GAME VALUE FOR A PURSUIT-EVASION DIFFERENTIAL GAME PROBLEM IN A HILBERT SPACE

ABBAS JA’AFARU BADAKAYA*1 AND AMINU SULAIMAN HALLIRU1

1Department of Mathematical Sciences
Bayero University, Kano, Nigeria
JAMILU ADAMU1,2
2Federal University, Gashua
Yobe State, Nigeria
(Communicated by Leon Petrosyan)

ABSTRACT. We consider a pursuit-evasion differential game problem with countable number pursuers and one evader in the Hilbert space $l_2$. Players’ dynamic equations described by certain $n^{th}$ order ordinary differential equations. Control functions of the players subject to integral constraints. The goal of the pursuers is to minimize the distance to the evader and that of the evader is the opposite. The stoppage time of the game is fixed and the game payoff is the distance between evader and closest pursuer when the game is stopped. We study this game problem and find the value of the game. In addition to this, we construct players’ optimal strategies.

1. Introduction. Since the inception of an idea that prosper and became a research area named differential games, efforts by many researchers resulted in bountiful literature in the research area. To mention but a few, the papers [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19] and some references therein are typical examples of works from literature on differential games.

Pursuit-evasion differential game problems that involve finding value of the game is our main focus in this paper. The papers [1, 2, 5, 6, 7, 9, 10, 11, 13, 18] and some references therein are examples of works that are concerned with this type of problem.

The paper [5] investigated a differential game problem in which finite number of dynamical objects pursue a single one. All players perform simple motion. The termination time of the game is fixed. Control functions of all players are subject to geometric constraints except that of some few finite number of pursuers whose control functions are subject to integral constraints. The payoff of the game is the distance between the evader and closest pursuer at the instant the game is over. Optimal strategies of the players are constructed and value of the game is found.

In the paper [6] by Ibragimov, a differential game of countably many pursuers and one evader is considered. Players perform simple motion with geometric constraints imposed on control functions of the players. Optimal strategies of the players are

2020 Mathematics Subject Classification. Primary: 91A23; Secondary: 49N75.
Key words and phrases. Value of the game, pursuer, evader, Hilbert space, integral constraints.
* Corresponding author: Abbas Ja’afaru Badakaya.
constructed and obtained value of the game. This problem with integral constraints on controls of the players is discussed in [10].

Ibragimov and Salimi [11] studied a pursuit-evasion differential game problem of fixed duration and infinitely many pursuers and one evader. Dynamic equations of the players are given by certain second order differential equations in the Hilbert space $l_2$. Control functions of the players are subject to integral constraints. Sufficient condition for finding value of the game is obtained and constructed optimal strategies of the players. Ibragimov et. al in [7] improved the result obtained in [11] by eliminating the condition under which the value of the game is obtained in the former paper. Furthermore, game problem studied in [11] but with geometric constraints on control functions of the players is studied in [9] and obtained the game value.

The paper [13] deals with the study of a differential game of fixed duration with many pursuers and one evader in the space $R^n$. Motions of the players are described by linear systems of differential equations of the same type. Integral constraints are imposed on control functions of the players. The game payoff is the minimum of the distances between the evader and the pursuers when the game terminates. An estimate of the game payoff is obtained, which can be guaranteed by players and explicitly describe the strategies of the players.

A differential game in which a finite or countable number of pursuers pursue a single evader in a Hilbert space is studied in [18]. The problem studied in the paper considered first order differential equations as the dynamic equations of the players. Control functions of the players satisfy integral constraints. The game value is found and optimal strategies of the players are constructed.

In view of the literature above, we attempt to generalize some of the existing results in the literature. This is by studying a pursuit-evasion differential game problem which is concern with finding the game value and construction of players’ optimal strategies. The players of the game consist of many pursuers and one evader. Players’ dynamic equations are certain $n^{th}$ order differential equations. Control functions of the players are subject to integral constraints. The problem is to be studied in the Hilbert space $l_2$.

2. Problem formulation. Let $P_i, i \in I = \{1, 2, 3, \ldots, m\}$ and $E$ represent finite or countable many pursuers and an evader respectively, in the Hilbert space $l_2$.

Consider the dynamics equations of the pursuers and evader defined by

\[ P_i : \frac{d^n x_i}{dt^n} = u_i(t), \quad x_i(0) = x^0_i, \quad \frac{dx_i}{dt}(0) = x^1_i, \ldots, \frac{d^{n-1} x_i}{dt^{n-1}}(0) = x^{n-1}_i, \quad i \in I \]
\[ E : \frac{d^n y}{dt^n} = v(t), \quad y(0) = y^0, \quad \frac{dy}{dt}(0) = y^1, \ldots, \frac{d^{n-1} y}{dt^{n-1}}(0) = y^{n-1}, \]

where $x_i, x^0_i, x^1_i, \ldots, x^{n-1}_i, u_i, y^0, y^1, y^2, \ldots, y^{n-1}, v \in l_2$, $u_i = (u_{i1}, u_{i2}, \ldots)$ is a control parameter of the pursuer $P_i$ and $v = (v_1, v_2, \ldots)$ is that of the evader $E$. Let the stoppage time of the game be denoted by fixed positive number $\theta$.

