The $\mathbb{Z}/(p)$-equivariant product-isomorphism
in fixed point Floer cohomology

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Abstract

Let $p \geq 2$ be a prime, and $\mathbb{F}_p$ be the field with $p$ elements. Extending a result of Seidel for $p = 2$, we construct an isomorphism between the Floer cohomology of an exact or Hamiltonian symplectomorphism $\phi$, with $\mathbb{F}_p$ coefficients, and the $\mathbb{Z}/(p)$-equivariant Tate Floer cohomology of its $p$-th power $\phi^p$. The construction involves a Kaledin-type quasi-Frobenius map, as well as a $\mathbb{Z}/(p)$-equivariant pants product: an equivariant operation with $p$ inputs and 1 output. Our method of proof involves a spectral sequence for the action filtration, and a local $\mathbb{Z}/p\mathbb{Z}$-equivariant coproduct providing an inverse on the $E^2$-page. This strategy has the advantage of accurately describing the effect of the isomorphism on filtration levels. We describe applications to the symplectic mapping class group, as well as develop Smith theory for persistence modules of Hamiltonian diffeomorphisms on symplectically aspherical symplectic manifolds. We illustrate the latter by giving a new proof of the celebrated no-torsion theorem of Polterovich. Along the way, we prove a sharpening of the classical Smith inequality for actions of $\mathbb{Z}/p\mathbb{Z}$.

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1 Introduction

Equivariant cohomology with respect to the action of the cyclic group of order $p$, and the resulting Smith-type inequalities (see [5,6,20] for the classical theory) have been obtained and used to great effect in various Floer theories in recent years, see for example [22,24,38,40,44]. In the case of symplectic fixed point Floer cohomology $HF^*(\phi)$ of a symplectic automorphism $\phi$ of a Liouville manifold, and the $\mathbb{Z}/2\mathbb{Z}$-equivariant cohomology of its second iterate $\phi^2$, it was most recently studied by Seidel by means of a remarkable cohomological operation coming from the $\mathbb{Z}/2\mathbb{Z}$ symmetry of the pair of pants with boundary conditions given by $\phi$ on the two input cylinders, and by $\phi^2$ on its output cylinder. In particular, he proves, under suitable assumptions that the analogue of the Smith inequality

$$\dim_{\mathbb{F}_2} HF^*(\phi) \leq \dim_{\mathbb{F}_2} HF^*(\phi^2)_{\mathbb{Z}/2\mathbb{Z}} \leq \dim_{\mathbb{F}_2} HF^*(\phi^2)$$

holds true for fixed point Floer cohomology.

In this paper, we extend the work of Seidel to all primes $p \geq 2$, to fixed point Floer cohomology in an action window $I = (a,b)$, where $a < b$ are generic values in $\mathbb{R} \cup \{\pm \infty\}$,
and to local Floer cohomology. In the latter case, a Smith-type inequality was obtained by more elementary methods in [47] during the preparation of this paper. We note that the case of action windows, and of local Floer homology, is of interest already for Hamiltonian diffeomorphisms of symplectically aspherical symplectic manifolds.

We record a sample application. Let \( \phi \) be a Hamiltonian diffeomorphism of a symplectically aspherical symplectic manifold, and \( \phi^p \) its \( p \)-th iterate. Assume that \( p \cdot a, p \cdot b \) are not critical values of the Hamiltonian action functional corresponding to \( \phi^p \) and the free homotopy class of contractible loops. Then the Floer cohomology of \( \phi \) in interval \( I = (a, b) \) and that of \( \phi^p \) in interval \( p \cdot I = (p \cdot a, p \cdot b) \) are related by the following Smith-type inequality:

\[
\dim_{\mathbb{F}_p} HF^*(\phi)^I \leq \dim_{\mathbb{F}_p} (HF^*(\phi^p)^{p \cdot I})_{\mathbb{Z}/p\mathbb{Z}} \leq \dim_{\mathbb{F}_p} HF^*(\phi^p)^{p \cdot I}.
\]

(2)

We remark that these cohomology groups are defined by perturbing \( \phi \) by a sufficiently \( C^2 \)-small Hamiltonian diffeomorphism to \( \phi_1 \), and using the fact that the endpoints of the interval are not in the spectrum. In this case, we can choose a perturbation \( \phi_1 \) so that \( \phi_1^p \) is a sufficiently \( C^2 \)-small Hamiltonian perturbation of \( \phi^p \).

There are two essentially immediate consequences of the above Smith-type inequalities \((1)\) and \((2)\). Similarly to the results obtained by Hendricks [22] and Seidel [38] in the case of \( p = 2 \), one first obtains the following corollary, that is proven as Corollary 17 below, for the \( p \)-th iterates of \( \phi \) in the symplectic mapping class group.

**Corollary 1.** Given an exact symplectic manifold \( W \) which is cylindrical at infinity, and a compactly supported exact symplectomorphism \( \phi \), if \( \dim_{\mathbb{F}_p} HF^*(\phi) > \dim_{\mathbb{F}_p} H^*(W) \), then \( [\phi^k] \neq 1 \) in the symplectic mapping class group of \( W \) for all \( k \geq 0 \).

Furthermore, with the help of the action-filtered version of the Smith-type inequality \((2)\), we also provide a new proof of a well-known theorem of Polterovich [30], stating that the group of Hamiltonian diffeomorphisms of a closed symplectically aspherical symplectic manifold contains no non-trivial torsion elements.

**Corollary 2** (Polterovich [30]). Let \( \phi \in \text{Ham}(M, \omega) \) be a Hamiltonian diffeomorphism of a symplectically aspherical symplectic manifold, such that \( \phi^k = 1 \) for some \( k \in \mathbb{Z}_{\geq 1} \). Then \( \phi = \text{id} \).

To prove our main theorem, we use a cohomological operation coming from a branched cover of a cylinder with \( p \) inputs and 1 output, and its \( \mathbb{Z}/p\mathbb{Z} \)-symmetry, as in [38] for \( p = 2 \). However, showing that this \( \mathbb{Z}/p\mathbb{Z} \)-equivariant product map is an isomorphism on the associated Tate cohomology groups requires substantially more complicated tools. Indeed, while for \( p = 2 \) the local contributions can be deduced from a few special cases, including the period-doubling bifurcation, as discussed in [38, Section 6], for \( p > 2 \) a more refined approach was found to be necessary. We proceed by providing a local inverse map for the product in terms of an equivariant coproduct operation with 1
input and $p$ outputs, in the spirit of [38] Remark 6.10]. It is curious to note that the classical Wilson theorem from number theory ultimately plays an important role in the calculation leading to local invertibility. The local-to-global argument then proceeds by the use of the action-filtration spectral sequence.

We record our main result, from which inequality (2) follows purely algebraically. In fact a stronger inequality follows (see Remark 16). We refer to the Sections below, for a detailed definition of all the notions involved in it.

**Theorem A.** Let $\phi$ be an exact symplectic automorphism of a Liouville domain, or a Hamiltonian diffeomorphism of a closed symplectically aspherical symplectic manifold. For a generic interval $I = (a, b)$ with $a < b$, $a, b \in \mathbb{R} \cup \{\pm \infty\}$, and a prime $p \geq 2$, working with coefficients in $\mathbb{F}_p$, there exists a quasi-Frobenius isomorphism of Tate cohomology groups

$$F : \hat{H}^*(\mathbb{Z}/p\mathbb{Z}, HF^*(\phi)^I) \to \hat{H}^*(\mathbb{Z}/p\mathbb{Z}, (CF^*(\phi)^{\otimes p})^p)$$

and a natural map between $\mathbb{Z}/p\mathbb{Z}$ group cohomology and the $\mathbb{Z}/p\mathbb{Z}$-equivariant Floer cohomology group

$$\mathcal{P} : H^*(\mathbb{Z}/p\mathbb{Z}, (CF^*(\phi)^{\otimes p}))^p \to HF_{\mathbb{Z}/p\mathbb{Z}}^*(\phi^p)^p$$

that becomes an isomorphism of Tate cohomology groups

$$\mathcal{P} : \hat{H}^*(\mathbb{Z}/p\mathbb{Z}, (CF^*(\phi)^{\otimes p}))^p \to \hat{HF}_{\mathbb{Z}/p\mathbb{Z}}^*(\phi^p)^p$$

after tensoring with $\mathbb{F}_p((u))$ over $\mathbb{F}_p[[u]]$. Here

$$\hat{H}^*(\mathbb{Z}/p\mathbb{Z}, HF^*(\phi)^I)^{(1)} \cong HF^*(\phi)^I \otimes_{\mathbb{F}_p} \mathbb{F}_p((u)) \langle \theta \rangle,$$

where $u$ and $\theta$ are formal variables, which can be thought to be of degree 2 and 1, $\mathbb{F}_p((u)) = \mathbb{F}_p[u^{-1}, u]$ denotes the formal Laurent power series in $u$, and $\langle \theta \rangle$ denotes an exterior algebra on $\theta$, and the superscript $(1)$ denotes the Tate twist.

We observe that the composition

$$\mathcal{P} \circ F : HF^*(\phi)^I \otimes_{\mathbb{F}_p} \mathbb{F}_p((u)) \langle \theta \rangle \to \hat{HF}_{\mathbb{Z}/p\mathbb{Z}}^*(\phi^p)^p$$

is in particular an isomorphism of $\mathbb{F}_p((u))$ vector spaces. Studying their dimensions, we deduce (2). Since the case of $p = 2$ amounts essentially to a repetition, with perhaps a very slight extension, of results of [38], for brevity we omit the discussion of this case. However we note that the methods used in this paper do apply for $p = 2$, and we obtain a slightly stronger result, in view of the inequality (115).

We add that a generalization of a part of the results of this paper to the monotone case, as well as further applications to dynamics, are forthcoming.


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2 Group cohomology and Tate cohomology

In this section, we recall the preliminary definitions of group cohomology and Tate cohomology for vector spaces and cochain complex endowed with $G = \mathbb{Z}/p\mathbb{Z}$ actions.

For a fixed prime $p$, we let $K$ be a field of characteristic $p$ and consider $\mathbb{Z}/2\mathbb{Z}$- or $\mathbb{Z}$-graded vector spaces or cochain complexes $V^*$ defined over a commutative ring $K$.

In this paper, we mainly consider the case when $K = \mathbb{F}_p$.

An action of the cyclic group $G = \mathbb{Z}/p\mathbb{Z}$ on a (graded) vector space $V$ is given by a (degree-preserving) linear transformation $\sigma: V \to V$ such that $\sigma^p = id$. Alternatively, the $G$-action on $V$ is equivalent to a (graded) $K\mathbb{[}G\mathbb{]}$-module structure on $V$. Given such a $G$-action, the $G$-invariants and the $G$-coinvariants are defined as follows

$$V^G = \{x \in V \mid g \cdot x = x \text{ for all } g \in G\}$$

$$V_G = V/\text{Im}(1-\sigma),$$

where $\text{Im}(1-\sigma) := (1-\sigma)V$ is the $K\mathbb{[}G\mathbb{]}$-submodule generated by $g \cdot x$ for $g = 1-\sigma$ and $x \in V$. The group homology and cohomology can be then defined as the (derived) $G$-invariants and $G$-coinvariants of the $G$-action

$$H_*(G; V) := \text{Tor}^*_{K\mathbb{[}G\mathbb{]}}(K, V);$$

$$H^*(G; V) := \text{Ext}^*_{K\mathbb{[}G\mathbb{]}}(K, V).$$

To compute these (co)homology groups explicitly for cyclic groups $G := \mathbb{Z}/p\mathbb{Z}$, one takes the free resolution of the ground field $K$ as a $K\mathbb{[}G\mathbb{]}$-module given by

$$\cdots \xrightarrow{\epsilon} K[G] \xrightarrow{1-\sigma} K[G] \xrightarrow{N} K[G] \xrightarrow{1-\sigma} K[G] \xrightarrow{\epsilon} K \to 0,$$

where $\epsilon(\sum_i a_i g_i) = \sum_i a_i$ is the augmentation map, and $N = id + \sigma + \sigma^2 + \cdots + \sigma^{p-1}$ is the norm map of the $G$-action. Let $P^\bullet := (\cdots \xrightarrow{N} K[G] \xrightarrow{1-\sigma} K[G] \xrightarrow{N} K[G] \xrightarrow{1-\sigma} K[G])$. The group homology and cohomology can be computed explicitly as the homology of $P^\bullet \otimes_{K[G]} V$ and $\text{Hom}_{K[G]}(P^\bullet, V).$
One the other hand, the Tate invariants and coinvariants are the kernels and cokernels of the norm map $N$,

$$
\hat{H}_0(G; V) = \text{Ker}(N)/\text{Im}(1-\sigma), \quad \hat{H}^0(G; V) = \text{Ker}(1-\sigma)/\text{Im}(N).
$$

Extending the previous free resolution two-periodically, one obtains the Tate resolution

$$
Q^\bullet := (\cdots \xrightarrow{N} \mathbb{K}[G] \xrightarrow{1-\sigma} \mathbb{K}[G] \xrightarrow{N} \mathbb{K}[G] \xrightarrow{1-\sigma} \mathbb{K}[G] \to \cdots).
$$

The Tate homology and cohomology are defined to be the (derived) Tate invariants and coinvariants,

$$
\hat{H}_i(G; V) = H_i(Q^\bullet \otimes_{\mathbb{K}[G]} V)
$$

$$
\hat{H}^i(G; V) = H_i(Hom_{\mathbb{K}[G]}(Q^\bullet, V)) \cong H_i((Q^\bullet)^{\vee} \otimes_{\mathbb{K}[G]} V),
$$

where $(Q^\bullet)^{\vee} := Hom_{\mathbb{K}[G]}(Q^\bullet, \mathbb{K}[G])$ is the dual resolution.

For a cochain complex $(V, d_V)$ of $\mathbb{K}[G]$-modules, the Tate (co)homology can be defined alternatively as the homology of the Tate (co)chain complex as

$$
\hat{C}_k(\mathbb{Z}/p\mathbb{Z}; V) = \bigoplus_{i+j=k} Q^i \otimes V^j, \quad \hat{d} = d + (-1)^i d_V,
$$

$$
\hat{C}^k(\mathbb{Z}/p\mathbb{Z}; V) = \bigoplus_{i+j=k} (Q^{\vee})^i \otimes V^j, \quad \hat{d} = d + (-1)^i d_V,
$$

where $d = d_0 := N = 1 + \sigma + \cdots + \sigma^{p-1}$ if $i$ is even and $d = d_1 := 1 - \sigma$ if $i$ is odd on $\hat{C}_k(\mathbb{Z}/p\mathbb{Z}; V)$. Whereas on $\hat{C}^k(\mathbb{Z}/p\mathbb{Z}; V)$ we have that $d^\vee = (d_0)^\vee = 1 - \sigma$ for $i$ even and $d^\vee = (d_1)^\vee = N$ for $i$ odd. One can therefore rewrite the Tate homology complex with coefficients in a cochain complex $(V, d_V)$ as

$$
(V \otimes_{\mathbb{K}} \mathbb{K}((u))/[\theta]/\theta^2, \hat{d}),
$$

where the differential $\hat{d}$ on each direct summand $V \otimes_{\mathbb{K}} 1$ and $V \otimes_{\mathbb{K}} \theta$ are

$$
\hat{d}(x \otimes 1) = \begin{pmatrix} d_V & 0 \\ N & d_V \end{pmatrix}, \quad \hat{d}(x \otimes \theta) = \begin{pmatrix} -d_V & u^{-1}(1-\sigma) \\ 0 & -d_V \end{pmatrix}.
$$

Similarly, the Tate cohomology complex with coefficients in $(V, d_V)$ is defined as the complex $V \otimes_{\mathbb{K}} \mathbb{K}((u))/[\theta]/\theta^2$ with the differential

$$
\hat{d}(x \otimes 1) = \begin{pmatrix} d_V & 0 \\ 1 - \sigma & d_V \end{pmatrix}, \quad \hat{d}(x \otimes \theta) = \begin{pmatrix} -d_V & uN \\ 0 & -d_V \end{pmatrix}.
$$

The Tate homology $\hat{H}_*(G; V)$ and cohomology $\hat{H}^*(G; V)$ are (co)homology of the above complexes respectively. In this paper, we mainly use Tate cohomology and we prove its functionality in the following lemma.
Lemma 3. Let \((V,d_V)\) and \((W,d_W)\) be cochain complexes over \(\mathbb{K}\) equipped with \(G\)-actions.

1. Suppose that \(H^\ast(V) \cong 0\), then the group and Tate cohomology groups are also zero
   \[ H^\ast(G;V) \cong 0 \text{ and } \hat{H}^\ast(G;V) \cong 0; \]

2. Suppose that there is a \(G\)-equivariant chain map \(f: (V,d_V) \to (W,d_W)\) that induces a quasi-isomorphism \(f^\ast: H^\ast(V) \xrightarrow{\cong} H^\ast(W)\), then one has
   \[ f^\ast: H^\ast(G;V) \xrightarrow{\cong} H^\ast(G;W) \text{ and } f^\ast: \hat{H}^\ast(G;V) \xrightarrow{\cong} \hat{H}^\ast(G;W). \]

3. Given a short exact sequence of cochain complexes and \(G\)-equivariant maps between them
   \[ 0 \to V_1 \to V_2 \to V_3 \to 0, \]
   there is an induced long exact sequence in group or Tate cohomology groups
   \[ \cdots \to H^\ast(G;V_1) \to H^\ast(G;V_2) \to H^\ast(G;V_3) \to H^{\ast+1}(G;V_1) \to \cdots \]
   \[ \cdots \to \hat{H}^\ast(G;V_1) \to \hat{H}^\ast(G;V_2) \to \hat{H}^\ast(G;V_3) \to \hat{H}^{\ast+1}(G;V_1) \to \cdots \]

Proof. For (1), one defines the zero chain maps \(F: C^\ast(G;V) \to 0\) and \(\hat{F}: \hat{C}^\ast(G;V) \to 0\). There are vertical filtrations on \(C^\ast(G;V)\) and \(\hat{C}^\ast(G;V)\) defined by
   \[ F^k C^\ast(G;V) = \bigoplus_{i \geq k,j} P^i \otimes_{R[G]} V^j \text{ and } \hat{F}^k \hat{C}^\ast(G;V) = \bigoplus_{i \geq k,j} Q^i \otimes_{R[G]} V^j \]

The condition \(H^\ast(V) \cong 0\) implies that the chain maps \(F\) and \(\hat{F}\) induces a quasi-isomorphisms on the associated spectral sequences which converging to \(H^\ast(G;V)\), \(\hat{H}^\ast(G;V)\) and zero group respectively. By comparison of spectral sequences, one obtains the result.

The proof of (2) follows similarly from the fact that the map of spectral sequences associated to the vertical filtrations on \(C^\ast(G;V)\) and \(C^\ast(G;W)\) induces a quasi-isomorphism on \(E^1\)-pages. 

Note that Lemma 3 implies that for a chain complex \(V\) over a field \(\mathbb{K}\), \(H^\ast(G,V^{\otimes p}) \cong H^\ast(G,H(V)^{\otimes p})\), where the action on \(V^{\otimes p}\) is by cyclically permuting the factors with suitable signs defined in \([13]\), and similarly for \(\hat{H}(V)^{\otimes p}\). Indeed as \(V\) is quasi-isomorphic to \(H(V)\), \(V^{\otimes p}\) is \(G\)-equivariantly quasi-isomorphic to \(H(V)^{\otimes p}\).
3 Quasi-Frobenius maps

Let \((V, d)\) be a graded chain complex over a perfect field \(K\) of characteristic \(p\). The Tate twist \(V^{(1)}\) of \(V\) is defined as a vector space over \(K\) with a different action of \(K\) by

\[
a: K \times V^{(1)} \rightarrow V^{(1)}, \quad a(k, x) = k^p \cdot x, \tag{17}
\]

where \(\cdot\) is the original action of \(K\) on \(V\). Having defined the Tate twist \(V^{(1)}\), one considers the Tate complexes associated to the trivial \(\mathbb{Z}/p\mathbb{Z}\)-action on \(V^{(1)}\) and the \(\mathbb{Z}/p\mathbb{Z}\) action \(V^{\otimes p}\) given by

\[
\sigma \cdot x_0 \otimes \cdots \otimes x_{p-1} = (-1)^{|x_{p-1}|(1+\cdots+p-2)} x_{p-1} \otimes x_0 \otimes \cdots \otimes x_{p-2}. \tag{18}
\]

There is an induced Tate map, or a quasi-Frobenius map on the associated Tate complexes

\[
F: \hat{C}^*(\mathbb{Z}/p\mathbb{Z}, V^{(1)}) \rightarrow \hat{C}^*(\mathbb{Z}/p\mathbb{Z}, V^{\otimes p}), \quad x \mapsto x^{\otimes p}. \tag{19}
\]

The name of the quasi-Frobenius map is originated the study of the non-commutative analogue of the Frobenius map \(a \mapsto a^p\) for associative algebras \([21]\), Section 4. At the first glance, the morphism \(F\) is not a chain map, because it fails to be additive. However, we will prove that this map descends to an isomorphism of \(K\)-modules in homology.

