Gravitational closure of matter field equations

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Abstract
We show how to extract the hidden information about gravity in one’s choice of matter dynamics. Restricting attention to canonically quantizable matter field equations, but therefore being able to admit any tensorial background geometry, one is left with very little choice for the dynamics of the geometry. Indeed, the physical requirement that the common canonical evolution of matter and geometry can start and end on shared Cauchy surfaces imposes consistency conditions so strong that the Lagrangian for the geometry arises as the solution of a particular system of linear partial differential equations. Through these equations, the Lagrangian for the geometry is thus determined by the stipulated matter field dynamics. In contrast to previous work, we now build the theory on a suitable associated bundle encoding the canonical configuration degrees of freedom, which allows to include necessary constraints on the geometry in practically tractable fashion.

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I. INTRODUCTION

It is probably fair to say that there remains an uncomfortable arbitrariness in the construction of modified gravity models, which even plagues the proposals that heed the currently known theoretical and observational constraints [1–3]. Since a finite number of experiments will not be able to discriminate against an infinity of models, bona fide physical input must likely be injected into the construction beforehand, instead of being left to discriminate against theories only afterwards.

In this article, we argue that such genuine physical input, which promises to effect a reduction of the current infinite ambiguity to a finite one, is provided by first prescribing the matter dynamics on a spacetime. The kinematics and, more remarkably, the dynamics of the underpinning spacetime geometry then both follow from the matter dynamics, namely by the requirement that they have a common canonical evolution. Thus there is no one-size-fits-all gravity theory that would apply independently of what types of matter dynamics are discovered. Instead, any new insights into the nature of matter imply new insights into gravity. Not too bad a perspective in the first place. And not a new one either, considering that it was the dynamics of matter, namely the classical electromagnetic field, that single-handedly led Einstein to the identification and kinematical interpretation of Lorentzian geometries and finally the field equations governing their dynamics.

Being quite general — admitting multiple matter fields and geometric backgrounds described by up to finitely many tensor fields of arbitrary valences and symmetries — we show that there is a clear calculational path from prescribed classical matter dynamics to the kinematics and dynamics that the underlying spacetime geometry must obey, in order for its canonical evolution to not clash with that of the specified matter theory. Such generality at one point usually requires to throw in a strong assumption elsewhere. The physically hardly negotiable condition that the chosen matter theory admit a quantum formulation, however, already implies all required mathematical assumptions. More precisely, this paper builds on the fact that the canonical quantizability of any classical dynamics

\[ S_{\text{matter}}[A; G] = \int d^4 x \mathcal{L}_{\text{matter}}[A; G], \]

for fundamental matter fields \( A \) on a smooth manifold \( M \), imposes severe algebraic restrictions \([4]\) on the background geometry \( G \). Throughout this paper, the notation \( F[A; B] \) indicates that a functional \( F \) depends on the field \( A \) and arbitrary derivatives of it, while
it only depends on the underived field $B$, with the semicolon separating groups of such fields. The restrictions from canonical quantizability are so strong that they dictate both, the kinematics and the dynamics of any priori arbitrary tensor field $G$ that features as the geometry. The underlying mechanism is constructive \cite{5}; starting from nothing more than the classical matter action, one can indeed calculate the most general Lagrangian density $\mathcal{L}_{\text{geometry}}$ for the geometry that provides the gravitational closure

$$
S_{\text{closed}}[A,G] = \int d^4x \left( \mathcal{L}_{\text{matter}}[A;G] + \mathcal{L}_{\text{geometry}}[G] \right)
$$

of the matter theory one started with \cite{6}. Only after this step is there a dynamically closed theory: variation with respect to the matter fields $A$ yields the matter field equations on the previously undetermined background $G$, while variation with respect to $G$ closes this system by providing equations of motion for the underlying geometry, sourced by the very same matter.

Before delving into the technical details of this paper, we wish to provide an intuition for how it is possible at all that the dynamics of matter already constrain the underlying geometry so much that the gravitational dynamics are essentially fixed. A central point is the required canonical quantizability of the matter equations, which need to be stipulated in the first place in order to start the gravitational closure program. Two severe algebraic restrictions \cite{4} on the background geometry $G$ follow from this canonical quantization condition. The first restriction is imposed by the requirement that there be a well-posed initial value formulation for the classical dynamics, because canonical quantization cannot even start otherwise \cite{7}. In particular, the existence of initial value surfaces for the matter dynamics requires that the associated field equations feature a principal polynomial $P_G$ that is hyperbolic \cite{8}. The second restriction follows from the need to split the classical field modes into such of positive and negative frequencies, in a way that any two inertial observers at the same point agree upon, even if this famously fails globally in quantum field theory \cite{9}. In particular, this requires that also the algebraic dual of the principal polynomial be hyperbolic \cite{4}. If both conditions are satisfied, the triple $(M,G,P_G)$ features strong algebraic properties that fix all key kinematical notions, such as initial data surface foliations and local observer frames. This is the first, kinematical, part of the story.

The dynamical part then simply employs this kinematical structure, in order to bring a well-known observation from metric geometrodynamics to new fruition in our general
context. For one finds that these extracted spacetime kinematics carry just enough structure for us to be left with no choice concerning the dynamics of the geometry \(^6\). This more recent result technically significantly extends a well-known observation, which was originally made by Hojman, Kuchař and Teitelboim for Lorentzian manifolds \(^{10}\), to any structure \((M, G, P_G)\) that arises from canonically quantizable matter dynamics. The key insight, which remains valid for our general case \(^{11}\), is that the commutator algebra of normal and tangential deformation operators of initial data surfaces must exactly mimic the constraint algebra of any diffeomorphism-invariant dynamics for the geometry. And since the algebra amounts to three functional differential equations for these constraints that can be solved up to typically only some constants of integration, the gravitational theory is completely determined after equally many independent experiments have been conducted to fix these constants. Having argued with reference to one particular choice of matter dynamics, it is probably worth mentioning that the coexistence of several matter field dynamics does not pose a problem, even if they do not share the same principal polynomial when considered separately. For once they are considered together, as they must be, they immediately imply one common principal polynomial \(P_G\); see section \(^{V B}\) for an illustrating example.

In this article, we present the complete set of equations that must be solved in order to obtain all possible Lagrangian densities \(\mathcal{L}_{\text{geometry}}\) for the geometry that describe a canonical evolution commensurate with that of a specified canonically quantizable matter field theory. The conceptual and technical developments presented in this article significantly improve on the results obtained in \(^6\), and spread over the four technical sections of this paper.

First, we condense the gravitationally relevant information from the equations of motion for canonically quantizable matter into the triple \((M, G, P_G)\). This is meaningful because the gravitational closure mechanism only requires the structural data that trickles down to this triple. Since the calculation of the principal polynomial for a system of partial differential equations is a standard task \(^{12}\), we only concisely review the required kinematical constructions based on this triple in section \(^{II}\) as they will be used throughout the article.

Secondly, and more importantly, we remove a theoretically inexistent, but practically almost prohibitive problem with the application of the results of \(^6\) to kinematical spacetime geometries for which the separation of lapse and shift from true dynamical degrees of freedom imposes non-linear algebraic conditions on the initial data surface geometry. For exactly as in classical mechanics, where the condition that a particle move on a non-linear submanifold
of Euclidean space is most effectively dealt with by introduction of generalized coordinates, we also employ generalized tensor field components (corresponding to points in a suitable associated bundle over the spacetime frame bundle), in order to directly deal only with the true degrees of freedom of the theory. The relevant technology, once set up, makes things quite simple and will be developed in detail in section III.

Thirdly, we now convert the entire constraint algebra for the gravitational dynamics into a countable set of linear homogeneous partial differential equations, for whose solution powerful methods are available [12]. Unlike the construction in [6], this reveals one single and immutable set of equations for the gravitational Lagrangian. Only the coefficient functions appearing in these partial differential equations vary with the choice of matter dynamics and can now be constructed swiftly according to simple rules, which are provided, together with these gravitational closure equations, in section IV.

Fourthly, we show in section IV E how to completely bypass the canonical formalism employed in the previous two sections in favor of a complete spacetime formulation. In particular, we provide a gravitational action functional $S_{\text{geometry}}[G]$ that depends on the spacetime geometry only, rather than geometric phase space variables. Addition of this spacetime action to the initially provided matter action and subsequent variation with respect to the tensor field $G$ then yields the complete gravitational field equations coupled to matter.

How truly simple it is to set up the gravitational closure equations for a variety of matter models on different tensorial geometries, is then illustrated by three prototypical examples in section V. In particular, we set up the gravitational closure equations for the case of standard model matter on a metric manifold, for two scalar fields on a bimetric background and for a refinement of Maxwell theory on a background that does not exclude birefringence a priori. We will, however, not solve the equations for any of these examples here. We conclude in section VI by spelling out the impact of our results for both fundamental and phenomenological questions and by pointing out several results we were able to obtain by building of the present article, including explicit solutions to gravitational closure equations.
II. REVISION OF SPACETIME KINEMATICS

This section collects the minimum technology that is required to gravitationally close a given set of matter field equations. The aim of gravitational closure is to generate further field equations, but now for the \textit{coefficients} of the matter field equations rather than for the matter fields themselves, in such a way that the coefficients and matter fields finally share a common canonical evolution. The entire information that is needed, to this end, from the matter field equations, is contained in the triple \((M, G, P_G)\), as discussed in the introduction. We will not be concerned with the standard task of how to obtain the principal polynomial \(P_G\) from matter field equations on a smooth manifold \(M\) with an arbitrary tensorial geometry \(G\). For canonically quantizable matter field equations, this triple allows to foliate any kinematical spacetime manifold into initial data hypersurfaces and then to also devise the projection of spacetime tensor fields to the respective leaves.

A. Kinematics from canonically quantizable matter

If the triple \((M, G, P_G)\) arises from canonically quantizable matter dynamics \(S_{\text{matter}}[A; G]\), as we will assume without further mention throughout this paper, the principal tensor field in particular satisfies salient algebraic properties \([4]\), which enable the following key constructions. We admit a tensor field \(G\) of arbitrary valence and algebraic symmetries, or even a collection of such, as the geometric background on which the dynamics for the matter field \(A\) are formulated. Notwithstanding this generality, there will be automatic restrictions on the geometry \(G\) from the said requirement that the matter field equations be canonically quantizable.

In order to keep the discussion simple, we consider only matter actions \(S[A; G]\) in which \(G\) appears underived and whose associated field equations are linear in the highest derivative of the matter fields \(A\), as is, for instance, the case for the standard model of particle physics. For then the principal polynomials

\[
P_G(x) : T^*_x M \longrightarrow \mathbb{R}, \quad k \mapsto P_G(x)(k)
\]

of the field equations at each point \(x\) of the spacetime manifold \(M\) are given entirely in terms of the underived tensor field \(G\) at this point. They are homogenous of some global degree
deg $P$ and all arise from one symmetric contravariant smooth vector field, whose components with respect to any frame we also denote by the kernel letter $P_G$, by virtue of

$$P_G(x)(k) = P_G^{a_1 \ldots a_{\deg P}}(x) k_{a_1} \cdots k_{a_{\deg P}}.$$ 

For the practical calculation of $P_G$ from given matter field equations, we refer the reader to the literature \[12\]. If the matter field equations are canonically quantizable, the following crucial constructions exist and are unique \[4\].

First, there always are several open and convex hyperbolicity cones \[13\] and a smooth choice of one such, $C_x$ say, in each cotangent space $T^*_x M$ physically corresponds to an energy orientation, as is explained in detail also in \[4\]. Technically, each such $C_x$ is a maximal connected set of covectors $h \in T^*_x M$ with $P_G(x, h) \neq 0$ such that for any further covector $q$ the condition $P_G(x, q + \lambda h) = 0$ has only real solutions $\lambda$. The physical meaning of the hyperbolicity cones is two-fold. On the one hand, they contain all possible conormals to initial data surfaces for the matter field theory. A hypersurface in $M$ whose conormals do not all lie in a smooth distribution of hyperbolicity cones cannot be an initial data surface for the matter field dynamics at hand \[13\]. On the other hand, the momenta of massive particles with positive energy are given by the covectors in $C_x$ \[4\].

