QCD improved determination
of the hadronic contribution to the
anomalous magnetic moment of the muon

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Abstract

We present the results of a new evaluation of the anomalous magnetic moment $a_\mu = (g_\mu - 2)/2$ of the muon where the role of input data needed in the evaluation is lowered in the interval between 1.2 and 3.0 GeV below charm threshold. This is achieved by decreasing the size of the weight function in the dispersion integral over the experimental ratio $R(s)$ by subtracting a polynomial from the weight function which mimics its energy dependence in that given energy interval. In order to compensate for this subtraction, the same polynomial weight integral is added again but is now evaluated on a circular contour in the complex plane using QCD and global duality. For the hadronic contribution to the shift in the anomalous magnetic moment of the muon we obtain $a_\mu^{\text{had}}(\text{LO}) = (701.3 \pm 6.4) \times 10^{-10}$ at leading order in the electromagnetic coupling. In addition, using the same procedure, we recalculate the next-to-leading contribution $a_\mu^{\text{had}}(\text{NLO}) = (-10.3 \pm 0.2) \times 10^{-10}$. Including QED, electroweak, and light-by-light contribution, we obtain a value $a_\mu = (11659.185.6 \pm 6.4_{\text{had}} \pm 3.5_{\text{LBL}} \pm 0.4_{\text{QED+EW}}) \times 10^{-10}$. 
1 Introduction

The anomalous magnetic moment $a_\mu$ of the muon is one of a few physical parameters which can be determined with high precision and therefore can serve as precision test for the Standard Model of elementary particle physics. In this paper we join the attempts to determine the hadronic contribution of the anomalous magnetic moment of the muon with a better accuracy (for an overview over the status of calculations of different collaborations two years ago see e.g. Ref. [1, 2]). While the dominant QED contribution $a_\mu^{\text{QED}} = (11 658 470.6 \pm 0.3) \times 10^{-10}$ and the weak contribution $a_\mu^{\text{weak}} = (15.4 \pm 0.1 \pm 0.2) \times 10^{-10}$ are known with very good accuracy (see e.g. Refs. [3, 4, 5] and references therein), the main uncertainty is given by the hadronic contribution. As an example for the actual calculations of the leading order hadronic contribution we cite the value $a_\mu^{\text{had}} = (683.1 \pm 5.9 \pm 2.0) \times 10^{-10}$ [6].

The calculation of the hadronic contribution to the anomalous magnetic moment of the muon mostly relies on experimental data in the $e^+e^-$ channel. In principle one can also make use of data from $\tau$-decay [7]. However, the inclusion of $\tau$-decay data introduces systematic uncertainties originating from isospin symmetry breaking effects which are difficult to estimate [8]. We have therefore decided to only include $e^+e^-$ data in our analysis.

There are different concepts for using the $e^+e^-$ data sets, either to its full extent [9] or in part by dividing the energy range into resonance regions close to the pair production thresholds where the experimental values are taken and those regions far from thresholds where perturbation theory is assumed to be valid. In this paper we present an approach which has been successfully applied for the determination of the running fine structure constant at the $Z^0$ boson resonance, $\alpha(M_Z^2)$ [10].

The subject of a precision determination of the anomalous magnetic moment became rather important again as a precision measurement for the positive muon could be accomplished at the Brookhaven National Laboratory (BNL) [11]. The value

$$a_\mu^{\text{exp}} = (11 659 202.3 \pm 15.1) \times 10^{-10}$$  (1)
cited in Ref. [11] showed a deviation from the Standard Model prediction at that time,

\[ a_{\mu}^{\text{SM}} = (11659.159.7 \pm 6.7) \times 10^{-10} \]  

as weighted average over the calculation results of different collaborations [2]. The deviation was 2.6 standard deviations which seemed to have opened the window for new physics. Many such suggestions were published, including concepts of supersymmetry, leptoquarks, lepton number violating models, technicolor models, string theory concepts, extra dimensions and so on. The discrepancy between measurement and the Standard Model prediction became smaller, though, when a sign error was discovered in the theoretical calculation of the light-by-light contribution [13]. Therefore, it is worth still to analyze the situation of the Standard Model prediction thoroughly, giving special emphasis on the error estimate, in order to compare it with the experimental world average

\[ a_{\mu}^{\text{exp}} = (11659.203 \pm 8) \times 10^{-10} \]  

which is dominated by the measurement of Ref. [12], as we will do in this paper.

