RICCI FLOW STARTING FROM AN EMBEDDED CLOSED
CONVEX SURFACE IN $\mathbb{R}^3$

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ABSTRACT. In this paper, we establish the existence and uniqueness of Ricci flow that admits an embedded closed convex surface in $\mathbb{R}^3$ as metric initial condition. The main point is a family of smooth Ricci flows starting from smooth convex surfaces whose metrics converge uniformly to the metric of the initial surface in intrinsic sense.

1. Introduction

Since Ricci flow was introduced by Hamilton [6] in 1982, it has many applications in differential geometry and topology, such as the solutions of Poincaré conjecture [8, 9] and differentiable sphere theorem [3] etc. Another important application of Ricci flow is the smoothing of initial condition. In [18, 19], Simon proved the existence of Ricci flow that admits a class of irregular metric spaces with dimension two or three as metric initial condition. This is an approximation of the metric space by Ricci flow. Based on Simon’s work, Richard [13, 14] studied the existence and uniqueness of Ricci flow whose metric initial condition is a closed Alexandrov surface with curvature bounded from below, which gives a canonical smoothing of such surface via Ricci flow. The works of Simon and Richard are related to the Gromov-Hausdorff convergence. In this paper, we consider stronger convergence of Ricci flow for embedded closed convex surface in $\mathbb{R}^3$ (see Theorem 1.3).

Before stating our results, we first recall Simon [18, 19] and Richard’s results [13, 14]. One key point in [19] is the following estimates.

Theorem 1.1. (Theorem 7.1 in [19]) Let $(M, g_0)$ be a complete smooth three (or two) manifold without boundary such that

\begin{align}
(a) \ & \text{Ricci}(g_0) \geq k; \\
(b) \ & \text{vol}(g_0 B_1(x)) \geq v_0 > 0 \text{ for all } x \in M; \\
(c) \ & \sup_M |\text{Riem}(g_0)| < \infty.
\end{align}

Then there are constants $c_1 = c_1(v_0, k) > 0$, $c_2 = c_2(v_0, k) > 0$, $S = S(v_0, k) > 0$ and $K = K(v_0, k)$ and a solution $(M, g(t))_{t \in [0, T)}$ to Ricci flow which satisfies $T \geq S$, and

\begin{align}
(a_t) \ & \text{Ricci}(g(t)) \geq -K^2; \\
(b_t) \ & \text{vol}(g B_1(x)) \geq \frac{v_0}{2} > 0 \text{ for all } x \in M \text{ and } t \in (0, T);
\end{align}

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Based on this result, the following existence and uniqueness of Ricci flow were proved in [19].

Theorem 1.2. (Theorem 1.9 in [19] and Theorem 0.5 in [14]) Let \((X, d)\) be a closed Alexandrov surface with curvature bounded from below by \(-K\). Then there is a smooth Ricci flow \((M, g(t))_{t \in (0,T)}\) admitting \((X, d)\) as metric initial condition in the sense that the Riemannian distances \(d_{g(t)}\) uniformly converge as \(t\) goes to 0 to a distance \(\bar{d}\) on \(M\) such that \((M, \bar{d})\) is isometric to \((X, d)\).

Moreover, if there is another smooth Ricci flow \((N, h(t))_{t \in (0,T)}\) also admitting \((X, d)\) as metric initial condition in above sense, then there is a diffeomorphism \(\varphi : M \to N\) such that \(g(t) = \varphi^* h(t)\).

Their ideas are as follows. They first construct a sequence of smooth manifolds \((M_i, g_i)\) that converges to \((X, d)\) in Gromov-Hausdorff distance and keeps the uniform properties in [17]. For every \(i\), there is a smooth Ricci flow \((M_i, g_i(t))_{t \in [0,T]}\) starting from \((M_i, g_i)\). Then by using Theorem 1.1 and taking the limit as \(i \to \infty\), they get the Ricci flow \((M, g(t))_{t \in (0,T)}\) that converge to \((X, d)\) in Gromov-Hausdorff distance.

A natural question is when such a Ricci flow will converge to the initial metric in classical sense, and what kind of uniqueness one can claim. In this paper, when the metric initial condition \((X, d)\) is an embedded closed convex surface in \(\mathbb{R}^3\), we prove that the Ricci flow will converge to the intrinsic sense to \((X, d)\) as \(t \to 0\) and that such flows keep the isometries between their metric initial conditions. We would like to remark that the convex surface in this paper is in the sense of Alexandrov (see section 2), unless otherwise specified.

Our first result is the following existence theorem.

Theorem 1.3. If \((X, d)\) is an embedded closed convex surface in \(\mathbb{R}^3\), then there exists a \(T > 0\) and a smooth Ricci flow \((X, g(t))_{t \in (0,T)}\) such that the distance functions \(d_{g(t)}\) induced by \(g(t)\) converge uniformly to \(d\) as \(t \to 0\), that is,

\[
\lim_{t \to 0} \max_{p, q \in X} |d_{g(t)}(p, q) - d(p, q)| = 0.
\]

The difference between Theorem 1.2 and Theorem 1.3 is that we remove the isometry between \((M, \bar{d})\) and \((X, d)\) in Theorem 1.2 when the metric initial condition \((X, d)\) is an embedded closed convex surface in \(\mathbb{R}^3\). This is due to the existence of smooth convex surfaces that approximate \((X, d)\) in Hausdorff distance (Lemma 3.1) instead of the Gromov-Hausdorff convergence in [13, 18, 19]. In fact, removing the isometry is crucial for proving the uniqueness of such Ricci flow and then study the rigidity problem of closed convex surfaces (see our project in Section 4).