**Lemma 4.** Let \((V, d)\) be a graded cochain complex over a field \(K\) of characteristic \(p\), there is an isomorphism of \(K[[u]](\theta)\)-modules

\[
F: \hat{H}^*(\mathbb{Z}/p\mathbb{Z}, V^{(1)}) \rightarrow \hat{H}^*(\mathbb{Z}/p\mathbb{Z}, V^{\otimes p})
\]

**Proof.** We will prove the case when \((V, d)\) has trivial differential, and the general case follows from Lemma 3 above. For a graded vector space \(V\) over \(K\), as the \(\mathbb{Z}/p\mathbb{Z}\)-action on \(V^{(1)}\) is trivial, it amounts to prove that for each degree \(i\), there is an isomorphism of \(K\)-modules

\[
V^{(1)} \cong \hat{H}^i(\mathbb{Z}/p\mathbb{Z}, V^{\otimes p}).
\]

Since for the Tate twist \(V^{(1)}\), the quasi-Frobenius map \(F\) is \(K\)-linear, it suffices to check that \(\text{Im}(F) \subset \text{Ker}(\hat{d})\) and it descends to an additive isomorphism in homology. First, one observes that \(\text{Im}(F) \subset \text{Ker}(\hat{d})\). If a basis \(\{v_i\}_{i=0}^{n-1}\) of \(V^{(1)}\) as a \(K\)-module is given, then the basis of \(V^{\otimes p}\) is of the form \(\{v_f = v_{f(0)} \otimes \cdots \otimes v_{f(p-1)} \mid f: \mathbb{Z}/p\mathbb{Z} \rightarrow \mathbb{Z}, f(1) \neq 0\}\). Denote by \([f]\) the equivalence class of \(f\) under the quotient \(F(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z})\) of \(\{f: \mathbb{Z}/p\mathbb{Z} \rightarrow \mathbb{Z}\}\) by \(\mathbb{Z}/p\mathbb{Z}\)-action of cyclically permuting the inputs. For \(i\) even, one has that

\[
\sum_{f: \mathbb{Z}/p\mathbb{Z} \rightarrow \mathbb{Z}, f \text{ is non-constant}} C_f \cdot v_{f(0)} \otimes \cdots \otimes v_{f(p-1)} \in \text{Ker}(1 - \sigma) \implies \sum_{f: \mathbb{Z}/p\mathbb{Z} \rightarrow \mathbb{Z}, f \text{ is non-constant}} C_f \cdot v_{f(0)} \otimes \cdots \otimes v_{f(p-1)} = N \left( \sum_{f \in F(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}) \setminus \text{f is non-constant}} C_f \cdot v_{f(0)} \otimes \cdots \otimes v_{f(p-1)} \right).
\]
The condition \( f \) is nonconstant is required, as \( v^{\otimes p} \in \text{Ker}(1 - \sigma) \) and \( v^{\otimes p} \notin \text{Im}(N) \). To see this holds similarly for \( i \) odd, one notices that \( (1 - \sigma)^{p-1} = N \) over a field of characteristic \( p \). This implies that on \( V^{\otimes p}_{\text{non-const}} = \{v \mid f : \mathbb{Z}/p\mathbb{Z} \to \mathbb{R} \text{ is non-constant}\} \), \( \text{Ker}(N) \cong \text{Im}(1 - \sigma) \) and \( \text{Im}(N) \cong \text{Ker}(1 - \sigma) \), and the proof when \( i \) is odd follows from the case when \( i \) is even. This shows that the quasi-Frobenius map \( F \) becomes a \( \mathbb{K} \)-linear map in homology and \( V^{(1)} \cong \hat{H}^i(\mathbb{Z}/p\mathbb{Z}; V) \) for all \( i \).

\[ \square \]

### 4 Morse functions on classifying spaces

Let us consider the Hilbert space

\[ \mathcal{H} := \{z = \langle z_k \rangle_{k \in \mathbb{Z}} \mid z_k \in \mathbb{C}, \sum_{k \in \mathbb{Z}} (1 + k^2)|z_k|^2 < \infty\}. \] (20)

There is a free action of \( \mathbb{Z}/p\mathbb{Z} \) on the subspace \( S^\infty := \{z \in \mathcal{H} \mid ||z||^2 := \sum_{k \in \mathbb{Z}} |z_k|^2 = 1\} \) given by

\[ (m \cdot z)_k = e^{2\pi im/p} \cdot z_k \text{ for } m \in \mathbb{Z}/p\mathbb{Z}. \] (21)

The standard Morse function

\[ f : \mathcal{H} \to \mathbb{R}, \quad f(z) := \sum_k k \cdot |z_k|^2, \] (22)

which is invariant under this action and descends to a Morse-Bott function on \( B\mathbb{Z}/p\mathbb{Z} := S^\infty /\mathbb{Z}/p\mathbb{Z} \) with critical submanifolds being the \( S^1 \)-fibers of the fibration \( \pi : B\mathbb{Z}/p\mathbb{Z} \to BS^1 \). For each \( i \in \mathbb{Z} \) the critical submanifold are precisely

\[ S^1_i = \{[\langle z_k \rangle_{k \in \mathbb{Z}}] \mid |z_i| = 1, z_j = 0, j \neq i\}. \] (23)

Note that the coindex of each critical submanifold is infinite, while its index is finite. There is an embedding of \( S^\infty \) into itself such that \( \tau^* f = f + 1 \) defined by

\[ \tau(z_0, z_1, z_2, \cdots) = (0, z_0, z_1, \cdots). \] (24)

This map is compatible with the \( \mathbb{Z}/p\mathbb{Z} \)-action on \( S^\infty \), which yields an automorphism of \( B\mathbb{Z}/p\mathbb{Z} \) that sends the critical submanifold \( S^1_i \) to \( S^1_{i+1} \). By choosing a small perturbation of the Morse-Bott function \( f \) near each critical submanifold \( S^1_i \), one obtains a \( \mathbb{Z}/p\mathbb{Z} \)-perfect Morse function \( F \) on \( B\mathbb{Z}/p\mathbb{Z} \). As the symmetry group \( \mathbb{Z}/p\mathbb{Z} \) is discrete, one can lift the perfect Morse function \( F \) to a \( \mathbb{Z}/p\mathbb{Z} \)-invariant Morse function \( \tilde{F} \) on \( S^\infty \) such that there are exactly \( p \) critical points of index \( i \) on \( S^\infty \), denoted as \( Z^0_i, \ldots, Z^{p-1}_i \), lying over each critical point of index \( i \) of \( F \). One can also choose a Riemmanian metric \( g \) on \( B\mathbb{Z}/p\mathbb{Z} \) and lift it to a \( \mathbb{Z}/p\mathbb{Z} \)-invariant metric \( \tilde{g} \) on \( S^\infty \). Due to the \( \mathbb{Z}/p\mathbb{Z} \)-invariance of the Morse function \( F \) and the Riemannian metric \( \tilde{g} \), there is a \( \mathbb{Z}/p\mathbb{Z} \)-action the critical
points of index $i$, given by $m \cdot Z^i \mapsto Z^{i+m} \mod p$, making each cochain group of degree $i$ into a free $\mathbb{Z}/p\mathbb{Z}$-module of rank 1. Under this identification, the Morse cochain complex of $\tilde{F}$ can be written as

$$0 \to \mathbb{F}_p[\mathbb{Z}/p\mathbb{Z}] \xrightarrow{1-\sigma} \mathbb{F}_p[\mathbb{Z}/p\mathbb{Z}] \xrightarrow{\eta} \mathbb{F}_p[\mathbb{Z}/p\mathbb{Z}] \xrightarrow{1-\sigma} \mathbb{F}_p[\mathbb{Z}/p\mathbb{Z}] \xrightarrow{\eta} \cdots$$  \hfill (25)

where $\mathbb{F}_p[\mathbb{Z}/p\mathbb{Z}]$ is the group ring of $\mathbb{Z}/p\mathbb{Z}$ with coefficients in $K = \mathbb{F}_p$, and $\sigma$ is the action of $1 \in \mathbb{Z}/p\mathbb{Z}$. The homology of this complex is hence $\mathbb{F}_p$ in degree 0, and vanishes in all other degrees. The Morse complex of $F$ is given by tensoring by $\mathbb{F}_p$ over $\mathbb{F}_p[\mathbb{Z}/p\mathbb{Z}]$:

$$0 \to \mathbb{F}_p \xrightarrow{0} \mathbb{F}_p \xrightarrow{0} \mathbb{F}_p[\mathbb{Z}/p\mathbb{Z}] \xrightarrow{0} \mathbb{F}_p[\mathbb{Z}/p\mathbb{Z}] \xrightarrow{0} \cdots$$ \hfill (26)

The cohomology ring can be identified as $R_p = \mathbb{F}_p[[u]] \langle \theta \rangle$, for a formal degree 2 variable $u$ and formal degree 1 variable $\theta$, so $\theta^2 = 0$. Here $u$ corresponds to the generator of $H^*(BS^1, \mathbb{F}_p)$ under $\pi^*$ and $\theta$ corresponds to the generator of $H^*(S^1, \mathbb{F}_p)$.

For the rest of the paper we set $R_p = \mathbb{K}[[u]] \langle \theta \rangle$ for the cohomology of $B(\mathbb{Z}/p\mathbb{Z})$, and $\hat{R}_p = \mathbb{K}((u)) \langle \theta \rangle = R_p \otimes_{\mathbb{K}[[u]]} \mathbb{K}((u))$ for its Tate version. Note that $\hat{R}_p$ is a vector space of dimension 2 over $K = \mathbb{K}((u))$.

We denote by $\mathcal{P}^{i,m}_0$ the moduli space of parametrized flow lines $w: \mathbb{R} \to S^\infty$ satisfying

$$\partial_s w(s) + \nabla \tilde{F}(w) = 0 \quad \hfill (27)$$

$$\lim_{s \to \infty} w(s) = Z_i^m, \quad \text{and} \quad \lim_{s \to -\infty} w(s) = Z_0^0.$$  

Similarly, we denote by $\mathcal{P}^{i,m}_1$ the parametrized flow lines that satisfy (27) and have asymptotic behaviors $\lim_{s \to \infty} w(s) = Z_i^m$ and $\lim_{s \to -\infty} w(s) = Z_1^0$. There are free $\mathbb{R}$ actions on $\mathcal{P}^{i,m}_0$, $\mathcal{P}^{i,m}_1$ defined by translations

$$r \cdot w(s) \mapsto w(s + r).$$ \hfill (28)

Their quotients are defined as $\mathcal{Q}^{i,m}_0 = \mathcal{P}^{i,m}_0/\mathbb{R}$ and $\mathcal{Q}^{i,m}_1 = \mathcal{P}^{i,m}_1/\mathbb{R}$ respectively.

The standard compactification of the moduli space of unparametrized Morse flow lines, together with the $\tau$-invariance, provides that

$$\overline{\mathcal{Q}}^{i,m}_{\alpha_0} = \bigsqcup \mathcal{Q}^{i_1,m_1}_{\alpha_1} \times \mathcal{Q}^{i_2,m_2}_{\alpha_2} \times \cdots \times \mathcal{Q}^{i_n,m_n}_{\alpha_n}.$$ \hfill (29)

where the union is taken over all tuples of triples $(i_1, \alpha_1, m_1), \ldots, (i_n, \alpha_n, m_n)$ such that $m_1 + m_2 + \ldots + m_n = m$ in $\mathbb{Z}/p\mathbb{Z}$, $\alpha_1 = \alpha_0$, $i_n = i$ (mod 2), and $\sum_{j=1}^n (i_j - \alpha_j) = i - \alpha_0$.

A parametrized flow line of the Morse function $\tilde{F}$ on $S^\infty$ is a unparametrized flow line of the Morse function $\Psi + \tilde{F}$ on $\mathbb{R} \times S^\infty$, where $\Psi: \mathbb{R} \to \mathbb{R}$ is a Morse function.
which has a unique maximum at \( r = 1 \) and a unique minimum at \( r = 0 \). Then compactifying the space of Morse flow lines on \( \mathbb{R} \times S^\infty \) between \((1, Z^{i,m})\) and \((0, Z^{0,0})\) yields a compactification of the moduli space of parametrized flow lines

\[
\mathcal{P}_{i,m} = \bigcup Q_{\alpha_1}^{i_1,m_1} \times \cdots \times Q_{\alpha_{d-1}}^{i_{d-1},m_{d-1}} \times \mathcal{P}_{\alpha_d}^{i_d,m_d} \times Q_{\alpha_{d+1}}^{i_{d+1},m_{d+1}} \times \cdots \times Q_{\alpha_n}^{i_n,m_n},
\]

(30)

where the union is taken over the same indexing set of triples \( \{(i_j, \alpha_j, m_j)\} \). In the subsequent sections, we will use these geometric moduli spaces to define the \( \mathbb{Z}/p\mathbb{Z} \)-equivariant Floer cohomology and the \( \mathbb{Z}/p\mathbb{Z} \)-equivariant product and coproduct maps correspondingly.

5 Fixed point Floer cohomology

Given a symplectomorphism \( \phi \) of a symplectic manifold \((M, \omega)\), we recall a few equivalent definitions of its fixed point Floer homology, that we shall use in this paper. While these definitions are equivalent, each one highlights different aspects of the theory, which turns out to be useful.

We assume throughout that our symplectic manifold \((M, \omega)\) is symplectically atoroidal, exact, or symplectically aspherical. In the case when \((M, \omega)\) is not closed, we assume that it is an exact Liouville domain. We consider a suitable class of symplectomorphisms in each case, where the conditions of the symplectic manifold and the symplectomorphism imply that we can work over a ground field \( \mathbb{K} \), without the presence of a Novikov field. We make no assumption on grading, since our main isomorphisms are essentially those of ungraded filtered homologies.

Similar definitions work in the case when \((M, \omega)\) is closed or tame at infinity and weakly monotone, and symplectomorphisms being monotone, as in [36], however these cases necessitate the introduction of Novikov coefficients, which is not the focus herein. We simply note that if \( M \) is monotone and simply-connected, then all symplectomorphisms of \( M \) are automatically monotone. A few references for this section are [7,8,35,37] and [29] for a more general setup.

We recall the above-mentioned conditions on \((M, \omega)\), and \( \phi \in \text{Symp}(M, \omega) \).

(a) (Exact) The symplectic form is \( \omega = d\theta_M \), and the Liouville vector field \( Z \), defined by \( i_Z \omega = \theta_M \), points strictly outwards along the boundary \( \partial M \), which is equivalently to that fact that \( \alpha := \theta_M |_{\partial M} \) is a contact form. The symplectomorphism \( \phi: M \to M \) is exact, that is, one has that \( \phi^* \theta_M = \theta_M + dG_\phi \) for some \( G_\phi \in C^\infty_c(M) \). For a small Hamiltonian perturbation \( H_\epsilon \), which we shall typically choose to be of the form \( \epsilon \cdot r \) near \( \partial M \), wherein \( r \) is the Liouville radial coordinate, one can always assume that \( \phi^H_{H_\epsilon} \circ \phi \) has nondegenerate isolated fixed points.

(b) (Symplectically aspherical) In this case \( M \) is closed and \( \omega(A) = 0 \) for all classes \( A \in H^S_2(M; \mathbb{Z}) \) in the image of the Hurewicz homomorphism \( \pi_2(M) \to H^S_2(M; \mathbb{Z}) \).
In this case we consider Hamiltonian symplectomorphisms \( \phi \) of \( M \), that is - the time-one maps of an isotopy generated by a Hamiltonian vector field \( X^H_t \) given by \( i_{X^H_t} \omega = -d(H(t, -)) \), for \( H \in \mathcal{H} = C^\infty([0, 1] \times M, \mathbb{R})/C^\infty([0, 1], \mathbb{R}) \), where we pick the normalization \( \int H(t, -) \omega^n = 0 \) for all \( t \in [0, 1] \). Furthermore, here we work with Floer cohomology in the contractible free homotopy class of loops. One may define fixed point Floer homology over \( K \) under less stringent assumptions, for example for those \( \phi \in \text{Symp}(M, \omega) \) for which \( \int_C \omega = 0 \) for all cylinders \( C : S^1 \times [0, 1] \rightarrow M \) with \( C(s, 0) = \phi(C(s, 1)) \) for all \( s \in S^1 \) and \( C(0, t) = x_0 \) for all \( t \in [0, 1] \) (such a cylinder represents a loop in the twisted loop space \( \mathcal{L}_\phi M \) below based at a fixed point \( x_0 \) of \( \phi \)). However, conditions of this kind are not in general preserved under iteration.

For a pair consisting of a symplectic manifold \((M, \omega)\) and symplectomorphism \( \phi \in \text{Symp}(M, \omega) \), as above, we will describe the fixed point Floer cohomology \( HF^*(\phi) \) in the following three ways. In case \( (b) \), the cohomology \( HF^*(\phi) \) is isomorphic to \( H^*(M; \mathbb{K}) \), and the main interest of our results lies in associated filtered homology theory.

### 5.1 Twisted loop space

First consider an exact symplectic manifold \((M, \omega)\) and \( \phi \in \text{Symp}(M) \), as above. The flow of the Liouville vector field \( Z \) near \( \partial M \) gives rise to a trivialization of the collar neighborhood of the boundary

\[
\Psi : (-\epsilon, 0] \times \partial M \rightarrow M, (r, y) \mapsto \phi^r_Z(y).
\]

This implies that for \( R = e^r \) one has \( R|_{\partial M} = 1 \) and \( Z \cdot R = R \) near \( \partial M \). On the conical neighborhood \((-\epsilon, 0] \times \partial M\), the symplectomorphism \( \phi \) satisfies

\[
\phi^* \theta_M = \theta_M = dG_{\phi},
\]

where \( G_{\phi} \) is a smooth function on \( M \) which vanishes on the boundary \( \partial M \), which ensures that near the boundary \( \partial M \) we have \( \phi^* R = R \). If one considers time-dependent compatible almost complex structures \( J_t \) satisfying \( J_t = \phi^* J_{t+1} \)

\[
dR \circ J_t = - \theta_M, \text{ on } (-\epsilon, 0] \times \partial M,
\]

then the condition \( (33) \) is preserved under \( \phi \). We denote the space of such almost complex structures as \( J_{\phi} \).

Given such a symplectomorphism \( \phi \in \text{Symp}(M) \), the mapping torus of \( \phi \) is defined by

\[
M_\phi := \mathbb{R} \times M/(t, x) \sim (t + 1, \phi(x)).
\]

By construction, there is a natural projection map \( \pi : M_\phi \rightarrow S^1 \). The twisted loop space is identified with the space of smooth sections of \( \pi \),

\[
\mathcal{L}_\phi M := \{ x \in C^\infty(\mathbb{R}, M) | x(t) = \phi(x(t + 1)) \}.
\]
Given \( \phi \in \text{Symp}(M) \), we can associate a 1-form \( \alpha_\phi \) on the twisted loop space

\[
\alpha_\phi(x)(\xi) = \int_0^1 \omega(\frac{\partial x}{\partial t}, \xi(t))dt. 
\]

In the exact case, this one-form is given as the differential of the action functional

\[
A_\phi : \mathcal{L}_\phi \to \mathbb{R},
\]

\[
A_\phi(x) = \int_0^1 x^* \theta_M + G_\phi(x(1)).
\]

The critical points of the action functional \( A_\phi \) are therefore the constant paths in \( \mathcal{L}_\phi \) at fixed points of \( \phi \), which we still call \( \text{Fix}(\phi) \).

We call the set of critical values \( A_\phi(\text{Fix}(\phi)) \) the spectrum \( \text{Spec}(\phi) \) of \( \phi = (\phi, G_\phi) \).

Each \( J_\ell \in \mathcal{J}_\phi \) defines a \( L^2 \)-metric on the twisted free loop space \( \mathcal{L}_\phi M \). With respect to this metric, negative gradient flow lines which are asymptotic to fixed points \( x_0, x_1 \) are in bijection with solutions \( u : \mathbb{R}^2 \to M \) to Floer’s equation

\[
\partial_s u + J_\ell \partial_t u = 0; \\
u(s, t) = \phi(u(s, t + 1)); \\
\lim_{s \to -\infty} u(s, t) = x_0(t), \quad \lim_{s \to +\infty} u(s, t) = x_1(t).
\]

The maximum principle applied to the subharmonic function \( R(u) \) ensures that no solutions of \( 37 \) reach the boundary \( \partial M \). We denote the solutions to \( 37 \) up to translations in \( s \)-direction by \( \mathcal{M}(x_0, x_1) \). For generic choice of \( J_\ell \), the moduli space \( \mathcal{M}(x_0, x_1) \) is a smooth finite dimensional manifold of dimension

\[
\dim \mathcal{M}(x_0, x_1) = |x_1| - |x_0| - 1
\]

The Floer cochain complex is now defined by

\[
CF^i(\phi) := \bigoplus_{|x|=i} \mathbb{K}\langle o_x \rangle, 
\]

with differential given by \( d(x_1) = \sum_{|x_1|=|x_0|+1} \# \mathcal{M}(x_0, x_1) \) \( x_0 \). Here the notation \( o_x \) is the orientation line associated to the generator \( x \in \text{Fix}(\phi) \) defined in Appendix Appendix A.

For questions regarding signs in the differential \( d \), we also refer the reader such section. It is readily verified that

\[
A_\phi(dx) < A_\phi(x),
\]

where

\[
A_\phi \left( \sum a_j x_j \right) = \max\{A_\phi(x_j) | a_j \neq 0 \}.
\]

In the case when the manifold \((M, \omega)\) is closed and symplectically aspherical, and \( \phi \in \text{Ham}(M, \omega) \), writing \( \phi = \phi^1_H \), for \( H \in \mathcal{H} \) the twisted loop space \( \mathcal{L}_\phi \) is identified
with the usual free loop space $\mathcal{L}M$ by the map $D_H : \mathcal{L}L_\phi \to \mathcal{L}M$, $z(t) \mapsto \phi_H^t z(t)$. It is easy to see that

$$(D_H^{-1})^* A_\phi = A_H,$$

for

$$A_H : \mathcal{L}M \to \mathbb{R},$$

$$A_H(x) = -\int_0^1 H(t, x(t)) \, dt + \int_\pi \omega,$$

where $\pi : \mathbb{D} \to M$ is a map with boundary values $\pi(e^{2\pi it}) = x(t)$. The unusual sign in the action functional is consistent with the above definition, and with the requirement that the differential in cohomology decrease the action filtration.

In both cases, we denote by $\phi$ the tuple consisting of $\phi$ and the data required to define the action functional $A_\phi$. In the first case, this means the primitive $G_\phi$ of $\phi^* \theta_M - \theta_M$, and in the second case it can either be considered to be the choice of a base-point of the connected component of $\mathcal{L}_\phi M$ corresponding to the component $\mathcal{L}_\phi M$ of contractible loops in $\mathcal{L}M$, or in the $\mathcal{L}M$ description: a choice of a Hamiltonian $H \in \mathcal{H}$ generating $\phi$, with the observation that for each contractible Hamiltonian loop $\eta : S^1 \to \text{Ham}(M, \omega)$, the map $D_\eta : \mathcal{L}M \to \mathcal{L}M$ satisfies $D_\eta^* A_H = A_{H#K}$, where $K \in \mathcal{H}$ is the normalized Hamiltonian generating the loop $\eta$.

Finally, for a non-degenerate $\phi$, that is if $\ker(D(\phi)_x - \text{id}) = 0$ for all $x \in \text{Fix}(\phi)$, then for an admissible action window, that is an interval $I = (a, b)$ with $a < b$, $a, b \in \mathbb{R} \setminus \text{Spec}(\phi) \cup \{\pm \infty\}$, we define $HF^*(\phi)^I$ as the homology of the quotient complex

$$CF^*(\phi)^I = CF^*(\phi)^{<b} / CF^*(\phi)^{<a},$$

where $CF^*(\phi)^{<c}$ is the subcomplex spanned by generators $x$ of action value $A(x) < c$. These chain complexes and homologies admit natural comparison maps for $I_1 = (a_1, b_1), I_2 = (a_2, b_2)$ with $a_1 \leq a_2, b_1 \leq b_2$. When $\phi$ is degenerate, then for $a, b$ as above, $\phi_{1,G} = \phi_G^1 \phi$ for a sufficiently $C^2$-small Hamiltonian $G$, still satisfies $a, b \in \mathbb{R} \setminus \text{Spec}(\phi) \cup \{\pm \infty\}$, and for all $G$ sufficiently $C^2$-small the homologies $HF^*(\phi_{1,G})^I$ are canonically isomorphic, whence we define $HF^*(\phi)^I$ as the colimit of the associated indiscrete groupoid.

To ensure that all the fixed points of $\phi$ are non-degenerate, we add a small Hamiltonian perturbation $\psi_G^1$ to the original symplectomorphism so that $\psi_G^1 \circ \phi$ has all non-degenerate fixed points (where $\phi_G^1$ is the time-1 map of some diffeomorphism $G$).

Considering Hamiltonian isotopies $\{\phi_{G'}^t\}$ induced by Hamiltonians $G'$, it is classical to show that $HF^*(\psi_G^1 \circ \phi)$ does not depend on $G$. This is due to the fact that each symplectic isotopy $\{\psi^t\}$ of $M$ generated by 1-forms given by $b_t$ such that $b_{t+1} = \phi^*(b_t)$ and whose flux

$$\int_0^1 b_t \, dt \text{ lies in } \text{Im}(\phi^* - \text{id}) \subset H^1(M; \mathbb{R}),$$

(42)
induces a canonical isomorphism between fixed point Floer cohomologies $HF^*(\psi^1 \circ \phi) \cong HF^*(\phi)$. When $\psi^1 = \psi^1_G$ for some Hamiltonian perturbation, this allows us to define $HF^*(\phi)$.

