Generating functions for weighted Hurwitz numbers

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Abstract

Weighted double Hurwitz numbers for \(n\)-sheeted, fixed genus branched coverings of the Riemann sphere are introduced, together with associated weighted paths in the Cayley graph of \(S_n\) generated by transpositions. The path weights are determined by a weight generating function, and depend only on their signature. The associated generating functions for such weighted Hurwitz numbers are shown to consist of 2D Toda \(\tau\)-functions of generalized hypergeometric type. Both the enumerative geometric and the combinatorial significance of these are derived. Four classical cases are detailed, in which the weighting is uniform: the classical double Hurwitz numbers, in which the ramification is simple at all but two specified branch points; the case of Belyi curves, with three branch points, two having fixed profiles, the third constrained only by the genus; the general case, with any specified number of branch points, two with fixed profiles, the rest again constrained only by the genus; and the signed enumeration case, with sign determined by the parity of the number of branch points. Using the exponentiated quantum dilogarithm function as weight generator, three new types of weighted enumerations are introduced. These determine \(q\)-deformed, quantum Hurwitz numbers, in which the branching profiles may be viewed as random variables. By suitable interpretation of the parameter \(q\), the statistical mechanics of such quantum weighted branched covers may be related to that of Bosonic gases. The standard double Hurwitz numbers are recovered in the classical limit.

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1 Introduction

In [8, 10], a method was developed for constructing parametric families of 2D Toda \( \tau \)-functions [20, 22, 21] of hypergeometric type [19] that serve as generating functions for Hurwitz numbers, counting various configurations of branched coverings of the Riemann sphere. A natural combinatorial construction was shown to yield an equivalent interpretation in terms of path-counting in the Cayley graph of the symmetric group \( S_n \) generated by all transpositions. All previously known cases were placed within this framework and several new examples were interpreted both from the enumerative geometric and the combinatorial viewpoint.

In the present work, this approach is extended to include general 1-parameter families of 2D Toda \( \tau \)-functions of hypergeometric type, determined by an associated weight generating
function $G(z)$. These $\tau$-functions may all be interpreted as generating functions for various types of weighted enumerations of branched covers of the Riemann sphere or, equivalently, paths in the Cayley graph. By suitably choosing $G(z)$, it is straightforward to recover all cases previously studied [18, 6, 8, 10, 23, 2, 1, 12] and add an infinite variety of new ones. In particular, the counting of Belyi coverings [23, 12, 2] of fixed genus, with two branch points of fixed ramification type, plus a third whose profile length is determined by the genus, was shown in [8, 10] to correspond to counting strictly monotonic paths in the Cayley graph, while the signed counting of branched covers, again with two ramification profiles specified plus an arbitrary number of further branch points constrained again by the Riemann-Hurwitz formula to provide fixed genus, was shown to correspond to counting weakly monotonic paths [6, 8, 10].

A further class of examples of special interest appears when the quantum dilogarithm function is used in the definition of the weight generating function $G(z)$. This leads to the notion of $q$-deformed, or quantum Hurwitz numbers. In sections 3 and 4, three variants are studied, which may be seen as $q$-deformations of the previously considered generating functions for strictly and weakly monotonic path counting. The classical limit is shown to reproduce the double Hurwitz numbers $\text{Cov}_{d}(\mu, \nu)$ introduced by Okounkov [18], which count branched covers having a pair of branch points with fixed ramification profiles $\mu$ and $\nu$ and a specified number of additional simple branch points.

In the general setting, the number of branch points may be viewed as a random variable, as can the Hurwitz numbers themselves. In the case of quantum Hurwitz numbers, the state space is identifiable with that of a Bosonic gas with linearly spaced energy eigenvalues and fixed total energy. If the energy is taken as proportional to the degree of degeneration of the covering over the various branch points, fixing the total energy corresponds to fixing the genus of the covering curve or, equivalently, the number of steps in the Cayley graph.

The relation between 2D Toda $\tau$-functions and weighted paths in the Cayley graph is $S_n$ is derived in In Section 2. The main result, showing the $\tau$-function to be a generating function for the weighted number $\tilde{F}_G^{d}(\mu, \nu)$ of $d$-step paths from the conjugacy class of cycle type $\mu$ to that of type $\nu$ is given in Theorem 2.4.

Section 3 deals with examples, showing how four previously studied classical cases may be recovered within the general approach, and introducing the three new examples in which the generating function $G(z)$ is defined in terms of the quantum dilogarithm function, leading to weighted paths involving the quantum deformation parameter $q$.

In Section 4, the weighted Hurwitz numbers for the $q$-deformed cases are interpreted as expectation values of Hurwitz numbers. Theorems 4.1, 4.3 and 4.5 give the detailed form of the generating functions for coverings with fixed genus, and a variable number of additional branch points counted either with positive weight factors, or with signed factors determined by the parity of the number of branch points. The quantum weight for any configuration of branch points may be related to the energy distribution function in a quantum Bose
gas with energy spectrum linear in the integers, if the energy is viewed as proportional
to the degeneracy of the covering; i.e., the sum of the complements $\ell^*(\mu) = |\mu| - \ell(\mu)$ of the
ramification profile lengths. By the Riemann-Hurwitz formula, fixing the total energy is thus
equivalent to fixing the genus of the covering curve.

2 Hypergeometric $\tau$-functions and weighted path enumeration

2.1 Weight generating functions, Jucys-Murphy elements and content products

Let

$$G(z) = \sum_{k=0}^{\infty} G_k z^k. \quad (2.1)$$

be the Taylor series of a complex analytic function in a neighbourhood of the origin, with
$G(0) = 1$. This will be referred to as the weight generating function since, in the following,
it will be used to define weights associated with paths in the Cayley graph of $S_n$ generated
by all transpositions, as well as with the ramification structure of branched covers of the
Riemann sphere. Developing further on the methods introduced in [8, 10], we will show
how to use such functions to construct 2D Toda $\tau$-functions [20, 22, 21] that are, in general,
generating functions for weighted Hurwitz numbers and also for associated weighted paths
in the Cayley graph. We refer the reader to [8, 10], for further details of the construction,
notation and additional examples.

Let $(ab) \in S_n$ denote the transposition interchanging the elements $a$ and $b$, and

$$\mathcal{J}_b := \sum_{a=1}^{b-1} (ab), \quad b = 1, \ldots, n \quad (2.2)$$

the Jucys-Murphy elements [13, 16, 3] of the group algebra $C[S_n]$, which generate a maximal
commutative subalgebra. We associate a 1-parameter family of elements $G(z, \mathcal{J})$ of the
center of the group algebra $\mathbb{Z}(C[S_n])$ by forming the product

$$G(z, \mathcal{J}) := \prod_{a=1}^{n} G(z \mathcal{J}_a). \quad (2.3)$$

Under multiplication, such elements determine endomorphisms of $\mathbb{Z}(C[S_n])$ that are diagonal
in the basis $\{F_\lambda\}$ of orthogonal idempotents, say

$$G(z, \mathcal{J}) F_\lambda = r_\lambda^G(z) F_\lambda, \quad (2.4)$$
where the eigenvalue is given by the parametric family of content product formulae [19, 8, 10],

\[ r_{\lambda}^{G(z)} := \prod_{(i,j) \in \lambda} G(z(j - i)) \]  

(2.5)
taken over the coordinates contained in the Young diagram of the partition \( \lambda \) of weight \( |\lambda| = n \).

Ref. [8] shows how to use such elements to define parametric families of 2D Toda \( \tau \)-functions of hypergeometric type [19] that serve as generating functions for combinatorial invariants enumerating certain paths in the Cayley graph of \( S_n \) generated by all transpositions. These may be expanded as a series summed over diagonal products of Schur functions,

\[ \tau^{G(z)}(\mathbf{t}, \mathbf{s}) = \sum_{\lambda} r_{\lambda}^{G(z)} S_{\lambda}(\mathbf{t}) S_{\lambda}(\mathbf{s}), \]  

(2.6)

where

\[ \mathbf{t} = (t_1, t_2, \ldots), \quad \mathbf{t} = (s_1, s_2, \ldots) \]  

(2.7)

are the 2D Toda flow variables, which may be identified in this notation with the power sum symmetric functions

\[ t_i = \frac{p_i}{i}, \quad s_i = \frac{p'_i}{i} \]  

(2.8)
in two independent sets of variables. (See [15] for notation and further definitions involving symmetric functions.)

**Remark 2.1.** No dependence on the lattice site \( N \in \mathbb{Z} \) is indicated in (2.6), since in the examples considered below only \( N = 0 \) is required. The \( N \) dependence is introduced in a standard way [19, 9], by replacing the factor \( G(z(j - i)) \) in the content product formula (2.5) by \( G(z(N + j - i)) \), and multiplying by an overall \( \lambda \)-independent factor \( r_G^{0(z)} \). This produces a lattice of 2D Toda \( \tau \)-functions \( \tau^{G(N, \mathbf{t}, \mathbf{s})} \) which, for all the cases considered below, may be explicitly expressed in terms of \( \tau^{G(0, \mathbf{t}, \mathbf{s})} : = \tau^{G(\mathbf{t}, \mathbf{s})} \) by applying a suitable transformation of the parameters involved [18, 10], and an explicit multiplicative factor depending only on \( N \).

