Categorical Results in the Theory of Two-Crossed Modules of Commutative Algebras

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Abstract

In this paper we explore some categorical results of 2-crossed module of commutative algebras extending work of Porter in [18]. We also show that the forgetful functor from the category of 2-crossed modules to the category of $k$-algebras, taking \{L, M, P, \partial_2, \partial_1\} to the base algebra $P$, is fibred and cofibred considering the pullback (coinduced) and induced 2-crossed modules constructions, respectively. Also we consider free 2-crossed modules as an application of induced 2-crossed modules.

Introduction

Crossed modules of groups were initially defined by Whitehead [23, 24] as models for (homotopy) 2-types. The commutative algebra case of crossed modules is contained in the paper of Lichtenbaum and Schlessinger [16] and also the work of Gerstenhaber [13] under different names. Some categorical results and Koszul complex link are also given by Porter [18, 19]. Conduché, [12], in 1984 described the notion of 2-crossed modules as a model for 3-types. The commutative algebra version of 2-crossed modules has been defined by Grandjéan and Vale [14]. Arvasi and Porter [5, 6] have important studies related with that construction.

The purpose of this paper is to investigate some categorical theory of 2-crossed modules. It is considered the results as easy to prove but nonetheless giving some functorial relations between the category $X_2\text{Mod}$ of 2-crossed modules of commutative algebras and other categories with some adjoint pairs of functors would seem to be important to find out the question as to whether or not $X_2\text{Mod}$ is a fibred category.

Here we will give the construction of the pullback and induced 2-crossed modules of commutative algebras extending results of Shammu in [21] and Porter in [18] for crossed modules of algebras. The construction of pullback and induced 2-crossed modules will give us a pair of adjoint functors $(\phi^*, \phi_*)$ where

$$\phi^* : X_2\text{Mod}/R \to X_2\text{Mod}/S \quad \text{and} \quad \phi_* : X_2\text{Mod}/S \to X_2\text{Mod}/R$$

with respect to an algebra morphism $\phi : S \to R$. Since we have pullback object of $X_2\text{Mod}$ along any arrow of $k\text{-Alg}$, we get the fact that $X_2\text{Mod}$ is a fibred category and also cofibred which is the dual of the fibred.
We end with an application which leads to link free 2-crossed modules with induced 2-crossed modules.

Conventions

Throughout this paper $k$ will be a fixed commutative ring and $R$ will be a $k$-algebra with identity. All algebras will be commutative and actions will be left and the right actions in some references will be rewritten by using left actions.

1 Two-Crossed Modules of Algebras

Crossed modules of groups were initially defined by Whitehead [23, 24] as models for (homotopy) 2-types. Conduché, [12], in 1984 described the notion of 2-crossed module as a model for 3-types. Both crossed modules and 2-crossed modules have been adapted for use in the context of commutative algebras in [14, 19].

A crossed module is an algebra morphism $\partial : C \to R$ with an action of $R$ on $C$ satisfying $\partial (r \cdot c) = r \partial (c)$ and $\partial (c) \cdot c' = cc'$ for all $c, c' \in C, r \in R$. When the first equation is satisfied, $\partial$ is called pre-crossed module.

If $(C, R, \partial)$ and $(C', R', \partial')$ are crossed modules, a morphism, $(\theta, \varphi) : (C, R, \partial) \to (C', R', \partial')$, of crossed modules consists of $k$-algebra homomorphisms $\theta : C \to C'$ and $\varphi : R \to R'$ such that

$\partial' \theta = \varphi \partial$ \hspace{1cm} (ii) $\theta (r \cdot c) = \varphi (r) \cdot \theta (c)$

for all $r \in R, c \in C$. We thus get the category $XMod$ of crossed modules.

Examples of crossed modules are:

(i) Any ideal, $I$, in $R$ gives an inclusion map $I \to R$, which is a crossed module then we will say $(I, R, i)$ is an ideal pair. In this case, of course, $R$ acts on $I$ by multiplication and the inclusion homomorphism $i$ makes $(I, R, i)$ into a crossed module, an “inclusion crossed module”. Conversely, given any crossed module, $\partial : C \to R$ one easily sees that the image $\partial (C)$ of $C$ is an ideal in $R$.

(ii) Any $R$-module $M$ can be considered as an $R$-algebra with zero multiplication and hence the zero morphism $0 : M \to R$ sending everything in $M$ to the zero element of $R$. Again conversely, any $(C, R, \partial)$ crossed module, $\ker \partial$ is an ideal in $C$ and inherits a natural $R$-module structure from $R$-action on $C$. Moreover, $\partial (C)$ acts trivially on $\ker \partial$, hence $\ker \partial$ has a natural $R/\partial (C)$-module structure.

(iii) Let $M(R)$ be multiplication algebra defined by Mac Lane [15] (see also [17]) as the set of all multipliers $\delta : R \to R$ such that for all $r, r' \in R$, $\delta (rr') = r \delta (r')$ where $R$ is a commutative $k$-algebra and $Ann (R) = 0$ or $R^2 = R$. Then $\mu : R \to M(R)$ is a crossed module given by $\mu (r) = \delta_r$ with $\delta_r (r') = rr'$ for all $r, r' \in R$. (See [2] for details).
(iv) Any epimorphism of algebras $C \rightarrow R$ with the kernel in the annihilator of $C$ is a crossed module, with $r \in R$ acting on $c \in C$ by $r \cdot c = \bar{c}c$, where $\bar{c}$ is any element in the pre-image of $r$.

Grandjean and Vale [14] have given a definition of 2-crossed modules of algebras. The following is an equivalent formulation of that concept.

A 2-crossed module of $k$-algebras consists of a complex of $P$-algebras $L \xrightarrow{\partial_2} M \xrightarrow{\partial_1} P$ together with an action of $P$ on all three algebras and a $P$-linear mapping

$\{-,-\} : M \times M \rightarrow L$

which is often called the Peiffer lifting such that the action of $P$ on itself is by multiplication, $\partial_2$ and $\partial_1$ are $P$-equivariant.

**PL1:** $\partial_2 \{m_0, m_1\} = m_0m_1 - \partial_1 (m_1) \cdot m_0$

**PL2:** $\{\partial_2 (l_0), \partial_2 (l_1)\} = l_0l_1$

**PL3:** $\{m_0, m_1m_2\} = \{m_0m_1, m_2\} + \partial_1 (m_2) \cdot \{m_0, m_1\}$

**PL4:** $\{m, \partial_2 (l)\} + \{\partial_2 (l), m\} = \partial_1 (m) \cdot l$

**PL5:** $\{m_0, m_1\} \cdot p = \{m_0 \cdot p, m_1\} = \{m_0, m_1 \cdot p\}$

for all $m, m_0, m_1, m_2 \in M, l, l_0, l_1 \in L$ and $p \in P$.

Note that we have not specified that $M$ acts on $L$. We could have done that as follows: if $m \in M$ and $l \in L$, define

$m \cdot l = \{m, \partial_2 (l)\}$.

From this equation $(L, M, \partial_2)$ becomes a crossed module. We can split **PL4** into two pieces:

**PL4**: (a) $\{m, \partial_2 (l)\} = m \cdot l$

$b. \{\partial_2 (l), m\} = m \cdot l - \partial_1 (m) \cdot l.$

We denote such a 2-crossed module of algebras by $\{L, M, P, \partial_2, \partial_1\}$.

A morphism of 2-crossed modules is given by the following diagram

```
 L \xrightarrow{\partial_2} M \xrightarrow{\partial_1} P
 \downarrow f_2 \quad \downarrow f_1 \quad \downarrow f_0
 L' \xrightarrow{\partial'_2} M' \xrightarrow{\partial'_1} P'
```

where $f_0 \partial_1 = \partial'_1 f_1, f_1 \partial_2 = \partial'_2 f_2$

$f_1 (p \cdot m) = f_0 (p) \cdot f_1 (m)$, $f_2 (p \cdot l) = f_0 (p) \cdot f_2 (l)$

for all $m \in M, l \in L, p \in P$ and

$\{-,-\} (f_1 \times f_1) = f_2 \{-,-\}$

We thus get the category of 2-crossed modules denoting it by $X_2\text{Mod}$ and when the morphism $f_0$ above is the identity we will get $X_2\text{Mod}/P$ the category of 2-crossed modules over fixed algebra $P$. 
Some remarks on Peiffer lifting of 2-crossed modules are: Suppose we have a 2-crossed module
\[ L \overset{\partial_2}{\to} M \overset{\partial_1}{\to} P \]
with trivial Peiffer lifting. Then
(i) There is an action of \( P \) on \( L \) and \( M \) and the \( \partial \)'s are \( P \)-equivariant. (This gives nothing new in our special case.)
(ii) \( \{-,-\} \) is a lifting of the Peiffer commutator so if \( \{m,m'\} = 0 \), the Peiffer identity holds for \( (M,P,\partial_1) \), i.e. that is a crossed module.
(iii) if \( l,l' \in L \), then \( 0 = \{\partial_2 l, \partial_2 l'\} = ll' \)
and,
(iv) as \( \{-,-\} \) is trivial \( \partial_1 (m) \cdot l = 0 \) so \( \partial M \) has trivial action on \( L \).
Axioms PL3 and PL5 vanish.
The above remarks are known for 2-crossed modules of groups. These are handled in recent book of Porter in [20].

