GLOBAL ESTIMATES OF RESONANCES FOR 1D DIRAC OPERATORS

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Abstract. We discuss resonances for 1D massless Dirac operators with compactly supported potentials on the line. We estimate the sum of the negative power of all resonances in terms of the norm of the potential and the diameter of its support.

Keywords: Resonances, 1D Dirac operator, estimates

1. Introduction and main results

In this paper we plan to determine global estimates of resonances in terms of the potential for massless Dirac operators $H$ acting in $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ and given by

$$H = -iJ \frac{d}{dx} + V,$$

Here $q$ is a complex-valued, integrable function with compact support $\text{supp } q \subset [0, \gamma]$ for some $\gamma > 0$. It is well known that the operator $H$ is self-adjoint (see Theorem 3.2 in [LM03]) and the spectrum of $H$ is purely absolutely continuous and covers the real line (see [DEGM82]).

Below we consider all functions and the resolvent in upper-half plane $\mathbb{C}^+$ and we will obtain their analytic extensions into the whole complex plane $\mathbb{C}$. Note that we can consider all functions and the resolvent in lower-half plane $\mathbb{C}^-$ and to obtain their analytic extensions into the whole complex plane $\mathbb{C}$. The Riemann surface of the resolvent for the Dirac operator consists of two not-connected sheets $\mathbb{C}$. In the case of the Schrödinger operator the corresponding Riemann surface is the Riemann surface of the function $\sqrt{\lambda}$.

We consider the fundamental solutions $\psi^\pm$ of the Dirac equation

$$-iJ f' + V(x)f = \lambda f$$

under the following conditions

$$\psi^\pm(x, \lambda) = e^{\pm i\lambda x} e_\pm, \quad x > \gamma; \quad \varphi^\pm(x, \lambda) = e^{\pm i\lambda x} e_\pm, \quad x < 0.$$  

(1.2)

where the vectors $e_+ = (1, 0)$ and $e_- = (0, 1)$. The scattering matrix $S(\lambda)$ for the pair $H$ and $H_0 = -iJ \frac{d}{dx}$ has the following form

$$S(\lambda) = \frac{1}{a}(1 - \frac{b}{a})(\lambda), \quad \lambda \in \mathbb{R},$$

(1.3)

see e.e. [DEGM82]. Here $\frac{1}{a}$ is the transmission coefficient and $-\frac{b}{a}$ (or $\frac{b}{a}$) is the right (left) reflection coefficient. We have

$$a(\lambda) = \det(\psi^+, \varphi^-) = \psi^+_+(0, \lambda), \quad b(\lambda) = -\psi^-_+(0, \lambda).$$

(1.4)

The function $a(\lambda)$ is analytic in the upper half-plane $\mathbb{C}^+$ and has an analytic extension in the whole complex plane $\mathbb{C}$. All zeros of $a$ lie in $\mathbb{C}^-$ (on the so-called non-physical sheet). We
denote by $(\lambda_n)_{n=1}^\infty$ the sequence of zeros of $a$ (multiplicities counted by repetition), so arranged that $0 < |\lambda_1| < |\lambda_2| < |\lambda_3| < \ldots$. By the definition, the zero $\lambda_n \in \mathbb{C}$ of $a$ is a resonance. The multiplicity of the resonance is the multiplicity of the corresponding zero of $a$.

We recall some facts from [IK12]. Let $R_0(\lambda) = (H_0 - \lambda)^{-1}$ and $\mathfrak{F}(\lambda) = V_1R_0(\lambda)|q|^{\frac{1}{2}}$ and $V = V_1|q|^{\frac{1}{2}}$. Note that $\mathfrak{F}(\lambda)$, $\text{Im} \lambda \neq 0$ is the Hilbert-Schmidt operator, but the operator $\mathfrak{F}(\lambda)$ is not trace class. In this case we define the modified Fredholm determinant $D(\lambda)$ by

$$D(\lambda) = \det \left[ (I + \mathfrak{F}(\lambda))e^{-\mathfrak{F}(\lambda)} \right], \quad \lambda \in \mathbb{C}_+.$$  

We formulate now some results about resonances from [IK12]: The determinant $D(\lambda)$ is analytic in $\mathbb{C}_+$ and has an analytic extension into the whole complex plane $\mathbb{C}$ and $D = a$. Thus all zeros of $D$ are zeros of $a$, lie in $\mathbb{C}_+$ and satisfy

$$\mathcal{N}(r, D) = \frac{2r}{\pi}(\gamma + o(1)) \quad \text{as} \quad r \to \infty,$$

where $\gamma > 0$ is a diameter of the support of the potential $q$. Here we denote the number of zeros of function $f$ having modulus $\leq r$ by $\mathcal{N}(r, f)$, each zero being counted according to its multiplicity.

