Global stability of combination of a viscous contact wave with rarefaction waves for the compressible fluid models of Korteweg type

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Received 12 January 2018, revised 22 September 2018
Accepted for publication 23 October 2018
Published 4 January 2019

Abstract
This paper is concerned with the large-time behavior of solutions to the Cauchy problem of the one-dimensional compressible fluid models of Korteweg type with density- and temperature-dependent viscosity, capillarity, and heat conductivity coefficients, which models the motions of compressible viscous fluids with internal capillarity. We show that the combination of a viscous contact wave with two rarefaction waves is asymptotically stable with a large initial perturbation if the strength of the composite wave and the capillarity coefficient satisfy some smallness conditions. The proof is based on some refined $L^2$-energy estimates to control the possible growth of the solutions caused by the high nonlinearity of the system, the interactions of waves from different families and large data, and the key ingredient is to derive the uniform positive lower and upper bounds on the specific volume and the temperature.

Keywords: compressible Korteweg model, viscous contact wave, rarefaction waves, global stability, large initial perturbation

Mathematics Subject Classification numbers: 35Q35, 35L65, 35B40
1. Introduction

This paper is concerned with the Cauchy problem of the non-isothermal compressible fluid models of Korteweg type in Lagrangian coordinates:

\[
\begin{aligned}
    v_t - u_x &= 0, \\
    u_t + p(v, \theta) &= \left( \frac{\mu(v, \theta) u_x}{v} \right)_x + K_x, \\
    (C_v - \frac{\theta}{2} \kappa v^2)_{\theta} + p(v, \theta) u_x &= \left( \frac{\alpha(v, \theta) u_{\theta}}{v} \right)_x + \frac{\mu(v, \theta) u_{\theta}}{v} + F,
\end{aligned}
\]

with the initial and far field conditions:

\[
\begin{aligned}
    (v, u, \theta)(0, x) &= (v_0, u_0, \theta_0)(x), & x \in \mathbb{R}, \\
    (v, u, \theta)(t, \pm \infty) &= (v_{\pm}, u_{\pm}, \theta_{\pm}), & t > 0.
\end{aligned}
\]

Here the unknown functions are the specific volume \(v(x, t)\), the velocity \(u(x, t)\), the temperature \(\theta(x, t)\), and the pressure \(p(v, \theta)\) of the fluid, respectively, while \(\mu(v, \theta), \kappa(v, \theta), \alpha(v, \theta)\) denote the viscosity coefficient, the capillarity coefficient and the heat conductivity coefficient respectively. \(C_v > 0, v_{\pm} > 0, u_{\pm} \) and \(\theta_{\pm} > 0\) are given constants, and we assume \((v_0, u_0, \theta_0)((\pm \infty) = (v_{\pm}, u_{\pm}, \theta_{\pm})\) as compatibility conditions. The Korteweg stress tensor \(K\) and the nonlinear terms \(F\) are given by

\[
\begin{aligned}
    K &= \frac{-\kappa(v, \theta) v_{\theta} + 5\kappa(v, \theta)v_{\theta}}{2v} v_x^2 - \kappa v_{\theta} v_{\theta}, \\
    F &= \theta_{\theta \alpha} v_{\theta} u_{\theta} + \frac{\kappa_{\alpha \beta} v_{\theta} - \kappa_{\alpha \theta} v_{\theta}}{2v} \theta u_{\alpha} v_{\beta}.
\end{aligned}
\]

Throughout this paper, we suppose that the pressure \(p(v, \theta)\) and the constant \(C_v\) are given by

\[
p(v, \theta) = \frac{R\theta}{v} = Av^{-\gamma} \exp \left[ \frac{\gamma - 1}{R} s \right], \quad C_v = \frac{R}{\gamma - 1},
\]

where \(s\) is the entropy of the fluid and \(\gamma > 1\). \(A\) and \(R\) are positive constants.

System (1.1) can be used to model the motions of compressible viscous fluids with internal capillarity. The formulation of the theory of capillarity with diffuse interface was first studied by Van der Waals [46] and Korteweg [34], and then derived rigorously by Dunn and Serrin [11]. Note that if the capillarity coefficient \(\kappa = 0\), the system (1.1) is reduced to the compressible Navier–Stokes system.

There have been extensive studies on the mathematical theory of the compressible fluid models of Korteweg type. For the case with small initial data, the results available now are almost complete. We briefly review some of them here. Hattori and Li [19, 20] proved the global existence of smooth solutions around constant states in Sobolev space. Wang and Tan [48] established the optimal time decay rate of the smooth solution obtained in [19]. Danchin and Desjardins [10] and Haspot [16, 17] discussed the global existence and uniqueness of strong solutions in Besov space. Kotschote [35] proved the exponential stability of a non-constant stationary solution in the phase space. The authors in [3, 36, 49] obtained the existence and nonlinear stability of non-constant stationary solutions in Sobolev space. Chen et al [7–9] studied the nonlinear stability of some single basic waves (such as rarefaction wave, viscous shock wave and viscous contact wave) in Sobolev space. And the global existence of weak solutions in the whole space \(\mathbb{R}^2\) was obtained by Danchin and Desjardins [10] and Haspot [15].

For the case with large initial data, Haspot [18] proved the global existence of strong solution for an isothermal fluid with density-dependent viscosity and capillarity coefficients in the
whole space $\mathbb{R}^N$ with $N \geq 2$. Bresch, Desjardins, and Lin [1] studied the global existence of weak solutions for an isothermal Korteweg system with a linearly density-dependent viscosity and a constant capillarity coefficient in a periodic domain $\mathbb{T}^d$ with $d = 2$ or 3. Then Haspot [15] and Jüngel [31] improved the results of [1] to some other types of density-dependent viscosity and capillarity coefficients. Tsyganov [45] discussed the global existence of weak solutions for an isothermal system with the viscosity coefficient $\mu(\rho) \equiv 1$ and the capillarity coefficient $\kappa(\rho) = \rho^{-5}$ on the interval $[0, 1]$. Germain and LeFloch [14] investigated the global existence of weak solutions for the isothermal Korteweg system with general density-dependent viscosity and capillarity coefficients in $\mathbb{R}$. The global existence of a large strong solution to an isothermal Korteweg system with the viscosity coefficient $\mu(\rho) = \varepsilon \rho$ and the capillarity coefficient $\kappa(\rho) = \varepsilon^2 \rho^{-1}$ in $\mathbb{R}$ ($\varepsilon$ is positive constant) was obtained by Charve and Haspot [2]. For the global existence of smooth large-amplitude solutions to the compressible Korteweg system in the whole space $\mathbb{R}$, we refer to [4, 6] for isothermal system with general density-dependent viscosity and capillarity coefficients, [5] for a non-isothermal system with general density-dependent viscosity and capillarity coefficients, and density- and temperature-dependent heat conductivity coefficient. The time-asymptotic nonlinear stability of strong rarefaction waves for the isothermal Korteweg system with large initial data was also obtained in [4, 6].

From the above results, it is seen that few results have been obtained on the global stability of basic waves for the compressible fluid models of Korteweg type so far. Here and hereafter, global stability means the nonlinear stability result with large initial perturbation. And if the initial perturbation is small, the nonlinear stability result is usually called local stability. A natural question is whether some global stability results of the composite waves for the one-dimensional compressible Korteweg system (1.1) with general density- and temperature-dependent viscosity, capillarity, and heat conductivity coefficients.

It is known [7–9] that the large-time behavior of solutions of the Cauchy problem (1.1) and (1.2) is closely related to the Riemann problem of the compressible Euler system

$$
\begin{cases}
v_t - u_x = 0, \\
u_t + p(v, \theta)_x = 0, \\
C_v \theta_t + p(v, \theta)u_x = 0,
\end{cases}
$$

with the Riemann initial data

$$
(v, u, \theta)(0, x) = \begin{cases}
(v_-, u_-, \theta_-), & x < 0, \\
(v_+, u_+, \theta_+), & x > 0.
\end{cases}
$$

The Euler system (1.5) is a strict hyperbolic system of conservation laws with three distinct eigenvalues:

$$
\lambda_1(v, \theta) = -\sqrt{\frac{\gamma p}{v}}, \quad \lambda_2 = 0, \quad \lambda_3(v, \theta) = \sqrt{\frac{\gamma p}{v}},
$$

which implies that the second characteristic field is linearly degenerate and the others are genuinely nonlinear. Then it is well known [44] that the Riemann problems (1.5) and (1.6) admit three basic wave patterns: the shock wave, rarefaction wave and contact discontinuity, and the Riemann solution to (1.5) has a wave pattern consisting of a linear combination of these three
basic waves. In particular, the contact discontinuity solution of the Riemann problems (1.5) and (1.6) take the form [44]

\[
(v^{cd}, u^{cd}, \theta^{cd})(t, x) = \begin{cases} (v_-, u_-, \theta_-), & x < 0, t > 0, \\
(v_+, u_+, \theta_+), & x > 0, t > 0,
\end{cases}
\]  

provided that

\[
u_- = u_+, \quad p_- \triangleq \frac{R \theta}{v_-} = \frac{R \theta_+}{v_+} \triangleq p_+.
\]  

The viscous contact wave \( (V, U, \Theta)(t, x) \) corresponding to the contact discontinuity \( (v^{cd}, u^{cd}, \theta^{cd})(t, x) \) for the compressible Navier–Stokes–Korteweg system (1.1) becomes smooth and behaviors as a diffusion waves due to the effect of heat conductivity. As in [8], we can define the viscous contact wave \( (V, U, \Theta)(t, x) \) as follows.

Since the pressure of the profile \( (V, U, \Theta)(t, x) \) is expected to be almost constant, we set

\[
P \coloneqq \frac{R \theta}{V} = p_+.
\]

Then the leading part of the energy equation (1.1)_3 is

\[
C_v \Theta_t + p_+ U_x = \left( \frac{\alpha(V, \Theta) \Theta_x}{V} \right)_x.
\]

Using the equations (1.9), \( V = U_t \) and (1.10), we get a nonlinear diffusion equation

\[
\begin{cases}
\Theta_t = a \left( \frac{\alpha(\Theta) \Theta_x}{\Theta} \right)_x, \\
a = \frac{p+ (\gamma - 1)}{\gamma R}.
\end{cases}
\]

where \( \alpha(\Theta) \triangleq \alpha(T, \Theta) \). Due to [12], (1.11) has a unique self-similar solution \( \Theta(t, x) = \Theta(\xi), \xi = \frac{ct - \gamma}{\sqrt{1 + \gamma}} \), which is a monotone function, increasing if \( \theta_+ > \theta_- \) and decreasing if \( \theta_+ < \theta_- \).

Once \( \Theta(t, x) \) is determined, the viscous contact wave \( (V, U, \Theta)(t, x) \) is defined by

\[
V = \frac{R \theta}{p_+}, \quad U = u_- + \frac{\gamma - 1}{\gamma R} \frac{\alpha(\Theta) \Theta_x}{\Theta}, \quad \Theta = \Theta(t, x),
\]

then it is easy to check that the viscous contact wave \( (V, U, \Theta)(t, x) \) satisfies

\[
\begin{cases}
V_t - U_x = 0, \\
U_t + \left( \frac{R \Theta}{p_+} \right)_x = \left( \frac{\mu(V, \Theta) U_t}{V} \right)_x + R_1, \\
\frac{R}{\gamma - 1} \Theta_t + p(V, \Theta) U_x = \left( \frac{\alpha(V, \Theta) \Theta_x}{V} \right)_x + \frac{\mu(V, \Theta) U_x}{V} + R_2
\end{cases}
\]

with

\[
R_1 = U_t - \left( \frac{\mu(V, \Theta) U_t}{V} \right)_x = O(1)(\gamma - 1)^{\frac{1}{2}}(1 + t)^{-\frac{1}{2}} e^{-\frac{\alpha c^2}{16(1 + t)^{1/2}}},
\]

\[
R_2 = -\mu(V, \Theta) \frac{U_t^2}{V} = O(1)(\gamma - 1)^{\frac{3}{2}}(1 + t)^{-\frac{3}{2}} e^{-\frac{\alpha c^2}{16(1 + t)^{1/2}}}. \tag{1.14}
\]

Our first theorem is concerned with the global stability of the single viscous contact wave \( (V, U, \Theta)(t, x) \), which is stated as follows.
Theorem 1.1 (Global stability of viscous contact wave). Let \((V, U, \Theta)(t,x)\) be the viscous contact wave defined in (1.12). Suppose that

(i) the given constants \(v_\pm, u_\pm, \Theta_\pm\) do not depend on \(\gamma - 1\), and satisfy (1.8). Moreover, \(|\theta_+ - \theta_-| \leq l_0(\gamma - 1)\) for some positive constant \(l_0\) independent of \(\gamma - 1\);

(ii) the initial data \((v_0, u_0, \Theta_0)\) satisfies that \(N_0 := \|v_0(\cdot) - V(0,\cdot)\|_{H^1(\mathbb{R})} + \|u_0(\cdot) - U(0,\cdot)\|_{H^1(\mathbb{R})} + \|\Theta_0(\cdot) - \Theta(0,\cdot)\|_{H^1(\mathbb{R})} + \|\Theta_0(\cdot) - \Theta(0,\cdot)\|_{H^1(\mathbb{R})} + \|\Theta_0(\cdot) - \Theta(0,\cdot)\|_{H^1(\mathbb{R})}\) is bounded by some constant independent of \(\gamma - 1\);

(iii) there exist positive constants \(\bar{V}, \bar{V}, \bar{\Theta}\) and \(\bar{\Theta}\) independent of \(\gamma - 1\) such that
\[ V \leq v_0(x), V(t,x) \leq \bar{V}, \quad \Theta \leq \Theta_0(x), \Theta(t,x) \leq \bar{\Theta}, \quad \forall (t,x) \in \mathbb{R}^+ \times \mathbb{R}; \tag{1.15} \]

(iv) the viscosity coefficient \(\mu(v, \theta)\), the capillarity coefficient \(\kappa(v, \theta)\) and the heat-conductivity coefficient \(\bar{\alpha}(v, \theta)\) are smooth positive functions of \(v > 0\) and \(\theta > 0\), and the following assumptions hold:

\[ \frac{\mu(v, \theta)}{\bar{\alpha}(v, \theta)} \leq L_0, \quad |\kappa\theta(v, \theta)| < \varepsilon, \quad \kappa\theta\theta(v, \theta) < 0, \tag{1.16} \]

where \(L_0 > 0\) is a uniform constant independent of \(\gamma - 1\) and \(\varepsilon > 0\) is small positive constant whose precise range can be specified in the proof of theorem 1.1.

(b) One of the following two conditions holds.

(b)_1 There exist constants \(a \geq 0\) and \(b \leq -\frac{1}{3}\) such that
\[ \mu_1(v) = \min_{\theta \in [\frac{2}{3}, 2\bar{\Theta}]} \{\mu(v, \theta)\} \sim \begin{cases} v^{-a}, & v \to 0^+, \\ v^{-b}, & v \to +\infty. \end{cases} \tag{1.17} \]

(b)_2 There exist constants \(c \geq -3\) and \(d \leq -2\) such that
\[ \kappa_1(v) = \min_{\theta \in [\frac{2}{3}, 2\bar{\Theta}]} \{\kappa(v, \theta)\} \sim \begin{cases} v^{-c}, & v \to 0^+, \\ v^{-d}, & v \to +\infty. \end{cases} \tag{1.18} \]

(c) The viscosity and capillarity coefficients are coupled by
\[ f(v, \theta) := -\frac{2}{3} \left( \frac{\sqrt{\mu\kappa}}{v^3} \right) \left( \frac{\sqrt{\mu\kappa}}{v^3} \right)_v + \left[ \left( \frac{\sqrt{\mu\kappa}}{v^3} \right)_v \right]^2 + \frac{1}{3} \left( \frac{\kappa}{v^3} \left( \frac{\mu}{v^3} \right)_v \right)_v \]
\[ \leq 0, \tag{1.19} \]

or
\[ g(v, \theta) := 3\kappa\mu + 2\nu\kappa\mu_\nu - \nu\kappa\nu_\nu = 0. \tag{1.20} \]

Then there exist positive constants \(\varepsilon_0 \ll 1, \delta_0 \ll 1\) and \(C_0\) that depend only on \(\bar{V}, \bar{V}, \bar{\Theta}, \bar{\Theta}\) and the initial data \(N_0\) such that if \(0 < \varepsilon < \varepsilon_0\) and \(0 < \delta := \gamma - 1 \leq \delta_0\), the Cauchy problem (1.1) and (1.2) admits a unique global-in-time solution \((v, u, \theta)(t,x)\) satisfying
\[ \begin{cases} (v - V, u - U, \theta - \Theta) \in C(0, +\infty; H^2(\mathbb{R}) \times H^1(\mathbb{R}) \times H^1(\mathbb{R})), \\ (v - V, u - U, \theta - \Theta) \in L^2(0, +\infty; H^2(\mathbb{R}) \times H^1(\mathbb{R}) \times H^1(\mathbb{R})), \end{cases} \tag{1.21} \]
\[ C_0^{-1} \leq v(t, x) \leq C_0, \quad \frac{\Theta}{2} \leq \theta(t, x) \leq 2\Theta, \quad \forall (t, x) \in [0, \infty) \times \mathbb{R}, \quad (1.22) \]

and

\[ \lim_{t \to +\infty} \sup_{x \in \mathbb{R}} \{|(v - V, u - U, \theta - \Theta)(t, x)|\} = 0. \quad (1.23) \]

**Remark 1.1.** Several remarks on theorem 1.1 are given as follows.

1. Since \( \theta_{\pm} = \frac{\alpha}{R} v_{\pm} \exp \left( \frac{x - s_{\pm}}{R} \right) \) and the constants \( \theta_{\pm}, v_{\pm}, s_{\pm} \) are assumed independent of \( \gamma - 1 \), it is easy to check that \( |\theta_+ - \theta_-| \leq l_0(\gamma - 1) \) for some constant \( l_0 \) independent of \( \gamma - 1 \).

2. A typical example of the physical coefficients \( \mu(v, \theta), \kappa(v, \theta) \) and \( \kappa(v, \theta) \) that satisfy the assumption (iv) of theorem 1.1 is \( \mu(v, \theta) = v^{-\alpha} \theta^{\lambda_1}, \kappa(v, \theta) = v^{-\beta} \theta^{\lambda_2}, \kappa(v, \theta) = v^{-\alpha} \theta^{\lambda_3} \), where \( \alpha, \beta, \lambda_1, \lambda_2 \) are real constants. Then the assumptions (iv)(a), (iv)(b) and (iv)(c) of theorem 1.1 are equivalent to

\[ L_0 = 1, \quad \lambda_2 \theta^{\lambda_1-1} v^{-\beta} < \varepsilon, \quad \lambda_2(\lambda_2 - 1) \theta^{\lambda_2-2} v^{-\beta} < 0; \]

\[ 0 \leq \alpha \leq \frac{1}{2} \quad \text{or} \quad -3 \leq \beta \leq -2; \]

and

\[ 3\alpha^2 + \beta^2 - 2\alpha \beta - 4\alpha + 6\beta + 8 \leq 0 \quad \text{or} \quad \beta = 2\alpha - 3, \]

respectively. Here the condition \( 3\alpha^2 + \beta^2 - 2\alpha \beta - 4\alpha + 6\beta + 8 \leq 0 \) is equivalent to

\[ \alpha - 3 - \sqrt{-2\alpha^2 - 2\alpha + 1} \leq \beta \leq \alpha - 3 + \sqrt{-2\alpha^2 - 2\alpha + 1} \]

or

\[ \frac{\beta + 2}{3} - \frac{1}{3} \sqrt{-2\beta^2 - 14\beta - 20} \leq \alpha \leq \frac{\beta + 2}{3} + \frac{1}{3} \sqrt{-2\beta^2 - 14\beta - 20}. \]

Thus, from the assumptions of theorem 1.1, we see that if the constants \( \alpha, \beta \) satisfy one of the following conditions:

1) \( 0 \leq \alpha \leq \frac{1}{2}, \quad \alpha - 3 - \sqrt{-2\alpha^2 - 2\alpha + 1} \leq \beta \leq \alpha - 3 + \sqrt{-2\alpha^2 - 2\alpha + 1}; \)

2) \( 0 \leq \alpha \leq \frac{1}{2}, \quad \beta = 2\alpha - 3; \)

3) \( -3 \leq \beta \leq -2, \quad \frac{\beta + 2}{3} - \frac{1}{3} \sqrt{-2\beta^2 - 14\beta - 20} \leq \alpha \leq \frac{\beta + 2}{3} + \frac{1}{3} \sqrt{-2\beta^2 - 14\beta - 20}; \)

4) \( -3 \leq \beta \leq -2, \quad \beta = 2\alpha - 3, \)

and the constants \( \lambda_1 \in \mathbb{R}, 0 < \lambda_2 < 2\lambda_2 - 1(\Theta^{\lambda_1-1} \theta)^{\lambda_2} \) is valid, then all the conclusions of theorem 1.1 hold.

3. The assumption (iv)(a) is used to control the possible growth of solutions to the Cauchy problems (1.1) and (1.2) induced by the nonlinearity of the system, (iv)(b) is used to derive the uniform-in-time lower and upper bounds for the specific volume \( v(t, x) \), and (iv)(c) is a technical condition in estimating \( \|\mu(v, \theta)\|_V(\tau)\| \) (see the proof of lemmas 3.3–3.5 for details).
(4) In theorem 1.1, although the initial perturbation \( \| \theta_0(x) - \Theta(0,x) \|_{(R)} \) is small when \( \gamma > 1 \)
when \( \gamma > 1 \) is close to 1, the initial perturbations \( \| v_0(x) - V(0,x) \|_{(R)} \) and \( \| u_0(x) - U(0,x) \|_{(R)} \)
can be arbitrarily large. This improves the main result of [8], where the nonlinear stability of viscous contact wave for the one-dimensional compressible fluid models of Korteweg type was obtained with all the initial perturbations are sufficiently small.

(5) From the proof of theorem 1.1, we see that \( \gamma - 1 \) needs to be sufficiently small such that
\( (\gamma - 1)F(N_0) < 1 \) with \( F(N_0) \) being a smooth increasing function on the initial data \( N_0 \)
(see (3.10) and (3.11)). Thus in this sense, theorem 1.1 is a Nishida–Smoller type result
[43] with large initial data.

