On the pseudospectra of Schrödinger operators on Zoll manifolds

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Abstract. We consider non-self-adjoint Schrödinger operators $\Delta + V$ where $\Delta$ is the Laplace-Beltrami operator on a Zoll manifold $X$ and $V \in C^\infty(X, \mathbb{C})$. We obtain asymptotic results on the pseudo-spectrum and numerical range of such operators.

1. Introduction and statement of results

Let $\Delta$ denote the Laplace-Beltrami operator on a manifold $X$ and let $V \in C^\infty(X, \mathbb{C})$. We will study asymptotic properties of the pseudospectrum and numerical range of the Schrödinger operator $\Delta + M_V$, where $M_V$ is the operator of multiplication by $V$. Here we are taking $\Delta$ as a self-adjoint operator with a suitable domain in $L^2(X)$.

Recall that, given an operator $P$ densely defined on some Hilbert space $H$, and $\epsilon > 0$, the $\epsilon$-pseudospectrum of $P$ is the set of complex numbers $\lambda$ such that

$$\|(P - \lambda I)^{-1}\| \geq \frac{1}{\epsilon},$$

where, by definition, the norm on the left-hand side is infinite if $\lambda$ is an eigenvalue of $P$. Equivalently, $\lambda$ is in the $\epsilon$-pseudospectrum of $P$ iff there exists $\psi \in H \setminus \{0\}$ such that

$$\frac{\|(P - \lambda I)\psi\|}{\|\psi\|} \leq \epsilon.$$

This motivates the following definition:

**Definition 1.1.** We say that a sequence of complex numbers $\{\lambda_k\}$ is in the asymptotic pseudospectrum of $\Delta + M_V$ if there exists a sequence $\{\psi_k\}$ of functions in $L^2(X)$ such that as $k \to \infty$,

$$\frac{\|(\Delta + M_V - \lambda_k I)\psi_k\|}{\|\psi_k\|} = O(k^{-\infty}) \quad \text{and} \quad \lim_{k \to \infty} |\lambda_k| = +\infty.$$

The sequence $\{\psi_k\}$ is referred to as an associated pseudo mode.
Note that (1.1) implies that, \( \forall \epsilon > 0 \), \( \lambda_k \) will be in the \( \epsilon \)-pseudospectrum of \( \Delta + M \) for all sufficiently large \( k \). The previous definition has an analogue in the semi-classical setting, see for example [18], §12.5 (and references therein). In the present case the role of Planck’s constant is played by \( \hbar = 1/k \).

It is well-known that the spectrum of a non-self-adjoint operator is very unstable. The pseudospectrum, however, is much more stable, which makes it a natural object to study. It is also known that points in the pseudospectrum do not have to be close to the spectrum, even in the semi-classical limit.

In this paper we consider the case where \( X \) is a closed, connected Zoll manifold. More specifically, we make the assumption there is a \( T > 0 \) such that each geodesic on \( X \) has minimal period \( T \) (we rule out the existence of exceptional short geodesics). Such manifolds are often called \( C_l \)-manifolds [1]. The simplest examples are the spheres with the round metric. It is known that the spectrum of the Laplacian \( \Delta \) on a Zoll manifold consists of clusters of eigenvalues of uniformly bounded width centered at the points

\[
\Lambda_k = \frac{4\pi^2}{T^2} \left( k + \frac{\beta}{4} \right)^2, \quad k = 1, 2, \ldots,
\]

where \( \beta \) is the Morse index of one periodic geodesic (hence all periodic geodesics). Specifically

(1.2) \[ \exists C > 0 \text{ such that } \sigma(\Delta) \subset \bigcup_{k \in \mathbb{N}} [\Lambda_k - C, \Lambda_k + C], \]

where \( \sigma(\Delta) \) denotes the spectrum of \( \Delta \). Note that the distance between the clusters \( \Lambda_k \) and \( \Lambda_{k+1} \) is linear in \( k \), which allows us to separate the eigenvalues of the Laplacian unambiguously into clusters, at least for large \( k \).

More precisely, as proved in [4], there exist self-adjoint, commuting pseudo-differential operators \( A \) of order one and \( Q_0 \) of order zero such that the spectrum of \( A \) consists of the eigenvalues \( k = 1, 2, \ldots \) and

(1.3) \[ \Delta = \frac{4\pi^2}{T^2} \left( A + \frac{\beta}{4} \right)^2 + Q_0. \]

In particular, we can take \( C \) equals to the operator norm \( ||Q_0|| \) with \( Q_0 \in \mathcal{B}(L^2(X)) \). Moreover, if for each \( k \) we let

(1.4) \[ E_k := \text{eigenspace of } A \text{ corresponding to the eigenvalue } k, \]

the eigenvalues in the \( k \)-th cluster consist of the eigenvalues of the restriction \( Q_0|_{E_k} \), shifted by \( \Lambda_k \).

The operator \( (A + \frac{\beta}{4})^2 \) has spectrum contained in \( (\mathbb{N} + \frac{\beta}{4})^2 \), and has the same principal and sub principal symbols as \( \Delta \) up to the factor \( \frac{4\pi^2}{T^2} \). For example, in case \( X = S^n \), the unit \( n \)-dimensional sphere, \( T = 2\pi \) and the eigenvalues of the Laplacian are \( k(k + n - 1) = (k + \frac{n-1}{2})^2 - \frac{(n-1)^2}{4} \). Therefore in this case

\[ \beta = 2(n-1) \quad \text{and} \quad Q_0 = \frac{(n-1)^2}{4} I. \]

For the round sphere the principal symbol of \( Q_0 \) is constant. In the general Zoll case the symbol of \( Q_0 \) is not constant (see [17], Theorem 3, for an expression for it), but it is always constant along geodesics since \( [Q_0, \Delta] = 0 \). The fact that the symbol if \( Q_0 \) is not constant corresponds to the non-trivial clustering phenomenon
of the Laplace eigenvalues seen in Zoll metrics, as the asymptotic behavior of the eigenvalue clusters is given to first order by the principal symbol of $Q_0$ via a Szegö limit theorem.

