SOLVABILITY OF DOUBLY NONLINEAR PARABOLIC EQUATION WITH $p$-LAPLACIAN

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Abstract. In this paper, we consider a doubly nonlinear parabolic equation
\[ \partial_t \beta(u) - \nabla \cdot \alpha(x, \nabla u) \ni f \quad (x, t) \in Q := \Omega \times (0, T), \]
\[ u(x, t) = 0 \quad (x, t) \in \partial \Omega \times (0, T), \]
where $\Omega \subset \mathbb{R}^d \ (d \geq 1)$ be a bounded domain with a sufficiently smooth boundary $\partial \Omega$. Throughout this paper, we impose the followings on $\alpha$ and $\beta$:

(H.\alpha) There exist a $C^1$-class function $a : \Omega \times \mathbb{R}^d \to \mathbb{R}$ such that $a(x, \cdot) : \mathbb{R}^d \to \mathbb{R}$ is convex and $\alpha(x, z) = D_z a(x, z)$ holds for every $z \in \mathbb{R}^d$ and almost every $x \in \Omega$. Moreover, $a$ and its derivative $\alpha : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ satisfy the followings with some exponent $p \in (1, \infty)$ and constants $c, C > 0$:

(1) \[ c|z|^p - C \leq a(x, z) \leq C(|z|^p + 1), \]

(2) \[ \alpha(x, z)| \leq C(|z|^{p-1} + 1), \quad \alpha(x, 0) = 0, \]

and

(3) \[ (\alpha(x, z_1) - \alpha(x, z_2)) \cdot (z_1 - z_2) \geq \frac{c|z_1 - z_2|^2}{|z_1|^{2-p} + |z_2|^{2-p} + C} \quad \text{if} \ 1 < p < 2, \]

for every $z, z_1, z_2 \in \mathbb{R}^n$ and almost every $x \in \Omega$.

1. Introduction

In this paper, we are concerned with the initial boundary value problem of the following doubly nonlinear equation:

\[ \{ \begin{align*}
\partial_t \beta(u(x, t)) - \nabla \cdot \alpha(x, \nabla u(x, t)) & \ni f(x, t) \quad (x, t) \in Q := \Omega \times (0, T), \\
u(x, t) & = 0 \quad (x, t) \in \partial \Omega \times (0, T),
\end{align*} \]

where $\Omega \subset \mathbb{R}^d \ (d \geq 1)$ be a bounded domain with a sufficiently smooth boundary $\partial \Omega$. Throughout this paper, we impose the followings on $\alpha$ and $\beta$:

(H.\alpha) There exist a $C^1$-class function $a : \Omega \times \mathbb{R}^d \to \mathbb{R}$ such that $a(x, \cdot) : \mathbb{R}^d \to \mathbb{R}$ is convex and $\alpha(x, z) = D_z a(x, z)$ holds for every $z \in \mathbb{R}^d$ and almost every $x \in \Omega$. Moreover, $a$ and its derivative $\alpha : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ satisfy the followings with some exponent $p \in (1, \infty)$ and constants $c, C > 0$:

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(3) \[ (\alpha(x, z_1) - \alpha(x, z_2)) \cdot (z_1 - z_2) \geq \frac{c|z_1 - z_2|^2}{|z_1|^{2-p} + |z_2|^{2-p} + C} \quad \text{if} \ 1 < p < 2, \]

for every $z, z_1, z_2 \in \mathbb{R}^n$ and almost every $x \in \Omega$.

2010 Mathematics Subject Classification. Primary 35K92; Secondary 35K61, 47J35, 34G25.

Key words and phrases. Doubly nonlinear equation, parabolic type, initial boundary value problem, $p$-Laplacian, well-posedness, entropy solution.

Supported by the Fund for the Promotion of Joint International Research (Fostering Joint International Research (B)) #18KK0073, JSPS Japan.
(H.β) $\beta : \mathbb{R} \to 2^\mathbb{R}$ is maximal monotone and satisfies $0 \in \beta(0)$.

A typical example of $Au(x) := -\nabla \cdot \alpha(x, \nabla u(x, t))$ is the so-called $p$-Laplacian $-\Delta_p u := -\nabla \cdot (|\nabla u|^{p-2} \nabla u)$, which satisfies (H.α) with $\alpha(x, z) = \frac{1}{p}|z|^p$ and $\alpha(x, z) = |z|^{p-2} z$. The sum of a finite number of $p_i$-Laplacian ($i = 1, 2, \ldots, n$) also fulfills (H.α) with $p = \max_{i=1,2,\ldots,n} p_i$.

Putting $\beta(s) = |s|^{r-2}s$ with $r > 1$ in (P), we obtain

$$\partial_t |u|^{r-2}u - \Delta_p u = f,$$

which has a huge amount of previous works, e.g., [9] [15] [33] [47] [53]. We also refer to the following equation as an example of (P):

$$\partial_t v - \Delta \log v = f,$$

which is considered in, for instance, [20] [33] for $d = 1$, [49] [54] [55] [56] for $d = 2$, and [26] [46] for higher dimensions. This equation with the boundary condition $v|_{\partial\Omega} = 1$ can be reduce to (P) by $p = 2$, $u := \log v$, and $\beta(u) := e^u - 1$. In Miyoshi–Tsustumi [45], they obtain the following logarithmic diffusion equation by passing to the singular limit of a generalized Carleman model:

$$\partial_t v - \nabla \cdot (|\nabla \log v|^{p-2} \nabla \log v) = f.$$

Compared with above two, there are very few investigations for this equation, e.g., in [34]. In order for (P) to cover such equations possessing strong nonlinearity, we need to mitigate the growth condition or coerciveness of $\beta$. Moreover, if one deal with Stefan problem [50], Hele-Shaw problem [30], constraint problem ($\beta$ is the subdifferential of an indicator function), or combination of them, one might face the case where $\beta$ is multi-valued and $D(\beta) \neq \mathbb{R}$.

The solvability for $p = 2$, i.e., $A = -\Delta$, can be derived from the abstract theory of evolution equations given by Bénilan [13] and Brézis [16]. Carrillo [23] and Kobayashi [59] discussed the existence and the uniqueness of solution to (P) with a hyperbolic term. As for the case of $p \neq 2$, Alt–Luckhaus [3] considered the system of quasilinear doubly nonlinear equations and assured the solvability of (P) where $\beta$ is single-valued or multi-valued with a growth condition for jump. We can find the solvability results of (P) in, e.g., [6] [14] [21] [24] [29] for the case where $\beta$ is single-valued and in [3] [7] [22] [32] [37] for multi-valued. In these study, they imposed some restriction on $\beta$ besides $0 \in \beta(0)$ instead of adopting generalized quasilinear terms $\alpha = \alpha(u, \nabla u)$ or $\alpha(x, u, \nabla u)$. In Akagi–Stefanelli [2], they proposed a different approach from those in the above, which are based on the time-discretization technique and $L^1$-contraction principle. They reduce (P) to $0 \in (A)^{-1}(f - \partial_\beta \xi) - \beta^{-1}(\xi)$, where $\xi \in \beta(u)$, and use the so-called Weighted Energy Dissipation (WED) method.

We next comment on the results for the abstract evolution equation in Banach spaces:

$$\partial_t Bu + Au \ni f,$$

where $A$ and $B$ are maximal monotone operators. Under some boundedness or coerciveness condition of operators, the solvability has been obtained by, e.g., [1] [12] [31] [35] [36] [38] [44] [51] [52] [57]. On the other hand, Barbu [10] removed such assumptions by using the angular condition between $A$ and the Yosida approximation of $B$ and proved the solvability of (P) with $p \geq 2$ for any maximal monotone graph satisfying $0 \in \beta(0)$. Here the condition $p \geq 2$ seems to be essential in this
result since it is hard to obtain the explicit formula of the Yosida approximation of \( \tilde{\beta} \), the realization of \( \beta \) in \( L'(\Omega) \) for \( r \neq 2 \).

Main purpose of this paper is to show the existence of solution to (P) without any restriction to the exponent \( p \in (1, \infty) \) and the nonlinearity \( \beta \) except \( 0 \in \beta(0) \). From viewpoint of application to specific physical models, such generalization might seem to be excessive. However, it is still important to complete the solvability result for arbitrary \( \beta \) as an auxiliary problem for the classification of \( \beta \) by occurrence of extinction phenomena (see, e.g., \cite{27, 28}). Another aim of this article is to establish better estimates and regularities of solution, which may be a useful tool for investigating time-global behavior and perturbation theory. Although the smoothing property is a fundamental result for standard quasilinear parabolic equations, it is not obvious that the smoothness of solution is inherited from the given data in the doubly nonlinear equation with general nonlinearity since the monotonicity between the derivatives of \( \tilde{\beta}(u) \) and \( u \) no longer holds. Instead of relaxing the requirement of \( \beta \), we impose some stricter condition on the given initial data, external force \( f \), and \( \alpha \) than those in the previous works given above. In the next section, we fix several notations and state our main results more precisely. Section 3 and 4 will be expended on the demonstration. We first deal with an elliptic equation affiliated with (P) in Section 3 and employ the standard time-discretization technique given by Raviart \cite{47} and Grange–Mignot \cite{35} in Section 4. In the final section, we discuss the uniqueness of solution by following the argument by Carrillo \cite{23}, where the properties of entropy solution (see Kružkov \cite{40, 41}) are neatly used. We here collect some notations and basic properties which will be used later (see e.g., \cite{8, 11, 18, 15}).

\section{Main Theorem}

\subsection{Definition and Notation.}

We here collect some notations and basic properties that will be used later (see e.g., \cite{8, 11, 18, 15}). Let \( X \) be a Banach space with norm \( \| \cdot \|_X \) and \( X^* \) be its dual space with norm \( \| \cdot \|_{X^*} \). Duality pairing between \( X \) and \( X^* \) is denoted by \( \langle \cdot, \cdot \rangle_X \). Moreover, \( D(A) \) and \( R(A) \) stand for the domain and the range of an operator \( A \), respectively. When \( A \) is multi-valued, we identify \( A \) with its graph \( G(A) \) and write \( [u,v] \in A \) to describe \( u \in D(A) \) and \( v \in Au \). An operator \( A : X \to 2^{X^*} \) (the power set of \( X^* \) ) is said to be monotone if \( \langle u_i^* - u_2^*, u_1 - u_2 \rangle_X \geq 0 \) holds for every \( [u_i, u_2^*] \in A \) \( (i = 1, 2) \) and a monotone operator \( A \) is said to be maximal monotone if there is no monotonic extension of \( A \). When \( X = X^* \) is a Hilbert space, maximality is equivalent to \( R(id + \lambda A) = X \) for any \( \lambda > 0 \), where \( id : X \to X \) stands for the identity mapping.

A typical example of maximal monotone operator is the subdifferential. Let \( \phi : X \to (-\infty, +\infty] \) be a proper \( (\phi \not\equiv +\infty) \) lower semi-continuous convex functional. We define the subdifferential operator \( \partial_X \phi : X \to 2^{X^*} \) associated with \( \phi \) by

\[
\partial_X \phi(u) := \{ u^* \in X^* \mid \langle u^*, v - u \rangle_X \leq \phi(v) - \phi(u) \quad \forall v \in D(\phi) \},
\]

where \( D(\phi) := \{ v \in X \mid \phi(v) < +\infty \} \) is called the effective domain of \( \phi \). Obviously \( 0 \in \partial \phi(u) \) implies that \( \phi \) attains its minimum at \( u \). Let \( V \) be another Banach space which is densely embedded in \( X \) and satisfies \( D(\phi) \subset V \). Then the restriction of \( \phi \) onto \( V \) can be regarded as a proper lower semi-continuous convex functional on \( V \).
and its subdifferential is denoted by
\[ \partial V \phi(u) := \{ u^* \in V^* : \langle u^*, v - u \rangle \leq \phi(v) - \phi(u) \quad \forall v \in D(\phi) \}. \]
In general, we have \( \partial X \phi \subset \partial V \phi \), namely, it holds that \( D(\partial X \phi) \subset D(\partial V \phi) \) and \( \partial X \phi(u) \subset \partial V \phi(u) \) for any \( u \in D(\partial X \phi) \). We also define
\[ \phi_\lambda(u) := \inf_{v \in X} \left\{ \frac{\|u - v\|_X^2}{2\lambda} + \phi(v) \right\} \quad \lambda > 0, \]
which is called the Moreau–Yosida regularization of \( \phi \). If \( X \) and \( X^* \) are strictly convex, \( \partial X \phi \) coincides with the Yosida approximation of \( \partial X \phi \) with parameter \( \lambda > 0 \). Furthermore, the Legendre–Fenchel transformation (conjugate) of \( \phi \) is defined by
\[ \phi^*(u^*) := \sup_{v \in X} \{ \langle u^*, v \rangle - \phi(v) \}. \]
Remark that \( \phi^* : X^* \to (-\infty, +\infty] \) is proper lower semi-continuous convex and \( \partial X^* \phi^* = (\partial X \phi)^{-1} \). Moreover, \( \xi \in \partial X \phi(u) \) holds if and only if \( \phi(u) + \phi^*(\xi) = \langle \xi, u \rangle_X \).