When the relation \( (1.8) \) fails, the basic theory of hyperbolic systems of conservation laws [44] tells us that for any given constant state \((v_-,u_-,\theta_-)\) with \( v_- > 0, u_- \in \mathbb{R} \)
and \( \theta_- > 0 \), there exists a neighborhood \( \Omega(v_-,u_-,\theta_-) \) of \((v_-,u_-,\theta_-)\) such that for any
\((v_+,u_+,\theta_+)\), the Riemann problem \((1.5) \) and \((1.6) \) has a unique solution. In this paper, we only consider the stability of the combination of the viscous contact wave and rarefaction waves. Consequently, we assume that

\[
(v_+,u_+;\theta_+) \in R_-C_+R_+(v_-,u_-,\theta_-) \subset \Omega(v_-,u_-,\theta_-), \quad |\theta_+ - \theta_-| \leq \delta_1,
\]

where \( \delta_1 \) is a positive constant, \( R_-, R_+, C_+ \) denote the 1-rarefaction wave, 3-rarefaction wave, and the contact wave curve respectively, and

\[
R_-C_+R_+(v_-,u_-,\theta_-) \triangleq \{ (v,u,\theta) \in \Omega(v_-,u_-,\theta_-) \mid s \neq s_- \},
\]

\[
u \geq u_- - \int_{s_0}^{s_0 + v_0(v_-,\theta_-,\gamma)} \lambda_-(\eta,s_-) \, d\eta, \quad u \geq u_- - \int_{s_0}^{s_0 + v_0(v_-,\theta_-,\gamma)} \lambda_+(\eta,s) \, d\eta \}
\]

Due to [44], if \( \delta_1 \) in \((1.24) \) is suitably small, then there exists a positive constant
\( C = C(\theta_-, \delta_1) \) and a pair of points \((v_0,m_0,\theta_0)\) and \((v_0',m',\theta_0')\) in \( \Omega(v_-,u_-,\theta_-) \) such that

\[
\frac{R\theta_0 n}{v_0} = \frac{R\theta_0}{v_0'} = p_0\gamma, \quad |v_0^m - v_0| + |u_0^m - u_0| + |\theta_0^m - \theta_0| \leq C|\theta_- - \theta_+|.
\]

Moreover, the states \((v_0,m_0,\theta_0)\) and \((v_0',m',\theta_0')\) belong to the 1-rarefaction wave curve
\( R_-(v_-,u_-,\theta_-) \) and 3-rarefaction wave curve \( R_+(v_+,u_+,\theta_+) \) respectively, where

\[
R_+(v_+,u_+,\theta_+) \triangleq \{ s = s_+, u = u_+ - \int_{s_0}^{v_+} \lambda_+(\eta,s_+) \, d\eta, \quad v > v_+ \}
\]

with

\[
s = \frac{R}{\gamma - 1} \ln \frac{R\theta}{A} + R \ln v, \quad s_+ = \frac{R}{\gamma - 1} \ln \frac{R\theta_+}{A} + R \ln v_+,
\]

\[
\lambda_+(v,s) = \pm \sqrt{A\gamma v^{-\gamma-1} e^{(\gamma-1)s/R}}.
\]

The contact discontinuity wave curve \( C_+ \) is defined by

\[
C_+(v_0^m,u_0^m,\theta_0^m) \triangleq \{ (v,u,\theta)(t,x) \mid u = u_0, p = p_0, v \neq v_0 \}.
\]

The 1-rarefaction wave \((v_1^r,u_1^r,\theta_1^r)\) (respectively the 3-rarefaction wave \((v_1^r,u_1^r,\theta_1^r)\)\) connecting \((v_-,u_-,\theta_-)\) and \((v_0^m,u_0^m,\theta_0^m)\) (respectively \((v_0^m,u_0^m,\theta_0^m)\) and \((v_+,u_+,\theta_+)\)) is the weak solution of the Riemann problem of the Euler system \((1.5) \) with the Riemann initial data:
\[(v_{\pm}, u_{\pm}, \theta_{\pm})(0, x) = \begin{cases} 
(v^m_{\pm}, u^m_{\pm}, \theta^m_{\pm}), & \pm x < 0, \\
(v^0_{\pm}, u^0_{\pm}, \theta^0_{\pm}), & \pm x > 0.
\end{cases} \tag{1.28}\]

The contact discontinuity wave \((v^{cd}, u^{cd}, \theta^{cd})(t, x)\) connecting \((v^m_{\pm}, u^m_{\pm}, \theta^m_{\pm})\) and \((v^m_{\pm}, u^m_{\pm}, \theta^m_{\pm})\) takes the form
\[(v^{cd}, u^{cd}, \theta^{cd})(t, x) = \begin{cases} 
(v^m_{\pm}, u^m_{\pm}, \theta^m_{\pm}), & x < 0, t > 0, \\
(v^m_{\pm}, u^m_{\pm}, \theta^m_{\pm}), & x > 0, t > 0.
\end{cases} \tag{1.29}\]

Since the rarefaction waves \((v^\pm_{\pm}, u^\pm_{\pm}, \theta^\pm_{\pm})(t, x)\) are not smooth enough, to study the stability problem, we need to construct their smooth approximations. As [41], the smooth approximate rarefaction waves \((V^\pm_\pm, U^\pm_\pm, \Theta^\pm_\pm)(t, x)\) can be defined by
\[
\begin{align*}
\lambda_\pm(V^\pm_\pm, x_{\pm}) &= w_{\pm}(t, x), \\
U^\pm_\pm(t, x) &= u_{\pm} - \int_{x_{\pm}}^x \lambda_\pm(\eta, s_{\pm}) d\eta, \\
\Theta^\pm_\pm &= \Theta_\pm(V^\pm_\pm)^{\gamma-1}(V^\pm_\pm)^{1-\gamma},
\end{align*}
\tag{1.30}\]

where \(w_-(t, x)\) (respectively \(w_+(t, x)\)) is the solution of the Cauchy problem of the Burger equation:
\[
\begin{align*}
w_t + w w_x &= 0, \quad x \in \mathbb{R}, t > 0, \\
w(0, x) &= \frac{u_0 + w_0}{2} + \frac{w_0 - u_0}{2} \tanh x
\end{align*}
\tag{1.31}\]

with \(w_t = \lambda_- (v_-, s_-)\) and \(w_r = \lambda_- (v^m_-, s_-)\) (respectively \(w_l = \lambda_+ (v^m_+, s_+)\) and \(w_r = \lambda_+ (v_+, s_+)\)).

Let \((V^r, U^r, \Theta^r)(t, x)\) be the viscous contact wave defined in (1.12) with \((v_{\pm}, u_{\pm}, \theta_{\pm})\) replaced by \((v^m_{\pm}, u^0_{\pm}, \theta^m_{\pm})(t, x)\) respectively. Set
\[
\begin{pmatrix} V \\ U \\ \Theta \end{pmatrix}(t, x) = \begin{pmatrix} V^r_\pm + V^c + V^0_\pm \\ U^r_\pm + U^c + U^0_\pm \\ \Theta^r_\pm + \Theta^c + \Theta^0_\pm \end{pmatrix}(t, x) - \begin{pmatrix} v^m_\pm + u^m_\pm \\ 2u^m_\pm \\ \theta^m_\pm + \theta^m_\pm \end{pmatrix},
\tag{1.32}\]

then our second main result is as follows.

**Theorem 1.2 (Global stability of composite waves).** Suppose that the constant states \((v_{\pm}, u_{\pm}, \theta_{\pm})\) satisfy (1.24) for some small constant \(\delta_1 > 0\), and \(\|	heta_+ - \theta_-\| \leq l_0(\gamma - 1)\) for some \((\gamma - 1)\)-independent positive constant \(l_0\). Let \((V, U, \Theta)(t, x)\) be the combination of the viscous contact wave and approximate rarefaction waves defined in (1.32), and the conditions (ii)-(iv) of theorem 1.1 hold. Then there exist positive constants \(\varepsilon_1 \ll 1, \delta_2 \ll 1\) and \(C_i\) that depend only on \(V, \overline{V}, \Theta, \overline{\Theta}\) and the initial data \(N_0\) such that if \(0 < \varepsilon \leq \varepsilon_1\) and \(0 < \delta := \gamma - 1 \leq \delta_2\), the Cauchy problem (1.1) and (1.2) admits a unique global-in-time solution \((v, u, \theta)(t, x)\) satisfying
\[
\begin{align*}
\{(v - V, u - U, \theta - \Theta) \in C(0, +\infty; H^2(\mathbb{R}) \times H^1(\mathbb{R}) \times H^1(\mathbb{R})), \\
(v - V, u - U, \theta - \Theta) \in L^2(0, +\infty; H^2(\mathbb{R}) \times H^1(\mathbb{R}) \times H^1(\mathbb{R})),
\end{align*}
\tag{1.33}\]

\[
C_1^{-1} \leq \nu(t, x) \leq C_1, \quad \frac{\Theta}{2} \leq \theta(t, x) \leq 2\overline{\Theta}, \quad \forall (t, x) \in [0, \infty) \times \mathbb{R},
\tag{1.34}\]

and
\[
\lim_{t \to +\infty} \sup_{x \in \mathbb{R}} \{|(v - V, u - U, \theta - \Theta)(t, x)|\} = 0.
\tag{1.35}\]
Remark 1.2. Two remarks on theorems 1.1 and 1.2 are listed below.

1. From lemma 2.3 and (1.35), we have also the following asymptotic behavior of solutions: 
\[
\lim_{t \to +\infty} \sup_{x \in \mathbb{R}} \left( \left| \left( v - v^r_v - V^c - \frac{v^m}{\gamma} + \frac{v^m}{\gamma} \right)(t, x) \right| \right) = 0,
\]
\[
\left| \left( u - u^r_v - u^c + u^m \right)(t, x) \right| \left| \left( \theta - \theta^r_v - \theta^c + \theta^m \right)(t, x) \right| \leq C(\gamma - 1, \theta^0),
\]
where \((v^r_v, u^r_v, \theta^r_v)(t, x)\) and \((v^c, u^c, \theta^c)(t, x)\) are the 1-rarefaction wave and 3-rarefaction wave uniquely determined by (1.5) and (1.28), respectively.

2. In theorems 1.1 and 1.2, the smallness of \(\gamma - 1\) plays an important role in our analysis. Recently, Huang and Wang [30] studied the global stability of the combination of viscous contact wave with rarefaction waves for the Cauchy problem of the 1D compressible Navier–Stokes system without the condition that \(\gamma\) is close to 1. However, it seems that the method of [30] cannot be applied to the non-isothermal compressible Navier–Stokes–Korteweg system (1.1) because of some difficult nonlinear terms caused by the Korteweg tensor. The problem on how to get the global stability of basic waves for the non-isothermal compressible fluid models of Korteweg type with general constant \(\gamma > 1\) is under our current research.
Throughout this paper, we denote \( V \) and \( L \) the heat kernel and a domain decomposition technique were also presented in this section.

The second difficulty is to control the growth of solutions caused by the viscous contact wave, which is quite different from the case of the compressible Navier–Stokes system [21, 23]. Due to the effect of the Korteweg tensor, a third order spatial derivative of the viscous contact wave \( \Theta_{\text{vcc}}(t, x) \) appears in the estimate of \( \int_0^t \| \phi_{\text{vcc}}(\tau) \|^2 \, d\tau \). Consequently, \( \int_0^t \| \phi_{\text{vcc}}(\tau) \|^2 \, d\tau \) may be bounded by \( C(\gamma - 1)^{-\frac{1}{2}} \) for some positive constant \( C \) independent of \( t \) and \( \gamma - 1 \) (see (3.93)). Moreover, \( \int_0^t \| \phi_{\text{vcc}}(\tau) \|^2 \, d\tau \) also presents as a remainder term in the estimate of \( \| \psi(t) \| \) (see (3.89)). Since \( \gamma - 1 \) should be small in our setting, the terms \( \int_0^t \| \phi_{\text{vcc}}(\tau) \|^2 \, d\tau \) and \( \| \psi(t) \| \) will grow as \( \gamma - 1 \) decrease. Thus a difficulty problem is how to control the growth of solutions induced by \( \Theta_{\text{vcc}}(t, x) \). To overcome such a difficulty, we make the \( a \ priori \) assumptions (3.7) and (3.8), which together with a careful continuation argument can yield the desired energy-type estimates of solutions to the Cauchy problems (3.1) and (3.2).

The proof of theorem 1.2 is similar to that of theorem 1.1, but with an additional difficulty of controlling the interactions of waves from different families. With the aid of a domain decomposition technique developed in [25], and the properties of the approximate rarefaction waves and viscous contact wave, we can successfully overcome this difficulty and finally get the desired \( a \ priori \) estimates for solutions of the Cauchy problems (4.5) and (4.6).

Before concluding this section, we should mention that the nonlinear stability of basic waves for the compressible Navier–Stokes equations has been studied by many authors. We refer to [24, 37, 39, 40] for the nonlinear stability of viscous shock waves, [38, 41, 42] for the nonlinear stability of rarefaction waves, [21, 25–30] for the nonlinear stability of contact discontinuity, and [13, 22, 23, 25, 30, 47] for the nonlinear stability of composite waves.

The rest of this paper is organized as follows. In section 2, we list some basic properties of the viscous contact wave and rarefaction waves for later use. An important lemma concerning the heat kernel and a domain decomposition technique were also presented in this section. Sections 3 and 4 are devoted to the proof the main theorems 1.1 and 1.2, respectively. In the final appendix, we present the proof of lemma 3.2, which is important in the energy estimates.

**Notations.** Throughout this paper, we denote \( \delta := \gamma - 1 \) for notational simplicity. \( O(1) \) stands for some generic positive constant that may depend on \( V, V, \Theta, \Theta, m_0 \) and \( M_0 \) (\( m_0, M_0 \) are positive constants given in proposition 3.2), and \( c, C \) are generic positive constants that may depend on \( V, V, \Theta, \Theta, \Theta \) and the initial data, but independent of \( m_0 \) and \( M_0 \). All the constants \( c, C \) and \( O(1) \) in this paper are independent of \( t \) and \( \delta \). If the dependence need to be explicitly pointed out, the notation \( C(\cdot, \cdot, \cdot, \cdot, \cdot, \cdot, \cdot) \), or \( c(\cdot, \cdot, \cdot, \cdot, \cdot, \cdot, \cdot) \), or \( C(\cdot, \cdot, \cdot, \cdot, \cdot) \) \((i \in \mathbb{N})\) is used. For function spaces, \( L^p(\mathbb{R}) \) \((1 \leq p \leq +\infty) \) denotes the standard Lebesgue space with the norm

\[
\| f \|_{L^p(\mathbb{R})} = \left( \int_{\mathbb{R}} |f(x)|^p \, dx \right)^{\frac{1}{p}},
\]

and \( H^l(\mathbb{R}) \) is the usual \( l \)th order Sobolev space with its norm

\[
\| f \|_{H^l} = \left( \sum_{i=0}^l \| \partial_x^i f \|^2 \right)^{\frac{1}{2}} \quad \text{with} \quad \| \cdot \| := \| \cdot \|_{L^2(\mathbb{R})}.
\]

Finally, \( \| \cdot \|_{L^p_{\infty}} \) stands for the norm \( \| \cdot \|_{L^\infty([0,T] \times \mathbb{R})} \).

---

2. The second difficulty is to control the growth of solutions caused by the viscous contact wave, which is quite different from the case of the compressible Navier–Stokes system [21, 23]. Due to the effect of the Korteweg tensor, a third order spatial derivative of the viscous contact wave \( \Theta_{\text{vcc}}(t, x) \) appears in the estimate of \( \int_0^t \| \phi_{\text{vcc}}(\tau) \|^2 \, d\tau \). Consequently, 

\[
\int_0^t \| \phi_{\text{vcc}}(\tau) \|^2 \, d\tau \text{ may be bounded by } C(\gamma - 1)^{-\frac{1}{2}} \text{ for some positive constant } C \text{ independent of } t \text{ and } \gamma - 1 \text{ (see (3.93)). Moreover, } \int_0^t \| \phi_{\text{vcc}}(\tau) \|^2 \, d\tau \text{ also presents as a remainder term in the estimate of } \| \psi(t) \| \text{ (see (3.89)). Since } \gamma - 1 \text{ should be small in our setting, the terms } \int_0^t \| \phi_{\text{vcc}}(\tau) \|^2 \, d\tau \text{ and } \| \psi(t) \| \text{ will grow as } \gamma - 1 \text{ decrease. Thus a difficulty problem is how to control the growth of solutions induced by } \Theta_{\text{vcc}}(t, x). To overcome such a difficulty, we make the } a \ priori \text{ assumptions (3.7) and (3.8), which together with a careful continuation argument can yield the desired energy-type estimates of solutions to the Cauchy problems (3.1) and (3.2).}
2. Preliminaries

The viscous contact wave \((V, U, \Theta)(t, x)\) defined in (1.12) has the following properties.

**Lemma 2.1** ([21]). Let \(|\theta_+ - \theta_-| \leq l_0\delta\), where \(\delta = \gamma - 1\) and \(l_0 > 0\) is a positive constant independent of \(\delta\), then it holds that

1. \(|V - v_\pm| + |\Theta - \theta_\pm| \leq C_1\delta e^{-\frac{\alpha x^2}{\delta (1 + t)}}
2. |\partial_t^k V| + |\partial_t^k \Theta| \leq C_2\delta^{\frac{k-2}{4}} (1 + t)^{-\frac{k}{4}} e^{-\frac{\alpha x^2}{\delta (1 + t)}}, \quad \forall k \in \mathbb{Z}^+
3. |\partial_t^{-\frac{1}{2}} U| \leq C_3\delta^2 |\partial_t^\frac{1}{2} \Theta| \leq C_4\delta^{\frac{k-2}{4}} (1 + t)^{-\frac{k}{4}} e^{-\frac{\alpha x^2}{\delta (1 + t)}}, \quad \forall k \in \mathbb{Z}^+

where \(C_i, i = 0, 1, 2, 3, 4\) are positive constants depending only on \(\theta_\pm\).

The following lemma on the heat kernel will play an important role in the analysis of this paper, whose proof can be found in [25].

For \(\alpha \in (0, \frac{1}{2})\), we define

\[
\begin{align*}
\omega(t, x) = (1 + t)^{-\frac{1}{2}} \exp \left\{ - \frac{\alpha x^2}{\delta (1 + t)} \right\},
& g(t, x) = \int_{-\infty}^{\infty} \omega(t, y) \, dy.
\end{align*}
\]

Then it is easy to check that

\[
4\alpha g_t = \delta \omega_x, \quad \|g(t, \cdot)\|_{L^\infty} = \sqrt{\pi} \alpha^{-1/2} \delta^{1/2},
\]

and we have the following lemma.

**Lemma 2.2** ([25]). For any \(0 < T \leq \infty\), suppose that \(h(t, x)\) satisfies

\[
h \in L^\infty(0, T; L^2(\mathbb{R})), \quad h_x \in L^2(0, T; L^2(\mathbb{R})), \quad h_t \in L^2(0, T; H^{-1}(\mathbb{R})).
\]

Then the following estimate holds:

\[
\int_0^T \int_{\mathbb{R}} h^2 \omega^2 \, dx \, dt \leq 4\pi \frac{\alpha}{\delta} \|h(0, \cdot)\|^2 + 4\pi \delta \alpha^{-1} \int_0^T \|h_x(\cdot)\|^2 \, d\tau + \frac{8\alpha}{\delta} \int_0^T \langle h_t, h \omega^2 \rangle \, d\tau,
\]

where \(\langle \cdot, \cdot \rangle\) denotes the inner product on \(H^{-1} \times H^1\).

The solution \(w(t, x)\) of the Cauchy problem (1.31) has the following properties.

**Lemma 2.3** ([41]). For given \(w_l \in \mathbb{R}\) and \(\bar{w} > 0\), let \(w_r \in \{w | 0 \leq \bar{w} \leq w \leq w_l \}\). Then the problem (1.31) has a unique global smooth solution satisfying the following.

1. If \(w_l < w(t, x) < w_r, \; w_l > 0, \; x \in \mathbb{R}, \; t > 0\).
2. For any \(p \in [1, +\infty]\), there exists some positive constant \(C_2 = C_2(p, w_l, \bar{w})\) such that for \(\bar{w} > 0\) and \(t > 0\),

\[
\|w_l(t)\|_{L^p(\mathbb{R})} \leq C_2 \min \{\bar{w}, \bar{w}^{2+1/p}\}, \quad \|w_{2l}(t)\|_{L^p(\mathbb{R})} \leq C_2 \min \{\bar{w}, \bar{w}^{-1}\}.
\]

3. If \(w_l > 0\), for any \((t, x) \in [0, +\infty) \times (-\infty, 0]\),

\[
|w(t, x) - w_l| \leq \bar{w} e^{-2(|t| + |w_l|)}, \quad |w_x(t, x)| \leq 2\bar{w} e^{-2(|t| + |w_l|)}.
\]

4. If \(w_r < 0\), for any \((t, x) \in [0, +\infty) \times [0, +\infty)\),

\[
|w(t, x) - w_r| \leq \bar{w} e^{-2(|t| + |w_r|)}, \quad |w_x(t, x)| \leq 2\bar{w} e^{-2(|t| + |w_r|)}.
\]
(v) Let \( w'\left(\frac{t}{x}\right) \) be the Riemann solution of the scalar equation (1.31) with the Riemann initial data

\[
w(0, x) = \begin{cases} 
    w_l, & x < 0, \\
    w_r, & x > 0,
\end{cases}
\]

then we have

\[
\lim_{t \to +\infty} \sup_{x \in \mathbb{R}} \left| w(t, x) - w'\left(\frac{t}{x}\right) \right| = 0.
\]

In order to use lemma 2.3 to study the properties of the smooth rarefaction waves \((V^\pm_\pm, U^\pm_\pm, \Theta^\pm_\pm)\) constructed in (1.30) and the viscous contact wave \((V^c, U^c, \Theta^c)(t,x)\), we divided the domain \(\mathbb{R} \times (0, t) = \Omega_- \cup \Omega_c \cup \Omega_+\) with \(\Omega_\pm = \{(x, t) | \pm 2x > \pm \lambda_\pm (v^m_\pm, s_\pm) t\}\). \(\Omega_c = \{(x, t) | \lambda_-(v^m_-, s_-) t \leq 2x \leq \lambda_+(v^m_+, s_+) t\}\).

**Lemma 2.4 (25).** Assume that (1.25) holds. Then the smooth rarefaction waves \((V^\pm_\pm, U^\pm_\pm, \Theta^\pm_\pm)\) constructed in (1.30) and the viscous contact wave \((V^c, U^c, \Theta^c)(t,x)\) satisfy the following.

(i) \( (U^\pm_\pm) \geq 0, x \in \mathbb{R}, t > 0 \).

(ii) For any \( p \in [1, +\infty] \), there exists a positive constant \( C = C(p, v_-, u_-, \theta_-, \delta, l_0) \) such that for \( \delta = \gamma - 1 \) and \( t \geq 0 \),

\[
\begin{align*}
\| (V^\pm_\pm)(t) \|_{L^p(\mathbb{R})} & \leq C \min \{ \delta, \delta^{\frac{1}{2}} t^{-1+\frac{1}{p}} \}, \\
\| (\partial_t V^\pm_\pm, \partial_t^k U^\pm_\pm, \partial_t^k \Theta^\pm_\pm)(t) \|_{L^p(\mathbb{R})} & \leq C \min \{ \delta, t^{-1} \}, \quad k = 2, 3.
\end{align*}
\]

(iii) There exists some positive constant \( C = C(p, v_-, u_-, \theta_-, \delta, l_0) \) such that for \( \delta = \gamma - 1 \) and

\[
c_3 = \frac{1}{20} \min \left\{ \left| \lambda_-(v^m_-, s_-) \right|, \lambda_+(v^m_+, s_+), c_0 \lambda_2^2 (v^m_-, s_-), c_0 \lambda_3^2 (v^m_+, s_+) \right\},
\]

we have in \( \Omega_- \) that

\[
\left| (V^\pm_\pm)(x, t) \right| + \left| \Theta^\pm_\pm \right| \leq C \delta e^{-c \delta (|x|+t)},
\]

and in \( \Omega_+ \),

\[
\begin{align*}
|V^\pm_\pm| + |\Theta^\pm_\pm| & \leq C \delta^{\frac{1}{2}} e^{-c \delta (|x|+t)}, \\
|V^c - v^m_\pm| + |\Theta^c - \theta^m_\pm| + |U^c_\pm| & \leq C \delta e^{-c \delta (|x|+t)},
\end{align*}
\]

(iv) It holds that

\[
\lim_{t \to +\infty} \sup_{x \in \mathbb{R}} \left| (V^\pm_\pm, U^\pm_\pm, \Theta^\pm_\pm)(t,x) - (v^\pm_\pm, u^\pm_\pm, \theta^\pm_\pm) \left( \frac{t}{x} \right) \right| = 0.
\]
3. Proof of theorem 1.1

This section is devoted to proving theorem 1.1. To do so, we first reformulate the original problem, and then perform energy estimates on solutions to the reformulated system.