In conclusion, our Schrödinger operator $\Delta + M_V$ has the form

$$\Delta + M_V = \frac{4\pi^2}{T^2} \left( A + \frac{\beta}{4} \right)^2 + Q_0 + M_V.$$ 

Thus we can think of $M_V$ as a non-self-adjoint perturbation added to $Q_0$.

1.1. Statements of our results. We can now state the main results of this paper.

1.1.1. On the spectrum. First we will show that the spectrum of $\Delta + M_V$ also has a cluster structure (at least if we are sufficiently far from the origin) generalizing (1.2) (see also [5]). Let us denote by $D$ the closed disk around the origin with radius $||Q_0 + M_V||$.

**Theorem 1.2.** Let $V : X \to \mathbb{C}$ be a bounded measurable function. Then the spectrum of $H := \Delta + M_V$ is contained in the union

$$\bigcup_{k=0}^{\infty} D_k, \quad D_k := \Lambda_k + D.$$ 

The spectrum of $H$ consists entirely of isolated eigenvalues with finite algebraic multiplicities and, for $k$ sufficiently large, the spectrum of $H$ in the disk $D_k$ consists of eigenvalues with total algebraic multiplicity equal to the dimension of $E_k$. Moreover, the generalized eigenspaces of $\Delta + M_V$ span $L^2(X)$.

Note that the sets $D_k$ are pairwise disjoint for large $k$.

1.1.2. Results on the asymptotic pseudospectrum. We now turn to the statement of our results on the asymptotic pseudospectrum of $\Delta + M_V$.

Let $H_0(x, \xi) = ||\xi||$, $H_0 \in C^\infty(T^*X \setminus \{0\})$. This Hamiltonian generates geodesic flow re-parametrized by arc length. In what follows we shall think of oriented geodesics as trajectories of the Hamiltonian $H_0$ on the unit tangent bundle of $X$. Let $O$ be the manifold of oriented geodesics on $X$. This is the symplectic manifold obtained by symplectic reduction of $T^*X \setminus \{0\}$ under the circle action generated by $H_0$. Given any smooth function $V$ on $X$, one can define its Radon transform to be the function $\tilde{V}$ on $O$ defined by

$$\tilde{V}(\gamma) = \frac{1}{T} \int_\gamma V \, ds,$$

where $s$ denotes arc length along the oriented geodesic $\gamma \in O$, and we are abusing the notation and continue to denote by $V$ the pull-back of the potential to the unit tangent bundle. Note that $O$ has an involution corresponding to reversing the orientation of the geodesics. This involution is anti-symplectic, and $\tilde{V}$ is even with respect to it.

Now recall that $Q_0$ commutes with $\Delta$ (and with $(A + \frac{\beta}{4})$). Therefore its principal symbol $q_0$ is invariant along geodesics and may thus be viewed as a function on $O$, also denoted $q_0$. We reviewed above that, in case $X$ is the unit sphere, $q_0 = -\frac{(n-1)^2}{4}$ (constant). With this notation, our result is as follows:
Theorem 1.3. (1) Let $\mu = (\tilde{V} + q_0)(\gamma)$, where $\gamma \in \mathcal{O}$ is such that
$$\{\Re \tilde{V} + q_0, \Im \tilde{V}\}(\gamma) < 0.$$ Then $\{\Lambda_k + \mu\}$ is in the asymptotic pseudospectrum of $\Delta + MV$. In fact, for each $m \in \gamma$, there exists a pseudo mode $\{\psi_k\}$ such that the sum $\sum_{k=1}^{\infty} \psi_k$ is a distribution whose wave-front set is the ray in $T^*X \setminus \{0\}$ through $m$.

(2) If $\{\Lambda_k + \mu\}$ is in the asymptotic pseudospectrum with $\mu = (\tilde{V} + q_0)(\gamma)$, and
$$\{\Re \tilde{V} + q_0, \Im \tilde{V}\}(\gamma) > 0,$$ then for any associated pseudo mode $\{\psi_k\}$, the wave-front set of $\sum_k \psi_k$ is disjoint from the conic set in $T^*X \setminus \{0\}$ generated by $\gamma$.

Note that although the potential $\tilde{V}$ is even under the involution of reversing the orientation of $\gamma$, the Poisson bracket $\{\Re \tilde{V} + q_0, \Im \tilde{V}\}(\gamma)$ is odd under the same involution. Together with part (1) of the theorem, this implies that the set of sequences of the form
$$\left\{\lambda_k + \tilde{V}(\gamma) ; \{\Re \tilde{V} + q_0, \Im \tilde{V}\}(\gamma) \neq 0\right\}$$
is contained in the asymptotic pseudospectrum. However, in general this set does not equal the asymptotic pseudospectrum. For example, in the case of spheres, if $V$ is odd (with respect to the antipodal map) then $\tilde{V}$ is identically zero. In that case, however, there is a substitute function for $\tilde{V}$, namely the function
$$(1.7) \quad \tilde{V} = \frac{1}{4} \tilde{V}^2 - \frac{1}{8\pi} \int_0^{2\pi} dt \int_0^t \{\phi^*_s V, \phi^*_s V\} ds$$
where $\phi_t$ is geodesic flow, see [14] (Theorem 3.2). More subtly we expect that one can have that $\{\Re \tilde{V} + q_0, \Im \tilde{V}\}(\gamma) = 0$ and have pseudo modes concentrated at $\gamma$ with pseudo eigenvalue $\tilde{V}(\gamma)$, in the spirit of Theorem 2.1 in [13], where non-vanishing of higher-order derivatives is assumed.

