Abstract

The purpose of this paper is to introduce the notion of fuzzy pairwise $R_0$ axiom in fuzzy bitopological spaces and study some of its properties. Several interesting results have been obtained viz. it satisfy hereditary, good extension, productive and projective properties.

Key words: Fuzzy $R_0$ topological space, Fuzzy pairwise $R_0$ bitopological spaces, Hereditary property, Good extension property, Productive and projective properties.

1. Introduction

Fuzzy $R_0$ spaces have been introduced and studied earlier by Hutton and Reilly and Srivastava et al. independently. We follow here the definition given by Srivastava et al. We introduce it as a generalization of F-$R_0$ spaces. We see that FP-$R_0$ spaces satisfy hereditary, productive and projective properties. We have also seen that it is good extension of P-$R_0$ in bitopological spaces.

2. Preliminaries:

Here we shall follow Lowen’s definition of a fuzzy topological spaces (in short, an fts). The symbol $[0,1]$ will denote the unit interval $[0,1]$ and $I^X$ denotes the set of all fuzzy sets in $X$. All other undefined concepts are taken from.

Definition 2.1. A triple $(X, \tau_1, \tau_2)$, where $X$ is a non empty set and $\tau_1, \tau_2$ are arbitrary fuzzy topologies on $X$, is called a fuzzy bitopological space (in short, fbts).

Definition 2.2. Let $(X_i, \tau_{1i}, \tau_{2i})$ be a family of fbts. Then the product $(X, \tau_1, \tau_2)$ is called the product of $(X_i, \tau_{1i}, \tau_{2i})$ with the underlying set $A$. Induced by $\tau_{1i}, i = 1, 2$, then $(A, \tau_{1A}, \tau_{2A})$ is called the subspace fuzzy topology on $A$.

Definition 2.3. Let $(X, \tau_{1}, \tau_{2})$ be a family of fbts. Then the space $(X, \tau_{1A}, \tau_{2A})$ is called their product where $\tau_{1A}$ denotes the usual product fuzzy topology of the family $(\tau_{1i}, i \in A)$ of fuzzy topologies on $X, k=1, 2$. 

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A property $P$ is called productive if the product fts $(X, \tau_1 \Pi \tau_2)$ has $P$ if each coordinate fuzzy space has $P$. A property $P$ is called projective if the product fts $(X, \tau_1 \Pi \tau_2)$ has $P$ implies that each coordinate fuzzy space has $P$.

A property $P$ of an fts $(X, \pi, \tau)$ is said to be hereditary if every subspace of the space possesses $P$.

The ‘good extension’ property in the sense of Lowen$^5$ has been extended to the case of fuzzy bitopological spaces as follows:

A fuzzy bitopological analogue $FP$ of a bitopological property $P$ is said to be a good extension of $P$ if for every bitopological space $(X, T_1, T_2)$ possesses $P$ if the fts $(X, \omega (T_1), \omega (T_2))$ possesses $FP$.

3. Fuzzy Pairwise $R_0$ bitopological spaces:

In this section, we introduce the concept of fuzzy $R_0$ bitopological spaces and study some of its properties.

Definition 3.1: A bitopological space $(X,T_1,T_2)$ is pairwise $R_0$ (in short, $P-R_0$) iff for all $x,y\in X, x\neq y$ whenever there is a $U\in T_1$ such that $U(x)=1, U(y)=0$ there is also $V\in T_2$ such that $V(y)=1, V(x)=0$.

Definition 3.2: A fuzzy bitopological space $(X,\tau_1,\tau_2)$ is fuzzy pairwise $R_0$ (in short, $FP-R_0$) iff for all $x, y\in X, x\neq y$ whenever there is a $U\in \tau_1$, such that $U(x)=1, U(y)=0$ there is also $V\in \tau_2$ such that $V(y)=1, V(x)=0$.

The definition of fuzzy pairwise $R_0$ bitopological spaces are good extension of pairwise $R_0$ as seen below:

Theorem 3.1: A bitopological space $(X,T_1,T_2)$ is $P-R_0$ iff the bitopological spaces $(X,\omega(T_1),\omega(T_2))$ is $FP-R_0$.

