ON INDICES OF THE DIRAC OPERATOR IN A NON-FREDHOLM CASE

ALEXANDER MOROZ*†

School of Physics and Space Research, University of Birmingham, Edgbaston, Birmingham B15 2TT, U. K.

ABSTRACT

The Dirac Hamiltonian with the Aharonov-Bohm potential provides an example of a non-Fredholm operator for which all spectral asymmetry comes entirely from the continuous spectrum. In this case one finds that the use of standard definitions of the resolvent regularized, the heat kernel regularized, and the Witten indices misses the contribution coming from the continuous spectrum and gives vanishing spectral asymmetry and axial anomaly. This behaviour in the case of the continuous spectrum seems to be general and its origin is discussed.

PACS numbers: 11.30.Pb, 02-30.Tb, 03-65.Bz
1 Introduction

Let us consider an abstract Dirac Hamiltonian $H$ in $L^2(\mathbb{R}^2)^2$ with supersymmetry. It can be written in the form [1]

$$H = Q + M \tau,$$

(1)

where $M$ is a positive self-adjoint “mass” operator, which commutes with $Q$ and $\tau$, and $Q$,

$$Q = \begin{pmatrix} 0 & D^\dagger \\ D & 0 \end{pmatrix}, \quad \{Q, \tau\} = 0,$$

(2)

is a supercharge with respect to the involution $\tau$,

$$\tau = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$ 

(3)

The physical examples are provided by the Dirac Hamiltonian with a scalar field, the Dirac Hamiltonian with and without anomalous electric and magnetic moments in electromagnetic fields, the Klein-Gordon equation in an external magnetic field, and Dirac type operators over Riemannian manifolds (see [1], pp. 151-154 and references therein).

For a supercharge $Q$, the “Fredholm index” is defined as

$$\text{ind} \ Q \equiv \dim \ker D - \dim \ker D^\dagger = \dim \ker D^\dagger D - \dim \ker DD^\dagger,$$

(4)

whenever this number exists. Operators $D^\dagger D$ and $DD^\dagger$ are “close” in the sense that their spectra coincide except for the zero energy. In fact, it can be shown that their respective restrictions on $(\ker D)^\perp$ and $(\ker D^\dagger)^\perp$ are unitary equivalent (see [1], p. 144). Therefore, in the case that the zero energy belongs to the point spectrum, ind $Q$ is nothing but

$$\Sigma = \lim_{\varepsilon \downarrow 0} \int_{0-}^{E} \sigma(\varepsilon) \ d\varepsilon,$$

(5)

where $\sigma(\varepsilon)$ is the spectral asymmetry. The latter can be defined as the difference between spectral densities $\rho(\varepsilon)$ for positive and negative energy continuums (cf. [2]),

$$\sigma(\varepsilon) \equiv \rho(\varepsilon) - \rho(-\varepsilon),$$

(6)

where $\varepsilon = |\varepsilon| \geq 0$, or, generally, as

$$\sigma(\varepsilon) \equiv -\frac{1}{\pi} \lim_{\varepsilon \downarrow 0} \text{ImTr} \left( \frac{1}{\varepsilon + i\varepsilon - H} + \frac{1}{\varepsilon + i\varepsilon + H} \right)$$

$$= -\frac{2\varepsilon}{\pi} \lim_{\varepsilon \downarrow 0} \text{ImTr} \left( \frac{1}{\varepsilon^2 + i\varepsilon - D^\dagger D} + \frac{1}{\varepsilon^2 + i\varepsilon - DD^\dagger} \right).$$

