CALCULATING THE VIRTUAL COHOMOLOGICAL DIMENSION
OF THE AUTOMORPHISM GROUP OF A RAAG.

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ABSTRACT. We describe an algorithm to find the virtual cohomological dimension of the automorphism group of a right-angled Artin group. The algorithm works in the relative setting; in particular it also applies to untwisted automorphism groups and basis-conjugating automorphism groups. The main new tool is the construction of free abelian subgroups of certain Fouxe-Rabinovitch groups of rank equal to their virtual cohomological dimension, generalizing a result of Meucci in the setting of free groups.

1. INTRODUCTION

Automorphism groups of right-angled Artin groups (or RAAGs) form a diverse and interesting family, encompassing the rich worlds of both integer matrix groups and automorphism groups of free groups. For any right-angled Artin group $A_\Gamma$, Laurence [18] gave a generating set for $\text{Aut}(A_\Gamma)$, and since this result authors have worked to understand higher finiteness properties of these groups. In particular, Charney and Vogtmann [7] showed that each outer automorphism group $\text{Out}(A_\Gamma)$ has finite virtual cohomological dimension (vcd). Given recent constructions of classifying spaces for untwisted subgroups [6] and the natural analogs of congruence kernels for these groups [11], it is natural to ask what $\text{vcd}(\text{Out}(A_\Gamma))$ actually is. Indeed, the construction of upper and lower bounds for specific examples and interesting subfamilies have been obtained in many cases [5, 6, 11, 21], giving the vcd when these bounds coincide.

In this paper we give an algorithm to compute $\text{vcd}(\text{Out}(A_\Gamma))$ for an arbitrary graph $\Gamma$. More generally, this algorithm gives the virtual cohomological dimension of any outer automorphism group of a right-angled Artin group relative to a collection of special subgroups. This includes the untwisted automorphism groups of $\mathbb{Z}$ and partially symmetric (or basis-conjugating) outer automorphism groups of RAAGs.

In [11], the first and third authors initiated a study of relative outer automorphism groups of right-angled Artin groups, which are affectionately known as RORGs. Such a group is defined by taking collections $\mathcal{G}, \mathcal{H}$ of special subgroups (a special subgroup is one of the form $A_\Delta$ given by an induced subgraph $\Delta \subset \Gamma$) of a right-angled Artin group $A_\Gamma$ and looking at the subgroup $\text{Out}(A_\Gamma; \mathcal{G}, \mathcal{H})$ of automorphisms that preserve each element of $\mathcal{G}$ and act trivially on each element of $\mathcal{H}$ (see Section 2.2). This approach is not an idle exercise in generalization if one wants to understand $\text{Out}(A_\Gamma)$. The main result of [11] uses RORGs to construct a subnormal series for $\text{Out}(A_\Gamma)$ (more generally, for an arbitrary RORG) such that the consecutive quotients of this series are either finite, free-abelian groups, copies of $\text{GL}(n, \mathbb{Z})$, or groups known as Fouxe-Rabinovitch groups. We call such a normal
series a decomposition series. We will see that the virtual cohomological dimension of \( \text{Out}(A_\Gamma) \) is the sum of the vcds of the consecutive quotients appearing in a decomposition series.

To make this process algorithmic, one needs to know how to find the vcd of a Fouxe-Rabinovitch group. Let us first recall the definition of these groups. Let \( G = G_1 * G_2 * \cdots * G_k * F_m \) be a free factor decomposition of a group \( G \). An element \( \Phi \in \text{Out}(G) \) belongs to the Fouxe-Rabinovitch group associated to this free factor decomposition if for each \( G_i \) there exists a representative \( \phi \in \Phi \) restricting to the identity on \( G_i \). For example, the basis-conjugating automorphism group of a free group is the Fouxe-Rabinovitch group given by the free-factor decomposition \( F_n = \mathbb{Z} * \mathbb{Z} * \cdots * \mathbb{Z} \). Going back to RAAGs, if each \( G_i = A_{\Delta_i} \) is a special subgroup, then the Fouxe-Rabinovitch group is the relative automorphism group \( \text{Out}(A_\Gamma; \{A_{\Delta_i}\}_i) \).

**Theorem A.** Let \( A_\Gamma = A_{\Delta_1} * A_{\Delta_2} * \cdots * A_{\Delta_k} * F_m \) be a free factor decomposition of a right-angled Artin group with \( k \geq 1 \). Let \( d(\Delta_i) \) be the size of a maximal clique in each \( \Delta_i \), and let \( z(\Delta_i) \) be the rank of the center of \( A_{\Delta_i} \). Then

\[
\text{vcd}(\text{Out}(A_\Gamma; \{A_{\Delta_i}\}_i)) = (k + 2m - 2) \cdot \max_i \{d(\Delta_i)\} + \sum_{i=1}^{k} (d(\Delta_i) - z(\Delta_i)).
\]

There exists a free abelian subgroup of \( \text{Out}(A_\Gamma; \{A_{\Delta_i}\}_i) \) of rank equal to the virtual cohomological dimension.

This generalizes a theorem of Meucci [20] on relative automorphism groups of free groups and Collins [9] on partially symmetric automorphism groups. To prove this theorem, we obtain a lower bound for the virtual cohomological dimension by constructing free-abelian subgroups of the appropriate rank. The upper bound is obtained by a careful analysis of simplex stabilizers for the action of the Fouxe-Rabinovitch group on the spine of relative Outer space (roughly speaking, we have to make sure that simplices of large dimension have small stabilizers). This uses work of Guirardel and Levitt [17].

