Against the Wheeler-DeWitt equation

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Abstract

The ADM approach to canonical general relativity combined with Dirac’s method of quantizing constrained systems leads to the Wheeler-DeWitt equation. A number of mathematical as well as physical difficulties that arise in connection with this equation may be circumvented if one employs a covariant Hamiltonian method in conjunction with a recently developed, mathematically rigorous technique to quantize constrained systems using Rieffel induction. The classical constraints are cleanly separated into four components of a covariant momentum map coming from the diffeomorphism group of spacetime, each of which is linear in the canonical momenta, plus a single finite-dimensional quadratic constraint that arises in any theory, parametrized or not.

The new quantization method is carried through in a minisuperspace example, and is found to produce a “wavefunction of the universe”. This differs from the proposals of both Vilenkin and Hartle-Hawking for a closed FRW universe, but happens to coincide with the latter in the open case.

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1 Introduction

Dirac’s theory of constrained systems in classical mechanics (cf. Gotay et al. (1978) for a modern geometric formulation) consists of two steps. Firstly, the constraints \( \Phi_a = 0 \) are imposed on the phase space \( S \) of the unconstrained system, singling out the constraint hypersurface \( C \subset S \) as their solution space. Secondly, one forms the quotient \( S^0 = C/\mathcal{F}_0 \) of \( C \) by the foliation \( \mathcal{F}_0 \) defined by the null directions of the induced symplectic form \( i^*\omega \) on \( S \) (here \( \omega \) is the symplectic form on \( S \) and \( i \) is the injection of \( C \) into \( S \)). This second step (which is absent if all constraints are second class) identifies physically equivalent points on \( C \). If \( S = T^*Q \) is a cotangent bundle and the constraints are components of a momentum map \( \Phi \) (cf. Marsden and Ratiu 1994) derived from an action of a group \( G \) on \( Q \), then all \( \Phi_a \) are necessarily linear in the momenta. The so-called Marsden-Weinstein quotient \( \Phi^{-1}(0)/G \) then coincides with the reduced phase space.

In canonical (ADM) classical gravity (cf. Fischer and Marsden 1979, Isham 1993, Kuchař 1992) the configuration space \( Q \) is taken to be the space of Riemannian 3-metrics (subject to certain regularity conditions) on a (Cauchy) hypersurface \( \Sigma \) (here assumed to be compact) in space-time \( X \). The \( \Phi_a \) are the (super) Hamiltonian- and momentum constraints \( \mathcal{H}_\alpha^x \) (one for each point \( x \) of \( \Sigma \); \( \alpha = 0, 1, 2, 3 \)), the first of which are quadratic in the canonical momenta. Also, the Poisson bracket of two Hamiltonian constraints is proportional to the inverse of the three-metric, which makes it impossible to formulate these constraints as the pull-back of a Poisson morphism \( \Phi \) from \( S \) into any Poisson manifold \( P \). (The super-momentum constraints, which are linear in the canonical momenta, are of such a nature, with \( P \) being the dual of the Lie algebra of \( \text{Diff}(\Sigma) \) and \( \Phi \) the equivariant momentum map corresponding to the natural action of this group on \( S \).) In particular, the connection between the Hamiltonian constraints and \( \text{Diff}(X) \) is obscure. Time-evolution is generated by the constraints smeared with arbitrary functions \( N(x, t) \); hence the dynamical evolution takes place along some of the null directions of \( i^*\omega \), and collapses to no evolution whatsoever on the physical phase space \( S^0 \). With obvious
modifications, this discussion applies to the Ashtekar variables as well.

In quantizing constrained systems, the two-step classical procedure is replaced by a single step. Dirac (1964) singled out the first step: if $\mathcal{H}$ is the Hilbert space of states of the unconstrained system (that is, the quantization of $S$) and $\hat{\Phi}_a$ are self-adjoint operators on $\mathcal{H}$ quantizing the classical constraints, then the quantum analogue $\mathcal{H}_D$ of the physical phase space $S^0$ is defined as $\mathcal{H}_D = \{ \psi \in \mathcal{H} | \hat{\Phi}_a \psi = 0 \ \forall a \}$. This space then inherits the inner product from $\mathcal{H}$, and is a Hilbert space in its own right. In other words, one imposes the constraints and that’s it. This only makes sense if all constraints have 0 as discrete eigenvalue, with common eigenspace. This condition is rarely satisfied in practice, and this has led to certain modifications of the Dirac proposal (Ashtekar and Tate 1994, Hájíček 1994, Ashtekar et.al. 1995, Marolf 1995b), in which one solves the constraints on a bigger space than $\mathcal{H}$. One thereby loses the inner product, and the main problem is then to construct an inner product on the solution space from scratch. In some examples, such methods lead to the same result as our approach, but in the former one still tries to mimic the first step of the classical reduction process. Moreover, it is easy to devise examples where such rigged Hilbert space techniques will not lead to the desired answer, see below.

