On stochastic heat equation with measure initial data

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Abstract

The aim of this short note is to obtain the existence, uniqueness and moment upper bounds of the solution to a stochastic heat equation with measure initial data, without using the iteration method in [1, 2, 3].

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1 Introduction

Consider the stochastic heat equation

\[ \frac{\partial u}{\partial t} = Lu + b(u) + \sigma(u) \dot{W} \]  

for \((t, x) \in (0, \infty) \times \mathbb{R}^d (d \geq 1)\) where \(L\) is the generator of a Lévy process \(X = \{X_t\}_{t \geq 0}.\) \(\dot{W}\) is a centered Gaussian noise with covariance formally given by

\[ \mathbb{E}(\dot{W}(t, x)\dot{W}(s, y)) = \delta(s - t)f(x - y), \]

where \(f\) is some nonnegative and nonnegative definite function whose Fourier transform is denoted by

\[ \hat{f}(\xi) := \int_{\mathbb{R}^d} f(x)e^{-ix\xi}dx \]

in distributional sense, and \(\delta\) denotes the Dirac delta function at 0. For some technical reasons, we will assume that \(f\) is lower semicontinuous (see Lemma 4 below).

Let \(\Phi\) be the Lévy exponent of \(X_t\), we will assume that

\[ \exp(-\text{Re}\Phi) \in L^t(\mathbb{R}^d) \text{ for all } t > 0. \]  

Thus according to Proposition 2.1 in [5], \(X_t\) has a transition function \(p_t(x)\) and we can (and will) find a version of \(p_t(x)\) which is continuous on \((0, \infty) \times \mathbb{R}^d\) and uniformly continuous for all \((t, x) \in [\eta, \infty) \times \mathbb{R}^d\) for every \(\eta > 0\), and that \(p_t\) vanishes at infinity for all \(t > 0\).
The initial condition $u(0, \cdot)$ is assumed to be a (positive) measure $\mu(\cdot)$ such that
\[
\int_{\mathbb{R}^d} p_t(x - y) \mu(dy) < \infty \quad \text{for all } t > 0 \text{ and } x \in \mathbb{R}^d.
\]
(3)

To avoid trivialities, we assume that $\mu(\cdot) \neq 0$.

Using iteration method, the existence, uniqueness and some moment bounds of the solution have been obtained in [1, 2, 3] for the case $b \equiv 0$ and for some specific choice of $L$. However, these approaches rely on the structure (or asymptotic structure) of $p_t(x)$. In this article, we will study the equation (1) with also a Lipschitz drift term $b$ and establish the existence, uniqueness and $p$-th moment upper bound, without using the iteration method in [1, 2, 3], also, our criteria only need some integrability of the Lévy exponent.

To state the result, let us recall that by a solution $u$ to (1) we mean a mild solution. That is, (i) $u$ is a predictable random field on a complete probability space $(\Omega, \mathcal{F}, P)$, with respect to the Brownian filtration generated by the cylindrical Brownian motion defined by $B_t(\phi) := \int_{[0,t] \times \mathbb{R}^d} \phi(y) W(ds, dy)$, for all $t \geq 0$ and measurable $\phi : \mathbb{R}^d \to \mathbb{R}$ such that \( \int_{\mathbb{R}^d \times \mathbb{R}^d} \phi(y) \phi(z) f(y - z) dydz < \infty \); and (ii) for any $(t, x) \in (0, \infty) \times \mathbb{R}^d$, the following equation holds a.s.
\[
\begin{align*}
\int_{\mathbb{R}^d} p_t(x - y) \mu(dy) + \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x - y) b(u(s, y)) dyds \\
+ \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x - y) \sigma(u(s, y)) W(ds, dy).
\end{align*}
\]
(4)
where $p_t(x)$ is the transition function for $X_t$ and the stochastic integral above is in the sense of Walsh [6]. The following theorem is the main result of this paper.

