Causal inference with misspecified exposure mappings

Fredrik Sävje

Yale University

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Abstract

Exposure mappings facilitate investigations of complex causal effects when units interact in experiments. Current methods assume that the exposures are correctly specified, but such an assumption cannot be verified, and its validity is often questionable. This paper describes conditions under which one can draw inferences about exposure effects when the exposures are misspecified. The main result is a proof of consistency under mild conditions on the errors introduced by the misspecification. The rate of convergence is determined by the dependence between units’ specification errors, and consistency is achieved even if the errors are large as long as they are sufficiently weakly dependent. In other words, exposure effects can be precisely estimated also under misspecification as long as the units’ exposures are not misspecified in the same way. The limiting distribution of the estimator is discussed. Asymptotic normality is achieved under stronger conditions than those needed for consistency. Similar conditions also facilitate conservative variance estimation.

Keywords: Causal inference, experiments, interference, spillover effects.

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1 Introduction

Experimenters use exposure mappings to investigate complex causal effects involving interference between units. An exposure mapping is a terse representation of the nominal treatments assigned to the units under study. The representation facilitates both definition and estimation of causally relevant exposure effects, presuming that the exposures are correctly specified. The exposures are correctly specified when they capture all causal information pertaining to the treatments. Recognizing that it is often difficult to construct correctly specified exposures, this paper considers estimation of exposure effects when the exposures are misspecified.

The paper builds on the insight that exposure mappings conventionally fill two separate roles. The first role is to capture aspects of treatment assignment deemed relevant or interesting for the question at hand. To serve this purpose, the exposure mappings do not need to be correctly specified, because they can successfully capture interesting aspects of the interference without necessarily capture all causal information. For example, in a study of whether a vaccine can prevent household transmission of some pathogen, we may use an exposure mapping to define household-level spillover effects without necessarily making a claim that no other spillover effects exist. The second role is to encode assumptions about the causal structure in the experiment.

It is convenient to have exposures that fill both of these roles simultaneously, because it allows experimenters to use standard causal inference techniques also in the presence of interference. If the exposures are correctly specified, there is no unmodeled interference, so experimenters can proceed as they typically would but with the exposures as treatments. However, it is rare that experimenters have sufficiently detailed knowledge about the causal structure in an experiment to construct an exposure mapping that fills the second role. In the vaccine example, it would require the experimenter to either assume there are no spillover effects between households, which generally is untenable, or extend the exposure mapping to include more information about the nominal treatments, which generally is impractical, and would make the results of the study difficult to interpret. Therefore, when experimenters try to construct exposures that fill both roles, they are forced to make unverified and often questionable assumptions.

This paper aims to separate the two roles, allowing experimenters to capture relevant aspects of the causal structure they want to investigate without making untenable assumptions on that structure. That is, the exposures are constructed to fill the first role but not necessarily the second one. This renders the exposures misspecified, motivating the investigation of estimation of exposure effects under misspecification that is the primary focus of the paper.

The investigation starts by recognizing that the conventional definition of exposure effects is not appropriate when the exposures are not correctly specified, and the paper
provides a definition of the effects that is robust to misspecification. The re-defined exposure effect is the average difference in expected outcomes for the two exposures under consideration. The definition has the advantage that it coincides with the conventional exposure effect when the exposures are correctly specified but remains well-defined when the exposures are misspecified.

The paper next considers estimation of the exposure effects, with a focus on conventional estimators, including the Horvitz–Thompson and Hájek estimators. The main contribution is a proof of consistency of the estimators for the exposure effects under mild conditions on the errors produced by the misspecification. The critical condition is that the dependence between units’ specification errors is sufficiently weak. Like the assumption of correctly specified exposure mappings, the weak dependence assumption is generally not testable. Its advantage is instead that it is considerably more tenable than the prevailing assumption; assuming the exposures are correctly specified is equivalent to assuming that the specification errors are uniformly zero. Weak dependence allows for potentially grave misspecification as long as the units’ exposures are not misspecified in the same way. The final contribution is a discussion about asymptotic normality and variance estimation. These tasks are less tractable than point estimation, but conditions allowing for some progress are discussed.

2 Related work

The idea of exposure mappings has its origin in Halloran and Struchiner (1995), who discuss causal inference under interference and provide some foundational definitions. This initial work was later developed by Sobel (2006) and Hudgens and Halloran (2008), who both consider effects that we today would recognize as exposure effects. The authors consider exposures based on proportions of treated units in disjoint groups of units. For example, an exposure effect in this setting could be when 75% of the units in the group are treated versus when only 25% are treated.

Hudgens and Halloran (2008) use two assumptions to investigate these effects: partial interference and stratified interference. The first assumption stipulates that only the treatment assignments of units in a unit’s own group affect its outcome. The second assumption stipulates that only the proportion of treated units affects the outcome. Taken together, the two assumptions amount to assuming that the group proportion of treated units together with a unit’s own treatment assignment is a complete description of the causal structure, or in other words, that the exposures are correctly specified. These early results were later extended to more intricate settings; for example, Toulis and Kao (2013) consider when interference is restricted to neighborhoods in known social networks, which may not be disjoint.

The exposure mapping idea was taken to full generality by Manski (2013) and by Aronow
and Samii (2017). Much of the following literature has been built on the latter paper, so it will be the focus here. Aronow and Samii (2017) recognize that the key methodological tool in Hudgens and Halloran (2008) was a terse description of the treatments assigned to all units. This insight suggests a generalization beyond proportions of treated units in some neighborhood to arbitrary summaries of the nominal treatments. The authors consider a function that maps from the full set of treatments to some low-dimensional representation thereof. The elements of the codomain of this function are given labels, referred to as “exposures,” and contrasts between outcomes under different exposures are interpreted as causal effects. The authors impose the assumption that the exposures are correctly specified. That is, they assume that the exposures provide a complete description of the causal structure. They then proceed as usual to estimate the exposure effect, but with the exposures substituted for the nominal treatments.

The assumption that the exposures are correctly specified is critical to prove the theoretical results in Manski (2013) and Aronow and Samii (2017). The assumption has a direct parallel to the no-interference assumption often used to facilitate inference about average treatment effects (Rubin, 1980). The average treatment effect may be interpreted as an exposure effect based on a simple exposure mapping with two level only depending on a unit’s own treatment assignment. The necessity of the no-interference in this setting has recently been investigated by Sävje, Aronow, and Hudgens (2021). The authors show that inferences can be drawn about average treatment effects even if the units’ treatments are misspecified, which provides inspiration for the results presented here. This paper connects these ideas with the literature discussed above, generalizing the results to exposure mappings of arbitrary complexity. The current paper also derives its results using quantitative interference measures, whereas Sävje et al. (2021) used a qualitative measure. That is, the interference measures used here take into account not only whether interference exists between units, but also the strength of that interference.

A parallel strand of the literature investigates interference under misspecification using ideas from Fisher (1935). The main insight here is that randomization tests can be constructed without the need for correctly specified exposures as long as the tested hypotheses are sufficiently precise (Aronow, 2012; Athey, Eckles, & Imbens, 2018; Basse, Feller, & Toulis, 2018; Bowers, Fredrickson, & Panagopoulos, 2013). The approach requires that the response of a subset of units are perfectly known under the null hypothesis for all possible treatment assignments for some subset of assignments. Rosenbaum (2007) uses similar ideas to show that certain test statistics can be inverted to form an estimate of the shift in the ranks of the outcomes for treated units relative to controls without any restrictions on the interference. Choi (2017) extends the approach to estimate the size, rather than rank shifts, of attributable effects of treatment under the assumption of non-negative effects.

The Fisherian approach does not easily accommodate estimation of exposure effects. Closer to the current investigation is a collection of papers using more conventional es-
timation approaches. Eckles, Karrer, and Ugander (2017) discuss strategies to minimize bias introduced by violations to no-interference assumptions. Basse and Airoldi (2018) and Karwa and Airoldi (2018) provide impossibility results for inference about causal quantities when no assumptions are made about the interference structure. Egami (2018) studies estimation of spillover effects in partially unobserved interference networks, which is a way to formalize misspecification. Building on the results in this paper, Leung (2019) develops techniques to estimate exposure effects under misspecification when partial information about the interference structure is available in the from of a graph on which the strength of interference is known to decay in the geodesic distance between vertices.

3 Misspecified exposures

3.1 Preliminaries

Consider a sample of $n$ units indexed by $\mathcal{U} = \{1, 2, \ldots, n\}$ and a set of treatments indexed by $\mathcal{Z} \subseteq \mathbb{N}$. Each unit $i \in \mathcal{U}$ is assigned one of the treatments $z_i \in \mathcal{Z}$. The assignments of all units are collected in $z = (z_1, \ldots, z_n)$, which is an element of the set of all possible assignments $\Omega = \mathcal{Z}^n$. It is common to consider binary treatments, in which case $\Omega = \{0, 1\}^n$, but the current discussion applies more generally.

The assignment of treatments potentially affects the units. This is captured by a function $y_i : \Omega \to \mathbb{R}$ that gives the observed outcome for unit $i$ under a specific, potentially counterfactual, assignment (Holland, 1986; Neyman, 1990). That is, $y_i(z)$ is the response of unit $i$ when the treatments are assigned as $z \in \Omega$. The elements of the image of the function are called potential outcomes. The potential outcomes are assumed to be well-defined throughout the paper. This requires that no hidden versions exist of the treatments in $\Omega$ and that the potential outcomes themselves are not inherently random. The outcomes are also assumed to be bounded. Boundedness can be weaken without materially changing the results, but the condition eases the exposition.

**Condition 1** (Bounded potential outcomes). For all $i \in \mathcal{U}$ and $z \in \Omega$, $|y_i(z)| \leq k_1 < \infty$.

The treatments are assigned at random according to some probability space built upon $\Omega$. Let $Z$ be a random variable denoting the randomly selected treatment vector. The distribution of $Z$ is called the assignment mechanism or design. The design is the sole source of randomness under consideration in this paper, and the sample of units is considered non-random and fixed. The observed outcome $Y_i$ for unit $i$ is defined as the potential outcome corresponding to the randomly selected intervention: $Y_i = y_i(Z)$. 
3.2 Exposures

The potential outcomes contain all causal information pertaining to the treatments, and any causal quantity can be expressed solely using them. However, definitions of such causal quantities may be complex, and it is often difficult to formulate effects that are relevant for theory or policy. Even when experimenters succeed in formulate relevant effects, readers often find it difficult to interpret such intricate effects.

Exposures and exposure mappings are used to make the definitions more intuitive. The idea is that sets of treatments often share similar causal interpretations, and the purpose of an exposure mapping is to encode this information. Two assignment vectors are mapped to the same exposure if they are deemed similar. In this way, the exposures are no more than labels on subsets of $\Omega$ that have the same causal interpretation.

To state this formally, consider a set of exposure labels indexed by $\Delta \subseteq \mathbb{N}$. The exposure mapping is then a function $d_i: \Omega \rightarrow \Delta$ for each unit that maps from all possible assignments to the exposures. The exposure of unit $i$ is $d_i(z)$ when the treatments are assigned according to $z$. If $d_i(z) = d_i(z')$, then $z$ has a similar causal interpretation as $z'$ with respect to unit $i$. For example, in a vaccine study, one exposure $a \in \Delta$ could be that 75% of unit $i$’s neighbors are vaccinated. In this case, if $d_i(z) = d_i(z') = a$ for two assignments $z$ and $z'$, then 75% of the unit’s neighbors are vaccinated under both assignments, but it may be different units being vaccinated.

The realized exposure is a random variable because the treatments are randomly assigned. Let $D_i = d_i(Z)$ denote the realized exposure for unit $i$, and let $\pi_i(d) = \Pr(D_i = d)$ be its marginal distribution. A positivity assumption will initially be made on the distribution of $D_i$. This is later relaxed in Section 7.

**Condition 2.** An exposure $d \in \Delta$ satisfies positivity if $1/\pi_i(d) \leq k_2 < \infty$ for all $i \in U$.

3.3 Conventional exposure effects

The current convention is to assume that exposure mappings are correctly specified. The assumption states that, for all units $i \in U$ and assignments $z, z' \in \Omega$,

$$y_i(z) = y_i(z') \quad \text{whenever} \quad d_i(z) = d_i(z').$$

Manski (2013) calls the assumption “constant treatment response,” and Aronow and Samii (2017) call it “properly specified exposure mappings.”

Correctly specified exposure mappings implies that each exposure corresponds to a unique and well-defined potential outcome for every unit. This facilitates a simple definition of exposure effects. Under the assumption, a function $\tilde{y}_i: \Delta \rightarrow \mathbb{R}$ exists for each unit such that $\tilde{y}_i(d_i(z)) = y_i(z)$ for all $z \in \Omega$. In other words, the assumption states that the exposures accurately capture the complete causal structure in the experiment. Since the
full treatment vector provides no causal information in addition to what a unit’s exposure already provides, we can use $\tilde{y}_i(d)$ defined on $\Delta$ rather than the more cumbersome potential outcomes $y_i(z)$ defined on the full $\Omega$. The reduction in complexity can be considerable, because $|\Omega|$ grows exponentially in $n$ while $|\Delta|$ typically is fixed.

Causal effects can then be defined in the usual manner as contrasts between potential outcomes produced by the exposures. For example, the average causal effect of exposure $a \in \Delta$ relative to $b \in \Delta$ is

$$\bar{\tau}(a, b) = \frac{1}{n} \sum_{i=1}^{n} [\tilde{y}_i(a) - \tilde{y}_i(b)].$$

The interpretation of these effects is generally straightforward, because the exposures are chosen to have natural causal interpretations.

3.4 Exposure effects under misspecification

The reason exposures are conventionally assumed to be correctly specified is because the construction of $\tilde{y}_i$ requires it. No function $\tilde{y}_i : \Delta \rightarrow \mathbb{R}$ exists such that $\tilde{y}_i(d_i(z)) = y_i(z)$ for all $z \in \Omega$ unless each exposure corresponds to a single, unique potential outcome. When this is not the case, so that $y_i(z) \neq y_i(z')$ and $d_i(z) = d_i(z')$ for some assignments $z, z' \in \Omega$, the exposure mapping is misspecified, and the conventional exposure effect is no longer well-defined. A solution must provide analogues of exposure-based potential outcomes that remain unambiguous even when the exposures are misspecified.

Let $\bar{y}_i : \Delta \rightarrow \mathbb{R}$ be a function such that $\bar{y}_i(d) = \mathbb{E}[y_i(Z) \mid D_i = d]$ where the expectation is taken over the design. The interpretation of $\bar{y}_i$ is essentially the same as for $\tilde{y}_i$. The function captures the expected potential outcome under each exposure for each unit, so $\bar{y}_i(d)$ is the potential outcome we expect to be realized when unit $i$ is assigned to exposure $d$. A definition of an exposure effect under misspecification is now immediate.

**Definition 1.** The expected exposure effect for exposures $a$ and $b$ is

$$\tau(a, b) = \frac{1}{n} \sum_{i=1}^{n} [\bar{y}_i(a) - \bar{y}_i(b)].$$

Effects building on this idea have previously been discussed in the literature. The earliest example is the effects introduced by Hudgens and Halloran (2008). The authors derive their main results assuming that the exposures are correctly specified (i.e., under partial and stratified interference). However, they define the effect assuming only partial interference, which implicitly allows from some misspecification. The way they proceed is exactly as in Definition 1; namely, they marginalize over all assignments that map to the same exposure.
The expected exposure effect is also related to a discussion in Aronow and Samii (2017). The authors derive the expectation of their estimator when their assumption that the exposures are correctly specified is relaxed. They show the expectation is a particular weighted average of the potential outcomes defined on the full treatment vector, and this weighted average can be shown to coincide with Definition 1.

A more distant parallel can be drawn with the definition of causal effects in settings without interference. An historically common assumption was that the treatment effects were constant between units, meaning that $\tilde{y}_i(a) - \tilde{y}_i(b)$ did not vary with $i$. This allowed experimenters to focus on a single, well-defined causal parameter applicable to all units in the experiment. In the case experimenters were interested in effect heterogeneity, a model was stipulated to capture the heterogeneity, and the model was generally assumed to be correctly specified.

Experimenters grew skeptical of assumptions of constant effects and correctly modeled heterogeneity, because it is not reasonable to expect that people have the level of structural knowledge required to motivate them. This skepticism prompted experimenters to redefine their inferential targets as average treatment effects. Here, the effect remains well-defined even in the face of essentially unlimited heterogeneity; experimenters acknowledge that unmodeled heterogeneity may exist, and focus instead on estimating the expectation of the distribution of the unit-level effects. In the case experimenters want to investigate effect heterogeneity directly, the targets are conditional averages, which can capture aspects of the heterogeneity that are of interest, while marginalizing over all irrelevant aspects. These conditional average causal effects are robust to heterogeneity, and they are a direct parallel to expected exposure effects, which are robust to misspecification of the exposures.

4 Specification errors

Misspecification introduces specification errors. The errors can be formalized as differences between the actual outcomes and the outcomes predicted by the exposures. Or, equivalently, as differences between the potential outcomes based on the full treatment vector and the potential outcomes based on the exposures.

**Definition 2** (Specification error). $\varepsilon_i = y_i(Z) - \tilde{y}_i(D_i)$.

The assumption that the exposures are correctly specified is the same as assuming that the specification errors are zero with probability one. This insight suggests a way to weaken the assumption. Rather than assuming that the specification errors are zero, it may be sufficient to ensure that they are small, or perhaps only that they are sufficiently controlled in some other way. This is the approach explored in this paper.

Small specification errors are indeed sufficient to achieve consistency, but such a condition is unnecessarily strong. Instead, the critical aspect is the dependence of errors between
units. The subsequent section shows that consistency is achieved if the dependence averaged over all pairs of units diminishes as the sample grows. The relevant pair-wise dependence concept is captured by $E[\varepsilon_i \varepsilon_j \mid D_i, D_j]$. However, this quantity may not provide much intuition about the sources of the dependence, and the remainder of this section explores a decomposition of this dependence measure that may be easier to interpret.

Dependence between errors can be separated into two components. The first is the conditioning event itself, capturing the fact that knowledge about $j$’s exposure can provide information about $i$’s outcome in excess of the information provided by $i$’s exposure. An example is when unit $j$ interferes with unit $i$ in a way that is not captured in $i$’s exposure. The second source is dependence in excess of what can be explained by the conditioning event. This captures the fact two units’ errors can be dependent if misspecified in the same way even if the exposures themselves provide no information about the outcomes.

We may gain a better understanding about the two components after realizing that the specification errors to some degree are in our control, because we decide how to define the exposures. Consider when $j$’s exposure provide information about $i$’s outcome in excess of the information provided by its own exposure, meaning that the conditioning event is informative. A simple way to eliminate this misspecification is to redefine $i$’s exposure to include also the exposure of $j$. If $i$’s redefined exposure is $(D_i, D_j)$, no part of $i$’s specification error can be explained by $j$’s exposure because $i$’s exposure already contains this information.

Conceptually, it is straightforward to gradually remove misspecification by redefining the exposures, but such an approach will often prove impractical. The exposures would in that case increasingly depart from their primary purpose of producing an effect that is interpretable and relevant for theory or policy. In particular, if applied to all units in the sample, the redefined exposures would be the intersection of all units’ nominal exposures, and much of the reduction in complexity the original exposures provided is lost. However, the idea of redefined exposures suggests a way to formalize the decomposition of the specification error that will prove useful.

Let $\bar{y}_{ij}: \Delta \times \Delta \to \mathbb{R}$ be a function such that $\bar{y}_{ij}(d_1, d_2) = E[y_i(Z) \mid D_i = d_1, D_j = d_2]$. That is, $\bar{y}_{ij}(d_1, d_2)$ is the potential outcome of unit $i$ when defined over the exposures of both $i$ and $j$. Such a function may not be unambiguously defined if the event $D_i = d_1$ and $D_j = d_2$ has measure zero. The concern is valid but technical, so it is ignored for the moment and is instead discussed in Section A3 in Supplement A.

Because the combination of $D_i$ and $D_j$ provides more information about the treatment assignments than $D_i$ alone, the potential outcome $\bar{y}_{ij}(d_1, d_2)$ based on the refined exposures is a more precise representation of unit $i$’s outcome than the potential outcome $\bar{y}_i(d_1)$ based on the original exposures. One may therefore interpret the difference between $\bar{y}_{ij}(d_1, d_2)$ and $\bar{y}_i(d_1)$ as the part of the specification error for unit $i$ explainable by $j$’s exposure.

**Definition 3** (Explainable specification error). $e_{ij}(d_1, d_2) = \bar{y}_{ij}(d_1, d_2) - \bar{y}_i(d_1)$. 

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While \( \bar{y}_{ij}(d_1, d_2) \) provides more information than \( \bar{y}_i(d_1) \), it will generally not be correctly specified. That is, \( \bar{y}_{ij}(d_1, d_2) \) will not provide complete causal information, in the sense that it does not provide the same information as \( y_i(z) \). The remaining error is that which cannot be explained by \( j \)'s exposure. This part is strictly speaking not unexplainable, because the full treatment vector will always perfectly explain the potential outcomes, but it is unexplainable with respect to pairwise refinements of the exposures. Similar to Definition 2, we may define the error not explainable by \( j \)'s exposure as the difference between the actual potential outcome and the outcome predicted by the redefined exposures.

