ANOTHER PROOF OF MASUOKA’S THEOREM FOR SEMISIMPLE IRREDUCIBLE HOPF ALGEBRAS

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Abstract. Masuoka proved (2009) that a finite-dimensional irreducible Hopf algebra $H$ in positive characteristic is semisimple if and only if it is commutative semisimple if and only if the Hopf subalgebra generated by all primitives is semisimple. In this paper, we give another proof of this result by using Hochschild cohomology of coalgebras.

1. Introduction

The classification of semisimple Hopf algebras over an algebraically closed field is an open question [1]. By a result of Larson and Radford [7], semisimple Hopf algebras are cosemisimple in characteristic zero. Nonetheless, there are plenty examples of semisimple Hopf algebras which are not cosemisimple in positive characteristic. For instance, suppose $p$ is a prime number and the base field $k$ has characteristic $p$. Then for any $p$-group $G$, $(kG)^*$ is semisimple but not cosemisimple. Next, we recall an old theorem of Hochschild about restricted Lie algebras. Still assume char $k = p$ and let $g$ be a restricted Lie algebra over $k$. Denote $u(g)$ as the restricted enveloping algebra of $g$. Hochschild [5] proves that $u(g)$ is semisimple if and only if $g$ is abelian and $g = k^p$. From this, we can obtain the following assertion:

**Theorem A** (Hochschild). Suppose char $k = p$ and $k = \bar{k}$. Let $H$ be a finite-dimensional cocommutative connected Hopf algebra over $k$. Denote $g$ as the primitive space of $H$ and further assume that $H$ is primitively generated by $g$. Then the following are equivalent:

1. $H$ is semisimple.
2. The restricted map ($p$-th power map) on $g$ is bijective.
3. $H \cong (kG)^*$, for $G = (C_p)^n$ and $C_p$ the cyclic group of order $p$.

In algebraic group theory, Nagata [11] proves that a fully reducible connected affine algebraic group in positive characteristic is a torus. Later, this result was generalized independently by Demazure-Gabriel [4] and Sweedler [14] to any connected fully reducible affine group schemes. Here we only recall the Hopf algebra version of the generalized Nagata’s theorem. Suppose char $k = p$ and $H$ is a commutative cosemisimple Hopf algebra over $k$, whose maximal separable subalgebra is $k$. Then $H$ is cocommutative. In finite-dimensional case, we can dualize the result for semisimple cocommutative Hopf algebras.

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**Theorem B** (Demazure-Gabriel and Sweedler). Suppose \( \text{char} k = p \) and \( k = \overline{k} \). Let \( H \) be a finite-dimensional cocommutative connected Hopf algebra over \( k \). Then the following are equivalent:

1. \( H \) is semisimple.
2. \( H \cong (kG)^* \), for \( G \) an abelian \( p \)-group.

As we can see, Theorem B is a generalization of Theorem A without the assumption that the cocommutative Hopf algebra is primitively generated. Masuoka [9] further extends Theorem B to include the case for all finite-dimensional connected Hopf algebras. Moreover, he generalizes the result by providing the criteria when these Hopf algebras are semisimple.

**Theorem C** (Masuoka). Suppose \( \text{char} k = p \) and \( k = \overline{k} \). Let \( H \) be a finite-dimensional connected Hopf algebra over \( k \). Denote \( K \) as the Hopf subalgebra generated by all primitive elements of \( H \). Then the following are equivalent:

1. \( H \) is semisimple.
2. \( K \) is semisimple.
3. \( K \cong (kN)^* \), for \( N \cong (C_p)^n \).
4. \( H \cong (kG)^* \), for \( G \) a \( p \)-group.

