Random path in negatively curved manifolds

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March 19, 2020

Abstract

In this article we consider sequences of random points on a complete, simply connected, negatively curved manifold. The sequence is a Markov process defined through a geometric rule. We study the asymptotic behaviour of the sequences. We obtain a spectral gap on the Markov operator and show the convergence almost surely in the Gromov boundary of the manifold.

1 Introduction

The study of random walks in groups has a long history. One of the first result concerns the random walk in abelian lattices, $\mathbb{Z}^n \subset \mathbb{R}^n$, where Polya showed that the simple random walk is transient if and only if $n \geq 3$ [Pol21]. We will come back to this setting in section 2.4.

With the development of geometric group theory, the simple random walk has been studied in many other cases, a classical theorem due to Kesten, asserts that the simple random walk on a locally compact, discrete group, is transient as soon as the group is not amenable, see [Kes59]. If the group is acting on a metric space $X$, then one could also look at the orbit $X_n = \gamma_1 \cdots \gamma_1 x_0$ for $x_0 \in X$, where $\gamma_i$ are random elements of the group. To only cite a few, one can see [BQ12] or the book [BQ16], for actions on homogeneous spaces, or [Mas95] for action on Teichmüller spaces.

In all the results previously mentioned, one is interested in a fixed probability measure $\mu$ on a group $G$, then one looks at the orbit $\gamma_n \cdots \gamma_1 (x_0)$ or the random trajectory $\gamma_n \cdots \gamma_1 x$, $x \in X$ where $\gamma_i$ are independent random variables distributed according to $\mu$. We insist on the fact that in both cases the measure $\mu$ is fixed once for all.

The problem we consider in this article is of a different kind: the probability distribution changes at each point with a geometric rule. It produces a sequence of random points $X_n$ which cannot be written as $\gamma_n \cdots \gamma_1 x_0$ and $\gamma_k$, i.i.d. random variables.

We can describe this process at two different levels. On a compact manifold or on its universal cover. The problem in this paper originally arose from the point of view of the quotient manifold, however we mainly look at the universal cover setting in order to answer it.

Let $M$ be a compact riemannian manifold. Let $x_n$ be a sequence of i.i.d. random points in $M$, with uniform distribution. For almost all pair of points in $M$ there is a unique minimizing geodesic joining them. Then for almost all sequence $x_n$, there is a unique minimizing geodesic between $x_i$ to $x_{i+1}$ for $i \in \mathbb{N}$. This process forms a random piecewise geodesic path and the problem is to understand the behaviour of this path when $n \to \infty$.

As we said, it can also be considered at the universal cover level. This is a more practical way to deal with this problem and we will focus from now on the process as described in the following paragraph.

Let $X$ be the universal cover of $M$. Let $\Gamma \subset \text{Isom}(X)$ be the deck transformations group, such that $X/\Gamma = M$. A central notion in this work is the one of Dirichlet domains centred at $x \in X$, defined by

$$D(x) := \{ z \in X \mid d(x, z) \leq d(x, \gamma z) \forall \gamma \in \Gamma \}.$$ 

We can now describe the random process on $X$: Pick a starting point $X_0 \in X$ then pick recursively a random point $X_n$ in $D(X_{n-1})$ following a uniform distribution on $D(X_{n-1})$. This gives a sequence...
of random points in $X$ or if we join $X_i$ to $X_{i+1}$ by a geodesic path, a sequence of random piecewise geodesic paths.

Dirichlet domain $D(X)$ has the remarkable property that for all $z$ in $D(x)$ the minimizing geodesic between $z$ and $x$ is the minimizing geodesic on the quotient space. This is the property making the bridge between the process at the universal cover level and on the compact manifold.

Finally we insist on the fact that it is not a trajectory of a simple random walk in some isometry subgroups of $X$. Even in the case of the symmetric homogeneous space $X = \mathbb{H}^n$, indeed if $y \notin \Gamma.x$ then there is no relation between $D(x)$ and $D(y)$ due to the fact that $\text{Isom}(X)$ is not abelian. This is in great contrast with the flat case where we will show that for flat tori, the problem reduces to a classical random walk in $\mathbb{R}^k$.

1.1 Statement of results

Let $X$ be the universal cover of a compact negatively curved manifold $M$. Let $\Gamma$ be the subgroup of isometries of $X$ such that $M = X/\Gamma$. For every point $x \in X$, consider the Dirichlet domain center at $x$, $D(x)$.

We define a sequence of random points by the following rule: Fix a point $o \in X$ the starting point of the path. Let $X_1$ be a random variable with value in $X$ with uniform distribution on $D(o)$. Then define by induction $X_n$ a random variable with value in $X$ with uniform distribution in $D(X_{n-1})$.

We will prove that the sequence of random points $X_n$ converges almost surely on the Gromov boundary of $X$.

First we consider the Markov operator of the random walk, defined by:

$$A : L^2(X) \to L^2(X),$$

$$A(f)(x) := \frac{1}{\text{Vol}(M)} \int_{D(x)} f(y)dy.$$ 

And we prove the following interesting result:

**Theorem 1.1.** The spectral radius of $A$ is strictly less than 1.

This will leads easily to the transience of the random process:

**Corollary 1.2.** The sequence $X_n$ is transient.

Then we will precise this statement, showing that the sequence $(X_n)_{n \in \mathbb{N}}$ is escaping linearly from the origin and will converge almost surely in the boundary:

**Theorem 1.3.** There exists $\ell > 0$ such that for almost all sequence $X_n$ one has:

$$\lim_{n \to \infty} \frac{d(o, X_n)}{n} = \ell.$$

And the sequence $X_n$ converges almost surely to a point in the geometric boundary $\partial X$.