**Definition 2.1.** An admissible control of the $i^{th}$ pursuer $P_i$ is the function $u_i(\cdot)$, $u_i : [0, \theta] \rightarrow l_2$, whose coordinates $u_{ik} : [0, \theta] \rightarrow R, k = 1, 2, \ldots$, are Borel measurable functions and

\[ \int_0^\theta \|u_i(t)\|^2 dt \leq \rho_i^2, \]

where $\rho_i$ is a positive number.
Definition 2.2. An admissible control of the evader $E$ is the function $v(\cdot)$, $v : [0, \theta] \to l_2$, whose coordinates $v_k : [0, \theta] \to R, k = 1, 2, \ldots$, are Borel measurable functions and
\[
\int_0^\theta \|v(t)\|^2 dt \leq \sigma^2,
\]
where $\sigma$ is a positive number.

The solutions to the dynamic equations (1) depend on the chosen admissible controls $u_i(\cdot), i \in I$ and $v(\cdot)$ by the pursuers and evader respectively. These solutions are defined by $x_i(t) = (x_{i1}(t), x_{i2}(t), \ldots), y(t) = (y_1(t), y_2(t), \ldots)$, where the coordinates are given by
\[
x_{ik}(t) = x_{ik}^0 + \int_0^t \int_0^{t_1} \int_0^{t_2} \cdots \int_0^{t_{n-1}} u_{ik}(s) dt_{n-1} \cdots dt_2 dt_1,
\]
\[
y_k(t) = y_k^0 + \int_0^t \int_0^{t_1} \int_0^{t_2} \cdots \int_0^{t_{n-1}} v_k(s) dt_{n-1} \cdots dt_2 dt_1.
\]
It is important to see and note that $x_i(\cdot), y(\cdot) \in C(0, \theta; l_2)$, where $C(0, \theta; l_2)$ is the space of functions $\alpha(t) = (\alpha_1(t), \alpha_2(t), \ldots) \in l_2, t \geq 0$, such that $\alpha(t)$ is a continuous function in the norm of the space $l_2$ and whose coordinates $\alpha_k(t), k = 1, 2, \ldots$ are absolutely continuous functions.

Definition 2.3. A strategy of the $i$th pursuer $P_i$ is a function $U_i(t, x_i, y, v), U_i : [0, \infty) \times l_2 \times l_2 \times l_2 \to l_2$, such that the system
\[
\frac{d^n x_i}{dt^n} = U_i(t, x_i, y, v), x_i(0) = x_{i0}, \frac{d^n x_i}{dt^n} (0) = x_{i1}, \ldots, \frac{d^{n-1} x_i}{dt^{n-1}}(0) = x_{i}^{n-1},
\]
\[
\frac{d^n y}{dt^n} = v, y(0) = y^0, \frac{d^n y}{dt^n} (0) = y^1, \ldots, \frac{d^{n-1} y}{dt^{n-1}}(0) = y^{n-1}
\]
has a unique solution $(x_i(\cdot), y(\cdot))$, with $x_i(\cdot), y(\cdot) \in C(0, \theta; l_2)$, for arbitrary admissible control $v(\cdot)$, of the evader $E$. A strategy $U_i$ is admissible if each control involved in the formation of this strategy is admissible.

Definition 2.4. A strategy of the evader $E$ is a function $V(t, x_1, x_2, \ldots, y, v), V : [0, \infty) \times l_2 \times l_2 \times l_2 \times l_2 \to l_2$, such that the system of equations
\[
\frac{d^n x_i}{dt^n} = u_i(t), x_i(0) = x_{i0}, \frac{d^n x_i}{dt^n} (0) = x_{i1}, \ldots, \frac{d^{n-1} x_i}{dt^{n-1}}(0) = x_{i}^{n-1}, i \in I,
\]
\[
\frac{d^n y}{dt^n} = V(t, x_1, x_2, \ldots, y, v), y(0) = y^0, \frac{d^n y}{dt^n} (0) = y^1, \ldots, \frac{d^{n-1} y}{dt^{n-1}}(0) = y^{n-1},
\]
has a unique solution $(x_1(\cdot), x_2(\cdot), \ldots, y(\cdot), v(\cdot))$, with $x_i(\cdot), y(\cdot) \in C(0, \theta; l_2)$, for arbitrary admissible controls $u_i(\cdot)$ of the pursuers $P_i$. The strategy $V(\cdot)$ is admissible if each control involved in the formation of this strategy is admissible.

Definition 2.5. Optimal strategies of the pursuers $P_i, i \in I$, are strategies $\hat{U}_i, i \in I$ such that
\[
\Psi_i(\hat{U}_1, \hat{U}_2, \ldots, \hat{U}_r, \ldots) = \inf_{u_1, u_2, \ldots, u_r, \ldots} \Psi_i(U_1, U_2, \ldots, U_r, \ldots)
\]
where $\Psi_i(U_1, U_2, \ldots, U_r, \ldots) = \sup_{v(\cdot)} \inf_{V(\cdot) \in \mathcal{U}_i} \|x_i(\theta) - y(\theta)\| ; U_i$ and $v(\cdot)$ are pursuers' admissible strategies and evader's admissible control respectively.
Definition 2.6. An optimal strategy of the evader $E$ is the strategy $\hat{V}$ such that

$$\Psi_2(\hat{V}) = \sup_V \Psi_2(V),$$

where $\Psi_2(V) = \inf_{u_1(\cdot), \ldots, u_r(\cdot)} \inf_{i \in I} \|x_i(\theta) - y(\theta)\| ; u_i(\cdot), i \in I$ and $V$ are pursuers' admissible controls and evader’s admissible strategy respectively.