In this short note, we give another proof of Theorem C different from the original one in [9]. Our approach is discussed in section \([ \underline{3} \) where our aim is to prove Lemma \([ \underline{3.7} \). Suppose \( \text{char} k = p \) and \( H \) is a finite-dimensional connected Hopf algebra over \( k \). Denote \( K \) as the Hopf subalgebra generated by all primitive elements of \( H \). Let \( L \) be a Hopf subalgebra of \( H \) such that \( K \subseteq L \). Then there exists some \( z \in H \setminus L \) such that the comultiplication of \( z \) satisfies

\[
\Delta(z) = z \otimes 1 + 1 \otimes z + u,
\]

where the term \( u \) represents a non-zero cohomological class in \( H^2(L, k) \), i.e., the Hochschild cohomology of \( L \) with coefficients in \( k \). Moreover, we say the extension \( L \subseteq H \) is essential if there is no proper Hopf subalgebras between them. Now assume \( H \) to be commutative and \( L = (kG)^* \) for a \( p \)-group \( G \). Lemma \([ \underline{3.7} \) says that in such essential extension \( L \subseteq H \), we can choose a particular \( z \) satisfying more nice conditions. As a consequence, \( H \) is semisimple. Now we filter \( H \) by a sequence of essential extensions starting from \( K \):

\[
K = F_0H \subseteq F_1H \subseteq \cdots \subseteq F_nH = H.
\]

In section \([ \underline{4} \), we will prove (2) implies (4) in Theorem C by induction on the length \( n \). The initial step for \( n = 0 \) is just Theorem A due to Hochschild. In the inductive process, we can assume that \( H \) is generated by \( F_{n-1}H \) and some \( z \) with the comultiplication described above. Notice that the main stumbling block is to show \( H \) is commutative. The key is to use Lemma \([ \underline{5.7} \) for each step \( F_{i-1}H \subseteq F_iH \) and prove that \( z \) commutes with \( F_i \) for all \( 0 \leq i \leq n - 1 \) inductively.

Following the main theorem \([ \underline{4.1} \) and removing the assumption \( k = \overline{k} \), we also prove two corollaries. Corollary \([ \underline{4.2} \) says that any semisimple connected Hopf algebra is commutative and Corollary \([ \underline{4.3} \) lists several equivalent conditions for a finite-dimensional connected Hopf
algebra to be semisimple. One consequence is that the semisimplicity of such Hopf algebra is completely captured by the semisimplicity of the Hopf subalgebra generated by the first term of its coradical filtration.

The application of Theorem C can be used to classify all $p^3$-dimensional connected Hopf algebras, which will be appearing in a forthcoming paper. Suppose $\text{char } k = p$ and $k = \overline{k}$. Let $H$ be a connected Hopf algebra of dimension $p^3$ over $k$. Assume $H$ is not primitively generated, otherwise it is isomorphic to $u(g)$ for some three-dimensional restricted Lie algebra $g$. Then we can find some Hopf subalgebra $L$ of dimension $p^2$ containing all the primitive elements of $H$. The structure of $L$ is already known, since all $p^2$-dimensional connected Hopf algebra has been classified in [16]. Moreover, $H$ can be obtained by adding one non-primitive element to $L$ whose comultiplication is described by $H^2(L, k)$. Now according to Theorem C, we not only know the structure of $H$ that is semisimple, but also know any nonsemisimple $H$ only arises from nonsemisimple $L$.

**Remark 1.1.** Any semisimple Hopf algebra is finite-dimensional by [15, pp. 107]. If furthermore it is connected, then the base field is of positive characteristic. Otherwise the Hopf subalgebra generated by all primitive elements is infinite-dimensional by [10, Theorem 5.6.5].

### 2. Preliminary

Throughout this paper, $k$ is a base field. Vector spaces, linear maps and tensor products are assumed to be over $k$ unless stated otherwise. Let $p$ be a prime number, $C_p^n$ is the cyclic group of order $p^n$ for a positive integer $n$. By standard notations, we use $(H, m, u, \Delta, \varepsilon, S)$ to denote an Hopf algebra. In this section, we recall some basic definitions and facts that will be used throughout the paper.

