Abstract Thermodynamically consistent fractional Burgers constitutive models for viscoelastic media, divided into two classes according to model behavior in stress relaxation and creep tests near the initial time instant, are coupled with the equation of motion and strain forming the fractional Burgers wave equations. The Cauchy problem is solved for both classes of Burgers models using an integral transform method, and an analytical solution is obtained as a convolution of the solution kernels and initial data. The form of the solution kernel is found to be dependent on model parameters, while its support properties imply infinite wave propagation speed for the first class and finite speed for the second class. Spatial profiles corresponding to the initial Dirac delta displacement with zero initial velocity display features which are not expected in wave propagation behavior.

1 Introduction

The fractional Burgers wave equation is written as the system of equations consisting of: equation of motion corresponding to a one-dimensional deformable body

\[
\frac{\partial}{\partial x} \sigma(x, t) = \rho \frac{\partial^2}{\partial t^2} u(x, t), \quad x \in \mathbb{R}, \ t > 0,
\]

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Fractional Burgers wave equation
where \( u \) and \( \sigma \) are displacement and stress, while \( \rho \) is constant material density; strain for small local deformations
\[
\varepsilon(x, t) = \frac{d}{dx} u(x, t), \quad x \in \mathbb{R}, \ t > 0;
\]  
and constitutive equation represented by the fractional Burgers model
\[
(1 + a_1 D_\tau^\alpha + a_2 D_\tau^\beta + a_3 D_\tau^\gamma) \sigma(x, t) = (b_1 D_\tau^\mu + b_2 D_\tau^\nu) \varepsilon(x, t), \quad x \in \mathbb{R}, \ t > 0,
\]  
having the model parameters \( a_1, a_2, a_3, b_1, b_2 > 0, \alpha, \beta, \mu \in [0, 1] \), with \( \alpha \leq \beta \), and \( \gamma, \nu \in [1, 2] \), while the operator of the Riemann–Liouville fractional derivative \( D_\tau^\xi \) of order \( \xi \in [n, n+1], n \in \mathbb{N}_0, \) is defined by
\[
D_\tau^\xi f(t) = \frac{d^{n+1}}{dt^{n+1}} \left( \frac{t-(\xi-n)}{\Gamma(1-(\xi-n))} * f(t) \right), \quad t > 0,
\]  
see [19], where \( * \) denotes the convolution in time: \( f(t) * g(t) = \int_0^t f(t') g(t-t') \, dt', \ t > 0. \)

Since the fractional Burgers model (3) is not thermodynamically consistent with the model parameters assumed above, the constitutive equation considered for modeling wave propagation is actually assumed either as (6), or as (7), given below, corresponding either to the first or to the second class of thermodynamically consistent fractional Burgers models. Note that these two classes contain eight thermodynamically consistent models, as will be described later.

In order to solve the Cauchy problem on the real line \( x \in \mathbb{R} \) and \( t > 0 \), the system of governing equations (1), (2), and (3) is subject to initial and boundary conditions:
\[
\begin{align*}
\lim_{x \to \pm \infty} u(x, t) &= u_0(x), \\
\lim_{x \to \pm \infty} \sigma(x, t) &= 0,
\end{align*}
\]  
where \( u_0 \) is the initial displacement and \( v_0 \) is the initial velocity.

Considering the rheological scheme of the classical Burgers model, with the dash-pot element replaced by the Scott-Blair (fractional) element, the fractional Burgers model (3) is derived in [27]. Moreover, using the requirement of storage and loss modulus nonnegativity, the analysis of thermodynamical consistency for the fractional Burgers model (3), conducted in [27], yielded that the orders of fractional derivatives \( \gamma, \nu \in [1, 2] \) cannot be independent of the orders of fractional derivatives \( \alpha, \beta, \mu \in [0, 1] \), and this led to the formulation of eight thermodynamically consistent fractional Burgers models, divided into two classes.

The first class contains five models, written as
\[
(1 + a_1 D_\tau^\alpha + a_2 D_\tau^\beta + a_3 D_\tau^\gamma) \sigma(t) = (b_1 D_\tau^\mu + b_2 D_\tau^\nu) \varepsilon(t)
\]  
in a unified manner, such that the highest fractional differentiation order of strain is \( \mu + \eta \in [1, 2] \), with \( \eta \in [\alpha, \beta] \), while the highest fractional differentiation order of stress is either \( \gamma \in [0, 1] \) in the case of Model I, with \( 0 \leq \alpha \leq \beta \leq \gamma \leq \mu \leq 1 \) and \( \eta \in [\alpha, \beta, \gamma] \), or \( \gamma \in [1, 2] \) in the case of Models II–V, with \( 0 \leq \alpha \leq \beta \leq \mu \leq 1 \) and \( (\eta, \gamma) \in (\alpha, 2\alpha), (\alpha, \alpha + \beta), (\beta, \alpha + \beta), (\beta, 2\beta) \). The fractional differentiation order of stress is less than the differentiation order of strain regardless on the interval [0, 1] or [1, 2].

The second class contains three models, written as
\[
(1 + a_1 D_\tau^\alpha + a_2 D_\tau^\beta + a_3 D_\tau^\gamma) \sigma(t) = (b_1 D_\tau^\mu + b_2 D_\tau^\nu) \varepsilon(t)
\]  
in a unified manner, such that \( 0 \leq \alpha \leq \beta \leq 1 \) and \( \beta + \eta \in [1, 2] \), with \( \eta = \alpha \), in the case of Model VI; \( \eta = \beta \) in the case of Model VII; and \( \alpha = \eta = \beta, a_1 = a_1 + a_2, \) and \( a_2 = a_3 \) in the case of Model VIII. Considering the interval [0, 1], the highest fractional differentiation orders of stress and strain are equal, which also holds true for the orders from interval [1, 2].

The responses in creep and stress relaxation tests for Models I–VIII are examined in [28]. Recall that creep compliance \( \varepsilon(t) \) is the strain (stress) history function obtained as a response to the stress (strain) assumed as the Heaviside step function. It is found that the models’ behavior near the initial time instant is different for the first and the second model class: Models I–V have zero glass compliance, i.e.,
The fractional Burgers wave equation, as the dimensionless system of equations:

\[
\varepsilon_{ct}^{(q)} = \varepsilon_{ct} (0) = 0 \quad \text{and thus infinite glass modulus, i.e., } \sigma_{ct}^{(q)} = \sigma_{ct} (0) = \infty, \quad \text{while Models VI–VIII have nonzero glass compliance } \varepsilon_{ct}^{(q)} = \frac{a_1}{b_2} \text{ implying the nonzero glass modulus } \sigma_{ct}^{(q)} = \frac{b_2}{a_2} \text{ as well. On the other hand, the equilibrium compliance is infinite, i.e., } \varepsilon_{ct} = \lim_{t \to \infty} \varepsilon_{ct} (t) = \infty, \quad \text{so that the equilibrium modulus is zero, i.e., } \sigma_{ct}^{(q)} = \lim_{t \to \infty} \sigma_{ct} (t) = 0 \text{ for both model classes, and therefore, all fractional Burgers models describe fluid-like materials. Note that if the equilibrium compliance is finite, the model would represent a solid-like material.}
\]

The implication proved in the present work is that fluid-like Burgers models belonging to the first class have infinite, while the ones belonging to the second class have finite wave propagation speed

\[
c = \sqrt{\frac{\sigma_{ct}^{(q)}}{\varepsilon_{ct}^{(q)}}} = \frac{1}{\sqrt{\varepsilon_{ct}^{(q)}}} = \sqrt{\frac{b_2}{a_3}},
\]

as in the case of thermodynamically consistent fractional models arising from the general fractional linear model

\[
\sum_{i=1}^{m} a_i \partial_{\alpha_i}^{D_i} \varepsilon (x, t) = \sum_{j=1}^{n} b_j \partial_{\beta_j}^{D_j} \varepsilon (x, t), \quad a_i, b_j > 0, \quad a_i, \beta_j \in (0, 1),
\]

obtained and analyzed in [2] for thermodynamical consistency and used in [22] as constitutive equations in wave propagation modeling. Namely, the results of [20,21], where the wave propagation speed is found via the conic solution support, i.e., \( |x| < ct \), in the case of the fractional Zener model and its generalization, respectively, given by

\[
\left( 1 + a_0 D_{\alpha} \right) \varepsilon (x, t) = E \left( 1 + b_0 D_{\beta} \right) \varepsilon (x, t), \quad 0 < a \leq b, \quad \alpha \in (0, 1),
\]

\[
\sum_{i=1}^{m} a_i \partial_{\alpha_i}^{D_i} \varepsilon (x, t) = \sum_{j=1}^{n} b_j \partial_{\beta_j}^{D_j} \varepsilon (x, t), \quad 0 \leq \alpha_1 \leq \cdots \leq \alpha_n < 1, \quad \frac{a_1}{b_1} \geq \cdots \geq \frac{a_n}{b_n} \geq 0,
\]

are extended in [22], using the same argumentation as in the previous work, to all four classes of thermodynamically consistent linear fractional models and moreover to the power-type distributed-order model assuming that the orders of fractional differentiation do not exceed one. In particular, it is found that both solid-like and fluid-like materials can have either infinite or finite wave speed. Singularity propagation properties of the memory and non-local type fractional wave equations are investigated in [17,18] using the tools of microlocal analysis, supporting the results obtained in [20].

