Lower Bounds of Dirichlet Eigenvalues for General Grushin Type Bi-Subelliptic Operators

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Abstract. Let \( \Omega \) be a bounded open domain in \( \mathbb{R}^n \) with smooth boundary \( \partial \Omega \). Let \( X = (X_1, X_2, \cdots, X_m) \) be a system of general Grushin type vector fields defined on \( \Omega \) and the boundary \( \partial \Omega \) is non-characteristic for \( X \). For \( \Delta_X = \sum^m_{j=1} X^2_j \), we denote \( \lambda_k \) as the \( k \)-th eigenvalue for the bi-subelliptic operator \( \Delta^2_X \) on \( \Omega \). In this paper, by using the sharp sub-elliptic estimates and maximally hypoelliptic estimates, we give the optimal lower bound estimates of \( \lambda_k \) for the operator \( \Delta^2_X \).

Key Words: Eigenvalues, degenerate elliptic operators, sub-elliptic estimate, maximally hypoelliptic estimate, bi-subelliptic operator.

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1 Introduction and main results

Let \( X = (X_1, X_2, \cdots, X_m) \) be the system of general Grushin type vector fields, which is defined on an open domain \( W \) in \( \mathbb{R}^n \) (\( n \geq 2 \)).

Let \( J = (j_1, \cdots, j_k), 1 \leq j_i \leq m \) be a multi-index, \( X^J = X_{j_1}X_{j_2}\cdots X_{j_k} \), we denote \( |J| = k \) be the length of \( J \), if \( |J| = 0 \), then \( X^J = id \). We introduce following function space (cf. [18, 21, 23]):

\[
H^2_X(W) = \{ u \in L^2(W) | X^J u \in L^2(W), |J| \leq 2 \}.
\]

It is well known that \( H^2_X(W) \) is a Hilbert space with norm \( \| u \|^2_{H^2_X(W)} = \sum_{|J| \leq 2} \| X^J u \|^2_{L^2(W)} \).

Assume the vector fields \( X = (X_1, X_2, \cdots, X_m) \) satisfy Hörmander’s condition:

Definition 1.1 (cf. [2, 12]). We say that \( X = (X_1, X_2, \cdots, X_m) \) satisfies the Hörmander’s condition in \( W \) if there exists a positive integer \( Q \), such that for any \( |J| = k \leq Q \), \( X \) together with all \( k \)-th repeated commutators

\[
X_J = [X_{j_1}, [X_{j_2}, [X_{j_3}, \cdots, [X_{j_{k-1}}, X_{j_k}]]]]
\]

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span the tangent space at each point of $W$. Here $Q$ is called the Hörmander index of $X$ in $W$, which is defined as the smallest positive integer for the Hörmander’s condition to be satisfied.

For any bounded open subset $\Omega \subset W$, we define the subspace $H^2_{X,0}(\Omega)$ to be the closure of $C^\infty_0(\Omega)$ in $H^2_X(W)$. Since $\partial \Omega$ is smooth and non characteristic for $X$, we know that $H^2_{X,0}(\Omega)$ is well defined and also a Hilbert space. In this case, we also say that $X$ satisfies the Hörmander’s condition on $\Omega$ with Hörmander index $1 \leq Q < +\infty$. Thus $X$ is a finitely degenerate system of vector fields on $\Omega$ and the finitely degenerate elliptic operator $\Delta_X = \sum_{i=1}^m X_i^2$ is a sub-elliptic operator.

The degenerate elliptic operator $\Delta_X$ has been studied by many authors, e.g., Hörmander [11], Jerison and Sánchez-Calle [13], Métivier [17], Xu [23]. More results for degenerate elliptic operators can be found in [2–6] and [9, 10, 12, 14].

In this paper, we study the following eigenvalues problem for bi-subelliptic operators in $H^2_{X,0}(\Omega)$:

$$\begin{cases}
\Delta_X^2 u = \lambda u & \text{in } \Omega, \\
u = 0, Xu = 0 & \text{on } \partial \Omega,
\end{cases}$$

(1.1)

where $X$ will be the following general Grushin type vector fields (see (1.5) and (1.7) below). In this case we know that for each $j$, $X_j$ is formally skew-adjoint, i.e., $X_j^* = -X_j$. Then there exists a sequence of discrete eigenvalues $\{\lambda_j\}_{j \geq 1}$ for the problem (1.1), which satisfying $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_k \cdots$ and $\lambda_k \to +\infty$ as $k \to +\infty$ (see Proposition 2.5 below).

In the classical case, if $X = (\partial_{x_1}, \cdots, \partial_{x_n})$, then $\Delta_X^2 = \Delta^2$ is the standard bi-harmonic operator. In this case our problem is motivated from the following classical clamped plate problem, namely

$$\begin{cases}
\Delta^2 u = \lambda u & \text{in } \Omega, \\
u = 0, \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial \Omega,
\end{cases}$$

(1.2)

where $\Delta = \partial^2_{x_1} + \partial^2_{x_2} + \cdots + \partial^2_{x_n}$, $\frac{\partial u}{\partial \nu}$ denotes the derivative of $u$ with respect to the outer unit normal vector $\nu$ on $\partial \Omega$.

For the eigenvalues of the clamped plate problem (1.2), Agmon [1] and Pleijel [20] showed the following asymptotic formula

$$\lambda_k \sim \frac{16\pi^4}{(B_n \text{vol}(\Omega))^{\frac{2}{n}}} k^\frac{4}{n} \text{ as } k \to +\infty,$$

(1.3)

where $B_n$ denotes the volume of the unit ball in $\mathbb{R}^n$. In 1985, Levine and Protter [15] proved that

$$\frac{1}{k} \sum_{i=1}^k \lambda_i \geq \frac{n}{n+4} \frac{16\pi^4}{(B_n \text{vol}(\Omega))^{\frac{2}{n}}} k^\frac{4}{n},$$

(1.4)
Later in 2012, Cheng and Wei [7] showed that the eigenvalues of the bi-harmonic operator satisfy

\[ \frac{1}{k} \sum_{i=1}^{k} \lambda_i \geq \frac{n}{n+4} \frac{16\pi^4}{(B_n \text{vol}(\Omega))^\frac{4}{n}} + \left( \frac{n+2}{12(n+4)} - \frac{1}{1152n^2(n+4)} \right) \frac{\text{vol}(\Omega)}{I(\Omega)} \frac{n}{n+2} \frac{4\pi^4}{(B_n \text{vol}(\Omega))^\frac{4}{n}} \]

\[ + \left( \frac{1}{576(n+4)} - \frac{1}{27648n^2(n+2)(n+4)} \right) \left( \frac{\text{vol}(\Omega)}{I(\Omega)} \right)^2, \]

where \( I(\Omega) \) is the moment of inertia of \( \Omega \).

