Abstract. We use a variation on the commutator collection process to characterize those pure braids which become trivial when any one strand is deleted, or, more generally, those pure braids which become trivial when all the strands in any one of a list of sets of strands is deleted.

0. INTRODUCTION

A Brunnian link is a tame link of \( n \) closed curves in \( \mathbb{R}^3 \) such that deleting any one of the curves results in a trivial link of \( n - 1 \) components. (See Rolfsen [11].) By analogy, we shall call a braid of \( n \) strands a **Brunnian braid** if deleting any one of the strands produces a trivial braid of \( n - 1 \) strands. A Brunnian braid on more than two strands must clearly be a pure braid, so we confine our attention to the pure braid group \( P_n \subset B_n \). Though the idea of “deleting a strand” is topological, we take here a purely algebraic approach.

Brunnian braids were considered by Levinson [7, 8] under the name “decomposable braids”. More generally, he considers “\( k \)-decomposable braids”, which become trivial when any \( k \) strands are deleted. In [7] he gives a geometric characterization of such braids, and in [8] he gives algebraic characterizations in the cases \((n, k) = (3, 1), (4, 1), (4, 2)\). We shall generalize these results below.

It is not hard to see that the set of \( n \)-strand Brunnian braids is a free normal subgroup of \( P_n \). In the Kourovka notebook [6], this was called the subgroup of “smooth” braids, and the problem was posed to give a set of free generators for smooth braids with a given number of strands. Johnson [5] used the Hall commutator collection process [4] to give a set of generators modulo any group of the lower central series. There has appeared in a conference proceedings [3] an abstract of a solution to the Kourovka problem, after which the problem was taken from the notebook. I have been unable to locate, however, a paper which follows through on the abstract.

In this paper, we give sets of generators for the Brunnian subgroups, the \( k \)-decomposable subgroups, and for a somewhat more general class of subgroups. Our generating sets are not minimal, so we do not address the Kourovka question. Our method is a finite variation on the Hall commutator collection process. A similar method was used in [10], in order to show that an element of an arbitrary group \( G \) is “\( n \)-trivial” if and only if it is in the \( n \)th group of the lower central series of \( G \).
We give our characterization of the Brunnian subgroup of $P_n$ (Corollary 2.3) in terms of monic commutators (Definition 1.3). We set $[x, y] = x^{-1}y^{-1}xy$ for any group elements $x, y$. Here are some monic commutators: $[p_{1,3}, p_{2,4}] \in P_4$, $[[p_{1,2}, p_{1,3}], p_{1,4}] \in P_4$, and $[[[p_{1,2}, p_{2,3}], p_{3,4}], [p_{4,5}, p_{5,6}]] \in P_6$. It is not hard to see that these particular examples are all Brunnian braids, since deleting any one strand trivializes at least one entry in the iterated commutator, which of course trivializes the whole commutator. We define (Definition 1.4 and Proposition 1.6) the support of a monic commutator to be the strands whose indices appear somewhere in the commutator. Thus the support of $[p_{1,2}, p_{1,4}] \in P_4$ is \{1, 2, 4\}. Deleting strands 1, 2, or 4 from this commutator trivializes it, but deleting strand 3 does nothing. We characterize the Brunnian subgroup of $P_n$ as being generated by all monic commutators whose support is the whole set of strands \{1, 2, 3, …, n\}. As noted below, the Brunnian subgroup is not finitely-generated (for $n > 2$), so its list of generators necessarily includes commutators of with an arbitrary number of brackets.

More generally, consider any finite collection of subsets $S_1, S_2, \ldots, S_m \subset \{1, 2, \ldots, n\}$. Our main theorem states that the subgroup of braids which become trivial when the strands in any one $S_i$ are cut is generated by the set of monic commutators whose support intersects each $S_i$ nontrivially. When $S_i = \{i\}$ we get the characterization of Brunnian braids, and when the $S_i$ consist of all subsets with $k$ elements, we get a characterization of $k$-decomposable braids.

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1. BASIC IDEAS

A standard reference on braids is Birman [2]. Recall that the pure braid group $P_n$ is generated by $p_{a,b}$ for $1 \leq a < b \leq n$, where $p_{a,b}$ is the braid which links strand $a$ and strand $b$ in front of the other strands. Artin’s [1] semidirect product decomposition may be used to find a finite presentation for $P_n$. Here is one version:

