INTUITIONISTIC LOGIC WITH TWO GALOIS CONNECTIONS
COMBINED WITH FISCHER SERVI AXIOMS

WOJCIECH DZIK, JOUNI JÄRVINEN, AND MICHIRO KONDO

Abstract. Earlier, the authors introduced the logic IntGC, which is an extension of intuitionistic propositional logic by two rules of inference mimicking the performance of Galois connections (Logic J. of the IGPL, 18:837-858, 2010). In this paper, the extensions Int2GC and Int2GC+FS of IntGC are studied. Int2GC can be seen as a fusion of two IntGC logics, and Int2GC+FS is obtained from Int2GC by adding instances of duality-like connections $\Diamond (A \rightarrow B) \rightarrow (\Box A \rightarrow \Diamond B)$ and $(\Diamond A \rightarrow \Box B) \rightarrow (\Box A \rightarrow B)$, introduced by G. Fischer Servi (Rend. Sem. Mat. Univers. Politecn. Torino, 42:179-194, 1984), for interlinking the two Galois connections of Int2GC. Both Kripke-style and algebraic semantics are presented for Int2GC and Int2GC+FS, and the logics are proved to be complete with respect to both of these semantics. We show that rough lattice-valued fuzzy sets defined on complete Heyting algebras are proper algebraic models for Int2GC+FS. We also prove that Int2GC+FS is equivalent to the intuitionistic tense logic IK$^t$, and an axiomatisation of IK$^t$ with the number of axioms reduced to the half of the number of axioms given by W. B. Ewald (J. Symb. Log, 51:166–179, 1986) is presented.

Key words and phrases: intuitionistic logic, Galois connections, information logic, rough lattice-valued fuzzy sets on complete Heyting algebras, Kripke semantics, intuitionistic tense logic, completeness theorems.

1. Introduction and Motivation

In [15], Information Logic of Galois Connections (ILGC) was introduced as classical propositional logic with a pair of unary connectives $\blacktriangle$ and $\blacktriangledown$ mimicking a Galois connection. Motivation for ILGC originates in rough set theory [16], where it is assumed that our knowledge about objects of a universe of discourse $U$ is expressed by an information relation $R$. An information relation may reflect similarity or difference between objects. For instance, $R$ can be defined on the set of all human beings in such a way that two persons are $R$-related if they are of the same gender and the difference of their ages is less than a year. Originally, Pawlak assumed information relations to be equivalences (reflexive, symmetric, and transitive binary relations), so called indiscernibility relations, but in the literature can be found numerous studies considering information relations of different type; see [4], for example.

In terms of an information relation $R$, we may define the upper approximation of a set $X \subseteq U$ as

$$X^\blacktriangle = \{ x \in U \mid (\exists y \in U) \ x R y \ \& \ y \in X \},$$

and the lower approximation of $X$ is

$$X^\blacktriangledown = \{ x \in U \mid (\forall y \in U) \ x R y \Rightarrow y \in X \}.$$

For instance, if $R$ is the information relation considered above, then $x \in X^\blacktriangledown$ if all the persons that are coarsely of the same age and are of the same gender as $x$ belong to $X$, and $x \in X^\blacktriangle$ if there exists at least one such person. Therefore, $\blacktriangledown$ may
be interpreted to represent certainty and ▲ possibility with respect to knowledge expressed by the relation R.

We may also define another pair of mappings φ(U) → φ(U) by reversing the relation R. For any set \( X \subseteq U \), let us define

\[
X^\Delta = \{ x \in U \mid (\exists y \in U) yRx & y \in X \}
\]
and

\[
X^\nabla = \{ x \in U \mid (\forall y \in U) yRx \Rightarrow y \in X \}.
\]

It is well-known that for any binary relation, the pairs \((\triangle, \nabla)\) and \((\nabla, \nabla)\) are order-preserving Galois connections \(\phi(U) \rightarrow \phi(U)\).

The logic ILGC was defined by adding to classical propositional logic two rules of inference:

\[
\begin{align*}
(GC \land \land) & : A \rightarrow \nabla B \\
& A \land A \rightarrow B
\end{align*}
\]

\[
\begin{align*}
(GC \lor \lor) & : A \rightarrow B \\
& A \lor A \rightarrow \nabla B
\end{align*}
\]

Another pair of connectives is introduced by De Morgan-type assertions:

\[
\begin{align*}
(\ast) & : \triangle A = \neg \nabla \neg A & \nabla A = \neg \triangle \neg A
\end{align*}
\]

For \( \triangle \) and \( \nabla \) the following rules are admissible in ILGC:

\[
\begin{align*}
(GC \land \land) & : A \rightarrow \nabla B \\
& \triangle A \rightarrow B
\end{align*}
\]

\[
\begin{align*}
(GC \lor \lor) & : A \rightarrow B \\
& \nabla A \rightarrow \nabla B
\end{align*}
\]

This means that in ILGC, we get another Galois connection \((\land, \lor)\) “for free”.

In [7], we introduced an intuitionistic propositional logic with a Galois connection (IntGC) and studied its main properties. In addition to the intuitionistic logic axioms and inference rule of Modus Ponens, IntGC contains rules \((GC \land \land)\) and \((GC \lor \lor)\). Since the base logic is changed from classical to intuitionistic, the classical-type axioms \((\ast)\) can not be used to introduce another Galois connection. More precisely, if we define the operators \(\triangle, \nabla\) from \(\nabla, \land\) in terms of intuitionistic negation, the pair \((\triangle, \nabla)\) does not form a Galois connection; see Lemma 3.3 in [7]. Therefore, to define an intuitionistic logic of two Galois connections, the other Galois connection must be defined by adding rules \((GC \land \land)\) and \((GC \lor \lor)\).

In Section 2 we define two intuitionistic logics with two Galois connections. The first one, called Int2GC, is obtained by extending intuitionistic propositional logic with the connectives \(\land, \lor, \triangle, \nabla\) and by rules \((GC \land \land)\), \((GC \lor \lor)\), \((GC \land \land)\), \((GC \lor \lor)\). In Int2GC, the two Galois connections \((\land, \lor)\) and \((\triangle, \nabla)\) are not connected with each other, and this means that Int2GC is simply the fusion of two IntGC logics, the first one having the operators \(\land \) and \(\lor\), and the second has \(\triangle \) and \(\nabla\). The logic Int2GC+FS is obtained by extending Int2GC with instances of the axioms \(\land (A \rightarrow B) \rightarrow (\square A \rightarrow \nabla B)\) and \(\nabla (A \rightarrow \square B) \rightarrow \square (A \rightarrow B)\) introduced by Fischer Servi [10]. This means that Int2GC+FS has duality-like connections \(\triangle (A \rightarrow B) \rightarrow (\nabla A \rightarrow \nabla B)\), \(\nabla (A \rightarrow B) \rightarrow (\triangle A \rightarrow \triangle B)\), \(\nabla (A \rightarrow \nabla B) \rightarrow \nabla (A \rightarrow B)\), and \(\nabla (A \rightarrow \nabla B) \rightarrow \nabla (A \rightarrow B)\). These axioms defining Int2GC+FS are referred to as (FS1), (FS2), (FS3), and (FS4), respectively. We show that in Int2GC, axioms (FS1) and (FS4) are equivalent, and the same holds with (FS2) and (FS3). This implies that we have several equivalent combinations of axioms to define Int2GC+FS.

Section 3 is devoted to H2GC- and H2GC+FS-algebras that are used for defining algebraic semantics for Int2GC and Int2GC+FS, respectively. H2GC-algebras are Heyting algebras equipped with two order-preserving Galois connections \(\langle \land, \lor \rangle\) and \(\langle \land, \lor \rangle\), and H2GC+FS-algebras are H2GC-algebras such that the operations \(\nabla\) and \(\nabla\) are connected by an identity corresponding to axiom (FS1), and \(\nabla\) and \(\nabla\) are connected by an equation that corresponds (FS2). In [8], J. M. Dunn studied distributive lattices with two operators \(\square\) and \(\land\). He introduced conditions \((D_\land)\ \land x \land \square y \leq \land (x \land y)\) and \((D_\lor)\ \square (x \lor y) \leq \land x \lor \land y\) for the interactions
between $\square$ and $\Diamond$. We show that $\text{H2GC}+\text{FS}$-algebras can be defined also as $\text{H2GC}$-algebras satisfying the identities corresponding $(D_\wedge)$, that is, $\text{H2GC}+\text{FS}$-algebras are $\text{H2GC}$-algebras satisfying $a^{\wedge} \wedge b^{\wedge} \to (a \wedge b)^{\wedge} = 1$ and $a^{\wedge} \wedge b^{\wedge} \to (a \wedge b)^{\wedge} = 1$. In Section 4.2, we consider rough fuzzy sets defined on complete Heyting algebras, and show how in this setting $\text{H2GC}+\text{FS}$-algebras arise naturally. Algebras of rough fuzzy sets satisfy $(D_\wedge)$ when $\Diamond$ and $\square$ are interpreted by $\wedge$ and $\vee$ (or $>^\wedge$ and $<^\wedge$), but condition $(D_\vee)$ is not satisfied. So, rough fuzzy sets are proper algebraic models for $\text{H2GC}+\text{FS}$. In Section 5, we introduce algebraic semantics for $\text{Int2GC}$ and $\text{Int2GC}+\text{FS}$ with respect to $\text{H2GC}$- and $\text{H2GC}+\text{FS}$-algebras, respectively, and present algebraic completeness theorems.

In Section 4, Kripke-semantics for $\text{Int2GC}$ and $\text{Int2GC}+\text{FS}$ are considered. We begin with recalling Kripke-frames and completeness for $\text{IntGC}$ from [7]  in Section 4.1. In addition, we introduce Kripke-frames and semantics for $\text{Int2GC}$ and $\text{Int2GC}+\text{FS}$, and soundness of both $\text{Int2GC}$ and $\text{Int2GC}+\text{FS}$ is proved. Canonical frames of $\text{H2GC}$-algebras are introduced and Kripke-completeness is proved. Section 4.1 ends by an example in which particular Kripke-frames for $\text{Int2GC}+\text{FS}$ frames of $\text{H2GC}$-algebras are introduced and Kripke-completeness is proved. Canonical frames and algebraic completeness result of $\text{Int2GC}+\text{FS}$.

It is proved in [15] that $\text{ILGC}$ is equivalent, with respect to provability, to the minimal (classical) tense logic $\text{K}_t$, that is, $\text{ILGC}$ can be viewed as a simple formulation of $\text{K}_t$. In Section 4, we prove that intuitionistic tense logic $\text{IK}_t$, introduced by Ewald [9], is equivalent syntactically to $\text{Int2GC}+\text{FS}$ when $\Box, \Diamond, \square, \triangledown$ are identified with tense operators $F, G, P, H$, respectively. In other words, in $\text{IK}_t$ and $\text{Int2GC}+\text{FS}$ exactly the same formulas can be proved. This then means that $\text{Int2GC}+\text{FS}$ can be seen as an alternative formulation of $\text{IK}_t$. In addition, we give an axiomatisation of $\text{IK}_t$ with the number of axioms reduced to half of the number of axioms of $\text{IK}_t$ (with the same rules) given by Ewald [9], and we present another definition of $\text{Int2GC}$ using only axioms of Ewald and rules admissible in $\text{IK}_t$.