Substituting the Frobenius character formula

\[ S_{\lambda} = \sum_{\mu, |\mu| = |\lambda|} Z_{\mu}^{-1} \chi_{\lambda}(\mu) P_{\mu}, \]  

(2.9)

into (2.6), where \( \chi_{\lambda}(\mu) \) is the character of the irreducible representation of type \( \lambda \) evaluated on the conjugacy class of type \( \mu \) and

\[ Z_{\mu} = \prod_{i=1}^{|\mu|} i^{m_i} (m_i)!, \quad m_i = \text{number of parts of } \mu \text{ equal to } i \]  

(2.10)
is the size of the stabilizer of the conjugacy class, we obtain an equivalent expansion in terms of products of power sum symmetric functions and a power series in $z$,

$$
\tau^G(z)(t, s) = \sum_{d=0}^{\infty} \sum_{\mu, \nu \mid |\mu| = |\nu|} F^d_G(\mu, \nu) P_\mu(t) P_\nu(s) z^d.
$$

The coefficients $F^d_G(\mu, \nu)$ will be interpreted in Theorem 2.4 below as weighted enumerations of paths in the Cayley graph starting in the conjugacy class of cycle type $\mu$ and ending in the class of type $\nu$. The geometric interpretation of $F^d_G(\mu, \nu)$ will also be given, in sections 3.1 and 3.2 below, as weighted Hurwitz numbers for enumerations of $n = |\mu| = |\nu|$ sheeted branched covers of the Riemann sphere. Various examples of such generating functions were studied in [6, 8, 10, 13, 15, 19], including cases that involve dependence on multiparameter families constructed in a multiplicative way.

### 2.2 Fermionic representation

Double KP $\tau$-functions of the form (2.6) also have a Fermionic representation [19, 8, 9, 10]

$$
\tau^G(t, s) = \langle 0 | \hat{\gamma}^+(t) \hat{C}_G \hat{\gamma}^-(s) | 0 \rangle
$$

where the Fermionic operator $\hat{C}_G$, $\hat{\gamma}^+(t)$ and $\hat{\gamma}^-(s)$ are defined by

$$
\hat{C}_G = e^{\sum_{j=-\infty}^{\infty} T^G_j(z) \psi_j \psi_j^\dagger}, \quad \hat{\gamma}^+(t) = e^{\sum_{i=1}^{\infty} t_i J_i}, \quad \hat{\gamma}^-(s) = e^{\sum_{i=1}^{\infty} s_i J_i}, \quad J_i = \sum_{k \in \mathbb{Z}} \psi_k \psi^\dagger_{k+i}, \quad i \in \mathbb{Z},
$$

in terms of the Fermionic creation and annihilation operators $\{\psi_i, \psi^\dagger_i\}_{i \in \mathbb{Z}}$, acting on the Fermionic Fock space,

$$
\mathcal{F} = \bigoplus_{N \in \mathbb{Z}} \mathcal{F}_N \quad (N = \text{vacuum charge}),
$$

satisfying the usual anticommutation relations

$$
[\psi_i, \psi_j^\dagger]_+ = \delta_{ij}
$$

and vacuum state $|0\rangle$ vanishing conditions

$$
\psi_i |0\rangle = 0, \quad \text{for } i < 0, \quad \psi_i^\dagger |0\rangle = 0, \quad \text{for } i \geq 0,
$$

and the parameters $T^G_j(z)$ are defined by

$$
T^G_j(z) = \sum_{k=1}^{j} \ln G(zk), \quad T^G_0(z) = 0, \quad T^G_{-j}(z) = -\sum_{k=0}^{j-1} \ln G(-zk) \quad \text{for } j > 0.
$$
This follows from the fact that $\hat{C}_G$ is diagonal in the basis $\{|\lambda; N\}^\ast$

$$\hat{C}_G|\lambda; N\rangle = r_\lambda^G(N)|\lambda; N\rangle \tag{2.19}$$

with eigenvalues

$$r_\lambda^G(N) := r_0^G(N) \prod_{(i,j) \in \lambda} G(z(N + j - i)), \tag{2.20}$$

$$r_0(N) = \prod_{j=1}^{N-1} G((N - j)z)^j, \quad r_0(0) = 1, \quad r_0(-N) = \prod_{j=1}^N G((j - N)z)^{-j}, \quad N > 1. \tag{2.21}$$

Eq. (2.19) means that the the map

$$\mathfrak{F}: \bigoplus_{n \geq 0} \mathcal{Z}(\mathbb{C}[S_n]) \rightarrow \mathcal{F}_0$$

$$\mathfrak{F}: F_\lambda \mapsto \frac{1}{h_\lambda}|\lambda; 0\rangle \tag{2.22}$$

where $h_\lambda$ is the product of the hook lengths of the partition $\lambda$ intertwines the action of the abelian group of elements of the form $\hat{C}_G$ on $\mathcal{F}_0$ with the action of the group of elements $G(z, J) \in \mathcal{Z}(C(S_n))$ by multiplication on the direct sum of the centers $\mathcal{Z}(C[S_n])$ of the $S_n$ group algebras [8].

More generally, using the charge $N$ vacuum state

$$|N\rangle = \psi_{N-1} \cdots \psi_0 |0\rangle, \quad |-N\rangle = \psi^\dagger_{N-1} \cdots \psi^\dagger_0 |0\rangle, \quad N \in \mathbb{N}^+, \tag{2.23}$$

we may define a 2D Toda lattice of $\tau$-functions by

$$\tau^G(N, t, s) = \langle N|\hat{\gamma}_+^N(t)\hat{C}_G\hat{\gamma}_-^N(s)|N\rangle \tag{2.24}$$

$$= \sum_\lambda r_\lambda^G(N)S_\lambda(t)S_\lambda(s). \tag{2.25}$$

These satisfy the infinite set of Hirota bilinear equations for the 2D Toda lattice hierarchy [22, 20, 21].

### 2.3 Weighted paths in the Cayley graph

For any partition $\lambda = (\lambda_1 \geq \cdots \lambda_{\ell(\lambda)} > 0)$, let

$$M_\lambda(J) = \sum_{b_1, \ldots, b_{\ell(\lambda)}} J_{b_1}^{\lambda_1} \cdots J_{b_{\ell(\lambda)}}^{\lambda_{\ell(\lambda)}}, \tag{2.26}$$

be the monomial sum symmetric function evaluated on the Jucys-Murphy elements, where the sum is over all sequences $(b_1, \ldots, b_{\ell(\lambda)})$ of distinct $b_i$’s, $1 \leq b_i \leq n$; that is, all distinct monomials in the elements $J_1, \ldots, J_n$ whose exponents are $\lambda_1, \ldots, \lambda_{\ell(\lambda)}$ in some order.
Lemma 2.1. For any weight generating function $G(z)$, we have the following expansion for $G(z, J)$:

$$G(z, J) = \sum_{\lambda} G_\lambda M_\lambda(J) z^{\lambda|},$$

where

$$G_\lambda := \prod_{i \geq 1} (G_i)^{m_i} = \prod_{i=1}^{\ell(\lambda)} G_{\lambda_i},$$

with $m_i$ the number of parts of $\lambda$ equal to $i$.

Proof.

$$G(z, J) = \prod_{a=1}^{n} \left( \sum_{k=0}^{\infty} G_k z^k J_a^k \right)$$
$$= \left( \sum_{k_1=0}^{\infty} G_{k_1} z^{k_1} J_1^{k_1} \right) \cdots \left( \sum_{k_n=0}^{\infty} G_{k_n} z^{k_n} J_n^{k_n} \right)$$
$$= \sum_{d=0}^{\infty} z^d \sum_{\lambda, |\lambda|=d} \sum_{b_1, \ldots, b_{\ell(\lambda)}} \left( \prod_{i=1}^{\ell(\lambda)} G_{\lambda_i} J_{b_i}^{\lambda_i} \right)$$
$$= \sum_{\lambda} G_\lambda M_\lambda(J) z^{\lambda|}.$$  

(2.29)

Let cyc($\mu$) $\subset S_n$ denote the conjugacy class consisting of elements with cycle lengths equal to the parts $\mu_i$ of the partition $\mu$. The number of elements in cyc($\mu$) is:

$$|\text{cyc}(\mu)| = \frac{|\mu|!}{Z_\mu}.$$  

(2.30)

Definition 2.1. A $d$-step path in the Cayley graph of $S_n$ (generated by all transpositions) is an ordered sequence

$$(h, (a_1 b_1)h, (a_2 b_2)(a_1 b_1)h, \ldots, (a_d b_d) \cdots (a_1 b_1)h)$$

(2.31)

of $d + 1$ elements of $S_n$, where consecutive elements differ by composition on the left with a transposition $(a_i b_i)$. This path is said to start at the permutation $h$ and stop at the permutation $g := (a_d b_d) \cdots (a_1 b_1)h$. If $h \in \text{cyc}(\mu)$ and $g \in \text{cyc}(\nu)$, the path is also said to be going from cyc($\mu$) to cyc($\nu$).

By convention, the transpositions $(a_i b_i)$ in (2.31) are written with $a_i < b_i$. 