Next, we consider the situation for the bi-subelliptic operators \( \Delta_X^2 \). Before we state our results, we need the following concepts:

**Definition 1.2.** If \( X \) satisfies the Hörmander’s condition in \( W \) with the Hörmander index \( Q \geq 1 \). Then for each \( 1 \leq j \leq Q \) and \( x \in W \), we denote \( V_j(x) \) as the subspace of the tangent space \( T_x(W) \) spanned by the vector fields \( X_J \) with \( |J| \leq j \). We say the system of the vector fields \( X \) satisfies Métivier’s condition on \( \Omega \) if the dimension of \( V_j(x) \) is constant \( v_j \) in a neighborhood of each \( x \in \Omega \), and in this case the Métivier index is defined as

\[ v = \sum_{j=1}^{Q} j(v_j - v_{j-1}), \quad \text{here} \quad v_0 = 0. \]

As it well-known that under the Métivier’s condition, we can get the asymptotic estimate for the eigenvalues of sub-elliptic operator \(-\Delta_X\) (cf. [17]). However, for most degenerate vector fields \( X \), the Métivier’s condition will be not satisfied. Thus we need to introduce the following generalized Métivier index.

**Definition 1.3.** If \( X \) satisfies the Hörmander’s condition in \( W \) with the Hörmander index \( Q \geq 1 \). Then for each \( 1 \leq j \leq Q \) and \( x \in W \), we denote \( V_j(x) \) as the subspace of the tangent space \( T_x(W) \) spanned by the vector fields \( X_J \) with \( |J| \leq j \). We denote that

\[ v(x) = \sum_{j=1}^{Q} j(v_j(x) - v_{j-1}(x)), \quad \text{with} \quad v_0(x) = 0, \]

where \( v_j(x) \) is the dimension of \( V_j(x) \). Then we define

\[ \tilde{\vartheta} = \max_{x \in \Omega} v(x), \]

as the generalized Métivier index. It is obvious that \( \tilde{\vartheta} = v \) if \( X \) satisfies the Métivier’s condition on \( \Omega \).
Recently, in case of $X$ to be some special Grushin vector fields Chen and Zhou [8] obtained lower bound estimates of eigenvalues for the bi-subelliptic operator $\Delta^2_X$. In this paper, we shall study the similar problem for more general Grushin type vector fields $X$. In the first part of this paper, we shall study the bi-subelliptic operators $\Delta^2_X$ in case of

$$X = (\partial_{x_1}, \ldots, \partial_{x_{n-1}}, f(\xi) \partial_{x_n}),$$

where $f(\xi) = \sum_{|\alpha| \leq k} a_{\alpha} \xi^\alpha$ is a multivariate polynomial of $\xi$ with order $s$, $\xi = (x_1, \ldots, x_{n-1})$, $\alpha = (\alpha_1, \ldots, \alpha_{n-1}) \in \mathbb{Z}_{+}^{n-1}$, $|\alpha| = \alpha_1 + \cdots + \alpha_{n-1}$, $a_\alpha$ are constants. We suppose that

$(H_1)$: If $f(\xi)$ has a unique zero point at origin $\xi = 0$ in $\Omega$ only, and there exists a unique multi-index $a_0$ with $|a_0| = s_0 \leq s$, satisfying $\partial^{|a_0|} f(\xi)|_{\xi = 0} \neq 0$ and $\partial^{|a_0|} f(\xi)|_{\xi = 0} = 0$ for any $|\alpha| < |a_0|$.

Thus we have the following result.

**Theorem 1.1.** Let $X = (\partial_{x_1}, \ldots, \partial_{x_{n-1}}, f(\xi) \partial_{x_n})$, $\xi = (x_1, x_2, \ldots, x_{n-1})$. Under the condition $(H_1)$ above, $X$ satisfies the Hörmander’s condition with its Hörmander index $Q = s_0 + 1$, and the generalized Métivier index of $X$ is $\bar{\vartheta} = Q + n - 1$. Suppose $\lambda_j$ is the $j$-th eigenvalue of the problem (1.1), then for all $k \geq 1$,

$$\sum_{j=1}^k \lambda_j \geq C(Q) k^{1 + \frac{\bar{\vartheta}}{Q}} - \frac{C_2(Q)}{C_1(Q)} k,$$

where

$$C(Q) = \frac{A_Q}{C_1(Q) n^2 (n + Q + 3)^{n+Q+1}} \left( \frac{(2\pi)^n}{Q \omega_{n-1} |\Omega|_n} \right)^{\frac{4}{n+Q+1}} (n + Q - 1)^{\frac{n+Q+3}{n+Q+1}},$$

and

$$A_Q = \begin{cases} \min\{1, n^{\frac{3-Q}{Q}}\}, & Q \geq 2, \\ n, & Q = 1. \end{cases}$$

Here $C_1(Q), C_2(Q)$ are the constants in Proposition 2.3 below, $\omega_{n-1}$ is the area of the unit sphere in $\mathbb{R}^n$, and $|\Omega|_n$ is the volume of $\Omega$.

**Remark 1.1.** (1) Since $k\lambda_k \geq \sum_{j=1}^k \lambda_j$, then Theorem 1.1 shows that the eigenvalues $\lambda_k$ satisfy

$$\lambda_k \geq C(Q) k^{1 + \frac{\bar{\vartheta}}{Q}} - \frac{C_2(Q)}{C_1(Q)}, \quad \text{for all } k \geq 1.$$

(2) If $Q \geq 1$, we can deduce from Definition 1.3 that $n + Q - 1 \leq \bar{\vartheta} \leq nQ$. Thus in our case in Theorem 1.1 $\bar{\vartheta} = n + Q - 1$ is the smallest. That means the lower bound estimates (1.6) will be optimal.

(3) If $f(\xi) = 1$ in Theorem 1.1, then $Q = 1$, $\Delta^2_X = \Delta^2$ is the standard bi-harmonic operator. Then $C_1(Q) = 1$, $C_2(Q) = 0$ and $C(Q) = \frac{16\pi^4}{n+4} \left( \omega_{n-1} |\Omega|_n \right)^{-4/n}$. Thus the result of Theorem 1.1 will be the same to the result of (1.4) in Levine and Protter [15].
In the second part, we shall study the bi-subelliptic operators $\Delta_{\tilde{X}}^2$ for more general cases, namely

$$X = (\partial x_1, \cdots, \partial x_{n-p}, f_1(\tilde{x}_{(p)}) \partial x_{n-p+1}, \cdots, f_p(\tilde{x}_{(p)}) \partial x_n),$$

(1.7)

where $\tilde{x}_{(p)} = (x_1, \cdots, x_{n-p})$,

$$f_j(\tilde{x}_{(p)}) = \sum_{|a| \leq s_j} a_j x_a^{(p)}, \quad (1 \leq j \leq p),$$

are multivariate polynomials of $\tilde{x}_{(p)}$ with order $s_j$. Thus $X$ is more general Grushin type degenerate vector fields with $p$ degenerate directions. We suppose that

$(H_2)$: For each $j, 1 \leq j \leq p < n$, if $f_j(\tilde{x}_{(p)})$ has a unique zero point at origin $\tilde{x}_{(p)} = 0$ in $\Omega$ only, and there exists a unique multi-index $a_{0j}$ with $|a_{0j}| = s_{0j} \leq s_j$, satisfying $\partial_{\tilde{x}_{(p)}}^{|a_{0j}|} f_j(\tilde{x}_{(p)})|_{\tilde{x}_{(p)} = 0} \neq 0$ and $\partial_{\tilde{x}_{(p)}}^{s_{0j}} f_j(\tilde{x}_{(p)})|_{\tilde{x}_{(p)} = 0} = 0$ for any $|a_1| < |a_{0j}|$.

