*}
\alpha &= \sigma(\beta, \beta\alpha, \alpha\beta), \\
\beta &= (1, \beta\alpha\beta, \alpha, 1), \\
\gamma &= (\gamma, \beta, \gamma, \beta), \\
P &= \pi, \\
S &= \sigma\pi(P\tau^{-1}, P, S^{-1}\tau^{-1}, S^{-1}),
\end{align*}
\]

where $\tau = \gamma\alpha\beta$.

Note that the subgroup $\langle \alpha, \beta, \gamma \rangle$ of $\text{IMG}(F)$ is self-similar. It is the subgroup generated by loops of the form $\ell\gamma\ell^{-1}$, where $\ell$ is a path, and $\gamma$ is a loop inside the plane $p = q$ for some $q \in \mathbb{C} \setminus \{0, 1\}$. We denote this subgroup $\mathcal{G}$.

**Proof.** Denote by $L_0$ the element $[\phi(1)1]$ of the biset $\mathfrak{M}_\phi$. Denote then

\[L_1 = \tau \cdot L_0, \quad R_0 = P \cdot L_0, \quad R_1 = P\tau \cdot L_0,\]

and order the basis of the biset associated with $\phi$ in the sequence $(L_0, L_1, R_0, R_1)$.

We have then

\[
\begin{align*}
\alpha \cdot L_0 &= \tau \cdot L_0 \cdot \phi(\tau^{-1}\alpha) = L_1 \cdot \phi(\beta\alpha\gamma\alpha) = L_1 \cdot \beta, \\
\alpha \cdot L_1 &= L_0 \cdot \phi(\alpha\tau) = L_0 \cdot \phi(\alpha\gamma^{-1}\alpha^{-1}\beta^{-1}) = L_0 \cdot \beta^{-1}, \\
\alpha \cdot R_0 &= P\tau \cdot L_0 \cdot \phi(\tau^{-1}P^{-1}\alpha P) = L_1 \cdot \phi(\beta\alpha\gamma\alpha^{-1}\beta^{-1}\alpha\beta) = L_1 \cdot \beta\alpha,
\end{align*}
\]

and

\[
\begin{align*}
\alpha \cdot R_1 &= P \cdot L_0 \cdot \phi(P^{-1}\alpha P\tau) = R_0 \cdot \phi(\alpha^{-1}\beta^{-1}\alpha\beta\alpha^{-1}\beta^{-1}) = R_0 \cdot \alpha^{-1}\beta^{-1}.\end{align*}
\]

Consequently,

\[\alpha = \sigma(\beta, \beta^{-1}, \beta\alpha, \alpha^{-1}\beta^{-1}).\]

We have

\[
\begin{align*}
\beta \cdot L_0 &= L_0 \cdot \phi(\beta) = L_0 \cdot 1, \\
\beta \cdot L_1 &= \tau \cdot L_0 \cdot \phi(\tau^{-1}\beta\tau) = L_1 \cdot \phi(\beta\alpha\gamma\beta^{-1}\alpha^{-1}\beta^{-1}) = L_1 \cdot \beta\alpha\beta^{-1}, \\
\beta \cdot R_0 &= P \cdot L_0 \cdot \phi(P^{-1}\beta P) = R_0 \cdot \phi(\alpha^{-1}\beta\alpha) = R_0 \cdot \alpha,
\end{align*}
\]

and

\[
\begin{align*}
\beta \cdot R_1 &= P\tau \cdot L_0 \cdot \phi(\tau^{-1}P^{-1}\beta P\tau) = R_1 \cdot \phi(\beta\alpha\gamma\alpha^{-1}\beta\alpha\gamma^{-1}\alpha^{-1}\beta^{-1}) = R_1 \cdot 1,
\end{align*}
\]

hence

\[\beta = (1, \beta\alpha\beta^{-1}, \alpha, 1).\]

We have

\[
\begin{align*}
\gamma \cdot L_0 &= L_0 \cdot \phi(\gamma) = L_0 \cdot \gamma, \\
\gamma \cdot L_1 &= \tau \cdot L_0 \cdot \phi(\tau^{-1}\gamma\tau) = L_1 \cdot \phi(\beta\alpha\gamma\alpha^{-1}\beta^{-1}) = L_1 \cdot \beta,
\end{align*}
\]

and also

\[\gamma \cdot R_0 = R_0 \cdot \gamma, \quad \gamma \cdot R_1 = R_1 \cdot \beta,
\]

since $P$ commutes with $\gamma$.

Since $\phi(P^2) = 1$ and $P$ commutes with $\tau$, we have

\[P = \pi.\]
Finally,
\[ S \cdot L_0 = P \tau \cdot L_0 \cdot \phi(\tau^{-1}P^{-1}S) = R_1 \cdot \phi(\beta \alpha \gamma \cdot P^{-2} \beta \alpha \cdot \alpha^{-1} \beta^{-1} P S \gamma^{-1} \cdot \gamma) = R_1 \cdot \phi(P^{-2} \beta \alpha \gamma \beta \alpha \cdot T^{-1} \cdot \gamma) = R_1 \cdot \beta \alpha P^{-1} \gamma = R_1 \cdot \tau^{-1}P^{-1}. \]

Since \( T \) commutes with \( \alpha \), we have:
\[ S \cdot L_1 = P \cdot L_0 \cdot \phi(P^{-1}S \tau) = R_0 \cdot \phi(P^{-2} \beta \alpha \cdot \alpha^{-1} \beta^{-1} P S \gamma^{-1} \cdot \alpha^{-1} \beta^{-1}) = R_0 \cdot \phi(P^{-2} \beta \alpha T^{-1} \alpha^{-1} \beta^{-1}) = R_0 \cdot \phi(\beta T^{-1} \beta^{-1}) = R_0 \cdot P^{-1}. \]

Since \( S \) commutes with \( \alpha \gamma \) and \( T \) commutes with \( \alpha \), we have:
\[ S \cdot R_0 = \tau \cdot L_0 \cdot \phi(\tau^{-1}SP) = L_1 \cdot \phi(\beta \alpha \gamma S^{-1} \cdot \alpha \gamma S^{-1} P^{-1} \beta \alpha \cdot \alpha^{-1} \beta^{-1} P^{-2}) = L_1 \cdot \phi(\beta S^2 \alpha \gamma \alpha^{-1} T \beta^{-1}) = L_1 \cdot \tau^{-1}S^{-1} P^{-1} P = L_1 \cdot \tau^{-1}S^{-1}, \]
and, since \( P \) and \( S \) commute with \( \tau \) and \( \gamma \) commutes with \( P \):
\[ S \cdot R_1 = L_0 \cdot \phi(SP \tau) = L_0 \cdot \phi(S^2 \gamma^{-1} \cdot \gamma S^{-1} P^{-1} \beta \alpha \cdot \alpha^{-1} \beta^{-1} P^2 \gamma^{-1} \alpha^{-1} \beta^{-1}) = L_0 \cdot \phi(S^2 \gamma^{-1} T \alpha^{-1} \beta^{-1} \gamma^{-1} \alpha^{-1} \beta^{-1} P^2) = L_0 \cdot \tau^{-1}S^{-1} P^{-1} \gamma^{-1} P \alpha^{-1} \beta^{-1} = L_0 \cdot \tau^{-1}S^{-1} \tau = L_0 \cdot S^{-1}, \]
which implies that
\[ S = \sigma \pi(P^{-1} \tau^{-1}, P^{-1}, S^{-1} \tau^{-1}, S^{-1}), \]
which finishes the proof. \( \square \)

Computation (for example using the GAP packages \([1] \) or \([12] \)) gives the following nucleus of IMG \((F)\):
\[ (10) \quad \{ 1, \alpha, \beta, \gamma, \alpha^\beta, \beta^\alpha, \gamma^\alpha, \gamma^\beta, \gamma^\alpha \beta, P, \gamma P, \gamma^\alpha \beta P \} \cup \{ \alpha \beta, \alpha \gamma, \beta \gamma, \tau, \alpha \tau, \beta \tau, S, \alpha S, S \beta, S \gamma, \alpha S \beta, S \gamma \beta, S \gamma \alpha, S \alpha \beta, \gamma \alpha \beta S, \tau S, \beta \tau S, \alpha \sigma \beta, Pa, P \beta, Pa \beta, P \tau \}^{\pm 1} \]
consisting of 60 elements.

3.2. **An index 2 extension of IMG \((F)\).**

**Definition 3.1.** Denote by \( \overline{G} \) the group generated by the elements
\[ \alpha = \sigma, \quad \beta = (1, \alpha, \alpha, 1), \quad \gamma = (\gamma, \beta, \gamma, \beta), \]
\[ a = \pi, \quad b = (a, a, a, a \alpha), \quad c = (\beta b, \beta c, \gamma c), \]
where \( \sigma = (12)(34) \) and \( \pi = (13)(24) \).

It is easy to check that \( a^2 = b^2 = c^2 = 1 \),
\[ a \beta a = \alpha \beta a, \quad b \gamma b = \beta \gamma b, \quad c \beta c = \gamma \beta \gamma, \]
and that \( a, b, c \) commute with the remaining generators \( \alpha, \beta, \gamma \).

Denote by \( \{ e_{00}, e_{01}, e_{10}, e_{11} \} \) the ordered basis of the biset in the definition of the group \( \overline{G} \). Then \( \sigma(e_{i,j}) = e_{i(1-j)} \) and \( \pi(e_{i,j}) = e_{(1-i),j} \).
Recall that by $G$ we denote the subgroup of $\text{IMG}(F)$ generated by $\alpha, \beta, \gamma$. We will see below that it is equivalent as a self-similar group to the subgroup of $G$ generated by $\alpha, \beta, \gamma$.

**Proposition 3.4.** The group $\text{IMG}(F)$ is isomorphic (as a self-similar group) to an index two subgroup of the group $G$. The isomorphism maps $\alpha, \beta, \gamma$ to the corresponding generators of $G$ and maps $S$ and $P$ to $ac\alpha\gamma$ and $\beta ba$, respectively. The bisets of the wreath recursions for $\text{IMG}(F)$ (as in Theorem 3.3) and $G$ (as in Definition 3.1) are identified with each other using the equalities

$$\{L_0 = e_{00}, \quad L_1 = e_{01} \cdot \beta, \quad R_0 = e_{10} \cdot a, \quad R_1 = e_{11} \cdot \beta a a\}.$$

**Proof.** Recall, that the group $G$ is generated by

$$\alpha = \sigma, \quad \beta = (1, \alpha, \alpha, 1), \quad \gamma = (\gamma, \beta, \gamma, \beta),$$

$$a = \pi, \quad b = (a, a, \alpha a, \alpha a), \quad c = (\beta b, \beta b, \gamma c, \gamma c).$$

We have

$$\beta^a = \alpha \beta \alpha, \quad \gamma^b = \beta \gamma \beta, \quad \beta^c = \gamma \beta \gamma,$$

and $a, b, c$ commute with the remaining generators $\alpha, \beta, \gamma$.

Conjugating the right hand side of the recursion by $(1, \beta, a, \beta \alpha a)$, we get

$$\alpha = \sigma(\beta, \beta, a \alpha \beta a, a \beta a a) = \sigma(\beta, \beta, \beta \alpha, \alpha \beta),$$

$$\beta = (1, \beta a \beta, a a a, 1) = (1, \beta a \beta, a, 1),$$

$$\gamma = (\gamma, \beta, a \gamma a, a \alpha \beta a a) = (\gamma, \beta, \gamma, \beta),$$

$$a = \pi(a, \alpha a, a, \alpha a),$$

$$b = (a, \beta a \alpha, a a a, a a \beta a \beta a a) = (a, \beta a \beta a a, a a, \alpha a),$$

$$c = (\beta b, \beta b \beta, a \gamma a a, a \alpha \beta a \gamma a a) = (\beta b, \beta a a \gamma a, a \gamma a a a a).$$

Let us show that $S = ac\alpha\gamma$, and $P = \beta ba$ satisfy the wreath recursion of Proposition 3.4.

We have, using commutation of the involutions $a$ and $\alpha$:

$$\beta ba = \pi(\alpha a a a a a, \alpha a a a, a a, \beta a \beta \alpha a \alpha a a) = \pi,$$

which agrees with the condition $P = \pi$.

We have

$$ac\alpha\gamma = \pi(a, \alpha a, a, \alpha a)(\beta b, \beta b, \gamma a a a, a \alpha \gamma a a a a)\sigma(\beta, \beta, \beta a, \alpha \beta)(\gamma, \beta, \gamma, \beta) = \pi \sigma(ab\gamma, ab\beta, \gamma a a a a, \gamma a a a a) = \pi \sigma(ab\gamma, ab\beta, \gamma a a a a, \gamma a a a a),$$

which also agrees with

$$S = \pi \sigma(P^{-1}r^{-1}, P^{-1}, S^{-1}r^{-1}, S^{-1}),$$

and finishes the proof. □
3.3. Properties of the groups $\mathcal{G}$, $\mathcal{G}$, and $\text{IMG}(F)$. Note that for every element $g \in \mathcal{G}$ and for every $e_{ij}$ we have $g \cdot e_{ij} = e_{ik} \cdot h$ for some $h \in \mathcal{G}$ and $k \in \{0, 1\}$. In other words, the self-similarity biset of $\mathcal{G}$ is a disjoint union ("direct sum") of the bisets $\mathcal{M}_0 = \{e_{00}, e_{01}\} \cdot \mathcal{G}$ and $\mathcal{M}_1 = \{e_{10}, e_{11}\} \cdot \mathcal{G}$.

Let us denote $E_i = \{e_{i0}, e_{i1}\}$ for $i \in \{0, 1\}$. We will also denote $E_0 = \{\emptyset\}$ and $E_{i_1, i_2, ..., i_n} = E_{i_1} E_{i_2} \cdots E_{i_n} \subset \{e_{00}, e_{01}, e_{10}, e_{11}\}^n$.

Then for every sequence $w = i_1 i_2 \ldots \{0, 1\}^\omega$ the subtree

$$T_w = \bigcup_{n \geq 0} E_{i_1, i_2, ..., i_n}$$

of the tree $T = \{e_{00}, e_{01}, e_{10}, e_{11}\}^*$ is invariant under the action of the group $\mathcal{G}$.