In what will follow, we shall refer to the game described by (1)-(3) as game $G^*$. Also in in $I_2$, we denote by $B_r(a)$, a ball with center at $a$ radius $r$. It is reported in [20] that the number $\delta$ is the value of the game $G^*$, if

$$\Psi_1(\hat{U}_1, \hat{U}_2, \ldots, \hat{U}_r, \ldots) = \delta = \Psi_2(\hat{V}).$$

Research problem. For the game $G^*$, we construct optimal strategies $\hat{U}_i, i \in I$ and $\hat{V}$ of the pursuers $P_i, i \in I$ and evader $E$ respectively. In addition, we find the value of the game at the stoppage time of the game.

3. Preliminary results. In this section, we present some results that are useful in the establishing the main result of the paper.

1. Solutions of the dynamic equations. The solution to the dynamic equation for the $i^{th}$ pursuer in (1) is given by

$$x_i(\theta) = x_{i0} + \int_0^\theta \int_0^{t_1} \cdots \int_0^{t_{n-1}} u_i(t)dt dt_{n-1} \cdots dt_2 dt_1$$

where $x_{i0} = x_i^0 + \theta x_i^1 + \frac{\theta^2}{2!} x_i^2 + \cdots + \frac{\theta^{n-1}}{(n-1)!} x_i^{n-1}$ and the expression with the multiple integrals in (4) can be reduced to the expression with single integral in (5). The detail method of the reduction is discussed in [21]. The equation (5) is called the state equation of the $i^{th}$ pursuer. In a similar way, we can also obtain the state equation of the evader from (1) as

$$y(\theta) = y_0 + \int_0^\theta (\theta - t)^{n-1} \frac{v(t)}{(n-1)!} dt,$$

where $y_0 = y^0 + \theta y_1^0 + \frac{\theta^2}{2!} y_2^0 + \cdots + \frac{\theta^{n-1}}{(n-1)!} y^{n-1}$.

Alternatively, in place of the players’ dynamic equations (1), we can consider the following first order differential equations:

$$\begin{cases}
P_i : \frac{dx_i}{dt} = \frac{(\theta - t)^{n-1}}{(n-1)!} u_i(t), x_i(0) = x_{i0}, \\
E : \frac{dy}{dt} = \frac{(\theta - t)^{n-1}}{(n-1)!} v(t), y(0) = y_0.
\end{cases}$$

where $x_{i0} = x_i^0 + \theta x_i^1 + \frac{\theta^2}{2!} x_i^2 + \cdots + \frac{\theta^{n-1}}{(n-1)!} x_i^{n-1}$, and $y_0 = y^0 + \theta y_1^0 + \frac{\theta^2}{2!} y_2^0 + \cdots + \frac{\theta^{n-1}}{(n-1)!} y^{n-1}$. Observe that the players’ state equations (5) and (6) can equally be obtained from (7). In view of this, studying the game $G^*$ is equivalent to the study of the game $G^*$ with the players’ dynamic equations (1) replaced by (7).

2. Attainability sets. The attainability sets of the players of the game $G^*$ are given in the proposition below
**Proposition 1.** The attainability sets of the pursuer \( P_\ast \) and evader \( E \) from their respective initial positions in the space \( l_2 \) at time \( t = 0 \) are the balls

\[
B_{R_{P_\ast}}(x_0) := \{ x \in l_2 : \| x - x_0 \| \leq R_{P_\ast} \},
\]

and

\[
B_{R_E}(y_0) := \{ y \in l_2 : \| y - y_0 \| \leq R_E \},
\]

where \( R_{P_\ast} = \left( \frac{\theta^{2n-1}}{2n-1} \right)^{\frac{1}{2}} \frac{\rho_i}{(n-1)!} \) and \( R_E = \left( \frac{\theta^{2n-1}}{2n-1} \right)^{\frac{1}{2}} \frac{\sigma}{(n-1)!} \), respectively.

3. **Game of two players.** We consider the game \( G^* \) with only one pursuer instead of countable number of pursuers. That is, game \( G^* \) with \( i \in I = \{1\} \).

The goal of the pursuer \( P \) is to attain the equation \( x(\tau) = y(\tau) \) for some \( \tau \in [0, \theta] \), and that of the evader \( E \) is the contrary. The question is ”under what condition can the pursuer achieve its goal?” To answer this question we define a set

\[
\varphi = \begin{cases} 
z \in l_2 : 2 (y_0 - x_0, z) \leq \xi (\rho^2 - \sigma^2) + \| y_0 \|^2 - \| x_0 \|^2, & \text{if } x_0 \neq y_0, \\
\{ z \in l_2 \} & \text{if } x_0 = y_0,
\end{cases}
\]

where \( \xi = \frac{4(n + 1)\theta^{2n-1}}{9\theta^2((n - 1)!)^2} \) and \( p_0 \) is an arbitrary fixed unit vector. We let the initial position of the pursuer to be in the set \( \varphi \). Then the following lemma gives sufficient conditions that ensure the pursuer to achieve its goal:

**Lemma 3.1.** If \( \rho \geq \sigma \) and \( y(\theta) \in \varphi \) then there exists pursuer’s strategy to ensure the equation \( x(\tau) = y(\tau) \) for some \( \tau \in [0, \theta] \).