Theorem 1.3 is analogous to Theorem 12.8 in [18] (originally due to Davies), where a similar Poisson bracket condition appears. It is also analogous in spirit to the main result in [2].

1.1.3. Results on the numerical range. Recall that the numerical range of an operator $\hat{H}$ on $L^2(X)$ with domain $\mathcal{D}(\hat{H})$ is the subset of the complex plane defined by:
$$\mathcal{R}(\hat{H}) = \left\{ \frac{\langle \hat{H}(\psi), \psi \rangle}{\langle \psi, \psi \rangle} ; \psi \in \mathcal{D}(\hat{H}) \right\}.$$ We are interested in the numerical range of $\hat{H} = \Delta + MV$. We will only consider the asymptotics of the sets of values
$$\left\{ \frac{\langle \hat{H}(\psi), \psi \rangle}{\langle \psi, \psi \rangle} ; \psi \in E_k \right\}.$$ To state our result we shift these sets by $\Lambda_k$:

Theorem 1.4. In case $X = S^n$ or $X$ a Zoll surface, the limit $\mathcal{R}_\infty$ of the sets
$$\mathcal{R}_k = \left\{ \frac{\langle (\Delta + MV)\psi, \psi \rangle}{\langle \psi, \psi \rangle} - \Lambda_k ; \psi \in E_k \setminus \{0\} \right\}$$
as $k$ tends to infinity is the convex hull of the image of $\tilde{V} + q_0 : \mathcal{O} \to \mathbb{C}$. 

For a similar result for Berezin-Toeplitz operators see [2].

2. Proof of theorem 1.2

Let $D$ be the operator $\frac{1}{\pi} \left( A + \frac{d}{2} \right)^2$. Let $z$ be in the resolvent set $\rho(D)$ of the operator $D$, with the distance $d(z, \sigma(D))$ from $z$ to the spectrum $\sigma(D)$ of $D$ greater than the norm $\|Q_0 + M_V\|$. Since $D$ is self adjoint then $||(D - z)^{-1}|| = 1/d(z, \sigma(D))$ which implies $||(Q_0 + M_V)(D - z)^{-1}|| < 1$. Therefore the operator $\left( I + (Q_0 + M_V)(D - z)^{-1} \right)$ is a bounded operator which in turn implies that $(\Delta + M_V - z)^{-1}$ exists and it is a bounded operator, i.e. $z$ is in the resolvent set $\rho(H)$ of $H$. Thus the spectrum $\sigma(H)$ of $H$ must be contained in the union of the closed disks $D_k$ indicated in (1.6).

Consider $z \in \mathbb{C}$ such that $d(z, \sigma(\Delta)) > 2\|M_V\|$. Since $\Delta$ has compact resolvent then writing $H - z = (I + M_V(\Delta - z)^{-1})(\Delta - z)$ we see that $(H - z)^{-1}$ exists and it must be compact (note that since $\Delta$ is self adjoint then $||(\Delta - z)^{-1}|| = 1/d(z, \sigma(\Delta))$). Thus we have that the spectrum of $\hat{H}$ consists entirely of eigenvalues with finite algebraic multiplicities, see reference [12], theorem III, section 6.8.

Let $g(k) = \Lambda_k - \Lambda_{k-1} = O(k)$ and let $C_k$ be the circle of radius $g(k)/2$ and center $\Lambda_k$. Note that, for $k$ sufficiently large, both the interval of radius $\|Q_0\|$ around $\Lambda_k$ and the disk $D_k$ are inside $C_k$. Moreover,

$$\forall z \in C_k, \quad d(z, \sigma(\Delta)) \geq g(k)/3 \quad \text{and} \quad d(z, \sigma(\hat{H})) \geq g(k)/3.$$ 

In particular, $C_k$ is contained in both $\rho(\Delta)$ and $\rho(H)$. Thus the following two projections are well defined:

$$P_k = \frac{1}{2\pi i} \oint_{C_k} (z - H)^{-1} \, dz \quad \text{and} \quad \Pi_k = \frac{1}{2\pi i} \oint_{C_k} (z - \Delta)^{-1} \, dz.$$ 

Since $\Delta$ is self adjoint then $||(z - \Delta)^{-1}|| = 1/d(z, \sigma(\Delta)) = O(k^{-1})$ uniformly in $C_k$. This implies for $k$ sufficiently large that $\|M_V(z - \Delta)^{-1}\| \leq 1/2$ which in turn implies, using a Neumann series, that

$$\| (z - H)^{-1} \| \leq \| (z - \Delta)^{-1} \| \left( I + M_V(z - \Delta)^{-1} \right)^{-1} \| \leq \frac{1}{d(z, \sigma(\Delta))} \frac{1}{1 - \|M_V(z - \Delta)^{-1}\|} = O(k^{-1})$$

uniformly in $C_k$.

