Quantum Fourier Transform in a Decoherence-Free Subspace

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Quantum Fourier transform is of primary importance in many quantum algorithms. In order to eliminate the destructive effects of decoherence induced by couplings between the quantum system and its environment, we propose a robust scheme for quantum Fourier transform over the intrinsic decoherence-free subspaces. The scheme is then applied to the circuit design of quantum Fourier transform over quantum networks under collective decoherence. The encoding efficiency and possible improvements are also discussed.

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I. INTRODUCTION

Quantum Fourier transform (QFT) plays essential roles in various quantum algorithms such as Shor’s algorithms [1, 2] and hidden subgroup problems [3]. Inspired by the exponential speed-up of Shor’s polynomial algorithm for factorization [1], many people investigated the problem of efficient realization of QFT in a quantum computer [2, 3, 4]. Up to now, many improvements have been made. In [2], Moore and Nilsson showed that QFT can be parallelized to linear depth in a quantum network, and upper bound of the circuit depth was obtained by Cleve and Watrous [3] for computing QFT with a fixed error. In [4], the actual time-cost for performing QFT in the quantum network was examined. Further, Blais [5] designed an optimized quantum network with respect to time-cost for QFT.

In practice, the decoherence problem induced by the unavoidable coupling of quantum system with the environment have to be considered in circuit design for QFT over a quantum network. If no measure is taken, decoherence will destroy the encoded quantum information. Many methods have been proposed to suppress decoherence in a quantum system, among which, an important scheme is to encode the quantum information into the decoherence-free subspaces or subsystems (DFS) [10, 11, 12, 13, 14, 15, 16, 17, 18] of quantum system. Theoretically, DFSs are completely isolated from the noises [10, 11, 12]. A large amount of discussions about DFS have appeared in the literature [14, 15, 16, 17, 18]. In this paper, we will take advantage of the decoherence-free subspaces to develop a novel scheme for performing QFT in a quantum computer. The circuits designed in this way have the robustness against noise in the procedure of implementing QFT. The paper is organized as follows: in Sec.II, some notations and the preliminary knowledge on DFS and QFT will be reviewed; in Sec.III, general method will be introduced for implementing QFT in DFS; in sec.IV, circuits will be designed to perform QFT in the DFS of a quantum network with respect to weak collective decoherence (WCD) and strong collective decoherence (SCD) respectively; in Sec.V, the efficiency of the circuit and possible improvements will be discussed; finally, a conclusion will be made in Sec.VI.

II. NOTATIONS AND PRELIMINARIES

Suppose the quantum system $S$ under consideration is coupled to an environment $E$. The overall system is governed by the Hamiltonian in the form of [10]:

$$H_{SE} = \sum_{\alpha \in \Lambda} S_{\alpha} \otimes E_{\alpha},$$

(1)

where $S_{\alpha}(E_{\alpha})$, $(\alpha \in \Lambda)$ are operators acting on the state space $\mathcal{H}_S(\mathcal{H}_E)$ of $S(E)$, and the index set $\Lambda$ contains all the possible couplings between the system and the environment. Assume $S_{\alpha}(\alpha \in \Lambda)$ span a $\mathcal{A}$-closed associate algebra $\mathcal{A}$. According to [10], $\mathcal{A}$ is isomorphic to a direct sum of $d_J \times d_J$ complex matrix algebras, each with multiplicity $n_J$ [15, 16]:

$$\mathcal{A} \cong \bigoplus_{J \in \mathcal{J}} I_{n_J} \otimes \mathcal{M}(d_J, \mathbb{C}),$$

(2)

where the index set $\mathcal{J}$ labels all the irreducible components of $\mathcal{A}$. Correspondingly, the system Hilbert space $\mathcal{H}_S$ can be decomposed into a similar form:

$$\mathcal{H}_S = \bigoplus_{J \in \mathcal{J}} C^{n_J} \otimes C^{d_J}.$$

(3)

All the subsystem spaces $C^{n_J}$ ($J \in \mathcal{J}$) in the right hand side of Eq. (3) correspond to decoherence-free subsystems of the quantum system $S$. Particularly, $C^{n_J}$ gives
a decoherence-free subspace of the quantum system $S$ when $d_j = 1$.