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Definition 2.2. If the sequence $b_1, b_2, \ldots, b_d$ is weakly/strictly increasing, then the path is said to be weakly/strictly monotonic.

Definition 2.3. The signature of the path (2.31) is the partition $\lambda$ of weight $|\lambda| = d$ obtained by counting how many times each particular number $b_i$ appears in the sequence $b_1, b_2, \ldots, b_d$ and ordering these counts in weakly decreasing order.

Let $\{C_\mu\}$ denote the basis of the center $\mathcal{Z}(C[S_n])$ of the group algebra $C[S_n]$ consisting of the sums over the elements of $\text{cyc}(\mu)$

$$C_\mu = \sum_{g \in \text{cyc}(\mu)} g. \quad (2.32)$$

Lemma 2.2. Multiplication by $M_\lambda(J)$ defines an endomorphism of $\mathcal{Z}(C[S_n])$ which, expressed in the $\{C_\mu\}$ basis, is given by

$$M_\lambda(J)C_\mu = \sum_{|\nu| = |\mu|} m^\lambda_{\mu\nu} \frac{Z_\nu}{|\nu|!} C_\nu, \quad (2.33)$$

where $m^\lambda_{\mu\nu}$ is the number of monotonic $|\lambda|$-step paths in the Cayley graph of $S_n$ from $\text{cyc}\mu$ to $\text{cyc}\nu$ with signature $\lambda$.

Remark 2.2. Note that in the expansion above, for the coefficients $m^\lambda_{\mu\nu}$, we must have $|\mu| = |\nu|$, but there is no restriction on $|\lambda|$.

Remark 2.3. The enumerative constants $m^\lambda_{\mu\nu}$ may be interpreted in another way, that is perhaps more natural, since it puts no restrictions on the monotonicity of the path. It is a fact that any path in the Cayley graph can be associated to a unique monotonic path with the same starting and stopping points and the same signature. By counting the number of distinct rearrangements of a sequence $b_1, b_2, \ldots, b_d$ with signature $\lambda$, it follows that the number of not necessarily monotonic $|\lambda|$-step paths in the Cayley graph from $\text{cyc}\mu$ to $\text{cyc}\nu$ with signature $\lambda$ is related to its monotonic counterpart by

$$\tilde{m}^\lambda_{\mu\nu} := \frac{|\lambda|!}{\prod_{i=1}^{\ell(\lambda)} \lambda_i!} m^\lambda_{\mu\nu}. \quad (2.34)$$

Assign weight

$$\tilde{G}_\lambda := \left(\prod_{i=1}^{\ell(\lambda)} \lambda_i! \right) G_\lambda$$

(2.35)

to any such path of signature $\lambda$ and $|\lambda| = d$. Then

$$\tilde{F}^d_G(\mu, \nu) := \sum_{\lambda, |\lambda| = d} \tilde{G}_\lambda \tilde{m}^\lambda_{\mu\nu} = d! F^d_G(\mu, \nu) \quad (2.36)$$
is the weighted sum over all $d$-step paths, where

$$F^d_G(\mu, \nu) := \frac{1}{|\nu|!} \sum_{\lambda, |\lambda|=d} G_\lambda m_{\mu\nu}^{\lambda}. \tag{2.37}$$

It then follows from lemmas 2.1 and 2.2 that

**Proposition 2.3.**

$$G(z, \mathcal{J})C_\mu = \sum_{d=0}^{\infty} z^d \sum_{\nu, |\nu|=|\mu|} F^d_G(\mu, \nu) Z_\nu C_\nu, \tag{2.38}$$

### 2.4 2D Toda $\tau$-functions $\tau^{G(z)}(t, s)$ as generating functions for weighted paths

For each choice of $G(z)$, we define a corresponding 2D Toda $\tau$-function of generalized hypergeometric type (for $N = 0$) by the formal series (2.6). It follows from general considerations [21, 19, 9] that this is indeed a double KP $\tau$-function that, when extended suitably to a lattice $\tau^{G(z)}(N, t, s)$ of such $\tau$ functions, satisfies the corresponding system of Hirota bilinear equations of the 2D Toda hierarchy [20, 22, 21].

Substituting the Frobenius character formula (2.9) for each of the factors $S_\lambda(t)S_\lambda(s)$ into (2.6), and the corresponding relation between the bases $\{C_{\mu}\}$ and $\{F_\lambda\}$

$$F_\lambda = h_\lambda^{-1} \sum_{\mu, |\mu|=|\lambda|} \chi_\lambda(\mu) C_\mu \tag{2.39}$$

where $h_\lambda$ is the product of the hook lengths of the partition $\lambda$

$$h_\lambda^{-1} = \det \left( \frac{1}{(\lambda_i - i + j)!} \right), \tag{2.40}$$

into eqs. (2.38) and (2.4), equating coefficients in the $C_\mu$ basis, and using the orthogonality relation for the characters

$$\sum_{\mu, |\mu|=|\lambda|} \chi_\lambda(\mu) \chi_\nu(\mu) = Z_\mu \delta_{\lambda\nu}, \tag{2.41}$$

we obtain the expansion

$$\tau^{G(z)}(t, s) = \sum_{d=0}^{\infty} \sum_{\mu, \nu, |\mu|=|\nu|} \frac{z^d}{d!} F^d_G(\mu, \nu) P_{\mu}(t) P_{\nu}(s). \tag{2.42}$$

This proves the following result.

**Theorem 2.4.** $\tau^{G(z)}(t, s)$ is the generating function for the numbers $F^d_G(\mu, \nu)$ of weighted $d$-step paths in the Cayley graph, starting at an element in the conjugacy class of cycle type $\mu$ and ending at the conjugacy class of type $\nu$, with weights of all weakly monotonic paths of type $\lambda$ given by $G_\lambda$. 

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3 Examples: classical and quantum

3.1 Classical counting of paths: generating functions for combinatorial Hurwitz numbers

The following four examples were studied in ref. [18, 6, 8, 10]. The interpretation of the associated \( \tau \)-functions as generating functions for weighted enumeration of paths and as Hurwitz numbers for various types of branched covers of \( \mathbb{C}P^1 \) of fixed genus will be recalled in what follows.

Example 3.1. Double Hurwitz numbers [18]. This case is Okounkov’s double Hurwitz numbers [18], which enumerate \( n = |\mu| = |\nu| \) sheeted branched coverings of the Riemann sphere, with ramification types \( \mu \) and \( \nu \) at 0 and \( \infty \), \( d \) additional simple branch points and genus \( g \) given by the Riemann-Hurwitz formula

\[
2 - 2g = \ell(\mu) + \ell(\nu) - d. \tag{3.1}
\]

The weight generating function \( G(z) \) in this case is just the exponential function. The parameters entering into the associated \( \tau \)-function are as follows.

\[
G(z) = \exp(z) := \sum_{i=1}^{\infty} \frac{z^i}{i!}, \quad \exp_t = \frac{1}{t!}, \quad \exp_\lambda = \frac{1}{\prod_{i=1}^{\ell(\lambda)} \lambda_i!} \tag{3.2}
\]

\[
\exp(z, \mathcal{J}) = \exp(z) \sum_{b=1}^{n} \mathcal{J}_b = \sum_{k=0}^{\infty} \frac{z^d}{d!} \left( \sum_{b=1}^{n} \mathcal{J}_b \right)^d \tag{3.3}
\]

\[
= \sum_{d=0}^{\infty} z^d \sum_{\lambda, |\lambda|=d} \left( \prod_{i=1}^{\ell(\lambda)} (\lambda_i)! \right)^{-1} M_\lambda(\mathcal{J}) \tag{3.4}
\]

\[
r_j^\exp = e^{zj}, \quad r_j^\exp(\lambda) = e^{z \sum_{i=1}^{\ell(\lambda)} \lambda_i (\lambda_i - 2i + 1)}, \quad T_j^\exp = \frac{1}{2} j(j+1) z. \tag{3.5}
\]

and

\[
\tilde{F}_d^\exp(\mu, \nu) = d! \sum_{\lambda, |\lambda|=d} \left( \prod_{j=1}^{\ell(\lambda)} (\lambda_j)! \right)^{-1} m_\lambda^{\mu\nu} = \sum_{\lambda, |\lambda|=d} \tilde{m}_\lambda^{\mu\nu} \tag{3.6}
\]

is the total number of \( (d+1) \)-term products \((a_1 b_1) \cdots (a_d b_d) g\) such that \( g \in \text{cyc}(\mu) \) and the product \((a_1 b_1) \cdots (a_d b_d) g \in \text{cyc}(\nu)\); i.e., the number of (unordered) sequences of \( d \) transpositions leading from the class of type \( \mu \) to the class of type \( \nu \). Equivalently, it may be viewed as the number of \( d \)-step paths in the Cayley graph of \( S_n \) generated by all transpositions, from the conjugacy class of cycle type \( \mu \) to the class of cycle type \( \nu \).
Example 3.2. Monotonic double Hurwitz numbers [8, 10].

\[ G(z) = E(z) := 1 + z, \quad E_i = \delta_{1i}, \text{ for } i \geq 1, \quad E_\lambda = \delta_{\lambda, (1^{\ell(\lambda)})} \]  
(3.7)

\[ E(z, J) = \prod_{a=1}^{n} (1 + z J_a), \]  
(3.8)

\[ r_j^{E} = 1 + zj, \quad r_\lambda^{E}(z) = \prod_{(i,j) \in \lambda} (1 + z(j-i)) = z^{\ell(\lambda)} (1/z)_\lambda \]  
(3.9)

\[ T_j^{E} = \sum_{i=1}^{j} \ln(1 + iz), \quad T_{-j}^{E} = -\sum_{i=1}^{j-1} \ln(1 - iz), \quad j > 0, \]  
(3.10)

where

\[ (u)_\lambda = \prod_{i=1}^{\ell(\lambda)} (u - i + 1)_{\lambda_i} \]  
(3.11)

is the multiple Pochhammer symbol corresponding to the partition \( \lambda \).

