A two-step Lagrange–Galerkin scheme for the shallow water equations with a transmission boundary condition and its application to the Bay of Bengal region. Part I: Flat bottom topography

Md Mamunur Rasid\textsuperscript{1,2}, Masato Kimura\textsuperscript{3}, Md Masum Murshed\textsuperscript{4}, Erny Rahayu Wijayanti\textsuperscript{5}, and Hirofumi Notsu\textsuperscript{3,∗}

\textsuperscript{1}Division of Mathematical and Physical Sciences, Kanazawa University, Kakuma, Kanazawa 920-1192, Japan
\textsuperscript{2}University of Rajshahi, Rajshahi-6205, Bangladesh
\textsuperscript{3}Faculty of Mathematics and Physics, Kanazawa University, Kakuma, Kanazawa 920-1192, Japan
\textsuperscript{4}Department of Mathematics, University of Rajshahi, Rajshahi-6205, Bangladesh
\textsuperscript{5}Department of Mathematics, Gadjah Mada University, Yogyakarta 55281, Indonesia

\texttt{mamun.math@stu.kanazawa-u.ac.jp, mkimura@se.kanazawa-u.ac.jp, mmmurshed82@gmail.com, wijayanti.erny@gmail.com, notsu@se.kanazawa-u.ac.jp}

Abstract

This study presents a two-step Lagrange–Galerkin scheme for the shallow water equations with a transmission boundary condition (TBC). Firstly, the experimental order of convergence of the scheme is shown to see the second-order accuracy in time. Secondly, the effect of the TBC on a simple domain is discussed; the artificial reflections are kept from the Dirichlet boundaries and removed significantly from the transmission boundaries. Thirdly, the scheme is applied to a complex practical domain, i.e., the Bay of Bengal region, which is non-convex and includes islands. The effect of the TBC is discussed again for the complex domain; the artificial reflections are removed significantly from the transmission boundaries, which are set on open sea boundaries. Based on the numerical results, it is revealed that the scheme has the following properties: (i) the same advantages of Lagrange–Galerkin methods (the CFL-free robustness for convection-dominated problems and the symmetry of the matrices for the system of linear equations); (ii) second-order accuracy in time; (iii) mass preservation of the function for the water level from the reference height (until the contact with the transmission boundaries of the wave); and (iv) no significant artificial reflection from the transmission boundaries. The numerical results by the scheme are presented in this paper for the flat bottom topography of the domain. In the next part of this work, Part II, the scheme will be applied to rapidly varying bottom surfaces and a real bottom topography of the Bay of Bengal region.

Keywords: Shallow water equations, two-step Lagrange–Galerkin scheme, second order in time, transmission boundary condition, Bay of Bengal.

∗Corresponding author
1 Introduction

The system of the shallow water equations (SWEs) is one of the most common models for describing fluid flow in rivers, channels, estuaries, and coastal areas and is often used for simulating tsunamis and storm surges in oceanic phenomena. Natural disasters like tsunamis, cyclones, and storm surges cause a tremendous loss of lives and properties in the coastal areas in several regions. According to [14], statistics show that about 5% of the global tropical cyclones form over the Bay of Bengal, and, on average, five to six storms form in this region every year, but with 80% of the global casualties. The significant factors behind the heavy casualties are the shallow coastal water, thickly populated low-lying islands, highly curved coastal and island boundaries, river discharge, high astronomical tidal range, and favorable cyclone track, cf. [13] and Figure 1. That is why an effective storm surge prediction model and method are highly desired for the coastal region of Bangladesh to minimize the resulting damage from storm surges.

![Figure 1: The Bay of Bengal region](image)

Studies focusing on the Bay of Bengal region are found in [13,14,20,33–35,39] and references therein. For open sea boundaries, almost all the researchers implemented SWEs with a radiation-type boundary condition, which is comparable to a transmission boundary condition (TBC) employed in [22, 27]. Although for real problems, the finite element method is more suitable than the finite difference method because of the advantages of handling complex physical domains, geometries, or boundary conditions, as far as we know, there is no study to solve SWEs employing a TBC for the Bay of Bengal region using the finite element method except [26].

The system of the SWEs consists of two equations, a pure convection equation for the total wave height and a modified Navier–Stokes momentum equation for the velocity derived by taking the average of function values in $x_3$-direction, cf. [21, 27], which include the material derivatives in conservative and non-conservative forms, respectively. For a time step size $\Delta t > 0$, let $t^n := n\Delta t$. The so-called Lagrange–Galerkin method is the finite element method combined with the idea of the method of characteristics;
the non-conservative and conservative material derivatives are discretized as, for a scalar-valued function $\phi$ and a velocity $u$, cf., e.g., [15,16,36,41],

$$\frac{\partial \phi}{\partial t} + u \cdot \nabla \phi \left( x, t^n \right) = \frac{\phi^n(x) - \phi^{n-1}(x - u^n(x) \Delta t)}{\Delta t} + O(\Delta t),$$

$$\frac{\partial \phi}{\partial t} + \nabla \cdot (u \phi) \left( x, t^n \right) = \frac{\phi^n(x) - \phi^{n-1}(x - u^n(x) \Delta t) \gamma^n(x)}{\Delta t} + O(\Delta t),$$

respectively, which are first-order approximations in time, where $x - u^n(x) \Delta t$ is an upwind point of $x$ with respect to $u^n(x)$ and $\gamma^n$ is the Jacobian determinant of the mapping $x - u^n(x) \Delta t$. In general, the Lagrange–Galerkin method has two advantages; (i) the CFL-free robustness for convection-dominated problems and (ii) the symmetry of the resulting coefficient matrices for the system of linear equations. In addition to the four pioneering works above, many authors have proposed the ideas of this type of approximations in the context of the finite element method, cf. [1–12, 17, 23–25, 28–32, 37, 38, 40, 42–45] and references therein. When we focus on the SWEs, to the best of our knowledge, Murshed et al. [27] and Murshed [26] firstly solved the SWEs with a TBC by a (single-step) Lagrange–Galerkin scheme of first-order in time for a flat bottom topography. Recently, a two-step mass-preserving Lagrange–Galerkin scheme of second order in time for conservative convection-diffusion problems has been proposed and analyzed with error estimates in [18].

In this paper, we present a new two-step Lagrange–Galerkin scheme to solve the SWEs together with a TBC, which is of second order in time and maintains the two advantages of the Lagrange–Galerkin methods, i.e., the CFL-free robustness and the symmetry of the resulting matrices. The two material derivatives are discretized based on the ideas of two-step methods proposed for the non-conservative form in [9,16,17,32] and the conservative form in [18]. Firstly, preparing an artificial exact solution, we observe our scheme’s experimental order of convergence (EOC) to see the second-order accuracy in time on a simple (square) domain. Since long (real-)time computations on a mesh refined locally are needed in practical problems, the CFL-free second-order accuracy in time of our scheme is a significant advantage, enabling us to employ a more extensive time increment compared with first-order numerical methods. Secondly, we observe the effect of the TBC on a simple (square) domain, and the artificial reflections are kept from the Dirichlet boundaries and removed significantly from the transmission boundaries. Thirdly, our scheme is applied to the Bay of Bengal region, which is non-convex, includes islands, and is, therefore, a complex domain. We again observe the effect of the TBC for this realistic domain. The artificial reflections are removed significantly from the transmission boundaries, which are set on open sea boundaries. We also study the effect of a position of an open sea boundary with the TBC and reveal that it is sufficiently small to neglect. In [27], energy estimates for the SWEs were given, where the $L^2$-norm of the water level from the reference height was an important value related to the potential energy. Focusing on the energy and the mass of the water level function, we observe the $L^2$-norm and the mass of the water level function, which show the effectiveness of the TBC.