Wave propagation phenomena in viscoelastic bodies, modeled by integer and fractional order models, including the question of wave speed and energy dissipation properties are analyzed in [8,9]. The wavefront expansion of solution, due to Buchen and Mainardi, is introduced in [7] to be later used in [11,12] when considering the wave equation in viscoelastic materials described by the Bessel as well as by the integer and fractional order Maxwell and Kelvin–Voigt models. The Bessel model for a viscoelastic body is introduced in [13] and analyzed in [10]. Features of the wave propagation in viscoelastic media, like the asymptotic behavior of the fundamental solution near the wavefront, dispersion, and attenuation, are examined in [14–16]. Wave propagation speed, reinterpreted as the fundamental solution’s peak propagation speed, is analyzed in [23–25]. Modeling viscoelastic materials using the fractional order models, as well as dispersion and attenuation effects described by the corresponding wave equations is reviewed in [26].

Fractional wave equations on bounded and semi-bounded domains are considered in [29–31] for different fractional models including the Zener, modified Zener, and modified Maxwell models, as well as in [4–6] in the case of power-type distributed-order model. Generalizations of the classical wave equations and corresponding problems are reviewed in [3,32].

2 Fractional Burgers model in wave propagation

The fractional Burgers wave equation, as the dimensionless system of equations:

\[
\frac{\partial}{\partial x} \varepsilon (x, t) = \frac{\partial^2}{\partial t^2} u (x, t), \quad \varepsilon (x, t) = \frac{\partial}{\partial x} u (x, t),
\]

(10)
and either
\[
(1 + a_1 0 D_t^\alpha + a_2 0 D_t^\beta + a_3 0 D_t^\gamma) \sigma (x, t) = \left( 0 D_t^\mu + b_0 D_t^{\mu + \eta} \right) \varepsilon (x, t)
\]
for the first class of Burgers models, or
\[
(1 + a_1 0 D_t^\alpha + a_2 0 D_t^\beta + a_3 0 D_t^{\beta + \eta}) \sigma (x, t) = \left( 0 D_t^\mu + b_0 D_t^{\mu + \eta} \right) \varepsilon (x, t),
\]
for the second class of Burgers models, subject to the initial and boundary conditions
\[
u(x, 0) = u_0(x), \quad \frac{\partial}{\partial t} u(x, 0) = v_0(x), \quad \sigma (x, 0) = 0, \quad \varepsilon (x, 0) = 0,
\]
\[
\lim_{x \to \pm \infty} u(x, t) = 0, \quad \lim_{x \to \pm \infty} \sigma (x, t) = 0,
\]
is obtained by introducing the dimensionless quantities
\[
\bar{x} = \frac{x}{U^*}, \quad \bar{t} = \frac{t}{U^*}, \quad \bar{u} = \frac{u}{U}, \quad \bar{u}_0 = \frac{u_0}{U}, \quad \bar{v}_0 = \frac{T^*}{U^*} v_0, \quad \bar{\sigma} = \frac{\sigma}{\sigma^*}, \quad \bar{a}_1 = \frac{a_1}{(T^*)^\alpha}, \quad \bar{a}_2 = \frac{a_2}{(T^*)^\beta},
\]
\[
T^* = \left( \frac{\rho U^2}{b_1} \right), \quad \sigma^* = \left( \frac{b_1^2}{(T^*)^\xi} \right), \quad \bar{b} = \frac{b_2}{b_1 (T^*)^\gamma},
\]
with \( \xi = \mu \) and \( \zeta = \gamma \) for the first class of Burgers models, \( \xi = \beta \) and \( \zeta = \beta + \eta \) for the second class, and \( \mathcal{U} = \sup_{x \in \mathbb{R}} |u_0 (x)| \), into system of governing equations (1), (2) and either (6) or (7), subject to (4), (5), and by subsequent omission of bars.

The models in dimensionless form, along with the corresponding thermodynamical restrictions, are listed below.

**Model I:**
\[
(1 + a_1 0 D_t^\alpha + a_2 0 D_t^\beta + a_3 0 D_t^\gamma) \sigma (t) = \left( 0 D_t^\mu + b_0 D_t^{\mu + \eta} \right) \varepsilon (t),
\]
\[
0 \leq \alpha \leq \beta \leq \gamma \leq \mu \leq 1, \quad 1 \leq \mu + \eta \leq 1 + \alpha, \quad b \leq a_1 \frac{\cos (\mu - \eta) \pi}{\cos (\mu + \eta) \pi};
\]
with \( (\eta, \iota) \in \{ (\alpha, 1), (\beta, 2), (\gamma, 3) \} \);

**Model II:**
\[
(1 + a_1 0 D_t^\alpha + a_2 0 D_t^\beta + a_3 0 D_t^{2\alpha}) \sigma (t) = \left( 0 D_t^\mu + b_0 D_t^{\mu + \alpha} \right) \varepsilon (t),
\]
\[
\frac{1}{2} \leq \alpha \leq \beta \leq \mu \leq 1, \quad \frac{a_3}{a_1} \frac{\sin (\mu - 2\alpha) \pi}{\sin \frac{\mu \pi}{2}} \leq b \leq a_1 \frac{\cos (\mu - \alpha) \pi}{\cos (\mu + \alpha) \pi};
\]

**Model III:**
\[
(1 + a_1 0 D_t^\alpha + a_2 0 D_t^\beta + a_3 0 D_t^{\alpha + \beta}) \sigma (t) = \left( 0 D_t^\mu + b_0 D_t^{\mu + \alpha} \right) \varepsilon (t),
\]
\[
0 \leq \alpha \leq \beta \leq \mu \leq 1, \quad \alpha + \beta \geq 1, \quad \frac{a_3}{a_2} \frac{\sin (\mu - \beta - \alpha) \pi}{\sin (\mu - \beta + \alpha) \pi} \leq b \leq a_1 \frac{\cos (\mu - \alpha) \pi}{\cos (\mu + \alpha) \pi};
\]

**Model IV:**
\[
(1 + a_1 0 D_t^\alpha + a_2 0 D_t^\beta + a_3 0 D_t^{\alpha + \beta}) \sigma (t) = \left( 0 D_t^\mu + b_0 D_t^{\mu + \beta} \right) \varepsilon (t),
\]
\[
0 \leq \alpha \leq \beta \leq \mu \leq 1, \quad 1 - \alpha \leq \beta \leq 1 - (\mu - \alpha), \quad \frac{a_3}{a_1} \frac{\sin (\mu - \alpha - \beta) \pi}{\sin (\mu - \alpha + \beta) \pi} \leq b \leq a_2 \frac{\cos (\mu - \beta) \pi}{\cos (\mu + \beta) \pi};
\]
Model V:
\[
\left( 1 + a_1 0D^\alpha_t + a_2 0D^\beta_t + a_3 0D^{2\beta}_t \right) \sigma(t) = \left( 0D^\alpha_t + b 0D^{\alpha+\beta}_t \right) \varepsilon(t),
\]
(23)
\[
0 \leq \alpha \leq \beta \leq \mu \leq 1, \quad \frac{1}{2} \leq \beta \leq 1 - (\mu - \alpha), \quad \frac{a_3}{a_2} \sin \left( \frac{(\mu-\beta)\pi}{2} \right) \leq b \leq a_2 \frac{\cos \left( \frac{(\mu-\beta)\pi}{2} \right)}{\cos \left( \frac{a_1}{a_2} \beta \pi \right)};
\]
(24)

Model VI:
\[
\left( 1 + a_1 0D^\alpha_t + a_2 0D^\beta_t + a_3 0D^{\alpha+\beta}_t \right) \sigma(t) = \left( 0D^\beta_t + b 0D^{\alpha+\beta}_t \right) \varepsilon(t),
\]
(25)
\[
0 \leq \alpha \leq \beta \leq 1, \quad \alpha + \beta \geq 1, \quad \frac{a_3}{a_2} \leq b \leq a_1 \frac{\cos \left( \frac{a_1}{a_2} \beta \pi \right)}{\cos \left( \frac{a_1}{a_2} \gamma \pi \right)_i};
\]
(26)

Model VII:
\[
\left( 1 + a_1 0D^\alpha_t + a_2 0D^\beta_t + a_3 0D^{2\beta}_t \right) \sigma(t) = \left( 0D^\beta_t + b 0D^{2\beta}_t \right) \varepsilon(t),
\]
(27)
\[
0 \leq \alpha \leq \beta \leq 1, \quad \frac{1}{2} \leq \beta \leq \frac{1+\alpha}{2}, \quad \frac{a_3}{a_2} \leq b \leq a_2 \frac{1}{\cos (\beta \pi)};
\]
(28)

Model VIII:
\[
\left( 1 + a_1 0D^\alpha_t + a_2 0D^\beta_t + a_3 0D^{2\beta}_t \right) \sigma(t) = \left( 0D^\beta_t + b 0D^{2\beta}_t \right) \varepsilon(t),
\]
(29)
\[
\frac{1}{2} \leq \alpha \leq 1, \quad \frac{a_2}{a_1} \leq b \leq \frac{1}{a_1 \cos (\alpha \pi)}.
\]
(30)