From now on, unless otherwise specified, by saying that \((X, g(t))_{t \in (0,T)}\) is a Ricci flow admitting an embedded closed convex surface \((X, d)\) in \(\mathbb{R}^3\) as metric initial condition, we mean that it is a Ricci flow in the sense of Theorem 1.3.
Since every closed convex surface can be embedded in $\mathbb{R}^3$ as the boundary of a convex body by using Alexandrov’s embedding theorem (Theorem 1.3). By Theorem 1.3, we have the following existence result.

**Corollary 1.4.** For any closed convex surface $(X, d)$, there exists a $T > 0$ and a smooth Ricci flow $g(t)$ with $t \in (0, T)$ admitting $(X, d)$ as metric initial condition in the sense that $(\tilde{X}, g(t))_{t \in (0, T)}$ is a Ricci flow admitting $(\tilde{X}, \tilde{d})$ as metric initial condition, where $(\tilde{X}, \tilde{d})$ is an isometric embedding of $(X, d)$ into $\mathbb{R}^3$.

The second result in this paper is the following uniqueness theorem.

**Theorem 1.5.** Assume that $(X_1, d_1)$ and $(X_2, d_2)$ are two non-degenerate embedded closed convex surfaces in $\mathbb{R}^3$ and $\varphi : (X_1, d_1) \to (X_2, d_2)$ is an isometry. Let $(X_1, g_1(t))_{t \in (0, T)}$ and $(X_2, g_2(t))_{t \in (0, T)}$ be Ricci flows admitting $(X_1, d_1)$ and $(X_2, d_2)$ as metric initial conditions respectively. Then $g_1(t) = f^* g_2(t)$.

Theorem 1.5 gives the exact expression of the diffeomorphism in Theorem 1.2 when the metric initial condition is an embedded closed convex surface in $\mathbb{R}^3$. This result means that Ricci flows obtained in Theorem 1.3 keep the isometries between their metric initial conditions. The point here is to prove that the isometry between the two metric initial conditions is differentiable (Theorem 3.7), which implies that the pull back metrics under this isometry still satisfy Ricci flow. Then Theorem 1.5 follows from Proposition 0.6 in [14].

If the metric initial conditions are closed convex surfaces, we have the following uniqueness result.

**Corollary 1.6.** Assume that $(X_1, d_1)$ and $(X_2, d_2)$ are two isometric closed convex surfaces with non-degenerate isometric embeddings in $\mathbb{R}^3$, and that $g_1(t)$ and $g_2(t)$ with $t \in (0, T)$ are Ricci flows admitting $(X_1, d_1)$ and $(X_2, d_2)$ as metric initial conditions in the sense of Corollary 1.4. Then $g_1(t)$ and $g_2(t)$ are isometric.

**Remark 1.7.** Let $\varphi_1 : (X_1, d_1) \to (\tilde{X}_1, \tilde{d}_1)$ and $\varphi_2 : (X_2, d_2) \to (\tilde{X}_2, \tilde{d}_2)$ be isometric embeddings of $(X_1, d_1)$ and $(X_2, d_2)$ into $\mathbb{R}^3$ respectively, and $f : (X_1, d_1) \to (X_2, d_2)$ be the isometry in Corollary 1.4. From Theorem 1.5, $F := \varphi_2 \circ f \circ \varphi_1^{-1}$ is the isometry between $g_1(t)$ and $g_2(t)$ in Corollary 1.6 that is, $g_1(t) = F^* g_2(t)$.

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2. Preliminaries

In this section, we recall some basic results about convex surfaces in the sense of Alexandrov. These are mainly taken from \([2,11]\), see also the Appendix of \([14]\).

Let \((X,d)\) be a metric space, it is called a geodesic metric space if any two points \(a\) and \(b\) in \(X\) can be connected by a continuous path of shortest length on \(X\). Suppose \((X_1,d_1)\) and \((X_2,d_2)\) are two metric spaces, an isometry between \(X_1\) and \(X_2\) is a bijection \(f : X_1 \to X_2\) such that

\[
d_2(f(a), f(b)) = d_1(a, b), \quad \text{for all } a, b \in X_1.
\]

Let \(a, b\) and \(c\) be three different points in a geodesic metric space \((X,d)\), we define the comparison angle \(\angle a\tilde{b}\tilde{c}\) as the angle at \(\tilde{a}\) of the comparison triangle \(\tilde{a}\tilde{b}\tilde{c}\) in \(S_0\) whose sides have length \(d_0(\tilde{a}, \tilde{b}) = d(a, b), d_0(\tilde{a}, \tilde{c}) = d(a, c)\) and \(d_0(\tilde{b}, \tilde{c}) = d(b, c)\), where \(S_0\) is the Euclidean space, and \(d_0\) is the standard distance in \(S_0\).