From the computations, we show that our new scheme has the following properties; (i) the same advantages of Lagrange–Galerkin methods; (ii) second-order accuracy in time; (iii) mass preservation of the function of the water level from the reference height (until the contact with the transmission boundaries of the wave); and (iv) no significant artificial reflection from the transmission boundaries. All of the numerical results in this paper, Part I, are for the flat bottom topography, and the non-homogeneous bottom topography will be studied in our forthcoming paper, Part II.
The outline of this paper is as follows. Section 2 presents a two-step Lagrange–Galerkin scheme for the SWEs together with a TBC, which is of second order in time. In Section 3, numerical results for simple square domains are shown to observe the second-order accuracy in time and the effect of TBC. In Section 4, our scheme is applied to the Bay of Bengal region, where the domain is non-convex and complex. In Section 5, conclusions are given. The data for choosing the constant $c_0$ required in the TBC is given in the Appendix.

2 A two-step Lagrange–Galerkin scheme

We introduce some notations to be used in this paper. $\Omega$ is a bounded spatial domain in $\mathbb{R}^2$, $\Gamma := \partial \Omega$ is the boundary of $\Omega$, and $(0, T)$ is a temporal domain in $\mathbb{R}_+$, i.e., $(x \in \mathbb{R}; x > 0)$ for a positive constant $T$. We use the Lebesgue space $L^2(\Omega)$ and the Sobolev space $H^1(\Omega)$. Let $(\cdot, \cdot)$ be the inner product in $L^2(\Omega)$, i.e., $(f, g) := \int_\Omega f(x)g(x)dx$ for $f, g \in L^2(\Omega)$. We employ the same notation $(\cdot, \cdot)$ to represent the $L^2(\Omega)$ inner product for scalar-, vector-, and matrix-valued functions. Let $A : B$ be the tensor product defined by $A : B := \sum_{i,j=1}^{2} A_{ij}B_{ij} = \text{tr}(AB^T)$ for $A, B \in \mathbb{R}^{2 \times 2}$.

2.1 Statement of the problem

Our problem is to find $(\phi, u) : \Omega \times (0, T) \to \mathbb{R} \times \mathbb{R}^2$ such that

\begin{align}
\frac{\partial \phi}{\partial t} + \nabla \cdot (u\phi) &= f & \text{ in } \Omega \times (0, T), \quad (2.1a) \\
\rho\phi \left[ \frac{\partial u}{\partial t} + (u \cdot \nabla)u \right] - 2\mu \nabla \cdot (\phi D(u)) + \rho g \phi \nabla \eta &= F & \text{ in } \Omega \times (0, T), \quad (2.1b) \\
\phi &= \eta + \zeta & \text{ in } \Omega \times (0, T), \quad (2.1c) \\
u &= 0 & \text{ on } \Gamma_D \times (0, T), \quad (2.1d) \\
\left( u \right) &= \left( c_0 \sqrt{g} \eta \right) & \text{ on } \Gamma_T \times (0, T), \quad (2.1e) \\
(\phi, u) &= (\phi^0, u^0) & \text{ in } \Omega, \text{ at } t = 0, \quad (2.1f)
\end{align}

where the total wave height and the velocity are denoted by $\phi$ and $u = (u_1, u_2)^T$, respectively, the water level from the reference height and the depth of water level from the reference height, i.e., bottom topography, are represented by $\eta : \Omega \times (0, T) \to \mathbb{R}$ and $\zeta : \Omega \to \mathbb{R}_+$, respectively, a pair of external forces is given by $(f, F) : \Omega \times (0, T) \to \mathbb{R} \times \mathbb{R}^2$, a pair of initial values is given by $(\phi^0, u^0) : \Omega \to \mathbb{R} \times \mathbb{R}^2$, density and viscosity constants of water are denoted by $\rho > 0$ and $\mu > 0$, the gravity constant is given by $g > 0$, the strain-rate tensor $D(u)$ is defined by

$$D(u) := \frac{1}{2} \left[ \nabla u + (\nabla u)^T \right],$$

and the outward unit normal vector is denoted by $n : \Gamma \to \mathbb{R}^2$, cf. Figure 2. We suppose that the boundary $\Gamma$ is divided into two non-overlapping parts, $\Gamma_D$ and $\Gamma_T$, i.e., $\Gamma = \overline{\Gamma_D} \cup \overline{\Gamma_T}$ and $\Gamma_D \cap \Gamma_T = \emptyset$, where the subscripts “D” and “T” imply Dirichlet and transmission boundaries, respectively. A positive constant $c_0$ is chosen suitably to remove the artificial reflection, and, throughout this paper, we employ $c_0 = 0.9$, which is determined based on numerical experiments given in Appendix. We consider homogeneous flat bottom topography in this paper, Part I, and non-homogeneous bottom topography in our forthcoming paper, Part II.
Now, we present our scheme for solving problem (2.1). Let problem (2.1) is to find \( \phi \) for a function \( \phi \) dependent function, \( G(\phi) = G(\phi; \eta) : \Gamma \rightarrow \mathbb{R}^2 \), defined by

\[
G(\phi) = G(\phi; \eta) := c_0 \sqrt{\kappa \zeta - \frac{\eta}{\phi}}.
\]

Assume \( \phi^0 \in \Psi, \eta^0 := \phi^0 - \zeta \in \Psi \) and \( u^0 \in V(G(\phi^0; \eta^0)) \). A weak formulation to problem (2.1) is to find \( \{(\phi, u(t)) \in \Psi \times V(G(\phi(t); \eta(t))) ; t \in (0, T)\} \) such that, for \( t \in (0, T) \),

\[
\begin{align*}
\left( \frac{\partial \phi}{\partial t} + \nabla \cdot (u \phi), \psi \right) &= (f, \psi) \quad \forall \psi \in \Psi, \quad (2.2a) \\
\rho \left( \phi \left( \frac{\partial u}{\partial t} + (u \cdot \nabla)u \right), v \right) + a(u, v; \phi) + b(\eta, v; \phi) &= (F, v) \quad \forall v \in V, \quad (2.2b) \\
\phi &= \eta + \zeta, \quad (2.2c)
\end{align*}
\]

with the initial condition \( (\phi, u(0)) = (\phi^0, u^0) \in \Psi \times V(G(\phi^0; \eta^0)) \), where the bilinear forms \( a(\cdot, \cdot; \phi) : \Psi \times Y \rightarrow \mathbb{R} \) and \( b(\cdot, \cdot; \phi) : \Psi \times Y \rightarrow \mathbb{R} \) are defined by

\[
a(u, v; \phi) := 2\mu(D(u), D(v)), \quad b(\eta, v; \phi) := \rho g(\phi \nabla \eta, v).
\]

Now, we present our scheme for solving problem (2.1). Let \( \mathcal{T}_h = \{ K \} \) be a partition of \( \Omega \) by triangular elements, \( h \) be the maximum diameter of \( K \in \mathcal{T}_h \), and \( \Omega_h := \text{int}(\bigcup_{K \in \mathcal{T}_h} K) \) be an approximated domain. Although it holds that \( \Omega \neq \Omega_h \) in general, we assume \( \Omega = \Omega_h \) throughout the paper to avoid the complexity of introducing many symbols. We define finite element spaces, \( \Psi_h, Y_h \) and \( V_h(G) \), corresponding to \( \Psi, Y \) and \( V(G) \) by

\[
\begin{align*}
\Psi_h &:= \{ \psi_h \in C^0(\Omega_h) ; \psi_h|_K \in P_1(K) \ \forall K \in \mathcal{T}_h \}, \\
Y_h &:= \{ v_h \in C^0(\Omega_h)^2 ; v_h|_K \in P_1(K)^2 \ \forall K \in \mathcal{T}_h \}, \\
V_h(G) &:= \{ v_h \in Y_h ; v_h = 0 \text{ on } \Gamma_D \text{ and } v_h = G \text{ on } \Gamma_T \},
\end{align*}
\]
and set \( V_h \coloneqq V_0(0) \), where the function \( G : \mathcal{T} \to \mathbb{R}^2 \) is assumed to be a piecewise linear function.

