EXISTENCE OF MINIMIZERS FOR SOME QUASILINEAR ELLIPTIC PROBLEMS

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Dedicated to Gisèle Ruiz Goldstein on the occasion of her 60th birthday

Abstract. The aim of this paper is investigating the existence of at least one weak bounded solution of the quasilinear elliptic problem
\[
\begin{cases}
- \text{div}(a(x,u,\nabla u)) + A_t(x,u,\nabla u) = f(x,u) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]
where $\Omega \subset \mathbb{R}^N$ is an open bounded domain and $A(x,t,\xi), f(x,t)$ are given real functions, with $A_t = \frac{\partial A}{\partial t}, a = \nabla \xi A$.

We prove that, even if $A(x,t,\xi)$ makes the variational approach more difficult, the functional associated to such a problem is bounded from below and attains its infimum when the growth of the nonlinear term $f(x,t)$ is "controlled" by $A(x,t,\xi)$. Moreover, stronger assumptions allow us to find the existence of at least one positive solution.

We use a suitable Minimum Principle based on a weak version of the Cerami–Palais–Smale condition.

1. Introduction. The aim of this paper is investigating the existence of solutions of the quasilinear elliptic problem
\[
(GP) \quad \begin{cases}
- \text{div}(a(x,u,\nabla u)) + A_t(x,u,\nabla u) = f(x,u) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]
where $\Omega \subset \mathbb{R}^N$ is an open bounded domain, $N \geq 2$, $f(x,t)$ is a given function on $\Omega \times \mathbb{R}$ and $A(x,t,\xi)$ is a function on $\Omega \times \mathbb{R} \times \mathbb{R}^N$ such that
\[
A_t(x,t,\xi) = \frac{\partial A}{\partial t}(x,t,\xi), \quad a(x,t,\xi) = (\frac{\partial A}{\partial \xi_1}(x,t,\xi), \ldots, \frac{\partial A}{\partial \xi_N}(x,t,\xi)).
\]

For example, a family of model problems is given by
\[
A(x,t,\xi) = \left( \sum_{i,j=1}^{N} a_{i,j}(x,t)\xi_i\xi_j \right)^{\frac{p}{2}},
\]
where $p > 1$ and $(a_{i,j}(x,t))_{1 \leq i,j \leq N}$ is an elliptic matrix.

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In particular, if \( \mathcal{A} : \Omega \times \mathbb{R} \to \mathbb{R} \) is a given function such that

\[
a_{i,j}(x,t) = \left( \frac{1}{p} \mathcal{A}(x,t) \right) \frac{i}{j^2},
\]

then \( \mathcal{A}(x,t,\xi) = \frac{1}{p} \mathcal{A}(x,t)|\xi|^p \) and \((GP)\) becomes

\[
(MP) \quad \begin{cases}
- \text{div}(\mathcal{A}(x,u)|\nabla u|^{p-2} \nabla u) + \frac{1}{p} \mathcal{A}_t(x,u)|\nabla u|^p = f(x,u) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

with related functional

\[
J_\mathcal{A}(u) = \frac{1}{p} \int_\Omega \mathcal{A}(x,u) |\nabla u|^p \, dx - \int_\Omega F(x,u) \, dx
\]

defined in a natural domain contained in \( W^{1,p}_0(\Omega) \), where \( F(x,t) = \int_0^t f(x,s) \, ds \).

We note that, if \( \mathcal{A}(x,t) \) is constant, e.g., \( \mathcal{A}(x,t) \equiv 1 \), and \( f(x,t) = \alpha(x)|u|^{q-2}u \) with \( \alpha \in L^\infty(\Omega) \), \( q \geq 1 \) and subcritical, then the equation in \((MP)\) reduces to the classical \( p \)-Laplacian problem

\[
- \Delta_p u = \alpha(x)|u|^{q-2}u \quad \text{in } \Omega 
\]

and the related functional

\[
J_0(u) = \frac{1}{p} \int_\Omega |\nabla u|^p \, dx - \frac{1}{q} \int_\Omega \alpha(x)|u|^q \, dx
\]

is defined in the whole Sobolev space \( W^{1,p}_0(\Omega) \).

On the other hand, if \( p \leq N \) and \( \mathcal{A}(x,t) \) depends on \( t \), even in the simplest case \( F(x,t) \equiv 0 \), with \( \mathcal{A}(x,t) \) smooth, bounded and far away from zero, functional \( J_\mathcal{A} \) is defined in \( W^{1,p}_0(\Omega) \) but is Gâteaux differentiable only along directions of the Banach space \( X = W^{1,p}_0(\Omega) \cap L^\infty(\Omega) \).

More in general, if \( \mathcal{A}(x,t,\xi) \) grows as \( |\xi|^p \) with respect to \( \xi \), where \( p > 1 \), suitable additional growth assumptions allow us to prove that problem \((GP)\) can be associated to the functional

\[
J(u) = \int_\Omega \mathcal{A}(x,u,\nabla u) \, dx - \int_\Omega F(x,u) \, dx
\]

whose natural domain \( D \) is a subset of \( W^{1,p}_0(\Omega) \) and contains \( X \).

