On a Four-Dimensional Formulation of Dimensionally Regulated Amplitudes

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and references therein.
Outline

The NLO computations of hard processes

Four Dimensional Feynman Rules

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Four point one-loop massless color ordered amplitudes

The all helicity-plus four gluons planar amplitude with a gluonic loop

The effective coupling of Higgs to gluons in NLO amplitudes

Conclusions and perspectives
The NLO computations of hard processes

A hard parton-level cross section $2 \rightarrow m$ at NLO is made by

$$\sigma^{NLO} = \int_m d\sigma^B + \int_m \left( d\sigma^V + \int_1 d\sigma^A \right) + \int_{m+1} \left( d\sigma^R - d\sigma^A \right)$$

- $d\sigma^B$ is the Born exclusive regularization scheme independent differential cross section ($\Sigma A^B A^{B*}$).
- $d\sigma^V$ is the virtual correction ($\Sigma \mathcal{R}[A^B A^{V*}]$). It involves loop diagrams whose UV (ultraviolet divergent) part is made finite in a given renormalization scheme and therefore the UV divergences are regularization scheme independent.
- $d\sigma^R$ is the real corrections, affected (together with $d\sigma^V$) by soft and collinear divergences.
- $d\sigma^A$ and $\int_1 d\sigma^A$ are the unintegrated and integrated counterterms (allowing to compute real emission of massless particles in 4 dimensions).
Four Dimensional Feynman Rules for gauge theories bare one-loop dimensionally regularized diagrams

The external legs are treated as usual four dimensional states.

- The pure Yang-Mills (YM) loop propagators in Feynman-'t Hooft gauge

\[
\begin{align*}
\begin{array}{c}
\includegraphics[width=1cm]{gluon} \\
\text{a, } a, \alpha \quad \text{b, } b, \beta
\end{array}
\end{align*}
\]

\[= -i \delta^{ab} \frac{g^{\alpha\beta}}{k^2 - \mu^2 + i\varepsilon} \quad \text{(gluon),}
\]

\[
\begin{align*}
\begin{array}{c}
\includegraphics[width=1cm]{ghost} \\
\text{a, } a \quad \text{b, } b
\end{array}
\end{align*}
\]

\[= i \delta^{ab} \frac{1}{k^2 - \mu^2 + i\varepsilon} \quad \text{(ghost),}
\]

\[
\begin{align*}
\begin{array}{c}
\includegraphics[width=1cm]{scalar} \\
\text{a, } a, A \quad \text{b, } b, B
\end{array}
\end{align*}
\]

\[= -i \delta^{ab} \frac{G^{AB}}{k^2 - \mu^2 + i\varepsilon} \quad \text{(scalar),}
\]

The scalars come from a dimensional reduction of \( D = 4 - 2\varepsilon \) dimensional gluons vector fields.
In $D = 4 - 2\epsilon$ dimensions we perform the decomposition of the loop momentum $\tilde{k}^\alpha$ in a 4-dimensional part $k^\alpha$ and in its orthogonal complement the $-2\epsilon$-dimensional fixed vector $\mu^\alpha$

$$\tilde{k}^\alpha = k^\alpha + \mu^\alpha \quad \mu^\alpha \mu_\alpha = -\mu^2$$

$$\bar{g}^{\alpha\beta} = g^{\alpha\beta} + \tilde{g}^{\alpha\beta} \quad \tilde{g}^{\alpha\beta} \rightarrow G^{AB} \quad \mu^\alpha \rightarrow i\mu Q^A$$

where the $A$ and $B$ label the components of the complementary space of dimension $D - 4$.

The metric $G^{AB}$ and the vector $Q^A$ needed to reformulate the Feynman rules satisfy

$$G^{AB} G^{BC} = G^{AC}, \quad G^{AA} = 0, \quad G^{AB} = G^{BA}$$

$$Q^A G^{AB} = Q^B, \quad Q^A Q^A = 1.$$
Fermion propagator in a loop

Dirac matrices have the following splitting

\[ \tilde{\gamma}^\alpha = \gamma^\alpha + \tilde{\gamma}^\alpha \]

and satisfy in \( D \) dimensions the Clifford algebra

\[ \{ \tilde{\gamma}^\alpha, \tilde{\gamma}^\beta \} = 2\tilde{g}^{\alpha\beta} \].