2. Intuitionistic logics with Galois connections and Fischer Servi axioms

In this section we introduce two modal logics $\text{Int2GC}$ and $\text{Int2GC}+\text{FS}$ based on intuitionistic propositional logic [3]. We begin with recalling the intuitionistic propositional logic with a Galois connection (IntGC) defined by the authors in [7]. The language of IntGC is constructed from an enumerable infinite set of propositional variables Var, the connectives $\neg, \lor, \land, \to$, and the unary operators $\Box$ and $\Diamond$. The constant $true$ is defined by setting $\top := p \to p$ for some fixed propositional variable $p \in \text{Var}$, and the constant $false$ is defined by $\bot := \neg \top$. We also set $A \leftrightarrow B := (A \to B) \land (B \to A)$. The logic IntGC is the smallest logic that contains intuitionistic propositional logic, and is closed under the rules of substitution, modus ponens, and rules (GC$\lor$) and (GC$\land$). The following rules are admissible in IntGC:

$(\text{RN}\lor) \quad \frac{A \lor \bot}{\bot}$

$(\text{RM}\lor) \quad \frac{A \to B \lor \bot \to \bot}{\bot}$

$(\text{RM}\land) \quad \frac{A \land B}{\land A \to \land B}$

In addition, the following formulas are provable:

(GC1) $A \to \Box A$ and $\Box \land A \to A$;
(GC2) $\Box A \leftrightarrow \Box \land A$ and $\land A \leftrightarrow \land \land A$;
(GC3) $\land \top$ and $\neg \Box \bot$;
The language of the logic \( \text{Int2GC} \) is the one of \( \text{IntGC} \) extended by two unary connectives \( \triangle \) and \( \nabla \), and the logic \( \text{Int2GC} \) is the smallest logic extending \( \text{IntGC} \) by rules (GC \( \nabla \triangle \)) and (GC \( \triangle \nabla \)). Obviously, in \( \text{Int2GC} \) also the rules:

\[
\begin{align*}
(GC4) \quad & \nabla (A \land B) \leftrightarrow \nabla A \land \nabla B \quad \text{and} \quad \Box (A \lor B) \leftrightarrow \Box A \lor \Box B; \\
(GC5) \quad & \nabla (A \rightarrow B) \rightarrow (\nabla A \rightarrow \nabla B).
\end{align*}
\]

are admissible, and the following formulas are provable:

\[
\begin{align*}
(GC1) & \quad A \rightarrow \nabla A \quad \text{and} \quad \nabla A \rightarrow A; \\
(GC2) & \quad \nabla A \leftrightarrow \nabla \nabla A \quad \text{and} \quad \nabla A \leftrightarrow \nabla \nabla A; \\
(GC3) & \quad \nabla \top \quad \text{and} \quad \neg \Delta \bot; \\
(GC4) & \quad \nabla (A \land B) \leftrightarrow \nabla A \land \nabla B \quad \text{and} \quad \nabla (A \lor B) \leftrightarrow \nabla A \lor \nabla B; \\
(GC5) & \quad \nabla (A \rightarrow B) \rightarrow (\nabla A \rightarrow \nabla B).
\end{align*}
\]

Intuitionistic modal logic \( \text{IK} \) was introduced by G. Fischer Servi in [10]. The logic \( \text{IK} \) is obtained by adding two modal connectives \( \Diamond \) and \( \Box \) to intuitionistic logic satisfying the following axioms:

\[
\begin{align*}
(IK1) & \quad \Diamond (A \land B) \rightarrow \Diamond A \land \Diamond B \\
(IK2) & \quad \Box A \land \Box B \rightarrow \Box (A \land B) \\
(IK3) & \quad \neg \Diamond \bot \\
(IK4) & \quad \Diamond (A \rightarrow B) \rightarrow (\Box A \rightarrow \Diamond B) \\
(IK5) & \quad (\Diamond A \rightarrow \Box B) \rightarrow (\Box A \rightarrow B)
\end{align*}
\]

In addition, the monotonicity rules for both \( \Diamond \) and \( \Box \) are admissible, that is:

\[
\begin{align*}
(RM\Diamond) & \quad A \rightarrow B \quad \Rightarrow \quad \Diamond A \rightarrow \Diamond B \\
(RM\Box) & \quad A \rightarrow B \quad \Rightarrow \quad \Box A \rightarrow \Box B
\end{align*}
\]

In this work, we call axioms (IK4) and (IK5) the Fischer Servi axioms, and they have a special role in interlinking the two Galois connections of \( \text{Int2GC} \). From (IK4) and (IK5) we can form the following four axioms by replacing \( \Box \) and \( \Diamond \) by \( \nabla \) and \( \Box \), and by \( \nabla \) and \( \Delta \), respectively:

\[
\begin{align*}
(FS1) & \quad \Box (A \rightarrow B) \rightarrow (\nabla A \rightarrow \Box B) \\
(FS2) & \quad \Delta (A \rightarrow B) \rightarrow (\nabla A \rightarrow \Delta B) \\
(FS3) & \quad (\Box A \rightarrow \nabla B) \rightarrow (\nabla (A \rightarrow B) \\
(FS4) & \quad (\Delta A \rightarrow \nabla B) \rightarrow (\nabla (A \rightarrow B)
\end{align*}
\]

**Proposition 2.1.** In \( \text{Int2GC} \), the following assertions hold:

(a) Axioms (FS1) and (FS4) are equivalent.

(b) Axioms (FS2) and (FS3) are equivalent.

**Proof.** We prove only assertion (a), because (b) can be proved analogously. Here \( \vdash A \) denotes that \( A \) is provable in \( \text{Int2GC} \).

(FS1)\Rightarrow (FS4): Let us set \( X := A, Y := \nabla \Delta A \) and \( Z := \Box \nabla B \) in the provable formula \( (X \rightarrow Y) \rightarrow ((Y \rightarrow Z) \rightarrow (X \rightarrow Z)) \). We get \( \vdash (\nabla \Delta A \rightarrow \Box \nabla B) \rightarrow (A \rightarrow \nabla \Delta B) \) by using also \( \vdash A \rightarrow \nabla \Delta A \). This is equivalent to \( \vdash A \land (\nabla \Delta A \rightarrow \Box \nabla B) \rightarrow \nabla \Delta B \). Because \( \vdash \nabla \Delta A \rightarrow \Box \nabla B \rightarrow (A \rightarrow B) \), this means \( \vdash (A \land (\nabla \Delta A \rightarrow \Box \nabla B) \rightarrow \nabla \Delta A \rightarrow \Box \nabla B) \rightarrow (A \rightarrow B) \). If we set \( A := \Delta A \) and \( B := \nabla B \) in (FS1), we obtain \( \vdash (\nabla \Delta A \rightarrow \Box \nabla B) \rightarrow (\nabla \Delta A \rightarrow \Box \nabla B) \), and so \( \vdash (\Delta A \rightarrow \nabla B) \rightarrow (A \rightarrow B) \). This implies \( \vdash (\Delta A \rightarrow \nabla B) \rightarrow \nabla (A \rightarrow B) \) by (GC \( \nabla \)).

(FS4)\Rightarrow (FS1): We set \( X := \Delta \nabla A, Y := A \) and \( Z := B \) in \( (X \rightarrow Y) \rightarrow ((Y \rightarrow Z) \rightarrow (X \rightarrow Z)) \). This gives \( \vdash (\Delta \nabla A \rightarrow A) \rightarrow ((A \rightarrow B) \rightarrow (\Delta \nabla A \rightarrow B)) \), and \( \vdash (A \rightarrow
have ▽ ▲ by (GC B algebras and give completeness theorems for ⊢ Int2GC+FS several equivalent axiomatisations of Int2GC+FS. In particular, if P these algebras.

L connection on ϕ (gc1) (HGC-algebras form an equational class. HGC-algebras are usually denoted by (H, ▽, ▲, ▵). Therefore, we obtain ⊢ △ ▼ A → △ ▼ A by (GC △ ▼).

\[ x \text{ is } \text{the greatest element, Heyting algebras are always distributive pseudocomplemented lattices such that the pseudocomplement } \neg a \text{ of } a \text{ is } a \to 0. \]

\[ \text{H = (H, } \lor, \land, \to, 0, \neg, ▽, ▲) \text{, which can be equationally defined as follows (see e.g. [1]):} \]

1. \[ x \land (x \to y) = x \land y \]
2. \[ x \land (y \to z) = x \land ((x \land y) \to (x \land z)) \]
3. \[ z \land (x \land y \to x) = x \]

Note also that if H is a Heyting algebra, then (gc2) can be written in the form (gc2)∗ a → ψ(φ(a)) = 1 and ψ(φ(a)) → a = 1.

An HGC-algebra is an algebra (H, ▽, ▲, ▵, △, ▼), where \( H = (H, \lor, \land, \to, 0) \) is a Heyting algebra and (▲, ▼) is a Galois connection on H. By the above, HGC-algebras form an equational class. HGC-algebras are usually denoted by (H, △, ▽, ▼). An H2GC-algebra \( H = (H, △, ▽, ▼) \) is such that \( H = (H, \lor, \land, \to, 0) \) is a Heyting algebra, and (H, △, ▽) and (H, △, ▼) are H2GC-algebras, meaning that...
\((\bullet, \circ)\) and \((\triangledown, \mathbin{\rhd})\) are Galois connections on \(H\). Also H2GC-algebras form an equational class. We denote H2GC-algebras simply by \((H, \bullet, \circ, \triangledown, \mathbin{\rhd})\).

For an H2GC-algebra \((H, \bullet, \circ, \triangledown, \mathbin{\rhd})\), we introduce the following identities corresponding to the instances (FS1)–(FS4) of the Fischer Servi axioms:

\begin{align*}
&\text{(fs1)} \quad (a \to b)\bullet \to (a \circ \to b) = 1 \\
&\text{(fs2)} \quad (a \to b)\circ \to (a \circ \to b) = 1 \\
&\text{(fs3)} \quad (a \triangledown \to b\triangledown) \to (a \to b)\circ = 1 \\
&\text{(fs4)} \quad (a \triangledown \to b\triangledown) \to (a \to b)\mathbin{\rhd} = 1
\end{align*}

In \([6]\), J. M. Dunn studied minimal positive modal logic \(K_+\) with the connectives \(\land, \lor, \Box,\) and \(\lozenge\). \(K_+\) can be described in algebraic terms as modal logic based on a distributive lattice with two operations \(\Box\) and \(\lozenge\), where \(\Box\) distributes over \(\land\), \(\lozenge\) distributes over \(\lor\), and the following two conditions hold:

\begin{align*}
&(D_\land) \quad a \land \Box b \leq \Box (a \land b) \\
&(D_\lor) \quad a \lor b \leq \Box (a \lor b)
\end{align*}

We introduce the instances of \((D_\land)\) as identities defined on an H2GC-algebra:

\begin{align*}
&\text{(d1)} \quad a^\bullet \land b^\circ \to (a \land b)^\circ = 1 \\
&\text{(d2)} \quad a^\circ \land b^\circ \to (a \land b)^\circ = 1
\end{align*}

Now we may write the following proposition.

**Proposition 3.1.** Let \((H, \bullet, \circ, \triangledown, \mathbin{\rhd})\) be an H2GC-algebra.

(a) **Identities** (fs1), (d1), and (fs4) are equivalent.

(b) **Identities** (fs2), (d2), and (fs3) are equivalent.

**Proof.** (a) Let us set \(a := a \to b \) and \(b := a\) in (d1). We obtain \((a \to b)\bullet \land a^\circ \leq (a \land (a \to b))^\circ\), because \(a \land (a \to b) \leq b\). This gives directly \((a \to b)\bullet \leq a^\circ \to b)\circ\), that is, \((a \to b)\bullet \to (a \to b)\circ = 1\), and so (d1) implies (fs1). Conversely, if we set \(b := a \land b\) in (fs1), we have \(b^\circ \leq (a \to b)\bullet \leq (a \to a \land b)^\circ \leq a^\circ \to (a \land b)^\circ\), because \(a \to a \land b = a \to b\) and \(b \leq a \to b\). This is equivalent to \(a^\circ \land b^\circ \leq (a \land b)^\circ\), and \((a^\circ \land b^\circ) \to (a \land b)^\circ = 1\). Thus, also (fs1) implies (d1), and (fs1) and (d1) are equivalent.

Suppose that (fs1) holds. Then \((a^\circ \to b^\circ)\bullet \leq a^\circ \to b^\circ\). Since \(a \leq a^\circ \) and \(b^\circ \leq b\), we have \((a^\circ \to b^\circ)\bullet \leq a \to b\). This is equivalent to \(a^\circ \to b^\circ \leq (a \to b)^\circ\) and \((a^\circ \to b^\circ) \to (a \to b)^\circ = 1\), that is, (fs4) is true. On the other hand, if (fs4) holds, then \(a \to b \leq a^\circ \to b^\circ \leq (a^\circ \to b^\circ)^\circ\), that is, \((a \to b)^\circ \leq (a^\circ \to b^\circ)^\circ\), \((a \to b)^\circ \leq (a^\circ \to b^\circ)^\circ = 1\), and (fs1) is true. Hence, (fs1) and (fs4) are equivalent.

