AN EXAMPLE OF A NON-COMMUTATIVE UNIFORM BANACH GROUP

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ABSTRACT. We construct a non-commutative uniform Banach group which has the free group of countably many generators as a dense subgroup.

INTRODUCTION

A uniform Banach group is a Banach space equipped with an additional group structure so that the group unit coincides with the Banach space zero and the group operations are uniformly continuous with respect to norm. Uniform Banach groups were introduced and studied by Enflo in [2], [3] with connection to the infinite-dimensional version of the Hilbert’s fifth problem. Typical example comes when we are given two Banach spaces $X$ and $Y$ and a uniform homeomorphism $\phi : X \to Y$ between them such that $\phi(0) = 0$. Then we can define a (commutative) group operation $\cdot$ on $X$ as follows: for $x, y \in X$ we set $x \cdot y = \phi^{-1}(\phi(x) + \phi(y))$. Note that unless $\phi$ is linear there is no a priori connection between the two group operations $+$ (resp. $+_{X}$) and $\cdot$.

A comprehensive source of information about uniform Banach groups is Chapter 17 in [1] where the following problem is mentioned. Does there exist a non-commutative uniform Banach group? This question was also asked by Prassidis and Weston in [4]. Here we give a positive answer to this question. The following is the main result.

Theorem 0.1. There exists an infinite dimensional separable Banach space $(X, +, 0, \| \cdot \|)$ equipped with an additional group structure $(\cdot, -1, 0)$ whose unit coincides with the Banach space zero, the group multiplication $\cdot$ is invariant with respect to the norm $\| \cdot \|$, and $F_{\infty}$, the free group of countably many generators, is a dense subgroup of $(X, \cdot, -1, 0)$.

In particular, there exists a non-commutative uniform Banach group.

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1. Preliminary discussion

Let us roughly explain the ideas behind the proof of Theorem 0.1. We shall construct a countable set $X$ equipped with two group structures. Under the first group structure, $X$ is isomorphic to the free group of countably many generators. Under the second one, $X$ is isomorphic to the minimal subgroup of the real vector space with countable Hamel basis that contains the elements of this Hamel basis and is closed under multiplication by scalars that are dyadic rationals. Moreover, $X$ will get equipped with a metric which is bi-invariant with respect to the non-commutative group operation and which behaves like a norm with respect to the latter group operation. In particular, the completion of $X$ with respect to this metric will become a real Banach space.

Let us mention two ‘peculiarities’ of the construction. First, $X$ is constructed so that there is no connection between these two algebraic structures (i.e. non-commutative group structure and commutative group structure with dyadic scalar multiplication) in the sense that for any $x, y \in X$, the elements $x, y, x \cdot y, x^{-1}$ are linearly independent (in the commutative structure), and similarly, for any $x, y \in X$ and dyadic rationals $\alpha, \beta$ we have that $\alpha x + \beta y$ is a new free group generator (in the non-commutative structure).

The second thing to mention is that the metric on $X$ is constructed by induction on finite fragments $X_n, n \in \mathbb{N}$, of $X$. This will give us a better control of the metric we are defining.

1.1. Some notation regarding free groups and vector spaces. Let $S$ be some non-empty set and let $w$ be a word over the alphabet $S \cup S^{-1} \cup \{e\}$, where $S^{-1} = \{s^{-1} : s \in S\}$ is a disjoint copy of $S$ interpreted as a set of inverses of elements from $S$ and $e$ is an element not belonging to either $S$ or $S^{-1}$ which shall be interpreted as a group unit. We say that $w = w_1 \ldots w_n$ is irreducible if either $n = 1$ and $w = w_1 = e$, or $n > 1$ and for every $i \leq n$, $w_i \neq e$ and there is no $i < n$ such that $w_i = w_{i+1}^{-1}$. If $w$ is a word that is not irreducible, then by $w'$ we shall denote the unique irreducible word obtained from $w$ by deleting each occurrence of the letter $e$ and each occurrence of neighbouring letters $a$ and $a^{-1}$ (if this procedure leads to an empty word, then we set $w'$ to be $e$).

