Relativistic Dissipative Fluid Dynamics from Non-Equilibrium Statistical Operator

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Abstract: We present a new derivation of the second-order relativistic dissipative hydrodynamics for quantum systems using Zubarev’s formalism of non-equilibrium statistical operator. In particular we discuss the second-order expression for the shear stress tensor and argue that the relaxation terms for the dissipative quantities arise from the memory effects of the statistical operator. We also identify the new transport coefficients which describe the relaxation of dissipative processes to second order and express them in terms of equilibrium correlation functions, thus establishing Kubo-type formulae for second-order transport coefficients.

Keywords: relativistic hydrodynamics, statistical operator, non-equilibrium states, transport coefficients, correlation functions

1. Introduction

Fluid dynamics (hydrodynamics) is a powerful tool to describe low-frequency and long-wave-length phenomena in statistical systems [1]. It finds numerous applications in astrophysics, cosmology, heavy-ion physics and other areas. In particular, it has been applied successfully to describe the quark-gluon-plasma (QGP) created in heavy-ion collision experiments at RHIC and LHC assuming an almost perfect-fluid behavior.

In this work we adopt the method of non-equilibrium statistical operator [2,3] to obtain the hydrodynamic equations of strongly correlated matter, such as the QGP, in the non-perturbative regime. The methods was applied to the quantum fields [4] and has been since extended to treat systems in strong magnetic fields [5]. It is based on a generalization of the Gibbs canonical ensemble to non-equilibrium states, i.e., the statistical operator is promoted to a non-local functional of the thermodynamic parameters and their space-time derivatives. Assuming that the thermodynamic parameters are sufficiently smooth over the correlation lengths characterizing the system, the statistical operator is expanded into series in gradients of these parameters to the desired order. The hydrodynamics equations for the dissipative fluxes emerge then after statistical averaging of the relevant quantum operators. An advantage of the method of non-equilibrium statistical operator (hereafter NESCO) is that the transport coefficients of the system are automatically obtained in the form of Kubo-type relations, i.e., they are related to certain correlation functions of the underlying field theory in the strong coupling regime.

This contribution provides a concise presentation of our recent work on derivation of the second-order fluid dynamics from the method of NESCO [6,7]. As well known, hydrodynamics describes the state of a relativistic fluid in terms of its energy-momentum tensor and currents of conserved charges, which in the relevant low-frequency and long-wavelength limit can be expanded around their equilibrium values. The zeroth-order expansion corresponds to the ideal
(non-dissipative) hydrodynamics. At the first-order, the dissipative relativistic hydrodynamics emerges from truncation that keeps the linear order terms in gradients [1,8]. Second-order relativistic theories were also constructed [9,10] to avoid the acausality of the first-order theory and the resulting numerical instabilities. In the second-order theories the dissipative fluxes satisfy relaxation equations, which describe the process of their relaxation towards their Navier-Stokes values at asymptotically large times. While the general structure of the second-order fluid dynamics is known, varying results have been obtained for the coefficients entering these equations, see e.g., [11,12]. The various versions of the second-order hydrodynamics and relaxation equations are reviewed and compared to each other in, e.g., in the review articles [13–15], to which we refer the reader for more detailed expositions.

This work is structured as follows. Section 2 gives a brief summary of the Zubarev’s formalism of the NESO [2,3]. Section 3 recapitulates the Navier-Stokes theory and the Kubo formulas for the first-order transport coefficients. The second-order transport equations are discussed in Section 4 and a summary is given in Section 5. We work below in the flat space described by the metric $g^{\mu\nu} = \text{diag}(+,−,−,−)$.

2. Non-equilibrium statistical operator and correlation functions

The hydrodynamic state of a relativistic quantum system is described by the operators of the energy-momentum tensor $\hat{T}^{\mu\nu}(x)$ and the particle current $\hat{N}^\mu(x)$. The equations of relativistic hydrodynamics consist of the covariant conservation laws for these quantities

$$\partial_\mu \hat{T}^{\mu\nu}(x) = 0, \quad \partial_\mu \hat{N}^\mu(x) = 0. \quad (1)$$

Here we assume that the fluid contains only one particle species. The generalization to the case of multiple conserved species is straightforward and will be given elsewhere [6].

Hydrodynamic description of fluids is based on the concept of local thermodynamic equilibrium. This allows one to introduce local thermodynamic parameters, such as temperature $T(x) \equiv \beta^{-1}(x)$, chemical potential $\mu(x)$ and fluid 4-velocity $u^\nu(x)$ as slowly varying functions of the space-time coordinates $x \equiv (x,t)$. In terms of these quantities we define new auxiliary functions

$$\beta^\nu(x) = \beta(x)u^\nu(x), \quad \alpha(x) = \beta(x)\mu(x). \quad (2)$$

Note that in this context “slowly” means that the characteristic macroscopic scales over which the hydrodynamic quantities change in space and time should be much larger compared to the characteristic microscopic scales of the system.