**Definition 2.1.** [10, Definitions 5.1.5, 5.2.1] The coradical $H_0$ of $H$ is the sum of all simple subcoalgebras of $H$. The Hopf algebra $H$ is connected if $H_0$ is one-dimensional. For each $n \geq 1$, inductively set

$$H_n = \Delta^{-1}(H \otimes H_{n-1} + H_0 \otimes H).$$

The chain of subcoalgebras $H_0 \subseteq H_1 \subseteq \ldots \subseteq H_{n-1} \subseteq H_n \subseteq \ldots$ is the coradical filtration of $H$.

**Definition 2.2.** [13, Lemma 1.1] Consider $k$ as a trivial $H$-bicomodule. We use $H^\bullet(k, H)$ to denote the Hochschild cohomology of $H$ with coefficients in $k$. It can be computed as the homology of the complex $(H^\otimes \bullet, d^\bullet)$, where $d^0 = 0$ and

$$d^n(x) = 1 \otimes x - (\Delta \otimes \text{Id}_{n-1})(x) + \cdots + (-1)^n(\text{Id}_{n-1} \otimes \Delta)(x) + (-1)^{n+1}x \otimes 1,$$

for $n \geq 1$ and any $x \in H^\otimes n$.

**Definition 2.3.** [8, V. §7 Definition 4] A restricted Lie algebra $g$ of characteristic $p$ is a Lie algebra of characteristic $p$ in which there is defined a map $a \rightarrow a^{[p]}$ such that

1. $(aa)^{[p]} = a^p a^{[p]}$, 

(2) \((a + b)^{[p]} = a^{[p]} + b^{[p]} + \sum_{i=1}^{p-1} s_i(a, b)\), where \(s_i(a, b)\) is the coefficient of \(\lambda^{i-1}\) in \(a \cdot (\lambda a + b))^{p-1}\),

(3) \([ab]^{[p]} = a^{[p]} b^{[p]}\),

for all \(a, b \in \mathfrak{g}\) and \(\alpha \in k\). Let \(\mathfrak{g}\) be restricted, and \(U(\mathfrak{g})\) be the usual enveloping algebra.

Suppose \(B\) is the ideal in \(U(\mathfrak{g})\) generated by all \(a^p - a^{[p]}\), \(x \in \mathfrak{g}\). Define \(u(\mathfrak{g}) = U(\mathfrak{g})/B\), which is called the \textit{restricted enveloping algebra} of \(\mathfrak{g}\).

\textbf{Definition 2.4.} \cite{10, Definitions 4.1.1, 7.1.1} A Hopf algebra \(H\) measures an algebra \(A\) if there is a linear map \(H \otimes A \rightarrow A\), given by \(h \otimes a \to h \cdot a\), such that \(h \cdot 1 = \varepsilon(h)1\) and \(h \cdot (ab) = \sum (h_1 \cdot a)(h_2 \cdot b)\), for all \(h \in H, a, b \in A\). Moreover, assume that \(\sigma\) is an invertible map in \(\text{Hom}_k(H \otimes H, A)\). The \textit{crossed product} \(A \#_\sigma H\) of \(A\) with \(H\) is the set \(A \otimes H\) as a vector space, with multiplication

\[(a \# h)(b \# k) = \sum a(h_1 \cdot b)\sigma(h_2, k_1) \# h_3 k_2,\]

for all \(h, k \in H\) and \(a, b \in A\).

\section{3. Extensions of connected Hopf algebras}

For the remaining of the paper, let \(\text{char } k = p\), and \(H\) be a finite-dimensional connected Hopf algebra. We use the following notations regarding \(H\):

1. The coradical filtration of \(H\) is denoted by \(\{H_n\}_{n \geq 0}\);
2. The Hopf subalgebra of \(H\) generated by \(H_1\) is denoted by \(K\);
3. The primitive space of \(H\) is denoted by \(\mathfrak{g}\);
4. The center of \(H\) is denoted by \(Z(H)\);
5. The augmented ideal of \(H\) is denoted by \(H^+ = \ker \varepsilon\);
6. Define \(\text{ad}(x)y = [x, y]\) and \(y(\text{ad} x) = [y, x]\) for any \(x, y \in H\).