In the beginning, the lack of regularity for \( J \) has been overcome by introducing suitable definitions of critical point (see, e.g., [1, 11] and also [2, 4] and references therein) or by using a suitable change of variable if \( \mathcal{A}(x,t) \) has a very particular form (see, e.g., [12, 16]). More recently, a different approach has been developed which exploits the interaction between two different norms on \( X \) (see [5]).

Following the ideas in [5], here we are able to prove that \( J \) is \( C^1 \) in \( X \) equipped with the “intersection norm” \( \| \cdot \|_X \) equal to the sum of the classical \( W^{1,p}_0 \)-norm, namely \( \| \cdot \|_W \), and the standard \( L^\infty \) one, namely \( \| \cdot \|_\infty \) (see Proposition 2.7), so problem \((GP)\) has a variational structure and its weak solutions are critical points of \( J \) in \( X \).

We recall that, looking for solutions of the classical \( p \)-Laplacian equation \((1.2)\), different variational arguments are required according to the growth exponent \( q \). In fact, in the sub-\( p \)-linear case \( 1 \leq q < p \) the related functional \( J_0 \) is bounded from below and minimization arguments can be used. On the contrary, in the super-\( p \)-linear case \( p < q < p^\ast \) (with \( p^\ast = \frac{pN}{N-p} \) if \( p < N \), \( p^\ast = +\infty \) otherwise), \( J_0 \) is
unbounded but, for example, the classical Ambrosetti–Rabinowitz Mountain Pass Theorem may apply (see, e.g., [13] and references therein). On the other hand, if \( p = q \) a careful analysis of the interaction of the coefficient \( \alpha(x) \) with the \( p \)-Laplacian spectrum is required in order to find critical points of \( J_0 \) (see, e.g., [3]).

Here, we deal with the existence of critical points of \( J \) in \( X \) when \( F(x,t) \) grows as \( |t|^q \), \( q \geq 1 \), and \( A(x,t,\xi) \) grows as \( |\xi|^p \) with respect to \( \xi \). In the corresponding super-\( p \)-linear case for problem \((GP)\), i.e. if \( p < q < p^* \), suitable hypotheses allow one to find a mountain pass critical point (see [5]). More recently, by strongly using the dependence of \( A(x,t,\xi) \) also on \( t \), namely requiring that it grows at least as \((1 + |t|^{ps})|\xi|^p \) with \( s > 0 \), if \( p < N \) in [7] we extend such a result to the range \( p^* \leq q < p^*(1 + s) \) (see also [2] with a different approach and [8, 10] for the related problem with lack of symmetry).

The aim of this paper is proving that, in the previous growth assumptions for \( A(x,t,\xi) \), \( J \) has a minimum critical point not only if \( 1 \leq q < p \), i.e. in the sub-\( p \)-linear case, but also for any \( p \leq q < p(1 + s) \) (see Sections 2 and 3).

Furthermore, in Section 4, again by minimization arguments, we prove the existence of a positive solution of problem \((GP)\) under stronger hypotheses (see [1] and [9] for the super-\( p \)-linear case).

We note that, in order to apply a Minimum Principle, a compactness assumption may be used, for example the classical Palais–Smale condition or its Cerami’s variant. Here, following [6], we consider a weaker version of the Cerami’s variant of the Palais–Smale condition (see Definition 3.1) and the related Minimum Principle (see Theorem 3.2).

### 2. Main theorem and variational principle.

From now on, let \( \Omega \subset \mathbb{R}^N \) be an open bounded domain, \( N \geq 2 \), and consider a function \( A : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R} \) with partial derivatives \( A_t(x,t,\xi) \) and \( a(x,t,\xi) \) according to the notation in (1.1).

Assume that a real number \( p > 1 \) and a radius \( R_0 \geq 1 \) exist such that the following assumptions hold:

\((H_0)\) \( A(x,t,\xi) \) is a \( C^1 \) Carathéodory function, i.e., \( A(t,\xi) : x \in \Omega \mapsto A(x,t,\xi) \in \mathbb{R} \) is measurable for all \((t,\xi) \in \mathbb{R} \times \mathbb{R}^N\), \( A(x,\cdot,\cdot) : (t,\xi) \in \mathbb{R} \times \mathbb{R}^N \mapsto A(x,t,\xi) \in \mathbb{R} \) is \( C^1 \) for a.e. \( x \in \Omega \);

\((H_1)\) some positive continuous functions \( \Phi_i, \phi_i : \mathbb{R} \rightarrow \mathbb{R} \), \( i \in \{1,2\} \), exist such that

\[
|A_t(x,t,\xi)| \leq \Phi_1(t) + \phi_1(t) |\xi|^p \quad \text{a.e. in } \Omega, \text{ for all } (t,\xi) \in \mathbb{R} \times \mathbb{R}^N,
\]

\[
|a(x,t,\xi)| \leq \Phi_2(t) + \phi_2(t) |\xi|^{p-1} \quad \text{a.e. in } \Omega, \text{ for all } (t,\xi) \in \mathbb{R} \times \mathbb{R}^N;
\]

