COMPOSITION OPERATOR FOR FUNCTIONS OF BOUNDED VARIATION

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Abstract. We study the optimal conditions on a homeomorphism $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ to guarantee that the composition $u \circ f$ belongs to the space of functions of bounded variation for every function $u$ of bounded variation. We show that a sufficient and necessary condition is the existence of a constant $K$ such that $|Df|(f^{-1}(A)) \leq K \mathcal{L}^n(A)$ for all Borel sets $A$. We also characterize homeomorphisms which maps sets of finite perimeter to sets of finite perimeter. Towards these results we study when $f^{-1}$ maps sets of measure zero onto sets of measure zero (i.e. $f$ satisfies the Lusin $(N^{-1})$ condition).

1. Introduction

In this paper we address the following issue. Suppose that $\Omega \subset \mathbb{R}^n$ is an open set, $f : \Omega \to \mathbb{R}^n$ is a homeomorphism and a function of bounded variation and $u$ is a function of $BV(f(\Omega))$. Under which conditions can we then conclude that $u \circ f \in BV(\Omega)$ or that $u \circ f$ is weakly differentiable in some weaker sense? Our main theorem gives a complete answer to this question.

Theorem 1.1. Let $\Omega_1, \Omega_2$ be open subsets of $\mathbb{R}^n$ and let $f \in BV_{loc}(\Omega_1, \Omega_2)$ be a homeomorphism. Suppose that there is a constant $K > 0$ such that

\begin{equation}
|Df|(f^{-1}(A)) \leq K \mathcal{L}^n(A) \text{ for all Borel sets } A \subset \Omega_2.
\end{equation}

Then the operator $T_f(u) = u \circ f$ maps functions from $BV(\Omega_2)$ into $BV(\Omega_1)$ and

\begin{equation}
|D(u \circ f)|(\Omega_1) \leq K|Du|(\Omega_2).
\end{equation}

On the other hand, if $f$ is a homeomorphism of $\Omega_1$ onto $\Omega_2$ such that the operator $T_f$ maps $C_0(\Omega_2) \cap BV(\Omega_2)$ into $BV(\Omega_1)$, then $f \in BV_{loc}(\Omega_1, \Omega_2)$ and there exists a constant $K > 0$ such that (1.1) holds.

The class of homeomorphisms that satisfy (1.1) forms a natural extension of a special class of mappings of finite distortion. More precisely: in the fourth chapter we show that the set of homeomorphisms in $W^{1,1}_{loc}$ with the property

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coincides with the known class of homeomorphisms with finite distortion satisfying that there exists a constant $K > 0$ such that 
\[ |Df(x)| \leq K |J_f(x)| \text{ for a.e. } x \in \Omega. \]
It is known that for this class of Sobolev homeomorphisms we have $T_f(u) := u \circ f \in W^{1,1}$ for all $u \in W^{1,1}$. See [6] or [14] for details. Hence naturally $T_f$ maps function from $W^{1,1}$ to $BV$.

Let us note that the morphism property of $T_f$ on $BV$ was also known under the assumption that the homeomorphism $f$ belongs to class of mappings with a Lipschitz inverse. This can be found in [2, Theorem 3.16], or [9]. We show that the above two classes of homeomorphisms differ (and our class contains both of them).

To prove Theorem 1.1 we need to know that $f$ satisfies the Lusin $(N^{-1})$ condition, i.e. preimages of sets of Lebesgue measure zero have measure zero. If the condition fails then there is a set $A \subset \Omega_1$ such that $\mathcal{L}^n(A) > 0$ and $\mathcal{L}^n(f(A)) = 0$. Then we can redefine $u$ on the null set $f(A)$ arbitrarily and the composed function may fail to be measurable. On the other hand, if $f$ satisfies the Lusin $(N^{-1})$ condition then the validity of our statement for one representative of $u$ implies the validity for all representatives, because the compositions only differ on a set of measure zero. The Lusin $(N^{-1})$ condition is well-studied in the Sobolev case (see [12] and references given there, [11]). We study this condition for functions of bounded variation in the third section. The proof of Theorem 1.1 is given in the fourth and fifth section. We also prove that it is enough to test $f$ on sets of finite perimeter.

**Theorem 1.2.** Let $\Omega_1, \Omega_2$ be open subsets of $\mathbb{R}^n$ and let $f$ be a homeomorphism $\Omega_1 \to \Omega_2$. Then the following conditions are equivalent:

1. There is a constant $K > 0$ such that $P(f^{-1}(A), \Omega_1) \leq KP(A, \Omega_2)$.
2. The function $f$ has locally bounded variation and there exists a constant $K > 0$ such that (1.1) holds.

Actually we prove more general statements of the theorems. We allow $f$ to fail to be a homeomorphism. Our mapping will be a general mapping of bounded variation (its multiplicity can be unbounded) with no jump part and satisfying (1.1) for some good representative of $f$.

### 2. Preliminaries

We use the usual convention that $C$ denotes a generic positive constant whose exact value may change from line to line. We denote by $\mathcal{L}^n$ the Lebesgue measure. The symbol $\nabla u(x)$ denotes the classical gradient of $u$ in $x$. By $Du$ we denote the distributional derivative.

Let $\Omega$ be an open subset of $\mathbb{R}^n$. We write $G \subset \subset \Omega$ if the closure $\overline{G}$ is compact and $\overline{G} \subset \Omega$. A function $u \in L^1(\Omega)$ whose partial derivatives in the sense of distributions are signed measures with finite total variation in $\Omega$ is called a function of
bounded variation. The vector space of functions of bounded variation is denoted by $BV(\Omega)$. We write $u \in BV(\Omega, \mathbb{R}^d)$ if $u_i \in BV(\Omega)$ for all $i \in \{1, \ldots, d\}$.

If $u \in BV(\Omega, \mathbb{R}^d)$, the total variation of the measure $Du$ is defined by

$$|Du|(E) = \sup \left\{ \sum_{i=1}^{m} \int_{E} u_i \text{div} \phi_i d\mathcal{L}^n : \phi_i \in C^1_c(\Omega, \mathbb{R}^{d \times n}), |\phi_i(x)| \leq 1 \text{ for } x \in \Omega \right\} < \infty.$$  

We write $u \in BV_{loc}(\Omega, \mathbb{R}^n)$ if for all $x \in \Omega$ there is a ball $B \ni x$ such that $u \in BV(B, \mathbb{R}^n)$.

Propositions 3.6 and 3.13 in [2] give us a simple characterization of BV functions.

**Theorem 2.1.** Suppose that $\Omega \subset \mathbb{R}^n$ is open and $u \in L^1(\Omega)$. Then $u \in BV(\Omega)$ if and only if there is a sequence $u_k \in W^{1,1}(\Omega)$ such that $u_k \to u$ in $L^1$ and $\sup_k \|Du_k\|_{L^1} < \infty$.

