Arithmetic on Moran sets

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Abstract

Let \((\mathcal{M}, c_k, n_k)\) be a class of Moran sets. We assume that the convex hull of any \(E \in (\mathcal{M}, c_k, n_k)\) is \([0, 1]\). Let \(A, B\) be two non-empty sets in \(\mathbb{R}\). Suppose that \(f\) is a continuous function defined on an open set \(U \subset \mathbb{R}^2\). Denote the continuous image of \(f\) by
\[
 f_U(A, B) = \{ f(x, y) : (x, y) \in (A \times B) \cap U \}.
\]
In this paper, we prove the following result. Let \(E_1, E_2 \in (\mathcal{M}, c_k, n_k)\). If there exists some \((x_0, y_0)\) \(\in (E_1 \times E_2) \cap U\) such that
\[
 \sup_{k \geq 1} \{ 1 - c_k n_k \} \times \frac{\partial_y f(x_0, y_0)}{\partial_x f(x_0, y_0)} < \inf_{k \geq 1} \left\{ \frac{c_k}{1 - n_k c_k} \right\}.
\]
Then \(f_U(E_1, E_2)\) contains an interior.

1 Introduction

Given two non-empty sets \(A, B \subset \mathbb{R}\). Define \(A * B = \{ x * y : x \in A, y \in B \}\), where \(*\) is \(+\), \(-\), \(\times\) or \(\div\) (when \(*\) = \(\div\), \(y \neq 0\)). We call \(A * B\) the arithmetic on \(A\) and \(B\). Generally, we may define the arithmetic on \(A\) and \(B\) in terms of some functions. Suppose that \(f\) is a continuous function defined on an open set \(U \subset \mathbb{R}^2\). Denote the continuous image of \(f\) by
\[
 f_U(A, B) = \{ f(x, y) : (x, y) \in (A \times B) \cap U \}.
\]
For simplicity, we still call \(f_U(A, B)\) the arithmetic on \(A\) and \(B\). Arithmetic on the fractal sets has strong connections with many different problems in geometry measure theory and dynamical systems \([30, 26]\). For instance, in geometry measure theory, the visible problem is related to the division on the fractals \([6, 11, 18]\). The main reason is due to the following observation. Let \(K \subset [0, 1]\) be a fractal set. Given \(\alpha \geq 0\), we say the line \(y = \alpha x\) is visible through \(K \times K\) if
\[
 \{(x, \alpha x) : x \in \mathbb{R} \setminus \{0\}\} \cap (K \times K) = \emptyset.
\]
It is easy to verify that the line \(y = \alpha x\) is visible through \(K \times K\) if and only if
\[
 \alpha \notin \frac{K}{K} := \left\{ \frac{x}{y} : x, y \in K, y \neq 0 \right\}.
\]

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The arithmetic sum of two Cantor sets was studied by many scholars. There are many results concerning with this topic, see [2] [3] [4] [8] [12] [15] [20] and references therein. It is an important problem in homoclinic bifurcations [19]. Palis [19] posed the following problem: whether it is true (at least generically) that the arithmetic sum of dynamically defined Cantor sets either has measure zero or contains an interval. This conjecture was solved in [2]. Motivated by Palis’ conjecture, it is natural to investigate when the sum of two Cantor sets contains some interiors. Newhouse [27] proved the following thickness theorem. Given any two Cantor sets $C_1$ and $C_2$, if $\tau(C_1)\tau(C_2) > 1$, where $\tau(C_i), i = 1, 2$ denotes the thickness of $C_i, i = 1, 2$, then $C_1 + C_2$ contains some interiors. However, Newhouse thickness theorem cannot handle a general function $f$, i.e. whether $f(C_1, C_2)$ contains an interior or not.