**Definition 4** (Unexplainable specification error). \( U_{ij} = y_i(Z) - \bar{y}_{ij}(D_i, D_j) \).

The overall specification error can now be decomposed using the explainable and unexplainable specification errors. In particular, we have \( \varepsilon_i = e_{ij}(D_i, D_j) + U_{ij} \) with probability one for any pair of units \( i \) and \( j \).

Definitions 2, 3 and 4 capture the specification errors pertaining to any particular unit. The following definition aggregates these errors to an overall description of the specification errors in the experiment as a whole.

**Definition 5.** The average explainable error dependence for exposure \( d \in \Delta \) is

\[
\bar{e}_d = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} e_{ij}(d, d)e_{ji}(d, d),
\]

and the average unexplainable error dependence for the same exposure is

\[
\bar{u}_d = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} \text{Cov}(U_{ij}, U_{ji} \mid D_i = D_j = d).
\]

Definition 5 captures pair-wise dependencies between errors of units. To understand the definitions, first consider the average explainable error dependence. If \( e_{ij}(a, b) = 0 \), then knowing that \( D_j = b \) provides no insights about \( Y_i \) in excess of knowing that \( D_i = a \). Thus, \( e_{ij}(d, d)e_{ji}(d, d) \) is non-zero only when the exposures of \( i \) and \( j \) both provide information about the other unit’s outcome. This means that the magnitude of the explainable errors \( e_{ij}(d, d) \) matters only insofar that the dependence make them large simultaneously. If the explainable errors are perfectly symmetric, so that \( e_{ij}(d, d) = e_{ji}(d, d) \) for all pairs of units, then \( \bar{e}_d \) collapses to a measure of magnitude. However, without perfect symmetry, \( \bar{e}_d \) is a measure of both magnitude and between-unit coordination in the explainable errors. Indeed, \( \bar{e}_d \) will be small, or even negative, if the pair-wise explainable errors tend to have opposite signs. These insights are perhaps made clear by the inequality

\[
\bar{e}_d \leq \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} [e_{ij}(d, d)]^2.
\]
which shows that the average explainable error dependence is upper bounded by the average magnitude of the explainable errors.

Consider a vaccination trial as an example. Unit $j$ in this trial is an asymptomatic potential carrier, meaning that $j$ would not get sick if infected but could potentially spread the pathogen to other units. Unit $i$ on the other hand will show symptoms if infected. Here, the exposure assigned to $j$ provides information about $i$’s outcome in excess of knowing $i$’s exposure, because $j$’s exposure provides information about whether unit $i$ is infected, and thus shows symptoms. Part of $i$’s error is thus explainable by $j$’s exposure, and $e_{ij}(d, d)$ is non-zero. However, $i$’s exposure contains no information about $j$’s outcome, because $j$ never shows symptoms, so $e_{ji}(d, d) = 0$. The lack of symmetry means that there is no dependence between the explainable errors of units $i$ and $j$ according to Definition 5.

Next, consider the average unexplainable error dependence. The fact that $\bar{u}_d$ captures dependence is immediate by the use of a covariance in its definition. To build intuition, consider the vaccine trial again. Consider when the exposures capture whether units close to the unit in question are vaccinated (e.g., in their household, or in a neighborhood in a social network). For illustration, assume that the experiment is so large that the vaccinations in the experiment have the potential to induce herd immunity. The exposures of any pair of units will in this case provide little information about whether herd immunity is achieved, because whether or not a particular household is vaccinated matters little in that context. However, if the design of the experiment induces variation in whether head immunity is achieved, then the units’ errors will exhibit great dependence even in cases where the explainable errors are small or zero, because pair-wise exposure cannot capture the global behavior. Generalizing from this example, the unexplainable error dependence captures whether the exposures are misspecified in the same way.

5  Finite sample behavior

5.1  Unbiasedness

Commonly used estimators for exposure effects build on ideas originally introduced in the survey sampling literature. Aronow and Samii (2017) focus on a version of the Horvitz–Thompson estimator (Horvitz & Thompson, 1952). They also discuss extensions to the Hájek estimator (Hájek, 1971) and various estimations facilitating covariate adjustments. Karwa and Airoldi (2018) show that the Horvitz–Thompson estimator is inadmissible when exposures are correctly specified, and they provide a large set of alternative estimators that take advantage of covariates and other auxiliary information.

The initial focus in this paper is the Horvitz–Thompson estimator. The analysis of this estimator can be extended to several other estimators, including the estimators discussed by Aronow and Samii (2017) and Karwa and Airoldi (2018). However, this extension does
not provide many new theoretical insights, and these results are relegated to Section A2 of Supplement A. While the Horvitz–Thompson estimator suffices to show the main ideas explored in this paper, experimenters will generally benefit from using one of the refined estimators discussed in the supplement.

**Definition 6.** The *Horvitz–Thompson estimator* for exposure effect $\tau(a, b)$ is

$$\hat{\tau}(a, b) = \frac{1}{n} \sum_{i=1}^{n} \frac{D_{ia}Y_i}{\pi_i(a)} - \frac{1}{n} \sum_{i=1}^{n} \frac{D_{ib}Y_i}{\pi_i(b)},$$

where $D_{id} = \mathbb{1}[D_i = d]$ is an indicator denoting whether unit $i$’s exposure is $d \in \Delta$.

The use of the Horvitz–Thompson estimator builds on the idea that the observed potential outcomes can be seen as a sample from a finite population consisting of the potential outcomes of all units in the study. Seen through the lens of survey sampling, an assignment mechanism that disproportionately assigns some exposures to some units will lead to oversampling of the corresponding potential outcomes, and those outcomes must be given less weight in the estimator. For the Horvitz–Thompson estimator, these weights are the reciprocal of the probability of observing the outcomes.

When exposures are correctly specified, the realized outcome is the same as the potential outcome corresponding to the realized exposure. The positivity assumption stated in Condition 2 thus ensures that the reweighed outcomes for each term is equal in expectation to the corresponding potential outcome. Under misspecification, the realized outcomes will vary even if the realized exposure is fixed, so the same logic does not apply. Instead, the expectation of each term is a linear combination of all potential outcomes under the same exposure label.

Two insights about the estimator makes this linear combination interpretable. First, $\pi_i(d)$ is fixed for all potential outcomes under the same exposure label, so each coefficient in the linear combination is proportional to the probability that the corresponding potential outcome is realized. Second, the expectation of $D_{id}$ is $\pi_i(d)$, so the coefficients sum to one. The resulting convex combination is thus a conditional expectation. More precisely, the unconditional expectation of each term of the estimator is equal to

$$E \left[ \frac{D_{id}Y_i}{\pi_i(d)} \right] = \frac{\Pr(D_i = d)}{\pi_i(d)} E[Y_i \mid D_i = d] = E[y_i(Z) \mid D_i = d] = \bar{y}_i(d),$$

which makes the following result immediate.

**Proposition 1.** Provided that Condition 2 holds for exposures $a$ and $b$, the Horvitz–Thompson estimator is unbiased for the expected exposure effect: $E[\hat{\tau}(a, b)] = \tau(a, b)$.
The proposition is essentially a rephrasing of Proposition 8.1 in Aronow and Samii (2017), but the implications of the result may be clearer when connected with an explicit target parameter as here. The proposition shows that the estimator is unbiased no matter how severe the misspecification is; no restrictions on the quantities in Definition 5 are needed.

Proposition 1 provides control over the location of the sampling distribution of the estimator. While this is not enough to meet the needs of experimenters, it is a comforting first step. The Horvitz–Thompson estimator is known to emphasize unbiasedness at the cost of precision, so if the estimator was shown not to control the bias, we would suspect its behavior more generally to be poor.

5.2 Controlling design dependence

Definition 5 provides a way to control the dependence introduced by the specification errors. Another channel through which dependence can be introduced is the exposures themselves. For example, if the exposures were defined as the proportion of treated unit in the whole sample, it would follow that \( D_1 = D_2 = \cdots = D_n \), and the Horvitz–Thompson estimator would exhibit considerable variability in large samples even if the exposures were correctly specified. To proceed, we must control the dependence between the exposures introduced by the design.

**Definition 7.** The average design dependence for exposure \( d \in \Delta \) is

\[
\bar{c}_d = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{i \neq j} |\text{Cov}(D_{id}, D_{jd})|.
\]

The average design dependence captures the dependence between units’ assigned exposures. It tells us how much information a unit’s exposure provides about other units’ exposures on average. This is in contrast to the explainable error dependence discussed in the previous section, which told us how much information a unit’s exposure provides about another unit’s specification error.

The definition may appear unfamiliar, but the concept it captures is not. It can be seen a measure of effective sample size with respect to the assignment mechanism. If \( \bar{c}_d \) diminishes in \( n \), the effective sample size grows with the nominal size. The definition has a direct parallel in Aronow and Samii (2017) where \( \bar{c}_d \) is implicitly defined as

\[
\frac{1}{n^2} \sum_{i=1}^{n} \sum_{i \neq j} \mathbb{1}[\text{Cov}(D_{id}, D_{jd}) = 0],
\]

where \( \mathbb{1}[\text{Cov}(D_{id}, D_{jd}) = 0] \) is an indicator taking the value one when \( D_{id} \) and \( D_{jd} \) are
uncorrelated. In other words, Aronow and Samii (2017) use a similar, but stronger, dependence concept as the one in Definition 7.

When particular designs, such as complete randomization, are studied in a setting without interference, there is no need to consider quantities like the one in Definition 7. This is because most common designs are constructed to ensure that the dependence between units’ treatment is small on average, so quantities like $\bar{c}_d$ are ensured to diminish in $n$ by construction. It is not generally possible to pre-specify a design when investigating arbitrary exposure mappings. Even if the design is simple at the level of the nominal treatments, the complexity of the exposure mappings often give rise to an intricate design at the level of the exposures. The role of Definition 7 is to measure how much dependence the design and exposure mappings introduce. Note, however, that the design and the exposure mappings are known, so $\bar{c}_d$ can be calculated, although it might not be computationally straightforward to do so.

5.3 Variance bound

We now have the components needed to characterize the behavior of the estimator beyond its expectation.

**Proposition 2.** Provided that Conditions 1 and 2 hold for exposures $a$ and $b$, the variance of the Horvitz–Thompson estimator is upper bounded by

$$\text{Var}(\hat{\tau}(a, b)) \leq 8k_1^2k_2/n + 20k_1^2k_2^2(\bar{e}_a + \bar{e}_b) + 4(\bar{e}_a + \bar{e}_b + \bar{u}_a + \bar{u}_b),$$

where $\bar{c}_d$, $\bar{e}_d$ and $\bar{u}_d$ are given by Definitions 5 and 7.

The bound demonstrates that three aspects are relevant for the variance of the estimator. The first term captures variability induced by the fact that the exposures are randomly assigned. That is, even when the exposures are independent and correctly specified, the estimator would still vary because different potential outcomes would be observed for different assignments.

The second term captures variability induced by dependence between exposures. That is, even when the exposures are correctly specified, the estimator tends to vary more when exposures are highly dependent. In some cases, such dependence can reduce the variance of the estimator, but to make that into a general statement requires additional restrictions on the potential outcomes (e.g., that they have the same sign) or on the design (e.g., that the dependence between exposures are only negative).

The final term in the bound captures variance stemming from misspecification. Recall that Definition 5 captures the dependence between the explainable and unexplainable specification errors. If the specification errors are strongly positively correlated, the estimator
will exhibit more variability. The bound makes clear that the magnitude of the specification errors is less of a concern. Large specification errors will affect the variance, but their effect is absorbed by the first term, so the large sample behavior is only affected by misspecification through the quantities given by Definition 5.

6 Large sample behavior

6.1 Asymptotic regime

The asymptotic regime used in the large sample investigation considers a sequence of fixed samples indexed by $n$. All quantities pertaining to the samples, such as the potential outcomes and designs, will thus have their own sequences also indexed by $n$. The indices are, however, suppressed when no confusion ensues.

The regime differs from the conventional setup in that the sample is fixed and no superpopulation exists in the usual sense. The asymptotic properties discussed below are therefore not with respect to some sampling scheme, but apply uniformly to all sequences of samples that satisfy the stated conditions. A consequence is that the samples need not be related in any specific way, and no assumptions about independent and identically distributed sampling or other stabilizing mechanisms are needed. This is particularly useful when studying interference because units tend to be neither independent nor identically distributed in that case.

This type of regime has been used extensively in the literature on design-based sampling (e.g., Isaki & Fuller, 1982). It has more recently seen extensive use in the design-based causal inference literature (e.g., Freedman, 2008; Lin, 2013).

6.2 Limiting behavior

Because the estimator is unbiased, the variance bound directly describes the estimator’s asymptotic behavior in mean square sense. Control over the terms in the bound thus provides consistency through Markov’s inequality. The following two conditions provide the control.

**Condition 3.** An exposure $d \in \Delta$ satisfies limited design dependence if $\bar{e}_d = o(1)$.

**Condition 4.** An exposure $d \in \Delta$ satisfies limited specification error dependence if there exists some positive sequence $a_n = o(1)$ such that $\bar{e}_d \leq a_n$ and $\bar{u}_d \leq a_n$.

**Proposition 3.** Provided that Conditions 1 and 2 hold for exposures $a$ and $b$, the stochastic order of the estimation error of the Horvitz–Thompson estimator is

$$
\tilde{\tau}(a,b) - \tau(a,b) = \mathcal{O}_p\left(n^{-0.5} + \bar{c}_a^{0.5} + \bar{c}_b^{0.5} + \bar{e}_a^{0.5} + \bar{e}_b^{0.5} + \bar{u}_a^{0.5} + \bar{u}_b^{0.5}\right).
$$
If Conditions 3 and 4 also hold, the estimator is consistent for the expected exposure effect.

The proposition states that the rate of convergence depends on the rate at which the three different dependence measures diminish. In a setting without interference, it is generally unproblematic to design the experiment so that $\bar{c}_d = \mathcal{O}(n^{-1})$. This may not be the case when complex exposure mappings are used. For example, if the exposures capture proportions of treated units in some neighborhood, then the order of $\bar{c}_d$ will depend on the rate at which the neighborhoods grow. If the neighborhoods are of fixed size in the asymptotic sequence, then $\bar{c}_d = \mathcal{O}(n^{-1})$ should still be attainable. However, if the neighborhoods grow with the sample size, $\bar{c}_d$ will generally diminish at a slower rate. If the neighborhoods grow proportionally to the sample size, $\bar{c}_d$ will not converge to zero, and the estimator is no longer consistent.

The rate of convergence of the average explainable error dependence $\bar{e}_d$ captures the direct interference between units as the sample grows. We expect $\bar{e}_d$ to diminish asymptotically if the interference is local, or when the interference is global but weak. If the interference is local, a unit’s exposure will be uninformative of the outcome of most other units, so $e_{ij}(d,d)$ will be zero for an increasingly proportion of the unit pairs in the sample, implying that $\bar{e}_d$ approaches zero. Similar to $\bar{c}_d$, the rate at which it approaches zero depends on how local the interference is.

If the interference is global, $e_{ij}(d,d)$ will be non-zero for most pairs of units also asymptotically. If the interference is both global and unrestricted, $\bar{e}_d$ will generally not converge to zero, and the estimator may not be consistent. However, the interference may be global as long as it is restricted. In particular, $\bar{e}_d$ may approach zero even when $e_{ij}(d,d)$ is non-zero if its magnitude diminishes in the sample size. An example is an experiment that takes place in a global market for some good. The actions of any customer or firm in the market will affect all other customers and firms, because the global market price acts as a coordinating mechanism. This means that $e_{ij}(d,d)$ could be non-zero for all pairs of units. However, unless the customer or firm is large, the effect will be negligible, so $\bar{e}_d$ will be close to zero in large samples.

The rate of convergence of the average unexplainable error dependence $\bar{u}_d$ captures other factors than the exposures that can coordinate the specification errors. The global market experiment is an illustrative example also in this case. If the market under study has two or more equilibria, then the assignment of treatments in the experiment may induce variation in which of the equilibria that is being realized. Because the behavior of most customers and firms will differ between equilibria in a way that is not generally captured by the exposures, the specification errors will be highly dependent between units in such a setting. In particular, the instability of the equilibrium induced by treatment assignment will act as a coordinating force for the specification errors, making $\bar{u}_d$ large also in large samples. This can be remedied either by ensuring that parts of the market are isolated, effectively acting as different markets with their own equilibria, or to pick an experimental
design that does not induce such variation, so that only one equilibrium is realized with high probability. These concerns and remedies about the unexplainable error dependence apply more generally to studies involving interference other than general equilibrium effects in markets.

6.3 Limiting distribution

The limiting distribution of the estimator is less tractable than its limit. A situation where progress can be made is when the specification errors are very small relative to the sample size. The following aggregated measure of the misspecification is a strengthening of the error dependence measures used for the consistency results.

**Definition 8.** The average total error dependence is

$$\bar{t}_d = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \{E[\varepsilon_i \varepsilon_j | D_i = D_j = d]\}^+, $$

where \(\{x\}^+ = \max(0, x)\) denotes the positive part of \(x\).

The definition is a strengthening of Definition 5 in two ways. First, the average total error dependence considers only the positive parts of the specification errors, while the dependence measures in Definition 2 includes the negative terms as well. It is possible that the dependence between some units’ errors is negative, and this will have a compensatory effect in Definition 2, making the average smaller. Definition 8 ignores any such compensatory effects. Second, unlike the previous dependence measures, the average total error dependence includes the diagonal elements of the double sum, \(E[\varepsilon_i^2 | D_i = d]\), which generally will be larger than the off-diagonal elements.

**Proposition 4.** Suppose Condition 2 holds and that the asymptotic distribution of the Horvitz–Thompson estimator when the exposures are correctly specified would be

$$R_n[\hat{\tau}(a, b) - \tau(a, b)] \xrightarrow{d} Q,$$

for some sequence \(R_n\) capturing the convergence rate of the estimator, and some random variable \(Q\). If the exposures are misspecified, but the misspecification is sufficiently weak so that \(\bar{t}_a + \bar{t}_b = o(R_n^{-2})\), then the asymptotic distribution of the Horvitz–Thompson estimator remains the same as when the exposures are correctly specified.

The proposition states that the limiting distribution of the estimator is unchanged if the specification errors are small. However, observe that the condition \(\bar{t}_a + \bar{t}_b = o(R_n^{-2})\) used in Proposition 4 is considerably stronger than Condition 4 used for the consistency results.
Indeed, while the previous conditions allowed for considerable misspecification also in large samples, this stronger condition is saying that any misspecification is negligible asymptotically relative to the variability induced by the randomization of treatments. Hence, the applicability of Proposition 4 is more limited than the applicability of Proposition 3, and experimenters should show caution before using the proposition to motivate any inferential statements. But, in the few situations where Proposition 4 is applicable, experimenters may find the following special case of the proposition particularly useful.

**Corollary 1.** *If the Horvitz–Thompson estimator would be root-n consistent and asymptotically normal when the exposures are correctly specified, then the estimator is root-n consistent and asymptotically normal also under misspecified exposures provided that \( \bar{t}_a + \bar{t}_b = o(n^{-1}) \).*

A last resort if no asymptotic approximation is available as a sound basis for hypothesis testing and interval estimation is Chebyshev’s inequality. The inequality guarantees 95% coverage rates for confidence intervals constructed as 4.47 standard errors wide windows on either side of the point estimate. Naturally, this interval estimator would be tremendously conservative in most situations, and it still requires access to a reasonable variance estimator, but it may be the only option in some settings.

7 Lack of positivity

Positivity conditions are often seen as innocuous in experiments because the experimenter controls the design and can ensure their validity. But this is rarely the case when estimating exposure effects. Exposure mappings tend to be complex, and it may not be feasible to construct a design that would induce a desired distribution over the exposures. Experimenters will instead settle for heuristic choices for the design at the treatment level, and this could induce violations of Condition 2.

The positivity condition can fail in two ways. The first is when it is fundamentally impossible for a unit to be assigned a particular exposure. For example, someone living in a single-person household cannot be assigned to an exposure requiring that at least two household members are vaccinated. This may be formalized by saying that there is an exposure \( d \in \Delta \) for which no \( z \in \Omega \) exists with \( d_i(z) = d \).

The consequences of such a failure are more than just statistical. If it is nonsensical to talk about some collection of units being assigned to a certain exposure, it is nonsensical to consider exposure effects that include those units in its average. Unless the experimenter is comfortable stipulating a metaphysical model allowing extrapolation to unrealizable potential outcomes, the only solution is to exclude such units from the average. The result may be that the number of units included in the analysis is fewer than the length of \( z \), but this is not an issue other than for efficiency. In the following discussion, it will be assumed
that such exclusions have been made if necessary. That is, if the aim is to estimate the effect of exposures \(a\) and \(b\), then \(\{a, b\} \subseteq \{d_i(z) : z \in \Omega\}\) for all units.

The second way the positivity condition can fail is through the design; assignments \(z \in \Omega\) exist so that \(d_i(z) = d\), but the design is such that \(\pi_i(d) = 0\). Statistical issues are the only sequela in this case, which all have cures. Two situations must be considered. The first is when the assignment probability for some exposure is exactly zero, \(\pi_i(d) = 0\). The second is when the probability approaches zero asymptotically. Both are problematic, but they have different solutions.