1.1 Functorial Relations with Some Other Categories

1. Let \( M \overset{\partial}{\to} P \) be a pre-crossed module. The Peiffer ideal \( \langle M, M \rangle \) is generated by the Peiffer commutators
\[ \langle m,m' \rangle = \partial m \cdot m' - mm' \]
for all \( m,m' \in M \). The pre-crossed modules in which all Peiffer commutators are trivial are precisely the crossed modules. Thus the category of crossed modules is the full subcategory of the category of pre-crossed modules whose objects are crossed modules. So we can define the following skeleton functor
\[ \text{Sk} : \text{PXMod} \to \text{X}_2\text{Mod} \]
by \( \text{Sk}(M,P,\partial_1) = \{\langle M, M \rangle, M, P, \partial_2, \partial_1\} \) as a 2-crossed module with the Peiffer lifting \( \{m,m'\} = \langle m,m' \rangle \). This functor has a right adjoint truncation functor:
\[ \text{Tr} : \text{X}_2\text{Mod} \to \text{PXMod} \]
given by \( \text{Tr} \{L, M, P, \partial_2, \partial_1\} = (M, P, \partial_1) \).
2. Any crossed module gives a 2-crossed module. If \( (M,P,\partial) \) is a crossed module, the resulting sequence
\[ L \to M \to P \]
is a 2-crossed module by taking \( L = 0 \). Thus we have
\[ \alpha : \text{XMod} \to \text{X}_2\text{Mod} \]
de fined by \( \alpha(M,P,\partial) = \{0, M, P, 0, \partial\} \) that is the adjoint of the following functor:
\[ \beta : \text{X}_2\text{Mod} \to \text{XMod} \]
given by \( \beta \{L, M, P, \partial_2, \partial_1\} = (M/\text{Im} \partial_2, P, \partial_1) \) where \( \text{Im} \partial_2 \) is an ideal of \( M \).
3. The functor \( \delta : \text{XMod} \to \text{k-Alg} \) which is given by \( \delta(C,R,\partial) = R \) has a right adjoint \( \gamma : \text{k-Alg} \to \text{XMod}, \gamma(A) = (A, A, i\bar{d}_A) \).
Proposition 1 Given $X_2\text{Mod} \xrightarrow{\beta} \text{XMod} \xrightarrow{\alpha} k\text{-Alg}$ adjoint functors as defined above. Then $(\delta \circ \beta, \alpha \circ \gamma)$ is a pair of adjoint functors.

Proof. For $\{L, M, P, \partial_2, \partial_1\} \in X_2\text{Mod}$ and $R \in k\text{-Alg}$, we have functorial isomorphisms:

$$k\text{-Alg}(\delta \beta (\{L, M, P, \partial_2, \partial_1\}), R) \simeq \text{XMod}(\beta \{L, M, P, \partial_2, \partial_1\}, \gamma (R)) \simeq X_2\text{Mod}(\{L, M, P, \partial_2, \partial_1\}, \alpha \gamma (R))$$

Now we will give the construction of pullback and induced 2-crossed modules. Similar constructions have appeared in several studies on crossed module of groups, algebras and 2-crossed modules of groups, e.g. [1, 7, 8, 10, 11].

2 The Pullback Two-Crossed Modules

The construction of “change of base” is well-known in a module theory. The higher dimension of this had been considered by Porter [18] and Shamm [21]. There are called (co)-induced crossed modules. The first author and Gürmen were also deeply analysed that in [3]. In this section the functor that is going to be the right adjoint of the induced 2-crossed module, the “pullback” will be defined. This is an important construction which, given a morphism of algebras $\phi : S \rightarrow R$, enables us to change of base of 2-crossed modules.

Definition 2 Given a crossed module $\partial : C \rightarrow R$ and a morphism of $k$-algebras $\phi : S \rightarrow R$, the pullback crossed module can be given by

(i) a crossed module $\phi^* (C, R, \partial) = (\partial^* : \phi^*(C) \rightarrow S)$

(ii) given

$$(f, \phi) : (B, S, \mu) \rightarrow (C, R, \partial)$$

crossed module morphism, then there is a unique $(f^*, \text{id}_S)$ crossed module morphism that commutes the following diagram:
or more simply as

\[
\begin{array}{c}
B \xrightarrow{f} C \\
\mu \downarrow \downarrow \phi^*(C) \xrightarrow{\phi} \partial \\
S \downarrow \downarrow \phi^*(C) \xrightarrow{\phi} \partial \\
\end{array}
\]

where \( \phi^*(C) = C \times_R S = \{(c,s) \mid \partial(c) = \phi(s)\} \), \( \partial^*(c,s) = s \), \( \phi'(c,s) = c \) for all \((c,s) \in \phi^*(C)\), and \( S \) acts on \( \phi^*(C) \) via \( \phi \) and the diagonal.

**Definition 3** Given a 2-crossed module \( \{C_2,C_1,R,\partial_2,\partial_1\} \) and a morphism of \( k \)-algebras \( \phi : S \to R \), the pullback 2-crossed module can be given by

(i) a 2-crossed module \( \phi^* \{C_2,C_1,R,\partial_2,\partial_1\} = \{C_2,\phi^*(C_1),S,\partial_2^*,\partial_1^*\} \)

(ii) given any morphism of 2-crossed modules

\( (f_2,f_1,\phi) : \{B_2,B_1,S,\partial_2',\partial_1'\} \to \{C_2,C_1,R,\partial_2,\partial_1\} \)

there is a unique \( (f_2^*,f_1^*,id_S) \) 2-crossed module morphism that commutes the following diagram:

\[
\begin{array}{c}
(B_2,B_1,S,\partial_2',\partial_1') \\
\xrightarrow{(f_2^*,f_1^*,id_S)} \\
(C_2,\phi^*(C_1),S,\partial_2^*,\partial_1^*) \\
\xrightarrow{(id_{C_2},\phi',\phi)} \\
(C_2,C_1,R,\partial_2,\partial_1) \\
\end{array}
\]
or more simply as

\[
\begin{array}{c}
\begin{array}{c}
\phi^*(C_1) = C_1 \times_R S = \{(c_1, s) \mid \partial_1(c_1) = \phi(s)\}
\end{array}
\end{array}
\]

which is usually pullback in the category of algebras. There is a commutative diagram

\[
\begin{array}{c}
\begin{array}{c}
\phi^*(C_1) \xrightarrow{\phi'} C_1 \\
\phi^*(C_1) \xrightarrow{\partial_1} C_1 \\
S \xrightarrow{\phi} R
\end{array}
\end{array}
\]

where \(\partial_1^*(c_1, s) = s, \phi'(c_1, s) = c_1\) and \(\partial_1^*\) is \(S\)-equivariant with the action \(s' \cdot (c_1, s) = (\phi(s') \cdot c_1, s')\) for all \((c_1, s) \in \phi^*(C_1)\) and \(s \in S\).

So we get a pre-crossed module \((\phi^*(C_1), S, \partial_1^*)\) which is called the pullback pre-crossed module of \((C_1, R, \partial_1)\) along \(\phi\). Then we can define a pullback of \(\partial_2 : C_2 \rightarrow C_1\) along \(\phi'\) as given in the following diagram

\[
\begin{array}{c}
\begin{array}{c}
\phi^*(C_2) \xrightarrow{\phi''} C_2 \\
\phi^*(C_1) \xrightarrow{\phi'} C_1
\end{array}
\end{array}
\]

in which

\[
\phi^*(C_2) = \{(c_2, (c_1, s)) \mid \partial_2(c_2) = \phi'(c_1, s) = c_1, \phi(s) = \partial_1(c_1)\}
\]

\[
= \{(c_2, (\partial_2(c_2), s)) \mid \phi(s) = \partial_1(\partial_2(c_2)) = 0\} \cong C_2 \times C_1 \times (\text{Ker} \partial_1 \times \text{Ker} \phi)
\]
for all $c_2 \in C_2$ and $(c_1, s) \in \phi^*(C_1)$.

Since pullback of a pullback is a pullback, we have already constructed the pullback composition

$$\phi^*(C_2) \xrightarrow{\partial_2^*} \phi^*(C_1) \xrightarrow{\partial_1^*} S$$

which is the pullback of $\partial_1 \partial_2 = 0$ by $\phi$.

On the other hand, we can construct directly the pullback of $\partial_1 \partial_2 = 0$ by $\phi$ as $\partial : B \to S$ where $B = \{(c_2, s) \mid \phi(s) = 0\} \cong C_2 \times \text{Ker} \phi$ and $\partial(c_2, s) = s$.

We can define the isomorphism $\Psi : \phi^*(C_2) \to B$, $\Psi(x) = (c_2, s)$ where $x = (c_2, (\partial_2(c_2), s)) \in \phi^*(C_2)$. So $\phi^*(C_2) \cong B$.