Consider now the NESO given by [5]

$$\hat{\rho}(t) = Q^{-1}e^{-\hat{A}+\hat{B}}, \quad Q = \text{Tr}e^{-\hat{A}+\hat{B}}, \quad (3)$$

where

$$\hat{A}(t) = \int d^3x \left[ \beta^\nu(x)\hat{T}_{\nu\mu}(x) - \alpha(x)\hat{N}^\mu(x) \right], \quad (4)$$
$$\hat{B}(t) = \int d^3x_1 \int_{-\infty}^t dt_1 e^{\epsilon(t_1-t)}\hat{C}(x_1), \quad (5)$$
$$\hat{C}(x) = \hat{T}^{\mu\nu}(x)\partial_\nu \beta^\nu(x) - \hat{N}^\mu(x)\partial_\mu \alpha(x), \quad (6)$$

with $\epsilon \rightarrow +0$ taken after the thermodynamic limit. The NESO satisfies the quantum Liouville equation with an infinitesimal source term $\epsilon$, which for positive values selects the retarded solution [2, 3]. The operators $\hat{A}(t)$ and $\hat{B}(t)$ correspond to the equilibrium and non-equilibrium parts of the statistical operator, where the operator $\hat{C}(t)$ stands for the thermodynamic “force” as it involves
the gradients of local thermodynamic variables, i.e., the temperature, chemical potential and fluid 4-velocity. We define also the local equilibrium statistical operator as

\[ \hat{\rho}_0(t) = Q_l^{-1} e^{-\hat{A}}, \quad Q_l = \text{Tr} e^{-\hat{A}}, \]  

which is the analogue of the Gibbs distribution involving local thermodynamic parameters.

Before proceeding, we remark that the thermodynamic variables are well-defined quantities only in equilibrium states. Their extension to non-equilibrium states requires a prescription that allows one to construct their non-equilibrium counterparts. For this purpose we define first the operators of the energy and particle densities via \( \hat{\epsilon} = u_\mu u^\mu \hat{T}^{\mu \nu} \) and \( \hat{n} = u_\mu \hat{N}^\mu \). These imply simply that \( \hat{\epsilon} \) and \( \hat{n} \) are the time-like eigenvalues of the energy-momentum tensor and the charge currents, respectively, measured by a local observer comoving with a fluid element. The local values of the Lorentz-invariant thermodynamic parameters \( \beta(x) \) and \( \alpha(x) \) can be fixed then by requiring that the average values of the operators \( \hat{\epsilon} \) and \( \hat{n} \) are match the local equilibrium values of these quantities. The matching conditions [2,3] are then written as

\[ \langle \hat{\epsilon}(x) \rangle = \langle \hat{\epsilon}(x) \rangle_1, \quad \langle \hat{n}(x) \rangle = \langle \hat{n}(x) \rangle_1, \]  

where for arbitrary operator \( \hat{X}(x) \) the non-equilibrium and local equilibrium statistical averages are defined as

\[ \langle \hat{X}(x) \rangle = \text{Tr}[\hat{\rho}(t) \hat{X}(x)], \quad \langle \hat{X}(x) \rangle_1 = \text{Tr}[\hat{\rho}_0(t) \hat{X}(x)]. \]  

The local equilibrium values \( \langle \hat{\epsilon} \rangle_1 \) and \( \langle \hat{n} \rangle_1 \) in Eq. (8) are evaluated formally at constant values of \( \beta \) and \( \mu \), which are identified then by matching \( \langle \hat{\epsilon} \rangle_1 \) and \( \langle \hat{n} \rangle_1 \) to the real values of these quantities \( \langle \hat{\epsilon} \rangle \) and \( \langle \hat{n} \rangle \) at any given point \( x \).

At this point it useful to note that the 4-velocity \( u^\mu \) acquires physical meaning after it is related to a particular physical current. For example, in the Landau-Lifshitz frame the fluid 4-velocity is parallel to the fluid 4-momentum or, equivalently, to the energy flow, i.e., \( u_\mu \langle \hat{T}^{\mu \nu} \rangle = \langle \hat{\epsilon} \rangle u^\nu \) [1]. In the Eckart frame the fluid velocity is associated with the particle flow via \( \langle \hat{N}^\mu \rangle = \langle \hat{n} \rangle u^\mu \) [8]. However, in the following, we will keep the fluid velocity generic without specifying any particular reference frame.