Let \( \Delta t \) be a time increment, \( N_{\tau} \coloneqq \lfloor T / \Delta t \rfloor \) a total number of time steps, and \( t_n \coloneqq n \Delta t \) a time at \( n \)-th time step. For \( v : \Omega \to \mathbb{R}^2 \), we define mappings \( X_1[v], \hat{X}_1[v] : \Omega \to \mathbb{R}^2 \) and \( \gamma_1[v], \hat{\gamma}_1[v] : \Omega \to \mathbb{R} \) by

\[
X_1[v](x) := x - \Delta t \, v(x), \quad \hat{X}_1[v](x) := x - 2 \Delta t \, v(x),
\]

\[
\gamma_1[v](x) := \det \left( \frac{\partial X_1[v]}{\partial x} (x) \right), \quad \hat{\gamma}_1[v](x) := \det \left( \frac{\partial \hat{X}_1[v]}{\partial x} (x) \right).
\]

For \( \{ \phi^n \}_{n=0}^{N_{\tau}} \) and \( \{ u^n \}_{n=0}^{N_{\tau}} \), we define an operator \( \mathcal{A}_\Delta[u] \phi^n \) by, for \( n = 1, \ldots, N_{\tau} \),

\[
\mathcal{A}_\Delta[u] \phi^n := \begin{cases} 
\mathcal{A}_\Delta^{(1)}[u] \phi^n & (n = 1), \\
\mathcal{A}_\Delta^{(2)}[u] \phi^n & (n \geq 2),
\end{cases}
\]

where

\[
\mathcal{A}_\Delta^{(1)}[u] \phi^n := \frac{\phi^n - \phi^{n-1} \circ X_1[u^{n-1}] \gamma_1[u^{n-1}]}{\Delta t},
\]

\[
\mathcal{A}_\Delta^{(2)}[u] \phi^n := \frac{3 \phi^n - 4 \phi^{n-1} \circ X_1[u^n] \gamma_1[u^n] + \phi^{n-2} \circ \hat{X}_1[u^{n-2}] \hat{\gamma}_1[u^{n-2}]}{2 \Delta t}.
\]

The composition of functions is represented by the symbol \( \circ \), i.e.,

\[
(\psi \circ X_1[v])(x) = \psi(X_1[v](x)),
\]

and the function \( u^{n+} : \Omega \to \mathbb{R}^2 \) is defined by

\[
u^{n+} := 2u^{n-1} - u^{n-2},
\]

which is a second-order temporal approximation of \( u^n \) if \( u \) is sufficiently smooth. We also define, for \( \{ w^n \}_{n=0}^{N_{\tau}} \),

\[
\mathcal{B}_\Delta[w] u^n := \begin{cases} 
\mathcal{B}_\Delta^{(1)}[w] u^n & (n = 1), \\
\mathcal{B}_\Delta^{(2)}[w] u^n & (n \geq 2),
\end{cases}
\]

where

\[
\mathcal{B}_\Delta^{(1)}[w] u^n := \frac{u^n - u^{n-1} \circ X_1[w^{n-1}]}{\Delta t},
\]

\[
\mathcal{B}_\Delta^{(2)}[w] u^n := \frac{3u^n - 4u^{n-1} \circ X_1[w^n] + u^{n-2} \circ \hat{X}_1[w^{n-2}]}{2 \Delta t}.
\]

The two-step Lagrange–Galerkin scheme is to find \( \{ (\phi^n_h, u^n_h) \in \mathcal{V}_h \times V_h(\phi^n_h : \eta^n_h) ; n = 1, \ldots, N_{\tau} \} \) such that, for \( n = 1, 2, \ldots, N_{\tau} \),

\[
\rho(\phi^n_h, \psi_h) \mathcal{A}_\Delta[u_h] \phi^n_h, \psi_h \mathcal{B}_\Delta[u_h] u^n_h, v_h + a(u^n_h, v_h; \phi^n_h) + b(\eta^n_h, v_h; \phi^n_h)
\]

\[
= (F^n, v_h) \quad \forall \psi_h \in \mathcal{V}_h, \quad (2.3a)
\]

\[
\rho(\phi^n_h, \psi_h) \gamma_h \mathcal{B}_\Delta[u_h] u^n_h, v_h + a(u^n_h, v_h; \phi^n_h) + b(\eta^n_h, v_h; \phi^n_h)
\]

\[
= (F^n, v_h) \quad \forall \psi_h \in \mathcal{V}_h, \quad (2.3b)
\]

\[
\phi^n_h = \eta^n_h + \Pi_h \xi, \quad (2.3c)
\]
Remark 2.1. (i) At each time step, we obtain \( (\phi^n_h, u^n_h) = (\Pi_h \phi^0, \Pi_h u^0) \in \Psi_h \times Y_h \), \( n = 1 \) with an initial condition \( u^0_h \in Y_h \), where the Lagrange interpolation operator is denoted by \( \Pi_h : C(\overline{\Omega}) \rightarrow \Psi_h \), which is also used for the vector-valued function \( u^0 \), i.e., \( \Pi_h u^0 \in Y_h \).

(ii) We need \( A^{(1)}_\Delta [u] \) and \( B^{(1)}_\Delta [u] \) due to the lack of the functions \( \phi^{n-1}_h \) and \( U^{n-1}_h \) for \( n = 1 \), which are used for \( A^{(2)}_\Delta [u_h] \phi^n_h \) and \( B^{(2)}_\Delta [u_h] u^n_h \) for \( n \geq 2 \).

(iii) The two-step methods in conservative and non-conservative forms, \( A^{(2)}_\Delta [u_h] \phi^n_h \) and \( B^{(2)}_\Delta [u_h] u^n_h \), are developed and analyzed for convection-diffusion problems in \([16, 18]\).

(iv) It is discussed in \([18, 32]\) that the one-time use of first-order single-step methods, \( A^{(1)}_\Delta [u_h] \phi^n_h \) and \( B^{(1)}_\Delta [u_h] u^n_h \), has no loss of convergence orders in discrete versions of \( L^\infty(0, T; L^2(\Omega)) \) and \( L^2(0, T; H^1(\Omega)) \)-norms for a convection-diffusion equation and the Navier–Stokes equations, respectively.

