Non-resonance $D$-modules over arrangements of hyperplanes

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1 Introduction

The aim of this note is a combinatorial description of a category of $D$-modules over an affine space, smooth along the stratification defined by an arrangement of hyperplanes. These $D$-modules are assumed to satisfy certain non-resonance condition. The main result, see Theorem 4.1, generalizes [3].

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2 Quivers of an arrangement

2.1. Let $A = \{H_i\}, i \in I$ be a collection of affine hyperplanes in a complex affine space $X = \mathbb{C}^N$. Hyperplanes $H_i$ define on $X$ the structure of a stratified space which we denote by the same letter $A$ or by $X_A$. The closure of a stratum $X_\alpha \subset \mathbb{C}^N$ is an intersection of some hyperplanes

$$X_\alpha = \bigcap_{i \in J \subseteq I} H_i ,$$

and the stratum $X_\alpha$ is an interior of $\overline{X}_\alpha$:

$$X_\alpha = \overline{X}_\alpha \setminus \bigcup_{X_\beta \subseteq X_\alpha, X_\beta \neq X_\alpha} \overline{X}_\beta .$$

The combinatorial structure of the adjacencies of the strata can be described by a connected oriented graph $\Gamma_A$. The vertices $\alpha$ of the graph correspond to the strata $X_\alpha$, the set of the vertices we denote by $\text{ver}(A)$; an arrow $\alpha \rightarrow \beta \in \text{arr}(A)$ corresponds to the adjacency in codimension 1:

$$\alpha \rightarrow \beta \quad X_\alpha \supset X_\beta , \quad \text{codim} \overline{X}_\alpha \overline{X}_\beta = 1 .$$

Inclusions of closed strata define the partial order in the set $\text{ver}(A)$: $\alpha > \beta \iff \overline{X}_\alpha \supset \overline{X}_\beta \iff$ there exists an oriented path from $\alpha$ to $\beta$ in graph $\Gamma_A$. This partial order is in agreement with integervalued function $\text{codim} : \Gamma_A \Rightarrow \mathbb{N}$, $\text{codim} \alpha = \text{codim} \mathbb{C}^N \overline{X}_\alpha$ which means that $\alpha > \beta \Rightarrow \text{codim} \alpha < \text{codim} \beta$. The partial order allows to attach to any vertex $\alpha$ of the graph $\Gamma_A$ the full subgraph $\Gamma_A^\alpha \subset \Gamma_A$ whose vertices are the vertices $\beta \in \text{ver}(A)$ which satisfy the condition $\beta < \alpha$. One can easily see that the graph $\Gamma_A^\alpha$ describes the combinatorial structure of affine subspace $\overline{X}_\alpha$ with induced stratification. By $X_\emptyset$ we denote the the complement to all hyperplanes $H_i$ which is an open in $\mathbb{C}^N$ stratum.
Definition 2.1 A quiver $V_A = \{V_\alpha, A_{\alpha,\beta}\}$ of the arrangement $A$ is a collection of linear spaces $V_\alpha$ over $\mathbb{C}$, $\alpha \in \text{ver}(A)$ together with a collection of linear maps $A_{\alpha,\beta} : V_\beta \to V_\alpha$, $\alpha, \beta \in \text{ver}(A)$ such that $A_{\alpha,\beta} = 0$ unless $\alpha$ and $\beta$ are connected by an arrow ($A_{\alpha,\beta} \neq 0 \Rightarrow \alpha \to \beta$ or $\beta \to \alpha$), and which satisfy the following quadratic relations:

$$\sum_\beta A_{\alpha,\beta} A_{\beta,\gamma} = 0 \quad \text{for any } \alpha, \gamma \in \text{ver}(A), \alpha \neq \gamma.$$  \hspace{1cm} (2.2)

Remark. One can see that the relation (2.2) is nontrivial only in two cases:

(i) $|\text{codim } \alpha - \text{codim } \gamma| = 2$ and $\alpha > \gamma$ or $\alpha < \gamma$;
(ii) $\text{codim } \alpha = \text{codim } \gamma$, $\text{codim } X_\alpha X_\alpha \cap X_\gamma = 1$.

In the last case there are one or two summands in the relation (2.2).

The quivers $V_A$ compose the category $\text{Qui}(A)$ with natural morphisms. In other words, $\text{Qui}(A)$ is the category of finite-dimensional modules over graded algebra $B_A$. The algebra $B_A$ is generated by orthogonal idempotents $e_\alpha$, $\alpha \in \text{ver}(A)$ of degree 0, $\sum e_\alpha = 1$ and by degree one generators $a_{\alpha,\beta}$, $\alpha, \beta \in \text{ver}(A)$ such that $a_{\alpha,\beta} = 0$ if $\alpha$ and $\beta$ are connected by an arrow in $\Gamma_A$ and, moreover,

$$a_{\alpha,\beta} e_\gamma = \delta_{\beta,\gamma} a_{\alpha,\beta}, \quad e_\gamma a_{\alpha,\beta} = \delta_{\gamma,\alpha} a_{\alpha,\beta},$$

$$\sum_\beta a_{\alpha,\beta} a_{\beta,\gamma} = 0, \quad \alpha \neq \gamma.$$

Let $V_A = \{V_\alpha, A_{\alpha,\beta}\}$ be a quiver of the arrangement $A$. We denote by $V_A^t = \{V_\alpha^t, A_{\alpha,\beta}^t\}$ the following dual quiver:

$$V_\alpha^t = V_\alpha^*, \quad A_{\alpha,\beta}^t \equiv (-1)^{(\text{codim } \alpha + \text{codim } \beta - 1)/2} A_{\beta,\alpha}^*.$$ \hspace{1cm} (2.3)

The correspondence $V_A \to V_A^t$ defines the contravariant functor of duality

$$^t : \text{Qui}(A) \to \text{Qui}(A).$$

3 Quiver $D$-modules

3.1. For any stratum $X_\alpha$ we denote by $\Omega_\alpha$ one-dimensional space of codim $X_\alpha$-forms $\omega_\alpha \in \det T^*_X \alpha$, which are constant along $X_\alpha$, and by $\hat{\Omega}_\alpha$ denote one-dimensional space of top degree forms of the conormal bundle of $X_\alpha$, $\hat{\omega}_\alpha \in T^*_{(X_\alpha \setminus X)}$ constant along $X_\alpha$.