**Proof.** Let the pursuer’s strategy be defined as follows: if \( x_0 = y_0 \), we set \( U(t) = v(t) \), otherwise we set

\[
U(t) = \begin{cases} 
v(t) - (v(t), e) e + \left( \frac{(n+1)(\rho^2 - \sigma^2)}{(\theta - t)^n \theta^{n+1}} + (v(t), e)^2 \right)^{\frac{1}{2}} e, & 0 \leq t \leq \tau \\
v(t), & \tau \leq t \leq \theta,
\end{cases}
\]

where \( e = \frac{y_0 - x_0}{\| y_0 - x_0 \|} \). This strategy is admissible. Indeed, for the case \( x_0 = y_0 \), it is trivial, because \( \rho \geq \sigma \). For the case \( x_0 \neq y_0 \), we have

\[
\int_0^\theta \| u(t) \|^2 \, dt = \int_0^\tau \| v(t) \|^2 \, dt + \int_0^\tau \left( \frac{(n+1)(\rho^2 - \sigma^2)}{(\theta - t)^n \theta^{n+1}} \right) dt + \int_\tau^\theta \| v(t) \|^2 \, dt \\
= \int_0^\theta \| v(t) \|^2 \, dt + \left( \frac{(n+1)(\rho^2 - \sigma^2)}{\theta^{n+1}} \right) \int_0^\theta (\theta - t)^n \, dt \\
\leq \sigma^2 + \rho^2 - \sigma^2 = \rho^2.
\]

Next, we show that if the pursuer uses the strategy (8) then the equation \( x(\tau) = y(\tau) \) for some \( \tau \in [0, \theta] \) is attainable. For the case \( x_0 = y_0 \), we have

\[
x(\theta) = x_0 + \int_0^\theta (\theta - s)^{n-1} u(s) \, ds = y_0 + \int_0^\theta (\theta - s)^{n-1} v(s) \, ds = y(\theta).
\]
For the other case $x_0 \neq y_0$, we have
\[
y(t) - x(t) = y_0 - x_0 + \int_0^t (\theta - s)^{n-1} (n-1)! v(s)ds - \int_0^t (\theta - s)^{n-1} (n-1)! v(s)ds + \int_0^t (\theta - s)^{n-1} (n-1)! \langle v(s), e \rangle ds
\]
\[
\quad - \int_0^t (\theta - s)^{n-1} (n-1)! \left( \frac{(n+1)(\rho^2 - \sigma^2)}{(\theta - s)^{-n} \theta^{n+1}} + \langle v(s), e \rangle \right)^{\frac{1}{2}} ds
\]
\[
= \left( \|x_0 - y_0\| + \int_0^t (\theta - s)^{n-1} (n-1)! \langle v(s), e \rangle ds \right) e
\]
\[
- \left( \int_0^t (\theta - s)^{n-1} (n-1)! \left( \frac{(n+1)(\rho^2 - \sigma^2)}{(\theta - s)^{-n} \theta^{n+1}} + \langle v(s), e \rangle \right)^{\frac{1}{2}} ds \right) e.
\]
This equation can be written as
\[
y(t) - x(t) = \|y_0 - x_0\| + \beta_1(t) - \beta_2(t) e, \tag{9}
\]
where
\[
\beta_1(t) = \int_0^t (\theta - s)^{n-1} (n-1)! \langle v(s), e \rangle ds,
\]
\[
\beta_2(t) = \int_0^t (\theta - s)^{n-1} (n-1)! \left( \frac{(n+1)(\rho^2 - \sigma^2)}{(\theta - s)^{-n} \theta^{n+1}} + \langle v(s), e \rangle \right)^{\frac{1}{2}} ds.
\]
Now, we estimate $\beta_1(t)$ and $\beta_2(t)$ at $t = \theta$. We begin with $\beta_1(\theta)$ and use the fact that $y(\theta) \in \varphi$. This means that
\[
2 \langle y_0 - x_0, y(\theta) \rangle \leq \left( \frac{4(n+1)\theta^{2n-1}}{9n^2((n-1)!)^2} \right) (\rho^2 - \sigma^2) + \|y_0\|^2 - \|x_0\|^2.
\]
This inequality implies that $\langle e, y(\theta) \rangle \leq \gamma$, where
\[
\gamma = \left( \frac{4(n+1)\theta^{2n-1}}{9n^2((n-1)!)^2} \right) (\rho^2 - \sigma^2) + \|y_0\|^2 - \|x_0\|^2 / 2 \|y_0 - x_0\|.
\]
Therefore, \[
\langle e, y(\theta) \rangle = \left\langle e, y_0 + \int_0^\theta (\theta - t)^{n-1} (n-1)! v(t)dt \right\rangle \leq \gamma. \tag{10}
\]
From this, we deduce that
\[
\int_0^\theta (\theta - t)^{n-1} (n-1)! \langle e, v(t) \rangle dt \leq \gamma - \langle y_0, e \rangle.
\]
This estimates $\beta_1(\theta)$. To estimate $\beta_2(\theta)$, we consider the following two-dimensional vector function:
\[
h(t) = \left( \frac{(n+1)(\rho^2 - \sigma^2)}{\theta^n + (\theta - t)^{2-3n}} \right)^{\frac{1}{2}}, \quad \left( \theta - t \right)^{\frac{1}{2}}
\]
\[
\beta_2(\theta) = \frac{1}{(n-1)!} \int_0^\theta \left( \frac{(n+1)(\rho^2 - \sigma^2)}{(\theta - t)^{2-3n} \theta^{n+1}} + \langle v(t), e \rangle \right)^{\frac{1}{2}} dt
\]
\[
= \frac{1}{(n-1)!} \int_0^\theta \left( \frac{(n+1)(\rho^2 - \sigma^2)}{(\theta - t)^{2-3n} \theta^{n+1}} + \langle v(t), e \rangle \right)^{\frac{1}{2}} dt.
\]
and using the inequality (10) in (12), we have it verifiable that the right hand is this inequality is zero. Therefore, we define a function

\[ t \in \mathbb{R} \]

for all \( \| y \|_p > 0 \). This implies that \( \| y \|_p \geq 1 \). Consequently, we have \( x(\theta) = y(\theta) \).

