RIGHT-TOPOLOGICAL SEMIGROUP OPERATIONS ON INCLUSION HYPERSPACES

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Abstract. We show that for any discrete semigroup $X$ the semigroup operation can be extended to a right-topological semigroup operation on the space $G(X)$ of inclusion hyperspaces on $X$. We detect some important sub-semigroups of $G(X)$, study the minimal ideal, the (topological) center, left cancelable elements of $G(X)$, and describe the structure of the semigroups $G(\mathbb{Z}_n)$ for small numbers $n$.

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INTRODUCTION

After the topological proof of Hindman theorem \cite{H1} given by Galvin and Glazer (unpublished, see \cite{HS}, p.102), \cite{H2}) topological methods become a standard tool in the modern combinatorics of numbers, see \cite{HS, P}. The crucial point is that the semigroup operation $\ast$ defined on any discrete space $S$ can be extended to a right-topological semigroup operation on $\beta S$, the Stone-Čech compactification of $S$. The product of two ultrafilters $U, V \in \beta S$ can be found in two steps: firstly for every element $a \in S$ of the semigroup we extend the left shift $L_a : S \to S$, $L_a : x \mapsto a \ast x$, to a continuous map $\beta L_a : \beta S \to \beta S$. In such a way, for every $a \in S$ we define the product $a \ast V = \beta L_a(V)$. Then, extending the function $R_V : S \to \beta S$, $R_V : a \mapsto a \ast V$, to a continuous map $\beta R_V : \beta S \to \beta S$, we define the product $U \circ V = \beta R_V(U)$. This product can be also defined directly: this is an ultrafilter with the base $\bigcup_{x \in U} x \ast V_x$ where $U \in U$ and $\{V_x\}_{x \in U} \subset V$. Endowed with so-extended operation the Stone-Čech compactification $\beta S$ becomes a compact Hausdorff right-topological semigroup. Because of the compactness the semigroup $\beta S$ has idempotents, minimal (left) ideals, etc., whose existence has many important combinatorial consequences.

The Stone-Čech compactification $\beta S$ can be considered as a subset of the double power-set $\mathcal{P}(\mathcal{P}(S))$. The power-set $\mathcal{P}(X)$ of any set $X$ (in particular, $X = \mathcal{P}(S)$) carries a natural compact Hausdorff topology inherited from the Cantor cube $\{0, 1\}^X$ after identification of each subset $A \subset X$ with its characteristic function. The power-set $\mathcal{P}(X)$ is a complete distributive lattice with respect to the operations of union and intersection.

The smallest complete sublattice of $\mathcal{P}(\mathcal{P}(S))$ containing $\beta S$ coincides with the space $G(S)$ of inclusion hyperspaces, a well-studied object in Categorical Topology. By definition, a family $A \subset \mathcal{P}(S)$ of non-empty subsets of $S$ is called an inclusion hyperspace if together with each set $A \in A$ the family $A$ contains all supersets of $A$ in $S$. In \cite{G1} it is shown that $G(S)$ is a compact Hausdorff lattice with respect to the operations of intersection and union.

Our principal observation is that the algebraic operation of the semigroups $S$ can be extended not only to $\beta S$ but also to the complete lattice hull $G(S)$ of $\beta S$ in $\mathcal{P}(\mathcal{P}(S))$. Endowed with so-extended operation, the space of inclusion hyperspaces $G(S)$ becomes a compact Hausdorff right-topological semigroup containing $\beta S$ as a closed subsemigroup. Besides $\beta S$, the semigroup $G(S)$ contains many other important spaces as closed subsemigroups: the superextension $\lambda S$ of $S$, the space $N_k(S)$ of $k$-linked inclusion hyperspaces, the space Fil($S$) of filters on $S$ (which contains an isomorphic copy of the global semigroup $\Gamma(S)$ of $S$), etc.
We shall study some properties of the semigroup operation on \( G(S) \) and its interplay with the lattice structure of \( G(S) \). We expect that studying the algebraic structure of \( G(S) \) will have some combinatorial consequences that cannot be obtained with help of ultrafilters, see [BGN] for further development of this subject.

1. Inclusion hyperspaces

In this section we recall some basic information about inclusion hyperspaces. More detail information can be found in the paper [G1].

1.1. General definition and reduction to the compact case. For a topological space \( X \) by \( \exp(X) \) we denote the space of all non-empty closed subspaces of \( X \) endowed with the Vietoris topology. By an inclusion hyperspace we mean a closed subfamily \( F \subset \exp(X) \) that is monotone in the sense that together with each set \( A \in F \) the family \( F \) contains all closed subsets \( B \subset X \) that contain \( A \).

By [G1], the closure of each monotone family in \( \exp(X) \) is an inclusion hyperspace. Consequently, each family \( B \subset \exp(X) \) generates an inclusion hyperspace \( \langle B \rangle \) denoted by \( B \). In this case \( B \) is called a base of \( F = \langle B \rangle \). An inclusion hyperspace \( \langle x \rangle \) generated by a singleton \( \{x\}, x \in X \), is called principal.

If \( X \) is discrete, then each monotone family in \( \exp(X) \) is an inclusion hyperspace, see [G1].

Denote by \( G(X) \) the space of all inclusion hyperspaces with the topology generated by the subbase

\[
U^+ = \{ A \in G(X) : \exists B \in A \text{ with } B \subset U \} \quad \text{and} \\
U^- = \{ A \in G(X) : \forall B \in A \text{ with } B \cap U \neq \emptyset \},
\]

where \( U \) is open in \( X \).

For a \( T_1 \)-space \( X \) the map \( \eta_X : X \to G(X), \eta_X(x) = \{ F \subset X : x \in F \} \), is an embedding (see [G1]), so we can identify principal inclusion hyperspaces with elements of the space \( X \).

For a \( T_1 \)-space \( X \) the space \( G(X) \) is Hausdorff if and only if the space \( X \) is normal, see [G1], [M]. In the latter case the map

\[
h : G(X) \to G(\beta X), \quad h(F) = cl_{\exp(\beta X)}(cl_{\beta X} F \mid F \in \mathcal{F}),
\]

is a homeomorphism, so we can identify the space \( G(X) \) with the space \( G(\beta X) \) of inclusion hyperspaces over the Stone-Čech compactification \( \beta X \) of the normal space \( X \), see [M]. Thus we reduce the study of inclusion hyperspaces over normal topological spaces to the compact case where this construction is well-studied.

In [G1] the inclusion hyperspace \( \langle B \rangle \) generated by a base \( B \) is denoted by \( \mathcal{B} \).

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1In [G1] the inclusion hyperspace \( \langle B \rangle \) generated by a base \( B \) is denoted by \( \mathcal{B} \).
For a (discrete) $T_1$-space the space $G(X)$ contains a (discrete and) dense sub-space $G^*(X)$ consisting of inclusion hyperspaces with finite support. An inclusion hyperspace $A \in G(X)$ is defined to have finite support in $X$ if $A = \langle F \rangle$ for some finite family $F$ of finite subsets of $X$.

An inclusion hyperspace $F \in G(X)$ on a non-compact space $X$ is called free if for each compact subset $K \subset X$ and any element $F \in F$ there is another element $E \subset F \setminus K$. By $G^o(X)$ we shall denote the subset of $G(X)$ consisting of free inclusion hyperspaces. By $\mathcal{G}$, for a normal locally compact space $X$ we shall denote the subset of $G(X)$ consisting of free inclusion hyperspaces. By [G1], for a normal locally compact space $X$ the subset $G^o(X)$ is closed in $G(X)$. In the simplest case of a countable discrete space $X = \mathbb{N}$ free inclusion hyperspaces (called semifilters) on $X = \mathbb{N}$ have been introduced and intensively studied in [BZ].

1.2. Inclusion hyperspaces in the category of compacta. The construction of the space of inclusion hyperspaces is functorial and monadic in the category $\text{Comp}$ of compact Hausdorff spaces and their continuous maps, see [TZ]. To complete $G$ to a functor on $\text{Comp}$ observe that each continuous map $f : X \to Y$ between compact Hausdorff spaces induces a continuous map $Gf : G(X) \to G(Y)$ defined by

$$Gf(A) = \langle f(A) \rangle = \{ B \subseteq Y : B \supseteq f(A) \text{ for some } A \in A \}$$

for $A \in G(X)$. The map $Gf$ is well-defined and continuous, and $G$ is a functor in the category $\text{Comp}$ of compact Hausdorff spaces and their continuous maps, see [TZ]. By Proposition 2.3.2 [TZ], this functor is weakly normal in the sense that it is continuous, monomorphic, epimorphic and preserves intersections, singletons, the empty set and weight of infinite compacta.

Since the functor $G$ preserves monomorphisms, for each closed subspace $A$ of a compact Hausdorff space $X$ the inclusion map $i : A \to X$ induces a topological embedding $Gi : G(A) \to G(X)$. So we can identify $G(A)$ with a subspace of $G(X)$. Now for each inclusion hyperspace $A \in G(X)$ we can consider the support of $A$

$$\text{supp} A = \cap \{ A \subset Y : A \in G(A) \}$$

and conclude that $A \in G(\text{supp} A)$ because $G$ preserves intersections, see [TZ] §2.4.

Next, we consider the monadic properties of the functor $G$. We recall that a functor $T : \text{Comp} \to \text{Comp}$ is monadic if it can be completed to a monad $T = (T, \eta, \mu)$ where $\eta : \text{Id} \to T$ and $\mu : T^2 \to T$ are natural transformations (called the unit and multiplication) such that $\mu \circ T(\mu_X) = \mu \circ \mu_{TX} : T^3X \to TX$ and $\mu \circ \eta_{TX} = \mu \circ T(\eta_X) = \text{Id}_{TX}$ for each compact Hausdorff space $X$, see [TZ].

For the functor $G$ the unit $\eta : \text{Id} \to G$ has been defined above while the multiplication $\mu = \{ \mu_X : G^2X \to G(X) \}$ is defined by the formula

$$\mu_X(\Theta) = \cup \{ \cap M | M \in \Theta \}, \Theta \in G^2X.$$
1.3. Some important subspaces of \( G(X) \). The space \( G(X) \) of inclusion hyperspaces contains many interesting subspaces. Let \( X \) be a topological space and \( k \geq 2 \) be a natural number. An inclusion hyperspace \( A \in G(X) \) is defined to be

- **\( k \)-linked** if \( \cap F \neq \emptyset \) for any subfamily \( F \subset A \) with \( |F| \leq k \);
- **centered** if \( \cap F \neq \emptyset \) for any finite subfamily \( F \subset A \);
- **a filter** if \( A_1 \cap A_2 \in A \) for all sets \( A_1, A_2 \in A \);
- **an ultrafilter** if \( A = A' \) for any filter \( A' \in G(X) \) containing \( A \);
- **maximal \( k \)-linked** if \( A = A' \) for any \( k \)-linked inclusion hyperspace \( A' \in G(X) \) containing \( A \).