In the case where \( k = 0 \), the virtual cohomological dimension of \( \text{Out}(F_m) \) was shown to be \( 2m - 3 \) in Culler and Vogtmann’s seminal paper on Outer space [10]. There, the lower bound is obtained by finding a copy of \( \mathbb{Z}^{2m-3} \) in \( \text{Out}(F_m) \) generated by Nielsen automorphisms. The abelian subgroups found in the Fouxe-Rabinovitch case are very similar and made up of transvections and partial conjugations (see Remark 3.6). On the other side of the RAAG spectrum, similar results hold for \( \text{GL}(n, \mathbb{Z}) \). Here the virtual cohomological dimension is equal to the Hirsch length of the polycyclic subgroup of upper triangular matrices. Given all of this, it is natural to conjecture that for an arbitrary RORG there is also a polycyclic subgroup of rank equal to the virtual cohomological dimension. Indeed, this conjecture holds in all known examples, but we cannot prove it in general. Luckily, we do not need explicit polycyclic subgroups to calculate vcd.

**Theorem B.** There is an algorithm which, given the input of a finite graph \( \Gamma \) and two collections of special subgroups \( \mathcal{G} \) and \( \mathcal{H} \) of \( A_\Gamma \), computes the virtual cohomological dimension of \( \text{Out}(A_\Gamma; \mathcal{G}, \mathcal{H}') \).

If one prefers to look at the absolute automorphism group, the vcds of \( \text{Aut}(A_\Gamma) \) and \( \text{Out}(A_\Gamma) \) differ by the dimension of \( A_\Gamma/Z(A_\Gamma) \) (see Remark 4.4).
The main idea behind the proof of Theorem 3 is as follows. The existence of the polycyclic subgroups above imply that in all of these examples, the rational cohomological dimension of each group is equal to the vcd. Although vcd is only subadditive with respect to exact sequences, rational cohomological dimension is additive (by a theorem of Bieri [1]). Therefore the vcd of a RORG is the sum of the vcds of the pieces that appear in its decomposition series.

**Structure of the paper.** We describe the relevant background material on cohomological dimension and automorphism groups in Section 2. In Section 3 we give a proof of Theorem A and in Section 4 describe how the decomposition series of a RORG can be found algorithmically and complete the proof of Theorem B.

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### 2. Background

#### 2.1. Cohomological dimension.

For a thorough treatment of cohomological dimension, the reader is referred to the books of Bieri [1] and Brown [3]. Let \( R \) be a unital commutative ring. For a group \( G \), the cohomological dimension of \( G \) over \( R \), denoted \( \text{cd}_R(G) \), is given by

\[
\text{cd}_R(G) = \max\{ n : H^n(G; R) \neq 0 \text{ for some } R\text{-module } M \}.
\]

The cohomological dimension of a group \( G \) is given by \( \text{cd}(G) = \text{cd}_Q(G) \). The cohomological dimension satisfies \( \text{cd}_R(G) \leq \text{cd}(G) \) for any ring \( R \). A group \( G \) is of finite type, or of type \( F \), if \( G \) is the fundamental group of an aspherical CW-complex with a finite number of cells. If \( G \) is of finite type, then to find \( \text{cd}_R(G) \) one only needs to look at the cohomology with coefficients in the group ring \( RG \), and

\[
\text{cd}_R(G) = \max\{ n : H^n(G; RG) \neq 0 \}.
\]

If \( 1 \to N \to G \to Q \to 1 \) is an exact sequence of groups, then

\[
\text{cd}_R(G) \leq \text{cd}_R(N) + \text{cd}_R(Q).
\]

However, equality does not hold in general. For instance, Dranishnikov [11] constructed a family of hyperbolic groups \( G_p \) such that \( \text{cd}(G_p) = 3 \) for all \( p \), but \( \text{cd}(G_p \times G_q) = 5 \) whenever \( p \neq q \). Roughly speaking, the failure of equality in (1) comes from torsion in the top cohomology group (this is explored in detail in [12]).

Over a field these difficulties disappear, so that one has the following:

**Theorem 2.1** ([1], Theorem 5.5). *If \( 1 \to N \to G \to Q \to 1 \) is an exact sequence of groups of finite type, then\*

\[
\text{cd}_Q(G) = \text{cd}_Q(N) + \text{cd}_Q(Q).
\]

Throughout this paper, we will be working with groups satisfying \( \text{cd}(G) = \text{cd}_Q(G) \), and will be able to make use of the following proposition.

**Proposition 2.2.** Let \( 1 \to N \to G \to Q \to 1 \) be an exact sequence of groups. Suppose that \( N \) and \( Q \) are groups of finite type, with

\[
\text{cd}_Q(N) = \text{cd}(N) \text{ and } \text{cd}_Q(Q) = \text{cd}(Q),
\]

then

\[
\text{cd}_Q(G) = \text{cd}(G) = \text{cd}(N) + \text{cd}(Q).
\]
Proof. By applying Theorem 2.1 and equation (1) we have
\[
\text{cd}(G) \geq \text{cd}_Q(G) = \text{cd}_Q(N) + \text{cd}_Q(Q) = \text{cd}(N) + \text{cd}(Q) \geq \text{cd}(G),
\]
so there is equality throughout. \(\square\)

Any group with torsion has infinite cohomological dimension. However, if \(G\) has a finite-index subgroup \(H\) with finite cohomological dimension, then a theorem of Serre ([22], or alternatively [3, VIII.3]) asserts that for any other torsion-free finite-index subgroup \(H'\) one has \(\text{cd}(H) = \text{cd}(H')\). It follows that if \(G\) contains a torsion-free subgroup of finite index, then the virtual cohomological dimension of \(G\) can be defined by
\[
\text{vcd}(G) = \{\text{cd}(H) : H \text{ is torsion free and } [G : H] < \infty\}.
\]