**Definition 2.1.** Let \((X,d)\) be a geodesic metric space, it is said to satisfy the convexity condition if for any point \(a \in X\), and any two shortest paths \((\gamma_1(s))_{s \in [0,T]}\) and \((\gamma_2(s))_{s \in [0,T]}\) in \(X\) parametrized by arc length issuing from \(a\), the comparison angle \(\angle \tilde{a}\gamma_1(s)\tilde{b}\gamma_2(t)\) is an non-increasing function of \(s\) and \(t\).

**Definition 2.2.** Let \((X,d)\) be a geodesic metric space, it is called a closed convex surface in the sense of Alexandrov, if it is at the same time a compact topological surface without boundary, and satisfies the convexity condition.

We also have the following equivalent definition.

**Definition 2.3.** A closed convex surface in the sense of Alexandrov is a geodesic metric space \((X,d)\) which is at the same time a compact topological surface without boundary and a metric space with non-negative curvature in the sense of Alexandrov.

A geodesic metric space has non-negative curvature in the sense of Alexandrov if its geodesic triangles are bigger than the geodesic triangles in \(S_0\). To be more precise, a geodesic metric space \((X,d)\) has non-negative curvature in the sense of Alexandrov if and only if the following condition is satisfied:

Let \(a, b\) and \(c\) be any three points in \((X,d)\), and \(m\) be any point on a shortest path from \(b\) to \(c\). Let \(\tilde{a}, \tilde{b}\) and \(\tilde{c}\) be points in \(S_0\) such that \(d_0(\tilde{a}, \tilde{b}) = d(a, b), d_0(\tilde{a}, \tilde{c}) = d(a, c)\) and \(d_0(\tilde{b}, \tilde{c}) = d(b, c)\). If \(\tilde{m}\) is a point on \(\tilde{b}\tilde{c}\) such that \(d_0(\tilde{b}, \tilde{m}) = d(b, m)\). Then \(d(a,m) \geq d_0(\tilde{a}, \tilde{m})\).

In the following, we call a closed convex surface in the sense of Alexandrov a closed convex surface if there is no confusion. By Toponogov’s theorem, every closed smooth surface with non-negative Gauss curvature is a closed convex surface. The boundary of a convex set with the induced metric in \(\mathbb{R}^3\) is also a closed convex surface (Theorem 10.2.6 in \([4]\)). Alexandrov proved the following isometric embedding theorem.

**Theorem 2.4.** (Page 269 in \([2]\)) Any closed convex surface \((X,d)\) can be isometrically embedded into \(\mathbb{R}^3\) as the boundary of a (possibly degenerate) convex body.

The following lemma will be used in the proof of Theorem 1.3. Its geometric meaning is that the Hausdorff distance of two convex surfaces in \(\mathbb{R}^3\) controls their intrinsic distance functions.
Lemma 2.5. (Theorem 2 in Chapter 3 of [2]) For every closed convex surface $F$ and for every $\varepsilon > 0$, there exists a $\delta > 0$ such that whenever the deviation of a closed convex surface $S$ from $F$ is less than $\delta$ and the distance of some points $X$ and $Y$ on $F$ from some points $A$ and $B$ on $S$ are also less than $\delta$, we have

$$|d_F(X,Y) - d_S(A,B)| < \varepsilon,$$

where $d_F$ and $d_S$ are the distance functions on $F$ and $S$ respectively.

3. Existence and Uniqueness of Ricci Flow

In this section, assuming that $(X, d)$ is an embedded closed convex surface in $\mathbb{R}^3$, we prove the existence and uniqueness of the Ricci flow that admits $(X, d)$ as a metric initial condition.

3.1. Existence of the Ricci flow. In this subsection, we prove the existence of Ricci flow $(X, g(t))_{t \in [0,T)}$ that admits $(X, d)$ as metric initial condition. First, note that, for every embedded closed convex surface $(X, d)$ in $\mathbb{R}^3$, it can be seen as the boundary of a convex body $K$. Then there is a sequence of smooth convex bodies $K_i$ that converges to $K$ in Hausdorff distance (Theorem 3.4.1 in [15]). In particular, $\partial K_i$ converges to $X$ in Hausdorff distance as well. By Lemma 2.5, $\partial K_i$ with induced metric in $\mathbb{R}^3$ (denoted by $(X_i, \hat{g}_i)$) converge to $(X, d)$ in intrinsic sense. More precisely, we have the following lemma.

Lemma 3.1. Let $(X, d)$ be an embedded closed convex surface in $\mathbb{R}^3$. Then there exists a sequence of closed smooth convex surfaces $\{(X_i, \hat{g}_i)\}_{i=1}^{\infty}$ and bijections $f_i : X \to X_i$ such that $d_i(x, y)$ converges to $d(x, y)$ uniformly for $x, y \in X$ as $i \to \infty$, where $d_i(x, y) = \hat{d}_i(f_i(x), f_i(y))$, and $\hat{d}_i$ is the distance on $X_i$ induced by $\hat{g}_i$.