By \( N_k(X) \), \( N_{<\omega}(X) \), and Fil\( ^0(X) \) we denote the subsets of \( G(X) \) consisting of \( k \)-linked, centered, and filter inclusion hyperspaces, respectively. Also by \( \beta(X) \) and \( \lambda_k(X) \) we denote the subsets of \( G(X) \) consisting of ultrafilters and maximal \( k \)-linked inclusion hyperspaces, respectively. The space \( \lambda(X) = \lambda_2(X) \) is called the *superextension* of \( X \).

The following diagram describes the inclusion relations between subspaces \( N_kX \), \( N_{<\omega}X \), Fil\( ^0(X) \), \( \lambda X \) and \( \beta X \) of \( G(X) \) (an arrow \( A \rightarrow B \) means that \( A \) is a subset of \( B \)).

\[
\begin{array}{c}
\text{Fil}(X) \rightarrow N_{<\omega}X \rightarrow N_kX \rightarrow N_2X \rightarrow G(X) \\
\beta X \quad \quad \quad \lambda X
\end{array}
\]

For a normal space \( X \) all the subspaces from this diagram are closed in \( G(X) \), see [G1].

For a non-compact space \( X \) we can also consider the intersections

\[
\begin{align*}
\text{Fil}^0(X) &= \text{Fil}(X) \cap G^0(X), \\
N^0_{<\omega}(X) &= N_{<\omega}(X) \cap G^0(X), \\
N^0_k(X) &= N_k(X) \cap G^0(X), \\
\lambda^0_k(X) &= \lambda_k(X) \cap G^0(X), \quad \text{and} \\
\beta^0(X) &= \beta X \cap G^0(X) = \beta X \setminus X.
\end{align*}
\]

Elements of those sets will be called free filters, free centered inclusion hyperspaces, free \( k \)-linked inclusion hyperspaces, etc. For a normal locally compact space \( X \) the subsets \( \text{Fil}^0(X), N^0_{<\omega}(X), N^0_k(X), \lambda^0(X) = \lambda^0_2(X), \) and \( \beta^0(X) \) are closed in \( G(X) \), see [G1]. In contrast, \( \lambda^0_k(\mathbb{N}) \) is not closed in \( G(\mathbb{N}) \) for \( k \geq 3 \), see [I].

1.4. The inner algebraic structure of \( G(X) \). In this subsection we discuss the algebraic structure of the space of inclusion hyperspaces \( G(X) \) over a topological space \( X \). The space of inclusion hyperspaces \( G(X) \) possesses two binary operations...
∪, ∩, and one unary operation
\[ \bot: G(X) \to G(X), \quad \bot: \mathcal{F} \mapsto \mathcal{F}^\bot = \{ E \subset X : \forall F \in \mathcal{F}, E \cap F \neq \emptyset \} \]
called the transversality map. These three operations are continuous and turn
\( G(X) \) into a symmetric lattice, see [G1].

**Definition 1.1.** A symmetric lattice is a complete distributive lattice \((L, \lor, \land)\)
edowed with an additional unary operation \( \bot: L \to L \), \( \bot: x \mapsto x^\perp \), that is an
involutive anti-isomorphism in the sense that

- \( x^{\perp \perp} = x \) for all \( x \in L \);
- \( (x \lor y)^\perp = x^\perp \land y^\perp \);
- \( (x \land y)^\perp = x^\perp \lor y^\perp \);

The smallest element of the lattice \( G(X) \) is the inclusion hyperspace \( \{ X \} \) while
the largest is \( \exp(X) \).

For a discrete space \( X \) the set \( G(X) \) of all inclusion hyperspaces on \( X \) is a
subset of the double power-set \( \mathcal{P}(\mathcal{P}(X)) \) (which is a complete distributive lattice)
and is closed under the operations of union and intersection (of arbitrary families
of inclusion hyperspaces).

Since each inclusion hyperspace is a union of filters and each filter is an inter-
section of ultrafilters, we obtain the following proposition showing that the lattice
\( G(X) \) is a rather natural object.

**Proposition 1.2.** For a discrete space \( X \) the lattice \( G(X) \) coincides with the small-
est complete sublattice of \( \mathcal{P}(\mathcal{P}(X)) \) containing all ultrafilters.

2. Extending algebraic operations to inclusion hyperspaces

In this section, given a binary (associative) operation \( * : X \times X \to X \) on a discrete
space \( X \) we extend this operation to a right-topological (associative) operation on
\( G(X) \). This can be done in two steps by analogy with the extension of the operation
to the Stone-Čech compactification \( \beta X \) of \( X \).

First, for each element \( a \in X \) consider the left shift \( L_a : X \to X, L_a(x) = a \ast x \) and extend it to a continuous map \( \tilde{L}_a : \beta X \to \beta X \) between the Stone-Čech compactifications of \( X \). Next, apply to this extension the functor \( G \) to obtain the
continuous map \( G\tilde{L}_a : G(\beta X) \to G(\beta X) \). Clearly, for every inclusion hyperspace \( \mathcal{F} \in G(\beta X) \) the inclusion hyperspace \( G\tilde{L}_a(\mathcal{F}) \) has a base \( \{a \ast F \mid F \in \mathcal{F} \} \). Thus, we
have defined the product \( a \ast \mathcal{F} = G\tilde{L}_a(\mathcal{F}) \) of the element \( a \in X \) and the inclusion
hyperspace \( \mathcal{F} \).

Further, for each inclusion hyperspace \( \mathcal{F} \in G(\beta X) = G(X) \) we can consider the
map \( R_\mathcal{F} : X \to G(\beta X) \) defined by the formula \( R_\mathcal{F}(x) = x \ast \mathcal{F} \) for every \( x \in X \).
Extend the map \( R_\mathcal{F} \) to a continuous map \( \tilde{R}_\mathcal{F} : \beta X \to G(\beta X) \) and apply to this
extension the functor $G$ to obtain a map $G\tilde{R}_F : G(\beta X) \to G^2(\beta X)$. Finally, compose the map $G\tilde{R}_F$ with the multiplication $\mu_X = \mu_G X : G^2 X \to G(X)$ of the monad $G = (G, \eta, \mu)$ and obtain a map $\mu_X \circ G\tilde{R}_F : G(\beta X) \to G(\beta X)$. For an inclusion hyperspace $U \in G(\beta X)$, the image $\mu_G X \circ G\tilde{R}_F(U)$ is called the product of the inclusion hyperspaces $U$ and $F$ and is denoted by $U \circ F$.

It follows from continuity of the maps $G\tilde{R}_F$ that the extended binary operation on $G(X)$ is continuous with respect to the first argument with the second argument fixed. We are going to show that the operation $\circ$ on $G(X)$ nicely agrees with the lattice structure of $G(X)$ and is associative if so is the operation $\ast$. Also we shall establish an easy formula

$$U \circ F = \bigcup_{x \in U} x \ast F_x : U \in U, \{F_x\}_{x \in U} \subset F$$

for calculating the product $U \circ F$ of two inclusion hyperspaces $U, F$. We start with necessary definitions.

**Definition 2.1.** Let $\ast : G(X) \times G(X) \to G(X)$ be a binary operation on $G(X)$. We shall say that $\ast$ respects the lattice structure of $G(X)$ if for any $U, V, W \in G(X)$ and $a \in X$

1. $(U \cup V) \ast W = (U \ast W) \cup (V \ast W)$;
2. $(U \cap V) \ast W = (U \ast W) \cap (V \ast W)$;
3. $a \ast (V \cup W) = (a \ast V) \cup (a \ast W)$;
4. $a \ast (V \cap W) = (a \ast V) \cap (a \ast W)$.

**Definition 2.2.** We will say that a binary operation $\ast : G(X) \times G(X) \to G(X)$ is right-topological if

- for any $U \in G(X)$ the right shift $R_U : G(X) \to G(X), R_U : F \mapsto F \ast U$, is continuous;
- for any $a \in X$ the left shift $L_a : G(X) \to G(X), L_a : F \mapsto a \ast F$, is continuous.

The following uniqueness theorem will be used to find an equivalent description of the induced operation on $G(X)$.

**Theorem 2.3.** Let $\ast, \circ : G(X) \times G(X) \to G(X)$ be two right-topological binary operations that respect the lattice structure of $G(X)$. These operations coincide if and only if they coincide on the product $X \times X \subset G(X) \times G(X)$.

**Proof.** It is clear that if these operations coincide on $G(X) \times G(X)$, then they coincide on the product $X \times X$ identified with a subset of $G(X) \times G(X)$. We recall that each point $x \in X$ is identified with the ultrafilter $\langle x \rangle$ generated by $x$.

Now assume conversely that $x \ast y = x \circ y$ for any two points $x, y \in X \subset G(X)$. First we check that $a \ast F = a \circ F$ for any $a \in X$ and $F \in G(X)$. Since the left
shifts $\mathcal{F} \mapsto a \star \mathcal{F}$ and $\mathcal{F} \mapsto a \circ \mathcal{F}$ are continuous, it suffices to establish the equality $a \star \mathcal{F} = a \circ \mathcal{F}$ for inclusion hyperspaces $\mathcal{F}$ having finite support in $X$ (because the set $G^*(X)$ of all such inclusion hyperspaces is dense in $G(X)$, see [GH]). Any such a hyperspace $\mathcal{F}$ is generated by a finite family of finite subsets of $X$.

If $\mathcal{F} = \langle F \rangle$ is generated by a single finite subset $F = \{a_1, \ldots, a_n\} \subset X$, then $\mathcal{F} = \bigcap_{i=1}^n \langle a_i \rangle$ is the finite intersection of principal ultrafilters, and hence

$$\langle a \rangle \star \mathcal{F} = \langle a \rangle \star \bigcap_{i=1}^n \langle a_i \rangle = \bigcap_{i=1}^n \langle a \rangle \star \langle a_i \rangle = \bigcap_{i=1}^n \langle a \rangle \circ \langle a_i \rangle = \langle a \rangle \circ \bigcap_{i=1}^n \langle a_i \rangle = \langle a \rangle \circ \mathcal{F}.$$  

If $\mathcal{F} = \langle F_1, \ldots, F_n \rangle$ is generated by finite family of finite sets, then $\mathcal{F} = \bigcup_{i=1}^n \langle F_i \rangle$ and we can use the preceding case to prove that

$$\langle a \rangle \star \mathcal{F} = \langle a \rangle \star \bigcup_{i=1}^n \langle F_i \rangle = \bigcup_{i=1}^n \langle a \rangle \star \langle F_i \rangle = \bigcup_{i=1}^n \langle a \rangle \circ \langle F_i \rangle = \langle a \rangle \circ \bigcup_{i=1}^n \langle F_i \rangle = \langle a \rangle \circ \mathcal{F}.$$  

Now fixing any inclusion hyperspace $\mathcal{U} \in G(X)$ by a similar argument one can prove the equality $\mathcal{F} \star \mathcal{U} = \mathcal{F} \circ \mathcal{U}$ for all inclusion hyperspaces $\mathcal{F} \in G^*(X)$ having finite support in $X$. Finally, using the density of $G^*(X)$ in $G(X)$ and the continuity of right shifts $\mathcal{F} \mapsto \mathcal{F} \circ \mathcal{U}$ and $\mathcal{F} \mapsto \mathcal{F} \star \mathcal{U}$ one can establish the equality $\mathcal{F} \star \mathcal{U} = \mathcal{F} \circ \mathcal{U}$ for all inclusion hyperspaces $\mathcal{F} \in G(X)$.  

