DUALITY FOR AUTOMORPHIC SHEAVES WITH NILPOTENT SINGULAR SUPPORT

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Abstract. We identify the category $\text{Shv}_{\text{Nilp}}(\text{Bun}_G)$ of automorphic sheaves with nilpotent singular support with its own dual, and relate this structure to the Serre functor on $\text{Shv}_{\text{Nilp}}(\text{Bun}_G)$ and miraculous duality.

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Introduction

0.1. What is this paper about? This paper is a sequel to the paper [AGKRRV] by the same set of authors. In loc. cit. a conjecture was suggested that the trace of the Frobenius endofunctor on the category of automorphic sheaves with nilpotent singular support is isomorphic to the space of (non-ramified) automorphic functions. This paper carries out the first step towards the proof of this conjecture.

0.1.1. Let $X$ be a smooth projective curve and $G$ a reductive group (over a ground field $k$). Let $\text{Bun}_G$ be the moduli stack of $G$-bundles on $X$. We consider the category of automorphic sheaves $\text{Shv}(\text{Bun}_G)$
The category Shv_{Nilp}(Bun_{G}) is compactly generated, and hence dualizable. So, given an endofunctor Φ of Shv_{Nilp}(Bun_{G}), it makes sense to consider its trace

$$\text{Tr}(\Phi, \text{Shv}_{\text{Nilp}}(\text{Bun}_{G})) \in \text{Vect},$$

where Vect is the DG category of chain complexes of vector space over the field $e$ of coefficients of our sheaf theory.

0.1.2. By definition, the trace of an endofunctor Φ of a dualizable category C is the endofunctor of Vect equal to the composition

$$\text{Vect} \xrightarrow{\text{unit}} C \otimes C^\vee \xrightarrow{\Phi \otimes \text{id}} C \otimes C^\vee \xrightarrow{\text{counit}} \text{Vect},$$

where $C^\vee$ is the dual category, and unit and counit are the unit and the counit of the duality, respectively.

Hence, in order to say something explicit about the trace Tr(Φ, C), one must first describe $C^\vee$ and the functors unit and counit.

This is what we do for the category Shv_{Nilp}(Bun_{G}).

0.1.3. From the above perspective, the main result of this note (Theorem 3.2.2) is easy to explain. It says that there exists a canonical equivalence

$$(0.1) \quad \text{Shv}_{\text{Nilp}}(\text{Bun}_{G})^\vee \simeq \text{Shv}_{\text{Nilp}}(\text{Bun}_{G}),$$

for which the counit of the duality is the functor

$$\text{Shv}_{\text{Nilp}}(\text{Bun}_{G}) \otimes \text{Shv}_{\text{Nilp}}(\text{Bun}_{G}) \to \text{Vect},$$

denoted $\text{ev}_{Bun_{G}^{\text{vir}}}$, given by

$$\mathcal{F}_1, \mathcal{F}_2 \mapsto C_{e}(\text{Bun}_{G}, \mathcal{F}_1 \otimes \mathcal{F}_2).$$

0.1.4. We should emphasize that the above pairing is really different from the usual Verdier duality pairing, which makes sense for a quasi-compact algebraic stack $Y$:

$$\text{ev}_{Y}(\mathcal{F}_1, \mathcal{F}_2) := C_{e}(\mathcal{F}_1 \otimes \mathcal{F}_2),$$

(here $C_{e}(\mathcal{F})$ is the renormalized version of the functor of sheaf cochains, see Sect. A.2.4).

However, Bun_{G} is not quasi-compact, so the pairing $\text{ev}_{Bun_{G}}$ would not make sense “as-is”. Rather, $\text{ev}_{Bun_{G}}$ defines a perfect pairing between Shv(Bun_{G}) (resp., Shv_{Nilp}(Bun_{G})) and its “co” version, denoted Shv(Bun_{G})_{co} (resp., Shv_{Nilp}(Bun_{G})_{co}), see Sect. 2.5.

The relationship between $\text{ev}_{Bun_{G}}^{\text{vir}}$ and $\text{ev}_{Bun_{G}}$ is the second main point of this paper, to be discussed below.

0.1.5. The assertion that $\text{ev}_{Bun_{G}}^{\text{vir}}$ defines a perfect pairing may seem innocuous enough, but it is actually very non-trivial. In fact, it uses the full strength of some of the key results of the paper [AGKRRV].

Namely, consider the embedding

$$\text{Shv}_{\text{Nilp} \times \text{Nilp}}(\text{Bun}_{G} \times \text{Bun}_{G}) \hookrightarrow \text{Shv}(\text{Bun}_{G} \times \text{Bun}_{G});$$

let

$$\text{ps-}u_{\text{Bun}_{G}} \in \text{Shv}_{\text{Nilp} \times \text{Nilp}}(\text{Bun}_{G} \times \text{Bun}_{G})$$

be the object equal to the value of the spectral projector $P_{\text{Bun}_{G} \times \text{Nilp}} \otimes \text{Id}_{\text{Bun}_{G}}$ (see Sect. 10.1) on the object

$$\mathcal{E}_{\text{Bun}_{G}} \in \text{Shv}(\text{Bun}_{G} \times \text{Bun}_{G}),$$

where $\mathcal{E}_{\text{Bun}_{G}} \in \text{Shv}(\text{Bun}_{G})$ is the constant sheaf.
The assertion of Theorem 3.2.2 is easily deduced from the fact that the above object $\text{ps-u}_{\text{Nilp}}$ belongs to the essential image of (what is a priori just a fully faithful functor):

\[(0.2) \quad \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \otimes \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \to \text{Shv}_{\text{Nilp} \times \text{Nilp}}(\text{Bun}_G \times \text{Bun}_G),\]

thereby providing the unit for the sought-for self-duality of $\text{Shv}_{\text{Nilp}}(\text{Bun}_G)$.

Now, it turns out that the functor (0.2) is actually an equivalence; this is the highly non-trivial [AGKRRV, Theorem 16.3.3]. We give a slightly different proof of this theorem in the present paper (see Theorem 1.6.7), but one which still uses the key results of [AGKRRV].

0.2. Relation to miraculous duality and the Serre functor.

0.2.1. Recall that there is a way to define a self-duality on the entire category $\text{Shv}(\text{Bun}_G)$, as well as its subcategory $\text{Shv}_{\text{Nilp}}(\text{Bun}_G)$.

Namely, we consider the category $\text{Shv}(\text{Bun}_G)^{\text{co}}$ (see Sect. 2.5 for the definition), and Verdier duality defines an identification

\[\text{Shv}(\text{Bun}_G)^{\vee} \simeq \text{Shv}(\text{Bun}_G)^{\text{co}},\]

with the counit defined by the pairing $\text{ev}_{\text{Bun}_G}$ mentioned above.

Now, in the paper [Ga1], it is shown that a certain canonically defined functor, denoted in this note by

\[\text{Mir}_{\text{Bun}_G} : \text{Shv}(\text{Bun}_G)^{\text{co}} \to \text{Shv}(\text{Bun}_G),\]

is an equivalence. The functor $\text{Mir}_{\text{Bun}_G}$, which makes sense for any algebraic stack, was proposed by Drinfeld; we refer to it as the \textbf{miraculous functor}.

Combining Verdier duality with the miraculous functor, we obtain an identification

\[(0.3) \quad \text{Shv}(\text{Bun}_G)^{\vee} \simeq \text{Shv}(\text{Bun}_G).\]

Following Drinfeld, we refer to (0.3) as the \textbf{miraculous self-duality} of $\text{Shv}(\text{Bun}_G)$.

0.2.2. One shows (see Corollary 2.9.7) that the identification (0.3) induces an identification

\[(0.4) \quad \text{Shv}_{\text{Nilp}}(\text{Bun}_G)^{\vee} \simeq \text{Shv}_{\text{Nilp}}(\text{Bun}_G).\]

Now, the second main result of this note, Corollary 3.3.7, says that the identification (0.4) equals (0.1) introduced in Sect. 0.1.3.

0.2.3. In the process of identifying (0.4) with (0.1) we relate $\text{Mir}_{\text{Bun}_G}$ to the \textbf{pseudo-identity} and \textbf{Serre} functors (see Sects. 5.2.3 and 5.1.4 for what we mean by the latter functors).

Namely, let $\mathcal{U}$ be a universally Nilp-\textit{contruncative} quasi-compact open substack of $\text{Bun}_G$ (see Sect. 7.1.3 for what this means; such substacks form a cofinal subset among all quasi-compact open substacks of $\text{Bun}_G$). Consider the endofunctor

\[\text{Mir}_{\mathcal{U}} : \text{Shv}(\mathcal{U}) \to \text{Shv}(\mathcal{U}),\]

and let us restrict it to $\text{Shv}_{\text{Nilp}}(\mathcal{U}) \subset \text{Shv}(\mathcal{U})$.

We show that this restriction sends $\text{Shv}_{\text{Nilp}}(\mathcal{U})$ to $\text{Shv}_{\text{Nilp}}(\mathcal{U})$. We show that the resulting endofunctor of $\text{Shv}_{\text{Nilp}}(\mathcal{U})$ is an equivalence, which identifies with $\text{Ps-Id}_{\text{Shv}_{\text{Nilp}}(\mathcal{U})}$, and also with the \textit{inverse} of the Serre functor $\text{Se}_{\text{Shv}_{\text{Nilp}}(\mathcal{U})}$. In particular, the categories $\text{Shv}_{\text{Nilp}}(\mathcal{U})$ are Serre (see Sect. 5.1.6 for what this means).

We should emphasize that the idea that for a Serre category, the Serre and pseudo-identity endofunctors are mutually inverse goes back to A. Yom Din (it was recorded in the paper [GaYo]).
0.2.4. Returning to the entire $\text{Bun}_G$, we show that the Serre functor on $\text{Shv}_{\text{Nilp}}(\text{Bun}_G)$ canonically factors as

$$\	ext{Shv}_{\text{Nilp}}(\text{Bun}_G) \xrightarrow{\text{Ser}_{\text{Shv}_{\text{Nilp}}(\text{Bun}_G),\text{co}}} \text{Shv}_{\text{Nilp}}(\text{Bun}_G)^{\text{co}} \xrightarrow{\text{Id}^\text{naive}_{\text{Bun}_G}} \text{Shv}_{\text{Nilp}}(\text{Bun}_G),$$

where the first arrow is an equivalence inverse to $\text{Mir}_{\text{Bun}_G|\text{Shv}_{\text{Nilp}}(\text{Bun}_G)} : \text{Shv}_{\text{Nilp}}(\text{Bun}_G)^{\text{co}} \rightarrow \text{Shv}_{\text{Nilp}}(\text{Bun}_G)$, and the second arrow is the tautological functor, denoted $\text{Id}^\text{naive}_{\text{Bun}_G}$, defined for any non-quasi-compact stack (see Sect. C.2.3).

Thus, we obtain that $\text{Shv}_{\text{Nilp}}(\text{Bun}_G)$ is “Serre, up to the issue of non-quasi-compactness”, which is compensated by replacing the target $\text{Shv}_{\text{Nilp}}(\text{Bun}_G)$ with $\text{Shv}_{\text{Nilp}}(\text{Bun}_G)^{\text{co}}$.

Remark 0.2.5. The functor $\text{Mir}_{\text{Bun}_G}^{-1} : \text{Shv}(\text{Bun}_G) \rightarrow \text{Shv}(\text{Bun}_G)^{\text{co}}$ has been recently studied by L. Chen in [Ch]. Namely, in loc. cit. it was shown that it is canonically isomorphic to the Deligne-Lusztig functor, the latter being an explicit complex whose terms are composites of Eisenstein and Constant Term functors.

A spectral counterpart of the Deligne-Lusztig functor, which is an endofunctor of the category $\text{IndCoh}_{\text{Nilp}}(\text{LocSys}_G(X))$ has been studied by D. Beraldo in [Be]. In loc. cit. it is shown that the spectral Deligne-Lusztig functor is the composition

$$\text{IndCoh}_{\text{Nilp}}(\text{LocSys}_G(X)) \rightarrow \text{QCoh}(\text{LocSys}_G(X)) \
\rightarrow \text{QCoh}(\text{LocSys}_G(X)) \hookrightarrow \text{IndCoh}_{\text{Nilp}}(\text{LocSys}_G(X)),$$

where the middle arrow $\text{QCoh}(\text{LocSys}_G(X)) \rightarrow \text{QCoh}(\text{LocSys}_G(X))$ is given by tensor product with an explicit object of $\text{QCoh}(\text{LocSys}_G(X))$, called the Steinberg object, which is in some sense “the structure sheaf of the locus of semi-simple local systems”.

0.3. An intrinsic characterization of $\text{Shv}_{\text{Nilp}}(\text{Bun}_G)$.

0.3.1. The isomorphism between the functors (0.4) with (0.1) has another, rather unexpected consequence:

It turns out that the subcategory $\text{Shv}_{\text{Nilp}}(\text{Bun}_G)^{\text{constr}}$ can be characterized intrinsically, without mentioning the singular support condition.

0.3.2. In Sect. B we recall the construction of a 2-category, whose objects are algebraic stacks, and whose category of morphisms for a given pair of stacks $Y_1$ and $Y_2$ is

$$\text{Shv}(Y_1 \times Y_2).$$

An object $\Omega \in \text{Shv}(Y_1 \times Y_2)$ gives rise to a functor

$$Q : \text{Shv}(Y_1) \rightarrow \text{Shv}(Y_2),$$

but the data of $\Omega$ carries more information. Namely, $\Omega$ gives rise to functors denoted

$$\text{Id}_Z \otimes Q : \text{Shv}(Z \times Y_1) \rightarrow \text{Shv}(Z \times Y_2)$$

for any algebraic stack $Z$ that commute appropriately with functors that relate different $Z$’s (see Sect. B.1.5).

0.3.3. Thus, given $\Omega \in \text{Shv}(Y_1 \times Y_2)$ one can ask whether it admits a right or a left adjoint within the above 2-category.

For example, $\Omega$ admits a right adjoint if and only if the above functors $\text{Id}_Z \otimes Q$ admit right adjoints for every $Z$ (this is not very restrictive), and if these right adjoints, denoted $(\text{Id} \otimes Q)^R$, themselves commute appropriately with functors that relate different $Z$’s (this is a really non-trivial condition).
0.3.4. Let us consider the case when $Y_1 = Y$ is a connected separated scheme and $Y_2 = \text{pt}$. Then an object $\mathcal{F} \in \text{Shv}(Y) = \text{Shv}(Y_1 \times Y_2)$ admits a right adjoint if and only if the following conditions are satisfied: (i) $Y$ is smooth and proper; (ii) $\mathcal{F} \in \text{Lisse}(Y)$ (i.e., $\mathcal{F}$ is bounded, and each of its cohomologies is a locally system of finite rank).

0.3.5. The third main result of this paper (Theorem 1.5.4) says that an object $\mathcal{F} \in \text{Shv(Bun}_G)$ admits a right adjoint if and only if it belongs to $\text{Shv}_{\text{Nilp}}(\text{Bun}_G)$.

This result suggests that in a certain sense, the stack $\text{Bun}_G$ (or any Nilp-cotruncative open substack thereof) together with the subset Nilp of its cotangent bundle, behaves similarly to a proper smooth scheme $Y$, with $\{0\} \subset \text{T}^* (Y)$, see also Sects. 5.5.6 and 4.6.8.

0.4. Organization of the paper. Let us briefly review the contents of this work.

In Sect. 1 we extend the results of [AGKRRV, Part III] concerning $\text{Shv}_{\text{Nilp}}(\text{Bun}_G)$ to the case when instead of $\text{Bun}_G$, we consider the relative situation, i.e., $Z \times \text{Bun}_G$, where $Z$ is an arbitrary algebraic stack.

In Sect. 2 we show that the category $\text{Shv}_{\text{Nilp}}(\text{Bun}_G)$ is self-dual via a combination of Verdier duality and the miraculous functor.

In Sect. 3 we prove the main result relevant to the trace calculation, namely, that the pairing $\text{ev}^j_{\text{Bun}_G}$ defines the counit of a self-duality on $\text{Shv}_{\text{Nilp}}(\text{Bun}_G)$.

In Sect. 4 we establish the intrinsic characterization of the subcategory $\text{Shv}_{\text{Nilp}}(\text{Bun}_G)^{\text{constr}} \subset \text{Shv}(\text{Bun}_G)$ as consisting of objects that admit right adjoints when viewed as kernels.

In Sect. 5 we relate the miraculous functor on $\text{Shv}_{\text{Nilp}}(\text{Bun}_G)$ to the Serre functor, and establish the Serre property of categories $\text{Shv}_{\text{Nilp}}(U)$, where $U$ is a universally Nilp-contruncative quasi-compact open substack of $\text{Bun}_G$.

In Sect. A we collect some basic facts pertaining to the category of sheaves on algebraic stacks; in particular to Verdier duality in this situation and the functor of renormalized direct image.

In Sect. B we review the theory of functors defined by kernels, their behavior with respect to adjunctions and relation to the miraculous functor.

In Sect. C we review the theory of sheaves on non quasi-compact stacks, the “co”-categories, and functors defined by kernels in this situation.

0.5. Conventions.

0.5.1. Algebraic geometry. Throughout the note we will work over a ground field $k$, assumed algebraically closed. The algebraic geometry over $k$ will be classical, i.e., non-derived.

Our algebraic-geometric objects will be either schemes of finite type or algebraic stacks locally of finite type over $k$. In this note we will not need more general prestacks.

All quasi-compact algebraic stacks that appear in this paper will be of the form $Z/H$, where $Z$ is a scheme (of finite type) and $H$ a linear algebraic group. All non quasi-compact algebraic stacks that appear in this paper will be unions of quasi-compact stacks of the above form.
0.5.2. Higher algebra. Let \( e \) be a field of coefficients, assumed algebraically closed and of characteristic 0. Our main objects of study are DG categories over \( e \). In our treatment of DG categories we follow the conventions of [AGKRRV].

In particular, we will denote by DGCat the \((\infty, 1)\)-category, whose objects are cocomplete DG categories, and whose 1-morphisms are colimit-preserving functors.

The category DGCat carries a symmetric monoidal structure, given by the Lurie tensor product; we denote it by

\[ C_1, C_2 \mapsto C_1 \otimes C_2. \]

The unit object for this symmetric monoidal structure is the DG category of chain complexes of \( e \)-vector spaces, denoted Vect.

In particular, we can talk about dualizable objects in DGCat. These are dualizable DG categories. For a dualizable DG category \( C \) we will denote by \( C^\vee \) its dual.

0.5.3. Compact generation. For a given DG category \( C \) and objects \( c_1, c_2 \in C \) we will denote by

\[ \text{Hom}_C(c_1, c_2) \in \text{Vect} \]

the object corresponding to the canonical enrichment of \( C \) over Vect.

For a given \( C \) we will denote by \( C^c \) the subcategory consisting of compact objects, i.e., those objects for which the functor

\[ \text{Hom}_C(c, -) : C \to \text{Vect} \]

preserves colimits.

A category \( C \) is said to be compactly generated if \( \text{Hom}_C(c, c') = 0 \) for all \( c \in C^c \) implies \( c' = 0 \). In this case, \( C \) can be recovered as the \textit{ind-completion} of its subcategory \( C^c \).

Furthermore, if \( C \) is compactly generated, it is dualizable, and its dual \( C^\vee \) can be described explicitly as follows: \( C^\vee \) is also compactly generated and its subcategory of compact objects \( (C^\vee)^c \) identifies with \( (C^c)^{op} \).

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1. The category ShvNilp(Bun_G) and its variants

In this section we extend the results of [AGKRRV Part III] concerning ShvNilp(Bun_G) to the case when instead of Bun_G, we consider the relative situation, i.e., \( Z \times \text{Bun}_G \), where \( Z \) is an arbitrary algebraic stack.

1.1. Hecke action in the relative situation.

1.1.1. For the duration of the paper we fix a smooth projective curve \( X \), and a reductive group \( G \) (both over \( k \)).

We let Bun_G denote the moduli stack of principal G-bundles on X.
1.1.2. For a finite set \( I \), we consider the \( I \)-legged Hecke stack \( \mathcal{H}_{G,X^I} \)
\[
\text{Bun}_G \xrightarrow{\mathbf{h}} \mathcal{H}_{G,X^I} \xrightarrow{\mathbf{h} \times s} \text{Bun}_G \times X^I.
\]

Let
\[
\text{Sat}_I : \text{Rep}(\hat{G})^{\otimes I} \to \text{Shv}(\mathcal{H}_{G,X^I})
\]
denote the geometric Satake functor.

We normalize it so that the value of \( \text{Sat}_I \) on the trivial representation \((1_{\text{Rep}(\hat{G})})^{\otimes I}\) is (the direct image of) \( \omega_{\text{Bun}_G \times X^I} \), where \( \text{Bun}_G \times X^I \hookrightarrow \mathcal{H}_{G,X^I} \)

is the unit section.

Remark 1.1.3. The entire \( \mathcal{H}_{G,X^I} \) is an ind-algebraic stack, i.e., a filtered union of algebraic stacks that map to each other by means of closed embeddings.

For each compact object \( V \in \text{Rep}(\hat{G})^{\otimes I} \), the object \( \text{Sat}_I(V) \in \text{Shv}(\mathcal{H}_{G,X^I}) \) is supported on one of the closed algebraic substacks of \( \mathcal{H}_{G,X^I} \).

Hence, in the discussion below, one can start by working with compact objects of \( \text{Rep}(\hat{G})^{\otimes I} \), and then deal with usual algebraic stacks, and then ind-extend to all of \( \text{Rep}(\hat{G})^{\otimes I} \).

1.1.4. Let \( Z \) be an arbitrary algebraic stack. We can consider Hecke functors, denoted
\[
\text{Id}_Z \boxtimes \mathbb{H}(-, -) : \text{Rep}(\hat{G})^{\otimes I} \otimes \text{Shv}(Z \times \text{Bun}_G) \to \text{Shv}(Z \times \text{Bun}_G \times X^I). \quad I \in \text{ISet}.
\]

Namely, for \( V \in \text{Rep}(\hat{G})^{\otimes I} \), the Hecke functor \( \text{Id}_Z \boxtimes \mathbb{H}(V, -) \) is given by
\[
(1.1) \quad \text{Id}_Z \boxtimes \mathbb{H}(V, \mathcal{F}) := ((\text{id}_Z \times (\overrightarrow{\mathbf{h}} \times s))^\ast \circ (\text{id}_Z \times (\overrightarrow{\mathbf{h}} \times s))^\ast \circ (\omega_{\text{Bun}_G \times X^I} \otimes (\text{Sat}_I(V))) , \quad \mathcal{F} \in \text{Shv}(Z \times \text{Bun}_G).
\]

Note that in the above formula, the functor \( (\text{id}_Z \times (\overrightarrow{\mathbf{h}} \times s))^\ast \) is canonically isomorphic to the functor \( (\text{id}_Z \times (\overrightarrow{\mathbf{h}} \times s))^\ast \), since the morphism \( \overrightarrow{\mathbf{h}} \times s \) is (ind)-schematic (it is actually (ind)-proper).

The functors \( \text{Id}_Z \boxtimes \mathbb{H}(V, -) \) introduced above fall into the paradigm of functors defined by kernels, see Sect. [B.3.4] for what this means (see also Sect. [C.3] for the formalism of functors defined by kernels on non quasi-compact stacks).

Thus, the notation \( \text{Id}_Z \boxtimes \mathbb{H}(V, -) \) is consistent with one in Sect. [B.1.3].

1.1.5. A key property of the objects of the form \( \text{Sat}_I(V) \), for \( V \in (\text{Rep}(\hat{G})^{\otimes I})^c \), is that they are ULA with respect to the map \( \overrightarrow{\mathbf{h}} \times s \) (and, by symmetry, also with respect to the map \( \overrightarrow{\mathbf{h}} \times s \)).

In particular, since \( X^I \) is smooth, the objects \( \text{Sat}_I(V) \), for \( V \in (\text{Rep}(\hat{G})^{\otimes I})^c \), are ULA with respect to the projection \( \overrightarrow{\mathbf{h}} \). This implies that there are canonical isomorphisms
\[
(\text{Id}_Z \times (\overrightarrow{\mathbf{h}} \times s))^\ast (\mathcal{F}) \circ (\omega_{\text{Bun}_G \times X^I}) \simeq (\text{Id}_Z \times (\overrightarrow{\mathbf{h}} \times s))^\ast (\mathcal{F} \otimes (\text{Sat}_I(V) \otimes (\overrightarrow{\mathbf{h}} \times s) \otimes (\text{Bun}_G))) , \quad \mathcal{F} \in \text{Shv}(Z \times \text{Bun}_G),
\]
where we note that the operation \( - \otimes (\overrightarrow{\mathbf{h}} \times s) \) amounts to the cohomological shift \([ -2 \dim(\text{Bun}_G) ] \).

Furthermore, the map \( \overrightarrow{\mathbf{h}} \times s \) is ind-proper. This implies that the functor \( \text{Id}_Z \boxtimes \mathbb{H}(V, -) \) can be rewritten as
\[
(\text{Id}_Z \times (\overrightarrow{\mathbf{h}} \times s))^\ast (\mathcal{F}) \circ (\omega_{\text{Bun}_G \times X^I}) \left[ -2 \dim(\text{Bun}_G) \right].
\]

This implies that the functors \( \text{Id}_Z \boxtimes \mathbb{H}(V, -) \) are both defined and codefined by kernels, see Sect. [B.3.4] for what this means. In particular, the functors \( \text{Id}_Z \boxtimes \mathbb{H}(V, -) \) commute with \( _s \)-pullbacks and \( ! \)-pushforwards along the \( Z \) variable.
1.1.6. Recall the monoidal category $\text{Rep}(\hat{G})_{\text{Ran}}$, see [AGKRRV, Sect. 11.1]. As in [AGKRRV, Sect. 15.1], the functors (1.1) give rise to a monoidal action of $\text{Rep}(\hat{G})_{\text{Ran}}$ on $\text{Shv}(\mathbb{Z} \times \text{Bun}_G)$.

We will denote this action by

$$V \in \text{Rep}(\hat{G})_{\text{Ran}} \rightsquigarrow V \star -.$$

For a fixed $V$, we will also denote the above functor by

$$(1.3) \quad \text{Id}_\mathbb{Z} \boxtimes H_V.$$

This is again consistent with the notation of Sect. 1.1.5 above, since these functors are defined by kernels, by construction.

We will refer to functors (1.3) as integral Hecke functors.

1.1.7. The next observation will play an important role in the sequel:

**Proposition 1.1.8.** The functors $H_V$ are defined and codefined by kernels.

The proof is given in the next subsection.

1.2. **Proof of Proposition 1.1.8**

1.2.1. We need to show that the functors $\text{Id}_\mathbb{Z} \boxtimes H_V$ of (1.3) commute with $\ast$-pullbacks and $!$-pushforwards along the $\mathbb{Z}$ variable. It is enough to check this assertion on the generators of the category $\text{Rep}(\hat{G})_{\text{Ran}}$.

Thus, we fix a finite set $I$, an object $V \in (\text{Rep}(\hat{G})^c)^I$ and $M \in \text{Shv}(X^I)^c$. The corresponding functor $\text{Id}_\mathbb{Z} \boxtimes H_V$ is given by

$$(1.4) \quad \mathcal{F} \mapsto (p_{\mathbb{Z} \times \text{Bun}_G})_\ast ((\text{Id}_\mathbb{Z} \boxtimes H(V, -))(\mathcal{F}) \otimes p^I_{X^I}(M)),$$

where

$$\mathbb{Z} \times \text{Bun}_G \xleftarrow{p_{\mathbb{Z} \times \text{Bun}_G}} \mathbb{Z} \times \text{Bun}_G \times X^I \xrightarrow{p^I_{X^I}} X^I$$

are the two projections.

We need to show that these functors commute with $\ast$-pullbacks and $!$-pushforwards along the $\mathbb{Z}$ variable. We will prove this by showing that the functor (1.4) is canonically isomorphic to the functor

$$(1.5) \quad \mathcal{F} \mapsto (p_{\mathbb{Z} \times \text{Bun}_G})_\ast ((\text{Id}_\mathbb{Z} \boxtimes H(V, -))(\mathcal{F}) \otimes p^I_{X^I}(M))[-2|I|].$$

From here, the commutation with $\ast$-pullbacks and $!$-pushforwards would then follow from the fact that the functor $H(V, -)$ is defined and codefined by a kernel, see Sect. 1.1.5 above.

We can assume that $\mathcal{F}$ is compact. Since the map $p_{\mathbb{Z} \times \text{Bun}_G}$ is proper, the isomorphism between (1.4) and (1.5) follows from the next assertion:

**Proposition 1.2.2.** An object of the form

$$(\text{Id}_\mathbb{Z} \boxtimes H(V, -))(\mathcal{F}) \in \text{Shv}(\mathbb{Z} \times \text{Bun}_G \times X^I)^c, \quad \mathcal{F} \in \text{Shv}(\mathbb{Z} \times \text{Bun}_G)^c$$

is ULA with respect to the projection $p_{X^I}: \mathbb{Z} \times \text{Bun}_G \times X^I \to X^I$. 

□[Proposition 1.1.8]
1.2.3. **Proof of Proposition 1.2.2.** Since the map
\[ \text{id}_Z \times (h \times s) : Z \times \mathcal{F}_{G,X} \to Z \times \text{Bun}_G \times X^I \]
is ind-proper, it suffices to show that the object
\[ (1.6) \quad (\text{id}_Z \times h)^! (F) \otimes (\omega_Z \boxtimes \text{Sat}_I(V)) \in \text{Shv}(Z \times \mathcal{F}_{G,X}) \]
is ULA with respect to the map
\[ p_{X_I} \circ (\text{id}_Z \times (h \times s)), \]
where we note that
\[ p_{X_I} \circ (\text{id}_Z \times (h \times s)) = s = p_{X_I} \circ (\text{id}_Z \times (h \times s)). \]

Thus, it suffices to show that (1.6) is ULA with respect to the projection \( p_{X_I} \circ (\text{id}_Z \times (h \times s)) \).

However, this follows from the fact that \( \text{Sat}_I(V) \) is ULA with respect to \( h \times s \) using the following general lemma:

**Lemma 1.2.4.** Let
\[ y_1 \xleftarrow{p_1} y \xrightarrow{p_2} y_2 \]
be a diagram of stacks. Let \( G \in \text{Shv}(y)^c \) be ULA with respect to the map
\[ (p_1, p_2) : y \to y_1 \times y_2. \]
Then for any \( G_1 \in \text{Shv}(y_1)^{\constr} \), the object
\[ G \otimes p_1^!(G_1) \in \text{Shv}(y)^{\constr} \]
is ULA with respect to \( p_2 \).

\[ \square \text{Proposition 1.2.2} \]

1.3. **Spectral decomposition in a relative situation.** Let
\[ \text{Nilp} \subset T^*(\text{Bun}_G) \]
be the nilpotent cone. It is known to be half-dimensional, under very mild assumptions on \( \text{char}(k) \), see [AGKRRV, Sect. D], which we impose from now on.

In [AGKRRV] Theorem 14.4.3], it was shown that the subcategory \( \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \subset \text{Shv}(\text{Bun}_G) \) can be characterized as that of “Hecke-lisse” objects. In this subsection we will establish an analog of this result in the relative situation.

1.3.1. For \( V \in \text{Rep}(\hat{G}) \), consider the corresponding Hecke functor
\[ \text{Id}_Z \boxtimes \mathbb{H}(V') : \text{Shv}(Z \times \text{Bun}_G) \to \text{Shv}(Z \times \text{Bun}_G \times X). \]

Let
\[ \text{Shv}(Z \times \text{Bun}_G)^{\text{Hecke-lisse}} \]
denote the full subcategory of \( \text{Shv}(Z \times \text{Bun}_G) \) that consists of objects \( \mathcal{F} \) such that for every \( V \in \text{Rep}(\hat{G}) \),
\[ \mathbb{H}(V, \mathcal{F}) \in \text{Shv}(Z \times \text{Bun}_G) \otimes \text{QLisse}(X) \subset \text{Shv}(Z \times \text{Bun}_G \times X). \]

As in [GKRv Proposition C.2.5], we obtain that the Hecke functors (1.1) give rise to a compatible family of functors
\[ (1.7) \quad \text{Rep}(\hat{G})^{\otimes I} \to \text{End}(\text{Shv}(Z \times \text{Bun}_G)^{\text{Hecke-lisse}}) \otimes \text{QLisse}(X)^{\otimes I}. \]

Thus, in the terminology of [AGKRRV] Sect. 8.4.2], we obtain that \( \text{Shv}(Z \times \text{Bun}_G)^{\text{Hecke-lisse}} \) acquires an action of the (symmetric) monoidal category \( \text{Rep}(\hat{G})^{\otimes X-\text{lisse}} \).
1.3.2. Let LocSys$^\text{restr}_G(X)$ be the prestack introduced in [AGKRRV, Sect. 1.4.2], and recall now (see [AGKRRV, Equation (8.10)]) that we have a canonically defined functor

$$\text{Rep}(\hat{G}) \otimes X\text{-lisse} \to \text{QCoh}(\text{LocSys}^\text{restr}_G(X)),$$

A key result [AGKRRV, Theorem 8.3.7] says that the functor (1.8) is an equivalence.

Thus, we obtain that the above Rep$(\hat{G}) \otimes X\text{-lisse}$-action on Shv$(Z \times \text{Bun}_G)_{\text{Hecke-lisse}}$ is obtained from a uniquely defined monoidal action of the category $\text{QCoh}(\text{LocSys}^\text{restr}_G(X))$ on Shv$(Z \times \text{Bun}_G)_{\text{Hecke-lisse}}$.

We will refer to this phenomenon as the spectral decomposition of Shv$(Z \times \text{Bun}_G)_{\text{Hecke-lisse}}$ along $\text{QCoh}(\text{LocSys}^\text{restr}_G(X))$.

1.3.3. Recall the symmetric monoidal functor

$$\text{Rep}(\hat{G})_{\text{Ran}} \to \text{Rep}(\hat{G}) \otimes X\text{-lisse},$$

see [AGKRRV, Sect. 11.2.3]. Its composition with (1.8) is the functor $\text{Loc} : \text{Rep}(\hat{G})_{\text{Ran}} \to \text{QCoh}(\text{LocSys}^\text{restr}_G(X))$ of [AGKRRV, Sect. 12.7.1].

Following [AGKRRV] Sect. 13.3.1, set

$$\text{Shv}(Z \times \text{Bun}_G)^{\text{spec}} := \text{Funct}_{\text{Rep}(\hat{G})_{\text{Ran}}}(\text{QCoh}(\text{LocSys}^\text{restr}_G(X)), \text{Shv}(Z \times \text{Bun}_G)).$$

Pre-composition with Loc defines a functor

$$\text{Shv}(Z \times \text{Bun}_G)^{\text{spec}} \to \text{Funct}_{\text{Rep}(\hat{G})_{\text{Ran}}}(\text{Rep}(\hat{G})_{\text{Ran}}, \text{Shv}(Z \times \text{Bun}_G)) = \text{Shv}(Z \times \text{Bun}_G),$$

which is fully faithful, according to [AGKRRV] Proposition 13.3.4.

Thus, we can view $\text{Shv}(Z \times \text{Bun}_G)^{\text{spec}}$ as a full subcategory of $\text{Shv}(Z \times \text{Bun}_G)$.

1.3.4. By construction, the action of $\text{Rep}(\hat{G})_{\text{Ran}}$ on $\text{Shv}(Z \times \text{Bun}_G)_{\text{Hecke-lisse}}$ factors via (1.9).

Hence, we obtain

$$\text{Shv}(Z \times \text{Bun}_G)_{\text{Hecke-lisse}} \subset \text{Shv}(Z \times \text{Bun}_G)^{\text{spec}}.$$

1.3.5. Consider also the subcategory

$$\text{Shv}_{\frac{1}{2}\text{-dim} \times \text{Nilp}}(Z \times \text{Bun}_G) \subset \text{Shv}(Z \times \text{Bun}_G),$$

see Sect. A.6.2 for the notation.

1.3.6. We will prove:

**Theorem 1.3.7.** For any stack $Z$ the following four full subcategories of $\text{Shv}(Z \times \text{Bun}_G)$ coincide:

(i) The essential image of $\text{Shv}(Z) \otimes \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \to \text{Shv}(Z \times \text{Bun}_G)$.

(ii) $\text{Shv}_{\frac{1}{2}\text{-dim} \times \text{Nilp}}(Z \times \text{Bun}_G)$;

(iii) $\text{Shv}(Z \times \text{Bun}_G)_{\text{Hecke-lisse}}$;

(iv) $\text{Shv}(Z \times \text{Bun}_G)^{\text{spec}}$.

**Remark 1.3.8.** We will eventually prove also that for a conical half-dimensional closed subset $N \subset T^*(Z)$, the fully faithful functor

$$\text{Shv}_N(Z) \otimes \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \to \text{Shv}_{N \times \text{Nilp}}(Z \times \text{Bun}_G)$$

is an equivalence, see Theorem 4.4.2.

1.4. Proof of Theorem 1.3.7
1.4.1. The inclusion (i) \( \subset \) (ii) is evident. Let us establish (ii) \( \subset \) (iii).

Let \( \mathcal{F} \) be an object in \( \text{Shv}_{\frac{1}{2} \cdot \dim \times \text{Nilp}}(Z \times \text{Bun}_G) \), and let \( V \) be an object of \( \text{Rep}(\hat{G}) \). It follows by the argument of [GKRV, Theorem B.5.2] that
\[
\text{H}(V, \mathcal{F}) \in \text{Shv}_{\frac{1}{2} \cdot \dim \times \text{Nilp} \times \{0\}}(Z \times \text{Bun}_G \times X).
\]

However, the subcategory \( \text{Shv}_{\frac{1}{2} \cdot \dim \times \text{Nilp} \times \{0\}}(Z \times \text{Bun}_G \times X) \) is contained in the essential image of
\[
\text{Shv}(Z \times \text{Bun}_G) \otimes \text{QLisse}(X) \subset \text{Shv}(Z \times \text{Bun}_G \times X),
\]
by Theorem A.6.5 (applied to \( Y_1 = X \)).

The inclusion (iii) \( \subset \) (iv) has been noted in Sect. 1.3.4. Thus, it remains to prove (iv) \( \subset \) (i).

1.4.2. Let \( Z_n f_n \to \text{LocSys}_{\text{restr}} \hat{G}(X) \) be as in [AGKRRV, Sect. 16.1.2]. Consider the category
\[
\text{Shv}(Z \times \text{Bun}_G)^{\text{spec}} \otimes_{\text{QCoh}(\text{LocSys}_{\text{restr}}(X))} \text{QCoh}(Z_n),
\]
equipped with the forgetful functor, denoted, \( \text{oblv}_{\text{Hecke}} \):
\[
\text{Shv}(Z \times \text{Bun}_G)^{\text{spec}} \otimes_{\text{QCoh}(\text{LocSys}_{\text{restr}}(X))} \text{QCoh}(Z_n) \xrightarrow{\text{Id} \otimes (f_n)_*} \to \text{Shv}(Z \times \text{Bun}_G)^{\text{spec}} \otimes_{\text{QCoh}(\text{LocSys}_{\text{restr}}(X))} \text{QCoh}(\text{LocSys}_{G}(X)) \cong \text{Shv}(Z \times \text{Bun}_G)^{\text{spec}} \hookrightarrow \text{Shv}(Z \times \text{Bun}_G).
\]

The functor \( \text{oblv}_{\text{Hecke}} \) admits a left adjoint, to be denoted
\[
(1.11) \quad (\text{Id} \boxtimes P_{Z_n})^{\text{enh}},
\]
see [AGKRRV, Corollary 13.5.4].

1.4.3. We will prove:

**Proposition 1.4.4.** Objects of the form
\[
(\text{Id}_Z \boxtimes P_{Z_n})^{\text{enh}}(F_Z \boxtimes \delta_y), \quad F_Z \in \text{Shv}(Z), \quad y \in \text{Bun}_G(k)
\]
generate the category \( \text{Shv}(Z \times \text{Bun}_G)^{\text{spec}} \otimes_{\text{QCoh}(\text{LocSys}_{\text{restr}}(X))} \text{QCoh}(Z_n) \).

Let us assume this proposition temporarily and finish the proof of the containment (iv) \( \subset \) (i) in Theorem 1.3.7.

1.4.5. Consider the functor
\[
\text{Shv}(Z \times \text{Bun}_G)^{\text{spec}} \otimes_{\text{QCoh}(\text{LocSys}_{\text{restr}}(X))} \text{QCoh}(Z_n) \xrightarrow{\text{Id} \otimes (f_n)_*} \to \text{Shv}(Z \times \text{Bun}_G)^{\text{spec}} \otimes_{\text{QCoh}(\text{LocSys}_{\text{restr}}(X))} \text{QCoh}(\text{LocSys}_{G}(X)) \cong \text{Shv}(Z \times \text{Bun}_G)^{\text{spec}}.
\]

As in [AGKRRV, Sect. 16.1.4], one shows that the union (over the index \( n \)) of the essential images of these functors generates \( \text{Shv}(Z \times \text{Bun}_G)^{\text{spec}} \).

Hence, applying Proposition 1.4.4 we obtain that in order to prove the containment (iv) \( \subset \) (i), it suffices to show that for a fixed index \( n \), the functor
\[
(1.12) \quad \text{Shv}(Z \times \text{Bun}_G) \xrightarrow{(\text{Id}_Z \boxtimes P_{Z_n})^{\text{enh}}} \to \text{Shv}(Z \times \text{Bun}_G)^{\text{spec}} \otimes_{\text{QCoh}(\text{LocSys}_{\text{restr}}(X))} \text{QCoh}(Z_n) \xrightarrow{\text{oblv}_{\text{Hecke}}} \text{Shv}(Z \times \text{Bun}_G)
\]
applied to objects of the form
\[ F \boxtimes \delta_y, \quad F \in \text{Shv}(Z), \quad y \in \text{Bun}_G(k) \]
maps to the essential image of
\[ \text{Shv}(Z) \otimes \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \to \text{Shv}(Z \times \text{Bun}_G). \]

1.4.6. Note that, as in [AGKRRV, Equations (15.15) and (15.16)], the composite functor (1.12) is given by
\[ \text{Id}_Z \boxtimes H_V \]
(see Sect. 1.1.6 for the notation), for
\[ V := (\text{Id}_{\text{Rep}(\hat{G})_{\text{Ran}}} \otimes \Gamma(Z_n, -)) (R_{Z_n}) \in \text{Rep}(\hat{G})_{\text{Ran}}, \]
where
\[ R_{Z_n} \in \text{Rep}(\hat{G})_{\text{Ran}} \otimes \text{QCoh}(Z_n) \]
is the object of [AGKRRV, Sect. 15.3.1].

However, for any \( V \in \text{Rep}(\hat{G})_{\text{Ran}}, \)
\[ (\text{Id}_Z \boxtimes H_V)(F \boxtimes \delta_y) \simeq F \boxtimes H_V(\delta_y). \]

Now, for \( V \) given by (1.13), we have
\[ H_V(\delta_y) \in \text{Shv}_{\text{Nilp}}(\text{Bun}_G), \]
(e.g., by [AGKRRV Equations (15.15) and (15.16) and Corollary 15.5.4]), which implies the desired containment.

1.5. Proof of Proposition 1.4.4

1.5.1. We will prove a slightly stronger assertion. Namely, let \( y_i \in \text{Bun}_G(k) \) be points chosen as in [AGKRRV Sect. 16.2.1-16.2.2]. We will show that the objects
\[ (\text{Id}_Z \boxtimes P_{Z_n})^\text{enh}(F \boxtimes \delta_{y_i}) \]
genenerate
\[ \text{Shv}(Z \times \text{Bun}_G)_{\text{spec}} \otimes_{\text{QCoh}(\text{LocSys}^{\text{rest}}(X))_{\hat{G}}} \text{QCoh}(Z_n). \]

1.5.2. By adjunction, it suffices to show that for every \( F \in \text{Shv}(Z \times \text{Bun}_G)_{\text{spec}}, \) we can find \( F \in \text{Shv}(Z) \) and an index \( i \) such that
\[ \mathcal{H}om(F \boxtimes \delta_{y_i}, F) \neq 0. \]

Let \( i_{y_i} \) be the map \( \text{pt} \to \text{Bun}_G \) corresponding to the point \( y_i \). Our assertion is equivalent to saying that there exists an index \( i \) such that
\[ (\text{id}_Z \times i_{y_i})^\dagger(F) \neq 0. \]

1.5.3. Let \( g : Z_1 \to Z_2 \) be a map between algebraic stacks. The pullback functor
\[ \text{Shv}(Z_2 \times \text{Bun}_G) \xrightarrow{(g \times \text{id})^\dagger} \text{Shv}(Z_1 \times \text{Bun}_G) \]
is compatible with the Hecke action. In particular, it sends
\[ \text{Shv}(Z_2 \times \text{Bun}_G)_{\text{spec}} \to \text{Shv}(Z_1 \times \text{Bun}_G)_{\text{spec}}. \]
1.5.4. Let $\text{Spec}(k') \to \mathbb{Z}$ be a geometric point such that the $!$-pullback of $\mathcal{F}$ to
\[ \text{Bun}_G := \text{Spec}(k') \times \text{Bun}_G \to \mathbb{Z} \times \text{Bun}_G \]
is non-zero (such a point exists by a Cousin argument).

We can view the above $!$-pullback as the $!$-pullback along
\[ \text{Bun}_G' := \text{Spec}(k') \times \text{Bun}_G \to \mathbb{Z} \times \text{Bun}_G \]
of the base change $\mathcal{F}'$ of $\mathcal{F}$ to $\mathbb{Z} \times \text{Spec}(k')$.