\((H_2)\) some constants \( \alpha_1 > 0 \) and \( s \geq 0 \) exist such that

\[
A(x,t,\xi) \geq \alpha_1 (1 + |t|^{ps}) |\xi|^p \quad \text{a.e. in } \Omega \text{ if } |(t,\xi)| \geq R_0;
\]

\((H_3)\) a constant \( \eta_1 > 0 \) exists such that

\[
|A(x,t,\xi)| \leq \eta_1 \quad \text{a.e. in } \Omega \text{ if } |(t,\xi)| \leq R_0;
\]

\((H_4)\) a constant \( \eta_2 > 0 \) exists such that

\[
A(x,t,\xi) \leq \eta_2 a(x,t,\xi) \cdot \xi \quad \text{a.e. in } \Omega \text{ if } |(t,\xi)| \geq R_0;
\]

\((H_5)\) a constant \( \alpha_2 > 0 \) exists such that

\[
a(x,t,\xi) \cdot \xi \geq \alpha_2 |\xi|^p \quad \text{a.e. in } \Omega \text{ if } |(t,\xi)| \leq R_0;
\]

\((H_6)\) a constant \( \alpha_3 > 0 \) exists such that

\[
a(x,t,\xi) \cdot \xi + A_t(x,t,\xi)t \geq \alpha_3 a(x,t,\xi) \cdot \xi \quad \text{a.e. in } \Omega \text{ if } |(t,\xi)| \geq R_0;
\]

\( i = 1,2 \), exist such that
(H₇) for all $\xi, \xi^* \in \mathbb{R}^N$, $\xi \neq \xi^*$, it is

$$|a(x, t, \xi) - a(x, t, \xi^*)| : |\xi - \xi^*| > 0 \quad \text{a.e. in } \Omega, \text{ for all } t \in \mathbb{R}.$$ 

Remark 2.1. From (H₀), (H₂) and (H₃) a constant $\eta_3 > 0$ exists such that

$$A(x, t, \xi) \geq \alpha_1 (1 + |t|^p) |\xi|^p - \eta_3$$

(2.1)
a.e. in $\Omega$, for all $(t, \xi) \in \mathbb{R} \times \mathbb{R}^N$.

On the other hand, from (H₀), (H₁), (H₃) and (H₄), direct computations imply that

$$A(x, t, \xi) \leq \eta_1 + \eta_2 \Phi_2(t) + \eta_2 (\Phi_2(t) + \phi_2(t)) |\xi|^p$$

a.e. in $\Omega$, for all $(t, \xi) \in \mathbb{R} \times \mathbb{R}^N$.

Remark 2.2. From (H₀), (H₂), (H₄), (H₅) and direct computations, it follows that

$$a(x, t, \xi) \cdot \xi \geq \alpha_4 (1 + |t|^p) |\xi|^p \quad \text{a.e. in } \Omega, \text{ for all } (t, \xi) \in \mathbb{R} \times \mathbb{R}^N,$$

(2.2)

with $0 < \alpha_4 \leq \min \{ \frac{\alpha_3}{1 + \eta_0}, \frac{\alpha_4}{n} \}$.

On the other hand, assume that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the following hypotheses:

(ⅰ) $f(x, t)$ is a Carathéodory function, i.e.,

$$f(\cdot, t) : x \in \Omega \mapsto f(x, t) \in \mathbb{R} \text{ is measurable for all } t \in \mathbb{R},$$

$$f(x, \cdot) : t \in \mathbb{R} \mapsto f(x, t) \in \mathbb{R} \text{ is continuous for a.e. } x \in \Omega;$$

(ⅱ) $a_1, a_2 > 0$ and $q \geq 1$ exist such that

$$|f(x, t)| \leq a_1 + a_2 |t|^{q - 1} \quad \text{a.e. in } \Omega, \text{ for all } t \in \mathbb{R}.$$

Remark 2.3. From (ⅰ)–(ⅱ) it follows that $a_3, a_4 > 0$ exist such that

$$|F(x, t)| \leq a_3 + a_4 |t|^q \quad \text{a.e in } \Omega, \text{ for all } t \in \mathbb{R}.$$ 

Finally, we can state our first main theorem.

Theorem 2.4. Assume that (H₀)–(H₇) and (h₀)–(h₁) hold. If

$$1 \leq q < p(1 + s),$$

(2.4)

then problem (GP) admits at least a weak bounded solution.

If we consider the model problem (MP), the assumptions of Theorem 2.4 can be simplified as follows.

Corollary 2.5. Let $u$ consider $A(x, t, \xi) = \frac{1}{p} A(x, t) |\xi|^p$ with $A : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfying the following hypotheses:

(H₀) $A(x, t)$ is a $C^1$ Carathéodory function;

(H₁) some positive continuous functions $\phi_i^* : \mathbb{R} \to \mathbb{R}$, $i \in \{1, 2\}$, exist such that

$$|A(x, t)| \leq \phi_1(t) \quad \text{a.e. in } \Omega, \text{ for all } t \in \mathbb{R},$$

$$|A_i(x, t)| \leq \phi_2(t) \quad \text{a.e. in } \Omega, \text{ for all } t \in \mathbb{R};$$

(H₂) some constants $\alpha_i^* > 0$ and $s \geq 0$ exist such that

$$A(x, t) \geq \alpha_1^* (1 + |t|^p) \quad \text{a.e. in } \Omega, \text{ for all } t \in \mathbb{R};$$

(H₃) some constants $\alpha_i^* > 0$ and $R_0^* > 0$ exist such that

$$pA(x, t) + A_i(x, t) t \geq \alpha_3^* A(x, t) \quad \text{a.e. in } \Omega \text{ if } |t| \geq R_0^*.$$

If (h₀)–(h₁) and (2.4) hold, then (MP) admits at least a weak bounded solution.

