Research Article
The Q-Time Deformed Wave Equation

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For $q \in (0,1)$, we introduce the $q$-time deformed wave equation; using the $q$-derivative, the solution of the $q$-time deformed wave equation is given. Also, we introduce the free-time wave equation which is a limit case $q \to 0$ of the $q$-time deformed wave equation.

1. Introduction

The wave equation is a second-order partial differential equation; it is used in many branches of sciences. It is plays essential part in physics. The equation,

$$u_{tt} = c^2 u_{xx}, \quad (1)$$

is standard example of wave equation. There are two real characteristic slopes at each point $(x,t)$. On the whole, this equation is famous by the wave equation, in one dimension, and describes the propagation (bi-directional) of waves with finite speed $\pm c$. The one-dimensional wave equation (1) can be solved exactly by the Jean-Baptiste and Rond d’Alembert method, using a Fourier transform method, or via separation of variables. On the contrary, in the domain of combinatorics and quantum calculus, the $q$-derivative is a $q$-analogue of the ordinary derivative, presented by Jackson Frank [1, 2]. The $q$-differentiation is parallel to ordinary derivatives, with inquisitive differences. In approximation theory, the uses of $q$-calculus are new domain in most recent 25 years. A few different scientists have proposed the exponential-type operators. This study is arranged as follows. In Section 2, we introduce the notion of the $q$-calculus. InSection 3, we introduce the $q$-time deformed wave equation for $q \in (0,1)$ and the solution of such equation is given. In Section 4, we study the free-time wave equation.

2. Preliminaries

Rendering a few essential notations of the parlance of $q$-calculus (see [3–8]), the $q$-deformation of natural number $n$ for $q \in (0,1)$ is

$$[n]_q = 1 + q + q^2 + \cdots + q^{n-1}, \text{ with } [0]_q = 0. \quad (2)$$

Sometimes, we will write $[\infty]_q$ for the limit of these numbers: $1/(1-q)$. One can get easily that

$$[n]_q = \frac{1-q^n}{1-q}, \quad q \in (0,1), \quad \forall n \in \mathbb{N}. \quad (3)$$

The $q$-factorials and $q$-binomial coefficients are known as follows:

$$[n]_q! = [1]_q \cdot [2]_q \cdots [n]_q, \text{ with } [0]_q! = 1, \quad (4)$$

$\forall q \in (0,1)$, and analytic $f$: $\mathbb{C} \to \mathbb{C}$ define operators $Z$ and $D_q$ by (see [9–15])

$$(Zf)(z) = zf(z),$$

$$(D_qf)(z) = \begin{cases} f(z) - f(qz), & z \neq 0, \\ f'(0), & \end{cases} \quad (5)$$
where $D_q$ has the following properties:

1. $\lim_{q \to 1} (D_q f)(z) = f'(z)$
2. $D_q(z^n) = [n]_q z^{n-1}$
3. $D_q(f(z)g(z)) = (D_q f)(z)g(z) + f(qz)(D_q g)(z)$
4. $D_q(f(z)/g(z)) = (D_q f)(z)/g(z) - f(z)(D_q g)(z)/g(z)g(qz)$

It is well known [8] that the operators $D_q$ and $Z$ satisfy

$$D_q Z - qZD_q = 1.$$  
(6)

One of the $q$-analogues of $e^x$ is

$$e_q(x) = \sum_{i=0}^{\infty} \frac{x^i}{(i)_q}.$$  
(7)

We see that

$$e_q(x) = \frac{1}{(1-(1-q)x)_q^{\infty}},$$  
(8)

where

$$(1-y)_q^{\infty} = \prod_{i=0}^{\infty} (1-q^i y).$$  
(9)

The $q$-exponential functions has the following property:

$$D_q e_q(x) = e_q(x).$$  
(10)

### 3. $q$-Time Deformed Wave Equation

Let $q \in (0,1)$. As a $q$-analogue of equation (11), we introduce the following equation:

$$D^2_{q,t} u = c^2 q^2 \frac{\partial^2 u}{\partial x^2},$$  
(11)

where

$$D_{q,t} u(t,x) = (D_q u(\cdot,x))(t).$$  
(12)

For $t > 0$, we study equation (11) which will be called $q$-time deformed wave equation.

**Theorem 1.** Let $0 < q < 1$. Then, we have

1. $u(t,x)$ is given by

$$u(t,x) = \left( A e^{-\sqrt{k} t} + B e^{\sqrt{k} t} \right) e_q(t \sqrt{k} c^2),$$  
(13)

which is the solution of equation (11) with any constants $A$ and $B$, where $k > 0$ and $j^2 = -1$.