Case (b) can be proved analogously.

\(\square\)

An H2GC+FS-algebra is an H2GC-algebra \((H, \bullet, \circ, \triangledown, \mathbin{\rhd})\) satisfying (fs1) and (fs2). By Proposition 3.1, H2GC+FS-algebras have several equivalent characterizations. Clearly, H2GC+FS-algebras form an equational class.

3.2. **Rough fuzzy sets on complete Heyting algebras.** We consider here rough lattice-valued fuzzy sets defined on complete Heyting algebras. These are also closely connected to fuzzy Galois connections studied, for instance, in \([2][11]\).

A complete Heyting algebra is a Heyting algebra such that its underlying ordered set is a complete lattice. A complete lattice \(L\) satisfies the **join-infinite distributive law** if for any \(S \subseteq L\) and \(x \in L\),

\[(JID) \quad x \land \left(\bigvee S\right) = \bigvee \{x \land y \mid y \in S\}.
\]

A complete lattice is a Heyting algebra if and only if it satisfies \((JID)\) (see e.g. \([13][17]\)). Thus, complete Heyting algebras are the complete lattices satisfying \((JID)\).

Rough fuzzy sets were generalized to \(L\)-fuzzy sets by J. A. Goguen in such a way that an \(L\)-fuzzy set \(\varphi\) on \(U\) is a mapping \(\varphi: U \to L\), where \(U\) is any set representing
objects of some universe of discourse and \( L \) is a partially ordered set \([12]\). The set \( L^U \) of all maps from \( U \) to \( L \) is then the set of all \( L \)-fuzzy sets on \( U \). The set \( L^U \) can be equipped whatever operators \( L \) has, and these induced operators obey any law valid in \( L \) which extends pointwise.

Here we assume that \( \mathbb{H} \) is a complete Heyting algebra, therefore we can make \( H^U \) a complete Heyting algebra by defining

\[
\left( \bigvee_{i \in I} \varphi_i \right)(a) = \bigvee_{i \in I} \varphi_i(a) \quad \text{and} \quad \left( \bigwedge_{i \in I} \varphi_i \right)(a) = \bigwedge_{i \in I} \varphi_i(a)
\]

for all \( \{ \varphi_i \}_{i \in I} \subseteq H^U \). The least element of \( H^U \) is \( 0 \): \( x \mapsto 0 \) and the greatest element of \( H^U \) is \( 1 \): \( x \mapsto 1 \). Furthermore, \( H^U \) is relatively pseudocomplemented in such a way that for all \( \varphi, \psi \in L^U \) and \( a \in U \),

\[
(\varphi \rightarrow \psi)(a) = \varphi(a) \rightarrow \psi(a)
\]

We denote this complete Heyting algebra by \( \mathbb{H}^U \). Elements of this Heyting algebra are called \( \mathbb{H} \)-\textit{sets}.

Dubois and Prade introduced \textit{rough fuzzy sets} in \([3]\). The idea is that the objects to be approximated are fuzzy sets, and the approximations are determined by means of fuzzy relations. Here we study \textit{rough} \( \mathbb{H} \)-\textit{sets}, which means that approximations of \( \mathbb{H} \)-\textit{sets} are determined by \( \mathbb{H} \)-fuzzy relations.

Let \( \varphi \) be an \( \mathbb{H} \)-set and let \( R \) be an \( \mathbb{H} \)-fuzzy relation on \( U \), that is, \( R \) is a mapping from \( U \times U \) to \( H \). Then, we may define the \( \mathbb{H} \)-sets \( \varphi^\downarrow \) and \( \varphi^\uparrow \) by setting

\[
\varphi^\downarrow(x) = \bigvee_{y \in U} \{ R(x, y) \land \varphi(y) \}
\]

\[
\varphi^\uparrow(x) = \bigwedge_{y \in U} \{ R(x, y) \rightarrow \varphi(y) \}
\]

for all \( x \in U \). The \( \mathbb{H} \)-sets \( \varphi^\downarrow \) and \( \varphi^\uparrow \) are called the upper and the lower approximations of \( \varphi \).

We can define another pair of mappings in terms of the inverse of \( R \) by setting

\[
\varphi^\downarrow(x) = \bigvee_{y \in U} \{ R(y, x) \land \varphi(y) \}
\]

\[
\varphi^\uparrow(x) = \bigwedge_{y \in U} \{ R(y, x) \rightarrow \varphi(y) \}
\]

for all \( x \in U \). It is clear that if \( \varphi \) is a two-valued set on \( U \) and \( R \) is a two-valued binary relation on \( U \), then the operations \( \downarrow, \uparrow, \downarrow, \uparrow \) coincide with the rough set operators defined by a binary relation.

\textbf{Proposition 3.2.} For any complete Heyting algebra \( \mathbb{H} \) and an \( \mathbb{H} \)-fuzzy relation \( R \) on \( U \), the algebra of rough \( \mathbb{H} \)-sets \( (\mathbb{H}^U, \downarrow, \uparrow, \downarrow, \uparrow) \) is an H2GC+FS-algebra.

\textit{Proof.} Suppose \( \varphi \) and \( \psi \) are \( \mathbb{H} \)-sets such that \( \varphi \leq \psi \). Then for all \( y \in U \), \( R(x, y) \land \varphi(y) \leq R(x, y) \land \psi(y) \) and this implies

\[
\varphi^\downarrow(x) = \bigvee_{y \in U} \{ R(x, y) \land \varphi(y) \} \leq \bigvee_{y \in U} \{ R(x, y) \land \psi(y) \} = \psi^\downarrow(x).
\]

Similarly, \( R(y, x) \rightarrow \varphi(y) \leq R(y, x) \rightarrow \psi(y) \) for all \( y \in U \). Thus,

\[
\varphi^\uparrow(x) = \bigwedge_{y \in U} \{ R(y, x) \rightarrow \varphi(y) \} \leq \bigwedge_{y \in U} \{ R(y, x) \rightarrow \psi(y) \} = \psi^\uparrow(x).
\]
Hence, for all \( y \),
\[ \varphi^\wedge(x) = \bigvee_{y \in U} \{ R(x, y) \land \varphi^\wedge(y) \} = \bigvee_{y \in U} \{ R(x, y) \land \bigwedge_{z \in U} \{ R(z, y) \to \varphi(z) \} \} \]
\[ \leq \bigvee_{y \in U} \{ R(x, y) \land (R(x, y) \to \varphi(x)) \} \leq \bigvee_{y \in U} \{ \varphi(x) \} = \varphi(x). \]

This means that \( \varphi^\wedge \leq \varphi \). Analogously, for any \( x \in U \),
\[ \varphi^\vee(x) = \bigwedge_{y \in U} \{ R(y, x) \to \varphi^\vee(y) \} = \bigwedge_{y \in U} \{ R(y, x) \to \bigvee_{z \in U} \{ R(y, z) \land \varphi(z) \} \} \]
\[ \geq \bigwedge_{y \in U} \{ R(y, x) \to \{ R(y, x) \land \varphi(x) \} \} \geq \bigwedge_{y \in U} \{ \varphi(x) \} = \varphi(x). \]

Thus, also \( \varphi \leq \varphi^\vee \). We have that \((\wedge, \vee)\) is a Galois connection, because \((gc1)\) and \((gc2)\) are satisfied. Similarly, we can show that \((\wedge, \vee)\) is a Galois connection.

Next we show that \((d1)\) holds. For all \( x, y \in U \), we have
\[ R(x, y) \land \varphi(y) \land \psi(x) = R(x, y) \land \varphi(y) \land \bigwedge_{z \in U} \{ R(x, z) \to \psi(z) \} \]
\[ \leq R(x, y) \land \varphi(y) \land (R(x, y) \to \psi(y)) \]
\[ = (R(x, y) \land (R(x, y) \to \psi(y))) \land \varphi(y) \]
\[ = R(x, y) \land \psi(y) \land \varphi(y) \]
\[ = R(x, y) \land (\varphi \land \psi)(y) \]
\[ \leq \bigvee_{z \in U} \{ R(x, z) \land (\varphi \land \psi)(z) \} \]
\[ = (\varphi \land \psi)^\wedge(x). \]

Hence, for all \( y \in U \),
\[ R(x, y) \land \varphi(y) \land \psi(x)^\vee \leq (\varphi \land \psi)^\wedge(x). \]

Because complete Heyting algebras satisfy the join-infinite distributive law, we have that for all \( x \in U \),
\[ (\varphi^\wedge \land \psi^\vee)(x) = \varphi^\wedge(x) \land \psi^\vee(x) = \bigvee_{y \in U} \{ R(x, y) \land \varphi(y) \} \land \psi^\vee(x) \]
\[ = \bigvee_{y \in U} \{ R(x, y) \land \varphi(y) \land \psi^\vee(x) \} \leq (\varphi \land \psi)^\wedge(x). \]

Thus, \( \varphi^\wedge \land \psi^\vee \leq (\varphi \land \psi)^\wedge \). Assertion \((d2)\) can be proved similarly. 

\[ \square \]

**Example 3.3.** The instances
\[ (a \lor b)^\wedge \leq a^\wedge \lor b^\wedge \quad \text{and} \quad (a \lor b)^\vee \leq a^\vee \lor b^\vee \]
of Dunn’s axiom \((D_\vee)\) are false in some \(H2G+FS\)-algebras of rough \(\mathbb{H}\)-sets.

Namely, let \( U = \{ x, y \} \) and consider the finite (and hence complete) Heyting algebra \( 2^2 \oplus 1 \), that is, \( \mathbb{H} = \{ 0, a, b, c, 1 \} \) the Heyting algebra with the order \( 0 < a, b < c < 1 \), where \( a \) and \( b \) are incomparable. Note that \(-a = b \) and \(-b = a \).

We define two \(\mathbb{H}\)-sets \( \varphi, \psi \) on \( U \) by setting \( \varphi(u) = 0 \) and \( \psi(u) = 1 \) for all \( u \in U \). An \(\mathbb{H}\)-fuzzy relation \( R: U \times U \to H \) is defined by \( R(x, x) = R(y, y) = a \) and
\[ R(x, y) = R(y, x) = b. \] Then,
\[(\varphi \lor \psi)^{(a)}(x) = \bigwedge_{u \in U} (R(x, u) \rightarrow (\varphi \lor \psi)(u)) = \bigwedge_{u \in U} (R(x, u) \rightarrow (\varphi(u) \lor \psi(u)))
\]
\[= \bigwedge_{u \in U} (R(x, u) \rightarrow 1) = (a \rightarrow 1) \land (b \rightarrow 1) = 1 \land 1 = 1,
\]
but
\[\varphi^<(x) \lor \psi^<(x) = \bigwedge_{u \in U} (R(x, u) \rightarrow \varphi(u)) \lor \bigvee_{u \in U} (R(x, u) \land \psi(u))
\]
\[= \bigwedge_{u \in U} (R(x, u) \rightarrow 0) \lor \bigvee_{u \in U} (R(x, u) \land 1)
\]
\[= \bigwedge_{u \in U} \neg R(x, u) \lor \bigvee_{u \in U} R(x, u)
\]
\[= (\neg R(x, x) \land \neg R(x, y)) \lor (R(x, x) \lor R(x, y))
\]
\[= (\neg a \land \neg b) \lor (a \lor b) = 0 \lor c = c.
\]
Hence condition \((\varphi \lor \psi)^{(a)} \leq \varphi^<( \lor \psi^<)\) is not satisfied, because \(c \notin c\). Similarly, we can show that
\[\varphi^<(y) = 1 \text{ and } \varphi^<(y) \land \psi^<(y) = 0 \lor c = c,
\]
that is, \((\varphi \lor \psi)^{(a)} \leq \varphi^<( \lor \psi^<)\) is not satisfied.

Hence, we may conclude this subsection by stating that the rough lattice-valued fuzzy sets defined on complete Heyting algebras are algebraic models for \(\text{Int2GC}+\text{FS}\).

