A lower bound for the mass of axisymmetric connected black hole data sets

Piotr T Chruściel\textsuperscript{1} and Luc Nguyen\textsuperscript{2}

\textsuperscript{1} Gravitational Physics, University of Vienna, Vienna, Austria
\textsuperscript{2} Department of Mathematics, Princeton University, Princeton, NJ, USA

E-mail: piotr.chrusciel@univie.ac.at

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Abstract

We present a generalization of the Brill-type proof of positivity of mass for axisymmetric initial data to initial data sets with black hole boundaries. The argument leads to a strictly positive lower bound for the mass of simply connected and connected axisymmetric black hole data sets in terms of the mass of a reference Schwarzschild metric.

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1. Introduction

In [6], the first author extended the validity of the axisymmetric positive mass theorems of Brill [4], Moncrief (unpublished), Dain (unpublished) and Gibbons and Holzegel [9] to all asymptotically flat initial data on $\mathbb{R}^3$ invariant under a U(1)-action with a positive Ricci scalar. The object of this work is to show how to adapt the analysis to the case where black hole boundaries are present in the initial data. This leads to a strictly positive lower bound for the mass of initial data sets containing a connected ‘non-degenerate horizon’.

Let $(M, g)$ be a simply connected three-dimensional Riemannian manifold with boundary $\partial M$ which admits a Killing vector field with periodic orbits and is the union of a compact set and of one asymptotically flat end.

By [6], $g$ admits a global coordinate system in which the metric takes the form

$$ds^2 = e^{-2U(\rho,z)}(d\rho^2 + dz^2) + \rho^2 e^{-2U}(d\varphi + \rho B d\rho + \Lambda dz)^2,$$

where $\partial_\rho$ is the Killing vector field, $\varphi \in [0, 2\pi)$, the coordinates $(\rho, z)$ cover $(0, \infty) \times \mathbb{R} \setminus \hat{K}$ for some compact set $K$ whose intersection with the axis $\{\rho = 0\}$ is connected and non-empty. Here $\hat{K}$ denotes the interior of $K$. The choice of $K$ is never unique. However, we show in this paper that it cannot be arbitrary either. In fact, if one lets $K'$ be the compact set obtained by adjoining to $K$ its reflection about the $\rho$ axis in the $(\rho, z)$-plane, then the logarithmic capacity of $K'$ (with respect to the $(\rho, z)$-plane) depends uniquely on the geometry of $(M, g)$.

The above property of $K$ leads to two canonical choices of $K$: one can use either a line segment of length $2m_1$ on the $\rho$-axis or a half-disc of radius $m_1$ centred on the axis, where
$m_1$ is twice the logarithmic capacity of $K'$. In the stationary vacuum case, those coordinate systems are respectively known as Weyl coordinates and isotropic (or spherical) coordinates. In the static case, i.e. $(M, g)$ is a Schwarzschild slice, $m_1$ coincides with the Schwarzschild mass. For the maximal slice of the Kerr metric, $m_1 = \sqrt{m^2 - a^2}$.

The main result of this paper is as follows.

**Theorem 1.1.** Let $(M, g)$ be a smooth simply connected, connected three-dimensional manifold which has a smooth connected compact boundary $\partial M$, is asymptotically flat with one end and satisfies (2.1) for some $k \geq 5$ and (2.2). Furthermore, assume that $(M, g)$ admits a Killing vector field with periodic orbits. If $M$ has a non-negative scalar curvature and the mean curvature of $\partial M$ with respect to the normal pointing towards $M$ is non-positive, then the ADM mass of $(M, g)$ satisfies

$$m > \frac{\pi}{4} m_1,$$

(1.2)

where $m_1$ is the positive constant obtained in theorems 2.2 and 2.4.

Even though the constant $m_1$ is uniquely determined, it should be admitted that it is not easy to determine $m_1$ if the metric is not given directly in the coordinate system (2.15) or (2.27). In the general case, one needs to solve a PDE on $M$, and then $m_1$ can be read off from the asymptotic behaviour of the solution at infinity, see proposition 2.6.

We note that the simple-connectedness of $M$ would be a consequence of the topological censorship theorem of [8] if $M$ were a Cauchy hypersurface for $J^+(M) \cap J^-(\partial M)$. We are grateful to G Galloway for pointing this out.

A satisfactory generalization of our result to degenerate horizons would require a thorough understanding of the behaviour of the metric near such horizons, a problem which is widely unexplored. We simply note that positivity of $m$ is easily established by similar methods if one assumes, e.g., that $(M, g)$ has no boundary but contains instead suitably defined asymptotically cylindrical ends. In this case, whenever a twist potential exists, one further has the stronger Dain-type inequality controlling the mass from below in terms of a positive quantity, which equals the square root of the length of the angular momentum for connected configurations.

We conjecture that the sharp inequality is

$$m \geq m_1,$$

(1.3)

with equality if and only if $M$ is a time-symmetric Cauchy hypersurface for the d.o.c. of the Schwarzschild–Kruskal–Szekeres spacetime.

Ideally, one would like to obtain a simple proof of the Penrose inequality in the current setting, but we have not been able to achieve this. We make some comments about that in the appendix.

We note the recent paper [1], where a lower bound for minimal-surface area in terms of angular momentum is established under a set of restrictive conditions; see also [11]. Our construction of global coordinates is relevant for the analysis in [1].

2. Axisymmetric black hole data sets

Let $(M, g)$ be a three-dimensional smooth simply connected, connected manifold with a smooth connected compact boundary $\partial M$. On $(M, g)$, we assume that there is a Killing vector field $\eta$ with periodic orbits, among which the principal ones are assumed, without loss of generality, to have period $2\pi$.

$(M, g)$ will be assumed to have one asymptotically flat end in the usual sense that there exists a region $M_{\text{ext}} \subset M$ diffeomorphic to $\mathbb{R}^3 \setminus B_R$, where $B_R$ is a coordinate ball of radius $R$,
such that $M \setminus M_{\text{ext}}$ is compact and in local coordinates on $M_{\text{ext}}$ obtained from $\mathbb{R}^3 \setminus B_R$ the metric satisfies the fall-off conditions, for some $k \geq 1$,

\begin{align}
&g_{ij} - \delta_{ij} = o_k(r^{-1/2}), \quad (2.1) \\
&\partial_k g_{ij} \in L^2(M_{\text{ext}}), \quad (2.2) \\
&R^i_{\ jkl} = o(r^{-5/2}), \quad (2.3)
\end{align}

where we write $f = o_k(r^\mu)$ if $f$ satisfies

\[\partial_{k_1 \ldots k_l} f = o(r^{\mu - l}) \quad \text{for } 0 \leq l \leq k. \quad (2.4)\]

The boundary behaviour near the event horizon of all functions of interest has been established in [7] in a closely related context. For completeness, and to make it clear that it applies to our setting, we outline the analysis of [7] in what follows.

### 2.1. Reduction to the boundaryless case

Since $M$ is simply connected and $\partial M$ is connected, reference [10, lemma 4.9] implies that $\partial M$ has the topology of a 2-sphere,

\[\partial M \approx \mathbb{S}^2, \quad (2.5)\]

and can be filled by a ball, say $B_2$. Moreover, the metric $g$ and the $U(1)$-action induced by $\eta$ on $(M, g)$ extend to the extended manifold $M_{\text{ext}} := M \cup B_2$.

Let

\[\mathcal{C}_f := \{ p \in M : g(\eta, \eta)|_p = 0 \}, \quad \mathcal{C}_{f_2} := \{ p \in M_2 : g(\eta, \eta)|_p = 0 \}.\]

Denote by $Q_1 := M_2/U(1)$, $Q := M / U(1)$ and $\partial M / U(1)$ the collections of the orbits of the group of isometries generated by $\eta$ on $M_2$, $M$, and $\partial M$, respectively. It is known that $Q_1$ is a manifold with boundary $\mathcal{C}_{f_2}/U(1) \approx \mathcal{C}_f$, while $Q$ is a manifold with boundary $(\mathcal{C}_f/U(1)) \cup (\partial M / U(1)) \approx \mathcal{C}_f \cup (\partial M / U(1))$.

As $\partial M$ is a sphere and invariant under $U(1)$, it contains exactly two fixed points, say $p_n$ and $p_s$ of $U(1)$. In other words, $\partial M \cap \mathcal{C}_f = \{ p_n, p_s \}$.

It is readily seen that $\partial M / U(1)$ is a smooth curve in $Q_1$ with endpoints $p_n$ and $p_s$. Additionally, at both $p_n$ and $p_s$, $\partial M / U(1)$ intersects $\mathcal{C}_{f_2}$ at a right angle.

