A RENORMALIZABLE COSMODYNAMIC MODEL

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ABSTRACT
The fermionic gyromagnetic ratio \( g = 2 \) of the Kerr-Newman spacetime cannot be a computational "coincidence". This naturally emerges in a four dimensional generally covariant modified Yang-Mills action, which depends on the lorentzian complex structure of spacetime and not its metric. This metric independence makes the model renormalizable. It is a counter example to the general belief that "string theory is the only selfconsistent quantum model which includes gravity". The other properties of the model are phenomenologically very interesting too. The modified Yang-Mills action generates a linear potential, instead of the Coulomb-like \( \frac{1}{r} \) potential of the ordinary action. Therefore the Yang-Mills excitations must be perturbatively confined. This separates the solutions of the model into the vacuum bosonic sector of the periodic configurations, the "leptonic" sector with fermionic solitons and their gauge field excitations, the "hadronic" sector. Simple integrability conditions of the pure geometric equations imply a limited number of "leptonic" and "hadronic" families. The geometric surfaces are generally inside the \( SU(2,2) \) classical domain. Soliton spin and gravity measure how much the surface penetrates inside the classical domain. The \( \text{i}^0 \) point of infinity breaks the \( SU(2,2) \) symmetry down to the Poincaré and dilation groups. A scaling breaking mechanism is presented. Hence the pure geometric modes and asymptotically flat solitons of the model must belong to representations of the Poincaré group. The metrics compatible to the lorentzian complex structure are induced by a Kaehler metric and the spacetime is a totally real lagrangian submanifold of a Kaehler manifold. This opens up the possibility to use the geometric quantization directly to the solitonic surfaces of the model, considering their corresponding Kaehler symplectic manifold as their phase space.
1 INTRODUCTION

The recent failure of ATLAS and CMS experiments to find minimal supersymmetry effects and (large) higher spacetime dimensions have severe consequences to the dominant theories of High Energy Physics. While the discovery of the Higgs particle confirms the minimal Standard Model, its proposed superstring extension does not seem to be compatible with the negative results on supersymmetry. The well-known 11 dimensional superstring model was considered as the dominant candidate for the extension of the Standard Model to include gravity. Many proponents of the superstring model claim that this is the unique quantum self-consistent model, which includes gravity. This statement is wrong. In the present review article, I will present a renormalizable 4-dimensional generally covariant quantum field theoretic model with first order derivatives.

The renormalizability of the model is implied by the metric independence of its lagrangian. Recall that the linearized string action

$$I_S = \frac{1}{2} \int d^2 \xi \sqrt{-\gamma} \gamma^{-\alpha \beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu \nu}$$

(1.1)

has exactly the same property. It does not essentially depend on the metric $\gamma_{\alpha \beta}$ of the 2-dimensional surface but on its complex structure. It depends on its structure coordinates $(z^0, \tilde{z}^0)$, because in these coordinates it takes the metric independent form

$$I_S = \int d^2 z \partial_0 X^\mu \partial_{\tilde{0}} X^\nu \eta_{\mu \nu}$$

(1.2)

All the wonderful properties of the string model are essentially based on this characteristic feature of the string action.

The plausible question and exercise is “what 4-dimensional action with first order derivatives depends on the complex structure, but it does not depend on the metric of the spacetime?”. The additional expectation is that such an action may be formally renormalizable, because the regularization procedure will not generate geometric counterterms. The term “formally” is used, because the 4-dimensional action may have anomalies, which could destroy renormalizability, as it happens in the string action.

The lorentzian signature of spacetime is not compatible with a real tensor (complex structure) $J^\mu_\nu$. Therefore Flaherty introduced a complex tensor to define the lorentzian complex structure, which he extensively studied. It can be shown that there is always a null tetrad $(\ell_\mu, n_\mu, m_\mu, \overline{m}_\mu)$ such that the metric tensor and the complex structure tensor take the form

$$g_{\mu \nu} = \ell_\mu n_\nu + n_\mu \ell_\nu - m_\mu \overline{m}_\nu - \overline{m}_\mu n_\nu$$

$$J^\mu_\nu = i(\ell_\mu n_\nu - n_\mu \ell_\nu - m_\mu \overline{m}_\nu + \overline{m}_\mu n_\nu)$$

(1.3)

The integrability condition of this complex structure implies the Frobenius integrability conditions of the pairs $(\ell_\mu, m_\mu)$ and $(n_\mu, \overline{m}_\mu)$.
\begin{align}
(\ell^\mu m^\nu - \ell^\nu m^\mu)(\partial_\mu \ell_\nu) = 0 \quad & , \quad (\ell^\mu m^\nu - \ell^\nu m^\mu)(\partial_\nu \ell_\mu) = 0 \\
(n^\mu m^\nu - n^\nu m^\mu)(\partial_\mu n_\nu) = 0 \quad & , \quad (n^\mu m^\nu - n^\nu m^\mu)(\partial_\nu n_\mu) = 0
\end{align}
\tag{1.4}

That is, only metrics with two geodetic and shear free congruences \((\kappa = \sigma = \lambda = \nu = 0)\) admit an integrable complex structure.

Frobenius theorem states that there are four complex functions 

\[ z^\alpha = (z^\alpha, z^{\bar{\alpha}}), \quad \alpha = 0, 1, \]

such that

\[ dz^\alpha = f_\alpha \ell^\mu dx^\mu + h_\alpha n^\mu dx^\mu, \quad dz^{\bar{\alpha}} = f_{\bar{\alpha}} n^\mu dx^\mu + h_{\bar{\alpha}} \mu^\mu dx^\mu \]  \tag{1.5}

These four functions are the structure coordinates of the (integrable) lorentzian complex structure. Notice that in the present case of lorentzian spacetimes, the coordinates \(z^{\bar{\alpha}}\) are not complex conjugate of \(z^\alpha\), because \(J^\nu_{\mu} \) is no longer a real tensor. It is exactly this complex property of \(J^\nu_{\mu}\) that implies the pair "particle" and "antiparticle".

Using these structure coordinates, a metric independent action takes the simple form

\[ I_G = \frac{1}{2} \int d^4z \det(g_{\alpha\beta}) \, g^{\alpha\beta} \gamma F_{\alpha\beta} + c. \text{ conj.} = \int d^4z \, F_{j01} F_{j0\bar{1}} + c. c. \]
\[ F_{j\mu\nu} = \partial_\mu A_{j\nu} - \partial_\nu A_{j\mu} - \gamma f_{jik} A_{ia} A_{kb} \]  \tag{1.6}

This transcription is possible because the metric takes the simple form \(ds^2 = 2g_{\alpha\beta} dz^\alpha dz^{\bar{\beta}}\) in the structure coordinate system.

The covariant null tetrad form of this action [17] is

\[ I_G = \int d^4x \, g^{\mu\nu} \{(\ell^\mu m^\nu F_{j\mu\nu}) (n^\rho \mu^\rho F_{j0\rho} + (\ell^\nu m^\rho F_{j\nu\rho}) (n^\nu m^\rho F_{j0\rho})\}
\]
\[ F_{j\mu\nu} = \partial_\mu A_{j\nu} - \partial_\nu A_{j\mu} - \gamma f_{jik} A_{ia} A_{kb} \]  \tag{1.7}

where \(A_{j\mu}\) is an \(SU(N)\) gauge field and \((\ell^\mu, n^\mu, m^\mu, \mu^\mu)\) is the special integrable null tetrad [13]. The difference between the present action and the ordinary Yang-Mills action becomes more clear in its following form

\[ I_G = -\frac{1}{8} \int d^4x \, \sqrt{-g} \left( 2g^{\mu\nu} g^{\rho\sigma} - J^{\mu\nu} J^{\rho\sigma} - \frac{J^{\mu\nu} J^{\rho\sigma}}{J^{\mu\nu} J^{\rho\sigma}} \right) F_{j\mu\rho} F_{j\nu\sigma} \]  \tag{1.8}

where \(g_{\mu\nu}\) is a metric derived from the null tetrad [13] and \(J^{\mu\nu}\) is the corresponding tensor of the integrable complex structure.

In the case of the string action [14] we do not need additional conditions, because any orientable 2-dimensional surface admits a complex structure. But in the case of 4-dimensional surfaces, the integrability of the complex structure
has to be imposed through precise conditions. These integrability conditions may be imposed using the ordinary procedure of Lagrange multipliers

\[ I_C = \int d^4x \left\{ \phi_0(\ell^\mu m^\nu - \ell^\nu m^\mu)(\partial_\mu \ell_\nu) + \phi_1(\ell^\mu m^\nu - \ell^\nu m^\mu)(\partial_\mu m_\nu) + \phi_0(n^\mu \overline{m}^\nu - n^\nu \overline{m}^\mu)(\partial_\mu n_\nu) + \phi_1(n^\mu \overline{m}^\nu - n^\nu \overline{m}^\mu)(\partial_\mu \overline{m}_\nu) + \text{c.conj.} \right\} \]

(1.9)

This technique makes the complete action \[ I = I_G + I_C \] self-consistent and the usual quantization procedures may be applied.

The model has been quantized using the canonical and the BRST procedure. After the expansion around the trivial null tetrad (light-cone coordinates), the first order one-loop diagrams have been computed using a convenient dimensional regularization and they were found to be finite. The model does not contain any dimensional constant, therefore the number of the counterterms is finite. Their independence from the metric tensor assures that there will not be geometric term between them. Therefore no \( R^2 \) term will appear in the present action unlike the ordinary Yang-Mills action. This assures the formal renormalizability of the model.

