CURVES OF CONSTANT BREADTH ACCORDING TO TYPE-2 BISHOP FRAME IN $E^3$

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ABSTRACT. In this paper, we study the curves of constant breadth according to type-2 Bishop frame in the 3-dimensional Euclidean Space $E^3$. Moreover some characterizations of these curves are obtained.

1. INTRODUCTION

In 1780, L. Euler studied curves of constant breadth in the plane [3]. Thereafter, this issue investigated by many geometers [2, 4, 12]. Constant breadth curves are an important subject for engineering sciences, especially, in cam designs [17]. M. Fujiwara introduced constant breadth for space curves and surfaces [4]. D. J. Struik published some important publications on this subject [16]. O. Kose expressed some characterizations for space curves of constant breadth in Euclidean 3-space [10] and M. Sezer researched space curves of constant breadth and obtained a criterion for these curves [15]. A. Magden and O. Kose obtained constant breadth curves in Euclidean 4-space [11]. Characterizations for spacelike curves of constant breadth in Minkowski 4-space were given by M. Kazaz et al. [9]. S. Yilmaz and M. Turgut studied partially null curves of constant breadth in semi-Riemannian space [18]. The properties of these curves in 3-dimensional Galilean space were given by D. W. Yoon [20]. H. Gun Bozok and H. Oztekin investigated an explicit characterization of mentioned curves according to Bishop frame in 3-dimensional Euclidean space [5]. The curve of constant breadth on the sphere studied by W. Blaschke [2]. Furthermore, the method related to the curves of constant breadth for the kinematics of machinery was given by F. Reuleaux [14].

L. R. Bishop defined Bishop frame, which is known alternative or parallel frame of the curves with the help of parallel vector fields [1]. Then, S. Yilmaz and M. Turgut examined a new version of the Bishop frame which is called type-2 Bishop frame [19]. Thereafter, E. Ozyilmaz studied classical differential geometry of curves according to type-2 Bishop trihedra [13].

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In this paper, we used the theory of the curves with respect to type-2 Bishop frame. Then, we gave some characterizations for curves of constant breadth according to type-2 Bishop frame.

2. Preliminaries

The standard flat metric of 3-dimensional Euclidean space $E^3$ is given by

$$\langle \cdot, \cdot \rangle : dx_1^2 + dx_2^2 + dx_3^2$$

(2.1)

where $(x_1, x_2, x_3)$ is a rectangular coordinate system of $E^3$. For an arbitrary vector $x$ in $E^3$, the norm of this vector is defined by $\|x\| = \sqrt{\langle x, x \rangle}$. $\alpha$ is called a unit speed curve, if $\langle \alpha', \alpha' \rangle = 1$. Suppose that $\{t, n, b\}$ is the moving Frenet–Serret frame along the curve $\alpha$ in $E^3$. For the curve $\alpha$, the Frenet-Serret formulae can be given as

\[
\begin{align*}
t' &= \kappa n \\
n' &= -\kappa t + \tau b \\
b' &= -\tau n
\end{align*}
\]

(2.2)

where

\[
\langle t, t \rangle = \langle n, n \rangle = \langle b, b \rangle = 1,
\]

\[
\langle t, n \rangle = \langle t, b \rangle = \langle n, b \rangle = 0.
\]

and here, $\kappa = \kappa(s) = \|t'(s)\|$ and $\tau = \tau(s) = -\langle n, b' \rangle$. Furthermore, the torsion of the curve $\alpha$ can be given

$$\tau = \frac{[\alpha', \alpha'', \alpha''']}{\kappa^2}.$$ 

Along the paper, we assume that $\kappa \neq 0$ and $\tau \neq 0$.

Bishop frame is an alternative approachment to define a moving frame. Assume that $\alpha(s)$ is a unit speed regular curve in $E^3$. The type-2 Bishop frame of the $\alpha(s)$ is expressed as [19]

\[
\begin{align*}
N'_1 &= -k_1 B, \\
N'_2 &= -k_2 B, \\
B' &= k_1 N_1 + k_2 N_2.
\end{align*}
\]

(2.3)

The relation matrix may be expressed as

\[
\begin{bmatrix}
t \\
n \\
b
\end{bmatrix} = \begin{bmatrix}
sin \theta(s) & -\cos \theta(s) & 0 \\
\cos \theta(s) & \sin \theta(s) & 0 \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
N_1 \\
N_2 \\
B
\end{bmatrix}.
\]