The above theorem will be applied to show that the operation $\circ : G(X) \times G(X) \to G(X)$ induced by the operation $\star : X \times X \to X$ coincides with the operation $\star : G(X) \times G(X) \to G(X)$ defined by the formula

$$\mathcal{U} \star \mathcal{V} = \bigcup_{x \in U} x \star V_x : U \in \mathcal{U}, \ \{V_x\}_{x \in U} \subset \mathcal{V}$$

for $\mathcal{U}, \mathcal{V} \in G(X)$.

First we establish some properties of the operation $\star$.

**Proposition 2.4.** The operation $\star$ commutes with the transversality operation in the sense that $(\mathcal{U} \star \mathcal{V})^\perp = \mathcal{U}^\perp \star \mathcal{V}^\perp$ for any $\mathcal{U}, \mathcal{V} \in G(X)$.

**Proof.** To prove that $\mathcal{U}^\perp \star \mathcal{V} \subset (\mathcal{U} \star \mathcal{V})^\perp$, take any element $A \in \mathcal{U}^\perp \star \mathcal{V}^\perp$. We should check that $A$ intersects each set $B \in \mathcal{U} \star \mathcal{V}$. Without loss of generality, the sets $A$ and $B$ are of the basic form:

$$A = \bigcup_{x \in F} x \star G_x \text{ for some sets } F \in \mathcal{U}^\perp \text{ and } \{G_x\}_{x \in F} \subset \mathcal{V}^\perp$$

and

$$B = \bigcup_{x \in U} x \star V_x \text{ for some sets } U \in \mathcal{U} \text{ and } \{V_x\}_{x \in U} \subset \mathcal{V}.$$  

Since $U \in \mathcal{U}$ and $F \in \mathcal{U}^\perp$, the intersection $F \cap U$ contains some point $x$. For this point $x$ the sets $V_x \in \mathcal{V}$ and $G_x \in \mathcal{V}^\perp$ are well-defined and their intersection
$V_x \cap G_x$ contains some point $y$. Then the intersection $A \cap B$ contains the point $x \ast y$ and hence is not empty, which proves that $A \in (U \ast V)^\perp$.

To prove that $(U \ast V)^\perp \subseteq U^\perp \ast V^\perp$, fix a set $A \in (U \ast V)^\perp$. We claim that the set $F = \{ x \in X : x^{-1}A \in V^\perp \}$ belongs to $U^\perp$ (here $x^{-1}A = \{ y \in X : x \ast y \in A \}$). Assuming conversely that $F \notin U^\perp$, we would find a set $U \in U$ with $F \cap U = \emptyset$. By the definition of $F$, for each $x \in U$ the set $x^{-1}A \notin V^\perp$ and thus we can find a set $V_x \in V$ with empty intersection $V_x \cap x^{-1}A$. By the definition of the product $U \ast V$, the set $B = \bigcup_{x \in U} x \ast V_x$ belongs to $U \ast V$ and hence intersects the set $A$. Consequently, $x \ast y \in A$ for some $x \in U$ and $y \in V_x$. The inclusion $x \ast y \in A$ implies that $y \in x^{-1}A \subset X \setminus V_x$, which is a contradiction proving that $F \in U^\perp$. Then the sets $A \supset \bigcup_{x \in F} x \ast x^{-1}A$ belong to $U^\perp \ast V^\perp$. \[\Box\]

**Proposition 2.5.** The equality $(U \cap V) \ast W = (U \ast W) \cap (V \ast W)$ holds for any $U, V, W \in G(X)$.

**Proof.** It is easy to show that $(U \cap V) \ast W \subset (U \ast W) \cap (V \ast W)$.

To prove the reverse inclusion, fix a set $F \in (U \ast W) \cap (V \ast W)$. Then $F \supset \bigcup_{x \in U} x \ast W_x'$ and $F \supset \bigcup_{y \in V} y \ast W_y''$ for some $U \in U$, $\{W_x'\}_{x \in U} \subset W$, and $V \in V$, $\{W_y''\}_{y \in V} \subset W$. Since $U, V$ are inclusion hyperspaces, $U \cup V \in U \cap V$. For each $z \in U \cup V$ let $W_z = W_z'$ if $z \in U$ and $W_z = W_z''$ if $z \notin U$. It follows that $F \supset \bigcup_{z \in U \cup V} z \ast W_z$ and hence $F \in (U \cap V) \ast W$. \[\Box\]

By analogy one can prove

**Proposition 2.6.** For any $U, V, W \in G(X)$ and $a \in X$ 

$$a \ast (V \cup W) = (a \ast V) \cup (a \ast W) \quad \text{and} \quad a \ast (V \cap W) = (a \ast V) \cap (a \ast W).$$

Combining Propositions 2.4 and 2.5 we get

**Corollary 2.7.** For any $U, V, W \in G(X)$ we get 

$$(U \cup V) \ast W = (U \ast W) \cup (V \ast W).$$

**Proof.** Indeed, 

$$(U \cup V) \ast W = ((U \cup V) \ast W)^\perp = ((U \cup V)^\perp \ast W^\perp)^\perp =
=(U^\perp \cap V^\perp) \ast W^\perp = ((U^\perp \ast W^\perp) \cap (V^\perp \ast W^\perp))^\perp =
=(U^\perp \ast W^\perp)^\perp \cup (V^\perp \ast W^\perp)^\perp = (U \ast W) \cup (V \ast W).$$

\[\Box\]
Proposition 2.8. The operation
\[ \star : G(X) \times G(X) \to G(X), \quad U \star V = \{ x \in U : \{ V_x \}_{x \in U} \subset V \}, \]
respects the lattice structure of \( G(X) \) and is right-topological.

Proof. Propositions 2.5, 2.6 and Corollary 2.7 imply that the operation \( \star \) respects the lattice structure of \( G(X) \).

So it remains to check that the operation \( \star \) is right-topological. First we check that for any \( U \in G(X) \) the right shift \( R_U : G(X) \to G(X), R_U : F \mapsto F \star U \), is continuous.

Fix any inclusion hyperspaces \( \mathcal{F}, \mathcal{U} \in G(X) \) and let \( W^+ \) be a sub-basic neighborhood of their product \( \mathcal{F} \star \mathcal{U} \). Find sets \( F \in \mathcal{F} \) and \( \{ U_x \}_{x \in F} \subset \mathcal{U} \) such that \( \bigcup_{x \in F} x \star U_x \subset W \). Then \( F^+ \) is a neighborhood of \( F \) with \( F^+ \star \mathcal{U} \subset W^+ \).

Now assume that \( \mathcal{F} \star \mathcal{U} \in W^- \) for some \( W \subset X \). Observe that for any inclusion hyperspace \( V \in G(X) \) we get the equivalences \( V \in W^- \iff W \in V^\perp \iff V^\perp \in W^+ \). Consequently, \( \mathcal{F} \star \mathcal{U} \in W^- \) is equivalent to \( \mathcal{F}^\perp \star \mathcal{U}^\perp \in W^+ \). The preceding case yields a neighborhood \( O(\mathcal{F}^\perp) \) such that \( O(\mathcal{F}^\perp) \star \mathcal{U}^\perp \in W^+ \). Now the continuity of the transversality operation implies that \( O(\mathcal{F}^\perp) \perp \) is a neighborhood of \( \mathcal{F} \) with \( O(\mathcal{F}^\perp)^\perp \star \mathcal{U} \in W^- \).

Finally, we prove that for every \( a \in X \) the left shift \( L_a : G(X) \to G(X), L_a : \mathcal{F} \mapsto a \star \mathcal{F} \), is continuous. Given a sub-basic open set \( W^+ \subset G(X) \) note that \( L_a^{-1}(W^+) \) is open because \( L_a^{-1}(W^+) = (a^{-1}W)^+ \) where \( a^{-1}W = \{ x \in X : a \star x \in W \} \). On the other hand, \( a \star \mathcal{F} \in W^- \) is equivalent to \( a \star \mathcal{F}^\perp = (a \star \mathcal{F})^\perp \in (W^-)^\perp = W^+ \) which implies that the preimage
\[ L_a^{-1}(W^-) = (L_a(W^+))^\perp \]
is also open. \( \square \)

The operation \( \circ \) has the same properties.

Proposition 2.9. The operation \( \circ : G(X) \times G(X) \to G(X), \mathcal{U} \circ \mathcal{V} = \mu_GX \circ \mu_GX_R(\mathcal{U}) \)
respects the lattice structure of \( G(X) \) and is right-topological.

Proof. For any \( \mathcal{U} \in G(X) \) the right shift \( R_\mathcal{U} = \mu_GX_R \circ \mu_GX_R : G(X) \to G(X), R_\mathcal{U} : \mathcal{F} \mapsto \mathcal{F} \circ \mathcal{U} \) is continuous being the composition of continuous maps. Next for any \( a \in X \) and \( \mathcal{F} \in G(X) \) we have \( L_a(\mathcal{F}) = \mathcal{a} \circ \mathcal{F} = \mu_GX(\langle a \rangle \circ \mathcal{F}) = \mu_GX(\langle a \star \mathcal{F} \rangle) = a \star \mathcal{F} = G \mathcal{L}_a(\mathcal{F}) \) and the map \( L_a : G \mathcal{L}_a \) is continuous.

It is known (and easy to verify) that the multiplication \( \mu_GX : G^2(X) \to G(X) \)
is a lattice homomorphism in the sense that
\[ \mu_GX(\mathcal{U} \cup \mathcal{V}) = \mu_GX(\mathcal{U}) \cup \mu_GX(\mathcal{V}) \quad \text{and} \quad \mu_GX(\mathcal{U} \cap \mathcal{V}) = \mu_GX(\mathcal{U}) \cap \mu_GX(\mathcal{V}) \]
for any \( U, V \in G(X) \). Then for any \( U, V, W \in G(X) \) and \( a \in X \) we get
\[
(U \cup V) \circ W = \mu_{G(X)}(G\overline{R}_W(U \cup V)) = \mu_{G(X)}(G\overline{R}_W(U)) \cup \mu_{G(X)}(G\overline{R}_W(V)) = (U \circ W) \cup (U \circ W)
\]
and similarly \((U \cap V) \circ W = (U \circ W) \cap (U \circ W)\).

Note that for any \( a \in X \)
\[
a \circ W = \mu_{G(X)}(\overline{R}_W(a)) = (\overline{R}_W(a)) = (\overline{R}_W(a)) = a \ast W.
\]
Consequently,
\[
a \circ (V \cup W) = a \ast (V \cup W) = (a \ast V) \cup (a \ast W) = (a \circ V) \cup (a \circ W)
\]
and similarly \( a \circ (V \cap W) = (a \circ V) \cap (a \circ W) \).

