A REMARK ON CUSPIDAL LOCAL SYSTEMS

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Abstract. In this note we show that all reductive groups are clean in characteristic $\geq 3$. In characteristic 2 there are two cuspidal local systems (one for $F_4$ and one for $E_8$) which cannot be handled by our method.

1. Introduction

1.1. Let $G$ be a connected reductive group over an algebraically closed field of characteristic $p \geq 0$. In the study of the character sheaves on the group $G$ [4] an important technical assumption is that the group $G$ is clean. This means that all cuspidal local systems (see [4]) on all Levi subgroups of $G$ are clean (that is the IC-extension of these local systems coincides with the extension by zero). It is expected that any group $G$ is clean; equivalently any cuspidal local system is clean for any reductive group $G$. It is known [5] that the last assertion is equivalent to the similar assertion with $G$ assumed to be almost simple. It was shown in [5] that any cuspidal local system is clean if

1) $G$ is of classical type and $p$ is arbitrary;
2) $G$ is of type $E_6$ and $p = 0$ or $p > 2$;
3) $G$ is of type $G_2$, $F_4$, $E_7$ and $p = 0$ or $p > 3$;
4) $G$ is of type $E_8$ and $p = 0$ or $p > 5$.

In [7, 8] T. Shoji used the Shintani descent theory to improve the bounds for $p$ above for $G$ of types $G_2$, $F_4$, $E_8$.

The aim of this note is to present a simple argument which proves

**Theorem 1.** Let $G$ be an almost simple group. Then any cuspidal local system on $G$ is clean except, possibly, two cases: $p = 2$ and $G$ is of type $F_4$ or $E_8$.

Moreover, in each unsettled case there is exactly one cuspidal local system which is not known to be clean. In view of results of Lusztig [5] and Shoji [7, 8] Theorem 1 is new only for $G$ of type $E_6$ and $p = 2$ and $G$ of type $E_7$ and $p = 3$.

1.2. Applications. Recall that Lusztig defined a class of perverse sheaves on group $G$ called the admissible complexes, see [4]. It is known that the character sheaves form a subset of admissible sheaves, see [5]. It is known in many cases that actually character sheaves coincide with admissible complexes, see [5, 7, 8].

**Theorem 2.** The class of character sheaves coincides with the class of admissible complexes for any $p \geq 0$.

In view of Lusztig’s results in [5] 7.1 Theorem 2 is an immediate consequence of Theorem 2.12(a) below.

Recall that Lusztig defined generalized Springer correspondence in [4] which is a bijection between the irreducible $Ad(G)$—equivariant local systems supported on the unipotent orbits and the irreducible representations of some collection of Coxeter groups. The generalized Springer correspondence is known explicitly [6, 9]...
in all cases with two very small gaps: in the case when the Coxeter group is of type $G_2$ there is an ambiguity in attaching the local systems to two-dimensional representations of this group for $G$ of type $E_6$ when $p = 2$ and $E_8$ when $p = 3$. This ambiguity can be now removed using the method used by Lusztig in [5] 24.10 to handle a similar problem for $E_6$, $p > 3$. We use below the notations from [9].

Proposition 1. Let $G$ be of type $E_6$ and $p = 2$ (respectively, of type $E_8$ and $p = 3$). Under the generalized Springer correspondence the reflection representation of $G_2$ corresponds to the local system supported on the orbit of type $A_5$ (respectively $E_7$).

We omit the proof since it coincides with [5] 24.10 (note that the calculation of the corresponding generalized Green functions are almost identical for $E_6$ and $E_8$).

1.3. Acknowledgment. I learned the definition of the automorphism $\Theta_F$ which is crucial for this paper from Roman Bezrukavnikov who in turn learned it from Vladimir Drinfeld. I am happy to thank both of them. I am deeply grateful to George Lusztig for very useful conversations. Thanks are also due to Toshiaki Shoji for interesting comments. This work was supported in part by NSF grants DMS-0098830 and DMS-0111298.

2. Proofs

2.1. Let $a, p : G \times G$ be the adjoint action and the second projection respectively:

$$a(g, x) = gxg^{-1}, \quad p(g, x) = x.$$ 

Let $\mathcal{F}$ be a complex of (constructible) sheaves on $G$ which is $Ad(G)$–equivariant in the naive sense: we are given an isomorphism $\xi : a^*\mathcal{F} \to p^*\mathcal{F}$ satisfying the cocycle relation, see e.g. [7]. Let $\Delta : G \to G \times G$ be the diagonal embedding. Obviously $p\Delta = a\Delta = \text{Id}$. The following definition is crucial for this note:

Definition 2.1. We define a canonical automorphism $\Theta_F$ of $\mathcal{F}$ as the composition:

$$\mathcal{F} = (a\Delta)^*\mathcal{F} \to \Delta^*a^*\mathcal{F} \xrightarrow{\Delta^*\xi} \Delta^*p^*\mathcal{F} = (p\Delta)^*\mathcal{F} = \mathcal{F}.$$ 

Remark 2.2. (i) The definition above makes sense in the case when $G$ is a finite group. In this case $\Theta_F$ is well known in the conformal field theory under the name of $T$–matrix, see e.g. [1].