Thus we have

**Theorem 1.2.** Under the condition $(H_2)$ above, the vector fields $X$ satisfies the Hörmander’s condition with its Hörmander index $Q = \max\{s_{01}, s_{02}, \cdots, s_{0p}\} + 1$, and the generalized Métivier index $\tilde{\sigma} = n + \sum_{j=1}^{p} s_{0j}$. Suppose $\lambda_j$ is the $j$-th eigenvalue of the problem (1.1), then for all $k \geq 1$,

$$\sum_{j=1}^{k} \lambda_j \geq \hat{C}(Q) k^{1 + \frac{4}{Q}} - \frac{C_4(Q)}{C_3(Q)} k,$$

(1.8)

where

$$\hat{C}(Q) = \frac{2^n}{5C_3(Q)n^{\frac{4 + \tilde{\sigma}}{n - \tilde{\sigma}}}} \left( \frac{\tilde{\sigma}}{\omega_{n-1} \prod_{j=1}^{p} (s_{0j} + 1) \Omega^n} \right)^{\frac{4 + \tilde{\sigma}}{n - \tilde{\sigma}}},$$

where $\tilde{\sigma} = n + \sum_{j=1}^{p} s_{0j}$, $C_3(Q)$ and $C_4(Q)$ are the corresponding sub-elliptic estimate constants in Proposition 2.4, $\omega_{n-1}$ is the area of the unit sphere in $\mathbb{R}^n$, $|\Omega|^n$ is the volume of $\Omega$.

**Remark 1.2.** Since $k \lambda_k \geq \sum_{j=1}^{k} \lambda_j$, then Theorem 1.2 shows that the eigenvalues $\lambda_k$ satisfy

$$\lambda_k \geq \hat{C}(Q) k^{\frac{4}{Q}} - \frac{C_4(Q)}{C_3(Q)}, \quad \text{for all} \quad k \geq 1.$$
2 Preliminaries

**Proposition 2.1.** Let the system of vector fields $X = (X_1, \cdots, X_m)$ satisfies the Hörmander’s condition on $\Omega$ with its Hörmander index $Q \geq 1$, then the following estimate

$$
\left\| \left| \nabla \right|^{\frac{3}{2}} u \right\|_{L^2(\Omega)}^2 \leq C(Q) \left\| \Delta_X u \right\|_{L^2(\Omega)}^2 + \tilde{C}(Q) \left\| u \right\|_{L^2(\Omega)}^2
$$

holds for all $u \in C^\infty_0(\Omega)$, where $\nabla = (\partial_{x_1}, \cdots, \partial_{x_m})$, $\left| \nabla \right|^{\frac{3}{2}}$ is a pseudo-differential operator with the symbol $\left| \xi \right|^{\frac{3}{2}}$, the constants $C(Q) > 0$, $\tilde{C}(Q) \geq 0$ depending on $Q$.

**Proof.** Refer to [12] and [21], the subelliptic operator $\Delta_X = \sum_{i=1}^m X_i^2$ satisfies the following sub-elliptic estimate for any $u \in C^\infty_0(\Omega)$,

$$
\left\| u \right\|_{(2\varepsilon)} \leq C_1 \left\| \Delta_X u \right\|_{L^2(\Omega)} + C_2 \left\| u \right\|_{L^2(\Omega)},
$$

with $\varepsilon = \frac{1}{Q}$, where $\left\| u \right\|_{(2\varepsilon)}$ is the Sobolev norm of order $2\varepsilon$. On the other hand, we have

$$
\left\| u \right\|_{(\frac{3}{2})} = \left( \int_\Omega \left( 1 + \left| \xi \right|^2 \right)^{\frac{3}{2}} |\hat{u}(\xi)|^2 d\xi \right)^{\frac{1}{2}} \geq \left( \int_\Omega \left| \xi \right|^{\frac{3}{2}} |\hat{u}(\xi)|^2 d\xi \right)^{\frac{1}{2}} = \left\| \nabla \frac{3}{2} u \right\|_{L^2(n)} = \left\| \nabla \frac{3}{2} u \right\|_{L^2(\Omega)}.
$$

By using the Cauchy-Schwarz inequality we get the following estimate

$$
\left\| \nabla \frac{3}{2} u \right\|_{L^2(\Omega)}^2 \leq C(Q) \left\| \Delta_X u \right\|_{L^2(\Omega)}^2 + \tilde{C}(Q) \left\| u \right\|_{L^2(\Omega)}^2.
$$

Thus, we complete the proof. \(\square\)

**Proposition 2.2.** (cf. [19, 21] and [22]) Let the system of vector fields $X = (X_1, \cdots, X_m)$ satisfies the Hörmander’s condition on $\Omega$, then the operator $\Delta_X = \sum_{i=1}^m X_i^2$ is maximally hypo-elliptic, i.e., there exists a constant $C > 0$, such that for any $u \in C^\infty_0(\Omega)$ we have the following maximally hypo-elliptic estimate

$$
\sum_{|\alpha| \leq 2} \left\| X^\alpha u \right\|_{L^2(\Omega)}^2 \leq C (\left\| \Delta_X u \right\|_{L^2(\Omega)}^2 + \left\| u \right\|_{L^2(\Omega)}^2),
$$

where $\alpha = (\alpha_1, \cdots, \alpha_m)$ is a multi-index with $|\alpha| = \alpha_1 + \cdots + \alpha_m$ and $X^\alpha = X_1^{\alpha_1} \cdots X_m^{\alpha_m}$. 
**Proposition 2.3.** Let $X = (\partial_{x_1}, \ldots, \partial_{x_{n-p}}, f_1, \partial_{x_{n-p+1}}, \ldots, f_p)$, where $f_j$ is a multivariate polynomial and satisfies the condition $(H1)$ above. Then $X$ satisfies the Hörmander’s condition with its Hörmander index $Q \geq 1$, and we can deduce the following sub-elliptic estimate

$$
\sum_{j=1}^{n-1} \|\partial^2_{x_j} u\|^2_{L^2(\Omega)} + \|\partial_{x_n}\hat{\xi} u\|^2_{L^2(\Omega)} \leq C_1(\Omega)\|\Delta_X u\|^2_{L^2(\Omega)} + C_2(\Omega)\|u\|^2_{L^2(\Omega)},
$$

(2.2)

for all $u \in C_0^\infty(\Omega)$, where $|\partial_{x_n}\hat{\xi}|$ is a pseudo-differential operator with the symbol $|\xi_n|\hat{\xi}$, $C_1(\Omega) > 0$, $C_2(\Omega) \geq 0$ are constants depending on $Q$.