(2.4)
where \( \theta (s) = \int_0^s \kappa (s) \, ds \). Then, type-2 Bishop curvatures can be defined in the following:

\[
\begin{align*}
    k_1 (s) &= -\tau (s) \cos \theta (s), \\
    k_2 (s) &= -\tau (s) \sin \theta (s).
\end{align*}
\]

On the other hand,

\[
\theta' = \kappa = \frac{\left( \frac{k_2}{k_1} \right)'}{1 + \left( \frac{k_2}{k_1} \right)^2}.
\]

The frame \( \{ N_1, N_2, B \} \) is properly oriented, \( \tau \) and \( \theta (s) = \int_0^s \kappa (s) \, ds \) are polar coordinates for the curve \( \alpha \). Then, \( \{ N_1, N_2, B \} \) is called type-2 Bishop trihedra and \( k_1, k_2 \) are called Bishop curvatures.

The characterizations of inclined curves in \( E^n \) is given [7] and [8] as follows:

**Theorem 1.** \( \alpha \) is an inclined curve in \( E^n \) \( \iff \sum_{i=1}^{n-2} H_i^2 = \text{const} \) and \( \alpha \) is an inclined curve in \( E^{n-1} \) \( \iff \det \left( V_1', V_2', ..., V_n' \right) = 0. \)

**Theorem 2.** Let \( M \subset E^3 \) is a curve given by \( (I, \alpha) \) chart. Then \( M \) is an inclined curve if and only if \( H(s) = \frac{k_1(s)}{k_2(s)} \) is constant for all \( s \in I. \)

3. **Curves of Constant Breadth According to Type-2 Bishop Frame in \( E^3 \)**

Let \( X = \bar{X} (s) \) be a simple closed curve in \( E^3 \). These curves will be denoted by \( (C) \). The normal plane at every point \( P \) on the curve meets the curve at a single point \( Q \) other than \( P \). The point \( Q \) is called the opposite point of \( P \). Considering a curve \( \alpha \) which have parallel tangents \( \bar{T} \) and \( \bar{T}^* \) in opposite points \( X \) and \( X^* \) of the curve as in [4]. A simple closed curve of constant breadth which have parallel tangents in opposite directions can be introduced by

\[
X^*(s) = X(s) + m_1(s) N_1 + m_2(s) N_2 + m_3(s) B \tag{3.1}
\]

where \( X \) and \( X^* \) are opposite points and \( N_1, N_2, B \) denote the type-2 Bishop frame in \( E^3 \) space. If \( N_1 \) is taken instead of tangent vector and differentiating equation (3.1) we have

\[
\frac{dX^*}{ds} = \frac{dX^*}{ds^*} \frac{ds^*}{ds} = N_1 \frac{ds^*}{ds} = \left( 1 + \frac{dm_1}{ds} + m_3 k_1 \right) N_1 + \left( \frac{dm_2}{ds} + m_3 k_2 \right) N_2 + \left( \frac{dm_3}{ds} - m_1 k_1 - m_2 k_2 \right) B \tag{3.2}
\]
where $k_1$ and $k_2$ are the first and the second curvatures of the curve, respectively [6]. Since $N_1^* = -N_1$, we obtain

$$\frac{ds^*}{ds} + \frac{dm_1}{ds} + m_3k_1 + 1 = 0,$$

$$\frac{dm_2}{ds} + m_3k_2 = 0,$$

$$\frac{dm_3}{ds} - m_1k_1 - m_2k_2 = 0. \quad (3.3)$$

Suppose that $\phi$ is the angle between the tangent of the curve $(C)$ at point $X(s)$ with a given fixed direction and $\frac{d\phi}{ds} = k_1$, then the equation (3.3) can be written as

$$\frac{dm_1}{d\phi} = -m_3 - f(\phi),$$

$$\frac{dm_2}{d\phi} = -\rho k_2 m_3,$$

$$\frac{dm_3}{d\phi} = m_1 + \rho k_2 m_2, \quad (3.4)$$

where $f(\phi) = \rho + \rho^*$, $\rho = \frac{1}{k_1}$ and $\rho^* = \frac{1}{k_1}$ denote the radius of curvatures at $X$ and $X^*$, respectively. If we consider equation (3.4), we get

$$\frac{k_1 m_1''}{k_2} + \left(\frac{k_1}{k_2}\right)' m_1'' + \left(\frac{k_1}{k_2} + \frac{k_2}{k_1}\right)' m_1' + \left(\frac{k_1}{k_2}\right)'' m_1 + \left(\frac{k_1}{k_2}\right)' f(\phi) + \left(\frac{k_1}{k_2}\right) f'(\phi) + \left(\frac{k_1}{k_2}\right) f(\phi) = 0 \quad (3.5)$$