**Proof.** Let $0 < q < 1$. Then, we obtain

$$D^{q}_{q,t} u(t,x) = \frac{u(t,x) - u(qt,x)}{t(1-q)} = F(t,x),$$  
(15)

$$D^{q}_{q,t} u(t,x) = \frac{F(t,x) - F(qt,x)}{t(1-q)}.$$  
(16)

Now, let

$$u(t,x) = f(t)g(x).$$  
(17)

Then, equation (11) is equivalent to

$$g(x)D^2_{q,t} f(t) = c^2 f(t)g''(x).$$  
(18)

Hence,

$$\frac{D^2_{q,t} f(t)}{c^2 f(t)} = \frac{g''(x)}{g(x)} = k,$$  
(19)

where $k$ is constant. We deduce that

$$g''(x) = k(g(x)),$$  
(20)

$$D^2_{q,t} f(t) = kc^2 f(t).$$  
(21)

We apply equation (16) in (21); we obtain

$$q f(t) - (1+q) f(qt) + f(q^2 t) = \frac{1}{q(t)(1-q)^2} = kc^2 f(t),$$  
(22)

$$q f(t) - (1+q) f(qt) + f(q^2 t) = q^2 (1-q)^2 kc^2 f(t).$$  
(23)

We obtain that

$$q(1-t^2 (1-q^2) kc^2) f(t) = (1+q) f(q t) - f(q^2 t).$$  
(24)

By recurrence relation, we obtain

$$q(1-t^2 (1-q^2) kc^2) f(q t) = (1-q) f(q^2 t) - f(q^3 t)$$

$$= (1-q) f(q^3 t) - f(q^4 t)$$

$$\vdots$$

$$q(1-t^2 (1-q^2) kc^2) f(q^{n+1} t) = (1+q) f(q^{n+1} t) - f(q^{n+2} t)$$  
(25)

$$q(1-t^2 (1-q^2) kc^2) f(q^{n+1} t) = (1+q) f(q^{n+1} t) - f(q^{n+2} t).$$  
(26)

Now, by simplification, we obtain
\[ q(1 - a_i) f(t) + q f(qt) - q^2 a_i f(qt) \]
\[ + q f(q^n t) - q^{2n+1} a_i f(q^n t) \]
\[ + q f(q^n t) - q^{2n+1} a_i f(q^{n+1} t) \]
\[ = f(qt) + q f(qt) - f(q^2 t) \]
\[ + f(q^2 t) + q f(q^2 t) - f(q^3 t) \]
\[ + f(q^3 t) + q f(q^3 t) - f(q^4 t) \]
\[ + f(q^4 t) + q f(q^4 t) - f(q^5 t) \]
\[ + f(q^5 t) + q f(q^5 t) - f(q^6 t) \]
\[ + f(q^6 t) + q f(q^6 t) - f(q^7 t) \]
\[ \vdots \]
\[ + f(q^{n-1} t) + q f(q^{n-1} t) - f(q^n t) \]
\[ + f(q^n t) + q f(q^n t) - f(q^{n+1} t) \]
\[ + f(q^{n+1} t) + q f(q^{n+1} t) - f(q^{n+2} t) \]
\[ q(1 - a_i) f(t) - a_i (q^3 f(qt)) + \cdots + q^{2n+3} f(q^{n+1} t) \]
\[ = f(qt) + q f(q^n t) - f(q^{n+1} t), \] \hspace{1cm} (25)

where
\[ a_i = t^2 (1 - q^2) kc^2. \] \hspace{1cm} (26)

Now, let \( f \) be given by
\[ f(t) = \sum_{i=0}^{\infty} a_i t^i. \] \hspace{1cm} (27)

Then, we have
\[ a_i (q^3 \sum a_i q^i t^i + \cdots + q^{2n+3} \sum a_i q^{(n+1)i} t^i) \]
\[ = q^3 (1 - q^2) kc^2 \left( \sum_{i=0}^{\infty} a_i q^i \left( 1 + q^i + \cdots + (q^{3i})^{n} \right) t^{i+2} \right) \]
\[ = q^3 (1 - q^2) kc^2 a_i q^i \left( 1 - \frac{q^{3i}}{1 - q^{3i}} \right) t^{i+2} = \Gamma \]
\[ \text{as } n \to \infty; \text{ the last terms becomes} \]
\[ \Gamma \to q^3 (1 - q^2) kc^2 \sum_{i=0}^{\infty} a_i t^{i+2} - \frac{q^i}{1 - q}. \] \hspace{1cm} (29)

Hence, equations (25) and (29) give
\[ q(1 - t^2 (1 - q^2) kc^2) \sum_{i=0}^{\infty} a_i t^i - \Gamma = (\sum_{i=0}^{\infty} a_i q^i t^i) + (q - 1) a_0. \] \hspace{1cm} (30)