3.1. Reformulation of the problem

First, we define the perturbation \((\phi, \psi, \zeta)(t, x)\) by

\[
\phi(t, x) = v(t, x) - V(t, x), \quad \psi(t, x) = u(t, x) - U(t, x), \quad \zeta(t, x) = \theta(t, x) - \Theta(t, x),
\]

then it follows from (1.1) and (1.13) that

\[
\begin{cases}
\phi_t - \psi_t = 0, \\
\alpha_t + \left(\frac{R(2+\zeta)}{\sqrt{\gamma - 1}} - \frac{8\Theta}{\sqrt{\gamma}}\right) = \frac{\mu(v, \psi) u_t}{v} - \frac{\mu(v, \Theta) U_t}{v} + K_t - R_1, \\
\left(\frac{\mu(v, \psi)}{v} - \frac{\alpha(v, \Theta)}{\sqrt{\gamma}}\right) + \frac{\mu(v, \Theta) e^2}{v} - \frac{\mu(v, \Theta) e^2}{v} - R_2 + F + \frac{\alpha k w(v, \Theta)^2}{\sqrt{\gamma}} \Theta,
\end{cases}
\]

where \(K\) and \(F\) are defined in (1.3). System (3.1) is supplemented with the following initial data and far-field end state:

\[
\begin{aligned}
&\left(\phi, \psi, \zeta\right)(0, x) = \left(\phi_0, \psi_0, \zeta_0\right)(x) = (v - V, u - U, \theta - \Theta)(0, x), \\
&\left(\phi, \psi, \zeta\right)(t, \pm \infty) = 0.
\end{aligned}
\]  

(3.2)

We seek the solutions of the Cauchy problems (3.1) and (3.2) in the following set of functions:

\[
X(0, T; m_1, M_1, m_2, M_2) = \left\{ (\phi, \psi, \zeta)(t, x) \in C(0, T; H^2(\mathbb{R}) \times H^1(\mathbb{R}) \times H^1(\mathbb{R})), \quad (\phi_0, \psi_0, \zeta_0)(t, x) \in L^2(0, T; H^2(\mathbb{R}) \times H^1(\mathbb{R}) \times H^1(\mathbb{R})), \quad m_1 \leq \phi(t, x) + V(t, x) \leq M_1, \quad m_2 \leq \zeta(t, x) + \Theta(t, x) \leq M_2, \right\}
\]

where \(m_1, m_2, M_1, M_2\) and \(0 \leq T < +\infty\) are some positive constants. Then to prove theorem 1.1, it suffices to show the following theorem.

**Theorem 3.1.** Under the assumptions of theorem 1.1, there exist two small positive constants \(\varepsilon_0\) and \(\delta_0\) depending only on \(V\), \(\bar{V}\), \(\Theta\), \(\bar{\Theta}\), \(\|\phi_0\|_2\) and \(\left(\|\psi_0, \frac{\zeta_0}{\sqrt{\Theta}}\|\right)_1\) such that if \(0 < \varepsilon < \varepsilon_0\) and \(0 < \delta := \gamma + 1 \leq \delta_0\), the Cauchy problems (3.1) and (3.2) admit a unique global-in-time solution \((\phi, \psi, \zeta)(t, x)\) satisfying

\[
C_0^{-1} \leq v(t, x) \leq C_0, \quad \Theta^{-1} \leq \theta(t, x) \leq 2\bar{\Theta}, \quad \forall (t, x) \in [0, \infty) \times \mathbb{R},
\]

\[
\|\phi(t)\|_2^2 + \left\| \left(\psi, \frac{\zeta}{\sqrt{\Theta}}\right) (t) \right\|_1^2 + \int_0^t \|\phi_t, \psi_t, \zeta_t\|_1^2 \, d\tau \leq C_3 \left(\|\phi_0\|_2^2 + \left\| \left(\psi_0, \frac{\zeta_0}{\sqrt{\Theta}}\right) \right\|_1^2\right), \quad \forall t > 0,
\]

(3.4)
and
\[ \int_0^t \|\phi_{xx}(\tau)\|^2 \, d\tau \leq C_4 \left(1 + \delta^{-1}\right), \quad \forall t > 0. \] \hfill (3.5)

Moreover, the following large-time behavior of solutions hold:
\[ \lim_{t \to +\infty} \sup_{x \in \mathbb{R}} \{ ||(\phi, \psi, \zeta)(t, x)|| \} = 0. \] \hfill (3.6)

Here \( C_0 \) is a positive constant depending only on \( V, \overline{V}, \Theta, \overline{\Theta}, \|\phi_0\|_1 \) and \( ||(\psi_0, \frac{\zeta_0}{\sqrt{\delta}})|| \) and \( C_3 \) and \( C_4 \) are positive constants depending only on \( V, \overline{V}, \Theta, \overline{\Theta}, \|\phi_0\|_2 \) and \( ||(\psi_0, \frac{\zeta_0}{\sqrt{\delta}})|| \).

In order to prove theorem 3.1, we first give the following local existence result.

**Proposition 3.1 (Local existence).** Under the assumptions of theorem 1.1, there exists a sufficiently small positive constant \( t_1 \) depending only on \( V, \overline{V}, \Theta, \overline{\Theta}, ||\phi_0||_1, \|\psi_0\|_2, ||(\psi_0, \frac{\zeta_0}{\sqrt{\delta}})|| \) such that the Cauchy problem (3.1) and (3.2) admits a unique smooth solution \( (\phi, \psi, \zeta)(t, x) \in \mathbb{R} \). Moreover, the following large-time behavior of solutions hold:
\[ \lim_{t \to +\infty} \sup_{x \in \mathbb{R}} ||(\phi, \psi, \zeta)(t, x)|| = 0. \]

Proposition 3.1 can be obtained by using the dual argument and iteration technique, the proof of which is similar to that of theorem 2.1 in [19] and thus omitted here for brevity.

Suppose that the local solution \( (\phi, \psi, \zeta)(t, x) \) obtained in proposition 3.1 has been extended to the time step \( t = T > t_1 \) for some positive constant \( T > 0 \). To prove the global existence of solutions to the Cauchy problems (3.1) and (3.2), by the standard continuation argument, we need to establish the following \textit{a priori} estimates.

**Proposition 3.2 (A priori estimates).** Under the assumptions of theorem 3.1, suppose that \( (\phi, \psi, \zeta)(t, x) \in \mathbb{R} \) is a solution of the Cauchy problems (3.1) and (3.2) for some positive constants \( T, m_0, M_0, \Theta_0, \Theta_1, \) and satisfies the following a priori assumptions:
\[ \sup_{t \in [0, T]} \left\{ ||\phi(t)||_2^2 + \left\| \left( \psi, \frac{\zeta}{\sqrt{\delta}} \right)(t) \right\|_1^2 \right\} + \int_0^T \left( \|\phi_{xx}(\tau)\|_1^2 + ||(\psi_0, \zeta_0)(\tau)||_1^2 \right) \, d\tau \leq \mathcal{N}_1^2, \] \hfill (3.7)
\[ \int_0^T \|\phi_{xxx}(\tau)\|^2 \, d\tau \leq \mathcal{N}_2^2 \] \hfill (3.8)

for some positive constants \( \mathcal{N}_1 \) and \( \mathcal{N}_2 \). Then there exists a smooth positive function \( \Xi_1(m_0, M_0; V, \overline{V}, \Theta, \overline{\Theta}, N_0) \) that is increasing on both \( (m_0)^{-1} \) and \( M_0 \) and a positive constant \( \Xi_2(V, \overline{V}, \Theta, \overline{\Theta}, N_0) \) with \( N_0 := ||\phi_0||_1 + ||(\psi_0, \zeta_0)||_1 \) such that if
\[
\begin{align*}
&\Xi_1(m_0, M_0; \psi, \Theta, \Xi, N_0)N_1^{1/2} < \frac{1}{\delta}, \\
&\Xi_2(\psi, \Theta, \Xi, N_0)N_1^{1/2} \epsilon < \frac{1}{\delta}, \\
&N_2^{1/2} \delta < 1,
\end{align*}
\] (3.9)

then the inequalities in (3.3)–(3.5) hold for all \((t, x) \in [0, T] \times \mathbb{R}\).

**Proof of theorem 3.1.** Based on propositions 3.1 and 3.2, we now use the continuation argument to extend the unique local solution \((\phi, \psi, \zeta)(t, x)\) to be a global one, i.e. \(T = +\infty\). First, we have from proposition 3.1 that \((\phi, \psi, \zeta)(t, x) \in X(0, t_1; m_0, M_0, \Theta_0, \Theta_1)\) with \(m_0 = \frac{\bar{M}}{2}, M_0 = 2\bar{V}, \Theta_0 = \frac{\Psi}{2}, \Theta_1 = 2\bar{\Theta}\) and the *a priori* assumptions (3.7) and (3.8) hold with

\[
N_1 = N_2 = \sqrt{b} \left( \| \phi_0 \|_2 + \left\| \psi_0, \frac{\zeta_0}{\sqrt{\delta}} \right\|_1 \right) := \sqrt{b} N_0
\]

for all \(t \in [0, t_1]\), where \(t_1 > 0\) is a small positive constant given in proposition 3.1. Then it is easy to find two small positive constants \(\delta_0 > 0\) and \(\epsilon_0 > 0\) depending only on \(\bar{V}, \bar{V}, \bar{\Theta}, \bar{\Theta}\) and \(N_0\) such that

\[
\begin{align*}
&\Xi_1 \left( \frac{\bar{M}}{2}, 2\bar{V}; \psi, \Theta, \Xi, N_0 \right) (\sqrt{b} N_0)^{1/2} \delta < \frac{1}{\delta}, \\
&\Xi_2 \left( \psi, \Theta, \Xi, N_0 \right) (\sqrt{b} N_0)^{1/2} \epsilon < \frac{1}{\delta}, \\
&(\sqrt{b} N_0)^{1/2} \delta < 1.
\end{align*}
\] (3.10)

Thus if \(0 < \delta < \delta_0\) and \(0 < \epsilon < \epsilon_0\), then the inequalities in (3.3)–(3.5) hold for all \((t, x) \in [0, t_1] \times \mathbb{R}\).

Now we take \((\phi, \psi, \zeta)(t_1, x)\) as initial data, then by proposition 3.1, we can extend the local solution \((\phi, \psi, \zeta)(t, x)\) to the time step \(t = t_1 + t_2\) for some suitably small constant \(t_2 > 0\) depending only on \(\bar{V}, \bar{V}, \bar{\Theta}, \bar{\Theta}\) and \(N_0\). Moreover, \((\phi, \psi, \zeta)(t, x) \in X(t_1, t_1 + t_2; m_0, M_0, \Theta_0, \Theta_1)\) with \(m_0 = \frac{\bar{M}}{2}, M_0 = 2\bar{V}, \Theta_0 = \frac{\Psi}{2}, \Theta_1 = 4\bar{\Theta}\) and the *a priori* assumptions (3.7) and (3.8) hold with

\[
N_1 = \sqrt{C_3 N_0}, \quad N_2 = \sqrt{C_4 \left( 1 + \delta^{-2} \right)}
\]

for all \(t \in [t_1, t_1 + t_2]\). Then there exist two small positive constants \(\delta_2 > 0\) and \(\epsilon_2 > 0\) depending only on \(\bar{V}, \bar{V}, \bar{\Theta}, \bar{\Theta}\) and \(N_0\) such that

\[
\begin{align*}
&\Xi_1 \left( \frac{C_3^{-1}}{2}, 2C_0; \psi, \Theta, \Xi, N_0 \right) (\sqrt{C_3} N_0)^{1/2} \delta_2 < \frac{1}{\delta}, \\
&\Xi_2 \left( \psi, \Theta, \Xi, N_0 \right) (\sqrt{C_3} N_0)^{1/2} \epsilon_2 < \frac{1}{\delta}, \\
&C_4 \left( 1 + \delta_2^{-2} \right) \delta_2 < 1.
\end{align*}
\] (3.11)

Consequently, if \(0 < \delta < \delta_2\) and \(0 < \epsilon < \epsilon_2\), the inequalities in (3.3)–(3.5) hold for all \((t, x) \in [t_1, t_1 + t_2] \times \mathbb{R}\). Letting \(\delta_0 = \min\{\delta_1, \delta_2\}\) and \(\epsilon_0 = \min\{\epsilon_1, \epsilon_2\}\), we see that if \(0 < \delta < \delta_0\) and \(0 < \epsilon < \epsilon_0\), then the solution \((\phi, \psi, \zeta)(t, x) \in X(0, t_1 + t_2; \bar{C}_0^{-1}, C_0, \frac{\Psi}{2}, 2\bar{\Theta})\).

Next, taking \((\phi, \psi, \zeta)(t_1 + t_2, x)\) as initial data and exploiting proposition 3.1 again, we can extend the local solution \((\phi, \psi, \zeta)(t, x)\) to the time step \(t = t_1 + 2t_2\). By repeating the above
procedure, we can thus extend the local solution \((\phi, \psi, \zeta)(t, x)\) step by step to a global one provided that \(0 < \delta < \delta_0\) and \(0 < \varepsilon < \varepsilon_0\). As a by-product, the inequalities in (3.3)–(3.5) hold for all \((t, x) \in [0, +\infty) \times \mathbb{R}\).

Finally, the estimate (3.4) and the system (3.1) imply that
\[
\int_0^{+\infty} \left( \| (\phi, \psi, \zeta)(t) \|^2 + \left| \frac{d}{dt} \left( \| (\phi, \psi, \zeta)(t) \|^2 \right) \right| \right) dt < \infty, \tag{3.12}
\]
which, together with (3.4) and the Sobolev inequality:
\[
\| f(t) \|_{L^\infty} \leq \| f(t) \|_{L^\frac{1}{2}} \| f_\xi(t) \|_{L^\frac{1}{2}}, \quad \forall f(t, \cdot) \in H^1(\mathbb{R}) \tag{3.13}
\]
leads to the asymptotic behavior (3.6). This completes the proof of theorem 3.1. \(\square\)

### 3.2. Energy estimates

In this subsection, we shall prove proposition 3.2. First of all, notice that (3.7) implies
\[
\| \zeta(t) \|_1 \leq N_1 \delta^\frac{1}{2} \text{ for all } t \in [0, T], \text{ thus if } \delta > 0 \text{ is sufficiently small such that } N_1 \delta^\frac{1}{2} < \frac{\Theta}{2}, \text{ then we have}
\]
\[
\| (\theta - \Theta)(t, x) \| \leq \| \zeta(t) \|_{L^\infty(\mathbb{R})} \leq \| \zeta(t) \|_1 \leq \frac{\Theta}{2}, \quad \forall (t, x) \in [0, T] \times \mathbb{R}. \tag{3.14}
\]
Consequently
\[
\frac{\Theta}{2} \leq \Theta(t, x) - \frac{\Theta}{2} \leq \theta(t, x) \leq \Theta(t, x) + \frac{\Theta}{2} \leq 2\Theta, \quad \forall (t, x) \in [0, T] \times \mathbb{R}. \tag{3.15}
\]
Throughout this subsection, we always assume \(N_1 \delta^\frac{1}{2} < \frac{\Theta}{2}\) so that (3.14) and (3.15) hold. Moreover, we denote
\[
N_0 := \| \phi_0 \|_2 + \left\| \left( \psi_0, \frac{\delta_0}{\sqrt{\delta}} \right) \right\|_1, \quad N_{01} := \| \phi_0 \|_1 + \left\| \left( \psi_0, \frac{\delta_0}{\sqrt{\delta}} \right) \right\|_1,
\]
and assume that \(N_1 \geq N_0 > 1, N_{01} > 1\) and \(\delta < 1\) without loss of generality. In this subsection, we also use \(C(m_0, M_0)\) to denote some generic positive constant that may depend on \(V, \tilde{V}, \tilde{\Theta}, \tilde{\Theta}, m_0\) and \(M_0\), and \(C\) to stand for some generic positive constant which may depend on \(V, \tilde{V}, \tilde{\Theta}, \tilde{\Theta}\) and the initial data, but independent of \(m_0\) and \(M_0\).

Proposition 3.2 will obtained by a series of lemmas below. The following basic energy estimates is key for the proof of proposition 3.2.

**Lemma 3.1 (Basic energy estimates).** Under the assumptions of proposition 3.2, there exists a positive constant \(C(V, \tilde{V}, \tilde{\Theta}, \tilde{\Theta})\) and a positive constant \(C_5\) depending only on \(V, \tilde{V}, \tilde{\Theta}, \tilde{\Theta}, m_0, M_0\) such that
\[
\int_0^T \left[ R \Theta \Phi \left( \frac{\psi}{\sqrt{\theta}} \right) + \frac{\psi^2}{2} + \frac{R}{\delta} \Theta \Phi \left( \frac{\theta}{\bar{\sigma}} \right) + \frac{n(v, \theta)\psi^2}{\sigma} \right] \, dt + \int_0^T \int_\mathbb{R} \left( \frac{\mu(v, \theta)\psi^2}{\theta v} + \frac{\tilde{\delta}(v, \theta)\Theta \psi^2}{\sigma} \right) \, dx \, dt \leq C(V, \tilde{V}, \tilde{\Theta}, \tilde{\Theta}) \left( \left\| \left( \phi_0, \phi_0, \psi_0, \frac{\delta_0}{\sqrt{\delta}} \right) \right\|_1^2 + \int_0^T (1 + r)^{-1} \int_\mathbb{R} \left( \phi^2 + \frac{\psi^2}{\sigma} \right) e^{-\frac{\psi^2}{\sigma}} \, dx \, dt \right.
\]
\[
+ \left. C_5 \left( N_0 + \int_0^T \left\| (\phi_0, \phi_0, \psi_0, \zeta_0)(r) \right\|_2^2 \, dr + N_{01} \delta \right). \tag{3.16}
\]
where the function \(\Phi(\cdot)\) is defined by \(\Phi(s) = s - 1 - \ln s\).
Proof. Multiplying (3.1) by $-R\Theta(v^{-1} - V^{-1})$, (3.1) by $\psi$, and (3.1) by $\zeta \theta^{-1}$, we can obtain by direct calculation that

$$\left[ R\Theta\Phi \left( \frac{v}{V} \right) \right]_t + p_+ \Phi \left( \frac{V}{v} \right) V_t + R\Theta \left( \frac{1}{v} - \frac{1}{V} \right) = 0,$$

$$\left( \psi^2 \right)_t + \left\{ (p - p_+)\psi - \left( \frac{\mu(v, \theta)}{v} u_x - \frac{\mu(V, \Theta)}{V} U_x \right) \psi \right\}_x + \frac{\mu(v, \theta)}{v} \psi^2 = 0,$$

$$= R\Theta \left( \frac{1}{v} - \frac{1}{V} \right) + \frac{R\zeta}{v} \psi_x - \left( \frac{\mu(v, \theta)}{v} - \frac{\mu(V, \Theta)}{V} \right) U_x \psi_x + K_x \psi - R_1 \psi,$$

$$\left[ \frac{R}{\delta} \Theta\Phi \left( \frac{\theta}{\Theta} \right) \right] = \frac{R}{\delta} \left( 1 - \frac{\Theta}{\Theta} \right) \zeta - \frac{R}{\delta} \Phi \left( \frac{\theta}{\Theta} \right) \Theta,$$

$$= \frac{\kappa_0 v_0(v, \theta)}{2v^3} \zeta^2 \frac{\Theta_0^2}{\Theta} - \frac{Rc}{v} \psi_x + (p_+ - p) \frac{\zeta}{v} U_x - \frac{\bar{\alpha}(v, \theta) \Theta \zeta^2}{v \theta^2} + \left( \frac{\bar{\alpha}(v, \theta)}{v} - \frac{\bar{\alpha}(V, \Theta)}{V} \Theta \right) \frac{\zeta}{\Theta_x} + \frac{\bar{\alpha}(v, \theta) \zeta \Theta_x}{\theta^2} + \left( \frac{\mu(v, \theta)}{v} (U_x + \psi_x)^2 - \frac{\mu(V, \Theta)}{V} U_x^2 \right) \frac{\zeta}{\Theta_x} + \left( 1 - \frac{\Theta}{\Theta} \right) \zeta.$$ (3.19)

Adding the above three equations (3.17)–(3.19) together, we have

$$\left\{ R\Theta\Phi \left( \frac{v}{V} \right) + \frac{\psi^2}{2} + \frac{R}{\delta} \Theta\Phi \left( \frac{\theta}{\Theta} \right) \right\} + E_x + \frac{\mu(v, \theta) \Theta \psi_x^2}{v \theta} + \frac{\bar{\alpha}(v, \theta) \Theta \zeta^2}{v \theta^2} = Q_0 + Q_1 + Q_2 + Q_3,$$ (3.20)

where

$$E = (p - p_+)\psi - \left( \frac{\mu(v, \theta)}{v} u_x - \frac{\mu(V, \Theta)}{V} U_x \right) \psi - \left( \frac{\bar{\alpha}(v, \theta)}{v} - \frac{\bar{\alpha}(V, \Theta)}{V} \Theta \right) \frac{\zeta}{\Theta_x},$$

$$Q_0 = -p_+ \Phi \left( \frac{V}{v} \right) U_x - \frac{p_+}{\delta} \Phi \left( \frac{\theta}{\Theta} \right) U_x - \left( \frac{\mu(v, \theta)}{v} - \frac{\mu(V, \Theta)}{V} \right) U_x \psi_x,$$

$$+ (p_+ - p) \frac{\zeta}{v} U_x + \frac{\bar{\alpha}(v, \theta) \zeta \Theta_x}{\theta^2} + \left( \frac{\bar{\alpha}(v, \theta)}{v} - \frac{\bar{\alpha}(V, \Theta)}{V} \right) \left( \frac{\zeta}{\Theta_x} - \frac{\zeta \Theta_x}{\Theta} \right) \Theta_x + \left( \frac{\mu(v, \theta)}{v} (U_x + \psi_x)^2 - \frac{\mu(V, \Theta)}{V} U_x^2 \right) \frac{\zeta}{\Theta_x},$$

$$Q_1 = K_x \psi, \quad Q_2 = \frac{F}{\Theta} \zeta - R_2 \frac{\zeta}{\Theta} \quad Q_3 = \frac{\kappa_0 v_0(v, \theta)}{2v^3} v^2 \theta \zeta.$$ (3.21)

Integrating (3.20) over $[0, t] \times \mathbb{R}$ yields
\[
\int_{\mathbb{R}} \left\{ R\Theta \Phi \left( \frac{\nu}{\Theta} \right) + \frac{\psi^2}{2} + \frac{R}{\delta} \Theta \Phi \left( \frac{\theta}{\Theta} \right) \right\} \, dx + \int_{0}^{t} \int_{\mathbb{R}} \left( \frac{\mu(v, \theta)}{v\theta} \psi_t^2 + \frac{\widehat{\alpha}(v, \theta)\psi_t^2}{v\theta^2} \psi_x^2 \right) \, dx \, d\tau \\
= \int_{\mathbb{R}} \left[ R\Theta_0 \Phi \left( \frac{v_0}{\Theta_0} \right) + \frac{\psi_0^2}{2} + \frac{R}{\delta} \Theta_0 \Phi \left( \frac{\theta_0}{\Theta_0} \right) \right] \, dx + \int_{0}^{t} \int_{\mathbb{R}} (Q_0 + Q_1 + Q_2 + Q_3) \, dx \, d\tau. \tag{3.22}
\]

where \( v_0 = V(x, t)|_{t=0}, \Theta_0 = \Theta(t, x)|_{t=0} \).