In what follows, we will assume that (2.1) holds for some $k \geq 5$. Note that this implies (2.3). By [6, theorem 2.7], the metric $g$ on $M_2$ admits the following representation:

\[g = e^{-2U_2(z_2)}(d\rho^2 + dz_2^2) + \rho^2 e^{-2U_1(\varphi)}(d\varphi + \rho \hat{B} \, d\rho + \tilde{A} \, dz_2)^2, \quad (2.6)\]

where $\partial_\varphi$ is the rotational Killing vector field, and $M_2$ can be identified with $\mathbb{R}^3$ on which $(\rho_2, z_2, \varphi)$ are its cylindrical coordinates. Furthermore, $U_2$, $\alpha_2$, $\hat{B}$ and $\tilde{A}$ are smooth functions on $M_2$ which are $\varphi$-independent and satisfy $\alpha_2 \equiv 0$ whenever $\rho_2 = 0$ and

\[U_2 = \alpha_2 = \alpha_4 = o_k(r^{-1/2}), \quad \hat{B} = o_k(z_2^{-5/2}), \quad \tilde{A} = o_k(z_2^{-3/2}) \quad \text{for } r_2 = \sqrt{\rho_2^2 + z_2^2} \to \infty. \quad (2.7)\]

It is useful to consider the manifolds $Q_1^{(2)}$ and $Q^{(2)}$ obtained by doubling $Q_1$ and $Q$ along $\mathcal{C}_{f_2}/U(1)$ and $\mathcal{C}_f/U(1)$, respectively. Naturally, $Q^{(2)}$ injects into $Q_1^{(2)}$. In addition, $\rho_2$ and $z_2$ can be extended naturally to $Q_1^{(2)}$ so as to make $\rho_2$ an odd function about $\mathcal{C}_{f_2}$ while $z_2$ an...
even function. Then \( Q_2 \) can be identified with the complex plane \( \mathbb{C}_z := \{ \zeta_z := \rho_z + iz_z \} \) in which
\[
Q_2 \approx \{ \zeta_z : \text{Re } \zeta_z \geq 0 \}, \quad \approx \{ \zeta_z : \text{Re } \zeta_z = 0 \},
\]
where \( \approx \) denotes ‘diffeomorphic to’. This implies in particular that \( Q_2 \) has a natural complex structure. Furthermore, in this picture, \( Q_2 \) is an unbounded (open) subset, denoted by \( \Omega_2 \), of \( \mathbb{C}_z \) whose boundary is a smooth connected closed curved
\[
Q \approx \Omega_2 \cap \{ \zeta_z : \text{Re } \zeta_z \geq 0 \} \quad \text{and} \quad \approx \Omega_2 \cap \{ \zeta_z : \text{Re } \zeta_z = 0 \}.
\]

2.2. Pseudo-spherical coordinates

We proceed to modify \((\rho_z, z_z, \varphi)\) to a coordinate system \((\rho_S, z_S, \varphi)\) on \( M \) such that \( \partial M \) corresponds to a sphere \( \{ \rho^2_S + z^2_S = \text{const} \} \). An approach to achieve this is to follow the procedure in [7] to first construct Weyl coordinate functions and then transform them to the desired form. We present here a simpler approach, directly tied to the theory of conformal mappings. As will be seen, this also provides an alternative to the construction of Weyl coordinates in [7].

Without loss of generality, we assume that \( \Omega_z \) does not contain the origin of \( \mathbb{C}_z \). Let \( \Theta \) denote the inversion map of \( \mathbb{C}_z \) about the unit circle \( \partial D_z(0, 1) \) and define \( G_z = \Theta(\Omega_z) \cup \{0\} \). Note that as \( \partial \Omega_z \) is a smooth simple closed curved, so is \( \partial G_z \). This implies that \( G_z \) is simply connected. Let \( h_1 \) be the solution to the problem
\[
\begin{align*}
\Delta^2 h_1 &= 0 & \text{in } G_z, \\
h_1 &= -\frac{i}{2} \log (\rho^2_z + z^2_z) & \text{on } \partial G_z,
\end{align*}
\]
where \( \Delta^2 \) is the Laplace operator of \( d\rho^2_z + dz^2_z \), and \( h_2 \) be a harmonic conjugate of \( h_1 \), i.e. \( h_2 \) satisfies
\[
\partial_{\rho_z} h_1 = \partial_z h_2, \quad \partial_z h_2 = -\partial_{\rho_z} h_1.
\]

Define
\[
\Psi := \psi_1 + i\psi_2 := \zeta_z \exp(h_1 + ih_2);
\]
recall that \( \zeta_z = \rho_z + iz_z \). Evidently, \( \Psi \) is holomorphic, fixes the origin, and maps \( \partial G_z \) to the unit circle \( \partial D_z(0, 1) \). Furthermore, by the definitions of \( h_1 \), \( h_2 \) and standard elliptic theory, \( \Psi \in C^\infty(G_z) \).

We claim that \( \Psi \) is ‘the’ Riemann map which maps \( G_z \) one-to-one and onto the unit disc \( D_z(0, 1) \). Indeed, let \( \tilde{\Psi} \) be a Riemann map of \( G_z \) which fixes the origin. Then \( \tilde{\Psi} = \zeta_z \tilde{H} \) for some holomorphic function \( \tilde{H} \). Additionally, as \( \tilde{\Psi} \) is one-to-one, \( \tilde{H} \) is nowhere vanishing. Since \( G_z \) is simply connected, this implies \( \tilde{H} = \exp \tilde{h} \) for some holomorphic function \( \tilde{h} \). As \( \tilde{\Psi}(\partial G_z) \subset \partial D_z(0, 1) \), it follows that
\[
\text{Re } \tilde{h} = -\log |\zeta_z| = -\frac{1}{2} \log (\rho^2_z + z^2_z) \quad \text{on } \partial G_z.
\]

By uniqueness of solutions of the Laplace equation, we thus have \( \text{Re } \tilde{h} \equiv h_1 \), which implies \( \text{Im } \tilde{h} \equiv h_1 + C \) for some constant \( C \). The claim follows.

As a Riemann map, \( \Psi \) has an inverse \( \Psi^{-1} : D_z(0, 1) \to G_z \). Since \( G_z \) is a Jordan domain, \( \Psi^{-1} \) extends to a homeomorphism of the closed domains, thanks to the Carathéodory theorem (see e.g. [15, theorem 14.19]). We claim that this extension is of \( C^\infty(D_z(0, 1)) \)-differentiability class, and in fact is a diffeomorphism up-to-boundary. By the inverse function theorem, it suffices to show that \( \Psi' \) is nowhere vanishing in \( G_z \). Furthermore, since \( \Psi \) is holomorphic and one-to-one in \( G_z \), it suffices to show that \( \Psi' \) does not vanish on \( \partial G_z \). Consider a point \( p \in \partial G_z \).
and let \( q = \Psi(p) \in \partial D_2(0, 1) \). Without loss of generality, we can assume that \( q = -i \). Pick a \( \delta > 0 \) sufficiently small such that
\[
\Psi(G_z \cap D_2(p, \delta)) \subset \tilde{D}_2(0, 1) \cap D_2 \left( q, \frac{1}{10} \right).
\] (2.8)
Since \( \psi_1^2 + \psi_2^2 = 1 \) on \( \partial G_z \) we find that, near \( p \), the function
\[
F := \psi_2 + \sqrt{1 - \psi_1^2}
\]
satisfies
\[
F > 0 \quad \text{in} \quad G_z \cap D_2(p, \delta),
\]
\[
F = 0 \quad \text{on} \quad \partial G_z \cap D_2(p, \delta).
\]
Since \( \psi_1 \) and \( \psi_2 \) are harmonic, we have
\[
\Delta^2 F = -\frac{1}{(1 - \psi_1^2)^{3/2}} |\nabla \psi_1|^2 \leq 0 \quad \text{in} \quad G_z \cap D_2(p, \delta).
\]
It hence follows from the Hopf lemma that
\[
\partial_n \psi_2(p) = \partial_n F(p) > 0,
\]
where \( \partial_n \) is the derivative in the direction of the inward pointing normal, which gives \( |\Psi'(p)| \geq |\partial_n \psi_2(p)| > 0 \). Since \( p \) is arbitrary, we thus conclude that \( \Psi' \) is always non-zero on \( \partial G_z \) and so on \( \bar{G}_z \), whence the claim.

Note that we have recovered the Kellogg–Warschawski theorem (see [14, theorem 3.6] or the original papers [13, 16, 17] of Kellogg and of Warschawski).

**Proposition 2.1.** Let \( G \subset \mathbb{C} \) be a simply connected bounded domain whose boundary \( \partial G \) is \( C^{k,\alpha} \)-regular for some \( k \geq 2, 0 < \alpha < 1 \), and \( \Psi : G \to D(0, 1) \) its Riemann map. Then \( \Psi \) extends to a map in \( C^{k,\alpha}(\bar{G}) \) and \( \Psi^{-1} \) extends to a map in \( C^{k,\alpha}(\bar{D}(0, 1)) \).

Define
\[
\rho_S + i z_S = \zeta_S := \Theta^{-1} \circ \left( \frac{1}{|\Psi'(0)|} \Psi \right) \circ \Theta.
\] (2.9)
Then \((\rho_S, z_S)\) maps \( \Omega_2 \) one-to-one and onto \( \mathbb{C}_S \setminus \bar{D}(0, \frac{m_1}{2}) \), where we use the symbol \( \mathbb{C}_S \) to denote the complex plane coordinatized by \((\rho_S, z_S)\), and where
\[ m_1 = 2 |\Psi'(0)| \]
is twice the logarithmic capacity of \( \partial \Omega_2 \). The constant \( m_1 \) is related to the Robin constant \( \gamma(\partial \Omega_2) \) of the boundary of \( \partial \Omega_2 \) by
\[ m_1 = 2 \exp(-\gamma(\partial \Omega_2)). \] (2.10)
Note also that by construction, as \( \rho_S^2 + z_S^2 \to \infty \), there holds
\[ (\rho_S, z_S) - (\rho_S, z_S) = O_l \left( (\rho_S^2 + z_S^2)^{-1/2} \right), \quad l \geq 0, \] (2.11)
where \( O_l \) is defined in a way analogous to (2.4).