We formulate our main result.

**Theorem 1.1.** Let the potential $q \in L^1(\mathbb{R})$ and let supp $q \subset [0, \gamma]$, but in no smaller interval. Then for each $p > 1$ the following estimate hold true:

$$\sum_{\text{Im} \lambda_n < 0} \frac{1}{|\lambda_n - i|^p} \leq \frac{C Y_p}{\log 2} \left( \frac{4\gamma}{\pi} + \int_\mathbb{R} |q(x)|dx \right),$$

where $C \leq 2^5$ is an absolute constant and $Y_p = \int_\mathbb{R} (1 + x^2)^{-p/2}dx$, $p > 1$.

**Remark.** 1) If $\gamma + \int_\mathbb{R} |q(x)|dx \to 0$, then all resonances go infinity.

2) The proof is based on analysis of the function $a$ and the Carleson measure arguments [C58], [C62]. We use harmonic analysis and Carleson’s Theorem (see Theorems 1.56 and 2.3.9 in [G81] and references therein) about the Carleson measure. In fact, we use the approach from [K12], where the estimates of resonances in terms of the norm of potentials for 1D Schrödinger operators were obtained. Note that in the case of the Dirac operator we obtain the sharper estimate (1.7).

3) $C$ is an absolute constant from Carleson’s Theorem ( [C58], [C62], see Theorems 1.56 and 2.3.9, [G81]), see also (2.10).

4) In fact, the estimate (1.7) gives a new global property of resonance stability.

5) The function $Y_p$ is strongly monotonic on $(1, \infty)$ and satisfies

$$Y_2 = \pi, \quad Y_p = \begin{cases} \frac{\sqrt{2\pi}}{\sqrt{p}} (1 + o(1)) & \text{as } p \to \infty \\ \frac{2^{p+1}}{p^{-p}} (1 + o(1)) & \text{as } p \to 1 \end{cases}.$$  

These properties of the function $Y_p$ is discussed in [K12]. In particular, asymptotics (1.8) are proved. Thus we can control the RHS of (1.7) at $p \to 1$ and large $p \to \infty$. Note we take $p > 1$, since the asymptotics (1.6) implies the simple fact $\sum_{n \geq 1} \frac{1}{p_n} = \infty$, see p. 17 in [L03].

Resonances for the multidimensional case were studied by Melrose, Sjöstrand, and Zworski and other, see [M83], [Z89], [SZ91] and references therein. We discuss the one dimensional case. A lot of papers are devoted to the resonances for the 1D Schrödinger operator, see Froese [F97], Simon [S00], Zworski [Z87], K11 and references therein. We recall that Zworski [Z87] obtained the first results about the asymptotic distribution of resonances for the Schrödinger
operator with compactly supported potentials on the real line. Different properties of resonances were determined in [H99], [S00], [Z87] and [K04], [K05], [K11]. Inverse problems (characterization, recovering, plus uniqueness) in terms of resonances were solved by Krotov for the Schrödinger operator with a compactly supported potential on the real line [K05] and the half-line [K04], see also [Z02] about uniqueness.

The "local resonance" stability problem was considered in [K04s]. It was proved that: if \( \kappa = (\kappa_n)_{n=1}^{\infty} \) is a sequence of eigenvalues and resonances of the Schrödinger operator with some compactly supported potential \( q \) on the half-line and \( \sum_{n=1}^{\infty} n^{2\varepsilon} |\kappa_n - \kappa_n|^2 < \infty \) for some sequence \( \tilde{\kappa} = (\tilde{\kappa}_n)_{n=1}^{\infty} \) and \( \varepsilon > 1 \), then \( \tilde{\kappa} \) is a sequence of eigenvalues and resonances of a Schrödinger operator with for some unique real compactly supported potential \( \tilde{q} \). Another type of the local resonance stability problem was studied in [MSW10].

