Dark Energy Accretion onto Van der Waal’s Black Hole*  

Sandip Dutta† and Ritabrat Biswas‡  
Department of Mathematics, The University of Burdwan, Golapbag Academic Complex, Burdwan-713104, Purba Bardhaman, West Bengal, India  
(Received May 8, 2018; revised manuscript received July 2, 2018)  

**Abstract** We consider the most general static spherically symmetric black hole metric. The accretion of the fluid flow around the Van der Waal’s black hole is investigated and we calculate the fluid’s four-velocity, the critical point and the speed of sound during the accretion process. We also analyze the nature of the universe’s density and the mass of the black hole during accretion of the fluid flow. The density of the fluid flow is also taken into account. We observe that the mass is related to redshift. We compare the accreting power of the Van der Waal’s black hole with Schwarzschild black hole for different accreting fluid.  

**DOI:** 10.1088/0253-6102/71/2/209  
**Key words:** black hole physics, thermodynamics, accretion disc, extended Chaplygin gas

1 Introduction  
On the Anti de-Sitter (AdS) boundary, the studies of strongly coupled thermal field theories led us to some interesting results regarding the physics of asymptotically AdS black holes. In Ref. [1], for Schwarzschild-AdS black hole space-time, a first order phase transition of thermal radiation/black hole is observed. The thermodynamic behaviours make after those of Van der Waal’s fluid if charge and rotation are incorporated. A more prominent analogy[6–7] is observed when cosmological constant is treated as a thermodynamic pressure, \( P \), given as

\[
P = -\frac{\Lambda}{8\pi} = \frac{3}{8\pi l^2}.
\]

This also extends the first law of black hole thermodynamics as \( \delta M = T\delta S + V\delta P + \cdots \). To do so we need to introduce a quantity, which is thermodynamically conjugated to \( P \) and is interpreted as a black hole thermodynamic volume[8–9] which is given as

\[
V = \left( \frac{\delta M}{\delta T} \right)_{S}.
\]

Immediately after the consideration of analogue pressure and volume, one may query about the nature of black hole’s equation of state. Comparison between the temperatures of the black holes and concerned fluid or the comparison between the volumes or the pressures of the black holes to the corresponding physical properties of concerned fluid (with which the black holes’ equation of state does resemble) can be built up. References like Ref. [10] have chosen the equation of state possessed by Van der Waal’s fluids. For one mole of such fluid, the equation of state turns to be

\[
T = \left( P + \frac{a}{\sqrt{v}} \right)(v - b),
\]

where \( v = V/N \), \( N \) being the fluid’s degrees of freedom. The attraction between two molecules of the concerned fluid is measured by the parameter “\( a \)” \( (a > 0) \). On the other hand, the volumetric measure is kept by the parameter “\( b \)”.

It is studied in Refs. [6–7] that the rotating (or nonrotating as well) AdS black holes’ thermodynamic natures resemble with this particular type of fluid to a large extent. Whenever a phase transition is observed in the black hole, a swallowtail catastrophe is exposed by Gibb’s free energy for both the fluid and the black hole. An exact match between the properties of Van der Waal’s fluid and a particular type of black holes are tried to be obtained in Refs. [11–12]. The static spherically symmetric black hole metric obtained by these article is given as

\[
d s^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2),
\]

with the lapse function

\[
f(r) = 2\pi a_v v - \frac{2M}{r} + \frac{r^2}{l^2} \left( 1 + \frac{3b_v v}{2r} \right) = -\frac{3\pi a_v b_v^2 v^2}{r(2r + 3b_v v)} - 4\pi a_v b_v \log \left( \frac{r}{b_v v} + \frac{3}{2} \right),
\]

where \( M \) is the mass of the Van der Waal’s black hole.

*This research is supported by the project grant of Government of West Bengal, Department of Higher Education, Science and Technology and Biotechnology (File No: -ST/P/8k/T/16G-19/2017). SD thanks Government of West Bengal, Department of Higher Education, Science and Technology and Biotechnology for non-NET fellowship
†E-mail: duttasandip.mathematics@gmail.com
‡E-mail: biswas.ritabrat@gmail.com
© 2019 Chinese Physical Society and IOP Publishing Ltd

http://www.iopscience.iop.org/ctp http://ctp.itp.ac.cn
tures, nowadays, where\textsuperscript{[13–14]} the thermodynamic natures of the Van der Waal’s black holes have been studied. A common conclusion drawn from all these articles is that the Van der Waal’s black hole solution is qualitatively analogical (on a thermodynamic perspective) with the Van der Waal’s fluid. Else we can treat it as just another type of static spherically symmetric black hole ansatz.

As we know, since almost twenty years from now, that our universe is experiencing a late time cosmic acceleration and to justify such kind of accelerated expansion, we have proposed many models, where a homogenous exotic fluid coined as “quintessence”, “dark energy” or “phantom field” etc. is assumed to be present in the universe (spreaded all over in it) and is exerting negative pressure.

Until now, many dark energy models have been speculated. Between these models, the most appealing one is the cosmological constant\textsuperscript{[15]} which is the simplest candidate of dark energy, distinguished by the equation of state $p = \omega \rho$ with $\omega = -1$. However, two problems raise for the cosmological constant model: the coincidence problem\textsuperscript{[16]} and the fine tuning problem. Some methods are given to solve these problems, such as considering the holographic principle,\textsuperscript{[17]} anthropic principle,\textsuperscript{[18]} invoking an interaction between dark matter and dark energy,\textsuperscript{[19]} and variable cosmological constant scenario.\textsuperscript{[20]} In this context, several well-known models such as phantom, quintom, Chaplygin gas, quintessence, geographic dark energy and holographic dark energy, etc., have been proposed.\textsuperscript{[21–28]} The Chaplygin gas is an exotic type of fluid whose energy density $\rho$ and pressure $p$ fit the equation of state $p = -B/\rho$, where $B$ is a positive constant.\textsuperscript{[29]} At large value of scale factor, the Chaplygin gas tends to accelerate the universe’s expansion, however, at small value of the scale factor it acts as pressureless fluid.