We show that \( m_1 \) is uniquely determined by \((M, g)\), i.e. independent of how we form \( M^\sharp \).
To see this, let \( \tilde{M}_2 = M \cup \tilde{B}_2 \) be a different way of extending \( M \). In \( M^\sharp \) and \( \tilde{M}_2 \), the regions representing \( M \) are isometric. Hence, if \((\tilde{\rho}_2, \tilde{z}_2, \tilde{\varphi})\) is the counterpart in \( \tilde{M}_2 \) of \((\rho_2, z_2, \varphi)\), then, by (2.6), the map \( T : (\rho_2, z_2) \mapsto (\tilde{\rho}_2, \tilde{z}_2) \) gives a conformal transformation of \( Q^{(2)} \). Furthermore, by (2.7), we also have
\[
\left| \frac{\partial (\tilde{\rho}_2, \tilde{z}_2)}{\partial (\rho_2, z_2)} \right| = 1 + O((\rho_2^2 + z_2^2)^{-1/2}) \quad \text{as} \quad \rho_2^2 + z_2^2 \to \infty.
\] (2.12)
Hence, if $Q^{(2)}$ is represented by $\Omega_1$ in the complex plane $\hat{\mathbb{C}}_1$ parameterized by $(\hat{\theta}, \hat{z}_1)$, then $T$ defines naturally a bijection of $\Omega_1$ and $\Omega_2$, with $T(\infty) = \infty$ and (by (2.12)) $|T'(\infty)| = 1$. It then follows that $\partial \Omega_1$ and $\partial \Omega_2$ have the same logarithmic capacity, and so $m_1$ is independent of the way that the metric has been extended to $B_\infty$.

Next, by the uniqueness property of the Laplace equation with Dirichlet boundary data, $h_1$ is even in the $\rho_2$-variable, which implies that, after shifting by a constant, $h_2$ is odd in the $\rho_2$-variable. Using this, one can check that $\rho_S$ is odd while $z_S$ is even in the $\rho_2$-variable. In particular, $\rho_S$ vanishes on $\{\rho_2 = 0\} \cap \Omega_1 \approx \mathbb{C} / U(1)$. This implies that $\rho^2_S$ is a smooth function which vanishes on $\mathbb{C} / U(1)$ and is even in the $\rho_2$-variable. Thus, there is a smooth function $\chi$ of $(\rho_2^2, z_2)$ such that

$$\rho_S^2 = \rho^2_S \chi(\rho_2^2, z_2).$$

Furthermore, as

$$(\partial_{\rho_S} \rho_S)^2 + (\partial_{z_S} \rho_S)^2 = \frac{\partial(\rho_S, z_S) \left| \partial(\rho_S, z_S) \right|}{\rho_S}$$

is nowhere vanishing in $\Omega_1$ by proposition 2.1, we also have

$$\chi(\rho_2^2, z_2) > 0 \text{ along } \{\rho_2 = 0\} \cap \overline{\Omega_1} \text{ and so in } \overline{\Omega_1}.$$  \hspace{1cm} (2.14)

We thus have

**Theorem 2.2.** Let $(M, g)$ be a three-dimensional smooth simply connected, connected manifold with a smooth connected compact boundary $\partial M$ and assume that $(M, g)$ admits a Killing vector field with periodic orbits. Furthermore, assume that $(M, g)$ has one asymptotically flat end where it satisfies (2.1) for some $k \geq 5$. Then, there exists a unique $m_1 > 0$ such that $M$ is diffeomorphic to $\mathbb{R}^3 \setminus B(0, \frac{m_1}{\sqrt{2}})$, and, in cylindrical-type coordinates $(\rho_S, z_S, \varphi)$ on $\mathbb{R}^3$, $g$ takes the form

$$g = e^{-2U_S + 2S}((d\rho_S^2 + dz_S^2) + \rho_S^2 e^{-2U_S}(d\varphi + \rho_S B \rho_S d\varphi + \bar{A}_S dz_S)^2),$$

where $\partial_S$ is the rotational Killing vector field, $U_S, \alpha_S, B_S$ and $\bar{A}_S$ are smooth functions on $M$ which are $\varphi$-independent and satisfy $\alpha_S = 0$ whenever $\rho_S = 0$ and

$$U_S = o_{k-3}(r_S^{-1/2}), \quad \alpha_S = o_{k-4}(r_S^{-1/2}), \quad B_S = o_{k-3}(r_S^{-5/2}), \quad \bar{A}_S = o_{k-3}(r_S^{-3/2}) \text{ for } r_S = \sqrt{\rho_S^2 + z_S^2} \to \infty.$$  \hspace{1cm} (2.16)

### 2.3. Weyl coordinates

We next construct the Weyl coordinates $(\rho, z, \varphi)$ so that $\rho$ vanishes on both the rotation axis $\mathbb{C} / U(1)$ and the boundary $\partial M$. This can be done using a (rotated) Joukovsky transformation

$$\rho + i z = \zeta := \zeta_S - \frac{m_1^2}{4\zeta_S}. \hspace{1cm} (2.17)$$

Componentwise, we have

$$\rho = \rho_S \frac{\rho_S^2 + z_S^2 - \frac{m_1}{2}}{\rho_S^2 + z_S^2}, \quad z = \zeta_S \frac{\rho_S^2 + z_S^2 + \frac{m_1}{2}}{\rho_S^2 + z_S^2}. \hspace{1cm} (2.18)$$

We now check that the map $\zeta_S \mapsto \zeta$ maps $\mathbb{C}_S \setminus \mathbb{D}(0, \frac{m_1}{2})$ one-to-one and onto $\mathbb{C} \setminus I$ where $I = \{ i z : -m_1 \leq z \leq m_1 \}$. In view of (2.18), to invert the map it suffices to solve for $|\zeta_S| > \frac{m_1}{2}$. First, note that by (2.17)

$$\zeta \pm i m_1 = \frac{1}{\zeta_S} \left( \zeta_S \pm i \frac{m_1}{2} \right)^2. \hspace{1cm} (2.19)$$
It follows that
\[(\xi + im_1)\xi = \frac{1}{|\xi|^2} \left( (|\xi|^2 - \frac{m^2}{4})^2 - m^2 \rho^2 \right).\]

Taking the real part and recalling (2.18) we get
\[|\xi|^2 - m^2 = \frac{1}{|\xi|^2} \left( (|\xi|^2 - \frac{m^2}{4})^2 - m^2 \rho^2 \right) = \frac{m^2 |\xi|^2}{(|\xi|^2 - \frac{m^2}{4})^2} \rho^2.
\]

This implies that
\[ \frac{|\xi|^2 - m^2}{|\xi|^2} = \frac{1}{2} \left[ |\xi|^2 - m^2 + \sqrt{(|\xi|^2 - m^2)^2 + 4 m^2 \rho^2} \right].\]

As \(|\xi| > m^2\), we thus get
\[\frac{|\xi|^2 - m^2}{|\xi|} = \frac{1}{\sqrt{2}} \left[ |\xi|^2 - m^2 + \sqrt{(|\xi|^2 - m^2)^2 + 4 m^2 \rho^2} \right]^{1/2},\]

which implies
\[|\xi| = \frac{1}{\sqrt{2}} \left[ \frac{(|\xi|^2 - m^2 + \sqrt{(|\xi|^2 - m^2)^2 + 4 m^2 \rho^2})^{1/2}}{2} \right].\]

From what has been said we see that the map \(\xi S \mapsto \xi \) maps \(C S \setminus D(0, \frac{m^2}{2})\) one-to-one and onto \(C \setminus I\). In fact, in view of (2.18), (2.21) and (2.22), its inverse is given by
\[\rho S = (\mu^{1/2} + 1) \frac{\rho}{2}, \quad z S = (\mu^{-1/2} + 1) \frac{z}{2},\]

where
\[\mu = \frac{|\xi|^2 + m^2 + \sqrt{(|\xi|^2 - m^2)^2 + 4 m^2 \rho^2}}{|\xi|^2 - m^2 + \sqrt{(|\xi|^2 - m^2)^2 + 4 m^2 \rho^2}}.\]

Recall that \(\rho S\) is odd in the \(\rho S\)-variable and so vanishes on \(\mathbb{S}^2 / U(1)\). Thus, by (2.18), \(\rho\) also vanishes on \(\mathbb{S}^2 / U(1)\). Also by (2.18), \(\rho\) vanishes on \(\mathbb{S}^2 / U(1)\). Moreover, by (2.17), as \(\rho S + z S \to \infty\), there holds
\[(\rho, z) - (\rho S, z S) = O(\epsilon (\rho S^2 + z S^2)^{-1/2}), \quad \epsilon > 0.\]

It thus follows that

**Theorem 2.3.** The map
\[(\rho S, z S) \mapsto (\rho, z)\]

defined by (2.9) and (2.17) provides a holomorphic diffeomorphism from \(\hat{Q}\) to the complex half-plane \([\xi = \rho + iz : \rho > 0]\).

In the \((\rho, z, \varphi)\) coordinate system the metric \(g\) on \(M\) again admits a representation of the form
\[g = e^{-2U + 2a}(d\rho^2 + dz^2) + \rho^2 e^{-2U}(d\varphi + \rho B_d d\rho + A_d dz)^2.\]


In the rest of this section, we will use theorem 2.3 to study the regularity properties of the functions involved.