Consider the Schrödinger operator \( H = -\Delta - V \) acting in \( L^2(\mathbb{R}^d) \), \( d \geq 1 \), where the potential \( V \geq 0 \) decreases sufficiently fastly at infinity. The negative part of the spectrum of \( H \) is discrete and let \( E_n < 0, n \geq 1 \) be the corresponding increasing sequence of eigenvalues, each eigenvalue is counted according to its multiplicity. This sequence is either finite or tends to zero. Lieb and Thirring [LT] proved inequalities of the type

\[
\sum_{n=1}^{\infty} |E_n|^\tau \leq C_{\tau,d} \int_{\mathbb{R}^d} V^{\frac{d}{2} + \tau} dx,
\]

for some positive \( \tau \). There are a lot of papers about such inequalities, see [LS10], [LW00] and references therein. In fact (1.7) is the Lieb-Thirring type inequalities for resonances of the Dirac operator.

2. Proof

2.1. Estimates for entire functions. An entire function \( f(z) \) is said to be of exponential type if there is a constant \( \alpha \) such that \( |f(z)| \leq \text{const} e^{\alpha |z|} \) everywhere. The infimum of the set of \( \alpha \) for which such inequality holds is called the type of \( f \).

**Definition.** Let \( \mathcal{E}_\gamma, \gamma > 0 \) denote the space of exponential type functions \( f \), which satisfy

\[
|f(\lambda)| \leq e^{A + \gamma(1 - \text{Im} \lambda)} \quad \forall \lambda \in \mathbb{C}, \tag{2.1}
\]

\[
|f(\lambda)| \geq 1 \quad \forall \lambda \in \mathbb{R}, \tag{2.2}
\]

for some constants \( A = A(f) \geq 0 \).

In the proof of Theorem 1.1 we need some properties of the zeros of \( f \in \mathcal{E}_\gamma \) in terms of the Carleson measure. Recall that a positive Borel measure \( M \) defined in \( \mathbb{C}_- \) is called the Carleson measure if there is a constant \( C_M \) such that for all \((r,t) \in \mathbb{R}_+ \times \mathbb{R} \)

\[
M(D_-(t,r)) \leq C_M r, \quad \text{where} \quad D_-(t,r) \equiv \{z \in \mathbb{C}_- : |z-t| < r\}, \tag{2.3}
\]

here \( C_M \) is the Carleson constant independent of \((t,r)\).

For an entire function \( f \) with zeros \( \lambda_n, n \geq 1 \) we define an associated measure by

\[
d\Omega(\lambda, f) = \sum_{\text{Im} \lambda_n < 0} \delta(\lambda - \lambda_n + i) d\mu d\eta, \quad \lambda = \mu + i\eta \in \mathbb{C}_-. \tag{2.4}
\]

In order to prove Theorem 1.1 we need following results.
Theorem 2.1. Let \( f \in \mathcal{E}, \gamma > 0 \). Then

i) for each \( r > 0 \) the following estimate hold true:

\[
\mathcal{N}(r, f) \leq \frac{1}{\log 2} \left( \frac{4r\gamma}{\pi} + A \right).
\]  

(2.5)

ii) \( d\Omega(\lambda, f) \) is the Carleson measure and satisfies

\[
\Omega(D_-(t, r), f) \leq \frac{r}{\log 2} \left( \frac{4\gamma}{\pi} + A \right) \quad \forall (r, t) \in \mathbb{R}_+ \times \mathbb{R}.
\]  

(2.6)

iii) For each \( p > 1 \) the following estimates hold true:

\[
\sum_{n \geq 1} \frac{1}{|\lambda_n - i|^p} \leq \frac{C Y_p}{\log 2} \left( \frac{4\gamma}{\pi} + A \right),
\]  

(2.7)

where \( C \leq 2^5 \) is an absolute constant and \( Y_p = \int_{\mathbb{R}} \frac{dx}{(1 + x^2)^{p/2}} \), \( p > 1 \).

Proof. i) Let \( F = e^{-i\lambda f}, \lambda = re^{i\phi} \). Then the Jensen formula implies (see 2 p. in [Koo88])

\[
\log |f(0)| + \int_0^r \frac{\mathcal{N}(t, f)}{t} dt = \frac{1}{2\pi} \int_0^{2\pi} \log |F(re^{i\phi})|d\phi,
\]  

(2.8)

since \( \mathcal{N}(t, f) = \mathcal{N}(t, F) \). Using the estimate (2.1) we obtain

\[
\log |F(re^{i\phi})| \leq \gamma r|\sin \phi| + A, \quad \lambda = re^{i\phi}, \quad \phi \in [0, 2\pi],
\]

which yields

\[
\frac{1}{2\pi} \int_0^{2\pi} \log |F(re^{i\phi})|d\phi \leq \frac{2\gamma}{\pi} r + A.
\]  