The next step is to expand the NESO around the local equilibrium value (7) treating the non-equilibrium part, which is described by the operator \( \hat{B} \), as a perturbation

\[ \hat{\rho} = \hat{\rho}_0 + \hat{\rho}_1 + \hat{\rho}_2, \]  

where the first-order term is given by

\[ \hat{\rho}_1(t) = \int d^4x_1 \int_0^1 d\tau \left[ \hat{C}_\tau(x_1) - \langle \hat{C}_\tau(x_1) \rangle_1 \right] \hat{\rho}_1, \]  

while the second order term by

\[ \hat{\rho}_2(t) = \frac{1}{2} \int d^4x_1 d^4x_2 \int_0^1 d\tau \int_0^1 d\lambda \left[ \hat{T} \{ \hat{C}_\lambda(x_1) \hat{C}_\tau(x_2) \} - \langle \hat{T} \{ \hat{C}_\lambda(x_1) \hat{C}_\tau(x_2) \} \rangle_1 \right. 
\]  
\[ \left. - \langle \hat{C}_\lambda(x_1) \rangle_1 \langle \hat{C}_\tau(x_2) \rangle_1 + \hat{C}_\lambda(x_1) \langle \hat{C}_\tau(x_2) \rangle_1 + 2 \langle \hat{C}_\lambda(x_1) \rangle_1 \langle \hat{C}_\tau(x_2) \rangle_1 \right] \hat{\rho}_1. \]  

Here \( \hat{T} \) is the anti-chronological operator acting on variables \( \tau \) and \( \lambda \) and we used the short-hand notations

\[ \int d^4x_1 = \int d^3x_1 \int_0^1 dt_1 u^\rho(t_1), \quad \hat{X}_\alpha = e^{-\alpha A} \hat{X} e^{\alpha A}, \quad \alpha \in \tau, \lambda \]  

\[ (12) \]
The expansion (10) implies that the statistical average of any operator \( \hat{X}(x) \) can be decomposed into three terms
\[
\langle \hat{X}(x) \rangle = \langle \hat{X}(x) \rangle_1 + \langle \hat{X}(x) \rangle_1 + \langle \hat{X}(x) \rangle_2,
\] (14)
where the first-order term is given by
\[
\langle \hat{X}(x) \rangle_1 = \int d^4x_1 \left( \hat{X}(x), \hat{C}(x_1) \right),
\] (15)
with
\[
\left( \hat{X}(x), \hat{Y}(x_1) \right) = \int_0^1 d\tau \langle \hat{X}(x) \rangle [\hat{Y}_\tau(x_1) - \langle \hat{Y}_\tau(x_1) \rangle_1]_l
\] (16)
being the two-point correlation function between two arbitrary operators \([4,5]\). The second-order term in Eq. (14) can be written as
\[
\langle \hat{X}(x) \rangle_2 = \int d^4x_1 d^4x_2 \left( \hat{X}(x), \hat{C}(x_1), \hat{C}(x_2) \right),
\] (17)
where we introduced the three-point correlation function of operators \( \hat{X}, \hat{Y} \) and \( \hat{Z} \) as
\[
\left( \hat{X}(x), \hat{Y}(x_1), \hat{Z}(x_2) \right) = \frac{1}{2} \int_0^1 d\tau \int_0^1 d\lambda \langle \hat{X}(x) \rangle \left[ \hat{Y}_\lambda(x_1) \hat{Z}_\tau(x_2) - \langle \hat{Y}_\lambda(x_1) \hat{Z}_\tau(x_2) \rangle_1 \right.
- \langle \hat{Y}_\lambda(x_1) \rangle_1 \hat{Z}_\tau(x_2) - \hat{Y}_\lambda(x_1) \langle \hat{Z}_\tau(x_2) \rangle_1
\] (21)
\[ 2 \langle \hat{Y}_\lambda(x_1) \rangle_1 \langle \hat{Z}_\tau(x_2) \rangle_1 \right].
\] (18)

