Statistical Inference Based on a New Weighted Likelihood Approach

Suman Majumder  
Indian Statistical Institute, Kolkata-700108, India  
e-mail: suman69.suman@gmail.com

Adhidev Biswas  
Interdisciplinary Statistical Research Unit  
Indian Statistical Institute, Kolkata-700108, India  
e-mail: adhidevbiswas@gmail.com

Tania Roy  
Department of Statistics, UC Davis  
One Shields Avenue, Davis, California 95616  
e-mail: tania112358@gmail.com

Subir Bhandari and Ayanendranath Basu  
Interdisciplinary Statistical Research Unit  
Indian Statistical Institute, Kolkata-700108, India  
e-mail: subir@isical.ac.in; ayanbasu@isical.ac.in

Abstract: We discuss a new weighted likelihood method for parametric estimation. The method is motivated by the need for generating a simple estimation strategy which provides a robust solution that is simultaneously fully efficient when the model is correctly specified. This is achieved by appropriately weighting the score function at each observation in the maximum likelihood score equation. The weight function determines the compatibility of each observation with the model in relation to the remaining observations and applies a downweighting only if it is necessary, rather than automatically downweighting a proportion of the observations all the time. This allows the estimators to retain full asymptotic efficiency at the model. We establish all the theoretical properties of the proposed estimators and substantiate the theory developed through simulation and real data examples.

Keywords and phrases: Weighted Likelihood, Asymptotic Efficiency, Robustness, Influence Function, Robust Regression.

1. Introduction

We consider a new approach to weighted likelihood estimation. A weighted likelihood estimating equation employs a reweighting of the components of the likelihood score equation. This method is useful when the model is in doubt or when outliers are present in the data. The weighted likelihood estimator considered here (obtained as a solution of the weighted likelihood estimating
equation) is asymptotically fully efficient in cases where the model is true, and in cases where the model is perturbed the proposed estimator works robustly, identifying the points in the data that are not in agreement with the model.

The method we discuss is based on a recent proposal by Biswas et al. (2015) [9]. The aforementioned work simply puts forward a proposal in a brief article; detailed numerical investigations or derivation of the theoretical properties of the method have not been undertaken. In the present paper we provide a comprehensive follow up of the proposal, derive its theoretical properties, describe the possible extension of the method to situations beyond the ordinary independent and identically distributed (i.i.d.) data model and consider extensive numerical explorations; overall we provide a general discussion of the scope of the application of the method in statistical inference.

Let, $X_1, X_2, ..., X_n$ be an i.i.d. random sample from a distribution function $G$ having the corresponding density $g$. We model $G$ by the parametric family $\mathcal{F} = \{F_\theta : \theta \in \Theta \subset \mathbb{R}^p\}$. Let $u_\theta(x) = \nabla \ln[f_\theta(x)]$ be the usual likelihood score function where $f_\theta$ is the density of $F_\theta$, $\nabla$ denotes differentiation with respect to $\theta$ and $\ln$ denotes natural logarithm. Under standard regularity conditions, the maximum likelihood estimator (MLE) of $\theta$ is obtained as a solution of the likelihood score equation $\sum_{i=1}^n u_\theta(X_i) = 0$.

For any given point $t$ in the sample space, we will construct a weight function $w(t, F_\theta, F_n)$ that will depend on the point $t$, the chosen model $F_\theta$ and the empirical distribution function $F_n$. By construction, the weights will be constrained to lie between 0 and 1. Ideally, the weights should be close to 1 for points where the data closely follow the model but should be substantially smaller when the two do not agree. We then consider the solution to the weighted likelihood estimating equation

$$\sum_{i=1}^n w_\theta(X_i)u_\theta(X_i) = 0 \quad (1)$$

to be our estimated value of $\theta$. Here $w_\theta(X_i) = w(X_i, F_\theta, F_n)$ is the weight attached to the score function of $X_i$. It is to be noted that the dependency of $\theta$ in $w_\theta(\cdot)$ comes solely from $F_\theta$, which is a component of the argument of $w(\cdot)$. As a function, $w(\cdot)$ is not dependent on $\theta$. To be consistent with the philosophy described here, the weights should be equal to 1 at the points where $F_\theta$ and $F_n$ are identical. Viewed as a function of the ratio of $F_n$ and $F_\theta$ we would want the weights to smoothly go down as the above ratio moves away from 1 in either direction.

This scheme of estimating the parameter values using the weighted likelihood estimating equation approach borders on the idea of minimum disparity estimation proposed by Lindsay (1994) [11] and Basu & Lindsay (1994) [7], Markatou et al. (1997,1998) [13, 14] considered a weighted likelihood approach to estimation based on weights that provide a quantification of the magnitude and sign of the Pearson residual, and are generally linked to a residual adjustment function employed in minimum disparity estimation. This is a very useful procedure that simplifies the estimation technique of minimum disparity estimation, specially in continuous models. In particular, the estimating function is reduced to
a sum over the observed data rather than an integral over the entire support. However the procedure still depends on the kernel density estimation method for the comparison of the data and the model densities and has to deal with issues of kernel and bandwidth selection. This method also has to contend with other problems such as the construction of the weights in models with bounded support and the slow convergence of the kernel in case of multivariate data. We trust that our residual function and our weights would provide improvements on these counts. Nevertheless, the Markatou et al. (1997, 1998) [13, 14] papers remain the pioneers in this particular area of research based on weighted likelihood. Gervini & Yohai (2002) [10] represent another work in the same spirit dealing with robust and asymptotically fully efficient estimators.

Claudio Agostinelli has extensively studied the form of the weighted likelihood estimators proposed by Markatou et al. (1998) [14] and applied them to different inference scenarios and generated useful robust estimators and other inference tools. See, for example, Agostinelli & Markatou (1998, 2001) [5, 6]; Agostinelli (2002a, 2002b, 2007) [1, 2, 3]; Agostinelli & Greco (2013) [4]. In all of these cases our residuals and weight functions will provide an alternative approach to the corresponding inference problem.

The rest of the paper is organized as follows. In Section 2, we describe the residual function and propose several weight functions. These weight functions represent a rich collection of shapes under the general framework described above and allow many other possibilities in comparison to the simplistic exponential weight function considered in Biswas et al. (2015) [9]. In Section 3, we illustrate the performance of the weighted likelihood methods in robust estimation problems through real data examples. We address a few computational issues that arise in the implementation of the above method in Section 4. A relevant simulation study is provided in Section 5. Theoretical properties of the estimator, including influence function, location-scale equivariance, consistency and asymptotic normality are discussed in Section 6. As the first order influence function turns out to be a poor descriptor of the robustness of our estimators, we take up a higher order influence function analysis of these weight functions in Section 7. The method is applied to bivariate data and robust regression problems in Section 8. Some concluding remarks are given in Section 9.

2. The Residual Function and The Weight Function

Here we describe the residual function as presented in Biswas et al. (2015) [9] and propose several weight functions for the construction of the weighted likelihood estimating equation.
2.1. The Residual Function

Let $I(A)$ represent the indicator function of the event $A$. We define $F_n$ and $S_n$ as

$$F_n(x) = \frac{1}{n} \sum_{i=1}^{n} I(X_i \leq x), \quad S_n(x) = \frac{1}{n} \sum_{i=1}^{n} I(X_i \geq x).$$

These represent the empirical distribution function and the empirical survival function of the data. Let $F_\theta(x) = P_\theta(X \leq x)$ and $S_\theta(x) = P_\theta(X \geq x)$ be the corresponding theoretical quantities. Now, the residual function as proposed by Biswas et al. (2015) [9] can be described through the following steps:

- Choose a suitable fraction $0 \leq p \leq 0.5$. This tuning parameter will determine the proportion of observations on either tail that will be subjected to (possible) downweighting.
- Assign weights $w_\theta(X_i) = 1$ to all observations satisfying $p < F_\theta(X_i) < 1 - p$. Thus all observations belonging to the central $100 \times (1 - 2p)$% of the distribution at the current value of the estimator will get weights equal to 1.
- For each value $X_i$ in the lower tail, i.e. with $F_\theta(X_i) \leq p$, we consider a possible downweighting of the observation as follows. Compute
  $$\tau_n(X_i) = \frac{F_n(X_i)}{F_\theta(X_i)} - 1$$
  which may be viewed as a standardized residual in comparing the two distribution functions. If the two values $F_n(X_i)$ and $F_\theta(X_i)$ are severely mismatched, i.e. the value of $\tau_n(X_i)$ differs substantially from 0 (in either direction), we treat it as a case that requires downweighting.
- For each value $X_i$ in the upper tail, i.e., with $F_\theta(X_i) \geq 1 - p$, we consider a similar downweighting, but in this case we construct the standardized residuals as
  $$\tau_n(X_i) = \frac{S_n(X_i)}{S_\theta(X_i)} - 1.$$  

Thus the final form of the residual function $\tau_n(X_i)$ is

$$\tau_n(X_i) = \begin{cases}  
   \frac{F_n(X_i)}{F_\theta(X_i)} - 1 & \text{if } 0 < F_\theta(X_i) \leq p, \\
   0 & \text{if } p < F_\theta(X_i) < 1 - p, \\
   \frac{S_n(X_i)}{S_\theta(X_i)} - 1 & \text{if } 1 - p \leq F_\theta(X_i) < 1.
\end{cases}$$

The rationale for defining the residual function in the above manner has been described in Biswas et al. (2015) [9]. It is clear that the consideration of the distribution function in the left tail and the survival function in the right tail helps highlight the mismatch of the data and the model in the respective tails.
2.2. The Weight Function

Now, that we have our residual function, the next objective is to construct suitable weight functions in line with the principles stated earlier. The weight given to an observation $X_i$ based on $n$ i.i.d. observations, is denoted by $w(X_i) = w(\tau_n(X_i))$ keeping the dependence on $\theta$ implicit. To adhere to our requirements, the weight function should have following properties:

1. $0 \leq w(x) \leq 1$, $w(0) = 1$.
2. $w(-1)$ is small, preferably close to zero; at any rate, substantially smaller than 1.
3. $\lim_{x \to \infty} w(x) = 0$.
4. $w$ is a smooth function such that $w$ admits two derivatives at zero with $w'(0) = 0$.

To determine a possible weight function having the above mentioned properties, we employ the following technique:

1. First we find a nonnegative function $g_\theta(x)$ which is a function from $\mathbb{R}^2 \to \mathbb{R}$. Suppose for any fixed $\theta$, $g_\theta(\cdot)$ has domain $[a, \infty)$, where $a$ is a finite real number with $g_\theta(a) = 0$. Suppose that the function $g_\theta(\cdot)$ has a unique mode in an interior point of the interval $[a, \infty)$, $g_\theta(x)$ is bounded, and $g_\theta(x)$ is twice differentiable in $\theta$ for each fixed value of $x$.
2. Then, if possible, we choose the parameter values such that, if the domain is $[a, \infty)$, the unique mode is at $a+1$. Let this function be $g_{\theta_0}$, where $\theta_0$ provides the required parametrization.
3. Once Step 2 is done, we define the weight function as $w(\tau_n(x)) = \frac{g_{\theta_0}(\tau_n(x) + a + 1)}{g_{\theta_0}(a + 1)}$. (2)

This procedure gives candidates for weight function such that $w(-1) = 0$, $w(0) = 1$ and $w'(0) = 0$.

Using this procedure, several weight functions are proposed.

2.2.1. Weight Function 1

We take $g_\theta(x)$ as the probability density function of a gamma random variable with scale parameter $\lambda$ and shape parameter $\alpha$, which has the form

$$g_{(\lambda,\alpha)}(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{(\alpha-1)} \exp(-\lambda x), \ x > 0.$$ 

It is bounded, twice differentiable, has a unique mode and the domain is $[0, \infty)$. To set the mode at 1, the parametrization needed is $\lambda = \alpha - 1$, $\alpha > 1$. So, the final weight function is

$$w_\alpha(\tau(x)) = \frac{g_{(\alpha-1,\alpha)}(\tau(x) + 1)}{g_{(\alpha-1,\alpha)}(1)}.$$
As we increase the value of $\alpha$, the amount of downweighting increases and as $\alpha \downarrow 1$ the weights tend to 1 for all values of $\tau(x)$, and will eventually lead to the original score equation $\frac{1}{n} \sum_{i=1}^{n} u_{\theta}(X_i) = 0$, which yields the maximum likelihood estimates for the data. The shapes of this weight function for different values of the tuning parameter $\alpha$ are presented in Figure 1.

