Asymptotic behavior of quantum walks on the line

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The notion of quantum walks, often called discrete time quantum random walks, was introduced by Aharonov-Davidovich-Zagury ([ADZ]) in 1993 as a quantum analogue of the classical random walks, and re-discovered in the area of computer science. In particular, Ambainis-Kempe-Rivosh ([AKR]) utilized two-dimensional quantum walks to improve Grover’s quantum search algorithm. In the talk, various local asymptotic formulas of transition probabilities of quantum walks on the one-dimensional integer lattice, obtained in [ST], will be given. In the present article, we just mention one of the formulas, which is a limit formula in a large deviation regime. To be precise, let us give a definition of quantum walks on the one-dimensional integer lattice. The quantum walks we consider in the talk is defined by a (special) unitary matrix,

$$A = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}, \quad a, b \in \mathbb{C}, \quad |a|^2 + |b|^2 = 1,$$

and its decomposition,

$$A = P + Q, \quad P = \begin{pmatrix} a & 0 \\ -\bar{b} & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & b \\ 0 & \bar{a} \end{pmatrix}.$$

Let $\ell^2(\mathbb{Z}) \otimes \mathbb{C}^2$ be the Hilbert space of square summable functions on $\mathbb{Z}$ with values in $\mathbb{C}^2$ whose inner product is given by

$$\langle f, g \rangle = \sum_{x \in \mathbb{Z}} \langle f(x), g(x) \rangle_{\mathbb{C}^2},$$

where $\langle \cdot, \cdot \rangle_{\mathbb{C}^2}$ denotes the standard inner product on $\mathbb{C}^2$. For any $u \in \mathbb{C}^2$ and $x \in \mathbb{Z}$, define $\delta_x \otimes u \in \ell^2(\mathbb{Z}) \otimes \mathbb{C}^2$ by

$$\langle \delta_x \otimes u \rangle(y) = \begin{cases} u & (y = x), \\ 0 & (y \neq x). \end{cases}$$

Then, the vectors, $\delta_x \otimes e_i$ ($i = 1, 2, x \in \mathbb{Z}$), where $\{e_1, e_2\}$ is the standard orthonormal basis in $\mathbb{C}^2$, form an orthonormal basis of $\ell^2(\mathbb{Z}) \otimes \mathbb{C}^2$. The unitary evolution, $U$, of the quantum walks on $\mathbb{Z}$ is a unitary operator on $\ell^2(\mathbb{Z}) \otimes \mathbb{C}^2$ defined as

$$U = P\tau^{-1} + Q\tau,$$

where $\tau$ is the shift operator on $\ell^2(\mathbb{Z}) \otimes \mathbb{C}^2$ defined by $\tau(\delta_x \otimes u) = \delta_{x+1} \otimes u$. The operator $U$ is indeed a unitary operator, and hence the function

$$p_n(\varphi; x) = \|U^n(\delta_0 \otimes \varphi)(x)\|_{\mathbb{C}^2}^2, \quad x \in \mathbb{Z}$$

defines a probability distribution on $\mathbb{Z}$ supported on $[-n, n]$ for any unit vector $\varphi$ in $\mathbb{C}^2$ and positive integer $n$, which we call the transition probability of the quantum walk $U$. The behavior of $p_n(\varphi; x)$ as $n \to \infty$ is one of main topics in the study of quantum walks. Indeed, as the following Figure 1 shows, it is drastically different from the behavior of transition probabilities of classical random walks. In Figure 1, the ‘wall’ is located at $x/n \sim \pm|a|$, where $a$ is a component of the given unitary matrix $A$. The behavior of $p_n(\varphi; x)$ heavily depends on the ‘normalized’ position $x/n$ according as

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1Figure 1 is due to Dr. Takuya Machida in Meiji University.
(1) \( x/n \) is inside the interval, \((-|a|, |a|)\),

(2) \( x/n \) stays around the ‘wall’, say, \( x/n \sim \pm |a| \), or

(3) \( x/n \) is outside the interval, say, \( |x/n| > |a| \).

Our analysis in [ST] gives precise asymptotic formulas in each regime (1) – (3). For instance, a corollary to our results is stated as follows.

**Corollary** Let \( \xi \in \mathbb{R} \) satisfy \( |\xi| < 1 \). Suppose that a sequence of integers, \( \{x_n\} \), satisfies

\[
x_n = n\xi + O(1) \quad (n \to \infty).
\]

If \( p_n(\varphi; x_n) \neq 0 \) for every sufficiently large \( n \), we have the following limit formula of the large deviation type.

\[
\lim_{n \to \infty} \frac{1}{n} p_n(\varphi; x_n) = -H_Q(\xi),
\]

where the function \( H_Q(\xi) \) is given by

\[
H_Q(\xi) = 2|\xi| \log \left( |b||\xi| + \sqrt{\xi^2 - |a|^2} \right) - 2 \log \left( |b| + \sqrt{\xi^2 - |a|^2} \right) + (1 - |\xi|) \log(1 - \xi^2) - 2|\xi| \log |a|.
\]

In the talk, after the explanation of backgrounds, properties and known results, such as a weak limit formula due to Konno ([K]), on the quantum walks on \( \mathbb{Z} \) comparing with classical random walks, our main results on the asymptotic formulas of \( p_n(\varphi; x) \) are introduced. According to our results, the asymptotic behavior of \( p_n(\varphi; x) \) has indeed a quantum mechanical nature. The resemblance of the asymptotic behavior of \( p_n(\varphi; x) \) and that of the Hermite functions will be pointed out by introducing the Plancherel-Rotach formula on asymptotic behavior of the Hermite functions.

**References**

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