Linear convergence of cyclic SAGA

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Abstract
In this work, we present and analyze C-SAGA, a (deterministic) cyclic variant of SAGA. C-SAGA is an incremental gradient method that minimizes a sum of differentiable convex functions by cyclically accessing their gradients. Even though the theory of stochastic algorithms is more mature than that of cyclic counterparts in general, practitioners often prefer cyclic algorithms. We prove C-SAGA converges linearly under the standard assumptions. Then, we compare the rate of convergence with the full gradient method, (stochastic) SAGA, and incremental aggregated gradient (IAG), theoretically and experimentally.

Keywords Cyclic updates · SAGA · IAG · Incremental methods · Just-in-time update · Linear convergence

1 Introduction

Consider the optimization problem

\[
\min_{x \in \mathbb{R}^d} f(x) := \frac{1}{n} \sum_{i=1}^{n} f_i(x).
\]

(P)

where \( f_i : \mathbb{R}^d \to \mathbb{R} \) for \( i = 1, \ldots, n \) are convex and differentiable. This finite sum structure commonly appears in machine learning problems minimizing empirical risk. When \( n \), the number of components, is small, we can solve (P) with the classical
gradient method, which computes (full) gradients of $f$ every iteration. When $n$ is large, however, the cost of evaluating a full gradient even once can be very expensive, and algorithms with lower per-iteration cost become appealing.

Recently, the class of incremental methods that solve (P) by computing the gradient of only one $f_i$ at each iteration has received much attention. Such methods include stochastic gradient methods \cite{11,17,20,21,26,27}, its variance reduced variants \cite{7,8,10,12–15,19,23–25,33,36}, and deterministic incremental methods \cite{2,3,9,16,29–32,35}. Among these methods, variance reduced gradient methods such as SAGA \cite{7} achieve a faster rate of convergence compared to classical gradient descent under certain assumptions.

Stochastic incremental methods randomly access one $\nabla f_i$ at each iteration. The randomness is essential to the theoretical analysis, and the theory for such stochastic incremental methods have significantly matured in the past five years. In contrast, shalev2016 deterministic incremental methods deterministically choose one $\nabla f_i$ at each iteration, and their theoretical analysis is much weaker than that of the stochastic incremental methods even though they often perform well empirically.

Nevertheless, many practitioners prefer cyclic methods, which access the components (deterministically) cyclically. Iterations of cyclic methods can be faster due to systemic reasons such as cache locality, and it is often important to guarantee all components are accessed once within each epoch (a pass through $n$ components) especially when only a few epochs are used. Whether practitioners should use cyclic methods is an interesting question in its own right. Regardless, analyzing the theoretical strength or weakness of the cyclic methods practitioners use is of practical interest.

In this paper, we present a cyclic variant of SAGA, which we call C-SAGA. We prove it converges linearly under the standard assumptions of strong convexity and smoothness. We theoretically and empirically compare the rate of convergence with other deterministic and stochastic incremental methods.

Main method C-SAGA has the same algorithmic structure as SAGA, except that the choice of $\nabla f_i$ is cyclic, not random. Define $[k]_n = \text{mod}(k, n) + 1$. We can think of $[\cdot]_n$ as the modulo operator with 1-based indexing. We can write C-SAGA as

$$x^{k+1} = x^k - \gamma \left( \nabla f_{[k]_n}(x^k) - \nabla f_{[k]_n}(x^{k-n}) + \frac{1}{n} \sum_{i=1}^{n} \nabla f_{[k-i]_n}(x^{k-i}) \right)$$

where $x^0, x^{-1}, x^{-2} \ldots, x^{-n}$ are some starting points. For computational efficiency, implementations of C-SAGA should store the $n$ most recent gradients. This way, the $x^{k+1}$-update will compute one new gradient $\nabla f_{[k]_n}(x^k)$ and use past gradients stored in memory.

Prior work The original analysis of SAGA by Defazio et al. \cite{7} relies crucially on the random index selection and is not adaptable to the deterministic cyclic setup. Ying et al. \cite{34} analyzed a random permutation variant of SAGA using a similar but different Lyapunov function and established the rate $1 - \frac{1}{108 \kappa^2} + O(1/k^4)$ for the random permutation setup.
Stochastic incremental methods such as Finito, MISO, SVRG, SAG, SAGA, and SDCA [7,8,10,12–15,19,23–25,33,36] have been an intense and fruitful area of research in recent years. These methods achieve a faster rate of convergence compared to classical gradient descent.

Classical stochastic and deterministic incremental gradient methods require diminishing stepsize for convergence [1,2]. The diminishing stepsize limits the rate of convergence to a sublinear rate, usually $O(1/k)$ or slower, even under the assumptions of strong convexity and smoothness.