A possible 4-dimensional representation of \( \tilde{\gamma} \) matrices is in terms of \( \gamma^5 \) by the replacement

\[ \tilde{\gamma}^\alpha \to \gamma^5 \Gamma^A. \]

By imposing the rule \( Q^A \Gamma^A = 1 \) needed to recover \( \mu \mu = -\mu^2 \),

\[ \begin{array}{c}
\ell \\
\tilde{j} \\
i
\end{array} = i \delta_{\tilde{j}}^i \left( \frac{k + m - i\mu\gamma^5}{k^2 - m^2 - \mu^2 + i\epsilon} \right). \]
Four Dimensional Interaction Vertices

\[ 2, b, \beta = -g f^{abc} \left[ (k_1 - k_2)\gamma g^{\alpha\beta} + (k_2 - k_3)\alpha g^{\beta\gamma} + (k_3 - k_1)\beta g^{\gamma\alpha} \right], \]

\[ = -g f^{abc} k_2^\alpha, \]

\[ = -ig f^{abc} \mu Q^A, \]

\[ = -g f^{abc} (k_2 - k_3)^\alpha G^{BC}, \]

\[ = \mp g f^{abc} (i\mu) g^{\gamma\alpha} Q^B \left( \tilde{k}_1 = 0, \quad \tilde{k}_3^\gamma = \pm \mu^\gamma \right) \]
\[ 1, a, \alpha, b, \beta, 2, 4, d, \delta, 3, c, \gamma \]

\[ = -ig^2 \left[ 
+ f_{\alpha \delta} \frac{f_{\gamma \beta}}{g_{\alpha \gamma} \gamma_\beta} 
+ f_{\alpha \gamma} \frac{f_{\gamma \beta}}{g_{\alpha \beta} \gamma_\beta} 
+ f_{\alpha \gamma} \frac{f_{\gamma \beta}}{g_{\alpha \beta} \gamma_\beta} 
\right] , \]

\[ = 2ig^2 \ g^{\alpha \delta} \left( f_{\alpha \beta} \frac{f_{\gamma \delta}}{g_{\alpha \gamma} \gamma_\delta} + f_{\alpha \gamma} \frac{f_{\gamma \delta}}{g_{\alpha \beta} \gamma_\delta} \right) G^{BC} , \]

\[ 2, b, \beta \]

\[ 1, j \]

\[ 3, i \]

\[ = -ig \ \left( t^b \right)^i \frac{\gamma_\beta}{\bar{j}} , \]

\[ 2, b, B \]

\[ 1, j \]

\[ 3, i \]

\[ = -ig \ \left( t^b \right)^i \gamma^5 \Gamma^B \]
Selection rules (−2ε SRs) or about the algebra in the \(D - 4\) dimensional complementary space.

In the \(−2\varepsilon\)-dimensional vector space the following rules

\[
G^{AB} G^{BC} = G^{AC}, \quad G^{AA} = 0, \quad G^{AB} = G^{BA}, \\
\Gamma^A G^{AB} = \Gamma^B, \quad \Gamma^A \Gamma^A = 0, \quad Q^A \Gamma^A = 1, \\
Q^A G^{AB} = Q^B, \quad Q^A Q^A = 1
\]

completely define our four dimensional formulation in agreement with the **Four Dimensional Helicity scheme** up to spurious terms as explicitly checked in reproducing the integrand numerator of QCD amplitudes of the following processes

\[
q \bar{q} \rightarrow t \bar{t}, \quad g g \rightarrow t \bar{t}, \quad t \bar{t} \rightarrow t \bar{t}, \\
g g \rightarrow g g, \quad q \bar{q} \rightarrow t \bar{t} g, \quad g g \rightarrow t \bar{t} g, \\
q \bar{q} \rightarrow t \bar{t} q' \bar{q}'.
\]
Generalized Internal legs

- Generalized subluminal Dirac equation.
  Given the $\ell$ four dimensional vector
  \[
  (\ell - i\mu\gamma^5 - m) \ u_\lambda (\ell) = 0, \\
  (\ell - i\mu\gamma^5 + m) \ v_\lambda (\ell) = 0, \\
  \ell^\mu = \ell^b\mu + \frac{m^2 + \mu^2}{2\ell \cdot q_\ell} q_\mu^\ell; \quad (\ell^b)^2 = 0 = q_\ell^2.
  \]