3 Numerical results in square domains

In this section, numerical results via FreeFem++ \([19]\) are presented to see the experimental order of convergence (EOC) and the effect of the TBC in square domains. We call scheme (2.3) LG2, and also call scheme (2.3) replacing \( A_\Delta \) and \( B_\Delta \) with \( A^{(1)}_\Delta \) and \( B^{(1)}_\Delta \), respectively, LG1 which is a (single-step) Lagrange–Galerkin scheme of first order in time.

3.1 Experimental order of convergence

We solve Examples 1 and 2 below by LG1 and LG2 and compare the experimental orders of convergence (EOCs).

Example 1 \((\Gamma = \Gamma_D)\). In problem (2.1), we set \( \Omega = (0, 1)^2 \), \( \Gamma = \Gamma_D \) \( (\Gamma_T = \emptyset) \), \( T = 1 \), \( \nu = \rho = \mu = \zeta = 1 \), and the function \( \eta^0_h, u^0, f \) and \( F \) are given so that the exact solution is

\[
\phi(x, t) = 1 + \frac{\sin \pi x_1 \sin \pi x_2 (2 + \sin \pi t)}{8}, \quad u(x, t) = \frac{\sin \pi x_1 \sin \pi x_2 (2 + \sin \pi t)}{3} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.
\]

Example 2 \((\Gamma = \Gamma_D \cup \Gamma_T)\). In Example 1, we replace \( \Gamma_T \) and \( \Gamma_D \) with \( \Gamma_T = \{ x \in \Gamma : x_1 = 0 \} \) and \( \Gamma_D = \Gamma \setminus \Gamma_T \), respectively.

For a numerical solution \( z_h = \{ z^n_h \}_{n=0}^{N_T} \) and its exact solution \( z = \{ z^n \}_{n=0}^{N_T} \), we introduce notations of errors, \( E_i(z) \), \( i = 0, 1 \), defined by

\[
E_0(z) := \| z_h - z \|_{L^2}, \quad E_1(z) := \| \nabla (z_h - z) \|_{L^2},
\]

where \( \| \cdot \|_{L^2} \) is a norm given by

\[
\| z \|_{L^2} := \max \{ \| z^n \|_{L^2} : n = 0, \ldots, N_T \}.
\]
Let $N$ be a division number of each side of the unit square domain $\Omega$ and $h := 1/N$ a representative mesh size. We prepare non-uniform triangulations of $\Omega$, $\mathcal{T}_h$, for $N = 8, 16, 32, 64, 128$ and 256, cf. Figure 3 for $N = 32$. Choosing $\Delta t = 0.25\sqrt{h}$, we compute the errors, $E_i(\eta)$ and $E_i(u)$, $i = 0, 1$, by LG1 and LG2. Figures 4 and 5 show graphs of the errors of $E_0(\cdot)$ and $E_1(\cdot)$, respectively, in logarithmic scale by LG1 for Example 1 (i) and Example 2 (ii), and by LG2 for Example 1 (iii) and Example 2 (iv), and the values of errors and their EOCs are given in Tables 1 and 2. We observe that LG2 is of second order in time numerically and that the order is higher than that of LG1. Although $E_1(\eta)$ is not of second order in time, it is natural as equation (2.1a) for $\phi (= \eta + \zeta)$ does not include any diffusion term.

3.2 Effect of the TBC

We consider the following example to see the effect of the TBC.

(i) LG1 for Ex.1  (ii) LG1 for Ex.2  (iii) LG2 for Ex.1  (iv) LG2 for Ex.2

Figure 4: Graphs of errors $E_0(\eta)$ and $E_0(u)$ in logarithmic scale by LG1 for Example 1 (i) and Example 2 (ii), and by LG2 for Example 1 (iii) and Example 2 (iv).

3.2 Effect of the TBC

We consider the following example to see the effect of the TBC.
Table 1: Values of $E_i(\eta)$ and $E_i(u)$, $i = 0, 1$, by schemes LG1 and LG2 for Example 1 ($\Gamma = I_0$).

| $N$   | $\Delta t$ | $E_i(\eta)$   | EOC | $E_i(u)$   | EOC |
|-------|------------|----------------|-----|------------|-----|
| LG1   |            |                |     |            |     |
| 8     | $8.84 \times 10^{-2}$ | $3.89 \times 10^0$ | -   | $3.78 \times 10^{-2}$ | -   |
| 16    | $6.25 \times 10^{-2}$ | $2.20 \times 10^0$ | 1.65 | $2.28 \times 10^{-2}$ | 1.45 |
| 32    | $4.42 \times 10^{-2}$ | $1.45 \times 10^0$ | 1.19 | $1.57 \times 10^{-2}$ | 1.09 |
| 64    | $3.13 \times 10^{-2}$ | $1.01 \times 10^0$ | 1.05 | $1.10 \times 10^{-2}$ | 1.03 |
| 128   | $2.21 \times 10^{-2}$ | $7.11 \times 10^{-1}$ | 1.01 | $7.77 \times 10^{-3}$ | 1.00 |
| 256   | $1.56 \times 10^{-2}$ | $5.02 \times 10^{-1}$ | 1.00 | $5.51 \times 10^{-3}$ | 0.99 |
| LG2   |            |                |     |            |     |
| 8     | $8.84 \times 10^{-2}$ | $6.81 \times 10^{-1}$ | -   | $1.71 \times 10^{-2}$ | -   |
| 16    | $6.25 \times 10^{-2}$ | $1.96 \times 10^{-1}$ | 3.60 | $7.03 \times 10^{-3}$ | 2.57 |
| 32    | $4.42 \times 10^{-2}$ | $8.53 \times 10^{-2}$ | 2.40 | $3.32 \times 10^{-3}$ | 2.16 |
| 64    | $3.13 \times 10^{-2}$ | $3.82 \times 10^{-2}$ | 2.32 | $1.64 \times 10^{-3}$ | 2.04 |
| 128   | $2.21 \times 10^{-2}$ | $1.87 \times 10^{-2}$ | 2.05 | $8.20 \times 10^{-4}$ | 1.99 |
| 256   | $1.56 \times 10^{-2}$ | $9.46 \times 10^{-3}$ | 1.97 | $4.17 \times 10^{-4}$ | 1.95 |

| $N$   | $\Delta t$ | $E_i(\eta)$   | EOC | $E_i(u)$   | EOC |
|-------|------------|----------------|-----|------------|-----|
| LG1   |            |                |     |            |     |
| 8     | $8.84 \times 10^{-2}$ | $3.97 \times 10^0$ | -   | $5.68 \times 10^{-2}$ | -   |
| 16    | $6.25 \times 10^{-2}$ | $2.24 \times 10^0$ | 1.65 | $2.90 \times 10^{-2}$ | 1.94 |
| 32    | $4.42 \times 10^{-2}$ | $2.00 \times 10^0$ | 0.33 | $1.20 \times 10^{-2}$ | 2.54 |
| 64    | $3.13 \times 10^{-2}$ | $1.64 \times 10^0$ | 0.57 | $6.72 \times 10^{-3}$ | 1.67 |
| 128   | $2.21 \times 10^{-2}$ | $1.17 \times 10^0$ | 0.97 | $3.23 \times 10^{-3}$ | 2.11 |
| 256   | $1.56 \times 10^{-2}$ | $8.64 \times 10^{-1}$ | 0.88 | $1.47 \times 10^{-3}$ | 2.28 |
Table 2: Values of $E_i(\eta)$ and $E_i(u)$, $i = 0, 1$, by schemes LG1 and LG2 for Example 2 ($I = T_D \cup T_T$).