By the convexity of \( \Phi(s) \) and the Cauchy inequality, we have

\[
|Q_0| \leq C(m_0, M_0) \left| \frac{\psi^2 U_x}{\delta} \right| + \left| \left( \frac{\zeta^2 U_x}{\delta} \right) \right|
+ |(\phi, \zeta) U_x \psi_x| + |(\phi, \zeta)| |U_x| \\
\leq \left( \frac{\mu(v, \theta)}{4v} \right) \psi^2 + \frac{\widehat{\alpha}(v, \theta)\psi_t^2}{v\theta^2} \psi_x^2 + C(m_0, M_0) \left( \sigma^2 + \left( \frac{\zeta^2}{\delta} \right) \right) \left( |U_x| + \Theta_1^2 \right), \tag{3.23}
\]

thus

\[
\left| \int_{0}^{t} \int_{\mathbb{R}} Q_0 \, dx \, d\tau \right| \leq \frac{1}{4} \int_{0}^{t} \int_{\mathbb{R}} \left( \frac{\mu(v, \theta)}{v\theta} \psi^2 \right) \, dx \, d\tau + \frac{1}{4} \int_{0}^{t} \int_{\mathbb{R}} \frac{\widehat{\alpha}(v, \theta)\psi_t^2}{v\theta^2} \psi_x^2 \, dx \, d\tau \\
+ C(m_0, M_0) \int_{0}^{t} \int_{\mathbb{R}} \left( \sigma^2 + \left( \frac{\zeta^2}{\delta} \right) \right) \left( |U_x| + \Theta_1^2 \right) \, dx \, d\tau. \tag{3.24}
\]

Using the definition of the Korteweg stress tensor \( K \), i.e. (1.3), and the continuity equations (3.1) and (1.1), we have by a direct computation that

\[
Q_1 = K_x \psi = \{ K \psi \}_x - K \psi_x \\
= \{ K \psi \}_x - \left\{ - \frac{\kappa(v, \theta)v_{xx}}{v^5} + \frac{5\kappa(v, \theta) - v_{xx}(v, \theta)}{2v^6} \psi_x^2 - \frac{\kappa_\theta(v, \theta)v_x\theta_x}{v^5} \right\} \psi_x \\
= \{ K \psi \}_x - \left\{ - \frac{\kappa(v, \theta)v_{xx}}{v^5} + \frac{5\kappa(v, \theta) - v_{xx}(v, \theta)}{2v^6} \psi_x^2 - \frac{\kappa_\theta(v, \theta)v_x\theta_x}{v^5} \right\} \phi_x \\
+ \left\{ - \frac{\kappa(v, \theta)v_{xx}}{v^5} + \frac{5\kappa(v, \theta) - v_{xx}(v, \theta)}{2v^6} \psi_x^2 - \frac{\kappa_\theta(v, \theta)v_x\theta_x}{v^5} \right\} \theta_x \\
+ \left\{ \frac{\kappa(v, \theta)}{v^5} - \frac{\kappa(v, \theta)}{v^5} + \frac{5\kappa(v, \theta) - v_{xx}(v, \theta)}{2v^6} \psi_x^2 - \frac{\kappa_\theta(v, \theta)v_x\theta_x}{v^5} \right\} V_x \\
- \kappa_v(v, \theta) \psi V_x + \kappa_\theta(v, \theta) \psi \theta V_x \\
+ \left\{ \frac{\kappa(v, \theta)}{v^5} - \frac{\kappa(v, \theta)}{v^5} + \frac{5\kappa(v, \theta) - v_{xx}(v, \theta)}{2v^6} \psi_x^2 - \frac{\kappa_\theta(v, \theta)v_x\theta_x}{v^5} \right\} V_t \\
= \{ \cdots \}_x - \left( \frac{\kappa(v, \theta)}{2v^5} \right) + \kappa_\theta \psi \theta \cdot \theta_x \\
+ \left( \frac{\kappa(v, \theta)}{2v^5} \right) + \frac{\kappa_\theta \psi \theta}{2v^5} \theta_x. \tag{3.25}
\]
Here and hereafter, \( \{\cdots\}_x \) denotes the terms that will disappear after integrating with respect to \( x \) and may change from line to line. Thus we have

\[
egin{align*}
\int_0^T \int_{\mathbb{R}} Q_1 \, dx \, d\tau &= - \int_{\mathbb{R}} \frac{\kappa(v, \theta)v^2}{2v^3} \, dx + \int_{\mathbb{R}} \frac{\kappa(v_0, \theta_0)v_0^2}{2v_0^3} \, dx + \int_0^T \int_{\mathbb{R}} \frac{\kappa_0\theta v^2}{2v^3} \, dx \, d\tau \\
&\quad + \int_0^T \int_{\mathbb{R}} \left\{ -\frac{\kappa(v, \theta)v_{x\tau}}{v^3} + \frac{5\kappa(v, \theta) - \kappa_0(v, \theta)}{2\sigma} v^2 \right\} V_1 \, dx \, d\tau \\
&:= - \int_{\mathbb{R}} \frac{\kappa(v, \theta)v^2}{2v^3} \, dx + \int_{\mathbb{R}} \frac{\kappa(v_0, \theta_0)v_0^2}{2v_0^3} \, dx + I_1 + I_2.
\end{align*}
\]

For \( I_1 \), notice that (3.3) implies

\[
\frac{\kappa_0\theta v^2}{2v^3} = \frac{\kappa_0v^2}{2v^3} \frac{1}{C_v - \frac{\theta}{2^\theta v^3} \frac{v^3}{C_v}} \left[ -p\theta_x + \left( \frac{\hat{\alpha}(v, \theta)\theta_x}{v} \right)_x \\
+ \frac{\mu(v, \theta)u_x^2}{v} + \frac{\theta_0\theta v_{x\tau}}{v^3} + \frac{\nu_0(v, \theta) - \kappa_0(v, \theta)}{2\sigma} \theta u_x v^2 \right] \\
- \frac{1}{(C_v - \frac{\theta}{2^\theta v^3} \frac{v^3}{C_v})^3} u_x - \frac{\hat{\alpha}\theta_x}{v} \left( 2\kappa_0 v_{x\tau} + \kappa_0 v_x + \kappa_0\theta v_x^2 \right) \\
+ \frac{\kappa v_x^2}{2v^3} \left( C_v - \frac{\theta}{2^\theta v^3} \frac{v^3}{C_v} \right) \frac{1}{2\sigma} \left[ \frac{\kappa_0\theta v_x^2}{2v^3} + \frac{\theta_0\theta v_x v_{x\tau}}{2v^3} \right] \\
+ \frac{\theta_0\theta v_{x\tau}v_x^2}{2v^3} - \frac{\theta_0\theta v_{x\tau}v_x^2}{v^3} - \frac{5\theta_0\theta v^2}{2v^3} \left[ \frac{\kappa_0\theta v_x^2}{2v^3} + \frac{\theta_0\theta v_x v_{x\tau}}{2v^3} \right] \\
\leq C(m_0, M_0) \delta \left( |v_x^2 u_x| + |v_x u_x^2| + |u_x v_x^2| + |v_x v_{x\tau}| \right) \\
+ \left( |v_x^4| + |v_x^4| + |v_x v_{x\tau}^2| + |v_x v_{x\tau}| + |v_x v_{x\tau}| \right) |u_x| \\
+ |v_x^2 u_x| + |v_x^2 v_x^2| + |v_x v_{x\tau}| + |v_x v_{x\tau}| + \{\cdots\}_x.
\]

where we have used \( \frac{1}{C_v - \frac{\theta}{2^\theta v^3} \frac{v^3}{C_v}} \leq \frac{1}{C_v} \leq C\delta \). Therefore,

\[
I_1 \leq C(m_0, M_0) \delta \int_0^T \int_{\mathbb{R}} \left( v_x^2 u_x + v_x^2 + v_x^4 \right) \, dx \, d\tau \\
+ C(m_0, M_0) \delta \int_0^T \int_{\mathbb{R}} \left( v_x^2 + 1 \right) \left( \left| v_x v_{x\tau} \theta_x \right| + |v_x^2 u_x| + |v_x v_{x\tau}^2| + \left| v_x v_{x\tau} \theta_x \right| \right) \, dx \, d\tau \\
+ C(m_0, M_0) \delta \int_0^T \int_{\mathbb{R}} \left( v_x^2 u_x^2 + (v_x^2 + v_x^4)|u_x v_{x\tau}| \right) \, dx \, d\tau \\
:= I_1^1 + I_1^2 + I_1^3.
\]
It follows from the Cauchy inequality, Young inequality, Sobolev inequality and lemma 2.1 that

\[
I_1^2 \leq C(m_0, M_0) \delta \int_0^t \left( 1 + \| \psi_x(\tau) \|_{L^\infty}^2 \right) \int_\mathbb{R} \left( \| \phi_x(\tau) \|_{L^\infty} \| \phi_x \psi_x + \phi_x^2 U_1^2 + V_1^2 \psi_1^2 + V_1^2 U_1 \| \right) \, d\tau d\tau \\
\leq C(m_0, M_0) \left( N_1^2 \delta \int_0^t \| (\phi_x, \psi_x)(\tau) \|^2 \, d\tau + N_1^2 \delta^2 \right).
\]

(3.29)

\[
I_1^2 \leq C(m_0, M_0) \delta \int_0^t \left( 1 + \| \psi_x(\tau) \|_{L^\infty}^2 \right) \int_\mathbb{R} \left( \| \phi_x(\tau) \|_{L^\infty} \| \phi_x \psi_x + \phi_x^2 U_1^2 + V_1^2 \psi_1^2 + V_1^2 U_1 \| \right) \, d\tau d\tau \\
\leq C(m_0, M_0) \left( N_1^2 \delta \int_0^t \| (\phi_x, \psi_x)(\tau) \|^2 \, d\tau + N_1^2 \delta^2 \right).
\]

(3.30)

Combining (3.28)–(3.31) yields

\[
I_1 \leq C(m_0, M_0) N_1^2 \delta \int_0^t \| (\phi_x, \psi_x, \psi_x, \phi_x)(\tau) \|^2 \, d\tau + C(m_0, M_0) N_1^2 \delta^2.
\]

(3.32)

Similarly, it holds that

\[
I_2 \leq C(m_0, M_0) \delta \int_0^t \int_\mathbb{R} \left( \| \phi_x \| + |V_{xx}| + |\phi_x^2| + |V_x^2 + \zeta_1^2 + \Theta_1^2| \right) \| \Theta_1 \| \, d\tau d\tau \\
\leq C(m_0, M_0) \delta \int_0^t \| (\phi_x, \phi_x, \zeta_1)(\tau) \|^2 \, d\tau + C(m_0, M_0) \delta^2.
\]

(3.33)

Substituting (3.32) and (3.33) into (3.26), we have

\[
\int_0^t \int Q_1 \, d\tau d\tau \leq - \int_\mathbb{R} \frac{\kappa(\nu, \theta) \psi_2^2}{2\eta^5} \, dx + \int_\mathbb{R} \frac{\kappa(\nu_0, \theta_0) \psi_2^2}{2\eta^5} \, dx \\
+ C(m_0, M_0) \left( N_1^2 \delta \int_0^t \| (\phi_x, \phi_x, \psi_x, \zeta_1)(\tau) \|^2 \, d\tau + N_1^2 \delta^2 \right).
\]

(3.34)

By using the estimate of $I_1$ and the fact that $\| \zeta(t) \|_{L^\infty} \leq \sup_{t \in [0, T]} \| \zeta(t) \|_1 \leq N_1 \sqrt{\delta},$
\[ \left| \int_0^t \int_R Q_3 \, dx \, d\tau \right| \leq C \left| \int_0^t \int_R \frac{\kappa_0 \tau \theta \psi^2_\tau}{2 \varepsilon^c} \, dx \, d\tau \right| \cdot \| \psi \|_{L^p_R}^2 \]
\[ \leq C(m_0, M_0) \left( N^0 \delta^{\frac{1}{2}} \int_0^t \| (\phi_\alpha, \phi_\xi, \psi_\tau) \|_R^2 \, d\tau + N^0 \delta^{\frac{1}{2}} \right). \]  
(3.35)

Finally, for the estimate of \( \int_0^t \int_R Q_2 \, dx \, d\tau \), notice that
\[ F \frac{\partial \zeta}{\partial \tau} = \left( \frac{\kappa_0 \tau \theta \psi_\tau - \mu \theta \psi_\tau}{\varepsilon^c} + \frac{\mu \theta \psi_\tau - \kappa_0 \theta \psi_\tau}{\varepsilon^c} \right) \zeta \]
\[ = \{ \cdots \} - \left( \frac{\kappa_0 \tau \theta \psi_\tau}{\varepsilon^c} \right) \alpha_\tau + \frac{\mu \theta \psi_\tau}{\varepsilon^c} \zeta \]
\[ = \{ \cdots \} - \frac{v^\varepsilon}{\varepsilon^c} (\kappa_0 \psi_\tau + \mu \theta \psi_\tau) + \kappa_0 \theta \psi_\tau + \frac{\mu \theta \psi_\tau}{\varepsilon^c} \zeta \]
\[ + \frac{1}{2} v^\varepsilon (\mu \theta \psi_\tau - \kappa_0 \theta \psi_\tau) \zeta^2 \frac{\partial \zeta}{\partial \tau} \]
\[ = \{ \cdots \} + O(1) \left( |\psi_\tau^2 \zeta| + |\theta \psi_\tau \psi_\xi| + |\psi_\tau \psi_\xi| + |\psi_\tau \psi_\tau| \right). \]

then similar to the estimates of (3.29)–(3.31), we obtain
\[ \left| \int_0^t \int_R F \frac{\partial \zeta}{\partial \tau} \, dx \, d\tau \right| \leq C(m_0, M_0) \int_0^t \int_R \left( \| \psi_\tau \|_R^2 + |\theta \psi_\tau \|_R^2 + |\psi_\xi \|_R^2 + \| \psi_\tau \|_R^2 \right) \, dx \, d\tau \]
\[ \leq C(m_0, M_0) \int_0^t \int_R \left( N^0 \delta^{\frac{1}{2}} (\psi_\tau^2 + \psi_\xi^2 + \theta \psi_\tau^2 + \psi_\xi^2) + \| \psi_\xi \|_R^2 \right) \, dx \, d\tau \]
\[ \leq C(m_0, M_0) N^0 \delta^{\frac{1}{2}} \int_0^t \int_R \left( \| \psi \|_R^2 \| \psi_\xi \|_R^2 + \| \psi_\tau \|_R^2 \| \psi_\xi \|_R^2 + \| \psi_\tau \|_R^2 \| \psi_\tau \|_R^2 \right) \, dx \, d\tau \]
\[ + C(m_0, M_0) \int_0^t \int_R \left( \| \psi \|_R^2 \| \psi_\tau \|_R^2 \| \psi_\xi \|_R^2 + \| \psi_\tau \|_R^2 \| \psi_\xi \|_R^2 \right) \, dx \, d\tau \]
\[ \leq C(m_0, M_0) \left( N^0 \delta^{\frac{1}{2}} \int_0^t \| (\psi, \psi_\xi, \psi_\tau) \|_R^2 \, d\tau + N^0 \delta^{\frac{1}{2}} \right). \]  
(3.36)

On the other hand, we derive from (1.14) that
\[ \left| \int_0^t \int_R \left( -R_1 \psi - \frac{R_2}{\varepsilon} \zeta \right) \, dx \, d\tau \right| \]
\[ \leq C \left| \int_0^t \int_R \left( |R_1 \psi| + |R_2 \zeta| \right) \, dx \, d\tau \right| \]
\[ \leq C \int_0^t \left( \| \psi(\tau) \|_R^2 \| \psi_\tau(\tau) \|_R^2 \| R_1 \|_{L^1} + \| \zeta(\tau) \|_R^2 \| \psi_\xi(\tau) \|_R^2 \| R_2 \|_{L^1} \right) \, d\tau \]
\[ \leq \frac{1}{4} \int_0^t \int_R \frac{\mu \theta \psi_\tau}{\varepsilon^c} \psi_\tau^2 \, dx \, d\tau + \frac{1}{4} \int_0^t \int_R \frac{\mu \theta \psi_\tau}{\varepsilon^c} \zeta^2 \, dx \, d\tau \]
\[ + \frac{1}{4} \int_0^t \int_R \left( \frac{\mu \theta \psi_\tau}{\varepsilon^c} \psi_\tau^2 + \frac{\mu \theta \psi_\tau}{\varepsilon^c} \zeta^2 \right) \, dx \, d\tau + C(m_0, M_0) N^0 \delta^{\frac{1}{2}}. \]  
(3.37)
Consequently,
\[
\left| \int_0^t \int_R Q_2 \, dx \right| \leq \left| \int_0^t \int_R \left( -R_1 \psi - \frac{R_2}{\theta} \right) \, dx \right| + \left| \int_0^t \int_R \frac{F}{\theta} \, dx \right|
\]
\[
\leq \frac{1}{4} \int_0^t \int_R \left( \frac{\mu(v, \theta) \Theta}{\theta v} \psi^2 + \frac{\tilde{\alpha}(v, \theta) \Theta}{v \theta^2} \zeta^2 \right) \, dx \, d\tau
\]
\[
+ C(m_0, M_0) \left( N_1^\delta \int_0^t \left\| (\phi_x, \phi_{xx}, \psi_x, \zeta_x) (\tau) \right\|^2 \, d\tau + N_1 \delta \right). \tag{3.38}
\]

Putting (3.24), (3.34), (3.35) and (3.38) into (3.22) we get (3.16) immediately. This finishes the proof of lemma 3.1.

For the remainder term \( \int_0^t (1 + \tau)^{-1} \int_R (\phi^2 + \frac{\zeta^2}{\delta}) \, e^{-\frac{\tau^2}{\pi (1 + \tau)}} \, dx \, d\tau \) in (3.16), we establish the following:

**Lemma 3.2.** Under the assumptions of proposition 3.2, there exist two positive constants \( C_6, C_7 > 1 \) depending only on \( V, \overline{V}, \Theta, \overline{\Theta}, m_0, M_0 \) such that
\[
C_6 N_1^\delta \delta \leq \frac{1}{2} \min \left\{ 2p_2^2, R^2, \Theta \right\},
\]
then it holds for all \( t \in [0, T] \)
\[
\int_0^t \int_R \left( \phi^2 + \psi^2 + \frac{\zeta^2}{\delta} \right) \, dx \, d\tau \leq C_7 N_1^\delta \delta^{- \frac{1}{2}} + C_7 N_1^\delta \delta^{- \frac{1}{2}} \int_0^t \left\| (\phi_x, \phi_{xx}, \psi_x, \zeta_x) (\tau) \right\|^2 \, d\tau. \tag{3.40}
\]

The proof of lemma 3.2 is given in the appendix, which is a technique similar to that of lemma 5 in [25].

Next, we estimate \( \left\| \frac{\mu(v, \theta) \phi_x}{v} (t) \right\| \).

**Lemma 3.3.** Under the assumptions of proposition 3.2, there exists a positive constant \( C(V, \overline{V}, \Theta, \overline{\Theta}) \) and a positive constant \( C_9 \) depending only on \( V, \overline{V}, \Theta, \overline{\Theta}, m_0, M_0 \) such that
\[
\int_R \frac{\mu^2(v, \theta)}{v} \phi^2 \, dx + \int_0^t \int_R \frac{\mu(v, \theta) \theta}{v^3} \phi^2 \, dx \, d\tau
\]
\[
+ \int_0^t \int_R \left( \left( \frac{\sqrt{\mu(v, \theta) \kappa(v, \theta)} \phi_{xx}}{v} \right)^2 + \frac{\kappa(v, \theta)}{\mu(v, \theta) v^2} \left[ \left( \frac{\mu(v, \theta) \phi_x}{v} \right)^2 \right] \right) \, dx \, d\tau
\]
\[
\leq C(V, \overline{V}, \Theta, \overline{\Theta}) \left\| \left( \phi_{t}, \psi_t, \frac{\phi_0}{\sqrt{\delta}}, \phi_{t}, \phi_{t}, \phi_{t} \right) \right\|^2 + C_9 \left( N_1^\delta \delta \int_0^t \left\| (\phi_x, \phi_{xx}, \psi_x, \zeta_x) (\tau) \right\|^2 \, d\tau \right). \tag{3.41}
\]

**Proof.** Using the equation:
\[
\left( \frac{\mu(v, \theta) u_a}{v} \right)_x = \left( \frac{\mu(v, \theta) v_x}{v} \right)_x = \left( \frac{\mu(v, \theta) V_x}{v} \right)_x + \frac{\mu(v, \theta) (v, \theta_i - \theta, u_a)}{v}.
\]
we can rewrite (3.1) as
\[
\left( \frac{\mu(v, \theta) \phi_x}{v} \right)_t - \psi_t + \frac{R\theta}{v^2} \phi_x = - \left( \frac{\mu(v, \theta) V_x}{v} \right)_t + \frac{R\zeta}{v^2} - \frac{R - p + \phi}{v} V_x + \frac{\mu_0 (v, \theta) (v_s \theta_t - \theta u_s)}{v} - K_x + U_t, \tag{3.42}
\]

Multiplying (3.42) by \(\frac{\mu(v, \theta) \phi_x}{v}\) yields
\[
\left( \frac{\mu^2(v, \theta) \phi_x^2}{2v^2} - \psi \frac{\mu(v, \theta) \phi_x}{v} \right)_t + \frac{R\mu(v, \theta) \phi_x^2}{v^2} = \{ \cdots \}_x + J_0 + J_1 + J_2 + J_3 + J_4, \tag{3.43}
\]

where
\[
J_0 = -\psi \phi_x \frac{\mu_0 (v, \theta)}{v} + \frac{\mu_0 \mu \theta}{v^2} V_x \phi_x - \frac{\mu_0 \mu \theta}{v^2} \theta \phi_x V_x,
J_1 = -\frac{\mu \mu_0 \theta u_x \phi_x}{v^2} + \frac{\mu_0 \theta \phi_x \psi_x}{v},
J_2 = \left( \frac{\mu}{v} - \frac{\mu_0}{v} \right) (-\psi \phi_x U_x + \psi \phi_x V_x) - \frac{\mu}{v} (R\zeta - p + \phi) V_x \phi_x
\]
\[
- \left( \frac{\mu(v, \theta)}{v} \right) \frac{\mu}{v} \phi_x V_x V_t - \frac{\mu^2}{v^2} \phi_x V_x U_t + \frac{\mu R}{v^2} \zeta \phi_x + \frac{\mu}{v} \phi_x U_t,
J_3 = -\left( \frac{\mu(v, \theta)}{v} \right) \frac{\mu}{v} \psi_x \phi_x V_x, \quad J_4 = -\frac{\mu}{v} \phi_x K_x.
\]

Integrating (3.43) with respect to \(t\) and \(x\) over \([0, t] \times \mathbb{R}\) gives
\[
\int_{\mathbb{R}} \int_0^t \int_{\mathbb{R}} R\mu(v, \theta) \phi_x^2 \text{d}x \text{d}t \text{d}\tau \leq \int_{\mathbb{R}} \int_{0}^{t} \frac{3\mu^2(v_0, \theta_0)}{4v_0^2} \phi_0^2 \text{d}x \text{d}t + \| \psi(t) \|^2 + \| \psi_0 \|^2 + \int_{\mathbb{R}} \int_0^t \int_{\mathbb{R}} \frac{\mu(v, \theta) \psi_x^2}{v} \text{d}x \text{d}t \text{d}\tau + \sum_{i=0}^4 \int_{\mathbb{R}} \int_{0}^{t} J_i \text{d}x \text{d}t \text{d}\tau, \tag{3.44}
\]

where we have used the following Cauchy inequalities:
\[
\int_{\mathbb{R}} \psi \frac{\mu(v, \theta) \phi_x}{v} \text{d}x \leq \| \psi(t) \|^2 + \frac{1}{4} \int_{\mathbb{R}} \frac{\mu^2(v, \theta) \phi_x^2}{v^2} \text{d}x,
\]
\[
\int_{\mathbb{R}} \psi_0 \frac{\mu(v_0, \theta_0) \phi_0}{v_0} \text{d}x \leq \| \psi_0 \|^2 + \frac{1}{4} \int_{\mathbb{R}} \frac{\mu^2(v_0, \theta_0) \phi_0^2}{v_0^2} \text{d}x.
\]