To date, there are not so many results concerning with the arithmetic on the fractal sets [11] [23] [24]. The first result of this direction, to the best of our knowledge, is due to Steinhaus [23] who proved the following interesting result: $C - C = [-1, 1]$, where $C$ is the middle-third Cantor set. Equivalently, Steinhaus proved that for any $x \in [-1, 1]$, there are some $x_1, x_2 \in C$ such that $x = x_1 - x_2$. Recently, Athreya, Reznick and Tyson [11] considered the multiplication on the middle-third Cantor set. They proved that $17/21 \leq L(C \cdot C) \leq 8/9$, where $L$ denotes the Lebesgue measure. Jiang and Xi [13] proved that $C \cdot C$ indeed contains infinitely many intervals. In [14], Jiang and Xi considered the representations of real numbers in $C - C$, i.e. let $x \in [-1, 1]$, define

$$S_x = \{(y_1, y_2) : y_1 - y_2 = x, (y_1, y_2) \in C \times C\}.$$ 

and

$$U_r = \{x : z(S_x) = r\}, r \in \mathbb{N}^+.$$ 

They proved that $\dim_H(U_r) = \frac{\log 2}{\log 3}$ if $r = 2^k$ for some $k \in \mathbb{N}$. Moreover,

$$0 < \mathcal{H}^s(U_1) < \infty, \mathcal{H}^s(U_{2^k}) = \infty, k \in \mathbb{N}^+,$$

where $s = \frac{\log 2}{\log 3}$, $U_{32^k} \cdot U_{32^k}$ is an infinitely countable set for any $k \geq 1$, where $\dim_H$ and $\mathcal{H}^s$ denote the Hausdorff dimension and Hausdorff measure, respectively. For more results, see [14]. In [25], Tian et al. defined a class of overlapping self-similar sets as follows: let $K$ be the attractor of the IFS

$$\{f_1(x) = \lambda x, f_2(x) = \lambda x + c - \lambda, f_3(x) = \lambda x + 1 - \lambda\},$$

where $f_1(I) \cap f_2(I) \neq \emptyset, (f_1(I) \cup f_2(I)) \cap f_3(I) = \emptyset$, and $I = [0, 1]$ is the convex hull of $K$. This class of self-similar set was investigated by many scholars, see [7] [9] [16] [17] [28] [29] [30]. Tian et al. $K \cdot K = [0, 1]$ if and only if $(1 - \lambda)^2 \leq c$. Equivalently, they gave a necessary and sufficient condition such that for any $x \in [0, 1]$ there exist some $y, z \in K$ such that $x = yz$. Moreover, Ren, Zhu, Tian and Jiang [21] proved that

$$\sqrt{K} + \sqrt{K} = [0, 2]$$

if and only if

$$\sqrt{c} + 1 \geq 2\sqrt{1 - \lambda},$$

where $\sqrt{K} + \sqrt{K} = \{\sqrt{x} + \sqrt{y} : x, y \in K\}$. If $c \geq (1 - \lambda)^2$, then

$$\frac{K}{K} = \left\{\frac{x}{y} : x, y \in K, y \neq 0\right\} = [0, \infty).$$
As a consequence, they proved that the following conditions are equivalent:

1. For any $u \in [0, 1]$, there are some $x, y \in K$ such that $u = x \cdot y$;
2. For any $u \in [0, 1]$, there are some $x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10} \in K$ such that
   \[ u = x_1 + x_2 = x_3 - x_4 = x_5 \cdot x_6 = x_7 \div x_8 = \sqrt{x_9} + \sqrt{x_{10}}; \]
3. $c \geq (1 - \lambda)^2$.