3.3. Algebraic Semantics and Completeness. As we already noted, \(\text{IntGC}\) is complete with respect to \(\text{HGC}\)-algebras. Here we show completeness of \(\text{Int2GC}\) and \(\text{IntGC}+\text{FS}\) with respect to \(\text{H2GC}\)- and \(\text{H2GC}+\text{FS}\)-algebras.

Let \((H, \lor, \land, \rightarrow, 0)\) be an \(\text{H2GC}\)-algebra, where \(H = (H, \lor, \land, \rightarrow, 0)\). A valuation is a function \(v: \text{Var} \rightarrow H\) assigning to each propositional variable \(p\) an element \(v(p)\) of \(H\). Let \(\Phi\) denote the set of well-formed \(\text{Int2GC}\)-formulas. Clearly, \(\Phi\) is the set of well-formed \(\text{Int2GC}+\text{FS}\)-formulas as well, because these logics have the same language. The valuation \(v\) can be extended to the set \(\Phi\) inductively\(^1\).

\[
v(\neg A) = v(A) \rightarrow 0 \quad v(A \rightarrow B) = v(A) \rightarrow v(B)
\]
\[
v(A \land B) = v(A) \land v(B) \quad v(A \lor B) = v(A) \lor v(B)
\]
\[
v(\square A) = v(A)^\uparrow \quad v(\Diamond A) = v(A)^\downarrow
\]
\[
v(\Delta A) = v(A)^\uparrow \downarrow \quad v(\nabla A) = v(A)^\downarrow \uparrow
\]
An \(\text{Int2GC}\)-formula \(A\) is valid if \(v(A) = 1\) for any valuation \(v\) on any \(\text{H2GC}\)-algebra. Similarly, we may define validity of \(\text{Int2GC}+\text{FS}\)-formulas over \(\text{H2GC}+\text{FS}\)-algebras.

**Theorem 3.4 (Soundness I).**

(a) Provable \(\text{Int2GC}\)-formulas are valid in \(\text{H2GC}\)-algebras

(b) Provable \(\text{Int2GC}+\text{FS}\)-formulas are valid in \(\text{H2GC}+\text{FS}\)-algebras.

**Proof.** The proof concerning intuitionistic logic is standard (see [17], for instance). As we have proved in [7] for \(\text{IntGC}\), rules (GC \(\lor\)) (GC \(\land\)), (GC \(\rightarrow\)), (GC \(\land\)) preserve validity. Thus, (a) holds. For (b), it is clear that axioms (FS1) and (FS2) are valid, because \(\text{H2GC}+\text{FS}\)-algebras satisfy identities (fs1) and (fs2).

\(^1\)Note that the idea is that the operations \(\uparrow, \downarrow, \uparrow \downarrow, \downarrow \uparrow\) may be obtained from their logical counterparts \(\land, \lor, \lor \land, \land \lor\) just by turning them 90 degrees clockwise.
To obtain completeness, we apply Lindenbaum–Tarski algebras. We denote by \( \Phi \) the algebra of \( \Phi \)-formulas, that is, the abstract algebra
\[
\Phi = (\Phi, \lor, \land, \rightarrow, \bot, \Box, \Diamond, \top, \lor, \land).
\]
We define two equivalences \( \equiv_1 \) and \( \equiv_2 \) on \( \Phi \):
\[
A \equiv_1 B \text{ if and only if } A \leftrightarrow B \text{ is provable in } \text{Int2GC};
\]
\[
A \equiv_2 B \text{ if and only if } A \leftrightarrow B \text{ is provable in } \text{Int2GC+FS}.
\]
Concerning \( \lor, \land, \) and \( \rightarrow \), the next result is known from the theory of intuitionistic logic, and for \( \Box, \Diamond, \top, \) and \( \lor \) the claim follows from monotonicity.

**Lemma 3.5.** The equivalences \( \equiv_1 \) and \( \equiv_2 \) are congruences on \( \Phi \).

For any \( A \in \Phi \), we denote by \([A]_1\) and \([A]_2\) the congruence class of \( A \) with respect to the congruences \( \equiv_1 \) and \( \equiv_2 \). The sets of \( \equiv_1 \)- and \( \equiv_2 \)-classes are denoted by \( \Phi/\equiv_1 \) and \( \Phi/\equiv_2 \). Next we define the quotient algebras of \( \Phi \) with respect to \( \equiv_1 \) and \( \equiv_2 \) by introducing the following operations on \( \Phi/\equiv_i \) for \( i = 1, 2 \):
\[
[A]_i \lor_1 [B]_i = [A \lor B]_i, \quad [A]_i \land_1 [B]_i = [A \land B]_i,
\]
\[
[A]_i \rightarrow_1 [B]_i = [A \rightarrow B]_i, \quad 0_1 = [\bot]_i,
\]
\[
[A]_i \Box_1 = [\Box A]_i, \quad [A]_i \land_1 = [\land A]_i.
\]

As we have noted, H2GC- and H2GC+FS-algebras form equational classes. By the theory of intuitionistic logic, \( \Phi/\equiv_1 \) satisfies the identities defining Heyting algebras. Also (gc1) and (gc2) hold by (GC1), (GC1)*, (GC4), (GC4)*. Since also identities (fs1) and (fs2) are the counterparts of axioms (FS1) and (FS2), we may write the following proposition.

**Proposition 3.6.**

(a) The algebra \( (\Phi/\equiv_1, \lor_1, \land_1, \rightarrow_1, 0_1, \Box_1, \land_1, \land_1) \) is an H2GC-algebra.

(b) The algebra \( (\Phi/\equiv_2, \lor_2, \land_2, \rightarrow_2, 0_2, \Box_2, \land_2, \land_2) \) is an H2GC+FS-algebra.

We define two valuations \( v_1 : \text{Var} \rightarrow \Phi/\equiv_1 \) and \( v_2 : \text{Var} \rightarrow \Phi/\equiv_2 \) by:
\[
v_1(p) = [p]_1 \quad \text{and} \quad v_2(p) = [p]_2.
\]
By a straightforward formula induction we see that \( v_1(A) = [A]_1 \) and \( v_2(A) = [A]_2 \) for all formulas \( A \in \Phi \). We can now write the following results.

**Lemma 3.7.** For any formula \( A \in \Phi \):

(a) \( A \) is provable in \( \text{Int2GC} \) if and only if \( v_1(A) = 1 \).

(b) \( A \) is provable in \( \text{Int2GC+FS} \) if and only if \( v_2(A) = 1 \).

**Theorem 3.8 (Completeness I).** For any formula \( A \in \Phi \):

(a) \( A \) is provable in \( \text{Int2GC} \) if and only if \( A \) valid in \( \text{H2GC-algebras} \).

(b) \( A \) is provable in \( \text{Int2GC+FS} \) if and only if \( A \) valid in \( \text{H2GC+FS-algebras} \).

**Proof.** (a) Suppose that \( A \) is valid in \( \text{H2GC-algebras} \). We have \( v_1(A) = 1 \) in \( \Phi/\equiv_1 \), that is, \( A \) is provable. The other direction is proved in Theorem 3.3. For (b), the proof is basically the same. \( \square \)

Clearly, rough \( H \)-sets considered in Section 3.2 provide algebraic models for \( \text{Int2GC+FS} \). Let us introduce axioms (D1) and (D2) corresponding to equations (d1) and (d2):

(D1) \( \Box A \land \Box B \rightarrow \Box(A \land B) \)

(D2) \( \triangle A \land \triangle B \rightarrow \triangle(A \land B) \)
By Proposition 3.1 and Theorem 3.8, we can write the following corollary giving additional ways to axiomatize \( \text{Int2GC}+\text{FS} \).

**Corollary 3.9.**

\( \text{Int2GC}+\text{FS} = \text{Int2GC} + \{(\text{FS1}) \text{ or } (\text{FS4}) \text{ or } (\text{D1})\} + \{(\text{FS2}) \text{ or } (\text{FS3}) \text{ or } (\text{D2})\} \)

By Example 3.3 and Theorem 3.8 formulas \( \Box (A \lor B) \rightarrow \Box \Diamond A \lor B \lor \Diamond A \lor \Box B \) corresponding to Dunn’s condition (D\(_v\)) are not provable in \( \text{Int2GC}+\text{FS} \).

### 4. Kripke semantics and completeness

#### 4.1. Kripke frames

In this section we consider Kripke frames and models for the three systems \( \text{IntGC} \), \( \text{Int2GC} \), and \( \text{Int2GC}+\text{FS} \), where the system \( \text{Int2GC}+\text{FS} \) will be shown in Section 5 to be equivalent to intuitionistic temporal logic \( \text{Ik}_t \).

**IntGC-frames.** A structure \( \mathcal{F} = (X, \leq, R) \) is called a Kripke frame of \( \text{IntGC} \) (an IntGC-frame, in short) \( \mathcal{F} \), if \( X \) is a non-empty set, \( \leq \) is a preorder (reflexive and transitive binary relation) on \( X \), and \( R \) is a relation on \( X \) such that:

\( (R1) \quad (\geq \circ R \circ \geq) \subseteq R. \)

Let \( v \) be a function \( v: P \rightarrow \wp(X) \) assigning to each propositional variable \( p \) a subset \( v(p) \) of \( X \) with the property that \( x \in v(p) \) and \( x \leq y \) imply \( y \in v(p) \), that is, \( v(p) \) is \( \leq \)-closed. Such functions are called *valuations* and the pair \( \mathcal{M} = (\mathcal{F}, v) \) is called an IntGC-model. For any \( x \in X \) and \( A \in \Phi \), we define the *satisfiability relation* in \( \mathcal{M} \) inductively by the following way:

\[
\begin{align*}
x \models p & \iff x \in v(p), \\
x \models A \land B & \iff x \models A \text{ and } x \models B, \\
x \models A \lor B & \iff x \models A \text{ or } x \models B, \\
x \models A \rightarrow B & \iff \text{for all } y \geq x, y \models A \text{ implies } y \models B, \\
x \models \neg A & \iff \text{for no } y \geq x \text{ does } y \models A, \\
x \models \Diamond A & \iff \text{exists } y \text{ such that } x R y \text{ and } y \models A, \\
x \models \forall A & \iff \text{for all } y, y R x \text{ implies } y \models A.
\end{align*}
\]

Note that the satisfiability relation \( \models \) is *persistent*, that is, for all formulas \( A \), if \( x \models A \) and \( x \leq y \), then \( y \models A \). An IntGC-formula \( A \) is *valid in a model* \( \mathcal{M} \), if \( x \models A \) for all \( x \in X \). The formula \( A \) is *valid in a frame* \( \mathcal{F} \), if \( A \) is valid in every model based on \( \mathcal{F} \). A formula is *Kripke-valid* if it is valid in every frame.

We noted in \( \mathcal{F} \) that IntGC is *Kripke-sound*, that is, every provable IntGC-formula is Kripke-valid. We also proved Kripke-completeness by applying canonical frames, and next we shortly recall these constructions, because a similar technique will be used later in cases of Int2GC and Int2GC+FS.

For an HGC-algebra \( (H, ◁, □) \), its canonical frame is a triple \( (X_H, \subseteq, R^H) \) such that \( X_H \) is the set of the prime filters of the lattice \( H \) and the relation \( R^H \) is defined by

\[
(x, y) \in R^H \iff y \subseteq [x]^{-1},
\]

where \([x]^{-1} = \{a \in H \mid a_r^{-1} \subseteq x\} \). The relation \( R^H \) can be described also in terms of the map \( □ \) by

\[
(x, y) \in R^H \iff [y]^{a^{-1}} \subseteq x,
\]

where \([y]^{a^{-1}} = \{a \in H \mid a^{\#} \in y\} \).

For a Heyting algebra \( \mathbb{H} \), we denote by \( \mathcal{O}(H) \) the set of all \( \leq \)-closed subsets.

**Lemma 4.1.** Let \( (\mathbb{H}, ◁, □) \) be an HGC-algebra. The pair \( (□^{-1}, ◁^{-1}) \) is a Galois connection on \( \mathcal{O}(H), \subseteq \).
Proof. Let \( x, y \in \mathcal{O}(H) \). Suppose \( [x]^{-1} \preceq y \). If \( a \in x \), then \( a \preceq a^{\downarrow} \) implies \( a^{\downarrow} \in x \), because \( x \in \mathcal{O}(H) \). This means that \( a^{\uparrow} \in [x]^{-1} \subseteq y \) and so \( a \in [y]^{-1} \).