For example, let \( X = L^r(\Omega) \) with \( r \in (1, \infty) \) and define a functional \( \varphi \) by
\[ \varphi(u) := \begin{cases} \int_{\Omega} a(x, \nabla u(x)) \, dx & \text{if } u \in L^r(\Omega) \cap W^{1,p}_0(\Omega), \\ +\infty & \text{if } u \in L^r(\Omega) \setminus W^{1,p}_0(\Omega). \end{cases} \]
From the assumption (H.1), \( \varphi \) is a lower semi-continuous and convex functional on \( L^r(\Omega) \). Since \( \mathcal{D}(\Omega) \) (the set of infinitely differentiable functions with compact support in \( \Omega \)) is dense in \( L^r(\Omega) \cap W^{1,p}_0(\Omega) \), \( \varphi \) is Gâteaux differentiable and \( \partial L^r \varphi(u) = Au \) with domain \( D(\partial L^r \varphi) = \{ u \in L^r(\Omega) \cap W^{1,p}_0(\Omega) : Au \in L^r(\Omega) \} \), where \( r' := r/(r-1) \) is the Hölder conjugate exponent of \( r \). Moreover, by letting \( r = p \) in \( \mathcal{H} \) and restricting \( \varphi \) onto \( V = W^{1,p}_0(\Omega) \), we can see that \( \partial W^{1,p}_0 \varphi(u) = Au \) with domain \( D(\partial W^{1,p}_0 \varphi) = \{ u \in W^{1,p}_0(\Omega) : Au \in W^{-1,p'}(\Omega) \} = W^{1,p}_0(\Omega) \) by \( \mathcal{H} \), where \( W^{-1,p'}(\Omega) \) is the dual of \( W^{1,p}_0(\Omega) \).

If \( X = X^* = \mathbb{R} \), the maximal monotone operator always possesses a primitive function. That is to say, if \( \beta : \mathbb{R} \to 2^{\mathbb{R}} \) is a maximal monotone graph, there exist a proper lower semi-continuous convex function \( j : \mathbb{R} \to (-\infty, +\infty] \) such that \( \beta = \partial j \). If \( 0 \in \beta(0) \) holds, we can assume \( j(0) = 0 \) and \( j \geq 0 \) without loss of generality. Let \( X = L^r(\Omega) \) with \( r \in (1, \infty) \), then
\[ \psi(u) := \begin{cases} \int_{\Omega} j(u(x)) \, dx & \text{if } u \in L^r(\Omega), \ j(u) \in L^1(\Omega), \\ +\infty & \text{otherwise}, \end{cases} \]
is proper lower semi-continuous convex on \( L^r(\Omega) \) and \( \xi \in \partial L^r \psi(u) \) if and only if \( \xi \in L^{r'}(\Omega) \) and \( \xi(x) \in \beta(u(x)) \) for a.e. \( x \in \Omega \). In this sense, we write \( \beta := \partial L^r \psi \) and call it the realization of \( \beta \) in \( L^r(\Omega) \). However, if one consider the subdifferential of the canonical restriction \( \psi \) onto \( V = W^{1,p}_0(\Omega) \), then \( \xi \in \partial W^{1,p}_0 \psi(u) \) does not necessarily imply \( \xi(x) \in \beta(u(x)) \) except the case of \( D(\beta) = \mathbb{R} \) (see Brézis [17]).

Define the resolvent of \( \beta \) by \( J_\lambda := (id + \lambda \beta)^{-1} \) and the Yosida approximation by \( \beta_\lambda := (id - J_\lambda)/\lambda \). Since \( \beta_\lambda \) is a Lipschitz continuous function on \( \mathbb{R} \), the realization of \( \beta_\lambda \) in \( L^r(\Omega) \) is Lipschitz continuous if \( r \geq 2 \). Note that this does not hold if \( r < 2 \) in general. We also remark that the Moreau–Yosida regularization of \( \psi \) given in \( \mathcal{H} \) does not coincide with \( u \mapsto \int_{\Omega} j_\lambda(u(x)) \, dx \) except the case of \( r = 2 \), where
Definition 1. Let \( (u_0, \xi_0) \) satisfy \( \xi_0(x) \in \beta(u_0(x)) \) for a.e. \( x \in \Omega \). Then \( (u, \xi) \) is said to be a solution to \( (P) \) with initial data \( (u_0, \xi_0) \) if it satisfies

\[
\begin{align*}
\xi & \in W^{1,\infty}(0, T; W^{-1,p'}(\Omega)) \cap L^\infty(0, T; L^p(\Omega)), \\
\xi(x, t) & \in \beta(u(x, t)) \text{ for a.e. } (x, t) \in Q, \\
\partial_t \xi(t) + A\xi(t) & = f(t) \text{ in } W^{-1,p'}(\Omega) \text{ for a.e. } t \in (0, T), \\
\xi(0) & = \xi_0.
\end{align*}
\]

Then our main results can be stated as follows:

Theorem 2.1. Assume (H.\( \alpha \)) and (H.\( \beta \)). Let

\[
f \in W^{1,p'}(0, T; W^{-1,p'}(\Omega)) \cap L^\infty(0, T; L^p(\Omega)) \cap L^q(\Omega)
\]

with some \( q \in [1, \infty] \). Then for any \( u_0 \in W^{-1,p'}_0(\Omega) \) and \( \xi_0 \in L^p(\Omega) \cap L^q(\Omega) \) such that \( \xi_0(x) \in \beta(u_0(x)) \) for a.e. \( x \in \Omega \), \( (P) \) possesses at least one solution with initial data \( (u_0, \xi_0) \) satisfying

\[
\begin{align*}
\sup_{0 \leq t \leq T} \|\xi(t)\|_{L^p} & \leq T \sup_{0 \leq t \leq T} \|f(t)\|_{L^{p'}} + \|\xi_0\|_{L^p}, \\
\sup_{0 \leq t \leq T} \|\xi(t)\|_{L^q} & \leq T \sup_{0 \leq t \leq T} \|f(t)\|_{L^q} + \|\xi_0\|_{L^q}.
\end{align*}
\]
**Theorem 2.2.** In addition to the assumptions of Theorem 2.1 let $Au_0 \in L^p(\Omega)$. Then there exist at least one solution to (P) with initial data $(u_0, \xi_0)$ satisfying (9) and

$$\|\xi(t) - \xi(s)\|_{L^1} \leq C|t - s| \quad \forall t, s \in [0, T],$$

where $C > 0$ is a constant.

Existence of solution satisfying (9) was obtained in Alt–Luckhaus [3] for Lipschitz continuous $\beta$ and Bamberger [4] for $\beta(s) = |s|^{r-2}r$ with $r > 1$ (see also Remark 3 below). We shall show that the time-Lipschitz continuity of solution can be assured for every maximal monotone graph $\beta$.

In order to state the next result, we here define

$$TV_\Omega(\xi) := \sup \left\{ \int_\Omega \xi(x) \nabla \cdot \zeta(x) dx ; \zeta = (\zeta_1, \zeta_2, \ldots, \zeta_d) \in \mathcal{D}(\Omega), \|\zeta\|_{L^\infty} \leq 1 \right\},$$

$$BV(\Omega) := \{ \xi \in L^1(\Omega) ; \|\xi\|_{BV} := \|\xi\|_{L^1} + TV_\Omega(\xi) < \infty \},$$

and $BV_0(\Omega)$ by the closure of $\mathcal{D}(\Omega)$ with respect to the norm $\|\cdot\|_{BV}$ (see [4] [59]).

**Theorem 2.3.** In addition to the assumptions of Theorem 2.1 (resp. Theorem 2.2), let $\xi_0 \in BV(\Omega)$ and $f \in L^\infty(0, T; BV_0(\Omega))$. Then there exist at least one solution to (P) with initial data $(u_0, \xi_0)$ satisfying

$$\sup_{0 \leq t \leq T} TV_\Omega(\xi(\cdot, t)) \leq T \sup_{0 \leq t \leq T} TV_\Omega(f(\cdot, t)) + TV_\Omega(\xi_0)$$

in addition to the properties in Theorem 2.1 (resp. Theorem 2.2).

Combining this fact with Theorem 2.2 we can assure the existence of solution belonging to $BV(Q)$ for general nonlinearity $\beta$ by choosing an appropriate initial value and external force.

Moreover, we can obtain

**Theorem 2.4.** If $u_0 \in L^\infty(\Omega)$ and $q = \infty$ in Theorem 2.1 and 2.2 then the solution to (P) satisfies $u \in L^\infty(Q)$.

Such $L^\infty$-estimate of $u$ is already given in Bénilan–Wittbold [14]. In this paper, we shall give a simple proof by standard Moser’s iteration.

**Remark 1.** Regularity given in Definition 11 is enough to apply Lemma 1.5 of Alt–Luckhaus [3]. Namely, it holds that

$$\int_\Omega j^*(\xi(x, t_2)) dx - \int_\Omega j^*(\xi(x, t_1)) dx = \int_{t_1}^{t_2} \langle \partial_t \xi(t), u(t) \rangle_{W^{1,p}} dt$$

for any $t_1, t_2 \in [0, T]$.

3. **Elliptic Problem**

We first deal with the following elliptic problem:

\[
\begin{cases}
\xi(x) - \nabla \cdot (\alpha(x, \nabla u(x))) = h(x) & x \in \Omega, \\
\xi(x) \in \beta(u(x)) & x \in \Omega, \\
u(x) = 0 & x \in \partial \Omega.
\end{cases}
\]

(E$_h$)

If $D(\beta) = \mathbb{R}$, we can easily obtain the solvability of (E$_h$) for any $h \in W^{-1,p'}(\Omega)$ by considering the variational problem of $I(u) := \psi(u) + \varphi(u) - \langle h, u \rangle_{W_0^{1,p}}$ on $W_0^{1,p}(\Omega)$.
and using the result by Brézis [17], where \( \varphi \) and \( \psi \) is defined in [4] and [3] with \( r = p \). Otherwise, however, \( \partial_{L^p} \varphi(u) \) is defined in the sense of distribution and we can not assure that \( \xi(x) \in \beta(u(x)) \) holds for each \( x \in \Omega \). To cope with this difficulty, we here slightly restrict the integrability of \( h \) and construct a strong solution to \( \text{[E]} \).

**Theorem 3.1.** Assume \((H.\alpha)\) and \((H.\beta)\) with \( p \in (1, \infty) \). Let \( h \in L^q(\Omega) \cap L^{r'}(\Omega) \) with some \( q \in [1, \infty] \). Then \( \text{[E]} \) possesses a unique solution \( u \in W^{1,p}_0(\Omega) \) such that \( \xi, Au \in L^q(\Omega) \cap L^{r'}(\Omega) \) and

\[
\|\xi\|_{L^q} \leq \|h\|_{L^r}, \quad \|\xi\|_{L^{r'}} \leq \|h\|_{L^{r'}},
\]

where \( Au(x) := -\nabla \cdot (\alpha(x, \nabla u(x))) \).

**Proof of Theorem 3.1.** The uniqueness of solution is shown by the standard energy estimate. Let \((u_1, \xi_1)\) \((i = 1, 2)\) be solutions to \( \text{[E]} \). Testing the difference of equations by \((u_1 - u_2)\), we have

\[
\int_{\Omega} (\alpha(x, \nabla u_1(x)) - \alpha(x, \nabla u_2(x))) (\nabla u_1(x) - \nabla u_2(x)) \, dx \leq 0.
\]

Then we obtain \( u_1 \equiv u_2 \) by [3], and immediately \( \xi_1 \equiv \xi_2 \). Hence we only have to discuss the existence of solution to \( \text{[E]} \).