**Proof.** From the Plancherel’s formula, we have

$$
\left\| \partial_{x_n}\hat{\xi} u \right\|^2_{L^2(\Omega)} = \left\| \xi_n\hat{\xi} u \right\|^2_{L^2(\mathbb{R}^n)} \leq \left\| \xi \hat{\xi} u \right\|^2_{L^2(\mathbb{R}^n)} = \left\| \nabla \hat{\xi} u \right\|^2_{L^2(\Omega)} = \left\| \nabla \hat{\xi} u \right\|^2_{L^2(\Omega)}.
$$

(2.3)

Also, from the maximally hypo-elliptic estimate of Proposition 2.2 we can deduce that

$$
\sum_{j=1}^{n-1} \|\partial^2_{x_j} u\|^2_{L^2(\Omega)} \leq \sum_{|\alpha| \leq 2} \|X^\alpha u\|^2_{L^2(\Omega)} \leq C(\|\Delta_X u\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)}).
$$

(2.4)

Combining (2.1), (2.3) and (2.4) we can deduce that

$$
\sum_{j=1}^{n-1} \|\partial^2_{x_j} u\|^2_{L^2(\Omega)} + \|\partial_{x_n}\hat{\xi} u\|^2_{L^2(\Omega)} \leq C_1(\Omega)\|\Delta_X u\|^2_{L^2(\Omega)} + C_2(\Omega)\|u\|^2_{L^2(\Omega)}.
$$

Thus, we complete the proof. □

**Proposition 2.4.** Let $X = (\partial_{x_1}, \ldots, \partial_{x_{n-p}}, f_1, f_2, \ldots, f_p)$, where $f_j$ is a multivariate polynomial which satisfies the condition $(H2)$ above. Then $X$ satisfies the Hörmander’s condition with its Hörmander index $Q \geq 1$, and we get the following sub-elliptic estimate

$$
\sum_{i=1}^{n-p} \|\partial^2_{x_i} u\|^2_{L^2(\Omega)} + \sum_{j=1}^{p} \left\| \partial_{x_{n-p+j}} \hat{\xi} u \right\|_{L^2(\Omega)}^2 \leq C_3(\Omega)\|\Delta_X u\|^2_{L^2(\Omega)} + C_4 \|u\|^2_{L^2(\Omega)},
$$

(2.5)

for all $u \in C_0^\infty(\Omega)$, where $|\partial_{x_i}|\hat{\xi}$ is a pseudo-differential operator with the symbol $|\xi_j|\hat{\xi}$, and the constants $C_3(\Omega) > 0$, $C_4(\Omega) \geq 0$ depending on $Q$. 


Proof. We consider the system of vector fields \( \tilde{X} = (\partial_{x_{1}}, \cdots, \partial_{x_{n-p}}, f_{j}(x_{p})\partial_{x_{n-p+j}}) \) (for \( 1 \leq j \leq p < n \)) defined on the projection \( \Omega_{e} \) of \( \Omega \) on the direction \( x'_{j} = (x_{1}, \cdots, x_{n-p}, x_{n-p+j}) \). Similar to Proposition 2.3, for all \( j \) (\( 1 \leq j \leq p \)), we have

\[
\sum_{i=1}^{n-p} \left\| \partial_{x_{i}}^{2} u \right\|_{L^{2}(\Omega_{e})}^{2} + \left\| \partial_{x_{n-p+j}} \left| \frac{2}{\alpha_{i}} \right| u \right\|_{L^{2}(\Omega_{e})}^{2} \leq \tilde{C}_{4}(Q)\|\Delta_{X} u\|_{L^{2}(\Omega_{e})}^{2} + \tilde{C}_{2}(Q)\|u\|_{L^{2}(\Omega_{e})}^{2}. 
\]

Then for all \( j \) (\( 1 \leq j \leq p \)), we have

\[
\sum_{i=1}^{n-p} \left\| \partial_{x_{i}}^{2} u \right\|_{L^{2}(\Omega)}^{2} + \left\| \partial_{x_{n-p+j}} \left| \frac{2}{\alpha_{i}} \right| u \right\|_{L^{2}(\Omega)}^{2} \leq \tilde{C}_{1}(Q)\|\Delta_{X} u\|_{L^{2}(\Omega)}^{2} + \tilde{C}_{2}(Q)\|u\|_{L^{2}(\Omega)}^{2}. 
\]

By using the Cauchy-Schwarz inequality and Proposition 2.2, there exists a constant \( C_{3} > 0 \) such that

\[
\|\Delta_{X} u\|_{L^{2}(\Omega)}^{2} \leq C_{3} \sum_{|a| \leq 2} \|X^{a} u\|_{L^{2}(\Omega)}^{2} \leq C_{3}C(\|\Delta_{X} u\|_{L^{2}(\Omega)}^{2} + \|u\|_{L^{2}(\Omega)}^{2}),
\]

where \( C \) is given in Proposition 2.2. Finally, we get the following sub-elliptic estimate from \( (2.6) \)

\[
\sum_{i=1}^{n-p} \left\| \partial_{x_{i}}^{2} u \right\|_{L^{2}(\Omega)}^{2} + \sum_{j=1}^{p} \left\| \partial_{x_{n-p+j}} \left| \frac{2}{\alpha_{i}} \right| u \right\|_{L^{2}(\Omega)}^{2} \leq C_{3}(Q)\|\Delta_{X} u\|_{L^{2}(\Omega)}^{2} + C_{4}(Q)\|u\|_{L^{2}(\Omega)}^{2}.
\]

Thus, we complete the proof. \( \square \)

Next, for the Dirichlet eigenvalues problem \( (1.1) \), we have

**Proposition 2.5.** The Dirichlet eigenvalues problem \( (1.1) \) has a sequence of discrete eigenvalues \( \{\lambda_{j}\}_{j \geq 1} \), which satisfying \( 0 < \lambda_{1} \leq \lambda_{2} \leq \lambda_{3} \leq \cdots \leq \lambda_{k} \cdots \) and \( \lambda_{k} \to +\infty \) as \( k \to +\infty \). Also, the corresponding eigenfunctions \( \{\phi_{k}(x)\}_{k \geq 1} \) constitute an orthonormal basis of \( L^{2}(\Omega) \) and an orthogonal basis of \( H_{\nabla,0}^{2}(\Omega) \).

The proof of Proposition 2.5 depends the following lemma:

**Lemma 2.1.** If \( u \in H_{\nabla,0}^{2}(\Omega) \), then for \( 1 \leq j \leq m \), \( X_{j} u \in H_{\nabla,0}^{1}(\Omega) \).

Proof. Since \( u \in H_{\nabla,0}^{2}(\Omega) \), we have \( X_{i}(X_{j} u) \in L^{2}(\Omega) \) for any \( 1 \leq i, j \leq m \), and \( (X_{j} u) \in L^{2}(\Omega) \). That implies \( X_{j} u \in H_{\nabla,0}^{1}(\Omega) \). Now, \( u \in H_{\nabla,0}^{2}(\Omega) \), then there exists a sequence \( \varphi_{j} \in C_{0}^{\infty}(\Omega) \) which converges to \( u \) in \( H_{\nabla,0}^{2}(\Omega) \). That means \( X_{j} \varphi_{j} \to X_{j} u \) in \( H_{\nabla,0}^{1}(\Omega) \). Observe that \( X_{j} \varphi_{j} \in H_{\nabla,0}^{1}(\Omega) \) and \( H_{\nabla,0}^{1}(\Omega) \) is a Hilbert space, thus we have \( X_{j} u \in H_{\nabla,0}^{1}(\Omega) \). \( \square \)
Proof of Proposition 2.5. We know that the definition domain of $\Delta_X^2$ is
\[ \text{dom}(\Delta_X^2) = \{ u \in H^2_{X,0}(\Omega) | \Delta_X^2 u \in L^2(\Omega) \}. \]