- Solutions of the generalized Dirac equation
  \[
  u_+ (\ell) = \ell^b \Bigg| \ell^b \Bigg\rangle - \frac{(m - i\mu)}{[\ell^b \ q_\ell]} \ |q_\ell\rangle, \\
  u_- (\ell) = \ell^b \Big| \ell^b \Big\rangle - \frac{(m + i\mu)}{\langle \ell^b \ q_\ell \rangle} \ |q_\ell\rangle, \\
  v_- (\ell) = \ell^b \Big| \ell^b \Big\rangle + \frac{(m - i\mu)}{[\ell^b \ q_\ell]} \ |q_\ell\rangle, \quad v_+ (\ell) = \ell^b \Bigg| \ell^b \Bigg\rangle + \frac{(m + i\mu)}{\langle \ell^b \ q_\ell \rangle} \ |q_\ell\rangle.
  \]
Polarization sum of the solutions of the generalized Dirac equation

\[ \sum_{\lambda = \pm} u_\lambda (\ell) \bar{u}_\lambda (\ell) = \ell - i \mu \gamma^5 + m, \]

\[ \sum_{\lambda = \pm} v_\lambda (\ell) \bar{v}_\lambda (\ell) = \ell - i \mu \gamma^5 - m. \]

**BCFW (Britto, Cachazo, Feng, Witten) recursive relations**

\[ A_n = \sum_{\text{partitions}} \sum_{\lambda} A_L(\hat{p}_i, \hat{P}^*) \frac{u_\lambda(\hat{P}) \bar{u}_\lambda(\hat{P})}{P^2 - m^2 - \mu^2} A_R(-\hat{P}^*, \hat{p}_j) \]

\[ = \sum_{\text{partitions}} A_L(\hat{p}_i, \hat{P}^*) \frac{\hat{P} - m - i \mu \gamma^5}{P^2 - m^2 - \mu^2} A_R(-\hat{P}^*, \hat{p}_j) \]

The hatted are the shifted complex momenta and \( P^* \) shows that the amplitude has been stripped of his external spinor wave function.
PROOF OF THE COMPLETENESS RELATIONS

Chirality projectors

\[ \omega_\pm = \frac{\mathbb{I} \pm \gamma^5}{2}, \]

and we show that:

\[
\frac{|q_\ell][\ell^b| - |\ell^b][q_\ell|}{[\ell^b q_\ell]} = \\
= \frac{|q_\ell\rangle\langle q_\ell|\ell^b| + |\ell^b]\langle \ell^b| q_\ell\rangle}{2\ell^b \cdot q_\ell}
\]

\[
= \frac{\omega_- q_\ell \omega_+ \ell^b + \omega_-\ell^b \omega_+ q_\ell}{2\ell^b \cdot q_\ell}
\]

\[
= \frac{\omega_\ell^2 \{q_\ell, \ell^b\}}{2\ell^b \cdot q_\ell} = \omega_-, \quad (7a)
\]

and similarly
\[
\frac{|\ell^b\rangle\langle q_e| - |q_e\rangle\langle \ell^b|}{\langle q_e \ell^b \rangle} = \omega_+.
\]

Therefore

\[
\sum_{\lambda=\pm} u_\lambda(\ell) \bar{u}_\lambda(\ell) =
\]

\[
\left( |\ell^b\rangle + \frac{(m - i\mu)}{[\ell^b q_e]} |q_e\rangle \right) \left( |\ell^b| + \frac{(m + i\mu)}{\langle q_e \ell^b \rangle} |q_e\rangle \right) +
\]

\[
\left( |\ell^b\rangle + \frac{(m + i\mu)}{\langle \ell^b q_e \rangle} |q_e\rangle \right) \left( |\ell^b| + \frac{(m - i\mu)}{[q_e \ell^b]} |q_e\rangle \right) =
\]

\[
= \ell^b + \frac{m^2 + \mu^2}{2\ell^b \cdot q_e} \phi_\ell + (m - i\mu) \frac{|q_e| [\ell^b| - |\ell^b][q_e|}{[\ell^b q_e]}
\]

\[
+ (m + i\mu) \frac{|\ell^b\rangle\langle q_e| - |q_e\rangle\langle \ell^b|}{\langle q_e \ell^b \rangle} =
\]