2.2.2. Weight Function 2

In this case we take $g_{\theta}(x)$ to be the probability density function of a Weibull distribution with scale parameter $\lambda$ and shape parameter $k$, which has the form

$$g_{(k,\lambda)}(x) = \frac{k}{\lambda} (x/\lambda)^{(k-1)} \exp \left[-\left(\frac{x}{\lambda}\right)^k\right], \quad x > 0.$$  

This function is bounded, twice differentiable and takes value in the interval $[0,\infty)$ and has a unique interior mode. To set the value of the mode at 1, the parametrization needed is $k > 1$ and $\lambda = \left(\frac{k-1}{k}\right)^{-1/k}$. So, the final weight...
The shapes of weight function 2 for different values of the tuning parameter $k$

function is

$$w_k(\tau(x)) = \frac{g_k(\frac{x-\mu}{\beta})^{-1/k}(\tau(x) + 1)}{g_k(\frac{x-\mu}{\beta})^{-1/k}(1)}.$$  

Here also, as we increase the value of $k$, the downweighting characteristics increase and as $k \downarrow 1$, the weights converge to 1 for all the observations pushing the procedure towards maximum likelihood. The changing pattern of the weight function for different choices of $k$ can be seen in Figure 2.

2.2.3. Weight Function 3

We take our $g_0(x)$ to be the probability density function of the generalized extreme value (GEV) distribution with location parameter $\mu$, scale parameter $\beta$ and shape parameter $\xi$ which has the form

$$g_{(\mu, \beta, \xi)}(x) = \frac{1}{\beta} t(x)^{(1+\xi)} \exp(-t(x)), \quad t(x) = \left(1 + \left(\frac{x - \mu}{\beta}\right)^{\frac{1}{\xi}}\right)^{-\frac{1}{\xi}}.$$  

To set the starting point of the domain at $-1$ and to set the mode at 0, we reparametrize the function as $\xi > 0$, $\mu = (1 + \xi)^\xi - 1$ and $\beta = \xi(1 + \xi)^\xi$. So, the
The final weight function is

$$w_\xi(\tau(x)) = \frac{g((1+\xi)(1+\xi)^{\xi}(\tau(x)))}{g((1+\xi)(1+\xi)^{\xi})(0)}.$$  

The variation in this weight function is shown in Figure 3. Here we see that as we increase the value of $\xi$, the downweighting decreases as opposed to the previous cases.

2.2.4. Weight Function 4

We now take $g_\theta(x)$ to be the function

$$g(d_1,d_2,a)(x) = \frac{1}{B(d_1/2,d_2/2)} \left( \frac{d_1/d_2}{d_1/2} (x/a)^{d_1/2} - 1 \right) \left( 1 + \frac{d_1x}{d_2a} \right)^{-(d_1+d_2)/2}$$

on the domain $[0,\infty)$, where the parameters $d_1, d_2, a$ are all positive and $B(\cdot, \cdot)$ is the beta function. The function is integrable and normalized. To fix the unique interior mode at 1, we parametrize the function as $a = \frac{d_1(d_2+2)}{(d_1-2)d_2}$, $d_1 > 2$, $d_2 > 0$. 

**Fig 3.** The shapes of weight function 3 for different values of the tuning parameter $\xi$
So, the final weight function is

\[ w_{d_1, d_2}(\tau(x)) = \frac{g\left(d_1, d_2, \frac{d_1(d_2+2)}{(d_1-2)d_2}\right) \left(\tau(x) + 1\right)}{g\left(d_1, d_2, \frac{d_1(d_2+2)}{(d_1-2)d_2}\right)(1)}. \]

As \( d_1 \downarrow 2 \), the weights tend to 1 and the weighted likelihood equation eventually reduces to the maximum likelihood score equation. So, in this case also, the MLE can be considered as a limiting case of the weight likelihood estimator.

We keep \( d_2 \) fixed, and change \( d_1 \) only. The change in the weight function is presented in Figure 4 where \( d_2 \) is fixed at 1, while Figure 5 shows the change in the weight function due to the change in \( d_2 \) while \( d_1 \) remains constant at 3.

We see that the second tuning parameter \( d_2 \) has no particular effect on the left tail when the other tuning parameter \( d_1 \) is kept constant. The parameter \( d_1 \) however has an effect on both the tails. This gives us the freedom to control the right tail of the weight function separately without affecting the left tail.
Fig 5. The shapes of weight function $w(\tau)$ for different values of the tuning parameter $d_2$ keeping the tuning parameter $d_1$ fixed.
3. Some Real Data Examples

3.1. Drosophila Data (Poisson Example)

Here we consider a chemical mutagenicity experiment. These data were analyzed previously by Simpson (1987) [18]. The details of the experimental protocol are available in Woodruff et al. (1984) [21]. In this experiment which involved drosophila, a variety of fruit flies, the experimenter exposed groups of male flies to different doses of a chemical to be screened. Subsequently each male was mated with unexposed females. Approximately 100 daughter flies were sampled for each male and the experimenter noted the number of daughters carrying a recessive lethal mutation on the X chromosome. The data set consisted of the observed frequencies of males having 0, 1, 2, \ldots recessive lethal daughters. For our purpose, we consider a specific experimental run, that on day 177. The dataset is presented in Table 1.

A Poisson($\theta$) model was fitted to these data. We set our parameter value for $p = 0.5$ for this purpose. The same value of $p$ has been used henceforth throughout the paper, unless otherwise mentioned. The values for the estimated means in various methods are given in Table 2. Here MLE-D represents the MLE after deleting the large outlier (91). Our proposed weighted likelihood estimators (WLE) successfully provide outlier resistant estimates of $\theta$ giving almost 0 weight to the outlier, unlike the MLE. In fact, the weighted likelihood estimators are seen to be very close to the outlier deleted MLE. The divergences HD, GKL$_{1/3}$ and RLD$_{1/3}$ represent the Hellinger distance, the generalized Kullback-Leibler divergence and the robustified likelihood disparity (with indicated tuning parameters if appropriate). The estimators corresponding to these three divergences are as reported by Basu et al. (2011) [8], Table 2.2. All the robust estimators considered in this table have values within a very small band, whereas the MLE clearly produces a nonsensical result.

| Method    | $\theta$ |
|-----------|----------|
| MLE       | 3.0588   |
| MLE-D     | 0.3939   |
| HD        | 0.3637   |
| GKL$_{1/3}$ | 0.3813  |
| RLD$_{1/3}$ | 0.3588  |
| WLE$_1(\alpha = 1.01)$ | 0.3948  |
| WLE$_2(k = 1.01)$ | 0.3948  |

### Table 1

**Drosophila data**

| No. of daughters | 0 | 1 | 2 | 3 | 4 | $\geq 5$ |
|------------------|---|---|---|---|---|---------|
| Observed frequency | 23 | 7 | 3 | 0 | 0 | 1(91) |

### Table 2

Parametric estimates obtained for the drosophila data using different methods
3.2. Newcomb’s Speed of Light Data

We consider Newcomb’s speed of light data obtained through an experiment done in 1882. These data are available in Stigler (1977) [20] and have been analyzed by several authors. The dataset consists of 66 observations. There are two distinct outliers at −44 and −2. A normal model would provide a nice fit to these data if the outliers are deleted. The estimates of $\mu$ and $\sigma^2$ under the normal model are presented in Table 3 for several of our weighted likelihood estimators together with their likelihood based competitors. All the weighted likelihood estimators presented here are successful in controlling the effect of the large outliers while the MLE is severely affected.

| Estimator  | $\hat{\mu}$ | $\hat{\sigma}^2$ |
|-----------|--------------|------------------|
| MLE       | 26.21212     | 113.7126         |
| MLE-D     | 27.75        | 25.4375          |
| WLE$_1(\alpha = 1.01)$ | 27.7581    | 25.3204          |
| WLE$_1(\alpha = 1.1)$  | 27.8460    | 23.9902          |
| WLE$_2(k = 1.05)$     | 27.7982    | 24.7364          |
| WLE$_2(k = 1.1)$      | 27.8722    | 23.6171          |
| WLE$_3(\xi = 5)$      | 27.8303    | 23.7256          |
| WLE$_3(\xi = 10)$     | 27.7891    | 24.6965          |

3.3. Melbourne’s Daily Rainfall Data

Here we take up a dataset that allows us to apply our method beyond the domain of the normal distribution. This dataset is on the daily rainfall in Melbourne, Australia for the months of June, July and August in the years 1981 to 1983 and presented in Staudte & Sheather (1990) [19]. On half of the days, there is no measurable rainfall in the area and so we only look at the days with rainfall and term them as ‘rain days’. As it is quite unrealistic to pretend that rainfall over successive days are independent, Staudte & Sheather took the measurements for every fourth rain day and assumed that they are independent as well as identically distributed. Under this assumption, we fit an exponential model to the data and estimate the rate parameter $\lambda$ for the exponential distribution with density $f_\lambda(x) = \lambda \exp(-\lambda x), \ x > 0$. Table 4 presents the estimated values of the parameter using the methods of maximum likelihood and weighted likelihood respectively. Also Fig. 6 presents the histogram and the densities fitted to these data. We see that the proposed estimator does well to overcome the effect of the large outlier in the right tail.
Table 4
The estimated parameter values for the Melbourne’s daily rainfall data

| Method      | $\lambda$ |
|-------------|-----------|
| MLE         | 0.2224    |
| WLE$_1(\alpha = 1.05)$ | 0.2786 |

Fig 6. Histogram and exponential densities fitted to the Melbourne’s daily rainfall data
4. Issues Regarding the Construction of Weights and Multiple Roots

4.1. Choice of the Tuning Parameter

An important issue regarding the construction of weights is the choice of appropriate tuning parameters. For the first two weight functions, increasing the value of the tuning parameter ($\alpha$ and $k$ respectively) leads to greater downweighting. To strike a balance between the degree of robustness and small sample efficiency, we propose, based on extensive numerical studies, the preferred ranges of 1.02 to 1.2 for $\alpha$ and 1.01 to 1.1 for $k$. Even smaller values may work for extreme outliers. For example, $\alpha = 1.01$ works well in the examples described by Tables 2 and 3.

For the third weight function the degree of downweighting decreases with the tuning parameter $\xi$. Based on our empirical investigations, we suggest the range $[3, 10]$ as being the set of reasonable tuning parameter values.

For the fourth weight function, we have two tuning parameters $d_1$ and $d_2$. We have already observed that increasing $d_1$ leads to stricter downweighting in both tails, whereas increasing $d_2$ leads to stricter downweighting in the right tail without affecting the nature of the weights in the left tail. Based on our experience, we suggest a range of 2.5 to 3.5 for $d_1$ and 1 to 2 for $d_2$.

The recommendations about the possible choices of reasonable tuning parameters as provided here have been primarily empirical. It will be of value to be able to come up with a more mathematical derivation of an “optimal” value of the tuning parameter based on some, possibly data based, objective criterion. This appears to be a difficult problem but solving this issue remains among our agenda for future work.

4.2. A Normal Mixture Study

As observed by Biswas et al. (2015) [9], the weighted likelihood estimating equations may have more than one solution, particularly when different parts of the sampled data appear to have been generated by different models. To gain more insight into the pattern of roots, we investigate the problem of the normal mixture density at different mixing weights. The unknown value of the normal mean is the target while the variance is assumed to be known and held fixed at $\sigma^2 = 1$.