In order to state the variational formulation of problem (GP), we denote by:
• $L^q(\Omega)$ the Lebesgue space with norm $|u|_q = (\int_\Omega |u|^q \, dx)^{1/q}$ if $1 \leq q < +\infty$;
• $L^\infty(\Omega)$ the space of Lebesgue–measurable and essentially bounded functions $u : \Omega \rightarrow \mathbb{R}$ with norm $|u|_\infty = \text{ess sup} |u|_{\Omega}$;
• $W^{1,p}_0(\Omega)$ the classical Sobolev space with norm $\|u\|_W = |\nabla u|_p$ if $p \geq 1$.

We set
$$X := W^{1,p}_0(\Omega) \cap L^\infty(\Omega), \quad \|u\|_X = \|u\|_W + |u|_\infty \quad (2.5)$$
(here and in the following, $| \cdot |$ will denote the standard norm on any Euclidean space as the dimension of the considered vector is clear and no ambiguity arises).

Moreover, from the Sobolev Embedding Theorem, for any $r \in [1,p^*]$ a constant $\sigma_r > 0$ exists, such that
$$|u|_r \leq \sigma_r \|u\|_W \quad \text{for all } u \in W^{1,p}_0(\Omega)$$
and the embedding $W^{1,p}_0(\Omega) \hookrightarrow L^r(\Omega)$ is compact.

By definition, $X \hookrightarrow W^{1,p}_0(\Omega)$ and $X \hookrightarrow L^\infty(\Omega)$ with continuous embeddings. We note that $X = W^{1,p}_0(\Omega)$ if $p > N$, as $W^{1,p}_0(\Omega) \hookrightarrow L^\infty(\Omega)$.

**Remark 2.6.** Taking $u \in X$, it results $|u|^s u \in W^{1,p}_0(\Omega)$ as
$$|\nabla (|u|^s u)|^p = (1+s)^p |u|^{ps} |\nabla u|^p \quad \text{a.e. in } \Omega. \quad (2.6)$$

Now, we consider the functional $J : X \rightarrow \mathbb{R}$ defined as in (1.3).
Taking any $u, v \in X$, by direct computations it follows that its Gâteaux differential in $u$ along the direction $v$ is
$$\langle dJ(u), v \rangle = \int_\Omega (a(x,u,\nabla u) \cdot \nabla v + A_t(x,u,\nabla u)v) \, dx - \int_\Omega f(x,u)v \, dx. \quad (2.7)$$

The regularity of $J$ in $X$ is stated in the following proposition.

**Proposition 2.7.** Let us assume that conditions $(H_0)$–$(H_4)$ and $(h_0)$–$(h_1)$ hold. Then, if $(u_n)_n \subset X$, $u \in X$ are such that
$$\|u_n - u\|_W \to 0, \quad u_n \to u \text{ a.e. in } \Omega \quad \text{if } n \to +\infty,$$
$$M > 0 \text{ exists so that } |u_n|_\infty \leq M \text{ for all } n \in \mathbb{N},$$
then
$$J(u_n) \to J(u) \quad \text{and} \quad \|dJ(u_n) - dJ(u)\|_{X'} \to 0 \quad \text{if } n \to +\infty.$$  
Hence, $J$ is a $C^1$ functional on $X$ with Fréchet differential operator as in (2.7).

**Proof.** The proof follows from Remark 2.1 and [7, Proposition 3.2].

**Remark 2.8.** From Proposition 2.7 it follows that the research of weak bounded solutions of $(GP)$ is reduced to the study of critical points of $J$ in $X$ if conditions $(H_0)$–$(H_4)$ and $(h_0)$–$(h_1)$ hold.

3. **Existence result.** Firstly, we recall some abstract tools.

We denote by $(X, \| \cdot \|_X)$ a Banach space with dual space $(X', \| \cdot \|_{X'} )$, by $(W, \| \cdot \|_W)$ another Banach space such that $X \hookrightarrow W$ continuously, and by $J : X \rightarrow \mathbb{R}$ a given $C^1$ functional.