\textbf{Remark 3.1.} Notice that \(K \cong u(\mathfrak{g})\) and \(\dim H = p^n\) for some integer \(n\) by \cite{16, Proposition 2.2(8)}.

\textbf{Definition 3.2.} Let \(L\) be a proper Hopf subalgebra of \(H\). Then \(L \subset H\) is called a \textit{weakly essential extension} if there is no proper Hopf subalgebra between them. If moreover \(K \subset L\), we say it is an \textit{essential extension}.

\textbf{Lemma 3.3.} Let \(L \subset H\) be an extension of connected Hopf algebras. If there exists some \(x \in H \setminus L\) satisfying \(\Delta(x) = x \otimes 1 + 1 \otimes x + u\), where \(u \in L \otimes L\). Then the subalgebra generated by \(L\) and \(x\) is a Hopf subalgebra of \(H\). Moreover if the extension is weakly essential, then \(L\) is equal to \(H\).

\textit{Proof.} First of all, the subalgebra generated by \(L\) and \(x\) is a sub-bialgebra of \(H\), which is connected by \cite{10, Lemma 5.2.12}. Therefore it is a Hopf subalgebra because of \cite{10, Lemma 2.10}. Moreover if the extension is weakly essential, then by definition it is equal to \(H\).

\textbf{Lemma 3.4.} If \(K\) is semisimple, then \(K \subset Z(H)\).
Finally, Lemma 3.6.

Proof. By Remark 3.1 and Hochschild’s result [10] Theorem 2.3.3, the restricted Lie algebra $g$ is abelian with $g = kq^p$. Fix a basis $\{x_i|1 \leq i \leq d\}$ for it. Denote $x = (x_1, x_2, \ldots, x_d)^T$ and $x^p = (x_1^p, x_2^p, \ldots, x_d^p)^T$. We can write the restricted map for $g$ in a matrix form:

$$x^p = RX,$$

for some $R \in GL_d(k)$. It suffices to prove inductively that $[H_n, g] = 0$ for all $n \geq 0$. Obviously, it is true for $H_0$ and $H_1$. In the following, suppose that $[H_n, g] = 0$ for some $n \geq 1$. By [10] Lemma 5.3.2(2), for any $z \in H_{n+1}$, we have $\Delta(z) = z \otimes 1 + 1 \otimes z + u$, where $u \in H_n \otimes H_n$. Therefore by induction

$$\Delta([z, x_i]) = [\Delta(z), \Delta(x_i)]$$

$$= [z \otimes 1 + 1 \otimes z + u, x_i \otimes 1 + 1 \otimes x_i]$$

$$= [z, x_i] \otimes 1 + 1 \otimes [z, x_i],$$

which yields that $[z, x_i]$ is primitive for all $1 \leq i \leq d$. Therefore we can write these relations in a matrix form as follows:

$$(\text{ad} z)x = Ax,$$

where $A \in M_d(k)$. Hence a simple calculation shows that:

$$(\text{ad} z)x^p = R(\text{ad} z)x = RAx.$$

By the definition of restricted Lie algebras, on the other hand, $[z, x_i^p] = z(\text{ad} x_i)^p = 0$ for all $1 \leq i \leq d$. Hence $RA = 0$ which implies that $A = 0$. \hfill \square

Lemma 3.5. Let $L \subseteq H$ be a proper Hopf subalgebra. Then there is a finite set $\{u_i\} \subseteq L \otimes L$, whose image is a basis in $H^2(k, L)$. Moreover if $K \subseteq L$, then there exists an element $z \in H \setminus L$ such that $\Delta(z) = z \otimes 1 + 1 \otimes z + \sum \alpha_i u_i$, where the $\alpha_i \in k$ are not all zero.

