Subquadratic harmonic functions on Calabi-Yau manifolds with maximal volume growth

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Abstract
On a complete Calabi-Yau manifold $M$ with maximal volume growth, a harmonic function with subquadratic polynomial growth is the real part of a holomorphic function. This generalizes a result of Conlon-Hein. We prove this result by proving a Liouville-type theorem for harmonic 1-forms, which follows from a new local $L^2$ estimate of the exterior derivative.

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1 | INTRODUCTION

In this paper, we define a Calabi-Yau manifold to be a Ricci-flat Kähler manifold. Let $(M, \omega)$ be a complete noncompact Calabi-Yau manifold of complex dimension $n$. We say $(M, \omega)$ has maximal volume growth if there exists $\nu > 0$ such that for $p \in M$ and $r > 0$, we have

$$\text{Vol}(B(p, r)) \geq \nu r^{2n}.$$ 

The study of complete noncompact Calabi-Yau manifolds dates back to the foundational papers of Tian-Yau [46, 47], in which they construct complete Ricci-flat Kähler metrics on the
complement of a neat, almost ample divisor in a projective variety. An important class of examples is asymptotically conical (AC) Calabi-Yau manifolds. An AC Calabi-Yau manifold is a complete noncompact Ricci-flat Kähler manifold such that outside a compact subset, the manifold is diffeomorphic to a Ricci-flat Kähler cone, and the metric on the manifold is (polynomially) asymptotic to the cone metric. Explicit important examples of AC Calabi-Yau manifolds include smoothing and small resolutions of the ordinary double point; see, for example, [6, 44]. For a Calabi-Yau manifold $M$ with maximal volume growth, Cheeger-Tian [12] show that if one tangent cone at infinity $M$ satisfies an integrability condition, which implies smoothness, then $M$ is AC Calabi-Yau. We refer the reader to Conlon-Hein [19–21] for important results of AC Calabi-Yau manifolds.

The AC condition is restrictive, as it implies the tangent cone at infinity is both smooth and unique. There are complete noncompact Calabi-Yau manifolds with maximal volume growth which are not AC Calabi-Yau. For example, Joyce’s QALE manifolds [33] have $\mathbb{C}^n/\Gamma$ as a tangent cone at infinity, where $\Gamma \subset SU(n)$ is a discrete subgroup which does not act freely on $\mathbb{C}^n$. Recently Li [37], Conlon-Rochon [22], and Székelyhidi [45] independently constructed non-flat Calabi-Yau metrics with maximal volume growth on $\mathbb{C}^3$; the last two groups have various generalizations of the construction to higher dimensions. These metrics are not AC, since the tangent cone at infinity of these metrics is $\mathbb{C} \times A_1$, where $A_1$ denotes the singularity

$$\mathbb{C}^2/\mathbb{Z}_2 \simeq \{z_1^2 + z_2^2 + z_3^2 = 0\} \subset \mathbb{C}^3,$$

equipped with the flat cone metric. This motivates us to try to understand the relationship between these metrics. More generally, we would like to study deformations of Calabi-Yau metrics on a Calabi-Yau manifold with maximal volume growth. Since such manifolds also serve as local models in various gluing constructions of compact Calabi-Yau manifolds, the question of uniqueness is also natural in this respect.

In Conlon-Hein [19], the rigidity of AC Calabi-Yau metrics is shown by first showing that a harmonic function with subquadratic growth is pluriharmonic. This particular result can be seen as the linearized version of the rigidity of the complex Monge-Ampère equation. In this paper, we generalize this result (see theorem 3.8 and corollary 3.9 in [19]) to the following:

**Theorem 1.1.** Let $M$ be a complete noncompact Ricci-flat manifold with maximal volume growth. Let $u$ be a harmonic 1-form on $M$, and by this we mean that $(dd^* + d^*d)u = 0$. Suppose $u$ has sublinear growth, that is there exist constants $C > 0$ and $s < 1$ such that

$$|u| \leq C(1 + r)^s,$$

where $r$ is the distance function from a fixed point $p \in M$. Then $u = df$, where $f$ is a subquadratic harmonic function.

Theorem 1.1 can be seen as an analogue of the Liouville theorem of Cheng [14] that every harmonic function with sublinear growth on a complete manifold with nonnegative Ricci curvature is constant. As a corollary, we have
**Theorem 1.2.** Let $M$ be a complete noncompact Calabi-Yau manifold with maximal volume growth. Then any subquadratic harmonic function on $M$ is the real part of a subquadratic holomorphic function. In particular, it is pluriharmonic.

**Remark 1.3.**

1. When $M$ is AC Calabi-Yau, the proof in [19] shows that actually we can replace the subquadratic growth condition with the weaker condition $o(r^2)$ as $r \to \infty$. See Remark 5.5 for details. It is interesting to know whether Theorem 1.2 still holds when we only assume $o(r^2)$.

2. Theorems 1.1 and 1.2 are false without the assumption of maximal volume growth. One counterexample is the Taub-NUT manifold, which is $\mathbb{C}^2$ equipped with a hyperkähler metric with cubic volume growth. There is a harmonic function with linear growth which is not pluriharmonic. See Remark 3.10 in [19] for more details.

**Proof of Theorem 1.2.** Let $f$ be a subquadratic harmonic function. Then $d^c f = i(\bar{\partial} - \partial)f$ is a sub-linear harmonic 1-form by the Cheng-Yau gradient estimate [13]. So $d^c f = dg$ by Theorem 1.1. It follows that $f$ is the real part of the subquadratic holomorphic function $f + ig$. □

In [19], the main idea is to compare the AC manifold with its tangent cone at infinity. A lemma of Cheeger-Tian [12], roughly speaking, says that Theorem 1.1 is true on any cone with nonnegative Ricci curvature. The comparison is then carried out using weighted function spaces and the polynomial convergence of the metrics. In general we do not have such linear theory, and the tangent cone at infinity might have non-isolated singularities.

One approach to tackle the general case of maximal volume growth would be to show that the spaces of harmonic functions with polynomial growth on the manifold have the same dimensions as the corresponding spaces on any tangent cone at infinity, and that the exterior derivative mapping subquadratic harmonic functions to sublinear harmonic 1-forms is surjective. In general, this is only possible if we assume unique tangent cone at infinity. However in the Kähler case, tangent cones at infinity are affine algebraic [40]. As a consequence, part of the harmonic spectrum is discrete, and this is sufficient for establishing Theorem 1.1 in the Kähler case. We refer to [15, sec. 4.2] for a proof following this line of thought.

In this paper we follow another approach, which is to show that any sublinear polynomial growth harmonic 1-form is both closed and co-closed. Once this is shown, then a result of Anderson [4] showing that $H^1(M) = 0$ implies that such harmonic 1-forms must be exact. As an analogue of the proof of the Liouville theorem for harmonic functions [14], we prove this Liouville-type theorem for harmonic 1-forms by establishing the following $L^2$ analogue of the Cheng-Yau gradient estimate [13]:

**Theorem 1.4.** Let $B(p, 2)$ be a metric ball with $\text{Vol}(B(p, 1)) \geq v > 0$ and $|\text{Ric}| \leq 1$. Then for any $\delta > 0$, there exists a constant $C > 0$ depending on $\delta$ and $v$ such that for any harmonic 1-form $u$ on $B(p, 1)$, we have

$$\int_{B(p, r)} |du|^2 + |d^* u|^2 \leq Cr^{-2\delta} \int_{B(p, 1)} |u|^2$$

for any $r \in (0, 1/2]$. 
Remark 1.5.

(1) Actually, we have a pointwise bound for $|d^* u|^2$ in terms of the $L^2$ average of $u$. See Remark 5.4.
(2) The above estimate misses the traceless symmetric part of the gradient $\nabla u$, but is enough for our purpose.

By a rescaling argument, we obtain Theorem 1.1 from Theorem 1.4. Another immediate corollary of Theorem 1.4, which might be of independent interest, is the following $L^2$ estimate of the complex Hessian of a harmonic function:

**Theorem 1.6.** Let $(X, \omega)$ be a Kähler manifold. Let $B(p, 2)$ be a metric ball in $X$ with $\text{Vol}(B(p, 1)) > \nu > 0$ and $|\text{Ric}| \leq 1$. Then for any $\delta > 0$, there exists a constant $C > 0$, depending on $\delta$ and $\nu$, such that for any harmonic function $f$ on $B(p, 1)$, we have

$$\int_{B(p, r)} |\bar{\partial} \partial f|^2 \leq Cr^{-2\delta} \int_{B(p, 1)} f^2$$

for any $r \in (0, 1/2]$.

Remark 1.7. The counterexample of the Taub-NUT manifold, as discussed in Remark 1.3, shows that the dependence on the volume lower bound $\nu$ in both Theorem 1.4 and Theorem 1.6 cannot be removed. This is because we can apply a rescaling argument to obtain Theorem 1.1 from either Theorem 1.4 or Theorem 1.6.

The proof of Theorem 1.4, roughly speaking, reduces to a statement about harmonic 1-forms on cones by Cheeger-Colding theory [7]. Thus the Cheeger-Tian lemma mentioned above plays a crucial role. Another technical ingredient is the monotonicity of frequency functions of harmonic forms on Ricci-flat cones (Proposition 3.2), which implies $L^2$ three circle theorems (Theorem 3.4) for harmonic functions and 1-forms. The frequency function is a direct generalization of the one defined in Colding-Minicozzi [17] on cones for the function case, which in turn is a generalization of the original version on $\mathbb{R}^n$ defined by Almgren [1]. Frequency functions have been studied and applied intensively [2, 27, 30], and so are various three-circle theorems [24, 26, 39, 48]. The relationship between the monotonicity of frequency functions and the three-circle theorem is made precise and used in Lin [38].

