ON AUTOMORPHISMS GROUPS OF STRUCTURES OF COUNTABLE COFINALITY

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ABSTRACT. In [2], Su Gao proves that the following are equivalent for a countable $\mathcal{M}$ (cf. theorem 1.2 too):

(I) There is no uncountable model of the Scott sentence of $\mathcal{M}$.

(II) There exists some $j \in \text{Aut}(\mathcal{M}) \setminus \text{Aut}(\mathcal{M})$, where $\text{Aut}(\mathcal{M})$ is the closure of $\text{Aut}(\mathcal{M})$ under the product topology in $\omega^\omega$.

(III) There is an $\mathcal{L}_{\omega_1,\omega}$-elementary embedding $j$ from $\mathcal{M}$ to itself such that $\text{range}(j) \subset M$.

We generalize his theorem to all cardinals $\kappa$ of cofinality $\omega$ (cf. theorem 4.2). The following are equivalent:

(I$^\ast$) There is a model of the Scott sentence of $\mathcal{M}$ of size $\kappa^+$.

(II$^\ast$) For all $\alpha < \beta < \kappa^+$, there exist functions $j_{\beta,\alpha}$ in $\text{Aut}(\mathcal{M})^T \setminus \text{Aut}(\mathcal{M})$, such that for $\alpha < \beta < \gamma < \kappa^+$,

\[ j_{\gamma,\beta} \circ j_{\beta,\alpha} = j_{\gamma,\alpha}, \]

where $\text{Aut}(\mathcal{M})^T$ is the closure of $\text{Aut}(\mathcal{M})$ under the product topology in $\kappa^+$.

(III$^\ast$) For every $\beta < \kappa^+$, there exist $\mathcal{L}^\text{fin}_{\omega_1,\kappa}$-elementary embeddings (cf. definition 2.5) $(j_\alpha)_{\alpha < \beta}$ from $\mathcal{M}$ to itself such that $\alpha_1 < \alpha_2 \Rightarrow \text{range}(j_{\alpha_1}) \subset \text{range}(j_{\alpha_2})$.

Theorem 4.2 holds both for countable and uncountable $\kappa$. Condition (I$^\ast$), which does not appear in the countable case, can not be removed when $\kappa$ is uncountable (cf. theorem 4.5).

Condition (II$^\ast$) imply the existence of at least $\kappa^\omega$ automorphisms of $\mathcal{M}$ (cf. corollary 4.6). It is unknown to the author whether a purely topological proof of corollary 4.6 exists.

1. Background

In [2], Su Gao proves the following

**Theorem 1.1** (Su Gao). The following are equivalent for a countable $\mathcal{M}$:

(I) There is no uncountable model of the Scott sentence of $\mathcal{M}$.

(II) $\text{Aut}(\mathcal{M})$ admits a compatible left-invariant complete metric.

(III) There is no $\mathcal{L}_{\omega_1,\omega}$-elementary embedding from $\mathcal{M}$ to itself which is not onto.

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It is also proven in [2] that condition (II) of theorem 1.1 is equivalent to

\[(II)' \quad \text{Aut}(\mathcal{M}) \text{ is closed in the Baire space } \omega^\omega.\]

Using \((II)'\) and taking negations, we rephrase theorem 1.1.

**Theorem 1.2** (Su Gao). The following are equivalent for a countable model \(\mathcal{M}\):

1. There is an uncountable model of the Scott sentence of \(\mathcal{M}\).
2. There exists some \(j \in \overline{\text{Aut}(\mathcal{M})} \setminus \text{Aut}(\mathcal{M})\), where \(\overline{\text{Aut}(\mathcal{M})}\) is the closure of \(\text{Aut}(\mathcal{M})\) under the product topology in \(\omega^\omega\).
3. There is an \(L_{\omega_1,\omega}\) -elementary embedding \(j\) from \(\mathcal{M}\) to itself such that \(\text{range}(j) \subset \mathcal{M}\).

**Notation:** Throughout the rest of the paper we assume that

1. \(\kappa\) is a cardinal of cofinality \(\omega\),
2. unless otherwise stated, \(\mathcal{M}, \mathcal{N}\) are models of size \(\kappa\) and
3. both \(\subset\) and \(\supset\) refer to strict subset and strict superset relations.

The main theorem is theorem 1.2 and generalizes theorem 1.1 to any cardinal of cofinality \(\omega\). Section 4 is devoted to the proof of the main theorem and a counterexample is provided that existence of \(\kappa^+\)-many (or even \(\kappa^\omega\)-many) elements in \(\overline{\text{Aut}(\mathcal{M})} \setminus \text{Aut}(\mathcal{M})\) does not imply the existence of a model of the Scott sentence of \(\mathcal{M}\) in \(\kappa^+\). This renders condition (\#) necessary. Section 2 contains some notions from infinitary logic \(L_{\infty,\kappa}\) and the definition of \(L_{\infty,\kappa,\text{fin}}\) -elementary embeddings, \(\prec_{\infty,\kappa,\text{fin}}\). Section 3 contains the main properties of \(\prec_{\infty,\kappa,\text{fin}}\). The last section 5 contains open questions, a couple of which are from General Topology.

