Towards a variational principle for motivated vehicle motion

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We deal with the problem of deriving the microscopic equations governing the individual car motion based on the assumptions about the strategy of driver behavior. We suppose the driver behavior to be a result of a certain compromise between the will to move at a speed that is comfortable for him under the surrounding external conditions, comprising the physical state of the road, the weather conditions, etc., and the necessity to keep a safe headway distance between the cars in front of him. Such a strategy implies that a driver can compare the possible ways of his further motion and so choose the best one. To describe the driver preferences we introduce the priority functional whose extremals specify the driver choice. For simplicity we consider a single-lane road. In this case solving the corresponding equations for the extremals we find the relationship between the current acceleration, velocity and position of the car. As a special case we get a certain generalization of the optimal velocity model similar to the “intelligent driver model” proposed by Treiber and Helbing.

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I. INTRODUCTION

The fundamentals of the traffic flow dynamics are far from being well established because up to now it is not evident what the specific form of the microscopic equations governing the individual car motion should be. The matter is that the car motion is controlled by the motivated driver behavior rather than obeys Newton’s laws. In fact, the behavior of a driver is due to a certain compromise between the will to move at a speed that is comfortable for him and attained on the empty road, on one side, and the need to avoid possible traffic accidents, on the other side. So, comparing the car ensembles with physical systems it is not obvious beforehand that there is a direct relationship between the acceleration of a given car and the positions and the velocities of the other cars as is the case for physical particles. In addition, the linear superposition typical in the interaction of physical particles is not self-evident to hold for the vehicle interaction.

By contrast, on macroscopic scales the car ensembles exhibit a wide class of critical and self-organization phenomena widely met in physical systems (for a review see Ref. 6). It should be pointed out that fish and bird swarms, colonies of bacteria, pedestrians, etc. also demonstrate similar cooperative motion (for a review see Ref. 6) and also Ref. 6. So the cooperative behavior of many particle systems, including social and biological ones, seems to be of more general nature then the mechanical laws and to find out what microscopic regularities are responsible for the cooperative phenomena in the general case is a challenge problem. The traffic dynamics is widely studied in this context also due to the large potential for industrial applications.

The currently adopted approach to specifying the microscopic governing equations of the individual car motion is the so-called social force model or the generalized force model. Its detailed motivation and description can be found in Ref. 6, here we only touch its basic ideas. At each moment \( t \) of time a given driver \( \alpha \) gets up or lets down the speed \( v_\alpha \) of his car, or keeps its value unchanged depending on the road conditions and the arrangement in the neighboring cars:

\[
\frac{dv_\alpha}{dt} = f_\alpha(v_\alpha) + \sum_{\alpha'} f_{\alpha\alpha'}(x_\alpha, v_\alpha; x_{\alpha'}, v_{\alpha'}). \tag{1}
\]

Here the term \( f_\alpha(v_\alpha) \) taken typically in the form

\[
f_\alpha(v_\alpha) = \frac{v^0_\alpha - v_\alpha}{\tau_\alpha}
\]

describes the driver tendency to move on the empty road at a certain fixed speed \( v^0_\alpha \) depending on the physical state of the road, weather conditions, the legal traffic regulations, etc. The relaxation time \( \tau_\alpha \) characterizes the acceleration capability of the given car as well as the delay in the driver control over the headway. The term \( f_{\alpha\alpha'}(x_\alpha, v_\alpha; x_{\alpha'}, v_{\alpha'}) \) describes the interaction of car \( \alpha \) with car \( \alpha' (\alpha' \neq \alpha) \) that is due to the necessity for

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driver \( \alpha \) to keep a certain safe headway distance between the cars. The force \( f_{\alpha \alpha'}(x_\alpha, v_\alpha \mid x_{\alpha'}, v_{\alpha'}) \) is assumed to depend directly only on the velocities \( v_\alpha, v_{\alpha'} \) and the position \( x_\alpha, x_{\alpha'} \) of this car pair and being of nonphysical nature does not meet the third Newton’s law, i.e., in the general case \( f_{\alpha \alpha'} \neq -f_{\alpha' \alpha} \). Paper \cite{1} presents and discusses possible generalized ansatzes of the dependence \( f_{\alpha \alpha'}(x_\alpha, v_\alpha \mid x_{\alpha'}, v_{\alpha'}) \).