This paper is organized as follows. In Section 2, we prove preliminary results about harmonic forms on limit Ricci-flat cones. In Section 3, we prove the monotonicity of Almgren’s frequency functions of harmonic $k$-forms on a limit Ricci-flat cone. In Section 4, we prove the aforementioned lemma of Cheeger-Tian [12] in our singular setting. Finally, in Section 5, we prove Theorem 1.4, and deduce Theorem 1.1 and Theorem 1.2 as corollaries.

We end this introduction by reviewing the notion of tangent cones at infinity. Let $(M, g)$ be a complete noncompact Riemannian manifold $M$ with nonnegative Ricci curvature and maximal volume growth, and let $p \in M$ be a fixed point. Given a sequence of positive number $r_i \to \infty$, we consider the sequence of pointed Riemannian manifolds $(M_i, p_i, g_i) = (M, p, r_i^{-2}g)$. By the Gromov compactness theorem [29], after passing to a subsequence, the sequence $(M_i, p_i, g_i)$ converges to a complete metric space $(T_\infty, d)$ in the pointed
Gromov-Hausdorff sense. Cheeger-Colding theory [8] tells us that the limit space \((T_\infty, d)\) is actually a metric cone. We call \((T_\infty, d)\) a tangent cone at infinity of \(M\). If the metric \(g\) is Ricci-flat, then the results of Anderson [3] and Cheeger-Naber [11] combined show the following:

- the regular set is open and dense,
- the singular set of \(T_\infty\) is closed of Minkowski codimension at least 4, and
- on the regular set, the metrics \(g_i\) converge in \(C^\infty\) to a Ricci-flat metric \(g_\infty\).

The above regularity properties, except for the Minkowski content estimate, were also proved in the Kähler setting by Cheeger-Colding-Tian [9]. Note that when we only assume nonnegative Ricci curvature, the tangent cones at infinity are in general not unique [42]. However, if the metric is Ricci-flat and if one tangent cone at infinity is smooth, then it is unique by Colding-Minicozzi [18].

A remark on notations. In this paper we always use the Hodge Laplacian \(\Delta = dd^* + d^*d\) for arbitrary \(k\)-forms, including \(k = 0\), so in particular the eigenvalues are nonnegative. A harmonic \(k\)-form is a \(k\)-form \(u\) such that \(\Delta u = 0\). What we mean by subquadratic is in the sense of \(O(r^s)\) for some \(s < 2\). The same goes for the term sublinear.

2  |  ANALYSIS ON LIMIT RICCI-FLAT CONES

Before moving on, we shall remark that all the results in this section also hold for smooth Riemannian cones.

**Definition 2.1.** We define a limit Ricci-flat cone \(C(Y)\) to be a metric cone over a compact metric space \(Y\) such that \(C(Y)\) is itself a pointed Gromov-Hausdorff limit of a noncollapsing sequence of complete Riemannian manifolds \((M_i, p_i, g_i)\) with \(|\text{Ric}(g_i)| \leq \varepsilon_i \to 0\).

In particular, a tangent cone at infinity of a Ricci-flat manifold with maximal volume growth is a limit Ricci-flat cone. The regularity results mentioned in the introduction enable us to extend various results in the smooth case to our singular setting. We remark that the condition \(\text{Ric}(g_i) \to 0\) in the definition, though weaker than the condition mentioned in the introduction, is strong enough to ensure that the limit metric is smooth and Ricci-flat on the regular set. This follows from elliptic regularity of the Ricci curvature equation in harmonic coordinates. See for example the argument in the proof of [5, main lemma 2.2].

A key consequence of these regularity results is the existence of good cutoff functions. First, let us note that a limit Ricci-flat cone \(C(Y)\) is an RCD\(^*\)\((0, m)\)-space and the cross section \(Y\) is an RCD\(^*\)\((m - 2, m - 1)\)-space [34], where \(m\) is the (real) dimension of \(C(Y)\). RCD spaces are generalizations of limits of complete Riemannian manifolds with Ricci curvature bounded from below. See [28] for an introduction. We have the following lemma.

**Lemma 2.2** Mondino-Naber [41]. Let \(X\) be a RCD\(^*\)\((K, N)\)-space for some \(K \in \mathbb{R}\) and \(N \in (1, \infty)\). Then for every \(x \in X\), \(R > 0\), \(0 < r < R\), there exists a Lipschitz function \(\psi^r : X \to \mathbb{R}\) satisfying
• supp $\psi^r \subset B(x, r)$.
• $\psi^r = 1$ on $B(p, r/2)$,
• $r^2|\Delta \psi^r| + r|\nabla \psi^r| \leq C(K, N, R)$.

Using the existence of good cutoff functions, we can construct cutoff functions that allow us to do analysis on spaces with codimension 4 singularities.

**Lemma 2.3.** Let $C(Y)$ be a limit Ricci-flat cone. Denote $\Sigma$ the singular set of $C(Y)$. Fix $p \in C(Y)$. Then for any $\epsilon > 0$, there exists a cutoff function $\phi_\epsilon$ on $B(p, 1)$ such that:

• supp $\phi_\epsilon \subset \Sigma_\epsilon$, where $\Sigma_\epsilon$ is the $\epsilon$-neighborhood of $\Sigma$,
• $\phi_\epsilon = 1$ in a neighborhood of $\Sigma$,
• $\|\nabla \phi_\epsilon\|_{L^2} \to 0$ as $\epsilon \to 0$,
• $\|\Delta \phi_\epsilon\|_{L^1} \to 0$ as $\epsilon \to 0$,
• $\|\Delta \phi_\epsilon\|_{L^2} < C$ for some constant $C > 0$ not depending on $\epsilon$.

The exact same construction also holds on the cross section $Y$.

**Proof.** The proof follows from a standard covering argument. See, for example, the proof of proposition 3.5 in [25].

We briefly recall some basic facts about the geometry of cones. Let $C(Y)$ be a limit Ricci-flat cone of dimension $m$. Thanks to the regularity results and the existence of good cutoff functions, we can do analysis on the regular set $R$ of $C(Y)$ or the regular set of $Y$ as in the smooth case. In the rest of this paper, unless otherwise stated, tensors, as well as operators acting on them, are defined on the regular set $R$. The Hodge Laplacian $\Delta : \Omega^k(R) \to \Omega^k(R)$ is defined as

$$\Delta = d^* d + dd^*.$$ 

Recall the Bochner formula. Let $\omega$ be a differential $k$-form. Then we have

$$\Delta \omega = \nabla^* \nabla \omega + \mathcal{R}(\omega),$$

where $\mathcal{R}$ is a 0th-order self-adjoint differential operator defined using the Riemann curvature tensor. If $k = 1$, then $\mathcal{R} = \text{Ric}$ is just the Ricci tensor.

Let $i, j, k$, etc., denote the indices of local coordinates on the regular part of the cross section $Y$. The coordinate vector fields $\delta_i$ on $Y$ extend trivially to vector fields on $R$. The cone metric on $R$ is given by $g_{C(Y)} = dr^2 + r^2 g_Y$. Here $r$ is the radial coordinate. The Christoffel symbols of $g_{C(Y)}$ with respect to these coordinates can be calculated as

$$\Gamma^k_{ij} = (\Gamma^Y)_{ij}^k, \quad \Gamma^k_{ir} = \frac{1}{r} \delta^k_i, \quad \Gamma^r_{ij} = -rg^Y_{ij}, \quad \Gamma^r_{ir} = \Gamma^i_{rr} = \Gamma^r_{rr} = 0.$$
Using these, the Riemann curvature tensor can be calculated as follows:

\[ R(\partial_i, \partial_r) = 0, \]
\[ R(\partial_i, \partial_j) \partial_k = R^Y(\partial_i, \partial_j) \partial_k - g^Y_{jk} \partial_i + g^Y_{ik} \partial_j, \]
\[ R(\partial_i, \partial_j) \partial_r = 0. \]

We note that the Riemann curvature tensor is homogeneous of degree \(-2\):

\[ \nabla_{\partial_r} R = -2R. \]

This follows from the fact that

\[ \nabla_{\partial_r} \partial_i = \partial_i, \quad \nabla_{\partial_r} dx^i = -dx^i. \]

We need the following definition:

**Definition 2.4.** Let \( C(Y) \) be a limit Ricci-flat cone, and let \( \mathcal{R} \) denote the regular set of \( C(Y) \). Let \( u \) be a harmonic \( k \)-form on \( \mathcal{R} \). We say \( u \) is locally \( L^p \) (resp., locally \( W^{1,2} \)) and write \( u \in L^p_{\text{loc}} \) (resp., \( u \in W^{1,2}_{\text{loc}} \)) if for any \( p \in C(Y) \), regular or not, and for any \( r > 0 \), we have

\[ \int_{B(p, r) \cap \mathcal{R}} |u|^p < \infty, \]
\[ \left( \text{resp.,} \int_{B(p, r) \cap \mathcal{R}} |u|^2 + \int_{B(p, r) \cap \mathcal{R}} |\nabla u|^2 < \infty \right). \]

For \( k \)-forms on a smooth Riemannian cone (not necessarily Ricci-flat), the same definition follows.