First, using \( g_{\phi\phi} = \rho^2 e^{-2U(\rho, z)} = \rho_S^2 e^{-2U_S(\rho_S, z_S)} \) and (2.18), the function \( U(\rho, z) \) is given by

\[
U(\rho, z) := U_S(\rho_S, z_S) - \log \frac{\rho_S}{\rho} = U_S(\rho_S, z_S) - \log \frac{\rho_S^2 + z_S^2}{\rho^2 + z^2} + m_1^2.
\]  

(2.28)

Recalling (2.22), the above relation can be rewritten as

\[
U(\rho, z) = \tilde{U}_S(\rho, z_S) + \frac{1}{2} \log \left[ \rho^2 + z^2 - m_1^2 + \sqrt{(\rho^2 + z^2 - m_1^2)^2 + 4m_1^2\rho^2} \right]
\]  

(2.29)

for some \( \tilde{U}_S \in C^\infty(\mathbb{C}_S \setminus D(0, \frac{m_1}{2})) \).

Next, by (2.17),

\[
d\rho^2 + dz^2 = \frac{|\xi_S^2 + m_1^2|^2}{|\xi_S|^2} (d\rho_S^2 + dz_S^2).
\]  

Thus, using (2.28) and

\[
e^{-2U(\rho, z) + 2\alpha(\rho, z)} (d\rho^2 + dz^2) = e^{-2U_S(\rho_S, z_S) + 2\alpha_S(\rho_S, z_S)} (d\rho_S^2 + dz_S^2)
\]  

in \( \tilde{Q}^2 \),

we get

\[
\alpha(\rho, z) = U(\rho, z) - U_S(\rho_S, z_S) + \alpha_S(\rho_S, z_S) + \log \frac{|\xi_S|^2}{|\xi_S|^2 + m_1^2},
\]  

(2.30)

(2.31)

Recalling (2.19), we can rewrite (2.31) as

\[
\alpha(\rho, z) = \tilde{\alpha}_S(\rho_S, z_S) + \frac{1}{2} \log \frac{\rho^2 + z^2 - m_1^2 + \sqrt{(\rho^2 + z^2 - m_1^2)^2 + 4m_1^2\rho^2}}{2\sqrt{\rho^2 + (z-m_1)^2\sqrt{\rho^2 + (z+m_1)^2}}}
\]  

(2.32)

for some \( \tilde{\alpha}_S \in C^\infty(\mathbb{C}_S \setminus D(0, \frac{m_1}{2})) \). Also, as \( \alpha_S \) vanishes on the axis \( SS \), (2.31) implies that

\[
\alpha(\rho, z) = 0 \quad \text{for} \quad \rho = 0, z \notin [-m_1, m_1].
\]  

(2.33)

We also need to understand the behaviour of the metric functions \( B_\rho \) and \( A_z \), keeping in mind that \( B_S \) and \( \tilde{A}_S \) are smooth up-to-boundary. Since

\[
\rho_S \tilde{B}_S(\rho_S, z_S) d\rho_S + \tilde{A}_S(\rho_S, z_S) dz_S = \rho B_\rho(\rho, z) d\rho + A_z(\rho, z) dz,
\]

\( B_\rho \) and \( A_z \) satisfy

\[
B_\rho(\rho, z) = \frac{1}{8} \left[ (\mu^1/2 + 1)^2 + \frac{1}{2} (\mu^{-1/2} + 1) \mu_{,\rho} \right] \tilde{B}_S(\rho_S, z_S) - \frac{1}{4} \rho^{-1} \mu^{-3/2} \mu_{,\rho} \tilde{A}_S(\rho_S, z_S),
\]  

(2.34)

\[
A_z(\rho, z) = \frac{1}{8} (\mu^{-1/2} + 1) \mu_{,z} \tilde{B}_S(\rho_S, z_S) + \frac{1}{2} \left[ (\mu^{-1/2} + 1) - \frac{1}{2} \mu^{-3/2} \mu_{,z} \right] \tilde{A}_S(\rho_S, z_S).
\]  

(2.35)

We compute from (2.25)

\[
\mu_{,\rho} = -\frac{4m_1^2\rho}{\sqrt{(\xi_S^2 + m_1^2)^2 + 4m_1^4\rho^2}}, \quad \frac{|\xi_S|^2 + m_1^2 + \sqrt{(\xi_S^2 - m_1^2)^2 + 4m_1^2\rho^2}}{2\sqrt{(\xi_S^2 + m_1^2)^2 + 4m_1^4\rho^2}}.
\]  

(2.36)

\[
\mu_{,z} = -\frac{4m_1^2}{\sqrt{(\xi_S^2 + m_1^2)^2 + 4m_1^4\rho^2}}, \quad \frac{1}{\sqrt{(\xi_S^2 - m_1^2)^2 + 4m_1^4\rho^2}}.
\]  

(2.37)
Also, note that by (2.20) and (2.21),
\[ |\zeta|^2 - m_1^2 + \sqrt{(|\zeta|^2 - m_1^2)^2 + 4m_1^2\rho^2} = \frac{2(|\zeta S|^2 - \frac{m_1}{4})}{|\zeta S|^2}, \]
\[ |\zeta|^2 + m_1^2 + \sqrt{(|\zeta|^2 - m_1^2)^2 + 4m_1^2\rho^2} = \frac{2(|\zeta S|^2 + \frac{m_1}{4})}{|\zeta S|^2}, \]
\[ \sqrt{(|\zeta|^2 - m_1^2)^2 + 4m_1^2\rho^2} = \frac{(|\zeta S|^2 - \frac{m_1}{4})}{|\zeta S|^2} + \frac{m_1^2|\zeta S|^2}{(|\zeta S|^2 - \frac{m_1}{4})^2}\rho^2. \]

Thus,
\[ \mu_{,\rho} = -\frac{2m_1^2|\zeta S|^4\rho}{(|\zeta S|^2 - \frac{m_1}{4})^2 + m_1^2|\zeta S|^4\rho^2} \left( \frac{|\zeta S|^2 + \frac{m_1}{4}}{|\zeta S|^2 - \frac{m_1}{4}} \right)^2, \] (2.36)
\[ \mu_{,z} = -\frac{2m_1^2|\zeta S|^4z}{(|\zeta S|^2 - \frac{m_1}{4})^2 + m_1^2|\zeta S|^4\rho^2}. \] (2.37)

We also have, from (2.25) and the before-last displayed equations, that
\[ \mu = \left( \frac{|\zeta S|^2 + \frac{m_1}{4}}{|\zeta S|^2 - \frac{m_1}{4}} \right)^2. \] (2.38)

Substituting (2.36), (2.37), (2.38) and (2.24) into (2.34) and (2.35) we obtain
\[ B_\rho(\rho, z) = \left[ \frac{|\zeta S|^4}{(|\zeta S|^2 - \frac{m_1}{4})^2} - \frac{m_1^2|\zeta S|^2\rho^2(|\zeta S|^2 + \frac{m_1}{4})}{2(|\zeta S|^2 - \frac{m_1}{4})^2 + m_1^2|\zeta S|^4\rho^2} \right] \tilde{B}_S(\rho S, z S)
+ \frac{m_1^2|\zeta S|^2z S}{2(|\zeta S|^2 - \frac{m_1}{4})^2 + m_1^2|\zeta S|^4\rho^2} \tilde{A}_S(\rho S, z S), \]
\[ A_\rho(\rho, z) = -\frac{m_1^2\rho^2z S}{2(|\zeta S|^2 - \frac{m_1}{4})^2 + m_1^2|\zeta S|^4\rho^2} \tilde{B}_S(\rho S, z S)
+ \left[ \frac{|\zeta S|^2}{|\zeta S|^2 + \frac{m_1}{4}} - \frac{m_1^2\rho^2(|\zeta S|^2 - \frac{m_1}{4})}{2(|\zeta S|^2 + \frac{m_1}{4})^2 + m_1^2|\zeta S|^4\rho^2} \right] \tilde{A}_S(\rho S, z S). \]

We thus write
\[ B_\rho(\rho, z) = \frac{|\zeta S|^4}{2(|\zeta S|^2 - \frac{m_1}{4})^2 + m_1^2|\zeta S|^4\rho^2} \tilde{B}_S(\rho S, z S)
= \frac{1}{\rho^2 + z^2 - m_1^2 + \sqrt{(\rho^2 + z^2 - m_1^2)^2 + 4m_1^2\rho^2}} \times \frac{1}{\sqrt{(\rho^2 + z^2 - m_1^2)^2 + 4m_1^2\rho^2}} \tilde{B}_S(\rho S, z S), \] (2.39)
\[ A_\rho(\rho, z) = \frac{|\zeta S|^2}{(|\zeta S|^2 - \frac{m_1}{4})^2 + m_1^2\rho^2} \tilde{B}_S(\rho S, z S)
= \frac{1}{\sqrt{(\rho^2 + z^2 - m_1^2)^2 + 4m_1^2\rho^2}} \tilde{A}_S(\rho S, z S). \] (2.40)
where $\tilde{B}_S, \tilde{A}_S \in C^\infty (C_S \setminus D(0, \frac{m_1}{2}))$.

