We obtain the wave functions associated to the quantum Newtonian universe with a cosmological constant which is described by the Schrödinger equation and discuss some aspects of its dynamics for all forms of energy density, namely, matter, radiation, vacuum, dark energy, and quintessence. These wave functions of the quantum Newtonian universe are obtained in terms of the Heun’s functions and the respective energy levels are shown. We use these solutions in order to investigate the expansion of the universe and found that the asymptotic behavior for the scale factor is $R \sim e^{t}$ for whatever the form of energy density is. We also analyze the behavior when the universe was at early stages.

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I. INTRODUCTION

A model of the universe constructed by combining Newtonian mechanics with the cosmological principle and assuming the fact that the matter is pressureless and our universe experiences an expansion was presented in the 1930’s by Milne and McCrea \[1, 2\]. Then, they derived the cosmological equation which is algebraically analogous to the Friedmann and thus, the descriptions obtained in this approach, with pressureless matter, and in relativistic cosmology are equivalent. In other words, the descriptions of the universe using Newtonian dynamics and gravitation and the Einstein equations, when pressure is zero and assuming the cosmological principle and that the universe expands, are exactly the same, and thus these classical descriptions are equivalent.

It is worth call attention to the fact that when pressure is different from zero, it is necessary to modify the equations to take into account the pressure \[3\] and adopt a series of assumptions \[4\] to guarantee that the Newtonian and Einstein theories give us analogous results.

In the context of quantum cosmology the universe should be described by a single wave function defined for the three of the Friedmann-Robertson-Walker spacetime geometries taking into account all matter contents. In this approach, the equation for the wave function is not the Schrödinger equation, but the Wheeler-DeWitt equation \[5, 6\].

During the 1980’s different investigations in quantum cosmology based on the Wheeler-DeWitt equation have been done with the proposal to determine the wave function of the universe \[7–19\].

Another context in which is possible to find the wave function of the universe corresponds to what is termed quantum Newtonian cosmology. In this approach the wave functions are solutions of the Schrödinger equation for the system under consideration \[20, 24\], which, in principle, is much more simple than by solving the Wheeler-DeWitt equation. On the other hand, the quantum approach to Newtonian cosmology could be a source of inspiration to construct a real quantum theory of gravity.

This paper is organized as follows. In Sec. II we solve the Schrödinger equation for different contents od matter and determine the energy spectrum. In Sec. III we analyze the behaviors of the scale factor for different scenarios. Finally, in Sec. IV we present the conclusions.
II. SCHRODINGER EQUATION IN A NEWTONIAN UNIVERSE: WAVE FUNCTIONS
AND ENERGY LEVELS

In a previous paper [25], we have found that the Hamiltonian operator for a particle (galaxy) moving in the Newtonian universe is given by

$$H = -\frac{\hbar^2}{2\mu} \frac{d^2}{dR^2} - \frac{GM\mu}{R} - \frac{1}{6}\Lambda\mu R^2,$$

(1)

where \(\mu\) is the mass of a particle, \(R\) is the scale factor, and \(\Lambda\) is the cosmological constant. The total mass \(M\) of the Newtonian universe (mass of the attractive sphere) is given by

$$M = \frac{4}{3}\pi R^3 \rho.$$

(2)

The density energy \(\rho\) can be expressed as

$$\rho = A_\omega R^{-3(\omega+1)},$$

(3)

where

$$A_\omega = \rho_{\omega 0} R_0^{3(\omega+1)},$$

(4)

and \(\rho_{\omega 0}\) stands for the value of \(\rho_\omega\) at present time, with the state parameter, \(\omega\), being given by

$$\omega = \begin{cases} 0 & \text{for matter } (\rho_m), \\ \frac{1}{3} & \text{for radiation } (\rho_r), \\ -1 & \text{for vacuum } (\rho_v), \\ -\frac{1}{3} & \text{for dark energy } (\rho_d), \\ -\frac{2}{3} & \text{for quintessence } (\rho_q), \\ \end{cases}$$