We consider the roots at the level of the true distribution rather than through simulated data. Thus the empirical data distribution is replaced by the mixture distribution $F_m(x) = (1 - \epsilon)F_{(0, 1)}(x) + \epsilon F_{(5, 1)}(x)$, where $F_{(\mu, \sigma^2)}(x)$ is the cumulative distribution function of a $N(\mu, \sigma^2)$ random variable at the point $x$. Let $f_m(x)$ be the corresponding density function. The true distribution is modeled by the one parameter $N(\mu, 1)$ distribution. Our weight functions are now applied on the residuals $\frac{F_m(x)}{F_{(\mu, 1)}(x)} - 1$ in the lower tail and the residuals $\frac{S_m(x)}{S_{(\mu, 1)}(x)} - 1$ in the upper tail, where $S_m(x) = (1 - \epsilon)S_{(0, 1)}(x) + \epsilon S_{(5, 1)}(x)$ and $S_{(\mu, \sigma^2)}(x)$ represents the probability $P(X \geq x)$ for a $N(\mu, \sigma^2)$ random variable at the point $x$. 
To give a specific illustration, we use our proposed weight of the first type with the tuning parameter set at $\alpha = 1.05$ and investigate the pattern of the roots.

We choose the tuning parameter $p = 1/2$ and plot the value of the weighted score function $\int_{-\infty}^{\infty} w(\tau(x)) u_\mu(x) f_m(x) dx$ against different values of $\mu$, where $f_m(x)$ is the density of the mixture population at the point $x$ and

$$\tau(x) = \begin{cases} 
F_m(x) / F(\mu, 1)(x) - 1 & \text{if } 0 < F(\mu, 1)(x) \leq 1/2, \\
S_m(x) / S(\mu, 1)(x) - 1 & \text{otherwise}.
\end{cases}$$

Figure 7 presents the different roots obtained for the weighted likelihood score equation $\int_{-\infty}^{\infty} w(\tau(x)) u_\mu(x) f_m(x) = 0$ at the mixture distribution for different values of $\epsilon$. As we see, there is only one root when $\epsilon = 0$, i.e., when the data are “pure”. For $\epsilon \geq 0.1$, the weighted likelihood estimator successfully recognizes three roots: one near 0, one near 5 and the third somewhere in the middle of the mixture. Clearly the two extreme roots describe the two components while the middle root is a “MLE like” root.

That the second component is recognized for fairly small values of $\epsilon$ is a consequence of the fact that the two mixing components are fairly well separated. If we consider the mixture $(1-\epsilon)F(0, 1) + \epsilon F(4, 1)$, larger values of $\epsilon$ are required to recognize the root near 4. In fact in this case, one requires $\epsilon$ to be 0.2 or higher to observe multiple roots. For brevity the corresponding figure is not presented here.

Next, we show, by a real data example, how the multiple root issue becomes relevant for practical problems.

**Lubischew Data Example** : We consider the flea-beetle data presented by Lubischew (1962) [12]. These data contain six measurements for each of three different species. We only use two species *Chaetocnema concinna* and *Chaetocnema heptapotamica* and one measurement per species, namely, the front angle of the aedeagus. There are 21 and 22 observations from the two species respectively. We model the data by an univariate normal distribution of unknown mean and variance. We calculate the MLE of these parameters from the data and also use the WLE method to estimate the parameters. We use the first of the proposed weight functions and set the tuning parameter $\alpha$ at 1.02.

We search for the different possible roots for these data and discover three distinct roots. One of them gives weights near 1 to the observations from the first species (*concinna*) while observations from the second one (*heptapotamica*) get almost zero weights. The second root has an exactly opposite behavior. The third root gives weights close to 1 to almost all observations, irrespective of species and thus behaves like the MLE. This root is referred to as the ‘MLE-like root’ in Figure 8 and Tables 5 and 6, which list all the roots and weights allotted to each observation under different roots, respectively. The ‘C’ and ‘H’ in the first column of Table 6 indicates the type of the corresponding observation, *concinna* and *heptapotamica*, respectively.
Fig 7. The roots obtained for the weighted likelihood score equation for the normal mixture population

Table 5

| Root Type              | $\mu$   | $\sigma^2$ |
|------------------------|---------|------------|
| MLE                    | 12.0465 | 4.9502     |
| MLE-like root          | 12.0483 | 4.8327     |
| Concinna root          | 14.0644 | 0.8239     |
| Heptapotamica root     | 10.0480 | 0.8479     |

From this example it is clear that presence of multiple roots is not necessarily a nuisance. In this case, for example, the presence of multiple roots is significant as it clearly shows us that the whole dataset is comprised of data taken from two different population and these roots represent them. The fact that the proposed estimation methodology can successfully identify the roots and the corresponding observations is evident from Tables 5 and 6.

5. Simulation Study

In this section, we will present the results of an extensive simulation study to numerically demonstrate the performance of the proposed weighted likelihood estimators in providing high efficiency simultaneously with strong robustness. As the greatest benefit of this method compared to the disparity based methods of inference is in the continuous model, we choose the normal and exponential models for illustration. For this purpose, we take the first weight function and examine the obtained results.
Table 6
Weights allotted to all the observations under different roots

| Observation No. | MLE-like Root | Concinna Root | Heptapotamica Root |
|-----------------|---------------|---------------|--------------------|
| 1C              | 0.9965        | 0.9899        | 0.0000             |
| 2C              | 0.9999        | 0.9494        | 0.0000             |
| 3C              | 0.9999        | 0.9959        | 0.0000             |
| 4C              | 0.9983        | 0.9992        | 0.0000             |
| 5C              | 0.9999        | 0.9494        | 0.0000             |
| 6C              | 0.9965        | 0.9899        | 0.0000             |
| 7C              | 0.9999        | 0.9959        | 0.0000             |
| 8C              | 0.9999        | 0.9959        | 0.0000             |
| 9C              | 0.9999        | 0.9959        | 0.0000             |
| 10C             | 0.9965        | 0.9899        | 0.0000             |
| 11C             | 0.9999        | 0.9494        | 0.0000             |
| 12C             | 0.9965        | 0.9899        | 0.0000             |
| 13C             | 0.9999        | 0.9494        | 0.0000             |
| 14C             | 0.9965        | 0.9899        | 0.0000             |
| 15C             | 0.9999        | 0.9959        | 0.0000             |
| 16C             | 0.9999        | 0.9959        | 0.0000             |
| 17C             | 0.9965        | 0.9899        | 0.0000             |
| 18C             | 0.9999        | 0.9959        | 0.0000             |
| 19C             | 0.9999        | 0.9494        | 0.0000             |
| 20C             | 0.9999        | 0.9494        | 0.0000             |
| 21C             | 0.9999        | 0.9959        | 0.0000             |
| 22H             | 0.9989        | 0.0000        | 0.9945             |
| 23H             | 0.9978        | 0.0000        | 0.9735             |
| 24H             | 0.9978        | 0.0000        | 0.9735             |
| 25H             | 0.9953        | 0.0000        | 0.9987             |
| 26H             | 0.9978        | 0.0000        | 0.9735             |
| 27H             | 0.9953        | 0.0000        | 0.9987             |
| 28H             | 0.9999        | 0.4832        | 0.5714             |
| 29H             | 0.9978        | 0.0000        | 0.9735             |
| 30H             | 0.9953        | 0.0000        | 0.9987             |
| 31H             | 0.9967        | 0.0000        | 0.9999             |
| 32H             | 0.9978        | 0.0000        | 0.9735             |
| 33H             | 0.9967        | 0.0000        | 0.9999             |
| 34H             | 0.9967        | 0.0000        | 0.9999             |
| 35H             | 0.9978        | 0.0000        | 0.9735             |
| 36H             | 0.9953        | 0.0000        | 0.9987             |
| 37H             | 0.9953        | 0.0000        | 0.9987             |
| 38H             | 0.9953        | 0.0000        | 0.9987             |
| 39H             | 0.9967        | 0.0000        | 0.9999             |
| 40H             | 0.9978        | 0.0000        | 0.9735             |
| 41H             | 0.9953        | 0.0000        | 0.9987             |
| 42H             | 0.9967        | 0.0000        | 0.9999             |
| 43H             | 0.9953        | 0.0000        | 0.9987             |
We shall consider the problem of estimating the mean parameter for a normal distribution. Table 7 represents the mean square error for the MLE and the WLE for the first of the proposed weight functions at \( \alpha = 1.01 \) and \( \alpha = 1.02 \) for a \( N(0,1) \) distribution contaminated by a \( N(0,25) \) distribution. We consider the level of contamination \( \epsilon \) to be from 0\% to 50\%, at intervals of 10\%.

Since the existence of multiple roots to the weighted likelihood estimating equations is a natural issue here, we do a bootstrap root search as proposed by Markatou et al. (1998) [14]. For our simulation study, for each of the 6 levels of contamination, we picked 1000 samples, each of size 30. Then we took 50 independent bootstrap samples of size 3 from each sample. Using the MLEs of these samples as starting values we obtained the weighted likelihood estimators for each such starting value and identified the unique roots. In presence of multiple roots, we followed the suggestion made by Biswas et al. (2015) [9] and picked the estimator for which the sum of weights is second highest, provided, the corresponding sum of weight was at least as high as 25\% of the total weight. After choosing the roots, we calculated the mean square error around 0. We see that the mean square error for the WLE’s are much smaller when compared to the MLE in presence of contamination.

Next we consider a scenario where a \( N(0,1) \) model is contaminated by a \( N(5,1) \) distribution. The contamination scheme of the previous paragraph represents a scale contamination, whereas this one is a location contamination. Table 8 represents the mean square errors in estimating the mean parameter. Since, the presence of multiple roots is highly likely in this case as well, we employed the exact same strategy as described in the previous paragraph and
Table 7
Mean square error of the proposed estimators: Scale contamination

| $\epsilon$ | MLE | WLE$_{1,\alpha=1.01}$ | WLE$_{1,\alpha=1.02}$ |
|------------|-----|-----------------------|-----------------------|
| 0%         | 0.03390 | 0.08464               | 0.04338               |
| 10%        | 0.11786 | 0.05259               | 0.03769               |
| 20%        | 0.19132 | 0.07035               | 0.07109               |
| 30%        | 0.28393 | 0.11469               | 0.10449               |
| 40%        | 0.36348 | 0.18997               | 0.10466               |
| 50%        | 0.45381 | 0.28766               | 0.23793               |

Note: the sample size for each of the cases was 30. Each mean square error is based on 1000 replications. $\epsilon$ is the level of contamination in the model $(1-\epsilon)N(0,1) + \epsilon N(0,25)$.

Table 8
Mean square error of the proposed estimators: Location contamination

| $\epsilon$ | MLE | WLE$_{1,\alpha=1.01}$ | WLE$_{1,\alpha=1.02}$ |
|------------|-----|-----------------------|-----------------------|
| 0%         | 0.03233 | 0.03557               | 0.04290               |
| 10%        | 0.36684 | 0.06305               | 0.05261               |
| 20%        | 1.14137 | 0.14865               | 0.09071               |
| 30%        | 2.46722 | 0.55077               | 0.47253               |
| 40%        | 4.34540 | 3.72141               | 3.48538               |
| 50%        | 6.46095 | 11.03328              | 10.70855              |

Note: the sample size for each of the cases was 30. Each mean square error is based on 1000 replications. $\epsilon$ is the level of contamination in the model $(1-\epsilon)N(0,1) + \epsilon N(5,1)$.

calculated the mean square error around 0.

As expected, in presence of contamination, the mean square error for MLE blows up. However, the WLE performs well in identifying the target value of zero and hence produces substantially smaller mean square error, at least for smaller values of contamination. It may be seen, however, that as the contamination proportion tends to the level of 50%, the performance of the WLE becomes poorer (in terms of increased mean square error), and this phenomenon demands an explanation. At 50% contamination, both the components of the mixture become equally strong, on the average, in terms of its representation in the sample, and the final selection of the root becomes a toss up between the means of the two components. In an ideal sense, therefore, the method chooses a root around 0 half of the time, and a root around 5 in the remaining half. Thus the empirical mean square error is supposed to be of the order $(5-0)^2/2 = 12.5$, which is what we approximately observe. In case of the MLE, however, the process throws out an estimator which is close to the average of the two component means, which is 2.5. Thus the mean square error in this case is of the order of $(2.5-0)^2 = 6.25$, which is close to the observed value in Table 8.