Before we proceed, the reader should note that if \(\phi_M\) is the Scott sentence of some model \(\mathcal{M}\), then the collection \(K = \{\mathcal{N} | \mathcal{N} \models \phi_M\} <_{\infty,\kappa,\text{fin}}\) is not an A.E.C. The reason is that \(\prec_{\infty,\kappa}\) and \(\prec_{\infty,\kappa,\text{fin}}\) are not closed under unions of length \(< \kappa^+\). This is a pretty well-known fact for \(\prec_{\infty,\kappa}\) and although not presented here, similar examples can be found for \(\prec_{\infty,\kappa,\text{fin}}\). To compensate for the lack of closedness under unions some extra work is required in Section 3. Nevertheless, many of the properties of \(K\) are also properties of A.E.C.  

2. Infinitary Logic

One obstacle in extending theorem 1.2 to uncountable cardinals is that if \(\mathcal{M}\) is a model of uncountable cardinality, there maybe no Scott sentence for

\[\text{If } \text{cf}(\kappa) = \omega, \text{ then } \kappa^\omega \geq \kappa^+.\]

\( \mathcal{M} \). If \( cf(|\mathcal{M}|) = \omega \), then the Scott sentence is guaranteed by the following theorem from \([1]\):

**Theorem 2.1** (C. C. Chang). Let \( \mathcal{N} \) be a model of cardinality \( \kappa \) with \( cf(\kappa) = \omega \). Then there is a sentence \( \phi_{\mathcal{N}} \) in \( L_{(\kappa, \kappa^+)}^{\kappa} \) such that:

1. For any \( \mathcal{N}' \) of cardinality \( \leq \kappa \), \( \mathcal{N}' \models \phi_{\mathcal{N}} \) iff \( \mathcal{N}' \cong \mathcal{N} \).
2. If \( \mathcal{N}' \) is any model (possibly of cardinality \( > \kappa \)), then \( \mathcal{N}' \models \phi_{\mathcal{N}} \) iff \( \mathcal{N}' \equiv_{\infty, \kappa} \mathcal{N} \).

For this theorem also follows **Theorem 2.2**.

Let \( \mathcal{N} \) be a model of cardinality \( \kappa \) with \( cf(\kappa) = \omega \) and let \( \mathcal{N}_0 \) be a subset of \( \mathcal{N} \) of size \( < \kappa \). Then there is a sentence \( \phi_{\mathcal{N}, \mathcal{N}_0} \) in \( L_{(\kappa, \kappa^+)}^{\kappa} \) such that:

1. If \( \mathcal{N}' \) is a model cardinality \( \leq \kappa \) and \( \mathcal{N}_0' \subset \mathcal{N}' \), then \( \mathcal{N}' \models \phi_{\mathcal{N}, \mathcal{N}_0}[\mathcal{N}_0'] \) iff there is an isomorphism \( i : \mathcal{N}' \cong \mathcal{N} \) with \( i(\mathcal{N}_0) = \mathcal{N}_0' \).
2. If \( \mathcal{N}' \) is a model of any cardinality (possibly \( > \kappa \)) and \( \mathcal{N}_0' \subset \mathcal{N}' \), then \( \mathcal{N}' \models \phi_{\mathcal{N}, \mathcal{N}_0}[\mathcal{N}_0'] \) iff \( (\mathcal{N}', \mathcal{N}_0') \equiv_{\infty, \kappa} (\mathcal{N}, \mathcal{N}_0) \).

In particular, the above theorem holds for \( \mathcal{N}_0 \) finite. The sentence \( \phi_{\mathcal{N}, \mathcal{N}_0} \) is called the Scott sentence of \( \mathcal{N}_0 \) (in \( \mathcal{N} \)).

**Definition 2.3.** \( A \) and \( B \) are \( \kappa \)-partially isomorphic, write \( A \cong_{\kappa} B \), if there is a non-empty set \( I \) of partial isomorphisms from \( A \) to \( B \) with the \( \kappa \)-back-and-forth property:

for any \( f \in I \) and \( C \subset A \) with \( |C| < \kappa \) (or \( D \subset B \) with \( |D| < \kappa \)), there is some \( g \in I \) that extends \( f \) and \( C \subset \text{dom}(g) \) (or \( D \subset \text{range}(g) \)).

The following two theorems are from \([6]\). The first is attributed to C. Karp.

**Theorem 2.4.** Let \( \kappa \geq \omega \). Then for any \( \mathcal{N}, \mathcal{N}' \),

1. \( \mathcal{N}' \equiv_{\infty, \kappa} \mathcal{N} \) iff \( \mathcal{N}' \cong_{\kappa} \mathcal{N} \)
2. \( \mathcal{N}' \equiv_{\infty, \kappa} \mathcal{N} \) iff for every \( \vec{a} \in \mathcal{N'}^{<\kappa} \), there is some \( \vec{b} \in \mathcal{N}^{<\kappa} \) such that \( (\mathcal{N'}, \vec{a}) \equiv_{\infty, \kappa} (\mathcal{N}, \vec{b}) \).

Note that in the second part we can switch the roles of \( \vec{a} \) and \( \vec{b} \), i.e.

\( \mathcal{N}' \equiv_{\infty, \kappa} \mathcal{N} \) iff for every \( \vec{b} \in \mathcal{N}^{<\kappa} \) there is some \( \vec{a} \in \mathcal{N'}^{<\kappa} \) such that \( (\mathcal{N'}, \vec{a}) \equiv_{\infty, \kappa} (\mathcal{N}, \vec{b}) \).