(5)

so that

$$\begin{align*}
A_m &= \rho_{m 0} R_0^2, \\
A_r &= \rho_{r 0} R_0^4, \\
A_v &= \rho_{v 0}, \\
A_d &= \rho_{d 0} R_0^2, \\
A_q &= \rho_{q 0} R_0.
\end{align*}$$

(6)

If matter, radiation, vacuum, dark energy and quintessence all contribute, the density is the sum

$$\rho = \rho_m + \rho_r + \rho_v + \rho_d + \rho_q.$$ 

(7)
Thus, substituting Eqs. (2)-(5) into Eq. (1), we obtain

$$H = -\frac{\hbar^2}{2\mu} \frac{d^2}{dR^2} - \frac{4\pi G \mu}{3} \left[ A_d + A_q R + \left( A_v + \frac{\Lambda}{8\pi G} \right) R^2 + \frac{A_m}{R} + \frac{A_r}{R^2} \right],$$

which corresponds to the generalized Hamiltonian operator for a particle moving in the Newtonian universe.

Let us solve the time independent Schrödinger equation $H\psi(R) = E\psi(R)$, where $\Psi(R) = \psi(R)e^{-iEt/\hbar}$. Instead of solve this equation separately for each value of the state parameter $\omega$, we will do this in a general way which is valid for any value of $\omega$. Therefore, we will write the Schrödinger equation for a particle moving in the Newtonian universe as

$$\frac{d^2\psi(R)}{dR^2} + \left( B_1 + B_2 R + B_3 R^2 + \frac{B_4}{R} + \frac{B_5}{R^2} \right) \psi(R) = 0,$$

where the coefficients $B_1, B_2, B_3, B_4,$ and $B_5$ are given by

$$B_1 = \frac{2\mu E}{\hbar^2} + \frac{8\pi G \mu^2}{3\hbar^2} A_d,$$

$$B_2 = \frac{8\pi G \mu^2}{3\hbar^2} A_q,$$

$$B_3 = \frac{8\pi G \mu^2}{3\hbar^2} \left( A_v + \frac{\Lambda}{8\pi G} \right),$$

$$B_4 = \frac{8\pi G \mu^2}{3\hbar^2} A_m,$$

$$B_5 = \frac{8\pi G \mu^2}{3\hbar^2} A_r.$$

Now, we define a new variable, $x$, such that

$$x = \tau R,$$

where the parameter $\tau$ is given by

$$\tau = (-B_3)^{\frac{1}{2}}.$$

Thus, with this new variable, we can write Eq. (9) as

$$\frac{d^2\psi(x)}{dx^2} + \left( b_1 + b_2 x - x^2 + \frac{b_4}{x} + \frac{b_5}{x^2} \right) \psi(x) = 0,$$
where the coefficients $b_1, b_2, b_4,$ and $b_5$ are written as

$$b_1 = \frac{B_1}{\tau^2},$$

$$b_2 = \frac{B_2}{\tau^3},$$

$$b_4 = \frac{B_1}{\tau},$$

$$b_5 = B_5.$$

In what follows we will solve Eq. (17) and conclude that this solution is given in terms of the biconfluent Heun functions, and determine the eigenvalues.

Equation (17) is a biconfluent Heun equation [27], which is a particular case of a second order linear differential equation with four singularities, called Heun equation. The confluent form of this equation is obtained when two of the singularities coalesce and at infinity there is an irregular point. The canonical form of the biconfluent Heun equation reads as

$$\frac{d^2y(x)}{dx^2} + \left(\frac{1 + \alpha}{x} - \beta - 2x\right)\frac{dy(x)}{dx} + \left\{ (\gamma - \alpha - 2) - \frac{1}{2} [\delta + (1 + \alpha)\beta] \frac{1}{x} \right\} y(x) = 0,$$

where $y(x) = \text{HeunB}(\alpha, \beta, \gamma, \delta; x)$ is the biconfluent Heun function. By using the approach described in [28], we can write Eq. (22) in the normal form as

$$\frac{d^2Y(x)}{dx^2} + \left[ \frac{1}{4}(4\gamma - \beta^2) - \beta x - x^2 - \frac{\delta/2}{x} - \frac{(\alpha^2 - 1)/4}{x^2} \right] Y(x) = 0,$$

where $Y(x) = x^{1/2(1+\alpha)} e^{-\frac{1}{2}(x^2+\beta x)} y(x)$.