Since both operations \( \circ \) and \( \ast \) are right-topological and respect the lattice structure of \( G(X) \) we may apply Theorem 2.3 to get

**Corollary 2.10.** For any binary operation \( \ast : X \times X \rightarrow X \) the operations \( \circ \) and \( \ast \) on \( G(X) \) coincide. Consequently, for any inclusion hyperspaces \( U, V \in G(X) \) their product \( U \circ V \) is the inclusion hyperspace
\[
\left( \bigcup_{x \in U} x \ast V_x : U \in \mathcal{U}, \{V_x\}_{x \in U} \subset \mathcal{V} \right) = \{ A \subset X : \{ x \in X : x^{-1}A \in \mathcal{V} \} \in \mathcal{U} \}.
\]

Having the apparent description of the operation \( \circ \) we can establish its associativity.

**Proposition 2.11.** If the operation \( \ast \) on \( X \) is associative, then so is the induced operation \( \circ \) on \( G(X) \).

**Proof.** It is necessary to show that \((U \circ V) \circ W = U \circ (V \circ W)\) for any inclusion hyperspaces \( U, V, W \).

Take any subset \( A \in (U \circ V) \circ W \) and choose a set \( B \in U \circ V \) such that \( A \supset \bigcup_{z \in B} z \ast W_z \) for some family \( \{W_z\}_{z \in B} \subset W \). Next, for the set \( B \in U \circ V \) choose a set \( U \in U \) such that \( B \supset \bigcup_{x \in U} x \ast V_x \) for some family \( \{V_x\}_{x \in U} \subset \mathcal{V} \). It is clear that for each \( x \in U \) and \( y \in V_x \) the product \( x \ast y \) is in \( B \) and hence \( W_{x \ast y} \) is defined. Consequently, \( \bigcup_{y \in V_x} y \ast W_{x \ast y} \in V \circ W \) for all \( x \in U \) and hence \( \bigcup_{x \in U} x \ast (\bigcup_{y \in V_x} y \ast W_{x \ast y}) \in U \circ (V \circ W) \). Since \( \bigcup_{x \in U} x \ast y \ast W_{x \ast y} \subset A \), we get \( A \in U \circ (V \circ W) \). This proves the inclusion \((U \circ V) \circ W \subset U \circ (V \circ W)\).

To prove the reverse inclusion, fix a set \( A \in U \circ (V \circ W) \) and choose a set \( U \in U \) such that \( A \supset \bigcup_{x \in U} x \ast B_x \) for some family \( \{B_x\}_{x \in U} \subset \mathcal{V} \circ W \). Next, for each \( x \in U \) find a set \( V_x \in \mathcal{V} \) such that \( B_x \supset \bigcup_{y \in V_x} y \ast W_{x,y} \) for some family \( \{W_{x,y}\}_{y \in V_x} \subset W \).

Let \( Z = \bigcup_{x \in U} x \ast V_x \). For each \( z \in Z \) we can find \( x \in U \) and \( y \in V_x \) such that \( z = x \ast y \).
and put $W_z = W_{x,y}$. Then $Z \in \mathcal{U} \circ \mathcal{V}$ and $\bigcup_{z \in Z} z \ast W_z \in (\mathcal{U} \circ \mathcal{V}) \circ \mathcal{W}$. Taking into account $\bigcup_{z \in Z} z \ast W_z \subset \bigcup_{x \in U} \bigcup_{y \in V_x} x \ast y \ast W_{x,y} \subset A$, we conclude $A \in (\mathcal{U} \circ \mathcal{V}) \circ \mathcal{W}$. □

3. HOMOMORPHISMS OF SEMIGROUPS OF INCLUSION HYPERSPACES

Let us observe that our construction of extension of a binary operation for $X$ to $G(X)$ works well both for associative and non-associative operations. Let us recall that a set $S$ endowed with a binary operation $\ast : X \times X \to X$ is called a groupoid. If the operation is associative, then $X$ is called a semigroup. In the preceding section we have shown that for each groupoid (semigroup) $X$ the space $G(X)$ is a groupoid (semigroup) with respect to the extended operation.

A map $h : X_1 \to X_2$ between two groupoids $(X_1, \ast_1)$ and $(X_2, \ast_2)$ is called a homomorphism if $h(x \ast_1 y) = h(x) \ast_2 h(y)$ for all $x, y \in X_1$.

**Proposition 3.1.** For any homomorphism $h : X_1 \to X_2$ between groupoids $(X_1, \ast_1)$ and $(X_2, \ast_2)$ the induced map $Gh : G(X_1) \to G(X_2)$ is a homomorphism of the groupoids $G(X_1), G(X_2)$.

**Proof.** Given two inclusion hyperspaces $\mathcal{U}, \mathcal{V} \in G(X_1)$ observe that

$$Gh(\mathcal{U} \circ \mathcal{V}) = Gh((\bigcup_{x \in U} x \ast_1 V_x : U \in \mathcal{U}, \{V_x\}_{x \in U} \subset \mathcal{V})) =$$

$$= (h(\bigcup_{x \in U} x \ast_1 V_x) : U \in \mathcal{U}, \{V_x\}_{x \in U} \subset \mathcal{V}) =$$

$$= (\bigcup_{x \in U} h(x) \ast_2 h(V_x) : U \in \mathcal{U}, \{V_x\}_{x \in U} \subset \mathcal{V}) =$$

$$= (\bigcup_{x \in h(U)} x \ast_2 h(V_x) : U \in \mathcal{U}, \{h(V_x)\}_{x \in U} \subset Gh(\mathcal{V})) =$$

$$= (h(U) : U \in \mathcal{U}) \circ_2 (h(V) : V \in \mathcal{V}) = Gh(\mathcal{U}) \circ_2 Gh(\mathcal{V}).$$

□

Reformulating Proposition 2.4 in terms of homomorphisms, we obtain

**Proposition 3.2.** For any groupoid $X$ the transversality map $\perp : G(X) \to G(X)$ is a homomorphism of the groupoid $G(X)$.

4. SUBGROUPOIDS OF $G(X)$

In this section we shall show that for a groupoid $X$ endowed with the discrete topology all (topologically) closed subspaces of $G(X)$ introduced in Section 1.3 are subgroupoids of $G(X)$. A subset $A$ of a groupoid $(X, \ast)$ is called a subgroupoid of $X$ if $A \ast A \subset A$, where $A \ast A = \{a \ast b : a, b \in A\}$.
We assume that \( * : X \times X \to X \) is a binary operation on a discrete space \( X \) and \( \circ : G(X) \times G(X) \to G(X) \) is the extension of \( * \) to \( G(X) \). Applying Proposition 3.2 we obtain

**Proposition 4.1.** If \( S \) is a subgroupoid of \( G(X) \), then \( S^\perp \) is a subgroupoid of \( G(X) \) too.

Our next propositions can be easily derived from Corollary 2.10.

**Proposition 4.2.** The sets \( \text{Fil}(X) \), \( N_{<\omega}(X) \) and \( N_k(X) \), \( k \geq 2 \), are subgroupoids in \( G(X) \).

**Proposition 4.3.** The Stone-\v{C}ech extension \( \beta X \) and the superextension \( \lambda X \) both are closed subgroupoids in \( G(X) \).

**Proof.** The superextension \( \lambda X \) is a subgroupoid of \( G(X) \) being the intersection \( \lambda(X) = N_2(X) \cap (N_2(X))^\perp \) of two subgroupoids of \( G(X) \). By analogy, \( \beta(X) = \text{Fil}(X) \cap \lambda(X) \) is a subgroupoid of \( G(X) \). \( \square \)

**Remark 4.4.** In contrast to \( \lambda X \) for \( k \geq 3 \) the subset \( \lambda_k(X) \) need not be a subgroupoid of \( G(X) \). For example, for the cyclic group \( \mathbb{Z}_5 = \{0, 1, 2, 3, 4\} \) the subset \( \lambda_3(\mathbb{Z}_5) \) of \( G(\mathbb{Z}_5) \) contains a maximal 3-linked system

\[
L = \langle \{0, 1, 2\}, \{0, 1, 4\}, \{0, 2, 4\}, \{1, 2, 4\} \rangle
\]

whose square

\[
L \ast L = \langle \{1, 2, 4, 5\}, \{0, 2, 3, 4\}, \{0, 1, 3, 4\}, \{0, 1, 2, 4\}, \{0, 1, 2, 3\} \rangle
\]

is not maximal 3-linked.

By a direct application of Corollary 2.10 we can also prove

**Proposition 4.5.** The set \( G^\ast(X) \) of all inclusion hyperspaces with finite support is a subgroupoid in \( G(X) \).

Finally we find conditions on the operation \( * \) guaranteeing that the subset \( G^\circ(X) \) of free inclusion hyperspaces is a subgroupoid of \( G(X) \).

**Proposition 4.6.** Assume that for each \( b \in X \) there is a finite subset \( F \subset X \) such that for each \( a \in X \setminus F \) the set \( a^{-1}b = \{x \in X : a \ast x = b\} \) is finite. Then the set \( G^\circ(X) \) is a closed subgroupoid in \( G(X) \) and consequently, \( \text{Fil}^\circ(X) \), \( \lambda^\circ(X) \), \( \beta^\circ(X) \) all are closed subgroupoids in \( G(X) \).

**Proof.** Take two free inclusion hyperspaces \( A, B \in G(X) \) and a subset \( C \in A \circ B \). We should prove that \( C \setminus K \in A \circ B \) for each compact subset \( K \subset X \). Without loss of generality, the set \( C \) is of basic form: \( C = \bigcup_{a \in A} a \ast B_a \) for some set \( A \in A \) and some family \( \{B_a\}_{a \in A} \subset B \).
Since \( X \) is discrete, the set \( K \) is finite. It follows from our assumption that there is a finite set \( F \subset X \) such that for every \( a \in X \setminus F \) the set \( a^{-1}K = \{ x \in X : a \ast x \in K \} \) is finite. The hyperspace \( \mathcal{A} \), being free, contains the sets \( A' = A \setminus F \). By the same reason, for each \( a \in A' \) the hyperspace \( \mathcal{B} \) contains the set \( B'_a = B_a \setminus a^{-1}K \). Since \( C \setminus K \supset \bigcup_{a \in A'} a \ast B'_a \in \mathcal{A} \circ \mathcal{B} \), we conclude that \( C \setminus K \in \mathcal{A} \circ \mathcal{B} \).

**Remark 4.7.** If \( X \) is a semigroup, then \( G(X) \) is a semigroup and all the sub-semigroups considered above are closed sub-semigroups in \( G(X) \). Some of them are well-known in Semigroup Theory. In particular, so is the semigroup \( \beta X \) of ultrapower and \( \beta^0(X) = \beta X \setminus X \) of free ultrafilters. The semigroup \( \text{Fil}(X) \) contains an isomorphic copy of the global semigroup of \( X \), which is the hyperspace \( \exp(X) \) endowed with the semigroup operation \( A \ast B = \{ a \ast b : a \in A, b \in B \} \).