\[
= \ell^b + \frac{m^2 + \mu^2}{2\ell^b \cdot q_e} \phi_\ell + (m - i\mu) \omega_- + (m + i\mu) \omega_+ =
\]

\[
= \ell + i\mu \gamma^5 + m.
\]
D-dimensional Polarization Vectors

In Arnowitt-Fickler gauge the helicity sum of the transverse D-dimensional polarization vectors is

$$
\sum_{i=1}^{D-2} \varepsilon^\alpha_{i(D)} (\vec{l}, \vec{\eta}) \varepsilon^{*\beta}_{i(D)} (\vec{l}, \vec{\eta}) = -\bar{g}^{\alpha\beta} + \frac{\bar{\ell}^\alpha \bar{\eta}_\beta + \bar{\ell}^\beta \bar{\eta}_\alpha}{\bar{\ell} \cdot \bar{\eta}} - \frac{\bar{\eta}^2 \bar{\ell}^\alpha \bar{\ell}^\beta}{(\bar{\eta} \cdot \bar{\ell})^2}
$$

$$\bar{\ell} \cdot \bar{\eta} \neq 0$$

From the gauge invariance in $D$ dimensions the choice of the fixed $D$-dimensional gauge vector

$$\bar{\eta}^\alpha = \mu^\alpha$$

allows for the disentanglement

$$
\sum_{i=1}^{D-2} \varepsilon^\alpha_{i(D)} (\vec{l}, \vec{\eta}) \varepsilon^{*\beta}_{i(D)} (\vec{l}, \vec{\eta}) = \left( -\bar{g}^{\alpha\beta} + \frac{\ell^\alpha \ell^\beta}{\mu^2} \right) - \left( \bar{g}^{\alpha\beta} + \frac{\mu^\alpha \mu^\beta}{\mu^2} \right).
$$
Generalized Polarization Vectors

Once again let us decompose the massive four-dimensional vector \((\ell^2 = \mu^2)\)

\[
\ell^\alpha = \ell^b_\alpha + \hat{q}_{\ell}^\alpha
\]

the \(\mu\)-massive polarizations vectors are

\[
\varepsilon_\pm^\alpha (\ell) = -\frac{[\ell^b | \gamma_\alpha | \hat{q}_{\ell}]}{\sqrt{2\mu}}, \quad \varepsilon^\alpha (\ell) = -\frac{\langle \ell^b | \gamma_\alpha | \hat{q}_{\ell} \rangle}{\sqrt{2\mu}},
\]

\[
\varepsilon_0^\alpha (\ell) = \frac{\ell^b_\alpha - \hat{q}_{\ell}^\alpha}{\mu}
\]

with the usual Proca’s completeness relation

\[
\sum_{\lambda=\pm,0} \varepsilon_\lambda^\alpha (\ell) \varepsilon^*_{\lambda} (\ell) = -g^{\alpha\beta} + \frac{\ell^\alpha \ell^\beta}{\mu^2}
\]

\[
\varepsilon_\pm^2 (\ell) = 0, \quad \varepsilon_\pm (\ell) \cdot \varepsilon_{\mp} (\ell) = -1,
\]

\[
\varepsilon_0^2 (\ell) = -1, \quad \varepsilon_{\pm} (\ell) \cdot \varepsilon_0 (\ell) = 0,
\]

\[
\varepsilon_\lambda (\ell) \cdot \ell = 0 \quad \lambda = \pm, 0.
\]
The numerator of cut propagator of the scalar can be expressed in terms of the \((-2\epsilon)\)-SRs:

\[
\tilde{g}^{\alpha\beta} + \frac{\mu^\alpha \mu^\beta}{\mu^2} \rightarrow \hat{G}^{AB} \equiv G^{AB} - Q^A Q^B.
\]

The factor \(\hat{G}^{AB}\) can be easily accounted for by defining the cut propagator as

\[
\begin{array}{ccc}
  a, A & | & b, B \\
  \Rightarrow & & \Rightarrow \\
  \hat{G}^{AB} \delta^{ab}.
\end{array}
\]

The previous Feynman rules and cuts prescriptions fully reconstruct the \(\mu^2\) dependence of the \(\epsilon\)—dimensional numerator of scattering amplitudes of renormalized gauge theories.
Four point massless one-loop color ordered amplitudes $A_4$