The Schrödinger equation for a particle moving in the Newtonian universe, given by Eq. (17) for an arbitrary $\omega$, can be written as Eq. (23), and therefore, its exact solution is given by

$$\psi(x) = C_1 x^{1/2(1+\alpha)} e^{-\frac{1}{2}(x^2+\beta x)} \text{HeunB}(\alpha, \beta, \gamma, \delta; x),$$

where $C_1$ is a constant to be determined, and the parameters $\alpha, \beta, \gamma, \text{and} \delta$ are identified as

$$\alpha = \sqrt{1 - 4b_5},$$

$$\beta = -b_2,$$

$$\gamma = b_1 + \frac{b_2^2}{4},$$

$$\delta = b_5.$$
TABLE I. The parameters $\alpha, \beta, \gamma,$ and $\delta$ for the biconfluent Heun function related to the $\omega$ predominance, where $y(x) = \text{HeunB}(\alpha, \beta, \gamma, \delta; x)$.

| $\omega$ | $\psi(x)$ | $\alpha$ | $\beta$ | $\gamma$ | $\delta$ |
|----------|------------|----------|---------|----------|---------|
| 0        | $xe^{-\frac{x^2}{2}} y(x)$ | 1        | 0       | $\frac{2\mu E}{\hbar^2}$ | $-\frac{16\pi G\mu^2 A}{3\hbar^2}$ |
| $\frac{1}{3}$ | $x^{\frac{1}{3}(1+\alpha)} e^{-\frac{x^2}{2}} y(x)$ | $\sqrt{1 - \frac{32\pi G\mu^2 A}{3\hbar^2}}$ | 0       | $\frac{2\mu E}{\hbar^2}$ | 0 |
| -1       | $xe^{-\frac{x^2}{2}} y(x)$ | 1        | 0       | $\frac{2\mu E}{\hbar^2}$ | 0 |
| $-\frac{1}{3}$ | $xe^{-\frac{x^2}{2}} y(x)$ | 1        | 0       | $\frac{2\mu E}{\hbar^2}$ | $\frac{8\pi G\mu^2 A}{3\hbar^2}$ | 0 |
| $-\frac{2}{3}$ | $xe^{-\frac{1}{3}(x^2+\beta x)} y(x)$ | 1 | $-\frac{8\pi G\mu^2 A}{3\hbar^2}$ | $\frac{2\mu E}{\hbar^2}$ | $\frac{1}{4}\left(\frac{8\pi G\mu^2 A}{3\hbar^2}\right)^2$ | 0 |

TABLE II. The energy levels related to the $\omega$ predominance.

| $\omega$ | $E_n$ |
|----------|-------|
| 0        | $\frac{h^2 x^2}{\mu} \left(n + \frac{3}{2}\right)$ |
| $\frac{1}{3}$ | $\frac{h^2 x^2}{\mu} \left(n + \frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{32\pi G\mu^2 A}{3\hbar^2}}\right)$ |
| -1       | $\frac{h^2 x^2}{\mu} \left(n + \frac{1}{2}\right)$ |
| $\frac{1}{3}$ | $\frac{h^2 x^2}{\mu} \left(n + \frac{3}{2} - \frac{4\pi G\mu^2 A}{3\hbar^2}\right)$ |
| $\frac{2}{3}$ | $\frac{h^2 x^2}{\mu} \left[n + \frac{3}{2} - \frac{1}{4}\left(\frac{8\pi G\mu^2 A}{3\hbar^2}\right)^2\right]$ |

$\delta = -2b_4$, \hspace{1cm} (28)

whose complete set of solutions, for different values of $\omega$, is summarized in Table I.