**Definition 2.5.** The \( (L_{\omega, \omega}) \)-embedding \( j : \mathcal{M} \rightarrow \mathcal{N} \) will be called a \( L_{\infty, \kappa}^{\text{fin}} \)-elementary embedding if for every formula \( \phi(\vec{x}) \in L_{\infty, \kappa} \) with finitely many free variables, and for every finite \( \vec{a} \in M \),

\( \mathcal{M} \models \phi[\vec{a}] \) iff \( \mathcal{N} \models \phi[j(\vec{a})] \).
Similarly we define $\mathcal{L}_{\infty, \kappa}^{\text{fin}}$-elementary substructures:

If $\mathcal{M} \subset \mathcal{N}$ we will call $\mathcal{M}$ a $\mathcal{L}_{\infty, \kappa}^{\text{fin}}$-elementary substructure of $\mathcal{N}$ and write $\mathcal{M} \prec_{\infty, \kappa}^{\text{fin}} \mathcal{N}$, if the inclusion map $\text{id} : \mathcal{M} \rightarrow \mathcal{N}$ is an $\mathcal{L}_{\infty, \kappa}^{\text{fin}}$-elementary embedding.

The difference than the regular definition of $\mathcal{L}_{\infty, \kappa}$-elementary embedding is that we restrict ourselves to finite $\vec{a}$ only. The motivation for this definition is from lemma 3.4. Observe also that if $\kappa > \omega$, the $\mathcal{L}_{\infty, \kappa}$-formulas with finitely many free variables are not closed under subformulas.

3. Properties of $\prec_{\infty, \kappa}^{\text{fin}}$

As we noted before, if $\phi_{\mathcal{M}}$ is the Scott sentence of some model $\mathcal{M}$, then the collection $K = \{ \mathcal{N} | \mathcal{N} \models \phi_{\mathcal{M}} \}$, $\prec_{\infty, \kappa}^{\text{fin}}$ is not an A.E.C., but many of the properties of $K$ proved in this section are also properties of A.E.C.

Lemma 3.1. $\prec_{\infty, \kappa}^{\text{fin}}$ is a reflexive and transitive relation.

Lemma 3.2. If $\mathcal{M}_0, \mathcal{M}_1 \prec_{\infty, \kappa}^{\text{fin}} \mathcal{M}_2$ and $\mathcal{M}_0 \subset \mathcal{M}_1$, then $\mathcal{M}_0 \prec_{\infty, \kappa}^{\text{fin}} \mathcal{M}_1$.

Proof. As in the first-order case. □

Lemma 3.3. Let $\phi$ be an $\mathcal{L}_{\infty, \kappa}$ sentence and $(\mathcal{M}_i : i < \lambda)$ be an increasing $\prec_{\infty, \kappa}^{\text{fin}}$-chain of models of $\phi$ with $cf(\lambda) \geq \kappa$. Then:

1. $\bigcup_{i < \lambda} \mathcal{M}_i$ is a model of $\phi$,
2. for each $i < \lambda$, $\mathcal{M}_i \prec_{\infty, \kappa}^{\text{fin}} \bigcup_{i < \lambda} \mathcal{M}_i$ and
3. if for some model $\mathcal{N}$ and for each $i < \lambda$, $\mathcal{M}_i \prec_{\infty, \kappa}^{\text{fin}} \mathcal{N}$, then $\bigcup_{i < \lambda} \mathcal{M}_i \prec_{\infty, \kappa}^{\text{fin}} \mathcal{N}$.

Proof. The argument for (2) is essentially the argument that proves that $\mathcal{L}_{\infty, \kappa}$ is closed under unions of size $\lambda$ with $cf(\lambda) \geq \kappa$ (cf. [6]).

Let $\mathcal{M}_\lambda = \bigcup_{i < \lambda} \mathcal{M}_i$. Let $\vec{a}$ be some (finite) tuple in some fixed $\mathcal{M}_i$. We prove by induction on $\phi \in \mathcal{L}_{\infty, \kappa}$ that

$$\mathcal{M}_i \models \phi[\vec{a}] \text{ iff } \mathcal{M}_\lambda \models \phi[\vec{a}].$$

If $\phi$ is atomic the result follows from $\mathcal{M}_i$ being a substructure of $\mathcal{M}_\lambda$. The cases where $\phi$ is a conjunction or disjunction are immediate. We prove the case where $\phi$ is of the form $\exists X \psi[\vec{a}, X]$, where $X$ is a subset of size $< \kappa$. The case $\forall X \psi[\vec{a}, X]$ is proved similarly. So, assume that $\mathcal{M}_\lambda \models \exists X \psi[\vec{a}, X]$ and fix some $X$ of size $< \kappa$ so that $\mathcal{M}_\lambda \models \psi[\vec{a}, X]$. Since $|X| < \kappa \leq cf(\lambda)$, there must be some $j$ with $i \leq j < \lambda$ so that $X \subset \mathcal{M}_j$. By the induction hypothesis, $\mathcal{M}_j \models \psi[\vec{a}, X]$, i.e. $\mathcal{M}_j \models \exists X \psi[\vec{a}, X]$. Since $\mathcal{M}_i \prec_{\infty, \kappa}^{\text{fin}} \mathcal{M}_j$, $\mathcal{M}_i \prec_{\infty, \kappa}^{\text{fin}} \mathcal{M}_j$, $\mathcal{M}_j \models \psi[\vec{a}, X]$. Since $\mathcal{M}_i \prec_{\infty, \kappa}^{\text{fin}} \mathcal{M}_j$, $\mathcal{M}_i \prec_{\infty, \kappa}^{\text{fin}} \mathcal{M}_j$.
it follows that also $\mathcal{M}_i \models \exists X \psi[a, X]$, i.e. $\mathcal{M}_i \models \phi[a]$. The left-to-right direction is immediate and we established (2).