2 Theoretical background

The leading hadronic contribution to the anomalous magnetic moment of the muon \(a_{\mu}^{\text{had}}\) can be extracted from the \(O(\alpha^2)\) vertex correction shown in Fig. 1. The analytic expression for \(a_{\mu}^{\text{had}}\) from this diagram is a weighted dispersion integral over the imaginary part of the vector part of the hadronic vacuum polarisation,

\[ a_{\mu}^{\text{had}} = \frac{4\alpha^2}{\pi} \int_{4m_e^2}^{\infty} \text{Im} \Pi^{\text{had}}(s) \frac{K(s)}{s} ds \]  

with the QED kernel

\[ K(s) = \ln(1+x) \frac{(1 + x^2)(1 + x)^2}{x^2} + x^2(1 + x) \frac{1 - x}{1 - x} \ln(x) \]

\[ + \frac{x^2}{2} (2 - x^2) + \frac{(1 + x^2)(1 + x)^2}{x^2} \left(-x + \frac{x^2}{2}\right). \]
Figure 1: $O(\alpha^2)$ correction to the hadronic part of $a_\mu$. The shaded bubble represents the current contributions to the vector part of the hadronic vacuum polarisation given in Refs. [14, 15].

As usual we make use of the kinematic variables

$$x(s) = \frac{\beta(s) - 1}{\beta(s) + 1}, \quad \beta(s) = \sqrt{1 - \frac{4m_\mu^2}{s}}. \quad (6)$$

$\text{Im}\, \Pi_{\text{had}}(s)$ is calculated from the hadronic current-current correlator,

$$\Pi_{\mu\nu}^{\text{had}}(q^2) = i \int e^{iqx} \langle 0 | T j_\mu(x) j_\nu(x') | 0 \rangle d^4x = \Pi^{\text{had}}(q^2) \left( q^2 g_{\mu\nu} - q_\mu q_\nu \right) \quad (7)$$

Higher perturbative QCD corrections are well-known up to $O(\alpha_s^2)$ with $O(m_q^{12}/q^{12})$ quark mass corrections which are supplemented by massless terms up to $O(\alpha_s^4)$ [14]. This perturbative part is denoted by $\Pi^P(q^2)$. For brevity we only cite the first few terms of this expansion in $\alpha_s$ and $m_q^2/q^2$,

$$\Pi^P(q^2) = \frac{3}{16\pi^2} \sum_{i=1}^{n_f} Q_i^2 \left[ \frac{20}{9} + \frac{4}{3}L + C_F \left( \frac{55}{12} - 4\zeta(3) + L \right) \frac{\alpha_s}{\pi} + O(\alpha_s^2, m_q^2/q^2) \right] \quad (8)$$

with $L = \ln(\mu^2/q^2)$. In our analysis we use the full $O(\alpha_s^2, m_q^{12}/q^{12})$ expression given in Ref. [14]. The number of active flavours is denoted by $n_f$ which changes according to the energy interval under consideration. For the zeroth order term in the $m_q^2/q^2$ expansion we have included the known higher order terms in $\alpha_s$,

$$\frac{3}{16\pi^2} \sum_{i=1}^{n_f} Q_i^2 \left[ \left( c_3 + k_2L + \frac{1}{2}(k_0\beta_1 + 2k_1\beta_0)L^2 + \frac{1}{3}k_0\beta_0^2L^3 \right) \left( \frac{\alpha_s}{\pi} \right)^3 + O(\alpha_s^4) \right] \quad (9)$$
with \( k_0 = 1, k_1 = 1.63982 \) and \( k_2 = 6.37101 \). We have denoted the yet unknown constant term in the four-loop contribution by \( c_3 \) which, however, does not contribute to our calculations since it has no absorption part.