Superficially, the first situation appears most acute, because the Horvitz–Thompson estimator then involves division by zero, rendering it ill-defined. However, for each term of the estimator with a zero denominator, the numerator is also zero with probability one. Hence, a straightforward solution is to define \(0/0\) as zero, and that is the solution that will be used here. However, to ensure that the estimator behaves well, the proportion of units with zero assignment probability must be small. To capture this, let \(s_i(d) = 1[\pi_i(d) = 0]\) denote whether unit \(i\) has a zero assignment probability, and let

\[
\bar{s}_d = \frac{1}{n} \sum_{i=1}^{n} s_i(d),
\]

be the proportion of such units in the sample.

As for assignment probabilities that approaches zero, we must consider the rate at which they do so. The following norm-like quantity captures the average rate of convergence towards zero:

\[
\Pi(d, p) = \left[ \frac{1}{n} \sum_{i=1}^{n} \frac{1 - s_i(d)}{\pi_i(d)^p + s_i(d)} \right]^{1/p}.
\]

The quantities \(\bar{s}_d\) and \(\Pi(d, p)\) allow us to weaken the positivity assumption in a controllable way. In particular, Condition 2 is the same as \(\bar{s}_d = 0\) and \(\lim_{p \to \infty} \Pi(d, p) \leq k_2 < \infty\). The following proposition shows that neither part is necessary for consistency. But the weakening comes at the cost of potentially slower convergence rates. This is captured by a strengthening of the definition of the design dependence, namely

\[
\bar{c}_d(q) = \left[ \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} \left| \text{Cov}(D_{id}, D_{jd}) \right|^q \right]^{1/q}.
\]

This extended definition collapses to Definition 7 when \(q = 1\), but generally \(\bar{c}_d = o(\bar{c}_d(q))\) when \(q > 1\). Thus, Condition 3 using \(\bar{c}_d(q)\) when \(q > 1\) is stronger than the original version. Extending the results in Delevoye and Sävje (2020) to a setting with interference, this additional machinery admits a proof of consistency without positivity.
Proposition 5. Suppose Conditions 1 and 4 hold, and that
\[ \Pi(d, p) \leq k_2 < \infty, \quad \bar{s}_d = o(1) \quad \text{and} \quad \bar{c}_d(p/(p - 2)) = o(1), \]
for \( d \in \{a, b\} \) and some \( p > 2 \). The Horvitz–Thompson estimator is then consistent for the expected exposure effect and converges at the rate
\[ \hat{\tau}(a, b) - \tau(a, b) = O_p\left(n^{-0.5} + \bar{s}_a + \bar{s}_b + \bar{c}_a + \bar{c}_b + \bar{e}_a + \bar{e}_b + \bar{u}_a + \bar{u}_b\right), \]
where \( \bar{c}_{dp} \) is short-hand for \( \bar{c}_d(p/(p - 2)) \).

The proposition states that we achieve consistency under misspecification even if positivity does not hold as long as the dependence between exposures, as captured by \( \bar{c}_d(q) \), is sufficiently weak. While experimenters still should try to ensure that their designs and exposure mappings satisfy positivity, Proposition 5 provides some reassurance that the results are not automatically invalidated in the case they are not perfectly successful and small violations to positivity occur.

8 Variance estimation

8.1 Variance estimation under correctly specified exposures

Variance estimation for exposure effect estimators is challenging because the variance consists of pair-wise products of potential outcomes, and some of those outcomes are not simultaneously observable. The issue is not unique to exposure effects, but exposure mappings tend to induce complex distributions on the exposures, which exacerbates the problem.

The solution suggested by Aronow and Samii (2017) is to use Young’s inequality for products to bound the unobservable parts of the variance expression. To better understand this idea, let \( \pi_{ij}(d_1, d_2) = \Pr(D_i = d_1, D_j = d_2) \) be the joint probability of unit \( i \) and \( j \)’s exposures. If \( \pi_{ij}(d_1, d_2) = 0 \), then the potential outcomes \( \bar{y}_i(d_1) \) and \( \bar{y}_j(d_2) \) are never observed simultaneously, which will complicate variance estimation because the variance depends on the product \( \bar{y}_i(d_1)\bar{y}_j(d_2) \). Aronow and Samii (2017) tackle these products by using the bound
\[ \bar{y}_i(d_1)\bar{y}_j(d_2) \leq \frac{[\bar{y}_i(d_1)]^2 + [\bar{y}_j(d_2)]^2}{2}. \]
After having applied the bound to all problematic terms in the variance of the point estimator, they arrive at the estimator
\[ \widehat{\text{Var}}_{\text{As}}(\hat{\tau}(a, b)) = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} (D_{ia} - D_{ib})(D_{ja} - D_{jb})P_{ij}(D_i, D_j)Y_i Y_j. \]
\[ + \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left[ \frac{D_{ia}}{\pi_i(a)} + \frac{D_{ib}}{\pi_i(b)} \right] \left[ s_{ij}(D_i, a) + s_{ij}(D_i, b) \right] Y_i^2, \]

where,

\[ P_{ij}(d_1, d_2) = \frac{\pi_{ij}(d_1, d_2) - \pi_i(d_1)\pi_j(d_2)}{\pi_{ij}(d_1, d_2)\pi_i(d_1)\pi_j(d_2) + s_{ij}(d_1, d_2)} \quad \text{and} \quad s_{ij}(d_1, d_2) = \mathbb{1}[\pi_{ij}(d_1, d_2) = 0]. \]

Aronow and Samii (2017) show that this variance estimator is conservative in expectation when exposures are correctly specified. However, what does not appear to be fully appreciated in the literature is that the bound on the problematic products may make the estimator excessively conservative. In fact, unless the assumption of correctly specified exposures is complemented with

\[ \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} [s_{ij}(a, a) + s_{ij}(b, b) + s_{ij}(a, b)] = \mathcal{O}(n^{-1}), \]

the normalized variance estimator \( n\hat{\text{Var}}_{as}(\hat{\tau}(a, b)) \) generally diverges to infinity. I will not offer a solution to this problem. The remark instead serves as an illustration of the difficulty of variance estimation for complex exposure effects. It also provides insights about the mechanics of the estimator, which will aid our understanding of its behavior under misspecification.

### 8.2 Variance estimation under misspecification

The analysis of the variance estimator by Aronow and Samii (2017) does not hold when the exposures are misspecified. The expectation of the variance estimator could both increase and decrease under misspecification relative to when the exposures are correctly specified. In some situations, the decrease is sizable, and the estimator may become anti-conservative, providing an unjustly optimistic estimate of the precision of the point estimator. The following proposition exactly characterizes the bias of the variance estimator under misspecification.

**Proposition 6.** Provided that Conditions 1 and 2 hold for exposures \( a \) and \( b \), the bias of the variance estimator described by Aronow and Samii (2017) is

\[
\text{E} \left[ \widehat{\text{Var}}_{as}(\hat{\tau}(a, b)) \right] - \text{Var}(\hat{\tau}(a, b)) = B_1(a, b) + B_2(a, b) + B_2(b, a) + B_3(a, b) + B_3(b, a) \\
+ 2B_4(a, b) - B_4(a, a) - B_4(b, b),
\]

21
where

\[ B_1(d_1, d_2) = \frac{1}{n^2} \sum_{i=1}^{n} [\bar{y}_i(d_1) - \bar{y}_i(d_2)]^2, \]

\[ B_2(d_1, d_2) = \frac{1}{2n^2} \sum_{i=1}^{n} \sum_{j \neq i} \left( s_{ij}(d_1, d_1)[\bar{y}_i(d_1) + \bar{y}_j(d_1)]^2 + s_{ij}(d_1, d_2)[\bar{y}_i(d_1) - \bar{y}_i(d_2)]^2 \right), \]

\[ B_3(d_1, d_2) = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} \left( s_{ij}(d_1, d_1) + s_{ij}(d_1, d_2) \right) \text{Var}(\varepsilon_i \mid D_i = d_1), \]

\[ B_4(d_1, d_2) = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} \left[ 1 - s_{ij}(d_1, d_2) \right] \left[ \bar{y}_i(d_1)e_{ji}(d_2, d_1) + \bar{y}_i(d_2)e_{ij}(d_1, d_2) \right. \]

\[ + e_{ij}(d_1, d_2)e_{ji}(d_2, d_1) + \text{Cov}(U_{ij}, U_{ji} \mid D_i = d_1, D_j = d_2) \right]. \]

The terms on the right-hand side capture aspects that introduce bias of the variance estimator. These biases help us understand when variance estimation is possible under misspecification. The term \( B_1(a, b) \) stems from what Holland (1986) describes as the fundamental problem of causal inference, namely that a unit cannot simultaneously be assigned to two different treatments. The joint distribution of the potential outcomes affects the variance, but the distribution can only be estimated if both potential outcomes are observed simultaneously. Such simultaneous observations would require simultaneous assignment of two different treatments to the same unit, but this is not possible. The first term captures the bias arising from our inability to estimate this aspect of the potential outcomes. The issue is not unique to the current setting, and similar bias terms arise for most causal inference problems in finite populations, including when the exposures are correctly specified.

As already noted, the distribution of the exposures is often complex, and the joint exposure probabilities may be zero for a considerable number of pairs of units. This issue is similar to the first source of bias, but it is now introduced by the design rather than being fundamental. The use of Young’s inequality to tackle this problem introduces bias, and this bias is captured by the terms \( B_2(a, b) \) and \( B_2(b, a) \) in Proposition 6. These biases arise also when the exposures are correctly specified.

At this point, we have replicated the result in Aronow and Samii (2017). In particular, the remaining terms are zero by construction when the exposures are correctly specified, \( B_3(d_1, d_2) = B_4(d_1, d_2) = 0 \). In other words, the bias of the variance estimator when the exposures are correctly specified is

\[ E \left[ \hat{\text{Var}}_{\text{AS}}(\hat{\tau}(a, b)) \right] - \text{Var}(\hat{\tau}(a, b)) = B_1(a, b) + B_2(a, b) + B_2(b, a), \]

which the same result as in Aronow and Samii (2017), but presented in a different form. The
fact that these three terms are non-negative by construction confirms that the estimator is conservative when the exposures are correctly specified. What Proposition 6 shows is that we need to consider additional sources of bias when the exposures are misspecified.

The remaining biases stem from two sources. The first also arises from the use of Young’s inequality in the construction of the estimator. If the probability that two units are simultaneously assigned to a certain combination of exposures is zero, then the corresponding specification errors cannot interact, and the errors do not affect the variance of the point estimator. Section A1 and the proof of Lemma A5 in Supplement A provide a deeper discussion about this fact. This means that we do not need to adjust for any dependence between such errors, because we know it is zero. However, these pairs of units are exactly the ones for which we apply the bound on the unobserved product of potential outcomes to ensure conservativeness under correctly specified exposures. This inadvertently leads to that the variance estimator is affected by the magnitude of the corresponding errors. The terms $B_3(a)$ and $B_3(b)$ capture this part of the bias. Like the previous terms, these terms are non-negative by construction.

The terms of real concern are the last three: $B_4(a,b)$, $B_4(a,a)$, and $B_4(b,b)$. These capture the bias introduced by our inability to estimate the dependence in the specification errors. Unlike the previous terms, the signs of the terms are unknown, so they may introduce negative bias. The consequence is that we may systematically underestimate the variance when the specification errors are negatively dependent, and our inferences would then be anti-conservative.

The problem has no immediate solution, but some progress can be made. Similar to the approach taken in Section 6.3, if the specification errors can be assumed to be negligible relative to the sample size, the terms $B_4(d_1, d_2)$ are negligible relative to the other terms, and the variance estimator is ensured to be asymptotically conservative.

An alternative approach is to incorporate more information about the structure of the interference in the variance estimator. In particular, the anti-conservative behavior of the estimator stems from negative interactions of errors in $B_4(d_1, d_2)$. One may remove such interactions by setting $s_{ij}(d_1, d_2) = 1$ for the corresponding pairs of units, even if $\pi_{ij}(d_1, d_2) > 0$ holds. This will move the corresponding terms from $B_4(d_1, d_2)$, where negative interactions are possible, to $B_3(d)$, where no interactions exist. It may be hard to discern whether the interaction between two specific units’ errors is negative or positive. A conservative approach is to set $s_{ij}(d_1, d_2) = 1$ for all pairs of units where an interaction of any type is suspected.

As an example, consider when units only interfere with each other within known disjoint groups (i.e., partial interference). One may here set $s_{ij}(d_1, d_2) = 1$ if either $\pi_{ij}(d_1, d_2)$ is zero or if unit $i$ and $j$ belong to the same group. A redefinition of $s_{ij}(d_1, d_2)$ along these lines would ensure that $B_4(d_1, d_2) = 0$, so the variance estimator remains conservative. Of course, such knowledge about the interference would allow for the definition of exposure
mappings that are correctly specified, which would remove any concerns about misspecification. However, experimenters may want to keep the main exposure mapping simple to facilitate interpretation. They can then proceed with misspecified exposures for point estimation, and use the more intricate information about the interference structure only when estimating variance.

A third approach is a combination of the previous two. One may set \( s_{ij}(d_1, d_2) = 1 \) for pairs of units where negative interaction are suspected to be particularly large. Even if one does not capture all pairs with negative interactions, the variance estimator will still be conservative as long as the missed interactions are small. In other words, setting \( s_{ij}(d_1, d_2) = 1 \) for the terms deemed most problematic makes the assumption that the remaining errors are small more reasonable. It should also be noted that the other bias terms tend to be large and positive, and they therefore provide considerable leeway in the case \( B_4(d_1, d_2) \) is negative.

The expectation of the variance estimator should be seen as a rough representation of its behavior more generally. The precision of the variance estimator will be poor if the joint exposure probabilities are small even if they are never exactly zero. Experimenters should be aware that the variance estimator could be imprecise even when the bias is positive, particularly when the number of exposures is large. This concern is not specific to misspecification.

9 Concluding remarks

The idea that motivated the investigation in this paper is that exposure mappings primarily are used to collect and describe sets of assignment vectors that share similar causal interpretation. It is rare that exposures that serve this purpose also are correctly specified in the sense that they provide a complete description of the causal structure in an experiment. The results herein show that conventional point estimators can perform well even if the exposures are misspecified, provided that the specification errors are only weakly dependent. This gives reassurance to experimenters studying complex causal effects under interference, including spillover effects, that their analyses remain informative also in the event their exposures miss some aspect of the causal structure.

These insights may prompt experimenters to focus on defining exposure mappings that are relevant for the substantive question at hand. If they instead followed the conventional recommendation to focus on making the exposures correctly specified, the exposures would often be too granular to be informative of theory or policy. Misspecified exposures may therefore make experimental studies more informative and interpretable.

The investigation in this paper highlights several open questions and problems. The first question is whether estimators can be constructed that fully separate the two purposes that exposure mappings traditionally have served. That is, estimators that can incorporate
knowledge about the interference structure in the estimation of an exposure effect without necessarily changing the effect that is being estimated. The results in this paper show that inference is possible without such knowledge, but our inferences may be improved if we can take advantage of all available information about the interference.

The second open question concerns the limiting distribution of the point estimator under misspecification. The result that the limiting distribution remains unchanged under misspecification in Section 6.3 requires considerably stronger conditions than the convergence results in Section 6.2. I conjecture that the limiting distribution can be characterized under weaker conditions, but perhaps not fully as weak as for the convergence results. But this remains to be shown.

A final set of open questions concerns variance estimation. As noted in Section 8, variance estimation under both correctly specified and misspecified exposures is difficult, because complex exposure mappings can make many joint exposure probabilities small or zero. Experimenters can address this problem by making the estimator conservative, but the estimator will often be excessively conservative, rendering hypothesis tests and confidence intervals uninformative. An open question is whether improved variance estimation is possible for complex and arbitrary designs.

Variance estimation is particularly challenging when the exposures are misspecified, as the units’ specification errors can be coordinated in such a way to make the estimator anti-conservative. Partial knowledge of the interference structure can mitigate the problem, but this is of little help when experimenters are completely oblivious to this structure. It therefore remains an open question whether a non-trivial variance estimator exists that is conservative under completely unknown and excessive interference. In the case the interference is restricted using the dichotomous interference dependence concept in Sävje et al. (2021), the same inflation adjustments used in that paper can be used also here. But it is unclear if a similar approach is possible under a quantitative notion of interference dependence, as used in this paper.

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Supplement A: Additional results and proofs

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A1 Rigorous definitions of the potential outcomes

Condition 2 ensures that $\bar{y}_i$ is uniquely defined by $\bar{y}_i(d) = E[y_i(Z) \mid D_i = d]$. This does not hold when the condition is relaxed in Section 7. In particular, $D_i = d$ can then be a measure zero event for some $i \in U$, and the definition provided in the paper is ambiguous. The issue was ignored in the main text to expedite the exposition, but it will be addressed here. In particular, let the full definition be

$$\bar{y}_i(d) = \begin{cases} E[y_i(Z) \mid D_i = d] & \text{if } \pi_i(d) > 0, \\ \left(\sum_{z \in \Omega} 1 \left[d_i(z) = d\right]\right)^{-1} \left(\sum_{z \in \Omega} 1 \left[d_i(z) = d\right] y_i(z)\right) & \text{if } \pi_i(d) = 0. \end{cases}$$

That is, if $\pi_i(d) = 0$, then $\bar{y}_i(d)$ is the arithmetic mean of all potential outcomes for which the corresponding $z$ maps to $d_i(z) = d$.

A similar issue arises for the definition of $\bar{y}_{ij}$ in Section 4. In particular, the function is not uniquely defined if $\pi_{ij}(d_1, d_2) = 0$ for some pairs of units. To accommodate such cases, let the full definition be

$$\bar{y}_{ij}(d_1, d_2) = \begin{cases} E[y_i(Z) \mid D_i = d_1, D_j = d_2] & \text{if } \pi_{ij}(d_1, d_2) > 0, \\ \bar{y}_i(d) & \text{if } \pi_{ij}(d_1, d_2) = 0, \end{cases}$$

It follows from this definition that $e_{ij}(d_1, d_2) = 0$ when $\pi_{ij}(d_1, d_2) = 0$. This captures the intuition that learning $D_j = d_2$ provides no information about $\bar{y}_i(d)$ when $D_i = d_1$ is not simultaneously possible with $D_j = d_2$. Similarly, set $\text{Cov}(U_{ij}, U_{ji} \mid D_i = D_j = d_1) = 0$ when $\pi_{ij}(d_1, d_2) = 0$.

A2 The behavior of other estimators

A2.1 The Hájek estimator

The main paper noted that the Horvitz–Thompson estimator rarely is a good choice in practice because of its instability in small samples. However, the analysis of the Horvitz–Thompson estimator serves as a foundation on which we can build understanding about the behavior of other estimators. This section investigates the behavior of common refinements of the Horvitz–Thompson estimator, which experimenters often prefer over the original estimator.

The first refinement accounts for the realized number of units assigned to the exposures of interest. The Hájek estimator (Hájek, 1971) does this by dividing each term in the estimator with the sum of the reciprocals of the assignment probabilities for the units assigned to the exposure, rather than dividing by $n$. The change can absorb some of the
variability in the estimator introduced by randomness in the number of units assigned to each exposure. The ratio structure introduces bias, but the bias is generally small enough to still grant improvements in mean square error sense. The denominator can generally be shown to be well-behaved, so the estimator’s limited behavior can be linked to the Horvitz–Thompson estimator through linearization.

**Definition A1.** The Hájek estimator for expected exposure effect \( \tau(a,b) \) is:

\[
\hat{\tau}_{Há}(a,b) = \left( \frac{\sum_{i=1}^{n} D_{ia}Y_i / \pi_i(a)}{\sum_{i=1}^{n} D_{ia} / \pi_i(a)} \right) - \left( \frac{\sum_{i=1}^{n} D_{ib}Y_i / \pi_i(b)}{\sum_{i=1}^{n} D_{ib} / \pi_i(b)} \right).
\]

**Proposition A1.** Provided that Conditions 1, 2, 3 and 4 hold for exposures \( a \) and \( b \), the Hájek estimator is consistent for the expected exposure effect and converges at the rate

\[
\hat{\tau}_{Há}(a,b) - \tau(a,b) = O_p \left( n^{-0.5} + \bar{c}_a^{0.5} + \bar{c}_b^{0.5} + \bar{e}_a^{0.5} + \bar{e}_b^{0.5} + \bar{u}_a^{0.5} + \bar{u}_b^{0.5} \right).
\]

Experimenters often use estimators that implicitly adjust for the assignment probabilities. One such example is the difference-in-means estimator. This estimator can be shown to coincide with the Hájek estimator whenever the assignment probabilities are the same for all units: \( \pi_i(d) = \pi_j(d) \) for all \( i, j \in U \). Proposition A1 thus implies that the difference-in-means estimator can be used in similar situations also under misspecification. Experimenters should, however, not blindly use the difference-in-means estimator for exposure effects because the exposure mappings may not induce equal assignment probabilities on the exposures even if they are equal for the nominal treatments. Another estimator coinciding with the Hájek estimator is the ordinary least squares (OLS) estimator. The unweighted version requires equal assignment probabilities just like the difference-in-means estimator, but a weighted OLS estimator is equivalent to the Hájek estimator also with unequal assignment probabilities.