Let us identify $T_w$, for $w = x_1 x_2 \ldots$, with the binary tree $\{0, 1\}^*$ by the isomorphism

$$e_{x_1, i_1} e_{x_2, i_2} \cdots e_{x_n, i_n} \mapsto i_1 i_2 \cdots i_n.$$ Denote by $\mathcal{G}_w$, the restriction of the action of $\mathcal{G}$ onto the subtree $T_w$, seen as an automorphism group of the binary tree.

Then it follows directly from the wreath recursion for $\mathcal{G}$ that the group $\mathcal{G}_w$ is generated by automorphisms $\alpha_w, \beta_w, \gamma_w$ (images of $\alpha, \beta, \gamma$) which are defined by the following recursions.

$$\alpha_w = \sigma, \ \gamma_w = (\gamma_\mathcal{W}, \beta_\mathcal{W}),$$

and

$$\beta_w = \begin{cases} (1, \alpha_\mathcal{W}) & \text{if } x_1 = 0, \\ (\alpha_\mathcal{W}, 1) & \text{if } x_1 = 1. \end{cases}$$

Here $\mathcal{W} = x_2 x_3 \ldots$ is the shift of $w$.

The group $\mathcal{G}$ is the universal group of the family $\{\mathcal{G}_w : w \in \{0, 1\}^\omega\}$, i.e., $\mathcal{G}$ is the quotient of the free group $\langle \alpha, \beta, \gamma \mid \emptyset \rangle$ by the normal subgroup $R = \bigcap_{w \in \{0, 1\}^\omega} R_w$, where $R_w$ is the kernel of the natural epimorphism $\alpha \mapsto \alpha_w, \ \beta \mapsto \beta_w, \ \gamma \mapsto \gamma_w$ of the free group $\langle \alpha, \beta, \gamma \mid \emptyset \rangle$ onto the group $\mathcal{G}_w$. This follows from the fact that the subtrees $T_w$ cover the tree $T$.

**Proposition 3.5.** The group $\mathcal{G}$ is contracting with the nucleus $\mathcal{N} = \{1, \alpha, \beta, \gamma\}$.

**Proof.** By [13] Lemma 2.11.2 we have to show that sections of $\mathcal{N} \cdot \{\alpha, \beta, \gamma\}$ eventually belong to $\mathcal{N}$. But sections of the elements $\alpha, \beta$ in words of length more than one are trivial, while $\gamma$ is of order two. Hence, sections of the elements of $\mathcal{N} \cdot \{\alpha, \beta, \gamma\}$ in words of length two belong to $\mathcal{N}$. \(\square\)

The following proposition is a direct corollary of the wreath recursion defining the groups $\mathcal{G}$ and $\mathcal{K}$.

**Proposition 3.6.** Denote by $\mathcal{K}$ the self-similar group generated by

$$\tilde{a} = \sigma, \ \tilde{b} = (\tilde{a}, \tilde{a}), \ \tilde{c} = (\tilde{b}, \tilde{c}),$$

where $\sigma$ is the transposition. The map $g \mapsto \tilde{g}$ from $\mathcal{G}$ to $\mathcal{K}$ defined by

$$a \mapsto \tilde{a}, \ \ b \mapsto \tilde{b}, \ \ c \mapsto \tilde{c}$$

and $g \mapsto 1$ for $g \in \mathcal{G}$ extends to an epimorphism $\mathcal{G} \twoheadrightarrow \mathcal{K}$. Together with the map $e_{ij} \mapsto j$ it generates an epimorphism of bisets.

The image of the subtree $T_w$ under the action of an element $h \in \mathcal{G}$ is the subtree $T_{h(w)}$. 
Proposition 3.7. For any element $g$ of the kernel of the epimorphism $\mathcal{G} \to K$ there exists $n$ such that $g|_v \in \mathcal{G}$ for all words $v$ of length greater than $n$.

Proof. It is easy to check that the wreath recursion

$$\tilde{a} = \sigma, \quad \tilde{b} = (\tilde{a}, \tilde{a}), \quad \tilde{c} = (\tilde{b}, \tilde{c})$$

is contracting on the group given by the presentation $\tilde{K} = \langle \tilde{a}, \tilde{b}, \tilde{c} \mid (\tilde{a})^2 = (\tilde{b})^2 = (\tilde{c})^2 = 1 \rangle$ with the nucleus $\{1, \tilde{a}, \tilde{b}, \tilde{c}\}$.

It follows from [13, Proposition 2.13.2] that a product $\tilde{g}$ of the generators $\tilde{a}, \tilde{b}, \tilde{c}$ is trivial in $K$ if and only if there exists $n$ such that $\tilde{g}$ belongs to the kernel of the $n$th iterate of the wreath recursion on $\tilde{K}$. Let $g$ be a product of the generators of $\overline{K}$ and let $\tilde{g}$ be the word obtained from $g$ by removing all generators $\alpha, \beta, \gamma$ and applying the homomorphism $h \mapsto \tilde{h}$ to every letter $a, b, c$. Then the word $\tilde{g}$ represents a trivial element of $\overline{K}$. Let $n$ by such that $\tilde{g}$ belongs to the kernel of the $n$th iterate of the wreath recursion on $\tilde{K}$. Then it follows from the wreath recursion defining $\mathcal{G}$ and normality of $\mathcal{G}$ in $\overline{G}$ that the sections of $g$ in all words of length $v$ belong to $\mathcal{G}$. \hfill \Box

It follows directly from the interpretation of the generators $S$ and $P$ of $IMG(F)$ that the image of the subgroup $IMG(F) < \mathcal{G}$ in the quotient $K$ is isomorphic the iterated monodromy group of $f(p) = \left(\frac{p-1}{p+1}\right)^2$. Consequently, $IMG(F)$ is isomorphic to the self-similar group generated by

$$S = \sigma(P, S^{-1}), \quad P = \sigma. \tag{12}$$

The corresponding epimorphism of the self-similarity bisets acts by the rule $L_i \mapsto L$ and $R_i \mapsto R$, where $(L,R)$ is the ordered basis associated with the above recursion for $IMG(F)$.

Proposition 3.8. The transformation $\kappa$ of the space $\{L, R\}^{-\omega}$ changing in every sequence $w \in \{L, R\}^{-\omega}$ each letter $L$ to $R$ and vice versa induces a homeomorphism of the limit space of $IMG(F)$, corresponding to the complex conjugation on the Julia set of $f$.

Proof. Let us compute the iterated monodromy group $IMG(F)$ directly, in order to understand the geometric meaning of the elements $L$ and $R$.

The post-critical set of $f$ is $\{\infty, 0, 1\}$. Take $-1$ as the basepoint. Let $S$ and $P$ be the loops going in the positive direction around $0$ and around both $0$ and $1$, respectively, as it is shown on the top part of Figure 10. Connect the basepoint $1$ with its preimages $\pm i$ by straight segments.

The bottom part of Figure 10 shows the inverse images of the generators under the action of the rational function. We see that if we label the path connecting the basepoint $-1$ to $i$ by $\ell_k$ and the path connecting $-1$ to $-i$ by $\ell_1$, then the associated biset is defined by

$$P \cdot L = R, \quad P \cdot R = L,$$

and

$$S \cdot L = P \cdot R, \quad S \cdot L = R \cdot S^{-1},$$

which agrees with the wreath recursion (12).

Recall that a sequence $\ldots X^{(1)}X^{(0)} \in \{L,R\}^{-\omega}$ represents the point of the Julia set equal to the limit of the path $\ell_0\ell_1 \ldots$, where $\ell_k$ is a continuation of the path
Figure 10. Computation of $\text{IMG}(f)$

$\ell_{k-1}$ and is a lift of the path $\ell_{X^{(k)}}$ by the $k$th iteration of the rational function. Since the rational function \( \left( \frac{p-1}{p+1} \right)^2 \) has real coefficients, the basepoint $-1$ is real, and complex conjugation permutes the paths $\ell_k$ and $\ell_{\bar{\ell}}$, the transformation $\alpha$ maps a sequence corresponding to a point $z$ to the sequence corresponding to the conjugate point $\overline{z}$. \quad \square

Recall that $J_1(q)$ denotes the intersection of the Julia set $J_1$ with the $z$-line $p = q$.

**Proposition 3.9.** Each connected component of the limit space $J_G$ of the group $G$ consists of points represented by the sequences of the form $\ldots X^{(2)}_{i_2} X^{(1)}_{i_1} \ldots$, where $w = \ldots X^{(2)} X^{(1)} \in \{R, L\}^{-\omega}$ is fixed and $i_k \in \{0, 1\}$ are arbitrary. The connected component corresponding to $w \in \{R, L\}^{-\omega}$ is homeomorphic $J_1(p_0)$, where $p_0$ is the point of the Julia set of $f$ encoded by the sequence $w$.

**Proof.** Since $G$ is a self-similar subgroup of $\text{IMG}(F)$, the equivalence relation associated with $G$ is a sub-relation of the asymptotic equivalence relation associated with $\text{IMG}(F)$. The group $G$ changes only the indices of the symbols $X^{(k)}_{i_k}$, hence $G$-equivalent sequences are of the form $\ldots X^{(2)}_{i_2} X^{(1)}_{i_1} \ldots, \ldots X^{(2)}_{j_2} X^{(1)}_{j_1} \ldots$ for some $i_k, j_k \in \{0, 1\}$ and $X^{(k)} \in \{L, R\}$. Since $G$ is level-transitive on each of the subtrees, the image of the set $\{\ldots X^{(2)}_{i_2} X^{(1)}_{i_1} : \ldots i_2 i_1 \in \{0, 1\}^{-\omega}\}$ in the limit space of $G$ is connected (by the argument similar to that of [13, Section 3.5]), hence is a connected component.
It remains to show that the equivalence relation associated with $\text{IMG}(F)$ restricted to the set $\{ \ldots X_{i_2}^{(2)}X_{i_1}^{(1)} : \ldots i_2 i_1 \in \{0,1\}^n \}$ coincides with the restriction of the equivalence associated with $G$, i.e., that the group $\text{IMG}(F)$ does not introduce new identifications inside the connected components of the limit space of $G$. Suppose that the sequences $\ldots X_{i_2}^{(2)}X_{i_1}^{(1)} \ldots X_{j_2}^{(2)}X_{j_1}^{(1)}$ are equivalent with respect to the action of $\text{IMG}(F)$. It means that there exists a sequence $g_k$ of elements of $\text{IMG}(F)$ assuming a finite set of values and such that $g_k \cdot X_{i_k}^{(k)} = X_{j_k}^{(k)} \cdot g_{k-1}$ for all $k \geq 1$. The limit space of $\text{IMG}(F)$ has no singular points, since the rational function $f(p) = \left( \frac{p-1}{p+1} \right)^2$ is hyperbolic. It follows that the images of $g_k$ in $K$ are trivial. But this implies, by Proposition 3.7, that the elements $g_k$ belong to $G$, i.e., that the sequences are equivalent with respect to the action of the group $G$. \qed

3.4. The Schreier graphs of the groups $G_w$. Recall that for $v = i_1 i_2 \ldots i_n \in \{0,1\}^n$, we denote by $E_v$, the set of words of the form $e_{i_1 j_1} e_{i_2 j_2} \ldots e_{i_n j_n}$, where $j_1 j_2 \ldots j_n \in \{0,1\}^n$. We will denote the word $e_{i_1 j_1} e_{i_2 j_2} \ldots e_{i_n j_n}$ just $j_1 j_2 \ldots j_n$, for simplicity of notation. This notation agrees with the interpretation of $E_v$ as the $n$th level of the tree on which the group $G_w$ acts.

Let $v \in \{0,1\}^*$. Denote by $\Gamma_v$ the Schreier graph of the action of $G$ on the set $E_v$, i.e., the graph with the set of vertices $E_v$, in which to elements $w_1, w_2 \in E_v$ are adjacent if and only if $g(w_1) = w_2$ for some $g \in \{\alpha, \beta, \gamma\}$. We label the corresponding edge of $\Gamma_v$ by $g$.

Note that the graph $\Gamma_v$ is the Schreier graph of the action of the group $G_w$ on the $n$th level of the tree, where $w$ is any infinite word starting with $v$, and $n$ is the length of $v$.

Note that for every generator $g \in \{\alpha, \beta, \gamma\}$ of $G$ there exists a unique word $z_{g,v} \in E_v$ of length $n$ and a generator $h \in \{\alpha, \beta, \gamma\}$ such that $h|_{z_{g,v}} = g$. For the remaining pairs $h \in \{\alpha, \beta, \gamma\}$ and $u \in E_v$ we have $h|_u = 1$.

Namely, we have, for $v \in \{0,1\}^n$:

$$z_{\alpha,v} = \underbrace{00\ldots0}_{n-2\text{ times}}1x' \quad z_{\beta,v} = \underbrace{00\ldots0}_{n-1\text{ times}}1 \quad z_{\gamma,v} = \underbrace{00\ldots0}_n,$$

where $x' = 1 - x$ is the letter different from the last letter $x$ of $v$. If the word $v$ has length less than 2, one has to take the endings of length $|v|$ in the right hand sides of the equalities.

**Proposition 3.10.** Let $v, u \in \{0,1\}^*$ be arbitrary finite words. Consider for each word $w \in \{0,1\}^{|v|}$ a copy $\Gamma_{u,w}$ of the edge-labeled graph $\Gamma_u$. Connect, for each $g \in \{\alpha, \beta, \gamma\}$ and $w \in \{0,1\}^{|v|}$ the copy of $z_{g,u}$ in $\Gamma_{u,w}$ with the copy of $z_{g,u}$ in $\Gamma_{u,g(w)}$ by an edge labeled by the element $h \in \{\alpha, \beta, \gamma\}$ such that $h|_{z_{g,u}} = g$. The obtained graph is isomorphic to $\Gamma_{uw}$.

In this graph the vertex $z_{g,uv}$ is the copy of $z_{h,u}$ in $\Gamma_{u,w}$ for $h \in \{\alpha, \beta, \gamma\}$ and $w \in \{0,1\}^{|v|}$ such that $h|_w = g$.

Note that the copies $\Gamma_{u,w}$ of $\Gamma_u$ are connected in $\Gamma_{uw}$ in the same way as the vertices $w$ are connected in the graph $\Gamma_v$.