(1) is a consequence of (2) and (3) is left to the reader. □

**Lemma 3.4.** Let $\kappa$ be a cardinal of cofinality $\omega$ and $\mathcal{M}$ a model of size $\kappa$. Equip $\kappa$ with the discrete topology and $\kappa^\kappa$ with the product topology, call it $T$. If $\overline{\text{Aut}(\mathcal{M})}^T$ is the closure of $\text{Aut}(\mathcal{M})$ under $T$, then $j \in \overline{\text{Aut}(\mathcal{M})}^T$ iff $j$ is an $L^\text{fin}_{\infty, \kappa}$-elementary embedding from $\mathcal{M}$ to itself.

**Proof.** Let $j$ be in $\overline{\text{Aut}(\mathcal{M})}^T$ and $\bar{a} \in \mathcal{M}^{<\omega}$. By the definition of the topology, there must be an automorphism $f \in \text{Aut}(\mathcal{M})$ such that,

$$f(\bar{a}) = j(\bar{a}).$$

Then $\phi^\kappa_{\mathcal{M}} = \phi^f_{\mathcal{M}} = \phi^j_{\mathcal{M}}$, which proves that $j$ is an elementary $L^\text{fin}_{\infty, \kappa}$-embedding from $\mathcal{M}$ to $\mathcal{M}$.

Conversely, assume that $j : \mathcal{M} \to \mathcal{M}$ is an $L^\text{fin}_{\infty, \kappa}$-elementary embedding. In particular, $j$ is an isomorphism between $\mathcal{M}$ and $j[\mathcal{M}]$, and if $\bar{a} \in \mathcal{M}^{<\omega}$, then

$$j[\mathcal{M}] \models \phi^\kappa_{\mathcal{M}}[j(\bar{a})].$$

By elementarity,

$$\mathcal{M} \models \phi^\kappa_{\mathcal{M}}[j(\bar{a})].$$

By theorem 2.2, there is an automorphism $f$ of $\mathcal{M}$ such that $f(\bar{a}) = j(\bar{a})$. Since this is true for any $\bar{a}$, $j$ is in the closure of $\text{Aut}(\mathcal{M})$ in the product topology $T$. □

The following is a Downward Lowenheim- Skolem- type theorem:

**Theorem 3.5.** Let $\kappa$ be a cardinal of cofinality $\omega$. Let $\mathcal{N}$ be a structure of cardinality $\kappa$ and $\phi_N$ its Scott sentence. If $\mathcal{A}$ is a model of $\phi_N$ of any size (possibly $> \kappa$) and $A_0$ is a subset of $\mathcal{A}$ of size $\leq \kappa$, then there is some $A_1 \subset \mathcal{A}$ such that $A_0 \subset A_1$, there exists some isomorphism $i : A_1 \cong \mathcal{N}$ and $A_1 \prec_{\text{fin}} \mathcal{A}$.

**Proof.** Without loss of generality assume that $\mathcal{A}$ has size $\kappa$. Since $\kappa$ has cofinality $\omega$, assume that $\kappa = \cup_n \kappa_n$. Therefore we can write $A_0$ as the union of $A_{0,n}$, for $n < \omega$, where $|A_{0,n}| = \kappa_n$. Similarly we can write $\mathcal{N}$ as the union of $\mathcal{N}_n$ where $|\mathcal{N}_n| = \kappa_n$.

By Theorem 2.2, $\mathcal{A}$ and $\mathcal{N}$ are $\equiv_{\infty, \kappa}$-equivalent. Using Theorem 2.4 part (2), we can give the usual back-and-forth argument to construct subsets $M_n \subset \mathcal{N}$ and $B_n \subset \mathcal{A}$, for all $n \in \omega$, with the following properties:

1. $M_n \subset M_m$, for $n < m$, 

...
(2) $B_n \subseteq B_m$, for $n < m$,
(3) $M_{2n} \supseteq N_n$,
(4) $B_{2n+1} \supseteq A_{0,n}$ and
(5) $(N, M_n) \equiv_{\kappa}(A, B_n)$.

By Theorem 2.4, it also follows that $(N, M_n) \cong_n (A, B_n)$, i.e. there is a partial isomorphism $i_n : M_n \rightarrow B_n$.

Taking unions $A_1 = \bigcup_n B_n$ and $i = \bigcup_n i_n$, we have $A_1 \subseteq A$, $A_1 \supseteq A_0$ and $i$ is an isomorphism between $N$ and $A_1$. Since every finite $\vec{a} \in N$ will be included in some $M_n$, it follows by Theorem 2.2 that $\phi_{\vec{a}}^N = \phi_{\vec{a}}^A$. Since $\phi_{i_n}^{A_1} = \phi_{\vec{a}}^N$, the result follows. \hfill $\square$

Notice that the assumption that $\vec{a}$ is finite, is crucial for the proof. Thus, Theorem 3.5 provides a second justification for the use of $L_{\kappa}^{\text{fin}}$.