As it was described in the paper [25], the biconfluent Heun function becomes a polynomial of degree $n$ if and only if the following conditions are fulfilled: $\gamma - \alpha - 2 = 2n$ and $C_{n+1} = 0$, where $n = 0, 1, 2, \ldots$, and $C_{n+1}$ is a polynomial in $\delta$. From the first condition, the energy levels for each value of $\omega$ are shown in Table I.

Now, we adapt the method developed by He et al. [29], based in the approach used by Vilenkin [30], who analyzed the dynamical interpretation of the wave function of the universe from the Wheeler-DeWitt equation in the minisuperspace model. Thus, we will use the exact solution of the Schrödinger equation for a particle moving in the Newtonian universe in order to study the dynamical interpretation of the Newtonian wave function and then discuss the boundary conditions in quantum Newtonian cosmology scenario.

As in Eq. (9) there is only one variable, namely, the scale factor $R$, the function $\psi(R)$ can be rewritten as

$$\psi(R) = F(R) e^{iS(R)}, \hspace{1cm} (29)$$
where $F$ and $S$ are real functions. In our case, the square modulus of the wave function of the Newtonian universe is given by

$$|\psi(R)|^2 = F^2(R).$$  \hfill (30)

From the quantum mechanics formalism [31], we have that the conserved probability current density is written as

$$\vec{j}(R, t) = i\hbar \mu \left[\Psi^* (\vec{\nabla} \Psi) - \Psi (\vec{\nabla} \Psi^*)\right],$$  \hfill (31)

in such a way to guarantee the validity of the continuity equation, namely,

$$\vec{\nabla} \cdot \vec{j}(R, t) = 0.$$

Substituting Eq. (29) into Eq. (31), we obtain

$$\vec{j} = -\frac{\hbar}{\mu} F^2 \frac{\partial S}{\partial R}.$$

On the other hand, integrating Eq. (32) we get

$$\vec{j} = C_0,$$

where $C_0$ is a constant. Thus, from Eqs. (33) and (34), we have

$$-\frac{\hbar}{\mu} F^2 \frac{\partial S}{\partial R} = C_0.$$

Now, we may use the Hamilton-Jacobi formalism of quantum mechanics in order to write the following relation between the action and the canonical momentum

$$p_R = \frac{\partial S}{\partial R} = \frac{\partial L}{\partial R} = \mu \dot{R},$$

where $L$ is the Lagrangian for the motion of a particle in the Newtonian universe, given by Eq. (1) in Ref. [25]. Thus, from Eqs. (35) and (36), we get

$$F^2 = -\frac{C_0}{\hbar \dot{R}}.$$

### III. THE EXPANSION OF THE UNIVERSE FROM THE QUANTUM NEWTONIAN COSMOLOGY

In this section we investigate the classical evolution laws of the universe, which were already obtained using the solutions of the Friedmann equation. However, we want to show that this can also be made from the quantum dynamical interpretation of the Newtonian universe.
Now, consider the wave function of the Schrödinger equation in the $R \gg 1$ limit, which implies that $x \gg 1$. The biconfluent Heun functions have the following asymptotic behaviors

$$\text{HeunB}(\alpha, \beta, \gamma, \delta; x) \sim \begin{cases} x^{-\gamma/2} \sum_{k \geq 0} a_k x^k, \\
 x^{-\gamma/2-\alpha} e^{x^2+\beta x} \sum_{k \geq 0} a_k x^k,
\end{cases}$$

where $|\arg x| \leq \frac{\pi}{2} - \epsilon$, $a_0 = 1$, and $e_0 = 1$. Then, the Newtonian wave function can be written as

$$\psi(R) \sim C_1 (\tau R)^{-\frac{1}{3} + \frac{\gamma}{2}},$$

and as a consequence

$$|\psi^2(R)| = \frac{C_1^2}{\tau R} = F^2,$$

where we have used the fact that $\gamma$ is an imaginary number. Thus, taking into account Eq. (37), we get

