Regular Representation
of the Quantum Heisenberg Double
\{ U_q(sl(2)), Fun_q(SL(2)) \}
( q is a root of unity).

D. V. Gluschenkov *†
A. V. Lyakhovskaya ‡

Institute of Theoretical Physics, Uppsala University,
Box 803 S-75108, Uppsala, Sweden.

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Abstract

Pairing between the universal enveloping algebra \( U_q(sl(2)) \) and the algebra
of functions over \( SL_q(2) \) is obtained in explicit terms. The regular representa-
tion of the quantum double is constructed and investigated. The structure of
the root subspaces of the Casimir operator is revealed and described in terms
of \( SL_q(2) \) elements.

*Supported in part by a Soros Foundation Grant awarded by the American Physical Society.
†On leave of absence from LOMI, Fontanka 27, St. Petersburg, Russia.
e-mail: snake@diada.spb.su and snake@lomi.spb.su
‡On leave of absence from LOMI, Fontanka 27, St. Petersburg, Russia.
e-mail: anechka@diada.spb.su and anechka@lomi.spb.su
Introduction.

Introduction contains the self-consistent definition of the problem solved in this paper.

The quantum double is a pair of dual Hopf algebras \([1]\).

Hopf algebra is an associative algebra \(A\), equipped by the following mappings \([1]\):

- homomorphism \(\Delta\), called comultiplication \(\Delta : A \rightarrow A \otimes A\), such that
  \[(\text{id} \otimes \Delta)\Delta = (\Delta \otimes \text{id})\Delta\]

- homomorphism \(\varepsilon\), called co-unit \([1]\) \(\varepsilon : A \rightarrow A\), such that
  \[(\varepsilon \otimes \text{id})\Delta = (\text{id} \otimes \varepsilon)\Delta = \text{id}\]

- antihomomorphism \(S\), called antipode \(S : A \rightarrow A\) such that
  \[m(S \otimes \text{id})\Delta = m(\text{id} \otimes S)\Delta = \eta \circ \varepsilon,\]
  where \(m : A \otimes A \rightarrow A\) is the associative multiplication.

The quantum deformation of the Lie algebra \(sl(2)\) (\([H, X_{\pm}] = \pm 2X_{\pm}, [X_{+}, X_{-}] = H\)) is an example of a Hopf algebra.

Let \(q\) be a complex number, \(|q| = 1\) and denote by \(K\) the following exponential expansion\([2]\)

\[K = q^{\frac{H}{2}},\]

then the Hopf algebra attributes look like

\[KX_{\pm} = q^{\pm 1}X_{\pm}K,\]
\[X_{+}X_{-} - X_{-}X_{+} = \frac{K^2 - K^{-2}}{q - q^{-1}};\]

\[\Delta K = K \otimes K,\]
\[\Delta X_{\pm} = X_{\pm} \otimes K^{-1} + K \otimes X_{\pm};\]

\[S(K) = K^{-1}, \quad S(X_{\pm}) = -K^{-1}X_{\pm}K;\]
\[\varepsilon(1) = 1, \quad \varepsilon(K) = 1, \quad \varepsilon(X_{\pm}) = 0.\]

This Hopf algebra is the “quantum universal enveloping algebra” \(U_q(sl(2))\) \([3]\), \([4]\).

\(^1\) For completeness we might add homomorphism \(\eta\), called unity, \(\eta : C \rightarrow A\) such that \(\eta(\lambda) = \lambda \cdot 1\) where \(\lambda\) is a complex number and \(1\) is the unity of the algebra \(A\).