Now we estimate the terms \(\int_{\mathbb{R}} J_i \text{d}x \text{d}t, i = 0, 1, 2, 3\) one by one. First, using (1.1)_3, we have
\[ J_0 = \{ \cdots \} + \left( -\psi_0 v - \frac{\mu_0}{v^2} v_0 \phi_0 - \frac{\mu_0}{v^2} \phi_0 V_x \right) \frac{1}{(C_v - \frac{\mu_0}{v^2})} \left( -p_{\theta \phi} + \frac{\mu(v, \theta) \alpha^2}{\alpha} \right) \\
- \left( -\psi_0 v - \frac{\mu_0}{v^2} v_0 \phi_0 - \frac{\mu_0}{v^2} \phi_0 V_x \right) \frac{1}{(C_v - \frac{\mu_0}{v^2})} \left( \frac{\alpha(v, \theta) \phi_x}{\theta} + \frac{\theta_{\phi x} V_x}{v^2} u_x \right) \\
+ \left( \psi_0 - \frac{\mu_0}{v^2} v_0 \phi_0 + \frac{\mu_0}{v^2} \phi_0 V_x \right) \left[ \left( \frac{1}{C_v - \frac{\mu_0}{v^2}} \right) \frac{\alpha \phi_x}{\theta} + \left( \frac{\theta_{\phi x} V_x}{(C_v - \frac{\mu_0}{v^2})} \right) u_x \right] \\
= \{ \cdots \} + O(1) \left( |\psi_0| + |V_x \phi_0| + |\phi_0^2| \right) \left( |v_0| + |U_x| + |\psi|^2 \right) \\
+ O(1) \left( |\psi_0 \phi_x| + |\psi_0 \phi_x + V_x| + |\psi_0 \phi_x + \psi_\theta + |\phi_0^2| + |\phi_0^2 V_x| + |\phi_x^2| + |\phi_0 \phi_x| + |V_x \phi_0| + |V_x \phi_0 + V_x \phi_x| \right) \\
+ O(1) \left( |\psi_0 \phi_x + |\phi_0^2| + |\phi_0 \phi_x| + |V_x \phi_0| + |V_x \phi_0 + V_x \phi_x| \right) \left( |\theta_\phi + |\psi_v U_x| \right) \\
+ O(1) \left( |\psi_0 \phi_x + |\phi_0^2| + |\phi_0 \phi_x + |V_x \phi_0| + |V_x \phi_0 + V_x \phi_x| \right) \left( |\theta_\phi + |\psi_v U_x| \right) .
\]

Then similar to the estimate of \( I_1 \), we obtain

\[
\int_0^t \int_R J_0 \mathrm{d}x \mathrm{d}\tau \leq C(m_0, M_0) N_\delta^2 \int_0^t \left( \phi_{\phi x}, \phi_x, \psi_x, \zeta_x \right) (\tau) \right) \right) \mathrm{d}\tau \\
+ C(m_0, M_0) \delta^2 \int_0^t \int_R \psi^2 \omega^2 \mathrm{d}x \mathrm{d}\tau + C(m_0, M_0) N_\delta^2 \delta^3. \tag{3.45}
\]

The terms \( \int_0^t \int_R |J_i| \mathrm{d}x \mathrm{d}\tau, i = 1, 2, 3 \) can be controlled by the Sobolev inequality, the Cauchy inequality, the Young inequality and lemma 2.1 that

\[
\int_0^t \int_R |J_i| \mathrm{d}x \mathrm{d}\tau \leq C(m_0, M_0) \int_0^t \int_R \left( |\theta_{\psi_v U_x} + |\theta_{\psi_\psi_\psi} \right) \mathrm{d}x \mathrm{d}\tau \\
\leq C(m_0, M_0) \int_0^t \int_R \left( |\zeta_\phi \psi_x | + |\zeta_\phi \psi_x U_x| + |\psi_\phi \phi_x \psi_x| \\
+ |\psi_\phi \phi_x \psi_x| + |\psi_\psi_\psi \psi_x| \right) \mathrm{d}x \mathrm{d}\tau \\
\leq C(m_0, M_0) \int_0^t \left\{ \left( |\phi_\phi (\tau)| \right) \left( |\phi_\phi (\tau)| \right) \left( |\zeta_\phi (\tau)| \right) \right\} \int_0^t \left( |\phi_\phi (\tau)| \right) \mathrm{d}\tau \\
+ C(m_0, M_0) \delta \int_0^t \int_R \psi^2 \omega^2 \mathrm{d}x \mathrm{d}\tau + C \delta^2 \int_0^t \left( 1 + \tau \right)^{-\frac{1}{2}} \mathrm{d}\tau \\
\leq C(m_0, M_0) N_\delta \delta^{\frac{1}{2}} \int_0^t \left( |\phi_\phi (\tau)| \right) \left( |\phi_\phi (\tau)| \right) \mathrm{d}\tau \\
+ C(m_0, M_0) \delta \int_0^t \int_R \psi^2 \omega^2 \mathrm{d}x \mathrm{d}\tau + C(m_0, M_0) \delta^2. \tag{3.46}
\]
\[
\int_0^t \int_{\mathbb{R}} |J_2| \, dx \, d\tau \leq C(m_0, M_0) \int_0^t \int_{\mathbb{R}} (|\psi \phi_{x} U_2| + |\psi \psi_{x} V_2| + |(\zeta, \phi) V_2 \phi_{x}| + |\phi_{x} V_2| + |\phi V_2| \\
+ |\phi_{x} U_{1}|) \, dx \, d\tau + \int_0^t \int_{\mathbb{R}} \left| \frac{\mu R}{v^2} \zeta_{\phi} \right| \, dx \, d\tau
\]
\[
\leq \frac{1}{2} \int_0^t \int_{\mathbb{R}} \frac{\mu (v, \Theta) \psi^2}{v \theta^2} \, dx \, d\tau + C(m_0, M_0) \left( \delta \int_0^t \int_{\mathbb{R}} \psi^2 w^2 \, dx \, d\tau + \delta \right).
\]
\[
\leq \frac{1}{2} \int_0^t \int_{\mathbb{R}} \frac{\mu (v, \Theta) \psi^2}{v \theta^2} \, dx \, d\tau + C(m_0, M_0) \left( \delta \int_0^t \int_{\mathbb{R}} \psi^2 w^2 \, dx \, d\tau + \delta \right)
\]
and
\[
\int_0^t \int_{\mathbb{R}} |J_3| \, dx \, d\tau \leq \int_0^t \int_{\mathbb{R}} \frac{\mu (v, \Theta) \psi^2}{v \theta} \, dx \, d\tau + C(m_0, M_0) \delta \int_0^t \int_{\mathbb{R}} \|(\phi_{x}, \psi_{x})(\tau)\|^2 \, d\tau.
\]
\[-\frac{\mu}{v} \phi_x^2 = -\left[\frac{\sqrt{\mu R}}{v^2} \phi_x\right]^2 + \left[\frac{2}{3} \frac{\sqrt{\mu R}}{v^3} \phi_x\right] + \left[\frac{\sqrt{\mu R}}{v^3} \phi_x\right] \right\} \phi_x^4 + 2 \left[\frac{\sqrt{\mu R}}{v^3} \phi_x\right] \theta_x \phi_x^2 + 2 \frac{\sqrt{\mu R}}{v^4} \theta_x \phi_x^2 + 2 \left[\frac{\sqrt{\mu R}}{v^3} \phi_x\right]
\left[\frac{2}{3} \frac{\sqrt{\mu R}}{v^3} \phi_x\right] \theta_x \phi_x^2 + \left[\frac{\sqrt{\mu R}}{v^3} \phi_x\right] \phi_x^2 + \cdots \right]. \tag{3.50}

Substituting (3.50) into (3.49) leads to
\[-\frac{\mu}{v} \phi_x K_x = \{ \cdots \} - \left[\frac{\sqrt{\mu R}}{v^3} \phi_x\right] \right\} \phi_x^4 + \frac{1}{2} \{ \theta_x \} \phi_x^2 + 2 \phi_x \phi_x^2 + \left[\frac{\sqrt{\mu R}}{v^3} \phi_x\right] \phi_x^2 + \cdots \right]. \tag{3.51}

where the function \( f(v, \theta) \) is defined in (1.19).

Thus by the assumption (1.19) and some similar estimates as (3.46)–(3.48), we obtain
\[
\int_0^t \int_{\mathbb{R}} J_x \, dx \, dt \leq -\int_0^t \int_{\mathbb{R}} \left[\left(\frac{\sqrt{\mu R}}{v^3} \phi_x\right) \right] \left[\phi_x^4 + \frac{1}{2} \{ \theta_x \} \phi_x^2 + 2 \phi_x \phi_x^2 + \left[\frac{\sqrt{\mu R}}{v^3} \phi_x\right] \phi_x^2 + \cdots \right] \, dx \, dt + C(m_0, M_0) \left( \int_0^t \frac{\phi_x^2}{(\phi_x^2 + \phi_x^2 + \cdots) \, d\tau + \delta \frac{1}{2}} \right). \tag{3.52}

On the other hand, we can also deal with \(-\frac{\mu}{v} \phi_x K_x\) by
\[
-\frac{\mu}{v} \phi_x K_x = \{ \cdots \} - \left[\frac{\mu}{v} \phi_x\right] \right\} \phi_x^4 + \frac{1}{2} \{ \theta_x \} \phi_x^2 + 2 \phi_x \phi_x^2 + \left[\frac{\sqrt{\mu R}}{v^3} \phi_x\right] \phi_x^2 + \cdots \right]. \tag{3.53}

where the function \( g(v, \theta) \) is defined in (1.20). Thus if the condition (1.20) holds, then
\[
\int_0^t \int_{\mathbb{R}} J_x \, dx \, dt \leq -\int_0^t \int_{\mathbb{R}} \left[\left(\frac{\mu}{v} \phi_x\right) \right] \left[\phi_x^4 + \frac{1}{2} \{ \theta_x \} \phi_x^2 + 2 \phi_x \phi_x^2 + \left[\frac{\sqrt{\mu R}}{v^3} \phi_x\right] \phi_x^2 + \cdots \right] \, dx \, dt + C(m_0, M_0) \left( \int_0^t \frac{\phi_x^2}{(\phi_x^2 + \phi_x^2 + \cdots) \, d\tau + \delta \frac{1}{2}} \right). \tag{3.53}
\]
Equation (3.41) thus follows from (3.44)–(3.48), (3.52) and lemmas 3.1 and 3.2. This completes the proof of lemma 3.3.

As a direct consequence of lemmas 3.1–3.3, we have

**Corollary 3.1.** There exist a constant $C_{10} > 0$ depending only on $V, \overline{V}, \Theta, \overline{\Theta}, m_0, M_0$ and a constant $C_{11} > 0$ depending only on $V, \overline{V}, \Theta, \overline{\Theta}$ such that

$$C_{10} N_1^1 \delta^2 < 1,$$  \hspace{1cm} (3.54)

then it holds for all $t \in [0, T]$ that

$$\int_{\mathbb{R}} \left[ R \Theta \Phi \left( \frac{\psi}{\sqrt{v}} \right) + \frac{\psi^2}{2} + \frac{R}{\theta} \Theta \Phi \left( \frac{\theta}{\Theta} \right) \right] dx + \int_{\mathbb{R}} \left( \frac{\kappa (\psi, \theta) v^2}{v^2} + \frac{\mu (\psi, \theta)}{v^2} \phi_x^2 \right) dx$$

$$+ \int_{0}^{t} \int_{\mathbb{R}} \left( \frac{\mu \Theta \psi_x^2}{\psi \theta} + \frac{\mu \Theta}{\psi \theta^2} \phi_x^2 + \left[ \left( \frac{\sqrt{\mu \rho}}{\psi} \phi_x \right)_x \right]^2 + \frac{\kappa}{\mu \rho^2} \left[ \left( \frac{\mu \rho}{\psi} \phi_x \right)_x \right]^2 \right) dx \, d\tau$$

$$\leq C_{11} \left\| \left( \phi_0, \psi_0, \frac{\zeta_0}{\sqrt{\delta}}, \phi_{0x} \right) \right\|^2.$$  \hspace{1cm} (3.55)

**Proof.** Notice that the *a priori* assumption $m_0 \leq v(t, x) \leq M_0$ and (3.15) imply that

$$\int_{0}^{t} \int_{\mathbb{R}} \left( \frac{\mu (v, \theta) \Theta \psi_x^2}{v \theta} + \frac{\alpha (v, \theta) \Theta \phi_x^2}{v \theta^2} + \frac{\mu (v, \theta) \theta}{v^2} \phi_x^2 \right) d\tau \geq C_{12} \int_{0}^{t} \left\| (\phi, \psi, \zeta) (\tau) \right\|^2 d\tau,$$  \hspace{1cm} (3.56)

where $C_{12}$ is a positive constant depending only $V, \overline{V}, \Theta, \overline{\Theta}, m_0, M_0$. Without loss of generality, we can assume $C_{12} < 1$.

Thus by adding (3.16), (3.40) and (3.41) together, and choosing $\delta > 0$ sufficiently small such that

$$C_{13} N_1^3 \delta^2 \leq \frac{C_{12}}{2}, \quad C_{13} := 2C_3 C_7 + C_9,$$  \hspace{1cm} (3.57)

we have

$$A(t) \leq C(V, \overline{V}, \Theta, \overline{\Theta}) \left\| \left( \phi_0, \psi_0, \frac{\zeta_0}{\sqrt{\delta}}, \phi_{0x} \right) \right\|^2 + 1 + 2C_{13} N_1^3 \delta^2 \int_{0}^{t} \left\| \phi_{0x} (\tau) \right\|^2 d\tau$$

$$\leq C(V, \overline{V}, \Theta, \overline{\Theta}) \left\| \left( \phi_0, \psi_0, \frac{\zeta_0}{\sqrt{\delta}}, \phi_{0x} \right) \right\|^2 + 2C_{13} N_1^3 \delta^2 \int_{0}^{t} \left\| \phi_{0x} (\tau) \right\|^2 d\tau,$$  \hspace{1cm} (3.58)

where $A(t)$ denotes the formula on the left-hand side of (3.55), and we have used the assumption that $N_{01} := \left\| \left( \phi_0, \psi_0, \frac{\zeta_0}{\sqrt{\delta}}, \phi_{0x} \right) \right\| > 1$ without loss of generality.

To estimate the reminder term $\int_{0}^{t} \left\| \phi_x (\tau) \right\|^2 d\tau$ in (3.58), we rewrite
\[\phi_{ax}^2 = \frac{v^6}{\mu_k} \left[ \left( \frac{\sqrt{v^2/|\theta|}}{|\phi|} \right) \right]^2 - \left( \frac{\sqrt{v^2/|\theta|}}{|\phi|} \right)^2 v_\phi^2 \phi_x^2 - \left( \frac{\sqrt{v^2/|\theta|}}{|\phi|} \right)^2 \phi_x^2 \]

\[-2 \left( \frac{\sqrt{v^2/|\theta|}}{|\phi|} \right) \left( \frac{\sqrt{v^2/|\theta|}}{|\phi|} \right) v_\phi \theta_x \phi_x^2 - 2 \sqrt{v^2/|\theta|} \left( \frac{\sqrt{v^2/|\theta|}}{|\phi|} \right) v_\phi \phi_x \phi_x^2 - 2 \sqrt{v^2/|\theta|} \phi_x \phi_x \phi_x \]

\[= O(1) \left[ \left( \frac{\sqrt{v^2/|\theta|}}{|\phi|} \right) \right]^2 + \phi_x^4 + \Theta_x^2 \phi_x^2 + \zeta^2 \phi_x^2 + |\phi_x \phi_x| + |\phi_x | + |\phi_x \phi_x| \right] .
\]

\[(3.59)\]

Consequently, it follows from (3.59), the Cauchy inequality, and the Sobolev inequalities: \(\|\phi_x(t)\|_{L^\infty(\mathbb{R})} \leq \|\phi_x(t)\|^2 \|\phi_x(t)\|^2\frac{1}{2} \) and \(\|\phi_x(t)\|_{L^1(\mathbb{R})} \leq \|\phi_x(t)\|^2 \|\phi_x(t)\|^2\frac{1}{2} \) that

\[\int_0^t \|\phi_x(\tau)\|^2 d\tau \leq C \int_0^t \left[ \left( \frac{\sqrt{v^2/|\theta|}}{|\phi|} \right) \right]^2 \|\phi_x(\tau)\|^2 d\tau + C \int_0^t \left[ \left( \frac{\sqrt{v^2/|\theta|}}{|\phi|} \right) \right]^2 \|\phi_x(\tau)\|^2 d\tau + CN_1 \int_0^t \|\phi_x(\tau)\|^2 d\tau,
\]

which implies that

\[\int_0^t \|\phi_x(\tau)\|^2 d\tau \leq C_{14} \int_0^t \left[ \left( \frac{\sqrt{v^2/|\theta|}}{|\phi|} \right) \right]^2 \|\phi_x(\tau)\|^2 d\tau + C_{14} N_1 \int_0^t \|\phi_x(\tau)\|^2 d\tau ,\]

\[(3.60)\]

where \(C_{14}\) is a positive constant depending only on \(v, \overline{v}, \Theta, \Theta_0, m_0, M_0\).

Inserting (3.60) into (3.58), then (3.55) holds provided that \(\delta\) is sufficiently small such that

\[2C_{13}C_{14}N_1^1 \delta \frac{1}{2} \leq \frac{C_{12}}{4} .\]

(3.61)

Letting \(C_{10} = \max \{ 8C_{12}^{-1}C_{13}C_{14}, 2C_6 \left( \min \{ 2p^2, R^2, R \} \right)^{-1} \} \), then we finish the proof of corollary 3.1. \(\square\)

Based on corollary 3.1, we now show the uniform lower and upper bounds on \(v(x, t)\) by using Kanel’s method [32].

**Lemma 3.4.** Under the assumptions of corollary 3.1, there exist a positive constant \(C_{15}\) depending only on \(v, \overline{v}, \Theta, \Theta_0, m_0, M_0\) and a constant \(C_0 > 0\) depending only on \(v, \overline{v}, \Theta, \Theta_0\) and \(||(\phi_0, \psi_0, \overline{\psi}_0, \phi_{ax})||\) such if

\[C_{15}N_1^1 \delta \frac{1}{2} < 1 ,\]

then it holds

\[C_0^{-1} \leq v(t, x) \leq C_0, \quad \forall (t, x) \in [0, T] \times \mathbb{R}.\]

(3.63)
Proof. First, (3.55) implies that
\[
\int_{\mathbb{R}} \frac{\mu_1^2(v)}{v^2} \phi_x^2 \, dx + \int_{\mathbb{R}} \frac{\kappa_1(v)}{v^2} v_t^2 \, dx \leq C(\|\phi_0, \psi_0, \frac{\zeta_0}{\sqrt{\delta}} , \phi_{0x}\|)^2,
\]
(3.64)
where \(\mu_1(v) := \min_{\theta \in [\frac{\sqrt{2}}{\sqrt{3}}, \frac{\sqrt{2}}{\sqrt{3}}]} \{\mu(v, \theta)\}\) and \(\kappa_1(v) = \min_{\theta \in [\frac{\sqrt{2}}{\sqrt{3}}, \frac{\sqrt{2}}{\sqrt{3}}]} \{\kappa(v, \theta)\}\).

Let \(\tilde{v} = \frac{v}{\sqrt{2}}, \) then under the assumptions (1.17) and (1.18), there exists a positive constant \(C(V, V, \theta, \overline{\theta})\) such that
\[
\mu_1(v) \geq C(V, V, \theta, \overline{\theta}) \mu_1(\tilde{v}), \quad \kappa_1(v) \geq C(V, V, \theta, \overline{\theta}) \kappa_1(\tilde{v}).
\]
(3.65)
Consequently, it follows from (3.64) that
\[
\int_{\mathbb{R}} \frac{\mu_1^2(v)}{v^2} \phi_x^2 \, dx + \int_{\mathbb{R}} \frac{\kappa_1(v)}{v^2} v_t^2 \, dx \leq C(V, V, \theta, \overline{\theta}) \left( \|\phi_0, \psi_0, \frac{\zeta_0}{\sqrt{\delta}} , \phi_{0x}\| \right)^2.
\]
(3.66)

Set
\[
\Phi(\tilde{v}(t, x)) = \int_{1}^{\tilde{v}} \sqrt{\Phi(\eta)} \eta \, d\eta, \quad \Phi(\eta) = \eta - 1 - \ln \eta,
\]
(3.67)
then by the assumption (1.17),
\[
|\Phi(\tilde{v}(t, x))| \geq \begin{cases} A_1 |\ln \tilde{v}| - A_2, & \tilde{v} \to 0^+, \\ A_1 (\tilde{v} \to b + \ln \tilde{v}) - A_2, & \tilde{v} \to +\infty, \end{cases}
\]
(3.68)
where \(b < \frac{1}{2}, \) \(A_1 > 0, A_2 > 0\) are positive constants. On the other hand, it holds that
\[
|\Phi(\tilde{v}(t, x))| = \left| \int_{-\infty}^{\tilde{v}} \Phi(\eta) \, d\eta \right|
\leq \int_{\mathbb{R}} \sqrt{\Phi(\tilde{v})} \mu_1(\tilde{v}) \tilde{v}_x \, dx
\leq \left\| \sqrt{\Phi(\tilde{v})} (t) \right\| \left\| \mu_1(\tilde{v}) \tilde{v}_x (t) \right\|
\leq C(V, V, \theta, \overline{\theta}) \left( \|\phi_0, \psi_0, \frac{\zeta_0}{\sqrt{\delta}} , \phi_{0x}\| \right)^2,
\]
(3.69)
where we have used (3.55) and the fact that
\[
\left\| \frac{\mu_1(\tilde{v}) \tilde{v}_x (t)}{\tilde{v}} \right\|
= \left\| \left( \frac{\mu_1(\tilde{v}) \phi_x}{\tilde{v}} + \mu_1(\tilde{v}) \left( \frac{V_x}{\tilde{v}} - \frac{V}{\tilde{v}} \right) \right) (t) \right\|
\leq \left\| \frac{\mu_1(\tilde{v}) \phi_x}{\tilde{v}} (t) \right\| + C_{16}(V, \overline{V}, m_0, M_0) \delta \frac{1}{2}
\leq C(V, V, \theta, \overline{\theta}) \left( \|\phi_0, \psi_0, \frac{\zeta_0}{\sqrt{\delta}} , \phi_{0x}\| + 1 \right)
\leq C(V, V, \theta, \overline{\theta}) \left( \|\phi_0, \psi_0, \frac{\zeta_0}{\sqrt{\delta}} , \phi_{0x}\| \right)
\]
(3.70)
due to (3.66), the smallness of $\delta$ such that $C_{16}(V, \overline{V}, m_0, M_0) \delta^{\frac{1}{2}} < 1$, and the assumption that $N_{01} := \| (\psi_0, \phi_0, \frac{\zeta_0}{\sqrt{\delta}}, \phi_0) \| > 1$ without loss of generality.

Then (3.68) and (3.69) lead to
\[
V \exp \left\{ -C_{17}N_{01}^2 \right\} \leq v(t,x) \leq V \exp \left\{ C_{19}N_{01}^2 \right\}
\]
for all $(t,x) \in [0,T] \times \mathbb{R}$, where $C_{17}, C_{18}, C_{19}$ are positive constants depending only on $V, \overline{V}, \Theta, \overline{\Theta}$.

Now we suppose the condition (1.18) holds. Since
\[
\int_{\tilde{v}(t_0,x)}^{\tilde{v}(t,x)} \frac{\sqrt{\kappa_1(\tilde{\nu})}}{\tilde{\nu}} d\tilde{\nu} = \int_{\nu(t_0,x)}^{\nu(t,x)} \frac{\sqrt{\kappa_1(\nu)}}{\nu} d\nu - \int_{\nu(t_0,x)}^{\nu(t,x)} \frac{\sqrt{\kappa_1(\nu)}}{\nu} d\nu,
\]
we have from (3.66) that
\[
\left\| \frac{\sqrt{\kappa_1(\tilde{v})}}{\tilde{v}} \right\| \leq C(V, \overline{V}) \left\| \frac{\sqrt{\kappa_1(\nu)}}{\nu} \right\| + C(V, \overline{V}, m_0, M_0) \| v(t) \|
\leq C(V, \overline{V}) \left\| \frac{\sqrt{\kappa_1(\nu)}}{\nu} \right\| + C_{20}(V, \overline{V}, m_0, M_0) \delta^{\frac{1}{2}}
\leq C(V, \overline{V}, \Theta, \overline{\Theta}) \left\| (\phi_0, \psi_0, \frac{\zeta_0}{\sqrt{\delta}}, \phi_0) \right\| + 1
\leq C(V, \overline{V}, \Theta, \overline{\Theta}) \left\| (\phi_0, \psi_0, \frac{\zeta_0}{\sqrt{\delta}}, \phi_0) \right\|,
\]
provided that $\delta$ is sufficiently small such that $C_{20}(V, \overline{V}, m_0, M_0) \delta^{\frac{1}{2}} < 1$.