Proof: Let us assume that $(X, T_1, T_2)$ is $P-R_0$. Take $x,y\in X, x\neq y$. Suppose $\exists \ U \in \omega(T_1)$ such that $U(x)=1, U(y)=0$. Then $U^{-1}[0,1] \in T_1$ and is such that $x \in U^{-1}(0,1], y \in U^{-1}(0,1]$. Now since $(X, T_1, T_2)$ is $R_0$ there is $V\in T_2$ such that $x \notin V, y \in V$. Now considering $V \in o(T)$ we found that $V(y)=1, V(x)=0$. Hence $(X, T_1, T_2)$ is fuzzy pairwise $R_0$. Conversely let $(X, \tau_1, \tau_2)$ be fuzzy pairwise $R_0$. Take $x,y\in X$ $x \neq y$ Then whenever $\exists U \in T_1$ such that $x \in U \text{ and } y \notin U$ then the crisp fuzzy open set $U$ is such that $U(x)=1, U(y)=0$ and since $(X, \omega(T_1), \omega(T_2))$ is fuzzy $R_0$ there is $V \in \omega(T)$ Then $V^{-1}(0,1] \in T_2$ such that $x \notin V^{-1}(0,1]$ and $y \in V^{-1}(0,1]$ implying that $(X, T_1, T_2)$ is pairwise $R_0$.

Next we show that $FP-R_0$ satisfy hereditary property as:

Theorem 3.2: Every subspace of fuzzy pairwise $R_0$ fuzzy bitopological space is also fuzzy pairwise $R_0$.

Proof: Let the fts $(X, \tau_1, \tau_2)$ be a fuzzy pairwise $R_0$. Let $(Y, \tau_1y, \tau_2y)$ be its subspace. Let $x, y \in Y \subseteq X, x \neq y$. Since $(X, \tau_1, \tau_2)$ is fuzzy pairwise $R_0$ whenever there is a $U \in \tau_1$ such that $U(x)=1, U(y)=0$ there is also $V \in \tau_2$ such that $V(y)=1, V(x)=0$. Now we see that whenever there is a $U_1(x)=U \cap Y(x)=1, U_2(y)=U \cap Y(y)=0$ there is also $V \in \tau_2$ such that $V(y)=V \cap Y(x)=1, V(x)=V \cap Y(x)=0$. This implying that $(Y, \tau_1y, \tau_2y)$ is also fuzzy pairwise $R_0$.

Proposition 3.1: An fts $(X, \tau_1, \tau_2)$ is fuzzy pairwise $R_0$ iff the fts $(X, \tau_1)$ and $(X, \tau_2)$ are fuzzy $R_0$.

Proof: First let us suppose that the fts $(X, \tau_1, \tau_2)$ is fuzzy pairwise $R_0$. Then for $x,y\in X, x\neq y$ whenever there is a $U_1 \in \tau_1$ such that $U_1(x)=1, U_1(y)=0$ there is also $V_1(y)=1$ and $V_1(x)=0$. Now if we take $y,x\in X, x\neq y$ then whenever there is a $U_2 \in \tau_1$ such that $U_2(x)=0, U_2(y)=1$ there is also $V_2 \in \tau_2$ such that $V_2(y)=1, V_2(x)=0$. Thus for $x,y\in X, x\neq y$ we have seen that whenever there is $\tau_1$-fuzzy sets $U_1$ and $U_2$ such that $U_1(x)=1, U_1(y)=0$ and $U_2(x)=0, U_2(y)=1$ there is $\tau_2$-fuzzy open set $V_1$ and $V_2$ such that $V_1(x)=0, V_1(y)=1$ and $V_2(x)=1, V_2(y)=0$. Thus $(X, \tau_1)$ and $(X, \tau_2)$ is fuzzy $R_0$.

Conversely suppose that $(X, \tau_1)$ and $(X, \tau_2)$ is fuzzy $R_0$. Then first taking $(X, \tau_1)$ fuzzy $R_0$, for $x,y\in X, x\neq y$ whenever there is a $U \in \tau_1$ such that $U(x)=1, U(y)=0$ there is also $V \in \tau_1$ such that $V(x)=0, V(y)=1$. Next taking $(X, \tau_2)$ fuzzy
R₀, for y, x ∈ X, y ≡ x whenever there is U ∈ τ₂ such that U(y) = 1, U(x) = 0 there is a V ∈ τ₂ such that V(x) = 0, V(y) = 1. Thus for x, y ∈ X, x ∼ y whenever there is U ∈ τ₁ such that U(x) = 1, U(y) = 0 there is also V ∈ τ₂ such that V(x) = 0, V(y) = 1. Imposing that (X, τ₁, τ₂) is fuzzy pairwise R₀.