\(^2\) It is more usual to denote \(K = q^H\), see for example papers \([3]\), \([4]\), \([5]\), devoted to the representation theory of quantum groups at roots of unity but we want \(K\) (not \(K^{1/2}\)) to appear in the comultiplication \([3]\).
Consider the matrix elements $a, b, c, d$ of $g \in SL(2)$

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$ 

They are simplest functions on the group. The elements of the quantized algebra of functions are linear combinations of monomials $a^ib^jc^kd^l$. The Hopf algebra structure of $SL_q(2)$ can be defined as follows [4], [5]:

- **Multiplication:**
  
  $$ad - qbc = 1, \quad ab = qba, \quad ac = qca, \quad bd = qdb, \quad cd = qdc, \quad bc = cb, \quad ad - da = (q - q^{-1})bc$$

- **Comultiplication:**
  
  $$\Delta a = a \otimes a + b \otimes c, \quad \Delta c = c \otimes a + d \otimes c, \quad \Delta b = a \otimes b + b \otimes d, \quad \Delta d = c \otimes b + d \otimes d$$

- **Antipode:**
  
  $$S(a) = d, \quad S(b) = -q^{-1}b, \quad S(c) = -qc, \quad S(d) = a.$$ 

- **Co-unit**
  
  $$\varepsilon(a) = 1, \quad \varepsilon(b) = 0, \quad \varepsilon(c) = 0, \quad \varepsilon(d) = 1.$$ 

Now we have two infinite dimensional Hopf algebras $U_q(sl(2))$ and $SL_q(2)$. In order to obtain a double, one have to construct a bilinear pairing

$$U_q(sl(2)) \otimes SL_q(2) \rightarrow \mathbb{C},$$

such that

$$\langle U, f_1 \cdot f_2 \rangle = \langle \Delta U, f_1 \otimes f_2 \rangle$$

and

$$\langle U_1 \cdot U_2, f \rangle = \langle U_1 \otimes U_2, \Delta f \rangle$$

in a compact form: $Rg^1g^2 = g^2g^1R$, where $R(q) = \begin{pmatrix} q^{-1/2} & 0 & 0 & 0 \\ 0 & q^{1/2} & q^{-1/2} - q^{3/2} & 0 \\ 0 & 0 & q^{1/2} & 0 \\ 0 & 0 & 0 & q^{-1/2} \end{pmatrix}$, while comultiplication, antipode and co-unit look like $\Delta g = g^1 \otimes g^2$, $S(g) = g^{-1}$, $\varepsilon(g) = 1$. 

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3. Multiplication can be written in a compact form: $Rg^1g^2 = g^2g^1R$, where $R(q) = \begin{pmatrix} q^{-1/2} & 0 & 0 & 0 \\ 0 & q^{1/2} & q^{-1/2} - q^{3/2} & 0 \\ 0 & 0 & q^{1/2} & 0 \\ 0 & 0 & 0 & q^{-1/2} \end{pmatrix}$, while comultiplication, antipode and co-unit look like $\Delta g = g^1 \otimes g^2$, $S(g) = g^{-1}$, $\varepsilon(g) = 1$. 

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for any \( U, U_1, U_2 \in U_q(sl(2)) \) and \( f, f_1, f_2 \in SL_q(2) \), where
\[
\langle U_1 \otimes U_2, f_1 \otimes f_2 \rangle \equiv \langle U_1, f_1 \rangle \cdot \langle U_2, f_2 \rangle.
\]

So, in quantum double, the elements of \( A^* = SL_q(2) \) are linear functionals on \( A = U_q(sl(2)) \) and vice-versa. In the next section we construct the pairing between \( U_q(sl(2)) \) and \( Fun_qSL(2) \) and reveal that the majority of functionals in both spaces are linearly dependent or equal to zero. Factorizing them out we obtain the Quantum Heisenberg double which turns out to be finite-dimensional.

1 Pairing

We assume that
\[
\langle K, a \rangle = \xi, \quad \langle K, b \rangle = 0, \quad \langle K, c \rangle = 0,
\]
\[
\langle X_+, b \rangle = \zeta, \quad \langle X_+, a \rangle = 0, \quad \langle X_+, c \rangle = 0,
\]
\[
\langle X_-, c \rangle = \vartheta, \quad \langle X_-, a \rangle = 0, \quad \langle X_-, b \rangle = 0.
\]

(12)

Proposition 1 Supposing that relations (10, 11) and (12) are valid, we derive the formula
\[
\langle X^k_{+} K_{+} X^l_{-}, a^n b^m c^p \rangle = \delta^k_l \delta^m_p \frac{q^{(n-k-1)n+2l}}{[i]! [k]!} \zeta^k \vartheta^l.
\]

(13)

and conclude that in (12)

- \( \zeta \) and \( \vartheta \) can be chosen arbitrarily (we fix \( \zeta = 1, \vartheta = 1 \)).
- but \( \xi \) must be set equal to \( q^{1/2} \).