A. $p_{a,b}p_{a,c}p_{b,c} = p_{a,c}p_{b,c}p_{a,b} = p_{b,c}p_{a,b}p_{a,c}$ for all $1 \leq a < b < c \leq n$

B. $p_{a,b}p_{c,d} = p_{c,d}p_{a,b}$ and $p_{a,d}p_{b,c} = p_{b,c}p_{a,d}$ for all $1 \leq a < b < c < d \leq n$

C. $p_{a,c}p_{b,c}^{-1}p_{b,d}p_{c,d} = p_{b,c}^{-1}p_{b,d}p_{c,d}p_{a,c}$ for all $1 \leq a < b < c < d \leq n.$

Each of the above relations corresponds to a “geometrically obvious” commutation relation. For the (B) relations this is immediate, and examples for the (A) and (C) relations are shown in the figure below.
Fix the positive integer \( n \), and let \( N = \{1, 2, \ldots, n\} \). Suppose \( S \subset n \). Denote by \( \overline{S} \) the complement of \( S \) in \( N \). Let \( P_S \) be subgroup of \( P_N = P_n \) generated by \( p_{a,b} \) with \( a \in S \) and \( b \in S \), and let \( Q_S \) be the subgroup generated by \( p_{a,b} \) such that either \( a \in S \) or \( b \in S \). The standard semidirect product decomposition states that there is a retraction homomorphism from \( P_N \) to \( P_{\overline{S}} \) whose image is \( P_N^{-1} \) and whose kernel is \( Q_{\{n\}} \). This homomorphism is accomplished geometrically by cutting or trivializing the \( n \)th string of a pure braid. If instead we cut all the strings in \( S \subset N \), then we still get a retraction map and a semidirect product decomposition.

**Proposition 1.1.** For each \( S \subset N \), there exists a retraction homomorphism \( \phi_S : P_n \to P_n \) whose image is \( P_{\overline{S}} \) and whose kernel is \( Q_S \). Moreover, \( \phi_{S_1} \circ \phi_{S_2} = \phi_{S_1 \cup S_2} \) for any \( S_1, S_2 \subset N \).

**Proof:** Define \( \phi_S(p_{a,b}) = p_{a,b} \) if \( a \notin S \) and \( b \notin S \), and \( \phi_S(p_{a,b}) = 1 \) otherwise. Then check using the presentation above that this defines a homomorphism with the required properties. \( \square \)

Now, given a set of subsets \( S_1, S_2, S_3, \ldots, S_m \subset n \), we are interested in characterizing \( \bigcap_{i=1}^{m} Q_{S_i} \). First note it follows from Proposition 1.1 that \( P_{S_1} \cap P_{S_2} = P_{S_1 \cap S_2} \). Inductively, we have

**Proposition 1.2.** Let \( S_1, S_2, \ldots, S_m \subset N \). Then \( \bigcap_{i=1}^{m} P_{S_i} = P_{\bigcap_{i=1}^{m} S_i} \).

By way of contrast, the subgroup \( Q = \bigcap_{i=1}^{m} Q_{S_i} \) is not in general finitely-generated. For example, consider \( Q_{\{2\}} \) and \( Q_{\{3\}} \) in \( P_3 \). Both of these subgroups are free on two generators, so \( Q_{\{2\}} \cap Q_{\{3\}} \) is also free. The retraction \( \phi_{\{3\}} : P_3 \to P_3 \) restricts to a retraction \( Q_{\{2\}} \to Q_{\{2\}} \), whose kernel is \( Q_{\{2\}} \cap Q_{\{3\}} \) and whose image is the infinite cyclic group \( \langle p_{1,2} \rangle \). Thus \( Q_{\{2\}} \cap Q_{\{3\}} \) has infinite index in \( Q_{\{2\}} \), and therefore \( Q_{\{2\}} \cap Q_{\{3\}} \) is not finitely-generated.

We shall show that \( \bigcap_{i=1}^{m} Q_{S_i} \) is generated by a subset of the monic commutators of \( P_n \).
**Definition 1.3.** A monic commutator is an element of $P_n$ defined recursively as follows

A. $p_{a,b}$ and $p_{a,b}^{-1}$ are monic commutators for all $1 \leq a < b \leq n$.

B. If $x$ and $y$ are monic commutators, and $[x, y] \neq 1$, then $[x, y]$ is a monic commutator.

**Definition 1.4.** If $x \in P_n$, then the support $\sigma(x)$ is the intersection of all $S \subset n$ such that $x \in P_S$.

We have $\sigma(p_{a,b}^{\pm 1}) = \{a, b\}$. Two other things are also immediate. First, $\sigma([x, y]) \subset \sigma(x) \cup \sigma(y)$. Second, if $\sigma(x) \cap S = \emptyset$, then $\phi_S(x) = x$. For monic commutators, we can say more:

**Proposition 1.5.** If $x$ is a monic commutator in $P_n$, then $\phi_S(x) = 1$ if and only if $\sigma(x) \cap S \neq \emptyset$.