Moreover,

$$|Du|(\Omega) = \inf \left\{ \sup_k \|Du_k\|_{L^1(\Omega)} : u_k \in L^1(\Omega), u_k \to u \text{ in } L^1(\Omega) \right\}.$$  

**Theorem 2.2.** Suppose that $\Omega \subset \mathbb{R}^n$ is open, $u_k \in BV(\Omega)$ and there is $u \in L^1(\Omega)$ such that $u_k \to u$ in $L^1(\Omega)$ and $\sup_k |Du_k|(\Omega) < \infty$. Then $u$ belongs to $BV(\Omega)$ and $u_k$ weakly* converges to $u$ in $BV(\Omega)$.

We say that $E \subset \Omega$ has finite perimeter if the characteristic function $\chi_E$ belongs to $BV(\Omega)$ and we set

$$P(E, \Omega) = |D\chi_E|(\Omega).$$

The following lemma gives us a connection between functions of bounded variation and sets of finite perimeter. (Theorem 3.39 in [2])

**Lemma 2.3.** Suppose that $\Omega \subset \mathbb{R}^n$ is open and $u \in L^1_{loc}(\Omega)$. Then

$$|Du|(\Omega) = \int_{-\infty}^{\infty} P\left(\{x : u(x) > t\}, \Omega\right) dt$$

We say that the approximate limit of $f \in L^1(\Omega, \mathbb{R}^m)$ exists at $x \in \Omega$ if there is $z \in \mathbb{R}^m$ such that

$$\lim_{r \to 0^+} \frac{1}{\mathcal{L}^n(B(x, r))} \int_{B(x, r)} |f(y) - z| dy = 0.$$  

We write $z = \text{app lim}_{y \to x} f(y)$. If $f$ is integrable then the set $S_f$ where the limit does not exists is $\mathcal{L}^n$-negligible and Borel and $\tilde{f} = \text{app lim} f$ is Borel measurable on $\Omega \setminus S_f$. (See Proposition 3.66 in [2].)

Let us note that slightly weaker definitions of approximate limits are available in literature. For instance in [5] $z \in \mathbb{R}^m$ is called the approximate limit of measurable function $f : \Omega \to \mathbb{R}^m$ at $x \in \Omega$ if all the sets

$$E_\varepsilon = \{y \in \Omega : |u(y) - z| > \varepsilon\}$$
has density 0 in $x$. In our paper we follow the notation from [2]. See the discussion which follows after Proposition 3.64 in [2] to find differences between these definitions.

The main tool is the analogy of the chain rule for the composition of a smooth function and a function of bounded variation, see [1] or Theorem 3.96 in [2].

**Theorem 2.4.** Suppose that $\Omega \subset \mathbb{R}^n$ is open, $f \in BV(\Omega, \mathbb{R}^n)$ and $u \in C^1(\mathbb{R}^n, \mathbb{R}^k)$. Then the composition $u \circ f$ belongs to $BV(\Omega)$ and

$$D(u \circ f) = \nabla u \circ f \cdot D^a f \mathcal{L}^n + \nabla u \circ \tilde{f} \cdot D^c f + [u(f^+) - u(f^-)] \otimes \nu_f \mathcal{H}^{n-1}|_J,$$

where

$$Df = D^a f \mathcal{L}^n + D^c f + \nu_f \mathcal{H}^{n-1}|_J$$

is the usual decomposition of $Df$ in its absolutely continuous part $D^a f$ with respect to the Lebesgue measure $\mathcal{L}^n$, its Cantor part $D^c u$ and its jump part, which is represented by the restriction of the $(n-1)$ dimensional Hausdorff measure to the jump set $J$. Moreover, $\nu_f$ denotes the measure theoretical unit normal to $J$, $\tilde{f}$ is the approximate limit and $f^+, f^-$ are the approximate limits from both sides of $J$.

We will work only with functions which have no jump part, i.e. $J = \emptyset$. In that case we have

$$D(u \circ f) = \nabla u \circ \tilde{f} \cdot Df.$$

### 2.1. Basic properties of measures.

If $u$ is a $\mu$-measurable function and $E$ is a $\mu$-measurable set then we denote by $\int_E u \, d\mu$ (or $\int_E u(x) \, d\mu(x)$ if we want to emphasize the variable) the integral of $u$ over $E$ with respect to the measure $\mu$. Instead of $d\mathcal{L}^n(x)$ we write shortly $dx$.

Given measure spaces $(X, \mathcal{A})$ and $(Y, \mathcal{B})$, a measurable mapping $f : X \rightarrow Y$ and a measure $\mu : \mathcal{A} \rightarrow [0, \infty]$, the image of $\mu$ is defined to be the measure $f(\mu) : \mathcal{B} \rightarrow [0, \infty]$ given by

$$(f(\mu))(A) = \mu(f^{-1}(A)) \text{ for } A \in \mathcal{B}.$$}

Sometimes $f(\mu)$ is called the pushforward of $\mu$.

**Theorem 2.5.** Let $X, Y, f, \mu$ be as above and $g : Y \rightarrow \mathbb{R}^n$ then we have that

$$\int_Y g \, d(f(\mu)) = \int_X g \circ f \, d\mu,$$

whenever one of the integrals is well-defined.

Let $\mu, \nu$ be measures defined on the same $\sigma$-algebra $\mathcal{A}$ of the space $X$. We say that $\mu$ is

- absolute continuous with respect to $\nu$ if

$$|\nu|(A) = 0 \Rightarrow |\mu|(A) = 0.$$
• singular with respect to $\nu$ if there are $X_a, X_s \in A$ such that $X = X_a \cup X_s$ and
  $$|\nu|(X_s) = 0 = |\mu|(X_a).$$

We set $\text{supp} \nu = X_s$.

For each pair of non-negative measures $\mu$ and $\nu$ on the same $\sigma$-algebra $A$ we can find a decomposition $\mu = \mu^a + \mu^s$ such that $\mu^a$ is absolute continuous with respect to $\nu$ and $\mu^s, \nu$ are singular.

**Theorem 2.6** (Radon-Nikodym). Let $\mu$ be a non-negative Borel measure on $\mathbb{R}^n$ and set

$$\frac{d\mu}{d\mathcal{L}^n}(x) = \lim_{r \to 0_+} \frac{\mu(B(x, r))}{\mathcal{L}^n(B(x, r))}.$$ 

Then $\frac{d\mu}{d\mathcal{L}^n}$ exists $\mathcal{L}^n$-a.e., $\frac{d\mu}{d\mathcal{L}^n}(x)$ is $\mathcal{L}^n$-measurable and

$$\int_A \frac{d\mu}{d\mathcal{L}^n}(x) dx \leq \mu(A)$$

for all Borel sets $A \subset G$.

Moreover, if $\mu$ is absolute continuous with respect to $\mathcal{L}^n$ then the above inequality holds as equality.

### 3. Lusin $(N^{-1})$ Condition

In this section we generalize the result of P. Koskela and J. Malý. In [12] they proved our Theorem 3.2 in the special case when $f$ is a Sobolev mapping.

The following lemma will be useful. See [12, Lemma 2.1].

