Structure of Lorentzian algebras and Conformal Field Theory

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Abstract

The main properties of indefinite Kac-Moody and Borcherds algebras, considered in a unified way as Lorentzian algebras, are reviewed. The connection with the conformal field theory of the vertex operator construction is discussed. By the folding procedure a class of subalgebras is obtained.

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1 Introduction

This paper is devoted to present the mathematical aspects of Lorentzian algebras with emphasis on the deep connections between their vertex operator realization and the conformal properties of the fields which are introduced in order to build up this representation.

Lorentzian algebras include the indefinite Kac-Moody algebras \([1]\) and the generalized Kac-Moody algebras introduced by Borcherds \([2]\) which we call Borcherds algebras.

For a review of the properties of Kac-Moody algebras (KMA) see \([3][4]\).

The history of Kac-Moody algebras development is linked to string theories and conformal field theory, therefore the main applications are in these context and make use of Affine KMA.

However the interest in physics for the indefinite KMA has been emphasized by many authors. In particular Julia \([5]\) has suggested that such algebras may appear in the dimensional reduction of supergravity models and recently Nicolai \([6]\) has shown that an hyperbolic extension of \(SL(2,\mathbb{R})\) appears in the dimensional reduction of \(N = 1\) supergravity from four to one dimension.

The main features of this algebra, which we call \(\hat{A}_1^{(1)}\), are recalled in Appendix A.

In string theories the Lorentzian algebras have appeared in different ways. One of this was suggested by Goddard and Olive \([7]\) in the context of a unification view of the heterotic string.

The algebras involved are very special kinds of Lorentzian algebras, one of these is \(E_{19}\) the natural candidate for unification of \(E(8) \otimes E(8)\) and \(SO(32)\) theories.

The use of Lorentzian algebras is necessary also in the fermionic string, see \([8]\) and references therein, in which the role of ghosts sector can be understood by means of these algebras.

Another way to introduce Lorentzian algebras in string theories consists in reversing the sign of hantiolomorphic fields \([8]\). There is a strange interplay between this aspect and the role of ghosts in previous application.

The energy operator in physical models, exhibiting an Aff-KMA as symmetry algebra, does not belong to the symmetry algebra but it can be identified with the zero generator of the infinite dimensional Virasoro algebra associated with the KMA. Slansky \([9]\) has recently suggested that the energy operator as well as the particle number operator can be included in the symmetry algebra by an extension of the last one to generalized Kac-Moody algebra.

Moreover the Monster Lie algebra \([10]\) can be seen as an infinite rank Borcherds
The strange coincidence that occurs between the theory of Monster and the string theory is simply astonishing, so many physicists undergo the charm of this fact.

We shall not discuss the Monster Lie algebra in this paper. For a discussion of the connection of this algebra with the abstract theory of vertex algebra and the conformal field theory see [12].

Nowadays most theoretical physicists are aware of the fact that the symmetries of string field theory are larger then the symmetries of the related conformal field theories ([13] and references therein), but it is not well know how this phenomenon can be realized.

We believe that all these, at present different and uncorrelated way to make use of Lorentzian algebras in string theories, may be unified and understood as a not yet discovered symmetry of the quantized string field theory.

This paper is organized in the following way. In sec. 2 we define a Lorentzian algebra and we recall its main features. In sec. 3 we discuss a vertex operator realization of a Lorentzian algebra, mainly of the indefinite KMA, and we emphasize the role of conformal symmetry.

In sec. 4 we discuss some class of not trivial subalgebras of the Lorentzian algebra. At the end we point out some of the many yet open problems.

In Appendix A we recall in some detail the vertex operators of the hyperbolic KMA $A_1^{(1)}$ which we use as an example to illustrate general features of the vertex realization of the Borcherds algebras as subalgebras.

In Appendix B we introduce an explicit system of simply roots for symmetric hyperbolic KMA of rank $\geq 3$. 


2 Lorentzian algebras

We call Lorentzian algebra (LA) a Lie algebra whose roots belong to a lattice in a D-dimensional space with indefinite metric \( g_{\mu\nu} = (-, \ldots, +, \ldots) \) with \( q \) signs - and \( p \) signs +. In the following we consider mainly the case \( p = D - 1, q = 1 \), but at the end we briefly comment on the general case. There are two main classes of LA: the indefinite KMA and the generalized KMA introduced by Borcherds which we shall call in the following Borcherds algebra (BA).

Firstly let us recall a few definitions which we need to define respectively a rank \( d \) KMA and BA \([1, 2]\). The rank \( d \) is always equal to the dimension \( D \) of the space for Hyp-KMA (see below for definition) while in the general case we have \( d \geq D \). In the case of \( D = 26 \) we are not considering the infinite rank Monster Lie algebra \([10]\) although this seems the most interesting case.

A \((d \times d)\) matrix \( A = [a_{ij}]\) is called a generalized Cartan matrix (GCM) \([4]\) if it satisfies the following conditions:

i) \( a_{ij} \in \mathbb{Z} \)

ii) \( a_{ii} = 2 \)

iii) \( a_{ij} \leq 0 \) \((i \neq j)\)

iv) \( a_{ij} = 0 \) implies \( a_{ji} = 0 \)

A matrix \( A \) is called symmetrisable if the matrix \( UA \) is symmetric, \( U \) being an invertible diagonal matrix. In the following we mainly consider symmetric GCM, i.e simply laced Lie algebras.

A \((d \times d)\) symmetric matrix \( A = [a_{ij}]\) is called a generalized symmetrized Cartan matrix (SCM) \([2]\) if it satisfies the above conditions i), iii), iv) while condition ii) is replaced by

iia) \( a_{ii} \in \mathbb{Z} \)

iib) \( 2a_{ij}/a_{ii} \) is integer if \( a_{ii} \) is positive

In the following we shall assume that

iic) \( a_{ii} \) is 0 or 2
A matrix $A$ is called indecomposable if it cannot be reduced to a block diagonal form by shuffling rows and columns. To a GCM or to a SCM we associate a Dynkin diagram (DD), denoted sometimes $S(A)$, with the following properties:

a) $S(A)$ has $d$ vertices

b) if $a_{ij}a_{ji} = n \leq 4$, the vertices $i$ and $j$ are joined by $|a_{ij}| \geq |a_{ji}|$ lines

c) if $|a_{ij}| \geq |a_{ji}|$, we put on the lines $(ij)$ an arrow pointing, resp., towards the vertex $j$ ($i$)

d) if $n > 4$ the vertices $i$ and $j$ are connected by a boldface line on which an ordered pair of integers, $|a_{ij}|$ and $|a_{ji}|$, is written.

e) if $a_{ii} = 2$ the $i$-th vertex will be denoted by a white circle; if $a_{ii} = 0$ the $i$-th vertex will be denoted by a crossed circle

Note that sometimes in the literature when condition b) is satisfied the vertices are joined by $n$ lines.

Clearly $A$ is indecomposable if and only if the corresponding $S(A)$ is a connected diagram.

To a given GCM $A$ we associate a complex Lie algebra defined by $3d$ generators, $E_i$, $F_i$ and $H_i$ which satisfy the following commutation and Serre relations

\[
[E_i, F_j] = a_{ij} H_i \quad (1)
\]
\[
[H_i, H_j] = 0 \quad (2)
\]
\[
[H_i, E_j] = a_{ij} E_j \quad (3)
\]
\[
[H_i, F_j] = -a_{ij} F_j \quad (4)
\]
\[
(ad E_i)^{1-a_{ij}} E_j = (ad F_i)^{1-a_{ij}} F_j = 0 \quad (i \neq j) \quad (5)
\]

To a given SCM $A$ we associate a complex Lie algebra defined by $3d$ generators, $E_i$, $F_i$ and $H_i$ which satisfy the above commutation and Serre relations if $a_{ii} \geq 0$ and by the following condition if $a_{ii} = 0$

\[
a_{ij} = 0 \rightarrow [E_i, E_j] = [F_i, F_j] = 0 \quad (6)
\]

The algebra can be written under the following form (triangular decomposition)

\[
G(A) = N_- \oplus H \oplus N_+ 
\]
where $H$ is the cartan subalgebra and $N_-(N_+)$ are resp. the linear span of $F_i(E_i)$.