Plan of the paper  Section 2 is devoted to reminders on the background needed for random walk, gromov boundary, and also explain how the flat torus case boils down to a classic random walk. In Section 3 we prove Theorem 1.1 This is the most technical part, we will use an isoperimetric inequality and follow the proof of Kesten. In the last section we apply Kingmann subadditive ergodic theorem to the function $f_n = d(o, X_n)$ to prove Theorem 1.3 the subtlety lies in the good way to define recursively the function $f_n$, in order to obtain the subadditivity property.

Acknowledgements  The author want to thank many supportive persons, Itai Benjamini, Adrien Boulanger, Gilles Courtois, Peter Haissinski and François Maucourant among others. And especially Pierre-Louis Blayac for his help in the resolution of the isoperimetric theorem.
2 Background

2.1 Dirichlet domains

One of a central object in our study is the Dirichlet fundamental domain. Let $\Gamma \subset \text{Isom}(X)$ be a\n\hspace{1cm}cocompact discrete subgroup of isometry. Then for all $x \in X$ we define:
\[ D(x) = \{ y \in X \mid d(x, y) \leq d(x, \gamma y) \forall \gamma \in \Gamma \}. \]

The following properties are well known, see for example [Bea93, Section 9.4]:

**Proposition 2.1.**

- For all $x \in X$, $D(x)$ is a fundamental domain.
- For all $x$, the diameter of $D(x)$ is less than the diameter of $M$.
- The boundary of $D(x)$ is of measure 0.
- The volume of $D(x)$ is equal to the volume of $M$.

The first property implies in particular that for all $z \in X$ there exists $\gamma \in \Gamma$ such that $\gamma z \in D(x)$.

The third property implies in particular, there exists a set $C(x)$ of measure 0 such that for all $y \not\in C(x)$, there is a unique $\gamma$ in $\Gamma$ such that $\gamma y \in D(x)$. In the quotient, this exactly means that for almost all pairs points $(a, b) \in M$ there is a unique minimizing geodesic between $a$ and $b$.

We will often use the following reflexivity property:

**Lemma 2.2.** For all $x, y \in X$, we have
\[ y \in D(x) \text{ if and only if } x \in D(y). \]

**Proof.** Let $y \in D(x) = \{ y \in X \mid d(x, y) \leq d(x, \gamma y) \forall \gamma \in \Gamma \}$. Then $d(y, x) \leq d(y, \gamma^{-1}x)$ for all $\gamma \in \Gamma$. Therefore, $x \in D(y)$. \qed

2.2 Random walks

Let $G$ be a locally compact topological group. Let $\mu$ be a probability measure on $G$, such that the\n\hspace{1cm}support of $\mu$ generates $G$. Consider $(g_i)_{i \in \mathbb{N}}$ a sequence of i.i.d. random variables with\n\hspace{1cm}distribution $\mu$. A random walk with respect to $\mu$ is the random product $X_i := g_1 \ldots g_i$.\n
One of the tool in order to study a random walk is the so called Markov operator associated to\n\hspace{1cm}$\mu$. It is an operator acting on $L^2(G)$ ($G$ endowed with its Haar measure) and defined by
\[ A(f)(x) := \int_{g \in G} f(gx) d\mu(x). \]

An important result in the theory of Markov operators is the following theorem due to Kesten, see [Qui14, Theorem 5.2] for this formulation:

**Theorem 2.3.** [Kes59] If $G$ is non amenable, then the spectral radius of the Markov operator is\n\hspace{1cm}strictly less than 1.

Consider the following random process: pick a point $X_0$ in $X$ then defined by induction $X_n$ to be a random point following a uniform distribution on the ball of radius $r > 0$ centered at $X_{n-1}$, $B(X_{n-1}, r)$. The following example shows that in the constant curvature case, this random\n\hspace{1cm}process correspond to a simple random walk: Identified $\mathbb{H}^n$ with the symmetric space: $\mathbb{H}^n \cong SO(n, 1)/SO(n)$. Consider the ball of radius $r \geq 0$ through this identification : $B(o, r) = \{ gk \mid g \in SO(n, 1), k \in SO(n), \kappa(g) \leq r \}$, where $\kappa(g)$ is the Cartan projection of $g$. Let $\mu$ be the uniform Haar\n\hspace{1cm}measure on $B(o, r) \subset SO(n, 1)$ normalized to be a probability measure. By Cartan decomposition,
we see that the support of \( \mu \) (ie. \( B(o, r) \)) generates \( G = \text{SO}(n, 1) \). Then, through this identification, Kesten’s theorem implies that the operator \( A_r : L^2(\mathbb{H}^n) \to L^2(\mathbb{H}^n) \) defined by

\[
A_r(f)(x) = \frac{1}{\text{Vol}(B(o, r))} \int_{y \in \mathbb{H}^n} f(x)1_{B(o, r)}(y) dy,
\]

has spectral radius strictly less than 1.

The principal difference with our case of study is the fact that the image of a ball by an isometry is a ball. This is not true for Dirichlet fundamental domains: the image of \( D(o) \) by an element of \( g \in \text{Isom}(\mathbb{H}^n) \) is a Dirichlet domain only for \( g \) in the centralizer of \( \Gamma \). Moreover since we consider non-constant curvature, the homogeneous setting is not appropriate to our problem.

### 2.3 Gromov hyperbolic geometry

We will make a short reminder on visual boundary, see [GdlH][Hai13] for more details, we follow the second reference. For this work, it is convenient to define this boundary in the following way. Let \( \langle x, y \rangle_o \) be the Gromov product of \( x \) and \( y \) seen from \( o \), that is:

\[
\langle x, y \rangle_o := \frac{1}{2} \left( d(x, o) + d(y, o) - d(x, y) \right).
\]

A sequence \( (x_n)_n \in X \) is said to be diverging if \( \lim \inf_{m,n \to \infty} \langle x_n, x_m \rangle_o = +\infty \). The Gromov boundary \( \partial X \) of \( X \) is defined to be the set of diverging sequence up to the equivalence \( (x_n) \sim (y_n) \) if and only if \( \lim \inf_{m,n \to \infty} \langle x_n, y_n \rangle_o = +\infty \).