In this case we have

\[ \sum_{\lambda, |\lambda| = d} E_\lambda M_\lambda(J) = \sum_{b_1 < \cdots < b_d} J_{b_1} \cdots J_{b_d}. \]  
(3.12)

and the coefficient \( F_d^d(\mu, \nu) \) is

\[ F_d^d(\mu, \nu) = m_{\mu, \nu}^{(1)d}, \]  
(3.13)

which enumerates all \( d \)-step paths in the Cayley graph of \( S_n \) starting at an element in the conjugacy class of cycle type \( \mu \) and ending in the class of type \( \nu \), that are strictly monotonically increasing in their second elements [8].

Equivalently [10], this equals the double Hurwitz numbers for Belyi curves, [23, 12, 2], which enumerate \( n \)-sheeted branched coverings of the Riemann sphere having three ramification points, with ramification profile types \( \mu \) and \( \nu \) at 0 and \( \infty \), and a single additional branch point, with ramification profile \( \mu^{(1)} \) having colength

\[ \ell^*(\mu^{(1)}) := n - \ell(\mu^{(1)}) = d \]  
(3.14)

i.e., with \( n - d \) preimages. The genus is again given by the Riemann-Hurwitz formula (3.1).

Example 3.3. Multimonotonic double Hurwitz numbers [10].

\[ G(z) = E(z)^k := (1 + z)^k, \quad E_i^k = \left(\begin{array}{c} k \\ i \end{array}\right), \quad E_\lambda^k = \prod_{i=1}^{\ell(\lambda)} \left(\begin{array}{c} k \\ \lambda_i \end{array}\right) \]  
(3.15)

\[ E(z, J)^k = \prod_{a=1}^{n} (1 + z J_a)^k, \]  
(3.16)

\[ r_j^{E_k} = (1 + zj)^k, \quad r_\lambda^{E_k}(z) = \prod_{(i,j) \in \lambda} (1 + z(j-i))^k = z^{k \ell(\lambda)} ((1/z)_\lambda)^k, \]  
(3.17)

\[ T_j^{E_k} = k \sum_{i=1}^{j} \ln(1 + iz), \quad T_{-j}^{E_k} = -k \sum_{i=1}^{j-1} \ln(1 - iz), \quad j > 0. \]  
(3.18)
\[ \sum_{\lambda, |\lambda| = d} E^k_\lambda M_\lambda(\mathcal{J}) = \sum_{\lambda, |\lambda| = d} \left( \prod_{i=1}^{n} \binom{\ell(\lambda)}{k} \right) M_\lambda(\mathcal{J}) = [z^d] \prod_{a=1}^{n} (1 + zJ_a)^k \]  

(3.19)

where \([z^d]\) means the coefficient of \(z^d\) in the polynomial.

The coefficient

\[ F^d_{E^k}(\mu, \nu) = \sum_{\lambda, |\lambda| = k} \left( \prod_{i=1}^{n} \binom{\ell(\lambda)}{k} \right) m^\lambda_{\mu\nu} \]  

(3.20)

is the number of \((d + 1)\)-term products \((a_1 b_1) \cdots (a_d b_d)g\) such that \(g \in \text{cyc}(\mu)\), while the product \((a_1 b_1) \cdots (a_d b_d)g \in \text{cyc}(\nu)\), and which consist of a product of \(k\) consecutive subsequences, each of which is strictly monotonically increasing in the second elements of each \((a_i b_i)\) \[8, 10\].

Equivalently, they are double Hurwitz numbers that enumerate \(n\)-sheeted branched coverings of the Riemann sphere with ramification profile types \(\mu\) and \(\nu\) at 0 and \(\infty\), and (at most) \(k\) additional branch points, such that the sum of the colengths of their ramification profile types (i.e., the “defect” in the Riemann Hurwitz formula (3.1)) is equal to \(d\):

\[ \sum_{i=1}^{k} \ell^*(\mu^{(i)}) = kn - \sum_{i=1}^{k} \ell(\mu^{(i)}) = d. \]  

(3.21)

This amounts to counting covers with the genus fixed by (3.1) and the number of additional branch points fixed at \(k\), but no restriction on their simplicity.

**Example 3.4. Weakly monotonic double Hurwitz numbers [6, 8].**

\[ G(z) = H(z) := \frac{1}{1 - z}, \quad H_i = 1 \text{ for } i \geq 1, \quad H_\lambda = 1 \]  

(3.22)

\[ H(z, \mathcal{J}) = \prod_{a=1}^{n} (1 - zJ_a)^{-1}, \]  

(3.23)

\[ r^H_j = (1 - zj)^{-1}, \quad r^H_\lambda(z) = \prod_{(i,j) \in \lambda} (1 - z(j - i))^{-1} = (-z)^{-|\lambda|} ((-1/z)_\lambda)^{-1}, \]  

(3.24)

\[ T^H_j = -\sum_{i=1}^{j} \ln(1 - iz), \quad T^E_{-j} = \sum_{i=1}^{j-1} \ln(1 + iz), \quad j > 0. \]  

(3.25)

We now have

\[ \sum_{\lambda, |\lambda| = d} G_\lambda M_\lambda(\mathcal{J}) = \sum_{b_1 \leq \cdots \leq b_d} \mathcal{J}_{b_1} \cdots \mathcal{J}_{b_d}. \]  

(3.26)

and

\[ F^d_{H}(\mu, \nu) = \sum_{\lambda, |\lambda| = k} m^\lambda_{\mu\nu} \]  

(3.27)
is the number of number of \((d+1)\)-term products \((a_1 b_1) \cdots (a_d b_d)g\) that are weakly monotonically increasing, such that \(g \in \text{cyc}(\mu)\) and \((a_1 b_1) \cdots (a_d b_d)g \in \text{cyc}(\nu)\). These enumerate \(d\)-step paths in the Cayley graph of \(S_n\) from an element in the conjugacy class of cycle type \(\mu\) to the class cycle type \(\nu\), that are weakly monotonically increasing in their second elements \([\text{8}]\).

Equivalently, they are double Hurwitz numbers for \(n\)-sheeted branched coverings of the Riemann sphere with branch points at \(0\) and \(\infty\) having ramification profile types \(\mu\) and \(\nu\), and an arbitrary number of further branch points, such that the sum of the colengths of their ramification profile lengths is again equal to \(d\)

\[
\sum_{i=1}^{k} \ell^* (\mu^{(i)}) = kn - \sum_{i=1}^{k} \ell (\mu^{(i)}) = d.
\]  

(3.28)

The latter are counted with a sign, which is \((-1)^{n+d}\) times the parity of the number of branch points \([\text{10}]\). The genus is again given by (3.1).

### 3.2 Quantum combinatorial Hurwitz numbers

In this subsection, we introduce three new examples involving weighted enumeration of paths in the Cayley graph of the symmetric group, which can also be interpreted as weighted enumeration of configurations of branched covers of the Riemann sphere. Each can be viewed as a \(q\)-deformation of one the classical examples. Because of the similarity of the weighting to that of a quantum Bosonic gas, the resulting weighted sums will be identified as quantum Hurwitz numbers.

**Example 3.5. \(E(q)\). Quantum Hurwitz numbers (i).**

\[
G(z) = \sum_{i=0}^{\infty} E_i(q) z^i, \quad E_i(q) = \frac{q^\frac{1}{2}(i-1)}{\prod_{j=1}^{n}(1-q^j)}, \quad i \geq 1, \quad E_{\lambda}(q) = \prod_{i=1}^{\ell(\lambda)} \frac{q^{\frac{1}{2}\lambda_i(\lambda_i-1)}}{\lambda_i \prod_{j=1}^{\lambda_i}(1-q^j)}
\]  

(3.29)

\[
E(q, J) = \prod_{a=1}^{n} \prod_{k=0}^{\infty} (1 + q^k z J_a),
\]  

(3.30)

\[
r_{E}(q) = \prod_{k=0}^{\infty} (1 + q^k z^j),
\]  

(3.31)

\[
r_{\lambda}^{E}(q)(z) = \prod_{k=0}^{\infty} \prod_{(i,j) \in \lambda} (1 + q^k z(j - i)) = \prod_{k=0}^{\infty} (zq^k)^{\lambda} (1/(zq^k))_{\lambda}
\]  

(3.32)

\[
T_{E}^{E(q)} = - \sum_{i=1}^{j} \text{Li}_2(q, -zi), \quad T_{-E}^{E(q)} = \sum_{i=0}^{j} \text{Li}_2(q, zi), \quad j > 0.
\]  

(3.33)

This weight generating function is related to the quantum dilogarithm function by

\[
E(q, z) = e^{-\text{Li}_2(q, -z)}, \quad \text{Li}_2(q, z) := \sum_{k=1}^{\infty} \frac{z^k}{k(1-q^k)}.
\]  

(3.34)
The coefficients $E_i(q)$ are themselves generating functions for the number of partitions having exactly $i$ or $i-1$ parts, all distinct.