**Proof.** Let $w_1 w_2 \in X_{uv}$ and $|w_1| = |u|$, $|w_2| = |v|$. It follows from the definition of the words $z_{u,v}, z_{h,u}, z_{\gamma,u}$ that a generator $g \in \{\alpha, \beta, \gamma\}$ changes the end of length $|w_2|$ in the word $w_1 w_2$ only when $w_1 = z_{h,u}$ for some $h \in \{\alpha, \beta, \gamma\}$, and then we have $g(w_1 w_2) = w_1 h(w_2)$. In all the other cases $g(w_1 w_2) = g(w_1) w_2$, since $g|_{w_1} = 1$. \qed
In the case $|v| = 1$ we get the following inductive rule of constructing the graphs $\Gamma_u$.

**Corollary 3.11.** In order to get $\Gamma_{ux}$ one has to take two copies $\Gamma^{(0)}_u$ and $\Gamma^{(1)}_u$ of $\Gamma_u$ and connect by an edge the copies of the vertices $z_{\alpha,u}$. The obtained graph is $\Gamma_{ux}$. The vertex $z_{\alpha,ux}$ is the copy of $z_{\beta,u}$ in $\Gamma^{(1-x)}_u$. The vertex $z_{\gamma,ux}$ is the copy of $z_{\gamma,u}$ in $\Gamma^{(0)}_u$.

In the opposite case (when $|u| = 1$) we get the following rule.

**Corollary 3.12.** In order to get $\Gamma_{xv}$ one has to replace in $\Gamma_v$ each vertex $w$ by a pair of vertices $0w$ and $1w$, connected by an edge (labeled by $\alpha$), connect $(1-x)w$ to $(1-x)\alpha(w)$ by an edge (labeled by $\beta$), connect $0w$ to $0\gamma(w)$ and $1w$ to $1\beta(w)$ by edges labeled by $\gamma$.

We get nice pictures of the graphs $\Gamma_v$ when we draw the edges labeled by $\alpha$, $\beta$, and $\gamma$ in such a way that they have equal length and for every vertex $w$ the edge labeled by $\beta$ incident with $w$ (if it exists) is obtained from the edge labeled by $\alpha$ by rotation by $\pi/2$ around $w$, while the edge labeled by $\gamma$ is obtained from the edge labeled by $\alpha$ by rotation by $-\pi/2$. We can also use the opposite agreement (changing the signs of $\pi/2$ and $-\pi/2$). See, for instance, Figure 11 where some graphs $\Gamma_v$ are constructed in this way. Figure 12 shows different graphs $\Gamma_v$ for $|v| = 6$.

![Figure 11. Graphs $\Gamma_v$](image)

The rule from Corollary 3.12 is shown then on Figure 13. Note that in the transition from $\Gamma_v$ to $\Gamma_{1v}$ the relative position of the edges labeled by $\alpha$ (connecting $0w$ with $1w$), $\beta$ (connecting $0w$ with $0\alpha(w)$) and $\gamma$ (connecting $0w$ with $0\gamma(w)$ and $1w$ with $1\beta(w)$) is inverted. This can be corrected by taking mirror image of $\Gamma_{1v}$.

The inductive rule, shown on Figure 13 can be used to prove many properties of the graphs $\Gamma_v$, but we will use a more unified approach later.
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Figure 12. Graphs $\Gamma_v$ for $|v| = 6$

Figure 13. Inductive construction of $\Gamma_v$

3.5. **External angles.** The group generated by the binary adding machine (odometer) $\tau = \sigma(1, \tau)$ is the iterated monodromy group of the polynomial $z^2$. For every quadratic polynomial $h(z)$ the loop around infinity generates a self-similar cyclic subgroup of $\text{IMG} (h)$ equivalent as a self-similar group to the group generated by
the adding machine. The obtained embedding $\text{IMG}(z^2) \rightarrow \text{IMG}(h)$ induces a surjection from the circle (the Julia set of $z^2$) onto the Julia set of $h$, which agrees with the dynamics (i.e., is a semiconjugacy). This semiconjugacy coincides with the classical Caratheodory loop, see [11], i.e., to the extension to the boundary of the biholomorphic conjugacy from the action of $z^2$ on the complement of the unit disc to the action of $h$ on the complement of its filled Julia set (i.e., of the set of points that have bounded $h$-orbits).

By analogy with the Caratheodory loop, let us consider the subgroup $\mathcal{R} = \langle P, S, \tau \rangle < \text{IMG}(F)$ and the semiconjugacy of the corresponding limit spaces. The generators of the subgroup $\mathcal{R}$ are given by the wreath recursion

$P = \pi,$

$S = \pi\sigma(P\tau^{-1}, P, S^{-1}\tau^{-1}, S^{-1}),$

$\tau = \sigma(1, \tau, 1, \tau).$

**Proposition 3.13.** The nucleus of the group $\mathcal{R} = \langle P, S, \tau \rangle$ is the set

$\mathcal{N} = \{1, S, S^{-1}, P, \tau, \tau^{-1}, S\tau, S^{-1}\tau^{-1}, P\tau, P\tau^{-1}\}.$

**Proof.** It follows directly from the recursion (and the fact that $\tau$ commutes with $P$ and $S$) that $\mathcal{N}$ is a symmetric state-closed set (a subset $A$ of a self-similar group is called state-closed if for every $g \in A$ and $x \in X$ we have $g|x \in A$). We have to prove that the sections of the elements

$\{S, S^{-1}, \tau, \tau^{-1}, S\tau, S^{-1}\tau^{-1}\} \cdot \{S, \tau\}$

eventually belong to $\mathcal{N}$ (sections of $P$ are trivial in non-empty words, so we do not have to consider it). But this follows from the equalities

$S^2 = (PS^{-1}\tau^{-1}, PS^{-1}\tau^{-1}, S^{-1}P\tau, S^{-1}P\tau),$

$S\tau^{-1} = \pi(P\tau^{-1}, P\tau^{-1}, S^{-1}\tau^{-1}, S^{-1}\tau^{-1}),$

$S^2\tau = \sigma(PS^{-1}\tau^{-1}, PS^{-1}, S^{-1}P\tau, S^{-1}P),$

$\tau^2 = (\tau, \tau, \tau, \tau),$

$S\tau^2 = \pi\sigma(P, P\tau, S^{-1}, S^{-1}\tau).$

\[\square\]

See the Moore diagram of the nucleus on Figure 14.

Let $w = \ldots X_{i_1}^{(3)} X_{i_2}^{(2)} X_{i_1}^{(1)}$ be an element of $\{L_0, L_1, R_0, R_1\}^{-\omega}$, where $X^{(k)} \in \{L, R\}$ and $i_k \in \{0, 1\}$. Denote then

$p(w) = \ldots X^{(3)} X^{(2)} X^{(1)} \in \{L, R\}^{-\omega},$

and

$\theta(w) = \sum_{k=1}^{\infty} \frac{i_k}{2^k} \in \mathbb{R}/\mathbb{Z}.$

**Proposition 3.14.** Two sequences are asymptotically equivalent with respect to $\mathcal{R}$ if and only if they are equal to sequences $w_1, w_2 \in \{L_0, L_1, R_0, R_1\}^{-\omega}$ such that one of the conditions is satisfied

$(1) \quad p(w_1) = p(w_2), \quad \theta(w_1) = \theta(w_2),$
Figure 14. Moore diagram of the nucleus

\( \begin{align*}
  p(w_1) &= (RL)^{-\omega}, & p(w_2) &= (LR)^{-\omega}, & \theta(w_1) &= \theta(w_2) + \frac{2}{3}, \\
  p(w_1) &= (RL)^{-\omega}L, & p(w_2) &= (LR)^{-\omega}R, & \theta(w_1) &= \theta(w_2) + \frac{1}{3}, \\
  \text{(4) there exists a non-empty word } v \in \{L, R\}^* \text{ such that } & p(w_1) = (RL)^{-\omega}Lv, & p(w_2) = (LR)^{-\omega}Rv', & \theta(w_1) &= \theta(w_2) + \frac{1}{2|v|3},
\end{align*} \)

Proof. Note that the wreath recursion for \( \tau \) and \( P \) is written in terms of the self-similarity biset as

\( \begin{align*}
  \tau \cdot X_0 &= X_1, & \tau \cdot X_1 &= X_0 \cdot \tau, \\
  P \cdot R_i &= L_i, & P \cdot L_i &= R_i,
\end{align*} \)

where \( X \) is one of the symbols \( L, R \) and \( i \) is one of the symbols \( 0, 1 \).

Note also that \( \tau S = \pi(P, P, S^{-1}, S^{-1}) \), hence

\( \begin{align*}
  S \cdot L_i &= \tau^{-1} \cdot R_i \cdot P, & S \cdot R_i &= \tau^{-1} \cdot L_i \cdot S^{-1},
\end{align*} \)
which implies
\[ S^{-1} \cdot L_i = \tau \cdot R_i \cdot S, \quad S^{-1} \cdot R_i = \tau \cdot L_i \cdot P, \]
since \( S \) and \( \tau \) commute.

Examining the nucleus of the group on Figure 14, we see that left-infinite (infinite in the past) paths labeled by \((w_1, w_2)\) in its Moore diagram belong to one of the following types:

**I** The path travels inside the set \(\{1, \tau, \tau^{-1}\}\). In this case the we have \(p(w_1) = p(w_2)\) and \(\theta(w_1) = \theta(w_2)\), and every pair \((w_1, w_2)\) satisfying these two equalities can be obtained in this way.

**II** Its vertices alternatively belongs to the sets \(\{S, S\tau\}\) and \(\{S^{-1}, S^{-1}\tau^{-1}\}\).

If the last vertex of the path belongs to \(\{S, S\tau\}\), then \(p(w_1) = (RL)^{-\omega}\) and \(p(w_2) = (LR)^{-\omega}\). If the last vertex belongs to \(\{S^{-1}, S^{-1}\tau^{-1}\}\), then \(p(w_1) = (LR)^{-\omega}\) and \(p(w_2) = (RL)^{-\omega}\), which is symmetric with the first case.

**III** The last vertex of the path belongs to \(\{P, P\tau, P\tau^{-1}\}\). Then either \(p(w_1) = (RL)^{-\omega}L\) and \(p(w_2) = (LR)^{-\omega}R\) (if the previous vertex belongs to \(\{S, S\tau\}\)), or \(p(w_1) = (LR)^{-\omega}R\) and \(p(w_2) = (RL)^{-\omega}L\) (otherwise).

**IV** One of the vertices of the path (but not the last one) belong to \(\{P, P\tau, P\tau^{-1}\}\). Then \(p(w_1) = (RL)^{-\omega}L\) and \(p(w_2) = (LR)^{-\omega}R\), or \(p(w_1) = (LR)^{-\omega}R\) and \(p(w_2) = (RL)^{-\omega}L\), where \(v\) is obtained from \(v\) by changing the first letter.

In the first case of **II**, if \(w_1 = \ldots R_{i_4}L_{i_3}R_{i_2}L_{i_1}\) and \(w_2 = \ldots L_{j_4}R_{j_3}L_{j_2}R_{j_1}\), then for any \(n\) either
\[ S \cdot R_{i_2n}L_{i_2n-1} \ldots R_{i_2}L_{i_1} = L_{j_2n}R_{j_2n-1} \ldots L_{j_2}R_{j_1} \cdot S, \]
or
\[ S \cdot R_{j_2n}L_{j_2n-1} \ldots R_{j_2}L_{j_1} = L_{j_2n}R_{j_2n-1} \ldots L_{j_2}R_{j_1} \cdot S\tau, \]
or
\[ \tau S \cdot R_{i_2n}L_{i_2n-1} \ldots R_{i_2}L_{i_1} = L_{j_2n}R_{j_2n-1} \ldots L_{j_2}R_{j_1} \cdot S, \]
or
\[ \tau S \cdot R_{i_2n}L_{i_2n-1} \ldots R_{i_2}L_{i_1} = L_{j_2n}R_{j_2n-1} \ldots L_{j_2}R_{j_1} \cdot S\tau. \]

This implies that either
\[ \tau^{-1} \cdot L_{i_2n} \cdot \tau \cdot R_{i_2n-1} \ldots \tau^{-1} \cdot L_{i_2} \cdot \tau \cdot R_{i_2} \cdot S = L_{j_2n}R_{j_2n-1} \ldots L_{j_2}R_{j_1} \cdot S, \]
or
\[ \tau^{-1} \cdot L_{i_2n} \cdot \tau \cdot R_{i_2n-1} \ldots \tau^{-1} \cdot L_{i_2} \cdot \tau \cdot R_{i_2} \cdot S = L_{j_2n}R_{j_2n-1} \ldots L_{j_2}R_{j_1} \cdot S\tau, \]
or
\[ L_{i_2n} \cdot \tau \cdot R_{i_2n-1} \ldots \tau^{-1} \cdot L_{i_2} \cdot \tau \cdot R_{i_2} \cdot S = L_{j_2n}R_{j_2n-1} \ldots L_{j_2}R_{j_1} \cdot S, \]
or
\[ L_{i_2n} \cdot \tau \cdot R_{i_2n-1} \ldots \tau^{-1} \cdot L_{i_2} \cdot \tau \cdot R_{i_2} \cdot S = L_{j_2n}R_{j_2n-1} \ldots L_{j_2}R_{j_1} \cdot S\tau. \]

In all cases, as \(n \to \infty\) we get
\[ \theta(w_2) = \theta(w_1) + 1/2 - 1/4 + 1/8 - 1/16 + \cdots = \theta(w_1) + 1/3 \pmod{1}, \]
i.e., \(\theta(w_1) = \theta(w_2) + 2/3\).

In the first case of **III** we have \(w_1 = \ldots R_{i_4}L_{i_3}R_{i_2}L_{i_1}\), and \(w_2 = \ldots L_{j_4}R_{j_3}L_{j_2}R_{j_1}\), and we have
\[ \tau^{k_1} \cdot S \cdot R_{i_2n+1}L_{i_2n} \ldots R_{i_2}L_{i_1} = L_{j_2n+1}R_{j_2n} \ldots L_{j_2}R_{j_1} \cdot P\tau^{k_2}. \]
for \( k_1 \in \{0, 1\} \) and \( k_2 \in \{0, -1\} \). Then

\[
\tau^{k_1 - 1} \cdot L_{2n+1} \cdot \tau \cdot R_{2n} \cdots \tau^{-1} \cdot L_{13} \cdot \tau \cdot R_{12} \cdot \tau^{-1} \cdot R_{11} \cdot P = L_{j2n+1} R_{j2n} \cdots L_{j3} R_{j2} R_{j1} \cdot P^{k_2},
\]

which implies

\[
\theta(w_2) = \theta(w_1) - 1/2 + 1/4 - 1/8 + \cdots = \theta(w_1) - 1/3 \pmod{1},
\]

which proves case (III) of the proposition.