Next we show that FP-R₀ satisfies productive and projective properties:

Theorem 3.3: Let \((X_i, τ_i, τ'_i), \text{i} ∈ A\) be a family of fts. Then the product fts \((ΠX_i, Πτ_i, Πτ'_i)\) is FP-R₀ iff each coordinate space \((X_i, τ_i, τ'_i)\) is FP-R₀.

Proof: First let each coordinate space \((X_i, τ_i, τ'_i), \text{i} ∈ A\) be FP-R₀. Then show that the product fts is FP-R₀, let x, y ∈ X, x ∼ y. Let x = \(Π_{x_i = x_i, y = y_i}\) where xᵢ ≠ yᵢ for some j ∈ A. Now take xᵢ, yᵢ ∈ Xᵢ. Since \((X_i, τ_i, τ'_i)\) is FP-R₀ whenever there is a Uᵢ ∈ τᵢ such that Uᵢ(xᵢ) = 1, Uᵢ(yᵢ) = 0 there is a Vⱼ ∈ τⱼ such that Vⱼ(yⱼ) = 1, Vⱼ(xⱼ) = 0. Now consider U = ΠUᵢ and V = ΠVⱼ, where Uᵢ = Vᵢ for i ≠ j, Uᵢ = Uᵢ and Vⱼ = Vⱼ then whenever there is a U in τ₁ such that U(x) = 1, U(y) = 0 there is a V in τ₂ such that V(y) = 1, V(x) = 0. Thus the product fuzzy bitopological space is FP-R₀.

Conversely, let \((ΠX_i, Πτ_i, Πτ'_i)\) be fuzzy pairwise R₀. Consider any coordinate spaces say \((X_i, τ_i, τ'_i)\) choose xᵢ, yᵢ ∈ Xᵢ, xᵢ ∼ yᵢ. Construct x, y ∈ X such that x = \(Πxᵢ = xᵢ\), y = \(Πyᵢ = yᵢ\) where \(⇒ \forall r < \inf Uᵢ(xᵢ)\) for all j ∈ A ⇒ \(∀ r < Uᵢ(xᵢ)\) for all j ∈ A ⇒ U(y) = 0 ⇒ ΠUᵢ(y) = 0. Then there is a fuzzy point yᵢ ∈ V such that there exist a basic fuzzy open set ΠVⱼ ∈ τⱼ such that yᵢ ∈ ΠVⱼ and \(⇒ s < Vⱼ(yⱼ)\) for all j ∈ A and \(ΠVⱼ(y) = 0\).

Now \(ΠUᵢ(y) = 0 \Rightarrow Uᵢ(yᵢ) = 0\). Hence \(Uᵢ(yᵢ) = Uᵢ(xᵢ)\) \(⇒ r\). Similarly \(ΠVⱼ(y) = 0 \Rightarrow Vⱼ(yⱼ) = 0\). Thus \(Vⱼ(xⱼ) = Vⱼ(yⱼ)\) \(⇒ s\). Consider \(Uᵢ = Uᵢ ∈ τᵢ\) and \(Sup Vⱼ = Vⱼ ∈ τⱼ\) whenever there is a \(U(xᵢ) = 1\), \(U(yᵢ) = 0\) there is a \(Vⱼ ∈ τⱼ\) such that \(Vⱼ(yⱼ) = 1\), \(Vⱼ(xⱼ) = 0\). Imposing that \((X_i, τ_i, τ'_i)\) is fuzzy pairwise R₀.

Conclusion

In this paper the concept of fuzzy pairwise R₀ axiom in a fuzzy bitopological spaces has been introduced. The appropriateness of our definition is established by proving some interesting relevant results. We proved that FP-R₀ in fuzzy bitopological spaces is good extension of the corresponding concept of P-R₀ in bitopological spaces. Our definition of FP-R₀ also satisfy hereditary, productive and projective properties.

References

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