(7)
In the most physically important case, $Q$ is not Fredholm since 0 is not an isolated eigenvalue and it lies in the essential spectrum. In this case, one must first define a regularized index. The common choice is to use either the “resolvent regularized index” or to define an index with the “heat-kernel regularization” \cite{1}. To define the regularized indices, we shall use the fact that the Hilbert space $H$ can be written as the direct sum of the “positive”, $H_+$, and “negative”, $H_-$, subspaces with respect to involution $\tau$, $H = H_+ \oplus H_-$. Accordingly, any operator $A$ in the Hilbert space can be written as

$$A = \begin{pmatrix} A_+ & A_- \\ A_- & A_+ \end{pmatrix}. \quad (8)$$

Then, according to Ref. \cite{1}, we shall denote by $\text{mtr}$ the diagonal sum (“the matrix trace”) of $A$,

$$\text{mtr} A \equiv A_+ + A_- \quad (9)$$

and by $\text{str}$ the “supertrace” of $A$,

$$\text{str} A \equiv \text{Tr} \text{mtr} A = \text{Tr} (A_+ + A_-), \quad (10)$$

where $\text{Tr}$ is the usual trace in the Hilbert space. Now, if $\text{mtr} \tau (z - Q^2)^{-1}$ is trace class for some $z \in C \setminus [0, \infty)$, the “resolvent regularized index”, $\text{ind}_z Q$, is defined as \cite{1},

$$\text{ind}_z Q \equiv z^{1/2} \text{str} (z^{1/2} - Q)^{-1} \equiv z \text{ str} (z - Q^2)^{-1}$$

$$= z \text{ Tr} [(z - DD^\dagger)^{-1} - (z - DD^\dagger)^{-1}]. \quad (11)$$

If $\exp (-Q^2t)$ is trace class for some $t > 0$, one defines $\text{ind}_t Q$, the index of $Q$ in the heat-kernel regularization, as \cite{1}

$$\text{ind}_t Q \equiv \text{str} \tau e^{-Q^2t}. \quad (12)$$

If $Q$ is not Fredholm, one cannot expect regularized indices $\text{ind}_z Q$ and $\text{ind}_t Q$ to be independent, respectively, of $z$ and of $t$. Therefore, one defines a parameter independent index as

$$W(Q) \equiv \lim_{z \to 0, \ |\arg z| > \epsilon > 0} \text{ind}_z Q = \lim_{t \to \infty} \text{ind}_t Q, \quad (13)$$

whenever either the first or the second limit exists. Index $W(Q)$ is called the Witten index \cite{1, 3}. This definition reduces to the notion of the Fredholm index whenever $Q$ is
Fredholm \( [1] \). The *axial anomaly* \( A(Q) \) can be defined in terms of \( \text{ind}_z Q \) as \([1]\)

\[
A(Q) \equiv - \lim_{z \to \infty} \frac{\text{ind}_z Q}{|\arg z| > \epsilon > 0}.
\] (14)

In the case that the spectral asymmetry \( \sigma(\mathcal{E}) \) comes from the point spectrum, the Witten index coincides with \( \Sigma \). However, we shall show that definition (13) of the Witten index is not equal to \( \Sigma \) if the spectral asymmetry comes entirely from the continuous part of the spectrum. In particular, in the case of the Dirac Hamiltonian in the presence of an Aharonov-Bohm potential, indices \( \text{ind}_z Q, \text{ind}_t Q, \) and \( W(Q) \), and consequently the axial anomaly \( A(Q) \) as defined by (14), turn out to be zero.

## 2 Krein’s spectral shift function

In order to calculate the resolvent regularized index, \( \text{ind}_z Q \), we shall use Krein’s formula \([4]\) which provides an efficient way of calculating traces such as the trace in (11). Let us consider a pair of operators \( T_1 \) and \( T_2 \) in a Hilbert space \( \mathcal{H} \), and define

\[
G_1(z) = \frac{1}{z - T_1}, \quad G_2(z) = \frac{1}{z - T_2}.
\] (15)