Second, there is an injective Legendre map at each point $x$ of the spacetime manifold $M$,

$$\ell_x : C_x \longrightarrow T^*_x M, \quad h \mapsto \ell_x(h) := \frac{P_G^{m_2 \ldots m_{\deg P}}(x) h_{m_2} \cdots h_{m_{\deg P}}}{P_G(x, h)} ,$$

with inverse $\ell_x^{-1} : \ell_x(C_x) \longrightarrow C_x$. Particularly the inverse Legendre map plays two important roles. On the one hand, it is instrumental in defining the action

$$S_{\text{massive particle}}[x] = m \int d\lambda P \left( x(\lambda), \ell_x^{-1}(x(\lambda)) \left( \frac{dx}{d\lambda}(\lambda) \right) \right) - \deg P ,$$

for the trajectories $x$ of point particles of positive mass $m$. Moreover, it defines the purely spatial directions $S \subset T^*_x M$ seen by an observer with worldline tangent $e_0$ at $x \in M$ by virtue of the conditions

$$e_0 \in C_x^\# \quad \text{and} \quad \ell_x^{-1}(e_0)(S) = 0 ,$$

where $C_x^\# \subseteq \ell_x(C_x)$ denotes a uniquely defined dual cone in tangent space, the open and convex observer cone. It can be shown that the observer cone is a hyperbolicity cone of the dual polynomial $P_G^\#(x) : T_x M \longrightarrow \mathbb{R}$, which is uniquely defined in terms of the principal
polynomial $P_G(x)$ up to an irrelevant factor [4]. Apart from providing the observer cone, its physical role is to determine the action for the trajectories $x$ of massless particles to

$$S_{\text{massless particle}}[x, \mu] = \int d\lambda \mu(\lambda) P^# \left( x(\lambda), \frac{dx}{d\lambda}(\lambda) \right).$$

For the developments of this article, however, it is the Legendre maps $\ell_x$ that play a key role, namely to provide a unique normal spacetime vector field $e_0 := \ell(e^0)$ transversal to the initial data hypersurface, taking as input the unique conormal field $e^0$ along the hypersurface that lies in the chosen hyperbolicity cone and has been normalized such that $P_x(e^0) = 1$. This will be exploited immediately in the following subsection.

**B. Canonical foliation and induced geometry**

Foliating the spacetime into leaves of initial data hypersurfaces and inducing a canonical geometry, familiar from standard general relativity, extends straightforwardly to manifolds $(M, G, P_G)$ whose structure arises from canonically quantizable matter field actions. In order to fix notation and to devise a way to project spacetime geometries $G$ of arbitrary valence to initial data surfaces, we quickly collect the relevant constructions.

Let $X_t : \Sigma \to M$ be a one-real-parameter family of maps embedding a three-dimensional manifold $\Sigma$ such that $M$ is foliated into hypersurfaces $X_t(\Sigma)$ with everywhere hyperbolic conormal $e^0(t, \sigma)$ for $\sigma \in \Sigma$. Employing coordinates $y^\alpha$ on $\Sigma$, we define the one-parameter families of spacetime vectors

$$e_0(t, \sigma) := \ell_{X_t(\sigma)}(e^0(t, \sigma)) \quad \text{and} \quad e_\alpha(t, \sigma) := X_t^*((\partial/\partial y^\alpha)_\sigma) \quad \text{for} \ t \in \mathbb{R}, \sigma \in \Sigma.$$

With the additional normalization condition $P(X_t(\sigma), e^0(t, \sigma)) = 1$, these provide the so-called orthogonal projection frame field along each embedded hypersurface $X_t(\Sigma)$. The frames $e_0(t, \sigma), \ldots, e_3(t, \sigma)$, together with their unique dual frames $e^0(t, \sigma), \ldots, e^3(t, \sigma)$, allow to project spacetime tensors of arbitrary valence to the manifold $\Sigma$.

In the context of this article, we will perform such projections for the spacetime tangent vector field $\dot{X}_t$ constructed from the family of embedding maps, the spacetime tensor field $G$ and the principal tensor $P_G$. We will discuss these, in turn, below. The manifold $\Sigma$ thus becomes a kind of three-dimensional cinema screen on which the evolution of the four-dimensional spacetime geometry is shown as a movie in the foliation parameter $t$. Note that
the entire construction is conceptually standard, but that the Legendre map \( \ell \) is generically non-linear for the spacetime geometries \((M, G, P_G)\) we consider.

Now more precisely, consider first the vector field \( \dot{X}_t \), which is the tangent vector field to the congruence of spacetime curves that correspond to points that do not move on the manifold \( \Sigma \) as the foliation parameter increases. Its projection to \( \Sigma \) gives rise to two one-parameter families of fields, namely the induced lapse and shift fields

\[
\mathbf{n}(t) := \epsilon^0(t)(\dot{X}_t) \quad \text{and} \quad \mathbf{n}^a(t) := \epsilon^a(t)(\dot{X}_t).
\]

Secondly, we perform the projection of the spacetime geometry \( G \) to several one-parameter families of tensors on \( \Sigma \), which is an important intermediate step towards setting up the gravitational closure equations for any \((M, G, P_G)\). Their components are practically obtained [14] by inserting either the frame field \( e_0(t, \sigma) \) or \( e_\alpha(t, \sigma) \) into a slot of \( G \) that requires a vector, and correspondingly either \( \epsilon^0(t, \sigma) \) or \( \epsilon^\alpha(t, \sigma) \) into a slot that requires a covector.

For instance, considering a spacetime geometry given by a \((1, 2)\)-tensor field \( G \), one obtains eight tensors of various valences on the manifold \( \Sigma \), one of which is the \((0, 1)\)-tensor field

\[
g^{0}_{\alpha 0}(t, \sigma) := G_{X(t, \sigma)}(\epsilon^0(t, \sigma), e_0(t, \sigma), e_\alpha(t, \sigma)),
\]

which generically differs from the correspondingly defined \( g^{0}_{\alpha 0}(t, \sigma) \), which is why we do not suppress the 0-indices in the notation. It is economical to define one single hyperindex that collects all index combinations for all resulting tensors on \( \Sigma \), in some chosen order, such as

\[
\mathcal{A} = \langle 0_0, 0_00, 0_\beta \alpha, 0_\beta \alpha_0, 0_\alpha \beta_2, 0\alpha_0 \beta_2, \alpha_0 \alpha_0, \alpha_0 \alpha_0 \beta_2 \rangle
\]

for our example. Note that we abstain from employing potential algebraic symmetries of the spacetime geometry \( G \), such as \( G^a_{\beta \gamma} = G^a_{[\beta \gamma]} \), which of course could be used to remove redundant information from the list \( g^{\mathcal{A}} \). These are most efficiently dealt with later, when identifying the canonical degrees of freedom of the geometry on the manifold \( \Sigma \).

Thirdly, we project the principal tensor field \( P_G \) from spacetime \( M \) to the manifold \( \Sigma \), resulting in the \( \deg P + 1 \) tensor fields

\[
p^{\alpha_1 \ldots \alpha_i}(t, \sigma) := P_{G_X(t, \sigma)}(\epsilon^{\alpha_1}(t, \sigma), \ldots, \epsilon^{\alpha_i}(t, \sigma), \epsilon^0(t, \sigma), \ldots, \epsilon^0(t, \sigma)) \quad \text{for} \ i = 0, \ldots, \deg P,
\]

where the total symmetry of \( P_G \) enables the simpler index notation chosen here for the various induced tensor fields \( p \). Due to the definition of the dual projection frame \( \epsilon^0, \ldots, \epsilon^3 \),
however, the first two fields of this set, for $i = 0, 1$, are trivial,

$$p(t, \sigma) = 1 \quad \text{and} \quad p^\alpha = 0.$$  \hspace{1cm} (1)

Finally note that for any fixed value of the foliation parameter $t$, all fields $p(t)$ and $g(t)$ present not only tensor fields on $\Sigma$, but, at the same time, are functionals of the embedding map $X_t$. This will become technically relevant in the following subsection.

C. Hypersurface deformation algebra

The kinematical information, encoded in the triple $(M, G, P_G)$ in general and the Legendre map $\ell$ in particular, takes its most useful form in the deformation algebra of hypersurfaces. The latter is the commutator algebra of the functional differential operators

\begin{align*}
H_t(n) &:= \int_{\Sigma} d^3 z \, n(z) e^a_0(t, z) \frac{\delta}{\delta X_t^a(z)} \quad \text{and} \quad D_t(\vec{n}) := \int_{\Sigma} d^3 z \, n^a(z) e^a_0(t, z) \frac{\delta}{\delta X_t^a(z)}
\end{align*}

acting on functionals of the the embedding maps $X_t : \Sigma \rightarrow M$ introduced in the previous subsection. We defined these operators for arbitrary test functions $n$ and $\vec{n}$ on the manifold $\Sigma$ in order to calculate their algebra below. Their geometric meaning, namely as normal and tangential deformation operators, is revealed, however, by letting $n := n$ and $\vec{n} := \vec{n}$, for the lapse and shift fields $n$ and $\vec{n}$ induced by the foliation.

Note that only the Legendre map $\ell$ enters here, namely implicitly in the definition of the normal vector field $e_0(t, z)$ along the hypersurface $X_t(\Sigma)$. As is straightforward to calculate \cite{6,10}, the commutator algebra of the above set of operators is

\begin{align*}
[H_t(n), H_t(m)] &= -D_t(\nabla P - 1) p^{\alpha \beta}_t (m \partial_\beta n - n \partial_\beta m) \partial_\alpha) , \hspace{1cm} (2) \\
[D_t(\vec{n}), H_t(m)] &= -H_t(\nabla \vec{m}) , \hspace{1cm} (3) \\
[D_t(\vec{n}), D_t(\vec{m})] &= -D_t(\nabla \vec{m}) . \hspace{1cm} (4)
\end{align*}

These equal parameter time commutation relations only depend on the components $p^{\alpha \beta}$ of the induced principal polynomial (with all lower and higher rank components $p^{\alpha_1 \ldots \alpha_n}_t$ with $n \neq 2$ not being relevant) and thus on the initially specified matter field dynamics and their geometric background. How the dynamics of the geometry arise as a representation of this hypersurface deformation algebra is the subject of chapter \[LV\]
III. CANONICAL KINEMATICS

In this section, we revert our previous perspective, where the spacetime geometry was considered as primary and then induced a geometry on the leaves of some foliation, to the opposite canonical point of view, which now considers the geometry on the leaves as primary and the spacetime geometry as only reconstructed from there by virtue of the foliation. Technically, this amounts to mimicking the induced geometry $g^A$ by independent tensor fields $g^A$ of the same tensor type. This change of perspective comes at the price that four generically non-linear conditions, which the $g^A$ satisfied by construction, must now be reinstated explicitly for the $g^A$. But in contrast to previous work, where these constraints had been left as almost intractable subsidiary conditions in the solution of the gravitational closure equations, we now devise an associated bundle technique that captures the constraints automatically. This is the conceptual and technical basis for the construction of the canonical phase space for the geometry, at the beginning of the next chapter and throughout the remainder of this paper.

A. Canonical geometry

The transition from the induced geometry to the canonical one is easily made precise as follows. If the geometry $g^A(t)$ is induced from a spacetime geometry $G$ by virtue of a foliation $X_t : \Sigma \rightarrow M$, together with an induced lapse $n(t)$ and induced shift $n^\alpha(t)$, then we introduce

$$g^A(t), \quad n(t), \quad n^\alpha(t)$$

as new, independent one-parameter families of tensor fields on $\Sigma$, which capture precisely the tensor structure of the fields $g^A(t)$, the lapse $n(t)$ and the shift $n^\alpha(t)$. Note that the construction of the induced tensor fields $g^A(t)$ automatically equips them with properties that are not captured by their mere tensor valence, while their valence is indeed the only information left after the transition to $g^A(t)$. How to reinstate the missing information will be remedied in the next subsection, and already leads to the associated bundle techniques mentioned above.

We will also need to translate quantities that were previously defined in terms of the induced geometry $g^A$, into corresponding quantities of the $g^A$. The most relevant such
transition for the purposes of this paper, is the one from the $p^{\alpha_1 \ldots \alpha_i}(t)$ to the new one-parameter families of fields

$$p^{\alpha_1 \ldots \alpha_i}(t) \quad \text{for } i = 0, \ldots, \deg P,$$

which are defined as precisely the same functions of $g^{ef}(t)$ as the $p^{\alpha_1 \ldots \alpha_i}(t)$ were of the $g^{ef}(t)$.

B. Frame conditions and symmetry conditions

The most relevant property of the induced geometry $g^{ef}$, which is not automatically captured by the canonical geometry $g^{ef}$, is the frame conditions [1]. While these are satisfied for the induced fields $p$ and $p^\alpha$ by construction, this information is lost when the functionals $g$ are replaced by the fields $g^{ef}$ that merely mimic their tensorial structure. Thus the normalization and annihilation conditions must be explicitly reinstated as

$$p(g)(t) = 1 \quad \text{and} \quad p^\alpha(g) = 0. \quad (5)$$

These conditions impose four generically non-linear conditions on the canonical geometry $g^{ef}$ and thus effectively remove four of their degrees of freedom. These non-linear relations are captured, beginning with the next subsection, by a suitable parametrization.