Applied to canonical gravity (Ehlers and Friedrich 1994), the Dirac procedure leads to the Wheeler-DeWitt equation, in which the quantized Hamiltonian constraints are imposed on the wavefunction $\psi$, which is a function(al) of the three-metric on $\Sigma$ (or of the corresponding Ashtekar variables). Quite apart from the fact to what extent $\mathcal{H}$ is well-defined (for considerable progress in this direction see Ashtekar et.al. 1994, 1995), this approach meets formidable obstacles (Kuchar 1992, Isham 1993).

The problems with the Wheeler-DeWitt equation can be traced back to (at least) two sources: i) the lack of covariance of the ADM approach, which is especially dangerous in connection with quantum field theory, where fields are not defined at sharp times; ii) the Dirac quantization method of constrained systems. It seems that one can do better on both accounts. Firstly, there exists a covariant Hamiltonian
formulation of classical field theory (Kijowksi 1973, Gotay 1991, Gotay et.al. 1993),
and secondly the author (Landsman 1995) has recently formulated a new method of
quantizing constrained systems, which avoids many problems in the Dirac approach.

2 Constrained quantization revisited

To start with the latter, the main idea is to implement a quantized version of the
second step of the classical constraint procedure, rather than the first one. The con-
straints then do not have to be implemented, avoiding the Wheeler-DeWitt equation
altogether. This is done by modifying the inner product ( , ) on \( \mathcal{H} \) (which is posi-
tive definite) into a positive semi-definite sesquilinear form ( , )\(_0\). The latter has a
nontrivial null space \( \mathcal{N}_0 = \{ \psi \in \mathcal{H} | (\psi, \psi)_0 = 0 \} \), in terms of which the state space
\( \mathcal{H}^0 \) of the constrained system is simply given by \( \mathcal{H}^0 = \mathcal{H}/\mathcal{N}_0 \). Crucially, \( \mathcal{H}^0 \) inherits
the form ( , )\(_0\), which is now positive definite since its null vectors have been thrown
away. Thus one has a bona fide inner product on \( \mathcal{H}^0 \), which can be used to calculate
physical amplitudes and probabilities.

To illustrate the method, consider a compact group \( G \), which acts on \( \mathcal{H} \) through
a unitary representation \( U \). If the constraints correspond to the generators \( T_a \) of (the
Lie algebra of) \( G \), the modified inner product turns out to be given by (Landsman
1995) \( (\psi, \varphi)_0 = \int_G dg (U(g)\psi, \varphi) \). One recognizes that the right-hand side coincides
with \( (P_0\psi, P_0\varphi) \), where \( P_0 \) is the orthogonal projector onto the subspace \( \mathcal{H}_0 \) of \( \mathcal{H} \)
which transforms trivially under \( G \). Hence \( \mathcal{N}_0 = \mathcal{H}_0^\perp \), so that \( \mathcal{H}^0 = \mathcal{H}/\mathcal{N}_0 \), which
coincides with \( \mathcal{H}_D \) of Dirac. The reason one can get away with not explicitly imple-
menting the constraints is that states not satisfying them are of zero norm in the
modified inner product, and therefore quotient to the null vector in \( \mathcal{H}^0 \). Also, the
action of a constraint on a vector maps it into a null vector (in other words, a gauge
transformation changes a vector only by a null vector).

Had \( G \) not been compact, the modified form would be ill-defined on some of
\( \mathcal{H} \), but our method is easily adapted to such instances: instead of all of \( \mathcal{H} \) one
finds a dense subspace \( L \subset \mathcal{H} \) on which ( , )\(_0\) is defined. The only change in the
procedure is then that the quotient \( L/\mathcal{N}_0 \) is not complete under the induced norm,
and has to be completed to form $\mathcal{H}^0$. In this way, constraints with continuous spectrum can be handled without any difficulty (Landsman 1995). The general way of finding the modified inner product is based on the mathematical analogy between the momentum map and generalized Marsden-Weinstein reduction in symplectic geometry, and Hilbert $C^*$-modules and Rieffel induction (Rieffel 1974) in operator algebra theory (cf. Landsman 1995 for details and references).