The local symmetries of the action \[ I = I_G + I_C \] are a) the well known local gauge transformations, b) the reparametrization symmetry as it is the case in any generally covariant action and c) the following extended Weyl transformation of the tetrad

\[ \ell'_\mu = \Lambda \ell_\mu \quad n'_\mu = N n_\mu \quad m'_\mu = M m_\mu \]

\[ \ell'^\mu = \Lambda^{-1} \ell^\mu \quad n'^\mu = N^{-1} n^\mu \quad m'^\mu = M^{-1} m^\mu \]

\[ \phi'_0 = \phi_0 \Lambda^{-1}N^{-1}M^{-1} \quad \phi'_1 = \phi_1 \Lambda^{-1}N^{-1}M^{-1} \]

\[ g' = g(\Lambda N M \bar{M})^2 \]

(1.10)

where \( \Lambda, N \) are real functions and \( M \) is a complex one. It is larger than the ordinary Weyl (conformal) transformation. I will call this tetrad-Weyl transformation. Then the conventional metric (1.3) takes the form

\[ g_{\mu\nu} = \Lambda N(\ell_\mu n_\nu + n_\mu \ell_\nu) - M \overline{M}(m_\mu \overline{m}_\nu - \overline{m}_\mu m_\nu) \]

(1.11)

The following dimensionless geometric action term is invariant under the tetrad-Weyl transformation.

\[ I_g = k \int d^4x \sqrt{-g}(\ell n \partial m)(\ell n \partial \overline{m})(m \overline{m} \partial \ell)(m \overline{m} \partial n) = k \int d^4x \sqrt{-g}(\tau + \pi)(\tau + \pi)(\rho - \beta)(\rho - \mu) \]

(1.12)
In the first line the null tetrad compact notation \( (e_a e_b \partial e_c) \equiv (e_a^\mu e_b^\nu - e_a^\nu e_b^\mu) \partial_\mu e_c^\nu \) is used and in the second line the Newman-Penrose (NP) spin coefficient formalism is used. Notice that this term is not affected (annihilated) by the integrability conditions of the complex structure. For the sake of completeness, I will consider this term in the derivation of the field equations, despite the fact that such a geometric term could not be a counterterm. In any case even with this term the model is renormalizable because \( k \) is a dimensionless constant.

The mathematical formalism of the model is heavily based on the Newman-Penrose formalism and the geometric properties of the geodetic and shear free congruences. Therefore in any section of the present work I have to include a short mathematical introduction in order to make it understandable to the high energy theoretical physicists.

The most interesting direct phenomenological effects of the model are: a) The natural emergence of the Poincaré group, instead of the BMS group in Einstein gravity. b) The bosonic modes of the vacuum sector of the model have 12 independent variables like the Standard Model. c) The modified Yang-Mills action generates a linear confining potential, instead of the Coulomb-like \( \frac{1}{r} \) potential of the ordinary action. Therefore the "colored" vacuum and solitonic excitations are perturbatively confined. d) The solitonic sector is separated into 3+1 "lepton families" and their confined excited colored "quark families".

It is well known that the equality of the inertial and gravitational mass is not an accident of nature. I think that the Einstein derivation of the equations of motion and the fermionic gyromagnetic ratio \( g = 2 \) of the Kerr-Newman manifold are not accidents of nature either. I think that the present quantum field theoretic model provides the solitonic framework for these results. Throughout this presentation I will use an elementary particle terminology in quotes, in order to stress the analogy with current phenomenology.

2 FIELD EQUATIONS

The tetrad is the set of two real \( e_0^\mu = \ell_\mu \), \( e_\tilde{0}^\mu = n_\mu \) and a complex vector \( e_1^\mu = m_\mu \), \( e_\tilde{1}^\mu = \overline{m}_\mu \) which are linearly independent. Its inverse \((e_0^\mu)^{-1} = e_0^\nu\) is denoted with \( e_0^\nu = n^\nu \), \( e_\tilde{0}^\nu = \ell^\nu \), \( e_1^\nu = -\overline{m}^\nu \), \( e_\tilde{1}^\nu = -m^\nu \). Every tetrad \( e_\mu^a \) defines a metric \( g_{\mu\nu} \) relative to which the tetrad is null. The metric (1.3) has the form \( g_{\mu\nu} = \eta_{ab} e_\mu^a e_\nu^b \) with

\[
\eta_{ab} = \eta^{ab} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}
\]

Notice that the tetrad defines the metric and the precise form of \( \eta_{ab} \) makes the tetrad null.
The NP spin coefficients\cite{14} are defined by the following relations

\[
\nabla_\mu \ell_\nu = (\gamma + \tau) \ell_\mu \ell_\nu - \tau\ell_\mu m_\nu + (\varepsilon + \tau)n_\mu \ell_\nu - \kappa n_\mu m_\nu m_\nu
- \rho m_\mu m_\nu - (\alpha + \beta)m_\mu \ell_\nu + \pi m_\mu m_\nu m_\nu + \rho m_\mu m_\nu + \pi m_\mu m_\nu + (\alpha + \beta)m_\mu n_\nu - \lambda m_\mu m_\nu
\]

\[
\nabla_\mu n_\nu = -(\gamma + \tau)\ell_\mu n_\nu + \nu\ell_\mu m_\nu + \pi\ell_\mu m_\nu - (\varepsilon + \tau)n_\mu n_\nu + \pi n_\mu m_\nu + \pi n_\mu m_\nu + (\alpha + \beta)m_\mu n_\nu - \lambda m_\mu m_\nu
- \mu n_\mu m_\nu - \lambda m_\mu m_\nu
\]

\[
\nabla_\mu m_\nu = \nu\ell_\mu m_\nu - \tau\ell_\mu n_\nu + (\gamma - \tau)\ell_\mu m_\nu + \pi n_\mu \ell_\nu - \kappa n_\mu n_\nu + \pi n_\mu m_\nu - \mu n_\mu m_\nu + \rho m_\mu n_\nu + (\beta - \alpha)m_\mu m_\nu
- \lambda m_\mu m_\nu + \sigma m_\mu n_\nu + (\beta - \alpha)m_\mu m_\nu
\]

For their computation, it is easier to use the relations

\[
\begin{align*}
\alpha &= \frac{1}{2} [m\ell m - (\ell m) + (\ell m) - (\ell m) - 2(\ell m) - 2(\ell m)] \\
\beta &= \frac{1}{2} [m\ell m - (\ell m) - (\ell m) - 2(\ell m)] \\
\gamma &= \frac{1}{2} [m\ell m - (\ell m) - (\ell m) + 2(\ell m)] \\
\varepsilon &= \frac{1}{2} [m\ell m - (\ell m) - (\ell m) + 2(\ell m)] \\
\mu &= \frac{1}{2} [m\ell m - (\ell m) - (\ell m) - 2(\ell m)] \\
\pi &= \frac{1}{2} [m\ell m - (\ell m) - (\ell m)] \\
\rho &= \frac{1}{2} [m\ell m - (\ell m) - (\ell m)] \\
\tau &= \frac{1}{2} [m\ell m - (\ell m) - (\ell m)] \\
\kappa &= (\ell m) - \sigma = (\ell m) \\
\lambda &= - (\ell m)
\end{align*}
\]

(2.3)

where the symbols (...\) have been previously defined. Notice that if the spin coefficients are defined with the last relations (2.3), they do not depend on a precise metric. This definition will be adopted in the present work.

We will also use below the following commutation relations

\[
D \equiv \ell^\mu \partial_\mu , \quad \Delta \equiv n^\mu \partial_\mu , \quad \delta \equiv \ell^\mu \partial_\mu
\]

\[
[\Delta, D] = (\gamma + \tau) D + (\varepsilon + \tau) \Delta - (\pi + \tau) \delta - (\tau + \pi) \delta
\]

\[
[\delta, D] = (\pi + \beta - \pi) D + \kappa \Delta - (\mu + \varepsilon - \pi) \delta - (\pi + \varepsilon) \delta
\]

(2.4)

\[
[\delta, \Delta] = -\tau D + (\tau - \pi - \beta) \Delta + (\mu - \gamma + \pi) \delta - (\tau + \beta) \delta
\]

\[
[\delta, \Delta] = (\pi - \mu) D + (\pi - \rho) \Delta + (\alpha - \overline{\beta}) \delta + (\alpha - \overline{\beta}) \delta
\]

Besides the Weyl tensor $C_{\mu\nu\rho\sigma}$ and the Ricci $R_{\mu\nu}$ tensor are used through their
tetrad components. The Weyl tensor components are
\[ \Psi_0 = -C_{\mu\nu\rho\sigma} \ell^\mu m^\nu \ell^\rho m^\sigma, \]
\[ \Psi_1 = -C_{\mu\nu\rho\sigma} \ell^\mu n^\nu \ell^\rho m^\sigma, \]
\[ \Psi_3 = -C_{\mu\nu\rho\sigma} \ell^\mu m^\nu \ell^\rho n^\sigma, \]
\[ \Psi_4 = -C_{\mu\nu\rho\sigma} \ell^\mu n^\nu \ell^\rho n^\sigma, \]
\[ \Psi_5 = -C_{\mu\nu\rho\sigma} n^\mu \ell^\nu \ell^\rho \ell^\sigma. \]

All the formulas of the NP formalism may be found in the modern books of general relativity\[^{[5]}\]. This formalism is very adequate to the study of the geodetic and shear free congruences (\(\kappa = 0 = \sigma\)) and (\(\nu = 0 = \lambda\)) determined by the vectors \(\ell^\mu\) and \(n^\mu\) respectively.

Variation of the action relative to the Lagrange multipliers gives back the complex structure integrability conditions. Using the NP spin coefficients they take the form
\[ \kappa = \sigma = \lambda = \nu = 0 \]

Variation of the action with respect to the gauge field \(A_{j\mu}\) gives the field equations
\[
D_\mu (\sqrt{-g}(\ell^\mu m^\nu - \ell^\nu m^\mu)(n^\nu \ell^\mu F_{j\rho\sigma}) + (n^\nu \ell^\mu - n^\mu \ell^\nu) (\ell^\rho m^\sigma F_{j\rho\sigma})) + \]
\[ + (\ell^\mu m^\nu - \ell^\nu m^\mu)(n^\nu m^\sigma F_{j\rho\sigma}) + (n^\nu m^\mu - n^\mu m^\nu)(\ell^\rho m^\sigma F_{j\rho\sigma}) \] = 0

where \(D_\mu = \delta_\mu^j \partial_\mu + \gamma f_{\ell jk} A_{k\mu}\) is the gauge symmetry covariant derivative and \(\gamma\) the coupling constant. In order to simplify the relations, I made the bracket notations \((e_a e_b F_j) \equiv e_a^\mu e_b^\nu F_{j\mu\nu}\) for the gauge field components. Multiplying with the null tetrad, these equations take the form
\[
\{m^\mu D_\mu + \pi - 2\tau\} (\ell m F_j) + \{\ell m D_\mu + \pi - 2\alpha\} (\ell m F_j) = 0
\]
\[
\{m^\mu D_\mu + 2\beta - \tau\} (n m F_j) + \{\ell m D_\mu + 2\beta - \tau\} (n m F_j) = 0
\]
\[
\{\ell^\mu D_\mu + 2\pi - \mu\} (n m F_j) + \{n^\mu D_\mu + \mu - 2\gamma\} (\ell m F_j) = 0
\]

I put in brackets \(\{\ldots\}\) the covariant derivatives of the primary quantities relative to the tetrad-Weyl transformations. The integrability conditions of these field equations are satisfied identically.