**A2.2 The difference estimator**

A disadvantage of both the Horvitz–Thompson and Hájek estimators is their inability to take advantage of auxiliary information. A modification of the Horvitz–Thompson estimator allows us to incorporate such information. The idea is that information beside the observed potential outcomes themselves might allow us to predict the potential outcomes we do not observe. If this prediction is sufficiently good, the predicted outcomes can be used to offset chance imbalances introduced by the randomization. Särndal, Swensson, and Wretman (1992) call it the difference estimator in a sampling setting, and the name will be used here as well.
Definition A2. The difference estimator for the expected exposure effect $\tau(a, b)$ is

$$
\hat{\tau}_{\text{de}}(a, b) = \frac{1}{n} \sum_{i=1}^{n} \left[ \hat{y}_i(a) - \hat{y}_i(b) \right] + \frac{1}{n} \sum_{i=1}^{n} \frac{(D_{ia} - D_{ib}) [Y_i - \hat{y}_i(D_i)]}{\pi_i(D_i)},
$$

where $\hat{y}_i(d)$ is a prediction of unit $i$'s potential outcome when assigned to $d \in \Delta$.

The definition of the estimator reveals the idea that motivate its use. The first term is simply the average difference in predicted potential outcomes. If the predictions are of high quality, this term will be an accurate estimator of the exposure effect. The issue is that the predictions may have systematic errors. The second term is included to ensure unbiasedness. If the predictions are of low quality, this term will compensate for the errors in the first term, and it ensures that the estimator performs well in expectation. The estimator bears a resemblance in this regard to the class of doubly robust estimators used in observational studies when the assignment mechanism is unknown (see, e.g., Robins & Rotnitzky, 2001).

The properties of the difference estimator depend on the way the predictions are constructed. In particular, the estimator can be shown to retain the advantageous properties of the Horvitz–Thompson estimator if the predictions are external to the study. External here means that they do not depend on the treatment assignment. As the only randomness under consideration here stems from the assignment mechanism, independence between $\hat{y}_i(d)$ and $Z$ implies that the predictions are non-random. The probability space can extended to accommodate random predictions if one wants to account for the consequences of external variability. Such variability could affect the rate of convergence if predictions are sufficiently dependent between units, but it is otherwise inconsequential to the results.

Definition A3. A prediction $\hat{y}_i(d)$ is external if it is independent of the treatment assignment vector: $\hat{y}_i(d) \perp \perp Z$.

Definition A4. The average prediction dependence for exposure $d \in \Delta$ is

$$
p_d = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} |\operatorname{Cov}(\hat{y}_i(d), \hat{y}_j(d))|.
$$

An alternative to focusing on the dependence between predictions is to consider their convergence. In particular, the average prediction dependence can be bounded by

$$
p_d = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} |\operatorname{Cov}(\hat{y}_i(d), \hat{y}_j(d))| \leq \left( \frac{1}{n} \sum_{i=1}^{n} \sqrt{\operatorname{Var}(\hat{y}_i(d))} \right)^2,
$$

which shows that Condition A2 is satisfied if the predictions on average converges in mean square.
Condition A1 (Prediction moments). $E[|\hat{y}_i(d)|^2] \leq k_3 < \infty$ for all $i \in U$ and $d \in \Delta$.

Condition A2. An exposure $d \in \Delta$ satisfies limited prediction dependence if $p_d = o(1)$.

Proposition A2. Provided that Condition 2 holds and that the predictions are external, the difference estimator is unbiased for the expected exposure effect: $E[\hat{\tau}_{de}(a,b)] = \tau(a,b)$.

Proposition A3. Provided that Conditions 1, 2, 3, 4, A1 and A2 hold and that the predictions are external, the difference estimator is consistent for the expected exposure effect and converges at the rate

$$\hat{\tau}_{de}(a,b) - \tau(a,b) = O_p\left(n^{-0.5} + \epsilon_a^{0.5} + \epsilon_b^{0.5} + \bar{e}_a^{0.5} + \bar{e}_b^{0.5} + \bar{u}_a^{0.5} + \bar{u}_b^{0.5} + p_a^{0.5} + p_b^{0.5}\right).$$

The difference estimator seemingly provides advantages at no cost. Good predictions of the potential outcomes confer improvements in finite samples, but the estimator has the same behavior as the Horvitz–Thompson estimator both in large samples. The no-cost advantages are superficial. The mean square error may increase when the predictions are poor, so investigators should use the difference estimator only when the predictions are expected to be of reasonably high quality.

However, the quality of the predictions is less of a concern than their construction. Covariate information can be used to make the predictions, but the assigned exposures and the observed outcomes can generally not be used because it would induce dependence between the predictions and $Z$. More precisely, if $x_i$ denotes a vector of covariates describing characteristics of unit $i$, we can form the predictions as $\hat{y}_i(d) = f(d, x_i)$ for some function $f$. The function $f$ can, however, not be constructed using $(Y_1, Y_2, \ldots, Y_n)$ or $(D_1, D_2, \ldots, D_n)$. This illustrates that the construction of $f$ truly needs to be external to treatment assignment when used for the predictions in the difference estimator. This severely limits its applicability. Split-sample or leave-one-out approaches (see, e.g., Williams, 1961) that often are used to solve the issue cannot be used here because the misspecification may induce dependence between subsamples that otherwise appear isolated.

A2.3 The generalized regression estimator

An estimator facilitating dependence between the predictions of the potential outcomes and the treatment assignments is inspired by the generalized regression estimator commonly used in the sampling literature. The estimator has received recent attention in the causal inference literature as well (see, e.g., Lin, 2013; Middleton, 2018).

The estimator uses a linear working model for the relationship between the potential outcomes and the covariates. The working model is used to construct the predictions. Generally, $\hat{y}_i(d) = x_i \beta(d)$ for some vector of coefficients $\beta(d)$ indexed by $d \in \Delta$, so different coefficients are used for different exposures. No assumptions are made about the validity
of the model, but the quality of the predictions are related to how well the model can
approximate the potential outcomes. It remains to pick the coefficients \( \beta(d) \). The general-
ized regression estimator allows of dependence between the coefficients and the treatment
assignments, so the coefficients can be estimated in the sample. For example, we may pick
them as the minimizing solution to \( \sum_{i=1}^{n} D_id[Y_i - x_i \beta(d)]^2 \) as is often done in applications.
But other choices exist, and the estimator is largely agnostic about how the coefficients
were constructed.

**Definition A5.** The generalized regression estimator for expected exposure effect is
\[
\hat{\tau}_{gr}(a, b) = \frac{1}{n} \sum_{i=1}^{n} x_i [\hat{\beta}(a) - \hat{\beta}(b)] + \frac{1}{n} \sum_{i=1}^{n} (D_{ia} - D_{ib}) \frac{[Y_i - x_i \hat{\beta}(D_i)]}{\pi_i(D_i)},
\]
where \( \hat{\beta}(a) \) and \( \hat{\beta}(b) \) are two random vectors of the same dimensions as \( x_i \).

The conventional approach to investigating the properties of the generalized regression
estimator is to assume that the vector of coefficients constructed in the sample convergences
to some fixed vector asymptotically. This ensures that the dependence between units’
predictions is small in large samples, which provides consistency. The assumption can be
weaken to only require that the magnitude of the vector of coefficients is asymptotically
bounded, thereby bypassing the need of assuming a well-defined limit.

**Condition A3** (Bounded covariates). \( x_i \in \mathcal{X} \) for some bounded \( \mathcal{X} \subset \mathbb{R}^p \).

**Condition A4** (Asymptotically bounded regression coefficients). \( \mathbb{E}[\|\hat{\beta}(d)\|] = O(1) \).

**Proposition A4.** Provided that Conditions 1, 2, 3, 4, A3 and A4 hold, the generalized
regression estimator is consistent for the expected exposure effect and converges at the rate
\[
\hat{\tau}_{gr}(a, b) - \tau(a, b) = O_p\left(n^{-0.5} + \epsilon_a^{0.5} + \epsilon_b^{0.5} + \bar{e}_a^{0.5} + \bar{e}_b^{0.5} + \bar{u}_a^{0.5} + \bar{u}_b^{0.5}\right).
\]

### A3 Proofs of propositions in main paper

#### A3.1 Miscellaneous lemmas

**Lemma A1.** For any \( N \) random variables \( X_1, \ldots, X_N \),
\[
\text{Var}(X_1 + \cdots + X_N) \leq \left(\sqrt{\text{Var}(X_1)} + \cdots + \sqrt{\text{Var}(X_N)}\right)^2.
\]

**Proof.** Write the variance of the sum as a double sum of covariances:
\[
\text{Var}\left(\sum_{i=1}^{N} X_i\right) = \sum_{i=1}^{N} \sum_{j=1}^{N} \text{Cov}(X_i, X_j).
\]
Separate the covariances using the Cauchy–Schwarz inequality and reorder:

\[
\sum_{i=1}^{N} \sum_{j=1}^{N} \text{Cov}(X_i, X_j) \leq \sqrt{\frac{N}{N} \sum_{i=1}^{N} \text{Var}(X_i)} \sqrt{\frac{N}{N} \sum_{j=1}^{N} \text{Var}(X_j)} = \left( \sum_{i=1}^{N} \sqrt{\text{Var}(X_i)} \right)^2.
\]

**Lemma A2.** For any \(N\) random variables \(X_1, \ldots, X_N\),

\[
\text{Var}(X_1 + \cdots + X_N) \leq N \text{Var}(X_1) + \cdots + N \text{Var}(X_N).
\]

**Proof.** Apply Lemma A1 to get

\[
\text{Var} \left( \sum_{i=1}^{N} X_i \right) \leq \left( \sum_{i=1}^{N} \sqrt{\text{Var}(X_i)} \right)^2 = N^2 \left( \frac{1}{N} \sum_{i=1}^{N} \sqrt{\text{Var}(X_i)} \right)^2.
\]

The square is a convex function, so Jensen’s inequality gives

\[
N^2 \left( \frac{1}{N} \sum_{i=1}^{N} \sqrt{\text{Var}(X_i)} \right)^2 \leq N^2 \left( \frac{1}{N} \sum_{i=1}^{N} \text{Var}(X_i) \right) = N \sum_{i=1}^{N} \text{Var}(X_i).
\]

**Lemma A3.** If Condition 1 holds, then for all \(i \in U\) and \(d \in \Delta\),

i. \(|\bar{y}_i(d)| \leq k_1\),

ii. \(E[Y_i^2 | D_i = d] \leq k_1^2\),

iii. \(|E[\varepsilon_i \varepsilon_j | D_i = D_j = d]| \leq 4k_1^2\).

**Proof.** Consider each statement in turn:

i. Recall the definition \(\bar{y}_i(d) = E[y_i(Z) | D_i = d].\) Condition 1 gives

\[
|\bar{y}_i(d)| = |E[y_i(Z) | D_i = d]| \leq E[|y_i(Z)| | D_i = d] \leq E[k_1 | D_i = d] = k_1.
\]

ii. Condition 1 ensures that \(|Y_i| = |y_i(Z)| \leq k_1\) with probability one. It follows that

\[
E[Y_i^2 | D_i = d] = E[(y_i(Z))^2 | D_i = d] \leq k_1^2.
\]

iii. Recall that \(\varepsilon_i = Y_i - \bar{y}_i(D_i)\), so we can write

\[
\varepsilon_i \varepsilon_j = Y_i Y_j - Y_i \bar{y}_j(D_j) - Y_j \bar{y}_i(D_i) + \bar{y}_i(D_i) \bar{y}_j(D_j).
\]

Condition 1 ensures that \(|Y_i| \leq k_1\) and \(|\bar{y}_i(D_i)| \leq k_1\) with probability one. It follows that

\[
|\varepsilon_i \varepsilon_j| \leq 4k_1^2.
\]
Hence,

\[ |E[\varepsilon_i \varepsilon_j | D_i = D_j = d] - E[|\varepsilon_i \varepsilon_j | D_i = D_j = d] | \leq 4k_1^2. \]

\[ \Box \]

**A3.2 Proposition 1: Unbiasedness**

**Proposition 1.** Provided that Condition 2 holds for exposures \( a \) and \( b \), the Horvitz–Thompson estimator is unbiased for the expected exposure effect: \( E[\hat{\tau}(a,b)] = \tau(a,b) \).

**Proof.** For any exposure \( d \in \Delta \) satisfying Condition 2, we have

\[
\frac{1}{n} \sum_{i=1}^{n} \frac{E[D_{id}Y_i]}{\pi_i(d)} = \frac{1}{n} \sum_{i=1}^{n} \frac{\pi_i(d)E[Y_i | D_i = d]}{\pi_i(d)} = \frac{1}{n} \sum_{i=1}^{n} \frac{\pi_i(d)\bar{y}_i(d)}{\pi_i(d)} = \frac{1}{n} \sum_{i=1}^{n} \bar{y}_i(d).
\]

It follows that

\[
E[\hat{\tau}(a,b)] = \frac{1}{n} \sum_{i=1}^{n} \frac{E[D_{ia}Y_i]}{\pi_i(a)} - \frac{1}{n} \sum_{i=1}^{n} \frac{E[D_{ib}Y_i]}{\pi_i(b)} = \frac{1}{n} \sum_{i=1}^{n} \bar{y}_i(a) - \frac{1}{n} \sum_{i=1}^{n} \bar{y}_i(b) = \tau(a,b). \]

\[ \Box \]

**A3.3 Proposition 2: Bound on variance**

**Proposition 2.** Provided that Conditions 1 and 2 hold for exposures \( a \) and \( b \), the variance of the Horvitz–Thompson estimator is upper bounded by

\[
\text{Var}(\hat{\tau}(a,b)) \leq 8k_1^2k_2/n + 20k_1^2k_2^2(\bar{e}_a + \bar{e}_b) + 4(\bar{e}_a + \bar{e}_b + \bar{u}_a + \bar{u}_b),
\]

where \( \bar{e}_d \), \( \bar{e}_d \) and \( \bar{u}_d \) are given by Definitions 5 and 7.

**Proof.** Recall the definition of the estimator:

\[
\hat{\tau}(a,b) = \frac{1}{n} \sum_{i=1}^{n} \frac{D_{ia}Y_i}{\pi_i(a)} - \frac{1}{n} \sum_{i=1}^{n} \frac{D_{ib}Y_i}{\pi_i(b)}.
\]

Apply Lemma A2 to get

\[
\text{Var}(\hat{\tau}(a,b)) \leq 2 \text{Var}\left( \frac{1}{n} \sum_{i=1}^{n} \frac{D_{ia}Y_i}{\pi_i(a)} \right) + 2 \text{Var}\left( \frac{1}{n} \sum_{i=1}^{n} \frac{D_{ib}Y_i}{\pi_i(b)} \right).
\]

Note that Definition 2 implies that \( Y_i = \bar{y}_i(D_i) + \varepsilon_i \), so for any exposure \( d \in \Delta \),

\[
\frac{1}{n} \sum_{i=1}^{n} \frac{D_{id}Y_i}{\pi_i(d)} = \frac{1}{n} \sum_{i=1}^{n} \frac{D_{id}\bar{y}_i(d)}{\pi_i(d)} + \frac{1}{n} \sum_{i=1}^{n} \frac{D_{id}\varepsilon_i}{\pi_i(d)}.
\]
Apply Lemma A2 again to get
\[
2 \text{Var} \left( \frac{1}{n} \sum_{i=1}^{n} \frac{D_{id}}{\pi_i(d)} Y_i \right) \leq 4 \text{Var} \left( \frac{1}{n} \sum_{i=1}^{n} \frac{D_{id}}{\pi_i(d)} \bar{y}_i(d) \right) + 4 \text{Var} \left( \frac{1}{n} \sum_{i=1}^{n} \frac{D_{id}}{\pi_i(d)} \varepsilon_i \right).
\]

We can write the two terms as
\[
4 \text{Var} \left( \frac{1}{n} \sum_{i=1}^{n} \frac{D_{id}}{\pi_i(d)} \bar{y}_i(d) \right) = \frac{4}{n^2} \sum_{i=1}^{n} \left[ \bar{y}_i(d) \pi_i(d) \right]^2 \text{Var}(D_{id})
\]
\[+ \frac{4}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} \frac{\bar{y}_i(d)\bar{y}_j(d)}{\pi_i(d)\pi_j(d)} \text{Cov}(D_{id}, D_{id}),
\]
\[
4 \text{Var} \left( \frac{1}{n} \sum_{i=1}^{n} \frac{D_{id}}{\pi_i(d)} \varepsilon_i \right) = \frac{4}{n^2} \sum_{i=1}^{n} \left[ \frac{1}{\pi_i(d)} \right]^2 \text{Var}(D_{id} \varepsilon_i)
\]
\[+ \frac{4}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} \frac{\text{Cov}(D_{id} \varepsilon_i, D_{jd} \varepsilon_j)}{\pi_i(d)\pi_j(d)},
\]

because \( \bar{y}_i(d) \) and \( \pi_i(d) \) are not random.

Lemma A4 implies that
\[
\frac{4}{n^2} \sum_{i=1}^{n} \left[ \frac{\bar{y}_i(d)}{\pi_i(d)} \right]^2 \text{Var}(D_{id}) + \frac{4}{n^2} \sum_{i=1}^{n} \left[ \frac{1}{\pi_i(d)} \right]^2 \text{Var}(D_{id} \varepsilon_i) = \frac{4}{n^2} \sum_{i=1}^{n} \left[ \frac{1}{\pi_i(d)} \right]^2 \text{Var}(D_{id} Y_i).
\]

Lemma A3ii gives
\[
\text{Var}(D_{id} Y_i) \leq \text{E}[D_{id} Y_i^2] = \pi_i(d) \text{E}[Y_i^2 | D_i = d] \leq \pi_i(d) k_1^2.
\]

Together with Condition 2, this implies
\[
\frac{4}{n^2} \sum_{i=1}^{n} \left[ \frac{1}{\pi_i(d)} \right]^2 \text{Var}(D_{id} Y_i) = \frac{4}{n^2} \sum_{i=1}^{n} k_1^2 k_2 = \frac{4k_1^2 k_2}{n}.
\]

Using Condition 2, Lemma A3i and Definition 7, we can bound the following term by
\[
\frac{4}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} \frac{\bar{y}_i(d)\bar{y}_j(d)}{\pi_i(d)\pi_j(d)} \text{Cov}(D_{id}, D_{id}) \leq \frac{4k_1^2 k_2}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} \text{Cov}(D_{id}, D_{id}) = 4k_1^2 k_2 \bar{c}_d.
\]

Using Lemma A5, we can bound the following term by
\[
\frac{4}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} \frac{\text{Cov}(D_{id}\varepsilon_i, D_{jd}\varepsilon_j)}{\pi_i(d)\pi_j(d)} \leq \frac{16k_1^2k_2^2}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} \left| \text{Cov}(D_{id}, D_{jd}) \right| \\
+ \frac{4}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} e_{ij}(d,d)e_{ji}(d,d) + \frac{4}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} \text{Cov}(U_{ij}, U_{ji} \mid D_i = D_j = d).
\]

Definitions 5 and 7 allow us to write this as

\[
\frac{4}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} \frac{\text{Cov}(D_{id}\varepsilon_i, D_{jd}\varepsilon_j)}{\pi_i(d)\pi_j(d)} \leq 16k_1^2k_2^2\bar{c}_d + 4\bar{e}_d + 4\bar{u}_d.
\]

The derivations for all four terms taken together yield

\[
2 \text{Var} \left( \frac{1}{n} \sum_{i=1}^{n} \frac{D_{id}}{\pi_i(d)} Y_i \right) \leq \frac{4k_1^2k_2}{n} + 20k_1^2k_2^2\bar{c}_d + 4\bar{e}_d + 4\bar{u}_d,
\]

which means that the variance of the estimator is bounded by

\[
\text{Var}(\hat{\tau}(a,b)) \leq \frac{8k_1^2k_2}{n} + 20k_1^2k_2^2(\bar{c}_a + \bar{c}_b) + 4(\bar{e}_a + \bar{e}_b + \bar{u}_a + \bar{u}_b).
\]

\textbf{Lemma A4.} For all \(i \in U\) and \(d \in \Delta\),

\[
\text{Var}(D_{id}Y_i) = [\bar{y}_i(d)]^2 \text{Var}(D_{id}) + \text{Var}(D_{id}\varepsilon_i).
\]

\textit{Proof.} Definition 2 implies that

\[
\text{E}[\varepsilon_i \mid D_i = d] = \text{E}[Y_i - \bar{y}_i(D_i) \mid D_i = d] = \text{E}[y_i(Z) \mid D_i = d] - \bar{y}_i(d) = 0.
\]

It follows that

\[
\text{E}\left[ D_{id}^2\bar{y}_i(d)\varepsilon_i \right] = \pi_i(d) \text{E}\left[ D_{id}^2\bar{y}_i(d)\varepsilon_i \mid D_i = d \right] = \bar{y}_i(d)\pi_i(d) \text{E}[\varepsilon_i \mid D_i = d] = 0,
\]

and

\[
\text{E}[D_{id}\varepsilon_i] = \pi_i(d) \text{E}[D_{id}\varepsilon_i \mid D_i = d] = \pi_i(d) \text{E}[\varepsilon_i \mid D_i = d] = 0.
\]

We therefore have

\[
\text{Cov}(D_{id}\bar{y}_i(d), D_{id}\varepsilon_i) = \text{E}\left[ D_{id}^2\bar{y}_i(d)\varepsilon_i \right] - \text{E}\left[ D_{id}\bar{y}_i(d) \right] \text{E}\left[ D_{id}\varepsilon_i \right] = 0.
\]

It follows that

\[
\text{Var}(D_{id}Y_i) = \text{Var}(D_{id}\bar{y}_i(d) + D_{id}\varepsilon_i) = [\bar{y}_i(d)]^2 \text{Var}(D_{id}) + \text{Var}(D_{id}\varepsilon_i).
\]
Lemma A5. If Conditions 1 and 2 hold, then for all \( i, j \in \mathcal{U} \) and \( d \in \Delta \),

\[
\frac{\text{Cov}(D_{id} \varepsilon_i, D_{jd} \varepsilon_j)}{\pi_i(d) \pi_j(d)} \leq 4k_1^2k_2^2 \text{Cov}(D_{id}, D_{jd}) + e_{ij}(d, d)e_{ji}(d, d) + \text{Cov}(U_{ij}, U_{ji} \mid D_i = D_j = d).
\]

Proof. Equation (A1) in the proof of Lemma A4 showed that \( \mathbb{E}[D_{id} \varepsilon_i] = 0 \), so we have

\[
\text{Cov}(D_{id} \varepsilon_i, D_{jd} \varepsilon_j) = \mathbb{E}[D_{id} \varepsilon_i D_{jd} \varepsilon_j]
\]

The rest of the proof proceeds separately for the cases \( \pi_{ij}(d, d) > 0 \) and \( \pi_{ij}(d, d) = 0 \).