Quantum network under collective decoherence (CD) provides a nice paradigm for the DFSSs. Roughly speaking, all qubits of a quantum network under CD are coupled to the environment in the same manner. In the literature\cite{10}, two types of CD, weak collective decoherence (WCD) and strong collective decoherence (SCD), are frequently discussed.

SCD is defined as the decoherence due to the interaction Hamiltonian

$$H_{SE} = \sum_{\alpha=x,y,z} S_\alpha \otimes E_\alpha,$$  \hspace{1cm} (4)

where $S_\alpha = \sum_{i=1}^{n} \sigma_\alpha^i$, and $\sigma_\alpha^i (\alpha = x, y, z)$ represents the Pauli matrix $\sigma_\alpha$ that corresponds to the local operation on the $i^{th}$ qubit.

If only one term appears in the right hand side of Eq.(4), i.e. the system is coupled to the environment only in one direction, the induced decoherence is called WCD. Without loss of generality, the Hamiltonian can be written as

$$H_{SE} = S_2 \otimes E_2.$$

Next, we give a brief description of QFT implemented over an $n$-qubit quantum network. Mathematically, the quantum Fourier transformation $QF_n$ can be expressed as \cite{8}: \hspace{1cm}

$$QF_n: |\phi\rangle \rightarrow \frac{1}{2^n} \sum_{\phi=0}^{2^n-1} e^{i2\pi\phi\phi}|\phi\rangle. \hspace{1cm} (6)$$

Denote the state of the quantum network by the qubit string $|s_n s_{n-1} \cdots s_1\rangle, (s_t \in \{0, 1\}, t = 1, 2, \cdots, n)$ in which the $t^{th}$ qubit is at the state $|s_t\rangle$. The transformation $QF_n$ can be realized by applying the following sequence of quantum gates (all the gate sequences in this paper are operated from the right to let one by one)

$$T^{(1)}T^{(2)} \cdots T^{(n-1)}T^{(n)},$$

where

$$T^{(k)} = P^{(1,k)}(\frac{\pi}{2^{k-1}})P^{(2,k)}(\frac{\pi}{2^{k-2}}) \cdots P^{(k-1,k)}(\frac{\pi}{2})H^{(k)}.$$

Eq.(5) includes two classes of elementary quantum gates, $H^{(k)}$ and $P^{(i,j)}(\theta)$. The local Hadamard gate $H^{(k)}$ represents

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \hspace{1cm} (9)$$

over $k^{th}$ qubit. The controlled-phase-shift gate $P^{(i,j)}(\theta)$ represents the action

$$P(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{i\theta} \end{bmatrix} \hspace{1cm} (10)$$

over $i^{th}$ (control) and $j^{th}$ (target) qubits.

In addition, there are three important elementary gates that will be used in this paper. The controlled-NOT gate $CN^{(i,j)}$ flips the $j^{th}$ qubit (target qubit) when the $i^{th}$ qubit (control qubit) is at the state $|1\rangle$, and nothing is done when the control qubit is at the state $|0\rangle$. The rotation gate $R^{(k)}(\alpha)$ realizes the unitary transformation over $k^{th}$ qubit:

$$|0\rangle \rightarrow \cos \alpha|0\rangle + \sin \alpha|1\rangle, \quad |1\rangle \rightarrow -\sin \alpha|0\rangle + \cos \alpha|1\rangle. \hspace{1cm} (11)$$

The controlled-rotation gate $CR^{(i,j)}(\beta)$ realizes the rotation $R^{(j)}(\beta)$ over $j^{th}$ qubit (target qubit) when the state on the $i^{th}$ qubit (control qubit) is $|1\rangle$, and nothing is done when the state on the $i^{th}$ qubit (control qubit) is $|0\rangle$.

More details about DFS and QFT can be found in ref.\cite{12, 13, 16} and the references therein.

**III. SCHEME FOR PERFORMING QFT IN A DFS**

In the following parts of this paper, we focus the study on implementing QFT over decoherence-free subspaces. If not claimed, the abbreviation DFS will indicate only the decoherence-free subspace. Suppose $H_{DFS}$ is an $n_J$ dimensional DFS of the quantum system $S$, then one can select $2^{[\log_2 n_J]}$ orthonormal states in $H_{DFS}$ to construct at most $[\log_2 n_J]$ new qubits. For clarity, we call these qubits logical-qubits, and the original qubits physical-qubits.