**Lemma 3.6** (Trevor Wilson\footnote{The proof of this corollary is due to Trevor Wilson who answered a corresponding question on MathOverflow. See \cite{4}}). If $|\operatorname{Aut}(M)|^T \setminus \operatorname{Aut}(M)| \geq 1$, then $|\operatorname{Aut}(M)|^T \setminus \operatorname{Aut}(M)| \geq |\operatorname{Aut}(M)|$.

**Proof.** Let $f \in \operatorname{Aut}(M)^T \setminus \operatorname{Aut}(M)$. Then $g \mapsto f \circ g$ is an injection from $\operatorname{Aut}(M)$ to $\operatorname{Aut}(M)^T \setminus \operatorname{Aut}(M)$. \hfill $\square$

**Lemma 3.7.** Let $\kappa$ be a cardinal of cofinality $\omega$ and $M$ a model of size $\kappa$. The following are equivalent:

1. There is a strictly increasing $\prec_{\text{fin}}^{\kappa}$ chain of models $(M_\alpha)_{\alpha < \kappa^+}$ of cardinality $\kappa$ such that $M_0 = M$.
2. There is a model of the Scott sentence of $M$ of size $\kappa^+$.

**Proof.** (2) $\Rightarrow$ (1). Assume that $N$ is a model of $\phi_M$ of size $\kappa^+$, where $\phi_M$ is the Scott sentence of $M$. Construct an increasing chain of models $(M_\alpha)_{\alpha < \kappa^+}$ by induction. Assume that for some $\alpha < \kappa^+$, there exists a chain $(M_\gamma)_{\gamma < \alpha}$ such that $M_0 = M$, $M_\alpha \cong M$ and $M_\gamma \prec_{\text{fin}}^{\kappa} N$, for all $\gamma < \alpha$. Extend this sequence to $M_\alpha$ and the construction works for both $\alpha$ successor and $\alpha$ limit ordinal. Let $U = \bigcup_{\gamma < \alpha} M_\gamma$ and $U$ has cardinality $\kappa$. Then there exists some $a \in N \setminus U$ and apply theorem 3.5 to find some $M_\alpha \prec_{\text{fin}}^{\kappa} N$ which contains $U \cup \{a\}$ and is isomorphic to $M$. It follows from lemma 3.2 that for all $\gamma < \alpha$, $M_\gamma \prec_{\text{fin}}^{\kappa} M_\alpha$.

(1) $\Rightarrow$ (2). By lemma 3.3. \hfill $\square$

**Definition 3.8.** Let $(M_\alpha : \alpha \leq \beta)$, $(N_\alpha : \alpha \leq \gamma)$ be two $\prec_{\text{fin}}^{\kappa}$-increasing sequences. The two sequences are called compatible if there is an isomorphism that maps the one sequence to an initial segment of the other. E.g. if...
\[ \beta < \gamma, \text{ there is an isomorphism } i : M_\beta \cong N_\beta \text{ such that } i[M_\alpha] = N_\alpha \text{ for all } \alpha < \beta. \] Similarly for the cases \( \gamma < \beta \) and \( \gamma = \beta \).

The following lemma is immediate.

**Lemma 3.9.** Let \( \beta < \gamma \) and \((M_\alpha)_{\alpha \leq \beta}, (N_\alpha)_{\alpha \leq \gamma}\) be two compatible sequences witnessed by \( i : M_\beta \cong N_\beta \). Then there exists a sequence \((M_\alpha)_{\alpha \leq \gamma}\) that extends \((M_\alpha)_{\alpha \leq \beta}\) and the compatibility of \((M_\alpha)_{\alpha \leq \gamma}\) and \((N_\alpha)_{\alpha \leq \gamma}\) is witnessed by some \( i' : M_\gamma \cong N_\gamma \) that extends \( i \).

**Lemma 3.10.** Let \( \kappa \) be a cardinal of cofinality \( \omega \), \( \beta, \gamma < \kappa^+ \), \( M_\beta \cong N_\gamma \) be two isomorphic models of size \( \kappa \) and \((M_\alpha)_{\alpha \leq \beta}, (N_\alpha)_{\alpha \leq \gamma}\) be two \( \preceq_{\text{fin}} \omega^{\omega \kappa^-} \) increasing sequences. Then \((M_\alpha)_{\alpha \leq \beta}\) can be extended to a sequence \((M_\alpha)_{\alpha \leq \beta+\gamma}\) such that \( M_{\beta+\gamma} \) has size \( \kappa \).

**Proof.** By theorem 2.1 \( N_0 \) is isomorphic to \( N_\gamma \), and therefore to \( M_\beta \). Then use lemma 3.9 to extend \( (M_\beta) \) to a sequence \((M_{\beta+\alpha})_{\alpha \leq \gamma}\) compatible to \((N_\alpha)_{\alpha \leq \gamma}\).

We now are ready to generalize theorem 1.2 to all cardinalities of cofinality \( \omega \).

### 4. Main Theorem

Recall theorem 1.2

**Theorem 4.1** (Su Gao). The following are equivalent for a countable model \( \mathcal{M} \):

(I) There is an uncountable model of the Scott sentence of \( \mathcal{M} \).

(II) There exists some \( j \in \text{Aut}(\mathcal{M}) \setminus \text{Aut}(\mathcal{M})^T \) is the closure of \( \text{Aut}(\mathcal{M}) \) under the product topology in \( \omega^\omega \).