Variation of the action \(I = I_G + I_C + I_g\) with respect to the tetrad, gives PDEs on the Lagrange multipliers. In order to preserve the relations between the tetrad and its inverse (the covariant and contravariant forms of the tetrad) we will use the identities
\[
\delta e^\mu_a = e^\lambda_b [-n^a \delta \ell_\lambda - \ell^a \delta m_\lambda + m^a \delta \ell_\lambda] \]
\[
\delta \sqrt{-g} = \sqrt{-g} [n^\lambda \delta \ell_\lambda + \ell^\lambda \delta m_\lambda - \ell^\lambda \delta \ell_\lambda - m^\lambda \delta m_\lambda] \]
Variation with respect to $\ell_\lambda$ gives the PDEs

$$\{m^\mu \partial_\mu + 3\beta - 2\tau + \pi\} \phi_0 + \{m^\mu \partial_\mu + 3\bar{\beta} - 2\bar{\tau} + \alpha\} \phi_0 - 2(nmF_j)(n\bar{m}F_j) + k(\tau + \pi)(\tau + \pi)(\bar{\mu} - \mu)^2 = 0$$

$$\{\ell^\mu \partial_\mu + 3\varepsilon + \pi - \rho\} \phi_0 + \phi_1[\tau + \pi] + (\ell nF_j)(\ell \bar{m}F_j) - k\{m^\mu \partial_\mu + \pi + \alpha + \bar{\beta} - 2\bar{\tau}\}[\tau + \pi](\tau + \pi)(\bar{\mu} - \mu) = 0$$

and the conserved current (integrability condition)

$$\nabla_\lambda\{\ell^\lambda[2(nmF_j)(n\bar{m}F_j) + \phi_0(\tau - \alpha - \beta) + \bar{\phi}_0(\tau + \beta)] + n^\lambda k(\tau + \pi)(\tau + \pi)(\bar{\mu} - \mu) + m^\lambda [(\ell nF_j)(\ell \bar{m}F_j) + \phi_0(\varepsilon + \tau) + \phi_1[\tau + \pi] + k(\tau + \pi)(\tau + \pi)(\bar{\mu} - \mu) + \bar{m}^\lambda [(\ell \bar{m}F_j)(n\ell F_j) + \phi_0(\varepsilon + \tau) + \phi_1(\tau + \pi) - k(\tau + \pi)(\tau + \pi)(\bar{\mu} - \mu)(\tau - \alpha - \bar{\beta})] = 0$$

Variation with respect to $n_\lambda$ gives the PDEs

$$\{m^\mu \partial_\mu - 3\alpha + 2\pi - \bar{\beta}\} \phi_0 + \{m^\mu \partial_\mu - 3\bar{\alpha} + 2\bar{\pi} - \beta\} \bar{\phi}_0 - 2(\ell mF_j)(\ell \bar{m}F_j) + k(\tau + \pi)(\tau + \pi)(\bar{\mu} - \mu)^2 = 0$$

$$\{n^\mu \partial_\mu - 3\gamma - \tau + \mu\} \phi_0 - \phi_1[\tau + \pi] - (\ell nF_j)(\ell mF_j) + k\{m^\mu \partial_\mu + 2\bar{\tau} - \tau - \bar{\alpha} - \beta\}(\tau + \pi)(\tau + \pi)(\bar{\mu} - \mu) = 0$$

and the corresponding conserved current is

$$\nabla_\lambda\{\ell^\lambda k(\tau + \pi)(\tau + \pi)(\bar{\mu} - \mu) + n^\lambda [2(\ell mF_j)(\ell \bar{m}F_j) + \phi_0(\alpha + \bar{\beta} - \pi) + \bar{\phi}_0(\bar{\alpha} + \beta - \pi)] - m^\lambda [(\ell nF_j)(\ell \bar{m}F_j) + \phi_0(\gamma + \bar{\pi}) + \phi_1(\tau + \pi) - k(\tau + \pi)(\tau + \pi)(\bar{\mu} - \mu)(\alpha + \beta - \pi)] - \bar{m}^\lambda [(\ell \bar{m}F_j)(\ell mF_j)] + \phi_0(\gamma + \tau) + \phi_1(\tau + \pi) + k(\tau + \pi)(\tau + \pi)(\bar{\mu} - \mu)(\tau + \pi)(\bar{\mu} - \mu) = 0$$

Variation with respect to $m_\lambda$ gives the PDEs

$$\{m^\mu \partial_\mu - 3\bar{\pi} + \beta + \pi\} \bar{\phi}_1 + \phi_0[\mu - \tau] - (\ell \bar{m}F_j)(m\ell F_j) - k\{\ell^\mu \partial_\mu + \varepsilon - \tau - 2\rho\}(\tau + \pi)(\bar{\mu} - \mu) = 0$$

$$\{m^\mu \partial_\mu + 3\beta - \tau - \rho\} \phi_1 + \phi_0[\rho - \bar{\tau}] - (\ell mF_j)(m\ell F_j) + k\{n^\mu \partial_\mu + 2\bar{\mu} + \mu + \gamma - \tau\}(\tau + \pi)(\bar{\mu} - \mu) = 0$$

$$\{\ell^\mu \partial_\mu + 3\varepsilon - 2\rho - \tau\} \phi_1 + \{n^\mu \partial_\mu - 3\bar{\tau} + 2\bar{\pi} + \gamma\} \bar{\phi}_1 - 2(\ell \bar{m}F_j)(\ell \bar{m}F_j) + k(\tau + \pi)^2(\bar{\mu} - \mu)(\bar{\mu} - \mu) = 0$$
and the corresponding conserved current is

$$\nabla_\lambda \{ \ell^\lambda [(m m F_j)(n m F_j) + \phi_0(\overline{\rho} - \rho) + \phi_1(\overline{\tau} - \beta) + k(\overline{\tau} + \pi)(\overline{\rho} - \rho)(\overline{\tau} - \gamma - \overline{\rho})] + n^\lambda [(m m F_j)(n m F_j) + \phi_0(\overline{\rho} - \mu) + \phi_1(\overline{\pi} - \beta) - k(\overline{\pi} + \mu)(\overline{\rho} - \rho)(\overline{\pi} - \epsilon + \rho)] - m^\lambda [2(\ell m F_j)(n m F_j) + \phi_1(\overline{\rho} - \epsilon + \rho) + \phi_1(\overline{\gamma} - \gamma - \overline{\rho})] + + m^\lambda k(\tau + \pi)(\tau - \rho)(\tau - \mu) \} = 0$$

(2.15)

The field equations of the model indicate the following process for their solution. One may first solve the pure geometric equations (2.6). These are the vacuum configurations and possible solitonic configurations with vanishing gauge field $F_{j\rho\sigma}$. I call this solitonic sector "leptonic". The form of the equations indicates that for each "leptonic" soliton, there may be solitons with non-vanishing gauge field $F_{j\rho\sigma}$. This solitonic sector will be called "hadronic". The reason for this name is the observation that the static potential of the gauge field equations (2.7) is linear[20] in $r$.

We consider the trivial spherical complex structure determined by the following (spherical) null tetrad in spherical coordinates $(t, r, \theta, \varphi)$

$$\ell_\mu = (1, -1, 0, 0)$$

$$n_\mu = \frac{1}{2} (1, 1, 0, 0)$$

$$m_\mu = \frac{1}{r \sqrt{2}} (0, 0, 1, i \sin \theta)$$

(2.16)

with its contravariant coordinates

$$\ell^\mu = (1, 1, 0, 0)$$

$$n^\mu = \frac{1}{2} (1, -1, 0, 0)$$

$$m^\mu = \frac{1}{r \sqrt{2}} (0, 0, 1, \frac{i}{\sin \theta})$$

(2.17)

If we expand the gauge field into the null tetrad

$$A_{j\mu} = B_{j1} \ell_\mu + B_{j2} n_\mu + B_{j3} m_\mu + B_{j4} m_\mu$$

(2.18)

we find the gauge field components $B_{j1}$, $B_{j2}$, $B_{j3}$. In the present null tetrad, the conjugate momenta of $B_{j1}$, $B_{j2}$ vanish. Therefore we must assume $B_{j1} = 0 = B_{j2}$. Assuming the convenient gauge condition

$$\overline{m}^\nu \partial_\nu (r \sin \theta m^\mu A_{j\mu}) + m^\nu \partial_\nu (r \sin \theta m^\mu A_{j\mu}) = 0$$

(2.19)

the field equation takes the form

$$\left( \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r^2} \right) (r m^\mu A_{j\mu}) = [source]$$

(2.20)

which apparently implies a linear "gluonic" potential for the field variable $(r m^\mu A_{j\mu})$. A more sophisticated calculation may be done[21].

9
3 THE POINCARE GROUP

In the spinor formalism the tetrad takes the form

\[ \ell^\mu = \frac{1}{\sqrt{2}} e^\mu_a \sigma_{A'A} T^{A'A} o^A \]

\[ n^\mu = \frac{1}{\sqrt{2}} e^\mu_a \sigma_{A'A} T^{A'A} \]

\[ m^\mu = \frac{1}{\sqrt{2}} e^\mu_a \sigma_{A'A} T^{A'A} o^A \] (3.1)

where \( e^\mu_a \) is any vierbein, \((o^A, \iota^A)\) is a spinor dyad (basis) normalized by the condition \( o^A B \epsilon_{AB} = 1 \) and \( \sigma^0_{A'A} \) is the identity and \( \sigma^i_{A'A}, i = 1, 2, 3 \) are the ordinary Pauli matrices.