For an interesting example which highlights the difference between our Rieffel induction method and rigged Hilbert space techniques, take $G$ a noncompact semisimple Lie group, and $S = T^*G$ with momentum map $\Phi$ defined by the natural right-action of $G$ on $S$. The classical reduced space $\Phi^{-1}(0)/G$ is a point. Quantization with our method proceeds by taking $\mathcal{H} = L^2(G)$ as the quantization of $S$, which carries the right-regular representation $U$ of $G$. Let $(\cdot, \cdot)_0$ on $L = C_c(G)$ be defined as in the compact case. The physical Hilbert space comes out as $\mathcal{H}^0 = \mathbb{C}$, which is obviously the correct answer. However, the trivial representation of $G$ is not weakly contained in the regular representation on $L^2(G)$, so using the standard rigged Hilbert space $S(G) \subset L^2(G) \subset S(G)'$, where $S(G)$ is the Harish-Chandra Schwartz space of $G$, would produce an empty physical state space. (Note that the word ‘rigged’ as used in Landsman (1995) to describe the modified inner product has nothing to do with the same word occurring in ‘rigged Hilbert space’.)

Let $V\psi \in \mathcal{H}^0$ be the image of $\psi \in L$ in the quotient $L/N_0$, so that $V : L \rightarrow \mathcal{H}^0$ satisfies $(V\psi, V\varphi)_{\mathcal{H}^0} = (\psi, \varphi)_0$. In our examples, $\mathcal{H}^0$ is of the form $L^2(\mathbb{R}^n)$ (in momentum space), and $V$ has the form $(V\psi)(p) = (\psi, f_p)$ for a family of functions $f_p$ lying in some appropriate dual of $L$; the r.h.s. can be calculated as if it were the inner product in $\mathcal{H}$, although $f_p$ is not in $\mathcal{H}$, cf. Poerschke and Stolz 1993.

Observables are those self-adjoint operators $A$ on $\mathcal{H}$ which satisfy $(A\psi, \varphi)_0 = (\psi, A\varphi)_0$ for all $\psi, \varphi$ in $L$; this condition is the quantum version of the classical condition that $\{f, \Phi_a\}$ vanish on $C$. In either case, the point is that observables are well-defined on the reduced spaces $\mathcal{H}^0$ or $S^0$, respectively. The physical action $\pi^0$ on $\mathcal{H}^0$ is given by $\pi^0(A)V\psi = VA\psi$. 
3 Covariant Hamiltonian method

For our quantization method to apply, the classical reduced space has to have the form of a generalized Marsden-Weinstein quotient. This renders our method inapplicable to ADM gravity. However, the covariant Hamiltonian approach advertised above creates a context in which our technique does apply. For the parametrized particle and minisuperspace examples we will have to deal with noncompact abelian groups, for which the Dirac method breaks down. Yet our method works, the modified inner product being given by an expression similar to that in the compact group example above, with due care taken about the choice of the dense subspace $L$.

To see how the covariant method operates in a system without constraints, consider an unparametrized relativistic particle, with Lagrangian $L = -m\sqrt{1 - \dot{x}^2}$, where $\mathbf{x} = (x^1, x^2, x^3)$ and $\dot{\mathbf{x}} = d\mathbf{x}/dt$. The covariant phase space $S_{\text{cov}}$ of this model (Kijowski 1973) is $\mathbb{R}^8$, with co-ordinates $\{t, p, x^i, p^i\}$. The covariant Legendre transform (Gotay 1991, Gotay et.al. 1993) leads to the primary constraint $\Phi = p - H = 0$, where $H = \sqrt{p^2 + m^2} \equiv \omega_p$. (Here $H$ happens to coincide with the usual Hamiltonian, but in field theory it would be a covariant generalization thereof.) The symplectic form on $S_{\text{cov}}$ is $\omega = -dp \wedge dt + dp^i \wedge dx^i$. The solutions to the equations of motion are precisely tangent to the null direction of $i^* \omega$ at any given point. The reduced phase space $S^0_{\text{cov}}$ of $S_{\text{cov}}$ with respect to the constraint $p - H$ is the usual phase space $T^*\mathbb{R}^3$. The co-ordinates $p$ and $t$ have been eliminated by the reduction procedure. Time evolution on $S^0$ may be described by any observable acting as a Hamiltonian on $S^0_{\text{cov}}$, such as $H$.