Application of the Fourier transform with respect to the spatial coordinate
\[
\hat{f}(\xi) = \mathcal{F}[f(x)](\xi) = \int_{-\infty}^{\infty} f(x) e^{-i\xi x} dx, \quad \xi \in \mathbb{R},
\]
and Laplace transform with respect to time
\[
\hat{f}(s) = \mathcal{L}[f(t)](s) = \int_{0}^{\infty} f(t) e^{-st} dt, \quad \Re s > 0,
\]
with initial (13) and boundary conditions (14) taken into account, transforms the system of governing equations (10) and either (11) or (12) into \((\xi \in \mathbb{R}, \Re s > 0)\)
\[
i \xi \hat{\sigma}(\xi, s) = s^2 \hat{u}(\xi, s) - s \hat{u}_0(\xi) + \hat{v}_0(\xi), \quad \hat{\varepsilon}(\xi, s) = i \xi \hat{\Sigma}(\xi, s),
\]
(31)
\[
\Phi_\sigma(s) \hat{\sigma}(\xi, s) = \Phi_\varepsilon(s) \hat{\varepsilon}(\xi, s),
\]
(32)
with either
\[
\Phi_\sigma(s) = 1 + a_1 s^\alpha + a_2 s^\beta + a_3 s^\gamma, \quad \Phi_\varepsilon(s) = s^\mu + b s^{\mu+\eta},
\]
(33)
in the case of the first class of Burgers equation (11), or
\[
\Phi_\sigma(s) = 1 + a_1 s^\alpha + a_2 s^\beta + a_3 s^{\beta+\eta}, \quad \Phi_\varepsilon(s) = s^\beta + b s^{\beta+\eta},
\]
(34)
in the case of the second class of Burgers equation (11).

It is obtained that
\[
\hat{u}(\xi, s) = \hat{K}(\xi, s) \left( \hat{u}_0(\xi) + \frac{1}{s} \hat{v}_0(\xi) \right), \quad \xi \in \mathbb{R}, \Re s > 0,
\]
(35)
with
\[
\hat{K}(\xi, s) = \frac{s \Phi_\sigma(s)}{\Phi_\varepsilon(s) \xi^2 + s^2 \Phi_\varepsilon(s)} \xi, \quad \xi \in \mathbb{R}, \Re s > 0,
\]
(36)
once the system of equations (31), (32) is solved with respect to displacement \( \hat{u} \), implying the solution to the fractional Burgers equation (10) and either (11), or (12), subject to (13) and (14), in the form

\[
u(x, t) = K(x, t) * u_0(x) \delta(t) + v_0(x) H(t),
\]

(37)

where \(*_x\) denotes the convolution with respect to the spatial variable: \( f(x) *_x g(x) = \int_{-\infty}^{\infty} f(x') g(x-x') \, dx' \), \( x \in \mathbb{R} \), after inverting Fourier and Laplace transforms in (35).

In order to calculate the solution kernel \( K \), the inversion of the Fourier transform is performed in (36) using a well-known inversion formula

\[
\mathcal{F}^{-1} \left[ \frac{1}{\xi^2 + \lambda} \right](x) = \frac{1}{2\sqrt{\lambda}} e^{-|x|\sqrt{\lambda}}, \quad x \in \mathbb{R}, \quad \lambda \in \mathbb{C}\setminus(-\infty, 0],
\]

(38)

implying

\[
\tilde{K}(x, s) = \frac{1}{2} \sqrt{\frac{\Phi_\sigma(s)}{\Phi_\varepsilon(s)}} e^{-|x|\sqrt{\frac{\Phi_\sigma(s)}{\Phi_\varepsilon(s)}}}, \quad x \in \mathbb{R}, \quad \text{Re} \, s > 0,
\]

(39)

provided that

\[
s^2 \frac{\Phi_\sigma(s)}{\Phi_\varepsilon(s)} \in \mathbb{C}\setminus(-\infty, 0] \iff \frac{\Phi_\sigma(s)}{\Phi_\varepsilon(s)} \left( s^2 + \xi^2 \frac{\Phi_\varepsilon(s)}{\Phi_\sigma(s)} \right) \neq 0, \quad \text{for} \quad \xi \in \mathbb{R}, \quad \text{Re} \, s > 0,
\]

(40)

which holds for all Models I–VIII, as proved in Appendix A. Further, inverting the Laplace transformation in (39) by the definition

\[
K(x, t) = \mathcal{L}^{-1} \left[ \tilde{K}(x, s) \right](t) = \frac{1}{2\pi i} \int_{\Gamma_0} \tilde{K}(x, s)e^{st} \, ds, \quad x \in \mathbb{R}, \quad t > 0,
\]

(41)

where \( \Gamma_0 \) is the Bromwich path, the two forms of solution kernel \( K \) are obtained in Appendix B depending on the number and position of branching points of function \( \tilde{K} \), given by (39), originating from the zeros of function \( \Phi_\sigma \), since \( \Phi_\varepsilon \), except for \( s = 0 \), has no other zeros in the principal Riemann plane, with \( \Phi_\sigma \) and \( \Phi_\varepsilon \) given by either (33) or (34). There are three possible cases, since, as shown in [28], function \( \Phi_\sigma \) can have no zeros, one negative real zero, or a pair of complex conjugated zeros having negative real part. However, the solution kernel has the same form in the first two cases, thus merged into Case 1 below, while the form of the solution kernel differs in the third case, thus being labeled as Case 2.

**Case 1.** If the function \( \tilde{K} \), except for \( s = 0 \), either has no branching points, or has a negative real branching point, then the function \( K \) is found as

\[
K(x, t) = \frac{1}{4\pi i} \int_0^\infty \left( \frac{\Phi_\sigma(\rho e^{-i\frac{\pi}{2}})}{\Phi_\varepsilon(\rho e^{-i\frac{\pi}{2}})} e^{-|x|\sqrt{\frac{\Phi_\sigma(\rho e^{-i\frac{\pi}{2}})}{\Phi_\varepsilon(\rho e^{-i\frac{\pi}{2}})}}} - \frac{\Phi_\sigma(\rho e^{i\frac{\pi}{2}})}{\Phi_\varepsilon(\rho e^{i\frac{\pi}{2}})} e^{-|x|\sqrt{\frac{\Phi_\sigma(\rho e^{i\frac{\pi}{2}})}{\Phi_\varepsilon(\rho e^{i\frac{\pi}{2}})}}} \right) e^{-\rho t} \, d\rho,
\]

(42)

either having support in \( \mathbb{R} \times [0, \infty) \) for the first class of fractional Burgers models, or having support in the conic domain \( |x| < \sqrt{\frac{4t}{\pi}}, \) for the second class.

**Case 2.** If the function \( \tilde{K} \), except for \( s = 0 \), has a pair of complex conjugated branching points with negative real part: \( s_0 = \rho_0 e^{i\phi_0} \) and \( \tilde{s}_0 = \rho_0 e^{-i\phi_0} \), then the function \( K \) is found as

\[
K(x, t) = \frac{1}{4\pi i} \int_0^\infty \left( \frac{\Phi_\sigma(\rho e^{i\phi_0})}{\Phi_\varepsilon(\rho e^{i\phi_0})} e^{-|x|\sqrt{\frac{\Phi_\sigma(\rho e^{i\phi_0})}{\Phi_\varepsilon(\rho e^{i\phi_0})}}} - \frac{\Phi_\sigma(\rho e^{-i\phi_0})}{\Phi_\varepsilon(\rho e^{-i\phi_0})} e^{-|x|\sqrt{\frac{\Phi_\sigma(\rho e^{-i\phi_0})}{\Phi_\varepsilon(\rho e^{-i\phi_0})}}} \right) \, d\rho.
\]

(43)

either having support in \( \mathbb{R} \times [0, \infty) \) for the first class of fractional Burgers models, or having support in the conic domain \( |x| < \sqrt{\frac{4t}{\pi}}, \) for the second class.
The solution support properties, in both cases of solution kernel, define the wave propagation speed: infinite if the support is $\mathbb{R} \times [0, \infty)$, obtained for the first class of Burgers models, and finite if the support is a conic domain $|x| < \sqrt{\frac{a_2}{b}} t$, obtained as

$$c = \sqrt{\frac{b}{a_3}}$$

(44)

for the second class of Burgers models. Since $\alpha^{(s)}_\mu = \frac{\alpha}{\mu}$, see [28, Eq. (57)], the wave propagation speed (44) is exactly the wave propagation speed (8) that is obtained in [22] for the constitutive models having fractional differentiation orders not exceeding one.

3 Numerical examples

Spatial profiles of the solution to the fractional Burgers wave equations, written as the system of equations (10) and either (11) or (12), subject to initial and boundary conditions (13) and (14), with the initial displacement being the Dirac delta distribution and initial velocity being zero, i.e., $u_0 = \delta$, and $v_0 = 0$, implying that the solution is equal to the solution kernel $K$, are depicted in Figs. 1, 2, and 3 for Model V, representing the first class of fractional Burgers models and in Figs. 4, 5, and 6 for Model VII, representing the second class. Recall that in the case of constitutive models belonging to the first class the wave propagation speed is infinite, while in the case of the second class the speed is finite and given by (8). Spatial profiles produced by using the analytical formula for the solution kernel $K$, given by either (42) or (43), are compared with the solution kernel numerically calculated by the fixed Talbot numerical Laplace inversion Mathematica function, developed by J. Abate and P. P. Valko according to [1] and available at: http://library.wolfram.com/infocenter/MathSource/4738/. In each of the numerical examples, good agreement between profiles obtained by these two methods is found.