Tautologically,
\[ \mathcal{F}' \in \text{Shv}(\mathbb{Z} \times \text{Spec}(k'))^{\text{spec}}. \]

By assumption, $(i'_z)^!(\mathcal{F})' \neq 0$, and by Sect. 1.5.3,
\[ (i'_z)^!(\mathcal{F})' \in \text{Shv}(\text{Bun}_G')^{\text{spec}}. \]

1.5.5. Let $i_{y_i} : \text{Spec}(k') \to \text{Bun}_G'$ denote the base change of the map $i_{y_i}$. We have
\[ (i_z)^! \circ (\text{id}_z \times i_{y_i})^!(\mathcal{F}) \simeq (i'_{y_i})^!(i'_z)^!(\mathcal{F})'. \]

Hence, it suffices to show that there exists an index $i$, such that
\[ (i'_{y_i})^!(i'_z)^!(\mathcal{F})' \neq 0. \]

However, by [AGKRRV, Proposition 15.4.4],
\[ \text{Shv}(\text{Bun}_G')^{\text{spec}} = \text{Shv}_{\text{Nilp}}(\text{Bun}_G'), \]
so
\[ 0 \neq (i'_z)^!(\mathcal{F})' \in \text{Shv}_{\text{Nilp}}(\text{Bun}_G'), \]
and the assertion follows from [AGKRRV, Sect. 16.2.2-16.2.3]. □ [Proposition 1.4.4]

1.6. The projector onto the category with nilpotent singular support. In [AGKRRV, Sect. 15.4.5], a particular object of $\text{Rep}(\check{G})_{\text{Ran}}$ was introduced whose action on $\text{Shv}(\text{Bun}_G)$ effects a projection onto the full subcategory $\text{Shv}_{\text{Nilp}}(\text{Bun}_G) \subset \text{Shv}(\text{Bun}_G)$.

In this subsection, we will establish an analog of this result in the relative situation.

1.6.1. Let $R \in \text{Rep}(\check{G})_{\text{Ran}}$ be the object introduced in [AGKRRV, Sect. 13.4.1].

Denote by $\text{Id}_Z \mapsto P$ the resulting endofunctor of $\text{Shv}(Z \times \text{Bun}_G)$, i.e.,
\[ P := H_R, \]
as functors defined by kernels.

In what follows we will also use the notation
\[ P_{\text{Bun}_G, \text{Nilp}} := P. \]

Remark 1.6.2. For $Z = \text{pt}$, the resulting functor endofunctor $P$ of $\text{Shv}(\text{Bun}_G)$ is the one considered in [AGKRRV, Sect. 15.4.7].
1.6.3. We claim:

**Theorem 1.6.4.** The endofunctor \( \text{Id}_{\mathcal{Z}} \boxtimes \mathcal{P} \) of \( \text{Shv}(\mathcal{Z} \times \text{Bun}_G) \) is a projector onto the full subcategory 
\[
\text{Shv}(\mathcal{Z}) \otimes \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \simeq \text{Shv}_{1\text{-dim} \times \text{Nilp}}(\mathcal{Z} \times \text{Bun}_G) \subset \text{Shv}(\mathcal{Z} \times \text{Bun}_G).
\]

**Proof.** Interpreting \( \text{Shv}_{1\text{-dim} \times \text{Nilp}}(\mathcal{Z} \times \text{Bun}_G) \) as \( \text{Shv}(\mathcal{Z} \times \text{Bun}_G)^{\text{spec}} \) (see Theorem 1.3.7), the assertion follows by [AGKRRV, Remark 13.4.8]. □

**Corollary 1.6.5.** The endofunctor \( \mathcal{P} \boxtimes \mathcal{P} \) of \( \text{Shv}(\text{Bun}_G \times \text{Bun}_G) \) is a projector onto the full subcategory 
\[
\text{Shv}_{\text{Nilp}}(\text{Bun}_G) \otimes \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \subset \text{Shv}(\text{Bun}_G \times \text{Bun}_G).
\]

**Proof.** Since \( \mathcal{P} \) acts as identity on \( \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \), it follows from (B.3) that \( \mathcal{P} \boxtimes \mathcal{P} \) acts as identity on \( \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \otimes \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \).

We need to show that the essential image of \( \mathcal{P} \boxtimes \mathcal{P} \) is contained in \( \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \otimes \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \). We write
\[
(1.15) \quad \mathcal{P} \boxtimes \mathcal{P} = (\mathcal{P} \boxtimes \text{Id}_{\text{Bun}_G}) \circ (\text{Id}_{\text{Bun}_G} \boxtimes \mathcal{P}).
\]

By Theorems 1.6.4 and 1.6.7, the essential image of \( \text{Id}_{\text{Bun}_G} \boxtimes \mathcal{P} \) is contained in \( \text{Shv}(\text{Bun}_G) \otimes \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \subset \text{Shv}(\text{Bun}_G) \otimes \text{Shv}(\text{Bun}_G) \subset \text{Shv}(\text{Bun}_G \times \text{Bun}_G) \).

By (B.3), the functor \( \mathcal{P} \boxtimes \text{Id}_{\text{Bun}_G} \), restricted to \( \text{Shv}(\text{Bun}_G) \otimes \text{Shv}(\text{Bun}_G) \), acts as \( \mathcal{P} \otimes \text{Id} \). Hence, the essential image of its further restriction to \( \text{Shv}(\text{Bun}_G) \otimes \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \) is contained in the subcategory \( \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \otimes \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \).

□

1.6.6. Finally, we observe that Theorem 1.6.4 gives us a (somewhat) alternative proof of the following result ([AGKRRV, Theorem 16.3.3]):

**Theorem 1.6.7.** The functor
\[
\text{Shv}_{\text{Nilp}}(\text{Bun}_G) \otimes \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \to \text{Shv}_{\text{Nilp} \times \text{Nilp}}(\text{Bun}_G \times \text{Bun}_G)
\]
is an equivalence.

**Proof.** It is enough to show that the functor \( \mathcal{P} \boxtimes \mathcal{P} \) acts as identity on \( \text{Shv}_{\text{Nilp} \times \text{Nilp}}(\text{Bun}_G \times \text{Bun}_G) \). Writing \( \mathcal{P} \boxtimes \mathcal{P} \) as in (1.15), it suffices to show that both \( \text{Id}_{\text{Bun}_G} \boxtimes \mathcal{P} \) and \( \mathcal{P} \boxtimes \text{Id}_{\text{Bun}_G} \) act as identity on \( \text{Shv}_{\text{Nilp} \times \text{Nilp}}(\text{Bun}_G \times \text{Bun}_G) \).

We have
\[
\text{Shv}_{\text{Nilp} \times \text{Nilp}}(\text{Bun}_G \times \text{Bun}_G) \subset \text{Shv}_{1\text{-dim} \times \text{Nilp}}(\text{Bun}_G \times \text{Bun}_G) \cap \text{Shv}_{\text{Nilp} 	imes 1\text{-dim}}(\text{Bun}_G \times \text{Bun}_G).
\]

Now, by Theorem 1.6.4, \( \text{Id}_{\text{Bun}_G} \boxtimes \mathcal{P} \) acts as identity on \( \text{Shv}_{1\text{-dim} \times \text{Nilp}}(\text{Bun}_G \times \text{Bun}_G) \) and \( \mathcal{P} \boxtimes \text{Id}_{\text{Bun}_G} \) acts as identity on \( \text{Shv}_{\text{Nilp} \times 1\text{-dim}}(\text{Bun}_G \times \text{Bun}_G) \).

□

1.7. (Universally) Nilp-cotruncative substacks. In [AGKRRV] Theorem 14.1.5, there was exhibited a collection of quasi-compact open substacks \( \mathcal{U} \subset \text{Bun}_G \) with particularly favorable properties vis-à-vis the subcategory \( \text{Shv}_{\text{Nilp}}(\mathcal{U}) \).

In this subsection, we will show that these properties carry over to the relative situation.
1.7.1. We shall say that an open substack \( \mathcal{U} \hookrightarrow \text{Bun}_G \) is Nilp-cotruncative if:

- It is cotruncative, i.e., if the functor \( j_! \) is defined by a kernel (see Sect. C.1.1);
- The functor \( j^* \) (equivalently, \( j_* \)) sends \( \text{Shv}_{\text{Nilp}}(\mathcal{U}) \) to \( \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \).

We shall say that an open substack \( \mathcal{U} \hookrightarrow \text{Bun}_G \) is universally Nilp-cotruncative if, moreover:

- For any algebraic stack \( Z \), the functor \( (\text{id} \times j)_! \) sends \( \text{Shv}_{\text{sh}}(Z \times \mathcal{U}) \) to \( \text{Shv}_{\text{sh}}(Z \times \text{Bun}_G) \).

We refer the reader to Sect. C.1.5 where the general notion of (universal) cotruncativeness relative to a subset in the cotangent bundle is discussed.

1.7.2. The following is an extension of the combination of \cite{DrGa1, Theorem 9.1.2} and \cite{AGKRRV, Theorem 14.1.5} (in fact, the arguments in \textit{loc. cit.} prove this strengthened statement):

**Theorem 1.7.3.** The stack \( \text{Bun}_G \) can be written as a filtered union of its quasi-compact universally Nilp-cotruncative open substacks.

In the terminology of Sect. C.1.6 the assertion of Theorem 1.7.3 is that \( \text{Bun}_G \) is universally Nilp-truncatable.

1.7.4. For the remainder of this subsection, we fix a universally Nilp-cotruncative quasi-compact open substack \( \mathcal{U} \hookrightarrow \text{Bun}_G \).

Let

\[
\text{Shv}_{\frac{1}{2}\text{-dim} \times \text{Nilp}}(Z \times \mathcal{U}) \subset \text{Shv}(Z \times \mathcal{U})
\]

be as in Sect. A.6.2.

The assumption that \( \mathcal{U} \) is universally Nilp-cotruncative implies that the functors \( (\text{id} \times j)_! \) and \( (\text{id} \times j)_* \) map

\[
\text{Shv}_{\frac{1}{2}\text{-dim} \times \text{Nilp}}(Z \times \mathcal{U}) \to \text{Shv}_{\frac{1}{2}\text{-dim} \times \text{Nilp}}(Z \times \text{Bun}_G)
\]

for any closed conical subset \( N \subset T^*(Z) \).

1.7.5. We claim:

**Corollary 1.7.6.** Let \( \mathcal{U} \hookrightarrow \text{Bun}_G \) be a universally Nilp-cotruncative open substack. Then for any stack \( Z \), the functor

\[
\text{Shv}(Z) \otimes \text{Shv}_{\text{Nilp}}(\mathcal{U}) \to \text{Shv}_{\frac{1}{2}\text{-dim} \times \text{Nilp}}(Z \times \mathcal{U})
\]

is an equivalence.

**Proof.** We only have to show that the functor in question is essentially surjective. This follows from the equivalence \((i) \Leftrightarrow (ii)\) in Theorem 1.3.7 and the fact that in the commutative diagram

\[
\begin{array}{ccc}
\text{Shv}(Z) \otimes \text{Shv}_{\text{Nilp}}(\text{Bun}_G) & \longrightarrow & \text{Shv}_{\frac{1}{2}\text{-dim} \times \text{Nilp}}(Z \times \text{Bun}_G) \\
\text{Id} \otimes j^* & \downarrow & \downarrow (\text{id} \times j)^* \\
\text{Shv}(Z) \otimes \text{Shv}_{\text{Nilp}}(\mathcal{U}) & \longrightarrow & \text{Shv}_{\frac{1}{2}\text{-dim} \times \text{Nilp}}(Z \times \mathcal{U})
\end{array}
\]

the vertical arrows are essentially surjective (which is in turn implied by the fact that they admit fully faithful left adjoints, due to the universal Nilp-cotruncativeness assumption).
1.8. **The projector on universally Nilp-cotruncative substacks.** In this subsection we fix a universally Nilp-cotruncative quasi-compact open substack 

\[ \mathcal{U} \hookrightarrow \text{Bun}_G. \]

We will show how the results pertaining to the projector \( P \) can be applied to \( \mathcal{U} \).

1.8.1. Denote by \( P_{\mathcal{U}, \text{Nilp}} \) the endofunctor of \( \text{Shv}(\mathcal{U}) \) defined by the kernel equal to 

\[ j^* \circ P_{\text{Bun}_G, \text{Nilp}} \circ j_* , \]

where we remind that \( P_{\text{Bun}_G, \text{Nilp}} \) is the same as the functor defined by a kernel, denoted \( P \), in Sect. 1.6.1

1.8.2. We claim:

**Corollary 1.8.3.** For a stack \( Z \), the endofunctor \( \text{Id} \lor P_{\mathcal{U}, \text{Nilp}} \) of \( \text{Shv}(Z \times \mathcal{U}) \) is a projector onto the full subcategory 

\[ \text{Shv}(Z) \otimes \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \simeq \text{Shv}_{\text{dim} \times \text{Nilp}}(Z \times \text{Bun}_G) \subset \text{Shv}(Z \times \mathcal{U}). \]

**Proof.** The fact that the endofunctor functor \( \text{Id} \lor P_{\mathcal{U}, \text{Nilp}} \) maps \( \text{Shv}(Z \times \mathcal{U}) \) to \( \text{Shv}_{\text{dim} \times \text{Nilp}}(Z \times \text{Bun}_G) \) follows from the fact that \( P_{\text{Bun}_G, \text{Nilp}} \) maps \( \text{Shv}(Z \times \text{Bun}_G) \) to \( \text{Shv}_{\text{dim} \times \text{Nilp}}(Z \times \text{Bun}_G) \) (see Theorem 1.6.4).

The fact that \( \text{Id} \lor P_{\mathcal{U}, \text{Nilp}} \) acts as identity on \( \text{Shv}_{\text{dim} \times \text{Nilp}}(Z \times \mathcal{U}) \) follows from the fact that \( (id \times j) \), maps \( \text{Shv}_{\text{dim} \times \text{Nilp}}(Z \times \mathcal{U}) \) to \( \text{Shv}_{\text{dim} \times \text{Nilp}}(Z \times \text{Bun}_G) \) and the fact that \( P_{\text{Bun}_G, \text{Nilp}} \) acts as identity on \( \text{Shv}_{\text{dim} \times \text{Nilp}}(Z \times \text{Bun}_G) \).

Furthermore, from Corollary 1.6.3 we obtain:

**Corollary 1.8.4.** The endofunctor \( P_{\mathcal{U}, \text{Nilp}} \lor P_{\mathcal{U}, \text{Nilp}} \) of \( \text{Shv}(\mathcal{U} \times \mathcal{U}) \) is a projector onto the full subcategory 

\[ \text{Shv}_{\text{Nilp}}(\mathcal{U}) \otimes \text{Shv}_{\text{Nilp}}(\mathcal{U}) \subset \text{Shv}_{\text{Nilp} \times \text{Nilp}}(\mathcal{U} \times \mathcal{U}) \]

**Proof.** Indeed, the functor \( P_{\mathcal{U}, \text{Nilp}} \lor P_{\mathcal{U}, \text{Nilp}} \) is the composition

\[
\text{Shv}(\mathcal{U} \times \mathcal{U}) \xrightarrow{\langle \text{id} \times j \rangle} \text{Shv}(\text{Bun}_G \times \text{Bun}_G) \xrightarrow{\text{PSP}} \\
\rightarrow \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \otimes \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \xrightarrow{j^* \otimes j^*} \text{Shv}_{\text{Nilp}}(\mathcal{U}) \otimes \text{Shv}_{\text{Nilp}}(\mathcal{U}).
\]

\[ \square \]

1.8.5. Finally, from Theorem 1.6.7 we obtain:

**Corollary 1.8.6.** The functor 

\[ \text{Shv}_{\text{Nilp}}(\mathcal{U}) \otimes \text{Shv}_{\text{Nilp}}(\mathcal{U}) \rightarrow \text{Shv}_{\text{Nilp} \times \text{Nilp}}(\mathcal{U} \times \mathcal{U}) \]

is an equivalence.

**Proof.** Follows from Theorem 1.6.7 in the same way as Corollary 1.7.6 follows from Theorem 1.8.7 \[ \square \]

2. **Verdier duality for \( \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \)**

This section is devoted to the study of the pattern of Verdier (i.e., usual) self-duality on \( \text{Bun}_G \) (and its quasi-compact open substacks) and how this self-duality interacts with the subcategory \( \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \subset \text{Shv}(\text{Bun}_G) \).

Since \( \text{Bun}_G \) is not quasi-compact, the category that is naturally dual to \( \text{Shv}(\text{Bun}_G) \) is \( \text{Shv}(\text{Bun}_G)_{\text{co}} \) (see Sect. 2.9). That said, the self-duality on \( \text{Shv}(\text{Bun}_G) \) by identifying \( \text{Shv}(\text{Bun}_G)_{\text{co}} \) using the miraculous functor \( \text{Mir}_{\text{Bun}_G} \), see Sect. 2.9.

This leads us to study the corresponding subcategory \( \text{Shv}_{\text{Nilp}}(\text{Bun}_G)_{\text{co}} \subset \text{Shv}(\text{Bun}_G)_{\text{co}} \).
2.1. The diagonal object. We begin this section by establishing the behavior of the diagonal object on Bun$_G$ with respect to the Hecke action. The result is quite obvious (Proposition 2.1.6), but it serves as a basis for our further study.

2.1.1. Let $\mathcal{Y}$ be a quasi-compact algebraic stack. Throughout this paper, we use the following notation:

$$u_{\mathcal{Y}} := (\Delta_{\mathcal{Y}})_*(\omega_{\mathcal{Y}}) \in \text{Shv}(\mathcal{Y} \times \mathcal{Y}).$$

2.1.2. This object should not be confused with the unit of the Verdier self-duality on $\text{Shv}(\mathcal{Y})$,

$$u_{\text{Shv}(\mathcal{Y})} \in \text{Shv}(\mathcal{Y}) \otimes \text{Shv}(\mathcal{Y}),$$

see Sect. A.4.3.

When we view $\text{Shv}(\mathcal{Y}) \otimes \text{Shv}(\mathcal{Y})$ as a full subcategory of $\text{Shv}(\mathcal{Y} \times \mathcal{Y})$ via (A.15), we have a tautological map

$$(2.1) \quad u_{\text{Shv}(\mathcal{Y})} \to u_{\mathcal{Y}}.$$ 

In fact, according to [AGKRRV, Sect. 22.2.4], $u_{\text{Shv}(\mathcal{Y})}$ isomorphic to the value on $u_{\mathcal{Y}}$ of the right adjoint functor to (A.15), and (2.1) is given by the counit of the adjunction.

2.1.3. Let now $\mathcal{Y}$ be not necessarily quasi-compact, but truncatable (see Sect. C.1.4 for what this means). In this case, one can still consider the object

$$(\Delta_{\mathcal{Y}})_*(\omega_{\mathcal{Y}}) \in \text{Shv}(\mathcal{Y} \times \mathcal{Y}),$$

but we will denote it by

$$u_{\mathcal{Y}\text{naive}}.$$ 

It has more refined variants, namely

$$(2.2) \quad u_{\mathcal{Y},co_1} \text{ and } u_{\mathcal{Y},co_2},$$

which are objects of the categories

$$(2.3) \quad \text{Shv}(\mathcal{Y} \times \mathcal{Y})_{co_1} \text{ and } \text{Shv}(\mathcal{Y} \times \mathcal{Y})_{co_2},$$

respectively, see Sects. C.4.5 and C.4.6 where the above categories and objects are defined, respectively.

By Sect. C.4.4 the categories in (2.3) correspond to functors defined by kernels

$$\text{Shv}(\mathcal{Y}) \to \text{Shv}(\mathcal{Y})$$

and

$$\text{Shv}(\mathcal{Y})_{co} \to \text{Shv}(\mathcal{Y})_{co},$$

respectively. The objects (2.2) correspond to the identity functor in both cases.

By contrast, objects of $\text{Shv}(\mathcal{Y} \times \mathcal{Y})$ correspond to functors defined by kernels

$$\text{Shv}(\mathcal{Y})_{co} \to \text{Shv}(\mathcal{Y}),$$

and the object $u_{\mathcal{Y}\text{naive}}$ corresponds to the functor

$$\text{Id}_{\mathcal{Y}\text{naive}} : \text{Shv}(\mathcal{Y})_{co} \to \text{Shv}(\mathcal{Y})$$

see Sect. C.2.3

2.1.4. The key player in this section is the object

$$u_{\text{Bun}_G}\text{naive} \in \text{Shv}(\text{Bun}_G \times \text{Bun}_G).$$
2.1.5. In what follows, we will need the following property of $u_B^{\text{naive}}$ vis-a-vis the Hecke action:

The category $\text{Rep}(\hat{G})$ carries a canonical involution, denoted $\tau$, given by the Chevalley involution on $\hat{G}$. We will denote it by $V \mapsto V^\tau$. Let

$$V \mapsto V^\tau$$

denote the induced involution on $\text{Rep}(\hat{G})_{\text{Ran}}$.

We claim:

**Proposition 2.1.6.** For $V \in \text{Rep}(\hat{G})_{\text{Ran}}$,

$$\sigma((\text{Id}_{\text{Bun}_G} \boxtimes \text{H}_V)(u_B^{\text{naive}})) \cong (\text{Id}_{\text{Bun}_G} \boxtimes \text{H}_{V^\tau})(u_B^{\text{naive}}),$$

where $\sigma$ is the transposition acting on $\text{Bun}_G \times \text{Bun}_G$.

**Proof.** Unwinding the construction of the functors $\text{H}_V$, we obtain that we need to show that for every $V \in \text{Rep}(\hat{G})^{\otimes I}$, we have

$$\sigma((\text{Id}_{\text{Bun}_G} \boxtimes \text{H}(V, -))(u_B^{\text{naive}})) \cong (\text{Id}_{\text{Bun}_G} \boxtimes \text{H}(V^\tau, -))(u_B^{\text{naive}})$$

as objects of

$$\text{Shv}(\text{Bun}_G \times \text{Bun}_G \times X^I),$$

where $\sigma$ is the transposition of the $\text{Bun}_G$ factors.

By the definition of the Hecke action, the object

$$(\text{Id}_{\text{Bun}_G} \boxtimes \text{H}(V, -))(u_B^{\text{naive}}) \in \text{Shv}(\text{Bun}_G \times \text{Bun}_G \times X^I)$$

identifies with

$$(h \times h \times s)_*(\text{Sat}_I(V)).$$

The assertion now follows from the fact that there is a canonical isomorphism

$$(2.4) \quad \sigma(\text{Sat}_I(V)) \cong \text{Sat}_I(V^\tau), \quad V \in \text{Rep}(\hat{G})^{\otimes I}.$$  

\qed

2.2. The projector and the diagonal. In this subsection we establish the behavior of the diagonal object $u_B^{\text{naive}}$ vis-a-vis the projector $P$ onto the subcategory with nilpotent singular support.

2.2.1. We claim:

**Proposition 2.2.2.** We have canonical isomorphisms

$$(P_{\text{Bun}_G, \text{Nilp}} \boxtimes \text{Id}_{\text{Bun}_G})(u_B^{\text{naive}}) \cong (P_{\text{Bun}_G, \text{Nilp}} \boxtimes P_{\text{Bun}_G, \text{Nilp}})(u_B^{\text{naive}}) \cong (\text{Id}_{\text{Bun}_G} \boxtimes P_{\text{Bun}_G, \text{Nilp}})(u_B^{\text{naive}}).$$

**Proof.** We will prove the second isomorphism; the first one would follow by symmetry.

In the notation of Sect. 1.6.1, we have

$$(2.5) \quad (P_{\text{Bun}_G, \text{Nilp}} \boxtimes P_{\text{Bun}_G, \text{Nilp}})(u_B^{\text{naive}}) = (H_R \boxtimes H_R)(u_B^{\text{naive}}) \cong (\text{Id}_{\text{Bun}_G} \boxtimes H_R) \circ (H_R \boxtimes \text{Id}_{\text{Bun}_G})(u_B^{\text{naive}}).$$

Note that

$$(H_R \boxtimes \text{Id}_{\text{Bun}_G})(u_B^{\text{naive}}) \cong \sigma((\text{Id}_{\text{Bun}_G} \boxtimes H_R)(u_B^{\text{naive}})),$$

and by Proposition 2.1.6 we can rewrite the latter as

$$(\text{Id}_{\text{Bun}_G} \boxtimes H_{R^\tau})(u_B^{\text{naive}}).$$

Hence, the expression in (2.5) can be rewritten as

$$(2.6) \quad (\text{Id}_{\text{Bun}_G} \boxtimes H_R) \circ (\text{Id}_{\text{Bun}_G} \boxtimes H_{R^\tau})(u_B^{\text{naive}}).$$

Now, the canonicity of the construction of the object $R$ implies that we have a canonical identification

$$R^\tau \cong R.$$  

Hence, we can further rewrite the expression in (2.6) as

$$(\text{Id}_{\text{Bun}_G} \boxtimes H_R) \circ (\text{Id}_{\text{Bun}_G} \boxtimes H_R)(u_B^{\text{naive}}) \cong (\text{Id}_{\text{Bun}_G} \boxtimes P_{\text{Bun}_G, \text{Nilp}}) \circ (\text{Id}_{\text{Bun}_G} \boxtimes P_{\text{Bun}_G, \text{Nilp}})(u_B^{\text{naive}}).$$
Now, the isomorphism with \((\text{Id}_{\text{Bun}_G} \boxtimes \mathcal{P}_{\text{Bun}_G, \text{Nilp}})(u_{\text{Bun}_G}^{\text{naive}}))\) follows from Theorem 1.6.4.

2.2.3. In what follows we will denote the object that appears in Proposition 2.2.2 by 
\(u_{\text{Bun}_G, \text{Nilp}}^{\text{naive}}\).

By Corollary 1.6.5, it belongs to the full subcategory
\[ \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \otimes \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \subset \text{Shv}(\text{Bun}_G \times \text{Bun}_G). \]

2.2.4. Let \(U_j \hookrightarrow \text{Bun}_G\) be a universally Nilp-cotruncative quasi-compact open prestack. Denote by \(u_{U_j, \text{Nilp}}\) the object 
\[ (j \times j)^* (u_{\text{Bun}_G, \text{Nilp}}^{\text{naive}}) \in \text{Shv}(U \times U). \]

Since \(u_{\text{Bun}_G, \text{Nilp}}^{\text{naive}} \in \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \otimes \text{Shv}_{\text{Nilp}}(\text{Bun}_G)\), we obtain that \(u_{U_j, \text{Nilp}}\) belongs to the full subcategory
\[ \text{Shv}_{\text{Nilp}}(U) \otimes \text{Shv}_{\text{Nilp}}(U) \subset \text{Shv}(U \times U). \]

**Proposition 2.2.5.** The object \(u_{U_j, \text{Nilp}}\) is isomorphic to 
\[ (\mathcal{P}_{U_j, \text{Nilp}} \boxtimes \text{Id}_U)(u_U), \ (\mathcal{P}_{U_j, \text{Nilp}} \boxtimes \mathcal{P}_{U_j, \text{Nilp}})(u_U) \text{ and } (\text{Id}_U \boxtimes \mathcal{P}_{U_j, \text{Nilp}})(u_U). \]

**Proof.** We have 
\[ (\mathcal{P}_{U_j, \text{Nilp}} \boxtimes \text{Id}_U)(u_U) \simeq ((j^* \circ \mathcal{P}_{\text{Bun}_G, \text{Nilp}} \circ j) \boxtimes \text{Id}_U)(u_U) \simeq ((j^* \circ \mathcal{P}_{\text{Bun}_G, \text{Nilp}} \circ \text{Id}_U) \circ (j^* \boxtimes \text{Id}_U))(u_U) \simeq \]
\[ \simeq ((j^* \circ \mathcal{P}_{\text{Bun}_G, \text{Nilp}}) \boxtimes \text{Id}_U) \circ (\text{Id}_U \boxtimes j^*)(u_{\text{Bun}_G}^{\text{naive}}) \simeq (j \times j)^* (u_{\text{Bun}_G}^{\text{naive}}) \simeq (j \times j)^* (u_{U_j, \text{Nilp}}^{\text{naive}}) = u_{U_j, \text{Nilp}}. \]

This proves the first isomorphism. The third isomorphism follows by symmetry. To prove the middle isomorphism, we note that
\[ (\mathcal{P}_{U_j, \text{Nilp}} \boxtimes \mathcal{P}_{U_j, \text{Nilp}})(u_U) \simeq (\text{Id}_U \boxtimes \mathcal{P}_{U_j, \text{Nilp}})(u_U) \simeq (\mathcal{P}_{U_j, \text{Nilp}} \boxtimes \text{Id}_U)(u_U) \simeq (\text{Id}_U \boxtimes \mathcal{P}_{U_j, \text{Nilp}})(u_U) \]
\[ \simeq (\text{Id}_U \boxtimes \mathcal{P}_{U_j, \text{Nilp}})(u_{U_j, \text{Nilp}}) = u_{U_j, \text{Nilp}}. \]

**2.3. Verdier self-duality for a universally Nilp-cotruncative substack.** Let \(U \hookrightarrow \text{Bun}_G\) be a universally Nilp-cotruncative quasi-compact open substack.

In this subsection we will show that the Verdier self-duality on \(\text{Shv}(U)\) restricts to a self-duality of the full subcategory \(\text{Shv}_{\text{Nilp}}(U) \subset \text{Shv}(U)\).

2.3.1. Throughout this paper, for a quasi-compact algebraic stack \(Y\) we denote by \(\text{ev}_Y\) the functor
\[ \text{Shv}(Y) \otimes \text{Shv}(Y) \xrightarrow{\text{ev}_Y} \text{Shv}(Y) \xrightarrow{\text{ev}_Y^{-1}} \text{Vect}. \]

Recall that according to Sect. A.4, the functor (2.7) defines the counit of a self-duality on \(\text{Shv}(Y)\), which we refer to as Verdier self-duality.
2.3.2. We claim:

**Proposition 2.3.3.** The functors

\[(2.8) \quad \text{Shv}_{\text{Nilp}}(\mathcal{U}) \otimes \text{Shv}_{\text{Nilp}}(\mathcal{U}) \to \text{Shv}(\mathcal{U}) \otimes \text{Shv}(\mathcal{U}) \overset{\text{ev}}{\to} \text{Vect} \]

and

\[u_{\text{Nilp}} : \text{Shv}_{\text{Nilp}}(\mathcal{U}) \otimes \text{Shv}_{\text{Nilp}}(\mathcal{U}) \]

define a datum of self-duality on \(\text{Shv}_{\text{Nilp}}(\mathcal{U})\).

*Proof.* Let \(\mathcal{F}\) be an object of \(\text{Shv}_{\text{Nilp}}(\mathcal{U})\). We have to construct a canonical isomorphism

\[(p_2)_! (p_1^!(\mathcal{F}) \otimes u_{\text{Nilp}}) \simeq \mathcal{F}.\]

Using Proposition 2.3.3 and interpreting \(u_{\text{Nilp}}\) as \((\text{Id}_U \boxtimes \text{P}_{U, \text{Nilp}})(u_U)\), we rewrite

\[(p_2)_! (p_1^!(\mathcal{F}) \otimes u_{\text{Nilp}}) \simeq \text{P}_{U, \text{Nilp}}((p_2)_! (p_1^!(\mathcal{F}) \otimes u_U)) \simeq \text{P}_{U, \text{Nilp}}(\mathcal{F}),\]

while the latter is isomorphic to \(\mathcal{F}\) by Corollary 1.8.3.

\[\square\]

**Corollary 2.3.4.** The pair \((\mathcal{U}, \text{Nilp})\) is duality-adapted.

2.3.5. We now claim:

**Proposition 2.3.6.** With respect to the self-duality

\[\text{Shv}(\mathcal{U})^\vee \simeq \text{Shv}(\mathcal{U})\]

of (A.12) and the self-duality

\[\text{Shv}_{\text{Nilp}}(\mathcal{U})^\vee \simeq \text{Shv}_{\text{Nilp}}(\mathcal{U})\]

of Proposition 2.3.3, the functor \(\text{P}_{U, \text{Nilp}} : \text{Shv}(\mathcal{U}) \to \text{Shv}_{\text{Nilp}}(\mathcal{U})\) identifies with the dual of the tautological embedding \(\text{u}_{\mathcal{U}} : \text{Shv}_{\text{Nilp}}(\mathcal{U}) \hookrightarrow \text{Shv}(\mathcal{U})\).

*Proof.* We need to establish a canonical isomorphism

\[(2.9) \quad (\text{Id}_{\text{Shv}(\mathcal{U})} \otimes \text{P}_{U, \text{Nilp}})(u_{\text{Shv}(\mathcal{U})}) \simeq u_{\text{Nilp}}\]

(here \(u_{\text{Shv}(\mathcal{U})}\) is as in Sect. 2.1.2) as objects of

\[\text{Shv}(\mathcal{U}) \otimes \text{Shv}_{\text{Nilp}}(\mathcal{U}) \subset \text{Shv}(\mathcal{U}) \otimes \text{Shv}(\mathcal{U}).\]

Let \(\mathcal{F}_1, \mathcal{F}_2\) be a pair of compact objects of \(\text{Shv}(\mathcal{U})\), and let us calculate

\[\text{Hom}_{\text{Shv}(\mathcal{U}) \otimes \text{Shv}(\mathcal{U})}(\mathcal{F}_1 \otimes \mathcal{F}_2, -)\]

into both sides of (2.9).

Note that since \(\mathcal{F}_1\) is compact, the functor

\[\text{Hom}_{\text{Shv}(\mathcal{U})}(\mathcal{F}_1, -) : \text{Shv}(\mathcal{U}) \to \text{Vect}\]

is continuous. Furthermore, the functor

\[\text{Hom}_{\text{Shv}(\mathcal{U})}(\mathcal{F}_1, -) \otimes \text{Id}_{\text{Shv}(\mathcal{U})} : \text{Shv}(\mathcal{U}) \otimes \text{Shv}(\mathcal{U}) \to \text{Shv}(\mathcal{U})\]

sends \(u_{\text{Shv}(\mathcal{U})}\) to \(\Box_{\text{Verdier}}(\mathcal{F}_1)\).

Hence, for the left-hand side of (2.9), we obtain

\[\text{Hom}_{\text{Shv}(\mathcal{U}) \otimes \text{Shv}(\mathcal{U})}(\mathcal{F}_1 \otimes \mathcal{F}_2, \text{Id}_{\text{Shv}(\mathcal{U})} \otimes \text{P}_{U, \text{Nilp}})(u_{\text{Shv}(\mathcal{U})}) \simeq\]

\[\simeq \text{Hom}_{\text{Shv}(\mathcal{U})}(\mathcal{F}_2, \text{Hom}_{\text{Shv}(\mathcal{U})}(\mathcal{F}_1, -) \otimes \text{Id}_{\text{Shv}(\mathcal{U})} \circ (\text{Id}_{\text{Shv}(\mathcal{U})} \otimes \text{P}_{U, \text{Nilp}})(u_{\text{Shv}(\mathcal{U})})) \simeq\]

\[\simeq \text{Hom}_{\text{Shv}(\mathcal{U})}(\mathcal{F}_2, \text{P}_{U, \text{Nilp}} \circ (\text{Hom}_{\text{Shv}(\mathcal{U})}(\mathcal{F}_1, -) \otimes \text{Id}_{\text{Shv}(\mathcal{U})})(u_{\text{Shv}(\mathcal{U}))} \simeq\]

\[\simeq \text{Hom}_{\text{Shv}(\mathcal{U})}(\mathcal{F}_2, \text{P}_{U, \text{Nilp}}(\Box_{\text{Verdier}}(\mathcal{F}_1))).\]

\[\text{Identical to Sect. 2.3.5 for what this means.}\]
Let us interpret \( u_{\mathcal{U}, \text{Nilp}} \) as \( (\text{Id}_\mathcal{U} \boxtimes \mathcal{P}_{\mathcal{U}, \text{Nilp}})(u_\mathcal{U}) \). Note that since \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) are compact, for any \( \mathcal{F} \in \text{Shv}(\mathcal{U} \times \mathcal{U}) \), we have

\[
\text{Hom}_{\text{Shv}(\mathcal{U} \times \mathcal{U})}(\mathcal{F}_1 \boxtimes \mathcal{F}_2, \mathcal{F}) \simeq \text{Hom}_{\text{Shv}(\mathcal{U})}(\mathcal{F}_2, (p_2)_! \left( (\mathcal{D}^\text{Verdier}(\mathcal{F}_1)) \circlearrowright \mathcal{F} \right)).
\]

Hence, for the right-hand side of \( (2.9) \), we obtain

\[
\text{Hom}_{\text{Shv}(\mathcal{U})}(\mathcal{F}_1 \boxtimes \mathcal{F}_2, u_{\mathcal{U}, \text{Nilp}}) \simeq \text{Hom}_{\text{Shv}(\mathcal{U} \times \mathcal{U})}(\mathcal{F}_1 \boxtimes \mathcal{F}_2, (\text{Id}_\mathcal{U} \boxtimes \mathcal{P}_{\mathcal{U}, \text{Nilp}})(u_\mathcal{U})) \simeq \text{Hom}_{\text{Shv}(\mathcal{U})}(\mathcal{F}_2, \mathcal{P}_{\mathcal{U}, \text{Nilp}} \left( (p_2)_! \left( (\mathcal{D}^\text{Verdier}(\mathcal{F}_1)) \circlearrowright u_\mathcal{U} \right) \right)) \simeq \text{Hom}_{\text{Shv}(\mathcal{U})}(\mathcal{F}_2, \mathcal{P}_{\mathcal{U}, \text{Nilp}}(\mathcal{D}^\text{Verdier}(\mathcal{F}_1))),
\]

as required.

\[ \square \]

**Corollary 2.3.7.** For \( \mathcal{F}_1 \in \text{Shv}_{\text{Nilp}}(\mathcal{U}) \) and \( \mathcal{F}_2 \in \text{Shv}(\mathcal{U}) \), we have a canonical isomorphism

\[
C_\bullet(\mathcal{U}, \mathcal{F}_1 \boxtimes \mathcal{F}_2) \simeq C_\bullet(\mathcal{U}, \mathcal{F}_1 \boxtimes \mathcal{P}_{\mathcal{U}, \text{Nilp}}(\mathcal{F}_2)).
\]

2.4. **Contraccessibility.** The material in this subsection is not needed for the main results of this paper, so it can be skipped on the first pass.

2.4.1. We claim:

**Proposition 2.4.2.** Let \( \mathcal{U} \subset \text{Bun}_G \) be a universally Nilp-cotrancative quasi-compact open substack. The following statements are equivalent:

(i) The functor \( t_{\mathcal{U}}^\text{fake-op} \) is continuous;

(ii) The functor \( \mathcal{P}_{\mathcal{U}, \text{Nilp}} \) provides a right adjoint to \( t_{\mathcal{U}} \).

**Proof.** Clearly, (ii) \( \Rightarrow \) (i). We will prove that (i) implies (ii) using the following general assertion:

**Lemma 2.4.3.** Let \( \iota : \mathcal{C}_1 \to \mathcal{C} \) be a fully faithful functor between compactly generated DG categories. Suppose that \( \iota \) preserves compactness, and let

\[
\iota^\text{fake-op} : \mathcal{C}_1^\vee \to \mathcal{C}^\vee,
\]

be the ind-extension of \( \iota^\text{op} : (\mathcal{C}_1^\vee)^\text{op} \to (\mathcal{C}^\vee)^\text{op} \). Then \( (\iota^\text{fake-op})^\vee \) is the right adjoint of \( \iota \).

We apply this lemma as follows: take \( \iota : \mathcal{C}_1 \to \mathcal{C} \) to be \( t_{\mathcal{U}} : \text{Shv}_{\text{Nilp}}(\mathcal{U}) \hookrightarrow \text{Shv}(\mathcal{U}) \).

Assumption (i) means that \( \text{Shv}_{\text{Nilp}}(\mathcal{U}) \) is contraccessible, and the self-duality of \( \text{Shv}_{\text{Nilp}}(\mathcal{U}) \) with counit \( (2.8) \), at the level of compact objects is induced by the Verdier duality functor \( (\wedge_\mathcal{U}) \). This implies that with respect to this self-duality of \( \text{Shv}_{\text{Nilp}}(\mathcal{U}) \) and the self-duality of \( \text{Shv}(\mathcal{U}) \) given by \( (\wedge_\mathcal{U}) \), the functor \( t_{\mathcal{U}}^\text{fake-op} \) identifies again with \( t_{\mathcal{U}} \).

Hence,

\[
t_{\mathcal{U}} \overset{\text{Lemma 2.4.3}}{\simeq} t_{\mathcal{U}}^\text{fake-op} \overset{\vee}{\simeq} t_{\mathcal{U}} \overset{\text{Proposition 2.4.2}}{\simeq} \mathcal{P}_{\mathcal{U}, \text{Nilp}}.
\]

\[ \square \]
2.4.4. From now on, for the duration of this subsection we will assume \[\text{AGKRRV} \ Conjecture 14.1.8\]. This conjecture says that the category Shv\(\text{Nilp}(Bun_G)\) is generated by objects that are compact in the ambient category Shv\(Bun_G\). (Recall also that this conjecture is known to hold in the de Rham and Betti contexts, see \[\text{AGKRRV} \ Theorems 16.4.3 and 16.4.10\].)

Given Theorem 1.7.3, this conjecture is equivalent to the statement that for every universally Nilp-cotruncative quasi-compact open substack \(U \subset Bun_G\), the pair \((U, \text{Nilp})\) is constraccessible\(^2\) (see \[\text{AGKRRV} \ Lemma F.8.10\]).

Since each Shv\(\text{Nilp}(U)\) is compactly generated (\[\text{AGKRRV} \ Corollary 16.1.8\]), the latter statement is equivalent to saying that the right adjoint of the tautological embedding
\[\iota_U : \text{ShvNilp}(U) \hookrightarrow \text{Shv}(U)\]
is continuous.

Hence, by Proposition 2.4.2, we obtain that the functor
\[P_{U, \text{Nilp}}\]
identifies with the right adjoint of \(\iota_U\).

In addition, according to \[\text{AGKRRV} \ Proposition 17.2.3\], the functor \(P_{Bun_G, \text{Nilp}}\), viewed as a functor \(\text{Shv}(Bun_G) \rightarrow \text{Shv}\text{Nilp}(Bun_G)\), is the right adjoint to the tautological embedding \(\iota\).

2.4.5. Let \(Z\) be an algebraic stack. Recall that the functors
\[\text{Id}_Z \boxtimes P_{Bun_G, \text{Nilp}}\]
and
\[\text{Id}_Z \boxtimes P_{U, \text{Nilp}}\]
takes values in the subcategories
\[\text{Shv}(Z) \otimes \text{Shv}\text{Nilp}(Bun_G) \subset \text{Shv}(Z \times Bun_G)\]
and
\[\text{Shv}(Z) \otimes \text{Shv}\text{Nilp}(U) \subset \text{Shv}(Z \times U),\]
respectively (see Theorem 1.6.4 and Corollary 1.8.3).

We claim:

**Proposition 2.4.6.** Assuming \[\text{AGKRRV} \ Conjecture 14.1.8\], for any algebraic stack \(Z\), the functors
\[\text{Id}_Z \boxtimes P_{Bun_G, \text{Nilp}}\]
and
\[\text{Id}_Z \boxtimes P_{U, \text{Nilp}}\]
are the right adjoints of the embeddings
\[\text{Shv}(Z) \otimes \text{Shv}\text{Nilp}(Bun_G) \hookrightarrow \text{Shv}(Z \times Bun_G)\]
and
\[\text{Shv}(Z) \otimes \text{Shv}\text{Nilp}(U) \hookrightarrow \text{Shv}(Z \times U),\]
respectively.

**Proof.** We will prove the assertion for \(Bun_G\); the case of \(U\) is similar.

First, it is easy to reduce the assertion to the case when \(Z\) is quasi-compact, which we will from now on assume.

We need to show that for \(\mathcal{F} \in \text{Shv}(Z)^c\), \(\mathcal{F}_{\text{Nilp}} \in \text{Shv}\text{Nilp}(Bun_G)\) and \(\mathcal{F} \in \text{Shv}(Z \times Bun_G)\) we have a canonical isomorphism
\[\mathcal{K}om(\mathcal{F} \boxtimes \mathcal{F}_{\text{Nilp}}, \mathcal{F}) \simeq \mathcal{K}om(\mathcal{F} \boxtimes \mathcal{F}_{\text{Nilp}}, (\text{Id}_Z \boxtimes P_{Bun_G, \text{Nilp}})(\mathcal{F})).\]

Set
\[\mathcal{F}' := (P_{Bun_G})_*(p^!_{Z}(\mathbb{D} \text{verdier}(\mathcal{F})) \otimes \mathcal{F}) \in \text{Shv}(Bun_G).\]
Then the left-hand side in (2.10) identifies with
\[\mathcal{K}om_{\text{Shv}(Bun_G)}(\mathcal{F}_{\text{Nilp}}, \mathcal{F}').\]

Set
\[\mathcal{F}'' := (P_{Bun_G})_*(p^!_{Z}(\mathbb{D} \text{verdier}(\mathcal{F})) \otimes (\text{Id}_Z \boxtimes P_{Bun_G, \text{Nilp}})(\mathcal{F})) \in \text{Shv}(Bun_G).\]
Then the right-hand side in (2.10) identifies with
\[\mathcal{K}om_{\text{Shv}(Bun_G)}(\mathcal{F}_{\text{Nilp}}, \mathcal{F}'').\]

\(^2\)See Sect. 3.5.4 for what this means.
However, since $\mathcal{F}_Z$ is compact (and hence so is $\mathcal{D}^{\text{verdier}}(\mathcal{F}_Z)$), the maps

$$(p_{\text{Bun}_G})_*(p_{\mathcal{F}_Z}^!(\mathcal{D}^{\text{verdier}}(\mathcal{F}_Z)) \otimes \mathcal{F}) \to (p_{\text{Bun}_G})_*(p_{\mathcal{F}_Z}^!(\mathcal{D}^{\text{verdier}}(\mathcal{F}_Z)) \otimes \mathcal{F})$$

and

$$(p_{\text{Bun}_G})_*(p_{\mathcal{F}_Z}^!(\mathcal{D}^{\text{verdier}}(\mathcal{F}_Z)) \otimes (\text{Id}_Z \boxtimes \mathcal{P}_{\text{Bun}_G,Nilp})(\mathcal{F})) \to (p_{\text{Bun}_G})_*(p_{\mathcal{F}_Z}^!(\mathcal{D}^{\text{verdier}}(\mathcal{F}_Z)) \otimes (\text{Id}_Z \boxtimes \mathcal{P}_{\text{Bun}_G,Nilp})(\mathcal{F}))$$

are isomorphisms.