We have for $LA \ detA \leq 0$. We recall that $\det A = 0$ with rank of $A$ equal to $d - 1$ and determinant of any leading principal submatrix positive, corresponds to affine KMA (Aff-KMA).

No general classification scheme exist for general Ind-KMA or BA.

The hyperbolic (Hyp) KMA algebras are a particular case of the Ind-KMA with the further condition that every leading submatrix decomposes into constituents of finite and/or affine type or, in equivalent way, if deleting a vertex of the corresponding $S(A)$ one gets DD of finite or affine KM algebras.

A classification of hyperbolic algebras has been made in refs. [14, 15] generalizing previous results obtained in [4]. In [14] all the DDs have been drawn, finding 238 DDs (of rank $\geq 3$) of which 142 DDs correspond to symmetric or symmetrisable GCM. The highest rank of the Hyp-KMA is 10 and to this class belongs $E_{10}$ which has appeared several time in the context of string theories. The determinant of a Hyp-KMA is always negative.

The root lattice $\Gamma$ is generated by the elements $\alpha_i$ ($i = 1, 2, .., d$), called simple roots (SR), and $\Gamma$ has a real-valued bilinear form defined by

$$a_{ij} = (\alpha_i, \alpha_j) = \alpha_i \cdot \alpha_j \quad (7)$$

We have:

**Proposition 1** The determinant of a symmetrizable Cartan matrix $A$ corresponding to a Lie algebra with no imaginary SR of rank 3 is given by

$$det|a_{ij}| = 2\{4 - (\alpha\beta\gamma)^{1/2} - (\alpha + \beta + \gamma)\} \quad (8)$$

where $\alpha, \beta, \gamma$ are, respectively, the number of lines joining the 3 vertices of $S(A)$.

**Proof:** From the relation between $S(A)$ and the matrix $A$ we have that $\alpha, \beta, \gamma$ are related to the off diagonal elements of the Cartan matrix by $\alpha = a_{12}a_{21}, \beta = a_{13}a_{31}, \gamma = a_{23}a_{32}$. Defining the quantity $\epsilon_{ij} = a_{ij}/a_{ji}$ which denotes the ratio between the lengths of the i-th and j-th SRs and satisfies $\epsilon_{ij} = \epsilon_{il}\epsilon_{lj}$ (no sum over $l$) and $\epsilon_{ij} = 1/\epsilon_{ji}$ we have

$$a_{12}a_{23}a_{31} = \left(\frac{\alpha\beta\gamma}{\epsilon_{21}\epsilon_{32}\epsilon_{13}}\right)^{1/2} = (\alpha\beta\gamma)^{1/2} \quad (9)$$

$$a_{13}a_{21}a_{32} = \left(\frac{\alpha\beta\gamma}{\epsilon_{31}\epsilon_{12}\epsilon_{23}}\right)^{1/2} = (\alpha\beta\gamma)^{1/2} \quad (10)$$
inserting the above equality in the expression of the determinant of a rank 3 matrix we find the quoted result. As the value of $det|a_{ij}|$ is an integer, in the case of loop DD we have some constraints as $(\alpha \beta \gamma)^{1/2}$ has to be an integer. Q.E.D.

A root is called positive if it is a sum with not negative integers of SR. A root $\alpha$ is called real if $(\alpha, \alpha)$ is positive, imaginary if $(\alpha, \alpha) \leq 0$.

Clearly the set of roots $\Delta$ satisfies

$$\Delta = \Delta^e \cup \Delta^im$$

where

$$\Delta^e = \{ r \in \Delta : (\alpha, \alpha) = 2 \}$$

$$\Delta^im = \{ r \in \Delta : (\alpha, \alpha) \leq 0 \}$$

The multiplicity ($mult$) of roots does not depend only from its length and it can be computed by means of the Peterson recursion formula for LA whose determinant of the defining GCM or SCM is negative:

$$(\alpha, \alpha - 2 \rho)C_\alpha = \sum_{\alpha' + \alpha'' = \alpha} (\alpha', \alpha'')C_{\alpha'}C_{\alpha''}$$

where

$$\alpha \in \Gamma^+ \quad \quad \Gamma^+ = \sum n_i \alpha_i \quad \quad n_i \in \mathbb{Z}^+$$

$\rho$ is the “unit” root

$$(\rho, \alpha_i) = 1/2(\alpha_i, \alpha_i) \quad \forall i \leq d$$

and

$$C_\alpha = \sum_{n \geq 1} (1/n)mult(\alpha/n)$$

The 23 DDs corresponding to symmetric Hyp-KMA of rank $d \geq 3$ are reported in Appendix B where an explicit SR systems is written.

Let us recall that an Aff-KMA can be obtained by adding to the lattice $\Gamma$ of roots of a finite Lie algebra $G_0$ (horizontal algebra) a lightlike vector $K^+$:

$$(K^+, K^+) = 0 \quad (K^+, \alpha_i) = 0$$

The affine root is obtained adding $K^+$ to the lowest root of $G_0$ (affinization procedure).

A class of Hyp-KMA is obtained adding to the lattice of roots of Aff-KMA an other lightlike root $K^-$ such that:
\[ (K^-, \alpha_i) = (K^-, K^-) = 0 \quad (K^-, K^+) = 1 \] (19)

We call hyperaffinization or double-affinization the procedure of adding to an Aff-KMA a SR containing \( K^- \) such that this new root has scalar product equal to \(-1\) with the affine root and zero with the other SRs.

A classification of double-affinized KMA's has been obtained by Ogg [25]. In the original paper the procedure has been called superaffinization but we prefer to change the name in hyper-affinization to avoid confusion with the procedure of affinization of superalgebras. One can show that the determinant of a hyper-affine LA is just the opposite of the determinant of the horizontal Lie algebra. Let us remark that in the case of Aff-KMA \( K^+ \) is a root (not simple), while in the case of Hyp KMA's \( K^- \) is not always a root.

We recall that a graded Lie algebra \( G = \bigoplus_{-\infty}^{\infty} G_i \), generated by \( G_0 \oplus G_1 \oplus G_{-1} \), simple (not containing non trivial homogeneous ideals) is said to be of finite growth [17] if the dimension of the space \( G_i \) grows as a power of \( |i| \). From Kac's theorem in [17] we see that the Hyp-KMA are of not finite growth.

Let us remark that the peculiar properties of the lightlike vectors \( K^\pm \) allow one to build trans-hyperbolic KMA which contains a “cluster” of Aff-KMA which can be obtained taking the direct sum of \( n \) affine algebras \( G_i \) (\( i=1,2,\ldots,n \)) then by adding an extra SR

\[ \alpha = -(K^+ + K^-) \] (20)

which will be simply connected with all the affine SR of any \( G_i \). The most celebrated example of such a “cluster” is the trans-hyperbolic algebra \( E_{19} \) introduced by Goddard-Olive which in a 19 rank Ind-KM which contains as subalgebras \( E_8(1) \times E_8(1) \) and \( SO^{(1)}(32) \). It has been argued the \( E_{19} \) could be the underlying unifying symmetry between the two consistent models of heterotic string. We remark that in \( 10 - \text{dim} \) Minkowskian space we can have a “cluster” which is a rank 11 trans-hyperbolic KMA which contains as subalgebras \( SO^{(1)}(8) \times SO^{(1)}(8) \) and \( SO^{(1)}(16) \). In general in a \( D - \text{dim} \) one may build up cluster of \( p \) copies of the same Aff-KMA \( G^{(1)} \) of rank \( d \) (\( pd = D - 2 \)) exhibiting a \( p \)-ality property. The determinant of the GCM A corresponding to a “cluster” is vanishing as one root is linearly dependent from the others, in fact we have a LA with rank equal to \( (D + 1) - \text{dim} \) space. Finally we recall a few definitions and properties of the Weyl group for KMA and BA.

We recall [10], that the Weyl group \( (W) \) of a KM algebra is a discrete group of isometries of the dual of the Cartan subalgebra generated by the reflections with respect to the SRs (fundamental reflections). For KMA The elements of \( W \) which
are obtained as a product of an even number of fundamental reflections form a
normal subgroup $W_0$ of $W$, called either conformal Weyl group or even subgroup.
Clearly, for Hyp KMAs, $W_0 \in SO^+(d - 1, 1)$, while $W \in O^+(d - 1, 1)$.