It is well-known and easy to observe that elements of $F(S)$, the free group of free generators coming from the set $S$ with $e$ as a unit, are in one-to-one correspondence with irreducible words over the alphabet $S \cup S^{-1} \cup \{e\}$. 
Let $n \in \mathbb{N}$. By $W_n(S)$ we shall denote the set of all irreducible words of length at most $n$ over the alphabet $S \cup S^{-1} \cup \{e\}$.

Let now similarly $B$ be some non-empty set not containing the distinguished element $e$. The vector space over some field $F$ with $B$ as the maximal linearly independent set and $e$ representing zero can be viewed as a set of all functions from $B$ to $K$ that have finite support, where $f$ has finite support if for all but finitely many $b \in B$ we have $f(b) = 0$.

In our case, we shall work with $F = \mathbb{R}$ (resp. $\mathbb{Q}$), however since we shall need to work with only finitely many vectors at any given time we restrict to functions whose range is some specified finite subset of $\mathbb{R}$ (also, the set $B$ will be always finite).

Let $K \subseteq \mathbb{R}$ some finite subset of reals. Then by $V_K(B)$ we shall denote the set of all functions from $B$ to $K$ with finite support. The requirement on finite support will be in our construction superfluous since $B$ will be at any given time finite as already mentioned.

2. The proof of Theorem 0.1

We start by describing the underlying countable set $X$ mentioned above in the preliminaries.

2.1. The underlying dense set. We now describe a countably infinite set $X$, constructed as an increasing union of finite sets $X_0 \subseteq X_1 \subseteq \ldots$, which will also carry a multiplicative group operation and the corresponding group inverse operation so that $X$ will be isomorphic to a free group of countably many generators, and it will also carry an additive (abelian) group operation together with multiplication by scalars that are dyadic rationals so that it is a proper subgroup of an infinite dimensional vector space over the rationals. Moreover, the unit for addition and multiplication will be the same.

Later, we shall define a metric on $X$ that will be invariant under both addition and multiplication and will preserve scalar multiplication. The completion then will be a Banach space over the reals which is also equipped with multiplication with free group as a dense part.

Let $n \geq 1$ be arbitrary. By $D_n$ we shall denote the set of dyadic rationals $\frac{a}{2^n}$, where $a \in [-2^{2n}, 2^{2n}]$. Clearly, $D = \bigcup_n D_n$ is the set of all dyadic rational numbers.

We set $X_0 = \{e\}$. Let $S_1 = \{x\}$ be some singleton. We set $X_1 = W_1(S_1) = \{e, x, x^{-1}\}$. Let $B_2 = X_1 \setminus X_0 = \{x, x^{-1}\}$. We set
Suppose now that $X_{2n}$ has been constructed. We need to construct $X_{2n+1}$ and $X_{2n+2}$. Set $S_{2n+1}$ to be $S_{2n-1} \cup (X_{2n} \setminus X_{2n-1})$. Then we set $X_{2n+1}$ to be $W_{2n+1}(S_{2n+1})$.

Next we set $B_{2n+2}$ to be $B_{2n} \cup (X_{2n+1} \setminus X_{2n})$. Then we set $X_{2n+2}$ to be $V_{D_{n+1}}(B_{2n+2})$.

This finishes the inductive construction. Note that $X = \bigcup_n X_n = \bigcup_n W_{2n+1}(S_{2n+1}) = \bigcup_n V_{D_{n+1}}(B_{2n+2})$. It follows that if we set $S = \bigcup_n S_{2n+1}$ and $B = \bigcup_n B_{2n+2}$, then $X$ is also naturally isomorphic to $F(S)$, the free group of countably many generators coming from the set $S$, and it is a (additive) proper subgroup of the rational (or real) vector space with $B$ as the maximal linearly independent set - the minimal subgroup that contains free abelian group with $B$ as a set of generators that is closed under multiplication by scalars from $D$.