From the reduction theorem a dimensionally regularized $A_4$ is decomposed in a cut-constructible part and in a rational part ($\mathcal{R}$) expressed in terms of scalar integrals in $D = 4 - 2\epsilon$ dimensions. The coefficients $c_i$ are rational functions of the external momenta and polarizations.

$$A_4 = \frac{1}{(4\pi)^{2-\epsilon}} \left[ c_{1|2|3|4;0} \right. $$

$$\left. l_{1|2|3|4} + \left( c_{12|3|4;0} \right. \right. $$

$$\left. l_{12|3|4} + c_{1|2|34;0} l_{1|2|34} + c_{1|23|4;0} l_{1|23|4} + c_{2|3|41;0} l_{2|3|41} \right) $$

$$+ \left( c_{12|34;0} l_{12|34} + c_{23|41;0} l_{23|41} \right) \right] + \mathcal{R} + O(\epsilon),$$

$$\mathcal{R} = \frac{1}{(4\pi)^{2-\epsilon}} \left[ c_{1|2|3|4;4} l_{1|2|3|4}[\mu^4] + \left( c_{12|3|4;2} l_{12|34}[\mu^2] \right. $$

$$\left. + c_{1|2|34;2} l_{1|2|34}[\mu^2] + c_{1|23|4;2} l_{1|23|4}[\mu^2] $$

$$+ c_{2|3|41;2} l_{2|3|41}[\mu^2] \right) $$

$$+ \left. \left( c_{12|34;2} l_{12|34}[\mu^2] + c_{23|41;2} l_{23|41}[\mu^2] \right) \right].$$
By the separation

\[ \int \frac{d^D \ell}{(2\pi)^D} = \int \frac{d^{-\epsilon}(\mu^2)}{(2\pi)^{-2\epsilon}} \int \frac{d^4 \ell}{(2\pi)^4}. \]

and using polar coordinates in the $-2\epsilon$ dimensional Euclidean vector space, all the integrals in $\mathcal{R}$ can be computed. In particular

\[ \lim_{\epsilon \to 0} I^{4-2\epsilon}_{1|2|3|4}[\mu^4] = \lim_{\epsilon \to 0} \left( \epsilon(\epsilon - 1) I^{8-2\epsilon}_{1|2|3|4} \right) = -\frac{1}{6}. \]

We found a way of computing the rational part of scattering amplitudes by unitarity cuts with loop momenta in $D = 4$. 
Inside our FDF scheme we are using the **generalized unitarity** to compute the full one loop amplitude including the **cut-constructible** and the full gauge invariant **rational part**.
The all helicity-plus four gluons planar amplitude with a gluonic loop

In order to reconstruct the full $\mu$ dependence and to obtain by cut construction the rational coefficients of the master integral decomposition, the following color-ordered trees amplitudes are needed. With all outgoing complex momenta $2^+ + 1^+ + 3^+ = 0$, $2^+ + 1^+ + 3^- - 3^+ = i g \left( \frac{[1^b|2] \hat{q}_1|2]}{\mu} + \frac{\langle r_2|1|2 \rangle}{\langle r_2|2 \rangle} \right)$, $2^+ + 1^0 - 3^+ = 0$. 
\[
\begin{align*}
\begin{array}{c}
1^0 \\
2^+ \\
3^-
\end{array} &= \frac{\sqrt{2}ig [\hat{q}_1 | 2]^2}{\mu}, \\
\begin{array}{c}
1^- \\
2^+ \\
3^-
\end{array} &= ig \frac{[\hat{q}_1 | 2][\hat{q}_3 | 2]}{\mu^2} \langle 1^b | 3^b \rangle, \\
\begin{array}{c}
1^0 \\
2^+ \\
3^0
\end{array} &= -ig \frac{\langle r_2 | 1 | 2 \rangle}{\langle r_2 | 2 \rangle} \left\{ 1 - \left(1 + \xi \right) \frac{1}{\xi \mu^2} \left( (1 + \xi) \mu^2 \\
+ \xi \langle \hat{q}_1 | 2 | \hat{q}_1 \rangle \right) \right\}, \\
\begin{array}{c}
1 \\
2^+ \\
3
\end{array} &= \frac{ig}{\sqrt{2}} (3 - 1)^\mu \varepsilon^+_{\mu} (2, r_2) G^{AB} \\
&= -ig \frac{\langle r_2 | 1 | 2 \rangle}{\langle r_2 | 2 \rangle} G^{AB},
\end{align*}
\]