Special cases of this model bear own names. In particular, for a single-lane road when all the cars can be ordered according to the position on the road in the car motion direction,

\[
\ldots < x_{a-2} < x_{a-1} < x_a < x_{a+1} < x_{a+2} < \ldots ,
\]

solely the interaction of the nearest neighboring cars \( \alpha \) and \( \alpha+1 \) is taken into account, i.e. \( f_{\alpha \alpha'} \neq 0 \) for \( \alpha' = \alpha+1 \) and may be, \( \alpha' = \alpha - 1 \) only. For this case Bando et al. \cite{12, 13} proposed the optimal velocity model that describes the individual car motion as

\[
\frac{dv_\alpha}{dt} = \frac{1}{\tau} \left[ \vartheta_{\text{opt}}(x_{\alpha+1} - x_\alpha) - v_\alpha \right], \quad (2)
\]

where \( \vartheta_{\text{opt}}(\Delta) \) is the steady-state velocity (the optimal velocity) chosen by drivers for the given headway distance \( \Delta = x_{\alpha+1} - x_\alpha \) between the cars. This model and its modification successfully were used to explain the properties of the “stop-and-go” waves developing in dense traffic on single-lane roads (see, e.g. Ref. \cite{3, 4, 5, 6, 11, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29}).

However, on multilane highways the behavior of traffic flow becomes sufficiently complex because of the strong correlations in the car motion at different lanes (for a review see Ref. \cite{3, 4, 5, 6, 7, 30}). In this case it is not sufficient to confine the consideration to the interaction between the nearest neighboring cars only and one has to specify several independent components of the social forces \( \{ f_{\alpha \alpha'}(x_\alpha, v_\alpha \mid x_{\alpha'}, v_{\alpha'}) \} \). As a result the number of essential fitting parameters entering the social forces increases substantially. A possible way to overcome this problem is to formulate a mathematical principle characterizing the strategy of driver behavior in terms of a certain functional quantifying the objectives pursued by drivers. Constructing this functional may be also at the phenomenological level can be lightened by the clear physical meaning of the driver objectives. Then, using standard techniques one will derive the required governing equations. The given work presents our first steps towards such an approach. It should be noted that the derivation of microscopic governing equations for systems with motivated behavior based on a certain “optimal self-organization” principle was discussed in Ref. \cite{3, 4, 11, 12, 30, 31}. The main idea of this approach is the assumption that individuals try to minimize the interaction strength or, what is actually the same, to optimize their own success and to minimize the efforts required for this.

Our approach is related to the concepts of mathematical economics, namely, to the concepts of preferences and utility (see, e.g. Ref. \cite{32}). We suppose that at each moment \( t_0 \) of time a driver plans his motion in a certain way in order, first, to move as fast as possible and, second, to prevent traffic accidents. In particular, in the previous paper \cite{33} using this idea we developed a model explaining in a simple way the experimentally observed sequence of the first order phase transitions from the “free flow” to the jam phase through the “synchronized mode” \cite{34, 35, 36}. Such a strategy actually implies that a driver evaluates any possible path of his further motion, \( \{ \chi(t), t > t_0 \} \), with respect to its preferability. In other words, a driver can compare any two paths \( \{ \chi_1(t), t > t_0 \} \) and \( \{ \chi_2(t), t > t_0 \} \) and decide which of them is more preferable, for example, \( \chi_1(t) \). The latter relation will be designated as \( \chi_1(t) > \chi_2(t) \). Obviously, the given relation exhibits the transitivity, i.e. if \( \chi_1(t) > \chi_2(t) \) and \( \chi_2(t) > \chi_3(t) \) then \( \chi_1(t) > \chi_3(t) \). In this case we may seek for a priority functional \( \mathcal{L} \{ \chi \} \) meeting the condition \( \mathcal{L} \{ \chi_1 \} > \mathcal{L} \{ \chi_2 \} \) when and only when \( \chi_1 > \chi_2 \). Finally the driver chooses the best path \( \chi_{\text{opt}}(t) \) of his further motion maximizing the priority functional \( \mathcal{L} \{ \chi \} \). So its extremals have to satisfy the desired microscopic governing equation of individual car motion. It should be noted that chosen path \( \chi_{\text{opt}}(t) \) of the planned motion specifies the acceleration at the current time moment \( t_0 \) rather than the real trajectory \( x(t) \) of the car motion because at the next time \( t > t_0 \) the driver again plans his motion in the same way, introducing the corrections caused by changes in the surroundings. The same concerns the car velocity and its acceleration, therefore to avoid possible misunderstanding we will designate the real velocity and acceleration of the car as \( v(t) \) and \( a(t) \) whereas these values corresponding to the optimal \( \chi_{\text{opt}}(t) \) will be labelled by \( \{ v(t), t > t_0 \} \) and \( \{ a(t), t > t_0 \} \), respectively.