Therefore, \( x \subseteq [y]^{-1} \). Conversely, assume that \( x \subseteq [y]^{-1} \). If \( a \in [x]^{-1} \), then \( a^{\downarrow} \in x \subseteq [y]^{-1} \), that is, \( a^{\downarrow} \in y \). Because \( a^{\downarrow} \leq a \), we have \( a \in y \), since \( y \in \mathcal{O}(H) \). Thus, \( [x]^{-1} \subseteq y \). \( \square \)

Let \( (H, \triangleright, \triangleleft) \) be an HGC-algebra. In [7], we showed that the canonical frame \( F^{H} = (X_{H}, \leq, R^{H}) \) is an \( \text{IntGC} \)-frame. Let \( v : \text{Var} \rightarrow H \) be a valuation on this HGC-algebra. We may now define a valuation \( v^{\star} : \text{Var} \rightarrow X_{H} \) for the canonical frame \( F^{H} \) by setting \( x \in v^{\star}(p) \) if and only if \( v(p) \in x \) for all \( p \in \text{Var} \). Obviously, for all \( x, y \in X_{H} \), \( y, y^{*} \in \text{Var} \), and \( x \in v^{*}(p) \) and \( x \subseteq y \) imply \( y \in v^{*}(p) \), so \( v^{\star} \) is really a valuation. In the canonical model \( (F^{H}, v^{\star}) \), we have \( x \models p \) if and only if \( v(p) \in x \) for all \( x \in X_{H} \). In [7], we proved by formula induction the Key Lemma stating that for any \( \text{IntGC} \)-formula \( A \) and \( x \in X_{H} \), \( x \models A \) if and only if \( v(A) \in x \). This enabled us to prove the Kripke-completeness, that is, an \( \text{IntGC} \)-formula is provable if and only if it is Kripke-valid.

**Int2GC-frames.** An \( \text{Int2GC} \)-frame (or a Kripke frame of \( \text{Int2GC} \)) is a quadruple \( F = (X, \leq, R_{1}, R_{2}) \) such that \( X \) is a non-empty set, \( \leq \) is a preorder on \( X \), and \( R_{1} \) and \( R_{2} \) are relations on \( X \) satisfying

\[
\begin{align*}
(R2) & \quad (\geq \circ R_{1} \circ \geq) \subseteq R_{1} \\
(R3) & \quad (\leq \circ R_{2} \circ \leq) \subseteq R_{2}.
\end{align*}
\]

Our next lemma is obvious.

**Lemma 4.2.** \( (X, \leq, R_{1}, R_{2}) \) is an \( \text{Int2GC} \)-frame if and only if \( (X, \leq, R_{1}) \) and \( (X, \leq, R_{2}^{-1}) \) are \( \text{IntGC} \)-frames.

In \( \text{Int2GC} \)-frames the valuations and the satisfiability relation \( \models \) for \( \vee, \wedge, \rightarrow, \) and \( \neg \) are defined as earlier, but satisfiability of formulas \( \Box A, \forall A, \triangle A, \) and \( \nabla A \) are defined by

\[
\begin{align*}
\models x \models \Box A & \iff \text{exists } y \text{ such that } x R_{1} y \text{ and } y \models A, \\
\models x \models \forall A & \iff \text{for all } y, y R_{1} x \text{ implies } y \models A, \\
\models x \models \triangle A & \iff \text{exists } y \text{ such that } y R_{2} x \text{ and } y \models A, \text{ and} \\
\models x \models \nabla A & \iff \text{for all } y, x R_{2} y \text{ implies } y \models A.
\end{align*}
\]

It is obvious that \( \text{Int2GC} \) is Kripke-sound, that is, every formula provable in \( \text{Int2GC} \) is Kripke valid.

We can introduce two \( \text{IntGC} \)-logics, one with the operators \( \Box \) and \( \forall \), and the other with \( \triangle \) and \( \nabla \). We denote these by \( \text{IntGC}_{1} \) and \( \text{IntGC}_{2} \), respectively. Next we show that \( \text{Int2GC} \) extends \( \text{IntGC}_{1} \) and \( \text{IntGC}_{2} \).

**Lemma 4.3.** Let \( F_{i} = (X, \leq, R_{i}) \) be a Kripke-frame for \( \text{IntGC}_{i} \) and let \( A_{i} \) be a well-formed formula of \( \text{IntGC}_{i} \), where \( i = 1, 2 \). Then, \( A_{i} \) is valid in \( F_{i} \) if and only if \( A_{i} \) is valid in \( F = (X, \leq, R_{1}, R_{2}^{-1}) \).

**Proof.** We prove the claim by formula induction. Concerning \( \text{IntGC}_{1} \)-frames and -formulas, the claim is obvious.

Let \( v \) be a valuation for the frame \( F_{2} = (X, \leq, R_{2}) \). Thus, \( M_{2} = (F_{2}, v) \) is an \( \text{IntGC}_{2} \)-model and \( M = (F, v) \) is an \( \text{Int2GC} \)-model.
Let $A_2$ be a formula of $\IntGC_2$ which is of the form $\triangle A$ for some $\IntGC_2$-formula $A$ having this property. Then, for all $x \in X$,

$$M_2, x \models \triangle A \iff (\exists y) x R_2 y \text{ and } M_2, y \models A \iff (\exists y) x R_2 y \text{ and } M, y \models A \iff (\exists y) y R_2^{-1} x \text{ and } M, y \models A \iff M, x \models \triangle A$$

Therefore, $A_2$ is valid in $F_2$ if and only if $A_2$ is valid in $F$. The claim concerning the operator $\mathbf{\downarrow}$ can be proved analogously.

The canonical $\Int2GC$-frame of an $H2GC$-algebra $(\mathbb{H}, \cdot, \triangleleft, \triangleright)$ is a structure $(X_H, \subseteq, R_1^H, R_2^H)$, where $X_H$ is the set of lattice-filters of $H$ and the relations $R_1^H$ and $R_2^H$ are defined by

$$(x, y) \in R_1^H \iff y \subseteq [x]^{\downarrow^{-1}} \quad \text{and} \quad (x, y) \in R_2^H \iff [x]^{\downarrow^{-1}} \subseteq y$$

where $[x]^{\downarrow^{-1}} = \{ a \in H \mid a^\downarrow \in x \}$. Equivalently, these relations can be defined as

$$(x, y) \in R_1^H \iff [y]^{\downarrow^{-1}} \subseteq x \quad \text{and} \quad (x, y) \in R_2^H \iff x \subseteq [y]^{\triangleright^{-1}}$$

in which $[y]^{\triangleright^{-1}} = \{ a \in H \mid a^\triangleright \in y \}$. The next lemma is obvious and its proof is omitted.

**Lemma 4.4.** $(\langle^{-1}, \cdot^{-1} \rangle)$ and $(\triangleright^{-1}, \langle^{-1} \rangle)$ are Galois connections on $(\mathcal{O}(H), \subseteq)$.

Similarly, as in the case of HGC-algebras, we can show that the canonical frame $F^H = (X_H, \subseteq, R_1^H, R_2^H)$ of any $H2GC$-algebra $(\mathbb{H}, \cdot, \triangleleft, \triangleright)$ is an $\Int2GC$-frame. For any valuation $v$ on $H$, we can define the valuation $\v^*$ for the canonical frame $F^H$ by setting $x \in v^*(p)$ if and only if $v(p) \in x$ for all propositional variables $p \in \mathrm{Var}$ and $x \in X_H$. As in case of $\IntGC$ (see Lemma 5.7 in [1]), we can prove by formula induction that the Key Lemma holds, that is, for any $\Int2GC$-formula $A$ and $x \in X_H$, $x \models A$ if and only if $v(A) \in x$. Therefore, we may state the Kripke-completeness presented in the next theorem.

**Theorem 4.5 (Completeness II for $\Int2GC$).** A formula $A$ is provable in $\Int2GC$ if and only if $A$ is Kripke-valid.

**Proof.** Suppose that an $\Int2GC$-formula $A$ is not provable. This means that there exists an $H2GC$-algebra $(\mathbb{H}, \cdot, \triangleleft, \triangleright)$ and a valuation $v : \mathrm{Var} \rightarrow H$ such that $v(A) \neq 1$. We construct the canonical frame $F^H$ and the valuation $\v^*$ as above. Because $v(A) \neq 1$, there exists a prime filter $x$ such that $v(A) \notin x$. By the Key Lemma, this means that $x \not\models A$ in the canonical model $(F^H, v^*)$. Therefore, $A$ is not Kripke-valid. 

**Int2GC+FS-frames.** An $\Int2GC+FS$-frame (of a Kripke frame of $\Int2GC+FS$) is a tripe $F = (X, \leq, R)$, where $X$ is a non-empty set, $\leq$ is a preorder on $X$, and $R$ is a relation on $X$ satisfying

$$(R4) \quad (R \circ \leq) \subseteq (\leq \circ R)$$

$$(R5) \quad (\geq \circ R) \subseteq (R \circ \geq)$$

**Lemma 4.6.** $(X, \leq, R)$ is an $\Int2GC+FS$-frame if and only if $(X, \leq, R \circ \geq, \leq \circ R)$ is an $\Int2GC$-frame.
Proof. Suppose \((X, \leq, R)\) is an Int2GC+FS-frame. Let \(R_1 = R \circ \geq\) and \(R_2 = \leq \circ R\). We show that the relations \(R_1\) and \(R_2\) satisfy conditions (R2) and (R3). Now, 

\[ R \circ \geq = (R \circ \geq) \circ \geq = (R \circ \geq) \circ \geq = R \circ \geq = R_1, \]

that is, (R2) is satisfied. Similarly, \(\leq \circ R_2 \leq \leq \circ (\leq \circ R) \leq = \leq \circ R \leq = \leq \circ R \leq = \leq \circ R \leq\), and also (R5) holds. Thus, \((X, \leq, R \circ \geq, \leq \circ R)\) is an Int2GC-frame.

Conversely, suppose that \((X, \leq, R \circ \geq, \leq \circ R)\) is an Int2GC-frame. We again put \(R_1 = R \circ \geq\) and \(R_2 = \leq \circ R\). Then, 

\[ R_1 = R_2 \iff 
\]

Thus, \((X, \leq, R \circ \geq)\) and \((X, \leq, R^{-1} \circ \geq)\) are IntGC-frames.

\[ (X, \leq, R) \text{ is an Int2GC+FS-frame if and only if } (X, \leq, R \circ \geq) \text{ and } (X, \leq, R^{-1} \circ \geq) \text{ are IntGC-frames} \]

\[ \square \]

Corollary 4.7. \((X, \leq, R)\) is an Int2GC+FS-frame if and only if \((X, \leq, R \circ \geq)\) and \((X, \leq, R^{-1} \circ \geq)\) are IntGC-frames.

Proof. The claim follows directly from Lemmas 4.2 and 4.6 because \((\leq \circ R)^{-1} = R^{-1} \circ \geq\).

\[ \square \]

Again, in Int2GC+FS-frames the valuations and the satisfiability relation \(\models\) for \(\lor, \land, \rightarrow, \neg\) are defined as earlier, and satisfiability of \(\Delta A, \forall A, \Delta A,\) and \(\forall A\) are defined by

\[ x \models \Delta A \iff \text{ exists } y \text{ such that } x(R \circ \geq) y \text{ and } y \models A \]

\[ x \models \forall A \iff \text{ for all } y, y(R \circ \geq) x \text{ implies } y \models A \]

\[ x \models \forall A \iff \text{ exists } y \text{ such that } y(\leq \circ R) x \text{ and } y \models A \]

\[ x \models \forall A \iff \text{ for all } y, x(\leq \circ R) y \text{ implies } y \models A \]

Lemma 4.8. For all Int2GC+FS-models \(M = (F, v)\) and formulas \(A \in \Phi:\)

\[ x \models A \text{ and } x \leq y \text{ imply } y \models A. \]

Proof. As an example, we show the claim for \(\Delta\) and \(\forall\).