This implies that

$$\mathcal{F}' \cong \mathcal{P}_{\text{Bun}_G,Nilp}(\mathcal{F}')$$

Hence, the assertion follows from the $(\iota, \mathcal{P}_{\text{Bun}_G,Nilp})$ adjunction.

\[\square\]

Iterating, from Proposition 2.4.6 we obtain:

**Corollary 2.4.7.** Assuming [AGKRRV] Conjecture 14.1.8], the functor

$$\mathcal{P}_{\text{Bun}_G,Nilp} \boxtimes \mathcal{P}_{\text{Bun}_G,Nilp} : \text{Shv}(\text{Bun}_G \times \text{Bun}_G) \to \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \otimes \text{Shv}_{\text{Nilp}}(\text{Bun}_G)$$

is the right adjoint of

$$\text{Shv}_{\text{Nilp}}(\text{Bun}_G) \otimes \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \xrightarrow{\boxminus} \text{Shv}(\text{Bun}_G) \otimes \text{Shv}(\text{Bun}_G) \xrightarrow{\boxplus} \text{Shv}(\text{Bun}_G \times \text{Bun}_G).$$

2.4.8. Note that by passing to right adjoints in the commutative diagrams

$$\xymatrix{ \text{Shv}_{\text{Nilp}}(\mathcal{U}) \ar[r]^{\iota|} & \text{Shv}(\mathcal{U}) \ar[d]^{j|} \ar[r]^{j|} & \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \ar[r]^\iota & \text{Shv}(\text{Bun}_G) }$$

and

$$\xymatrix{ \text{Shv}_{\text{Nilp}}(\mathcal{U}) \ar[r]^{\iota|} & \text{Shv}(\mathcal{U}) \ar[r]^{j*} & \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \ar[r]^\iota & \text{Shv}(\text{Bun}_G) }$$

we obtain the isomorphisms

$$j* \circ \mathcal{P}_{\text{Bun}_G,Nilp} \cong \mathcal{P}_{\mathcal{U},Nilp} \circ j*$$

and

$$j* \circ \mathcal{P}_{\mathcal{U},Nilp} \cong \mathcal{P}_{\text{Bun}_G,Nilp} \circ j*.$$  

We claim that a stronger assertion holds:

**Corollary 2.4.9.** Assuming [AGKRRV] Conjecture 14.1.8], the isomorphisms 2.11] and 2.12 hold as functors defined by kernels, where we view $\mathcal{P}_{\text{Bun}_G,Nilp}$ and $\mathcal{P}_{\mathcal{U},Nilp}$ as endofunctors defined by kernels of $\text{Shv}(\text{Bun}_G)$ and $\text{Shv}(\mathcal{U})$, respectively.

**Proof.** We will prove the assertion for 2.11; the assertion for 2.12 is similar.

We have to construct an isomorphism

$$\text{Id}_Z \boxtimes (j* \circ \mathcal{P}_{\mathcal{U}}) \cong \text{Id}_Z \boxtimes (\mathcal{P}_{\mathcal{U}} \circ j*)$$

for any algebraic stack $Z$.

This follows formally from Proposition 2.4.6 by passing to right adjoints of the composition

$$\text{Shv}(\mathcal{Z}) \otimes \text{Shv}_{\text{Nilp}}(\mathcal{U}) \xrightarrow{\text{Id} \otimes j} \text{Shv}(\mathcal{Z}) \otimes \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \xrightarrow{\iota} \text{Shv}(\mathcal{Z} \times \text{Bun}_G),$$

which is the same as

$$\text{Shv}(\mathcal{Z}) \otimes \text{Shv}_{\text{Nilp}}(\mathcal{U}) \xrightarrow{\iota} \text{Shv}(\mathcal{Z} \times \mathcal{U}) \xrightarrow{(\text{id} \times j)^*} \text{Shv}(\mathcal{Z} \times \text{Bun}_G).$$
2.5. The “co” category for $\text{Bun}_G$ with nilpotent singular support. In order to talk about Verdier duality on the (non quasi-compact) algebraic stack $\text{Bun}_G$, we need to study the category $\text{Shv}(\text{Bun}_G)_{\text{co}}$, see Sect. 2.3.

In this subsection we will discuss the pattern of Hecke action on $\text{Shv}(\text{Bun}_G)_{\text{co}}$.

We also introduce the “co”-version of the subcategory with nilpotent singular support

$$\text{Shv}_{\text{Nilp}}(\text{Bun}_G)_{\text{co}} \subset \text{Shv}(\text{Bun}_G)_{\text{co}}.$$ 

2.5.1. Recall the category $\text{Shv}(\text{Bun}_G)_{\text{co}}$, defined as

$$\text{colim}_U \text{Shv}(U),$$

where the colimit is taken over the poset of cotruncative quasi-compact open substacks of $\text{Bun}_G$, see Sect. 2.2.2.

2.5.2. The Hecke action on $\text{Shv}(\text{Bun}_G)$ gives rise to a Hecke action on $\text{Shv}(\text{Bun}_G)_{\text{co}}$:

Let $V$ be an object of $\text{Rep}(\tilde{G})^\otimes I$. Note that the Hecke functor

$$H(V, -) : \text{Shv}(\text{Bun}_G) \to \text{Shv}(\text{Bun}_G \times X^I)$$

is defined by the kernel

$$(\text{Id}_{\text{Bun}_G} \boxtimes H(V, -))(u_{\text{Bun}_G, co_1}) \in \text{Shv}(\text{Bun}_G \times \text{Bun}_G \times X^I)_{co_1}.$$ 

Consider now the object

$$\sigma ((\text{Id}_{\text{Bun}_G} \boxtimes H(V^\tau, -))(u_{\text{Bun}_G, co_1})) \in \text{Shv}(\text{Bun}_G \times \text{Bun}_G \times X^I)_{co_2},$$

where $\sigma$ is the transposition of the two factors of $\text{Bun}_G$.

We let

$$H(V, -)_{\text{co}} : \text{Shv}(\text{Bun}_G)_{\text{co}} \to \text{Shv}(\text{Bun}_G \times X^I)_{\text{co}}$$

be the functor defined by the kernel (2.15).

2.5.3. Recall the functor

$$\text{Id}^\text{naive}_{\text{Bun}_G} : \text{Shv}(\text{Bun}_G)_{\text{co}} \to \text{Shv}(\text{Bun}_G),$$

see Sect. 2.1.3. We claim:

**Lemma 2.5.4.** The diagram (of functors defined by kernels)

$$\begin{array}{ccc}
\text{Shv}(\text{Bun}_G)_{\text{co}} & \xrightarrow{H(V, -)_{\text{co}}} & \text{Shv}(\text{Bun}_G \times X^I)_{\text{co}} \\
\downarrow_{\text{Id}^\text{naive}_{\text{Bun}_G}} & & \downarrow_{\text{Id}^\text{naive}_{\text{Bun}_G} \boxtimes \text{Id}_{X^I}} \\
\text{Shv}(\text{Bun}_G) & \xrightarrow{H(V, -)} & \text{Shv}(\text{Bun}_G \times X^I)
\end{array}$$

commutes.

**Proof.** Follows by diagram chase from Proposition 2.1.6. 

□
2.5.5. A similar discussion applies to the action of $\text{Rep}(\tilde{G})_{\text{Ran}}$. Namely, for $V \in \text{Rep}(\tilde{G})_{\text{Ran}}$ we have a functor defined by a kernel

$$H_{V,\text{co}} : \text{Shv}(\text{Bun}_G)_{\text{co}} \to \text{Shv}(\text{Bun}_G)_{\text{co}},$$

and the diagram (of functors defined by kernels)

$$\begin{array}{ccc}
\text{Shv}(\text{Bun}_G)_{\text{co}} & \xrightarrow{H_{V,\text{co}}} & \text{Shv}(\text{Bun}_G)_{\text{co}} \\
\downarrow & & \downarrow \\
\text{Shv}(\text{Bun}_G) & \xrightarrow{H_V} & \text{Shv}(\text{Bun}_G)
\end{array}$$

(2.16)
commutes (indeed, this formally follows from Lemma 2.5.4).

2.5.6. We claim:

**Lemma 2.5.7.** For $V \in \text{Rep}(\tilde{G})_{\text{Ran}}$, we have a canonical isomorphism

$$(H_{V,\text{co}} \boxtimes \text{Id}_{\text{Bun}_G})(u_{\text{Bun}_G,\text{co}_1}) \simeq (\text{Id}_{\text{Bun}_G} \boxtimes H_V)(u_{\text{Bun}_G,\text{co}_1})$$

as objects of $\text{Shv}(\text{Bun}_G \times \text{Bun}_G)_{\text{co}_1}$.

**Proof.** The assertion of the lemma is equivalent to an isomorphism

$$\sigma((\text{Id}_{\text{Bun}_G} \boxtimes H_V)(u_{\text{Bun}_G,\text{co}_1})) \simeq (\text{Id}_{\text{Bun}_G} \boxtimes H_V)(u_{\text{Bun}_G,\text{co}_1}).$$

The fact that the images of both sides under the forgetful functor

$$\text{Id}^\text{naive}_{\text{Bun}_G} \boxtimes \text{Id}_{\text{Bun}_G} : \text{Shv}(\text{Bun}_G \times \text{Bun}_G)_{\text{co}_1} \to \text{Shv}(\text{Bun}_G \times \text{Bun}_G)$$

are canonically isomorphic is the content of Proposition 2.4.6 (using the commutative diagram (2.16) as functors defined by kernels).

In order to upgrade this isomorphism to an isomorphism that takes place in $\text{Shv}(\text{Bun}_G \times \text{Bun}_G)_{\text{co}_1}$, we argue as follows:

We can assume that the object $V \in \text{Rep}(\tilde{G})_{\text{Ran}}$ is compact. It suffices to establish a compatible family of isomorphisms

$$(\text{Id}_{\text{Bun}_G} \times j)^*(H_{V,\text{co}} \boxtimes \text{Id}_{\text{Bun}_G})(u_{\text{Bun}_G,\text{co}_1}) \simeq (\text{Id}_{\text{Bun}_G} \times j)^*((\text{Id}_{\text{Bun}_G} \boxtimes H_V)(u_{\text{Bun}_G,\text{co}_1}))$$

as objects in $\text{Shv}(\text{Bun}_G \times \mathcal{U})_{\text{co}_1}$ for quasi-compact open substacks $\mathcal{U} \subset \subset \text{Bun}_G$.

However, for $V$ compact and a fixed $\mathcal{U}$, both sides in (2.17) lie in the essential image of the (fully faithful) functor

$$(j' \times \text{id}_{\mathcal{U}})_{*,\text{co}} : \text{Shv}(\mathcal{U}' \times \mathcal{U}) \to \text{Shv}(\text{Bun}_G \times \mathcal{U})_{\text{co}}$$

for some quasi-compact open $\mathcal{U}' \subset \subset \text{Bun}_G$.

Hence, it is enough to show that both sides in (2.17) become isomorphic after applying the functor

$$(j' \times \text{id}_{\mathcal{U}})_{*,\text{co}} : \text{Shv}(\text{Bun}_G \times \mathcal{U})_{\text{co}} \to \text{Shv}(\mathcal{U}' \times \mathcal{U}).$$

However,

$$(j' \times \text{id}_{\mathcal{U}})_{*,\text{co}} \circ (\text{Id}_{\text{Bun}_G} \times j)^* \simeq (j' \times j)^* \circ (\text{Id}^\text{naive}_{\text{Bun}_G} \boxtimes \text{Id}_{\text{Bun}_G})$$

and the assertion follows. □
2.5.8. Following Sect. C.2.4, we define the category $\text{Shv}_{\text{Nilp}}(\text{Bun}_G)_{\text{co}}$ as
\[
\text{colim}_U \text{Shv}_{\text{Nilp}}(U),
\]
where the colimit is taken over the poset of universally Nilp-cotruncative quasi-compact open substacks of $\text{Bun}_G$, and the transition functors are given by $\sim$.

Since the above poset is filtered, the functor $\iota_{\text{co}} : \text{Shv}_{\text{Nilp}}(\text{Bun}_G)_{\text{co}} \to \text{Shv}(\text{Bun}_G)_{\text{co}}$ comprised of the functors $\iota_U : \text{Shv}_{\text{Nilp}}(U) \to \text{Shv}(U)$, is fully faithful.

2.5.9. The following results from the definitions:

**Lemma 2.5.10.** The functor $\text{Id}_{\text{naive}}^{\text{Bun}_G}$ sends $\text{Shv}_{\text{Nilp}}(\text{Bun}_G)_{\text{co}}$ to $\text{Shv}_{\text{Nilp}}(\text{Bun}_G)$.

2.6. Duality between $\text{Shv}_{\text{Nilp}}(\text{Bun}_G)_{\text{co}}$ and $\text{Shv}_{\text{Nilp}}(\text{Bun}_G)$. In this subsection we will extend the Verdier self-duality of $\text{Shv}_{\text{Nilp}}(U)$ (where $U$ is a universally Nilp-contruncative quasi-compact open substack of $\text{Bun}_G$) to obtain a duality between $\text{Shv}_{\text{Nilp}}(\text{Bun}_G)$ and $\text{Shv}_{\text{Nilp}}(\text{Bun}_G)_{\text{co}}$.

2.6.1. Recall that according to Sect. C.3.3, there is a canonical identification
\[
\text{Shv}(\text{Bun}_G)_{\text{co}} \cong \text{Shv}(\text{Bun}_G)^{\vee},
\]
so that the pairing
\[
\text{Shv}(\text{Bun}_G) \otimes \text{Shv}(\text{Bun}_G)_{\text{co}} \to \text{Vect},
\]
denoted $\text{ev}_{\text{Bun}_G}$, is given by
\[
\mathcal{F}_1 \in \text{Shv}(\text{Bun}_G), \quad \mathcal{F}_2 \in \text{Shv}(\text{Bun}_G)_{\text{co}} \mapsto C_{\bullet}(\text{Bun}_G, \mathcal{F}_1 \otimes \mathcal{F}_2),
\]
where we view $\mathcal{F}_1 \otimes \mathcal{F}_2$ as an object of $\text{Shv}(\text{Bun}_G)_{\text{co}}$, and $C_{\bullet}(\text{Bun}_G, -)$ denotes the corresponding functor $\text{Shv}(\text{Bun}_G) \to \text{Vect}$, see Sect. C.3.3.

2.6.2. From Lemma 2.5.7 and Sect. B.1.2 we obtain:

**Corollary 2.6.3.** With respect to the duality (2.18), the dual of the functor $H^\vee$ identifies with $H^\vee_{\text{V}^\bullet,\text{co}}$.

2.6.4. Combining Sect. C.3.5 and Corollary 2.3.4, we obtain:

**Corollary 2.6.5.** The restriction of the pairing (2.19) along $\text{Shv}_{\text{Nilp}}(\text{Bun}_G)_{\text{co}} \otimes \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \to \text{Shv}(\text{Bun}_G)_{\text{co}} \otimes \text{Shv}(\text{Bun}_G)$ defines an equivalence
\[
\text{Shv}_{\text{Nilp}}(\text{Bun}_G)_{\text{co}} \cong \text{Shv}_{\text{Nilp}}(\text{Bun}_G)^{\vee}.
\]

2.6.6. Our next step it to define a projector from $\text{Shv}(\text{Bun}_G)_{\text{co}}$ onto $\text{Shv}_{\text{Nilp}}(\text{Bun}_G)_{\text{co}}$.

Let $P_{\text{co}}$ denote the endofunctor of $\text{Shv}(\text{Bun}_G)_{\text{co}}$ equal to $H_{\text{R},\text{co}}$ (see Sect. 2.5.5 for the notation), where $\text{R} \in \text{Rep}(\mathcal{G})_{\text{Ran}}$ is as in Sect. 1.6.1. We claim:

**Proposition 2.6.7.** With respect to the identifications (2.18) and (2.20), we have:

(i) The functor dual to $P : \text{Shv}(\text{Bun}_G) \to \text{Shv}_{\text{Nilp}}(\text{Bun}_G)$ identifies with $\iota_{\text{co}} : \text{Shv}_{\text{Nilp}}(\text{Bun}_G)_{\text{co}} \to \text{Shv}(\text{Bun}_G)_{\text{co}}$.

(ii) The essential image of the functor $P_{\text{co}}$ lies in $\text{Shv}_{\text{Nilp}}(\text{Bun}_G)_{\text{co}}$, and the resulting functor $P_{\text{co}} : \text{Shv}(\text{Bun}_G)_{\text{co}} \to \text{Shv}_{\text{Nilp}}(\text{Bun}_G)_{\text{co}}$ is the functor dual of $\iota : \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \to \text{Shv}(\text{Bun}_G)$. 
Proof. Point (i) follows by unwinding the definitions from Corollary 2.3.7.

For point (ii), given point (i), it suffices to show that the dual of $P$, viewed as an endofunctor of $\text{Shv}(\text{Bun}_G)$, is $P_{\text{co}}$, viewed as an endofunctor of $\text{Shv}(\text{Bun}_G)_{\text{co}}$.

Since $P$ is defined by the kernel

$$(\text{Id}_{\text{Bun}_G} \boxtimes \mathcal{H}_R)(u_{\text{Bun}_G,\text{co}_1}) \in \text{Shv}(\text{Bun}_G \times \text{Bun}_G)_{\text{co}_1},$$

from Corollary 2.6.3, we obtain that its dual is the functor $H_{\mathcal{R},\text{co}}$.

Now the required assertion follows from the isomorphism $R^\tau \simeq R$. \hfill $\square$

**Corollary 2.6.8.** The endofunctor $P_{\text{co}}$ of $\text{Shv}(\text{Bun}_G)_{\text{co}}$ is the projector onto the full subcategory $\text{Shv}_{\text{Nilp}}(\text{Bun}_G)_{\text{co}} \subset \text{Shv}(\text{Bun}_G)_{\text{co}}$.

### 2.7. The projector and the diagonal, revisited.

In this subsection, we will refine the results of Sect. 2.2, when instead of the object $u_{\text{Bun}_G}^{\text{naive}} \in \text{Shv}(\text{Bun}_G \times \text{Bun}_G)$,

we consider its refined version, namely,

$$u_{\text{Bun}_G,\text{co}_2} \in \text{Shv}(\text{Bun}_G \times \text{Bun}_G)_{\text{co}_2}.$$

#### 2.7.1. First, we claim:

**Proposition 2.7.2.** The objects

$$(\text{Id}_{\text{Bun}_G} \boxtimes P_{\text{co}})(u_{\text{Bun}_G,\text{co}_2}), \ (P \boxtimes P_{\text{co}})(u_{\text{Bun}_G,\text{co}_2}), \ (P \boxtimes \text{Id}_{\text{Bun}_G})(u_{\text{Bun}_G,\text{co}_2})$$

are isomorphic.

**Proof.** The isomorphism follows from Lemma 2.5.7 using the isomorphism $R^\tau \simeq R$ in the same way as Proposition 2.2.2 follows from Proposition 2.1.6. \hfill $\square$

#### 2.7.3. Denote the object appearing in Proposition 2.7.2 by $u_{\text{Bun}_G,\text{Nilp},\text{co}_2}$. We claim:

**Proposition 2.7.4.** The object $u_{\text{Bun}_G,\text{Nilp},\text{co}_2}$ belongs to

$$\text{Shv}_{\text{Nilp}}(\text{Bun}_G) \otimes \text{Shv}_{\text{Nilp}}(\text{Bun}_G)_{\text{co}} \subset \text{Shv}(\text{Bun}_G) \otimes \text{Shv}(\text{Bun}_G)_{\text{co}} \subset \text{Shv}(\text{Bun}_G \times \text{Bun}_G)_{\text{co}_2}.$$

**Proof.** We first show that $(P \boxtimes \text{Id}_{\text{Bun}_G})(u_{\text{Bun}_G,\text{co}_2})$ belongs to

$$\text{Shv}_{\text{Nilp}}(\text{Bun}_G) \otimes \text{Shv}(\text{Bun}_G)_{\text{co}} \subset \text{Shv}(\text{Bun}_G \times \text{Bun}_G)_{\text{co}_2}.$$

For that, it suffices to show that for every quasi-compact cotruncative open substack $\mathcal{U} \xrightarrow{j} \text{Bun}_G$, the object

$$(\text{Id}_{\text{Bun}_G} \boxtimes j^\tau) \circ (P \boxtimes \text{Id}_{\text{Bun}_G})(u_{\text{Bun}_G,\text{co}_2}) \simeq (P \boxtimes j^\tau)(u_{\text{Bun}_G,\text{co}_2}) \simeq (P \boxtimes \text{Id}_{\text{Bun}_G}) \circ (\text{Id}_{\text{Bun}_G} \boxtimes j^\tau)(u_{\text{Bun}_G,\text{co}_2})$$

belongs to

$$\text{Shv}_{\text{Nilp}}(\text{Bun}_G) \otimes \text{Shv}(\mathcal{U}) \subset \text{Shv}(\text{Bun}_G \times \mathcal{U}),$$

where $j^\tau$ is the functor defined by a kernel as in Sect. 1.3.7. However, this follows from Theorems 1.6.4 and 1.3.7.

Now, the fact that

$$(P \boxtimes P_{\text{co}})(u_{\text{Bun}_G,\text{co}_2}) \simeq (\text{Id}_{\text{Bun}_G} \boxtimes P_{\text{co}}) \circ (P \boxtimes \text{Id}_{\text{Bun}_G})(u_{\text{Bun}_G,\text{co}_2})$$

belongs to

$$\text{Shv}_{\text{Nilp}}(\text{Bun}_G) \otimes \text{Shv}_{\text{Nilp}}(\text{Bun}_G)_{\text{co}} \subset \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \otimes \text{Shv}(\text{Bun}_G)_{\text{co}}$$

follows from Corollary 2.6.8. \hfill $\square$
2.7.5. We now claim:

**Proposition 2.7.6.** The object

\[ u_{\text{Bun}_G, \text{Nilp}, \text{co}} \in \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \otimes \text{Shv}_{\text{Nilp}}(\text{Bun}_G)_{\text{co}} \]

is the unit of the duality (2.20).

**Proof.** Follows formally from Proposition 2.7.2 in the same way as in the proof of Proposition 2.3.3. \(\square\)

2.8. The miraculous functor on \(\text{Bun}_G\). We now input another ingredient into our story, the miraculous functor, see Sects. B.4 and C.4.7.

2.8.1. Consider the object

\[ \text{ps-u}_{\text{Bun}_G} := (\Delta_{\text{Bun}_G} h)_{\text{Bun}_G} \in \text{Shv}(\text{Bun}_G \times \text{Bun}_G) \]

Following Sect. C.4.7, we denote the resulting functor

\[ \text{Shv}(\text{Bun}_G)_{\text{co}} \to \text{Shv}(\text{Bun}_G) \]

by \(\text{Mir}_{\text{Bun}_G}\).

2.8.2. First, we claim:

**Lemma 2.8.3.** For a finite set \(I\) and \(V \in \text{Rep}(\hat{\mathcal{G}})^{\otimes I}\), the diagram

\[
\begin{array}{ccc}
\text{Shv}(\text{Bun}_G)_{\text{co}} & \xrightarrow{H(V,-)_{\text{co}}} & \text{Shv}(\text{Bun}_G \times X^I)_{\text{co}} \\
\text{Mir}_{\text{Bun}_G} & & \downarrow \text{Mir}_{\text{Bun}_G} \otimes \text{Id}_{X^I} \\
\text{Shv}(\text{Bun}_G) & \xrightarrow{H(V,-)} & \text{Shv}(\text{Bun}_G \times X^I)
\end{array}
\]

commutes, as functors defined by kernels.

**Proof.** Using Sect. 1.1.5 and (2.2), we obtain that both circuits of the diagram are functors defined by the kernel

\[(h \times h \times s)_{\text{Sat}_I(V)}[-2 \dim(\text{Bun}_G)] \in \text{Shv}(\text{Bun}_G \times \text{Bun}_G \times X^I).\]

\(\square\)

**Corollary 2.8.4.** The functor \(\text{Mir}_{\text{Bun}_G}\) intertwines the actions of \(\text{Rep}(\hat{\mathcal{G}})_{\text{Ran}}\) on \(\text{Shv}(\text{Bun}_G)_{\text{co}}\) and \(\text{Shv}(\text{Bun}_G)\), respectively, as functors defined by kernels.

In particular, we obtain:

**Corollary 2.8.5.** The functor \(\text{Mir}_{\text{Bun}_G}\) intertwines the action of \(\mathcal{P}_{\text{co}}\) on \(\text{Shv}(\text{Bun}_G)_{\text{co}}\) and the action of \(\mathcal{P}\) on \(\text{Shv}(\text{Bun}_G)\).

2.8.6. We now claim:

**Proposition 2.8.7.** The functor \(\text{Mir}_{\text{Bun}_G}\) sends \(\text{Shv}_{\text{Nilp}}(\text{Bun}_G)_{\text{co}}\) to \(\text{Shv}_{\text{Nilp}}(\text{Bun}_G)\).

**Proof.** Follows from Corollary 2.8.3 combined with Corollary 2.6.8 and Theorem 1.6.4 (for \(Z = \text{pt}\)). \(\square\)

2.8.8. Let \(\mathfrak{U} \xhookrightarrow{j} \text{Bun}_G\) be a quasi-compact open substack. Consider the corresponding endofunctor

\[ \text{Mir}_{\mathfrak{U}} : \text{Shv}(\mathfrak{U}) \to \text{Shv}(\mathfrak{U}), \]

see Sect. B.4.1.

Assume that \(\mathfrak{U}\) is cotruncative. Then, according to Sect. C.4.8, we have

\[
\text{Mir}_{\text{Bun}_G} \circ j_{\ast,\text{co}} \simeq j_! \circ \text{Mir}_{\mathfrak{U}}.
\]
2.8.9. We claim:

**Proposition 2.8.10.** Assume that \( U \) is \( \text{Nilp}^{\text{-cotruncative}} \). Then the functor \( \text{Mir}_U \) sends \( \text{Shv}_{\text{Nilp}}(U) \to \text{Shv}_{\text{Nilp}}(U) \).

**Proof.** We have

\[
\mathcal{F} \in \text{Shv}_{\text{Nilp}}(U) \iff j_\ast, co(\mathcal{F}) \in \text{Shv}_{\text{Nilp}}(\text{Bun}_G)_{\text{co}}
\]

and

\[
\text{Mir}_U(\mathcal{F}) \in \text{Shv}_{\text{Nilp}}(U) \iff j \circ \text{Mir}_U(\mathcal{F}) \in \text{Shv}_{\text{Nilp}}(\text{Bun}_G).
\]

Using (2.22), the assertion follows from Proposition 2.8.8. \( \square \)

2.9. **The miraculous property of \( \text{Bun}_G \).**

2.9.1. We now recall that in the paper \( [\text{Ga1}] \) it was shown that the functor \( \text{Mir}_{\text{Bun}_G} : \text{D-mod}(\text{Bun}_G)_{\text{co}} \to \text{D-mod}(\text{Bun}_G) \) is an equivalence. However, the proof in loc. cit. applies in any sheaf-theoretic context. Furthermore, the same proof shows that \( \text{Id}_Z \boxtimes \text{Mir}_{\text{Bun}_G} : \text{Shv}(Z \times \text{Bun}_G)_{\text{co}} \to \text{Shv}(Z \times \text{Bun}_G)_{\text{co}} \) is an equivalence for any algebraic stack \( Z \).

This implies that \( \text{Mir}_{\text{Bun}_G} \) admits an inverse as a functor defined by a kernel. Hence, it makes sense to consider

\[
(\text{Id}_Z \boxtimes \text{Mir}_{\text{Bun}_G}^{-1}) : \text{Shv}(Z \times \text{Bun}_G) \to \text{Shv}(Z \times \text{Bun}_G)_{\text{co}},
\]

for any algebraic stack \( Z \). In other words, \( \text{Bun}_G \) is *miraculous* in the terminology of Sect. C.5.1.

2.9.2. Following Sect. C.5.4, let us denote by \( \text{ev}_{\text{Mir}_{\text{Bun}_G}} \) the composite functor

\[
(2.23) \quad \text{Shv}(\text{Bun}_G) \otimes \text{Shv}(\text{Bun}_G) \xrightarrow{\text{Id} \otimes \text{Mir}_{\text{Bun}_G}^{-1}} \text{Shv}(\text{Bun}_G) \otimes \text{Shv}(\text{Bun}_G)_{\text{co}} \xrightarrow{\text{ev}_{\text{Bun}_G}} \text{Vect},
\]

Combining Sects. 2.9.1 and 2.6.1 we obtain that the functor (2.23) is the counit of a self-duality on \( \text{Shv}(\text{Bun}_G) \).

We refer to it as the *miraculous self-duality* of \( \text{Shv}(\text{Bun}_G) \).

2.9.3. We claim:

**Proposition 2.9.4.** The functors \( \text{Mir}_{\text{Bun}_G} \) and \( \text{Mir}_{\text{Bun}_G}^{-1} \) send the subcategories \( \text{Shv}_{\text{Nilp}}(\text{Bun}_G)_{\text{co}} \subset \text{Shv}(\text{Bun}_G)_{\text{co}} \) and \( \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \subset \text{Shv}(\text{Bun}_G) \) to one another.

**Proof.** Follows from the fact that the functor \( \text{Mir}_{\text{Bun}_G} \) intertwines the functors \( P \) and \( P_{\text{co}} \) (see Corollary 2.8.5), combined with Theorem 1.6.4 (for \( Z = \text{pt} \)) and Corollary 2.6.8. \( \square \)

**Corollary 2.9.5.** The functor \( \text{Mir}_{\text{Bun}_G} \) induces an equivalence

\[
\text{Shv}_{\text{Nilp}}(\text{Bun}_G)_{\text{co}} \to \text{Shv}_{\text{Nilp}}(\text{Bun}_G).
\]

2.9.6. Combining Corollaries 2.9.5 with 2.6.5 we obtain:

**Corollary 2.9.7.** The composition

\[
\text{Shv}_{\text{Nilp}}(\text{Bun}_G) \otimes \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \leftarrow \text{Shv}(\text{Bun}_G) \otimes \text{Shv}(\text{Bun}_G) \xrightarrow{\text{ev}_{\text{Bun}_G}} \text{Vect}
\]

is the counit of a self-duality on \( \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \).

We refer to the self-duality of \( \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \) established in Corollary 2.9.7 as the *miraculous self-duality* of \( \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \).
2.9.8. The fact that Bun$_G$ is miraculous formally implies that any cotruncative quasi-compact open substack $U \subset$ Bun$_G$ is also miraculous, see Sect. C.5.2.

In particular, the functor
\[ \text{Mir}_U : \text{Shv}(U) \to \text{Shv}(U) \]
is invertible also as a functor defined by a kernel, and so we can consider the functor
\[ (\text{Id}_Z \otimes \text{Mir}^{-1}_U) : \text{Shv}(Z \times U) \to \text{Shv}(Z \times U), \]
for any $Z$.

Similarly to Sect. 2.9.2, we obtain that the functor, denoted $\text{ev}^{\text{Mir}}_U$,
\[ \text{Shv}(U) \otimes \text{Shv}(U) \xrightarrow{\text{Id} \otimes \text{Mir}^{-1}_U} \text{Shv}(U) \otimes \text{Shv}(U) \xrightarrow{\text{ev}_U} \text{Vect} \]
is the counit of a self-duality on Shv(U).

2.9.9. We claim:

**Proposition 2.9.10.** For a quasi-compact Nilp-cotruncative $U \hookrightarrow$ Bun$_G$, the functor $\text{Mir}^{-1}_U$ sends Shv$_{\text{Nilp}}(U)$ to Shv$_{\text{Nilp}}(U)$.

**Proof.** Follows from Proposition 2.9.4 in the same way as Proposition 2.8.10 follows from Proposition 2.8.7. \qed

Finally, similarly to Corollary 2.9.7, we obtain:

**Corollary 2.9.11.** The composition
\[ \text{Shv}_{\text{Nilp}}(U) \otimes \text{Shv}_{\text{Nilp}}(U) \hookrightarrow \text{Shv}(U) \otimes \text{Shv}(U) \xrightarrow{\text{ev}_U^{\text{Mir}}} \text{Vect} \]
is the counit of a self-duality on Shv$_{\text{Nilp}}(U)$.

We refer to the self-duality of Shv$_{\text{Nilp}}(U)$ established in Corollary 2.9.7 as the miraculous self-duality of Shv$_{\text{Nilp}}(U)$.

3. The non-standard duality for Shv$_{\text{Nilp}}$(Bun$_G$)

In this section we will prove what can be considered as the main result of this paper, Theorem 3.2.2. It states that the category Shv$_{\text{Nilp}}$(Bun$_G$) is self-dual via a certain non-standard procedure.

The proof of this theorem will be rather straightforward, modulo the work done in [AGKRRV].

We will then connect this non-standard duality with the miraculous self-duality of Sect. 2.9.6.

3.1. The projection of the pseudo-diagonal object. In this subsection we study the object obtained by applying the projector $P = P_{\text{Bun}_G, \text{Nilp}}$ to the pseudo-diagonal object on Bun$_G$.

3.1.1. Let $P_{\text{Bun}_G, \text{Nilp}}$ is as in Sect. 1.6.1, i.e., this is the idempotent on Shv(Bun$_G$) that corresponds to the precomposition of the fully faithful embedding
\[ \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \hookrightarrow \text{Shv}(\text{Bun}_G) \]
with its left inverse.

Recall that $P_{\text{Bun}_G, \text{Nilp}}$ is a functor defined by a kernel. In particular, it makes sense to consider the endofunctors of Shv(Bun $\times$ Bun$_G$) given by
\[ P_{\text{Bun}_G, \text{Nilp}} \boxtimes \text{Id}_{\text{Bun}_G}, \ P_{\text{Bun}_G, \text{Nilp}} \boxtimes P_{\text{Bun}_G, \text{Nilp}} \text{ and } \text{Id}_{\text{Bun}_G} \boxtimes P_{\text{Bun}_G, \text{Nilp}}. \]
3.1.2. Let
\[ \text{ps-u}_{\text{Bun}_G, \text{Nilp}} \in \text{Shv} (\text{Bun}_G \times \text{Bun}_G) \]
denote the object
\[ (P_{\text{Bun}_G, \text{Nilp}} \boxtimes P_{\text{Bun}_G, \text{Nilp}})(\text{ps-u}_{\text{Bun}_G}) , \]
where
\[ \text{ps-u}_{\text{Bun}_G} := (\Delta_{\text{Bun}_G})! (e_{\text{Bun}_G}) , \]
see Sect. 2.8.1.
Note that by Corollary 1.6.5, the functor \( P_{\text{Bun}_G, \text{Nilp}} \boxtimes P_{\text{Bun}_G, \text{Nilp}} \) is the idempotent corresponding to the embedding
\[ \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \otimes \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \to \text{Shv}(\text{Bun}_G \times \text{Bun}_G) . \]
Hence, we can think of \( \text{ps-u}_{\text{Bun}_G, \text{Nilp}} \) as an object of
\[ \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \otimes \text{Shv}_{\text{Nilp}}(\text{Bun}_G) . \]

3.1.3. We claim:

**Proposition 3.1.4.** We have canonical isomorphisms
\[ (P_{\text{Bun}_G, \text{Nilp}} \otimes \text{Id}_{\text{Bun}_G})(\text{ps-u}_{\text{Bun}_G}) \simeq (P_{\text{Bun}_G, \text{Nilp}} \otimes P_{\text{Bun}_G, \text{Nilp}})(\text{ps-u}_{\text{Bun}_G}) \simeq (\text{Id}_{\text{Bun}_G} \otimes P_{\text{Bun}_G, \text{Nilp}})(\text{ps-u}_{\text{Bun}_G}) . \]

**Proof.** Repeats the proof of Proposition 2.2.2, taking into account Proposition 1.1.8. □

3.2. **The main result.** In this subsection we state and prove the main result of this paper, Theorem 3.2.2.

3.2.1. Let \( \text{ev}_{\text{Bun}_G} \) denote the functor
\[ \text{Shv}(\text{Bun}_G) \otimes \text{Shv}(\text{Bun}_G) \to \text{Vect} , \quad \mathcal{F}_1, \mathcal{F}_2 \mapsto C_c (\text{Bun}_G, \mathcal{F}_1 \otimes \mathcal{F}_2) . \]

The main result of this paper is:

**Theorem 3.2.2.** The functors
\[ \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \otimes \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \hookrightarrow \text{Shv}(\text{Bun}_G) \otimes \text{Shv}(\text{Bun}_G) \xrightarrow{\text{ev}_{\text{Bun}_G}^{-1}} \text{Vect} \]
and
\[ \text{ps-u}_{\text{Bun}_G, \text{Nilp}} \in \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \otimes \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \]
define a duality datum.

**Remark 3.2.3.** As we shall see, the proof of Theorem 3.2.2 will be essentially a formal manipulation (modulo Proposition 3.1.4), given the highly non-trivial fact that \( \text{ps-u}_{\text{Bun}_G, \text{Nilp}} \) actually belongs to \( \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \otimes \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \). The latter result uses the full strength of the main results of [AGKRRV].

**Remark 3.2.4.** The peculiarity of Theorem 3.2.2 is that is straddles two different paradigms in which duality takes place:

In the constructible sheaf-theoretic contexts and for D-modules, for a scheme \( Y \) we have the perfect pairing
\[ \text{ev}_Y : \text{Shv}(Y) \otimes \text{Shv}(Y) \to \text{Vect} , \quad \mathcal{F}_1, \mathcal{F}_2 \mapsto C (Y, \mathcal{F}_1 \otimes \mathcal{F}_2) , \]
where the corresponding contravariant equivalence
\[ (\text{Shv}(Y))^\vee \to \text{Shv}(Y)^\vee \]
is given by Verdier duality. (The same is valid when \( Y \) is a Verdier-compatible quasi-compact algebraic stack, but we need to replace \( C (Y, -) \) by \( C^*_\bullet (Y, -) \).)
In the Betti context, for the category $\text{Shv}^{\text{all}}(\mathcal{Y})$ of all sheaves of $\mathbf{e}$-vector spaces (not necessarily constructible ones), we have a perfect pairing

$$
(3.2) \quad \text{ev}^\mathcal{Y}_f : \text{Shv}^{\text{all}}(\mathcal{Y}) \otimes \text{Shv}^{\text{all}}(\mathcal{Y}) \to \text{Vect}, \quad \mathcal{F}_1, \mathcal{F}_2 \mapsto C_c(\mathcal{Y}, \mathcal{F}_1 \otimes \mathcal{F}_2).
$$

The content of Theorem 3.2.2 is that an analog of the latter pairing defines a self-duality on $\text{Shv}_{\text{Nilp}}(\text{Bun}_G)$. We will see that this non-standard duality actually coincides with the miraculous duality of Sect. 2.9.6.

**Remark 3.2.5.** It is easy to see that if $\mathcal{Y}$ is a smooth and proper scheme, then the pairing (3.2) does induce a perfect pairing

$$
\text{Shv}_{(0)}(\mathcal{Y}) \otimes \text{Shv}_{(0)}(\mathcal{Y}) \to \text{Vect},
$$

where $\text{Shv}_{(0)}(-)$ is the same as $\text{QLisse}(-)$.

Further, in Sect. 3.2.6 we will see that if $\mathcal{Y}$ is a (quasi-compact) stack with finitely many isomorphism classes of $k$-points, then (3.2) is a perfect perfect

$$
\text{Shv}(\mathcal{Y}) \otimes \text{Shv}(\mathcal{Y}) \to \text{Vect}.
$$

Thus, the pair $(\text{Bun}_G, \text{Nilp})$ behaves in a way analogous to both $(\mathcal{Y}, \{0\})$ for a smooth and proper $\mathcal{Y}$ and $(\mathcal{Y}, T^*\mathcal{Y})$ for $\mathcal{Y}$ with finitely many isomorphism classes of $k$-points.

In fact, this analogy is even stronger, see Sect. 4.5.8.

**3.2.6. Proof of Theorem 3.2.2** In order to prove Theorem 3.2.2 we need to show that for $\mathcal{F} \in \text{Shv}_{\text{Nilp}}(\text{Bun}_G)$, the object

$$
(3.3) \quad (\text{Id} \otimes \text{ev}^\mathcal{Y}_f)(\text{ps-u}_{\text{Bun}_G, \text{Nilp}} \otimes \mathcal{F}) \simeq (\text{Id} \otimes C_c(\text{Bun}_G, -)) \circ (\text{Id} \otimes (\Delta_{\text{Bun}_G})^*)(\text{ps-u}_{\text{Bun}_G, \text{Nilp}} \boxtimes \mathcal{F}) \in \text{Shv}_{\text{Nilp}}(\text{Bun}_G)
$$

identifies canonically with $\mathcal{F}$, where in the right-hand side of the above formula we consider $\text{ps-u}_{\text{Bun}_G, \text{Nilp}} \boxtimes \mathcal{F}$ as an object of the category

$$
\text{Shv}(\text{Bun}_G) \otimes \text{Shv}(\text{Bun}_G \times \text{Bun}_G).
$$

Using Proposition 3.1.4 we interpret $\text{ps-u}_{\text{Bun}_G, \text{Nilp}}$ as $(P_{\text{Bun}_G, \text{Nilp}} \boxtimes \text{Id}_{\text{Bun}_G})(\text{ps-u}_{\text{Bun}_G})$. We now use the fact that the functor $P_{\text{Bun}_G, \text{Nilp}}$ is codelined by a kernel, see Proposition 1.1.8.

Let us view $C_c(\text{Bun}_G, -)$ and $(\Delta_{\text{Bun}_G})^*$ also as functors codelined by kernels. Hence, we can rewrite the right-hand side in (3.3) as

$$
(\text{Id}_{\text{Bun}_G} \boxtimes C_c(\text{Bun}_G, -)) \circ (\text{Id}_{\text{Bun}_G} \boxtimes (\Delta_{\text{Bun}_G})^*) \circ (P_{\text{Bun}_G, \text{Nilp}} \boxtimes \text{Id}_{\text{Bun}_G} \boxtimes \text{Id}_{\text{Bun}_G})(\text{ps-u}_{\text{Bun}_G} \boxtimes \mathcal{F}),
$$

where we now view $\text{ps-u}_{\text{Bun}_G} \boxtimes \mathcal{F}$ as an object of $\text{Shv}(\text{Bun}_G \times \text{Bun}_G \times \text{Bun}_G)$.

We can further rewrite the above expression as

$$
P_{\text{Bun}_G, \text{Nilp}} \circ (\text{Id}_{\text{Bun}_G} \boxtimes C_c(\text{Bun}_G, -)) \circ (\text{Id}_{\text{Bun}_G} \boxtimes (\Delta_{\text{Bun}_G})^*)(\text{ps-u}_{\text{Bun}_G} \boxtimes \mathcal{F}).
$$

We note that we have, tautologically,

$$(\text{Id}_{\text{Bun}_G} \boxtimes C_c(\text{Bun}_G, -)) \circ (\text{Id}_{\text{Bun}_G} \boxtimes (\Delta_{\text{Bun}_G})^*)(\text{ps-u}_{\text{Bun}_G} \boxtimes \mathcal{F}) \simeq \mathcal{F}.
$$

Finally, we have

$$
P_{\text{Bun}_G, \text{Nilp}}(\mathcal{F}) \simeq \mathcal{F},
$$

since $\mathcal{F} \in \text{Shv}_{\text{Nilp}}(\text{Bun}_G)$.

\[\square\] Theorem 3.2.2

**3.3. Relation to the miraculous functor.** In this subsection, we will relate the non-standard duality of Theorem 3.2.2 with the miraculous duality of Sect. 2.9.6.
3.3.1. On the one hand, according to Theorem 3.2.2 we have a canonical identification
\[ \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \cong \text{Shv}_{\text{Nilp}}(\text{Bun}_G)^\vee. \]

On the other hand, according to Corollary 2.6.5 we have a canonical identification
\[ \text{Shv}_{\text{Nilp}}(\text{Bun}_G)_{\text{co}} \cong \text{Shv}_{\text{Nilp}}(\text{Bun}_G)^\vee. \]

3.3.2. We claim:

**Theorem 3.3.3.** The identifications (3.4) and (3.5) are intertwined by the miraculous functor
\[ \text{Mir}_{\text{Bun}_G} : \text{Shv}_{\text{Nilp}}(\text{Bun}_G)_{\text{co}} \sim \text{Shv}_{\text{Nilp}}(\text{Bun}_G). \]

**Proof.** It suffices to show that the functor
\[ \text{Id}_{\text{Shv}_{\text{Nilp}}(\text{Bun}_G)} \otimes \text{Mir}_{\text{Bun}_G} : \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \otimes \text{Shv}_{\text{Nilp}}(\text{Bun}_G)_{\text{co}} \rightarrow \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \otimes \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \]

sends the unit of the (3.5) duality to the unit of the (3.4) duality.