We define inductively a rank function $r : X \to \omega$. For any $x \in X$ we set $r(x) = 0$ if neither $x$ nor $x^{-1}$ is possible to write as $\alpha_1 y_1 + \ldots + \alpha_m y_m$, where $\alpha_i > 0$, for every $i$, and $\alpha_1 + \ldots + \alpha_m = 1$.

If $x$ or $x^{-1}$ is possible to write as $\alpha_1 y_1 + \ldots + \alpha_m y_m$, where $\alpha_i > 0$, for every $i$, and $\alpha_1 + \ldots + \alpha_m = 1$, then we set $r(x) = \max\{r(y_i) : i \leq m\} + 1$.

2.2. Construction of the metric. We shall now define a metric $\rho$ and a norm $\| \cdot \|$ on $X$. Actually, the metric and the norm will be one and the same in the sense that for any $x, y \in X$ we shall have $\rho(x, y) = \| x - y \|$ and $\| x \| = \rho(x, e)$. The distinguishing is done only for practical notational reasons since we understand $X$ as both a free group and a subgroup of a vector space. In the former case, it is more natural to consider a metric there, while in the latter to consider a norm there. By induction, we shall define functions $\rho_n : X_n^2 \to \mathbb{R}$ (the range will actually be a subset of non-negative rationals) for odd $n$ and functions $\| \cdot \|_n : X_n \to \mathbb{R}$ for even $n$ that satisfy the following properties:

1. for every odd $n$ we have that $\rho_n$ is a symmetric function that is equal to zero only on diagonal, i.e. $\rho_n(x, y) = 0$ iff $x = y$ for $x, y \in X_n$; similarly, for every even $n$ we have that $\| x \|_n = 0$ iff $x = e$ for $x \in X_n$;

2. for every even $n$, $\| \cdot \|_n$ extends $\rho_{n-1}$, i.e. for every $a, b \in X_{n-1}$ we have $\| a - b \|_n = \rho_{n-1}(a, b)$,
We define $\rho_n$ for every $\alpha, \beta \in 2$. Obviously, this satisfies all the requirements.

Let us now define $\| \cdot \|_n$ for every odd $n$ over the alphabet $S_n \cup S_n^{-1} \cup \{e\}$ such that $w_m, v_1, v_2$ and $v'_1, v'_2$ are elements of $X_n$, we have

$$\rho_n(w_m, v_1) = \rho_n(w_m, v_2) \leq \rho_n(w_m, v'_1) + \rho_n(w_m, v'_2),$$

and for every $a, b \in X_n$ we have

$$\rho_n(a, b) = \rho_n(a^{-1}, b^{-1}),$$

(4) for every odd $n$ and for every $a \in X_n$ and for every $b \in X_n$ such that $b = \alpha_1 c_1 + \cdots + \alpha_m c_m$, where $\alpha_i \geq 0$ and $c_i \in B_{n-1}$, for every $i$, and $\alpha_1 + \cdots + \alpha_m = 1$, we have

$$\rho_n(a, b) \leq \alpha_1 \cdot \rho_n(a, c_1) + \cdots + \alpha_m \cdot \rho_n(a, c_m),$$

(5) for every even $n$ and for every $a \in X_n$ and any scalar $\alpha$ such that $\alpha a \in X_n$ we have

$$\|\alpha a\|_n = |\alpha| \cdot \|a\|_n,$$

and for every $a, b \in X_n$ such that also $a + b \in X_n$ we have

$$\|a + b\|_n \leq \|a\|_n + \|b\|_n,$$

(6) for every odd $n$, for every $a \in X_n$ and for every $b \in X_n$ such that $b = (\alpha_1 c_1 + \cdots + \alpha_m c_m)^{-1}$, where $\alpha_i \geq 0$ and $c_i \in B_{n-1}$, for every $i$, and $\alpha_1 + \cdots + \alpha_m = 1$, we have

$$\rho_n(a, b) \leq \alpha_1 \cdot \rho_n(a, c_1^{-1}) + \cdots + \alpha_m \cdot \rho_n(a, c_m^{-1}),$$

and similarly, for every even $n$, for every $a \in X_n$ and for every $b \in X_{n-1}$ such that $b = (\alpha_1 c_1 + \cdots + \alpha_m c_m)^{-1}$, where $\alpha_i \geq 0$ and $c_i \in B_{n-2}$, for every $i$, and $\alpha_1 + \cdots + \alpha_m = 1$, we have

$$\|a - b\|_n \leq \alpha_1 \cdot \|a - c_1^{-1}\|_n + \cdots + \alpha_m \cdot \|a - c_m^{-1}\|_n.$$