We consider the exponential distribution for our next simulation study. Smoothing based on usual kernels produce nonnegative estimated densities for part of the negative side of the real line, and more sophisticated kernels are needed for this case, complicating the theory. Such a difficulty does not arise in this case, and the estimation method can easily proceed as in the normal case. We
Table 9
Mean square error of the proposed estimators: Exponential model

| ε     | MLE  | WLE_{P=1.01} | WLE_{α=1.02} |
|-------|------|--------------|--------------|
| 0%    | 0.03374 | 0.03919 | 0.04672 |
| 10%   | 0.09971 | 0.06601 | 0.06244 |
| 20%   | 0.19193 | 0.15571 | 0.15252 |
| 30%   | 0.27974 | 0.19973 | 0.20942 |
| 40%   | 0.35634 | 0.19973 | 0.20942 |
| 50%   | 0.42225 | 0.37637 | 0.39666 |

NOTE: the sample size for each of the cases was 30. Each mean square error is based on 1000 replications. ϵ is the level of contamination for the model $(1-ϵ)$ exponential $(1) + ϵ$ exponential $(1/5)$.

employed the same scheme for choosing the root as before. Table 9 presents the mean square errors for an exponential $(1)$ distribution contaminated by an exponential $(1/5)$ distribution. The model is assumed to be exponential $(λ)$ with the probability density function

$$f_λ(x) = λ \exp(-λx), \ x > 0.$$  

Here also, we see that in presence of contamination, the WLE outperforms MLE in terms of mean square error.

6. Theoretical Properties of The Weighed Likelihood Estimator

In this section, we discuss some properties of the proposed weighted likelihood estimator that deal with the robustness and the asymptotic efficiency of these estimators for the case $p = 1/2$.

6.1. The Influence Function of the Weighted Likelihood Estimators

The influence function is an important heuristic tool for assessing the robustness of an estimator. To determine the influence function of the proposed weighted likelihood estimators, consider the following setup. Let $\mathcal{F} = \{F_θ : θ ∈ Θ ⊂ \mathbb{R}\}$ be the parametric model and let $T : G → Θ$ be a functional from a relevant class of distribution functions to the parameter space. We assume Fisher consistency, i.e. $T(F_θ) = θ$. To find the influence function of the proposed estimators, we consider the $ϵ$ contaminated version of the true distribution function $G$ given by

$$G_ϵ(x) = (1-ϵ)G(x) + ϵΛ_y(x)$$  \hspace{1cm} (3)

where $Λ_y(x)$ is the distribution function of $χ_y$, the random variable which puts all its mass on $y$. Denote by $Λ^*_y(x) = P(χ_y ≥ x)$ and $G^*(x) = P(X ≥ x)$, with $X$ being the random variable having the distribution function $G$. We consider a general distribution $G$, not necessarily in the model.
Theorem 1. The influence function of the proposed estimator is

\[ T'(y) = \frac{\partial}{\partial \epsilon} \theta_{\epsilon} \bigg|_{\epsilon=0} = D^{-1}N \]  

where \( \theta^g = T(G) \), \( \theta_{\epsilon} \) is the functional corresponding to the contaminated distribution in (3) and

\[ D = \left[ \int_{X_1} w'(\tau(x)) u_{\theta^g}(x) \frac{\nabla F_{\theta^g}(x)}{F_{\theta^g}(x)} (\tau(x) + 1) dG(x) \right. \\
+ \int_{X_2} w'(\tau(x)) u_{\theta^g}(x) \frac{\nabla S_{\theta^g}(x)}{S_{\theta^g}(x)} (\tau(x) + 1) dG(x) \\
+ \int w(\tau(x)) \nabla(-u_{\theta^g}(x)) dG(x) \right], \\
N = \left[ w(\tau(y)) u_{\theta^g}(y) + \int_{X_1} w'(\tau(x)) \Lambda_{\theta^g}(x) \frac{u_{\theta_{\epsilon}}(x)}{F_{\theta^g}(x)} dG(x) \\
+ \int_{X_2} w'(\tau(x)) \bar{\Lambda}_{\theta^g}(x) \frac{u_{\theta_{\epsilon}}(x)}{S_{\theta^g}(x)} dG(x) \\
- \int w'(\tau(x))(\tau(x) + 1) u_{\theta^g}(x) dG(x) \right], \\
\]

where \( X_1 = \{ x \in X : F_{\theta^g}(x) \leq 1/2 \} \) and \( X_2 = \{ x \in X : F_{\theta^g}(x) > 1/2 \} \) and \( X \) being the support of the distribution. Note that \( X_1 \) and \( X_2 \) are disjoint and \( X = X_1 \cup X_2 \). When the true distribution \( G \) belongs to the model, then \( G(x) = F_{\theta}(x) \) for some \( \theta \in \Theta \) and the Influence function takes the simple form

\[ T'(y) = \left[ \int -\nabla u_{\theta}(x) dF_{\theta} \right]^{-1} = I^{-1}(\theta)u_{\theta}(y) \]

which is the same as the influence function of the maximum likelihood estimator.

Proof. Let our residual function be defined as:

\[ \tau_{\epsilon}(x) = \begin{cases} 
G_{\epsilon}(x) \\
F_{\theta_{\epsilon}}(x) 
\end{cases} - 1 \text{ if } 0 < F_{\theta_{\epsilon}}(x) \leq 1/2, \\
\begin{cases} 
G^*_\epsilon(x) \\
S_{\theta_{\epsilon}}(x) 
\end{cases} - 1 \text{ if } 1/2 < F_{\theta_{\epsilon}}(x) < 1, \]

where \( G^*_\epsilon(x) = (1-\epsilon)G^*(x) + \epsilon \bar{\Lambda}_{\theta^g}(x) \). We want to differentiate the weighted likelihood estimating equation

\[ \int_X w(\tau_{\epsilon}(x)) u_{\theta_{\epsilon}}(x) dG_{\epsilon}(x) = 0 \]  

(5)

with respect to \( \epsilon \). We assume that the the underlying distribution has a probability density function \( g \) and assume that the range of the distribution is free of
\(\theta\) and hence free of \(\epsilon\).

\[
\frac{\partial}{\partial \epsilon} \int_{X} w(\tau_{x}(x))u_{\theta}(x)g_{\epsilon}(x)dx = \frac{\partial}{\partial \epsilon} \int_{X_{1}} w(\tau_{x}(x))u_{\theta}(x)g_{\epsilon}(x)dx + \frac{\partial}{\partial \epsilon} \int_{X_{2}} w(\tau_{x}(x))u_{\theta}(x)g_{\epsilon}(x)dx,
\]

where \(g_{\epsilon}(x)\) is the density function corresponding to the contaminated distribution in (3). Let \(\mu_{\epsilon}\) be the median of the contaminated distribution. Then the above expression can be written as

\[
\frac{\partial}{\partial \epsilon} \int_{a}^{b} w(\tau_{x}(x))u_{\theta}(x)g_{\epsilon}(x)dx = \frac{\partial}{\partial \epsilon} \int_{\mu_{\epsilon}}^{b} w(\tau_{x}(x))u_{\theta}(x)g_{\epsilon}(x)dx
\]

where \(X' = (a, b)\).

Now applying Leibniz integral rule, we get the above expression to be

\[
\int_{a}^{\mu_{\epsilon}} \frac{\partial}{\partial \epsilon} (w(\tau_{x}(x))u_{\theta}(x)g_{\epsilon}(x))dx + w(\tau_{x}(\mu_{\epsilon}))u_{\theta}(\mu_{\epsilon})g_{\epsilon}(\mu_{\epsilon}) \left( \frac{\partial}{\partial \epsilon} \mu_{\epsilon} \right) + \int_{\mu_{\epsilon}}^{b} \frac{\partial}{\partial \epsilon} (w(\tau_{x}(x))u_{\theta}(x)g_{\epsilon}(x))dx - w(\tau_{x}(\mu_{\epsilon}))u_{\theta}(\mu_{\epsilon})g_{\epsilon}(\mu_{\epsilon}) \left( \frac{\partial}{\partial \epsilon} \mu_{\epsilon} \right)
\]

which leads to the expression

\[
\int_{X_{1}} w'(|\tau_{x}(x)|)\tau'_{x}(x)u_{\theta}(x)dG_{\epsilon}(x) + \theta_{\epsilon}' \left[ \int_{X_{1}} w(\tau_{x}(x))\nabla u_{\theta}(x)dG_{\epsilon}(x) \right] + \int_{X_{1}} w(\tau_{x}(x))u_{\theta}(x)d(\Lambda y - G)(x) + \int_{X_{2}} w'(|\tau_{x}(x)|)\tau'_{x}(x)u_{\theta}(x)dG_{\epsilon}(x) + \theta_{\epsilon}' \left[ \int_{X_{2}} w(\tau_{x}(x))\nabla u_{\theta}(x)dG_{\epsilon}(x) \right] + \int_{X_{2}} w(\tau_{x}(x))u_{\theta}(x)d(\Lambda y - G)(x).
\]

We set the expression to be equal to 0 to get an equation.

For \(x \in X_{1}\), note that

\[
\tau'_{x}(x) |_{\epsilon=0} = \frac{1}{F_{\theta y}(x)}[\Lambda y(x) - G(x) - \frac{\nabla F_{\theta y}(x)}{F_{\theta y}(x)}T'_{y}(y)G(x)].
\]

And for \(x \in X_{2}\), note that

\[
\tau'_{x}(x) |_{\epsilon=0} = \frac{1}{S_{\theta y}(x)}[\Lambda y(x) - G^{\ast}(x) - \frac{\nabla S_{\theta y}(x)}{S_{\theta y}(x)}T'_{y}(y)G^{\ast}(x)].
\]

When the expression for \(\tau'_{x}(x) |_{\epsilon=0}\) is substituted in the above equation, we get

\[
T'_{y}\left[ \int_{X_{1}} w'(|\tau(x)|)u_{\theta y}(x)\frac{\nabla F_{\theta y}(x)}{F_{\theta y}(x)} G(x)F_{\theta y}(x)dG(x) + \right.
\]

\[
\int_{X_2} w'(\tau(x))u_{\theta}(x) \nabla S_{\theta}(x) G^*(x) dG(x) - \int w(\tau(x))\nabla u_{\theta}(x) dG(x)
\]

\[
= w(\tau(y))u_{\theta}(y) + \int_{X_1} w'(\tau(x))[\Lambda y - G(x)] \frac{u_{\theta_{\theta}}(x)}{F_{\theta_{\theta}}(x)} dG(x) +
\]

\[
\int_{X_2} w'(\tau(x))[\tilde{\Lambda} y - G^*(x)] \frac{u_{\theta_{\theta}}(x)}{S_{\theta_{\theta}}(x)} dG(x).
\]

Since \( \tau(x) + 1 = \frac{G(x)}{F_{\theta}(x)} \) when \( x \in X_1 \) and \( \tau(x) + 1 = \frac{G^*(x)}{S_{\theta}(x)} \) when \( x \in X_2 \), we get the required result.

When \( G = F_{\theta} \) is the true distribution function, we get \( \theta_{\theta} = \theta \) and \( \tau(x) = 0 \).
We see that \( w(0) = 1 \) and \( w'(0) = 0 \). Substituting these values, we get the simple form of the influence function

\[
T'_{\theta}(y) = \left[ \int -\nabla u_{\theta}(x) dF_{\theta}(x) \right]^{-1} u_{\theta}(y) = I^{-1}(\theta)u_{\theta}(y)
\]

which is the same as the influence function of the maximum likelihood estimator at the model.

### 6.2. Location-Scale Equivariance

We now look into the location-scale equivariance of the proposed weighted likelihood estimator. We consider a location-scale family characterized by either of the following equivariant formulations,

1. \( f_{(\mu, \sigma)}(x) = \frac{1}{\sigma} f_{(0,1)} \left( \frac{x-\mu}{\sigma} \right) \),
2. \( F_{(\mu, \sigma)}(x) = F_{(0,1)} \left( \frac{x-\mu}{\sigma} \right) \),

and \( \theta = (\mu, \sigma) \) represents our parameter of interest. Consider i.i.d observations \( Z_1, Z_2, \ldots, Z_n \) from a location-scale family \( F_{(\mu, \sigma)} \). The corresponding estimate for the parameter vector \( (\mu, \sigma) \) obtained by the proposed weighted likelihood method is \( (\hat{\mu}, \hat{\sigma}) \), say. Consider the transformation

\[
X_i = a + bZ_i, \quad a \in \mathbb{R}, b > 0, \quad i = 1, 2, \ldots, n.
\]

Then our weighted likelihood estimator is location-scale equivariant in the sense \((a + b\hat{\mu}, b\hat{\sigma})\) is the estimated parameter vector for the transformed data.