\((\Delta A)\) Suppose \(x \models \Delta A\) and \(x \leq y\). Then, there exists \(z\) such that \(z(\leq \circ R)x\) and \(z \models A\). Thus, there is \(w\) such that \(z \leq w, wRx\), and \(z \models A\). Now \(wRx\) and \(x \leq y\) imply \(w(\leq \circ R)y\). From frame condition (R4), we get \(w(\leq \circ R)y\). Now \(z \leq w\) implies \(z(\leq \circ R)y\). Since \(z \models A\), we have \(y \models \Delta A\).

\((\forall A)\) Assume that \(x \models \forall A, x \leq y,\) but \(y \not\models \forall A\). Then, there exists \(z\) such that \(y(\leq \circ R) z\) and \(z \not\models A\). Since \(x \leq y\), we have \(x(\leq \circ R) z\). By \(z \not\models A\), we get \(x \not\models \forall A\), a contradiction.

\[ \square \]

Our next lemma showing a connection between validity in Int2GC+FS-frames and IntGC-frames is obvious and thus its proof is omitted.

Lemma 4.9. Let \(A \in \Phi\). Then, \(A\) is valid in the Int2GC+FS-frame \((X, \leq, R)\) if and only if \(A\) is valid in the IntGC-frame \((X, \leq, R \circ \geq, \leq \circ R)\).

Theorem 4.10 (Soundness II for Int2GC+FS). Every formula provable in Int2GC+FS is Kripke-valid.

Proof. Suppose that \(\Delta A \rightarrow B\) is valid in a Int2GC+FS-frame \((X, \leq, R)\). Then, by Lemma 4.9 \(\Delta A \rightarrow B\) is valid in the Int2GC-frame \((X, \leq, R \circ \geq, \leq \circ R)\). This implies that \(A \rightarrow \forall B\) is valid in the Int2GC-frame \((X, \leq, R \circ \geq, \leq \circ R)\), because Int2GC-preserves validity of the Galois connection rules. By Lemma 4.9, \(A \rightarrow \forall B\) is valid in the Int2GC+FS-frame \((X, \leq, R)\). Thus, (GC \(\forall\)) preserves validity. Rules (GC \(\forall\)), (GC \(\forall\)), and (GC \(\forall\)) may be considered similarly.

We show that axiom (D1) is a valid formula. Validity of (D2) can be proved analogously. By Corollary 3.3 this gives that the axioms of Int2GC+FS are valid.

Suppose \(x \models \Delta A \land \forall B\). Then, \(x \models \Delta A\) and \(x \models \forall B\). So, there exists \(y\) such that \(x(R \circ \geq)y\) and \(y \models A\). Thus, there is \(w\) such that \(xRw\) and \(w \geq y\). Because
of persistency, we have \( w \models A \). Now \( x \leq x \) and \( xRw \) imply \( x(\leq \circ R)w \). The fact \( x \models \nabla B \) means that for all \( z \), \( x(\leq \circ R)z \) implies \( z \models B \). Therefore, \( w \models B \) and thus \( w \models A \land B \). Because \( xRw \) and \( w \geq w \), we have \( x(R \circ \geq)w \) implying \( x \models \bigtriangleup(A \land B) \). So, (D1) is a valid formula.

Example 4.11. We present an application showing how preference relations may be used for obtaining particular Kripke-frames of Int2GC+FS. Several definitions of preference structures can be found in the literature; see [14]. There are two fundamental relations, namely “better” (strict preference) and “similar” (indifference). Here we denote “\( b \) is better than \( a \)” by \( a \prec b \) and \( a \sim b \) denotes that \( a \) and \( b \) are similar. Usually, it is assumed that \( \prec \) and \( \sim \) have at least the following properties:

(i) \( a \prec b \) implies \( b \nprec a \) (asymmetry of \( \prec \))
(ii) \( a \sim a \) (reflexivity of \( \sim \))
(iii) \( a \sim b \) implies \( b \sim a \) (symmetry of \( \sim \))
(iv) \( a \prec b \) implies \( a \nprec b \) (incompatibility of \( \prec \) and \( \sim \))

Suppose now that \( \prec \) is a transitive strict preference relation on some universe of discourse \( U \). Transitivity is a quite natural property of strict preference, because if \( a \) is better than \( b \) and \( b \) is better than \( c \), also \( a \) should be better than \( c \).

Let us denote by \( \preceq \) the relation \( \prec \cup \Delta_U \), where \( \Delta_U \) is the identity relation of \( U \), that is, \( \Delta_U = \{(x, x) \mid x \in U\} \). The relation \( \preceq \) is obviously a preorder. Note that since \( \prec \) is assumed to be asymmetric, then \( a \leq b \) and \( b \leq a \) imply \( a = b \). This means that \( \preceq \) is a partial order on \( U \). Assume also that \( \preceq \) and \( \sim \) are connected by conditions (R4) and (R5), that is, \((\sim \circ \preceq) \subseteq (\preceq \circ \sim) \) and \((\preceq \circ \sim) \subseteq (\sim \circ \preceq) \), where \( \preceq \) is the inverse relation of \( \preceq \). These assumptions hold for instance in such object sets which can organized in “levels” as in Figure 1 – elements in the same level are all similar with respect to their properties, and the elements in an upper level are better than the lower ones.

![Figure 1](image)

Hence, the triple \((U, \leq, \sim)\) can be viewed as an Int2GC+FS-frame. Because the relation \( \sim \) is symmetric, \( \nabla A \) and \( \forall A \) have equal interpretations, and the same holds for \( \bigtriangleup A \) and \( \bigtriangledown A \). This means that \( \nabla A \leftrightarrow \forall A \) and \( \bigtriangleup A \leftrightarrow \bigtriangledown A \) are valid formulas in any Kripke-model based on the frame \((U, \leq, \sim)\). This implies, for instance, that \( A \rightarrow \nabla \bigtriangleup A \) and \( \bigtriangledown \nabla A \rightarrow A \) are valid in all such Kripke-models for all Int2GC+FS-formulas \( A \).

Additionally, because \( \sim \) and \( \leq \) are reflexive, we have that \( A \rightarrow \bigtriangleup A \) and \( \nabla A \rightarrow A \) are valid in all Kripke-models based on \((U, \leq, \sim)\). Let the formula \( A \) represent some property, that is, \( x \models A \) means that the object \( x \in U \) has this property. The formulas \( \bigtriangleup A \) and \( \nabla A \) have the following interpretations:
(i) \( x \models △ A \) if there exist \( y, z \in U \) such that \( x \sim y, y \succeq z \) and \( z \models A \), that is, 
\( x \) is similar to an object that is better than or equal to an object having the property \( A \).

(ii) \( x \models ▽ A \iff \) for all \( y, z \), \( x \leq y \) and \( y \sim z \) imply \( z \models A \), that is, all objects similar to the objects being better or equal to \( x \) have the property \( A \).

Thus, the semantics based on preference \( \preceq \) admit structure (\( X \)).

Before that, we present some results and observations that are needed for our proofs. We denote by \( [S] \) the lattice-filter generated by \( S \subseteq H \). It is well known that \( [S] \) is the set of all elements \( a \in H \) such that \( a \wedge \cdots \wedge a_n \leq a \) for some elements \( a_1, \ldots, a_n \in S \). We also denote for any \( x \in X_H \):

\[
[x]^* = \{ a^* \mid a \in x \}; \quad [x]^{-} = \{ a^{-} \mid a \in x \}; \quad [x]^{-*} = \{ a^{-*} \mid a \in x \}.
\]

**Lemma 4.12.** Let \((H, \triangleright, \triangleleft, \triangleright, ^*)\) be an \( H2GC+FS \)-algebra. If \( k \) is a filter and \( y \) is a prime filter such that \( k \cap [-y]^* = \emptyset \), then there exists a prime filter \( u \) such that \( k \subseteq u \) and \( u \cap [-y]^* = \emptyset \).

**Proof.** Let us denote \( \Gamma = \{ t \mid t \text{ is a filter, } k \subseteq t, \text{ and } t \cap [-y]^* = \emptyset \} \). Clearly \( \Gamma \neq \emptyset \) and, by Zorn’s Lemma, \( \Gamma \) has a maximal element \( u \). Then, \( u \) is a filter, \( k \subseteq u \), and \( u \cap [-y]^* = \emptyset \).

Assume that \( u \) is not a prime filter. Then there exists two elements \( a, b \in H \) such that \( a \vee b \in u \), but \( a, b \notin u \). By maximality of \( u \), this implies that \( [u \cup \{ a \}] \) and \( [u \cup \{ b \}] \) are not in \( \Gamma \). Therefore, we must have that \( [u \cup \{ a \}] \cap [-y]^* \neq \emptyset \) and \( [u \cup \{ b \}] \cap [-y]^* \neq \emptyset \). So, there exists \( c, d \in [-y]^* \) such that \( c \in [u \cup \{ a \}] \) and \( d \in [u \cup \{ b \}] \). Because \( u \) is a filter, this implies that there exist \( c, f \in u \) such that \( e \wedge a \leq c \) and \( f \wedge b \leq d \). Now \( e \wedge f \in u \) and \( a \wedge b \in u \) imply \( (e \wedge f) \vee (a \wedge b) \in u \). Since

\[
(e \wedge f) \wedge (a \vee b) = (e \wedge f \wedge a) \vee (e \wedge f \wedge b) \leq e \vee d,
\]

we obtain \( e \vee d \in u \). Now the exist \( c_1, d_1 \in [-y] \) such that \( c = c_1^* \) and \( d = d_1^* \). Because \( e \vee d \in u \) and \( c \vee d = (c_1^* \vee d_1^*) \leq (c_1 \vee d_1)^* \), we have \( (c_1 \vee d_1)^* \in u \). On the other hand, \( c_1, d_1 \notin y \) implies \( c_1 \vee d_1 \notin y \), because \( y \) is a prime filter. Thus, \( c_1 \vee d_1 \notin y \) implies \( (c_1 \vee d_1)^* \in [-y]^* \). But \( u \cap [-y]^* = \emptyset \), a contradiction. Thus, \( u \) is a prime filter.
Let $S$ be a non-empty subset of a lattice $L$ such that $a \lor b \in S$ implies $a \in S$ or $b \in S$ for all $a, b \in L$. It is easily seen that such sets $S$ can be characterised as the sets whose set-theoretical complement $\neg S$ is a $\lor$-subsemilattice of $L$. In [7], we proved the following lemma.

**Lemma 4.13.** Let $L$ be a distributive lattice. If $x$ is a filter and $u$ is a superset of $x$ such that its set-theoretical complement $\neg u = -u$ is a $\lor$-subsemilattice of $L$, then there exists a prime filter $\mathcal{z}$ such that $x \subseteq \mathcal{z} \subseteq u$.

Our next proposition shows that the canonical frames are Int2GC+FS-frames.

**Proposition 4.14.** If $(\mathbb{H}, \land, \lor, \land^+, \lor^+)$ is an H2GC+FS-algebra, then $(X^H, \subseteq, R^H)$ is an Int2GC+FS-frame.

**Proof.** Assume that $x (R^H \circ \subseteq) y$. This implies that there exists $z \in X_H$ such that $x R^H z$ and $z \subseteq y$. Then, $[x]^{-1} \subseteq z \subseteq [y]^{-1}$. Let $k = [x \cup [y]^{+}]$ be the filter generated by $x \cup [y]^{+}$. We show first that $k \cap [-y]^{+} = \emptyset$. If $k \cap [-y]^{+} \neq \emptyset$, then there exists an element $a$ such that $a \in k$ and $a \in [-y]^{+}$. Since $a \in k = [x \cup [y]^{+}]$, there are $b \in x$ (recall that $x$ is a filter) and $c_1, \ldots, c_n \in y$ such that $b \land c_1^{+} \land \cdots \land c_n^{+} \leq a$. Let us denote $c = c_1 \land \cdots \land c_n \in y$. Because the map $\land$ is order-preserving, we have $c^{+} \leq c_1^{+} \land \cdots \land c_n^{+}$. This means that $b \land c^{+} \leq a$ and $b \leq c^{+} \rightarrow a$. Now $a = d^{+}$ for some $d \in -y$, and so $b \leq c^{+} \rightarrow d^{+} \leq (c \rightarrow d)^{+}$ by (fs3). Since $b \in x$ and $x$ is a filter, we have $(c \rightarrow d)^{+} \in x$. From this we get $c \rightarrow d \subseteq [x]^{-1} \subseteq z \subseteq y$. Because $c \in y$ and $y$ is a filter, also $c \cap (c \rightarrow d) \subseteq d$, so we have $d \in y$, a contradiction. Hence, $k \cap [-y]^{+} = \emptyset$.