**Theorem 1.** Assume that the initial condition satisfies (3) and assume that
\[
\Upsilon(\beta) := \sup_{t > 0} \int_0^t \int_{\mathbb{R}^d} \exp \left[ -2s \text{Re} \Phi \left( (1 - \frac{s}{t}) \xi \right) - 2(t - s) \text{Re} \Phi \left( \frac{s}{t} \xi \right) \right] e^{-2\beta(t-s)} \hat{f}(\xi) d\xi ds < \infty
\]
and
\[
\bar{\Upsilon}(\beta) := \int_{\mathbb{R}^d} \frac{\hat{f}(\xi) d\xi}{\beta + \text{Re} \Phi(\xi)} < \infty
\]
for any $\beta > 0$. And assume that $\sigma$ and $b$ are Lipschitz functions with Lipschitz coefficients $L_{\sigma}, L_b > 0$ respectively. Then there exists a unique mild solution to equation (1). Moreover, define
\[
\bar{\gamma}(p) := \limsup_{t \to \infty} \frac{1}{t} \sup_{x \in \mathbb{R}^d} \log \left\| \frac{u(t, x)}{\tau + p_t * \mu(x)} \right\|_{L^p(\Omega)},
\]
(7)
where
\[
\tau = \max \left\{ \frac{|b(0)|}{L_b}, \frac{|\sigma(0)|}{L_{\sigma}} \right\}.
\]
(8)

Then,
\[
\bar{\gamma}(p) \leq \inf \{ \beta > 0 : B(\beta, p) < 1 \} \quad \text{for all integers } p \geq 2,
\]
(9)
where
\[ B(\beta, p) := \frac{L_b}{\beta} + \frac{z_p L_{\sigma}}{(2\pi)^{d/2}} \left( \sqrt{\frac{\tilde{Y}(\beta)}{2}} + \sqrt{Y(\beta)} \right), \]
and \( z_p \) denotes the largest positive zero of the Hermite polynomial \( H_{\nu} \).

**Remark 2.** If we choose \( \mathcal{L} \) to be the generator of an \( a \)-stable Lévy process \( D_\theta^a \) for \( 1 < a < 2 \), where \( \theta \) is the skewness and \( |\theta| < 2-a \) (see [2]), or the Laplacian \( \frac{1}{2} \Delta \) \( (a = 2) \), then the classical Dalang’s condition
\[ \int \hat{f}(d\xi) \frac{1}{1 + |\xi|^a} < \infty \] \tag{11}
implies condition (5), since in this case \( \text{Re}\Phi(\xi) = C|\xi|^a \) for some \( C > 0 \). Also, in the case \( d = 1 \) and \( \hat{W} \) is a space-time white noise, that is, \( f(\xi) \equiv 1 \), condition (6) clearly guarantees that (2) holds.

**Remark 3.** (Borrowed from [5, Remark 1.5]). Recall that
\[ H_k(x) = 2^{-k/2} H_k(x/\sqrt{2}) \quad \text{for all integers } k \geq 0 \text{ and } x \in \mathbb{R}, \]
where \( \{H_k\}_{k=0}^\infty \) is defined uniquely via the following:
\[ e^{-2xt-t^2} = \sum_{k=0}^\infty \frac{t^k}{k!} H_k(x) \quad (t > 0, x \in \mathbb{R}). \]

## 2 Proof of Theorem 1

In the proof of Theorem 1 we will need two results about taking Fourier transforms, which we now state.

**Lemma 4** (Corollary 3.4 in [5]). Assume that \( f \) is lower semicontinuous, then for all Borel probability measures \( \nu \) on \( \mathbb{R}^d \),
\[ \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x-y) \nu(dx) \nu(dy) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \hat{f}(\xi) |\hat{\nu}(\xi)|^2 d\xi. \]

**Lemma 5.** If \( f \) is lower semicontinuous, then
\[ \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} p_{t-s}(x-y_1)p_s * \mu(y_1)p_{t-s}(x-y_2)p_s * \mu(y_2)f(y_1-y_2)dy_1dy_2 \]
\[ \leq \frac{[p_t * \mu(x)]^2}{(2\pi)^d} \int_{\mathbb{R}^d} \exp \left[ -2s\text{Re}\Phi \left( (1 - \frac{s}{t})\xi \right) - 2(t-s)\text{Re}\Phi \left( \frac{s}{t}\xi \right) \right] \hat{f}(\xi) d\xi. \]