In general terms, when such exotic fluids violate the strong energy condition $3p + \rho > 0$, we call them to be in the quintessence era. But as they violate the weak energy condition, i.e., $p + \rho > 0$ we say that these fluids are in phantom era. In phantom era, a future singularity named “Big Rip” may occur where all the four fundamental forces might be defeated by the dark energy and every matter will be destroyed. Now it was a challenging question that the compact objects like black holes will exist at big rip or not. Babichev et al.\textsuperscript{[30]} for the first time in literature, have general relativistically studied the phantom energy accretion on Schwarzschild black hole and shown that if the black hole is surrounded by a fluid, which is following the equation $p + \rho < 0$ then the mass of the black hole will be decreased by the effect of such kind of fluid’s accretion. Dark energy accretion and its properties for different black holes as central engine and different equation of states of accreting fluid are studied in different Refs. [20, 31–36].

The general overview is that the black holes will loss mass due to such kind of exotic matter accretion. A series of pseudo Newtonian studies of accretion disc nature for dark energy accretion are done in Refs. [37–40]. It is speculated that dark energy accretion strengthens the wind branch and weakens the accretion branch which as a result faints the accretion disc, i.e., it weakens the feeding process of the black holes. Effects of different modified gravity parameters on accretion are interesting topics to study.

Van der Waal’s black hole is a thermodynamically modified version of static spherically symmetric black hole which incorporates extra parameters $a_{\nu\nu}$ and $b_{\nu\nu}$ which modify the black hole’s gravitating power. As a fluid, Van der Waal’s gas incorporates the mutual attractions between molecules and the pressure exerted by molecules on the container’s wall. It is expected that the Van der Waal’s black hole will take into account the mutual interactions of microstates in it. This mutual attraction might have effect on the external power of the black hole. Our motivation for this paper is to study how does the black hole accretion disc behave if both the central black hole is modified as a Van der Waal’s black hole and the accreting fluid is taken to be an exotic one. This may give rise to a new construction of black hole’s mass reduction formula. Besides the nature of the mass of the black holes, the density variation near the black hole will be studied. This may speculate how much of negative pressure generating nature is carried out by the dark energy models near this particular type of compact objects.

In the present paper, we consider most general static spherically symmetric black hole solution in Sec. 2. An investigation regarding the accretion of any general kind of fluid flow around the black hole is done in the same section. Next, we analyze the accretion of the fluid flow around Van der Waal’s black hole in Sec. 3. Here we calculate the existence(s) of critical point(s), velocity of sound and the fluid’s four velocity during the process of accretion. Finally, we briefly conclude through a discussion.

2 Accretion onto a General Static Spherically Symmetric Black Hole

We will consider a general static spherically symmetric metric\textsuperscript{[11–12]} written as Eq. (3), where $f(r)(>0)$ considered as a function of $r$ and $M$ as the mass of the black hole.

For the accreting fluid, energy-momentum tensor is given by (8$\pi G = c = 1$)

$$T_{\mu\nu} = (\rho + p)u_{\mu}u_{\nu} + pg_{\mu\nu},$$

where $\rho$ and $p$ are the pressure and energy density of the fluid. Also, $u^\mu = dx^\mu/ds = (u^0, u^1, 0, 0)$ is the four-velocity vector of the fluid flow, where $u^0$ and $u^1$ are the components (non-zero) of velocity vector satisfying

$$u_\mu u^\mu = -1 \Rightarrow g_{00}u_0^0 u^0 + g_{11}u_1^1 u^1 = -1 \Rightarrow (u^0)^2 = f + (u^1)^2/f.$$  

Let us consider the radial velocity of the flow $u^1 = u$. Therefore, $u_0 = g_{00}u^0 = \sqrt{u^2 + f}$ where $\sqrt{-g} = r^2 \sin \theta.$
From Eq. (5), we get \( T^0_0 = (\rho + p)u_0u_t \). For inward flow, assuming \( u < 0 \) (as the fluid flows towards the black hole).

For the fluid flow, we may take that the fluid is any kind of dark energy or dark matter. When a static spherically symmetric black hole is considered, a proper dark-energy accretion model should be gained by generalizing Michell’s theory.\(^{[41]}\) Babichev et al.\(^{[30,42]}\) have performed the generalization of the dark energy accretion onto Schwarzschild black hole. \( T^0_0 = 0 \) is the energy-momentum conservation law for the relativistic Bernoulli equation (the time component).

Taking the radial temporal component of relativistic energy momentum conservation equation, we have \((d/dr)(T^0_0) = 0\), which provides the first integral \((\rho + p)u_tu^t = C_1\). Hence,

\[
u u^2 (\rho + p) \sqrt{u^2 + f} = C_1,
\]

where \(C_1\) is an integrating constant, having the dimension of the energy density. For the energy momentum tensor, the energy flux equation can be defined by the projection of the conservation law, i.e., \( u_t T^0_0 = 0 \Rightarrow u^2 \rho_0 + (\rho + p)u_t^2 = 0 \). From this, we get (taking \( \mu = 1 \)),

\[
u u^2 \exp \left[ \int_{\rho_0}^{u_t} \frac{\rho}{\rho + p(\rho)} \right] = -C,
\]

where \(C\) is a constant of integration constant and for convenience the minus sign is taken. Moreover, \( \rho_0 \) and \( \rho_t \) denote the the energy densities at infinite distance from the black hole and at the black hole horizon respectively. From Eqs. (6) and (7), we get,

\[
u u^2 \exp \left[ \int_{\rho_0}^{u_t} \frac{\rho}{\rho + p(\rho)} \right] \sqrt{u^2 + f} = C_2,
\]

where \(C_2 = -C_1/C = \rho_0 + p(\rho_0)\). Also \( J^\mu_\mu = 0 \) is the equation of mass which gives \( (d/dr)(J^1_\sqrt{-g}) = 0 \Rightarrow \rho u^1 \sqrt{-g} = A_1 \) and yields

\[
u u^2 \sqrt{u^2 + f} = C_3,
\]

where \(C_3\) is another integrating constant. From Eqs. (6) and (9), we get,

\[
u\frac{\rho + p}{\rho} \sqrt{u^2 + f} = \frac{C_1}{C_3} = C_4 = \text{constant},
\]