Similarly, any algebraic symmetry of the spacetime geometry $G$ is automatically inherited to the induced tensor fields $g^{ef}$, but must again be explicitly reinstated for the canonical geometry $g^{ef}$ by additional, now however linear and homogeneous, conditions

$$(\delta^{ef}_{\beta} - \Pi^{ef}_{\beta})g^{\beta} = 0 \quad (6)$$

for suitable projectors $\Pi$. These additional symmetry conditions can be implemented without extra effort alongside the generically non-linear frame conditions by the method developed in the following subsection. The possibility of such a combined treatment was one reason for withholding the implementation of symmetry conditions before.

C. Parametrization of the canonical geometry

The configuration variables of the gravitational dynamics we are about to construct parameterize, without further constraints, canonical geometries $g^{ef}$ that respect the frame and
symmetry conditions identified in the previous subsection. But because of their generic overall non-linearity, these conditions cannot be implemented by simply cutting away some tensor field components among the $g^{\alpha \beta}$ while keeping others. In fact, the situation is pretty much the same as for a particle in Euclidean space that is conditioned to move on an embedded submanifold, such as a circle. One cannot simply cut away one of the Cartesian coordinates, as one could if the particle was constrained to a linear subspace instead. The conceptually and technically best solution in classical mechanics is to introduce generalized coordinates. The same idea applies here. We require exactly as many configuration variables $\varphi^1, \ldots, \varphi^F$ as are needed to bijectively parametrize the tensor fields $g^{\alpha \beta}$ such that the frame conditions (5) and symmetry conditions (6) are met by construction. Technically, this is achieved by choosing a suitable $F$-dimensional manifold $\Phi$ and smooth maps $\hat{g}^{\alpha \beta} : \Phi \to \mathbb{R}$ such that any canonical geometry $g^{\alpha \beta}$ generated by $\hat{g}^{\alpha \beta}(\varphi^1, \ldots, \varphi^F)$ satisfies the conditions

$$(\delta^{\alpha \beta} - \Pi^{\alpha \beta})\hat{g}^{\alpha \beta}(\varphi(t)) = 0,$$

$$p(g(\varphi(t, \sigma))) = 1 \quad \text{and} \quad p^\alpha(g(\varphi(t, \sigma))) = 0$$

for any $\sigma \in \Sigma$ and any real $t$ in the range of the foliation parameter. If one single map $\hat{g}^{\alpha \beta}$ does not suffice to cover the required range, the usual chart transition constructions can be invoked. The number $F$ of configuration variables is the total number of all $g^{\alpha \beta}$ minus the normalization condition minus the three annihilation conditions and minus the dimension of the eigenspace $\text{Eig}_1(\Pi)$ of the projector $\Pi$. For instance, when the triple $(M, G_{\text{metric}}, G_{\text{metric}}^{-1})$ induced by a Lorentzian metric $G_{\text{metric}}$, there are $F = 16 - 3 - 1 - 6 = 6$ configuration field variables, which, due to all constraints being linear in this case, can coincidentally be written as a not further constrained metric tensor on the three-dimensional manifold $\Sigma$.

Conversely, we require the existence of inverse maps $\hat{\varphi}^A$ that send any collection $g^{\alpha \beta}$ (even if the frame and symmetry conditions are not met) to a real number, but which are constructed such that

$$(\hat{\varphi}^A \circ \hat{g})(\varphi) = \varphi^A \quad \text{for } A = 1, \ldots, F.$$
throughout the theory is then made by the maps
\[ \frac{\partial \hat{g}^A}{\partial \hat{g}^B}(\hat{g}(\varphi)) \quad \text{and} \quad \frac{\partial \hat{g}^A}{\partial \varphi^B}(\varphi), \]
as they emerge as intertwiners between the components of the canonical geometry, labeled by \( \mathcal{A} \), and the components of the configuration variables, labeled by \( A \). The above defining conditions for the maps \( \hat{g} \) and \( \hat{\varphi} \) immediately imply the important and heavily used completeness relations
\[ \frac{\partial \hat{g}^A}{\partial \hat{g}^B}(\hat{g}(\varphi)) \frac{\partial \hat{g}^B}{\partial \varphi^C}(\varphi) = \delta^A_C \quad \text{and} \quad \frac{\partial \hat{g}^A}{\partial \varphi^B}(\varphi) \frac{\partial \hat{g}^B}{\partial \hat{g}^C}(\hat{g}(\varphi)) = \mathcal{T}^A_{\mathcal{B}}(\varphi), \]
where \( \mathcal{T}^A_{\mathcal{B}}(\varphi) \) is defined by the left hand side and is easily seen to be a projector.

IV. CANONICAL DYNAMICS

Employing the technology developed in the previous section, we now significantly improve and extend the results of [6]. The crucial technical difference is the identification of the geometric phase space with the non-tensorial configuration variables and canonically conjugate momentum densities, whose transformation behaviour already captures the non-linear constraints on the canonical geometry that was left to be implemented only afterwards in previous treatments. The complete determination of the gravitational Hamiltonian, conditioned to yield a diffeomorphism invariant theory whose canonical evolution is compatible with that of the initially specified matter theory, then finally leads to the gravitational construction equations. This is a countably infinite set of partial differential equations that needs to be solved in order to obtain the gravitational Hamiltonian or, equivalently, Lagrangian density.

A. Canonical phase space

Having identified the unconstrained geometric configuration variables \( \varphi^A \) for a space-time geometry \( G \) in the previous section, we are now in the position to adjoin canonically conjugate momentum fields \( \pi_A \) with respect to the field-theoretic Poisson bracket
\[ \{ F, G \} := \int_{\Sigma} d^3 z \left( \frac{\delta F}{\delta \varphi^A(z)} \frac{\delta G}{\delta \pi_A(z)} - \frac{\delta G}{\delta \varphi^A(z)} \frac{\delta F}{\delta \pi_A(z)} \right), \]
which is to be evaluated on any two functionals $F[\varphi, \pi]$ and $G[\varphi, \pi]$ of the canonical configuration variables $\varphi^A$ and the associated canonical momenta $\pi_A$. We remark in passing that, as usual, there is an ambiguity in the choice of the canonical momenta for some given set of configuration variables $\varphi^A$. For if $\pi_A$ presents a possible choice, then so does $\pi_A + \delta \Lambda / \delta \varphi^A$ for any functional $\Lambda$ of the configuration variables only.

From the obvious requirement that this bracket be well-defined (namely independent of the choice of coordinates $z$ on the manifold $\Sigma$), we can derive the precise mathematical nature of the momenta. Technically, the key observation is that the $F$ configuration variables $\varphi^A$ are a section of an $F$-dimensional $\Phi$-fibre bundle over $\Sigma$, which is an associated bundle with respect to the frame bundle $L\Sigma$ by virtue of the (generically non-linear) group action $\rho : GL(3) \times \Phi \rightarrow \Phi$ that is enforced by the way the $\varphi^A$ transform under coordinate transformations:

$$\rho^A\left(\frac{\partial \tilde{z}}{\partial z}, \varphi\right) := \tilde{\varphi}^A\left(\mathcal{R}^B_{\Phi}(\frac{\partial \tilde{z}}{\partial z}) \tilde{g}^B(\varphi^1, \ldots \varphi^F)\right),$$

where $\mathcal{R}^B_{\Phi}(\frac{\partial \tilde{z}}{\partial z})$ denotes the standard tensorial action of the $GL(3)$-transformation $\partial \tilde{z} / \partial z$ on the various tensors on $\Sigma$ which we collectively labeled by $\Phi$. Note that the above transformation behavior of configuration variables are not a postulate, but directly follows from our choice of parameterization map $\tilde{\varphi}$ and its inverse $\tilde{g}$, on which the group action then naturally depends. But with the transformation behavior of the configuration variables thus under control, we can now straightforwardly read off the $GL(3)$ group action that defines a further associated $\Pi$-fibre bundle over the manifold $\Sigma$, of which the canonical momenta $\pi_A$ shall constitute a section. In order for the Poisson bracket above to be well-defined, we then need to impose the group action $\rho^* : GL(3) \times \Pi \rightarrow \Pi$

$$\rho_A^*\left(\frac{\partial \tilde{z}}{\partial z}, \pi\right) := \left(\det \frac{\partial \tilde{z}}{\partial z}\right) \frac{\partial \tilde{g}^B}{\partial \varphi^A}\mathcal{R}^{-1}_{\Phi}(\frac{\partial \tilde{z}}{\partial z}) \frac{\partial \tilde{g}^A}{\partial \pi_B} \tilde{\pi}_B.$$

Indeed, it is easy to see that then the Poisson bracket is well-defined, because the functional derivative $\delta F / \delta \varphi^A(z)$ has density weight one (since $\varphi^A$ has density weight zero), while the fact that $\pi_A$ already has density weight one cancels the density weight from the functional differentiation in $\delta G / \delta \pi_A(z)$, rendering the latter of weight zero. Thus the integrand of the Poisson bracket can be shown to be a scalar density of weight one and thus the integral to be well-defined.
B. Constraint algebra

In this section, we identify the two key properties that we require of dynamics for the geometric configuration variables $\varphi^A$. On the one hand, we wish to devise a theory that is invariant under spacetime diffeomorphisms. On the other hand, we technically implement the condition that it share its initial data surfaces with those of the previously specified matter theory, in order to ensure common canonical evolution.

So first, in order for gravitational dynamics to be invariant under spacetime diffeomorphisms, the pertinent Hamiltonian must be of the totally constrained form

$$H[\varphi, \pi; n, \vec{n}] = \mathcal{H}(n) + \mathcal{D}(\vec{n}),$$

with a so-called superhamiltonian and supermomentum

$$\mathcal{H}(n) := \int_\Sigma d^3z n(z) \mathcal{H}[\varphi(z), \pi(z)] \quad \text{and} \quad \mathcal{D}(\vec{n}) := \int_\Sigma d^3z n^\alpha(z) \mathcal{D}_\alpha[\varphi(z), \pi(z)]$$

that depend linearly on the underived lapse and shift fields, such that the latter act as Lagrange multipliers.

Secondly, in order to implement canonical dynamics for the geometry that consistently start and end on the same initial data surfaces as the previously specified matter dynamics, we follow in imposing that the superhamiltonian $\mathcal{H}(n)$ evolve canonical geometric data $g^{\text{cf}}$ in precisely the same fashion in phase space as the normal deformation operator $\mathcal{H}(n)$ from section II C evolves the induced geometry $g$ on spacetime, which amounts to requiring

$$\mathcal{H}_t(n)g^{\text{cf}} = -\left\{\mathcal{H}(n), g^{\text{cf}}\right\}_t,$$

where the equal sign is to be understood in the sense that the right hand side is the same function of the canonical geometry $g^{\text{cf}}$ as the left hand side is of the induced geometry $g$. Similarly, we require

$$\mathcal{D}_t(\vec{n})g^{\text{cf}} = -\left\{\mathcal{D}(\vec{n}), g^{\text{cf}}\right\}_t.$$

In order to ensure that these two requirements are consistent throughout, we additionally stipulate that the constraint algebra of the superhamiltonian and supermomentum mimic the commutator algebra in section II C, so that

$$\left\{\mathcal{H}(n), \mathcal{H}(m)\right\} = \mathcal{D} \left((\text{deg}P - 1)p^\alpha\beta(m \partial_\beta n - n \partial_\beta m)\partial_\alpha\right),$$

$$\left\{\mathcal{D}(\vec{n}), \mathcal{H}(m)\right\} = \mathcal{H}(\mathcal{L}_{\vec{n}} m),$$

$$\left\{\mathcal{D}(\vec{n}), \mathcal{D}(\vec{m})\right\} = \mathcal{D}(\mathcal{L}_{\vec{n}} \vec{m}).$$
Together with the conditions (8) and (9), this constraint algebra provides functional
differential conditions on \( H(n) \) and \( D(\vec{n}) \) that turn out to be so strong as to determine
the superhamiltonian and supermomentum from it. The resulting Hamiltonian \( \mathcal{H}(n) \) then
generates the evolution of phase space curves \((\varphi^A(t), \pi_A(t))\) with respect to what, from a
spacetime point of view, is the foliation parameter \( t \). The thus generated “geometry movie”
on the manifold \( \Sigma \) can then be embedded, frame by frame, into the spacetime manifold by
virtue of the one-parameter family \( X_t : \Sigma \to M \) by letting \( n := n \) and \( \vec{n} := \vec{n} \), which results
in the immutable spacetime geometry \( G \). This is the mechanism behind the possibility to
attain the dynamical closure of prescribed canonically quantizable matter dynamics.

