Higgs Algebraic Symmetry in Two-Dimensional Dirac Equation

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The dynamical symmetry algebra of the two-dimensional Dirac Hamiltonian with equal scalar and vector Smorodinsky-Winternitz potentials is constructed. It is the Higgs algebra, a cubic polynomial generalization of SU(2). With the help of the Casimir operators, the energy levels are derived algebraically.

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The concept of dynamical symmetry (DS) is essential and prevalent both in classical and quantum mechanics [1]. Hydrogen atom and isotropic harmonic oscillator are two relatively simple model with DS, whose classical orbits of motion are closed [2]. For simplicity, we only concern with the bound states of the Coulomb problem in this report. In addition to the orbit angular momentum corresponding to the rotational symmetry, there exist more constants of motion in these systems. They have been proved to be the Rung-Lenz vector [3, 4] and the second order tensors [5], in the hydrogen atom and harmonic oscillator respectively. The algebraic relations of these conserved quantities reveal the SO(4) symmetry in the hydrogen atom, and SU(3) in harmonic oscillator. They are called DSs, because the nature of them are not geometrical but the symmetries in the phase space. These symmetries lead to an algebraic approach to determine the energy levels. Generally, the N-dimensional (ND) hydrogen atom has the SO(N+1) and the oscillator has the SU(N) symmetry.

Starting from the feature of the classical orbits, Higgs [6] introduced a generalization of the hydrogen atom and harmonic oscillator in a spherical space. The conserved quantities of them construct a cubic polynomial generalization of SU(2), which is called the Higgs algebra now. Its increasing applications have been the focus of very active research in recent years [7, 8, 9]. Especially, Floranini and his colleagues [10] find the DS of the two-body Calogero model can be described by the Higgs algebra. Under an orthogonal transformation (Eq. (11) in [10]), their Hamiltonian of the two-body Calogero model is equivalent to the 2D Smorodinsky-Winternitz (SW) system [11, 12, 13]. This indicates that the DSs, especially which are described by the polynomial Lie algebra, exist not only in the quantum mechanics systems with rotation symmetry but also in some non-central superintegrable potentials [14]. Moreover, the two-body Calogero model is shown related to the concept of hidden nonlinear supersymmetry [15, 16]. This nonlinear generalization of supersymmetry is investigated in many systems in recent years [17, 18, 19, 20, 21, 22, 23].

In the relativistic quantum mechanics, the motion of spin-1/2 particle satisfies the Dirac equation, which predicts the intrinsic magnet moment naturally. The spin-orbit coupling leads to the breaking of DSs in the Dirac hydrogen atom [24] and the Dirac oscillator [25].

Recently, in his illuminating work [26, 27], Ginocchio has found the U(3) and pseudo-U(3) symmetry in the Dirac equation with scalar and vector harmonic oscillator potentials of equal magnitude. The Dirac Hamiltonian, with scalar and vector potentials of equal magnitude (SVPEM), is said to have the spin or pseudospin symmetry corresponding to the same or opposite sign [28]. In the spherical potentials, the total angular momentum can be divided into conserved orbital and spin parts, which form the SU(2) algebra separately. Take the spin symmetry case as an example, the two conserved parts are given by [29]

\[
\vec{L} = \left[ \vec{l} \begin{array}{c} 0 \\ U_p \vec{l} U_p^\dagger \end{array} \right], \quad \vec{s} = \left[ \vec{s} \begin{array}{c} 0 \\ U_p \vec{s} U_p^\dagger \end{array} \right],
\]

where \(\vec{l} = \vec{r} \times \vec{p}, \vec{s} = \vec{\sigma}\) are the usual spin generators, \(\vec{\sigma}\) are the Pauli matrices, and \(U_p = U_p^\dagger = \frac{2}{p^{\dagger}}\) is the helicity unitary operator [30]. The sum of them equals to the total angular momentum on account of \(U_p \vec{l} U_p^\dagger = \vec{l} + \vec{s}\). Ginocchio has proved the conserved orbit momentum \(\vec{L}\) in Eq. (11) to be three of the eight generators of the SU(3) symmetry group. And, the spin part has no influence on the Hamiltonian, which behaves like the spin in the non-relativistic harmonic oscillator. We have applied Ginocchio’s approach in [27] to study the Coulomb potential problem, and found the SO(4) DS in the Dirac hydrogen atom with spin symmetry [31]. A 2D version of this approach also has been introduced to investigate the SU(2) DS in the 2D Dirac equations with equal scalar and vector oscillator potentials, and SO(3) for the Coulomb case [32].