We define $\rho_1$ on $X_1$ by setting $\rho_1(x, e) = \rho_1(x^{-1}, e) = 1$ and $\rho_1(x, x^{-1}) = 2$. Obviously, this satisfies all the requirements.

Let us now define $\| \cdot \|_2$ on $X_2$, i.e. we have to define $\|\alpha x + \beta x^{-1}\|_2$ for every $\alpha, \beta \in D_1$. We set

$$\|\alpha x + \beta x^{-1}\|_2 = \min\{\gamma_1 \cdot \rho_1(x, e) + \gamma_2 \cdot \rho_1(x^{-1}, e) + \gamma_3 \cdot \rho_1(x, x^{-1}) : \alpha x + \beta x^{-1} = \gamma_1 x + \gamma_2 x^{-1} + \gamma_3 (x - x^{-1}), \gamma_1, \gamma_2, \gamma_3 \in \mathbb{R}\}.$$
Condition (6) is automatically satisfied as there is no $b \in X_1$ of the form $(\alpha_1 c_1 + \ldots + \alpha_m c_m)^{-1}$ for appropriate $\alpha$’s and $c$’s. Condition (1) is also obvious. Thus we need to check the conditions (2) and (5).

Let us do the former. We need to check that $\| \cdot \|_2$ extends $\rho_1$. We shall check that $\|x\|_2 = \rho_1(x,e)$. The other cases are similar. Clearly, $\|x\|_2 \leq \rho_1(x,e)$. Suppose that $\|x\|_2 = |\gamma_1| \cdot \rho_1(x,e) + |\gamma_2| \cdot \rho_1(x^{-1}, e) + |\gamma_3| \cdot \rho_1(x, x^{-1})$, where $x = \gamma_1 x + \gamma_2 x^{-1} + \gamma_3 (x - x^{-1})$. It follows that necessarily $\gamma_2 = \gamma_3$ and $\gamma_1 + \gamma_3 = 1$. By triangle inequality we have $|\gamma_2| \cdot \rho_1(x^{-1}, e) + |\gamma_2| = |\gamma_3| \cdot \rho_1(x, x^{-1}) \geq |\gamma_2| \cdot \rho_1(x,e)$, thus $|\gamma_1| \cdot \rho_1(x,e) + |\gamma_2| \cdot \rho_1(x^{-1}, e) + |\gamma_3| \cdot \rho_1(x, x^{-1}) \geq |\gamma_1| \cdot \rho_1(x,e) + |\gamma_2| \cdot \rho_1(x,e) \geq |\gamma_1 + \gamma_2| \cdot \rho_1(x,e) = \rho_1(x,e).

We now check (5). Fix some $a \in X_2$ and $\alpha$ such that $a a \in X_2$. Suppose that $\|a\| = |\gamma_1| \cdot \rho_1(x,e) + |\gamma_2| \cdot \rho_1(x^{-1}, e) + |\gamma_3| \cdot \rho_1(x, x^{-1})$, where we have $a = \gamma_1 x + \gamma_2 x^{-1} + \gamma_3 (x - x^{-1})$. Then since $\alpha a = \alpha \cdot \gamma_1 x + \alpha \cdot \gamma_2 x^{-1} + \alpha \cdot \gamma_3 (x - x^{-1})$ we have that $\|a a\|_2 \leq |\alpha \gamma_1| \cdot \rho_1(x,e) + |\alpha \gamma_2| \cdot \rho_1(x^{-1}, e) + |\alpha \gamma_3| \cdot \rho_1(x, x^{-1}) = |\alpha| \cdot \|a\|_2$. The other inequality is analogous. By a similar argument one can also show that for any $a, b \in X_2$ such that also $a + b \in X_2$ we have $\|a + b\|_2 \leq \|a\|_2 + \|b\|_2$ and we leave this to the reader. In fact, we note here that this definition of $\| \cdot \|_2$ is equivalent with that one that says that $\| \cdot \|_2$ is the greatest function that satisfies condition (4) and such that $\|x\|_2 \leq \rho_1(x,e)$, $\|x^{-1}\|_2 \leq \rho_1(x^{-1}, e)$ and $\|x - x^{-1}\|_2 \leq \rho_1(x, x^{-1})$.