5.2 Symplectic fibrations

Given a symplectic manifold $(M, \omega)$ we let a symplectic fibration $\pi : E \to B$ over a base manifold $B$ with fiber $(M, \omega)$ be a smooth fibration with a closed two-form $\Omega$ on $E$, such that for all $z \in B$, setting $E_z = \pi^{-1}(z)$, $(E_z, \Omega|_{E_z})$ is a symplectic manifold symplectomorphic to $(M, \omega)$. This is not quite the standard terminology: for example in [27, Chapter 8] symplectic fibrations satisfy a more general condition, while the fibrations we consider are called locally Hamiltonian. Denote by $\text{Vert} = \ker(D\pi) \subset TE$ the vertical subbundle of vectors tangent to the fibers. Furthermore, let $\text{Hor} = \{ v \in TE : \iota_v \Omega|_{\text{Vert}} = 0 \}$ be the horizontal subbundle of $TE$. It is transverse to $\text{Vert}$ and isomorphic to $\pi^*(TB)$ via $D\pi$, whence it induces an Ehresmann connection on $E \to B$. An important feature of such fibrations is that the holonomy of this connection over each loop in the base is a symplectomorphism of the fiber, that is Hamiltonian if the loop is contractible.

We call an almost complex structure $J$ on $E$ compatible with the fibration, and more specifically with $\Omega$, if $J(\ker d\pi) = \ker d\pi$ and $d\pi \circ J = j \circ d\pi$, that is $J$ preserves the fibers and makes the projection map holomorphic with respect to the complex structure $j$ on $S$, and $J|_{\ker d\pi}$ is compatible with the symplectic form $\Omega|_{\ker d\pi}$.

We note that the mapping torus $M_{\phi} \to S^1$ of a symplectomorphism $\phi \in \text{Symp}(M, \omega)$ is a symplectic fibration over $S^1$ : the form $\omega$ on $M$ naturally extends to a closed form $\Omega = \omega_\phi$ on $M_{\phi}$, with $\Omega|_{(M_{\phi})_t} = \omega$ for all $t \in S^1$, $(M_{\phi})_t = M$. Furthermore, $\text{Fix}(\phi)$ is in bijective correspondence with the flat sections of $M_{\phi} \to S^1$.

Following [35], we can define 

$$CF^i(\phi) = \bigoplus_{|x| = i} \mathbb{K}\langle o_x \rangle,$$

where now $x$ ranges over the flat sections of $M_{\phi} \to S^1$, and setting $Z = \mathbb{R} \times S^1$ with the standard complex structure, and letting $\pi_{E_{\phi}} : E_{\phi} = \mathbb{R} \times M_{\phi} \to Z$ be the pullback symplectic fibration of $M_{\phi} \to S^1$ by the natural projection $Z \to S^1$, the differential counts isolated solutions (modulo the $\mathbb{R}$-action by translation) of finite energy $\int_Z u^*\Omega$ to the equation

$$u : Z \to E_{\phi}, \quad \pi_{E_{\phi}} \circ u = \text{id}_Z$$

$$(du)^{(0,1)} = 0$$

$$u(s, t) \xrightarrow{s \to \pm \infty} \sigma_x(\pm t),$$

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where $\sigma_x$ for $x$ a flat section of $M_\phi \to S^1$ is the induced flat section of $E_\phi$, convergence being exponential in suitable trivializations over the ends, and the $(0,1)$-part being taken with respect to an $\Omega$-compatible almost complex structure. The comparison between this definition and the one in Section 5.1 is rather straightforward: it essentially amounts to the well-known Gromov graph trick [27, Chapter 8].

Similarly to the previous section, one can consider a class of perturbations to the symplectic connection associated to the symplectic fibration $\pi E_\phi$, which is induced by smooth families of 1-forms $b_t$ satisfying $b_{t+1} = \phi^* b_t$ and the condition (42). Every such family $b_t$ is equivalent to a family of exact 2-forms $B_t = dt \wedge b_t$ on $\mathbb{R} \times M_\phi$. Now in the case that $\phi$ has degenerate fixed points, there is an open dense subset $B_{reg}$ of such $B_t$ considered above such that if we define the new symplectic fibration to be

$$(\mathbb{R} \times M_\phi, \tilde{\Omega} = \Omega + B_t),$$

then the condition (42) implies the monodromy of this symplectic fibration becomes $\psi_t \circ \phi$. Then the map

$$\Psi: \mathcal{P}_\phi \to \mathcal{P}_{\psi_t \circ \phi}, \quad \Psi(\gamma) = \psi_1 \circ \gamma$$

gives rise to the canonical isomorphism $HF^*(\phi) \cong HF^*(\Psi_1 \circ \phi)$. Again, it suffices in our case to choose some Hamiltonian isotopy $\psi_t := \psi_1 K$ generated by some $K \in C^\infty([0,1] \times M, \mathbb{R})$ to ensure the non-degeneracy. In this case $b_t = -d(K_t)$.

### 5.3 Lagrangian graph construction

Alternatively, we may consider the symplectic manifold $M^- \times M$ and let

$$HF^*(\phi) = HF(\text{graph}(\phi), \Delta),$$

where

$$\text{graph}(\phi) = \{(\phi(x), x) \mid x \in M\} \subset M^- \times M,$$

$$\Delta = \text{graph}(\text{id}_M) \subset M^- \times M$$

are (weakly) exact Lagrangian submanifolds. The comparison between this approach and the one in Section 5.1 is again straightforward, and has to do with choosing product type almost complex structures on $M^- \times M$. We refer for example to [22] for details.

For the above Floer cohomology to be well-defined, one needs that the graph of $\phi$ intersects the diagonal Lagrangian transversally. This can always be achieved for example by adding a small Hamiltonian perturbation to $\phi$, or by introducing a small Hamiltonian perturbation into the Floer equation for the differential. Since the non-degeneracy of the fixed points of $\psi_1 K \circ \phi$ is equivalent to the graph of the perturbed symplectomorphism $\psi_1 K \circ \phi$ is transverse to the diagonal Lagrangian $\Delta$ in $M^- \times M$.  

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5.4 The $\mathbb{Z}/k\mathbb{Z}$-action on $HF(\phi^k)$

For any integer $k$, we define the fixed point Floer cohomology associated to $\phi^k$ following [38]. Given then function $G_\phi$ such that $\phi^* \theta_M - \theta_M = dG_\phi$, one can choose the corresponding function for $\phi^k$ to be

$$G_{\phi^k} = (\phi^*)^k G_\phi + (\phi^*)^{k-1} G_\phi + \cdots + G_\phi.$$  \hfill (44)

so that $\phi^k \theta_M - \theta_M = dG_{\phi^k}$. On the twisted loop space

$$\mathcal{L}_{\phi^k} M := \{ x \in C^\infty(\mathbb{R}, M) \mid x(t) = \phi^k(x(t+k)) \},$$  \hfill (45)

ones define the action functional $\mathcal{A}_{\phi^k} : \mathcal{L}_{\phi^k}(M) \to \mathbb{R}$ in equation [38]. There is a $\mathbb{Z}_k$ action on $\mathcal{L}_{\phi^k}(M)$ defined by

$$\mathbb{Z}_k \times \mathcal{L}_{\phi^k}(M) \to \mathcal{L}_{\phi^k}(M), (m, x(t)) \mapsto \phi^m(x(t+m)).$$  \hfill (46)

The fixed point set of the $\mathbb{Z}_k$-action is precisely $\mathcal{L}_\phi(M)$. For $x(t) \in \mathcal{L}_\phi(M)$, we have

$$\mathcal{A}_{\phi^k}(x(t)) = k \mathcal{A}_\phi(x(t)).$$  \hfill (47)

We denote the space of compatible almost complex structures corresponding to $\phi^k$ by

$$\mathcal{J}_{\phi^k} = \{(J_t)_{t \in \mathbb{R}} \mid J_t is \ \omega\text{-compatible and } J_t = \phi^k(J_{t+k}) \}. \hfill (48)$$

For any integer $k \in \mathbb{Z}$, there is an action of the finite group $\mathbb{Z}/k\mathbb{Z}$ on $\mathcal{J}_{\phi^k}$ defined as follows

$$\rho: \mathbb{Z}/k\mathbb{Z} \times \mathcal{J}_{\phi^k} \to \mathcal{J}_{\phi^k}, \ (m, J_t) \mapsto (\phi^m)_*(J_{t+m}), \forall m \in \mathbb{Z}. \hfill (49)$$

We call the almost complex structure $J$ in $\mathcal{J}_{\phi^k}$ symmetric if it is invariant under the $\mathbb{Z}/k\mathbb{Z}$-action defined above.

Next we describe a $\mathbb{Z}/k\mathbb{Z}$-action in Floer cohomology $HF^*(\phi^k)$ for each $k \in \mathbb{Z}$. Similar to the $\mathbb{Z}/2\mathbb{Z}$ case in [38] Section 4 and the $\mathbb{Z}/k\mathbb{Z}$ case considered in [31, 32, 50], and in [43, 17] Section 3, where the signs and orientations were made explicit, the chain map which induces the generator of the $\mathbb{Z}/k\mathbb{Z}$-action in cohomology is the following composition of maps

$$\sigma: (CF^*(\phi^k), d_{J_{\phi^k}}) \xrightarrow{c} (CF^*(\phi^k), d_{\rho_* J_{\phi^k}}) \xrightarrow{\rho} (CF^*(\phi^k), d_{J_{\phi^k}}),$$

where $d_{J_{\phi^k}}$ and $d_{\rho_* J_{\phi^k}}$ denotes the differentials of the fixed point Floer cochain complex defined using the almost complex structure $J_{\phi^k} \in \mathcal{J}_{\phi^k}$ and its image under the $\mathbb{Z}/k\mathbb{Z}$ action $\rho_* J_{\phi^k}$. The first cochain map $c : (CF^*(\phi^k), d_{J_{\phi^k}}) \to (CF^*(\phi^k), d_{\rho_* J_{\phi^k}})$ is given by a standard continuation map for the almost complex structures $J_{\phi^k}$ and $\rho_* J_{\phi^k}$, and the second map is induced by applying $\rho$ in [43] to the Floer data defining $(CF^*(\phi^k), d_{\rho_* J_{\phi^k}})$ and identify the generators with those of $(CF^*(\phi^k), d_{J_{\phi^k}})$. It can be checked that both
of these maps are isomorphisms and the composition is a chain map. We denote the induced map in cohomology by
\[ \sigma_* : HF^* (\phi^k) \rightarrow HF^* (\phi^k) \text{ and } (\sigma^*)^k = id. \]

The same definition goes through for cohomology groups in all admissible action windows $I$, giving a map
\[ \sigma_* : HF^* (\phi^k)^I \rightarrow HF^* (\phi^k)^I \text{ and } (\sigma^*)^k = id, \]
commuting with the interval comparison maps.

In fact, the cochain-level map $\sigma$ will define a homotopical action of $\mathbb{Z}/k\mathbb{Z}$ on $CF^* (\phi^k)$.

To describe the action of $\mathbb{Z}/k\mathbb{Z}$ on $HF^* (\phi^k)$, we study the equation that defines the map $\sigma$ in more detail. One first considers the interpolation of almost complex structures to define the continuation map above $c$,

\[ J_{s,t} = \begin{cases} J_{\phi^k}, & s \rightarrow \infty \\ \rho_* J_{\phi^k}, & s \rightarrow -\infty \end{cases} \]

Then one solve the following continuation equation for $u : Z_k := \mathbb{R} \times \mathbb{R} / k\mathbb{Z} \rightarrow \mathbb{R} \times M_{\phi^k}$ such that
\[
\begin{cases}
(du)_{s,t}^{(0,1)} = 0 \\
\lim_{s \rightarrow \infty} u(s,t) = x_0(t) \quad \text{and} \quad \lim_{s \rightarrow -\infty} u(s,t) = x_1(t+1)
\end{cases}
\]

When $|x_0| = |x_1|$, one defines the continuation map which is obtained by counting rigid solutions to the above moduli space. Now suppose that $x_0 = x_1 = \gamma^k$ for some $k$-fold iterates of a simple Reeb orbits $\gamma$ in the symplectization of the mapping tori $\mathbb{R} \times M_{\phi^k}$. It has been shown in Lemma 6.7 \[49\] that depending on the parity of the index difference $\mu_{CZ}(\gamma^k) - \mu_{CZ}(\gamma)$ and parity of $k$, the induced map in cohomology $\sigma_*$ can have non-trivial signs or not. Specifically, as the linearization of the equation (50) is given by the following operator
\[
T : W^{1,p}(Z_k, \mathbb{R}^{2n+2}) \rightarrow L^p(Z_k, \mathbb{R}^{2n+2}), \quad \partial_s + J_0 \partial_t + S_k(t + \beta(s)),
\]

where $S_1(t)$ is the loop of symmetric matrices associated to the orbit $\gamma$ defined in equation (118) in the Appendix and $S_k(t) = S_1(kt)$, and the function $\beta : \mathbb{R} \rightarrow [0, 1]$ is a smooth cut-off function such that
\[
\lim_{s \rightarrow -\infty} \beta(s) = 1, \quad \lim_{s \rightarrow \infty} \beta(s) = 0.
\]

Now it is shown in \[49\] Lemma 6.7 that this operator acts as an orientation revering map on $\text{Det}(\mathcal{O}(S_k, S))$ \[118\] if and only if $k$ is even and the difference of Conley-Zehnder indices $\mu_{CZ}(\gamma^k) - \mu_{CZ}(\gamma)$ is odd, that is to say, the Reeb orbit $\gamma$ is bad.

\[1\]The definition of the determinant line associated to the Fredholm operator $T$ is in Appendix A.
6 The $\mathbb{Z}/p\mathbb{Z}$-equivariant Floer cohomology

6.1 Definitions

The $\mathbb{Z}/2\mathbb{Z}$-equivariant Floer cohomology $HF^*_{\mathbb{Z}/2\mathbb{Z}}(\phi^2)$ has been considered in [38]. In this section, we fix a prime number $p$ and consider the analogous constructions for the $\mathbb{Z}/p\mathbb{Z}$-equivariant Floer cohomology $HF^*_{\mathbb{Z}/p\mathbb{Z}}(\phi^p)$.

Given a pair $(M, \phi)$ satisfying conditions (a) and (b) listed in Section 5, the mapping torus of $\phi$ is defined as

$$M_\phi := \mathbb{R} \times M/(t, x) \sim (t + 1, \phi(x)).$$

The symplectic form $\omega$ on $M$ descends to a 2-form of maximal rank on $M_\phi$, which we still denote as $\omega$. We may consider $\mathbb{R} \times M_\phi$ as a locally Hamiltonian fibration over $Z = \mathbb{R} \times S^1$, as above, and discuss holomorphic sections. However, it turns out beneficial to use an equivalent description in a language that has been more developed for our purposes.

The pair $(\omega, dt)$ defines a stable Hamiltonian structure on the mapping torus $M_\phi$. The symplectization of this stable Hamiltonian structure is

$$(\mathbb{R} \times M_\phi, \tilde{\omega} = \omega + ds \wedge dt)$$

(51)

For any parametrized flow line $w$ on $S^\infty$, one notices that the families of time-dependent almost complex structures $J_t$ extended to almost complex structures $\tilde{J}_t$ on $\mathbb{R} \times M_\phi$ canonically by seeing the symplectization as

$$\mathbb{R} \times M_\phi = \mathbb{C} \times M/(s, t, x) \sim (s, t + 1, \phi(x)).$$

(52)

We denote by $\mathcal{J}(M_\phi)$ be the space of $\tilde{\omega}$-compatible almost complex structures $\tilde{J}_t$ obtained by such extension. As fixed points of $\phi^k$ are in bijection with the Reeb orbits of the stable Hamiltonian structure $M_\phi$ of period $k$, one considers a class of Hamiltonian perturbations $\mathcal{H}^l(M_\phi)$ defined on $\mathbb{R} \times M_\phi$ by the following conditions

- $\text{Supp}(H_l) \subset \bigcup_{\gamma: \gamma(t+l) = \gamma(t)} N(\gamma)$, where $\gamma$ is any Reeb orbit of period $l$ and $N(\gamma)$ is a tubular neighborhood of such Reeb orbit $\gamma$.
- The norm of the Hamiltonian $||H_l||_{C^\infty} \leq \epsilon_l$ for some sufficiently small constant $\epsilon_l > 0$ that depends on $l$.

Given a fixed prime number $p$ and a pair of Floer data $(\tilde{J}_t \times \tilde{H}_l)$ in $\mathcal{J}_{M_{p\phi}} \times \mathcal{H}(M_{p\phi})$, one can extend $\tilde{J}_t$ to an almost complex structure $\tilde{J}_{t,z} = \tilde{J}_{s,t,x,z}$ on $\mathbb{R} \times M_{p\phi} \times S^\infty$ satisfying the following properties

- (Locally constant at critical points): $\tilde{J}_{t,z} = \tilde{J}_t$ for $z$ in a neighbourhood of $Z_i^0$ for each $i$. 

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For fixed a non-negative integer $i$ and $m$ in $\mathbb{Z}/p\mathbb{Z}$, the moduli space $\tilde{M}_{\alpha}^{i,m}(x_0, x_1)$ consists of solutions $u: \mathbb{R} \times \mathbb{R}/p\mathbb{Z} \to \mathbb{R} \times M_{\phi^p}$ and $w: \mathbb{R} \to S^\infty$ to the following parametrized J-holomorphic equations

$$
\begin{cases}
\langle (du - X_{H_t} \otimes dt) \circ j = J_{t,w(s)} \circ (du - X_{H_t} \otimes dt), \\
\partial_s w(s) + \nabla \tilde{F}(w) = 0,
\end{cases}
$$

with asymptotic behavior

$$
\lim_{s \to -\infty} (u(s, \cdot), w(s)) = (\phi^{-m}(x_0), Z^m_i), \quad \lim_{s \to \infty} (u(s, \cdot), w(s)) = (x_1, Z^0_\alpha), \quad \text{where } \alpha = 0, 1.
$$

Note that since $\phi^p(x_0) = x_0$, $\phi^{-m}(x_0)$ is well-defined for $m \in \mathbb{Z}/p\mathbb{Z}$.

There is a free $\mathbb{R}$-action on $\tilde{M}_{\alpha}^{i,m}(x_0, x_1)$ given by

$$
r \cdot (u(s, t), w(s)) \mapsto (u(s + r, t), w(s + r))
$$

We denote its quotient by $M_{\alpha}^{i,m}(x_0, x_1) := \tilde{M}_{\alpha}^{i,m}(x_0, x_1)/\mathbb{R}$. For generic choice of almost complex structure $J_t$ on $S^1 \times M \times S^\infty$ described previously, the moduli space is a smooth finite dimensional manifold with

$$
\dim M_{\alpha}^{i,m}(x_0, x_i) = |x_1| - |x_0| + i - \alpha - 1 \text{ for all } \alpha.
$$

For $|x_1| = |x_0| + i + \alpha + 1$, one can define $d_{0,i}^{i,m}: CF^*(\phi^p) \to CF^{*+1-i+\alpha}(\phi^p)$ by

$$
d_{0,i}^{i,m}(x_1) = \sum_{x_0:|x_1| = |x_0| + i+1} \# M_{0,i}^{i,m}(x_0, x_1)x_0,
$$

$$
d_{1,i}^{i,m}(x_1) = \sum_{x_0:|x_1| = |x_0| + i+1} \# M_{1,i}^{i,m}(x_0, x_1)x_0.
$$

Let $u$ be a formal variable of degree 2 and $\theta$ be a formal variable of degree 1 so that $\theta^2 = 0$ as in Section 2. We set $d_\alpha^0 = d_{0}^\alpha + d_{1}^\alpha + \cdots + d_{p-1}^\alpha$. The equivariant differential $d^{\mathbb{Z}/p\mathbb{Z}}: CF^*(\phi^p)[[u]](\theta) \to CF^{*+1}(\phi^p)[[u]](\theta)$ can be written as

$$
d^{\mathbb{Z}/p\mathbb{Z}}(x \otimes 1) = d_{0}^0(x) \otimes 1 + u d_{0}^1(x) \otimes 1 + u^2 d_{0}^2(x) \otimes 1 + \ldots + d_{i}^0(x) \otimes \theta + u d_{i}^1(x) \otimes \theta + u^2 d_{i}^2(x) \otimes \theta + \ldots + u d_{i}^i(x) \otimes 1 + u^2 d_{i}^{i+1}(x) \otimes 1 + \ldots
$$

$$
d^{\mathbb{Z}/p\mathbb{Z}}(x \otimes \theta) = d_{1}^0(x) \otimes \theta + u d_{1}^1(x) \otimes \theta + u^2 d_{1}^2(x) \otimes \theta + \ldots + u d_{i}^i(x) \otimes 1 + u^2 d_{i}^{i+1}(x) \otimes 1 + \ldots
$$

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There is a natural projection map
\[ M_{\alpha}^{i,m}(x_0, x_i) \to Q_{\alpha}^{i,m} \] induced by \((w, u) \mapsto w\. \tag{59} \]

Considering the codimension 1 strata in the compactifications of the unparametrized moduli spaces \(Q_{\alpha}^{i,m}\) described in \[31\], gives, by standard Gromov-Floer compactness, transversality, and gluing arguments, that \((d^{\mathbb{Z}/p\mathbb{Z}})^2 = 0\) (for a discussion the analytic issues see \[38\) Section 4.2).\]

We call \(CF_{\mathbb{Z}/p\mathbb{Z}}^*(\phi^p) = CF^*(\phi^p) \oplus R_p\) with \(d^{\mathbb{Z}/p\mathbb{Z}}\) the \(\mathbb{Z}/p\mathbb{Z}\)-equivariant cochain complex of \(\phi^p\), and its homology \(HF_{\mathbb{Z}/p\mathbb{Z}}^*(\phi^p)\) the \(\mathbb{Z}/p\mathbb{Z}\)-equivariant cohomology of \(\phi^p\). Furthermore, we observe that \(CF_{\mathbb{Z}/p\mathbb{Z}}^*(\phi^p)\) and \(HF_{\mathbb{Z}/p\mathbb{Z}}^*(\phi^p)\) are modules over \(R_p\). We call the homology \(\tilde{HF}_{\mathbb{Z}/p\mathbb{Z}}^*(\phi^p)\) of \(CF_{\mathbb{Z}/p\mathbb{Z}}^*(\phi^p)\) \(= CF_{\mathbb{Z}/p\mathbb{Z}}^*(\phi^p) \otimes_R \hat{R}_p\), with \(d^{\mathbb{Z}/p\mathbb{Z}} = d^{\mathbb{Z}/p\mathbb{Z}} \otimes \text{id}\) the \(\mathbb{Z}/p\mathbb{Z}\)-equivariant Tate homology of \(\phi^p\).

We outline a selected technical aspect.

**Lemma 5.** For generic choice of almost complex structures, all elements in \(\tilde{M}_{\alpha}^{i,m}(x_0, x_1)\) are regular for all \(i, m\) and \(\alpha\).