(III) There is an \( L_{\omega_1, \omega} \) - elementary embedding \( j \) from \( \mathcal{M} \) to itself such that \( \text{range}(j) \subset \mathcal{M} \).

We prove the following:

**Theorem 4.2.** Let \( \kappa \) be an uncountable cardinal of cofinality \( \omega \) and \( \mathcal{M} \) a model of size \( \kappa \). The following are equivalent:

(I*) There is a model of the Scott sentence of \( \mathcal{M} \) of size \( \kappa^+ \).

(II*) For all \( \alpha < \beta < \kappa^+ \), there exist functions \( j_{\beta, \alpha} \) in \( \text{Aut}(\mathcal{M}) \setminus \text{Aut}(\mathcal{M})^T \) such that for \( \alpha < \beta < \gamma < \kappa^+ \),

\[
j_{\gamma \beta} \circ j_{\beta \alpha} = j_{\gamma \alpha},
\]
where $\overline{\text{Aut}(M)}^T$ is the closure of $\text{Aut}(M)$ under the product topology in $\kappa^\kappa$.

(III*) For every $\beta < \kappa^+$, there exist $\mathcal{L}_{\kappa,\kappa}^{\text{fin}}$-elementary embeddings $(j_\alpha)_{\alpha<\beta}$ from $M$ to itself such that $\alpha_1 < \alpha_2 \Rightarrow \text{range}(j_{\alpha_1}) \subset \text{range}(j_{\alpha_2})$.

Proof. (I*) $\Rightarrow$ (II*). By lemma 3.7, (I*) is equivalent to the existence of an increasing chain of models $(M_\alpha)_{\alpha<\kappa^+}$ of cardinality $\kappa$ such that $M_0 = M$ and for all $\alpha < \beta$, $M_\alpha \prec_{\text{fin}^\kappa,\kappa} M_\beta$. First observe that for all $\alpha < \kappa^+$, $M_0 = M \equiv_{\text{fin}^\kappa,\kappa} M_\alpha$ implies $M \cong M_\alpha$. Thus, we can find isomorphisms $i_\alpha : M_\alpha \cong M$, for each $\alpha < \kappa^+$. Let

$$j_{\beta,\alpha} = i_\beta \circ i^{-1}_\alpha,$$

for $\alpha < \beta < \kappa^+$. This is a well-defined function from $M$ to $M$ since $\alpha < \beta$ implies $M_\alpha \subset M_\beta$, but $j_{\beta,\alpha}$ fails to be onto. On the other hand, $M_\alpha \prec_{\text{fin}^\kappa,\kappa} M_\beta$ implies that $j_{\beta,\alpha}$ is an $\mathcal{L}_{\kappa,\kappa}^{\text{fin}}$-elementary embedding from $M$ to itself.

By lemma 3.4, $j_{\beta,\alpha}$ is in $\text{Aut}(M)^T \setminus \text{Aut}(M)$ and it remains to prove that for $\alpha < \beta < \gamma < \kappa^+$,

$$j_{\gamma,\beta} \circ j_{\beta,\alpha} = j_{\gamma,\alpha}.$$

This follows immediately by the definition:

$$j_{\gamma,\beta} \circ j_{\beta,\alpha} = i_\gamma \circ i^{-1}_\beta \circ i_\beta \circ i^{-1}_\alpha = i_\gamma \circ i^{-1}_\alpha = j_{\gamma,\alpha}.$$

(II*) $\Rightarrow$ (III*). Assume the existence of $j_{\beta,\alpha}$'s as in (II*). First observe that by lemma 3.4, every $j_{\beta,\alpha}$ is an $\mathcal{L}_{\kappa,\kappa}^{\text{fin}}$-embedding, but not onto. Then by (I*) it follows that $\text{range}(j_{\beta,\alpha}) \subset \text{range}(j_{\beta,\alpha})$, whenever $\alpha_1 < \alpha_2 < \beta$.

(III*) $\Rightarrow$ (I*) By lemma 3.7, it suffices to prove that there is a strictly increasing $\prec_{\text{fin}^\kappa,\kappa}$-chain of models $(M_\alpha)_{\alpha<\kappa^+}$ of cardinality $\kappa$ such that $M_0 = M$.

Assume $\beta < \kappa^+$ and there exists an $\prec_{\text{fin}^\kappa,\kappa}$-increasing sequence $(M_\alpha)_{\alpha \leq \beta}$ of models of size $\kappa$. Fix an ordinal $\gamma$, $0 < \gamma < \kappa$. We extend $(M_\alpha)_{\alpha \leq \beta}$ to an increasing sequence of length $\beta + \gamma$.

By (III*), there are some $\mathcal{L}_{\kappa,\kappa}^{\text{fin}}$-elementary embeddings $(j_\alpha)_{\alpha \leq \gamma}$ such that $\alpha_1 < \alpha_2 \Rightarrow \text{range}(j_{\alpha_1}) \subset \text{range}(j_{\alpha_2})$. Let $N_\alpha = \text{range}(j_\alpha)$ and by lemma 3.2 $\alpha_1 < \alpha_2 \Rightarrow N_{\alpha_1} \prec_{\text{fin}^\kappa,\kappa} N_{\alpha_2}$. Since $N_\gamma \cong M \cong M_\beta$, the result follows from lemma 3.10.