Now we prove Theorem 1.3 by using Lemma 3.1.

Proof of Theorem 1.3. First, for a sequence of subsets in the same metric space, Gromov-Hausdorff distance by definition is not greater than Hausdorff distance. Thus the Hausdorff convergence of $(X_i, \hat{g}_i)$ to $(X, d)$ in Lemma 3.1 implies the Gromov-Hausdorff convergence of $(X_i, \hat{g}_i)$ to $(X, d)$. Suppose

$$diam(X, d) \leq D \quad \text{and} \quad vol(X) \geq \tilde{v}_0 > 0$$

for some positive constants $D$ and $\tilde{v}_0$. Since the diameter and volume are continuous with respect to Gromov-Hausdorff convergence with sectional curvature bounded from below, and all the surfaces $(X_i, \hat{g}_i)$ and $(X, d)$ are convex, we have

$$diam(X_i, \hat{g}_i) \leq 2D \quad \text{and} \quad \hat{g}_i vol(X_i) \geq \frac{\tilde{v}_0}{2} > 0$$

for $(X_i, \hat{g}_i)$ with large $i$. By Bishop-Gromov comparison theorem, we get

$$\frac{vol(\hat{g}_i B_1(x))}{vol(B_1(x))} \geq \frac{vol(\hat{g}_i B_2D(x))}{vol(B_2D(x))} = \frac{\hat{g}_i vol(X_i)}{vol(B_2D(x))} \geq \frac{\tilde{v}_0}{2 vol(B_2D(x))},$$

which implies that

$$vol(\hat{g}_i B_1(x)) \geq \tilde{v}_0 \frac{vol(B_1(x))}{2 vol(B_2D(x))} := \nu_0 > 0.$$
Due to Theorem 1.1, we know that there are smooth Ricci flows \((X, g_i(t))_{t \in [0, T)}\) with \(g_i(0) = \tilde{g}_i\) satisfying

\[
\begin{align*}
(a_i') & \text{ Ricci}(g_i(t)) \geq 0 \text{ for all } t \in [0, T); \\
(b_i') & \text{ vol}(^p(t) B_1(x)) \geq \frac{\nu_0}{2} > 0, \text{ for all } x \in X \text{ and } t \in [0, T); \\
(c_i') & \sup_{X_i} |\text{Riem}(g_i(t))| \leq \frac{K}{t} \text{ for all } t \in [0, T); \\
(d_i') & d_{g_i(t)}(p, q) - c_2(\sqrt{t} - \sqrt{s}) \leq d_{g_i(t)}(p, q) \leq e^{c_1(t-s)}d_{g_i(t)}(p, q), \\
& \quad \text{ for all } 0 \leq s \leq t \in [0, T) \text{ and } p, q \in X_i,
\end{align*}
\]

where \(K = K(\nu_0), c_1 = c_1(\nu_0), c_2 = c_2(\nu_0)\) and \(T = T(\nu_0)\) are constants independent of \(i\). Combining \((c_i')\) and Shi’s higher derivative estimates \([10, 17]\) with Arzela-Ascoli theorem, there exists a subsequence (which we also denote by \(g_i(t)\)) converges to a metric \(g(t)\) on \(X\), and \((X, g(t))_{t \in (0, T)}\) is a smooth Ricci flow. Let \(s \to 0\) in \((d_i')\), for all \(t \in (0, T)\) and \(p, q \in X\), we have

\[
d_i(t) = d_{f_i(t)}(p, f_i(q)) - c_2(\sqrt{t} - \sqrt{s}) \leq d_{g_i(t)}(p, q) \leq c_1 d_i(t) - c_2(\sqrt{t} - \sqrt{s}),
\]

Since \(X_i\) converges to \(X\) in Hausdorff distance and \(g_i(t)\) converges to \(g(t)\) in local smooth sense of \((0, T)\), by letting \(i \to \infty\) and using Lemma 3.11 we have

\[
d(p, q) - c_2\sqrt{t} \leq d_{g(t)}(p, q) \leq c_1 d(p, q).
\]

Then the uniform convergence of \(d_{g(t)}\) to \(d\) follows by letting \(t \to 0\) in \((3.10)\). \(\square\)

3.2. Uniqueness. In this subsection, we prove the uniqueness of Ricci flow that admits a non-degenerate embedded closed convex surface in \(\mathbb{R}^3\) as a metric initial condition. The point is to give an exact expression of the initial metric \((\text{Theorem } 3.6)\) firstly, and then to prove that the isometry between the two metric initial conditions is differentiable (Theorem 3.27).