Proof. By the definition of the complex $(L^{\otimes \cdot}, d^\cdot)$, $H^2(k, L) = \ker d^2/\text{Im} d^1$. Hence there exists a set $\{u_i\} \subseteq \ker d^2 \subseteq L \otimes L$, whose image is a basis in $H^2(k, L)$. It is finite since $L$ is finite-dimensional. Furthermore we assume that $K \subseteq L$. As a result of [10] Theorem 5.2.2(1)], there is a minimal number $d$ ($\geq 2$) such that $L_d \neq H_d$, whence we choose some $z \in H_d \setminus L_d \subseteq H \setminus L$. By [10] Lemma 5.3.2(2)], $\Delta(z) = z \otimes 1 + 1 \otimes z + u$, where $u \in H_{d-1} \otimes H_{d-1} = L_{d-1} \otimes L_{d-1} \subseteq L \otimes L$. Consider $(L^{\otimes \cdot}, d^\cdot)$ as a subcomplex of $(H^{\otimes \cdot}, d^\cdot)$. Hence $d^2(u) = d^2d^1(-z) = 0$ and $u$ is a 2-cocycle in the subcomplex. In regard to the chosen basis $\{u_i\}$ in $H^2(k, L)$, there are coefficients $\alpha_i \in k$ and an element $y \in L$ such that

$$u - \sum \alpha_i u_i = d^1(y) = 1 \otimes y - \Delta(y) + y \otimes 1;$$

$$\Delta(z + y) = (z + y) \otimes 1 + 1 \otimes (z + y) + \sum \alpha_i u_i.$$

Finally, $z + y \notin L$ because of $z \notin L$ and $y \in L$. The $\alpha_i$ are not all zero, otherwise $z + y \in H_1 = K_1 \subseteq L$. \hfill \square

Lemma 3.6. Let $H = (kG)^*$, where $G$ is a $p$-group. Then $H$ is connected and there exists a set $\{u_i|1 \leq i \leq n\} \subseteq H \otimes H$ satisfying
(1) The image of \( \{u_i\} \) is a basis in \( H^2(k, H) \);

(2) \( u_i^p = u_i \) in \( H \otimes H \) for all \( 1 \leq i \leq n \).

**Proof.** Consider \( k \supset \mathbb{F}_p \) as a field extension. It is well known that \( \mathbb{F}_p G \) and \( kG \) are scalar local because \( G \) is a \( p \)-group. Therefore by [10, Proposition 5.2.9(2)], their duals are connected. As shown in [10, Example 1.3.6], \( L := (\mathbb{F}_p G)^* \) has a basis \( \{f_x|x \in G\} \), where \( f_x \) is the characteristic function on the element \( x \in G \) and its multiplication and comultiplication structures are described as follows:

\[
f_x f_y = \delta_{x,y} f_x, \quad \Delta(f_x) = \sum_{uv=x} f_u \otimes f_v.
\]

Obviously, there exists a set \( \{u_i|1 \leq i \leq n\} \subseteq L \otimes L \), whose image becomes a basis in \( H^2(\mathbb{F}_p, L) \). Moreover each \( u_i = \sum \alpha_i f_{x_i} \otimes f_{y_i} \) for some \( \alpha_i \in \mathbb{F}_p \). Hence

\[
u_i^p = \left( \sum \alpha_i f_{x_i} \otimes f_{y_i} \right)^p = \sum \alpha_i^p f_{x_i}^p \otimes f_{y_i}^p = \sum \alpha_i f_{x_i} \otimes f_{y_i} = u_i
\]

for all \( i \). Furthermore, \( H^2(\mathbb{F}_p, L) \otimes k \) is naturally isomorphic to \( H^2(k, H) \) as \( k \)-vector spaces because the base change functor \( \otimes_{\mathbb{F}_p} k \) is exact. Then \( \{u_i\} \) becomes a basis for the latter through the natural isomorphism. \( \square \)

**Lemma 3.7.** Let \( H \) be commutative, and \( L \subsetneq H \) be an essential extension. Furthermore assume that \( L = (kG)^* \) for some \( p \)-group \( G \). Then there exists some element \( z \in H \) satisfying the following conditions:

1. \( L \) and \( z \) generate \( H \) as an algebra.
2. \( \Delta(z) = z \otimes 1 + 1 \otimes z + u \), where \( u \) is a 2-cocycle in the complex \( (L^{\otimes n}, d^*) \).
3. \( z^p \not\in L \) for all \( n \geq 0 \).
4. \( z \) satisfies some relation: \( z^p + \lambda_{l-1} z^{p^{l-1}} + \cdots + \lambda_1 z + a = 0 \) in \( H \), where \( \lambda_i \in k \) for all \( 1 \leq i \leq l - 1 \) with \( \lambda_1 \neq 0 \) and \( a \in L \).