By Proposition 2.7.6, the unit of the (3.5) duality is the object
\[ u_{\text{Bun}_G, \text{Nilp}, \text{co}^2} \cong (P \boxtimes \text{Id}_{\text{Bun}_G})(u_{\text{Bun}_G, \text{co}^2}), \]

and the unit of the duality (3.4) is the object
\[ \text{ps-u}_{\text{Bun}_G, \text{Nilp}} \cong (P \boxtimes \text{Id}_{\text{Bun}_G})(\text{ps-u}_{\text{Bun}_G}). \]

Hence, it suffices to establish an isomorphism
\[ (\text{Id}_{\text{Bun}_G} \boxtimes \text{Mir}_{\text{Bun}_G}) \circ (P \boxtimes \text{Id}_{\text{Bun}_G})(u_{\text{Bun}_G, \text{co}^2}) \cong (P \boxtimes \text{Id}_{\text{Bun}_G})(\text{ps-u}_{\text{Bun}_G}). \]

The latter isomorphism follows from the tautological isomorphism
\[ (\text{Id}_{\text{Bun}_G} \boxtimes \text{Mir}_{\text{Bun}_G})(u_{\text{Bun}_G, \text{co}^2}) \cong \text{ps-u}_{\text{Bun}_G}. \]

\[ \square \]

3.3.4. From Theorem 3.3.3 we obtain:

**Corollary 3.3.5.** The diagram
\[
\begin{array}{ccc}
\text{Shv}_{\text{Nilp}}(\text{Bun}_G) \otimes \text{Shv}_{\text{Nilp}}(\text{Bun}_G)_{\text{co}} & \longrightarrow & \text{Shv}(\text{Bun}_G) \otimes \text{Shv}(\text{Bun}_G)_{\text{co}} \\
\text{ev}_{\text{Bun}_G} & & \text{ev}_{\text{Bun}_G} \\
\text{Id} & \downarrow & \text{Id} \\
\text{Shv}_{\text{Nilp}}(\text{Bun}_G) \oplus \text{Shv}_{\text{Nilp}}(\text{Bun}_G) & \longrightarrow & \text{Shv}(\text{Bun}_G) \oplus \text{Shv}(\text{Bun}_G) \\
\end{array}
\]

commutes.

3.3.6. From Corollary 3.3.5 we obtain:

**Corollary 3.3.7.** The pairing
\[ \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \otimes \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \leftrightarrow \text{Shv}(\text{Bun}_G) \otimes \text{Shv}(\text{Bun}_G) \]

identifies canonically with
\[ \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \otimes \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \leftrightarrow \text{Shv}(\text{Bun}_G) \otimes \text{Shv}(\text{Bun}_G) \]

In other words, this corollary says that the non-standard duality of Theorem 3.2.2 identifies canonically with the miraculous duality of Sect. 2.9.6.

3.4. **Pairings against sheaves with nilpotent singular support.** In this subsection we will amplify the assertion of Corollary 3.3.7.
3.4.1. We claim:

**Theorem 3.4.2.** The pairings $\text{ev}^l_{\text{Bun}_G}$ and $\text{ev}^\text{Mir}_{\text{Bun}_G}$

$$\text{Shv}(\text{Bun}_G) \otimes \text{Shv}(\text{Bun}_G) \to \text{Vect}$$

agree on the subcategories

$$\text{Shv}_{\text{Nilp}}(\text{Bun}_G) \otimes \text{Shv}(\text{Bun}_G) \subset \text{Shv}(\text{Bun}_G) \otimes \text{Shv}(\text{Bun}_G) \supset \text{Shv}(\text{Bun}_G) \otimes \text{Shv}_{\text{Nilp}}(\text{Bun}_G).$$

**Remark 3.4.3.** We can reformulate Theorem 3.4.2 as saying that the following diagrams commute:

\[
\begin{array}{ccc}
\text{Shv}(\text{Bun}_G) \otimes \text{Shv}_{\text{Nilp}}(\text{Bun}_G) & \rightarrow & \text{Shv}(\text{Bun}_G) \otimes \text{Shv}(\text{Bun}_G) \\
\text{id} \otimes \text{Mir}_{\text{Bun}_G} & \downarrow & \downarrow \text{id} \\
\text{Shv}(\text{Bun}_G) \otimes \text{Shv}_{\text{Nilp}}(\text{Bun}_G) & \rightarrow & \text{Shv}(\text{Bun}_G) \otimes \text{Shv}(\text{Bun}_G)
\end{array}
\]

and

\[
\begin{array}{ccc}
\text{Shv}_{\text{Nilp}}(\text{Bun}_G) \otimes \text{Shv}(\text{Bun}_G) & \rightarrow & \text{Shv}(\text{Bun}_G) \otimes \text{Shv}(\text{Bun}_G) \\
\text{id} \otimes \text{Mir}_{\text{Bun}_G} & \downarrow & \downarrow \text{id} \\
\text{Shv}_{\text{Nilp}}(\text{Bun}_G) \otimes \text{Shv}(\text{Bun}_G) & \rightarrow & \text{Shv}(\text{Bun}_G) \otimes \text{Shv}(\text{Bun}_G)
\end{array}
\]

I.e., unlike the commutative diagram in Corollary 3.3.5, we only need one of the factors to have singular support in Nilp.

3.4.4. The rest of this subsection is devoted to the proof of Theorem 3.4.2. Since both pairings are swap-equivariant (see Sects. 3.4.3 and 3.5.3), it suffices to show that the two pairings agree on

$$\text{Shv}_{\text{Nilp}}(\text{Bun}_G) \otimes \text{Shv}(\text{Bun}_G) \subset \text{Shv}(\text{Bun}_G) \otimes \text{Shv}(\text{Bun}_G).$$

3.4.5. By Proposition 2.6.7(i), for $\mathcal{F}_1 \in \text{Shv}_{\text{Nilp}}(\text{Bun}_G)$ and $\mathcal{F}_2 \in \text{Shv}(\text{Bun}_G)$, we have

$$\text{ev}^\text{Mir}_{\text{Bun}_G}(\mathcal{F}_1, \mathcal{F}_2) := \text{ev}_{\text{Bun}_G}^l(\text{Mir}^{-1}_\text{Bun}_G(\mathcal{F}_1), \mathcal{F}_2) \simeq \text{ev}_{\text{Bun}_G}^l(\text{Mir}^{-1}_\text{Bun}_G(\mathcal{F}_1), \mathcal{P}(\mathcal{F}_2)),$$

and by Corollary 3.3.7 the latter is canonically isomorphic to

$$\text{ev}^l_{\text{Bun}_G}(\mathcal{F}_1, \mathcal{P}(\mathcal{F}_2)).$$

Hence, in order to prove Theorem 3.4.2 it suffices to show that for $\mathcal{F}_1 \in \text{Shv}_{\text{Nilp}}(\text{Bun}_G)$ and $\mathcal{F}_2 \in \text{Shv}(\text{Bun}_G)$, we have a canonical isomorphism

$$\text{ev}^l_{\text{Bun}_G}(\mathcal{F}_1, \mathcal{F}_2) \simeq \text{ev}^l_{\text{Bun}_G}(\mathcal{F}_1, \mathcal{P}(\mathcal{F}_2)).$$

This would follow from the next assertion:

**Proposition 3.4.6.** For any $\mathcal{F}_1, \mathcal{F}_2 \in \text{Shv}(\text{Bun}_G)$, we have a canonical isomorphism

$$\text{ev}^l_{\text{Bun}_G}(\mathcal{P}(\mathcal{F}_1), \mathcal{F}_2) \simeq \text{ev}^l_{\text{Bun}_G}(\mathcal{F}_1, \mathcal{P}(\mathcal{F}_2)).$$

3.4.7. **Proof of Proposition 3.4.6.** Recall that $\mathcal{P} = H_{\text{R}}$ for $\mathcal{R}$ as in Sect. 1.6.1. Recall also that $\mathcal{R} \simeq \mathbb{R}^*$. Hence, Proposition 3.4.6 follows from the next general assertion:

**Lemma 3.4.8.** For $\mathcal{F}_1, \mathcal{F}_2 \in \text{Shv}(\text{Bun}_G)$ and $\mathcal{V} \in \text{Rep}(\hat{G})_{\text{Ran}}$, we have a canonical isomorphism

$$\text{ev}^l_{\text{Bun}_G}(H_{\mathcal{V}}(\mathcal{F}_1), \mathcal{F}_2) \simeq \text{ev}^l_{\text{Bun}_G}(\mathcal{F}_1, H_{\mathcal{V}}(\mathcal{F}_2)).$$
Proposition 1.4.6]). In particular, we have a canonical fully faithful embedding.

3.5.1. Let $\mathcal{F}_1, \mathcal{F}_2 \in \text{Shv}(\text{Bun}_G)$, we have

$$eV_{\text{Bun}}(\mathcal{F}_1, \mathcal{F}_2) \cong \text{Hom}_{\text{Shv}(\text{Bun}_G \times \text{Bun}_G)}(\mathcal{F}_1 \boxtimes \mathcal{F}_2, u_{\text{Bun}}^{\text{naive}}).$$

Recall (see [AGKRRV Sect. 11.3]) that the category $\text{Rep}(\hat{G})_{\text{Ran}}$ is rigid. Let $V^\vee$ denote the monoidal dual of $V$.

We have

\[ \text{Hom}_{\text{Shv}(\text{Bun}_G \times \text{Bun}_G)}(\mathcal{F}_1 \boxtimes \mathcal{F}_2, u_{\text{Bun}}^{\text{naive}}) \cong \text{Hom}_{\text{Shv}(\text{Bun}_G \times \text{Bun}_G)}(\mathcal{F}_1 \boxtimes \mathcal{F}_2, (H_{V^\vee} \boxtimes \text{Id}_{\text{Bun}_G})(u_{\text{Bun}}^{\text{naive}})) \]

and

\[ \text{Hom}_{\text{Shv}(\text{Bun}_G \times \text{Bun}_G)}(\mathcal{F}_1 \boxtimes H_{V^\vee}(\mathcal{F}_2), u_{\text{Bun}}^{\text{naive}}) \cong \text{Hom}_{\text{Shv}(\text{Bun}_G \times \text{Bun}_G)}(\mathcal{F}_1 \boxtimes \mathcal{F}_2, (\text{Id}_{\text{Bun}_G} \boxtimes H_{V^\vee})(u_{\text{Bun}}^{\text{naive}})). \]

Note also that $V^\vee \cong V^{\tau V}$.

Hence, the assertion of the lemma follows from the isomorphism

\[ (H_{V^\vee} \boxtimes \text{Id}_{\text{Bun}_G})(u_{\text{Bun}}^{\text{naive}}) \cong (\text{Id}_{\text{Bun}_G} \boxtimes H_{V^\vee})(u_{\text{Bun}}^{\text{naive}}) \]

of Proposition 2.1.6.

3.5. Pairings of cuspidal sheaves. In this subsection we will discuss an application of Corollary 3.3.5 to two ways to define pairings on cuspidal objects of $\text{Shv}(\text{Bun}_G)$.

3.5.1. Let

$$\text{Shv}(\text{Bun}_G)_{\text{cusp}} \hookrightarrow \text{Shv}(\text{Bun}_G)$$

be the full subcategory of cuspidal objects, see [DrGa2 Sect. 1.4].

Recall that every object of $\text{Shv}(\text{Bun}_G)_{\text{cusp}}$ is a clean extension from a particular open substack $\mathfrak{u}_0 \subset \text{Bun}_G$, whose intersection with every connected component of $\text{Bun}_G$ is quasi-compact (see [DrGa2 Proposition 1.4.6]). In particular, we have a canonical fully faithful embedding

$$\text{Shv}(\text{Bun}_G)_{\text{cusp}} \hookrightarrow \text{Shv}(\text{Bun}_G)_{\text{co}}.$$

Furthermore, in [Ga1 Theorem 2.2.7 and Corollary 3.3.2], the following isomorphism is established:

\[ \text{Mir}_{\text{Bun}_G} \circ \mathfrak{J}_{\text{co}} \simeq \mathfrak{j}([-2(\dim(\text{Bun}_G)) - \dim(Z_G))], \]

where $\dim(Z_G)$ is the center of $G$.

3.5.2. Combining (3.6) with Corollary 3.3.3, we obtain

**Corollary 3.5.3.** Let $\mathcal{F}_1, \mathcal{F}_2 \in \text{Shv}_{\text{Nilp}}(\text{Bun}_G)$ be two objects with

$$\mathcal{F}_1 = \mathfrak{j}(\mathcal{F}_{1, \text{cusp}}), \quad \mathcal{F}_{1, \text{cusp}} \in \text{Shv}(\text{Bun}_G)_{\text{cusp}}.$$

Then there is a natural isomorphism

$$C_\mathfrak{i}(\text{Bun}_G, \mathcal{F}_1 \boxtimes \mathcal{F}_2)[-2(\dim(\text{Bun}_G)) - \dim(Z_G))] \simeq C_{\mathfrak{j}}(\text{Bun}_G, \mathfrak{J}_{\text{co}}(\mathcal{F}_{1, \text{cusp}}) \boxtimes \mathcal{F}_2).$$

**Remark 3.5.4.** Corollary 3.5.3 justifies a degree of agnosticism that some authors used when considering pairings on automorphic sheaves: do we want to use

$$C_\mathfrak{i}(\text{Bun}_G, - \boxtimes -),$$

which is more natural in the Betti setting, or

$$C_{\mathfrak{j}}(\text{Bun}_G, - \boxtimes -),$$

which is more natural in the de Rham setting?
Now, Corollary 3.5.3 says that as long as one of the objects is cuspidal, the two give the same result (up to a cohomological shift).

Remark 3.5.5. Similarly to Remark 3.4.3 for the validity of Corollary 3.5.3 it is sufficient to require that only one of the objects $\mathcal{F}_1, \mathcal{F}_2$ have nilpotent singular support.

3.6. **Restricting to quasi-compact substacks of $\text{Bun}_G$.** In this subsection we will discuss a variant of Corollary 3.3.5 for quasi-compact open substacks of $\text{Bun}_G$.

3.6.1. Let $U_j \hookrightarrow \text{Bun}_G$ be a Nilp-cotruncative quasi-compact open substack $\text{Bun}_G$.

Set $\text{ps}_u(U_j, \text{Nilp}) := \left( j \times j \right)^* (\text{ps}_u(\text{Bun}_G, \text{Nilp})) \in \text{Shv}(U \times U).

Since $\text{ps}_u(\text{Bun}_G, \text{Nilp})$ belongs to $\text{Shv}_\text{Nilp}(\text{Bun}_G) \otimes \text{Shv}_\text{Nilp}(\text{Bun}_G)$, we obtain that $\text{ps}_u(U_j, \text{Nilp})$ belongs to the full subcategory $\text{Shv}_\text{Nilp}(U) \otimes \text{Shv}_\text{Nilp}(U) \subset \text{Shv}(U \times U)$.

3.6.2. Let $\text{ev}^i_U : \text{Shv}(U) \otimes \text{Shv}(U) \rightarrow \text{Vect}$ denote the functor $F_1 \otimes F_2 \mapsto \text{ev}^i_U(F_1, F_2, \text{Nilp})$.

From Theorem 3.2.2, we obtain:

**Corollary 3.6.3.** The functors $\text{Shv}_\text{Nilp}(U) \otimes \text{Shv}_\text{Nilp}(U) \rightarrow \text{Shv}(U) \otimes \text{Shv}(U) \xrightarrow{\text{ev}^i_U} \text{Vect}$ and $\text{ps}_u(U, \text{Nilp}) \in \text{Shv}_\text{Nilp}(U) \otimes \text{Shv}_\text{Nilp}(U)$ define a duality datum.

**Proof.** We need to show that for $\mathcal{F} \in \text{Shv}_\text{Nilp}(U)$,

\[(\text{ev}^i_U \otimes \text{Id})(\mathcal{F} \otimes \text{ps}_u(U, \text{Nilp})) \simeq \mathcal{F}.
\]

We have

\[(\text{ev}^i_U \otimes \text{Id})(\mathcal{F} \otimes \text{ps}_u(U, \text{Nilp})) = (\text{ev}^i_U \otimes \text{Id})(\mathcal{F} \otimes (j^* \otimes j^*)(\text{ps}_u(\text{Bun}_G, \text{Nilp}))) \simeq (\text{ev}^i_{\text{Bun}_G} \otimes \text{Id})(j_!(\mathcal{F}) \otimes (\text{Id} \otimes j^*)(\text{ps}_u(\text{Bun}_G, \text{Nilp}))) \simeq j^* \left( ((\text{ev}^i_{\text{Bun}_G} \otimes \text{Id})(j_!(\mathcal{F}) \otimes \text{ps}_u(\text{Bun}_G, \text{Nilp})) \right)^\text{Theorem 3.2.2} \simeq j^* \circ j_!(\mathcal{F}) \simeq \mathcal{F}.
\]

3.6.4. Parallel to Theorem 3.3.3 we have:

**Corollary 3.6.5.** Suppose that $U$ is universally Nilp-cotruncative (in particular, the category $\text{Shv}_\text{Nilp}(U)$ is self-dual via Verdier duality). Then:

(a) The two identifications $\text{Shv}_\text{Nilp}(U) \xrightarrow{\sim} (\text{Shv}_\text{Nilp}(U))^\vee$ of Proposition 3.3.5 and Corollary 3.6.3 respectively, are intertwined by the miraculous functor $\text{Mir}_U$.

(b) The diagram $\text{Shv}_\text{Nilp}(U) \otimes \text{Shv}_\text{Nilp}(U) \xrightarrow{\text{Id} \otimes \text{Mir}_U} \text{Shv}(U) \otimes \text{Shv}(U) \xrightarrow{\text{ev}^i_U} \text{Vect} \xrightarrow{\text{Id}} \text{Shv}_\text{Nilp}(U) \otimes \text{Shv}_\text{Nilp}(U) \xrightarrow{\text{ev}^i_U} \text{Vect}$.
commutes.

(c) The self-duality of point (a) identifies with the miraculous self-duality of Corollary 2.9.11.

Proof. Point (a) follows in the same way as Theorem 3.3.3. Point (b) follows formally from point (a). Point (c) follows formally from point (b).

\[ \square \]

3.6.6. Finally, we claim that we have the following analog of Theorem 3.4.2:

**Corollary 3.6.7.** The pairings \( ev^U_{\text{nil}} \) and \( ev^\text{Mir}_U \)

\[ \text{Shv}(U) \otimes \text{Shv}(U) \rightarrow \text{Vect} \]

agree on the subcategory \( \text{Shv}_{\text{nilp}}(U) \otimes \text{Shv}(U) \subset \text{Shv}(U) \otimes \text{Shv}(U) \).

**Proof.** Consider objects \( F_1 \in \text{Shv}_{\text{nilp}}(U) \) and \( F_2 \in \text{Shv}(U) \). We have

\[
ev^U_{\text{nil}}(F_1, F_2) \cong ev^\text{Bun}_G(j_!(F_1), j_!(F_2)) \cong ev^\text{Bun}_G(j_!(F_1), \text{Mir}^{-1}_U j_!(F_2)) \cong ev_U(j^* \circ j_!(F_1), \text{Mir}^{-1}_U(\text{Mir}^{-1}_U F_2)),
\]

as required.

\[ \square \]

**Remark 3.6.8.** As in Remark 3.4.3, we can reformulate Corollary 3.6.7 as saying that the following diagram commutes:

\[
\begin{array}{ccc}
\text{Shv}_{\text{nilp}}(U) \otimes \text{Shv}(U) & \longrightarrow & \text{Shv}(U) \otimes \text{Shv}(U) \\
\text{Id} \otimes \text{Mir}_U & & \downarrow \text{Id} \\
\text{Shv}_{\text{nilp}}(U) \otimes \text{Shv}(U) & \longrightarrow & \text{Shv}(U) \otimes \text{Shv}(U) \rightarrow ev^U_{\text{nil}} \rightarrow \text{Vect}.
\end{array}
\]

I.e., unlike the commutative diagram in Corollary 3.6.5(b), we only need one of the factors to have singular support in Nilp.

4. An intrinsic characterization of sheaves with nilpotent singular support

The goal of this section is to show that the full subcategory \( \text{Shv}_{\text{nilp}}(\text{Bun}_G)^{\text{constr}} \subset \text{Shv}(\text{Bun}_G) \) can be characterized by an intrinsic categorical property (see Theorem 4.5.4):

This theorem says that a constructible object of \( \text{Shv}(\text{Bun}_G) \) belongs to \( \text{Shv}_{\text{nilp}}(\text{Bun}_G) \) if and only if the functor

\[ \mathcal{F} \mapsto C_\lambda(\text{Bun}_G; \mathcal{F} \otimes \mathcal{F}') \]

admits a right adjoint as a functor defined by a kernel.\[ ^3 \]

4.1. Comparison of two pairings, revisited. In this subsection we revisit the isomorphism of Corollary 3.4.3(b) from the point of view of the material in Sect. 13.

\[ ^3 \text{In the case of D-modules, the above condition is equivalent to the functor in question preserving compactness.} \]
4.1.1. Recall that, according to Sect. B.4.5, if \( \mathcal{Y} \) is a quasi-compact algebraic stack, we have a natural transformation
\[
(4.1) \quad \text{ev}_Y^l(\mathcal{F}_1, \text{Mir}_Y(\mathcal{F}_2)) \rightarrow \text{ev}_Y(\mathcal{F}_1, \mathcal{F}_2)
\]
as functors
\[
\text{Shv}(\mathcal{Y}) \otimes \text{Shv}(\mathcal{Y}) \rightarrow \text{Vect}.
\]
Using Sect. C.4.8, this natural transformation automatically extends to the case when \( \mathcal{Y} \) is not necessarily quasi-compact, when we view both sides as functors
\[
\text{Shv}(\mathcal{Y}) \otimes \text{Shv}(\mathcal{Y}) \rightarrow \text{Vect}.
\]
In particular, we obtain a natural transformation
\[
(4.2) \quad \text{ev}_B^l(\mathcal{F}_1, \text{Mir}_{B^G}(\mathcal{F}_2)) \rightarrow \text{ev}_{B^G}(\mathcal{F}_1, \mathcal{F}_2), \quad \mathcal{F}_1 \in \text{Shv}(\text{Bun}_{B^G}), \quad \mathcal{F}_2 \in \text{Shv}(\text{Bun}_{B^G})_{\text{co}}.
\]

4.1.2. Recall now that according to Corollary 3.3.5, we have a canonical isomorphism
\[
(4.3) \quad C_\cdot c(B_{B^G}, \mathcal{F}_1^* \otimes \text{Mir}_{U}(\mathcal{F}_2)) \cong C_\cdot \Delta^*(B_{B^G}, \mathcal{F}_1 \otimes \mathcal{F}_2), \quad \mathcal{F}_1 \in \text{Shv}_{\text{Nilp}}(\text{Bun}_{B^G}), \quad \mathcal{F}_2 \in \text{Shv}_{\text{Nilp}}(\text{Bun}_{B^G})_{\text{co}}.
\]
The goal of this subsection is to prove the following assertion:

**Proposition 4.1.3.** For \( \mathcal{F}_1 \in \text{Shv}_{\text{Nilp}}(\text{Bun}_{B^G}) \) and \( \mathcal{F}_2 \in \text{Shv}_{\text{Nilp}}(\text{Bun}_{B^G})_{\text{co}} \), the map
\[
(4.2)
\]
identifies with the isomorphism
\[
(4.3)
\]

Note that Proposition 4.1.3 formally implies:

**Corollary 4.1.4.** Let \( \mathcal{U} \) be a universally Nilp-cotruncative quasi-compact open substack of \( \text{Bun}_{B^G} \). Then for \( \mathcal{F}_1, \mathcal{F}_2 \in \text{Shv}_{\text{Nilp}}(\mathcal{U}) \), the map
\[
(4.4) \quad \text{ev}_{\mathcal{U}}^l(\mathcal{F}_1, \text{Mir}_Y(\mathcal{F}_2)) \rightarrow \text{ev}_{\mathcal{U}}(\mathcal{F}_1, \mathcal{F}_2)
\]
of (4.1) identifies with the isomorphism of Corollary 3.6.5(b).

The rest of this subsection is devoted to the proof of Proposition 4.1.3.

4.1.5. According to Sect. B.4.5, the map (4.2) is given by
\[
\text{ev}_{B^G}^l(\mathcal{F}_1, \text{Mir}_{B^G}(\mathcal{F}_2)) \rightarrow \text{ev}_{B^G}(\mathcal{F}_1, \mathcal{F}_2), \quad \mathcal{F}_1 \in \text{Shv}(\text{Bun}_{B^G}), \quad \mathcal{F}_2 \in \text{Shv}(\text{Bun}_{B^G})_{\text{co}}.
\]
where
\[
\text{ev}_{B^G}^l := C_\cdot (\text{Bun}_{B^G}, -) \circ \Delta^*_B \text{ and } \text{ev}_{B^G} := C_\Delta (\text{Bun}_{B^G}, -) \circ \Delta^*_B,
\]
as functors
\[
\text{Shv}(\text{Bun}_{B^G} \times \text{Bun}_{B^G}) \rightarrow \text{Shv}(\text{pt}) \quad \text{and} \quad \text{Shv}(\text{Bun}_{B^G} \times \text{Bun}_{B^G})_{\text{co}} \rightarrow \text{Shv}(\text{pt}),
\]
codefined and defined by kernels, respectively.

Consider now the corresponding map
\[
\text{ev}_{B^G}^l(\mathcal{F}_1, \text{Mir}_{B^G}(\mathcal{F}_2)) \rightarrow \text{ev}_{B^G}(\mathcal{F}_1, \mathcal{F}_2), \quad \mathcal{F}_1 \in \text{Shv}_{\text{Nilp}}(\text{Bun}_{B^G}), \quad \mathcal{F}_2 \in \text{Shv}_{\text{Nilp}}(\text{Bun}_{B^G})_{\text{co}}.
\]
We claim that we have a commutative diagram
with vertical arrows being isomorphisms.

The existence of this commutative diagram implies the assertion of Proposition 4.1.3. Indeed, unwinding the definitions, we obtain that slanted bottom arrow in (4.5) fits into the commutative diagram

\[\text{ev}_{\text{Bun}_G}^l \circ (\text{Id}_{\text{Bun}_G \times \text{Bun}_G} \boxtimes \text{ev}_{\text{Bun}_G})(\mathcal{F}_1 \boxtimes \text{ps-u}_{\text{Bun}_G} \boxtimes \mathcal{F}_2)\]

4.1.6. Thus, it remains to establish the existence of (4.5).

Let $\mathcal{F}$ be an arbitrary object of Shv(\text{Bun}_G), and let $\mathcal{V}_1, \mathcal{V}_2$ be two objects of $\text{Rep}(\mathcal{G})_{\text{Ran}}$. Since the Hecke functors $H_{\mathcal{V}_1}$ and $H_{\mathcal{V}_2}$ are defined and codefined by kernels, we have a commutative diagram

\[C_{\bullet}(\text{Bun}_G, \mathcal{F}_1 \otimes \text{Mir}_{\text{Bun}_G}(\mathcal{F}_2))\]
Taking $\mathcal{F} = \text{ps-u}_{\text{Bun}_G}$ and $\mathcal{V}_1 = \mathcal{V}_2 = \mathbb{R}$, we obtain that the terms in (4.6) identify with the terms in (4.5), establishing the existence of the latter diagram. 

**Remark 4.1.7.** Note that the same argument shows that the map (4.2) identifies with the isomorphism of Theorem 3.4.2 when one of the objects $F_1$ and $F_2$ has nilpotent singular support. The same remark applies when instead of $\text{Bun}_G$ we take a universally contruncative open substack $U \subset \text{Bun}_G$.

### 4.2. Kernels defined by objects from $\text{Shv}_{\text{Nilp}}(\text{Bun}_G)$

In this subsection we will combine Proposition 4.1.3 with Theorem B.8.8 and deduce that (constructible) objects from $\text{Shv}_{\text{Nilp}}(\text{Bun}_G)$ admit right adjoints, when viewed as functors defined by kernels.

#### 4.2.1. Let $U \subset \text{Bun}_G$ be a universally Nilp-contruncative quasi-compact open substack. We are going to prove:

**Theorem 4.2.2.** Let $\mathcal{F} \in \text{Shv}(U)$ be an object contained in $\text{Shv}_{\text{Nilp}}(U)^{\text{constr}}$. Then $\mathcal{F}$ admits a right adjoint, viewed as a functor

$$\text{Shv}(U) \to \text{Vect},$$

defined by a kernel.

**Proof.** We will apply the criterion of Theorem B.8.8 to $\mathcal{F}$. (Note that the condition in Sect. B.8.3 holds automatically, since $\mathcal{V}_2 = \text{pt}$.)
By the equivalence (i) ⇔ (iii) in Theorem B.8.8 it suffices to show that for any $F' \in \text{Shv}_{\text{Nilp}}(U)$, the map

$$\text{ev}_U(F, F') \xrightarrow{\text{B.20}} \text{ev}_U(F, \text{Id}_{\text{L}}(F'))$$

is an isomorphism.

Recall that the stack $U$ is miraculous and that the functor $\text{Mir}_U$ defines a self-equivalence on $\text{Shv}_{\text{Nilp}}(U)$. Hence, we can assume that $F'$ is of the form $\text{Mir}_U(F'')$ for $F'' \in \text{Shv}_{\text{Nilp}}(U)$.

Consider the composition

$$\text{ev}_U(F, F') = \text{ev}_U(F, \text{Mir}_U(F'')) \xrightarrow{\text{B.20}} \text{ev}_U(F, \text{Id}_{\text{L}} \circ \text{Mir}_U(F'')) \xrightarrow{\text{B.21}} \text{ev}_U(F, F'').$$

By Sect. B.4.9, this composite map identifies with the map (4.4), and hence is an isomorphism by Corollary 4.1.4. Now, the map

$$\text{Id}_{\text{L}} \circ \text{Mir}_U(F'') \to F''$$

is an isomorphism since $U$ is miraculous (see Sect. B.7.4).

This implies that the map

$$\text{ev}_U(F, \text{Mir}_U(F'')) \xrightarrow{\text{B.20}} \text{ev}_U(F, \text{Id}_{\text{L}} \circ \text{Mir}_U(F''))$$

is an isomorphism, as required.

4.2.3. Let $F$ be as in Theorem 4.2.2. Let us describe the right adjoint $F^R$ explicitly. Namely, by Theorem B.8.8 we have

$$F^R \simeq \text{Id}_U \circ D_{\text{Verdier}}(F) \simeq \text{Mir}_U^{-1} \circ D_{\text{Verdier}}(F),$$

which also identifies with

$$D_{\text{Verdier}} \circ \text{Mir}_U(F),$$

see Remark B.8.2.

The unit of the adjunction is the map

$$u_U \to F \boxtimes (\text{Id}_U \circ D_{\text{Verdier}}(F))$$

(4.7)

obtained from the tautological map

$$\text{ps}-u_U \to F \boxtimes D_{\text{Verdier}}(F)$$

by applying the functor $\text{Id}_U \boxtimes \text{Id}_U'$.

The counit is the map

$$\text{ev}_U(F, \text{Mir}_U^{-1} \circ D_{\text{Verdier}}(F)) \xrightarrow{\text{Corollary 4.6.6 B)}} \simeq \text{ev}_U(F, \text{Mir}_U \circ \text{Mir}_U^{-1} \circ D_{\text{Verdier}}(F)) = \text{ev}_U(F, D_{\text{Verdier}}(F)) \to e,$$

where the last arrow is obtained by adjunction.

4.2.4. We now claim:

Corollary 4.2.5. For any $F \in \text{Shv}_{\text{Nilp}}(U)$, the functor

$$F : \text{Shv}_{\text{Nilp}}(U) \to \text{Vect}, \quad F' \mapsto C_\bullet(U, F \boxtimes F')$$

is defined and codefined by a kernel. The codefining object identifies canonically with

$$\mathcal{G} := \text{Mir}_U(F) \in \text{Shv}_{\text{Nilp}}(U) \subset \text{Shv}(U).$$
Proof. If \( \mathcal{F} \) is constructible, the assertion follows by combining Theorems 4.2.2 and 5.3.3 a).

We will now show how to reduce the assertion to the case when \( \mathcal{F} \) is constructible.

We need check that the map

\[
(\text{Id} \boxtimes G^!)(\mathcal{F}') \to \text{Id} \boxtimes \mathcal{F}(\mathcal{F}'),
\]

is an isomorphism for \( \mathcal{F}' \in \text{Shv}(\mathbb{Z} \times \mathcal{U}) \).

Note that both sides in (4.9) commute with colimits in \( \mathcal{F} \) and \( \mathcal{F}' \). Hence, we can assume that \( \mathcal{F}' \) is bounded above.

If \( \mathcal{F}' \) is bounded above, both sides of (4.9) maps isomorphically to the limits over \( n \in \mathbb{N} \) of the corresponding expressions, when we replace \( \mathcal{F} \) by \( \tau \geq -n(\mathcal{F}) \).

Thus, we can assume that \( \mathcal{F} \) is bounded below. In this case, \( \mathcal{F} \) is a colimit of constructible objects in \( \text{Shv}_{\text{nilp}}(\mathcal{U}) \). Hence, the assertion that (4.9) is an isomorphism follows from the constructible case, by passing to colimits in the \( \mathcal{F} \) argument.

\( \square \)

4.2.6. We now consider the situation with all of \( \text{Bun}_G \). We claim:

**Theorem 4.2.7.** Let \( \mathcal{F} \in \text{Shv}(\text{Bun}_G) \) be an object contained in \( \text{Shv}_{\text{nilp}}(\text{Bun}_G)_{\text{constr}} \). Then it admits a right adjoint, when viewed as a functor

\[
\text{Shv}(\text{Bun}_G)_{\text{co}} \to \text{Vect}.
\]

**Proof.** Set

\[
\mathcal{F}^R := \text{Mir}^{-1}_{\text{Bun}_G} \circ \mathcal{D} \text{Verdier}(\mathcal{F}) \in \text{Shv}(\text{Bun}_G)_{\text{co}},
\]

where we view \( \mathcal{D} \text{Verdier} \) as a functor

\[
(\text{Shv}(\text{Bun}_G)_{\text{constr}})^{\text{op}} \to \text{Shv}(\text{Bun}_G)_{\text{constr}}.
\]

We will show that \( \mathcal{F}^R \) provides an adjoint of \( \mathcal{F} \). The unit of the adjunction is a map

\[
(4.10) \quad u_{\text{Bun}_G, \text{co}^2} : \mathcal{F} \boxtimes \mathcal{F}^R \to \mathcal{F}^R
\]

constructed as follows:

For every universally \( \text{Nilp} \)-contruncative quasi-compact open substack \( \mathcal{U} \hookrightarrow \text{Bun}_G \), the \( j^* \boxtimes j^! \) restriction of (4.10) is the map

\[
(j^* \boxtimes j^!)(u_{\text{Bun}_G, \text{co}^2}) \simeq (\text{Id}_{\text{Bun}_G} \boxtimes j^*)(u_{\text{Bun}_G, \text{co}^2}) \simeq (\text{Id}_{\text{Bun}_G} \boxtimes j^!) \circ (\text{Id}_{\mathcal{U}} \boxtimes j_{\mathcal{U}, \text{co}})(u_{\mathcal{U}}) \simeq (\text{Id}_{\text{Bun}_G} \boxtimes (j^! \circ j_*))(u_{\mathcal{U}}) \simeq u_{\mathcal{U}} \boxtimes j^!(\mathcal{F}) \boxtimes (\text{Mir}_{\mathcal{U}}^{-1} \circ \mathcal{D} \text{Verdier} \circ j^!(\mathcal{F})) \simeq j^*(\mathcal{F}) \boxtimes (j^! \circ \text{Mir}_{\mathcal{U}}^{-1} \circ \mathcal{D} \text{Verdier} \circ (\mathcal{F})) \simeq (j^* \boxtimes j^!)(\mathcal{F} \boxtimes \mathcal{F}^R).
\]

The counit of the adjunction is a map

\[
(4.11) \quad C_\bullet(\text{Bun}_G; \mathcal{F} \boxtimes \mathcal{F}^R) \to e
\]

is equal to the composition

\[
C_\bullet(\text{Bun}_G; \mathcal{F} \boxtimes \mathcal{F}^R) \simeq C_\bullet(\text{Bun}_G; \mathcal{F} \boxtimes \mathcal{D} \text{Verdier}(\mathcal{F})) \to e,
\]

where the latter map is obtained by adjunction from the map

\[
\mathcal{F} \boxtimes \mathcal{D} \text{Verdier}(\mathcal{F}) \to (\Delta_{\text{Bun}_G})_*(\omega_{\text{Bun}_G}).
\]

\( \square \)
Corollary 4.2.8. For any $\mathcal{F} \in \text{Shv}_{\text{Nilp}}(\text{Bun}_G)_{\text{co}}$, the functor

$$
\mathcal{F} : \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \to \text{Vect}, \quad \mathcal{F}' \mapsto C^1(\text{Bun}_G, \mathcal{F} \otimes \mathcal{F}')
$$

is defined and codelined by a kernel. The codefining object identifies canonically with

$$
\mathcal{G} := \text{Mir}_{\text{Bun}_G}(\mathcal{F}) \in \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \subset \text{Shv}(\text{Bun}_G).
$$

Proof. Repeats that of Corollary 4.2.5 (indeed, it is easy to see that Theorem B.6.3 extends from the case of quasi-compact algebraic stacks to truncatable ones). \hfill \Box

4.3. Pairings against sheaves with nilpotent singular support, revisited.

4.3.1. Let $\mathcal{U}$ be a universally Nilp-contruncative quasi-compact open substack of $\text{Bun}_G$.

Let $\mathcal{Y}$ be an algebraic stack. Note that for $\mathcal{F} \in \text{Shv}(\mathcal{U})$ and $\mathcal{F}' \in \text{Shv}(\mathcal{Y} \times \mathcal{U})$, we have a canonically defined map

$$
(p_y)_!(\mathcal{F}' \circ p_\mathcal{U}^!(\text{Mir}_\mathcal{U}(\mathcal{F}))) \to (p_y)_! (\mathcal{F}' \circ p_\mathcal{Y}^!(\mathcal{F}))
$$

as functors $\text{Shv}(\mathcal{Y} \times \mathcal{U}) \to \text{Shv}(\mathcal{Y})$, see (B.17).

From Corollary 4.2.5, we obtain:

Corollary 4.3.2. The natural transformation (4.12) is an isomorphism if $\mathcal{F} \in \text{Shv}_{\text{Nilp}}(\mathcal{U})$.

Remark 4.3.3. Recall that Corollary 3.6.7 says that we have a canonical isomorphism

$$
(p_y)_!(\mathcal{F}' \circ p_\mathcal{U}^!(\text{Mir}_\mathcal{U}(\mathcal{F}))) \to (p_y)_! (\mathcal{F}' \circ p_\mathcal{Y}^!(\mathcal{F})),
$$

when $\mathcal{Y} = \text{pt}$. In fact, the same proof shows that there exists an isomorphism as in (4.13) for any $\mathcal{Y}$.

The additional information provided by Corollary 4.3.2 is that this isomorphism comes from the generally defined natural transformation (B.17).

4.3.4. We now claim:

Corollary 4.3.5. There exists a canonical isomorphism

$$
(p_y)_!(\text{Id}_\mathcal{Y} \boxtimes \text{Mir}_\mathcal{U})(\mathcal{F}') \circ p_\mathcal{U}^!(\mathcal{F})) \simeq (p_y)_! (\mathcal{F}' \circ p_\mathcal{Y}^!(\mathcal{F}))
$$

as functors $\text{Shv}(\mathcal{Y} \times \mathcal{U}), \mathcal{F} \in \text{Shv}_{\text{Nilp}}(\mathcal{U})$.

Proof. Write

$$
(p_y)_!(\text{Id}_\mathcal{Y} \boxtimes \text{Mir}_\mathcal{U})(\mathcal{F}') \circ p_\mathcal{U}^!(\mathcal{F})) = (p_y)_! \left( (\text{Id}_\mathcal{Y} \boxtimes \text{Mir}_\mathcal{U})(\mathcal{F}') \circ p_\mathcal{U}^!(\text{Mir}_\mathcal{U}^{-1}(\mathcal{F})) \right)
$$

Corollary 4.3.2

$$
\simeq (p_y)_! (\text{Id}_\mathcal{Y} \boxtimes \text{Mir}_\mathcal{U})(\mathcal{F}') \circ p_\mathcal{U}^!(\text{Mir}_\mathcal{U}^{-1}(\mathcal{F})).
$$

However, since the object $\text{ps-}u_\mathcal{U}$ is swap-equivariant, the latter expression identifies with

$$
(p_y)_! \left( (\mathcal{F}' \circ p_\mathcal{U}^!(\text{Mir}_\mathcal{U}^{-1}(\mathcal{F}))) \right) \simeq (p_y)_! (\mathcal{F}' \circ p_\mathcal{Y}^!(\mathcal{F})),
$$

as required. \hfill \Box

4.3.6. We now revisit the pairings on all of $\text{Bun}_G$. Let $\mathcal{Y}$ be as above. For $\mathcal{F} \in \text{Shv}(\text{Bun}_G)_{\text{co}}$ consider the natural transformation

$$
(p_y)_!(\mathcal{F}' \circ p_{\text{Bun}_G}^!(\text{Mir}_{\text{Bun}_G}(\mathcal{F}))) \to (p_y)_! (\mathcal{F}' \circ p_{\text{Bun}_G}^!(\mathcal{F})),
$$

as functors $\text{Shv}(\mathcal{Y} \times \text{Bun}_G) \to \text{Shv}(\mathcal{Y})$, see (B.17).

From Corollary 4.2.8, we obtain:

Corollary 4.3.7. The natural transformation (4.14) is an isomorphism if $\mathcal{F} \in \text{Shv}_{\text{Nilp}}(\text{Bun}_G)$. 

4.3.8. Finally, we claim:

**Corollary 4.3.9.** For $\mathcal{F} \in \text{Shv}_{\text{Nilp}}(\text{Bun}_G)$ and any algebraic stack $\mathcal{Y}$, there is a canonical isomorphism isomorphism of functors $\text{Shv}(\mathcal{Y} \times \text{Bun}_G)_{\text{co}} \rightarrow \text{Shv}(\mathcal{Y})$

\[(4.15) \quad (p_\mathcal{Y})_!( (\text{Id}_\mathcal{Y} \boxtimes \text{Mir}_{\text{Bun}_G}) (\mathcal{F}' \boxtimes p^*_\text{Bun}_G(\mathcal{F}))) \simeq (p_\mathcal{Y})_! (\mathcal{F}' \boxtimes p^*_\text{Bun}_G(\mathcal{F})), \quad \mathcal{F}' \in \text{Shv}(\mathcal{Y} \times \text{Bun}_G).\]

**Proof.** Follows from Corollary 4.3.7 in the same way as Corollary 4.3.5 follows from Corollary 4.3.2. \[\square\]

**Remark 4.3.10.** Note that in the case when $\mathcal{Y} = \text{pt}$, the assertion of Corollaries 4.3.9 and 4.3.7 has been already established in Theorem 3.4.2. Similarly to Remark 4.3.3, the proof of Theorem 3.4.2 can be extended to include the case of an arbitrary $\mathcal{Y}$. Furthermore, one can show that the resulting isomorphisms identify with those of Corollaries 4.3.9 and 4.3.7.

4.4. A refined version of Theorem 1.3.7. In this subsection we discuss an amplification of the categorical K"unneth formula, Theorem 1.3.7.

4.4.1. In this subsection, we will use Theorem 4.2.2 to prove the following result:

**Theorem 4.4.2.** Let $\mathcal{Z}$ be an algebraic stack, and let $\mathcal{N}$ be a conical half-dimensional closed subset of $T^*(\mathcal{Z})$. Then the functor $\text{Shv}_\mathcal{N}(\mathcal{Z}) \otimes \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \rightarrow \text{Shv}_{\mathcal{N} \times \text{Nilp}}(\mathcal{Z} \times \text{Bun}_G)$ is an equivalence.

By passing to the limit, the statement of Theorem 4.4.2 is obtained from the following:

**Theorem 4.4.3.** Let $\mathcal{Z}$ be an algebraic stack, and let $\mathcal{N}$ be a half-dimensional closed subset of $T^*(\mathcal{Z})$. Then for any universally Nilp-cotruncative open substack $\mathcal{U} \subset \text{Bun}_G$, the functor $\text{Shv}_\mathcal{N}(\mathcal{Z}) \otimes \text{Shv}_{\text{Nilp}}(\mathcal{U}) \rightarrow \text{Shv}_{\mathcal{N} \times \text{Nilp}}(\mathcal{Z} \times \mathcal{U})$ is an equivalence.

We proceed to the proof of Theorem 4.4.3.

4.4.4. Given Corollary 1.7.6 in order to prove Theorem 1.3.7 we have to show the following:

Let $\mathcal{F}'$ be an object in $\text{Shv}(\mathcal{Z}) \otimes \text{Shv}_{\text{Nilp}}(\mathcal{U})$, whose image along the functor

\[(4.16) \quad \text{Shv}(\mathcal{Z}) \otimes \text{Shv}_{\text{Nilp}}(\mathcal{U}) \xrightarrow{\text{ev}_\mathcal{U}} \text{Shv}(\mathcal{Z} \times \mathcal{U})
\]

is contained in $\text{Shv}_{\mathcal{N} \times \text{Nilp}}(\mathcal{Z} \times \mathcal{U})$. Then $\mathcal{F}' \in \text{Shv}_\mathcal{N}(\mathcal{Z}) \otimes \text{Shv}_{\text{Nilp}}(\mathcal{U})$.