Let \((X, g(t))_{t \in (0, T)}\) be the Ricci flow obtained in subsection 3.1. It is easy to see that its Gaussian curvature \(K_{g(t)}\) is positive. In fact, the non-negativity of the Gaussian curvature \(K_{\tilde{g}_i}\) of the smooth convex surfaces \((X_i, \tilde{g}_i)\) implies that the Gauss curvature \(K_{g_i(t)}\) along Ricci flow \((X_i, g_i(t))_{t \in [0, T)}\) is also non-negative by applying maximum principle to the evolution equation of \(K_{g_i(t)}\),

\[
\frac{\partial}{\partial t} K_{g_i(t)} = \Delta_{g_i(t)} K_{g_i(t)} + |\text{Ric}_{g_i(t)}|^2_{g_i(t)}.
\]

From Gauss-Bonnet theorem for \(g(t)\) and the fact that \(g_i(t)\) converges to \(g(t)\) smoothly, for \(t \in (0, T)\), \(K_{g(t)}\) is non-negative and satisfies

\[
\int_X K_{g(t)} dV_{g(t)} = 4\pi.
\]

Hence there must be a point \(x_0\) such that \(K_{g(t)}\) is positive at \(x_0\). Then strong maximum principle implies that \(K_{g(t)}\) is positive everywhere.

Fix \(t_0 \in (0, T)\) and denote \((X, g(t_0)) = (X, g_{t_0})\). By the Uniformization theorem, there is a conformal equivalence (holomorphic isomorphism) \(\Phi : (S^2, \tilde{h}) \to (X, g_{t_0})\), where \(\tilde{h}\) is a smooth metric of positive constant curvature, i.e. \(\Phi^*(g_{t_0}) = e^{\tilde{u}(t_0, x)} \tilde{h}(x)\) for some smooth function \(\tilde{u}(t_0, x)\) on \(S^2\). On the other hand, the 2-dimensional Ricci flow can be written as

\[
\frac{\partial}{\partial t} g(t) = -R_{g(t)} g(t).
\]
Hence we can write \( g(t) = e^{-\int_0^t R_s(\gamma) ds} g_{t_0} := \hat{\omega}(t, x) g_{t_0} \), which implies that
\[
g(t) = \hat{\omega}(t, x) (\Phi^{-1})^* (e^t \hat{\omega}(t, x) \bar{h}(x)) = \hat{\omega}(t, x) e^{\hat{\omega}(t, \Phi^{-1}(x))} \bar{h}(\Phi^{-1}(x)) := e^{2u(t, x)} \bar{h}(x).
\]
where \( h(x) := \bar{h}(\Phi^{-1}(x)) \) and \( u(t, x) := \frac{\ln(\hat{\omega}(t, x)) + \hat{\omega}(t, \Phi^{-1}(x))}{2} \). We call \( u(t) := u(t, x) \) the conformal potential along Ricci flow \( (X, g(t))_{t \in (0, T)} \). From (3.13), the evolution equation of \( u(t) \) reads
\[
\frac{\partial u(t)}{\partial t} = e^{-2u(t)} (\Delta_h u(t) - K_h) = -K_{g(t)}.
\]
Since \( K_{g(t)} \) is positive, \( u(t) \) increases as \( t \) decreases to 0, and \( u(t) \geq u(T) \) for \( t \in (0, T) \). It is proved in Lemma 2.2 of [14] that \( u(t) \) is uniformly bounded in \( L^1 \)-sense and converges to an integrable function \( u_0(x) \) in \( L^1 \)-sense. Here, we prove that \( u(t) \) is uniformly bounded for \( t \in (0, T] \) in the classical sense (so \( u_0(x) \) is also bounded in the classical sense).

Before starting the proof, we remark that every Ricci flow \( (X, g(t))_{t \in (0, T)} \) admitting \( (X, d) \) as metric initial condition can be obtained through the process in subsection 3.1. This is due to Proposition 0.6 in [14]. Hence we only need to consider the uniqueness for the Ricci flow obtained in subsection 3.1.

**Lemma 3.2.** Assume that \( (X, d) \) is a non-degenerate embedded closed convex surface in \( \mathbb{R}^3 \). Let \( (X, g(t))_{t \in (0, T)} \) be a Ricci flow admitting \( (X, d) \) as metric initial condition. Then the conformal potential \( u(t) \) along \( (X, g(t))_{t \in (0, T)} \) is uniformly bounded. Thus, \( u_0(x) \) is bounded on \( X \).

To prove this lemma, we first prove that the smooth approximating metrics \( \tilde{g}_i \) obtained in Lemma 3.1 are uniformly equivalent to the standard metric \( \delta \) on the unit sphere \( S^2 \). Let \( (X, g) \) be a smooth embedded closed convex surface in \( \mathbb{R}^3 \), \( \rho \) be the radial function of \( (X, g) \), and \( v = \frac{1}{\rho} \). Then the induced metric on \( X \) and the second fundamental form can be written as
\[
g_{ij} = \rho^2 \delta_{ij} + \rho \delta_{ij},
\]
\[
h_{ij} = \frac{1}{\sqrt{\rho^2 + |\nabla \rho|^2}} \left( \rho^2 \delta_{ij} + 2 \rho_i \rho_j - \rho \delta_{ij} \right)
\]
\[
= \frac{\rho^3}{\sqrt{\rho^2 + |\nabla \rho|^2}} (v_{ij} + v \delta_{ij}),
\]
where the derivatives are taken with respect to the connection of \( (S^2, \delta) \).