Then, the following weaker version of the Cerami’s variant of Palais–Smale condition can be introduced.
Definition 3.1. The functional $J$ satisfies the weak Cerami–Palais–Smale condition at level $\beta$ ($\beta \in \mathbb{R}$), briefly $(wCPS)_{\beta}$ condition, if taking any sequence $(u_n)_n \subset X$ such that
\[
\lim_{n \to +\infty} J(u_n) = \beta \quad \text{and} \quad \lim_{n \to +\infty} \|dJ(u_n)\|_{X'} (1 + \|u_n\|_{X}) = 0,
\]
then a point $u \in X$ exists, such that
\begin{enumerate}[(i)]
    
    \item $\lim_{n \to +\infty} \|u_n - u\|_W = 0$ \quad (up to subsequences),
    
    \item $J(u) = \beta$, $dJ(u) = 0$.
\end{enumerate}

Condition $(wCPS)_{\beta}$ implies that the set of critical points of $J$ at level $\beta$ is compact with respect to $\|\cdot\|_W$ and this weaker “compactness” assumption allows one to prove a Deformation Lemma and then some abstract critical point theorems (see [6]). In particular, the following Minimum Principle can be stated (for the proof, see [6, Theorem 1.6]).

Theorem 3.2 (Minimum Principle). If $J \in C^1(X, \mathbb{R})$ is bounded from below in $X$ and $(wCPS)_{\beta}$ holds at level $\beta = \inf_X J$, then $J$ attains its infimum, i.e., $u_0 \in X$ exists such that $J(u_0) = \beta$.

From now on, let $(X, \|\cdot\|_X)$ be defined as in (2.5), $W = W^{1,p}_0(\Omega)$ and $J = J$ as in (1.3).

Lemma 3.3. Assume that hypotheses $(H_0)$, $(H_2)$–$(H_3)$ and $(h_0)$–$(h_1)$ are satisfied. Then, if also (2.4) holds, some positive constants $c_1$ and $c_2$ exist such that
\[
J(u) \geq \alpha_1 \|u\|_W^p + \frac{\alpha_1}{(1 + s)^p} |||u|_s u||_W^p - c_1 |||u|_s u||_W^{\frac{p}{1 + s}} - c_2.
\]  
Hence,
\[
\inf_{u \in X} J(u) > -\infty.
\]

Proof. Taking $u \in X$, from (2.1), (2.3) and (2.6) a constant $b_1 > 0$ exists such that
\[
J(u) \geq \alpha_1 \int_\Omega |\nabla u|^p dx + \frac{\alpha_1}{(1 + s)^p} \int_\Omega |\nabla (|u|_s u)|^p dx - a_4 \int_\Omega |||u|_s u||_W^\frac{p}{1 + s} dx - b_1;
\]
then (3.1) follows from (2.4) and the Sobolev Embedding Theorem.

Now, we note that (3.1) implies
\[
J(u) \geq \frac{\alpha_1}{(1 + s)^p} |||u|_s u||_W^p - c_1 |||u|_s u||_W^{\frac{p}{1 + s}} - c_2,
\]
then (3.2) follows from (2.4).

Remark 3.4. In the hypotheses of Lemma 3.3, from (2.4), (3.1) and direct computations it follows that
\[
J(u) \geq \alpha_1 \|u\|_W^p + c_3 |||u|_s u||_W^p - c_4
\]
for suitable constants $c_3$, $c_4 > 0$.

In order to prove that functional $J$ satisfies the $(wCPS)$ condition in $\mathbb{R}$, we need the following results.
Lemma 3.5. Fix $s \geq 0$ and let $(u_n)_n \subset X$ be a sequence such that
\[
\left( \int_{\Omega} (1 + |u_n|^p) |\nabla u_n|^p \, dx \right)_n \text{ is bounded. (3.3)}
\]
Then, $u \in W^{1,p}_0(\Omega)$ exists such that $|u|^s u \in W^{1,p}_0(\Omega)$, too, and, up to subsequences, if $n \to +\infty$ we have
\[
\begin{align*}
& u_n \rightharpoonup u \text{ weakly in } W^{1,p}_0(\Omega), \\
& |u_n|^s u_n \rightharpoonup |u|^s u \text{ weakly in } W^{1,p}_0(\Omega), \\
& u_n \to u \text{ a.e. in } \Omega, \\
& u_n \to u \text{ strongly in } L^r(\Omega) \text{ for each } r \in [1, p^+(s + 1)]. (3.7)
\end{align*}
\]

Proof. For the proof, see [7, Lemma 3.8]. □

Lemma 3.6. Let $p, r$ be so that $1 < p \leq r < p^*, p < N$ and take $v \in W^{1,p}_0(\Omega)$. Assume that $\bar{a} > 0$ and $k_0 \in \mathbb{N}$ exist such that the inequality
\[
\int_{\Omega_k^+} |\nabla v|^p \, dx \leq \bar{a} \left( |\Omega_k^+| + \int_{\Omega_k^+} v^r \, dx \right)
\]
holds for all $k \geq k_0$, with $\Omega_k^+ = \{ x \in \Omega : v(x) > k \}$. Then, ess sup $v$ is bounded from above by a positive constant which can be chosen so that it depends only on $|\Omega|, N, p, r, \bar{a}, k_0, |v|_{p^*}$.

Proof. For the proof, see [14, Theorem II.5.1] stated in a more general setting. □

Proposition 3.7. In the hypotheses of Theorem 2.4 the functional $J$ satisfies the (wCPS) condition in $\mathbb{R}$.