If \(P\) is a torsion-free polycyclic group, then \(\text{cd}(P) = \text{cd}_Q(P) = h(P)\), where \(h(P)\) is the Hirsch length of \(P\)—the number of infinite cyclic factors in a normal series for \(P\) (this follows from Proposition 2.2). If \(P\) is a subgroup of a group \(G\) then \(\text{cd}_R(P) \leq \text{cd}_R(G)\), so polycyclic groups can be used to find lower bounds for (rational) cohomological dimension. In particular, one has:

**Proposition 2.3.** Suppose that \(G\) acts properly and cocompactly on a contractible complex of dimension \(d\) and contains a polycyclic subgroup \(P\) with Hirsch length \(h(P) = d\). Then for any finite-index, torsion-free subgroup \(H\) of \(G\), one has
\[
\text{cd}_Q(H) = \text{cd}(H) = d.
\]
In particular, if \(G\) has a finite-index torsion-free subgroup then \(\text{vcd}(G) = d\). \(\square\)

Culler and Vogtmann [10] use the spine of Outer space and the existence of a free abelian subgroup of rank \(2n - 3\) to show that for any torsion-free, finite-index subgroup \(H\) of \(\text{Out}(F_n)\) one has
\[
\text{cd}_Q(H) = \text{cd}(H) = 2n - 3.
\]
Similarly, by combining Borel and Serre’s calculation of the vcd [2] with the upper-triangular matrices in \(\text{GL}(n, \mathbb{Z})\), we see that
\[
\text{cd}_Q(H) = \text{cd}(H) = \frac{n(n - 1)}{2},
\]
for any torsion-free, finite index subgroup \(H\) of \(\text{GL}(n, \mathbb{Z})\).

2.2. RAAGs and RORGs. Let \(A_\Gamma\) be the right-angled Artin group determined by a finite graph \(\Gamma\). If \(\Delta\) is a full subgraph of \(\Gamma\) we use \(A_\Delta\) to denote the special subgroup generated by the vertices contained in \(\Delta\). An outer automorphism \(\Phi\) of \(A_\Gamma\) preserves \(A_\Delta\) if there exists a representative \(\phi \in \Phi\) that restricts to an automorphism of \(A_\Delta\). An outer automorphism \(\Phi\) acts trivially on \(A_\Delta\) if there exists a representative \(\phi \in \Phi\) acting as the identity on \(A_\Delta\). If \(\mathcal{G}, \mathcal{H}\) are collections of special subgroups of \(A_\Gamma\), then the relative outer automorphism group \(\text{Out}(A_\Gamma; \mathcal{G}, \mathcal{H}')\) consists of automorphisms that preserve each \(A_\Delta \in \mathcal{G}\) and act trivially on each \(A_\Delta \in \mathcal{H}\).
An important feature of RORGs is the fact that if $A_\Delta$ is invariant under the group $\text{Out}(A_T; G, H^t)$ then there is a restriction homomorphism

$$R_\Delta : \text{Out}(A_T; G, H^t) \to \text{Out}(A_\Delta)$$

such that the image and kernel of $R_\Delta$ are also RORGs [11, Theorem E]. This allows us to study RORGs using inductive methods.

If $G$ is a collection of special subgroups then a $G$-path in $\Gamma$ is a sequence of vertices $v_1, \ldots, v_k$ of $\Gamma$ such that each pair $(v_i, v_{i+1})$ either span an edge of $\Gamma$ or are contained in some common element of $G$. A $G$-component of a subgraph $\Delta \subset \Gamma$ is a maximal subgraph $\Lambda \subset \Delta$ with the property that any two vertices in $\Lambda$ are connected by a $G$-path in $\Lambda$. Given a vertex $v \in \Gamma$ and a collection of special subgroups $G$ we define $G^v$ to be the subset of $G$ consisting of special subgroups that do not contain $v$, so that:

$$G^v = \{ A_\Delta \in G : v \notin \Delta \}.$$

There is a finite-index subgroup of the relative outer automorphism group, denoted $\text{Out}^0(A_T; G, H^t)$, that is generated by the inversions, transvections, and extended partial conjugations $\text{Out}(A_T; G, H^t)$ contains (see Theorem D of [11]). These are defined below.

- **Inversions.** For a vertex $v$ in $\Gamma$, the inversion of $v$ is the automorphism $\iota_v$ of $A_T$ that sends $v$ to $v^{-1}$ and fixes all other generators. An inversion $[\iota_v]$ is in $\text{Out}(A_T; G, H^t)$ if and only if $v$ is not contained in any element of $H$.

- **Transvections.** Given vertices $v, w$ in $\Gamma$, the right (respectively, left) transvection of $w$ on $v$ is the automorphism $\rho_w^v$ (respectively, $\lambda_w^v$) that sends a vertex $w$ to $vw$ (respectively, $vw$) and fixes all other generators. This is a well-defined automorphism of $A_T$ if $\text{lk}(w) \subset \text{st}(v)$. A transvection $[\rho_w^v]$ is in $\text{Out}(A_T; G, H^t)$ if and only if $w$ is not contained in any element of $G^v \cup H$.

- **Extended partial conjugations.** Fix a vertex $v$. If $C$ is a union of components of $\Gamma - \text{st}(v)$ then we define the extended partial conjugation $\pi_\Delta^v$ to send each $w \in C$ to $vwv^{-1}$ and fix all other generators. The automorphism $[\pi_\Delta^v]$ belongs to $\text{Out}(A_T; G, H^t)$ if and only if $C$ is a union of $(G^v \cup H)$-components of $\Gamma - \text{st}(v)$.

The condition on extended partial conjugations is stated slightly differently to the one seen in [11, Proposition 3.9], where for developing the theory it was convenient to assume $H$ (and often all special subgroups of groups in $H$) were contained in $G$. With the aim of simplifying computations, the above characterizations work in general.