IAG, which can be viewed as a cyclic variant of SAG [22], was the first deterministic incremental method to achieve a linear rate of convergence. Blatt et al. [4] first proved linear convergence for quadratic components, and Gürbüzbalaban et al. [9] recently proved the linear (epoch-by-epoch) rate with factor $1 - O(1/n\kappa^2)$ for the general strongly convex case. Mokhtari et al. [16] presented DIAG, an incremental method that can be viewed as a cyclic variant of Finito, and proved a linear (epoch-by-epoch) rate with factor $1 - O(1/\kappa)$. Other IAG-type methods include [16,28–31,35].

2 Convergence analysis

We now analyze the convergence of C-SAGA. The main result of this work is stated as Theorem 1, which we prove in several steps.

**Theorem 1** Assume $f_i$ is $\mu$-strongly convex and $\nabla f_i$ is $L$-Lipschitz continuous for all $i = 1, \ldots, n$. Define $V^k = \|x^k - x^*\|^2 + \frac{1}{n} \sum_{j=1}^{n} \|x^k - x^{k-j}\|^2$. For $0 < \gamma \leq \frac{\mu}{65 \sqrt{n(n+1)L^2}}$, $V^k$ converges linearly. In particular, for $\gamma = \frac{\mu}{130 \sqrt{n(n+1)L^2}}$,

$$V^{k+n} \leq \left(1 - \frac{1}{368 \kappa^2}\right)V^k,$$

where $\kappa = L/\mu$.

The goal of the analysis is to establish a rate for $\|x^k - x^*\|^2 \to 0$. However, directly using $\|x^k - x^*\|^2$ as a Lyapunov function in the analysis seems difficult, as it does not monotonically decrease. We therefore use the Lyapunov function $V^k$ which does monotonically decrease.

IAG and C-SAGA are similar incremental and aggregated-type algorithms with essentially the same computational cost per iteration. However, the rate for IAG shown in Gürbüzbalaban et al. [9] is $1 - O(1/n\kappa^2)$ while the rate for C-SAGA shown in Theorem 1 is $1 - O(1/\kappa^2)$, which is better. One explanation for this discrepancy is that IAG is indeed slower (in the worst case) than C-SAGA, and our computational experiments support this possibility. Another possibility is that the analysis for IAG is not tight.

In [16], Mokhtari et al. showed an even better rate of $1 - O(1/\kappa)$ for their method DIAG. However, DIAG, which can be viewed as a cyclic variant of Finito, cannot take advantage of “just-in-time” updates, a technique applicable to SAG, SAGA, IAG, and C-SAGA that reduces the computational cost when the gradients are sparse [7,22].
For certain subtle reasons, this technique does not work with Finito, as Defazio et al. acknowledge in their work in saying “We do not recommend the usage of Finito when gradients are sparse.” [8] and, by extension, to DIAG. So when the gradients are sparse, an iteration of C-SAGA with just-in-time updates runs faster than an iteration of DIAG, and C-SAGA can still be theoretically competitive with DIAG.

Finally, we point out that Theorem 1 combined with the inequality
\[
f(x^k) - f(x^*) \leq \frac{L}{2} \left( 1 - \frac{1}{368k^2} \right)^k V^0
\]
leads to a linear rate of convergence on function values.

**Corollary 1** In the setup of Theorem 1, if \( x^* \) is the solution, then
\[
f(x^{kn}) - f(x^*) \leq \frac{L}{2} \left( 1 - \frac{1}{368k^2} \right)^k V^0
\]
for \( k = 0, 1, \ldots \).

More specifically, applying the result \( n \) times with \( V^0, \ldots, V^{n-1} \) establishes R-linear convergence for the whole sequence \( f(x^k) - f(x^*) \) for \( k = 0, 1, \ldots \).

### 2.1 Main proof

We now present the main theoretical contribution, the proof. We first start by stating a few inequalities that are well-known. Throughout this section, we write \([k]\) in place \([k]_n\) for the sake of brevity. Throughout this section, assume \( f_i \) is \( \mu \)-strongly convex and \( \nabla f_i \) is \( L \)-Lipschitz continuous for all \( i = 1, \ldots, n \).

Let \( v_1, \ldots, v_n \in \mathbb{R}^d \). Then

\[
\left\| \frac{1}{n} \sum_{i=1}^{n} v_i \right\|_2^2 \leq \frac{1}{n} \sum_{i=1}^{n} \| v_i \|_2^2,
\]

which follows from Jensen’s inequality. For any \( a, b \in \mathbb{R}^d \) and \( \beta > 0 \), we have
\[
\| a + b \|_2^2 \leq (1 + \beta)\| a \|_2^2 + (1 + \beta^{-1})\| b \|_2^2,
\]

which can be found in references like [7]. Young’s inequality states
\[
2\langle a, b \rangle \leq \varepsilon \| a \|_2^2 + \varepsilon^{-1} \| b \|_2^2,
\]