Starting with the case \( \pi_{ij}(d, d) > 0 \), note that we can write the expression to be bounded as

\[
\frac{\text{Cov}(D_{id} \varepsilon_i, D_{jd} \varepsilon_j)}{\pi_i(d) \pi_j(d)} = \frac{\mathbb{E}[D_{id} D_{jd} \varepsilon_i \varepsilon_j]}{\pi_i(d) \pi_j(d)} = \frac{\mathbb{P}(D_i = D_j = d)}{\pi_i(d) \pi_j(d)} \mathbb{E}[\varepsilon_i \varepsilon_j \mid D_i = D_j = d]
\]

\[
= \frac{\mathbb{P}(D_i = D_j = d) - \pi_i(d) \pi_j(d) + \pi_i(d) \pi_j(d)}{\pi_i(d) \pi_j(d)} \mathbb{E}[\varepsilon_i \varepsilon_j \mid D_i = D_j = d]
\]

\[
= \frac{\text{Cov}(D_{id}, D_{jd})}{\pi_i(d) \pi_j(d)} \mathbb{E}[\varepsilon_i \varepsilon_j \mid D_i = D_j = d] + \mathbb{E}[\varepsilon_i \varepsilon_j \mid D_i = D_j = d],
\]

because \( \pi_{ij}(d, d) > 0 \) ensures that \( \pi_i(d) > 0 \) and \( \pi_j(d) > 0 \). Consider the first term through the lens of Condition 2 and Lemma A3iii:

\[
\frac{\text{Cov}(D_{id}, D_{jd})}{\pi_i(d) \pi_j(d)} \mathbb{E}[\varepsilon_i \varepsilon_j \mid D_i = D_j = d] \leq 4k_1^2k_2^2 \text{Cov}(D_{id}, D_{jd})).
\]

For the second term, let \( E_{ij} = e_{ij}(D_i, D_j) \), so that \( \varepsilon_i = E_{ij} + U_{ij} \). Then,

\[
\mathbb{E}[\varepsilon_i \varepsilon_j \mid D_i = D_j = d] = \mathbb{E}[(E_{ij} + U_{ij})(E_{ji} + U_{ji}) \mid D_i = D_j = d]
\]

\[
= \mathbb{E}[E_{ij}E_{ji} + E_{ij}U_{ji} + U_{ij}E_{ji} + U_{ij}U_{ji} \mid D_i = D_j = d].
\]

Note that \( E_{ij} = e_{ij}(D_i, D_j) \) and \( E_{ji} = e_{ij}(D_j, D_i) \) are constant conditional on \( D_i = D_j = d \), and that

\[
\mathbb{E}[U_{ij} \mid D_i, D_j] = \mathbb{E}[Y_i - \bar{y}_{ij}(D_i, D_j) \mid D_i, D_j] = \mathbb{E}[Y_i \mid D_i, D_j] - \bar{y}_{ij}(D_i, D_j) = 0.
\]

It follows that

\[
\mathbb{E}[E_{ij}U_{ji} \mid D_i = D_j = d] = e_{ij}(d, d) \mathbb{E}[U_{ji} \mid D_i = D_j = d] = 0,
\]

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and
\[
E[\varepsilon_i \varepsilon_j \mid D_i = D_j = d] = e_{ij}(d,d)e_{ji}(d,d) + E[U_{ij}U_{ji} \mid D_i = D_j = d] = e_{ij}(d,d)e_{ji}(d,d) + \text{Cov}(U_{ij}, U_{ji} \mid D_i = D_j = d).
\]

Taken together we have showed that, when \( \pi_{ij}(d, d) > 0 \),
\[
\text{Cov}
\left( \frac{D_{id} \varepsilon_i, D_{jd} \varepsilon_j}{\pi_i(d)\pi_j(d)} \right)
\leq 4k_1^2k_2^2|\text{Cov}(D_{id}, D_{jd})| + e_{ij}(d,d)e_{ji}(d,d)
\]
\[+ \text{Cov}(U_{ij}, U_{ji} \mid D_i = D_j = d).\]

Continuing with the case \( \pi_{ij}(d, d) = 0 \), note that \( D_{id}D_{jd} \) is zero with probability one in this case, so
\[\frac{\text{Cov}(D_{id} \varepsilon_i, D_{jd} \varepsilon_j)}{\pi_i(d)\pi_j(d)} = \frac{E[D_{id}D_{jd} \varepsilon_i \varepsilon_j]}{\pi_i(d)\pi_j(d)} = 0.\]

Recall that the rigorous definitions in Section A1 imply that, when \( \pi_{ij}(d, d) = 0 \),
\[e_{ij}(d,d) = 0 \quad \text{and} \quad \text{Cov}(U_{ij}, U_{ji} \mid D_i = D_j = d) = 0.\]

It follows that, when \( \pi_{ij}(d, d) = 0 \),
\[\frac{\text{Cov}(D_{id} \varepsilon_i, D_{jd} \varepsilon_j)}{\pi_i(d)\pi_j(d)} \leq 4k_1^2k_2^2|\text{Cov}(D_{id}, D_{jd})| + e_{ij}(d,d)e_{ji}(d,d)
\]
\[+ \text{Cov}(U_{ij}, U_{ji} \mid D_i = D_j = d). \quad \square\]

**A3.4 Proposition 3: Consistency**

**Proposition 3.** Provided that Conditions 1 and 2 hold for exposures \( a \) and \( b \), the stochastic order of the estimation error of the Horvitz–Thompson estimator is
\[\hat{\tau}(a,b) - \tau(a,b) = O_p\left( n^{-0.5} + \bar{e}_a^{0.5} + \bar{e}_b^{0.5} + \bar{e}_a^{0.5} + \bar{e}_b^{0.5} + \bar{u}_a^{0.5} + \bar{u}_b^{0.5} \right).\]

If Conditions 3 and 4 also hold, the estimator is consistent for the expected exposure effect.

**Proof.** The proof presumes that \( \bar{e}_a, \bar{e}_b, \bar{u}_a \) and \( \bar{u}_b \) are non-negative. If they are not, the contravening quantities can be set to zero, and the proof would apply.

Consider the root mean square error:
\[\sqrt{E\left[\left(\hat{\tau}(a,b) - \tau(a,b)\right)^2\right]} = \sqrt{\text{Var}(\hat{\tau}(a,b))}\]

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\[
\leq \sqrt{8k_1^2k_2/n + 20k_1^2k_2^2[\bar{c}_a + \bar{c}_b] + 4[\bar{e}_a + \bar{e}_b + \bar{u}_a + \bar{u}_b]}
\]

where the equality follows from Proposition 1, and the inequality from Proposition 2. Concavity of the square root gives
\[
\sqrt{\text{Var}(\hat{\tau}(a,b)) \leq 3k_1k_2^{0.5}/n^{0.5} + 5k_1k_2[\bar{c}_a^{0.5} + \bar{c}_b^{0.5}] + 2[\bar{e}_a^{0.5} + \bar{e}_b^{0.5} + \bar{u}_a^{0.5} + \bar{u}_b^{0.5}]
\]

which gives the rate of convergence in the \(L^2\)-norm:
\[
\sqrt{\mathbb{E}\left[(\hat{\tau}(a,b) - \tau(a,b))^2\right]} = \mathcal{O}(n^{-0.5} + \bar{c}_a^{0.5} + \bar{c}_b^{0.5} + \bar{e}_a^{0.5} + \bar{e}_b^{0.5} + \bar{u}_a^{0.5} + \bar{u}_b^{0.5}).
\]

In turn, this gives the rate of convergence in probability by Markov’s inequality. Conditions 3 and 4 complete the proof.

A3.5 Proposition 4: Limiting distribution

**Proposition 4.** Suppose Condition 2 holds and that the asymptotic distribution of the Horvitz–Thompson estimator when the exposures are correctly specified would be
\[
R_n[\hat{\tau}(a,b) - \tau(a,b)] \xrightarrow{d} Q,
\]

for some sequence \(R_n\) capturing the convergence rate of the estimator, and some random variable \(Q\). If the exposures are misspecified, but the misspecification is sufficiently weak so that \(\bar{t}_a + \bar{t}_b = o(R_n^{-2})\), then the asymptotic distribution of the Horvitz–Thompson estimator remains the same as when the exposures are correctly specified.

**Proof.** Note that Definition 2 allows us to write the observed outcome as \(Y_i = \bar{y}_i(D_i) + \varepsilon_i\), which means that the estimator can be written as
\[
\hat{\tau}(a,b) = \hat{\tau}(a,b) + \hat{\varepsilon}(a,b),
\]

where
\[
\hat{\tau}(a,b) = \frac{1}{n} \sum_{i=1}^{n} \frac{D_{ia}\bar{y}_i(a)}{\pi_i(a)} - \frac{1}{n} \sum_{i=1}^{n} \frac{D_{ib}\bar{y}_i(b)}{\pi_i(b)} \quad \text{and} \quad \hat{\varepsilon}(a,b) = \frac{1}{n} \sum_{i=1}^{n} \frac{D_{ia}\varepsilon_i}{\pi_i(a)} - \frac{1}{n} \sum_{i=1}^{n} \frac{D_{ib}\varepsilon_i}{\pi_i(b)}.
\]

The random variable \(\hat{\tau}(a,b)\) is the Horvitz–Thompson estimator we would use if we observed \(\bar{y}_i(d)\) when \(D_i = d\). That is, if we observed the expected potential outcome without the specification error. The random variable \(\hat{\varepsilon}(a,b)\) is the Horvitz–Thompson estimator of the treatment effect on the expected specification error.
The decomposition of the estimator allows us to write
\[ R_n[\hat{\tau}(a, b) - \tau(a, b)] = R_n[\tilde{\tau}(a, b) - \tau(a, b)] + R_n\hat{\varepsilon}(a, b). \]

The premise of the proposition thus give us that
\[ R_n[\tilde{\tau}(a, b) - \tau(a, b)] \xrightarrow{d} Q, \]
because we have \( \hat{\varepsilon}(a, b) = 0 \) with probability one when the exposures are correctly specified. Lemma A6 gives
\[ R_n\hat{\varepsilon}(a, b) = \mathcal{O}_p(R_n\bar{t}_a^{0.5} + R_n\bar{t}_b^{0.5}) = o_p(1). \]

As a result, we can apply Slutsky’s theorem to get
\[ R_n[\hat{\tau}(a, b) - \tau(a, b)] \xrightarrow{d} Q. \]

**Lemma A6.** If Condition 2 holds, then
\[ \hat{\varepsilon}(a, b) = \frac{1}{n} \sum_{i=1}^{n} D_{ia}\varepsilon_i - \frac{1}{n} \sum_{i=1}^{n} D_{ib}\varepsilon_i = \mathcal{O}_p(\bar{t}_a^{0.5} + \bar{t}_b^{0.5}). \]

**Proof.** Note that the expectation of \( \hat{\varepsilon}(a, b) \) is zero:
\[ E[\hat{\varepsilon}(a, b)] = \frac{1}{n} \sum_{i=1}^{n} E[\varepsilon_i | D_i = a] - \frac{1}{n} \sum_{i=1}^{n} E[\varepsilon_i | D_i = b] = 0, \]
because
\[ E[\varepsilon_i | D_i = a] = E[Y_i - \bar{y}_i(D_i) | D_i = a] = E[Y_i | D_i = a] - \bar{y}_i(a) = 0. \]

Using Lemma A2, the variance can be bounded as
\[ \text{Var}(\hat{\varepsilon}(a, b)) \leq \frac{2}{n^2} \text{Var} \left( \sum_{i=1}^{n} \frac{D_{ia}\varepsilon_i}{\pi_i(a)} \right) + \frac{2}{n^2} \text{Var} \left( \sum_{i=1}^{n} \frac{D_{ib}\varepsilon_i}{\pi_i(b)} \right). \]

Each of the two terms can be written as
\[ \frac{2}{n^2} \text{Var} \left( \sum_{i=1}^{n} \frac{D_{id}\varepsilon_i}{\pi_i(d)} \right) = \frac{2}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\text{Cov}(D_{id}\varepsilon_i, D_{jd}\varepsilon_j)}{\pi_i(d)\pi_j(d)}. \]
\[
\frac{2}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\Pr(D_i = D_j = d)}{\pi_i(d)\pi_j(d)} \mathbb{E}[\varepsilon_i\varepsilon_j \mid D_i = D_j = d],
\]

because \(\mathbb{E}[D_i\varepsilon_i] = 0\). Using Condition 2, this may in turn be bounded by

\[
\frac{2}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\Pr(D_i = D_j = d)}{\pi_i(d)\pi_j(d)} \mathbb{E}[\varepsilon_i\varepsilon_j \mid D_i = D_j = d] \leq \frac{2}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \mathbb{E}[\varepsilon_i\varepsilon_j \mid D_i = D_j = d]^{+} = 2k_2\bar{t}_d.
\]

Hence, the variance is bounded by

\[
\text{Var}(\hat{\tau}(a, b)) \leq 2k_2(\bar{t}_a + \bar{t}_b).
\]

This provides convergence to zero in mean square:

\[
\sqrt{\mathbb{E}[\left(\hat{\tau}(a, b)\right)^2]} = O(\bar{t}_a^{0.5} + \bar{t}_b^{0.5}),
\]

which in turn provides convergence to zero in probability by Markov’s inequality. \qed

### A3.6 Proposition 5: Consistency without positivity

**Proposition 5.** Suppose Conditions 1 and 4 hold, and that

\[
\Pi(d, p) \leq k_2 < \infty, \quad \bar{s}_d = o(1) \quad \text{and} \quad \bar{c}_d(p/(p - 2)) = o(1),
\]

for \(d \in \{a, b\}\) and some \(p > 2\). The Horvitz–Thompson estimator is then consistent for the expected exposure effect and converges at the rate

\[
\hat{\tau}(a, b) - \tau(a, b) = O_p\left(n^{-0.5} + \bar{s}_a + \bar{s}_b + \bar{c}_a^{0.5} + \bar{c}_b^{0.5} + \bar{e}_a^{0.5} + \bar{e}_b^{0.5} + \bar{u}_a^{0.5} + \bar{u}_b^{0.5}\right),
\]

where \(\bar{c}_{dp}\) is short-hand for \(\bar{c}_d(p/(p - 2))\).

**Proof.** Decompose the mean square error of the estimator into its squared bias and variance:

\[
\mathbb{E}\left[\left(\hat{\tau}(a, b) - \tau(a, b)\right)^2\right] = \left(\mathbb{E}[\hat{\tau}(a, b)] - \tau(a, b)\right)^2 + \text{Var}(\hat{\tau}(a, b)).
\]
Applying the bounds in Lemmas A7 and A9, we get
\[
E \left[ (\hat{\tau}(a, b) - \tau(a, b))^2 \right] \leq k^2_1 (s_a + s_b)^2 + \frac{4k^2_1}{n} [\Pi(a, p) + \Pi(b, p)] \\
+ 20k^2_1 \left\{ [\Pi(a, p)]^2 \bar{c}_{ap} + [\Pi(b, p)]^2 \bar{c}_{bp} \right\} + 4(\bar{e}_a + \bar{e}_b) + 4(\bar{u}_a + \bar{u}_b).
\]

A premise of the proposition was \( \Pi(a, p) \leq k \) and \( \Pi(b, p) \leq k \), which implies
\[
E \left[ (\hat{\tau}(a, b) - \tau(a, b))^2 \right] \leq k^2_1 (s_a + s_b)^2 + \frac{8kk^1}{n} + 20k^2_1(\bar{c}_{ap} + \bar{c}_{bp}) \\
+ 4(\bar{e}_a + \bar{e}_b) + 4(\bar{u}_a + \bar{u}_b).
\]

Because the square root is a concave function on the positive number line, we can apply Jensen’s inequality to get
\[
\sqrt{E \left[ (\hat{\tau}(a, b) - \tau(a, b))^2 \right]} \leq k_1 (s_a + s_b) + \frac{3kk_1}{n} + 5kk_1(\bar{c}_{ap}^{0.5} + \bar{c}_{bp}^{0.5}) \\
+ 2(\bar{e}_a^{0.5} + \bar{e}_b^{0.5}) + 2(\bar{u}_a^{0.5} + \bar{u}_b^{0.5}),
\]
which implies that the root mean square error is asymptotically bounded as
\[
\sqrt{E \left[ (\hat{\tau}(a, b) - \tau(a, b))^2 \right]} = \mathcal{O}(n^{-0.5} + s_a + s_b + \bar{c}_{ap}^{0.5} + \bar{c}_{bp}^{0.5} + \bar{e}_a^{0.5} + \bar{e}_b^{0.5} + \bar{u}_a^{0.5} + \bar{u}_b^{0.5}).
\]

Thus, the estimator is consistent in mean square under the conditions of the proposition. Markov’s inequality then gives consistency and the rate of convergence in probability. □

**Lemma A7.** Provided that Condition 1 holds for exposures \( a \) and \( b \), the absolute bias of the Horvitz–Thompson estimator is upper bounded by
\[
|E[\hat{\tau}(a, b)] - \tau(a, b)| \leq k_1 (s_a + s_b).
\]

**Proof.** Write the estimator as in Lemma A8, and take its expectation
\[
E[\hat{\tau}(a, b)] = \frac{1}{n} \sum_{i=1}^{n} \frac{[1 - s_i(a)] E[D_{ia} Y_i]}{\pi_i(a) + s_i(a)} - \frac{1}{n} \sum_{i=1}^{n} \frac{[1 - s_i(b)] E[D_{ib} Y_i]}{\pi_i(b) + s_i(b)} \\
= \frac{1}{n} \sum_{i=1}^{n} [1 - s_i(a)] y_i(a) - \frac{1}{n} \sum_{i=1}^{n} [1 - s_i(b)] y_i(b).
\]
The bias is therefore
\[ E[\hat{\tau}(a, b)] - \tau(a, b) = -\frac{1}{n} \sum_{i=1}^{n} s_i(a) \bar{y}_i(a) + \frac{1}{n} \sum_{i=1}^{n} s_i(b) \bar{y}_i(b). \]

Using Condition 1, we can upper bound the absolute value of the bias as
\[ |E[\hat{\tau}(a, b)] - \tau(a, b)| \leq \frac{1}{n} \sum_{i=1}^{n} s_i(a) |\bar{y}_i(a)| + \frac{1}{n} \sum_{i=1}^{n} s_i(b) |\bar{y}_i(b)| = k_1 (\bar{s}_a + \bar{s}_b). \]

where \( \bar{s}_d \) is defined in Section 7.

**Lemma A8.** When 0/0 is defined to be zero, the Horvitz–Thompson estimator can be written as
\[ \hat{\tau}(a, b) = \frac{1}{n} \sum_{i=1}^{n} \frac{[1 - s_i(a)] D_{ia} Y_i}{\pi_i(a) + s_i(a)} - \frac{1}{n} \sum_{i=1}^{n} \frac{[1 - s_i(b)] D_{ib} Y_i}{\pi_i(b) + s_i(b)}, \]

where \( s_i(d) = 1[\pi_i(d) = 0] \).