**Theorem 2.** The proposed weighted likelihood estimators are location-scale equivariant.

**Proof.** We have \( Z_1, Z_2, \ldots, Z_n \) independent and identically distributed observations from a location scale family with probability density function \( \frac{1}{\sigma} f_{(0,1)}(\frac{x-\mu}{\sigma}) \), parametrized by \( (\mu, \sigma) \). We know that \((\hat{\mu}, \hat{\sigma})\) are weighted likelihood estimators of \((\mu, \sigma)\) obtained form solving the equation

\[
-\frac{1}{n} \sum_{i=1}^{n} w(\tau_{(\mu, \sigma)}(Z_i))u_{(\mu, \sigma)}(Z_i) = 0.
\]
We intend to show that for \( a \in \mathbb{R} \) and \( b > 0 \) and for \( X_i = a + bZ_i, i = 1, 2, \ldots, n; \) \((a + b\mu, b\sigma)\) will be the weighted likelihood estimators of \((a + b\mu, b\sigma)\).

In order to obtain the said estimators, we need to show

\[
\frac{1}{n} \sum_{i=1}^{n} w(\tau^X_{(a+b\mu,b\sigma)}(X_i)) w_{(a+b\mu,b\sigma)}(X_i) = 0.
\]

We will show that for any choice of \( a, \mu \) and \( b > 0, \sigma > 0 \) and for any \( z \in \mathbb{R} \) and \( x = a + bz \),

\[
w(\tau^X_{(a+b\mu,b\sigma)}(x)) = w(\tau^Z_{(\mu,\sigma)}(z)),
\]

\[
u_{(a+b\mu,b\sigma)}(x) = \nu_{(\mu,\sigma)}(z).
\]

We first see that \( F_{(a+b\mu,b\sigma)}(x) = F_{(\mu,\sigma)}(z) \) and \( S_{(a+b\mu,b\sigma)}(x) = S_{(\mu,\sigma)}(z) \). Also note that,

\[
F^Z_n(z) = \frac{1}{n} \sum_{i=1}^{n} I(Z_i \leq z) = \frac{1}{n} \sum_{i=1}^{n} I(X_i \leq x) = F^X_n(x),
\]

\[
S^Z_n(z) = \frac{1}{n} \sum_{i=1}^{n} I(Z_i \geq z) = \frac{1}{n} \sum_{i=1}^{n} I(X_i \geq x) = S^X_n(x).
\]

Now, when \( F_{(a+b\mu,b\sigma)}(x) = F_{(\mu,\sigma)}(z) \leq 1/2, \)

\[
\tau^X_{(a+b\mu,b\sigma)}(x) = \frac{F^X_n(x)}{F_{(a+b\mu,b\sigma)}(x)} - 1 = \frac{F^Z_n(z)}{F_{(\mu,\sigma)}(z)} - 1 = \tau^Z_{(\mu,\sigma)}(z).
\]

Otherwise,

\[
\tau^X_{(a+b\mu,b\sigma)}(x) = \frac{S^X_n(x)}{S_{(a+b\mu,b\sigma)}(x)} - 1 = \frac{S^Z_n(z)}{S_{(\mu,\sigma)}(z)} - 1 = \tau^Z_{(\mu,\sigma)}(z).
\]

Note that the function \( w(\cdot) \) is not inherently dependent on the parameter \( \theta \), rather it is the residual function, which is the argument of \( w(\cdot) \), that induces the dependence on \( \theta \). And since we have \( \tau^X_{(a+b\mu,b\sigma)}(x) = \tau^Z_{(\mu,\sigma)}(z) \), we can conclude

\[
w(\tau^X_{(a+b\mu,b\sigma)}(x)) = w(\tau^Z_{(\mu,\sigma)}(z)).
\]

As to show the equality of the score functions, we note that score function is simply the derivative of the natural logarithm of the corresponding probability density function with respect to the parameters. So, we now proceed to show the equality of the derivatives involved. Since we are dealing with a location scale family, the probability density functions of \( Z \) and \( X \) are of the form

\[
f_{(\mu,\sigma)}(z) = \frac{1}{\sigma} f_{(0,1)} \left( \frac{z - \mu}{\sigma} \right)
\]

and

\[
f_{(a+b\mu,b\sigma)}(x) = \frac{1}{b\sigma} f_{(0,1)} \left( \frac{x - a - b\mu}{b\sigma} \right) = \frac{1}{b\sigma} f_{(0,1)} \left( \frac{z - \mu}{\sigma} \right)
\]
respectively. Taking natural logarithm we get
\[
\ln f(\mu, \sigma)(z) = -\ln \sigma + \ln f_{0,1} \left( \frac{z - \mu}{\sigma} \right),
\]
\[
\ln f(a + b\mu, b\sigma)(x) = -\ln b - \ln \sigma + \ln f_{0,1} \left( \frac{z - \mu}{\sigma} \right).
\]
Since the two functions differ only by a constant (in this case \(-\ln b\)), their derivatives with respect to both \(\mu\) and \(\sigma\) will be the same, thereby assuring
\[
u_{a + b\mu, b\sigma}(x) = \nu_{(\mu, \sigma)}(z).
\]
Now that we have the two desired results, we see that
\[
\frac{1}{n} \sum_{i=1}^{n} w(\tau_{(a + b\mu, b\sigma)}(X_i)) u_{(a + b\mu, b\sigma)}(X_i) = 0.
\]
Thus we complete the proof of the theorem. \(\Box\)

6.3. Consistency and Asymptotic Normality

In this section, we discuss the consistency and asymptotic efficiency of the proposed weighted likelihood estimators. We first present some regularity conditions. While the general case involving multiple parameters can be handled by making the conditions more complicated and by routinely extending the proof, in the following we consider the case of a scalar parameter.

(C1) The weight function \(w(\tau)\) is nonnegative, bounded above by 1 and twice differentiable with respect to \(\tau\); \(w(0) = 1\) and \(w'(0) = 0\).

(C2) The function \(w'(\tau)(1 + \tau)\) is bounded, where \(w'(\tau)\) is the derivative of \(w(\tau)\) with respect to \(\tau\).

(C3) The functions \(w''(\tau)\) and \(w''(\tau)(1 + \tau)^2\) are bounded.

(C4) For every \(\theta_0 \in \Theta\), there is a neighborhood \(N(\theta_0)\) such that for every \(\theta \in N(\theta_0)\), the quantities \(|\tilde{u}_\theta(x)\nabla u_{\theta}(x)|\), \(|\tilde{u}_\theta^2(x)u_{\theta}(x)|\), \(|\nabla u_{\theta}(x)u_{\theta}(x)|\) and \(|\nabla u_{\theta}(x)\nabla u_{\theta}(x)|\) are bounded by \(K_1(x), K_2(x), K_3(x)\) and \(K_4(x)\) respectively, where \(\nabla_u\) represents second derivative with respect to \(\theta\) and \(E_{\theta_0}[K_i(X)] < \infty\) for \(i = 1, 2, 3, 4\).

(C5) \(E_{\theta_0}[\tilde{u}_\theta^2(X)u_{\theta}(X)] < \infty\).

(C6) The Fisher Information \(I(\theta) = E_\theta[u_{\theta}^2(X)]\) is nonzero and finite for any \(\theta \in \Theta\).

Here, \(\tilde{u}_\theta(x)\) is defined as:
\[ \hat{u}_\theta(x) = \begin{cases} \frac{\nabla F_\theta(x)}{F_\theta(x)} & \text{if } 0 < F_\theta(x) \leq 1/2, \\ \frac{\nabla S_\theta(x)}{S_\theta(x)} & \text{if } 1/2 < F_\theta(x) < 1. \end{cases} \]

All the proposed weight functions follow the conditions (C1) to (C3) and conditions (C4) to (C6), we assume to be true.

**Theorem 3.** Let the true distribution belong to the model, \( \theta_0 \) be the true parameter and let \( \hat{\theta}_w \) be the weighted likelihood estimator. Under conditions (C1) - (C6) the following results hold:

1. The convergence
   \[ \sqrt{n}|A_n - \frac{1}{n} \sum_{i=1}^{n} u_{\theta_0}(X_i)| \to 0 \]
   holds in probability, where \( A_n = \frac{1}{n} \sum_{i=1}^{n} w_{\theta_0}(\tau_n(X_i))u_{\theta_0}(X_i), \) and \( w_{\theta_0}(\tau_n(X_i)) \) are weights based on the residual function \( \tau_n(X_i). \)

2. The convergence
   \[ |B_n - \frac{1}{n} \sum_{i=1}^{n} \nabla u_{\theta_0}(X_i)| \to 0 \]
   holds in probability, where \( B_n = \frac{1}{n} \sum_{i=1}^{n} \nabla (w_{\theta}(\tau_n(X_i))u_{\theta}(X_i))|_{\theta=\theta_0}. \)

3. \( C_n = O_p(1) \), where \( C_n = \frac{1}{n} \sum_{i=1}^{n} \nabla_2 (w_{\theta}(\tau_n(X_i))u_{\theta}(X_i))|_{\theta=\theta'}. \) Here \( \theta' \) is in between \( \theta_0 \) and \( \hat{\theta}_w \) and \( \nabla_2 \) represents second derivative with respect to \( \theta. \)

In the proof of the above theorem which follows shortly, we modify our residual functions slightly, such that the modified residual function is asymptotically same as the residual function we defined. We modify the residual function \( \tau_n \) the following way

\[ \tilde{\tau}_{n,k}(X_i) = \begin{cases} \frac{F_n(X_i)}{k^{-1}n^{-3/4}} - 1 & \text{if } 0 \leq F_\theta(X_i) \leq k^{-1}n^{-3/4}, \\ \frac{F_n(X_i)}{F_\theta(X_i)} - 1 & \text{if } k^{-1}n^{-3/4} < F_\theta(X_i) \leq 1/2, \\ \frac{S_n(X_i)}{S_\theta(X_i)} - 1 & \text{if } 1/2 < F_\theta(X_i) \leq 1 - k^{-1}n^{-3/4}, \\ \frac{S_n(X_i)}{k^{-1}n^{-3/4}} - 1 & \text{if } 1 - k^{-1}n^{-3/4} < F_\theta(X_i) \leq 1. \end{cases} \]

This modification is intended to keep the residuals from blowing up. Note that the regions where the residual function is redefined shrinks at a rate faster than \( n^{-1/2} \) and therefore this redefinition does not interfere with the asymptotic results proved here.

Having defined this new residual function, we will now proceed to establish the required results for the residual \( \tilde{\tau}_{n,k}(\cdot) \) in place of \( \tau_n(\cdot). \) To avoid cumbersome
notations, we proceed with the notation $\tau_n$ although the residual function being implied is actually $\tilde{\tau}_{n,k}$.

**Remark 1.** We have seen that for all fixed, positive values of $k$, the limiting distribution of $\tilde{\tau}_{n,k}$, as $n \to \infty$, remains same. The modified residual function $\tilde{\tau}_{n,k}$ has been introduced to avoid the unboundedness of $\tau_n$. The redefinition only affects the extreme tails in the $\tau$ scale, and the redefined residuals are smaller than $o_p(n^{-1/2})$ terms. For the purpose of simulation, we preassign large values of $k$.