The Dirac Hamiltonian with SVPEM are derived from the investigation of the dynamics between a quark and an antiquark [29, 33, 34, 35, 36]. Many researches about this type Dirac equations are reported in recent years [28, 37, 38, 39, 40, 41, 42, 43]. The very lately studies [42, 43] have revealed that, the motion of a spin-1/2 particle with SVPEM satisfies the same differential equation
and has the same energy spectrum as a scalar particle. Furthermore, the relativistic energy spectra of the Klein-Gordon equation with SVPEM are shown to have a one-to-one relationship with its nonrelativistic limits \([14]\). It is worth while to note that, these results and the concept of the spin or pseudospin symmetry are independent of the shape of the potentials: radial or non-central. This leads us to foretell that the Dirac equation with SVPEM have the same DS with its nonrelativistic limit, no matter its potential is spherical or not. In other words, the conservation of the deformed orbit momentum \(\hat{L}\) in Eq. \([1]\) is not necessary for the presence of DS in the the Dirac Hamiltonian with SVPEM, although it is the first step in Ginocchio’s approach to deal with the harmonic oscillator [27] and the Coulomb potentials [31].

As the first trial of our conjecture above, we consider a 2D Dirac system with equal scalar and vector potentials (ESVP) for simplicity. Comparing the results in \([32]\) with \([27, 31]\), one can find the correspondence between the 2D and 3D cases is straightforward. The non-central potential we choosing is the SW potential mentioned above, \(V_{sw}(\vec{r}) = \frac{1}{2}(x_1^2 + x_2^2 + k/x_2^2)\) with \(k > 0\). The 2D Dirac Hamiltonian with ESVP, in the relativistic units, \(\hbar = c = m = 1\), takes the form

\[
H = \vec{\alpha} \cdot \vec{p} + \beta + (1 + \beta)\frac{1}{2}V(\vec{r}),
\]

where \(\vec{\alpha} = (\sigma_1, \sigma_2)\) and \(\beta = \sigma_3\) are the Pauli matrices. As shown in \([32]\), when the potential \(V(\vec{r})\) in Eq. \([2]\) is radial, \(H\) commutes with a 2D version definition of the conserved orbital angular momentum as

\[
L = \begin{bmatrix}
0 & 1 \\
1 & 0 \\
B^\dagger B & B^\dagger p B
\end{bmatrix},
\]

where \(B = p_1 - ip_2\), \(B^\dagger = p_1 + ip_2\), and \(l = x_1 p_2 - x_2 p_1\) is the usual orbital angular momentum. To derive other additional conserved quantities in the Coulomb and harmonic oscillator potentials in \([32]\), we assume the constants of motion take the form as

\[
Q = \begin{bmatrix}
Q_{11} & Q_{12} B \\
B^\dagger Q_{21} & B^\dagger Q_{22} B
\end{bmatrix}.
\]

The commutation relation \([Q, H] = 0\) requires the matrix elements must satisfy the equations:

\[
[Q_{12}, V(\vec{r})] + [Q_{12}, p^2] = 0,
\]

\[
[Q_{12}, V(\vec{r})] + [Q_{22}, p^2] = 0,
\]

\[
[Q_{11} = Q_{12}(2 + V(\vec{r})) + Q_{22} p^2.
\]

They are the same as the 3D case \([27]\).

When the potential in Eq. \([2]\) takes the SW form, \(V(\vec{r}) = V_{sw}(\vec{r})\), the deformed orbit momentum \(L\) in Eq. \([3]\) is no longer a constant of motion. But, one can notice the derivative process, from the ansatz form of \(Q\) to the conditions of its element in Eq. \([5]\), is not relying on the form of \(V(\vec{r})\). In addition, \(L\) also satisfies the conditions when the potential \(V(\vec{r})\) is radial symmetric. Therefore, we presume all the generators of the symmetry algebra of the Hamiltonian \(H\) with the SW potential can be determined from the conditions.

Three solutions of Eq. \([5]\) with \(V(\vec{r}) = V_{sw}(\vec{r})\) are found, whose independent elements are given by

\[
D_1 : \quad D_{12}^{(1)} = x_1^2 \left( x_2^2 - \frac{k}{x_2^2} \right) \left[ 2 + V_{sw}(\vec{r}) \right] - 2l^2 + 2p_1^2 \frac{k}{x_2^2} + 2 x_1 x_2 p_1 p_2 + 2 p_1 p_2 x_1 x_2,
\]

\[
D_{22}^{(1)} = x_2^2 \left( x_1^2 - \frac{k}{x_1^2} \right) + 4 p_1^2 p_2^2; \quad D_{22}^{(2)} = x_1^2 (x_2 p_2 + p_2 x_2) - (x_2^2 - \frac{k}{x_2^2}) (x_1 p_1 + p_1 x_1),
\]

\[
D_{22}^{(2)} = \frac{2}{p^2} \left[ p_2^2 (x_1 p_1 + p_1 x_1) - p_1^2 (x_2 p_2 + p_2 x_2) \right];
\]

\[
Q_3 : \quad Q_{12}^{(3)} = \frac{1}{2} \left( x_1^2 - x_2^2 - \frac{k}{x_2^2} \right),
\]

\[
Q_{22}^{(3)} = \frac{p_1^2 - p_2^2}{p^2}.
\]