**Proof.** The estimator as written in Definition 6 is
\[ \hat{\tau}(a, b) = \frac{1}{n} \sum_{i=1}^{n} \frac{D_{ia} Y_i}{\pi_i(a)} - \frac{1}{n} \sum_{i=1}^{n} \frac{D_{ib} Y_i}{\pi_i(b)}. \]

Note that \( D_{id} \) is zero with probability one when \( s_i(d) = 1 \). Furthermore, \( \pi_i(d) + s_i(d) \) is one when \( \pi_i(d) = 0 \), and \( \pi_i(d) + s_i(d) = \pi_i(d) \) when \( \pi_i(d) > 0 \). Taken together, it follows that
\[ \frac{D_{ia} Y_i}{\pi_i(a)} = \frac{D_{ia} Y_i}{\pi_i(d) + s_i(d)} \]

with probability one, because both expressions are zero when \( s_i(d) = 1 \) given that we define 0/0 to be zero. The fact that \( D_{id} \) is zero when \( s_i(d) = 1 \) also implies that
\[ D_{id} Y_i = [1 - s_i(d)] D_{id} Y_i. \]

**Lemma A9.**
\[ \text{Var}(\hat{\tau}(a, b)) \leq 4 k_1^2 \left[ \frac{1}{n} \left[ \Pi(a, p) + \Pi(b, p) \right] + 20 k_1^2 \left[ \left[ \Pi(a, p) \right]^2 \bar{c}_{ap} + \left[ \Pi(b, p) \right]^2 \bar{c}_{bp} \right] \right] + 4 (\bar{e}_a + \bar{e}_b) + 4 (\bar{u}_a + \bar{u}_b), \]

where \( \bar{c}_{dp} = \bar{c}_d(p/(p-2)) \).
Proof. Use Lemmas A2 and A8 to bound the variance of the estimator as
\[
\text{Var}(\hat{\pi}(a, b)) \leq 2 \text{Var} \left( \frac{1}{n} \sum_{i=1}^{n} \frac{[1 - s_i(a)]D_{id}Y_i}{\pi_i(a) + s_i(a)} \right) + 2 \text{Var} \left( \frac{1}{n} \sum_{i=1}^{n} \frac{[1 - s_i(b)]D_{id}Y_i}{\pi_i(b) + s_i(b)} \right).
\]

Using a similar decomposition as in the proof of Proposition 2, we can bound each term of this expression as
\[
2 \text{Var} \left( \frac{1}{n} \sum_{i=1}^{n} \frac{[1 - s_i(d)]D_{id}Y_i}{\pi_i(d) + s_i(d)} \right) \leq 4 \sum_{i=1}^{n} \left[ \frac{1 - s_i(d)}{\pi_i(d) + s_i(d)} \right]^2 \text{Var}(D_{id}Y_i) + 4 \sum_{i=1}^{n} \sum_{j \neq i} \left[ \frac{1 - s_i(d)}{\pi_i(d) + s_i(d)} \right] \left[ \frac{1 - s_j(d)}{\pi_j(d) + s_j(d)} \right] \text{Cov}(D_{id}, D_{jd}) + 4 \sum_{i=1}^{n} \sum_{j \neq i} \left[ \frac{1 - s_j(d)}{\pi_j(d) + s_j(d)} \right] \left[ \frac{1 - s_i(d)}{\pi_i(d) + s_i(d)} \right] \text{Cov}(D_{id}\tilde{e}_i, D_{jd}\tilde{e}_j).
\]
The terms are bounded, respectively, by Lemmas A10, A11 and A12, resulting in
\[
2 \text{Var} \left( \frac{1}{n} \sum_{i=1}^{n} \frac{[1 - s_i(d)]D_{id}Y_i}{\pi_i(d) + s_i(d)} \right) \leq 4k_1^2 \frac{\Pi(d, p)}{n} + 20k_1^2 [\Pi(d, p)]^2 \tilde{c}_d + 4\tilde{c}_d + 4\tilde{u}_d
\]
The lemma follows when this bound is applied to the two terms in the first expression. \(\square\)

Lemma A10.
\[
\frac{4}{n^2} \sum_{i=1}^{n} \left[ \frac{1 - s_i(d)}{\pi_i(d) + s_i(d)} \right]^2 \text{Var}(D_{id}Y_i) \leq \frac{4k_1^2 \Pi(d, p)}{n}.
\]

Proof. The expression can be bounded as
\[
\frac{4}{n^2} \sum_{i=1}^{n} \left[ \frac{1 - s_i(d)}{\pi_i(d) + s_i(d)} \right]^2 \text{Var}(D_{id}Y_i) \leq \frac{4}{n^2} \sum_{i=1}^{n} \left[ \frac{1 - s_i(d)}{\pi_i(d) + s_i(d)} \right]^2 \text{E}[D_{id}Y_i^2]
\]
\[
= \frac{4}{n^2} \sum_{i=1}^{n} \left[ \frac{1 - s_i(d)}{\pi_i(d) + s_i(d)} \right]^2 \pi_i(d) \text{E}[Y_i^2 \mid D_i = d] \leq \frac{4}{n^2} \sum_{i=1}^{n} \left[ \frac{1 - s_i(d)}{\pi_i(d) + s_i(d)} \right] k_1^2,
\]
where the last inequality follows from Lemma A3ii and the fact that
\[
\left[ \frac{1 - s_i(d)}{\pi_i(d) + s_i(d)} \right]^2 = \begin{cases} [\pi_i(d)]^{-2} & \text{if } s_i(d) = 0, \\ 0 & \text{if } s_i(d) = 1. \end{cases}
\]
When \( p \geq 1 \), we can use Jensen’s inequality to get

\[
\frac{1}{n} \sum_{i=1}^{n} \left[ 1 - s_i(d) \right] \left[ \frac{1}{\pi_i(d) + s_i(d)} \right] \leq \left[ \frac{1}{n} \sum_{i=1}^{n} \left[ 1 - s_i(d) \right] \frac{1}{\pi_i(d) + s_i(d)} \right]^{1/p} = \Pi(d, p),
\]

where \( \Pi(d, p) \) is defined in Section 7. \( \square \)

**Lemma A11.**

\[
\frac{4}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} \left[ 1 - s_i(d) \right] \left[ \frac{\bar{y}_i(d)}{\pi_i(d) + s_i(d)} \right] \left[ 1 - s_j(d) \right] \left[ \frac{\bar{y}_j(d)}{\pi_j(d) + s_j(d)} \right] \text{Cov}(D_{id}, D_{jd}) \leq 4k_1^2 \left[ \Pi(d, p) \right]^2 \bar{c}_{dp},
\]

where \( \bar{c}_{dp} = \bar{c}_d(p/(p - 2)) \).

**Proof.** Using Lemma A3i, the expression can be bounded as

\[
\frac{4}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} \left[ 1 - s_i(d) \right] \left[ \frac{\bar{y}_i(d)}{\pi_i(d) + s_i(d)} \right] \left[ 1 - s_j(d) \right] \left[ \frac{\bar{y}_j(d)}{\pi_j(d) + s_j(d)} \right] \text{Cov}(D_{id}, D_{jd})
\]

\[
\leq \frac{4k_1^2}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} \left[ 1 - s_i(d) \right] \left[ \frac{1}{\pi_i(d) + s_i(d)} \right] \left[ 1 - s_j(d) \right] \left[ \frac{1}{\pi_j(d) + s_j(d)} \right] \left| \text{Cov}(D_{id}, D_{jd}) \right|
\]

Apply Hölder’s inequality with conjugates \( p/2 \) and \( p/(p - 2) \) to get

\[
\frac{4k_1^2}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} \left[ 1 - s_i(d) \right] \left[ \frac{1}{\pi_i(d) + s_i(d)} \right] \left[ 1 - s_j(d) \right] \left[ \frac{1}{\pi_j(d) + s_j(d)} \right] \left| \text{Cov}(D_{id}, D_{jd}) \right|
\]

\[
\leq \frac{4k_1^2}{n^2} \left[ \sum_{i=1}^{n} \sum_{j \neq i} \left[ 1 - s_i(d) \right] \left[ \frac{1}{\pi_i(d) + s_i(d)} \right] \right]^{p/2} \left[ \sum_{i=1}^{n} \sum_{j \neq i} \left[ 1 - s_j(d) \right] \left[ \frac{1}{\pi_j(d) + s_j(d)} \right] \right]^{p/2} 2^{p/2}
\]

\[
\times \left[ \sum_{i=1}^{n} \sum_{j \neq i} \left| \text{Cov}(D_{id}, D_{jd}) \right| \right]^{p/(p - 2)} (p - 2)/p
\]

\[
\leq 4k_1^2 \left[ \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} \left[ 1 - s_i(d) \right] \left[ \frac{1}{\pi_i(d) + s_i(d)} \right] \right]^{p/2} \left[ \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} \left[ 1 - s_j(d) \right] \left[ \frac{1}{\pi_j(d) + s_j(d)} \right] \right]^{p/2} 2^{p/2}
\]

\[
\times \left[ \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} \left| \text{Cov}(D_{id}, D_{jd}) \right| \right]^{p/(p - 2)} (p - 2)/p
\]
The middle factor can be bounded as
\[
\left[ \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} \left[ \frac{1 - s_i(d)}{\pi_i(d) + s_i(d)} \right]^{p/2} \left[ \frac{1 - s_j(d)}{\pi_j(d) + s_j(d)} \right]^{p/2} \right]^{2/p} \leq \left[ \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left[ \frac{1 - s_i(d)}{\pi_i(d) + s_i(d)} \right]^{p/2} \left[ \frac{1 - s_j(d)}{\pi_j(d) + s_j(d)} \right]^{p/2} \right]^{2/p} = \left( \frac{1}{n} \sum_{i=1}^{n} \left[ \frac{1 - s_i(d)}{\pi_i(d) + s_i(d)} \right]^{p/2} \right)^2 \].

Apply Jensen’s inequality on the square to get
\[
\left[ \left( \frac{1}{n} \sum_{i=1}^{n} \left[ \frac{1 - s_i(d)}{\pi_i(d) + s_i(d)} \right]^{p/2} \right)^2 \right]^{2/p} \leq \left[ \frac{1}{n} \sum_{i=1}^{n} \left[ \frac{1 - s_i(d)}{\pi_i(d) + s_i(d)} \right]^{p/2} \right]^{p/2} = [\Pi(d, p)]^{2/p}.
\]

The final factor is
\[
\tilde{c}_{dp} = \tilde{c}_d(p/(p-2)) = \left[ \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} |\operatorname{Cov}(D_{id}, D_{jd})|^{p/(p-2)} \right]^{(p-2)/p},
\]
so when taken together with the bound on the middle factor, we get

\[
\frac{4}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} \left[ \frac{1 - s_i(d)}{\pi_i(d) + s_i(d)} \right] \left[ \frac{1 - s_j(d)}{\pi_j(d) + s_j(d)} \right] \operatorname{Cov}(D_{id}, D_{jd}) \leq 4k_1^2 [\Pi(d, p)]^2 \tilde{c}_{dp}. \quad \square
\]

**Lemma A12.**

\[
\frac{4}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} \left[ \frac{1 - s_i(d)}{\pi_i(d) + s_i(d)} \right] \left[ \frac{1 - s_j(d)}{\pi_j(d) + s_j(d)} \right] \operatorname{Cov}(D_{id} \varepsilon_i, D_{jd} \varepsilon_j) \leq 16k_1^2 [\Pi(d, p)]^2 \tilde{c}_{dp} + 4\tilde{c}_d + 4\tilde{u}_d,
\]

where \( \tilde{c}_{dp} = \tilde{c}_d(p/(p-2)) \).

**Proof.** If either \( s_i(d) = 1 \) or \( s_j(d) = 1 \), then
\[
\left[ \frac{1 - s_i(d)}{\pi_i(d) + s_i(d)} \right] \left[ \frac{1 - s_j(d)}{\pi_j(d) + s_j(d)} \right] \operatorname{Cov}(D_{id} \varepsilon_i, D_{jd} \varepsilon_j) = 0.
\]

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Furthermore, if \( s_i(d) = 0 \) and \( s_j(d) = 0 \), but \( s_{ij}(d, d) = 1 \), then

\[
\text{Cov}(D_{id}\varepsilon_i, D_{jd}\varepsilon_j) = E[D_{id}D_{jd}\varepsilon_i\varepsilon_j] - E[D_{id}\varepsilon_i] E[D_{jd}\varepsilon_j] = 0,
\]

because \( D_{id}D_{jd} \) is then zero with probability one, and \( E[D_{id}\varepsilon_i] = 0 \) as shown in the proof of Lemma A4. It follows that the terms in the double sum are nonzero only when \( s_{ij}(d, d) = 0 \). Note that \( s_{ij}(d, d) = 0 \) implies that \( s_i(d) = 0 \) and \( s_j(d) = 0 \), which means that we can rewrite the expression to be bounded as

\[
\frac{4}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} \left[ \frac{1 - s_i(d)}{\pi_i(d) + s_i(d)} \right] \left[ \frac{1 - s_j(d)}{\pi_j(d) + s_j(d)} \right] \text{Cov}(D_{id}\varepsilon_i, D_{jd}\varepsilon_j)
\]

\[
= \frac{4}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} \left[ \frac{1 - s_{ij}(d, d)}{\pi_i(d)\pi_j(d) + s_{ij}(d, d)} \right] \text{Cov}(D_{id}\varepsilon_i, D_{jd}\varepsilon_j).
\]

When \( s_{ij}(d, d) = 0 \), we can use the same decomposition as in the proof of Lemma A5 to get

\[
\frac{\text{Cov}(D_{id}\varepsilon_i, D_{jd}\varepsilon_j)}{\pi_i(d)\pi_j(d)} = \frac{\text{Cov}(D_{id}, D_{jd})}{\pi_i(d)\pi_j(d)} E[\varepsilon_i\varepsilon_j \mid D_i = D_j = d] + E[\varepsilon_i\varepsilon_j \mid D_i = D_j = d]
\]

\[
\leq \frac{4k^2|\text{Cov}(D_{id}, D_{jd})|}{\pi_i(d)\pi_j(d)} + E[\varepsilon_i\varepsilon_j \mid D_i = D_j = d]
\]

where the final inequality follows from Lemma A3iii. This means that the expression can be bounded as

\[
\frac{4}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} \left[ \frac{1 - s_{ij}(d, d)}{\pi_i(d)\pi_j(d) + s_{ij}(d, d)} \right] \text{Cov}(D_{id}\varepsilon_i, D_{jd}\varepsilon_j)
\]

\[
\leq \frac{16k^2}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} \left[ \frac{1 - s_{ij}(d, d)}{\pi_i(d)\pi_j(d) + s_{ij}(d, d)} \right] |\text{Cov}(D_{id}, D_{jd})|
\]

\[
+ \frac{4}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} [1 - s_{ij}(d, d)] E[\varepsilon_i\varepsilon_j \mid D_i = D_j = d].
\]

Because \( s_{ij}(d, d) = 0 \) implies that \( s_i(d) = 0 \) and \( s_j(d) = 0 \), we have

\[
\left[ \frac{1 - s_{ij}(d, d)}{\pi_i(d)\pi_j(d) + s_{ij}(d, d)} \right] \leq \left[ \frac{1 - s_i(d)}{\pi_i(d) + s_i(d)} \right] \left[ \frac{1 - s_j(d)}{\pi_j(d) + s_j(d)} \right],
\]

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so we also have

$$\frac{16k^2}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} \left[ \frac{1 - s_{ij}(d,d)}{\pi_i(d)\pi_j(d) + s_{ij}(d,d)} \right] \left| \text{Cov}(D_{id}, D_{jd}) \right| \leq \frac{16k^2}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} \left[ \frac{1 - s_i(d)}{\pi_i(d) + s_i(d)} \right] \left[ \frac{1 - s_j(d)}{\pi_j(d) + s_j(d)} \right] \left| \text{Cov}(D_{id}, D_{jd}) \right| .$$

By applying Hölder’s inequality in the same way as in the proof of Lemma A11, we get

$$\frac{16k^2}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} \left[ \frac{1 - s_i(d)}{\pi_i(d) + s_i(d)} \right] \left[ \frac{1 - s_j(d)}{\pi_j(d) + s_j(d)} \right] \left| \text{Cov}(D_{id}, D_{jd}) \right| \leq 16k^2 [\Pi(d,p)]^2 \bar{c}_dp,$$

Continuing, we can write

$$[1 - s_{ij}(d,d)] \text{E}[\varepsilon_i \varepsilon_j | D_i = D_j = d] = e_{ij}(d,d)e_{ji}(d,d) + \text{Cov}(U_{ij}, U_{ji} | D_i = D_j = d),$$

because we can use the same derivation as in the proof of Lemma A5 when $s_{ij}(d,d) = 0$, and we have

$$e_{ij}(d,d) = \text{Cov}(U_{ij}, U_{ji} | D_i = D_j = d) = 0,$$

when $s_{ij}(d,d) = 1$. Therefore,

$$\frac{4}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} [1 - s_{ij}(d,d)] \text{E}[\varepsilon_i \varepsilon_j | D_i = D_j = d]$$

$$= \frac{4}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} e_{ij}(d,d)e_{ji}(d,d) + \frac{4}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} \text{Cov}(U_{ij}, U_{ji} | D_i = D_j = d),$$

which is equal to $4\bar{c}_d + 4\bar{u}_d$ as defined by Definitions 5.  

\[ \square \]

### A3.7 Proposition 6: Expectation of the variance estimator

**Proposition 6.** Provided that Conditions 1 and 2 hold for exposures $a$ and $b$, the bias of the variance estimator described by Aronow and Samii (2017) is

$$\text{E}[\hat{\text{Var}}_a \hat{\text{Var}}_b (\hat{\tau}(a,b))] - \text{Var}(\hat{\tau}(a,b)) = B_1(a,b) + B_2(a,b) + B_3(a,b) + B_3(b,a) + 2B_4(a,b) - B_4(a,a) - B_4(b,b),$$
where

\[ B_1(d_1, d_2) = \frac{1}{n^2} \sum_{i=1}^{n} \left[ \bar{y}_i(d_1) - \bar{y}_i(d_2) \right]^2, \]

\[ B_2(d_1, d_2) = \frac{1}{2n^2} \sum_{i=1}^{n} \sum_{j \neq i} s_{ij}(d_1, d_1) \left[ \bar{y}_i(d_1) + \bar{y}_j(d_1) \right]^2 + s_{ij}(d_1, d_2) \left[ \bar{y}_i(d_1) - \bar{y}_i(d_2) \right]^2, \]

\[ B_3(d_1, d_2) = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} [s_{ij}(d_1, d_1) + s_{ij}(d_1, d_2)] \operatorname{Var}(\varepsilon_i | D_i = d_1), \]

\[ B_4(d_1, d_2) = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} [1 - s_{ij}(d_1, d_2)] \left[ \bar{y}_i(d_1) e_{ji}(d_2, d_1) + \bar{y}_i(d_2) e_{ij}(d_1, d_2) + e_{ij}(d_1, d_2) e_{ji}(d_2, d_1) + \operatorname{Cov}(U_{ij}, U_{ji} | D_i = d_1, D_j = d_2) \right]. \]

\[ \text{Proof.} \] Recall that the variance estimator is

\[ \widehat{\text{Var}}_{\text{AS}}(\hat{\tau}(a, b)) = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} (D_{ia} - D_{ib})(D_{ja} - D_{jb}) P_{ij}(D_i, D_j) Y_i Y_j \]

\[ + \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left[ \frac{D_{ia}}{\pi_i(a)} + \frac{D_{ib}}{\pi_i(b)} \right] \frac{s_{ij}(D_i, a) + s_{ij}(D_i, b)}{\pi_i(a) \pi_i(b)} Y_i^2, \]

where

\[ P_{ij}(d_1, d_2) = \frac{\pi_{ij}(d_1, d_2) - \pi_i(d_1) \pi_j(d_2)}{\pi_{ij}(d_1, d_2) \pi_i(d_1) \pi_j(d_2) + s_{ij}(d_1, d_2)}; \quad s_{ij}(d_1, d_2) = 1[\pi_{ij}(d_1, d_2) = 0], \]

and \( \pi_{ij}(d_1, d_2) = \Pr(D_i = d_1, D_j = d_2) \). Taking expectations of each terms and applying Lemmas A13 and A14 gives

\[ \operatorname{E}\left[ \widehat{\text{Var}}_{\text{AS}}(\hat{\tau}(a, b)) \right] \]

\[ = \operatorname{E}\left[ (\hat{\tau}(a, b))^2 \right] + B_3(a, b) + B_3(b, a) + 2B_4(a, b) - B_4(a, a) - B_4(b, b) \]

\[ + \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} [s_{ij}(a, a) + s_{ij}(a, b)] [\bar{y}_i(a)]^2 - \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} [1 - s_{ij}(a, a)] [\bar{y}_i(a)] [\bar{y}_j(a) \bar{y}_j(a)] \]

\[ + \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} [s_{ij}(b, a) + s_{ij}(b, b)] [\bar{y}_i(b)]^2 - \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} [1 - s_{ij}(b, b)] [\bar{y}_i(b)] [\bar{y}_j(b)] \]
\[
+ \frac{2}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} [1 - s_{ij}(a, b)] \bar{y}_i(a) \bar{y}_j(b)
\]

Applying Lemma A15 on the five last terms gives

\[
\text{E} \left[ \text{Var}_{\text{AS}}(\hat{\tau}(a, b)) \right] = \text{E} \left[ (\hat{\tau}(a, b))^2 \right] - (\tau(a, b))^2 + B_1(a, b) + B_2(a, b)
+ B_3(a, b) + B_3(b, a) + 2B_4(a, b) - B_1(a, a) - B_4(b, b).
\]

The proof is completed by

\[
\text{Var}(\hat{\tau}(a, b)) = \text{E} \left[ (\hat{\tau}(a, b))^2 \right] - \left( \text{E} \left[ \hat{\tau}(a, b) \right] \right)^2 = \text{E} \left[ (\hat{\tau}(a, b))^2 \right] - (\tau(a, b))^2,
\]

where the final equality follows from Proposition 1.