| $N$     | $\Delta t$ | $E_0(\eta)$ | EOC | $E_0(u)$ | EOC | $E_1(\eta)$ | EOC | $E_1(u)$ | EOC |
|---------|-------------|--------------|-----|----------|-----|--------------|-----|----------|-----|
| LG1     |             |              |     |          |     |              |     |          |     |
| 8       | $8.84 \times 10^{-2}$ | $3.88 \times 10^9$ | -   | $3.86 \times 10^{-2}$ | -   |              |     |          |     |
| 16      | $6.25 \times 10^{-2}$ | $2.19 \times 10^9$ | 1.65 | $2.33 \times 10^{-2}$ | 1.46 |              |     |          |     |
| 32      | $4.42 \times 10^{-2}$ | $1.45 \times 10^9$ | 1.9 | $1.58 \times 10^{-2}$ | 1.11 |              |     |          |     |
| 64      | $3.13 \times 10^{-2}$ | $101 \times 10^8$  | 1.05 | $1.11 \times 10^{-2}$ | 1.03 |              |     |          |     |
| 128     | $2.21 \times 10^{-2}$ | $7.09 \times 10^8$  | 1.01 | $7.82 \times 10^{-3}$ | 1.01 |              |     |          |     |
| 256     | $1.56 \times 10^{-2}$ | $5.01 \times 10^8$  | 1.00 | $5.53 \times 10^{-3}$ | 1.00 |              |     |          |     |
| LG1     |             |              |     |          |     |              |     |          |     |
| 8       | $8.84 \times 10^{-2}$ | $2.95 \times 10^9$  | -   | $7.80 \times 10^{-2}$ | -   |              |     |          |     |
| 16      | $6.25 \times 10^{-2}$ | $1.71 \times 10^9$  | 1.57 | $4.64 \times 10^{-2}$ | 1.50 |              |     |          |     |
| 32      | $4.42 \times 10^{-2}$ | $1.24 \times 10^9$  | 0.94 | $2.95 \times 10^{-2}$ | 1.31 |              |     |          |     |
| 64      | $3.13 \times 10^{-2}$ | $9.78 \times 10^8$  | 0.67 | $2.03 \times 10^{-2}$ | 1.07 |              |     |          |     |
| 128     | $2.21 \times 10^{-2}$ | $6.42 \times 10^8$  | 1.21 | $1.41 \times 10^{-2}$ | 1.04 |              |     |          |     |
| 256     | $1.56 \times 10^{-2}$ | $4.34 \times 10^8$  | 1.13 | $9.96 \times 10^{-3}$ | 1.01 |              |     |          |     |
| LG2     |             |              |     |          |     |              |     |          |     |
| 8       | $8.84 \times 10^{-2}$ | $6.70 \times 10^9$  | -   | $1.75 \times 10^{-2}$ | -   |              |     |          |     |
| 16      | $6.25 \times 10^{-2}$ | $1.95 \times 10^9$  | 3.56 | $7.23 \times 10^{-3}$ | 2.55 |              |     |          |     |
| 32      | $4.42 \times 10^{-2}$ | $8.58 \times 10^8$  | 2.37 | $3.37 \times 10^{-3}$ | 2.20 |              |     |          |     |
| 64      | $3.13 \times 10^{-2}$ | $3.97 \times 10^7$  | 2.22 | $1.67 \times 10^{-3}$ | 2.03 |              |     |          |     |
| 128     | $2.21 \times 10^{-2}$ | $1.87 \times 10^7$  | 2.17 | $8.37 \times 10^{-4}$ | 2.00 |              |     |          |     |
| 256     | $1.56 \times 10^{-2}$ | $9.54 \times 10^6$  | 1.94 | $4.25 \times 10^{-4}$ | 1.96 |              |     |          |     |
| LG2     |             |              |     |          |     |              |     |          |     |
| 8       | $8.84 \times 10^{-2}$ | $3.89 \times 10^9$  | -   | $5.70 \times 10^{-2}$ | -   |              |     |          |     |
| 16      | $6.25 \times 10^{-2}$ | $2.21 \times 10^9$  | 1.63 | $2.93 \times 10^{-2}$ | 1.92 |              |     |          |     |
| 32      | $4.42 \times 10^{-2}$ | $1.98 \times 10^9$  | 0.32 | $1.24 \times 10^{-2}$ | 2.49 |              |     |          |     |
| 64      | $3.13 \times 10^{-2}$ | $1.65 \times 10^9$  | 0.54 | $6.90 \times 10^{-3}$ | 1.69 |              |     |          |     |
| 128     | $2.21 \times 10^{-2}$ | $1.17 \times 10^9$  | 0.97 | $3.26 \times 10^{-3}$ | 2.16 |              |     |          |     |
| 256     | $1.56 \times 10^{-2}$ | $8.62 \times 10^{-1}$ | 0.89 | $1.48 \times 10^{-3}$ | 2.27 |              |     |          |     |
Example 3. In problem (2.1), we set $\Omega = (0, 10)^2$, $T = 100$, $g = \rho = \mu = \zeta = 1$, $(f, F) = (0, 0)$, $\eta^0 = c \exp(-100|x-p|^2)$, $c = 10^{-3}$, $p = (5, 5)^T$, and $u^0 = 0$. We consider five cases of $\Gamma_\mathcal{T}$,
(a) $\Gamma_\mathcal{T} = \emptyset$, i.e., $\Gamma = \Gamma_\mathcal{D}$,
(b) $\Gamma_\mathcal{T} = \{x \in \Gamma; x_2 = 0\}$ (bottom), $\Gamma_\mathcal{D} = \Gamma \setminus \Gamma_\mathcal{T}$,
(c) $\Gamma_\mathcal{T} = \{x \in \Gamma; x_1 = 10, x_2 = 0\}$ (right and bottom), $\Gamma_\mathcal{D} = \Gamma \setminus \Gamma_\mathcal{T}$,
(d) $\Gamma_\mathcal{T} = \{x \in \Gamma; x_1 = 10, x_2 = 0, 10\}$ (right, bottom and top), $\Gamma_\mathcal{D} = \Gamma \setminus \Gamma_\mathcal{T}$,
(e) $\Gamma_\mathcal{T} = \Gamma$.

We solve Example 3 by LG2. Figure 6 shows the color contours of $\eta^n$ for $t = 25k$, $k = 0, \ldots, 4$, cf. (i)-(v), for the five cases, (a)-(e). We can see the effect of the boundary conditions; the artificial reflection is observed and removed significantly when the wave touches the Dirichlet ($\Gamma_\mathcal{D}$) and the transmission ($\Gamma_\mathcal{T}$) boundaries, respectively. Thus, LG2 works well for the SWEs with and without the TBC in the simple square domain.

4 Application to the Bay of Bengal

In this section, we apply LG2, i.e., scheme (2.3) discussed in Subsection 2.2, to a computational domain of the Bay of Bengal region, cf. Figure 7, which is an approximate domain of the original, cf. Figure 1. All the computations are performed via FreeFem++ [19].