For the nonperturbative part the operator product expansion leads to

\[
\Pi^{NP}(q^2) = \frac{1}{18(q^2)^2} \left( 1 + \frac{7\alpha_s}{6\pi} \right) (\frac{\alpha_s}{\pi} G^2) + \frac{8}{9(q^2)^2} \left( 1 + \frac{\alpha_s}{4\pi} C_F + \ldots \right) \langle m_u \bar{u} u \rangle + \frac{2}{9(q^2)^2} \left( 1 + \frac{\alpha_s}{4\pi} C_F + \ldots \right) \langle m_d \bar{d} d \rangle + \frac{2}{9(q^2)^2} \left( 1 + \frac{\alpha_s}{4\pi} C_F + (5.8 + 0.92L) \frac{\alpha_s^2}{\pi^2} \right) \langle m_s \bar{s} s \rangle + \frac{\alpha_s^2}{9\pi^2(q^2)^2} (0.6 + 0.333L) \langle m_u \bar{u} u + m_d \bar{d} d \rangle - \frac{C_A m_s^4}{36\pi^2(q^2)^2} \left( 1 + 2L + (0.7 + 7.333L + 4L^2) \frac{\alpha_s}{\pi} \right) - \frac{448\pi}{243(q^2)^5} \alpha_s |\langle \bar{q} q \rangle|^2 + \frac{1}{(q^2)^4} (0.48 \text{ GeV})^8 - \frac{1}{(q^2)^5} (0.3 \text{ GeV})^{10} + O((q^2)^{-6})
\]

where \( C_F = 4/3, C_A = 3, T_F = 1/2 \) are \( SU(3) \) colour factors. We will use these results for the evaluation of the theoretical contributions.

### 3 Experimental contributions

The evaluation of the integral involves experimental data via the optical theorem which connects the imaginary part of the vacuum polarisation with the ratio

\[
R(s) = \frac{\sigma(e^+e^- \rightarrow \text{hadrons})}{\sigma(e^+e^- \rightarrow \mu^+\mu^-)} = 12\pi \text{Im} \Pi^{\text{had}}(s).
\]

Especially in the MeV and the low GeV energy range and in regions of resonances where perturbative QCD cannot be applied, data sets taken from experimental measurements are crucial. For our evaluation we have used the combined \( e^+e^- \) data sets from Ref.\[9, 19\] and complement them in the dominant low energy range from 610 MeV to 961 MeV by recent two pion data \[16\]. By comparing with the new data for three and four pion decays \[17, 18\] one ensures that their influence for the sub GeV-range where the two pion data were recorded is negligible.
All data sets are combined by weighting their relative contribution to the total experimental value with the corresponding statistic and systematic errors.

4 Introduction of the method

Global duality states that QCD can be used in weighted integrals over a spectral function if the spectral function is multiplied by polynomial functions. However, this does not work if the polynomial function is replaced by a singular function such as the weight $K(s)/s$ in the present case. Nevertheless, local duality is expected to hold for large values of $s$ far from resonances and threshold regions, i.e. $\text{Im} \Pi^\text{had}(s) \simeq \text{Im} \Pi^\text{QCD}(s)$.