**Lemma A13.**

\[
\text{E} \left[ \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} (D_{ia} - D_{ib})(D_{ja} - D_{jb})P_{ij}(D_i, D_j)Y_iY_j \right]
\]

\[
= \text{E} \left[ (\hat{\tau}(a, b))^2 \right] - \frac{1}{n^2} \sum_{i=1}^{n} \left( \text{E} \left[ Y_i^2 \mid D_i = a \right] + \text{E} \left[ Y_i^2 \mid D_i = b \right] \right)
\]

\[
- \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} [1 - s_{ij}(a, a)] \bar{y}_i(a) \bar{y}_j(a) - B_4(a, a)
\]

\[
- \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} [1 - s_{ij}(b, b)] \bar{y}_i(b) \bar{y}_j(b) - B_4(b, b)
\]

\[
+ \frac{2}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} [1 - s_{ij}(a, b)] \bar{y}_i(a) \bar{y}_j(b) + 2B_4(a, b).
\]

**Proof.** Note that when \(s_{ij}(d_1, d_2) = 0\), we have

\[
P_{ij}(d_1, d_2) = \frac{1}{\pi_i(d_1)\pi_j(d_2)} - \frac{1}{\pi_{ij}(d_1, d_2)},
\]

and when \(s_{ij}(d_1, d_2) = 1\), we have

\[
P_{ij}(d_1, d_2) = -\pi_i(d_1)\pi_j(d_2).
\]

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Hence, for \( s_{ij}(d_1, d_2) \in \{0, 1\} \), we have

\[
P_{ij}(d_1, d_2) = \frac{1 - s_{ij}(d_1, d_2)}{\pi_i(d_1)\pi_j(d_2)} - \frac{1 - s_{ij}(d_1, d_2)}{\pi_{ij}(d_1, d_2) + s_{ij}(d_1, d_2)} - s_{ij}(d_1, d_2)\pi_i(d_1)\pi_j(d_2).
\]

This allows us to decompose the first term of the variance estimator as

\[
\frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{(D_{ia} - D_{ib})(D_{ja} - D_{jb})P_{ij}(D_i, D_j)Y_iY_j}{\pi_i(D_i)\pi_i(D_j)} - \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{(D_{ia} - D_{ib})(D_{ja} - D_{jb})[1 - s_{ij}(D_i, D_j)]Y_iY_j}{\pi_{ij}(D_i, D_j) + s_{ij}(D_i, D_j)} - \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} (D_{ia} - D_{ia})(D_{ja} - D_{ja})s_{ij}(D_i, D_j)\pi_i(D_i)\pi_i(D_j)Y_iY_j
\]

Note that \( s_{ij}(D_i, D_j) = 0 \) with probability one by construction. It follows that

\[
\frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{(D_{ia} - D_{ib})(D_{ja} - D_{jb})[1 - s_{ij}(D_i, D_j)]Y_iY_j}{\pi_i(D_i)\pi_i(D_j)} = \left( \frac{1}{n} \sum_{i=1}^{n} \frac{(D_{ia} - D_{ib})Y_i}{\pi_i(D_i)} \right)^2 = \left( \frac{1}{n} \sum_{i=1}^{n} \frac{D_{ia}Y_i}{\pi_i(a)} - \frac{D_{ib}Y_i}{\pi_i(b)} \right)^2 = (\hat{\tau}(a, b))^2.
\]

Also because \( s_{ij}(D_i, D_j) = 0 \) with probability one, the third term is zero:

\[
\frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} (D_{ia} - D_{ia})(D_{ja} - D_{ja})s_{ij}(D_i, D_j)\pi_i(D_i)\pi_i(D_j)Y_iY_j = 0.
\]

Next, consider the expectation of the second term:

\[
\frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \mathbb{E} \left[ \frac{(D_{ia} - D_{ib})(D_{ja} - D_{jb})[1 - s_{ij}(D_i, D_j)]Y_iY_j}{\pi_{ij}(D_i, D_j) + s_{ij}(D_i, D_j)} \right].
\]

Using the law of total expectation, we can write

\[
\mathbb{E} \left[ \frac{(D_{ia} - D_{ib})(D_{ja} - D_{jb})[1 - s_{ij}(D_i, D_j)]Y_iY_j}{\pi_{ij}(D_i, D_j) + s_{ij}(D_i, D_j)} \right]
\]
Note that the expression reduces to
\[ E \left[ \frac{(D_{ia} - D_{ib})(D_{ja} - D_{jb})[1 - s_{ij}(D_i, D_j)]Y_iY_j}{\pi_{ij}(D_i, D_j) + s_{ij}(D_i, D_j)} \right] = E[Y_i^2 \mid D_i = a] + E[Y_i^2 \mid D_i = b]. \]

This means that we can write
\[
\frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} E \left[ \frac{(D_{ia} - D_{ib})(D_{ja} - D_{jb})[1 - s_{ij}(D_i, D_j)]Y_iY_j}{\pi_{ij}(D_i, D_j) + s_{ij}(D_i, D_j)} \right] \\
= \frac{1}{n^2} \sum_{i=1}^{n} \left( E[Y_i^2 \mid D_i = a] + E[Y_i^2 \mid D_i = b] \right) \\
+ \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} [1 - s_{ij}(a, a)] E[Y_iY_j \mid D_i = a, D_j = a] \\
+ \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} [1 - s_{ij}(b, b)] E[Y_iY_j \mid D_i = b, D_j = b] \\
- \frac{2}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} [1 - s_{ij}(a, b)] E[Y_iY_j \mid D_i = a, D_j = b].
\]

Note that \( Y_i = \tilde{y}_i(D_i) + E_{ij} + U_{ij} \). Therefore, provided that \( i \neq j \) and \( s_{ij}(d_1, d_2) = 0 \), we can write
\[
E[Y_iY_j \mid D_i = d_1, D_j = d_2] \\
= E \left[ (\tilde{y}_i(D_i) + E_{ij} + U_{ij})(\tilde{y}_j(D_j) + E_{ji} + U_{ji}) \right] \mid D_i = d_1, D_j = d_2 \\
= \tilde{y}_i(d_1)\tilde{y}_j(d_2) + \tilde{y}_i(d_2)E_{ij}(d_1, d_2) + \tilde{y}_j(d_2)E_{ji}(d_1, d_2) \mid D_i = d_1, D_j = d_2 \\
+ \tilde{y}_i(d_1)e_{ji}(d_2, d_1) + e_{ij}(d_1, d_2)e_{ji}(d_2, d_1) + e_{ji}(d_2, d_1)E[U_{ij} \mid D_i = d_1, D_j = d_2] \\
+ \tilde{y}_i(d_1)E[U_{ji} \mid D_i = d_1, D_j = d_2] + e_{ij}(d_1, d_2)E[U_{ji} \mid D_i = d_1, D_j = d_2] \\
+ E[U_{ij}U_{ji} \mid D_i = d_1, D_j = d_2]
\]
Recall that $U_{ij} = Y_i - \bar{y}_{ij}(D_i, D_j)$ and $\bar{y}_{ij}(d_1, d_2) = E[Y_i \mid D_i = d_1, D_j = d_2]$, so

$$E[U_{ij} \mid D_i = d_1, D_j = d_2] = E[Y_i - \bar{y}_{ij}(D_i, D_j) \mid D_i = d_1, D_j = d_2]$$

$$= E[Y_i \mid D_i = d_1, D_j = d_2] - \bar{y}_{ij}(d_1, d_2) = 0.$$ 

It follows that

$$E[U_{ij}U_{ji} \mid D_i = d_1, D_j = d_2] = \text{Cov}(U_{ij}, U_{ji} \mid D_i = d_1, D_j = d_2),$$

and

$$E[Y_iY_j \mid D_i = d_1, D_j = d_2] = \bar{y}_i(d_1)\bar{y}_j(d_2) + \bar{y}_i(d_1)e_{ji}(d_2, d_1) + \bar{y}_j(d_2)e_{ij}(d_1, d_2) + e_{ij}(d_1, d_2)e_{ji}(d_2, d_1) + \text{Cov}(U_{ij}, U_{ji} \mid D_i = d_1, D_j = d_2).$$

Recall the definition of $B_4(d_1, d_2)$:

$$B_4(d_1, d_2) = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} [1 - s_{ij}(d_1, d_2)] \left[ \bar{y}_i(d_1)e_{ji}(d_2, d_1) + \bar{y}_j(d_2)e_{ij}(d_1, d_2) + e_{ij}(d_1, d_2)e_{ji}(d_2, d_1) + \text{Cov}(U_{ij}, U_{ji} \mid D_i = d_1, D_j = d_2) \right],$$

which means we can write

$$\frac{1}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} [1 - s_{ij}(d_1, d_2)] E[Y_iY_j \mid D_i = d_1, D_j = d_2]$$

$$= \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} [1 - s_{ij}(d_1, d_2)] \bar{y}_i(d_1)\bar{y}_j(d_2) + B_4(d_1, d_2).$$

Taken together, this allows us to write

$$\frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} E \left[ \frac{(D_{ia} - D_{ib})(D_{ja} - D_{jb})[1 - s_{ij}(D_i, D_j)]Y_iY_j}{\pi_{ij}(D_i, D_j) + s_{ij}(D_i, D_j)} \right]$$

$$= \frac{1}{n^2} \sum_{i=1}^{n} \left( E[Y_i^2 \mid D_i = a] + E[Y_i^2 \mid D_i = \bar{b}] \right)$$

$$+ \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} [1 - s_{ij}(a, a)] \bar{y}_i(a)\bar{y}_j(a) + B_4(a, a)$$

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\[ + \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j \neq i}^{n} \left[ 1 - s_{ij}(b, b) \right] \bar{y}_i(b) \bar{y}_j(b) + B_4(b, b) \]
\[ - \frac{2}{n^2} \sum_{i=1}^{n} \sum_{j \neq i}^{n} \left[ 1 - s_{ij}(a, b) \right] \bar{y}_i(a) \bar{y}_j(b) - 2B_4(a, b). \]

**Lemma A14.**

\[
\mathbb{E} \left[ \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left[ \frac{D_{ia}}{\hat{\pi}_i(a)} + \frac{D_{ib}}{\hat{\pi}_i(b)} \right] \left[ s_{ij}(D_i, a) + s_{ij}(D_i, b) \right] Y_i^2 \right]
= \frac{1}{n^2} \sum_{i=1}^{n} \left( \mathbb{E}[Y_i^2 \mid D_i = a] + \mathbb{E}[Y_i^2 \mid D_i = b] \right)
+ \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j \neq i}^{n} \left[ s_{ij}(a, a) + s_{ij}(a, b) \right] \bar{y}_i(a) [\bar{y}_i(b)]^2 + B_3(a, b)
+ \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j \neq i}^{n} \left[ s_{ij}(b, a) + s_{ij}(b, b) \right] [\bar{y}_i(b)]^2 + B_3(b, a).
\]

**Proof.** The law of total expectation gives

\[
\mathbb{E} \left[ \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left[ \frac{D_{ia}}{\hat{\pi}_i(a)} + \frac{D_{ib}}{\hat{\pi}_i(b)} \right] \left[ s_{ij}(D_i, a) + s_{ij}(D_i, b) \right] Y_i^2 \right]
= \pi_i(a) \mathbb{E} \left[ \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left[ \frac{D_{ia}}{\hat{\pi}_i(a)} + \frac{D_{ib}}{\hat{\pi}_i(b)} \right] \left[ s_{ij}(D_i, a) + s_{ij}(D_i, b) \right] Y_i^2 \mid D_i = a \right]
+ \pi_i(b) \mathbb{E} \left[ \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left[ \frac{D_{ia}}{\hat{\pi}_i(a)} + \frac{D_{ib}}{\hat{\pi}_i(b)} \right] \left[ s_{ij}(D_i, a) + s_{ij}(D_i, b) \right] Y_i^2 \mid D_i = b \right]
= \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left[ s_{ij}(a, a) + s_{ij}(a, b) \right] \mathbb{E}[Y_i^2 \mid D_i = a]
+ \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left[ s_{ij}(b, a) + s_{ij}(b, b) \right] \mathbb{E}[Y_i^2 \mid D_i = b]
\]

Recall from the proof of Lemma A13 that \( s_{ii}(a, a) = s_{ii}(b, b) = 0 \) due to marginal positivity as stipulated by Condition 2, and that \( s_{ii}(a, b) = 1 \) by the fundamental problem of causal
inference. This allows us to write
\[
\mathbb{E} \left[ \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left[ \frac{D_{ia}}{\pi_i(a)} + \frac{D_{ib}}{\pi_i(b)} \right] [s_{ij}(D_i, a) + s_{ij}(D_i, b)] Y_i^2 \right]
\]
\[
= \frac{1}{n^2} \sum_{i=1}^{n} \left( \mathbb{E}[Y_i^2 \mid D_i = a] + \mathbb{E}[Y_i^2 \mid D_i = b] \right)
\]
\[
+ \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} [s_{ij}(a, a) + s_{ij}(a, b)] \mathbb{E}[Y_i^2 \mid D_i = a]
\]
\[
+ \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} [s_{ij}(b, a) + s_{ij}(b, b)] \mathbb{E}[Y_i^2 \mid D_i = b]
\]

Note that \(Y_i^2 = (\bar{y}_i(D_i) + \epsilon_i)^2\) and \(\mathbb{E}[\epsilon_i \mid D_i = d] = 0\), so
\[
\mathbb{E}[Y_i^2 \mid D_i = d] = [\bar{y}_i(d)]^2 + 2\bar{y}_i(d) \mathbb{E}[\epsilon_i \mid D_i = d] = [\bar{y}_i(d)]^2 + \text{Var}(\epsilon_i \mid D_i = d),
\]
and
\[
\frac{1}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} [s_{ij}(a, a) + s_{ij}(a, b)] \mathbb{E}[Y_i^2 \mid D_i = a]
\]
\[
= \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} [s_{ij}(a, a) + s_{ij}(a, b)] [\bar{y}_i(a)]^2 + B_3(a, b)
\]

where
\[
B_3(d_1, d_2) = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} [s_{ij}(d_1, d_1) + s_{ij}(d_1, d_2)] \text{Var}(\epsilon_i \mid D_i = d_1).
\]

Taken together, we have
\[
\mathbb{E} \left[ \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left[ \frac{D_{ia}}{\pi_i(a)} + \frac{D_{ib}}{\pi_i(b)} \right] [s_{ij}(D_i, a) + s_{ij}(D_i, b)] Y_i^2 \right]
\]
\[
= \frac{1}{n^2} \sum_{i=1}^{n} \left( \mathbb{E}[Y_i^2 \mid D_i = a] + \mathbb{E}[Y_i^2 \mid D_i = b] \right)
\]
\[
+ \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} [s_{ij}(a, a) + s_{ij}(a, b)] [\bar{y}_i(a)]^2 + B_3(a, b)
\]

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Proof. Recall that Lemma A15.

\[ \sum_{i=1}^{n} \sum_{j \neq i} \left[ s_{ij}(b, a) + s_{ij}(b, b) \right] [\bar{y}_j(b)]^2 + B_3(b, a). \]

Lemma A15.

\[ B_1(a, b) + B_2(a, b) + B_2(b, a) - (\tau(a, b))^2 \]
\[ = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} \left[ s_{ij}(a, a) + s_{ij}(a, b) \right] [\bar{y}_i(a)]^2 - \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} \left[ 1 - s_{ij}(a, a) \right] \bar{y}_i(a) \bar{y}_j(a) \]
\[ + \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} \left[ s_{ij}(b, a) + s_{ij}(b, b) \right] [\bar{y}_i(b)]^2 - \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} \left[ 1 - s_{ij}(b, b) \right] \bar{y}_i(b) \bar{y}_j(b) \]
\[ + \frac{2}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} \left[ 1 - s_{ij}(a, b) \right] \bar{y}_i(a) \bar{y}_j(b) \]

Proof. Recall that \( s_{ij}(d_1, d_2) = 1[\pi_{ij}(d_1, d_2) = 0] \), so \( s_{ij}(d, d) = s_{ji}(d, d) \). It follows that

\[ \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} s_{ij}(d, d) [\bar{y}_i(d)]^2 = \frac{1}{2n^2} \sum_{i=1}^{n} \sum_{j \neq i} \left[ s_{ij}(d, d) + s_{ji}(d, d) \right] [\bar{y}_i(d)]^2 \]
\[ = \frac{1}{2n^2} \sum_{i=1}^{n} \sum_{j \neq i} s_{ij}(d, d) \left[ [\bar{y}_i(d)]^2 + [\bar{y}_j(d)]^2 \right], \]

where the final equality follows from a reversal of the indices \( i \) and \( j \) for one of the terms. Note that

\[ [\bar{y}_i(d) + \bar{y}_j(d)]^2 = [\bar{y}_i(d)]^2 + [\bar{y}_j(d)]^2 + 2\bar{y}_i(d)\bar{y}_j(d), \]

so we can write

\[ \frac{1}{2n^2} \sum_{i=1}^{n} \sum_{j \neq i} s_{ij}(d, d) \left[ [\bar{y}_i(d)]^2 + [\bar{y}_j(d)]^2 \right] = \frac{1}{2n^2} \sum_{i=1}^{n} \sum_{j \neq i} s_{ij}(d, d) \left[ [\bar{y}_i(d) + \bar{y}_j(d)]^2 \right] \]
\[ - \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} s_{ij}(d, d) \bar{y}_i(d) \bar{y}_j(d). \]

Next, note that \( s_{ij}(d_1, d_2) = s_{ji}(d_2, d_1) \) for the same reason as above. This allows us to write

\[ \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} s_{ij}(b, a) [\bar{y}_i(b)]^2 = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} s_{ij}(a, b) [\bar{y}_j(b)]^2, \]
which means that

\[
\frac{1}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} s_{ij}(a, b)[\bar{y}_i(a)]^2 + \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} s_{ij}(b, a)[\bar{y}_i(b)]^2
\]

\[
= \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} s_{ij}(a, b)[\bar{y}_i(a)]^2 + [\bar{y}_i(b)]^2.
\]

Note that

\[
[\bar{y}_i(a) - \bar{y}_j(b)]^2 = [\bar{y}_i(a)]^2 + [\bar{y}_j(b)]^2 - 2\bar{y}_i(a)\bar{y}_j(b),
\]

so

\[
\frac{1}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} s_{ij}(a, b)[\bar{y}_i(a)]^2 + [\bar{y}_j(b)]^2.
\]

Taken together, we can write

\[
\frac{1}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} [s_{ij}(a, a) + s_{ij}(a, b)][\bar{y}_i(a)]^2 + \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} [s_{ij}(b, a) + s_{ij}(b, b)][\bar{y}_i(b)]^2
\]

\[
= \frac{1}{2n^2} \sum_{i=1}^{n} \sum_{j \neq i} s_{ij}(a, a)[\bar{y}_i(a) + \bar{y}_j(a)]^2 - \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} s_{ij}(a, b)\bar{y}_i(a)\bar{y}_j(a)
\]

\[
+ \frac{1}{2n^2} \sum_{i=1}^{n} \sum_{j \neq i} s_{ij}(b, b)[\bar{y}_i(b) + \bar{y}_j(b)]^2 - \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} s_{ij}(b, b)\bar{y}_i(b)\bar{y}_j(b)
\]

\[
+ \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} s_{ij}(a, b)[\bar{y}_i(a) - \bar{y}_j(b)]^2 + \frac{2}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} s_{ij}(a, b)\bar{y}_i(a)\bar{y}_j(b).
\]

Recall that

\[
B_2(d_1, d_2) = \frac{1}{2n^2} \sum_{i=1}^{n} \sum_{j \neq i} \left( s_{ij}(d_1, d_1)[\bar{y}_i(d_1) + \bar{y}_j(d_1)]^2 + s_{ij}(d_1, d_2)[\bar{y}_i(d_1) - \bar{y}_i(d_2)]^2 \right),
\]

which allows us to write

\[
\frac{1}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} [s_{ij}(a, a) + s_{ij}(a, b)][\bar{y}_i(a)]^2 + \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} [s_{ij}(b, a) + s_{ij}(b, b)][\bar{y}_i(b)]^2
\]

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\[
= B_2(a, b) + B_2(b, a) + \frac{2}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} s_{ij}(a, b) \bar{y}_i(a) \bar{y}_j(b) \\
- \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} s_{ij}(a, a) \bar{y}_i(a) \bar{y}_j(a) - \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} s_{ij}(b, b) \bar{y}_i(b) \bar{y}_j(b).
\]

In turn, this gives

\[
\frac{1}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} [s_{ij}(a, a) + s_{ij}(a, b)] \bar{y}_i(a) \bar{y}_j(a) - \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} [1 - s_{ij}(a, a)] \bar{y}_i(a) \bar{y}_j(a) \\
+ \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} [s_{ij}(b, a) + s_{ij}(b, b)] \bar{y}_i(b) \bar{y}_j(b) - \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} [1 - s_{ij}(b, b)] \bar{y}_i(b) \bar{y}_j(b) \\
+ \frac{2}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} [1 - s_{ij}(a, b)] \bar{y}_i(a) \bar{y}_j(b)
\]

\[
= B_2(a, b) + B_2(b, a) + \frac{2}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} s_{ij}(a, b) \bar{y}_i(a) \bar{y}_j(b) \\
- \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} s_{ij}(a, a) \bar{y}_i(a) \bar{y}_j(a) - \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} s_{ij}(b, b) \bar{y}_i(b) \bar{y}_j(b) \\
- \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} [1 - s_{ij}(a, a)] \bar{y}_i(a) \bar{y}_j(a) - \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} [1 - s_{ij}(b, b)] \bar{y}_i(b) \bar{y}_j(b) \\
+ \frac{2}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} [1 - s_{ij}(a, b)] \bar{y}_i(a) \bar{y}_j(b)
\]

\[
= B_2(a, b) + B_2(b, a) - \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} [\bar{y}_i(a) \bar{y}_j(a) + \bar{y}_i(b) \bar{y}_j(b) - 2 \bar{y}_i(a) \bar{y}_j(b)]
\]

Note that

\[
B_1(d_1, d_2) = \frac{1}{n^2} \sum_{i=1}^{n} [\bar{y}_i(d_1) - \bar{y}_i(d_2)]^2 = \frac{1}{n^2} \sum_{i=1}^{n} [\bar{y}_i(d_1) \bar{y}_i(d_1) - 2 \bar{y}_i(d_1) \bar{y}_i(d_2) + \bar{y}_i(d_2) \bar{y}_i(d_2)],
\]

so

\[
- \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} [\bar{y}_i(a) \bar{y}_j(a) + \bar{y}_i(b) \bar{y}_j(b) - 2 \bar{y}_i(a) \bar{y}_j(b)]
\]

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\[ B_1(a, b) - \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left( \bar{y}_i(a) \bar{y}_j(a) + \bar{y}_i(b) \bar{y}_j(b) - 2 \bar{y}_i(a) \bar{y}_j(b) \right). \]

Finally, note that
\[
\frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left[ \bar{y}_i(a) \bar{y}_j(a) + \bar{y}_i(b) \bar{y}_j(b) - 2 \bar{y}_i(a) \bar{y}_j(b) \right]
= \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} [\bar{y}_i(a) \bar{y}_j(a) + \bar{y}_i(b) \bar{y}_j(b) - \bar{y}_i(a) \bar{y}_j(b) - \bar{y}_i(b) \bar{y}_j(a)]
\]

and
\[
\bar{y}_i(a) \bar{y}_j(a) + \bar{y}_i(b) \bar{y}_j(b) - \bar{y}_i(a) \bar{y}_j(b) - \bar{y}_i(b) \bar{y}_j(a) = \left[ \bar{y}_i(a) - \bar{y}_i(b) \right] \left[ \bar{y}_j(a) - \bar{y}_j(b) \right],
\]
which implies that
\[
\frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} [\bar{y}_i(a) \bar{y}_j(a) + \bar{y}_i(b) \bar{y}_j(b) - 2 \bar{y}_i(a) \bar{y}_j(b)] = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left[ \bar{y}_i(a) - \bar{y}_i(b) \right] \left[ \bar{y}_j(a) - \bar{y}_j(b) \right] = \left( \frac{1}{n} \sum_{i=1}^{n} [\bar{y}_i(a) - \bar{y}_i(b)] \right)^2 = (\tau(a, b))^2,
\]
which completes the proof. \(\square\)

## A4 Proofs of propositions in supplement

### A4.1 Proposition A1: The Hájek estimator

The linearization used to prove consistency for the Hájek estimator requires an alternative representation of the estimator.