Finally, by (2.16) and (2.26) and the above regularity justification, we have
\begin{align*}
U &= \alpha_k^{a_3} (r^{-1/2}), \
\alpha &= \alpha_k^{a_4} (r^{-1/2}), \
B_\rho &= \alpha_k^{a_3} (r^{-5/2}), \
A_z &= \alpha_k^{a_3} (r^{-3/2}).
\end{align*}
(2.41)

We have thus shown

**Theorem 2.4.** Let $(M, g)$ be a three-dimensional smooth simply connected, connected manifold with a smooth connected compact boundary $\partial M$ and assume that $(M, g)$ admits a Killing vector field with periodic orbits. Furthermore, assume that $(M, g)$ has one asymptotically flat end where it satisfies (2.1) for some $k \geq 5$. Then there exists a unique $m_1 > 0$ such that $M$ is diffeomorphic to $\mathbb{R}^3 \setminus I$ for some line interval $I$ of length $2m_1$, and, in cylindrical coordinates $(\rho, z, \phi)$ of $\mathbb{R}^3$ aligned so that $I = [-m_1, m_1]$ lies on the $z$-axis, the metric $g$ takes the form (2.27), $\partial_\phi$ is the rotational Killing vector field of $M$, and $U, \alpha, A_\rho$ and $B_z$ satisfy (2.29), (2.32), (2.33), (2.39), (2.40) and (2.41).

**Remark 2.5.** The above analysis can be carried out with some additional work to take care of the case where $\partial M$ is disconnected. The only delicate point is the construction of the coordinates $(\rho_S, z_S)$ such that, in the $(\rho_S, z_S)$-plane, $\partial M$ corresponds to a union of a finite number of disjoint circles. An alternative way is to first construct the $(\rho, z)$ coordinates as in [7, section 6.3], and use our analysis here to derive the behaviour near each component of $\partial M$ of the functions of interest. This approach simplifies the analysis in [7, section 6.5].

### 2.4. The constant $m_1$

We showed earlier that $m_1$ is uniquely determined by the geometry of $(M, g)$. Here we give a more explicit description of $m_1$.

Recall that $Q^{(2)}$ is represented by $\Omega_2$ in $(\rho_z, z_z)$-coordinates and that $m_1$ can be expressed in terms of the Robin constant $\gamma(\partial/\Omega_1)$ of $\partial/\Omega_1$ by (2.10). By definition, if $\Gamma = \Gamma_{\partial/\Omega_1}$ is the unique harmonic function in $C_\#$ (with the flat metric) which vanishes at $\partial/\Omega_1$ and is asymptotic to $\frac{1}{2} \log (\rho_z^2 + z_z^2)$ at infinity, then
\begin{align*}
\Gamma(\rho_z, z_z) &= \frac{1}{2} \log (\rho_z^2 + z_z^2) + \gamma(\partial/\Omega_1) + O((\rho_z^2 + z_z^2)^{-1/2}).
\end{align*}

Let $\{y^1, y^2, y^3 = \phi\}$ be a coordinate system on $M$ such that $\{y^1, y^2\}$ is a coordinate system on $Q^{(2)}$. In what follows, indices $a$ and $b$ range over $\{1, 2\}$, while Greek indices range over $\{1, 2, 3\}$. The induced quotient metric on $Q^{(2)}$ is given by
\begin{align*}
q_{ab} &= g_{ab} - K g_{\phi\phi} g_{\phi a} g_{\phi b},
\end{align*}
where $K g_{\phi\phi} = \frac{1}{\sqrt{\det g}}$. Note that, by (2.6),
\begin{align*}
q &= e^{-2U + 2a_0} (d\rho_z^2 + dz_z^2).
\end{align*}

Thus, as a function on $Q^{(2)}$, $\Gamma$ is harmonic with respect to the metric $q$, i.e.
\begin{align*}
\partial_{\phi} \left( \sqrt{\det q} q_{ab} \partial_{\phi} \Gamma \right) = 0 \quad \text{in} \quad \tilde{Q}^{(2)}.
\end{align*}

Since $\Gamma$ is $\phi$-independent and $\partial_\phi$ is Killing, this implies that as a function on $M$, $\Gamma$ satisfies
\begin{align*}
\partial_{y^\mu} \left( \frac{\det g}{g_{\phi\phi}} g^{\mu\nu} \partial_{y^\nu} \Gamma \right) = 0 \quad \text{in} \quad \tilde{M} \setminus \partial_{y}.
\end{align*}
(2.42)
We thus conclude that $\Gamma_1$ satisfies
\[
\begin{align*}
L \Gamma_1 & = \Delta g \log g \Gamma_1^\phi, \\
\Gamma_1 & = 0 \quad \text{on} \quad \partial M, \\
\Gamma_1 & = \log r + O(1) \quad \text{as} \quad r \to \infty,
\end{align*}
\] (2.43)
where $r$ is the coordinate radius in the asymptotic region. Moreover, by construction, $\Gamma_1$ is the unique solution to (2.43) satisfying $\partial_\phi \Gamma_1 = 0$.

We thus have

**Proposition 2.6.** The constant $m_1$ is given by
\[
m_1 = 2 \exp \left( - \lim_{r \to \infty} (\Gamma_1 - \log r) \right),
\] (2.44)
where $\Gamma_1$ is the unique axially symmetric smooth solution to (2.43).

3. The ADM mass

In this section, we compute the ADM mass $m$ of $g$ as a volume integral over $\mathbb{R}^3 \setminus B(0, \frac{m_1}{2})$ and then use it to prove theorem 1.1. We have
\[
m = \lim_{R \to \infty} \frac{1}{16\pi} \int_{S_R} (g_{ij,j} - g_{jj,i}) \nu_i \, d\sigma,
\]
where the metric components are computed in a coordinate system satisfying (2.1)–(2.3), $d\sigma$ is the surface area form on $S_R$, and $S_R$ can be taken to be any piecewise differentiable surface homologous to a coordinate sphere of radius $R$ with
\[
\inf \{ r(p) : p \in S_R \} \to R \to \infty \infty
\]
That the ADM mass is well defined is well known, see [2, 5].

3.1. Mass in pseudo-spherical coordinates

Define
\[
x^1 = x_S = \rho_S \cos \varphi, \quad x^2 = y_S = \rho_S \sin \varphi, \quad x^3 = z_S.
\]
Using (2.16), we can write the metric (2.15) as
\[
g = e^{-2U_S} (dx^2_S + dy^2_S) + e^{-2U_S} \left( e^{2\alpha_S} - 1 \right) (x_S \, dx_S + y_S \, dy_S)^2
+ e^{-2U_S + 2\alpha_S} \, dz^2_S + 2(x_S \, dy_S - y_S \, dx_S) (\bar{B}_S(x_S \, dx_S + y_S \, dy_S) + \bar{A}_S \, dz_S) + o_1(r^{-1}).
\] (3.1)
Here $r$ denotes the coordinate radius $r = \sqrt{x^2_S + y^2_S + z^2_S}$.

In the following computation, $S_R$ is the sphere of coordinate radius $R$. Obviously, the error terms in (3.1) has no contribution to the mass integral. A straightforward computation using (2.15) shows that the terms involving $\bar{B}_S$ and $\bar{A}_S$ also give zero contribution to the mass integral.

The rest of the mass integrand is then found to be
\[
\begin{align*}
\partial_{x_S} \left( e^{-2U_S} \left( e^{2\alpha_S} - 1 \right) \frac{x_S y_S}{\rho_S} \right) - \partial_{y_S} \left( e^{-2U_S} + e^{-2U_S} \left( e^{2\alpha_S} - 1 \right) \frac{y^2_S}{\rho_S^2} \right) - \partial_{z_S} e^{-2U_S + 2\alpha_S} \left( \frac{x_S}{r} \right) \\
+ \partial_{x_S} \left( e^{-2U_S} \left( e^{2\alpha_S} - 1 \right) \frac{x_S y_S}{\rho_S} \right) - \partial_{y_S} \left( e^{-2U_S} + e^{-2U_S} \left( e^{2\alpha_S} - 1 \right) \frac{y^2_S}{\rho_S^2} \right) - \partial_{z_S} e^{-2U_S + 2\alpha_S} \left( \frac{y_S}{r} \right) \\
+ \partial_{z_S} \left( e^{-2U_S} + e^{-2U_S} \left( e^{2\alpha_S} - 1 \right) \frac{x^2_S}{\rho_S^2} \right) - \partial_{x_S} \left( e^{-2U_S} + e^{-2U_S} \left( e^{2\alpha_S} - 1 \right) \frac{y^2_S}{\rho_S^2} \right) - \partial_{y_S} e^{-2U_S + 2\alpha_S} \left( \frac{z_S}{r} \right).
\end{align*}
\]
Upon simplifying this gives
\[\left\{ \partial_{y^S} \left( e^{-2U_S} \left( e^{2\alpha_S} - 1 \right) \frac{x^S y^S}{\rho_S^2} \right) - \partial_{x^S} \left( e^{-2U_S} \frac{x^2 y^S}{\rho_S^2} \right) \right\} \frac{x^S}{r} \]
\[+ \left\{ \partial_{x^S} \left( e^{-2U_S} \left( e^{2\alpha_S} - 1 \right) \frac{x^S y^S}{\rho_S^2} \right) - \partial_{y^S} \left( e^{-2U_S} \frac{x^2 y^S}{\rho_S^2} + e^{-2U_S+2\alpha_S} \right) \left( 1 + \frac{x^2}{\rho_S^2} \right) \right\} \frac{y^S}{r} \]
\[- \partial_{z^S} \left( e^{-2U_S} \left( e^{2\alpha_S} + 1 \right) \right) \frac{z^S}{r}.\]

Expanding using (2.16) we obtain
\[\partial R \left( 2U_S - \alpha_S \right) + \frac{\alpha_S}{r} + o(r^{-2}).\]

We thus arrive at
\[m = \frac{1}{4\pi} \lim_{R \to \infty} \int_{S_R} \partial_r \left( \frac{U_S - 1}{2} \alpha_S \right) d\sigma + \frac{1}{2R} \int_{S_R} \alpha_S d\sigma. \tag{3.2}\]

(This is similar to a formula derived in [6], but the integrations are over different sets, which requires the new derivation above. The current expression is more convenient for our purposes.)