In general the Weyl group is generated by the $d$ fundamental reflexions whose
action on the SRs is given by:

$$w_{\alpha_i}\alpha_j = \alpha_j - a_{ij}\alpha_i$$

Let us remind a few important properties of the Weyl group:

- the bilinear form $(\cdot, \cdot)$, the sets $\Delta$ and $\Delta^\text{im}$ are invariant under the action of $W$;
- the dimension of the space of a root is equal to the dimension of the space of
  the reflected root;
- contrary to what happens in the case of finite Lie algebras, in the root space
  of a KMA or of a BA may exist fixed points.

3 Vertex construction and Virasoro algebra

In this section we shall consider the relationship between vertex construction of
Lorentzian algebras and conformal field theory (CFT)[18].

Our motivation for doing this is to prove that the symmetries generated by
the Virasoro algebra that we associate to LA are necessary to obtain the correct
commutation relations. In fact we shall shown that a subalgebra of the full conformal
algebra is relevant in the structure of the vertex construction of LA.

The understanding of the role of these symmetries is an important point in
physical applications of LA because the very strange character of these algebras
links many different and apparently uncorrelated topics in physics, as we have briefly
recalled in the Introduction, so it is tempting to suppose that many yet unknown
links exist between these topics.
We consider in particular a bosonic CFT defined on a Lorentzian lattice and then we construct the associate Virasoro algebra.

3.1 Virasoro algebra on Lorentzian lattice

In the vertex construction of Lorentzian algebras we use a bosonic conformal field theory on lattice, therefore in this section we consider a generic stress tensor $T(z)$. Expanding in Laurent serie we obtain:

$$T(z) = \sum_m L_m z^{-m-2} \quad (22)$$

where:

$$L_m = \frac{1}{2\pi i} \oint_{C_0} T(z) z^{m+1} dz = \frac{1}{2} \sum_n \alpha_n \cdot \alpha_{m-n} \quad (23)$$

with

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{D}{12} m(m^2-1)\delta_{m+n} \quad (24)$$

We consider also the subalgebra of projective transformations (Witt algebra):

$$[L_m, L_n] = (m-n)L_{m+n} \quad \forall \ m, n \in \{\pm1, 0\} \quad (25)$$

and the relative representations.

These are the only globally invertible transformations on the Riemann sphere.

A primary field $\phi(z)$ (PF) satisfies the relations:

$$[L_m, \phi(z)] = z^{m+1} \frac{d}{dz} \phi(z) + (m+1)hz^m \phi(z) \quad \forall \ m \quad (26)$$

then the relative coefficients of the Laurent expansion:

$$\phi(z) = \sum_n A_n z^{-n-h} \quad (27)$$

have the following commutation relations with Virasoro algebra:

$$[L_m, A_n] = [m(h-1) - n] A_{n+m} \quad \forall \ m, n \quad (28)$$

A quasiprimary field (QPF) satisfies the same relations only with the projective transformations subalgebra.
The fields that do not satisfy the eq. (26) and eq. (28) relations are denoted as a secondary (SF).

Let us define also a quasisecondary (QSF) field, provided it does not satisfy the relations of eq. (26) restricted to $m = \{\pm 1, 0\}$.

Any QSF can be constituted by a set of coordinative derivates of QPF. Expanding this basic derivates fields in Laurent serie we define:

$$i \frac{d^l}{dz^l} \phi(z) = \sum_n A_n^{(l)} z^{-n-l-h}$$

(29)

The $A_n^{(l)}$ coefficients are related to $A_n$ by means of the relations:

$$i \frac{d^l}{dz^l} \phi(z) = i \sum_n l! \left( \begin{array}{c} -n - h \\ l \end{array} \right) A_n z^{-n-l-h}$$

(30)

$$A_n = \frac{1}{2\pi i l!} \left( \begin{array}{c} -n - h \\ l \end{array} \right) \oint_{C_0} dz z^{n+l+h-1} \frac{d^l}{dz^l} \phi(z)$$

(31)

Therefore we can extract the relations:

$$A_n^{(l)} = il! \left( \begin{array}{c} -n - h \\ l \end{array} \right) A_n$$

$$A_n^{(l)} = 0 \quad \forall \quad n : \quad -h - l + 1 \leq n \leq -h$$

(32)

$$[L_m, A_n^{(l)}] = [m(h - 1) - n] A_{n+m}^{(l)} \quad \forall \quad n \in [-h - l + 1, -h]$$

(33)

Then the $A_0$ operators relative to PFs with conformal dimension $h = 1$ commute with full Virasoro algebra, instead, the $A_0$ operators relative QPFs with $h = 1$ commute only with the $m = \{\pm 1, 0\}$ $L_m$ generators.

Note that all the $A_0^{(l)}$ operators relative to QSF of dimension $h + l = 1$ are vanish. These are the zero modes of quasinull fields (QNFs) and in the following we shall show its role in the vertex construction.

In sec. (3.2) we construct the generators of a Lorentzian algebra by means of the zero modes $A_0$ relative to fields of $h = 1$ but as we have showed that only QPFs give contribution to $A_0$ operators, therefore, we can conclude that this realization of Lorentzian algebras commute with the subalgebra of projective transformations.

If the full algebra is built up by $A_0$ operators relative to PFs of $h = 1$ the full Virasoro algebra commute with it, therefore, in this case the vertex realization
conserves the conformal properties. We shall see that such a construction cannot be obtained, except in the case of $D = 26$, for any Lorentzian algebra.

Indeed, it is possible to prove this property in the case of 26 dimension by means of the no-ghost theorem [19].

### 3.2 Vertex operator construction

The well-known vertex construction of an Aff KMA can be generalized to the case of Ind KMAs along the lines of the covariant construction of Goddard and Olive [7, 20].

Note that this construction does not apply to BA in general as eq.(6) is not satisfied. The construction can be carried on when the conditions of eq.(6) are never verified, e. g. when one imaginary SR is connected with all the other real SRs. In sec.4 we shall discuss in detail this point in a specific case.

In this section we examine this construction in particular and introduce a formalism in which the remarkable properties of Lorentzian algebras are more evident.

We study a bosonic string in a compactified space-time background in which we consider only the holomorphic modes. For an introduction to vertex algebra see [12].

The structure of the vertex representation depend only by the value of the scalar products on the lattice, therefore we can generalize our consideration to the algebras with root lattices in Minkowskian space $R^{(q,p)}$.

We restrict the discussion to the $q = 1$ case.

Let us introduce $D$ Fubini-Veneziano fields ($\mu = 1, .., D$)

$$Q^\mu(z) = q^\mu - ip^\mu ln z + i \sum_{n \neq 0} \frac{\alpha^\mu_n}{n} z^{-n}$$

where

$$[q^\mu, p^\nu] = ig^{\mu\nu} \quad sign(g) = (-, +, +, .., +)$$

$$[\alpha_n^\mu, \alpha_m^\nu] = n\delta_{n+m,0}g^{\mu\nu} \quad \alpha^\mu_0 = \alpha_{-m}$$

The above fields are defined on a $D$-dim Minkowskian torus and satisfy periodical boundary conditions on the lattice $\Gamma$.

In our construction we consider only symmetric Cartan matrix, therefore, this can be used directly as metric tensor.
\[ a_{ij} = \alpha_i \cdot \alpha_j \]

We introduce a stress tensor:

\[ T(z) = \frac{1}{2} \sum_i D : \alpha_i \cdot Q^{(1)}(z) \alpha^i \cdot Q^{(1)}(z) : \]

\[ = \frac{1}{2} : \alpha^i \cdot Q^{(1)}(z) a_{ij} \alpha^j \cdot Q^{(1)}(z) : \quad (37) \]

where we introduce a set of \( D \) independent roots \( \{\alpha_i\} \) and the dual basis \( \{\alpha^i\} \).

Note that in the case \( d > D \) we must consider only a subset of simple roots that are associate to a submatrix of rank \( D \) of the Cartan matrix.