Thus we obtain

$$\|P_k - \Pi_k\| = \left\| \frac{1}{2\pi i} \oint_{C_k} (z - H)^{-1} M_V(z - \Delta)^{-1} \, dz \right\| \leq \frac{1}{2\pi} \oint_{C_k} \| (z - H)^{-1} \| \| M_V \| \| (z - \Delta)^{-1} \| \, dz \|

\leq \frac{1}{2\pi} (2\pi g(k)) O(k^{-1}) \| M_V \| O(k^{-1}) = O(k^{-1}).$$

Therefore, for $k$ sufficiently large, the norm $\|P_k - \Pi_k\|$ is less than one which in turn implies that the dimension of the range of $P_k$, $\dim(\text{Ran}(P_k))$, and the
dimension of the range of $\Pi_k$, $\dim(\text{Ran}(\Pi_k))$, must be the same, see reference \[12\], chapter I, section 4.6. Moreover, $\dim(\text{Ran}(P_k))$ must be finite which implies that the spectrum of $H$ inside $C_k$ consists only of eigenvalues, see reference \[12\], chapter III, section 6.5. We conclude that the total algebraic multiplicity of the eigenvalues of $H$ inside $D_k$ must be the same as the initial geometric multiplicity of the cluster of eigenvalues of $\Delta$ in the interval $[\Lambda_k - \|Q_0\|, \Lambda_k + \|Q_0\|]$, namely the dimension of $E_k$.

Finally, for $1 < p < \infty$, let us denote by $S_p$ the space of all completely continuous operators $A$ for which $\sum_{j=1}^{\infty} s_j^p(A) < \infty$ where $s_j(A)$ are the eigenvalues of $|A| \equiv \sqrt{A^*A}$. The fact that the generalized eigenspaces of $\Delta + M_V$ span $L^2(X)$ is a consequence of theorem 10.1 in reference \[5\] noting that $\Delta$ has discrete spectrum and for all $\lambda$ in the resolvent set of $\Delta$, $(\Delta - \lambda)^{-1}V(\Delta - \lambda)^{-1}$ belongs to a Schatten class $S_p$ for some $1 < p < \infty$. This last fact is a consequence of (i) the property that $S_p$ is an ideal in $B(L^2(X))$, (ii) $V(\Delta - \lambda)^{-1}$ is a bounded operator and (iii) $(\Delta - \lambda)^{-1}$ belongs to $S_p$ for some $1 < p < \infty$ (the multiplicity of the eigenvalues of $\Delta$ in the interval $[\Lambda_k - C, \Lambda_k + C]$ is given by a polynomial in $k$ for $k$ sufficiently large, see reference \[4\]).

3. Proof of theorem 1.3

3.1. The averaging method. We begin by recalling the “averaging method” (\[16, 8\]):

Proposition 3.1. If $V : X \to \mathbb{C}$ is smooth, there exists a pseudo-differential operator of order $(-1)$, $F$, such that

$$e^F (\Delta + M_V) e^{-F} = e^{F \left( \frac{4\pi^2}{T^2} (A + \frac{\beta}{4})^2 + Q_0 + M_V \right)} e^{-F} = \frac{4\pi^2}{T^2} (A + \frac{\beta}{4})^2 + B + S,$$

where $B$ is a pseudo-differential operator of order zero such that $[A, B] = 0$, and $S$ is a smoothing operator. Moreover, the symbol of $B$ is $\widetilde{V} + q_0$, regarded as a function on $T^*X^n \setminus \{0\}$.

Proof. We’ll sketch the proof, for completeness. Let

$$\tilde{A} = \frac{2\pi}{T} (A + \frac{\beta}{4});$$

then $\tilde{A}^2 + Q_0 = \Delta$, and $\tilde{A}$ takes the value $\sqrt{\Lambda_k}$ on the space $E_k$. We proceed inductively, with $F_1$ an operator or order $(-1)$ solving the equation

$$[F_1, \tilde{A}^2] = B_0 - (M_V + Q_0) + \text{order} (1),$$

where $B_0$ is the operator

$$B_0 = \frac{1}{T} \int_0^T e^{it\tilde{A}} (M_V + Q_0) e^{-it\tilde{A}} dt.$$

It is easy to see that $B_0$ commutes with $\tilde{A}$ and hence with $A$.

For $F_1$ we take an operator with principal symbol

$$f_1 := -\frac{H_0^{-1}}{2T} \int_0^T dt \int_0^t \phi_s^*(V + q_0) ds.$$
where $H_0(x, \xi) = ||\xi||$ and $\phi_\epsilon$ is its Hamilton flow. Note that the principal symbol of $A^2$ is the same as the principal symbol of $\Delta$, i.e. $H_0^2$. The symbol of $[F_1, A^2]$ is

$$\{f_1, H_0^2\} = 2H_0\{f_1, H_0\} = -\frac{1}{T} \int_0^T (\phi_t^\dagger (V + q_0) - (V + q_0)) \, dt = V - V_{\text{ave}},$$

where $V_{\text{ave}} + q_0 = \frac{1}{T} \int_0^T \phi_t^\dagger (V) \, dt + q_0$ is the principal symbol of $B_0$. Note that the factors of $q_0$ cancel since $\phi_t^\dagger (q_0) = q_0$ (i.e. $q_0$ is already averaged over the flow). It follows that

$$e^{F_1} (A^2 + Q_0 + M_V) e^{-F_1} = \tilde{A}^2 + B_0 + R_{-1},$$

where $R_{-1}$ is an operator of order $(-1)$. Using the same constructions, one can modify $F_1$ by an operator $G_1$ of order $(-2)$ so that, if $F_2 = F_1 + G_1$,

$$e^{F_2} (A^2 + Q_0 + M_V) e^{-F_2} = \tilde{A}^2 + B_0 + B_{-1} + R_{-2}$$

where $R_{-2}$ is now an operator of order $(-2)$ and both $B_0$ and $B_{-1}$ commute with $A$. This process can be continued indefinitely, and after a Borel summation we can find an operator $F_\infty$, with the same principal symbol as $F_1$, such that

$$e^{F_\infty} (A^2 + Q_0 + M_V) e^{-F_\infty} = \tilde{A}^2 + B + S,$$

where $[A, B] = 0$ and $S$ is a smoothing operator, and where the principal symbol of $B$ is the same as that of $B_0$. □