One can extend the Gromov product to the boundary, by \( \langle \xi, \eta \rangle_o := \inf \lim \inf_{n \to \infty} \langle x_n, y_n \rangle_o \) where the infimum is taken over all representative of \( \xi \) and \( \eta \).

A visual metric is a distance on \( \partial X \) comparable to \( e^{-\epsilon \langle \xi, \eta \rangle_o} \) for some \( \epsilon > 0 \) and \( o \in X \). For \( \epsilon > 0 \) small enough there exists visual metric. In fact, as it is shown in [Hai13] Lemma 2.2, one can find a distance \( d_o \) on \( X \cup \partial X \) which is bilipschitz to the function \( q \) defined by \( q(x, y) = e^{-\epsilon \langle \xi, \eta \rangle_o} \) if \( x \neq y \) and \( q(x, x) = 0 \). We even have a better comparison due to Bonk, Heinonen and Koskela:

**Theorem 2.4. [BHK]** There exists \( \epsilon, C > 0 \) and a distance \( d_o \) on \( X \cup \partial X \) such that for all \( x, y \in X \cup \partial X \):

\[
\frac{1}{C} d_o(x, y) \leq e^{-\epsilon \langle x, y \rangle_o} \min(1, |x - y|) \leq C d_o(x, y)
\]

### 2.4 The torus case

We explain in this section how the problem described in the introduction reduces to a simple random walk on \( \mathbb{R}^k \) when we consider the flat torus case.

Let \( T^k \) be a flat torus of dimension \( k \). Let \( \Gamma \subset \mathbb{R}^k \) be such that \( T^k \) is isometric to \( \mathbb{R}^k / \Gamma \). The group \( \Gamma \) is isomorphic to \( \mathbb{Z}^k \) and acts by translations on \( \mathbb{R}^k \). Let \( \tau_z \) be the translation of vector \( x \in \mathbb{R}^k \). The reason that the geometric problem is the same as the usual random walk is because Dirichlet domains ”commutes” with the translations:

**Lemma 2.5.** For all \( x, y \in \mathbb{R}^k \),

\[
\tau_y D(x) = D(\tau_y x) = D(y + x).
\]

**Proof.** Let \( z \in \tau_y D(x) \), we have for all \( \gamma \in \Gamma \): \( d(x, \tau_y^{-1} z) \leq d(x, \gamma \tau_y^{-1} z) \). Since \( \Gamma \) is a group of translation, we see that \( \gamma \) is acting as \( \tau_v \) for some \( v \in \mathbb{R}^n \). In particular \( \gamma \tau_y^{-1} = \tau_v \tau_y^{-1} \tau_v^{-1} = \tau_y^{-1} \gamma \) Therefore for all \( \gamma \in \Gamma \) we have: \( d(\tau_y x, z) \leq d(\tau_y x, \gamma z) \). This proves the lemma.

Let \( d\mu(x) := 1_{D(o)}(x)dx \). We have \( (\tau_y)^* (d\mu(x)) = 1_{D(o)}(x - y)dx = 1_{D(\tau_o y)}(x)dx \). Let \( Y_n \) be a sequence of iid random variables with distribution \( \mu \), and consider \( S_n := \sum_{k=1}^n Y_k \). Consider also the sequence \( X_n \) of random variables, defined recursively, by \( X_0 \) having distribution \( \mu(x) \) and \( X_n \) having distribution \( 1_{D(X_{n-1})} dx \).
Theorem 2.6. The two sequences $S_n$ and $X_n$ have the same law.

Proof. This is true for $n = 1$ by definition.

Suppose that $S_n$ and $X_n$ have the same law. Then for all $x \in \mathbb{R}^k$ and all measurable sets $A \subset \mathbb{R}^k$

\[
P(S_{n+1} \in A \mid S_n = x) = P(S_n + Y_{n+1} \in A \mid S_n = x)
= P(Y_{n+1} \in A - x)
= \mu(A - x) = \tau_x^{*} \mu(A)
\]

and

\[
P(X_{n+1} \in A \mid X_n = x) = \int_A 1_{D(x)}(y)dy = \tau_x^{*} \mu(A).
\]

By the classical result of Polya on symmetric random walks in $\mathbb{R}^k$ we have:

Corollary 2.7. The sequence $X_n$ on $\mathbb{R}^k$ is transient if and only if $k \geq 3$.

3 Transience

In this section we turn back to the negative curvature setting and we prove Theorem 1.2. For this we consider the Markov operator:

\[
A : L^2(X) \to L^2(X),
A(f)(x) := \frac{1}{\text{Vol}(M)} \int_{D(x)} f(y)dy.
\]

and study its spectral properties. The transience will follow directly from the fact that $A$ is a contraction.

We use the following lemma, sometimes referred to Schur’s Test:

Theorem 3.1 (Schur’s Test). Let $X$ be a measurable space and $K$ be in $L^1(X^2)$. Let $P$ be the integral operator

\[
P(f)(x) := \int_X K(x,y)f(y)dy.
\]

Let $C_1 := \sup_x \int_X |K(x,y)|dx$ and $C_2 := \sup_x \int_X |K(x,y)|dy$. Then

\[
\|P\| \leq \sqrt{C_1 C_2}.
\]

We apply this lemma to $K(x,y) = \frac{1}{\text{Vol}(M)} 1_{D(x)}(y)$. We have $C_1 = \sup_y \int_X \frac{1}{\text{Vol}(M)} 1_{D(x)}(y)dx = 1$, where we used Lemma 2.2. Similarly we have, $C_2 = 1$. This implies:

Corollary 3.2. The $L^2$ operator norm of $A$ is less than 1.