If the curvature operator \( \mathfrak{R} \) on \( C(Y) \) is nonnegative, then by the Bochner formula \(|u|\) is subharmonic for any harmonic \( k \)-form \( u \). We can then deduce locally \( L^\infty \) from locally \( L^1 \) or locally \( L^2 \) by the following mean value inequality on limit Ricci-flat cones.

**Proposition 2.5.** Let \( C(Y) \) be a limit Ricci-flat cone of dimension \( m \). Then there exists a constant \( C > 0 \) depending on \( m \) such that the following holds. Let \( f \in L^2_{\text{loc}} \cap C^\infty(\mathcal{R}) \) be a subharmonic function, that is, \(-\Delta f \geq 0 \) on \( \mathcal{R} \). Then for any \( x \) in \( \mathcal{R} \), we have

\[ f(x) \leq C \int_{B(x, 1)} |f| \leq C \sqrt{\int_{B(x, 1)} f^2}. \]

**Proof.** For a proof on smooth manifolds, see [35, theorem 1.2]. Fix \( x \in \mathcal{R} \) a regular point. Let \( H(x, y, t) \) be the heat kernel on \( C(Y) \). Since \( H \) is the uniform limit of the heat kernels on manifolds along the sequence [23], \( H \) satisfies the Gaussian bounds and time derivative bounds in [43, theorem 5.4.12]. Using the Li-Yau inequality [36], we also obtain a Gaussian upper bound for
In sum, we have bounds
\[ |H(x, y, t)| + \sqrt{t} |\nabla_y H(x, y, t)| \leq \frac{C'}{\text{Vol}(B(x, \sqrt{t}))} \exp \left( - \frac{c'd(x, y)^2}{t} \right) \]
for \( t \in (0, 1] \). Here \( C', c' > 0 \) are dimensional constants. Let \( \eta \) be a cutoff function on \( B(x, 1) \) given in Lemma 2.2 with \( r = 1/2 \). Recall that \( \|\nabla\eta\|_{L^\infty}, \|\Delta\eta\|_{L^\infty} < C \) where \( C > 0 \) depends only on the dimension \( m \). Now we compute
\[
\frac{\partial}{\partial t} \int_{B(x,1)} Hf\eta = \int_{B(x,1)} (-\Delta H)f\eta
\]
\[
= \int_{B(x,1)} -f\Delta(H\eta) + fH\Delta\eta - 2f\nabla\eta \cdot \nabla H
\]
\[
\geq - \int_{B(x,1)} f\Delta(H\eta) - C''\int_{B(x,1)} |f|, \quad (2.1)
\]
where \( C'' > 0 \) is a dimensional constant. Here in the last inequality we have used the uniform bounds for \( \nabla\eta, \Delta\eta, H, \) and \( VH \) on the annulus \( A(x, 1/2, 1) \), which contains the support of \( \nabla\eta \) and \( \Delta\eta \), and volume comparison.

It remains to conclude that
\[
- \int_{B(x,1)} f\Delta(H\eta) \geq 0 \quad (2.2)
\]
for all \( t > 0 \). Let \( \phi_\varepsilon \) be the cutoff function given in Lemma 2.3, and set \( \psi_\varepsilon = (1 - \phi_\varepsilon) \). Recall that \( \|\nabla\psi\|_{L^2} \to 0 \) and \( \|\Delta\psi\|_{L^2} \leq C \) for a constant \( C \) independent of \( \varepsilon \). We compute
\[
- \int_{B(x,1)} \psi_\varepsilon f\Delta(H\eta)
\]
\[
= - \int_{B(x,1)} f\Delta(\psi_\varepsilon H\eta) - \int_{B(x,1)} fH\eta\Delta\psi_\varepsilon - 2\int_{B(x,1)} f\nabla(H\eta) \cdot \nabla\psi_\varepsilon
\]
\[
\geq - \int_{B(x,1)} fH\eta\Delta\psi_\varepsilon - 2\int_{B(x,1)} f\nabla(H\eta) \cdot \nabla\psi_\varepsilon
\]
Here in the second inequality we perform an integration by parts and use the fact that \(-\Delta f \geq 0\). Using the Cauchy-Schwarz inequality and the properties of \( \psi_\varepsilon \), we conclude (2.2) by letting \( \varepsilon \to 0 \).

Finally, integrating (2.1) on \([0,1]\) and using the Gaussian upper bound again, we get
\[
C'\int_{B(x,1)} f \geq \int_{B(x,1)} H(x, y, 1)f\eta \geq f(x) - C''\int_{B(x,1)} |f|.
\]
This concludes the proof. \( \square \)
Proposition 2.6. On a limit Ricci-flat cone $C(Y)$ with nonnegative curvature operator $\mathcal{R}$, a harmonic $k$-form which is locally $L^\infty$ is locally $W^{1,2}$.

Proof. Let $\eta$ be a cutoff function given in Lemma 2.2 on $B_2$ such that $\eta = 1$ on $B_1$. Let $\phi_\varepsilon$ be a cutoff function given in Lemma 2.3. For a locally $L^\infty$ harmonic $k$-form $u$,

$$\int_{B_1} (1 - \phi_\varepsilon) |\nabla u|^2 \leq \int_{B_2} \eta (1 - \phi_\varepsilon) |\nabla u|^2 \leq \int_{B_2} -\frac{1}{2} \eta (1 - \phi_\varepsilon) \Delta |u|^2$$

$$= \int_{B_2} -\frac{1}{2} [(1 - \phi_\varepsilon) \Delta \eta + 2 \nabla \eta \cdot \nabla \phi_\varepsilon - \eta \Delta \phi_\varepsilon] |u|^2$$

$$\leq C \int_{B_2} (1 - \phi_\varepsilon) |u|^2 + C \int_{B_2} |\nabla \phi_\varepsilon||u|^2 + C \int_{B_2} |\Delta \phi_\varepsilon||u|^2.$$

By the Cauchy-Schwarz inequality and locally $L^\infty$ assumption on $u$, the last two terms on the right-hand side vanish as $\varepsilon \to 0$. \hfill \Box

The following is needed in order to bound $|\nabla_{r\partial} u|$.

Proposition 2.7. Let $C(Y)$ be a limit Ricci-flat cone with nonnegative curvature operator $\mathcal{R}$. Then $|\nabla_{r\partial} u|$ is subharmonic for any harmonic $k$-form $u$ on $C(Y)$.

Proof. Using the Bochner formula and the cone structure of $C(Y)$, by a straightforward calculation, we get

$$-\Delta |\nabla_{r\partial} u|^2 = 2\mathcal{R}(\nabla_{r\partial} u, \nabla_{r\partial} u) + 2|\nabla \nabla_{r\partial} u|^2 \geq 2|\nabla \nabla_{r\partial} u|^2.$$

Now set $h = |\nabla_{r\partial} u|$. From the above we have

$$-\Delta h^2 = 2|\nabla h|^2 - 2h^2 \geq 2|\nabla h|^2,$$

and so $-\Delta h \geq 0$. Let $p \in \mathcal{R}$. If $h(p) = 0$, then necessarily $-\Delta h(p) \geq 0$, for $p$ is a minimum. If $h(p) > 0$, then we also have $-\Delta h(p) \geq 0$. The result follows. \hfill \Box

Lemma 2.8. Let $C(Y)$ be a limit Ricci-flat cone and let $u$ be a locally $L^\infty$ harmonic 1-form. Then locally $\sup |\nabla_{r\partial} u| < \infty$.

Proof. This follows from Proposition 2.5, Proposition 2.6, and Proposition 2.7. \hfill \Box

3 | MONOTONICITY OF THE FREQUENCY FUNCTION

Fix for now a (smooth) Riemannian cone $C(Y)$ of real dimension $m \geq 2$. Later on, we will generalize what we obtained for smooth cones to limit Ricci-flat cones using cutoff functions. In this section, $B_r$ will denote the open ball of radius $r$ centered at the vertex of $C(Y)$, and
∂B_r = \{r\} \times Y will denote the boundary of B_r. Let u be a harmonic k-form such that locally 
\sup |u| + \sup |\nabla_{r3} u| < \infty. Define

\[ D(r) = \int_{B_r} |\nabla u|^2 + \Re(u, u) \quad \text{and} \quad H(r) = \int_{\partial B_r} |u|^2. \]

In analogy with the frequency of harmonic functions [31], we define

\[ N(r) = \frac{rD(r)}{H(r)} \]

whenever \( H(r) > 0 \). \( N(r) \) is called the frequency function of \( u \).

By the Bochner formula,

\[ \frac{1}{2} \nabla \cdot \nabla |u|^2 = |\nabla u|^2 + \Re(u, u). \]

So

\[ D(r) = \frac{1}{2} \int_{B_r} \nabla \cdot \nabla |u|^2 = \int_{\partial B_r} u \cdot \nabla_{r3} u, \quad (3.1) \]

Note that unlike in the \( \mathbb{R}^m \) case, the vertex of the cone is a singular point. To justify the integration by parts in (3.1), we can cut out a small ball centered at the vertex of the cone and then letting the radius of the small ball go to 0. Hence we need the assumption that locally \( \sup |u| + \sup |\nabla_{r3} u| < \infty \).