Let us assume

\[

V^2 = \frac{d\ln(\rho + p)}{d\ln\rho} - 1,
\]

From Eqs. (9), (10), and (11), we get,

\[

\left[ V^2 - \frac{u^2}{u^2 + f} \right] \frac{du}{u} = \left[ -2V^2 + \frac{rf'}{2(u^2 + f)} \right] \frac{dr}{r} = 0.
\]

If one or the other of the bracketed factors in Eq. (12) is terminated, we obtain a turn-around point and for this case, the solutions will give two values in either \( r \) or \( u \). The solutions are passing through a critical point that assembles the material falling out (or flowing into) and along with the particle trajectory the object has monotonically increasing velocity. Critical point is a point where the speed of the flow is equal to the speed of sound inside the fluid. Assuming at \( r = r_c \), where the critical point of accretion is located, which can be obtained by assuming the two bracketed terms (the coefficients of \( dr \) and \( du \)) in Eq. (12) to be zero. Therefore, at \( r = r_c \), we get,

\[

V_c^2 = \frac{u_c^2}{u_c^2 + f(r_c)} \quad \text{and} \quad \frac{4V_c^2}{r_c} = \frac{f'(r_c)}{u_c^2 + f(r_c)},
\]

where \( u_c \) is the critical speed of the flow at \( r = r_c \) (at the critical point) and the subscript \( c \) is denoting the critical value. From Eq. (13), we get

\[

u_c^2 = \frac{r_c f'(r_c)}{4},
\]

\[

V_c^2 = \frac{r_c f'(r_c)}{4f(r_c) + r_c f'(r_c)}.
\]

At \( r = r_c \), the sound speed can be obtained by

\[

c_s^2 = \frac{dp}{d\rho} \bigg|_{r=r_c} = \frac{C_4(V_c^2 + 1)}{u_c} - 1.
\]

The solutions are physically admissible if \( u_c^2 > 0 \) and \( V_c^2 > 0 \).

### 3 Accretion onto a Van der Waal’s Black Hole

Considering that the fluid flow accretes upon the Van der Waal’s black hole, we will compute the expressions of \( u_c^2 \), \( V_c^2 \) and \( c_s^2 \) at \( r = r_c \) (i.e., at the critical point). We get (using Eqs. (14) and (15)):

\[

u_c^2 = \frac{2}{4} + \frac{1}{2} + \frac{2M}{r_c^2} + \frac{2(1 + 3b_{uw}/2r_c)}{l^2} + \frac{6a_{uw}b_{uw}^2\pi}{r_c(2r_c + 3b_{uw})} + \frac{3\pi a_{uw}b_{uw}^2}{r_c^2(2r_c + 3b_{uw})} - \frac{4a_{uw}b_{uw}\pi}{r_c(2r_c + b_{uw})} + \frac{4a_{uw}b_{uw}\pi}{r_c^2(2r_c + b_{uw})},
\]

\[

V_c^2 = \left\{ 1 + \frac{4}{r_c} + \frac{2M}{r_c^2} + \frac{2(1 + 3b_{uw}/2r_c)}{l^2} + \frac{6a_{uw}b_{uw}^2\pi}{r_c(2r_c + 3b_{uw})} + \frac{3\pi a_{uw}b_{uw}^2}{r_c^2(2r_c + 3b_{uw})} - \frac{4a_{uw}b_{uw}\pi}{r_c(2r_c + b_{uw})} + \frac{4a_{uw}b_{uw}\pi}{r_c^2(2r_c + b_{uw})} \right\}^{-1},
\]

and \( c_s^2 \) can be obtained by using Eqs. (16), (17), and (18).

The physically admissible solutions of the above equations are obtained if \( u_c^2 > 0 \) and \( V_c^2 > 0 \), i.e.,

\[

\frac{2M}{r_c^2} + \frac{2(1 + 3b_{uw}/2r_c)}{l^2} + \frac{6a_{uw}b_{uw}^2\pi}{r_c(2r_c + 3b_{uw})} + \frac{3\pi a_{uw}b_{uw}^2}{r_c^2(2r_c + 3b_{uw})} + \frac{4a_{uw}b_{uw}\pi}{r_c(2r_c + b_{uw})} + \frac{4a_{uw}b_{uw}\pi}{r_c^2(2r_c + b_{uw})}.
\]
called quasi equilibrium. In this work, we have considered the black hole nature in local cells. Such type of equilibrium is governed by the black hole with very close to the black hole, there exists a local equilibrium in the vicinity of the black hole. Again, black hole simultaneously radiates energy when the mass of the black hole is greater than a certain critical value. The equation related to the mass of the black hole is given by [31–33] (satisfying the holographic equation of state). The equation related to the mass of the black hole is given by [31–33] (satisfying the holographic equation of state).

\[ M = M_0 + 4\pi C (\rho_\infty + p(\rho_\infty)) (t - t_0) \]  

The result (22) can be written for any general \( \rho \) and \( p \) as done in Refs. [31–33] (satisfying the holographic equation of state and violating weak energy condition), i.e., can be written as

\[ \dot{M} = 4\pi C (\rho + p) . \]