A couple of notes: Although theorem 4.2 holds true for both $\kappa$ countable and uncountable, it is of interest mainly in the uncountable case. If $\kappa$ is countable, theorem 1.2 provides sharper equivalent conditions. On the other
hand, if \( \kappa \) is an uncountable cardinal, (*) can not be omitted from (II*). I.e. mere existence of \( \kappa^+ \) many elements in \( \overline{Aut(M)}^T \setminus Aut(M) \), or even existence of \( \kappa^\omega \) many such elements, is not sufficient to prove the existence of a model of the Scott sentence of \( M \) in \( \kappa^+ \). This is proved in theorem 4.5 and the argument is based on the following theorem of Kueker (cf. [6,7]).

**Theorem 4.3** (Kueker). Let \( \kappa \) be an uncountable cardinal of cofinality \( \omega \) and \( M \) a model of size \( \kappa \). Then the following implications (1) \( \Rightarrow \) (2) \( \Rightarrow \) (3) hold true, but there exist counterexamples for the inverse implications:

1. There is a model \( N \) of size \( > \kappa \) such that \( M \equiv_{\infty, \kappa} N \).
2. For every \( \vec{a} \in M^{< \kappa}, (M, \vec{a}) \) has a proper automorphism.
3. \( M \) has at least \( \kappa^\omega \) automorphisms.

**Lemma 4.4.** There is a model \( N \) of size \( \kappa \) such that there exists an \( L^{\text{fin}}_{\infty, \kappa} \)-embedding \( j : N \rightarrow N \) which is not onto and \( N \) has trivial automorphism group.

**Proof.** i.e. Take \( N \) to be the cardinal \( \kappa \) with its well-ordering \(<\). Then \( (\kappa, <) \) has only the trivial automorphism and for each \( n \in \omega \), the embedding \( j \) that sends 0 to \( n \) is as needed. \( \square \)

Notice that by lemma 3.4 \( j \) is in \( \overline{Aut(N)}^T \setminus Aut(N) \).

**Theorem 4.5.** Let \( \kappa \) be an uncountable cardinal of cofinality \( \omega \). Then there exists a model \( M \) of size \( \kappa \) such that:

1. there is no model of the Scott sentence of \( M \) of size \( \kappa^+ \)
2. there exists at least \( \kappa^\omega \) many elements in \( \overline{Aut(M)}^T \setminus Aut(M) \).

**Proof.** By theorem 4.3 there exist some model \( M \) of size \( \kappa \) which has at least \( \kappa^\omega \) automorphisms and for some \( \vec{a} \in M^{< \kappa} \), \( (M, \vec{a}) \) has no proper automorphisms. If there exists at least one element in \( \overline{Aut(M)}^T \setminus Aut(M) \), then the results follows by lemma 3.6 If not, consider \( A \) to be the disjoint union of \( M \) and \( N \), \( N \) given by lemma 4.4. It follows that \( A \) has the same number of automorphisms as \( M \), \( \overline{Aut(A)}^T \setminus Aut(A) \) is not empty and \( (A, \vec{a}) \) has no proper automorphisms. The result follows by lemma 3.6 and theorem 4.3 \( \square \)

**Corollary 4.6.** Let \( \kappa \) be a cardinal of cofinality \( \omega \) and \( M \) a model of size \( \kappa \). Assume that for all \( \alpha < \beta < \kappa^+ \), there exist functions \( j_{\beta, \alpha} \) in \( \overline{Aut(M)}^T \setminus Aut(M) \), such that for \( \alpha < \beta < \gamma < \kappa^+ \),

\[
(*) \quad j_{\gamma, \beta} \circ j_{\beta, \alpha} = j_{\gamma, \alpha},
\]
where $\overline{\text{Aut}(M)}^p$ is the closure of $\text{Aut}(M)$ under the product topology in $\kappa^\kappa$.

Then there are at least $\kappa^\omega$ many automorphisms of $M$.

Proof. By theorems 4.3 and 4.2.

Definition 4.7. Let $\kappa$ be an infinite cardinal and $M$ a structure of size $\kappa$. $M$ will be called $\kappa$-homogeneous if every isomorphism between substructures of $M$ generated by $(<\kappa)$-many elements, can extend to an automorphism of $M$.

Theorem 4.8. Let $M$ be a model of size $\kappa$ and $N \supset M$ an $\omega$-homogeneous model (possibly of size $>\kappa$). Then $M \equiv_{\infty,\kappa} N$ iff $M \prec^{\text{fin}}_{\infty,\kappa} N$.

Proof. The right-to-left implication is immediate. So, assume that $M \equiv_{\infty,\kappa} N$. Let $\vec{a} \in M^{<\omega}$ and let $\phi_{\vec{a}}^M \in \mathcal{L}_{\infty,\kappa}$ be the Scott sentence of $\vec{a}$ in $M$. We must prove that $N \models \phi_{\vec{a}}^M[\vec{a}]$. By theorem 2.4, there exists some $\vec{b} \in N^{<\omega}$ such that $(M, \vec{a}) \equiv_{\infty,\kappa} (N, \vec{b})$. In particular, $N \models \phi_{\vec{a}}^M[\vec{b}]$ and there exists an isomorphism $p$ between $\vec{a}$ and $\vec{b}$. By $N$ being $\omega$-homogeneous, $p$ can be extended to some automorphism $j$ of $N$ so that $j[\vec{a}] = \vec{b}$. It follows that $N \models \phi_{\vec{a}}^M[\vec{a}]$ as desired.