Thus, for $X_j$ to be formally skew-adjoint, then for any function $u \in C_0^\infty(\Omega)$ and $v \in \text{dom}(\Delta_X^2)$, we have
\[ \int_{\Omega} u \Delta_X^2 v \, dx = \int_{\Omega} v \Delta_X^2 u \, dx = \int_{\Omega} v \Delta_X (\Delta_X u) \, dx = \sum_{j=1}^m \int_{\Omega} v \cdot X_j^2 (\Delta_X u) \, dx. \]

Since $v \in H^2_{X,0} \subset H^1_{X,0}(\Omega)$, and from the result of Lemma 2.1, $X_j v \in H^1_{X,0}(\Omega)$. Then the equation above gives
\[ \int_{\Omega} u \Delta_X^2 v \, dx = -\sum_{j=1}^m \int_{\Omega} X_j v \cdot X_j (\Delta_X u) \, dx = \sum_{j=1}^m \int_{\Omega} X_j^2 v \cdot (\Delta_X u) \, dx, \]

that gives the following Green formula:
\[ \int_{\Omega} u \Delta_X^2 v \, dx = \int_{\Omega} \Delta_X u \cdot \Delta_X v \, dx, \quad \text{for} \quad u \in H^2_{X,0}(\Omega), \quad v \in \text{dom}(\Delta_X^2). \quad (2.7) \]

On the other hand, for $u \in H^2_{X,0}(\Omega)$,
\[ \| u \|_{H^2_X}^2 = \| u \|_{L^2(\Omega)}^2 + \sum_{i=1}^m \| X_i u \|_{L^2(\Omega)}^2 + \sum_{i,j=1}^m \| X_i X_j u \|_{L^2(\Omega)}^2. \]

Thus we have
\[ \| u \|_{H^2_X}^2 \geq \| u \|_{L^2(\Omega)}^2 + \sum_{j=1}^m \| X_j^2 u \|_{L^2(\Omega)}^2 \geq \| \Delta_X u \|_{L^2(\Omega)}^2. \quad (2.8) \]

By maximally hypoellipticity of $\Delta_X$ (also see Proposition 2.2 above), we have following estimate for any $u \in H^2_{X,0}(\Omega)$,
\[ \| u \|_{H^2_X}^2 = \sum_{|\alpha| \leq 2} \| X^\alpha u \|_{L^2(\Omega)}^2 \leq C \| \Delta_X u \|_{L^2(\Omega)}^2 + \| u \|_{L^2(\Omega)}^2. \quad (2.9) \]

Furthermore, the Poincaré inequality gives
\[ \| u \|_{L^2(\Omega)}^2 \leq C_1 \| Xu \|_{L^2(\Omega)}^2 \leq C_1 |(\Delta_X u,u)| \leq C_1 \| \Delta_X u \|_{L^2(\Omega)} \cdot \| u \|_{L^2(\Omega)}. \]

Thus for any $0 < \epsilon < 1$ there is $C_\epsilon > 0$, such that
\[ \| \Delta_X u \|_{L^2(\Omega)} \cdot \| u \|_{L^2(\Omega)} \leq C_\epsilon \| \Delta_X u \|_{L^2(\Omega)}^2 + \epsilon \| u \|_{L^2(\Omega)}^2. \]
That means from (2.9) that there exists $C_2 > 0$, such that
\[
\|u\|_{H^2_X}^2 \leq C_2 \|\Delta_X u\|_{L^2(\Omega)}^2.
\] (2.10)
Hence from (2.8) and (2.10) one has for any $u \in H^2_{X,0}(\Omega)$,
\[
\|\Delta_X u\| \leq \|u\|_{H^2_X} \leq C_3 \|\Delta_X u\|.
\] (2.11)
Thus we define that
\[
[u, \varphi] = (\Delta_X u, \Delta_X \varphi),
\] (2.12)
then $[\cdot, \cdot]$ is another inner product, and $H^2_{X,0}(\Omega)$ with this inner product is complete.

Now, we choose $u, v \in \text{dom}(\Delta^2_X)$, then
\[
(\Delta^2_X u, v) = (\Delta_X u, \Delta_X v) = (\Delta^2_X v, u).
\]
Hence, $\Delta^2_X$ is symmetric operator in $\text{dom}(\Delta^2_X)$. Also
\[
(\Delta^2_X u, u) = (\Delta_X u, \Delta_X u) \geq 0,
\]
which implies that $\Delta^2_X$ is positive in $\text{dom}(\Delta^2_X)$.

Next, for any given $f \in L^2(\Omega)$ and any $\varphi \in H^2_{X,0}(\Omega)$, we define a functional $f(\varphi) = (f, \varphi)$. Since
\[
|\langle f, \varphi \rangle| \leq \|f\|_{L^2(\Omega)} \cdot \|\varphi\|_{L^2(\Omega)} \leq \|f\|_{L^2(\Omega)} \cdot \|\varphi\|_{H^2_X(\Omega)},
\]
then the functional $\langle f, \varphi \rangle$ is a continuous linear functional on Hilbert space $H^2_{X,0}(\Omega)$. By Riesz representation theorem, there exists a unique $u \in H^2_{X,0}(\Omega)$ such that
\[
\langle f, \varphi \rangle = [u, \varphi] = (\Delta_X u, \Delta_X \varphi).
\]
Thus the Green formula (2.7) gives that
\[
(\Delta^2_X u, \varphi) = (\Delta_X u, \Delta_X \varphi) = (f, \varphi)
\] (2.13)
holds for any $\varphi \in C^\infty_0(\Omega)$. That implies $\Delta^2_X u = f$, i.e., $u \in \text{dom}(\Delta^2_X)$. This proves the existence of the resolvent operator $R := (\Delta^2_X)^{-1}$, and $Rf = u$.

On the other hand, if we choose $\varphi = u$ in (2.13), then $\langle Rf, f \rangle = (u, f) = \|\Delta_X u\|_{L^2(\Omega)}^2 \geq 0$.

$R$ is positive in $L^2(\Omega)$. Meanwhile we have
\[
\|Rf\|_{L^2(\Omega)}^2 = \|u\|_{L^2(\Omega)}^2 \leq C\|f\|_{L^2(\Omega)} \|Rf\|_{L^2(\Omega)},
\]
this implies that $R$ is bounded in $L^2(\Omega)$. In order to prove the operator $R$ is self-adjoint, it suffices to prove that $R$ is symmetric, i.e.,
\[
\langle Rf, g \rangle = (f, Rg) \quad \text{for all} \quad f, g \in L^2(\Omega).
\]
Let \( Rf = u, Rg = v \), and choosing \( \phi = v \) in (2.13), we obtain
\[
(\Delta_X u, \Delta_X v) = (f, Rg).
\]

Since the left hand side is symmetric in \( u \) and \( v \), we conclude that the right side is symmetric in \( f \) and \( g \). That implies that \( R \) is symmetric. Also, we know that the operator \( R^{-1} := \Delta_X^2 \) is a self-adjoint on \( \text{dom}(\Delta_X^2) \).