4.1 Numerical simulation with and without TBC

We set the following example.
Figure 6: Color contours of $\eta^n_h$ by LG2 with and without the TBC for the five cases, (a)-(e), in Example 3.
Example 4. Let $\Omega$ be the domain shown in Figure 7. The domain is considered from 0 to 1051.4 [km] in the horizontal direction and 0 to 889.59 [km] in the vertical direction. We employ two boundary conditions, the Dirichlet boundary condition on $\Gamma_D$ and the TBC on $\Gamma_T$, cf. Figure 7. We set $\Gamma_D$ on the coastal and island boundaries and $\Gamma_T$ on the artificial boundaries for the open sea. As shown in Figure 7, there are three artificial boundaries on the open sea, i.e., $\Gamma_T = \Gamma_{T1} \cup \Gamma_{T2} \cup \Gamma_{T3}$. In problem (2.1), we set $T = 5,000$ [s], $\xi = 2$ [km], $\eta^0(x) = c_1 \exp(-0.04|x-p|^2)$ [km], $c_1 = 0.01$ [–], $p = (559.56, 430.02)^T$, $u^0 = 0$, $\mu = 1$ [Pa s], $\rho = 10^{12}$ [kg/km$^3$], $g = 9.8 \times 10^{-3}$ [km/s$^2$] and $(f, F) = (0, 0)$.

We prepare a triangular mesh of the domain as shown in Figure 8, where the numbers of elements and nodal points are 60,619 and 31,120, respectively. Then, a numerical simulation is done by LG2 with $\Delta t = 0.2$ [s]. The results at $t = 0, 2,500, 3,000, 4,000, 4,500$ and 5,000 [s] are presented in Figures 9 and 10. In the figures, for comparison to see the effect of the TBC, we compute Example 4 by replacing $\Gamma_T$ with $\Gamma_D$ and put it on the left. From Figure 9, we can see that a circular wave is created at around the point $p$, that it propagates towards the boundary over time, that reflections are found when the wave touches $\Gamma_D$, and that the results with $\Gamma = \Gamma_D$ (left) and $\Gamma = \Gamma_D \cup \Gamma_T$ (right) are similar. From Figure 10, we can observe that artificial reflections on the open sea boundaries are significantly removed when the wave touches $\Gamma_T$, cf. the right figures. Thus, LG2 works well for a simple (square) domain and this complex domain, the Bay of Bengal region, which is non-convex and includes islands.
Figure 8: The mesh for the Bay of Bengal region used for Example 4.

For any (smooth) solution to problem (2.1), we define the total energy $E(t)$ by

$$E(t) := E_1(t) + E_2(t) := \int_{\Omega} \frac{\rho}{2} |u|^2 \, dx + \int_{\Omega} \frac{\rho g |\eta|^2}{2} \, dx,$$

where $E_1(t)$ is the kinetic energy, and $E_2(t)$ is the potential energy. Then, it is worthy to note that the following energy estimate holds, cf. [27, Corollary 3.3-(i)],

$$\frac{d}{dt} E(t) = -\rho \int_{\Gamma_T} \phi |u|^2 (u \cdot n) \, ds - \rho g \int_{\Gamma_T} \phi \eta(u \cdot n) \, ds$$

$$+ 2\mu \int_{\Gamma_T} \phi ([D(u)n] \cdot u) \, ds - 2\mu \int_{\Omega} \phi |D(u)|^2 \, dx.$$

Here, focusing on $E_2(t) = \int_{\Omega} \rho g |\eta|^2 \, dx$ and the mass of $\eta$, i.e., $\int_{\Omega} \eta \, dx$, we present the values of the $L^2(\Omega)$-norm of $\eta^n_t$, i.e., $||\eta^n_t||_{L^2(\Omega)}$, and the mass of $\eta^n_t$, i.e., $\int_{\Omega} \eta^n_t \, dx$, in Figures 11 and 12, respectively. In principle, we can say that the TBC works well numerically if $||\eta^n_t||_{L^2(\Omega)}$ and $\int_{\Omega} \eta^n_t \, dx$ decrease around the time that the wave touches the transmission boundaries. Figure 11 shows graphs of $||\eta^n_t||_{L^2(\Omega)}$ for the two cases, with and without the transmission boundaries, i.e., $\Gamma = \Gamma_D \cup \Gamma_T$ and $\Gamma = \Gamma_D$ ($\Gamma_T = \emptyset$), respectively. Figure 12 shows the graphs of $\int_{\Omega} \eta^n_t \, dx$ for the four cases of (transmission) boundaries, (i) no transmission boundary, i.e., $\Gamma_T = \emptyset$, (ii) one transmission boundary, i.e., $\Gamma_T = \Gamma_2$, (iii) two transmission boundaries, i.e., $\Gamma_T = \Gamma_1 \cup \Gamma_3$, and (iv) three
transmission boundaries, i.e., $\Gamma_T = T_1 \cup T_2 \cup T_3$. From Figures 11 and 12, we can see that there are decreasing phenomena of the value of $L^2(\Omega)$-norm as well as the value of the mass when the TBC is imposed. From Figure 9, we can see that the wave touches the transmission boundary $T_2$ at time around $t = 3,000$ [s]; that is why, the mass of $\eta^n_{h}$ decreases drastically from around 3,000 [s] to 3,200 [s], cf. Figure 12 (yellow and green lines). Again, the mass started to decrease between the period from around 4,000 [s] to 4,500 [s], cf. Figure 12, since the wave reached the transmission boundary $T_1$ and $T_3$, cf. Figure 10.

Figure 9: Contour plot of $\eta^n_{h}$ by LG2 with $\Gamma = T_D$ (left) and $\Gamma = D \cup T$ (right) on the Bay of Bengal for $t = 0, 2,500$ and 3,000.
4.2 Effect of position of a transmission boundary

We consider Example 4 again to see the effect of the TBC with an extension of the domain ($\Omega$), where the size of the domain in the vertical direction is extended from 889.59 [km] to 989.59 [km], i.e., 100 [km] extension. We employ the same boundary conditions on $\Gamma = \Gamma_D \cup \Gamma_T$ for both original and extended domains, where $\Gamma_T = \Gamma_{T1} \cup \Gamma_{T2} \cup \Gamma_{T3}$. We compare the numerical results for the extended domain with the ones for the original domain, cf. Figures 13 and 14, where the left and right figures
Figure 11: Graphs of $\|\eta_h^n\|_{L^2(D)}$ with respect to time ($t = t^n$) for Example 4 with $I_T$ ($\Gamma = I_D \cup I_T$) and without $I_T$ ($\Gamma = I_D$).

Figure 12: Graphs of the mass of $\eta_h^n$ with respect to time ($t = t^n$) for Example 4 with the following four settings, (i) no transmission boundary, i.e., $I_T = \emptyset$ (purple), (ii) one transmission boundary, i.e., $I_T = I_{T2}$ (green) (iii) two transmission boundaries (blue), i.e., $I_T = I_{T1} \cup I_{T3}$ (blue), and (iv) three transmission boundaries, i.e., $I_T = I_{T1} \cup I_{T2} \cup I_{T3}$ (yellow).
show the results for the extended and original domains, respectively. It is observed that there is no significant effect of the vertical position of the bottom transmission boundary $I_{T2}$. We also computed the mass of $\eta$ for both domains, cf. Figure 15. From Figure 15, we can see that the mass of $\eta_b^k$ started to decrease at time $t = 3,000$ for the original domain, cf. Figure 13-(c2), while the mass of $\eta_b^k$ started to decrease at time $t = 4,000$ for the extended domain, cf. Figure 14-(e1), because the wave touches the boundary $I_{T2}$ at these times ($t = 3,000$ and $t = 4,000$) for the original and extended domains, respectively. A similar decreasing property of mass of $\eta_b^k$ can be observed from Figure 15 when the wave touches the transmission boundaries. The results confirm that the TBC works well numerically and that we can choose the vertical position of the bottom transmission boundary $I_{T2}$ without significant effect.