**Lemma 3.3.** There is a uniform constant \( C \) such that
\[
\frac{1}{C} \delta \leq \tilde{g}_i \leq C \delta \quad \text{for large } i.
\]

**Proof.** Let \( \rho \) be the radial function of \( (X, \tilde{g}_i) \) and \( v = \frac{1}{\rho} \). We claim that
\[
\max_{S^2} (|\nabla \delta \ i v|^2 + \ i v^2) \leq \max_{S^2} \ i v^2.
\]
Define \( f = |\nabla \delta \ i v|^2 + (1 + \eta) \ i v^2 \) for \( \eta > 0 \). At the maximum point of \( f \), we have
\[
0 = \nabla \delta f = \nabla \delta (|\nabla \delta \ i v|^2 + (1 + \eta) \ i v^2) = 2 \ i v_j (\ i v_j + (1 + \eta) \ i v \delta_{ij}).
\]
Since \( (X, \tilde{g}_i) \) is convex, \( \ i h_{ij} \geq 0 \) and then \( \ i v_j + \ i v \delta_{ij} \geq 0 \). Then we have \( \ i v_j + (1 + \eta) \ i v \delta_{ij} > 0 \). Hence \( \nabla \delta v = 0 \) at the maximum point of \( f \), which implies
\[
\max_{S^2} (|\nabla \delta \ i v|^2 + (1 + \eta) \ i v^2) \leq (1 + \eta) \max_{S^2} \ i v^2.
\]
Let $\eta \to 0$, we complete the proof of the claim.

Since $(X, d)$ is non-degenerate and $(X_i, \tilde{g}_i)$ converges to $(X, d)$ in Hausdorff sense, there exists a constant $C$ such that for large $i$,

\[
\frac{1}{C} \leq i \rho \leq C.
\]

From the claim, $|\nabla_\delta i \rho|^2$ are also uniformly bounded for large $i$. Taking trace with respect to $\delta$ on both sides of $(\tilde{g}_i)_{lj} = i \rho_i^2 \delta_{ij} + i \rho_i \rho_j$, we conclude that there exists uniform constant $C$ such that

\[
tr_\delta \tilde{g}_i \leq C \quad \text{and} \quad tr_\delta \tilde{g}_i \geq \frac{1}{C} > 0 \quad \text{for large } i.
\]

Therefore, there is a uniform constant $C$ such that

\[
\frac{1}{C} \delta \leq \tilde{g}_i \leq C \delta
\]

for large $i$.

Next, we extend this equivalence to Ricci flow.

**Lemma 3.4.** Let $(X, g(t))_{t \in [0, T]}$ be a 2-dimensional smooth Ricci flow with initial metric $g_0$, then we have

\[
g(t) \leq e^{L_1 T} g_0 \quad \text{and} \quad tr_{g(t)} g_0 \leq L^2 (\frac{dV_{g_0}}{dV_{g(t)}})^2,
\]

where $-L_1$ is the lower bound of $R_{g_0}$, and $L_2$ depends on $L_1$, $T$.

**Proof.** Applying maximum principle to the evolution equation of the scalar curvature $R_{g(t)}$

\[
\frac{\partial}{\partial t} R_{g(t)} = \Delta_{g(t)} R_{g(t)} + 2 |\text{Ric}_{g(t)}|_{g(t)}^2,
\]

we get

\[
R_{g(t)} \geq R_{g_0} \geq -L_1.
\]

In dimension 2, Ricci flow can be written as

\[
\frac{\partial}{\partial t} g(t) = -R_{g(t)} g(t) \leq L_1 g(t).
\]

Hence we have $g(t) \leq e^{L_1 t} g_0$ and get the first estimate in \[3.23\].

For the second estimates, we need the following inequality.

\[
n \left( \frac{\det g_1}{\det g_2} \right)^{\frac{1}{n}} \leq tr_{g_2} g_1 \leq n \left( \frac{\det g_1}{\det g_2} \right) (tr_{g_1} g_2)^{n-1},
\]

where $g_1$ and $g_2$ are any two smooth $n$-dimensional metrics. In our case,

\[
tr_{g(t)} g_0 \leq 2 \left( \frac{\det g_0}{\det g(t)} \right) (tr_{g_0} g(t)) = 2 \left( \frac{dV_{g_0}}{dV_{g(t)}} \right)^2 (tr_{g_0} g(t)).
\]

Hence we only need to prove that $dV_{g(t)}$ is bounded from below uniformly. The volume form evolves as

\[
\frac{\partial}{\partial t} dV_{g(t)} = -R dV_{g(t)} \leq L_1 dV_{g(t)},
\]
which implies that \( e^{-L_1 t} dV_{g(t)} \) decrease and then \( dV_{g(t)} \geq e^{-L_1 (T-t)} dV_{g(T)} \geq e^{-L_1 T} dV_{g(T)} \). So we have

\[
tr_{g(t)} g_0 \leq 4 \left( \frac{dV_{g_0}}{e^{-L_1 T} dV_{g(T)}} \right)^2 e^{L_1 T}.
\]

Let \( L_2 = 4e^{3L_1 T} \), we complete this Lemma.