In this way the problem of specifying many independent components of the social forces is replaced by constructing the priority functional describing the driver compromise between the will to move as fast as being allowed by the physical state of the road and the necessity to avoid possible traffic accidents. So to obtain the priority functional we may apply to the general assumptions about the driver behavior.

II. VARIATIONAL PRINCIPLE FOR THE INDIVIDUAL CAR MOTION

At the first step we should determine the collection of phase variables characterizing the quality of a given car motion. We note that for the driver under consideration the neighboring car arrangement and its evolution should be regarded as given beforehand. Indeed, it cannot be directly controlled by him and, so, has to be treated as the external conditions. Formulating this problem we actually assume the existence of a certain collection of variables \textit{taken at the current time moment} \( t \) that completely quantify the priority measure of the car motion.
at the same time. Adopting the latter assumption we may construct the priority functional \( \mathcal{L} \{ \chi \} \) in terms of a certain integral of a function \( F \) of the phase variables with respect to time.

Applying to the conventional driver experience we will characterize the individual car motion at each time moment \( t \) by its position on the road \( x(t) \), the velocity \( v(t) \), and the acceleration \( a(t) \). For a multilane highway, for example, the position of a car should also bear the information about the lane occupied by the car, but this problem will be considered elsewhere and in the present paper we confine our consideration to a single-lane road only. In this property a car is distinct from a physical particle because the motion of the latter is completely determined by the current position and velocity. The variables \( x(t) \), \( v(t) \), and \( a(t) \), however, exhibit different behavior. The coordinate \( x(t) \) and the velocity \( v(t) \) of the car vary continuously, i.e. the driver cannot change them immediately. In contrast, the acceleration \( a(t) \) may exhibit sharp jumps because it is the acceleration that is controlled directly by the driver without remarkable delay. In such an analysis it is quite reasonable to ignore the short physiological delay in the driver behavior to changes in the surroundings, allowing sharp jumps in the dependence \( a(t) \). Therefore, planning his further path of motion the driver regards the position \( x_0 := x(t_0) \) and the velocity \( v_0 := v(t_0) \) of the car at the current moment of time \( t_0 \) as the initial data.

Now let us write the general form of the priority functional \( \mathcal{L} \{ \chi \} \) for a trial path \( \{ \chi(t), t > t_0 \} \) of the further motion:

\[
\mathcal{L} \{ \chi \} = - \int_{t_0}^{\infty} dt \exp \left( -\frac{x(t) - x_0}{\ell} \right) F(t, \chi, \nu, \varepsilon), \tag{3}
\]

where \( F(t, \chi, \nu, \varepsilon) \) is the density of the path priority measure, \( \nu := d\chi/dt \) and \( \varepsilon := d^2\chi/dt^2 \), and the exponential cofactor reflects the fact that drivers can monitor the traffic flow state and so plan the motion only inside a certain region of length \( \ell \) in front of them. In addition, the leading minus has been chosen in order to reduce finding the maximum of the functional \( \mathcal{L} \{ \chi \} \) to getting the minimum of integral \( (3) \) as it is the typical case in physical theories. The direct dependence of the functional \( F(t, \chi, \nu, \varepsilon) \) on the time \( t \) reflects the effect of the surroundings, i.e. the physical road state and the neighboring car arrangement on the driver planning.

According to the adopted assumption the driver chooses the path \( \chi_{\text{opt}}(t) \) of his further motion that maximizes the functional \( \mathcal{L}(t, \chi, \nu, \varepsilon) \) and at the current time \( t_0 \) together with other all trial paths \( \{ \chi(t) \} \) meets the conditions

\[
\chi(t_0) = x_0, \quad \nu(t_0) = v_0. \tag{4}
\]

Besides, the present paper analyzes the car motion in the traffic flow, i.e. does not consider any way by which a fixed car leaves the traffic flow, for example, to stop. So we assume all the trial paths to exhibit bounded variations, i.e. there are such constants \( C_{\chi}, C_{\nu}, C_{\varepsilon} \), and \( C_{\ell} \) that

\[
C_{\chi}(t - t_0) < \chi(t) < C_{\chi}(t - t_0),
\]

\[
\nu(t) < C_{\nu}, \quad |\varepsilon(t)| < C_{\varepsilon}. \tag{5}
\]