Moreover \( H \) is semisimple.

**Proof.** Choose a finite set \( \{u_i\} \subseteq L \otimes L \), which represents a basis in \( H^2(k, L) \). By Lemma 3.6, we can assume that \( u_i^p = u_i \) for all \( i \). Moreover apply Lemma 3.5 to \( L \subsetneq H \), there exists an element \( z \in H \setminus L \) such that

\[
\Delta(z) = z \otimes 1 + 1 \otimes z + \sum \alpha_i u_i,
\]
where the $\alpha_i \in k$ are not all zero. Because of Lemma 3.3, $H$ is generated by $L$ and $z$ as an algebra. A simple calculation shows that:

$$\Delta(z^n) = z^n \otimes 1 + 1 \otimes z^n + \left( \sum \alpha_i u_i \right)^n_n$$

for all $n \geq 0$. By definition $d^1(z^n) = -\sum \alpha_i^n u_i$, which is never a 2-coboundary in the complex $(L^\otimes, d^\bullet)$. Therefore $z^n \notin L$ for all $n \geq 0$. Furthermore by [16, Theorem 4.5], we see $z$ satisfies some relation

$$z^{pl} + \lambda_l z^{l-1} + \cdots + \lambda_1 + a = 0,$$

in $H$, where all $\lambda_i \in k$ with $a \in L$. The condition (3) ensures that there exists a smallest index $m$ such that $\lambda_m \neq 0$. Replace $z$ with $z^m$, which still satisfies all the conditions (1) to (4) in the assertion.

Finally, $H$ is isomorphic as an algebra to a certain crossed product $L#_\sigma(H/L^+H)$ by [3, Theorem 7.2.11]. The quotient Hopf algebra $H/L^+H$ is generated by the image $\bar{z}$ chosen above satisfying

$$\bar{z}^{p^l} + \lambda_l \bar{z}^{l-1} + \cdots + \lambda_1 \bar{z} + \varepsilon(a) = 0,$$

where all $\lambda_i \in k$ with $\lambda_1 \neq 0$. Hence it is a polynomial with distinctive roots in $\bar{k}$. Then the quotient Hopf algebra is semisimple and so is $H$ by [2, Theorem 2.6].

4. Main Results

Recall that char $k = p$, and $H$ is a finite-dimensional connected Hopf algebra. We use $K$ to denote the Hopf subalgebra generated by the primitive space $\mathfrak{g}$ of $H$.

**Theorem 4.1.** Suppose $k$ is algebraically closed. The following are equivalent:

1. $H$ is semisimple.
2. $K$ is semisimple.
3. $K \cong (kN)^*$, for $N \cong (C_p)^n$.
4. $H \cong (kG)^*$, for $G$ a $p$-group.

**Proof.** Condition (1) implies (2) because of [10, Corollary 2.2.2(2)] and Nichols-Zoeller Theorem [12]. By [10, Corollary 2.3.5], (2) implies (3). Since group algebras are cosemisimple, their duals are semisimple. Hence (3) implies (2) and (4) imply (1). Because $H$ is finite-dimensional, there exists a finite chain of Hopf subalgebras

$$K = F_0 H \subsetneq F_1 H \subsetneq \cdots \subsetneq F_n H = H,$$

where each step is an essential extension. We prove (3) implies (4) by induction on the length $n$. When $K = H$, there is nothing to prove. We assume the statement is true for $n \leq d$ and suppose that $n = d + 1$. By induction when $1 \leq i \leq d$,

$$F_i H \cong (kG_i)^*,$$
for $G_i$ a $p$-group. Furthermore if $H$ is commutative, by Lemma 3.7, $H$ is semisimple. Hence it is the dual of a group algebra by [10, Theorem 2.3.1]. Remark 3.1 gives the order of the group. Thus the inductive step is complete.