To this end, we consider the variational problem for \( I_1(u) := \psi_1(u) + \varphi(u) - \langle h, u \rangle_{L^p} \) in \( L^p(\Omega) \), where \( \varphi \) is given in [4] with \( r = p \) and

\[
\psi_1(u) := \begin{cases}
\int_{\Omega} j_\lambda(u(x)) \, dx & \text{if } u \in L^p(\Omega), \\
+\infty & \text{otherwise}.
\end{cases}
\]

Obviously, \( \varphi \) and \( \psi_1 \) are proper lower semi-continuous convex on \( L^p(\Omega) \). Remark that \( \partial_{L^p} \psi_1 \) may not coincide with the subdifferential of the Moreau–Yosida regularization of \( \xi \), i.e., the Yosida approximation of \( \beta \) when \( p \neq 2 \) (for instance, let \( \beta = \text{id} \). Then the realization of \( J_\lambda = \beta_\lambda = 1/(1 + \lambda) \) is not Lipschitz continuous on \( L^p(\Omega) \) into \( L^{p'}(\Omega) \) if \( p < 2 \). Since \( I_1 \) is bounded from below for any \( u \in L^p(\Omega) \), there exist a global minimizer, denoted by \( u_\lambda \).

If \( p \geq 2 \), the Lipschitz continuity of \( \beta_\lambda \) yields \( D(\widetilde{\beta}_\lambda) = L^p(\Omega) \). By Theorem 2.10 of [10], we have \( \partial_{L^p} I_1 = \partial_{L^p} \psi_1 + \partial_{L^{p'}} \varphi - h \), which implies that the minimizer \( u_\lambda \) becomes a unique solution to

\[
\begin{cases}
\beta_\lambda(u_\lambda(x)) + Au_\lambda(x) = h(x) & x \in \Omega, \\
u_\lambda(x) = 0 & x \in \partial \Omega,
\end{cases}
\]

such that \( u_\lambda \in W^{1,p}_0(\Omega) \) and \( \widetilde{\beta}_\lambda(u_\lambda), Au_\lambda \in L^{p'}(\Omega) \).

Testing \( \text{[E]} \) by \( u_\lambda \), we have

\[
c\|\nabla u_\lambda\|_{L^{p'}}^{p'-1} \leq C_1 \|h\|_{L^{p'}}
\]

by \( \beta_\lambda(0) = 0 \), [24], and [3], where \( C_1 \) is a constant arising from Poincaré’s inequality. Next we multiply \( \text{[E]} \) by \( k_m^r(\beta_\lambda(u_\lambda)) \), where \( m > 0 \) and

\[
k_m^r(s) := \begin{cases}
|s|^{-r} & \text{if } |s| > m, \\
m^{-r} & \text{if } |s| \leq m.
\end{cases}
\]
Since $\beta_\lambda$ and $k^r_m$ with $r \leq 2$ are Lipschitz continuous and $k^r_m(\beta_\lambda(0)) = 0$, we have by (2)
\[
\int_\Omega k^q_m(\beta_\lambda(u_\lambda(x)))Au_\lambda(x)dx \geq 0.
\]
Moreover, from the fact that
\[
\int_\Omega k^q_m(\beta_\lambda(u_\lambda))dx
\]
\[
= \int_{\{x \in \Omega: |\beta_\lambda(u_\lambda(x))| \geq m\}} |\beta_\lambda(u_\lambda)|^p dx + \int_{\{x \in \Omega: |\beta_\lambda(u_\lambda(x))| \leq m\}} m^p - 2|\beta_\lambda(u_\lambda)|^2 dx
\]
\[
\geq \int_{\{x \in \Omega: |\beta_\lambda(u_\lambda(x))| \geq m\}} k^q_m(\beta_\lambda(u_\lambda))|p dx + \int_{\{x \in \Omega: |\beta_\lambda(u_\lambda(x))| \leq m\}} m^p - 2|\beta_\lambda(u_\lambda)|^p dx
\]
\[
= \int_\Omega |k^q_m(\beta_\lambda(u_\lambda))|^p dx,
\]
we can derive
\[
\left(\int_\Omega |k^q_m(\beta_\lambda(u_\lambda))|^p dx\right)^{1/p'} \leq \|h\|_{L^{p'}}.
\]
Passing the limit as $m \to 0$ and using Fatou’s lemma, we obtain
\[
\|\tilde{\beta}_\lambda(u_\lambda)\|_{L^{p'}} \leq \|h\|_{L^{p'}}.
\]
By the same reasoning, it holds that
\[
\|\tilde{\beta}_\lambda(u_\lambda)\|_{L^q} \leq \|h\|_{L^q}
\]
for $q \in [1, 2)$ (substitute $\text{sgn}_m(\beta_\lambda(u_\lambda))$ for $k^q_m(\beta_\lambda(u_\lambda))$ if $q = 1$). When $q \geq 2$, we replace $k^q_m$ with $K^q_M$, where $M > 0 and
\[
K^q_M(s) := \begin{cases}
|s|^{r-2}s & \text{if } |s| \leq M, \\
M^{r-1}\text{sgn }s & \text{if } |s| > M.
\end{cases}
\]
Since $K^r_M$ is Lipschitz continuous for $r \geq 2$ and satisfies $K^r_M(0) = 0$, we have by (2)
\[
\int_\Omega K^q_M(\beta_\lambda(u_\lambda(x)))Au_\lambda(x)dx \geq 0
\]
and
\[
\int_\Omega K^q_M(\beta_\lambda(u_\lambda))dx
\]
\[
= \int_{\{x \in \Omega: |\beta_\lambda(u_\lambda(x))| \leq M\}} |\beta_\lambda(u_\lambda)|^q dx + \int_{\{x \in \Omega: |\beta_\lambda(u_\lambda(x))| > M\}} M^{q-1}|\beta_\lambda(u_\lambda)|dx
\]
\[
\geq \int_{\{x \in \Omega: |\beta_\lambda(u_\lambda(x))| \leq M\}} |\beta_\lambda(u_\lambda)|^q dx + \int_{\{x \in \Omega: |\beta_\lambda(u_\lambda(x))| > M\}} M^q dx
\]
\[
= \int_\Omega |K^q_M(\beta_\lambda(u_\lambda))|^q dx,
\]
which implies (15) with $q \in [2, \infty)$. 
By uniform boundedness given above, there exist a subsequence \( \{ \lambda_n \}_{n \in \mathbb{N}} \) of \( \{ \lambda \} \) such that \( \lambda_n \to 0 \) and
\[
\begin{cases}
  u_{\lambda_n} \to \exists u & \text{strongly in } L^p(\Omega) \text{ and weakly in } W^{1,p}_0(\Omega), \\
  \beta_{\lambda_n}(u_{\lambda}) \to \exists \xi & \text{weakly in } L^q(\Omega) \text{ and } L^{p'}(\Omega), \\
  Au_{\lambda_n} \to \exists \eta & \text{weakly in } L^q(\Omega) \text{ and } L^p(\Omega),
\end{cases}
\]
as \( n \to \infty \) (we use the Dunford-Pettis theorem for \( q = 1 \)). The demi-closedness of maximal monotone operator leads to \( n = Au \). Since \( J_{\lambda_n} u_{\lambda_n} - J_{\lambda_n} u \to 0 \) strongly in \( L^p(\Omega) \) and \( J_{\lambda_n} u_{\lambda_n} - u_{\lambda_n} \to 0 \) strongly in \( L^{p'}(\Omega) \) by the Lipschitz continuity of \( J_\lambda \) and uniform boundedness of \( \| \beta_{\lambda}(u_{\lambda}) \|_{L^{p'}} \), we can see that \( J_{\lambda_n} u \to u \) strongly in \( L^p(\Omega) \) (remark that \( p' \leq p \)). Then from Lebesgue’s dominated convergence theorem, it follows that \( J_{\lambda_n} u_{\lambda_n} \to u \) strongly in \( L^p(\Omega) \). Hence by the maximal monotonicity of \( \beta \) and the fact that \( \beta_{\lambda}(u_{\lambda}(x)) \in \beta(J_{\lambda} u_{\lambda}(x)) \) for a.e. \( x \in \Omega \), we can see that \( \xi \in \beta(u) \) in \( L^p(\Omega) \), i.e., \( \xi(x) \in \beta(u(x)) \) for a.e. \( x \in \Omega \). Therefore the equation of \((13)\) weakly converges to the equation of \((16)\) in \( L^p(\Omega) \) and the limit \( u \) is a (unique) solution to \( (16) \). Moreover, \((12)\) can be obtained as the limit of \( (15) \) as \( \lambda \to 0 \).

Next suppose that \( 1 < p < 2 \), where we cannot confirm whether \( D(\partial_{L^p} \psi_1) \) coincides with \( L^p(\Omega) \). To cope with this difficulty, we begin with the variational problem of \( I_2(u) := \frac{\varepsilon}{2} \| u \|_{L^2}^2 + \psi_1(x) + \varphi(u) - \langle h, u \rangle_{L^2} \) (recall \( h \in L^p(\Omega) \cap L^q(\Omega) \subset L^2(\Omega) \)), where \( \varepsilon > 0 \), \( \varphi \) is given in \((11)\) with \( r = 2 \), and \( \psi_1 \) in \((13)\) with \( L^p \) replaced by \( L^2 \). Thanks to the approximation term \( \frac{\varepsilon}{2} \| u \|_{L^2}^2 \), \( I_2 \) attains its minimum in \( L^2(\Omega) \). Since \( D(\partial_{L^2} \psi_1) = L^2(\Omega) \), the global minimizer \( u_\varepsilon \) is a (unique) solution to the following Euler-Lagrange equation:
\[
\begin{cases}
  \varepsilon u_\varepsilon + \beta_{\lambda}(u_\varepsilon(x)) + Au_\varepsilon(x) = h(x) & x \in \Omega, \\
  u_\varepsilon(x) = 0 & x \in \partial \Omega,
\end{cases}
\]
where \( u_\varepsilon \in D(\partial_{L^2} \psi_1) \cap D(\partial_{L^2} \varphi) \), i.e., \( u_\varepsilon \in L^2(\Omega) \cap W^{1,p}_0(\Omega) \) and \( \beta_{\lambda}(u_\varepsilon) \), \( Au_\varepsilon \in L^2(\Omega) \).

Multiplying \((16)\) by \( u_\varepsilon \), we get
\[
\frac{\varepsilon}{2} \| u_\varepsilon \|_{L^2}^2 + c \| \nabla u_\varepsilon \|_{L^p}^p \leq C_2 \| h \|_{L^{p'}}^p,
\]
where \( C_2 > 0 \) is a general constant independent of \( \varepsilon \). Testing \((16)\) by \( K_M^{p'}(u_\varepsilon) \), we have
\[
\varepsilon \| K_M^{p'}(u_\varepsilon) \|_{L^{p'}} = \| h \|_{L^{p'}} \| K_M^{p'}(u_\varepsilon) \|_{L^{p'}},
\]
which implies
\[
\varepsilon \| u_\varepsilon \|_{L^{p'}} = \varepsilon \| u_\varepsilon \|_{L^{p'}} \leq \| h \|_{L^{p'}}.
\]
Repeating the same procedures as that for \( p \geq 2 \), we obtain
\[
\| \beta_{\lambda}(u_\varepsilon) \|_{L^{p'}} \leq \| h \|_{L^{p'}} \quad \| \beta_{\lambda}(u_\varepsilon) \|_{L^q} \leq \| h \|_{L^q}.
\]
Returning to \((10)\), we get \( \| Au_\varepsilon \|_{L^{p'}} \leq 3 \| h \|_{L^{p'}} \). These estimates imply that \( u_\varepsilon \) belongs to \( D(\partial_{L^{p'}} \varphi) \cap D(\partial_{L^{p'}} \psi_1) \), which is included in \( D(\partial_{L^2} \varphi) \cap D(\partial_{L^2} \psi_1) \). Hence
we can extract a subsequence of \( \{ \varepsilon \} \) (we omit relabeling) such that

\[
\begin{cases}
\varepsilon u_\varepsilon \to 0 & \text{strongly in } L^2(\Omega) \text{ and weakly in } L^p'(\Omega), \\
u_\varepsilon \to \exists u_\lambda & \text{strongly in } L^p(\Omega) \text{ and weakly in } W^{1,p}_0(\Omega), \\
\beta_\lambda(u_\varepsilon) \to \exists \xi_\lambda & \text{weakly in } L^q(\Omega) \text{ and } L^p'(\Omega), \\
A u_\varepsilon \to \exists \eta_\lambda & \text{weakly in } L^q(\Omega) \text{ and } L^p(\Omega).
\end{cases}
\]

The maximal monotonicity of \( \partial L_p \varphi \) and \( \partial L_p \psi \) yield \( \eta_\lambda = A u_\lambda \) and \( \xi_\lambda = \tilde{\beta}_\lambda(u_\lambda) \). Therefore the limit \( u_\lambda \in W^{1,p}_0(\Omega) \) is a (unique) solution to \( (14) \) for \( 1 < p < 2 \).