5. **Ideals and zeros in \( G(X) \)**

A non-empty subset \( I \) of a groupoid \( (X, \ast) \) is called an ideal (resp. right ideal, left ideal) if \( I \ast X \cup X \ast I \subset I \) (resp. \( I \ast X \subset I, X \ast I \subset I \)). An element \( O \) of a groupoid \( (X, \ast) \) is called a zero (resp. left zero, right zero) in \( X \) if \( \{ O \} \) is an ideal (resp. right ideal, left ideal) in \( X \). Each right or left zero \( z \in X \) is an idempotent in the sense that \( z \ast z = z \).

For a groupoid \( (X, \ast) \) right zeros in \( G(X) \) admit a simple description. We define an inclusion hyperspace \( \mathcal{A} \in G(X) \) to be shift-invariant if for every \( A \in \mathcal{A} \) and \( x \in X \) the sets \( x \ast A \) and \( x^{-1}A = \{ y \in X : x \ast y \in A \} \) belong to \( \mathcal{A} \).

**Proposition 5.1.** An inclusion hyperspace \( \mathcal{A} \in G(X) \) is a right zero in \( G(X) \) if and only if \( \mathcal{A} \) is shift-invariant.

**Proof.** Assuming that an inclusion hyperspace \( \mathcal{A} \in G(X) \) is shift-invariant, we shall show that \( \mathcal{B} \circ \mathcal{A} = \mathcal{A} \) for every \( \mathcal{B} \in G(X) \). Take any set \( F \in \mathcal{B} \circ \mathcal{A} \) and find a set \( B \in \mathcal{B} \) and a family \( \{ A_x \}_{x \in B} \subset \mathcal{A} \) such that \( \bigcup_{x \in B} x \ast A_x \subset F \). Since \( \mathcal{A} \in G(X) \) is shift-invariant, \( \bigcup_{x \in B} x \ast A_x \in \mathcal{A} \) and thus \( F \in \mathcal{A} \). This proves the inclusion \( \mathcal{B} \circ \mathcal{A} \subset \mathcal{A} \). On the other hand, for every \( F \in \mathcal{A} \) and every \( x \in X \) we get \( x^{-1}F \in \mathcal{A} \) and thus \( F \supset \bigcup_{x \in X} x \ast x^{-1}F \in \mathcal{B} \circ \mathcal{A} \). This shows that \( \mathcal{A} \) is a right zero of the semigroup \( G(X) \).

Now assume that \( \mathcal{A} \) is a right zero of \( G(X) \). Observe that for every \( x \in X \) the equality \( \langle x \rangle \circ \mathcal{A} = \mathcal{A} \) implies \( x \ast A \in \mathcal{A} \) for every \( A \in \mathcal{A} \).

One the other hand, the equality \( \{ X \} \circ \mathcal{A} = \mathcal{A} \) implies that for every \( A \in \mathcal{A} \) there is a family \( \{ A_x \}_{x \in X} \subset \mathcal{A} \) such that \( \bigcup_{x \in X} x \ast A_x \subset \mathcal{A} \). Then for every \( x \in X \) the set \( x^{-1}A = \{ z \in X : x \ast z \in A \} \supset A_x \in \mathcal{A} \) belongs to \( \mathcal{A} \) witnessing that \( \mathcal{A} \) is shift-invariant. \(\square\)
By $\widehat{G}(X)$ we denote the set of shift-invariant inclusion hyperspaces in $G(X)$. Proposition 5.1 implies that $\mathcal{A} \circ \mathcal{B} = \mathcal{B}$ for every $\mathcal{A}, \mathcal{B} \in \widehat{G}(X)$. This means that $\widehat{G}(X)$ is a rectangular semigroup.

We recall that a semigroup $(S, \ast)$ is called *rectangular* (or else a semigroup of *right zeros*) if $x \ast y = y$ for all $x, y \in S$.

**Proposition 5.2.** The set $\widehat{G}(X)$ is closed in $G(X)$, is a rectangular subsemigroup of the groupoid $G(X)$ and is closed complete sublattice of the lattice $G(X)$ invariant under the transversality operation. Moreover, if $\widehat{G}(X)$ is non-empty, then it is a left ideal that lies in each right ideal of $G(X)$.

**Proof.** If $\mathcal{A} \in G(X) \setminus \widehat{G}(X)$, then there exists $x \in X$ and $A \in \mathcal{A}$ such that $x \ast A \notin \mathcal{A}$ or $x^{-1} A \notin \mathcal{A}$. Then

$$O(\mathcal{A}) = \{ A' \in G(X) : A \in \mathcal{A}' \text{ and } (x \ast A \notin \mathcal{A}' \text{ or } x^{-1} A \notin \mathcal{A})\}$$

is an open neighborhood of $\mathcal{A}$ missing the set $\widehat{G}(X)$ and witnessing that the set $\widehat{G}(X)$ is closed in $G(X)$.

Since $\mathcal{A} \circ \mathcal{B} = \mathcal{B}$ for every $\mathcal{A}, \mathcal{B} \in \widehat{G}(X)$, the set $\widehat{G}(X)$ is a rectangular subsemigroup of the groupoid $G(X)$.

To show that $\widehat{G}(X)$ is invariant under the transversality operation, note that for every $\mathcal{A} \in G(X)$ and $Z \in \widehat{G}(X)$ we get $\mathcal{A} \circ Z^\perp = (A^\perp \circ Z)^\perp = Z^\perp$ which means that $Z^\perp$ is a right zero in $G(X)$ and thus belongs to $\widehat{G}(X)$ according to Proposition 5.1.

To show that $\widehat{G}(X)$ is a complete sublattice of $G(X)$ it is necessary to check that $\widehat{G}(X)$ is closed under arbitrary unions and intersections. It is trivial to check that arbitrary union of shift-invariant inclusion hyperspaces is shift-invariant, which means that $\bigcup_{\alpha \in A} Z_\alpha \in \widehat{G}(X)$ for any family $\{Z_\alpha\}_{\alpha \in A} \subset \widehat{G}(X)$. Since $G(X)$ is closed under the transversality operation we also get

$$\bigcap_{\alpha \in A} Z_\alpha = (\bigcup_{\alpha \in A} Z_\alpha^\perp)^\perp \in \widehat{G}(X)^\perp = \widehat{G}(X).$$

If $\widehat{G}(X)$ is not empty, then it is a left ideal in $G(X)$ because it consists of right zeros. Now take any right ideal $I$ in $G(X)$ and fix any element $\mathcal{R} \in I$. Then for every $Z \in \widehat{G}(X)$ we get $Z = \mathcal{R} \circ Z \in I$ which yields $\widehat{G}(X) \subset I$. \hfill \Box

**Proposition 5.3.** If $X$ is a semigroup and $\widehat{G}(X)$ is not empty, then $\widehat{G}(X)$ is the minimal ideal of $G(X)$.

**Proof.** In light of the preceding proposition, it suffices to check that $\widehat{G}(X)$ is a right ideal. Take any inclusion hyperspaces $\mathcal{A} \in \widehat{G}(X)$ and $\mathcal{B} \in \widehat{G}(X)$ and take any set
$F \in A \circ B$. We need to show that the sets $x \ast F$ and $x^{-1}F$ belong to $A \circ B$. Without loss of generality, $F$ is of the basic form:

$$F = \bigcup_{a \in A} a \ast B_a$$

for some set $A \in A$ and some family $\{B_a\}_{a \in A} \subset B$. The associativity of the semigroup operation on $S$ implies that

$$x \ast F = \bigcup_{a \in A} x \ast a \ast B_a = \bigcup_{z \in x \ast A} z \ast B_{a(z)} \in A \circ B$$

where $a(z) \in \{a \in A : x \ast a = z\}$ for $z \in x \ast A$. To see that $x^{-1}F \in A$ observe that the set $A' = \bigcup_{z \in x^{-1}A} z \ast B_{xz}$ belongs to $A$ and each point $a' \in A'$ belongs to the set $z \ast B_{xz}$ for some $z \in x^{-1}A$. Then $x \ast a' \in x \ast z \ast B_{xz} \subset F$ and hence $A \ni A' \subset x^{-1}F$, which yields the desired inclusion $x^{-1}F \in A$. \hfill $\Box$

Now we find conditions on the binary operation $\ast : X \times X \rightarrow X$ guaranteeing that the set $\overrightarrow{G}(X)$ is not empty. By $\min \overrightarrow{G}X = \{X\}$ and $\max \overrightarrow{G}X = \{A \subset X : A \neq \emptyset\}$ we denote the minimal and maximal elements of the lattice $\overrightarrow{G}(X)$.

**Proposition 5.4.** For a groupoid $(X, \ast)$ the following conditions are equivalent:

1. $\min \overrightarrow{G}X \in \overrightarrow{G}(X)$;
2. $\max \overrightarrow{G}X \in \overrightarrow{G}(X)$;
3. for each $a, b \in X$ the equation $a \ast x = b$ has a solution $x \in X$.

**Proof.** (1) $\Rightarrow$ (3) Assuming that $\min \overrightarrow{G}X \in \overrightarrow{G}(X)$ and applying Proposition 5.1 observe that for every $a \in X$ the equation $\langle a \rangle \circ \{X\} = \{X\}$ implies that for every $b \in X$ the equation $a \ast x = b$ has a solution.

(3) $\Rightarrow$ (1) If for every $a, b \in X$ the equation $a \ast x = b$ has a solution, then $a \ast X = X$ and hence $\mathcal{F} \circ \{X\} = \{X\}$ for all $\mathcal{F} \in \overrightarrow{G}(X)$. This means that $\{X\} = \min \overrightarrow{G}X$ is a right zero in $\overrightarrow{G}(X)$ and hence belongs to $\overrightarrow{G}(X)$ according to Proposition 5.1.

(2) $\Rightarrow$ (3) Assume that $\max \overrightarrow{G}X \in \overrightarrow{G}(X)$ and take any points $a, b \in X$. Since $\langle a \rangle \circ \max \overrightarrow{G}X = \max \overrightarrow{G}X \ni \{b\}$, there is a non-empty set $X_a \in \max \overrightarrow{G}X$ with $a \ast X_a \subset \{b\}$. Then any $x \in X_a$ is a solution of $a \ast x = b$.