Figures 1, 2, and 3 present spatial profiles for Model V in cases when the function $\tilde{K}$, given by (39), except for $s = 0$ does not have other branching points, has one negative real, and has a pair of complex conjugated branching points, respectively. Different number and position of the branching points is a consequence of the change of a single parameter $\beta$. Apart from the main peak originating from the propagation of the initial Dirac delta displacement, there is a noticeable additional peak that is more prominent for small times and ceasing as time passes. As the parameter $\beta$ increases, the change of the nature (number and position) of the branching points from no branching points to a pair of complex conjugated ones implies the growth of prominence of the additional peak. During the propagation, due to energy dissipation, the height of the main peak decreases, while the width of the profile is increasing, while the propagation itself is rather slow.

The wave propagation speed is finite for the second class of fractional Burgers models, and in Figs. 4, 5, and 6, presenting spatial profiles for Model VII, it is underlined by denoting the ending points of the solution by dots.

Fig. 1 Spatial profiles of solution $u$, represented by solid line—analytical solution, and squares—numerical solution, at different time instances for Model V with parameters: $a_1 = 0.075$, $a_2 = 0.8$, $a_3 = 1.14$, $b = 1.39$, $\alpha = 0.4$, $\beta = 0.6$, and $\mu = 0.7$, when, except for $s = 0$, there are no other branching points.
Fig. 2 Spatial profiles of solution $u$, represented by solid line—analytical solution, and squares—numerical solution, at different time instances for Model V with parameters: $a_1 = 0.075$, $a_2 = 0.8$, $a_3 = 1.14$, $b = 1.39$, $\alpha = 0.4$, $\beta = 0.63138$, and $\mu = 0.7$, when, except for $s = 0$, there is one real branching point.

Fig. 3 Spatial profiles of solution $u$, represented by solid line—analytical solution, and squares—numerical solution, at different time instances for Model V with parameters: $a_1 = 0.075$, $a_2 = 0.8$, $a_3 = 1.14$, $b = 1.39$, $\alpha = 0.4$, $\beta = 0.685$, and $\mu = 0.7$, when, except for $s = 0$, there is a pair of complex conjugated branching points.

Support by circles. It is also noticeable that during the propagation, due to energy dissipation, the height of the peak decreases, while its width increases.

Figure 4 presents spatial profiles depending on the nature of the branching points, different than $s = 0$, of the function $\tilde{K}$ given by (39) in three cases obtained as a consequence of changing parameter $\beta$: Fig. 4a represents case when there are no other branching points, Fig. 4b when there is one negative real branching point, and Fig. 4c when there is a pair of complex conjugated branching points. For small times, the profile shapes are considerably different, while as time passes the profile shapes become alike. In all cases, there are jumps at the ending points of solution support: in Figs. 4a, b the displacement jumps from a positive value to zero, while in Fig. 4c the displacement jumps from a negative value to zero.

When compared to the profiles from Fig. 4a, where the displacement jumps to zero at the ending point of the solution support, the displacements plotted in Fig. 5, representing also the case when there are no other branching points than $s = 0$, tend smoothly to zero at the ending points of solution support. Profiles from Fig. 5 are similar to the profiles obtained in [20–22] for fractional constitutive models used wave propagation modeling in viscoelastic dissipative media.
Fractional Burgers wave equation

(a) $\beta = 0.7$ - no branching points

(b) $\beta = 0.76976$ - one real branching point

(c) $\beta = 0.79$ - pair of complex conjugated branching points

Fig. 4 Spatial profiles of solution $u$, represented by solid line—analytical solution, and squares—numerical solution, while circles represent ending points of solution support, at different time instances for Model VII with parameters: $a_1 = 1.25$, $a_2 = 1.5$, $a_3 = 2.825$, $b = 1.885$, and $\alpha = 0.6$. There are three cases corresponding to different number of branching points, except $s = 0$, depending on $\beta$.

Fig. 5 Spatial profiles of solution $u$, represented by solid line—analytical solution, and squares—numerical solution, at different time instances for Model VII with parameters: $a_1 = 0.25$, $a_2 = 0.75$, $a_3 = 0.15$, $b = 1.25$, $\alpha = 0.2$, and $\beta = 0.59$, when, except for $s = 0$, there are no other branching points.

Figure 6 presents spatial profiles in another case of model parameters yielding existence of a pair of complex conjugated branching points (apart of $s = 0$) which differ from the ones presented in Fig. 4c, since it seems that peaks are situated at zero, while displacement seems to converge to infinity at the ending point of solution support.
4 Conclusion

Fractional Burgers wave equations, considered as a dimensionless system of equation of motion and strain (10), coupled with the constitutive Burgers models either of the first class (11) or of the second class (12), are solved for the Cauchy initial value problem, and their solutions as a response to the initial Dirac delta displacement with zero initial velocity are qualitatively analyzed through numerical examples. The method of Fourier, with respect to space, and Laplace transform with respect to time is used in order to obtain an analytical solution as a convolution of the solution kernels and initial data. The form of the solution kernel proved to be dependant on model parameters, so that if the parameters yield, except for \( s = 0 \), either no branching points, or one negative real branching point of the Laplace transform of the solution kernel, then the solution kernel takes the form (42), while if, except for \( s = 0 \), the Laplace transform of the solution kernel has a pair of complex conjugated branching points, then the solution kernel takes form (43).

Arising from the solution support properties, in both cases of solution kernel, infinite wave propagation speed is obtained for the first class of Burgers models and finite speed for the second class. Moreover, the obtained wave propagation speed is consistent with the one obtained for the wave equations involving fractional linear models with differentiation orders below one.

A qualitative analysis has shown the dissipative behavior for both classes of Burgers wave equations, as expected from thermodynamically consistent constitutive laws for the viscoelastic body. However, spatial profile shapes differ for the different nature of the branching points. The features of the spatial profiles include the jumps from a finite value of displacement to zero at the ending points of the solution support, as well as profiles that are not expected in wave propagation behavior, like occurrence of the additional peaks and peaks situated at zero.

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A Justification for using the Fourier inversion formula

The solution kernel is obtained by the Fourier and Laplace transforms as (36), and in order to apply the Fourier transform inversion formula (38), condition (40), i.e.,

\[
\frac{\Phi_\sigma(s)}{\Phi_\varepsilon(s)} \left( s^2 + \xi^2 \frac{\Phi_\varepsilon(s)}{\Phi_\sigma(s)} \right) \neq 0, \quad \text{for } \xi \in \mathbb{R}, \ \text{Re } s > 0,
\]

must be fulfilled.
The functions $\Phi_\sigma$ and $\Phi_\epsilon$, given by (33) in the case of the first, or by (34) in the case of the second model class, are never zero for $\Re s > 0$. Namely, it is well known that function $\Phi_\epsilon$, except for $s = 0$, does not have other zeros in the principal Riemann branch $\arg s \in (-\pi, \pi)$, while for function $\Phi_\sigma$ it is proved in [28] that if it has zeros, then they lie in the left complex half-plane.

Therefore, it is left to prove that

$$\psi(s) = s^2 + s^2 \Phi_\sigma(s) \neq 0, \quad \text{for } s \in \Re, \Re s > 0. \quad (45)$$

It is clear that if $s = \rho > 0$, then

$$\psi(\rho) = \rho^2 + \xi^2 \rho^\mu \frac{1 + b \rho^\eta + a_1 \rho^\alpha + a_2 \rho^\beta + a_3 \rho^\gamma}{1 + a_1 \rho^\alpha + a_2 \rho^\beta + a_3 \rho^\gamma} > 0.$$ 

Further, by substituting $s = \rho e^{i\varphi}, \varphi \in (-\frac{\pi}{2}, \frac{\pi}{2})$, into (45) one obtains

$$\text{Im} \psi(\rho, \varphi) = \rho^2 \sin(2\varphi) + \frac{\xi^2 \rho^\mu}{|\Phi_\sigma(\rho, \varphi)|^2} f_\rho(\varphi),$$

with

$$f_\rho(\varphi) = \sin(\mu \varphi) + b \rho^\eta \sin((\mu + \eta) \varphi) + a_1 \rho^\alpha \sin((\mu - \alpha) \varphi) + a_1 \rho^\alpha \sin((\mu + \eta - \alpha) \varphi) + a_2 \rho^\beta \sin((\mu - \beta) \varphi) + a_2 \rho^\beta \sin((\mu + \eta - \beta) \varphi) + a_3 \rho^\gamma \sin((\mu - \gamma) \varphi) + a_3 \rho^\gamma \sin((\mu + \eta - \gamma) \varphi). \quad (46)$$

that will for each fractional Burgers model prove to be strictly positive if $\varphi \in (0, \frac{\pi}{2})$ implying that $\psi$, given by (45) cannot be zero for $\Re s > 0$. Since $\Im \psi(\rho, -\varphi) = -\Im \psi(\rho, \varphi)$, note that $\Im \psi(\rho, \varphi) < 0$ if $\varphi \in (-\frac{\pi}{2}, 0)$.