To proceed, we recall a formula for the scalar curvature on \(M\) from [9]:
\[R_{\text{g}} = 4e^{-2U_S} - 2\alpha_S \left[ \Delta \left( U_S - 1 \right) - \frac{1}{2} \left| \nabla U_S \right|^2 + \frac{1}{2\rho_S} \partial_{\rho_S} \alpha_S - \frac{1}{8} \rho_S^2 e^{-2\alpha_S} \left( \rho_S \partial_{z^S} \bar{B}_S - \partial_{\rho_S} \bar{A}_S \right)^2 \right]. \tag{3.3}\]

Here \(\Delta\) and \(\nabla\) are the Laplacian and the gradient operator taken with respect to the flat metric in \(\mathbb{R}^3\).

Using (3.3), we can convert (3.2) into the volume integral form. Note that if \(\Phi\) is a function defined on \(\mathbb{R}^3 \setminus B \left( 0, \frac{m_1}{2} \right) \) satisfying
\[\Phi > 0 \quad \text{in} \quad \mathbb{R}^3 \setminus B \left( 0, \frac{m_1}{2} \right) \quad \text{and} \quad \Phi = 1 + o(r^{-1/2}) \quad \text{as} \quad r \to \infty, \tag{3.4}\]
then
\[\lim_{R \to \infty} \int_{S_R} \partial_r \left( U_S - 1 \right) \frac{1}{2} \alpha_S d\sigma = \lim_{R \to \infty} \int_{S_R} \Phi \partial_r \left( U_S - 1 \right) \frac{1}{2} \alpha_S d\sigma. \tag{3.5}\]

By the divergence theorem, we have
\[\int_{S_R} \Phi \partial_r \left( U_S - 1 \right) \frac{1}{2} \alpha_S d\sigma = \int_{B(0,R) \setminus B(0,\frac{m_1}{2})} \left[ \nabla \Phi \cdot \nabla \left( U_S - 1 \right) \right] d^3x \]
\[+ \int_{\partial B(0,\frac{m_1}{2})} \Phi \partial_r \left( U_S - 1 \right) \frac{1}{2} \alpha_S d\sigma. \]

Hence, by (3.3),
\[\int_{S_R} \Phi \partial_r \left( U_S - 1 \right) \frac{1}{2} \alpha_S d\sigma = \int_{B(0,R) \setminus B(0,\frac{m_1}{2})} \left[ \nabla \Phi \cdot \nabla \left( U_S - 1 \right) \right] d^3x \]
\[+ \frac{1}{2} \Phi \left| \nabla U_S \right|^2 - \frac{1}{2\rho_S} \partial_{\rho_S} \alpha_S \Phi + \frac{1}{4} e^{-2U_S+2\alpha_S} \Phi R_S \]
\[+ \frac{1}{8} \rho_S^2 e^{-2\alpha_S} \left( \rho_S \partial_{z^S} \bar{B}_S - \partial_{\rho_S} \bar{A}_S \right)^2 \right] d^3x \]
\[+ \int_{\partial B(0,\frac{m_1}{2})} \Phi \partial_r \left( U_S - 1 \right) \frac{1}{2} \alpha_S d\sigma. \tag{3.6}\]
To get rid of the terms involving gradients of $\alpha_S$ we choose $\Phi$ to satisfy

$$\frac{1}{\rho_S} \text{div}(\rho_S \nabla \Phi) = \text{div}(\nabla \Phi + \Phi \nabla \log \rho_S) = 0 \quad \text{in} \quad \mathbb{R}^3 \setminus B \left(0, \frac{m_1}{2}\right). \quad (3.7)$$

Note that if we view $\Phi$ as a function defined in $\mathbb{R}^4 \setminus B(0, \frac{m_1}{2})$ invariant under $SO(2)$ and assume that $\Phi$ is locally bounded, then

$$\Delta^{(4)} \Phi = 0 \quad \text{in} \quad \mathbb{R}^4 \setminus B \left(0, \frac{m_1}{2}\right). \quad (3.8)$$

In particular, this implies that $\frac{1}{\rho_S} \partial_{\rho_S} \Phi$ is locally bounded, and $\partial \Phi = O(r^{-3})$ for large $r$. Thus, as $\alpha_S$ vanishes wherever $\rho_S = 0$, an application of the divergence theorem gives

$$\int_{B(0,R) \setminus B(0,\frac{m_1}{2})} \left[ \nabla \cdot (\nabla \alpha_S + \frac{1}{\rho_S} \partial_{\rho_S} \alpha_S \Phi) \right] d^3x$$

$$= \int_{B(0,R) \setminus B(0,\frac{m_1}{2})} \nabla \alpha_S \cdot (\nabla \Phi + \Phi \nabla \log \rho_S) d^3x$$

$$= \int_{S_\alpha} \alpha_S (\partial_{\sigma} \log \rho_S + O(R^{-3})) d\sigma - \int_{B(0,\frac{m_1}{2})} \alpha_S (\partial_r \Phi + \Phi \partial_r \log \rho_S) d\sigma$$

$$= \frac{1}{R} \int_{S_\alpha} \alpha_S d\sigma + o(R^{-3/2}) - \int_{B(0,\frac{m_1}{2})} \alpha_S \left(\partial_r \Phi + \frac{2}{m_1} \Phi\right) d\sigma.$$ 

Substituting the above into $(3.6)$ yields

$$\int_{S_\alpha} \Phi \partial_r \left( U_S - \frac{1}{4} \alpha_S \right) d\sigma = \int_{B(0,R) \setminus B(0,\frac{m_1}{2})} \left\{ \nabla \Phi \cdot \nabla U_S + \frac{1}{2} \Phi |\nabla U_S|^2 
+ \frac{1}{4} e^{-2U_S+2\alpha_S} \Phi R_S + \frac{1}{8} \rho_S^2 e^{-2\alpha_S} \Phi (\rho_S \partial_{\sigma} B_S - \partial_{\rho_S} A_S)^2 \right\} d^3x$$

$$- \frac{1}{2R} \int_{S_\alpha} \alpha_S d\sigma + o(R^{-3/2}) + \int_{B(0,\frac{m_1}{2})} \left\{ \Phi \partial_r \left( U_S - \frac{1}{4} \alpha_S \right) + \frac{1}{2} \alpha_S \left( \partial_r \Phi + \frac{2}{m_1} \Phi \right) \right\} d\sigma.$$

Recalling $(3.2)$ and $(3.5)$, we arrive at

$$m = \frac{1}{4\pi} \int_{\mathbb{R}^3 \setminus B(0,\frac{m_1}{2})} \left\{ \nabla \Phi \cdot \nabla U_S + \frac{1}{2} \Phi |\nabla U_S|^2 
+ \frac{1}{4} e^{-2U_S+2\alpha_S} \Phi R_S + \frac{1}{8} \rho_S^2 e^{-2\alpha_S} \Phi (\rho_S \partial_{\sigma} B_S - \partial_{\rho_S} A_S)^2 \right\} d^3x$$

$$+ \frac{1}{4\pi} \int_{B(0,\frac{m_1}{2})} \left\{ \Phi \partial_r \left( U_S - \frac{1}{4} \alpha_S \right) + \frac{1}{2} \alpha_S \left( \partial_r \Phi + \frac{2}{m_1} \Phi \right) \right\} d\sigma. \quad (3.9)$$

Next, if $\Psi$ is a function defined on $\mathbb{R}^3 \setminus \bar{B}(0,\frac{m_1}{2})$ such that

$$\begin{cases} 
\Delta \Psi = 0 & \text{in} \quad \mathbb{R}^3 \setminus \bar{B}(0,\frac{m_1}{2}), \\
\Psi = \text{Const} + O(r^{-1}) & \text{as} \quad r \to \infty,
\end{cases} \quad (3.10)$$

then

$$\int_{\mathbb{R}^3 \setminus B(0,\frac{m_1}{2})} \nabla \Psi \cdot \nabla U_S d^3x = - \int_{B(0,\frac{m_1}{2})} U_S \partial_r \Psi d\sigma. \quad (3.11)$$
Using the above identity in (3.9) yields

\[
m = \frac{1}{4\pi} \int_{\mathbb{R}^3 \setminus B(0, \frac{m_1}{2})} \left\{ \frac{1}{2} \Phi |\nabla U_S|^2 + \nabla U_S \cdot \nabla (\Phi - \Psi) \right. \\
+ \frac{1}{4} e^{-2U_S + 2\alpha S} \Phi R_\phi + \frac{1}{8} \rho_S^2 e^{-2\alpha S} \Phi (\rho_S \partial_{\nu_2} B_S - \partial_{\rho_2} \bar{A}_S)^2 \right\} d^3 x \\
+ \frac{1}{4\pi} \int_{\partial B(0, \frac{m_1}{2})} \left\{ \Phi \partial_r \left( U_S - \frac{1}{2} \alpha_S \right) + \frac{1}{2} \alpha_S \left( \partial_r \Phi + \frac{2}{m_1} \Phi \right) + U_S \partial_r \psi \right\} d\sigma.
\]

(3.12)

To conclude, we have shown

**Proposition 3.1.** Under the hypotheses of theorem 2.2 and (2.2), the ADM mass of \((M, g)\) is well defined and satisfies (3.12) for any \(\Phi\) and \(\Psi\) satisfying (3.4), (3.7) and (3.10).