But, we find that the pullback $\phi^*(C_2) \xrightarrow{\partial_2^*} \phi^*(C_1) \xrightarrow{\partial_1^*} S$ is not a complex of $S$-algebras unless $\phi$ is a monomorphism. To see this, note that for $(c_2, s) \in C_2 \times \text{Ker} \phi$,

$$\partial_1^* \partial_2^*(c_2, s) = \partial_1^*(\partial_2(c_2), s) = s.$$

This last expression is equal to 0 if $\phi$ is a monomorphism. So $\phi^*(C_2) \cong C_2$.

Thus, we can give the pullback 2-crossed module of $\phi : S \to R$ along $\phi$ as follows.

**Proposition 4** If $C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} R$ is a 2-crossed module and if $\phi : S \to R$ is a monomorphism of $k$-algebras then

$$C_2 \xrightarrow{\partial_2^*} \phi^*(C_1) \xrightarrow{\partial_1^*} S$$

is a pullback 2-crossed module where $\partial_2^*(c_2) = (\partial_2(c_2), 0)$ and $\partial_1^*(c_1, s) = s$ and the action of $S$ on $\phi^*(C_1)$ and $C_2$ by $s \cdot (c_1, s') = (\phi(s) \cdot c_1, ss')$ and $s \cdot c_2 = \phi(s) \cdot c_2$ respectively.

**Proof.** Since

$$\partial_2^*(s \cdot c_2) = (\partial_2(s \cdot c_2), 0) = (\partial_2(\phi(s) \cdot c_2), 0) = (\phi(s) \cdot \partial_2(c_2), 0) = s \cdot (\partial_2(c_2), 0) = s \cdot \partial_2^*(c_2),$$

$\partial_2^*$ is $S$-equivariant. Also, we have seen above that $\partial_1^*$ is $S$-equivariant and $C_2 \to \phi^*(C_1) \to S$ is a complex of $S$-algebras.

The Peiffer lifting

$$\{-,-\} : \phi^*(C_1) \times \phi^*(C_1) \to C_2$$

is given by $\{(c_1, s_1)(c_1', s_1')\} = \{c_1, c_1'\}$.

**PL1:**

$$(c_1, s_1)(c_1', s_1') - (c_1, s_1) \cdot \partial_1^*(c_1', s_1') = (c_1c_1', s_1s_1') - (c_1, s_1) \cdot s_1'$$

$$= (c_1c_1', s_1s_1') - (c_1 \cdot \phi(s_1'), s_1s_1')$$

$$= (c_1c_1', s_1s_1') - (c_1 \cdot \phi(s_1'), s_1s_1')$$

$$= (c_1c_1' - c_1 \cdot \phi(s_1'), 0)$$

$$= (c_1c_1' - c_1 \cdot \partial_l(c_1'), 0)$$

$$= \{\partial_2\{c_1, c_1'\}, 0\}$$

$$= \partial_2^* \{(c_1, s_1)(c_1', s_1')\}.$$
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PL2:
$$\{\partial_2^\ast (c_2), \partial_2^\ast (c_2')\} = \{(\partial_2 (c_2), 0), (\partial_2 (c_2'), 0)\} = \{\partial_2 (c_2), \partial_2 (c_2')\} = c_2 c_2'.$$

The rest of axioms of 2-crossed module is given in appendix.

$$(id_{C_2}, \phi', \phi) : \{C_2, \phi^\ast (C_1), S, \partial_2^\ast, \partial_1^\ast\} \to \{C_2, C_1, R, \partial_2, \partial_1\}$$

or diagrammatically,

\[
\begin{array}{ccc}
C_2 & \xrightarrow{id_{C_2}} & C_2 \\
\downarrow{\partial_2^\ast} & & \downarrow{\partial_2} \\
\phi^\ast (C_1) & \xrightarrow{\phi'} & C_1 \\
\downarrow{\partial_1^\ast} & & \downarrow{\partial_1} \\
S & \xrightarrow{\phi} & R
\end{array}
\]

is a morphism of 2-crossed modules. (See appendix.)

Suppose that $$(f_2, f_1, \phi) : \{B_2, B_1, S, \partial'_2, \partial'_1\} \to \{C_2, C_1, R, \partial_2, \partial_1\}$$
is any 2-crossed module morphism

\[
\begin{array}{ccc}
B_2 & \xrightarrow{\partial'_2} & B_1 \\
\downarrow{f_2} & & \downarrow{\phi} \\
C_2 & \xrightarrow{\partial_2} & C_1 \\
\downarrow{\partial'_1} & & \downarrow{\partial_1} \\
S & \xrightarrow{R}
\end{array}
\]

Then we will show that there is a unique 2-crossed module morphism

$$(f_2^\ast, f_1^\ast, id_S) : \{B_2, B_1, S, \partial'_2, \partial'_1\} \to \{C_2, \phi^\ast (C_1), S, \partial_2^\ast, \partial_1^\ast\}$$

\[
\begin{array}{ccc}
B_2 & \xrightarrow{\partial'_2} & B_1 \\
\downarrow{f_2^\ast} & & \downarrow{\partial'_1} \\
C_2 & \xrightarrow{\phi^\ast (C_1)} & S \\
\end{array}
\]

where $f_2^\ast (b_2) = f_2 (b_2)$ and $f_1^\ast (b_1) = (f_1 (b_1), \partial'_1 (b_1))$ which is an element in $\phi^\ast (C_1)$. First, let us check that $$(f_2^\ast, f_1^\ast, id_S)$$ is a morphism of 2-crossed modules. For $b_1 \in B_1, b_2 \in B_2, s \in S$

\[
f_2^\ast (s \cdot b_2) = f_2 (s \cdot b_2) = \phi (s) \cdot f_2 (b_2) = \phi (s) \cdot f_2^\ast (b_2) = s \cdot f_2^\ast (b_2) = id_S (s) \cdot f_2^\ast (b_2),
\]
similarly \( f^*_1(s \cdot b_1) = \text{id}_S(s) \cdot f^*_1(b_1) \), also the above diagram is commutative and
\[
\{ -, - \} (f^*_1 \times f^*_1)(b_1, b'_1) = \{ -, - \} (f^*_1(b_1), f^*_1(b'_1)) \\
= \{ -, - \} ((f_1(b_1), \partial'_1(b_1)), (f_1(b'_1), \partial'_1(b'_1))) \\
= \{(f_1(b_1), \partial'_1(b_1)), (f_1(b'_1), \partial'_1(b'_1))\} \\
= \{f_1(b_1), f_1(b'_1)\} \\
= \{ -, - \} (f_1(b_1), f_1(b'_1)) \\
= \{ -, - \} (f_1 \times f_2)(b_1, b'_2) \\
= f_2(\{ -, - \} (b_1, b'_1)) \\
= f_2(b_1, b'_1) \\
= f^*_2(\{b_1, b'_1\}) \\
= f^*_2(\{ -, - \} (b_1, b'_1)) \\
\]
for all \( b_1, b'_1 \in B_1 \). So \((f^*_2, f^*_1, \text{id}_S)\) is a morphism of 2-crossed modules.

Furthermore; following equations are easily verified:
\[
\text{id}_{C_2} f^*_2 = f_2 \quad \text{and} \quad \phi' f^*_1 = f_1.
\]

Thus we get a functor
\[
\phi^* : X_2\text{Mod}/R \to X_2\text{Mod}/S
\]
which gives our pullback 2-crossed module.

**Remark 5** These functors have the property that for any monomorphisms \( \phi \) and \( \phi' \) there are natural equivalences \( \phi^* \phi'^* \simeq (\phi' \phi)^* \).

### 2.1 The Examples of Pullback Two-Crossed Modules

Given 2-crossed module \( \{(0), I, R, 0, i\} \) where \( i \) is an inclusion of an ideal. The pullback 2-crossed module is
\[
\phi^* \{(0), I, R, 0, i\} = \{0, \phi^*(I), S, \partial'_2, \partial'_1\} \\
= \{0, \phi^{-1}(I), S, \partial'_2, \partial'_1\}
\]
as,
\[
\phi^*(I) = \{(a, s) \mid \phi(s) = i(a) = a, s \in S, a \in I\} \\
\cong \{s \in S \mid \phi(s) = a\} = \phi^{-1}(I) \subseteq S.
\]
The pullback diagram is
\[
\begin{array}{ccc}
\{0\} & \xrightarrow{\partial'_2} & \{0\} \\
\downarrow{\phi^{-1}(I)} & & \downarrow{0} \\
S & \xrightarrow{\phi} & R.
\end{array}
\]
Particularly if \( I = \{0\} \), since \( \phi \) is monomorphism we get
\[
\phi^* (\{0\}) \cong \{ s \in S \mid \phi(s) = 0 \} = \ker \phi \cong \{0\}
\]
and so \( \{0\}, \{0\}, S, 0, 0 \) is a pullback 2-crossed module.

Also if \( \phi \) is an isomorphism and \( I = R \), then \( \phi^*(R) = R \times S \).

Similarly when we consider examples given in Section 1, the following diagrams are pullbacks.

![Diagram](image)

3 The Induced Two-Crossed Modules

We will define a functor \( \phi^*_s \) left adjoint to the pullback \( \phi^* \) given in the previous section. The “induced 2-crossed module” functor \( \phi^*_s \) is defined by the following universal property, developing works in [2] and [18].