Again, black hole simultaneously radiates energy when it accretes fluid. This radiation is known as Hawking radiation. The black hole evaporates for this radiation, which is balanced by the accretion of matter into the black hole and as an outcome the total system is guessed to be under equilibrium. But when we examine the parameters (e.g., temperature) of the accretion fluid at very far from the black hole with very close to the black hole, there will be a big difference. But the parameters show equilibrium nature in local cells. Such type of equilibrium is called quasi equilibrium. In this work, we have considered large black holes (in general). For small black holes, the relation between temperature and mass is given by

\[ T = \left( \frac{8\pi M}{\dot{M}} \right)^{-1} . \]

For this reason, the black holes radiate more following to the standard fourth order rule of black body radiation. The accretion radiation equilibrium may not be the equilibrium one under such large amount of Hawking radiation. For this type of cases we will unable to talk whether the accretion process is at all dependent of the mass or not. The process of accretion for very small black holes is still a fact to research with. From Eqs. (4) and (10), we have (with \( r_0 = 1 \)) the index of the equation of state as

\[ \omega_D = -1 + C_4 \left\{ u^2 + 2\pi a_{vw} - \frac{2M}{r} + \frac{r^2}{l^2} \left( 1 + \frac{3b_{vw}}{2r} \right) - \frac{3\pi a_{vw}b_{vw}}{2r(2r + 3b_{vw})} - \frac{4\pi a_{vw}b_{vw}}{r} \log \left( \frac{r}{b_{vw}} + \frac{3}{2} \right) \right\}^{-1/2} . \]

Note: \( \omega_D > 0 \) or \( \omega_D < 0 \) depends on the sign of the constant \( C_4 \).

4 Thermodynamic Analysis of Accreting Matter on Van der Waal’s Black Hole

Now, we will discuss about the thermodynamics of the dark energy accretion. The equation related to the thermodynamic studies is given by the equation of state \( p = \omega_D \rho \). First, we wish to evaluate the value of \( C \) such that the sign of \( M \) can be determined and secondly, we verify the exactness of the generalized second law of thermodynamics which is an invariant law and will search any limitation on the equation of state \( \omega_D \) from thermodynamic point of view. The energy supply vector \( \psi_i \) and the work density \( W \) are defined as

\[ W = -\frac{1}{2} \text{Trace} (T_j) = \frac{1}{2} (\rho - 3p) \]  

and

\[ \psi_i = T^j_j \partial_j r + W \partial_i r , \]

i.e.,

\[ \psi_0 = T^0_0 = -u(\rho + p) \sqrt{u^2 + 2\pi a_{vw} - \frac{2M}{r} + \frac{r^2}{l^2} \left( 1 + \frac{3b_{vw}}{2r} \right) - \frac{3\pi a_{vw}b_{vw}}{2r(2r + 3b_{vw})} - \frac{4\pi a_{vw}b_{vw}}{r} \log \left( \frac{r}{b_{vw}} + \frac{3}{2} \right) } \]

and

\[ \psi_1 = T^1_1 + W = \rho \left\{ \frac{1}{2} + \frac{2\pi a_{vw} - \frac{2M}{r} + \frac{r^2}{l^2} \left( 1 + \frac{3b_{vw}}{2r} \right) - \frac{3\pi a_{vw}b_{vw}}{2r(2r + 3b_{vw})} - \frac{4\pi a_{vw}b_{vw}}{r} \log \left( \frac{r}{b_{vw}} + \frac{3}{2} \right) } \right\} . \]
\[ p\left( -\frac{1}{2} + \frac{1}{2\pi a^2} - \frac{2M}{r} + \frac{r^2}{2} \left( 1 + \frac{3\omega}{2} \right) - \frac{u^2}{\left( 2 + 3\omega \right) r} \right) \] 

where \( T^i_j \) is the projected energy-momentum tensor (normal to the 2-sphere). Therefore, the change of energy (across the event horizon) is given by\(^4\)

\[ -dE = -A\psi = -A[\psi_0 dt + \psi_1 dr]. \]

The amount of energy crossing the event horizon is\(^5\)\(^6\)\(^7\)\(^8\)

\[ dE = 4\pi u^2 (p + \rho) dt. \]  \hspace{1cm} (26)

From Eqs. (22) and (26) (as \( c = 1 \) and \( E = mc^2 \)), we get, the arbitrary constant \( C \), given by

\[ C = u^2, \text{ i.e., } M = 4\pi u^2 (p + \rho). \]

In quintessence era, we can say that \( \dot{M} > 0 \), i.e., the black hole mass is increasing although the rate of increment is slowly decreasing as we move to the line of phantom barrier, whereas, in phantom era, \( M < 0 \), i.e., the black hole mass is decreasing. The holographic energy density given by

\[ \rho = \frac{3c^2}{R_h^4} \]  \hspace{1cm} (27)

where \( R_h = a \int_0^\infty dt/a = a \int_0^\infty da / Ha^2 \) which directs to result balanced with observations. Here “\( u \)” is the scale factor of the background metric of the universe and \( H \) is the Hubble parameter.

We can identify the dimensionless dark energy density parameter as:

\[ \Omega_h = \frac{\rho}{3H^2} = \frac{c^2}{R_h^4 H^2}. \]  \hspace{1cm} (28)

For a dark energy subjected universe, dark energy enlarge similar to the conservation law

\[ \dot{\rho} + 3H(p + \rho) = 0, \]  \hspace{1cm} (29)

or identically\(^9\)

\[ \dot{\Omega}_h = H\Omega_h(1 - \Omega_h) \left( 1 + 2\sqrt{\Omega_h} / c \right), \]  \hspace{1cm} (30)

where \( \rho = \omega_D \rho \) is the equation of state.

Also, the equation of state of the index is of the form\(^10\)

\[ \omega_D = -\frac{1}{3} \left( 1 + 2\sqrt{\Omega_h} / c \right). \]  \hspace{1cm} (31)

Here, \( \omega_D \) depends on the parameter \( c \). Since, the observation predicts\(^11\) \( \Omega_h \rightarrow 1 \) for the present time, therefore, at \( c = 1, \omega_D \rightarrow -1 \), i.e., our model acts like cosmological constant. Also for \( c > 1 \), we get, \(-1 < \omega_D < -1/3 \), i.e., our model shows the quintessence region and if \( c < 1 \), we get, \( \omega_D < -1 \), i.e., the phantom type behaviour occurs.