The aim of the next sections is to show that the norm is strictly less than 1. We follow a document due to Lalley [Lal14], this is where we found the different tricks of Sections 3.2, 3.3. The main technical difficulty is to generalize an equivalent of an isoperimetric inequality (which in the case of non-ameanable group corresponds to Følner criterion), this is done in the next section. The rest of the proof is a sequence of analysis tricks on operators related to $A$ and seems to be fairly well known. We provide the proofs to be as self-contained as possible.

3.1 Isoperimetric inequality

We denote by $p(x,y) := \frac{1}{\text{Vol}(M)} 1_{D(x)}(y)$. Clearly we have $p(x,y) = p(y,x)$ and $\int_{y \in X} p(x,y)dy = 1$.

Proposition 3.3. There exists $\alpha > 0$ such that for all relatively compact open sets $U \subset X$:

\[
\int_{x \in U} \int_{y \in U^c} p(x,y)dydx \geq \alpha \text{Vol}(U).
\]

The proof is rather technical. We will decompose the set $U$ into annulus of size $\epsilon$ and prove by induction that the inequality holds.
Remark. We call this inequality an isoperimetric inequality because we had the following intuition in mind. The integral \( \frac{1}{\Vol(M)} \int_{x \in U} \Vol(B(x, \epsilon_0) \cap U^c)dx \) should be larger than the volume of the \( \epsilon \)-neighbourhood of \( U \), itself larger than \( \epsilon \Vol \partial U \), which by a classical isoperimetric inequality \cite{Yau75} is larger than \( c \Vol(U) \). However this intuition is false, and the term isoperimetric is somehow misleading. Indeed, it is not true that there exist \( c > 0 \) such that \( \int_{x \in U} \int_{y \in U^c} p(x, y)dydx \geq c \Vol(\partial U) \). If it were, since we can find sequence \( U_n \) such that \( \Vol(\partial U_n) \to +\infty \) and \( \Vol(U_n) \) stay bounded, it would be in contradiction with the obvious upper bound \( \Vol(U) \geq \frac{1}{\Vol(M)} \int_{x \in U} \Vol(B(x, \epsilon_0) \cap U^c)dx \).

Before the proof, let us introduce some notations. Let \( \epsilon_0 > 0 \) be the injectivity radius of \( M \) and \( \epsilon = \epsilon_0/10 \). Let \( A_n \) be the annulus

\[ A_n := B(o, (n + 1)\epsilon) \setminus B(o, n\epsilon). \]

Let \( S = T^1_o X \) be the unit tangent sphere at \( o \in X \). We call a cone of base \( E \subset S \) the subset of \( X \) defined by:

\[ C(E) := \{ \exp(tv) \mid v \in E, t \in \mathbb{R}^+ \}. \]

We will use the following proposition, that gives a partition of each \( A_n \) by domain of the same measure and with bounded diameter:

**Proposition 3.4.** For each \( n \in \mathbb{N} \), there is a partition of \( A_n \), \( A_n = \bigcup_{i \in I_n} A_n(i) \) such that:

1. \( A_n(i) \) is a part of a cone: there exists \( E_n(i) \subset S \) such that \( A_n(i) = C(E_n(i)) \cap A(n) \).
2. The volume of \( A_n(i) \) is independent of \( i \in I_n \), denoted by \( v_n \).
3. \( v_n \) is increasing.
4. For all \( i \in I_n \), \( \text{Diam}(A_n(i)) \leq \epsilon \).

There have been similar works on equal volume partition, we will use the following result of Feige-Schechtman,

**Theorem 3.5.** \cite{FS02} Lemma 21] For all \( \alpha \in (0, \pi/2) \) the sphere \( S^{d-1} \) can be partitioned into \( N = (O(1)/\alpha)^d \) regions of equal volume, each of diameter at most \( \alpha \).

Gigante-Leopardi \cite{GL16} extends this result for any connected Ahlfors-regular metric, measure space. T

**Theorem 3.6.** \cite{GL16} Theorem 2] Let \( (X, \rho, \mu) \) be a connected Ahlfors regular metric measure space of dimension \( d \) and finite measure. Then there exist positive constant \( c > 0 \) such that for every sufficiently large \( N \), there is a partition of \( X \) into \( N \) regions of measure \( \mu(X)/N \) each contained in a ball of radius \( cN^{-1/d} \).

Their techniques can be extended to a sequence of uniformly Ahlfors-regular metric spaces. That is a sequence of \( (X, d_n, \mu_n) \) of metric measure spaces of finite volume, for which there is a constant \( c > 0 \) and \( \delta > 0 \) such that for all \( x \in X \), all \( \epsilon > 0 \) and for all \( n \): \( \frac{1}{c}e^\delta \leq \mu_n(B_n(x, \epsilon)) \leq ce^\delta \).

**Proof of Proposition 3.4.** Consider \( \pi_n : S(o, n) \to S \) to be the projection of the sphere of radius \( n \) onto \( S \), and \( p_n : A(n) \to S \) to be the projection of \( A(n) \) onto \( S \). Denote by \( d_n \) the push forward metric by \( \pi_n \) and \( \mu_n \) the push forward measure of the annulus \( A(n) \) onto \( S \). Since \( X \) is the universal cover of a compact Ahlfors-regular metric space, the sequence of metric spaces \( (S, d_n, \mu_n) \) is uniformly Ahlfors-regular.