Let $V_A$ be a quiver of an arrangement $A$. We relate to it a $D_X$-module $EV_A$ by the following rules:

(i) $EV_A$ is generated by the linear space $\Theta_\alpha(V_\alpha \otimes \Omega_\alpha)$ as $D_X$-module,

(ii) for any $\alpha$ and for any vectorfield $L_\alpha$ along $X_\alpha$ with constant coefficients

$$L_\alpha(v_\alpha \otimes \omega_\alpha) = \sum_{\beta : \alpha \to \beta} \frac{L_\alpha(\hat{\omega}_\beta) \wedge \omega_\alpha}{\hat{\omega}_\beta \wedge \omega_\beta} A_{\beta,\alpha} v_\alpha \otimes \omega_\beta, \quad v_\alpha \in V_\alpha, \omega_\gamma \in \Omega_\gamma, \hat{\omega}_\gamma \in \hat{\Omega}_\gamma.$$ \hspace{1cm} (3.1)
For any $\alpha$ and for any affine function $f_\alpha$, $f_\alpha|_{X_\alpha} = 0$

$$f_\alpha(v_\alpha \otimes w_\alpha) = \sum_{\beta: \beta \rightarrow \alpha} \frac{df_\alpha \wedge \hat{\omega}_\beta \wedge \omega_\beta}{\hat{\omega}_\beta \wedge \omega_\beta} A_{\beta, \alpha} v_\alpha \otimes \omega_\beta, \quad v_\alpha \in V_\alpha, \omega_\gamma \in \Omega_\gamma, \hat{\omega}_\gamma \in \hat{\Omega}_\gamma, \quad (3.2)$$

In other words, $EV_A$ is a factor of free $D_X$-module $D_X \otimes (\oplus_\alpha V_\alpha \otimes \Omega_\alpha)$ modulo the relations $(3.1)$–$(3.2)$.

The closure $X_\alpha$ of any stratum is affine subspace of $X = \mathbb{C}^N$. We can choose local affine coordinates $\{z^\alpha, w^\alpha\} = \{z_1^\alpha, \ldots, z_{\text{dim}X_\alpha}^\alpha, w_1^\alpha, \ldots, w_{\text{codim}X_\alpha}^\alpha\}$ such that $X_\alpha$ is defined by the equations:

$$w_1^\alpha = w_2^\alpha = \ldots = w_{\text{codim}X_\alpha}^\alpha = 0. \quad (3.3)$$

**Proposition 3.1**

(i) $EV_A$ is holonomic $D_X$-module with regular singularities flat along the stratification $A$ (that is, $ss.EV_A \subset \bigcup_\alpha T^*_\alpha X_\alpha$);

(ii) The global sections of $EV_A$ are:

$$\Gamma(EV_A) \equiv \oplus_\alpha \mathbb{C}[\{z^\alpha\}, \{\frac{\partial}{w^\alpha}\}]V_\alpha \otimes \Omega_\alpha. \quad (3.4)$$

**Proof.** (ii) Due to the relations $(3.1)$ and $(3.2)$ the space of global sections is isomorphic to a factorspace of linear space

$$U = \oplus_\alpha U^\alpha = \oplus_\alpha \mathbb{C}[\{z^\alpha\}, \{\frac{\partial}{w^\alpha}\}]V_\alpha \otimes \Omega_\alpha.$$

So it is sufficient to define the structure of $D_X$ module in linear space $U$. We can equip $U$ with a structure of graded space by the relations $\deg (v_\alpha \otimes w_\alpha) = 0$, $\deg (z_i^\alpha) = \deg (\partial w_j^\alpha) = 1$ ($\partial w_j^\alpha = \frac{\partial}{w_j^\alpha}$) and define an action of the generators $x_i, \partial x_i$ of the ring $D_X$ by induction over the grading. The generators $\partial z_i^\alpha$ and $w_j^\alpha$ act on the elements of degree $0$ $v_\alpha \otimes \omega_\alpha$ according to the relations $(3.1)$ and $(3.2)$, whereas the generators $\partial w_j^\alpha$ and $z_i^\alpha$ act on them by free multiplication. If we already define the action of the generators $x_i, \partial x_i$ on the elements of degree $\leq (n - 1)$, the action of $x_i, \partial x_i$ on the elements of degree $n$ from $U_\alpha$ is uniquely defined by the following conditions:

(a) the elements $\partial w_j^\alpha$ and $z_i^\alpha$ act on subspace $U^\alpha$ by free multiplication, in particular,

$$[z_i^\alpha, z_k^\alpha] = [z_i^\alpha, \partial w_j^\alpha] = [\partial w_j^\alpha, \partial w_k^\alpha] = 0, \quad (3.5)$$

(b) the operators $\partial z_i^\alpha$ and $w_j^\alpha$ commute with $\partial w_j^\alpha$ and $z_i^\alpha$ according to the relations in $D_X$, that is,

$$[\partial z_i^\alpha, \partial w_j^\alpha] = [w_j^\alpha, z_i^\alpha] = 0, \quad [\partial z_i^\alpha, z_k^\alpha] = \delta_{i,k}, \quad [w_j^\alpha, \partial w_k^\alpha] = -\delta_{j,k}. \quad (3.6)$$

We have to prove that this action satisfied all the relation in $D_X$, so we are to check the relations

$$[\partial w_i^\alpha, \partial w_j^\alpha] = [w_i^\alpha, w_j^\alpha] = [\partial w_i^\alpha, w_j^\alpha] = 0. \quad (3.7)$$

The relations $(3.5)$ and $(3.6)$ allow to reduce the check of $(3.7)$ to the space of degree zero, where it is valid due to $(3.1)$, $(3.2)$ and $(2.1)$ (this is a simple but crucial calculation).
(i) The description of the global sections of the module $EV_A$ enables one to describe the concrete good filtration of $EV_A$. Remind that a filtration $M_k$ of $D_X$-module $X$ is good, if it is an arrangement with a filtration $D_k$ of $D_X$ over the degree of differential operator: $D_kM_l \subset M_{k+l}; D_kM_l = M_{k+l}$ for all $l \geq l_0, k \geq 0$; and all $M_k$ are finite $O_X$-modules.

Let, as before $U^\alpha = \oplus_\alpha C[[\tau^\alpha], \{\frac{\partial}{\partial \tau^\alpha}\}]V_\alpha \otimes \Omega_\alpha$. We showed that, globally, $\Gamma(EV_A) = U = \oplus_\alpha U_\alpha$. We equip $U_\alpha$ with a filtration putting $\deg(V_\alpha \otimes \Omega_\alpha) = \text{codim}\alpha$, $\deg z^\alpha_i = 0$, $\deg \frac{\partial}{\partial \tau^\alpha} = 1$, $U^\alpha_k = \{m \in U_\alpha, \deg m \leq k\}$ and let $U_k = \oplus_\alpha U_k^\alpha$. Then, due to the relations (3.1) and (3.2), the filtration $0 \subset U_0 \subset \ldots \subset U$ is a good filtration. Moreover, the same relations show that the product

$$\prod_\alpha \left( \prod_i p_i^\alpha \cdot \prod_j w_j^\alpha \right)$$

annihilate $EV_A$ (here $p_i^\alpha = \text{gr} \frac{\partial}{\partial \tau^\alpha}$). It means that $EV_A$ is holonomic $D_X$ module with singular support contained in $\bigcup X^\alpha \Gamma(T^*_n|C^N)$. It has regular singularities due to (3.1), (3.2).

Thus we have a functor

$$E: Quii(A) \longrightarrow M(A), \quad (3.8)$$

where $M(A)$ is a category of holonomic RS $D_X$-modules flat among the stratification $A$. Note that the presentation of the module $EV_A$ allows one in addition easily describe the sections over the open sets $X_H = X \setminus H$

where $H$ is a hyperplane. Let, for instance, $H_\beta$ be a generic hyperplane which contains the stratum $X_\beta$. Then the global sections over $X^\beta_H$ constitute the linear space

$$U_\beta = \oplus_{\alpha: \beta \neq \alpha} U^\alpha = \oplus_\alpha C[[\tau^\alpha], \{\frac{\partial}{\partial \tau^\alpha}\}]V_\alpha \otimes \Omega_\alpha,$$

where the polynomial over $z^\alpha_i$ should be replaced by the regular functions of $\overline{X_\beta \cap U}$.