Let us define

\[ U^r(z) = e^{i r \cdot Q(z)} : (38) \]

and

\[ r \cdot Q^{(n)}(z) = \frac{i d^n}{n! d z^n} r \cdot Q(z) \quad (39) \]

where \( : : \) denotes normal ordered product and the ‘polarization’ \( r \) is in \( \Gamma \). In general \( r \) is an element of the lattice \( \Gamma \), so a linear combination of \( \alpha_i \), but not necessarily a root. In the expression of the VO of simple root \( r \) denotes the SR \( \alpha_i \).

\( U^r(z) \) are the vertex operator (VO) which are introduced for the vertex realization of an Aff KMA. It is possible introduce generalized vertex operator (GVO), see Borcherds \[11\] and \[19\] by means of the following ordered product

\[ : r_1 \cdot Q^{(n_1)}(z) r_2 \cdot Q^{(n_2)} \ldots r_N \cdot Q^{(n_N)}(z) U^r(z) : \quad (40) \]

where \( n_i \in Z_+ \). It is convenient to express a GVO in a different basis \[21\] introducing a set of Schur polynomials, which are defined by the following formal expansion

\[ exp(\sum_{m>0} c_m z^{-m}) = \sum_n P_n(c) z^{-n} \quad (41) \]

where \( c_m \) are commuting variables. So we have
\[ P_n(c) = \sum_{k_l} \frac{1}{k_l!} c_l^{k_l} \sum_l lk_l = n \quad (42) \]

We use a new basis for the derivative fields in which they are represented as Schur polynomials in the fields \( r \cdot Q^{(l)}(z) \) (1 \( \leq l \leq n \)):

\[
P_n(r \cdot Q^{(l)}(z)) = \left( \frac{1}{n!} \frac{d^n}{dz^n} U^r(z) \right) U^{-r}(z) = \lim_{z \to z_1} \frac{1}{n!} \left[ \frac{\partial^n}{\partial z_i^n} U^r(z_1) U^{-r}(z) \right]. \quad (43)\]

The above equation has to be read as a Schur polynomial, the variable \( c_l \) being now replaced by the field \( r \cdot Q^{(l)} \). It follows that a GVO is an ordered product of Schur polynomials and standard VO:

\[ U^{\{r,r_i\}}_{\{(n_i)\}}(z) = \prod_i P_{n_i}(r_i \cdot Q^{(l_i)}(z)) U^r(z) : \quad (44) \]

which can be explicitly written in the following form:

\[
U^{\{r,r_i\}}_{\{(n_i)\}}(z) = \prod_i \lim_{z_i \to z} \frac{1}{n_i!} \frac{\partial^{n_i}}{\partial z_i^{n_i}} U^{r_i}(z_i) U^{-r_i}(z) U^r(z) = \prod_i U^{r_i(n_i)}(z) U^{-r_i}(z) U^r(z) : \quad (45)\]

To any root of length \( L = r^2 \) we associate a GVO with conformal weight \( h = 1 \) such that

\[ h = \frac{r^2}{2} + \sum_i n_i = 1 \quad (46) \]

So for \( L = 2 \) the corresponding GVO is the standard VO while for \( L = 0 \) it is the photonic VO \[7\].

Let us point out that the eq.(46) is connected with the conformal symmetry of the GVOs, moreover not all the set of \( (r_i, n_i) \), which satisfy eq.(46), really appear in the construction of the algebra, see below.

We can make a Laurent expansion of a GVO

\[ U^{\{r,r_i\}}_{\{(n_i)\}}(z) = \sum_m A_{m\{(n_i)\}}^{\{r,r_i\}} z^{-m-1} \quad (47) \]
We select the terms in eq.(47) with \( m = 0 \)
\[
A_{\{r,r_1,\ldots,r_N\}}^{\{(n_1),\ldots,(n_N)\}} = \frac{1}{2\pi i} \oint_{C_0} dz U_{\{(n_1),\ldots,(n_N)\}}^{\{r,r_1,\ldots,r_N\}}(z)
\]  
(48)
where the integral is performed along a closed path \( C_0 \), including the point \( z = 0 \).

We can summarize the relevant commutation relations in the following formula, where suitable cocycles are supposed to have been included in the l.h.s. of the equation
\[
\left[ A_{\{r,r_i\}}^{\{(n_i)\}}, A_{\{s,s_j\}}^{\{(n_j)\}} \right] = \chi_{\{(r_i,s_j),(n_i)\}}^{\{(n_j)\}} r \cdot p
\]  
(49)
In particular if \( s = -r \) and \( s_i = -r_i \) we get
\[
\left[ A_{\{r,r_1,\ldots,r_N\}}^{\{(n_1),\ldots,(n_N)\}}, A_{\{-r,-r_1,\ldots,-r_N\}}^{\{(n_1),\ldots,(n_N)\}} \right] = \chi_{\{(r_i,s_j),(n_i)\}}^{\{(n_j)\}} r \cdot p
\]  
(50)
The \( \chi \) coefficients can be explicitly computed as a combination of factorials
\[
\chi_{\{(r_i,s_j),(n_i)\}}^{\{(n_j)\}} = \prod_i \prod_j \sum_{k_1} \sum_{l_1=0}^{k_1} (-1)^{l_1} \left( \begin{array}{c} s_j \cdot (r_i) \\ k_j - l_j \end{array} \right) \left( \begin{array}{c} r_i \cdot s_j \\ l_i \end{array} \right) \\
\left( \begin{array}{c} r_i \cdot (s - s_j) \\ k_i - l_i \end{array} \right) \left( \begin{array}{c} r_i \cdot s_j - l_i \\ l_j \end{array} \right)
\]  
(51)
To compute the r.h.s. of the eq.(49) we have to evaluate the residues at poles of order
\[
p = \sum k_i + \sum k_j - r \cdot s \quad \text{for} \quad z = \xi
\]  
(52)
in the OPE.
From eq.(48) for roots \( r \) and \( s \) it follows that the r.h.s. of eq.(49) has poles
\[
\frac{4 - (r + s)^2}{2} \geq p \geq -r \cdot s \geq 0
\]  
(53)
therefore, there are no poles for
\[
(r + s)^2 \geq 4
\]  
(54)
so we obtain a condition on the length of roots which insures the absence of poles.
When eq.(54) is satisfied the r.h.s. of eq.(49) vanishes.

Let us remark that not all the operators are relevant for the construction of the algebra as there is a class of GVOs which vanishes. In fact if \( U_{(n_i)}^{(r,r_i)}(z) \) can be written as a total \( z \) derivative of a GVO, then \( A_{(n_i)}^{(r,r_i)} = 0. \)

This property is a consequence of the quasi-conformal symmetry, in fact the above fields are QNFs.

We can also show that the commutator of two GVOs that satisfy the eq. (46) is a linear combination of GVOs that satisfy the same relation. In fact, in the r.h.s of eq.(49) we extract the zero modes from a linear combination of GVOs in which the conformal weight is:

\[
h = \frac{(r + s)^2}{2} + \sum_j n_j + \sum_i n_i - r \cdot s - 1 \quad (55)
\]

therefore by means of eq.(46):

\[
h = h_r + h_s - 1 = 1 \quad (56)
\]

From eq.(50) we see the Cartan generators \( H_i \) are given by \( \alpha_i \cdot p \). Note that for \( n_i = 0 \), for \( r_i = 0 \) and \( r^2 = 2 \) the \( \chi \) coefficient in the r.h.s. is equal to 1.

Note that in the LAs the level \( K \) of the Aff-KM subalgebras is not a c-number but simply a weight, therefore the vertex construction cannot depend by the particular value of \( K \) so as it happen in the Fenkel-Kac-Siegel construction.

This aspect is pointed out in [22] where is showed that the auxiliary parafermionic fields, that are necessary in the case of \( K \neq 1 \), are naturally obtained by an opportune factorization of the vertex operators.

### 3.3 The structure of representations space

In this section we shall discuss the structure of CFT Fock space in relation with the structure of the representations of Lorentzian algebras and the conformal properties.

The whole representations space is composed by the Fock space built up by the creation operators \( a_n^\dagger \) with \( n < 0 \) and the vector space of the eigenstates of momentum operator with eigenvalue on Lorentzian lattice.

This space can be decomposed in representations of Virasoro algebra associated to the Lorentzian algebra.