The proof is based on inductive method and involves the formulae (10, 11) in turn. Paring (13) for \( q^N = 1 \) obeys the periodicity conditions:
\[
\langle X^k_{+} K_{+}^{2N} X^l_{-}, f \rangle = \langle X^k_{+} K_{+} X^l_{-}, f \rangle
\]
\[
\langle U, a^{n+2N} b^k c^l \rangle = \langle U, a^n b^k c^l \rangle
\]

(14)

for any \( f \in A^* = SL_q(2), \quad U \in A = U_q(sl(2)), \) and for any \( k, i, t, l, m, n \)

In order to eliminate the identical functionals we factorize over the relations
\[
K^{2N} = 1, \quad a^{2N} = 1.
\]

(15)

It is easy to check that this factorization does not violate the structures (2)-(8) of the Hopf algebras \( A \) and \( A^* \). Note, that \( K^N, a^N \) belong to the centers of the corresponding algebras, while we have to factorize over \( K^{2N}, a^{2N} \).

4 \( [n]! \equiv q^{(q^{n-1} - 1)(q^{n-1} - 2) \cdots (q^{n-1})} \).

5 Note that if one uses \( q_U \) in the relations of \( U_q(sl(2)) \) and \( q_f \) in those of \( SL_q(2) \), he will inevitably come to the equations \( \langle K, a \rangle = q^{1/2}_U \) and \( \langle K, a \rangle = q^{1/2}_f \), so that \( q_U \) and \( q_f \) must be taken equal.
The cases of odd and even $N$ turn out to be different.

**For odd $N$** we have

$$\langle X_+^k K^t X_-^i, a^n b^l c^m \rangle = 0 \text{ for } k \geq N \text{ or } i \geq N, \text{ or } l \geq N, \text{ or } m \geq N,$$

and any $t,n,l,m$ \hspace{1cm} (16)

since

$[n]! = 0$ for $n \geq N$.

In order to exclude the zero functionals we factorize over the following relations

$$X_+^N = 0, \ b^N = c^N = 0. \hspace{1cm} (17)$$

This factorization is compatible with the Hopf algebra requirements (2) – (8).

The finite-dimensional dual bases can be introduced in $U_q(sl(2)$ and $SL_q(2)$:

$$\langle \frac{1}{[k][l]} q^{-\frac{k(l-k)}{2}} X_+^k K^t X_-^l, b^l f_n(a) c^m \rangle = \delta^{kl} \delta^{im} \delta^{t-k+i,n}, \hspace{1cm} (18)$$

where

$$k,i,l,m = 1, \ldots N, \ t,n = 1, \ldots, 2N, \hspace{1cm} (19)$$

and we use the notation

$$f_n(a) = \frac{1}{2N} \sum_{j=0}^{2N-1} q^{-\frac{n}{2} j} a^j.$$  

The dimensions of $A$ and $A^*$ are $\dim A = \dim A^* = 2N^3$.

At the same time **for even $N$** $[\frac{N}{2}]! = 0$ and we obtain

$$\langle X_+^k K^t X_-^i, a^n b^l c^m \rangle = 0, \ k \geq \frac{N}{2} \text{ or } i \geq \frac{N}{2}, \text{ or } l \geq \frac{N}{2}, \text{ or } m \geq \frac{N}{2}, \hspace{1cm} (20)$$

Therefore we must factorize over

$$X_+^{N \over 2} = 0, \ b^{N \over 2} = c^{N \over 2} = 0. \hspace{1cm} (21)$$

Although the elements $X_+^{N \over 2}$, $b^{N \over 2}$, $c^{N \over 2}$ do not belong to the centers, the factorization is still selfconsistent.