Using Eqs. (27) and (31), we get

\[ M = 8\pi u^2 \frac{c^2}{R_h^2} \left( 1 - \frac{1}{R_h H} \right) \]

\[ \Rightarrow \frac{dM}{dR_h} = 8\pi u^2 \frac{c^2}{R_h^3} \left( \frac{3}{H} - 2R_h \right). \]  \hspace{1cm} (32)

If \( R_h < (3/2)R_H \), then \( \dot{M} \) increases where \( R_H \) is the Hubble radius and \( R_h \) is the radius of the event horizon. If \( R_h > (3/2)R_H \), then \( \dot{M} \) decreases.

5 Dark Energy Accretion upon Van der Waal’s Black Hole

Here, we will discuss about dark energy model such as extended Chaplygin gas. We consider the spatially flat, homogeneous and isotropic Friedman–Robertson–Walker (FRW) model of the universe is described by the following metric

\[ \dot{a}/a = -\omega_D(t)/a \]  \hspace{1cm} (33)

\[ \frac{H^2}{(1 + \omega_D)} = \frac{1}{3}\rho, \]  \hspace{1cm} (34)

It is also assumed that the total matter and energy are conserved with the following conservation equation (29). Now, we consider the extended Chaplygin gas\(^12\) as dark energy model. The equation of state is given by

\[ \rho = \sum_n A_n\rho^n - \frac{B}{\rho^3}. \]  \hspace{1cm} (35)

(i) For \( n = 1 \)

Special case of \( n = 1 \) reduces Eq. (35) to the modified Chaplygin gas equation of state with the density (using Eq. (29))

\[ \rho = \left( \frac{B}{1 + A} + \frac{C}{a^{3(1+\alpha)(1+\alpha)}} \right)^{1/(1+\alpha)}, \]  \hspace{1cm} (36)

where \( C > 0 \) is an integration constant.

Therefore, the current value of the energy density

\[ \rho_0 = \left( \frac{B}{1 + A} + C \right)^{1/(1+\alpha)}. \]  \hspace{1cm} (37)

For MCG model, we get

\[ c_s^2 = A + \frac{\alpha B}{\rho^{(1+\alpha)}}, \]  \hspace{1cm} (38)
\[ \frac{2 \pi a_{vw}}{r_c} - \frac{2M}{r_c} + \frac{r_c^2}{2M} \left( 1 + \frac{3b_{vw}}{2r_c} \right) - \frac{3 \pi a_{vw} b_{vw}^2}{r_c (2r_c + 3b_{vw})} - \frac{4 \pi a_{vw} b_{vw} \log \left( \frac{r_c}{b_{vw}} + \frac{2}{r_c} \right)}{r_c (2r_c + 3b_{vw})} \right) \]

\[ \times \left\{ \left( 1 + 3 \frac{b_{vw}}{2r_c} \right) r_c + \frac{6 a_{vw} b_{vw}}{r_c (2r_c + 3b_{vw})} + \frac{3 \pi a_{vw} b_{vw}^2}{r_c (2r_c + 3b_{vw})} \right\}^{1/(1+\alpha)}. \tag{39} \]

In Fig. 1(a) we have plotted \( \rho \) vs. \( M \), the mass of the central gravitating object. Accretion has been considered. The plots show if the mass of the central engine is increased, the density of the accreting dark energy is reduced. Not only that but also we observe that the density profile is high if \( \alpha \) is low. So dark energy, whenever is strongly repulsive (i.e., \( \alpha \) is high) we find it to reduce the density to be accreted in. Super massive black holes are less capable to accrete a strongly dense dark energy flow towards it than the local stellar mass black holes can do. For \( \alpha = 0 \), we get back the barotropic fluid accretion. We see for small mass of the black hole, the accretion is very high (than dark energy cases) and as black hole’s mass is increased, density of the exotic fluid decreases. The negative pressure of dark energy faints the strength of accretion.

\[ V_c^2 = \sum_{i=1}^{n} \frac{i A_i \rho_i^{(1-\alpha)}}{1 + \sum_{i=1}^{n} A_i \rho_i^{(1-\alpha)}} = V_c^2 = 1 + \frac{4}{r_c} \]

(ii) For \( \alpha = -1 \)

Here, we assume the last term of expression in EoS (35) is dominant. In this case, we can write the energy density in terms of the scale factor as (using Eqs. (29))

\[ \rho = \left[ \frac{A}{B - 1} + \frac{C}{a^3(B-1)(n-1)} \right]^{1/(1-n)}, \tag{40} \]

where \( C \) is an arbitrary integrating constant.

Therefore, the current value of the energy density looks like

\[ \rho_0 = \left[ \frac{A}{B - 1} + C \right]^{(1/(1-n))}. \tag{41} \]

Also, we get,

\[ c_s^2 = \sum_{i=1}^{n} i A_i \rho_i^{(1-\alpha)} - B. \tag{42} \]
As we increase the value of \( n \) where \( \phi \) is decreasing.

Therefore, the energy density is given by

\[
\rho = \phi_1(a),
\]

where \( \phi_1 \) can be evaluated by expressing \( \rho \) as a function of \( a \) by using Eq. (45).

Therefore, the current value of the energy density is \( \rho_0 = \phi_1(1) \).