**Proof.** For proving the first result, we do a Taylor series expansion of $w_{\theta_0}(\tau_n(X_i))$ around 0 which gives

$$w_{\theta_0}(\tau_n(X_i)) - 1 = w_{\theta_0}(0) + w'_{\theta_0}(0)\tau_n(X_i) + \frac{w''_{\theta_0}(\xi_i)}{2}\tau_n^2(X_i) - 1.$$  

From the given conditions, $w_{\theta_0}(0) = 1$, $w'_{\theta_0}(0) = 0$ and $|w''_{\theta_0}(x)| \leq M$ for some large enough $M \in \mathbb{R}$ and for all $x \in [-1, \infty)$. Using these facts, we establish that

$$(w_{\theta_0}(\tau_n(X_i)) - 1)^2 \leq M^2\tau_n^4(X_i)/4.$$  

We now proceed to calculate the expected value of $(w_{\theta_0}(\tau_n(X_i)) - 1)^2$. Clearly,

$$E((w_{\theta_0}(\tau_n(X_i)) - 1)^2) \leq M^2E(\tau_n^4(X_i))/4.$$  

To calculate the above, we notice that, it is now of our interest to calculate $E(\tau_n^4(X_i)) = E(E(\tau_n^4(X_i)|X_i = x))$. Since $X_1, X_2, \ldots, X_n$ are supposed to be i.i.d. observations coming from a continuous distribution, conditioning by $X_i = x$ is equivalent to conditioning by $F_{\theta_0}(X_i) = F_{\theta_0}(x)$.

It is to be noted here that $nF_{\theta_0}(X_i) - 1$ is a binomial random variable with parameters $n - 1$ and $p_x = F_{\theta_0}(x)$ and $nS_n(X_i) - 1$ is another binomial random variable with parameters $n - 1$ and $q_x = (1 - p_x) = 1 - F_{\theta_0}(x)$, conditional on $X_i = x$. Using this, we compute $E(\tau_n^4(X_i))$ and we find that the leading term is of the order $n^{-5/4}$. Hence, we establish that

$$E(w_{\theta_0}(\tau_n(X_i)) - 1)^2 \leq \frac{M_k}{n^{5/4}}$$  

for an appropriate choice of $M_k$.

Having established this, we now have

$$E(|w_{\theta_0}(\tau_n(X_i)) - 1|u_{\theta_0}(X_i)) \leq E((w_{\theta_0}(\tau_n(X_i)) - 1)^2)^{1/2}E(u_{\theta_0}^2(X_i))^{1/2} \leq \frac{M_2}{n^{5/8}}.$$
And so, we have
\[
\begin{align*}
E \left| \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (w_{\theta_0}(\tau_n(X_i)) - 1) u_{\theta_0}(X_i) \right| & \leq \frac{1}{\sqrt{n}} \sum_{i=1}^{n} E \left| (w_{\theta_0}(\tau_n(X_i)) - 1) u_{\theta_0}(X_i) \right| \\
& \leq \frac{1}{\sqrt{n}} \sum_{i=1}^{n} M_2 n^{5/8} \\
& = \frac{M_2}{n^{1/8}} \\
& \to 0.
\end{align*}
\]

And thus we prove that \( \sqrt{n} |A_n - \frac{1}{n} \sum_{i=1}^{n} u_{\theta_0}(X_i) | \overset{L_1}{\to} 0 \) and hence
\[
\sqrt{n} |A_n - \frac{1}{n} \sum_{i=1}^{n} u_{\theta_0}(X_i) | \overset{P}{\to} 0.
\]

This completes the proof of part 1.

We now proceed to prove part 2. Using an approach similar to the proof of part 1, we get
\[
\begin{align*}
B_n - \frac{1}{n} \sum_{i=1}^{n} \nabla u_{\theta_0}(X_i) & \leq \frac{1}{n} \sum_{i=1}^{n} |w_{\theta_0}'(\tau_n(X_i))(1 + \tau_n(X_i))\tilde{u}_{\theta_0}(X_i)u_{\theta_0}(X_i)| \\
& + \frac{1}{n} \sum_{i=1}^{n} |(w_{\theta_0}(\tau_n(X_i)) - 1)\nabla u_{\theta_0}(X_i)|.
\end{align*}
\]

We shall show that both the dominating terms go to 0 in probability. For the second term on the right, we already know that
\[
E((w_{\theta_0}(\tau_n(X_i)) - 1)^2) \leq \frac{M_k}{n^{5/4}}
\]
and as per our assumption, \( E(\nabla u_{\theta_0}^2(X_i)) \) is finite. Combining these, we get
\[
E \left( \frac{1}{n} \sum_{i=1}^{n} |(w_{\theta_0}(\tau_n(X_i)) - 1)\nabla u_{\theta_0}(X_i)| \right) \leq \frac{\tilde{M}_3}{n^{5/8}},
\]
thus leading to the conclusion that \( \frac{1}{n} \sum_{i=1}^{n} |(w_{\theta_0}(\tau_n(X_i)) - 1)\nabla u_{\theta_0}(X_i)| \overset{P}{\to} 0. \)

In order to show convergence of the first term, we take a Taylor series expansion of \( w_{\theta_0}'(\tau_n(X_i)) \) around 0 to get
\[
\begin{align*}
w_{\theta_0}'(\tau_n(X_i)) &= w_{\theta_0}'(0) + \tau_n(X_i) w_{\theta_0}''(\xi_i),
\end{align*}
\]
where \( \xi_i \) is in between 0 and \( \tau_n(X_i) \). Since \( w_{\theta_0}'(0) = 0 \) and \( |w_{\theta_0}''(\xi_i)| \leq \tilde{M} \) for some appropriately large \( \tilde{M} \), we have
\[
E((w_{\theta_0}'(\tau_n(X_i))(1 + \tau_n(X_i))))^2) \leq \tilde{M}^2 E(\tau_n^2(X_i) + 2\tau_n(X_i) + \tau_n^4(X_i)).
\]
As before, we proceed to compute this expected value and see that the leading term is again of the order $n^{-5/4}$ and exactly like before, we have

$$\frac{1}{n} \sum_{i=1}^{n} |w_{\theta_0}'(\tau_n(X_i))(1 + \tau_n(X_i))\tilde{u}_{\theta_0}(X_i)\tilde{u}_{\theta_0}(X_i)| \leq \frac{M_4}{n^{5/8}}$$

for some large $M_4$. And thus, this convergence also holds in probability. This yields

$$\left| B_n - \frac{1}{n} \sum_{i=1}^{n} \nabla u_{\theta_0}(X_i) \right| \overset{P}{\to} 0.$$

In order to prove the result regarding the third term, note that

$$C_n = \frac{1}{n} \sum_{i=1}^{n} \nabla_2 (w_{\theta}(\tau_n(X_i))u_{\theta}(X_i))|_{\theta=\theta'}$$

$$= \frac{1}{n} \sum_{i=1}^{n} w_{\theta'}(\tau_n(X_i))\nabla_2 u_{\theta'}(X_i)$$

$$- \frac{2}{n} \sum_{i=1}^{n} w_{\theta'}(\tau_n(X_i))(1 + \tau_n(X_i))\tilde{u}_{\theta'}(X_i)\nabla u_{\theta'}(X_i)$$

$$- \frac{1}{n} \sum_{i=1}^{n} w_{\theta'}(\tau_n(X_i))(1 + \tau_n(X_i))\nabla \tilde{u}_{\theta'}(X_i)u_{\theta}(X_i)$$

$$+ \frac{1}{n} \sum_{i=1}^{n} w_{\theta'}(\tau_n(X_i))(1 + \tau_n(X_i))\tilde{u}_{\theta'}^2(X_i)u_{\theta'}(X_i)$$

$$+ \frac{1}{n} \sum_{i=1}^{n} w_{\theta'}(\tau_n(X_i))(1 + \tau_n(X_i))^2 \tilde{u}_{\theta'}^2(X_i)u_{\theta'}(X_i).$$

Each of the term in the right hand side is bounded as per our assumption and as a result, we can conclude

$$C_n = O_p(1)$$

and this completes the proof of the theorem.

Using the above results, the consistency of the proposed weighted likelihood estimator $\hat{\theta}_w$ follows from Serfling (1980, pp. 146-148) [17]. A straightforward Taylor Series expansion of the weighted likelihood estimating equation

$$\frac{1}{n} \sum_{i=1}^{n} w_{\hat{\theta}_w}(\tau_n(X_i))u_{\hat{\theta}_w}(X_i) = 0$$

around $\hat{\theta}_w = \theta_0$ leads to the relation

$$\hat{\theta}_w - \theta_0 = -\frac{\sqrt{n}A_n}{B_n + \frac{1}{2}C_n}$$

(7)
which, altogether with the above results, immediately yields
\[ \sqrt{n}(\hat{\theta}_w - \theta_0) \xrightarrow{d} Z^* \sim N(0, I^{-1}(\theta_0)). \]

**Remark 2.** $F_\theta(X_i)$ is a function of $\theta$ and we assume it to be twice differentiable with respect to $\theta$. $\tau_n$ is a function of $F_\theta$. $\tilde{\tau}_{n,k}$ is a modification of $\tau_n$, changing it beyond certain constant (given $n$) values. $\tilde{\tau}_{n,k}$ may not be differentiable with respect to $\theta$ at (only) $F_\theta(x) = k^{-1}n^{-3/4}$ and $1 - k^{-1}n^{-3/4}$. To bypass this inconvenience, let us define $\hat{\tau}_{n,k}$ in the following way:

\[
\hat{\tau}_{n,k}(X_i) = \begin{cases} 
\frac{F_n(X_i)}{k^{-1}n^{-3/4}} - 1 & \text{if } 0 \leq F_\theta(X_i) \leq k^{-1}n^{-3/4}, \\
t_{n,\theta}(X_i) & \text{if } k^{-1}n^{-3/4} < F_\theta(X_i) \leq k^{-1}n^{-3/4} + \delta_n, \\
\frac{F_n(X_i)}{F_\theta(X_i)} - 1 & \text{if } k^{-1}n^{-3/4} + \delta_n < F_\theta(X_i) \leq 1/2, \\
S_n(X_i) & \text{if } 1/2 < F_\theta(X_i) \leq 1 - k^{-1}n^{-3/4} - \delta_n, \\
\frac{S_n(X_i)}{S_\theta(X_i)} - 1 & \text{if } 1 - k^{-1}n^{-3/4} - \delta_n < F_\theta(X_i) \leq 1 - k^{-1}n^{-3/4}, \\
\frac{S_n(X_i)}{k^{-1}n^{-3/4}} - 1 & \text{if } 1 - k^{-1}n^{-3/4} < F_\theta(X_i) \leq 1,
\end{cases}
\]

where $t_{n,\theta}(x)$ and $s_{n,\theta}(x)$ are functions such that the first and second derivative of $\hat{\tau}_{n,k}$ both exist and the choice of $\delta_n$ is made in such a way, that the ensuing changes do not affect the “in probability convergence” results.

7. Higher Order Influence Function Analysis

In Section 6.1 we have seen that the proposed weighted likelihood estimator has the same influence function as that of the maximum likelihood estimator. So, the influence function approach will not predict the estimators to be any more robust than the MLE. But in reality, as we have already seen in real data examples and simulations, and as we will see later in higher dimensions and regression scenarios, the proposed estimator exhibits strong robustness properties in contrast to the maximum likelihood estimator. This indicates that the influence function, an extensively used tool for ‘measuring’ robustness, is not quite useful in this case. Noticing that the influence function represents a first order approximation, one could, to break the tie, opt for a higher order analysis of robustness in a situation where the data are contaminated at a single point. In this section, we discuss the second order influence function analysis for our proposed estimator. If the second order bias prediction turns out substantially smaller than the first order prediction, it not only demonstrates the robustness of the estimator, but also indicates the inadequacy of the first order influence function approach in this case.
As in Section 6.1, we take $G_\epsilon = (1-\epsilon)G + \epsilon \Lambda_y$ to be the distribution function $G$ contaminated at a point $y$ by an infinitesimally small proportion $\epsilon$, with $\Lambda_y$ being the distribution function of the random variable degenerate at $y$. $\Lambda_y^*$ and $G^*$ are also accordingly defined. Let $T(G_\epsilon) = T((1-\epsilon)G + \epsilon \Lambda_y)$ with $T$ being the functional of interest as before. The influence function of the functional $T(\cdot)$ is given by

$$T'(y) = \frac{\partial T(G_\epsilon)}{\partial \epsilon}|_{\epsilon=0}. $$

Viewed as a function of $\epsilon$, $\Delta T(\epsilon) = T(G_\epsilon) - T(G)$ quantifies the amount of bias and describes how the bias changes with contamination. The second order Taylor series expansion gives,

$$T(G_\epsilon) - T(G) \approx \epsilon T'(y) + \frac{\epsilon^2}{2} T''(y), $$

where $T''(y)$ is the second derivative of $T(G_\epsilon)$ evaluated at $\epsilon = 0$. We will find an expression for $T''(y)$ and then ascertain the expected behavior of the proposed estimator with changes in the level of contamination using these expressions.