In the case of a flat spacetime and the cartesian coordinates the spinorial integrability conditions become the Kerr differential equations

\[
(\partial_0' \lambda) + \lambda(\partial_{0'} 1 \lambda) = 0 \quad \text{and} \quad (\partial_{1'} \lambda) + \lambda(\partial_{1'} 1 \lambda) = 0 \] (3.2)

where \( \xi^A = [1, \lambda] \) and the spinorial notation is used with

\[ x^{A'} = x^\mu \sigma_{A'A} = \begin{pmatrix} x^0 + x^3 \\ (x^1 - ix^2) / x^0 - x^3 \end{pmatrix} \]

\[ x_{A'} = \begin{pmatrix} x^0 - x^3 \\ -(x^1 + ix^2) / x^0 + x^3 \end{pmatrix} \] (3.3)

\[ \partial_{A'} = \frac{\partial}{\partial x^{A'}} = \sigma^\mu_{A'A} \partial_\mu = \begin{pmatrix} \partial_0 + \partial_3 & \partial_1 - i \partial_2 \\ \partial_1 + i \partial_2 & \partial_0 - \partial_3 \end{pmatrix} \]

In this notation primed and unprimed indices are interchanged unlike the Penrose notation\[14\]. Kerr’s theorem states\[9\] that a general solution of these equations is any function \( \lambda(x^{A'B}) \), which satisfies a relation of the form

\[ K(\lambda, x_{0'0} + x_{0'1} \lambda, x_{1'0} + x_{1'1} \lambda) = 0 \] (3.4)

where \( K(\cdot, \cdot, \cdot) \) is an arbitrary function.

In the general case, the integrability conditions can be formally solved too. In every coordinate neighborhood of the spacetime, the reality relations of the tetrad combined with \[13\] imply the following conditions

\[ dz^0 \wedge dz^1 \wedge dz^2 \wedge dz^3 = 0 \]

\[ dz^5 \wedge dz^0 \wedge dz^\bar{0} \wedge dz^\bar{3} = 0 \] (3.5)

\[ dz^5 \wedge dz^0 \wedge dz^\bar{0} \wedge dz^\bar{3} = 0 \]

for the structure coordinates \( z^b \equiv (z^\alpha, z^{\bar{\alpha}}), \alpha = 0, 1 \). Hence we may conclude that there are two real functions \( \rho_{11}, \rho_{22} \) and a complex one \( \rho_{12} \), defined in neighborhoods of \( \mathbb{C}^4 \), such that
\[ \rho_{11}(\overline{z^\alpha}, z^n) = 0 \quad , \quad \rho_{12}(\overline{z^\alpha}, \overline{z^\alpha}) = 0 \quad , \quad \rho_{22}(\overline{z^\alpha}, \overline{z^\alpha}) = 0 \quad (3.6) \]

Notice the special dependence of the defining functions on the structure coordinates. These conditions permit us to get off the weak notion of the lorentzian complex structure and pose it in the context of CR structure\cite{10} and real submanifolds of complex manifolds. The conditions \((3.6)\) define totally real submanifolds\cite{3} of \(\mathbb{C}^4\).

Let \(z^0 = u + iU\) and \(z^\alpha = v + iV\). Using the coordinates \((u, v, \zeta = z^1)\), the first real condition \(\rho_{11}(\overline{z^\alpha}, z^n) = 0\) determines \(U\) and the last condition \(\rho_{22}(\overline{z^\alpha}, \overline{z^\alpha}) = 0\) determines \(V\). In the Penrose terminology the scri+ boundary of the spacetime is \(J^+ = \{ v \to \infty \mid u, \zeta = \text{finite} \}\) and the scri- boundary is \(J^- = \{ u \to -\infty \mid v, \zeta = \text{finite} \}\). Asymptotic flatness of the complex structure has to be defined with the assumptions that \(\rho_{11}(\overline{z^\alpha}, z^n) = 0\) and \(\rho_{22}(\overline{z^\alpha}, \overline{z^\alpha}) = 0\) are compatible with flat spacetime geodetic and shear free congruences.

Penrose noticed\cite{14} that the general solution of the Kerr theorem takes the form
\[ \overline{X^m} E_{mn} X^n = 0 \quad , \quad K(X^m) = 0 \quad (3.7) \]
where \(X^n\) is an element of \(CP^3\) and \(K(X^m)\) is a homogeneous function. Therefore, for an asymptotically flat complex structure in \(J^+\) and \(J^-\) the conditions \((3.7)\) may take the form
\[ \overline{X^{m_1}} E_{mn} X^{n_1} = 0 \quad , \quad K_1(X^{m_1}) = 0 \]
\[ \overline{X^{m_1}} E_{mn} X^{n_2} = \Omega(X^{m_1}, X^{n_2}) \]
\[ \overline{X^{m_2}} E_{mn} X^{n_2} = 0 \quad , \quad K_2(X^{m_2}) = 0 \quad (3.8) \]
where \(X^{mi}\) are \(4 \times 2\) matrices of rank-2, in order to assure the non-degeneracy of the complex structure tensor and \(\Omega(X^{m_1}, X^{n_2})\) is a homogeneous function. The structure coordinates \(z^\alpha\) are then two independent functions of \(X^{m_1}/X^{m_2}\) and \(\overline{z^\alpha}\) are two independent functions of \(X^{m_2}/X^{m_1}\).

In order to make things clear I find necessary to make very brief review of the of the Grassmannian manifold \(G_{2,2}\). We consider the set of the \(4 \times 2\) complex matrices of rank 2
\[ T = \begin{pmatrix} T_1 \\ T_2 \end{pmatrix} \quad (3.9) \]
with the equivalence relation \(T \sim T'\) if there exists a \(2 \times 2\) invertible matrix \(S\) such that
\[ T' = TS \quad (3.10) \]
This is the \( G_{2,2} \) Grassmannian manifold with coordinates
\[
z = T_2 T_1^{-1} \quad (3.11)
\]
which completely determine the points of the set. The coordinates \( T \) are called homogeneous coordinates and the coordinates \( z \) are called projective coordinates. Under a general linear \( 4 \times 4 \) transformation
\[
\begin{pmatrix}
    T'_1 \\
    T'_2
\end{pmatrix} = \begin{pmatrix}
    A_{11} & A_{12} \\
    A_{21} & A_{22}
\end{pmatrix}
\begin{pmatrix}
    T_1 \\
    T_2
\end{pmatrix}
\quad (3.12)
\]
the inhomogeneous coordinates transform as
\[
z' = (A_{21} + A_{22} z) (A_{11} + A_{12} z)^{-1} \quad (3.13)
\]
which is called fractional transformation and it preserves the compact manifold \( G_{2,2} \).

The points of \( G_{2,2} \) with positive definite \( 2 \times 2 \) matrix
\[
\begin{pmatrix}
    0 & 0 \\
    0 & I
\end{pmatrix}
\quad (3.14)
\]
is the bounded \( SU(2, 2) \) classical domain\[15, 26\] because it is bounded in the general \( z \)-space and it is invariant under the \( SU(2, 2) \) transformation
\[
\begin{pmatrix}
    T'_1 \\
    T'_2
\end{pmatrix} = \begin{pmatrix}
    A_{11} & A_{12} \\
    A_{21} & A_{22}
\end{pmatrix}
\begin{pmatrix}
    T_1 \\
    T_2
\end{pmatrix}
\quad (3.15)
\]
\[
z' = (A_{21} + A_{22} z) (A_{11} + A_{12} z)^{-1}
\]
\[
A_{11}^\dagger A_{11} - A_{21}^\dagger A_{21} = I \quad , \quad A_{12}^\dagger A_{12} - A_{22}^\dagger A_{22} = 0
\]
\[
A_{12}^\dagger A_{22} - A_{11}^\dagger A_{21} = I
\]

The characteristic (Shilov) boundary of this domain is the \( S^1 \times S^3 = U(2) \) manifold with \( z^\dagger z = I \).

In the homogeneous coordinates
\[
\begin{pmatrix}
    H_1 \\
    H_2
\end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix}
    I & -I \\
    I & I
\end{pmatrix}
\begin{pmatrix}
    T_1 \\
    T_2
\end{pmatrix}
\quad (3.16)
\]
\[
T = \begin{pmatrix}
    T_1 \\
    T_2
\end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix}
    I & I \\
    -I & I
\end{pmatrix}
\begin{pmatrix}
    H_1 \\
    H_2
\end{pmatrix}
\]
because we have
\[
\begin{pmatrix}
    0 & I \\
    I & 0
\end{pmatrix} = \frac{1}{2} \begin{pmatrix}
    I & -I \\
    I & I
\end{pmatrix} \begin{pmatrix}
    I & 0 \\
    0 & -I
\end{pmatrix} \begin{pmatrix}
    I & I \\
    -I & I
\end{pmatrix}
\quad (3.17)
\]
and the positive definite condition takes the form
\[
\begin{pmatrix}
    H_1 \\
    H_2
\end{pmatrix} \begin{pmatrix}
    0 & I \\
    I & 0
\end{pmatrix} \begin{pmatrix}
    H_1 \\
    H_2
\end{pmatrix} > 0 \iff -i(r - r^\dagger) = y > 0 \quad (3.18)
\]
where the projective coordinate \( r_{A'B} = x_{A'B} + iy_{A'B} \) is defined as \( r_{A'B} = iH_2H_1^{-1} \), which implies \( H_2 = -irH_1 \) and

\[
    r = i(I + z)(I - z)^{-1} = i(I - z)^{-1}(I + z)
\]
\[
    z = (r - iI)(r + iI)^{-1} = (r + iI)^{-1}(r - iI) \tag{3.19}
\]

The fractional transformations which preserve the unbounded domain are

\[
    \begin{pmatrix}
    H_1' \\
    H_2'
    \end{pmatrix} = \begin{pmatrix}
    B_{11} & B_{12} \\
    B_{21} & B_{22}
    \end{pmatrix} \begin{pmatrix}
    H_1 \\
    H_2
    \end{pmatrix}
\]

\[
    r' = (B_{22} r + iB_{21})(B_{11} - iB_{12} r)^{-1} \tag{3.20}
\]

\[
    B_{11}B_{22} + B_{12}B_{21} = I, \quad B_{11}B_{21} + B_{12}B_{22} = 0
\]

In this "upper plane" realization of the classical domain, the homogeneous coordinates take the form \( X^{mi} \)

\[
    X^{mi} = \begin{pmatrix}
    \lambda^A_i \\
    -ir_{A'B} \lambda^B_i
    \end{pmatrix} \tag{3.21}
\]

and the characteristic boundary is the "real axis"

\[
    y = 0 \tag{3.22}
\]

The precise form of the surfaces implies that the asymptotically flat complex structures respect the \( SU(2,2) \) group. This group preserves the characteristic (Shilov) boundary \( (\Omega = 0) \) of the classical domain. From the Penrose conformal representation of the Minkowski spacetime, we know that it is a submanifold of this boundary. Permitting the spacetime to have a singularity at the point \( i^0 \) of the boundary, the \( SU(2,2) \) group is broken down to its Poincaré dilation group. We will see below how the scaling group is expected to be spontaneously broken.