By tracing the same procedures as for \( p \geq 2 \), we obtain

\[
\| \nabla u_\lambda \|_{L^{p-1}}^{p-1} \leq C_2 \| h \|_{L^{p'}} \| \tilde{\beta}_\lambda(u_\lambda) \|_{L^{p'}} \leq \| h \|_{L^{p'}} \| \tilde{\beta}_\lambda(u_\lambda) \|_{L^q} \leq \| h \|_{L^q},
\]

and

\[
\begin{cases}
u_\lambda \to \exists u & \text{strongly in } L^p(\Omega) \text{ and weakly in } W^{1,p}_0(\Omega), \\
\tilde{\beta}_\lambda(u_\lambda) \to \exists \xi & \text{weakly in } L^q(\Omega) \text{ and } L^p'(\Omega), \\
A u_\lambda \to \exists A u & \text{weakly in } L^q(\Omega) \text{ and } L^p(\Omega),
\end{cases}
\]

where \( u_\lambda \) is a solution to \( (14) \) with \( 1 < p < 2 \). Furthermore, \( J_\lambda u_\lambda - \tilde{J}_\lambda u \to 0 \) in \( L^p(\Omega) \) and \( J_\lambda u_\lambda - u_\lambda \to 0 \) in \( L^p(\Omega) \) directly lead to \( J_\lambda u_\lambda \to u \) strongly in \( L^p(\Omega) \) since \( p < p' \). Therefore we can assure that \( \xi \in \tilde{\beta}(u) \) and the limit \( u \) is a required solution to \( (E) \).

If \( h \in L^\infty(\Omega) \), we obtain \( (12) \) with \( q = \infty \) by taking the limit as \( q \to \infty \) (see Ch.1 §3 Theorem 1 of [58]). \( \square \)

**Remark 2.** For later use, we here establish several a priori estimates of \( u \) and \( \xi \) with some restrictions on \( h, \alpha \) and \( \beta \).

1. If \( h \in L^\infty(\Omega) \), we obtain \( u \in L^\infty(\Omega) \) by Moser’s iteration technique. Indeed, testing \( (E_\alpha) \) by \( K_M^{p(r-1)+2}(u) \) with \( r \geq 1 \) and using \( (2) \), we have

\[
|\Omega|^{1 - \frac{p(r-2)+1}{p}} \| h \|_{L^\infty} \| u \|_{L^{p(r-1)+1}} \geq cr^{-p} \lim_{M \to \infty} \| K_M^{r+1}(u) \|_{L^{p+1}}^p \geq cr^{-p} \| u \|_{L^{p+1}}^p.
\]

where \(|\Omega|\) is the measure of \( \Omega \) and \( \mu > 1 \) is an exponent arising from Sobolev’s inequality. Hence putting \( r := \mu^l \) with \( l = 0, 1, \ldots \) and letting \( l \to \infty \), we obtain \( \| u \|_{L^\infty} \leq C \), where \( C \) is depending only on \( \| h \|_{L^\infty}, \mu \), and \(|\Omega|\).

2. If \( \alpha(x,z) = \alpha(z) \) and \( h \in Z(\Omega) \), we obtain \( \xi \in BV(\Omega) \) (see Theorem 6 of [23]). We first prove it for the case where \( \beta \) is Lipschitz continuous. Define

\[
\beta^c(s) := \begin{cases}
\beta(s) - \beta(\varepsilon) & \text{if } s \geq \varepsilon, \\
0 & \text{if } -\varepsilon \leq s \leq \varepsilon, \quad \varepsilon > 0, \\
\beta(s) - \beta(-\varepsilon) & \text{if } s \leq -\varepsilon,
\end{cases}
\]

Clearly, \( \beta^c \) is Lipschitz continuous with the same constant as \( \beta \) and converges to \( \beta \) uniformly in \( \mathbb{R} \). Let \( u^c \in L^\infty(\Omega) \cap W^{1,p}(\Omega) \) be a solution to \( \beta^c(u^c) - Au^c = h \).

Since \( h - \beta^c(u^c) \in L^\infty(\Omega) \), \( u^c \) is continuous on \( \overline{\Omega} \) (see Ch. 4 Theorem 1.1 of [12]). By homogeneous Dirichlet boundary condition, the set \( \omega^c := \{ x \in \Omega; |u^c(x)| \geq \varepsilon \} \) is compact and included in \( \Omega \). Since \( \beta^c(u^c) = 0 \) in \( \Omega \setminus \omega^c \), it holds that \( \text{supp} \, Au^c \subset (\text{supp} \, \beta^c(u^c) \cup \text{supp} \, h) \equiv : \Omega^c \equiv \Omega \). Then for any \( c \in \mathbb{R}^d \) such that
\[ \lim_{\lambda \to 0} \int_0^\infty (\alpha(\nabla u^\varepsilon(x) + e) - \alpha(\nabla u^\varepsilon(x))) \cdot \nabla \text{sgn}_\lambda (u^\varepsilon(x) + e - u^\varepsilon(x)) \, dx \]
\[ = - \int_\Omega |\beta^\varepsilon(u^\varepsilon(x) + e) - \beta^\varepsilon(u^\varepsilon(x))| \, dx + \int_\Omega |h(x + e) - h(x)| \, dx, \]
which implies \( TV_\Omega(\beta^\varepsilon(u)) \leq TV_\Omega(h). \) Repeating a priori estimates above, we have the uniform boundedness of \( \|u^\varepsilon\|_{W^{1,p}} \), \( \|A u^\varepsilon\|_{L^\infty} \), and \( \|\beta^\varepsilon(u^\varepsilon)\|_{L^\infty} \) independent of \( \varepsilon \), which yields \( u_\varepsilon \to u \) strongly in \( L^p(\Omega) \) and \( A u^\varepsilon \to A u \ast\text{-weakly in} \ L^\infty(\Omega) \) as \( \varepsilon \to 0 \). Moreover, we obtain \( \beta^\varepsilon(u^\varepsilon) \to \beta(u) \ast\text{-weakly in} \ L^\infty(\Omega) \) by the definition of \( \beta^\varepsilon \). Therefore the limit \( h \) coincides with the unique solution to (14) and satisfies \( TV_\Omega(\beta(u)) \leq TV_\Omega(h) \) by the lower semi-continuity of \( TV_\Omega \) (more precisely, we have \( \|\nabla \beta(u)\|_{L^1} \leq \|\nabla h\|_{L^1} \) since \( \beta(u) \in W^{1,p}_0(\Omega) \)).

If \( \beta \) is not Lipschitz continuous, we use this fact to (14). Hence by letting \( \lambda \to 0 \), we obtain
\[ TV_\Omega(\xi) \leq \liminf_{\lambda \to 0} TV_\Omega(\beta_\lambda(u_\lambda)) \leq TV_\Omega(h), \]
which still holds if \( h \in BV_0(\Omega) \).

In order to prove Theorem 3.2, we prepare the following:

**Theorem 3.2.** Let \( h_i \in L^p(\Omega) \) and \( (u_i, \xi_i) \) be the solution to \((E_{h_i})\), where \( i = 1, 2 \). Then it holds that
\[ \|\xi_1 - \xi_2\|_{L^1} \leq \|h_1 - h_2\|_{L^1}. \]

**Proof of Theorem.** Let \( u_{\lambda i} \) \((i = 1, 2)\) be a solution to (14) with \( h = h_i \). Since \( \text{sgn}_\lambda \) is Lipschitz continuous and (9) is assumed, we have
\[ \int_\Omega \text{sgn}_\lambda (u_{\lambda 1}(x) - u_{\lambda 2}(x)) (A u_{\lambda 1}(x) - A u_{\lambda 2}(x)) \, dx \geq 0. \]
Hence multiplying the difference of equations by \( \text{sgn}_\lambda (u_{\lambda 1} - u_{\lambda 2}) \) and taking the limit as \( \lambda \to +0 \), we obtain
\[ \int_{\{x \in \Omega : u_{\lambda 1}(x) \neq u_{\lambda 2}(x)\}} |\beta_\lambda(u_{\lambda 1}(x)) - \beta_\lambda(u_{\lambda 2}(x))| \, dx \leq \|h_1 - h_2\|_{L^1}. \]
The L.H.S. of (19) coincides with \( \|\beta_\lambda(u_{\lambda 1}) - \beta_\lambda(u_{\lambda 2})\|_{L^1} \) since \( \beta_\lambda \) is single-valued.

We here recall that there exist a subsequence of \((u_{\lambda i}, \beta_\lambda(u_{\lambda i}))\) which converges to the solution of \((E_{h_i})\). By the uniqueness of solution to \((E_{h_i})\), that is, the fact that the limit is determined independently of the choice of subsequence, the original sequence \((u_{\lambda i}, \beta_\lambda(u_{\lambda i}))\) tends to \((u_i, \xi_i)\) without extracting a subsequence. Moreover, by (15), the Dunford-Pettis theorem is applicable to the sequence \([\beta_\lambda(u_{\lambda i}) - \beta_\lambda(u_{\lambda i})]_{\lambda > 0} \). Thus (18) follows from the limit of (19) as \( \lambda \to 0 \).

**Remark 3.** Define a multi-valued operator \( A_q : L^1(\Omega) \to 2^{L^1(\Omega)} \) by
\[ A_q \xi = A \beta^{-1}(\xi) = -\nabla \cdot \alpha(\cdot, \nabla \beta^{-1}(\xi(\cdot))) \]
with domain
\[ D(A_q) := \left\{ \xi \in L^q(\Omega) ; \exists u \in W^{1,p}_0(\Omega) \text{ s.t. } \xi(x) \in \beta(u(x)) \text{ a.e. } x \in \Omega, A u \in L^q(\Omega) \right\}. \]
Indeed, by testing the difference of (14) with \( h \) converges to the integral solution in see that any approximate solution of suitable time-discretization of (20) strongly satisfies (9) if \( f \) \( \in \text{Lip}(0,T;\text{L}^1(\Omega)) \) (see [13], [26], or Theorem 4.2 of [14]). Moreover, by lower semi-continuity of \( \xi \in C([0,T];\text{L}^1(\Omega)) \) for any \( \xi_0 \in \text{L}^1(\Omega) \) and \( f \in \text{L}^1(Q) \). Furthermore, we can see that any approximate solution of suitable time-discretization of (20) strongly converges to the integral solution in \( L^\infty(0,T;\text{L}^1(\Omega)) \) (see [13], [26], or Theorem 4.2 of [14]). Moreover, by lower semi-continuity of \( \xi \in C([0,T];\text{L}^1(\Omega)) \) for any \( \xi_0 \in \text{L}^1(\Omega) \) and \( f \in \text{L}^1(Q) \) (remark that it is not trivial that \( A_q \) satisfies \( L^1 \)-closedness) and (7) is not enough to obtain such a time-Lipschitz continuity in abstract setting.