3. Relativistic fluid dynamics at first order

To examine specific dissipative processes, e.g., conduction of heat, shear and bulk stresses, etc., the energy-momentum tensor and the particle current are decomposed as
\[
\hat{T}^{\mu\nu} = \dot{\epsilon} u^\mu u^\nu - \hat{\rho} \Delta^{\mu\nu} + \hat{\sigma}^{\mu\nu} + \hat{\tau}^\mu u^\nu + \hat{\lambda}^{\mu\nu},
\] (19)
\[
\hat{N}^\mu = \hat{n} u^\mu + \hat{j}^\mu,
\] (20)
where the fluid velocity \( u^\mu \) is normalized as \( u_\mu u^\mu = 1 \), and \( \Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu \) is the projection operator onto the 3-space orthogonal to \( u^\mu \). Clearly, these decompositions have the most general form that can be constructed from the fluid velocity and the tensor \( \Delta^{\mu\nu} \). The operators of the physical quantities on right-hand sides of Eqs. (19) and (20) can be written as certain projections of the energy-stress tensor and particle current
\[
\dot{\epsilon} = u_\mu u^\nu \hat{T}^{\mu\nu}, \quad \hat{n} = u_\mu \hat{N}^\mu, \quad \hat{\rho} = -\frac{1}{3} \Delta^{\mu\nu} \hat{T}^{\mu\nu},
\] (21)
\[
\hat{\tau}^{\mu\nu} = \Delta^{\mu\nu} \hat{\tau}^{\alpha\beta}, \quad \hat{\sigma}^\mu = u_\alpha \Delta_\mu^{\alpha\beta} \hat{\tau}^{\alpha\beta}, \quad \hat{j}^\nu = \Delta^\nu \hat{N}^\mu,
\] (22)
where
\[
\Delta_{\mu\nu\rho\sigma} = \frac{1}{2} \left( \Delta_{\mu\rho} \Delta_{\nu\sigma} + \Delta_{\mu\sigma} \Delta_{\nu\rho} \right) - \frac{1}{3} \Delta_{\mu\nu} \Delta_{\rho\sigma}
\] (23)
is a forth-rank traceless projector orthogonal to \( u^\mu \). It is seen from Eq. (21) that \( \dot{\epsilon} \) is the energy density operator, \( \hat{n} \) is the operator of particle density, \( \hat{\rho} \) is the operator of pressure, and the dissipative terms \( \hat{\tau}^{\mu\nu}, \hat{\sigma}^\mu \) and \( \hat{j}^\nu \) in (22) are the shear stress tensor, energy diffusion flux and particle diffusion flux, respectively. These satisfy the conditions
\[
u_\nu \hat{\tau}^\nu = 0, \quad u_\nu \hat{\tau}^\nu = 0, \quad u_\nu \hat{\tau}^{\mu\nu} = 0, \quad \hat{\tau}^\mu_\mu = 0.
\] (24)
In local equilibrium the averages of these operators vanish [16]:
\[
(\hat{q}^\mu)_1 = 0, \quad (\hat{j}^\mu)_1 = 0, \quad (\hat{\pi}^{\mu\nu})_1 = 0,
\]
and one recovers the limit of ideal hydrodynamics. The local equilibrium pressure is given by the equation of state, i.e., \( \langle \hat{p} \rangle_1 \equiv p = p(\epsilon, n) \), which closes the set of ideal hydrodynamics equations.