Here I want to point out that in Einstein’s gravity (with the metric been the fundamental quantity) the asymptotically flat spacetimes belong to representations of the BMS group, which does not appear in nature.

4 THE TRAJECTORY OF A "LEPTON"

It is clear that a pure geometric solution \( F_{j\mu\nu} = 0 \) should be viewed the way Einstein-Infeld-Hoffman considered a spacetime in order to derive the equations of motion of two bodies. In this context the particle appears as a "concentrated gravity tube" of the Einstein tensor \( E_{\mu\nu} \) around a trajectory being the center-line of the tube. I want to point out that solitonic configurations must be regular. The equation of motion of two such "particles" is derived from the
self-consistency identity $\nabla_\mu E^{\mu\nu} \equiv 0$ on the two coordinate neighborhoods and the definitions of center of mass and the momenta. This great success generated the geometrodynamics ideas of Misner and Wheeler [11].

Newman has defined a complex trajectory determined by the annihilation of the asymptotic shear of a spacetime. Recall that a spacetime-solution of the present model has at least two geodetic and shear free congruences. Recently Newman and collaborators [2] derived equations of motion for this trajectory. The imaginary part of this complex trajectory has been related to the “spin” of the “particle”. They also rederived the peculiar result [4], [13] that an asymptotically Kerr-Newman spacetime has the electron gyromagnetic ratio $g = 2$.

In the formula (3.8) the Kerr conditions assure the annihilation of the shear of $\ell^\mu$ and $n^\mu$. One can easily check that if the $G_{2,2}$ homogeneous coordinates $X^{\alpha i}$ take the form

$$X^{\alpha i} = \left( \lambda^A_i \right)$$

where $\xi^i_{A'B}(\tau_i)$, $i = 1, 2$ are two complex trajectories in the Grassmannian manifold $G_{2,2}$, two Kerr functions are derived. A combination of this parametrization with the Grassmannian one (3.21) implies the two conditions det $[r_A^B - \xi^i_{A'B}(\tau_i)] = 0$ for the two linear equations $[r_A^B - \xi^i_{A'B}(\tau_i)]\lambda^B_i = 0$ to admit non-vanishing solutions. In this notation, the structure coordinates are

$$z^0 = \tau_1 \quad , \quad z^1 = \lambda_1^{01} \quad , \quad z^0 = \tau_2 \quad , \quad z^0 = -\lambda_2^{01}$$

I do not actually know whether all the Kerr function conditions are generated by complex trajectories, but it is clear that the present definition of the complex trajectories is equivalent to the Newman asymptotic definition. The $\xi^i_{A'B}(\tau_1)$ is the trajectory viewed from $J^+$ and the second $\xi^i_{A'B}(\tau_2)$ is the trajectory viewed from $J^-$. If these two trajectories coincide, then the lorentzian complex structure may be called "simple".

5 STATIC "LEPTONIC SOLITONS"

The knowledge of the Poincaré group permit us to look for stationary (static) axisymmetric solitonic complex structures, which will be interpreted as particles of the model with precise mass and angular momentum. In the case of vanishing gauge field, we may use the general solutions (3.8) to find special solutions which respect some symmetries. In this case the convenient coordinates are

$$z^0 = u + iU \quad , \quad z^1 = \zeta \quad , \quad z^0 = v + iV \quad , \quad z^0 = -W^2 \zeta$$

where $u = t - r$, $v = t + r$ and $t \in R$, $r \in R$, $\zeta = e^{i\phi} \tan \frac{\theta}{2} \in S^2$ are assumed to be the four coordinates of the spacetime surface. Assuming the definitions

$$z^0 = i \frac{X^{21}}{X^{01}} \quad , \quad z^1 = \frac{X^{11}}{X^{01}} \quad , \quad z^0 = i \frac{X^{32}}{X^{12}} \quad , \quad z^0 = -\frac{X^{02}}{X^{12}}$$
we look for massive solutions such that
\[ \delta X^m = i e^0 [P_0]^m X^n \]
where \( P_\mu = -\frac{i}{2} \gamma_\mu (1 + \gamma_3) \). It implies
\[ \delta X^{0i} = 0 , \quad \delta X^{1i} = 0 \]
\[ \delta X^{2i} = -i e^0 X^{0i} , \quad \delta X^{3i} = -i e^0 X^{1i} \]
The above definition of the structure coordinates implies
\[ \delta z^0 = e^0 , \quad \delta z^1 = 0 \]
\[ \delta \tilde{z}^0 = e^0 , \quad \delta \tilde{z}^1 = 0 \]
and consequently
\[ \delta u = e^0 , \quad \delta U = 0 \]
\[ \delta v = e^0 , \quad \delta V = 0 \]
\[ \delta \zeta = 0 , \quad \delta W = 0 \]
This procedure gives stable (time independent) solutions. We may look for solutions, which are “eigenstates” of the z-component of the spin too. That is they are axisymmetric. In this case the homogeneous coordinates must also satisfy the following transformations
\[ \delta X^m = i e^{12} [\Sigma_{12}]^m X^n \]
where \( \Sigma_{\mu\nu} = \frac{1}{2} \sigma_{\mu\nu} = \frac{1}{2} (\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu) \). That is we have
\[ \delta X^{0i} = -i e^{12} X^{0i} , \quad \delta X^{1i} = i e^{12} X^{1i} \]
\[ \delta X^{2i} = -i e^{12} X^{2i} , \quad \delta X^{3i} = i e^{12} X^{3i} \]
The above definition of the structure coordinates implies
\[ \delta \tilde{z}^0 = 0 , \quad \delta \tilde{z}^1 = i e^{12} \tilde{z}^1 \]
\[ \delta \tilde{z}^0 = 0 , \quad \delta \tilde{z}^1 = -i e^{12} \tilde{z}^1 \]
and consequently
\[ \delta u = 0 , \quad \delta U = 0 \]
\[ \delta v = 0 , \quad \delta V = 0 \]
\[ \delta \zeta = i e^{12} \zeta , \quad \delta W = 0 \]
A general solution, which satisfies these symmetries, is given by the relations

\[ U = U \left[ z^1 \bar{z}^1 \right], \quad V = V \left[ z^1 \bar{z}^1 \right] \]  
(5.11)

\[ W = W \left[ v - u - i(V + U) \right] \]

Looking for an actually symmetric static quadratic polynomial Kerr function, I found the following form

\[ Z^1 Z^2 - Z^0 Z^3 + 2a Z^0 Z^1 = 0 \]  
(5.12)

This Kerr function is generated by the static trajectory

\[ \xi^a (\tau) = (\tau, 0, 0, ia) \]  
(5.13)

The asymptotic flatness condition (3.8) implies

\[ U = -2a \frac{z^1 \bar{z}^1}{1 + z^1 \bar{z}^1}, \quad V = 2a \frac{z^1 \bar{z}^1}{1 + z^1 \bar{z}^1} \]  
(5.14)

A quite general solution is found if \( W W = 1 \) \((V + U) = 0\). In this case we have

\[ U = -2a \sin^2 \theta, \quad V = 2a \sin^2 \theta \]

\[ W = \frac{r - ia}{r + ia} e^{-2if(r)} \]  
(5.15)

One may easily compute the corresponding tetrad up to their arbitrary factors \( N_1, N_2 \) and \( N_3 \).

\[ \ell = N_1 \left[ dt - dr - a \sin^2 \theta \ d\varphi \right] \]

\[ n = N_2 \left[ dt + \left( \frac{r^2 + a^2 \cos^2 \theta - 2a \sin^2 \theta \ \frac{df}{d\tau}}{r^2 + a^2} \right) dr - a \sin^2 \theta \ d\varphi \right] \]  
(5.16)

\[ m = N_3 \left[ -ia \sin \theta \ (dt - dr) + (r^2 + a^2 \cos^2 \theta) d\theta + i(r^2 + a^2) \sin \theta d\varphi \right] \]

The corresponding projective coordinates are

\[ r_{0'0} = i \frac{X^{11} X^{12} - X^{11} X^{22}}{X^{11} X^{12} - X^{11} X^{22}} = \frac{z^0 + (z^0 - 2ia) z^1 \bar{z}^1}{1 + z^1 \bar{z}^1} \]

\[ r_{0'1} = i \frac{X^{01} X^{22} - X^{21} X^{02}}{X^{01} X^{22} - X^{21} X^{02}} = \frac{z^0 - \bar{z}^0 + 2ia \bar{z}^1}{1 + z^1 \bar{z}^1} \]

\[ r_{1'0} = i \frac{X^{31} X^{12} - X^{11} X^{32}}{X^{31} X^{12} - X^{11} X^{32}} = \frac{z^0 - \bar{z}^0 + 2ia \bar{z}^1}{1 + z^1 \bar{z}^1} \]

\[ r_{1'1} = i \frac{X^{01} X^{32} - X^{31} X^{02}}{X^{01} X^{32} - X^{31} X^{02}} = \frac{z^0 + (z^0 + 2ia) z^1 \bar{z}^1}{1 + z^1 \bar{z}^1} \]  
(5.17)

If these projective coordinates become a Hermitian matrix \( x_{A'A} \), then the complex structure is compatible with the Minkowski metric. Otherwise, it is a
curved spacetime complex structure. The form (5.15) has been chosen such that for \( f(r) = 0 \) the complex structure becomes compatible with the Minkowski metric.