**Definition 6** For any crossed module \( \mathcal{D} = (D, S, \partial) \) and any homomorphism \( \phi : S \to R \) the crossed module induced by \( \phi \) from \( \partial \) should be given by:

1. A crossed module \( \phi^*_s(D) = (\phi^*_s(D), R, \phi^*_s \partial) \),
2. A morphism of crossed modules \( (f, \phi) : \mathcal{D} \to \phi^*_s(D) \), satisfying the dual universal property that for any morphism of crossed modules,
   \[
   (h, \phi) : \mathcal{D} \to \mathcal{B}
   \]

there is a unique morphism of crossed modules \( h' : \phi^*_s(D) \to \mathcal{B} \) such that the diagram

![Diagram](image)

commutes.

**Definition 7** For any 2-crossed module \( D_2 \overset{\partial_2}{\to} D_1 \overset{\partial_1}{\to} S \) and a morphism \( \phi : S \to R \) of \( k \)-algebras, the induced 2-crossed can be given by
Categorical Results in the Theory of Two-Crossed Modules of Commutative Algebras

(i) a 2-crossed module \( \phi^* \{D_2, D_1, S, \partial_2, \partial_1\} = \{\phi^*(D_2), \phi^*(D_1), R, \partial_2, \partial_1\} \)

(ii) given any morphism of 2-crossed modules

\[ (f_2, f_1, \phi) : \{D_2, D_1, S, \partial_2, \partial_1\} \to \{B_2, B_1, R, \partial'_2, \partial'_1\} \]

then there is a unique \((f_2, f_1, \text{id}_R)\) 2-crossed module morphism that commutes the following diagram:

Proposition 8 Let \( D_2 \xrightarrow{\partial_2} D_1 \xrightarrow{\partial_1} S \) be a 2-crossed module and \( \phi : S \to R \) be a morphism of \( k \)-algebras. Then \( \phi_*(D_2) \xrightarrow{\partial_2} \phi_*(D_1) \xrightarrow{\partial_1} R \) is the induced 2-crossed module where \( \phi_*(D_1) \) is generated as an algebra, by the set \( D_1 \times R \) with defining relations

\[
\begin{align*}
(d_1, r_1)(d'_1, r'_1) &= (d_1d'_1, r_1r'_1), \\
(d_1, r) - (d'_1, r) &= (d_1 - d'_1, r), \\
(s, d_1, r) &= (d_1, \phi(s) r)
\end{align*}
\]
and $\phi_*(D_2)$ is generated as an algebra, by the set $D_2 \times R$ with defining relations

\[
\{(d_1, r_1), (d_1', r_1')\} = \{(d_1, r_1), (d_1', r_1')\} = \{(d_1, r_1), (d_1', r_1')\} = \\
\{(d_1, r_1), (d_1', r_1')\} = \{(d_1, r_1), (d_1', r_1')\} = \\
(d_2, r_2)(d_2', r_2') = (d_2 d_2', r_2 r_2'), \\
(d_2, r) + (d_2', r) = (d_2 + d_2', r), \\
(s \cdot d_2, r) = (d_2, \phi(s)r)
\]

for any $d_1, d_1', d_2', d_2' \in D_1$, $d_2, d_2' \in D_2$, $s \in S$ and $r, r_1, r_1', r_2, r_2' \in R$. The morphism $\partial_2 : \phi_*(D_2) \to \phi_*(D_1)$ is given by $\partial_2, (d_2, r) = (\partial_2(d_2), r)$ the action of $\phi_*(D_1)$ on $\phi_*(D_2)$ by $\partial_1, (d_1, r_1) = (d_1, r_1) , r_2, r_2' \in R$. The morphism $\partial_1 : \phi_*(D_1) \to \phi_*(D_2)$ is given by $\partial_1, (d_1, r) = \phi \partial_1(d_1)r$, the action of $R$ on $\phi_*(D_1)$ and $\partial_1, (d_1, r) = (d_1, r r_1)$ and $r \cdot (d_2, r') = (d_2, r r')$ respectively.

**Proof.** (i) As $\partial_1, (\partial_2, (d_2, r)) = \partial_1, (\partial_2(d_2), r) = \phi (\partial_1(\partial_2(d_2))) = \phi(0) = 0,

\[
\phi_*(D_2) \stackrel{\partial_2}{\to} \phi_*(D_1) \to \phi_*(D_2)
\]

is a complex of $k$-algebras. The Peiffer lifting

\[
\{-,-\} : \phi_*(D_1) \times \phi_*(D_1) \to \phi_*(D_2)
\]

is given by $\{(d_1, r_1), (d_1', r_1')\} = \{(d_1, d_1'), r_1 r_1'\}$ for all $(d_1, r_1), (d_1', r_1') \in \phi_*(D_1)$.

**PL1:**

\[
\partial_2, \{(d_1, r_1), (d_1', r_1')\} = \partial_2, \{(d_1, d_1'), r_1 r_1'\} = \\
= \partial_2 \{d_1, d_1', r_1 r_1'\} = \\
= (d_1 d_1' - d_1 \cdot \partial_1(d_1'), r_1 r_1') = \\
= (d_1 d_1', r_1 r_1') - (d_1 \cdot \partial_1(d_1'), r_1 r_1') = \\
= (d_1 d_1', r_1 r_1') - (d_1, \phi \partial_1(d_1')) r_1 r_1' = \\
= (d_1 d_1', r_1 r_1') - (d_1, r_1) \cdot \partial_1(d_1, r_1') = \\
= (d_1, r_1)(d_1', r_1') - (d_1, r_1) \cdot \partial_1(d_1', r_1').
\]

**PL2:**

\[
\{\partial_2, (d_2, r_2), \partial_2, (d_2', r_2')\} = \{(\partial_2(d_2), r_2), (\partial_2(d_2'), r_2')\} = \\
= \{(\partial_2(d_2), \partial_2(d_2')), r_2 r_2'\} = \\
= (d_2 d_2', r_2 r_2') = \\
= (d_2, r_2)(d_2', r_2')
\]

for all $(d_1, r_1), (d_1', r_1') \in \phi_*(D_1), (d_2, r_2), (d_2', r_2') \in \phi_*(D_2)$.

The rest of axioms of 2-crossed module is given in appendix.

(ii) It is clear that

\[
(\phi'', \phi', \phi) : \{D_2, D_1, S, \partial_2, \partial_1\} \to \{\phi_*(D_2), \phi_*(D_1), R, \partial_2, \partial_1\}
\]
or diagrammatically,

\[
\begin{array}{ccc}
D_2 & \xrightarrow{\phi''} & \phi_*(D_2) \\
\downarrow{\partial_2} & & \downarrow{\partial_{2*}} \\
D_1 & \xrightarrow{\phi'} & \phi_*(D_1) \\
\downarrow{\partial_1} & & \downarrow{\partial_{1*}} \\
S & \xrightarrow{\phi} & R \\
\end{array}
\]

is a morphism of 2-crossed modules.

Suppose that

\[(f_2, f_1, \phi) : \{D_2, D_1, S, \partial_2, \partial_1\} \rightarrow \{B_2, B_1, R, \partial'_2, \partial'_1\}\]

is any 2-crossed module morphism. Then we will show that there is a 2-crossed module morphism

\[(f_2, f_1, id_R) : \{\phi_*(D_2), \phi_*(D_1), R, \partial_{2*}, \partial_{1*}\} \rightarrow \{B_2, B_1, R, \partial'_2, \partial'_1\}\]

where \(f_2, (d_2, r_2) = r_2 \cdot f_2(d_2)\) and \(f_1, (d_1, r_1) = r_1 \cdot f_1(d_1)\). First we will check that \((f_2, f_1, id_R)\) is a 2-crossed module morphism. We can see this easily as follows:

\[f_2, (r'_2 \cdot (d_2, r_2)) = f_2, (d_2, r'_2 r_2) = r'_2 \cdot (r_2 \cdot f_2(d_2)) = r'_2 \cdot f_2, (d_2, r_2)\]

Similarly \(f_1, (r'_1 \cdot (d_1, r_1)) = r'_1 \cdot f_1, (d_1, r_1)\),

\[(f_1, \partial_{2*}) (d_2, r_2) = f_1, (\partial_{2*} (d_2, r_2)) = f_1, (\partial_2 (d_2), r_2) = r_2 \cdot (f_1, (\partial_2 (d_2))) = r_2 \cdot (\partial'_2 (f_2, (d_2))) = \partial'_2 (r_2 \cdot f_2, (d_2)) = \partial'_2 (f_2, (d_2)),\]

and \(\partial'_1 f_1 = id_R \partial_{1*}\) for all \((d_1, r_1) \in \phi_*(D_1), (d_2, r_2) \in \phi_*(D_2), r'_1, r'_2 \in R\) and

\[\{ -, - \} (f_1, f_1) = f_2^* \{ -, - \} .\]

So we get the induced 2-crossed module functor

\[\phi_* : X_2\text{Mod}/S \rightarrow X_2\text{Mod}/R.\]

We can give the following naturality condition for \(\phi_*\) similar to remark.
Remark 9 They satisfy the “naturality condition” that there is a natural equivalence of functors
\[ \phi'_* \phi_* \simeq (\phi')_* . \]

Theorem 10 For any morphism of k-algebras \( \phi : S \to R \), \( \phi_* \) is the left adjoint of \( \phi^* \).