$$\frac{\dot{R}}{R} = -\frac{C_0 \tau}{C_1 \hbar},$$

where $C_0 < 0$ [29]. Rewriting this formula as

$$dR = \frac{|C_0| \tau}{C_1 \hbar} dt,$$

the integration gives us the following asymptotic behavior for the scale factor

$$R \propto e^{t + t_0}.$$  

The evolution law of the universe from the quantum Newtonian cosmology in the classical limit ($R \gg 1$) is completely consistent with the solution of the Friedmann equation for the dark energy predominance, which means that the universe will behave according to the dark energy, whatever the dominant form of energy. In fact, the dark sector is the main form of energy in the present universe and will continue to be at the end of the expansion. Furthermore, our result is independent of the form of energy density and hence it is general than the ones found in the literature.

If we try to fix the constants $C_0$ and $C_1$ in such a way that turn the exponential dimensionless, we find

$$C_1 = \sqrt{\frac{|C_0| \mu}{\hbar^2 \tau}},$$

where we have used the fact that $E_n \sim \hbar^2 \tau^2 / \mu$. In this way, we can write

$$R(t) = e^{\frac{\hbar^2}{\mu} (t + t_0)}.$$
Next, consider the wave function of the Schrödinger equation in the $R \ll 1$ limit, which implies that $x \ll 1$. In this case, the biconfluent Heun functions have the following asymptotic form

$$\text{HeunB}(\alpha, \beta, \gamma, \delta; x) = \sum_{s \geq 0} D_s \frac{x^s}{(1 + \alpha)_s s!},$$

where $D_0 = 1$, and

$$(1 + \alpha)_s = \frac{\Gamma(s + 1 + \alpha)}{\Gamma(1 + \alpha)}.$$ (47)

Thus, the Newtonian wave function given by Eq. (24) for the small scale factor $R \ll 1$ can be rewritten as

$$\psi(R) \sim \sqrt{|C_0| \mu \frac{\hbar}{\hbar^2 \tau}} (\tau R)^{1+\alpha},$$

where we have used the same value for $C_1$. Thus, the squared modulus of the wave function is given by

$$\psi^2(R) = \frac{|C_0| \mu}{\hbar^2 \tau} (\tau R)^{1+\alpha}.$$ (49)

Now, taking into account once more Eq. (37), we get

$$R^{1+\alpha} \, dR = \frac{\hbar}{\mu \tau^\alpha} \, dt.$$ (50)

Therefore, in this limit, the scale factor has the following asymptotic form

$$R(t) \sim \left[ \frac{(2 + \alpha)\hbar}{\mu \tau^\alpha} \right]^{\frac{1}{1+\alpha}} \frac{t^{\frac{1}{1+\alpha}}}{t^{\frac{1}{1+\alpha}}},$$

which means that when the universe was very small, its behavior depends on the parameter $\alpha$.

**IV. CONCLUSIONS**

In this work we generalized the previous results for the quantum Newtonian cosmology in the sense that we have now the analytical solutions for each value of the form of energy density.

The Newtonian wave function is given in terms of the biconfluent Heun functions and obeys the appropriate boundary conditions. From the polynomial condition for the biconfluent Heun equation, we have obtained the energy levels related to each value of the parameter $\omega$.

We present a dynamical interpretation for the wave function of the Newtonian universe. This dynamical interpretation gives the behavior of the scale factor at the end of the expansion, which is
in accordance with the correspondence principle applied to quantum cosmology, namely, we recover
the classical limit.

On the other hand, when the universe was very small, the quantum effects on the scale factor
are given in terms of the parameter $\alpha$, which depends on the form of the energy density.

Therefore, as the cosmological equation in the Newtonian cosmology, the wave function of the
Newtonian universe has a different meaning when compared with the relativistic cosmology in
the sense that here we recover just the behavior of the scale factor at the end of the expansion.

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