**Proof.** We begin by noting that
\[ \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} p_{t-s}(x-y_1)p_s * \mu(y_1)p_{t-s}(x-y_2)p_s * \mu(y_2)f(y_1-y_2)dy_1dy_2 \]
\[ = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{p_{t-s}(x-y_1)p_s(y_1-z_1)p_{t-s}(x-y_2)p_s(y_2-z_2)}{p_t(x-z_1)p_t(x-z_2)} f(y_1-y_2)dy_1dy_2 \]
\[ \times p_t(x-z_1)p_t(x-z_2)\mu(dz_1)\mu(dz_2), \]
We first show that whenever \( \beta \) fields

\[
\text{Proof of Theorem}
\]

and as a function of \( y \), the quotient \( \frac{p_{t-s}(x-y)p_s(y-z)}{p_t(x-z)} \) is the probability density of the Lévy bridge \( \tilde{X}_{z,t} = \{\tilde{X}_{z,t}(s)\}_{0 \leq s \leq t} \) which is at \( z \) when \( s = 0 \) and at \( x \) when \( s = t \). Actually, \( \tilde{X}_{z,t}(s) \) can be written as

\[
\tilde{X}_{z,t}(s) = X_s - \frac{s}{t}X_t + z + \frac{s}{t}(x-z)
\]

hence by the independence of increment of Lévy process, we have

\[
\text{E}e^{i\xi \tilde{X}_{z,t}(s)} = \exp \left( -s\Phi \left( \frac{(1 - \frac{s}{t})}{\xi} \right) - (t-s)\Phi \left( \frac{-\frac{s}{t}}{\xi} \right) \right) e^{i(z + \frac{s}{t}(x-z))}.
\]

Thus, an application of Lemma 4 to \( \nu_j(dy) = \frac{p_{t-s}(x-y)p_s(y-z)}{p_t(x-z)} \) for \( j = 1, 2 \), yields

\[
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{p_{t-s}(x-y_1)p_s(y_1-z_1)}{p_t(x-z_1)} \frac{p_{t-s}(x-y_2)p_s(y_2-z_2)}{p_t(x-z_2)} f(y_1-y_2)dy_1dy_2
\]

\[
= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \text{E}e^{i\xi \tilde{X}_{z,t}(s)} \text{E}e^{i\xi \tilde{X}_{z,t}(s)} f(\xi) d\xi
\]

\[
\leq \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \exp \left[ -2s\text{Re}\Phi \left( \frac{(1 - \frac{s}{t})}{\xi} \right) - 2(t-s)\text{Re}\Phi \left( \frac{-\frac{s}{t}}{\xi} \right) \right] f(\xi) d\xi,
\]

which proves the lemma.

To prove Theorem 1, we first define a norm for all \( \beta, p > 0 \) and all predictable random fields \( v := v(t, x) \),

\[
\|v\|_{\beta,p} = \sup_{t>0} \sup_{x \in \mathbb{R}^d} ||v(t, x)||_{L_p(\Omega)}.
\]

(12)

Let \( B_{\beta,p} \) denote the collection of all predictable random fields \( v := \{v(t, x)\}_{t \geq 0, x \in \mathbb{R}^d} \) such that \( \|v\|_{\beta,p} < \infty \). We note that after the usual identification of a process with its modifications, \( B_{\beta,p} \) is a Banach space (see Section 5 in [5]).