So, we obtain
\[ \left( \sum_{i=0}^{\infty} q a_i t^i \right) - q (1 - q^2) kc^2 \sum_{i=0}^{\infty} a_i t^{i+2} - \Gamma \]
\[ = \left( \sum_{i=0}^{\infty} a_i q^i t^i \right) + (q - 1) a_0, \] \hspace{1cm} (31)

and this implies
\[ qa_i - q(1 - q^2) kc^2 a_i - q(1 - q^2) kc^2 \frac{q^i}{1 - q} a_i = a_i q^i. \] \hspace{1cm} (32)

which gives
\[ qa_i - q(1 - q^2) kc^2 a_i - q(1 - q^2) kc^2 \frac{q^i}{1 - q} a_i = a_i q^i. \] \hspace{1cm} (33)

Then,
\[ q(1 - q^{-1}) a_i = q(1 - q^2) kc^2 \left( 1 + \frac{q^i}{1 - q} \right) a_i - a_i. \] \hspace{1cm} (34)

Therefore,
\[ q [i - 1] q a_i = q(1 - q) kc^2 \left( -1 + \frac{q^i}{1 - q} \right) a_i - a_i. \] \hspace{1cm} (35)

Then, we obtain
\[ [i - 1] q a_i = kc^2 \frac{1}{[i - 1] q} a_i - a_i. \] \hspace{1cm} (36)

This gives
\[ [i - 1] q a_i = kc^2 \frac{1}{[i - 1] q} a_i - a_i. \] \hspace{1cm} (37)

Then, we obtain
\[ a_i = \frac{kc^2}{[i - 1] q} a_i - a_i, \quad i \geq 2. \] \hspace{1cm} (38)

Note that, equation (38) can be found as follows:
\[ D_{a_i} f(t) = \sum_{i=1}^{\infty} a_i [i]_q t^{i-1}, \]
\[ D_{a_i}^2 f(t) = a_i [i]_q [i - 1]_q t^{i-2}. \] \hspace{1cm} (39)

Therefore, equation (21) becomes
\[ \sum_{i=1}^{\infty} a_i [i]_q [i - 1]_q t^{i-2} = kc^2 \sum_{i=0}^{\infty} a_i t^i, \]
\[ \sum_{i=0}^{\infty} a_{i+2} [i + 2]_q [i + 1]_q t^i = kc^2 \sum_{i=0}^{\infty} a_i t^i. \] \hspace{1cm} (40)

So, we take
\[ a_{i+2} [i + 2]_q [i + 1]_q = kc^2 a_i. \] \hspace{1cm} (41)

Finally, we obtain
\[ a_{i+2} = \frac{kc^2}{[i + 2]_q [i + 1]_q} a_i, \quad i \geq 0. \] \hspace{1cm} (42)

Now, for \( k > 0 \), let
\[ a_i = \left( \frac{\sqrt{kc^2}}{i} \right)^i. \]  

(43)

Then, we obtain

\[ a_{i+2} = \frac{(kc^2)^{i+2}}{[i + 2]_q} = \frac{(kc^2)^i}{[i]_q} \cdot \frac{(kc^2)^2}{[i + 2]_q[i + 1]_q}, \]

(44)

\[ = \frac{kc^2}{[i + 2]_q[i + 1]_q}, \]

which verifies equation (42). In this case, we obtain

\[ f(t) = \sum_{i=0}^{\infty} a_i f^i = \sum_{i=0}^{\infty} \left( \frac{\sqrt{kc^2}}{i} \right)^i = e_q \left( t\sqrt{kc^2} \right). \]

(45)

Now, for \( k < 0 \), let

\[ a_i = \left( \frac{j\sqrt{kc^2}}{i} \right)^i, \quad j = \sqrt{-1}. \]

(46)

Then, we obtain

\[ a_{i+2} = (j)^{i+2} \left( \frac{\sqrt{kc^2}}{i} \right)^{i+2} \]

\[ = \frac{(-kc^2)}{[i + 2]_q[i + 1]_q} = \frac{kc^2}{[i + 2]_q[i + 1]_q} a_i, \]

which verifies equation (42). In this case, we obtain

\[ f(t) = \sum_{i=0}^{\infty} a_i f^i = e_q \left( tj\sqrt{kc^2} \right). \]

(47)

(48)

Finally, we get, for \( k > 0 \),

\[ u(t, x) = \left( Ae^{-\sqrt{k}x} + Be^{\sqrt{k}x} \right) e_q \left( t\sqrt{kc^2} \right). \]

(49)

For \( k < 0 \), we obtain

\[ u(t, x) = (A \cos(\sqrt{-k}x) + B \sin(\sqrt{-k}x)) e_q \left( tj\sqrt{kc^2} \right), \]

(50)

with any constants \( A \) and \( B \). This completes the proof. \( \square \)

4. Free-Time Wave Equation

Now, we will study the free-time wave equation:

\[ D_{0,t}^2 u = c^2 \frac{\partial^2 u}{\partial x^2}, \]

(51)

where

\[ D_{0,t} u(x, y) = \frac{u(x, y) - u(0, y)}{x}, \quad x \neq 0. \]

(52)

Note that, since the limit (when \( q \to 0 \)) of \( D_{0,t} \) is \( D_{0,t} \), the free-time wave equation is the limit case (when \( q \to 0 \)) of the \( q \)-time deformed wave equation (11).