Corollary 4.9. If in the statement of theorem 4.2 we add the assumption that $M$ is an $\omega$-homogeneous model, then condition (III*) can be relaxed to the following:

(III*)' For every $\beta < \kappa^+$, there exist $(M_\alpha)_{\alpha < \beta}$ such that $\alpha_1 < \alpha_2 \Rightarrow M_{\alpha_1} \subset M_{\alpha_2} \subset M$ and every $M_\alpha$ satisfy the Scott sentence of $M$.

Proof. By theorem 2.1 $M_\alpha \equiv_{\infty,\kappa} M$. Using $\omega$-homogeneity and theorem 4.8 (III*)' is equivalent to (III*), which finishes the proof.

5. Open Questions

We mention some open questions relating to the results in this paper, or extensions of them.

Open Question 1. Can the results of this paper extend to the case where $\kappa$ is a successor cardinal? It seems that Ehrenfeucht-Fraisse games, or equivalently the infinitely deep languages $M_{\kappa^+,\kappa}$, must be used instead of $\mathcal{L}_{(\kappa^{<\kappa})^+,\kappa}$.

Open Question 2. Can we prove corollary 4.6 directly, without using theorem 4.3?
under composition. If $\mathcal{M}$ is a model of size $\kappa$, then it follows by lemma 3.4, $\text{Aut}(\mathcal{M})$ is a closed subgroup of $S_\kappa$.

The following property is inspired by (II*).

**Definition 5.1.** Let $G$ be a closed subgroup of $S_\kappa$. We say that $G$ has large closure if for all $\alpha < \beta < \gamma < \kappa^+$ there exist $j_{\beta,\alpha} \in \overline{G} \setminus G$ such that $j_{\gamma,\beta} \circ j_{\beta,\alpha} = j_{\gamma,\alpha}$, where $\overline{G}$ is the closure of $G$ in $\kappa^\kappa$ under the product topology.

**Lemma 5.2.** $S_\kappa$ has large closure.

**Proof.** Let $\alpha$ be an ordinal such that $\kappa \leq \alpha < \kappa^+$. Let $i_\alpha$ be a bijection from $\kappa$ to $\alpha$. Let $j_{\beta,\alpha} = i_{\beta}^{-1} \circ i_\alpha$, where $\alpha < \beta$. The reader can verify that these $j_{\beta,\alpha}$’s are 1-1, but not onto functions, and they witness the large closure of $S_\kappa$. □

The following two open questions are motivated by similar results in [2] for $\kappa$ countable.

**Open Question 3.** Let $G$ be a closed subgroup of $S_\kappa$ and there exists a continuous onto homomorphism $p : G \to S_\kappa$. Can we conclude that $G$ has large closure?

**Open Question 4.** Let $G$ be a closed subgroup of $S_\kappa$ and let $H$ be a closed normal subgroup of $G$. Then $G$ has large closure iff $H$ has large closure or $G/H$ has large closure.

An positive answer to open question [4] implies a positive answer to open question [3]. Indeed, since $S_\kappa = p(G)$ has large closure, the same is true for $G/Ker(p)$ and by a positive answer to question [4], the same is true for $G$.

If $\kappa$ is countable, both questions [3] and [4] have positive answers as proved in [2]. In [3], Hjorth used these positive answers to prove theorem 5.4. We need a definition before we can state the theorem.

**Definition 5.3.** Let $P$ be a unary predicate and let $\mathcal{M}$ be a model in a language that contains $P$. Then $P$ is homogeneous for $\mathcal{M}$, if $P(\mathcal{M})$ is infinite and every permutation of it extends to an automorphism of $\mathcal{M}$.

**Theorem 5.4 (Hjorth).** Let $\mathcal{M}$ be a countable model in a language that contains a unary predicate $P$. If $P$ is homogeneous for $\mathcal{M}$, then the Scott sentence of $\mathcal{M}$ has a model of size $\aleph_1$.

Assuming an affirmative answer to open question [3] we can generalize theorem 5.4 to uncountable structures of cofinality $\omega$. 

Theorem 5.5. Assume an affirmative answer to open question 3. Let \( \kappa \) be an infinite cardinal of cofinality \( \omega \) and let \( \mathcal{M} \) be a model of size \( \kappa \) in a language that contains a unary predicate \( P \). If \( P \) is homogeneous for \( \mathcal{M} \) and \( P(\mathcal{M}) \) has size \( \kappa \), then the Scott sentence of \( \mathcal{M} \) has a model of size \( \kappa^+ \).

Proof. Let \( \sigma \in \text{Aut}(\mathcal{M}) \). Since \( P(\mathcal{M}) \) has size \( \kappa \), we can assume that \( P(\mathcal{M}) = \kappa \) and consider \( \sigma|_{P(\mathcal{M})} \) as a function from \( \kappa \) to \( \kappa \). By the homogeneity of \( P \), the map \( \sigma \mapsto \sigma|_{P(\mathcal{M})} \) from \( \text{Aut}(\mathcal{M}) \) to \( S_\kappa \) is onto and the reader can verify that it is also a continuous homomorphism. Assuming an affirmative answer to open question 3, \( \text{Aut}(\mathcal{M}) \) has large closure, which by theorem 4.2 implies that the Scott sentence of \( \mathcal{M} \) has a model of size \( \kappa^+ \). \( \square \)

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