**Lemma 3.1.** There is a constant $\tau = \tau(n)$ with the following property: For each atomless probability Borel measure $\mu$ on $\mathbb{R}^n$ there is a point $y \in \mathbb{R}^n$ and a radius $R > 0$ such that

$$\mu(B(y, 2R)) \geq \tau \quad \text{and} \quad \mu(\mathbb{R}^n \setminus B(y, 3R)) \geq \tau.$$ 

**Theorem 3.2.** Let $\Omega \subset \mathbb{R}^n$ be connected open set, $f \in BV(\Omega, \mathbb{R}^n)$ have no jump part. Suppose that

$$(3.1) \quad |Df|(\tilde{f}^{-1}(A)) \leq \int_A K(y) dy \text{ for all Borel set } A \subset \mathbb{R}^n,$$

where $K(y) \in L^p$ for some $p \in [1, n]$, $p' = \frac{p}{p-1}$. If $f$ is not constant then $f$ satisfies Lusin $(N^{-1})$ condition, i.e. for any set $E \subset \mathbb{R}^n$ we have

$$\mathcal{L}^n(E) = 0 \Rightarrow \mathcal{L}^n(f^{-1}(E)) = 0.$$

**Proof.** Step 1. Without loss of generality we may assume that $K$ is a Borel function and if $p = 1$ we have $K(y) = \text{esssup}_{z \in \mathbb{R}^n} K(z)$ for all $y \in \mathbb{R}^n$.

We first prove an auxiliary estimate. With the help of Theorem 2.5 and the fact that for the image of measure $Df$ we have

$$\tilde{f}(|Df|)(A) = |Df|(\tilde{f}^{-1}(A)) \leq \int_A K(y) dy$$
we obtain for each nonnegative Borel measurable function \( g \) and Borel set \( A \subset \mathbb{R}^n \) that
\[
(3.2) \quad \int_{f^{-1}(A)} g(\tilde{f}(x)) \, d|Df|(x) = \int_A g(y) \, d\tilde{f}(\|Df\|)(y) \leq \int_A g(y)K(y) \, dy.
\]

Note that the set
\[
N = \{ x \in \Omega : K(\tilde{f}(x)) = 0 \} = \tilde{f}^{-1}(\{ y \in \mathbb{R}^n : K(y) = 0 \})
\]
has \( |Df| \)-measure zero. Let \( E \subset \Omega \) be a measurable set. Consider a smooth function \( u \) with a compact support in \( \mathbb{R}^n \).

With the help of Theorem \[2,3\] Hölder inequality and \[3.2\] we can estimate
\[
(3.3) \quad |D(u \circ f)|(E) \leq \int_E |\nabla u(\tilde{f}(x))| \, d|Df|(x)
\]
\[
= \int_{E \cap N} |\nabla u(\tilde{f}(x))|K(\tilde{f}(x))^{-\frac{1}{p}}K(\tilde{f}(x))^{\frac{n}{p}} \, d|Df|(x)
\]
\[
\leq \left( \int_{E \cap N} |\nabla u(\tilde{f}(x))|^p K(\tilde{f}(x))^{-1} \, d|Df|(x) \right)^{1/p} \left( \int_E K(\tilde{f}(x))^{\frac{n}{n'}} \, d|Df|(x) \right)^{1/n'}
\]
\[
\leq \left( \int_{\mathbb{R}^n} |\nabla u(y)|^n \, dy \right)^{1/n} \left( \int_E K(\tilde{f}(x))^{\frac{n}{n'}} \, d|Df|(x) \right)^{1/n'}
\].

Step 2. We claim that
\[
(3.4) \quad y_0 \in \mathbb{R}^n \Rightarrow \mathcal{L}^n(f^{-1}(\{ y_0 \})) = 0.
\]

For this, consider an arbitrary ball \( B \subset \subset \Omega \) and \( y_0 \in \mathbb{R}^n \). Suppose that \( f \) differs from \( y_0 \) on a set of positive measure in \( B \). Then there is \( R > 0 \) such that
\[
(3.5) \quad \kappa := \mathcal{L}^n(B \setminus f^{-1}(B(y_0, R))) > 0.
\]

Since singletons have zero \( n \)-capacity, given \( \varepsilon > 0 \) there is a smooth function \( u \) on \( \mathbb{R}^n \) such that
\[
\text{supp} \, u \subset B(y_0, R), \; u(y_0) = 1 \quad \text{and} \quad \int_{\mathbb{R}^n} |\nabla u|^n \, d\mathcal{L}^n < \varepsilon^n.
\]

Then
\[
\min\{ \mathcal{L}^n(B \cap f^{-1}(\{ y_0 \})), \kappa \} \leq Cr|D(u \circ f)|(B).
\]

For this we used the well-known trick
\[
(3.6) \quad 1/2 \min\{ \mathcal{L}^n(B \cap \{ v \leq 0 \}), \mathcal{L}^n(B \cap \{ v \geq 1 \}) \} \leq \inf_{c \in \mathbb{R}} \int_B |v - c| \, d\mathcal{L}^n \leq Cr|Dv|(B),
\]

based on the Poincaré inequality, where the hypothesis is that \( v \in BV \). Note that from \[3.2\] it follows for \( p > 1 \) that
\[
\int_{\Omega} K(\tilde{f}(x))^{\frac{n}{n'-1}} \, d|Df|(x) \leq \int_{\mathbb{R}^n} K(y)^{p'} \, dy < \infty.
\]
We know that $|Df|$ is a finite measure because $f \in BV(\Omega)$. Hence $K(\tilde{f}(x))^{\frac{1}{p-1}} \in L^1(\Omega, |Df|)$. Trivially this relation holds even for the case when $p = 1$.

Together with (3.3) we obtain

$$\min\{|B \cap f^{-1}\{y_0\}|, \kappa\} \leq \left( \int_{\mathbb{R}^n} |\nabla u|^n d\mathcal{L}^n \right)^{1/n} \left( \int_B K(\tilde{f}(x))^{\frac{1}{p-1}} d|Df|(x) \right)^{1/n'} \leq C\varepsilon \left( \int_B K(\tilde{f}(x))^{\frac{1}{p-1}} d|Df|(x) \right)^{1/n'}.$$

Letting $\varepsilon \to 0$ and using (3.5) we obtain that $\mathcal{L}^n(B \cap f^{-1}\{y_0\}) = 0$ whenever $f$ differs from $y_0$ on a set of positive measure in $B$. Hence (3.4) follows by taking the connectedness of $\Omega$ and the assumption that $f$ is not constant into account.