According to Krein’s formula \([4]\),

\[
\text{Tr} [G_1(z) - G_2(z)] = \int_{-\infty}^{\infty} \frac{\xi_{T_1T_2}(\lambda)}{(\lambda - z)^2} d\lambda,
\] (16)

where \( \xi_{T_1T_2}(\lambda) \) is a bounded function, called Krein’s *spectral displacement operator* for the pair \( T_1 \) and \( T_2 \). Krein’s formula (16) is valid whenever \( T_1 \) and \( T_2 \) are trace comparable, which means essentially that the operation on the left hand side in (16) has meaning. Both regularized indices (11) and (12) can be expressed in terms of Krein’s spectral shift function \( \xi \) \([5]\),

\[
\text{ind}_z Q = \int_0^{\infty} \frac{z \xi(\lambda)}{(\lambda - z)^2} d\lambda,
\] (17)

\[
\text{ind}_t Q = -t \int_0^{\infty} \xi(\lambda) e^{-\lambda t} d\lambda.
\] (18)

The Witten index and the axial anomaly are then given as \([1]\)

\[
W(Q) = -\xi(0) \quad \text{and} \quad A = \xi(\infty).
\] (19)
In the case that $T_1 = H_0 + V$ and $T_2 = H_0$ are, respectively, perturbed and unperturbed Hamiltonians, the spectral displacement operator $\xi$ is given as

$$\xi_{T_1 T_2}(\lambda) = \frac{i}{2\pi} \ln \det S(\lambda),$$

(20)

where $S(\lambda)$ is the on-the-energy-shell S matrix [6]. We have used formulas (16) and (20) to calculate the change in the density of states for various physical systems, including the gravitational vortex [2], electromagnetic waves [7], the Pauli, the Schrödinger [8, 9], the Dirac, and the Klein-Gordon Hamiltonians [2] in the presence of the Aharonov-Bohm potential.

An important point is that validity of Krein’s formula (16) together with representation (20) of Krein’s spectral displacement operator $\xi(\lambda)$ is not restricted to a particular operator, such as the Schrödinger operator or the Dirac operator, but can be applied to any pair of operators $T_1$ and $T_2$ (subject to the condition that they are trace comparable). In the latter case, the formal S matrix can be defined in the same way as the “physical” S matrix. Let $\psi_{T_1}$ and $\psi_{T_2}$ be asymptotic eigenstates of operators $T_1$ and $T_2$. Then, for the purpose of calculating the trace in (11), the formal S matrix associated with the pair $T_1$ and $T_2$ can be defined as

$$\psi_{T_1} = S \psi_{T_2}.$$  

(21)

For example, in the case that both $T_1$ and $T_2$ are rotationally symmetric, the S matrix in the $l$th channel, where $l$ is the angular momentum, will be

$$S_l(\lambda) = \exp \left[ 2i\delta_{l: T_1/T_2}(\lambda) \right],$$

(22)

where $\delta_{l: T_1/T_2}(\lambda)$ is the relative phase shift of the scattering solution of $T_1$ with respect to the scattering solution of $T_2$ in $l$th channel. Then, according to (20),

$$\xi(\lambda)_{T_1 T_2} = -\frac{1}{\pi} \sum_{l=-\infty}^{\infty} \delta_{l: T_1/T_2}(\lambda).$$

(23)

In order to calculate the trace in (11), the relevant pair of operators is

$$T_1 = D^\dagger D, \quad T_2 = DD^\dagger.$$  

(24)

Since spectra of operators $D^\dagger D$ and $DD^\dagger$ coincide except for the zero energy, one expects that the relative S matrix for the pair of operators $D^\dagger D$ and $DD^\dagger$ will be relatively simple. We shall confirm this expectation by explicit calculations.
3 The Dirac Hamiltonian in the Aharonov-Bohm potential

Let us illustrate the above procedure for the case of the Dirac Hamiltonian in the presence of an Aharonov-Bohm (AB) potential \( A(r) \) \[10\] which, in the radial gauge, is given by

\[
A_r = 0, \quad A_\phi = \frac{\Phi}{2\pi r} = \frac{\alpha}{2\pi r} \Phi_0,
\]

where \( \Phi = \alpha \Phi_0 \) is the total flux through the flux tube and \( \Phi_0 = \hbar c/|e| \). Let us write \( \alpha = n + \eta \) where \( n \) and \( \eta \) are, respectively, the integer and the fractional part of \( \alpha \).