The other elements, \(D_{11}^{(1)}, D_{11}^{(2)}\) and \(Q_{11}^{(3)}\) \((i = 1, 2)\), can be obtained easily from the first and the last relations in Eq. \([5]\). They combine into the three constants of motion in the form of Eq. \([4]\) as

\[
T_i = \begin{bmatrix}
T_{11}^{(i)} & T_{12}^{(i)} B \\
B^\dagger T_{21}^{(i)} & B^\dagger T_{22}^{(i)} B
\end{bmatrix},
\]

with \(T = D\) for \(i = 1, 2\) and \(T = Q\) for \(i = 3\). We can define the normalized generators, \(D_\pm = D_1 + \pm \sqrt{\mathcal{G}} D_2\) and \(D_3 = 4\mathcal{G}^{-1} Q_3\), with \(\mathcal{G} = 2(H + 1)\) being a constant for a fixed energy level. They satisfy the Higgs algebra relations

\[
[D_3, D_\pm] = \pm D_\pm,
\]

\[
[D_+, D_-] = c_3 D_3^3 + c_1 D_3 + c_0,
\]

where \(c_3 = -1024 G^2\), \(c_1 = 64(F - 2G) G + 32kG^3\), \(c_0 = 8k G^2 \sqrt{(F + G) G} F = (H^2 - 1)^2 - \mathcal{G}\). The Casimir operator of the Higgs algebra can be obtained immediately \([43, 46]\)

\[
\mathcal{C} = \{ D_+, D_- \} + \frac{c_6}{2} D_3^4 + (c_1 + \frac{c_3}{2} D_3^2 + 2 c_0 D_3
\]

\[
= 2 F(F - 8G) - 2k^2 G(F + 4G).
\]

In the basis of the common eigenstates of \(H\) and \(D_3, H[E, m] = E[E,m], D_3[E, m] = m[E, m]\), \(D_\pm\) are the ladder operators of \(D_3\). \(D_3 D_\pm |E, m\rangle = (m \pm 1)|D_\pm |E, m\rangle\). There exist a highest and a lowest weights for a fixed energy level, denoted as \(D_+ |E, m\rangle = 0\) and
\( D_\pm |E, m\rangle = 0 \). Let \( S_+ = 2D_- D_+ \) and \( S_- = 2D_+ D_- \), one can obtain from the relations in Eq. \([4]\) and \([7]\)

\[
S_\pm = C \left[ \frac{c_3}{2} D_\pm^2 (D_\pm \pm 1)^2 + c_1 D_\pm (D_\pm \pm 1) + c_0 (2D_\pm \pm 1) \right].
\]

Operating them on \( |E, m\rangle \) and \( |E, m\rangle \) respectively, we obtain

\[
\begin{align*}
\bar{m} &= \frac{1}{8} \left[ 4 - 4 \sqrt{4 + 8k(E + 1) + \sqrt{2(E + 1)(E - 1)}} \right], \\
\tilde{m} &= \frac{1}{8} \left[ \lambda - \sqrt{2(E + 1)(E - 1)} \right],
\end{align*}
\]

where \( \lambda = 2 \) or 6. The degeneracy of states in a given energy level should be a natural number, which leads to \( \bar{m} - \tilde{m} = n = 0, 1, 2, \ldots \). Substituting it into Eq. \([5]\), we find the eigenvalues of \( H \) satisfy the equation

\[
\sqrt{\frac{1}{2}(E + 1)(E - 1)} = N - \frac{3}{2} \sqrt{1 + k \frac{2}{E + 1}},
\]

where \( N = 2n \) or \( 2n + 1 \) corresponding to the different values of \( \lambda \) in Eq. \([3]\). The degeneracy of the energy level is the same as the two-body Calogero model given in \([10]\), \( d = n + 1 = \lfloor N/2 \rfloor + 1, \lfloor x \rfloor \) being the integer part of \( x \).

When the parameter \( k \to 0 \), the energy levels became the results in the 2D harmonic oscillator \([32]\), but different in the value range of the total quantum number. The difference is derived from the fact that the limit of the SW potential is a harmonic oscillator potential, whereas in the nonrelativistic limit of the energy levels is given by \( E \to -1, \) the nonrelativistic limit of the energy levels is given by \( E = N + 3/2 + \sqrt{k + 1/4} \), which agrees with the nonrelativistic results \([12]\).

In summary, we have shown that the 2D Dirac system with equal scalar and vector SW potentials has a DS described by the Higgs algebra. The three generators are derived by using a 2D version of the ansatz form given by Ginocchio. The relation of the Casimir operator of the Higgs algebra and the Hamiltonian leads to an algebraic solution of the relativistic energy spectrum.

This is our first attempt to investigate the DS in noncentral Dirac system. As and we know, it is also the first example in the Dirac quantum mechanics, whose dynamical symmetry is described by the Higgs algebra. Actually, the 3D Dirac equations with spin symmetry or pseudospin symmetry are more real, which exist frequently in antinucleon and nucleon spectra \([28]\). Whereas, it is natural to generalize our treatment to the 3D case to study the dynamical symmetries in some noncentral potentials, such as the anisotropic harmonic oscillator with rational frequency ratio and the caged anisotropic oscillator potentials \([12]\).

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