Then using the standard technique we get the governing equation for the extremals of the priority functional \( \mathcal{L} \{ \chi \} \) following from the condition \( \delta \mathcal{L} \{ \chi \} = 0 \) at \( \chi(t) = \chi_{\text{opt}}(t) \):

\[
\frac{d^2}{dt^2} \left\{ \exp \left[ -\frac{\chi}{\ell} \right] \frac{F(t, \chi, \nu, \varepsilon)}{\partial \varepsilon} \right\} - \frac{d}{dt} \left\{ \exp \left[ -\frac{\chi}{\ell} \right] \frac{\partial F(t, \chi, \nu, \varepsilon)}{\partial \nu} \right\} + \frac{\partial}{\partial \chi} \left\{ \exp \left[ -\frac{\chi}{\ell} \right] F(t, \chi, \nu, \varepsilon) \right\} = 0. \tag{6}
\]

By virtue of (3) the function \( F(t, \chi, \nu, \varepsilon) \) exhibits bounded variations, which enables us to integrate equation (6) twice with respect to time \( t \) reducing it to the following

\[
\frac{\partial F(t, \chi(t), \nu(t), \varepsilon(t))}{\partial \varepsilon(t)} = -\int_{t_0}^{\infty} dt' \exp \left[ -\frac{\chi(t') - \chi(t)}{\ell} \right] \frac{\partial F(t', \chi(t'), \nu(t'), \varepsilon(t'))}{\partial \nu(t')} - \int_{t_0}^{\infty} dt' \int_{t'}^{\infty} dt'' \frac{\partial}{\partial \chi(t'')} \left\{ \exp \left[ -\frac{\chi(t'') - \chi(t)}{\ell} \right] F(t'', \chi(t''), \nu(t''), \varepsilon(t'')) \right\}. \tag{7}
\]

Equation (7) relates the planned acceleration \( \varepsilon \) to the car position \( \chi \) and the velocity \( \nu \). Subjecting this equation to the initial conditions (4) we can find the optimal path \( \chi_{\text{opt}}(t) \) and then setting \( t = t_0 \) we will get the desired relationship of the real current position \( x(t_0) \) and velocity \( v(t_0) \) of the car with the acceleration \( a(t_0) \) that driver takes under these conditions. Exactly the expression to be obtained in such a way is the microscopic governing
equation of the individual car motion. In addition, it should be noted that equation (1) is of fourth order as it must because a trial path is fixed in part by the position and the velocity at the initial and terminal points. However, in the case under consideration the characteristics of the terminal point are replaced by conditions (3), allowing us to reduce the order.

Now let us demonstrate the proposed approach analyzing a simple example.

### III. THE GENERALIZED OPTIMAL VELOCITY MODEL

In constructing the priority functional we assumed that in the steady-state traffic flow a driver prefers to move at a certain speed $\vartheta_{\text{opt}}(t,x)$ depending on the surroundings and given beforehand. Besides, we consider the car motion without acceleration to be the best way of driving. Therefore we adopt the following ansatz:

$$ F(t,x,v,a) = \frac{1}{2} [v - \vartheta_{\text{opt}}(t,x)]^2 + \frac{1}{2} \tau^2 a^2, \quad (8) $$

where the time scale $\tau$ characterizes the acceleration capability of the given car. The driver monitoring the car arrangement in front of him can predict the situation development, which is described in terms of the linear dependence of the optimal velocity $\vartheta_{\text{opt}}(t,x)$ on the time $t$ and the distance $x$:

$$ \vartheta_{\text{opt}}(t,x) = \vartheta_{\text{opt}}^0 \left[ 1 + \varepsilon_t \frac{t - t_0}{\tau} + \varepsilon_x \frac{x - x_0}{\ell} \right], \quad (9) $$

Multiplying equation (10) by the factor $\exp(-\vartheta_{\text{opt}}^0/t/\ell)$, differentiating the obtained result with respect to $t$, and solving the equation subject to the initial condition (1) we get the dependence $\nu(t)$ obeying (10). The differentiating $\nu(t)$ at the current moment $t_0$ of time point we obtain the desired microscopic governing equation for the individual car motion:

$$ \frac{dv}{dt} = -\frac{1}{\tau} \kappa \left( v - \vartheta_{\text{opt}}^0 \left[ 1 + \kappa (\varepsilon_t + 2\varepsilon_x) \right] \right), \quad (11) $$

where the constants

$$ \kappa = \left( \sigma + \sqrt{\sigma^2 + 1} \right)^{-1} \quad \text{and} \quad \sigma = \frac{\tau \vartheta_{\text{opt}}^0}{2 \ell}. $$