In our approach we attempt to reduce the influence of experimental $R(s)$ data in regions where the data has large uncertainties. Specifically, this holds for the interval between 1.2 and 3.0 GeV. The essence of our method is to diminish the magnitude of the weight function by subtracting a polynomial function which mimics the weight function at those energies. In order to compensate for this subtraction, the same polynomial function is added again, but now its contribution is evaluated by using global duality on a circular contour in the complex plane, according to

$$\frac{\alpha^2}{3\pi^2} \int_{s_1}^{s_2} R(s) \frac{K(s)}{s} ds = \frac{\alpha^2}{3\pi^2} \int_{s_1}^{s_2} R(s) \left( \frac{K(s)}{s} - P_n(s) \right) ds$$

$$+ \frac{4\alpha^2}{\pi} \left[ \frac{1}{2\pi i} \oint_{|s|=s_1} \Pi^\text{had}(s) P_n(s) ds - \frac{1}{2\pi i} \oint_{|s|=s_2} \Pi^\text{had}(s) P_n(s) ds \right]. \quad (12)$$

5 Application of the method to $a^\text{had}_\mu$

The usual procedure to evaluate Eq. (4) is to use the experimental cross section up to some high momentum transfer and to calculate the remaining part from QCD using local duality.

Indeed, in the energy region up to 1.2 GeV we use the $e^+e^-$ cross section data set from Refs. [9] and [16]. The corresponding dispersion integral reads

$$(\Delta a^\text{had}_\mu)_1 = \frac{\alpha^2}{3\pi^2} \int_{4m^2_e}^{(1.2\text{GeV})^2} \left[ R^{e^+e^-}(s) + R^{\text{BW}}(s) \right] \frac{K(s)}{s} ds. \quad (13)$$
Narrow resonances are added in explicit form. In Eq. (13) the missing resonances are parametrized by a Breit-Wigner fit $R_{BW}(s)$ and added to the remaining data sets.

In the subsequent interval from 1.2 to 3.0 GeV the relevant $e^+e^-$ data from Refs. [9], combined with data sets from the BES Collaboration [19], can be efficiently replaced by theoretical input from QCD. By subtracting a polynomial function from the weight function $K(s)/s$ we reduce the experimental input in this energy interval. Since the polynomial function is an analytic function in the whole complex plane, the remaining contribution can be evaluated off the real axis on a circular contour in the complex plane using perturbative and nonperturbative QCD results,

$$
\begin{align*}
(a_{\mu}^{\text{had}})_{2} &= \frac{\alpha^2}{3\pi^2} \int_{s_1=(1.2\text{ GeV})^2}^{s_2=(3.0\text{ GeV})^2} \Re e^+e^-(s) \left( \frac{K(s)}{s} - P_n(s) \right) ds \\
&\quad - \frac{4\alpha^2}{\pi} \frac{1}{2}\pi i \oint_{|s|=1.2\text{ GeV}} \left[ \Pi^P(s) + \Pi^NP(s) \right] P_n(s) ds \\
&\quad + \frac{4\alpha^2}{\pi} \frac{1}{2}\pi i \oint_{|s|=1.2\text{ GeV}} \left[ \Pi^P(s) + \Pi^NP(s) \right] P_n(s) ds \\
&= (a_{\mu}^{\text{exp}})_{2} + (a_{\mu}^{\text{the}})_{2}
\end{align*}
$$

where $n_f = 3$ is taken for this interval.

At first glance the polynomial degree $n$ seems to be arbitrary. However, if $n$ is chosen too large, both the theoretical error from the strong coupling $\alpha_s(M_Z)$ and from unknown nonperturbative contributions arising from higher order condensates lead to larger uncertainties.

Moreover, we have to ensure that the calculated value is stable with respect of the variations of the lower polynomial degrees $n$ where the less known contributions from condensates are negligible. This point will help us find an optimal energy domain for our method.