Consider next the fluid dynamics at first order. Quite generally the hydrodynamic quantities \( \pi^{\mu\nu}, q^\mu \) and \( j^\mu \) are obtained as the statistical averages of the corresponding operators over the NESO according to Eqs. (14)–(18). Keeping only the first-order terms in Eq. (14) we obtain the relativistic Navier-Stokes equations
\[
\tau_{\mu\nu} = 2\eta \sigma_{\mu\nu}, \quad \Pi = -\zeta \theta, \quad \mathcal{J}_\mu = \kappa \left( \frac{nT}{h} \right)^2 \nabla_\mu \alpha,
\]
where \( \pi_{\mu\nu} \equiv \langle \hat{\pi}_{\mu\nu} \rangle \), etc., the bulk viscous pressure \( \Pi = \langle \hat{p} \rangle - \langle \hat{p} \rangle_1 \) is the difference between the first-order average of the pressure operator and the local value of pressure, \( h = \epsilon + p \) is the enthalpy and
\[
\mathcal{J}_\mu = j_\mu - \frac{n}{h} q_\mu
\]
is the irreversible particle flow, i.e., the particle flow with respect to the energy flow \([9,10]\). On the right-hand side of Eq. (26) \( \tau_{\mu\nu} = \partial_{\varphi^{\mu}} \pi^{\nu>}_\varphi \) is the velocity stress tensor defined through the short-hand notation \( A_{\mu>\nu} = \Delta_{\mu>\nu}A_{\beta>\gamma}, \theta = \partial_{\mu} u^{\mu} \) is the fluid expansion rate, \( \nabla_\alpha = \Delta_\alpha \partial_\alpha \) is the covariant spatial derivative. The coefficients \( \eta, \zeta \) and \( \kappa \) have the usual meaning of the transport coefficients of the shear and bulk viscosities and thermal conductivity, respectively. These transport coefficients can be expressed through two-point correlation functions via the following Kubo formulas \([4,5]\)
\[
\eta = \frac{\beta}{10} \int d^4x_1 \left( \hat{\pi}_{\alpha\nu}(x), \hat{\pi}^{\alpha\nu}(x_1) \right), \quad (28)
\]
\[
\zeta = \beta \int d^4x_1 \left( \hat{\rho}^\gamma(x), \hat{\rho}^\gamma(x_1) \right), \quad (29)
\]
\[
\kappa = -\frac{\beta^2}{3} \int d^4x_1 \left( \hat{\mu}(x), \hat{\mu}(x_1) \right), \quad (30)
\]
where
\[
\hat{\rho}^\gamma = \hat{\rho} - \gamma \hat{\epsilon} - \delta \hat{n}, \quad \hat{\mu}^\mu = \hat{q}^\mu - \frac{h}{n} \hat{p}^\mu,
\]
and
\[
\gamma = \left( \frac{\partial p}{\partial \epsilon} \right)_n, \quad \delta = \left( \frac{\partial p}{\partial n} \right)_\epsilon.
\]
The correlation functions in Eqs. (28)–(30) can be evaluated already with the thermodynamic parameters of uniform background matter, i.e., as if the system was in global thermal equilibrium. They can be expressed in terms of the two-point retarded equilibrium Green functions as
\[
\eta = -\frac{1}{10} \left. \frac{d}{d\omega} \text{Im} G^R_{\hat{\pi}_{\alpha\nu}\hat{\pi}^{\alpha\nu}}(\omega) \right|_{\omega=0}, \quad (33)
\]
\[
\zeta = -\left. \frac{d}{d\omega} \text{Im} G^R_{\hat{\rho}\hat{\rho}}(\omega) \right|_{\omega=0}, \quad (34)
\]
\[
\kappa = \left. \frac{1}{3T} \frac{d}{d\omega} \text{Im} G^R_{\hat{\mu}\hat{\mu}}(\omega) \right|_{\omega=0}, \quad (35)
\]
where for any two operators $\hat{X}$ and $\hat{Y}$

$$G_{\hat{X}\hat{Y}}^R(\omega) \equiv -i \int_0^\infty dt e^{i\omega t} \int d^3x \langle [\hat{X}(x, t), \hat{Y}(0, 0)] \rangle_t.$$  \hspace{1cm} (36)

Equations (33)–(35) represent a particularly suitable form of Kubo formulas which lend themselves to evaluation with methods of the equilibrium finite-temperature field theory. Before closing this section it is useful to clarify the relation between the expansions in the powers of thermodynamic forces and the Knudsen number $K = l/L$, where $l$ and $L$ are typical microscopic and macroscopic length scales. To obtain the relations (26) from Eq. (15) we used Curie’s theorem. It states that in isotropic medium the correlations between operators of different rank vanish [17]. The integrands in Eqs. (28)–(30) are mainly concentrated within the range $|x_1 - x| \lesssim l$, where $l$ is the mean correlation length, which in the weak coupling limit is of the order of the particle mean free path. Hydrodynamic regime implies $l \ll L$, where $L$ is the typical length scale over which the parameters $\beta^\mu$ and $\alpha$ vary in space. Therefore, the thermodynamic forces $\partial^\mu \beta^\nu$ and $\partial^\mu \alpha$ involved in Eq. (6) can be factored out from the integral (15) with their average values at $x$, i.e., the non-locality of the thermodynamic forces can be neglected in this approximation. Because $|\sigma^\mu| \simeq |\mu^\mu|/L$, relations (26) obtained from the gradient expansion (10) of the NESCO are consistent with the expansion scheme in the powers of the Knudsen number.

4. Relativistic fluid dynamics at second order

We have computed systematically all second-order corrections to the dissipative quantities $\pi^\mu_\nu$, $\Pi$ and $\mathcal{J}^\mu$ on the basis of Eqs. (14)–(18) [6,7]. Before presenting the results, we note that the second-order contributions arise not only from Eq. (17), which is quadratic in the thermodynamic force $\tilde{C}$, but also from Eq. (15), where the non-local nature of the thermodynamic forces in space and time should be carefully taken into account. The non-local effects generate finite relaxation terms in the hydrodynamic equations, which are required for the causality. To see that these corrections are of the second order in the Knudsen number, note that they involve the differences of the thermodynamic forces, e.g., $\partial^\mu \beta^\nu(x)$, at the points $x_1$ and $x$ [see Eqs. (6) and (15)]. Therefore, we can approximate $\partial^\mu \beta^\nu(x_1) - \partial^\mu \beta^\nu(x) \simeq \partial_1 \partial^\mu \beta^\nu(x)(x_1 - x)^i \sim K\delta^\mu \beta^\nu(x)$, because $x_1 - x \sim l$ and $\partial \sim L^{-1}$, as already done in Sec. 3. Thus, these corrections contain an additional power of the Knudsen number $K$ as compared to the first-order expressions (26), and, therefore, are of second order.