**Model I** is obtained for $\eta \in \{\alpha, \beta, \gamma\}$, so that function $f_\rho$, given by (46), reads

$$f_\rho(\varphi) = \sin(\mu \varphi) + a_1 \rho^\alpha \sin((\mu - \alpha) \varphi) + a_2 \rho^\beta \sin((\mu - \beta) \varphi) + a_3 \rho^\gamma \sin((\mu - \gamma) \varphi)$$

$$+ a_2 \rho^\beta \sin((\mu - \beta) \varphi) + a_2 \rho^\beta \sin((\mu + \eta - \beta) \varphi) + a_3 \rho^\gamma \sin((\mu - \gamma) \varphi) + a_3 \rho^\gamma \sin((\mu + \eta - \gamma) \varphi). \quad (47)$$

The thermodynamical restrictions (16) imply the positivity of all terms in (47), yielding $f_\rho(\varphi) > 0$ if $\varphi \in (0, \frac{\pi}{2})$.

**Model II** is obtained for $\gamma = 2\alpha$ and $\eta = \alpha$, so that function $f_\rho$, given by (46), reads

$$f_\rho(\varphi) = \sin(\mu \varphi) + a_1 \rho^\alpha \sin((\mu + \alpha) \varphi) + a_1 \rho^\alpha \sin((\mu - \alpha) \varphi) + a_2 \rho^\beta \sin((\mu - \beta) \varphi)$$

$$+ a_2 \rho^\beta \sin((\mu - \beta + \alpha) \varphi) + a_3 \rho^\gamma \sin((\mu - \alpha) \varphi)$$

$$+ a_1 \rho^2 \sin(\mu \varphi) \left( b - \frac{a_3}{a_1} \frac{\sin((\mu - 2\alpha) \varphi))}{\sin(\mu \varphi)} \right). \quad (48)$$

Consider the function $g$ and its first derivative $g'$:

$$g(\varphi) = \frac{\sin(\xi \varphi)}{\sin(\xi \varphi)} \quad \text{and} \quad g'(\varphi) = \frac{\xi \varphi \cos(\xi \varphi) \cos(\xi \varphi)}{\varphi \sin^2(\xi \varphi)} \left( \frac{\tan(\xi \varphi)}{\xi \varphi} - \frac{\tan(\xi \varphi)}{\xi \varphi} \right), \quad (49)$$

on the interval $\varphi \in (0, \frac{\pi}{2})$. Let $0 < \xi < \xi < 1$. Since function $\tan\frac{\xi}{x}$ is monotonically increasing for $x \in (0, \frac{\pi}{2})$, one has $g'(\varphi) > 0, \varphi \in (0, \frac{\pi}{2})$, implying that function $g$ is an increasing function on the same interval and therefore

$$g(\varphi) < g\left(\frac{\pi}{2}\right), \quad \text{for } \varphi \in \left(0, \frac{\pi}{2}\right). \quad (50)$$
The thermodynamical restriction (18) yields $0 < 2\alpha - \mu < \mu < 1$, so that by setting $\zeta = 2\alpha - \mu$ and $\xi = \mu$ in the function $g$ given by (49), using (50) one has
\[
\frac{\sin (2(\alpha - \mu) \phi)}{\sin (\mu \phi)} < \frac{\left| \sin \left( \frac{(\mu - 2\alpha)\pi}{2} \right) \right|}{\sin \left( \frac{2\pi}{2} \right)}.
\]
Therefore, again by (18), one has that $b - \frac{a_1}{a_1} \frac{|\sin((\mu - 2\alpha)\phi)|}{\sin(\mu \phi)} > 0$, which, along with the positivity of all other terms in (51), implies that $f_{\rho}(\phi) > 0$ if $\phi \in \left(0, \frac{\pi}{2}\right)$.

Model III is obtained for $\gamma = \alpha + \beta$ and $\eta = \alpha$, so that the function $f_{\rho}$, given by (46), reads
\[
f_{\rho}(\phi) = \sin (\mu \phi) + b_{\rho} \alpha \sin ((\mu + \alpha) \phi) + a_1 \alpha^\rho \sin ((\mu - \alpha) \phi) + a_1 \beta \rho \sin ((\mu - \beta) \phi) + a_3 \beta \rho^\alpha \sin ((\mu - \beta) \phi) + a_2 \beta \rho^\beta \sin ((\mu - \beta + \alpha) \phi) \left( b - \frac{a_1}{a_2} \frac{|\sin((\mu - \beta - \alpha)\phi)|}{\sin(\mu \phi)} \right).
\]
The thermodynamical restriction (20) yields $0 < \alpha - (\mu - \beta) < \alpha + (\mu - \beta) < 1$, so that by setting $\zeta = \alpha - (\mu - \beta)$ and $\xi = \alpha + (\mu - \beta)$ in the function $g$ given by (49), using (50) one has
\[
\frac{\sin ((\alpha + \beta - \mu) \phi)}{\sin ((\mu + \beta + \alpha) \phi)} < \frac{\left| \sin \left( \frac{\mu - (\alpha - \beta)\pi}{2} \right) \right|}{\sin \left( \frac{\mu + (\alpha + \beta)\pi}{2} \right)}.
\]
Therefore, again by (20), one has that $b - \frac{a_1}{a_1} \frac{|\sin((\mu - (\alpha - \beta)\phi)|}{\sin((\mu + (\alpha + \beta)\phi)} > 0$, which, along with the positivity of all other terms in (51), implies that $f_{\rho}(\phi) > 0$ if $\phi \in \left(0, \frac{\pi}{2}\right)$.

Model IV is obtained for $\gamma = \alpha + \beta$ and $\eta = \beta$, so that the function $f_{\rho}$, given by (46), reads
\[
f_{\rho}(\phi) = \sin (\mu \phi) + b_{\rho} \beta \sin ((\mu + \beta) \phi) + a_1 \beta^\rho \sin ((\mu - \beta) \phi) + a_2 \beta^\beta \sin ((\mu - \beta) \phi) + a_3 \beta^\alpha \sin ((\mu - \beta) \phi) + a_2 \beta \rho^\beta \sin ((\mu - \beta + \alpha) \phi) \left( b - \frac{a_3}{a_1} \frac{|\sin((\mu - (\beta - \alpha)\phi)|}{\sin(\mu \phi)} \right).
\]
The thermodynamical restriction (22) yields $0 < \beta - (\mu - \alpha) < \beta + (\mu - \alpha) < 1$, so that by setting $\zeta = \beta - (\mu - \alpha)$ and $\xi = \beta + (\mu - \alpha)$ in the function $g$ given by (49), using (50) one has
\[
\frac{\sin ((\alpha + \beta - \mu) \phi)}{\sin ((\mu + \beta + \alpha) \phi)} < \frac{\left| \sin \left( \frac{(\mu - \beta)\pi}{2} \right) \right|}{\sin \left( \frac{(\mu - \beta + \alpha)\pi}{2} \right)}.
\]
Therefore, again by (22), one has that $b - \frac{a_1}{a_1} \frac{|\sin((\mu - (\beta - \alpha)\phi)|}{\sin((\mu + (\beta + \alpha)\phi)} > 0$, which, along with the positivity of all other terms in (52), implies that $f_{\rho}(\phi) > 0$ if $\phi \in \left(0, \frac{\pi}{2}\right)$.

Model V is obtained for $\gamma = 2\beta$ and $\eta = \beta$, so that function $f_{\rho}$, given by (46), reads
\[
f_{\rho}(\phi) = \sin (\mu \phi) + b_{\rho} \beta \sin ((\mu + \beta) \phi) + a_1 \beta^\rho \sin ((\mu - \beta) \phi) + a_2 \beta \rho \sin ((\mu - \beta) \phi) + a_3 \beta \rho^\beta \sin ((\mu - \beta) \phi) + a_2 \beta^\beta \sin ((\mu - \beta) \phi) + a_2 \beta \rho^\beta \sin ((\mu - \beta) \phi) \left( b - \frac{a_1}{a_2} \frac{|\sin((\mu - (\beta - \alpha)\phi)|}{\sin(\mu \phi)} \right).
\]
The thermodynamical restriction (24) yields $0 < 2\beta - \mu < \mu < 1$, so that by setting $\zeta = 2\beta - \mu$ and $\xi = \mu$ in function $g$ given by (49), using (50) one has
\[
\frac{\sin ((2\beta - \mu) \phi)}{\sin (\mu \phi)} < \frac{\left| \sin \left( \frac{2\beta\pi}{2} \right) \right|}{\sin \left( \frac{\mu \pi}{2} \right)}.
\]
Therefore, again by (24), one has that $b - \frac{a_1}{a_2} \frac{|\sin((2\beta - \mu)\phi)|}{\sin(\mu \phi)} > 0$, which, along with the positivity of all other terms in (53), implies that $f_{\rho}(\phi) > 0$ if $\phi \in \left(0, \frac{\pi}{2}\right)$. 
Model VI is obtained for \( \gamma = \alpha + \beta, \mu = \beta, \) and \( \eta = \alpha, \) so that function \( f_\rho, \) given by (46), reads
\[
f_\rho (\varphi) = \sin (\beta \varphi) + b \rho^\alpha \sin ((\alpha + \beta) \varphi) + a_1 \rho^\alpha \sin ((\beta - \alpha) \varphi) + a_1 b \rho^{2\alpha} \sin (\beta \varphi) + a_2 \rho^{\alpha+\beta} \sin (\alpha \varphi) \left( b - \frac{a_3}{a_2} \right). \tag{54}\]
The thermodynamical restriction (26) yields \( b - \frac{a_3}{a_2} > 0, \) which, along with the positivity of all other terms in (54), implies that \( f_\rho (\varphi) > 0 \) if \( \varphi \in \left( 0, \frac{\pi}{2} \right) \).