Define
\[
\Psi(\tilde{v}(t,x)) = \int_1^{\tilde{v}} \sqrt{\frac{\Phi(\eta)}{\eta^2}} \sqrt{\kappa_1(\eta)} d\eta,
\]
then the assumption (1.18) implies
\[
| \Psi(\tilde{v}(t,x)) | \geq \begin{cases} A_1 \ln \tilde{v} - A_4, & \tilde{v} \rightarrow 0^+, \\ A_3 (\tilde{v}^{1-\frac{d}{2}} + \ln \tilde{v}) - A_4, & \tilde{v} \rightarrow +\infty, \end{cases}
\]
where $d < -2, A_1 > 0, A_2 > 0$ are positive constants.

On the other hand, it follows from (3.55), (3.66) and (3.72) that
\[
|\Psi(\tilde{v}(t,x))| = \left| \int_{-\infty}^{\tilde{v}(t,x)} \Psi(y) \, dy \right| \\
\leq \int_{\mathbb{R}} \frac{\sqrt{\Phi(v)}}{v^2} \sqrt{\kappa_1(v)} \nu_1 \, dx \\
\leq \left\| \sqrt{\Phi(v)(t)} \right\| \left\| \frac{\sqrt{\kappa_1(v)} \nu_1(t)}{v^2} \right\| \\
\leq C(V, \nabla, \Theta, \bar{\Theta}) \left\| \left( \phi_0, \psi_0, \frac{\zeta_0}{\sqrt{\delta}}, \phi_0 \right) \right\|^2. \tag{3.74}
\]

Equation (3.73) together with (3.74) implies
\[
V \exp \{ -C_{21}N_{01}^2 \} \leq \nu(t,x) \leq C_{22}N_{01}^{-\frac{2}{m}} + \bar{V} \exp \{ C_{23}N_{01}^2 \}, \tag{3.75}
\]
where \( C_{21}, C_{22}, C_{23} \) are positive constants depending only on \( V, \nabla, \Theta, \bar{\Theta} \). Letting
\[
C_0 = \max \left\{ V^{-1} \exp \{ C_{17}N_{01}^2 \}, C_{18}N_{01}^{-\frac{2}{m}} + \bar{V} \exp \{ C_{19}N_{01}^2 \} \right\},
\]
and \( C_{15} = \max \{ C_{10}, C_{16}, C_{20} \} \), then we can get (3.63). This completes the proof of lemma 3.4. \( \square \)

Lemmas 3.1–3.4 imply the following corollary.

**Corollary 3.2.** Under the assumptions of lemma 3.4, there exists a positive constant \( C_{24} \) depending only on \( V, \nabla, \Theta, \bar{\Theta} \) and \( \|(\phi_0, \psi_0, \frac{\zeta_0}{\sqrt{\delta}}, \phi_0)\| \) such that
\[
\left\| \left( \phi, \psi, \frac{\zeta}{\sqrt{\delta}}, \phi_0 \right) (t) \right\|^2 + \int_0^t \| \left( \phi_x, \phi_{xx}, \psi_x, \zeta_x \right)(\tau) \|^2 \, d\tau \leq C_{24} \left\| \left( \phi_0, \psi_0, \frac{\zeta_0}{\sqrt{\delta}}, \phi_0 \right) \right\|^2. \tag{3.76}
\]

**Proof.** First, it is easy to see from corollary 3.1 and lemma 3.4 that
\[
\left\| \left( \phi, \psi, \frac{\zeta}{\sqrt{\delta}}, \phi_0 \right) (t) \right\|^2 + \int_0^t \| \left( \psi_x, \zeta_x \right)(\tau) \|^2 \, d\tau \leq C_{25} \left\| \left( \phi_0, \psi_0, \frac{\zeta_0}{\sqrt{\delta}}, \phi_0 \right) \right\|^2, \tag{3.77}
\]
where \( C_{25} \) is a positive constant depending only on \( V, \nabla, \Theta, \bar{\Theta} \) and \( \|(\phi_0, \psi_0, \frac{\zeta_0}{\sqrt{\delta}}, \phi_0)\| \).

On the other hand, it follows from (3.55) and (3.77), the Cauchy inequality, the Young inequality and the Sobolev inequality that
\[
\|\phi_s(t)\|^2 + \int_0^t \|\phi_s(\tau)\|^2 \, d\tau 
\leq C_{26} \left( \left\| \left( \phi_{0s}, \psi_{0s}, \theta_{0s}, \phi_{0s} \right) \right\| \right)^2 + C_{26} \int_0^t \int \left( |\phi_s^3 \theta_s| + |\phi_s^3 \zeta_s| + |\phi_s^3 \Theta_s| \right) \, d\tau + |\phi_s^3 \zeta_s| + |\phi_s^3 \Theta_s| + |\phi_s^3 \phi_s(\Theta_s)| + |\theta_s(\phi_s \phi_s)|),\right) \, d\tau
\leq C_{26} \left( \left\| \left( \phi_{0s}, \psi_{0s}, \theta_{0s}, \phi_{0s} \right) \right\| \right)^2 + C_{26} \int_0^t \left( \|\phi_s(\tau)\|^2 \|\phi_s(\tau)\| + \|\Theta_s(\tau)\|^2 \|\phi_s(\tau)\| \right) \, d\tau + \int_0^t \sup_{|\tau| \leq t} \{ \|\phi_s(\tau)\|^2 \|\zeta_s(\tau)\|^2 \} \, d\tau + C_{26} \left( \left\| \left( \phi_{0s}, \psi_{0s}, \theta_{0s}, \phi_{0s} \right) \right\| \right)^2 + C_{26} \int_0^t \left( \|\phi_s(\tau)\|^2 \|\phi_s(\tau)\| \right) \, d\tau + \int_0^t \sup_{|\tau| \leq t} \{ \|\phi_s(\tau)\|^2 \|\zeta_s(\tau)\|^2 \} \, d\tau + C_{26} \left( \left\| \left( \phi_{0s}, \psi_{0s}, \theta_{0s}, \phi_{0s} \right) \right\| \right)^2 + C_{26} \int_0^t \left( \|\phi_s(\tau)\|^2 \|\phi_s(\tau)\| \right) \, d\tau + \int_0^t \sup_{|\tau| \leq t} \{ \|\phi_s(\tau)\|^2 \|\zeta_s(\tau)\|^2 \} \, d\tau + C_{26} N_{01}^6.
\]

which implies that
\[
\|\phi_s(t)\|^2 + \int_0^T \|\phi_s(\tau)\|^2 \, d\tau \leq C_{27} N_{01}^6.
\]

Here \( C_{26}, C_{27} \) are positive constants depending only on \( V, V, \Theta, \Theta, \) and \( N_{01} \), and we have used the smallness of \( \delta \) such that \( N_1^2 \delta < 1 \) in the last step of (3.78), and the assumptions that \( N_{01} := \|\phi_0\|_1 + \|\psi_0, \theta_0\|_2 \| \) > 1 without loss of generality in (3.78) and (3.79).

Letting \( C_{24} = \max\{ C_{25}, C_{27} N_{01}^4 \} \), then (3.76) follows from (3.77) and (3.79) immediately. This completes the proof of corollary 3.2.

The following lemma gives the estimate on \( \| (\psi_s, \zeta_s, \sqrt{\delta}) (t) \| \).

**Lemma 3.5.** Under the assumptions of lemma 3.4, there exists a positive constant \( C_{28} \) depending only on \( V, V, \Theta, \Theta, \) and \( N_{01} \) and a positive constant \( C_{29} \) depending only on \( V, V, \Theta, \Theta, \) and \( N_{02} \) such that if
\[
C_{28} N_1^4 \delta^2 < \frac{1}{3}, \quad C_{28} N_1^6 \eta^2 < \frac{1}{3}, \quad N_2^2 \delta^2 < 1,
\]
then it holds for all \( t \in [0, T] \) that
\[
\left( \left( \phi_{ss}, \psi_{ss}, -\zeta_s, \sqrt{\delta} \right) (t) \right)^2 + \int_0^t \| (\psi_{ss}, \zeta_{ss}) (\tau) \|^2 \, d\tau \leq C_{28} N_{02}^2.
\]
Proof. Multiplying (3.1) by \(-\psi_{xx}\) and using (3.1), we have
\[
\left(\frac{\psi_{x}^{2}}{2}\right) + \left(\frac{\kappa(\psi, \theta)\phi_{xx}^{2}}{2v^{2}}\right) + \frac{\mu(\psi, \theta)}{v}\psi_{xx} = \{\cdot\}_{x} + J_{5} + J_{6} + J_{7} + J_{8}.
\] (3.82)

where
\[
J_{5} = -\left(\frac{\mu(\psi, \theta)}{v}\right)\psi_{xx} - \left(\frac{\mu(\psi, \theta)}{v}\right)_{\theta}\psi_{xx} + \frac{R\phi_{x} - p_{\psi}\phi_{x}}{v^{2}}\psi_{xx}

+ \frac{R\phi_{x} - p_{\psi}\phi_{x}}{v^{2}}\psi_{xx} - \left[\left(\frac{\mu(\psi, \theta)}{v}\right)\phi_{x}\psi_{xx} \right] + \frac{R\phi_{x} - p_{\psi}\phi_{x}}{v^{2}}\psi_{xx}.
\]

Integrating (3.82) over \([0, \tau] \times \mathbb{R}\) gives
\[
\frac{1}{2} \int_{\mathbb{R}} \left(\psi_{x}^{2} + \frac{\kappa(\psi, \theta)\phi_{xx}^{2}}{v^{2}}\right) \, dx + \int_{0}^{\tau} \int_{\mathbb{R}} \frac{\mu(\psi, \theta)}{v}\psi_{xx}^{2} \, dx \, dt

= \frac{1}{2} \int_{\mathbb{R}} \left(\psi_{0x}^{2} + \frac{\kappa(\psi_{0}, \theta_{0})\phi_{0xx}^{2}}{v_{0}^{2}}\right) \, dx + \int_{0}^{\tau} \sum_{l=5}^{8} J_{l} \, dx \, dt.
\] (3.83)

We derive from the Cauchy inequality, the Sobolev inequality and lemmas 2.1 and 3.4 that
\[
\left|\int_{0}^{\tau} J_{5} \, dx \right| \leq \eta \int_{0}^{\tau} \left\|\psi_{x}(\tau)\right\|^{2} \, d\tau + C_{\eta} \int_{0}^{\tau} \int_{\mathbb{R}} \left(v_{0}^{2}\psi_{x}^{2} + \theta_{x}^{2}\psi_{x}^{2} + |(\phi, \zeta)|U_{x}^{2} + |(\phi, \zeta)\theta|U_{x}^{2}

+ |(\phi, \zeta)U_{xx}|^{2} + \zeta_{x}^{2} + \phi_{x}^{2} + |(\phi, \zeta)|^{2}\phi_{x}^{2} + |(\phi, \zeta)|^{2}\phi_{xx}^{2} + R_{x}^{2}\right) \, dx \, dt

\leq \eta \int_{0}^{\tau} \left\|\psi_{x}(\tau)\right\|^{2} \, d\tau + C_{\eta} \int_{0}^{\tau} \int_{\mathbb{R}} \left\{\left\|\psi_{x}(\tau)\right\|^{2} + \left\|\psi_{x}(\tau)\right\|^{2} \sup_{0 \leq \tau < t} \left\|\phi_{x}(\tau)\right\|^{2} + \left\|\phi_{x}, \psi_{x}, \zeta_{x}\right\|(\tau)\right\}^{2}

\right. + \left. \left\|\psi_{x}(\tau)\right\|^{2} \sup_{0 \leq \tau < t} \left\|\zeta_{x}(\tau)\right\|^{2} + \left\|\phi_{x}, \psi_{x}, \zeta_{x}\right\|(\tau)\right\}^{2} + \left\|U_{x}(\tau)\right\|^{2} \right\} \, d\tau

+ C_{\eta} \int_{0}^{\tau} \int_{\mathbb{R}} \left\|(\phi, \zeta)\right\|^{2} w_{x} \, dx \, dt + C_{\eta} \sup_{0 \leq \tau < t} \left\|\phi_{x}, \psi_{x}, \zeta_{x}\right\|(\tau)\right\}^{2} \, d\tau

\leq 2\eta \int_{0}^{\tau} \left\|\psi_{x}(\tau)\right\|^{2} \, d\tau + C_{\eta}N_{01}^{2} \int_{0}^{\tau} \left\|\phi_{x}, \psi_{x}, \zeta_{x}\right\|(\tau)\right\}^{2} \, d\tau

+ C_{\eta} \int_{0}^{\tau} \int_{\mathbb{R}} \left\|(\phi, \zeta)\right\|^{2} w_{x} \, dx \, dt + C_{\eta}N_{01}^{2} \delta^{2},
\] (3.84)
\[ \left| \int_0^T \int_R J_0 \, dx \, d\tau \right| \leq C \int_0^T \left\{ \left( (|v| + |U_c|)|\phi_a|^2 + (|v| + |v|_\infty)|\psi_a| + (|v|_\theta + |v|_\theta^2)|\psi_a| \right) \right\} \, dx \, d\tau \\
\leq C \int_0^T \left\{ \left( \|v\|_\infty \|\phi_a\|_\infty^2 + \|\phi_a\|_\infty \|\phi_a\|_\infty^2 \right) \right\} \, dx \, d\tau \\
+ \frac{n}{2} \int_0^T \|\phi_a\|_\infty^2 \, dx \, d\tau + C \int_0^T \left( \phi_a^2 + V_a^2 + \phi_a^2 \phi_a^2 + V_a^2 \phi_a^2 + V_a^2 \phi_a^2 + V_a^2 \phi_a^2 \\
+ \phi_a^2 \phi_a^2 + \phi_a^2 \phi_a^2 + \phi_a^2 \phi_a^2 + \phi_a^2 \phi_a^2 + \phi_a^2 \phi_a^2 \right) \, dx \, d\tau \\
\leq \eta \int_0^T \|\phi_a\|_\infty^2 \, dx \, d\tau + C \eta \int_0^T \left( \|\phi_a\|_\infty^4 + \|\phi_a\|_\infty^2 \right) \, dx \, d\tau \\
+ C \int_0^T \left( \|\phi_a\|_\infty^4 + \|\phi_a\|_\infty^2 \right) \, dx \, d\tau + C \delta^2. \tag{3.85} \]

For \( \int_0^T \int_R J_0 \, dx \, d\tau \), notice that
\[ |\theta_l| = \left| \frac{1}{C_v - \frac{2}{\eta_0} \phi_a \frac{v}{\eta_0}} \right| \left| -pu + \langle \gamma(v, \theta) \theta_l \rangle \right| + \frac{\mu(v, \theta) u^2}{\nu} + \frac{v \phi_{\theta l} - \eta_0}{2 \nu} \phi_{\theta l} + \frac{\theta \phi_{\theta l} u_{\eta l}}{\nu} \right| \\
\leq C \delta (|u| + |\theta_a| + |v|_\alpha + |\theta_a^2| + u^2 + |u|_\alpha^2 + |v|_\alpha^2), \]

thus we have
\[ \left| \int_0^T \int_R J_0 \, dx \, d\tau \right| \leq C \delta \int_0^T \int_R \left( |u| + |\theta_a| + |v|_\alpha + |\theta_a^2| + u^2 + |u|_\alpha^2 + |v|_\alpha^2 \right) \, dx \, d\tau \\
\leq C \delta \int_0^T \int_R \left( \|v\|_\infty \|\phi_a\|_\infty^2 + \|\phi_a\|_\infty \|\phi_a\|_\infty^2 \right) \, dx \, d\tau \\
\leq C \delta \int_0^T \left( \|\phi_a\|_\infty^4 + \|\phi_a\|_\infty^2 \right) \, dx \, d\tau. \tag{3.86} \]

For (3.86), we only deal with the most difficulty terms \( \int_0^T \int_R \|\phi_a \phi_a^2 \| \, dx \, d\tau \) and \( \int_0^T \int_R \|\phi_a \phi_a \phi_a \| \, dx \, d\tau \), the other terms can be estimated similarly as (3.84) and (3.85). In fact, 
\[ \int_0^T \int_R \|\phi_a \phi_a \| \, dx \, d\tau \leq \int_0^T \left( \|\phi_a\|_\infty \|\phi_a\|_\infty \|\phi_a\|_\infty \right) \sup_{0 \leq \tau \leq T} \left\{ \|\phi_a\|_\infty \right\} \, dx \, d\tau \\
\leq \sup_{0 \leq \tau \leq T} \left\{ \|\phi_a\|_\infty \right\} \int_0^T \left( \|\phi_a\|_\infty \|\phi_a\|_\infty \right) \, dx \, d\tau \\
\leq \sup_{0 \leq \tau \leq T} \left\{ \|\phi_a\|_\infty \right\} \int_0^T \left( \|\phi_a\|_\infty \|\phi_a\|_\infty \right) \, dx \, d\tau \leq \int_0^T \|\phi_a\|_\infty \, dx \, d\tau. 
\]
Consequently, it holds that
\[
\left| \int_0^t \int_\mathbb{R} J_7 \, dx \, d\tau \right| \leq CN_1^2 \delta \int_0^t \left( \| \psi_{xx}, \zeta_{xx}, \psi_x(\tau) \| \right)^2 \, d\tau + CN_1^4 \delta \int_0^t \| \phi_{xx}(\tau) \| \, d\tau. \tag{3.87}
\]

Similarly,
\[
\left| \int_0^t \int_\mathbb{R} J_8 \, dx \, d\tau \right| \leq \eta \int_0^t \| \psi_{xx}(\tau) \| \, d\tau + C_\eta \varepsilon \int_0^t \int_\mathbb{R} \left( \phi_{xx}^2 \delta^2 + \phi_{xx}^2 \Theta^2_\delta + V_{xx}^2 \phi_{xx}^2 + \phi_{xx}^2 \Theta^2_\delta + V_{xx}^2 \phi_{xx}^2 + V_{xx}^2 \Theta^2_\delta \right) \, dx \, d\tau
\leq \eta \int_0^t \| \psi_{xx}(\tau) \| \, d\tau + C_\eta \int_0^t \left( \| \phi_{xx}, \phi_x, \zeta_x, \sqrt{\delta} \zeta_{xx}(\tau) \| \right)^2 \, d\tau + C_\eta \delta
+ C_\eta \varepsilon \int_0^t \left\{ \| \zeta_x(\tau) \| \| \zeta_{xx}(\tau) \| \| \phi_{xx}(\tau) \| + \| \phi_x(\tau) \| \| \phi_{xx}(\tau) \| \| \zeta_{xx}(\tau) \| \right\} \, d\tau
\leq \eta \int_0^t \| \psi_{xx}(\tau) \| \, d\tau + C_\eta \int_0^t \left( \| \phi_{xx}, \phi_x, \zeta_x \| \right)^2 \, d\tau + C_\eta \delta
+ C_\eta \varepsilon \int_0^t \| \phi_{xx}(\tau) \| \delta^2 \, d\tau + C_\eta N_1^2 \delta \int_0^t \| \zeta_{xx}(\tau) \| \, d\tau \tag{3.88}
\]
where we have used the assumption (1.16)_2.

Combining (3.83)–(3.88) and using corollary 3.2, the \emph{a priori} assumptions (3.7) and (3.8) and the smallness of \( \eta \), we obtain
\[
\left\| (\phi_{xx}, \psi_x)(t) \right\|^2 + \int_0^t \| \psi_{xx}(\tau) \| \, d\tau \leq C \left\| (\psi_{xx}, \phi_{xx})(0) \right\|^2 + N_0^6 \int_0^t \left( \| \phi_x, \psi_x, \zeta_x, \phi_{xx} \| \right)^2 \, d\tau + C_30 N_1^2 \delta \int_0^t \| \phi_{xx}(\tau) \| \, d\tau
+ C_30 N_1^4 \delta \int_0^t \| \phi_{xx}(\tau) \| \, d\tau + C_30 \int_0^t \| \phi_{xx}(\tau) \| \delta^2 \, d\tau
\leq C \left\| (\phi_{xx}, \psi_x, \phi_{xx}, \psi_{xx})(0) \right\|^2 + N_0^6 \left( \delta + \varepsilon + N_0^2 \delta \right) + C_30 \int_0^t \| \phi_{xx}(\tau) \| \delta^2 \, d\tau
\leq C \left\| (\phi_{xx}, \psi_x, \phi_{xx}, \psi_{xx})(0) \right\|^2 + N_0^6 \delta + C_38 \int_0^t \| \phi_{xx}(\tau) \| \delta^2 \, d\tau \tag{3.89}
\]
for all \( t \in [0, T] \), provided that
\[
C_30 N_1^4 \delta^2 < \frac{1}{3}, \quad C_30 N_1^4 \varepsilon < \frac{1}{3}, \quad N_0^2 \delta^2 < 1
\]
holds, where \( C_30 \) is a positive constant depending only on \( V, V, \Theta, \bar{\Theta}, \bar{\Theta} \) and \( N_0 \).

Then Gronwall’s inequality implies that
\[
\| (\phi_{xx}, \psi_x)(t) \|^2 + \int_0^t \| \psi_{xx}(\tau) \|^2 \, d\tau \leq C_{31} N_0^2 \exp(C_{32} N_0^2), \tag{3.90}
\]
where \( C_{31}, C_{32} \) are positive constants depending only on \( V, V, \Theta, \bar{\Theta}, \bar{\Theta} \) and \( N_0 \).
Next, we give the estimate of $\|\zeta(t)\|$. For this, we multiply (3.1) by $-\zeta_{xx}$ to get

$$\frac{R}{2\delta} (\zeta_{t}^2) + \frac{\alpha(v, \theta)\zeta}{v} \zeta_{xx} = \{ \cdots \} \zeta + J_9 + J_{10} + J_{11},$$  

(3.91)

where

$$J_9 = p(v, \theta) \psi_x \zeta_{xx} + \left( \frac{p(v, \theta)}{V} - P(V, \Theta) \right) U_x \zeta_{xx} - \left( \frac{\alpha(v, \theta)}{v} \right) \psi \zeta_{xx} - \left( \frac{\alpha(v, \theta)}{V} \right) \Theta \zeta_{xx} + \frac{\mu(V, \Theta)}{V} U_z \zeta_{xx}$$

$$+ R_{2\zeta_{xx}} - \frac{\mu(v, \theta)}{v} \nu^2 \zeta_{xx}.$$  

$$J_{10} = -F \zeta_{xx}, \quad J_{11} = -\frac{\theta}{2} \kappa \theta \nu^2 \zeta_{xx}.$$

Integrating (3.91) over $[0, t] \times \mathbb{R}$ gives

$$\left\| \frac{\zeta(t)}{\sqrt{\delta}} \right\|^2 + \int_0^t \| \zeta_{xx}(\tau) \|^2 \text{d} \tau \leq C_{33} \left\| \frac{\zeta_0}{\sqrt{\delta}} \right\|^2 + C_{33} \left| \int_0^t \sum_{i=9}^{11} J_i \text{d} \tau \right|,$$  

(3.92)

where $C_{33}$ is a positive constant depending only on $V, \bar{V}, \Theta, \bar{\Theta}$ and $N_{01}$.