Also, we get

\[
c^2_s = A_1 + 2A_2\rho + \frac{B}{2\rho_{1/2}},
\]

\[
V_c^2 = \frac{A_2\rho + (3/2)(B/\rho_{1/2})}{1 + A_1 + A_2\rho - B/\rho_{1/2}} \Rightarrow V_c^2 = 1 + \frac{4}{r_c}
\]

\[
\times \frac{2\pi a_{uv} - \frac{2M}{r_c} + r_c^2}{\frac{1}{r_c} + \frac{3}{2r_c}} \left( 1 + \frac{3b_{uv}}{2r_c} \right) - \frac{3\pi a_{uv}b_{uv}}{r_c(2r_c + 3b_{uv})} = \frac{4\pi a_{uv}b_{uv}}{r_c} \log \left( \frac{r_c}{b_{uv}} + \frac{3}{2} \right) r_c^2.
\]

In this case the EoS (35) can be written as,

\[
p = A_1\rho + A_2\rho^2 - \frac{B}{\sqrt{\rho}}.
\]

Using Eq. (29), we get

\[
\ln(a) = - \int \frac{d\rho}{3 \{ (1 + A_1)\rho + A_2\rho^2 - B/\sqrt{\rho} \}}.
\]

Therefore, the energy density is given by

\[
\rho = \phi_1(a),
\]

where \( \phi_1 \) can be evaluated by expressing \( \rho \) as a function of \( a \) by using Eq. (45).

Therefore, the current value of the energy density is \( \rho_0 = \phi_1(1) \).

Also, we get

\[
c^2_s = A_1 + 2A_2\rho + \frac{B}{2\rho_{1/2}},
\]

\[
V_c^2 = \frac{A_2\rho + (3/2)(B/\rho_{1/2})}{1 + A_1 + A_2\rho - B/\rho_{1/2}} \Rightarrow V_c^2 = 1 + \frac{4}{r_c}
\]

\[
\times \frac{2\pi a_{uv} - \frac{2M}{r_c} + r_c^2}{\frac{1}{r_c} + \frac{3}{2r_c}} \left( 1 + \frac{3b_{uv}}{2r_c} \right) - \frac{3\pi a_{uv}b_{uv}}{r_c(2r_c + 3b_{uv})} = \frac{4\pi a_{uv}b_{uv}}{r_c} \log \left( \frac{r_c}{b_{uv}} + \frac{3}{2} \right) r_c^2.
\]

In this case the Eq. (35) can be written as,

\[
p = A_1\rho + A_2\rho^2 + A_3\rho^3 - \frac{B}{\sqrt{\rho}}.
\]

Using Eq. (29), we get

\[
\ln(a) = - \int \frac{d\rho}{3 \{ (1 + A_1)\rho + A_2\rho^2 + A_3\rho^3 - B/\sqrt{\rho} \}}.
\]

Therefore, the current value of the energy density is \( \rho_0 = \phi_2(1) \).

Also, we get

\[
c^2_s = A_1 + 2A_2\rho + 3A_3\rho^2 + \frac{B}{2\rho_{3/2}},
\]

\[
V_c^2 = \frac{A_2\rho + 3A_3\rho^2 + \frac{3B}{2\rho_{3/2}}}{1 + A_1 + A_2\rho + A_3\rho^2 - \frac{B}{\rho_{3/2}}} \Rightarrow V_c^2 = 1 + \frac{4}{r_c}
\]

\[
\times \frac{2\pi a_{uv} - \frac{2M}{r_c} + r_c^2}{\frac{1}{r_c} + \frac{3}{2r_c}} \left( 1 + \frac{3b_{uv}}{2r_c} \right) - \frac{3\pi a_{uv}b_{uv}}{r_c(2r_c + 3b_{uv})} = \frac{4\pi a_{uv}b_{uv}}{r_c} \log \left( \frac{r_c}{b_{uv}} + \frac{3}{2} \right) r_c^2.
\]

Density vs. mass for \( n = 3 \) and \( \alpha = 1/2 \) case has been plotted in Fig. 1(d). The basic features do match

\[
M = -\frac{4\pi u^2}{\sqrt{3} \sqrt{\rho}},
\]
which implies
\[ M = M_0 - \frac{8\pi u^2}{\sqrt{3}} (\sqrt{\rho} - \sqrt{\rho_0}) , \]
(54)
where \( M_0 \) is the current value of the Van der Waal’s black hole’s mass. If \( a \) is very large \((z \to -1)\), i.e., at the last stage of the universe, the mass of the black hole will be
\[ M = M_0 + \frac{8\pi u^2}{\sqrt{3}} \sqrt{\rho_0} . \]

Using the Solution of \( \rho \) in Eq. (54), the black hole mass \( M \) can be written in terms of scale factor \( a \) and then using \( z = 1/a - 1 \), the formula of redshift, \( M \) will be written in terms of redshift \( z \).

For \( n = 1 \), \( M \) can be written as
\[ M = M_0 + \frac{8\pi u^2}{\sqrt{3}} \left[ (C + \frac{B}{1 + A})^{1/2(1+n)} \right] \]

We will now compare Chaplygin gas accretion for Van der Waal’s black hole and Schwarzschild black hole. A quantitative study of Chaplygin gas accretion and detailed density profile are studied in Ref. [41]. For a nonrotating \((j = 0)\) i.e., Schwarzschild black hole, we observe that if the specific angular momentum \((\lambda_c)\) of the accretion is 2.7, for adiabatic accretion (Fig. 2(a)) the density of accretion near to the black hole is very high \((\sim 10^{-13} \text{ gm/cc})\), whereas, at thousand Schwarzschild radius this will be of the order of \(10^{-27} \text{ gm/cc} \). As we increase \( x \), we observe that the density become asymptotic to the distance axis.

If we observe the properties of Van der Waal’s fluid besides the ideal fluid, we see that the Van der Waal’s fluid incorporates the volume occupied by the molecules and the increment in pressure due to their collisions. This is analogous to the effects created by the microstates of the black holes. It is expected that the existence of different microstates will reduce the “as a whole accreting power” of a black hole (due to their mutual interactions). We observe in Figs. 1(a)-1(d) that as we introduce the effects of Van der Waal’s black hole very prominently, the accreting density reduces after a particular measurement of black hole mass \( M \) (distance = \( x(GM/c^2) \)). This means the Van der Waal’s parameters increase the mutual interactions between the microstates, which result reduction in accreting power. This makes the accretion disk of a finite length.