Similarly, we can prove that the inverse operator \( (\Delta_X^2 + \alpha \cdot id)^{-1} \) exists and is bounded for any \( \alpha \geq 0 \). We see that \(-\alpha\) is a regular value of \( \Delta_X^2 \), hence \( \text{spec}(\Delta_X^2) \subset (0, +\infty) \). Moreover, we can deduce that \( R : L^2(\Omega) \rightarrow H^2_{X,0}(\Omega) \) is continuous, this is because that
\[
\|Rf\|_{H^2_X}^2 \leq C(\|\Delta_X(Rf)\|_{L^2(\Omega)}^2) \leq C(f, Rf) \leq C\|f\|_{L^2(\Omega)}\|Rf\|_{L^2(\Omega)} \leq C\|f\|_{L^2(\Omega)}\|Rf\|_{H^2_X(\Omega)}.
\]

By using the subelliptic estimate, we know that \( H^2_{X,0} \) can be continuously embedded into the standard Sobolev space \( H^2(\Omega) \), and \( H^2(\Omega) \) can be compactly embedded into \( L^2(\Omega) \). Hence \( R \) is a compact operator from \( L^2(\Omega) \) to \( L^2(\Omega) \). By spectral theory we know that \( R \) has positive discrete eigenvalues \( \mu_i, \mu_1 \geq \mu_2 \geq \cdots \geq \mu_k \geq \cdots \) and \( \mu_k \rightarrow 0 \) as \( k \rightarrow +\infty \); and the corresponding eigenfunctions \( \phi_i \) of \( R \) form an orthonormal basis of \( L^2(\Omega) \), namely
\[
R\phi_i = \mu_i \phi_i.
\]

That means the eigenfunctions \( \{\phi_i\}_{i \geq 1} \) will be the orthogonal basis of \( H^2_{X,0}(\Omega) \). Finally we let \( \lambda_i = \mu_i^{-1} \), then \( \lambda_i \) are the Dirichlet eigenvalues of \( \Delta_X^2 \) which will be discrete and satisfying \( 0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots \), and \( \lambda_k \rightarrow +\infty \) as \( k \rightarrow +\infty \). The proof of Proposition 2.5 is completed. \( \square \)

3 Proof of Theorem 1.1

\textbf{Lemma 3.1 (cf. [3, 16]).} For the system of vector fields \( X = (X_1, \cdots, X_m) \), if \( \{\phi_j\}^{k}_{j=1} \) are the set of orthonormal eigenfunctions corresponding to the eigenvalues \( \{\lambda_j\}^{k}_{j=1} \). Define
\[
\Phi(x,y) = \sum_{j=1}^{k} \phi_j(x)\phi_j(y).
\]

Then for \( \hat{\Phi}(z,y) = (2\pi)^{-n/2} \int_{\mathbb{R}^{m}} \Phi(x,y)e^{-ix\cdot z}dx \) to be the partial Fourier transformation of \( \Phi(x,y) \) with respect to the \( x \)-variable, we have
\[
\int_{\Omega} \int_{\mathbb{R}^n} \left| \Phi(z,y) \right|^2 dzdy = k \quad \text{and} \quad \int_{\Omega} \left| \hat{\Phi}(z,y) \right|^2 dy \leq (2\pi)^{-n} |\Omega|_n.
\]
Lemma 3.2 (cf. [8]). Let \( f \) be a real-valued function defined on \( \mathbb{R}^n \) with \( 0 \leq f \leq M_1 \), and for \( Q \in \mathbb{N}^+ \),
\[
\int_{\mathbb{R}^n} \left( \sum_{j=1}^{n-1} z_j^2 + |z_n|^{2} \right)^2 f(z) \, dz \leq M_2.
\]
Then
\[
\int_{\mathbb{R}^n} f(z) \, dz \leq \left( Q M_1 \omega_{n-1} \right)^{n+Q-1} \left( n(n+Q+3) \right)^{n+Q-1} \frac{A_Q}{M_2^{n+Q-1}},
\]
where \( \omega_{n-1} \) is the area of the unit sphere in \( \mathbb{R}^n \), and
\[
A_Q = \begin{cases} 
\min \{1, n^{\frac{2}{n-Q}}\}, & Q \geq 2, \\
n, & Q = 1.
\end{cases}
\]

Proof of Theorem 1.1. From the results of Proposition 2.5, let \( \{\lambda_k\}_{k \geq 1} \) be a sequence of the eigenvalues for the problem (1.1), and \( \{\phi_k(x)\}_{k \geq 1} \) be the corresponding eigenfunctions, then \( \{\phi_k(x)\}_{k \geq 1} \) constitute an orthogonal basis of \( H^2_{X,0}(\Omega) \).

Let
\[
\Phi(x,y) = \sum_{j=1}^{k} \phi_j(x)\phi_j(y),
\]
by Cauchy-Schwarz inequality we have
\[
\int_{\mathbb{R}^n} \int_{\Omega} \left( \sum_{j=1}^{n-1} z_j^2 + |z_n|^{2} \right)^2 \left| \Phi(z,y) \right|^2 \, dydz 
\leq n \int_{\mathbb{R}^n} \int_{\Omega} \left( \sum_{j=1}^{n-1} z_j^4 + |z_n|^{4} \right) \left| \Phi(z,y) \right|^2 \, dydz.
\] (3.1)