Proof. From the naturality conditions given earlier, it is immediate that for any 2-crossed modules \( D = \{ D_2, D_1, S, \partial_2, \partial_1 \} \) and \( B = \{ B_2, B_1, R, \partial_2, \partial_1 \} \) there are bijections
\[
(X_2\text{Mod}/S)(D, \phi^*(B)) \cong \{(f_2, f_1) \mid (f_2, f_1) : D \to B \text{ is a morphism in } X_2\text{Mod}\}
\]
as proved in proposition 4 and
\[
(X_2\text{Mod}/R)(\phi_*(D), B) \cong \{(f_2, f_1) \mid (f_2, f_1) : D \to B \text{ is a morphism in } X_2\text{Mod}\}
\]
as given in the proposition 8.

Their composition gives the bijection needed for adjointness. ■

Next if \( \phi : S \to R \), is an epimorphism the induced 2-crossed module has a simpler description.

Proposition 11 Let \( D_2 \xrightarrow{\partial_2} D_1 \xrightarrow{\partial_1} S \) is a 2-crossed module, \( \phi : S \to R \) is an epimorphism with \( \ker \phi = K \). Then
\[ \phi_*(D_2) \cong D_2/KD_2 \quad \text{and} \quad \phi_*(D_1) \cong D_1/KD_1, \]
where \( KD_2 \) denotes the ideal of \( D_2 \) generated by \( \{ k \cdot d_2 \mid k \in K, d_2 \in D_2 \} \) and \( KD_1 \) denotes the ideal of \( D_1 \) generated by \( \{ k \cdot d_1 \mid k \in K, d_1 \in D_1 \} \).

Proof. As \( \phi : S \to R \) is an epimorphism, \( R \cong S/K \). Since \( K \) acts trivially on \( D_2/KD_2, D_1/KD_1, R \cong S/K \) acts on \( D_2/KD_2 \) by \( r \cdot (d_2 + KD_2) = (s + K) \cdot (d_2 + KD_2) = s \cdot d_2 + KD_2 \) and \( R \cong S/K \) acts on \( D_1/KD_1 \) by \( r \cdot (d_1 + KD_1) = (s + K) \cdot (d_1 + KD_1) = s \cdot d_1 + KD_1 \).

\[ D_2/KD_2 \xrightarrow{\partial_2} D_1/KD_1 \xrightarrow{\partial_1} R \]
is a 2-crossed module where
\[ \partial_2^*(d_2 + KD_2) = \partial_2(d_2) + KD_1, \partial_1^*(d_1 + KD_1) = \partial_1(d_1) + K \]
and \( D_1/KD_1 \) acts on \( D_2/KD_2 \) by \( (d_1 + KD_1) \cdot (d_2 + KD_2) = d_1 \cdot d_2 + KD_2 \).

As
\[ \partial_1^*, (\partial_2^*(d_2 + KD_2)) = \partial_1(\partial_2(d_2) + KD_1) = \partial_1(\partial_2(d_2)) + K = 0 + K \cong 0_R, \]
\[ D_2/KD_2 \xrightarrow{\partial_2} D_1/KD_1 \xrightarrow{\partial_1} R \]
is a complex of k-algebras.
The Peiffer lifting

\{-, -\} : D_1/KD_1 \times D_1/KD_1 \rightarrow D_2/KD_2

is given by \{d_1 + KD_1, d'_1 + KD_1\} = \{d_1, d'_1\} + KD_2.

**PL1:**

\[
\partial_2, \{d_1 + KD_1, d'_1 + KD_1\} = \partial_2, (\{d_1, d'_1\}) + KD_1
\]

\[
= (d_1 d'_1 - d_1 \cdot \partial_1(d'_1)) + KD_1
\]

\[
= (d_1 d'_1 + KD_1) - (d_1 \cdot \partial_1(d'_1) + KD_1)
\]

\[
= (d_1 + KD_1)(d'_1 + KD_1) - (d_1 + KD_1) \cdot \partial_1(d'_1 + KD_1).
\]

**PL2:**

\[
\{\partial_2, (d_2 + KD_2), \partial_2, (d'_2 + KD_2)\} = \{\partial_2(d_2) + KD_1, \partial_2(d'_2) + KD_1\}
\]

\[
= \{\partial_2(d_2), \partial_2(d'_2)\} + KD_2
\]

\[
= d_2 d'_2 + KD_2
\]

\[
= (d_2 + KD_2)(d'_2 + KD_2).
\]

The rest of axioms of 2-crossed module is given by in appendix.

\((\phi'', \phi', \phi) : \{D_2, D_1, S, \partial_2, \partial_1\} \rightarrow \{D_2/KD_2, D_1/KD_1, R, \partial_2, \partial_1\}\)

or diagrammatically,

\[
\begin{array}{ccc}
D_2 & \xrightarrow{\phi''} & D_2/KD_2 \\
\partial_2 & & \partial_2 \\
D_1 & \xrightarrow{\phi'} & D_1/KD_1 \\
\partial_1 & & \partial_1 \\
S & \xrightarrow{\phi} & R \\
\end{array}
\]

is a morphism of 2-crossed modules. (See appendix.)

Suppose that

\((f_2, f_1, \phi) : \{D_2, D_1, S, \partial_2, \partial_1\} \rightarrow \{B_2, B_1, R, \partial'_2, \partial'_1\}\)

is any 2-crossed module morphism. Then we will show that there is a unique 2-crossed module morphism

\((f_2, f_1, \text{id}_R) : \{D_2/KD_2, D_1/KD_1, R, \partial_2, \partial_1\} \rightarrow \{B_2, B_1, R, \partial'_2, \partial'_1\}\)
where $f_2, (d_2 + KD_2) = f_2(d_2)$ and $f_1, (d_1 + KD_1) = f_1(d_1)$. Since

$$f_2(k \cdot d_2) = \phi(k) \cdot f_2(d_2) = 0_R \cdot f_2(d_2) = 0_{B_2}$$

and similarly $f_1(d_1 + KD_1) = 0_{B_1}$, $f_2(KD_2) = 0_{B_2}$ and $f_1(KD_1) = 0_{B_1}$, $f_2$, and $f_1$, are well defined.

First let us check that $(f_2, f_1, id_R)$ is a 2-crossed module morphism. For $d_2 + KD_2 \in D_2/KD_2, d_1 + KD_1 \in D_1/KD_1$ and $r \in R$,

$$f_2, (r \cdot (d_2 + KD_2)) = f_2, ((s + K) \cdot (d_2 + KD_2))$$

similarly $f_1, (r \cdot (d_1 + KD_1)) = r \cdot f_1, (d_1 + KD_1)$,

$$f_1, \partial_2, (d_2 + KD_2) = f_1, (\partial_2 (d_2) + KD_2)$$

Similarly $\partial_1^* f_1, = id_R \partial_1$, and

$$f_2, \{-, -\} (d_1 + KD_1, d'_1 + KD_1) = f_2, \{d_1 + KD_1, d'_1 + KD_1\}$$

So $(f_2, f_1, id_R)$ is a morphism of 2-crossed modules. Furthermore; following equations are verified.

$$f_2, \phi'' = f_2 \text{ and } f_1, \phi' = f_1.$$ 

So given any morphism of 2-crossed modules

$$(f_2, f_1, \phi) : \{D_2, D_1, S, \partial_2, \partial_1\} \rightarrow \{B_2, B_1, R, \partial'_2, \partial'_1\}$$

then there is a unique $(f_2, f_1, , id_R)$ 2-crossed module morphism that commutes the following diagram:

$$\begin{CD}
(D_2, D_1, S, \partial_2, \partial_1) @> (f_2, f_1, \phi) >> (B_2, B_1, R, \partial'_2, \partial'_1)
\end{CD}$$

$$(D_2/KD_2, D_1/KD_1, R, \partial_2, \partial_1) \rightarrow (D_2, D_1, S, \partial_2, \partial_1)$$
4 Fibrations and Cofibrations of Categories

The notion of fibration of categories is intended to give a general background to constructions analogous to pullback by a morphism. It seems to be a very useful notion for dealing with hierarchical structures. A functor which forgets the top level of structure is often usefully seen as a fibration or cofibration of categories.

We rewrite from [9] and [22] the definition of fibration and cofibration of categories and some propositions.

Definition 12 Let $\Phi : X \to B$ be a functor. A morphism $\varphi : Y \to X$ in $X$ over $u := \Phi(\varphi)$ is called cartesian if and only if for all $v : K \to J$ in $B$ and $\theta : Z \to X$ with $\Phi(\theta) = uv$ there is a unique morphism $\psi : Z \to Y$ with $\Phi(\psi) = v$ and $\theta = \varphi \psi$.