**Remark 3.1.** The definition of the quantum dilogarithm is not uniform in the literature. What is referred to in [4] as the quantum dilogarithm is

$$\Psi(z) := E(q, -z) = e^{-\text{Li}_2(q,z)}.$$  

(3.36)

The notation $\text{Li}_2(q,z)$ used here is natural since, for $q = e^{-\epsilon}$, $|q| < 1$, the leading term as $\epsilon \to 0$ coincides with the classical dilogarithm in the rescaled argument $\frac{z}{\epsilon}$:

$$\text{Li}_2(q,z) \sim \sum_{m=1}^{\infty} \frac{(\frac{z}{\epsilon})^m}{m^2} = \text{Li}_2\left(\frac{z}{\epsilon}\right).$$

(3.37)

A slight modification of this is obtained by removing the $q^0$ term in the product, giving the weight generating function

$$E'(q,z) := \prod_{k=1}^{\infty} (1 + q^k z).$$

(3.38)

The coefficient $F_{E(q)}^d(\mu, \nu)$ is

$$F_{E(q)}^d(\mu, \nu) = \sum_{\lambda, |\lambda|=d} E_\lambda(q) m_\mu^\lambda = (d!)^{-1} \sum_{\lambda, |\lambda|=d} \tilde{E}_\lambda(q) \tilde{m}_\mu^\lambda$$

(3.39)

Its combinatorial interpretation follows from Theorem 2.3 as a weighted enumeration of paths in the Cayley graph of $S_n$ from the conjugacy class of type $\mu$ to the class $\nu$, where paths of signature $\lambda$ have weighting factor $E_\lambda(q)$.

The geometric interpretation will be detailed in Theorem 4.1. It may be viewed as weighted sums over branched covers, in which the weights are closely related to distributions for Bosonic gases, with the parameter $q$ interpreted as

$$q = e^{-\beta \hbar \omega}, \quad \beta = \frac{1}{k_B T}$$

(3.40)

for some fundamental frequency $\omega$ and linear energy spectrum, with the energy levels proportional to the total ramification defect. The case $E'(q)$ is obtained by removal of the zero energy levels, giving a distribution that more closely resembles that of the Bosonic gas. In the classical limit $q \to 1$, we recover Example 3.1.
Example 3.6. H(q). Quantum Hurwitz numbers (ii).

\[ G(z) = H(q, z) := \prod_{k=0}^{\infty} (1 - q^k z)^{-1} = e^{\text{Li}_2(qz)} = \sum_{i=0}^{\infty} H_i(q) z^i, \]  
(3.41)

\[ H_i(q) := \frac{1}{\prod_{j=1}^{i}(1 - q^j)}, \quad H_\lambda(q) = \prod_{i=1}^{\ell(\lambda)} \frac{1}{\prod_{j=1}^{\lambda_j}(1 - q^j)} \]  
(3.42)

\[ H(q, J) = \prod_{k=0}^{\infty} \prod_{a=1}^{n} (1 - q^k z^a)^{-1}, \]  
(3.43)

\[ r_j^H(q) = \prod_{k=0}^{\infty} (1 - q^k z^j)^{-1}, \]  
(3.44)

\[ r_\lambda^H(q)(z) = \prod_{k=0}^{\infty} \prod_{(i,j) \in \lambda} (1 - q^k z(j - i))^{-1} = \prod_{k=0}^{\infty} (-1/(q^k))^{|\lambda|} \]  
(3.45)

\[ T_j^H(q) = \sum_{i=1}^{j} \text{Li}_2(q, zi), \quad T_{-j}^H(q) = -\sum_{i=1}^{j-1} \text{Li}_2(q, -zi), \quad j > 0. \]  
(3.46)

The coefficients \( H_i(q) \) of the weight generating function in this case are generating functions for the number of partitions having at most \( i \) parts, which need not be distinct. The modification corresponding to removing the zero energy level state is based similarly on the generating function \( H'(q, z) := \prod_{k=1}^{\infty} (1 - q^k z)^{-1}. \)  
(3.47)

The coefficient \( F^d_{H}(\mu, \nu) \) is

\[ F^d_{H}(\mu, \nu) = \sum_{\lambda, \ |\lambda|=d} H_\lambda(q) m_\lambda^{\mu \nu} = (d!)^{-1} \sum_{\lambda, \ |\lambda|=d} \tilde{H}_\lambda(q) \tilde{m}_\lambda^{\mu \nu} \]  
(3.48)

Its combinatorial interpretation again follows from Theorem 2.3 as the weighted enumeration of paths in the Cayley graph of \( S_n \) from the conjugacy class of type \( \mu \) to the class \( \nu \) where paths of signature \( \lambda \) have weighting factor \( H_\lambda(q) \).

The geometric interpretation is detailed in Theorem 4.3. It may be viewed as a signed version of the weighted Hurwitz numbers associated to the Bose gas interpretation, with the sign again determined by the parity of the number of branch points. In the classical limit \( q \to 1 \), we again recover Example 3.1.

Example 3.7. Q(q,p). Double quantum Hurwitz numbers.

\[ G(z) = Q(q, p, z) := E(q, z) H(p, z) = \prod_{k=0}^{\infty} (1 + q^k z)(1 - p^k z)^{-1} = \sum_{i=0}^{\infty} Q_i(q, p) z^i, \]  
(3.49)

\[ Q_i(q, p) := \sum_{m=0}^{i} q^{\frac{1}{2}m(m-1)} \left( \prod_{j=1}^{m} (1 - q^j) \prod_{j=1}^{i-m} (1 - p^j) \right)^{-1}, \quad Q_\lambda(q, p) = \prod_{i=1}^{\ell(\lambda)} Q_{\lambda_i}(q, p), \]
\[ Q(q, p, J) = E(q, z)H(p, z, J), \]  
\[ r_jQ(q, p) = \prod_{k=0}^{\infty} \frac{1 + q^k z^j}{1 - p^k z^j}, \]  
\[ r_\lambda Q(q, p)(z) = \prod_{k=0}^{\infty} \prod_{(i, j) \in \lambda} \frac{1 + q^k z(j - i)}{1 - p^k z(j - i)} = \prod_{k=0}^{\infty} (-q/p)^{k|\lambda|} (1/(zq^k))_\lambda, \]  
\[ T_jQ(q, p) = \sum_{i=1}^{j} \text{Li}_2(p, zi) - \sum_{i=1}^{j} \text{Li}_2(q, -zi), \]  
\[ T_{-j}Q(q, p) = -\sum_{i=1}^{j-1} \text{Li}_2(p, -zi) + \sum_{i=1}^{j-1} \text{Li}_2(q, zi), \quad j > 0. \]  

The coefficient \( F^d_{Q(q, p)}(\mu, \nu) \) is

\[ F^d_{Q(q, p)}(\mu, \nu) = \sum_{\lambda, |\lambda| = d} Q_\lambda(q, p) m_\mu^\lambda = (d!)^{-1} \sum_{\lambda, |\lambda| = d} \tilde{Q}_\lambda(q, p) \tilde{m}_\mu^\lambda. \]  

Its combinatorial and geometric interpretations are given in Theorem 2.3 and in Theorem 4.5.

Geometrically, these are the composite of two types of weighted enumerations; i.e., two species of branch points, one of which is counted with the weight corresponding to a Bosonic gas as in Example 3.5, the other counted, as in Example 3.6, with signs determined by the parity of the number of such branch points. In the classical limit \( q \to 1, p \to 1 \), we again recover Example 3.1.

### 3.3 Generating functions for Hurwitz numbers: classical counting of branched covers

For Example 3.1, the generating \( \tau \)-function is [18]

\[ \tau^{\text{exp}}(z)(t, s) = \sum_{\lambda} e^{\frac{z}{2} \sum_{n=1}^{\infty} \lambda_n (\lambda_n - 2n + 1)} S_\lambda(t) S_\mu(s) 
= \sum_{d=0} \sum_{|\mu| = |\nu|} F^d_{\text{exp}}(\mu, \nu) z^d P_\mu(t) P_\nu(s), \]  

where

\[ F^d_{\text{exp}}(\mu, \nu) = H^d_{\text{exp}}(\mu, \nu) = \frac{1}{d!} H(\mu^{(1)}) = (2, 1^{n-2}), \ldots, \mu^{(d)} = (2, 1^{n-2}), \mu, \nu \]  

is \( \frac{1}{d!} \) times Okounkov’s double Hurwitz number \( \text{Cov}_d(\mu, \nu) \) [18]; that is, the number of \( n = |\mu| = |\nu| \) sheeted branched covers with branch points of ramification type \( \mu \) and \( \nu \) at the points 0 and \( \infty \), and \( d \) further simple branch points.