In this paper, we shall consider similar problems on the Moran sets. The Moran sets are, in certain sense, random. Nevertheless, any self-similar set with the open set condition is a Moran set \cite{10}. Now we give the definition of a class of Moran set. Let $\{n_k\} \subset \mathbb{N}^+$ be a sequence (we assume that $n_k \geq 2$). For any $k \in \mathbb{N}^+$, write

\[ D_k = \{(\sigma_1, \cdots, \sigma_k) : \sigma_j \in \mathbb{N}^+, 1 \leq \sigma_j \leq n_j, 1 \leq j \leq k \}. \]

Define

\[ D = \bigcup_{k \geq 0} D_k. \]

We call $\sigma \in D$ a word. For simplicity, we let $D_0 = \emptyset$. If $\sigma = (\sigma_1, \cdots, \sigma_k) \in D_k$, $\tau = (\tau_1, \cdots, \tau_m) \in D_m$, then we define the concatenation $\sigma \ast \tau = (\sigma_1, \cdots, \sigma_k, \tau_1, \cdots, \tau_m) \in D_{k+m}$. Let $T = [0, 1]$ and $\{c_k\}$ be a positive real sequence with $c_k n_k < 1$, $k \in \mathbb{N}^+$, we say the class

\[ \mathcal{F} = \{T_\sigma \subset T : \sigma \in D\} \]

has the Moran structure if the following conditions are satisfied:

1. for any $\sigma \in D$, $T_\sigma$ is similar to $T$, i.e. there exists a similitude $S_\sigma : \mathbb{R} \rightarrow \mathbb{R}$ such that $S_\sigma(T) = T_\sigma$;
2. for any $k \geq 0$ and $\sigma \in D_k$, $T_{\sigma_1}, T_{\sigma_2}, \cdots, T_{\sigma n_k+1}$ is a subset of $T_\sigma$ and
   \[ \text{int}(T_{\sigma i}) \cap \text{int}(T_{\sigma j}) = \emptyset, i \neq j, \]
   where $\text{int}(A)$ denotes the interior of $A$, for simplicity, we denote by $\tilde{T_\sigma} = \bigcup_{i=1}^{n_k+1} T_{\sigma i}$;
3. for any $k \geq 1$ and $\sigma \in D_{k-1}$, $\frac{|T_{\sigma i}|}{|T_\sigma|} = c_k$, and the convex hull of $T_{\sigma i}$ and $T_\sigma$ coincide for any $1 \leq i \leq n_k$, where $|A|$ denotes the diameter of $A$.

Suppose $\mathcal{F} = \{T_\sigma \subset T : \sigma \in D\}$ has the Moran structure, then we call

\[ E = \bigcap_{k \geq 1} \bigcup_{\sigma \in D_k} T_\sigma \]

a Moran set. We denote by $(\mathcal{M}, c_k, n_k)$ all the Moran sets generated by the Moran structure $\mathcal{F}$. By the third condition, it is easy to see that the convex hull of any $E$ from $(\mathcal{M}, c_k, n_k)$ is $[0, 1]$.

Now we are ready to state the main result of this paper.
Theorem 1.1. Let \( E_1, E_2 \in (\mathcal{M}, c_k, n_k) \). If there exists some \((x_0, y_0) \in (E_1 \times E_2) \cap U\) such that
\[
\sup_k \{1 - c_k n_k\} < \frac{|\partial_y f|_{(x_0, y_0)}}{|\partial_x f|_{(x_0, y_0)}} < \inf_k \left\{ \frac{c_k}{1 - n_k c_k} \right\},
\]
Then \( f_U(E_1, E_2) \) contains an interior.

The paper is arranged as follows. In section 2, we prove two basic lemmas and give a proof of Theorem 1.1. In section 3, we give some remarks.

2 Proof of Theorem 1.1

In this section, we shall prove Theorem 1.1. First, we give some definitions and prove two useful lemmas.