Here we restrict ourselves to the second-order expression for the shear stress tensor and compare it with results of Refs. [11,18].

4.1. Second-order corrections to the shear stress tensor

As explained above, we now keep the NESCO at second order in small perturbation from local equilibrium and, in addition, we retain terms which are of second order in the gradients of thermodynamic forces. In this manner we find the shear stress tensor at second order as

$$\pi^\mu_\nu = 2\eta \sigma^\mu_\nu - 2\eta \tau_\pi (\chi^\mu_\nu + \gamma \theta \sigma^\mu_\nu) + \lambda_\pi \sigma^\alpha_\nu c_\nu^\alpha + 2\lambda_\pi \Pi \theta \sigma^\mu_\nu + \lambda_\pi \mathcal{J} \nabla c_\nu^\nu a,$$  \hspace{1cm} (37)

where $\chi^\mu_\nu \equiv \Delta^\mu_\nu D^\nu \sigma^\sigma_\sigma$, with $D = u^\mu \partial_\mu$ being the covariant time-derivative, and $\tau_\pi$, $\lambda_\pi$, $\lambda_\pi \Pi$, $\lambda_\pi \mathcal{J}$ and representing four new coefficients associated with the second-order corrections to the shear stress. The first term on the right-hand side of Eq. (37) is easily recognized as the first-order (Navier-Stokes) contribution. The second-order terms collected in the parentheses (i.e., those $\propto \tau_\pi$) represent the non-local corrections to Eq. (15), whereas the last three terms stand for the nonlinear corrections arising from the three-point correlation functions in Eq. (17). It is easy to see that the first non-local correction describes memory effects due to its non-locality in time. The relevant transport coefficient $\tau_\pi$, which has the dimension of time, measures how long the information remains in the “memory” of the shear stress tensor $\pi^\mu_\nu$. Therefore, it is natural to associate it with the relaxation time of the shear stresses towards their asymptotic Navier-Stokes values. The second term involves a
product of the thermodynamic force $\sigma_{\mu\nu}$ with the velocity 4-divergence $\theta = \partial_{\mu}u^{\mu}$ and can be regarded as a (scalar) measure of the spatial “non-locality” in the fluid velocity field. This term describes how the shear stress tensor is distorted by uniform expansion or contraction of the fluid.

We find that the relaxation time $\tau_\eta$ is related to the frequency-derivative of the corresponding first-order transport coefficient, $i.e.,$ the shear viscosity, by a Kubo formula

$$\eta \tau_\eta = -i \frac{d}{d\omega} \eta(\omega) \bigg|_{\omega=0} = \frac{1}{10} \frac{d^2}{d\omega^2} \text{Re} \left[ G_{\eta,\eta}^{R}(\omega) \right] \bigg|_{\omega=0},$$

where $\eta \equiv \eta(0)$ is given by Eq. (33); the retarded Green’s function $G_{\eta,\eta}^{R}$ is defined in Eq. (36) and the frequency-dependent shear viscosity $\eta(\omega)$ is given by Eq. (33).

The physical meaning of the formula (38) for $\tau_\eta$ is easy to understand. As we mentioned, the relaxation terms originate from the non-local (memory) effects encoded in the non-equilibrium statistical operator. In the case where these memory effects are neglected (first-order theory), the proportionality between $\pi_{\mu\nu}$ and $\sigma_{\mu\nu}$ is given by the zero-frequency (static) limit of the shear viscosity, as seen from Eqs. (26) and (33). The memory effects imply time-delay which translates into frequency-dependence in the shear viscosity. At the leading order this is accounted for by the first-order frequency-derivative of $\eta(\omega)$ as Eq. (38) demonstrates.