For Example 3.2, the generating \( \tau \)-function is [8, 10]

\[ \tau^{E}(z)(t, s) = \sum_{\lambda} z^{|\lambda|} (z^{-1})_\lambda S_\lambda(t) S_\mu(s) \]
where

\[ H^d_E(\mu, \nu) = \sum_{\mu^{(1)}, \ell^*(\mu_1) = d} H(\mu^{(1)}, \mu, \nu) \]

is now interpreted as the number of \( n = |\mu| = |\nu| = |\mu^{(1)}| \) sheeted branched covers with branch points of ramification type \( \mu \) and \( \nu \) at 0 and \( \infty \), and one further branch point, with colength \( \ell^*(\mu^{(1)}) = d \); i.e., the case of Belyi curves [23, 12, 10] or dessins d’enfants.

For Example 3.3, the generating \( \tau \)-function is [10] is

\[ \tau^{E_k(z)}(t, s) = \sum_{\lambda} z^{\lambda} (1/z)^\lambda S_\lambda(t)S_\mu(s) \]

\[ = \sum_{d=0}^{\infty} z^d \sum_{\mu, \nu, |\mu| = |\nu| = n} H^d_E(\mu, \nu) P_\mu(t)P_\nu(s), \]

where

\[ H^d_E(\mu, \nu) = \sum_{\mu^{(1)}, \ldots, \mu^{(k)}} H(\mu^{(1)}, \ldots, \mu^{(k)}, \mu, \nu) \]

is now interpreted [10] as the number of \( n = |\mu| = |\nu| = |\mu^{(i)}| \) sheeted branched covers with branch points of ramification type \( \mu \) and \( \nu \) at 0 and \( \infty \), and (at most) \( k \) further branch points, the sums of whose colengths is \( d \).

For Example 3.4, the generating \( \tau \)-function is [8, 10]

\[ \tau^{H_k(z)}(t, s) = \sum_{\lambda} (-z)^{\lambda} (-z^{-1})^\lambda S_\lambda(t)S_\mu(s) \]

\[ = \sum_{d=0}^{\infty} z^d \sum_{\mu, \nu, |\mu| = |\nu|} H^d_H(\mu, \nu) P_\mu(t)P_\nu(s), \]

where

\[ H^d_H(\mu, \nu) = (-1)^{n+d} \sum_{j=1}^{\infty} (-1)^j \sum_{\mu^{(1)}, \ldots, \mu^{(j)}} H(\mu^{(1)}, \ldots, \mu^{(j)}, \mu, \nu) \]

is now interpreted as the signed counting of \( n = |\mu| = |\nu| \) sheeted branched covers with branch points of ramification type \( \mu \) and \( \nu \) at 0 and \( \infty \), and any number further branch points, the sum of whose colengths is \( d \), with sign determined by the parity of the number of branch points [10].

We thus have, for each of the four cases \( G = \exp, E, E_k \) and \( H \) shown the equality

\[ F^d_G(\mu, \nu) = H^d_G(\mu, \nu) \] (3.64)
between the combinatorial weighted path enumeration and the weighted (signed) branched covering enumeration.

### 3.4 The \(\tau\)-functions \(\tau^E(q,z)\), \(\tau^H(q,z)\) and \(\tau^Q(q,p,z)\) as generating functions for enumeration of \(q\)-weighted paths

The particular cases

\[
\tau^E(q,z)(t,s) := \sum_{\lambda} r^E_{\lambda}(z) S_{\lambda}(t) S_{\lambda}(s) \tag{3.65}
\]

\[
= \sum_{d=0}^{\infty} \sum_{|\mu|=|\nu|} z^d F^d_E(\mu, \nu) P_\mu(t) P_\nu(s), \tag{3.66}
\]

\[
\tau^H(q,z)(t,s) := \sum_{\lambda} r^H_{\lambda}(z) S_{\lambda}(t) S_{\lambda}(s) \tag{3.67}
\]

\[
= \sum_{d=0}^{\infty} \sum_{|\mu|=|\nu|} z^d F^d_H(\mu, \nu) P_\mu(t) P_\nu(s), \tag{3.68}
\]

\[
\tau^Q(q,p,z)(t,s) := \sum_{\lambda} r^Q_{\lambda}(q,p)(z) S_{\lambda}(t) S_{\lambda}(s) \tag{3.69}
\]

\[
= \sum_{d=0}^{\infty} \sum_{|\mu|=|\nu|} z^d F^d_Q(q,p)(\mu, \nu) P_\mu(t) P_\nu(s). \tag{3.70}
\]

may be viewed as special \(q\)-deformations of the generating functions associated to examples Example 3.1, Example 3.2, with \(G(z) = 1 + z\), \(G(z) = (1 - z)^{-1}\) respectively, and the hybrid combination generated by the ratio \(\frac{1 + z}{1 - z}\). The former were considered previously in \([8, 10]\), and given both combinatorial and geometric interpretations in terms of weakly or strictly monotonic paths in the Cayley graph.

**Remark 3.2.** Note that, for the special values of the flow parameters \((t,s)\) given by trace invariants of a pair of commuting \(M \times M\) matrices, \(X\) and \(Y\),

\[
t_i = \frac{1}{i} \text{tr}(X^i), \quad s_i = \frac{1}{i} \text{tr}(Y^i), \tag{3.71}
\]

with eigenvalues \((x_1, \ldots, x_M), (y_1, \ldots, y_M)\), these may be viewed as special cases of the two types of basic hypergeometric functions of matrix arguments \([7, 19]\).

### 3.5 Classical limits of examples \(E(q), H(q)\) and \(Q(q,p)\)

Setting \(q = e^\epsilon\) for some small parameter, and taking the leading term contribution in the limit \(\epsilon \to 0\), we obtain

\[
\lim_{\epsilon \to 0} E(q, \epsilon z) = e^z \tag{3.72}
\]
and therefore, taking the scaled limit with $z \to \epsilon z$, we obtain
\[
\lim_{\epsilon \to 0} \tau^{E(q, \epsilon z)}(t, s) = \tau^{\exp(z)}(t, s) \quad (3.73)
\]
Similarly, we have
\[
\lim_{\epsilon \to 0} H(q, \epsilon z) = e^z \quad (3.74)
\]
and hence
\[
\lim_{\epsilon \to 0} \tau^{H(q, \epsilon z)}(t, s) = \tau^{\exp(z)}(t, s). \quad (3.75)
\]
And finally, for the double quantum Hurwitz case, Example 3.7, setting
\[
q = e^\epsilon, \quad p = e^{\epsilon'} \quad (3.76)
\]
and replacing $z$ by $z(\frac{1}{\epsilon} + \frac{1}{\epsilon'})$, we get
\[
\lim_{\epsilon, \epsilon' \to 0} Q(q, p, z\epsilon\epsilon') = e^z \quad (3.77)
\]
and hence
\[
\lim_{\epsilon, \epsilon' \to 0} \tau^{Q(q, p, z\epsilon\epsilon')}(t, s) = \tau^{\exp(z)}(t, s). \quad (3.78)
\]
Thus, we recover Okounkov’s classical double Hurwitz number generating function $\tau^{\exp(z)}(t, s)$ as the classical limit in each case.

## 4 Weighted and quantum Hurwitz numbers

We proceed to the interpretation of the quantities $F^{k}_{E(q,\epsilon z)}(\mu, \nu)$, $F^{k}_{H(q,\epsilon z)}(\mu, \nu)$ and $F^{k}_{Q(q, p, z\epsilon\epsilon')}(\mu, \nu)$ as weighted enumerations of branched coverings of the Riemann sphere. The key is the Frobenius-Schur-Burnside formula [14, Appendix A], [5],

\[
H(\mu^{(1)}, \ldots, \mu^{(k)}) = \sum_{\lambda} h_{\lambda}^{k-2} \prod_{i=1}^{k} \frac{\chi_{\lambda}(\mu^{(i)})}{Z_{\mu^{(i)}}}. \quad (4.1)
\]

expressing the Hurwitz number $H(\mu^{(1)}, \ldots, \mu^{(k)})$, which counts the number of inequivalent $n$-sheeted branched coverings of the Riemann sphere with $k$ distinct branch points whose ramification profiles are given by the $k$ partitions $\mu^{(1)}, \ldots, \mu^{(k)}$ with weight $|\mu^{(i)}| = n$ in terms of the characters of the symmetric group.
4.1 Symmetrized monomial sums and \(q\)-weighted Hurwitz sums

Before proceeding, we recall three symmetrized monomial summation formulae that will be needed in what follows. These have various applications in combinatorics and are easily proved (e.g., by recursive diagonal summation of the geometric series involved).