**Step 3.** Let us prove that there is some $c > 0$ such that

$$\lim_{r \to 0^+} \frac{\int_{B(x_0, r)} K(\tilde{f}(x))^{\frac{1}{p-1}} d|Df|(x)}{\mathcal{L}^n(B(x_0, r))} > c$$

for a.e. $x_0$ in $\Omega$. Fix a ball $B(x_0, r) \subset \subset \Omega$. Consider the Borel measure $\mu$ defined by

$$\mu(A) = \frac{\mathcal{L}^n(B(x_0, r) \cap f^{-1}(A))}{\mathcal{L}^n(B(x_0, r))}, \quad A \subset \mathbb{R}^n.$$

From Step 2 we know that $\mu$ does not have atoms. By Lemma (3.1) we find a point $y \in \mathbb{R}^n$ and a radius $R > 0$ such that

$$\mu(B(y, 2R)) \geq \tau \quad \text{and} \quad \mu(\mathbb{R}^n \setminus B(y, 3R)) \geq \tau.$$

Find a smooth function $u$ such that

$$u(x) = 1 \text{ on } B(y, 2R), \quad u = 0 \text{ outside } B(y, 3R) \text{ and } \int_{\mathbb{R}^n} |\nabla u|^n d\mathcal{L}^n \leq C(n).$$

The function $v = u \circ f$ belongs to $BV(\Omega)$ and we have

$$\frac{\mathcal{L}^n(B(x_0, r) \cap \{v = 1\})}{\mathcal{L}^n(B(x_0, r))} \geq \frac{\mathcal{L}^n(B(x_0, r) \cap f^{-1}(B(y, 2R)))}{\mathcal{L}^n(B(x_0, r))} = \mu(B(y, 2R)) \geq \tau,$$

$$\frac{\mathcal{L}^n(B(x_0, r) \cap \{v = 0\})}{\mathcal{L}^n(B(x_0, r))} \geq \frac{\mathcal{L}^n(B(x_0, r) \setminus f^{-1}(B(y, 3R)))}{\mathcal{L}^n(B(x_0, r))} = \mu(\mathbb{R}^n \setminus B(y, 3R)) \geq \tau.$$

By (3.9), (3.6) and (3.3) we have

$$1 \leq C r^{1-n} |D(u \circ f)|(B(x_0, r)) \leq C r^{1-n} \left( \int_{B(x_0, r)} K(\tilde{f}(x))^{\frac{1}{p-1}} d|Df|(x) \right)^{1/n'} = C \left( \frac{1}{|B(x_0, r)|} \int_{B(x_0, r)} K(\tilde{f}(x))^{\frac{1}{p-1}} d|Df|(x) \right)^{1/n'}. $$
Step 4. From Step 3 we know that the Radon-Nikodym derivative of the measure \( \nu = K(\tilde{f}(x)) \frac{|Df|}{|Df|} \) with respect to \( \mathcal{L}^n \) is greater than some \( c > 0 \). Let \( E \) be an arbitrary set of measure zero. Take \( \tilde{E} \supset E \) a Borel set of measure zero. It follows from Radon-Nikodym theorem \( \text{2.6} \) and (3.2) that

\[
 c \mathcal{L}^n(f^{-1}(\tilde{E})) = \int_{f^{-1}(\tilde{E})} c \, d\mathcal{L}^n \leq \int_{f^{-1}(\tilde{E})} \frac{d\nu}{d\mathcal{L}^n} \, d\mathcal{L}^n \leq \int_{f^{-1}(\tilde{E})} K(\tilde{f}(x)) \frac{1}{|Df|} \, d|Df|(x) \leq \int_{\tilde{E}} K(y) \frac{1}{n} \, dy = 0.
\]

Hence \( f^{-1}(E) \) is a subset of a set of measure zero and it has measure zero. \( \square \)

**Theorem 3.3.** Let \( f \) satisfy the assumption of Theorem \( \text{3.2} \) for \( p = 1 \) (i.e. the function \( K \) is in \( L^\infty(\mathbb{R}^n) \)). Then the operator \( T_f \) defined by \( T_f(u)(x) = u(f(x)) \) maps \( L^1(\mathbb{R}^n) \) to \( L^1(\Omega) \) boundedly.

**Proof.** First note that from Theorem we know that \( \text{DODELAT} \) without loss of generality we may assume that \( K(y) \leq K \) everywhere and then we obtain by (3.11) that

\[
 1 \leq C \mathcal{L}^n \left( \frac{1}{|\mathcal{L}^n(B(x_0, r))|} \int_{B(x_0, r)} |Df| \right)^{1/n'}.
\]

Thus we proved that the Radon-Nikodym derivative of measure \( \nu = |Df| \) with respect to \( \mathcal{L}^n \) is greater than some \( c > 0 \). It follows from Radon-Nikodym theorem \( \text{2.6} \) and (3.2) that

\[
 c \int_{\Omega} |u \circ f| \, d\mathcal{L}^n \leq \int_{\Omega} |u \circ \tilde{f}| \frac{|Df|}{|D\mathcal{L}^n|} \, d\mathcal{L}^n \leq \int_{\int_{\mathbb{R}^n} |u(y)| |K(y)| \, dy \leq K \int_{\mathbb{R}^n} |u(y)| \, dy.
\]

\( \square \)

**Remark 3.4.** Analogously to Theorem \( \text{3.2} \) it is possible to show that for such mapping \( f \) its operator \( T_f \) maps any rearrangement invariant space \( X(\mathbb{R}^n) \) to \( X(\Omega) \) and \( \|u \circ f\|_{X(\Omega)} \leq c\|u\|_{X(\mathbb{R}^n)} \). To get the sufficient estimate on the level set use (3.11) on \( u = \chi_{\{u \geq \alpha\}} \).

The conditions on \( f \) in Theorem \( \text{3.2} \) are sharp. For all \( p > n \) there is a Sobolev self-homeomorphism of \((0, 1)^n\) such that \( K(y) \in L^p \) but Lusin \((N^{-1})\) condition fails. Indeed, in \( \text{11} \) we constructed a homeomorphism of finite distortion such that \( |Df(x)|^p \leq L(x)J_f(x) \) a.e. with \( L(x) \in L^\infty \), but Lusin \((N^{-1})\) condition fails. Let us show that (3.11) from Theorem \( \text{3.2} \) is satisfied for \( K(y) = L(f^{-1}(y))^{\frac{1}{p}}J_f(f^{-1}(y))^{\frac{1}{p'}} \).
Denote by $N$ a set of measure zero such that $f|_{(0, 1)^n \setminus N}$ satisfies the Lusin $(N)$ condition. Set $Z = \{x : J_f(x) = 0 \text{ or does not exist}\}$. Then with the help of area formula (see [7, Theorem 2]) we easily obtain

$$|Df|(E) \leq \int_E L(x)^\frac{p}{p-1} J_f^\frac{1}{p}(x) \, dx = \int_{E \setminus (\mathbb{Z} \cup N)} L(x)^\frac{p}{p-1} J_f^\frac{1}{p}(x) \, dx$$

$$= \int_{f(E \setminus (\mathbb{Z} \cup N))} L(f^{-1}(y))^\frac{p}{p-1} J_f(f^{-1}(y))^\frac{1}{p} \, \frac{dy}{y}.$$ 