Next, by using integration-by-parts, we have
\[
\sum_{j=1}^{k} \lambda_j = \sum_{j=1}^{k} \int_{\Omega} \lambda_j \phi_j(x) \cdot \phi_j(x) \, dx = \sum_{j=1}^{k} \int_{\Omega} \Delta^2_X \phi_j(x) \cdot \phi_j(x) \, dx
\]
\[
= \sum_{j=1}^{k} \int_{\Omega} X(\Delta_X \phi_j(x)) \cdot X \phi_j(x) \, dx = \sum_{j=1}^{k} \int_{\Omega} \Delta_X \phi_j(x) \cdot \Delta_X \phi_j(x) \, dx
\]
\[
= \int_{\Omega} \int_{\Omega} \sum_{j=1}^{k} |\Delta_X \phi_j(x) \cdot \phi_j(y)|^2 \, dx dy = \int_{\Omega} \int_{\Omega} |\Delta_X \Phi(x,y)|^2 \, dx dy.
\] (3.2)
Then by using Plancherel’s formula and Proposition 2.3, we have
\[
\int_{R^n} \int_{\Omega} \left( \sum_{j=1}^{n-1} z_j^2 + |z_n|^2 \right) \left| \hat{\Phi}(z,y) \right|^2 dy dz \\
\leq n \int_{R^n} \int_{\Omega} \left( \sum_{j=1}^{n-1} z_j^4 + |z_n|^4 \right) \left| \hat{\Phi}(z,y) \right|^2 dy dz \\
= n \int_{\Omega} \int \left( \sum_{j=1}^{n-1} |\partial_{x_j}^2 \Phi(x,y)|^2 + |\partial_{x_n} |^2 \Phi(x,y)|^2 \right) dy dx \\
= n \int_{\Omega} \int \left( \sum_{j=1}^{n-1} |\partial_{x_j}^2 \Phi(x,y)|^2 + |\partial_{x_n} |^2 \Phi(x,y)|^2 \right) dy dx \\
\leq n \left[ C_1(Q) \int_{\Omega \setminus Q} |\Delta x \Phi(x,y)|^2 dxdy + C_2(Q) \int_{\Omega \setminus Q} |\Phi(x,y)|^2 dxdy \right].
\]
Then from (3.2) and Lemma 3.1 above, we can deduce that
\[
\int_{\Omega} \left( \sum_{j=1}^{n-1} z_j^2 + |z_n|^2 \right) \left| \hat{\Phi}(z,y) \right|^2 dy dz \leq n \left( C_1(Q) \sum_{j=1}^{k} \lambda_j + C_2(Q)k \right).
\]
Next, we choose
\[
f(z) = \int_{\Omega} \left| \hat{\Phi}(z,y) \right|^2 dy, \quad M_1 = (2\pi)^{-n} |\Omega|, \quad M_2 = n \left( C_1(Q) \sum_{j=1}^{k} \lambda_j + C_2(Q)k \right).
\]
Then from the result of Lemma 3.2, we know that for any \( k \geq 1, \)
\[
k \leq \frac{Q \omega_{n-1} (2\pi)^{-n} |\Omega|}{n + Q - 1} \left( \frac{n(n+Q+3)}{(2\pi)^{-n} |\Omega| Q A_Q \omega_{n-1}} \right)^{\frac{n+Q-1}{n+Q+3}} \left( n \left( C_1(Q) \sum_{j=1}^{k} \lambda_j + C_2(Q)k \right) \right)^{\frac{n+Q-1}{n+Q+3}}.
\]
This means, for any \( k \geq 1, \)
\[
\sum_{j=1}^{k} \lambda_j \geq \frac{\tilde{C}(Q) k^{1+\frac{2}{Q}} - C_2(Q)}{C_1(Q) k},
\]
with
\[
\tilde{C}(Q) = \frac{A_Q}{C_1(Q)n^2(n+Q+3)} \left( \frac{(2\pi)^n}{Q \omega_{n-1} |\Omega|} \right)^{\frac{1}{n+Q+3}} (n+Q-1)^{\frac{n+Q+3}{n+Q+3}}.
\]
The proof of Theorem 1.1 is completed. \( \square \)
4 Proof of Theorem 1.2

Lemma 4.1. Let \( f \) be a real-valued function defined on \( \mathbb{R}^n \) with \( 0 \leq f \leq M_1 \), and for \( p, q \in \mathbb{N}^+ \),

\[
\int_{\mathbb{R}^n} \left( \sum_{i=1}^{n-p} z_i^2 + \sum_{j=1}^p |z_{n-p+j}| \frac{2}{s_0+j} \right)^2 f(z) \, dz \leq M_2.
\]

Then

\[
\int_{\mathbb{R}^n} f(z) \, dz \leq \omega_{n-1} \prod_{j=1}^p (s_j + 1) \left( \frac{5n^{\frac{4q}{2n}}}{2^n} \right)^{\frac{s_j}{4q}} M_1^{\frac{4q}{2n}} M_2^{\frac{4q}{2n}},
\]

where \( \omega_{n-1} \) is the area of the unit sphere in \( \mathbb{R}^n \).

Proof. First, we choose \( R \) such that

\[
\int_{\mathbb{R}^n} \left( \sum_{i=1}^{n-p} z_i^2 + \sum_{j=1}^p |z_{n-p+j}| \frac{2}{s_0+j} \right)^2 g(z) \, dz = M_2,
\]

where

\[
g(z) = \begin{cases} 
M_1, & \sum_{i=1}^{n-p} z_i^2 + \sum_{j=1}^p |z_{n-p+j}| \frac{2}{s_0+j} \leq R^2, \\
0, & \sum_{i=1}^{n-p} z_i^2 + \sum_{j=1}^p |z_{n-p+j}| \frac{2}{s_0+j} > R^2.
\end{cases}
\]

Then

\[
\left[ \left( \sum_{i=1}^{n-p} z_i^2 + \sum_{j=1}^p |z_{n-p+j}| \frac{2}{s_0+j} \right)^2 - R^4 \right] (f(z) - g(z)) \geq 0.
\]

Hence we have

\[
R^4 \int_{\mathbb{R}^n} (f(z) - g(z)) \, dz \leq \int_{\mathbb{R}^n} \left( \sum_{i=1}^{n-p} z_i^2 + \sum_{j=1}^p |z_{n-p+j}| \frac{2}{s_0+j} \right)^2 (f(z) - g(z)) \, dz \leq 0.
\]

That means

\[
\int_{\mathbb{R}^n} f(z) \, dz \leq \int_{\mathbb{R}^n} g(z) \, dz. \tag{4.1}
\]

Now we have

\[
M_2 = \int_{\mathbb{R}^n} \left( \sum_{i=1}^{n-p} z_i^2 + \sum_{j=1}^p |z_{n-p+j}| \frac{2}{s_0+j} \right)^2 g(z) \, dz = M_1 \int_{\mathbb{R}^n} \left( \sum_{i=1}^{n-p} z_i^2 + \sum_{j=1}^p |z_{n-p+j}| \frac{2}{s_0+j} \right)^2 \, dz,
\]
Then we have

$$B_R = \left\{ z \in \mathbb{R}^n, \sum_{i=1}^{n-p} z_i^2 + \sum_{j=1}^{p} |z_{n-p+j}|^{\frac{2}{s_0+1}} \leq R^2 \right\}.$$ 

Next, we change the variables as follows,

$$z_i = z_i' \quad (i = 1, 2, \cdots, n-p), \quad z_{n-p+j} = \text{sgn}(z_{n-p+j}') |z_{n-p+j}'|^{s_0+1}, \quad (j = 1, 2, \cdots, p).$$

Then we have the following determinant of Jacobian,

$$|\det(\frac{\partial (z_1', \cdots, z_n')}{\partial (z_1, \cdots, z_n)} )| = \prod_{j=1}^{p} (s_0+1) |z_{n-p+j}'|^{s_0}.$$ 

Hence

$$M_2 = M_1 \int_{B_R} \left( \sum_{i=1}^{n-p} z_i^2 + \sum_{j=1}^{p} |z_{n-p+j}|^{\frac{2}{s_0+1}} \right)^2 \, dz$$

$$= M_1 \prod_{j=1}^{p} (s_0+1) \int_{B_R} |z|^4 \prod_{j=1}^{p} |z_{n-p+j}|^{s_0} \, dz$$

$$\geq M_1 \prod_{j=1}^{p} (s_0+1) \int_{A_R} |z|^4 \prod_{j=1}^{p} |z_{n-p+j}|^{s_0} \, dz,$$

where

$$B_R = \{ z \in \mathbb{R}^n, |z| \leq R \}, \quad A_R = \{ z \in \mathbb{R}^n, |z| \leq \frac{R}{\sqrt{n}}, j = 1, \cdots, n \}.$$ 