4. Some important lemmas. In part of this section, we present some existing established results that are also useful in the proof of the main result of the paper.

Lemma 3.2. [6] Suppose that there exists a nonzero vector \( p_0 \) such that
\[ \langle y_0 - x_{i0}, p_0 \rangle \geq 0, \quad \text{for all } i \in I, \]
and let
\[ X_i = \begin{cases} \{ z \in l_2 : 2 (y_0 - x_{i0}, z) \leq R_i^2 - r^2 + \| y_0 \|^2 - \| x_{i0} \|^2 \} & \text{if } x_{i0} \neq y_0, \\ \{ z \in l_2 : 2 (p_0, z - y_0) \leq R_i \} & \text{if } x_{i0} = y_0. \end{cases} \]

We then have the following:
1. If $B_r(y_0) \subset \bigcup_{i \in I} B_{R_i}(x_{i0})$ then $B_r(y_0) \subset \bigcup_{i \in I} X_i$.

2. If for any $\epsilon \in (0, R_0)$, where $0 < R_0 = \inf_{i \in I} R_i$, the set $\bigcup_{i \in I} B_{R_i-\epsilon}(x_{i0})$ does not contain the ball $B_r(y_0)$, then there exists a point $\hat{y} \in S_r(y_0) = \{y \in l_2 : \|y - y_0\| = r\}$ such that $\|\hat{y} - x_{i0}\| \geq R_i$, for all $i \in I$.

4. **Main result.** The theorem below is the main result of the paper and it provides value of the game $G^*$ subject to some conditions.

**Theorem 4.1.** Suppose that and there exists a nonzero vector $p_0$ such that

\[ \langle y_0 - x_{i0}, p_0 \rangle \geq 0 \quad \text{for all } i \in I, \]  

and that $\sigma \leq \rho_i + \left(\frac{2n-1}{\theta^{n-1}}\right)^{\frac{1}{2}} \frac{\delta}{(n-1)!}, \quad \text{for all } i \in I,$

then the number

\[ \delta := \inf \left\{ l > 0 : B_{R_0}(y_0) \subset \bigcup_{i \in I} B_{R_{i0}+l}(x_{i0}) \right\} \quad (13) \]

is the value of the game $G^*$.

**Proof.** To prove this theorem, we first introduce dummy pursuers whose dynamic equations are given by

\[ \frac{dz_i}{dt} = \frac{(\theta - t)^{n-1}}{(n-1)!} \nu_i^*(t), \quad z_i(0) = x_{i0}, \quad i \in I, \]

with the control parameter $\nu_i^*(t)$ is such that

\[ \int_0^\theta \|\nu_i^*(t)\|^2 dt \leq \tilde{\rho}_i^2(\epsilon), \quad \text{where } \tilde{\rho}_i(\epsilon) = \rho_i + \frac{(n-1)^\frac{1}{2}}{k_i} \left(\frac{2n-1}{\theta^{n-1}}\right)^{\frac{1}{2}} \frac{\delta}{(n-1)!} \frac{1}{k_i}, \]

$k_i = \max\{1, \rho_i\}$ and $\epsilon$ is a constant such that $\epsilon \in (0, 1)$. In accordance with the attainability set of the real pursuers, we define that of the $i^{th}$ dummy pursuer as the ball $B_r(x_{i0})$, with $r = \left(\frac{2n-1}{\theta^{n-1}}\right)^{\frac{1}{2}} \frac{\tilde{\rho}_i^2(\epsilon)}{\theta^{n-1}}$.

We define the dummy pursuers’ strategies as follow: For all $i \in I$ and if $x_{i0} = y_0$, we set $\nu_i^*(t) = v(t)$, otherwise we set

\[ \nu_i^*(t) = \frac{v(t) - \langle v(t), e_i \rangle e_i}{v(t)} + \left(\frac{(n+1)(\tilde{\rho}_i^2(\epsilon) - \sigma^2)}{(\theta - t)^{n-1}} + \langle v(t), e_i \rangle^2\right)^{\frac{1}{2}} e_i, \quad 0 \leq t \leq \tau_i, \]

\[ \tau_i \leq t \leq \theta, \quad (14) \]

where $e_i = \frac{y_0 - x_{i0}}{\|y_0 - x_{i0}\|}$, when the $i^{th}$ dummy pursuer uses this strategy, $\tau_i$ is the time at which the equation $z_i(\tau_i) = y(\tau_i)$ is achieved. The admissibility of this strategy can be established in a similar way we established the admissibility of the strategy (8). We now let the strategies of the real pursuers be defined as

\[ U_i(t) = \frac{\rho_i}{\tilde{\rho}_i(0)} \nu_i^0(t), \quad 0 < t < \theta, \quad (15) \]

The admissibility of this strategy follows from the fact that

\[ \int_0^\theta \|\nu_i^0(t)\|^2 dt \leq \tilde{\rho}_i^2(0). \]