It remains to show that $L(f^{-1}(y))^\frac{p}{p-1} J_f(f^{-1}(y))^\frac{1}{p} \chi_{(0, 1)^n \setminus f(\mathbb{Z} \cup N)}$ is in $L^{p'}$. This follows since

$$\int_{(0, 1)^n} \left( L(f^{-1}(y))^\frac{1}{p} J_f(f^{-1}(y))^\frac{1}{p-1} \chi_{(0, 1)^n \setminus f(\mathbb{Z} \cup N)} \right)^{p'} \, dy$$

$$= \int_{(0, 1)^n \setminus f(\mathbb{Z} \cup N)} L(f^{-1}(y))^{p-1} J_f(f^{-1}(y))^{-1} \, dy = \int_{(0, 1)^n} L(x)^{p-1} \, dx < \infty.$$

Surprisingly it is not enough to control by the absolute continuous part of the derivative $Df$. Indeed, it is possible to construct a homeomorphism $f$ such that for any constant $K \in \mathbb{R}$ we have

$$(3.12) \quad |D^a f|(x) \leq K J_f(x) \text{ for a.e. } x$$

and Lusin $(N^{-1})$ condition fails. In [8] we can find a Sobolev homeomorphism $g$ of $(0, 1)^n$ such that $J_g = 0$ a.e. and $|Dg| \in L^{n-1}$. The homeomorphism $g$ maps a set of full measure into a set of measure zero and a set of measure zero into a set of full measure. Let us show that $f = g^{-1}$ satisfies $|D^a f|(x) = 0$ and hence also (3.12).

It follows from Lemma 4.3 in [3] and Theorem 3.8 [13] that $f = g^{-1} \in BV((0, 1)^n, (0, 1)^n)$ and

$$|Df|(f^{-1}(G)) \leq C \int_G |\text{adj } Dg| \, d\mathcal{L}^n$$

holds for every open $G \subset (0, 1)^n$ where $C$ depends only on $n$. Hence it also holds for each Borel set $A$ and we have

$$|Df|(f^{-1}(A)) \leq C \int_A |\text{adj } Dg| \, d\mathcal{L}^n.$$ 

Denote by $N$ a Borel set $N \subset (0, 1)^n$ such that $\mathcal{L}^n(N) = 0$ and $g(N) = f^{-1}(N) = \mathcal{L}^n((0, 1)^n)$. Then

$$|D^a f|((0, 1)^n) = |D^a f|(f^{-1}(N)) \leq |Df|(f^{-1}(N)) \leq C \int_N |\text{adj } Dg|^{n-1} \, d\mathcal{L}^n = 0.$$ 

Thus $|D^a f| = 0$ a.e. and the inequality (3.12) trivially holds.
4. Sufficient condition

**Theorem 4.1.** Let $\Omega_1, \Omega_2$ be open subsets of $\mathbb{R}^n$ and let $f \in BV_{\text{loc}}(\Omega_1, \Omega_2)$ have no jump part. Suppose that $f$ is not constant on any component of $\Omega$ and there is a constant $K > 0$ such that

$$|Df|(\tilde{f}^{-1}(A)) \leq K\mathcal{L}^n(A) \text{ for all Borel sets } A \subset \Omega_2.$$  

Then the operator $T_f(u)(x) = u(f(x))$ maps functions from $BV(\Omega_2)$ into $BV(\Omega_1)$ and

$$|D(u \circ f)|(\Omega_1) \leq K|Du|(\Omega_2).$$

**Proof.** Suppose that $u \in BV(\Omega_2)$. Let be $u_k$ an approximation of $u$ from Theorem 2.1 and $G \subset \subset \Omega_1$ be an open set. We prove that $u_k \circ f$ is a good approximation of $u \circ f$ on $G$.

It follows from Theorem 3.3 that $u \circ f \in L^1(\Omega)$ and due to (3.11) we obtain

$$\|u_k \circ f - u \circ f\|_{L^1(\Omega_1)} \leq C\|u_k - u\|_{L^1(\Omega_2)},$$

Thus $u_k \circ f \to u \circ f$ in $L^1(G)$. Let us note that Theorem 3.2 is key for us. It gives us validity of Lusin ($N^{-1}$) condition for the function $f$ and hence the composition $u \circ f$ is a well-defined function.

By Theorem 2.4 we have that $u_k \circ f$ belongs to $BV(G)$ and $D(u_k \circ f)(x) = \nabla u_k(\tilde{f}(x)) \cdot Df(x)$. As in (3.2) we can with the help of Theorem 2.5 and the fact that $\tilde{f}(|Df|)(A) \leq K\mathcal{L}^n(A)$ estimate

$$|D(u_k \circ f)|(G) \leq \int_G |\nabla u_k(\tilde{f}(x))| \, d|Df|(x) \leq K \int_{\Omega_2} |\nabla u_k| \, d\mathcal{L}^n.$$ 

Lemma 2.2 gives us that $u \circ f$ has bounded variation on $G$. Moreover, using semi-continuity of the variation we obtain

$$|D(u \circ f)|(G) \leq \inf \left\{ \sup_k \|Dv_k\|_{L^1} : v_k \in L^1(G), v_k \to u \circ f \text{ in } L^1(G) \right\}$$

$$\leq \inf \left\{ \sup_k \|D(u_k \circ f)\|_{L^1} : u_k \in C^\infty, u_k \to u \text{ in } L^1(\Omega_2) \right\}$$

$$\leq K \inf \left\{ \sup_k \|Du_k\|_{L^1} : u_k \in C^\infty, u_k \to u \text{ in } L^1(\Omega_2) \right\}$$

$$= K|Du|(\Omega_2).$$

To prove (4.2) find open sets $G_k \subset \subset \Omega$ such that $G_k \subset G_{k+1}$ and $\Omega_1 = \bigcup_{k=1}^\infty G_k$ then

$$|D(u \circ f)|(\Omega_1) = \lim_{k \to \infty} |D(u \circ f)|(G_k) \leq K|Du|(\Omega_2).$$

In the case when $f$ is constant on some component $G$ of $\Omega$ the composition $u \circ f$ may fail to be well-defined. If we take a representative of $u$ such that $\tilde{u}(x) = 0$
for all $x$ such that there is a component $G$ of $\Omega$ satisfying $f(G) = \{x\}$ then for this representative we have $\hat{u} \circ f \in BV(\Omega_1)$ and (4.2) again holds.

By applying Theorem 4.1 on characteristic functions of sets we easily obtain the following corollary.

**Corollary 4.2.** Let $f$ satisfy the assumptions of Theorem 4.1. Then for any set of finite perimeter $E \subset \Omega_2$ the preimage $f^{-1}(E)$ is a set of finite perimeter in $\Omega_1$ and

$$P(f^{-1}(E), \Omega_1) \leq KP(E, \Omega_2).$$

**Remark 4.3.** The condition (4.1) can be rewritten as

$$\int_{f^{-1}(A)} |D^a f| \, d\mathcal{L}^n + \int_{f^{-1}(A)} d|D^c f| \leq K \mathcal{L}^n(A),$$

which is equivalent to existence of constants $C_1, C_2 \in \mathbb{R}$ such that

$$\int_{f^{-1}(A)} |D^a f| \, d\mathcal{L}^n \leq C_1 \mathcal{L}^n(A)$$

and

$$\int_{f^{-1}(A)} d|D^c f| \leq C_2 \mathcal{L}^n(A).$$

The second condition (4.5) implies that $|D^c f|(\hat{f}^{-1}(A)) = 0$ whenever $A \subset \Omega_2$ has measure zero.