The formulae (18) for dual bases are still valid, but (19) must be substituted by

$$k,i,l,m = 1, \ldots N \over 2, \ t,n = 1, \ldots, 2N. \hspace{1cm} (22)$$

The dimensions of $A$ and $A^*$ are $\dim A = \dim A^* = 2N \cdot (N \over 2)^2 = N^3$.

**Proposition 2** The pairing (9) satisfies the relations (14) and (17).

The relation (14) can be proved by direct calculation, and it takes slightly more time and accuracy in the case of (14).

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6 For example, $[X_+^{N \over 2}, X_-] = 0$ for even $N$. 
2 \quad A = U_q(sl(2)) \text{ acts on itself.}

The elements of $A = U_q(sl(2))$ can act on $A$ by left and right multiplication.

To describe the structure of this representation, we need some extra objects (see [4] for details):

1. The Casimir operator

$$C = X_+X_- + \frac{q^{-1}K^2+qK^{-2}}{(q-q^{-1})^2} = X_-X_+ + \frac{qK^2+q^{-1}K^{-2}}{(q-q^{-1})^2}.$$  \hspace{1cm} (23)

2. The representation $v_j$ of $U_q(sl(2))$:

$$Ke^m_j = q^m e^m_j,$$  \hspace{1cm} (24)

$$X_{\pm}e^m_j = \sqrt{(j \mp m)q(j \pm m + 1)q}e^{m \pm 1}_j$$

with the basis $e^m_j$ ($m = -j, -j+1, \ldots, j$) [4]

For odd $N$, $0 \leq j \leq \frac{N-1}{2}$,

$\dim v_j = 2j + 1$;

For even $N$, $0 \leq j \leq \frac{N/2-2}{2}$, $\dim v_j = 2j + 1$. The representation $v_j$ is shown at Figure [4]. Each point denotes one vector of the basis; $X_+$ moves up and $X_-$ moves down.

3. The representation $V_j$ of $U_q(sl(2))$.

The representation $V_j$ described by the same formulae (24) corresponds to a negative $j$ (see Figure [4]):

For odd $N$, $-\frac{N}{2} \leq j \leq -\frac{1}{2}$, $\dim V_j = N + 2j + 1$;

For even $N$, $-\frac{N}{4} \leq j \leq -\frac{1}{2}$, $\dim V_j = \frac{N}{2} + 2j + 1$.

The two series $v_j$ and $V_j$ form the complete list of irreducible representations of $U_q(sl(2))$.

4. The representation $w_j$.

For odd $N$, $0 \leq j \leq \frac{N}{2} - 1$, $\dim w_j = 2N$;

For even $N$, $0 \leq j \leq \frac{N}{4} - 1$, $\dim w_j = N$.

The structure of this representation is shown on Figure [2].

The following should be mentioned:

a) Each letter denotes the array of vectors $e_m$ in the space of the corresponding representation: $e_m$ under $e_{m+1}$ and so on. $X_+$ moves up and $X_-$ moves down. One might add a vertical coordinate axis on the figure and call it $\ln K$.

\[ (x)_q = \frac{q^x-q^{-x}}{q-q^{-1}} \]
Figure 1: $v_j$ and $V_j$

\[
\begin{align*}
\text{v}_j & \\
\ln K & \downarrow e_j \\
\vdots & \text{\vdots} \\
\vdots & \text{\vdots} \\
\vdots & \text{\vdots} \\
\text{e}_{-j} & \\
\text{e}_{-j} &
\end{align*}
\]

$V_j$

\[
\begin{align*}
\text{lnK} & \\
\text{e}_{-j-1} & \\
\text{e}_{-j-1} & \\
\text{e}_{-N-j}$ for odd $N$ \\
\text{e}_{-\frac{N}{2}-j}$ for even $N$
\]