### 3.2. Constructing the pseudo mode.

We now proceed to construct a pseudo mode for $\Delta + M_V$, under the hypotheses of part (1) of Theorem 1.3. We will do this by first constructing a pseudo mode for $A^2 + B$, then showing that it is also a pseudo mode for $A^2 + B + S$, and finally applying the averaging lemma.

If $S^* X$ denotes the unit cosphere bundle of $X$, one has a diagram

\[
\begin{array}{ccc}
t : S^* X & \hookrightarrow & T^* X \setminus \{0\} \\
\pi & \downarrow & \\
\mathcal{O} & \end{array}
\]

(3.1)

and $\pi^* \tilde{V}$ is the symbol of $B$ restricted to $S^* M$. Let us assume the hypotheses of part (1) of the theorem, taking without loss of generality $\sigma_B(m) = \mu = 0$. Thus, if $m \in \gamma$

\[
\{ \mathbb{R} \tilde{V}, 3 \tilde{V} \}(m) < 0.
\]

(3.2)

Let $\mathcal{R} \subset T^* X \setminus \{0\}$ be the ray through $m$. $\mathcal{R}$ is a conic isotropic submanifold of $T^* X$. Associated to any such manifold are spaces of distributions $J^k(X, \mathcal{R})$, the Hermite distributions of Boutet de Monvel and Guillemin [3].

**Lemma 3.2.** For every $\ell$ there exists $\psi \in J^\ell(X, \mathcal{R})$ (with non-zero principal symbol) such that $B(\psi) \in C^\infty(X)$.

**Proof.** The construction is symbolic. The symbol of an element $\psi \in J^\ell(X, \mathcal{R})$ at $m \in \mathcal{R}$ is an object in the space

$$\sigma_m \in \bigwedge^{1/2} \mathcal{R}_m \otimes S(\Sigma_m),$$

where:

(1) $\mathcal{R}_m = T_m \mathcal{R}$ and $\bigwedge^{1/2}$ stands for half forms on it,
Let $\psi$ be as in the lemma and $\psi = \sum_{k=1}^{\infty} \psi_k$ be its decomposition into sums of eigenfunctions corresponding to the eigenvalue clusters. Since $[\Delta, B] = 0$, it follows that

$$B(\psi) = \sum_{k=1}^{\infty} B(\psi_k)$$

is the cluster decomposition of $B(\psi)$, and therefore

$$\|B(\psi_k)\| = O(k^{-\infty})$$

since $B(\psi)$ is smooth. Since $\psi \in J^t(X, R)$, proceeding similarly as in [2], one can prove that the sequence of norms $\{\psi_k\}$ has an asymptotic expansion as $k \to \infty$ in decreasing powers of $k$ with non-trivial leading term. It follows that $\{\psi_k\}$ is a pseudo mode for $\Delta + B$ with pseudo-eigenvalues $\lambda_k = \Lambda_k$.

At this point, we should have a pseudo mode for $\tilde{A}^2 + B$ with $\psi_k$ an eigenfunction of $\tilde{A}^2$ for each $k$, with eigenvalue precisely $\Lambda_k$.

Of course, the averaging lemma related $\Delta + M_V$ to $\tilde{A}^2 + B + S$, not to $\tilde{A}^2 + B$. However:

**Lemma 3.3.** $\frac{\|S\psi_k\|}{\|\psi_k\|} = O(k^{-\infty})$.

**Proof.** For any $N \in \mathbb{N}$, the operator $S\tilde{A}^N$ is smoothing (since $S$ is) and hence is bounded on $L^2$ by some constant $C_N$. Then $\|S\tilde{A}^N\psi_k\| \leq C_N\|\psi_k\|$. On the
other hand, \( \|S\Lambda_k^{-N/2}\psi_k\| = \|S\Lambda_k^{-N/2}\psi_k\| = \Lambda_k^{-N/2}\|\psi_k\| \). We conclude that \( \|S\psi_k\| \leq C_N\Lambda_k^{-N/2}\|\psi_k\| \); however, \( \Lambda_k \) is a quadratic polynomial in \( k \) with positive leading coefficient. Since \( N \) is arbitrary, this completes the proof. \( \square \)

**Corollary 3.4.** The sequence \( \{\psi_k\} \) is also a pseudo mode for \( \tilde{A}^2 + B + S \).

**Proof.** We may as well assume \( \psi_k \) are \( L^2 \)-normalized; the proof then follows immediately from the previous lemma. \( \square \)