We shall show below how appropriate choices of \(\Phi\) and \(\Psi\) allow one to control the mass. For further reference we note

**Corollary 3.2.** If \((\Phi_1, \Psi_1)\) and \((\Phi_2, \Psi_2)\) satisfy (3.4), (3.7) and (3.10) then

\[
0 = \frac{1}{4\pi} \int_{\mathbb{R}^3 \setminus B(0, \frac{m_1}{2})} \left\{ \frac{1}{2} (\Phi_1 - \Phi_2)|\nabla U_S|^2 + \nabla U_S \cdot \nabla (\Phi_1 - \Phi_2 - \Psi_1 + \Psi_2) \right. \\
+ \frac{1}{4} e^{-2U_S + 2\alpha S} (\Phi_1 - \Phi_2) R_\phi + \frac{1}{8} \rho_S^2 e^{-2\alpha S} (\Phi_1 - \Phi_2)(\rho_S \partial_{\nu_2} B_S - \partial_{\rho_2} \bar{A}_S)^2 \right\} d^3 x \\
+ \frac{1}{4\pi} \int_{\partial B(0, \frac{m_1}{2})} \left\{ (\Phi_1 - \Phi_2) \partial_r \left( U_S - \frac{1}{2} \alpha_S \right) \\
+ \frac{1}{2} \alpha_S \left( \partial_r \Phi_1 + \frac{2}{m_1} \Phi_1 - \partial_r \Phi_2 - \frac{2}{m_1} \Phi_2 \right) - U_S (\partial_r \psi_1 - \partial_r \psi_2) \right\} d\sigma.
\]

(3.13)

### 3.2. Lower bound for the ADM mass

In this section, we prove theorem 1.1. We now assume that

\[
R_\phi \geq 0 \quad \text{in} \quad M,
\]

(3.14)

together with a Riemannian version of the condition that \(\partial M\) is weakly outer trapped, namely

the mean curvature of \(\partial M\) is non-positive.

(3.15)

Here the mean curvature is computed with respect to the normal pointing towards \(M\). By a direct computation, (3.15) is equivalent to

\[
\bar{\partial}_s \left( U_S - \frac{1}{2} \alpha_S \right) \geq \frac{2}{m_1} \quad \text{on} \quad \partial B \left( 0, \frac{m_1}{2} \right).
\]

(3.16)

**Proof of theorem 1.1.** Under (3.14) and (3.15), (3.12) implies, keeping in mind that \(\Phi\) is positive, and completing the square in the volume integral when passing from the first to the second inequality,

\[
m \geq \frac{1}{4\pi} \int_{\mathbb{R}^3 \setminus B(0, \frac{m_1}{2})} \left\{ \frac{1}{2} \Phi |\nabla U_S|^2 + \nabla U_S \cdot \nabla (\Phi - \Psi) \right\} d^3 x \\
+ \frac{1}{4\pi} \int_{\partial B(0, \frac{m_1}{2})} \left\{ \Phi \frac{2}{m_1} + \frac{1}{2} \alpha_S \left( \partial_s \Phi + \frac{2}{m_1} \Phi \right) - U_S \partial_s \psi \right\} d\sigma.
\]
\begin{align*}
\geq -\frac{1}{8\pi} \int_{\mathbb{R}^3 \backslash B(0, \frac{m_1}{2})} \frac{1}{\Phi} |\nabla (\Phi - \Psi)|^2 d^3x \\
+ \frac{1}{4\pi} \int_{\partial B(0, \frac{m_1}{2})} \left( \Phi \frac{2}{m_1} + \frac{1}{2} \alpha_S \left( \partial_r \Phi + \frac{2}{m_1} \Phi \right) - U_S \partial_r \Psi \right) d\sigma.
\end{align*}

(3.17)

To continue, we specialize the choice of \( \Phi \) and \( \Psi \) by taking
\[ \Psi \equiv 1 \quad \text{and} \quad \Phi \equiv 1 + \frac{m_1^2}{4r^2}. \]

Then (3.17) gives
\begin{align*}
m \geq m_1 - \frac{1}{8\pi} \int_{\mathbb{R}^3 \backslash B(0, \frac{m_1}{2})} \frac{m_1^4}{r^4 \left( 4r^2 + m_1^2 \right)} d^3x \\
= \frac{\pi}{4} m_1.
\end{align*}

(3.18)

Next, assume that \( m = \frac{\pi}{4} m_1 \). Then, we must have
\begin{align*}
R_g = \rho \partial_r \bar{S} - \partial_\rho \bar{A} \equiv 0 \quad \text{in} \quad \mathbb{R}^3 \backslash B \left( 0, \frac{m_1}{2} \right),
\end{align*}

(3.19)

\begin{align*}
\nabla U_S \equiv -\frac{1}{\Phi} \nabla (\Phi - \Psi) = -\frac{1}{\Phi} \nabla \Phi \quad \text{in} \quad \mathbb{R}^3 \backslash B \left( 0, \frac{m_1}{2} \right),
\end{align*}

(3.20)

\begin{align*}
\partial_r \left( U_S - \frac{1}{2} \sigma_S \right) \equiv \frac{2}{m_1} \quad \text{on} \quad \partial B \left( 0, \frac{m_1}{2} \right).
\end{align*}

(3.21)

By (2.16), the second relation implies that
\[ \nabla U_S \equiv \frac{2m_1^2}{r \left( 4r^2 + m_1^2 \right)} \partial_r \quad \text{in} \quad \mathbb{R}^3 \backslash B \left( 0, \frac{m_1}{2} \right), \]
and so, since \( U_S \) is assumed to asymptote to zero at infinity,
\[ U_S \equiv \log \frac{4r^2}{4r^2 + m_1^2} \quad \text{in} \quad \mathbb{R}^3 \backslash B \left( 0, \frac{m_1}{2} \right). \]

(3.22)

Taking (3.3), (3.21) and (3.22) into account we get
\[ \begin{cases} 
\Delta \alpha_S - \frac{1}{\rho_S} \partial_\rho \alpha_S = -\frac{16m_1^2}{\left( 4r^2 + m_1^2 \right)^2} < 0 \quad \text{in} \quad \mathbb{R}^3 \backslash B \left( 0, \frac{m_1}{2} \right), \\ 
\partial_r \alpha_S = 0 \quad \text{on} \quad \partial B \left( 0, \frac{m_1}{2} \right), \\ 
\alpha_S = o(r^{-1/2}) \quad \text{as} \quad r \to \infty.
\end{cases} \]

Since \( \alpha_S \) is \( \phi \)-independent, this implies
\[ \begin{cases} 
\partial_\rho^2 \alpha_S + \partial_\phi^2 \alpha_S < 0 \quad \text{in} \quad \mathbb{R}^2 \backslash \bar{D} \left( 0, \frac{m_1}{2} \right), \\ 
\partial_r \alpha_S = 0 \quad \text{on} \quad \partial D \left( 0, \frac{m_1}{2} \right), \\ 
\alpha_S = o(r^{-1/2}) \quad \text{as} \quad r \to \infty.
\end{cases} \]

This is impossible by Hadamard’s three-circle theorem, proving that the equality cannot hold in (3.18). We conclude the proof of theorem 1.1. \( \square \)
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Appendix. A remark on the axisymmetric Penrose inequality

In [9] a proof of the Penrose inequality for axisymmetric initial data sets with a positive scalar curvature has been given, under however undesirably stringent conditions on the geometry near the horizon. It seems therefore of interest to attempt to remove the overly restrictive conditions. In particular one can enquire whether our arguments above can be adapted to obtain the Penrose inequality. In this appendix we provide an argument that gives a result stronger than that in [9], but fails to provide the full Penrose inequality.

We will always assume (3.14), i.e.

\[ R_g \geq 0 \text{ in } M. \]

Furthermore, we will assume that \( \partial M \) is minimal, i.e.

\[ \partial \left( \Phi \right) = \left( \Phi \right) \partial \left( \frac{1}{2} \right) \text{ on } \partial B \left( 0, \frac{m_1}{2} \right). \]

(A.1)

By the first inequality in (3.17), we have

\[ m \geq \frac{1}{4\pi} \int_{\mathbb{R}^3 \setminus B(0, \frac{m_1}{2})} \left\{ \frac{1}{2} \Phi \left| \nabla U_S \right|^2 + \nabla U_S \cdot \nabla (\Phi - \Psi) \right\} \, d^3x \]

\[ + \left( \frac{1}{4\pi} \int_{\partial B(0, \frac{m_1}{2})} \left\{ \frac{2}{m_1} \Phi + \frac{1}{2} \alpha S \left( \partial_r \Phi + \frac{2}{m_1} \Phi \right) - U_S \partial_r \Psi \right\} \, d\sigma \right) \]  

(A.2)

for any \( \Phi \) and \( \Psi \) satisfying (3.4), (3.7) and (3.10). Moreover, this is an equality iff \( R_g \equiv 0 \equiv \rho_S \partial_z B_S - \partial \rho_S \bar{A}_S \).

