Chern-Simons States
and Topologically Massive Gauge Theories

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Abstract

In an abelian topologically massive gauge theory, any eigenstate of the Hamiltonian can be decomposed into a factor describing massive propagating gauge bosons and a Chern-Simons wave function describing a set of nonpropagating “topological” excitations. The energy depends only on the propagating modes, and energy eigenstates thus occur with a degeneracy that can be parametrized by the Hilbert space of the pure Chern-Simons theory. We show that for a nonabelian topologically massive gauge theory, this degeneracy is lifted: although the Gauss law constraint can be solved with a similar factorization, the Hamiltonian couples the propagating and nonpropagating (topological) modes.

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Three-dimensional topologically massive gauge theories \([1,2]\) have recently attracted considerable attention as useful models for condensed matter physics \([3,4,5,6]\) and string theory \([7,8]\). With actions that combine an ordinary Yang-Mills kinetic term and a Chern-Simons term, these models exhibit a hybrid behavior: they contain massive propagating “photons” or “gluons,” but also display such “topological” features as statistical transmutation.

The propagating sector of a topologically massive gauge theory is relatively easy to understand in the language of ordinary field theory. The behavior of the “topological” sector is more subtle. For an abelian theory, it is known that energy eigenstates are degenerate, with a set of ground states that are in precise one-to-one correspondence with the states of the associated pure Chern-Simons theory \([3,9,10]\). These degenerate states determine the “topological” features of the theory: in the spectral decomposition of the propagator, for instance, they account for the long range Aharonov-Bohm interaction that leads to fractional spin and statistics \([11]\). (In absence of matter fields, the degeneracy associated with the topological modes disappears in the infinite area limit, however \([12]\).)

It is an interesting open question whether the correspondence between degenerate energy eigenstates and states of a Chern-Simons theory continues to hold in the nonabelian case. The purpose of this Letter is to demonstrate that for nonabelian theories a simple splitting of wave functions into “propagating” and “topological” factors is no longer possible, and that the self-interaction of the gauge field lifts this degeneracy.

Let us consider a topologically massive gauge theory with gauge group \(G\) on a three-manifold \(M = \mathbb{R} \times \Sigma\), where \(\Sigma\) is an arbitrary Riemann surface. In terms of a Lie algebra-valued gauge potential \(A_\mu\), our action is

\[
S = \int d^3 x \text{Tr} \left\{ -\frac{1}{4\gamma} \sqrt{|g|} F_{\mu\nu} F^{\mu\nu} + \frac{k}{8\pi} \epsilon^{\mu\nu\rho} \left( A_\mu (\partial_\nu A_\rho - \partial_\rho A_\nu) + \frac{2}{3} A_\mu [A_\nu, A_\rho] \right) \right\}. \tag{1}
\]

It is convenient to adopt Gaussian normal coordinates for \(M\), for which the metric takes the form

\[
ds^2 = dt^2 - h_{ij} dx^i dx^j. \tag{2}
\]

(Here, \(D_i\) is the gauge-covariant derivative and \(\epsilon^{ij} = \epsilon^{0ij} / \sqrt{h}\).) We shall further choose a set of local complex coordinates for \(\Sigma\),

\[
h_{ij} dx^i dx^j = 2h_{zz} dz d\bar{z}, \tag{3}
\]

in part to simplify an eventual comparison with standard Chern-Simons results. In a \((2 + 1)\)-dimensional splitting, the action then becomes

\[
S = \int dt \int_\Sigma d^2 x \sqrt{h} \text{Tr}\left\{ -\frac{1}{4\gamma} F_{ij} F^{ij} - \frac{1}{2\gamma} F_{i0} F^{i0} - \frac{k}{4\pi} \epsilon^{ij} A_i \partial_0 A_j + \frac{k}{4\pi} \epsilon^{ij} A_0 F_{ij} \right\}. \tag{4}
\]

where

\[
F_{ij} = \partial_i A_j - \partial_j A_i - [A_i, A_j], \quad F_{i0} = D_i A_0 - \partial_0 A_i. \tag{5}
\]

For an abelian theory, the degeneracy of energy eigenstates is already apparent at this stage. In the Hodge decomposition of the gauge potential,

\[
A = A_i dx^i = a + d\phi + *d\varphi, \quad da = d* a = 0, \tag{6}
\]
it is easy to check that the harmonic component \( a \) decouples from \( \phi \) and \( \varphi \). In fact, the dynamics of the harmonic field \( a \) is described by an effective quantum mechanical action equivalent to the Landau action for a charged particle in a constant magnetic field \([3, 10]\), and the degeneracy of the Landau states gives rise to a corresponding degeneracy of states in the full theory. For \( G = U(1) \), the space of gauge orbits is

\[
\mathcal{M}^{U(1)}_\Sigma = \mathbb{Z} \times S^1 \times \cdots \times S^1 \times P(H),
\]

where \( g \) is the genus of the surface \( \Sigma \) and \( P(H) \) is the projective space associated with the transverse field \( \varphi \). The \( \mathbb{Z} \) connected components of \( \mathcal{M}^{U(1)}_\Sigma \) are parametrized by the magnetic monopole charge of the magnetic field, and the \( (S^1)^{2g} \) describe the harmonic modes \([13]\).