We now prove the monotonicity of frequency functions.

**Proposition 3.1.** Let \( C(Y) \) be a smooth Riemannian cone of dimension \( m \geq 2 \), let \( u \) be a harmonic k-form on \( C(Y) \) such that locally \( \sup |u| + \sup |\nabla_{r3} u| < \infty \), and let \( N(r) \) be the frequency function of \( u \). Then \( N'(r) \geq 0 \). \( N(r) \) is constant if and only if

\[ \nabla_{r3} u = h(r)u \]

for some function \( h(r) \).

The following proof is a slight modification of the case of harmonic functions on \( \mathbb{R}^m \). See [31] for comparison.

**Proof.** By direct differentiation,

\[ N'(r) = N(r) \left\{ \frac{1}{r} + \frac{D'(r)}{D(r)} - \frac{H'(r)}{H(r)} \right\}. \]

So our goal is to show that

\[ \frac{1}{r} + \frac{D'(r)}{D(r)} - \frac{H'(r)}{H(r)} \geq 0. \]
First we calculate $D'(r)$.

$$D'(r) = \int_{\partial B_r} |\nabla u|^2 + \Re(u, u) = \frac{1}{r} \int_{\partial B_r} (|\nabla u|^2 + \Re(u, u)) r \partial_r \cdot \partial_r$$

$$= \frac{1}{r} \int_{B_r} \nabla \cdot (|u|^2 + \Re(u, u)) r \partial_r = \frac{1}{r} \int_{B_r} \nabla \cdot (|u|^2 + \Re(u, u)) r \partial_r.$$

The divergence inside the integral can be calculated as follows. Fix normal coordinates on $Y$, and denote the indices of the normal coordinates by $i, j$. In particular, $\partial_r, r^{-1} \partial_i$ form a local orthonormal frame on $C(Y)$. We have

$$\nabla \cdot (|\nabla u|^2 + \Re(u, u)) r \frac{\partial}{\partial r} = \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial r} \left( \sqrt{\det g} \left( |\nabla u|^2 + \Re(u, u) \right) r \right)$$

$$= m (|\nabla u|^2 + \Re(u, u)) + r \partial_r (|u|^2 + \Re(u, u)).$$

For the third term on the right-hand side, we have

$$r \partial_r |\nabla u|^2 = r \partial_r (|\nabla \partial_r u|^2 + r^{-2} |\nabla_i u|^2)$$

$$= 2r \nabla \partial_r \nabla \partial_r u \cdot \nabla \partial_r u - 2(|\nabla u|^2 - |\nabla \partial_r u|^2) + 2r^{-2} \nabla \partial_r \nabla_i u \cdot \nabla_i u$$

$$= 2 \nabla \partial_r \nabla \partial_r u \cdot \nabla \partial_r u - 2r \partial_r (|\nabla u|^2 + 2r^{-2} \nabla \partial_r \nabla_i u \cdot \nabla_i u).$$

Using the fact that $\nabla \partial_r \nabla_i = \nabla_i \nabla \partial_r$ and combining the first and third terms on the right-hand side, we get

$$r \partial_r |\nabla u|^2 = 2 \nabla \partial_r \nabla_i u \cdot \nabla u - 2 |\nabla u|^2.$$

By homogeneity of the curvature tensor, we also have

$$r \partial_r \Re(u, u) = -2 \Re(u, u) + 2 \Re(\nabla \partial_r \nabla_i u, u).$$

Combining these, the divergence term gives

$$\nabla \cdot (|\nabla u|^2 + \Re(u, u)) r \partial_r = (m - 2) (|\nabla u|^2 + \Re(u, u))$$

$$+ 2 (\nabla \partial_r \nabla_i u \cdot \nabla u + \Re(\nabla \partial_r \nabla_i u, u)).$$

Finally,

$$D'(r) = \frac{m - 2}{r} D(r) + \frac{2}{r} \int_{B_r} \nabla \partial_r \nabla_i u \cdot \nabla u + \Re(\nabla \partial_r \nabla_i u, u)$$

$$= \frac{m - 2}{r} D(r) + 2 \int_{\partial B_r} |\nabla \partial_r u|^2.$$  (3.2)

The last equality follows from the Bochner formula and $u$ being harmonic.
Next we work on $H(r)$.

\[ H'(r) = \frac{(m-1)}{r} H(r) + 2 \int_{\partial B_r} u \cdot \nabla_{\partial_r} u. \]

So

\[
\frac{1}{r} + \frac{D'(r)}{D(r)} = \frac{H'(r)}{H(r)} = 2 \left( \frac{\int_{\partial B_r} |\nabla_{\partial_r} u|^2}{\int_{\partial B_r} u \cdot \nabla_{\partial_r} u} - \frac{\int_{\partial B_r} u \cdot \nabla_{\partial_r} u}{\int_{\partial B_r} |u|^2} \right) \geq 0
\]

by the Cauchy-Schwarz inequality. Equality holds if and only if

\[ \nabla_{r_\partial} u = h(r) u \]

for some function $h(r)$. \qed

We now turn to the singular setting. Suppose now that $C(Y)$ is a limit Ricci-flat cone. By defining $D(r) = \int_{\partial B_r} u \cdot \nabla_{\partial_r} u$ as in (3.1), we see that $D(r)$ is well-defined provided that the harmonic $k$-form $u$ is such that locally $\sup |u| + \sup |\nabla_{r_\partial} u| < \infty$. The integration by parts in (3.1) is then justified using cutoff functions in Lemma 2.3. The precise argument is similar to the proof of the proposition below, so we omit it. We now generalize Proposition 3.1 to the case of singular cross sections.

**Proposition 3.2.** Let $C(Y)$ be a limit Ricci-flat cone of dimension $m \geq 2$. Let $u$ be a harmonic $k$-form on $C(Y)$ such that locally $\sup |u| + \sup |\nabla_{r_\partial} u| < \infty$. Then $N(r)$ is well-defined and differentiable. Moreover, $N'(r) \geq 0$. $N(r)$ is constant if and only if $\nabla_{r_\partial} u = h(r) u$ for some function $h(r)$.

**Proof.** We need to show that both $H(r)$ and $D(r)$ are differentiable, using cutoff functions. We will focus on $D(r)$ as it is more involved. Let $\phi_\varepsilon$ be the cutoff function on $Y$ given in Lemma 2.3. Define

\[ u_\varepsilon = (1 - \phi_\varepsilon) u, \]

and define $D_\varepsilon(r)$ to be $D(r)$ with $u$ replaced by $u_\varepsilon$. From a direct calculation and using the properties of the cutoff functions $\phi_\varepsilon$, we see that $\lim_{\varepsilon \to 0} D_\varepsilon(r) = D(r)$. The goal is to show that $D'(r)$ exists by showing that $D'(r) = \lim_{\varepsilon \to 0} D'_\varepsilon(r)$. We now calculate $D'_\varepsilon(r)$. From (3.2), we have

\[ D'_\varepsilon(r) = \frac{m-2}{r} D_\varepsilon(r) + 2 \int_{B_r} \nabla \nabla_{r_\partial} u_\varepsilon \cdot \nabla u_\varepsilon + R(\nabla_{r_\partial} u_\varepsilon, \nabla_{r_\partial} u_\varepsilon). \]
We cannot apply (3.3) since $u_\varepsilon$ is no longer harmonic. To estimate the second term on the right-hand side, we integrate by parts and use the Bochner formula:

$$\frac{1}{r} \int_{B_r} \nabla \nabla \partial_r u_\varepsilon \cdot \nabla u_\varepsilon + \Re(\nabla \partial_r u_\varepsilon, u_\varepsilon)$$

$$= \int_{\partial B_r} |\nabla \partial_r u_\varepsilon|^2 + \frac{1}{r} \int_{B_r} \nabla \partial_r u_\varepsilon \cdot \nabla^* \nabla u_\varepsilon + \Re(\nabla \partial_r u_\varepsilon, u_\varepsilon)$$

$$= \int_{\partial B_r} (1 - \phi_\varepsilon)^2 |\nabla \partial_r u|^2 - \frac{1}{r} \int_{B_r} (1 - \phi_\varepsilon) \Delta \phi_\varepsilon \nabla \partial_r u \cdot u$$

$$+ \frac{2}{r} \int_{B_r} \nabla \partial_r u \cdot ((1 - \phi_\varepsilon) \nabla \phi_\varepsilon \cdot \nabla u).$$

Here the integration by parts in the first equality is justified using the fact that locally $\sup |u| + \sup |\nabla \partial_r u| < \infty$ as in (3.1). To conclude that $\lim_{\varepsilon \to 0} D'_\varepsilon(r)$ exists and satisfies the right equation, it is enough to show that the last two terms on the right-hand side vanish as $\varepsilon \to 0$. By the Cauchy-Schwarz inequality and the assumptions that locally $\sup |u| + \sup |\nabla \partial_r u| < \infty$, the problem reduces to showing that

$$\lim_{\varepsilon \to 0} \int_Y |\nabla \phi_\varepsilon|^2 = 0, \quad \lim_{\varepsilon \to 0} \int_Y |\Delta \phi_\varepsilon| = 0.$$

But these are the properties of $\phi_\varepsilon$. In sum, we have shown that

$$\lim_{\varepsilon \to 0} D'_\varepsilon(r) = \frac{m - 2}{r} D(r) + \int_{\partial B_r} |\nabla \partial_r u|^2 = D'(r).$$

The rest follows from the proof of Proposition 3.1.