The last three terms in Eq. (37) contain all possible combinations of the thermodynamic forces $\sigma_{\mu\nu}$, $\theta$ and $\nabla_{\mu}a_{\nu}$ which are allowed by the symmetries to quadratic order. These are $\theta\sigma_{\mu\nu}$, $\sigma_{\nu<\mu}\hat{\sigma}_{>\nu}^{\mu}$, and $\nabla_{<\mu}a_{\nu}\nabla_{>\nu}a_{\mu}$. The second-order transport coefficients associated with each of these terms can be expressed via three-point correlation functions according to

$$\lambda_{\pi} = \frac{12}{35} \beta^2 \int d^4x_1 d^4x_2 \left( \hat{\pi}_{\mu}^{\nu}(x), \hat{\pi}_{\nu}^{\mu}(x_1), \hat{\pi}_{\nu}^{\mu}(x_2) \right),$$

$$\lambda_{\pi\Pi} = \frac{\beta^2}{5} \int d^4x_1 d^4x_2 \left( \hat{\pi}_{\mu\nu}(x), \hat{\pi}_{\mu
u}(x_1), \hat{\pi}_{\mu
u}^{\ast}(x_2) \right),$$

$$\lambda_{\pi}^{\Phi} = \frac{1}{5} \int d^4x_1 d^4x_2 \left( \hat{\pi}_{\mu\nu}(x), \hat{\pi}_{\mu
u}^{\ast}(x_1), \hat{\pi}_{\mu
u}^{\ast}(x_2) \right),$$

where $\hat{\sigma}_{\mu}^{\nu}$ is the operator corresponding to the 4-current (27). In analogy with the leading order coefficient $\eta$, which is given by the two-point correlation in the stress-tensor, the second-order coefficient $\lambda_{\pi}$ is given by the three-point correlation of the shear-stress tensor. The coefficient $\lambda_{\pi\Pi}$ describes the nonlinear coupling between the shear and the bulk viscous processes and is given by a three-point correlation function between two shear stresses and the bulk viscous pressure. Finally, the coefficient $\lambda_{\pi}^{\Phi}$ describes the nonlinear coupling between the shear and the diffusion processes. Similarly, this coefficient is given by a three-point correlation function between two diffusion currents and the shear stress tensor. Note that in Eq. (37) the term $\propto \lambda_{\pi\Pi}$ and the second term in the parenthesis have the same gradient structure, but they have different origins and physical interpretation. The term $\propto \tau_{\pi}$ originates from non-local effects in the statistical distribution, whereas the term $\propto \lambda_{\pi\Pi}$ stands purely for nonlinear coupling between the bulk and the shear viscous effects. In this sense it is natural to regard as nonlinear only the term $\propto \lambda_{\pi\Pi}$ but not the term $\propto \tau_{\pi}$. A similar classification of the second-order terms was earlier suggested in Ref. [12].

4.2. Comparison with previous studies

For the sake of simplicity we will consider here a fluid without conserved charges. In this case Eq. (32) implies $\gamma \equiv c_s^2$, where $c_s$ is the speed of sound, and Eq. (37) reduces

$$\pi_{\mu\nu} = 2\eta \sigma_{\mu\nu} - 2\eta \tau_{\pi}(\dot{\sigma}_{\mu\nu} + c_s^2 \theta \sigma_{\mu\nu}) + \lambda_{\pi} \sigma_{\nu<\mu} \sigma_{>\nu}^{\mu} + 2\lambda_{\pi\Pi} \theta \sigma_{\mu\nu}.$$  

(42)
Ref. [11] finds in this case and for conformal fluids\(^1\)

\[
\pi_{\mu\nu}^B = 2\eta\sigma_{\mu\nu} - 2\eta\tau_\pi \left( \sigma_{\mu\nu} + \frac{1}{3} \theta\sigma_{\mu\nu} \right) + \lambda_1 \sigma_{<\mu<\nu>} \tag{43}
\]

where we have dropped the terms involving the vorticity tensor \(w_{\alpha\beta} = (\nabla_\alpha u_\beta - \nabla_\beta u_\alpha)/2\). Because \(c_s^2 = 1/3\) for a conformal fluid, we recover from Eq. (42) the term involving \(\tau_\pi\) in Eq. (43). Furthermore, because the conformal invariance implies vanishing bulk viscous pressure, the correlations involving the operator \(\hat{\rho}^*\) [see Eqs. (29) and (31)] vanish, i.e., \(\lambda_{\pi\Pi} = 0\) in this case. Finally we see that \(\lambda_1 \equiv \lambda_\pi\).

In the case of non-conformal fluids the second-order expression for the shear stress tensor was found, e.g., in Ref. [18] in the absence of conserved charges. Again neglecting the vorticity tensor and assuming flat space-time

\[
\pi_{\mu\nu}^B = 2\eta\sigma_{\mu\nu} - 2\eta\tau_\pi \left( \sigma_{\mu\nu} + \frac{1}{3} \theta\sigma_{\mu\nu} \right) + \lambda_1 \sigma_{<\mu<\nu>} + \lambda_2 \nabla_\mu \ln s \nabla_\nu \ln s. \tag{44}
\]

The term \(\propto \tau_\pi^*\) has the same gradient structure as the non-local term \(-2\eta\tau_\pi^* \theta\sigma_{\mu\nu}/3\). Comparing Eq. (44) with our expression (42), we identify \(\tau_\pi^* = \tau_\pi(3c_s^2 - 1) - 3\lambda_{\pi\Pi}/\eta\), and \(\lambda_2 = 0\).