Extending the metric.

Suppose we have defined $\| \cdot \|_n$ for some even $n \geq 2$. We now define $\rho_{n+1}$ on $X_{n+1}$. First we inductively define an auxiliary function $\delta$ on $X_{n+1}^2$. Fix a pair $x, y \in X_{n+1}$. If $r(x) = r(y) = 0$, then we set

$$\delta(x, y) = \min\{\|a_1 - b_1\|_n + \ldots + \|a_m - b_m\|_n : x = (a_1 \ldots a_m)', y = (b_1 \ldots b_m)', \forall i \leq m(a_i, b_i, a_i - b_i \in X_n)\}.$$  

Note that the minimum is indeed attained as $X_n$ is finite. Note again that for any $z \in X_{n+1}$ if $r(z) > 0$ we have that either $z$ or $z^{-1}$ belongs to $X_n$. So if $r(x) > 0, r(y) > 0$, then we set

$$\delta(x, y) = \begin{cases} 
\|x - y\|_n & \text{if } x, y \in X_n \\
\|x^{-1} - y^{-1}\|_n & \text{if } x^{-1}, y^{-1} \in X_n \\
\min\{\|x - z\| + \|z^{-1} - y\| : z, z^{-1} \in X_n\} & \text{if } x, y^{-1} \in X_n \\
\min\{\|x^{-1} - z^{-1}\| + \|z - y\| : z, z^{-1} \in X_n\} & \text{if } x^{-1}, y \in X_n.
\end{cases}$$

Now we suppose that for one of the elements, say $x$, we have $r(x) = 0$, and for the other one we have $r(y) > 0$. The following is done by
induction on \( r(y) \). First suppose that \( y = \alpha_1 z_1 + \ldots + \alpha_m z_m \), where \( \alpha_i > 0 \), for all \( i \), and \( \alpha_1 + \ldots + \alpha_m = 1 \). Then we set

\[ \delta(x, y) = \alpha_1 \cdot \delta(x, z_1) + \ldots + \alpha_m \cdot \delta(x, z_m). \]

Similarly, if \( y = (\alpha_1 z_1 + \ldots + \alpha_m z_m)^{-1} \), where \( \alpha_i > 0 \), for all \( i \), and \( \alpha_1 + \ldots + \alpha_m = 1 \), then we set

\[ \delta(x, y) = \alpha_1 \cdot \delta(x, z_1^{-1}) + \ldots + \alpha_m \cdot \delta(x, z_m^{-1}). \]

We are now ready to define \( \rho_{n+1} \). Thus fix now again a pair \( x, y \in X_{n+1} \) and set

\[ \rho_{n+1}(x, y) = \min \{ \delta(a_1, b_1) + \ldots + \delta(a_m, b_m) : x = (a_1 \ldots a_m)', y = (b_1 \ldots b_m)', a_1, \ldots, a_m, b_1, \ldots, b_m \in X_{n+1} \}. \]

First notice that since \( X_{n+1} \) is finite the minimum is attained, thus in particular we have for \( x \neq y \) that \( \rho_{n+1}(x, y) > 0 \). Since \( \rho_{n+1} \) is clearly symmetric we get it satisfies the condition (1).