Model VII is obtained for \( \gamma = 2\beta \) and \( \mu = \eta = \beta, \) so that function \( f_\rho, \) given by (46), reads
\[
f_\rho (\varphi) = \sin (\beta \varphi) + b \rho^{2\beta} \sin (2\beta \varphi) + a_1 \rho^\alpha \sin ((\beta - \alpha) \varphi) + a_1 b \rho^{2\alpha} \sin ((2\beta - \alpha) \varphi) + a_2 \rho^{2\beta} \sin (\beta \varphi) \left( b - \frac{a_3}{a_2} \right). \tag{55}\]
The thermodynamical restriction (28) yields \( b - \frac{a_3}{a_2} > 0, \) which, along with the positivity of all other terms in (55), implies that \( f_\rho (\varphi) > 0 \) if \( \varphi \in \left( 0, \frac{\pi}{2} \right) \).

Model VIII is obtained for \( \gamma = 2\alpha, \beta = \mu = \eta = \alpha, a_1 + a_2 = \tilde{a}_1, \) and \( a_3 = \tilde{a}_2, \) so that function \( f_\rho, \) given by (46), reads
\[
f_\rho (\varphi) = \sin (\alpha \varphi) + b \rho^\alpha \sin (2\alpha \varphi) + \tilde{a}_1 \rho^{2\alpha} \sin (\alpha \varphi) \left( b - \frac{\tilde{a}_2}{\tilde{a}_1} \right). \tag{56}\]
The thermodynamical restriction (30) yields \( b - \frac{\tilde{a}_2}{\tilde{a}_1} > 0, \) which, along with the positivity of all other terms in (56), implies that \( f_\rho (\varphi) > 0 \) if \( \varphi \in \left( 0, \frac{\pi}{2} \right) \).

B Calculation of the solution kernel

In order to obtain the Cauchy integral formulas, given by (42) and (43), the inverse Laplace transform (41) will be calculated using the Cauchy integral formula
\[
\oint_\Gamma \tilde{K} (x, s) e^{st} \, ds = 0, \quad x \in \mathbb{R}, \quad t > 0, \tag{57}\]
where \( \Gamma \) is a closed curve containing the Bromwich path \( \Gamma_0 \) from the Laplace inversion formula (41) and chosen differently depending on the number and position of the branching points of function \( \tilde{K}, \) given by (39).

Branching points of function \( \tilde{K} \) are points in which the function under the square root is zero, i.e., in (39) either \( \Phi_\sigma (s) = 0 \) or \( \Phi_\rho (s) = 0, \) \( s \in \mathbb{C}, \) with \( \Phi_\sigma \) and \( \Phi_\rho \) given by (33) in the case of the first or by (34) in the case of the second model class. Function \( \Phi_\rho, \) except for \( s = 0, \) does not have other zeros in the principal Riemann plane \( \arg s \in (-\pi, \pi), \) since
\[
\sum_{i=1}^{N} a_i s^{\alpha_i} \neq 0, \quad s \in \mathbb{C}, \quad a_i \geq 0, \quad a_i \in [0, 1),
\]
as proved in [22]. Zeros of function
\[
\Phi_\sigma (s) = 1 + a_1 s^\alpha + a_2 s^\beta + a_3 s^\gamma, \quad s \in \mathbb{C},
\]
with \( a_1, a_2, a_3 > 0, \alpha, \beta, \gamma \in (0, 1), \) and \( \alpha < \beta < \gamma, \) are analyzed in [28], where it is found that if \( \gamma \in (0, 1), \) then function \( \Phi_\sigma \) has no zeros in the complex plane, which is valid for Model I, while if \( \gamma \in (1, 2), \) then the number and position of zeros of function \( \Phi_\sigma \) is as follows:

- if \( \text{Re} \, \Phi_\sigma (\rho^s) < 0, \) then \( \Phi_\sigma \) has no zeros in the complex plane;
- if \( \text{Re} \, \Phi_\sigma (\rho^s) = 0, \) then \( \Phi_\sigma \) has one negative real zero \( -\rho^s; \)
- if \( \text{Re} \, \Phi_\sigma (\rho^s) > 0, \) then \( \Phi_\sigma \) has a pair of complex conjugated zeros \( s_0 \) and \( \bar{s}_0 \) having negative real part.
where
\[
\text{Re } \Phi_\sigma (\rho^*) = 1 + a_1 (\rho^*)^\alpha \cos (\alpha \pi) + a_2 (\rho^*)^\beta \cos (\beta \pi) + a_3 (\rho^*)^\gamma \cos (\gamma \pi),
\]
with \( \rho^* \) determined from \( \text{Im } \Phi_\sigma (\rho^*) = 0 \), i.e.,
\[
\frac{a_1 \sin (\alpha \pi)}{a_3 |\sin (\gamma \pi)|} + \frac{a_2 \sin (\beta \pi)}{a_3 |\sin (\gamma \pi)|} (\rho^*)^{\beta-\alpha} = (\rho^*)^{\gamma-\alpha},
\]
which is valid for Models II–VII. In the case of Model VIII, zeros of function
\[
\Phi_\sigma (s) = 1 + \tilde{a}_1 s^\alpha + \tilde{a}_2 s^{2\alpha}, \quad s \in \mathbb{C},
\]
are as follows:

- if \( \left( \frac{\tilde{a}_1}{\tilde{a}_2} \right)^2 \geq \frac{1}{\tilde{a}_2} \), or
- if \( \left( \frac{\tilde{a}_1}{\tilde{a}_2} \right)^2 < \frac{1}{\tilde{a}_2} \) and \( \frac{\tilde{a}_1}{\tilde{a}_2} \) is not in the interval \( \left( \frac{\cos(\alpha \pi)}{\sin(\alpha \pi)} \right) \sqrt{\frac{1}{\tilde{a}_2} - \left( \frac{\tilde{a}_1}{\tilde{a}_2} \right)^2} \), then \( \Phi_\sigma \) has no zeros in the complex plane;

- if \( \left( \frac{\tilde{a}_1}{\tilde{a}_2} \right)^2 < \frac{1}{\tilde{a}_2} \) and \( \frac{\tilde{a}_1}{\tilde{a}_2} \) is in the interval \( \left( \frac{\cos(\alpha \pi)}{\sin(\alpha \pi)} \right) \sqrt{\frac{1}{\tilde{a}_2} - \left( \frac{\tilde{a}_1}{\tilde{a}_2} \right)^2} \), then \( \Phi_\sigma \) has one negative real zero \( -\rho^* \);

- if \( \left( \frac{\tilde{a}_1}{\tilde{a}_2} \right)^2 < \frac{1}{\tilde{a}_2} \) and \( \frac{\tilde{a}_1}{\tilde{a}_2} \) is not in the interval \( \left( \frac{\cos(\alpha \pi)}{\sin(\alpha \pi)} \right) \sqrt{\frac{1}{\tilde{a}_2} - \left( \frac{\tilde{a}_1}{\tilde{a}_2} \right)^2} \), then \( \Phi_\sigma \) has a pair of complex conjugated zeros \( s_0 \) and \( \tilde{s}_0 \) having negative real part,

with \( \rho^* \) determined by
\[
\rho^* = \left( \frac{b}{\sin (\alpha \pi)} \right)^{\frac{1}{\alpha}}.
\]

Note that the branching point \( s = 0 \) is due to the differentiation of fractional order and that function \( \tilde{K} \) does not have any singularities other than branching points, justifying the use of the Cauchy integral formula.