The monotonicity of the frequency implies the following $L^2$ three-circle theorem:

**Theorem 3.3.** Let $C(Y)$ be a limit Ricci-flat cone of dimension $m \geq 2$. Let $u$ be a harmonic 1-form on $C(Y)$ such that locally $\sup |u| + \sup |\nabla \partial_r u| < \infty$. Then $H(r)$ is log-convex with respect to $\log r$. Equality holds if and only if $\nabla \partial_r u = h(r) u$ for some function $h(r)$.

**Proof.**

$$\frac{d \log H(r)}{d \log r} = \frac{r H'(r)}{H(r)} = (n - 1) + 2N(r).$$

So

$$\frac{d^2 \log H(r)}{d(\log r)^2} = 2 \frac{dN(r)}{d \log r} = 2r N'(r) \geq 0.$$
Let

\[ F(r) = \int_{B_r} u^2. \]

Integrating \( H(r) \), we see that \( F(r) \) also satisfies the three-circle theorem:

**Theorem 3.4.** Let \( C(Y) \) be a limit Ricci-flat cone of dimension \( m \geq 2 \). Let \( u \) be a harmonic 1-form on \( C(Y) \) such that locally \( \sup |u| + \sup |
\n\n∇_r \partial_r u| < \infty \). Then

\[ F(r) = \int_{B_r} u^2 \]

is log-convex with respect to \( \log r \). Equality holds if and only if \( \nabla_{r_3} u = h(r)u \) for some function \( h(r) \).

**Proof.** Let \( 0 < r_1 < r < r_2 \) and let

\[ \frac{1}{p} = \log \frac{r_2 - r}{\log r_2 - \log r_1}, \quad \frac{1}{q} = \log \frac{r_2 - r_1}{\log r_2 - \log r_1}. \]

Then we compute

\[ F(r) = \int_{0}^{r} H(s)ds \]

\[ \leq \int_{0}^{r} H(r_1 s/r)^{1/p} H(r_2 s/r)^{1/q}ds \]

\[ \leq \left( \int_{0}^{r} H(r_1 s/r)ds \right)^{1/p} \left( \int_{0}^{r} H(r_2 s/r)ds \right)^{1/q} \]

\[ = F(r_1)^{1/p} F(r_2)^{1/q} \left( \frac{r}{r_1} \right)^{1/p} \left( \frac{r}{r_2} \right)^{1/q} \]

\[ = F(r_1)^{1/p} F(r_2)^{1/q}. \]

\[ \square \]

4 | **HOMOGENEOUS HARMONIC 1-FORMS**

Let us now focus on the case when the frequency function is constant. In the case of harmonic 0-forms, that is, when \( u \) is a harmonic function, the homogeneous condition \( \nabla_{r_3} u = h(r)u \) is equivalent to

\[ u = r^s g, \]

where \( g \) is an eigenfunction on the cross section \( Y \) with eigenvalue \( s(s + m - 2) \). Thus the set of possible degrees \( s \), denoted as \( D(C(Y)) \), is determined by the spectrum of the cross section \( Y \). The following lemma, based on a lemma of Cheeger-Tian [12, lemma 7.27], characterizes the homogeneous condition in the case of harmonic 1-forms.
Lemma 4.1. Let $u$ be harmonic 1-form on a limit Ricci-flat cone $C(Y)$ of dimension $m$ such that locally $\sup |u| + |\nabla_{r^2} u| < \infty$. Suppose

$$\nabla_{r^2} u = h(r)u$$

for some function $h(r)$. Then up to linear combination, $u$ can be written as one of the following:

(I) $u = rdr$ or $r^{-(m-1)}dr$.

(II) $u = d(r^{s+1}g(x))$, where $g(x)$ is an eigenfunction on $Y$ with eigenvalue $(s+1)(s+m-1)$.

(III) $u = r^s g(x) dr - \frac{r^{s+1}}{s+m-3} dg(x)$, where $g(x)$ is an eigenfunction on $Y$ with eigenvalue $(s-1)(s+m-3)$.

(IV) $u = r^{s+1}\eta(x)$, where $\eta(x)$ is a co-closed eigen 1-form on $Y$ with eigenvalue $(s+1)(s+m-3)$.

(V) $u = (\log r) d(r^{-(m-4)/2}g(x))$, where $g(x)$ is an eigenfunction on $Y$ with eigenvalue $(-m^2/4) + m$.

(VI) $u = r^{-(m-2)/2}(\log r)\eta(x)$, where $\eta(x)$ is a cocilosed eigen 1-form on $Y$ with eigenvalue $(-m^2/4) + 2m - 4$.

Note that type (V), (VI) are ruled out by the bounds on $u$. They are listed here for completeness. If $u$ is one of the above types, the power of $r$ in $|u|$ is called the growth rate of $u$. In (I), the growth rate is 1 or $-(m-1)$. In (II), (III), and (IV), the growth rate is $s$.

Proof. Let $C = C(Y)$ be a limit Ricci-flat cone of real dimension $m$. Any 1-form $u$ on $C$ can be written as

$$u = \kappa(r, x) dr + \eta(r, x), \tag{4.1}$$

where $\eta$ is the part tangent to the cross section $Y$. Following the calculation in appendix B of Hein-Sun [32], denote prime as the derivative with respect to $r$ and denote anything with a tilde the operators on $Y$. The operators and tensors on $Y$ extend trivially to $C(Y)$ by scaling. We have

$$\Delta u = \left(\frac{1}{r^2} \Delta \kappa - \kappa'' - \frac{m-1}{r} \kappa' + \frac{m-1}{r^2} \kappa - \frac{2}{r^3} d^* \eta\right) dr$$

$$+ \frac{1}{r^2} \Delta \eta - \eta'' - \frac{m-3}{r} \eta' - \frac{2}{r} d\kappa.$$ 

Suppose $\Delta u = 0$. Then this is equivalent to

$$\frac{1}{r^2} \Delta \kappa - \kappa'' - \frac{m-1}{r} \kappa' + \frac{m-1}{r^2} \kappa - \frac{2}{r^3} d^* \eta = 0 \tag{4.2}$$

and

$$\frac{1}{r^2} \Delta \eta - \eta'' - \frac{m-3}{r} \eta' - \frac{2}{r} d\kappa = 0. \tag{4.3}$$

Suppose $u$ satisfies the following condition:

$$\nabla_{r^2} u = h(r)u. \tag{4.4}$$
Using the decomposition \((4.1)\) and
\[
\nabla_{\partial r} dx^i = -dx^i,
\]
\[(4.4)\] becomes
\[
rx' = h x, \quad r\eta'_i = (h + 1)\eta_i,
\]
where \(\eta = \eta_i(r, x)dx^i\). We can solve these ODEs in \(r\) and get
\[
x = f(r)g(x), \quad \eta_i = f(r)rh_i(x)
\]
for some functions \(g(x), h_i(x)\) on \(Y\), where
\[
f(r) = e^{\int \frac{h(r)}{r} dr}.
\]
In sum, \(u\) can be written as
\[
u = f(r)g(x)dr + f(r)r\eta_1(x),
\]
where \(\eta_1(x)\) is a 1-form on \(Y\). Now we plug in \((4.5)\) into \((4.2)\) and \((4.3)\). After rearranging, \((4.2)\) becomes
\[
\ddot{\Delta}g - 2d^*\eta_1 g = r^2f'' + (m-1)rf' - (m-1)f = c_1,
\]
where \(c_1\) is a constant. We thus have an ODE of \(f\):
\[
r^2f''+(m-1)rf'-(c_1+m-1)f = 0.
\]
The equation of indicial roots, that is, plugging into \(f = r^s\) to the ODE, is
\[
s^2 + (m-2)s - (c_1 + m - 1) = 0.
\]
Without further assumptions on the geometry, the indicial roots could be distinct or repeated. In the former case, \(f\) can be written as a linear combination of
\[
r^s = \text{where } s_\pm = \frac{-(m-2) \pm \sqrt{m^2 + 4c_1}}{2}.
\]
In the latter case, \(f\) can be written as a linear combination of
\[
\frac{r^{m-2}}{2}, \frac{r^{m-2}}{2} \log r.
\]
Plugging the solution \(f\) into the ODE \((4.7)\) to \((4.3)\) and then canceling the \(f\)'s and \(r\)'s, we get
\[
\ddot{\Delta}\eta_1 - (c_1 + 2m - 4)\eta_1 - 2d\eta = 0.
\]
We now have a system of equations on \(Y\):
\[
\ddot{\Delta}g - 2d^*\eta_1 = c_1 g,
\]
\[(4.9)\]
Note that at this point, we see that we can decompose the harmonic 1-form $u$ according to the decomposition of $f$ into powers of $r$. So we may assume that $f = r^s$. The case when $f = r^{-(m-2)/2} \log r$ can be worked out similarly.