We define the partial preorder $\leq_{(G, H)}$ on $V(\Gamma)$ by saying that $w \leq_{(G, H)} v$ if and only if $\text{lk}(w) \subset \text{st}(v)$ and $w \notin G^v \cup H$ (equivalently $[\rho_w^v] \in \text{Out}(A_T; G, H^t)$). Given a subgraph $\Delta \subset \Gamma$ we say that $\Delta$ is upwards closed under $\leq_{(G, H)}$ if $w \in \Delta$ and $w \leq_{(G, H)} v$ implies that $v \in \Delta$. We say that $\Delta$ is $(G, H)$-star-separated by a vertex $v$ if $\Delta$ intersects more than one $(G^v \cup H)$-component of $\Gamma - \text{st}(v)$. This is equivalent to the existence of an extended partial conjugation $[\pi_\Delta^v] \in \text{Out}(A_T; G, H^t)$ which acts on $A_\Delta$ as a non-inner automorphism.

By using the above generating set of $\text{Out}^0(A_T; G, H^t)$, one can show the following (cf. [11, Lemma 2.2]).
Proposition 2.4. Let $A_\Delta$ be a special subgroup of $A_\Gamma$.

- $A_\Delta$ is invariant under $\text{Out}^0(A_\Gamma; \mathcal{G}, \mathcal{H})$ if and only if $\Delta$ is upwards closed under $\leq_{(\mathcal{G}, \mathcal{H})}$ and $\Delta$ is not $(\mathcal{G}, \mathcal{H})$-star-separated by a vertex $v \in \Gamma - \Delta$.

- The group $\text{Out}^0(A_\Gamma; \mathcal{G}, \mathcal{H})$ acts trivially on $A_\Delta$ if and only if every $v \in \Delta$ is contained in some element of $\mathcal{H}$ and $\Delta$ is not $(\mathcal{G}, \mathcal{H})$–star-separated by any vertex of $\Gamma$.

Given the sets $(\mathcal{G}, \mathcal{H})$, we can use Proposition 2.4 to find the invariant special subgroups of $\text{Out}^0(A_\Gamma; \mathcal{G}, \mathcal{H})$.

3. The virtual cohomological dimension of a Fouxe-Rabinovitch group

In this section we use the relative outer space of Guirardel and Levitt to find the virtual cohomological dimension of a Fouxe-Rabinovitch group associated to a free factor decomposition of a right-angled Artin group.

3.1. Fouxe-Rabinovitch groups and congruence subgroups. Let $G = G_1 * G_2 * \cdots * G_k * F_m$ be a free factor decomposition of a group. We let $\mathcal{G} = \{G_i\}$ and define the Fouxe-Rabinovitch group associated to this free factor decomposition to be

$$\text{FR}_G = \text{Out}(G; G_i).$$

This is the subgroup of $\text{Out}(G)$ acting trivially on each $G_i$ in the decomposition. We do not assume the free factor decomposition is maximal: it need not be the Grushko decomposition of $G$. We do, however, require that this free factor decomposition is nontrivial in the sense that $k \geq 1$ and $k + m \geq 2$.

The level 3 congruence subgroup of $\text{FR}_G$ is defined in the same way as the subgroups of $\text{GL}(n, \mathbb{Z})$ of the same name. It is the finite-index subgroup $\text{FR}_G^{[3]}$ acting trivially on $H_1(G; \mathbb{Z}/3\mathbb{Z})$. As the action of $\text{FR}_G$ on each $G_i$ is trivial, this is the same as the subgroup acting trivially on $H_1(F_m; \mathbb{Z}/3\mathbb{Z})$ via the quotient map $\text{FR}_G \to \text{Out}(F_m)$. If each $G_i$ and each $G_i/\mathbb{Z}(G_i)$ is torsion-free, then so is each level 3 congruence subgroup of $\text{FR}_G$ (see [17, Theorem 5.2]).

3.2. Relative Outer space. Following the work of Guirardel and Levitt in [16] [15], we recall the definition of the spine of the relative outer space given by a free factor decomposition of a group.

Definition 3.1. A Grushko $\mathcal{G}$-tree is a minimal action of $G$ on a simplicial tree $T$ with trivial edge stabilizers such that each element of $\mathcal{G}$ is elliptic in $T$ and each vertex stabilizer is either trivial or conjugate to an element of $\mathcal{G}$. Two Grushko $\mathcal{G}$-trees $T_1$ and $T_2$ are equivalent if there is a $\mathcal{G}$-equivariant homeomorphism $f : T_1 \to T_2$.

The set of Grushko $\mathcal{G}$-trees forms a poset, where $T_1 < T_2$ if there is a $(\mathcal{G}$-equivariant) subforest in $T_2$ which collapses to give the action of $G$ on $T_1$. The geometric realization of this poset is called the spine of relative Outer space and we will denote it by $X_\mathcal{G}$.

By a theorem of Guirardel and Levitt [16], the spine $X_\mathcal{G}$ is contractible. The spine admits an action of $\text{Out}(G; \mathcal{G})$ by precomposing the action of $G$ on a tree $T$ with the automorphism. Automorphisms may act nontrivially on each $G_i$, but this action restricts to an action of $\text{FR}_G$. 
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Figure 1. A rose with $m$ petals and $k-1$ leaves, whose Bass–Serre tree gives a vertex in the spine of relative Outer space.

Each $n$-simplex corresponds to a chain $T_0 < T_1 < \cdots < T_n$ of Grushko $G$-trees. As the action of $\text{FR}_G$ preserves the number of edge orbits in a Grushko $G$-tree, the action of $\text{FR}_G^{[3]}$ on $X_G$ is rigid (any automorphism preserving a simplex fixes it pointwise). The lemma below uses ideas from the proof of Proposition 3.7 of [17] and gives a description of simplex stabilizers for the action.