(ii) The automorphism $\Theta_F$ was used in [2, 7] to prove that the characteristic functions of character sheaves are eigenvectors for Shintani descent.

We are going to apply this definition for two kinds of complexes: the usual constructible sheaves and the perverse sheaves.

Lemma 2.3. Let $\mathcal{F}$ be a simple $Ad(G)$–equivariant perverse sheaf. Then $\Theta_F = \theta_F\text{Id}_F$ for some scalar $\theta_F$.

Proof. This is an immediate consequence of the Schur’s Lemma. □

Let $\mathcal{O}$ be an adjoint orbit in $G$ and let $x \in \mathcal{O}$. Recall that the functor $\mathcal{L} \mapsto \mathcal{L}_x$ defines an equivalence $\{ G$–equivariant local systems $\mathcal{L}$ on $\mathcal{O} \} \to \{ \text{Representations of } A_G(x) := Z_G(x)/Z_G(x)^0 \}$. Let $\bar{x}$ be the class of $x \in Z_G(x)$ in the group $A_G(x)$. Observe that $\bar{x} \in A_G(x)$ is central.

Lemma 2.4. Under the equivalence above we have $\Theta_{\mathcal{L}} = \bar{x}|_{\mathcal{L}_x}$.

Proof. This is a direct consequence of definition. □
Combining Lemmas \[24\] and \[25\] one can calculate the number \(\theta_\mathcal{F}\) for an irreducible \(Ad(G)\)-equivariant perverse sheaf \(\mathcal{F}\) in the following way: take any point \(x \in G\) such that \(\mathcal{F}_x \neq 0\), then \(\hat{x} \in A_G(x)\) acts on \(\mathcal{F}_x\) via the scalar \(\theta_\mathcal{F}\).

Now let \(P\) be a parabolic subgroup of \(G\) with Levi quotient \(L\). Recall (see \[5\]) that the induction functors \(\text{Ind}_P : \{ Ad(L)\text{-equivariant sheaves on } L\} \to \{ Ad(G)\text{-equivariant sheaves on } G\}\) is defined as follows: consider the variety \(\tilde{G}_P = \{(x, gP) \in G \times G/P | g^{-1} x g \in P\}\). The group \(G\) acts on \(\tilde{G}_P\) in the following way: \(h \cdot (x, gP) = (hxh^{-1}, hgP)\). It is easy to see that we have an equivalence: \(\nu : \{ G\text{-equivariant sheaves on } \tilde{G}_P\} \simeq \{ Ad(P)\text{-equivariant sheaves on } P\}\). Let \(m : P \to L\) be the canonical projection and let \(n : \tilde{G}_P \to G\) be defined by \(n(x, gP) = x\). Then the functor \(\text{Ind}_P := n\nu^{-1}m^*\) (we use here just naive notion of the equivariance but a similar construction holds for example in the equivariant derived category).

**Lemma 2.5.** The automorphism \(\Theta_\mathcal{F}\) commutes with the induction functor: 
\[
\text{Ind}_P(\Theta_\mathcal{F}) = \Theta_{\text{Ind}_P(\mathcal{F})}.
\]

**Proof.** Easy. \(\square\)

**Remark 2.6.** A statement similar to Lemma \[26\] involving the Shintani descent twisting operator instead of \(\Theta_\mathcal{F}\) is contained in \[27\].

### 2.2. Calculation of \(\theta_\mathcal{F}\) for some unipotent cuspidal pairs

We refer the reader to \[4\] for the definition of the cuspidal pair \((C, \mathcal{L})\) for the group \(G\) (recall only that here \(C\) is some inverse image of conjugacy class under projection \(G \to G/Z_G^0\) and \(\mathcal{L}\) is some \(Ad(G)\)-equivariant local system on \(G\)). In this section we consider the case when \(G\) is semisimple and \(C\) is an unipotent class (such cuspidal pairs are called unipotent) and calculate \(\theta_\mathcal{F}\) for \(\mathcal{F} = \mathcal{L}\) (extended by zero to \(G\)). First note that

**Lemma 2.7.** Assume that the characteristic of \(k\) is good for \(G\). Then \(\theta_\mathcal{L} = 1\) for any unipotent cuspidal pair \((C, \mathcal{L})\).