For any \( k \geq 1 \), denote by \( E_k \) the union of basic intervals when we construct a Moran set \( E \), i.e.
\[
E_k = \bigcup_{\sigma \in D_k} T_\sigma, \quad E = \cap_{k=1}^{\infty} E_k,
\]
where \( T_\sigma \) is called a basic interval with rank \( k \). It is easy to check that the length of any basic interval with rank \( k \) is \( c_1 c_2 \cdots c_k \). Let \([A, B] \subset [0, 1]\), where \( A \) and \( B \) are the left and right endpoints of some basic intervals in \( E_k \) for some \( k \geq 1 \), respectively. \( A \) and \( B \) may not in the same basic interval. In the following lemma, we choose \( A \) and \( B \) in this way. Let \( F_k \) be the collection of all the basic intervals in \([A, B]\) with length \( c_1 c_2 \cdots c_k, k \geq k_0 \) for some \( k_0 \in \mathbb{N}^+ \), i.e. the union of all the elements of \( F_k \) is denoted by \( G_k = \bigcup_{t=1}^{k} I_{k,i} \), where \( t_k \in \mathbb{N}^+ \), \( I_{k,i} \subset E_k \cap [A, B] \). Clearly, by the definition of \( G_n \), it follows that \( G_{n+1} \subset G_n \) for any \( n \geq k_0 \).

Lemma 2.1. Let \( E_1, E_2 \in (\mathcal{M}, c_k, n_k) \), i.e.
\[
E_1 = \cap_{k=1}^{\infty} E_k^{(1)}, \quad E_2 = \cap_{k=1}^{\infty} E_k^{(2)}.
\]
Assume \( F : \mathbb{R}^2 \to \mathbb{R} \) is a continuous function. Suppose \( A \) and \( B \) (\( C \) and \( D \)) are the left and right endpoints of some basic intervals in \( E_k^{(1)}(E_k^{(2)}) \) for some \( k_0 \geq 1 \), respectively. Then \( E_1 \cap [A, B] = \cap_{n=k_0}^{\infty} G_n^{(1)}, E_2 \cap [C, D] = \cap_{n=k_0}^{\infty} G_n^{(2)} \). Moreover, if for any \( n \geq k_0 \) and any basic intervals \( I_1 \subset G_n^{(1)}, I_2 \subset G_n^{(2)} \), we have
\[
F(I_1, I_2) \subset F(I_1, I_2),
\]
then \( F(E_1 \cap [A, B], E_2 \cap [C, D]) = F(G_n^{(1)}, G_n^{(2)}) \).

Proof. We assume that \( G_n^{(i)} = \bigcup_{1 \leq j \leq n} I_{n,i} \), \( i = 1, 2 \). By the construction of \( G_n^{(i)}, i = 1, 2 \), it is clear that \( G_n^{(i)} \subset G_n^{(i)} \) for any \( n \geq 1 \). Therefore,
\[
E_1 \cap [A, B] = \cap_{n=k_0}^{\infty} G_n^{(1)}, E_2 \cap [C, D] = \cap_{n=k_0}^{\infty} G_n^{(2)}.
\]
In terms of the continuity of \( F \), we conclude that
\[
F(E_1 \cap [A, B], E_2 \cap [C, D]) = \cap_{n=k_0}^{\infty} F(G_n^{(1)}, G_n^{(2)}).
\]
Therefore,

\[
F(G_n^{(1)}, G_n^{(2)}) = \bigcup_{1 \leq i \leq n, 1 \leq j \leq n} F(I_{n,i}, J_{n,j})
\]

\[
= \bigcup_{1 \leq i \leq n, 1 \leq j \leq n} F(\tilde{I}_{n,i}, \tilde{J}_{n,j})
\]

\[
= F(\bigcup_{1 \leq i \leq n} \tilde{I}_{n,i}, \bigcup_{1 \leq j \leq n} \tilde{J}_{n,j})
\]

\[
= F(G_{n+1}^{(1)}, G_{n+1}^{(2)}).
\]

Consequently, \(F(E_1 \cap [A, B], E_2 \cap [C, D]) = F(G_{k_0}^{(1)}, G_{k_0}^{(2)})\) follows immediately from the identity (1) and \(F(G_n^{(1)}, G_n^{(2)}) = F(G_{n+1}^{(1)}, G_{n+1}^{(2)})\) for any \(n \geq k_0\). \(\square\)

