AN INEQUALITY FOR THE MAXIMUM CURVATURE THROUGH A GEOMETRIC FLOW

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Abstract. We provide a new proof of the following inequality: the maximum curvature $k_{\text{max}}$ and the enclosed area $A$ of a smooth Jordan curve satisfy $k_{\text{max}} \geq \sqrt{\pi/A}$. The feature of our proof is the use of the curve shortening flow.

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The aim of the present note is to give a new proof for the following inequality: if $\gamma \subset \mathbb{R}^2$ is a smooth Jordan curve, then

$$k_{\text{max}} \geq \sqrt{\pi/A},$$

where $k_{\text{max}}$ is the maximum curvature and $A$ is the enclosed area, and the equality holds iff $\gamma$ is a circle. We remark that here and later on we work with the signed curvature, in particular, the curvature is non-negative iff the curve is convex. The inequality (1) follows from a result by Pestov-Ionin on inscribed disks [7]: If the curve of a smooth Jordan curve does not exceed some positive $\kappa > 0$, then the interior of the curve contains a disk of radius $1/\kappa$. Hence, the comparison of the areas gives $\pi/\kappa^2 \leq A$, and (1) is obtained for $\kappa = k_{\text{max}}$. The original work [7] is hardly available, and a complete proof can be found e.g. in [5, Proposition 2.1]. We are going to show that the inequality (1) can be alternatively deduced from the properties of the curve shortening flow [3, 4]. The use of geometric flows for isoperimetric inequalities is a well established machinery, see e.g. [9, 8], but the link to the inequality (1) seems to be new. We also mention that Eq. (1) plays a role for Faber-Krahn-type inequalities for some eigenvalue problems [6]. Our proof naturally splits in several parts.

A. Uniqueness. We remark first that if the inequality (1) is proved, then one may show in a standard way that the equality holds only for the circles, see e.g. [6, Proposition 7]; we include the argument for the sake of completeness. By contradiction, assume that one has the equality in (1) for some $\gamma$ different from a circle. At some point of $\gamma$ the curvature is strictly smaller than $k_{\text{max}}$, and by a small local deformation of $\gamma$ near such a point we may construct a new smooth Jordan curve $\gamma'$ having the same maximum curvature $k_{\text{max}}'$ but enclosing a strictly smaller area $A'$, which gives $k_{\text{max}}' < \sqrt{\pi/A'}$ and contradicts (1).

B. The inequality holds for the star-shaped curves. It is elementary to show (1) for star-shaped curves, cf. e.g. [6, Proposition 7], and we include the proof for convenience. Assume that $\gamma$ is star-shaped with respect to the origin and denote by $\ell$ its length. Let $\Gamma: \mathbb{R}/\ell \mathbb{Z} \to \gamma \subset \mathbb{R}^2$ be a properly oriented arc-length parametrization, then the Frenet formula $\Gamma'' = -kn$, where $k$ is the curvature and $n$ is the outer unit normal, and the integration by parts give

$$\int_0^\ell k \Gamma \cdot n \, ds = -\int_0^\ell \Gamma \cdot \Gamma'' \, ds = \int_0^\ell \|\Gamma''\|^2 \, ds = \ell.$$

As $\gamma$ is star-shaped, we have $\Gamma \cdot n \geq 0$ and

$$\ell = \int_0^\ell k \Gamma \cdot n \, ds \leq k_{\text{max}} \int_0^\ell \Gamma \cdot n \, ds = k_{\text{max}} \int_\gamma x_1 \, dx_2 - x_2 \, dx_1 = 2k_{\text{max}}A,$$

and (1) follows from the classical isoperimetric inequality $\ell^2 \geq 4\pi A$, see e.g. [2, §2.10].

C. Some properties of the flow by curvature. The study of general curves will be reduced to the star-shaped ones using the flow by curvature (also called the curve shortening flow). Denote $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ and let $C(\cdot, 0): \mathbb{T} \to \mathbb{R}^2$ be a smooth embedded curve. By [4, Main theorem and introduction], there exist $T > 0$ and $C : \mathbb{T} \times [0, T) \to \mathbb{R}^2$ such that, for any $t$, $C(\cdot, t)$ is a smooth embedded curve and

$$\frac{\partial C(x, t)}{\partial t} = -k(x, t)n(x, t),$$

where $n(x, t)$ and $k(x, t)$ are respectively the outer unit normal and the curvature of the curve $C(\cdot, t)$ at the point $C(x, t)$, and the limiting shape is a round point, with convergence in $C^\infty$ norm, and, in particular, there exists

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\( \tau \in [0, T) \) such that \( C(\cdot, t) \) is convex for \( t \in [\tau, T) \). We remark that a compact proof can be found in [1]. The following properties will be used, see Section 1 in [4]: the area \( A(t) \) enclosed by the curve \( C(\cdot, t) \) is

\[ A(t) = A(0) - 2\pi t, \]

hence, \( T = A(0)/(2\pi) \), and the curvature satisfies

\[ \frac{\partial k}{\partial t} = \frac{\partial^2 k}{\partial s^2} + k^3, \]

where \( \partial/\partial s \) means the derivative with respect to the arc-length on \( C(\cdot, t) \).

D. Proof of (1) for general curves. By using a suitable scaling we may assume that \( A = \pi \), then the sought inequality becomes \( k_{\text{max}} \geq 1 \). By contradiction, assume that for some \( \gamma \) we have

\[ k_{\text{max}} < 1 \]

and construct a family \( C(\cdot, t), t \in [0, 1/2) \), of curves evolving by curvature as in the part C with \( C(\cdot, 0) = \gamma \). By (2), the curve \( C(\cdot, t) \) encloses the area \( \pi(1 - 2t) \), hence, the enlarged curves

\[ \Sigma(t) := \frac{1}{\sqrt{1 - 2t}} C(\cdot, t) \]

enclose the constant area \( \pi \), and the curvature \( K \) on \( \Sigma(\cdot, t) \) is \( K = \sqrt{1 - 2t} k \). Using the equality (3) we arrive at

\[ \frac{\partial K}{\partial t} = \sqrt{1 - 2t} \frac{\partial^2 k}{\partial s^2} - \frac{1}{1 - 2t} K(1 - K^2). \]

By (4), there is \( M \in (0, 1) \) with \( K(x, 0) < M \) for all \( x \in T \). Let us show that

\[ K(x, t) < M < 1 \]

for all \( (x, t) \). Assume by contradiction that the inequality (6) is false, then there exists a minimal value \( t_* \in (0, 1/2) \) for which one can find \( x_* \in T \) with \( K(x_*, t_*) = M \), and then \( x_* \) is a maximum of \( K(\cdot, t_*) \). As a consequence it is also a maximum of \( k(\cdot, t_*) \), in particular, \( \partial^2 k/\partial s^2(x_*, t_*) \leq 0 \), and the equality (5) gives

\[ \frac{\partial K}{\partial t}(x_*, t_*) \leq -\frac{1}{1 - 2t_*} M(1 - M^2) < 0. \]

It follows that for small positive \( \varepsilon \) one has \( K(x_*, t_* - \varepsilon) > M \), which contradicts the above choice of \( t_* \). Hence, the claim (6) holds. On the other hand, by part C, for some \( \tau > 0 \) the curve \( \Sigma(\cdot, \tau) \) is convex, and, by part B, for some \( x \in T \) we have \( K(x, \tau) \geq 1 \), which contradicts the inequality (6). Therefore, the condition (4) cannot be satisfied.

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