Consider now case (IV). We have \( w_1 = \ldots R_i L_i R_i L_i \cdots L_1 u \) and \( w_2 = \ldots L_j R_j L_j R_j I_1 u \) for some \( u, u' \in \{L_0, L_1, R_0, R_1\}^\ast \), and for every \( n \) we have

\[
\tau^{k_1} S \cdot R_{i2n+1} \cdots R_{i2} \cdots L_i \cdots L_1 u = L_{j2n+1} R_{j2n} \cdots L_{j3} R_{j2} R_{j1} u',
\]

for some \( k_1 \in \{0, 1\} \), hence

\[
\tau^{k_1 - 1} \cdot L_{2n+1} \cdot \tau \cdot R_{2n} \cdots \tau^{-1} \cdot L_{13} \cdot \tau \cdot R_{12} \cdot \tau^{-1} \cdot R_{11} \cdot P \cdot u = L_{j2n+1} R_{j2n} \cdots L_{j3} R_{j2} R_{j1} u'.
\]

This implies that

\[
\theta(w_2) = \theta(w_1) + \frac{1}{2|u|} \left( -\frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots \right) = \theta(w_1) - \frac{1}{2|u| \cdot 3},
\]

which finishes the proof. \( \square \)

Recall that the iterated monodromy group \( IMG(f) \) of \( f(p) = \left( \frac{p-\gamma}{p-1} \right)^2 \) is the image of the group \( G = \langle \alpha, \beta, \gamma, R, S \rangle \) under the natural epimorphism \( G \to K \) of self-similar groups described in Proposition 3.14. Note that the image of the group \( \mathcal{R} \) under this epimorphism is also \( IMG(f) \). The asymptotic equivalence relation defined by \( IMG(f) \) is generated by the identifications

\[
(RL)^{-\omega} \sim (LR)^{-\omega}, \quad (RL)^{-\omega} v \sim (LR)^{-\omega} v',
\]

where \( v \in \{L, R\}^\ast \) is arbitrary and \( v' \) is obtained from \( v \) by changing the first letter. This follows also from Proposition 3.14 just by ignoring the indices, i.e., the map \( \theta \).

The natural projection \( p : \{L_0, L_1, R_0, R_1\}^{-\omega} \to \{L, R\}^{-\omega} \) agrees with the equivalence relations defined by the group \( \mathcal{R} \) and its quotient \( IMG(f) \), so that \( p \) induces a surjective continuous map \( \tilde{p} : J_\mathcal{R} \to J_{IMG(f)} \) of the limit spaces. The fibers of the map \( \tilde{p} \) are circles by Proposition 3.14. It follows that the limit space of \( \mathcal{R} \) can be interpreted as the bundle over the Julia set of \( f \) of the Caratheodory loops around the \( p \)-slices \( J_1(p) \) of the Julia set of \( F \).

Let us describe the limit space of the group \( IMG(f) \) following Section 3.10. As the zero step approximation of the tile of the group take a rectangle. The vertices of the rectangle (which will correspond to the boundary points of the tile) are labeled by the sequences \((RL)^{-\omega} L, (LR)^{-\omega} R, (LR)^{-\omega}, (RL)^{-\omega}\) in the given cyclic order counterclockwise. Hence, the zero step approximation of the limit space will be the rectangle with two pairs of vertices identified. In order to get the next approximation of the tile one has to take two copies of the previous approximation, append \( R \) to end of the labels of one of them and append \( L \) to the labels of the other. After that one has to identify the point labeled by \((LR)^{-\omega} RR\) with the point labeled by \((RL)^{-\omega} LL\) and the point labeled by \((LR)^{-\omega} RL\) with the point labeled by \((RL)^{-\omega} LR\).

See the sixth approximation of the tile on the middle picture of Figure 15. The two pairs of the boundary points of the tile, which are identified in the limit space, are drawn close to each other, so that we get a picture approximating the limit
space. The left-hand side part of Figure 15 shows the indentifications of the vertices of 64 rectangles made in the process of construction the approximation of the limit space. Compare the obtained pictures with the Julia set of the rational function \( u \mapsto \frac{u^2+1}{u^2-1} \), shown on the right-hand side of Figure 15. This rational function is conjugate to \( f : p \mapsto \left( \frac{1-p}{1+p} \right)^2 \) via the identification \( p = \frac{u-1}{u+1} \).

Figure 15. The Julia set of \((p-1)^2/(p+1)^2\) and its combinatorial model.

If we apply just the identifications (1) of Proposition 3.14 to the space \( \{L_0, L_1, R_0, R_1\}^{-\omega} \), i.e., if we consider the limit space of \( \langle \tau \rangle \), then we will get the direct product of the Cantor set \( \{L, R\}^{-\omega} \) with the circle \( \mathbb{R}/\mathbb{Z} \).

Figure 16 shows the remaining identifications producing the limits space of \( R \). The arrows show which sequences \( w \in \{L, R\}^{-\omega} \) are identified, while the labels are the rotations applied to the corresponding circles. Namely, if we have an arrow from \( w_1 \) to \( w_2 \) labeled by \( \theta_0 \), then each point \( \theta \) of the circle above \( w_1 \) is identified with the point \( \theta + \theta_0 \) of the circle above \( w_2 \). The limit dynamical system acts on the Julia set of \( f \) as \( f \) (equivalently, as the shift on \( \{L, R\}^{-\omega} \)), and on the circles as the map \( \theta \mapsto 2\theta \). Note that the identifications described by Proposition 3.14 and Figure 16 are such that the resulting map on the limit space of \( R \) is well defined.

Figure 16. Building the space of external angles.
The embedding \( R < \text{IMG}(F) \) induces a semiconjugacy of the limit spaces
\[
\Phi : \mathcal{J}_R \to \mathcal{J}_{\text{IMG}(F)} = J_1.
\]

We call points of \( J_R \) external rays. We say that an external ray \( \zeta \in J_R \) lands on \((z, p) \in J_1\) if \( \Phi(\zeta) = (z, p) \). Points of \( J_R \) are encoded by pairs \((\theta, w)\), where \( \theta \in \mathbb{R}/\mathbb{Z} \) and \( w \in \{L, R\}^{-\omega} \) is a sequence representing a point \( p \in J_1 \). The coordinate \( \theta \) is called the angle of the external ray. Note that angle of an external ray may be not uniquely defined, since a point of the Julia set of \( f \) may be represented by different sequences. On the other hand, difference between angles of two external rays above the same point of the Julia set of \( f \) is well defined, since two circles are pasted to each other (in Proposition 3.14) using a rotation.

**Proposition 3.15.** Denote by \( q_1 \) the fixed point \( \approx 0.2956 \) of \( f(p) = \left(\frac{1-p}{1+p}\right)^2 \). If \( p_0 \) belongs to the backward orbit \( \bigcup_{n \geq 0} f^{-n}(q_1) \) of \( q_1 \), then there are two external rays landing on \((p_0, p_0)\). The difference of angles of these external rays is equal to \( \frac{1}{2^{n+1}} \), where \( k \) is the smallest integer such that \( f^k(p_0) = q_1 \). In all the other cases there is a unique ray landing on \((p_0, p_0)\).

Recall that the line \( z = p \) is contained in \( J_1 \) and is an \( F \)-invariant subset of the post-critical locus of \( F \).

**Proof.** It follows from the description of the asymptotic equivalence relation of the group \( \text{IMG}(f) \) that the fixed points of \( f \) are encoded in the limit space by the sequences \( R^{-\omega}, L^{-\omega} \) and \( (RL)^{-\omega} \sim (LR)^{-\omega} \). The transformation \( \kappa \) permutes the first two sequences and fixes the last one. Since \( \kappa \) corresponds to complex conjugation (see Proposition 3.8), we conclude that the real fixed point of \( f \) is encoded by the sequences \( (RL)^{-\omega} \sim (LR)^{-\omega} \). Hence, the points of the backward orbit of \( q_1 \) are the points encoded by the sequences of the form \( (RL)^{-\omega} v \), for \( v \in \{R, L\}^* \).

It follows from the dynamics on the post-critical set of \( F \) that the points of the line \( z = p \) are singular with the isotropy group a conjugate of \( \langle \gamma \rangle \). The wreath recursion in Theorem 3.3 implies that the points encoded by the sequences \( \ldots X_0^{(2)} X_0^{(1)} \) for \( X^{(k)} \in \{L, R\} \) have isotropy group \( \langle \gamma \rangle \), hence these sequences encode the points \( z = p_0 \), where \( p_0 \) is encoded by \( \ldots X^{(2)} X^{(1)} \) in the limit space of \( \text{IMG}(f) \).

Let us see to which sequences of the form \( \ldots X^{(2)} X^{(1)} \) the sequence \( \ldots X_0^{(2)} X_0^{(1)} \) can be equivalent. By Proposition 3.9 such two sequences, if they are equivalent with respect to \( \text{IMG}(F) \), then they are equivalent with respect to \( G \).

The nucleus of \( G \) for the wreath recursion of Theorem 3.3 is equal to
\[
\{\alpha, \beta, \gamma, (\alpha \beta)^\pm 1, \alpha^\beta, (\alpha \gamma)^\pm 1, (\beta \gamma)^\pm 1, \gamma^\alpha, \gamma^\beta, \gamma^\alpha \beta, \tau^\pm 1, (\alpha \tau)^\pm 1, (\beta \tau)^\pm 1\},
\]
see [10] on page 18.

We are interested in the left-infinite paths in the Moore diagram of the nucleus with the arrows labeled by pairs of the form \( (X_0, X_i) \) for \( X \in \{L, R\} \) and \( i \in \{0, 1\} \). Removing all the other arrows and removing all arrows which do not belong to any left-infinite path, we get the graph shown on Figure 17.

It follows that we have only the following non-trivial identifications
\[
(L_0 R_0)^{-\omega} R_0 X_0^{(n)} \ldots X_0^{(2)} X_0^{(1)} \sim (L_1 R_0)^{-\omega} R_1 X_1^{(n)} \ldots X_1^{(2)} X_1^{(1)},
\]
\[
(R_0 L_0)^{-\omega} L_0 X_0^{(n)} \ldots X_0^{(2)} X_0^{(1)} \sim (R_0 L_1)^{-\omega} L_0 X_0^{(n)} \ldots X_0^{(2)} X_0^{(1)},
\]
their shifts and the identification

\[ \ldots X_0^{(2)} X_0^{(1)} \sim \ldots X_1^{(2)} X_1^{(1)} \].

The last identification is trivial in terms of external angles.

It follows that the point \( z = p_0 \) is a landing point of one external ray to the slice \( p = p_0 \) of the Julia set of \( F \) except when \( p_0 \) is in the backward orbit of the fixed point \( q_1 \), when it is a landing point of exactly two external rays.

The remaining statements follow from Proposition \( 3.14 \). \( \square \)

Note that the points of the backward orbit of \( q_1 \) are precisely the points where different Fatou components of \( f \) touch each other, i.e., the points belonging to boundaries of two Fatou components of \( f \). This follows from the fact that the fixed point \( q_1 \) belongs to the boundaries of the Fatou components containing 0 and 1, and that every Fatou component of \( f \) is mapped by some iterations of \( f \) onto the Fatou components containing 0 and 1.

See Figure 18 where the external rays to the point \((q_1, q_1)\) are shown.

Figure 17.

Figure 18. External rays landing at \((q_1, q_1)\)
4. Matings

4.1. An amalgam of $\mathcal{G}$ with itself. Consider a copy $\mathcal{G}_1$ of the group $\mathcal{G}$ generated by

\begin{align}
\alpha_1 &= \sigma(\beta_1, \beta_1, \beta_1 \alpha_1, \alpha_1 \beta_1), \\
\beta_1 &= (1, \beta_1 \alpha_1 \beta_1, \alpha_1, 1), \\
\gamma_1 &= (\gamma_1, \beta_1, \gamma_1, \beta_1),
\end{align}

as in Theorem 3.3.

Let us conjugate the right hand side of the recursion defining $\alpha_1, \beta_1, \gamma_1$ by $\pi = (1,3)(2,4)$ (which corresponds to changing each $L_i$ by $R_i$ and vice versa). We get then an equivalent copy $\mathcal{G}_2$ of $\mathcal{G}$:

\begin{align}
\alpha_2 &= \sigma(\beta_2 \alpha_2, \alpha_2 \beta_2, \beta_2, \beta_2), \\
\beta_2 &= (\alpha_2, 1, 1, \beta_2 \alpha_2 \beta_2), \\
\gamma_2 &= (\gamma_2, \beta_2, \gamma_2, \beta_2).
\end{align}

Note that

$$\gamma_1 \alpha_1 \beta_1 = \sigma(1, \gamma_1 \alpha_1 \beta_1, 1, \gamma_1 \alpha_1 \beta_1).$$

Similarly,

$$\gamma_2 \alpha_2 \beta_2 = \sigma(1, \gamma_2 \alpha_2 \beta_2, 1, \gamma_2 \alpha_2 \beta_2),$$

which implies that $\gamma_1 \alpha_1 \beta_1 = \gamma_2 \alpha_2 \beta_2 = \pi$.

Denote by $\hat{\mathcal{G}}$ the group generated by the set $\mathcal{G}_1 \cup \mathcal{G}_2$.

Lemma 4.1. The elements $\beta_1$ and $\beta_2$ act non-trivially on disjoint sets of words and hence commute. The same is true for $\gamma_1$ and $\gamma_2$.