A state is called primary (PS), quasiprimary (QPS), secondary (SS) and so on, if it is created by the action of the corresponding field on the vacuum state

\[
| \alpha > = \phi_\alpha(0) | 0 > \quad (57)
\]
A PS satisfies

\[ L_n | \alpha > = 0 \quad \forall n > 0 \]
\[ L_0 | \alpha > = h | \alpha > \]  \hspace{1cm} (58)

while a QPS satisfies

\[ L_1 | \alpha >= 0 \quad L_0 | \alpha >= h | \alpha > \]  \hspace{1cm} (59)

A secondary state (SS) (or quasisecondary state (QSS)) that satisfies the same PS (or QPS) relations is denoted as a null state (NS) (or quasinull state (QNS)).

In sec. 3.1 we have shown that the Lorentzian algebra is built up by \( A_0 \) operators relative to QPFs of conformal dimension \( h = 1 \), therefore we have

\[ [L_m, A^{(r, r_i)}_{\{n_i\}}] = 0 \quad \forall m \in \{\pm 1, 0\} \]  \hspace{1cm} (60)

From eq.(60) it follows that the application of generator \( A^{(r, r_i)}_{\{n_i\}} \) on a state of a representation of Lorentzian algebra cannot change the conformal properties of the state restricted to projective properties.

So we can associate a QPS to every operator \( A^{(r, r_i)}_{\{n_i\}} \) by the relation:

\[ U^{(r, r_i)}_{\{n_i\}}(0) | 0 > = | r, r_i(n_i) > \]  \hspace{1cm} (61)

with the properties:

\[ L_0 | r, r_i(n_i) > = \left( \frac{r^2}{2} + \sum_i n_i \right) | r, r_i(n_i) > = | r, r_i(n_i) > \]
\[ L_1 | r, r_i(n_i) > = 0 \]  \hspace{1cm} (62)

We denote this space as \( P^{(1)} \).

This space include also the QNS of dimension \( h+l = 1 \), but we have showed that the relative operator is vanishing, so we must consider the quotient space \( P^{(1)} / P^{(1)}_N \) where \( P^{(1)}_N \) indicate the space of null states.

Note that the QSFs associated to QNSs are fundamental to obtain a correct commutation relations in fact the symmetric part of the OPE of the fields in the r.h.s. of eq.(49) is a QNF that correspond to a QNS so we can have a commutator only on the quotient space.

The projective symmetry assure us that these states are orthogonal to all QPSs.
In the case of 26 dimension is possible to extend the above considerations to the whole Virasoro algebra by means of no-ghost theorem \[19\], so we must consider the quotient \( P^1/P^1_N \) (where \( P^1, P^1_N \) is the space of PS and NS of dimension \( h = 1 \)).

We can introduce also a GVO representation of highest weight states

\[
| \Lambda > = \sum_i m_i | \Lambda_i >
\]

where \( \Lambda_i \) are the fundamental weights.

These states are created by the tachionic vertex operator:

\[
V^{(\Lambda)} = \frac{1}{2\pi i} \oint \frac{dz}{z} U^{(\Lambda)}(z)
\]

where \( \Lambda \) belongs to the weight lattice \( \Gamma^* \)

The highest weight representations of Lorentzian algebras are built up by lowering operators acting on the highest weight states (see \[4\]). To any highest weight \( \Lambda \) we associate the PS of conformal dimension \( h = \Lambda^2/2 \), so it is also a highest weight state also for the Virasoro algebra (and for the projective subalgebra).

This conformal weight is, generally, not an integer value. This can be a problem if we consider the properties of the corresponding conformal field theories, but we do not discuss this topic which is linked to the fusion rules of vertex operator algebras.

Considerable effort is spent on this subject by physicists and mathematicians in particular for the study of the Moonshine module.

Moreover, in the physical applications, we must use CFTs that are unitary and with local OPE properties so we must combine these theories in a opportune way.

A general state of a highest weight representation is obtained by application of lowering operators so we have:

\[
\lambda = \Lambda - \sum_i r_i
\]

where \( r_i \) are roots.

The corresponding state have the form

\[
| \lambda, \lambda_j(n_j) >= \lim_{z \to 0} U^{(\lambda,\lambda_j)}_{(n_j)}(z) | 0 >
\]

where \( \lambda_j \) are polarizations

therefore the relative GVO is:

\[
V^{(\lambda,\lambda_j)}_{(n_j)} = \frac{1}{2\pi i} \oint \frac{dz}{z} U^{(\lambda,\lambda_j)}_{(n_j)}(z)
\]
The action of the Lorentzian algebras on the representation GVO is a generalization of the commutation relation eq. (19)

\[
\left[ A^{\{r, r_i\}}_{\{n_i\}}, V^{\{\lambda, \lambda_j\}}_{\{n_j\}} \right] = \frac{1}{(2\pi i)^2} \oint_0 \frac{dz}{z} \oint_0 \frac{d\xi}{\xi} U^{\{r, r_i\}}_{\{n_i\}}(z) U^{\{\lambda, \lambda_j\}}_{\{n_j\}}(\xi) \tag{68}
\]

where we use the fact that \( \lambda \) is in the dual-lattice so we have an integer scalar product with the root \( r \) and the cocycle relations are extended also to the dual-lattice.

Then this operator creates states that transform under the representation of highest weight \( \Lambda \).

The conformal properties of the \( V^{\{\lambda, \lambda_j\}}_{\{n_j\}} \) are a consequence of general properties of the GVO and can be obtained immediately if we recognize that this vertex is the therm \(-h\) in the expansion of the vertex field \( U^{\{\lambda, \lambda_j\}}_{\{n_j\}}(z) \)

\[
U^{\{\lambda, \lambda_j\}}_{\{n_j\}}(z) = \sum_m A^{\{\lambda, \lambda_j\}}_{\{n_j\}}(z) z^{-m-h} \quad m \in \mathbb{Z} + h \tag{69}
\]

where \( h \) is the conformal weight and \( V^{\{\lambda, \lambda_j\}}_{\{n_j\}} = A^{\{\lambda, \lambda_j\}}_{\{n_j\}} - h \) so the commutation relations with the Virasoro algebra generators are

\[
\left[ L_m, V^{\{\lambda, \lambda_j\}}_{\{n_j\}} \right] = [(m + 1)h - m] A^{\{\lambda, \lambda_j\}}_{\{n_j\}} \tag{70}
\]

where we must consider only the value of \( m \in \{0, \pm 1\} \).

By the action of Lorentzian algebra on this GVO we deduce the following relation

\[
h = \frac{\lambda^2}{2} + \sum_j n_j = \frac{\Lambda^2}{2} \tag{71}
\]

in fact in the eq. (68) we extract the \(-h\) mode from the OPE of the fields with conformal weights

\[
h_r = \frac{r^2}{2} + \sum_i n_i = 1 \quad h_\lambda = \frac{\lambda^2}{2} + \sum_j n_j \tag{72}
\]

and the weight of the resulting field is

\[
h_{r+\lambda} = h_r + h_\lambda - 1 = h_\lambda \tag{73}
\]

Therefore the conformal weight of the representations are invariant for the algebra, but they are not invariant for the action of Virasoro algebra.

Note that \( h \leq 0 \) for the fundamental representations of the hyperbolic algebras while the Virasoro algebra invariance needs the condition \( h = 1 \) so we cannot have conformal invariant representations in this case.
4 Subalgebras of Ind-KMA

There is no general classification scheme for subalgebras of Ind-KMA or BA. Of course by deleting one or more vertex of a S(A) associated to a LA one finds regular subalgebras and, in particular, all the finite and affine subalgebras of Hyp-KMA have been classified in [15]. We want to show that in complete analogy with the Fin- and Aff-KMA cases the procedure of “folding”, which exploits the symmetry of the DD, can be applied to the case of LA and in the particular case of Hyp-KMA gives singular subalgebras of the same type. In fact we have

**Proposition 1** The DD obtained by folding the DD of a Hyp-KMA corresponds to a Hyp-KMA.

*Proof:* Let us recall that by folding of a Fin- or Aff-KMA an algebra of the same type is obtained and that by deleting one or more dots of a Hyp-KMA a Fin- or Aff-KMA is obtained. Let DD’ be the Dynkin diagram obtained by folding the DD of a Hyp-KMA (see below for a list), then it is evident that by deleting one dot of DD’ a diagram describing a Fin- or Aff-KMA is obtained. *Q.E.D.*

Even if the Hyp-KMA have been completely classified, at our knowledge, there is no generally accepted convention to identify them. We propose the following convention: to identify a rank $d$ Hyp-KMA we delete the $d$-th vertex from the corresponding DD, so we get a Fin- or Aff-KMA which can be identified by the standard Cartan notation, and then we add in square brackets the number(s) identifying the vertex(s) which are connected with the $d$-th vertex with an exponent(s) (omitted if equal to 1) denoting the number of lines in the connection. We omit to write the label $d$ in the square bracket if the $d$-th SR has the same length than the connected SR(s), otherwise we write $d$ before (after) the number(s) identifying the other SR(s) if the $d$-th is greater (smaller) than the connected SR. This convention is obviously not unique as it depends on the labelling of the SRs in the Hyp-, Fin- and Aff-KMA. In the case of hyperaffine KMA we identify the algebra by putting a hat on the notation of the corresponding Aff-KMA.