**Lemma 3.2.** Let $\sigma$ be a simplex in $X_G$ corresponding to the chain $T_0 < T_1 < \cdots < T_n$ of Grushko $G$-trees. Then the stabilizer of $\sigma$ in $\text{FR}_G^{[3]}$ is a finite-index subgroup of

$$\bigoplus_{i=1}^k G_i^{v_i}/Z(G_i),$$

where $v_i$ is the number of $G_i$-orbits of edges at the vertex fixed by $G_i$ in $T_n$ and $Z(G_i)$ is embedded in $G_i^{v_i}$ diagonally. Furthermore,

$$\sum_{i=1}^k v_i \leq 2(m + k - 1) - n.$$

**Proof.** Firstly, $\text{FR}_G^{[3]}$ maps to a torsion-free subgroup of $\text{Out}(F_m)$, automorphisms in $\text{FR}_G^{[3]}$ preserving a Grushko $G$-tree $T$ act trivially on the quotient graph $T/G$, and therefore preserve all collapses of $T$. This can be seen since the leaf vertices in $T/G$ must have a non-trivial stabilizer in $G$, so must be in $G$. Hence the action fixes these vertices. Any such action induces a finite order element of $\text{Out}(\pi_1(T/G)) \cong \text{Out}(F_m)$, but since our image in here is torsion-free, the action must be trivial (note that if $T/G$ is a circle, all vertices are fixed). As each $T_j$ is a collapse of $T_n$, any automorphism in $\text{FR}_G^{[3]}$ that fixes $T_n$ in $\text{FR}_G^{[3]}$ also fixes each $T_j$ with $j < n$, so that:

$$\text{Stab}_{\text{FR}_G^{[3]}}(\sigma) = \text{Stab}_{\text{FR}_G}(T_n) \cap \text{FR}_G^{[3]},$$

where $\text{Stab}_{\text{FR}_G}(T_n)$ denotes the stabilizer of $T_n$ in $\text{FR}_G$ that acts trivially on $T_n/G$. This is the group of twists of the splitting [19] Section 2.4 and satisfies

$$\text{Stab}_{\text{FR}_G}(T_n) \cong \bigoplus_{i=1}^k G_i^{v_i}/Z(G_i),$$

where, as in the hypothesis, each $v_i$ is the number of $G_i$-orbits of edges in $T_n$ at the vertex fixed by $G_i$.

It remains to justify the final inequality. Note that we have to collapse at least $n$ orbits of edges in $T_n$ to obtain $T_0$, so we may assume $T_n$ has $N \geq n$ orbits of vertices with trivial stabilizer. In total, the quotient graph $T_n/G$ has $N+k$ vertices.
and $N + k + m - 1$ edges (as the fundamental group is $F_m$). There are at least 3 half-edges adjacent to each of the $N$ vertices with trivial stabilizer. Therefore:

$$\sum_{i=1}^{k} v_i \leq 2(N + k + m - 1) - 3N$$

$$= 2(m + k - 1) - N$$

$$\leq 2(m + k - 1) - n,$$

as required.

As each $v_i \geq 1$, the inequality in Lemma 3.2 shows that $n \leq 2m + k - 2$. It is not hard to check that the dimension of the spine is equal to $2m + k - 2$ by exhibiting a graph of groups decomposition of $G$ with $2m + k - 2$ trivalent vertices with trivial stabilizers, trivial edge groups, and each nontrivial vertex group corresponding to a $G_i$ (see Figure 2).

![Figure 2](image-url)

**Figure 2.** A graph of groups decomposition of $G$ with a maximal number of edge orbits in the case that $k = m = 4$.

The above work allows one to bound the geometric dimension of a Foux-Rabinovitch group via the following theorem:

**Theorem 3.3** ([14], Theorem 7.3.3). Let $X$ be a contractible, rigid $G$-CW complex with $\dim(X) \leq N$. For each $n$, suppose that the stabilizer of each $n$-cell in $X$ has geometric dimension at most $d_n$. Then $G$ has geometric dimension

$$\text{gd}(G) \leq \max\{d_n + n : 0 \leq n \leq N\}.$$

We will apply this to the specific case of right-angled Artin groups below.

### 3.3. Free product decompositions of RAAGs.

For a finite graph $\Gamma$, we define $d(\Gamma)$ to be the size of the largest clique in $\Gamma$. This is the same as the dimension of the Salvetti complex of $A_\Gamma$. We define $z(\Gamma)$ to be the number of vertices in $\Gamma$ that are adjacent to every other vertex. This is the same as the rank of the center of $A_\Gamma$, which is a finitely-generated free-abelian group. As $\text{FR}_{G}$ is finite-index in $\text{FR}_{N}$, the next theorem and its corollary imply Theorem A from the introduction.
Theorem 3.4. Let FR\(_\mathcal{G}\) be the Fouxe-Rabinovitch group associated to a nontrivial free factor decomposition

\[ A_\Gamma = A_{\Delta_1} \ast A_{\Delta_2} \ast \cdots \ast A_{\Delta_k} \ast F_m \]

of a right-angled Artin group, and let FR\(_\mathcal{G}[3]\) be its level 3 congruence subgroup. Then

\[ \text{gd}(\text{FR}\_\mathcal{G}[3]) = (k + 2m - 2) \cdot \max_i \{d(\Delta_i)\} + \sum_{i=1}^{k} (d(\Delta_i) - z(\Delta_i)). \]