Figure 2: $w_j$ and $W_j$

\[
\begin{align*}
\text{w}_j & \\
V_{-j-1} & \downarrow \quad \quad \downarrow \\
v_j & \quad \quad \quad v_j \\
\downarrow & \quad \quad \quad \uparrow \\
V_{-j-1} & \downarrow
\end{align*}
\]

$W_j$

\[
\begin{align*}
\text{w}_{-j-1} & \\
\text{v}_{-j-1} & \downarrow \quad \quad \downarrow \\
V_j & \quad \quad \quad V_j \\
\downarrow & \quad \quad \quad \uparrow \\
v_{-j-1} & \downarrow
\end{align*}
\]
b) The meaning of the arrows:

\[ X_+ (\text{the highest vector of the left copy of } v_j) = \]
\[ (\text{the lowest vector of the upper copy of } V_j) \]

\[ X_- (\text{the lowest vector of the upper copy of } V_j) = \]
\[ (\text{the highest vector of the right copy of } v_j) \]

\[ X_- (\text{the lowest vector of the left copy of } v_j) = \]
\[ (\text{the highest vector of the lower copy of } V_j) \]

\[ X_+ (\text{the highest vector of the lower copy of } V_j) = \]
\[ (\text{the lowest vector of the right copy of } v_j) \]

\[ X_- (\text{the highest vector of the lower copy of } V_j) = \]
\[ (\text{the lowest vector of the left copy of } v_j) \]

c) Both \( V_j \) and the right \( v_j \) are the eigenspaces of the Casimir operator with the eigenvalue

\[ \lambda_j = \frac{q^{2j+1}+q^{-2j-1}}{(q+q^{-1})^2} = -\frac{\cos \frac{\pi}{N}(2j+1)}{2\sin^2 \frac{\pi}{N}} \] (25)

*Each vector of the left copy of \( v_j \) is the adjoint vector of the Casimir operator.*
The corresponding eigenvector is located in the right copy of \( v_j \) at the same "level" of ln\( K \) scale.

Thus, we have three types of vectors: adjoint vectors, their eigenvectors and the eigenvectors which do not have adjoint vectors.

5. The representation \( W_j \).

\[ \text{For odd } N \quad -\frac{N}{2} \leq j \leq -1, \quad \dim W_j = 2N \; ; \]
\[ \text{for even } N \quad -\frac{N}{4} \leq j \leq -1, \quad \dim W_j = N \; . \]

The structure of \( W_j \) is shown on Figure 2. Here again each vector of the left copy of \( V_j \) is the adjoint vector of the Casimir operator.

Now we concentrate on the structure of the regular representation of \( U_q(sl(2)) \).
The Casimir operator \( \text{(23)} \) decomposes the functional space into root subspaces of two different types.

1. **Exceptional subspaces.**

   For odd \( N \) there is one exceptional subspace. It corresponds to \( \lambda_{j=-1/2} \) in \( \text{(23)} \).

   For exceptional \( j, j = -1/2 \) the representation \( V_{-1/2} \) has the maximal possible dimension equal to \( N \).

   This exceptional subspace consists of \( 2N^2 \) Casimir eigenvectors, which, at the same time, are eigenvectors for both left and right action of the operator \( K \).

   These vectors may be represented as two \( N \times N \) squares on the plane with coordinates \( \ln K^L, \ln K^R \).

   Each column of circles at the diagram for \( N = 3 \) (Figure 3, left part) denotes the representation \( V_{j=-1/2} \) of the left \( U_q(sl(2)) \) action. Each row of circles is the same representation of the right \( U_q(sl(2)) \) action. The other part of the subspace is represented by hearts. Each row of hearts is the representation \( \tilde{V}_{j=-1/2} \), which is described by the formulae:

   \[
   K e_j^m = q^{m+N/2} e_j^m, \\
   X_{\pm} e_j^m = -\sqrt{(j \mp m)q(j \pm m + 1)} q e_j^{m\pm 1}.
   \]

   Thus the subspace consists of two parts which cannot be connected by \( U_q(sl(2)) \) action.