(3) $\Rightarrow$ (2) Assume that for every $a, b \in X$ the equation $a \ast x = b$ has a solution. To show that $\mathcal{F} \circ \max \overrightarrow{G}X = \max \overrightarrow{G}X$ it suffices to check that $\max \overrightarrow{G}X \subset \mathcal{F} \circ \max \overrightarrow{G}X$. Take any set $B \in \max \overrightarrow{G}X$ and any set $\mathcal{F} \in \mathcal{F}$. For every $a \in F$ find a point $x_a \in X$ with $a \ast x_a \in B$. Then the sets $\bigcup_{a \in F} a \ast \{x_a\} \subset B$ belong to $\mathcal{F} \circ \max \overrightarrow{G}X$, which yields the desired inclusion $\max \overrightarrow{G}X \subset \mathcal{F} \circ \max \overrightarrow{G}X$. \hfill $\Box$

By analogy we can establish a similar description of zeros and the minimal ideal in the semigroup $G^a(X)$ of free inclusion hyperspaces.
Proposition 5.5. Assume that \((X, \ast)\) is an infinite groupoid such that for each \(b \in X\) there is a finite subset \(F \subset X\) such that for each \(a \in X \setminus F\) the set \(a^{-1}b = \{x \in X : a \ast x = b\}\) is finite and not empty. Then

1. \(G^\circ(X)\) is a closed subgroupoid of \(G(X)\);
2. \(G^\circ(X)\) is a left ideal in \(G(X)\) provided if for each \(a, b \in X\) the set \(a^{-1}b\) is finite;
3. the set \(\overline{G^\circ(X)} = \overline{G(X)} \cap G^\circ(X)\) of shift-invariant free inclusion hyperspaces is the minimal ideal in \(G^\circ(X)\);
4. the set \(\overline{G^\circ(X)}\) is a rectangular subsemigroup of the groupoid \(G(X)\) and is closed complete sublattice of the lattice \(G(X)\) invariant under the transversality map.

Remark 5.6. It follows from Propositions 5.2 and 5.3 that the minimal ideals of the semigroups \(G(\mathbb{Z})\) and \(G^\circ(X)\) are closed. In contrast, the minimal ideals of the semigroups \(\beta\mathbb{Z}\) and \(\beta^\circ\mathbb{Z} = \beta\mathbb{Z} \setminus \mathbb{Z}\) are not closed, see [HS §4.4].

Minimal left ideals of the semigroup \(\beta^\circ(\mathbb{Z})\) play an important role in Combinatorics of Numbers, see [HS]. We believe that the same will happen for the semigroup \(\lambda^\circ(\mathbb{Z})\). The following proposition implies that minimal left ideals of \(\lambda^\circ(\mathbb{Z})\) contain no ultrafilter!

Proposition 5.7. If a groupoid \(X\) admits a homomorphism \(h : X \to \mathbb{Z}_3\) such that for every \(y \in \mathbb{Z}_3\) the preimage \(h^{-1}(y)\) is not empty (is infinite) then each minimal left ideal \(I\) of \(\lambda(X)\) (of \(\lambda^\circ(X)\)) is disjoint from \(\beta(X)\).

Proof. It follows that the induced map \(\lambda h : \lambda(X) \to \lambda(\mathbb{Z}_3)\) is a surjective homomorphism. Consequently, \(\lambda h(I)\) is a minimal left ideal in \(\lambda(\mathbb{Z}_3)\). Now observe that \(\lambda(\mathbb{Z}_3)\) consists of four maximal linked inclusion hyperspaces. Besides three ultrafilters there is a maximal linked inclusion hyperspace \(\mathcal{L}_\Delta = \langle\{0, 1\}, \{0, 2\}, \{1, 2\}\rangle\) where \(\mathbb{Z}_3 = \{0, 1, 2\}\). One can check that \(\{\mathcal{L}_\Delta\}\) is a zero of the semigroup \(\lambda(\mathbb{Z}_3)\). Consequently, \(\lambda h(I) = \{\mathcal{L}_\Delta\}\), which implies that \(I \cap \beta(X) = \emptyset\).

Now assume that for every \(y \in \mathbb{Z}_3\) the preimage \(h^{-1}(y)\) is infinite. We claim that \(\lambda h(\lambda^\circ(X)) = \lambda(\mathbb{Z}_3)\). Take any maximal linked inclusion hyperspace \(\mathcal{L} \in \lambda(\mathbb{Z}_3)\). If \(\mathcal{L}\) is an ultrafilter supported by a point \(y \in \mathbb{Z}_3\), then we can take any free ultrafilter \(U\) on \(X\) containing the infinite set \(h^{-1}(y)\) and observe that \(\lambda h(U) = \mathcal{L}\). It remains to consider the case \(\mathcal{L} = \mathcal{L}_\Delta\). Fix free ultrafilters \(U_0, U_1, U_2\) on \(X\) containing the sets \(h^{-1}(0), h^{-1}(1), h^{-1}(2)\), respectively. Then \(\mathcal{L} = (U_0 \cap U_1) \cup (U_0 \cap U_2) \cup (U_1 \cap U_2)\) is a free maximal linked inclusion hyperspace whose image \(\lambda h(\mathcal{L}X) = \mathcal{L}_\Delta\).

Given any minimal left ideal \(I \subset \lambda^\circ(X)\) we obtain that the image \(\lambda h(I)\), being a minimal left ideal of \(\lambda(\mathbb{Z}_3)\) coincides with \(\{\mathcal{L}_\Delta\}\) and is disjoint from \(\beta(\mathbb{Z}_3)\). Consequently, \(I\) is disjoint from \(\beta(X)\).  \(\square\)
6. The center of $G(X)$

In this section we describe the structure of the center of the groupoid $G(X)$ for each (quasi)group $X$. By definition, the center of a groupoid $X$ is the set

$$C = \{ x \in X : \forall y \in X \ xy = yx \}.$$ 

A groupoid $X$ is called a quasigroup if for every $a, b \in X$ the system of equations $a \ast x = b$ and $y \ast a = b$ has a unique solution $(x, y) \in X \times X$. It is clear that each group is a quasigroup. On the other hand, there are many examples of quasigroups, not isomorphic to groups, see [Pf], [CPS].

**Theorem 6.1.** Let $X$ be a quasigroup. If an inclusion hyperspace $C \in G(X)$ commutes with the extremal elements $\max G(X)$ and $\min G(X)$ of $G(X)$, then $C$ is a principal ultrafilter.

**Proof.** By Proposition 5.4, the inclusion hyperspaces $\max G(X)$ and $\min G(X)$ are right zeros in $G(X)$ and thus $\max G(X) \circ C = C \circ \max G(X) = \max G(X)$ and $\min G(X) \circ C = C \circ \min G(X) = \min G(X)$. It follows that for every $b \in X$ we get $\{b\} \in \max G(X) = \max G(X) \circ C$, which means that $a \ast C \subset \{b\}$ for some $C \in \mathcal{C}$ and some $a \in X$. Since the equation $a \ast y = b$ has a unique solution $y \in X$, the set $C$ is a singleton, say $C = \{c\}$. It remains to prove that $C$ coincides with the principal ultrafilter $\langle c \rangle$ generated by $c$. Assuming the converse, we would conclude that $X \setminus \{c\} \in \mathcal{C}$. By our hypothesis, the equation $y \ast c = c$ has a unique solution $y_0 \in X$. Since the equation $y_0 \ast x = c$ has a unique solution $x = c$, $y_0 \ast (X \setminus \{c\}) \subset X \setminus \{c\}$. Letting $C_x = \{c\}$ for all $x \in X \setminus \{y_0\}$ and $C_x = X \setminus \{c\}$ for $x = y_0$, we conclude that $X \setminus \{c\} \supset \bigcup_{x \in X} x \ast C_x \in \min G(X) \circ C = C \circ \min G(X) = \min G(X)$, which is not possible. \hfill $\square$

**Corollary 6.2.** For any quasigroup $X$ the center of the groupoid $G(X)$ coincides with the center of $X$.

**Proof.** If an inclusion hyperspace $C$ belongs to the center of the groupoid $G(X)$, then $C$ is a principal ultrafilter generated by some point $c \in X$. Since $C$ commutes with all the principal ultrafilters, $C$ commutes with all elements of $X$ and thus $c$ belongs to the center of $X$.

Conversely, if $c \in X$ belongs to the center of $X$, then for every inclusion hyperspace $F \in G(X)$ we get

$$c \circ F = \{c \ast F : F \in \mathcal{F} \} = \{ F \ast c : F \in \mathcal{F} \} = F \circ c,$$

which means that (the principal ultrafilter generated by) $c$ belongs to the center of the groupoid $G(X)$. \hfill $\square$
Remark 6.3. It is interesting to note that for any group $X$ the center of the semigroup $\beta X$ also coincides with the center of the group $X$, see Theorem 6.54 of [HS].

Problem 6.4. Given a group $X$ describe the centers of the subsemigroups $\lambda(X)$, $\text{Fil}(X)$, $N_<(X)$, $N_k(X)$, $k \geq 2$ of the semigroup $G(X)$. Is it true that the center of any subsemigroup $S \subset G(X)$ with $\beta(X) \subset S = S^\perp$ coincides with the center of $X$?

Remark 6.5. Let us note that the requirement $S = S^\perp$ in the preceding question is essential: for any nontrivial group $X$ the center of the (non-symmetric) subsemigroup $X \cup \text{max} G(X)$ of $G(X)$ contains $\text{max} G(X)$ and hence is strictly larger than the center of the group $X$.

Problem 6.6. Given an infinite group $X$ describe the centers of the semigroups $G^\circ(X)$, $\lambda^\circ(X)$, $\text{Fil}^\circ(X)$, $N_<(X)$, $N_k(X)$, $k \geq 2$. (By Theorem 6.54 of [HS], the center of the semigroup of free ultrafilters $\beta^\circ(X)$ is empty).

7. The topological center of $G(X)$

In this section we describe the topological center of $G(X)$. By the topological center of a groupoid $X$ endowed with a topology we understand the set $\Lambda(X)$ consisting of all points $x \in X$ such that the left and right shifts

$$l_x : X \to X, \quad l_x : z \mapsto xz, \quad \text{and} \quad r_x : X \to X, \quad r_x : z \mapsto zx$$

both are continuous.

Since all right shifts on $G(X)$ are continuous, the topological center of the groupoid $G(X)$ consists of all inclusion hyperspaces $\mathcal{F}$ with continuous left shifts $l_\mathcal{F}$.

We recall that $G^\ast(X)$ stands for the set of inclusion hyperpsaces with finite support.

Theorem 7.1. For a quasigroup $X$ the topological center of the groupoid $G(X)$ coincides with $G^\ast(X)$.

Proof. By Proposition 2.3 the topological center $\Lambda(GX)$ of $G(X)$ contains all principal ultrafilters and is a sublattice of $G(X)$. Consequently, $\Lambda(GX)$ contains the sublattice $G^\ast(X)$ of $G(X)$ generated by $X$.

Next, we show that each inclusion hyperspace $\mathcal{F} \in \Lambda(GX)$ has finite support and hence belongs to $G^\ast(X)$. By Theorem 9.1 of [GI], this will follow as soon as we check that both $\mathcal{F}$ and $\mathcal{F}^\perp$ have bases consisting of finite sets.