3.2. One can also localize the whole construction to the complement of certain hyperplanes, that is, take as an initial space $X$ the complement to some hyperplanes instead of affine space. For instance, the configuration $\{A, H\}$ of hyperplanes $A = \{H_i\}, i \in I$ in $C^N \setminus H$ defines the quivers which consist of linear spaces $V_\alpha, \alpha \in \Gamma_A$, together with linear maps $A_{\alpha, \beta}: V_\beta \rightarrow V_\alpha$, which are zero unless $\alpha$ and $\beta$ are connected by an arrow, and linear maps $N_\alpha: V_\alpha \rightarrow V_\alpha$ which are not zero only for the strata $X_\alpha$ which intersect with $H$ over codimension one. These operators are subjected to the relations:

$$\sum_\beta A_{\alpha, \beta} A_{\beta, \gamma} = 0 \quad \text{for any } \alpha, \gamma \in \text{ver}(A), \alpha \neq \gamma. \quad (3.9)$$

$$N_\beta A_{\alpha, \beta} = A_{\beta, \alpha} \left( N_\alpha + \sum_{\gamma \neq \beta} A_{\alpha, \gamma} A_{\gamma, \alpha} \right), \quad \alpha \rightarrow \beta, \quad (3.10)$$

$$A_{\alpha, \beta} N_\beta = \left( N_\alpha + \sum_{\gamma \neq \beta} A_{\alpha, \gamma} A_{\gamma, \alpha} \right) A_{\alpha, \beta}, \quad \alpha \rightarrow \beta, \quad (3.11)$$

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The corresponding $D_X$-module is described by analogous relations:

\[ L_\alpha(v_\alpha \otimes \omega_\alpha) = \sum_{\beta : \alpha \to \beta} \frac{L_\alpha(\hat{\omega}_\beta) \wedge \omega_\alpha}{\hat{\omega}_\beta \wedge \omega_\beta} A_{\beta,\alpha} v_\alpha \otimes \omega_\beta + \]

\[ + \phi_{L_\alpha}(v_\alpha \otimes \omega_\alpha), \quad v_\alpha \in V_\alpha, \omega_\gamma \in \Omega_\gamma, \hat{\omega}_\gamma \in \hat{\Omega}_\gamma , \]

(3.12)

for any $\alpha$ and for any vectorfield $L_\alpha$ along $X_\alpha$ with constant coefficients

\[ f_\alpha(v_\alpha \otimes w_\alpha) = \sum_{\beta, \alpha \to \gamma} \frac{df_\alpha \wedge \hat{\omega}_\alpha \wedge \omega_\gamma}{\hat{\omega}_\alpha \wedge \omega_\gamma} A_{\beta,\alpha} v_\alpha \otimes \omega_\beta, \quad v_\alpha \in V_\alpha, \omega_\gamma \in \Omega_\gamma, \hat{\omega}_\gamma \in \hat{\Omega}_\gamma , \]

(3.13)

for any $\alpha$ and for any affine function $f_\alpha, f_\alpha|_{X_\alpha} = 0.$

The summand $\phi_{L_\alpha}(v_\alpha \otimes \omega_\alpha)$ is nonzero only for those $X_\alpha$ which intersect with $H$ over in the codimension 1 and looks like

\[ \phi_{L_\alpha}(v_\alpha \otimes \omega_\alpha) = \frac{L_\alpha(f)}{f} (N_\alpha v_\alpha \otimes \omega_\alpha - \frac{df \wedge \hat{\omega}_\alpha \wedge \omega_\gamma}{\hat{\omega}_\alpha \wedge \omega_\gamma} A_{\beta,\alpha} v_\alpha \otimes \omega_\beta, \quad v_\alpha \in V_\alpha, \omega_\gamma \in \Omega_\gamma, \hat{\omega}_\gamma \in \hat{\Omega}_\gamma , \]

(3.14)

Here $\omega_{Ho}$ is an arbitrary top form on $H \cap X_\alpha$ with constant coefficients, the symbol $\delta$ stands for a stratum which coincides with $X_\alpha + X_\gamma$; if $X_\alpha + X_\gamma$ is not a stratum then the corresponding summand in r.h.s. of the above formula is zero ($X_\alpha + X_\gamma$ means a sum of linear subspaces in linear space with an origin in $X_\alpha \cap X_\gamma$). The statement of Proposition remains true for such $D$-modules.

### 3.3. A free resolution of quiver $D$-modules.

Let $T_\alpha = T'_{\alpha} \oplus T''_{\alpha}$, $\dim T_\alpha = N$ be a direct sum of the space of the sections of tangent bundle $T_{X_\alpha}$ to a stratum $X_\alpha$ constant along $X_\alpha$ and of the space of sections of conormal bundle $T'_{(X_\alpha|\mathbb{C}^N)}$ constant along $X_\alpha$. In local coordinates $\bar{x}^\alpha, \bar{w}^\alpha$ (3.3) the spaces $T'_{\alpha}$ and $T''_{\alpha}$ are: $T'_{\alpha} = \mathbb{C} < \partial_{\bar{z}^\alpha} >, \ T''_{\alpha} = \mathbb{C} < dw^\alpha >$. In other way the space $T_\alpha$ can be identyf with the space of constant sections of tangent bundle $T'_{(X_\alpha|\mathbb{C}^N)}$ to the space of conormal bundle $T'_{(X_\alpha|\mathbb{C}^N)} \subset T_{\mathbb{C}^N}.$

There is a natural embedding $I_\alpha$ of the space $T_\alpha$ into the space of first order differential operators:

\[ I_\alpha : T_\alpha \to D_{\leq 1} . \]

In local coordinates (3.3) $I_\alpha(\partial_{\bar{z}_i}) = \frac{\partial}{\partial \bar{z}_i}$, $I_\alpha(dw^\alpha) = w^\alpha$. The map $I_\alpha$ does not depend on the chiose of affine coordinates $\bar{x}^\alpha, \bar{w}^\alpha$ satisfying (3.3).