By a direct calculation, we have

$$\int_{A_R} |z|^4 \prod_{j=1}^{p} |z_{n-p+j}|^{s_0} \, dz$$

$$\geq \int_{A_R} |z|^4 \prod_{j=1}^{p} |z_{n-p+j}|^{s_0} \, dz$$

$$= 2 \int_{0}^{\frac{R}{\sqrt{n}}} |z|^4 \, dz \times \prod_{j=1}^{p} \left( 2 \int_{0}^{\frac{R}{\sqrt{n}}} |z_{n-p+j}|^{s_0} \, dz \right) \times \left( 2 \int_{0}^{\frac{R}{\sqrt{n}}} 1 \, dz \right)$$

$$= \frac{2^n}{5} \frac{1}{\prod_{j=1}^{p} (s_0+1)} \frac{n^{n+1} \prod_{j=1}^{p} (s_0)^j}{2} R^{n+4} \sum_{j=1}^{p} s_0 j \quad n \geq 4.$$ 

Then we have

$$M_2 \geq \frac{2^n M_1}{5} n^{-\frac{4k\theta}{2}} R^{4+\theta}. \quad (4.2)$$
From the definition of \( g(z) \), we know that

\[
\int_{\mathbb{R}^n} g(z) \, dz = M_1 \int_{B_R} \left| \frac{\sum_{j=1}^{p} s_{0j}}{\sum_{j=1}^{p} |s_{0j}|} \right| \, dz = M_1 \int_{B_R} \left| \frac{\sum_{j=1}^{p} s_{0j}}{\sum_{j=1}^{p} |s_{0j}|} \right| \, dz
\]

\[
\leq M_1 \prod_{j=1}^{p} (s_{0j} + 1) \int_{B_R} \left| \frac{\sum_{j=1}^{p} s_{0j}}{\sum_{j=1}^{p} |s_{0j}|} \right| \, dz = M_1 \frac{\prod_{j=1}^{p} (s_{0j} + 1)}{\sum_{j=1}^{p} s_{0j}} \int_{\mathbb{R}} \omega_{n-1} r^{n-1} \, dr
\]

\[
= \frac{M_1 \omega_{n-1} \prod_{j=1}^{p} (s_{0j} + 1)}{n + \sum_{j=1}^{p} s_{0j}} = \frac{M_1 \omega_{n-1} \prod_{j=1}^{p} (s_{0j} + 1)}{\tilde{\nu}} \frac{5^{n+\tilde{\nu}}}{2^n} M_1^{\tilde{\nu}} M_2^{\tilde{\nu}}, \quad (4.3)
\]

From (4.1), (4.2) and (4.3), we obtain

\[
\int_{\mathbb{R}^n} f(z) \, dz \leq \int_{\mathbb{R}^n} g(z) \, dz \leq \frac{\omega_{n-1} \prod_{j=1}^{p} (s_{0j} + 1)}{\tilde{\nu}} \frac{5^{n+\tilde{\nu}}}{2^n} M_1^{\tilde{\nu}} M_2^{\tilde{\nu}},
\]

where \( \tilde{\nu} = n + \sum_{j=1}^{p} s_{0j} \). Lemma 4.1 is proved. \( \square \)

**Proof of Theorem 1.2.** Let \( \{\lambda_k\}_{k \geq 1} \) be a sequence of the eigenvalues for the problem (1.1), \( \{\Phi_k(x)\}_{k \geq 1} \) be the corresponding eigenfunctions. Then \( \{\phi_k(x)\}_{k \geq 1} \) constitute an orthogonal basis of \( H_{\lambda,0}^2(\Omega) \).

Let \( \Phi(x, y) = \sum_{j=1}^{k} \phi_j(x) \phi_j(y) \). Thus, by using the Cauchy-Schwarz inequality, we have

\[
\int_{\mathbb{R}^n} \int_{\Omega} \left( \sum_{j=1}^{n-p} z_j^2 + \sum_{j=1}^{p} |z_{n-p+j}|^{\frac{2}{|\nu|+1}} \right)^2 |\Phi(z,y)|^2 \, dy \, dz
\]

\[
\leq n \int_{\mathbb{R}^n} \int_{\Omega} \left( \sum_{j=1}^{n-p} z_j^4 + \sum_{j=1}^{p} |z_{n-p+j}|^{\frac{4}{|\nu|+1}} \right) |\Phi(z,y)|^2 \, dy \, dz. \quad (4.4)
\]

Similar to the result of (3.2), we obtain that

\[
\sum_{j=1}^{k} \lambda_j = \int_{\mathbb{R}^n} \int_{\Omega} |\Delta x \Phi(x,y)|^2 \, dx \, dy. \quad (4.5)
\]

Then by using Plancherel’s formula and Proposition 2.4, we have

\[
\int_{\mathbb{R}^n} \int_{\Omega} \left( \sum_{j=1}^{n-p} z_j^2 + \sum_{j=1}^{p} |z_{n-p+j}|^{\frac{2}{|\nu|+1}} \right)^2 |\Phi(z,y)|^2 \, dy \, dz
\]

\[
\leq n \int_{\mathbb{R}^n} \int_{\Omega} \left( \sum_{j=1}^{n-p} z_j^4 + \sum_{j=1}^{p} |z_{n-p+j}|^{\frac{4}{|\nu|+1}} \right) |\Phi(z,y)|^2 \, dy \, dz
\]
Thus from (4.5) and Lemma 3.1 above, we can deduce that

\[ \int_{\Omega} \left( \sum_{j=1}^{n-p} |\partial_{y_j} \Phi(x,y)|^2 + \sum_{j=1}^{p} |\partial_{x_{n-p+j}} \Phi(x,y)|^2 \right) \, dy \, dx \leq n \left[ C_3(Q) \int_{\Omega} |\Delta_X \Phi(x,y)|^2 \, dx \, dy + C_4(Q) \int_{\Omega} |\Phi(x,y)|^2 \, dx \, dy \right]. \]

The proof of Theorem 1.2 is completed.

Finally, we choose

\[ f(z) = \int_{\Omega} |\tilde{\Phi}(z,y)|^2 \, dy, \quad M_1 = (2\pi)^{-n} |\Omega|, \quad M_2 = n \left( C_3(Q) \sum_{j=1}^{k} \lambda_j + C_4(Q) k \right). \]

Then from the Lemma 4.1, we have for any \( k \geq 1 \),

\[ k \leq \frac{\omega_{n-1} \prod_{j=1}^{p} (s_{0j}+1)}{\vartheta \left( (2\pi)^{-n} |\Omega| \right)^{\frac{4+\delta}{n}}} \left( \frac{5n^{\frac{4+\delta}{n}}}{2^{n}} \right)^{\frac{1}{4+\delta}} \left( n \left( C_3(Q) \sum_{j=1}^{k} \lambda_j + C_4(Q) k \right) \right)^{\frac{n}{4+\delta}}. \]

This means, for any \( k \geq 1 \),

\[ \sum_{j=1}^{k} \lambda_j \geq \hat{C}(Q) k^{1+\frac{1}{\vartheta}} - \frac{C_4(Q)}{C_3(Q)} k, \]

where \( \vartheta = n + \sum_{j=1}^{p} s_{0j} \), and

\[ \hat{C}(Q) = \frac{2^n}{5C_3(Q)n^{\frac{4+\delta}{n}}} \left( \frac{\omega_{n-1} \prod_{j=1}^{p} (s_{0j}+1)}{(2\pi)^n |\Omega|} \right)^{\frac{4+\delta}{n}}. \]

The proof of Theorem 1.2 is completed.

\[ \Box \]

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