Take any set $F \in \mathcal{F}$, choose any point $e \in X$, and consider the inclusion hyperspace $\mathcal{U} = \{ U \subset X : e \in F \ast U \}$. Since for every $f \in F$ the equation $f \ast u = e$
has a solution in $X$, we conclude that $\{e\} \in \mathcal{F} \circ \mathcal{U}$ and by the continuity of the left shift $l_x$, there is an open neighborhood $\mathcal{O}(\mathcal{U})$ of $\mathcal{U}$ such that $\{e\} \in \mathcal{F} \circ A$ for all $A \in \mathcal{O}(\mathcal{U})$. Without loss of generality, the neighborhood $\mathcal{O}(\mathcal{U})$ is of basic form

$$
\mathcal{O}(\mathcal{U}) = U_1^+ \cap \cdots \cap U_n^+ \cap V_1^- \cap \cdots \cap V_m^-
$$

for some sets $U_1, \ldots, U_n, V_1, \ldots, V_m \in \mathcal{U}$. Take any finite set $A \subset F^{-1}e = \{x \in X : e \in F \ast x\}$ intersecting each set $U_i$, $i \leq n$, and consider the inclusion hyperspace $A = (A)^\perp$. It is clear that $A \subset U_1^+ \cap \cdots \cap U_n^+$. Since each set $V_j$, $j \leq m$, contains the set $F^{-1}e \supset A$, we get also that $A \in V_1^- \cap \cdots \cap V_m^-$. Then $\mathcal{F} \circ A \ni \{e\}$ and hence there is a set $E \in \mathcal{F}$ and a family $\{A_x\}_{x \in E} \subset A$ with $\bigcup_{x \in E} x \ast A_x \subset \{e\}$. It follows that the set $E \subset eA^{-1} = \{x \in X : \exists a \in A \text{ with } xa = e\}$ is finite. We claim that $E \subset F$. Indeed, take any point $x \in E$ and find a point $a \in A$ with $x \ast a = e$. Since $A \subset F^{-1}e$, there is a point $y \in F$ with $e = y \ast a$. Hence $xa = ya$ and the right cancellativity of $X$ yields $x = y \in F$. Therefore, using the continuity of the left shift $l_x$, for every $F \in \mathcal{F}$ we have found a finite subset $E \in \mathcal{F}$ with $E \subset F$. This means that $\mathcal{F}$ has a base of finite sets.

The continuity of the left shift $l_x$ and Proposition 2.1 imply the continuity of the left shift $l_{x^{-1}}$. Repeating the preceding argument, we can prove that the inclusion hyperspace $\mathcal{F}^\perp$ has a base of finite sets too. Finally, applying Theorem 9.1 of [CH], we conclude that $\mathcal{F} \in G^\bullet(X)$. \hfill $\square$

**Problem 7.2.** Given an infinite group $G$ describe the topological center of the subsemigroups $\lambda(X)$, $Fil(X)$, $N_{<\omega}(X)$, $N_k(X)$, $k \geq 2$, of the semigroup $G(X)$. Is it true that the topological center of any subsemigroup $S \subset G(X)$ containing $\beta(X)$ coincides with $S \cap G^\bullet(X)$? (This is true for the subsemigroups $S = G(X)$ (see Theorem 7.1) and $S = \beta(X)$, see Theorems 4.24 and 6.54 of [HS]).

**Problem 7.3.** Given an infinite group $X$ describe the topological centers of the semigroups $G^\circ(X)$, $\lambda^\circ(X)$, $Fil^\circ(X)$, $N_{\leq\omega}^\circ(X)$, and $N_k^\circ(X)$, $k \geq 2$. (It should be mentioned that the topological center of the semigroup $\beta^\circ(X)$ of free ultrafilters is empty $[P_2]$).

8. LEFTCancelable ELEMENTS OF $G(X)$

An element $a$ of a groupoid $S$ is called left cancelable (resp. right cancelable) if for any points $x, y \in S$ the equation $ax = ay$ (resp. $xa = ya$) implies $x = y$. In this section we characterize left cancelable elements of the groupoid $G(X)$ over a quasigroup $X$.

**Theorem 8.1.** Let $X$ be a quasigroup. An inclusion hyperspace $\mathcal{F} \in G(X)$ is left cancelable in the groupoid $G(X)$ if and only if $\mathcal{F}$ is a principal ultrafilter.
Proof. Assume that $\mathcal{F}$ is left cancelable in $G(X)$. First we show that $\mathcal{F}$ contains some singleton. Assuming the converse, take any point $x_0 \in X$ and note that $F*(X \setminus \{x_0\}) = X$ for any $F \in \mathcal{F}$. To see that this equality holds, take any point $a \in X$, choose two distinct points $b, c \in F$ and find solutions $x, y \in X$ of the equation $b*x = a$ and $c*y = a$. Since $X$ is right cancellative, $x \neq y$. Consequently, one of the points $x$ or $y$ is distinct from $x_0$. If $x \neq x_0$, then $a = b*x \in F*(X \setminus \{x_0\})$. If $y \neq x_0$, then $a = c*y \in F*(X \setminus \{x_0\})$. Now for the inclusion hyperspace $\mathcal{U} = (X \setminus \{x_0\}) \neq \min G(X)$, we get $\mathcal{F} \circ \mathcal{U} = \min G(X) = \mathcal{F} \circ \min G(X)$, which contradicts the choice of $\mathcal{F}$ as a left cancelable element of $G(X)$.

Thus $\mathcal{F}$ contains some singleton $\{c\}$. We claim that $\mathcal{F}$ coincides with the principal ultrafilter generated by $c$. Assuming the converse, we would conclude that $X \setminus \{c\} \in \mathcal{F}$. Let $\mathcal{A} = (X \setminus \{c\})\perp$ be the inclusion hyperspace consisting of subsets that meet $X \setminus \{c\}$. It is clear that $\mathcal{A} \neq \max G(X)$. We claim that $\mathcal{F} \circ \mathcal{A} = \max G(X) = \mathcal{F} \circ \max G(X)$ which will contradict the left cancelability of $\mathcal{F}$. Indeed, given any singleton $\{a\} \in \max G(X)$, consider two cases: if $a \neq c \ast c$, then we can find a unique $x \in X$ with $c \ast x = a$. Since $x \neq c$, $\{x\} \in \mathcal{A}$ and hence $\{a\} = c \ast \{x\} \in \mathcal{F} \circ \mathcal{A}$. If $a = c \ast c$, then for every $y \in X \setminus \{c\}$ we can find $a_y \in X$ with $y \ast a_y = a$ and use the left cancelativity of $X$ to conclude that $a_y \neq c$ and hence $\{a_y\} \in \mathcal{A}$. Then $\{a\} = \bigcup_{y \in X\setminus\{c\}} y \ast \{a_y\} \in \mathcal{F} \circ \mathcal{A}$.

Therefore $\mathcal{F} = \langle c \rangle$ is a principal ultrafilter, which proves the “only if” part of the theorem. To prove the “if” part, take any principal ultrafilter $\langle x \rangle$ generated by a point $x \in X$. We claim that two inclusion hyperspaces $\mathcal{F}, \mathcal{U} \in G(X)$ are equal provided $\langle x \rangle \circ \mathcal{F} = \langle x \rangle \circ \mathcal{U}$. Indeed, given any set $F \in \mathcal{F}$ observe that $x \ast F \in \langle x \rangle \circ \mathcal{F} = \langle x \rangle \circ \mathcal{U}$ and hence $x \ast F = x \ast U$ for some $U \in \mathcal{U}$. The left cancelativity of $X$ implies that $F = U \in \mathcal{U}$, which yields $\mathcal{F} \subset \mathcal{U}$. By the same argument we can also check that $\mathcal{U} \subset \mathcal{F}$.

Problem 8.2. Given an (infinite) group $X$ describe left cancelable elements of the subsemigroups $\lambda(X), \Fil(X), N_{<\omega}(X), N_k(X), k \geq 2$ (and $G^\circ(X), \lambda^\circ(X), \Fil^\circ(X), N_{<\omega}^\circ(X), N_k^\circ(X)$, for $k \geq 2$).

Remark 8.3. Theorem [HS] implies that for a countable Abelian group $X$ the set of left cancelable elements in $G(X)$ coincides with $X$. On the other hand, the set of (left) cancelable elements of $\beta(X)$ contains an open dense subset of $\beta^\circ(X)$, see Theorem 8.34 of [HS].

9. Right cancelable elements of $G(X)$

As we saw in the preceding section, for any quasigroup $X$ the groupoid $G(X)$ contains only trivial left cancelable elements. For right cancelable elements the situation is much more interesting. First note that the right cancelativity of an
inclusion hyperspace $F \in G(X)$ is equivalent to the injectivity of the map $\mu_X \circ G\tilde{R}_F : G(X) \to G(X)$ considered at the beginning of Section 2. We recall that $\mu_X : G^2(X) \to G(X)$ is the multiplication of the monad $G = (G, \mu, \eta)$ while $\tilde{R}_F : \beta X \to G(X)$ is the Stone-Čech extension of the right shift $R_F : X \to G(X)$, $R_F : x \mapsto x * F$. The map $\tilde{R}_F$ certainly is not injective if $R_F$ is not an embedding, which is equivalent to the discreteness of the indexed set $\{x * F : x \in X\}$ in $G(X)$. Therefore we have obtained the following necessary condition for the right cancelability.

**Proposition 9.1.** Let $X$ be a groupoid. If an inclusion hyperspace $F \in G(X)$ is right cancelable in $G(X)$, then the indexed set $\{x F : x \in X\}$ is discrete in $G(X)$ in the sense that each point $x F$ has a neighborhood $O(xF)$ containing no other points $yF$ with $y \in X \setminus \{x\}$.

Next we give a sufficient condition of the right cancelability.

**Proposition 9.2.** Let $X$ be a groupoid. An inclusion hyperspace $F \in G(X)$ is right cancelable in $G(X)$ provided there is a family of sets $\{S_x\}_{x \in X} \subset F \cap F^\perp$ such that $x S_x \cap y S_y = \emptyset$ for any distinct $x, y \in X$.

**Proof.** Assume that $A \circ F = B \circ F$ for two inclusion hyperspaces $A, B \in G(X)$. First we show that $A \subset B$. Take any set $A \in A$ and observe that the set $\bigcup_{a \in A} a S_a$ belongs to $A \circ F = B \circ F$. Consequently, there is a set $B \in B$ and a family of sets $\{F_b\}_{b \in B} \subset F$ such that

$$\bigcup_{b \in B} b F_b \subset \bigcup_{a \in A} a S_a.$$

It follows from $S_b \in F^\perp$ that $F_b \cap S_b$ is not empty for every $b \in B$.

Since the sets $a S_a$ and $b S_b$ are disjoint for different $a, b \in X$, the inclusion

$$\bigcup_{b \in B} b (F_b \cap S_b) \subset \bigcup_{b \in B} b F_b \subset \bigcup_{a \in A} a S_a$$

implies $B \subset A$ and hence $A \subset B$.

By analogy we can prove that $B \subset A$. $\square$

Propositions 9.1 and 9.2 imply the following characterization of right cancelable ultrafilters in $G(X)$ generalizing a known characterization of right cancelable elements of the semigroups $\beta X$, see [HS, 8.11].