This is illustrated by the following diagram:

If $Y \to X$ is a cartesian arrow of $X$ mapping to an arrow $\Phi(Y) \to \Phi(X)$ of $B$, we also say that $Y$ is a pullback of $X$ to $\Phi(Y)$.
A morphism \( \alpha : Z \to Y \) is called vertical (with respect to \( \Phi \)) if and only if \( \Phi(\alpha) \) is an identity morphism in \( B \). In particular, for \( I \in B \) we write \( X/I \), called the fibre over \( I \), for the subcategory of \( X \) consisting of those morphisms \( \alpha \) with \( \Phi(\alpha) = id_I \).

**Definition 13** The functor \( \Phi : X \to B \) is a fibration or category fibred over \( B \) if and only if for all \( u : J \to I \) in \( B \) and \( X \in X/I \) there is a cartesian morphism \( \varphi : Y \to X \) over \( u \). Such a \( \varphi \) is called a cartesian lifting of \( X \) along \( u \).

In other words, in a category fibred over \( B \), \( \Phi : X \to B \), we can pull back objects of \( X \) along any arrow of \( B \).

Notice that cartesian liftings of \( X \in X/I \) along \( u : J \to I \) are unique up to vertical isomorphism: given two pullbacks \( \varphi : Y \to X \) and \( \bar{\varphi} : \bar{Y} \to X \) of \( X \) to \( I \), the unique arrow \( \theta : \bar{Y} \to Y \) that fits into the diagram

\[
\begin{array}{ccc}
\bar{Y} & \xrightarrow{\bar{\varphi}} & X \\
\downarrow \theta & & \downarrow \varphi \\
Y & \xrightarrow{\varphi} & X \\
J & \xrightarrow{id} & \downarrow \Rightarrow I
\end{array}
\]

is an isomorphism; the inverse is the arrow \( Y \to \bar{Y} \) obtained by exchanging \( Y \) and \( \bar{Y} \) in the diagram above. In other words, a pullback is unique, up to a unique isomorphism.

The following results in the case of crossed modules of groupoids and commutative algebras have appeared in [9] and [3], respectively.

**Proposition 14** The forgetful functor \( p : X_2\text{Mod} \to k\text{-Alg} \) which sends \( \{C_2, C_1, R, \partial_2, \partial_1\} \mapsto R \) is fibred.

**Proof.** It is enough to consider the pullback construction from proposition 4 to prove that \( p \) is a fibred. Thus the morphism \( (id_{C_2}, \phi', \phi) : \{C_2, \phi^*(C_1), S, \partial_2^*, \partial_1^*\} \to \{C_2, C_1, R, \partial_2, \partial_1\} \) of \( X_2\text{Mod} \) is cartesian. Because for any morphism

\[
(f_2, f_1, \phi) : \{B_2, B_1, S, \partial_2', \partial_1'\} \to \{C_2, C_1, R, \partial_2, \partial_1\}
\]

in \( X_2\text{Mod} \) and any morphism

\[
id_S : p(\{B_2, B_1, S, \partial_2', \partial_1'\}) \to p(\{C_2, \phi^*(C_1), S, \partial_2^*, \partial_1^*\})
\]

in \( k\text{-Alg} \) with \( p(id_{C_2}, \phi', \phi) \circ id_S = p(f_2, f_1, \phi) \), there exists a unique arrow \( (f_2', f_1', id_S) \) with \( p(f_2', f_1', id_S) = id_S \) and \( (id_{C_2}, \phi', \phi) \circ (f_2', f_1', id_S) = (f_2, f_1, \phi) \).
as in the commutative diagram

In considering the functor \( p : \mathcal{X}_2\text{Mod} \to \text{k-Alg} \) as a fibration, if we fix the monomorphism \( \phi : S \to R \) in \( \text{k-Alg} \), the cartesian lifting \( \{ C_2, \phi^\ast(C_1), R, \partial_2, \partial_1 \} \) → \( \{ C_2, C_1, R, \partial_2, \partial_1 \} \) along \( \phi \) for \( \{ C_2, C_1, R, \partial_2, \partial_1 \} \in \mathcal{X}_2\text{Mod}/R \) gives a so-called reindexing functor

\[
\phi^* : \mathcal{X}_2\text{Mod}/R \to \mathcal{X}_2\text{Mod}/S
\]

given in section 2 and defined as objects by \( \{ C_2, C_1, R, \partial_2, \partial_1 \} \mapsto \{ C_2, \phi^\ast(C_1), R, \partial_2, \partial_1 \} \) and the image of a morphism \( \phi^\ast(\alpha, \beta, id_R) = (\alpha, \phi^\ast(\beta), id_S) \) the unique arrow commuting the quadrangles in the following diagram:

We can use this reindexing functor to get an adjoint situation for each monomorphism \( \phi : S \to R \) in \( \text{k-Alg} \).

**Proposition 15** Let \( p : \mathcal{X}_2\text{Mod} \to \text{k-Alg} \) be the forgetful functor with \( p\{ C_2, C_1, R, \partial_2, \partial_1 \} \mapsto R \), monomorphism \( \phi : S \to R \) be in \( \text{k-Alg} \), and \( \phi^* : \mathcal{X}_2\text{Mod}/R \to \mathcal{X}_2\text{Mod}/S \) be reindexing functor, pullback. Then there is a bijection

\[
\mathcal{X}_2\text{Mod}/S(\mathcal{B}, \phi^\ast(C)) \cong \mathcal{X}_2\text{Mod}/\phi(\mathcal{B}, C)
\]

natural in \( \mathcal{B} \in \mathcal{X}_2\text{Mod}/S \), \( C \in \mathcal{X}_2\text{Mod}/R \) where \( \mathcal{B} = \{ B_2, B_1, S, \partial_2, \partial_1 \} \), \( C = \{ C_2, C_1, R, \partial_2, \partial_1 \} \) and \( \mathcal{X}_2\text{Mod}/\phi(\mathcal{B}, C) \) consists of those morphisms \( (f_2, f_1, \phi) \in \mathcal{X}_2\text{Mod}(\mathcal{B}, C) \) with \( p(f_2, f_1, \phi) = \phi \).
Proof. Since \((id_{C_2}, \phi', \phi) : \{C_2, \phi^*(C_1), S, \partial^*_2, \partial^*_1\} \to \{C_2, C_1, R, \partial_2, \partial_1\}\) is a cartesian morphism over \(\phi\) as may be seen with the proof of proposition \(\text{14}\), there is a unique morphism

\[
\Psi : \{B_2, B_1, S, \partial'_2, \partial'_1\} \to \{C_2, \phi^*(C_1), S, \partial^*_2, \partial^*_1\}
\]

with \(p\Psi = id_S\) for \((f_2, f_1, \phi) : \{B_2, B_1, S, \partial'_2, \partial'_1\} \to \{C_2, C_1, R, \partial_2, \partial_1\}\) with \(p((f_2, f_1, \phi)) = \phi\).

On the other hand; it is clear that there is a morphism \(\phi' \circ \gamma : \{B_2, B_1, S, \partial'_2, \partial'_1\} \to \{C_2, C_1, \phi^*(C_1), S, \partial^*_2, \partial^*_1\}\).

For composable monomorphism \(\phi : S \to R\) and \(\phi' : R \to T\), there is a natural equivalence

\[
c_{\phi', \phi} : \phi^* \phi'^* \cong (\phi' \phi)^*
\]

but not equality as shown in the following diagram in which the morphisms \((id_{C_2}, h, \phi), (id_{C_2}, g, \phi)\) and \((id_{C_2}, g', \phi')\) are cartesian and so the composition \((id_{C_2}, g, \phi) \circ (id_{C_2}, g', \phi')\) is cartesian and \((id_{C_2}, k, id_S)\) is the unique vertical morphism from proposition \(\text{14}\) making the diagram commute:

Definition 16 Let \(\Phi : X \to B\) be a functor. A morphism \(\psi : Z \to Y\) in \(X\) over \(v := \Phi(\psi)\) is called cocartesian if and only if for all \(u : J \to I\) in \(B\) and \(\theta : Z \to X\) with \(\Phi(\theta) = uv\) there is a unique morphism \(\varphi : Y \to X\) with
\( \Phi(\varphi) = u \) and \( \theta = \varphi \psi \). This is illustrated by the following diagram:

\[
\begin{array}{c}
\text{\( \Phi(Z) \)} \\
\downarrow \quad \downarrow \\
\Phi(X) \\
\downarrow \quad \downarrow \\
\Phi(Y).
\end{array}
\]

The functor \( \Phi : X \to B \) is a cofibration or category cofibred over \( B \) if and only if for all \( v : K \to J \) in \( B \) and \( Z \in X/K \) there is a cartesian morphism \( \psi : Z \to Z' \) over \( v \). Such a \( \psi \) is called a cartesian lifting of \( Z \) along \( v \).

The cartesian liftings of \( Z \in X/K \) along \( v : K \to J \) are also unique up to vertical isomorphism.

It is interesting to get a characterisation of the cofibration property for a functor that already is a fibration. The following is a useful weakening of the condition for cartesian in the case of a fibration of categories.