**Proof.** All solutions to equation \((55)\) except the constant solutions can be made regular by choosing generic almost complex structure as in \[26\) Proposition 6.7.7]. For an energy zero solution \(w = (a, u) : \mathbb{R} \times \mathbb{R}/p\mathbb{Z} \to \mathbb{R} \times M_{\phi^p}\), after composing to the projection to \(M\), the map \(\pi_M(w) = u\) is the constant map whose image is some fixed point \(x\) of \(\phi^p\). The linearization of the Cauchy-Riemann equation of \(u\) at \(x \in M\) is of the form
\[ D_u : W^{k,p}(\mathbb{R}^2, T_x M) \to W^{k-1,p}(\mathbb{R}^2, T_x M), \quad D_u(\xi) = \partial_s \xi + J_{t,w(s)} \partial_t \xi \tag{60} \]
for some \(\xi : \mathbb{R} \to T_x M\) satisfying \(\xi(s, t) = D\phi^p_x(\xi(s, t + p))\). As \(x\) is a non-degenerate fixed point of \(\phi^p\), the linear operator \(D_u\) is Fredholm. The index of \(D_u\) is
\[ \dim \ker(D_u) - \dim \operatorname{coker}(D_u) + i = |x| - |x| + i = i. \tag{61} \]

To achieve surjectivity of \(D_u\), it suffices to show that the linearization \(D_u\) is injective for any almost complex structure \(J_{t,w(s)}\). Suppose \(D_u \xi = 0\), in particular we have \(|D_u \xi||_{W^{1,2}(\mathbb{R} \times [0, 1])} = 0\) if \((k, p) = (2, 2)\) in \[30\). With respect to the metric on \(T_x M\) defined by \(J_{t,w(s)}\) that we choose, one computes
\[ 0 = \int_{\mathbb{R} \times [0, 1]} |D_u \xi|^2 + \int_{\mathbb{R} \times [0, 1]} |\xi|^2 = \int_{\mathbb{R} \times [0, 1]} |\partial_s \xi|^2 + |\partial_t \xi|^2 + \int_{\mathbb{R} \times [0, 1]} \omega^s_x \xi \tag{62} \]
The second term \(\int_{\mathbb{R} \times [0, 1]} \omega_x \xi\) vanishes by Stokes’ Theorem. Hence we can conclude that \(\xi = 0\) by above equation, which completes the proof. \(\square\)

By \((59)\) and unwinding the definition of \(d^{\mathbb{Z}/p\mathbb{Z}}\), the following statement is evident.
Lemma 6. The differentials $d_0^I, d_1^I : CF^*(\phi^p) \to CF^*(\phi^p)$ are chain maps, and induce
\[
[d_0^I] = (1 - \sigma_*) : HF^*(\phi^p) \to HF^*(\phi^p),
\]
\[
[d_1^I] = N_* = (1 + \sigma_* + \ldots + \sigma_*^{p-1}) : HF^*(\phi^p) \to HF^*(\phi^p),
\]
on cohomology.

Finally, all constructions and statements in this section, have analogues for admissible action windows $I$ for $\phi^p$. Indeed, by an index argument and the finiteness of Fix($\phi$), in the non-degenerate case, $d^{Z/pZ}$ contains only a finite number of terms. Hence for each admissible window $I = (a, b)$ we can choose the perturbation data for the equivariant differential in such a way as to have $CF_{Z/pZ}^*(\phi^p)^<a$, $CF_{Z/pZ}^*(\phi^p)^<b$ be subcomplexes with respect to $d^{Z/pZ}$, and then set $HF_{Z/pZ}^*(\phi^p)^I$ as the homology of the quotient complex
\[
CF_{Z/pZ}^*(\phi^p)^I = CF_{Z/pZ}^*(\phi^p)^<a/CF_{Z/pZ}^*(\phi^p)^<b.
\]
One shows that this does not depend on the choice perturbations, provided that they are sufficiently small. Finally, for $\phi^p$ degenerate, we perturb it as $\phi_1^p$, so that $\phi_1, \phi_1^p$ are non-degenerate, and sufficiently $C^2$ close to $\phi, \phi^p$. Then the interval stays admissible, and making the perturbations for the differential sufficiently small, we can show that $HF_{Z/pZ}^*(\phi^p)^I$ does not depend, up to canonical isomorphism, on $\phi_1$ provided that it is sufficiently close to $\phi$. The same applies to $\tilde{HF}_{Z/pZ}^*(\phi_1^p)^I$.

6.2 The algebraic spectral sequence

For the purposes of this section, consider the following algebraic degree on $CF^*(\phi^p) \otimes R_p$. The algebraic degree of 1 is 0, that of $u$ is 2, that of $\theta$ is 1, and elements of $CF^*(\phi^p)$ have degree zero. Requiring that degrees of products add up, we extend this degree to $CF^*(\phi^p) \otimes R_p$.

Consider the filtration $\mathcal{F}_{\text{alg}}^k = \mathcal{F}_{\text{alg}}^k(CF^*(\phi^p) \otimes R_p)$ generated by the elements of $CF^*(\phi^p) \otimes R_p$ of degree at least $k$. It is easy to check that for all $k \geq 0$,
\[
d^{Z/pZ}(\mathcal{F}_{\text{alg}}^k) \subset \mathcal{F}_{\text{alg}}^k,
\]
and hence $\mathcal{F}_{\text{alg}}^k$ forms a decreasing filtration on $(CF^*(\phi^p) \otimes R_p, d^{Z/pZ})$, which is complete. We call it the algebraic filtration. The same applies when we fix an action window $I = (a, b)$, $-\infty \leq a < b \leq \infty$, and consider the algebraic filtration on the complex $(CF^*(\phi^p) \otimes R_p^I, d^{Z/pZ})$ in action window $I$. In each case, this filtration, being complete and exhaustive, gives a regular spectral sequence which converges to $HF_{Z/pZ}^*(\phi^p)^I$, as $R_p$ is already complete with respect to this filtration. We start by describing the $E^1$-page of this algebraic spectral sequence. Indeed the only terms in $d^{Z/pZ}$
that do not increase filtration are $d_0^1$ and $d_1^1$, both corresponding to the usual Floer differential on $CF(\phi^p)$. Hence the $E_1$-page is given by

$$E_1 = HF(\phi^p) \otimes R_p.$$  

Furthermore, the differential in this page is given by

$$d_{E_1}(x \otimes 1) = [d_0^1](x) \otimes \theta,$$

and

$$d_{E_1}(x \otimes \theta) = u[d_1^1](x) \otimes 1,$$  

observing that $(d^\mathbb{Z}/p\mathbb{Z})^2 = 0$ implies that $d_0^1$ and $d_1^1$ are in fact chain maps. Furthermore, by Lemma [6], $[d_0^1] = 1 - \rho$, and $[d_1^1] = 1 + \rho + \ldots + \rho^{p-1}$, for $\rho : HF(\phi^p) \to HF(\phi^p)$, the generator of the $\mathbb{Z}/p\mathbb{Z}$-action. This means that the $E_2$-page of the spectral sequence is given by

$$E_2 = H^*(\mathbb{Z}/p\mathbb{Z}, HF(\phi^p)),$$

as an $R_p$-module. We claim that there exists a differential $d_{2,\infty}$ on $H^*(\mathbb{Z}/p\mathbb{Z}, HF(\phi^p))$ of $R_p$-modules, whose homology calculates $H^*_\mathbb{Z}/p\mathbb{Z}(\phi^p)$. Indeed, by an application of the homological perturbation lemma [25] to the initial complex, making the homology subspaces and projection operators invariant with respect to multiplication by $u$, there exists a differential $d_{1,\infty}$ on $E_1$ of the form

$$d_{1,\infty} = d_{E_1} + D_1,$$

where $D_1$ consists of maps of order $\geq 2$ in the algebraic filtration. In a similar way, a second application of the homological perturbation lemma with respect to the splitting (65), produces the required differential. Consequently, tensoring with $K((u))$ over $K[[u]]$, we obtain a differential $\tilde{d}_{2,\infty}$ of $\tilde{R}_p$-modules on $\tilde{H}^*(\mathbb{Z}/p\mathbb{Z}, HF(\phi^p))$, whose homology is $\tilde{H}^*_{\mathbb{Z}/p\mathbb{Z}}(\phi^p)$. In particular, we obtain that

$$\dim_{K((u))} \tilde{H}^*_{\mathbb{Z}/p\mathbb{Z}}(\phi^p) \leq \dim_{K((u))} \tilde{H}^*(\mathbb{Z}/p\mathbb{Z}, HF(\phi^p)).$$  

### 7 Action and energy estimates

In this section we collect two classical results on action and energy of solutions to the Floer equation that we resort to throughout the paper.

Let $\bar{S}$ be a Riemann surface with complex structure $j$, and with $k_- + k_+$ punctures $\Gamma : I \to \bar{S}$, where $I = I_- \sqcup I_+$, $I_- = \{1, \ldots, k_-\}$, $I_+ = \{1, \ldots, k_+\}$. We denote $S = \bar{S} \setminus \Gamma(I)$. Let $S$ be equipped with holomorphically embedded cylindrical ends $\epsilon^-_j : (-\infty, -1] \times S^1 \to S$ around $\Gamma(j)$, $j \in I_-$, and $\epsilon^+_j : [1, \infty) \times S^1 \to S$ around $\Gamma(j)$, $j \in I_+$.

Let $u : S \to E$ be a section of an locally Hamiltonian symplectic fibration $\pi : E \to S$, with cylindrical trivializations over the cylindrical ends of $E$ and connection form $\Omega$.  

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Assume that \( u \) converges to sections \( x_j^\pm \) of \( E_j^\pm \to S^1 \) for \( j \in I_\pm \) in the cylindrical ends corresponding to \( \Gamma(j) \). Assume that \( u \) satisfies the Floer equation \((du)^{(0,1)} = 0\) with respect to an \( \Omega \)-compatible almost complex structure on \( E \). Then the energy \( E(u) = \int_S u^*\Omega^v \) satisfies
\[
E(u) = \int_S u^*\Omega - \int_S u^*R(\Omega),
\]
where \( R(\Omega) \) is the curvature form of the locally Hamiltonian fibration. Furthermore, if we equip \( E_j^\pm \to S^1 \) for \( j \in I_\pm \) with the structure of filtered symplectic brane, in a way that is compatible with \( (E, \Omega) \), so that
\[
\int_S u^*\Omega = \sum_{j \in I_-} A_j(x_j^-) - \sum_{j \in I_+} A_j(x_j^+),
\]
we obtain the action identity
\[
\sum_{j \in I_-} A_j(x_j^-) - \sum_{j \in I_+} A_j(x_j^+) = E(u) + \int_S u^*R(\Omega). \tag{67}
\]
This way, lower bounds on \( E(u) \), the most elementary of which is of course \( E(u) \geq 0 \), combined with lower bounds on the curvature term \( \int_S u^*R(\Omega) \) uniform in \( u \), yield upper bounds on the action shift \( \sum_{j \in I_+} A_j(x_j^+) - \sum_{j \in I_-} A_j(x_j^-) \). Typically, we will be able to make our curvature terms arbitrarily small.

Secondly, to describe a slightly more sophisticated lower bound on \( E(u) \), we adapt a well-known monotonicity lemma [41, Proposition 4.3.1.(ii)] to show the following quite general "crossing energy" type argument. One may prove the same statement in a different way (following [17][18][28]), applying the target-local Gromov compactness of Fish [10], however we choose to present a more elementary argument, which is sufficient for our purposes.

Suppose that our Riemann surface \( S \), as well as the (locally Hamiltonian) symplectic fibration \( E \) over it (see Section 5.2), is obtained by a branched cover from an \( s \)-invariant fibration over the standard cylinder \( Z = \mathbb{R} \times S^1 \). By \( s \)-invariance we mean that given a symplectic fibration \( E_0 \to S^1 \), the fibration \( E_1 \to Z \) satisfies \( E_1 = \pi_{S^1}^*E_0 \), for \( \pi_S : Z \to S^1 \) the projection to the \( S^1 \)-factor. Furthermore, we assume that the Hamiltonian perturbation term is sufficiently small. Then, if the asymptotic conditions of a solution \( u \) are distinct, this solution must satisfy an energy monotonicity statement: its energy is bounded from below by \( \epsilon > 0 \), depending only on \( J \), the asymptotic Floer data, and on the isolating neighborhoods of the asymptotics. These estimates shall be used later in defining various flavors of local Floer homology and related operations.

**Proposition 7.** Let \( p : S \to Z = \mathbb{R} \times S^1 \) be a branched covering with finite branch set. Let \( (E, \Omega) \) be a symplectic fibration over a surface \( S \), obtained by base-change by \( p \) from a symplectic fibration on \( Z \) of the form \( (E_1, \Omega_1) = \pi_{S^1}^*(E_0, \Omega_0) \) over \( Z \), where
Let us first deal with the case of at least two different asymptotics

Proof. By definition, it is clear that \( \{ \} \) have disjoint images and so do \( \sigma \) asymptotic conditions on \( j \) where \( u \) is a solution to the Floer equation \( du^{(0,1)} = 0 \) with respect to \( (J, K) \), satisfies

\[
E(u) \geq \epsilon > 0.
\]

Furthermore, there exists a small neighborhood \( U_x \) of \( \sigma_x \), such that if all the asymptotics of \( u \) are at \( x \), then the image of \( u \) is contained in \( U_x \).

Proof. Let us first deal with the case of at least two different asymptotics \( x, y \). To begin, we assume that \( (J, K) = (J_0, 0) \). Since our fibration is obtained from \( (E_1, \Omega_1) = \pi_{\mathbb{S}^3}(E_0, \Omega_0) \), the sections \( x, y \) of \( (E_0, \Omega_0) \) give us flat sections \( \sigma_x, \sigma_y : S \to E \). The asymptotic conditions on \( u \) yield that \( u \) is asymptotic to \( \sigma_x \) on a cylindrical end \( \epsilon_x \) of \( S \), and to \( \sigma_y \) on cylindrical end \( \epsilon_y \) of \( S \). Clearly, as \( x, y \) are flat and different, they have disjoint images and so do \( \sigma_x, \sigma_y \). Consider a small \( 2\delta \)-neighborhood \( U_x(2\delta) \) of \( \text{Im}(\sigma_x) \) inside \( E \) that is disjoint from \( \text{Im}(\sigma_y) \), and admits a trivialization \( \Phi : U_x(2\delta) \to D^{2n}(2\delta) \times S \) as a symplectic fibration, with \( \Phi(\sigma_x(z)) = (0, z) \) for all \( z \in S \). Let \( S_x(\delta) = \overline{U_x(\frac{3}{2}\delta)} \setminus U_x(\frac{1}{2}\delta) \subset U_x(2\delta) \) be a shell around \( \text{Im}(\sigma_x) \). Note that \( S_x(\delta) = \Phi^{-1}(S^{2n}(\delta) \times S) \), where \( S^{2n}(\delta) = \overline{D^{2n}(\frac{3}{2}\delta)} \setminus D^{2n}(\frac{1}{2}\delta) \). We claim that there exists \( \delta > 0 \), and \( \epsilon > 0 \), such that

\[
E(u) \geq \int_{\phi^{-1}(S_x(\delta))} u^*\Omega_v \geq \epsilon > 0.
\]

We start with the obvious observation that

\[
\int_{\phi^{-1}(S_x(\delta))} u^*\Omega_v = \int_{(\phi \circ u)^{-1}(S^{2n}(\delta) \times S)} (\phi \circ u)^*\Omega_v^{\text{triv}},
\]

where now \( \Omega_v^{\text{triv}} = \Phi_*(\Omega_v) \) is the symplectic connection form on the trivial fibration \( D^{2n}(2\delta) \times S \). Note that at the points of \( \{0\} \times S \), \( \Omega_v^{\text{triv}} = \omega \oplus 0 \), since \( \sigma_x \) was a flat section. This means \( \Omega_v^{\text{triv}} = \omega \oplus \eta \), for a two-form \( \eta \) on \( D^{2n}(2\delta) \times S \) such that \( ||\eta || \leq C_4 \cdot |w| \) for \( (w, z) \in D^{2n}(2\delta) \times S \), and \( C_4 \geq 0 \) a constant.

Set \( v = \Phi \circ u \). Note that \( J'_0 = \Phi_*(J_0) \) preserves the fibers, and is of the form

\[
J'_0(w, z) = \begin{bmatrix} j_{w, z}^w & A_{w, z} \\ 0 & j_z \end{bmatrix},
\]

where \( j \) is a complex structure on \( S \), \( J_{w, z}^w \) is a \( 2 \)-dependent almost complex structure on \( D^{2n} \) compatible with \( \omega \), and \( A_{w, z} \) a \( (j_z, J_{w, z}^w) \) anti-complex map \( T_z S \to T_w D^{2n}(2\delta) \). By definition, it is clear that \( \{J_{w, z}^w\}_{z \in S} \) is contained in a compact set of almost complex structures on \( D^{2n} \). Consider the standard Riemannian metric \( g \) on \( D^{2n} \). Then there
exists a uniform constant $C_1$ such that $\omega(\xi, J^v_x, \xi) \geq C_1 \cdot g_x(\xi, \xi)$ and $||\omega|| \leq C_2$ for all $z \in S, w \in D^{2n}$. Furthermore, as $A_{0,z} = 0$ for all $z \in S$, again by definition, there is a uniform constant $C_3$ such that $||A_{w,z}|| \leq C_3 \cdot |w|$. Furthermore, we may choose these constants for $\delta = \delta_0$, and keep them for all $\delta \leq \delta_0$. Choosing locally two vectors $\partial_s, \partial_t$ tangent to $S$ with $j\partial_s = \partial_t$, we obtain that

$$v^*\Omega^v_{triv}(\partial_s, \partial_t) = v^*\Omega^v_{triv}(\partial_s, j\partial_s) =$$

$$= \omega_{v(x)}(\partial_s v, J^v_x(\partial_s v, J^v_x(\partial_s v, A_x v)) + \eta_{z,v}(\partial_s v, (J^v_x + A_x v))\partial_s v) \geq$$

$$\geq C(\delta)|\partial_s v|^2,$$

where $C(\delta) = C_1 - C_2 C_3 \delta - 2 C_2 C_4^{-1} C_4 \delta - 4 C_3 C_4 \delta^2$.

Similarly,

$$v^*\Omega^v_{triv}(\partial_s, \partial_t) \geq C(\delta)|\partial_t v|^2.$$

Set $\delta_1 = \min\{\delta_0, \delta_s\}$, where $C(\delta) \geq C_1/2$ for all $\delta \leq \delta_s$ and choose $\delta \in \left[\frac{1}{2} \delta_1, \delta_1\right]$. Now for a compact submanifold with boundary $B \subset v^{-1}(S^{2n}(\delta) \times S)$, we have $\text{Area}_g(v|_B) \leq C_5 \int_B v^*\Omega^v_{triv}$, for a suitable constant $C_5$. From now on, for generic $\delta \in \left[\frac{1}{2} \delta_1, \delta_1\right]$, the argument of Sikorav [41] applies without change to show that $\int_{v^{-1}(S^{2n}(\delta) \times S)} v^*\Omega^v_{triv} \geq \epsilon > 0$, for $\epsilon$ that depends only on the geometric situation: $x, y, J, E$.

To obtain the result for general data $(J, K)$ sufficiently close to $(J_0, 0)$ one may for example apply the appropriate version of Gromov compactness [35, Section 12].

Finally, for the case of all asymptotics identical, if $u$ is not contained in $U_x = U_x(2\delta)$, then the same argument as above applies to show that $E(u) \geq \epsilon > 0$. However, $E$ being a branched cover of a cylinder, the curvature of $\Omega$ is zero. Therefore, for $(J, K)$ close to $(J_0, 0)$, by formula [57] the energy $E(u)$ is arbitrarily close to 0. This is a contradiction.

\section{$\mathbb{Z}/p\mathbb{Z}$-equivariant pants product and coproduct}

Let $h: S \rightarrow \mathbb{R} \times S^1$ be the branch cover of $\mathbb{R} \times S^1$ at $(0, 0) \in \mathbb{R} \times S^1$ of ramification index $p$ defined explicitly via the commutative diagram

$$\begin{array}{ccc}
S_p & \longrightarrow & \mathbb{C} \setminus \{e^{\frac{2\pi i m}{p}}\} \\
\downarrow h & & \downarrow y=1-\frac{1}{2^p} \\
\mathbb{R} \times S^1 & \xrightarrow{(s,t) \mapsto e^{-2\pi i (s+it)}} & \mathbb{C}^*,
\end{array}$$

The covering transformation group $\mathbb{Z}/p\mathbb{Z}$ acts on $S_p$. For each positive puncture of $S_p$, there is a trivialization of the cylindrical ends of $S_p$ over $s \geq 1$,

$$\varepsilon^+_i: [1, \infty) \times S^1 \rightarrow S_p, \quad i \in \mathbb{Z}/p\mathbb{Z}, \quad h(\varepsilon^+_i(s,t)) = (s,t) \in \mathbb{R} \times S^1, \quad m \cdot (\varepsilon^+_i(s,t)) = \varepsilon^+_{i+m \mod p}(s,t) \text{ for } m \in \mathbb{Z}/p\mathbb{Z}. \quad (68)$$

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For negative punctures, one also has cylindrical trivializations for $s \leq -1$ given by
\begin{equation}
\epsilon_i^- : (-\infty, -1] \times S^1 \to S_\mathcal{P} , \quad i \in \mathbb{Z}/p\mathbb{Z},
\end{equation}
\begin{equation}
h(\epsilon_i^-(s,t)) = (s,t), \quad m \cdot (\epsilon_i^-(s,t)) = \epsilon_i^-(s, t + m) \text{ for } m \in \mathbb{Z}/p\mathbb{Z}.
\end{equation}

Explicitly, choosing a branch of the logarithm on $\mathbb{C}^*$ around $z = 1$ for (71) and around $z = -1$ for (72), we write
\[
S_{\mathcal{P}} = \{(s,t,y) \in \mathbb{R} \times S^1 \times \mathbb{C} | y^p = 1 - e^{-2\pi(s+it)}\}
\]
\[
\epsilon_i^+(s,t) = (s,t,e^{2\pi ki/p}(1 - e^{-2\pi(s+it)})^{1/p});
\]
\[
\epsilon_i^-(s,t) = (s,t,e^{2\pi ki/p}e^{-2\pi(s+it)/p}(e^{2\pi(s+it)} - 1)^{1/p});
\]
\[
\alpha(s,t,y) = (s,t+1,y), \quad \tilde{\alpha}(s,t,y) = (s,t,e^{2\pi i/p} \cdot y).
\]

Having defined the domain Riemann surface $S_{\mathcal{P}}$, we prescribe the families of $\mathbb{Z}/p\mathbb{Z}$-equivariant Floer data
\[
(J_{w,z}, Y_z) \in C^\infty(S_{\mathcal{P}} \times S^\infty, \mathcal{J}(M_{\phi})) \times \Omega^1(S_{\mathcal{P}}, T(\mathbb{R} \times M_{\phi}))
\]
that are parametrized by $S_{\mathcal{P}} \times S^\infty$ and $S_{\mathcal{P}}$ respectively. One first chooses $(\tilde{J}_t, \tilde{H}_t)$ in $\mathcal{J}(M_{\phi}) \times \mathcal{J}(M_{\phi})$ and $(\tilde{J}_t^i, \tilde{H}_t^i)$ in $\mathcal{J}(M_{\phi}) \times \mathcal{H}(M_{\phi})$ for $i = 0, 1, \cdots, p - 1$ as the initial Floer data. Then one can define almost complex structures $J^{-}_{s,t,w}$ and $J^{+}_{s,t,w}$ parametrized by $\mathbb{R} \times S^1 \times S^\infty$ as follows. For $s \leq 1$, one requires $J^{-}_{s,t,w}$ to satisfy the following properties

- (Constant on the cylindrical ends): $J^{-}_{s,t,w} = \tilde{J}_t$ for $s \leq -2$ and $w \in S^\infty$.
- (Interpolation): $J^{-}_{s,t,w} \in \mathcal{J}_{\phi}$ for $s \in [-2, -1]$.
- ($\mathbb{Z}/p\mathbb{Z}$-equivariance): $J^{-}_{s,t,m,w} = \phi_m J^{-}_{s,t+m,w}$ for all $m \in \mathbb{Z}/p\mathbb{Z}$.
- (Invariance under shift): $J^{-}_{s,t,w} = J^{-}_{s+\pi(w),w}$ for all $w \in S^\infty$.