For any quiver $V_\lambda$ the relations (3.1) and (3.2) define also a natural map

\[ Q_\lambda : T_\alpha \otimes V_\alpha \otimes \Omega_\alpha \to \oplus_\beta V_\beta \otimes \Omega_\beta \]

which looks as follows:

\[ Q_\lambda(\xi \otimes v_\alpha \otimes \omega_\alpha) = \sum_{\beta: \alpha \to \beta} \frac{\xi(\hat{\omega}_\beta) \wedge \omega_\alpha}{\hat{\omega}_\beta \wedge \omega_\beta} A_{\beta,\alpha} v_\alpha \otimes \omega_\beta, \quad \xi \in T'_{\alpha}, \hat{\omega}_\beta \in \hat{\Omega}_\beta , \]

(3.14)
\[ Q_\alpha(df_\alpha \otimes v_\alpha \otimes \omega_\alpha) = \sum_{\beta, \beta \to \alpha} \frac{df_\alpha \wedge \hat{\omega}_\beta \wedge \omega_\alpha}{\hat{\omega}_\beta \wedge \omega_\beta} A_{\beta, \alpha} v_\alpha \otimes \omega_\beta, \quad df_\alpha \in T^{\prime\prime}_\alpha, \hat{\omega}_\beta \in \hat{\Omega}_\beta. \] (3.15)

Let \( E^n V_A, n = 1, 2, \ldots, N \) be a free left \( D \)-module
\[ E^n V_A = \bigoplus \alpha D \otimes \left( \bigwedge^n (T_\alpha) \otimes V_\alpha \otimes \Omega_\alpha \right) \]
and \( d^{(n)} : E^n V_A \to E^{n-1} V_A \) be the following \( D \)-linear map of free \( D \)-modules:
\[ d^{(n)}(1 \otimes t_1 \wedge \ldots \wedge t_n \otimes v_\alpha \otimes \omega_\alpha) = \sum_{k=1}^n (-1)^{k+1} I_\alpha(t_k) \otimes t_1 \wedge \ldots \wedge \hat{t}_k \wedge \ldots \wedge t_n \otimes v_\alpha \otimes \omega_\alpha - \]
\[ - \sum_{k=1}^n (-1)^{k+1} \otimes t_1 \wedge \ldots \wedge \hat{t}_k \wedge \ldots \wedge t_n \otimes Q_\alpha(t_k \otimes v_\alpha \otimes \omega_\alpha). \] (3.16)

We state that \( d^{(n-1)}d^{(n)} = 0 \) and

**Proposition 3.2** \( (E^* V_A, d^{(\cdot)}) \) is a free resolution of \( D \)-module \( EV_A \).

**Proof.** The relation \( d^{(n-1)}d^{(n)} = 0 \) is equivalent to basic relations (2.2) on the operators \( A_{\alpha, \beta} \). The proof of the exactness of the complex is the same as for standard Koszul resolution of the ring \( O_X \).

**Proposition 3.3** The dual complex
\[ \bigwedge^N (T^*_C N)^{-N} \otimes \text{Hom}_D \left( E^* V_A, d^{(\cdot)} \right) \]
is a free resolution of the module \( EV_A^t \).

**Proof.** One can see that
\[ \text{Hom}_D(D \otimes \bigwedge^n (T_\alpha) \otimes V_\alpha \otimes \Omega_\alpha, D) \approx D \otimes \bigwedge^{N-n} (T_\alpha) \otimes V^*_\alpha \otimes \Omega_\alpha \otimes \bigwedge^N (T^*_C N) \] (3.17)
and the dual maps \( d^{(n)*} \) of right \( D \)-modules \( D \otimes \bigwedge^{N-n} (T_\alpha) \otimes V^*_\alpha \otimes \Omega_\alpha \otimes \bigwedge^N (T^*_C N) \) are of the form analogous to (3.16):
\[ d^{(n)*}(1 \otimes t_1 \wedge \ldots \wedge t_n \otimes v^*_\alpha \otimes \omega_\alpha \otimes \pi_\theta) = \sum_{k=1}^n (-1)^{k+1} I_\alpha(t_k) \otimes t_1 \wedge \ldots \wedge \hat{t}_k \wedge \ldots \wedge t_n \otimes v^*_\alpha \otimes \omega_\alpha \otimes \pi_\theta - \]
\[ - \sum_{k=1}^n (-1)^{k+1} \otimes t_1 \wedge \ldots \wedge \hat{t}_k \wedge \ldots \wedge t_n \otimes Q^*_\alpha(t_k \otimes v^*_\alpha \otimes \omega_\alpha) \otimes \pi_\theta , \]
where \( Q^*_\alpha \) is given by the relations (3.14), (3.13) with \( v^*_\alpha, v^*_\beta \) and \( A^*_{\beta, \alpha} \) substituted instead of \( v_\alpha, v_\beta \) and \( A_{\alpha, \beta}, \pi_\theta \) is an \( N \)-vector on \( C^N \). The passage from the right \( D \)-modules to the left \( D \)-modules gives the sign \((-1)^{(\text{codim } \alpha - \text{codim } \beta - 1)/2} \) before \( A^*_{\beta, \alpha} \).

**Corollary 3.1** The \( D \)-module \( EV_A^t \) is the dual to \( EV_A \).

Note that the appearence of \( N \)-vector in (3.17) instead of the top form shows that it is more natural to treat the quiver’s \( D \)-modules as right \( D \)-modules but the presence of affine structure in the stratified space allows one to pass from the left to right \( D \)-modules just by simple change of signes in their defining relations.
4 Weighted quivers and weighted $D$-modules

4.1. The Category $\text{Qui}_{\lambda}(\mathcal{A})$ of weighted quivers.

Let us assign to any hyperplane $H_i$ of the arrangement $\mathcal{A}$ a complex number (weight) $\lambda_i$, and set

$$\lambda_\alpha = \sum_{i: H_i \supset X_\alpha} \lambda_i$$

(4.1)

for any stratum $X_\alpha$ of the arrangement, and

$$\lambda_{\alpha, \beta} = \sum_{i: H_i \supset X_\beta, H_i \not\supset X_\alpha} \lambda_i$$

(4.2)

for any arrow $\alpha \to \beta$ of the graph $\Gamma_\mathcal{A}$. Let us fix a stratum $X_\alpha$. Then we can attach to this stratum a subalgebra $B^0_\alpha \subset B_\mathcal{A}$ of the path algebra $B_\mathcal{A}$ (see section 1) generated by the elements $A_{\beta, \alpha} = A_{\alpha, \beta} A_{\beta, \alpha}$, where $\beta$ runs all the vertices of $\Gamma_\mathcal{A}$ satisfying the condition $\alpha \to \beta$. The algebra $B^0_\alpha$ can be described as an algebra with generators $A_{\beta, \alpha}$ subject to quadratic relations

$$[\sum_{\beta, \alpha \to \gamma} A_{\alpha, \beta}, A_{\beta, \gamma}] = 0$$

(4.3)

which are enumerated by the flags $\alpha \to \delta \to \gamma$. One can easily see that for any quiver $V_\mathcal{A}$ the linear space $V_\alpha$ is equipped with a structure of $B^0_\alpha$ module. We define a subcategory $\text{Qui}_\lambda(\mathcal{A}) \subset \text{Qui}(\mathcal{A})$ as follows:

Definition 4.1 A quiver $V_\mathcal{A} \in \text{Qui}(\mathcal{A})$ belongs to the subcategory $\text{Qui}_\lambda(\mathcal{A})$ if for any stratum $X_\alpha$, the linear space $V_\alpha$ considered as a $B^0_\alpha$-module admits a finite filtration by $B^0_\alpha$-submodules with the factors isomorphic to $C_{\lambda, \alpha}$, where $C_{\lambda, \alpha}$ is one-dimensional $B^0_\alpha$-module in which the generators $A_{\beta, \alpha}$ act by multiplication over $\lambda_{\alpha, \beta}$:

$$\rho_{\lambda, \alpha}(A_{\beta, \alpha}) = \lambda_{\alpha, \beta} \cdot \text{Id}.$$  

(4.4)

In other words, the action of the operators $A_{\alpha, \beta}$ in the space $V_\alpha$ can be given in some basis by triangular matrices with diagonal $\lambda_{\alpha, \beta} \cdot \text{Id}$. The proposition below describes the properties of the category $\text{Qui}_\lambda(\mathcal{A})$.