An optimal choice for the polynomial degree should guarantee a good approximation of the polynomial function to the weight function and thus effectively replace the experimental data by theoretical input while reducing the total uncertainty from this region to a minimum.
Table 1: Contributions to \((a_{\mu}^{\text{had}})_2\) for different fitting polynomial functions \(P_n(s)\) with degree \(n\). The purely experimental and theoretical contributions according to Eq. (14) are listed separately. Because the error estimate becomes worse for \(n = 3\) (cf. Table 2), polynomial degrees \(n = 1\) and \(n = 2\) are used to obtain a mean value and a methodical error estimate, \((71.55 \pm 0.13) \times 10^{-10}\).

Above 3.0 GeV the \(e^+e^-\) data in Ref. [9, 19] show low uncertainties and can be integrated directly,

\[
(a_{\mu}^{\text{had}})_3 = \frac{\alpha^2}{3\pi^2} \int_{s_2=(3.0\text{GeV})^2}^{(40\text{GeV})^2} \left[ R_{e^+e^-}(s) + R_{\text{BW}}(s) \right] \frac{K(s)}{s} ds. 
\] (15)

In the last step the high energy tail starting from 40 GeV can be calculated from theory using local duality. This is reasonable since there are no resonance contributions above 40 GeV. In this region it is sufficient to use the lowest order contribution given by

\[
\text{Im} \Pi^{\text{had}}(s) = 3 \sum_{\text{flavours}} Q_f^2 \frac{\alpha}{12\pi} \sqrt{1 - \frac{4m_f^2}{s}} \left( 1 + \frac{2m_f^2}{s} \right). 
\] (16)

### 6 Numerical results

Using the recent \(e^+e^-\) data from Refs. [9] and [16] we obtain for the dominant low energy component of \(a_{\mu}^{\text{had}}\),

\[
(a_{\mu}^{\text{had}})_1 = (518.1 \pm 4.8) \times 10^{-10}. 
\] (17)

We have added the statistical and point-to-point systematic errors in quadrature. In addition to this experimental value we have to consider contributions from the \(\omega\) and \(\phi\)
Table 2: Theoretical uncertainties of \( a^\text{had}_{\mu} \) from QCD parameters and condensates. Because the uncertainty is worse for higher polynomial degrees, the degrees \( n = 1 \) and \( n = 2 \) are selected for the analysis. The error estimate is given by the average value \( 3.80 \times 10^{-10} \).

The uncertainty due to \( \alpha_s |\langle q\bar{q} \rangle|^2 \) is less than \( 10^{-14} \) in all cases and is therefore omitted in the table.

Resonances for which we obtain the values \((38.9 \pm 1.4) \times 10^{-10}\) and \((40.4 \pm 1.3) \times 10^{-10}\), respectively, from an integration over the Breit–Wigner distribution function where the values for the parameters are taken from Ref. [20].

Applying our polynomial technique for the range between 1.2 and 3.0 GeV, we incorporate QCD-expressions for the hadronic current-current correlator with corresponding uncertainties for the parameters occuring in these expressions. The errors from the masses of the light quarks \( u, d, \) and \( s \) can be neglected at energies above 1 GeV. Thus the theoretical uncertainty is dominated by the strong coupling \( \alpha_s(M_Z) \) whose error originates from the uncertainty in the QCD scale \( \Lambda_{\text{MS}} = (380 \pm 60) \text{ MeV} \). For the condensate terms we assign generous errors of 100\%. Thus we have

\[
\langle \frac{\alpha_s}{\pi} G^2 \rangle = (0.04 \pm 0.04) \text{ GeV}^4, \quad \alpha_s |\langle q\bar{q} \rangle|^2 = (4 \pm 4) \times 10^{-4} \text{ GeV}^6.
\]

With increasing polynomial degrees the condensate error contributions grow fast. However, as can be clearly seen in Tab. 1 or in Fig. 2 good approximations to the kernel of the dispersion integral and consequently a very high reduction of the experimental data influence in the interval between 1.2 and 3.0 GeV can be achieved even by low degrees \( n = 1 \) and \( n = 2 \). This can be understood by the fact that one is far away from the singularity at \( s = 0 \) and thus the weight function can be well approximated by low degree polynomials.
Figure 2: The QED kernel $K(s)/s$ with fitting polynomial functions $P_n(s)$ of different degrees $n$. The polynomial functions were calculated by performing a least squares fit with no further constraints.