I have already showed [18, 20] that this tetrad takes the following Kerr-Schild form (in the Lindquist coordinates)

\[
\ell_\mu = L_\mu, \quad m_\mu = M_\mu, \quad n_\mu = N_\mu + \frac{h(r)}{2(r^2 + a^2 \cos^2 \theta)} L_\mu \tag{5.18}
\]

where the null tetrad \((L_\mu, N_\mu, M_\mu, \overline{M}_\mu)\) determines the following integrable flat complex structure

\[
L_\mu dx^\mu = dt - dr - a \sin^2 \theta d\varphi
\]

\[
N_\mu dx^\mu = \frac{-1}{\sqrt{2(r^2 + a^2 \cos \theta)}} \left[ -ia \sin \theta (dt - dr) + (r^2 + a^2 \cos^2 \theta) d\theta + i \sin \theta (r^2 + a^2) d\varphi \right]
\]

\[
M_\mu dx^\mu = \frac{-1}{\sqrt{2(r^2 + a^2 \cos \theta)}} \left[ ia \sin \theta \left( \frac{r^2 + a^2}{r + ia \cos \theta} \right) \right]
\]

Notice that for \( h(r) = -2mr + e^2 \) the Kerr-Newman space-time is found. The Kerr-Schild form has the following NP spin coefficients

\[
\begin{aligned}
\alpha &= \frac{ia(1 + \sin^2 \theta)}{2 \sqrt{2(r^2 + a^2 \cos \theta)}} - r \cos \theta \\
\beta &= \frac{ia \sin \theta}{2 \sqrt{2(r^2 + a^2 \cos \theta)}} \\
\gamma &= \frac{a^2 + ia r \cos \theta + h}{2 \mu^2 (r^2 + a^2 \cos \theta)} + \frac{h'}{4 \mu^2}, \quad \epsilon = 0 \\
\mu &= \frac{1}{\sqrt{2(r^2 + a^2 \cos \theta)}} \\
\rho &= \frac{1}{\sqrt{2(r^2 + a^2 \cos \theta)}} \\
\kappa &= 0, \quad \sigma = 0, \quad \nu = 0, \quad \lambda = 0
\end{aligned}
\tag{5.20}
\]

The soliton form factor \( f(r) \) is expected to be fixed by Quantum Theory, but I have not yet found the precise procedure. The massive configuration (5.18) with spin \( S_z = ma = \frac{1}{2} \) has \( g = 2 \) gyromagnetic ratio. Notice that the fact that the lorentzian complex structure is a complex tensor implies that the complex conjugate structure defines an independent lorentzian complex structure with the same mass, which I will call ”antiparticle”. This natural differentiation between ”particles” and ”antiparticles” makes the complex structure more convenient than the metric to describe elementary particles. On the other hand we know that the common point \( i \) of the \( J^+ \) and \( J^- \) at infinity is a singular point[14]. This implies that these configurations will belong into two representations of the Poincaré group.

### 6 THE ”LEPTONIC FAMILIES”

The integrability conditions of the complex structure can be formulated in the spinor formalism. They imply that both spinors \( o^A \) and \( i^A \) of the dyad satisfy the same PDE.
\[ \xi^A \xi^B \nabla_{\xi A} \xi_B = 0 \]  \hspace{1cm} (6.1)

where \( \nabla_{\xi A} \) is the covariant derivative connected to the vierbein \( e_a^\mu \). The integrability condition of these relations is

\[ \Psi_{ABCD} \xi^A \xi^B \xi^C \xi^D = 0 \]  \hspace{1cm} (6.2)

In a curved spacetime, which admits a complex structure, the geodetic and shear free congruences are determined by the solutions of the above 4th degree polynomial, which satisfy the integrability conditions (6.1). Or vice-versa, the geodetic and shear free congruences must coincide with the principal directions of the Weyl spinor \( \Psi_{ABCD} \), because in this tetrad we have \( \Psi_0 = 0 = \Psi_4 \). Hence a spacetime with non-vanishing Weyl tensor may admit a limited number of complex structures and their classification coincides with the well-known Petrov classification restricted to spacetimes which admit two geodetic and shear free congruences. The number of \( \xi^A (x) \) sheets, that a regular manifold admits, is a topological invariant. Taking into account that we need two sheets \((\sigma^A, \nu^A)\) to determine a complex structure, we have the following four cases

- **Case I**: \( \Psi_1 \neq 0 \), \( \Psi_2 \neq 0 \), \( \Psi_3 \neq 0 \)
- **Case II**: \( \Psi_1 \neq 0 \), \( \Psi_2 \neq 0 \), \( \Psi_3 = 0 \)
- **Case III**: \( \Psi_1 \neq 0 \), \( \Psi_2 = 0 \), \( \Psi_3 = 0 \)
- **Case IV**: \( \Psi_1 = 0 \), \( \Psi_2 \neq 0 \), \( \Psi_3 = 0 \)

(6.3)

The type N spacetimes do not admit a complex structure.

The four sheets on the regular spacetimes are expected to generate branch "surfaces" where the geodetic congruences will pass from the one to the other. The well known ring singularity of the Kerr-like complex structures are such branch "surfaces".

## 7 VACUUM AND SOLITON SECTORS

Recall that the two dimensional \( \phi^4 \)-model\(^7\) has two vacua with \( \phi = \pm \frac{\mu}{\sqrt{\lambda}} \). It is well known that the vacuum configurations are periodic, while the soliton configurations are not periodic. This characteristic difference will be used in the present model. The kink configuration and its excitations satisfy the boundary conditions \( \phi_{kink}(\pm \infty, t) = \pm \frac{\mu}{\sqrt{\lambda}} \) and the antikink configuration the opposite ones.

The vacuum sector of the model are spacetimes which admit two geodetic and shear free congruences (GSFC) which become periodic after the identification of \( \mathcal{J}^+ \) and \( \mathcal{J}^- \). Minkowski spacetime and its smooth deformations satisfy this periodicity criterion. Non periodic spacetimes constitute the solitonic sectors of the model, which we will call "leptons".
We must be careful to apply the periodicity criterion to the lorentzian complex structure (the GSFC) and not a precise metric of the spacetime. It is well known that a mass term implies non-periodicity of the metric. But it does not mean that $\ell^\mu, n^\mu$ of the two GSFCs are not periodic up to a tetrad-Weyl and diffeomorphic transformation. Typical examples are the massive spherically symmetric metrics which are not periodic. But these spacetimes are equivalent to Minkowski spacetime up a tetrad-Weyl transformation. Hence their GSFCs are periodic.

In order to make things explicit the Kerr-Newman integrable null tetrad will be used as an example. Around $\mathcal{I}^+$ the coordinates $(u, w = \frac{1}{r}, \theta, \varphi)$ are used, where the integrable tetrad takes the form

$$\ell = du - a \sin^2 \theta \, d\varphi$$

$$n = \frac{1 - 2mw + c^2 w^2 + a^2 w^2}{2w^2(1 + a^2 w^2 \cos^2 \theta)} \left[ w^2 \, du - \frac{2(1 + a^2 w^2 \cos^2 \theta)}{1 - 2mw + c^2 w^2 + a^2 w^2} \, dw - aw^2 \sin^2 \theta \, d\varphi \right]$$

$$m = \frac{1}{\sqrt{2w(1 + a^2 w^2 \cos \theta)}} \left[ iaw^2 \sin \theta \, du - (1 + a^2 w^2 \cos^2 \theta) \, d\theta - \frac{2a}{r^2 - 2mw + c^2 + a^2} \, dr \right]$$

The physical space is for $w > 0$ and the integrable tetrad is regular on $\mathcal{I}^+$ up to a factor, which does not affect the congruence, and it can be regularly extended to $w < 0$. Around $\mathcal{I}^-$ the coordinates $(v, w', \theta', \varphi')$ are used with

$$dv = du + \frac{2(r^2 + a^2)}{r^2 - 2mr + c^2 + a^2} \, dr$$

$$dw' = -dw, \quad d\theta' = d\theta$$

$$d\varphi' = d\varphi + \frac{2a}{r^2 - 2mr + c^2 + a^2} \, dr$$

and the integrable tetrad takes the form

$$\ell = \frac{1}{w'} \left[ w'^2 \, dv - \frac{2(1 + a^2 w'^2 \cos^2 \theta')}{1 + 2mw + c^2 w'^2 + a^2 w'^2} \, dw' - aw'^2 \sin^2 \theta' \, d\varphi' \right]$$

$$n = \frac{1 + 2mw' + c^2 w'^2 + a^2 w'^2}{2(1 + a^2 w'^2 \cos^2 \theta')} \left[ dv - a \sin^2 \theta' \, d\varphi' \right]$$

$$m = \frac{1}{\sqrt{2w'(1 - iaw' \cos \theta')}} \left[ iaw'^2 \sin \theta \, dv - (1 + a^2 w'^2 \cos^2 \theta') \, d\theta' - \frac{2a}{r^2 - 2mw' + c^2 + a^2} \, dr \right]$$

The physical space is for $w < 0$ and the integrable tetrad is regular on $\mathcal{I}^-$ up to a factor, which does not affect the congruence, and it can be regularly extended to $w > 0$. If the mass term vanishes the two regions $\mathcal{I}^+$ and $\mathcal{I}^-$ can be identified and the $\ell^\mu$ and $n^\mu$ congruences are interchanged, when $\mathcal{I}^+ (\equiv \mathcal{I}^-)$ is crossed. When $m \neq 0$ these two regions cannot be identified and the complex structure cannot be extended across $\mathcal{I}^+$ and $\mathcal{I}^-$. 19
In the present model the two real $\ell^\mu$, $n^\mu$ and the complex $m^\mu$ vector fields of the tetrad characterize the lorentian complex structure. We already know that the excitation modes must belong into unitary representations of the Poincaré group. I have not yet found a formal definition of the excitation modes, but they must not have more than 12 independent variables. Notice that the Standard Model bosonic modes (Higgs’s particle, $\gamma$, $Z$, $W$) are exactly 12.