C. Functional differential reformulation

We now solve the autonomous third constraint equation (12) for the supermomentum
\( \mathcal{D}(\vec{n}) \) and are thus able to reformulate the first two constraint equations (10) and (11) as
linear functional differential equations for a suitable Lagrangian functional.

Carefully taking into account the parametrization \( \hat{g}^{\sigma \sigma} \) of the canonical geometry \( g^{\sigma \sigma} \) in
terms of the configuration variables \( \varphi^A \), one finds that the constraint algebra equation (10)
and the compatibility condition (9) together already completely determine the constraint
functional to be

\[
\mathcal{D}(\vec{n}) = \int_{\Sigma} \mathrm{d}z \pi_A(z) \frac{\partial \hat{\varphi}^A}{\partial g^{\sigma \sigma}(\hat{g}(\varphi(z)))} \left( \mathcal{L}_{\vec{n}} \hat{g}(\varphi) \right)^{\sigma \sigma} (z),
\]

with the only, but significant, novelty compared to (4.16) of \[6\] being the appearance of the
intertwiner map \( \partial_{\hat{\varphi}^A}/\partial g^{\sigma \sigma} \) and the parametrization map \( \hat{g}^{\sigma \sigma} \). Likewise, the first compati-
bility condition (8) leads to a crucial partial determination

\[
\mathcal{H}(n) = \int_{\Sigma} d^3z n(z) \left[ \mathcal{H}_{\text{local}}[\varphi; \pi](z) - \partial_{\gamma} \left( M^A_A(\varphi) \pi_A \right)(z) \right]
\]

of the functional \( \mathcal{H} \) in terms of the coefficient \( M^A_A(\varphi) := \partial_{\hat{\varphi}^A}/\partial g^{\sigma \sigma}(\hat{g}(\varphi)) e^a_0(t, \sigma) \frac{\partial g^{\sigma \sigma}}{\partial X^a_{\gamma}}(t, \sigma), \)

where the last factor is easily calculated from the definition of the \( g^{\sigma \sigma} \) using the identities

\[
\frac{\partial e^m_0}{\partial X^a_{\gamma}} = -(\deg P - 1) e^m_\sigma e^0_\sigma^\rho \sigma^\gamma \quad \text{and} \quad \frac{\partial e^m}{\partial X^a_{\gamma}} = \delta^m_a \delta^\gamma_{\rho},
\]
for the tangent frame fields, and
\[
\frac{\partial \epsilon_m^0}{\partial \gamma X^a} = -\epsilon_a^\gamma \epsilon_m^0 \quad \text{and} \quad \frac{\partial \epsilon_m^\mu}{\partial \gamma X^a} = -\epsilon_a^\mu \epsilon_m^0 + (\deg P - 1)\epsilon_m^0 \epsilon_a^\rho \Pi^\gamma
\]
for the cotangent fields.

Thus the gravitational Hamiltonian (7) is already determined up to a functional \( H_{\text{local}}[\varphi; \pi] \), which is a function of the canonical momenta and arbitrarily high derivatives of the configuration variables. The determination of this remaining piece of the superhamiltonian, however, requires significant work and will finally lead to the gravitational closure equations in the next subsection. We prepare the derivation of these closure equations by following again closely [6] in applying a trick due to Kuchař [15], which converts both the first two constraint equations into linear equations.

To this end, we define the generalized velocity fields
\[ k^A[\varphi; \pi] := \frac{\partial H_{\text{local}}}{\partial \pi_A}[\varphi; \pi] \]
and subsequently perform a formal Legendre transformation on the \( \pi_A \), rewriting
\[ H_{\text{local}}[\varphi; \pi] = \pi_A k^A[\varphi; \pi] - \mathcal{L}[\varphi, k[\varphi, \pi]) \]
and thus trading the unknown density \( H_{\text{local}}[\varphi, \pi] \) for another unknown density \( \mathcal{L}[\varphi, k) \). The benefit of this trade, however, is that the quadratic condition (10) on \( H_{\text{local}} \) is converted into a merely linear homogeneous functional differential equation for the functional \( \mathcal{L} \). Indeed, using the same idea as in [6], but now employing the parameterization \( \hat{g}^A \) of the canonical geometry in terms of the configuration variables \( \varphi \), one picks up crucial additional terms and finally obtains the functional differential form of (10). More precisely, using the

shorthand notation \( Q_{A_1\ldots A_N} := \frac{\partial Q}{\partial \varphi_{A_1\ldots A_N}} \),

to denote partial derivatives with respect to partial derivatives of configuration variables over \( \Sigma \), where \( Q \) is any differentiable function of the configuration variables and their partial derivatives on \( \Sigma \), this functional differential equation equivalent to the bracket (10) reads
\[
0 = -k^B(y) \left( \frac{\partial \mathcal{L}(x)}{\partial \varphi^B(z)} \right)(y) + (\partial_x \delta_x)(y) k^B(y) M^{A_B}(x) \frac{\partial \mathcal{L}}{\partial k^A}(x) + \partial_{\mu} \left( \frac{\partial \mathcal{L}(x)}{\partial \varphi^\mu(z)} M^B_{\mu} \right)(y) \\
+ \partial_{\mu} \left( \frac{\partial \mathcal{L}}{\partial k^A}(x) \right) \left[ (\deg P - 1)p^\rho{}^\nu F_{\rho}{}^{\nu} - M^B[\mu]M^A[\nu]_{B} \right](x) (\partial_x \delta_x)(y) \\
- \frac{\partial \mathcal{L}}{\partial k^A}(x) \left[ (\deg P - 1)p^\rho{}^\nu \left( E_{\rho}^{\nu} + F_{\rho}^{\nu} \right) + \partial_{\mu} \left( M^B[\mu]M^A[\nu]_{B} \right) \right](x) (\partial_x \delta_x)(y) - (x \leftrightarrow y)
\]
with the new coefficients $E^A_\mu$ and $F^A_{\mu\nu}$ defined by
\[
\frac{\partial \tilde{\phi}^A}{\partial g^{ij}} (\mathcal{L}_\phi \tilde{g}(\varphi))^{ij} =: n^\mu E^A_\mu - \partial_\gamma n^\mu F^A_{\mu\gamma}.
\] (14)

Similarly, one rewrites the second constraint algebra relation \( \Pi \) as an additional linear homogeneous functional differential equation for the density $\mathcal{L}$, namely
\[
0 = \left( \frac{\partial \mathcal{L}}{\partial k^B} \right)(y) k^A(y) \left[ E^B_{\mu;A}(y) \delta_y(x) + \left( E^B_{\mu;A} + F^B_{\mu;A} \right)(y)(\partial_\gamma \delta_y)(x) \right]
- k^A(y)(\partial_\gamma \frac{\partial \mathcal{L}}{\partial k^B})(y) F^B_{\mu;A}(y) \delta_y(x)
- \left( k^A \frac{\partial \mathcal{L}}{\partial k^A} - \mathcal{L} \right)(y)(\partial_\mu \delta_y)(x) + \partial_\mu \left( k^A \frac{\partial \mathcal{L}}{\partial k^A} - \mathcal{L} \right)(y) \delta_y(x)
+ \left( E^A_\mu + F^A_{\mu;\gamma} \right)(x) \frac{\delta \mathcal{L}(y)}{\delta \varphi^A(x)} + F^A_{\mu;\gamma}(x) \partial_\gamma \left( \frac{\delta \mathcal{L}(y)}{\delta \varphi^A(\cdot)} \right)(x).
\]

The coefficients $E^A_\mu$, $F^A_{\mu;\gamma}$, $M^{B\mu}$ and $p^{\alpha\beta}$ are completely determined by the triple $(M, G, P_G)$, and need to be provided as input when solving the functional differential equations (or the indeed equivalent closure equations derived in the next subsection) for the only remaining unknown functional $\mathcal{L}$. We will therefore refer to these coefficients as the \textit{input coefficients} from now on. They are always directly calculated from the initially specified matter dynamics and their background geometry $G$.

**D. Gravitational closure equations**

The gravitational closure equations, for any given canonically quantizable matter field theory, are the countably infinite set of partial differential equations for the sequence of coefficient functionals
\[
C[\varphi], \quad C_{A_1}[\varphi], \quad C_{A_1A_2}[\varphi], \quad \ldots
\]
in terms of which one may expand the to-be-determined functional as the series
\[
\mathcal{L}[\varphi; K] = \sum_{N=0}^{\infty} C_{A_1\ldots A_N}[\varphi] k^{A_1} \ldots k^{A_N}.
\]
while this reformulation of the two functional differential equations we found for $\mathcal{L}$ in the previous subsection comes at the price of now having to solve countably many equations, this step makes the problem accessible to a properly developed machinery for the study of systems of partial differential equations \cite{12}.
The derivation of these linear homogeneous equations is a painstaking exercise. With two crucial exceptions, it however proceeds technically fully analogously to the steps performed in [6]. The first exception is that we now employ the parametrization \( \hat{g}^{\varphi} \) of the canonical geometry \( g^{\varphi} \) in terms of the unconstrained configuration variables \( \varphi^A \), which captures the generically non-linear polynomial frame conditions and any additional symmetry conditions for the tensor fields \( g^{\varphi} \). The second exception is that we now convert also the second functional differential equation for \( \mathcal{L} \) into a set of partial differential equations, since the workaround taken before is not longer available for the generalized tensor components we now use as configuration degrees of freedom [16]. Since it is ultimately straightforward to adapt the calculations of [6] to the new technical developments of this paper, we content ourselves with displaying the resulting set of linear homogeneous partial differential equations in terms of the

seven individual equations \((C1)\) to \((C7)\) and

fourteen sequences of equations \((C8_N)\) to \((C21_N)\) for \( N \geq 2 \)

displayed on the last two landscape pages of this article, which are the gravitational closure equations. Set up by provision of the matter-determined input coefficients \( E^A_{\mu}, \ F^A_{\mu \nu}, \ M^B_{\mu} \), their solution yields the sequence of output coefficients \( \{C_{A_1...A_N}[\varphi]\}_{N \geq 0} \) and thus the dynamics for the spacetime geometry.

The explicit form of these construction equations, as they are listed on the last two pages of this article, have already been simplified in so far as their derivation yields that, for \( N \geq 2 \), all output coefficients are functions

\[
C_{A_1A_2...A_N}(\varphi, \partial\varphi, \partial\partial\varphi),
\]

i.e., they only depend on at most second partial derivatives of the configuration variables \( \varphi^A \) with respect to the base manifold \( \Sigma \). A weaker result applies to the first two output coefficients

\[
C[\varphi] \quad \text{and} \quad C_A[\varphi],
\]

namely that if \( C_A \) depends on partial derivatives of the \( \varphi \) up to the \( D \)-th order, then \( C \) depends on partial derivatives up to order \( \max\{2, D+1\} \). Thus the first question, which one has to address in solving the gravitational closure equations for specific input coefficients, is whether \( C_A \) depends on at most finite order derivatives of \( \varphi \) and, if so, what this highest order \( D \) is.


E. Canonical equations and equivalent spacetime action

A practically most convenient result is turned up by translation of our results from the canonical picture back to a spacetime formulation. Indeed, the construction equations immediately provide a perfectly simple, ready-to-use spacetime action that just needs to be varied with respect to the components of the spacetime geometry in order to obtain the gravitational field equations, as usual.

In the canonical picture, it is the Hamiltonian \( H \) that determines the evolution of our canonical configuration and momentum degrees of freedom according to

\[
\dot{\varphi}_A(y) = \{ \varphi^A(y), H(n, \tilde{n}) \}_t \quad \text{and} \quad \dot{\pi}_A(y) = \{ \pi_A(y), H(n, \tilde{n}) \}_t,
\]

where the dot denotes the derivative with respect to the foliation parameter \( t \). The parameter \( t \) as well as the lapse \( n_t \) and shift \( \tilde{n}_t \) precisely parametrize the possible choices one could make to embed the three-dimensional manifold \( \Sigma \), on which the canonical dynamics play out, into the four-dimensional spacetime. The required diffeomorphism invariance of the theory is precisely the freedom to choose this embedding without changing the contents of the theory.