The papers previously cited give for all \( n \in \mathbb{N} \) a \( N_n \in \mathbb{N} \) such that \( \frac{\mu_n(S)}{N_n} \leq \inf_{x \in S, n \in \mathbb{N}} \mu_n(B_n(x, \epsilon/8)) \), a partition of \( S \) of size \( N_n \), each element of the partition of diameter less than \( \epsilon \).

The only thing to check is that one can take \( N_n \) big enough such that \( v_n = \frac{\mu_n(S)}{N_n} \) is increasing while keeping \( \frac{\mu_n(S)}{N_n} \leq \inf_{x \in S, n \in \mathbb{N}} \mu_n(B_n(x, \epsilon/8)) \). Taking by induction \( N_{n+1} := \lceil \frac{n+1}{n} \frac{\mu_n(S)}{\mu_{n-1}(S) N_n} \rceil \), will
ensure that \(v_n\) is increasing and one can check that if \(N_0\) is sufficiently large then \(N_n \geq \mu_n(S)(N_0 - K)\) for some \(K > 0\) independent of \(n\) (follows from tedious comparison with geometric sequences). This will in turn ensure the bound on \(\frac{\mu_n(S)}{n}\).

From now on we fix a partition satisfying the condition of Proposition 3.4. For \(x \in X\) we define three sets, \(A(x), A^+(x)\) and \(A^{-1}(x)\) such that if \(x \in A(n)\), we have \(A(x) \subset A_n\), \(A^+(x) \subset A_{n+1}\) and \(A^{-1}(x) \subset A_{n-1}\).

Definition 3.7. • \(A(x)\) is the element of the partition containing \(x\),

• \(A^+(x)\) the element of the partition in the "next" annulus intersecting the geodesic ray from \(o\) to \(x\). Formally, let \(v \in S\) and \(t \in \mathbb{R}^+\) such that \(x = \exp(tv)\), then \(A^+(x) = A(\exp((t + \epsilon)v)\)

• \(A^{-1}(x)\) be the set \(A^{-1}(x) := \{y | x \in A(y)\}\).

Remark that \(A^{-1}\) is not an element of the partition.

Due to negative curvature we have the following:

Proposition 3.8. There exists \(\lambda < 1\) such that for all \(n \in \mathbb{N}^*\), for all \(E \subset S\):

\[
\text{Vol}(C(E) \cap A_n) \leq \lambda \text{Vol}(C(E) \cap A_{n+1}).
\]

Proof. Let \((r, \theta)\) be polar coordinates centered at \(o \in X\). Let \(a(r, \theta)drd\theta\) the volume form and denote by \(-\kappa^2 < 0\), \(\kappa > 0\) the upper bound of the curvature. Let \(a'(r, \theta)\) be the partial derivative with respect to \(r\). Then we have for all \(r > 0\) for all \(\theta \in T^1_oX\): \(a'(r, \theta) \geq (n-1)\kappa\), see [GIL 4.B]. Therefore, we can integrate between \(R\) and \(R + \epsilon\) and we get:

\[
a(R + \epsilon, \theta) \geq \exp((n-1)\kappa)a(R, \theta).
\]

Let \(E\) be a measurable subset of \(S\) we get:

\[
\text{Vol}(C(E) \cap A_{n+1}) = \int_E \int_{\epsilon(n+1)}^{\epsilon(n+1)} a(R + \epsilon, \theta)drd\theta
\]

\[
\geq \exp((n-1)\kappa) \int_E \int_{\epsilon(n+1)}^{\epsilon(n+1)} a(R, \theta)drd\theta
\]

\[
\geq \exp((n-1)\kappa) \text{Vol}(C(E) \cap A_n)
\]

In particular there exists \(\lambda < 1\) such that for all \(x \in X\):

\[
\text{Vol} A^{-1}(x) \leq \lambda \text{Vol} A(x) \leq \lambda \text{Vol} A^+(x) \tag{1}
\]

Proof of Proposition 3.3. Remark that since \(A^+(x) \subset D(x)\) since we chose \(\epsilon\) small compare to the injectivity radius \(i_M\) of \(M\), and since for all \(x\), \(B(\epsilon, i_M/2) \subset D(x)\) we have:

\[
\int_{x \in U} \int_{y \in U^c} p(x, y)dydx = \int_{x \in U} \text{Vol}(D(x) \cap U^c)dx
\]

\[
\geq \int_{x \in U} \text{Vol}(A^+(x) \cap U^c)dx
\]

We denote by \(B(n) := B(o, \epsilon n)\). We will prove by induction that there exists \(\alpha > 0\) such that for all \(n\), for all open set \(U\) contained in \(B(n)\) that:

\[
\int_{x \in U} \text{Vol}(A^+(x) \cap U^c)dx \geq \alpha \text{Vol}(U).
\]
We first prove the result for any open set contained in $B(1)$. Indeed, for any $x \in B(1)$ we have $A^+(x) \cap U^c = A^+(x)$ and then,

$$\int_{x \in U} \Vol(A^+(x) \cap U^c)dx = \Vol(U)v_1.$$ 

The result follows for $n = 1$ with $\alpha := v_1(1 - \lambda)$ (We will see at the very end of the induction why we need this constant, and not only $v_1$.)

Suppose now the result true for all open sets contained in $B(n)$, and consider an open set $U$ contained in $B(n+1)$. Denote by $I := \int_{x \in U} \Vol(A^+(x) \cap U^c)dx$ and decompose $U$ by the following disjoint union: $U = (U \cap B(n)) \cup (U \cap A_n)$,

$$I = \int_{x \in U \cap B(n)} \Vol(A^+(x) \cap U^c)dx + \int_{x \in U \cap A_n} \Vol(A^+(x) \cap U^c)dx. \tag{2}$$

For all $x \in A_n$, $A^+(x) \subset A_{n+1}$ therefore we have $A^+(x) \subset U^c$ and:

$$\int_{x \in U \cap A_n} \Vol(A^+(x) \cap U^c)dx = \int_{x \in U \cap A_n} \Vol(A^+(x))dx.$$

For the first term in Equation 2 remark that we have the disjoint union $U^c \cup (U \cap A_n) = (U \cap B(n))^c$ (see Figure 1).