To prove this, by Proposition 2.3.3 it suffices to show that for every $\mathcal{F} \in \text{Shv}_{\text{Nilp}}(\mathcal{U})$, the object

\[\text{(Id} \otimes \text{ev}_\mathcal{U})(\mathcal{F}' \otimes \mathcal{F}) \in \text{Shv}(\mathcal{Z})\]

belongs to $\text{Shv}_\mathcal{N}(\mathcal{Z})$.

We rewrite

\[\text{(Id} \otimes \text{ev}_\mathcal{U})(\mathcal{F}' \otimes \mathcal{F}) \simeq (p_1)_!( \mathcal{F}' \otimes p^*_2(\mathcal{F}))\]

where in the right-hand side, we denote by the same character $\mathcal{F}'$ the image of $\mathcal{F}'$ along (4.16).

Thus, it suffices to show that for $\mathcal{F}' \in \text{Shv}_{\mathcal{N} \times T^*(\text{Bun}_G)}(\mathcal{Z} \times \mathcal{U})$ and $\mathcal{F} \in \text{Shv}_{\text{Nilp}}(\mathcal{U})$, the object

\[(\text{Id}_\mathcal{Z} \otimes \mathcal{F})(\mathcal{F}') \in \text{Shv}(\mathcal{Z})\]

belongs to $\text{Shv}_\mathcal{N}(\mathcal{Z})$, where $\mathcal{F}$ is the functor defined by $\mathcal{F}$ as a kernel.

According to Corollary 1.2.5 the functor $\mathcal{F}$ is defined and codefined by a kernel. The required assertion follows now from Corollary 1.2.10.\[\square\]
4.5. **A converse statement.** In this subsection we will state an assertion that provides a converse to Corollary 4.2.8 and as a result also to Theorem 4.2.7.

4.5.1. We claim:

**Theorem 4.5.2.** Let $\mathcal{F}$ be an object of $\text{Shv}(\text{Bun}_G)_{\text{co}}$ such that the corresponding functor

$$F := C_\bullet(\text{Bun}_G, \mathcal{F} \otimes -), \quad \text{Shv}(\text{Bun}_G) \to \text{Vect}$$

is defined and codefined by a kernel. Then $\mathcal{F} \in \text{Shv}_{\text{Nilp}}(\text{Bun}_G)_{\text{co}}$.

This theorem will be proved in Sect. 4.7. In the rest of this subsection we will derive some corollaries.

4.5.3. As a consequence, we obtain:

**Theorem 4.5.4.** Let $\mathcal{F}$ be an object of $\text{Shv}(\text{Bun}_G)^{\text{constr}}$ such that the corresponding functor

$$F := C_\bullet(\text{Bun}_G, \mathcal{F} \otimes -), \quad \text{Shv}(\text{Bun}_G)_{\text{co}} \to \text{Vect}$$

admits a right adjoint as a functor defined by a kernel. Then $\mathcal{F} \in \text{Shv}_{\text{Nilp}}(\text{Bun}_G)$.

**Proof.** Consider the object

$$\mathcal{G} := \text{Mir}^{-1}_{\text{Bun}_G}(\mathcal{F}) \in \text{Shv}(\text{Bun}_G)_{\text{co}}.$$  

The assumption on $\mathcal{F}$ implies that $\mathcal{G}$, viewed as a functor $\text{Shv}(\text{Bun}_G) \to \text{Vect}$ defined by a kernel, also admits a right adjoint. By Theorem B.6.3(a) (see Remark B.6.10), the resulting functor $\mathcal{G} : \text{Shv}(\text{Bun}_G) \to \text{Vect}$ is defined and codefined by a kernel.

Hence, from Theorem 4.5.2 we obtain that $\mathcal{G} \in \text{Shv}_{\text{Nilp}}(\text{Bun}_G)_{\text{co}}$. Hence,

$$\mathcal{F} = \text{Mir}_{\text{Bun}_G}(\mathcal{G}) \in \text{Shv}_{\text{Nilp}}(\text{Bun}_G).$$

\[\square\]

4.5.5. Combining Theorems 4.5.4, 4.2.7 and B.8.8 (see Remark B.8.9), we obtain:

**Corollary 4.5.6.** For $\mathcal{F} \in \text{Shv}(\text{Bun}_G)^{\text{constr}}$ the following conditions are equivalent:

(i) $\mathcal{F} \in \text{Shv}_{\text{Nilp}}(\text{Bun}_G)$;

(ii) The functor $F$ admits a right adjoint as a functor defined by a kernel.

(iii) The map

$$C_\bullet(\text{Bun}_G, \mathcal{F} \otimes D(\mathcal{F})) \to C_\bullet(\text{Bun}_G, \mathcal{F} \otimes (\text{Mir}^{-1}_{\text{Bun}_G} \circ D_{\text{Verdier}}(\mathcal{F})))$$

of (4.2) is an isomorphism, where $D_{\text{Verdier}}$ is understood as a functor $\text{Shv}(\text{Bun}_G)^{\text{constr}} \to \text{Shv}(\text{Bun}_G)^{\text{constr}}$.

4.5.7. Let us momentarily place ourselves in the context of (non-necessarily holonomic) D-modules. I.e., $\text{Shv}(\quad)$ will denote the category of ind-holonomic D-modules, viewed as a subcategory of the category $\text{D-mod}(\quad)$ of all D-modules. Note that the inclusion

$$\text{Shv}_{\text{Nilp}}(\text{Bun}_G) \subset \text{D-mod}_{\text{Nilp}}(\text{Bun}_G)$$

is an equivalence, since $\text{Nilp} \subset \mathcal{T}^\ast(\text{Bun}_G)$ is Lagrangian.

In this case, the condition that the functor $F$ admits a right adjoint as a functor defined by a kernel is equivalent to the condition that it admits a continuous right adjoint as a plain functor

$$\text{D-mod}(\text{Bun}_G)_{\text{co}} \to \text{Vect},$$

which is in turn equivalent to the condition that it sends compact objects to finite-dimensional vector spaces.
4.5.8. We claim:

**Theorem 4.5.9.** For \( \mathcal{F} \in \text{Shv}(\text{Bun}_G)^\text{constr} \subset \text{D-mod}(\text{Bun}_G) \) the following conditions are equivalent:

(i) \( \mathcal{F} \in \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \);

(ii) For any \( \mathcal{F}' \in \text{D-mod}(\text{Bun}_G)^c \), the object

\[
\mathbb{C}_\bullet(\text{Bun}_G, \mathcal{F}^! \otimes \mathcal{F}') \in \text{Vect}
\]

is finite-dimensional.

(ii') For any \( \mathcal{F}' \in \text{D-mod}(\text{Bun}_G)^c \), the object

\[
\mathcal{H}om_{\text{D-mod}(\text{Bun}_G)}(\mathcal{F}', \mathcal{F}) \in \text{Vect}
\]

is finite-dimensional.

(ii'') For any \( \mathcal{F}' \in \text{D-mod}(\text{Bun}_G)^{c}\text{co} \), the object

\[
\mathcal{H}om_{\text{D-mod}(\text{Bun}_G)}(\mathcal{F}, \text{Id}_{\text{naive}}(\text{Bun}_G(\mathcal{F}'))) \in \text{Vect}
\]

is finite-dimensional.

**Proof.** The equivalence of (i) and (ii) follows from the combination of Theorems 4.2.7 and 4.5.4. The equivalence of (ii) and (ii') is formal: for \( \mathcal{F}' \in \text{D-mod}(\text{Bun}_G)^c \)

\[
\mathcal{H}om_{\text{D-mod}(\text{Bun}_G)}(\mathcal{F}', \mathcal{F}) \simeq \mathbb{C}_\bullet(\text{Bun}_G, \mathcal{F}^! \otimes \mathbb{D}^{\text{Verdier}}(\mathcal{F}')), 
\]

where \( \mathbb{D}^{\text{Verdier}} \) is understood as an equivalence

\[
(\text{D-mod}(\text{Bun}_G)^c)^{\text{op}} \rightarrow \text{D-mod}(\text{Bun}_G)^{c}\text{co}.
\]

The equivalence of (ii') and (ii'') follows by viewing \( \mathbb{D}^{\text{Verdier}} \) as an equivalence

\[
(\text{Shv}(\text{Bun}_G)^{\text{constr}})^{\text{op}} \simeq \text{Shv}(\text{Bun}_G)^{\text{constr}}.
\]

\( \square \)

4.6. A version for a universally Nilp-cotruncative quasi-compact open substack.

4.6.1. Let \( \mathcal{U} \) be a universally Nilp-cotruncative quasi-compact open substack of \( \text{Bun}_G \). As a formal corollary of Theorem 4.5.2 we obtain:

**Corollary 4.6.2.** Let \( \mathcal{F} \) be an object of \( \text{Shv}(\mathcal{U}) \) such that the corresponding functor

\[
\mathcal{F} := \mathbb{C}_\bullet(\mathcal{U}, \mathcal{F}^! \otimes -), \quad \text{Shv}(\mathcal{U}) \rightarrow \text{Vect}
\]

is defined and codefined by a kernel. Then \( \mathcal{F} \in \text{Shv}_{\text{Nilp}}(\mathcal{U}) \).

From Theorem 4.5.4 we obtain:

**Corollary 4.6.3.** Let \( \mathcal{F} \) be an object of \( \text{Shv}(\mathcal{U})^{\text{constr}} \) such that the corresponding functor

\[
\mathcal{F} := \mathbb{C}_\bullet(\mathcal{U}, \mathcal{F}^! \otimes -), \quad \text{Shv}(\mathcal{U}) \rightarrow \text{Vect}
\]

admits a right adjoint defined by a kernel. Then \( \mathcal{F} \in \text{Shv}_{\text{Nilp}}(\mathcal{U}) \).

4.6.4. Observe that by combining Corollary 4.6.2 and Theorems 4.5.2 and 3.8.8 we obtain:

**Corollary 4.6.5.** Let \( \mathcal{U} \subset \text{Bun}_G \) be a universally Nilp-cotruncative quasi-compact open substack. Then for \( \mathcal{F} \in \text{Shv}(\mathcal{U})^{\text{constr}} \) the following conditions are equivalent:

(i) \( \mathcal{F} \in \text{Shv}_{\text{Nilp}}(\mathcal{U}) \);

(ii) The functor \( \mathcal{F} \) admits a right adjoint as a functor defined by a kernel.

(iii) The map

\[
\mathbb{C}_\bullet(\mathcal{U}, \mathcal{F}^! \otimes \mathbb{D}(\mathcal{F})) \rightarrow \mathbb{C}_\bullet(\mathcal{U}, \mathcal{F}^! \otimes (\text{Mir}_{\mathcal{U}}^1 \circ \mathbb{D}(\mathcal{F})))
\]

of (4.1) is an isomorphism.
4.6.6. Let \( Y \) be an arbitrary quasi-compact algebraic stack. Let \( \mathcal{F} \) be an object of \( \text{Shv}(\hat{Y})^{\text{constr}} \).

First, recall that according to Theorem 4.5.6, conditions (ii) and (iii) in Corollary 4.6.5 are equivalent.

The statement of Corollary 4.6.5 suggests the following question:

**Question 4.6.7.** Under what conditions on \( \hat{Y} \) does there exist a subset \( \mathcal{N} \subset T^*(\hat{Y}) \) so that conditions (ii) or (iii) as in Corollary 4.6.5 are equivalent to the condition that \( \mathcal{F} \in \text{Shv}(\mathcal{N}) \)?

4.6.8. Example. Let \( Y = Y^1 \) be a proper smooth scheme. Then it is easy to see that the assertion of Corollary 4.6.5 holds for \( N = \{0\} \).

So, in some ways the subset \( \text{Nilp} \subset T^*(\text{Bun}_G) \) plays the same role as the zero-section \( \{0\} \subset T^*(Y) \) for a proper smooth scheme \( Y \).

The other extreme case is when \( Y \) is an algebraic stack with finitely many isomorphism classes of points. Then the assertion of Corollary 4.6.5 holds for \( N = T^*(\hat{Y}) \).

Note that in both of the above examples, the pair \( (\hat{Y}, \mathcal{N}) \) is Serre; see Definition 5.3.4 for what this means.

4.7. Proof of Theorem 4.5.2

4.7.1. We will deduce Theorem 4.5.2 from the combination of the following two statements:

**Proposition 4.7.2.** Assume that \( \mathcal{F} \) is such that the functor (1.17) is defined and codefined by a kernel. Then for any \( \mathcal{F}' \in \text{Shv}(\text{Bun}_G) \) and \( V \in \text{Rep}(\mathcal{G}) \),

\[
(p_2)_* \left( \text{H}(V, \mathcal{F}) \right) \cong \text{QLisse}(X).
\]

**Proposition 4.7.3.** Let \( \hat{Y} \) be a scheme, and let \( \mathcal{F}_y \in \text{Shv}(\hat{Y}) \) be an object, such that for every geometric point \( i_y : \text{Spec}(k') \to \hat{Y} \), the object

\[
(i_y \times \text{id})^!(\mathcal{F}_y) \in \text{Shv}(\hat{X}', X' := \text{Spec}(k') \times_{\text{Spec}(k)} X
\]

belongs to \( \text{QLisse}(X') \). Then \( \mathcal{F}_y \) lies in the essential image of the functor

\[
\text{Shv}(\hat{Y}) \otimes \text{QLisse}(X) \hookrightarrow \text{Shv}(\hat{Y} \times X).
\]

4.7.4. Let us assume these propositions temporarily, and prove Theorem 4.5.2.

Let

\[
\mathcal{G} := \text{Mir}_{Bun}(\mathcal{F}) \in \text{Shv}(\text{Bun}_G).
\]

By Theorem 4.5.7 for \( Z = \text{pt} \) (which is AGKRIV Theorem 14.4.3), it suffices to show that for every \( V \in \text{Rep}(\mathcal{G}) \), the object \( \text{H}(V, \mathcal{G}) \) belongs to the essential image of

\[
\text{Shv}(\text{Bun}_G) \otimes \text{QLisse}(X) \hookrightarrow \text{Shv}(\text{Bun}_G \times X).
\]

By Proposition 4.7.2, it suffices to show that for every geometric point \( i_y : \text{Spec}(k') \to \hat{Y} \), the object

\[
(i_y \times \text{id})^!(\text{H}(V, \mathcal{G})) \in \text{Shv}(\hat{X}', X' := \text{Spec}(k') \times_{\text{Spec}(k)} X
\]

belongs to \( \text{QLisse}(X') \).

Changing base from \( k \) and \( k' \), we can assume that \( y \) is a closed point. We can rewrite

\[
(i_y \times \text{id})^!(\text{H}(V, \mathcal{G})) \cong (p_2)_* (\text{H}(V, \mathcal{G}) \otimes (i_y)_* (e)),
\]

and further as

\[
(p_2)_* \left( \text{Mir}_{Bun}(\hat{Y}, \mathcal{F}) \otimes (i_y)_* (e) \right) \cong (p_2)_* \left( \text{H}(V, \mathcal{F}) \otimes (i_y)_* (\text{Mir}_{Bun}(\hat{Y}, \mathcal{F})) \right).
\]

The required assertion follows now from Proposition 4.7.2 applied to

\[
\mathcal{F}' := \text{Mir}_{Bun}(\hat{Y}, \mathcal{F}).
\]
4.7.5. **Proof of Proposition 4.7.2.** We rewrite
\[(p_2)_! (H(V, \mathcal{F})_{\text{co}} \otimes p_1^!(\mathcal{F}')) \simeq (p_2)_! (p_1^!(\mathcal{F}) \otimes H(V', \mathcal{F}')).\]

With no restriction of generality, we can assume that \(V \in \text{Rep}(\hat{G})^c\) and \(\mathcal{F}' \in \text{Shv}(\text{Bun}_\mathcal{G})^\text{const}\). In particular, \(\mathcal{F}'\) is constructible, and so is \(H(V', \mathcal{F}')\).

Recall now that objects of the form
\[H(V', \mathcal{F}') \in \text{Shv}(\text{Bun}_\mathcal{G} \times X), \quad V' \in \text{Rep}(\hat{G})^c, \quad \mathcal{F}' \in \text{Shv}(\text{Bun}_\mathcal{G})^\text{const}\]
are ULA with respect to the projection
\[p_2 : \text{Bun}_\mathcal{G} \times X \to X.\]

Applying Corollary 4.10.10, we obtain that
\[(p_2)_! (p_1^!(\mathcal{F}) \otimes H(V', \mathcal{F}')) \in \text{QLisse}(X),\]
as required. \(\Box\)

4.7.6. **Proof of Proposition 4.7.3.** The functor \(4.19\) is fully faithful, and admits a continuous right adjoint, explicitly described as follows:

Identify \(\text{QLisse}(X)\) with its own dual via pairing \(\text{ev}_X\) (here we use the fact that the pair \((X, \{0\})\) is duality-adapted, see Sects. A.5.4 and A.5.5). Then the resulting functor
\[\text{Shv}(\mathcal{Y} \times X) \otimes \text{QLisse}(X) \simeq \text{Shv}(\mathcal{Y} \times X) \otimes \text{QLisse}(X)^{\vee} \to \text{Shv}(\mathcal{Y})\]
is given by
\[\mathcal{F}_y \otimes E_X \in \text{Shv}(\mathcal{Y} \times X) \otimes \text{QLisse}(X) \mapsto (p_1)_!(\mathcal{F}_y \otimes p_2^!(E_X)) \in \text{Shv}(\mathcal{Y}).\]

Note also that every object in the essential image of \(4.19\) satisfies the condition of Proposition 4.7.3. Hence, it suffices to show that if \(\mathcal{F}_y\) is such that it satisfies the condition of Proposition 4.7.3 and
\[(p_1)_!(\mathcal{F}_y \otimes p_2^!(E_X)) = 0, \quad \forall E_X \in \text{QLisse}(X),\]
then \(\mathcal{F}_y = 0\).

To check this, it suffices to show that
\[(1_y \times \text{id})^!(\mathcal{F}_y) = 0\]
for every geometric point \(1_y : \text{Spec}(k') \to \mathcal{Y}\). By the assumption on \(\mathcal{F}_y\), it suffices to show that
\[C(X', (1_y \times \text{id})^!(\mathcal{F}_y) \otimes E_{X'}) = 0\]
for every \(E_{X'} \in \text{QLisse}(X')\).

Note that the base change functor \(\text{QLisse}(X) \to \text{QLisse}(X')\) is an equivalence. Hence, the object \(E_{X'} \in \text{QLisse}(X')\) in the above formula is the base change of some \(E_X \in \text{QLisse}(X)\), and hence
\[C(X', (1_y \times \text{id})^!(\mathcal{F}_y) \otimes E_{X'}) \simeq 1_y \left( (p_1)_!(\mathcal{F}_y \otimes p_2^!(E_X)) \right),\]
while the latter vanishes, by assumption. \(\Box\)

5. **Serre functor on \(\text{Shv}_{\text{Nilp}}(\text{Bun}_\mathcal{G})\)**

In this section we will recast the results of Sect. 4 in a different light, by relating the miraculous functor on \(\text{Shv}_{\text{Nilp}}(\text{Bun}_\mathcal{G})\) to the **Serre functor**.

We will show that the category \(\text{Shv}_{\text{Nilp}}(\text{Bun}_\mathcal{G})\) is Serre (see Sect. 5.1.3 for what this means), up to the \(\text{Shv}_{\text{Nilp}}(\text{Bun}_\mathcal{G}) \sim \text{Shv}_{\text{Nilp}}(\text{Bun}_\mathcal{G})_{\text{co}}\) replacement.
5.1. **The Serre functor.** In this subsection we recall the basic definitions pertaining to the Serre functor.

5.1.1. Let $C$ be a dualizable category. Throughout this paper we denote by

$$u_C \in C \otimes C^\vee$$

the unit of the duality.

We denote by

$$ev_C : C \otimes C^\vee \to \text{Vect}$$

the canonical pairing.

5.1.2. From now on we will assume that $C$ is compactly generated. We will identify the dual $C^\vee$ of $C$ with $\text{Ind}((C^c)^{\text{op}})$. We will denote by $D$ the resulting contravariant equivalence $(C^c)^{\text{op}} \to (C^c)\vee$.

Under this identification, for $c \in C^c$ and $c' \in C$,

$$ev_C(c', D(c)) \simeq \text{Hom}_C(c, c')$$

5.1.3. Recall that $C$ is said to be proper if the functor $ev_C$ preserves compactness\(^4\). This is equivalent to the assumption that the functor

$$\text{Vect} \xrightarrow{u_C} C \otimes C^\vee$$

admits a left adjoint, to be denoted

\[(5.1)\quad u_C^L : C \otimes C^\vee \to \text{Vect}.\]

5.1.4. Assume that $C$ is proper. The Serre endofunctor of $C$, denoted $\text{Se}_C$, is defined by the formula

$$\text{Hom}_C(c_1, \text{Se}_C(c)) := \text{Hom}_C(c, c_1)^\vee, \quad c, c_1 \in C^c.$$ 

The following results from the definitions:

**Lemma 5.1.5.** Let $C$ be proper. Then the functor $u_C^L$ identifies canonically with

$$C \otimes C^\vee \xrightarrow{\text{Se}_C \otimes \text{id}} C \otimes C^\vee \xrightarrow{ev_C} \text{Vect}.$$

5.1.6. A DG category $C$ is said to be *Serre* if the functor $\text{Se}_C$ is a self-equivalence.

From Lemma 5.1.5, we obtain:

**Corollary 5.1.7.** The following conditions are equivalent:

(i) $C$ is Serre;
(ii) The functor $u_C^L$ is the counit of another duality between $C$ and $C^\vee$.

**Corollary 5.1.8.** Let $C$ be proper (and hence so is $C^\vee$). Then $C$ is Serre if and only if $C^\vee$ is Serre.

5.1.9. Let $C$ be Serre. Thus, we obtain a *new* identification between the dual of $C$ and $C^\vee$. In particular, we obtain a new contravariant equivalence

$$(C^c)^{\text{op}} \to (C^c)^\vee,$$

which we will denote by $D^{\text{new}}$.

By definition,

\[(5.2)\quad D^{\text{new}} \simeq D \circ \text{Se}_C.\]

\[^4\text{The more familiar formulation of this is that } \text{Hom}_C(-, -) \text{ between compact objects is compact.} \]
5.1.10. Example. Let \( C = \text{QCoh}(Y) \), where \( Y \) is a smooth and proper scheme. We identify 
\[
\text{QCoh}(Y) \simeq \text{QCoh}(Y)^\vee
\]
via the Serre pairing. I.e., the unit is given by 
\[
(\Delta_Y)_* (\omega_Y) \in \text{QCoh}(Y \times Y) \simeq \text{QCoh}(Y) \otimes \text{QCoh}(Y)
\]
(here \( \omega_Y \) is the dualizing sheaf) and the counit by 
\[
\text{QCoh}(Y) \otimes \text{QCoh}(Y) \simeq \text{QCoh}(Y \times Y) \xrightarrow{\Delta_Y^*} \text{QCoh}(Y) \xrightarrow{\Gamma(Y, -)} \text{Vect}.
\]
Then the functor 
\[
\mathbb{D} : (\text{Perf}(Y))^{\text{op}} \to \text{Perf}(Y)
\]
is 
\[
\mathcal{F} \mapsto \mathcal{F}^\vee \otimes \omega_Y,
\]
where 
\[
\mathcal{F}^\vee := \text{Hom}(\mathcal{F}, \mathcal{O}_Y).
\]
The Serre functor on \( \text{QCoh}(Y) \) is given by 
\[
\mathcal{F} \mapsto \mathcal{F} \otimes \omega_Y,
\]
in particular \( \text{QCoh}(Y) \) is Serre.

The functor \( \mathbb{D}^{\text{new}} \) is given by 
\[
\mathcal{F} \mapsto \mathcal{F}^\vee,
\]
and the new self-duality on \( \text{QCoh}(Y) \) has as counit 
\[
\text{QCoh}(Y) \otimes \text{QCoh}(Y) \simeq \text{QCoh}(Y \times Y) \xrightarrow{\Delta_Y^*} \text{QCoh}(Y) \xrightarrow{\Gamma(Y, -)} \text{Vect},
\]
and as unit 
\[
(\Delta_Y)_* (\mathcal{O}_Y) \in \text{QCoh}(Y \times Y) \simeq \text{QCoh}(Y) \otimes \text{QCoh}(Y).
\]

5.2. Serre vs pseudo-identity. In this subsection we reproduce some of the results of [GaYo], which relate the Serre functor to the pseudo-identity functor.

5.2.1. Let \( C \) be a compactly generated category.

Applying the right Kan extension to \( \mathbb{D} \) along \( (C^\text{op})^\text{op} \to C^\text{op} \), and extend \( \mathbb{D} \) to a discontinuous functor 
\[
C^\text{op} \to C^\vee.
\]
Explicitly, for 
\[
c = \colim_\alpha c_\alpha,
\]
we have 
\[
\mathbb{D}(c) = \lim_\alpha \mathbb{D}(c_\alpha).
\]

5.2.2. We identify 
\[
(C \otimes C^\vee)^\vee \simeq C^\vee \otimes C,
\]
and define the object 
(5.3) \( \text{ps-u}_C \in C \otimes C^\vee \)
to be \( \mathbb{D}(\text{u}_C) \).
5.2.3. We identify
\[
C \otimes C^\vee \simeq \text{Funct}_{\text{cont}}(C, C).
\]
Under the identification the object \(u_C \in C \otimes C^\vee\) corresponds to \(\text{Id}_C \in \text{Funct}_{\text{cont}}(C, C)\).

We define the pseudo-identity endofunctor of \(C\)
\[
\text{Ps-Id}_C \in \text{Funct}_{\text{cont}}(C, C)
\]
as the object corresponding under (5.4) to \(\text{ps-u}_C \in C \otimes C^\vee\).

Note that we have:
\[
(\text{Ps-Id}_C)^\vee \simeq \text{Ps-Id}_{C^\vee}.
\]

5.2.4. Let \(C\) be Serre. In particular, by Corollary 5.1.7, the functor
\[
C \otimes C^\vee u_L C \to \text{Vect}
\]
is the counit of a duality.

We claim:

**Proposition 5.2.5.** The unit of the duality (5.5) is given by \(\text{ps-u}_C\).

**Proof.** We have to show that there exists a canonical isomorphism
\[
\text{Hom}_{C \otimes C^\vee}(c_1 \otimes D(\text{Sec}(c_2)), \text{ps-u}_C) \simeq \text{Hom}_C(c_1, c_2), \quad c_1, c_2 \in C^c.
\]

The left-hand side identifies, by definition, with
\[
\text{Hom}_C(c_1, c) \otimes \text{Sec}(c_2), \quad c \in C^c,
\]
which is the same as the space of natural transformations
\[
c \to \text{Hom}_C(c_1, c) \otimes \text{Sec}(c_2), \quad c \in C^c,
\]
We rewrite the latter as the space of natural transformations
\[
\text{Hom}_C(c_1, c)^\vee \to \text{Hom}_C(c, \text{Sec}(c_2)), \quad c \in C^c,
\]
i.e.,
\[
\text{Hom}_C(c, c)^\vee \to \text{Hom}_C(c_2, c)^\vee, \quad c \in C^c,
\]
which is the same as
\[
\text{Hom}_C(c_2, c) \to \text{Hom}_C(c_1, c), \quad c \in C^c.
\]

By Yoneda, the latter is the same as \(\text{Hom}_C(c_1, c_2)\), as required. \(\square\)

5.2.6. To summarize, if \(C\) is Serre, we have the commutative diagrams
\[
\begin{array}{ccc}
C \otimes C^\vee & \xrightarrow{u_C} & \text{Vect} \\
\text{Sec} \otimes \text{Id} & & \text{Id} \\
\downarrow & & \downarrow \\
C \otimes C^\vee & \xrightarrow{u_C} & \text{Vect}
\end{array}
\]
and
\[
\begin{array}{ccc}
\text{Vect} & \xrightarrow{u_C} & C \otimes C^\vee \\
\text{Id} & & \text{Sec} \otimes \text{Id} \\
\downarrow & & \downarrow \\
\text{ Vect} & \xrightarrow{\text{ps-u}_C} & C \otimes C^\vee
\end{array}
\]
where the horizontal arrows are the counits and units of the corresponding dualities, respectively.

In particular, we recover the following result of [GaYo, Proposition 1.5.2]:

**Corollary 5.2.7.** Assume that \(C\) is Serre. Then the functor \(\text{Ps-Id}_C\) is the inverse of \(\text{Sec}_C\).
5.2.8. Thus, if $C$ is Serre, along with diagram (5.6) we also obtain the commutative diagram
\[
\begin{array}{c}
C \otimes C^\vee \xrightarrow{\psi_C} \text{Vect} \\
\downarrow \text{Ps-Id}_C \oplus \text{Id} \quad \downarrow \text{Id} \\
C \otimes C^\vee \xrightarrow{\psi_C} \text{Vect},
\end{array}
\]

5.2.9. Recall the functor $
\mathbb{D}^\text{new} : (C^c)^{\text{op}} \to (C^\vee)^c,$
see (5.2). We have
\[
\mathbb{D} \simeq \mathbb{D}^\text{new} \circ \text{Ps-Id}_C.
\]

5.3. **Duality-adapted pairs, complements.** In this subsection we collect some background material on the notion of duality-adapted pair, see Sect. [A.5.4].

5.3.1. Let $Y$ be a quasi-compact algebraic stack (see Sect. [0.5.1] for our assumptions). Let $N$ be a closed conical subset in $T^*(Y)$. Consider the corresponding full subcategory
\[
\text{Shv}_N(Y) \xrightarrow{\iota_Y} \text{Shv}(Y).
\]
We will assume that the pair $(Y, N)$ is duality-adapted, see Sect. [A.5.4], i.e., the functor
\[
\text{Shv}_N(Y) \otimes \text{Shv}_N(Y) \to \text{Shv}(Y) \otimes \text{Shv}(Y) \xrightarrow{\psi_Y} \text{Vect}
\]
is the counit of a self-duality
\[
(5.10) \quad \text{Shv}_N(Y)^\vee \simeq \text{Shv}_N(Y)
\]

5.3.2. Let
\[
\mathcal{P}_{Y,N} : \text{Shv}(Y) \to \text{Shv}_N(Y)
\]
denote the functor dual to the embedding
\[
\iota_Y : \text{Shv}_N(Y) \hookrightarrow \text{Shv}(Y)
\]
with respect the self-dualities
\[
\text{Shv}(Y)^\vee \simeq \text{Shv}(Y) \text{ and } \text{Shv}_N(Y)^\vee \simeq \text{Shv}_N(Y)
\]
of (A.12) and (5.10), respectively.
We will sometimes view $\mathcal{P}_{Y,N}$ as an endofunctor of $\text{Shv}(Y)$, by composing it with the embedding $\iota_Y$.

5.3.3. Let
\[
u_{Y,N} \in \text{Shv}_N(Y) \otimes \text{Shv}_N(Y)
\]
be the unit of the self-duality (5.10) on $\text{Shv}_N(Y)$.
We will sometimes view $\nu_{Y,N}$ as an object of $\text{Shv}(Y \times Y)$ via the embedding
\[
\text{Shv}_N(Y) \otimes \text{Shv}_N(Y) \xrightarrow{\iota_Y \otimes Y} \text{Shv}(Y) \otimes \text{Shv}(Y) \xrightarrow{\text{Id}} \text{Shv}(Y \times Y).
\]

**Remark 5.3.4.** Note that when $N = T^*(Y)$, we have
\[
\nu_{Y,N} = \nu_{\text{Shv}(Y)}.
\]
5.3.5. It follows from the definitions that $P_{y,N}$, viewed as an endofunctor of $Shv(y)$, identifies with 

$$\mathcal{F} \mapsto (p_2)_! \left( p_1^! (\mathcal{F}) \otimes u_{y,N} \right).$$

Thus, we extend $P_{y,N}$ to a functor defined by a kernel in the sense of Sect. 4.3.3 (with the kernel being $u_{y,N}$, viewed as an object of $Shv(y \times y)$). In particular, we obtain the endofunctor

$$Id_Z \boxtimes P_{y,N}$$

of $Shv(Z \times y)$ for any algebraic stack $Z$, so that for $Z = pt$ we recover the original $P_{y,N}$.

**Proposition 5.3.6.** The endofunctor $Id_Z \boxtimes P_{y,N}$ is the projector onto the full subcategory

$$Shv(Z) \otimes Shv_N(y) \hookrightarrow Shv(Z) \otimes Shv(y) \hookrightarrow Shv(Z \times y).$$

**Proof.** The fact that $u_{y,N}$ belongs to

$$Shv_N(y) \otimes Shv_N(y) \subset Shv(y) \otimes Shv_N(y)$$

implies that the essential image of $Id_Z \boxtimes P_{y,N}$ belongs to

$$Shv(Z) \otimes Shv_N(y) \subset Shv(Z \times y).$$

For any functor $Q$ defined by a kernel $Q \in Shv(y_1 \times y_2)$, the restriction of $Id_Z \boxtimes Q$ to

$$Shv(Z) \otimes Shv(y_1) \subset Shv(Z \times y_2)$$

identifies with

$$Id_{Shv(Z)} \otimes Q,$$

viewed as a functor

$$Shv(Z) \otimes Shv(y_1) \to Shv(Z) \otimes Shv(y_2) \subset Shv(Z \times y_2),$$

see (B.3.3).

This implies that $Id_Z \boxtimes P_{y,N}$ acts as identity when restricted to $Shv(Z) \otimes Shv_N(y)$. □

**Corollary 5.3.7.** The endofunctor $P_{y,N} \boxtimes P_{y,N}$ of $Shv(y \times y)$ is a projector onto

$$Shv_N(y) \otimes Shv_N(y) \subset Shv(y \times y).$$

**Proof.** Same as that of Corollary 1.6.5 □

5.3.8. We now claim:

**Lemma 5.3.9.** The object $u_{y,N}$ identifies with each of the following:

$$(P_{y,N} \boxtimes Id_y)(u_y), \ (P_{y,N} \boxtimes P_{y,N})(u_y), \ (Id_y \boxtimes P_{y,N})(u_y).$$

**Proof.** The isomorphism

$$u_{y,N} \simeq (Id_y \boxtimes P_{y,N})(u_y)$$

is tautological, and the isomorphism with $(P_{y,N} \boxtimes Id_y)(u_y)$ follows by symmetry.

We have

$$(P_{y,N} \boxtimes P_{y,N})(u_y) \simeq (P_{y,N} \boxtimes Id_y) \circ (Id_y \boxtimes P_{y,N})(u_y) \simeq (P_{y,N} \boxtimes Id_y)(u_{y,N}),$$

and the latter is isomorphic to $u_{y,N}$, since $P_{y,N} \boxtimes Id_y$ acts as identity on $Shv_N(y) \otimes Shv(y) \subset Shv(y \times y)$. □

5.4. Constraccessible pairs, complements. In this subsection we collect some background material on the notion of constraccessible pair, see Sect. A.3.2

In this subsection we assume that $y$ is quasi-compact.
5.4.1. Assume now that the pair \((\mathcal{Y}, N)\) is constraccessible, see Sect. 5.2. In particular, it is duality-adapted, and the equivalence
\[
(\text{Shv}_N(\mathcal{Y})^c)^{\text{op}} \to \text{Shv}_N(\mathcal{Y})^c,
\]
corresponding to (5.10), is obtained by restriction from the Verdier duality functor
\[
\mathbb{D}^{\text{Verdier}} : (\text{Shv}(\mathcal{Y})^c)^{\text{op}} \to \text{Shv}(\mathcal{Y})^c.
\]

5.4.2. We claim:

**Lemma 5.4.3.** For a quasi-compact algebraic stack \(Z\), the functor
\[
\text{Id} \otimes \mathcal{P}_{\mathcal{Y}, N} : \text{Shv}(Z \times \mathcal{Y}) \to \text{Shv}(Z) \otimes \text{Shv}_N(\mathcal{Y}),
\]
is the right adjoint of
\[
\text{Shv}(\mathcal{Z}) \otimes \text{Shv}_N(\mathcal{Y}) \xrightarrow{\text{Id} \otimes \iota_{\mathcal{Y}}} \text{Shv}(\mathcal{Z}) \otimes \text{Shv}(\mathcal{Y}) \hookrightarrow \text{Shv}(\mathcal{Z} \times \mathcal{Y}).
\]

**Proof.** Follows from Lemma 2.4.3, since the constraccessibility assumption implies that
\[
(\mathcal{Z} \otimes (\text{Id} \otimes \iota_{\mathcal{Y}}))^\text{fake-op} \simeq (\mathcal{Z} \otimes (\text{Id} \otimes \iota_{\mathcal{Y}})).
\]
□

**Corollary 5.4.4.** The endofunctor \(\mathcal{P}_{\mathcal{Y}, N} \otimes \mathcal{P}_{\mathcal{Y}, N}\) of \(\text{Shv}(\mathcal{Y} \times \mathcal{Y})\), viewed as a functor
\[
\text{Shv}(\mathcal{Y} \times \mathcal{Y}) \to \text{Shv}_N(\mathcal{Y}) \otimes \text{Shv}_N(\mathcal{Y})
\]
identifies with the right adjoint of the tautological embedding.

**Proof.** Obtained by applying Lemma 5.4.3 to
\[
\text{Id}_\mathcal{Y} \otimes \mathcal{P}_{\mathcal{Y}, N} : \text{Shv}(\mathcal{Y} \times \mathcal{Y}) \to \text{Shv}(\mathcal{Y}) \otimes \text{Shv}_N(\mathcal{Y})
\]
and
\[
\mathcal{P}_{\mathcal{Y}, N} \otimes \text{Id} : \text{Shv}(\mathcal{Y}) \otimes \text{Shv}_N(\mathcal{Y}) \to \text{Shv}_N(\mathcal{Y}) \otimes \text{Shv}_N(\mathcal{Y}).
\]
□

**Corollary 5.4.5.** The object \(u_{\mathcal{Y}, N}\) identifies with the value on \(u_{\mathcal{Y}}\) of the right adjoint to the embedding (5.11)
\[
\text{Shv}_N(\mathcal{Y}) \otimes \text{Shv}_N(\mathcal{Y}) \hookrightarrow \text{Shv}(\mathcal{Y} \times \mathcal{Y}).
\]

5.5. **Serre pairs.** In this subsection we introduce the notion of what it means for a pair \((\mathcal{Y}, N)\) to be Serre.

5.5.1. Let \(\mathcal{Y}\) be an algebraic stack (for now we are not assuming that \(\mathcal{Y}\) is quasi-compact). Let \(N\) be a subset in \(T^*(\mathcal{Y})\).

Recall the notation
\[
\text{ps-}u_{\mathcal{Y}} := \Delta_!(\mathbb{1}_\mathcal{Y}) \in \text{Shv}(\mathcal{Y} \times \mathcal{Y}).
\]

Let
\[
\text{ps-}u_{\mathcal{Y}, N} \in \text{Shv}_N(\mathcal{Y}) \otimes \text{Shv}_N(\mathcal{Y})
\]
be the value on \(\text{ps-}u_{\mathcal{Y}}\) of the right adjoint of the embedding
\[
\text{Shv}_N(\mathcal{Y}) \otimes \text{Shv}_N(\mathcal{Y}) \hookrightarrow \text{Shv}(\mathcal{Y}) \otimes \text{Shv}(\mathcal{Y}) \xrightarrow{\mathcal{E}} \text{Shv}(\mathcal{Y} \times \mathcal{Y}).
\]

The counit of the adjunction defines a map
\[
(5.12) \quad \text{ps-}u_{\mathcal{Y}, N} \to \text{ps-}u_{\mathcal{Y}}.
\]
5.5.2. Recall that
\[ \text{ev}^l_y : \text{Shv}(Y) \otimes \text{Shv}(Y) \to \text{Vect} \]
denoted the pairing
\[ \mathcal{F}_1, \mathcal{F}_2 \mapsto C_{\ast}(Y, \mathcal{F}_1 \otimes \mathcal{F}_2). \]
By a slight abuse of notation, we will denote by the same symbol \( \text{ev}^l_y \) its restriction along \( \text{Shv}(\mathcal{N}) \to \text{Shv}(Y) \otimes \text{Shv}(Y) \).

5.5.3. The map (5.12) gives rise to a natural transformation from the functor
\[
\text{Shv}(Y) \xrightarrow{\text{Id} \otimes \text{ps-u}_Y, \mathcal{N}} \text{Shv}(Y \times Y \times Y) \xrightarrow{\text{ev}^l_y \otimes \text{Id} \mathcal{N}} \text{Shv}(Y),
\]
to the composition
\[
\text{Shv}(Y) \xrightarrow{\text{Id} \otimes \text{ps-u}_Y} \text{Shv}(Y \times Y \times Y) \xrightarrow{\text{ev}^l_y \otimes \text{Id} \mathcal{N}} \text{Shv}(Y),
\]
which is the identity functor.

Restricting to \( \text{Shv}(Y) \), we obtain a natural transformation
\[
(\text{ev}^l_y \otimes \text{Id}) \circ (\text{Id} \otimes \text{ps-u}_Y, \mathcal{N}) \to \text{Id}
\]

**Definition 5.5.4.** We shall say that the pair \((Y, \mathcal{N})\) is Serre if the natural transformation (5.13) becomes an isomorphism when restricted to \( \text{Shv}(\mathcal{N}) \).

5.5.5. Thus, by definition, if \((Y, \mathcal{N})\) is Serre, then the object
\[ \text{ps-u}_Y, \mathcal{N} \in \text{Shv}(Y) \otimes \text{Shv}(\mathcal{N}) \]
and the pairing
\[ \text{ev}^l_y : \text{Shv}(Y) \otimes \text{Shv}(\mathcal{N}) \to \text{Shv}(Y) \]
define the unit and the counit for a self-duality on \( \text{Shv}(Y) \).

5.5.6. **Examples.** Let \( Y \) be a smooth proper scheme. Assume that the pair \((Y, \{0\})\) is constraccessible. Then the pair \((Y, \{0\})\) is Serre.

Let us now consider another extreme: let \( Y \) be an algebraic stack with finitely many isomorphism classes of points (e.g., we can take \( Y \) to be \( N\backslash G/B \) or an open substack \( \text{Bun}_G \) for a curve of genus 0). We claim that in case the pair \((Y, T^1(Y))\) is Serre. Indeed, this follows from the fact that for \( Y \) of the above form the functor
\[ \text{Shv}(Y) \otimes \text{Shv}(Y) \to \text{Shv}(Y \times Y) \]
is an equivalence, so
\[ \text{ps-u}_Y, \mathcal{N} \simeq \text{ps-u}_Y. \]

5.5.7. Recall that the category \( \text{Shv}(Y) \) is compactly generated by objects of the form \( g_i(\mathcal{F}) \), for
\[ g : S \to Y, \]
where \( S \) is an affine scheme, \( g \) is smooth and \( \mathcal{F} \in \text{Shv}(Z)^c = \text{Shv}(Z)^{\text{constr}} \) (see Sect. A.1.6). It is easy to see that the \( \text{Hom} \) space between objects of the above form is compact. This implies that the category \( \text{Shv}(Y) \) is proper.

Assume that the pair \((Y, \mathcal{N})\) is constraccessible, i.e., \( \text{Shv}(Y) \) is generated by objects that are compact in \( \text{Shv}(Y) \). We obtain that the category \( \text{Shv}(\mathcal{N}) \) is also proper. Hence, we can consider the Serre functor \( S \text{eShv}(\mathcal{N}) \).
5.5.8. We claim:

**Theorem 5.5.9.** Assume that $\mathcal{Y}$ is quasi-compact, and that $(\mathcal{Y}, N)$ is constraccessible and Serre. Then the category $\text{Shv}_N(\mathcal{Y})$ is Serre. Moreover under the identification $\text{Shv}_N(\mathcal{Y})^\vee \simeq \text{Shv}_N(\mathcal{Y})$ of (5.10), the object

$$\text{ps-u}_{\mathcal{Y}, N} \in \text{Shv}_N(\mathcal{Y}) \otimes \text{Shv}_N(\mathcal{Y})$$

corresponds to the object

$$\text{ps-u}_{\text{Shv}_N(\mathcal{Y})} \in \text{Shv}_N(\mathcal{Y}) \otimes \text{Shv}_N(\mathcal{Y})^\vee$$

of (5.8), and the pairing

$$\text{ev}_L^\mathcal{Y} : \text{Shv}_N(\mathcal{Y}) \otimes \text{Shv}_N(\mathcal{Y}) \to \text{Vect}$$

corresponds to the pairing

$$u^L_{\text{Shv}_N(\mathcal{Y})} : \text{Shv}_N(\mathcal{Y}) \otimes \text{Shv}_N(\mathcal{Y})^\vee \to \text{Vect}$$

of (5.1).

**Proof.** By Corollary 5.1.7 and Proposition 5.2.5, we only have to show that the functor $\text{ev}_L^\mathcal{Y}$ is the left adjoint of

$$\text{Vect} \xrightarrow{u_{\mathcal{Y}, N}} \text{Shv}_N(\mathcal{Y}) \otimes \text{Shv}_N(\mathcal{Y}).$$

However, this is formal from Corollary 5.4.5. $\square$

5.6. **Relation to the miraculous functor.** Let $\mathcal{Y}$ be quasi-compact, and let $(\mathcal{Y}, N)$ be constraccessible.

In this subsection we will relate the Serre functor on $\text{Shv}_N(\mathcal{Y})$ to the miraculous functor.