**Theorem 2.** (1) We have

\[ u(t, x) = \frac{a_0}{1 - kc^2 t^2} \left( A \cos \sqrt{-k}x + B \sin \sqrt{-k}x \right), \]

(53)

is the solution of equation (51), with any constants \( a_0, A \) and \( B \), where \( k < 0 \).

(2) We have

\[ u(t, x) = \begin{cases} 
\frac{a_0}{1 - kc^2 t^2} \left( A e^{-\sqrt{k}x} + B e^{\sqrt{k}x} \right), & \text{if } t \neq \frac{1}{\sqrt{kc^2}}, \\
f(t) \left( A e^{-\sqrt{k}x} + B e^{\sqrt{k}x} \right), & \text{if } t = \frac{1}{\sqrt{kc^2}},
\end{cases} \]

(54)

which is the solution of equation (51), with any constants \( a_0, A, A, \) and \( B \), where \( f(0) = 0 \) and \( k > 0 \).

**Proof.** Let \( u(t, x) \) be given by

\[ u(t, x) = f(t) g(x). \]

(55)

Then, equation (51) is equivalent to

\[ g(x) D_{0,t}^2 f(t) = c^2 f(t) g''(x). \]

(56)

Hence, we obtain

\[ \frac{D_{0,t}^2 f(t)}{c^2 f(t)} = \frac{g''(x)}{g(x)} = k, \quad k \text{ is constant.} \]

(57)

From equation (57), we obtain

\[ g''(x) = k g(x), \]

(58)

\[ D_{0,t}^2 f(t) = k c^2 f(t). \]

(59)

Equation (58) has a solution. If \( k > 0 \), we obtain

\[ g(x) = Ae^{-\sqrt{k}x} + Be^{\sqrt{k}x}, \]

(60)

and if \( k < 0 \), we obtain

\[ g(x) = A \cos(\sqrt{-k}x) + B \sin(\sqrt{-k}x), \]

(61)

with any \( A \) and \( B \). On the contrary, we have

\[ D_{0,t} f(t) = \frac{f(t) - f(0)}{t} = F_0(t). \]

(62)

Then, we obtain

\[ D_{0,t}^3 f(t) = \frac{F_0(t) - F_0(0)}{t} = \frac{(f(t) - f(0))/t) - 0}{t^2} \]

(63)
Then, equation (59) becomes
\[
\frac{f'(t) - f(0)}{t^2} = kc^2 f(t).
\] (64)

This gives
\[
f(t) - f(0) = kc^2 t^2 f(t),
\] (65)

which implies that
\[
(1 - kc^2 t^2) f(t) = f(0).
\] (66)

This gives
\[
f(t) = \frac{f(0)}{1 - kc^2 t^2}, \quad \text{for } k < 0.
\] (67)

For \( k > 0 \), we obtain
\[
f(t) = \frac{f(0)}{1 - kc^2 t^2}, \quad \text{if } t \neq \frac{1}{\sqrt{kc^2}},
\] (68)

and for \( t = 1/\sqrt{kc^2} \), we should take \( f(0) = 0 \). Hence, we obtain
\[
u(t, x) = \frac{a_0}{1 - kc^2 t^2} \left( A \cos(\sqrt{k} x) + B \sin(\sqrt{k} x) \right),
\] (69)

with any constants \( a_0, A, \) and \( B \), where \( k < 0 \). On the contrary, we obtain
\[
u(t, x) = \begin{cases} 
\frac{a_0}{1 - kc^2 t^2} \left( A e^{-\sqrt{k} x} + B e^{\sqrt{k} x} \right), & \text{if } t \neq \frac{1}{\sqrt{kc^2}}, \\
f(t) \left( A e^{-\sqrt{k} x} + B e^{\sqrt{k} x} \right), & \text{if } t = \frac{1}{\sqrt{kc^2}}.
\end{cases}
\] (70)

with any constants \( a_0, A, \) and \( B \), where \( f(0) = 0 \) and \( k > 0 \). This completes the proof. \( \Box \)

5. Conclusion

In this study, the \( q \)-time deformed wave equation as well as the free-time wave equation are studied. We expect to study the quantum white noise [16–20] case which is now attractive in mathematical physics area.

Data Availability

No data were used to support this study.

Conflicts of Interest

All authors declare that they have no conflicts of interest.

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