Proposition 4.1 (i) The category $\text{Qui}_\lambda(\mathcal{A})$ is a full abelian subcategory of $\text{Qui}(\mathcal{A})$ closed under extensions;

(ii) The category $\text{Qui}_\lambda(\mathcal{A})$ is stable with respect to the involution $^t$;

(iii) The category $\text{Qui}_\lambda(\mathcal{A})$ has finite number of irreducibles; the irreducibles $L_{\lambda, \alpha}$ are labeled by the vertices $\alpha$ of the graph $\Gamma_\mathcal{A}$ and are supported on the subgraph $\Gamma^\alpha_\mathcal{A}$ of the closure of the stratum $X_\alpha$, that is $(L_{\lambda, \alpha})_\beta \neq 0 \Rightarrow \alpha > \beta$ and $(L_{\lambda, \alpha})_\alpha \neq 0$.

The conjecture that the category $\text{Qui}_\lambda(\mathcal{A})$ contains the projective covers and has homological dimension at most $N$ also makes sense but we did not try to prove it.

Proof. The statements (i) and (ii) follow directly from the definitions and from the description of the involution. The most nontrivial statement (iii) is based on the explicit
construction of an analog of Verma modules $M_{\lambda, \alpha} \subset \text{Qui}_\lambda (A)$ which we give below. The quiver $M_\lambda = M_{\lambda, \emptyset}$ has the unique irreducible factor $L_\lambda = L_{\lambda, \emptyset}$ and is characterized by the property

$$\text{Hom}_{\text{Qui}(A)} (M_\lambda, V_A) \approx \text{Hom}_{B_0^\emptyset} (C_{\lambda, \emptyset}, (V_A)_\emptyset)$$

(4.5)

for any quiver $V_A \in \text{Qui}(A)$. The $B_0^\emptyset$ character of $C_{\lambda, \emptyset}$ is defined by the relation (4.4):

$$\rho_{\lambda, \emptyset} (A_{\emptyset, i}, A_{i, \emptyset}) = \lambda_i \cdot \text{Id}.$$  

(4.6)

The construction of the quivers $M_{\lambda, \alpha}$ and of their irreducible factors uses the inclusions $j_\alpha : X_\alpha \hookrightarrow C_N$. The inclusion $j_\alpha$ defines the induced stratification $A_\alpha$ in affine space $X_\alpha$ which is described by the graph $\Gamma_\alpha$. It also defines the full exact functor (which we denote by the same letter)

$$j_\alpha : \text{Qui}(A_\alpha) \rightarrow \text{Qui}(A).$$

Thus the irreducible quiver $L_{\lambda, \alpha}$ with a support $\Gamma_\alpha$ is

$$L_{\lambda, \alpha} = j_\alpha (L_{\lambda^\alpha})$$

and the proof of the statement (iii) of the proposition follows by induction over the codimension of the support of the irreducibles using (4.5). Let us describe now the quiver $M_\lambda$.

**Construction of the quiver $M_\lambda$.**

The quiver $M_\lambda$ which represents the functor

$$F(V_A) = \text{Hom}_{B_0^\emptyset} (C_{\lambda, \emptyset}, (V_A)_\emptyset)$$

can be defined in terms of the flag complex $\square$.

**Definition 4.2** A (complete) flag $L_{\alpha_0, \ldots, \alpha_n}$ in the stratified space $X_A$ is a chain of the closures of strata

$$X_{\alpha_0} \supset X_{\alpha_1} \supset \ldots \supset X_{\alpha_n}, \quad \text{codim} X_{\alpha_j} = j.$$  

(4.8)

In terms of the graph $\Gamma_A$, the complete flag $L_{\alpha_0, \ldots, \alpha_n}$ is a chain $\alpha_0 \rightarrow \ldots \rightarrow \alpha_n$, $\alpha_0 = \emptyset$. The linear space $(M_\lambda)_\gamma$ of the quiver $M_\lambda$ we identify with the factorspace of linear space with a basis given by all flags $L_{\alpha_0, \ldots, \alpha_n}$, $\alpha_n = \gamma$ modulo the relations

$$\sum_{\beta: \alpha_k \rightarrow \beta \rightarrow \alpha_{k+1}} L_{\alpha_0, \ldots, \alpha_k, \ldots, \alpha_n} = 0$$

(4.9)

valid for all partial flags $\alpha_0 \rightarrow \ldots \rightarrow \alpha_k \rightarrow \alpha_{k+1} \rightarrow \ldots \rightarrow \alpha_n$, $\alpha_0 = \emptyset$, $\alpha_n = \gamma$, $\text{codim} \alpha_{k+1} = \text{codim} \alpha_k + 2$. For any arrow $\alpha \rightarrow \beta$ the linear operators

$$A_{\beta, \alpha} : (M_\lambda)_\alpha \rightarrow (M_\lambda)_\beta, \quad A_{\alpha, \beta} : (M_\lambda)_\beta \rightarrow (M_\lambda)_\alpha$$

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have the following form:

\[ A_{\beta,\alpha}(L_{\alpha_0,\ldots,\alpha_{n-1},\alpha}) = L_{\alpha_0,\ldots,\alpha_{n-1},\alpha,\beta}, \quad (4.10) \]

\[ A_{\alpha,\beta} L_{\alpha_0,\ldots,\alpha_n} = \sum_{k=0}^{n} (-1)^k \lambda_{\alpha_{n-k},\alpha_{n-k+1}} \sum_{\alpha'_{n-k+1},\ldots,\alpha'_{n-k+1},\alpha'_{n-k+1} \neq \alpha_j, \alpha'_{n-k+1} = \beta} L_{\alpha_0,\ldots,\alpha_{n-k},\alpha'_{n-k+1},\ldots,\alpha'_{n}}. \quad (4.11) \]

One can verify by linear algebra calculations that the operators (4.10) and (4.11) are correctly determined maps of corresponding factorspaces and define a quiver from \( Q_{\lambda}(A) \) satisfying the condition (4.5).

### 4.2. The category \( \mathcal{M}_\lambda(A) \) of weighted \( D \)-modules.

For any collection \( \lambda = \{\lambda_i, i \in I\} \) of weights we define also a category \( \mathcal{M}_\lambda(A) \subset \mathcal{M}(A) \) of weighted \( D \)-modules.

Let \( f_\gamma, \gamma \in \Gamma_A \) stands for a generic linear (affine) function over \( \mathbb{C}^N \), such that \( f_\gamma|_{X_\alpha} = 0 \).

Let \( \Phi_{f_\alpha}(M)|_{X_\alpha} \) be the restriction to the open part \( X_\alpha \) of affine subspace \( X_\alpha \) of \( D \)-module of vanishing cycles \( \Phi_{f_\alpha}(M) \). If \( M \in \mathcal{M}(A) \) then \( \Phi_{f_\alpha}(M)|_{X_\alpha} \) is a flat \( DX_\alpha \)-module which one can describe by a flat connection in trivial bundle over \( X_\alpha \) with regular singularities on \( X_\alpha \).

Let us fix for a moment a collection of generic functions \( f_\alpha, \alpha \in \Gamma_A \).