For the sequence of smooth Ricci flow \((X_i, g_i(t))_{t \in [0, T]} \) with \( \tilde{g}_i \) as initial condition, we have by Lemma 3.3 and Lemma 3.4, that

\[
g_i(t) \leq C \delta \quad \text{and} \quad tr_{g_i(t)} \delta \leq 4C^3 \left( \frac{dV_\delta}{dV_{g_i(T)}} \right)^2,
\]

where \( C \) is the constant in Lemma 3.3. Letting \( i \to \infty \) gives

\[
g(t) \leq C \delta \quad \text{and} \quad tr_{g(t)} \delta \leq 4C^3 \left( \frac{dV_\delta}{dV_{g(T)}} \right)^2 \quad \text{for} \quad t \in (0, T],
\]

which is equivalent to

\[
\frac{1}{A} \delta \leq g(t) \leq C \delta \quad \text{for} \quad t \in (0, T],
\]

where \( A \) depends on \( C \) and \( T \). In fact, we proved the following Lemma.

**Lemma 3.5.** Assume that \((X, d)\) is a non-degenerate embedded closed convex surface in \( \mathbb{R}^3 \). Let \((X, g(t))_{t \in [0, T]} \) be a Ricci flow admitting \((X, d)\) as a metric initial condition. Then there exists a constant \( C \) such that

\[
\frac{1}{C} \delta \leq g(t) \leq C \delta \quad \text{for} \quad t \in (0, T].
\]

Now Lemma 3.2 follows immediately.

**Proof of Lemma 3.2.** Since \( g(t) = e^{2u(t)} h(x) \) is uniform equivalent to \( \delta \) for \( t \in (0, T] \), \( u(t) \) is uniformly bounded. Since \( u(t) \) increases to \( u_0 \) as \( t \) decreases to 0, \( u_0 \) is also bounded.

Next, for an \( L^1 \)-function \( u \) and a smooth Riemannian metric \( h \) on \( M \), there is a metric \( d_{h,u} \) defined as

\[
d_{h,u}(x, y) = \inf_{\gamma \in \Gamma(x, y)} \int_0^1 e^{u(\gamma(t))} |\dot{\gamma}(t)|_h dt,
\]

where \( \Gamma(x, y) \) is the space of \( C^1 \) paths \( \gamma \) from \([0, 1]\) to \( M \) with \( \gamma(0) = x \) and \( \gamma(1) = y \). This metric was studied by Reshetnyak [12]. For more details, please see the appendix of [14].

In [14], when \((X, d)\) is a compact Alexandrov surface with curvature bounded from below, the metric \( d_{g(t)} \) induced by \( g(t) \) (here, \( g(t) \) is a Ricci flow on \( M \) with metric initial condition \((X, d)\) in the sense of Theorem 1.2) converges to \( d_{h,u_0} \) uniformly, where \( u_0 \) is the \( L^1 \)-limit of the conformal potential \( u(t) \) along Ricci flow \((M, g(t))_{t \in [0, T]} \) as \( t \to 0 \) in [14]. But \( d \) may not be \( d_{h,u_0} \) there. In fact, we can only conclude that \((M, d_{h,u_0})\) is isometric to \((X, d)\) from the Lemma 2.4 in [14]. In our case when \((X, d)\) is a non-degenerate embedded closed convex surface in \( \mathbb{R}^3 \), we prove that indeed \( d = d_{h,u_0} \).
**Theorem 3.6.** Assume that $(X,d)$ is a non-degenerate embedded closed convex surface in $\mathbb{R}^3$. Let $(X, g(t))_{t \in [0,T]}$ be the Ricci flow admitting $(X,d)$ as a metric initial condition and $u(t)$ be the conformal potential along $(X, g(t))_{t \in [0,T]}$. Then

\[ d = d_{h,u_0}, \]

where $u_0(x)$ is the pointwise limit of $u(t,x)$ as $t \to 0$.

**Proof.** By definition \((3.35)\), and the definition of conformal potential $g(t) = e^{2u(t,x)}h(x)$, we have

\[
d_{h,u(t)}(x,y) = \inf_{\gamma \in \Gamma(x,y)} \int_0^1 e^{u(t,\gamma(\tau))}|\dot{\gamma}(\tau)|h\,d\tau
\]

\[(3.37)\]

By Lemma 2.4 in [14], $d_{h,u(t)}$ converges to $d_{h,u_0}$ uniformly as $t \to 0$. Since we have proved that $d_{g(t)}$ converges to $d$ uniformly as $t \to 0$ on $X$, then \((3.36)\) follows by letting $t \to 0$ on both sides of \((3.37)\). \qed

We now prove a regularity theorem for the isometry between the metric spaces $(X_1,e^{2u_1}h_1)$ and $(X_2,e^{2u_2}h_2)$, where $u_1$ and $u_2$ are two bounded functions, and $h_1$ and $h_2$ are pull back metrics of the metrics on $S^2$ with constant Gaussian curvature.

**Theorem 3.7.** Assume that $F : (X_1,e^{2u_1}h_1) \to (X_2,e^{2u_2}h_2)$ is an isometry. Then $F$ is differentiable, where $u_1$ and $u_2$ are two bounded functions, and $h_1$ and $h_2$ are two pull back metrics of the metrics on $S^2$ with constant Gaussian curvature.