In accordance with the payoff of the game, the number $\delta$ defined by (13) is the value of the game $G^*$, if the following inequalities hold

\[ \sup_{v(\cdot) \in I} \inf_{x_i(\cdot): x_i(\theta) = x_i(\theta)} \|y(\theta) - x_i(\theta)\| \leq \delta \leq \inf_{u_1(\cdot), u_2(\cdot), \ldots} \inf_{x_i(\cdot): x_i(\theta) = x_i(\theta)} \|x_i(\theta) - y(\theta)\|. \quad (16) \]
Now we establish the inequalities (16). Firstly, we show the left hand side inequality. Indeed, by the definition of $\delta$ in (13), we have $B_{R_\rho}(y_0) \subset \bigcup_{i \in I} B_{R_\rho + \delta}(x_{i0}) \subset \bigcup_{i \in I} B_{R_i}(x_{i0})$, where $R_i(\epsilon) = R_{\rho_i} + \delta + \frac{\epsilon}{\rho_i}$. Then by the part 1 of the Lemma 3.2, we have $B_{R_\rho}(y_0) \subset \bigcup_{i \in I} X_{R_i}(x_{i0})$, where

$$X_i^* = \begin{cases} \{z \in l_2 : 2\langle y_0 - x_{i0}, z \rangle \leq R_\rho^2(\epsilon) - R_{\rho_i}^2 + \|y_0\|^2 - \|x_{i0}\|^2 \}, & \text{if } x_{i0} \neq y_0, \\ \{z \in l_2 : 2\langle p_0, z - y_0 \rangle \leq R_i(\epsilon) \}, & \text{if } x_{i0} = y_0. \end{cases}$$

On this account, we have the point $y(\theta) \in B_{R_{\rho_i}}(y_0)$ belongs to the set $X_i^*$, for some $s \in I$.

By hypothesis of the Theorem 4.1, we have $\sigma \leq \hat{\rho}_i(\epsilon)$, then it follows from the Lemma 3.1 that if the dummy pursuer with state $z_i$, uses the strategy (14), then $y(\theta) = z_s(\theta)$. Mindful of this and if the real pursuers uses the strategy (15), we have

$$\|y(\theta) - x_s(\theta)\| = \|z_s(\theta) - x_s(\theta)\|$$

$$= \left\| \int_0^\theta \frac{(\theta - t)^{n-1}}{(n-1)!} \left( \nu_s^0(t) - \frac{\rho_s}{\hat{\rho}_s(0)} \nu_s^0(t) \right) dt \right\|$$

$$\leq \int_0^\theta \left\| \frac{(\theta - t)^{n-1}}{(n-1)!} \left( \nu_s^0(t) - \nu_s^0(t) \right) \right\| dt$$

$$+ \int_0^\theta \frac{(\theta - t)^{n-1}}{(n-1)!} \left\| \nu_s^0(t) - \frac{\rho_s}{\hat{\rho}_s(0)} \nu_s^0(t) \right\| dt. \quad (17)$$

Now we estimate each of the two expressions in the right-hand side of the last inequality. Firstly, we consider the case $x_{i0} = y_0$. In view of the dummy pursuers’ strategies, we have $\nu_i^0(t) = \nu_s^0(t) = v(t)$, and therefore for all $i \in I$, we have the first expression equal to zero. For any $i \in I$, the estimate of the second expression is given below.

$$\int_0^\theta \left\| \frac{(\theta - t)^{n-1}}{(n-1)!} \left( \nu_i^0(t) - \frac{\rho_i}{\tilde{\rho}_i(0)} \nu_i^0(t) \right) \right\| dt$$

$$= \int_0^\theta \left\| \frac{(\theta - t)^{n-1}}{(n-1)!} \left( 1 - \frac{\rho_i}{\tilde{\rho}_i(0)} \right) \nu_i^0(t) \right\| dt$$

$$\leq \frac{\tilde{\rho}_i(0) - \rho_i}{\tilde{\rho}_i(0)(n-1)!} \left( \int_0^\theta (\theta - t)^{2n-2} dt \right)^{\frac{1}{2}} \left( \int_0^\theta \| \nu_i^0(t) \|^2 dt \right)^{\frac{1}{2}}$$

$$\leq \left( \frac{2n - 1}{\theta^{2n-1}} \right)^{\frac{1}{2}} \frac{\delta}{\tilde{\rho}_i(0)} \left( \frac{\theta^{2n-1}}{2n - 1} \right)^{\frac{1}{2}} \left( \int_0^\theta \| \nu_i^0(t) \|^2 dt \right)^{\frac{1}{2}}$$

$$\leq \left( \frac{2n - 1}{\theta^{2n-1}} \right)^{\frac{1}{2}} \frac{\delta}{\tilde{\rho}_i(0)} \left( \frac{\theta^{2n-1}}{2n - 1} \right)^{\frac{1}{2}} \tilde{\rho}_i = \delta. \quad (18)$$

Note that this estimate is also true the for the case $x_{i0} \neq y_0$. This proves the left hand inequality in (16) for the case $x_{i0} = y_0$.