4.3. Relaxation equation for the shear stress tensor

Now a relaxation-type equation for \(\pi_{\mu\nu}\) can be derived from Eq. (37). For this purpose we replace \(2\sigma_{\mu\nu} \rightarrow \eta^{-1} \pi_{\mu\nu}\) in the second term on the right-hand-side of Eq. (37), as has also been done in Refs. [11, 19, 20]. We then obtain

\[
-2\eta\tau_\pi \sigma_{\mu\nu} \simeq -\tau_\pi \pi_{\mu\nu} + \tau_\pi \beta \eta^{-1} \left( \gamma \frac{\partial \eta}{\partial \beta} - \delta \frac{\partial \eta}{\partial \alpha} \right) \theta \pi_{\mu\nu} \simeq -\tau_\pi \pi_{\mu\nu} + 2\tau_\pi \beta \left( \gamma \frac{\partial \eta}{\partial \beta} - \delta \frac{\partial \eta}{\partial \alpha} \right) \theta \sigma_{\mu\nu}, \tag{45}
\]

where \(\pi_{\mu\nu} = \Delta_{\mu\nu\rho\sigma} D_\pi^{\rho\sigma}\). The terms in brackets contain the corresponding partial derivatives of \(\eta\), which in general are not small and should not be neglected. In Eq. (45) we employed the relations \(D_\beta = \beta \theta \gamma\) and \(D_\alpha = -\beta \theta \delta\) [5]. Combining Eqs. (37) and (45) and introducing the coefficient

\[
\lambda = \lambda_{\pi\Pi} - \gamma \eta \tau_\pi + \tau_\pi \beta \left( \gamma \frac{\partial \eta}{\partial \beta} - \delta \frac{\partial \eta}{\partial \alpha} \right), \tag{46}
\]

we finally obtain

\[
\tau_\pi \pi_{\mu\nu} + \pi_{\mu\nu} = 2\eta \sigma_{\mu\nu} + 2\lambda \theta \sigma_{\mu\nu} + \lambda_{\pi\rho\sigma} \rho_{<\mu<\nu>} + \lambda_{\pi} \rho \nabla_\mu \alpha \nabla_\nu \tau_\alpha. \tag{47}
\]

The time-derivative term on the left-hand side describes the relaxation of the shear stress tensor toward its Navier-Stokes value on characteristic time scale \(\tau_\pi\). Indeed, for vanishing right-hand side the relaxation is exponential, \(\pi_{\mu\nu} \propto \exp(-t/\tau_\pi)\), with a characteristic relaxation time-scale \(\tau_\pi\). Similar expressions for the relaxation times were obtained previously in Refs. [11, 18, 21, 22].

5. Summary

This work concisely presents the derivation of second-order relativistic fluid dynamics within the Zubarev’s NESCO formalism - a method which is well-suited for treatments of strongly correlated systems. The simple case of one-component fluid without electromagnetic fields or vorticity in flat space time was considered.

\(^1\) Note that Refs. [11, 18] use metric convention opposite to ours, and their definition of the shear viscosity differs from ours by a factor of 2.
Our analysis shows that the new terms in the second-order hydrodynamics arise from: (i) the quadratic terms in the Taylor expansion of the statistical operator; (ii) the linear terms of the same expansion which include memory and non-locality in space. In particular, we find that the type (ii) terms describe the relaxation in time of the dissipative fluxes, which is essential for the causality of the hydrodynamic theory.

Using the NESO method and the example of shear stresses we demonstrated that the second-order transport coefficients can be expressed in terms of certain two- and three-point equilibrium correlation functions. A discussion of the transport coefficients associated with other thermodynamic fluxes can be found elsewhere [7]. Furthermore, we have shown that Kubo-type formulas for the relaxation times of dissipative fluxes can be obtained within the NESO formalism [see Eq. (38)]. These are given by the zero-frequency limit of the derivatives of the corresponding first-order transport coefficients with respect to the frequency. These can be computed from the equilibrium theory of quantum fields at non-zero temperature as, for example, was done by us for quark-gluon plasma in heavy-ion collisions within the Nambu-Jona–Lasinio model [23,24].

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