Similarly if $s \geq -1$, we ask that

- (Constant on the cylindrical ends): $J^{+}_{s,t,w} = \tilde{J}_t^i$ for $s \leq -2$ and for all $w \in S^\infty$.
- (Interpolation): $J^{+}_{s,t,w} \in \mathcal{J}_{\phi}$ for $s \in [-2, -1]$ and all $i \in \mathbb{Z}/p\mathbb{Z}$, $w \in S^\infty$.
- ($\mathbb{Z}/p\mathbb{Z}$-equivariance): $J^{+}_{s,t,m,w} = \phi_m J^{+}_{s,t+m,w}$ for all $m \in \mathbb{Z}/p\mathbb{Z}$, $i \in \mathbb{Z}/p\mathbb{Z}$ and $w \in S^\infty$.
- (Invariance under shift): $J^{+}_{s,t,w} = J^{+}_{s+\pi(w),w}$ for all $w \in S^\infty$ and all $i \in \mathbb{Z}/p\mathbb{Z}$.

Given a parametrized Morse flow line $w : \mathbb{R} \to S^\infty$ of $\tilde{F} : S^\infty \to \mathbb{R}$ in $\mathcal{P}_{\alpha}^{i,m}$ defined in Section 4 for $\alpha = 0, 1$. One can then define a family $J_{z,w}$ of domain-dependent almost
complex structures parametrized by $S_P \times S^\infty$ by setting
\[ J_{z,w} = J^{+ \iota}_{s,t,w(s)} = J^{- \iota}_{s,t,w(s)} \text{ if } z \in \pi^{-1}([-1, 1] \times S^1); \] (71)
\[ J_{z,w} = J^{+ \iota}_{s,t,w(s)} \text{ if } z = \varepsilon^+_i(s, t) \text{ and for all } i = 0, 1, \ldots, p - 1; \] (72)
\[ J_{z,w} = J^{- \iota}_{s,t,m,w(s)} \text{ for all } m \in \mathbb{Z}/p\mathbb{Z} \text{ and } z = \varepsilon^-_m(s, t). \] (73)

Similarly, one chooses the domain-dependent perturbation term $Y_z \in \Omega^1(S_P, T(R \times M_\phi))$ that satisfy the following conditions

- (Constant on the cylindrical ends): $Y_{\varepsilon^+_i(s, t)} = X_{H_{t^+}^i} \times dt$ and $Y_{\varepsilon^-_m(s, t)} = X_{H_{t^-}^m} \otimes dt$ for fixed $H_{t^+}^i \in H(M_\phi)$ and $H_{t^-}^m \in H(M_{\phi^r})$.
- (Compactly supported near critical points): $Y_z \equiv 0$ outside the images of the cylindrical parametrizations $\varepsilon^+_i$ and $\varepsilon^-_j$ for $i, j \in \mathbb{Z}/p\mathbb{Z}$.
- ($\mathbb{Z}/p\mathbb{Z}$-invariance): $Y_{\lambda \cdot z} = Y_z$ where $\lambda$ is the generator of the deck transformation group for the covering space $h$.

For any Morse flow line $w : \mathbb{R} \to S^\infty$ that is asymptotic to $Z^m_1$ at $-\infty$ and $Z^0_1$ at $\infty$ for $\alpha = 0, 1$, one can choose $J_{z,w}$ and $Y_z$ as above and consider the moduli space $\mathcal{M}_{\mathcal{P},1}(x^-, x^+_0, \ldots, x^+_p)$ of solutions $u : S_P \to \mathbb{R} \times M_\phi$ to the parametrized Cauchy-Riemann equation
\[ (du - Y_z) \circ j = J_{z,w} \circ (du - Y_z), \] (74)
with asymptotic behaviors
\[ \lim_{s \to -\infty} u(\varepsilon^-_i(s, t)) = x^-(t + i), \quad \lim_{s \to -\infty} u(\varepsilon^+_k(s, t)) = x^+_k(t), \text{ for } k \in \mathbb{Z}/p\mathbb{Z}. \] (75)

As noticed in [38, Section 3c], for constant solutions of (74) one cannot achieve transversality by varying the almost complex structures $J_{z,w}$ purely. This is the reason that we also introduce the Hamiltonian perturbation term $Y_z$ in equation (74). For generic choice of $J_{z,w}$ and $Y_z$, the moduli space $\mathcal{M}_{\mathcal{P},1}(x^-, x^+_0, \ldots, x^+_p)$ can be shown to be a smooth finite dimensional manifold of dimension
\[ \dim \mathcal{M}_{\mathcal{P},1}(x^-, x^+_0, \ldots, x^+_p) = |x^-| - \sum_k |x^+_k| + i - \alpha. \] (76)

For each $m \in \mathbb{Z}/p\mathbb{Z}$ and $|x^-| = \sum_k |x^-_k| - i + \alpha$, one can define an operation by
\[ \mathcal{P}_{\alpha}^{i,m} : CF^s(\phi)^{\otimes p} \to CF^{s-i+m}(\phi^p) \] (77)
\[ \mathcal{P}_{\alpha}^{i,m}(x^+_0, \ldots, x^+_p) = \sum_{x^-} \# \mathcal{M}_{\mathcal{P},1}(x^-; x^+_0, \ldots, x^+_p) x^- \] (78)

As before if we set $\mathcal{P}_i = \sum_{m \in \mathbb{Z}/p\mathbb{Z}} \mathcal{P}_{\alpha}^{i,m}$, then the $\mathbb{Z}/p\mathbb{Z}$-equivariant product can be written as
\[ \mathcal{P} : CF^s(\phi)^{\otimes p}[u][\theta] \to CF^s(\phi^p)[[u]][\theta] \] (79)
\[ \mathcal{P}(- \otimes 1) = (\mathcal{P}_0^{0} + u\mathcal{P}_0^{2} + \ldots) \otimes 1 + (\mathcal{P}_1^{1} + u\mathcal{P}_0^{3} + \ldots) \otimes \theta, \] (80)
\[ \mathcal{P}(- \otimes \theta) = (u\mathcal{P}_1^{2} + u^2\mathcal{P}_1^{4} + \ldots) \otimes 1 + (\mathcal{P}_1^{1} + u\mathcal{P}_1^{3} + \ldots) \otimes \theta. \] (81)
To define the $\mathbb{Z}/p\mathbb{Z}$-equivariant coproduct, we first define a Riemann surface with one positive puncture and $p$ negative punctures by the following fiber diagram

$$
\begin{array}{ccc}
S_C & \longrightarrow & \mathbb{C} \setminus \{ e^{\frac{2\pi i m}{p}} \} \\
\downarrow & & \downarrow \\
\mathbb{R} \times S^1 & \longrightarrow & \mathbb{C}^*,
\end{array}
$$

Similar to the case of $\mathbb{Z}/p\mathbb{Z}$-equivariant product, one chooses $p$ positive and $p$ negative cylindrical trivializations as $\tilde{e}_i^+$ and $\tilde{e}_j^-$ for $i, j = 0, 1, \cdots, p - 1$. Then one choose similar families of Floer data $(J_{i,m}, J_{j,m})$ parametrized by $S_C \times S^1$ and $S_C$ separately and the directions of $m$-parameter reversed. Given a gradient flow line $w : \mathbb{R} \to S^\infty$ that is asymptotic to $Z_i^m$ at $-\infty$ and $Z_0^m$ at $\infty$. One considers the moduli space $\mathcal{M}_{C, \alpha}^i(x_0^+, \cdots, x_{p-1}^+; x^+)$ of solutions $u : S_C \to \mathbb{R} \times M_{\theta}$ to the parametrized perturbed Cauchy-Riemann equation

$$(du - Y_z) \circ j = J_{z,w} \circ (du - Y_z), \quad (82)$$

with asymptotic behaviors

$$\lim_{s \to -\infty} (u(\tilde{e}_i^-)(s, t)) = x_i^-(t), \quad \lim_{s \to -\infty} (u(\tilde{e}_i^-)(s, t)) = x^+(t + k) \quad \text{for } k \in \mathbb{Z}/p\mathbb{Z}. \quad (83)$$

For generic choice of $J_{z,w}$ and $H^w$, the moduli space of non-constant solutions to $\mathcal{M}_{C, \alpha}^i(x_0^+, \cdots, x_{p-1}^+; x^+)$ is a smooth finite dimensional manifold of dimension

$$\sum_{k \in \mathbb{Z}/p\mathbb{Z}} |x_k^-| - |x^+| + i - \alpha - 2n(p - 1) \quad (84)$$

For each $m \in \mathbb{Z}/p\mathbb{Z}$ and $\sum_{k=0}^{p-1} |x_k^-| = |x^+| - i$, one can define an operation by

$$C_{\alpha}^{i,m} : C^*(\mathbb{Z}/p\mathbb{Z}; C,F^*(\phi^p)) \to C^{*-1-\alpha+2n(p-1)}(\mathbb{Z}/p\mathbb{Z}; C,F^*(\phi)^p) \quad (85)$$

$$C_{\alpha}^{i,m}(x^+) = \sum_{x^-} \#\mathcal{M}_{C, \alpha}^i(x_0^-, \cdots, x_{p-1}^-; x^+) (x_0^- \otimes \cdots \otimes x_{p-1}^-). \quad (86)$$

As before if we set $C_{\alpha}^i = \sum_{m} C_{\alpha}^{i,m}$, then the $\mathbb{Z}/p\mathbb{Z}$-equivariant product is given by

$$C : CF^*(\phi^p)[[u]][\theta] \to CF^*(\phi)^p[[u]][\theta] \quad (87)$$

$$C(- \otimes 1) = (C_0^0 + uC_0^2 + \cdots) \otimes 1 + (C_1^0 + uC_0^2 + \cdots) \otimes \theta, \quad (88)$$

$$C(- \otimes \theta) = (uC_1^2 + u^2C_1^4 + \cdots) \otimes 1 + (C_1^1 + uC_1^3 + \cdots) \otimes \theta. \quad (89)$$

Our next goal is to show that the $\mathbb{Z}/p\mathbb{Z}$-equivariant produce and coproduct maps $\mathcal{P}$ and $\mathcal{C}$ define chain maps. Both cases are treated similarly, so we focus on $\mathcal{P}$. The chain map relation again follows, by standard Gromov-Floer compactness, transversality, and gluing arguments, from looking at compactifications of the 1-dimensional moduli spaces $\mathcal{M}_{\mathcal{P}, \alpha}^i(x^-; x_0^+, \cdots, x_{p-1}^+)$, coming from either Floer breaking in the interior, or from codimension 1 strata of $\mathcal{Q}_{\mathcal{P}, \alpha}$ at the boundaries (see Section 4.3 for a discussion of the analytical issues).
9 Local Floer cohomology and the action filtration

We describe local Floer cohomology at an isolated fixed point of a symplectomorphism, and its $\mathbb{Z}/p\mathbb{Z}$-equivariant version. Furthermore, we discuss the action spectral sequence starting with the direct sum of local Floer cohomologies and converges to the total Floer cohomology. This spectral sequence shall subsequently be used to prove that the $\mathbb{Z}/p\mathbb{Z}$-equivariant pants product is a filtered chain-homotopy equivalence between the two relevant filtered complexes, and in the proofs of our main applications. We refer to [16, 19] for more details on local Floer cohomology.

9.1 Local Floer cohomology

Let $\phi$ be a symplectomorphism of a symplectic manifold $M$ as specified in Section 5. Given an isolated fixed point $x$ of $\phi$, there exists an isolating neighborhood $U$ of $x$ (more precisely, of the image of the flat section $\sigma_x$ in $M_{\phi}$) for Floer cohomology. In particular, all Floer trajectories of each sufficiently $C^2$ small non-degenerate Hamiltonian perturbation $\phi'$ of $\phi$ between generators in $U$ are contained in $U$, and the resulting Floer cohomology as computed inside $U$, is well-defined and independent of the perturbation. This cohomology is called the local Floer cohomology $HF^{\text{loc}}(\phi, x)$ of $\phi$ at $x$. Whenever the local Floer cohomology is considered as an ungraded $\mathbb{K}$-module it depends on no additional data. A similar statement and definition applies to an isolated Morse-Bott submanifold $X$ of fixed points $\phi$ (see e.g. [11, 12, 28, 33]).

We recall the following additional properties of $HF^{\text{loc}}(\phi, x)$. First if $x$ is non-degenerate as a fixed point of $\phi$, then as $\mathbb{K}$-modules,

$$HF^{\text{loc}}(\phi, x) \cong \mathbb{K}.$$  

Second, let $c \in \text{Spec}(\phi)$ be an isolated action value, such that all $x \in \text{Fix}(\phi)$ with $A_\phi(x) = c$ are isolated. Furthermore, in view of Section 7 for two distinct fixed points $x, y \in \text{Fix}(\phi)$, there exists $\epsilon_0 > 0$, such that all Floer trajectories, or product structures considered in this paper, with $x, y$ among their asymptotics, carry energy of at least $\epsilon_0$. Hence for $\epsilon > 0$ sufficiently small,

$$HF(\phi)^{(c-\epsilon, c+\epsilon)} \cong \bigoplus_{x \in \text{Fix}(\phi), A_\phi(x) = c} HF^{\text{loc}}(\phi, x).$$

Finally, the above "building block" property implies that if all $x \in \text{Fix}(\phi)$ are isolated, and $M$ is aspherical or exact, then for each $a, b \in \mathbb{R} \setminus \text{Spec}(\phi) \cup \{\pm\infty\}$, $a < b$, there is a spectral sequence arising from the action filtration, that converges to $HF(\phi)^{(a,b)}$ and has $E_1$-page given by

$$\bigoplus_{x \in \text{Fix}(\phi), a < A_\phi(x) < b} HF^{\text{loc}}(\phi, x),$$
The above situation readily extends to the case of equivariant Floer cohomology. Indeed, supposing that all fixed points \( \text{Fix}(\phi^p) \) of \( \phi^p \) are isolated (or more generally belong to isolated connected Morse-Bott submanifolds), there is an upper and a lower bound depending only on \( \phi \) and \( \dim M \) on the possible indices of the fixed points of a sufficiently \( C^2 \)-small non-degenerate Hamiltonian perturbation of \( \phi^p \). We choose this perturbation to be of the form \( \phi^p_1 \), where \( \phi_1 \) is a sufficiently \( C^2 \)-small non-degenerate perturbation of \( \phi \). In particular, the terms \( d^i_\alpha \) of the equivariant differential vanish for all \( i > i_0(\phi) \), independently of the choice of perturbation data. Therefore, it is sufficient to control the perturbation data \( J_{t,w}, H_t \) in a compact family corresponding to \( w \in S^{10}(\phi) \). Hence, Proposition 7 ensures that the perturbation data can be chosen in such a way that the trajectories of the equivariant differential between generators inside a sufficiently small isolating neighborhood \( U_x \) of \( x \in \text{Fix}(\phi^p) \) stay inside \( U \). Furthermore, the same is true for neighborhoods \( U_{\phi^p x} \) of \( \phi^p x \) for \( x \in \mathbb{Z}/p\mathbb{Z} \). Therefore, by definition of the equivariant differential, gluing and compactness, the critical points of \( \phi^p \) in \( U = \bigcup_{\sigma \in \mathbb{Z}/p\mathbb{Z}} U_{\phi^p x} \) form a complex, and the cohomology of this complex is independent of the Hamiltonian perturbation \( \phi^p \). We call this cohomology the equivariant local Floer cohomology \( HF^\text{loc}_{\mathbb{Z}/p\mathbb{Z}}(\phi^p, \mathbb{Z}/p\mathbb{Z} x) \) of the orbit \( \mathbb{Z}/p\mathbb{Z} x \). If \( x \) is an iterated fixed point, that is \( \phi(x) = x \), or \( \mathbb{Z}/p\mathbb{Z} x = \{ x \} \) then only one isolating neighborhood \( U \) of \( x \) with respect to \( \phi^p \) is necessary, and we abbreviate by \( HF^\text{loc}_{\mathbb{Z}/p\mathbb{Z}}(\phi^p, x^{(p)}) \).

The local equivariant Floer cohomology enjoys properties similar to those of usual local Floer cohomology. First if \( x \) is non-degenerate as a fixed point of \( \phi^p \), then if \( x \) is iterated, we have

\[
HF^\text{loc}_{\mathbb{Z}/p\mathbb{Z}}(\phi^p, x^{(p)}) \cong H^*(\mathbb{Z}/p\mathbb{Z}, \mathbb{K}) = R_p
\]

as \( R_p \)-modules, and if \( x \) is simple, then then \( \mathbb{Z}/p\mathbb{Z} \)-action on \( \mathbb{Z}/p\mathbb{Z} x \) is free and transitive, and

\[
HF^\text{loc}_{\mathbb{Z}/p\mathbb{Z}}(\phi^p, \mathbb{Z}/p\mathbb{Z} x) \cong H^*(\mathbb{Z}/p\mathbb{Z}, \mathbb{K}[\mathbb{Z}/p\mathbb{Z}]) = \mathbb{K} = R_p/\langle u, \theta \rangle.
\]

Second, let \( c \in \text{Spec}(\phi) \) be an isolated action value, such that all \( x \in \text{Fix}(\phi^p) \) with \( A_\phi(x) = c \) are isolated. In view of Section 7 we again obtain that for \( \epsilon > 0 \) sufficiently small,

\[
HF^*_{\mathbb{Z}/p\mathbb{Z}}(\phi^p)^{(c-\epsilon, c+\epsilon)} \cong \bigoplus HF^\text{loc}_{\mathbb{Z}/p\mathbb{Z}}(\phi^p, \mathbb{Z}/p\mathbb{Z} x),
\]

the sum running over the orbits of the \( \mathbb{Z}/p\mathbb{Z} \)-action on \( \{ x \in \text{Fix}(\phi^p) \mid A_{\phi^p}(x) = c \} \).

Furthermore, if all \( x \in \text{Fix}(\phi^p) \) are isolated, and \( M \) is aspherical or exact, then for each \( a, b \in \mathbb{R} \setminus \text{Spec}(\phi) \cup \{ \pm \infty \}, a < b \), there is a spectral sequence arising from the action

filtered by \( A_\phi \).
filtration, that converges to \( HF^*_\mathbb{Z}/p\mathbb{Z}(\phi^p)^{(a,b)} \) and has \( E_1 \)-page given by

\[
\bigoplus_{O \in \{ x \in \text{Fix}(\phi^p), a < A_\phi(x) < b \}/\mathbb{Z}/p\mathbb{Z}} HF^\text{loc}_{\mathbb{Z}/p\mathbb{Z}}(\phi, O),
\]

filtered by \( A_{\phi^p} \).

Finally, tensoring with \( K = \mathbb{K}((u)) \) over \( \mathbb{K}[[u]] \) everywhere we obtain a similar spectral sequence for the Tate cohomology groups with \( E_1 \)-page given in terms of the local Tate cohomology groups.

### 9.3 Local product and coproduct operations

Consider an isolated fixed point \( x \in \text{Fix}(\phi) \), with isolating neighborhood \( U \) of \( \sigma_x \) in \( M_\phi \) that extends to a neighborhood \( U_P \) of \( \sigma_x \) in \( E_P \) and a neighborhood \( U_C \) of \( \sigma_x \) in \( E_C \).

By Proposition 7 and the argumentation of Section 9.2 for a sufficiently \( C^2 \)-small Hamiltonian perturbation of \( \phi \), the moduli spaces defining the product and coproduct operations with all inputs and outputs restricted to lie inside \( U \) involve only sections that lie inside \( U_P \) and \( U_C \) respectively. Furthermore, they define chain maps on the suitable local cohomology groups, and hence operations

\[
\mathcal{P}_x : H^*(\mathbb{Z}/p\mathbb{Z}; CF^\text{loc}(\phi, x)^{\otimes p}) \to HF^\text{loc}_{\mathbb{Z}/p\mathbb{Z}}(\phi^p, x^{(p)}),
\]

\[
\mathcal{C}_x : HF^\text{loc}_{\mathbb{Z}/p\mathbb{Z}}(\phi^p, x^{(p)}) \to H^*(\mathbb{Z}/p\mathbb{Z}; CF^\text{loc}(\phi, x)^{\otimes p}).
\]

Finally, choosing sufficiently small isolating neighborhoods, it is straightforward to deduce that the local Floer cohomology, its equivariant version at an iterated fixed point, as well as the local products and coproducts \( \mathcal{P}_x, \mathcal{C}_x \) depend only on the germ of \( \phi \) at \( x \).

### 10 The local coproduct-product is invertible

We first prove the assertion of Theorem A in the case of a non-degenerate fixed point \( x \) of \( \phi \). The case of local Floer cohomology, as well as the general symplectically aspherical and exact cases, will follow directly by a spectral sequence argument.

Fix for the duration of this section the non-degenerate fixed point \( x \in \text{Fix}(\phi) \), and an isolating neighborhood \( U \) of \( x \) for Floer cohomology, and \( U^{(p)} \) of \( x^{(p)} \) for \( \mathbb{Z}/p\mathbb{Z} \)-equivariant Floer cohomology. Recall that

\[
HF^\text{loc}(\phi, x) \cong \mathbb{K},
\]

32
\[ HF^*(\mathbb{Z}/p\mathbb{Z}; CF_{\text{loc}}^\phi(x)^{\otimes p}) \cong R_p, \]
\[ HF_{\mathbb{Z}/p\mathbb{Z}}^{*,\text{loc}}(\phi^p, x^{(p)}) \cong R_p. \]

Consider the local product and coproduct operators

\[ P_{x}^{\text{loc}} : HF^*(\mathbb{Z}/p\mathbb{Z}; CF_{\text{loc}}^\phi(x)^{\otimes p}) \to HF_{\mathbb{Z}/p\mathbb{Z}}^{*,\text{loc}}(\phi^p, x^{(p)}), \]
\[ C_{x}^{\text{loc}} : HF_{\mathbb{Z}/p\mathbb{Z}}^{*,\text{loc}}(\phi^p, x^{(p)}) \to HF^*(\mathbb{Z}/p\mathbb{Z}; CF_{\text{loc}}^\phi(x)^{\otimes p}). \]

In the subsections below we show that

\[ C_{x}^{\text{loc}} \circ P_{x}^{\text{loc}} = (-1)^n u^{(p-1)n} \cdot \text{id}, \quad (90) \]

and hence it becomes invertible after tensoring with \( \mathcal{K} \). Indeed, \( u^{(p-1)n} \) is a unit in \( \mathcal{K} \). Since \( \dim_{\mathcal{K}} R_p < \infty \), this implies that \( P_{x}^{\text{loc}} \) becomes invertible after extending coefficients to \( \mathcal{K} \).

### 10.1 Invariance properties

First, using invariance properties of \( C_{x}^{\text{loc}} \circ P_{x}^{\text{loc}} \) under isolated deformations, we show that

\[ C_{x}^{\text{loc}} \circ P_{x}^{\text{loc}} = c_n \cdot u^{(p-1)2n} \cdot \text{id}, \quad (91) \]

for a constant \( c_n \in \mathbb{K} \). In Section 10.3 we calculate that \( c_n = (-1)^n \), using a reduction to the Morse-theoretic model of Betz-Cohen [4] type.