![Figure 1: U is delimited by the red curve. The blue part correspond to $U^c$. The red part is $U \cap A_n$. Finally the white part is $U \cap B(n)$.](image)

$$\Vol(A^+(x) \cap U^c) = \Vol(A^+(x) \cap (U \cap B(n))^c) - \Vol(A^+(x) \cap (U \cap A_n)).$$

By induction, we have:

$$\int_{x \in U \cap B(n)} \Vol(A^+(x) \cap (U \cap B(n))^c)dx \geq \alpha \Vol(U \cap B(n)),$$

and we get:

$$I \geq \alpha \Vol(U \cap B(n)) - \int_{x \in U \cap B(n)} \Vol(A^+(x) \cap (U \cap A_n))dx + \int_{x \in U \cap A_n} \Vol(A^+(x))dx.$$
Now we can switch the order of integration in the second term and we obtain:

\[
\int_{x \in U \cap B(n)} \text{Vol}(A^+(x) \cap (U \cap A_n)) dx = \int_x \int_y 1_{U \cap B(n)}(x)1_{A^+(x)}(y)1_{A_n \cap U}(y) dy dx
\]
\[
= \int_y 1_{A_n \cap U} \int_x 1_{U \cap B(n)}(x)1_{A^+(x)}(y) dx dy
\]
\[
= \int_y 1_{A_n \cap U} \int_x 1_{U \cap B(n)}(x)1_{A^-(y)}(x) dx dy
\]
\[
= \int_y 1_{A_n \cap U} \text{Vol}(U \cap B(n) \cap A^-(y)) dy
\]
\[
\leq \int_y 1_{A_n \cap U} \text{Vol}(A^-(y)) dy
\]

Where we used for the third equality: \( y \in A^+(x) \) if and only if \( x \in A^-(y) \).

With the previous computations we get:

\[
I \geq \alpha \text{Vol}(U \cap B(n)) + \int_{U \cap A_n} \text{Vol}(A^+(x)) - \text{Vol}(A^-(x)) dx
\]

Finally Equation (1) implies:

\[
I \geq \alpha \text{Vol}(U \cap B(n)) + \int_{U \cap A_n} (1 - \lambda) v_n
\]
\[
\geq \alpha \text{Vol}(U \cap B(n)) + (1 - \lambda) v_1 \text{Vol}(U \cap A_n)
\]
\[
\geq \alpha \text{Vol}(U).
\]

3.2 Sobolev inequality

We define the following quantity

\[
S(f) := \frac{1}{2} \int_{x,y} |f(x) - f(y)|p(x,y) dxdy.
\]

**Proposition 3.9.** There exists \( \alpha > 0 \) such that for all \( f \in L_1(X) \):

\[
S(f) \geq \alpha \|f\|_1.
\]

**Proof.** Without loss of generality we suppose that \( f \geq 0 \), with compact support. We have:

\[
S(f) = \int_{x,y,f(x) > f(y)} (f(x) - f(y)) p(x,y) dxdy
\]
\[
= \int_0^\infty \int_{x,y} 1_{\{t, f(x) < t < f(y)\}}(t) p(x,y) dxdy dt.
\]

Now we apply the isoperimetric inequality, Proposition 3.3, to the set \( U_t := \{y, t < f(y)\} \) we get:

\[
S(f) = \int_0^\infty \int_{y \in U_t} \int_{x \in U_t^c} p(x,y) dxdy dt
\]
\[
\geq \int_0^\infty \alpha \text{Vol}(U_t) dt
\]
\[
\geq \alpha \|f\|_1.
\]

\[\square\]
3.3 Spectral radius gap

Define the Dirichlet form \( D(f, f) := \frac{1}{2} \iint (f(y) - f(x))^2 p(x, y) dxdy \).

**Proposition 3.10.**

\[ D(f, f) = \langle (I - A) f , f \rangle. \]

**Proof.**

\[
\langle (I - A) f , f \rangle = \int_{x \in X} (I - A)(f)(x) f(x) dx
\]
\[ = \int_{x \in X} \left( f^2(x) - f(x) \int_{y \in X} f(y) p(x, y) dy \right) dx \]
\[ = \int_{x \in X} f^2(x) dx - \int_{x,y \in X} f(x) f(y) p(x, y) dy dx \]
\[ = \frac{1}{2} \left( \int_{x \in X} f^2(x) dx - 2 \int_{x,y \in X} f(x) f(y) p(x, y) dy dx + \int_{y \in X} f^2(y) dy \right) \]
\[ = \frac{1}{2} \left( \int_{x,y \in X} (f^2(x) - 2 f(x) f(y) p(x, y) + f^2(y)) p(x, y) dxdy \right) \]
\[ = D(f, f). \]

**Proposition 3.11.**

\[ S(f^2) \leq \sqrt{D(f, f)} \| f \|^2 \]

**Proof.** Applying successively, Cauchy-Schwarz inequality and the classical \((a + b)^2 \leq 2(a^2 + b^2)\) we get:

\[
S(f^2) = \frac{1}{2} \iint_{x,y} (f(x) - f(y)) \sqrt{p(x, y)} (f(x) + f(y)) \sqrt{p(x, y)} dxdy 
\]
\[ \leq \frac{1}{2} \sqrt{\iint_{x,y} (f(x) - f(y))^2 p(x, y) dxdy} \sqrt{\iint_{x,y} (f(x) + f(y))^2 p(x, y) dxdy} \]
\[ \leq \frac{1}{2} \sqrt{D(f, f)} \sqrt{2 \iint_{x,y} (f(x)^2 + f(y)^2) p(x, y) dxdy} \]
\[ \leq \frac{1}{2} \sqrt{D(f, f)2 \| f \|^2} = \sqrt{D(f, f)} \| f \|^2 \]