for any \( \varepsilon > 0 \). If \( f \) is \( \mu \)-strongly convex and \( \nabla f \) is \( L \)-Lipschitz,
\[
\langle x - y, \nabla f(x) - \nabla f(y) \rangle \geq \frac{1}{L + \mu} \| \nabla f(x) - \nabla f(y) \|_2^2 + \frac{L\mu}{L + \mu} \| x - y \|_2^2
\]

for any \( x, y \) [18, Theorem 2.1.5].
Lemma 1 Let $\sigma_0 = 1, \tau_0 = 0$, and

$$
\begin{bmatrix}
\sigma_{k+1} \\
\tau_{k+1}
\end{bmatrix}
= 
\begin{bmatrix}
1 + c_1 & 1 \\
c_2 & 1
\end{bmatrix}
\begin{bmatrix}
\sigma_k \\
\tau_k
\end{bmatrix}
$$

for $k = 0, 1, \ldots$. Assume $c_1, c_2 \geq 0$ and $1 + c_1 \geq c_2$. Then

$$
\begin{bmatrix}
\sigma_k \\
\tau_k
\end{bmatrix}
\leq 
\lambda_1^k
\begin{bmatrix}
1 + \frac{c_1}{2\sqrt{c_1^2 + 4c_2}} \\
\frac{2\sqrt{c_1^2 + 4c_2}}{2c_2}
\end{bmatrix}
\begin{bmatrix}
\sigma_k \\
\tau_k
\end{bmatrix}
$$

where $\lambda_1 = 1 + \frac{c_1 + \sqrt{c_1^2 + 4c_2}}{2}$.

Proof. The eigenvalues and eigenvectors of $A = 
\begin{bmatrix}
1 + c_1 & 1 \\
c_2 & 1
\end{bmatrix}$ are

$$
\lambda_1 = 1 + \frac{c_1 + \sqrt{c_1^2 + 4c_2}}{2} \quad \lambda_2 = 1 + \frac{c_1 - \sqrt{c_1^2 + 4c_2}}{2}
$$

and

$$
v_1 = \begin{bmatrix}
c_1 + \sqrt{c_1^2 + 4c_2} \\
2c_2
\end{bmatrix} \quad v_2 = \begin{bmatrix}
c_1 - \sqrt{c_1^2 + 4c_2} \\
2c_2
\end{bmatrix}.
$$

It is simple to verify $\lambda_1, \lambda_2 \geq 0$ under $1 + c_1 \geq c_2$. From the following decomposition

$$
\begin{bmatrix}
\sigma_0 \\
\tau_0
\end{bmatrix}
= 
\begin{bmatrix}
1 \\
0
\end{bmatrix}
= 
\frac{1}{2\sqrt{c_1^2 + 4c_2}} \begin{bmatrix}
1 \\
0
\end{bmatrix}
= 
\frac{1}{2\sqrt{c_1^2 + 4c_2}} v_1
$$

we have

$$
\begin{bmatrix}
\sigma_k \\
\tau_k
\end{bmatrix}
= A^k 
\begin{bmatrix}
\sigma_0 \\
\tau_0
\end{bmatrix}
= \frac{1}{2\sqrt{c_1^2 + 4c_2}} \left( \lambda_1^k v_1 - \lambda_2^k v_2 \right)
$$

$$
= (\lambda_1^k - \lambda_2^k) \left[ \begin{bmatrix}
\frac{c_1}{2\sqrt{c_1^2 + 4c_2}} \\
\frac{2\sqrt{c_1^2 + 4c_2}}{2c_2}
\end{bmatrix} \right] + (\lambda_1^k + \lambda_2^k) \left[ \begin{bmatrix}
\frac{1}{2} \\
0
\end{bmatrix} \right]
$$

$$
\leq \lambda_1^k \left[ \begin{bmatrix}
1 + \frac{c_1}{2\sqrt{c_1^2 + 4c_2}} \\
\frac{2\sqrt{c_1^2 + 4c_2}}{2c_2}
\end{bmatrix} \right]
$$

$\square$
Define
\[ \tilde{g}^k = \frac{1}{n} \sum_{i=1}^{n} \nabla f_{[k-i]}(x^{k-i}), \]
so we can write \( x^{k+1} = x^k - \gamma (\nabla f_k(x^k) - \nabla f_k(x^{k-n}) + \tilde{g}^k) \).