We follow the arguments of Seidel [38, Section 6], combined with the additional flexibility provided by an alternative interpretation of \( C_{x}^{\text{loc}} \circ P_{x}^{\text{loc}} \) as an operation

\[ Z_{x}^{\text{loc}} : HF^*(\mathbb{Z}/p\mathbb{Z}; CF_{\text{loc}}^\phi(x)^{\otimes p}) \to HF^*(\mathbb{Z}/p\mathbb{Z}; CF_{\text{loc}}^\phi(x)^{\otimes p}) \]

obtained by counting \( p \)-tuples of Floer cylinders with a diagonal-type constraint. This allows us to show a more general isolated-deformation invariance that that of \( P_{x}^{\text{loc}} \): indeed, chambers in the linear symplectic group defined by excluding \( p \)-th roots of unity as eigenvalues, as in (110), play no role for this new map.

**Definition 8.** The chain-level operation

\[ Z_{x}^{\text{loc}} : CF^*(\mathbb{Z}/p\mathbb{Z}; CF_{\text{loc}}^\phi(x)^{\otimes p}) \to CF^*(\mathbb{Z}/p\mathbb{Z}; CF_{\text{loc}}^\phi(x)^{\otimes p}) \]
is defined as follows. Consider $p$ cylinders $C^m = \mathbb{R} \times S^1$, $m \in \mathbb{Z}/p\mathbb{Z}$. Choose $p$ almost complex structures $J^m_{s,t,w} \in \mathcal{J}$ each depending on the point $(s, t) \in C^m$, and $w \in S^\infty$. We require that

\begin{align*}
J^m_{s,t,\tau(w)} &= J^m_{s,t,\tau(w)}, \text{ for all } w \in S^\infty \quad (92) \\
J^m_{s,t,\pm} &= J^m_{s,t,\pm} \quad (93) \\
J^m_{s,t,w} &= J_t, \text{ for } |s| \geq 2 \quad (94)
\end{align*}

For a Morse flow-line $w: \mathbb{R} \to S^\infty$ in $\mathcal{P}^{i,m}_{\alpha}$, for $\alpha \in \{0, 1\}$, as defined in Section 4, we set $J^m_{s,t,w} = J^m_{s,t,u(s)}$.

Now as in the definitions in Section 8, introducing $C^2$-small Hamiltonian perturbations $H^m_{s,t}$ with corresponding perturbation form $Y^m_{\phi} = Y^m_{s,t} \otimes dt$ compactly supported away from $(0, 0) \in C^m$, we look at the moduli spaces $\mathcal{M}^m_{\phi, Z, m}(x^-_0, \ldots, x^-_{p-1}; x^+_0, \ldots, x^+_{p-1})$ of solutions $(w, (u_m)_{m \in \mathbb{Z}/p\mathbb{Z}})$, to the following parametric Floer equation. Let $\tilde{C}^m = \mathbb{R} \times \mathbb{R}$ be the universal cover of $C^m$, and $\lambda(s, t) = (s, t+1)$ the Deck transformation. Identifying $\mathbb{Z}/p\mathbb{Z} = \{0, 1, \ldots, p - 1\}$ as a set, $u_m: \tilde{C}^m \to M$, and $w \in \mathcal{P}^{i,m}_{\alpha}$ satisfy

\begin{equation}
\begin{cases}
(du_m - Y^m_{\phi}) \left( (0, 1) \right) = 0 \\
u_m(z) = \phi(u_m(\lambda(z))) \\
u_0(0, 0) = u_1(0, 0) = \ldots = u_{p-1}(0, 0).
\end{cases}
\end{equation}

with asymptotic conditions

\begin{equation}
\lim_{s \to -\infty} u_k(s, t) = x^-_k(t), \quad \lim_{s \to \infty} u_k(s, t) = x^+_k(t)
\end{equation}

where $\phi(x^+_k(t+1)) = x^+_k(t)$ are the suitable fixed points, considered as twisted loops.

**Remark 9.** It is not difficult to be define an equivalent equation alternatively in terms of suitable symplectic fibrations. Indeed, by Section 5, we may consider each $u_m$ as a section of a copy of $\mathbb{R} \times M$, with suitable perturbation and boundary data. Furthermore, the fiber over $(0, 0) \in \mathbb{R} \times S^1$ of $M$ is naturally identified with $M$. This allows us to write the necessary incidence condition. To work locally, we restrict attention to a neighborhood of the flat section $\sigma_x$ by Section 7.

It is straightforward to show that transversality for the moduli spaces obtained thus can be achieved by generic choice of $J^m_{s,t,z}$ and $H^m_{s,t}$, the latter being $C^2$-small, making $\mathcal{M}^m_{\mathbb{Z}, m}(x^-_0, \ldots, x^-_{p-1}; x^+_0, \ldots, x^+_{p-1})$ a smooth manifold of dimension

\[\sum_k |x^-_k| - \sum_k |x^+_k| + i - \alpha - 2n(p-1).\]

Furthermore, the dimension 0 moduli spaces, corresponding to the condition $\sum_k |x^-_k| = \sum_k |x^+_k| - i + \alpha + 2n(p-1)$ are compact, by a standard Gromov-Floer compactness
argument, since in the local, and the weakly exact, cases there are no holomorphic curves present. Hence they consist of a finite number of points.

Then $\mathcal{Z}_x^{\text{loc}}$, which we abbreviate as $\mathcal{Z}$, is given by the collection of operations

$$
\mathcal{Z}_{\alpha}^{i,m} : C^*([\mathbb{Z}/p\mathbb{Z}]^v; CF^* (\phi)^{\otimes p}) \to C^{*-i+\alpha+2n(p-1)}([\mathbb{Z}/p\mathbb{Z}]^v; CF^* (\phi)^{\otimes p})
$$

(97)

$$
\mathcal{Z}_{\alpha}^{i,m}(x_0^+ \otimes \cdots \otimes x_{p-1}^+) = \sum \# \mathcal{M}_{\alpha}^{i,m}(x_0^-, \cdots , x_{p-1}^-; x_0^+, \cdots, x_{p-1}^+)(x_0^- \otimes \cdots \otimes x_{p-1}^-)
$$

(98)

Following the usual recipe, if we set $\mathcal{Z}_\alpha^i = \sum_m \mathcal{Z}_{\alpha}^{i,m}$, then the $\mathbb{Z}/p\mathbb{Z}$-equivariant $p$-cylinder map is given by

$$
\mathcal{Z} : CF^* (\phi)^{\otimes p}[[u]](\theta) \to CF^* (\phi)^{\otimes p}[[u]](\theta)
$$

(99)

$$
\mathcal{Z}(\emptyset \otimes 1) = (\mathcal{Z}_0^0 + u\mathcal{Z}_0^2 + \ldots) \otimes 1 + (\mathcal{Z}_1^0 + u\mathcal{Z}_1^3 + \ldots) \otimes \theta,
$$

(100)

$$
\mathcal{Z}(\emptyset \otimes \theta) = (u\mathcal{Z}_1^2 + u^2\mathcal{Z}_1^4 + \ldots) \otimes 1 + (\mathcal{Z}_1^1 + u\mathcal{Z}_1^3 + \ldots) \otimes \theta.
$$

(101)

As for $C, \mathcal{P}$, the chain map relation for $\mathcal{Z}$ follows, by Section 7 and standard Gromov-Floer compactness, transversality, and gluing arguments, from looking at compactifications of the 1-dimensional moduli spaces $\mathcal{M}_{\alpha}^{i,m}(x_0^-, \cdots , x_{p-1}^-; x_0^+, \cdots, x_{p-1}^+)$. Furthermore the following identity holds on the chain level in the isolated non-degenerate case.

**Lemma 10.** $C_x^{\text{loc}} \circ \mathcal{P}_x^{\text{loc}} = \mathcal{Z}_x^{\text{loc}}$

**Proof.** The idea behind this proof consists in a degeneration-gluing argument. However, to carry it out, we must replace the $p$ trajectories $u_m : C_m \to M$, incident at $u_m(0, 0), \ m \in \mathbb{Z}/p\mathbb{Z},$ by a map $u : \bar{C} \to M$, where now $C$ is the nodal curve consisting of the curves $C_m, \ m \in \mathbb{Z}/p\mathbb{Z},$ and one genus zero curve $S \cong \mathbb{C}P^1 \cong \mathbb{C} \cup \infty,$ with nodes given by indetifying $(0, 0) \in C_m$ with $e^{2\pi mi/p} \in S$. Of course in the local, and the weakly exact, cases the restriction $u_0$ of the holomorphic curve $u$ to $S$ will be constant, however this turns out to be the right domain to perform gluing. Note that $C$ admits a homomorphic $\mathbb{Z}/p\mathbb{Z}$-action, given by cyclically permuting the $C_m$ under the natural identification, and rotating $S$ by the corresponding $p$-th roots of unity.

Consider the following family $\mathcal{R} \cong (0, 1)$ of Riemann surfaces with $\mathbb{Z}/p\mathbb{Z}$-action. By choosing cylindrical ends $\epsilon^m_+ : [1, \infty) \times S^1 \to S$, at the points $e^{2\pi mi/p} \in S$, equivariant with respect to the $\mathbb{Z}/p\mathbb{Z}$-action, $\tau \cdot \epsilon^m_+(\zeta) = \epsilon^{m+\tau}_+(\zeta)$ for all $\zeta \in [1, \infty) \times S^1$, and $\tau \in \mathbb{Z}/p\mathbb{Z}$, and $\epsilon^m_- : (-\infty, 1] \times S^1 \to C^m$, that are identified under the isomorphisms $C^m \cong \mathbb{R} \times S^1$, and performing gluing with parameter $l_+ \in [1, +\infty)$ we obtain one part of the family, $[1, \infty) \to \mathcal{R}$. Another part of the family, $(-\infty, 1] \to \mathcal{R}$ is given by gluing $S_0$ and $S_1$ along $\epsilon_0^- : (-\infty, 1] \times S^1 \to S_0$ and $\epsilon_0^+ : [1, \infty) \times S^1 \to S_1$ with gluing parameter $-l_- \in [1, \infty)$, where $l_- \in (-\infty, 1]$. Of course, by equivariance, this gives the same Riemann surface as gluing along $\epsilon_i^+, \epsilon_i^-$ would give, for all $i \in \mathbb{Z}/p\mathbb{Z}$. Finally, for $r \in [-1, 1]$, we choose an extension of $\{ |r| \geq 1 \} \to \mathcal{R}$, up to reparametrization,
to a smooth map \( \mathbb{R} \to \mathcal{M}_{\mathbb{Z}/p\mathbb{Z}} \) to the moduli space of genus 0 curves with \( \mathbb{Z}/p\mathbb{Z} \)-action, with \( p \) negative ordered marked points, and \( p \) positive ordered marked points, each \( p \)-tuple being \( \mathbb{Z}/p\mathbb{Z} \)-equivariant. This is indeed possible, by direct construction involving hyperbolic polygons: for example, representing each such complex structure by a hyperbolic metric with cusps at the marked points, and requiring that the metric be invariant under the \( \mathbb{Z}/p\mathbb{Z} \)-action (which, we recall, acts freely transitively on the set of negative cusps, and also on the set of positive cusps), as well as under the orientation-reversing involution obtained from complex conjugation on \( S \) and \((s, t) \mapsto (-s, t) \) on each \( C^m \), the parameter \( r \in \mathbb{R} \equiv \mathcal{R} \), is given, up to reparametrization, by \( r = -\log(l) \), where \( l \) is the length of the closed geodesic in the homotopy class determined by \( e^0_\ell(\text{pt} \times S^1) \).

Denote by \( \mathcal{S} \to \mathcal{R} \) the universal Riemann surface. Note that \( \mathcal{R} \) admits a natural compactification \( \overline{\mathcal{R}} \cong [0, 1] \), where \( \overline{\mathcal{R}}_0 \) is given by \( \mathcal{C} \), and \( \overline{\mathcal{R}}_1 \) is given by a nodal surface with the complement of the node given by two connected components isomorphic to \( S_C \) and \( S_P \) respectively. Finally, we note that each \( \mathcal{S}_r, r \in \mathcal{R} \), admits a holomorphic map \( \pi_r : \mathcal{S}_r \to Z = \mathbb{R} \times S^1 \), that is a branched cover with branch locus consisting of two points in \( Z \). Moreover we may choose cylindrical ends \( e^m_+ : (-\infty, -1] \times \mathcal{R} \to \mathcal{S} \), \( e^m_+ : [1, \infty) \times \mathcal{R} \to \mathcal{S} \), for \( m \in \mathbb{Z}/p\mathbb{Z} \), so that for each \( r \in \mathcal{R} \), \( e^m_{+, r} = e^m_-(-, r) \), and \( e^m_{+, r} = e^m_+(-, r) \) satisfy \( \pi_r \circ e^m_{+, r} = \text{id}_{(-\infty, -1] \times S^1} \), \( \pi_r \circ e^m_{+, r} = \text{id}_{[1, \infty) \times S^1} \), for all \( m \in \mathbb{Z}/p\mathbb{Z} \).

Counting solutions to the parametric Floer equation on the family \( \mathcal{S} \to \mathcal{R} \) of Riemann surfaces, with Floer data depending on points in \( S^\infty \), as above, provides a chain homotopy between \( \mathcal{Z}^\text{loc} \) and \( \mathcal{C}^\text{loc} \circ \mathcal{P}^\text{loc} \). We sketch the technical details below.

We choose Floer data \( \{ J_{z, w, r} \}_{r \in \mathcal{R}}, \{ Y_{z, r} \}_{r \in \mathcal{R}} \), depending on \( z \in \mathcal{S}_r \), and \( w \in S^\infty \), that with respect to the \( z, w \) coordinates are constant on the cylindrical ends, and satisfy the interpolation (with respect to the \( \pi_r \) map), \( \mathbb{Z}/p\mathbb{Z} \)-invariance, and shift-invariance axioms (see Sections 5, 8). Furthermore, as in [39, Chapter 9], we choose this Floer data compatible with the compactification \( \overline{\mathcal{R}} \) of \( \mathcal{R} \), and choices of Floer data for \( S_P, S_C \) and for \( \mathcal{C} \). Note that in fact, we should also take our fibrations compatible with the compactification. This does not present a difficulty, as the fibrations in the definitions are merely auxiliary, and all the Floer equations we consider can be written in terms of maps from suitable surfaces to \( M \).

Then for \( w : \mathbb{R} \to S^\infty \) denoting a gradient flow line that is asymptotic to \( Z^m_1 \) at \(-\infty \) and \( Z^m_0 \) at \( \infty \), one considers the moduli space \( \mathcal{M}^m_{S, \alpha}(x_0^-, \ldots, x_{p-1}^-, x_0^+, \ldots, x_{p-1}^+) \) of solutions \( (r, w, u : \mathcal{S}_r \to \mathbb{R} \times M_\alpha) \), where \( r \in \mathcal{R} \), to the parametrized perturbed Cauchy-Riemann equation

\[
(du - Y_{z, r}) \circ j = J_{z, w, r} \circ (du - Y_{z, r}),
\]

with asymptotic behaviors

\[
\lim_{s \to -\infty} u(\xi_k^-(s, t)) = x_k^-(t), \quad \lim_{s \to -\infty} u(\xi_k^+(s, t)) = x_k^+(t) \text{ for } k \in \mathbb{Z}/p\mathbb{Z}.
\]

For generic choice of \( J^w_z \) and \( H^w \), the moduli space of non-constant solutions to (102)
$\mathcal{M}^{i,m}_{c,\alpha}(x_0^+, \cdots, x_{p-1}^+; x^+)$ is a smooth finite dimensional manifold of dimension
\begin{equation}
\sum_k |x_k^-| - \sum_k |x_k^+| + i - \alpha - 2n(p - 1) + 1.
\end{equation}

For each $m \in \mathbb{Z}/p\mathbb{Z}$ and $\sum_{k=0}^{p-1} |x_k^-| = \sum_{k=0}^{p-1} |x_k^+| - i + \alpha - 1$, one can define an operation by
\begin{equation}
\mathcal{K}^{i,m}_{\alpha} : C^*(\mathbb{Z}/p\mathbb{Z}; CF^*(\phi) \otimes \mathbb{Z}/p) \to C^* / \mathbb{Z}/p \mathbb{Z}; CF^*(\phi) \otimes \mathbb{Z}/p
\end{equation}
\begin{equation}
\mathcal{K}^{i,m}_{\alpha}(x_0^+ \otimes \cdots \otimes x_{p-1}^+) = \sum \# \mathcal{M}^{i,m}_{c,\alpha}(x_0^-, \cdots, x_{p-1}^-, x_0^+, \cdots, x_{p-1}^+)(x_0^- \otimes \cdots \otimes x_{p-1}^-)
\end{equation}

Again, we set $\mathcal{K}^i_{\alpha} = \sum_m \mathcal{K}^{i,m}_{\alpha}$, and consider the map
\begin{equation}
\mathcal{K} : CF^*(\phi^p)[[u]](\theta) \to CF^*(\phi) \otimes \mathbb{Z}/p[[u]](\theta)
\end{equation}
\begin{equation}
\mathcal{K}(- \otimes 1) = (\mathcal{K}^0 + u\mathcal{K}^2 + \ldots) \otimes 1 + (\mathcal{K}^1 + u\mathcal{K}^3 + \ldots) \otimes \theta,
\end{equation}
\begin{equation}
\mathcal{K}(- \otimes \theta) = (u\mathcal{K}^2 + u^2\mathcal{K}^4 + \ldots) \otimes 1 + (\mathcal{K}^1 + u\mathcal{K}^3 + \ldots) \otimes \theta.
\end{equation}

By standard compactness, transversality, and gluing arguments, we obtain that
\begin{equation}
C^1_x \circ \mathcal{P}^{loc}_x - \mathcal{Z}^{loc}_x = d_{x/\mathbb{Z}, loc}^Z \mathcal{K} - \mathcal{K}_{d_{x/\mathbb{Z}, loc}}^Z,
\end{equation}
as required.

**Remark 11.** The same definition of $\mathcal{Z}$ works in the general context of local Floer cohomology, as well as globally, in the symplectically aspherical and exact cases, and the above argument can be modified to prove that $\mathcal{Z}$ is chain-homotopic to $\mathcal{C} \circ \mathcal{P}$. In general, however, these maps differ on the chain level. In case of Hamiltonian diffeomorphisms, this point is not necessary for us, since it is straightforward to see that $HF_{\mathbb{Z}/p\mathbb{Z}}(\phi^p) \cong H^*(M) \otimes R_p$ by either a continuation map argument, or by a suitable equivariant PSS map (see [46]). Moreover, we require the local version of this map for our main invertibility argument. We shall, however, require the fact that $\mathcal{Z}$ is a chain map in a very particular local case for the proof of Lemma [42].

**Remark 12.** The above curve $C$ suggests that the configurations one should consider in the presence of holomorphic spheres (in the non-local, non weakly exact case). We believe that at least when the manifold is monotone this can be carried out, however we opt to leave this point for discussion elsewhere.

For a matrix $A \in \text{Sp}(2n, \mathbb{R})$ and $\lambda \in \mathbb{C}$ set $\rho_\lambda(A) = \det(A - \lambda \cdot \text{id})$, and denote
\begin{equation}
\text{Sp}(2n, \mathbb{R})^* = \text{Sp}(2n, \mathbb{R}) \setminus \rho^{-1} \{0\}.
\end{equation}
Note that \( \text{Sp}(2n, \mathbb{R})^* \) has two connected components, and
\[
\text{sign}(\rho_1) : \pi_0(\text{Sp}(2n, \mathbb{R})^*) \to \{\pm 1\}
\]
determines an isomorphism of sets. Similarly, let
\[
\text{Sp}(2n, \mathbb{R})^{p*} = \text{Sp}(2n, \mathbb{R}) \setminus \bigcup_{\lambda^p = 1} \rho^{-1}_\lambda(\{0\}).
\] (110)

**Lemma 13.** \( Z_{x}^{\text{loc}} = c_{n, \pm} \cdot u^{(p-1)n} \cdot \text{id} \), where \( c_{n, \pm} \) depends only on the connected component of \( D\phi_x \in \text{Sp}(2n, \mathbb{R})^* \).

**Proof.** We follow [38, Section 6], the difference being that we consider the operator \( Z_{x}^{\text{loc}} \), and hence we may deform the differential of the symplectomorphism inside \( \text{Sp}(2n, \mathbb{R})^* \).

First of all, by degree reasons, and since \( CF^*(\phi)^{\text{loc}}_{\phi} \) has, in the non-degenerate case that we consider, the unique generator \( x \),
\[
Z_{x}^{\text{loc}}(x \otimes p \otimes 1) = \left( \sum \# M_{\mathbb{Z}, 0}^{2n(p-1), m} (x, \ldots, x; x, \ldots, x) \right) u^{n(p-1)}x \otimes p \otimes 1.
\]

It is sufficient to show that for an isolated deformation of the germ of \( \phi \) at \( x \), keeping \( x \) a non-degenerate fixed point, the count \( \sum \# M_{\mathbb{Z}, 0}^{2n(p-1), m} (x, \ldots, x; x, \ldots, x) \) remains invariant. This is carried out precisely as in [38, Section 6], by a cobordism argument, which ultimately works because the structure of the compactification of the spaces of Morse flow lines \( Q_{a_0}^i \) yields cancellations of the boundary points of the compactified one-dimensional parametric moduli spaces.

**Lemma 14.** We have \( c_{n, +} = c_{n, -} \).

**Proof.** Here we follow [38, Section 7], for \( Z_{x}^{\text{loc}} \). Consider the case when locally in a ball \( B \subset (\mathbb{R}^{2n}, \omega_{\text{st}}) \), \( \phi \) has two non-degenerate fixed points \( y, z \) with \( dy = z \). For example, we may take a small Morse function \( H \) with \( y, z \) being critical points, that lie in \( B \), with one gradient trajectory from \( y \) to \( z \), and let \( \phi \) be the Hamiltonian flow of \( H \) for a small positive time \( \epsilon > 0 \).

Hence \( HF^{\text{loc}}(\phi, B) = 0 \), and hence \( H^*(\mathbb{Z}/p\mathbb{Z}, CF^{\text{loc}}(\phi, B)) = 0 \). Moreover, clearly, for the Tate cohomology \( \hat{H}^*(\mathbb{Z}/p\mathbb{Z}, CF^{\text{loc}}(\phi, B)) = 0 \). Now, the Tate cohomology can be computed by the action spectral sequence, whose \( E^{p+1}\text{-st} \) page is given by the map induced from \( d : \mathbb{K}(y) \to \mathbb{K}(z) \), which is the identity map, in the natural bases \( \{y\}, \{z\} \), by the natural quasi-Frobenius isomorphisms (see Lemma 4)
\[
\hat{H}^*(\mathbb{Z}/p\mathbb{Z}, CF^{\text{loc}}_{y}(\phi)) \cong \hat{R}_p \otimes_{K} \mathbb{K}(y)(1),
\]
\[
\hat{H}^*(\mathbb{Z}/p\mathbb{Z}, CF^{\text{loc}}_{z}(\phi)) \cong \hat{R}_p \otimes_{K} \mathbb{K}(z)(1).
\]
We proceed to note that $D\phi(y), D\phi(z)$ lie in different components of $\text{Sp}(2n, \mathbb{R})^*$. Furthermore, since $Z$ induces a chain map between the Tate complexes, it induces a map of the action spectral sequences, for sufficiently small perturbation data, and in particular it induces a chain map on the $E^{0+1}$-st page. This immediately yields $c_{n,+} = c_{n,-}$. 

