Non-secret correlations can be used to distribute secrecy

Joonwoo Bae, Toby Cubitt and Antonio Acín

1School of Computational Sciences, Korea Institute for Advanced Study, Seoul 150-032, Korea
2Department of Mathematics, University of Bristol, University Walk, BS8 1TW, UK
3ICFO-Institut de Ciències Fotòniques, Mediterranean Technology Park, 08860 Castelldefels (Barcelona), Spain
4ICREA-Institució Catalana de Recerca i Estudis Avançats, Lluis Companys 23, 08010 Barcelona, Spain

(Dated: June 10, 2008)

A counter-intuitive result in entanglement theory was shown in [PRL 91 037902 (2003)], namely that entanglement can be distributed by sending a separable state through a quantum channel. In this work, following an analogy between the entanglement and secret key distillation scenarios, we derive its classical analog: secrecy can be distributed by sending non-secret correlations through a private channel. This strengthens the close relation between entanglement and secrecy.

PACS numbers:

I. INTRODUCTION

Entangled and secret bits are different information resources that turn out to be closely connected. An entangled bit, or ebit, corresponds to a maximally entangled state of two qubits,

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle),$$

(1)

and represents the basic unit of bipartite entanglement [1]. A standard problem in entanglement theory is, given an arbitrary bipartite quantum state, \(\rho_{AB}\), to determine how many ebits are required for its formation or can be distilled out of it by local operations and classical communication (LOCC).

On the other hand, secret bits, or sbits, are the basic unit of classical secret correlations. These arise when two honest parties, Alice and Bob, share correlated random variables, A and B, whereas the eavesdropper, Eve, holds a third random variable E. The total correlations are then described by a tripartite probability distribution, \(P(A, B, E)\). This distribution is a perfect sbit whenever

$$P(A = B = 0) = P(A = B = 1) = \frac{1}{2}$$

$$P(A, B, E) = P(A, B)P(E).$$

Note that Alice and Bob’s variables are perfectly correlated, while Eve gets no information whatsoever about them from her outcome. Similarly to the case of quantum states, a basic question is to quantify the number of sbits that are required to create a given tripartite probability distribution \(P(A, B, E)\), or that can be distilled out of it by local operations and public communication (LOPC).

There exist several analogies between the entanglement properties of quantum states and the cryptographic properties of the classical probability distributions derived from them by local measurements [2, 3]. In order to construct these analogies, one has to explicitly introduce a third party in the quantum scenario. This can easily be done by noting that any bipartite mixed state \(\rho_{AB}\) can be seen as a tripartite pure state \(|\psi\rangle_{ABE}\), such that \(\text{tr}_E|\psi\rangle\langle\psi|_{ABE} = \rho_{AB}\). Indeed, the environment (that is, the part of the global system that is not under the honest parties’ control) can naturally be associated with an adversary party, the eavesdropper. The goal is then to connect the entanglement properties of \(\rho_{AB}\) to the cryptographic properties of those probability distributions \(P(A, B, E)\) that can be written as

$$P(A, B, E) = \text{tr}(M_A \otimes M_B \otimes M_E|\psi\rangle\langle\psi|_{ABE}).$$

(2)

\(M_A, M_B\) and \(M_E\) are positive operators defining a quantum measurement in each local space, i.e., \(\sum_i M_i = \mathbb{I}_i\) with \(i = A, B, E\). Of course, the same rule can be applied in a multipartite scenario, where the quantum state \(\rho_{ABC...}\) is shared among \(N\) parties.