Proof. As above, let \(X_\mathcal{G}\) be the spine of relative Outer space and let \(\sigma\) be a simplex of dimension \(n\) given by the chain of trees \(T_0 < T_1 < \cdots < T_n\). By Lemma 3.2, the stabilizer \(\text{Stab}_{\text{FR}\_\mathcal{G}[3]}(\sigma)\) of \(\sigma\) is a finite index subgroup of

\[ \bigoplus_{i=1}^{k} A_{\Delta_i}/Z(A_{\Delta_i}) \cong \bigoplus_{i=1}^{k} (A_{\Delta_i}^{v_i-1} \oplus A_{\Delta_i}/Z(A_{\Delta_i})), \]

and \(\sum_{i=1}^{k} (v_i - 1) \leq 2m + k - 2 - n\). As the geometric dimension of \(A_{\Delta_i}\) is \(d(\Delta_i)\) and the geometric dimension of \(A_{\Delta_i}/Z(A_{\Delta_i})\) is \(d(\Delta_i) - z(\Delta_i)\), it follows that

\[ \text{gd}(\text{Stab}_{\text{FR}\_\mathcal{G}[3]}(\sigma)) \leq (2m + k - 2 - n) \cdot \max_i \{d(\Delta_i)\} + \sum_{i=1}^{k} (d(\Delta_i) - z(\Delta_i)) \]

\[ \leq [(2m + k - 2) \cdot \max_i \{d(\Delta_i)\} + \sum_{i=1}^{k} (d(\Delta_i) - z(\Delta_i))] - n. \]

Therefore Theorem 3.3 implies that

\[ \text{gd}(\text{FR}\_\mathcal{G}[3]) \leq (2m + k - 2) \cdot \max_i \{d(\Delta_i)\} + \sum_{i=1}^{k} (d(\Delta_i) - z(\Delta_i)). \]

To establish equality it is enough to find a free abelian subgroup of \(\text{FR}\_\mathcal{G}\) of rank equal to the right hand side of this equation. If we reorder the vertices so that \(A_{\Delta_i}\) has maximal dimension, we can find such a group inside the stabilizer of (the tree corresponding to) the rose given in Figure 1. In this case, Lemma 3.2 tells us that the stabilizer of the 0–cell given by the rose in \(X_\mathcal{G}\) under \(\text{FR}\_\mathcal{G}[3]\) is a finite index subgroup of

\[ A_{\Delta_i}^{2m+k-1}/Z(A_{\Delta_i}) \oplus_{i=2}^{k} A_{\Delta_i}/Z(A_{\Delta_i}) \cong A_{\Delta_i}^{2m+k-2} \oplus_{i=1}^{k} A_{\Delta_i}/Z(A_{\Delta_i}), \]

which contains a free-abelian subgroup of the desired rank. \(\square\)

The following corollary is immediate from the proof:

Corollary 3.5. There exists a free abelian subgroup of rank equal to \(\text{gd}(\text{FR}\_\mathcal{G}[3])\), so that

\[ \text{gd}(\text{FR}\_\mathcal{G}[3]) = \text{cd}(\text{FR}\_\mathcal{G}[3]) = \text{cd}_{\mathbb{Q}}(\text{FR}\_\mathcal{G}[3]). \]

\(\square\)

Remark 3.6. The free abelian subgroup used in the proof of Theorem 3.4 can be given quite explicitly. Firstly, order the factors of the decomposition so that \(A_{\Delta_1}\) has maximal dimension and let \(A_i\) be the vertices of a maximal clique in \(\Delta_i\). Let \(X\) be the set of vertices generating the free factor \(F_m\). Then, take all left and right transvections \(\rho_x^a\) and \(\lambda_x^a\) for \(x \in X\) and \(a \in A_1\). Adding the partial conjugations of the subgroups \(A_{\Delta_j}\), for \(j > 1\), by elements of \(A_1\) gives a free abelian group of rank \((2m + k - 1) \cdot d(\Delta_1)\) in \(\text{Aut}(A_\Gamma)\).
For each \( i = 1, \ldots, k \), and each vertex \( v \in A_i \), add the partial conjugation \( \pi_v^{\Delta_i} \). Such partial conjugations are trivial if \( v \) is in the center of \( A_{\Delta_i} \), and since a maximal clique in \( \Delta_i \) must contain all vertices in the center, this gives us \( d(\Delta_i) - z(\Delta_i) \) partial conjugations for each \( i \). One can check that all of automorphisms above generate a free abelian subgroup of \( \text{Aut}(A_\Gamma) \) of rank
\[
(2m + k - 1) \cdot d(\Delta_1) + \sum_{i=1}^{k} (d(\Delta_i) - z(\Delta_i)).
\]

The only inner automorphisms that appear in the above group come from products of generators with acting letter \( a \in A_1 \), so that the intersection of this subgroup with the inner automorphisms has rank \( d(\Delta_1) \). Subtracting this gives the rank in \( \text{Out}(A_\Gamma) \).

4. Calculating the vcd

We now give the details of the algorithm to compute the vcd of a RORG. As a first step, we explain how the decomposition procedure for a RORG given in [11] is algorithmic.

4.1. Dismantling RORGs. The finite-index subgroup \( \text{Out}^0(A_\Gamma; G, \mathcal{H}^t) \) of a RORG can be broken up in the following way:

**Theorem 4.1** ([11, Theorem A]). The group \( \text{Out}^0(A_\Gamma; G, \mathcal{H}^t) \) admits a subnormal series
\[
1 = H_0 \leq H_1 \leq H_2 \leq \cdots \leq H_K = \text{Out}^0(A_\Gamma; G, \mathcal{H}^t)
\]
such that each quotient \( Q_i = H_i/H_{i-1} \) is either:

- (D1) a finitely generated free abelian group,
- (D2) isomorphic to \( GL(m, \mathbb{Z}) \), or
- (D3) a Fouxe-Rabinovitch group given by a free factor decomposition of a special subgroup of \( A_\Gamma \).

Note that \( \text{Out}(F_m) \) may arise as a quotient via case (D3). As in the introduction, we call such a subnormal series a decomposition series for the group.