**Definition 4.3** A \( D \)-module \( M \) from \( \mathcal{M}(A) \) belongs to the category \( \mathcal{M}_\lambda(A) \), if for any stratum \( X_\alpha \) \( DX_\alpha \)-module \( \Phi_{f_\alpha}(M)|_{X_\alpha} \) admits \( DX_\alpha \)-filtration with factors isomorphic to the modules given by a flat connection in one-dimensional bundle with a form

\[ \nu_\alpha = \sum_{\beta : \alpha \rightarrow \beta} \lambda_{\alpha,\beta} d \log f_\beta \]

In other words, for some affine coordinates \( \alpha \)-coordinates \( D \)-module \( \Phi_{f_\alpha}(M)|_{X_\alpha} \) is given by a connection in trivial bundle with a form

\[ \nu_\alpha = \sum_{\beta : \alpha \rightarrow \beta} L_{\alpha,\beta} d \log f_\beta \]

where \( L_{\alpha,\beta} \) is a triangular matrix with constant coefficients and diagonal entries being equal to \( \lambda_{\alpha,\beta} \). One can see that the definition of the category \( \mathcal{M}_\lambda(A) \) does not depend on the choice of generic linear functions \( f_\alpha, \alpha \in \Gamma_A \) and depends only on the projection \( \tilde{\pi}(\lambda) \) of the collections of weights to \( \mathbb{C}/\mathbb{Z}, \tilde{\pi}(\lambda) = \exp(2\pi i \lambda) \).

**Definition 4.4** A collection \( \{\lambda\} \) of weights is called non-resonance if for any \( \alpha, \beta, \gamma, \delta \in \Gamma_A \), the following conditions hold:

\[ \lambda_{\alpha,\beta} \notin \mathbb{Z} \setminus \{0\} \quad \text{and} \quad \lambda_{\alpha,\beta} - \lambda_{\gamma,\delta} \notin \mathbb{Z} \quad (4.12) \]

The non-resonance property also essentially depends only on the exponents of weights in a sense that one can or cannot get the condition (4.12) by shifting some of \( \lambda_i \) by an integer. One of the main properties of weighted \( D \)-modules is given in the following theorem.
Theorem 4.1 For any non-resonance collection \( \{ \lambda \} \) of weights, the functor \( E \) establishes an equivalence of categories \( \mathrm{Qui}_\lambda (A) \) and \( \mathcal{M}_\lambda (A) \).

The statement of the theorem is a direct generalization of the results of [3]. Let us describe the scheme of the proof. It consists of the inductive (over the codimension of strata) construction of an inverse functor \( \tilde{E} : \mathcal{M}_\lambda (A) \to \mathrm{Qui}_\lambda (A) \) with simultaneous proof of basic statement that the functor \( E \) coincides with Beilinson’s functor [4] of gluing the \( D \)-modules. The functor \( \tilde{E} \) is as follows. The linear space \( (\tilde{E}M)_\alpha \) for \( M \in \mathcal{M}_\lambda (A) \) is isomorphic to the space of flat sections of \( \Phi_{fa}(M)|_{X_\alpha} \) and as \( B_\alpha^0 \)-module the space \( (\tilde{E}M)_\alpha \) is equivalent to \( \Phi_{fa}(M)|_{X_\alpha} \) as \( \pi_1(X_\alpha \setminus X_\alpha) \)-module, the operators \( A_{\alpha,\beta} \) and \( A_{\beta,\alpha} \), \( \alpha \to \beta \) are proportional to canonical maps \( \text{can} \) and \( \text{var} \) between \( D_{X_\alpha} \)-modules \( \Phi_{fa}(M)|_{X_\alpha} \) and \( \Psi_{fa}(M)|_{X_\alpha} \). Thus, the crucial calculations are the calculation of the vanishing cycles functor \( \Psi_{fa}(M)|_{X_\alpha} \) with the action of monodromy operator on it. These calculations were made in [3], under unipotent monodromy assumption, that is, for zero weights \( \lambda \).

As a result of a more general calculation of the specialization functors, we show in the next section that the vanishing cycles calculations remain valid also under a weaker non-resonance assumption. This allows to establish an equivalence of categories for non-resonance weighted \( D \)-modules as well.

5 Specialization of quivers and \( D \)-modules

5.1. Specialization of quivers.

Let \( X_\alpha \) be a stratum of stratification \( X_A \). Its closure \( \overline{X}_\alpha \) is an affine subspace of \( X = \mathbb{C}^N \), and the total space of normal bundle \( T_{\overline{X}_\alpha} X \) inherits the affine structure and can be naturally identified with a product of the affine space \( \overline{X}_\alpha \) and the linear space \( X/\overline{X}_\alpha : T_{\overline{X}_\alpha} X \approx \overline{X}_\alpha \times X/\overline{X}_\alpha \). Both \( \overline{X}_\alpha \) and \( X/\overline{X}_\alpha \) are stratified spaces: the strata of \( \overline{X}_\alpha \) are \( X_\beta, X_\beta \subset \overline{X}_\alpha, \beta \in \Gamma_A \), and the strata of \( X/\overline{X}_\alpha \) are the projections \( p_\alpha(X_\beta), \beta \in \Gamma_A, X_\beta \cap \overline{X}_\alpha \neq 0 \). Here \( p_\alpha : \mathbb{C}^N = X \to X/\overline{X}_\alpha \) is a natural projection. Let us denote strata of \( \overline{X}_\alpha \) as \( X_\beta \) and strata of \( X/\overline{X}_\alpha \) as \( X_\beta' \). Note that the stratification of \( X/\overline{X}_\alpha \) is linear: the closure of any stratum contains zero vector.

The stratification \( T_{\overline{X}_\alpha} A \) of the space \( T_{\overline{X}_\alpha} X \) identified as affine space with \( \overline{X}_\alpha \times X/\overline{X}_\alpha \) is described by certain direct products of strata of \( \overline{X}_\alpha \) and \( X/\overline{X}_\alpha \): \( (X_\beta' \times X_\beta'') \in T_{\overline{X}_\alpha} A \) if there exists such a stratum \( X_\beta \) of stratification \( A \) for \( \mathbb{C}^N \) that

\[
X_\beta' = \overline{X}_\alpha \cap X_\beta, \quad X_\beta'' = p_\alpha(X_\beta).
\]

One can see that the stratification \( T_{\overline{X}_\alpha} A \) is also described by intersections of hyperplanes \( \tilde{H}_i, i \in I : \tilde{H}_i = (H_i \cap \overline{X}_\alpha) \times p_\alpha(H_i) \). The passage from stratification \( A \) to \( T_{\overline{X}_\alpha} A \) is illustrated by a two-dimensional example below.
Since the stratification of the space $\overline{X}_\alpha$ and of the graph $\Gamma'_{\overline{X}_\alpha}$ is also defined on the weights. For a collection of weights $\alpha \in X_{\alpha} = X_{\beta', p_{\alpha}(X_{\beta})} = X_{\beta''}$. A map $S_{P_{\alpha}}$ is also defined on the weights. For a collection of weights $\{\lambda\} = \lambda_i = \lambda(H_i)$ of stratification $\mathcal{A}$ the collection $\{S_{P_{\alpha}}(\lambda)\}$ is defined by the rule $S_{P_{\alpha}}(\lambda)_i = \sum_{j: \tilde{H}_j = \tilde{H}_i} \lambda(H_i)$.

