A Subfamily of Univalent Functions Associated with $q$-Analogue of Noor Integral Operator

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The main objective of the present paper is to define a new subfamily of analytic functions using subordinations along with the newly defined $q$-Noor integral operator. We investigate a number of useful properties such as coefficient estimates, integral representation, linear combination, weighted and arithmetic means, and radius of starlikeness for this class.

1. Introduction and Definitions

In recent years, $q$-analysis ($q$-calculus) has motivated the researchers a lot due to its numerous applications in mathematics and physics. Jackson [1, 2] was the first to give some application of $q$-calculus and also introduced the $q$-analogue of derivative and integral operator. Later on, Aral and Gupta [3, 4] defined the $q$-Baskakov-Durrmeyer operator by using $q$-beta function while in papers [5, 6] the authors discussed the $q$-generalization of complex operators known as $q$-Picard and $q$-Gauss-Weierstrass singular integral operators. Using convolution of normalized analytic functions, Kanas and Raducanu [7] defined $q$-analogue of Ruscheweyh differential operator and studied some of its properties. The application of this differential operator was further studied by Aldweby and Darus [8] and Mahmood and Sokół [9]. The aim of the current paper is to define a $q$-analogue of the Noor integral operator involving convolution concepts and then give some interesting applications of this operator.

Let us denote the open unit disk by $\mathcal{D} = \{z \in \mathbb{C} : |z| < 1\}$ and the symbol $\mathcal{A}$ denotes the family of those analytic functions $f$ which has the following Taylor series representation:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad (z \in \mathcal{D}). \quad (1)$$

For two functions $f$ and $g$ that are analytic in $\mathcal{D}$ and have the form (1), we define the convolution of these functions by

$$f(z) * g(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k, \quad (z \in \mathcal{D}). \quad (2)$$

For $0 < q < 1$, the $q$-derivative of a function $f \in \mathcal{A}$ is defined by

$$\partial_q f(z) = \frac{f(qz) - f(z)}{z(q-1)}, \quad (z \neq 0). \quad (3)$$

It can easily be seen that for $n \in \mathbb{N} := \{1, 2, 3, \ldots\}$ and $z \in \mathcal{D}$

$$\partial_q \left\{ \sum_{n=1}^{\infty} a_n z^n \right\} = \sum_{n=1}^{\infty} [n, q] a_n z^{n-1}, \quad (4)$$

where

$$[n, q] = \frac{1 - q^n}{1 - q} = 1 + \sum_{l=1}^{n-1} q^l, \quad [0, q] = 0. \quad (5)$$

For any nonnegative integer $n$, the $q$-number shift factorial is defined by

$$[n, q]! = \begin{cases} 1, & n = 0, \\ [1, q] [2, q] [3, q] \cdots [n, q], & n \in \mathbb{N}. \end{cases} \quad (6)$$
Also the $q$-generalized Pochhammer symbol for $x > 0$ is given by
\[
[x,q]_n = \begin{cases} 
1, & n = 0, \\
[x,q][x+1,q] \cdots [x+n-1,q], & n \in \mathbb{N}.
\end{cases}
\]  
(7)

For $\mu > 1$, we define the function $\mathcal{F}_q^{-1}(\mu, z)$ by
\[
\mathcal{F}_q^{-1}(\mu, z) f(z) = \sum_{n=1}^{\infty} \binom{\mu, q}{n} \frac{a_n}{n!} z^n,
\]
(8)

where the function $\mathcal{F}_q^{\mu}(z)$ is given by
\[
\mathcal{F}_q^{\mu}(z) = z + \sum_{n=1}^{\infty} \binom{\mu+1, q}{n-1} \frac{n!}{(n-1)!} a_n z^n,
\]
(9)

In (9), it is quite clear that the series defined is convergent absolutely in $\mathfrak{D}$. Using the definition of $q$-derivative along with the idea of convolutions, we now define the integral operator $\mathcal{F}_q^{\mu} : \mathfrak{A} \to \mathfrak{A}$ by
\[
\mathcal{F}_q^{\mu} f(z) \overset{\text{def}}{=} \mathcal{F}_q^{-1}(\mu, z) f(z) = z + \sum_{n=1}^{\infty} \binom{\mu+1, q}{n-1} \frac{n!}{(n-1)!} a_n z^n,
\]
(10)

with
\[
\psi_n = \frac{[n,q]!}{[\mu+1,q]_{n-1}}.
\]
(11)

From (10), we can easily get the identity
\[
[\mu+1,q] \mathcal{F}_q^{\mu} f(z) = [\mu,q] \mathcal{F}_q^{\mu+1} f(z)
\]
(12)

We note that $\mathcal{F}_q^{\mu} f(z) = z \mathcal{D}_q f(z)$, and
\[
\lim_{q \to 1^-} \mathcal{F}_q^{\mu} f(z) = z + \sum_{n=1}^{\infty} \frac{n!}{(n-1)!} a_n z^n.
\]
(13)

This shows that, by taking $q \to 1^-$, the operator defined in (10) reduces to the familiar Noor integral operator introduced in [10, 11]. Also for more details on the $q$-analogue of differential and integral operators, see the work [12–14].