We thus take the values $(a_{\mu}^{\text{had}})_2$ for $n = 1$ and $n = 2$ and calculate the algebraic mean. Therefore, we have to take into account a small methodical error due to the variation of the polynomial degree,

$$(a_{\mu}^{\text{meth}})_2 = \pm 0.06 \times 10^{-10}. \quad (19)$$

Summing up the error contributions in quadrature and the algebraic mean uncertainty (for $n = 1$ and $n = 2$) from Table 2 one obtains

$$(a_{\mu}^{\text{had}})_2 = (71.55 \pm 3.83) \times 10^{-10}. \quad (20)$$

In the interval following 3.0 GeV up to the end of the $e^+e^-$ data set at 40 GeV as given in Ref. [9] we obtain

$$(a_{\mu}^{\text{had}})_3 = (23.2 \pm 0.2) \times 10^{-10}. \quad (21)$$

The contributions from narrow charmonium and bottonium resonances, $(8.8 \pm 0.6) \times 10^{-10}$ and $(0.11 \pm 0.01) \times 10^{-10}$, respectively, are taken by integrating over the narrow resonance distribution function where values for the parameters are again taken from Ref. [20].
We finally obtain a tiny continuum contribution from the last interval,
\[(a_{\mu}^{\text{had}})_4 = \frac{4\alpha^2}{\pi} \int_{(40 \text{GeV})^2}^{\infty} \text{Im} \Pi^{\text{had}}(s) \frac{K(s)}{s} ds = 0.15 \times 10^{-10}. \tag{22}\]

Summing up the contributions from the different energy intervals as shown in Table 3, we finally obtain
\[a_{\mu}^{\text{had}} = (701.3 \pm 6.4) \times 10^{-10} \tag{23}\]
for the hadronic contribution to \(a_{\mu}\). We have thereby assumed that errors with different origin are uncorrelated and have to be added quadratically in order to obtain the total error.
Table 3: The different contributions to the hadronic part of the anomalous magnetic moment $a_{\mu}^{\text{had}}$ of the muon.

### 7 Higher order correction

In the next-to-leading order (NLO) we have to consider three types of diagrams, those of type (2a) with an additional photon exchange, those of type (2b) with an electron loop inserted in one of the photon lines in Fig. 1, and finally the one of type (2c) with two correlator functions included in the photon line. For the former two the contribution reads

$$a_{\mu}^{\text{had}}(\text{NLO}) = \frac{1}{3} \left( \frac{\alpha}{\pi} \right)^3 \int_{4m_{\mu}^2}^{\infty} \frac{ds}{s} K^{(2)}(s) R(s).$$  

For numerical purposes it is convenient to represent the kernel functions $K^{(2a)}(s)$ and $K^{(2b)}(s)$ in terms of power series expansions in terms of $m^2/s$ \[23\] ($m = m_{\mu} = 105.6583568 \pm 0.0000052$ MeV \[20\] is the mass of the muon). One has

$$K^{(2a)}(s) = \frac{2m^2}{s} \left\{ \frac{223}{54} - \frac{\pi^2}{3} - \frac{23}{36} \ln \left( \frac{s}{m^2} \right) \right\}$$

$$+ \frac{m^2}{s} \left( \frac{8785}{1152} - \frac{37\pi^2}{48} - \frac{367}{216} \ln \left( \frac{s}{m^2} \right) + \frac{19}{144} \ln^2 \left( \frac{s}{m^2} \right) \right)$$  