Inclusion of matter, with a Hamiltonian \( H_{\text{matter}}[A; \varphi, n, n^\alpha] \) that does not depend on derivatives of the \( \varphi^A, n \) and \( n^\alpha \), thus leads to the geometric evolution equations

\[
\frac{\delta H_{\text{matter}}}{\delta \varphi(x)} = - \left[ \partial_t - n^\mu \partial_\mu - \partial_\mu n^\mu + (\partial_\gamma n) \frac{\partial M^{B\gamma}}{\partial \varphi} - (\partial_\gamma n^\mu) \frac{\partial F^{B\gamma}_\mu}{\partial \varphi^A} \right] \frac{\partial \mathcal{L}}{\partial k^A}(x) + \int d^3 y n(y) \frac{\delta \mathcal{L}(y)}{\delta \varphi(x)},
\]

and the two constraint equations

\[
\frac{\delta H_{\text{matter}}}{\delta n(x)} = - \left[ k^A - \partial_\gamma M^{A\gamma} - M^{A\gamma} \partial_\gamma \right] \frac{\partial \mathcal{L}}{\partial k^A}(x) + \mathcal{L}(x)
\]

and

\[
\frac{\delta H_{\text{matter}}}{\delta n^\mu(x)} = - \left[ \partial^\gamma \varphi^A + \partial_\gamma F^{A\mu}_\gamma + F^{A\mu}_\gamma \partial_\gamma \right] \frac{\partial \mathcal{L}}{\partial k^A}(x),
\]

in all three of which the \( k^A \) are to be replaced by

\[
k^A(x) = \frac{1}{n(x)} \left[ \partial_t \varphi - (\partial_\gamma n) M^{A\gamma} + n^\mu \partial_\mu \varphi^A - (\partial_\gamma n^\alpha) F^{A\gamma}_\mu \right](x)
\]

after previous execution of all related derivatives. The constraints are thus manifestly of at most first derivative order in the foliation parameter \( t \), and the evolution equations of
at most second derivative order in $t$, with respect to any chosen foliation. So there are, in particular, no Ostrogradsky ghosts [17]. The Helmholtz action giving rise to these canonical equations of motion is simply

$$S[\varphi, \pi, n, n^\alpha] = \int dt \left\{ -H_t[\varphi, \pi, n, n^\alpha] + \int_{\Sigma} dz \left( \pi_A \dot{\varphi}^A \right)(z) \right\},$$

but, remarkably, can be expressed directly in terms of the functional $L$ that follows from a solution of the construction equations. To see this, one uses the first Hamiltonian equation of motion above to express the derivative of the configuration variables with respect to the foliation parameter as

$$\dot{\varphi}^A = n^A[\varphi; \pi] + (\partial_\gamma n) M^{A\gamma}(\varphi) + \frac{\partial \hat{\varphi}^A}{\partial g^{\gamma\delta}}(\varphi) (L_{\vec{\epsilon}} \hat{g}(\varphi))^A. \quad (15)$$

Upon insertion of this expression and the partially determined Hamiltonian (13) into the above Helmholtz action, one immediately observes that all terms but the ones coming from the local superhamiltonian drop out. Finally converting the $k^A$ back to the $\dot{\varphi}^A$ by use of the first Hamiltonian equation, one obtains an equivalent action

$$S[\phi, N, N^\alpha] = \int dt \int_{\Sigma} dz \mathcal{L}_{\text{geometry}}[\phi, N, N^\alpha](t, z)$$

where the capitalized quantities

$$\phi(t, z) := \varphi_t(z), \quad N(t, z) := n_t(z), \quad N^\alpha(t, z) := n^\alpha_t(z)$$

numerically precisely coincide with the configuration variables, the lapse and the shift, but are now all considered as spacetime quantities, rather than one-parameter families on the manifold $\Sigma$. In particular, functionals of the capitalized quantities may now include time derivatives, such as the Lagrangian density obtained by simple multiplication of the lapse $N$ with the solution $L$ of the construction equations,

$$\mathcal{L}_{\text{geometry}}[\phi, N, N^\alpha] = N \cdot L\left[\phi, \frac{1}{N} \left( (\dot{\varphi}^A - (\partial_\gamma n) M^{A\gamma}(\varphi) - \frac{\partial \hat{\varphi}^A}{\partial g^{\gamma\delta}}(\varphi) (L_{\vec{\epsilon}} \hat{g}(\varphi))^A \right) \right]. \quad (16)$$

Indeed, it is quickly checked that varying the thus obtained total action

$$S_{\text{geometry}}[\phi, N, N^\alpha] + S_{\text{matter}}[A; \phi, N, N^\alpha]$$

with respect to the $\phi$, $N$ and $N^\alpha$ in a way that properly includes also time derivatives in the variations, yields a set of equations equivalent to the canonical gravitational evolution equations above.
V. EXAMPLES: MATTER ON METRIC, BI-METRIC AND HIGHER-RANK GEOMETRIES

How truly simple it is now — due to the new parameterization technology of any canonical geometry $g^{\alpha\beta}$ in terms of non-tensorial configuration variables $\varphi^A$ — to set up the gravitational closure equations for any given canonically quantizable matter action on any tensorial background, is illustrated by the three hopefully instructive examples presented in this last section. The first one, in section V A is a warm-up that starts from standard model matter, new only in that it uses non-tensorial configuration variables as the simplest illustration of how the latter are employed in practice. An illustration of how unexpectedly non-trivial the gravitational closure can turn out to be is then provided by the second example, which starts from an innocent-looking set of two scalar fields on a bi-metric background as the prescribed matter theory, for which the corresponding closure equations are set up in section V B. The last section V C finally presents the gravitational closure equations for a gravity theory of some phenomenological interest, namely the one underpinning the most general birefringent linear electrodynamics. The seriously involved construction equations for this theory are solved perturbatively in [18].

A. Gravitational closure of Klein-Gordon theory on a metric geometry

The arguably simplest canonically quantizable matter field theory we can formulate on a metric background $(M, G)$ is the Klein-Gordon action for a scalar field $\phi$,

$$S[A; G] = \int d^4x \sqrt{-\det G(x)} \left[ G^{ab}(x) \partial_a \phi(x) \partial_b \phi(x) - m^2 \phi^2(x) \right],$$

whose principal polynomial can be read off directly from the highest order derivative term of the associated field equations and has the components

$$P_G^{ij} = G^{ij}.$$ 

All matter dynamics of the standard model of particle physics are constructed such that they feature this principal polynomial. Thus the above Klein-Gordon theory, standard abelian and non-abelian gauge theory and indeed Dirac fields (the latter precisely because the Dirac algebra $\gamma^a \gamma^b = G^{ab}$ recovers again the same principal polynomial) all produce the same
triple $(M,G,G^{-1})$, where the canonical quantizability requires the metric $G$ to have Lorentzian signature.

We now quickly rush through the steps described in this paper to set up the gravitational closure equations. First, the induced geometry is calculated to be

$$g^{00} := G(e^0, e^0), \quad g^{0\alpha} := G(e^0, e^\alpha), \quad g^{\alpha0} := G(e^0, e^\alpha), \quad g^{\alpha\beta} := G(e^\alpha, e^\beta).$$

The associated frame conditions

$$\mathbf{p} = g^{00} \equiv 1 \quad \text{and} \quad \mathbf{p}^\alpha = g^{\alpha0} \equiv 0$$

are obviously linear. Transition from the induced geometry to the corresponding 16 independent tensor field components $g$, $g^\alpha$ and $g^{\alpha\beta}$, and subsequent implementation of the above frame constraints plus the automatically linear symmetry constraints

$$g^{[\alpha\beta]} = 0 \quad \text{and} \quad g^{[\alpha0]} = 0,$$

removes $4 + 3 + 3$ tensor components, namely the fields $g$ and $g^\alpha$. Thus, we are only left with the canonical geometry $g^{\alpha\beta}$ of the metric that can be parametrized employing 6 in terms of configuration variables $\varphi^A$. Of the infinity of possible parameterizations, we choose the parametrization maps

$$\hat{g}^{\alpha\beta}(\varphi) := \mathcal{T}^{\alpha\beta}_A \varphi^A \quad \text{and} \quad \hat{\varphi}^A(g) := \mathcal{T}_A^{\alpha\beta} g^{\alpha\beta},$$

where the respective constant intertwining matrices need to satisfy the two conditions

$$\mathcal{T}^{\alpha\beta}_A \mathcal{T}^B_{\alpha\beta} = \delta^A_B \quad \text{and} \quad \mathcal{T}^{\gamma\delta}_A \mathcal{T}^A_{\alpha\beta} = \delta^{(\gamma}_{\alpha} \delta^\delta_{\beta)},$$

in order to render the above pair a valid parametrization, and an explicit choice is

$$\mathcal{T}^{\alpha\beta}_A := \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} 0 0 0 0 0 \\ 0 1 0 0 0 0 \\ 0 0 1 0 0 0 \\ 0 0 0 \sqrt{2} 0 0 \\ 0 0 0 0 1 0 \\ 0 0 0 0 0 \sqrt{2} \end{bmatrix}^{\alpha\beta}_A \quad \text{and} \quad \mathcal{T}^A_{\alpha\beta} := \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} 0 0 0 0 0 0 0 0 \\ 0 1 0 0 0 0 0 0 0 \\ 0 0 1 0 0 0 0 0 0 \\ 0 0 0 \sqrt{2} 0 0 0 0 0 \\ 0 0 0 0 1 0 0 0 0 \\ 0 0 0 0 0 \sqrt{2} 0 0 0 \\ 0 0 0 0 0 0 1 0 0 \\ 0 0 0 0 0 0 0 1 0 \end{bmatrix}^{\alpha\beta}_A,$$

which however is rarely needed explicitly. With this parametrization at hand, the input coefficients defining the specific gravitational closure equations for this case are quickly calculated to be given by

$$p^{\alpha\beta} = g^{\alpha\beta}, \quad E^A_{\mu} = \varphi^A_{,\mu}, \quad F^A_{\mu\gamma} = 2 \mathcal{T}^A_{\mu\alpha} \mathcal{T}^\alpha_{\beta} \varphi^B, \quad M^{A\mu} = 0.$$
Solving the resulting construction equations yields the only non-vanishing dynamical potentials

\[ C[\varphi] = -\frac{1}{2\kappa} \frac{1}{\sqrt{-\det \tilde{g}(\varphi)}} (R[\tilde{g}(\varphi)] - 2\Lambda), \]

\[ C_{AB}(\varphi) = \frac{1}{8\kappa} \frac{1}{\sqrt{-\det \tilde{g}(\varphi)}} T^{\alpha\beta}_{\Lambda} T^{\mu\nu}_{B} \left( \tilde{g}_{\alpha\mu}(\varphi) \tilde{g}_{\beta\nu}(\varphi) - \tilde{g}_{\alpha\beta}(\varphi) \tilde{g}_{\mu\nu}(\varphi) \right), \]

where \( R[g] \) denotes the Ricci curvature scalar built from an inverse three-dimensional metric \( g \) and \( \tilde{g}_{\alpha\beta}(\varphi) \) denotes the matrix inverse of \( \tilde{g}^{\alpha\beta}(\varphi) \). But this is exactly the 3+1 decomposition of the Einstein-Hilbert action

\[ S_{\text{geometry}}[G] = \frac{1}{2\kappa} \int d^4x \sqrt{-\det G} \cdot (R[G] + 2\Lambda) \]

with the gravitational constant \( \kappa \) and cosmological constant \( \Lambda \) having emerged as undetermined constants of integration. It should be noted that since nothing in our set-up has been designed to arrive at this result, the above is a successful test of the gravitational closure approach, as we know that the Einstein-Hilbert action is consistent with standard model matter. As indicated above, this result as such has been derived a long time ago by Kuchař and, indeed, our parametrization of the canonical geometry \( g^A \) in terms of non-tensorial configuration variables \( \varphi^A \) was a sledgehammer used to crack a nut, since the frame conditions were merely linear. But this will change dramatically already for the next, at first sight quite innocent-looking, example of two free scalar fields coupled to two different metrics.