**Proposition 3.12.** The operator \( A \) is self-adjoint.

**Proof.** Let \( f, g \) be two functions in \( L^2(X) \). We look at the scalar product of \( A(f) \) and \( g \)

\[
\langle A(f) | g \rangle = \frac{1}{\text{Vol}(M)} \int_{x \in X} \int_{y \in D(x)} f(y) dy g(x) dx.
\]
\[ = \frac{1}{\text{Vol}(M)} \int_{x,y \in X} g(x) f(y) dxdy. \]
\[ = \frac{1}{\text{Vol}(M)} \int_{y \in X} \int_{x \in D(y)} g(x) dx f(y) dy.
\]
\[ = \langle f | A(g) \rangle. \]

Where the third equality comes from the fact that \( y \in D(x) \) if and only if \( x \in D(y) \), Lemma 2.2.
Theorem 3.13. The $L^2$ operator norm of $A$, satisfies $\|A\| < 1$.

Proof. Since $A$ is self-adjoint, we have $\|A\| = \sup_{f \in L^2(X)} \frac{(Af,f)}{\|f\|^2} = 1 - \sup_{f \in L^2(X)} \frac{D(f,f)}{\|f\|^2}$.

Therefore we only need to prove that there exists $\epsilon > 0$ such that

$$D(f,f) \geq \epsilon \|f\|^2.$$

Applying Proposition 3.9 we have $S(f^2) \geq \alpha \|f\|^2_1 = \alpha \|f\|^2_2$ and therefore by Proposition 3.11:

$$D(f,f) \geq \frac{S(f^2)^2}{\|f\|^2_2} \geq \alpha^2 \|f\|^2_2.$$

Before proving that the process is transient we need a last lemma bounding $A$ pointwise:

Lemma 3.14. For all $f \in L^2(X)$ and for all $x \in X$:

$$|A(f)(x)| \leq \frac{1}{\sqrt{\text{Vol}(M)}} \|f\|_2.$$

Proof. This is a simple consequence of Cauchy-Schwarz inequality:

$$A(f)(x) = \frac{1}{\text{Vol}(M)} \int_y 1_{D(x)}(y)f(y)dy$$

$$\leq \frac{1}{\text{Vol}(M)} \sqrt{\int_y 1_{D(x)}(y)dy} \sqrt{\int_y f(y)^2dy}$$

$$\leq \frac{1}{\sqrt{\text{Vol}(M)}} \|f\|_2.$$

Corollary 3.15. The random process $X_n$ is transient.

Proof. There exists $h > 0$ such that the map $f : x \mapsto e^{-hd(o,x)}$ belongs to $L^2(X)$ and we have:

$$E_o(f(X_n)) = A^n(f)(o) \leq \frac{1}{\sqrt{\text{Vol}(M)}} \|A^{n-1}(f)\|_2 \leq C(1-\epsilon)^n \|f\|_2,$$

for some $C, \epsilon > 0$, where we have applied successively Lemma 3.14 and Theorem 3.13.

Now we use Markov inequality:

$$\mathbb{P}_o(e^{-hd(o,X_n)} \geq e^{-cn}) \leq \frac{E_o(f(X_n))}{e^{-cn}},$$

which implies:

$$\mathbb{P}_o(hd(o, X_n) \leq cn) \leq Ce^{(\log(1-\epsilon)+c)n},$$

Choosing $c$ such that $0 < c < -h \log(1-\epsilon)$, we see that there exists $\alpha > 0$ such that

$$\mathbb{P}_o(d(o, X_n) \leq cn) \leq Ce^{-\alpha n}.$$

The corollary follows by Borel-Cantelli lemma.
4 Convergence in the boundary

Let $\tilde{\Omega} := M^\mathbb{N}$ and $\mu := \text{Leb}^{\otimes \mathbb{N}}$. The subset of $\tilde{\Omega}$ for which there exists $n \in \mathbb{N}$, $\omega_n \in C_{\omega_{n-1}}$ where $C_{\omega_{n-1}}$ denotes the cut locus of $\omega_{n-1}$ on $M$ is of measure 0. We restrict ourselves to $\Omega \subset \tilde{\Omega}$ the sequence for which every points satisfies, $\omega_n \notin C_{\omega_{n-1}}$.

We fix a point $o \in \mathbb{H}^n$ and define:

$$ R_0 : \Omega \to \mathbb{H}^n $$

$\omega \mapsto$ the unique lift of $\omega_0$ in $D(o)$,

and by induction

$$ R_n : \Omega \to \mathbb{H}^n $$

$\omega \mapsto$ the unique lift of $\omega_n$ in $D(R_{n-1}(\omega))$.

We then define the maps:

$$ f_n : \Omega \to \mathbb{R} $$

$\omega \mapsto d(o, R_n(\omega))$.

Let $T : \Omega \to \Omega$ be the shift operator. It is a classical fact from dynamical system that $T$ is ergodic, see for example [Cou16, Proposition 3.2].

**Proposition 4.1.** For all $n, m \in \mathbb{N}^2$, one has:

$$ f_{n+m} \leq f_m \circ T^n + f_n + D \tag{3} $$

where $D$ is the diameter of $M$.