By Lemma 4.11 there exists a prime filter $\mathcal{z}$ such that $k \subseteq \mathcal{z}$ and $\mathcal{z} \subseteq u$. If $a \in [u]^{-1}$, then $a^{+} \in u$ which gives $a^{+} \notin [-y]^{+}$, because $u \cap [-y]^{+} = \emptyset$. Thus, $a \notin [-y]$, that is, $a \in y$. Now $x \subseteq u$ and $u R_H y$ give $x (\subseteq \circ R^H) y$.

Assume $x (\supseteq \circ R^H) y$. Then for some $w \in X_H$, $x \supseteq w$ and $[w]^{-1} \subseteq y \subseteq [w]^{+}$. Hence, $y \subseteq [x]^{+}$, because $\supseteq$ is order-preserving. To show that $x (R^H \circ \supseteq) y$, we need to find a prime filter $z \in X_H$ such that $[x]^{-1} \subseteq z \subseteq [x]^{+}$ and $z \supseteq y$. Consider the filter $k = [y \cup [x]^{-1}]$). We show first that $k \subseteq [x]^{-1}$. Assume $a \in k$.

Then, there exists $c \in y$ and $d \in [x]^{-1}$ such that $c \land d \leq a$ (note that $y$ is a filter and $[x]^{-1}$ is closed under meets). Hence, $c \leq d \rightarrow a$ and $c^{+} \leq (d \rightarrow a)^{+} \leq d^{+} \rightarrow a$ by (fs1). Since $c \in y \subseteq [x]^{-1}$, we have $c^{+} \in x$ and $d^{+} \rightarrow a^{+} \in x$. Because $d^{+} \in x$, we obtain $a^{*} \in x$, that is, $a \in [x]^{+}$ as required.

Because $k \subseteq [x]^{+}$, $k$ is a filter, and $[x]^{+}$ is a set such that is set-theoretical complement is a $\lor$-subsemilattice of $L$, by Lemma 4.13 there exists $z \in X_H$ such that $k \subseteq z \subseteq [x]^{+}$. Combining the above observations, we have $z \supseteq k \supseteq y$ and $[x]^{-1} \subseteq k \subseteq z \subseteq [x]^{+}$, that is, $x R_H z$ and $z \supseteq y$.

Let $(\mathbb{H}, \land, \lor, \land^+, \lor^+)$ be an H2GC+FS-algebra. Let $v: \text{Var} \rightarrow H$ be a valuation. We may now define a valuation $v^{*}: \text{Var} \rightarrow X_H$ for the canonical frame $F_{H} = (X_H, \subseteq, R^H)$ by setting $x \in v^{*}(p)$ if and only if $v(p) \in x$ for all $p \in \text{Var}$. Hence, in the canonical model $(F^H, v^{*})$, we have $x \models p$ if and only if $v(p) \in x$ for all $x \in X_H$. We show that an analogous condition holds for all formulas $A$.

**Lemma 4.15 (Key Lemma).** Let $(\mathbb{H}, \land, \lor, \land^+, \lor^+)$ be an H2GC+FS-algebra and $v: \text{Var} \rightarrow H$ a valuation. In the canonical model $(F^H, v^{*})$, we have $x \models A$ if and only if $v(A) \in x$ for all $x \in X_H$ and $A \in \Phi$.

**Proof.** We prove the result by formula induction. For the connectives $\lor, \land, \rightarrow$, and $\neg$ the result is well known from the theory of intuitionistic logic. In addition, we
only show the proofs for formulas $\triangle A$ and $\nabla A$, since for $\blacklozenge A$ and $\nabla A$ the proofs are analogous.

$(\blacklozenge A)$ Suppose that $x \models \blacklozenge A$. This means that there exists a prime filter $y$ such that $x (R^H \circ \geq) y$ and $y \models A$. By the induction hypothesis, we have that $y \in v(A)$. In addition, there exists a prime filter $u$ such that $x R^H u$ and $y \subseteq u$, that is, $[x]^u \subseteq y \subseteq [x]^{u^{-1}}$. We obtain directly that $v(A) \in y \subseteq u \subseteq [x]^{u^{-1}}$, which means that $v(A) = v(\blacklozenge A) \in x$.

Conversely, suppose $v(\blacklozenge A) = v(A)^+ \in x$. Let us consider the filter $k = [[x]^{u^{-1}} \cup \{v(A)\}]$. First we show that $k \subseteq [x]^{u^{-1}}$. Assume that $a \in k$. Then there exists $b \in [x]^{u^{-1}}$ such that $b \land v(A) \leq a$ (note that $[x]^{u^{-1}}$ is closed under finite meets). We have that $v(A) \leq b \rightarrow a$ and $v(A)^+ \leq (b \rightarrow a)^+ \leq b^+ \rightarrow a^+$ by (ls1). This implies $b^+ \rightarrow a^+ \in x$. Because $b^+ \in x$, we obtain $a^+ \in x$ and $a \in [x]^{u^{-1}}$. Hence, $k \subseteq [x]^{u^{-1}}$. Because $k$ is a filter and $[x]^{u^{-1}}$ is a set such that is set-theoretical complement is a $\vee$-subsemilattice of $H$, we have by Lemma 4.13 that there exists a prime filter $y$ such that $k \subseteq y \subseteq [x]^{u^{-1}}$. By the definition of $k$, $[x]^{u^{-1}} \subseteq k \subseteq y$ and $v(A) \in k \subseteq y$. We have $[x]^{u^{-1}} \subseteq y \subseteq [x]^{u^{-1}}$, that is, $x R^H y$. By the induction hypothesis, $y \models A$. Since $y \geq y$ holds trivially, we have $x (R^H \circ \geq) y$ implying $x \models \blacklozenge A$.

$(\forall A)$ Suppose that $v(\forall A) = v(A)^\bullet \in x$. Let $y \in X_H$. If $y (R^H \circ \geq) x$, then there exists $z$ such that $y R^H z$ and $z \geq x$. Now $y R^H z$ is equivalent to $[z]^x \subseteq y \subseteq [z]^{x^{-1}}$. Therefore, $v(A)^\bullet \in x \subseteq z$ gives $v(A) \in [z]^{x^{-1}} \subseteq y$. By the induction hypothesis, $y \models A$ and hence $x \models \forall A$.

For the other direction, assume $v(\forall A) = v(A)^\bullet \notin x$, that is, $v(A) \notin [x]^{u^{-1}}$. It is easy to observe that $[x]^{u^{-1}}$ is a filter. Then, by the Prime Filter Theorem of distributive lattices (see [7] Lemma 5.4, for instance), there exists a prime filter $u$ such that $v(A) \notin u$ and $[x]^{u^{-1}} \subseteq u$.

Let us consider the filter $k = [x \cup [u]^\bullet]$. We first show that $k \cap [-u]^\bullet = \emptyset$. If $k \cap [-u]^\bullet \neq \emptyset$, then there exists $a \in k \cap [-u]^\bullet$. Because $a \in k = [x \cup [u]^\bullet]$, there are $b \in x$ and $c_1, \ldots, c_n \in u$ such that $b \land c_1^\bullet \land \cdots \land c_n^\bullet \leq a$ (recall that $x$ is a filter). Let us denote $c = c_1 \land \cdots \land c_n \in u$. Hence, $c^\bullet \leq c_1^\bullet \land \cdots \land c_n^\bullet$ and $b \land c^\bullet \leq a$. But now $a = d^\bullet$ for some $d \notin u$. So, $b \land c^\bullet \leq d^\bullet$. This gives that $b \leq c^\bullet \rightarrow d^\bullet \leq (c \rightarrow d)^\bullet$ by (is4). Because $b \in x$, we have $(c \rightarrow d)^\bullet \in x$. This means that $c \rightarrow d \in [x]^{u^{-1}} \subseteq u$. Now $c \in u$ implies $d \in u$, a contraction. Therefore, $k \cap [-u]^\bullet = \emptyset$.

By Lemma 4.12 there exists a prime filter $y$ such that $k \subseteq y$ and $y \cap [-u]^\bullet = \emptyset$. Since $k \subseteq y$, we have $x \subseteq y$ and $[u]^y \subseteq y$ meaning $u \subseteq [y]^{y^{-1}}$. The fact that $y \cap [-u]^y = \emptyset$ implies $[y]^{y^{-1}} \subseteq u$, because if $a \in [y]^{y^{-1}}$, then $a^\bullet \in y$. This gives $a^\bullet \notin [-u]^\bullet$, $a \notin -u$, and $a \in u$. By combining our observations, we have $[y]^{y^{-1}} \subseteq u \subseteq [y]^{y^{-1}}$, that is, $u R^H y$ and $y \geq y$. Thus, $u (R^H \circ \geq) x$. Because $v(A) \notin u$, we have $u \not\models A$ by the induction hypothesis. Hence, $x \not\models \forall A$. 

As in case of Theorem 4.15, we may prove the following completeness result by applying the Key Lemma.

**Theorem 4.16 (Completeness II for Int2GC+FS).** A formula $A$ is provable in Int2GC+FS if and only if $A$ is Kripke-valid.

5. Connections to intuitionistic tense logic

Intuitionistic tense logic $\mathbf{LT}_t$ was introduced by Ewald [9] by extending the language of intuitionistic propositional logic with the usual temporal expressions $FA$ (A is true at some future time), $PA$ (A was true at some past time), $GA$ (A will
be true at all future times), and $HA$ ($A$ has always been true in the past). The Hilbert-style axiomatisation of $IK_t$ can be found in [9, p. 171]:

1. All axioms of intuitionistic logic
2. $G(A \rightarrow B) \rightarrow (GA \rightarrow GB)$
3. $G(A \land B) \iff GA \land GB$
4. $F(A \lor B) \iff FA \lor FB$
5. $G(A \rightarrow B) \rightarrow (FA \rightarrow FB)$
6. $GA \land FB \rightarrow F(A \land B)$
7. $G\neg A \rightarrow \neg FA$
8. $FA \rightarrow A$
9. $A \rightarrow HFA$
10. $(FA \rightarrow GB) \rightarrow G(A \rightarrow B)$
11. $F(A \rightarrow B) \rightarrow (GA \rightarrow FB)$

The rules of inference are modus ponens (MP), and

$$\text{(RH)} \quad \frac{A}{HA} \quad \text{(RG)} \quad \frac{A}{GA}$$

Our next proposition shows that if we identify $\land$, $\lor$, $\Delta$, $\forall$ with $F$, $G$, $P$, $H$, respectively, then $Int2GC+FS$ and $IK_t$ will become syntactically equivalent. Recall that $Int2GC+FS = Int2GC + \{(FS1) or (FS4) or (D1)\} + \{(FS2) or (FS3) or (D2)\}$, and $Int2GC$ is obtained by extending intuitionistic logic with rules $(GC\land\forall)$, $(GC\lor\exists)$, $(GC\forall\lor)$, and $(GC\lor\forall)$. Theorem 5.1. $IK_t = Int2GC+FS$.

Proof. First we show that all axioms $IK_t$ are provable in $Int2GC+FS$, and all rules of $IK_t$ are admissible in $Int2GC+FS$. In this first part, let $\vdash$ denote that a formula $A$ is provable in $Int2GC+FS$. As noted in Section 2, axioms (2), (2'), (3), (3') (4), (4'), (8), (8'), (9), (9') are provable even in $Int2GC$. Additionally, rules (MP), (RH), and (RG) are admissible in $Int2GC$. Axioms (10), (10'), (11), (11') are the Fischer Servi axioms (FS3), (FS4), (FS1), (FS2), so they are provable in $Int2GC+FS$.