5.6.1. Consider the object

$$\text{ps-u}_{\mathcal{Y}, N} \in \text{Shv}_N(\mathcal{Y}) \otimes \text{Shv}_N(\mathcal{Y}) \subset \text{Shv}(\mathcal{Y} \times \mathcal{Y}),$$

see Sect. 5.5.1. According to Corollary 5.4.4 we have

$$\text{ps-u}_{\mathcal{Y}, N} \simeq (\text{P}_{\mathcal{Y}, N} \otimes \text{P}_{\mathcal{Y}, N})(\text{ps-u}_{\mathcal{Y}}).$$

5.6.2. Let

$$\text{Mir}_\mathcal{Y} : \text{Shv}(\mathcal{Y}) \to \text{Shv}(\mathcal{Y})$$

be the miraculous functor, i.e., the functor defined by the kernel

$$\text{ps-u}_{\mathcal{Y}} \in \text{Shv}(\mathcal{Y} \times \mathcal{Y}),$$

see Sect. B.4.1.

5.6.3. From Sect. 5.6.1 we obtain:

**Lemma 5.6.4.** The endofunctor of $\text{Shv}(\mathcal{Y})$ defined by $\text{ps-u}_{\mathcal{Y}, N}$ as a kernel is the composite

$$\text{Shv}(\mathcal{Y}) \xrightarrow{\text{P}_{\mathcal{Y}, N}} \text{Shv}(\mathcal{Y}) \xrightarrow{\text{Mir}_\mathcal{Y}} \text{Shv}(\mathcal{Y}) \xrightarrow{\text{P}_{\mathcal{Y}, N}} \text{Shv}(\mathcal{Y}).$$

5.6.5. We give the following definition:

**Definition 5.6.6.** We will say that $N \subset T^*(\mathcal{Y})$ is miraculous-compatible if the functor $\text{Mir}_\mathcal{Y}$ preserves the subcategory $\text{Shv}_N(\mathcal{Y}) \subset \text{Shv}(\mathcal{Y})$.

If $N$ is miraculous-compatible, we will denote by $\text{Mir}_{\mathcal{Y}, N}$ the resulting endofunctor of $\text{Shv}_N(\mathcal{Y})$. 
5.6.7. From Lemma 5.6.4 we obtain:

**Corollary 5.6.8.** Suppose that $\mathcal{N}$ be miraculous-compatible. Then:

(a) The endofunctor of $\text{Shv}_N(\mathfrak{Y})$ defined by $\text{ps-u}_{\mathfrak{Y},N}$ identifies canonically with $\text{Mir}_{\mathfrak{Y},N}$.

(b) We have canonical isomorphisms:

$$(\mathcal{P}_{\mathfrak{Y},N} \otimes \text{Id}_Y)(\text{ps-u}_{\mathfrak{Y}}) \simeq \text{ps-u}_{\mathfrak{Y},N} \simeq (\text{Id}_Y \otimes \mathcal{P}_{\mathfrak{Y},N})(\text{ps-u}_{\mathfrak{Y}}).$$

(c) We have canonical isomorphisms of endofunctors of $\text{Shv}(\mathfrak{Y})$

$$\text{Mir}_Y \circ \mathcal{P}_{\mathfrak{Y},N} \simeq \mathcal{P}_{\mathfrak{Y},N} \circ \text{Mir}_Y.$$  

**Proof.** Point (a) is immediate from Lemma 5.6.4. For point (b), by symmetry, it suffices to prove the first isomorphism. Since $(\mathcal{P}_{\mathfrak{Y},N} \otimes \text{Id}_Y)(\text{ps-u}_{\mathfrak{Y}}) \in \text{Shv}_N(\mathfrak{Y}) \otimes \text{Shv}(\mathfrak{Y})$ (by Lemma 5.4.3), it suffices to show that

$$(\text{ev}_Y \otimes \text{Id})(\mathcal{F} \otimes (\mathcal{P}_{\mathfrak{Y},N} \otimes \text{Id}_Y)(\text{ps-u}_{\mathfrak{Y}})) \simeq (\text{ev}_Y \otimes \text{Id})(\mathcal{F} \otimes (\mathcal{P}_{\mathfrak{Y},N} \otimes \mathcal{P}_{\mathfrak{Y},N})(\text{ps-u}_{\mathfrak{Y}})), \quad \mathcal{F} \in \text{Shv}_N(\mathfrak{Y}).$$

The left-hand side identifies with

$$(\text{ev}_Y \otimes \text{Id})(\mathcal{F} \otimes \text{ps-u}_{\mathfrak{Y}}) \simeq \text{Mir}_Y(\mathcal{F}),$$

while the right-hand side also identifies with $\text{Mir}_Y(\mathcal{F})$, by point (a).

Point (c) is formal from point (b).

\[\square\]

5.6.9. Finally, we claim:

**Corollary 5.6.10.** Suppose that $\mathcal{N}$ is miraculous-compatible and that $(\mathfrak{Y}, \mathcal{N})$ is Serre. Then:

(a) $\text{Mir}_{\mathfrak{Y},N} \simeq \text{Ps-Id}_{\text{Shv}_N(\mathfrak{Y})}$;

(b) $\text{Mir}_{\mathfrak{Y},N} \simeq (\text{Shv}_{\mathcal{N}}(\mathfrak{Y}))^{-1}$.

(c) The diagram

$$\begin{array}{ccc}
\text{Shv}_N(\mathfrak{Y}) \otimes \text{Shv}_N(\mathfrak{Y}) & \xrightarrow{\text{ev}_Y} & \text{Vect} \\
\text{Mir}_{\mathfrak{Y},N} \otimes \text{Id} & & \text{Id} \\
\text{Shv}_N(\mathfrak{Y}) \otimes \text{Shv}_N(\mathfrak{Y}) & \xrightarrow{\text{ev}_Y} & \text{Vect}
\end{array}$$

commutes.

**Proof.** Point (a) follows by combining Corollary 5.6.3(a) and Theorem 5.5.7. Point (b) follows from Corollary 5.2.7. Point (c) follows from Sect. 5.2.8.

\[\square\]

**Corollary 5.6.11.** Suppose that $\mathcal{N}$ is miraculous-compatible and that $(\mathfrak{Y}, \mathcal{N})$ is Serre. Then the endofunctor $\text{Mir}_{\mathfrak{Y},N}$ of $\text{Shv}_N(\mathfrak{Y})$ is an equivalence.

**Remark 5.6.12.** Suppose that in the setting of Corollary 5.6.10 above, the stack $\mathfrak{Y}$ is miraculous (see Sect. C.7.1 for what this means). Recall the notation

$$\mathcal{D}^\text{Mir} : (\text{Shv}(\mathfrak{Y}))^{\mathfrak{Y}^{\mathfrak{Y}}} \rightarrow \text{Shv}(\mathfrak{Y})^c,$$

see Sect. 15.7.2. Recall also the notation $\mathcal{D}^\text{new}$, see Sect. 5.1.9. We obtain:

$$\mathcal{D}^\text{Mir} \simeq \mathcal{D}^\text{new}.$$  

5.7. **Relation to the miraculous functor, continued.** We retain the assumptions of the previous subsection, and additionally require that $\mathcal{N}$ is miraculous-compatible and that $(\mathfrak{Y}, \mathcal{N})$ is Serre.
5.7.1. Note that Corollary 5.6.10 implies that we have a canonical isomorphism
\[(5.15) \quad \text{ev}_Y(F_1, \text{Mir}_Y(F_2)) \simeq \text{ev}_Y(F_1, F_2), \quad F_1, F_2 \in \text{Shv}_N(Y).\]

We claim:

**Proposition 5.7.2.** The isomorphism (5.15) identifies with the natural transformation (4.1).

**Proof.** Repeats the proof of Proposition 4.1.3, with the difference that the vertical arrows in the counterpart of diagram (4.5) are furnished by that natural transformation \(P_{Y, \text{Nilp}} \rightarrow \text{Id},\) given by the counit of the adjunction. \(\Box\)

5.7.3. As in Theorem 4.2.2, from Proposition 5.7.2 we obtain:

**Corollary 5.7.4.** Let \(F\) be an object in \(\text{Shv}_N(Y)^{\text{const}}\). Then the functor
\[F : \text{Shv}(Y) \rightarrow \text{Vect}\]
that it defines admits a right adjoint, as a functor defined by a kernel.

Further, as in Corollary 4.2.5, we obtain:

**Corollary 5.7.5.** Let \(F\) be an object in \(\text{Shv}_N(Y)\). Then the functor
\[F : \text{Shv}(Y) \rightarrow \text{Vect}\]
that it defines, is defined and codefined by a kernel.

As in Corollary 4.3.2, we obtain:

**Corollary 5.7.6.** Let \(Y\) be miraculous, and let \(F\) be an object in \(\text{Shv}_N(Y)\). Then for an algebraic stack \(Z\), the natural transformation
\[(p_Z)_!(\text{Id}_Z \boxtimes \text{Mir}_Y)(F') \otimes p_Z^!(F)) \simeq (p_Z)_!(\text{Id}_Z \boxtimes \text{Mir}_Y)(F') \otimes p_Z^!(F), \quad F' \in \text{Shv}(Z \times Y)\]
is an isomorphism.

Finally, as in Corollary 4.3.5, we obtain:

**Corollary 5.7.7.** Let \(Y\) be miraculous, and let \(F\) be an object in \(\text{Shv}_N(Y)\). Then for an algebraic stack \(Z\), we have a canonical isomorphism the natural transformation
\[(p_Z)_!(\text{Id}_Z \boxtimes \text{Mir}_Y)(F') \otimes p_Z^!(F)) \simeq (p_Z)_!(\text{Id}_Z \boxtimes \text{Mir}_Y)(F') \otimes p_Z^!(F), \quad F' \in \text{Shv}(Z \times Y)\]

5.8. **The (non)-Serre property of \(\text{Bun}_G\).** In this subsection we will study the Serre property of the category \(\text{Shv}_{\text{Nilp}}(\text{Bun}_G)\), and also of \(\text{Shv}_{\text{Nilp}}(U)\), for universally \(\text{Nilp}\)-cotruncative quasi-compact open substacks \(U \subset \text{Bun}_G\).

5.8.1. Let \(U \subset \text{Bun}_G\) be a universally \(\text{Nilp}\)-cotruncative quasi-compact open substack. Note that it follows from Proposition 2.3.3 that the object that was denoted \(u_{U, \text{Nilp}}\) in Sect. 2.3.3 is the same as the object that appears in Sect. 5.3.3 (for \(Y = U\) and \(N = \text{Nilp}\)).

Furthermore, it follows from Proposition 2.2.3 that the endofunctor of \(\text{Shv}(U)\) denoted \(P_{U, \text{Nilp}}\) in Sect. 1.8.1 identifies, as an endofunctor defined by a kernel, with the endofunctor that appears in Sect. 5.3.5 (for \(Y = U\) and \(N = \text{Nilp}\)).

Hence, the notations \(u_{U, \text{Nilp}}\) and \(P_{U, \text{Nilp}}\) are unambiguous.
5.8.2. From now on, for the duration of this subsection, we will assume \[\text{AGKRRV Conjecture 14.1.8}\]. I.e., we will assume that \(\text{Shv}_{\text{Nilp}}(\Bun_G)\) is generated by objects that are compact in \(\text{Shv}(\Bun_G)\). By \[\text{AGKRRV Lemma F.8.10}\], this implies, that for every \(U\) as above, the pair \((U, \text{Nilp})\) is constructable.

Applying Corollary 2.4.9, we obtain that the object \(ps_u \cdot U, \text{Nilp}\) introduced in Sect. 3.6.1 identifies with each of the following objects

\[(P \cdot U, \text{Nilp} \otimes \text{Id}_U)(ps_u \cdot U, \text{Nilp}), (P \cdot U, \text{Nilp} \otimes P \cdot U, \text{Nilp})(ps_u \cdot U, \text{Nilp}), (\text{Id}_U \otimes P \cdot U, \text{Nilp})(ps_u \cdot U, \text{Nilp}).\]

Combining with Sect. 5.6.1, we obtain that \(ps_u \cdot U, \text{Nilp}\) of Sect. 3.6.1 is the same as the object that appears in Sect. 5.5.1 (for \(\mathcal{Y} = U\) and \(N = \text{Nilp}\)). Hence, the notation \(ps_u \cdot U, \text{Nilp}\) is also unambiguous.

5.8.3. We now claim:

Corollary 5.8.4. The pair \((U, \text{Nilp})\) is Serre.

Proof. By Corollary 3.6.3, we know that \(ev_U \cdot \text{Nilp} : \text{Shv}_{\text{Nilp}}(U) \otimes \text{Shv}_{\text{Nilp}}(U) \to \text{Vect}, \mathcal{F}_1, \mathcal{F}_2 \mapsto C_u(U, \mathcal{F}_1 \otimes \mathcal{F}_2)\) and the object \(ps_u \cdot U, \text{Nilp}\) define a duality datum.

In particular, we obtain a canonical isomorphism

\[(ev_U \cdot \text{Id}) \circ (\text{Id} \otimes ps_u \cdot U, \text{Nilp}) \simeq \text{Id}.\]

However, unwinding the definitions, one shows that the above isomorphism agrees with the natural transformation in Definition 5.5.4.

5.8.5. We now consider the non quasi-compact stack \(\Bun_G\). From Corollary 2.4.7 we obtain:

Corollary 5.8.6. The object \(ps_u \cdot \Bun_G, \text{Nilp}\) of Sect. 3.1.2 identifies canonically with the value of the right adjoint to \(\text{Shv}_{\text{Nilp}}(\Bun_G) \otimes \text{Shv}_{\text{Nilp}}(\Bun_G) \to \text{Shv}(\Bun_G \times \Bun_G)\) on \(ps_u \cdot \Bun_G\).

Hence, as in Corollary 5.8.4, from Theorem 3.2.2, we obtain that the pair \((\Bun_G, \text{Nilp})\) is Serre.

5.5.7. The assumption that \[\text{AGKRRV Conjecture 14.1.8}\] holds implies, in particular, that the category \(\text{Shv}_{\text{Nilp}}(\Bun_G)\) is proper. Hence, we can consider the Serre functor \(S_{\text{Shv}_{\text{Nilp}}(\Bun_G)} : \text{Shv}_{\text{Nilp}}(\Bun_G) \to \text{Shv}_{\text{Nilp}}(\Bun_G)\).

We claim:

Theorem 5.8.8. The functor \(S_{\text{Shv}_{\text{Nilp}}(\Bun_G)}\) is canonically isomorphic to the composition

\[\text{Shv}_{\text{Nilp}}(\Bun_G) \xrightarrow{\text{Mir}_{\Bun_G}^{-1}} \text{Shv}_{\text{Nilp}}(\Bun_G)_{\text{co}} \xrightarrow{\text{Id}_{\text{naive}}^{\Bun_G}} \text{Shv}_{\text{Nilp}}(\Bun_G).\]

Proof. Let \(U \xrightarrow{j} \Bun_G\) be a universally Nilp-cotruncative quasi-compact open substack. Unwinding the definitions, we obtain a canonical isomorphism

\[S_{\text{Shv}_{\text{Nilp}}(U)} \circ j \simeq j_* \circ S_{\text{Shv}_{\text{Nilp}}(U)} .\]

Furthermore, for an inclusion \(U_1 \xrightarrow{j_{1,2}} U_2\), we have

\[S_{\text{Shv}_{\text{Nilp}}(U_2)} \circ (j_{1,2})_* \simeq (j_{1,2})_* \circ S_{\text{Shv}_{\text{Nilp}}(U_1)} ,\]

compatible with (5.16).

By (2.22), we also have a system of isomorphisms

\[\text{Id}_{\Bun_G}^{\text{naive}} \circ \text{Mir}_{\Bun_G}^{-1} \circ j_* \simeq j_* \circ \text{Id}_U^{-1} ,\]
compatible with
\[ \text{Mir}_{U_2}^{-1} \circ (j_{1,2})_* \simeq (j_{1,2})_* \circ \text{Mir}_{U_1}^{-1}. \]

Hence, it suffices to show that the system of isomorphisms
\[ \text{Se}_{\text{ShvNilp}}(u_1) \simeq \text{Mir}^{-1}_{U_1,\text{Nilp}} \]
of Corollary 5.6.10(b) makes the system of diagrams
\[
\begin{array}{ccc}
\text{Se}_{\text{ShvNilp}}(u_2) \circ (j_{1,2})! & \longrightarrow & (j_{1,2})_* \circ \text{Se}_{\text{ShvNilp}}(u_1) \\
\downarrow & & \downarrow \\
\text{Mir}_{U_2}^{-1} \circ (j_{1,2})! & \longrightarrow & (j_{1,2})_* \circ \text{Mir}_{U_1}^{-1}.
\end{array}
\]

However, this follows by unwinding the definitions. □

5.8.9. It follows from Theorem 5.8.8 that the functor \( \text{Se}_{\text{ShvNilp}}(\text{Bun}_G) \) sends compact objects of \( \text{ShvNilp}(\text{Bun}_G) \) to objects that lie in the essential image of the \textit{fully faithful} functor
\[ (\text{ShvNilp}(\text{Bun}_G)_c) \hookrightarrow \text{ShvNilp}(\text{Bun}_G) \xrightarrow{\text{Id}_{\text{Bun}_G}} \text{ShvNilp}(\text{Bun}_G). \]

Denote by
\[ \text{Se}_{\text{ShvNilp}}(\text{Bun}_G)_c \]
the ind-extension of the resulting functor
\[ \text{ShvNilp}(\text{Bun}_G) \rightarrow \text{ShvNilp}(\text{Bun}_G)_c. \]

From Theorem 5.8.8 we obtain:

**Corollary 5.8.10.** The functor \( \text{Se}_{\text{ShvNilp}}(\text{Bun}_G)_c \) identifies canonically with \( (\text{Mir}^{-1}_{\text{Bun}_G})|_{\text{ShvNilp}(\text{Bun}_G)}. \)

**Appendix A. Sheaves on stacks**

In this section we review the theory of \( \text{Shv}(-) \) on stacks, especially the aspects that have to do with Verdier duality, used in the bulk of the paper.

**A.1. The basics.** In this subsection we name the main players in \( \text{Shv}(-) \) on stacks.

A.1.1. In this paper we work with a constructible sheaf theory, denoted \( \text{Shv} \), which we view as a functor
\[ (\text{Sch}_{\text{aff}}^f_{/k})^{\text{op}} \rightarrow \text{DGCat}, \]
see [AGKRRV, Sects. 1.1], where for \( f : S_1 \rightarrow S_2 \), the functor \( \text{Shv}(S_2) \rightarrow \text{Shv}(S_1) \) is \( f^! \).

For a given affine scheme \( S \), the category \( \text{Shv}(S) \) is defined as the ind-completion of the category \( \text{Shv}(S)^{\text{const}} \) of constructible sheaves, so that \( \text{Shv}(S)^c \) recovers \( \text{Shv}(S)^{\text{const}} \).

A.1.2. We extend \( \text{Shv}(-) \) from affine schemes to algebraic stacks by the procedure of right Kan extension. Explicitly, for an algebraic stack \( \mathcal{Y} \),
\[ \text{Shv}(\mathcal{Y}) := \lim_{S \rightarrow \mathcal{Y}}^{\text{op}} \text{Shv}(S), \]
where the index category is any of the following:
- All affine schemes \( S \) of finite type over \( \mathcal{Y} \);
- Affine schemes \( S \) that map smoothly to \( \mathcal{Y} \), and all maps \( f : S_1 \rightarrow S_2 \);
- Affine schemes \( S \) that map smoothly to \( \mathcal{Y} \), and smooth maps \( f : S_1 \rightarrow S_2 \).

In the formation of the limit, the transition functors \( \text{Shv}(S_2) \rightarrow \text{Shv}(S_1) \) are given by \( f^! \).
A.1.3. One can also rewrite $\text{Shv}(\mathcal{Y})$ as
\[
\lim_{S \to \mathcal{Y}}^* \text{Shv}(S),
\]
over the same choice of index categories, but with transition functors $\text{Shv}(S_2) \to \text{Shv}(S_1)$ given by $f^*$. 

The two limits are equivalent via the following procedure: use the third index category in both cases. Applying the cohomological shift by $[2 \dim(S/Y)]$ on each $\text{Shv}(S)$, we can isomorph the limit (A.2) to (A.3)
\[
\lim_{S \to \mathcal{Y}}^* \text{shift} \text{Shv}(S),
\]
where the transition functors are now $f^*[2 \dim(S_2/S_1)]$.

Now, the limit (A.3) is equivalent to (A.1) term-wise.

A.1.4. For a stack $\mathcal{Y}$, we let $\text{Shv}(\mathcal{Y})^{\text{constr}}$ be the full subcategory of $\text{Shv}(\mathcal{Y})$ equal to $\lim_{S \to \mathcal{Y}} \text{Shv}(S)^{\text{constr}}$ (with respect to any of the above three index categories and either $-!$ or $-*$ version).

For an algebraic stack $\mathcal{Y}$, we let $\mathcal{E}_\mathcal{Y} \in \text{Shv}(\mathcal{Y})^{\text{constr}}$ denote the constant sheaf.

A.1.5. Built into the definition of $\text{Shv}(\mathcal{Y})$ are the functors
\[
f_!^*, f_*^* : \text{Shv}(\mathcal{Y}_2) \to \text{Shv}(\mathcal{Y}_1)
\]
for a morphism $f : \mathcal{Y}_2 \to \mathcal{Y}_1$ between algebraic stacks.

In addition, the $(-!, -^*)$ (resp., $(-*, -^*)$) base change isomorphisms for maps between schemes give rise to functors
\[
f_!^*, f_*^* : \text{Shv}(\mathcal{Y}_1) \to \text{Shv}(\mathcal{Y}_2)
\]
when $f$ is schematic. It follows from the construction that $f_!$ (resp., $f_*$) is the left (resp., right) adjoint of $f^!$ (resp., $f^*$).

We will explain how extend this definition for maps that are not necessarily schematic in Sect. A.1.7 below.

A.1.6. As is explained in [AGKRRV, Sect. F.1.1], for any algebraic stack $\mathcal{Y}$, the category $\text{Shv}(\mathcal{Y})$ is compactly generated. Namely, we can take as generators objects of the form $g_!(\mathcal{F})$ for $g : S \to \mathcal{Y}$, $\mathcal{F} \in \text{Shv}(S)^{\text{constr}}$.

We have an inclusion
\[
\text{Shv}(\mathcal{Y})^{\text{constr}} \subset \text{Shv}(\mathcal{Y})^{\text{constr}},
\]
but this inclusion is in general not an equality. For example, the object $\mathcal{E}_0$ is not compact for $\mathcal{Y} = B\mathbb{G}_m$.

It easy to see, however, that the subcategory $\text{Shv}(\mathcal{Y})^{\text{constr}}$ is preserved by the $*$ operation with objects in $\text{Shv}(\mathcal{Y})^{\text{constr}}$ (see (A.5) below).

A.1.7. Let $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ be a morphism between algebraic stacks. We define the functor
\[
f_! : \text{Shv}(\mathcal{Y}_1) \to \text{Shv}(\mathcal{Y}_2)
\]
as the left adjoint of $f^!$.

To show that $f_!$ exists, it is enough to show that it is defined on the generators of $\text{Shv}(\mathcal{Y}_1)$. Taking the generators described in Sect. A.1.6, we have
\[
f_!(g_!(\mathcal{F})) \simeq (f \circ g)_!(\mathcal{F}),
\]
where the right-hand side is well-defined because the morphism $f \circ g$ is schematic.

The same argument shows that the $-!$ pushforward satisfies base change against the $-^*$ pullback.
A.1.8. For \( f \) as above, we define the functor
\[
f_* : \text{Shv}(\mathcal{Y}_1) \to \text{Shv}(\mathcal{Y}_2)
\]
as the right adjoint of \( f^* \).

This functor exists for general reasons\footnote{I.e., Lurie’s Adjoint Functor Theorem, which in particular says that any colimit preserving functor between presentable DG categories admits a right adjoint.}, however it may be discontinuous (because the functor \( f^* \) does not necessarily preserve compactness).

However, \( -_* \) pushforward does satisfy base change against \( -^! \) pullback. This follows by passing to right adjoints from the \((-1,-^*)\) base change.

A.1.9. For any algebraic stack \( \mathcal{Y} \), Verdier duality defines a contravariant self-equivalence
\[
\mathbb{D}^{\text{Verdier}} : (\text{Shv}(\mathcal{Y})^{\text{constr}})^{\text{op}} \to \text{Shv}(\mathcal{Y})^{\text{constr}}.
\]

Its basic property is that for \( \mathcal{F} \in \text{Shv}(\mathcal{Y})^{\text{constr}} \)
\[
\mathcal{K} \text{Hom}_{\text{Shv}(\mathcal{Y})}(\mathcal{F} \boxtimes \mathcal{F}_1, \mathcal{F}_2) \simeq \mathcal{K} \text{Hom}_{\text{Shv}(\mathcal{Y})}(\mathcal{F}_1, \mathbb{D}^{\text{Verdier}}(\mathcal{F}) \boxtimes \mathcal{F}_2), \quad \mathcal{F}_1, \mathcal{F}_2 \in \text{Shv}(\mathcal{Y}).
\]

We will denote by \( \omega_\mathcal{Y} \in \text{Shv}(\mathcal{Y}) \) the dualizing sheaf on \( \mathcal{Y} \):
\[
\mathbb{D}^{\text{Verdier}}(\omega_\mathcal{Y}).
\]

A.2. Verdier-compatible stacks. In this subsection we will assume that algebraic stacks are quasi-compact.

We review the notion of what it means for an algebraic stack to be Verdier compatible. On the one hand, this property confers some particularly favorable properties on to \( \text{Shv}(\mathcal{Y}) \). On the other hand, it is quite ubiquitous.

A.2.1. It is not clear whether for a general algebraic stack \( \mathcal{Y} \), the Verdier involution \( \mathbb{D}^{\text{Verdier}} \) always sends \( \text{Shv}(\mathcal{Y})^{c} \) to \( \text{Shv}(\mathcal{Y})^{c} \).

If this is the case, following \[AGKRRV, \text{Sect. F.2.6}\], we shall say that \( \mathcal{Y} \) is \textit{Verdier compatible}. This property always holds under the assumption on algebraic stacks imposed in Sect. 0.5.1, see \[AGKRRV, \text{Theorem F.2.8}\].

Thus, we will assume that all stacks involved are Verdier compatible.

A.2.2. Since Verdier duality swaps \( -_* \) and \( -^! \), we obtain that objects of the form \( g_*(\mathcal{F}) \), where
\[
g : S \to \mathcal{Y}, \quad \mathcal{F} \in \text{Shv}(S)^c, \quad S \in \text{Sch}_{\text{aff}}/_{/k}
\]
are compact in and generate \( \text{Shv}(\mathcal{F}) \).

This implies that for a morphism \( f : \mathcal{Y}_1 \to \mathcal{Y}_2 \), the functor \( f_* \) sends \( \text{Shv}(\mathcal{Y}_1)^c \) to \( \text{Shv}(\mathcal{Y}_2)^c \).

This also implies that the subcategory \( \text{Shv}(\mathcal{Y})^c \) is preserved by the \( - \boxtimes - \) operation with objects in \( \text{Shv}(\mathcal{Y})^{\text{constr}} \).
A.2.3. For a map between stacks \( f : Y_1 \to Y_2 \), the usual functor of direct image \( f_* : \text{Shv}(Y_1) \to \text{Shv}(Y_2) \) is not in general colimit-preserving. We define the renormalized functor of direct image, denoted \( f^\bullet : \text{Shv}(Y_1) \to \text{Shv}(Y_2) \), to be the unique colimit-preserving functor such that restricts to \( f_* \) on \( \text{Shv}(Y_1)^c \). We have a tautologically defined natural transformation \( f^\bullet \to f_* \), which is an isomorphism if \( f \) is schematic.

If follows from Sect. A.2.2 that the functor \( f^\bullet \) preserves compactness and satisfies the projection formula. This in turn implies that for a pair of composable morphisms

\[
y_1 \xrightarrow{f_{1,2}} y_2 \xrightarrow{f_{2,3}} y_3,
\]

the natural transformation

\[
(f_{2,3})^\bullet \circ (f_{1,2})^\bullet \to (f_{2,3} \circ f_{1,2})^\bullet
\]

is an isomorphism.

A.2.4. For \( Y_1 = Y \) and \( Y_2 = \text{pt} \), we obtain the renormalized functor of sheaf cochains, denoted

\[
C^\bullet(Y, -) : \text{Shv}(Y) \to \text{Vect}.
\]

By Sect. A.2.3 for a map \( f : Y_1 \to Y_2 \), we have

\[
C^\bullet(Y_1, -) \simeq C^\bullet(Y_2, -) \circ f^\bullet.
\]

A.2.5. For future reference, we claim:

**Lemma A.2.6.** The functor \( f^\bullet : \text{Shv}(Y_1) \to \text{Shv}(Y_2) \) has a cohomological amplitude bounded on the right.

**Proof.** Covering \( Y_2 \) by a scheme (and using base change, see Sect. A.3.1 below), we can assume that \( Y_2 \) is a scheme \( Y_2 \).

By the assumption in Sect. 0.5.1 we can assume that \( Y_1 \) is of the form \( Y_1/H \), where \( Y_1 \) is a (quasi-compact) scheme and \( H \) is an algebraic group. We can factor the map \( f \) as

\[
Y_1/H \to Y_2/H = Y_2 \times BH \to Y_2.
\]

The arrow \( Y_1/H \to Y_2/H \) is schematic, so the \( -^\bullet \) functor is the same as \( -_* \), and so has a finite cohomological amplitude. Hence, it remains to consider the morphism of the form

\[
p_Y : Y \times BH \to Y.
\]

Any object \( \mathcal{F} \in \text{Shv}(Y \times BH) \) admits a finite canonical filtration with subquotients of the form

\[
p_Y^* \circ q_Y^*(\mathcal{F})[n],
\]

where \( q_Y \) is the map \( Y \to Y \times BH \). Hence, it remains to show that

\[
C^\bullet(BH, e_{BH}) \in \text{Vect}
\]

is bounded above.

However, this is a computation performed in [DrGa0, Example 9.1.6].
A.2.7. in general, the functors $f_*$ and $f^!$ do not preserve constructibility. However, the argument in Lemma [A.2.6] shows that both these functors send constructible objects to objects with constructible perverse cohomologies.

Furthermore, we have the following assertion:

**Proposition A.2.8.** For a constructible object $\mathcal{F} \in \text{Shv}(Y_1)$, the following conditions are equivalent:

(i) The map $f^!(\mathcal{F}) \to f_*(\mathcal{F})$ is an isomorphism;
(ii) The object $f^!(\mathcal{F}) \in \text{Shv}(Y_2)$ is cohomologically bounded on the left;
(ii) The object $f_*(\mathcal{F}) \in \text{Shv}(Y_2)$ is cohomologically bounded on the right.

**Proof.** Since both objects $f^!(\mathcal{F})$ and $f_*(\mathcal{F})$ have constructible cohomologies, it is enough to show that for every $k$-point $pt \xrightarrow{i_{Y_2}} Y_2$,

the following conditions are equivalent:

(i) The map $i_{Y_2} \circ f^!(\mathcal{F}) \to i_{Y_2} \circ f_*(\mathcal{F})$ is an isomorphism;
(ii) The object $i_{Y_2} \circ f^!(\mathcal{F}) \in \text{Vect}$ is cohomologically bounded on the left;
(ii) The object $i_{Y_2} \circ f_*(\mathcal{F}) \in \text{Vect}$ is cohomologically bounded on the right.

Note that both functors $f^!$ and $f_*$ satisfy base change against $!$-pullbacks: this is obvious for $f_*$, e.g., by passing to left adjoints, and for $f^!$, this is established in Sect. [A.3.1] below.

This reduces the proposition to the case when $Y_2 = pt$. Write $Y_1 := Y = Z/H$, where $Z$ is a scheme, and let is factor the projection $Y \to pt$ as

$\bar{g} : Y \to pt/H \to pt$.

The map $g$ schematic, and hence $g^* \to g_*$ is an isomorphism. This allows us to replace $Y$ by $pt/H$. However, in the latter case, the required assertion was established in [DrGa0, Proposition 10.4.7]. □

**A.3. Base change maps.** Let

\[
\begin{align*}
Y'_1 & \xrightarrow{f'} Y'_2 \\
\downarrow g_1 & \downarrow g_2 \\
Y_1 & \xrightarrow{f} Y_2
\end{align*}
\]

be a Cartesian diagram of quasi-compact algebraic stacks.

In this subsection we will construct certain natural transformations that have to do with applying various direct/inverse image functors along the arrows in the above diagram.

**A.3.1.** First, we claim that there is a canonical isomorphism

\[g_2^! \circ f^! \simeq f^* \circ g_1^!\]  \hfill (A.6)

First, we consider the case when $g_2$ is schematic (and hence so is $g_1$).

Note that for a schematic map $g : Y' \to Y$ the functor $g^*$ preserves compactness. Indeed, its right adjoint $g_*$ is continuous. Hence, $g^!$ also preserves compactness, by Verdier-compatibility.

Hence, (A.6) is obtained by ind-extending the restriction to compact objects the usual base change isomorphism

\[g_2^! \circ f^! \simeq f_! \circ g_1^!\].
A.3.2. For a general $g_2$ we proceed as follows.

It suffices to construct a compatible system of isomorphisms

$$h_2^2 \circ g_2 \circ f^*_\bullet \simeq h_1^2 \circ f^*_\bullet \circ g_1^1$$

in the Cartesian diagrams

\[
\begin{array}{ccc}
\mathcal{Y}_2'' & \xrightarrow{f''} & \mathcal{Y}_2'' \\
\downarrow h_1 & & \downarrow h_2 \\
\mathcal{Y}_2' & \xrightarrow{f'} & \mathcal{Y}_2 \\
\downarrow g_1 & & \downarrow g_2 \\
\mathcal{Y}_1 & \xrightarrow{f} & \mathcal{Y}_2,
\end{array}
\]

where $h_2 : \mathcal{Y}_2'' \to \mathcal{Y}_2'$ is a smooth map from an affine scheme.

We have

$$h_2^2 \circ g_2 \circ f^*_\bullet \simeq (g_2 \circ h_2)^2 \circ f^*_\bullet \circ g_2 \circ h_2 \text{ is schematic} \simeq f''_\bullet \circ (g_1 \circ h_1)^1 \simeq f''_\bullet \circ h_1^1 \circ g_1 \simeq (f^*_\bullet \circ g_1 \circ h_1)^1 \simeq h_2^1 \circ f^*_\bullet \circ g_1^1.$$  

A.3.3. Next, we claim that there is natural transformation

$$(g_2) \circ f'' \to f^*_\bullet \circ (g_1).$$

Indeed, it is obtained by ind-extending the restriction to compact objects the natural transformation

$$(g_2) \circ f'' \to f^*_\bullet \circ (g_1):$$

arising by adjunction from the base change isomorphism $f''_\bullet \circ g_1^1 \simeq g_2 \circ f_\bullet.$

A.3.4. Finally, we claim that there is a natural transformation

$$g_2^2 \circ f^*_\bullet \to f^*_\bullet \circ (g_1)^{!*}.$$  

First, the if $g_2$ is schematic (and hence so is $g_1$, and so the functors $g_2^*$ and $g_1^*$ preserve compactness), the natural transformation \[A.3\] is obtained by ind-extending the restriction to compact objects the natural transformation

$$g_2^* \circ f^*_\bullet \to f''_\bullet \circ g_1^1$$

arising by adjunction from the isomorphism $f^*_\bullet \circ (g_1)^{!*} \simeq (g_2)^{!*} \circ f''_\bullet.$

Note that \[A.3\] is an isomorphism if $g_2$ is smooth.

A.3.5. For a general $g_2$ we proceed as follows:

It suffices to construct a compatible system of natural transformations

$$h_2^2 \circ g_2 \circ f^*_\bullet \to h_2^2 \circ f^*_\bullet \circ g_1^1$$

for $h_2$ as in Sect. A.8

We rewrite the left-hand side in \[A.10\] as $(g_2 \circ h_2)^* \circ f^*_\bullet$, and it admits a natural transformation to $f''_\bullet \circ (g_1 \circ h_1)^{!*}$, since the map $g_2 \circ h_2$ is schematic.

Finally, we rewrite $f''_\bullet \circ (g_1 \circ h_1)^{!*}$ as

$$f''_\bullet \circ h_1^1 \circ g_1^1 \simeq h_2^2 \circ f^*_\bullet \circ g_1^1,$$

which is the right-hand side in \[A.10\].

A.4. **Verdier self-duality.** In this subsection we continue to assume that our substacks are quasi-compact (and Verdier-compatible).

We will show how the Verdier (anti)-involution $\operatorname{Shv}(\mathcal{Y})^{*}$ gives rise to a self-duality of $\operatorname{Shv}(\mathcal{Y})$ as a DG category.
A.4.1. By assumption, the functor \(A.3\) induces an equivalence
\[
D^{\text{Verdier}} : (\text{Shv}(\mathcal{Y}))^{\text{op}} \to \text{Shv}(\mathcal{Y})\).
\]
Hence, we obtain an identification
\[
\text{Shv}(\mathcal{Y})^\vee \simeq \text{Shv}(\mathcal{Y}),
\]
see Sect. \[0.5.3\].

A.4.2. We claim that the counit of the duality \(A.12\) is given by the functor \(\text{ev}_Y\)
\[
\text{Shv}(\mathcal{Y}) \otimes \text{Shv}(\mathcal{Y}) \xrightarrow{\text{ev}_Y} \text{Shv}(\mathcal{Y}) \xrightarrow{C_G(Y,-)} \text{Vect}.
\]
Indeed, this follows from \(A.5\).

A.4.3. The unit of the duality \(A.12\) is an object that we denote
\[
u_{\text{Shv}(\mathcal{Y})} \in \text{Shv}(\mathcal{Y}) \otimes \text{Shv}(\mathcal{Y}).
\]
This object should not be confused with the object
\[
u_Y = (\Delta_Y)_*(\omega_Y) \in \text{Shv}(\mathcal{Y} \times \mathcal{Y}),
\]
see Sect. \[2.1.2\].

A.4.4. It follows from the projection formula that for a map \(f : \mathcal{Y}_1 \to \mathcal{Y}_2\) between algebraic stacks, we have
\[
(f^! \vee \simeq f_*,
\]
where we identify
\[
\text{Shv}(\mathcal{Y}_1)^\vee \simeq \text{Shv}(\mathcal{Y}_1) \text{ and } \text{Shv}(\mathcal{Y}_2)^\vee \simeq \text{Shv}(\mathcal{Y}_2)
\]
by means of \(A.12\).

A.5. **Specifying singular support.** In this subsection we will see how the notions reviewed in the previous subsection interact with the condition imposed by singular support.

A.5.1. Let \(N\) be a conical Zariski-closed subset of \(T^* \mathcal{Y}\) (see \[AGKRRV\ Sects. F.6.1-F.6.2\]). In this case, following \[AGKRRV\ Sect. F.6.3\], we define a full subcategory
\[
\text{Shv}_N(\mathcal{Y}) \subset \text{Shv}(\mathcal{Y}).
\]
For \(N\) being the zero-section \(\{0\}\) and \(\mathcal{Y}\) smooth, we will use the notation
\[
\text{QLisse}(\mathcal{Y}) := \text{Shv}_{\{0\}}(\mathcal{Y}).
\]

A.5.2. Following \[AGKRRV\ Sect. F.7\], we shall say that the pair \((\mathcal{Y}, N)\) is **constraccessible** if \(\text{Shv}_N(\mathcal{Y})\) is generated by objects that are compact in \(\text{Shv}(\mathcal{Y})\).

In general, it may happen that \(\text{Shv}_N(\mathcal{Y})\) is compactly generated as an abstract DG category, but its compact generators are not compact as objects of \(\text{Shv}(\mathcal{Y})\). For example, this happens for \(\mathcal{Y} = \mathbb{P}^1\), \(N = \{0\}\), see \[AGKRRV\ Sect. E.2.6\].

**Remark A.5.3.** The above definition of constraccessibility is slightly more restrictive than the one given in \[AGKRRV\ Definition F.7.5\]. However, the two definitions agree under our assumptions on algebraic stacks (see Sect. \[0.5.1\] by \[AGKRRV\ Proposition F.7.9\].

If \(\mathcal{Y}\) is non-quasi-compact, but is \(N\)-truncatable (see Sect. \[C.1.6\] below), then the two definitions agree by \[AGKRRV\ Corollary F.8.11\].
A.5.4. Assume for a moment that \( Y \) is quasi-compact.

We shall say that the pair \((Y,N)\) is \textit{duality-adapted} if the restriction of (A.13) along

\[
\operatorname{Shv}_N(Y) \otimes \operatorname{Shv}_N(Y) \to \operatorname{Shv}(Y) \otimes \operatorname{Shv}(Y)
\]
defines the counit of a self-duality.

This is always the case when \((Y,N)\) is constraccessible: indeed, the resulting contravariant self-equivalence

\[
(\operatorname{Shv}_N(Y)^c)^{\text{op}} \to \operatorname{Shv}_N(Y)^c
\]
is induced by restricting the Verdier duality functor \( D_{\text{Verdier}} \)

\[
\operatorname{Shv}(Y)^c \simeq \operatorname{Shv}_N(Y) \cap \operatorname{Shv}(Y)^c \subset \operatorname{Shv}(Y)^{\text{constr}}.
\]

However, the pair \((Y,N)\) may be duality-adapted without being constraccessible, see again [AGKRRV, Sect. E.2.6 and Corollary E.4.7].

A.5.5. According to [AGKRRV, Corollary E.4.7], if \( X \) is a smooth algebraic curve, then the pair \((X,\{0\})\) is duality-adapted.

A.6. \textbf{Categorical Künneth formulas.} Let \( Y_1 \) and \( Y_2 \) be a pair of algebraic stacks. External tensor product defines a functor

(A.15)

\[
\operatorname{Shv}(Y_1) \otimes \operatorname{Shv}(Y_2) \xrightarrow{\cong} \operatorname{Shv}(Y_1 \times Y_2),
\]

which is easily seen to be fully faithful. However, (A.15) is rarely an equivalence.

Categorical Künneth formulas are assertions to the effect that (A.15) becomes an isomorphism after adjusting both sides. One such assertion is formulated here as Theorem A.6.5.

A.6.1. Let \( N_i \subset T^*(Y_i) \) be conical Zariski-closed subsets. Then the functor (A.15) gives rise to a functor

(A.16)

\[
\operatorname{Shv}_{N_i}(Y_1) \otimes \operatorname{Shv}_{N_2}(Y_2) \to \operatorname{Shv}_{N_1 \times N_2}(Y_1 \times Y_2).
\]

If one of the categories \( \operatorname{Shv}_{N_i}(Y_1) \) or \( \operatorname{Shv}_{N_2}(Y_2) \) is dualizable, then (A.16) is also fully faithful.

A.6.2. Throughout the paper, we use the following notation. Let \( N_1 \subset T^*(Y_1) \) be a closed conical subset.

Let

\[
\operatorname{Shv}_{N_1 \times \frac{1}{2} \text{-dim}}(Y_1 \times Y_2) \subset \operatorname{Shv}(Y_1 \times Y_2)
\]

be the full subcategory consisting of objects \( F \) with the property that for every \( m \) and every constructible sub-object \( F' \) of \( H^m(F) \), the singular support of \( F' \) is contained in a subset of the form

\[
N_1 \times N_2 \subset T^*(Y_1 \times Y_2),
\]

where \( N_2 \subset T^*(Y_2) \) is \textit{half-dimensional}.

A.6.3. We have the obvious inclusion

\[
\operatorname{Shv}_{N_1 \times \frac{1}{2} \text{-dim}}(Y_1 \times Y_2) \subset \operatorname{Shv}_{N_1 \times T^*(Y_2)}(Y_1 \times Y_2).
\]

If \( N_1 \) is Lagrangian \textit{and} \( \text{char}(k) = 0 \), then the above inclusion is an equality: this follows from the Lagrangian property of the singular support. But if \( \text{char}(k) \neq 0 \), this may be a proper inclusion.

Note that the image of the (fully faithful) functor

\[
\operatorname{Shv}_{N_1}(Y_1) \otimes \operatorname{Shv}(Y_2) \to \operatorname{Shv}(Y_1 \times Y_2)
\]
is contained in \( \operatorname{Shv}_{N_1 \times \frac{1}{2} \text{-dim}}(Y_1 \times Y_2) \).
A.6.4. Assume now that $Y_1 = Y_2$ is a smooth and proper scheme. In this case, we have the following assertion (see [AGKRRV, Theorems E.9.5 and F.9.7]):

**Theorem A.6.5.**

(a) Assume that $QLisse(Y_1)$ is dualizable as a DG category. Then the functor

$$QLisse(Y_1) \otimes \text{Shv}(y_2) \to \text{Shv}_{(0) \times \frac{1}{2}\text{-dim}}(Y_1 \times y_2)$$

is an equivalence.

(b) Assume that the pair $(Y_1, \{0\})$ is duality-adapted. Then for a half-dimensional closed conical subset $N_2 \subset T^*(y_2)$, the functor

$$QLisse(Y_1) \otimes \text{Shv}_{N_2}(y_2) \to \text{Shv}_{(0) \times N_2}(Y_1 \times y_2)$$

is an equivalence.

**Appendix B. Functors defined by kernels and miraculous duality**

Some of the key methods employed in the bulk of this paper have to do with the formalism of functors defined by kernels. In this subsection we review the relevant theory.

We should also mentioned that for the main results of this paper, all we need is the contents of Sect. [B.1]. Other subsections of this section are needed for the material in Sect. [4].