**Proposition 5.1** (i) The relations (5.2)–(5.3) define an exact functor

$$S_{P_{\alpha}} : Qui(\mathcal{A}) \to Qui(T_{\overline{X}_\alpha} \mathcal{A})$$

which commutes with the duality $^t$;

(ii) For any collection $\lambda$ of weights for the configuration $\mathcal{A}$,

$$S_{P_{\alpha}}(M_{\lambda}) = M_{S_{P_{\alpha}}(\lambda)} \ .$$

**Proof.** In order to check (i) it is sufficient to check that the operators defined by the relations (5.2) and (5.3) satisfy the relations (2.2) for the configuration $T_{\overline{X}_\alpha} \mathcal{A}$. These are
linear algebra calculations. Analogously, for the proof of (ii) one has to check the formulas (4.10), (4.11) or to use the functorial definition of the modules $M_\lambda$ and $M_{Sp_\lambda(\lambda)}$.

Let now $\overline{X}_{\alpha_1} \subset \overline{X}_{\alpha_2} \subset X = \mathbb{C}^N$ be a flag of inclusions of closed strata to $X$. Then the spaces $T_{\overline{X}_{\alpha_1}}(\overline{X}_{\alpha_2})$ and $T_{\overline{X}_{\alpha_1}} T_{\overline{X}_{\alpha_2}}(T_{\overline{X}_{\alpha_1}} X)$ are canonically isomorphic to a direct product $\overline{X}_{\alpha_1} \times \overline{X}_{\alpha_2}/\overline{X}_{\alpha_1} \times X/\overline{X}_{\alpha_2}$. Let us call it the space of flags of normal covectors and denote by $T_{\overline{X}_{\alpha_2}}(\overline{X}_{\alpha_2})$. It is a stratified space with a stratification analogous to described above. We denote it by $\overline{T}_{\overline{X}_{\alpha_2}}(\overline{X}_{\alpha_2})$. One can easily see that the compositions

$$Sp_{\alpha_1} Sp_{\alpha_2} : \text{Qui}(\mathcal{A}) \rightarrow \text{Qui}(T_{\overline{X}_{\alpha_2}}(\overline{X}_{\alpha_2}) \mathcal{A})$$

and

$$Sp_{\alpha_2} Sp_{\alpha_1} : \text{Qui}(\mathcal{A}) \rightarrow \text{Qui}(T_{\overline{X}_{\alpha_2}}(\overline{X}_{\alpha_2}) \mathcal{A})$$,

are canonically isomorphic. Here the functor $Sp_{\alpha_2}$ in the second composition is the functor of specialization to the stratum $\overline{X}_{\alpha_1} \times \overline{X}_{\alpha_2}/\overline{X}_{\alpha_1}$.

Analogously, for a three-flag $\overline{X}_{\alpha_1} \subset \overline{X}_{\alpha_2} \subset \overline{X}_{\alpha_3} \subset X = \mathbb{C}^N$ the morphism of associativity is identical on the linear spaces of a quiver, so for any flag

$$\overline{X}_{\alpha_1} \subset \overline{X}_{\alpha_2} \subset \ldots \subset \overline{X}_{\alpha_n} \subset X = \mathbb{C}^N$$

of closed strata we have a specialization functor

$$Sp_{\alpha_1, \ldots, \alpha_n} : \text{Qui}(\mathcal{A}) \rightarrow \text{Qui}(T_{\overline{X}_{\alpha_1} \subset \ldots \subset \overline{X}_{\alpha_n}})$$

which is described by iterated relations (5.1)–(5.3) and satisfy the properties mentioned in Proposition 4.1.

5.2. Specialization of quiver $D$-modules.

Let us first consider the case of linear stratification in linear space $\mathbb{C}^N$, that is we suppose that all hyperplanes $H_i$ contain an origin. Let $I_\alpha$ be an ideal of $\overline{X}_{\alpha}$: $f \in I_\alpha \iff f|_{\overline{X}_{\alpha}} = 0$ and let $F^*_\alpha(D_X)$ be a filtration of the ring $D_X$ corresponding to ideal $I_\alpha$ [3]:

$$F^k_\alpha(D_X) = \{ P \in D_X : P(I^j_\alpha \subset I^{j+k}_\alpha \text{ for any } j}\} .$$

Here $I^k_\alpha = \mathcal{C}[X]$ for $k \leq 0$. Then $F^k_\alpha(D_X)/F^{k+1}_\alpha(D_X)$ is isomorphic to a space of differential operators over $T_{\overline{X}_{\alpha}} X$ homogenous of order $k$ and the ring $Gr F_\alpha(D_X)$ is isomorphic to the ring $D_{T_{\overline{X}_{\alpha}} X}$.

Let now $V_\mathcal{A}$ be a quiver from $Qui(\mathcal{A})$ and $EV_\mathcal{A}$ be corresponding $D_X$-module. We define a filtration $F^*_\alpha EV_\mathcal{A}$ of $EV_\mathcal{A}$ in the following way:

$$F^*_\alpha EV_\mathcal{A} = \sum_{k=0}^{N} \sum_{\beta : \text{codim}_{X_{\beta} X_{\beta \cap \overline{X}_{\alpha}} = k}} F^{n-k}_\alpha(D_X) \cdot (V_\beta \otimes \Omega_\beta) .$$

(5.5)

One can check that the filtration $F^*_\alpha EV_\mathcal{A}$ is good, that is satisfy the following conditions:

(i) $F^d_\alpha(D_X) : F^d_\alpha EV_\mathcal{A} \subset F^{k+j}_\alpha EV_\mathcal{A}$;

(ii) $F^k_\alpha(D_X) \cdot F^d_\alpha EV_\mathcal{A} = F^{k+j}_\alpha EV_\mathcal{A}$ for $j, k \geq 0$ and for $j \leq -N, k \leq 0$;
We well known equivalence between analytic and algebraic 

The following proposition describes \( \text{Gr } F_a(EV_A) \) as \( D_{\overline{X}} \) module. It follows directly from the definition of quiver’s \( D \)-module \( EV_A \) and from the definition of the filtration.