### B. Gravitational closure of two Klein-Gordon fields on a bi-metric geometry

A veritable surprise is in store when we consider the case of a bimetric geometry, featuring two Lorentzian metrics \( G \) and \( H \). In order to equip this geometry with specific kinematical meaning, we inject the physical information contained in the matter action

\[ S[\phi, \psi; G, H] := \int d^4x \left[ \sqrt{-(\det G)(x)} G^{ab} \partial_a \phi(x) \partial_b \phi(x) + \sqrt{-(\det H)(x)} H^{ab} \partial_a \psi(x) \partial_b \psi(x) \right], \]

for scalar fields \( \phi \) and \( \psi \), where additional terms giving rise to first and zeroth derivative order terms at the level of the associated equations of motion could be added at will, since they will not influence the principal polynomial. The principal polynomial of this matter theory is simply the totally symmetrized product of the principal polynomials of the two
individual Klein-Gordon fields, i.e.,

\[ P_{G,H}^{ijkl} = G^{(ij} H^{kl)} , \]

which determines the triple

\[ (M, (G, H), P_{G,H}) . \]

This, incidentally, neatly illustrates that a multitude of matter fields, and even a multitude of geometric tensors, still results — as it must — in one and only one principal polynomial, which captures the information about the shared initial data surfaces. Moreover, it is straightforward to see that the principal tensor \( P_G \) provided by the above product principal polynomial has the algebraic dual

\[ P^\#_{ijkl} = G_{(ij} H_{kl)} \]

and that \( P \) and \( P^\# \) are both hyperbolic, as is required by the canonical quantizability of the matter action, if and only if both metrics \( G \) and \( H \) have Lorentzian signature.

The induced geometry is constructed similarly to the case of one Lorentzian spacetime metric. However, there are now twice as many fields as there are two Lorentzian metrics \( g^{\alpha\beta} \) and \( h^{\alpha\beta} \), yielding

\[
\begin{align*}
g^{00} & := G(e^0, e^0), \\
g^{0\alpha} & := G(e^0, e^\alpha), \\
g^{\alpha0} & := G(e^\alpha, e^0), \\
g^{\alpha\beta} & := G(e^\alpha, e^\beta), \\
h^{00} & := H(e^0, e^0), \\
h^{0\alpha} & := H(e^0, e^\alpha), \\
h^{\alpha0} & := H(e^\alpha, e^0), \\
h^{\alpha\beta} & := H(e^\alpha, e^\beta).
\end{align*}
\]

which satisfy, as always by construction of the employed frames, the frame conditions, which in this case read

\[
p = g^{00} \cdot h^{00} \overset{!}{=} 1 \quad \text{and} \quad p^\alpha = \frac{1}{2} (h^{00} g^{\alpha0} + g^{00} h^{\alpha0}) \overset{!}{=} 0 .
\]

Transition to the corresponding canonical geometry \( g, g^\alpha, g^{\alpha\beta}, h, h^\alpha, h^{\alpha\beta} \) requires to explicitly impose twelve symmetry conditions

\[
g^{[\alpha\beta]} = 0 , \quad h^{[\alpha\beta]} = 0 , \quad g^{[\alpha0]} = 0 , \quad h^{[\alpha0]} = 0 ,
\]

in addition to the frame conditions above, which reduce to requiring that

\[
h^{00} = \frac{1}{g^{00}} \quad \text{and} \quad h^{\alpha0} = -\frac{1}{(g^{00})^2} g^{\alpha0} .
\]
One thus finds that only 16 of the 32 components of the canonical geometry are independent. Unlike in the mono-metric case, however, the parametrization of the canonical geometry in terms of non-tensorial configuration variables, as developed in this paper, is now seriously needed, since the frame conditions are non-linear. Since it helps to group the relevant expressions, it is convenient to introduce the notation

$$(\varphi^1, \ldots, \varphi^{16}) =: (\vec{\varphi}, \vec{\varphi}^1, \vec{\varphi}^2, \vec{\varphi}^3, \ldots, \vec{\varphi}^{14}, \ldots, \vec{\varphi}^6, \ldots, \vec{\varphi}^6, \ldots, \vec{\varphi}^6, \ldots, \vec{\varphi}^6)$$

with the various groups of configuration variables mirroring the corresponding groups of tensors making up the canonical geometry, which we choose to parametrize as

$$\hat{g}(\varphi) := \vec{\varphi}, \quad \hat{g}^\alpha(\varphi) := \mathcal{I}^\alpha_a \vec{\varphi}^a, \quad \hat{g}^{\alpha\beta}(\varphi) := \mathcal{I}^{\alpha\beta} A \vec{\varphi}^A, \quad \hat{h}^{\alpha\beta}(\varphi) := \mathcal{I}^{\alpha\beta} A \vec{\varphi}^A,$$

where a lowercase Latin index $a$ ranges over 1, 2, 3, while an uppercase Latin index $A$ ranges over 1, $\ldots$, 6. The constant intertwining matrices $\mathcal{I}^{\alpha a} \varphi$ and $\mathcal{I}^{a \alpha} \varphi$ are as in the previous example of a mono-metric geometry, while

$$\mathcal{I}^{\alpha a} := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} a$$

and $\mathcal{I}^{a \alpha} := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} a$, and thus satisfy $\mathcal{I}^{\alpha b} \mathcal{I}^{a \alpha} = \delta^a_b$ and $\mathcal{I}^{a \alpha} \mathcal{I}^{\alpha a} = \delta^a_b$. The input coefficients are then straightforwardly calculated. Whenever it is convenient to keep terms and notation short and clear, the split of the configuration variables devised above will also be used in the expressions for the input coefficients. The non-vanishing input coefficients are

$$p^{\alpha\beta} = \frac{1}{6 \vec{\varphi}} \mathcal{I}^{\alpha\beta} A \left( \vec{\varphi}^A + \vec{\varphi} \vec{\varphi}^A \right) - \frac{2}{3 \vec{\varphi}^2} \mathcal{I}^{\alpha a} \mathcal{I}^{\beta b} \vec{\varphi} b \vec{\varphi}^b,$$

$$E^A_{\mu} = \varphi^{A, \mu},$$

$$F_{\mu}^{\varphi} = 0, \quad F_{\mu}^{\varphi} = \mathcal{I}^\alpha_a \mathcal{I}^\gamma_b \vec{\varphi}^{a \gamma}, \quad F_{\mu}^{A} = 2 \mathcal{I}^\alpha_a \mathcal{I}^{\gamma\alpha} B \vec{\varphi}^B, \quad F_{\mu}^{\varphi} = 2 \mathcal{I}^\alpha_a \mathcal{I}^{\gamma\alpha} B \vec{\varphi}^B,$$

$$M^{\gamma} = -2 \mathcal{I}^\alpha_a \vec{\varphi}^{a \gamma},$$

$$M^A = \frac{1}{2} \mathcal{I}^\alpha_a \mathcal{I}^\gamma B \left( \vec{\varphi}^B + (\vec{\varphi})^2 \vec{\varphi}^B \right) - \frac{2}{\vec{\varphi}} \mathcal{I}^\gamma_b \vec{\varphi}^a \vec{\varphi}^b,$$

$$M^{A} = \frac{1}{2} \mathcal{I}^\alpha_a \mathcal{I}^{\gamma\beta} \varphi \left( \vec{\varphi}^B + (\vec{\varphi})^2 \vec{\varphi}^B \right) - \frac{4}{\vec{\varphi}^2} \mathcal{I}^\gamma_a \mathcal{I}^{\beta c} \varphi \vec{\varphi} \vec{\varphi} \vec{\varphi} \vec{\varphi},$$

$$M^{A} = - \frac{1}{(\vec{\varphi})^3} \mathcal{I}^{\alpha a \beta} \mathcal{I}^{\beta \gamma} B \varphi \left( \vec{\varphi}^B + (\vec{\varphi})^2 \vec{\varphi}^B \right) + \frac{4}{(\vec{\varphi})^4} \mathcal{I}^{\alpha a \beta} \mathcal{I}^{\beta \gamma} B \varphi \vec{\varphi} \vec{\varphi} \vec{\varphi} \vec{\varphi} \vec{\varphi}.$$

With these input coefficients, the gravitational closure equations can be set up.
It is evident that the case of a bi-metric spacetime does not decompose into two separate metric sectors, as is often intuitively assumed, since then the fact that one shared principal polynomial is required in order to allow for a common evolution from common initial data surfaces would not be taken into account. Finding the most general Lagrangian for bi-metric gravity is as complicated as solving the above found construction equations. Their explicit solution is an open problem to be solved if one proposes such a theory. The linearized gravitational field equations, however, have already been obtained \[20\].

C. Gravitational closure of general linear electrodynamics

We finally set up the gravitational closure equations for the refinement of Maxwell theory that equips an abelian gauge covector field $A$ with the dynamics

$$S[A; G] = \int d^4x (\epsilon_{pqrs} G^{pqrs})^{-1} G^{abcd} F_{ab} F_{cd}$$

on an “area metric” geometry provided by a fourth rank contravariant tensor field $G$ featuring the algebraic symmetries $G^{abcd} = G^{cdab}$ and $G^{abcd} = G^{[ab][cd]}$. The principal tensor of this theory has been calculated first by Rubilar \[21, 22\] and takes the form

$$P^{ijkl} = -\frac{1}{24} ( \frac{1}{24} \epsilon_{abcd} G^{abcd} )^{-2} \epsilon_{mnpq} \epsilon_{rstu} G^{mnrt}(iG^{j[ps]}G^{l]qtu}) ,$$

whose non-polynomial dependence of the geometric tensor $G$ presents a technically particularly involved kinematical structure. The requirement that the above general linear electrodynamics are canonically quantizable requires that $G$ lie in one of seven (out of a total 23) algebraic classes \[23\]. The induced geometry features fields with antisymmetric index pairs, which we can dualize using the volume form density on $\Sigma$, arriving at the set

$$\bar{g}^{\alpha\beta} := -G(\epsilon^0, \epsilon^\alpha, \epsilon^0, \epsilon^\beta),$$

$$\bar{g}_{\alpha\beta} := \left( \frac{1}{4 \det \bar{g}} \right) \epsilon_{\alpha\mu\nu} \epsilon_{\beta\rho\sigma} G(\epsilon^\mu, \epsilon^\nu, \epsilon^\rho, \epsilon^\sigma),$$

$$\bar{\bar{g}}_{\alpha\beta} := (\bar{g}^{-1})_{\alpha\mu} \left( \frac{1}{2 \sqrt{\det \bar{g}}} \right) \epsilon_{\beta\kappa\lambda} G(\epsilon^0, \epsilon^\mu, \epsilon^\kappa, \epsilon^\lambda) - \delta^0_\beta .$$

The frame conditions for the employed frames, expressed in terms of the induced fields, are

$$\bar{g}^{\alpha\beta} \bar{g}_{\alpha\beta} = 0 \quad \text{and} \quad \bar{\bar{g}}_{[\alpha, \beta]} = 0 .$$
Transition to the corresponding canonical geometry $g_{\alpha\beta}, g'_{\alpha\beta}, \bar{g}_{\alpha\beta}$ thus requires to explicitly enforce these four conditions, together with the remaining symmetry conditions by requiring that

\[ g^{\alpha\beta} g_{\alpha\beta} = 0, \quad g'_{[\alpha\beta]} = 0, \quad \bar{g}^{[\alpha\beta]} = 0, \quad \bar{g}_{[\alpha\beta]} = 0, \]

reducing the a priori 27 independent entries of the tensor fields that make up the canonical geometry by 10. In order to account for these conditions, we thus need to choose 17 unconstrained configuration variables. It is convenient to denote them by

\[ \phi^A := (\phi^1, \ldots, \phi^6, \bar{\phi}^1, \ldots, \bar{\phi}^1, \ldots, \bar{\phi}^6, \bar{\phi}^1, \ldots, \bar{\phi}^5) \]

and to construct the parametrization maps $(a, b, c = 1 \ldots 6$ and $m, n = 1 \ldots 5)$

\[
\begin{align*}
\hat{g}^{\alpha\beta}(\phi) &:= T^{\alpha\beta}_{\ A} \phi^A, \\
\hat{g}'_{\alpha\beta}(\phi) &:= T^a_{\alpha\beta} \Delta_{\ a\ b} \phi^b, \\
\hat{g}_{\alpha\beta}(\phi) &:= T^a_{\alpha\beta} \left( \delta^b_c - \frac{n_a \bar{\phi}^b}{n_c \bar{\phi}} \right) \epsilon_{(m)b} \bar{\phi}^m, \\
\end{align*}
\]

where $\Delta_{\ a\ b}$ are the constant components of the standard inner product on $\mathbb{R}^6$, and $t^a, e^{(1)a}, \ldots, e^{(5)a}$ are the components of constant orthonormal basis vectors chosen such that $T^a_{\alpha\beta} \Delta_{\ a\ b}$ is a positive definite matrix. Note that $n_a := \Delta_{\ a\ b} t^b, \epsilon_{(1)a} := \Delta_{\ a\ b} e^{(1)b}, \ldots, \epsilon_{(5)a} := \Delta_{\ a\ b} e^{(5)b}$ is then the dual basis. Conversely, extraction of the configuration variables $\phi^A$ from the tensor fields $g^{\alpha\beta}$ constituting the canonical geometry is achieved by the maps