Below, we apply a suitable Floer-to-Morse reduction to show that $c_{n,+} = (-1)^n$, and hence by Lemma 14 $c_{n,\pm} = (-1)^n$. We could also calculate $c_{n,-}$ separately in the same way, proving our result without the above lemma.

### 10.2 Local Floer-to-Morse collapse

Section 10.1 allows one to reduce the consideration of $Z^\text{loc}_x$ to the local case of a fixed ball $B \subset (\mathbb{C}^n, \omega_{st})$ of radius 1, and symplectomorphism $\phi$ generated by Hamiltonian $H = \epsilon \cdot |z|^2$, where $|z|^2 = \frac{1}{2} \sum_{j=1}^n |z_j|^2$, and $1 \gg \epsilon > 0$ chosen arbitrarily. Note that we may choose the complex structure to coincide with the standard one $J_{st}$ outside $\frac{19}{20}B$, and to be $C^2$-close to $J_{st}$ on $B$. In this section we reduce the calculation in this case to Morse theory.

We start with the classical observation that the local Floer complex of $\phi^1_H$ at 0, as computed with an $\omega_{st}$-compatible almost complex structure $J$ coinciding with $J_{st}$ outside $\frac{19}{20}B$, is isomorphic to the local Lagrangian Floer complex $CF^\text{loc}(\Delta, H \oplus 0, x)$ of the Lagrangian diagonal

$$\Delta \subset X = \mathbb{C}^n \times (\mathbb{C}^n)^-,$$

at $(\phi^1_H \times \text{id})^{-1} \Delta \cap \Delta = \{ x = (0, 0) \}$, with Hamiltonian perturbation $H \oplus 0$, and almost complex structure $J \oplus -J$. Furthermore, denoting by $\Delta_H = (\phi^1_{H \oplus 0})^{-1} \Delta$, under this identification

$$Z^\text{loc}_x = Z^\text{loc,Lagr}_x : C^*(\mathbb{Z}/p\mathbb{Z}; CF^\text{loc}(\Delta, \Delta_H, x)^{\oplus p}) \to C^*(\mathbb{Z}/p\mathbb{Z}; CF^\text{loc}(\Delta, \Delta_H, x)^{\oplus p}),$$

the latter being defined analogously to $Z^\text{loc}_x$, yet in terms of Lagrangian Floer homology.

Finally, we note that the complex structure $J_{st} \oplus -J_{st}$ on $X(\mathbb{C}^n)^- \cong T^*\Delta$, coincides with the complex structure on $X$, induced by the standard Riemannian metric $g$ on $\Delta$. We recall that a Riemannian metric induces an $\omega_{\text{can}} = d(\lambda_{\text{can}})$-compatible almost complex structure by identifying $T_{(p,q)}(T^*\Delta) \cong T^*_q \Delta \oplus T_q \Delta$, via the Levi-Civita connection, and acting by $(\alpha, \xi) \mapsto (-G\xi, G^{-1}\alpha)$, where $G : T_q \Delta \to T^*_q \Delta$ is the isomorphism $G(\xi) = g_q(\xi, -)$. Furthermore, we recall that the symplectomorphism $\Theta : X \to T^*\Delta$, with the standard symplectic structures, is given by the symplectic Cayley transform $(z, w) \mapsto (p, q), p = \frac{w - \bar{z}}{i\sqrt{2}}, q = \frac{z + \bar{w}}{\sqrt{2}}$.

Hence we rewrite the $Z^\text{loc}_x$ map yet again as

$$Z^\text{loc}_x = Z^\text{loc,cot}_x : C^*(\mathbb{Z}/p\mathbb{Z}; CF^\text{loc}(\Delta, \pi^*F, x)^{\oplus p}) \to C^*(\mathbb{Z}/p\mathbb{Z}; CF^\text{loc}(\Delta, \pi^*F, x)^{\oplus p})$$
with $F \in C^\infty(\Delta, \mathbb{R})$ such that

$$\Gamma_{dF} = \Delta_H = (\phi_{H \equiv 0}^1)^{-1}\Delta.$$ 

The Morse function $F$ is given in terms of $H$ by the Cayley transform. In particular, like $H$, $F$ has a unique non-degenerate minimum at $x$. More precisely, for $q \in \Delta \cong \mathbb{C}^n$,

$$F(q) = \Theta^\# H(q) = \int_0^1 \langle (\pi|_{\Delta_H})^{-1}(t \cdot q), q \rangle \, dt.$$ 

In other words $\Theta^\# H(q) = \int_{\{t \in [0, 1] \mid (t \cdot q) \}} \lambda$. This formula establishes a bijective correspondence between functions $H'$ that are $C^2$-close to $H$ and functions $F'$ that are $C^2$-close to $F$, that is continuous in the $C^2$-topology. Moreover, the last observation applies to $H = 0$. Therefore by making $H$ sufficiently $C^2$ small in a suitable neighborhood of $0$ we can make $F$ as $C^2$-small as necessary in a neighborhood of $x$. Hence we may consider the latter model for all our purposes.

The above Lagrangian reformulation allows us somewhat more freedom in the choice of almost complex structures, while the reformulation in terms of the function $F$ allows us to relate our constructions to Morse theory. We shall use both, together with a classical convexity argument of Floer [11] to show the following result.

**Lemma 15.** There exists $\epsilon_0 > 0$ such that for Hamiltonian perturbations $F_s$ of $F$ that have $C^2$ norm smaller than $\epsilon_0$, and coincide with $F$ outside $\frac{1}{20}B$, all continuation maps

$$C(\{F_s\}) : C\overline{F}_{loc}(\Delta, \pi^* F, x) \to C\overline{F}_{loc}(\Delta, \pi^* F, x)$$

along a family of Hamiltonians $\{F_s\}_{s \in \mathbb{R}}$, with $F_s = F$ for $|s| \gg 1$, and almost complex structures $J_s$ suitably chosen, in fact coincide with the Morse continuation maps along $\{F_s\}$ considered as Morse functions, with suitable Riemannian metrics $g_s$.

As a consequence, we obtain a Morse-theoretical description of $Z_{x, \text{cot}}^{\text{loc}}$, which we detail in slightly larger generality in Section 10.3, and use in Section 10.4.

**Proof of Lemma 15.** We begin by describing the class of almost complex structures $J_s$ that we consider. Firstly, as long as a complex structure is sufficiently $C^2$-close to $J_{st} \oplus -J_{st}$, a classical monotonicity argument [11 Section 4.3], together with standard action estimates (similar to [67] see [39] Section (8g)]) show that all relevant curves do not escape a given fixed neighborhood $U \subset B \times B$ of $x$. We take as $U$ a symplectically embedded copy of $D^*(\sqrt{2}B)$, the unit co-disk bundle taken with respect to the standard metric on $B$ (indeed $x \in U$).

Hence, given $g_s$ sufficiently $C^2$-close to $g$, we may require that in $U$, $J_s$ concides at the zero section with the almost complex structure on the co-disk bundle induced by $g_s$. In this case, each trajectory $\gamma : \mathbb{R} \to \mathbb{C}^n$ satisfying the Morse continuation equation

$$\partial_s \gamma + (\nabla F_s) \circ \gamma = 0,$$ 

(111)
and inputs and one output, and call $\Gamma_p$ the analogue of Fukaya and Betz-Cohen operations defined by counting of Morse trees with a $\mathbb{Z}$-number $p$ of Wilkins [45] recently.

We let $C^\ast(M) := C^\ast_{\text{Morse}}(M)$ be the Morse cochain complex defined by $f$, which means that we grade the critical points of $f$ by $|x| = n - \text{Ind}_M(f)$. For each prime $p$, one can define the $p$th product map on the Morse cochain complex by $P_M : C^\ast(M)^{\otimes p} \to C^\ast(M)$ by

$$P_M(x_1 \otimes x_2 \cdots \otimes x_p) = \sum_{x_0 \in \text{Crit}(f)} \#\mathcal{M}(\Gamma_p; f^i_s)x_0,$$

(113)
where \( \#\overline{\mathcal{M}}(\Gamma_p; f^i_s) \) denotes the signed counts of the virtual dimension zero part of the moduli space of Morse trajectories \( u: \Gamma_p \to M \) satisfy the following conditions

1. \( d(u|_{c_i})/ds = -\nabla f^i_s \) for all \( i = 0, 1, \ldots, p \);
2. \( \lim_{s \to -\infty} u|_{c_{\text{out}}}(s) = x_0 \);
3. \( \lim_{s \to \infty} u|_{c_{\text{in}}}(s) = x_0 \) for all \( j = 1, 2, \ldots, p \).

We remark that our perturbations \( f^i_s \) of \( f \) should be chosen so that the corresponding vector fields \( -\nabla f^i_s \) are transverse at the point \( u(0) \), which is equivalent to the condition that the vectors \( -\nabla \text{grad}(f^i_s)(u(0)) \) span the tangent space \( T_{u(0)}M \). For such perturbations, the moduli space of Morse trajectories \( \mathcal{M}(\Gamma_p; f^i_s) \) can be shown to be regular. Now if all the inputs are the same, that is, one has \( x_1 = x_2 = \cdots = x_p \), then the domain graph \( \Gamma_p \) admits a free and transitive action by \( \mathbb{Z}/p\mathbb{Z} \), preserving the boundary conditions. We would like to study the equivariant operation that this symmetry produces.

A priori if one takes the perturbation \( f^i_1 = \cdots = f^i_p \) respectively, the moduli space \( \mathcal{M}(\Gamma_p; f^i_s) \) also admits a free and transitive action by \( \mathbb{Z}/p\mathbb{Z} \), which one may want to study its quotient and define the \( \mathbb{Z}/p\mathbb{Z} \)-equivariant product by counting virtual dimension zero part of \( \mathcal{M}(\Gamma_p; f^i_s)/\mathbb{Z}/p\mathbb{Z} \). However, there is a major problem that \( \mathcal{M}(\Gamma_p; f^i_s) \) cannot be made regular for \( \mathbb{Z}/p\mathbb{Z} \)-symmetric perturbations \( f^i_s \), since one cannot achieve \( \mathbb{Z}/p\mathbb{Z} \) transversality for elements in \( \mathcal{M}(\Gamma_p; f^i_s) \). As a solution, we consider \( \mathbb{Z}/p\mathbb{Z} \)-equivariant perturbations \( \{f^i_s,u\} \) parametrized by \( w \in S^\infty \) as before. For a given Morse trajectory \( w: \mathbb{R} \to S^\infty \) which is asymptotic to a critical point \( Z^m_i \) of index \( i \) when \( s \to -\infty \) and another critical point \( Z^0_\alpha \) of index \( \alpha \in \{0,1\} \) as \( s \to \infty \), one has that the \( \mathbb{Z}/p\mathbb{Z} \)-equivariant perturbation satisfy

1. (Diagonal \( \mathbb{Z}/p\mathbb{Z} \)-action) \( f^m_{s,m(w)} \) for all \( i = 1,2,\ldots,p \) and \( f^0_{s,z} = f_{s,m(w)} \) for all \( m \in \mathbb{Z}/p\mathbb{Z} \) and \( w \in S^\infty \).
2. (Invariant under shift) \( f^i_{s,r(w)} = f^i_{s,u} \) for all \( i \) and \( w \in S^\infty \).

Given any parametrized Morse flow line \( w \in \mathcal{P}^{i,m}_\alpha \) for fixed \( i \) and \( m \in \mathbb{Z}/p\mathbb{Z} \), we define the operation \( \mathcal{P}^{i,m}_{M,\alpha}: C^*(M)^{\otimes p} \to C^{*-i+\alpha}(M) \) by

\[
\mathcal{P}^{i,m}_{M,\alpha}(x_1 \otimes \cdots \otimes x_p) = \sum_{x_0} \#\overline{\mathcal{M}}^{i,m}_\alpha(M; f^i_{s,z}) x_0,
\]

where \( \#\overline{\mathcal{M}}^{i,m}_\alpha(M; f^i_{s,z}) \) denotes the signed counts of the virtual dimension zero part of the moduli space of pairs \((u,w)\) satisfying \( w \in \mathcal{P}^{i,m}_\alpha \)

\[
\begin{cases}
d(u|_{c_i})/ds = -\nabla f^i_{s,z} \text{ for all } i = 0, 1, \ldots, p; \\
\lim_{s \to -\infty} u|_{c_{\text{out}}}(s) = x_0; \\
\lim_{s \to \infty} u|_{c_{\text{in}}}(s) = x_0 \text{ for all } j = 1, 2, \ldots, p.
\end{cases}
\]

(114)
It is then straightforward to check that, as in the definitions of Section 8, the operations \( \mathcal{P}_{M,\alpha} \) combine to give a chain map
\[
\mathcal{P}_M: C^*(\mathbb{Z}/p\mathbb{Z}, CM^*(f)) \to CM^*(pf) \otimes R_p.
\]

10.4 Morse coproduct, and the local case

Similarly to the operation \( \mathcal{P}_M \) defined above, we define operations
\[
\mathcal{C}_M: CM^*(f) \otimes R_p \to C^*(\mathbb{Z}/p\mathbb{Z}, CM^*(f)^{\otimes p}),
\]
given by the graph \( \Gamma_p \) with one input and \( p \) outputs and
\[
\mathcal{Z}_M: C^*(\mathbb{Z}/p\mathbb{Z}, CM^*(f)^{\otimes p}) \to C^*(\mathbb{Z}/p\mathbb{Z}, CM^*(f)^{\otimes p}),
\]
given by a graph with \( p \) inputs and \( p \) outputs. Moreover, the analysis in Section 10.1 above shows that \( \mathcal{Z}_M \) is chain homotopic to \( \mathcal{C}_M \circ \mathcal{P}_M \).

Considering the case of local Morse homology of a function \( f(z) = \epsilon |z|^2 \) on \( \mathbb{C}^n \) at \( 0 \in \mathbb{C}^n \), we get that under the natural isomorphisms of the equivariant cohomology groups with \( R_p \), we have
\[
\mathcal{Z}_M = (-1)^n u^{n(p-1)} \cdot \text{id}.
\]

In this case the construction of Section 10.3 for \( \mathcal{Z}_M \) is in fact given, in the topological model of equivariant cohomology, by multiplication by the Euler class of \( E \), given by intersection product with the zero section in the total space of \( E \), taking into account that the latter is deformation equivalent to the base (see also [4]).

We calculate this Euler class as follows. Using the standard isomorphism \( \mathbb{C}^n \cong \mathbb{R}^{2n} \) of real vector spaces, we endow \( \mathbb{R}^{2n} \) with a complex structure and note that the \( \mathbb{Z}/p\mathbb{Z} \)-action on \( (\mathbb{R}^{2n})^p \) is in fact complex-linear. Hence we may consider \( E, E^{\mathbb{Z}/p\mathbb{Z}} \) and \( V \) as complex vector bundles of ranks \( np, n, n(p-1) \) respectively. In particular the Euler class of \( V \) is given by its top Chern class \( c_{n(p-1)}(V) \). Considering \( (\mathbb{C}^n)^p \) as a complex \( \mathbb{Z}/p\mathbb{Z} \)-representation and splitting it into isotypical components, we obtain the isomorphism of complex vector bundles
\[
V \cong \left( \bigoplus_{\chi \in (\mathbb{Z}/p\mathbb{Z})^\vee \setminus \{1\}} L_{\chi} \right)^{\otimes n},
\]
where \( L_{\chi} = \mathbb{C} \times_{\mathbb{Z}/p\mathbb{Z}} S^\infty \) is the complex line bundle over \( B(\mathbb{Z}/p\mathbb{Z}) \) associated to the character \( \chi \in (\mathbb{Z}/p\mathbb{Z})^\vee = \text{hom}(\mathbb{Z}/p\mathbb{Z}, \mathbb{C}^\times) \). Note that the sum does not include the
component of the trivial character, since we took the quotient by the \( \mathbb{Z}/p\mathbb{Z} \)-invariants. Now by [2, Section 8], the \((\mathbb{F}_p\text{-reductions of})\) \(c_1(L_\chi)\) for \(\chi \in (\mathbb{Z}/p\mathbb{Z})^\vee\) are given by \(a_\chi u\) for different invertible elements \(a_\chi\) in \(\mathbb{F}_p\), where \(u\) is the standard generator of \(H^2(B(\mathbb{Z}/p\mathbb{Z}), \mathbb{F}_p) \cong \mathbb{F}_p\), coming from \(H^2(\mathbb{C}P^\infty, \mathbb{F}_p)\). In view of the Whitney sum formula, we obtain

\[
c_{n(p-1)}(V) = \left( \prod_{\chi \in (\mathbb{Z}/p\mathbb{Z})^\vee \setminus \{1\}} a_\chi \right)^n u^{n(p-1)} = (-1)^n u^{n(p-1)},
\]

the last step being Wilson’s theorem, whereby

\[
\prod_{\chi \in (\mathbb{Z}/p\mathbb{Z})^\vee \setminus \{1\}} a_\chi = -1.
\]

However, we saw in Section 10.2 that in our particular local case, for \(x = 0\) a fixed point of the Hamiltonian flow \(\phi = \phi^1_H\) of \(H(z) = f(z)\),

\[
\mathcal{Z}^\text{loc}_x = \mathcal{Z}_M.
\]

This finishes the proof.

11 Proof of Theorem A: spectral sequence argument

First let us consider the case when \(\phi\) and \(\phi^p\) as above have all fixed points non-degenerate. Consider the equivariant product map

\[
\mathcal{P} : C^*(\mathbb{Z}/p\mathbb{Z}, CF^*(\phi)^{\otimes p}) \to CF^*_\mathbb{Z}/p\mathbb{Z}(\phi^p).
\]

It induces a map on the action spectral sequences associated to the action filtrations on both sides. Tensoring with \(\mathcal{K}\) over \(\mathbb{K}[[u]]\), we obtain a map of the corresponding spectral sequences for Tate cohomology groups. In view of Sections 2 and 9, the \(E_1\)-page on the left is given by

\[
\bigoplus \mathcal{H}^*(\mathbb{Z}/p\mathbb{Z}, HF^\text{loc}(\phi, x)^{\otimes p})
\]

and on the right by

\[
\bigoplus HF^\text{loc}_{\mathbb{Z}/p\mathbb{Z}}(\phi^p, x^{(p)}),
\]

the sum running over all the fixed points \(x\) of \(\phi\), and furthermore \(\mathcal{P}\) induces the map

\[
\oplus \mathcal{P}^\text{loc}_x
\]

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on these $E_1$ pages. By Section 11, this map on the $E_1$ page is invertible, and hence by the spectral sequence comparison argument, we obtain that

$$P : \widehat{H}^*(\mathbb{Z}/p\mathbb{Z}, HF^*(\phi)^{\otimes p}) \to \widehat{HF}_*^{\mathbb{Z}/p\mathbb{Z}}(\phi^p)$$

is an isomorphism.

Now, we first note that the same argument, after choosing a small isolating neighborhood and a small Hamiltonian perturbation non-degenerate therein, applies to prove that

$$P_{\text{loc}} : \widehat{H}^*(\mathbb{Z}/p\mathbb{Z}, HF_{\text{loc}}^*(\phi, x)^{\otimes p}) \to \widehat{HF}_{\mathbb{Z}/p\mathbb{Z}}(\phi^p, x^{(p)})$$

is an isomorphism for an isolated fixed point of $\phi$ that is isolated as a fixed point of $\phi^p$ as well.

Similarly, choosing an interval $I = (a, b)$ with $pa, pb \in \mathbb{R}\setminus(\text{Spec}(\phi^p) \cup \text{Spec}(\phi^+p))$, (where for a subset $A \subset \mathbb{R}$ we set $A^{+p} = \{a_1 + \ldots + a_p | a_j \in A, 1 \leq j \leq p\}$) the same argument, after choosing sufficiently small non-degenerate Hamiltonian perturbations $\phi_1, (\phi_1)^p$ of $\phi, \phi^p$, we obtain that

$$P^{p,I} : \widehat{H}^*(\mathbb{Z}/p\mathbb{Z}, CF^*(\phi)^{\otimes p})^{p,I} \to \widehat{HF}_*^{\mathbb{Z}/p\mathbb{Z}}(\phi^p)^{p,I}$$

is an isomorphism, where $\widehat{H}^*(\mathbb{Z}/p\mathbb{Z}, CF^*(\phi)^{\otimes p})^{p,I} = \widehat{H}^*(\mathbb{Z}/p\mathbb{Z}, (CF^*(\phi)^{\otimes p})^{p,I})$, for the quotient complex

$$(CF^*(\phi)^{\otimes p})^{p,I} \cong (CF^*(\phi)^{\otimes p})(CF^*(\phi)^{\otimes p})^{p,a}/(CF^*(\phi)^{\otimes p})^{p,b},$$

where for $x_1 \otimes \ldots \otimes x_p \in CF^*(\phi)^{\otimes p}$, we set

$$A_{\phi, \otimes p}(x_1 \otimes \ldots \otimes x_p) = \sum A_{\phi}(x_j).$$

Observe that there is a natural inclusion of complexes

$$(CF^*(\phi)^I)^{\otimes p} \to (CF^*(\phi)^{\otimes p})^{p,I},$$

such that the $\mathbb{Z}/p\mathbb{Z}$-action on the quotient is free. Therefore by Lemma 3 it induces a natural isomorphism

$$\widehat{H}^*(\mathbb{Z}/p\mathbb{Z}, (CF^*(\phi)^I)^{\otimes p}) \cong \widehat{H}^*(\mathbb{Z}/p\mathbb{Z}, (CF^*(\phi)^{\otimes p})^{p,I}).$$

In all cases, we finish the proof by an application of Lemma 4.
12 Applications and discussion

First, in the exact case, we prove the following version of the Smith inequality for fixed point Floer cohomology, extending the work of Seidel to prime orders $p > 2$.