B.1. **Functors defined by kernels.** In this subsection we introduce what we mean by functors defined by kernels.

B.1.1. Let $Y_1$ and $Y_2$ be quasi-compact algebraic stacks. The category of functors $\text{Shv}(Y_1) \to \text{Shv}(Y_2)$ defined by kernels is by definition

$$\text{Shv}(Y_1 \times Y_2).$$

An object $Q \in \text{Shv}(Y_1 \times Y_2)$ defines an actual functor

$$(B.1) \quad Q : \text{Shv}(Y_1) \to \text{Shv}(Y_2), \quad \mathcal{F} \mapsto (p_2)_!(p_1^!(\mathcal{F}) \otimes Q).$$

We shall say that a given functor $\text{Shv}(Y_1) \to \text{Shv}(Y_2)$ is defined by a kernel if it comes from an object in $\text{Shv}(Y_1 \times Y_2)$ by the above procedure.

B.1.2. For an object $Q \in \text{Shv}(Y_1 \times Y_2)$ we will denote by $Q^o$ the object of $\text{Shv}(Y_2 \times Y_1)$ obtained by swapping the two factors.

The corresponding functor $Q^o$ is the dual functor of $Q$ with respect to the Verdier self-duality of $\text{Shv}(Y_1)$:

$$\text{ev}_{y_2}(\mathcal{F}_1 \otimes Q^o(\mathcal{F}_2)) \simeq \text{ev}_{y_2}(Q(\mathcal{F}_1) \otimes \mathcal{F}_2)$$

(see [A.13] for the notation $\text{ev}_Y$).

B.1.3. More generally, for any algebraic stack $Z$ we have a well-defined functor

$$(B.2) \quad \text{Id}_Z \otimes Q : \text{Shv}(Z \times Y_1) \to \text{Shv}(Z \times Y_2), \quad \mathcal{F} \mapsto (p_{Z,y_2})_!(p_{Z,y_1}^!(\mathcal{F}) \otimes p_{y_1,y_2}(Q)),$$

where $p_{Z,y_1}$, $p_{Z,y_2}$ and $p_{y_1,y_2}$ are the maps

$$Z \times y_1 \times y_2 \to y_1 \times y_2, \quad Z \times y_1 \times y_2 \to Z \times Y_1, \quad Z \times Y_1 \times y_2 \to Z \times Y_2,$$

respectively.

Note that we have a commutative diagram

$$(B.3) \quad \begin{array}{ccc} \text{Shv}(Z) \otimes \text{Shv}(Y_1) & \longrightarrow & \text{Shv}(Z \times Y_1) \\
\text{Id}_{\text{Shv}(Z)} \otimes Q & & \text{Id}_{\text{Shv}(Z)} \otimes Q \\
\text{Shv}(Z) \otimes \text{Shv}(Y_2) & \longrightarrow & \text{Shv}(Z \times Y_2). \end{array}$$
B.1.4. It follows from the description of compact generators in Sect. [A.2.2] that if \( Q \) is compact, then the functors \( \text{Id}_Z \otimes Q \) send compact objects to compact objects.

B.1.5. Using Sect. [A.3.1] we obtain that the functors \( \text{Id}_Z \otimes Q \) commute with the following operations:

- For a map \( f : Z' \to Z \), the diagram
  \[
  \begin{array}{ccc}
  \text{Shv}(Z' \times Y_1) & \xrightarrow{\text{Id}_{Z'} \otimes Q} & \text{Shv}(Z' \times Y_2) \\
  (f \times \text{id}) & \downarrow & (f \times \text{id}) \\
  \text{Shv}(Z \times Y_1) & \xrightarrow{\text{Id}_Z \otimes Q} & \text{Shv}(Z \times Y_2)
  \end{array}
  \]
  commutes.

- For a map \( f : Z' \to Z \), the diagram
  \[
  \begin{array}{ccc}
  \text{Shv}(Z' \times Y_1) & \xrightarrow{\text{Id}_{Z'} \otimes Q} & \text{Shv}(Z' \times Y_2) \\
  (f \times \text{id})' & \uparrow & (f \times \text{id})' \\
  \text{Shv}(Z \times Y_1) & \xrightarrow{\text{Id}_Z \otimes Q} & \text{Shv}(Z \times Y_2)
  \end{array}
  \]
  commutes.

- For \( F \in \text{Shv}(Z) \), the diagram
  \[
  \begin{array}{ccc}
  \text{Shv}(Z \times Z' \times Y_1) & \xrightarrow{\text{Id}_{Z \times Z'} \otimes Q} & \text{Shv}(Z \times Z' \times Y_2) \\
  \cong & \uparrow & \cong \\
  \text{Shv}(Z' \times Y_1) & \xrightarrow{\text{Id}_{Z'} \otimes Q} & \text{Shv}(Z' \times Y_2)
  \end{array}
  \]
  commutes.

Vice versa, a system of functors \( \text{Id}_Z \otimes Q \) as in (B.2) that satisfies the above three properties, satisfying an appropriate system of compatibilities for diagrams of morphisms between the \( Z \)'s, comes from a uniquely defined object \( Q \in \text{Shv}(Y_1 \times Y_2) \). Namely, \( Q \) is recovered as

\[
(\text{Id}_{Y_1} \otimes Q)(u_{Y_1}),
\]
where \( u_{Y_1} \) is as in (A.1.4).

B.1.6. Let \( Q \in \text{Shv}(Y_1 \times Y_2) \) and \( Q' \in \text{Shv}(Y'_1 \times Y'_2) \) be a pair of objects. We can consider

\[
(Q \boxtimes Q')^{\otimes 3} \in \text{Shv}((Y_1 \times Y'_1) \times (Y_2 \times Y'_2)).
\]

We will denote the corresponding functor \( \text{Shv}(Y_1 \times Y'_1) \to \text{Shv}(Y_2 \times Y'_2) \) by \( Q \boxtimes Q' \). We have

\[
(Q \boxtimes \text{Id}_{Y'_2}) \circ (\text{Id}_{Y_1} \boxtimes Q') \simeq Q \boxtimes Q' \simeq (\text{Id}_{Y_2} \boxtimes Q') \circ (Q \boxtimes \text{Id}_{Y'_1}).
\]

B.1.7. For \( Q_{1,2} \in \text{Shv}(Y_1 \times Y_2) \) and \( Q_{2,3} \in \text{Shv}(Y_2 \times Y_3) \) we have a naturally defined composition

\[
Q_{1,3} := Q_{2,3} \ast Q_{1,2} := (p_{1,3})_1(p_{2,3} \triangleleft Q_{1,2}) \in \text{Shv}(Y_1 \times Y_3).
\]

From Sect. [A.3.1] we obtain that for any \( Z \), we have

\[
(\text{Id}_Z \otimes Q_{1,3}) \simeq (\text{Id}_Z \otimes Q_{2,3}) \circ (\text{Id}_Z \otimes Q_{1,2}).
\]

Note that we have

\[
(Id_{Y_1} \otimes Q_{2,3})(Q_{1,2}) \simeq Q_{1,3} \simeq (Q_{1,2} \otimes \text{Id}_{Y_3})(Q_{2,3}).
\]
B.1.8. We can form an \((\infty, 2)\)-category whose objects are algebraic stacks, and whose categories of 1-morphisms are \(\text{Shv}(\mathcal{Y}_1 \times \mathcal{Y}_2)\), with composition functors defined as above. (For the purposes of this paper, we only need the homotopy category of this \((\infty, 2)\)-category, i.e., we can replace the spaces of 2-morphisms by their sets of connected components.)

For a given \(\mathcal{Y}\), the unit 1-morphism in the above 2-category is given by the object
\[
\text{(B.6)} \quad u_{\mathcal{Y}} = (\Delta_{\mathcal{Y}}, \omega_{\mathcal{Y}}) \in \text{Shv}(\mathcal{Y} \times \mathcal{Y})
\]
of (A.14).

B.2. A discontinuous version.

B.2.1. Note that in addition to the functor \((\text{B.1})\), one can consider the functor
\[
\text{(B.7)} \quad \mathcal{Q} \text{disc} : \text{Shv}(\mathcal{Y}_1) \to \text{Shv}(\mathcal{Y}_2), \quad \mathcal{F} \mapsto (p_2)^*(p_1^!(\mathcal{F}) \otimes \mathcal{Q}).
\]

We have a tautologically defined map
\[
\text{(B.8)} \quad \mathcal{Q} \to \mathcal{Q} \text{disc}.
\]

It follows from the description of compact generators of \(\text{Shv}(\ )\) in Sect. A.2.2 and the projection formula that the natural transformation \((\text{B.8})\) is an isomorphism when evaluated on compact objects.

In particular, both sides of \((\text{B.8})\) send compact objects to constructible ones.

Similarly, \((\text{B.8})\) is an isomorphism if \(\mathcal{Q}\) is compact as an object of \(\text{Shv}(\mathcal{Y}_1 \times \mathcal{Y}_2)\).

If \(\mathcal{Q}\) is just constructible, in general, neither side in \((\text{B.8})\) preserves constructibility. However, the functor \(\mathcal{Q}\text{disc}\) has a cohomological amplitude bounded on the left, and both functors send constructible objects to objects with constructible perverse cohomologies, see Sect. A.2.7.

B.2.2. We can similarly consider the (a priori, discontinuous) functors
\[
\text{Id}_Z \boxtimes \mathcal{Q} \text{disc} : \text{Shv}(Z \times \mathcal{Y}_1) \to \text{Shv}(Z \times \mathcal{Y}_2),
\]

namely
\[
(\text{Id}_Z \boxtimes \mathcal{Q} \text{disc})(\mathcal{F}) := (p_{Z, \mathcal{Y}_2})^*(p_{Z, \mathcal{Y}_1}^!(\mathcal{F}) \otimes p_{Y_1, \mathcal{Y}_2}(\mathcal{Q})).
\]

The functors \(\text{Id}_Z \boxtimes \mathcal{Q} \text{disc}\) make the diagrams similar to ones in Sect. B.1.5 commute, where in the first diagram we use \(-^\bullet\) instead of \(-^\Delta\).

We have the natural transformations
\[
\text{(B.9)} \quad \text{Id}_Z \boxtimes \mathcal{Q} \to \text{Id}_Z \boxtimes \mathcal{Q} \text{disc}.
\]

B.2.3. We shall say that \(\mathcal{Q}\) is safe (as a functor defined by a kernel) if the natural transformations \((\text{B.9})\) are isomorphisms.

By definition, \(\mathcal{Q}\) is safe if and only if the functors \(\text{Id}_Z \boxtimes \mathcal{Q} \text{disc}\) are actually continuous.

For example, if \(\mathcal{Q}\) is compact as an object of \(\text{Shv}(\mathcal{Y}_1 \times \mathcal{Y}_2)\), then \(\mathcal{Q}\) is safe.

In addition, \(\mathcal{Q}\) is automatically safe if \(\mathcal{Y}_1\) is a scheme.

B.2.4. It follows from Proposition A.2.8 that if \(\mathcal{Q}\) and \(\mathcal{F}\) are both constructible, the following conditions are equivalent:

(i) The natural transformation \((\text{B.9})\) becomes an isomorphism when evaluated on \(\mathcal{F}\);
(ii) The object \((\text{Id}_Z \boxtimes \mathcal{Q})(\mathcal{F})\) is cohomologically bounded on the left;
(iii) The object \((\text{Id}_Z \boxtimes \mathcal{Q} \text{disc})(\mathcal{F})\) is cohomologically bounded on the right.

In particular, we obtain that if \(\mathcal{Q}\) is constructible and \(\mathcal{Q}\) safe, then the functors in \((\text{B.9})\) preserve constructibility and are of finite cohomological dimension.
B.2.5. Given two objects $\Omega_{1,2} \in \text{Shv}(Y_1 \times Y_2)$ and $\Omega_{2,3} \in \text{Shv}(Y_2 \times Y_3)$ we can form

$$\Omega_{1,3} := \Omega_{2,3} \star_{\text{disc}} \Omega_{1,2} := (p_{1,3})_*((p_{2,3})_*(\Omega_{2,3}) \otimes p_{1,2}^*(\Omega_{1,2})) \in \text{Shv}(Y_1 \times Y_3).$$

We have

$$\text{Id}_2 \boxtimes (Q_{1,3})_{\text{disc}} \simeq (\text{Id}_2 \boxtimes (Q_{2,3})_{\text{disc}}) \circ (\text{Id}_2 \boxtimes (Q_{1,2})_{\text{disc}}).$$

The natural transformation (B.9) gives rise to a map

$$\Omega_{2,3} \star \Omega_{1,2} \rightarrow \Omega_{2,3} \star_{\text{disc}} \Omega_{1,2},$$

which is an isomorphism if either $\Omega_{2,3}$ or $\Omega_{1,2}$ is safe.

B.3. Functors co-defined by kernels. In this subsection we explore the Verdier-dual notion: that of functors codefined by kernels.

A special family of interest is formed by functors that are both defined and codefined by kernels.

B.3.1. Starting from an object $P \in \text{Shv}(Y_1 \times Y_2)$ we can produce another family of functors, denoted

$$(B.10) \quad \text{Id}_2 \boxtimes P^i : \text{Shv}(Z \times Y_1) \rightarrow \text{Shv}(Z \times Y_2), \quad \mathcal{F} \mapsto (p_{2,3})_*(\mathcal{F} \otimes p_{1,2}^*(P)).$$

We will refer to this family as functors codefined by $P$. They satisfy compatibility conditions parallel to ones in Sect. B.1.5 with $-$ replaced by $-$ and $\boxtimes$ replaced by $\boxtimes$.

B.3.2. Let $P_{1,2} \in \text{Shv}(Y_1 \times Y_2)$ and $P_{2,3} \in \text{Shv}(Y_2 \times Y_3)$ be a pair of objects. Define

$$P_{1,3} := P_{2,3} \otimes P_{1,2} := (p_{1,3})_*((p_{2,3})_*(\mathcal{F} \otimes p_{1,2}^*(P_{1,2})) \in \text{Shv}(Y_1 \times Y_3).$$

We have

$$\text{Id}_2 \boxtimes P^i_{1,3} \simeq (\text{Id}_2 \boxtimes P^i_{2,3}) \circ (\text{Id}_2 \boxtimes P^i_{1,2}).$$

Furthermore,

$$(B.11) \quad (\text{Id}_{Y_1} \boxtimes P^i_{2,3})(P_{1,2}) \simeq P_{1,3} \simeq (P^i_{1,2} \boxtimes \text{Id}_{Y_3})(P_{2,3}).$$

B.3.3. Let $Q \in \text{Shv}(Y_1 \times Y_2)$ be constructible, and set $P := \mathbb{D}^{\text{Verdier}}(Q)$. Let $\mathcal{F} \in \text{Shv}(Z \times Y_1)^{\text{constr}}$ be an object satisfying the equivalent conditions of Sect. B.2.4

We obtain that in this case

$$(\text{Id}_2 \boxtimes P^i) \circ (\mathbb{D}^{\text{Verdier}}(\mathcal{F})) \simeq \mathbb{D}^{\text{Verdier}}((\text{Id}_2 \boxtimes Q_{\text{disc}})(\mathcal{F})) \simeq \mathbb{D}^{\text{Verdier}}((\text{Id}_2 \boxtimes Q)(\mathcal{F})).$$

B.3.4. Let us be given two objects

$$Q \in \text{Shv}(Y_1 \times Y_2) \text{ and } \tilde{P} \in \text{Shv}(\tilde{Y}_1 \times \tilde{Y}_2).$$

From Sects. A.3.3 A.3.4 we obtain a natural transformation

$$(B.12) \quad (\text{Id}_{Y_2} \boxtimes \tilde{P}^i) \circ (Q \boxtimes \tilde{\text{Id}}_{\tilde{Y}_1}) \rightarrow (Q \boxtimes \text{Id}_{\tilde{Y}_2}) \circ (\text{Id}_{Y_1} \boxtimes \tilde{P}^i), \quad \text{Shv}(Y_1 \times \tilde{Y}_1) \rightarrow \text{Shv}(Y_2 \times \tilde{Y}_2),$$

i.e., a 2-morphism in the diagram

$$(B.13) \quad \begin{array}{ccc}
\text{Shv}(Y_1 \times \tilde{Y}_1) & \xrightarrow{Q \boxtimes \text{Id}_{\tilde{Y}_1}} & \text{Shv}(Y_2 \times \tilde{Y}_1) \\
\text{Id}_{Y_1} \boxtimes \tilde{P}^i & \downarrow & \text{Id}_{Y_1} \boxtimes \tilde{P}^i \\
\text{Shv}(Y_1 \times \tilde{Y}_2) & \xrightarrow{Q \boxtimes \text{Id}_{\tilde{Y}_2}} & \text{Shv}(Y_2 \times \tilde{Y}_2) \\
\end{array}$$

B.3.5. Similarly (but more tautologically), we obtain a natural transformation

$$(B.14) \quad (\text{Id}_{Y_2} \boxtimes \tilde{P}^i) \circ (Q_{\text{disc}} \boxtimes \tilde{\text{Id}}_{\tilde{Y}_1}) \rightarrow (Q_{\text{disc}} \boxtimes \text{Id}_{\tilde{Y}_2}) \circ (\text{Id}_{Y_1} \boxtimes \tilde{P}^i).$$
B.3.6. In particular, given a functor $Q$ defined by a kernel, the diagrams in Sect. B.1.5 with $-^!$ replaced by $-^*$ and $-^!$ replaced by $-^\bullet$, commute up to natural transformations.

We shall say that $Q$ is defined and codefined by kernels if these natural transformations are isomorphisms. In particular, a functor defined and codefined by a kernel gives rise to a functor codefined by a kernel, i.e., it corresponds to an object $\mathcal{P} \in \text{Shv}(Y_1 \times Y_2)$, so that

\[(B.15) \quad \text{Id}_Z \boxtimes Q \simeq \text{Id}_Z \boxtimes \mathcal{P}^! \]

for all $Z$, in a way compatible with $!$- and $*$-pullbacks and $!$- and $\Delta$-pushforwards.

Note that in this case, the natural transformations (B.12) are the isomorphisms

\[(\text{Id}_Y \boxtimes \tilde{P}^!_1) \circ (\mathcal{P}^! \boxtimes \text{Id}_{Y_2}) \simeq \mathcal{P}^! \boxtimes \tilde{P}^! \simeq (\text{Id}_Y \boxtimes \tilde{P}^!)_1 \circ (\text{Id}_{Y_2} \boxtimes \tilde{P}^!).\]

In what follows we will give an explicit description of $\mathcal{P}$ in terms of $Q$, see Proposition B.4.4.

B.3.7. Similarly, given a functor $\mathcal{P}^!$ codefined by a kernel $\mathcal{P} \in \text{Shv}(Y_1 \times Y_2)$, it may have the property that it is defined and codefined by kernels, and thus corresponds to an object $Q \in \text{Shv}(Y_1 \times Y_2)$.

B.4. The miraculous functor. In this subsection we recall the construction of the miraculous endofunctor of $\text{Shv}(Y)$, defined for any algebraic stack.

Apart from its other applications in the bulk of the paper, the miraculous functor allows us to relate functors defined and codefined by kernels.

B.4.1. Let $Y$ be a quasi-compact algebraic stack. The miraculous functor on $Y$, denoted $\text{Mir}_Y$, is the functor defined by the kernel

\[(B.16) \quad \text{ps-u}_Y := (\Delta_Y)_{!}(e_Y).\]

B.4.2. For $Q \in \text{Shv}(Y_1 \times Y_2)$, set

\[\mathcal{P} \simeq (\text{Mir}_{Y_1} \boxtimes \text{Id}_{Y_2})(Q) \in \text{Shv}(Y_1 \times Y_2).\]

We claim that there is a canonically defined natural transformation

\[(B.17) \quad \text{Id}_Z \boxtimes \mathcal{P}^! \to \text{Id}_Z \boxtimes Q.\]

Indeed, the value of (B.17) on a given $\mathcal{T} \in \text{Shv}(Z \times Y_1)$ is the value on $\text{ps-u}_{Y_1}$ of the natural transformation \[(B.13)\] in the diagram

\[
\begin{array}{cccc}
\text{Shv}(Y_1 \times Y_1) & \xrightarrow{\text{Id}_{Y_1} \boxtimes Q} & \text{Shv}(Y_1 \times Y_2) \\
\text{Shv}(Z \times Y_1) & \xrightarrow{\mathcal{T} \boxtimes \text{Id}_{Y_2}} & \text{Shv}(Z \times Y_2)
\end{array}
\]

where $(\mathcal{T}^!)^!$ is the functor codefined by $\mathcal{T}^! \in \text{Shv}(Y_1 \times Z)$.

B.4.3. The following is immediate from the above diagram:

Proposition B.4.4.

(a) A functor defined by $Q$ is defined and codefined by a kernel if and only if the natural transformations (B.17) are isomorphisms (for all $Z$).

(b) In the situation of point (a), the co-defining object is given by $\mathcal{P} \simeq (\text{Mir}_{Y_1} \boxtimes \text{Id}_{Y_2})(Q)$. 


B.4.5. Note that in the particular case of
\[ Y_1 = Y, \quad Y_2 = Z = \text{pt}, \quad \mathcal{Q} = \mathcal{F} \in \text{Shv}(Y), \]
the natural transformation (B.17) gives rise to a map
\[ \text{ev}^Y_1(\mathcal{F}, \text{Mir}_Y(\mathcal{F})) \to \text{ev}_Y(\mathcal{F}, \mathcal{F}), \quad \mathcal{F}' \in \text{Shv}(Y), \]
where we recall that \( \text{ev}^Y_1(\mathcal{F}, \mathcal{G}) \) and \( \text{ev}_Y(\mathcal{F}, \mathcal{G}) \) denote the pairings
\[ C_c(Y, - \star -) \quad \text{and} \quad C_\bullet(Y, - \otimes -), \]
respectively.

Let us write down the map (B.18) explicitly. Namely, it is obtained by applying the natural transformation
\[ ((C_c(Y, -) \circ \Delta_Y) \boxtimes \text{Id}_Y) \circ \text{Id}_{Y \times Y} \boxtimes (C_\bullet(Y, -) \circ \Delta_Y) \circ ((C_c(Y, -) \circ \Delta_Y) \boxtimes \text{Id}_{Y \times Y}) \]
of (B.12) to the object
\[ \mathcal{F} \boxtimes \text{ps-u}_Y \boxtimes \mathcal{F} \in \text{Shv}(Y \times Y \times Y \times Y). \]

B.4.6. Note also that in the situation of Sects. B.1.7 and B.3.2, for 
\[ \mathcal{Q}_1, \mathcal{Q}_2 \in \text{Shv}(Y_1 \times Y_2) \quad \text{and} \quad \mathcal{Q}_2, \mathcal{Q}_3 \in \text{Shv}(Y_2 \times Y_3) \]
we have a map
\[ \mathcal{Q}_2, \mathcal{Q}_3 \mapsto ((\text{Id}_{Y_1} \boxtimes \text{Mir}_{Y_2})(\mathcal{Q}_1, 2)) \to \mathcal{Q}_2, \mathcal{Q}_3 \quad \text{(B.19)} \]

It follows from Proposition B.4.4 that if \( \mathcal{Q}_1, 2 \) is defined and codefined by a kernel, the map (B.19) is an isomorphism.

B.4.7. Consider the endofunctor codefined by \( Y \); denote it by \( \text{Id}_Y \).

Let \( \mathcal{P} \) be an object of \( \text{Shv}(Y_1 \times Y_2) \) and set
\[ \Omega := (\text{Id}_{Y_1} \boxtimes \text{Id}_{Y_2})(\mathcal{P}). \]

As in Sect. B.4.2 we produce a natural transformation
\[ \text{Id}_{\mathcal{Q}} \xrightarrow{(B.20)} \text{Id}_{\mathcal{Q}} \xrightarrow{(B.21)} \text{Id}_{\mathcal{Q}}. \]

As in Proposition B.4.4 we have:

**Proposition B.4.8.**

(a) A functor codefined by \( \mathcal{P} \) is defined and codefined by a kernel if and only if the natural transformations (B.20) are isomorphisms (for all \( \mathcal{Q} \)).

(b) In the situation of point (a), the defining object is given by
\[ \Omega \simeq (\text{Id}_{\mathcal{Q}} \boxtimes \text{Id}_{\mathcal{Q}})(\mathcal{P}). \]

B.4.9. The natural transformations (B.17) and (B.20) are compatible as follows. Start with an object 
\( \mathcal{Q} \in \text{Shv}(Y_1 \times Y_2) \), and set
\[ \mathcal{P} := (\text{Mir}_{Y_1} \boxtimes \text{Id}_{Y_2})(\mathcal{Q}) \quad \text{and} \quad \mathcal{Q}' := (\text{Id}_{Y_1} \boxtimes \text{Id}_{Y_2})(\mathcal{P}). \]

Using adjunction (see Sect. B.6.1 below) and the natural transformation (B.9), we obtain a map
\[ \Omega' \to \Omega. \]

It is straightforward to check that the composition
\[ \text{Id}_{\mathcal{Q}} \xrightarrow{(B.20)} \text{Id}_{\mathcal{Q}} \xrightarrow{(B.21)} \text{Id}_{\mathcal{Q}} \]
identifies with the map (B.17).

In particular, if \( \mathcal{Q} \) is defined and codefined by a kernel, all of the above maps are isomorphisms, including the map (B.21).
B.4.10. As in Sect. B.4.6, for $P_{1,2}, P_{2,3} \in \text{Shv}(Y_1 \times Y_2)$ and $P_{2,3} \in \text{Shv}(Y_2 \times Y_3)$, we have a map

$$P_{2,3} \overset{l \star}{\to} P_{1,2} \overset{\star}{\to} (\text{Id}_{Y_1} \boxtimes \text{Id}_{Y_2})(P_{1,2}).$$

If $P_{1,2}$ is defined and codelineared by a kernel, the map (B.22) is an isomorphism.

B.5. **Adjunctions for functors defined by kernels.** In this subsection we focus on the phenomenon of adjunction of functors defined by kernels.

B.5.1. Consider the notion of adjunction of 1-morphisms in the 2-category introduced in Sect. B.1.8. Explicitly, given a pair of objects $Q_{1,2} \in \text{Shv}(Y_1 \times Y_2)$ and $Q_{2,1} \in \text{Shv}(Y_2 \times Y_1)$, a datum that makes $(Q_{1,2}, Q_{2,1})$ into an adjoint pair is a pair of morphisms $u_{Q_1} \to Q_{2,1} \star Q_{1,2}$ and $Q_{1,2} \star Q_{2,1} \to u_{Q_2}$ that satisfy the usual axioms.

B.5.2. Note that in this case, for any $Z$, the corresponding functors

$$\text{id} \boxtimes Q_{1,2}, \text{id} \boxtimes Q_{2,1}$$

form an adjoint pair.

Vice versa, the datum of adjunction $(Q_{1,2}, Q_{2,1})$ is equivalent to the datum of adjunction of the functors (B.23) compatible with the isomorphisms from Sect. B.1.5.

**Remark** B.5.3. In what follows we will sometimes abuse the terminology as follows: for a functor $Q : \text{Shv}(Y_1) \to \text{Shv}(Y_2)$ defined by a kernel we will say that it admits an adjoint as a functor defined by the kernel if the corresponding object $Q \in \text{Shv}(Y_1 \times Y_2)$ has this property.

B.5.4. Note that if $(Q_{1,2}, Q_{2,1})$ is an adjoint pair, then so is

$$(Q_{2,1}^r, Q_{1,2}^r).$$

B.5.5. We claim:

**Proposition B.5.6.** Suppose that $Q \in \text{Shv}(Y_1 \times Y_2)$ admits a right adjoint, to be denoted $Q^R \in \text{Shv}(Y_2 \times Y_1)$. Then both $Q$ and $Q^R$ are constructible.

**Proof.** First, observe that the statement for $Q$ implies that for $Q^R$ by Sect. B.3.4.

Choose a smooth covering by an affine scheme $f : S_1 \to Y_1$. In order to show that $Q$ is constructible, it suffices to show that $(f \times \text{id})^!(Q)$ is compact.

Since the diagonal morphism $\Gamma_f : S_1 \to S_1 \times Y_1$

is schematic, the object $$(\Gamma_f)^!\omega_{S_1}) \in \text{Shv}(S_1 \times Y_1)$$

is compact. Note that

$$(f \times \text{id})^!(Q) \simeq (\text{Id}_{S_1} \boxtimes Q)((\Gamma_f)^!\omega_{S_1})).$$

Now, since the functor $\text{Id}_{S_1} \boxtimes Q$ admits a continuous right adjoint, we obtain that

$$(\text{Id}_{S_1} \boxtimes Q)((\Gamma_f)^!\omega_{S_1})) \in \text{Shv}(S_1 \times Y_2)$$

is compact, as required.

B.6. **Adjunctions and Verdier duality.** In this subsection we relate the phenomenon of adjunction of functors to Verdier duality of kernels that define them.
B.6.1. Let \( Q \in \text{Shv}(Y_1 \times Y_2) \) be constructible; in particular \( D_{\text{Verdier}}(Q) \) is well-defined. Denote \( P := D_{\text{Verdier}}(Q)^{\sigma} \in \text{Shv}(Y_2 \times Y_1) \).

Note that for an individual \( Z \), the functor
\[
\text{Id}_Z \boxtimes P^i : \text{Shv}(Z \times Y_2) \to \text{Shv}(Z \times Y_1)
\]
is the left adjoint of the functor
\[
\text{Id}_Z \boxtimes Q_{\text{disc}} : \text{Shv}(Z \times Y_1) \to \text{Shv}(Z \times Y_2),
\]
see Sect. B.2.2 for the notation.

B.6.2. We claim:

**Theorem B.6.3.** Suppose that \( Q \) admits a right adjoint \( Q^R \) as a functor defined by a kernel with the corresponding object of \( \text{Shv}(Y_2 \times Y_1) \) denoted by \( Q^R \). Denote
\[
P := D_{\text{Verdier}}(Q^R)^{\sigma}.
\]

Then:

(a) The functor \( Q \) is defined and codefined by a kernel, with codefining object \( P \).

(b) The functor \( Q^R \) is safe.⁶

Let us emphasize that point (a) of the theorem says that \( \text{Id}_Z \boxtimes Q \simeq \text{Id}_Z \boxtimes P^i \) for \( P \) codefined by the object \( P \) above.

**Proof.** It suffices to prove point (a): indeed, point (b) would follow from Sects. B.5.2 and B.6.1, since the functors \( \text{Id}_Z \boxtimes Q^R^\text{disc} \) and \( \text{Id}_Z \boxtimes Q^R \) are both right adjoint to \( \text{Id}_Z \boxtimes Q \simeq \text{Id}_Z \boxtimes P^i \).

Let \( f : Y_2 \to Y_1 \) be a smooth cover by an affine scheme. In order to prove point (a), it suffices to construct isomorphisms
\[(B.24) \quad (id \times f)^* \circ (\text{Id}_Z \boxtimes Q) \simeq (id \times f)^* \circ (\text{Id}_Z \boxtimes P^i)
\]
as functors
\[
\text{Shv}(Z \times Y_1) \to \text{Shv}(Z \times Y_2).
\]

Note that since \( f \) is smooth, the functor \( f^* \) is both defined and codefined by a kernel so that
\[
(id \times f)^* \simeq (\text{Id}_Z \boxtimes f^*).
\]

Denote \( \tilde{Q} = (id \times f)^*(Q) \), so that
\[
\text{Id}_Z \boxtimes \tilde{Q} \simeq (\text{Id}_Z \boxtimes f^*) \circ (\text{Id}_Z \boxtimes Q).
\]

Since the \( f^* \) admits a right adjoint as a functor defined by a kernel (namely, \( f_* \)), do so does the functor \( \tilde{Q} \). Note that the corresponding functor \( \tilde{P}^i \) is also given by \( f^* \circ P^i \), as a functor codefined by a kernel.

Hence, the sought-for isomorphism \((B.24)\) becomes
\[
\text{Id}_Z \boxtimes \tilde{Q} \simeq \text{Id}_Z \boxtimes \tilde{P}^i.
\]

In other words, we have reduced the verification of point (a) to the case when \( Y_2 \) is an affine scheme, which we will from now on assume.

By Sect. B.6.1, the functors \( \text{Id}_Z \boxtimes P^i, \text{Id}_Z \boxtimes Q^R_{\text{disc}} \) form an adjoint pair. Note, however, that since \( Y_2 \) is an affine scheme, the functor \( \text{Id}_Z \boxtimes P^i \) preserves compactness (this is true for any object codefined by a constructible kernel with target a scheme, see Sect. A.1.6). Hence, its right adjoint is continuous. This implies that \( \text{Id}_Z \boxtimes Q^R_{\text{disc}} \) is continuous, so \( Q^R \) is safe. (Note that this proves point (b) in the case when \( Y_2 \) is an affine scheme.)

⁶See Sect. B.2.3 for what this means.
In particular, we obtain that \((\Id\otimes\mathcal{P}^l, \Id\otimes\mathcal{Q}^R)\) form an adjoint pair. However, by Sect. B.5.2, the functors \((\Id\otimes\mathcal{Q}, \Id\otimes\mathcal{Q}^R)\) also form an adjoint pair. This implies
\[
\Id\otimes\mathcal{Q} \simeq \Id\otimes\mathcal{P}^l,
\]
i.e., the isomorphism of point (a).

\[\square\]

B.6.4. We also have the following partial converse of Theorem B.6.3:

**Proposition B.6.5.** Let \(\mathcal{Q}\) be constructible, and suppose that the functor \(\mathcal{Q}\) is defined and codefined by a kernel with codefining object \(\mathcal{P}\), i.e., \(\Id\otimes\mathcal{Q} \simeq \Id\otimes\mathcal{P}^l\). Suppose also that the functors \(\Id\otimes\mathcal{Q}\) preserve compactness. Then:

(a) The object \(\mathcal{P}\) is constructible.

(b) The functor \(\mathcal{Q}\) admits a right adjoint as a functor defined by a kernel with the corresponding object of \(\text{Shv}(\mathcal{Y}_2 \times \mathcal{Y}_1)\) being \(\Omega^R := \mathbb{D}^{\text{Verdier}}(\mathcal{P})^\sigma\).

(c) The functor \(\mathcal{Q}^R\) is safe.

**Proof.** The fact that \(\mathcal{P}\) is constructible follows as in Proposition B.5.6.

By Sect. B.6.1, the functor \(\Id\otimes\mathcal{Q}^R_{\text{disc}}\) is the right adjoint of \(\Id\otimes\mathcal{P}^l\), and since the latter preserves compactness, we obtain that \(\Id\otimes\mathcal{Q}^R_{\text{disc}}\) is continuous. Hence, \(\mathcal{Q}^R\) is safe; this proves point (c).

Thus, the natural transformation \(\Id\otimes\mathcal{Q}^R \to \Id\otimes\mathcal{Q}^R_{\text{disc}}\) is an isomorphism, we obtain that \(\Id\otimes\mathcal{Q}\) is the right adjoint of \(\Id\otimes\mathcal{P}^l\). This proves point (b).

\[\square\]

**Remark B.6.6.** Note that the assertion of Proposition B.6.5 would be false without the assumption that the functors \(\Id\otimes\mathcal{Q}\) preserve compactness.

For example, take \(\mathcal{Y}_1 = \text{pt}, \mathcal{Y}_2 = B\mathbb{G}_m\) and let \(\mathcal{Q}\) be the constant sheaf.

B.6.7. Combining Theorem B.6.3 and Proposition B.4.4, we obtain:

**Corollary B.6.8.** Assume that a functor \(\mathcal{Q}\) defined by the kernel \(\Omega \in \text{Shv}(\mathcal{Y}_1 \times \mathcal{Y}_2)\) admits a right adjoint as a functor defined by a kernel. Denote the corresponding object by \(\Omega^R \in \text{Shv}(\mathcal{Y}_2 \times \mathcal{Y}_1)\). Then
\[
\mathbb{D}^{\text{Verdier}}(\Omega^R) \simeq ((\text{Mir}_{\mathcal{Y}_1} \otimes \Id_{\mathcal{Y}_2})(\Omega))^\sigma
\]
as objects of \(\text{Shv}(\mathcal{Y}_1 \times \mathcal{Y}_2)\). In particular, the object
\[
\mathcal{P} := (\text{Mir}_{\mathcal{Y}_1} \otimes \Id_{\mathcal{Y}_2})(\Omega)
\]
is constructible.

The next assertion reproduces [Ga2, Theorem 6.3.2]:

**Corollary B.6.9.** In the situation of Corollary B.6.8 we have
\[
\mathbb{D}^{\text{Verdier}}(\Omega) \simeq (\text{Mir}_{\mathcal{Y}_1} \otimes \Id_{\mathcal{Y}_2})((\Omega^R)^\sigma).
\]

**Proof.** Follows by combining Sect. B.6.4 and Corollary B.6.8.

\[\square\]

**Remark B.6.10.** At one point on the main body of the paper (namely, Theorem B.5.4), we use an extension of Theorem B.6.3 to the case when the stack \(\mathcal{Y}_1\) is not necessarily quasi-compact, but is truncatable (see Sect. C.1.4 for what this means).

In this case, \(\Omega\) is an object of the category \(\text{Shv}(\mathcal{Y}_1 \times \mathcal{Y}_2)_{\text{co}}\) (see Sect. C.2.2 for the notation), such that for any cotruncative quasi-compact open substack \(\mathcal{U}_1 \xrightarrow{j} \mathcal{Y}_1\), the object
\[
(j^!)^!(\Omega) \in \text{Shv}(\mathcal{U}_1 \times \mathcal{Y}_2)
\]
is constructible.
The right adjoint $\Omega^R$ is characterized by the property that $((j \times \text{id})^*(\Omega))$ is the right adjoint of $(j^\natural \boxtimes \text{id})(\Omega)$. The codefining object

$$T \in \text{Shv}(Y_1 \times Y_2)$$

is related to $\Omega^R$ by the same formula as in Theorem [B.6.3].

The statement of Corollary [B.6.8] remains unchanged, where we now understand $\text{Mir}_{Y_1} \boxtimes \text{Id}_{Y_2}$ as a functor $\text{Shv}(Y_1 \times Y_2) \rightarrow \text{Shv}(Y_1 \times Y_2)$.

**Remark B.6.11.** Let us continue to be in the setting of Remark [B.6.10]. Let $\Omega$ be a constructible object of $\text{Shv}(Y_1 \times Y_2)$, and let us view $\Omega$ as a functor $\text{Shv}(Y_1)_{\text{co}} \rightarrow \text{Shv}(Y_2)_{\text{co}}$, defined by a kernel (see Sect. [C.4]).

Suppose that $\Omega$ admits a right adjoint $\Omega^R$, which is now an object $\Omega^R \in \text{Shv}(Y_2 \times Y_1)_{\text{co}}$.

We have

$$((j^\natural \boxtimes \text{id})(\Omega^R)) \simeq ((j \times \text{id})^*(\Omega))^R,$$

for $u_1 \hookrightarrow Y_1$ as in Remark [B.6.10].

In particular, Corollary [B.6.9] is applicable. Explicitly, we have

$$\Omega^R \simeq (\text{Id}_{Y_1} \boxtimes \text{Id}_{Y_2})(\text{D}_{\text{Verdier}}(\Omega))^\sigma,$$

where we view $\text{Id}_{Y_1}$ as a functor $\text{Shv}(Y_1) \rightarrow \text{Shv}(Y_1)_{\text{co}}$, defined by a kernel, see Sect. [C.4.9].

Note, however, that in this case, there is no sense in which we can talk about $\Omega$ being codefined by a kernel.

**B.7. Miraculous stacks.** In this subsection we review what it means for a quasi-compact algebraic stack to be miraculous.

**B.7.1.** Let $Y$ be a quasi-compact stack. We shall say that $Y$ is *miraculous* if the functor $\text{Mir}_Y$ is invertible, as a functor defined by a kernel.

Using Sect. [B.1.5] this is equivalent to the requirement that

$$\text{Id}_Z \boxtimes \text{Mir}_Y : \text{Shv}(Z \times Y) \rightarrow \text{Shv}(Z \times Y)$$

is an equivalence for every $Z$.

**B.7.2.** Assume that $Y$ is miraculous. In particular, $\text{Mir}_Y$, viewed as a plain endofunctor of $\text{Shv}(Y)$ is a self-equivalence.

In this case, we can define a *new self-duality* on $\text{Shv}(Y)$ with counit given by

$$\text{Shv}(Y) \boxtimes \text{Shv}(Y) \xrightarrow{\text{Mir}_Y^{-1} \boxtimes \text{Id}} \text{Shv}(Y) \boxtimes \text{Shv}(Y) \xrightarrow{\text{ev}_Y} \text{Vect};$$

we denote it by $\text{ev}_{\text{Mir}_Y}$. We will refer to it as the *miraculous self-duality* of $\text{Shv}(Y)$. The corresponding contravariant self-equivalence, denoted

$$\text{D}^{\text{Mir}} : (\text{Shv}(Y)^e)^{\text{op}} \rightarrow (\text{Shv}(Y)^e)^{\text{op}},$$

is given by

$$\text{D}^{\text{Mir}} \simeq \text{D}_{\text{Verdier}} \circ \text{Mir}_Y^{-1}.$$

**B.7.3.** Since the object $\text{ps-u}_Y$ is swap-equivariant, we have

$$(\text{Mir}_Y)^\vee \simeq \text{Mir}_Y,$$

and hence

$$(\text{Mir}_Y^{-1})^\vee \simeq \text{Mir}_Y^{-1}.$$ 

From here we obtain that the pairing $\text{ev}_{\text{Mir}_Y}$ is swap-equivariant. Hence, the functor $\text{D}^{\text{Mir}}$ is involutive.\footnote{In the sense of 2-category of Sect. [I.4.8].}
B.7.4. Let $\mathcal{Y}$ be miraculous. In particular, we obtain that the functor $\text{Mir}_Y$ admits a left adjoint as a functor defined by a kernel.

Applying Theorem B.6.3(b) we obtain that the functor $\text{Mir}_Y$ is safe. In particular, it preserves constructibility.

Furthermore, from Theorem B.6.3(a) we obtain that the inverse of $\text{Mir}_Y$, viewed as a functor code-defined by a kernel is given by $\text{Id}_Y$, see Sect. B.4.7 for the notation.

B.8. A criterion for admitting an adjoint.

B.8.1. Let $\mathcal{Q} \in \text{Shv}(\mathcal{Y}_1 \times \mathcal{Y}_2)$ be a constructible object. Denote
\[ (B.25) \quad \mathcal{Q}^R := (\text{Id}_{\mathcal{Y}_2} \boxtimes \text{Id}_{\mathcal{Y}_1})(\mathcal{D}^\text{Verdier}(\mathcal{Q}))^\sigma. \]

Remark B.8.2. Assume for a moment that $\mathcal{Q}$ admits a right adjoint so that $\mathcal{D}^\text{Verdier}(\mathcal{Q}^R) \simeq ((\text{Mir}_{\mathcal{Y}_1} \boxtimes \text{Id}_{\mathcal{Y}_2})(\mathcal{Q}))^\sigma$.

Since $\mathcal{D}^\text{Verdier}(\mathcal{Q}^R)$ is constructible, it follows from Sect. B.3.3 that in this case we have $\mathcal{Q}^R \simeq \mathcal{Q}^R$.

B.8.3. Assume that $\mathcal{Q}^R$ is safe.

Note that this assumption holds if $\mathcal{Q}$ admits a right adjoint. Indeed, this follows from Theorem B.6.3 and Remark B.8.2 above.

Note also that the safety condition is automatic whenever $\mathcal{Q}^R$ is compact or $\mathcal{Y}_2$ is a scheme, see Sect. B.2.3.

B.8.4. We claim that there exists a canonically defined map
\[ u_{\mathcal{Y}_1} \rightarrow \mathcal{Q}^R \ast \mathcal{Q}. \]

We start with the map
\[ (B.27) \quad \text{ps-}u_{\mathcal{Y}_1} \rightarrow \mathcal{D}^\text{Verdier}(\mathcal{Q})^\sigma \ast_{\text{disc}} \mathcal{Q}, \]

arising by adjunction, see Sect. B.2.5 for the notation $\ast_{\text{disc}}$.

We apply to both sides the functor $\text{Id}_{\mathcal{Y}_1} \boxtimes \text{Id}_{\mathcal{Y}_1}$, and we obtain a map
\[ u_{\mathcal{Y}_1} \rightarrow (\text{Id}_{\mathcal{Y}_1} \boxtimes \text{Id}_{\mathcal{Y}_1})(\mathcal{D}^\text{Verdier}(\mathcal{Q})^\sigma \ast_{\text{disc}} \mathcal{Q}), \]

and we follow it by the map
\[ (\text{Id}_{\mathcal{Y}_1} \boxtimes \text{Id}_{\mathcal{Y}_1})(\mathcal{D}^\text{Verdier}(\mathcal{Q})^\sigma \ast_{\text{disc}} \mathcal{Q}) \rightarrow (((\text{Id}_{\mathcal{Y}_1} \boxtimes \text{Id}_{\mathcal{Y}_1})(\mathcal{D}^\text{Verdier}(\mathcal{Q})^\sigma)) \ast_{\text{disc}} \mathcal{Q} = \mathcal{Q}^R \ast_{\text{disc}} \mathcal{Q}. \]

Finally, using the assumption that $\mathcal{Q}^R$ is safe, we obtain that the map $\mathcal{Q}^R \ast \mathcal{Q} \rightarrow \mathcal{Q}^R \ast_{\text{disc}} \mathcal{Q}$ is an isomorphism.