The most natural way to find the consecutive quotients in a decomposition series is to first build a decomposition tree for \( \text{Out}^0(A_\Gamma; G, \mathcal{H}^t) \). This is a rooted tree where every internal vertex is labelled by a group of the form \( \text{Out}^0(\Gamma_v; G_v, \mathcal{H}_v^t) \), with \( \Gamma_v \) a subgraph of \( \Gamma \) and \( G_v, \mathcal{H}_v \) sets of special subgroups of \( A_{\Gamma_v} \). Our initial group is at the root. Each internal vertex \( G_v \) has two descendants \( K_v \) and \( I_v \) forming an exact sequence
\[
1 \rightarrow K_v \rightarrow G_v \rightarrow I_v \rightarrow 1.
\]

Every leaf of this tree is labelled by a group of the form (D1), (D2), or (D3) and one can show (e.g. using induction on the size of the tree) that the leaves of the tree give consecutive quotients in a subnormal series for the root. An example of such a tree is given in [11, Figure 6].

**Proposition 4.2.** There is an algorithm that produces a decomposition tree for \( \text{Out}^0(A_\Gamma; G, \mathcal{H}^t) \).

**Proof.** The process for obtaining a tree is iterative. Given a vertex \( v \) in the tree, labelled by \( \text{Out}^0(\Gamma_v; G_v, \mathcal{H}_v^t) \), we describe below how to either

1. recognise \( \text{Out}^0(\Gamma_v; G_v, \mathcal{H}_v^t) \) as a group of type (D1), (D2), or (D3), or
If a new vertex has not been recognised as \(D\) further from the root, this algorithm will terminate. Because the complexity of RORGs decreases as we get further from the root, this algorithm will terminate.

Given \(\text{Out}^0(\Gamma_v; G_v, H_v^t)\), the first step is to extend \(G_v\) to its saturation \(G_v'\), relative to \((G_v, H_v)\), which is the collection \(G_v'\) of all special subgroups that are invariant under \(\text{Out}^0(\Gamma_v; G_v, H_v^t)\). The invariant special subgroups can be determined from the input using Proposition 2.4.

Now assume that \(G_v\) is saturated with respect to \((G_v, H_v)\). By [11] Theorem E, for each special subgroup \(A_\Delta\), the image of \(\text{Out}^0(\Gamma_v; G_v, H_v^t)\) under the restriction map \(R_\Delta\) is equal to \(\text{Out}^0(A_\Delta; (G_v)_\Delta, (H_v)_\Delta^t)\), where

\[
(G_v)_\Delta = \{A_{\Delta \Theta} : A_\Theta \in G_v\} - \{A_\Delta\},
\]

and \((H_v)_\Delta^t\) is defined similarly. This image is nontrivial if and only if there is an inversion, extended partial conjugation, or transvection with nontrivial image under \(R_\Delta\). This is a finite list of elements, and checking if each one has nontrivial image is a simple process.

We now divide into cases according to the nature of the images of restriction maps.

**Case 1.** There is a restriction map \(R_\Delta\) with nontrivial image.

In this case we use the exact sequence

\[
1 \to \text{Out}^0(A_{\Gamma_v}; G_v, (H_v \cup \{A_\Delta\})^t) \to \text{Out}^0(A_v; G_v, H_v^t) \xrightarrow{R_\Delta} \text{Out}^0(A_\Delta; (G_v)_\Delta, (H_v)_\Delta^t) \to 1.
\]

given by [11] Theorem E. As per the proof of [11] Theorem 5.9, the complexity of the RORG in the kernel and quotient is strictly lower than that of \(\text{Out}^0(\Gamma_v; G_v, H_v^t)\). In this case we add two new descendants below \(v\) in the tree, with vertices labelled by the kernel and image above.

**Case 2.** All restriction maps have trivial image.

As in [11] Section 5, we can break into five subcases.

**Case 2a.** \(\Gamma_v\) is disconnected and \(\Gamma_v\) is \(G_v\)-disconnected.

Here \(\text{Out}^0(\Gamma_v; G_v, H_v^t)\) is a Fouxe-Rabinovitch group where \(F_m\) is a free group on \(m\) isolated vertices not contained in any element of \(G_v\) and the \(A_\Delta\)'s are the remaining \(G_v\)-connected components ([11] Proposition 5.2]).

**Case 2b.** \(\Gamma_v\) is disconnected and \(\Gamma_v\) is \(G_v\)-connected.

The vertices which \((G_v, H_v)\)-star-separate form a complete graph \(\Theta\) and, as per the proof of [11] Proposition 5.2, \(\text{Out}^0(\Gamma_v; G_v, H_v^t)\) is a free-abelian group of rank equal to \(|\Theta|\).

**Case 2c.** \(\Gamma_v\) is connected and the center \(Z(A_{\Gamma_v})\) of \(A_{\Gamma_v}\) is trivial.

In this case \(\text{Out}^0(\Gamma_v; G_v, H_v^t)\) is generated by commuting partial conjugations with acting letters \(v\) that have \(N_v\) \((G, H)\)-connected components. It is not hard to check (e.g. using the first Johnson homomorphism [23]) that these elements form a free-abelian group of rank \(\sum(N_v - 1)\).
Case 2d. $\Gamma_v$ is connected and $Z(A_{\Gamma_v})$ is a proper, nontrivial subgroup.
If $\Delta = \Gamma_v - Z(\Gamma_v)$ we apply [11, Proposition 5.6]. There is a projection homomorphism $P_\Delta$ with image $Out^0(A_{\Delta}; (G_v)_{\Delta})$ (with $(G_v)_{\Delta}$ as defined above) whose kernel is a free abelian group with basis given by the leaf transvections in $Out^0(A_{\Gamma}; G, H')$. These are transvections $\rho_{w}^{u}$ with $w \in Z(\Gamma)$ and $u \notin Z(\Gamma)$, see [8]. It is a quick check on each transvection to determine if it is a leaf transvection. We therefore add two descendants below $v$, one labelled by a free abelian group of the appropriate rank, and the other labelled by $Out^0(A_{\Delta}; (G_v)_{\Delta})$.