Motivated from the work studied in [7, 15–17], we now define subfamilies of the set $\mathfrak{A}$ by using the operator $\mathcal{F}_q^{\mu}$ as follows.

**Definition 1.** Let $-1 < B < A < 1$ and $0 < q < 1$. Then the function $f \in \mathfrak{A}$ is in the class $\mathcal{Q}_q(\mu, A, B)$ if it satisfies
\[
\frac{z \partial_q \left( \mathcal{F}_q^{\mu} f(z) \right)}{\mathcal{F}_q^{\mu} f(z)} \leq \frac{1 + Az}{1 + Bz}, \quad (z \in \mathfrak{D}),
\]
(14)

where the notion “$<$” denotes the familiar subordinations.

Equivalently, a function $f \in \mathfrak{A}$ is in the class $\mathcal{Q}_q(\mu, A, B)$, if and only if
\[
\frac{z \partial_q \left( \mathcal{F}_q^{\mu} f(z) \right)}{\mathcal{F}_q^{\mu} f(z)} \leq \frac{1 + Az}{1 + Bz}, \quad (z \in \mathfrak{D}).
\]
(15)

We will assume throughout our discussion, unless otherwise stated, that
\[
\mu > -1, \quad -1 \leq B < A \leq 1, \quad 0 < q < 1
\]

and all coefficients $a_n$ are positive.

We need the following result in the proof of a result.

**Lemma 2** (see [18]). Let $-1 \leq B_2 \leq B_1 < A_1 \leq A_2 \leq 1$. Then
\[
\frac{1 + A_1 z}{1 + B_1 z} < \frac{1 + A_2 z}{1 + B_2 z}.
\]
(17)

**2. Main Results**

**Theorem 3.** Let $f \in \mathfrak{A}$ be given by (1). Then the function $f$ is in the family $\mathcal{Q}_q(\mu, A, B)$, if and only if
\[
\sum_{n=0}^{\infty} \{|n,q| \left( 1 - B - 1 + A \right) a_n \psi_{n-1} \} < (A - B).
\]
(18)

**Proof.** Let us assume first that inequality (18) holds. To show $f \in \mathcal{Q}_q(\mu, A, B)$, we only need to prove the inequality (15).

For this, consider
\[
\frac{z \partial_q \left( \mathcal{F}_q^{\mu} f(z) \right)}{\mathcal{F}_q^{\mu} f(z)} \leq \frac{1 + Az}{1 + Bz}, \quad (z \in \mathfrak{D}).
\]
(19)

where we have used (4), (10), and (18) and this completes the direct part. Conversely, let $f \in \mathcal{Q}_q(\mu, A, B)$ be of the form (1). Then from (15) along with (10), we have, for $z \in \mathfrak{D}$,
\[
\frac{z \partial_q \left( \mathcal{F}_q^{\mu} f(z) \right)}{\mathcal{F}_q^{\mu} f(z)} \leq \frac{1 + Az}{1 + Bz}, \quad (z \in \mathfrak{D}).
\]
(20)

Since $|\Re z| < |z|$, we have
\[
\Re \left\{ \sum_{n=1}^{\infty} \psi_{n-1} \left[ \left| n,q \right| - 1 \right] a_n z^n \right\} < 1.
\]
(21)
Now we choose values of \( z \) on the real axis such that \( z \partial_q (\mathfrak{H}_q f(z)) / \mathfrak{H}_q f(z) \) is real. Upon clearing the denominator in (21) and letting \( z \to 1 \) through real values, we obtain the required inequality (18).}

**Theorem 4.** Let \( f \in \mathfrak{C}_q(\mu, A, B) \). Then

\[
\mathfrak{H}_q f(z) = \exp \int_0^z \frac{1}{t} \left( \frac{1 - A\phi(t)}{1 - B\phi(t)} \right) dt,
\]

with \( |\phi(z)| < 1 \) and \( z \in \mathfrak{D} \).