We now finish the proof of the first part of Theorem 1.3. Let us define \( \varphi_k := e^{-F}\psi_k \) where \( \{\psi_k\} \) is a pseudo mode as in the previous corollary such that, without loss of generality, \( \|\psi_k\| \equiv 1 \). Then

\[
\| (\Delta + M_V - \Lambda_k I)\varphi_k \| = \| e^{-F}(\tilde{A}^2 + B + S - \Lambda_k I)(\psi_k) \|;
\]

since \( e^{-F} \) is a bounded operator and and \( \{\psi_k\} \) is a pseudomode for \( \tilde{A}^2 + B + S \), this is \( O(k^{-\infty}) \). On the other hand, since \( F \) is of order \((-1)\),

\[
(e^{-F})^*e^{-F} = I + T
\]

where \( T \) is an operator of order \((-1)\). Therefore

\[
\|\varphi_k\|^2 = \|\psi_k\|^2 + \langle T\psi_k, \psi_k \rangle.
\]

Since the second term on the right-hand side tends to zero as \( k \to \infty \), \( \{\varphi_k\} \) is a pseudo mode for \( \Delta + M_V \) with pseudo-eigenvalues \( \lambda_k = \Lambda_k \). (Recall that we assumed without loss of generality that \( \mu = 0 \).) This proves the first part of Theorem 1.3.

**3.3. Proof of part 2.** Let us assume that \( \{\Lambda_k + \mu\} \), where \( \mu = (\tilde{V} + q_0)(\gamma) \), is in the asymptotic pseudospectrum, By definition, there exists a pseudo-mode \( \{\psi_k \in E_k\} \), a sequence that satisfies \( \|\psi_k\| = 1 \) for all \( k \) and

\[
\| (M^\text{ave}_V + Q_0 - \mu)\psi_k \| = O(k^{-\infty}).
\]

Let \( \psi \) be the distribution \( \psi = \sum_{k=1}^{\infty} \psi_k \). This is clearly non-smooth, while

\[
(M^\text{ave}_V + Q_0 - \mu I)\psi \in C^\infty(X)
\]

by the Sobolev embedding theorem.

Assume now that

\[
\{[\tilde{V} + q_0, 3V] \} (\gamma) > 0.
\]

By Theorem 27.1.11 of [11], the operator \( M^\text{ave}_V + Q_0 - \mu I \) is microlocally subelliptic on the cone over the geodesic \( \gamma \subset S^*X \), with loss of \( 1/2 \) derivatives. Given (3.3), it follows that the wave-front set of \( \psi \) must be disjoint from this cone.

**4. Proof of Theorem 1.4**

Recall our notation: \( X \) is a Zoll manifold, and let \( A \) be the operator appearing in [13]. \( A \) is a first-order pseudo-differential operator with symbol \( \sigma_A(x, \xi) = \|\xi\|_x \) and spectrum the eigenvalues \( k = 0, 1, \ldots \). Let \( L^2(X) = \bigoplus_{k=0}^{\infty} E_k \) be the decomposition of \( L^2(X) \) into eigenspaces of \( A \).

Referring to [13] note first that, for each \( k \),

\[
\mathcal{R}_k = \left\{ \frac{\langle (Q_0 + M_V)\psi, \psi \rangle}{\langle \psi, \psi \rangle} ; \psi \in E_k \setminus \{0\} \right\}
\]

that is, \( \mathcal{R}_k \) is the numerical range of the operator \( Q_0 + M_V + R_{-1} \) restricted to the finite-dimensional subspace \( E_k \). It is clear that \( \mathcal{R}_k \) is closed for each \( k \) and,
by the Toeplitz-Hausdorff theorem, \( R_k \) is convex. An elementary argument shows that the limit \( R_\infty \) is also closed and convex.

### 4.1. Existence of modes with microsupport on geodesics

The purpose of this section is to prove:

**Proposition 4.1.** Let \( X \) be either a standard \( S^n \) or a Zoll surface, and let \( \gamma \subset S^*X \) be a geodesic. Then there exist sequences \( \{u_k\} \) of functions such that:

1. \( \forall k \ u_k \in E_k \) and \( \|u_k\| = 1 \),
2. the semi-classical wave front set of the sequence is equal to \( \gamma \), and
3. For all pseudo-differential operators \( Q \) of order zero on \( X \) one has

\[
\langle Q(u_k), u_k \rangle = \frac{1}{2\pi} \int_\gamma \sigma_Q \, ds + O(1/\sqrt{k})
\]

where \( \sigma_Q \) is the principal symbol of \( Q \).

**Proof.** First suppose \( X \) is a sphere. Since the construction is \( \text{SO}(n+1) \) equivariant, without loss of generality \( \gamma \) corresponds to the intersection \( \gamma_0 \) of \( S^n \subset \mathbb{R}^{n+1} \) with the \( x_1 x_2 \) plane, where \((x_1, \ldots, x_{n+1})\) are the standard coordinates in \( \mathbb{R}^{n+1} \). Then one can take

\[
u_k(x) = a_k (x_1 + ix_2)^k
\]

where \( a_k \in \mathbb{R} \) is chosen so that the \( L^2 \) norm of \( u_k \) is equal to one. It is known that these functions have the required properties (see Proposition 7.6 in [7]).

Now assume that \( X \) is a Zoll surface, which we normalize so that its geodesics have length \( 2\pi \). Then it is known that there exists an invertible Fourier integral operator \( U : L^2(X) \to L^2(S^2) \) such that

\[
U \Delta U^{-1} = \Delta_0 + R,
\]

where \( \Delta_0 \) is the standard Laplacian on \( S^2 \) and \( R \) is a pseudo-differential operator of order zero. (See [15].) The FIO \( U \) is associated to a homogeneous canonical transformation \( T : T^*S^2 \setminus \{0\} \to T^*X \setminus \{0\} \) that intertwines the geodesic flows. Applying once again the averaging method (more precisely Lemma 1 in [8]) to \( \Delta_0 + R \), one can assume without loss of generality that \( [\Delta_0, R] = 0 \). (The result cited says that any operator of the form \( \Delta_0 + R \) with \( R \) self-adjoint pseudo-differential of order zero can be conjugated, this time by a unitary pseudo-differential operator, to an operator of the same form but now such that \( [\Delta_0, R] = 0 \).) For each \( k \), the operator \( U \) maps the space of spherical harmonics of degree \( k \) onto \( E_k \).