As before, we denote $T(G_\epsilon) = \theta_\epsilon$ for a parametric family governed by the scalar parameter $\theta$. Let $\tau_\epsilon$ be as defined in Section 6.1. Our estimating equation is

$$\int_X w(\tau_\epsilon(x))u_{\theta_\epsilon}(x)dG_\epsilon(x) = 0.$$

Successive differentiation of the estimating equation with respect to $\epsilon$ gives,

$$\int_X w''(\tau_\epsilon)(\tau_\epsilon')^2u_{\theta_\epsilon}dG_\epsilon + \int_X w'(\tau_\epsilon)\tau_\epsilon''u_{\theta_\epsilon}dG_\epsilon + \theta_\epsilon'\int_X w'(\tau_\epsilon)\tau_\epsilon''u_{\theta_\epsilon}dG_\epsilon + \theta_\epsilon''\int_X w'(\tau_\epsilon)\tau_\epsilon'\nabla u_{\theta_\epsilon}dG_\epsilon + (\theta_\epsilon')^2\int_X w(\tau_\epsilon)\nabla^2 u_{\theta_\epsilon}dG_\epsilon + \theta_\epsilon'\int_X w(\tau_\epsilon)\nabla u_{\theta_\epsilon}d(\Lambda_y - G) + \theta_\epsilon''\int_X w(\tau_\epsilon)\nabla u_{\theta_\epsilon}d(\Lambda_y - G) = 0.$$

We plug in the values of $\tau'_\epsilon(x)$ and $\tau''_\epsilon(x)$ in the above equation.

$$\tau'_\epsilon(x) = \frac{1}{F_{\theta_\epsilon}(x)}[\Lambda_y(x) - G(x) - (\tau_\epsilon + 1)\nabla F_{\theta_\epsilon}(x)\theta'_\epsilon]$$

or,

$$\tau'_\epsilon(x) = \frac{1}{S_{\theta_\epsilon}(x)}[\Lambda_y(x)^* - G^*(x) - (\tau_\epsilon + 1)\nabla S_{\theta_\epsilon}(x)\theta'_\epsilon]$$

according to whether $x \in \mathcal{X}_1$ or $x \in \mathcal{X}_2$. And,

$$\tau''_\epsilon = 2(\tau_\epsilon + 1)\left(\frac{\nabla F_{\theta_\epsilon}}{F_{\theta_\epsilon}}\right)^2 (\theta'_\epsilon)^2 - 2\frac{(\Lambda_y - G)}{F_{\theta_\epsilon}} \frac{\nabla F_{\theta_\epsilon}}{F_{\theta_\epsilon}} \theta'_\epsilon$$
or,
\[ \tau''_e = 2(\tau_e + 1) \left( \frac{\nabla S_{\theta_e}}{S_{\theta_e}} \right)^2 (\theta'_e)^2 - 2 \left( \frac{\bar{\lambda}^*_y - G^*}{S_{\theta_e}} \right) \frac{\nabla S_{\theta_e}}{S_{\theta_e}} \theta'_e \]
\[ - (\tau_e + 1) \frac{\nabla S_{\theta_e}}{S_{\theta_e}} (\theta'_e)^2 - (\tau_e + 1) \frac{\nabla S_{\theta_e}}{S_{\theta_e}} \theta''_e \]

according to whether \( x \in X_1 \) or \( x \in X_2 \).
Plugging these values and after a simple although tedious manipulation, we get
\[ \theta''_e |_{\tau=0} = T''(y) = D_0^{-1} [(T'(y))^2 N_1 + 2T'(y)N_2 + N_3] \] (9)
where,
\[ D_0 = \int_{X_1} w(\tau) \nabla u_{\theta} dG - \int_{X_2} w(\tau)u_{\theta}(\Lambda_y - G) \frac{\nabla F_{\theta}}{F_{\theta}} dG \]
\[ + \int_{X_2} w(\tau)u_{\theta}(\Lambda^*_y - G^*) \frac{\nabla S_{\theta}}{S_{\theta}} dG, \]
\[ N_1 = 2 \int_{X_1} w'(\tau) \nabla u_{\theta}(\tau + 1) \frac{\nabla F_{\theta}}{F_{\theta}} dG + \int_{X_2} w'(\tau) \nabla u_{\theta}(\tau + 1) \frac{\nabla S_{\theta}}{S_{\theta}} dG \]
\[ - \int_{X_1} w''(\tau)u_{\theta}(\tau + 1)^2 \left( \frac{\nabla F_{\theta}}{F_{\theta}} \right)^2 F_{\theta} dG - \int_{X_2} w''(\tau)u_{\theta}(\tau + 1)^2 \left( \frac{\nabla S_{\theta}}{S_{\theta}} \right)^2 S_{\theta} dG \]
\[ - \int_{X_1} w'(\tau)u_{\theta}(\tau + 1) \left[ 2 \left( \frac{\nabla F_{\theta}}{F_{\theta}} \right)^2 - \frac{\nabla S_{\theta}}{S_{\theta}} \right] dG - \int_{X_2} w'(\tau) \nabla u_{\theta} dG, \]
\[ N_2 = \int_{X_1} w'(\tau)u_{\theta}(\tau + 1) \frac{\nabla F_{\theta}}{F_{\theta}} dG + \int_{X_2} w'(\tau)u_{\theta}(\tau + 1) \frac{\nabla S_{\theta}}{S_{\theta}} dG \]
\[ - \int_{X_1} w'(\tau) \nabla u_{\theta} \frac{\Lambda_y - G}{F_{\theta}} dG - \int_{X_2} w'(\tau) \nabla u_{\theta} \frac{\bar{\lambda}^*_y - G^*}{S_{\theta}} dG \]
\[ + \int_{X_1} w'(\tau)u_{\theta}(\Lambda_y - G) \frac{\nabla F_{\theta}}{F_{\theta}} dG + \int_{X_2} w'(\tau)u_{\theta}(\bar{\lambda}^*_y - G^*) \frac{\nabla S_{\theta}}{S_{\theta}} dG \]
\[ + \int_{X_1} w''(\tau)u_{\theta} \frac{\nabla F_{\theta}}{F_{\theta}} (\tau + 1)(\Lambda_y - G) dG + \int_{X_2} w''(\tau)u_{\theta} \frac{\nabla S_{\theta}}{S_{\theta}} (\tau + 1)(\bar{\lambda}^*_y - G^*) dG \]
\[ - \int w(\tau) \nabla u_{\theta} d\Lambda_y + \int w(\tau) \nabla u_{\theta} dG, \]
\[ N_3 = - \int_{X_1} w''(\tau) \left[ \frac{u_{\theta}}{S_{\theta}} \right]^2 (\Lambda_y - G)^2 dG - \int_{X_2} w''(\tau) \left[ \frac{u_{\theta}}{S_{\theta}} \right]^2 (\bar{\lambda}^*_y - G^*)^2 dG \]
We now apply the above and demonstrate how the second order influence function analysis leads to different result. We consider a $N(0,1)$ population contaminated at $y=10$ and plot the bias against the level of contamination ($\epsilon$).

Consider weight function 1. When $G = F_\theta g$, then $\tau = 0$, $w(0) = 1$, $w’(0) = 0$ and $w''(0) = 1 - \alpha$. Then, plugging in these values in the expression for $T''(y)$, we get

$$T''(y) = I^{-1}(\theta) \left[ \int_{X_1} (1 - \alpha) \frac{u_\theta}{F_\theta} (\Lambda_y - G)^2 dG + \int_{X_2} (1 - \alpha) \frac{u_\theta}{S_\theta} (\Lambda^*_y - G^*)^2 dG ight. \\
+ 2T'(y) \left( \int_{X_1} (\alpha - 1) u_\theta \frac{\nabla F_\theta}{F_\theta} (\Lambda_y - G) dG + \int_{X_2} (\alpha - 1) u_\theta \frac{\nabla S_\theta}{S_\theta} (\Lambda^*_y - G^*) dG ight) \\
+ \nabla u_\theta(y) + I(\theta)) + (T'(y))^2 \left( \int \nabla_2 u_\theta dG + \int_{X_1} (1 - \alpha) u_\theta \left( \frac{\nabla F_\theta}{F_\theta} \right)^2 F_\theta dG + \\
\int_{X_2} (1 - \alpha) u_\theta \left( \frac{\nabla S_\theta}{S_\theta} \right)^2 S_\theta dG \right].$$

For the MLE the first order linear approximation is exact, and the second order approximation does not alter it. But the robust weighted likelihood estimators, particularly those leading to sharper downweighting of large residuals...
lead to significantly smaller bias predictions using (8). This is demonstrated in Figure 9.

Now consider weight function 4. When $G = F_{\theta_y}$, then $\tau = 0$, $w(0) = 1$, $w'(0) = 0$ and $w''(0) = \frac{(2-d_1)(d_2+2)}{2(d_1+d_2)}$. Then, if we plug these values in the expression for $T''(y)$, we get,

$$T''(y) = I^{-1}(\theta) \left[ \int_{X_1} \frac{(2-d_1)(d_2+2)}{2(d_1+d_2)} \frac{u_\theta}{F_\theta} (\Lambda_y - G)^2 dG + \int_{X_2} \frac{(2-d_1)(d_2+2)}{2(d_1+d_2)} \frac{u_\theta}{S_\theta} (\bar{\Lambda}_y - G^*)^2 dG \right. \\
+ 2T'(y) \left( \int_{X_1} \frac{(d_1-2)(d_2+2)}{2(d_1+d_2)} u_\theta \nabla \frac{F_\theta}{F_\theta} (\Lambda_y - G) dG \\
+ \int_{X_2} \frac{(d_1-2)(d_2+2)}{2(d_1+d_2)} \frac{u_\theta}{S_\theta} \nabla S_\theta (\bar{\Lambda}_y - G^*) dG + \nabla u_\theta(y) + I(\theta) \right) \\
\left. + (T'(y))^2 \left( \int_{X_1} \nabla^2 u_\theta dG + \int_{X_1} \frac{(2-d_1)(d_2+2)}{2(d_1+d_2)} \frac{u_\theta}{F_\theta} \left( \frac{\nabla F_\theta}{F_\theta} \right)^2 F_\theta dG \\
+ \int_{X_2} \frac{(2-d_1)(d_2+2)}{2(d_1+d_2)} \frac{u_\theta}{S_\theta} \left( \frac{\nabla S_\theta}{S_\theta} \right)^2 S_\theta dG \right) \right].$$

Now, as in the earlier scenario, we present the predicted, second order bias for weight function 4 in Figure 10. Here also we can see that the bias for MLE increases linearly with respect to $\epsilon$; however for the proposed WLE, the increase in bias flattens out very quickly.

8. Extending the Method to Other Scenarios

We would now like to further demonstrate the use of the proposed weighted likelihood estimation method in other scenarios. A natural extension will be to the case of multivariate data. Another obvious one is the application of the method in the regression problem, where the observations are no longer identically distributed.