In particular, let the optimal velocity $\vartheta_{\text{opt}}(t,x) = \vartheta_{\text{opt}}(\Delta)$ be specified completely by the headway distance $\Delta = x_{\alpha+1} - x_\alpha$ between the given car $\alpha$ and the nearest one $\alpha + 1$ in front of it. Then within the linear approximation of the situation development the driver of the car $\alpha$ can anticipate that the headway distance will change in time as

$$ \Delta(t) = \Delta(t_0) + [v_{\alpha+1}(t_0) - v_\alpha(t_0)](t - t_0). $$

This expression together with the dependence $\vartheta_{\text{opt}}(\Delta)$ enables us to calculate the specific value of the constant

where $\vartheta_{\text{opt}}^0 = \vartheta_{\text{opt}}(t_0,x_0)$ and $\varepsilon_t$, $\varepsilon_x$ are constants regarded here as small parameters of the same order. In addition, the difference $v(t)/\vartheta_{\text{opt}}^0 - 1$ is also assumed to be of order of $\varepsilon_t \sim \varepsilon_x$. We have adopted the linear dependence of $\vartheta_{\text{opt}}(t,x)$ on $t$ and $x$ because it seems quite reasonable that a driver uses the linear approximation in estimating the position of the cars in front of him.

Substituting (3) and (4) into (7) and truncating all the terms whose order exceeds $\varepsilon_t \sim \varepsilon_x \sim (v(t)/\vartheta_{\text{opt}}^0 - 1)$ we get

$$ \varepsilon_t \text{ and then to rewrite formula (11) as} $$

$$ \frac{dv}{dt} = -\frac{1}{\tau} \kappa \left[ v - \vartheta_{\text{opt}}(\Delta) - \kappa \tau \delta v \frac{d\vartheta_{\text{opt}}(\Delta)}{d\Delta} \right], \quad (12) $$

where we have introduced the relative velocity $\delta v = v_{\alpha+1} - v_\alpha$ of the car $\alpha + 1$ with respect to the given car $\alpha$ and omitted the argument $t_0$ assuming all the values to be taken at the current moment of time.

It should be noted that the obtained expression (12) is similar to the phenomenological dependence $\vartheta_{\text{opt}}(\Delta, \delta v)$ generalizing the standard optimal velocity model (3), the “intelligent driver model” proposed by Treiber and Helbing [39] (see also Ref. 40).

### IV. CONCLUSION

Concluding the present paper we once more remember its main key points.

We deal with the problem of deriving microscopic equations governing the motion of individual vehicles. The currently adopted approaches similar to the social force model relate in the spirit of Newton’s law the acceleration of a given car to the position and velocities of the
neighboring cars. In order to apply such models to analysis of the traffic dynamics one has to specify all the essential components of the corresponding effective forces acting between the cars. However when the vehicle interaction becomes sufficiently complex as it is the case, for example, for dense traffic on multilane highways such an approach meets the problem of large number of fitting parameters.

The present paper proposed a possible way to avoid the aforementioned difficulty. The main idea is to describe at the first step the strategy of the driver behavior determined by the compromise between the driver will to move as fast as possible on the given road, on one hand, and the necessity to keep a safe headway distance and not to interfere with cars moving at neighboring lanes, on the other hand. This assumption actually implies that a driver can compare various ways of his further motion with respect their relative preference and choose the best (optimal) one at each time moment. This choice gives the relationship between the acceleration of the car under consideration and the arrangement of neighboring cars.

Following the concepts of mathematical economics we have introduced a priority functional in order to quantify the driver choice. The extremals of this priority functional describe the optimal path of the driver further motion. At the present paper we have considered traffic flow on a single lane road, constructed in this case the general form of the priority functional, and derived the equations for its extremals corresponding to the car motion with traffic flow. The latter means that here we do not analyze the paths by which a car enters or leave traffic flow on the given road because the question deserves an individual investigations.

By way of example, we have considered a special case leading to an expression relating the current acceleration of a fixed car to the headway distance between this car and one in front of it as well as their relative velocity. The obtained equation turned out to be similar to the “intelligent driver model” by Treiber and Helbing.
[38] B. S. Kerner and H. Rehborn, Phys. Rev. Lett. 79, 4030 (1997).

[39] M. Treiber and D. Helbing, *Explanation of observed features of self-organization in traffic flow*, e-print: cond-mat/9901239.

[40] M. Treiber, A. Hennecke, and D. Helbing, in: *Traffic and Granular Flow’99*, edited by D. Helbing, H. J. Herrmann, M. Schreckenberg and D. E. Wolf (Springer-Verlag, Singapore, 2000), p. 365.