For a nonabelian theory, this argument fails: the existence of a Gribov problem prevents a natural generalization of the Hodge decomposition, and the structure of the space of gauge orbits is more complex. In particular, there is no splitting between the topological modes (flat connections) and the propagating (transverse) modes \([3]\); indeed, the space of flat connections is not even a linear subspace of the space of nonabelian gauge fields. On the other hand, we know that a nonabelian topologically massive gauge theory should reproduce the corresponding Chern-Simons theory in the \( \gamma \to \infty \) limit \([14]\), and various other arguments suggest a role for Chern-Simons wave functions even when \( \gamma \) is finite \([15, 16]\).

To understand this situation more clearly, we shall examine topologically massive gauge theory in the "functional Schrödinger picture" (see, for instance, \([16]\)), in which wave functions are functionals \( \Psi = \Psi[A_i] \) and

\[
\Pi^z = -i \frac{\delta}{\delta A^z}, \quad \Pi^\bar{z} = -i \frac{\delta}{\delta A^\bar{z}}.
\]

We begin by reexpressing the action \((4)\) in canonical form. The momenta conjugate to \( A_i \) are

\[
\Pi^i = \frac{1}{\gamma} F^{i0} + \frac{k}{4\pi} \epsilon^{ij} A_j,
\]

while the momentum conjugate to \( A_0 \) vanishes identically, giving us the nonabelian Gauss law constraint

\[
\frac{1}{\gamma} D_i F^{i0} + \frac{k}{4\pi} \epsilon^{ij} F_{ij} = 0.
\]

This constraint is simple enough that no elaborate technology is needed to understand the canonical theory. In particular, the Hamiltonian density is

\[
H = \text{Tr} \left\{ \Pi^i \partial_0 A_i \right\} - L
\]

\[
= -\text{Tr} A_0 \left\{ D_i \left( \Pi^i - \frac{k}{4\pi} \epsilon^{ij} A_j \right) + \frac{k}{4\pi} \epsilon^{ij} F_{ij} \right\}
\]

\[
+ \frac{1}{8\gamma} \text{Tr} \left\{ \epsilon^{ij} F_{ij} \right\}^2 + \frac{\gamma}{2} \text{Tr} \left\{ h_{ij} \left( \Pi^i - \frac{k}{4\pi} \epsilon^{ik} A_k \right) \left( \Pi^j - \frac{k}{4\pi} \epsilon^{jl} A_l \right) \right\},
\]

up to total derivatives that will vanish when we integrate to obtain the Hamiltonian. The first term in \( H \) is simply the Gauss law constraint, and vanishes for physical wave functions; the remainder is of the form \( B^2 + E^2 \).
Let us first investigate the constraint, the generator of gauge transformations of Schrödinger picture states, which now takes the form

\[
\left\{ D_z \left[ \left( -i \frac{\delta}{\delta A_z} \right) + \frac{k}{4\pi} \epsilon^{zz} A_z \right] + D_{\bar{z}} \left[ \left( -i \frac{\delta}{\delta A_{\bar{z}}} \right) - \frac{k}{4\pi} \epsilon^{z\bar{z}} A_{\bar{z}} \right] + \frac{k}{2\pi} \epsilon^{z\bar{z}} F_{z\bar{z}} \right\} \Psi[A_z, A_{\bar{z}}] = 0. \tag{12}
\]

This expression should be contrasted with the constraint for a pure Yang-Mills theory,

\[
\left\{ D_{\bar{z}} \frac{\delta}{\delta A_{\bar{z}}} + D_z \frac{\delta}{\delta A_z} \right\} \Phi[A_z, A_{\bar{z}}] = 0, \tag{13}
\]

which has as its solution any gauge-invariant functional \( \Phi[A_z, A_{\bar{z}}] \). The extra terms in (12) imply that states in the topologically massive theory are not exactly gauge-invariant \[17\], transforming instead with a one-cocycle \[18\]. But we can separate out the noninvariant part of the wave function — essentially integrating the cocycle condition \[19\] — by writing

\[
\Psi[A_z, A_{\bar{z}}] = \exp \left\{ -\frac{ik}{4\pi} \int d^2x \sqrt{h} \epsilon^{zz} A_z A_{\bar{z}} \right\} \chi[A_z] \Phi[A_z, A_{\bar{z}}], \tag{14}
\]

where the “nontopological” component \( \Phi[A_z, A_{\bar{z}}] \) satisfies the standard constraint (13) and the “topological” factor \( \chi[A_z] \) is an arbitrary solution of the equation

\[
\left\{ D_z \frac{\delta}{\delta A_z} - \frac{ik}{2\pi} \epsilon^{zz} \partial_z A_z \right\} \chi[A_z] = 0. \tag{15}
\]