We list below the singular subalgebras which are obtained by “folding” the DD of the symmetric Hyp-KMA (see Appendix B). We follow the labelling of Appendix B and in the other cases we follow the convention of ref.[3], labelling however the affine root by the $d$ label rather than by the 0 label. In the following list the first number denotes the DD identifying the algebra in App. B and then we use the above convention

- 2) $A_1^{(1)}[(2)^2] \supset A_1^{2,4}$
where we introduce a notation to individuate the Hyp-KMAs of rank 2 in which we write the absolute values of the GCM elements $a_{12}$ and $a_{21}$ on the symbol $A_1$.

The procedure of folding can also be applied to transhyperbolic KMA, e.g. by folding $E_{19}$ we $E^{(1)}_8[(10,9)^2]$. The folding can also be applied to BA and may give algebras of the same type with associated Cartan matrix which are no more symmetric.

To give an example let us consider the following BA associated with the SCM

\[
A = \begin{pmatrix}
0 & -1 & 0 \\
-1 & 2 & -1 \\
0 & -1 & 0
\end{pmatrix}
\]
the subalgebra generated by

\[ E_{\tilde{\alpha}_1} = E_{\alpha_1} + E_{\alpha_3} \]  
\[ E_{\tilde{\alpha}_2} = E_{\alpha_2} \]  
\[ H_1 = H_1 + H_3 \]  
\[ H_2 = H_2 \]

which can be obtained by folding the roots \( \alpha_1 \) and \( \alpha_3 \), satisfies all the defining eqs. (74)-(77) with the following Cartan matrix

\[ A = \begin{pmatrix} 0 & -2 \\ -1 & 2 \end{pmatrix} \]

Note that one could have chosen also the second root as a vanishing one.

It is interesting to remark that the procedure of folding does not require the two roots \( \alpha_1 \) and \( \alpha_2 \) to be of the same type, i.e. both imaginary or real.

In the example considered with \( \alpha_3 \) real by folding we get \( B_2 \).

Finally we remark that some BA may appear as subalgebra of Ind-KMA. In fact the set of root of BA may be included in the set of roots of a Ind-KMA, an essential difference being the fact that imaginary roots appear as simple roots in BA while they appear at some height in the Ind-KMA. We are not able to make a general discussion of this topic and we discuss this point in some detail in a specific example, namely the case of the Hyp-KMA \( \tilde{A}^{(1)}_1 \) of which we recall in Appendix A the main properties and the VO realization. In this specific case we illustrate also the difficulties which appear in the construction of a VO for general BA.

Let us consider the Borcherds algebra, that we denote as \( A^{(0)}_1 \), associated to the \( 2 \times 2 \) SCM

\[ A = \begin{pmatrix} 0 & -1 \\ -1 & 2 \end{pmatrix} \]

and the corresponding \( S(A) \) is:
The SRs are $\hat{\alpha}_1, \hat{\alpha}_2$ which can be expressed as linear combination of the SRs of $\hat{A}^{(1)}_1$, denoted by $\alpha_i$, see App. A

\[
\hat{\alpha}_1 = \alpha_1 + \alpha_2 = K^+ \equiv [1, 1, 0]^0
\]
\[
\hat{\alpha}_2 = \alpha_3 = -(K^+ + K^-) \equiv [0, 0, 1]^2
\] 

(78) (79)

From the identification of the BA $A^{(0)}_1$ as a subalgebra of an Hyp-KMA we can get a VO realization of the algebra and we can discuss the structure of the fundamental representations. In order to achieve this we have to show that with this choice we really build up a BA, i.e. we have to verify that the defining eqs. (1) - (6) are satisfied when we identify $E_i$ with $A^{\hat{\alpha}_i}$ which is given by the eq.(99) that in this case read

\[
E_1 = A^{(\alpha_1 + \alpha_2, \alpha_1 - \alpha_2)} = \frac{1}{2\pi i} \oint_{C_0} dz : \left( \frac{d}{dz} - \frac{d}{d\xi} \right) U^{\alpha_1}(z) U^{\alpha_2}(\xi) : |_{z=\xi}
\] 

(80)

\[
E_2 = A^{\alpha_3} = \frac{1}{2\pi i} \oint_{C_0} dz U^{\alpha_2}(z)
\] 

(81)

Eq.(1) $(i \neq j)$ is immediately verified just remarking that $(\alpha_i, -\alpha_j) = 1$. Eq.(2) $(i = j)$ is easily verified just remarking that we have

\[
H_i = \hat{\alpha}_i \cdot p = \hat{\alpha}_{\mu\nu} p^\mu
\]

and $[p^\mu, p^\nu] = 0$. Eqs.(3)-(11) follow from the property
\[ [A, B] = c \rightarrow [A, \exp B] = c \exp B \quad c \in \mathbb{C} \] (83)

and from eqs.(94) and (95). Eqs.(8) (the Serre relations) can be verified in a straightforward way and have been explicitly computed in [23]. Let us remark that the fact that the elements of \( \hat{A}^{(1)}_1 \) above specified behave just as the generators of a BA can be shown quite in general without any use of the VO realization.

A few words on the representations of this BA. The fundamental weights are:

\[
\Lambda_1 = n\alpha - K^+ + K^- = (1, 0) \quad (n \in \mathbb{Z}_+)
\]
\[
\Lambda_2 = n\alpha - K^+ = (0, 1)
\] (84)

With a suitable choice of \( n \) \((n = 0)\) we can find the fundamental representations of \( A^{(0)}_1 \) as subrepresentations of the fundamental representations \((0, 1, 0)\) and \((0, 0, 1)\) of \( \hat{A}^{(1)}_1 \) discussed in [23].

\[
(1, 0) \subset (0, 1, 0) \quad (0, 1) \subset (0, 0, 1)
\] (85)

A few states of the \((1, 0)\) representation are given in fig.1. We can easily interpret the structure of the representation. The generator \( E^{-\hat{\alpha}_1} \) corresponds to the grade changing operator, while the \( E^{-\hat{\alpha}_2} \) is the lowering generator of \( A_1 \). The operator \( N \) associated to the commutator of the elements \( E^+, E^- \) \((r = -2\hat{\alpha}_1 - \hat{\alpha}_2 = -K^+ + K^-)\) is the intertwining operator which connects different \( A_1 \) representations. The multiplicity of the weights can be computed by the Freudental recursion formula and it has been explicitly computed for the lowest multiplicity by Slansky [9].

Let us remark that the choice we have done for the SRs is not unique. Indeed we could have made another choice for the 2nd SR, e.g. \( \alpha_2 + \alpha_3 \) instead of \( \alpha_2 \). Therefore \( \hat{A}^{(1)}_1 \) contains many BAs of the same type as subalgebras. Moreover one can also realize the the BA \( A^{(0)}_1 \) in a form which is not subalgebra of \( \hat{A}^{(1)}_1 \). In particular the structure of this BA which is discussed in [3] and which is obtained by adding to the algebra \( A_1 \) a suitable null root can be specified by the SRs:

\[
\hat{\alpha}_1 = \alpha
\]
\[
\hat{\alpha}_2 = 1/2(-\alpha + K^+ - K^-)
\] (86)

These roots do not belong to the set \( \Delta^+ \) of the roots of \( \hat{A}^{(1)}_1 \). A VO realization of this algebra should be of the form

\[
V^{\{\hat{\alpha}_2\}}(z) = r \cdot Q^{(1)}(z)U^{\{\hat{\alpha}_2\}}(z)
\] (87)
The polarization vector \( r = n\alpha + n_+ K^+ + n_- K^- \) is undetermined. By conformal properties we may require
\[
 r \cdot \alpha_2 = 0 \tag{88}
\]
However this choice still leaves some arbitrariness for \( r \). The \( N \) operator should be identified with an element of the Cartan subalgebra, but there is no affine subalgebra contained in this realization, as \( K^+ \) does not belong to the set of roots.