Case 2e. $\Gamma_v$ is complete and $A_{\Gamma_v} = Z_n$ for some $n$.
It is described in [11, Proposition 5.8] how the group fits in the exact sequence
\[
1 \to A \to Out^0(\Gamma_v; G_v, H'_v) \to GL(m, \mathbb{Z}) \to 1,
\]
where $A$ is a finitely generated free abelian group of matrices, so that the rank is easy to compute. We thus add two descendants below $v$, one labelled by $A$ and the other by $GL(m, \mathbb{Z})$.

Note that the construction of a decomposition tree involves many choices, as at each step we only pick some invariant special subgroup $A_\Delta$ for which there is a restriction map.

Question 4.3. Does the set of consecutive quotients in a decomposition series depend on the set of choices made to dismantle $Out^0(A_{\Gamma}; G, H')$?

In this direction, Brück [4, Section 7] uses careful choices of restriction maps to construct a decomposition tree for $Out^0(A_{\Gamma}; G, H')$ where the leaves can be described quite explicitly. As a trade-off, the leaves that appear in the decomposition tree of Brück are slightly more general (there are groups generated by partial conjugations that are not necessarily of type (D1), (D2), or (D3)).

4.2. Completing the proof of Theorem B. To complete the proof of Theorem B, we describe how to compute the vcd of a RORG step-by-step:

Step 1: Build a decomposition tree for $Out^0(A_{\Gamma}; G, H')$. This is detailed in Proposition 4.2.

Step 2: Find the vcd of each leaf. Each leaf is free-abelian, a copy of $GL(n, \mathbb{Z})$, or a Fouxe-Rabinovitch group, so this can be read off via the calculations of Borel–Serre [2] and Culler–Vogtmann [10] discussed in Section 2.1 and Theorem A.

Step 3: Add the vcds of the leaves to find the vcd of the root. We do not need to explain how to carry out this step, but we should justify why it works. This is where the discussion of rational cohomological dimension given in Section 2.1 comes into play. The key point here is that we can restrict to the congruence subgroup $Out^0(A_{\Gamma}; G, H')$ of $Out^0(A_{\Gamma}; G, H')$. This is the torsion-free, finite-index subgroup given by the elements acting trivially on $H_1(A_{\Gamma}; \mathbb{Z}/3\mathbb{Z})$. By [11 Theorem 4.8], the short exact sequence
\[
1 \to Out^0(A_{\Gamma_v}; G_v, (H_v \cup \{A_{\Delta}\})^f) \to Out^0(A_{\Gamma_v}; G_v, H'_v) \xrightarrow{R_\Delta} Out^0(A_{\Delta}; (G_v)_{\Delta}, (H_v)_{\Delta})^f \to 1,
\]
coming from each projection map restricts to a short exact sequence
\[ 1 \rightarrow \text{Out}^3(A_{\Gamma_v}; G_v, (H_v \cup \{A_\Delta\})^t) \rightarrow \text{Out}^3(A_{\Gamma_v}; G_v, H_v^t) \xrightarrow{R_\Delta} \text{Out}^3(A_\Delta; (G_v)_\Delta, (H_v)_\Delta^t) \rightarrow 1. \]
for congruence subgroups. Similar behaviour happens with the projection maps that appear in Case 2d and Case 2e during the construction of the decomposition tree (one can see this as both of the projection maps split). As a result, one obtains an analogous decomposition tree for \( \text{Out}^3(A_{\Gamma_v}; G_v, H_v^t) \) where each vertex is a level three congruence subgroup of the corresponding vertex in the decomposition tree for \( \text{Out}^0(A_{\Gamma_v}; G_v, H_v^t) \). This gives a subnormal series
\[ 1 = H_0 < H_1 < H_2 < \cdots < H_K = \text{Out}^3(A_{\Gamma_v}; G_v, H_v^t) \]
of \( \text{Out}^3(A_{\Gamma_v}; G_v, H_v^t) \) where the consecutive quotients are congruence subgroups of the leaves of the decomposition tree for \( \text{Out}^0(A_{\Gamma_v}; G_v, H_v^t) \) (and the leaves given by free-abelian groups are still free-abelian of the same rank). Some leaves, in particular those isomorphic to \( \text{GL}(1, \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} \), will now be trivial. All of these groups are of finite type, and have rational cohomological dimension equal to their cohomological dimension (using either the discussion in Section 2.1 or Theorem A). By Bieri’s theorem (Theorem 2.1) and Proposition 2.2, the sum of the (rational) cohomological dimensions of the leaves is equal to the cohomological dimension of \( \text{Out}^3(A_{\Gamma_v}; G_v, H_v^t) \), justifying the calculation of the vcd of \( \text{Out}^0(A_{\Gamma_v}; G_v, H_v^t) \) given above.

Remark 4.4. The above work shows that the rational cohomological dimension of a (relative) outer automorphism group is the same as its cohomological dimension. As the inner automorphisms are isomorphic to \( A_{\Gamma_v}/Z(A_{\Gamma_v}) \), the same is true for \( \text{Inn}(A_{\Gamma_v}) \). Bieri’s theorem implies that the vcds of \( \text{Out}(A_{\Gamma_v}) \) and \( \text{Aut}(A_{\Gamma_v}) \) differ by the dimension of \( A_{\Gamma_v}/Z(A_{\Gamma_v}) \).

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