Similar to the proof of (3.90), we have

$$\left| \int_0^T \int_{\mathbb{R}} \text{d} x \text{d} \tau \right| \leq \eta \int_0^T \| \zeta_{xx}(\tau) \|^2 \text{d} \tau + C_{33} \eta \exp(C_{33} N_{01}^2) \left( \int_0^T \| \phi_x, \psi_x, \phi_{xx}, \psi_{xx}, \zeta \|^2 \text{d} \tau + \delta^1 \right),$$

$$\left| \int_0^T \int_{\mathbb{R}} \text{d} x \text{d} \tau \right| \leq \eta \int_0^T \| \zeta_{xx}(\tau) \|^2 \text{d} \tau + C_{33} \eta \exp(4C_{33} N_{01}^2) \left( \int_0^T \| \psi_x \|^2 \text{d} \tau + \delta^2 \right),$$

$$\left| \int_0^T \int_{\mathbb{R}} \text{d} x \text{d} \tau \right| \leq C_{34} \delta \int_0^T \| \zeta_{xx}(\tau) \|^2 \text{d} \tau + C_{34} \delta \exp(8C_{34} N_{01}^2) \left( \int_0^T \| \phi_x, \psi_x, \phi_{xx}, \psi_{xx}, \zeta \|^2 \text{d} \tau + \delta^1 \right).$$

where $C_{34}$ is a positive constant depending only on $V, \bar{V}, \Theta, \bar{\Theta}$ and $N_{01}$.

Putting the estimates of $\int_0^T \int_{\mathbb{R}} J_i \text{d} x \text{d} \tau, i = 8, 9, 10$ into (3.92), and using the smallness of $\eta, \delta$ such that

$$C_{33} C_{34} \delta < \frac{1}{2}, \quad 2C_{33} \eta < \frac{1}{2},$$

we obtain

$$\left\| \frac{\zeta(t)}{\sqrt{\delta}} \right\|^2 + \int_0^T \| \zeta_{xx}(\tau) \|^2 \text{d} \tau \leq C_{35} N_{01}^6 \exp(8C_{35} N_{01}^2), \quad \forall \ t \in [0, T],$$

where $C_{35}$ is a positive constant depending only on $V, \bar{V}, \Theta, \bar{\Theta}$ and $N_{01}$. Letting $C_{28} = \max\{ C_{30}, C_{33}, C_{34} \}$ and $C_{29} = C_{35} N_{01}^6 \exp(8C_{35} N_{01}^2)$, then we can get (3.81) and hence finish the proof lemma 3.5. \(\Box\)

Finally, we give the estimate of $\int_0^T \| \phi_{xx}(\tau) \|^2 \text{d} \tau$. 

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Lemma 3.6. Under the assumptions of proposition 3.2, there exists a positive constant $C_{36}$ depending only on $V$, $\nabla$, $\Theta$, $\Sigma$ and $N_0$ such that
\[
\|\phi_{\alpha x}(t)\|^2 + \int_0^t \|\phi_{\alpha x}(\tau)\|^2 \, d\tau \leq C_{36} \left(1 + \delta^{-\frac{1}{2}}\right). \tag{3.93}
\]

Proof. Differentiating (3.1) with respect to $x$ once, then multiplying the resultant equation by $\frac{\phi_{\alpha x}}{v}$ and using (3.1), we have
\[
\left(\frac{\mu(x, \theta)}{v^2} - \psi_{\alpha x} \frac{\phi_{\alpha x}}{v} \right)_t + p_0 \phi_{\alpha x}^2 \frac{\kappa}{v^6} = \{\ldots\}_x + J_{12} + J_{13} + J_{14} + J_{15}, \tag{3.94}
\]
where
\[
\begin{align*}
J_{12} & = \frac{\psi_{\alpha x}^2}{v} - \psi_{\alpha x} \psi_{\alpha xx} \frac{v_x}{v^2} + \psi_{\alpha x} \phi_{\alpha x} (v_x + U_x), \\
J_{13} & = \left(\frac{R_\alpha}{v} - 2(R_\alpha' - p_0 \phi_x) \frac{v_x}{v^2} - \frac{(R_\alpha - p_0 \phi) v_{xx}}{v^3} + 2 \frac{(R_\alpha - p_0 \phi) v}{v^4} \right) \phi_{\alpha x}, \\
J_{14} & = \frac{\mu v}{2v^2} \psi_{\alpha x} \phi_{\alpha x} - \frac{\mu v}{2v^2} \psi_{\alpha x} \phi_{\alpha x} - \frac{\mu(v, \theta)}{v^2} \psi_{\alpha x} \phi_{\alpha x} - \frac{\mu(v, \theta)}{v^2} \psi_{\alpha x} \phi_{\alpha x} + U_x \phi_{\alpha x} \\
& \quad - \left(\psi_{\alpha x} \frac{v}{v} + 2 \mu \nu \psi_{\alpha x} \psi_{\alpha x} + \mu \mu \theta \psi_{\alpha x} + \mu \nu \psi_{\alpha x} + \mu \nu \psi_{\alpha x} + 2 \mu \nu \psi_{\alpha x} \psi_{\alpha x} + \mu \mu \theta \psi_{\alpha x} \psi_{\alpha x} \right) \phi_{\alpha x} \\
J_{15} & = \frac{\kappa V_{\alpha x} \phi_{\alpha x}}{v^3} + \frac{\left(\frac{\kappa v_x}{v^3} - \frac{\kappa \theta_x v_{xx}}{v^3}\right) \phi_{\alpha x}}{v} \\
& \quad + \frac{\left(\frac{\kappa v_x}{v^3} - \frac{\kappa \theta_x v_{xx}}{v^3}\right)}{v} \phi_{\alpha x} + \left(\frac{\kappa v_{xx} \phi_{\alpha x}}{v^3} - \frac{5 \kappa v_x}{2v^6} - \frac{5 \kappa - \nu v_x}{2v^6} + \frac{\kappa \theta_x v_{xx}}{v^3}\right) \phi_{\alpha x} \\
& \quad : = \frac{\kappa V_{\alpha x} \phi_{\alpha x}}{v^3} + J_{15}.
\end{align*}
\]

Similar to (3.84)–(3.88), we have
\[
\begin{align*}
\left|\int_0^t \int_R J_{12} \, dx \, d\tau\right| & \leq C_{N_0}^2 \exp(C_1 N_0^2) \int_0^t \|(\phi_{\alpha x}, \psi_{\alpha x}, \psi_{\alpha x})\|^2 \, d\tau, \\
\left|\int_0^t \int_R J_{13} \, dx \, d\tau\right| & \leq C_{N_0}^2 \exp(2C_2 N_0^2) \left(\int_0^t \|(\phi_{\alpha x}, \psi_{\alpha x}, \psi_{\alpha x})\|^2 \, d\tau + \delta^{1/2}\right), \\
\left|\int_0^t \int_R (J_{14} + J_{15}) \, dx \, d\tau\right| & \leq C_{N_0}^6 \exp(6C_3 N_0^2) \left(\int_0^t \|(\phi_{\alpha x}, \psi_{\alpha x}, \psi_{\alpha x})\|^2 \, d\tau + \delta^{1/2}\right) \\
& \quad + \frac{1}{4} \int_0^t \int_R \left(\frac{p_0 \phi_{\alpha x}^2}{v^2} + \frac{\kappa \phi_{\alpha x}}{v^6}\right) \, dx \, d\tau.
\end{align*}
\]
\[
\left| \int_0^t \int_{\mathbb{R}} \frac{K V_{xx} \phi_{xx}}{v} \, dx \, dt \right| \leq \frac{1}{4} \int_0^t \int_{\mathbb{R}} \frac{K \phi_{xx}^2}{v^6} \, dx \, dt + C \int_0^t \| \Theta_{xx}(\tau) \|^2 \, d\tau \\
\leq \frac{1}{4} \int_0^t \int_{\mathbb{R}} \frac{K \phi_{xx}^2}{v^6} \, dx \, dt + C t^{-\frac{1}{2}}. \quad (3.95)
\]

Integrating (3.94) over \([0, t] \times \mathbb{R}\) and using the estimates of \(J_i, i=11,12,13, J'_i\) and (3.95), then we can obtain (3.93) by corollary 3.2 and lemma 3.5. The proof of lemma 3.6 is completed.

Now we are in a position to prove proposition 3.2.

**Proof of proposition 3.2.** With the above lemmas in hand, we set
\[
\begin{align*}
\Xi_1 (m_0, M_0; V, \Theta, \bar{\Theta}, N_{01}) &= \max \{ C_{15}, C_{28} \}, \\
\Xi_2 (V, \bar{V}, \Theta, \bar{\Theta}, N_{01}) &= C_{28}, \quad (3.96)
\end{align*}
\]
where \(C_{15}\) and \(C_{28}\) are positive constants given in (3.62) and (3.80), respectively. Notice that
\[
\begin{align*}
C_{15} &= \max \{ C_{10}, C_{16}, C_{20} \}, \\
C_{10} &= \max \{ 8C_{12}^{-1} C_{13} C_{14}, 2C_5 \left( \min \{ 2p_+^2, R^2, \bar{\Theta} \} \right)^{-1} \}, \\
C_{13} &= 2C_3 C_7 + C_9,
\end{align*}
\]
and the constants \(C_i, i=5,6,7,9,14,16,20\) given in (or in the proof of) the previous lemmas are increasing functions on both \(m_0^{-1}\) and \(M_0\), while the constant \(C_{12}\) defined in (3.56) is decreasing on both \(m_0\) and \(M_0^{-1}\). Consequently, the function \(\Xi_1 (m_0, M_0; V, \Theta, \bar{\Theta}, N_{01})\) defined in (3.96) is increasing on both \(m_0^{-1}\) and \(M_0\). Thus if (3.9) holds, then all the conditions on \(\delta\) and \(\varepsilon\) listed in lemmas 3.1–3.6 and corollaries 3.1 and 3.2 satisfy. Letting \(C_3 := \max \{ C_{24}, C_{29} \}\) and \(C_4 := C_{36}\), we conclude from corollary 3.2 and lemmas 3.5 and 3.6 that (3.4) and (3.5) hold for all \((t, x) \in [0, T] \times \mathbb{R}\). Finally, from (3.63) and (3.15), it easy to see that the inequalities in (3.3) holds for all \((t, x) \in [0, T] \times \mathbb{R}\). This completes the proof of proposition 3.2.

### 4. Proof of theorem 1.2

This section is devoted to proving theorem 1.2. Since the viscous contact wave \((V^*, U^*, \Theta^*)\) satisfy (1.13), and the rarefaction waves \((V^+, U^+, \Theta^+_i)(x, t)\) solve the Euler equations
\[
\begin{align*}
(V^+)_{,t} + (U^+)_{,x} &= 0, \\
(U^+)_{,t} + p(V^+)_{,x} &= 0, \\
\frac{K}{\gamma-1} (\Theta^+)_{,t} + p(V^+)_{,x} (U^+)_{,x} &= 0,
\end{align*}
\]
(4.1)
it is easy to check that the composite wave \((V, U, \Theta)(x, t)\) defined in (1.32) satisfy
\[
\begin{align*}
V_t - U_x &= 0, \\
U_t + P_x &= \left( \frac{\mu(\Theta) U_t}{V} \right)_x + G, \\
\frac{K}{\gamma-1} \Theta_t + PU_x &= \left( \frac{\bar{g}(V, \Theta) \Theta}{V} \right)_x + \frac{\mu(\Theta) U^2}{V} + H,
\end{align*}
\]
(4.2)
where
\[ G = (P - P_+ - P_-)_x + \left( U'_r - \left( \frac{\mu(V, \Theta)U_r}{V} \right)_x \right), \quad P = \frac{R\Theta}{V}, P\pm = \frac{R\Theta^\pm}{V^\pm} \] (4.3)

and
\[ H = \left[ (P - p^m)U^m_r + (P - P_-)(U'_r)_x + (P - P_+)(U'_r)_x \right] \]
\[ - \frac{\mu(V, \Theta)U^2_r}{V} + \left( \frac{\widetilde{\alpha}(V, \Theta)\Theta'_r}{V'} - \frac{\widetilde{\alpha}(V, \Theta)\Theta}{V} \right)_x \] (4.4)

representing the interaction and error terms coming from different wave patterns.

Let
\[ (\phi, \psi, \zeta)(t, x) = (v - V, u - U, \theta - \Theta)(t, x), \]

then we have from (1.1) and (4.1) that
\[
\begin{aligned}
\phi_x - \psi_t &= 0, \\
\psi_t + (v^{-1}(R\zeta - P\phi))_x &= \frac{\mu(v, \theta)u_x}{v} - \frac{\mu(v, \theta)U'_r}{V} + K_x - G_x, \\
\left( \frac{\kappa}{2} - \frac{\alpha}{2} \right) \psi_t + p(v, \theta)u_t - p(v, \Theta)U'_r &= \left( \frac{\widetilde{\alpha}(v, \theta)\Theta'_r}{V'} - \frac{\widetilde{\alpha}(v, \Theta)\Theta}{V} \right)_x + \frac{\mu(v, \theta)u^2}{v} - \frac{\mu(v, \Theta)U^2_r}{V} + F - H + \frac{\beta}{2} R\theta(v, \Theta) \frac{\Theta'_r}{V'} \Theta, \\
\end{aligned}
\] (4.5)

with the following initial and far field conditions
\[
\begin{aligned}
(\phi, \psi, \zeta)(0, x) &= (\phi_0, \psi_0, \zeta_0)(x), \\
(\phi, \psi, \zeta)(t, \pm \infty) &= 0,
\end{aligned}
\] (4.6)

where the nonlinear terms \( K, F, G \) and \( H \) are given by (1.3)_1, (1.3)_2, (4.3) and (4.4), respectively.

The local existence result of the Cauchy problems (4.4) and (4.5) is similar to proposition 3.1. Then in order to extend the local solution to be a global one, one needs to derive the \emph{a priori} estimates as proposition 3.2. To do so, we first show the temporal decay estimate of the interactions terms and error terms coming from different wave patterns, i.e. the decay of \((G, H)\).

**Lemma 4.1.** There exists a uniform constant \( C > 0 \) such that
\[
\| (G, H)(t) \|_{L^1} \leq C\delta^{\frac{1}{2}} (1 + t)^{-\frac{5}{14}}.
\] (4.7)

**Proof.** We only give the detailed proof of the estimate \( \| G(t) \|_{L^1} \), the other one can be treated similarly. For simplicity, we denote the first and second terms on the right-hand side of (4.3) by \( G_1 \) and \( G_2 \) respectively. Then a direct calculation yields that
\[
\begin{aligned}
G_1 &= \left( \frac{\Theta}{V} - \frac{\Theta'_r}{V'_r} - \frac{\Theta'_r}{V'_r} \right)_x \\
&= (\Theta'_r)_x (V^{-1} - (V'_r)^{-1}) + (\Theta'_r)_x (V^{-1} - (V'_r)^{-1}) \\
&+ \Theta'_r (V^{-1} - (V'_r)^{-1}) + (V'_r)_x \left( \frac{\Theta'_r}{(V'_r)^2} - \frac{\Theta}{V^2} \right) \\
&+ (V'_r)_x \left( \frac{\Theta'_r}{(V'_r)^2} - \frac{\Theta}{V^2} \right) + V'_r \left( \frac{\Theta'_r}{(V'_r)^2} - \frac{\Theta}{V^2} \right).
\end{aligned}
\]
We deduce from (1.32) and lemma 2.4 that
\[
|[(\Theta^c_k)_{\pm} (V^1 - (V^c)_{\pm}^{-1})]| \leq C(|(\Theta^c_k)_{\pm})| (|V^1_+ - v^m_+| + |V^1_- - v^m_-|)
\]
\[
\quad \leq \varepsilon (|(\Theta^c_k)_{\pm})| (|V^1_+ - v^m_+| + |V^1_- - v^m_-|) \mid_{\Omega_+}
\]
\[
\quad + C(|(\Theta^c_k)_{\pm})| (|V^1_+ - v^m_+| + |V^1_- - v^m_-|) \mid_{\Omega_+ \cup \Omega_-}
\]
\[
\quad \leq C\delta \left( (|V^1_+ - v^m_+| + |V^1_- - v^m_-|) \mid_{\Omega_+} + (|(\Theta^c_k)_{\pm})|_{\Omega_+ \cup \Omega_-} \right)
\]
\[
\leq C\delta^2 e^{-\gamma(|s|+t)}.
\]

Similarly, it holds that
\[
|[(\Theta^c_k)_{\pm} (V^1 - (V^c)_{\pm}^{-1})]| \leq C\delta^2 e^{-\gamma(|s|+t)}
\]
\[
|\Theta^c_k (V^1 - (V^c)^{-1})| + |V^c_k \left( \frac{\Theta^c_k}{(V^c)^2} - \frac{\Theta}{V^2} \right)| \leq C\delta^2 e^{-\gamma(|s|+t)}.
\]

Thus we have
\[
|G_1| \leq C\delta^2 e^{-\gamma(|s|+t)}.
\]

For \(G_2\), we have
\[
|G_2| \leq C \left( (|U^1_+| + |U^1_-|)_{xL} + (|U^1_+|_{xL} + |U^1_-|_{xL}) \right)
\]
\[
\phantom{=} + C \left( (|U^1_+|_{xL} + |U^1_-|_{xL}) (|U^1_+|_{xL} + |U^1_-|_{xL}) + |V^1_+|_{xL} + |V^1_-|_{xL} \right)
\]
\[
:= G_2^1 + G_2^2.
\]

Then it follows from lemmas 2.1 and 2.4, and the inequality: \(\min\{a, b\} \leq a^\theta b^{1-\theta}, \forall \theta \in [0, 1]\) that
\[
\|G_2^1\|_{L^1} \leq C \left( \delta^\frac{1}{2} (1 + t)^{-\frac{1}{2}} + \delta (1 + t)^{-1} \right).
\]

Similar to (4.8), we get
\[
\|G_2^2\|_{L^1} \leq C \left( (|U^1_+|_{xL})_{xL} + (|U^1_-|_{xL})_{xL} + (|V^1_+|_{xL})_{xL} + (|V^1_-|_{xL})_{xL} \right)
\]
\[
\leq C\delta^2 e^{-\gamma(|s|+t)}.
\]

On the other hand, by using lemmas 2.1 and 2.4 again, we have
\[
\|U^1_+ V^1_+\|_{L^1} \leq C\delta (1 + t)^{-1},
\]
\[
\|U^1_- V^1_-\|_{L^1} \leq C\delta^2 (1 + t)^{-\frac{1}{2}}.
\]

Consequently,
\[
\|G_2^2\|_{L^1} \leq \|G_2^2\|_{L^1} + \|(U^1_+ V^1_+\)\|_{L^1} + \|(U^1_- V^1_-\)\|_{L^1} \leq C\delta (1 + t)^{-1}.
\]

Then the assertion for \(G\) in (4.7) follows from (4.9)–(4.12). This completes the proof of lemma 4.1.
With lemma 4.1 in hand, by using the same argument as the proof of lemma 3.1 before and lemma 4.1 of [30], one can obtain the following $L^2$-estimate of $(\phi, \psi, \zeta)(t, x)$. The details are omitted here for brevity.

**Lemma 4.2.** There exist a positive constant $C(V, V, \Theta, \Theta)$ and a constant $C_{37} > 0$ depending only $V, V, \Theta, \Theta$ and $m_0, M_0$ such that

\[
\begin{align*}
&\int_{\mathbb{R}} \left[ R\Phi \left( \frac{v}{V} \right) + \frac{\psi^2}{2} + \frac{R}{\gamma - 1} \Phi \left( \frac{\theta}{\Theta} \right) \right] \, dx + \int_{\mathbb{R}} \frac{\kappa(v, \theta) v^2}{\nu^2} \, dx \\
&+ \int_{0}^{t} \int_{\mathbb{R}} S_0 \left( (U^+), s \right) \, dx \, d\tau + \int_{0}^{t} \int_{\mathbb{R}} \left( \mu(v, \theta) \psi^2 - \frac{\alpha(v, \theta) \Theta_c^2}{\nu^2} \right) \, dx \, d\tau \\
\leq & C(V, V, \Theta, \Theta) \left[ \left( \phi_0, \phi_0, \psi_0, \frac{\delta_0}{\sqrt{\delta}} \right) \right]^2 + C_{37} \int_{0}^{t} \int_{\mathbb{R}} (1 + \tau)^{-1} \left( \phi^2 + \frac{\zeta^2}{\delta} \right) e^{-\frac{\alpha x^2}{\nu^2}} \, dx \, d\tau \\
&+ C_{37} \left( N_1^2 \delta_1 \int_{0}^{t} \left( \phi_x, \phi_x, \psi_x, \zeta_x \right)(\tau) \right)^2 \, d\tau + N_1^2 \delta_1 \left( \phi_x, \phi_x, \psi_x, \zeta_x \right)(\tau), \tag{4.13}
\end{align*}
\]

where $\Phi(s) = s - 1 - \ln s$ and $S_0 = P \left( \Phi \left( \frac{v}{V} \right) + \gamma \Phi \left( \frac{\theta}{\Theta} \right) \right)$ are positive functions.

For the remainder term $\int_{0}^{t} \int_{\mathbb{R}} (1 + \tau)^{-1} \left( \phi^2 + \frac{\zeta^2}{\delta} \right) e^{-\frac{\alpha x^2}{\nu^2}} \, dx \, d\tau$ in (4.13), similar to lemma 3.2, one can show that there exist two positive constants $C_{38}, C_{39}$ depending only on $V, V, \Theta, \Theta, m_0, M_0$ such that if

\[
C_{38} N_1^2 \delta_1^2 < \frac{1}{2} \min\{2P_+, R^2, \Theta\},
\]

it holds for all $t \in [0, T]$ that

\[
\begin{align*}
&\int_{0}^{t} \int_{\mathbb{R}} \left( \phi^2 + \psi^2 + \frac{\zeta^2}{\delta} \right) w^2 \, dx \, d\tau \leq C_{39} N_1^2 \delta_1^2 \int_{0}^{t} \left( \phi_x, \psi_x, \zeta_x \right)(\tau) \right)^2 \, d\tau \\
&+ C_{39} N_1^2 \delta_1 \int_{0}^{t} \int_{\mathbb{R}} \left( \phi^2 + \frac{\zeta^2}{\delta} \right) \left( (U^+), s \right) \, dx.
\end{align*}
\]

Moreover, it is easy to check that some similar estimates as lemmas 3.3–3.6 and corollaries 3.1 and 3.2 still hold for the solutions to the Cauchy problems (4.5) and (4.6). Thus we can get the desired $a$ priori estimates as proposition 3.2. Then similar to the proof of theorem 3.1, the global-in-time solutions to problem (4.5) and (4.6) can also be obtained. Hence the proof theorem 1.2 is completed.

**Acknowledgment**

The authors would like to thank the anonymous referees for their many valuable comments and suggestions on the draft version of this manuscript. This work was supported by the National Natural Science Foundation of China (Grant No. 11501003), and the Cultivation Fund of Young Key Teacher at Anhui University.
Appendix

The proof of lemma 3.2. The proof of (3.40) is divided into the following two parts:

\[
\int_0^t \int_R (R\zeta + \delta p + \phi)^2 w^2 \, dx \, dt \leq C_6 N_1^2 \delta^\frac{3}{4} \int_0^t \int_R \left( \phi^2 + \frac{\zeta^2}{\delta} \right) w^2 \, dx \, dt \\
+ C_6 N_1^2 \delta^\frac{3}{4} \int_0^t \| (\phi_x, \phi_{xx}, \psi_x, \zeta_x)(\tau) \|^2 d\tau + C_6 N_1^2 \delta, \tag{A.1}
\]

and

\[
\int_0^t \int_R \left( (R\zeta - p + \phi)^2 + \frac{\psi^2}{v^2} \right) w^2 \, dx \, dt \leq C_8 N_5^4 \delta^\frac{1}{2} + C_8 N_6^4 \delta^\frac{1}{2} \int_0^t \| (\phi_x, \phi_{xx}, \psi_x, \zeta_x)(\tau) \|^2 d\tau. \tag{A.2}
\]

Here \(C_6\) and \(C_8\) are two positive constants depending only on \(V, \nabla, \Theta, \nabla, m_0\) and \(M_0\), and without loss of generality, we may assume that \(C_6 \geq d \frac{m_0 - 1}{\delta}\) with \(d\) being a positive constant given in (A.14) below.