Proof of Theorem 1. We use Picard iteration. Set

\[
u^0(t, x) := p_t * \mu(x),
\]

\[
u^{n+1}(t, x) := p_t * \mu(x) + \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x-y)b(u^n(s, y))dyds
\]

\[
+ \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x-y)\sigma(u^n(s, y))W(ds, dy).
\]

We first show that whenever \( \beta \) is chosen such that \( B(\beta, p) < 1 \), where \( B(\beta, p) \) is defined in (10), then, for any \( n \geq 1 \),

\[
\left\| \frac{\tau + |u^n|}{\tau + p * \mu} \right\|_{\beta, p} < \infty.
\]

(13)

Note that by the dominated convergence theorem, the condition \( B(\beta, p) < 1 \) can be achieved if \( \beta \) is sufficiently large.
Recall that \( \tau \) is defined in (8). We start with the inequality
\[
\frac{\tau + |u^{n+1}(t, x)|}{\tau + p_t \ast \mu(x)} \leq 1 + \int_0^t \int \frac{p_{t-s}(x-y)\tau + p_s \ast \mu(y)}{\tau + p_t \ast \mu(x)} \frac{b(u^n(s, y))}{\tau + p_s \ast \mu(y)} dy ds \\
+ \int_0^t \int \frac{p_{t-s}(x-y)\tau + p_s \ast \mu(y)}{\tau + p_t \ast \mu(x)} \frac{\sigma(u^n(s, y))}{\tau + p_s \ast \mu(y)} W(ds, dy).
\]

(13) is clearly true for \( n = 0 \). Using induction, assume (13) is true for some \( n \), using \( \text{Burkholder inequality} \) (see [4]) and the assumption on \( \sigma \) and \( b \), we obtain
\[
\left\| \frac{\tau + |u^{n+1}(t, x)|}{\tau + p_t \ast \mu(x)} \right\|_{L^p(\Omega)} \leq 1 + L \int_0^t \int \frac{p_{t-s}(x-y)\tau + p_s \ast \mu(y)}{\tau + p_t \ast \mu(x)} \left\| \frac{\tau + |u^n(s, y)|}{\tau + p_s \ast \mu(y)} \right\|_{L^p(\Omega)} dy ds \\
+ z_p L_\sigma \left( \int_0^t \int \int \frac{p_{t-s}(x-y)\tau + p_s \ast \mu(y_1)}{\tau + p_t \ast \mu(x)} \frac{p_{t-s}(x-y_1)\tau + p_s \ast \mu(y_1)}{\tau + p_t \ast \mu(x)} \frac{p_{t-s}(x-y_1)\tau + p_s \ast \mu(y_1)}{\tau + p_t \ast \mu(x)} \left\| \frac{\tau + |u^n(s, y_1)|}{\tau + p_s \ast \mu(y_1)} \right\|_{L^p(\Omega)} \left\| \frac{\tau + |u^n(s, y_2)|}{\tau + p_s \ast \mu(y_2)} \right\|_{L^p(\Omega)} f(y_1 - y_2) dy_1 dy_2 ds \right)^{1/2},
\]

multiplying both sides by \( e^{-\beta t} \) and applying Minkowski’s inequality to the third summand above we obtain
\[
e^{-\beta t} \left\| \frac{\tau + |u^{n+1}(t, x)|}{\tau + p_t \ast \mu(x)} \right\|_{L^p(\Omega)} \leq 1 + L \int_0^t \int \frac{e^{-\beta(t-s)}p_{t-s}(x-y)\tau + p_s \ast \mu(y)}{\tau + p_t \ast \mu(x)} dy ds \\
+ z_p L_\sigma \left( \int_0^t \int \int e^{-2\beta(t-s)}p_{t-s}(x-y_1)p_{t-s}(x-y_2)f(y_1 - y_2) dy_1 dy_2 ds \right)^{1/2} \\
+ z_p L_\sigma \left( \int_0^t \int \int e^{-2\beta(t-s)}p_{t-s}(x-y_1)p_{t-s}(x-y_2)p_{t-s}(x-y_2)f(y_1 - y_2) dy_1 dy_2 ds \right)^{1/2} \\
\cdot \left( \int_0^t \int \int e^{-2\beta(t-s)}p_{t-s}(x-y_1)p_{t-s}(x-y_2)p_{t-s}(x-y_2)f(y_1 - y_2) dy_1 dy_2 ds \right)^{1/2} \\
:= 1 + I_1 + I_2 + I_3,
\]

where in obtaining \( I_2 \) and \( I_3 \) above, we have used the bound
\[
\frac{p_{t-s}(x-y)\tau}{\tau + p_t \ast \mu(x)} \leq p_{t-s}(x-y) \quad \text{and} \quad \frac{p_{t-s}(x-y)p_s \ast \mu(y)}{\tau + p_t \ast \mu(x)} \leq \frac{p_{t-s}(x-y)p_s \ast \mu(y)}{p_t \ast \mu(x)}.
\]