**Proof.** Obviously it is enough to prove the Lemma for simply connected almost simple groups. For groups of type \(A\) the order of \(A_G(u)\) for any unipotent element \(u\) is relatively prime to the characteristic of \(k\) and hence \(\bar{u} = 1\) (since the order of \(u\) is some power of the characteristic); the result follows from Lemma \[24\] and \[25\] for other classical groups \(A_G(u)\) is a 2-group; thus \(\bar{u} \in A_G(u)\) is trivial. Similarly, for groups of type \(E_6, E_7\) the order of \(A_G(u)\) is always relatively prime with the characteristic (assumed to be good). If the group \(G\) is of type \(G_2, F_4, E_8\) then the unipotent cuspidal local system \((C, \mathcal{L})\) is unique. In these cases for \(u \in C\) one has \(A_G(u) = S_3, S_4, S_5\) respectively. Thus \(A_G(u)\) has trivial center, hence \(\bar{u} = 1\). \(\square\)

Now we are going to calculate \(\theta_\mathcal{L}\) for unipotent cuspidal pairs in the exceptional groups. We are going to use the following fact due to T. Springer and B. Lou \[10\] \[9\]:

**Theorem 2.8.** Let \(u \in G\) be a regular unipotent element and let \(U \subset G\) be the maximal unipotent subgroup containing \(u\). Then \(Z_G(u) = Z_G \times Z_U(u)\); moreover the group \(Z_G(u)/Z_U^0(u)\) is cyclic and is generated by \(\bar{u}\).

The list of unipotent cuspidal pairs for exceptional groups is given in \[9\]. We give the values of \(\theta_\mathcal{L}\) in all bad characteristic cases using the notations of \textit{loc. cit.} In the table below \(\zeta\) (respectively \(\xi\)) is a fixed primitive root of unity of degree 3 (respectively 5).
Comments on the calculation. In cases 1,3,5,9,11,13,14,15,23 the calculation is immediate from 2.8; in cases 2,4,7,8,10,19,20,22,24 the calculation is immediate from the fact that $\overline{u}$ is central in $A_G(u)$. The calculation in case 12 is as follows: assume that $\overline{u} \neq 1$ in this case, then for the representation $\phi' = -1$ of $A_G(u)$ we will have $\phi'(\overline{u}) = -1$; this is a contradiction since the local system $L'$ corresponding to $\phi'$ appears in the principal series (see [9]) and thus we have $\theta_{L'} = 1$ by Lemma 2.5; thus $\overline{u} = 1$ and the result follows. The similar method applies to cases 16,17,18,21.

Finally, for the case 6 see [7] 7.2 (it is stated there that $\overline{u}$ is nontrivial in this case; one way to see this is an explicit calculation; one can also use the results on Shintani descent, see loc. cit.).

Remark 2.9. For types $G_2$, $F_4$, $E_8$ the numbers in the table were computed by T. Shoji [7, 8] as the eigenvalues of the twisting operator.

The important consequence of the calculation above is the following

Corollary 2.10. Let $L_1 \neq L_2$ be two cuspidal local systems on a simple group $G$. Then $\theta_{L_1} \neq \theta_{L_2}$ except for char($k$) = 2 and $G$ is of type $F_4$ or $E_8$.

Remark 2.11. Note that the results above together with the explicit knowledge of the generalized Springer correspondence for exceptional groups [9] allow to determine $\overline{u}$ for all unipotent elements $u$ in these groups.
2.3. Cleanness of cuspidal local systems. Recall that a local system $\mathcal{L}$ on a locally closed subset $U$ of a variety $X$ is called clean if $j_! \mathcal{L} = j_* \mathcal{L}$ where $j : U \subset X$ is the obvious embedding. It is expected that all cuspidal local systems are clean. Here is a main result of this note:

**Theorem 2.12.**

(a) Any cuspidal sheaf is a character sheaf.

(b) Assume that $G$ is an almost simple exceptional group. Let $(C, \mathcal{L})$ be a cuspidal pair for $G$. Then $\mathcal{L}$ is clean except possibly two cases: $k$ is of characteristic 2 and

1) $G$ is of type $F_4$ and $C$ is unipotent orbit of type $F_4(a_2)$ (there is a unique such cuspidal pair);

2) $G$ is of type $E_8$ and $C$ is unipotent orbit of type $D_8(a_1)$ (there is a unique such cuspidal pair).

**Proof.** The theorem is known to be true for classical groups and for exceptional groups in good characteristic [5]. The proof for exceptional groups in bad characteristic is quite similar to proofs in [5]. First one shows that any cuspidal character sheaf is clean (except, possibly, two cases in (b)) using Proposition 7.9 of [5] III with the action of the center replaced by the automorphism $\Theta_F$ and using Corollary 2.10 (recall that it is enough to consider the cuspidal local systems supported on the unipotent orbits, see [5] 7.11). Then the results of [5] provide a classification of character sheaves. Finally one compares the list of cuspidal character sheaves with the list of cuspidal local systems (known from [4]) and deduces (a). The case when $p = 2$ and $G$ is of type $F_4$ or $E_8$ requires additional arguments, see [7] 7.3 and [8] 5.3.

\[ \square \]

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