**Lemma 2** Let \( x \in \mathbb{R}^d \) be arbitrary. Then
\[ \|\tilde{g}^k\|_2^2 \leq \frac{2L^2}{n} \left( n \|x - x^*\|^2 + \sum_{i=1}^{n} \|x^{k-i} - x\|^2 \right). \]

**Proof**
\[
\begin{align*}
\|\tilde{g}^k\|_2^2 &= \left\| \frac{1}{n} \sum_{i=1}^{n} \nabla f_{[k-i]}(x^{k-i}) - \nabla f_{[k-i]}(x^*) \right\|^2 \\
&\leq \frac{1}{n} \sum_{i=1}^{n} \left\| \nabla f_{[k-i]}(x^{k-i}) - \nabla f_{[k-i]}(x^*) \right\|^2 \\
&\leq \frac{L^2}{n} \sum_{i=1}^{n} \|x^{k-i} - x^*\|^2 \\
&\leq \frac{2L^2}{n} \left( n \|x - x^*\|^2 + \sum_{i=1}^{n} \|x^{k-i} - x\|^2 \right). 
\end{align*}
\]

Here the first equality follows from the optimality condition, the first inequality follows from (1), the second inequality follows from \( L \)-Lipschitz gradient assumption, and the third inequality follows from (2).

Define
\[
\sigma_1 = 1 + \beta + 4(1 + \beta^{-1})\gamma^2 L^2, \quad \tau_1 = \frac{1}{n}(1 + \beta^{-1})\gamma^2 L^2,
\]
and
\[
\sigma_j = \sigma_{j-1}\sigma_1 + \tau_{j-1}, \quad \tau_j = \sigma_{j-1}\tau_1 + \tau_{j-1},
\]
for \( j = 2, 3, \ldots \). Note this follows the recurrence defined in Lemma 1 when \( c_1 = \beta + 4(1 + \beta^{-1})\gamma^2 L^2 \) and \( c_2 = \frac{1}{n}(1 + \beta^{-1})\gamma^2 L^2 \) are specified.

**Lemma 3** Let \( x \in \mathbb{R}^d \). Then
\[
\begin{align*}
\|x^k - x\|^2 &\leq \sigma_1 \|x^{k-1} - x\|^2 + n\tau_1 \|x^* - x\|^2 \\
&\quad + n\tau_1 \|x^{k-1-n} - x\|^2 + \tau_1 \sum_{i=2}^{n+1} \|x^{k-i} - x\|^2.
\end{align*}
\]
Proof

\[ \|x^k - x\|^2 = \|x^{k-1} - x - \gamma (\nabla f_k(x^{k-1}) - \nabla f_k(x^{k-1} - n) + g^k - 1)\|^2 \]
\[ \leq (1 + \beta) \|x^{k-1} - x\|^2 + 2(1 + \beta^{-1})\gamma^2\|\nabla f_k(x^{k-1}) - \nabla f_k(x^{k-1} - n)\|^2 \]
\[ + 2(1 + \beta^{-1})\gamma^2\|g^k - 1\|^2 \leq (1 + \beta) \|x^{k-1} - x\|^2 + 2(1 + \beta^{-1})\gamma^2 L^2 \|x^{k-1} - x^{k-1} - n\|^2 \]
\[ + 4(1 + \beta^{-1})\gamma^2 L^2 \|x - x^{k-1} - n\|^2 + 2(1 + \beta^{-1})\gamma^2 \sum_{i=1}^{n} \|x^{k-1-i} - x\|^2 \]
\[ = \sigma_1 \|x^{k-1} - x\|^2 + n\tau_1 \|x - x^{k-1} - n\|^2 + n\tau_1 \|x - x^*\|^2 + \tau_1 \sum_{i=1}^{n} \|x^{k-1-i} - x\|^2 . \]

Here, the first inequality uses (2) twice, one with \( \beta > 0 \) and the other with \( \beta = 1 \). The second inequality uses \( L \)-Lipschitz and the third inequality uses (2) and Lemma 2.

(For Lemmas 4, 5, and 6, we use the same \( \gamma \) and \( c \).)

Lemma 4 Let \( \gamma = c/(L\sqrt{n(n + 1)}) \), where \( c > 0 \). Let \( 1 \leq j \leq n \). Then

\[ \|x^{k+j} - x^k\|^2 \leq \frac{33c^2}{n} \exp(8c^2) \sum_{i=1}^{n} \|x^{k-i} - x^k\|^2 + 16.5c^2 \exp(8c^2) \frac{j}{n} \|x^* - x^k\|^2 . \]