**Proof.** Let \( f \in \mathfrak{C}_q(\mu, A, B) \) and setting

\[
\frac{z \partial_q (\mathfrak{H}_q f(z))}{\mathfrak{H}_q f(z)} = v(z),
\]

with

\[
v(z) < \frac{1 + Az}{1 + Bz},
\]

equivalently, we can write

\[
\frac{v(z) - 1}{A - Bv(z)} < 1,
\]

or in other way, we have

\[
v(z) - 1 = \phi(z),
\]

\[
|\phi(z)| < 1,
\]

\[
(z \in \mathfrak{D}).
\]

Thus we can rewrite

\[
\frac{z \partial_q (\mathfrak{H}_q f(z))}{\mathfrak{H}_q f(z)} = \left( \frac{1 - A\phi(t)}{1 - B\phi(t)} \right),
\]

and further by simple computation of integration, the proof is completed. \( \square \)

**Theorem 5.** Let \( f_i \in \mathfrak{C}_q(\mu, A, B) \) and have the form

\[
f_i(z) = z + \sum_{k=1}^{\infty} a_{ij} z^k, \quad \text{for } i = 1, 2, \ldots, l.
\]

Then \( F \in \mathfrak{C}_q(\mu, A, B) \), where

\[
F(z) = \sum_{i=1}^{l} c_i f_i(z) \quad \text{with} \quad \sum_{i=1}^{l} c_i = 1.
\]

**Proof.** By the virtue of Theorem 3, one can write

\[
\sum_{n=2}^{\infty} \left( \left[ n, q \right] (1 - B) - 1 + A \right) \psi_{n-1} a_{nj} < 1.
\]

Therefore

\[
F(z) = \sum_{i=1}^{l} c_i \left( z + \sum_{n=2}^{\infty} a_{ij} z^n \right) = z + \sum_{i=1}^{l} \sum_{n=2}^{\infty} c_i a_{nj} z^n
\]

\[
= z + \sum_{n=2}^{\infty} \left( \sum_{i=1}^{l} c_i a_{nj} \right) z^n;
\]

however

\[
\sum_{n=2}^{\infty} \left( \left[ n, q \right] (1 - B) - 1 + A \right) \psi_{n-1} \left( \sum_{i=1}^{l} a_{nj} \right) \leq 1;
\]

then \( F \in \mathfrak{C}_q(\mu, A, B) \). Hence the proof is complete. \( \square \)

**Theorem 6.** If \( f \) and \( g \) belong to \( \mathfrak{C}_q(\mu, A, B) \), then their weighted mean \( h_j \) is also in \( \mathfrak{C}_q(\mu, A, B) \), where \( h_j \) is defined by

\[
h_j(z) = \left\{ \frac{\left( 1 - j \right) f(z) + \left( 1 + j \right) g(z)}{2} \right\}.
\]

**Proof.** From (33), we can easily write

\[
h_j(z) = z + \sum_{n=2}^{\infty} \left( \frac{1 - j}{2} a_n + \frac{1 + j}{2} b_n \right) z^n.
\]

To prove that \( h_j \in \mathfrak{C}_q(\mu, A, B) \), we need to show that

\[
\sum_{n=2}^{\infty} \left( \left[ n, q \right] (1 - B) - 1 + A \right) \psi_{n-1} a_{nj} < 1.
\]

For this, consider

\[
\sum_{n=2}^{\infty} \left( \left[ n, q \right] (1 - B) - 1 + A \right) \psi_{n-1} a_{nj} = \left( \frac{1 - j}{2} \right)
\]

\[
\sum_{n=2}^{\infty} \left( \frac{1 - j}{2} a_n + \frac{1 + j}{2} b_n \right) \psi_{n-1} < 1.
\]

where we have used inequality (18). Hence the result follows. \( \square \)
Theorem 7. Let \( f_i \) with \( i = 1, 2, \ldots, \lambda \) belong to the class \( \mathcal{C}_q(\mu, A, B) \). Then the arithmetic mean \( h \) of \( f_i \) is given by
\[
h(z) = \frac{1}{\lambda} \sum_{i=1}^{\lambda} f_i(z) \tag{37}
\]
and is also in the class \( \mathcal{C}_q(\mu, A, B) \).