Proof: If $\sigma(x) \cap S = \emptyset$, then $\phi_S(x) = x \neq 1$ by definition. For the converse, it suffices by Proposition 1.1 to show that for a monic commutator, $i \in \sigma(x)$ implies that $\phi_{\{i\}}(x) = 1$. This is certainly true when $x = p_{a,b}^{\pm 1}$. Suppose that it is true for two monic commutators $x$ and $y$. If $i \in \sigma([x, y])$, then $i \in \sigma(x)$ or $i \in \sigma(y)$. Then $\phi_{\{i\}}(x) = 1$ or $\phi_{\{i\}}(y) = 1$, and in either case $\phi_{\{i\}}([x, y]) = 1$. □

**Proposition 1.6.** If $x, y$, and $[x, y]$ are monic commutators, then $\sigma([x, y]) = \sigma(x) \cup \sigma(y)$.

Proof: If $i \notin \sigma([x, y])$ then $\phi_{\{i\}}([x, y]) \neq 1$, and therefore $i \notin \sigma(x) \cup \sigma(y)$. □

2. THE THEOREM

**Theorem 2.1.** For $1 \leq i \leq m$, let $S_i \subset N$. Then $\cap_{i=1}^m Q_{S_i}$ is generated by the set of monic commutators $x$ such that $\sigma(x) \cap S_i \neq \emptyset$ for all $1 \leq i \leq m$.

We obtain as corollaries characterizations of $k$-decomposable braids and of Brunnian braids, generalizing results of Levinson [8].

**Corollary 2.2.** The normal subgroup of $k$-decomposable $n$-strand braids is generated in $P_n$ by all monic commutators whose support has cardinality at least $n-k+1$.

**Corollary 2.3.** The normal subgroup of all Brunnian braids in $P_n$ is generated by all monic commutators whose support is $N = \{1, 2, \ldots, n\}$.

**Corollary 2.4.** The subgroup of $(n-2)$-decomposable braids is the commutator subgroup of $P_n$.

Corollary 2.4 may also be proved directly by observing that if $|S| = n - 2$ then the image of $\phi_S$ is a two-strand pure braid group, isomorphic to $\mathbb{Z}$. More specifically, if $S = N - \{i, j\}$, then $\phi_S(x)$ can be taken to be the integer which measures the linking number of strands $i$ and $j$ in the braid $x$. Then $x$ is in the commutator subgroup of $P_n$ if and only if all these linking numbers vanish.

**Proof of Theorem 2.1:** If $x$ is a monic commutator which intersects each $S_i$ nontrivially, then $\phi_{S_i}(x) = 1$ for all $i$, and therefore $x \in \cap_{i=1}^m Q_{S_i}$. We need to show that any element
in $\cap_{i=1}^{m} Q_{S_i}$, can be written as a product of such commutators. First we will describe a type of commutator collection process, where any element of $P_n$ may be written as a product of monic commutators such that the monic commutators with common support are grouped together. When we apply this to an element of $\cap_{i=1}^{m} Q_{S_i}$, we will find that all the monic commutators whose support misses one of the $S_i$ will drop out.

Let $q \in P_n$ be given as a word in the $p_{i,j}^{\pm 1}$. Fix a total ordering $T_1, T_2, \ldots T_{2^n}$ of the subsets of $N$, arbitrary except that we require that if $T_i \subset T_j$ then $i < j$. For notational convenience we shall write $T_i < T_j$ if $i < j$. We claim that there exist elements $q_1, q_2, \ldots q_{2^k} \in P_n$ such that $q = q_1 q_2 \ldots q_{2^k}$, and such that each $q_i$ a product of monic commutators of support $T_i$, where an empty product is taken to be $1 \in P_n$. Suppose inductively that we have written $q = q_1 q_2 \ldots q_{r} s$, where $q_i$ is a product of monic commutators of support $T_i$ for $1 \leq i \leq r$, and $s$ is a product of monic commutators, each of support $\geq T_r$. We want to find all the monic commutators in $s$ of support $T_r$, and move them back into $q_r$. We may do this by inserting commutators, each of which has support $> T_r$. More precisely, let $s = x_1 x_2 \ldots x_t y z$, where each $x_i$ is a monic commutator of support $> T_r$, $y$ is monic commutator of of support $T_r$, and $z$ is a product of monic commutators of support $\geq T_r$. We may then write $s = y x_1 [y, x_1] x_2 [y, x_2] x_3 \ldots x_t [y, x_t]$, and then move $y$ into $q_r$. Even though this increases the number of monic commutators in $s$, it decreases by one the number of them that have support $T_r$. Thus it is possible to continue until all the monic commutators with support $T_r$ are contained in $q_r$, and inductively we can continue until $q = q_1 q_2 \ldots q_{2^k}$.

Now let $q \in \cap_{i=1}^{m} Q_{S_i}$ and $q = q_1 q_2 \ldots q_{2^k}$ as above. Fix an arbitrary $i$ between 1 and $n$. We need to show that if $T_j \cap S_i = \emptyset$, then $q_j = 1$. This is done by induction on $|T_j|$. Suppose that $q_{j'} = 1$ for all $T_j' \subset T_j$. By Proposition 1.1, $\phi_{T_j}$ factors through $\phi_{S_i}$, so

$$1 = \phi_{T_j}(q) = \phi_{T_j}(q_1) \phi_{T_j}(q_2) \ldots \phi_{T_j}(q_{2^n})$$

and the only possible nontrivial element of this product is $\phi_{T_j}(q_j)$, which must then be trivial as well. □
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