In fact, notice that

\[
(R\zeta + \delta p + \phi)^2 + (R\zeta - p + \phi)^2 \geq \delta \left( 2p^2 + \frac{(R\zeta)^2}{\delta} \right) \geq \min \{2p^2, R^2\} \delta \left( \phi^2 + \frac{\zeta^2}{\delta} \right). \tag{A.3}
\]

then adding (A.1) onto (A.2), we can get (3.40) by the assumption (3.39).

To prove (A.3), we denote

\[
f = \int_{-\infty}^{x} w^2 \, dy,
\]

then it holds that

\[
\| f(t) \|_{L^\infty} \leq C \delta^{\frac{1}{4}} (1 + t)^{-\frac{1}{2}}, \quad \| f_t(t) \|_{L^\infty} \leq C \delta^{\frac{1}{4}} (1 + t)^{-\frac{3}{2}}. \tag{A.4}
\]

We rewrite (3.1) as

\[
\psi_x + \left( \frac{R\zeta - p + \phi}{v} \right)_x = \left( \frac{\mu(v, \theta) \psi_x}{v} \right)_x + K_x + G, \quad G = -U_t + \left( \frac{\mu(v, \theta) U_x}{v} \right)_x. \tag{A.5}
\]

Multiplying (A.5) by \((R\zeta - p + \phi)vf\) and integrating the resulting equation over gives

\[
\int_R \psi_x (R\zeta - p + \phi)vf \, dx + \int_R \left( \frac{R\zeta - p + \phi}{v} \right)_x \psi_x (R\zeta - p + \phi)vf \, dx \\
= \int_R \left( \frac{\mu(v, \theta) \psi_x}{v} \right)_x (R\zeta - p + \phi)vf \, dx + \int_R K_x (R\zeta - p + \phi)vf \, dx + \int_R G (R\zeta - p + \phi)vf \, dx.
\]

Using integrating by parts, we have
\[
\frac{1}{2} \int_\mathbb{R} (R \zeta - p_+ \phi)^2 v^2 \, dx = \int_\mathbb{R} \psi(R \zeta - p_+ \phi)vf \, dx - \int_\mathbb{R} (R \zeta - p_+ \phi)^2 v^{-1}vf \, dx \\
+ \int_\mathbb{R} \mu(v, \theta) \psi(v^{-1}[R \zeta - p_+ \phi])vf \, dx \\
- \int_\mathbb{R} K_\epsilon(R \zeta - p_+ \phi)vf \, dx - \int_\mathbb{R} G(R \zeta - p_+ \phi)vf \, dx \\
= \left( \int_\mathbb{R} (R \zeta - p_+ \phi)vf \, dx \right)_i - \int_\mathbb{R} (R \zeta - p_+ \phi)vf \, dx \\
- \int_\mathbb{R} (R \zeta - p_+ \phi)^2 v^{-1}vf \, dx + \int_\mathbb{R} \mu(v, \theta) \psi(v^{-1}[R \zeta - p_+ \phi])vf \, dx \\
- \int_\mathbb{R} K_\epsilon(R \zeta - p_+ \phi)vf \, dx - \int_\mathbb{R} G(R \zeta - p_+ \phi)vf \, dx \\
= \left( \int_\mathbb{R} (R \zeta - p_+ \phi)vf \, dx \right)_i + \sum_{i=3}^{9} I_i, \tag{A.6}
\]

Now we estimate the terms \(I_3, I_4, \cdots I_9\) one by one. First, the equations (3.1)_1 and (3.1)_2 imply
\[
\left( \frac{R}{\gamma - 1} \zeta + p_+ \phi \right)_i = - \frac{R \zeta - p_+ \phi}{\gamma} (\psi_x + U_x) + \frac{\theta}{2} \kappa \delta_\theta (v, \theta) \frac{v^2}{\gamma^3} \theta_t + \left( \frac{\tilde{\alpha}(v, \theta) \theta_x}{\gamma} - \frac{\tilde{\alpha}(V, \theta) \Theta_x}{V} \right)_x \\
+ \frac{\mu(v, \theta) (\psi_x + U_x)^2}{\gamma} + F, \tag{A.7}
\]

thus it holds that
\[
I_3 = - \delta \int_\mathbb{R} \psi \left( \frac{R}{\gamma} \zeta + p_+ \phi \right)_i vf \, dx + \gamma p_+ \int_\mathbb{R} \psi v f \psi_x \, dx \\
= \delta \int_\mathbb{R} \psi f (R \zeta - p_+ \phi) (\psi_x + U_x) \, dx - \delta \int_\mathbb{R} \psi \left( \frac{\tilde{\alpha}(v, \theta) \theta_x}{\gamma} - \frac{\tilde{\alpha}(V, \theta) \Theta_x}{V} \right)_x \, dx \\
- \delta \int_\mathbb{R} \psi f F \, dx + \gamma p_+ \int_\mathbb{R} \psi \left( \frac{v^2}{2} \right)_x \, dx \\
:= \sum_{i=1}^{6} I_i, \tag{A.8}
\]

Using (A.4), the Cauchy inequality, the Young inequality and the Sobolev inequality, we have
\[
|I_4| + |I_5| \leq C \delta^2 (1 + t)^{-\frac{1}{2}} \|\psi(t)\|^{\frac{1}{2}} \|\psi_x(t)\|^{\frac{1}{2}} \|\psi(t)\| \|\psi_x(t)\| \|\psi(t)\| \|\psi_x(t)\| \leq C N_1^2 \delta^2 \left( \|\psi(t)\|^2 + (1 + t)^{-\frac{1}{2}} \right). \tag{A.9}
\]

From the estimate of \(I_i\), we obtain
\[ |I^2_1| \leq \delta \left\| \psi f \frac{\theta \kappa_0 (v, \theta)}{2 \kappa_0} \right\|_{L^\infty} \left( \int_\mathbb{R} \frac{\kappa_0 \theta v^2_1}{2 v^6} \, dx \right) \]

\[ \leq C(m_0, M_0) \left( \delta^5 (1 + t)^{-\frac{1}{2}} N^3_1 \| \phi_{\alpha x}, \phi_{\alpha}, \psi_x, \zeta_x \| (t)^2 + N^2_1 \delta^3 (1 + t)^{-\frac{1}{2}} \right). \quad (A.10) \]

Similarly, it holds that

\[ |I^2_3| = \left| \delta \int_\mathbb{R} (\psi f) x \left( \frac{\tilde{\alpha}(v, \theta) \zeta_x}{v} + \left( \frac{\tilde{\alpha}(v, \theta)}{v} - \frac{\tilde{\alpha}(V, \Theta)}{V} \right) \Theta_x \right) \, dx \right| \]

\[ \leq C(m_0, M_0) \delta \int_\mathbb{R} \left( |\psi_x |f + |\psi x f| + |\psi x f| \right) |\zeta_x| + |\phi \Theta_x| + |\zeta \Theta_x| \, dx \]

\[ \leq C(m_0, M_0) \delta N^3_1 \| \phi_{\alpha x}, \psi_x, \zeta_x \| (t)^2 + (1 + t)^{-\frac{1}{2}}, \quad (A.11) \]

\[ |I^2_4| \leq C(m_0, M_0) \delta \| \psi(t) \|_{L^\infty} \| f(t) \|_{L^\infty} \| (\psi, U_x)(t) \|^2 \]

\[ \leq C(m_0, M_0) \left( N^2_1 \delta^2 \| \psi_x(t) \|^2 + N_1 \delta^3 (1 + t)^{-\frac{1}{2}} \right), \quad (A.12) \]

\[ |I^2_5| = \delta \int \psi_x v \psi f \, dx + \psi f x \psi f \theta \kappa_0 + \psi f x v \psi f \theta \kappa_0 \]

\[ + \psi f x v \psi f \theta \kappa_0 \, dx \]

\[ \leq C(m_0, M_0) \delta \left( \| v(t) \|_{L^\infty} \| f(t) \|_{L^\infty} \| (\psi, U_x)(t) \| \right) \]

\[ \leq C(m_0, M_0) \delta \left( N^2_1 \delta^2 \| (\phi_{\alpha x}, \phi_x, \psi_x) \| (t)^2 + N_1 \delta^3 (1 + t)^{-\frac{1}{2}} \right), \quad (A.13) \]

and

\[ I^2_6 = - \gamma_0 p_+ \int \psi^2 \, dx \]

\[ \leq \frac{\gamma_0 p_+}{2} \int |\phi_x| \| f \| \psi^2 \, dx - \frac{\gamma_0 p_+}{2} \int v v^2 \psi^2 \, dx \]

\[ \leq \frac{\gamma_0 p_+}{2} \int \psi^2 \, dx - \frac{\gamma_0 p_+}{2} \int v v^2 \psi^2 \, dx + C \delta^2 \| \phi_x \| \| \psi \| \| \psi_x \| \| (1 + t)^{-\frac{1}{2}} \]

\[ \leq - \gamma_0 p_+ \int v v^2 \psi^2 \, dx + C N^2_1 \delta^2 \| (\phi_{\alpha x}, \psi_x) \| (t)^2 + (1 + t)^{-\frac{1}{2}}. \quad (A.14) \]
Here the constant \( d > 0 \) depends only on \( \theta_{\pm} \), and satisfies \(|V_x f| \leq d \delta u^2\). Moreover, we have used the smallness of \( \delta \) such that \( dm_0^{-1} \delta < \frac{1}{4} \) in (A.14).

Similar to the estimates as above, we also have

\[
|I_5| \leq C(m_0, M_0) \| \psi(t) \| \| (\zeta, \phi)(t) \| \| f(t) \|_{L^\infty} \leq C(m_0, M_0) N_1^2 \delta^\frac{1}{2} (1 + t)^{-\frac{1}{2}},
\]

(A.15)

\[
|I_7| \leq C(m_0, M_0) \left( N_1 \delta^\frac{1}{2} \| (\phi_x, \psi_x, \zeta_x)(t) \|^2 + N_1 \delta^\frac{1}{2} (1 + t)^{-\frac{1}{2}} \right),
\]

(A.16)

\[
|I_8| = \left| \int_{\mathbb{R}} K[(R\zeta - p_+ \phi)vf], dx \right|
\leq C(m_0, M_0) \int_{\mathbb{R}} \left( |\phi_{xx}| + |V_{xx}| + |\phi_x|^2 + |V_x|^2 + |\zeta_x|^2 + \Theta_x^2 \right)
\times \left( |(\phi_x, \zeta_x)f| + |(\phi, \zeta)(\phi_x + V_x)f| + |(\phi, \zeta)f_x| \right) dx
\leq C(m_0, M_0) \left( N_1^2 \delta^\frac{1}{2} \| (\phi_{xx}, \phi_x, \zeta_x)(t) \|^2 + N_1^2 \delta^\frac{1}{2} (1 + t)^{-\frac{1}{2}} \right),
\]

(A.17)

\[
|I_9| \leq C(m_0, M_0) \int_{\mathbb{R}} |G| |(\phi, \zeta)| |f| dx
\leq C(m_0, M_0) \int_{\mathbb{R}} (|U_x| + |U_{xx}| + |V_x U_x| + |\theta_x U_x|) |(\phi, \zeta)| |f| dx
\leq C(m_0, M_0) N_1 \delta^\frac{1}{2} \left( \| (\phi_x, \zeta_x)(t) \|^2 + (1 + t)^{-\frac{1}{2}} \right)
\]

(A.18)

and

\[
|I_6| \leq C(m_0) \int_{\mathbb{R}} \| (\phi, \zeta) \|^2 |\phi_x| |f| dx + \int_{\mathbb{R}} V^{-1}(R\zeta - p_+ \phi)^2 V_x f dx
\leq C(m_0) \| (\phi, \zeta)(t) \|^2 \| (\phi_x, \zeta_x)(t) \|^2 \| \phi_x(t) \|^2 \| \zeta_x(t) \|^2 (1 + t)^{-\frac{1}{2}} + dm_0^{-1} \delta \int_{\mathbb{R}} |R\zeta - p_+ \phi|^2 w^2 dx
\leq C(m_0) \left( N_1^2 \delta^\frac{1}{2} \| (\phi_x, \zeta_x)(t) \|^2 + N_1^2 \delta^\frac{1}{2} (1 + t)^{-\frac{1}{2}} \right) + \frac{1}{4} \int_{\mathbb{R}} |R\zeta - p_+ \phi|^2 w^2 dx
\]

(A.19)

provided that \( \delta \) is sufficiently small such that \( dm_0^{-1} \delta < \frac{1}{4} \).

Combining (A.6) and (A.8)-(A.18) and integrating the resulting equation over \([0, \tilde{t}]\), we can get (A.2) by the smallness of \( \delta \).

Next, we prove (A.1). Let \( h = R\zeta + \delta p_+ \phi \) and using (A.7), we have
\[ \frac{1}{\delta} (\eta, h^2)_{H^{1-\delta}} = \int_{\mathbb{R}} \left( \frac{R}{1 - 1} \zeta + \mu_{\pm} \phi \right) h^2 \, dx \]

\[ = - \int_{\mathbb{R}} \frac{R \zeta - \mu_{\pm} \phi}{v} \psi \zeta h^2 \, dx - \int_{\mathbb{R}} \frac{R \zeta - \mu_{\pm} \phi}{v} U_h h^2 \, dx \]

\[ + \int_{\mathbb{R}} 2(\varphi, \vartheta) \int_{\mathbb{R}} (\varphi, \vartheta) \sqrt{2} h^2 \, dx + \int_{\mathbb{R}} \left( \frac{\varphi}{v} \right) \frac{\varphi}{v} \, dx \]

\[ + \int_{\mathbb{R}} \mu(\varphi, \vartheta) (\psi, h^2) \, dx + \int_{\mathbb{R}} (\zeta, h^2) \, dx \]

\[ := \sum_{i=10}^{15} I_i. \quad (A.20) \]

Since \(|U_1| \leq C_\delta w^2, h = R \zeta + \delta p + \phi = O(1) \delta^\frac{1}{2} (\phi + \sqrt{\delta})\), it holds that

\[ |I_1| \leq C(m_0) \int_{\mathbb{R}} \delta^\frac{1}{2} \left( \phi, \frac{\zeta}{\sqrt{\delta}} \right) |U_1||(\zeta, \phi)||g||_{L^\infty} \, dx \leq C(m_0) \delta^\frac{1}{2} \int_{\mathbb{R}} \left( \phi, \frac{\zeta}{\sqrt{\delta}} \right) \, w^2 \, dx, \quad (A.21) \]

\[ |I_2| \leq C(m_0, M_0) \int_{\mathbb{R}} \frac{K\nu_2}{2\nu_2} \delta \eta \, dx \left| h^2 \right|_{L^\infty} \]

\[ \leq C(m_0, M_0) \left( \delta^\frac{1}{2} \left( \phi, \zeta \right) + \delta \left( \phi, \frac{\zeta}{\sqrt{\delta}} \right) \right) \left| g \right|_{L^\infty} \, dx \]

\[ \leq C(m_0, M_0) \left( \delta^\frac{1}{2} \left( \phi, \zeta \right) + \delta \left( \phi, \frac{\zeta}{\sqrt{\delta}} \right) \right) \left| g \right|_{L^\infty} \, dx \quad (A.22) \]

\[ |I_3| \leq C(m_0, M_0) \int_{\mathbb{R}} \left( |\psi, U_1| \right) \left| h^2 \right|_{L^\infty} \, dx \leq C(m_0, M_0) \int_{\mathbb{R}} \left( |\psi, U_1| \right) \delta^\frac{1}{2} \left( \phi, \frac{\zeta}{\sqrt{\delta}} \right) \left| g \right|_{L^\infty} \, dx \]

\[ \leq C(m_0, M_0) \left( N_1 \delta^\frac{1}{2} \left( \phi, \zeta \right) + N_1 \delta^\frac{1}{2} \left( \phi, \zeta \right) \right) \left| g \right|_{L^\infty} \, dx \quad (A.24) \]

Similar to the estimate of \( I_3 \), we obtain

\[ |I_4| \leq C(m_0, M_0) \int_{\mathbb{R}} \left( \mu_{\pm} \phi \right) u_1 \mu_{\pm} \phi \, dx \leq \int_{\mathbb{R}} \left( \mu_{\pm} \phi \right) u^2 \, dx \]

\[ \leq C(m_0, M_0) \left( N_1 \delta^\frac{1}{2} \left( \phi, \zeta \right) + N_1 \delta^\frac{1}{2} \left( \phi, \zeta \right) \right) \left| g \right|_{L^\infty} \, dx \quad (A.25) \]

For the estimate of \( I_{10} \), we compute from (A.7) that
\[-2 I_{10} = 2 \int_{\mathbb{R}} v^{-1} (h^2 - \gamma p_+ \phi) \phi_x \phi_y^2 dx \]
\[= \int_{\mathbb{R}} (2v^{-1} h^2 \phi_x - \gamma p_+ v^{-1} h g^2 \phi_x^2) dx \]
\[= \left( \int_{\mathbb{R}} v^{-1} h^2 \phi_x (2h - \gamma p_+ \phi) dx \right) - 2 \int_{\mathbb{R}} v^{-1} h g \phi (2h - \gamma p_+ \phi) g dx \]
\[+ \int_{\mathbb{R}} v^{-2} v_x g^2 h \phi_x (2h - \gamma p_+ \phi) dx - \int_{\mathbb{R}} v^{-1} g^2 \phi_4 (4h - \gamma p_+ \phi) h dx \]
\[= \left( \int_{\mathbb{R}} v^{-1} h^2 \phi_x (2h - \gamma p_+ \phi) dx \right) - \frac{\delta}{2v} \int_{\mathbb{R}} v^{-1} h g \phi (2h - \gamma p_+ \phi) w_x dx \]
\[+ \int_{\mathbb{R}} v^{-2} u_x g^2 \phi_x [h (2h - \gamma p_+ \phi) + \delta (4h - \gamma p_+ \phi) (R \zeta - p_+ \phi)] dx \]
\[= - \delta \int_{\mathbb{R}} v^{-1} g^2 \phi_4 (4h - \gamma p_+ \phi) \frac{\theta}{\sqrt{v^2 \theta}} dx \]
\[+ \frac{\delta}{2v} \int_{\mathbb{R}} v^{-1} g^2 \phi_4 (2h - \gamma p_+ \phi) \left( \frac{\dot{\alpha}(v, \theta) \theta_x}{v} - \frac{\dot{\alpha}(V, \Theta) \Theta_x}{V} \right) dx \]
\[- \delta \int_{\mathbb{R}} v^{-1} g^2 \phi_4 (4h - \gamma p_+ \phi) \left( \frac{\mu(v, \theta)}{v} \right) (\psi_x + U_x) dx \]
\[= \delta \int_{\mathbb{R}} v^{-1} g^2 \phi_4 (4h - \gamma p_+ \phi) F dx \]
\[= \left( \int_{\mathbb{R}} v^{-1} h^2 \phi_x (2h - \gamma p_+ \phi) dx \right) + \sum_{i=1}^{k} I_{10}^{(i)}. \tag{A.26} \]

By employing the Sobolev inequality, the Young inequality, the a priori assumption (3.7) and lemma 2.1, we can control the terms \( I_{10}^{(i)}, i = 1, \ldots, 6 \) as follows.

\[|I_{10}^{(i)}| \leq C (m_0) \delta (1 + t)^{-1} \int \| \phi \overline{\psi}_x \|^2 dx \leq C (m_0) \delta (1 + t)^{-1} \| \phi \zeta \|^2 (t) \| \phi_x \zeta_x \|^2 (t) \]
\[\leq C (m_0) \delta \left( \| \phi_x \zeta_x \|^2 (t) + N^2 \| (1 + t)^{-\frac{1}{2}} \right). \tag{A.27} \]

\[|I_{10}^2| \leq C (m_0) \int \| (U_x, \psi_x) \| (|\phi|^3 + |\zeta|^3) dx \leq C (m_0) \| \phi \zeta \|^2 (t) \| \phi_x \zeta_x \|^2 (t) \]
\[\leq C (m_0) \| \psi_x \|^2 \left( \| \phi_x \zeta_x \| (t) \| (1 + t)^{-\frac{1}{2}} \right), \tag{A.28} \]

\[|I_{10}^3| \leq C (m_0, M_0) \delta \int \| (\phi \zeta) \| (t) \| \psi_x \| \| \phi_x \zeta_x \| (t) \| \psi_x \| \| \phi_x \zeta_x \| (t) \| \psi_x \| \| \phi_x \zeta_x \| (t) \]
\[\leq C (m_0, M_0) \| \psi_x \|^2 \left( \| \phi_x \zeta_x \| (t) \| (1 + t)^{-\frac{1}{2}} \right), \tag{A.29} \]

\[|I_{10}^4| \leq C (m_0, M_0) \delta \int \| (v^{-1} g^2 \phi (4h - \gamma p_+ \phi)) \| (|\zeta| + |(\phi \zeta) \Theta_x|) dx \]
\[\leq C (m_0, M_0) \delta \int \left( \| \phi \| \phi_x \zeta_x \| (t) + \| \zeta \| \| \psi_x \| \| \phi_x \zeta_x \| (t) \right) (|\zeta| + |(\phi \zeta) \Theta_x|) dx \]
\[\leq C (m_0, M_0) \left( \delta \| \phi_x \zeta_x \|^2 (t) + N^2 \| \phi_x \zeta_x \|^2 \right) \int \left( \frac{\phi \zeta}{\sqrt{\delta}} \right)^2 \| (\phi \zeta \Theta_x) \| \| w \|^2 dx \right), \tag{A.30} \]
\[ | \tilde{I}_{t0}^{\delta} | \leq C(m_0, M_0) \delta^2 \int_{\mathbb{R}} |(\phi, \zeta)|^2 |(\psi_x, U_x)|^2 \, dx \leq C(m_0, M_0) N_1^2 \delta^2 \left( \| \psi_x(t) \|^2 + \delta^2 (1 + t)^{-\frac{3}{2}} \right), \tag{A.31} \]

\[ | \tilde{I}_{t0}^{\phi} | \leq \left| \int_{\mathbb{R}} (v^{-1} \phi g^2(4h - \gamma \rho + \phi)) \frac{\eta g^2}{v^2} u_x \, dx \right| + \gamma \int_{\mathbb{R}} (v^{-1} \phi g^2(4h - \gamma \rho + \phi)) v_x \, dx \leq C(m_0, M_0) \delta \int_{\mathbb{R}} |(\phi, \zeta)|^2 g^2 v_x^2 |u_x| + |\phi g^2| |(\phi, \zeta)||v_x u_x| + |\phi g^2| |(\phi, \zeta)||v_x u_x| + |\phi g^2| |(\phi, \zeta)||v_x u_x| \, dx \]

\[ \leq C(m_0, M_0) N_1^2 \delta^2 \left( N_1 \| (\phi_x, \phi, \psi_x, \zeta) \|^2 \right) + \int_{\mathbb{R}} \left| \left( \phi, \frac{\zeta}{\sqrt{\delta}} \right) \right|^2 w^2 \, dx + \delta (1 + t)^{-\frac{3}{2}}, \tag{A.32} \]

where in (A.27), we have used \(|w_t| \leq C \delta^{-\frac{3}{2}} (1 + t)^{-1}.

Combining (A.20)–(A.32) and integrating the resultant equation over \([0, t]\) yields

\[ \frac{1}{\delta} \int_0^t \left< h, \eta g^2 \right>_{H^{-1} \times H^1} \, d\tau \leq C(m_0, M_0) N_1^2 \delta^2 \int_0^t \| (\phi_x, \phi, \psi_x, \zeta) \|^2 \, d\tau \]

\[ + C(m_0, M_0) \left( N_1^2 \delta^2 \int_0^t \int_{\mathbb{R}} \left| \left( \phi, \frac{\zeta}{\sqrt{\delta}} \right) \right|^2 w^2 \, dx + N_1^2 \delta \right). \tag{A.33} \]

Then (A.1) follows from (A.33) and lemma 2.2 immediately. This completes the proof of lemma 3.2.

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