**Lemma 4.4.** Let $f$ belong to $BV(\Omega, \mathbb{R}^n)$, have no jump part and satisfy $f(z) = \text{app lim}_{x \to z} f(x)$ whenever $z \in \text{supp} |D^c f|$ and the limit exists. Then (4.1) holds if and only if

$$|Df|(f^{-1}(A)) \leq K \mathcal{L}^n(A) \text{ for all Borel sets } A \subset \Omega_2.$$

**Proof.** Because $f^{-1}(A)$ and $\hat{f}^{-1}(A)$ differ only by a set of $\mathcal{L}^n$ measure zero we have

$$\int_{f^{-1}(A)} |D^a f| \, d\mathcal{L}^n = \int_{\hat{f}^{-1}(A)} |D^a f| \, d\mathcal{L}^n.$$

For the second part we will use facts which can be found in Chapter 3 in [2]. The set $S_f$ where the approximate limit does not exists is $\mathcal{H}^{n-1}$-negligible [2, Theorem 3.76]. Thus $\hat{f}^{-1}(A) \cap \text{supp} |D^c f|$ and $f^{-1}(A) \cap \text{supp} |D^c f|$ are equal up to a set of $\mathcal{H}^{n-1}$-Hausdorff measure zero. Because $Du$ does not see sets of $\mathcal{H}^{n-1}$ measure zero [2, Lemma 3.76], we have

$$\int_{f^{-1}(A) \cap \text{supp} |D^c f|} d|D^c f| = \int_{\hat{f}^{-1}(A) \cap \text{supp} |D^c f|} d|D^c f|.$$ These two equalities together with (4.3) give us (4.6). □
Thus we may take in Theorem \ref{thm:main} the natural representative satisfying $f(x) = \lim_{r \to 0^+} \frac{1}{\mathcal{L}^n(B(x,r))} \int_{B(x,r)} f(z) \, dz$ and demand the condition \ref{eq:cond}.

**Lemma 4.5.** Assume that $f$ is a homeomorphism of bounded variation. Then the inequality \ref{eq:cond2} is equivalent to

\[ |D^a f(x)| \leq C_1 |J_f|(x) \text{ for a.e. } x \in \Omega_1. \tag{4.9} \]

**Proof.** It easily follows from \ref{eq:cond2} and Area formula (see \cite{[7]} Theorem 2) that we have

\[ \int_{f^{-1}(A)} |D^a f| \, d\mathcal{L}^n \leq C_1 \int_{f^{-1}(A)} |J_f| \, d\mathcal{L}^n \leq C_1 \mathcal{L}^n(A). \]

To prove the second implication let us assume that $x$ is a Lebesgue point of $J_f$ and $D_f^c$. Find a Borel set $N$ of measure zero such that $f|_{\Omega_1 \setminus N}$ satisfies Lusin ($N$) condition. It follows by \ref{eq:cond2} that

\[ \int_{B(x,r)} |D^a f| \, d\mathcal{L}^n = \int_{B(x,r) \setminus N} |D^a f| \, d\mathcal{L}^n \leq C_1 |f(B(x,r) \setminus N)| = C_1 \int_{B(x,r)} |J_f| \, d\mathcal{L}^n. \]

By dividing the both sides by $\mathcal{L}^n(B(x,r))$ and sending $r \to 0$ we get \ref{eq:cond2}. \hfill \Box

If we assume that $f$ is a Sobolev homeomorphism then $D^c f = 0$.

**Corollary 4.6.** If $f$ is a homeomorphism in $W^{1,1}_{loc}(\Omega_1, \mathbb{R}^n)$, then \ref{eq:cond} is equivalent to

\[ |Df(x)| \leq K |J_f|(x) \text{ for a.e. } x \in \Omega_1. \tag{4.10} \]

The simplest way to obtain the condition \ref{eq:cond} is to check the integrability of the inverse.

**Lemma 4.7.** Let $\Omega_1, \Omega_2 \subset \mathbb{R}^n$ and let $f : \Omega_1 \to \Omega_2$ be a mapping such that $f^{-1}$ is Lipschitz. Then \ref{eq:cond} holds.

**Proof.** It follows from Lemma 4.3 in \cite{[3]} and Theorem 3.8 in \cite{[13]} that $f \in BV^{loc}(\Omega_2, \Omega_1)$ and

\[ |Df|(f^{-1}(G)) \leq C \int_{G} |\text{adj} D(f^{-1})| \, d\mathcal{L}^n, \tag{4.11} \]

where $C$ depends only on $n$. Hence \ref{eq:cond} holds for all Borel sets and we have

\[ |Df|(f^{-1}(A)) \leq C \int_{A} |\text{adj} D(f^{-1})| \, d\mathcal{L}^n \leq C \|D(f^{-1})\|_{L^n}^{p^{-1}} \mathcal{L}^n(A). \]

\hfill \Box

**Example 4.8.** There is a homeomorphism $f$ such that \ref{eq:cond} holds but $f \notin W^{1,1}_{loc}$.
Proof. Consider the usual Cantor ternary function $u$ on the interval $(0,1)$. And set $g(x) = u(x) + x$. This function is continuous, increasing and fails to be absolutely continuous. Moreover, $g$ does not belong to $W^{1,1}_{loc}$. On the other hand, the inverse function $g^{-1}$ is Lipschitz and maps $(0,2)$ homeomorphically onto $(0,1)$. If we set

$$f(x_1, \ldots, x_n) = (g(x_1), x_2, \ldots, x_n)$$

then obviously $f$ fails to belong to $W^{1,1}_{loc}((0,1)^n, \mathbb{R}^n)$, and $f^{-1}$ is a Lipschitz function. Due to Lemma 4.7 the function $f$ satisfies $(4.1)$.

□

In the special case when $n = 2$ we obtain the equivalence in Lemma 4.7.

**Lemma 4.9.** Let $\Omega_1, \Omega_2 \subset \mathbb{R}^2$ and $f : \Omega_1 \to \Omega_2$ be a homeomorphism. Then $f^{-1} \in W^{1,\infty}(\Omega_2, \Omega_1)$ if and only if $f \in BV_{loc}(\Omega_1, \Omega_2)$ and $(4.1)$ holds.

**Proof.** It remains to prove the second implication. Let $f \in BV_{loc}(\Omega_1, \Omega_2)$. It follows from [13] that $f^{-1}$ is in $BV_{loc}(\Omega_2, \Omega_1)$ and

$$|D(f^{-1})|(\Omega_2) = |Dg|(|f^{-1}(\Omega_2)|)$$

and

$$|D(f^{-1})|(\Omega_2) = |Df||f^{-1}(\Omega_2)|.$$ 

It holds for all open set $\Omega_2$ thus we have for all Borel sets $A \subset \Omega_2$

$$|D(f^{-1})|(A) \leq 2|Df||f^{-1}(A)|.$$ 

By combining with (4.1) we have $|D(f^{-1})|(A) \leq 2K_L^n(A)$ and thus the measure $D(f^{-1})$ is absolutely continuous with respect to $\mathcal{L}^n$ and its Radon-Nikodym derivative with respect to $\mathcal{L}^n$ is in $L^\infty$.