Let \( A \) be the area of \( \partial M \). Then

\[ A = \int_{\partial B(0, \frac{m_1}{2})} e^{\alpha S - 2U_S} \, d\sigma. \]

(A.3)

According to Bray, Huisken, and Ilmanen [3, 12] one has

\[ m \geq \sqrt{\frac{A}{16\pi}}. \]

(A.4)

Hence, under the stated hypotheses and that \( \mathbb{R}^3 \setminus B(0, \frac{m_1}{2}) \) with the metric (2.15) contains no compact minimal surfaces other than its boundary, one would naively expect that it must hold that

\[ J_{\Phi, \Psi}(U_S, \alpha_S) := \frac{1}{4\pi} \int_{\mathbb{R}^3 \setminus B(0, \frac{m_1}{2})} \left\{ \frac{1}{2} \Phi \left| \nabla U_S + \frac{1}{\Phi} \nabla (\Phi - \Psi) \right|^2 - \frac{1}{2\Phi} \left| \nabla (\Phi - \Psi) \right|^2 \right\} \, d^3x \]

\[ + \left( \frac{1}{4\pi} \int_{\partial B(0, \frac{m_1}{2})} \left\{ \frac{2}{m_1} \Phi + \frac{1}{2} \alpha S \left( \partial_r \Phi + \frac{2}{m_1} \Phi \right) - U_S \partial_r \Psi \right\} \, d\sigma \right) \]

\[ - \left( \frac{1}{16\pi} \int_{\partial B(0, \frac{m_1}{2})} e^{\alpha S - 2U_S} \, d\sigma \right) \geq 0 \]

(A.5)
for some well-chosen $\Phi$ and $\Psi$. Moreover, equality should only hold for the Schwarzschild solution.

For a fixed $m_1$ this is thus a variational inequality: if the infimum over $U_S$ and $\alpha_S$ as described above of $J_{\Phi, \Psi}(U_S, \alpha_S)$ is zero, then the axisymmetric Riemannian Penrose inequality would follow.

A natural choice for $\Phi_1$ and $\Psi_1$ is to use functions which make the first volume integrand in (A.5) vanish for the Schwarzschild solution:

$$\nabla U_{S,\text{Schw}} + \frac{1}{\Phi_1} \nabla (\Phi_1 - \Psi_1) \equiv 0,$$

where $U_{S,\text{Schw}}$ is the ‘$U_S$’ of the Schwarzschildian slice

$$U_{S,\text{Schw}} = -2 \log \frac{2r + m_1}{2r}.$$ 

This leads to $\Phi_1 = 1 + \frac{am_1^2}{4r}$ and $\Psi_1 = \frac{bm_1}{r}$. (Here we have used equations (3.8) and (3.10).)

Entering this into (A.6), we obtain $a = -1$ and $b = -2$.

There is a special case where the expected inequality holds.

**Proposition A.1.** Let $(U_s, \alpha_s)$ be as in Theorem 2.2. Assume further that $(U_s, \alpha_s)$ satisfies (3.14), (3.16) and

$$U_s - \frac{1}{2} \alpha_s \equiv C_H \geq -2 \log 2 \quad \text{on} \quad \partial B \left(0, \frac{m_1}{2}\right),$$

there holds

$$J_{\Phi, \Psi}(U_s, \alpha_s) \geq 0,$$

where $\Phi_s = 1 - \frac{m_1^2}{2r}$ and $\Psi_s = -\frac{m_1}{r}$. Moreover, equality holds if and only if the metric (2.15) is that of a Schwarzschildian slice.

It should be noted that the existence of admissible data verifying (A.7) (other than the Schwarzschildian slice) is not clear. We also note that the requirement that $\partial M$ be the outermost minimal surface is not necessary, but rather $\partial M$ being merely weakly outer trapped is sufficient.

Proposition A.1 should be compared with a result in [9], where equality in (A.7) is assumed together with the supplementary requirement that $A$, as defined by (A.3), equals $16\pi m_1^2$.

**Proof.** We will only sketch the proof. Using the explicit form of $(\Phi_*, \Psi_*)$, one finds

$$J_{\Phi, \Psi}(U_s, \alpha_s) = -m_1 (2 \log 2 - 1) - m_1 C_H - \frac{1}{4} m_1 e^{-C_H} + \frac{1}{8\pi} \int_{\mathbb{R}^3 \setminus B(0, \frac{m_1}{2})} \Phi |V_S|^2 \, d^3x,$$

where

$$V_S = \nabla U_s + \frac{1}{\Phi_s} \nabla (\Phi_s - \Psi_s).$$

Next, set $\Sigma = \frac{m_1}{2r}$. Applying corollary (3.2) to $(\Phi_1, \Psi_1) = (\Phi_* + \Sigma, 0)$ and $(\Phi_2, \Psi_2) = (\Phi_*, \Psi_*)$ and noting (3.14) and (3.16), we find

$$0 = \frac{1}{4\pi} \int_{\mathbb{R}^3 \setminus B(0, \frac{m_1}{2})} \left\{ \frac{1}{2} \Sigma |\nabla U_s|^2 + \nabla U_s \cdot \nabla (\Sigma + \Psi_s) + \nabla U_s \right\} \, d^3x$$

$$+ \frac{1}{4} e^{-2U_s + 2\alpha_s} \Sigma R_s + \frac{1}{8} \rho_s^2 e^{-2\alpha_s} (\rho_s \partial_{\bar{z}} \tilde{B}_S - \partial_{\rho_s} \tilde{A}_S)^2 \, d^3x.$$
Proof. Using the explicit expressions for \( \Xi \) and \( \Psi_\ast \), we then get

\[
-m_1 C_H = \frac{1}{4\pi} \int_{\partial B(0, \frac{m_1}{2})} \left\{ \Xi \partial_r \left( U_S - \frac{1}{2} \alpha_S \right) + \frac{1}{2} \alpha_S \left( \partial_r \Xi + \frac{2}{m_1} \Xi \right) + U_S \partial_r \Psi_\ast \right\} d\sigma
\]

\[
\geq \frac{1}{4\pi} \int_{\partial B(0, \frac{m_1}{2})} \frac{2}{m_1} \Xi \partial_r \left( U_S - \frac{1}{2} \alpha_S \right) + \frac{1}{2} \alpha_S \left( \partial_r \Xi + \frac{2}{m_1} \Xi \right) + U_S \partial_r \Psi_\ast \right\} d\sigma.
\]

Using the explicit expressions for \( \Xi \) and \( \Psi_\ast \), we get

\[
-m_1 C_H \geq \frac{1}{4\pi} \int_{\partial B(0, \frac{m_1}{2})} \frac{2}{m_1} \Xi \partial_r \left( U_S - \frac{1}{2} \alpha_S \right) + \frac{1}{2} \alpha_S \left( \partial_r \Xi + \frac{2}{m_1} \Xi \right) + U_S \partial_r \Psi_\ast \right\} d\sigma
\]

Recalling that \( \nabla U_S = V_S - \frac{1}{\Phi_0} \nabla (\Phi_\ast - \Psi_\ast) \), we thus have

\[
-m_1 C_H \geq \frac{1}{4\pi} \int_{\partial B(0, \frac{m_1}{2})} \frac{2}{m_1} \Xi \partial_r \left( U_S - \frac{1}{2} \alpha_S \right) + \frac{1}{2} \alpha_S \left( \partial_r \Xi + \frac{2}{m_1} \Xi \right) + U_S \partial_r \Psi_\ast \right\} d\sigma
\]

Using the explicit expressions for \( \Phi_\ast, \Psi_\ast \) and \( \Xi \) again, one arrives at

\[
-m_1 C_H \geq 2m_1 \log 2 - \frac{m_1}{8\pi} \int_{\mathbb{R}^3 \setminus B(0, \frac{m_1}{2})} \Phi |V_S|^2 \, d^3 x. \tag{A.9}
\]

Define

\[
t = -C_H - 2 \log 2 \quad \text{and} \quad l := \frac{1}{8\pi m_1} \int_{\mathbb{R}^3 \setminus B(0, \frac{m_1}{2})} \Phi |V_S|^2 \, d^3 x.
\]

Then, by (A.8) and (A.9) and as \( l \geq 0 \) and \( t \leq 0 \) (by (A.7)),

\[
\frac{1}{m_1} J_{\Phi_\ast, \Psi_\ast}(U_S, \alpha_S) = 1 + t - e^l + l
\]

\[
\geq 1 - \sqrt{l} - e^{-\sqrt{l}} + l \geq 0.
\]

This finishes the proof. \( \Box \)

The bad news for the above program arises from the following.

**Proposition A.2.** There exists an ‘admissible’ \( (U_S, \alpha_S) = (U_S, 0) \) such that

1. \( U_S - \frac{1}{2} \alpha_S \equiv C_H < -2 \log 2 \) on \( \partial B(0, \frac{m_1}{2}) \),
2. \( J_{\Phi_\ast, \Psi_\ast}(U_S, \alpha_S) < 0 \).

**Proof.** The example is provided by conformally Schwarzschildian metrics, i.e. \( \alpha_S \equiv 0 \). \( U_S \) is given by

\[
U_S = U_S^{(k)} = -2 \log \left[ 1 + \frac{6k + m_1^2}{2m_1 r} - \frac{k}{2r^2} \right], \quad k \geq 0.
\]
For $k = 0$, this gives exactly the Schwarzschild metric. For $k > 0$, the scalar curvature is readily seen to be positive as $e^{-U_S/2}$ is super-harmonic (with respect to the flat metric). One can check directly from (A.1) that $\partial M$ is minimal. In fact, for $k < \frac{n^2}{6}$, $\partial M$ is outermost minimal. (An easy way to see that is to check that for those values of $k$, the coordinate spheres provide a foliations of $M$ by constant positive mean curvature surfaces.) The rest of the argument is to use (A.8) to verify that $J_{\phi_-,\psi_+}(U_S, \alpha_S)$ is negative for sufficiently small $k > 0$. □

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