Proof. The first statement follows directly from the wreath recursion. The second statement follows from the first. \hfill \square

Computer computation using GAP shows that $\hat{\mathcal{G}}$ is contracting with the following nucleus of 122 elements

$$\{1, \alpha_i, \beta_i, \gamma_i, \alpha_i^{\beta_i}, \gamma_i^{\alpha_i}, \beta_i^{\gamma_i}, \beta_i^{\alpha_i}, \gamma_i^{\beta_i}, \beta_i^{\alpha_i \beta_i},$$

$$\alpha_i^{\beta_i}, \beta_i^{\gamma_i}, \gamma_i^{\alpha_i \beta_i}, \beta_i^{\gamma_i}, \gamma_i^{\beta_i}, \beta_i^{\alpha_i \gamma_i},$$

$$\alpha_i^{\beta_i}, \alpha_i^{\gamma_i}, \beta_i^{\alpha_i}, \beta_i^{\gamma_i}, \gamma_i^{\alpha_i}, \beta_i^{\gamma_i}, \alpha_i^{\gamma_i},$$

$$\tau, \beta_i \alpha_i \beta_i, \beta_i \alpha_i \alpha_i, \alpha_i \beta_i \alpha_i \beta_i, \beta_i \gamma_i \alpha_i, \alpha_i \tau, \beta_i \tau, \alpha_i B, \beta_i B, \beta_i \alpha_i B,$$

$$\beta_i \tau \beta_i, \tau, \alpha_i C, \tau, \alpha_i C \tau, \beta_i C \tau \}^{\pm 1}.$$

Here $\{i, j\} = \{0, 1\}$ and $B = \beta_1 \beta_2, C = \gamma_1 \gamma_2$.

For the definition of $J_1(p)$ see the remark before Proposition 3.9.

Proposition 4.2. The connected components of the limit space of the group $\hat{\mathcal{G}}$ are obtained by taking the slices $J_1(p)$ and $J_1(\overline{p})$ of the Julia set of $F$ and gluing one to the other along the Carathéodory loop, where the external ray landing on $(p, p)$ is identified with the external ray landing on $(\overline{p}, \overline{p})$. If there are two external rays landing on $(p, p)$ (i.e., when $p$ belongs to the grand orbit of the real fixed point $q_1$ of $f$), then the Carathéodory loops are aligned in such a way that only one external ray landing on $(p, p)$ is identified with the external ray landing on $(\overline{p}, \overline{p})$. 

Equivalently, the connected components of the limit space of \( \hat{G} \) are obtained by taking two copies of \( J_1(p) \), and gluing the Caratheodory loop around one copy of \( J_1(p) \) to its mirror reflection along the diameter containing a ray landing on \((p,p)\).

The image of the Caratheodory loop will be a curve going through every point of the pillowcase \( \mathbb{C}/\mathcal{H} \) induced by the embedding \( \langle \tau \rangle < \text{IM}(F) \).

**Proof.** It follows from Propositions 3.9 and 3.8 that the connected components of the limit space of \( \hat{G} \) are obtained by gluing together the slice \( J_1(p) \) of the Julia set of \( F \) with the slice \( J_1(\bar{p}) \). Since \( \gamma_1 \alpha_1 \beta_1 = \gamma_2 \alpha_2 \beta_2 = \tau \), the Caratheodory loop around \( J_1(p) \) is identified with the Caratheodory loop around \( J_1(\bar{p}) \) by the map induced by the map \( \cdots X_t^{(2)} X_{t_1}^{(1)} \mapsto \cdots Y_t^{(2)} Y_{t_1}^{(1)} \) on the corresponding sets of sequences. Here \( \cdots i_{j_1} \in \{0,1\}^{-\omega} \) encodes the points of the circle \( J_{\tau_j} \) and \( \cdots X^{(2)} X^{(1)} = \kappa(\cdots Y^{(2)} Y^{(1)}) \) is the sequence encoding the point \( p \). The identification rule of the circles of external rays follows then from Proposition 3.14. \( \square \)

Classically (see [11]) the identifications described in Proposition 4.2 are called “matings”. The only difference is that in the case of the classical mating the polynomials are monic, and the corresponding Caratheodory loops are reflected with respect to the real axis (which corresponds to the angle 0 external ray of a special fixed point of the polynomial). In our case we reflect the Caratheodory loop with respect to the diameter containing a ray landing on the points of the invariant line \((p,p)\). Since there can be two rays landing on \((p,p)\), there are two possible “rotated matings”.

As particular cases of components described in Proposition 4.2 we get the mating of the polynomial \( h_{q_0}(z) = \left( \frac{z}{q_0 + 1} - 1 \right)^2 \), for \( q_0 \approx -0.6478 + 1.7214i \), with itself (see a detailed analysis of this mating in [11]), and two rotated matings of the polynomial \( h_{q_1}(z) = \left( \frac{z}{q_1 + 1} - 1 \right)^2 \), for \( q_1 \approx 0.2956 \) with itself.

### 4.2. The self-similarity biset of \( \hat{G} \)

Let \( (\hat{L}_0, \hat{L}_1, \hat{R}_0, \hat{R}_1) \) be the ordered basis of the self-similarity \( \hat{G} \)-biset corresponding to the original wreath recursion (13)–(18).

Then the \( G_1 \)-biset \( \{\hat{L}_0, \hat{L}_1, \hat{R}_0, \hat{R}_1\} \cdot G_1 \) is naturally isomorphic to the self-similarity biset of \( G \) (if we identify \( G_1 \) with \( G \) in the natural way). The isomorphism is given by the map

\[
\hat{L}_0 \mapsto L_0, \quad \hat{L}_1 \mapsto L_1, \quad \hat{R}_0 \mapsto R_0, \quad \hat{R}_1 \mapsto R_1,
\]

where \( \{L_0, L_1, R_0, R_1\} \) is the usual basis of the self-similarity biset of \( G \).

The \( G_2 \)-biset \( \{\hat{L}_0, \hat{L}_1, \hat{R}_0, \hat{R}_1\} \cdot G_2 \) is also isomorphic to the self-similarity biset of \( G \) via the mapping

\[
\hat{L}_0 \mapsto R_0, \quad \hat{L}_1 \mapsto R_1, \quad \hat{R}_0 \mapsto L_0, \quad \hat{R}_1 \mapsto L_1.
\]

The self-similarity biset of \( \hat{G} \) is a direct sum (i.e., disjoint union) of the biset \( \mathcal{L} = \{\hat{L}_0, \hat{L}_1\} \cdot \hat{G} \) and \( \mathcal{R} = \{\hat{R}_0, \hat{R}_1\} \cdot \hat{G} \). Also denote for \( i = 1, 2 \)

\[
\mathcal{L}_i = \{\hat{L}_0, \hat{L}_1\} \cdot G_i, \quad \mathcal{R}_i = \{\hat{R}_0, \hat{R}_1\} \cdot G_i.
\]

Let us identify \( G_1 \) and \( G_2 \) with \( G \) in a natural way, so that \( \mathcal{L}_i \) and \( \mathcal{R}_i \) become \( G \)-bisets. Note that then \( \mathcal{L}_1 = \mathcal{R}_2 \) and \( \mathcal{L}_2 = \mathcal{R}_1 \).

Let \( a = \pi(\alpha a, \beta a, \gamma a, \delta a) \), which is the element \( a = \pi \) of \( \overline{G} \) written with respect to the basis \( L_0 = e_{00}, L_1 = e_{01} \cdot \beta, R_0 = e_{10} \cdot a, R_1 = e_{11} \cdot \beta \alpha_1 a \), see Proposition 3.4.
Then \( a \) induces an automorphism of \( \mathcal{G} \) by conjugation:

\[
\alpha^a = \alpha, \quad \beta^a = \beta^\alpha, \quad \gamma^a = \gamma.
\]

Let \( \mathcal{M}_0 = \{e_{00}, e_{01}\} \cdot \mathcal{G} \) and \( \mathcal{M}_1 = \{e_{10}, e_{11}\} \cdot \mathcal{G} \) be the natural \( \mathcal{G} \)-bisets, see Subsection 3.3.

**Proposition 4.3.** Let \( v = X^{(1)} X^{(2)} \ldots X^{(n)} \in \{\mathcal{L}, \mathcal{R}\}^n \), denote

\[ x_i = \begin{cases} 
0 & \text{if } X^{(i)} = \mathcal{L}, \\
1 & \text{if } X^{(i)} = \mathcal{R}.
\end{cases} \]

Then the biset \( X^{(1)}_1 \otimes X^{(2)}_1 \otimes \cdots \otimes X^{(n)}_1 \) is isomorphic to the biset

\[ \mathcal{M}_{x_1} \otimes \mathcal{M}_{x_1+x_2} \otimes \mathcal{M}_{x_2+x_3} \otimes \cdots \otimes \mathcal{M}_{x_{n-1}+x_n} \cdot a^\gamma, \]

where addition of indices is modulo two.

The \( \mathcal{G} \)-biset \( X^{(1)}_2 \otimes X^{(2)}_2 \otimes \cdots \otimes X^{(n)}_2 \) is isomorphic to the biset

\[ \mathcal{M}_{1+x_1} \otimes \mathcal{M}_{1+x_1+x_2} \otimes \mathcal{M}_{x_2+x_3} \otimes \cdots \otimes \mathcal{M}_{x_{n-1}+x_n} \cdot a^{1+x_n}. \]

**Proof.** We have \( L_0 = e_{00}, L_1 = e_{01} \cdot \beta_1 \), so \( \mathcal{L}_1 = \mathcal{M}_0 \). We have \( R_0 = e_{10} \cdot a \) and \( L_1 = e_{11} \cdot \beta_0 a \), hence \( \mathcal{R}_1 \) is identified with \( \mathcal{M}_1 \cdot a \). Consequently, \( \mathcal{L}_2 \) is isomorphic to \( \mathcal{M}_1 \cdot a \), and \( \mathcal{R}_2 \) is isomorphic to \( \mathcal{M}_0 \).

Note that it follows from the wreath recursions defining \( \mathcal{G} \) that \( a \cdot \mathcal{M}_0 \cong \mathcal{M}_1 \) and \( a \cdot \mathcal{M}_1 \cong \mathcal{M}_0 \).

Consequently, the biset \( X^{(1)}_1 \otimes X^{(2)}_1 \otimes \cdots \otimes X^{(n)}_1 \) is isomorphic to the biset

\[ \mathcal{M}_{x_1} \cdot a^{x_1} \otimes \mathcal{M}_{x_2} \cdot a^{x_2} \otimes \cdots \otimes \mathcal{M}_{x_n} \cdot a^{x_n}, \]

which is isomorphic to \( \mathcal{M}_{x_1} \otimes \mathcal{M}_{x_1+x_2} \otimes \mathcal{M}_{x_2+x_3} \otimes \cdots \otimes \mathcal{M}_{x_{n-1}+x_n} \cdot a^{x_n}. \)

Similarly, the \( \mathcal{G} \)-biset \( X^{(1)}_2 \otimes X^{(2)}_2 \otimes \cdots \otimes X^{(n)}_2 \) is isomorphic to \( \mathcal{M}_{1+x_1} \cdot a^{1+x_1} \otimes \mathcal{M}_{1+x_2} \cdot a^{1+x_2} \otimes \cdots \otimes \mathcal{M}_{x_n} \cdot a^{1+x_n}, \)

which is isomorphic to \( \mathcal{M}_{1+x_1} \otimes \mathcal{M}_{1+x_2} \otimes \mathcal{M}_{x_2+x_3} \otimes \cdots \otimes \mathcal{M}_{x_{n-1}+x_n} \cdot a^{x_n}. \)

\[ \square \]

**4.3. A virtually abelian subgroup of \( \mathcal{G} \).** Denote \( A = \alpha_2, B = \beta_1 \beta_2, \) and \( C = \gamma_1 \gamma_2 \). We have then \( \alpha_1 = CAB = BAC. \)

Let us pass to the basis

\[ x_1 = \hat{L}_0, \quad x_2 = \hat{L}_1 \cdot \beta_1, \quad x_3 = \hat{R}_0, \quad x_4 = \hat{R}_1 \cdot \beta_2, \]

i.e., conjugate the wreath recursion defining \( \mathcal{G} \) by \( (1, \beta_1, 1, \beta_2) \). We get then

\[
CAB = \alpha_1 = \sigma(1, 1, AC, CA),
\]

\[
\beta_1 = (1, CAB, CAB, 1),
\]

\[
\gamma_1 = (\gamma_1, \beta_1, \gamma_1, \beta_1),
\]

\[
A = \alpha_2 = \sigma(BA, AB, 1, 1),
\]

\[
\beta_2 = (A, 1, 1, A),
\]

\[
\gamma_2 = (\gamma_2, \beta_2, \gamma_2, \beta_2).
\]

It follows that

\[
A = \sigma(BA, AB, 1, 1),
\]

\[
B = (A, CAB, CAB, A),
\]

\[
C = (C, B, C, B).
\]
Proposition 4.4. The subgroup $\mathcal{H} = \langle A, B, C \rangle$ of $\hat{G}$ is isomorphic as a self-similar group to the group of affine transformations of $C$ of the form $z \mapsto \pm z + q$, where $q \in \mathbb{Z}[i]$. The isomorphism identifies $A, B$ and $C$ with the affine transformations

$$z \cdot A = -z + 1, \quad z \cdot B = -z + 1 + i, \quad z \cdot C = -z,$$

the basis of the self-similarity biset is identified with the affine transformations

\begin{align*}
    z \cdot x_1 &= \frac{1}{1 + i} z = \frac{1 - i}{2} z, \\
    z \cdot x_2 &= \frac{1}{1 + i}(-z + i) = -\frac{1 - i}{2} z + \frac{1 + i}{2}, \\
    z \cdot x_3 &= \frac{1}{1 - i} z = \frac{1 + i}{2} z, \\
    z \cdot x_4 &= \frac{1}{1 - i}(-z + 1) = -\frac{1 + i}{2} z + \frac{1 + i}{2}.
\end{align*}

For identification of permutational bisets with sets of (partial) transformations, see Subsection 2.2. The biset structure comes from pre- and post-composition with the group action.

Proof. We have $x_4 = A \cdot x_3$ and $A \cdot x_4 = x_3$,

$$z \cdot A \cdot x_1 = -\frac{1 - i}{2} z + \frac{1 - i}{2} = -\left( -\left( -\frac{1 - i}{2} z + \frac{1 + i}{2} \right) + 1 + i \right) + 1 = x_2 \cdot BA,$$

hence $A \cdot x_1 = x_2 \cdot BA$ and $A \cdot x_2 = x_1 \cdot AB$, which agrees with the wreath recursion.