Finally, in Fig. 2(b), we draw the Chaplygin gas accretion on Schwarzschild black hole. Here, we see effect of dark energy accretion terminates the accretion disc very near to the black hole.

So we can speculate that the Chaplygin gas type dark energy accretion on the Van der Waal’s black hole is stronger than the dark energy accretion on Schwarzschild black hole (i.e., the Van der Waal’s black hole is able to even attract dark energy from distant regions) and is weaker than an adiabatic fluid accretion on Schwarzschild black hole (i.e., Schwarzschild black hole is able to accrete adiabatic fluid from distant region but Van der Waal’s black hole is unable do that).

Now \( M \) vs. \( z \) is drawn in Figs. 3(a), 3(b), and 3(c). Since our solution for extended Chaplygin gas model produces only quintessence, so from the figures, we can say that the mass \( M \) of the Van der Waal’s black hole always increases with decreasing \( z \). So we conclude that the mass of the Van der Waal’s black hole increases if the extended Chaplygin gas accretes onto the Van der Waal’s black hole. Also, if we fix \( C \) and vary \( \alpha \) then increment of \( \alpha \) decreases the value of the mass. However, if we fix \( \alpha \) and vary \( C \) then increasing \( C \) increases the value of the mass.

For \( \alpha = -1 \), \( M \) can be written as,
\[ M = M_0 + \frac{8\pi u^2}{\sqrt{3}} \left[ \left( C + \frac{A}{B-1} \right)^{1/2(1-n)} \right] - \left \{ C(1+z)^{(3-B-1)(n-1)} + \frac{A}{B-1} \right \}^{1/2(1-n)} \]
(56)
The basic features of Figs. 3(d), 3(e), and 3(f) do match with Figs. 3(a), 3(b), and 3(c). So we conclude that the mass of the Van der Waal's black hole increases if the extended Chaplygin gas accretes onto the Van der Waal's black hole. Also, if we fix $C$ and vary $n$, then increment of $n$ decreases the value of the mass. However, if we fix $n$ and vary $C$ then increment of $C$ increases the value of the mass.

For $n = 2$ and $\alpha = 1/2$, $M$ can be written as,

$$M = M_0 - \frac{8\pi u^2}{\sqrt{3}} \left[ \sqrt{\phi_1(a)} - \sqrt{\phi_1(1)} \right],$$  \hspace{1cm} (57)

where $\phi_1$ can be evaluated by expressing $\rho$ as a function of scale factor “$a$” by using Eq. (45).

For $n = 3$ and $\alpha = 1/2$, $M$ can be written as,

$$M = M_0 - \frac{8\pi u^2}{\sqrt{3}} \left[ \sqrt{\phi_2(a)} - \sqrt{\phi_2(1)} \right],$$  \hspace{1cm} (58)

where $\phi_2$ can be evaluated by expressing $\rho$ as a function of scale factor “$a$” by using Eq. (45).

6 Discussions

In this work, first we have considered the most general static spherically symmetric black hole metric. Then we have studied the accretion onto the Van der Waal’s black hole and found some inequalities for physical validation. Next, we have analyzed the thermodynamics of accreting matter around the black hole.

We can say that the mass is increasing, i.e., $\dot{M} > 0$ in quintessence era, however when we move towards the phantom barrier line, the rate of increment is slowing down, and in phantom era the mass of the black hole is decreasing, i.e., $\dot{M} < 0$, which is a point of interest. Finally, we have discussed about dark energy model such as extended Chaplygin gas and the nature of the universe’s density. For special case of the modified Chaplygin gas,
we have seen that the universe was infinitely dense at its beginning but when the scale factor has turned higher, the universe has started to grow in size. For $\alpha = -1$ in extended Chaplygin gas, the density of accreting fluid is increasing with a steep slope firstly and then the slope is reduced down. In extended Chaplygin gas for $n = 2$, $\alpha = 1/2$ and $n = 3$, $\alpha = 1/2$ we obtain an identical nature of the accretion density, i.e., increment in scale factor causes a loss of the density of the accreting fluid. Another reason for decrease of density of the accreting dark energy is increasing mass of the central engine. Since in our solution of modified Chaplygin gas, this model generates only quintessence dark energy and so the Van der Waal’s black hole mass increases during the whole evaluation of the accelerating universe. We find that the strength of dark energy accretion process on Van der Waal’s black hole lies between dark energy accretion on Schwarzschild black hole and adiabatic accretion on Schwarzschild black hole. We speculate that the mutual interactions of microstates of Van der Waal’s black hole reduce the outward acting attracting power of it as compared to Schwarzschild black hole when both of them are to attract adiabatic fluid. But when the accreting fluid is Chaplygin gas, due to its negative pressure, Van der Waal’s black hole attracts it more than the Schwarzschild one. The accreting power of Van der Waal’s black hole lies somewhere between different extremities. It is not so high or not so fainted for different cases like Schwarzschild black hole.

Acknowledgments

RB thanks Inter University Center for Astronomy and Astrophysics (IUCAA), Pune, India for Visiting Associateship. Authors thank Mr. Prasanta Choudhury, M.Sc (Mathematics), Sidhu Kanho Birsa University, West Bengal, India for primary construction of the problem.