**Proposition 17** Let \( \Phi : X \to B \) be a fibration of categories. Then \( \psi : Z \to Y \) in \( X \) over \( v : K \to J \) in \( B \) is cartesian if only if for all \( \theta' : Z \to X' \) over \( v \) there is a unique morphism \( \psi' : Y \to X' \) in \( X/J \) with \( \theta' = \psi' \psi \).

If we take the fibration \( p : X_2 \text{Mod} \to k-\text{Alg} \) and reindexing functor \( \phi^* : X_2 \text{Mod}/R \to X_2 \text{Mod}/S \) for monomorphism \( \phi : S \to R \), we get the morphism

\[
\phi_{\{D_2, D_1, S, \partial_2, \partial_1\}} : \{D_2, D_1, S, \partial_2, \partial_1\} \to \phi_{\{D_2, D_1, S, \partial_2, \partial_1\}}
\]

which is cartesian over \( \phi \), for all \( \{D_2, D_1, S, \partial_2, \partial_1\} \in X_2 \text{Mod}/S \).

Because there is the functor \( \phi_* : X_2 \text{Mod}/S \to X_2 \text{Mod}/R \) which is left adjoint to \( \phi^* \) as mentioned in theorem 10 and also by proposition 17 the adjointness gives the bijection required for cartesian property. Thus \( \phi_{\{D_2, D_1, S, \partial_2, \partial_1\}} \) is cartesian over \( \phi \).

So, by constructing the adjoint \( \phi_* \) of \( \phi^* \) for \( \phi \), the fibration \( p \) is also a cofibration.

## 5 Application: Free 2-Crossed Modules

The definition of a free 2-crossed module is similar in some ways to the corresponding definition of a free crossed module. We recall the definition of a free crossed module and a free 2-crossed module from [4].

Let \( (C, R, \partial) \) be a pre-crossed module, let \( Y \) be a set and let \( \vartheta : Y \to C \) be a function, then \( (C, R, \partial) \) is said to be a free pre-crossed module with basis \( \vartheta \) or, alternatively, on the function \( \partial \vartheta : Y \to R \) if for any pre-crossed module \( (C', R, \partial') \) and function \( \vartheta' : Y \to C' \) such that \( \partial' \vartheta' = \partial \vartheta \), there is a unique morphism

\[
\phi : (C, R, \partial) \to (C', R, \partial')
\]
such that \( \phi \vartheta = \vartheta' \).

Let \( \{C_2, C_1, C_0, \partial_2, \partial_1\} \) be a 2-crossed module, let \( Y \) be a set and let \( \vartheta : Y \to C_2 \) be a function, then \( \{C_2, C_1, C_0, \partial_2, \partial_1\} \) is said to be a free 2-crossed module with basis \( \vartheta \) or, alternatively, on the function \( \partial_2 \vartheta : Y \to C_1 \), if for any 2-crossed module \( \{C'_2, C_1, C_0, \partial'_2, \partial_1\} \) and function \( \vartheta' : Y \to C'_2 \) such that \( \partial_2 \vartheta = \partial'_2 \vartheta' \), there is a unique morphism \( \Phi : C_2 \to C'_2 \) such that \( \partial'_2 \Phi = \partial_2 \).

The following proposition is an application of induced 2-crossed modules using the universal properties of free 2-crossed modules and of universal morphism of 2-crossed modules.

**Proposition 18** Suppose \( \phi : S \to R \) is a \( k \)-algebra morphism, then the 2-crossed module \( \{C_2, C_1, R, \partial_2, \partial_1\} \) is the free 2-crossed module on \( \partial_2 \vartheta \) with the morphism \( \vartheta : Y \to C_2 \) if and only if the morphism \( (\vartheta, \partial_2 \vartheta, \phi) : \{Y, Y, S, id_Y, 0\} \to \{C_2, C_1, R, \partial_2, \partial_1\} \) of 2-crossed modules is a universal morphism, i.e. for any 2-crossed module morphism

\[
(\vartheta', \partial_2 \vartheta, \phi) : \{Y, Y, S, id_Y, 0\} \to \{C'_2, C_1, R, \partial'_2, \partial_1\}
\]

there exists a unique \((\Phi, id_{C_1}, id_R)\) 2-crossed module morphism such that \( \Phi \vartheta = \vartheta' \).

**Proof.** Suppose \((\vartheta, \partial_2 \vartheta, \phi)\) is a universal morphism of 2-crossed modules. Let

\[
(\vartheta', \partial_2 \vartheta, \phi) : \{Y, Y, S, id_Y, 0\} \to \{C'_2, C_1, R, \partial'_2, \partial_1\}
\]

be a 2-crossed module morphism. Then there exists a unique

\[
(\Phi, id_{C_1}, id_R) : \{C_2, C_1, R, \partial_2, \partial_1\} \to \{C'_2, C_1, R, \partial'_2, \partial_1\}
\]

2-crossed module morphism such that \( \Phi \vartheta = \vartheta' \). This description gives the required free 2-crossed module \( \{C_2, C_1, R, \partial_2, \partial_1\} \) on \( \partial_2 \vartheta \).

On the other hand, let \( \{C_2, C_1, R, \partial_2, \partial_1\} \) be a free 2-crossed module on \( \partial_2 \vartheta \), \( \{C'_2, C_1, R, \partial'_2, \partial_1\} \) be a 2-crossed module and \( \vartheta' : Y \to C'_2 \) be an algebra morphism such that \( \partial_2 \vartheta = \partial'_2 \vartheta' \). Then by using the universal property of a free
2-crossed module, we get a unique morphism $\Phi : C_2 \to C_2'$ such that $\partial_2'\Phi = \partial_2$.

This proves that $$(\vartheta, \partial_2\vartheta, \phi) : \{Y, Y, S, \text{id}_Y, 0\} \to \{C_2, C_1, R, \partial_2, \partial_1\}$$
is a universal morphism of 2-crossed modules.

![Diagram](image-url)

This proposition leads to link free 2-crossed modules with induced 2-crossed modules.

Given any $k$-algebra morphism $\phi : S \to R$, we note that $Y \xrightarrow{id_Y} Y \xrightarrow{0} S$ is a 2-crossed module and form the induced 2-crossed module $\{\phi_*(Y), \phi_*(Y), R, \text{id}_{Y*}, 0_*\}$ as described in section 3.

Proposition 18 implies that the free 2-crossed module on the morphism $Y \to \phi_*(Y)$ is the induced 2-crossed module on $\phi$.

Note that the definition of free 2-crossed modules has been chosen to tie in with the definition of freeness given in other more general context.

**Proposition 19** Let $\{C_2, C_1, C_0, \partial_2, \partial_1\}$ be a free 2-crossed module on $f$ and $f = 0$, then $C_2$ is a free $C_1$-module on $f$.

**Proof.** Given any free 2-crossed module $\{C_2, C_1, C_0, \partial_2, \partial_1\}$ on $f$, then $(C_2, C_1, \partial_2)$ is a free crossed module on $f$. Since $(C_2, C_1, \partial_2)$ is free on $f$, then there exists a function $\vartheta : Y \to C_2$ such that $\partial_2\vartheta = f = 0$. Let $C'_2$ be a $C_1$-module and $\vartheta' : Y \to C'_2$ be a function into $C'_2$. Form the crossed module $(C'_2, C_1, 0)$. Since $\partial_2\vartheta = f = 0 = 0\vartheta'$, thus there is a unique morphism $\Phi : C_2 \to C'_2$ such that $\Phi\vartheta = \vartheta'$. Therefore $C_2$ is free $C_1$-module on $f$. ■
6 Appendix

The proof of proposition[4]

PL3:
\[
\{ (c_1, s), (c'_1, s') (c''_1, s'') \} = \{ (c_1, s), (c'_1 s''_1, s') \} = \{ c_1, c'_1 \} = \{ c_1 c'_1, c''_1 \} + \partial_1 (c''_1) \cdot \{ c_1, c'_1 \} = \{ c_1 c'_1, c''_1 \} + \phi (s'') \cdot \{ c_1, c'_1 \} = \{ c_1, c'_1 \} + s'' \cdot \{ c_1, c'_1 \} = \{ c_1 c'_1, c''_1 \} + \partial_1 (c''_1, s'') \cdot \{ c_1, c'_1 \} = \{ (c_1 c'_1, ss'), (c''_1, s'') \} + \partial_1 (c''_1, s'') \cdot \{ (c_1, s), (c'_1, s') \} = \{ (c_1, s) (c'_1, s'), (c''_1, s'') \} + \partial_1 (c''_1, s'') \cdot \{ (c_1, s), (c'_1, s') \}.
\]

PL4:
\[
\{ (c_1, s), \partial_2^* (c_2) \} = \{ \partial_1 (c_2), 0 \} + \{ (c_1, s), 0 \} = \{ c_1, \partial_2 (c_2), c_1 \} = \{ c_1, \partial_2 (c_2) \} + \{ \partial_2 (c_2), c_1 \} = \partial_1 (c_1) \cdot c_2 = \phi (s) \cdot c_2 = s \cdot c_2 = \partial_1^*(c_1, s) \cdot c_2.
\]

PL5:
\[
\{ (c_1, s) \cdot s'', (c'_1, s') \} = \{ (c_1 \cdot \phi(s''), s''), (c'_1, s') \} = \{ c_1, \phi(s'') \}, c'_1 \} = \{ c_1, c'_1 \} \cdot \phi(s'') = \{ c_1, c'_1 \} = s'' \cdot \{ (c_1, s), (c'_1, s') \} \cdot s''.
\]

\[
\{ (c_1, s), (c'_1, s') \cdot s'' \} = \{ (c_1, s) (c'_1 \cdot \phi(s''), s's'') \} = \{ c_1, c'_1 \cdot \phi(s'') \} = \{ c_1, c'_1 \} \cdot \phi(s'') = \{ (c_1, s), (c'_1, s') \} \cdot s''
\]

for all \((c_1, s), (c'_1, s'), (c''_1, s'') \in \phi^*(C_1), c_2 \in C_2 \) and \( s'' \in S \).