Proof Apply Lemma 3 with \( x = x^{k-j} \) recursively to get

\[ \|x^k - x^{k-j}\|^2 \leq \sigma_1 \|x^{k-1} - x^{k-j}\|^2 + n\tau_1 \|x^* - x^{k-j}\|^2 + n\tau_1 \|x^{k-1} - n - x^{k-j}\|^2 \]
\[ + \tau_1 \sum_{i=2}^{n+1} \|x^{k-i} - x^{k-j}\|^2 \]
\[ + n\tau_1 \sum_{i=3}^{j} \|x^{k-i} - x^{k-j}\|^2 + \tau_1 \sum_{i=j+1}^{n+2} \|x^{k-i} - x^{k-j}\|^2 \]
\[ + \sigma_1 \tau_1 \sum_{i=3}^{j} \|x^{k-i} - x^{k-j}\|^2 + \sigma_1 \tau_1 \sum_{i=j+1}^{n+2} \|x^{k-i} - x^{k-j}\|^2 \]
\[ + nt_1(1 + \sigma_1) \|x^* - x^{k-j}\|^2 + n\tau_1 \left( \|x^{k-1} - n - x^{k-j}\|^2 + \sigma_1 \|x^{k-2} - n - x^{k-j}\|^2 \right) \]
\[ = \sigma_2 \|x^{k-2} - x^{k-j}\|^2 + \tau_2 \sum_{i=3}^{j} \|x^{k-i} - x^{k-j}\|^2 + \tau_1 \sum_{i=j+1}^{n+2} \|x^{k-i} - x^{k-j}\|^2 \]
\[ + n\tau_1(\sigma_0 + \sigma_1) \|x^* - x^{k-j}\|^2 + n\tau_1 \sum_{i=1}^{2} \sigma_{\ell-1} \sum_{i=j+1}^{n+2} \|x^{k-i} - x^{k-j}\|^2 . \]
We now do this recursively $j$ times to get

\[
\| x^k - x^{k-j} \|^2 \leq \tau_1 \sum_{\ell=1}^{j} \sigma_{\ell-1} \sum_{i=j+1}^{n+\ell} \| x^{k-i} - x^{k-j} \|^2 \\
+ n\tau_1 (\sigma_0 + \cdots + \sigma_{j-1}) \| x^* - x^{k-j} \|^2 + n\tau_1 \sum_{\ell=1}^{j} \sigma_{\ell-1} \| x^{k-\ell-n} - x^{k-j} \|^2 \\
\leq \tau_1 n\sigma_{n-1} \sum_{i=j+1}^{n+j} \| x^{k-i} - x^{k-j} \|^2 \\
+ n\tau_1 j\sigma_{j-1} \| x^* - x^{k-j} \|^2 + n\tau_1 \sum_{\ell=1}^{j} \| x^{k-\ell-n} - x^{k-j} \|^2 \\
\leq 2n\tau_1 \sigma_{n-1} \sum_{i=j+1}^{n+j} \| x^{k-i} - x^{k-j} \|^2 + n\tau_1 j\sigma_{j-1} \| x^* - x^{k-j} \|^2. 
\]

Finally, we shift the indices to get

\[
\| x^{k+j} - x^k \|^2 \leq 2n\tau_1 \sigma_{n-1} \sum_{i=1}^{n} \| x^{k-i} - x^k \|^2 + n\tau_1 j\sigma_{j-1} \| x^* - x^k \|^2. 
\] (5)

Now we bound $\sigma_{n-1}$ and specify $\tau_1$ with the choice $\beta = 1/n$. Write

\[
c_1 = \beta + 4(1 + \beta^{-1})\gamma^2L^2 = \frac{1}{n} + \frac{4c^2}{n}, \quad c_2 = (4/n)(1 + \beta^{-1})\gamma^2L^2 = \frac{4c^2}{n^2}.
\]

We use Lemma 1 to get

\[
\lambda_1 = 1 + \frac{1}{2n} + \frac{1}{2n} \left(4c^2 + \sqrt{1 + 24c^2 + 16c^4}\right) \\
\leq 1 + \frac{1}{2n} + \frac{1}{2n} \left(4c^2 + \sqrt{(1 + 12c^2)^2}\right) \\
= 1 + \frac{1}{n} + \frac{8c^2}{n} \leq \exp \left( \frac{1}{n} + \frac{8c^2}{n} \right) 
\]

and

\[
1 + \frac{c_1}{2\sqrt{c_1^2 + 4c_2}} = 1 + \frac{1 + 4c^2}{2\sqrt{(1 + 4c^2)^2 + 16c^2}} \leq 1.5.
\]
So for any \( i = 1, \ldots, n \), we have

\[
\sigma_i \leq \lambda_1^i \left( 1 + \frac{c_1}{2\sqrt{c_1^2 + 4c_2}} \right) \leq 1.5 \exp \left( \frac{i}{n} + \frac{8c_2^2 i}{n} \right) \leq 1.5 \exp(1) \exp\left( 8c^2 \right).
\]

Combine this bound in the inequality above and the fact \( \tau_1 = c_2 \) to get the stated result. \( \square \)

**Lemma 5** Let \( \gamma = c/(L\sqrt{n(n + 1)}) \), where \( c > 0 \). Let \( 1 \leq j \leq n \). Then

\[
\|x^{k+n} - x^{k+j}\|^2 \leq 33c^2\left(1 + 50c^2\right)\exp(16c^2) \left( \|x^* - x_k\|^2 + \frac{2}{n} \sum_{i=1}^{n} \|x^{k-i} - x^k\|^2 \right)
\]