**Remark 4.** By the same procedure as the proof for Theorem 3.2, we can obtain

\[
\int_{\Omega} (\xi_1(x) - \xi_2(x))_+ dx \leq \int_{\Omega} (h_1(x) - h_2(x))_+ dx,
\]

where \((\cdot)_+\) denotes the positive part, i.e.,

\[
(s)_+ = \begin{cases} 
  s & \text{if } s \geq 0, \\
  0 & \text{if } s < 0.
\end{cases}
\]

Indeed, by testing the difference of (14) with \( h_i \) by \( H_\lambda(u_{\lambda_1} - u_{\lambda_2}) \), we have

\[
\int_{\{x \in \Omega; u_{\lambda_1}(x) > u_{\lambda_2}(x)\}} (\beta_\lambda(u_{\lambda_1}(x)) - \beta_\lambda(u_{\lambda_2}(x))) dx \leq \int_{\Omega} (h_1(x) - h_2(x))_+ dx.
\]

Since \( \beta_\lambda \) is single-valued, this inequality is equivalent to

\[
\int_{\Omega} (\beta_\lambda(u_{\lambda_1}(x)) - \beta_\lambda(u_{\lambda_2}(x)))_+ dx \leq \int_{\Omega} (h_1(x) - h_2(x))_+ dx.
\]

Therefore by lower semi-continuity of \( v \mapsto \int_{\Omega} (v)_+ dx \), we obtain (21) by taking the limit as \( \lambda \to 0 \).

**Remark 5.** We can assure that there exist a pair of initial data \((u_0, \xi_0)\) satisfying \( \xi_0(x) \in \beta(u(x)) \) and the requirements in Theorem 2.1–2.4 by solving (2.1) with \( h \) belonging to \( L^q(\Omega), L^\infty(\Omega) \) or \( BV_0(\Omega) \).

### 4. Existence

Let \( N \in \mathbb{N} \) and define \( \tau := T/N \). As the standard approximation of (P), we consider the following time-discretization problem:

\[
(P_{\tau,n}) \quad \begin{cases} 
  \xi_{\tau}^{n+1}(x) - \xi_{\tau}^n(x) = \tau^{\alpha} \partial_x \partial_t u_{\tau}^n(x) = f_{\tau}^n(x) \quad x \in \Omega, \\
  \xi_{\tau}^{n+1}(x) \in \beta(u_{\tau}^{n+1}(x)) \\
  u_{\tau}^{n+1}(x) = 0 \quad x \in \partial \Omega,
\end{cases}
\]

where \( u_0^\tau := u_0, \xi_0^\tau := \xi_0 \) and \( f_{\tau}^n := \frac{1}{\tau} \int_{n\tau}^{(n+1)\tau} f(\cdot, s) ds \). By using Theorem 3.1 with \( h = \xi_{\tau}^n + \tau f_{\tau}^n \), we can inductively determine \( u_{\tau} = \{u_{\tau}^0, u_{\tau}^1, \ldots, u_{\tau}^N\} \) and \( \xi_{\tau} = \)
\( \{ \xi_0^n, \xi_1^n, \ldots, \xi_N^n \} \) satisfying \( u^n_\tau \in W^{1,p}_0(\Omega) \), \( Au^n_\tau, \xi^n_\tau \in L^{p'}(\Omega) \cap L^{q}(\Omega) \) for each \( n = 1, 2, \ldots, N \), and

\[
\| \xi_\tau^{n+1} \|_{L^{p'}} \leq \tau \left\| u^n_\tau \right\|_{L^{p'}} + \| \xi^n_\tau \|_{L^{p'}} \leq \tau \sup_{0 \leq t \leq T} \| f(t) \|_{L^{p'}} + \| \xi^n_\tau \|_{L^{p'}},
\]

for \( n = 0, 1, \ldots, N - 1 \). Similarly, if \( f \in L^\infty(0, T; BV_0(\Omega)) \), it holds that

\[
TV_\Omega(\xi_\tau^{n+1}) \leq \tau \sup_{0 \leq t \leq T} TV_\Omega(f(\cdot, t)) + TV_\Omega(\xi^n_\tau).
\]

By repeating exactly the same procedures as those in [35] and [47], we establish a priori estimates of \( u_\tau \) and \( \xi_\tau \). Multiplying \( (P_{\tau,n}) \) by \( u^n_\tau \), we have

\[
\int_\Omega j^*(\xi^{n+1}_\tau(x)) dx - \int_\Omega j^*(\xi_\tau(x)) dx + c_3 \tau \sum_{m=1}^{n+1} \| \nabla u^m_\tau \|_{L^p}^p \leq C_3 \tau \sup_{0 \leq t \leq T} \| f(t) \|_{W^{-1,p'}}^p
\]

for \( n = 0, 1, \ldots, N - 1 \) (recall \( n\tau \leq N\tau = T \)). Here and henceforth, \( c_3, C_3 \geq 0 \) stands for general constants independent of \( \tau \) and \( N \). By using

\[
\int_\Omega j^*(\xi^{n+1}_\tau(x)) dx - \int_\Omega j^*(\xi_\tau(x)) dx
\]

\[
\int_\Omega u^{n+1}_\tau(x) \xi^{n+1}_\tau dx - \int_\Omega j(u^{n+1}_\tau(x)) dx - \int_\Omega j^*(\xi_\tau(x)) dx
\]

\[
\geq - \int_\Omega j(0) dx - \int_\Omega j^*(\xi_\tau(x)) dx,
\]

we obtain

\[
\tau \sum_{n=1}^{N} \| \nabla u^n_\tau \|_{L^p}^p \leq C_3.
\]

Next testing \( (P_{\tau,n}) \) by \( (u^{n+1}_\tau - u^n_\tau) \), we have for \( n = 1, 2, \ldots, N - 1 \)

\[
\int_\Omega a(x, \nabla u^{n+1}_\tau(x)) dx - \int_\Omega a(x, \nabla u^n_\tau(x)) dx
\]

\[
\leq \sum_{m=0}^{n} \int_\Omega f^m_\tau(x)(u^{m+1}_\tau(x) - u^m_\tau(x)) dx
\]

\[
= \sum_{m=1}^{n} \int_\Omega (f^{m+1}_\tau(x) - f^m_\tau(x)) u^m_\tau(x) dx + \int_\Omega (f^m_\tau(x) u^{m+1}_\tau(x) - f^0_\tau(x) u^0_\tau(x)) dx
\]

\[
\leq C_3 \left\| \frac{df}{ds}(s) \right\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))} + \tau \sum_{m=1}^{N} \| \nabla u^m_\tau \|_{L^p}
\]

\[+ \sup_{0 \leq t \leq T} \| f(t) \|_{W^{-1,p'}} \left( \| \nabla u^0_\tau \|_{L^p} + \| \nabla u^{n+1}_\tau \|_{L^p} \right),
\]

which implies

\[
\sup_{n=0,1,\ldots,N} \| \nabla u^n_\tau \|_{L^p} \leq C_3.
\]

From [2], we can deduce \( \| a(\cdot, \nabla u^n_\tau) \|_{L^{p'}} \leq C_3 \) and

\[
\sup_{n=0,1,\ldots,N} \| Au^n_\tau \|_{W^{-1,p'}} \leq C_3.
\]
Immediately, we get

\[(27) \quad \sup_{n=0,1,\ldots,N-1} \left\| \frac{\xi^{n+1}_\tau - \xi^n_\tau}{\tau} \right\|_{W^{-1,\rho'}} \leq C_3.\]

Multiplying (26) by \(u^n_\tau\) and \(u^{n+1}_\tau\) again, we have

\[\frac{1}{\tau} \left( \int_{\Omega} j^*(\xi^n_\tau(x))dx - \int_{\Omega} j^*(\xi^{n+1}_\tau(x))dx \right) \geq - \|Au^{n+1}_\tau\|_{W^{-1,\rho'}}\|\nabla u^n_\tau\|_{L^p} - \|f^n_\tau\|_{W^{-1,\rho'}}\|\nabla u^{n+1}_\tau\|_{L^p}\]

and

\[\frac{1}{\tau} \left( \int_{\Omega} j^*(\xi^n_\tau(x))dx - \int_{\Omega} j^*(\xi^{n+1}_\tau(x))dx \right) \leq \|f^n_\tau\|_{W^{-1,\rho'}}\|\nabla u^{n+1}_\tau\|_{L^p}.\]

From (28) and (29), it follows that

\[(28) \quad \sup_{n=0,1,\ldots,N-1} \left| \int_{\Omega} j^*(\xi^{n+1}_\tau(x))dx - \int_{\Omega} j^*(\xi^n_\tau(x))dx \right| \leq \tau C_3.\]

If \(Au_0 \in L^p(\Omega)\), we can define \(\xi^{-}_\tau \in L^p(\Omega)\) by

\[\xi^{-}_\tau := \xi^0_\tau + \tau Au_0 - \tau f^{-}_\tau \iff \frac{\xi^0_\tau - \xi^{-}_\tau}{\tau} + Au_0 = f^{-}_\tau,\]

where \(f^{-}_\tau \equiv f(,0)\). Then \(u^n_\tau = u_0\) and \(\xi^n_\tau = \xi_0\) can be regarded as a unique solution to (26) with \(h = \xi^{-}_\tau + \tau f^{-}_\tau \in L^p(\Omega)\). Theorem 3.2 assures that

\[\|\xi^{n+1}_\tau - \xi^n_\tau\|_{L^1} \leq \|\xi^n_\tau - \xi^{n-1}_\tau\|_{L^1} + \tau \|f^n_\tau - f^{n-1}_\tau\|_{L^1},\]

\[\leq \|\xi^n_\tau - \xi^{n-1}_\tau\|_{L^1} + \tau \int_{(n-1)\tau}^{n\tau} \left\| \frac{f(t + \tau) - f(t)}{\tau} \right\|_{L^1} dt\]

for any \(n = 0, 1, \ldots, N - 1\). Therefore, we obtain

\[(29) \quad \sup_{n=0,1,\ldots,N-1} \left\| \frac{\xi^{n+1}_\tau - \xi^n_\tau}{\tau} \right\|_{L^1} \leq \|f^{-}_\tau - Au_0\|_{L^1} + \int_0^T \left\| \frac{df(t)}{dt} \right\|_{L^1} dt \leq C_3.\]

We here put \(f(,t) \equiv f(,0)\) for \(t < 0\).

We now discuss the convergence as \(\tau \to 0\). If there is no confusion, a subsequence may be denoted again by the same symbol as the original sequence. For a given sequence \(w_\tau := \{w^0_\tau, w^1_\tau, \ldots, w^n_\tau\}\), we set

\[\Pi_\tau w_\tau(t) := \begin{cases} 
  w^{n+1}_\tau & \text{if } t \in (n\tau, (n+1)\tau], \\
  w^n_\tau & \text{if } t = 0
\end{cases}\]

\[\Lambda_\tau w_\tau(t) := \begin{cases} 
  w^{n+1}_\tau - \frac{w^n_\tau}{\tau}(t-n\tau) + w^n_\tau & \text{if } t \in (n\tau, (n+1)\tau], \\
  w^n_\tau & \text{if } t = 0.
\end{cases}\]

Then (26)–(29) imply

\[(30) \quad \sup_{0 \leq t \leq T} \|\Pi_\tau u_\tau\|_{W^{1,\rho'}} + \sup_{0 \leq t \leq T} \|\Lambda_\tau u_\tau\|_{W^{-1,\rho'}} \leq C_3,\]

\[(31) \quad \sup_{0 \leq t \leq T} \|\partial_t \Lambda_\tau \xi_\tau\|_{W^{-1,\rho'}} \leq C_3,\]

\[(32) \quad \sup_{0 \leq t \leq T} \left| \int_{\Omega} j^*(\xi_\tau(x))dx \right| \leq C_3,\]
Hence we can extract a suitable subsequence such that we obtain a suitable subsequence which converges strongly in \( C^0 (\Omega) \). By (32), Ascoli–Arzela’s theorem assures that there is a subsequence \( \{ A_\tau \xi_\tau \} \) for a.e. \( (x, t) \in Q \) which converges strongly to \( \xi \). Therefore we obtain (36) by applying Dunford–Pettis’s theorem to (34). Moreover, if \( \xi_0 \in BV (\Omega) \) and \( f \in L^\infty (0, T; BV_0 (\Omega)) \), we obtain (35) \( TV_\Omega (\Pi_\tau \xi_\tau (t)) \leq T TV_\Omega (f(t)) + TV_\Omega (\xi_0) \).