Let \(\overline{C[x_1, \ldots, x_k]}\) be the completion of the field extension of \(C\) by \(k\) indeterminates, viewed as a normed vector space with norm \(|\cdot|\) and \(0 < |x_i| < 1\), so that the corresponding geometric series converge

\[
\sum_{m=0}^{\infty} x_i^m = \frac{1}{1 - x_i}, \quad i = 1, \ldots, k. \tag{4.2}
\]

Then

\[
\sum_{\sigma \in S_k} \sum_{0 \leq i_1 < \cdots < i_k} x_{\sigma(1)}^{i_1} \cdots x_{\sigma(k)}^{i_k} = \sum_{\sigma \in S_k} \frac{x_{\sigma(1)}^{k-1} x_{\sigma(2)}^{k-2} \cdots x_{\sigma(k-1)}^1}{(1 - x_{\sigma(1)})(1 - x_{\sigma(1)} x_{\sigma(2)}) \cdots (1 - x_{\sigma(1)} \cdots x_{\sigma(k)})} \tag{4.3}
\]

\[
\sum_{\sigma \in S_k} \sum_{1 \leq i_1 < \cdots < i_k} x_{\sigma(1)}^{i_1} \cdots x_{\sigma(k)}^{i_k} = \sum_{\sigma \in S_k} \frac{x_{\sigma(1)}^{k} x_{\sigma(2)}^{k-1} \cdots x_{\sigma(k)}^1}{(1 - x_{\sigma(1)})(1 - x_{\sigma(1)} x_{\sigma(2)}) \cdots (1 - x_{\sigma(1)} \cdots x_{\sigma(k)})} \tag{4.4}
\]

\[
\sum_{\sigma \in S_k} \sum_{0 \leq i_1 \leq \cdots \leq i_k} x_{\sigma(1)}^{i_1} \cdots x_{\sigma(k)}^{i_k} = \sum_{\sigma \in S_k} \frac{1}{(1 - x_{\sigma(1)})(1 - x_{\sigma(1)} x_{\sigma(2)}) \cdots (1 - x_{\sigma(1)} \cdots x_{\sigma(k)})} \tag{4.5}
\]

In what follows, we let \((\mu^{(1)}, \ldots, \mu^{(k)})\) denote a set of partitions of weight \(|\mu^{(i)}| = n\), and choose the \(x_i\)'s to be

\[
x_i := q^{\ell^*(\mu^{(i)})}. \tag{4.6}
\]

For the generating functions \(E(q), E'(q)\) and \(H(q)\), we define the following weighting factors

\[
W_{E(q)}(\mu^{(1)}, \ldots, \mu^{(k)}) := \frac{1}{k!} \sum_{\sigma \in S_k} \sum_{0 \leq i_1 < \cdots < i_k} q^{i_1 \ell^*(\mu^{(\sigma(1))})} \cdots q^{i_k \ell^*(\mu^{(\sigma(k))})} \tag{4.7}
\]

\[
= \frac{1}{k!} \sum_{\sigma \in S_k} \frac{q^{(k-1)\ell^*(\mu^{(1)})} \cdots q^{\ell^*(\mu^{(k-1)})}}{(1 - q^{\ell^*(\mu^{(\sigma(1))})}) \cdots (1 - q^{\ell^*(\mu^{(\sigma(k))})})}, \tag{4.8}
\]

\[
W_{E'(q)}(\mu^{(1)}, \ldots, \mu^{(k)}) := \frac{1}{k!} \sum_{\sigma \in S_k} \sum_{1 \leq i_1 < \cdots < i_k} q^{i_1 \ell^*(\mu^{(\sigma(1))})} \cdots q^{i_k \ell^*(\mu^{(\sigma(k))})} \tag{4.9}
\]

\[
= \frac{1}{k!} \sum_{\sigma \in S_k} \frac{q^{k\ell^*(\mu^{(\sigma(1))})} \cdots q^{\ell^*(\mu^{(\sigma(k))})}}{(1 - q^{\ell^*(\mu^{(\sigma(1))})}) \cdots (1 - q^{\ell^*(\mu^{(\sigma(k))})})}, \tag{4.10}
\]

\[
= \frac{1}{k!} \sum_{\sigma \in S_k} \frac{1}{(q^{-\ell^*(\mu^{(\sigma(1))})} - 1) \cdots (q^{-\ell^*(\mu^{(\sigma(k))})} - 1)} \tag{4.11}
\]
\[ W_{H(q)}(\mu^{(1)}, \ldots, \mu^{(k)}) := \frac{1}{k!} \sum_{\sigma \in S_k} \sum_{0 \leq i_1 < \cdots < i_k} q^{i_1 \ell^*(\mu^{(i_1)})} \cdots q^{i_k \ell^*(\mu^{(i_k)})} \]  
\[ = \frac{1}{k!} \sum_{\sigma \in S_k} \frac{1}{(1 - q^{\ell^*(\mu^{(i_1)})}) \cdots (1 - q^{\ell^*(\mu^{(i_k)})})}. \]  

4.2 Quantum Hurwitz numbers: the case of \( E(q) \)

It follows from the Frobenius character formula (2.9) that the Pochhammer symbol \((z)_{\lambda}\) may be written

\[ (z)_{\lambda} = 1 + h_{\lambda} \sum'_{\mu, |\mu| = |\lambda|} \frac{\chi_{\lambda}(\mu)}{Z_{\mu}} z^{\ell^*(\mu)}. \]  

where \(\sum'\) means the sum over partitions other than the cycle type of the identity element, \((1)^{|\lambda|}\), and

\[ \ell^*(\mu) = |\mu| - \ell(\mu) \]

is the colength of the partition \(\mu\). The content product formula (3.34) for this case may therefore be written as

\[ r_{\lambda}^{E(q)}(z) = \prod_{k=0}^{\infty} \left( 1 + h_{\lambda} \sum'_{\mu, |\mu| = |\lambda|} \frac{\chi_{\lambda}(\mu)}{Z_{\mu}} (zq)^k z^{\ell^*(\mu)} \right) \]  
\[ = \sum_{k=0}^{\infty} \sum'_{\mu^{(1)}, \ldots, \mu^{(k)}} \sum_{|\mu^{(i)}| = |\lambda|} \prod_{j=1}^{k} h_{\lambda} \chi_{\lambda}(\mu^{(j)}) \frac{z^{\ell^*(\mu^{(j)})}}{Z_{\mu^{(j)}}} q^{i_j \ell^*(\mu^{(j)})} \]  
\[ = \sum_{k=0}^{\infty} \sum'_{\mu^{(1)}, \ldots, \mu^{(k)}} W_{E(q)}(\mu^{(1)}, \ldots, \mu^{(k)}) \prod_{j=1}^{k} h_{\lambda} \chi_{\lambda}(\mu^{(j)}) \frac{z^{\sum_{i=1}^{k} \ell^*(\mu^{(i)})}}{Z_{\mu^{(j)}}} \]  

Substituting this into (3.65) and using the Frobenius character formula (2.9) for each of the factors \(S_{\lambda}(t)S_{\lambda}(s)\) gives

**Theorem 4.1.**

\[ \tau_{E(q,z)}(t,s) = \sum_{d=0}^{\infty} z^d \sum_{|\mu| = |\nu|} H_{E(q)}^d(\mu, \nu) P_\mu(t)P_\nu(s), \]  

where

\[ H_{E(q)}^d(\mu, \nu) := \sum_{k=0}^{\infty} \sum'_{\mu^{(1)}, \ldots, \mu^{(k)}} W_{E(q)}(\mu^{(1)}, \ldots, \mu^{(k)}) H(\mu^{(1)}, \ldots, \mu^{(k)}, \mu, \nu) \]
are the weighted (quantum) Hurwitz numbers that count the number of branched coverings with genus \( g \) given by (3.1) with weight \( W_{E(q)}(\mu^{(1)}, \ldots, \mu^{(k)}) \) for every branched covering of type \( (\mu^{(1)}, \ldots, \mu^{(k)}, \mu, \nu) \).

From eq. (3.66) follows:

**Corollary 4.2.** The weighted (quantum) Hurwitz number for the branched coverings of the Riemann sphere with genus given by (3.1) is equal to the combinatorial Hurwitz number given by formula (3.39) enumerating weighted paths in the Cayley graph:

\[
H^d_{E(q)}(\mu, \nu) = F^d_{E(q)}(\mu, \nu). \tag{4.21}
\]

### 4.3 Signed quantum Hurwitz numbers: the case of \( H(q) \)

We proceed similarly for this case. The content product formula (3.46) for this case may be written as

\[
r^{H(q)}_\lambda(z) = \prod_{k=0}^{\infty} \left( 1 + \sum_{\mu, |\mu|=|\lambda|} \frac{\chi_\lambda(\mu)}{Z^\mu} (-zq^k)^{\epsilon^*(\mu)} \right)^{-1} \tag{4.22}
\]

\[
= \sum_{k=0}^{\infty} \sum_{\mu^{(1)}, \ldots, \mu^{(k)}} \sum_{\substack{i_1 \leq \cdots \leq i_k \leq |\mu^{(i)}| = |\lambda|}} (-1)^k \prod_{j=1}^{k} \frac{h_\lambda(\mu^{(j)})}{Z^{\mu^{(j)}}} (-z)^{\epsilon^*(\mu^{(j)})} q^{i_j \epsilon^*(\mu^{(j)})} \tag{4.23}
\]

\[
= \sum_{k=0}^{\infty} (-1)^k \sum_{\mu^{(1)}, \ldots, \mu^{(k)}} \sum_{\substack{i_1 \leq \cdots \leq i_k \leq |\mu^{(i)}| = |\lambda|}} \prod_{j=1}^{k} \frac{h_\lambda(\mu^{(j)})}{Z^{\mu^{(j)}}} (-z)^{\sum_{i=1}^{k} \epsilon^*(\mu^{(j)})} \tag{4.24}
\]