\[
\begin{align*}
\hat{\phi}^a(g) &:= T^a_{\alpha\beta} \hat{g}^{\alpha\beta}, \\
\hat{\phi}^0(g) &:= \Delta_{\ a\ b} T^b_{\alpha\beta} \hat{g}_{\alpha\beta}, \\
\hat{\phi}^m(g) &:= T^a_{\alpha\beta} e^{(m)a} \hat{g}_{\alpha\beta},
\end{align*}
\]

which indeed recover precisely the configuration variables employed in the parametrization, as one readily checks. It is clear by construction that the three maps $\hat{g}, \hat{g}', \hat{g}$ produce symmetric tensor fields $g^{\alpha\beta}$ constituting the canonical geometry by 10. The intertwiners associated with this parametrization are then readily calculated as

\[
\begin{align*}
\hat{\partial} g^{\alpha\beta} = T^{\alpha\beta}_{\ A} \phi^A, \\
\hat{\partial} g_{\alpha\beta} = \Delta_{\ a\ b} T^b_{\alpha\beta}, \\
\hat{\partial} \bar{g}_{\alpha\beta} = T^a_{\alpha\beta} \left( \delta^b_c - \frac{n_a \bar{\phi}^b}{n_c \bar{\phi}} \right) \epsilon_{(m)b} \bar{\phi}^m = \epsilon_{(m)a} \bar{\phi}^m - \epsilon_{(m)a} \bar{\phi}^m = 0, \\
\end{align*}
\]

shows that also the first condition above is satisfied. The intertwiners associated with this parametrization are then readily calculated as

\[
\begin{align*}
\frac{\partial \hat{g}^{\alpha\beta}}{\partial \phi^A} &:= T^{\alpha\beta}_{\ A} \phi^A, \\
\frac{\partial \hat{g}_{\alpha\beta}}{\partial \phi^A} &:= \Delta_{\ a\ b} T^b_{\alpha\beta}, \\
\frac{\partial \bar{g}_{\alpha\beta}}{\partial \phi^A} &:= T^a_{\alpha\beta} \left( \delta^b_c - \frac{n_a \bar{\phi}^b}{n_c \bar{\phi}} \right) \epsilon_{(m)b} \bar{\phi}^m, \\
\frac{\partial \hat{g}_{\alpha\beta}}{\partial \phi^m} &:= T^b_{\alpha\beta} n_b \frac{\epsilon_{(m)c} \bar{\phi}^m}{(n_c \bar{\phi})^2} - T^b_{\alpha\beta} n_b \frac{\epsilon_{(m)c} \bar{\phi}^m}{n_c \bar{\phi}},
\end{align*}
\]
\[ \frac{\partial \hat{\cal g}^a}{\partial g_{\alpha\beta}} = T^a_{\alpha\beta}, \quad \frac{\partial \hat{\cal g}^a}{\partial g_{\alpha\beta}} = \Delta^{ab} T^b_{\alpha\beta}, \quad \frac{\partial \hat{\cal g}^a}{\partial g_{\alpha\beta}} = T^a_{\alpha\beta} e^{(m)a}. \]

The input coefficients for the gravitational closure equations are therefore

\[ p^{\alpha\beta} = \frac{1}{6} \left( \hat{\cal g}^{\gamma\delta} \hat{\cal g}^{\beta\alpha} \hat{\cal g}^{\gamma\delta} - \hat{\cal g}^{\alpha\beta} \hat{\cal g}^{\gamma\delta} \hat{\cal g}^{\gamma\delta} + 2 \hat{\cal g}^{\alpha\beta} \hat{\cal g}^{\gamma\delta} \hat{\cal g}^{\beta\alpha} \hat{\cal g}^{\gamma\delta} \right), \]

\[ \overline{E}_a^{\alpha} = \varphi_a^{\alpha}, \quad \overline{E}_a^{\alpha} = \hat{\varphi}_a^{\alpha}, \quad \overline{E}_a^{\alpha} = \Phi_a^{\alpha}, \]

\[ \overline{F}_a^{\alpha\beta} = 2 \hat{\cal g}^{\alpha\beta} \hat{\cal g}^{\gamma\delta} \hat{\cal g}^{\beta\alpha} \hat{\cal g}^{\gamma\delta} - 2 \hat{\cal g}^{\alpha\beta} \hat{\cal g}^{\gamma\delta} \hat{\cal g}^{\beta\alpha} \hat{\cal g}^{\gamma\delta} + 3 \hat{\cal g}^{\alpha\beta} \hat{\cal g}^{\gamma\delta} \hat{\cal g}^{\beta\alpha} \hat{\cal g}^{\gamma\delta}, \]

\[ M_a^{\gamma} = 2 \left( \det \hat{\cal g}(\varphi) \right)^{1/2} T^a_{\alpha\beta} T^\nu(\alpha)_b \epsilon_{\nu\gamma} \partial_{\varphi_a^{\nu\gamma}} \hat{\varphi} - \bar{\varphi}, \]

\[ M_a^{\gamma} = 6 \left( \det \hat{\cal g}(\varphi) \right)^{1/2} \epsilon_{\alpha\mu\nu} \Delta^{ab} T^a_{\alpha\beta} T^\lambda(\nu)_b \epsilon_{\lambda\gamma} \partial_{\varphi_a^{\lambda\gamma}} \hat{\varphi} - \bar{\varphi}, \]

\[ M_a^{\gamma} = - \left( \det \hat{\cal g}(\varphi) \right)^{1/2} \epsilon_{\mu\nu\gamma} \left( \hat{\cal g}^{-1} \right)_{\mu\alpha} \partial_{\varphi_a^{\mu\nu}} \left( \hat{\cal g} \right)_{\alpha\beta} T^\gamma(\beta)_b \epsilon_{\gamma\phi} \partial_{\varphi_a^{\gamma\phi}} \hat{\varphi} - \bar{\varphi} + T^a_{\beta\nu} \Delta_{bc} \hat{\varphi}. \]

The corresponding gravitational closure equations differ significantly from those proposed for this case in [6], because now the non-linear frame conditions are already taken care of by our use of non-tensorial configuration degrees of freedom, while previously they had to be added by hand and thus made the problem of solving the equations prohibitively difficult. While an exact general solution is still quite difficult, one can resort, for instance, to a perturbative solution. This has been worked out in detail in [18] in order to obtain the most general Lagrangian \( \mathcal{L}_{\text{geometry}} \) of second order in perturbations of the unconstrained configuration variables, which thus provides the most general linearized equations of motion for the geometry. The weak gravitational dynamics that can underpin this refined, most general type of linear electrodynamics, were thus calculated based on the technology developed in the present article.
VI. CONCLUSIONS

We showed how to gravitationally close a given set of matter field equations, in the sense of providing equations of motion for the background geometry on which the matter dynamics have been formulated in the first place. Practically, this is done by following the concrete calculational sequence

\[ \text{matter equations} \rightarrow \text{input coefficients} \rightarrow \text{output coefficients} \rightarrow \text{gravity equations}. \]

The first step, at its core, is a straightforward standard calculation in the theory of partial differential equations, namely the calculation of the principal polynomial of the matter field equations one starts from. It is then easy to identify the canonical geometry and to parameterize the latter in terms of non-tensorial configuration variables such that generically non-linear frame conditions are automatically captured and thus need not be worried about anymore in the remaining course of the treatment. If the matter dynamics are canonically quantizable, the previously calculated principal polynomial features all the properties needed to calculate the input coefficients that are required to set up the gravitational closure equations. The second step then consists in solving the closure equations for the output coefficients. Depending on the complexity assumed by the gravitational closure equations for the input coefficients at hand, one may either be able to find their general solution, or have to resort to perturbative techniques, or consider a suitable symmetry reduction. The third step is again straightforward, as it merely consists in employing the output coefficients to compose the gravitational action, whose variation with respect to the configuration variables then yields one side of the thus defined gravitational field equations. The other side is of course provided by the same variation, but applied to the matter action from which the entire construction started.

There are only a few routes by which one can escape the gravitational closure mechanism when presented with a matter theory coupled to some geometry. One is to introduce, in addition to the geometry employed in the matter dynamics, additional gravitational degrees of freedom to which none of the matter fields couple directly; this allows for arbitrary modifications to the gravitational dynamics, and thus comes at the cost of needing an infinite number of experiments to determine the constants of the theory before it becomes numerically predictive. The other circumvention would be to drop the canonical quantizability condition for at least some of the matter that inhabits the universe one wishes to model.
While this may be consistent when dealing with effective field theories, the description of fundamental matter dynamics requires, for all we know, the consideration of quantizable matter dynamics.

Fundamentally, the ability to perform the gravitational closure of canonically quantizable matter dynamics allows us to inject our current and future knowledge about matter directly into the construction of gravity theories. Additional constraints, such as the absence of ghosts, can and should be employed to further reduce the linear solution space of the gravitational closure equations. However, it is typically the specific gravitational closure equations as they follow from concrete matter dynamics — and not sweeping theoretical constraints — that effectively reduce the spectrum of possible gravity theories, such that, at best, only a finite number of constants are left to be determined by observation. Indeed, even decisive generic requirements, such as diffeomorphism invariance or ghost-freedom, do generally not achieve that.

Phenomenologically, one can now ask questions that hitherto were not systematically accessible, since they require bridging the gap between a hypothesis about matter and the resulting gravitational implications. For instance, a systematic exploration of the simple question whether there is any evidence for birefringence of light in vacuo, compels one to forsake the assumption of a metric background geometry in a favor of a refinement [24–28] that can be written by a fourth rank tensor $G$, such that Maxwell’s action is refined to the general linear electrodynamics, whose we discussed in section [24–28]. The refined Maxwell theory is canonically quantizable [29–31] and thus provides valid input coefficients for the pertinent gravitational closure equations. The temptation to discard such a refinement a priori is quite delusive. For even if coarse geometric optics effects are undetectable, the above action still predicts accumulative modifications for the way electromagnetic field energy is transported [5]. These result in a potentially measurable modification of Etherington’s distance duality relation [32] already in a weak gravitational field that admits birefringence [18], and which may also address otherwise inexplicable magnification anomalies [33].

Based on the results of the present paper, we believe that the construction of gravity theories must consider the dynamics of all matter fields that will populate a spacetime right from the start. The gravitational closure equations enable one to put this insight to immediate practical use. Either as a complete consistency check for an existing gravity theory, or for its derivation.
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[1] T. Clifton, P. G. Ferreira, A. Padilla, and C. Skordis, Phys. Rept. 513, 1 (2012), arXiv:1106.2476 [astro-ph.CO].

[2] A. Joyce, B. Jain, J. Khoury, and M. Trodden, Phys. Rept. 568, 1 (2015), arXiv:1407.0059 [astro-ph.CO].

[3] K. Koyama, Rept. Prog. Phys. 79, 046902 (2016) arXiv:1504.04623 [astro-ph.CO].

[4] D. Raetzel, S. Rivera, and F. P. Schuller, Phys. Rev. D83, 044047 (2011), arXiv:1010.1369 [hep-th].

[5] C. Witte, Gravity actions from matter actions, Ph.D. thesis, HU Berlin (2014).

[6] K. Giesel, F. P. Schuller, C. Witte, and M. N. R. Wohlfarth, Phys. Rev. D85, 104042 (2012), arXiv:1202.2991 [gr-qc].

[7] P. A. M. Dirac, Lectures on Quantum mechanics (Belfer Graduate School of Science, 1964).

[8] L. Hörmander, Linear Partial Differential Operators (Springer, 1963).

[9] N. D. Birrel and P. C. W. Davies, Quantum Fields in Curved Space (Cambridge University Press, 1982).

[10] S. A. Hojman, K. Kuchar, and C. Teitelboim, Annals Phys. 96, 88 (1976).

[11] S. Rivera Hernandez, Tensorial spacetime geometries carrying predictive, interpretable and quantizable matter dynamics, Ph.D. thesis, University Potsdam (2012).

[12] W. Seiler, Involution - The Formal Theory of Differential Equations and its Application in Computer Algebra (Springer, 2010).

[13] L. Gårding, Acta Math. 85, 1 (1951).

[14] F. P. Schuller and C. Witte, Phys. Rev. D89, 104061 (2014), arXiv:1402.6548 [gr-qc].