**Corollary 9.3.** Let $X$ be a countable groupoid. For an ultrafilter $U$ on $X$ the following conditions are equivalent:

1. $U$ is right cancelable in $G(X)$;
2. $U$ is right cancelable in $\beta X$;
3. the indexed set $\{x U : x \in X\}$ is discrete in $\beta X$;
(4) there is an indexed family of sets \( \{U_x\}_{x \in X} \subset \mathcal{U} \) such that for any distinct \( x, y \in X \) the shifts \( xU_x \) and \( yU_y \) are disjoint.

This characterization can be used to show that for any countable group \( X \) the semigroup \( \beta \mathbb{G}(X) \) of free ultrafilters contains an open dense subset of right cancelable ultrafilters, see [HS, 8.10]. It turns out that a similar result can be proved for the semigroup \( \mathbb{G}^\circ(X) \).

**Proposition 9.4.** For any countable quasigroup, the groupoid \( \mathbb{G}^\circ(X) \) contains an open dense subset of right cancelable free inclusion hyperspaces.

**Proof.** Let \( X = \{x_n : n \in \omega\} \) be an injective enumeration of the countable quasigroup \( X \). Given a free inclusion hyperspace \( \mathcal{F} \in \mathbb{G}^\circ(X) \) and a neighborhood \( O(\mathcal{F}) \) of \( \mathcal{F} \) in \( \mathbb{G}^\circ(X) \), we should find a non-empty open subset in \( O(\mathcal{F}) \). Without loss of generality, the neighborhood \( O(\mathcal{F}) \) is of basic form:

\[
O(\mathcal{F}) = \mathbb{G}^\circ(X) \cap U_0^+ \cap \cdots \cap U_n^+ \cap \mathbb{G}^\circ(X) \cap \cdots \cap U_{m-1}^-
\]

for some sets \( U_1, \ldots, U_{m-1} \) of \( X \). Those sets are infinite because \( \mathcal{F} \) is free. We are going to construct an infinite set \( C = \{c_n : n \in \omega\} \subseteq X \) that has infinite intersection with the sets \( U_i, i < m \), and such that for any distinct \( x, y \in X \) the intersection \( xC \cap yC \) is finite. The points \( c_k, k \in \omega \), composing the set \( C \) will be chosen by induction to satisfy the following conditions:

- \( c_k \in U_j \) where \( j = k \mod m \);
- \( c_k \) does not belong to the finite set

\[
F_k = \{z \in X : \exists i, j \leq k \exists l < k \ (x_iz = x_jc_l)\}.
\]

It is clear that the so-constructed set \( C = \{c_k : k \in \omega\} \) has infinite intersection with each set \( U_i, i < m \). Since \( X \) is right cancellative, for any \( i < j \) the set \( Z_{i,j} = \{z \in X : x_iz = x_jz\} \) is finite. Now the choice of the points \( c_k \) for \( k > j \) implies that \( x_1C \cap x_jC \subset x_i(Z_{i,j} \cup \{c_l : l \leq j\}) \) is finite.

Now let \( \mathcal{C} \) be the free inclusion hyperspace on \( X \) generated by the sets \( C \) and \( U_0, \ldots, U_n \). It is clear that \( \mathcal{C} \in O(\mathcal{F}) \) and \( \mathcal{C} \in \mathcal{C} \cap \mathcal{C}^\perp \). Consider the open neighborhood

\[
O(\mathcal{C}) = O(\mathcal{F}) \cap C^+ \cap (C^+)^\perp
\]

of \( \mathcal{C} \) in \( \mathbb{G}^\circ(X) \).

We claim that each inclusion hyperspace \( \mathcal{A} \in O(\mathcal{C}) \) is right cancelable in \( G(X) \). This will follow from Proposition 9.2 as soon as we construct a family of sets \( \{A_i\}_{i \in \omega} \in \mathcal{A} \cap \mathcal{A}^\perp \) such that \( x_iA_i \cap x_jA_j = \emptyset \) for any numbers \( i < j \). The sets \( A_i, i \in \omega \), can be defined by the formula

\[
A_k = C \setminus F_k
\]

where

\[
F_k = \{c \in C : \exists i < k \text{ with } x_kc = x_iC\}
\]
Problem 9.5. Given an (infinite) group $X$ describe right cancelable elements of the subsemigroups $\lambda(X)$, $\text{Fil}(X)$, $N_{<\omega}(X)$, $N_k(X)$, $k \geq 2$ ($\lambda^c(X)$, $\text{Fil}^c(X)$, $N_{<\omega}^c(X)$, $N_k^c(X)$, for $k \geq 2$).

10. The structure of the semigroups $G(H)$ over finite groups $H$

In Proposition 5.7 we have seen that the structural properties of the finite semigroup $\lambda(\mathbb{Z}_3)$ has non-trivial implications for the essentially infinite object $\lambda^c(\mathbb{Z})$. This observation is a motivation for more detail study of spaces $G(H)$ over finite Abelian groups $H$. In this case the group $H$ acts on $G(H)$ by right shifts:

$$s : G(H) \times H \rightarrow G(H), \ s : (A, h) \mapsto A \circ h.$$ 

So we can speak about the orbit $A \circ H = \{A \circ h : h \in H\}$ of an inclusion hyperspace $A \in G(H)$ and the orbit space $G(H)/H = \{A \circ H : A \in G(X)\}$. By $\pi : G(H) \rightarrow G(H)/H$ we denote the quotient map which induces a unique semigroup structure of $G(H)/H$ turning $\pi$ into a semigroup homomorphism.

We shall say that the semigroup $G(H)$ is splittable if there is a semigroup homomorphism $s : G(H)/H \rightarrow G(H)$ such that $\pi \circ s$ is the identity homomorphism of $G(H)/H$. Such a homomorphism $s$ will be called a section of $\pi$ and the semigroup $T(H) = s(G(H)/H)$ will be called a $H$-transversal semigroup of $G(H)$. It is clear that a $H$-transversal semigroup $T(H)$ has one-point intersection with each orbit of $G(H)$.

If the semigroup $G(H)$ is splittable, then the structure of $G(H)$ can be described as follows.

Proposition 10.1. If the semigroup $G(H)$ is splittable and $T(H)$ is the transversal semigroup of $G(H)$, then $T(H)$ is isomorphic to $G(H)/H$ and $G(H)$ is the quotient semigroup of the product $T(H) \times H$ under the homomorphism $h : T(H) \times H \rightarrow G(H)$, $h : (A, h) \mapsto A \circ h$.

It turns out that the semigroup $G(\mathbb{Z}_n)$ is splittable for $n \leq 3$ and not splittable for $n = 5$ (the latter follows from the non-splittability of the semigroup $\lambda(\mathbb{Z}_5)$ established in [BCN]). So below we describe the structure of the semigroups $G(\mathbb{Z}_n)$ and their transversal semigroup $T(\mathbb{Z}_n)$ for $n \leq 3$.

For a group $X$ we shall identify the elements $x \in X$ with the ultrafilters they generate. Also we shall use the notations $\land$ and $\lor$ to denote the lattice operations $\cap$ and $\cup$ on $G(X)$, respectively.

The semigroup $G(\mathbb{Z}_2)$. For the cyclic group $\mathbb{Z}_2 = \{e, a\}$ the lattice $G(\mathbb{Z}_2)$ contains four inclusion hyperspaces: $e, a, e \land a, e \lor a$, and is shown at the picture:
The semigroup $G(Z_2)$ has a unique $Z_2$-transversal semigroup

$$T(Z_2) = \{ e \land a, e, e \lor a \}$$

with two right zeros: $e \land a$, $e \lor a$ and one unit $e$.

The semigroup $G(Z_3)$ over the cyclic group $Z_3 = \{ e, a, a^{-1} \}$ contains 18 elements:

$$a \lor e \lor a^{-1},
\quad a \lor a^{-1}, \quad a \lor e, \quad e \lor a^{-1},
\quad a \lor (e \land a^{-1}), \quad e \lor (a \land a^{-1}), \quad a^{-1} \lor (a \land e),
\quad a, e, a^{-1},
\quad (a \lor e) \land (a \lor a^{-1}) \land (e \lor a^{-1}),
\quad a \land (e \lor a^{-1}), \quad e \land (a \lor a^{-1}), \quad a^{-1} \land (a \lor e),
\quad a \land a^{-1}, \quad a \land e, \quad e \land a^{-1},
\quad a \land e \land a^{-1}$$

divided into 8 orbits with respect to the action of the group $Z_3$.

The semigroup $G(Z_3)$ has 9 different $Z_3$-transversal semigroups one of which is drawn at the picture:

$$T(Z_3)$$

The semigroup $G(Z_3)$ has 3 shift-invariant inclusion hyperspaces which are right zeros: $a \land e \land a^{-1}$, $a \lor e \lor a^{-1}$ and $(a \lor e) \land (e \lor a^{-1}) \land (a \lor a^{-1})$. Besides right zeros $G(Z_3)$ has 3 idempotents: $e$, $e \lor (a \land a^{-1})$ and $e \land (a \lor a^{-1})$. The element $e$ is the unit of the semigroup $G(Z_3)$. 
The complete information on the structure of the $\mathbb{Z}_3$-transversal semigroup $T(\mathbb{Z}_3)$ (which is isomorphic to the quotient semigroup $G(\mathbb{Z}_3)/\mathbb{Z}_3$) can be derived from the Cayley table

| $\circ$ | $x_{-3}$ | $x_{-2}$ | $x_{-1}$ | $x_0$ | $x_1$ | $x_2$ | $x_3$ |
|---------|----------|----------|----------|-------|-------|-------|-------|
| $x_{-3}$ | $x_{-3}$ | $x_{-3}$ | $x_0$   | $x_0$ | $x_0$ | $x_3$ |       |
| $x_{-2}$ | $x_{-3}$ | $x_{-2}$ | $x_0$   | $x_0$ | $x_1$ | $x_3$ |       |
| $x_{-1}$ | $x_{-3}$ | $x_{-1}$ | $x_0$   | $x_0$ | $x_2$ | $x_3$ |       |
| $x_0$   | $x_{-3}$ | $x_0$   | $x_0$   | $x_0$ | $x_3$ |       |       |
| $x_1$   | $x_{-3}$ | $x_2$   | $x_0$   | $x_1$ | $x_3$ |       |       |
| $x_2$   | $x_{-1}$ | $x_0$   | $x_0$   | $x_2$ | $x_3$ |       |       |
| $x_3$   | $x_{-3}$ | $x_0$   | $x_0$   | $x_3$ | $x_3$ |       |       |

of its linearly ordered subsemigroup $T(\mathbb{Z}_3) \setminus \{e\}$ having with 7-elements:

- $x_{-3} = e \wedge a \wedge a^{-1}$,
- $x_{-2} = e \wedge a$,
- $x_{-1} = e \wedge (a \vee a^{-1})$,
- $x_0 = (e \vee a) \wedge (e \vee a^{-1}) \wedge (a \vee a^{-1})$,
- $x_1 = e \vee (a \wedge a^{-1})$,
- $x_2 = e \vee a$,
- $x_3 = e \vee a \vee a^{-1}$.

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