We will estimate \( I_1, I_2, I_3 \) separately. For \( I_1 \), the semigroup property of \( p_t(x) \) yields
\[
I_1 \leq \frac{L}{\beta} \left\| \frac{\tau + |u^n|}{\tau + p \ast \mu} \right\|_{L^p(\Omega)}.
\]
For $I_2$, an application of Lemma 4 to $\nu(dy) = p_{t-s}(x-y)dy$ yields
\[
\int_0^t \int_{\mathbb{R}^d} e^{-2\beta(t-s)} p_{t-s}(x-y_1)p_{t-s}(x-y_2)f(y_1-y_2)dy_1dy_2ds
= \frac{1}{(2\pi)^d} \int_0^t \int_{\mathbb{R}^d} e^{-2\beta(t-s)} Re\Phi(\xi) \hat{f}(\xi) d\xi e^{-2\beta(t-s)} ds \leq \frac{1}{2(2\pi)^d} \int_{\mathbb{R}^d} \hat{f}(\xi) d\xi \beta + Re\Phi(\xi),
\]
thus we obtain
\[
I_2 \leq z_p L_\sigma \left( \frac{1}{2(2\pi)^d} \tilde{\Upsilon}(\beta) \right)^{1/2} \left\| \frac{\tau + |u^n|}{\tau + p*\mu} \right\|_{\beta,p}.
\]
Finally, an application of Lemma 5 yields
\[
I_3 \leq z_p L_\sigma \left\| \frac{\tau + |u^n|}{\tau + p*\mu} \right\|_{\beta,p} \left( \frac{1}{2(2\pi)^d} \Upsilon(\beta) \right)^{1/2}.
\]
Combining the estimates for $I_1, I_2, I_3$, we arrive at
\[
\left\| \frac{\tau + |u^{n+1}|}{\tau + p*\mu} \right\|_{\beta,p} \leq 1 + B(\beta,p) \left\| \frac{\tau + |u^n|}{\tau + p*\mu} \right\|_{\beta,p},
\]
where $B(\beta,p)$ is defined in (10). Using the iteration, we see that (13) holds for all $n \geq 1$ if $B(\beta,p) < 1$.

The same technique applied to $\frac{u^{n+1}(t,x) - u^n(t,x)}{\tau + pt*\mu(x)}$ yields that
\[
\left\| \frac{u^{n+1} - u^n}{\tau + p*\mu} \right\|_{\beta,p} \leq B(\beta,p) \left\| \frac{u^n - u^{n-1}}{\tau + p*\mu} \right\|_{\beta,p},
\]
and if $\beta$ is chosen such that $B(\beta,p) < 1$, we will obtain that
\[
\sum_{n=1}^{\infty} \left\| \frac{u^n - u^{n-1}}{\tau + p*\mu} \right\|_{\beta,p} < \infty.
\]
Therefore, we can find a predictable random field $u^\infty \in \mathcal{B}_{\beta,p}$ such that $\lim_{n \to \infty} u^n = u^\infty$ in $\mathcal{B}_{\beta,p}$. It is easy to see that this $u^\infty$ is a solution to equation (4), and uniqueness is checked by a standard argument.

To prove (9), we note that since $u \in \mathcal{B}_{\beta,p}$ for those $\beta$ such that $B(\beta,p) < 1$,
\[
\sup_{x \in \mathbb{R}^d} \left\| \frac{u(t,x)}{\tau + pt*\mu(x)} \right\|_{L^p(\Omega)} \leq \sup_{x \in \mathbb{R}^d} \frac{\tau}{\tau + pt*\mu(x)} + Ce^{\beta t}
\]
for some $C > 0$ which does not depend on $t$, thus (9) is proved and the proof of Theorem 1 is complete.

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