We will need to compare the lifts in different fundamental domains. Remark, that $R_0(T^n \omega)$ and $R_n(\omega)$ are lifts of the same elements $\omega_n$. We will use the following lemma:

**Lemma 4.2.** Let $\gamma \in \Gamma$ such that $\gamma R_0(T^n \omega) = R_n(\omega)$. Then for all $m \in \mathbb{N}$ one has:

$$ \gamma R_m(T^n \omega) = R_{n+m}(\omega). $$

**Proof.** The proof is by induction on $m \in \mathbb{N}$. For $m = 0$ the statement is clear. Let $m \geq 0$, one has:

$$ \gamma R_{m+1}(T^n \omega) = \gamma \left( \text{the lift of } (T^n \omega)_{m+1} \text{ in } D(R_m(T^n \omega)) \right) $$

$$ = \text{the lift of } \omega_{n+m+1} \text{ in } \gamma D(R_m(T^n \omega)) $$

$$ = \text{the lift of } \omega_{n+m+1} \text{ in } D(\gamma R_m(T^n \omega)) \text{ by } \Gamma \text{ equivariance} $$

$$ = \text{the lift of } \omega_{n+m+1} \text{ in } D(R_{n+m}(\omega)) \text{, by induction} $$

$$ = R_{n+m+1}(\omega) $$

We can now prove Proposition 4.1.

**Proof of Proposition 4.1.** Let us recall what are the definitions of the different terms in Equation (3):

$$ f_{n+m}(\omega) = d(o, R_{n+m}(\omega)),$$

$$ f_n(\omega) = d(o, R_n(\omega)),$$

and finally:

$$ f_m \circ T^n(\omega) = d(o, R_m(T^n \omega)).$$
Let $\gamma \in \Gamma$ be such that $\gamma R_0(T^n \omega) = R_n(\omega)$.

\[
d(o, R_m(T^n \omega)) = d(\gamma o, \gamma R_m T^n(\omega)) = d(\gamma o, R_{n+m}(\omega)) \quad \text{By Lemma 4.2}
\]
\[
\geq d(R_n(\omega), R_{n+m}(\omega)) - d(\gamma o, R_n(\omega)) \quad \text{using the triangle inequality.}
\]

Remark that $o$ lies at distance at most $D$ of $R_0(x)$ for all $x \in M$, therefore $\gamma o$ is at distance at most $D$ of $\gamma R_0(T^n \omega) = R_n(\omega)$. One gets finally:

\[
f_m \circ T^n(\omega) = d(o, R_m(T^n \omega)) \geq d(R_n(\omega), R_{n+m}(\omega)) - D.
\]

This finishes the proof by the triangle inequality:

\[
f_{n+m}(\omega) = d(o, R_{n+m}(\omega)),
\]
\[
\leq d(o, R_n(\omega)) + d(R_n(\omega), R_{n+m}(\omega)),
\]
\[
\leq f_n(\omega) + f_m \circ T^n(\omega) + D.
\]

\[\blacktriangleleft\]

**Theorem 4.3.** There exist $\ell > 0$ such that for almost all trajectory $(X_n)_{n \in \mathbb{N}}$ one has:

\[
\lim_{n \to \infty} \frac{d(o, X_n)}{n} = \ell
\]

**Proof.** Consider the function $\tilde{f}_n := f_n + D$. By Proposition 4.1 the function $\tilde{f}_n$ is subadditive (ie. $\tilde{f}_{n+m} \leq \tilde{f}_m \circ T^n + \tilde{f}_n$). Using Kingmann ergodic theorem, this implies that for almost all $\omega \in \Omega$:

\[
\lim_{n \to \infty} \frac{\tilde{f}_n(\omega)}{n} = \ell.
\]

And we have $\frac{\tilde{f}_n(\omega)}{n} = \frac{f_n(\omega) + D}{n} = \frac{d(o, X_n) + D}{n}$. Passing to the limit proves that for almost all trajectories:

\[
\lim_{n \to \infty} \frac{d(o, X_n)}{n} = \ell.
\]

We now show that $\ell > 0$. In the proof of 3.15 we have shown that there exists $\alpha > 0$ such that for all $c > 0$:

\[
\mathbb{P}(e^{-d(o, X_n)} \geq e^{-cn}) \leq e^{(-\alpha/2)n}
\]

Choosing $0 < c < \alpha/4$ shows that

\[
\mathbb{P}(d(o, X_n) \leq cn) \leq e^{(-\alpha/2)n},
\]

By Borel-Cantelli lemma, this implies that for almost all trajectories, we have $\lim_{n \to \infty} \frac{d(o, X_n)}{n} \geq c > 0$. This concludes the proof.

\[\blacktriangleleft\]

**Corollary 4.4.** $X_n$ converges almost surely in the geometric boundary $\partial X$.

**Proof.** We will estimate the Gromov product $\langle X_n, X_{n+p} \rangle$, for $n \in \mathbb{N}$ large and every $p \in \mathbb{N}$.

By Theorem 4.3 for $\ell/2 > \eta > 0$, and $n \in \mathbb{N}$ sufficiently large,

\[
|d(o, X_n) - \ell n| \leq \eta n.
\]

Therefore,

\[
\langle X_n, X_{n+1} \rangle_o = \frac{1}{2} \left( d(o, X_n) + d(o, X_{n+1}) - d(X_n, X_{n+1}) \right) \geq (\ell - \eta) n - D.
\]

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This implies that the visual distance of parameter $\epsilon$, $d_\epsilon(X_n, X_{n+1}) \leq e^{-\epsilon(X_n, X_{n+1})}\leq Ce^{-\epsilon(\ell - \eta)n}$

By the triangle inequality, we get:

$$d_\epsilon(X_n, X_{n+p}) \leq \sum_{k=0}^{p-1} Ce^{-\epsilon(\ell - \eta)(n+k)} \leq C' e^{-\epsilon(\ell - \eta)n}.$$ 

By compactness of $X \cup \partial X$, this implies that $X_n$ converges in $\partial X$. 

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