We claim that \( \rho_{n+1} \) is the greatest function satisfying:

(a) \( \rho_{n+1}(x, y) \leq \|x - y\|_n \) for every \( x, y \in X_n \) such that \( x - y \in X_n \),
(b) \( \rho_{n+1}(x, y) = \rho_{n+1}(x^{-1}, y^{-1}) \) for every \( x, y \in X_{n+1} \),
(c) \( \rho_{n+1}(ab, cd) \leq \rho_{n+1}(a, c) + \rho_{n+1}(b, d) \) for every \( a, b, c, d \in X_{n+1} \) such that \( ab, cd \in X_{n+1} \),
(d) \( \rho_{n+1}(x, (\alpha_1 z_1 + \ldots + \alpha_m z_m)) \leq \alpha_1 \cdot \rho_{n+1}(x, z_1) + \ldots + \alpha_m \cdot \rho_{n+1}(x, z_m), \)
where \( \alpha_i > 0 \), for all \( i \), \( \alpha_1 + \ldots + \alpha_m = 1 \) and \( \alpha_1 z_1 + \ldots + \alpha_m z_m \in X_{n+1} \),
(e) \( \rho_{n+1}(x, (\alpha_1 z_1 + \ldots + \alpha_m z_m)^{-1}) \leq \alpha_1 \cdot \rho_{n+1}(x, z_1^{-1}) + \ldots + \alpha_m \cdot \rho_{n+1}(x, z_m^{-1}), \)
where \( \alpha_i > 0 \), for all \( i \), \( \alpha_1 + \ldots + \alpha_m = 1 \) and \( (\alpha_1 z_1 + \ldots + \alpha_m z_m)^{-1} \in X_{n+1} \).

First of all, it is clear from the definitions of \( \rho_{n+1} \) (and of \( \delta \)) that \( \rho_{n+1} \) satisfies all these conditions. Thus in particular, we get that \( \rho_{n+1} \) satisfies the conditions (3), (4) and (6). Next, if \( \xi \) is any other function satisfying conditions (a)-(e), then it is readily checked that \( \xi \leq \delta \) and because of (c) also \( \xi \leq \rho_{n+1} \).

We shall now conclude from that that \( \rho_{n+1} \) also satisfies (2). Indeed, let \( X'_n \subseteq X_n \) be such that for every \( x, y \in X'_n \) we have \( x - y \in X_n \). Then consider the metric \( \xi \) on \( X'_n \) defined as \( \xi(x, y) = \|x - y\|_n \). We claim it satisfies the conditions (a)-(e) above. Condition (a) is satisfied since \( \xi(x, y) = \|x - y\|_n \) for appropriate \( x, y \). Take some \( x, y \in X'_n \) such that \( x^{-1}, y^{-1} \in X'_n \). Necessarily \( x, y, x^{-1}, y^{-1} \in X_{n-1} \) and since \( \| \cdot \|_n \) extends \( \rho_{n-1} \) we get \( \xi(x, y) = \rho_{n-1}(x, y) = \rho_{n-1}(x^{-1}, y^{-1}) = \xi(x^{-1}, y^{-1}) \). Take now some \( a, b, c, d \in X'_n \) such that \( ab, cd \in X'_n \). We again necessarily
have that $a, b, c, d, ab, cd \in X_{n-1}$ and since $\| \cdot \|_n$ extends $\rho_{n-1}$ we again obtain $\xi(ab, cd) = \rho_{n-1}(ab, cd) \leq \rho_{n-1}(a, c) + \rho_{n-1}(b, d) = \xi(a, c) + \xi(b, d)$. We have verified conditions (b) and (c). Condition (d) follows since $\| \cdot \|_n$ satisfies the condition (5) further above. Finally, take some $x, (\alpha_1 z_1 + \ldots + \alpha_m z_m)^{-1} \in X'_n$, where $\alpha_i > 0$, for all $i$, $\alpha_1 + \ldots + \alpha_m = 1$. We have $\xi(x, (\alpha_1 z_1 + \ldots + \alpha_m z_m)^{-1}) = \| x - (\alpha_1 z_1 + \ldots + \alpha_m z_m)^{-1}\|_n \leq \alpha_1 \cdot \| x - z_1^{-1}\|_n + \ldots + \alpha_m \cdot \| x - z_m\|_n = \alpha_1 \cdot \xi(x, z_1) + \ldots + \alpha_m \cdot \xi(x, z_m)$, where the middle inequality follows from the property (6) above.