We have

$$z \cdot B \cdot x_1 = \frac{1 - i}{2}(-z + 1 + i) = -\frac{1 - i}{2} z + 1 = z \cdot x_1 \cdot A,$$

$$z \cdot B \cdot x_2 = -\frac{1 - i}{2}(-z + 1 + i) + \frac{1 + i}{2} = \frac{1 - i}{2} z - \frac{1 - i}{2} = z \cdot x_2 \cdot CAB,$$

since $z \cdot CAB = -z + i$,

$$z \cdot B \cdot x_3 = \frac{1 + i}{2}(-z + 1 + i) = -\frac{1 + i}{2} z + i = z \cdot x_3 \cdot CAB,$$

and

$$z \cdot B \cdot x_4 = -\frac{1 + i}{2}(-z + 1 + i) + \frac{1 + i}{2} = \frac{1 + i}{2} z + \frac{1 - i}{2} = z \cdot x_4 \cdot A,$$

which also agrees with the wreath recursion.

Finally, it is easy to check that $z \cdot C \cdot x_1 = z \cdot x_1 \cdot C$, $z \cdot C \cdot x_3 = z \cdot x_3 \cdot C$ and

$$z \cdot C \cdot x_2 = \frac{1 - i}{2} z + \frac{1 + i}{2} = z \cdot x_2 \cdot B,$$

and

$$z \cdot C \cdot x_4 = \frac{1 + i}{2} z + \frac{1 + i}{2} = z \cdot x_4 \cdot B.$$

$\square$
4.4. The limit dynamical system of \( \mathcal{H} \).

**Proposition 4.5.** The limit \( \mathcal{H} \)-space \( \mathcal{X}_\mathcal{H} \) is homeomorphic to the direct product of \( \mathbb{C} \) with the Cantor set \( \{L, R\}^{-\omega} \) with the natural (right) action of \( \mathcal{H} \) on \( \mathbb{C} \) and trivial action on \( \{L, R\}^{-\omega} \). The self-similarity structure is given by

\[
(z, \ldots y_2 y_1) \otimes x_1 = \left( \frac{1 - i}{2} z, \ldots y_2 y_1 L \right),
\]

\[
(z, \ldots y_2 y_1) \otimes x_2 = \left( \frac{1 - i}{2} z + \frac{1 + i}{2}, \ldots y_2 y_1 L \right),
\]

\[
(z, \ldots y_2 y_1) \otimes x_3 = \left( \frac{1 + i}{2} z, \ldots y_2 y_1 R \right),
\]

\[
(z, \ldots y_2 y_1) \otimes x_4 = \left( \frac{1 + i}{2} z + \frac{1 + i}{2}, \ldots y_2 y_1 R \right).
\]

**Proof.** Direct corollary of Proposition 4.4 and Theorem 2.1. \( \square \)

The orbispace \( \mathbb{C}/\mathcal{H} \) is a flat surface homeomorphic to the sphere with four singular points, which are the images of the fixed points \( 1/2, (1 + i)/2, 0, \) and \( i/2 \) of the transformations \( A, B, C \) and \( CAB \), respectively. Let us denote these singular points by \( Z_A, Z_B, Z_C \) and \( Z_{CAB} \), respectively. A fundamental domain \( D \) of \( \mathcal{H} \) is the rectangle with the vertices \( i/2, 0, 1 \) and \( 1 + i/2 \).

The natural map \( D \to \mathbb{C}/\mathcal{H} \) folds this rectangle along the segment connecting \( 1/2 \) and \( (1+i)/2 \) in two, so that we get a “pillowcase”, whose vertices are the points \( Z_A, Z_B, Z_C \) and \( Z_{CAB} \), see Figure 19.

![Figure 19. Fundamental domain of \( \mathcal{H} \)](image)

The following is a direct corollary of the description of the limit \( \mathcal{H} \)-space given in Proposition 4.5.

**Corollary 4.6.** The limit space \( \mathcal{J}_\mathcal{H} \) is homeomorphic to the direct product \( \mathbb{C}/\mathcal{H} \times \{L, R\}^{-\omega} \). The shift map \( s : \mathcal{J}_\mathcal{H} \to \mathcal{J}_\mathcal{H} \) acts by the rule

\[
s(z, \ldots y_2 y_1) = \begin{cases} 
((1 - i) z, \ldots y_3 y_2), & \text{if } y_1 = L, \\
((1 + i) z, \ldots y_3 y_2), & \text{if } y_1 = R.
\end{cases}
\]

Here \( z, (1 - i) z, \) and \( (1 + i) z \) are complex numbers representing the corresponding points of the orbispace \( \mathbb{C}/\mathcal{H} \).
4.5. Schreier graphs of $\mathcal{H}$ and $\hat{\mathcal{G}}$. Consider the natural (right) action of $\mathcal{H}$ on $\mathbb{C}$. Take the basepoint $\xi = \frac{1+i}{2}$. It has trivial stabilizer in $\mathcal{H}$, hence we can consider the orbit of $\xi$ as a vertex set of the left Cayley graph $\Gamma_{\mathcal{H}}$ of $\mathcal{H}$ with respect to the generating set $A, B, C, CAB$. See the Cayley graph on Figure 20. Here edges corresponding to the generators $A, B, C$, and $CAB$ are orange, blue, green, and red, respectively.

![Figure 20. Cayley graph of $\mathcal{H}$](image)

Denote by $\mathcal{H}_n$ the subgroup of $\mathcal{H}$ consisting of the affine transformations of the form $z \mapsto \pm z + q$, where $q \in \mathbb{Z}[i]$ is a Gaussian integer divisible by $(1+i)^n$.

It follows then from Proposition 4.4 that the Schreier graph of the action of $\mathcal{H}$ on the $n$th level of the tree consists of $2^n$ copies of the graph $\Gamma_n(\mathcal{H}) := \Gamma_{\mathcal{H}}/\mathcal{H}_n$. A fundamental domain of $\mathcal{H}_n$ is the rectangle with vertices $0, (1+i)^n, i(1+i)^n/2$, and $(1+i/2)(1+i)^n$. Note that its sides are either parallel to the real and imaginary axis (for even $n$) or parallel to the diagonals $\Re(z) = \Im(z)$ and $\Re(z) = -\Im(z)$.

See Figure 21 for the Schreier graphs $\Gamma_5(\mathcal{H})$ and $\Gamma_4(\mathcal{H})$. Note that they have four loops, which we will usually omit in the sequel, since we will consider simplicial Schreier graphs only.

Figure 22 shows a more convenient way of drawing the Schreier graphs $\Gamma_n(\mathcal{H})$ and their subgraphs. Here the graphs $\Gamma_5(\mathcal{H})$ and $\Gamma_4(\mathcal{H})$ are drawn inside fundamental domains of the action of $\mathcal{H}_n$ on $\mathbb{C}$. In order to get the Schreier graphs one has to fold the rectangle into a square pillowcase (which corresponds to taking the quotient $\mathbb{C}/\mathcal{H}_n$).

Denote for $v = X^{(1)}X^{(2)}\ldots X^{(n)} \in \{L,R\}^n$ by $\Gamma_{1,v}$ and $\Gamma_{2,v}$ the corresponding connected components of the Schreier graphs of the actions of $\mathcal{G}_1$ and $\mathcal{G}_2$ on the $n$th level of the tree. More precisely, they are the Schreier graphs of the actions of $\mathcal{G}_i$ on the spaces of right orbits of the bisets

$$\{X_0^{(1)}, X_1^{(1)}\} \otimes \{X_0^{(2)}, X_1^{(2)}\} \otimes \cdots \otimes \{X_0^{(n)}, X_1^{(n)}\} \cdot \mathcal{G}_i.$$
The graphs $\Gamma_{1,v}$ and $\Gamma_{2,v}$ are isomorphic to the Schreier graphs $\Gamma_{w_1}$ and $\Gamma_{w_2}$ of the group $G$, where $w_1, w_2 \in \{0,1\}^*$ are determined by the rules given in Proposition 4.3. Note that $w_2$ is obtained from $w_1$ by changing the first letter.

By Lemma 4.1, the graphs $\Gamma_{1,v}$ and $\Gamma_{2,v}$ have disjoint sets of edges such that their union is the set of edges of $\Gamma_n(H)$ (if we ignore the loops). Namely, the red edges of Figure 20 correspond to $\alpha_1$, the orange ones to $\alpha_2$; each blue edge corresponds either to $\beta_1$ or to $\beta_2$, each green edge either to $\gamma_1$ or to $\gamma_2$.

See Figure 23 for an example of the subgraphs $\Gamma_{1,v}$ and $\Gamma_{2,v}$ (colored red and black) of $\Gamma_6(H)$. We removed the edges corresponding to loops in $\Gamma_6(H)$.

Note that each edge of $\Gamma_H$ is a diagonal of a square with sides of length $1/2$ parallel to the real and imaginary axes. These squares tile the plane and the pillowcases $C/H_n$ and each square of the tiling has precisely one diagonal belonging to the Cayley graph $\Gamma_H$. By coloring the squares containing the edges of $\Gamma_{1,v}$ and $\Gamma_{2,v}$, in different colors (e.g., black and white), we get a nice visualization of the partition of $\Gamma_n(H)$ into the trees $\Gamma_{1,v}$ and $\Gamma_{2,v}$, see Figure 24. Here the squares whose diagonals are loops of $\Gamma_n(H)$ are colored blue. We will call them singular.
Let us denote by $K_{1,v}$ the union of the squares whose diagonals belong to $\Gamma_{1,v}$ and by $K_{2,v}$ the union of the squares whose diagonals belong to $\Gamma_{2,v}$ and of the singular squares.

**Proposition 4.7.** Boundary between $K_{1,v}$ and $K_{2,v}$ is a closed broken line $\lambda_v$ describing the action of $\tau$ on the vertex set of $\Gamma_n(H)$. Namely, for every vertex $u$ there are no vertices of $\Gamma_n(H)$ on $\lambda_v$ between $u$ and $\tau(u)$.

Note that our choice to include the singular squares in $K_{2,v}$ is not very important. It will change only the side on which the path $\lambda_v$ goes around the singular point of $\mathbb{C}/H_n$.

**Proof.** We have $\tau = \gamma_1 \alpha_1 \beta_1 = \gamma_2 \alpha_2 \beta_2$. Consider the little squares of $\Gamma_n(H)$. Two of their sides (opposite to each other) correspond to $\alpha_1 = CAB$ and $\alpha_2 = A$, the other two sides correspond to $B$ and $C$. Each of the latter two edges may belong either to $\Gamma_{1,v}$ or to $\Gamma_{2,v}$. Figure 25 shows all four possible cases. Note that if an edge corresponding to $B$ or $C$ belongs to $\Gamma_{1,v}$, then its endpoints are fixed under the action of $\beta_{1-i}$, respectively. This information makes it possible to determine for one of the pairs of vertices of the square that one is the image of the other under the action of $\tau$ as it is shown by black arrows on Figure 25. If one of the edges of the squares is a loop of $\Gamma_n(H)$, then we assume that it belongs to $\Gamma_{2,v}$ (according to our convention about the set $K_{2,v}$). Note that different agreement does not change the order in which $\lambda_v$ connects the vertices of $\Gamma_n(H)$. 

**Figure 23.** A component of a Schreier graph of $\widehat{G}$

**Figure 24.** Partition into sets $K_{i,v}$

**Figure 25.**
We see that the arrows describing the action of $\tau$ belong to the boundary $\lambda_v$ between the sets $K_{1,v}$ and $K_{2,v}$. \qed

Consequently, partition of the pillowcase $\mathbb{C}/\mathcal{H}_n$ into the sets $K_{1,v}$ and $K_{2,v}$ is an approximation of the mating described in Proposition 4.2. The boundary $\lambda_v$ between the sets converges (as $v$ converges to a left-infinite sequence $w \in \{L, R\}^{-\omega}$) to the map from the circle to a connected component of $\mathcal{J}_{\hat{G}}$ induced by the inclusion of $(\tau) < \hat{G}$.

See Figure 26 where two examples of the partition are given.

We orient $\lambda_v$ according to the action of $\tau$, so that oriented segments go from $u$ to $\tau(u)$. Then $\lambda_v$ goes around $\Gamma_{1,v}$ in positive direction, and around $\Gamma_{2,v}$ in the negative direction (if we orient $\mathbb{C}$ in the standard way), see Figure 25.

4.6. **Recursive rule of constructing** $\Gamma_{1,v}$ and $\Gamma_{2,v}$. It follows from Proposition 4.3 that the graphs $\Gamma_{1,v}$ and $\Gamma_{2,v}$ for $w = z_1 z_2 \ldots z_n \in \{L, R\}^n$ are isomorphic to the Schreier graphs $\Gamma_{w_1}$ and $\Gamma_{w_2}$ of $\hat{G}$, where $w_1 = x_1 x_2 \ldots x_n$, $w_2 = y_1 y_2 \ldots y_n \in \{0, 1\}^n$ are defined by the rule:

$$x_k = y_k = \begin{cases} 0 & \text{if } z_{k-1} = z_k, \\ 1 & \text{otherwise}, \end{cases}$$

for $k \geq 2$, while $x_1$ and $y_1$ are defined by the same rule with the assumption $z_0 = L$ and $z_0 = R$, respectively.
Let us translate now the recursive rule from Corollary 3.11 of construction of the graphs $\Gamma_v$ in terms of the sequences over the alphabet $\{L,R\}$ and subgraphs of the graph $\Gamma_H$.

Note that since the graphs $\Gamma_{i,v}$ are trees (as they are isomorphic to $\Gamma_w$ for some $w$), they can be lifted by the natural quotient map $\Gamma_H \to \Gamma_H/\mathcal{H}_n$ to a subgraph of the Cayley graph $\Gamma_H$ of $\mathcal{H}$.

On the initial step (for the empty word $v$) the graphs $\Gamma_{i,\emptyset}$ consist of one vertex only, which is marked by $z_{\alpha,\emptyset}, z_{\beta,\emptyset},$ and $z_{\gamma,\emptyset}$ simultaneously. Choose a point in $\Gamma_H$, which will be the lift of the graphs $\Gamma_{\emptyset,i}$. We will add, for convenience, halves of the incident edges of $\Gamma_H$, corresponding to $CAB, B, C$ for $\Gamma_{1,\emptyset}$ and $A, B, C$ for $\Gamma_{2,\emptyset}$.

Let us denote the obtained graphs by $\Delta_{1,\emptyset}$ and $\Delta_{2,\emptyset}$, respectively.