This equation is a characterization for $X^*$. If the distance between the opposite points of $(C)$ and $(C^*)$ is constant, then

$$\|X^* - X\|^2 = m_1^2 + m_2^2 + m_3^2 = l^2, \quad l \in \mathbb{R}.$$  

Hence, we write

$$m_1 \frac{dm_1}{d\phi} + m_2 \frac{dm_2}{d\phi} + m_3 \frac{dm_3}{d\phi} = 0 \quad (3.6)$$

By considering system (3.4), we obtain

$$m_1 \left(\frac{dm_1}{d\phi} + m_3\right) = 0. \quad (3.7)$$

Thus we can write $m_1 = 0$ or $\frac{dm_1}{d\phi} = -m_3$. Then, we consider these situations with some subcases.
Case 1. If \( \frac{dm_1}{d\phi} = -m_3 \), then \( f(\phi) = 0 \). So, \((C^*)\) is translated by the constant vector 
\[
 u = m_1N_1 + m_2N_2 + m_3B
\]
of \((C)\). Here, let us solve the equation \((3.5)\), in some special cases.

Case 1.1 Let \( X \) be an inclined curve. Then the equation \((3.5)\) can be written as follows,
\[
d^3 m_1 + \left(1 + \frac{k_2^2}{k_1^2}\right) \frac{dm_1}{d\phi} = 0.
\]
The general solution of this equation is
\[
m_1 = c_1 + c_2 \cos \sqrt{1 + \frac{k_2^2}{k_1^2} \phi} + c_3 \sin \sqrt{1 + \frac{k_2^2}{k_1^2} \phi}
\]
and therefore, we have \( m_2 \) and \( m_3 \), respectively,
\[
m_2 = \frac{k_2}{k_1} \left( c_2 \cos \sqrt{1 + \frac{k_2^2}{k_1^2} \phi} \right) + \frac{k_2}{k_1} \left( c_3 \sin \sqrt{1 + \frac{k_2^2}{k_1^2} \phi} \right)
\]
\[
m_3 = c_2 \sqrt{1 + \frac{k_2^2}{k_1^2} \sin \sqrt{1 + \frac{k_2^2}{k_1^2} \phi} - c_3 \sqrt{1 + \frac{k_2^2}{k_1^2} \cos \sqrt{1 + \frac{k_2^2}{k_1^2} \phi}}}
\]
where \( c_1 \) and \( c_2 \) are real numbers.

Corollary 1. Position vector of \( X^* \) can be formed by the equations \((3.10)\), \((3.11)\) and \((3.12)\). Also the curvature of \( X^* \) is obtained as
\[
k_1^* = -k_1.
\]

Case 2. \( m_1 = 0 \). Then, considering equation \((3.5)\) we get
\[
\left( \frac{k_1}{k_2} \right) f(\phi)'' + \left( \frac{k_1}{k_2} \right)' f(\phi)' + \left( \frac{k_2}{k_1} \right) f(\phi) = 0
\]
Case 2.1 Suppose that \( X \) is an inclined curve. The equation \((3.14)\) can be rewrite as
\[
f(\phi)'' + \left( \frac{k_2}{k_1} \right)^2 f(\phi) = 0.
\]
So, the solution of above differential equation is
\[
f(\phi) = L_1 \cos \frac{k_2}{k_1} \phi + L_2 \sin \frac{k_2}{k_1} \phi
\]
where \( L_1 \) and \( L_2 \) are real numbers. Using above equation we obtain

\[
m_2 = L_1 \sin \frac{k_2}{k_1} \phi - L_2 \cos \frac{k_2}{k_1} \phi
\]

\[
m_3 = -L_1 \cos \frac{k_2}{k_1} \phi - L_2 \sin \frac{k_2}{k_1} \phi = -\rho^* \quad (3.17)
\]

And therefore the curvature of \( X^* \) is obtained as

\[
k_1^* = \frac{1}{L_1 \cos \frac{k_2}{k_1} \phi + L_2 \sin \frac{k_2}{k_1} \phi - \frac{\rho}{k_1}} \quad (3.19)
\]

And distance between the opposite points of \((C)\) and \((C^* )\) is

\[
\|X - X^*\| = L_1^2 + L_2^2 = \text{const.} \quad (3.20)
\]

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