A first rather trivial analogy follows from the fact that one sbit can directly be obtained by measuring one ebit in, say, the computational basis. This simple observation is behind some of the security proofs of quantum key distribution protocols [1]. Beyond this basic analogy, other classical analogs of quantum information phenomena have been derived, and vice-versa. For instance, the results of [3] on the existence of what is called negative quantum information were translated into the classical scenario, obtaining analogous results for the secret-key rate [4]. In [3], a systematic way of mapping any entangled state onto a probability distribution containing secret correlations was derived. One of the nicest concepts in this direction is the existence of a cryptographic analog of bound entanglement, known as bound information, first conjectured in [5]. Recall that a quantum state is bound entangled when, despite being entangled, it is impossible to distill pure ebits out of it by LOCC. The existence and activation of non-distillable secret correlations has been demonstrated in [3] for the multipartite scenario, adapting some known results for multipartite bound entangled states. The existence of bipartite bound information remains an open question. Other results have followed the opposite direction, going from classical to quantum: the so-called squashed entanglement is an entanglement measure whose construction was inspired by a known upper bound on the secret-key rate [3]. In
general, the connection between entangled states and secret correlations is a useful tool in the study of ebits and sbits, since it provides much insight into these two fundamental resources. Note, however, that this analogy is not a strict correspondence: there are “exceptions”, such as the existence of bound entangled states that can be mapped into distillable probability distributions \[\text{[10]}\].

A remarkable result in entanglement theory was obtained in \[\text{[11]}\], where it was shown that entanglement can be distributed by sending a separable, non-entangled state through a quantum channel. The scope of this work is to study whether a similar result holds for probability distributions. Following the analogy between ebits and sbits, we indeed prove that this is the case: secret correlations can be distributed by sending non-secret correlations through a private channel.

The article is structured as follows. In the next section, we introduce the basic rules that apply when translating results from the quantum to the classical scenario, and vice-versa. Section \[\text{III}\] briefly reviews the results of \[\text{[11]}\], showing how to entangle two distant parties by sending a separable state. The main results are given in section \[\text{IV}\] where we show how to distribute secrecy by sending non-secret correlations. Finally, we discuss some relevant issues and conclude.

**II. ENTANGLEMENT VS. SECRET CORRELATIONS**

A standard scenario in entanglement theory consists of \(N\) distant parties, \(A, B, C, \ldots\), who share quantum correlations described by a state \(\rho\). The state may be mixed due to coupling to the environment, \(E\), the overall state being \(|\Psi\rangle\), where \(\rho = \text{tr}_E|\Psi\rangle \langle \Psi|\). The two main questions in this scenario are: (i) is the preparation of \(\rho\) possible by LOCC? and (ii) if not, can pure ebits be distilled from \(\rho\) by LOCC? These two questions define the separability and distillability problems. When considering the key-agreement scenario, many similarities appear (see for instance \[\text{[2, 3]}\]). Now, \(N\) distant honest parties and an eavesdropper share correlated random variables, described by a probability distribution \(P(A, B, C, \ldots, E)\). The corresponding questions are: (i) can these correlations be established by LOPC? and (ii) if not, can pure sbits be distilled by LOPC?

Most of the analogies between the two scenarios can be summarised as follows:

| Quantum entanglement | Secret correlations |
|----------------------|---------------------|
| Quantum communication | Private communication |
| Classical communication | Public communication |
| Local operations | Local actions |

Here a private channel is a classical channel that is only accessible to the honest parties. Using these intuitive rules, one can often adapt results from entanglement theory to the secret correlations scenario and vice-versa. For instance, a state is bound entangled whenever its formation by LOCC is impossible but nevertheless it cannot be transformed into pure ebits by LOCC. The corresponding concept for secret correlations, known as bound information, is simply given by a probability distribution that, despite its formation by LOPC being impossible, cannot be transformed into pure sbits by LOPC.

More quantitative statements can be made in the bipartite case. In the case of quantum states, the ebit represents the basic unit of entanglement. The number of ebits per copy that can be distilled out of copies of a given quantum state by LOCC defines the distillable entanglement, \(E_D\) \[\text{[12]}\]. The corresponding classical analog is the secret-key rate \(S(A : B|E)\) which gives the number of sbits distillable from \(P(A, B, E)\) by LOPC \[\text{[13]}\]. In a similar way, the number of ebits required per copy for the formation of an entangled state defines the entanglement cost, \(E_C\). The so-called information of formation, \(I_c(A ; B|E)\), introduced in \[\text{[14]}\], represents its classical analog. A probability distribution contains secret correlations if, and only if, its information of formation is non-zero \[\text{[14]}\].