In order to emphasize that, even if the set of roots of \( BA \) is contained in the set of an Ind-KMA, the \( BA \) not always can be considered as a subalgebra, let us consider the \( BA \), that we denote as \( A_2^{(0)} \) associated to the \( 3 \times 3 \) SCM
\[
 A = \begin{pmatrix}
 0 & -1 & 0 \\
 -1 & 2 & -1 \\
 0 & -1 & 2 \\
\end{pmatrix}
\]
and the corresponding \( S(A) \) is:

\[
\begin{array}{c}
\oplus \\
1 \hspace{1cm} 2 \hspace{1cm} 3 \\
\end{array}
\]

The simple roots can be chosen as:
\[
\begin{align*}
\hat{\alpha}_1 &= \alpha_1 + \alpha_2 = K^+ \equiv [1, 1, 0]^0 \\
\hat{\alpha}_2 &= \alpha_3 = -(K^+ + K^-) \equiv [0, 0, 1]^2 \\
\hat{\alpha}_3 &= \alpha_2 = -\alpha + K^+ \equiv [0, 1, 0]^2 \\
\end{align*}
\tag{89}
\]

The \( BA \) can be build up adding to the (real) SRs of the Lie algebra \( A_2 \) an imaginary (null) root. The VOs describing the elements of \( \hat{A}_1^{(1)} \) corresponding to the above

25
(not simple) roots do not satisfy the defining eqs. (1)-(6) of a BA so they do not form a realization of $A_2^{(0)}$. In fact, e.g., we have

$$[E^\hat{\alpha}_1, E^\hat{\alpha}_3] \neq 0 \quad a_{13} = 0 \quad (90)$$

$$[E^\hat{\alpha}_1, E^{-\hat{\alpha}_3}] \neq 0 \quad (91)$$

Indeed as $E^{\hat{\alpha}_i}$ are described by GVOs, their commutator is not vanishing even if the scalar product of the corresponding roots is vanishing, as explicitly remarked in the case of $\hat{A}_1^{(1)}$. This difficulty is overcome in the case of $A_1^{(0)}$ as there is no SR unconnected with the imaginary SR.

Of course BA may contain as subalgebra KMA. We do not discuss this point as, missing a VO or other type of representation of a BA, it is not really useful.
5 Conclusions

We have presented an essential review of the Lorentzian algebras with emphasis on the connection between the conformal field theory language and the properties of the fields which appear in the VO. The discussion of the conformal behaviour of the field requires the construction of the Virasoro algebra associated with the KMA, which has been discussed, in full generality, by Borcherds in [11]. The conformal structure of the fields in the vertex operator construction is extremely relevant for physical applications and it depends on the value of the dimension $d$. An other relevant aspect, which is related with the previous one, is the action of the GVOs on the Fock space of the representation. Many problems are still opened. One fundamental point still missing is the relation between a current algebra realization and the Lorentzian algebras. Moreover while the interpretation of the VO corresponding to imaginary negative roots as corresponding to massive states in the string theory language is clear, one may wonder if there is a deeper and more essential connection to be discovered between Lorentzian algebras and CFT. In sec. 1 we recalled that one of the motivation for the study of Lorentzian algebra has been based on the need in the case of superstring to introduce fermionic ghosts in order to insure the correct conformal behaviour and then to extend the space of the lattice to a Lorentzian one. It is well known that ghosts are essential ingredients in the quantization procedure. So it is tempting to think that Lorentzian algebras are more closely connected with the quantization procedure. Many speculations in this direction have appeared, but there is no clear convincing argument for this conjecture. Very likely this point is not completely unrelated from the previous one.

The bosonic VO construction we have presented in sec. 3 is completely general and it applies to any Ind-KMA. In this paper we have discussed the case of $D-dim$ space with indefinite metric $g_{\mu,\nu} = (-, + \ldots +)$. The generalization to the case with more then one $(-)$ sign can be done without difficulty just by a suitable redefinition of eqs.(35) and (36).

It is natural to argue that also for this algebras a fermionic VO construction can be obtained which can be essential for the case of not simply-laced algebras and, may be, to get general VO representations of the BA.

The procedure of folding has allowed us to find a class of subalgebras and, in analogy with the case of Aff-KMA [25], may give a method to build VO realization of not symmetric LA. A difficulty which has to be overcome is connected to the construction of suitable auxiliary fields to obtain the fermionic nature of the fields connected to the short roots and of the necessary cocycles.

The general structure of an IR for any Hyp KMA can be inferred from our
discussion in sec. 4. However many open problems are still present. For instance a
general proof of the complete reducibility of a h.w. IR in terms of IRs of the affine
subalgebra and a formula (at least formal) giving the decomposition of an IR of
Hyp KMA with respect to the affine subalgebra are missing. Moreover the string
functions, which in the case of Aff KMAs allow the computation of the multiplicity
of the weights, are not known for the Hyp KMAs.

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Appendix A

The symmetric GCM defining the $\hat{A}_1^{(1)}$ algebra is [4, 24]:

$$A = \begin{pmatrix}
2 & -2 & 0 \\
-2 & 2 & -1 \\
0 & -1 & 2
\end{pmatrix}$$

and the corresponding $S(A)$ is:

The simple roots are, denoting by $\alpha$ the root of $A_1$, $(\alpha_i, \alpha_i) = 2$:

$$\begin{align*}
\alpha_1 &= \alpha \\
\alpha_2 &= -\alpha + K^+ \\
\alpha_3 &= -K^+ - K^-
\end{align*}$$

(92)

The algebra is defined by eq.(1÷5), where now $E_i(F_i)$ (i=1,2,3) correspond resp. to $\alpha_i(-\alpha_i)$ and $a_{ij}$ is given in terms of the SR by eq.(7).

The set of roots $r$ ( $\Delta = \{r\}$ ) [24] is given by $r = \sum_{i=1}^{3} k_i \alpha_i$ where the triple of integers is constrained by:

$$(k_1^2 + k_2^2 + k_3^2) - 2k_1k_2 - k_2k_3 \leq 1$$

(93)

In the following we will specify a root by a triple of integers $[k_1, k_2, k_3]$, denoting by a superscript on the triple its length. In [23] a formula for computing the triple
of (finite) numbers \([k_1, k_2, k_3]\) in function of the height \((ht)\) \((ht = k_1 + k_2 + k_3)\) and of the length \((L)\) of the roots is given.

The roots of low height \((ht)\) are:

\[
ht = 1 \ \{[1, 0, 0]^2; [0, 1, 0]^2; [0, 0, 1]^2\}
\]

\[
ht = 2 \ \{[1, 1, 0]^2; [0, 1, 1]^2\}
\]

\[
ht = 3 \ \{[2, 1, 0]^2; [1, 2, 0]^2; [0, 2, 1]^0\}
\]

\[
ht = 4 \ \{[2, 2, 0]^0; [2, 1, 1]^2; [1, 2, 1]^0\}
\]

\[
ht = 5 \ \{[3, 2, 0]^2; [2, 3, 0]^2; [2, 2, 1]^{-2}; [1, 2, 2]^2\}
\]

Remark the first vanishing (negative) length root does appear, respectively at \(ht\) = 2, 5.

Let us explicitly write a few relevant commutation relations for the simple roots of \(\hat{A}_1^{(1)}\).

We have \((i=1,2,3)\)

\[
[A^{\alpha_i}, A^{-\alpha_i}] = \alpha_i \cdot p \tag{94}
\]

with

\[
[\alpha_i \cdot p, A^{\pm \alpha_j}] = \pm (\alpha_i \cdot \alpha_j) A^{\pm \alpha_j} = \pm a_{ij} A^{\pm \alpha_j} \tag{95}
\]

where are been made use of eqs.(42), (33) and (31).