Hence we can extract a suitable subsequence such that we obtain as \( \tau \to 0 \)

\[
\Pi_\tau u_\tau \rightharpoonup \exists u \quad \text{*-weakly in } L^\infty (0, T; W^{1,p} (\Omega)),
\]

\[
\Lambda_\tau \xi_\tau \rightharpoonup \exists \xi \quad \text{strongly in } C([0, T]; W^{-1,p'} (\Omega)),
\]

\[
\partial_t A_\tau \xi_\tau \to \partial_t \xi \quad \text{*-weakly in } L^\infty (0, T; L^p (\Omega)) \cap L^\infty (0, T; L^q (\Omega)),
\]

\[
A_\Pi \xi_\tau = \partial \xi \quad \text{*-weakly in } L^\infty (0, T; W^{-1,p'} (\Omega)).
\]

as \( \tau \to 0 \). Clearly, the limit inferior of (33) as \( \tau \to 0 \) yields (8). If \( \mathcal{A}_0 \in L^p (\Omega) \), we obtain (31) by applying Dunford–Pettis’s theorem to (34). Moreover, if \( \xi_0 \in BV (\Omega) \) and \( f \in L^\infty (0, T; BV_0 (\Omega)) \), (10) can be derived from (35).

According to Bénilan (see also Theorem 4.2 of Barbu), the sequence \( \{ \Pi_\tau \xi_\tau \} \) strongly converges to the integral solution in \( L^\infty (0, T; L^1 (\Omega)) \). Evidently, the limit coincides with \( \xi \). Furthermore, since

\[
||| \Pi_\tau \xi_\tau (t) - \xi (t) |||_{L^r (\Omega)} \leq \left( \int_\Omega ||| \Pi_\tau \xi_\tau (x, t) - \xi (x, t) |||_{L^r (\Omega)} dx \right)^{1/r - 1} \left( \int_\Omega ||| \Pi_\tau \xi_\tau (x, t) - \xi (x, t) |||_{L^r (\Omega)} dx \right)^{1/r}
\]

where \( s := 1/(r - 1)(p - 1) \) belongs to \( (1, \infty) \) if \( r \in (1, p') \), we have

\[
(36) \quad \Pi_\tau \xi \to \xi \quad \text{strongly in } L^\infty (0, T; L^r (\Omega)) \forall r \in (1, p').
\]

Now we show that \( \eta = \mathcal{A} u \) and \( \xi (x, t) \in \beta (u(x, t)) \) for a.e. \( (x, t) \in Q \) by reprising the argument of (33). By (31), (33) and the compactness of \( L^p (\Omega) \) \( \hookrightarrow \) \( W^{-1,p'} (\Omega) \), Ascoli’s theorem is applicable to \( \{ \Lambda_\tau \xi_\tau \} \) and there is a subsequence which strongly converges (to \( \xi \), obviously) in \( C([0, T]; W^{-1,p'} (\Omega)) \). Remark that \( \Pi_\tau \xi_\tau \to \Lambda_\tau \xi_\tau \to 0 \), i.e., \( \Pi_\tau \xi_\tau \to \xi \) holds strongly in \( L^\infty (0, T; W^{-1,p'} (\Omega)) \) by (27). Therefore we obtain (36) \( \Pi_\tau \xi_\tau \to \xi \) weakly in \( L^p (0, T; W^{0,1,p}_0 (\Omega)) \), which implies

\[
\int_Q \Pi_\tau \xi_\tau (x, t) \Pi_\tau u_\tau (x, t) dx dt \to \int_Q \xi (x, t) u(x, t) dx dt.
\]

Thus Lemma 1.2 in Brézis–Crandall–Pazy (19) leads to \( \xi \in \tilde{\beta} (u) \) in \( L^\infty (0, T; L^p (\Omega)) \).

By (32), Ascoli–Arzela’s theorem assures that \( \{ \Lambda_\tau \mathcal{J}^* (\xi_\tau) \} \) possesses a subsequence which converges strongly in \( C([0, T]; \mathbb{R}) \). Let \( \Theta \) be its limit. Since
\(\Pi_\tau \xi_\tau(T) \to \xi(T)\) weakly in \(L^p(\Omega)\), we have \(\Theta(T) \geq \int_\Omega j^*(\xi(T))dx\) by lower semi-continuity of \(\int_\Omega j^*(\cdot)dx\). Then by (37) and \(\Theta(0) = \int_\Omega j^*(\xi_0(x))dx\), we have

\[
\int_0^T \langle \partial_t \xi(t), u(t) \rangle_{W_0^{1,p}} dt = \int_\Omega j^*(\xi(x,T))dx - \int_\Omega j^*(\xi_0(x))dx, \\
\leq \Theta(T) - \Theta(0).
\]

Multiplying \(P_{\tau,n}\) by \(u_{\tau,n}^{n+1}\) and integrating over \((n\tau, (n+1)\tau)\), we get

\[
\int_\Omega j^*(\xi_{\tau,n}^{n+1}(x))dx - \int_\Omega j^*(\xi_\tau^{n}(x))dx + \int_{n\tau}^{(n+1)\tau} \langle A\Pi_\tau u_\tau(t), \Pi_\tau u_\tau(t) \rangle_{W_0^{1,p}} dt \\
\leq \int_{n\tau}^{(n+1)\tau} \int_\Omega f(x,t)\Pi_\tau u_\tau(x,t)dxdt.
\]

Calculating \(\sum_{n=0}^{N-1}\) and taking its limit as \(\tau \to 0\), we can derive from (37)

\[
\limsup_{\tau \to 0} \int_0^T \langle A\Pi_\tau u_\tau(t), \Pi_\tau u_\tau(t) \rangle_{W_0^{1,p}} dt \\
\leq \int_0^T \int_\Omega f(x,t)u(x,t)dxdt - \Theta(T) + \Theta(0) \\
\leq \int_0^T \int_\Omega f(x,t)u(x,t)dxdt - \int_\Omega j^*(\xi(x,T))dx + \int_\Omega j^*(\xi_0(x))dx \\
= \int_0^T \int_\Omega \langle f(t) - \partial_t \xi(t), u(t) \rangle_{W_0^{1,p}} dt = \int_0^T \int_\Omega \langle \eta(t), u(t) \rangle_{W_0^{1,p}} dt.
\]

By virtue of Lemma 1.2 of [19], we can assure that \(\eta = Au\), whence it follows Theorem 2.1, 2.2, and 2.3.

**Remark 6.** By the same way as [23], it can be shown that

\[
\Pi_\tau u_\tau \to u \quad \text{strongly in } L^p(0,T;W_0^{1,p}(\Omega)).
\]

Indeed, multiplying \(P_{\tau,n}\) by \(u_{\tau,n}^{n+1} - u(t)\) with \(t \in (n\tau, (n+1)\tau)\), we have

\[
\int_\Omega f_\tau^n(x,t)(u_{\tau,n}^{n+1} - u(x,t))dx \\
\geq \frac{1}{\tau} \left( \int_\Omega j^*(\xi_{\tau,n}^{n+1}(x))dx - \int_\Omega j^*(\xi_\tau^{n}(x))dx \right) - \left\langle \frac{\xi_{\tau,n}^{n+1} - \xi_\tau^{n}}{\tau}, u(t) \right\rangle_{W_0^{1,p}} \\
+ \langle Au_{\tau,n}^{n+1} - Au, u_{\tau,n}^{n+1} - u(t) \rangle_{W_0^{1,p}} + \langle Au, u_{\tau,n}^{n+1} - u(t) \rangle_{W_0^{1,p}},
\]

which yields

\[
\int_Q \Pi_\tau f_\tau^n(x,t)(\Pi_\tau u_\tau(x,t) - u(x,t))dx \\
\geq \left( \int_\Omega j^*(\xi_\tau^{N}(x))dx - \int_\Omega j^*(\xi_0(x))dx \right) + \int_0^T \langle \partial_t \Lambda_\tau, \xi_\tau(t), u(t) \rangle_{W_0^{1,p}} dt \\
+ \int_0^T \left( \langle A\Pi_\tau u_\tau(t) - Au(t), \Pi_\tau u_\tau(t) - u(t) \rangle_{W_0^{1,p}} + \langle Au(t), \Pi_\tau u_\tau(t) - u(t) \rangle_{W_0^{1,p}} \right) dt.
\]
Taking the limit as $\tau \to 0$ and using (37), we obtain
\[
0 \geq \Theta(T) - \Theta(0) + \int_0^T \langle \partial_t \xi(t), u(t) \rangle_{W_0^{1,p}} dt \\
+ \limsup_{\tau \to 0} \int_0^T \langle A\Pi_{\tau} u_{\tau}(t) - A(u(t), \Pi_{\tau} u_{\tau}(t) - u(t)) \rangle_{W_0^{1,p}} dt \\
\geq \int_\Omega j^*(\xi(x, T)) dx - \int_\Omega j^*(\xi_0(x)) dx + \int_0^T \langle \partial_t \xi(t), u(t) \rangle_{W_0^{1,p}} dt \\
+ \limsup_{\tau \to 0} \int_0^T \langle A\Pi_{\tau} u_{\tau}(t) - A(u(t), \Pi_{\tau} u_{\tau}(t) - u(t)) \rangle_{W_0^{1,p}} dt \\
= \limsup_{\tau \to 0} \int_0^T \langle A\Pi_{\tau} u_{\tau}(t) - A(u(t), \Pi_{\tau} u_{\tau}(t) - u(t)) \rangle_{W_0^{1,p}} dt.
\]
Hence (38) follows from (3).

In order to prove Theorem 2.4 we use Lemma 4 of Carrillo [23]: let $(u, \xi) \in \mathbb{R}$ and $\gamma : \mathbb{R} \to \mathbb{R}$ be non-decreasing Lipschitz continuous. If $\zeta \in C^1(\overline{\Omega})$ satisfies $\gamma(u) \in L^\infty(0, T; W_0^{1,p}(\Omega))$, then
\[
\int_\Omega \Gamma(\xi(x, t)) \zeta(x, t) dx - \int_\Omega \Gamma(\xi_0(x)) \zeta(x, 0) dx \\
= \int_0^t \langle \partial_t \xi(t), \gamma(u(t)) \zeta(t) \rangle_{W_0^{1,p}} dt + \int_0^t \int_\Omega \Gamma(\xi(x, t)) \partial_t \zeta(x, t) dx dt
\]
holds for a.e. $t \in (0, T)$, where
\[
\Gamma(s) := \begin{cases} 
\int_0^s \gamma((\beta^{-1})^{-1}(\sigma)) d\sigma & \text{if } s \in D(\gamma \circ \beta^{-1}), \\
+\infty & \text{otherwise.}
\end{cases}
\]
We use this formula with $\zeta \equiv 1$ and $\gamma = K_M^{p(r-1)+2}$. Since
\[
0 \leq \Gamma(\xi(x, t)) \leq \xi(x, t) K_M^{p(r-1)+2}(u(x, t)),
\]
we can obtain
\[
C r^{-p} \|u\|^p_{L^{p(r+1)}(\Omega)} \leq \|f\|_{L^\infty(\Omega)} \|u\|_{L^{p(r-1)+1}(\Omega)}^{p(r-1)+1} + \int_\Omega \Gamma(\xi_0(x)) dx \\
\leq \|f\|_{L^\infty(\Omega)} \|u\|_{L^{p(r-1)+1}(\Omega)}^{p(r-1)+1} + \|\xi_0\|_{L^\infty(\Omega)} \|u_0\|_{L^{p(r-1)+1}(\Omega)},
\]
with some $\mu > 1$. Therefore by Morse's iteration, we can deduce
\[
\|u\|_{L^\infty(\Omega)} \leq C = C(\|f\|_{L^\infty(\Omega)}, \|u_0\|_{L^\infty(\Omega)}, \|\xi_0\|_{L^\infty(\Omega)}, |I|, |Q|).
\]

5. Uniqueness

Throughout this section, we assume that
\[
\alpha(x, z) = \alpha(z)
\]
and initial data and external force satisfy
\[
f \in W^{1,p'}(0, T; W^{-1,p'}(\Omega)) \cap L^\infty(0, T; L^p(\Omega)), \\
u_0 \in W_0^{1,p}(\Omega), \quad \xi_0 \in L^p(\Omega), \quad \xi_0(x) \in \beta(u_0(x)) \text{ for a.e. } x \in \Omega.
\]
To discuss the uniqueness of solution, we rewrite our problem. Let \( v := u + \xi \), where \((u, \xi)\) is a solution to (P). Since \( \beta \) and \( \beta^{-1} \) are maximal monotone, \( g := (\text{id} + \beta^{-1})^{-1} \) and \( b := (\text{id} + \beta)^{-1} \) are Lipschitz continuous and satisfy \( g(v) = \xi, \ b(v) = u \), and \( g \circ b^{-1} = \beta, \ b \circ g^{-1} = \beta^{-1} \). In this manner, we obtain the following, which is equivalent to the original initial boundary value problem of (P):

\[
\begin{cases}
\partial_t g(v(x, t)) - \nabla \cdot \alpha(\nabla b(v(x, t))) = f(x, t) & (x, t) \in Q, \\
b(u(x, t)) = 0 & (x, t) \in \partial \Omega \times (0, T), \\
g(v(x, 0)) = \xi_0(x) & x \in \Omega.
\end{cases}
\]  

(43)

Theorem 2 assures the existence of a solution to (43) in the following sense:

**Theorem 2.1.**

A function \( v \in L^1(Q) \) is said to be a weak solution to (43) if

- \( b(v) \in L^\infty(0, T; W^{1,p}_0(\Omega)) \),
- \( g(v) \in W^{1,\infty}(0, T; W^{-1,p'}(\Omega)) \cap L^\infty(0, T; L^{p'}(\Omega)) \),
- \( \partial_t g(v) - \nabla \cdot \alpha(\nabla b(v)) = f \) in \( W^{-1,p'}(\Omega) \) for a.e. \( t \in (0, T) \),
- \( g(v(\cdot, 0)) = \xi_0(\cdot) \).