**B.1 Case 1**

*Function \( \tilde{K} \), except for \( s = 0 \), has no other branching points*

If function \( \tilde{K} \) (39), except for \( s = 0 \), has no other branching points, then the contour \( \Gamma \) appearing in the Cauchy integral formula (57) is chosen as in Fig. 7 and parametrized as in Table 1.
Table 1 Parametrization of integration contour $\Gamma$

| $\Gamma_0$ | Bromwich path, $s = p + i R$, $p \in [0, p_0]$, $p_0 \geq 0$ arbitrary. |
| $\Gamma_1$ | $s = R e^{\beta \pi}$, $\beta \in \left[\frac{1}{2}, \pi\right]$. |
| $\Gamma_2$ | $s = R e^{\beta \pi}$, $\beta \in [r, R]$. |
| $\Gamma_3$ | $s = \rho e^{\epsilon \pi}$, $\rho \in [-\pi, \pi]$. |
| $\Gamma_4$ | $s = \rho e^{\epsilon \pi}$, $\rho \in [r, R]$. |
| $\Gamma_5$ | $s = \rho e^{\epsilon \pi}$, $\rho \in [-\pi, -\frac{\pi}{2}]$. |
| $\Gamma_6$ | $s = R e^{\beta \pi}$, $p \in [0, p_0]$, $p_0 \geq 0$ arbitrary. |
| $\Gamma_7$ | $s = p - i R$. |

The integrals along contours $\Gamma_3$, $\Gamma_5$, and $\Gamma_0$, calculated as

$$\lim_{R \to \infty} \left| \int_{\Gamma_3} \tilde{K}(x, s) e^{st} ds \right| = \frac{1}{2} \int_0^\infty \left| \frac{\Phi_\sigma(\rho e^{\epsilon \pi})}{\Phi_\epsilon(\rho e^{\epsilon \pi})} \right| \rho^\frac{\mu + \eta - \gamma}{2} e^{-\frac{\mu + \eta - \gamma}{2} \rho^2} e^{-\rho^2} d\rho,$$

(60)

$$\lim_{R \to \infty} \left| \int_{\Gamma_5} \tilde{K}(x, s) e^{st} ds \right| = -\frac{1}{2} \int_0^\infty \left| \frac{\Phi_\sigma(\rho e^{-\epsilon \pi})}{\Phi_\epsilon(\rho e^{-\epsilon \pi})} \right| \rho^\frac{\mu + \eta - \gamma}{2} e^{-\frac{\mu + \eta - \gamma}{2} \rho^2} e^{-\rho^2} d\rho,$$

(61)

$$\lim_{R \to \infty} \left| \int_{\Gamma_0} \tilde{K}(x, s) e^{st} ds \right| = \frac{2\pi i K(x, t)},$$

(62)

yield the solution kernel $K$ in form (42) when used in the Cauchy integral formula (57), since the integrals along all other contours will prove to be zero.

The following estimates will be used. According to (33), respectively (34), after the substitution $s = \rho e^{i\varphi}$ is made, it is obtained that

$$\sqrt{\frac{\Phi_\sigma(s)}{\Phi_\epsilon(s)}} \sim \left\{ \begin{array}{ll}
\frac{\sqrt{\alpha}}{\sqrt{\beta}} \rho^\frac{\mu + \eta - \gamma}{2} e^{-\frac{\mu + \eta - \gamma}{2} \rho^2} & \text{for the first model class,} \\
\frac{\sqrt{\alpha}}{\sqrt{\beta}} e^{-\frac{\mu + \eta - \gamma}{2} \rho^2} & \text{for the second model class,}
\end{array} \right. \quad \text{as } \rho \to \infty,$$

(63)

and therefore

$$\arg \left( \frac{\Phi_\sigma(s)}{\Phi_\epsilon(s)} \right) \sim \left\{ \begin{array}{ll}
\frac{\sqrt{\alpha}}{\sqrt{\beta}} \rho^\frac{\mu + \eta - \gamma}{2} & \text{for the first model class,} \\
0 & \text{for the second model class,}
\end{array} \right. \quad \text{as } \rho \to \infty. \quad (64)$$

The integral along contour $\Gamma_1$ reads

$$\int_{\Gamma_1} \tilde{K}(x, s) e^{st} ds = \frac{1}{2} \int_{p_0}^0 \frac{\Phi_\sigma(p + i R)}{\Phi_\epsilon(p + i R)} e^{-|x|(p + i R)} \frac{\Phi_\sigma(\rho e^{\epsilon \pi})}{\Phi_\epsilon(\rho e^{\epsilon \pi})} \rho^\frac{\mu + \eta - \gamma}{2} e^{-\frac{\mu + \eta - \gamma}{2} \rho^2} e^{-\rho^2} d\rho,$$

and since $p + i R \sim R e^{i\frac{\pi}{2}}$, as $R \to \infty$, one has

$$\lim_{R \to \infty} \left| \int_{\Gamma_1} \tilde{K}(x, s) e^{st} ds \right| \leq \frac{1}{2} \lim_{R \to \infty} \int_{p_0}^0 \frac{\Phi_\sigma(R e^{i \frac{\pi}{2}})}{\Phi_\epsilon(R e^{i \frac{\pi}{2}})} e^{-|x|R} \frac{\Phi_\sigma(\rho e^{\epsilon \pi})}{\Phi_\epsilon(\rho e^{\epsilon \pi})} \rho^\frac{\mu + \eta - \gamma}{2} e^{-\frac{\mu + \eta - \gamma}{2} \rho^2} e^{-\rho^2} d\rho.$$

(65)

The use of (63) and (64) in (65), due to $0 < \frac{\mu + \eta - \gamma}{2} < 1$, yields

$$\lim_{R \to \infty} \left| \int_{\Gamma_1} \tilde{K}(x, s) e^{st} ds \right| \leq \frac{1}{2} \sqrt{\frac{\alpha}{\beta}} \lim_{R \to \infty} \int_{p_0}^0 R^{-\frac{\mu + \eta - \gamma}{2}} e^{-|x|R} \frac{\Phi_\sigma(R e^{i \frac{\pi}{2}})}{\Phi_\epsilon(R e^{i \frac{\pi}{2}})} \rho^\frac{\mu + \eta - \gamma}{2} e^{-\frac{\mu + \eta - \gamma}{2} \rho^2} \cos \left( 1 - \frac{\mu + \eta - \gamma}{2} \right) e^{-\rho^2} d\rho = 0,$$
for the first model class and choosing \( p_0 = 0 \)
\[
\lim_{R \to \infty} \left| \int_{\Gamma_1} \tilde{K}(x, s)e^{st}ds \right| \leq \frac{1}{2} \sqrt{\frac{a_3}{b}} \lim_{R \to \infty} \int_0^{p_0} e^{pt}dp = 0,
\]
for the second model class. Similar argumentation is valid for the integral along \( \Gamma_1 \).

The integral along contour \( \Gamma_2 \) takes the form
\[
\int_{\Gamma_2} \tilde{K}(x, s)e^{st}ds = \frac{1}{2} \int_{\frac{\pi}{2}}^{\pi} \frac{\Phi_\sigma (Re^{i\psi})}{\Phi_\epsilon (Re^{i\psi})} \frac{\sqrt{\frac{\Phi_\sigma (Re^{i\psi})}{\Phi_\epsilon (Re^{i\psi})}}}{\sqrt{\frac{\Phi_\sigma (Re^{i\psi})}{\Phi_\epsilon (Re^{i\psi})}}} e^{Re^{i\psi}t}e^{i\psi}d\psi,
\]
so that
\[
\lim_{R \to \infty} \left| \int_{\Gamma_2} \tilde{K}(x, s)e^{st}ds \right| \leq \frac{1}{2} \lim_{R \to \infty} \int_{\frac{\pi}{2}}^{\pi} R \frac{\Phi_\sigma (Re^{i\psi})}{\Phi_\epsilon (Re^{i\psi})} \left| R \frac{\Phi_\sigma (Re^{i\psi})}{\Phi_\epsilon (Re^{i\psi})} \right| R \left( \cos \psi - |x| \sqrt{\frac{\Phi_\sigma (Re^{i\psi})}{\Phi_\epsilon (Re^{i\psi})}} \cos \left( \psi + \arg \frac{\Phi_\sigma (Re^{i\psi})}{\Phi_\epsilon (Re^{i\psi})} \right) \right) d\psi = 0.
\]

Using (63) and (64) in (66), due to \( 0 < \frac{\mu + \eta - \gamma}{2} < 1 \) and \( \cos \psi < 0 \) for \( \psi \in \left[ \frac{\pi}{2}, \pi \right] \), yields
\[
\lim_{R \to \infty} \left| \int_{\Gamma_2} \tilde{K}(x, s)e^{st}ds \right| \leq \frac{1}{2} \sqrt{\frac{a_3}{b}} \lim_{R \to \infty} \int_{\frac{\pi}{2}}^{\pi} R^{1-\frac{\mu + \eta - \gamma}{2}} e^{Re^{i\psi}t}e^{i\psi}d\psi = 0,
\]
for \( (x, t) \in \mathbb{R} \times [0, \infty) \), in the case of the first model class and
\[
\lim_{R \to \infty} \left| \int_{\Gamma_2} \tilde{K}(x, s)e^{st}ds \right| \leq \frac{1}{2} \sqrt{\frac{a_3}{b}} \lim_{R \to \infty} \int_{\frac{\pi}{2}}^{\pi} R e^{Re^{i\psi}t} \cos \psi d\psi = 0, \quad |x| < \sqrt{\frac{b}{a_3}t},
\]
for the second model class. Similar argumentation is valid for the integral along \( \Gamma_6 \).