B.8.5. Note also that by adjunction, we have a canonically defined map
\[ (B.29) \quad \mathcal{Q} \ast \mathcal{D}^\text{Verdier}(\mathcal{Q})^\sigma \rightarrow u_{\mathcal{Y}_2}. \]

B.8.6. Recall now that we have a canonically defined map
\[ (B.30) \quad \mathcal{Q} \ast \mathcal{D}^\text{Verdier}(\mathcal{Q})^\sigma \rightarrow \mathcal{Q} \ast \mathcal{Q} \ast_{\text{disc}} \mathcal{Q}, \]

see (B.22).

\footnote{In the sense of 2-category of Sect. B.1.8.}
We claim:

**Theorem B.8.8.** For a constructible object $Q \in \text{Shv}(Y_1 \times Y_2)$, the following conditions are equivalent:

(i) The object $Q$ admits a right adjoint;

(ii) The map (B.26) is the unit of an adjunction;

(iii) The map (B.30) is an isomorphism;

(iv) The map (B.30) is an isomorphism, and the map (B.29) precomposed with the inverse of the isomorphism (B.30) is the counit of an adjunction;

(v) The map (B.30) is an isomorphism, and the map (B.29) precomposed with the inverse of the isomorphism (B.30) is the counit of an adjunction, with the unit being (B.26).

**Remark B.8.9.** At one place in the main body of the paper (namely, Corollary 4.5.6), we use the extension of Theorem B.8.8 to the case when $Y_1$ is not necessarily quasi-compact, but truncatable.

We view $Q$ as an object of $\text{Shv}(Y_1 \times Y_2)$, and we view $Q$ as a functor $\text{Shv}(Y_1) \to \text{Shv}(Y_2)$, defined by a kernel (see Sect. C.4).

We view $\text{Id}_{Y_1}$ as a functor $\text{Shv}(Y_1) \to \text{Shv}(Y_1)$, codelefined by a kernel, so $Q^R$ is an object of $\text{Shv}(Y_1 \times Y_2)$, see Sect. C.4.

The proof of Theorem B.8.8 extends to this case in a straightforward way (instead of Theorem B.6.3 we use its variant described in Remark B.6.11).

**B.9. Proof of Theorem B.8.8**

B.9.1. We have the tautological implications $(v) \Rightarrow (ii) \Rightarrow (i)$ and $(v) \Rightarrow (iv) \Rightarrow (iii)$.

B.9.2. Assume that $Q$ admits a right adjoint. Then the map (B.30) is an isomorphism by Theorem B.8.8 and Sect. B.4.10 (i.e., we have $(i) \Rightarrow (iii)$).

Furthermore, by Remark B.8.2 in this case $Q^R \simeq Q^R$. Unwinding the identification of Theorem B.6.3(a), we obtain that the unit of the $(Q, Q^R)$-adjunction is given by (B.26) and the counit is given by the map (B.30) precomposed with the isomorphism (B.30).

Thus, we obtain that $(i) \Rightarrow (v)$.

B.9.3. It remains to show that $(iii) \Rightarrow (i)$ or equivalently $(iii) \Rightarrow (v)$. As a first step, we will reduce the assertion to the case when $Y_2$ is a scheme.

Choose a smooth cover $f : \bar{Y}_2 \to Y_2$, where $\bar{Y}_2$ is a scheme. Denote

$$\bar{Q} := (\text{id} \times f)^*(Q) \in \text{Shv}(Y_1 \times \bar{Y}_2).$$

Note that we have a canonical identification

$$\bar{Q}^R \simeq Q^R \circ f_*,$$

as functors defined by kernels.

Furthermore, it is easy to see that condition (iii) for $Q$ implies the corresponding condition for $\bar{Q}$.

Assume that the implication $(iii) \Rightarrow (v)$ holds for $\bar{Q}$. Let us show that this implies condition $(i)$ for the original $Q$.

By assumption, the map

$$\text{Hom}_{\text{Shv}(Z \times \bar{Y}_2)}((\text{id}_Z \boxtimes \bar{Q})(\mathcal{F}), \bar{T}') \to \text{Hom}_{\text{Shv}(Z \times Y_1)}((\text{id}_Z \boxtimes Q^R)(\mathcal{F}'), \mathcal{T}') \in \text{Shv}(Z \times \bar{Y}_2),$$

induced by (B.26), is an isomorphism.

---

9 In the statements below we refer to adjunctions in the 2-category of Sect. B.8.8.
We have a commutative diagram

\[
\begin{array}{ccc}
\text{Hom}_{\text{Shv}(\mathbb{Z} \times Y_2)}((\text{Id}_{\mathbb{Z}} \boxtimes \tilde{\mathbb{Q}}^R)(F), \tilde{F}') & \longrightarrow & \text{Hom}_{\text{Shv}(\mathbb{Z} \times Y_1)}((\text{Id}_{\mathbb{Z}} \boxtimes \tilde{\mathbb{Q}}^R)(F), \tilde{F}') \\
\sim & & \sim \\
\text{Hom}_{\text{Shv}(\mathbb{Z} \times Y_2)}((\text{Id}_{\mathbb{Z}} \boxtimes \tilde{\mathbb{Q}})(F), (\text{id} \times f)_*(\tilde{F}')) & \longrightarrow & \text{Hom}_{\text{Shv}(\mathbb{Z} \times Y_1)}((\text{Id}_{\mathbb{Z}} \boxtimes \tilde{\mathbb{Q}}^R)(((\text{id} \times f)_*(\tilde{F}'))) \\
\end{array}
\]

Hence, we obtain that the map

(B.31) \[ \text{Hom}_{\text{Shv}(\mathbb{Z} \times Y_2)}((\text{Id}_{\mathbb{Z}} \boxtimes \tilde{\mathbb{Q}})(F), F') \rightarrow \text{Hom}_{\text{Shv}(\mathbb{Z} \times Y_1)}((\text{Id}_{\mathbb{Z}} \boxtimes \tilde{\mathbb{Q}}^R)(F), (\text{id} \times f)_*(\tilde{F}')) \]

is an isomorphism, whenever \( F' \in \text{Shv}(\mathbb{Z} \times Y_2) \) lies in the essential image of the functor \( (\text{id} \times f)_* : \text{Shv}(\mathbb{Z} \times Y_2) \rightarrow \text{Shv}(\mathbb{Z} \times Y_2) \).

We claim that this implies that (B.31) is an isomorphism for all \( F' \in \text{Shv}(\mathbb{Z} \times Y_2) \), which would mean that (i) holds.

Indeed, by the assumption in Sect. B.8.3, the functor \( \text{Id}_{\mathbb{Z}} \boxtimes \tilde{\mathbb{Q}}^R \) maps isomorphically to \( \text{Id}_{\mathbb{Z}} \boxtimes \tilde{\mathbb{Q}}^R_{disc} \) and the latter functor commutes with limits. Now, every object in \( F' \in \text{Shv}(\mathbb{Z} \times Y_2) \) can be written as a totalization of a cosimplicial object, whose terms belong to the essential image of \( (\text{id} \times f)_* \).

B.9.4. Thus, we can assume that \( Y_2 \) is a scheme. (Note that in this case, the assumption in Sect. B.8.3 holds automatically.)

We will show that (iii) implies (v). Namely, we will show that if (iii) holds, then the morphisms (B.26) and (B.29) satisfy the adjunction axioms.

We will show that the composition

(B.32) \[ Q \simeq Q \ast u_{Y_1} \overset{\text{(B.26)}}{\longrightarrow} Q \ast (Q^R \ast Q) \simeq (Q \ast Q^R) \ast Q \simeq (Q \ast \mathbb{D}^{\text{Verdier}}(Q)^\sigma) \ast Q \overset{\text{(B.29)}}{\longrightarrow} u_{Y_2} \ast Q \simeq \Omega \]

is the identity map. The other adjunction axiom is checked similarly.

B.9.5. Since \( Y_2 \) is a scheme, the map

\[ \mathbb{D}^{\text{Verdier}}(Q)^\sigma \ast Q \rightarrow \mathbb{D}^{\text{Verdier}}(Q)^\sigma \ast_{disc} Q \]

is an isomorphism.

Hence, we can view the map (B.27) as a map

(B.33) \[ ps-u_{Y_1} \rightarrow \mathbb{D}^{\text{Verdier}}(Q)^\sigma \ast Q. \]

Similarly, we can view the map (B.28) as a map

(B.34) \[ u_{Y_1} \rightarrow (\text{Id}_{Y_1} \boxtimes \text{Id}_{Y_1})((\mathbb{D}^{\text{Verdier}}(Q)^\sigma \ast Q), \]
B.9.6. For any \( Q \in \text{Shv}(Y_1 \times Y_2)^{\text{constr}} \), we have a commutative diagram

\[
\begin{array}{ccc}
Q & \xrightarrow{id} & Q \\
\downarrow & & \downarrow \\
\text{id} \quad \text{ps-}\text{u}_{y_1} & \xrightarrow{\text{id}} & Q \quad \text{u}_{y_1} \\
\downarrow & & \downarrow \\
\text{id} \quad \text{Verdier}(\mathcal{O})^* \quad \mathcal{O} & \xrightarrow{\text{id}} & Q \quad (\text{Id}_{y_1} \boxtimes \text{Id}_{y_1})(\text{Verdier}(\mathcal{O})^* \quad \mathcal{O}) \\
\downarrow & & \downarrow \\
\text{id} \quad \text{Verdier}(\mathcal{O})^* \quad \mathcal{O} & \xrightarrow{\text{id}} & (\mathcal{O} \text{R}^* \quad \mathcal{O}) \\
\downarrow & & \downarrow \\
\text{u}_{y_2} \quad \mathcal{O} & \xrightarrow{\text{id}} & (\mathcal{O} \text{R}^* \quad \mathcal{O}) \\
\downarrow & & \downarrow \\
\mathcal{O} & & \mathcal{O}
\end{array}
\]

If (iii) holds, then the bottom horizontal arrow in the above diagram is an isomorphism. In this case, by construction, the composition \( (B.32) \) identifies with the composite left vertical arrow in this diagram.

However, it is a straightforward verification that this composite map is the identity.

B.10. An example: the ULA property. In this subsection we will recast the property of a sheaf to be ULA (with respect to a given map) in terms of adjunction of functors given by kernels.

B.10.1. Let \( f : Y \to S \) be a map between algebraic stacks, where \( S \) is a separated scheme. Let \( \mathcal{F} \) be an object of \( \text{Shv}(Y) \).

**Definition B.10.2.** We shall say that \( \mathcal{F} \) is ULA with respect to \( f \) if the functor

\[
\mathcal{F} : \text{Shv}(S) \to \text{Shv}(Y), \quad \mathcal{G} \mapsto \mathcal{F} \boxtimes f^!(\mathcal{G}),
\]

viewed as a functor defined by the kernel

\[
(\text{Graph}_f)_* \mathcal{F} \in \text{Shv}(S \times Y),
\]

is defined and codefined by a kernel.

**Remark B.10.3.** Usually, one defines the notion of ULA for objects that are constructible. The above definition is justified by the following fact (see [Ga3]):

An object \( \mathcal{F} \in \text{Shv}(Y) \) is ULA with respect to \( f : Y \to S \) if and only if for every \( k \), every constructible sub-object \( \mathcal{F}' \subset H^k(\mathcal{F}) \) has this property.

For constructible objects, the above notion of ULA coincides with the classical one, see [BG, Appendix B].

**Remark B.10.4.** Assume for a moment that \( \mathcal{F} \) is compact. From Proposition B.6.5, we obtain that \( \mathcal{F} \) is ULA if and only if the above functor \( \mathcal{F} \) admits a right adjoint, as a functor defined by a kernel.
Remark B.10.5. We claim that in order to check that a given object $F \in \operatorname{Shv}(Y)^{constr}$ is ULA with respect to $f$, it is enough to check that one particular morphism taking place in $\operatorname{Shv}(\begin{array}{c} Y \\ S \end{array} \times Y)$ is an isomorphism (cf. [HS Proposition 3.3]).

This morphism in question is

$\left(\Delta_S^!(F \boxtimes \mathbb{D}^{\text{Verdier}}(F)) \rightarrow \Delta_S^!(F \boxtimes (f^*\mathbb{e}_S) \boxtimes \mathbb{D}^{\text{Verdier}}(F))\right)$,

where $\Delta_S$ is the map

$\begin{array}{c} Y \times Y \\ S \end{array} \rightarrow \begin{array}{c} Y \times Y \\ S \end{array}$.

Indeed, since the ULA condition is local on $Y$, we can assume that $Y$ is a scheme, and hence $F$ is compact. Then our assertion follows from Remark B.10.4 and the equivalence (i) $\Leftrightarrow$ (iii) in Theorem B.8.8.

B.10.6. We will now prove a statement that the property of being ULA is preserved by certain functors.

Let $S$ be as in Sect. B.10.1 and let $f_i : Y_i \to S$, $i = 1, 2$ be a pair of algebraic stacks over $S$. Let $Q$ be an object of $\operatorname{Shv}(Y_1 \times Y_2)$, such that the corresponding functor $Q$ is defined and codefined by a kernel. Assume that $Q$ is supported on

$\begin{array}{c} Y_1 \times Y_2 \\ S \end{array} \subset \begin{array}{c} Y_1 \times Y_2 \\ S \end{array}$.

We claim:

Corollary B.10.7. Under the above circumstances, the functor $Q : \operatorname{Shv}(Y_1) \to \operatorname{Shv}(Y_2)$ maps objects that are ULA with respect to $f_1$ to objects that are ULA with respect to $f_2$.

Proof. Let $F$ be an object of $\operatorname{Shv}(Y_1)$ that is ULA over $S$. Consider the object

$(\operatorname{Graph}_{f_1})_* (F) \in \operatorname{Shv}(S \times Y_1)$

and the corresponding functor $F : \operatorname{Shv}(S) \to \operatorname{Shv}(Y_1)$.

By assumption, the functor $F$ is defined and codefined by a kernel. By assumption, the same is true for the composition $Q \circ F$.

The functor $Q \circ F$ is defined by the kernel

$(\operatorname{Id}_S \boxtimes \mathbb{Q})(\bar{\operatorname{Graph}}_{f_1}(F)) \in \operatorname{Shv}(S \times \begin{array}{c} Y_1 \\ S \end{array})$.

Now, the fact that $Q$ is supported on $\begin{array}{c} Y_1 \times Y_2 \\ S \end{array}$ implies that

$(\operatorname{Id}_S \boxtimes \mathbb{Q})(\bar{\operatorname{Graph}}_{f_1}(F)) \simeq (\operatorname{Graph}_{f_2})_* (Q(F))$.

Hence, $Q(F)$ is ULA with respect to $f_2$, as desired. 

B.10.8. An example of the situation described in Sect. B.10.6 is when $\begin{array}{c} Y_1 \\ S \end{array} \simeq \begin{array}{c} Y_1' \\ S \end{array}$, and $Q$ comes from an object $Q' \in \operatorname{Shv}(\begin{array}{c} Y_1' \times Y_2' \\ S \end{array})$ such that the corresponding functor $Q'$ is defined and codefined by a kernel.
B.10.9. In the main body of the paper, we use the following particular case of Corollary \[B.10.7\] in the setting of Sect. \[B.10.8\].

Let \( Z \) be an algebraic stack, and let \( N \) be a conical Zariski-closed subset of \( T^*(Z) \). Let \( Y \) be another algebraic stack, and let \( F \in \text{Shv}(Y) \) be an object such that the corresponding functor \( F : \text{Shv}(Y) \to \text{Vect} \) is defined and codefined by a kernel.

We claim:

**Corollary B.10.10.** Under the above circumstances, the functor

\[
\text{Id}_Z \boxtimes F : \text{Shv}(Z \times Y) \to \text{Shv}(Z)
\]

sends the subcategory

\[
\text{Shv}_{N \times T^*(Y)}(Z \times Y) \subset \text{Shv}(Z \times Y)
\]

to

\[
\text{Shv}_N(Z) \subset \text{Shv}(Z).
\]

**Proof.** By the definition of singular support in \([Bei]\), and up to base changing \( Z \) to a scheme, we have to show that for a map of schemes \( f : S \to S' \), if an object \( G \in \text{Shv}_{N \times T^*(S)}(S \times Y)^{\text{constr}} \) is ULA with respect to the composition \( S \times Y \to S \xrightarrow{f} S' \),

then \( (\text{Id}_S \boxtimes F)(G) \in \text{Shv}(S) \) is ULA with respect to \( f \).

However, this follows from Corollary \[B.10.7\].

\[ \square \]

**Appendix C. Sheaves on non quasi-compact algebraic stacks**

The theory reviewed in Sects. \[A-B\] mostly pertains to quasi-compact algebraic stacks. In this section we explore the adjustments once needs to make in order to extend the theory to the non quasi-compact case.

**C.1. Cotruncative substacks.** In this subsection, we review, mostly following \([DrGa1]\), the notion of cotruncativeness of an open embedding, as well associated notions in the presence of a singular support condition.

**C.1.1.** Let

\( j : U_1 \hookrightarrow U_2 \)

be an open embedding of (quasi-compact) algebraic stacks.

We shall say that \( U_1 \) is a *cotruncative* substack of \( U_2 \) if the functor \( j_* \), *viewed as a functor defined by a kernel*, admits a right adjoint, to be denoted \( j^r \). Let \( \Omega \in \text{Shv}(U_2 \times U_1) \) be the kernel of this right adjoint.

It follows from Sect. \[B.5.4\] that the object \( \Omega^r \in \text{Shv}(U_1 \times U_2) \) defines the *left* adjoint to \( j^* \), as a functor defined by a kernel. In particular, the system of functors

\[
(id \times j)_! : \text{Shv}(Z \times U_1) \to \text{Shv}(Z \times U_2)
\]

satisfies the compatibilities of Sect. \[B.1.5\].

As in \([DrGa1]\) Lemma 3.7.1, one shows:

**Lemma C.1.2.** If \( U_1' \subset U_2 \) and \( U_2' \subset U_2 \) are cotruncative, then so is their union.

\[ \text{Remark B.10.3}. \]

---

\[10\] This definition is applicable for objects of \( \text{Shv}(-) \) rather than \( \text{Shv}(-)^{\text{constr}} \) using Remark \[B.10.3\].
C.1.3. Let 
\[ j : \mathcal{U} \hookrightarrow \mathcal{Y} \]
be an open embedding of not necessarily quasi-compact algebraic stacks. We say that \( \mathcal{U} \) is a cotruncative substack of \( \mathcal{Y} \) if for every quasi-compact open \( \mathcal{U}' \subset \mathcal{Y} \), the intersection \( \mathcal{U} \cap \mathcal{U}' \) is cotruncative as an open substack of \( \mathcal{U}' \).

C.1.4. We shall say that a (not necessarily quasi-compact) algebraic stack \( \mathcal{Y} \) is truncatable if it can be written as a union of quasi-compact cotruncative open substacks.

If \( \mathcal{Y} \) is truncatable, by Lemma [C.1.2] the poset of its quasi-compact cotruncative open substacks is filtered and cofinal in the poset of all quasi-compact open substacks.

C.1.5. Let \( j : \mathcal{U} \hookrightarrow \mathcal{Y} \) be as above. Let \( N \) be a conical Zariski closed subset of \( T^* (\mathcal{Y}) \). By a slight abuse of notation we will denote by the same symbol \( N \) its restriction to \( \mathcal{U} \).

We shall say that \( \mathcal{U} \) is \( N \)-cotruncative if it is cotruncative and the extension functor \( j_! \) (equivalently, \( j^* \)) sends \( \text{Shv}_N (\mathcal{U}) \rightarrow \text{Shv}_N (\mathcal{Y}) \).

We shall say that \( \mathcal{U} \) is universally \( N \)-cotruncative if it is cotruncative and for any algebraic stack \( Z \) and a closed conical subset \( N_Z \subset T^* (\mathcal{Z}) \), the extension functor \( (\text{id} \times j)_! \) (equivalently, \( (\text{id} \times j)^* \)) sends \( \text{Shv}_{N \times N}(Z \times \mathcal{U}) \rightarrow \text{Shv}_{N \times N}(Z \times \mathcal{Y}) \).

C.1.6. We shall say that \( \mathcal{Y} \) is \( N \)-truncatable (resp., universally \( N \)-truncatable) if it can be written as a filtered union of its \( N \)-cotruncative (resp., universally \( N \)-cotruncative) quasi-compact open substacks.

C.2. The non-quasi-compact case: the “co”-category.

If \( \mathcal{Y} \) is a non quasi-compact algebraic stack, Verdier duality is no longer a self-duality on \( \text{Shv}(\mathcal{Y}) \), but rather a duality between \( \text{Shv}(\mathcal{Y}) \) and another category, denoted \( \text{Shv}(\mathcal{Y})_{\text{co}} \).

In this subsection we introduce the category \( \text{Shv}(\mathcal{Y})_{\text{co}} \) and study its basic properties.

C.2.1. Let \( \mathcal{Y} \) be a (not necessarily quasi-compact) algebraic stack. By definition, \( \text{Shv}(\mathcal{Y}) \) is
\[ \lim \overset{\rightarrow}{\text{Shv}(\mathcal{U})}, \]
where the index category is the poset of quasi-compact open substacks \( \mathcal{U} \subset \mathcal{Y} \), and for \( \mathcal{U}_1 \rightarrow \mathcal{U}_2 \) the transition functor \( \text{Shv}(\mathcal{U}_1) \rightarrow \text{Shv}(\mathcal{U}_2) \) is \( j^* \).

By [Dr Ga 1] Proposition 1.7.5, we can rewrite the above limit as a colimit
\[ \underset{\mathcal{U}}{\text{colim}} \text{Shv}(\mathcal{U}), \]
where the transition functors are \( \text{Shv}(\mathcal{U}_1) \rightarrow \text{Shv}(\mathcal{U}_2) \) are \( j^* \).

C.2.2. We define the category \( \text{Shv}(\mathcal{Y})_{\text{co}} \) as
\[ \underset{\mathcal{U}}{\text{colim}} \text{Shv}(\mathcal{U}), \]
where the transition functors are \( \text{Shv}(\mathcal{U}_1) \rightarrow \text{Shv}(\mathcal{U}_2) \) are \( j^* \).

For a given quasi-compact
\[ U \rightarrow \mathcal{Y}, \]
we will denote by \( j^*_{\text{co}} \) the tautologically defined functor
\[ \text{Shv}(\mathcal{U}) \rightarrow \text{Shv}(\mathcal{Y})_{\text{co}}. \]

Note that when \( \mathcal{Y} \) is truncatable, we can replace the poset of all \( \mathcal{U} \) by the cotruncative ones. In this case, we can write \( \text{Shv}(\mathcal{Y})_{\text{co}} \) also as
\[ \lim \overset{\rightarrow}{\text{Shv}(\mathcal{U})}, \]
where the transition functors are \( \text{Shv}(\mathcal{U}_2) \rightarrow \text{Shv}(\mathcal{U}_1) \) are \( j^* \).
C.2.3. We have a tautologically defined functor
\[ \text{Id}^\text{naive} : \text{Shv}(Y)_{\text{co}} \to \text{Shv}(Y). \]
It corresponds to the compatible family of functors
\[ (U \hookrightarrow Y) \rightsquigarrow (j_* : \text{Shv}(U) \to \text{Shv}(Y)). \]

For a quasi-compact \( U \hookrightarrow Y \) we have
\[ \text{Id}^\text{naive} \circ j_* \cong j_* . \]

In addition, we have a naturally defined monoidal action of \( \text{Shv}(Y) \) on \( \text{Shv}(Y)_{\text{co}} \) given by \( \otimes \).

C.2.4. Let \( \mathcal{N} \) be a closed conical subset in \( T^*(Y) \), and assume that \( Y \) is \( \mathcal{N} \)-truncatable.

In this case we define the category \( \text{Shv}_N(Y)_{\text{co}} \) as
\[ \text{colim}_U \text{Shv}_N(U), \]
where the colimit is taken over the poset of \( \mathcal{N} \)-cotruncative quasi-compact open substacks of \( \text{Bun}_G \).

The fact that the above poset is filtered implies that the tautologically defined functor
\[ \text{Shv}_N(Y)_{\text{co}} \to \text{Shv}(Y)_{\text{co}} \]
is fully faithful.

Furthermore, the functor \( \text{Id}^\text{naive} \) sends
\[ \text{Shv}_N(Y)_{\text{co}} \to \text{Shv}_N(Y). \]

C.3. **Verdier duality in the non-quasi-compact case.** In this subsection we establish a duality between \( \text{Shv}(Y) \) and \( \text{Shv}(Y)_{\text{co}} \).

C.3.1. Recall the following paradigm:

Let
\[ i \mapsto C_i, \quad i \in I \]
be a diagram of DG categories. Denote
\[ C := \text{colim}_{i \in I} C_i. \]

Suppose that for each arrow \( i_1 \to i_2 \), the transition functor
\[ \phi_{i_1,i_2} : C_{i_1} \to C_{i_2} \]
adopts a continuous right adjoint. Suppose also that each \( C_i \) is dualizable.

We can form a new family
(C.1) \[ i \mapsto C_i^\vee, \quad i \in I, \]
where the transition functor \( C_i^\vee \to C_{i_2}^\vee \) is \( (\phi_{i_1,i_2}^R)^\vee \).

Denote
\[ \tilde{C} := \text{colim}_{i \in I} C_i^\vee. \]

Supposed for simplicity that the index category \( I \) is sifted, and that the transition functors \( \phi_{i_1,i_2} \) are fully faithful.

Then we have a naturally defined pairing
(C.2) \[ C \otimes \tilde{C} \to \text{Vect}. \]

Namely, using the siftedness assumption on \( I \), we write
\[ C \otimes \tilde{C} \simeq \text{colim}_{i \in I} C_i \otimes C_i^\vee. \]
and the sought-for pairing is given by the tautological pairings $C_i \otimes C_i^\vee \to \text{Vect}$.

The following is a particular case of [DrGa1, Proposition 1.8.3]:

**Proposition C.3.2.** The functor (C.2) identifies $\tilde{C}$ with the dual of $C$.

C.3.3. We apply the paradigm of Sect. C.3.1 as follows. Let $\mathcal{Y}$ be a (not necessarily quasi-compact) algebraic stack. Consider the index set consisting of its quasi-compact open substacks, and consider the functor

$$
\mathcal{U} \rightsquigarrow \text{Shv}(\mathcal{U}), \quad (\mathcal{U}_1 \xrightarrow{j_1} \mathcal{U}_2) \rightsquigarrow (j_1)_!.
$$

By Sect. C.2.1,

$$
\text{colim}_\mathcal{U} \text{Shv}(\mathcal{U}) =: \text{colim}_\mathcal{U} \text{Shv}(\mathcal{U}) \cong \text{Shv}(\mathcal{Y}).
$$

Recall (see Sect. A.4.1) that for a quasi-compact stack the category of sheaves is naturally self-dual via Verdier duality.

The corresponding dual family (C.1) identifies with

$$
\mathcal{U} \rightsquigarrow \text{Shv}(\mathcal{U}), \quad (\mathcal{U}_1 \xrightarrow{j_1} \mathcal{U}_2) \rightsquigarrow (j_1)^*,
$$

and its colimit

$$
\text{colim}_\mathcal{U} \text{Shv}(\mathcal{U})
$$

is by definition $\text{Shv}(\mathcal{Y})_{\text{co}}$.

Applying Proposition C.3.2, we obtain a canonical identification

(C.3) $\text{Shv}(\mathcal{Y})^\vee \cong \text{Shv}(\mathcal{Y})_{\text{co}}$.

C.3.4. Unwinding the definitions, we obtain that the counit of the duality (C.3) is given by the functor $\text{ev}_\mathcal{Y}$, i.e.,

$$
\text{Shv}(\mathcal{Y}) \otimes \text{Shv}(\mathcal{Y})_{\text{co}} \xrightarrow{1} \text{Shv}(\mathcal{Y})_{\text{co}} \xrightarrow{C_{\mathcal{Y}}(\mathcal{Y}, -)} \text{Vec},
$$

where

$$
C_{\mathcal{Y}}(\mathcal{Y}, -) : \text{Shv}(\mathcal{Y})_{\text{co}} \to \text{Vec}, \quad \mathcal{U} \text{ is a quasi-compact open in } \mathcal{Y}.
$$

The corresponding contravariant self-equivalence

$$
\mathbb{D}^{\text{Verdier}} : (\text{Shv}(\mathcal{Y})^c)^{\text{op}} \to (\text{Shv}(\mathcal{Y})_{\text{co}})^c
$$

sends

$$
j_!(\mathcal{F}_U) \mapsto j^*_{\text{co}}(\mathbb{D}^{\text{Verdier}}(\mathcal{F}_U)), \quad \mathcal{F}_U \in \text{Shv}(\mathcal{U})^c
$$

for a quasi-compact open $\mathcal{U} \hookrightarrow \mathcal{Y}$.

C.3.5. Let now $N \subset T^\ast(\mathcal{Y})$ be a conical Zariski-closed subset. Assume that $\mathcal{Y}$ is (universally) Nilpot-truncatable.

Let us suppose now that for every (universally) $N$-cotruncative quasi-compact open substack $\mathcal{U} \subset \mathcal{Y}$, the pair $(\mathcal{U}, N)$ is duality-adapted.

It then follows from Proposition C.3.2 that the restriction of $\text{ev}_\mathcal{Y}$ along

$$
\text{Shv}_N(\mathcal{Y}) \otimes \text{Shv}_N(\mathcal{Y})_{\text{co}} \to \text{Shv}(\mathcal{Y}) \otimes \text{Shv}(\mathcal{Y})_{\text{co}}
$$

defines an identification

$$
\text{Shv}_N(\mathcal{Y})^\vee \cong \text{Shv}_N(\mathcal{Y})_{\text{co}}.
$$

C.4. **Functors defined by kernels in the non-quasi-compact case.** In this subsection we explain modifications to the formalism of functors defined by kernels, required in order to consider the non quasi-compact case.
C.4.1. We first consider the formalism of functors defined by kernels. This theory requires no modifications from the quasi-compact case. Namely, an object

\[ \mathcal{P} \in \text{Shv}(\mathcal{Y}_1 \times \mathcal{Y}_2), \]

defines a family of functors

\[ \text{Id}_{\mathcal{Z}} \otimes \mathcal{P}^j : \text{Shv}(\mathcal{Z} \times \mathcal{Y}_1) \to \text{Shv}(\mathcal{Z} \times \mathcal{Y}_2), \]

by the same formula

\[ (\text{Id}_{\mathcal{Z}} \otimes \mathcal{P}^j)(\mathcal{F}) = (p_1^*)(\mathcal{F}) \otimes \mathcal{P}. \]

The observation in Sect. B.1.5 applies with \(-\mathbf{1}\) and \(-1^*\) replaced by \(-1\) and \(1^*\), respectively.

C.4.2. We now consider functors defined by kernels.

Let \(\mathcal{Y}\) and \(\mathcal{Z}\) be a pair of (not necessarily quasi-compact) algebraic stacks. In this case, in addition to

\[ \text{Shv}(\mathcal{Z} \times \mathcal{Y}) \text{ and } \text{Shv}(\mathcal{Z} \times \mathcal{Y})_{\text{co}}, \]

one can define the mixed categories

\[ \text{Shv}(\mathcal{Z} \times \mathcal{Y})_{\text{co} \mathcal{Z}} \text{ and } \text{Shv}(\mathcal{Z} \times \mathcal{Y})_{\text{co} \mathcal{Y}}. \]

Namely,

\[ \text{Shv}(\mathcal{Z} \times \mathcal{Y})_{\text{co} \mathcal{Z}} := \lim_{\mathcal{Z} \subset \mathcal{Y}}^* \text{Shv}(\mathcal{Z} \times \mathcal{U}_y)_{\text{co}} \simeq \lim_{\mathcal{U}_y \subset \mathcal{Z}}^* \text{colim}_{\mathcal{Y}} \text{Shv}(\mathcal{U}_z \times \mathcal{U}_y) \]

and

\[ \text{Shv}(\mathcal{Z} \times \mathcal{Y})_{\text{co} \mathcal{Y}} := \lim_{\mathcal{U}_z \subset \mathcal{Z}}^* \text{Shv}(\mathcal{U}_z \times \mathcal{Y})_{\text{co}} \simeq \lim_{\mathcal{U}_y \subset \mathcal{Y}}^* \text{colim}_{\mathcal{U}_z} \text{Shv}(\mathcal{U}_z \times \mathcal{U}_y), \]

where \(\mathcal{U}_y\) and \(\mathcal{U}_z\) are quasi-compact open substacks of \(\mathcal{Y}\) and \(\mathcal{Z}\), respectively.

We have naturally defined functors

\[ \text{Id}_{\mathcal{Z}} \otimes \text{Id}_{\mathcal{Y}}^{\text{naive}} : \text{Shv}(\mathcal{Z} \times \mathcal{Y})_{\text{co} \mathcal{Z}} \to \text{Shv}(\mathcal{Z} \times \mathcal{Y})_{\text{co} \mathcal{Z}} \]

and

\[ \text{Id}_{\mathcal{Z}}^{\text{naive}} \otimes \text{Id}_{\mathcal{Y}} : \text{Shv}(\mathcal{Z} \times \mathcal{Y})_{\text{co} \mathcal{Z}} \to \text{Shv}(\mathcal{Z} \times \mathcal{Y})_{\text{co} \mathcal{Z}}, \]

whose composition is the functor

\[ \text{Id}_{\mathcal{Z} \times \mathcal{Y}} : \text{Shv}(\mathcal{Z} \times \mathcal{Y})_{\text{co} \mathcal{Z}} \to \text{Shv}(\mathcal{Z} \times \mathcal{Y}). \]

C.4.3. Assume now that \(\mathcal{Y}\) is truncatable. Then we claim that we can rewrite \(\text{Shv}(\mathcal{Z} \times \mathcal{Y})_{\text{co} \mathcal{Z}}\) also we as

\[ \text{colim}_{\mathcal{U}_z \subset \mathcal{Z}}^* \text{Shv}(\mathcal{U}_z \times \mathcal{U}_y) \simeq \text{colim}_{\mathcal{U}_z \subset \mathcal{Z}}^* \text{Shv}(\mathcal{U}_z \times \mathcal{Y}). \]

Indeed, we note that for a cotruncative

\[ \mathcal{U}_{1, y} \xrightarrow{\mathcal{E}_y} \mathcal{U}_{2, y} \]

and any \(\mathcal{U}_{1, z} \xrightarrow{\mathcal{E}_z} \mathcal{U}_{2, z}\), the diagram

\[ \begin{array}{ccc}
\text{Shv}(\mathcal{U}_{1, 1} \times \mathcal{U}_{1, z}) & \xrightarrow{(\mathcal{E}_y \times \text{id})_*} & \text{Shv}(\mathcal{U}_{2, 1} \times \mathcal{U}_{1, z}) \\
(id \times j_{\mathcal{Z}})_* & & (id \times j_{\mathcal{Z}})_* \\
\text{Shv}(\mathcal{U}_{1, 1} \times \mathcal{U}_{2, z}) & \xrightarrow{(\mathcal{E}_y \times \text{id})_*} & \text{Shv}(\mathcal{U}_{2, 1} \times \mathcal{U}_{2, z})
\end{array} \]

commutes.

Hence, we can rewrite

\[ \lim_{\mathcal{U}_y \subset \mathcal{Y}}^* \text{colim}_{\mathcal{U}_z \subset \mathcal{Z}} \text{Shv}(\mathcal{U}_z \times \mathcal{U}_y) \simeq \text{colim}_{\mathcal{U}_z \subset \mathcal{Z}} \lim_{\mathcal{U}_y \subset \mathcal{Y}}^* \text{Shv}(\mathcal{U}_z \times \mathcal{U}_y) \simeq \text{colim}_{\mathcal{U}_z \subset \mathcal{Z}} \text{Shv}(\mathcal{U}_z \times \mathcal{U}_y) \simeq \text{colim}_{\mathcal{U}_z \subset \mathcal{Z}} \lim_{\mathcal{U}_y \subset \mathcal{Y}}^* \text{Shv}(\mathcal{U}_z \times \mathcal{U}_y). \]
C.4.4. The material in Sect. B.1 goes through with the following modifications:

Let \( Y_1 \) and \( Y_2 \) be a pair of truncatable stacks, and let \( Z \) be another algebraic stack.

- An object \( Q \in \text{Shv}(Y_1 \times Y_2) \) gives rise to functors
  \[ \text{Id}_Z \otimes Q : \text{Shv}(Z \times Y_1) \rightarrow \text{Shv}(Z \times Y_2) \]
  \[ \text{Id}_Z \otimes Q : \text{Shv}(Z \times Y_1) \rightarrow \text{Shv}(Z \times Y_2) \]
  \[ \text{Id}_Z \otimes Q : \text{Shv}(Z \times Y_1) \rightarrow \text{Shv}(Z \times Y_2) \]
- An object \( Q \in \text{Shv}(Y_1 \times Y_2) \) gives rise to functors
  \[ \text{Id}_Z \otimes Q : \text{Shv}(Z \times Y_1) \rightarrow \text{Shv}(Z \times Y_2) \]
  \[ \text{Id}_Z \otimes Q : \text{Shv}(Z \times Y_1) \rightarrow \text{Shv}(Z \times Y_2) \]
- An object \( Q \in \text{Shv}(Y_1 \times Y_2) \) gives rise to functors
  \[ \text{Id}_Z \otimes Q : \text{Shv}(Z \times Y_1) \rightarrow \text{Shv}(Z \times Y_2) \]

C.4.5. When \( Z = Y \), we will use the notations \( \text{Shv}(Y \times Y) \) and \( \text{Shv}(Y \times Y) \) for the corresponding mixed categories.

Let \( F \) be an object of \( \text{Shv}(Y) \). We have the usual diagonal object
\[ (\Delta_Y)^* (F) \in \text{Shv}(Y \times Y). \]

In addition, we can consider the objects
\[ (\Delta_Y)^* (F)_{\text{co1}} \in \text{Shv}(Y \times Y)_{\text{co1}} \]
and
\[ (\Delta_Y)^* (F)_{\text{co2}} \in \text{Shv}(Y \times Y)_{\text{co2}}, \]
defined as follows:

The restriction of \( (\Delta_Y)^* (F)_{\text{co1}} \) to \( Y \times U \) for a quasi-compact open \( U \) is
\[ (j \times \text{id})_{\text{co}} \circ (\Delta_U)^* \circ j^* (F) \in \text{Shv}(Y \times U)_{\text{co}}, \]
and similarly for \( (\Delta_Y)^* (F)_{\text{co2}} \).

By Sect. C.4.4, the object \( (\Delta_Y)^* (F)_{\text{co1}} \) defines a functor
\[ F : \text{Shv}(Y) \rightarrow \text{Shv}(Y), \]
and the object \( (\Delta_Y)^* (F)_{\text{co2}} \) defines a functor
\[ F : \text{Shv}(Y)_{\text{co}} \rightarrow \text{Shv}(Y)_{\text{co}}. \]

In both contexts, the functor \( F \) is given by \( F \otimes - \).

Let now \( F \) be an object of \( \text{Shv}(Y)_{\text{co}} \). Then \( (\Delta_Y)^* (F) \) is naturally an object of \( \text{Shv}(Y \times Y)_{\text{co}} \). The resulting functor
\[ F : \text{Shv}(Y) \rightarrow \text{Shv}(Y)_{\text{co}} \]
is given by \( - \otimes F \).
C.4.6. When \( F = \omega_Y \in \text{Shv}(Y) \), we will use the notation
\[
u_{Y, \text{naive}} := (\Delta_Y)_*(\omega_Y), \quad \nu_{Y, co_1} := (\Delta_Y)_*(\omega_Y)_{co_1}, \quad \nu_{Y, co_2} := (\Delta_Y)_*(\omega_Y)_{co_2}.
\]

In terms of Sect. C.4.4, the object \( \nu_{Y, \text{naive}} \) defines the functors
\[
\text{Id}_Z \boxtimes \text{Id}_{\nu_{Y, \text{naive}}} : \text{Shv}(Z \times Y)_{co} \to \text{Shv}(Z \times Y)_{co}
\]
and
\[
\text{Id}_Z \boxtimes \text{Id}_{\nu_{Y, co_1}} : \text{Shv}(Z \times Y)_{co_1} \to \text{Shv}(Z \times Y).
\]

The object \( \nu_{Y, co_1} \) defines the identity functors
\[
\text{Shv}(Z \times Y)_{co} \to \text{Shv}(Z \times Y)_{co}
\]
and \( \text{Shv}(Z \times Y)_{co_1} \to \text{Shv}(Z \times Y)_{co_1} \).

C.4.7. We now consider the object
\[
\text{ps-u}_Y := (\Delta_Y)_!(\epsilon_Y) \in \text{Shv}(Y \times Y).
\]
By Sect. C.4.4, it defines a functor
\[
\text{Shv}(Y)_{co} \to \text{Shv}(Y).
\]
This is the miraculous functor, to be denoted \( \text{Mir}_Y \).

C.4.8. Let \( Y' \hookrightarrow Y \) be a cotruncative open embedding. It follows formally that we have
\[
(C.4) \quad j_! \circ \text{Mir}_Y' \simeq \text{Mir}_Y \circ j^*_*,
\]
as functors given by kernels.

C.4.9. We will now consider one more variant of the construction of functors attached to sheaves on the product of two stacks.

Let \( Y_1 \) and \( Y_2 \) be truncatable, and let \( P \) be an object of \( \text{Shv}(Y_1 \times Y_2)_{co_2} \). Then for any \( Z \), we have a well-defined functor
\[
\text{Id}_Z \boxtimes P^! : \text{Shv}(Z \times Y_1) \to \text{Shv}(Z \times Y_2)_{co_2}.
\]

C.5. Miraculous stacks—the non quasi compact case. In this subsection we define what it means for a not necessarily quasi-compact algebraic stack to be miraculous.

C.5.1. Let now \( Y \) by a not necessarily quasi-compact truncatable stack.

We shall say that \( Y \) is miraculous if \( \text{Mir}_Y \) is invertible, as a functor defined by a kernel. I.e., if there exists an object
\[
Q \in \text{Shv}(Y \times Y)_{co}
\]
such that
\[
Q \star \text{ps-u}_Y \simeq u_{Y, co_2}
\]
as objects of \( \text{Shv}(Y \times Y)_{co_2} \) and
\[
\text{ps-u}_Y \star Q \simeq u_{Y, co_1}
\]
as objects of \( \text{Shv}(Y \times Y)_{co_1} \).

C.5.2. It is easy to see that \( Y \) is miraculous if and only if every cotruncative quasi-compact open substack \( U \subset Y \) is miraculous:

Indeed, the “if” direction follows from (C.4). For the “only if” direction, if \( Y \) is miraculous and \( U \subset Y \) is cotruncative, formula (C.4) implies that \( \text{Id}_U \boxtimes \text{Mir}_U \) is fully faithful. The essential surjectivity follows from the formula obtained from (C.4) by duality:
\[
\text{Mir}_U \circ j^*_U \simeq j^* \circ \text{Mir}_Y,
\]
where \( j^* \) is the right adjoint of \( j_*, co \).
C.5.3. Still equivalently, $Y$ is miraculous if and only if the functor

$$(\text{Id}_Z \boxtimes \text{Mir}_Y) : \text{Shv}(Z \times Y)_{\text{co}} \to \text{Shv}(Z \times Y)$$

is an equivalence for every quasi-compact $Z$.

If $Y$ is miraculous, then the functor

$$(\text{Id}_Z \boxtimes \text{Mir}_Y) : \text{Shv}(Z \times Y)_{\text{co}} \to \text{Shv}(Z \times Y)$$

and

$$(\text{Id}_Z \boxtimes \text{Mir}_Y) : \text{Shv}(Z \times Y)_{\text{co}} \to \text{Shv}(Z \times Y)_{\text{co}}$$

are equivalences for an arbitrary $Z$.

C.5.4. Let $Y$ be miraculous. In particular, $\text{Mir}_Y$, viewed just as a functor $\text{Shv}(Y)_{\text{co}} \to \text{Shv}(Y)_{\text{co}}$, is an equivalence.

In this case we define the miraculous self-duality of $\text{Shv}(Y)$ to have as counit the functor

$$\text{Shv}(Y) \otimes \text{Shv}(Y) \xrightarrow{\text{Mir}_Y^{-1} \otimes \text{Id}} \text{Shv}(Y)_{\text{co}} \otimes \text{Shv}(Y) \xrightarrow{\text{ev}_Y} \text{Vect},$$

which we denote $\text{ev}^\text{Mir}_Y$.

The corresponding contravariant self-equivalence, denoted

$$\mathcal{D}_Y : (\text{Shv}(Y)^c)^{\text{op}} \to \text{Shv}(Y)^c,$$

is given by

$$\mathcal{D}_Y \simeq \mathcal{D}_{\text{Verdier}} \circ \text{Mir}_Y^{-1}.$$

Thus, explicitly, for a cotruncative quasi-compact $U \xleftarrow{j} \text{Bun}_G$ and $F_U \in \text{Shv}(U)^c$, using [Ga1], we obtain

$$\mathcal{D}_Y(j_!(F_U)^c) \simeq j_! \left( \mathcal{D}_{\text{Verdier}} \circ \text{Mir}_U^{-1}(F_U) \right).$$

As in Sect. [B.7.3] one shows that the pairing $\text{ev}^\text{Mir}_Y$ is swap-equivariant, and hence the functor $\mathcal{D}_Y$ is involutive.

Remark C.5.5. Note that unlike the quasi-compact case, Verdier duality is not a self-duality of $\text{Shv}(Y)$, but rather a duality between $\text{Shv}(Y)$ and $\text{Shv}(Y)_{\text{co}}$. So, a priori, miraculous duality is our only option for a self-duality of $\text{Shv}(Y)$.

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