A useful upper bound for \(S(A : B|E)\) is given by the so-called intrinsic information \[\text{[13]}\]. The intrinsic information between \(A\) and \(B\) given \(E\) is defined as:

\[
I(A : B | E) = \min_{E \rightarrow \tilde{E}} I(A : B|\tilde{E}), \tag{3}
\]

where the minimisation runs over all possible stochastic maps \(P(E|\tilde{E})\) defining a new random variable \(\tilde{E}\). The quantity \(I(A : B|E)\) is the mutual information between \(A\) and \(B\) conditioned on \(E\). It can be written as

\[
I(A : B|E) = H(A, E) + H(B, E) - H(A, B, E) - H(E),
\]

where \(H(X)\) is the Shannon entropy of the random variable \(X\). It also gives a lower bound on the information of formation \[\text{[14]}\], thus

\[
S(A : B | E) \leq I(A : B | E) \leq I_c(A ; B|E). \tag{4}
\]

In fact, \(I_c(A ; B|E) > 0\) if, and only if, \(I(A : B | E) > 0\) \[\text{[14]}\]. The intrinsic information plays a key role in the proof of our results.

**III. QUANTUM SCENARIO**

Before presenting our results, we summarise the findings of Ref. \[\text{[11]}\] on the distribution of entanglement by means of a separable state. The scenario consists of two initially uncorrelated distant parties who are connected by a classical and a quantum channel \[\text{[17]}\]. In order to entangle two distant qubits, \(A\) and \(B\), the parties must use the quantum channel, since no entanglement can be created by LOCC. Thus one of the parties, say Alice, should prepare an additional qubit, \(C\), and send it to Bob. Clearly, a sufficient condition for entanglement distribution is that the mediating quantum particle \(C\) is entangled with Alice’s quantum system \(A\), so that Bob becomes entangled with her after receiving it. Intuitively,
one would expect that this is also a necessary condition. Remarkably, this is not the case, as shown in Ref. [11], where an explicit counterexample is provided in which Alice distributes entanglement to Bob by sending a qubit $C$ through the quantum channel that is never entangled across the partition $C – AB$.

The example works as follows. Alice holds two qubits, $A$ and $C$, while Bob has one qubit, $B$, in the initial state

$$\rho_{ABC} = \frac{1}{6} \sum_{k=0}^{3} |\Psi_k, \Psi_{-k}, 0\rangle \langle \Psi_k, \Psi_{-k}, 0| + \sum_{i=0}^{6} \frac{1}{6} |i, i, 1\rangle \langle i, i, 1|$$

where $|\Psi_k\rangle = (|0\rangle + e^{\pm i k/2}|1\rangle)/\sqrt{2}$. This state is fully separable across all partitions, so it can be prepared by LOCC. Alice now applies a controlled-NOT (CNOT) operation to her qubits, where $A$ ($C$) is the control (target) qubit, resulting in the state

$$\sigma_{ABC} = \frac{1}{3} |\Psi_{GHZ}\rangle \langle \Psi_{GHZ}| + \sum_{i,j,k=0}^{1} \beta_{ijk} |ijk\rangle \langle ijk|$$

where $|\Psi_{GHZ}\rangle_{ABC} = (|000\rangle + |111\rangle)/\sqrt{2}$, $\beta_{001} = \beta_{010} = \beta_{101} = \beta_{110} = 1/6$, and all other $\beta_{ijk} = 0$. This state is still separable across the $C – AB$ partition [16]. Alice now sends $C$ to Bob, who applies a CNOT with $B$ ($C$) as the control (target) qubit. After all these steps, Alice and Bob share a state

$$\tau_{ABC} = \frac{1}{3} |\Phi^+\rangle \langle \Phi^+|_{AB} \otimes |0\rangle \langle 0|_C + \frac{2}{3} |\Phi\rangle \langle \Phi|_AB \otimes |1\rangle \langle 1|_C,$$

where Bob has both $B$ and $C$. This state is distillable. Indeed, by measuring particle $C$ in the computational basis, Alice and Bob’s systems are projected into a maximally entangled state of two qubits with probability $1/3$.