**Proposition 5.2** There is a canonical isomorphism of \( D_{\overline{X}} \) modules:

\[
\text{Gr } F_a(EV_A) \approx ESP_{\alpha} V_A.
\]  

(5.6)

As it follows from the relations (5.1)–(5.3), the isomorphism (5.6) means that \( D_{\overline{X}} \) module \( \text{Gr } F_a(EV_A) \) is generated by vectors \( v_\gamma \otimes \omega_\beta \otimes \omega_\beta \) where \( \beta \in \text{ver}(\Gamma_A) \), \( v_\gamma \in V_\beta \), \( \omega_\beta \) is a top form for \( X_\beta \cap \overline{X}_\alpha \), \( \omega_\beta \) is a top form for \( X_\beta / X_\beta \cap \overline{X}_\alpha = p_\alpha(X_\beta) \) with the following relations:

\[
L_\beta'(v_\gamma \otimes \omega_\beta \otimes \omega_\beta) = \sum_{\gamma, \beta \rightarrow \gamma} \frac{L_\gamma'(\hat{\omega}_\gamma') \otimes \omega_\gamma'}{\hat{\omega}_\gamma' \otimes \omega_\gamma'} A_{\gamma, \beta}(v_\delta) \otimes \omega_\gamma \otimes \omega_\gamma' (5.7)
\]

where \( L_\beta' \) is linear vectorfield along stratum \( X_\beta \cap \overline{X}_\alpha \), \( \hat{\omega}_\gamma' \) is a top form for \( \overline{X}_\alpha / \overline{X}_\gamma \cap \overline{X}_\alpha \), \( \omega_\gamma' \) is a top form for \( \overline{X}_\gamma \cap \overline{X}_\alpha \);

\[
L_\gamma''(v_\gamma \otimes \omega_\gamma \otimes \omega_\gamma) = \sum_{\gamma, \beta \rightarrow \gamma} \frac{L_\gamma''(\hat{\omega}_\gamma') \otimes \omega_\gamma'}{\hat{\omega}_\gamma' \otimes \omega_\gamma'} A_{\gamma, \beta}(v_\gamma) \otimes \omega_\gamma \otimes \omega_\gamma' (5.8)
\]

where \( L_\gamma'' \) is linear vectorfield in \( X / \overline{X}_\alpha \) along \( X_\beta / X_\beta \cap \overline{X}_\alpha \), \( \gamma_\gamma' \) is a top form for \( (X / \overline{X}_\alpha) / (X_\gamma / \overline{X}_\gamma \cap \overline{X}_\alpha) \), \( \omega_\gamma'' \) is a top form for \( X_\gamma / \overline{X}_\gamma \cap \overline{X}_\alpha \);

\[
f_\beta'(v_\gamma \otimes \omega_\gamma \otimes \omega_\gamma) = \sum_{\gamma, \beta \rightarrow \gamma} \frac{df_\gamma' \otimes \hat{\omega}_\gamma' \otimes \omega_\gamma'}{\hat{\omega}_\gamma' \otimes \omega_\gamma'} A_{\gamma, \beta}(v_\gamma) \otimes \omega_\gamma \otimes \omega_\gamma' (5.9)
\]

where \( f_\beta' \) is linear function on \( \overline{X}_\alpha \) equal to zero on \( X_\beta \cap \overline{X}_\alpha \), with the same \( \hat{\omega}_\gamma', \omega_\gamma' \) as for (5.7);

\[
f_\gamma''(v_\gamma \otimes \omega_\gamma \otimes \omega_\gamma) = \sum_{\gamma, \beta \rightarrow \gamma} \frac{df_\gamma'' \otimes \hat{\omega}_\gamma'' \otimes \omega_\gamma'}{\hat{\omega}_\gamma'' \otimes \omega_\gamma'} A_{\gamma, \beta}(v_\gamma) \otimes \omega_\gamma \otimes \omega_\gamma'' (5.10)
\]

where \( f_\gamma'' \) is linear function on \( X / \overline{X}_\alpha \) equal to zero on \( X_\beta / X_\beta \cap \overline{X}_\alpha \) with the same \( \hat{\omega}_\gamma'', \omega_\gamma'' \) as in (5.8).

We can drop the assumption of linearity for the stratification if we pass first to analytic \( D \)-modules, and thus kill the strata which do not intersect with \( \overline{X}_\alpha \), and then use the well known equivalence between analytic and algebraic \( D \)-modules. Alternatively, we could first localize the initial \( D \)-module to a Zariski open set which does not contain strata not intersecting with \( \overline{X}_\alpha \), and then use the construction above.
In order to compare $D_{\mathcal{T}_\alpha}X$-module $Gr F_\alpha(EV_A)$ with the Malgrange-Kashiwara specialization of the $D_X$-module $EV_A$, we should investigate the eigenvalue of the target vectorfield $\theta^\alpha$ which satisfy the relations $\theta^\alpha|_{I_\alpha/I_\alpha^2} = id$. In local coordinates $\{z^\alpha, w^\alpha\} = \{z^\alpha_1, \ldots, z^\alpha_{\dim X_\alpha}, w^\alpha_1, \ldots, w^\alpha_{\codim X_\alpha}\}$ satisfying (3.3) it looks like $\theta^\alpha = \sum_j w^\alpha_j \frac{\partial}{\partial w^\alpha_j}$. Let also $\lambda = \{\lambda_1, \ldots, \lambda_n\}$ be a collection of weights and $M_\lambda$ be a quiver constructed in section 3.1 (see (4.3)).

**Proposition 5.3**

$$\theta^\alpha|_{Gr \alpha} EM_\lambda = \lambda + k.$$  \hspace{1cm} (5.11)

The proof follows from the relations (5.7), (5.8).

Correspondingly, for $D_X$-module $E(M_{\lambda,\beta})$ supported on the stratum $\overline{X}_\beta$ ($\overline{X}_\beta \cap \overline{X}_\alpha \neq 0$), we have

$$\theta^\alpha|_{Gr \alpha} EM_{\lambda,\alpha} = \lambda_{\beta',\beta} + k,$$

where $X_{\beta'} = X_\beta \cap \overline{X}_\alpha$.

**Corollary 5.1** (i) Let the number $\lambda_\alpha = \sum_{i:H_i \supset X_\alpha} \notin \mathbb{Z} \backslash 0$. Then

$$Sp_{\overline{X}_\alpha} E(M_\lambda) \approx E Sp_\alpha (M_\lambda);$$  \hspace{1cm} (5.12)

(ii) Let the nonresonance condition takes place: $\lambda_{\beta,\gamma} - \lambda_{\nu,\xi} \notin \mathbb{Z} \backslash 0$. Then

$$Sp_{\overline{X}_\alpha} E(M) \approx E Sp_\alpha (M);$$  \hspace{1cm} (5.13)

for any $M \in Qui(A)$.

Here $Sp_{\overline{X}_\alpha}$ denotes the Malgrange-Kashiwara specialization functor, [2].

**Proof of Theorem 4.1.** Following the scheme of [3], we choose a collection of generic linear functions $f_\alpha$, $f_\alpha|_{\overline{X}_\alpha} = 0$ and perform the glueing procedure starting from a local system on the open stratum $X_\beta$ and adding step by step strata of bigger codimension. In order to glue the stratum $X_\alpha$ we add hyperplane $f_\alpha = 0$ to the stratification with zero weight and with a condition that the support of $D$-module intersects with hyperplane $f_\alpha = 0$ only by stratum $\overline{X}_\alpha$. Part (ii) of the Corollary 4.1 allows to calculate the linear algebra data

$$\Psi_{f_\alpha}(M) \leftarrow \Phi_{f_\alpha}(M)$$

in terms of Malgrange-Kashiwara specializazion functor assuming we start from quiver’s $D$-modules with nonresonance condition.

Corollary 4.1 guarantees that they coincide with the linear algebra data calculated by means of specialization functors in category $Qui_\lambda(A)$ (and thus coincide with those calculated by Beilinson’s technique for the unipotent case in [3]). Then we can refer to [3] since we drop unipotent restriction. Alternatively, one can get an independent proof after the formulation of an analog of the Beilinson-Kashiwara equivalence theorem for the category $Qui(A)$ and functors $Sp_\alpha$ from section 4.1, which is purely combinatorial.
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