where \( \hat{q}_3 = \xi \hat{q}_1 \).
The box coefficients are obtained by the following attaching procedure, with the external legs of the trees on the generalized mass-shell

\[ C_{1|2|3|4}^{[0]} = C_{1|2|3|4}^{[0]} + C_{1|2|3|4}^{[0]} + C_{1|2|3|4}^{[0]}, \]

\[ c_{1|2|3|4; 4}^{[0]} = 3g^4 i \frac{[12] [34]}{\langle 12 \rangle \langle 34 \rangle} \]

in which the relations

\[ \langle j^b | \hat{q}_j \rangle = [\hat{q}_j | j^b] = \mu \]

allow to obtain a polynomial numerator in \( \mu \).
\[ C_{1234}^{[1]} = \sum_{h_i=\pm,0} \mathcal{T}_1 + \text{c.p.}, \]

\[ C_{1234}^{[2]} = \sum_{h_i=\pm,0} \mathcal{T}_1^2 + \mathcal{T}_2 + \text{c.p.}, \]

\[ C_{1234}^{[3]} = \sum_{h_1=\pm,0} \mathcal{T}_3 + \text{c.p.}, \]
\[ C_{1|2|3|4}^{[4]} = \mathcal{T}_4 \]

\[
\mathcal{T}_1 = Q^A \hat{G}^{AB} Q^B = 0, \\
\mathcal{T}_2 = Q^A \hat{G}^{AB} G^{BC} \hat{G}^{CD} Q^D = 0, \\
\mathcal{T}_3 = Q^A \hat{G}^{AB} G^{BC} \hat{G}^{CD} G^{DE} \hat{G}^{EF} Q^F = 0, \\
\mathcal{T}_4 = \text{tr} \left( G \hat{G} G \hat{G} G \hat{G} G \hat{G} G \hat{G} \right) = -1.
\]

\[
c_{1|2|3|4}^{[4]} = c_{1|2|3|4}^{[0]} + c_{1|2|3|4}^{[4]} = 3g^4 i \frac{[12][34]}{\langle 12 \rangle \langle 34 \rangle} - ig^4 \frac{[12][34]}{\langle 12 \rangle \langle 34 \rangle},
\]

\[
A_{4}^{\text{1-loop}} (1^+_g, 2^+_g, 3^+_g, 4^+_g) = \frac{2ig^4}{16\pi^2} \times \left(-\frac{1}{6}\right) \times \frac{[12][34]}{\langle 12 \rangle \langle 34 \rangle}.
\]
The effective coupling of Higgs to gluons in NLO amplitudes

- For 2 gluons \(\rightarrow\) Higgs we use an effective operator with \(m_{\text{top}} \rightarrow \infty\).

\[
L_{\text{int}} = \frac{C}{2} H \text{Tr} F_{\mu\nu} F^{\mu\nu}
\]

- The leading tree-level color ordered amplitude \(0 \rightarrow gggH\)

\[
A_{4,H}^{\text{tree}}(1^- 2^+ 3^+ H) = i \frac{[23]^4}{[12] [23] [31]}. 
\]

- As an example of application of our regularization scheme to an effective field theory consider at NLO in the large \(m_{\text{top}}\) limit the color ordered primitive amplitude

\[
A_{4,H}^{1\text{-loop}}(1^- 2^+ 3^+ H)
\]
To see just how the procedure works consider firstly some quadruple cuts:

\[ C_{1|2|3|H} = \begin{array}{c}
\begin{tikzpicture}
  \begin{scope}
    \node (n1) at (0,0) {1};
    \node (n2) at (1,0) {2};
    \node (n3) at (2,0) {3};
    \node (n4) at (3,0) {H};
    \draw (n1) -- (n2);
    \draw (n2) -- (n3);
    \draw (n3) -- (n4);
    \draw (n4) -- (n1);
  \end{scope}
\end{tikzpicture}
\end{array} + \begin{array}{c}
\begin{tikzpicture}
  \begin{scope}
    \node (n1) at (0,0) {1};
    \node (n2) at (1,0) {2};
    \node (n3) at (2,0) {3};
    \node (n4) at (3,0) {H};
    \draw (n1) -- (n2);
    \draw (n2) -- (n3);
    \draw (n3) -- (n4);
    \draw (n4) -- (n1);
  \end{scope}
\end{tikzpicture}
\end{array},
\right.