Let us check that \( \{ id_{C_2}, \phi, \phi : \{ C_2, \phi^*(C_1), S, \partial_2^*, \partial_1^* \} \rightarrow \{ C_2, C_1, R, \partial_2, \partial_1 \} \) is a morphism of 2-crossed modules where \( \phi'(c_1, s) = c_1 \).

\[
id_{C_2} (s \cdot c_2) = s \cdot c_2 = \phi(s) \cdot c_2 = \phi(s) \cdot id_{C_2}(c_2)
\]

and similarly \( \phi'(s \cdot (c_1, s')) = \phi(s) \cdot \phi'(c_1, s') \)

\[
(\phi' \partial_2^*) (c_2) = \phi' (\partial_2 (c_2), 0) = \partial_2 (c_2) = \partial_2 id_{C_2} (c_2)
\]
and $\phi \partial_1^* = \partial_1 \phi'$ for all $(c_1, s') \in \phi^* (C_1), c_2 \in C_2$ and $s \in S$.

$$id_{C_2} \{ - , - \} ((c_1, s), (c'_1, s')) = id_{C_2} \{ ((c_1, s), (c'_1, s')) \}$$

$$= \{ c_1, c'_1 \}$$

$$= \{ - , - \} (c_1, c'_1)$$

$$= \{ - , - \} (\phi' (c_1, s), \phi' (c'_1, s'))$$

$$= \{ - , - \} (\phi' \times \phi') ((c_1, s), (c'_1, s'))$$

for all $(c_1, s), (c'_1, s') \in \phi^* (C_1)$.

### The proof of proposition

#### PL3:

$$\{ (d_1, r), (d'_1, r') (d''_1, r'') \}$$

$$= \{ (d_1, r), (d'_1, r'), (d''_1, r'') \}$$

$$= \{ (d_1, d'_1, r''), (r r' r'') \}$$

$$= \{ (d_1, d'_1, r''), r (r' r'') \}$$

$$= \{ \phi (d_1, d'_1) r', \phi (d''_1) r'' \}$$

#### PL4:

$$\{ (d_1, r), \partial_2 (d_2, r') \}$$

$$= \{ (d_1, r), (d_2, r'), (d_1, r) \}$$

$$= \{ (d_1, d_2, r'), (r r') \}$$

$$= \{ (d_1, d_2), r' \}$$

$$= \{ (d_1, r), (d_2, r') \}$$

$$= \phi (d_1) r'$$

$$= \partial_1 (d_1, r) \cdot (d_2, r').$$

#### PL5:

$$\{ (d_1, r) \cdot r'', (d'_1, r') \} = \{ (d_1, r), (d'_1, r') \}$$

$$= \{ (d_1, d'_1), r''r' \}$$

$$= \{ (d_1, d'_1), r''r' \}$$

$$= \{ (d_1, r), (d'_1, r') \} \cdot r''.$$

$$\{ (d_1, r), (d'_1, r') \cdot r'' \} = \{ (d_1, r), (d'_1, r'') \}$$

$$= \{ (d_1, d'_1), r'r'' \}$$

$$= \{ (d_1, d'_1), r'r'' \}$$

$$= \{ (d_1, r), (d'_1, r') \} \cdot r''$$

for all $(d_1, r), (d'_1, r'), (d''_1, r'') \in \phi_4 (D_1), (d_2, r') \in \phi_4 (D_2)$ and $r'' \in R$. 

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The proof of proposition 11

PL3:

\[
\{d_1 + K D_1, (d_1' + K D_1) (d_1'' + K D_1)\} \\
= \{d_1 + K D_1, (d_1', d_1'') + K D_2\} \\
= \{(d_1 d_1', d_1'') + \partial_1 (d_1'') \cdot \{d_1, d_1'\} + K D_2\} \\
= \{(d_1 d_1', d_1'') + K D_2 + (\partial_1 (d_1'') + K) \cdot \{d_1, d_1'\} + K D_2\} \\
= \{(d_1 d_1', d_1'') + K D_2 + \partial_1 (d_1'') \cdot \{d_1, d_1'\} + K D_2\} \\
= \{(d_1 d_1', d_1'') + K D_2 + \partial_1 (d_1'') \cdot \{d_1 + K D_1, d_1' + K D_1\} \\
= \{(d_1 + K D_1) (d_1', d_1'') + K D_2 + \partial_1 (d_1'') \cdot \{d_1 + K D_1, d_1' + K D_1\}\}
\]

PL4:

\[
\{d_1 + K D_1, \partial_2, (d_2 + K D_2)\} + \{\partial_2, (d_2 + K D_2), d_1 + K D_1\} \\
= \{(d_1, \partial_2 (d_2) + K D_1) + \{\partial_2 (d_2), d_1 + K D_1\} + K D_2\} \\
= \{(d_1, \partial_2 (d_2)) + K D_2 + \{\partial_2 (d_2), d_1 + K D_2\}\} \\
= \{(d_1, d_2) + K D_2\} \\
= \{(d_1 + K D_1) \cdot (d_2 + K D_2)\} \\
= \{(d_1 + K D_1) \cdot (d_2 + K D_2)\}.
\]

PL5:

\[
\{d_1 + K D_1, d_1' + K D_1\} \cdot r = \{d_1 + K D_1, d_1' + K D_1\} \cdot (s + K) \\
= \{(d_1, d_1') + K D_2\} \cdot (s + K) \\
= \{(d_1, d_1') \cdot s + K D_2\} \\
= \{(d_1 \cdot s, d_1') + K D_2\} \\
= \{(d_1 + K D_1) \cdot (s + K), d_1' + K D_1\} \\
= \{(d_1 + K D_1) \cdot r, d_1' + K D_1\}.
\]

\[
\{d_1 + K D_1, d_1' + K D_1\} \cdot r = \{d_1 + K D_1, d_1' + K D_1\} \cdot (s + K) \\
= \{(d_1, d_1') + K D_2\} \cdot (s + K) \\
= \{(d_1, d_1') \cdot s + K D_2\} \\
= \{(d_1, d_1' \cdot s) + K D_2\} \\
= \{d_1 + K D_1, (d_1' \cdot s) + K D_1\} \\
= \{d_1 + K D_1, (d_1' + K D_1) \cdot (s + K)\} \\
= \{d_1 + K D_1, (d_1' + K D_1) \cdot r\}.
\]

for all \(d_1 + K D_1, d_1' + K D_1, d_1'' + K D_1 \in D_1/KD_1, d_2 + K D_2 \in D_2/KD_2, r \in R\) and \(s + K \in S/K\).

Let us check that \((\phi'', \phi', \phi) : \{D_2, D_1, S, \partial_2, \partial_1\} \rightarrow \{D_2/KD_2, D_1/KD_1, R, \partial_2, \partial_1\}\) where \(\phi'' (d_2) = d_2 + K D_2\) and \(\phi'' (d_1) = d_1 + K D_1\) is a morphism of 2-crossed modules.
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\[
\begin{array}{ccc}
D_2 & \xrightarrow{\partial_2} & D_1 \\
\downarrow{\phi''} & & \downarrow{\phi'} \\
D_2/KD_2 & \xrightarrow{\partial_2} & D_1/KD_1 \\
\end{array}
\]

\[
\phi''(s \cdot d_2) = (s \cdot d_2) + KD_2 \\
\phi'(s \cdot d_1) = (s \cdot d_1) + KD_1 \\
\phi(s) \cdot \phi''(d_2) = (s + K) \cdot (d_2 + KD_2) \\
\phi'(s \cdot d_1) = (s + K) \cdot (d_1 + KD_1) \\
\partial_2, (\phi''(d_2)) = \partial_2, (d_2 + KD_2) \\
= \partial_2(d_2) + KD_1 \\
= \phi'(\partial_2(d_2))
\]

similarly \(\partial_1, \phi' = \phi \partial_1\) for all \(d_1 \in D_1, d_2 \in D_2\) and \(s \in S\).

\[
\{\{\cdot, \cdot\}\} (\phi' \times \phi') (d_1, d'_1) = \{\{\cdot, \cdot\}\} (d_1 + KD_1, d'_1 + KD_1) \\
= \{d_1, d'_1\} + KD_2 \\
= \phi''(\{d_1, d'_1\}) \\
= (\phi''\{\cdot, \cdot\}\} (d_1, d'_1).
\]

for all \(d_1, d'_1 \in D_1\).

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