After separation of variables in polar coordinates, the Dirac Hamiltonian \( H(A) \) reduces to the direct sum, \( H(A) = \oplus_l h_{m,l} \), of radial channel operators \( h_{m,l} \) in \( L^2[(0, \infty), rdr] \),

\[
h_{m,l} = \begin{bmatrix}
m & -i \left( \partial_r + \nu + \frac{1}{r} \right) \\
-i \left( \partial_r - \nu \right) & -m
\end{bmatrix},
\]

where \( \nu = l + \alpha \) \[11\]. The supercharge in the \( l \)th channel is

\[
Q_l = \begin{bmatrix}
-1 & -i \left( \partial_r + \frac{\nu + 1}{r} \right) \\
-i \left( \partial_r - \nu \right) & 0
\end{bmatrix},
\]

and

\[
D^+ D = -\frac{d^2}{dr^2} - \frac{1}{r} \frac{d}{dr} + \frac{\nu^2}{r^2} + g_m \frac{\alpha}{r} \delta(r), \quad \nu_+ = l + \alpha,
\]

\[
DD^+ = -\frac{d^2}{dr^2} - \frac{1}{r} \frac{d}{dr} + \frac{\nu^2}{r^2} - g_m \frac{\alpha}{r} \delta(r), \quad \nu_- = l + 1 + \alpha,
\]

with \( g_m = 1 \) \[1, 12\].

3.1 Spectrum and scattering phase shifts

The point spectrum of the Dirac Hamiltonian \( H \) in the AB potential is empty \[4\]. There are no threshold modes \( \equiv \) zero modes in the case that \( M = 0 \) in \( \Phi \) in the spectrum. For a threshold state at \( E = m \) to exist, the lower component \( \chi_2 \) of the Dirac spinor \( \chi = (\chi_1, \chi_2) \) must be zero and the upper component \( \chi_1 \) has to obey

\[
\left( \partial_r - \frac{\nu}{r} \right) \chi_1(r) = 0.
\]
Similarly, at the threshold \( E = -m \), \( \chi_1 \) must vanish and \( \chi_2 \) has to satisfy equation
\[
\left( \partial_r + \frac{\nu + 1}{r} \right) \chi_2(r) = 0.
\] (31)
These two equations can be easily integrated. Their respective solutions are
\[
\chi_1(r) = r^\nu \quad \text{and} \quad \chi_2(r) = r^{-(1+\nu)}.
\] (32)
Obviously, neither \( \chi_1 \) nor \( \chi_2 \) are in \( L^2([0, \infty), r\,dr] \) for any \( l \). If they are square integrable at infinity they are not so at the origin and vice versa.

Continuous spectrum of the Dirac Hamiltonian \( H \) for \( E > m \) in the AB potential is given in terms of Bessel functions \([11, 13]\),
\[
\Psi_{\mathcal{E},l} = \chi(r) e^{i\varphi} e^{-i\mathcal{E}/\hbar},
\] (33)
where
\[
\chi(r) = \frac{1}{N} \left( \frac{\sqrt{\mathcal{E} + m}}{i \sqrt{\mathcal{E} - m}} (\varepsilon_l)^l J_{\varepsilon_l}(kr) - (\varepsilon_l)^{l+1} J_{\varepsilon_{l+1}}(kr)e^{i\varphi} \right).
\] (34)
\( N \) is a normalization factor, and \( \varepsilon_l = \pm 1 \). For \( E = -\mathcal{E} < -m \), the scattering states are given by
\[
\Psi_{-\mathcal{E},l}(t, r, \varphi) = \Psi_{\mathcal{E},l}^*(t, r, \varphi)|_{m \to -m}.
\] (35)
In what follows, \( \mathcal{E} \) will stand for \( |E| \). The square integrability at the origin fixes the sign of \( \varepsilon_l \) except for the channel \( l = -n - 1 \) \([11]\). Except for the channel \( l = -n - 1 \), phase shifts of up and down components of \( \chi(r) \) are invariant under the change of the sign of the energy, \( E \to -E \), and are given by
\[
\delta^u_l = \delta^d_l = \left\{ \begin{array}{ll}
-\pi\alpha, & l > -n - 1, \\
\pi\alpha, & l < -n - 1.
\end{array} \right.
\] (36)
The crucial point for our discussion is that two-component solutions of the massive Dirac equation have only one degree of freedom that is reflected in the equality of up and down phase shifts \([13]\). The ambiguity in the channel \( l = -n - 1 \), which is relevant when different self-adjoint extensions of the Dirac Hamiltonian in the Aharonov-Bohm potential are discussed \([3, 8, 9, 11, 13, 14]\), does not play any role here, because, despite that the actual value of the phase shift depends on the particular self-adjoint extension, up and down phase shifts in a given self-adjoint extension are always equal \([3]\) (cf. Refs.
Therefore, in all channels the relative phase shift for the pair of operators $D^\dagger D$ and $DD^\dagger$ equals to zero,