Taking $\tilde{d}^*$ of (4.10) and setting $g_1 = \tilde{d}^* \eta_1$, the system becomes

$$
\tilde{\Delta}g - 2g_1 = c_1 g, \quad (4.11)
$$

$$
\tilde{\Delta}g_1 - (c_1 + 2m - 4)g_1 = 2\tilde{\Delta}g. \quad (4.12)
$$

Substituting the $g_1$’s in (4.12) with (4.11) and completing the square, we get

$$
(\tilde{\Delta} - (c_1 + m))^2 g = (m^2 + 4c_1)g. \quad (4.13)
$$

Let

$$
g = \sum_{\lambda} g_\lambda
$$

be the spectral decomposition of $g$ with respect to the Laplacian $\tilde{\Delta}$. See, for example, [23] and the references therein. Then (4.13) yields the relation

$$
\lambda = (c_1 + m) \pm \sqrt{m^2 + 4c_1}
$$

$$
= (s + m - 1)(s + 1) \text{ or } (s - 1)(s + m - 3)
$$

$$
= \lambda_{\pm}.
$$

Thus

$$
g = g_+ + g_- \quad \text{where } \tilde{\Delta}g_{\pm} = \lambda_{\pm} g_{\pm}.
$$

For now, we assume that $\lambda_{\pm} \neq 0$. Any $\eta_1$ that satisfies

$$
d^* \eta_1 = \frac{\lambda_+ - c_1}{2} g_+ + \frac{\lambda_- - c_1}{2} g_-
$$

solves our system of equations. A particular solution is

$$
\frac{\lambda_+ - c_1}{2\lambda_+} d g_+ + \frac{\lambda_- - c_1}{2\lambda_-} d g_- = \frac{1}{s + 1} d g_+ - \frac{1}{s + m - 3} d g_-.
$$

Setting $\eta_2 = \eta_1 - \frac{1}{s+1} d g_+ + \frac{1}{s+m-3} d g_-$, we get

$$
u = \frac{1}{s + 1} d (r^{s+1} g_+) + r^{s+1} \eta_2 + r^s g_- dr - \frac{r^{s+1}}{s + m - 3} d g_-.
$$

(4.15)

It follows that $r^{s+1} \eta_2$ is harmonic and $\eta_2$ is a $\tilde{d}^*$-closed eigen 1-form:

$$
\tilde{\Delta} \eta_2 = (s + 1)(s + m - 3) \eta_2.
$$

(4.16)
The case when one of $\lambda_{\pm}$ is 0 can be reduced to the special case when $\lambda_{\pm} = \lambda = 0$ using the calculation above. When $\lambda = 0$, $g$ is a constant function. The case when $g = 0$ is already covered above. We may assume $g = 1$. By (4.11), (4.12), $c_1 = 0$ or $-(2m - 4)$. $c_1 = 0$ implies $s = 1$ or $-(m - 1)$. The case $c_1 = -(2m - 4)$ is not possible when $m \neq 2$, since a locally $L^\infty$, locally $W^{1,2}$ harmonic 1-form on a closed manifold with singularities in codimension 4 is automatically co-closed (and also closed). This can be seen easily using the cutoff functions in Lemma 2.3 and an integration by parts.

Remark 4.2. (1) The lemma actually holds for any smooth Riemannian cones without the bounds on $u$. (2) The main difference between the proof in [32] and our proof is that our proof avoids the use of spectral decomposition for co-closed 1-forms. This allows our proof to work in our singular setting.

A special case of Lichnerowicz theorem also holds in our singular setting.

**Lemma 4.3** [32, Lemma B.2]. Suppose $C(Y)$ is a limit Ricci-flat cone of real dimension $m \geq 3$, so that we have $\text{Ric}_Y = (m - 2)g_Y$. Let $\eta$ be a locally $L^\infty$, locally $W^{1,2}$ co-closed 1-form on $R \cap Y$. If $\Delta \eta = \lambda \eta$ for some $\lambda \in \mathbb{R}$, then $\lambda \geq 2m - 4$. When $\lambda = 2m - 4$, $\eta$ is dual to a Killing vector field. Alternatively, suppose $\eta$ is a closed 1-form with $\Delta \eta = \lambda \eta$. Then $\lambda \geq m - 1$.

**Proof.** We prove the co-closed case. The closed case is entirely similar. Let $\phi_\varepsilon$ be the cutoff function supported in the $\varepsilon$-neighborhood of the singular set of $Y$ as given in Lemma 2.3. Let $\eta$ be a co-closed eigen 1-form on (the regular set of) $Y$ with eigenvalue $\lambda$. By the Bochner formula, we compute

$$
\int_Y \lambda(1 - \phi_\varepsilon)|\eta|^2 = \int_Y \langle \Delta \eta, (1 - \phi_\varepsilon)\eta \rangle
$$

$$
= \int_Y (1 - \phi_\varepsilon)|\nabla \eta|^2 + \int_Y (m - 2)(1 - \phi_\varepsilon)|\eta|^2
$$

$$
- \int_Y \langle \nabla \eta, \nabla \phi_\varepsilon \otimes \eta \rangle.
$$

On the other hand, since $\eta$ is co-closed,

$$
\int_Y \lambda(1 - \phi_\varepsilon)|\eta|^2 = \int_Y \langle d^* d\eta, (1 - \phi_\varepsilon)\eta \rangle
$$

$$
= \int_Y (1 - \phi_\varepsilon)|d\eta|^2 - \int_Y \langle d\eta, d\phi_\varepsilon \wedge \eta \rangle.
$$

As in [32], we can decompose $\nabla \eta$ into traceless symmetric, trace, and skew-symmetric parts as

$$
|\nabla \eta|^2 = |\nabla^\text{sym}_0 \eta|^2 + \left| \frac{\text{tr}(\nabla \eta)}{m - 1} \right|^2 + |\nabla^\text{skew} \eta|^2.
$$

Note that $|\nabla^\text{skew} \eta|^2 = \frac{1}{2}|d\eta|^2$. Thus

$$
\int_Y (1 - \phi_\varepsilon)|\nabla \eta|^2 \geq \frac{1}{2} \int_Y (1 - \phi_\varepsilon)|\eta|^2,
$$

□
and we have
\[
\int_Y \lambda(1 - \phi_\epsilon)|\eta|^2 \geq \int_Y 2(m - 2)(1 - \phi_\epsilon)|\eta|^2 + 2\int_Y \langle d\eta, d\phi_\epsilon \wedge \eta \rangle \\
- 2\int_Y \langle \nabla \eta, \nabla \phi_\epsilon \otimes \eta \rangle.
\]
Thus it’s enough to show that the last two terms tend to 0 as \( \epsilon \to 0 \). But this follows from the Cauchy-Schwarz inequality, \( u \) is locally \( L^\infty \), \( u \) is locally \( W^{1,2} \) (Proposition 2.6), and
\[
\lim_{\epsilon \to 0} \int_Y |\nabla \phi_\epsilon|^2 = 0.
\]
Finally, the equality holds precisely when \( \nabla \eta \) is anti-symmetric; that is, the dual of \( \eta \) is Killing. \( \square \)

We can now rule out the unwanted parts in the decomposition in Lemma 4.1.

**Corollary 4.4.** Let \( C(Y) \) be a limit Ricci-flat cone of real dimension at least 4. Let \( u \) be a locally \( L^\infty \) harmonic 1-form on \( C(Y) \) satisfying
\[
\nabla_{r\partial r} u = h(r)u
\]
for some smooth function \( h \). Suppose further that the growth rate of \( u \) is less than 1, that is,
\[
|u| \leq C(1 + r)^\bar{s}
\]
for some \( \bar{s} < 1 \). Then \( u \) is actually exact:
\[
u = d(r^{s+1}g(x))
\]
for some \( 0 \leq s \leq \bar{s} \), where \( g \) is an eigenfunction on \( Y \) with eigenvalue \((s + 1)(s + m - 1)\). Note that \( r^{s+1}g(x) \) is a harmonic function on \( C(Y) \).

Assume furthermore that \( C(Y) \) is Kähler, that is, \( C(Y) \) is Calabi-Yau. Then any locally \( L^2 \) harmonic function \( f \) on \( C(Y) \) with
\[
|f| \leq C(1 + r)^{s'}, \quad 0 \leq s' < 2,
\]
is the real part of a holomorphic function. In particular, \( f \) is pluriharmonic.

**Proof.** Assume \( C(Y) \) is a limit Ricci-flat cone of real dimension at least 4. To prove the first part, we rule out types (III)–(VI). Types (V) and (VI) are ruled out by the assumption on \( u \). For type (III), Lichnerowicz implies that \( s \geq 2 \) or \( s \leq 2 - m \). The first case violates the growth assumption, while the second case violates the \( L^\infty_{\text{loc}} \) assumption. For type (IV), we have \( s \geq 2 \) or \( s \leq 2 - m \). Again, these are ruled out by our assumption.

Now, assume \( C(Y) \) is Calabi-Yau. Let \( f \) be a locally \( L^2 \) harmonic function on \( C(Y) \). By the spectral decomposition, we may assume \( f \) is homogeneous. Thus \( d^c f \) is a homogeneous harmonic 1-form and has growth rate less than 1. From the above we see that \( d^c f = dh \) for some harmonic function \( h \). So \( f \) is pluriharmonic. \( \square \)
5 | A LOCAL $L^2$ ESTIMATE FOR THE EXTERIOR DERIVATIVE

In this section, we prove Theorem 1.4. One key ingredient is Lemma 5.3, which roughly states that on a Ricci-flat metric ball which is close to a limit Ricci-flat cone, a harmonic 1-form that is orthogonal to exact 1-forms must grow at least linearly. This is true when the ball actually lies in a Ricci-flat cone and the 1-form is homogeneous, as we have seen in Corollary 4.4. To obtain the local $L^2$ estimate for the exterior derivative, we use the fact that all but finitely many scales in a Ricci-flat metric ball $B(p, 1)$ are close to a Ricci-flat cone in the Gromov-Hausdorff sense. This is a direct consequence of the Cheeger-Colding cone rigidity theorem [7]. At each scale $2^{-k}$ such that the manifold is actually close to a cone, we apply Lemma 5.3 to obtain a growth estimate for the orthogonal projection with respect to the $L^2$ inner product on the $(2^{-k})$-ball. Together with an integration by parts, we then concatenate these growth estimates above to conclude the proof of Theorem 1.4.