Pre-composing \( U \) with a rotation, we can also assume that \( T \) maps the geodesic \( \gamma_0 \) to \( \gamma \). We then define \( u_k \) to be the result of applying the operator \( U \) to the right-hand side of (4.2). Since the desired properties are equivariant with respect to actions of unitary FIOs, we are done.

\[\square\]

This existence result immediately implies:

**Corollary 4.2.** The image of \( \tilde{V} + q_0 \), and therefore its convex hull, are contained in \( R_\infty \).

**Proof.** Given \( \gamma \in \mathcal{O} \), let \( \{u_k\} \) be as in the previous Proposition. By (4.1), the limit of the matrix coefficients

\[
\langle (Q_0 + M_V)u_k, u_k \rangle \in R_k
\]
as \( k \to \infty \) is precisely \((q_0 + \tilde{V})(\gamma)\).

### 4.2. The converse.

We now prove that \( \mathcal{R}_\infty \) is contained in the convex hull of the image of \( \tilde{V} + q_0 \). (The proof is modeled on the proof of [2] Prop. 5.2.) To show this, we will show that for all lines in \( \mathbb{C} \) such that one of the half planes cut out by the line contains the range of \( \tilde{V} + q_0 \), every element of \( \mathcal{R}_\infty \) is contained in the same half plane. This will show that \( \mathcal{R}_\infty \) is contained in, hence equal to, the convex hull in question.

We will use the following Lemma (closely related to the sharp Gårding inequality):

**Lemma 4.3.** Let \( Q \) be a zeroth order, self-adjoint pseudo-differential operator on \( S^n \). Assume that its principal symbol \( q \in C^\infty(T^*S^n \setminus \{0\}) \) is non-negative: \( q \geq 0 \). Let \( \{\psi_k \in E_k\} \) be a sequence of spherical harmonics such that \( \|\psi_k\| = 1 \) for all \( k \). Assume furthermore that

\[ \lim_{k \to \infty} \langle Q(\psi_k), \psi_k \rangle = \ell \in \mathbb{R}. \]

Then \( \ell \geq 0 \).

**Proof.** By the sharp Gårding inequality (see [10], Exercise 4.9), there exists a pseudo-differential operator \( B \) of order \(-1\) such that

\[ \forall u \in L^2(S^n) \quad \langle Q(u), u \rangle \geq -\langle B(u), u \rangle. \]

As before, let \( A \) be the operator which is equal to multiplication by \( k \) when restricted to \( E_k \). Recall that \( A \) is first-order pseudo-differential, so that \( BA \) is of order zero and therefore bounded in \( L^2 \). Therefore \( \exists C > 0 \) such that, with \( \psi_k \) as in the hypotheses of the Lemma,

\[ k |\langle B(\psi_k), \psi_k \rangle| = |\langle BA(\psi_k), \psi_k \rangle| \leq C. \]

Therefore \( \forall k \)

\[ \langle Q(\psi_k), \psi_k \rangle \geq \frac{C}{k}. \]

Taking limits as \( k \to \infty \) yields the desired result. \( \square \)

To finish the proof of the converse, consider a line

\[ \{u + iv \in \mathbb{C}; \ au + bv = c, \ a, b, c \in \mathbb{R}\} \]

in \( \mathbb{C} \), and assume that the range of \( \tilde{V} + q_0 \) is contained in the region \( au + bv \geq c \). Let \( u_0 + iv_0 \in \mathcal{R}_\infty \). We need to show that \( au_0 + bv_0 \geq c \). By definition of \( \mathcal{R}_\infty \), there exists a sequence \( \{\psi_k \in E_k\} \) such that \( \forall k \|\psi_k\| = 1 \) and such that the sequence of complex numbers

\[ u_k + iv_k := \langle (Q_0 + M^aw)\psi_k, \psi_k \rangle \]

converges to \( u_0 + iv_0 \) as \( k \) goes to infinity. Consider now the operator

\[ Q = a(Q_0 + M^aw) + bM^aw - cI. \]

This is a self-adjoint pseudo-differential operator of order zero with non-negative symbol, and

\[ \langle Q(\psi_k), \psi_k \rangle = au_k + bv_k - c \xrightarrow{k \to \infty} au_0 + bv_0 - c. \]

By the Lemma this limit is non-negative, and the proof is complete.
5. Examples on $S^2$

5.1. Preliminaries on the averaging operator. We present here preliminary results on the operator $V \mapsto \tilde{V}$ on the two sphere $S^2 \subset \mathbb{R}^3$ (see also the appendix in [6]). The space, $\mathcal{O}$, of oriented geodesics on $S^2$ can be identified with a copy of $S^2$. Indeed an oriented great circle on $S^2$ is the intersection of $S^2$ with an oriented plane through the origin:

$$\gamma = S^2 \cap \pi_\gamma.$$ 

The identification $\mathcal{O} \cong S^2$ is via the map $\gamma \mapsto$ the unit normal vector to $\pi_\gamma$ defining the orientation. Clearly this identification is equivariant with respect to the action of SO(3). We can therefore regard the averaging operator as an operator on $C^\infty(S^2)$:

$$S^2 \xrightarrow{\text{ave}} S^2, \quad V \mapsto \tilde{V},$$

where $\tilde{V}(\gamma) = \frac{1}{2\pi} \int_{\gamma} V \, ds$. This operator is obviously SO(3) equivariant, and therefore it maps each space of spherical harmonics, $E_k$, into itself. Furthermore, by Schur’s lemma, it is a constant $c_k$ times the identity on $E_k$.