$$\delta_{\ell;D^\dagger D/DD^\dagger} = \left( \delta_{-n-1;D^\dagger D} - \delta_{-n-1;DD^\dagger} \right) = 0,$$

and, consequently, the relative $S$ matrix is an identity operator,

$$S_{\ell}(\lambda) = \exp \left[ 2i\delta_{\ell;D^\dagger D/DD^\dagger}(\lambda) \right] = 1.$$  

Then, according to (20), Krein’s spectral shift function $\xi(\lambda) \equiv 0$, and by using (17) and (18) one has

$$\text{ind}_x Q = \text{ind}_t Q = 0.$$  

Similarly, by using definitions (13-14), or, relation (19), one has

$$W(Q) = A(Q) = 0,$$

which contradicts the result for the axial anomaly of the Dirac Hamiltonian in the Aharonov-Bohm potential (see, for example, [2, 16]).

## 4 Discussion and conclusions

We have shown, by analyzing the example of the Dirac Hamiltonian in the Aharonov-Bohm potential, that standard definitions (11-14) of the resolvent regularized, the heat kernel regularized, and the Witten indices miss the contribution coming from the continuous spectrum (cf. also Ref. [17]). This behaviour is not restricted to the special case of the Aharonov-Bohm potential but is valid for a general potential. The crucial point, as it has been said above, is that two-component solutions of the massive Dirac equation have only one degree of freedom that is reflected in the equality of up and down phase shifts [13]. Therefore, by using Krein’s formulas (17), (18), and (20), the contribution of the continuous spectrum to indices will always be zero. Indeed, definitions (11-14) were tailored to count the difference in number of modes having, respectively, the upper and the lower component identically equal to zero. The latter are the standard threshold (zero) modes. However, both components of the Dirac spinor of a scattering mode are not identically zero, and definitions (11-14) are not sensitive enough to reflect a nonzero spectral asymmetry in this case. The latter statement can be rephrased as follows: definitions
(11–14) are not sensitive to the change $E \rightarrow -E$ in the sign of the energy: in the example discussed here, phase shifts in the critical channel $l = -n - 1$ are not invariant under the sign reversal of the energy, nevertheless definitions (11–14) give a vanishing answer. As a result, in the non-Fredholm and the noncompact cases one must use definition (7) for the spectral asymmetry in order to obtain nonzero axial anomaly and calculate Krein’s displacement operator $\xi_{T_1,T_2}$ for a corresponding choice of operators $T_1$ and $T_2$ (see, for example, [2]). Otherwise, if one makes choice (24) for $T_1$ and $T_2$, the definition of the spectral asymmetry [1] in terms of Krein’s displacement operator $\xi_{T_1,T_2}$,

$$\sigma(H) = \lim_{t \to 0} m \int_0^\infty \xi(\lambda) \frac{d}{d\lambda} e^{-t(\lambda + m^2)} \frac{e^{-t(\lambda + m^2)}}{\sqrt{\lambda + m^2}} = -\frac{m}{2} \int_0^\infty \xi(\lambda) (\lambda + m^2)^{-3/2} d\lambda,$$

(41)
gives zero.