Proof. Let $\beta \in \mathbb{R}$ be fixed and consider a sequence $(u_n)_n \subset X$ such that
\[
\begin{align*}
J(u_n) \to \beta \quad \text{and} \quad \|dJ(u_n)\|_{X'}(1 + \|u_n\|_X) & \to 0. \quad (3.8)
\end{align*}
\]
From (2.4), (2.6), Remark 3.4 and assumption (3.8) it follows that (3.3) holds; thus, from Lemma 3.5 a function $u \in W^{1,p}_0(\Omega)$ exists such that $|u|^s u \in W^{1,p}_0(\Omega)$ and (3.4)–(3.7) are satisfied, up to subsequences.

Now, from (2.2), (2.4) and Lemma 3.6, reasoning as in the proof of Step 2 in [7, Proposition 3.10] it follows that $u \in L^\infty(\Omega)$; hence, $u \in X$.

The last part of the proof can be obtained reasoning as in the proof of Steps 3-5 in [5, Proposition 4.6] but using (2.4) and (3.7) instead of [5, (4.15)]. □

Proof of Theorem 2.4. From Proposition 2.7, Lemma 3.3 and Proposition 3.7 we have that Theorem 3.2 applies and $J$ admits a minimum point in $X$; hence, the proof follows from Remark 2.8. □

Proof of Corollary 2.5. It is enough to note that, if $A(x, t)$ verifies $(H_0)$–$(H_3)$, then the function $A(x, t, \xi) = \frac{1}{p} A(x, t) |\xi|^p$ satisfies $(H_0)$–$(H_7)$. □

4. Existence of a positive solution. In this section we prove the existence of a weak bounded positive solution of problem (GP), i.e., a nontrivial bounded weak solution of problem
\[
(GP)_+ \quad \begin{cases}
- \text{div}(a(x, u, \nabla u)) + A_t(x, u, \nabla u) = f(x, u) & \text{in } \Omega, \\
u \geq 0 & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
\]
To this aim, we replace \((H_1)\) and \((H_3)\) with the following stronger conditions:\n
\[(H_1')\] some positive continuous functions \(\phi_1, \phi_2, \Phi_2 : \mathbb{R} \to \mathbb{R}\) exist such that
\[
|A_t(x, t, \xi)| \leq \phi_1(t) \, |\xi|^p \quad \text{a.e. in } \Omega, \text{ for all } (t, \xi) \in \mathbb{R} \times \mathbb{R}^N,
\]
\[
|a(x, t, \xi)| \leq \Phi_2(t) + \phi_2(t) \, |\xi|^{p-1} \quad \text{a.e. in } \Omega, \text{ for all } (t, \xi) \in \mathbb{R} \times \mathbb{R}^N;
\]
\[(H_3')\] a constant \(\eta_4 > 0\) exists such that
\[
|A(x, t, \xi)| \leq \eta_4 |\xi|^p \quad \text{a.e. in } \Omega \text{ if } |t| \leq R_0 \text{ for all } \xi \in \mathbb{R}^N,
\]
with \(R_0\) as in \((H_2)\).

On the other hand, we assume that \(f(x, t)\) satisfies the following hypotheses:
\[(h_0') f(x, t)\) is a Carathéodory function such that \(f(x, 0) = 0\) for a.e. \(x \in \Omega;\)
\[(h_1') a_1, a_2 > 0 \text{ and } q \geq 1 \text{ exist such that}\]
\[
|f(x, t)| \leq a_1 + a_2 t^{p-1} \quad \text{a.e. in } \Omega, \text{ for all } t \geq 0;
\]
\[(h_2') \text{ a constant } \lambda > p \eta_4 \lambda_1 \text{ exists such that}\]
\[
\lim_{t \to 0^+} \frac{f(x, t)}{t^{p-1}} = \lambda \quad \text{uniformly a.e. in } \Omega,
\]
where \(\eta_4\) is as in \((H_4')\) and \(\lambda_1\) is the smallest eigenvalue of \(-\Delta_p\) in \(W_0^{1, p}(\Omega)\).

**Theorem 4.1.** Assume that \((H_0), (H_1'), (H_2), (H_4'), (H_5)\) and \((h_0'), (h_1'), (h_2')\) are verified. If \((2.4)\) holds, then problem \((GP)_+ \) admits at least a weak bounded solution \(u^*\). Moreover, if also \((h_2')\) is satisfied, then \(u^* \neq 0\).

**Corollary 4.2.** In the assumptions of Theorem 4.1 but replacing \((H_1')\) with the stronger condition
\[(H_1'')\] some positive continuous functions \(\phi_1, \phi_2, \Phi_2 : \mathbb{R} \to \mathbb{R}\) exist such that
\[
|A_t(x, t, \xi)| \leq \phi_1(t) \, |\xi|^p \quad \text{a.e. in } \Omega, \text{ for all } (t, \xi) \in \mathbb{R} \times \mathbb{R}^N,
\]
\[
|a(x, t, \xi)| \leq \Phi_2(t) |t|^p + \phi_2(t) \, |\xi|^{p-1} \quad \text{a.e. in } \Omega, \text{ for all } (t, \xi) \in \mathbb{R} \times \mathbb{R}^N;
\]
then problem \((GP)_+ \) has at least one weak bounded solution \(u^* > 0\) in \(\Omega\).