   One more comment must be add here. Due to the condition \( K^{2N} = 1 \) we have the torus \( (lnK^L, lnK^R) \) instead of the plane. Let us substitute dots by
lowercase letters and hearts by capital letters. (See the left side of Figure 4.) If we want the torus to be presented as a square with the periodicity condition, we obtain the more precise (but less transparent) diagram at the right side of Figure 4. This periodicity must be taken into account while examining more complicated diagrams (the right sides of Figures 3 and 5) but will not be mentioned below.

For even $N$, there is one more exceptional subspace. It corresponds to $\lambda_{N/2-1}$, where $N/2$ is the maximum dimension of $v_j$ for even $N$. The structure of both exceptional subspaces is the same and it is presented at the left side of Figure 5 for $N = 6$. There are only two differences between this scheme and the left side of Figure 3. First, the size of each square now equals to $N/2$ (not to $N$), though it still coincides with the maximum possible dimension of $V_j$ and $v_j$. Second, the relative location of the squares is different.

2. Typical subspaces.

- For odd $N$ there are $(N+1)/2$ different eigenvalues (23), one exceptional subspace and $(N-1)/2$ typical subspaces.
- For even $N$ there are $(N+2)/2$ different eigenvalues (25), two exceptional subspaces and $(N-2)/2$ typical subspaces.
The sum of dimensions of the partner representations $v_j$ and $V_{j-1}$ in $w_j$ is always equal to the same number $M$ ($M = N$ if $N$ is odd, $M = N/2$ for even $N$). The same is valid for $V_j$ and $v_{j-1}$ in $W_j$. We also have

$$\lambda_j = \lambda_{-j-1}$$

Therefore, the eigenvalues of the Casimir operator coincide for $w_j$ and $W_{j-1}$ ($j \geq 0$), and these representations are embedded into the same root subspace.

The structure of the typical root subspaces is illustrated by the right sides of Figures 3 and 5 for odd and even $N$ respectively. Each column corresponds to the representation $w_j$ or $W_{j-1}$ of the left action of $U_q(sl(2))$. Each row denotes the representation $w_j$ or $W_{j-1}$ of the right action of $U_q(sl(2))$. Each white object represents one eigenvector, while each black object represents a pair: an adjoint vector and its eigenvector. (Remember the description of $w_j$ and $W_j$).

There is a principal difference between odd and even cases:

8 Only $v_{M-1}$, $\dim v_{M-1} = M$ does not have a partner, since the dimension of this unexisting partner would have been equal to zero. Therefore $w_{M-1}$ does not exist. For the same reason $V_{-1}$, $\dim V_{-1} = M$ does not have a partner, and there is no $W_{-1}$. These exceptions correspond to exceptional subspaces.

9 while for exceptional values of $j$ we obtain simply: $-j - 1 = j \pmod{N}$. 

Figure 5: Basis for even $N$, $q^6 = 1$
In the case of odd $N$ the subspace consists of two parts which cannot be connected by $U_q(sl(2))$ action: any vector in one part being chosen as the initial will never be transferred to the second part with the help of $U_q(sl(2))$ action. Starting from spades, you’ll never get to circles, neither white nor black ones.

In the case of even $N$ there is no such decomposition. If you start from an adjoint vector in the spade area, you easily get to some eigenvectors (not adjoint ones!), located in black circles area - via eigenvectors, represented by white circles.

The following rules describe the multiplication table for the vectors of the space of $U_q(sl(2))$ representation. The zero products are not mentioned.

1. The eigenvalue of $K^R$ of the left multiple must coincide with the eigenvalue of $K^L$ of the right multiple in order to get a non-zero product. In terms of picture objects this means that the column containing the left multiple must intersect with the row containing the right multiple at the point where the eigenvalues of $K^R$ and $K^L$ coincide (i.e., on the common diagonal of the squares).