Substituting this into (3.67) and using the Frobenius character formula (2.9) for each of the factors \( S_\lambda(\mathbf{t})S_\lambda(\mathbf{s}) \) gives:

**Theorem 4.3.**

\[
\tau^{H(q,z)}(\mathbf{t}, \mathbf{s}) = \sum_{d=0}^{\infty} z^d \sum_{|\mu|=|\nu|} H^d_{H(q)}(\mu, \nu) P_\mu(\mathbf{t}) P_\nu(\mathbf{s}), \tag{4.26}
\]

where

\[
H^d_{H(q)}(\mu, \nu) := \sum_{k=0}^{\infty} (-1)^{k+d} \sum_{\mu^{(1)}, \ldots, \mu^{(k)}} \sum_{\substack{i_1 \leq \cdots \leq i_k \leq |\mu^{(i)}| = d}} \prod_{j=1}^{k} \frac{h_\lambda(\mu^{(j)})}{Z^{\mu^{(j)}}} (-z)^{\sum_{i=1}^{k} \epsilon^*(\mu^{(j)})} \tag{4.27}
\]

are the weighted, signed (quantum) Hurwitz numbers that count the number of branched coverings with genus \( g \) given by (3.1) and sum of colengths \( k \), with weight \( W_{H(q)}(\mu^{(1)}, \ldots, \mu^{(k)}) \) for every branched covering of type \( (\mu^{(1)}, \ldots, \mu^{(k)}, \mu, \nu) \).
From eq. (3.66) it follows that:

**Corollary 4.4.** The weighted (quantum) Hurwitz number for the branched coverings of the Riemann sphere with genus given by (3.1) is again equal to the combinatorial Hurwitz number given by formula (3.48) enumerating weighted paths in the Cayley graph:

\[ H_{H(q)}^k(\mu, \nu) = F_{H(q)}^k(\mu, \nu). \]  \hspace{1cm} (4.28)

### 4.4 Double quantum Hurwitz numbers: the case of \( Q(q, p) \)

This case can be understood by combining the results for the previous two multiplicatively. Since

\[ r^Q_{\lambda}(z) = r^E(q)_\lambda(z) r^H(p)_\lambda(z), \]  \hspace{1cm} (4.29)

it follows that:

**Theorem 4.5.**

\[ \tau^{Q(q,p,z)}(t, s) = \sum_{d=0}^{\infty} z^d \sum_{|\mu| = |\nu|} H^d_{Q(q,p)}(\mu, \nu) P_\mu(t) P_\nu(s), \]  \hspace{1cm} (4.30)

where

\[ H^d_{Q(q,p)}(\mu, \nu) := \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} (-1)^{m+\sum_{i=1}^{\ell} \epsilon(\nu(i))} \sum_{\mu^{(1)}, \ldots, \mu^{(k)}, \nu^{(1)}, \ldots, \nu^{(m)}} W_{E(q)}(\mu^{(1)}, \ldots, \mu^{(k)}) \times W_{H(p)}(\nu^{(1)}, \ldots, \nu^{(m)}) H(\mu^{(1)}, \ldots, \mu^{(k)}, \nu^{(1)}, \ldots, \nu^{(m)}, \mu, \nu) \]  \hspace{1cm} (4.31)

are the weighted (quantum) Hurwitz numbers that count the number of branched coverings with genus \( g \) given by (3.1) and sum of colengths \( d \), with two mutually independent species of branch points, the first \( (\mu^{(1)}, \ldots, \mu^{(k)}) \) having weight \( W_{E(q)}(\mu^{(1)}, \ldots, \mu^{(k)}) \), the second \( (\nu^{(1)}, \ldots, \nu^{(m)}) \), signed weight \( (-1)^{m+\sum_{i=1}^{\ell} \epsilon(\nu(i))} W_{H(q)}(\nu^{(1)}, \ldots, \nu^{(m)}) \) for every branched covering of type \( (\mu^{(1)}, \ldots, \mu^{(k)}, \nu^{(1)}, \ldots, \nu^{(m)}, \mu, \nu) \).

It follows from eq. (3.66) that:

**Corollary 4.6.** The weighted (quantum) Hurwitz number for the branched coverings of the Riemann sphere with genus given by (3.1) is again equal to the combinatorial Hurwitz number given by formula (3.55) enumerating weighted paths in the Cayley graph:

\[ H^d_{Q(q,p)}(\mu, \nu) = F^d_{Q(q,p)}(\mu, \nu). \]  \hspace{1cm} (4.32)
4.5 Bose gas model

A slight modification of Example 3.5 consists of replacing the generating function \( E(q, z) \) defined in eq. (3.29) by \( E'(q, z) \), as defined in (3.38) and (3.47). The effect of this is simply to replace the weighting factors \( \frac{1}{1-q^{\ell(\mu)}} \) in eq. (4.20) by \( \frac{1}{q^{\ell(\mu)}-1} \).

If we identify
\[
q := e^{-\beta\hbar\omega}, \quad \beta = k_B T,
\]
where \( \omega \) is the lowest frequency excitation in a gas of identical Bosonic particles and assume the energy spectrum of the particles consists of integer multiples of \( \hbar\omega \)
\[
\epsilon_k = k\hbar\omega,
\]
the relative probability of occupying the energy level \( \epsilon_k \) is
\[
\frac{q^k}{1-q^k} = \frac{1}{e^{\beta\epsilon_k}-1},
\]
which is the energy distribution of a Bosonic gas. If we assign the energy
\[
\epsilon(\mu) := \epsilon_{\ell(\mu)} = \hbar\ell(\mu)\omega
\]
and assign a weight to a configuration \( (\mu^{(1)}, \ldots, \mu^{(k)}, \mu, \nu) \) that corresponds to the Bosonic gas weight for a state with total energy that of the additional \( k \) branch points
\[
\epsilon(\mu^{(1)}, \ldots, \mu^{(k)}) = \sum_{i=1}^{k} \epsilon(\mu^{(i)})
\]
we obtain the weight
\[
W(\mu^{(1)}, \ldots, \mu^{(k)}) = \frac{1}{e^{\beta\epsilon(\mu^{(1)}, \ldots, \mu^{(k)})}-1}.
\]
From eq. (4.11), the weighting factor \( W_{E'(q)}(\mu^{(1)}, \ldots, \mu^{(k)}) \) for \( k \) branch points with ramification profiles \( (\mu^{(1)}, \ldots, \mu^{(k)}) \) is thus the symmetrized product.
\[
W_{E'(q)}(\mu^{(1)}, \ldots, \mu^{(k)}) = \frac{1}{k!} \sum_{\sigma \in S_k} W(\mu^{(\sigma(1))}) \cdots W(\mu^{(\sigma(k))}).
\]
If we associate the branch points to the states of the gas and view the Hurwitz numbers \( H(\mu^{(1)}, \ldots, \mu^{(k)}, \mu, \nu) \) as independent, identically distributed random variables, the weighted Hurwitz numbers are given, as in eq. (4.20), by
\[
H^d_{E'(q)}(\mu, \nu) := \sum_{k=0}^{\infty} \sum_{\mu^{(1)}, \ldots, \mu^{(k)} \atop \sum_{i=1}^{k} \epsilon(\mu^{(i)})=d} W_{E'(q)}(\mu^{(1)}, \ldots, \mu^{(k)}) H(\mu^{(1)}, \ldots, \mu^{(k)}, \mu, \nu)
\]
Normalizing by the canonical partition function for fixed total energy \( d\hbar\omega \),

\[
Z_{E'(q)}^d := \sum_{k=0}^{\infty} \sum_{\mu^{(1)}, \ldots, \mu^{(k)}} W_{E'(q)}(\mu^{(1)}, \ldots, \mu^{(k)}),
\]

we may therefore interpret this as an expectation value of the Hurwitz numbers associated to the Bose gas, as

\[
\langle H_{E'(q)}^d(\mu, \nu) \rangle = \frac{H_{E'(q)}^d(\mu, \nu)}{Z_{E'(q)}^d},
\]

and view the corresponding \( \tau \)-function

\[
\frac{\tau^{E'(q,z)}(t,s)}{Z_{E'(q)}^d} = \sum_{d=0}^{\infty} z^d \sum_{|\mu|=|\nu|} \langle H_{E'(q)}^d(\mu, \nu) \rangle P_\mu(t)P_\nu(s),
\]

as a generating function for these expectation values.

### 4.6 Multiparameter extensions

By combining these cases multiplicatively, a multiparameter family of generating functions may be obtained, for which the underlying generator is the product

\[
G(q, w, z) := \prod_{a=1}^{k} E(q, w_a) \prod_{b=1}^{m} H(q, z_b).
\]

The interpretation of these multiparametric quantum Hurwitz numbers, both in terms of weighted enumeration of branched covers, and weighted paths in the Cayley graph will be the subject of [11].

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