We thus get $\| x - y\|_n = \xi(x, y) \leq \rho_{n+1}(x, y)$ for every $x, y \in X_n$ such that $x - y \in X_n$. Since by assumption $\rho_{n+1}(x, y) \leq \| x - y\|_n$ we get $\rho_{n+1}(x, y) = \| x - y\|_n$ and we are done.

**Extending the norm.**

Now suppose we have defined $\rho_n$ on $X_n$ for some odd $n > 2$. We define $\| \cdot \|_{n+1}$ on $X_{n+1}$. We again at first define, inductively, an auxiliary function $\gamma : X_{n+1} \to \mathbb{R}$. First, for every $x, y \in X_n$ we set $\gamma(x - y) = \rho_n(x, y)$.

Next, for every $x, y \in X_{n+1}$ such that $x - y \in X_{n+1}$ and $x \in X_{n+1} \setminus X_n$ we define $\gamma(x - y)$ by induction on $r(y)$. If $r(y) = 0$ then we set

$$\gamma(x - y) = \min\{ |\beta_i| \cdot \gamma(v_i) + \ldots + |\beta_i| \cdot \gamma(v_i) : x - y = \beta_i v_1 + \ldots + \beta_i v_i, \forall j \leq i \exists a_j, b_j \in X_n(v_j = a_j - b_j)\}.$$ 

If $r(y) > 0$ and $y = (\alpha_1 z_1 + \ldots + \alpha_m z_m)^{-1}$, where $\alpha_i > 0$, for every $i$, and $\alpha_1 + \ldots + \alpha_m = 1$, then we set $\gamma(x - y) = \alpha_1 \cdot \gamma(x - z_1^{-1}) + \ldots + \alpha_m \cdot \gamma(x - z_m^{-1})$.

Finally, for any $x \in X_{n+1}$ we set $\| x\|_{n+1} = \min\{ \gamma(y_1) + \ldots + \gamma(y_i) : x = y_1 + \ldots + y_i, y_1, \ldots, y_i \in X_{n+1}\}$.

First thing to observe is again that for any $x \in X_{n+1}$ we have $\| x\|_{n+1} = 0$ iff $x = 0$. It follows that condition (1) is satisfied for $\| \cdot \|_{n+1}$.

Next we claim that $\| \cdot \|_{n+1}$ is the greatest function satisfying:

(a) $\| x - y\|_{n+1} \leq \rho_n(x, y)$ for every $x, y \in X_n$,
(b) $\| \alpha x + \beta y\|_{n+1} \leq |\alpha| \cdot \| x\|_{n+1} + |\beta| \cdot \| y\|_{n+1}$ for $x, y \in X_{n+1}$ such that also $\alpha x + \beta y \in X_{n+1}$,
(c) $\| x - (\alpha_1 z_1 + \ldots + \alpha_m z_m)^{-1}\|_{n+1} \leq \alpha_1 \cdot \| x - z_1^{-1}\|_{n+1} + \ldots + \alpha_m \cdot \| x - z_m^{-1}\|_{n+1}$, where $\alpha_i > 0$, for every $i$, and $\alpha_1 + \ldots + \alpha_m = 1$.

First, it follows from the definitions of $\gamma$ and $\| \cdot \|_{n+1}$ that $\| \cdot \|_{n+1}$ satisfies all these conditions, thus we have that it satisfies the conditions (5) and (6) further above. If $\xi$ is any other functions satisfying
conditions (a)-(c) then it again follows that necessarily $\xi \leq \gamma$ and thus also $\xi \leq \|\cdot\|_{n+1}$.