2. The eigenvalue of $K^R$ of the product equals to that of the right multiple, and the eigenvalue of $K^L$ of the product equals to that of the left multiple. This means the product will be placed at the junction of the left multiple row and the right multiple column.

\[
\begin{align*}
(\text{adjoint vector}) \cdot (\text{adjoint vector}) &= (\text{adjoint vector}) \\
(\text{eigenvector}) \cdot (\text{adjoint vector}) &= (\text{eigenvector}) \\
(\text{eigenvector}) \cdot (\text{eigenvector}) &= (\text{eigenvector}) \\
(\text{eigenvector which has an adjoint one}) \cdot (\text{eigenvector}) &= 0
\end{align*}
\]

3. **Regular representation of $U_q(sl(2))$.**

We investigate the regular representation of $U_q(sl(2))$ on the space of functions over $SL_q(2)$.

The elements of $A = U_q(sl(2))$ can act

1. on $A$ by left and right multiplication,

2. on $A^*$, according to the following definition:

\[
\begin{align*}
U^L(f) &= \langle U, \Delta f \rangle_1 \quad \text{left action} \\
U^R(f) &= \langle U, \Delta f \rangle_2 \quad \text{right action}
\end{align*}
\] (27)

But if one identifies each element of $f_j \in A^*$ with the corresponding element $V_i \in A$: $\langle V_i, f_j \rangle = \delta_{ik}$ then he finds that these two representations are conjugate:

\[
\begin{align*}
U^L(V_a) &= V_b \quad \Rightarrow \quad U^L(f_b) = f_a \\
U^R(V_a) &= V_b \quad \Rightarrow \quad U^R(f_b) = f_a
\end{align*}
\] (28)
Therefore the space of the representation of $U_q(sl(2))$ can be regarded, first, as $A$, and second, as $A^*$. The action of $A = U_q(sl(2))$ on $A$ is already described in the previous section. The regular representation of $A = U_q(sl(2))$ illustrated on Figures 1, 2 acts on the elements of $SL_q(2)$ and has evidently the same structure. Now we investigate this representation in terms of elements of $SL_q(2)$.

Applying the definition (27) and taking into account pairing (13) one easily obtains:

$$X^L a = 0, \quad K^L a = q^{1/2} a$$  \hspace{1cm} (29)

Acting by the Casimir operator (23) on monomials $a^k, k \leq 2N$, we see that these monomials are eigenvectors with eigenvalues $\lambda_j = k^2 - 1$. These eigenvalues coincide if $k = k' \pmod{N}$, or $k + k' = 2 \pmod{N}$. \hspace{1cm} (30)

For example, for $N = 3$, $a^2$ belongs to the exceptional subspace $j = -1/2$, and $1 = a^0$ and $a^1$ are contained in the typical subspace $j = 0, -1$, unique for the given $N$.

Figure 6 represents the circles from the Figure 3 (adjoint vectors are enclosed in square brackets.). Consider the table $3 \times 3$. Since the representation under discussion is conjugate to one considered in the previous section, $X^L$ now goes up and kills the upper row of the table. This follows from the relations $X^L a = 0, X^L b = 0$. The $3 \times 3$ table is a good illustration of the general situation: for any $N$ and any root subspace the upper row of such a square of eigenvectors consists of $a^m b^l$, the left column consists of $a^m c^l$, and so on. Correspondence between the power of $a$ in the upper-left corner and the eigenvalue was mentioned above. Starting from this $a^k$ and moving it down by $X^L_+$ and right by $X^R_-$ one can derive the whole square of eigenvectors.

Now consider the adjoint vector $q^2 abc$ corresponding to the eigenvector $a$. Due to the relations

$$K^L(f \cdot (bc)) = K^L(f), \quad K^R(f \cdot (bc)) = K^R(f),$$

the eigenvalues of $K^L, K^R$ remain unchangeable, but $c$ enables $X^L$ to make \textit{one and only one} step up ( $a$ and $b$ would be killed ) and to get to the eigenvector $q^2 a^2 b$ (next step up leads to zero). At the same time the second power of $c$ in the adjoint vector $b^2 c^2 + bc$ enables $X^L_-$ to make \textit{two} steps up. This is the general rule: an adjoint vector is simply equal to its eigenvector, multiplied by a polynomial in $bc$ ($\alpha_0 + \alpha_1 bc + \cdots + \alpha_s (bc)^s$), where the highest power $s$ is constant for the whole square with the eigenvector $a^k$ in the upper-left corner, and the value of $s$ allows to perform the required number of steps: $s = N - k - 1 \text{ for odd } N; \quad s = N/2 - k - 1 \text{ for even } N$.