We need the following Green’s formula on limit Ricci-flat cones:

**Lemma 5.1.** Let $C(Y)$ be a limit Ricci-flat cone of dimension $m \geq 2$. Suppose $u, v$ are harmonic 1-forms on $B_r = B(o, r) \subset C(Y)$ such that locally $\sup(|u| + |\nabla \partial_r u| + |v| + |\nabla \partial_r v|) < \infty$. Then

$$\int_{\partial B_r} \langle \nabla \partial_r u, v \rangle = \int_{\partial B_r} \langle u, \nabla \partial_r v \rangle.$$

**Proof.** If the cone $C(Y)$ is smooth, then this follows from Green’s formula and the fact that both $u, v$ are harmonic. Let $\phi_\varepsilon$ be the good cutoff function on $B(o, r)$ supported outside the $\varepsilon$-neighborhood of the singular set of $B(o, r)$. Define $u_\varepsilon = \phi_\varepsilon u$. Then by Green’s formula,

$$\int_{\partial B_r} \langle \nabla \partial_r u_\varepsilon, v \rangle - \int_{\partial B_r} \langle u_\varepsilon, \nabla \partial_r v \rangle = \int_{B_r} \langle \nabla \cdot \nabla u_\varepsilon, v \rangle - \int_{B_r} \langle u_\varepsilon, \nabla \cdot \nabla v \rangle.$$

By the Bochner formula and the Ricci-flat condition, the second term of the right-hand side is 0, and the first term on the right-hand side can be computed as

$$\int_{B_r} \langle \nabla \cdot \nabla u_\varepsilon, v \rangle = \int_{B_r} \langle (-\Delta \phi_\varepsilon) u + 2 \nabla \phi_\varepsilon \cdot \nabla u, v \rangle.$$

Note that

$$\left| \int_{B_r} \langle (-\Delta \phi_\varepsilon) u, v \rangle \right| \leq C \int_{B_r} |\Delta \phi_\varepsilon| \to 0$$

as $\varepsilon \to 0$, and that

$$\left| \int_{B_r} \langle \nabla \phi_\varepsilon \cdot \nabla u, v \rangle \right| \leq C \sqrt{\int_{B_r} |\nabla \phi_\varepsilon|^2} \to 0$$

as $\varepsilon \to 0$ by the Cauchy-Schwarz inequality. □

The Green’s formula implies the following orthogonality property of homogeneous harmonic 1-forms.
**Proposition 5.2.** Let $C(Y)$ be a limit Ricci-flat cone of dimension $m$. Suppose $u, v$ are harmonic 1-forms on $B_1 = B(o, 1) \subset C(Y)$ such that both locally $\sup(|u| + |\nabla_{r \delta} u| + |v| + |\nabla_{r \delta} v|) < \infty$. Suppose furthermore that $v$ is homogeneous of degree $s$, that is,

$$\nabla_{r \delta} v = sv.$$ 

Then there exists a constant $C \in \mathbb{R}$ such that

$$\int_{\partial B_r} \langle u, v \rangle = Cr^{2s+m-1}$$

for all $r \in [0, 1]$.

**Proof.** Write

$$I(r) = \int_{\partial B_r} \langle u, v \rangle.$$ 

Then by the previous lemma and a cutoff function argument, we have

$$r \delta I(r) = \int_{\partial B_r} \langle \nabla_{r \delta} u, v \rangle + \int_{\partial B_r} \langle u, \nabla_{r \delta} v \rangle$$

$$= 2 \int_{\partial B_r} \langle u, \nabla_{r \delta} v \rangle$$

$$= 2s I(r).$$

Integrating, we get

$$I(r) = C r^{2s}$$

for some constant $C \in \mathbb{R}$. This completes the proof. $\square$

We are ready to state our key lemma.

**Lemma 5.3.** For any $\delta > 0$ and $v > 0$, there exists $\epsilon > 0$ with the following significance: let $B(p, 2)$ be a Riemannian metric ball such that $\text{Vol}(B(p, 1)) > v$ and $|\text{Ric}(B(p, 2))| < \epsilon$, and let $B(o, 2) \subset C(Y)$ be a limit Ricci flat cone such that

$$d_{GH}(B(p, 2), B(o, 2)) < \epsilon.$$ 

Let $u$ be a harmonic 1-form on $B(p, 1)$ such that $u$ is $L^2$-orthogonal to the space of closed and co-closed harmonic 1-forms on $B(p, 1)$. Then $u$ grows “almost linearly” in the following sense:

$$\int_{B(p, 1)} |u|^2 \geq 2^{2(1-\delta)} \int_{B(p,1/2)} |u|^2.$$ 

**Proof.** We argue by contradiction. Let $B(p_i, 2)$ be a sequence of Riemannian metric balls with $\text{Vol}(B(p_i, 1)) > v$ and $|\text{Ric}(B(p_i, 2))| < \epsilon_i$, and let $C(Y_i)$ be a sequence of limit Ricci-flat cones.
with vertex $o_i$. Suppose that
\[ d_{GH}(B(p_i, 2), B(o_i, 2)) < \varepsilon_i \]
for each $i$, where $\varepsilon_i \to 0$.

Suppose for contradiction that there exist $\delta > 0$ and a sequence $u_i$ of harmonic 1-forms on $B(p_i, 1)$ satisfying the $L^2$-orthogonal condition on $B(p_i, 1)$ such that
\[ \int_{B(p_i, 1)} |u_i|^2 < 2^{2(1-\delta)} \int_{B(p_i,1/2)} |u_i|^2. \]

After passing to a subsequence, we may assume that both $B(p_i, 2)$ and $B(o_i, 2)$ converge to $B(o, 2)$ in a limit Ricci-flat cone $C(Y)$. We may normalize $u_i$ so that
\[ \int_{B(p_i, 1/2)} |u_i|^2 = 1. \]

So
\[ \int_{B(p_i, 1)} |u_i|^2 < 2^{2(1-\delta)} \]
is uniformly bounded. Thus by passing to a subsequence, we may assume that $u_i$ converges to a nonzero harmonic 1-form $u$ on $B(o, 1)$. The convergence is smooth on any compact subset of the regular set of $B(o, 1)$, and both $u_i$ and $u$ are uniformly bounded on $B(o, 1/2)$ by Proposition 2.5. Taking the limit of the above inequality, we see that
\[ \int_{B(o,1)} |u|^2 \leq 2^{2(1-\delta)}. \]

Note that since $u$ is uniformly bounded and the convergence is smooth away from the singular set, we have
\[ \int_{B(o,1/2)} |u|^2 = \lim_{i \to \infty} \int_{B(p_i,1/2)} |u_i|^2 = 1. \]

This follows from an argument similar to [26, lemma 2.16] using Colding’s volume convergence theorem [16].

By Lemma 2.8, we have that locally $|\nabla_{\partial_2} u| < \infty$. Applying the $L^2$ three-circle theorem 3.4, we have
\[ \int_{B(o,2^{-i})} |u|^2 \leq 2^{2(1-\delta)} \int_{B(o,2^{-i-1})} |u|^2. \] (5.1)

We can extract the lowest-order term of $u$ as follows. Let
\[ u_i = \frac{\phi_i^* u}{\|\phi_i^* u\|_{L^2(B(o, 1))}}, \]
where $\phi_i : C(Y) \to C(Y)$ is the scaling by $2^{-i}$. Thus $u_i$ is uniformly bounded on any open subsets of the regular set of $B(o, 1/2)$. After passing to a subsequence, $u_i$ converges to a nonzero harmonic 1-form $v$ on $B(o, 1/2)$. By the growth estimate (5.1) we see that the $L^2$ ratio of $v$ is constant. So $v$
is homogeneous by Theorem 3.4. By Corollary 4.4, \( v = df \), where \( f \) is a harmonic function with growth rate \( 0 \leq s < 2 \). Since

\[
\int_{B(o,1/2)} \langle v, v \rangle > 0,
\]

we have, for large enough \( i \),

\[
0 < \int_{B(o,1/2)} \langle \phi_i^* u, v \rangle = 2^{mi} \int_{B(o,2^{-i-1})} \langle u, (\phi_i^{-1})^* v \rangle
= 2^{mi+s} \int_{B(o,2^{-i-1})} \langle u, v \rangle,
\]

where \( m \) denotes the dimension of \( C(Y) \), \( s \) denotes the degree of \( v \), and we use the homogeneity of \( v \) in the last equality. By Proposition 5.2, it follows that

\[
\int_{B(o,1)} \langle u, v \rangle > 0.
\]

On the other hand, by [24, theorem 2.1] (see also [48, lemma 4.1]), after passing to a subsequence there exists a sequence of harmonic functions \( f_i \) on \( B(p_i, 2) \) such that \( f_i \) converges to \( f \) uniformly in the Gromov-Hausdorff sense. By the Schauder estimates, it follows that \( f_i \) converges to \( f \) in \( C^\infty \) on compact subsets of \( B(o,1) \cap R \) in the Gromov-Hausdorff sense. Also by the Cheng-Yau gradient estimate, \( df_i \) and \( df \) are uniformly bounded on \( B(p_i, 1) \) and \( B(o, 1) \), respectively. It follows from an argument similar to [26, lemma 2.16] that we can take the limit of the \( L^2 \) orthogonal condition and get

\[
0 = \lim_{i \to \infty} \int_{B(p_i,1)} \langle u_i, df_i \rangle = \int_{B(o,1)} \langle u, df \rangle,
\]

which is a contradiction. \( \square \)

We can now prove our main theorem for harmonic 1-forms.