We here introduce the definition of entropy solution

**Definition 2.**

A function \( v \) is said to be an entropy solution if

\[
\int_\Omega H^0(v - s) \{ \alpha(\nabla b(v)) \cdot \nabla \zeta - (g(v) - g(s)) \partial_t \zeta - f \zeta \} \, dx dt
\]

\[
- \int_\Omega (\xi_0 - g(s))_+ \zeta(\cdot, 0) \, dx \leq 0,
\]

(45)

\[
\int_\Omega H^0(-s - v) \{ \alpha(\nabla b(v)) \cdot \nabla \zeta - (g(v) - g(-s)) \partial_t \zeta - f \zeta \} \, dx dt
\]

\[
+ \int_\Omega (g(-s) - \xi_0)_+ \zeta(\cdot, 0) \, dx \geq 0,
\]

(46)

for any \( s \in \mathbb{R} \) and \( \zeta = \zeta(x, t) \) such that \( \zeta \geq 0 \) and either of

- \( i) \ s \geq 0 \) and \( \zeta \in D([0, T) \times \overline{\Omega}) \),
- \( ii) \ s \in \mathbb{R} \) and \( \zeta \in D([0, T) \times \Omega) \).

Main assertion of this section is that the solution constructed in the previous section meets the requirements of entropy solution. To this end, we return to the elliptic problem:

**Lemma 1.**

Assume (H.\( \alpha \)), (H.\( \beta \)), (H.\( h \)), and \( h \in L^{p'}(\Omega) \). Let \( w \in L^1(\Omega) \) be a solution to

\[
\begin{cases}
g(w(x)) - \nabla \cdot \alpha(\nabla b(w(x))) = h(x) & x \in \Omega, \\
b(w(x)) = 0 & x \in \partial \Omega,
\end{cases}
\]

(48)

such that \( g(w), Ab(w) \in L^{p'}(\Omega) \) and \( b(w) \in W^{1,p}_0(\Omega) \). Then \( w \) is an entropy solution to (48), i.e., it holds that

\[
\int_\Omega H^0(w - s) \{ (g(w) - h) \zeta + \alpha(\nabla b(w)) \cdot \nabla \zeta \} \, dx \leq 0
\]

(49)

and

\[
\int_\Omega H^0(-w - s) \{ (g(w) - h) \zeta + \alpha(\nabla b(w)) \cdot \nabla \zeta \} \, dx \geq 0
\]

(50)
for every \((s, \zeta)\) satisfying \(\zeta \geq 0\) and either of
\[
\tag{51}
i) \ s \geq 0 \quad \text{and} \quad \zeta \in \mathcal{D}(\Omega), \quad \text{ii) } s \in \mathbb{R} \quad \text{and} \quad \zeta \in \mathcal{D}(\Omega).
\]

**Proof.** Remark that if \(w\) is a solution to \([48]\), then \(-w\) becomes a solution to \([48]\) with \(g\) replaced by \(\hat{g}(\sigma) := -g(-\sigma)\), \(b\) by \(\hat{b}(\sigma) := -b(-\sigma)\) (namely \(\beta\) by \(\beta(\sigma) := -\beta(-\sigma)\), which is still maximal monotone), \(\alpha\) by \(\hat{\alpha}(z) := -\alpha(-z)\), and \(h\) by \(-h\). Hence it is sufficient to prove \([49]\).

Multiplying \([48]\) by \(H_\lambda(b(w) - b(s))\zeta \in W^{1,p}_0(\Omega)\) with \((s, \zeta)\) satisfying i) or ii) in Lemma [1] and letting \(\lambda \to +0\), we get
\[
\int_\Omega H^\circ(b(w) - b(s)) \{(g(w) - h)\zeta + \alpha(\nabla b(w) \cdot \nabla \zeta)\} \, dx \leq 0.
\]

Here we assume that \(b(s) \notin E\), where
\[
\tag{52} E := \{s \in R(b); \ b^{-1}(s) \text{ is multi-valued}\} = \{s \in D(\beta); \ \beta(s) \text{ is multi-valued}\}.
\]

Since \(b(\sigma) = b(s)\) is attained only by \(\sigma = s\), we have \(H_0(w(x, t) - s) = H_0(b(w(x, t)) - b(s))\) for a.e. \((x, t) \in Q\), which immediately leads to \([49]\). Especially, if \(\beta\) is single-valued, \(E\) is empty and \([49]\) holds for every \(s\).

To prove the general case, we consider the following approximate problem:
\[
\tag{53}
\begin{cases}
g \circ b^{-1}_\varepsilon(u_\varepsilon(x)) - \nabla \cdot \alpha(\nabla u_\varepsilon(x)) = h_n(x) & x \in \Omega, \\
u_\varepsilon(x) = 0 & x \in \partial \Omega,
\end{cases}
\]

where \(b_\varepsilon = \varepsilon \text{id} + b\) with \(\varepsilon > 0\) and \(\{h_n\}_{n \in \mathbb{N}}\) is a sequence in \(\mathcal{D}(\Omega)\) converging to \(h\) in \(L^p(\Omega)\). By Theorem [34] there is a unique solution \(u_\varepsilon \in W^{1,p}_0(\Omega)\) such that \(\xi_\varepsilon := g \circ b^{-1}_\varepsilon(u_\varepsilon), \nabla \cdot \alpha(\nabla u_\varepsilon) \in L^p(\Omega)\). Moreover, by a priori estimates and Remark [2] (use here \([44]\)), we have
\[
\|\xi_\varepsilon\|_{L^\infty}, \ |\xi_\varepsilon|_{W^{1,1}}, \ |u_\varepsilon|_{W^{1,p}}, \ |u_\varepsilon|_{L^\infty} \leq C_n,
\]
where \(C_n\) is the general constant which is independent of \(\varepsilon > 0\). By Rellich-Kondrachov’s theorem, we can extract a subsequence such that
\[
u_\varepsilon \to \exists u_n \quad \text{strongly in } L^p(\Omega),
\]
\[
\text{weakly in } W^{1,p}_0(\Omega),
\]
\[
\ast\text{-weakly in } L^\infty(\Omega),
\]
\[
\xi_\varepsilon \to \exists \xi_n \quad \text{strongly in } L^r(\Omega) \quad \text{for any } r \in (1, \infty),
\]
\[
\ast\text{-weakly in } L^\infty(\Omega),
\]
\[
\nabla \cdot \alpha(\nabla u_\varepsilon) \to \exists \eta_n \quad \text{strongly in } L^r(\Omega) \quad \text{for any } r \in (1, \infty),
\]
\[
\ast\text{-weakly in } L^\infty(\Omega).
\]

Let \(w_\varepsilon := b^{-1}_\varepsilon(u_\varepsilon)\), then \(\|g(w_\varepsilon)\|_{L^\infty} \leq C_n\) and \(\|b(w_\varepsilon)\|_{L^\infty} \leq \|u_\varepsilon\|_{L^\infty} \leq C_n\). Remark that \(g + b\) is surjective and \(w_\varepsilon = g(w_\varepsilon) + b(w_\varepsilon)\) holds by the maximal monotonicity of \(\beta\). Hence
\[
w_\varepsilon \to \exists w_n \quad \text{strongly in } L^p(\Omega),
\]
\[
\ast\text{-weakly in } L^\infty(\Omega),
\]
\[
\varepsilon w_\varepsilon \to 0 \quad \text{strongly in } L^\infty(\Omega).
\]
Since $b$ and $g$ are Lipschitz continuous, we obtain $u_n = b(w_n)$, $\xi_n = g(w_n)$, $w_n = g(w_n) + b(w_n)$, and $\eta_n = \nabla \cdot \alpha(\nabla u_n)$. Testing the difference of (53) by $b_{\varepsilon_1}(w_{\varepsilon_1}) - b_{\varepsilon_2}(w_{\varepsilon_2})$ and using (50), we have

$$\mathcal{C}\|\nabla b_{\varepsilon_1}(w_{\varepsilon_1}) - \nabla b_{\varepsilon_2}(w_{\varepsilon_2})\|_{L^p}^p \leq (\varepsilon_2 - \varepsilon_1) \int_{\Omega} (g(w_{\varepsilon_1}) - g(w_{\varepsilon_2})) w_{\varepsilon_2} \, dx,$$

which yields $b_{\varepsilon}(w) \to b(w_n)$ strongly in $W_0^{1,p}(\Omega)$. By Lebesgue’s dominated convergence theorem and continuity of $\alpha$, we have $\alpha(\nabla b_{\varepsilon}(w)) \to \alpha(\nabla b(w_n))$ strongly in $L^{p'}(\Omega)$. Therefore, the limit of solution to (53) as $\varepsilon \to 0$ coincides with a unique solution to

$$(54) \begin{cases} g(w_n(x)) - \nabla \cdot \alpha(\nabla b(w_n(x))) = h_n(x) & x \in \Omega, \\ b(w_n(x)) = 0 & x \in \partial \Omega. \end{cases}$$

Moreover, since $g \circ b^{-1}_{\varepsilon}$ is single-valued, $w$ is an entropy solution to (53), i.e.,

$$\int_{\Omega} H^\varepsilon(w - s) \{(g(w) - h_n)\zeta + \alpha(\nabla b(w)) \cdot \nabla \zeta\} \, dx \leq 0$$

holds for every $(s, \zeta)$ in Lemma 1. By the maximal monotonicity of $\tilde{H}$ and strong convergence of $w_n$ in $L^p(\Omega)$, there is a subsequence of $H^\varepsilon(w_n - s)$ which $*$-weakly converges in $L^\infty(Q)$ and its limit $\chi_s$ belongs to $H(w_n - s)$ for a.e. $(x, t) \in Q$. Hence we have for any $s$

$$\int_{\Omega} \chi_s \{(g(w_n) - h_n)\zeta + \alpha(\nabla b(w_n)) \cdot \nabla \zeta\} \, dx \leq 0.$$

Here put $\{\sigma_i\}_{i \in \mathbb{N}}$ such that $\sigma_i \searrow s$ as $i \to \infty$. Then $\chi_{\sigma_i} \in H(w_n - \sigma_i)$ satisfies $\chi_{\sigma_i} \nearrow H^\varepsilon(w_n - s)$ for a.e. $(x, t) \in Q$. Hence we can derive

$$(55) \int_{\Omega} H^\varepsilon(w_n - s) \{(g(w_n) - h_n)\zeta + \alpha(\nabla b(w)) \cdot \nabla \zeta\} \, dx \leq 0$$

for any $(s, \zeta)$, i.e., $w_n$ is an entropy solution to (54).