[15] K. Kuchar, J. Math. Phys. 15, 708 (1974).
[16] N. Stritzelberger, *Perturbative canonical gravitational dynamics for birefringent spacetimes*, MSc thesis, University Erlangen-Nuremberg (2016).

[17] R. P. Woodard, Scholarpedia **10**, 32243 (2015) arXiv:1506.02210 [hep-th].

[18] J. Schneider, F. P. Schuller, N. Stritzelberger, and F. Wolz, in preparation.

[19] D. A. Reiss, *Covariance equations for tensors with symmetries: a general and practical approach via intertwiners*, BSc thesis, University Erlangen-Nuremberg (2014).

[20] U. Beier, F. P. Schuller, and F. Wolz, in preparation.

[21] G. F. Rubilar, *Annalen Phys.* **11**, 717 (2002) arXiv:0706.2193 [gr-qc].

[22] G. F. Rubilar, Y. N. Obukhov, and F. W. Hehl, *Int. J. Mod. Phys.* **D11**, 1227 (2002) arXiv:gr-qc/0109012 [gr-qc].

[23] F. P. Schuller, C. Witte, and M. N. R. Wohlfarth, *Annals Phys.* **325**, 1853 (2010) arXiv:0908.1016 [hep-th].

[24] F. W. Hehl and Y. N. Obukhov, *Foundations of Classical Electrodynamics* (Birkhäuser, 2003).

[25] A. Gross and G. F. Rubilar, *Phys. Lett.* **A285**, 267 (2001) arXiv:gr-qc/0103016 [gr-qc].

[26] F. W. Hehl, Y. N. Obukhov, and G. F. Rubilar, *General relativity, cosmology and relativistic astrophysics. Proceedings, Conference, Dublin, Ireland, September 6-8, 2001*, *Int. J. Mod. Phys.* **A17**, 2695 (2002) arXiv:gr-qc/0203105 [gr-qc].

[27] C. Lämmerzahl and F. W. Hehl, *Phys. Rev.* **D70**, 105022 (2004) arXiv:gr-qc/0409072 [gr-qc].

[28] F. W. Hehl and Y. N. Obukhov, *339th WE Heraeus Seminar on Special Relativity: Will It Survive the Next 100 Years? Potsdam, Germany, February 13-18, 2005*, *Lect. Notes Phys.* **702**, 163 (2006) arXiv:gr-qc/0508024 [gr-qc].

[29] S. Rivera and F. P. Schuller, *Phys. Rev.* **D83**, 064036 (2011) arXiv:1101.0491 [hep-th].

[30] C. Pfeifer and D. Siemssen, *Phys. Rev.* **D93**, 105046 (2016) arXiv:1602.00946 [math-ph].

[31] S. Grosse-Holz, F. P. Schuller, and R. Tanzi, (2017) arXiv:1703.07183 [hep-ph].

[32] F. P. Schuller and M. C. Werner, *Universe* **3**, 52 (2017) arXiv:1707.01261 [gr-qc].

[33] Z. Chu, G. L. Li, W. P. Lin, and H. X. Pan, *Mon. Not. Roy. Astron. Soc.* **461**, 4466 (2016) arXiv:1604.08339 [astro-ph.CO].
GRavitational Closure Equations

Input coefficients defined by

\[ N^\mu E^A_\mu [\varphi] - \partial_\gamma N^\mu F^A_\mu \gamma (\varphi) := \frac{\partial \tilde{\varphi}^A}{\partial g^{\gamma\sigma}} (\mathcal{L}_N \tilde{g}(\varphi))^{\gamma\sigma} \quad \text{and} \quad M^A_\gamma (\varphi) := \frac{\partial \tilde{\varphi}^A}{\partial g^\gamma} (\tilde{g}(\varphi)) \epsilon_0^a(t, \sigma) \frac{\partial g^\gamma}{\partial \partial_\gamma X^a}(t, \sigma) \]

Output coefficients defining

\[ \mathcal{L}[\varphi; K] := \sum_{N=0}^{\infty} C_{A_1 \ldots A_N}[\varphi] K^{A_1 \ldots A_N} \]

The seven individual equations

(C1) \[ 0 = -C \delta_{\mu}^\gamma + \sum_{K=0}^{\infty} (K+1) C_{A}^{\gamma_1 \ldots \gamma_K} \left( E^A_{\mu,\gamma_1 \ldots \gamma_K} + F^A_\mu \alpha_{K+1,\gamma_1 \ldots \gamma_K} \right) \quad \text{and} \quad M^A_\gamma (\varphi) := \frac{\partial \tilde{\varphi}^A}{\partial g^\gamma} (\tilde{g}(\varphi)) \epsilon_0^a(t, \sigma) \frac{\partial g^\gamma}{\partial \partial_\gamma X^a}(t, \sigma) \]

(C2) \[ 0 = -C_{A} \left( E^A_{\mu,\nu} + F^A_\mu \rho \right) + \sum_{K=0}^{\infty} C_{B,A}^{\gamma_1 \ldots \gamma_K} \left( E^A_{\mu,\gamma_1 \ldots \gamma_K} + F^A_\mu \alpha_{K+1,\gamma_1 \ldots \gamma_K} \right) \quad \text{and} \quad M^A_\gamma (\varphi) := \frac{\partial \tilde{\varphi}^A}{\partial g^\gamma} (\tilde{g}(\varphi)) \epsilon_0^a(t, \sigma) \frac{\partial g^\gamma}{\partial \partial_\gamma X^a}(t, \sigma) \]

(C3) \[ 0 = 2 (\deg P - 1) E^A_{\rho} + \sum_{K=0}^{\infty} C_{B,A}^{\gamma_1 \ldots \gamma_K} \left( M^A_\rho + \sum_{K=0}^{\infty} \frac{(-1)^{K+2}}{K} \frac{\partial^{K+2}}{\partial \partial_\gamma X^a}(t, \sigma) \frac{\partial g^\gamma}{\partial \partial_\gamma X^a}(t, \sigma) \right) \]

(C4) \[ 0 = 2 (\deg P - 1) C_{AB} \left( E^A_{\nu} - F^A_\nu \gamma \right) - C_{A} \left( M^A_\nu - \sum_{K=0}^{\infty} C_{B,A}^{\gamma_1 \ldots \gamma_K} M^A_\nu,\alpha_1 \ldots \alpha_K \right) \quad \text{and} \quad M^A_\gamma (\varphi) := \frac{\partial \tilde{\varphi}^A}{\partial g^\gamma} (\tilde{g}(\varphi)) \epsilon_0^a(t, \sigma) \frac{\partial g^\gamma}{\partial \partial_\gamma X^a}(t, \sigma) \]

(C5) \[ 0 = 2 \partial_\mu \left( C_{A} M^A_{\mu|B} M^B|\gamma \right) - 2 (\deg P - 1) p E^A_{\rho} \left( C_{A} F^A_\rho + \partial_\mu \left( C_{A} F^A_\rho \right) \right) \quad \text{and} \quad M^A_\gamma (\varphi) := \frac{\partial \tilde{\varphi}^A}{\partial g^\gamma} (\tilde{g}(\varphi)) \epsilon_0^a(t, \sigma) \frac{\partial g^\gamma}{\partial \partial_\gamma X^a}(t, \sigma) \]

(C6) \[ 0 = 6 (\deg P - 1) C_{AB} \left( E^A_{\nu} - F^A_\nu \gamma \right) - 4 C_{A} M^A_{\nu} - 2 C_{B_1 B_2} M^A_{\nu} - 2 C_{B_1 B_2} \alpha M^A_{\mu,\alpha} - 2 C_{B_1 B_2} \alpha \beta M^A_{\mu,\alpha \beta} \]

(C7) \[ 0 = \sum_{K=2}^{\infty} \sum_{J=2}^{\infty} (-1)^{J+1} \frac{K}{J} \frac{\partial^{J+1}}{\partial \partial_\gamma X^a}(t, \sigma) \frac{\partial g^\gamma}{\partial \partial_\gamma X^a}(t, \sigma) \]
The fourteen sequences of equations for $N \geq 2$

(C8) \[ 0 = \sum_{K=0}^{\infty} \binom{K + N}{K} \left[ C_{A}^{{\beta}_{1}...{\beta}_{N}{\alpha}_{1}...{\alpha}_{K}} \left( E^{A}_{\mu,\alpha_{1}...\alpha_{K}} + F^{A}_{\mu} \right) \right] \]

(C9) \[ 0 = \sum_{K=0}^{\infty} \binom{K + N}{K} \left[ C_{B;A}^{{\beta}_{1}...{\beta}_{N}{\alpha}_{1}...{\alpha}_{K}} \left( E^{A}_{\mu,\alpha_{1}...\alpha_{K}} + F^{A}_{\mu} \right) \right] \]

(C10) \[ 0 = -C_{B_{1}...B_{N}} \delta_{\mu}^{\gamma} - NC_{A(B_{1}...B_{N-1})} \gamma^{A}_{\mu} + C_{B_{1}...B_{N}A} \gamma^{A}_{\mu} \]

(C11) \[ 0 = C_{B_{1}...B_{N}A} \beta_{1} \beta_{2} \gamma^{A}_{\mu} - C_{B_{1}...B_{N}A} \beta_{1} \gamma^{A}_{\mu} \]

(C12) \[ 0 = C_{B_{1}...B_{N}A} \alpha \gamma^{A}_{\mu} \]

(C13) \[ 0 = C_{B_{1}...B_{N}A} \left( \mu \nu \right) \]

(C14) \[ 0 = C_{A;B_{1}...B_{N-1}} \left( B^{\mu}_{\nu} M^{A}_{\nu};_{B} + (deg P - 1) p^{\mu}_{\nu} \right) \]

(C15) \[ 0 = C_{B_{1}...B_{N+1};_{B_{1}}...B_{N+1}} \mu^{\nu} - C_{B_{1}...B_{N+1}} \mu^{\nu} \]

(C16) \[ 0 = N \cdot (N + 1) \cdot (deg P - 1) C_{A;B_{1}...B_{N}} p^{\mu|\nu} \gamma^{A}_{\mu} + N C_{A;B_{1}...B_{N}} \alpha^{\mu|\nu} \gamma^{A}_{\mu} + 2 N C_{A;B_{1}...B_{N}} \alpha^{\mu|\nu} \gamma^{A}_{\mu} \]

(C17) \[ 0 = (N + 2) \cdot (N + 1) \cdot (deg P - 1) C_{A;B_{1}...B_{N+1}} \left( p^{\mu|\nu} E^{A}_{\nu} - p^{\mu|\nu} \gamma^{A}_{\nu} \right) - (N + 1) \cdot (deg P - 1) C_{A;B_{1}...B_{N+1}} M^{A}_{\mu} \]

(C18) \[ 0 = C_{A;B}^{\mu_{1}...\mu_{N}} - \sum_{K=0}^{\infty} (-1)^{K+N} \binom{K + N}{K} \left( \partial_{\alpha_{1}...\alpha_{K}} C_{B;A}^{\alpha_{1}...\alpha_{K} \mu_{1}...\mu_{N}} \right) \]

(C19) \[ 0 = \sum_{K=0}^{\infty} \binom{K + N}{K} \left( C_{B;A}^{\alpha_{1}...\alpha_{K} (\mu_{1}...\mu_{N})} \right) \left( M^{A}_{\mu_{N+1}} \right) + \sum_{K=0}^{\infty} (-1)^{K+N} \left( \binom{K + N + 1}{N + 1} \right) \left( \partial_{\alpha_{1}...\alpha_{K}} C_{B;A}^{\alpha_{1}...\alpha_{K} \mu_{1}...\mu_{N+1}} \right) \]

(C20) \[ 0 = \sum_{J=N}^{\infty} \sum_{J=N+1}^{J+1} (-1)^{J} \binom{J}{N} \partial_{\alpha_{1}...\alpha_{J-N}} \left( C_{A}^{\beta_{J}...\beta_{K} (\alpha_{1}...\alpha_{J-N} \mu_{1}...\mu_{N-1})} \right) \]

(C21) \[ 0 = 2 \sum_{K=N-1}^{\infty} \binom{K + 1}{N - 1} C_{A}^{\beta_{N}...\beta_{K} (\mu_{1}...\mu_{N})} + \sum_{K=N}^{\infty} \sum_{J=N+1}^{J+1} (-1)^{J} \binom{J}{N} \partial_{\alpha_{1}...\alpha_{J-N}} \left( C_{A}^{\beta_{J}...\beta_{K} (\alpha_{1}...\alpha_{J-N} \mu_{1}...\mu_{N-1})} \right) \]