It remains to show that $H$ is commutative. We complete it by proving $F_j H \subseteq Z(H)$ inductively again for all $j$. When $j = 0$, we have $F_0 H = K \subseteq Z(H)$ by Lemma 3.4. Now assume it is true for $j = m$ and let $j = m + 1$. In the essential extension $F_d H \subseteq H$, by Lemma 3.3 there exists some $z \in H \setminus F_d H$ satisfying: (1) $z$ and $F_d H$ generate $H$; (2) $\Delta(z) = z \otimes 1 + 1 \otimes z + u$, where $u \in F_d H \otimes F_d H$. Similarly by Lemma 3.7, $F_{m+1} H$ is generated by $F_m H$ and some $y \in F_{m+1} H \setminus F_m H$ such that: (1) $\Delta(y) = y \otimes 1 + 1 \otimes y + v$, where $v \in F_m H \otimes F_m H$; (2) $y$ satisfies some relation $y^p + \lambda_{i-1} y^{p-1} + \cdots + \lambda_1 y + a = 0$, where $\lambda_i \in k$ with $\lambda_1 \neq 0$ and $a \in F_m H$. By induction, $F_{m+1} H \subseteq Z(H)$ if and only if $[F_{m+1}, z] = 0$ if and only if $[y, z] = 0$. Moreover since $F_m \subseteq Z(H)$ and $F_d H$ is commutative, then

$$\Delta([y, z]) = [\Delta(y), \Delta(z)] = [y \otimes 1 + 1 \otimes y + v, z \otimes 1 + 1 \otimes z + u] = [y, z] \otimes 1 + 1 \otimes [y, z].$$

Hence we can write $[y, z] = x$ for some primitive element $x \in K$ and

$$0 = [y^p + \lambda_{i-1} y^{p-1} + \cdots + \lambda_1 y + a, z] = (ady)^p(z) + \lambda_{i-1} (ady)^{p-1}(z) + \cdots + \lambda_1 [y, z] + [a, z] = \lambda_1 x.$$ Therefore $[y, z] = 0$ for $\lambda_1 \neq 0$. □

**Corollary 4.2.** If $H$ is semisimple, then $H$ is commutative.

**Proof.** By [10, Corollary 2.2.2], $H$ is a separable $k$-algebra. Without loss of generality, by we can assume $k$ to algebraically closed of characteristic $p \neq 0$ by a base field extension. Then the result follows from Theorem 4.1. □

**Corollary 4.3.** The following are equivalent:

1. $H$ is semisimple.
2. $\varepsilon(f_H^r) \neq 0$.
3. $\varepsilon(f_H^l) \neq 0$.
4. $K$ is semisimple.
5. $g$ is abelian and $g = kg^p$.
6. $\varepsilon(f_K^r) \neq 0$.
7. $\varepsilon(f_K^l) \neq 0$.

**Proof.** The equivalence of conditions (1), (2) and (3) is Maschke’s Theorem [8]. That (4) is equivalent to (5) is Hochschild’s result. Let $E \supseteq k$ be a field extension, and $J$ be the
Jacobson radical of $H^*$. We see $J \otimes E$ is the Jacobson radical of $(H \otimes E)^*$ because it is nilpotent and has codimension one. Therefore by [10, Proposition 5.2.9(2)], $(H \otimes E)_n = H_n \otimes E$ for any $n \geq 0$. By Maschke’s theorem [8] again, the semisimplicity of Hopf algebras is preserved by base change [10, Corollary 2.2.2]. Therefore we can extend the base filed $k$ to its algebraically closure and apply Theorem 4.1. □

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