(25)
\[ + \frac{m^4}{s^2} \left\{ \frac{13072841}{432000} - \frac{883\pi^2}{240} - \frac{10079}{3600} \ln \left( \frac{s}{m^2} \right) + \frac{141}{80} \ln^2 \left( \frac{s}{m^2} \right) \right\} + \ldots \}

\[ K^{(2b)}(s) = 2 \frac{m^2}{s} \left\{ \left( -\frac{1}{18} + \frac{1}{9} \ln \left( \frac{s}{m^2} \right) \right) \right\}

\[ + \frac{m^2}{s} \left( -\frac{55}{48} + \frac{\pi^2}{18} + \frac{5}{9} \ln \left( \frac{s}{m^2} \right) + \frac{5}{36} \ln \left( \frac{m^2}{m_f^2} \right) - \frac{1}{6} \ln^2 \left( \frac{s}{m^2} \right) + \frac{1}{6} \ln^2 \left( \frac{m^2}{m_f^2} \right) \right) \]

\[ + \frac{m^4}{s^2} \left( -\frac{11299}{1800} + \frac{\pi^2}{3} + \frac{10}{3} \ln \left( \frac{s}{m^2} \right) - \frac{1}{10} \ln \left( \frac{m^2}{m_f^2} \right) - \ln^2 \left( \frac{s}{m^2} \right) + \ln^2 \left( \frac{m^2}{m_f^2} \right) \right) + \ldots \} \]

where for \( m_f \) we used the electron mass, \( m_f = 510.998902 \pm 0.000021 \) keV \([20]\), since the contributions from the muon is already included in the contribution \((2a)\). The contribution of the \( \tau \) is suppressed by \( m^2_\tau/m^2_\mu \) and numerically negligible. The kernels of the different orders in the electromagnetic coupling, \( K(s) \) on the one hand and \( K^{(2a)}(s) \) and \( K^{(2b)}(s) \) on the other hand, are compared in Fig. 4. For illustration, we like to sum the results from the evaluation of Eq. \([21]\) which was performed in a completely analogous way as in the previous chapter. The final result \((-21.07 \pm 0.21) \times 10^{-10} \) in case of \( K^{(2a)}(s) \) (cf. Table 4) and \((10.78 \pm 0.08) \times 10^{-10} \) in case of \( K^{(2b)}(s) \) (Table 5) are in good agreement with the literature \([22]\). Finally, for the small contribution from diagrams of type \((2c)\) with two correlator functions included into the photon line we take the value \((0.27 \pm 0.01) \times 10^{-10} \) from Ref. \([23]\). We have checked that to the required accuracy a recalculation using the method presented in this paper is not necessary. The total hadronic NLO contribution then sums up to a value of \((-10.3 \pm 0.2) \times 10^{-10} \), as compared to the value \((-10.1 \pm 0.6) \times 10^{-10} \) given in Ref. \([23]\). Our central value is 3% smaller than that given in Ref. \([23]\) and the error is reduced from \(0.6 \times 10^{-10} \) to \(0.2 \times 10^{-10} \). Finally, summing up leading and next-to-leading order contributions together with the afore-mentioned QED and weak contributions and the so-called light-by-light contribution \( a_\mu^{\text{had}(LBL)} = (8.6 \pm 3.5) \times 10^{-10} \) \([24, 25, 26]\), we obtain \( a_\mu = (11 659 185.6 \pm 6.4_{\text{had}} \pm 3.5_{\text{LBL}} \pm 0.4_{\text{QED+EW}}) \times 10^{-10} \).
### Table 4: The different contributions to the hadronic part of the anomalous magnetic moment $a_{\mu}^{\text{had}}$ of the muon for the kernel $K^{(2a)}$.