□

**Remark 4.10.** This is not true in higher dimensions. Let $n \geq 3$. There exists a Sobolev homeomorphism $f$ on $[-1,1]^n$ onto $[-1,1]^n$ such that $(4.10)$ (thus also (4.11)) is satisfied but $f^{-1}$ is not even a Sobolev function.

This function is constructed in Example 6.3. in [10]. They construct a Sobolev homeomorphism which maps one Cantor set on another one and it is piecewise affine on the complement of the Cantor set. For arbitrary $\varepsilon \in (0, n-2)$ choose an big enough parameter $l \in \mathbb{N}$ such that $\varepsilon l > 2(n - 1 - \varepsilon)$. Then for their function $f$ the following holds $Df \approx k^l$ and $J_f \approx k^{l-1}k^{(n-2)k^{-1}}$ on $A_k$, $k \in \mathbb{N}$, where $A_k$ are pairwise disjoint sets of positive measure whose union has the full measure of $[-1,1]^n$. Then

$$\frac{|Df|}{J_f} \approx \frac{k^l}{k^{l-1}k^{(n-2)k^{-1}}} = \frac{1}{k^{(n-2)-2}} \xrightarrow{k \to \infty} 0.$$ 

Hence $f$ satisfies (4.10) (even stronger condition $|Df|^{n-1-\varepsilon} \leq C J_f$ for some $C > 0$). But they show that $f^{-1}$ is continuous but not ACL.
5. A necessary condition

**Theorem 5.1.** Let \( \Omega_1, \Omega_2 \) be open subsets of \( \mathbb{R}^n \) and let \( f \in BV_{loc}(\Omega_1, \Omega_2) \) have no jump part and suppose that the operator \( T_f \) maps functions from \( C_c^\infty(\Omega_2) \) into \( BV_{loc}(\Omega_1) \) and there is a constant \( K \in \mathbb{R} \) such that for all \( u \in C_c^\infty(\Omega_2) \) we have

\[
|D(u \circ f)|(\Omega_1) \leq K|Du|(\Omega_2). 
\]

Then for all Borel set \( A \subset \Omega_2 \) we have

\[
|Df|(f^{-1}(A)) \leq 16nKL^n(A).
\]

**Proof.** We may assume that \( f = \tilde{f} \). (We change \( u \circ f \) only on a set of measure zero.) Take \( A \subset \Omega_2 \) a Borel set. Suppose that \( |Df|(f^{-1}(A)) \neq 0 \), otherwise there is nothing to prove. Let \( t > 0 \) and \( 0 < L < |Df|(f^{-1}(A)) \) be arbitrary real numbers and fix \( i \in \{1, \ldots, n\} \) such that

\[
|D(f_i)|(f^{-1}(A)) \geq \frac{1}{n} |Df|(f^{-1}(A)) > \frac{1}{n} L.
\]

Find an open set \( G \subset \Omega_2 \) such that \( A \subset G \) and \( L^n(G) \leq L^n(A) + t \). Then

\[
A = \bigcup_k A_k = \bigcup_k \{ x \in A \cap B(0, k) : \text{dist}(x, \partial G) \geq 1/k \}.
\]

Choose \( k \in \mathbb{N} \) big enough such that

\[
|Df|(f^{-1}(A_k)) > \frac{1}{n} L.
\]

Find a cut-off function \( \eta \in C_c^\infty(\Omega_2) \) satisfying

\[
\text{supp } \eta \subset G, \ 0 \leq \eta \leq 1 \text{ and } \eta = 1 \text{ on } A_k.
\]

Take \( m \) such that \( m \geq 8 \) and \( \|\nabla \eta\|_{\infty} \leq m \). Choose \( E \) among the sets

\[
E^{\sin} = \{ x \in f^{-1}(A_k) : \cos^2(m^2 f_i(x)) \geq \frac{1}{2} \}, \\
E^{\cos} = \{ x \in f^{-1}(A_k) : \sin^2(m^2 f_i(x)) \geq \frac{1}{2} \}
\]

such that

\[
|D(f_i)|(E) \geq \frac{1}{2} |D(f_i)|(f^{-1}(A_k))
\]

and set

\[
u(y) = \begin{cases} 
\frac{1}{m^2} \eta(y) \sin(m^2 y_i) & \text{if } E = E^{\sin} \\
\frac{1}{m^2} \eta(y) \cos(m^2 y_i) & \text{if } E = E^{\cos}.
\end{cases}
\]

First consider \( E = E^{\sin} \). Obviously \( u \in C_c^\infty(\Omega_2) \) and

\[
|\nabla u(y)| = |1/m^2 \nabla \eta(y) \sin(m^2 y_i) + \eta(y) \cos(m^2 y_i) e_i| \leq 2 \text{ for all } y \in \Omega_2.
\]
By the product rule from Theorem 2.4 it easily follows
\[ |D(u \circ f)|(E) = \int_E |\nabla u|(\tilde{f}(x)) \, d|Df_i| \]
\[ \geq \int_E \left( |\eta(f)\cos(m^2 f_i)e_i| - \frac{1/m^2}{m^2} \nabla \eta(f) \sin(m^2 f_i) \right) \, d|Df_i| \]
\[ \geq \int_E \left( \frac{1}{\sqrt{2}} - \frac{1}{m} \right) \, d|Df_i| \geq \frac{1}{4} |Df_i|(E) \]
\[ \geq \frac{1}{8} |Df_i|(f^{-1}(A_k)) \geq \frac{1}{8n} L. \]

Thus together with (5.1), supp \( u \subset G \) and \( |\nabla u| \leq 2 \) we estimate
\[ L \leq 8n|D(u \circ f)|(\Omega_1) \leq 8n K|Du|(\Omega_2) \leq 8nK^2 \cdot L^n(\mathcal{G}) \leq 16Kn(L^n(A) + t). \]

By taking supremum over all \( L \leq |Df|(f^{-1}(A)) \) and letting \( t \to 0 \) we obtain (5.2).

The case when \( E = E^{\cos} \) is analogous. \( \Box \)

The following corollary gives us that we may only assume that \( f^{-1} \) maps sets of finite perimeter onto sets of finite perimeter.

**Corollary 5.2.** Let \( \Omega_1, \Omega_2 \) be open subsets of \( \mathbb{R}^n \) and let \( f \in BV_{loc}(\Omega_1, \Omega_2) \) have no jump part. If for all sets of finite perimeter \( E \subset \Omega_2 \) sets \( f^{-1}(E) \) have finite perimeter and
\[ P(f^{-1}(E), \Omega_1) \leq KP(E, \Omega_2). \]
Then (5.2) holds.