To quantize, we replace $S_{\text{cov}}$ by the covariant Hilbert space $\mathcal{H}_{\text{cov}} = L^2(\mathbb{R}^4)$ (with $\psi$ a function of $(t, \mathbf{x})$). According to our method, we have

$$ (\psi, \varphi)_0 = \int_{-\infty}^{\infty} d\lambda (\exp(-i\lambda \hat{\Phi})\psi, \varphi). $$

We find $\mathcal{H}^0 = L^2(\mathbb{R}^3; d^4p/(2\pi)^3)$, and $V$ defined by $f_p(t, \mathbf{x}) = \exp(-i\omega_p t + i\mathbf{p} \cdot \mathbf{x})$. These functions are a complete set of generalized solutions of the constraints, which explains in part why alternative methods (e.g., Hájíček 1994) are equally well capable of handling this example. This is not generic, however (cf. the last example below).
4 The Polyakov particle

To mimic minisuperspace, we now treat the relativistic particle in the parametrized Polyakov formalism (cf. Gotay et.al. 1993 for the classical part). The Lagrangian is

\[ L = -\frac{1}{2}\sqrt{g}(g^{00}\dot{x}^\mu\dot{x}_\mu + m^2), \]

where \( \dot{x} = dx/d\tau \), and \( g = g_{00} \) is a metric on the real line. The covariant phase space is parametrized by \( \{\tau, p, x^\mu, p_\mu, g, \pi\} \), subject to the primary constraints \( p - H = 0 \), with

\[ H = -\frac{1}{2}\sqrt{g}(p_\mu p^\mu - m^2), \]

and \( \pi = 0 \). The symplectic form on \( S_{\text{cov}} \) is \( \omega = -dp \wedge dt - dp_\mu \wedge dx^\mu + d\pi \wedge dg \).

According to the general method of Gotay et.al. (1993) to handle parametrized theories, the secondary constraints are picked up by the covariant momentum map, in this case with respect to the action of \( \text{Diff}^+(\mathbb{R}) \), under which the Lagrangian is covariant. If \( N(\tau)d/d\tau \) is an arbitrary element of the Lie algebra of this group (that is, a vector field on \( \mathbb{R} \)), this momentum map is given by \( \Phi_N = -pN - 2\pi gN \), which is linear in the canonical momenta, as it should, and generates diffeomorphisms in the appropriate manner.

The complete set of constraints, then, is \( p - H = \pi = \Phi_N = 0 \) (the latter for all \( N \)). In full general relativity, there would be analogous constraints, with \( g \) and \( \pi \) replaced by the lapse and shift components of the metric, and the diffeomorphism constraints now being of the form \( \int_\Sigma \sigma^*\Phi_N = 0 \) (here \( \sigma \) is a section of the covariant primary constraint bundle (Gotay et.al. 1993), and \( \Sigma \) is a Cauchy surface). In any case, the diffeomorphism constraints are linear in the canonical momenta also in that case. The primary constraints, i.e., \( \pi = 0 \) and \( p = H \), are imposed on the finite-dimensional covariant phase space. It is only when \( p = H \) is substituted into the diffeomorphism constraints (and evaluated at a fixed time) that one quarter of the latter become quadratic in the canonical momenta, ruining their key property of generating four-dimensional diffeomorphisms, as well as the property that their Poisson algebra represents the Lie algebra of the group. Such a substitution would, in fact, lead to the ADM formalism.
In the covariant Hamiltonian formalism, on the other hand, the constraints are of the type we can handle with our quantization method, since the diffeomorphism constraints are essentially the (covariant) momentum map, and the finite-dimensional constraint $p = H$, which is typical to the covariant method, and independent of the diffeomorphism invariance, can be thought of as a momentum map for an action of $\mathbb{R}$. Also, the formalism clearly distinguishes between the lapse $g$ and the infinitesimal diffeomorphism $N(\tau)d/d\tau$.

In the present case, instead of reducing with respect to $\text{Diff}^+(\mathbb{R})$, the same reduced phase space is obtained if we reduce with respect to an arbitrary one-parameter subgroup generated by some $N$, as long as $N > 0$. Also, since $g > 0$, we can rewrite the constraints as $\Phi_1 = p = 0$, $\Phi_2 = \pi = 0$, and $\Phi_3 = -\frac{1}{2}(p_\mu p^\mu - m^2) = 0$.