8 THE DILATION BREAKING MECHANISM

It has already pointed out\cite{12,1} that the explicit conditions $\rho_{11}(\overline{z^\alpha}, z^\alpha) = 0$, $\rho_{22}(\overline{\tilde{z}^\tilde{\alpha}}, \tilde{z}^\tilde{\alpha}) = 0$ and the corresponding holomorphic transformations $z^{\alpha} = f^\alpha(z^\alpha)$ and $\tilde{z}^{\tilde{\alpha}} = f^{\tilde{\alpha}}(\tilde{z}^\tilde{\alpha})$ which preserve the lorentian complex structure, are exactly those of the 3-dimensional CR structures\cite{?}. Therefore we may use the Moser procedure for the classification\cite{24} of the lorentian complex structures. For each hypersurface type CR nondegenerate structure we consider the following Moser expansions

\begin{align}
U &= z^{1}\overline{z^{1}} + \sum_{k \geq 2, j \geq 2} N_{jk}(u)(z^{1})^{j}(\overline{z^{1}})^{k} \\
N_{22} &= N_{32} = N_{33} = 0 \\
V &= \tilde{z}^{1}\overline{\tilde{z}^{1}} + \sum_{k \geq 2, j \geq 2} \tilde{N}_{jk}(v)(\overline{\tilde{z}^{1}})^{j}(\overline{\tilde{z}^{1}})^{k} \\
\tilde{N}_{22} &= \tilde{N}_{32} = \tilde{N}_{33} = 0
\end{align}

(8.1)

where $z^0 = u + iU$, $\tilde{z}^0 = v + iV$ and the functions $N_{jk}(u)$, $\tilde{N}_{jk}(v)$ characterize the lorentian complex structure. By their construction these functions belong into representations of the isotropy subgroup of SU(1, 2) symmetry group of the hyperquadric. Notice that the corresponding Moser chains are determined by $n^\alpha \frac{\partial}{\partial n^\alpha}$ and $\ell^\alpha \frac{\partial}{\partial \ell^\alpha}$.

These Moser expansions hide a dilation symmetry breaking, because the coefficients of the first term $z^{1}\overline{z^{1}}$ in the $U$ ($V$) expansion is assumed to be non-vanishing. This is implied by the nondegeneracy condition on CR structure. It is known that these coefficients are relative invariants of the corresponding CR structures. On the other hand these coefficients have the "length" dimension. Therefore, expanding a nondegenerate CR structure we have to fix these dimensional parameters, which implies scaling symmetry breaking.

In order to make things clear I will approach the same problem using the
dℓ = (ε + τ)n ∧ ℓ + (τ − α − β)m ∧ ℓ + (τ − β − δ)m ∧ ℓ + 
(ρ − p)m ∧ ℓ − n + m + m ∧ ℓ + (m ∧ ℓ)

dn = −(γ + τ)ℓ ∧ n + (α + 3 − β)m ∧ n + (τ + β + 3)m ∧ n + 
(μ − p)m ∧ ℓ + n + (n ∧ ℓ + m − ℓ ∧ m)

dm = (γ + τ)ℓ ∧ m + (ε − τ − δ)m ∧ m + (τ − β)m ∧ m − 
(τ + δ)ℓ ∧ n + m ∧ ℓ − σ ∧ m

It is integrable if κ = σ = λ = ν = 0.

Under tetrad Weyl transformations the Newman-Penrose spin coefficients[5] transform as follows

\[
\begin{align*}
\alpha' &= \frac{1}{N} \alpha + \frac{M}{N} \frac{\Lambda + \xi}{AN} (\tau + \pi) + \frac{1}{4M} \beta \ln \frac{\Lambda}{M^2} \\
\beta' &= \frac{1}{N} \beta + \frac{M}{N} \frac{\Lambda + \xi}{AN} (\tau + \pi) + \frac{1}{4M} \delta \ln \frac{\Lambda}{M^2} \\
\gamma' &= \frac{1}{N} \gamma + \frac{M}{N} \frac{\Lambda + \xi}{AN} (\tau + \pi) + \frac{1}{4M} \Delta \ln \frac{M}{N^2} \\
\rho' &= \frac{1}{N} \rho + \frac{M}{N} \frac{\Lambda + \xi}{AN} (\tau + \pi) + \frac{1}{4M} D \ln \frac{MN}{M^2} \\
\mu' &= \frac{1}{N} \mu + \frac{M}{N} \frac{\Lambda + \xi}{AN} (\mu + \rho) + \frac{1}{4M} \Delta \ln (M \Lambda) \\
\sigma' &= \frac{1}{N} \sigma + \frac{M}{N} \frac{\Lambda + \xi}{AN} (\mu + \rho) + \frac{1}{4M} D \ln (M \Lambda) \\
\pi' &= \frac{1}{N} \pi + \frac{M}{N} \frac{\Lambda + \xi}{AN} (\tau + \pi) + \frac{1}{4M} \Delta \ln (AN) \\
\tau' &= \frac{1}{N} \tau + \frac{M}{N} \frac{\Lambda + \xi}{AN} (\tau + \pi) + \frac{1}{4M} \Delta \ln (AN) \\
\kappa' &= \frac{1}{N} \kappa + \frac{M}{N} \sigma, \quad \sigma' = \frac{1}{N} \sigma, \quad \nu' = \frac{1}{N} \nu, \quad \lambda' = \frac{M}{N} \lambda
\end{align*}
\]

We see that (ρ − p), (μ − p), (τ + π) undergo the multiplicative transformations

\[
\begin{align*}
\rho' − \rho &= \frac{1}{M} (\rho − p) \\
\mu' − \mu &= \frac{1}{M} (\mu − p) \\
\tau' + \tau &= \frac{1}{M} (\tau + \pi)
\end{align*}
\]

It implies that the vanishing or not of (ρ − p), (μ − p), (τ + π) are relative invariants of the lorentzian complex structure. If these quantities vanish, the complex structure is kaehlerian, and the vectors of the null tetrad are hypersurface orthogonal. That is the lorentzian complex structure is trivial and apparently compatible with the Minkowski metric.

Taking into account the [length] dimensionality of (ρ − p), (μ − p), (τ + π) we may conclude that if the vacuum lorentzian complex structure has at least one of them which does not vanish, the scaling symmetry is broken.

As I have already pointed out, the tetrad-Weyl transformation should be considered the natural extension in four dimensions of the powerful two dimensional conformal transformation. Therefore it would be interesting to find an analogous formulation in four dimensions. Under tetrad-Weyl transformation a primary field \( \phi(x) \) of weight \( w = (w_1, w_2, w_3, w_4) \) transforms as

\[
\phi' = \Lambda^{w_1} N^{w_2} M^{w_3} \bar{M}^{w_4} \phi
\]
The covariant derivative is defined with the aid of two real \( Z_{1\mu} \), \( Z_{2\mu} \) and a complex vector field \( Z_{\mu} \) such that

\[
\hat{D}_\mu \phi = (\partial_\mu - w_1 Z_{1\mu} - w_2 Z_{2\mu} - w_3 Z_\mu - w_4 Z_{\mu}) \phi \tag{8.6}
\]

and their transformations are

\[
Z'_{1\mu} = Z_{1\mu} - \partial_\mu \Lambda, \quad Z'_{2\mu} = Z_{2\mu} - \partial_\mu N, \quad Z'_\mu = Z_\mu - \partial_\mu M \tag{8.7}
\]

The geometric combinations of the spin coefficients which satisfy the gauge transformations are

\[
Z_{1\mu} = (\theta_1 + \mu + \bar{\mu})\ell_\mu + (\varepsilon + \bar{\varepsilon})n_\mu - (\alpha + \bar{\alpha} - \beta - \bar{\beta})m_\mu - (\bar{\gamma} + \gamma)\bar{m}_\mu
\]

\[
Z_{2\mu} = - (\gamma + \bar{\gamma})\ell_\mu + (\theta_2 - \rho - \bar{\rho})n_\mu - (\pi - \alpha - \beta)\bar{m}_\mu
\]

\[
Z_\mu = (\gamma - \bar{\gamma} + \bar{\mu})\ell_\mu + (\varepsilon - \bar{\varepsilon} - \rho)\bar{n}_\mu - (\bar{\theta}_1 + \bar{\pi} - \tau)\bar{m}_\mu - (\beta - \bar{\alpha})\bar{m}_\mu \tag{8.8}
\]

where the additional geometric quantities are

\[
\theta_1 = n^\alpha \partial_\mu \ln \frac{\partial \rho_1}{\partial z_\alpha} \quad \theta_2 = \ell^\alpha \partial_\mu \ln \frac{\partial \rho_2}{\partial \bar{z}_\alpha} \quad \theta = \overline{m}^\alpha \partial_\mu \ln (\tau + \bar{\tau}) \tag{8.9}
\]

Notice their logarithmic dependence and that the field equations do not contain these quantities.