**Definition A6** (Components for the Hájek estimator).

\[
\mu_d = \sum_{i=1}^{n} \bar{y}_i(d), \quad \hat{\mu}_d = \sum_{i=1}^{n} \frac{D_{id} Y_i}{\pi_i(d)}, \quad \hat{n}_d = \sum_{i=1}^{n} \frac{D_{id}}{\pi_i(d)}.
\]

**Lemma A16.** If Condition 1 holds, then \( \mu_d = \mathcal{O}(n) \) for all \( d \in \Delta \).
Proof. By Lemma A3i,
\[ \mu_d = \sum_{i=1}^{n} \bar{y}_i(d) \leq \sum_{i=1}^{n} |\bar{y}_i(d)| \leq \sum_{i=1}^{n} k_1 = k_1 n. \]

Lemma A17. If Condition 1 and 2 hold, then
\[ (\hat{\mu}_d - \mu_d)/n = O_p(n^{-0.5} + \bar{c}_d^{0.5} + \bar{e}_d^{0.5} + \bar{u}_d^{0.5}). \]

Proof. Note that \( \hat{\tau}(a, b) = (\hat{\mu}_a - \hat{\mu}_b)/n \) and \( \tau(a, b) = (\mu_a - \mu_b)/n \), so the proofs of Propositions 1 and 2 can be copied almost in verbatim to show
\[ E[\hat{\mu}_d] = \mu_d \quad \text{and} \quad \frac{\text{Var}(\hat{\mu}_d)}{n^2} \leq \frac{2k_1^2k_2}{n} + 10k_1^2k_2^2c_d + 2\bar{e}_d + 2\bar{u}_d. \]
The logic of the proof of Proposition 3 then gives
\[ \sqrt{E[(\hat{\mu}_d - \mu_d)^2/n^2]} = O(n^{-0.5} + \bar{c}_d^{0.5} + \bar{e}_d^{0.5} + \bar{u}_d^{0.5}). \]
Markov’s inequality completes the proof. \( \square \)

Lemma A18. If Condition 2 holds, then \( (\hat{n}_d - n)/n = O_p(n^{-0.5} + \bar{c}_d^{0.5}). \)

Proof. The first step is to show that \( E[\hat{n}_d] = n \) when \( d \) satisfies Condition 2:
\[ E[\hat{n}_d] = \sum_{i=1}^{n} \frac{E[D_{id}]}{\pi_i(d)} = \sum_{i=1}^{n} \frac{\pi_i(d)}{\pi_i(d)} = n. \]
Next consider the variance:
\[ \text{Var}(\hat{n}_d) = \sum_{i=1}^{n} \frac{\text{Var}(D_{id})}{[\pi_i(d)]^2} + \sum_{i=1}^{n} \sum_{j \neq i} \frac{\text{Cov}(D_{id}, D_{jd})}{\pi_i(d)\pi_j(d)}. \]
By Condition 2,
\[ \frac{\text{Var}(D_{id})}{[\pi_i(d)]^2} = \frac{\pi_i(d)(1 - \pi_i(d))}{[\pi_i(d)]^2} \leq k_2 \quad \text{and} \quad \frac{\text{Cov}(D_{id}, D_{jd})}{\pi_i(d)\pi_j(d)} \leq k_2^2 |\text{Cov}(D_{id}, D_{jd})|, \]
so \( \text{Var}(\hat{n}_d)/n^2 \leq k_2 n^{-1} + k_2^2 \bar{c}_d. \) The logic of the proof of Proposition 3 then gives
\[ \sqrt{E[(\hat{n}_d - n)^2/n^2]} = O(n^{-0.5} + \bar{c}_d^{0.5}), \]
and Markov’s inequality completes the proof. \( \square \)
Proposition A1. Provided that Conditions 1, 2, 3 and 4 hold for exposures a and b, the Hájek estimator is consistent for the expected exposure effect and converges at the rate
\[ \hat{\tau}_{\text{Hájek}}(a, b) - \tau(a, b) = O_p(n^{-0.5} + \varepsilon_a^0.5 + \varepsilon_b^0.5 + \bar{\varepsilon}_a^0.5 + \bar{\varepsilon}_b^0.5 + \bar{\mu}_a^0.5 + \bar{\mu}_b^0.5). \]

Proof. Note that \( \hat{\tau}_{\text{Hájek}}(a, b) = \hat{\mu}_a/\hat{n}_a - \hat{\mu}_b/\hat{n}_b \) and \( \tau(a, b) = \mu_a/n - \mu_b/n \), so we can write
\[ \hat{\tau}_{\text{Hájek}}(a, b) - \tau(a, b) = \left( \frac{\hat{\mu}_a}{\hat{n}_a} - \frac{\mu_a}{n} \right) - \left( \frac{\hat{\mu}_b}{\hat{n}_b} - \frac{\mu_b}{n} \right). \]

For any \( d \in \{a, b\} \), consider
\[ \frac{\hat{\mu}_d - \mu_d}{\hat{n}_d/n} = \frac{\hat{\mu}_d/n - (\mu_d/n)(\hat{n}_d/n)}{\hat{n}_d/n} = \frac{(\hat{\mu}_d - \mu_d)/n}{\hat{n}_d/n} - \frac{(\mu_d/n)(\hat{n}_d - n)/n}{\hat{n}_d/n}, \]
where Lemma A18 ensures that we can ignore the event \( \hat{n}_d = 0 \).

Let \( f(x, y) = x/y \) and consider a Taylor expansion of the two terms around \((0, 1)\):
\[ \frac{(\hat{\mu}_d - \mu_d)/n}{\hat{n}_d/n} = f((\hat{\mu}_d - \mu_d)/n, \hat{n}_d/n) = (\hat{\mu}_d - \mu_d)/n + r_1 \]
\[ \frac{(\mu_d/n)(\hat{n}_d - n)/n}{\hat{n}_d/n} = f((\mu_d/n)(\hat{n}_d - n)/n, \hat{n}_d/n) = (\mu_d/n)(\hat{n}_d - n)/n + r_2 \]
where \( r_1 = o_p((\hat{\mu}_d - \mu_d)/n + (\hat{n}_d - n)/n) \) and \( r_2 = o_p((\mu_d/n)(\hat{n}_d - n)/n + (\hat{n}_d - n)/n) \) because Lemmas A16, A17 and A18 give convergence of \((\hat{\mu}_d - \mu_d)/n \) and \((\mu_d/n)(\hat{n}_d - n)/n \) to zero and of \( \hat{n}_d/n \) to one. Lemma A16 gives \((\mu_d/n)(\hat{n}_d - n)/n = O_p((\hat{n}_d - n)/n)\), so by Lemmas A17 and A18
\[ \hat{\tau}_{\text{Hájek}}(a, b) - \tau(a, b) = O_p(n^{-0.5} + \varepsilon_a^0.5 + \varepsilon_b^0.5 + \bar{\varepsilon}_a^0.5 + \bar{\varepsilon}_b^0.5 + \bar{\mu}_a^0.5 + \bar{\mu}_b^0.5). \] \[ \square \]

A4.2 Propositions A2 and A3: The difference estimator

Proposition A2. Provided that Condition 2 holds and that the predictions are external, the difference estimator is unbiased for the expected exposure effect: \( \mathbb{E}[\hat{\tau}_{\text{DR}}(a, b)] = \tau(a, b) \).

Proof. Write the estimator as
\[ \hat{\tau}_{\text{DR}}(a, b) = \frac{1}{n} \sum_{i=1}^{n} \left[ \hat{y}_i(a) - \hat{y}_i(b) \right] + \frac{1}{n} \sum_{i=1}^{n} \frac{(D_{ia} - D_{ib})[Y_i - \hat{y}_i(D_i)]}{\pi_i(D_i)} \]
\[ = \frac{1}{n} \sum_{i=1}^{n} \left[ \hat{y}_i(a) - \hat{y}_i(b) \right] + \frac{1}{n} \sum_{i=1}^{n} \left[ \frac{D_{ia}Y_i}{\pi_i(a)} - \frac{D_{ib}Y_i}{\pi_i(b)} - \frac{D_{ia}\hat{y}_i(a)}{\pi_i(a)} + \frac{D_{ib}\hat{y}_i(b)}{\pi_i(b)} \right] \]

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= \hat{\tau}(a, b) + \frac{1}{n} \sum_{i=1}^{n} [\hat{y}_i(a) - \hat{y}_i(b)] - \frac{1}{n} \sum_{i=1}^{n} \left[ \frac{D_{ia}\hat{y}_i(a)}{\pi_i(a)} - \frac{D_{ib}\hat{y}_i(b)}{\pi_i(b)} \right].

Take expectations to get

\[ E[\hat{\tau}_{de}(a, b)] = \tau(a, b) + \frac{1}{n} \sum_{i=1}^{n} \left[ E[\hat{y}_i(a)] - E[\hat{y}_i(b)] \right] - \frac{1}{n} \sum_{i=1}^{n} \left[ \frac{E[D_{ia}\hat{y}_i(a)]}{\pi_i(a)} - \frac{E[D_{ib}\hat{y}_i(b)]}{\pi_i(b)} \right]. \]

We have

\[ \frac{E[D_{id}\hat{y}_i(d)]}{\pi_i(d)} = \frac{E[D_{id}] E[\hat{y}_i(d)]}{\pi_i(d)} = \frac{\pi_i(d) E[\hat{y}_i(d)]}{\pi_i(d)} = E[\hat{y}_i(d)], \]

because the predictions are external and Condition 2. As a result, the two last terms in the expectation cancel.

\[ \square \]

**Proposition A3.** Provided that Conditions 1, 2, 3, 4, A1 and A2 hold and that the predictions are external, the difference estimator is consistent for the expected exposure effect and converges at the rate

\[ \hat{\tau}_{de}(a, b) - \tau(a, b) = \mathcal{O}_p \left( n^{-0.5} + \bar{c}_a^{0.5} + \bar{c}_b^{0.5} + \bar{e}_a^{0.5} + \bar{e}_b^{0.5} + \bar{u}_a^{0.5} + \bar{u}_b^{0.5} + p_a^{0.5} + p_b^{0.5} \right). \]

**Proof.** Write the estimator as

\[ \hat{\tau}_{de}(a, b) = \hat{\tau}(a, b) + \frac{1}{n} \sum_{i=1}^{n} [\hat{y}_i(a) - \hat{y}_i(b)] - \frac{1}{n} \sum_{i=1}^{n} \left[ \frac{D_{ia}\hat{y}_i(a)}{\pi_i(a)} - \frac{D_{ib}\hat{y}_i(b)}{\pi_i(b)} \right]. \]

Apply Lemma A2 to get

\[ \text{Var}(\hat{\tau}_{de}(a, b)) \leq 5 \text{Var}(\hat{\tau}(a, b)) + \frac{5}{n^2} \text{Var} \left( \sum_{i=1}^{n} \hat{y}_i(a) \right) + \frac{5}{n^2} \text{Var} \left( \sum_{i=1}^{n} \hat{y}_i(b) \right) + \frac{5}{n^2} \text{Var} \left( \sum_{i=1}^{n} \frac{D_{ia}\hat{y}_i(a)}{\pi_i(a)} \right) + \frac{5}{n^2} \text{Var} \left( \sum_{i=1}^{n} \frac{D_{ib}\hat{y}_i(b)}{\pi_i(b)} \right). \]

The first term is bounded by Proposition 2 as

\[ 5 \text{Var}(\hat{\tau}(a, b)) \leq 40k_1^2k_2/n + 100k_1^2k_2^2[\bar{c}_a + \bar{c}_b] + 20[\bar{e}_a + \bar{e}_b + \bar{u}_a + \bar{u}_b]. \]

Consider the two subsequent terms:

\[ \frac{5}{n^2} \text{Var} \left( \sum_{i=1}^{n} \hat{y}_i(d) \right) = \frac{5}{n^2} \sum_{i=1}^{n} \text{Var}(\hat{y}_i(d)) + \frac{5}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} \text{Cov}(\hat{y}_i(d), \hat{y}_j(d)) \leq \frac{5k_3}{n} + 5p_d, \]
where Condition A1 and Definition A4 were applied in the last inequality.

Next, consider the last two terms in the variance expression:

$$\frac{5}{n^2} \text{Var} \left( \sum_{i=1}^{n} \frac{D_{id} \hat{y}_i(d)}{\pi_i(d)} \right) = \frac{5}{n^2} \sum_{i=1}^{n} \frac{\text{Var}(D_{id} \hat{y}_i(d))}{[\pi_i(d)]^2} + \frac{5}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} \frac{\text{Cov}(D_{id} \hat{y}_i(d), D_{jd} \hat{y}_j(d))}{\pi_i(d) \pi_j(d)}.$$ 

Consider the first term, and recall that the predictions are external (i.e., independent of the assignments), so

$$\frac{5}{n^2} \text{Var}(D_{id} \hat{y}_i(d)) = \frac{[\pi_i(d)]^2 \text{Var}(\hat{y}_i(d)) + (E[\hat{y}_i(d)])^2 \text{Var}(D_{id}) + \text{Var}(D_{id}) \text{Var}(\hat{y}_i(d))}{{[\pi_i(d)]^2}}.$$ 

Note that $\text{Var}(D_{id})/\pi_i(d) \leq 1$ because $D_{id}$ is binary. Furthermore, Condition A1 gives

$$\text{Var}(\hat{y}_i(d)) \leq E\left[(\hat{y}_i(d))^2\right] \leq k_3 \quad \text{and} \quad (E[\hat{y}_i(d)])^2 \leq E\left[(\hat{y}_i(d))^2\right] \leq k_3,$$

where the second result uses Jensen’s inequality. Together with Condition 2, it follows that

$$\frac{5}{n^2} \sum_{i=1}^{n} \frac{\text{Var}(D_{id} \hat{y}_i(d))}{[\pi_i(d)]^2} \leq \frac{5}{n^2} \sum_{i=1}^{n} \left( k_3 + 2k_2^2k_3 \right) \leq \frac{5k_3}{n} + \frac{10k_2k_3}{n}.$$ 

Next, consider the second term. Recall again that the predictions are external and apply the covariance decomposition in Bohrnstedt and Goldberger (1969) to get

$$\text{Cov}(D_{id} \hat{y}_i(d), D_{jd} \hat{y}_j(d)) = \pi_i(d)\pi_j(d) \text{Cov}(\hat{y}_i(d), \hat{y}_j(d)) + E[\hat{y}_i(d)] E[\hat{y}_j(d)] \text{Cov}(D_{id}, D_{jd})$$

$$+ \text{Cov}(D_{id}, D_{jd}) \text{Cov}(\hat{y}_i(d), \hat{y}_j(d))$$

Note that

$$0 \leq \pi_i(d)\pi_j(d) + \text{Cov}(D_{id}, D_{jd}) = E[D_{id}D_{jd}] \leq 1,$$

so

$$\pi_i(d)\pi_j(d) \text{Cov}(\hat{y}_i(d), \hat{y}_j(d)) + \text{Cov}(D_{id}, D_{jd}) \text{Cov}(\hat{y}_i(d), \hat{y}_j(d)) \leq |\text{Cov}(\hat{y}_i(d), \hat{y}_j(d))|.$$ 

By Jensen’s inequality and Condition A1,

$$E[\hat{y}_i(d)] E[\hat{y}_j(d)] \leq \sqrt{E\left[(\hat{y}_i(d))^2\right]} E\left[(\hat{y}_j(d))^2\right] \leq \sqrt{k_3^2} = k_3,$$

so

$$E[\hat{y}_i(d)] E[\hat{y}_j(d)] \text{Cov}(D_{id}, D_{jd}) \leq k_3 |\text{Cov}(D_{id}, D_{jd})|.$$ 

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Together with Condition 2 and Definitions 7 and A4, this yields

\[
\frac{5}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} \frac{\text{Cov}(D_i \hat{y}_i(d), D_j \hat{y}_j(d))}{\pi_i(d) \pi_j(d)} \leq 5k_2^2 p_d + 5k_2^2 k_3 \bar{c}_d.
\]

Together with the bound on the first term, we have

\[
\frac{5}{n^2} \text{Var} \left( \sum_{i=1}^{n} \frac{D_i \hat{y}_i(d)}{\pi_i(d)} \right) \leq \frac{5k_3}{n} + \frac{10k_2 k_3}{n} + 5k_2^2 p_d + 5k_2^2 k_3 \bar{c}_d
\]

It follows that

\[
\text{Var}(\hat{\tau}_{de}(a, b)) \leq \frac{40k_1^2 k_2 + 20k_2 k_3 + 20k_3}{n} + \left( 100k_1^2 k_2^2 + 5k_2^2 k_3 \right) [\bar{c}_a + \bar{c}_b]
\]

\[
+ 20 [\bar{e}_a + \bar{e}_b + \bar{u}_a + \bar{u}_b] + (5 + 5k_3^2) [p_a + p_b]
\]

Together with Proposition A2, this gives convergence in mean square:

\[
\sqrt{\mathbb{E} \left[ (\hat{\tau}_{de}(a, b) - \hat{\tau}(a, b))^2 \right]} = O(n^{-0.5} + \bar{c}_a^{0.5} + \bar{e}_b^{0.5} + \bar{e}_a^{0.5} + \bar{u}_a^{0.5} + \bar{u}_b^{0.5} + p_a^{0.5} + p_b^{0.5}),
\]

and Markov’s inequality completes the proof. \(\square\)

### A4.3 Proposition A4: The generalized regression estimator

**Proposition A4.** Provided that Conditions 1, 2, 3, 4, A3 and A4 hold, the generalized regression estimator is consistent for the expected exposure effect and converges at the rate

\[
\hat{\tau}_{GR}(a, b) - \tau(a, b) = O_p \left( n^{-0.5} + \bar{c}_a^{0.5} + \bar{e}_b^{0.5} + \bar{e}_a^{0.5} + \bar{u}_a^{0.5} + \bar{u}_b^{0.5} \right).
\]

**Proof.** Write the estimator as

\[
\hat{\tau}_{GR}(a, b) = \hat{\tau}(a, b) + \left[ \frac{1}{n} \sum_{i=1}^{n} \left( x_i - \frac{D_{ia} x_i}{\pi_i(a)} \right) \right] \hat{\beta}(a) - \left[ \frac{1}{n} \sum_{i=1}^{n} \left( x_i - \frac{D_{ib} x_i}{\pi_i(b)} \right) \right] \hat{\beta}(b).
\]

Using the same approach as in the proofs of Propositions 1, 2 and 3, it can be shown that

\[
\left[ \frac{1}{n} \sum_{i=1}^{n} \left( x_i - \frac{D_{id} x_i}{\pi_i(d)} \right) \right] = O_p \left( n^{-0.5} + \bar{c}_d^{0.5} \right).
\]
By Markov’s inequality, \( \mathbb{E}[\|\hat{\beta}(d)\|] = \mathcal{O}(1) \) implies \( \hat{\beta}(d) = \mathcal{O}_p(1) \), so

\[
\frac{1}{n} \sum_{i=1}^{n} \left( x_i - \frac{D_{id}x_i}{\pi_i(d)} \right) \hat{\beta}(d) = \mathcal{O}_p(n^{-0.5} + \tilde{c}_d^{0.5}).
\]

The proof is then completed by Proposition 3. \(\square\)