Having reduced by $\Phi_1$ and $\Phi_2$, we then simply have to reduce $T^*\mathbb{R}^4$ by $\Phi_3$, which of course leads to the correct result for the reduced phase space; the restriction to $p_0 > 0$ has to be imposed by hand (cf. Landsman 1995).

The quantization of $S_{\text{cov}}$ is $\mathcal{H}_{\text{cov}} = L^2(\mathbb{R}^6)$, with $\psi$ a function of $(\tau, x^\mu, \log(g))$. The modified inner product is given by

$$ (\psi, \varphi)_0 = \int d\lambda_1 d\lambda_2 d\lambda_3 (\exp[-i \sum_{a=1}^3 \lambda_a \hat{\Phi}_a] \psi, \varphi), $$

defined on $\psi, \varphi \in L = C_c(\mathbb{R}^6)$. Imposing $p_0 > 0$, it follows that the physical Hilbert space $\mathcal{H}^0$ and the map $V$ are as in the previous treatment (the $f_p$ are now regarded as functions of $(\tau, x^\mu, \log(g))$, which happen to be independent of $\tau$ and $g$). Again, one easily verifies that observables such as the generators of the Poincaré group act in the correct way.

5 A minisuperspace example

We finally turn to a simple minisuperspace example (cf. Halliwell 1990). The Lagrangian is

$$ L = -\frac{1}{2}\sqrt{g}e^{3\alpha}(g^{00}(\dot{\alpha}^2 - \dot{\phi}^2) - \kappa e^{-2\alpha}), $$

which describes an FRW universe with radius $\exp(\alpha)$, filled with a homogeneous massless scalar field $\phi$ (here $\kappa = 0, -1$ (open), or $+1$ (closed)). The covariant phase
space \( S_{\text{cov}} \) is parametrized by \( \{\tau, p, \alpha, p_\alpha, \phi, p_\phi, g, \pi\} \). The constraints are as in the previous example, except that the Hamiltonian is now given by

\[
H = \frac{1}{2}\sqrt{g}e^{-3\alpha}(-p_\alpha^2 + p_\phi^2 - \kappa e^{4\alpha}).
\]

After elimination of \( \tau \) and \( g \), the physical phase space is obtained by reducing \( T^*\mathbb{R}^2 \) by the constraint \( H = 0 \).

In a suitable parametrization (of which the reduced space is evidently independent), which corresponds to putting \( \Phi_3 = \frac{1}{2}(-p_\alpha^2 + p_\phi^2 - \kappa \exp(4\alpha)) \), for \( \kappa = 1 \) the flow generated by this constraint is given by \( \phi(\lambda) = \phi(0) + p_\phi \lambda, \ p_\phi(\lambda) = p_\phi(0); \ \alpha(\lambda) = \frac{1}{2}\log[\sqrt{2E}/\cosh(2\sqrt{2E}(\lambda - \lambda_0))], \ p_\alpha(\lambda) = \sqrt{2E}\tanh(2\sqrt{2E}(\lambda - \lambda_0)), \) where \( E = \frac{1}{2}(p_\alpha(0)^2 + \kappa \exp(4\alpha(0))) \) and \( \lambda_0 \) is determined by \( \alpha(0) = \frac{1}{2}\log[\sqrt{2E}/\cosh(2\sqrt{2E}\lambda_0)] \) (or \( p_\alpha(\lambda_0) = 0 \)). Observables are, for example, the functions \( F_1 = p_\phi \) and \( F_2 = \phi - p_\phi(\text{arctanh}(p_\alpha/\sqrt{2E}))/2\sqrt{2E} \), which in fact project to canonical coordinates \( f_1, f_2 \) on the reduced space \( S^0 \). These are observables of DeWitt type, e.g., \( F_2(z) \) is the value of \( \phi(z(\lambda)) \) at the \( \lambda \) for which \( p_\alpha(z(\lambda)) = 0, \ z \in T^*\mathbb{R}^2 \). For \( \kappa = -1 \) the above expression for the flow is formally correct as well, but in that case the classical motion is incomplete at infinity and a special interpretation is necessary (cf. Feinberg and Peleg 1995 for similar cases).