Next, we will discuss how to realize a robust QFT algorithm over these logical-qubits. Similar to \cite{8}, we define the QFT over a DFS by

$$QF_n(DFS): |\phi\rangle \rightarrow \frac{1}{2^n} \sum_{\phi=0}^{2^n-1} e^{i2\pi\phi\phi}|\phi\rangle,$$

where the states $|\phi\rangle = |s_n s_{n-1} \cdots s_1\rangle, s_t \in \{0, 1\}, t = 1, 2, \cdots, n$ are basis states of the logical-qubits.

The basic idea for realizing $QF_n(DFS)$ is as follows. Notice that the two classes of gates $H^{(k)}$ and $P^{(i,j)}(\theta)$ play the central roles in the QFT\cite{5, 12, 13, 16}. We will construct correspondingly two similar classes of quantum gates for implementing QFT in a DFS, denoted by one-logical-qubit gate $H^{(k)}_{DFS}$ acting on the $k^{th}$ logical-qubit and two-logical-qubit gate $P^{(i,j)}_{DFS}(\theta)$ acting on the $i^{th}$ and
quantum gates $T_{qubits}$. Similarly to the general QFT described in Eq. (6), we can realize $QF_{n(DFS)}$ by applying the following sequence of quantum gates

$$T^{(1)}_{DFS}T^{(2)}_{DFS} \cdots T^{(n-1)}_{DFS}T^{(n)}_{DFS}, \quad (13)$$

where

$$T^{(k)}_{DFS} = P^{(1,k)}_{DFS}(\frac{\pi}{2k-1})P^{(2,k)}_{DFS}(\frac{\pi}{2k-2}) \cdots P^{(k-1,k)}_{DFS}(\frac{\pi}{2})H^{(k)}_{DFS}. \quad (14)$$

Gate sequence (13) provides us the general strategy for designing a circuit to implement QFT over the DFS in a quantum system. Concretely, let $|\tilde{l}\rangle, l = 0, 1, \cdots, n_J - 1$ be $n_J$ orthonormal states in the DFS $H_{DFS}$, and $|\hat{l}\rangle, l = n_J, n_J + 1, \cdots, 2^n - 1$ be $2^n - n_J$ orthonormal states in the orthogonal complementary space of $H_{DFS}$. Then between $|\tilde{l}\rangle$ and the natural basis $|s_n,s_{n-1} \cdots s_2 s_1\rangle, s_t \in \{0,1\}, t = 1, 2, \cdots, n$ of space $H_{S}$ there exists an unitary transformation $U$, i.e.

$$|s_n,s_{n-1} \cdots s_2 s_1\rangle = U|\tilde{l}\rangle, \quad (15)$$

where

$$l = s_n \cdot 2^{n-1} + s_{n-1} \cdot 2^{n-2} + \cdots + s_1 \cdot 2^0. \quad (16)$$

Let $m$ be an integer no greater than $\log_2 n_J$. Here we choose $|\tilde{l}\rangle, l = 0, 1, \cdots, 2^m - 1$ to construct $m$ logical-qubits for performing $m$-qubit QFT over the DFS, and rewrite them as

$$|\tilde{l}\rangle = |\hat{s}_m \hat{s}_{m-1} \cdots \hat{s}_1\rangle, \hat{s}_t \in \{0,1\}, t = 1, 2, \cdots, m, \quad (17)$$

where

$$l = s_m \cdot 2^{m-1} + s_{m-1} \cdot 2^{m-2} + \cdots + s_1 \cdot 2^0. \quad (18)$$