Axiom (FS1) is equivalent to $\land(A \rightarrow B) \land \forall A \rightarrow A B$. If we set $B := A \land B$ in this formula, we have $\vdash (\land(A \rightarrow A \land B) \land \forall A) \rightarrow A B$. Because $A \rightarrow A \land B$ is equivalent to $A \rightarrow B$, and $\vdash B \rightarrow (A \rightarrow B)$ gives $\vdash \land(B \rightarrow A B)$ by the monotonicity of $\land$, we obtain $\vdash \land B \land \forall A \rightarrow (A \land B)$ and thus (6) is provable in $Int2GC+FS$. Provability of (6') can be shown similarly.

Because $\vdash \lor\forall(A \rightarrow B) \rightarrow (A \rightarrow B)$, we have $\vdash \lor\forall(A \rightarrow B) \land A \rightarrow B$ and $\vdash \Delta(\lor\forall(A \rightarrow B) \land A) \rightarrow \Delta B$. Let us set $A := \lor\forall(A \rightarrow B)$ and $B := A$ in axiom (6') (which we just showed to be provable in $Int2GC+FS$). We obtain $\vdash \lor\forall\forall(A \rightarrow B) \land A \rightarrow \Delta B$. Thus, $\vdash \lor\forall\forall(A \rightarrow B) \land A \rightarrow \Delta B$. Because $\vdash \lor\forall(A \rightarrow B) \rightarrow \lor\forall\forall(A \rightarrow B)$, we have $\vdash \lor\forall(A \rightarrow B) \land A \rightarrow \Delta B$. This is equivalent to $\vdash \lor(A \rightarrow B) \rightarrow (\Delta A \land B)$. Hence, (5') is provable in $Int2GC+FS$, and provability of (5) can be showed in an analogous manner.

If we set $B := \bot$ in (5), we get $\vdash \lor\forall(A \rightarrow \bot) \rightarrow (\forall A \rightarrow A \land \bot)$. Because $\forall \bot$ is equivalent to $\bot$, we have $\vdash \lor\forall\forall\rightarrow A \rightarrow \lor\forall A$. This means that (7) and (7') are provable.

Because axioms (10), (10'), (11), (11') are the Fischer Servi axioms, for the other direction is enough to show admissibility of rules $(GC\land\forall)$, $(GC\lor\exists)$, $(GC\forall\lor)$, $(GC\lor\forall)$ in $IK_t$. First, we show admissibility of the rules of monotonicity, that is, if $A \rightarrow B$ is provable, then $HA \rightarrow HB$, $PA \rightarrow PB$, $GA \rightarrow GB$, and $FA \rightarrow FB$ are provable.
Here \( \vdash A \) denotes that the formula \( A \) is provable in \( \text{IK}_t \). Assume \( \vdash A \rightarrow B \). By (RG), \( \vdash G(A \rightarrow B) \). Now \( \vdash GA \rightarrow GB \) follows by (2), and from \( \vdash G(A \rightarrow B) \), we obtain also \( \vdash FA \rightarrow FB \) by (5). Similarly, \( \vdash A \rightarrow B \) implies \( \vdash HA \rightarrow HB \) and \( \vdash PA \rightarrow PB \) by applying (RH), (2'), and (5').

Next we prove admissibility of \((\text{GC} \square \triangle)\). Assume that \( \vdash A \rightarrow HB \). Then, \( FA \rightarrow FHB \) by monotonicity of \( F \). Because \( \vdash FHB \rightarrow B \) by (8), we obtain \( \vdash FA \rightarrow B \). Similarly, by (8') and monotonicity of \( P \), \( A \rightarrow GB \) implies \( PA \rightarrow B \), that is, \((\text{GC} \bigtriangledown \bigtriangleup)\) is admissible in \( \text{IK}_t \). Monotonicity of \( H \) and axiom (9) yield \( FA \rightarrow B \) implies \( A \rightarrow HB \), and monotonicity of \( G \) and (9') give that \( PA \rightarrow B \) implies \( A \rightarrow BG \). Thus, rules \((\text{GC} \bigtriangleup \bigtriangledown)\) and \((\text{GC} \bigtriangledown \bigtriangleup)\) are admissible.

**Remark 5.2.** It is proved in \cite{15} that \( \text{ILGC} \) is equivalent, with respect to provability, to the minimal (classical) tense logic \( \text{IK}_t \), that is, \( \text{ILGC} \) can be viewed as a simple formulation of \( \text{IK}_t \). The same analogy applies here, because \( \text{Int}2\text{GC+FS} \) can be seen as an alternative formulation of \( \text{IK}_t \).

It should be noted that with respect to Kripke-semantics, \( \text{IK}_t \) and \( \text{Int}2\text{GC+FS} \) are quite different. A Kripke-frame of \( \text{IK}_t \) consists of a partially-ordered set \( (\Gamma, \leq) \) (the “states-of-knowledge”), family of sets \( T_\gamma \), where \( \gamma \in \Gamma \) (times known at state-of-knowledge \( \gamma \)), such that \( \gamma \leq \varphi \) implies \( T_\gamma \subseteq T_\varphi \), meaning that advancing in knowledge retains what is known about times and their temporal ordering, and a collection of binary relations \( \mu_\gamma \) on \( T_\gamma \) (the temporal ordering of \( T_\gamma \) as it is understood at state-of-knowledge \( \gamma \)) \cite{9}, whereas \( \text{Int}2\text{GC+FS} \) is conceived as an information logic such that its frames \( (X, \leq, R) \) are such that \( X \) forms the universe of discourse, and \( \leq \) and \( R \) are relations reflecting relationships between the objects in \( X \), such as preference and indifference of objects (see Example \cite{111}).

It is also obvious and well-known that the axiomatisation of Ewald is not minimal, because several axioms can be deduced from the other axioms. We present a reduced axiomatisation, in which the number of axioms is the half of the size of the axiomatisation in \cite{9}.

**Proposition 5.3.** \( \text{IK}_t \) can be axiomatised by adding (2), (2'), (5), (5'), (8), (8'), (9), (9'), (11), (11') to the axioms of intuitionistic logic together with rules (MP), (RH), and (RG).

**Proof.** As shown in the proof of Theorem \ref{thm:5.1} if \( \square, \bigtriangledown, \triangle, \bigtriangledown \) are identified with \( F, G, P, H \), then axioms (2), (2'), (5), (5'), (8), (8'), (9), (9') with rules (MP), (RH) and (RG) are enough to show that rules \((\text{GC} \bigtriangledown \bigtriangleup), (\text{GC} \bigtriangledown \bigtriangledown), (\text{GC} \bigtriangledown \bigtriangledown), (\text{GC} \bigtriangledown \bigtriangleup)\) are admissible. Axioms (11), (11') coincide with (FS1) and (FS2), so the proof is complete, because \( \text{IK}_t = \text{Int}2\text{GC+FS} \). \( \square \)

In the next proposition, we present another axiomatisation of \( \text{Int}2\text{GC} \) using axioms of intuitionistic tense logic.

**Proposition 5.4.** \( \text{Int}2\text{GC} \) can be axiomatised by adding (2), (2'), (8), (8'), (9), (9') to the axioms of intuitionistic logic together with rules (MP), (RH), (RG), and rules:

\[
\text{(RMF)} \quad \frac{A \rightarrow B}{FA \rightarrow FB} \quad \text{(RMP)} \quad \frac{A \rightarrow B}{PA \rightarrow PB}
\]

**Proof.** Monotonicity of \( G \) and \( H \) follow from (RG), (RH), (2), and (2'). Because all operators are thus monotone, admissibility of rules \((\text{GC} \bigtriangledown \bigtriangleup), (\text{GC} \bigtriangledown \bigtriangledown), (\text{GC} \bigtriangledown \bigtriangleup), (\text{GC} \bigtriangledown \bigtriangledown)\) follow easily from (8), (8'), (9), (9').

On the other hand, in Section \ref{sec:2} we have noted that axioms (2), (2'), (8), (8'), (9), (9') are provable in \( \text{Int}2\text{GC} \) and rules (RH), (RG), (RMF), (RMP) are admissible. \( \square \)
Let us observe that Int2GC cannot be axiomatised by using only axioms and rules of Ewald’s system. The reason for this is that monotonicity of operators $P$ and $F$ need to be added, since rules (RMF) and (RMP) do not belong to the system by Ewald as “initial rules”, even they are admissible in $I K_t$. On the other hand, monotonicity of $P$ and $F$ could be obtained by adding axioms (5) and (5′) to the system of Proposition 5.4 (without monotonicity of $F$ and $P$), but then we have a logic which is too strong, since (5) and (5′) cannot be proved in Int2GC – this is because Galois connections $(\wedge, \vee)$ and $(\Delta, \nabla)$ are “independent”, that is, operations $\wedge$ and $\vee$ are not in anyway connected with each other. For instance, consider an $H_{2GC}$-algebra on the three element chain $0 < u < 1$ such that $x^{\wedge} = 0$ and $x^{\vee} = 1$ for all $x \in \{0, u, 1\}$. Then $(1 \rightarrow u)^{\wedge} = 1$, but $1^{\wedge} \rightarrow u^{\wedge} = 1 \rightarrow u = u$. This actually means that we have an “intermediate logic” Int2GC + $\{(5), (5')\}$ situated between Int2GC and Int2GC + FS. However, the study of Int2GC + $\{(5), (5')\}$ is confined outside of the scope of this work.

References

[1] R. Balbes and Ph. Dwinger, Distributive lattices, University of Missouri Press, Columbia, Missouri, 1974.
[2] R. Bělohlávek, Fuzzy Galois connections, Mathematical Logic Quarterly 45 (1999), 497–504.
[3] D. van Dalen, Intuitionistic logic, The Blackwell guide to philosophical logic, 2001, pp. 224–257.
[4] S. P. Demri and E. S. Orłowska, Incomplete information: Structure, inference, complexity, Springer-Verlag, Berlin, Heidelberg, 2002.
[5] D. Dubois and H. Prade, Rough fuzzy sets and fuzzy rough sets, International Journal of General Systems 17 (1990), 191–209.
[6] J. M. Dunn, Positive modal logic, Studia Logica 55 (1995), 301–317.
[7] W. Dzik, J. Järvinen, and M. Kondo, Intuitionistic propositional logic with Galois connections, Logic Journal of the IGPL 18 (2010), 837–858.
[8] M. Erné, J. Koslowski, A. Melton, and G. E. Strecker, A primer on Galois connections, Annals of the New York Academy of Sciences 704 (1993), 103–125.
[9] W. B. Ewald, Intuitionistic tense and modal logic, The Journal of Symbolic Logic 51 (1986), 166–179.
[10] G. Fischer Servi, Axiomatizations for some intuitionistic modal logics, Rendiconti del Seminario Matematico dell' Università Politecnica di Torino 42 (1984), 179–194.
[11] G. Georgescu and A. Popescu, Non-dual fuzzy connections, Archive for Mathematical Logic 43 (2004), 1009–1039.
[12] J. A. Goguen, L-fuzzy sets, Journal of Mathematical Analysis and Applications 18 (1967), 145–174.
[13] G. Grätzer, General lattice theory, 2nd ed., Birkhäuser, Basel, 1998.
[14] S. O. Hansson and T. Grün-Yanoff, Preferences, The Stanford encyclopedia of philosophy, 2009, [http://plato.stanford.edu/archives/spr2009/entries/preferences/](http://plato.stanford.edu/archives/spr2009/entries/preferences/)
[15] J. Järvinen, M. Kondo, and J. Kortelainen, Logics from Galois connections, International Journal of Approximate Reasoning 49 (2008), 595–606.
[16] Z. Pawlak, Rough sets, International Journal of Computer and Information Sciences 11 (1982), 341–356.
[17] H. Rasiowa and R. Sikorski, The mathematics of metamathematics, 2nd ed., PWN-Polish Scientific Publishers, Warsaw, 1968.

Wojciech Dzik, Institute of Mathematics, University of Silesia, ul. Bankowa 12, 40-007 Katowice, Poland
E-mail address: dzikw@silesia.top.pl

Jouni Järvinen, Turku, Finland
E-mail address: Jouni.Kalervo.Jarvinen@gmail.com

Michiro Kondo, School of Information Environment, Tokyo Denki University, Inzai, 270-1382, Japan
E-mail address: kondo@sie.dendai.ac.jp