This last expression may be recognized as the functional Schrödinger equation for a Chern-Simons wave function \[19, 20, 21\]. Its solutions are well-understood: \( \chi[A_z] \) can be described as a partition function for a suitably gauged chiral Wess-Zumino-Witten model on \( \Sigma \) with gauge group \( G \) \[22\], and the independent solutions are in one-to-one correspondence with the conformal blocks of this model. It should be stressed that the dependence of \( \chi \) on \( A_z \) and \( A_{\bar{z}} \) is invariantly determined by the metric \( h_{ij} \).

The appearance of Chern-Simons wave functions in the solution of the constraint is a first indication of the structure of our Hilbert space. The crucial question, however, is whether the factorization (14) is respected by the Hamiltonian. To explore this question, observe that the first term in \( \hat{H} \) is proportional to the constraint, and vanishes on physical wave functions. The second term is simply multiplicative. The third term, on the other hand, is now

\[
\gamma \frac{2}{\gamma} \text{Tr} \left\{ h_{ij} \left( \Pi^i - \frac{k}{4\pi} \epsilon^{ik} A_k \right) \left( \Pi^j - \frac{k}{4\pi} \epsilon^{jl} A_l \right) \right\} \Phi[A_z, A_{\bar{z}}] = \gamma h_{zz} \exp \left\{ -\frac{ik}{4\pi} \int d^2x \sqrt{h} \epsilon^{zz} A_z A_{\bar{z}} \right\} \chi[A_z] \text{Tr} \left\{ \left( \frac{\delta}{\delta A_z} - \frac{ik}{2\pi} \epsilon^{zz} A_z + \langle J^z \rangle \chi \right) \frac{\delta}{\delta A_z} \right\} \Phi[A_z, A_{\bar{z}}], \tag{16}
\]

where

\[
\langle J^z \rangle \chi = \chi^{-1} \frac{\delta \chi}{\delta A_z}. \tag{17}
\]
We can interpret $\langle J^z \rangle_\chi$ as the expectation value of the Kač-Moody current of the associated gauged WZW model; as $\chi$ varies over the Hilbert space of the Chern-Simons theory, $\langle J^z \rangle_\chi$ will vary over the corresponding space of current blocks.

In general, the Hamiltonian thus couples the “nontopological” wave function $\Phi[A_z, A_{\bar{z}}]$ to $\chi[A_z]$. This mixing will be absent only if the term

$$\langle J^z \rangle_\chi \frac{\delta \Phi}{\delta A_{\bar{z}}}$$

is independent of the choice of the Chern-Simons state $\chi$. Now, from (15) we have

$$D_z \langle J^z \rangle_\chi = \frac{ik}{2\pi} \epsilon^{\bar{z}z} \partial_{\bar{z}} A_z,$$

which determines $\langle J^z \rangle_\chi$ up to a factor proportional to the zero-modes of $D_z$. Letting $\phi^i_{\bar{z}}$ be a complete set of these zero-modes, we can thus write

$$\langle J^z \rangle_\chi = \hat{J}^z + \sum_i \left( \int d^2 w \sqrt{h} \text{Tr} \left\{ \langle J^w \rangle_\chi \phi^i_w \right\} \right) h^{\bar{z}z} \phi^i_{\bar{z}}$$

where $\hat{J}^z$ is independent of $\chi$. Equations (14), (16) and (19) then tell us that

$$H \Psi[A_z, A_{\bar{z}}] = \left( \exp \left\{ -\frac{ik}{4\pi} \int d^2 x \sqrt{h} \epsilon^{\bar{z}z} A_z A_{\bar{z}} \right\} \chi[A_z] \right) H_0 \Phi[A_z, A_{\bar{z}}]$$

with

$$H_0 = \text{Tr} \left\{ -\gamma h_{\bar{z}z} \left( \frac{\delta}{\delta A_{\bar{z}}} - \frac{ik}{2\pi} \epsilon^{\bar{z}z} A_{\bar{z}} + \hat{J}^z \right) \frac{\delta}{\delta A_z} + \frac{1}{2\gamma} (\epsilon^{\bar{z}z} F_{z\bar{z}})^2 \right\}$$

$$-\gamma \sum_i \left( \int d^2 w \sqrt{h} \text{Tr} \left\{ \langle J^w \rangle_\chi \phi^i_w \right\} \right) \text{Tr} \frac{\delta}{\delta A_z},$$

and the coupling of $\Phi$ and $\chi$ is now isolated in the last term.