5 Conclusions

We have presented a two-step Lagrange–Galerkin scheme for the shallow water equations with a TBC. For the scheme, the EOCs have been computed (cf. Examples 1 and 2 in Subsection 3.1) and the second-order accuracy in time has been confirmed. From numerical experiments on a simple square domain (cf. Example 3 in Subsection 3.2), it has been observed that the effect of the TBC works well. Our scheme has been applied to a realistic domain, the Bay of Bengal, and numerical experiments have been performed for two different types of boundary conditions, i.e., with and without the TBC (cf. Subsection 4.1). There have been no significant reflections from $I_T$ and the wave has passed through $I_T$ while reflections have been observed from $I_D$, and, in the graphs of $\|\eta_0^k\|_{L^2(G)}$ and the mass of $\eta_0^k$ (cf. Figures 11 and 12), natural decays of the values of $\|\eta_0^k\|_{L^2(G)}$ as well as the mass of $\eta_0^k$ have been observed when the TBC is imposed. In addition, for the domain extended by 100 [km] in the vertical direction, it has been confirmed that there is no significant effect of changing the position of the transmission boundary (cf. Subsection 4.2). From these numerical experiments, we conclude that our two-step Lagrange–Galerkin scheme, cf. (2.3), works well numerically not only for a simple domain but also for a complex domain with the TBC if the bottom topography is flat. In our forthcoming paper, Part II, the scheme will be applied to rapidly varying bottom surfaces and a real bottom topography of the Bay of Bengal region to investigate the effect of non-homogeneity of the bottom topography.

Acknowledgements

M.M.R. is supported by the MEXT scholarship. This work is partially supported by JSPS KAKENHI Grant Numbers JP20KK0058, JP21H00999, JP20H00117, JP20H01812, JP18H01135, JP21H04431, and JP20H01823, and JST CREST Grant Number JP-MJCR2014.

Appendix

A.1 Choice of $c_0$

Based on [27], focusing on the potential energy $E_2(t)$, cf. (4.1), we perform numerical experiments for the choice of $c_0$ for two cases with the following settings:

Case I (the square domain). In problem (2.1), we set $\Omega = (0,10)^2$, $T = 100$, $g =$
Figure 13: Contour plot of $\eta_n$ by LG2 with $\Gamma = \hat{\Gamma}_D \cup \hat{\Gamma}_T$ for the extended domain (left) and for the original domain (right) on the Bay of Bengal for $t = 0, 2,500$ and $3,000$. 
Figure 14: Contour plot of $\eta^D$ by LG2 with $\Gamma = \bar{\Gamma}_D \cup \bar{\Gamma}_T$ for the extended domain (left) and for the original domain (right) on the Bay of Bengal for $t = 3,500, 4,000$ and $5,000$. 
9.8 \times 10^{-3}, \rho = 10^{12}, \mu = \zeta = 1, (f, F) = (0, 0), c = 10^{-3}, \eta^0 = c \exp(-100|x - p|^2), p = (5.5)^{1/2}, u^0 = 0 \text{ and } \Gamma = \Gamma_T (\Gamma_D = \emptyset). \text{ We employ discretization parameters, } N = 200 (h = 1/N), \text{ and } \Delta t = 0.25 \sqrt{h}.

**Case II** (the Bay of Bengal). The parameters are the same as Example 4 except the value of \(c_0\). We employ the same mesh and \(\Delta t = 0.2\) in Section 4.

For \(\eta_h = \{\eta^n_h\}_{n=1}^{N_T}\), let \(\|\eta_h\|_{L^2(T; L^2)}\) be a norm of \(\eta_h\) defined by

\[
\|\eta_h\|_{L^2(T; L^2)} := \sqrt{\Delta t \sum_{n=1}^{N_T} \|\eta^n_h\|_{L^2(T; L^2)}^2} = \|\eta\|_{L^2(0, T; L^2(D))}.
\]

We compute the two cases for \(c_0 = 0.5, 0.6, \ldots, 1.2\). The results are shown in Table A.1 and imply that, for both cases, we have minimum values of \(\|\eta_h\|_{L^2(T; L^2)}\) for \(c_0 = 0.9\).

| Value of \(c_0\) | Case I (the square domain) | Case II (the Bay of Bengal) |
|------------------|----------------------------|----------------------------|
| 0.5              | 8.16 \times 10^{-2}        | 13.55                      |
| 0.6              | 8.08 \times 10^{-2}        | 13.54                      |
| 0.7              | 8.03 \times 10^{-2}        | 13.5342                    |
| 0.8              | 8.002 \times 10^{-2}       | 13.5323                    |
| 0.9              | 7.997 \times 10^{-2}       | 13.5319                    |
| 1.0              | 8.006 \times 10^{-2}       | 13.5328                    |
| 1.1              | 8.02 \times 10^{-2}        | 13.5354                    |
| 1.2              | 8.05 \times 10^{-2}        | 13.5375                    |
References

[1] Achdou, Y., Guermond, J.L.. Convergence analysis of a finite element projection/Lagrange–Galerkin method for the incompressible Navier–Stokes equations. SIAM Journal on Numerical Analysis 2000;37:799–826.

[2] Benítez, M., Bermúdez, A.. A second order characteristics finite element scheme for natural convection problems. Journal of Computational and Applied Mathematics 2011;235:3270–3284.

[3] Benítez, M., Bermúdez, A.. Numerical analysis of a second order pure Lagrange–Galerkin method for convection-diffusion problems. Part I: Time discretization. SIAM Journal on Numerical Analysis 2012a;50:858–882.

[4] Benítez, M., Bermúdez, A.. Numerical analysis of a second order pure Lagrange–Galerkin method for convection-diffusion problems. Part II: Fully discretized scheme and numerical results. SIAM Journal on Numerical Analysis 2012b;50:2824–2844.

[5] Bermejo, R., Saavedra, L.. Modified Lagrange–Galerkin methods of first and second order in time for convection-diffusion problems. Numerische Mathematik 2012;120:601–638.

[6] Bermejo, R., Gálan del Sastre, P., Saavedra, L.. A second order in time modified Lagrange–Galerkin finite element method for the incompressible Navier–Stokes equations. SIAM Journal on Numerical Analysis 2012;50:3084–3109.

[7] Bermúdez, A., Nogueiras, M.R., Vázquez, C.. Numerical analysis of convection-diffusion-reaction problems with higher order characteristics/finite elements, part i: Time discretization. SIAM Journal on Numerical Analysis 2006a;44(5):1829–1853.

[8] Bermúdez, A., Nogueiras, M.R., Vázquez, C.. Numerical analysis of convection-diffusion-reaction problems with higher order characteristics/finite elements, part ii: Fully discretized scheme and quadrature formulas. SIAM Journal on Numerical Analysis 2006b;44(5):1854–1876.

[9] Boukir, K., Maday, Y., Métivet, B., Razafindrakoto, E.. A high-order characteristics/finite element method for the incompressible Navier–Stokes equations. International Journal for Numerical Methods in Fluids 1997;25:1421–1454.