Proof. From (37), we can write
\[
h(z) = \frac{1}{\lambda} \sum_{i=1}^{\lambda} \left( z + \sum_{n=2}^{\infty} a_{n,i} z^n \right) = z + \sum_{n=2}^{\infty} \left( \frac{1}{\lambda} \sum_{i=1}^{\lambda} a_{n,i} \right) z^n. \tag{38}
\]
Since \( f_i \in \mathcal{C}_q(\mu, A, B) \) for every \( i = 1, 2, \ldots, \lambda \), using (38) and (18), we have
\[
\sum_{n=2}^{\infty} \psi_{n-1} \left[ [n, q] (1 - B) - 1 + A \right] \left( \frac{1}{\lambda} \sum_{i=1}^{\lambda} a_{n,i} \right)
= \frac{1}{\lambda} \sum_{i=1}^{\lambda} \left( \sum_{n=2}^{\infty} \psi_{n-1} \left[ [n, q] (1 - B) - 1 + A \right] a_{n,i} \right)
\leq \frac{1}{\lambda} \sum_{i=1}^{\lambda} (A - B) = (A - B),
\]
and this completes the proof. \( \square \)

Theorem 8. Let \( f \in \mathcal{C}_q(\mu, A, B) \). Then \( f \) is in the family \( \Delta^*(\beta) \) of starlike functions of order \( \beta \) \((0 \leq \beta < 1)\) for \(|z| < r_1\), where
\[
r_1 = \left( \frac{(1 - \beta) ([n, q] (1 - B) - 1 + A) [n, q]!}{(n - \beta)(A - B) [\mu + 1, q]_{n-1}} \right)^{1/(n-1)}. \tag{40}
\]
Proof. Let \( f \in \mathcal{C}_q(\mu, A, B) \). To prove \( f \in \Delta^*(\beta) \), we only need to show
\[
\left| \frac{z f'(z) f(z) - 1}{z f'(z) f(z) + 1 - 2\beta} \right| < 1. \tag{41}
\]
Using (1) along with some simple computation yields
\[
\sum_{n=2}^{\infty} \left( \frac{n - \beta}{1 - \beta} \right) |a_n| |z|^{n-1} < 1. \tag{42}
\]
Since \( f \in \mathcal{C}_q(\mu, A, B) \), from (18), we can easily obtain
\[
\sum_{n=2}^{\infty} [n, q]! \left( \frac{n, q] (1 - B) - 1 + A}{(A - B)} \right) |a_n| < 1. \tag{43}
\]
Now inequality (42) will be true, if the following holds:
\[
\sum_{n=2}^{\infty} \left( \frac{n - \beta}{1 - \beta} \right) |a_n| |z|^{n-1} < \sum_{n=2}^{\infty} [n, q]! \left( \frac{n, q] (1 - B) - (1 - A)}{A - B} \right) |a_n|, \tag{44}
\]
which implies that
\[
|z|^{\mu-1} < \frac{(1 - \beta) ([n, q] (1 - B) - 1 + A) [n, q]!}{(n - \beta)(A - B) [\mu + 1, q]_{n-1}}, \tag{45}
\]
and thus we get the needed result. \( \square \)

Theorem 9. Let \(-1 \leq B_2 < B_1 < A_1 \leq A_2 \leq 1 \) and \( \mathcal{A}^q_{\mu+1} f(z) \neq 0 \) in \( D \), and this satisfies
\[
\frac{[\mu + 1, q]}{q^\mu} \left( \frac{\mathcal{A}^q f(z) - [\mu, q]}{\mathcal{A}^q_{\mu+1} f(z) - [\mu, q]} \right) < \frac{1 + A_1 z}{1 + B_1 z}. \tag{46}
\]
Then \( f \in \mathcal{C}_q(\mu + 1, A_2, B_2) \).

Proof. Since \( \mathcal{A}^q_{\mu+1} f(z) \neq 0 \) in \( D \), therefore let us define the function \( p(z) \) by
\[
\frac{z \mathcal{A}^q f(z)}{\mathcal{A}^q_{\mu+1} f(z)} = p(z) \quad (z \in D). \tag{47}
\]
By the virtue of identity (12), we obtain
\[
\frac{1}{q^\mu} \left( [\mu + 1, q] \frac{\mathcal{A}^q f(z)}{\mathcal{A}^q_{\mu+1} f(z)} - [\mu, q] \right) = p(z). \tag{48}
\]
Therefore, using (46), we have
\[
\frac{z \mathcal{A}^q f(z)}{\mathcal{A}^q_{\mu+1} f(z)} = p(z) < \frac{1 + A_1 z}{1 + B_1 z}, \tag{49}
\]
and now, using Lemma 2, we have \( f \in \mathcal{C}_q(\mu + 1, A_2, B_2) \). \( \square \)

Conflicts of Interest
The authors agree with the contents of the manuscript and there are no conflicts of interest among the authors.

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