**Lemma 2.2.** Let \(I = [a, a + t], J = [b, b + t]\) be two basic intervals in \(G_k^{(1)}\) and \(G_k^{(2)}\), respectively. If there exists some \((x_0, y_0) \in (E_1 \times E_2) \cap (I \times J) \cap U\) such that

\[
\sup_k \{1 - c_k n_k\} < \left| \frac{\partial_y f |_{(x_0, y_0)}}{\partial_x f |_{(x_0, y_0)}} \right| < \inf_k \left\{ \frac{c_k}{1 - n_k c_k} \right\}.
\]

Then \(f(I, J) = f(\tilde{I}, \tilde{J})\).

**Proof.** Without loss of generality, we assume that \(\partial_x f |_{(x_0, y_0)} > 0, \partial_y f |_{(x_0, y_0)} > 0\). For other cases, we may consider the new function \(F(x, y) = f(x, 1 - y)\) or \(-f(x, y)\). By the definition of \(\tilde{I}\) and \(\tilde{J}\), we have

\[
\tilde{I} = \bigcup_{i=1}^{n_k} I_i, \tilde{J} = \bigcup_{j=1}^{n_k} J_j.
\]

Moreover, \(t = |I| = |J| = c_1 \cdots c_{k-1}\), where \(|A|\) denotes the length of \(A\). Therefore, we have

\[
f(\tilde{I}, \tilde{J}) = \bigcup_{i=1}^{n_k} \bigcup_{j=1}^{n_k} f(I_i, J_j).
\]

We first prove that for any \(1 \leq i \leq n_k, \bigcup_{j=1}^{n_k} f(I_i, J_j)\) is an interval. By the construction of Moran set, it suffices to prove that \(f(P_1) \geq f(P_2)\), see the second picture of Figure 1, that is, it remains to prove that there exists some \((\xi, \eta) \in E_1 \times E_2\) contained in the neighbour of \((x_0, y_0)\) such that

\[
(c_1 \cdots c_k) \partial_x f(\xi, \eta) \geq (c_1 c_2 \cdots c_{k-1} - n_k c_1 \cdots c_k) \partial_y f(\xi, \eta).
\]
However, this is clear due to the condition
\[
\frac{\partial_y f}{\partial_x f}(x_0, y_0) < \inf_k \left\{ \frac{c_k}{1 - n_k c_k} \right\},
\]
and the assumption \( \partial_x f, \partial_y f \) are continuous. Next, we prove that
\[
\bigcup_{i=1}^{n_k} \bigcup_{j=1}^{n_k} f(I_i, J_j)
\]
is an interval. Analogously, we need to show that \( f(P_3) \geq f(P_4) \), see the third picture of Figure 1. Indeed, it only remains to prove that there is some \((\xi_1, \eta_1) \in E_1 \times E_2\) which lies in the neighbour of \((x_0, y_0)\) such that
\[
(c_1 \cdots c_{k-1})\partial_y f(\xi_1, \eta_1) \geq (c_1 c_2 \cdots c_{k-1} - n_k c_1 \cdots c_k)\partial_x f(\xi_1, \eta_1).
\]
However, the above inequality follows from the condition
\[
\sup_k \{1 - c_k n_k\} < \frac{\partial_y f(x_0, y_0)}{\partial_x f(x_0, y_0)},
\]
and \( \partial_x f, \partial_y f \) are continuous. Therefore, we have proved that \( f(I, J) = f(\tilde{I}, \tilde{J}) \).

Proof of Theorem 1.1. Theorem 1.1 follows immediately from Lemmas 2.1 and 2.2.

3 Final remark

In Lemma 2.1, we note that if some basic intervals of \( E_k \) intersects, then similar result as Theorem 1.1 can be obtained. We leave it to the readers.

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