**Proof.** Since $F$ is an isometry, $F$ is bi-Lipschitz and then it is differentiable almost everywhere. Write locally $h_1 = \lambda_1(du^2+dv^2)$ and $h_2 = \lambda_2(dx^2+dy^2)$ for some positive functions $\lambda_1$ and $\lambda_2$. At differentiable point of $F$, we have

\[
e^{2u_1}\lambda_1(du^2+dv^2) = e^{2u_2}\lambda_2(dx^2+dy^2)
\]

\[= e^{2u_2}\lambda_2(F\circ F)((x_u du + x_v dv)^2 + (y_u du + y_v dv)^2)
\]

\[= e^{2u_2}\lambda_2(F\circ F)((x_u^2 + y_u^2)du^2 + (x_v^2 + y_v^2)dv^2 + 2(x_u x_v + y_u y_v)du dv)
\]

Since $u_1$ and $u_2$ are bounded functions, we have

\[(3.38)\]

\[x_u x_v + y_u y_v = 0 \quad \text{and} \quad x_u^2 + y_u^2 = x_v^2 + y_v^2,
\]

which is equivalent to the fact that

\[(3.39)\]

\[x_u = -y_v \quad \text{and} \quad x_v = y_u \quad \text{or} \quad x_u = y_v \quad \text{and} \quad x_v = -y_u.
\]

Then we know that $F$ is differentiable everywhere by using the analytic extension theorem in [11]. \qed

Since $u(t)$ increases to $u_0$ and both of them are bounded, by using \((3.37)\) and Lebesgue’s monotone convergence theorem, we conclude that $d = d_{h,u_0}$ is the metric induced by $e^{2u_0}h$.

**Proof of Theorem 3.6** By Theorem 3.6 $d_i = d_{h_i,u_i}$, (i = 1, 2) is the metric induced by $e^{2u_i}h_i$, where $u_i = \lim_{t \to 0} u_i(t)$, $u_i(t)$ is the conformal potential along $(X_i,g_i(t))_{t \in [0,T]}$ and $h_i$ is the pull back metric of a metric on $S^2$ with constant Gaussian curvature.

From Theorem 3.7 the isometry $f$ is differentiable. Then $(X_1,F^*g_2(t))_{t \in (0,T)}$ is also a Ricci flow admitting $(X_1,d_i)$ as metric initial condition in the sense that the
distance function induced by $f^*g_2(t)$ converges uniformly to $d_1$ as $t \to 0$. By using Proposition 0.6 in [14], we have $g_1(t) = f^*g_2(t)$. \hfill $\Box$

4. FURTHER DISCUSSIONS

In this section, we introduce our project which aims to study Pogorelov’s uniqueness theorem by Ricci flow.

Pogorelov’s uniqueness theorem [11] states that any two closed isometric convex surfaces with induced metrics in $\mathbb{R}^3$ are congruent. It is a generalization of the classical Cohn-Vesson’s rigidity theorem [5].

Our basic idea to study this theorem is to use Ricci flow to construct two families of smooth convex surfaces approximating the two isometric closed convex surfaces in Pogorelov’s theorem. We then apply Cohn-Vesson’s rigidity theorem to the smooth surfaces and take a limit as $t \to 0$ to see if “the limit of Cohn-Vesson’s rigidity theorem will imply Pogorelov’s rigid theorem”. More precisely, given two isometric embedded closed convex surfaces $(X_1, d_1)$ and $(X_2, d_2)$ in $\mathbb{R}^3$, Theorem 1.3 implies that there are two Ricci flows $(X_1, g_1(t))_{t \in (0, T)}$ and $(X_2, g_2(t))_{t \in (0, T)}$ admitting $(X_1, d_1)$ and $(X_2, d_2)$ as metric initial conditions. Then for every positive time $t$, we embed the Ricci flow $(X_1, g_1(t))$ and $(X_2, g_2(t))$ smoothly and isometrically into $\mathbb{R}^3$ as $(X_1^t, G_1(t))$ and $(X_2^t, G_2(t))$ respectively. The validity for these embeddings is due to the fact that $(X_1, g_1(t))$ and $(X_2, g_2(t))$ are smooth strictly convex surfaces and the solvability of Weyl’s problem proved by Nirenberg [7]. By Theorem 1.5 $(X_1^t, G_1(t))$ and $(X_2^t, G_2(t))$ are isometric. Then Cohn-Vesson’s rigidity theorem implies that there is a congruence $F(t) \in O(3)$ between them. We hope to investigate the limits of $(X_1^t, G_1(t))$ and $(X_2^t, G_2(t))$ as $t \to 0$. If $(X_1^t, G_1(t))$ and $(X_2^t, G_2(t))$ converge to $(X_1, d_1)$ and $(X_2, d_2)$ in Hausdorff distance up to an isometry in $O(3)$ respectively, the compactness of $O(3)$ will imply the congruence between $(X_1, d_1)$ and $(X_2, d_2)$.

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