In the model case \(A(x, t, \xi) = \frac{1}{p} |A(x, t)| |\xi|^p\), problem \((GP)_+ \) becomes
\[
(MP)_+ \quad \begin{cases} 
- \text{div} (A(x, u) |\nabla u|^{p-2} \nabla u) + \frac{1}{p} A_t(x, u) |\nabla u|^p = f(x, u) & \text{in } \Omega, \\
\quad u \geq 0 & \text{in } \Omega, \\
\quad u = 0 & \text{on } \partial \Omega.
\end{cases}
\]
Thus, Theorem 4.1 and Corollary 4.2 reduce to the following statement.

**Corollary 4.3.** Let \(A : \Omega \times \mathbb{R} \to \mathbb{R}\) be such that assumptions \((H_0)-(H_3)\) are satisfied. If \((h_0'), (h_1')-(h_2')\) and \((2.4)\) hold, then problem \((MP)_+ \) has at least one weak bounded solution \(u^* > 0\) in \(\Omega\).

**Remark 4.4.** In general, under the hypotheses of Theorem 4.1 the constant function \(u \equiv 0\) does not solve problem \((GP)_+ \). On the contrary, it is always a solution in the stronger assumption \((H_1'')\) required in Corollary 4.2.

From now on, assume that hypotheses \((H_0), (H_1'), (H_2), (H_4'), (H_5)\) and \((h_0'), (h_1'), (h_2')\) are verified.
In order to prove Theorem 4.1, we introduce the new function \( f_+ : \Omega \times \mathbb{R} \to \mathbb{R} \) defined as

\[
f_+(x,t) = \begin{cases} f(x,t) & \text{for a.e. } x \in \Omega \text{ and all } t \geq 0, \\ 0 & \text{for a.e. } x \in \Omega \text{ and all } t < 0,
\end{cases}
\]

and the related primitive

\[
F_+(x,t) = \int_0^t f_+(x,s) ds = \begin{cases} F(x,t) & \text{for a.e. } x \in \Omega \text{ and all } t \geq 0, \\ 0 & \text{for a.e. } x \in \Omega \text{ and all } t < 0.
\end{cases}
\]

**Remark 4.5.** By assumptions \((h_0^+), (h_1^+)\) it follows that \( f_+(x,t) \) satisfies conditions \((h_0)\) and \((h_1)\).

From Remark 4.5 and Proposition 2.7 the corresponding functional

\[
\mathcal{J}_+(u) = \int_{\Omega} A(x,u,\nabla u) dx - \int_{\Omega} F_+(x,u) dx
\]

is of class \( C^1 \) on the Banach space \( X \) in (2.5), where for any \( u, v \in X \) it is

\[
\langle d\mathcal{J}_+(u), v \rangle = \int_{\Omega} (a(x,u,\nabla u) \cdot \nabla v + A_i(x,u,\nabla u)v) dx - \int_{\Omega} f_+(x,u)v dx.
\]

**Proposition 4.6.** If \( u \in X \) is a critical point of \( \mathcal{J}_+ \), then \( u \geq 0 \) a.e. in \( \Omega \). Hence, \( \mathcal{J}(u) = \mathcal{J}_+(u) \) and \( d\mathcal{J}(u) = 0 \).

**Proof.** For the proof see [1, Lemma 1.3] and [9, Proposition 4.5]. \( \square \)

Now, we recall a Harnack type inequality for weak solutions of \( p \)-Laplacian type equations (see [17, Theorem 1.1]).

**Lemma 4.7.** Let \( u \in W^{1,p}_0(\Omega) \) be a weak solution of the equation

\[
- \text{div}(a(x,u,\nabla u)) = h(x,u,\nabla u) \quad \text{in a cube } K(3r) \subset \Omega. \tag{4.1}
\]

Assume that \( M > 0 \) exists such that \( 0 \leq u(x) < M \) for all \( x \in K(3r) \). If (2.2) holds and two positive continuous functions \( \phi, \Phi : \mathbb{R} \to \mathbb{R} \) and some positive constants \( d_i \) exist such that

\[
|a(x,t,\xi)| \leq \Phi(t)|\xi|^p + \phi(t)|\xi|^{p-1} \quad \text{a.e. in } \Omega, \text{ for all } (t,\xi) \in \mathbb{R} \times \mathbb{R}^N,
\]

\[
|h(x,t,\xi)| \leq d_1|\xi|^p + d_2|\xi|^{p-1} + d_3|t|^{p-1} \quad \text{a.e. in } \Omega, \text{ for all } (t,\xi) \in ]-M,M[ \times \mathbb{R}^N,
\]

then

\[
\max_{x \in K(r)} u(x) \leq C \min_{x \in K(r)} u(x),
\]

where \( C \) depends only on \( p, N, M \) and the constants which appear in the hypotheses.

**Remark 4.8.** A statement similar to Lemma 4.7 holds for any weak bounded solution \( u \in W^{1,p}_0(\Omega) \) of (4.1) which is \( u \leq 0 \) a.e. in \( \Omega \).

Now, we are ready to prove Theorem 4.1.