**Proof** Again, apply Lemma 3 recursively \( n \) times to get

\[
\|x^{k+n} - x^{k+j}\|^2 \leq 2\tau_1\sigma_{n-1} \sum_{i=1}^{n} \|x^{k-i} - x^{k+j}\|^2 + \tau_1 n^2 \sigma_{n-1} \|x^* - x^{k+j}\|^2
\]

\[
\leq 4\tau_1\sigma_{n-1} \sum_{i=1}^{n} \left( \|x^{k-i} - x^k\|^2 + \|x^* - x^{k+j}\|^2 \right)
\]

\[
+ 2\tau_1 n^2 \sigma_{n-1} \left( \|x^* - x^k\|^2 + n^2 \tau_1 \sigma_{n-1} \|x^* - x^k\|^2 \right)
\]

\[
= 4\sigma_{n-1} \tau_1 (1 + 2\sigma_{n-1} n^2 \tau_1 + \sigma_{n-1} n^2 \tau_1) \sum_{i=1}^{n} \|x^{k-i} - x^k\|^2
\]

\[
+ 2\sigma_{n-1} n^2 \tau_1 (1 + 2\sigma_{n-1} n^2 \tau_1 + \sigma_{n-1} n^2 \tau_1) \|x^* - x^k\|^2
\]

\[
\leq 2\sigma_{n-1} n^2 \tau_1 (1 + 2\sigma_{n-1} n^2 \tau_1 + \sigma_{n-1} n^2 \tau_1) \left( \|x^* - x^k\|^2 + \frac{2}{n} \sum_{i=1}^{n} \|x^{k-i} - x^k\|^2 \right),
\]

where we plugged in (5). As in Lemma 4, choose \( \beta = 1/n \) and we have

\[
2\sigma_{n-1} n^2 \tau_1 (1 + 2\sigma_{n-1} n^2 \tau_1 + \sigma_{n-1} n^2 \tau_1)
\]

\[
\leq 33c^2 \exp(8c^2) \left( 1 + \frac{33c^2 \exp(8c^2)}{n} + 16.5c^2 \exp(8c^2) \right)
\]

\[
\leq 33c^2 \exp(16c^2) \left( 1 + 16.5c^2 + \frac{33c^2}{n} \right)
\]

\[
= 33c^2 (1 + 50c^2) \exp(16c^2).
\]
Lemma 6 Define

\[ \Delta^k = (x^{k+n} - x^k + n \gamma \nabla f(x^k))/n\gamma. \]

Let \( \gamma = c/(L\sqrt{n(n+1)}) \), where \( c > 0 \). Then

\[
\|\Delta^k\|^2 \leq 50c^2 \exp\left(8c^2\right) L^2 \|x^k - x^*\|^2 + \left(4 + 200c^2 \exp\left(8c^2\right)\right) \frac{L^2}{n} \sum_{i=1}^{n} \|x^{k-i} - x^k\|^2.
\]

Proof Note that

\[
x^{k+n} - x^k = \sum_{j=0}^{n-1} x^{k+j+1} - x^{k+j}
\]

\[
= -\gamma \sum_{j=0}^{n-1} \left( \nabla f[k+j](x^{k+j}) - \nabla f[k+j](x^{k+j-n}) + \frac{1}{n} \sum_{i=1}^{n} \nabla f[k+j-i](x^{k+j-i}) \right)
\]

Define

\[
\Delta_j^k = \nabla f[k+j](x^{k+j}) - \nabla f[k+j](x^k)
\]

\[
+ \left(1 - \frac{j+1}{n}\right) \left( \nabla f[k+j](x^{k+j}) - \nabla f[k+j](x^{k+j-n}) \right).
\]

Then we have

\[
\Delta^k = -\frac{1}{n} \sum_{j=0}^{n-1} \Delta_j^k.
\]

Using the fact that \((1 - (j + 1)/n) \leq 1\), we get

\[
\|\Delta_j^k\|^2 \leq 2L^2 \|x^{k+j} - x^k\|^2 + 2L^2 \|x^{k+j} - x^{k+j-n}\|^2
\]

\[
\leq 2L^2 \|x^{k+j} - x^k\|^2 + 4L^2 \|x^{k+j} - x^k\|^2 + 4L^2 \|x^{k+j-n} - x^k\|^2
\]

\[
= 6L^2 \|x^{k+j} - x^k\|^2 + 4L^2 \|x^{k+j-n} - x^k\|^2
\]

\[
\leq \frac{j}{n} 100c^2 L^2 \exp(8c^2) \|x^k - x^*\|^2
\]

\[
+ 200 \frac{c^2 L^2}{n} \exp(8c^2) \sum_{i=1}^{n} \|x^{k-i} - x^k\|^2 + 4L^2 \|x^{k+j} - x^k\|^2.
\]
So

\[ \|\Delta^k\|^2 \leq \frac{1}{n} \sum_{j=1}^n \|\Delta_j^k\|^2 \]

\[ \leq 50^2 L^2 \exp(8c^2) \|x^k - x^*\|^2 + (4 + 200c^2 \exp(8c^2)) \frac{L^2}{n} \sum_{i=1}^n \|x^{k-i} - x^k\|^2, \]

where we used (1) and \(\sum_{j=0}^{n-1} j < n^2/2.\) \(\square\)