**Proof.** We show that the assumptions of Theorem 5.1 are satisfied. Let \( u \in C_c^\infty(\Omega_2) \) then \( u \circ f \in L^\infty(\Omega_1) \) and with the help of Lemma 2.3 we get
\[ |D(u \circ f)|(\Omega_1) = \int_{-\infty}^{\infty} P(\{x : u(f(x)) > t\}, \Omega_1) \, dt \]
\[ = \int_{-\infty}^{\infty} P(f^{-1}(\{y : u(y) > t\}), \Omega_1) \, dt \]
\[ \leq K \int_{-\infty}^{\infty} P(\{y : u(y) > t\}, \Omega_2) \, dt \leq K|Du|(\Omega_2). \]
Thus \( u \circ f \in BV(\Omega_1) \) and (5.1) holds. \( \Box \)

**Proof of Theorem 1.1.** The first part follows directly form Theorem 1.1. Let us prove the second part. First note that \( f \in BV_{loc}(\Omega_1, \Omega_2) \). To see it take an arbitrary ball \( B \subset \subset \Omega \). Then \( f(B) \subset \subset \Omega_2 \) and hence we can find a smooth cutoff function \( \Phi \) such that \( \Phi = 1 \) on \( f(B) \) and supp \( \Phi \subset \subset \Omega_2 \). It follows that \( u = e_i \Phi, i \in (1, \ldots, n) \) are suitable test function and \( u \circ f = f_i \) on \( B \). Thus each component \( f_i \) belongs to \( BV_{loc}(\Omega_1) \).
Suppose that (1.1) does not hold. Then there are Borel sets $G_k, k \in \mathbb{N}$ such that
\begin{equation}
|Df|(f^{-1}(G_k)) > k \mathcal{L}^n(G_k).
\end{equation}

Because the Lebesgue measure is regular we may assume $G_k$ are open. Moreover, we may assume that $|Df|(f^{-1}(G_k)) < \infty$, otherwise we would replace $G_k$ by $G_k \cap \{x \in B(0, R), \text{dist}(x, \partial \Omega_1) < 1/R\}$ for some $R$ big enough. We claim that is possible to find pairwise disjoint open sets $G_k$ satisfying (5.4).

Let $l \in \mathbb{N}, G_k$ satisfies (5.4), $G_1, \ldots G_{l-1}$ are pairwise disjoint and
\[
\bigcup_{i=1}^{l-1} G_i \cap \bigcup_{i=l}^{\infty} G_i = \emptyset.
\]
We describe how to construct $\tilde{G}_k$ which has properties of $G_k$ but additionally $\tilde{G}_i \cap \bigcup_{i=l+1}^{\infty} \tilde{G}_i = \emptyset$. Fix some $m \geq l/\tau$, where $\tau$ is from Lemma 3.1. Due to the non-atomicity of the measure $|Df|$ we may use Lemma 3.1 on the measure
\[
\mu(A) = \frac{|Df|(f^{-1}(A \cap G_m))}{|Df|(f^{-1}(G_m))}
\]
to find open sets $P_1 = B(y, 2R), P_2 = \mathbb{R}^n \setminus \overline{B(y, 11/4R)}, R_1 = B(y, 10/4R), R_2 = \mathbb{R}^n \setminus \overline{B(y, 9/4R)}$ such that
\[
\mu(P_1), \mu(P_2) \geq \tau, P_1 \cap R_2 = \emptyset = P_2 \cap R_1, R_1 \cup R_2 = \mathbb{R}^n.
\]
Then we obtain for all $i \in \mathbb{N}$ that
\[
|Df|(f^{-1}(G_i \cap R_1)) + |Df|(f^{-1}(G_i \cap R_2)) \geq |Df|(f^{-1}(G_i)) > i \mathcal{L}^n(G_i) \geq i/2 \mathcal{L}^n(G_i \cap R_1) + i/2 \mathcal{L}^n(G_i \cap R_2).
\]
Hence at least one of the sets
\[
C = \{i \in \mathbb{N}, i > m, |Df|(f^{-1}(G_i \cap R_1)) > i/2 \mathcal{L}^n(G_i \cap R_1)\},
\]
\[
D = \{i \in \mathbb{N}, i > m, |Df|(f^{-1}(G_i \cap R_2)) > i/2 \mathcal{L}^n(G_i \cap R_2)\}
\]
has to be infinite. First consider the case when $C$ is infinite. Let $c_i, i \in \mathbb{N}$ be an increasing sequence containing all elements of $C$. Set $\tilde{G}_i = G_i$ for $i < l$,
\[
\tilde{G}_l = G_m \cap P_2 \text{ and } G_i = G_{c_2i} \cap R_l \text{ for } i > l.
\]
Then obviously $\tilde{G}_i, i \in \{1, \ldots l\}$ are pairwise disjoint and
\[
\bigcup_{i=1}^{l} \tilde{G}_i \cap \bigcup_{i=l+1}^{\infty} \tilde{G}_i = \emptyset.
\]
It remains to verify (5.4). It follows that
\[
|Df|(f^{-1}(\tilde{G}_l)) = \mu(P_2) \cdot |Df|(f^{-1}(G_m)) > \tau m \mathcal{L}^n(G_m) \geq l \mathcal{L}^n(\tilde{G}_l)
\]
and for all $i > l$ we have

$$|Df| (f^{-1}(\tilde{G}_i)) = |Df| (f^{-1}(G_{c_2i} \cap R_1)) > 1/2 \cdot c_2i \mathcal{L}^n(G_{c_2i} \cap R_1) \geq i \mathcal{L}^n(\tilde{G}_i).$$

The case when $D$ is infinite is analogous. Thus we may iterate this construction to obtain $G_1, G_2, \ldots$ pairwise disjoint.

Because $f$ has no jump part and (5.2) does not hold on $G_k$ with $K = \frac{1}{16n}k$ it follows from Theorem 5.1 that there are $u_k \in C^\infty_c(G_k)$ such that

$$|Du_k \circ f|(\Omega_1) > \frac{1}{16n}k|Du_k|(G_k).$$

Replace $u_k$ by its constant multiple to obtain $|Du_k|(G_k) = 1$. Due to the fact that $|Dv| = |Dv|$ for any function $v$ of bounded variation we may assume that $\|u_k\|_{L^\infty} \leq 1$ (Otherwise we can iterate replacing $u_k$ by function $\tilde{u}_k = ||u_k| - 1/2\|u_k\|_{L^\infty}$, which has the same total variation of the distributional derivative and its maximum is half of the maximum of $u_k$.)

Set

$$u = \sum_{k=1}^{\infty} \frac{1}{k^2} u_k.$$

Obviously $u \in C_0 \cap BV(\Omega_2)$ and

$$|Du \circ f|(\Omega_1) = \sum_{k=1}^{\infty} \frac{1}{k^2} |Du_k \circ f|(f^{-1}(G_k)) = \sum_{k=1}^{\infty} Ck \frac{1}{k^2} = \infty.$$

Proof of Theorem 1.2. To prove first part we can follow the proof of Corollary 5.2 and instead of using Theorem 5.1 we use Theorem 1.1.

The second implication follows directly from Corollary 4.2. □

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