Then

$$U^{-1}H^{(k)}U|\hat{s}_m \hat{s}_{m-1} \cdots \hat{s}_1\rangle = U^{-1}H^{(k)}|0 \cdots 0 s_m s_{m+1} 0 s_{k-1} \cdots s_1\rangle$$

$$= U^{-1} \frac{1}{\sqrt{2}}(0 \cdots 0 s_m s_{m+1} 0 s_{k-1} \cdots s_1)$$

$$+ (0 \cdots 0 s_m s_{m+1} s_{k-1} \cdots s_1)$$

$$= \frac{1}{\sqrt{2}}(|\hat{s}_m \hat{s}_{k+1} 0 s_{k-1} \cdots s_1\rangle$$

$$+ |\hat{s}_m \hat{s}_{k+1} 1 s_{k-1} \cdots s_1\rangle). \quad (19)$$

Similarly, we have

$$U^{-1}H^{(k)}U|\hat{s}_m \hat{s}_{m-1} \cdots \hat{s}_1\rangle = \frac{1}{\sqrt{2}}(|\hat{s}_m \hat{s}_{k+1} 0 s_{k-1} \cdots s_1\rangle$$

$$- |\hat{s}_m \hat{s}_{k+1} 1 s_{k-1} \cdots s_1\rangle), \quad (20)$$

and

$$U^{-1}P^{(i,j)}(\theta)|\hat{s}_m \hat{s}_{m-1} \cdots \hat{s}_1\rangle$$

$$= \left\{ \begin{array}{ll}
|\hat{s}_m \hat{s}_{m-1} \cdots \hat{s}_1\rangle, & \text{if } |\hat{s}_i \hat{s}_j\rangle \in \{ |0\rangle, |1\rangle, |i\rangle \} \\
\exp(\theta)|\hat{s}_m \hat{s}_{m-1} \cdots \hat{s}_1\rangle, & \text{if } |\hat{s}_i \hat{s}_j\rangle = |\hat{1}\rangle \\
\end{array} \right. \quad (21)$$

where $1 \leq i \neq j \leq m$.

From Eqs. (19, 21), we can see that if $U$ can be constructed by elementary gates, then $U^{-1}H^{(k)}U$ and $U^{-1}P^{(i,j)}(\theta)U$ are feasible realizations for the two gates $H^{(k)}_{DFS}$ and $P^{(i,j)}(\theta)_{DFS}$. Thus the realization of the unitary transformation $U$ is crucial for building the circuits to implement QFT in a DFS.

The remainder tasks, then, are to find the transformation $U$ in Eq. (13) and build a circuit to realize it. From the theory of universal quantum computation [20], any unitary operator can be constructed by a sequence of universal elementary gates. In most cases it is not easy to obtain such explicit decompositions. Whereas, as will be shown in the next section, it is possible to build up a circuit for $QF_{n(DFS)}$ over the quantum network under collective decoherence with a finite number of elementary gates.

IV. CIRCUITS FOR QFT OVER QUANTUM NETWORKS UNDER COLLECTIVE DECOHERENCE

A. The weak collective decoherence case

In the quantum networks under WCD, nontrivial DFS exists only when the original network has no less than two physical-qubits [16]. For the simplest case, the DFS in a two-qubit quantum network under WCD is spanned by the orthonormal states $|01\rangle$ and $|10\rangle$ [16], with which one can build up one logical-qubit, i.e.

$$|\hat{0}\rangle = |01\rangle, \quad (22)$$

and

$$|\hat{1}\rangle = |10\rangle. \quad (23)$$

For a $2n$-qubit quantum network under WCD, we use the orthonormal states $|s_n\rangle \otimes |s_{n-1}\rangle \otimes \cdots \otimes |s_1\rangle, (s_t \in \{0,1\}, t = 1, 2, \cdots, n)$, where $|s_t\rangle$ represents the $t^{th}$ logical-qubit extracted from the $(2t-1)^{th}$ and $(2t)^{th}$
physical-qubits, to construct the circuit for robust QFT. It can be verified that all these states are contained in the biggest DFS.