**Theorem B.** Let $\phi$ be an exact symplectomorphism of a Liouville domain $W$, cylindrical at infinity. Then the ranks of its fixed point Floer cohomology, and that of its $p$-th iterate $\phi^p$ satisfy the following inequality:

$$\dim_{F_p} HF^*(\phi) \leq \dim_{F_p} HF^*(\phi^p) \leq \dim_{F_p} HF^*(\phi^p).$$

**Proof.** Observe that by the main result, Theorem A, we obtain

$$2 \dim_{F_p} HF^*(\phi) = \dim_{K((u))} \hat{H}^*(\mathbb{Z}/p\mathbb{Z}, HF^*(\phi)) = \dim_{K((u))} \hat{H}^*_{\mathbb{Z}/p\mathbb{Z}}(\phi^p).$$

However by estimate (66) the latter dimension satisfies

$$\dim_{K((u))} \hat{H}^*_{\mathbb{Z}/p\mathbb{Z}}(\phi^p) \leq \dim_{K((u))} \hat{H}^*(\mathbb{Z}/p\mathbb{Z}, HF^*(\phi^p)),$$

whence we obtain the bound

$$2 \dim_{F_p} HF^*(\phi) \leq \dim_{K((u))} \hat{H}^*(\mathbb{Z}/p\mathbb{Z}, HF^*(\phi^p)) \tag{115}$$

By the structure theorem for modules over PID, and the observation that $K[\mathbb{Z}/p\mathbb{Z}] \cong K[t]/\langle t^p \rangle$, we have that $HF^*(\phi^p)$, as a $K[\mathbb{Z}/p\mathbb{Z}]$-module, splits into a direct sum

$$HF^*(\phi^p) = \bigoplus_{0 \leq k \leq p} (K[t]/\langle t^k \rangle)^{\otimes m_k},$$

for multiplicities $m_k \geq 0$. It is easy to calculate that

$$\dim_{K((u))} \hat{H}^*(\mathbb{Z}/p\mathbb{Z}, HF^*(\phi^p)) = 2(m_0 + \ldots + m_{p-1}),$$

while $2 \dim_{F_p} HF^*(\phi^p)_{\mathbb{Z}/p\mathbb{Z}} = 2(m_0 + \ldots + m_{p-1} + m_p).$ \hfill \Box

**Remark 16.** Note that if $m_p > 0$, then the bound (115) is strictly stronger than the classical Smith inequality. We observe that the algebraic methods used in the proof of this bound work in the classical setting of the Smith inequality (see [5, 6, 20]), for example when the space in question is a finite CW complex, and provide a sharpening thereof. We have not found this sharper version in the literature. Of course in the classical setting, quite a lot more than the Smith inequality is known by now. For example, the cohomology of the fixed point set of a $\mathbb{Z}/p\mathbb{Z}$-action was completely described in terms the associated equivariant cohomology, as a module over both $R_p$ and the Steenrod algebra [9]. It would be very interesting to find an analogue of the latter result in Floer theory.
Corollary 17. In particular, if \( \dim_{\mathbb{F}} HF^*(\phi) > \dim_{\mathbb{F}} H^*(W) \) then \([\phi^p] \neq 1\) in the symplectic mapping class group of \( W \) for all \( k \geq 0 \).

This corollary is immediate, since \( HF^*(id) \cong HF^*(W) \) and \( HF^*(\phi) \) is invariant under isotopies of exact symplectomorphisms of \( W \) cylindrical at infinity. Furthermore, iterating the inequality of Theorem [3] we obtain that \( \dim_{\mathbb{F}} HF^*(\phi^p) \) is a non-decreasing function of \( k \in \mathbb{Z}_{\geq 0} \).

Furthermore, in both the exact and the symplectically aspherical setting, denoting for an open interval \( I \subset \mathbb{R} \), of the form \((a, b), a, b \in \mathbb{R}, \) or \((-\infty, b), b \in \mathbb{R}, \) by \( HF^*(\phi)^I \)

the cohomology of the subcomplex generated by points of action value in \( I \), and by \( k \cdot I \) for \( k > 0 \), the interval \((ka, kb)\) in the first case and \((-\infty, kb)\) in the second case, we obtain the following sharpening of Theorem [3]. We always assume that the finite endpoints of an interval are not in the spectrum of the associated Hamiltonian diffeomorphism.

We note that it is interesting for Hamiltonian diffeomorphisms on symplectically aspherical manifolds, even though the total cohomology is trivial, in the sense that in this case \( HF^*(\phi) \cong HF^*(id) \cong H^*(M) \) for all \( \phi \in \text{Ham}(M, \omega) \).

Theorem C. Let \( \phi \) be an exact symplectomorphism of a Liouville domain \( W \), cylindrical at infinity or a Hamiltonian diffeomorphism of a closed symplectically aspherical manifold. Then the ranks of its fixed point Floer cohomology in action window \( I \), and that of its \( p \)-th iterate \( \phi^p \) in action window \( p \cdot I \) satisfy the following inequality:

\[
\dim_{\mathbb{F}} HF^*(\phi)^I \leq \dim_{\mathbb{F}} (HF^*(\phi^p)^{p \cdot I})^{\mathbb{Z}/p\mathbb{Z}} \leq \dim_{\mathbb{F}} HF^*(\phi^p)^{p \cdot I}.
\]

We remark that this inequality is invariant with respect to shifts of the relevant action functionals, and therefore makes sense independently of them.

In both settings, we can consider the system \( (HF^*(\phi)^{< t}, \pi^{s,t}) \) where \( t \in \mathbb{R}, s \leq t, \) and \( HF^*(\phi)^{< t} := HF^*(\phi)^{(-\infty, t)} \), while \( \pi^{s,t} : HF^*(\phi)^{< s} \to HF^*(\phi)^{< t} \) is the map induced by inclusions. In case \( \phi \) is non-degenerate, it was observed in [31] that this system is a persistence module with certain additional constructibility properties. It therefore has an associated finite barcode, determined uniquely up to permutation: a finite multisets \( \mathcal{B}(\phi) = \{ (I_j, m_j) \} \) of intervals of the form \( I_j = (a_j, b_j] \) or \((a_j, \infty)\), and multiplicities \( m_j \in \mathbb{Z}_{> 0} \). One of the properties of such barcodes is that

\[
\dim HF^*(\phi)^{(-\infty, t)} = \sum_{t \in I_j} m_j,
\]

\[
\dim HF^*(\phi)^{(a, b)} = \sum_{b \in I_j, a \notin I_j} m_j.
\]
Furthermore, as an application, Theorem C gives a new proof of a celebrated "no-torsion" theorem of Polterovich for the Hamiltonian group of closed symplectically aspherical manifolds. Further applications to Hamiltonian dynamics are forthcoming.

We normalize actions in such a way that the barcode of id consists of $\dim HF^*(\phi)$ infinite bars starting at 0. In the closed symplectically aspherical case this is ensured by requiring that the Hamiltonian $H \in \mathcal{H}$ generating $\phi$ have zero mean over $M$ for all $t \in [0,1]$.

**Theorem D** (Polterovich [30]). Let $\phi \in \text{Ham}(M,\omega)$ be a Hamiltonian diffeomorphism of a symplectically aspherical symplectic manifold, such that $\phi^k = 1$ for some $k \in \mathbb{Z}_{>1}$. Then $\phi = id$.

Our new proof proceeds as follows. Let $p$ be a prime dividing $k$. Then $\phi_1 = \phi^{k/p}$ satisfies $\phi_1^p = id$. By induction it is therefore enough to prove the theorem for all $k = p$ prime. Fix a prime $p$ and suppose $\phi \neq id$, $\phi^p = id$. Fix $\mathbb{F}_p$ as the coefficient field for Floer cohomology. Let $c_+$ and $c_-$ be the maximal and minimal starting point of an infinite bar. By [31] we have $\phi \neq id$ if and only if $c_+(\phi) > c_-(\phi)$. Therefore there exists an interval $I = (a,b)$ with closure contained in $\mathbb{R} \setminus \{0\}$, such that $a, b$ are not in the spectrum of $\phi$, $pa, pb$ are not in the spectrum of $\phi^p$, and

$$\dim_{\mathbb{F}_p} HF^*(\phi)^I > 0.$$ 

Then by Theorem C we obtain $\dim_{\mathbb{F}_p} HF^*(\phi^p)^{pI} \geq 1$. However, since $\phi^p = id$, and $0 \notin p \cdot I$,

$$\dim_{\mathbb{F}_p} HF^*(\phi^p)^{pI} = 0.$$ 

This is a contradiction that finishes the proof.

### A Orientations and signs

As we need to work with $\mathbb{F}_p$ coefficients when defining the relevant Floer cochain groups and various operations, in this appendix we discuss how to choose consistent orientations on our moduli spaces in order to obtain well-defined signed counts in Sections 5, 6, 8.

Let $\hat{S} = S - (Z^+ \cup Z^-)$ be a punctured Riemann surface, where $Z^\pm = \bigcup_{i=1}^p \{z_i^\pm\}$ are finite subsets of a closed Riemann surface $S$. One can equip it with cylindrical ends

$$\epsilon_i^+: [1, +\infty) \times S^1 \to \hat{S} \text{ and } \epsilon_i^-: (-\infty, -1] \times S^1 \to \hat{S} \text{ for } i = 1, \cdots, p^+, \ j = 1, \cdots, p^-.$$ 

We consider the solutions $u: \hat{S} \to \mathbb{R} \times M_{\phi}$ to the perturbed Cauchy-Riemann equation of the form

$$\left(du - Y_z\right) \circ j = J_{w(s),t,z} \circ (du - Y_z), \quad (116)$$

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where \( w(s) \in P_{\alpha,m} \) is a parametrized Morse flow line of the \( \mathbb{Z}/p\mathbb{Z} \)-equivariant Morse function on \( S^\infty \) and \( Y_z \in \Omega^1(S^\infty, T(\mathbb{R} \times M_\phi)) \). Furthermore, the solution \( u \) satisfies the asymptotic conditions

\[
\lim_{s \to \pm} u(\epsilon_i^\pm(s,t)) = \gamma_i^\pm(t), \quad i = 1, \cdots, p^\pm
\]

for some sets of Reeb orbits \( \Gamma^\pm = \bigcup_{i=1}^{p^\pm} \gamma_i^\pm \) at the positive and the negative ends of the symplectization \( \mathbb{R} \times M_\phi \) respectively. We denote by \( M(\Gamma^+, \Gamma^-) \) the moduli space of solutions to the equation (116) modulo automorphisms. We will first explain how to orient this moduli space as follows. Given any path \( \Psi(t) \) in \( Sp(2n) \) such that \( \Psi(0) = I \) and \( \det(I - \Psi(1)) \neq 0 \), one can reparametrize \( \Psi(t) \) and associate it with a loop of symmetric matrices \( S(t) \) that satisfies

\[
\dot{\Psi}(t) = J_0 S(t) \cdot \Psi(t).
\]

Such a loop \( S(t) \) will be called nondegenerate if \( \det(I - \Psi(1)) \neq 0 \). For nondegenerate \( S(t) \), we denote by \( O_+(S) \) and \( O_-(S) \) the spaces of all operators of the form

\[
D_\psi: W^{1,p}(\mathbb{C}, \mathbb{R}^{2n}) \to L^p(\mathbb{C}, \mathbb{R}^{2n}), \quad p > 2
\]

where \( S \in C^0(\mathbb{C}, \mathfrak{gl}(2n)) \) is required to satisfy

\[
S(e^{s+2\pi i t}) = S(t) \text{ for } s \gg 0, \text{ if } D_\psi \in O_+(S),
\]

\[
S(e^{-s-2\pi i t}) = S(t) \text{ for } s \ll 0, \text{ if } D_\psi \in O_-(S).
\]

Similarly, for loops of nondegenerate symmetric matrices

\[
\Gamma^- = \bigcup S_i^-(t) \text{ and } \Gamma^+ = \bigcup S_j^+(t)
\]

corresponding to paths \( \Psi_i^-(t) \) and \( \Psi_j^+(t) \) in \( Sp(2n) \) for \( i = 1, \cdots, p^- \) and \( j = 1, \cdots, p^+ \) respectively, one defines \( O(\Gamma^-, \Gamma^+) \) as the space of all operators

\[
D: W^{1,p}(\hat{S}, \mathbb{R}^{2n}) \to L^p(\hat{S}, \mathbb{R}^{2n}), \quad p > 2
\]

such that there is \( S \in C^0(\hat{S}, \mathfrak{gl}(2n)) \) satisfying the following conditions on all the cylindrical ends

\[
S(\epsilon_i^-(s,t)) = S_i^-(t) \text{ for } s \ll 0 \text{ and } S(\epsilon_j^+(s,t)) = S_j^+(t) \text{ for } s \gg 0,
\]

and the operator \( D \) coincide with \( \partial_s X + J_0 \partial_t X + S_i^\pm \cdot X \) on the positive and negative ends respectively.

One can verify that \( O_+(S) \), \( O_-(S) \) and \( O(\Gamma^-, \Gamma^+) \) consist of Fredholm operators. There are determinant line bundles \( \text{Det}(O_+(S)) \), \( \text{Det}(O(\Gamma^-, \Gamma^+)) \) defined over the spaces \( O_+(S) \)
and \(O(\Gamma^-, \Gamma^+)\), whose fibers over an element \(D\) in \(O_\pm(S)\) or \(O(\Gamma^-, \Gamma^+)\) to be the determinant line bundle \(\det(D)\) of the Fredholm operator \(D\). If we fix nondegenerate asymptotic data \(S(t)\) and \(\Gamma^\pm\), the spaces \(O_\pm(S)\) and \(O(\Gamma^-, \Gamma^+)\) are contractible. This implies that line bundles \(\text{Det}(O_\pm(S))\) and \(\text{Det}(O(\Gamma^-, \Gamma^+))\) are trivial for fixed loops of symmetric matrices \(S_\pm(t)\).

For given nondegenerate asymptotic data \(\Gamma^-\) and \(\Gamma^+\), we consider the following Fredholm operators

\[
K \in O(\Gamma^-, \Gamma^+) \quad \text{and} \quad L_j^+ \in O_-(S_{j}^+) \quad \text{for} \quad j = 1, \ldots, p^+,
\]

or \(L_j^- \in O_-(S_{j}^-)\) and \(K \in O(\Gamma^-, \Gamma^+)\) for \(i = 1, \ldots, p^-\).

or \(K_1 \in O(\Gamma_1^-, \Gamma_1^+)\) and \(K_2 \in O(\Gamma_2^-, \Gamma_2^+)\) and \(p_1^+ = p_2^-\)

There is a linear gluing operation, denoted as \(K \#_\rho L\), for Fredholm operators \(K\) and \(L\) defined on the glued Riemann surfaces \(\bigcup_{i=1}^p \mathbb{C} \#_\rho \hat{S}, \hat{S} \#_\rho \bigcup_{i=1}^p \mathbb{C} \) and \(\hat{S}_1 \#_\rho \hat{S}_2\) under the identifications

\[
\epsilon_1^- ([\rho, 2\rho] \times S^1) \subset \hat{S} \rightarrow \{z \mid e^\rho \leq |z| \leq e^{2\rho}\} \subset \mathbb{C}:
(s, t) \mapsto e^{3\rho - s - 2\pi it},
\]

\[
\epsilon_j^+ ([\rho, 2\rho] \times S^1) \subset \hat{S} \rightarrow \{z \mid e^\rho \leq |z| \leq e^{2\rho}\} \subset \mathbb{C}:
(s, t) \mapsto e^{s + 2\pi it}.
\]

For \(\rho \gg 0\), we then obtain the glued operators \(K \#_\rho \{L_j^+\}_{j=1}^p, \{L_j^-\}_{j=1}^p \#_\rho K\) and \(K_1 \#_\rho K_2\) in \(O_-(S^-), O_+(S^+)\) and \(O(\Gamma_1^-, \Gamma_1^+)\) in each case. With respect to this gluing operation, it is shown in [13, Proposition 9] that there is a canonical isomorphisms

\[
\det(K \#_\rho \{L_j^+\}_{j=1}^p) \cong \det(K) \otimes \det(L_1^+) \otimes \cdots \otimes \det(L_p^+),
\]

\[
\det(\{L_j^-\}_{j=1}^p \#_\rho K) \cong \det(L_1^-) \otimes \cdots \otimes \det(L_p^-) \otimes \det(K),
\]

\[
\det(K_1 \#_\rho K_2) \cong \det(K_1) \otimes \det(K_2),
\]

up to multiplication by a positive real number.

We now proceed to various operations. For any \(J_t \in \mathcal{J}(M_\phi)\), the complex vector bundle \(\langle \gamma^*T(\mathbb{R} \times M_\phi), J_t \rangle\) over \(S^1\) can be trivialized. We choose a trivialization along the Reeb orbit \(\gamma(t)\) given by \(\xi(t): \mathbb{R}^{2n} \rightarrow T_{\gamma(t)}(\mathbb{R} \times M_\phi)\) and obtain a path \(\Psi_x(t)\) in \(\text{Sp}(2n)\) as the composition of

\[
\mathbb{R}^{2n} \xrightarrow{\xi(0)} T_{\gamma(0)}M \xrightarrow{\text{def}_{\gamma(t)}} T_{\gamma(t)}M \xrightarrow{\xi^{-1}(t)} \mathbb{R}^{2n}.
\]

Given Reeb orbits \(\gamma_i^- (t)\) and \(\gamma_j^+ (t)\) in the positive and the negative ends of \(\mathbb{R} \times M_\phi\), we denote by \(S_i^- (t)\) and \(S_j^+ (t)\) the loops of symmetric matrices which generate \(\Psi_i^- (t)\) and \(\Psi_j^+ (t)\), respectively. The construction in [118] yields operators \(D_{\Psi_i^-} (t)\) and \(D_{\Psi_j^+} (t)\) in \(O_-(S_i^-)\) and \(O_+(S_j^+)\) respectively. We introduce the notation

\[
o_{\gamma_i} := |\det(D_{\Psi_i^-})| \quad \text{for} \quad i = 0, 1,
\]
where $|\cdot|$ denotes the graded abelian group generated by the two orientations of $\det(D_{\Psi_{\gamma_i}})$ and modulo the relation that the sum vanishes. In the case that there is a free $\mathbb{R}$ action on the moduli space $\wt{\mathcal{M}}(\Gamma^-,\Gamma^+)$ of solutions to (116). We take $u \in \wt{\mathcal{M}}(\Gamma^-,\Gamma^+)$ and define $o_u := |\det(D_u)|$, where $D_u \in \mathcal{O}(\Gamma^-,\Gamma^+)$ is the linearization of the Floer equation at $u$ with respect to a trivialization of $u^*(T\wt{M}) \to \mathbb{R} \times S^1$ that agree with the trivializations of $x_i^*(T\wt{M}) \to S^1$ as $s \to \pm\infty$. For different choices of trivializations, Lemma 13 in [13] and Proposition 1.4.10 in [1] show that the corresponding determinant line bundles $\det(D_{\Psi_{\gamma_i}})$ and $\det(D_u)$ are isomorphic. This implies that $o_{\gamma^-_i}$, $o_{\gamma^+_i}$ and $o_u$ are well-defined for $i = 1, \ldots, p^-$ and $j = 1, \ldots, p^+$. By the gluing property (119), we have a canonical isomorphism

$$o_u \otimes o_{\gamma^-_1} \otimes \cdots \otimes o_{\gamma^-_{p^-}} \cong o_{\gamma^+_1} \otimes \cdots \otimes o_{\gamma^+_{p^+}}.$$  

Together with the fact that $o_u \cong |\mathbb{R}\partial_s| \otimes |\mathcal{M}(\Gamma^-,\Gamma^+)|$, we obtain an isomorphism

$$o_{\gamma^+_1} \otimes \cdots \otimes o_{\gamma^+_{p^+}} \cong |\mathbb{R}\partial_s| \otimes |\mathcal{M}(\Gamma^-,\Gamma^+)| \otimes o_{\gamma^-_1} \otimes \cdots \otimes o_{\gamma^-_{p^-}}, \quad (120)$$

where $\mathbb{R}\partial_s$ is the 1-dimensional subspace of $\ker(D_u)$ spanned by translation in positive $s$-direction. (If there is no translation automorphism for $u$ in the case of the continuation maps, then we set $o_u \cong |\mathcal{M}(\Gamma^-,\Gamma^+)|$.) For $\sum_i |\gamma^-_i| = \sum_j |\gamma^+_j| + 1$, we have that $TM(\Gamma^-,\Gamma^+)$ is canonically trivial as $\mathcal{M}(\Gamma^-,\Gamma^+)$ is a 0-dimensional manifold. By comparing the fixed orientations on both sides of (120), we obtain an isomorphism

$$\epsilon_u : o_{\gamma^+_1} \otimes \cdots \otimes o_{\gamma^+_{p^+}} \to o_{\gamma^-_1} \otimes \cdots \otimes o_{\gamma^-_{p^-}}. \quad (121)$$

Now the linearization of the equation (116) yields a linear map

$$D : \mathbb{R} \oplus T_z(W^u(Z^m_i) \cap W^s(Z^0_\alpha)) \oplus W^{1,p}(\mathbb{R} \times S^1, u^*(T\wt{M})) \to L^p(\mathbb{R} \times S^1, u^*(T\wt{M})),$$

where $p > 2$ and $W^u(Z^m_i)$ is the unstable manifold of the critical point $Z^m_i$ of index $i$ on $S^\infty$, and similarly $W^s(Z^0_\alpha)$ is the stable manifold of $Z^0_\alpha$ inside $S^N$ for $N$ large enough. The latter orientations can be chosen compatibly with the inclusions $S^N \to S^{N'}$, $N < N'$. The Floer data $(J_w,s,t,H_{w,s,t})$ is regular if the linear map $D$ is surjective. This is equivalent to surjectivity of the linear map

$$T_z(W^u(Z^m_i) \cap W^s(Z^0_\alpha)) \to \text{Coker}(D_u),$$

where $D_u$ is the linearized operator associated to an element $(u,w)$ in $\mathcal{M}_i(\Gamma^-,\Gamma^+)$. There is an short exact sequence

$$0 \to T_u\wt{\mathcal{M}}_i(\Gamma^-,\Gamma^+) \to T_z(W^u(Z^m_i) \cap W^s(Z^0_\alpha) \oplus \ker(D_u) \to \text{Coker}(D_u) \to 0, \quad (122)$$

which induces an isomorphism of determinant lines

$$\det(T_u\mathcal{M}_i(\Gamma^-,\Gamma^+)) \cong \det(D_u) \otimes \det(T_z(W^u(Z^m_i) \cap W^s(Z^0_\alpha))). \quad (123)$$

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There is another isomorphism given by gluing theory.

\[
|\det(T_uM_i(\Gamma^-, \Gamma^+))| \otimes o_{\gamma^+_1} \otimes \cdots \otimes o_{\gamma^+_p} \cong o_{\gamma^-_1} \otimes \cdots \otimes o_{\gamma^-_p}
\]  

(124)

We choose a coherent orientation for each unstable manifold \(W^u(Z^m_i)\) of a critical point \(Z^m_i\) on \(S^\infty\), and on the unstable manifolds of \(Z^0_\alpha\) in \(S^N\) for various \(N\), compatibly with the inclusions. This induces a coherent system of orientations on the spaces \(P^{i,m}_\alpha\).

For \(|\gamma_0| = |\gamma_1| - i - 1 + \alpha\), the moduli space \(M^{i,m}_\alpha(\Gamma^-, \Gamma^+)\) is zero-dimensional and \(T_uM^{i,m}_\alpha(\Gamma^-, \Gamma^+)\) is canonically trivial. By comparing (123) and (124), one obtains an isomorphism

\[
\epsilon^{i,m}_\alpha(u): o_{\gamma^+_1} \otimes \cdots \otimes o_{\gamma^+_p} \rightarrow o_{\gamma^-_1} \otimes \cdots \otimes o_{\gamma^-_p}
\]

for each \(u \in M^{i,m}_\alpha(\Gamma^-, \Gamma^+)\). When \(p^+ = p\) and \(p^- = 1\), then this gives the signs in the definition of the \(\mathbb{Z}/p\mathbb{Z}\)-equivariant product map, and similarly when \(p^+ = 1\) and \(p^- = p\), this defines the signed count for the coproduct map. For example, the operations \(P^{i,m}_\alpha\) that define the \(\mathbb{Z}/p\mathbb{Z}\)-equivariant product map can be then rewritten as

\[
\begin{align*}
P^{i,m}_\alpha &: CF^*(\phi) \otimes p \rightarrow CF^{*-i+\alpha}(\phi^p) \\
P^{i,m}_\alpha|_{o_{\gamma^+_0} \otimes \cdots \otimes o_{\gamma^+_p}} &= \bigoplus_{|\gamma^-|=\sum |\gamma^-_i| - i + \alpha} \sum u \in M^{i,m}_\alpha(\Gamma^-, \Gamma^+)
\end{align*}
\]

(125)

(126)

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