**Proof of Theorem 4.1.** From Remark 4.5, all the assumptions of Theorem 2.4 are satisfied. Hence, Lemma 3.3 and Propositions 2.7 and 3.7 apply to \( \mathcal{J}_+ \) in \( X \); thus from Theorem 3.2 functional \( \mathcal{J}_+ \) has a minimum point \( u^* \) in \( X \).

Then, Proposition 4.6 implies \( u^* \geq 0 \) a.e. in \( \Omega \), so it is a weak solution of \((GP)_+\).

Now, in order to prove the second part of the statement of Theorem 4.1, we claim that

\[
\mathcal{J}_+(u^*) = \min_{u \in X} \mathcal{J}_+(u) < 0. \tag{4.2}
\]
In fact, from \((h_1^+)\), taking \(\tilde{\lambda} \) such that \(p\eta_4\lambda_1 < \tilde{\lambda} < \lambda\), a constant \(\delta > 0\) exists so that
\[ f(x, t) > \tilde{\lambda} t^{p-1} \quad \text{a.e. in } \Omega, \text{ for all } 0 < t \leq \delta; \]
whence,
\[ F(x, t) > \frac{\tilde{\lambda}}{p} t^p \quad \text{a.e. in } \Omega, \text{ for all } 0 < t \leq \delta. \quad (4.3) \]
Now, let \(\varphi_1 \in X\) be the unique eigenfunction associated to \(\lambda_1\) such that \(\varphi_1 > 0\) a.e. in \(\Omega\) and \(|\varphi_1|_p = 1\), hence \(\|\varphi_1\|_W^p = \lambda_1\) (cf., e.g., [15]). Fixing \(\tau > 0\) such that \(\tau \leq \frac{1}{|\varphi_1|_{H^1}} \min\{R_0, \delta\}\), it is \(0 < \tau \varphi_1(x) \leq R_0\) and \(0 < \tau \varphi_1(x) \leq \delta\) a.e. in \(\Omega\). Then, from \((H'_2)\) and \((4.3)\) it results
\[ \mathcal{J}_+(\tau \varphi_1) \leq \left( \eta_4 \lambda_1 - \frac{\tilde{\lambda}}{p} \right) \tau^p \]
which is strictly negative from the choice of \(\tilde{\lambda}\) and so \((4.2)\) holds. Then, since \((H'_4)\) and the definition of \(F_+(x, t)\) imply \(\mathcal{J}_+(0) = 0\), it has to be \(u^* \neq 0\). \(\square\)

**Proof of Corollary 4.2.** Without loss of generality, assume that \(\Omega\) is a connected bounded domain in \(\mathbb{R}^N\). Then, from Theorem 4.1 a nontrivial weak solution \(u^* \in X\) of \((GP)_+\) exists, i.e., \(u^* \in X\) is a weak solution of
\[-\text{div}(a(x, u, \nabla u)) + A_i(x, u, \nabla u) = f_+(x, u) \quad \text{in } \Omega, \quad u^* \geq 0 \text{ in } \Omega.\]
Now, taking \(h(x, t, \xi) = -A_i(x, t, \xi) + f_+(x, t)\) and \(M > |u^*|_\infty\), from \((H'_1), (h_1^+), (h_2^+)\) and direct computations it follows that Lemma 4.7 applies and standard arguments imply \(u^* > 0\) in \(\Omega\). \(\square\)

**Proof of Corollary 4.3.** It is enough to note that \((H_0)-(H_3)\) imply not only the hypotheses \((H_0)-(H_7)\) but also \((H'_0)\) and \((H'_3)\). \(\square\)

**Remark 4.9.** By replacing assumptions \((h_1^+)\) and \((h_2^+)\) with the corresponding conditions
\((h_1)\) \(a_1, a_2 > 0\) and \(q \geq 1\) exist such that
\[ |f(x, t)| \leq a_1 + a_2|t|^{q-1} \quad \text{a.e. in } \Omega, \text{ for all } t \leq 0; \]
\((h_2)\) a constant \(\lambda > p\eta_4\lambda_1\) exists such that
\[ \lim_{t \to 0^-} \frac{f(x, t)}{|t|^{p-2}t} = \lambda \quad \text{uniformly a.e. in } \Omega; \]
and by using arguments similar to those ones in the proof of Theorem 4.1, respectively of Corollary 4.2, we are able to prove the existence of at least one nontrivial weak bounded solution of problem \((GP)\) which is negative, respectively strictly negative, in \(\Omega\).

**Remark 4.10.** In the hypotheses \((H_0), (H'_0), (H_2), (H'_2), (H_4)-(H_7), (2.4), (h'_0), (h_1)\) and both \((h_2^+)\) and \((h_2^-)\), i.e.,
\((h_2)\) a constant \(\lambda > p\eta_4\lambda_1\) exists such that
\[ \lim_{t \to 0} \frac{f(x, t)}{|t|^{p-2}t} = \lambda \quad \text{uniformly a.e. in } \Omega, \]
then from Corollary 4.2 and Remark 4.9 it follows that \(\mathcal{J}\) has two critical points \(u^+, u^- \in X\), such that \(u^+ > 0\) and \(u^- < 0\) a.e. in \(\Omega\), which are both local minimum points.
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