The integral along contour \( \Gamma_4 \):
\[
\int_{\Gamma_4} \tilde{K}(x, s)e^{st}ds = \frac{1}{2} \int_{-\pi}^{\pi} \sqrt{\frac{\Phi_\sigma (re^{i\psi})}{\Phi_\epsilon (re^{i\psi})}} e^{-|x|re^{i\psi}} \sqrt{\frac{\Phi_\sigma (re^{i\psi})}{\Phi_\epsilon (re^{i\psi})}} e^{re^{i\psi}i\psi}e^{i\psi}d\psi
\]
tends to zero when \( r \to 0 \), since
\[
\lim_{r \to 0} \left| \int_{\Gamma_4} \tilde{K}(x, s)e^{st}ds \right| \leq \frac{1}{2} \lim_{r \to 0} \int_{-\pi}^{\pi} \frac{\Phi_\sigma (re^{i\psi})}{\Phi_\epsilon (re^{i\psi})} \left| \frac{\Phi_\sigma (re^{i\psi})}{\Phi_\epsilon (re^{i\psi})} \right| e^{r^t \cos \psi} d\psi
\]
\[
\leq \frac{1}{2} \begin{cases} \lim_{r \to 0} \int_{-\pi}^{\pi} \frac{\Phi_\sigma (re^{i\psi})}{\Phi_\epsilon (re^{i\psi})} e^{r^t \cos \psi} d\psi = 0, & \text{for the first model class,} \\ \lim_{r \to 0} \int_{-\pi}^{\pi} \frac{\Phi_\sigma (re^{i\psi})}{\Phi_\epsilon (re^{i\psi})} e^{r^t \cos \psi} d\psi = 0, & \text{for the second model class,} \end{cases}
\]
due to \( \beta, \mu < 1 \) and
\[
\frac{\Phi_\sigma (s)}{\Phi_\epsilon (s)} \sim \begin{cases} r^{-\frac{\mu}{2}}, & \text{for the first model class,} \\ r^{-\frac{\beta}{2}}, & \text{for the second model class,} \end{cases} \quad \text{as } r \to 0,
\]
\[
\arg \frac{\Phi_\sigma (s)}{\Phi_\epsilon (s)} \sim \begin{cases} -\frac{\mu \psi}{2}, & \text{for the first model class,} \\ -\frac{\beta \psi}{2}, & \text{for the second model class,} \end{cases} \quad \text{as } \rho \to \infty.
\]
Fig. 8 Integration contour $\Gamma$

Table 2 Parametrization of integration contour $\Gamma$

| $\Gamma_0$ | Bromwich path, $s = p + iR$, $\rho \in [0, p_0]$, $p_0 \geq 0$ arbitrary, $\psi \in [\frac{\pi}{2}, \pi]$, $\rho = [\rho^* + r, R]$, $\phi, \phi \in [-\pi, \pi]$. |
| $\Gamma_1$ | $s = Re^{i\psi}$, $\rho \in [0, \pi]$, $\phi, \phi \in [0, \pi]$. |
| $\Gamma_3$ | $s = Re^{i\psi}$, $\rho \in [\rho^* + r, R]$, $\phi, \phi \in [-\pi, \frac{\pi}{2}]$. |
| $\Gamma_5$ | $s = Re^{i\psi}$, $\rho \in [0, \pi]$, $\phi, \phi \in [0, \pi]$. |
| $\Gamma_7$ | $s = p - iR$, $\rho \in [0, p_0]$, $p_0 \geq 0$ arbitrary, $\phi, \phi \in [0, \pi]$. |
| $\Gamma_8$ | $s - \rho^*e^{-i\pi} = re^{i\psi}$, $\rho \in [0, \pi]$, $\phi, \phi \in [0, \pi]$. |
| $\Gamma_9$ | $s - \rho^*e^{-i\pi} = re^{i\psi}$, $\rho \in [0, \pi]$, $\phi, \phi \in [0, \pi]$. |

Function $\tilde{K}$, except for $s = 0$, has a negative real branching point

If function $\tilde{K}$ (39), except for $s = 0$, has a negative real branching point $-\rho^*$, determined by (58) or (59), then the contour $\Gamma$ appearing in the Cauchy integral formula (57) is chosen as in Fig. 8 and parametrized as in Table 2.

The integrals along contours $\Gamma_3 \cup \Gamma_5$, $\Gamma_5 \cup \Gamma_7$, and $\Gamma_0$, when $r \to 0$ and $R \to \infty$, are the same integrals as (60), (61), and (62), thus yielding the solution kernel $K$ in form (42) when used in the Cauchy integral formula (57), since the integrals along contours $\Gamma_1$, $\Gamma_2$, $\Gamma_4$, $\Gamma_6$, and $\Gamma_7$ already proved to be zero, while the integrals along $\Gamma_8$ and $\Gamma_9$ will prove to be zero.

Namely, the integral along $\Gamma_8$ reads

$$\int_{\Gamma_8} \tilde{K}(x, s)e^{is} ds = \frac{1}{2} \int_0^\pi \Phi_\sigma(\rho^*e^{i\pi} + re^{i\psi}) \exp\left(-\frac{|s - \rho^*e^{i\pi} + re^{i\psi}|}{\Phi_\sigma(\rho^*e^{i\pi} + re^{i\psi})} \Phi_\epsilon(\rho^*e^{i\pi} + re^{i\psi}) e(\rho^*e^{i\pi} + re^{i\psi}) \int e^{i\psi} d\psi,\right.$$  

so that

$$\lim_{r \to 0} \int_{\Gamma_8} \tilde{K}(x, s)e^{is} ds = \frac{1}{2} e^{-\rho^*} \lim_{r \to 0} \int_0^\pi \Phi_\sigma(\rho^*e^{i\pi} + re^{i\psi}) \exp\left(\frac{|s - \rho^*e^{i\pi} + re^{i\psi}|}{\Phi_\sigma(\rho^*e^{i\pi} + re^{i\psi})} \Phi_\epsilon(\rho^*e^{i\pi} + re^{i\psi}) e(\rho^*e^{i\pi} + re^{i\psi}) \int e^{i\psi} d\psi = 0,$$  

since

$$\lim_{r \to 0} \frac{\Phi_\sigma(\rho^*e^{i\pi} + re^{i\psi})}{\Phi_\epsilon(\rho^*e^{i\pi} + re^{i\psi})} = \frac{\Phi_\sigma(\rho^*e^{i\pi})}{\Phi_\epsilon(\rho^*e^{i\pi})} = 0,$$  

because of $-\rho^*$ being zero of function $\Phi_\sigma$. Similar argumentation is valid for the integral along $\Gamma_9$.  

Fig. 9 Integration contour $\Gamma$

Table 3 Parametrization of integration contour $\Gamma$

| $\Gamma_0$   | $\Gamma_1$   | $\Gamma_2$   | $\Gamma_3$   | $\Gamma_4$   | $\Gamma_5$   |
|--------------|--------------|--------------|--------------|--------------|--------------|
| $s = p + iR$ | $s = R e^{i\phi}$ | $s = e^{i\phi}$ | $s = e^{i\phi}$ | $s = e^{-i\phi}$ | $s = R e^{i\phi}$ |
| $p \in [0, p_0]$ | $\phi \in [\frac{\pi}{2}, \phi_0]$ | $\rho \in [\rho_0 + r, R]$ | $\rho \in [r, \rho_0 - r]$ | $\rho \in [r, \rho_0 - r]$ | $\rho \in [\rho_0 + r, R]$ |

B.2 Case 2

Function $\tilde{K}$, except for $s = 0$, has a pair of complex conjugated branching points

If function $\tilde{K}$, except for $s = 0$, has a pair of complex conjugated branching points with negative real part: $s_0 = \rho_0 e^{i\phi_0}$ and $\tilde{s}_0 = \rho_0 e^{-i\phi_0}$, then the contour $\Gamma$ appearing in the Cauchy integral formula (57) is chosen as in Fig. 9 and parametrized as in Table 3.

The solution kernel $K$ in form (43) is obtained when the integrals along contours $\Gamma_3$ and $\Gamma_5$ are used in the Cauchy integral formula (57), since the integrals along contours $\Gamma_1$, $\Gamma_2$, $\Gamma_4$, $\Gamma_6$, and $\Gamma_7$ already proved to be zero, while the integrals along $\Gamma_8$ and $\Gamma_9$ will prove to be zero.
The integral along $\Gamma_8$ reads

$$\int_{\Gamma_8} \tilde{K}(x, s)e^{st}ds = \frac{1}{2} \int_{\gamma_0}^{-\pi+\psi_0} \frac{\Phi_\sigma(s_0 + re^{i\psi})}{\Phi_\sigma(s_0 + re^{i\psi})} e^{-|x|s_0 + re^{i\psi}} \sqrt{\frac{\Phi_\epsilon(s_0 + re^{i\psi})}{\Phi_\epsilon(s_0 + re^{i\psi})}} i r e^{i\psi}d\psi,$$

so that

$$\lim_{r \to 0} \int_{\Gamma_8} \tilde{K}(x, s)e^{st}ds = \frac{1}{2} e^{st} \lim_{r \to 0} \int_{\gamma_0}^{-\pi+\psi_0} \frac{\Phi_\sigma(s_0 + re^{i\psi})}{\Phi_\sigma(s_0 + re^{i\psi})} e^{-|x|s_0 + re^{i\psi}} \sqrt{\frac{\Phi_\epsilon(s_0 + re^{i\psi})}{\Phi_\epsilon(s_0 + re^{i\psi})}} i r e^{i\psi}d\psi = 0,$$

since

$$\lim_{r \to 0} \frac{\Phi_\sigma(s_0 + re^{i\psi})}{\Phi_\sigma(s_0 + re^{i\psi})} = \frac{\Phi_\sigma(s_0)}{\Phi_\sigma(s_0)} = 0,$$

because of $s_0$ being zero of function $\Phi_\sigma$. Similar argumentation is valid for the integral along $\Gamma_9$.

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