Suppose that we have constructed the graphs $\Delta_{1,v}$ and $\Delta_{2,v}$, which are lifts of the graphs $\Gamma_{i,v}$ and $\Gamma_{2,v}$, respectively, with three marked vertices $z_{\alpha,v}, z_{\beta,v}$, and $z_{\gamma,v}$ and halves of some edges of $\Gamma_H$ attached to the marked vertices. Let $z_{\alpha,v}, z_{\beta,v}$, and $z_{\gamma,v}$ be the other (“hanging”) vertices of the half-edges.

Then the graphs $\Delta_{i,vx}$ for $i \in \{1, 2\}$, $x \in \{L,R\}$ together with the marking are obtained by the following rule.

Denote $\Delta_{i,v,0} = \Delta_{i,v}$ and let $\Delta_{i,v,1}$ be $\Delta_{i,v}$ rotated by $180^\circ$ around $z'_{\alpha,v}$. Take the union of $\Delta_{i,v,0}$ with the $\Delta_{i,v,1}$, connecting in this way the respective copies of $z_{\alpha,v}$ by an edge.

The copy of $z_{\gamma,v}$ in $\Delta_{i,v,0}$ is the vertex $z_{\gamma,v}$. The copy of $z_{\gamma,v}$ in $\Delta_{i,v,1}$ is the vertex $z'_{\gamma,v}$. The last letter of $v$ coincides with $x$ (or if $v = \emptyset$, $x = L$, $i = 1$, or $v = \emptyset, x = R, i = 2$), then the copy of $z_{\beta,v}$ in $\Delta_{i,v,1}$ is $z_{\alpha,vx}$, otherwise $z_{\alpha,vx}$ is the copy of $z_{\beta,v}$ in $\Delta_{i,v,0}$. Remove the half-edge attached to the other (unmarked) copy of $z_{\beta,v}$. The obtained graph is $\Delta_{i,vx}$. The graph $\Gamma_{i,vx}$ is obtained from it by removing the three half-edges attaching to the marked vertices.

See the first three steps of this recursion (for $i = 1$) on Figure 27.

Note that it follows directly from the construction, that the points $z'_{\alpha,v}, z'_{\beta,v}$ and $z'_{\gamma,v}$ are vertices of a right isosceles triangle. Orientation of the triangles $z'_{\alpha,v}, z'_{\beta,v}, z'_{\gamma,v}$ depends on the last letter of $v$: it is counterclockwise if it is $L$ and clockwise if it is $R$.

5. Paper-folding curves

5.1. Mazes associated with graphs $\Gamma_{1,v}$. Consider again the Cayley graph $\mathcal{H}$ drawn in $\mathbb{C}$, as on Figure 20. Consider the half-integral grid on $\mathbb{C}$, i.e., the tiling of the plane by the parallel translations by the elements of $\mathbb{Z}[i]/2$ of the square with the vertices $0, 1/2, i/2$, and $1/2 + i/2$. The group $\mathcal{H}$ acts freely on the set of these squares with two orbits (corresponding to the two colors of the checkerboard coloring). We get in this way a checkerboard coloring of the pillowcases $\mathbb{C}/\mathcal{H}_n$. Let $Q_n$ be the graph consisting of the sides of the half-integral grid on $\mathbb{C}/\mathcal{H}_n$.

The vertices of the graph $\Gamma_{1,v}$ are centers of squares of one color in the checkerboard coloring of $\mathbb{C}/\mathcal{H}_n$. Since $\Gamma_{1,v}$ is a tree, there is a closed Eulerian path $\rho_v$ in $Q_n$ without transversal self-intersections, which goes around $\Gamma_{1,v}$, i.e., does not intersect it transversally, see Figure 28, where the squares containing $\Gamma_{1,v}$ are colored red, and the other squares are white. The path $\rho_v$ is the boundary of the white region (after we glue the picture into a pillowcase).
Note that, unlike for the path $\lambda_v$, there are no problems in the definition of $\rho_v$ concerning the singular points.

See more examples of the paths $\rho_v$ on Figure 29, where their connected lifts to $\mathbb{C}$ are shown.

Recall that the curve $\lambda_v$ describes the action of $\tau$ on the vertices of $\Gamma_{1,v}$. It is easy to check that $\rho_v$ is obtained from $\lambda_v$ by the replacements of segments of $\lambda_v$ between vertices of $\Gamma_{1,v}$ by curves shown on Figure 30. In particular, the curves $\rho_v$ also approximate the plane-filling curves coming from the matings described in Proposition 4.2.

5.2. Another pair of Schreier graphs. On Figure 28 the curve $\rho_n$ goes around $\Gamma_{1,v}$ bounding the red cells of the checkerboard tiling of the pillowcase $\mathbb{C}/\mathcal{H}_n$. It
also goes around the white cells, and these cells are arranged into a tree around which \( \rho_v \) travels. Let us try to interpret this “white” tree in terms of the group \( G \).

Let \( \zeta_1 = 1/4 + i/4 \) and \( \zeta_2 = 1/4 - i/4 \). Let \( \Sigma_1 = \Gamma_H \) be the left Cayley graph of \( H \) with the set of vertices \( \zeta_1 \cdot H \) and the edges corresponding to the generators \( A, B, C, CAB \). Let \( \Sigma_2 \) be the left Cayley graph of \( H \) with the set of vertices \( \zeta_2 \cdot H \) and the edges corresponding to the generators \( A, ABA, C, ABC \).

See Figure 31 for the graphs \( \Sigma_1, \Sigma_2 \). Note that each edge of \( \Sigma_1 \) intersects exactly one edge of \( \Sigma_2 \) and vice versa. Namely, the edges of \( \Sigma_1 \) corresponding to \( A, B, C, \) and \( CAB \) intersect the edges of \( \Sigma_2 \) corresponding to \( A, ABA, C, \) and \( ABC \), respectively.

For a given word \( v = X^{(1)}X^{(2)}\ldots X^{(n)} \in \{L, R\}^* \) of length \( n \), denote by \( \Sigma_{1, v} \) and \( \Sigma_{2, v} \) the Schreier graphs of the groups \( G_1 \) and \( G_2 \) acting on the set \( X^{(1)} \times X^{(2)} \times \cdots \times X^{(n)} \) (i.e., on the set of the right orbits of the bisets \( X_1^{(1)} \otimes X_1^{(2)} \otimes \cdots \otimes X_1^{(n)} \) and \( X_2^{(1)} \otimes X_2^{(2)} \otimes \cdots \otimes X_2^{(n)} \), respectively) defined with respect to the generating sets \( \{\alpha_1 = CAB, \beta_1, \gamma_1\} \) and \( \{\alpha_2 = A, \alpha_2 \beta_2 \alpha_2, \gamma_2\} \). We identify them with the corresponding sub-graphs of the graphs \( \Sigma_1/\mathcal{H}_n \) and \( \Sigma_2/\mathcal{H}_n \), respectively.

**Proposition 5.1.** The graphs \( \Sigma_{1, v} \) and \( \Sigma_{2, v} \) do not intersect (as subsets of \( \mathbb{C}/\mathcal{H}_n \)). In each pair of intersecting edges of \( \Sigma_1/\mathcal{H}_n \) and \( \Sigma_2/\mathcal{H}_n \) one edge belongs to one of the graphs \( \Sigma_{1, v} \) and \( \Sigma_{2, v} \) and the other edge does not belong to either graphs.
It follows that \( \rho_v \) separates the trees \( \Sigma_1, v = \Gamma_1, v \) and \( \Sigma_2, v \), and that the graphs \( \Sigma_1, v \) and \( \Sigma_2, v \) describe adjacency of the cells on the corresponding side of the curve \( \rho_v \).

**Proof.** Connect the basepoints \( \zeta_1 \) and \( \zeta_2 \) by a straight segment \( \ell \). The points \( \zeta_1 \) and \( \zeta_2 \) as vertices of the Cayley graphs \( \Sigma_1 \) and \( \Sigma_2 \) correspond to the identity in the group \( H \). It follows that the images of \( \ell \) under the action of \( H \) connect the vertices of \( \Sigma_1 \) and \( \Sigma_2 \) corresponding to the same elements of \( H \). It follows now from the construction of the graphs \( \Sigma_i \) (see Figure 5.2) that the edge connecting \( h \in H \) to \( Ch \) in \( \Sigma_1 \) intersects with the edge connecting the corresponding vertices in \( \Sigma_2 \). The same statement for the edges connecting \( h \) to \( Ah \) is true. The edge in \( \Sigma_1 \) connecting \( Ah \) to \( BAh \) intersects the edge in \( \Sigma_2 \) connecting \( h \) to \( ABAh \).

If the graph \( \Sigma_1, v \) contains an edge from \( w \) to \( \gamma_1(w) = C(w) \), then the graph \( \Sigma_2, v \) does not contain an edge connecting \( w \) to \( \gamma_2(w) \), and vice versa. The edge \( w \) to \( \gamma_1(w) \) coincides with the edge from some \( h \) to \( Ch \) in \( \Sigma_1 \), while the edge from \( w \) to \( \gamma_2(w) \) coincides with the edge from the same element \( h \) to \( Ch \) in \( \Sigma_2 \). This settles the statement for the edges \( (w, \gamma_i(w)) \).

The edge from \( w \) to \( \alpha_2(w) \) corresponds to the edge from \( h \) to \( Ah \), which is not included into \( \Sigma_1 \). Similarly, the edge from \( w \) to \( \alpha_1(w) = CAB(w) \) is not included into \( \Sigma_2 \).

If \( \Sigma_2(w) \) contains an edge from \( w \) to \( \alpha_2 \beta_2 \alpha_2(w) = ABA(w) \), then \( w \neq \alpha_2 \beta_2 \alpha_2(w) \), i.e., \( \alpha_2(w) \neq \beta_2 \alpha_2(w) \), which is equivalent to the condition that the graph \( \Sigma_1(w) \) does not have an edge from \( A(w) \) to \( BA(w) \).

**Proposition 5.2.** The graphs \( \Sigma_{1,v} \) and \( \Sigma_{2,v} \) are isomorphic for every word \( v \).

**Proof.** We know, see 4.3, that for every finite sequences \( v = X_1 X_2 \ldots X_n \in \{L, R\}^* \) the \( \mathcal{G}_1 \cong \mathcal{G} \)-biset \( X_1 \otimes X_2 \otimes \cdots \otimes X_n \cdot \mathcal{G}_1 \) is isomorphic to the \( \mathcal{G}_2 \cong \mathcal{G} \)-biset \( a \cdot X_1 \otimes X_2 \otimes \cdots \otimes X_n \cdot \mathcal{G}_2 \cdot a \). We have \( \alpha = a \alpha a, \alpha \beta \alpha = a \beta a \) and \( \gamma = a \gamma a \), which implies that the map

\[ w \mapsto a(w) \]
is an isomorphism of the Schreier graphs $\Sigma_{2,v} \to \Sigma_{1,v}$. □

5.3. Paper-folding. Consider a lift $\Sigma_{1,v}'$ of the graph $\Sigma_{1,v}$ to $\mathbb{C}$, and let $\rho_v'$ be the corresponding lift of the path $\rho_v$ (i.e., a closed path going around the lift of the tree $\Sigma_{1,v}$).

Since the singular points $Z_A$, $Z_B$, $Z_C$, and $Z_{CAB}$ of the orbifold $\mathbb{C}/\mathcal{H}_n$ belong to the graph $Q_n$ for every $n$, the path $\rho_v$ also passes through these points. The lift $\rho_v'$ will pass through preimages $Z_{A}', Z_{B}', Z_{C}'$ and $Z_{CAB}'$ of the singular points.

**Proposition 5.3.** The path $\rho_v'$ consists of four pieces: $\rho_v(A,B)$ from $Z_A'$ to $Z_B'$, $\rho_v(B,CAB)$ from $Z_B'$ to $Z_{CAB}'$, $\rho_v(CAB,C)$ from $Z_{CAB}'$ to $Z_C'$, and $\rho_v(C,A)$ from $Z_C'$ to $Z_A'$.

If the last letter of $v$ is $L$, or if $v$ is empty (resp., if the last letter of $v$ is $R$), then the path, going consecutively through $\rho_v(A,B)$, $\rho_v(B,CAB)$, $\rho_v(CAB,C)$, and $\rho_v(C,A)$ goes in the positive (resp., negative) direction around $\Sigma_{1,v}'$. Each path $\rho_v(X_1,X_2)$ is equal to the image of the previous path $\rho_v(X_0,X_1)$ under rotation by $-\pi/2$ (resp. $\pi/2$) around $Z_{X_1}'$.

The path $\rho_{vL}(C,A)$ (resp. $\rho_{vR}(C,A)$) can be taken equal to the union of the path $\rho_v(C,A) \cup \rho_v(A,B)$ with its image under rotation by $-\pi/2$ (resp. $\pi/2$) around $Z_{B}'$.

**Proof.** Straightforward induction, using the inductive rule of constructing the graphs $\Gamma_{1,v}$. See Figure 33 for the inductive step. The points $z_{\alpha,v}', z_{\beta,v}'$ and $z_{\gamma,v}'$ will coincide with the points $Z_{CAB}', Z_B'$ and $Z_{C}'$ (which are denoted $CAB,B$ and $C$ on the Figure 33).

Consider a strip of paper of length $2^{n-1}$. Let us denote one end of the strip by $C$. For a given word $v = X_1X_2 \ldots X_n$ of letters $X_i \in \{L,R\}$ fold the strip in two, fixing $C$ and moving the other end of the strip to $C$ on the left side, if $X_n = L$, or on the right side, if $X_n = R$ (see Figure 34). Repeat now the procedure for the word $X_1X_2 \ldots X_{n-1}$. After $n$ steps unfold the strip so that all bends are at right angles. We get in this way a broken line $P_v$. Take a copy of $P_v$, rotate it by $180^\circ$ and connect its endpoints with $P_v$. We get a closed broken line $\overline{P_v}$. 