Over these logical-qubits, it is observed that \( H_{DFS}^{(k)} \) and \( P_{DFS}^{(i,j)}(\theta) \) can be directly constructed from a sequence of elementary gates as follows (the circuits are given in Fig.1 and Fig.2):

\[
H_{DFS}^{(k)} = CN^{(2k,2k-1)} H^{(2k)} CN^{(2k,2k-1)}, \tag{24}
\]

\[
P_{DFS}^{(i,j)}(\theta) = (CN^{(2i,2i-1)} CN^{(2j,2j-1)}) P^{(2i,2j)}(\theta) \times (CN^{(2i,2i-1)} CN^{(2j,2j-1)}).	ag{25}
\]

Let \( U_n = CN^{(2,1)} CN^{(4,3)} \ldots CN^{(2n,2n-1)} \). Observing that the term \( CN^{(2i,2i-1)} \) commutes with \( H^{(2k)} \) when \( t \neq k \) and commutes with \( P^{(2i,2j)}(\theta) \) when \( t \neq i \) or \( j \), we have

\[
U_n^{-1} H^{(2k)} U_n = CN^{(2k,2k-1)} H^{(2k)} CN^{(2k,2k-1)} = H_{DFS}^{(k)}, \tag{26}
\]

and

\[
U_n^{-1} P^{(2i,2j)}(\theta) U_n = (CN^{(2i,2i-1)} CN^{(2j,2j-1)}) P^{(2i,2j)}(\theta) \times (CN^{(2i,2i-1)} CN^{(2j,2j-1)}) = P_{DFS}^{(i,j)}(\theta). \tag{27}
\]

Therefore, we can choose \( U_n \) as the unitary transformation \( U \) in Eq. (15):

\[
U = U_n = CN^{(2,1)} CN^{(4,3)} \ldots CN^{(2n,2n-1)}. \tag{28}
\]

Consider the three-qubit QFT as a simple example, the transformation \( QF_{3(DFS)} \) can be realized by applying \( H_{DFS}^{(k)} \) and \( P_{DFS}^{(i,j)}(\theta) \) in the sequence as follows (see the circuit in Fig.3):

\[
H_{DFS}^{(1)} P_{DFS}^{(1,2)}(\frac{\pi}{2}) H_{DFS}^{(2)} P_{DFS}^{(1,3)}(\frac{\pi}{4}) P_{DFS}^{(2,3)}(\frac{\pi}{2}) H_{DFS}^{(3)}. \tag{29}
\]

![FIG. 1: Circuit for the gate \( H_{DFS}^{(k)} \) in quantum network under WCD. The element with \( \oplus \) corresponds to a controlled-NOT gate with control on the filled circle and target on the \( \Theta \). (In this paper, all the different logical-qubits are labelled by numbers in the first column of the figures, while the individual physical-qubits are labelled by the numbers in the second column.)](image)

![FIG. 2: Circuit for the controlled-phase-shift gate \( P_{DFS}^{(i,j)}(\theta) \) over the \( i^{th} \) and \( j^{th} \) logical-qubits in quantum network under WCD.](image)

![FIG. 3: Circuit for realizing three-qubit QFT on a six-physical-qubit quantum network under WCD. \( \theta = \frac{\pi}{4} \); the gates \( H_{DFS}^{(k)} \) and \( P_{DFS}^{(i,j)}(\theta) \) are those given in Fig.1 and Fig.2.](image)

**B. The strong collective decoherence case**

It is more complicated to design the circuit for QFT over quantum networks under SCD than WCD. The corresponding condition for the existence of a DFS is more critical. Quantum network with four physical-qubits is of the smallest scale to ensure the existence of a nontrivial DFS, which is spanned by two orthonormal states

\[
|\tilde{0}\rangle = \frac{1}{\sqrt{2}} (|01\rangle - |10\rangle) \tag{30}
\]

and
where $\alpha$ is the circuits for the transformation according to Sec.III. $H$ (Fig.4): $U$ extracted from the $(4n)$-th logical-qubit network under SCD.

Under SCD are easy to be obtained (the corresponding $Q$-th physical-qubits, to construct $n$ logical-qubits in a $4n$-physical-qubit quantum network under SCD.

To perform QFT over the logical-qubits obtained above, it is still crucial to design the circuits for the corresponding two classes of gates $H_{DFS}$ and $P_{DFS}(\theta)$. Here we directly give the form of unitary transformation $U$ in Eq.15, then the gates $H_{DFS}$ and $P_{DFS}(\theta)$ are obtained according to Sec.III.