### IV. A TRANSLATED CLASSICAL SCENARIO

We now translate the previous quantum result to the key-agreement scenario. Namely, we show that secret correlations can be distributed by sending through the private channel a random variable that does not have secret correlations with either Alice and/or Bob. For the construction of the example, we can follow the “rules” given in section [11]. Following [2], the initial quantum state [3] is replaced by the probability distribution obtained by measuring in the computational bases:

$$A \quad B \quad C \quad E \quad P(A, B, C, E)$$

|       |       |       |       |                 |
|-------|-------|-------|-------|------------------|
| 0 0 0 | $e_0$ | 1/6   | (8)   |
| 0 1 0 | $e_{01}$ | 1/6 |
| 1 0 0 | $e_{10}$ | 1/6 |
| 1 1 0 | $e_0$   | 1/6 |
| 0 0 1 | $f_0$   | 1/6 |
| 1 1 1 | $f_1$   | 1/6 |

Recall that for any separable state there exists an optimal measurement by Eve such that the intrinsic information for the obtained distribution [2] is zero for all choices of measurements by Alice and Bob [3]. However, Eve’s optimal measurement is not necessarily in the computational basis. Thus, it is not immediate that the distribution [3] contains no secret correlations. However, it is possible to prove that this is indeed the case: the distribution has zero intrinsic information across the bipartition $AC – B$, since $I(AC : B|E) = 0$, which does imply that Alice and Bob do not share secret correlations.

Now, Alice performs the (classical) CNOT operation on $A$ and $C$, and sends $C$ through the private channel to Bob, who performs the CNOT operation on $B$ and $C$. After Alice’s CNOT, the probability distribution is

$$A \quad B \quad C \quad E \quad P(A, B, C, E)$$

|       |       |       |       |                 |
|-------|-------|-------|-------|------------------|
| 0 0 0 | $e_0$ | 1/6   |
| 0 1 0 | $e_{01}$ | 1/6 |
| 1 0 1 | $e_{10}$ | 1/6 |
| 1 1 1 | $e_0$   | 1/6 |
| 0 0 1 | $f_0$   | 1/6 |
| 1 1 0 | $f_1$   | 1/6 |

This distribution has zero intrinsic information across the partition $C – AB$. Indeed, consider the map $E \rightarrow \overline{E}$ in which Eve replaces $f_0$ and $f_1$ by $e_0$, but leaves everything else untouched. The resulting probability distribution has $I(A : B|E) = 0$, thus $I(AB : C \perp E) = 0$ for [3]. That is, the $C$ that is sent through the private channel does not share secret correlations with $A$ and/or $B$.

The final probability distribution between Alice and Bob, after Bob’s CNOT, is

$$A \quad B \quad C \quad E \quad P(A, B, C, E)$$

|       |       |       |       |                 |
|-------|-------|-------|-------|------------------|
| 0 0 0 | $e_0$ | 1/6   |
| 0 1 0 | $e_{01}$ | 1/6 |
| 1 0 1 | $e_{10}$ | 1/6 |
| 1 1 0 | $e_0$   | 1/6 |
| 0 0 1 | $f_0$   | 1/6 |
| 1 1 1 | $f_1$   | 1/6 |