We are ready to verify the remaining condition (2) for $\|\cdot\|_{n+1}$ that it extends $\rho_n$. Let $X'_{n+1} \subseteq X_{n+1}$ be the set $\{x - y : x, y \in X_n\}$. Define $\xi$ on $X'_{n+1}$ as follows: $\xi(x - y) = \rho_n(x, y)$ for every $x - y \in X'_{n+1}$. Then $\xi$ satisfies (a) since $\xi(x - y) = \rho_n(x, y)$ for appropriate $x, y$. Next we check (b). Take some $\alpha x + \beta y$. If $\alpha = 1$ and $\beta = -1$ (or vice versa), then we have $\xi(x - y) = \rho_n(x, y) \leq \rho_n(x, e) + \rho_n(e, y) = \xi(x) + \xi(y)$, where the middle inequality follows from the condition (3) that $\rho_n$ satisfies (which implies the triangle inequality). Finally, for $(\alpha_1 z_1 + \ldots + \alpha_m z_m)^{-1}$, where $\alpha_i > 0$, for every $i$, and $\alpha_1 + \ldots + \alpha_m = 1$; we have $\xi(x - (\alpha_1 z_1 + \ldots + \alpha_m z_m)^{-1}) = \rho_n(x, (\alpha_1 z_1 + \ldots + \alpha_m z_m)^{-1}) \leq \alpha_1 \cdot \rho_n(x, z_1^{-1}) + \ldots + \alpha_m \cdot \rho_n(x, z_m^{-1}) = \alpha_1 \cdot \xi(x - z_1^{-1}) + \ldots + \alpha_m \cdot \xi(x - z_m^{-1})$, since $\rho_n$ satisfies the condition (6), and we are done.

Now we can define $\|\cdot\|$ on $X$ by putting

$$\|\cdot\| = \bigcup_i \|\cdot\|_i$$

and analogously we can define $\rho$ on $X$ by putting

$$\rho = \bigcup_i \rho_i.$$

By the condition (2) we have that for any $x, y \in X$ we have $\|x - y\| = \rho(x, y)$. By (1) we have that for $x \neq y \in X$ we have $\rho(x, y) > 0$ and equivalently, for any $x \neq e \in X$ we have $\|x\| > 0$.

Let us check that $\rho$ is a bi-invariant metric on $X$ when considered as a (free) group. By (1) is symmetric. We use the following simple fact.

**Fact 2.1.** Let $G$ be a group equipped with a symmetric function $d : G^2 \to \mathbb{R}_0^+$ that is equal to 0 only on diagonal. Then $d$ is a bi-invariant metric if and only if for every $x, y, v, w \in G$ we have $d(x \cdot y, v \cdot w) \leq d(x, v) + d(y, w)$.

**Proof.** If $d$ is a bi-invariant metric then the inequality readily follows from bi-invariance and using triangle inequality.

So suppose that $d$ satisfies such an inequality for every $x, y, v, w \in G$. Fix $a, b, c \in G$. Then

$$d(a, c) = d(a \cdot b^{-1} \cdot b, b \cdot b^{-1} \cdot c) \leq d(a, b) + d(b^{-1}, b^{-1}) + d(b, c) = d(a, b) + d(b, c),$$

so $d$ is a metric. Now $d(a \cdot b, a \cdot c) \leq d(a, a) + d(b, c) = d(b, c) = d(a^{-1} \cdot a, a^{-1} \cdot a \cdot c) \leq d(a^{-1}, a^{-1}) + d(a, a \cdot c) = d(a, a, c)$ which shows the left-invariance. The right invariance is done analogously. $\square$
However, $\rho$ does satisfy the condition from the statement of the fact since it satisfies the condition (3).

Similarly, for every $x, y$ and $\alpha, \beta \in D$ (recall that $D$ denotes the dyadic rationals) we have $\|\alpha x + \beta y\| \leq |\alpha| \cdot \|x\| + |\beta| \cdot \|y\|$ since $\| \cdot \|$ satisfies the condition (5).

Denote now by $\mathbb{X}$ the completion of $X$ with respect to $\rho$, or equivalently, with respect to $\| \cdot \|$. Both the multiplicative and additive group operations extend to the completion as well as the scalar multiplication by dyadic rationals. Moreover, since the dyadic rationals are dense in $\mathbb{R}$, $\mathbb{X}$ has well-defined scalar multiplication by all the reals. Thus $\mathbb{X}$ is a Banach space.

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