| interval for $\sqrt{s}$ | contributions to $a_{\mu}^{\text{had}}$ | comments |
|-------------------------|------------------------------------------|-----------|
| [0.28 GeV, 1.2 GeV]    | $(−14.04 \pm 0.12) \times 10^{-10}$    | $e^+e^−$ cross section data |
| $\omega$ resonance     | $(−1.11 \pm 0.04) \times 10^{-10}$    | Breit-Wigner |
| $\phi$ resonances      | $(−1.29 \pm 0.04) \times 10^{-10}$    | narrow resonances |
| [1.2 GeV, 3.0 GeV]     | $(−2.84 \pm 0.14) \times 10^{-10}$    | polynomial method |
| $J/\psi$ resonances    | $(−0.44 \pm 0.03) \times 10^{-10}$    | narrow resonances |
| [3.0 GeV, 40 GeV]      | $(−1.35 \pm 0.09) \times 10^{-10}$    | $e^+e^−$ annihilation data |
| $\Upsilon$ resonances  | $< 10^{-13}$                           | narrow resonances |
| [40 GeV, $\infty$]    | $< 10^{-12}$                           | theory |
| top quark contr.       | $< 10^{-14}$                           | theory |
| hadronic contr.        | $(−21.07 \pm 0.21) \times 10^{-10}$  |           |

### Table 5: The different contributions to the hadronic part of the anomalous magnetic moment $a_{\mu}^{\text{had}}$ of the muon for the kernel $K^{(2b)}$.

| interval for $\sqrt{s}$ | contributions to $a_{\mu}^{\text{had}}$ | comments |
|-------------------------|------------------------------------------|-----------|
| [0.28 GeV, 1.2 GeV]    | $(7.98 \pm 0.07) \times 10^{-10}$    | $e^+e^−$ cross section data |
| $\omega$ resonance     | $(0.59 \pm 0.02) \times 10^{-10}$    | Breit-Wigner |
| $\phi$ resonances      | $(0.62 \pm 0.02) \times 10^{-10}$    | narrow resonances |
| [1.2 GeV, 3.0 GeV]     | $(1.09 \pm 0.01) \times 10^{-10}$    | polynomial method |
| $J/\psi$ resonances    | $(0.14 \pm 0.01) \times 10^{-10}$    | narrow resonances |
| [3.0 GeV, 40 GeV]      | $(0.364 \pm 0.003) \times 10^{-10}$  | $e^+e^−$ annihilation data |
| $\Upsilon$ resonances  | $< 10^{-13}$                           | narrow resonances |
| [40 GeV, $\infty$]    | $< 10^{-12}$                           | theory |
| top quark contr.       | $< 10^{-14}$                           | theory |
| hadronic contr.        | $(10.78 \pm 0.08) \times 10^{-10}$  |           |
Figure 4: Comparison of the LO and NLO kernels $K(s)$ and $K^{(2a,2b)}(s)$ normalized at $s = 4m_{\mu}^2$. The functional behaviour of the three weight functions is quite similar.

8 Conclusion

We have presented an alternative determination of the hadronic contribution to the anomalous magnetic moment of the muon where we have made use of theoretical QCD results to reduce the influence of the poor experimental data in the range between 1.2 and 3.0 GeV. Our analysis includes the leading and next-to-leading contribution.

Using the polynomial method in the range between 1.2 and 3.0 GeV, we could suppress the influence of experimental data effectively and thereby reduce the error on the determination of $a_{\mu}^{\text{had}}$. Note that using this method we were able to use a value for the lower limit of this range lower than usually used for pure QCD methods.

Using only $e^+e^-$ data and QCD as input, we obtain a result which is $1.6\sigma$ away from the measured world average, stating that the deviation between theory and experiment may be smaller than commonly suggested. Other recent results to compare with are $(11659.180.9\pm7.2_{\text{had}}\pm3.5_{\text{LBL}}\pm0.4_{\text{QED+EW}}) \times 10^{-10}$ [27] and $(11659.166.9\pm7.4) \times 10^{-10}$ [6].
Acknowledgements

We would like to thank K. Schilcher for discussions. S.G. acknowledges a grant given by the DFG, Germany through the Graduiertenkolleg “Eichtheorien” at the University of Mainz.

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