\[ c_{1|2|3|H;0} = -\frac{1}{2}A_{4,H}^{\text{tree}}s_{12}s_{23}, \]

\[ c_{1|2|3|H;4} = 0; \]

\[ C_{1|2|H|3} = \begin{array}{c}
\begin{tikzpicture}
  \begin{scope}
    \node (n1) at (0,0) {1};
    \node (n2) at (1,0) {2};
    \node (n3) at (2,0) {3};
    \node (n4) at (3,0) {H};
    \draw (n1) -- (n2);
    \draw (n2) -- (n3);
    \draw (n3) -- (n4);
    \draw (n4) -- (n1);
  \end{scope}
\end{tikzpicture}
\end{array} + \begin{array}{c}
\begin{tikzpicture}
  \begin{scope}
    \node (n1) at (0,0) {1};
    \node (n2) at (1,0) {2};
    \node (n3) at (2,0) {3};
    \node (n4) at (3,0) {H};
    \draw (n1) -- (n2);
    \draw (n2) -- (n3);
    \draw (n3) -- (n4);
    \draw (n4) -- (n1);
  \end{scope}
\end{tikzpicture}
\end{array},
\right.

\[ c_{1|2|H|3;0} = -\frac{1}{2}A_{4,H}^{\text{tree}}s_{13}s_{12}, \]

\[ c_{1|2|H|3;4} = 0. \]
......and by omitting for reasons of time many other contributions.....and just considering some among the double cuts

\[ C_{23|H1} = \]

\[ c_{23|H1; 0} = 0 \]

\[ c_{23|H1; 2} = 4A_{4,H}^{\text{tree}} \frac{s_{12}s_{13}}{s_{23}}. \]

By collecting all master integrals coefficients we recognize a full agreement with the the full Feynman diagrams calculations in the FDH scheme performed by Schmidt in 1997, the result is
\[ A_{4}^{1-loop} (1^-, 2^+, 3^+, H) = r_{1} A_{4}^{tree} \times \]
\[
\left\{ \frac{1}{\varepsilon^2} \left[ (-s_{12})^{-\varepsilon} + (-s_{13})^{-\varepsilon} + (-s_{23})^{-\varepsilon} \right] - \frac{\pi^2}{2} \right. \\
+ \left[ 2 \text{Li}_2 \left( 1 - \frac{s_{12}}{m_{H}^2} \right) + 2 \text{Li}_2 \left( 1 - \frac{s_{13}}{m_{H}^2} \right) + 2 \text{Li}_2 \left( 1 - \frac{s_{23}}{m_{H}^2} \right) \right] \\
+ \left[ \log \left( \frac{s_{12}}{m_{H}^2} \right) \log \left( \frac{s_{23}}{m_{H}^2} \right) + \log \left( \frac{s_{12}}{m_{H}^2} \right) \log \left( \frac{s_{13}}{m_{H}^2} \right) \\
+ \log \left( \frac{s_{13}}{m_{H}^2} \right) \log \left( \frac{s_{23}}{m_{H}^2} \right) \right] \\
- \frac{1}{3} \frac{s_{12}s_{13} + s_{12}s_{23} + s_{13}s_{23}}{m_{H}^4} + 1 \right\}
\]
Conclusions and perspectives

- A four-dimensional formulation (FDF) of dimensional regularization of one-loop scattering amplitudes has been applied to generalized unitarity techniques. At one loop the cut-constructible part and the rational part of scattering amplitudes have been computed by the same on-shell methods.

- The FDF Feynman rules have been extended to the recursive methods for generating the integrand of one-loop amplitudes.

- The inclusion of the fermion mass for a one loop amplitude like $0 \rightarrow ggt\bar{t}$ at one loop in FDF will be analysed.

- More loops and more jets in FDF is another goal to achieve.

- An important issue is to apply FDF for real corrections and corresponding subtraction terms of infrared divergences.