Secondly, we prove left hand inequality in (16) for the case if $x_{i0} \neq y_0$. By the strategy (14) and using the fact that $\epsilon \in (0, 1)$ and $k_i = \max\{1, \rho_i\}$, we have
\begin{align}
\int_0^\theta \left\| \frac{(\theta - t)^{n-1}}{(n-1)!} (\nu_i(t) - \nu_0(t)) \right\| \, dt &= \int_0^{\tau_i} \left\| \frac{(\theta - t)^{n-1}}{(n-1)!} (\nu_i(t) - \nu_0(t)) \right\|^2 \, dt \\
&\leq \frac{1}{(n-1)!} \left( \int_0^{\tau_i} (\theta - t)^{2n-2} \, dt \right)^{\frac{1}{2}} \left( \int_0^{\tau_i} \left\| \nu_i(t) - \nu_0(t) \right\|^2 \, dt \right)^{\frac{1}{2}} \\
&= \frac{1}{(n-1)!} \left( \frac{\theta^{2n-1}}{2n-1} \right)^{\frac{1}{2}} \left( \int_0^{\tau_i} \left\| \nu_i(t) - \nu_0(t) \right\|^2 \, dt \right)^{\frac{1}{2}} \\
&\leq \left( \frac{\theta^{2n-1}}{2n-1} \right)^{\frac{1}{2}} \left( \left( \frac{2n-1}{\theta^{2n-1}} \right)^{\frac{1}{2}} + (2\delta + 1) \left( \frac{2n-1}{\theta^{2n-1}} \right) (n-1)! \right) \epsilon. \quad (19)
\end{align}

We obtained the last inequality by using the following

\begin{align}
\left( \int_0^{\tau_i} \left\| \nu_i(t) - \nu_0(t) \right\|^2 \, dt \right)^{\frac{1}{2}} \\
&\leq \int_0^{\tau_i} \left( \left( \frac{(n+1)(\hat{\rho}_i^2(\epsilon) - \sigma^2)}{(\theta - t)^n \theta^{n+1}} \right)^{\frac{1}{2}} - \left( \frac{(n+1)(\hat{\rho}_0^2(0) - \sigma^2)}{(\theta - t)^n \theta^{n+1}} \right)^{\frac{1}{2}} \right)^2 \, dt \\
&\leq \int_0^\theta \left( \left( \frac{(n+1)}{(\theta - t)^n \theta^{n+1}} \right)^{\frac{1}{2}} \left( (\hat{\rho}_i^2(\epsilon) - \sigma^2)^{\frac{1}{2}} - (\hat{\rho}_0^2(0) - \sigma^2)^{\frac{1}{2}} \right) \right)^2 \, dt \\
&= \left( \left( \frac{2\hat{\rho}_i(0) \epsilon}{k_i ((n-1)!)^{\frac{1}{2}}} \right)^{\frac{1}{2}} + \frac{2n-1}{\theta^{2n-1}} \left( \epsilon (n-1)! \right)^{\frac{1}{2}} \right) \left( \frac{2n-1}{\theta^{2n-1}} \right)^{\frac{1}{2}} + \frac{2n-1}{\theta^{2n-1}} \left( \epsilon (n-1)! \right)^{\frac{1}{2}} \\
&\leq \left( \frac{2n-1}{\theta^{2n-1}} \right)^{\frac{1}{2}} \left( \epsilon (n-1)! \right)^{\frac{1}{2}} + (2\delta + 1) \left( \frac{2n-1}{\theta^{2n-1}} \right) (n-1)! \epsilon \\
\end{align}

Consequently, using (18) and (19) in (17), we have \( \| y(\theta) - x_0(\theta) \| \leq \delta + C \epsilon \), where

\[ C = \left( \frac{\theta^{2n-1}}{2n-1} \right)^{\frac{1}{2}} \left( \frac{2n-1}{\theta^{2n-1}} \right)^{\frac{1}{2}} + (2\delta + 1) \left( \frac{2n-1}{\theta^{2n-1}} \right) (n-1)!^2 \] is a constant and \( \epsilon \in (0, 1) \). This proves the left hand inequality in (16). Next we show the right hand inequality in (16). The inequality holds, if \( \delta = 0 \). We now consider the only other possibility that \( \delta > 0 \). According to the definition of \( \delta \), we have for any \( \epsilon > 0 \), the set \( \bigcup_{i \in I} B_{R_{k_i} + \delta - \epsilon(x_{10})} \), does not contain the ball \( B_{R_{k_i}(y_{10})} \). With this, part 2 of Lemma...
3.2 guaranteed the existence of a point \( \tilde{y} \in S_r(y_0) = \{ y \in l_2 : \| y - y_0 \| = r \} \), where \( r = R_E \). That is, \( \| \tilde{y} - y_0 \| = \left( \frac{\sigma^{2n-1}}{2n-1} \right)^{\frac{1}{2}} \frac{\rho_i}{(n-1)!} \), such that

\[
\| \tilde{y} - x_{i0} \| \geq \left( \frac{\sigma^{2n-1}}{2n-1} \right)^{\frac{1}{2}} \frac{\rho_i}{(n-1)!} + \delta. \tag{20}
\]

Besides this, using (4) and Cauchy-Schwartz inequality, we have

\[
\| x_i(\theta) - x_{i0} \| \leq \left\| \int_0^\theta (\theta - t)^{n-1} u_i(t) dt \right\| \leq \frac{1}{(n-1)!} \left( \int_0^\theta (\theta - t)^{2n-2} dt \right)^{\frac{1}{2}} \left( \int_0^\theta \| u_i(t) \|^2 dt \right)^{\frac{1}{2}} \leq \left( \frac{\sigma^{2n-1}}{2n-1} \right)^{\frac{1}{2}} \frac{\rho_i}{(n-1)!}. \tag{21}
\]

Thus, using (20) and (21), we have

\[
\| \tilde{y} - x_i(\theta) \| \geq \| \tilde{y} - x_{i0} \| - \| x_i(\theta) - x_{i0} \| \geq \left( \frac{\sigma^{2n-1}}{2n-1} \right)^{\frac{1}{2}} \frac{\rho_i}{(n-1)!} + \delta - \left( \frac{\sigma^{2n-1}}{2n-1} \right)^{\frac{1}{2}} \frac{\rho_i}{(n-1)!} = \delta.
\]

By this, if the evader can reach the point \( \tilde{y} \) at the time \( \theta \), then the right hand inequality in (16) is proved. Indeed, the evader’s control below will do this job:

\[
v(t) = \sigma \left( \frac{2n-1}{\sigma^{2n-1}} \right)^{\frac{1}{2}} (\theta - t)^{n-1} e, \quad 0 \leq t \leq \theta,
\]

where \( e = \frac{\tilde{y} - y_0}{\| \tilde{y} - y_0 \|} \). Thus, we proved the two inequalities in (16). Therefore, the proof of the theorem is complete.