**Proof of Theorem 1.4.** Let \( \epsilon > 0 \) be given as in Lemma 5.3. We first prove the case when \( |\text{Ric}| \leq \epsilon \).

From Cheeger-Colding’s cone rigidity theorem [7] and volume comparison (see, for example, [10, p. 334] or [15, prop. 2.29] for details), there exists a number \( N(v, \epsilon) \) such that for all but \( N(v, \epsilon) \) of \( k \), \( \delta_{GH}(B(p, 2^{-k+1}), B(o_k, 2^{-k+1})) \leq \epsilon 2^{-k+1} \) for some \( B(o_k, 2^{-k+1}) \) inside a metric cone \( C(Y_k) \). By a compactness argument, we can assume these metric cones \( C(Y_k) \) are limit Ricci-flat cones. Set \( u_0 = u \). We define \( u_k \) on \( B(p, 2^{-k}) \) inductively.

For each of the good \( k \), let \( u_k \) be the \( L^2 \) orthogonal projection of \( u_{k-1} \) onto the \( L^2 \) complement of the space of closed, co-closed 1-forms on \( B(o, 2^{-k}) \). Note that this space is closed in \( L^2 \); let \( w_i \) be a sequence of closed, co-closed \( L^2 \) 1-forms on \( B(p, 1) \) such that \( w_i \rightarrow w \) in \( L^2 \) for some \( L^2 \) 1-form on \( B(p, 1) \). Then this implies that \( dw = 0 \) and \( d^* w = 0 \) in the weak sense. Elliptic regularity then implies that \( dw = 0 \) and \( d^* w = 0 \) in the strong sense.
By Lemma 5.3,
\[ \int_{B(p,2^{-k-1})} |u_{k+1}|^2 \leq 2^{-2(1-\delta)} \int_{B(p,2^{-k})} |u_k|^2. \]
For finitely many bad \( k \), set \( u_k = u_{k-1} \). The following inequality holds by volume comparison:
\[ \int_{B(p,2^{-k-1})} |u_{k+1}|^2 \leq 2^{-2(1-\delta)} C \int_{B(p,2^{-k})} |u_k|^2 \]
for some constant \( C \) depending on \( \delta \) and \( v \). Concatenating the above two types of inequalities, we get
\[ \int_{B(p,2^{-k})} |u_k|^2 \leq C(\epsilon, \delta, v) 2^{-2k(1-\delta)} \int_{B(p,1)} |u|^2. \]

Now, pick \( 2^{-k-1} \leq 2r \leq 2^{-k} \), and we have
\[ \int_{B(p,2r)} |u_k|^2 \leq C \int_{B(p,2^{-k})} |u_k|^2. \]
Since \( \Delta |u_k|^2 = -2|
abla u_k|^2 - \text{Ric}(u_k, u_k) \), an integration by parts with a good cutoff function gives
\[ \int_{B(p,r)} |
abla u_k|^2 \leq \left( \frac{\epsilon}{2} + C r^{-2} \right) \int_{B(p,2r)} |u_k|^2. \]
As \( r \leq 1/2 \), we may absorb \( \epsilon/2 \) into \( C r^{-2} \) on the right-hand side of the above inequality. We also have the pointwise inequality
\[ |du|^2 + |d^* u|^2 = |du_k|^2 + |d^* u_k|^2 \leq 2|\nabla u_k|^2. \]
The desired estimate then follows by combining the above four inequalities.

Now we prove the case when \( |\text{Ric}| \leq 1 \). Let \( r \in (0, 1/2] \). First let us assume \( r \leq \sqrt{\epsilon}/2 \); that is, \( r \) is small. By rescaling the metric \( g = \epsilon \tilde{g} \), we can apply what we just proved to get
\[ \int_{B(p,r)} |du|^2 + |d^* u|^2 \leq C' \epsilon^{\delta-1} r^{-2\delta} \int_{B(p,\sqrt{\epsilon})} |u|^2 \leq C''(\delta, v, \epsilon) r^{-2\delta} \int_{B(p,1)} |u|^2, \]
where the last inequality follows from volume comparison.

Now let us assume \( r > \sqrt{\epsilon}/2 \). In this case, we get a better estimate. Integrating by parts with a good cutoff function gives
\[ \int_{B(p,r)} |du|^2 + |d^* u|^2 \leq 2 \int_{B(p,r)} |
abla u|^2 \leq \frac{C_1}{r^2} \int_{B(p,2r)} |u|^2. \]
So
\[ \int_{B(p,r)} |du|^2 + |d^* u|^2 \leq \frac{C_1}{r^2} \frac{\text{Vol}(B(p,1))}{\text{Vol}(B(p,r))} \int_{B(p,1)} |u|^2 \leq C_2(\delta, v, \epsilon) \int_{B(p,1)} |u|^2 \]
by volume comparison. \( \square \)
Remark 5.4. We have a pointwise bound for $d^*u$. Since $|d^*u|$ is subharmonic, we can apply the Li-Schoen mean value inequality to show that $|d^*u|^2$ is bounded by the $L^2$ average of $|d^*u|$, which in turn is bounded by the $L^2$ average of $u$ by an integration by parts with a good cutoff functions.

Proof of Theorem 1.6. Let $f$ be a harmonic function on $B(p, 1)$. Applying Theorem 1.4 to the harmonic 1-form $u = d^c f$ gives

$$
\int_{B(p, r)} |\bar{\partial} \partial f|^2 \leq C r^{-25} \int_{B(p, 3/4)} |d^c f|^2 = C r^{-25} \int_{B(p, 3/4)} |df|^2 \leq C' r^{-25} \int_{B(p, 1)} f^2,
$$

where the last inequality follows from an integration by parts with a good cutoff function and a volume comparison. □

Now that we have the analogue of the gradient estimate, we are ready to prove the Liouville-type theorem for harmonic 1-forms.

Proof of Theorem 1.1. It is enough to prove that for every $p \in M$,

$$\int_{B(p, 1)} |u|^2 = 0.$$

Let $r < 1/2$. We rescale the metric by $r^2$, use Theorem 1.1, then rescale back by $1/r^2$ and get

$$\int_{B(p, 1)} |du|^2 + |d^*u|^2 \leq C r^{-2-2\delta} \int_{B(p, 1/r)} |u|^2 \leq C r^{2-25-2s}.$$

We choose $\delta > 0$ such that $2 - 2\delta - 2s > 0$. Note that since $M$ has maximal volume growth, the constant $C$ only depends on a fixed volume lower bound and $\delta$. Hence it does not depend on $r$. Letting $r \to 0$, we get that $u$ is both closed and co-closed. On the other hand, by corollary 1.5(3) of Anderson [4], we have that $H^1(M) = 0$. So $u = df$ for some function $f$. Since $d^*u = 0$, $f$ is harmonic. □

Remark 5.5. As mentioned in the introduction, using the method in Conlon-Hein [19], we can relax the subquadratic growth condition to $o(r^2)$ in the AC case. Actually, we will show that we can replace the sublinear condition in Theorem 1.1 by $o(r)$, assuming the manifold $M$ is AC Calabi-Yau. The key is that the metric $g$ converges to the cone metric $g_0$ on the tangent cone at infinity $C$ in $O(r^{-\epsilon})$ for some $\epsilon > 0$ by definition. This implies that the Laplacian $\Delta$ with respect to $g$ and the Laplacian $\Delta_0$ with respect to $g_0$ vary in $O(r^{-\epsilon})$ in operator norm. Let $u$ be a harmonic 1-form on $M$ such that $u = o(r)$. We show that $u = O(r^s)$ for some $s < 1$. Since $M$ and $C$ are diffeomorphic outside compact subsets, we may transplant $u$ to a 1-form $u_0$ on $C$ such that $u = u_0$ outside a compact subset. Note that $\Delta_0 u_0 = (\Delta_0 - \Delta) u_0 = O(r^{-1-\epsilon})$ for $r$ large. Then we solve the equation $\Delta_0 u_0 = \Delta_0 u_0$ on $C$ by solving ODEs. Thus $v_0 = O(r^{1-\epsilon})$. Since $u_0 - v_0 = o(r)$ is harmonic, it follows that $u_0 - v_0 = O(r^{s'})$ for some $s < 1$ by the Cheeger-Tian lemma (Corollary 4.4). So $u = (u_0 - v_0) + v_0 = O(r^{s'})$ for $r$ large, where $s' = \max\{1 - \epsilon, s\}$. 

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