Then we have:

\[
[A^{\alpha_1}, A^{\alpha_3}] = 0 \quad (a_{13} = 0) \tag{96}
\]

\[
[A^{\alpha_2}, A^{\alpha_3}] = A^{\alpha_2 + \alpha_3} \quad (a_{23} = -1) \tag{97}
\]

\[
[A^{\alpha_2}, [A^{\alpha_2}, A^{\alpha_3}]] = 0 \tag{98}
\]

as \((\alpha_2 \cdot (\alpha_2 + \alpha_3) = 1)\)

\[
[A^{\alpha_1}, A^{\alpha_2}] = \\
= \chi^{\{\alpha_1, \alpha_2\}}_{(1)} \oint_{C_0} dz : \frac{1}{2} \left( \frac{d}{dz} - \frac{d}{d\xi} \right) U^{\alpha_1}(z) U^{\alpha_2}(\xi) : |_{z=\xi} \]

\[
= \frac{1}{2} A^{\{\alpha_1 + \alpha_2, \alpha_1 - \alpha_2\}}_{(1)} \quad (a_{12} = -2) \tag{99}
\]
\[ [A^\alpha_1, [A^\alpha_1, A^\alpha_2]] = [A^\alpha_1, \frac{1}{2}A^{(\alpha_1+\alpha_2,\alpha_1-\alpha_2)}] = \]
\[ = \frac{1}{2}(\alpha_1 \cdot (\alpha_2 - \alpha_1))A^{2\alpha_1+\alpha_2} \]  \hspace{1cm} (100)

Let us remark that the above commutator is not vanishing in spite of the vanishing of the scalar product \(\alpha_1 \cdot (\alpha_1 + \alpha_2) = 0\) as the VO corresponding to the root \(\alpha_1 + \alpha_2\) is a GVO and then in the commutator a pole of order \(\alpha_1 \cdot (\alpha_1 + \alpha_2) - 1 = -1\) appears, see eq.(44). The commutator of \(A^\alpha_1\) with \(A^{2\alpha_1+\alpha_2}\) vanishes as \(\alpha_1 \cdot (2\alpha_1 + \alpha_2) = 2\).
Appendix B

We write a system of simple roots for all symmetric hyperbolic algebras \((d \geq 3)\) in which we decompose the Lorentzian lattice in a transverse Euclidean and a longitudinal-timelike lattice.

For any algebra we report the Dynkin diagram and we write the simple roots in terms of the simple roots of the related affine subalgebra while the extended roots is written in terms of two light like vectors \(K^\pm\) in the longitudinal-time like space. In the following we denote with \(HR\) the highest root of the horizontal finite Lie algebra

1. \(\hat{A}_1^{(1)}\)
   \[ r_i \in A_1^{(1)} \quad \text{for } r_i \in \{1, \ldots, 2\}; \quad r_3 = -K^+ - K^- \]

2. \(A_1^{(1)}[(2)^2]\)
   \[ r_i \in A_1^{(1)} \quad \text{for } r_i \in \{1, \ldots, 2\}; \quad r_3 = -\frac{1}{2}K^+ - 2K^- \]

3. \(A_1^{(1)}[1, (2)^2]\)
   \[ r_i \in A_1^{(1)} \quad \text{for } r_i \in \{1, \ldots, 2\}; \quad r_3 = -\lambda_s - \frac{1}{2}K^+ - 3K^- \]

4. \(A_1^{(1)}[1, 2]\)
   \[ r_i \in A_1^{(1)} \quad \text{for } r_i \in \{1, \ldots, 2\}; \quad r_3 = -\lambda_s - \frac{3}{8}K^+ - 2K^- \]

5. \(A_1^{(1)}[(1)^2, (2)^2]\)
   \[ r_i \in A_1^{(1)} \quad \text{for } r_i \in \{1, \ldots, 2\}; \quad r_3 = -HR - 4K^- \]

6. \(\hat{A}_2^{(1)}\)
   \[ r_i \in A_2^{(1)} \quad \text{for } r_i \in \{1, \ldots, 3\}; \quad r_4 = -K^+ - K^- \]

7. \(A_2^{(1)}[2, 3]\)
   \[ r_i \in A_2^{(1)} \quad \text{for } r_i \in \{1, \ldots, 3\}; \quad r_4 = -HR - 2K^- \]

8. \(A_2^{(1)}[1, 2, 3]\)
   \[ r_i \in A_2^{(1)} \quad \text{for } r_i \in \{1, \ldots, 3\}; \quad r_4 = -HR - 3K^- \]

9. \(\hat{A}_3^{(1)}\)
   \[ r_i \in A_3^{(1)} \quad \text{for } r_i \in \{1, \ldots, 4\}; \quad r_5 = -K^+ - K^- \]
10. $A_3^{(1)}[1, 3]$
   $r_i \in A_3^{(1)}$ for $r_i \in \{1, \ldots, 4\}; \quad r_5 = -HR - 2K^-$
11. $\hat{A}_4^{(1)}$
   $r_i \in A_4^{(1)}$ for $r_i \in \{1, \ldots, 5\}; \quad r_6 = -K^+ - K^-$
12. $\hat{D}_4^{(1)}$
   $r_i \in D_4^{(1)}$ for $r_i \in \{1, \ldots, 5\}; \quad r_6 = -K^+ - K^-$
13. $D_4^{(1)}[1]$
   $r_i \in D_4^{(1)}$ for $r_i \in \{1, \ldots, 5\}; \quad r_6 = -HR - 2K^-$
14. $\hat{A}_5^{(1)}$
   $r_i \in A_5^{(1)}$ for $r_i \in \{1, \ldots, 6\}; \quad r_7 = -K^+ - K^-$
15. $\hat{D}_5^{(1)}$
   $r_i \in D_5^{(1)}$ for $r_i \in \{1, \ldots, 6\}; \quad r_7 = -K^+ - K^-$
16. $\hat{A}_6^{(1)}$
   $r_i \in A_6^{(1)}$ for $r_i \in \{1, \ldots, 7\}; \quad r_8 = -K^+ - K^-$
17. $\hat{E}_6^{(1)}$
   $r_i \in E_6^{(1)}$ for $r_i \in \{1, \ldots, 7\}; \quad r_8 = -K^+ - K^-$
18. $\hat{D}_6^{(1)}$
   $r_i \in D_6^{(1)}$ for $r_i \in \{1, \ldots, 7\}; \quad r_8 = -K^+ - K^-$
19. $\hat{A}_7^{(1)}$
   $r_i \in A_7^{(1)}$ for $r_i \in \{1, \ldots, 8\}; \quad r_9 = -K^+ - K^-$
20. $\hat{E}_7^{(1)}$
   $r_i \in E_7^{(1)}$ for $r_i \in \{1, \ldots, 8\}; \quad r_9 = -K^+ - K^-$
21. $\hat{D}_7^{(1)}$
   $r_i \in D_7^{(1)}$ for $r_i \in \{1, \ldots, 8\}; \quad r_9 = -K^+ - K^-$
22. $\hat{D}_8^{(1)}$
\[ r_i \in D_8^{(1)} \text{ for } r_i \in \{1, \ldots, 9\}; \quad r_{10} = -K^+ - K^- \]

23. $\hat{E}_8^{(1)}$
\[ r_i \in E_8^{(1)} \text{ for } r_i \in \{1, \ldots, 9\}; \quad r_{10} = -K^+ - K^- \]
FIGURES CAPTIONS

Fig. 1 The first states of the first three levels of \((1, 0)\) representation of the BA subalgebra in the \((0, 1, 0)\) of the Hyp-KMA \(\hat{A}_1^{(1)}\) are reported.

A state is denoted by a dot, the full and dotted arrows denote resp. the action of the BA and Hyp-KMA generators.

Vertical (horizontal) full arrow corresponds, respectively, to the action of \(E^{−\hat{\alpha}_1}\) \((E^{−\hat{\alpha}_2})\). Dotted arrow pointing to the right (left) corresponds, respectively, to the action of \(E^{−\alpha_2}\) \((E^{−\alpha_1})\).

The integers in brackets gives the weight of the states respect the BA subalgebra.
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