By repeating a priori estimates given in Section 3, we get

$$\|g(w_n)\|_{L^{p'}} + \|b(w_n)\|_{W^{1,p}} + \|\nabla \cdot \alpha(\nabla b(w_n))\|_{L^{p'}} \leq C$$

and

$$\|\nabla b(w_n) - \nabla b(w_m)\|_{L^{p-1}}^{p-1} \leq C \|h_n - h_m\|_{L^{p'}}$$

for any $n, m \in \mathbb{N}$, where $C$ is suitable constant independent of $n$. Moreover, Theorem 4.2 yields

$$\|g(w_n) - g(w_m)\|_{L^1} + \|\nabla \cdot \alpha(\nabla b(w_n)) - \nabla \cdot \alpha(\nabla b(w_m))\|_{L^1} \leq 3 \|h_n - h_m\|_{L^1}.$$

Hence we can see that

$$w_n \to w \text{ strongly in } L^r(\Omega) \quad \forall r \in (1, \min\{p', p\}),$$

$$b(w_n) \to b(w) \text{ strongly in } W_0^{1,p}(\Omega),$$

$$g(w_n) \to g(w) \text{ strongly in } L^1(\Omega) \text{ and weakly in } L^{p'}(\Omega),$$

$$\alpha(\nabla b(w_n)) \to \alpha(\nabla b(w)) \text{ strongly in } L^{p'}(\Omega),$$

$$\nabla \cdot \alpha(\nabla b(w_n)) \to \nabla \cdot \alpha(\nabla b(w)) \text{ strongly in } L^1(\Omega) \text{ and weakly in } L^{p'}(\Omega),$$

and $w$ is a unique solution to (48). By taking the limit of (55) as $n \to \infty$ and repeating the argument above, we can assure that $w$ satisfies (19). □
Let \( u_r := \{ u^0_r, u^1_r, \ldots, u^N_r \} \) and \( u_r := \{ \xi^0_r, \xi^1_r, \ldots, \xi^N_r \} \) be sequences determined by (36) and define \( v_r \) by \( v^n_r := u^n_r + \xi^n_r \) (remark that \( \xi^n_r = g(v^n_r) \) and \( u^n_r = b(v^n_r) \)). Note that (30) and (38) imply

\[
(56) \quad \Pi_r v_r \to v = u + \xi \quad \text{strongly in} \quad L^p(0, T; L^p(\Omega)) \quad \forall r \in (1, \min\{p, p'\}).
\]

Hence applying Lemma 1 to (43) and using (36), (38), (56), we have (let \( \Pi_r v_r(t) = v_0 \) for \( t < 0 \))

\[
0 \geq \int_Q H^u(\Pi_r v_r(x, t) - s) \{ \alpha(\nabla \Pi_r b(v_r(x, t))) \cdot \nabla \zeta(x, t) + \nabla(\Pi_r v_r(x, t)) \nabla(\Pi_r v_r(x, t)) - \Pi_r f_r \} \zeta(x, t) \right] dx \right) dtdx
\]

\[
\geq \int_Q H^u(\Pi_r v_r(x, t) - s) \{ \alpha(\nabla \Pi_r b(v_r(x, t))) \cdot \nabla \zeta(x, t) - \Pi_r f_r \zeta(x, t) \right] dtdx
\]

\[
+ \frac{1}{\tau} \int_0^\tau \int_\tau^T H^u(\Pi_r v_r(x, t) - s) (g(\Pi_r v_r(x, t)) - g(s)) \zeta(x, t) dx \right) dtdx
\]

\[
- \frac{1}{\tau} \int_{-\tau}^0 \int_\tau^T H^u(\Pi_r v_r(x, t) - s) (g(\Pi_r v_r(x, t)) - g(s)) \zeta(x, t + \tau) dx \right) dtdx
\]

\[
+ \int_{T-\tau}^T \int_\tau^T H^u(\Pi_r v_r(x, t) - s) (g(\Pi_r v_r(x, t)) - g(s)) \zeta(x, t) - \zeta(x, t + \tau) dx \right) dx \right) dtdx
\]

\[
\to \int_Q \alpha(\nabla b(v(t))) \cdot \nabla \zeta - (g(v) - g(s)) \partial_t \zeta - f \zeta dx \right) dtdx
\]

By putting \( \{ \sigma_i \}_{i \in \mathbb{N}} \) such that \( \sigma_i \searrow s \) as \( i \to \infty \), we obtain (45) for every \( (s, \zeta) \) in Definition 3. Immediately, we have (46) by replacing \( v \) with \( -v \), which is a solution to (33) with \( \beta'(\alpha) := -\beta(-\sigma) \) substituted for \( \beta \), \( \tilde{a}(z) := -\alpha(-z) \) for \( \alpha \), \( \tilde{f} := -f \) for \( f \), and \( \tilde{\xi}_0 := -\xi_0 \) for \( \xi_0 \). Thus it follows that

**Theorem 5.1.** Assume (H.\( \alpha \)), (H.\( \beta \)), and (13). Then (13) possesses at least one entropy solution.

If \( \beta \) is single-valued, we can show that any weak solution to (13) is entropy solution by the following lemma:

**Lemma 2.** Assume (H.\( \alpha \)), (H.\( \beta \)), and let \( v \) be a weak solution to (13). Then \( v \) satisfies

\[
\begin{align*}
&\int_Q H^u(v - s) \{ \alpha(x, \nabla b(v)) \cdot \nabla \zeta - (g(v) - g(s)) \partial_t \zeta - f \zeta \} dx dt
\end{align*}
\]

\[
(57) \quad - \int_\Omega (\xi_0 - g(s)) \zeta(x, 0) dx
\]

\[
= - \lim_{\lambda \to 0} \int_Q \alpha(x, \nabla b(v)) \cdot \nabla b(v) H^\lambda_{\alpha}(b(v) - b(s)) \zeta dx dt,
\]
for any \( (s, \zeta) \) which fulfills \( b(s) \not\in E \), \( \zeta \geq 0 \), and either of i) or ii) in \((57)\) and
\[
\int_Q H^\circ(-s - v) \{ \alpha(x, \nabla b(v)) \cdot \nabla \zeta - (g(v) - g(-s)) \partial_t \zeta - f \zeta \} \, dx dt
\]
\[(58)\]
\[+ \int_\Omega (g(-s) - \xi_0) + \zeta(0) \, dx = \lim_{\lambda \to 0} \int_\Omega \alpha(x, \nabla b(v)) \cdot \nabla b(v) H_\lambda'(b(-s) - b(v)) \zeta \, dx dt,
\]
for any \( (s, \zeta) \) which fulfills \(-b(-s) \not\in E \), \( \zeta \geq 0 \), and either of i) or ii) in \((47)\).

**Proof.** It is sufficient to prove \((57)\) for any \( (s, \zeta) \) such that \( b(s) \not\in E \). Multiplying \((43)\) by \( H_\lambda(b(v) - b(s)) \zeta \in L^\infty(0, T; W_0^{1,p}(\Omega)) \) and applying Lemma 4 of [23] (recall \((39)\)), we have
\[
\int_Q \Gamma(g(v)) \partial_t \zeta \, dx dt + \int_\Omega \Gamma(\xi_0) \zeta(0) \, dx
\]
\[(59)\]
\[= \int_\Omega \alpha(x, \nabla b(v)) \cdot \nabla (H_\lambda(b(v) - b(s)) \zeta) \, dx dt - \int_Q f H_\lambda(b(v) - b(s)) \zeta \, dx dt,
\]
where \( \Gamma \) is defined by \((40)\) with \( \gamma(\sigma) := H_\lambda(b(v) - b(s)) \). By \( \beta^{-1} = b \circ g^{-1} \), we get
\[
\Gamma(g(v)) := \int_0^{g(v)} H_\lambda((\beta^{-1})^\circ \sigma - b(s)) \, d\sigma = \int_0^{g(v)} H_\lambda((\beta^{-1})^\circ \sigma - b(s)) \, d\sigma
\]
\[\to (g(v) - g(s))_+ = H_0(v - s)(g(v) - g(s)),
\]
\[
\Gamma(\xi_0) = \int_{g(s)}^{\xi_0} H_\lambda((\beta^{-1})^\circ \sigma - b(s)) \, d\sigma
\]
\[\to (\xi_0 - g(s))_+
\]
Therefore by letting \( \lambda \to 0 \), we obtain \((57)\) since \( H_0(v(x, t) - s) = H_0(b(v(x, t)) - b(s)) \) holds by \( b(s) \not\in E \). \( \square \)

Consequently, we can derive the following, which implies the comparison principle and the uniqueness of entropy solution.

**Theorem 5.2.** Assume (H.\( \alpha \)), (H.\( \beta \)), and \((11)\). Let \( v_i \) \( (i = 1, 2) \) be an entropy solution to \((13)\) with \( v_i(\cdot, 0) = v_{i0}, \xi_{i0} = g(v_{i0}), u_{i0} = b(v_{i0}) \) and \( f_i \) satisfying \((12)\). Then it holds that
\[
\int_\Omega (g(v_1(x, t)) - g(v_2(x, t)))_+ \, dx
\]
\[(60)\]
\[\leq \int_\Omega (\xi_{10}(x) - \xi_{20}(x))_+ \, dx + \int_0^t \int_\Omega (f_1 - f_2)_+ \, dx dt
\]
for every \( t \in [0, T] \), and therefore
\[
\| g(v_1(t)) - g(v_2(t)) \|_{L^1} \leq \| \xi_{10} - \xi_{20} \|_{L^1} + \int_0^t \| f_1(t) - f_2(t) \|_{L^1} \, dt
\]
for every \( t \in [0, T] \).
**Proof.** Let \((y, s) \in Q\) and \((x, t) \in Q\) be variables used for \(i = 1\) and \(2\), respectively. Namely, set \(v_1 = v_1(y, s)\), \(f_1 = f_1(y, s)\), and \(v_2 = v_2(x, t)\), \(f_2 = f_2(x, t)\). Define a non-negative smooth function \(Z_1 = Z_1(y, s, x, t)\) on \(Q \times Q\) such that

\[
(y, s) \mapsto Z_1(y, s, x, t) \in \mathcal{D}(\{0, T\} \times \Omega) \quad \text{for each fixed } (x, t) \in Q,
\]

\[
(x, t) \mapsto Z_1(y, s, x, t) \in \mathcal{D}(\{0, T\} \times \Omega) \quad \text{for each fixed } (y, s) \in Q.
\]

Then by the definition of entropy solution and Lemma 2, we have

\[
\int_{Q \times Q} H^\alpha(v_1 - v_2) \left\{ - (g(v_1) - g(v_2))(\partial_t Z_1 + \partial_s Z_1) \right. \\
+ (\alpha(\nabla_x b(v_1)) - \alpha(\nabla_y b(v_2))) \cdot (\nabla_x Z_1 + \nabla_y Z_1) - (f_1 - f_2) Z_1 \left. \right\} \mathrm{d}y \mathrm{d}s \mathrm{d}x \mathrm{d}t
\]

\[
- \int_Q \int_\Omega (\xi_{01}(y) - g(v_2(x, t))) Z_1(y, 0, x, t) \mathrm{d}y \mathrm{d}x \mathrm{d}t
\]

\[
- \int_Q \int_\Omega (g(v_1(y, s)) - \xi_{02}(x)) Z_1(y, s, x, 0) \mathrm{d}y \mathrm{d}x \mathrm{d}s
\]

\[
\leq - \lim_{\lambda \to 0} \int_{(Q \setminus Q_1) \times (Q \setminus Q_2)} H_\lambda'(b(v_1^+) - b(v_2^+)) Z_1
\]

\[
\times (\alpha(\nabla_x b(v_1^+)) - \alpha(\nabla_y b(v_2^+))) \cdot (\nabla_x b(v_1^+) - \nabla_y b(v_2^+)) \mathrm{d}y \mathrm{d}s \mathrm{d}x \mathrm{d}t
\]

\[
\leq 0,
\]

where \(w^+\) and \(w^-\) denote the positive part and negative part of \(w\).
Therefore we can follow the argument via the convergence of mollification given in §5 of Carrillo [23] and obtain
\[
\int_Q H^\circ (v_1 - v_2) \left\{ -(g(v_1) - g(v_2)) \partial_t \zeta \\
+ (\alpha(\nabla b(v_1)) - \alpha(\nabla b(v_2))) \cdot \nabla \zeta - (f_1 - f_2) \zeta \right\} \, dx \, dt \\
- \int_\Omega (\xi_{01} - \xi_{02}) + \zeta(x, 0) \, dx \leq 0
\]
for every \( \zeta \in \mathcal{D}(\mathbb{R}^n) \). By putting \( \zeta(x, t) = \zeta(t) \) in (61), we derive (60) (see Corollary 10 of [23]). □

Especially, we can show that

**Corollary 1.** Assume (H.\( \alpha \)), (H.\( \beta \)), and \( \beta \) is single-valued. Then the solution to (P) given in Theorem 2.1 is unique.

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