We now show how Alice and Bob can distill 1 sbit from this distribution. Bob holds two of the random variables, $B$ and $C$. He receives $C = 0$ and $C = 1$ with probabilities, $1/3$ and $2/3$ respectively. By LOPC, Alice and Bob keep their outcome whenever $C = 1$, otherwise they reject the instance. Thus, with probability $1/3$, Alice and Bob (and Eve) are correlated according to:

$$A \quad B \quad C \quad E \quad P(A, B, C, E)$$

|       |       |       |       |                 |
|-------|-------|-------|-------|------------------|
| 0 0 0 | $e_0$ | 1/2   |
| 1 1 0 | $e_0$   | 1/2 |

which defines a perfect sbit. Thus, Alice and Bob are able to distribute distillable secret correlations by send-
ing a random variable, that does not itself have secret
correlations, through a private channel.

V. CONCLUDING REMARKS

In this work, we have constructed the cryptographic
analog of the distribution of (distillable) entanglement by
a separable state: (distillable) secrecy can be distributed
by sending non-secret correlations. This result is com-
pletely equivalent to its entanglement analog: the use of
the quantum (private) channel is essential for the success-
ful entanglement (secrecy) distribution, even though the
mediating particle (random variable) never has quantum
(secret) correlations with the sender and/or receiver.

At first sight, the existence of this cryptographic ana-
log suggests some interesting possibilities. For instance,
one might imagine that secrecy could be distributed by
an untrusted messenger, Charlie, who after transmitting
the relevant information could not break the established
secret key. Clearly, this is not the case if the transmit-
ter can later collaborate with the eavesdropper. Indeed,
since the information is classical, Charlie can keep a per-
fect copy of the transmitted random variable, $C$, and
give it to Eve. The channel is no longer private, and it
is known that the distribution of secret correlations by
LOPC is impossible.

One could however consider a less restrictive scenario,
in which the transmitter is still untrusted, but it is as-
sumed that he does not collaborate with Eve. Can Alice
and Bob use the above protocol to distill a secret key
against Eve and Charlie separately? Indeed, they can.
However, a much simpler protocol already achieves this.
Alice, Bob and Eve initially share a public perfectly cor-
related bit, $P(a = b = e = 0) = P(a = b = e = 1) = 1/2$.
Of course, no key extraction is possible. Then, Alice
generates a random bit which she sends to Bob, via the
messenger Charlie. Charlie leaves and may try to break
the key, but he is not allowed to collaborate with Eve.
When he delivers the random bit, Alice, Bob and Charlie
also share a perfectly correlated bit. It is clear that Alice
and Bob can distill a key against Charlie and Eve (if they
don’t collaborate), by taking the XOR of their two bits.

VI. ACKNOWLEDGEMENTS

This work is supported by the EU QAP project, the
Spanish MEC, under FIS2007-60182 and Consolider-
Ingenio QOIT projects, the Caixa Manresa, the Generali-
tat de Catalunya, and the IT R&D program of MIC/ITA
[2005-Y-001-04 , Development of next generation security
technology].

[1] C. H. Bennett, H. J. Bernstein, S. Popescu, and B. Schu-
macher, Phys. Rev. A 53, 2046 (1996).
[2] D. Collins and S. Popescu, Phys. Rev. A. 65 032321
(2002).
[3] N. Gisin and S. Wolf, Algorithmica 34 389-412 (2002).
[4] P. Shor and J. Preskill, Phys.Rev.Lett. 85 441-444
(2000).
[5] M. Horodecki, J. Oppenheim and A. Winter, Nature 436
673-676 (2005).
[6] J. Oppenheim, R. Spekkens and A. Winter, quant-ph/0511247
[7] A. Acín and N. Gisin, Phys. Rev. Lett. 94, 020501 (2005).
[8] A. Acín, J. I. Cirac and Ll. Masanes, Phys. Rev. Lett. 91
107903 (2004); Ll. Masanes and A. Acin, cs.CR/0501008
[9] to appear in IEEE Trans. Inf. Theory.
[10] M. Christandl and A. Winter, J. Math. Phys. 45, No 3,
829-840 (2004).