**Figure 32.** Proof of Proposition 5.1
The following statement is a direct corollary of Proposition 5.3.

**Corollary 5.4.** The broken line $P_v$ is isometric to the path $\rho'_v$ going around the graph $\Sigma_{1,v}'$. 
6. Boundaries of Fatou components of \( f \) and rotated tunings

Consider the group \( B = \langle S, \gamma \alpha, \beta \rangle = \langle S \rangle \times \langle \gamma \alpha, \beta \rangle \). Let us write the images of the generators under the wreath recursion.

\[
S = \sigma \pi (P \beta \alpha \gamma, P, S^{-1} \beta \alpha \gamma, S^{-1}), \\
\gamma \alpha = \sigma (1, \gamma \beta, \alpha, \gamma \alpha \beta), \\
\beta = (1, \beta \alpha \beta, \alpha, 1).
\]

Conjugate the right hand side of the recursion by (34)\((\beta, \beta, P \beta \alpha \gamma, P \beta)\):

\[
S = (13)(24)(1,1,T,T), \\
\gamma \alpha = \sigma (1, \beta \gamma, 1, \beta \gamma), \\
\beta = (1, \alpha, 1, \alpha).
\]

We see that in the right hand side of the recursion we get elements of the group \( A = \langle T, \beta \gamma, \alpha \rangle = \langle T \rangle \times \langle \beta \gamma, \alpha \rangle \). Moreover, the corresponding \((B-A)\)-biset \( M_{B,A} \) is the direct product of the \((\langle S \rangle - \langle T \rangle)\)-biset given by the binary recursion

\[
S = \sigma (1, T),
\]

with the \((\langle \gamma \alpha, \beta \rangle - \langle \beta \gamma, \alpha \rangle)\)-biset given by the recursion

\[
(19) \quad \gamma \alpha = \sigma (1, \beta \gamma), \\
(20) \quad \beta = (1, \alpha).
\]

The following proposition is proved by direct computation.

**Proposition 6.1.** The biset over free groups of rank two given by the wreath recursion \( (19)-(20) \) is isomorphic to the biset associated with the partial covering

\[
\mathbb{C} \setminus \{0, 1\} \supset \mathbb{C} \setminus \{1, 0, 2\} \rightarrow \mathbb{C} \setminus \{0, 1\}
\]

defined by the polynomial \((1-z)^2\), where \( \gamma \alpha \) and \( \beta \) correspond to the loops around punctures 0 and 1, respectively, and the generators \( \beta \gamma \) and \( \alpha \) correspond to the loops around 1 and 0, respectively.

The generators of the group \( A \) are decomposed (in the original recursion) as follows:

\[
T = (P, \beta P \beta, \gamma S \gamma, S), \\
\beta \gamma = (\gamma, \beta \alpha, \alpha \gamma, \beta), \\
\alpha = \sigma (\beta, \beta \alpha, \alpha \beta).
\]

Conjugate the wreath recursion by \((1, \beta, \alpha, \beta)\):

\[
T = (P, P, S, S), \\
\beta \gamma = (\gamma, \alpha \beta, \gamma \alpha, \beta), \\
\alpha = \sigma.
\]

Restricting to the last two coordinates of the wreath recursion (i.e., to the biset \( B_2 \)) we get a biset \( M_{A,B} \), which is a direct product of the biset defined by the isomorphism \( T \mapsto S \) and the \((\langle \beta \gamma, \alpha \rangle - \langle \gamma \alpha, \beta \rangle)\)-biset given by the recursion

\[
(21) \quad \beta \gamma = (\gamma, \alpha \beta), \\
(22) \quad \alpha = \sigma.
\]

The proof of the following proposition is also straightforward.
Proposition 6.2. The biset over the free groups defined by the wreath recursion \((21)-(22)\) is isomorphic to the biset associated with the partial covering
\[
C \setminus \{0,1\} \supset C \setminus \{0,1,1/2\} \to C \setminus \{0,1\},
\]
given by the polynomial \((2z-1)^2\), where the generators \(\alpha\) and \(\beta\gamma\) correspond to the loops around 0 and 1, while the generators \(\gamma\) and \(\alpha\beta\) correspond to the loops around 1 and 0, respectively.

Taking tensor products \(\mathcal{M}_{\mathcal{B},A} \otimes_{A} \mathcal{M}_{A,B} \) and \(\mathcal{M}_{A,B} \otimes_{B} \mathcal{M}_{B,A}\) we see that \(A\) and \(B\) are self-similar subgroups of \(\text{IMG}(F^{\circ 2})\).

Corollary 6.3. The bisets \(\mathcal{M}_{\mathcal{B},A} \otimes_{A} \mathcal{M}_{A,B} \) and \(\mathcal{M}_{A,B} \otimes_{B} \mathcal{M}_{B,A}\) are isomorphic to the bisets associated with the post-critically finite polynomials \((2(1-z)^2-1)^2 = (2z^2-4z+1)^2\) and \((1-(2z-1)^2)^2 = 16z^2(1-z)^2\), respectively.

Proof. It follows from Propositions 6.1, 6.2. \(\square\)

Note that the polynomials \(16z^2(1-z)^2\) and \((2z^2-4z+1)^2\) coincide with the restrictions of the second iteration of the endomorphism \(F\) of \(\mathbb{P}C^2\) to the post-critical lines \(p = 1\) and \(p = 0\), respectively. The action of \(F\) is written in homogeneous coordinates as

\[
[z : p : u] \mapsto [(2z-p-u)^2 : (p-u)^2 : (p+u)^2],
\]
hence its restriction to the line \(p = u\) is

\[
[z : p : p] \mapsto [(2z-2p)^2 : 0 : 4p^2],
\]
so that it acts on the first coordinate (in non-homogeneous coordinates) as

\[
z \mapsto (z-1)^2.
\]

Restriction of \(F\) onto the line \(p = 0\) is

\[
[z : 0 : u] \mapsto [(2z-u)^2 : u^2],
\]
i.e.,

\[
z \mapsto (2z-1)^2.
\]

Proposition 6.4. The limit spaces of \((A, \mathcal{M}_{A,B} \otimes \mathcal{M}_{B,A})\) and \((B, \mathcal{M}_{B,A} \otimes \mathcal{M}_{A,B})\) are direct products of circles with the Julia sets of the polynomials \(16z^2(1-z)^2\) and \((2z^2-4z+1)^2\), respectively. The images of \(\mathcal{J}_A\) and \(\mathcal{J}_B\) in \(\mathcal{J}_{\text{IMG}(F)}\) are identified with the subsets of the Julia set projected by \((z,p) \mapsto p\) to the boundaries of the Fatou components of \(f(p) = \left(\frac{p-1}{p+1}\right)^2\), containing 1 and 0, respectively.

Proof. Description of the limit space follows directly from the structure of the wreath recursion and Corollary 6.3. The groups \(\langle \alpha, \beta\gamma \rangle\) and \(\langle \beta, \gamma\alpha \rangle\) are level-transitive, which implies that the map from the limit spaces of \(A\) and \(B\) to the limit space of \(\text{IMG}(F)\) is surjective on the fibers of the natural projection \(\mathcal{J}_{\text{IMG}(F)} \to \mathcal{J}_{\text{IMG}(F)}\).

It follows from the post-critical dynamics of \(f\) and the interpretation of the maps \(S\) and \(P\) as loops in the space \(\mathbb{C} \setminus \{0,1\}\) (see Proposition 3.3) that the images of the limit spaces \(\mathcal{J}_A\) and \(\mathcal{J}_B\) in the Julia set of \(f\) are the boundaries of the Fatou components of 1 and 0, respectively. \(\square\)
See Figure 35 where the Julia sets of the polynomials $16z^2(1-z)^2$ and $(2z^2 - 4z + 1)^2$ are shown.

The natural map from $J_A$ and $J_B$ to the Julia set of $F$ are not injective. Let us see which points of the limit space are identified under these maps.

Consider the group $B_1 = \langle \gamma, \alpha, S \rangle$. Its generators are written as

\[
\begin{align*}
\alpha &= \sigma(\beta, \beta, \beta \alpha, \alpha \beta), \\
\gamma &= (\gamma, \beta, \gamma, \beta), \\
S &= \pi \sigma(P \beta \alpha \gamma, P, S^{-1} \beta \alpha \gamma, S^{-1}).
\end{align*}
\]

Conjugating the right-hand side by $(1, \beta, P, P \beta \alpha)$ we get

\[
\begin{align*}
\alpha &= \sigma, \\
\gamma &= (\gamma, \beta, \gamma, \beta), \\
S \gamma \alpha &= \pi(1, 1, T \beta \gamma, T \beta \gamma).
\end{align*}
\]

Consider now the group $A_1 = \langle \gamma, \beta, T \rangle$. We have

\[
\begin{align*}
\beta &= (1, \beta \alpha \beta, \alpha, 1), \\
\gamma &= (\gamma, \beta, \gamma, \beta), \\
T &= (P, \beta P, \alpha S, S), \\
T \beta \gamma &= (P \gamma, \beta P, S \gamma, S \beta).
\end{align*}
\]

Restriction to the third coordinate of the wreath recursion is the homomorphism

\[
\begin{align*}
\beta &\mapsto \alpha \\
\gamma &\mapsto \gamma \\
T \beta \gamma &\mapsto S \gamma \alpha.
\end{align*}
\]
Note that the subgroups $\langle \gamma\alpha, S \rangle$ and $\langle \beta\gamma, T \rangle$ are also subgroups of $\mathcal{B}$ and $\mathcal{A}$, respectively. The corresponding wreath recursions for these groups are

\[
\begin{align*}
\gamma\alpha &= \sigma(1, \beta\gamma), \\
S &= \sigma(1, T), \\
S\gamma\alpha &= (T, \beta\gamma),
\end{align*}
\]

(here we use a conjugated version (19)–(20)), and

\[
\begin{align*}
\beta\gamma &\mapsto \gamma\alpha, \\
T &\mapsto S, \\
T\beta\gamma &\mapsto S\gamma\alpha.
\end{align*}
\]

It follows from the recursions and post-critical dynamics of the polynomials $16z^2(1-z)^2$ and $(2z^2 - 4z + 1)^2$ that the limit space of $\langle \gamma\alpha, S \rangle$ with respect to these recursions is the natural direct product of the boundary of the Fatou component of $f$ containing 0 (which is the limit space of the subgroup $\langle S \rangle$) and the boundary of the Fatou component of $(2z^2 - 4z + 1)^2$ containing 0 (the limit space of $\langle \gamma\alpha \rangle$). Similarly, the limit space of $\langle \beta\gamma, T \rangle$ is the direct product of the the boundary of the Fatou component of $f$ containing 1 (the limit space of $\langle T \rangle$) with the boundary of the Fatou component of $16z^2(1-z)^2$ containing 1 (the limit space of $\langle \beta\gamma \rangle$). Hence, both limit spaces are tori.

In particular, the natural maps from the limit spaces of the groups $\langle \gamma\alpha, S \rangle$ and $\langle \beta\gamma, T \rangle$ to $\mathcal{J}_B$ and $\mathcal{J}_A$, respectively, are injective.

Let us compare the limit spaces of these groups with the limit spaces of their extensions $\mathcal{B}_1$ and $\mathcal{A}_1$. We have $\mathcal{B}_1 = \langle \alpha, \gamma \rangle \times \langle S\gamma\alpha \rangle$, $\mathcal{A}_1 = \langle \beta, \gamma \rangle \times \langle T\beta\gamma \rangle$. The direct factors $\langle \alpha, \gamma \rangle$ and $\langle \beta, \gamma \rangle$ are infinite dihedral with the wreath recursions

\[
\begin{align*}
\alpha &= \sigma, \quad \gamma = (\gamma, \beta),
\end{align*}
\]

and

\[
\begin{align*}
\beta &\mapsto \alpha, \quad \gamma &\mapsto \gamma.
\end{align*}
\]

It follows that the limit spaces of the subgroups $\langle \alpha, \gamma \rangle$ and $\langle \beta, \gamma \rangle$ are segments with singular endpoints (one with the isotropy group $\langle \alpha \rangle$ or $\langle \beta \rangle$ and the other with the isotropy group $\langle \gamma \rangle$).

Consequently, the limit spaces of $\mathcal{B}_1$ and $\mathcal{A}_1$ are annuli (direct products of a circle and a segment) and that the natural map from the limit spaces of $\langle \gamma\alpha, S \rangle$ and $\langle \beta\gamma, T \rangle$ to the limit spaces of $\mathcal{B}_1$ and $\mathcal{A}_1$ “flattens” the tori into annuli, by flattening the circles $\mathcal{J}_{(\gamma\alpha)}$ and $\mathcal{J}_{(\beta\gamma)}$ to the segments $\mathcal{J}_{(\gamma, \alpha)}$ and $\mathcal{J}_{(\beta, \gamma)}$. Note that the natural direct product decomposition of the spaces $\mathcal{J}_{(\gamma\alpha, S)}$ and $\mathcal{J}_{(\beta\gamma, T)}$ comes from the direct product decompositions $\langle \gamma\alpha \rangle \times \langle S \rangle$ and $\langle \beta\gamma \rangle \times \langle T \rangle$, while we have decompositions of self-similar groups $\mathcal{B}_1 = \langle \gamma, \alpha \rangle \times \langle S\gamma\alpha \rangle$ and $\mathcal{A}_1 = \langle \beta, \gamma \rangle \times \langle T\beta\gamma \rangle$. Consequently, the preimages of the boundaries of the annuli in the tori are diagonals with respect to which we flatten the boundary of the Fatou component of the polynomials $16z^2(1-z)^2$ and $(2z^2 - 4z + 1)^2$ rotates as $p$ travels along the boundary of the Fatou component of $f$.

This flattening of the circles into segments can be interpreted as a “rotated tuning” of the polynomials $16z^2(1-z)^2$ and $(2z^2 - 4z + 1)^2$ by the polynomial $z^2 - 2$ (which is the quadratic polynomial with the dihedral iterated monodromy
group and the Julia set a segment). It can be nicely illustrated by the slices of the Julia set of $F$ as $p$ travels close to the boundary of the Fatou component of $f$, but stays inside in it. Then the slices are still homeomorphic to the Julia sets of $16z^2(1 - z)^2$ and $(2z^2 - 4z + 1)^2$, but are close to the dendrite slices of the Julia set of $F$ along to boundary of the Fatou component of $f$.

See Figure 36 where slices of the Julia set of $F$ are shown when $p$ is traveling close to the boundary of the Fatou component of $f$ containing 1 (top half of the figure). Corresponding slices for $p$ on the boundary of the Fatou component are shown in the bottom part of the figure.

![Figure 36. Rotated tuning](image)

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