Proof (Theorem 1) We define

\[ x^{k+n} = x^k - n\gamma \nabla f(x^k) + n\gamma \Delta^k \]

We use (2) and (3) to get

\[ \|x^{k+n} - x^*\|^2 = \|x^k - x^*\|^2 - 2n\gamma \langle x^k - x^*, \nabla f(x^k) \rangle + 2n\gamma \langle x^k - x^*, \Delta^k \rangle \]

\[ + n^2\gamma^2 \|\nabla f(x^k) - \Delta^k\|^2 \]

\[ \leq \|x^k - x^*\|^2 - 2n\gamma \langle x^k - x^*, \nabla f(x^k) \rangle + \epsilon n\gamma \|x^k - x^*\|^2 \]

\[ + n\gamma/\epsilon \|\Delta^k\|^2 + 2n^2\gamma^2 \|\nabla f(x^k)\|^2 + 2n^2\gamma^2 \|\Delta^k\|^2, \]

for any \(\epsilon > 0.\) Apply (4) to get

\[ \|x^{k+n} - x^*\|^2 \leq (1 - 2\gamma n \frac{\mu L}{\mu + L} + n\gamma \epsilon) \|x^k - x^*\|^2 \]

\[ - \frac{n\gamma}{L + \mu} (1 - 2n\gamma (L + \mu)) \|\nabla f(x^k)\|^2 + (n\gamma/\epsilon + 2n^2\gamma^2) \|\Delta^k\|^2. \]

Since \(\kappa \geq 1,\) we have \(c < 1/65.\) So \(1 - 2n\gamma (L + \mu) < 1\) and we have

\[ \|x^{k+n} - x^*\|^2 \leq (1 - 2\gamma n \frac{\mu L}{\mu + L} + \gamma \epsilon) \|x^k - x^*\|^2 + n\gamma (1/\epsilon + 2n\gamma) \|\Delta^k\|^2 \]

\[ \leq (1 - \gamma n\mu + n\gamma \epsilon) \|x^k - x^*\|^2 + n\gamma (1/\epsilon + 2n\gamma) \|\Delta^k\|^2, \quad (6) \]

where the last inequality holds due to \(\frac{\mu L}{\mu + L} = \frac{\mu}{\kappa^{-1} + 1} \geq \frac{\mu}{\kappa^2}.\)

Using the assumption \(c < 1/65,\) we simplify Lemma 6 into

\[ \|\Delta^k\|^2 \leq 52c^2 L^2 \|x^k - x^*\|^2 + 5\frac{L^2}{n} \sum_{j=1}^n \|x^k - x^{k-j}\|^2 \quad (7) \]
and Lemma 5 into

$$\|x^{k+n} - x^k\|^2 \leq 35c^2 \left( \|x^k - x^\star\|^2 + \frac{2}{n} \sum_{j=1}^{n} \|x^k - x^{k-j}\|^2 \right).$$  \hspace{1cm} (8)

We plug in (6), (8) and (7) to have the following

$$V^{k+n} = \|x^{k+n} - x^\star\|^2 + \frac{1}{n} \sum_{j=1}^{n} \|x^{k+n} - x^{k+n-j}\|^2$$

$$\leq \left( 1 - \gamma n \mu + \gamma \epsilon + (n \gamma / \epsilon + 2n^2 \gamma^2)52c^2 L^2 + 35c^2 \right) \|x^k - x^\star\|^2$$

$$+ \left( \frac{5(n \gamma / \epsilon + 2n^2 \gamma^2)L^2 + \frac{70c^2}{n}}{n} \right) \frac{1}{n} \sum_{j=1}^{n} \|x^k - x^{k-j}\|^2$$

$$\leq \left( 1 - \gamma n \mu \left( 1 - \frac{\epsilon}{\mu} - \frac{52c^2 L^2}{\epsilon \mu} - \frac{104c^2n \gamma L^2}{\mu} - \frac{50cL}{\mu} \right) \right) \|x^k - x^\star\|^2$$

$$+ \left( \frac{5(n \gamma / \epsilon + 2n^2 \gamma^2)L^2 + \frac{70c^2}{n}}{n} \right) \frac{1}{n} \sum_{j=1}^{n} \|x^k - x^{k-j}\|^2$$