Let $U^{(k)}$ be an unitary transformation on the physical-qubits from $(4k-3)$-th to $(4k)$-th, which is realized by applying the sequence of elementary gates as follows (see the circuits for the transformation $U^{(k)}$ and its inverse in Fig.4):

$$U^{(k)} = \begin{vmatrix} C(4k,4k-2)C(4k,4k-3)C(4k,4k-4) \\ \times R(4k-2) \end{vmatrix}$$

where $\alpha = \pi - \arcsin \frac{1}{\sqrt{3}}$, $\beta_1 = - \pi + \arcsin \frac{1}{\sqrt{3}}$, $\beta_2 = - \frac{\pi}{4}$. Then, in a $4n$-physical-qubit quantum network, one of the feasible realization of the transformation $U$ is:

$$U = U^{(n)}U^{(n-1)} \cdots U^{(1)}$$

With the help of the unitary transformation $U$, the fundamental gates $H_{DFS}^{(k)}$ and $P_{DFS}^{(i,j)}(\theta)$ for performing $n$-qubit QFT over the DFS of a $4n$-qubit quantum network under SCD are easy to be obtained (the corresponding circuits are given in Fig.5 and Fig.6 respectively):

$$H_{DFS}^{(k)} = U^{-1}H^{(4k)}U = U^{(k)-1}H^{(4k)}U^{(k)},$$

$$P_{DFS}^{(i,j)}(\theta) = U^{-1}P^{(4i,4j)}(\theta)U$$

where the gates $H_{DFS}^{(k)}$ and $P_{DFS}^{(i,j)}(\theta)$ satisfy the requirements given section III:

$$H_{DFS}^{(k)}|0\rangle_k = \frac{1}{\sqrt{2}}(|0\rangle_k + |\bar{1}\rangle_k)$$

$$H_{DFS}^{(k)}|1\rangle_k = \frac{1}{\sqrt{2}}(|0\rangle_k - |\bar{1}\rangle_k)$$

V. THE EFFICIENCY AND OPTIMIZATION

The encoding efficiency of quantum algorithms over the DFS of an $n$-qubit quantum network, say $\eta(n)$, is defined as the ratio of the number of logical-qubits to that of physical-qubits. The efficiency depends on the selection of DFS and the way of building logical-qubits. From
FIG. 6: Circuit for the controlled-phase-shift gate \( P_{DFS}^{(i,j)}(\theta) \) over the \( i \)th and \( j \)th logical-qubits in quantum network under SCD.

In section III, it is obvious that the maximum encoding efficiency is:

\[
\eta_{\text{max}}(n) = \frac{\max_j \log_2(n_j)}{n}. \tag{38}
\]

It has been derived in \cite{12,16} that the efficiency \( \eta_{\text{max}}(n) \) of the quantum network under collective decoherence approaches to 1 when \( n \to \infty \). For the circuits we designed for QFT over the quantum network under collective decoherence, the encoding efficiency \( \eta(n) = \frac{1}{2} \) for WCD and \( \eta(n) = \frac{1}{4} \) for SCD. Therefore, it is possible to design a more efficient circuit for realizing QFT over the DFS of some quantum network under collective decoherence. However, our circuits are scalable for they are relatively easy to be realized for large scale robust QFT over quantum networks. Consequently, there is a trade-off between the encoding efficiency and circuit complexity.

For example, if we want to implement \( m \)-qubit QFT in a DFS of some quantum network under collective decoherence, then at least

\[
r = \min \{ n | \max_j \log_2(n_j) \geq m \} \tag{39}
\]

physical-qubits are required. The corresponding circuit for QFT over this \( r \)-qubit quantum network is the most efficient, but it will become much more complicated in using more elementary gates. The circuit design will be a formidable task.

VI. CONCLUSION

In this paper, strategies for performing QFT in a quantum network coupled with the environment are discussed. We propose a scheme for noise-isolated QFT over the decoherence-free subspaces. Following the scheme, circuits for implementing QFT are designed in quantum network under collective decoherence. The efficiency of these circuits and some possible improvements are discussed as well.

In the future, a general designing methodology needs to be found for more efficient QFT over arbitrary quantum network. Also, it is worthwhile to reduce the number of elementary gates using in the relevant quantum circuits. Moreover, it is interesting and useful to extend the problem from the decoherence-free subspaces to decoherence-free subsystems.

Acknowledgments

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