8.1. Extension to Bivariate Scenario

8.1.1. Method of Estimation

We illustrate the procedure for extending the method to multivariate scenarios by considering the bivariate case. Higher dimensions can be similarly approached. We begin by setting up a few definitions in the spirit of the univariate case.
Let \((X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)\) be paired i.i.d observations drawn from a bivariate population. Define

\[
F_n(x, y) = \frac{1}{n} \sum_{i=1}^{n} I(X_i \leq x)I(Y_i \leq y) \quad i, j = 1, 2, \ldots, n;
\]

\[
F_\theta(x, y) = P_{\theta}(X \leq x, Y \leq y);
\]

\[
S_n(x, y) = \frac{1}{n} \sum_{i=1}^{n} I(X_i \geq x)I(Y_i \geq y) \quad i, j = 1, 2, \ldots, n;
\]

and

\[
S_\theta(x, y) = P_{\theta}(X \geq x, Y \geq y).
\]

Here \(I(A)\) represents the indicator function for the event \(A\). Under these notations, we define

\[
\tau_n(X_i, Y_i) = \begin{cases} 
\frac{F_n(X_i, Y_i) - 1}{F_\theta(X_i, Y_i)} & \text{if } 0 < F_\theta(X_i, Y_i) \leq p \\
0 & \text{if } p < F_\theta(X_i, Y_i) < 1 - p \\
\frac{S_n(X_i, Y_i) - 1}{S_\theta(X_i, Y_i)} & \text{if } 1 - p \leq F_\theta(X_i, Y_i) < 1.
\end{cases}
\]

Then the weight \(w(\tau_n(X_i, Y_i))\) is constructed exactly as in the univariate case. With \(u_\theta(X_i, Y_i)\) representing the bivariate score function, our estimator of the
parameter $\theta$ is now obtained as the solution of the equation

$$\frac{1}{n} \sum_{i=1}^{n} w_{\theta}(\tau_n(X_i, Y_i))u_{\theta}(X_i, Y_i) = 0. \quad (10)$$

### 8.1.2. A Bivariate Example

We use the Hertzsprung-Russell dataset [16] which contains observations for the luminosity of stars versus their effective temperature in the logarithmic scale. There are four large outliers in the upper left hand corner of the scatter plot (Figure 11), as well as a few minor outliers. We treat these data as a sample from a bivariate normal population $BVN(\mu_1, \mu_2, \sigma^2_1, \sigma^2_2, \rho)$.

The parameter estimates obtained by the maximum likelihood and weighted likelihood estimation methods are displayed in Table 10. Clearly there is a major change in the scale of the log-temperature variable as well as in the covariance component. To visually demonstrate the effect of weighting the datapoints, we have presented the 95% concentration ellipses based on the parameter values obtained by the MLE and the WLEs presented in Figure 11. It is evident that the weighted likelihood scheme produces much more meaningful concentration ellipses covering the majority of the observed data and sacrificing the outlying points. The outlier resistant property of the weighted likelihood estimators cause the corresponding ellipses to shrink to sharper measures compared to the highly liberal practically useless ellipse produced by maximum likelihood estimation.

### 8.1.3. The Multiple Root Issue

Our next example illustrates an interesting phenomenon linked with the multiple root issue. We consider the data presented by Lubischew (1962) [12] on three species of beetles. We only use the observations from the first two species, i.e., *Chaetocnema concinna* and *Chaetocnema heptapotamica* and two measurements per species, namely the maximal width of the aedeagus in the fore-part and the front angle of the aedeagus. The data are presented in Figure 12, where the solid dots represent the 22 bivariate observations of *Chaetocnema heptapotamica*, while the 21 hollow dots represent the *Chaetocnema concinna* observations. The two populations are close but do have a clear separation from one another. As in the previous example, we model the entire data as coming from a single bivariate normal population.

| Method     | $\hat{\mu}_1$ | $\hat{\mu}_2$ | $\hat{\sigma}^2_1$ | $\hat{\sigma}^2_2$ | $\hat{\rho}$ |
|------------|----------------|----------------|---------------------|---------------------|--------------|
| MLE        | 4.3100         | 5.0121         | 0.0846              | 0.3263              | -0.2104      |
| WLE$_{\alpha=1.001}$ | 4.3998         | 4.9283         | 0.01482             | 0.2516              | 0.6813       |
| WLE$_{\alpha=1.01}$   | 4.4065         | 4.9435         | 0.0125              | 0.2425              | 0.6624       |
Fig 11. The 95% concentration ellipses for MLE and the proposed WLE at $\alpha = 1.001$ and $\alpha = 1.01$ for the Hertzsprung-Russell data.
Table 11
Different roots for the Lubischew beetle data

| Root-type        | Parameter | Initial Values               | Obtained Roots     |
|------------------|-----------|------------------------------|--------------------|
| MLE-type         | Mean      | (142.1395, 12.0468)         | (142.1268, 12.0457) |
|                  | Variance  | 39.6944                      | 38.6439            |
|                  | Covariance Matrix | 7.3981, 4.9502        | 7.2017, 4.8378     |
| Concinna root    | Mean      | (146.1905, 14.0952)         | (146.5985, 14.1968) |
|                  | Variance  | 31.6619                      | 29.6639, −1.1053   |
|                  | Covariance Matrix | −0.9690, 0.7905     | −1.1053, 0.7175    |
| Heptapotamica root | Mean     | (138.2727, 10.0909)         | (139.3396, 10.5952) |
|                  | Variance  | 17.1602                      | 7.3738, −2.4967    |
|                  | Covariance Matrix | −0.5022, 0.9437     | −2.4967, 2.5340    |

We use the first weight function for our analysis with the tuning parameter value set at $\alpha = 1.02$. We start with three different sets of initial values. The first one is the MLE of the entire combined data. With this starting value, the obtained root to the estimating equation is a solution which is quite close to the MLE. In this case, the final fitted weights are close to 1 for most of the observations in either group.

Next we start with the MLE of the 21 observations from Chaetocnema concinna species as the initial value and run the method on the entire dataset. In this case we obtain a root which gives final fitted weights that are close to 1 for most of the concinna observations, but are close to zero for most of the heptapotamica observations. Clearly this root represents the concinna component of the data set. The reverse appears to hold when the heptapotamica MLE is used as the initial value, although in this case the root downweights a few outlying heptapotamica observations as well.

Thus in this example there are at least three different roots for the weighted likelihood estimating equation. These results are summarized in Table 11. Figure 12 presents the 95% concentration ellipses for each of the three roots. As one would imagine, the 95% concentration ellipse for the MLE-like root is a huge, inclusive ellipse with a large variability which almost completely subsumes the other two smaller ellipses; the latter figures may be viewed as robust representations of the distributions of the individual species.

As it turns out, in this case the multiple roots help us to identify disjoint parts of the data which are generated by different models. They also indicate that in this case there is no single root which is representative of the entire data under the bivariate normal model. Clearly the presence of multiple roots need not be a nuisance all the time, and further review based on the roots obtained can uncover interesting features in the data. The beetle data were also analyzed by Markatou et al. (1998) [14] who had similar conclusions.

8.2. Use in Regression Scenario

In this section, we apply the weighted likelihood estimator in the regression problem, where the data points are independent but not identically distributed.
Fig 12. The 95% concentration ellipses for the three roots in the beetle data example.
8.2.1. Methodology

Let us consider a homoscedastic linear regression model
\[ y_i = \beta_0 + \beta_1 x_i + \epsilon_i \quad i = 1, 2, \ldots, n. \]

We also assume that the errors are independent and \( \epsilon_i \sim N(0, \sigma^2) \) for all \( i = 1, 2, \ldots, n. \)

Now, since \( \epsilon_i = y_i - \beta_0 - \beta_1 x_i \), then
\[ y_i - \beta_0 - \beta_1 x_i \sim N(0, \sigma^2), \]
\[ \Rightarrow \frac{y_i - \beta_0 - \beta_1 x_i}{\sigma} \sim N(0, 1). \]

Let \( Z_i = (Y_i - \beta_0 - \beta_1 X_i)/\sigma \quad i = 1, 2, \ldots, n. \) We define
\[ F_{n,\theta}(z) = \frac{1}{n} \sum_{i=1}^{n} I(Z_i \leq z), \quad S_{n,\theta}(z) = \frac{1}{n} \sum_{i=1}^{n} I(Z_i \geq z) \]
where \( \theta = (\beta_0, \beta_1, \sigma). \) If the true value of the parameters are specified, \( F_{n,\theta}(z) \) converges to \( \Phi(z) \), the standard normal cumulative distribution function at \( z \).

With this, we now define
\[ \tau_n(z) = \begin{cases} 
\frac{F_{n,\theta}(z)}{\Phi(z)} - 1 & \text{if } 0 < \Phi(z) \leq p, \\
0 & \text{if } p < \Phi(z) < 1 - p, \\
\frac{S_{n,\theta}(z)}{1 - \Phi(z)} - 1 & \text{if } 1 - p \leq \Phi(z) < 1.
\end{cases} \]

Now, \( w(\tau_n(z)) \) is defined as before in the univariate case. We obtain the estimates of \( \theta = (\beta_0, \beta_1, \sigma) \) by solving the weighted likelihood score equation
\[ \frac{1}{n} \sum_{i=1}^{n} w_{\theta}(Z_i) u_{\theta}(Z_i) = 0. \]

The method can obviously be extended to the multiple regression case in a routine manner.

8.2.2. Example

To illustrate the use of the weighted likelihood estimator in the regression scenario, we use the Animals data [16]. These data consist of 28 observations of the body weights and brain weights of different land animals. The model used for regression is
\[ \log y_i = \beta_0 + \beta_1 \log x_i + \epsilon_i \]
where \( y_i \) and \( x_i \) are brain and body weights of the \( i \)th animal and \( \epsilon_i \sim N(0, \sigma^2). \)

Table 12 contains the estimated values of the regression parameters obtained
Table 12: estimates obtained for the Animals data

| Method     | \( \hat{\beta}_0 \) | \( \hat{\beta}_1 \) | \( \hat{\sigma} \) |
|------------|----------------------|----------------------|---------------------|
| MLE        | 2.5549               | 0.4960               | 1.5320              |
| WLE\(_{d_1=2.5,d_2=1}\) | 1.7858               | 0.7785               | 0.1575              |

Figure 13: The scatter plot of the animals data and the OLS and weighted likelihood regression lines from the ordinary least squares method (which are the ML estimates under normality of errors) and the ones obtained from the weighted likelihood estimation method.

Figure 13 displays the different regression lines together with the scatter plot of the data. Clearly the weighted likelihood method keeps the effect of the outliers in check which the ordinary least squares (OLS) estimators fail to do. The fit obtained by the weighted likelihood method is definitely a better fit.

8.2.3. Choice of Initial Value

We have seen earlier in Section 4.2 that the choice of the initial value for the estimating equation plays an important role in determining the estimate in simple estimation scenarios. For regression analysis also, the same is true. In a mixed population, where the components are significantly different, the different choices of initial values lead to different values of the regression estimate. As an example we use the voltage drop data (Montgomery et al. 2012, p. 232) [15] which has 41 observations as depicted in the scatter plot of Figure 14.

Clearly the data cannot be appropriately modeled by a single regression line.
The OLS line passes through the center of the scatter plot without providing a meaningful description of the data. However, our weighted likelihood procedure, applied here using the fourth weight function with $d_1 = 2.5$ and $d_2 = 1$, clearly identifies three roots. While one root is essentially a MLE like root, the other two indicate that very different regressions would be appropriate for the first part and the second part of the data. The coefficients are presented in Table 13, and the fitted lines are displayed in Figure 14.

![Figure 14. Voltage Drop Data](image)

**Table 13**

| Method     | $\hat{\beta}_0$   | $\hat{\beta}_1$   | $\sigma$   |
|------------|--------------------|--------------------|------------|
| MLE        | 9.48547           | 0.18599           | 2.33       |
| WLE root 1 | 9.473935          | 0.186711          | 2.265886   |
| WLE root 2 | 5.4565171         | 0.9335482         | 0.3853895  |
| WLE root 3 | 23.08469          | -0.6587310        | 0.00007865 |

9. Conclusion

Our work is inherently linked to the work of Biswas et al. (2015) [9]. Here we note that our effort to estimate a parameter robustly by weighted likelihood methodology is not the first of its kind. Our work comes following the lead of Lindsay (1994) [11] and Markatou et al. (1998) [14]. While Lindsay’s proposal of distance based approach is very useful in discrete case, in continuous case, there are obvious problems which were later partially resolved in Markatou et al. (1998) [14]. However, this method involves appropriate nonparametric smoothing methods and so it still has to deal with bandwidth selection and other problems; models with bounded support could be an irritant. The weighted
likelihood estimation approach discussed in this paper provides a simple solution to such problems.

Although we have mainly focused on normally distributed models in the continuous cases, we have provided simulation studies in other models such as the exponential. We have demonstrated its extension to multivariate problems, as well as to the regression situation. Many other scenarios where the methods of Agostinelli and colleagues have found application would be accessible to our method. On the whole we expect that it will be an useful tool for the practitioners.

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