$$\leq \left( 1 - \gamma n \mu \left( 1 - \frac{\epsilon}{\mu} - \frac{52c^2 L \kappa}{\epsilon} - \frac{104c^3 \kappa}{\mu} - \frac{50c \kappa}{\mu} \right) \right) \|x^k - x^\star\|^2$$

$$+ \left( \frac{5(cL/\epsilon + 2c^2) + \frac{70c^2}{n}}{n} \right) \frac{1}{n} \sum_{j=1}^{n} \|x^k - x^{k-j}\|^2,$$

where the second and third inequality holds due to the fact that $\sqrt{n(n+1)} \leq \sqrt{2n}$ and $n \gamma = \frac{\sqrt{n}}{\sqrt{n+1}L} \leq \frac{\xi}{L}$.

For the choice of $\epsilon = \sqrt{52c}L$, since

$$\frac{\epsilon}{\mu} + \frac{52c^2 L^2}{\epsilon \mu} = 2\sqrt{52c} \kappa$$

holds, the inequality above becomes

$$V^{k+n} \leq \left( 1 - \gamma n \mu \left( 1 - 2\sqrt{52c} \kappa (1 + \sqrt{52c^2} + 3.5) \right) \right) \|x^k - x^\star\|^2$$

$$+ \left( \frac{5}{\sqrt{52}} + 10c^2 + \frac{70c^2}{n} \right) \frac{1}{n} \sum_{j=1}^{n} \|x^k - x^{k-j}\|^2.$$
Since $\kappa \geq 1$,

\[
V^{k+n} \leq \left(1 - \gamma n \mu (1 - 2\sqrt{52}c\kappa (1 + \sqrt{52}c^2\kappa^2 + 3.5))\right) \|x^k - x^*\|^2 \\
+ \left(\frac{5}{\sqrt{52}} + 10c^2\kappa^2 + \frac{70c^2\kappa^2}{n}\right) \frac{1}{n} \sum_{j=1}^{n} \|x^k - x^{k-j}\|^2.
\]

Under $c\kappa < 1/65$, it is simple to check

\[
2\sqrt{52}c\kappa (1 + \sqrt{52}c^2\kappa^2 + 3.5) < 1,
\]

and

\[
\frac{5}{\sqrt{52}} + 10c^2\kappa^2 + \frac{70c^2\kappa^2}{n} < 1.
\]

This gives us a contraction. For $c\kappa = 1/130$, the contraction factor is

\[
\max\left\{1 - \frac{\gamma n \mu}{2}, \frac{5}{\sqrt{52}} + 10c^2\kappa^2 + \frac{10c^2\kappa^2}{n}\right\} \leq 1 - \frac{1}{368\kappa^2},
\]

where again we use the fact that $\sqrt{n(n+1)} \leq \sqrt{2}n$ and $\kappa \geq 1$. \qed

### 3 Experiments

Figure 1 shows the experimental results. Overall, we observe that C-SAGA performs better than IAG, which is consistent with the theory. We also observe that C-SAGA is slower than DIAG in iteration count but is faster in wall clock time due to the acceleration just-in-time updates provide. For the RCV dataset with $n = 20,242$ and $m = 47,237$, DIAG and Finito (the randomized version of DIAG) took more than 10 h while the other methods took less than 5 s. For the experiments, we modified Defazio’s code [6], which implements SAG and SAGA, but not Finito. The dataset is a selection of commonly used datasets from the LIBSVM repository [5]. For IAG, C-SAGA, SAG, and SAGA, we use just-in-time updates (implemented by Defazio) to accelerate the computation as explained in Section 4.1 of [22]. As discussed in Sect. 2, just-in-time updates are not applicable to DIAG and Finito. For each algorithm, we ran experiments for a wide range of stepsizes from 8192 to $10^{-4}$ and chose the best one. The time measurements were taken on a MacBook Air with a 1.3 GHz Intel Core i5 CPU.

### 4 Conclusion

In this work, we analyzed C-SAGA and compared C-SAGA to existing methods, theoretically and experimentally. As an aside, we observed that the random permu-
Fig. 1 Function suboptimality vs. epoch: from top to bottom, the rows correspond to the AUSTRALIAN, COVTYPE, MUSHROOM, and RCV1 dataset. From left to right, each column corresponds to a 5, 10, 100% subsampling of the dataset. For the bottom right corner setup (the entire RCV1 dataset) Finito and DIAG took more than 10 h, while the other four methods took less than 5 s due to just-in-time updates.

...tion variant of SAGA outperforms SAGA and C-SAGA. Our experimental results conform with the curiously persistent phenomenon that, among index selection rules, random permutation outperforms IID, which in turn outperforms cyclic. Investigating the effect of random permutations on incremental methods and systematically comparing the deterministic and randomized complexity of finite-sum optimization problems is an interesting direction of future work.

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