Let’s turn to the description of eigenvectors which do not have adjoint ones. There are two upper-right corners of “wings” on figure 6. The two corner vectors must be killed by the action of corresponding operators (moving them up or to the right), and consequently each of them must look like

$$a^\alpha b^{N-1} = a^{2N-\alpha} b^{N-1}$$  \hspace{1cm} (31)
For example, when going right $ab^{N-1}$ is transfered to $b^N = 0$. At the same time after some steps left we must obtain the eigenvector $a^k$. Or after some steps down we must obtain $d^k$. This determines $\alpha$ from (31, 32): 

$$\alpha + (N - 1) = k - 1 \pmod{2N}$$

for odd $N$, and 

$$\alpha + (N/2 - 1) = k - 1 \pmod{2N}$$

for even $N$. All the other vectors of upper-right wings can be obtained simply using the formulae for $X^{L,R}_\pm$ action. These vectors can also be calculated acting on the adjoint vectors in the square, but in this case the calculations are more complicated.

The vectors already described in terms of $a, b, c, d$ (Figure 3) are shown by black and white circles on Figure 3. The second half of the vectors can be obtained multiplying the expressoins from (Figure 6) by $a^N$ and changing the signes of chess-like located elements. This follows from the relation 

$$X_\pm v = u \Rightarrow X_\pm a^N v = q^{\frac{N}{2}} a^N u = -a^N u.$$
The eigenvalues of the operator $K$ are multiplied by $q^N$:

$$Ku = \lambda u \Rightarrow Ka^N u = q^{N/2} \lambda u.$$

Consequently, the second copies must be shifted by $N/2$ in the scale of $lnK^L$ and $lnK^R$.

Now we have the complete description of the regular representation of $U_q(sl(2))$. Let us turn to the joint representation of $U_q(sl(2))$ and $Fun_q(SL(2))$.

4 Joint representation of $U_q(sl(2))$ and $Fun_q(SL(2))$.

The elements of $A^* = SL_q(2)$ can act

1. on $A^*$ by left and right multiplication,

2. on $A$, according to the following definition:

$$\begin{align*}
f^L(U) &= \langle \Delta U, f \rangle_1 \quad \text{left action} \\
f^R(U) &= \langle \Delta U, f \rangle_2 \quad \text{right action}
\end{align*}$$

The last proposition should complete the investigation of the regular representation of the double.

**Proposition 3** The joint regular representation of the double $(U_q(sl(2)), SL_q(2))$ on the space of functions over $SL_q(2)$ is irreducible.

This statement can be proved in three steps: 1. Any polynomial of $a, b, c, d$ can be transfered to the combination $\sum \gamma_k a^k$ by the action of suitable $X^{L,R}_\pm$.

2. The action of $(X^{L}_\pm)^l$ on this combination leads to $c^l$, $l$ is the largest power in the sum $\sum \gamma_k a^k$. Then the action of $X^{-}_L$ draws it back to $a^l$.

3. The monomial $a^l$ can be transmitted to the sector of given Casimir eigenvalue multiplying it by $a^i$, then, multiplying $a^i$ by $(\alpha_0 + \alpha_1 bc + \cdots + \alpha_s(bc)^s) \in SL_q(2)$, one obtains the adjoint vector. Finally, the action of $X^{L,R}_\pm$ can transmit it to any vector in the wings.

Thus, any function of $a, b, c, d$ can be obtained from any fixed function by the action of the regular representation. This proves the Proposition.
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