6. Conclusion. We successfully obtained game value for a pursuit-evasion differential game problem formulated in the Hilbert space \( l_2 \). Certain \( n \)th order differential equations described the dynamics of the players with players’ control functions subject to integral constraints. We used modified strategy of parallel approach, which was first constructed by Petrosyan [15], in establishing our result. The result of this paper is a generalization of some results in the literature. For example, the results of the papers [11] and [10] are corollaries to this paper when we set \( n = 2 \) and \( n = 1 \) in the players’ dynamic equations (1) respectively.

Acknowledgments. We thank the referees of the draft manuscript. Indeed, your valuable comments and observations added quality to the result and the entire write up.

REFERENCES

[1] J. Adamu, K. Muangchoo, A. J. Badakaya and J. Rilwan, On pursuit-evasion differential game problem in a Hilbert space, *AIMS Math.*, 5 (2020), 7467–7479.
[2] A. J. Badakaya, Value of a differential game problem with multiple players in a certain Hilbert space, *J. Nigerian Math. Soc.*, 36 (2017), 287–305.
[3] E. Bakolas and P. Tsiotras, Relay pursuit of a maneuvering target using dynamic Voronoi diagrams, *Automatica J. IFAC*, 48 (2012), 2213–2220.
[4] M. Chen, Z. Zhou and C. J. Tomlin, Multiplayer reach-avoid games via pairwise outcomes, *IEEE Trans. Automat. Control*, 62 (2017), 1451–1457.
[5] G. I. Ibragimov, On a game of optimal pursuit of one evader by several pursuers, *J. Appl. Math. Mech.*, **62** (1998), 187–192.

[6] G. I. Ibragimov, Optimal pursuit of an evader by countably many pursuers, *Differ. Equ.*, **41** (2005), 627–635.

[7] G. Ibragimov, N. Abd Rasid, A. Kuchkarov and F. Ismail, Multi pursuer differential game of optimal approach with integral constraints on control of players, *Taiwanese J. Math.*, **19** (2015), 963–976.

[8] G. Ibragimov, I. A. Alias, U. Waziri and A. B. Ja’afaru, Differential game of optimal pursuit for an infinite system of differential equations, *Bull. Malays. Math. Sci. Soc.*, **42** (2019), 391–403.

[9] G. Ibragimov and N. A. Hussin, A Pursuit-evasion differential game with many pursuers and one evader, *Malaysian J. Math. Sci.*, **4** (2010), 183–194.

[10] G. I. Ibragimov and A. S. Kuchkarov, Fixed duration pursuit-evasion differential game with integral constraints, *J. Physics: Conference Series*, **435** (2017).

[11] G. I. Ibragimov and M. Salimi, Pursuit-evasion differential game with many inertial players, *Math. Probl. Eng.*, **2009** (2009), 15pp.

[12] A. B. Ja’afaru and G. I. Ibragimov, On some pursuit and evasion differential game problems for an infinite number of first-order differential equations, *J. Appl. Math.*, **2012** (2012), 13pp.

[13] A. S. Kuchkarov, G. I. Ibragimov and M. Khakestari, On a linear differential game of optimal approach of many pursuers with one evader, *J. Dyn. Control Syst.*, **19** (2013), 1–15.

[14] A. Y. Levchenkov and A. G. Pashkov, Differential game of optimal approach of two inertial pursuers to a noninertial evader, *J. Optim. Theory Appl.*, **65** (1990), 501–518.

[15] L. A. Petrosyan, Differential pursuit games, Izdat. Leningrad. Univ., Leningrad, 1977, 222pp.

[16] M. V. Ramana and M. Kothari, Pursuit-evasion games of high speed evader, *J. Intell. Robot. Syst.*, **85** (2017), 293–306.

[17] M. V. Ramana and M. Kothari, Pursuit strategy to capture high-speed evaders using multiple pursuers, *J. Guidance Control Dyn.*, **49** (2017), 139–149.

[18] M. Salimi and M. Ferrara, Differential game of optimal pursuit of one evader by many pursuers, *Internat. J. Game Theory*, **48** (2019), 481–490.

[19] N. Satimov, B. B. Rikhsiev and A. A. Khamdamov, A pursuit problem for linear differential and discrete n-person games with integral constraints, *Mat. Sb. (N.S.)*, **118(160)** (1982), 456–469.

[20] A. I. Subbotin and A. G. Chentsov, *Guaranteed Optimization in Control Problems*, Nauka, Moscow, 1981, 288pp.

[21] A.-M. Wazwaz, *Linear and Nonlinear Integral Equations. Methods and Applications*, Higher Education Press, Beijing; Springer, Heidelberg, 2011.

Received December 2020; revised April 2021.

E-mail address: ajbadakaya.mth@buk.edu.ng
E-mail address: aminuhalliru1@gmail.com
E-mail address: jamilluadamu88@gmail.com