9 THE KAHLER AMBIENT MANIFOLD

The four real conditions \((3.6)\) imply that the spacetime, which admits an integrable lorentzian complex structure, is a CR manifold with codimension four. Following the ordinary procedure\[3\] we can find the corresponding four real forms. It is convenient to use the notation \( \partial f = \frac{\partial f}{\partial z^\alpha} dz^\alpha \) and \( \bar{\partial} f = \frac{\partial f}{\partial \bar{z}^\alpha} d\bar{z}^\alpha \). Assuming a restriction to the submanifold we find

\[
\ell = 2i \partial \rho_{11} |_S = i(\partial - \bar{\partial})\rho_{11} |_S = -2i \bar{\rho}_{11} |_S
\]

\[
n = 2i \bar{\partial} \rho_{22} |_S = i(\bar{\partial} - \partial)\rho_{22} |_S = -2i \rho_{22} |_S
\]

\[
m_1 = i(\partial + \bar{\partial} - \bar{\partial} - \partial)\rho_{12} \sqrt{\frac{\nu_{12}}{2}} |_S
\]

\[
m_2 = i(\partial + \bar{\partial} - \bar{\partial} - \partial)\rho_{12} \sqrt{-\frac{\nu_{12}}{2}} |_S
\]

These forms restricted on the manifold are real, because of \( d\rho_{ij} = 0 \) and the special dependence of each function on the structure coordinates \((z^\alpha, \bar{z}^\beta)\). The relations become simpler if we use the complex form

\[
m = m_1 + im_2 = 2i \partial \rho_{12} = -2i \bar{\rho}_{12} = i(\partial - \bar{\partial})\rho_{12} \tag{9.2}
\]
Notice that these forms coincide with the null tetrad up to a multiplicative factor. The tetrad-Weyl transformation is implied by the following permitted transformation
\[ \rho'_1 = \Lambda \rho_1, \quad \rho'_2 = N \rho_2, \quad \rho'_3 = M \rho_3 \] (9.3)
Where \( \Lambda, N \) and \( M \) are general functions which do not vanish in the definition neighborhood. The general CR transformation is actually restricted to a factor, because the dimension of the manifold coincides with its codimension 3.

In the interesting generic case, the conditions (9.6) define a maximally totally real submanifold \( S \) of \( \mathbb{C}^4 \). For every such condition and in the corresponding neighborhood of \( \mathbb{C}^4 \) we may define a kählerian metric, which turns out to be lorentzian on the manifold \( S \).

I consider the following Kaehler metric
\[ ds^2 = \sum_{a,b} \frac{\partial^2 \rho}{\partial z^a \partial \bar{z}^b} dz^a d\bar{z}^b \] (9.4)
where
\[ \rho = \rho_1 \rho_2 - \rho_3 \rho_4 \] (9.5)
A straightforward calculation gives
\[ f^a_{\bar{b}} = \rho_2 \frac{\partial^2 \rho_1}{\partial z^a \partial \bar{z}^b} + \frac{\partial \rho_1}{\partial z^a} \frac{\partial \rho_2}{\partial \bar{z}^b} + \frac{\partial \rho_2}{\partial z^a} \frac{\partial \rho_1}{\partial \bar{z}^b} + \rho_1 \frac{\partial^2 \rho_2}{\partial z^a \partial \bar{z}^b} - \frac{\partial \rho_1}{\partial z^a} \frac{\partial \rho_2}{\partial \bar{z}^b} - \frac{\partial \rho_2}{\partial z^a} \frac{\partial \rho_1}{\partial \bar{z}^b} \] (9.6)
On the surface \( (\rho_{ij} = 0) \) the metric takes the lorentzian form
\[ ds^2 \big|_S = 2(\frac{\partial \rho_1}{\partial z^a} \frac{\partial \rho_2}{\partial \bar{z}^b} + \frac{\partial \rho_2}{\partial z^a} \frac{\partial \rho_1}{\partial \bar{z}^b} - \frac{\partial \rho_1}{\partial z^a} \frac{\partial \rho_2}{\partial \bar{z}^b} - \frac{\partial \rho_2}{\partial z^a} \frac{\partial \rho_1}{\partial \bar{z}^b}) dz^a d\bar{z}^b = 2(\ell \otimes n - m \otimes \bar{m}) \] (9.7)
where \( \ell, n \) and \( m \) are defined in (9.1).

Using the \( G_{2,2} \) homogeneous coordinates \( X^{ni} \), the general defining relations take the form
\[ \rho = X^T \lambda - \begin{pmatrix} G_{11} & G_{12} \\ G_{12} & G_{22} \end{pmatrix} = 0 \] (9.8)
from which we find
\[ y^a = \frac{1}{2\sqrt{2}} [G_{22} N^a + G_{11} L^a - G_{12} M^a - G_{12} M^a] \] (9.9)
where \( y^a \) is the imaginary part of \( r^a = x^a + iy^a \) defined by the relation \( r_{A'B'} = r^a \sigma_a A'B' \) and the null tetrad is
\[ L^a = \frac{1}{\sqrt{2}} \lambda A1 \lambda B1 \sigma_a A'B' \quad N^a = \frac{1}{\sqrt{2}} \lambda A2 \lambda B2 \sigma_a A'B' \quad M^a = \frac{1}{\sqrt{2}} \lambda A1 \lambda B1 \sigma_a A'B' \]
\[ \epsilon_{AB} \lambda A1 \lambda B2 = 1 \] (9.10)
If we substitute the normalized $\lambda^{Ai}$ as functions of $r^a$, using the Kerr conditions $K_i(X^{mi})$, these relations turn out to be four real functions of $x^a$ and $y^a$. The implicit function theorem assures the existence of a solution $y^a = h^a(x)$ for (9.9). It is clear that in the context of the present model gravity is a manifestation of the penetration of the surface (spacetime) inside the classical domain and not of the metric tensor. There are spacetimes with nonvanishing curvature tensor which are compatible with a flat spacetime complex structure.

Notice that the structure coordinates $z^a$ are related to the $G_{2,2}$ projective coordinates $\tau^b$ with holomorphic transformations. Therefore in the case of the simple condition $X^1EX = 0$ we can always choose a tetrad-Weyl transformation such that the Kaehler metric takes the form

$$ds^2 = \frac{1}{2} \sum_{a,b} \partial^2 (\frac{-r^c -(r^c)^2}{6\tau^a \tau^b}) dr^a dr^b = \eta_{ab} dr^a dr^b \quad (9.11)$$

which apparently becomes the Minkowski metric on the surface $y^a = \text{Im}(r^a) = 0$.

Therefore we conclude that the spacetime with two geodetic and shear free congruences is always a submanifold of a Kaehler manifold. In fact it is a lagrangian submanifold of the corresponding symplectic manifold. This opens up a way to apply geometric quantization directly to the surfaces, without reference to the conventional Dirac or BRST quantization of the model.

The relation (9.9) implies that $y^ay^b\eta_{ab} < 0$ for any asymptotically flat spacetime. But taking into account the regularity of the surface and its holomorphic translation inside the classical (Siegel) domain with a complex time translation

$$\begin{pmatrix} \lambda^{Aj} \\ w^A_{ Bj} \end{pmatrix} = \begin{pmatrix} I & 0 \\ dI & I \end{pmatrix} \begin{pmatrix} \lambda^{Aj} \\ w^A_{ Bj} \end{pmatrix} \quad (9.12)$$

we may restrict the phase space (the Kaehler manifold) to the $SU(2, 2)$ classical domain. This permit us to define the necessary finite measure state line bundle.$^{[25]}$

10 PERSPECTIVES

Renormalizability seems to be the cornerstone for the unification of gravity with the other forces of nature. The Einstein action is not renormalizable, while actions with higher order derivatives are not unitary. The quite extended hope that superstring model would describe phenomenology was based on its consistency with Quantum Theory. On the other hand conventional Quantum Field Theoretic models, where every elementary particle is represented by a field with a corresponding quadratic term in the action, seems to have reached its limitations with the Standard Model. Any attempt to make it generally covariant introduces the metric which generates geometric counterterms in the action. My proposal is to skip from the metric to the complex structure of the spacetime. The present model is an example of a four dimensional renormalizable model
which depends on the lorentzian complex structure of the spacetime and not on its compatible metrics (??). This step opens up a new branch in four dimensional Quantum Field Theory, which is analogous to the two dimensional Conformal Field Theories.

The "colorless particles" of the model are special four dimensional open surfaces inside the $SU(2,2)$ classical domain. The surfaces with at least two geodetic and shear free congruences (that is an integrable tetrad) which are periodic (by identifying $J^+$ and $J^-$) represent the vacuum sector of the model. These are the 12 variables of the lorentzian complex structure arranged into representations of the Poincaré group. I think the most important step of this kind of complex structure based models would be the explicit (formal) derivation of the field representations of these modes. The non-periodic solitonic configurations constitute the "leptonic" sector of the model. The ordinary Kerr-Newman lorentzian complex structure cannot be an acceptable solitonic solution, because it is singular at $r = 0$. But a simple asymptotic calculation indicates that its mass and spin measure the non-periodicity of the tetrad.

Minkowski spacetime coincides with the characteristic (Shilov) boundary of the classical domain. The spin and the gravity of the surface measures how much deep inside the classical domain the surface penetrates. The lorentzian complex structure does not uniquely determine the metric of the surface. But the metric, which admits a lorentzian complex structure, determines it through the algebraic condition $\Psi_0 = \Psi_4 = 0$. I proved that every surface is a totally real lagrangian submanifold of a Kaehler (symplectic) ambient manifold. The permitted metrics of the surface are restrictions of corresponding Kaehler potentials. The consequences of the geometric quantization of some special surfaces is under investigation. My expectation is that this quantization will fix the spontaneous breaking of the tetrad-Weyl symmetry. That is, it will fix the Kaehler potential and subsequently the spacetime metric.

A static soliton generated by one simple complex trajectory represents a "particle" and its complex conjugate structure defines its "antiparticle". The $g = 2$ gyromagnetic ratio assures that the solitonic particle is fermionic. A surface with at least two coordinate neighborhoods and particle-like asymptotic behavior, represents the scattering of these "particles". Recall that it is exactly the Einstein point of view that generated the equations of motion. This implies that the Einstein equations should be viewed as the definition of the energy. But it is not yet clear how Quantum Mechanics breaks the tetrad-Weyl symmetry to provide a unique tetrad and subsequently metric to the surface.

Hence we conclude that the string model is not unique. Therefore the nonobservance of supersymmetric particles is not a setback to the process of unifying gravity with the other forces in nature. Besides the present model is more conventional and it has many more natural features than the string model.
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