(2.9)

Substituting the estimate (2.9) into the identity (2.8) together with the simple estimate

\[
\int_0^r \frac{\mathcal{N}(t)dt}{t} \geq \mathcal{N}(\frac{r}{2}) \int_{r/2}^r \frac{dt}{t} = \mathcal{N}(\frac{r}{2}) \log 2,
\]

we obtain (2.5), since \( |f(0)| \geq 1 \).

ii) Let \( r \leq 1, t \in \mathbb{R} \). Then by the construction of \( \Omega(\cdot, f) \), we obtain \( \Omega(D_-(t, r), f) = 0 \).

Let \( r > 1, t \in \mathbb{R} \). Then due to (2.5), the measure \( \Omega(\cdot, f) \) satisfies

\[
\Omega(D_+(t, r), f) \leq \mathcal{N}(r, f(t + \cdot)) \leq \frac{1}{\log 2} \left( \frac{4r\gamma}{\pi} + A \right) \leq \frac{r}{\log 2} \left( \frac{4\gamma}{\pi} + A \right).
\]

Thus \( \Omega(\cdot, f) \) is the Carleson measure with the Carleson \( C_\Omega = \frac{1}{\log 2} \left( \frac{4\gamma}{\pi} + A \right) \).

iii) In order to show (2.7) we recall the Carleson result (see p. 63, Theorem 3.9, [G81]):

Let \( f \) be analytic on \( \mathbb{C}_- \). For \( 0 < p < \infty \) we say \( f \in \mathcal{H}_p = \mathcal{H}_p(\mathbb{C}_-) \) if

\[
\sup_{y < 0} \int_{\mathbb{R}} |f(x + iy)|^p dx = \|f\|_{\mathcal{H}_p} < \infty
\]

Note that the definition of the Hardy space \( \mathcal{H}_p \) involve all \( y > 0 \), instead of small only value of \( y \), like say, \( y \in (0, 1) \). We define the Hardy space \( \mathcal{H}_p \) for the case \( \mathbb{C}_- \), since below we work with functions on \( \mathbb{C}_- \).
If $M$ is a Carleson measure, then the following estimate holds:

$$
\int_{C_{-}} |f|^{p}dM \leq CC_{M}\|f\|_{\mathcal{F}_{p}}^{p} \quad \forall \quad f \in \mathcal{F}_{p}, \quad p \in (0, \infty),
$$

(2.10)

where $C_{M}$ is the so-called Carleson constant in (2.3) and $C \leq 2^{5}$ is an absolute constant.

In order to prove (2.7) we take the functions $f(\lambda) = \frac{1}{\lambda-1}$. Estimate (2.10) yields

$$
\int_{C_{-}} |f(\lambda)|^{p}dM = \sum_{n \geq 1} \frac{1}{|\lambda_{n} - i|^{p}} \leq CC_{M}\|f\|_{\mathcal{F}_{p}}^{p}, \quad p \in (1, \infty).
$$

(2.11)

where we have the simple identity

$$
\|f\|_{\mathcal{F}_{p}}^{p} = \int_{\mathbb{R}} \frac{dt}{|t-i|^{p}} = \int_{\mathbb{R}} \frac{dt}{(t^{2} + 1)^{p/2}} = Y_{p}.
$$

(2.12)

Combine (2.11) and (2.12) we obtain (2.7). 

2.1. Estimates of resonances. We consider the function $a$ given by (1.4) under the condition that the potential $q$ satisfies $\text{supp } q \subset [0, \gamma]$. The solution $\psi^{+} = (\psi_{1}^{+}, \psi_{2}^{+})$ satisfy the integral equations

$$
\psi^{+}(x, \lambda) = e^{i\lambda x}e_{1} + \int_{x}^{\gamma} iJe^{i\lambda(x-t)J}V(t)\psi^{+}(t, \lambda)dt,
$$

(2.13)

where

$$
iJe^{i\lambda(x-t)J}V(t) = i \begin{pmatrix} 0 & q(t)e^{i\lambda(x-t)} \\ -\overline{q(t)}e^{-i\lambda(x-t)} & 0 \end{pmatrix}.
$$

(2.14)