References

[1] S. Hawking and D. N. Page, Common. Math. Phys. 87 (1983) 577.
[2] A. Chamblin, et al., Phys. Rev. D 60 (1999) 064018, arXiv:hep-th/9902170.
[3] M. Cvetic, et al., J. High Energy Phys. 9904 (1999) 024, arXiv:hep-th/9902195.
[4] M. M. Caldarelli, G. Cognola, and D. Klemm, Class. Quantum Gravity 17 (2000) 399420, arXiv:hep-th/9908022.
[5] C. Niu, Y. Tian, and X. N. Wu, Phys. Rev. D 85 (2012) 024017, arXiv:1104.3066.
[6] D. Kubiznak and R. B. Mann, J. High Energy Phys. 1207 (2012) 033, arXiv:1205.0559.
[7] S. Gunasekaran, R. B. Mann, and D. Kubiznak, J. High Energy Phys. 1211 (2012) 110, arXiv:1208.6251.
[8] D. Kastor, S. Ray, and J. Traschen, Class. Quantum Gravity 26 (2009) 195011, arXiv:0904.2765.
[9] B. Dolan, Class. Quantum Gravity 28 (2011) 125020, arXiv:1008.5023.
[10] A. Rajgopal, et al., Phys. Lett. B 737 (2014) 277.
[11] U. Debnath, Eur. Phys. J. C 75 (2015) 129; arXiv:1503.01645[gr-qc].
[12] U. Debnath, Eur. Phys. J. C 75 (2015) 449; arXiv:1409.4651[physics.gen-ph].
[13] M. Cadoni, E. Franzin, and M. Tuveri, Phys. Lett. B 768 (2017) 393.
[14] K. Bhattacharya, B. R. Majhi, and S. Samanta, Phys. Rev. D 96 (2017) 084037.
[15] T. Padmanabhan, AIP Conf. Proc. 861 (2006) 179; arXiv:0603114v4[astro-ph].
[16] E. J. Copeland, M. Sami, and S. Tsujikawa, Int. J. Mod. Phys. D 15 (2006) 1753; arXiv:0603057v3[hep-th].
[17] G. ’t Hooft, Conf. Proc. C 931006 (1993) 284; arXiv:gr-qc/9310026v1.
[18] F. Wilczek, In Carr, Bernard (ed.): Universe or multiverse (2004) 151; arXiv:0408167[hep-ph].
[19] S. D. Campo, R. Herrera, G. Olivares, and D. Pavon, Phys. Rev. D 74 (2006) 023501; arXiv:0606520v1[astro-ph].
[20] M. Jamil, F. Rahaman, and M. Kalam, Eur. Phys. J. C 60 (2009) 149; arXiv:0809.314v3[gr-qc].
[21] C. Wetterich, Nucl. Phys. B 302 (1988) 668; arXiv:1711.0384v1v1[hep-th].
[22] R. R. Caldwell, Phys. Lett. B 545 (2002) 23; arXiv:9908168v2[astro-ph].
[23] E. Elizalde, S. Nojiri, and S. D. Odintsov, Phys. Rev. D 70 (2004) 043539; arXiv:0405034[hep-th].
[24] P. Wu and H. Yu, JCAP 0703 (2007) 015; arXiv:0701463v3[astro-ph].
[25] M. Li, Phys. Lett. B 603 (2004) 1; arXiv:0403127v4[hep-th].
[26] R. G. Cai, Phys. Lett. B 657 (2007) 228; arXiv:0701494v3[hep-th].
[27] E. K. Li, Y. Zhang, J. L. Geng, and P. F. Duan, Gen. Relativ. Gravit. 47 (2015) 136; arXiv:1602.00581v1[gr-qc].
[28] E. K. Li, Y. Zhang, and J. L. Geng, Phys. Rev. D 90 (2014) 083534; arXiv:1412.5482v1[gr-qc].
[29] A. Kamenshchik, U. Moschella, and V. Pasquier, Phys. Lett. B 511 (2001) 265; arXiv:0103004v2[gr-qc].
[30] E. Babichev, V. Dokuchaev, and Yu. Eroshenko, Phys. Rev. Lett. 93 (2004) 021102; gr-qc/0402089.
[31] M. Jamil, M. A. Rashid, and A. Qadir, Eur. Phys. J. C 58 (2008) 325; arXiv:0808.115v2v1[astro-ph].
[32] M. Jamil and I. Hussain, Int. J. Theor. Phys. 50 (2011) 465; arXiv:1101.153v1[astro-ph.CO].
[33] M. Jamil and M. Akbar, Gen. Relt. Grav. 43 (2011) 1061; arXiv:1005.3444v2[gr-qc].
[34] R. Biswas, N. Mazumder, and S. Chakraborty, Astro. Space Sci. 335 (2011) 603; arXiv:1006.3130v2[gr-qc].
[35] Q. G. Huang and M. Li, JCAP 0408 (2004) 013; arXiv: 0404229v3[astro-ph].

[36] U. Debnath, A. Banerjee, and S. Chakraborty, Class. Quant. Grav. 21 (2004) 5609; arXiv: 0411015v1[gr-qc].

[37] R. Biswas, et al., Classical and Quantum Gravity 28 (2011) 035005, arXiv:astro-ph/1101.4602.

[38] R. Biswas, Europhys. Lett. 96 (2011) 49001.

[39] S. Dutta, R. Biswas, Euro. Phys. J. C 77 (2017) 717 [arXiv:1705.11058/astro-ph.HE].

[40] S. Dutta, R. Biswas, submitted to Euro. Phys. J. C.

[41] F. C. Michel, Astrophys. Space Sci. 15 (1972) 153; Preprint: Not available.

[42] E. Babichev, V. Dokuchaev, and Y. Eroshenko, J. Exp. Theor. Phys. 100 (2005) 528; arXiv:0505618v1[astro-ph].

[43] A. J. John, S. G. Ghosh, and S. D. Maharaj, Phys. Rev. D 88 (2013) 104005; arXiv:1310.7831v1[gr-qc].

[44] L. Susskind, Nucl. Phys. B 382 (1992) 123; arXiv: 9203054[hep-th].

[45] R. G. Cai and L. M. Cao, Phys. Rev. D 75 (2007) 064008; arXiv: 0611071v2[gr-qc].

[46] S. Chakraborty, R. Biswas, and N. Mazumder, Nuovo Cim. B 125 (2011) 1209; arXiv:1006.1169v1[gr-qc].

[47] N. Mazumder and S. Chakraborty, Class. Quant. Grav. 26 (2009) 195016.