The idea of taking the matter field \( \phi \) (which is not an observable) as a time variable (cf. Isham 1993) is implemented in a parametrization-invariant way by using \( f_1 \) as the physical Hamiltonian of the model. If \( a_0(z) \) is the value of \( \exp(\alpha(z(\lambda))) \) at the \( \lambda \) for which \( \phi(z(\lambda)) = 0, \ z \in T^*\mathbb{R}^2 \). For \( \kappa = -1 \) the above expression for the flow is formally correct as well, but in that case the classical motion is incomplete at infinity and a special interpretation is necessary (cf. Feinberg and Peleg 1995 for similar cases).

The wavefunction of the universe

We quantize on \( \mathcal{H}_{\text{cov}} = L^2(\mathbb{R}^4) \), with \( \psi \) a function of \( (\tau, \alpha, \phi, \log(g)) \). The modified inner product is defined as in the previous example. For \( \kappa = 0, 1 \), \( \hat{\Phi}_3 \) is the unique self-adjoint extension of the usual Schrödinger quantization of \( \Phi_3 \) on the core \( C^\infty_c(\mathbb{R}^4) \). For \( \kappa = 0 \) the analysis is trivial, leading to \( \mathcal{H}_0 = \bigoplus \mathbb{R}_+ L^2(\mathbb{R}; dk/2\pi) \), and
\[ V^{(0)} = V^{(0)}_+ \oplus V^{(0)}_- \] defined through \( f^{(0)}_{\pm}(\tau, \alpha, \phi, \log(g)) = \exp(ik(\alpha \pm \phi)) \). For \( \kappa = 1 \) the physical Hilbert space is \( \mathcal{H}^0_1 = L^2(\mathbb{R}; dk/2\pi) \), and some functional analysis (for which cf. Leis 1979, Picard 1989) shows that one has

\[ f^{(1)}_k(\tau, \alpha, \phi, \log(g)) = \pi^{-1}e^{i\phi k} \sqrt{\sinh(\pi|k|/2)} K_{|k|/2}(\frac{1}{2}e^{2\alpha}). \]

For \( \kappa = -1 \) the operator \( \Phi_3 \) is not essentially self-adjoint on \( C_c^\infty(\mathbb{R}^4) \); its deficiency indices are \((1, 1)\), and one obtains a one-parameter family of self-adjoint extensions by specifying boundary conditions at infinity. For all of these, \( \mathcal{H}^0_{-1} = \mathcal{H}^0_1 \), and the wavefunction of the universe (see below) turns out not to be affected by the particular choice of the extension. For simplicity we choose the boundary condition specified in Picard (1989). This leads to

\[ f^{(-1)}_k(\tau, \alpha, \phi, \log(g)) = \frac{1}{2}e^{i\phi k} \sqrt{\cosech(\pi|k|/2)}(J_{i|k|/2} + J_{-i|k|/2})(\frac{1}{2}e^{2\alpha}). \]

(Different boundary conditions at infinity would have led to a relative phase between \( J_{i|k|/2} \) and \( J_{-i|k|/2} \).)

These are solutions to the Wheeler-DeWitt equations of the model (which do not lie in \( \mathcal{H}_{\text{cov}} \)), but the difficulty lies in the fact that now there are other linearly independent solutions as well. Our technique gives a procedure of choosing between them. In alternative quantization methods, the boundary conditions (at zero radius) singling out the relevant solution are lost.

One may compare these solutions with proposed ‘wavefunctions of the universe’ \( \Psi \) (cf. Halliwell 1990). This comparison is feasible if one has a rationale for letting \( k \to 0 \) in \( f^{(\kappa)}_k \), which might be provided if the physical Hamiltonian is chosen to be \( \hat{p}_\phi \), as in the classical case. Up to a constant normalization factor, the Hartle-Hawking proposal here gives \( J_0 \) rather than our \( K_0 \) for \( \kappa = 1 \), and agrees with our \( J_0 \) for \( \kappa = -1 \) (see Zhuk 1992). Vilenkin’s wavefunction, which is always complex, differs from ours as well as Hartle-Hawking’s in both cases. But note that according to Halliwell (1990) both proposals are somewhat ambiguous and not always well-defined - in fact, one can find contradictory statements in the literature concerning these proposals. Our expansion functions \( f^{(\kappa)} \), on the other hand, only depend
on the domain chosen for the quantum constraints, which at least for \( \kappa = 0,1 \) is uniquely given (within the bounds of reason).

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