To proceed further, let $(x, y, z)$ denote the ambient $(\mathbb{R}^3)$ coordinates and let $\zeta = x + iy$. Note that, for each $k$,

$$\zeta^k \in E_k$$

(we will abuse the notation and denote the restrictions of $x, y, z$ to the sphere by the same letters). Indeed $(x + iy)^k$ is a homogeneous polynomial of degree $k$ and it is harmonic, by the Cauchy-Riemann equations. Following the identification $\mathcal{O} \cong S^2$ one finds that

$$c_k = \zeta^k(1, 0, 0) = \frac{1}{2\pi} \int_0^{2\pi} (i \cos(t))^k \, dt = \begin{cases} 0 & \text{if } k \text{ is odd} \\ (-1)^l \frac{(2l-1)(2l-3)\ldots3\cdot1}{(2l)(2l-2)\ldots4\cdot2} & \text{if } k = 2l \end{cases}$$

Using Wallis’ formula for $\pi$ it follows that $c_{2l} \sim (-1)^l (l\pi)^{-1/2}$.

5.2. Analytic functions. Our first example is actually a class of examples, namely potentials of the form

$$V(x, y, z) = f(\zeta), \quad f(\zeta) = \sum_{l=1}^{\infty} a_l \zeta^{2l}$$

where the series is assumed to have a radius of convergence $> 1$. In §5 of [9] it was shown that the eigenvalues of $\Delta + M_V$ are exactly the same as those of $\Delta$ (note that there is no constant term in the series), the intuition being that the average of the operator $M_V$, restricted to $E_k$, is nilpotent for each $k$.

To see what Theorem 1.3 says about the asymptotic pseudospectrum in this example, we need to investigate the Poisson bracket condition on $\tilde{V}$. Note that, by the previous discussion,

$$\tilde{V} = \tilde{f}(\zeta) = \sum_{l=1}^{\infty} a_l c_{2l} \zeta^{2l}$$

which shows that $\tilde{V}$ is of the same form as $V$. 
Lemma 5.1. Let $\tilde{V} = F + iG$ with $F$ and $G$ real-valued. Then
\begin{equation}
\{F, G\} = z\left(F_x^2 + F_y^2\right).
\end{equation}

Proof. Denote by $(\theta, z)$ toric coordinates on $\mathcal{O} \cong S^2$, so that the symplectic form is $dz \wedge d\theta$. Then
\begin{align*}
\{F, G\} &= F_z G_\theta - F_\theta G_z = \\
&= (xz F_x + yz F_y)(x_\theta G_x + y_\theta G_y) - (x_\theta F_x + y_\theta F_y)(xz G_x + yz G_y).
\end{align*}

Using the Cauchy-Riemann equations and simplifying one obtains
\begin{align*}
\{F, G\} &= (xz y_\theta - x_\theta y_z)(F_x^2 + F_y^2).
\end{align*}

But $\{x, y\} = z$. \hfill \Box

As the pre-image under $\tilde{V}$ of any complex number contains points whose $z$ coordinates differ by a sign, we get:

Corollary 5.2. For each $\mu \in \{\tilde{f}(\zeta) : |\zeta| < 1$ and $\tilde{f}'(\zeta) \neq 0\}$ the sequence $\{\Lambda_k + \mu\}$ is in the asymptotic pseudospectrum of $\Delta + M_V$.

5.3. Quadratic examples. In this section we take $V$ of the form $V = (ax + iy)^2$, where $a$ is a real constant not equal to either zero or one. If $\Delta$ denotes the (negative) Laplacian on $\mathbb{R}^3$, then $\Delta V = 2(a^2 - 1)$. It follows that
\begin{align*}
h &= (ax + iy)^2 + \frac{1}{3}(1 - a^2)(x^2 + y^2 + z^2)
\end{align*}
is a harmonic, homogeneous polynomial on $\mathbb{R}^3$. Therefore, the decomposition of $V$ into spherical harmonics is
\begin{align*}
V|_{S^2} = h|_{S^2} - \frac{1}{3}(1 - a^2).
\end{align*}
This allows us to compute $\tilde{V}$, which is, up to constants, basically $V$ itself:
\begin{align*}
\tilde{V} = -\frac{1}{2}h - \frac{1}{3}(1 - a^2) \Rightarrow \tilde{V} = -\frac{1}{2}V - \frac{1}{2}(1 - a^2).
\end{align*}
Therefore
\begin{align*}
\{\Re \tilde{V}, \Im \tilde{V}\} &= \frac{1}{4}\{a^2 x^2 - y^2, 2axy\} = \frac{a}{2}\{a^2 x^2, xy\} - \{y^2, xy\} = \\
&= \frac{a}{2}(2a^2 x^2 - 2y^2(-z)) = az(a^2 x^2 + y^2).
\end{align*}

Corollary 5.3. For each $\mu \in \{\tilde{V}(\zeta) : 0 < |\zeta| < 1\}$ the sequence $\{\Lambda_k + \mu\}$ is in the asymptotic pseudospectrum of $\Delta + M_{(ax+iy)^2}$. 
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