[10] Chrysafinos, K., Walkington, N.J.. Lagrangian and moving mesh methods for the convection diffusion equation. ESAIM: Mathematical Modelling and Numerical Analysis 2008;42:25–55.

[11] Colera, M., Carpio, J., Bermejo, R.. A nearly-conservative high-order Lagrange–Galerkin method for the resolution of scalar convection-dominated equations in non-divergence-free velocity fields. Computer Methods in Applied Mechanics and Engineering 2020;372:113366.

[12] Colera, M., Carpio, J., Bermejo, R.. A nearly-conservative, high-order, forward Lagrange–Galerkin method for the resolution of scalar hyperbolic conservation laws. Computer Methods in Applied Mechanics and Engineering 2021;376:113654.
[13] Das, P.K.. Prediction model for storm surges in the Bay of Bengal. Nature 1972;239(5369):211–213.

[14] Debsarma, S.K.. Simulations of storm surges in the Bay of Bengal. Marine Geodesy 2009;32(2):178–198.

[15] Douglas, J.J., Russell, T.F.. Numerical methods for convection-dominated diffusion problems based on combining the method of characteristics with finite element or finite difference procedures. SIAM Journal on Numerical Analysis 1982;19(5):871–885.

[16] Ewing, R., Russell, T.. Multistep Galerkin methods along characteristics for convection-diffusion problems. In: Vichnevetsky, R., Stepleman, R., editors. Advances in Computer Methods for Partial Differential Equations IV. IMACS; 1981. p. 28–36.

[17] Ewing, R., Russell, T., Wheeler, M.. Simulation of miscible displacement using mixed methods and a modified method of characteristics. In: Proceedings of the Seventh Reservoir Simulation Symposium. Society of Petroleum Engineers of AIME; 1983. p. 71–81.

[18] Futai, K., Kolbe, N., Notsu, H., Suzuki, T.. A mass-preserving two-step Lagrange–Galerkin scheme for convection-diffusion problems. Journal of Scientific Computing 2022;92(2):37.

[19] Hecht, F.. New development in FreeFem++. Journal of Numerical Mathematics 2012;20(3-4):251–265.

[20] Johns, B.. Numerical simulation of storm surges in the Bay of Bengal. Monsoon Dynamics 1981;:689–706.

[21] Kanayama, H., Dan, H.. A finite element scheme for two-layer viscous shallow-water equations. Japan Journal of Industrial and Applied Mathematics 2006;23(2):163–191.

[22] Kanayama, H., Dan, H.. Tsunami propagation from the open sea to the coast. Tsunami 2016;.

[23] Lukáčová-Medvid’ová, M., Mizerová, H., Notsu, H., Tabata, M.. Numerical analysis of the Oseen-type Peterlin viscoelastic model by the stabilized Lagrange–Galerkin method, Part I: A linear scheme. ESAIM: M2AN 2017a;51:1637–1661.

[24] Lukáčová-Medvid’ová, M., Mizerová, H., Notsu, H., Tabata, M.. Numerical analysis of the Oseen-type Peterlin viscoelastic model by the stabilized Lagrange–Galerkin method, Part II: A nonlinear scheme. ESAIM: M2AN 2017b;51:1663–1689.

[25] Lukáčová-Medvidová, M., Notsu, H., She, B.. Energy dissipative characteristic schemes for the diffusive Oldroyd-B viscoelastic fluid. International Journal for Numerical Methods in Fluids 2015;.

[26] Murshed, M.M.. Theoretical and Numerical Studies of the Shallow Water Equations with a Transmission Boundary Condition. Ph.D. thesis; Kanazawa University, Japan; 2019.
[27] Murshed, M.M., Futai, K., Kimura, M., Notsu, H.. Theoretical and numerical studies for energy estimates of the shallow water equations with a transmission boundary condition. Discrete and Continuous Dynamical Systems - S 2021;14(3):1063–1078.

[28] Notsu, H.. Numerical computations of cavity flow problems by a pressure stabilized characteristic-curve finite element scheme. Transactions of Japan Society for Computational Engineering and Science 2008;2008:20080032.

[29] Notsu, H., Rui, H., Tabata, M.. Development and L2-analysis of a single-step characteristics finite difference scheme of second order in time for convection-diffusion problems. Journal of Algorithms & Computational Technology 2013;7(3):343–380.

[30] Notsu, H., Tabata, M.. Error estimates of a pressure-stabilized characteristics finite element scheme for the oseen equations. Journal of Scientific Computing 2015:65(3):940–955.

[31] Notsu, H., Tabata, M.. Error estimates of a stabilized Lagrange–Galerkin scheme for the Navier–Stokes equations. ESAIM: Mathematical Modelling and Numerical Analysis 2016a;50(2):361–380.

[32] Notsu, H., Tabata, M.. Error estimates of a stabilized Lagrange–Galerkin scheme of second-order in time for the Navier–Stokes equations. Mathematical Fluid Dynamics, Present and Future Springer Proceedings in Mathematics & Statistics 2016b;:497–530.

[33] Paul, G.C., Ismail, A.I.M.. Tide–surge interaction model including air bubble effects for the coast of Bangladesh. Journal of the Franklin Institute 2012;349(8):2530–2546.

[34] Paul, G.C., Ismail, A.I.M.. Contribution of offshore islands in the prediction of water levels due to tide–surge interaction for the coastal region of Bangladesh. Natural Hazards 2013;65(1):13–25.

[35] Paul, G.C., Senthilkumar, S., Pria, R.. Storm surge simulation along the Meghna estuarine area: an alternative approach. Acta Oceanologica Sinica 2018;37(1):40–49.

[36] Pironneau, O.. On the transport-diffusion algorithm and its applications to the Navier-Stokes equations. Numerische Mathematik 1982;38(3):309–332.

[37] Pironneau, O.. Finite Element Methods for Fluids. Chichester: John Wiley & Sons, 1989.

[38] Pironneau, O., Tabata, M.. Stability and convergence of a Galerkin-characteristics finite element scheme of lumped mass type. International Journal for Numerical Methods in Fluids 2010;64:1240–1253.

[39] Roy, G., Kabir, A.H., Mandal, M., Haque, M.. Polar coordinates shallow water storm surge model for the coast of Bangladesh. Dynamics of Atmospheres and Oceans 1999;29(2-4):397–413.

[40] Rui, H., Tabata, M.. A second order characteristic finite element scheme for convection-diffusion problems. Numerische Mathematik 2002;92(1):161–177.
[41] Rui, H., Tabata, M.. A mass-conservative characteristic finite element scheme for convection-diffusion problems. Journal of Scientific Computing 2010;43:416–432.

[42] Süli, E.. Convergence and nonlinear stability of the Lagrange-Galerkin method for the Navier-Stokes equations. Numerische Mathematik 1988;53(4):459–483.

[43] Tabata, M., Uchiumi, S.. A genuinely stable Lagrange–Galerkin scheme for convection-diffusion problems. Japan Journal of Industrial and Applied Mathematics 2016;33:121–143.

[44] Tabata, M., Uchiumi, S.. An exactly computable Lagrange–Galerkin scheme for the Navier–Stokes equations and its error estimates. Mathematics of Computation 2018;87:39–67.

[45] Uchiumi, S.. A viscosity-independent error estimate of a pressure-stabilized Lagrange–Galerkin scheme for the Oseen problem. Journal of Scientific Computing 2019;80:834–858.