For $\chi = e^{-i\lambda x}\psi^{+}_{1}(x, \lambda)$ using (2.14) we obtain

$$
\chi(x, \lambda) = 1 + i \int_{x}^{\gamma} e^{-i\lambda s}q(s)\psi^{+}_{2}(s, \lambda)ds,
$$

$$
\psi_{2}^{+}(s, \lambda) = -i \int_{s}^{\gamma} e^{i\lambda(2t-s)\overline{q(t)}}\chi(t, \lambda)dt.
$$

Then $\chi(x, \lambda)$ satisfies the following integral equation

$$
\chi(x, \lambda) = 1 + \int_{x}^{\gamma} q(t_{1})dt_{1}\int_{t_{1}}^{\gamma} e^{2i\lambda(t_{2} - t_{1})\overline{q(t_{2})}}\chi(t_{2}, \lambda)dt_{2}.
$$

(2.15)

We have the standard formal iterations given by

$$
\chi(x, \lambda) = 1 + \sum_{n \geq 1} \chi_{n}(x, \lambda), \quad \chi_{n}(x, \lambda) = \int_{x}^{\gamma} q(t_{1})dt_{1}\int_{t_{1}}^{\gamma} e^{2i\lambda(t_{2} - t_{1})\overline{q(t_{2})}}\chi_{n-1}(t_{2}, \lambda)dt_{2},
$$

(2.16)

where $\chi_{0}(\cdot, \lambda) = 1$. Due to (1.4) we get $a(\lambda) = \chi(0, \lambda)$, which yields

$$
a(\lambda) = 1 + \sum_{n \geq 1} a_{n}(\lambda), \quad a_{n}(\lambda) = \chi_{n}(0, \lambda).
$$

(2.17)

We will describe these iterations and the function $a$.

Lemma 2.2. Let $q \in L^{1}(\mathbb{R})$ and $\text{supp } q \subset [0, \gamma]$. Then the function $a \in \mathcal{E}_{\gamma}$ and satisfies

$$
|a_{n}(\lambda)| \leq e^{\gamma(|n| - \eta)}\|q\|_{1}^{2n} \quad \forall \quad n \geq 1,
$$

(2.18)

$$
|a(\lambda)| \leq e^{\gamma(|n| - \eta)} ch \|q\|_{1},
$$

(2.19)
where \( \eta = \text{Im} \lambda \) and \( \|q\|_1 = \int \|q(x)\|dx \).

**Proof.** Let \( \eta_- = \frac{(|\eta| - \eta)}{2} \). Then using (2.16) we obtain

\[
|a_n(\lambda)| \leq \int_{0=0}^{t_0<0} \left( \prod_{1 \leq j \leq n} \left| q(t_{2j-1})q(t_{2j})e^{2\eta_-(t_{2j}-t_{2j-1})} \right| \right) dt_1 dt_2 ... dt_{2n},
\]

which yields

\[
|a_n(\lambda)| \leq \int_{0=0}^{t_0<0} \left( \prod_{1 \leq j \leq n} \left| q(t_{2j-1})q(t_{2j}) \right| e^{2\eta_- t_{2j-1}} dt_1 dt_2 ... dt_{2n} \leq e^{\gamma(|\eta| - \eta)} \int_{0=0}^{t_0<0} \left| q(t_1)q(t_2) ... q(t_{2n}) \right| dt_1 dt_2 ... dt_{2n} \leq e^{\gamma(|\eta| - \eta)} \||q\|^2_{2n} \right/(2n)!,
\]

which yields (2.18).

This shows that the series (2.17) converges uniformly on bounded subsets of \( \mathbb{C} \). Each term of this series is an entire function. Hence the sum is an entire function. Summing the majorants we obtain estimates (2.19) and (2.20). Thus the function \( a \) is entire.

The S-matrix is unitary, see [DEGM82], then from (1.3) we have the well-known fact \( |a(\lambda)|^2 = |b(\lambda)|^2 + 1 \geq 1 \) for all \( \lambda \in \mathbb{R} \). Then due to (2.20) we deduce that \( a \) belongs to \( \mathcal{E}_\gamma \).

**Proof of Theorem 1.1.** Recall that by Lemma 2.2 the function \( a \in \mathcal{E}_\gamma \) with \( A = \|q\|_1 \), since \( a \) satisfies \( |a(\lambda)| \leq e^{\gamma(|\eta| - \eta) + \|q\|_1} \), for all \( \lambda \in \mathbb{C} \) with \( \eta = \text{Im} \lambda \). Thus estimate (2.7) gives the main estimate (1.7) in Theorem 1.1. ■

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