We hope that we have sufficiently demonstrated the efficiency of Krein’s formulas (16) and (20) for calculation of traces such as the trace in (11) in the case when scattering phase shifts are known, and we think that Krein’s formulas could be a useful substitute to the path integral calculation of indices of the Dirac Hamiltonian [18].

I thank M. Cederwall for calling my attention to reference [17]. This work was supported by EPSRC grant number GR/J35214.

References

[1] B. Thaller, The Dirac Equation (Springer, New York, 1992) Chap. 5.

[2] A. Moroz, Phys. Lett. B 358, 305 (1995); Report IPNO-TH/89-94.

[3] E. Witten, Nucl. Phys. B 188, 513 (1981); J. Diff. Geom. 17, 661 (1982).

[4] J. M. Lifschitz, Usp. Matem. Nauk 7, 170 (1952); M. G. Krein, Matem. Sbornik 33, 597 (1953); J. Friedel, Nuovo Cimento Suppl. 7, 287 (1958); J. S. Faulkner, J. Phys. C 10, 4661 (1977).

[5] In the case of ind$_t$ $Q$, it is required that ind$_t$ $Q$ exists for some $t_0$. Then (18) is valid for all $t \geq t_0$ and the corresponding functions $\xi$ coincide. See [1] for more details.

[6] M. L. Birman and M. G. Krein, Sov. Math.-Dokl. 3, 740 (1962).
[7] A. Moroz, Phys. Rev. B 51, 2068 (1995).

[8] A. Moroz, Mod. Phys. Lett. B 9, 1407 (1995).

[9] A. Moroz, Phys. Rev. A 53, 669 (1996).

[10] Y. Aharonov and D. Bohm, Phys. Rev. 115, 485 (1959); since in the $l = -(n + 1)$-th channel the orders of the Bessel functions for the up/down component are respectively 1 and 0 when $\eta = 0$, we have taken $\nu = \eta - 1$ instead of $\nu = -\eta$.

[11] P. de Sousa Gerbert, Phys. Rev. D 40, 1346 (1989). Since in the $l = -(n + 1)$-th channel the orders of the Bessel functions for the up/down component are respectively 1 and 0 when $\eta = 0$, we have taken $\nu = \eta - 1$ instead of $\nu = -\eta$.

[12] C. R. Hagen, Phys. Rev. Lett. 64, 503 (1990).

[13] P. de Sousa Gerbert and R. Jackiw, Commun. Math. Phys. 124, 229 (1989).

[14] C. Manuel and R. Tarrach, Phys. Lett. B 301, 72 (1993).

[15] Although our conclusions are valid for any linear combination of the two possible solutions in the critical channel $l = n - 1$, this case is not relevant for our (supersymmetric) discussion. If one starts with a flux tube with a finite radius $R$, then, to obtain the nontrivial linear combination in the limit $R \to 0$, an attractive potential must be placed inside the flux tube (see Refs. [2, 8, 9, 11, 12, 14]), which is not the case considered here. Moreover, then the full Hilbert space contains a bound state which explicitly breaks the supersymmetry [2].

[16] T. Jaroszewicz, Phys. Rev. D 34, 3128 (1986); R. Musto, L. O’Raifeartaigh, and A. Wipf, Phys. Lett. B 175, 433 (1986); A. Comtet and S. Ouvry, Phys. Lett. B 225, 272 (1989).

[17] E. Weinberg, Nucl. Phys. B 203, 445 (1982).

[18] L. Alvarez-Gaume, Commun. Math. Phys. 90, 161 (1983); D. Friedan and P. Windey, Nucl. Phys. B 235, 395 (1984); N. V. Borisov and K. N. Ilinski, J. Sov. Math., 156 (1994).