The presence of a Chern-Simons wave function in our decomposition will thus give rise to degenerate energy eigenstates only in the subspace of states for which

$$P^i \Phi = \int d^2 z \sqrt{h} \text{Tr} \left\{ \frac{\delta}{\delta A_{\bar{z}}} \phi^i_{\bar{z}} \right\} \Phi = 0, \quad [H_0, P^i] \Phi = 0,$$

where the second condition is necessary to ensure that the Hamiltonian leaves us within the appropriate space of states. The operator $P^i$ can be interpreted as the generator of the transformation

$$A_z \rightarrow A_z, \quad A_{\bar{z}} \rightarrow A_{\bar{z}} + \phi^i_{\bar{z}},$$

where the single condition on $\phi^i_{\bar{z}}$ is that $D_z \phi^i_{\bar{z}} = 0$. Hence $\Phi[A_z, A_{\bar{z}}]$ will be annihilated by $P^i$ only if $A_{\bar{z}}$ appears solely in the combination $\partial_z A_z + [A_z, A_{\bar{z}}] = F_{z\bar{z}} + \partial_{\bar{z}} A_z$; that is, we must require that

$$\Phi = \Phi[A_z, F_{z\bar{z}}].$$
If equation (24) is satisfied, we can set
\[
\frac{\delta \Phi}{\delta A_z} = -D_z \frac{\delta \Phi}{\delta F_{z \bar{z}}},
\] (25)
and by a simple calculation, the Hamiltonian density becomes
\[
H_0 = \text{Tr} \left\{ -\gamma h_{z \bar{z}} \left( D_z \frac{\delta}{\delta A_z} - \frac{i k}{2\pi} \epsilon^{z \bar{z}} F_{z \bar{z}} \right) \frac{\delta}{\delta F_{z \bar{z}}} + \frac{1}{2\gamma} (\epsilon^{z \bar{z}} F_{z \bar{z}})^2 \right\}.
\] (26)
Using the constraint (13), we can write this expression as
\[
H_0 = \text{Tr} \left\{ -\gamma h_{z \bar{z}} \left( D_z D_z \frac{\delta}{\delta F_{z \bar{z}}} - \frac{i k}{2\pi} \epsilon^{z \bar{z}} F_{z \bar{z}} \right) \frac{\delta}{\delta F_{z \bar{z}}} + \frac{1}{2\gamma} (\epsilon^{z \bar{z}} F_{z \bar{z}})^2 \right\}.
\] (27)
It remains for us to check the commutator $[H_0, P^i]$. Since $P^i F_{z \bar{z}} = 0$, the only contribution to this commutator comes from the explicit dependence of the covariant derivative $D_z$ in (27) on $A_z$, and a simple computation gives
\[
[H_0, P^i] = -\gamma h_{z \bar{z}} c_{abc} \bar{\phi}^a \frac{\delta}{\delta F_{z \bar{z}}} D_z \frac{\delta}{\delta F_{z \bar{z}}}
\] (28)
where $c_{abc}$ are the structure constants of the group $G$. For a generic gauge potential, this expression does not vanish, and $H_0$ takes us out of the space of states annihilated by $P^i$. This means that $H_0$ typically has no eigenstates that satisfy $P^i \Phi = 0$, and the total Hamiltonian thus couples the topological and propagating modes.

If $G$ is abelian, of course, the situation is different: the structure constants vanish, and $[H_0, P^i] = 0$. In that case, we can write
\[
\Psi_{\bar{n}}[A_z, A_{\bar{z}}] = \exp \left\{ -\frac{i k}{4\pi} \int d^2 x \sqrt{\hbar} \epsilon^{z \bar{z}} A_z A_{\bar{z}} \right\} \chi_{\bar{n}}[A_z] \Phi[B], \quad \bar{n} \in (\mathbb{Z}_k)^g,
\] (29)
with
\[
\Phi[B] = \begin{cases} 
\exp \left\{ -\frac{k}{4\pi} \int d^2 x \sqrt{\hbar} B \Delta^{-1} B \right\} \xi[B] & \text{if } \int d^2 x \sqrt{\hbar} B = 0 \\
0 & \text{if } \int d^2 x \sqrt{\hbar} B \neq 0 \end{cases},
\] (30)
where $B = -\epsilon^{z \bar{z}} F_{z \bar{z}}$ is the magnetic field and $\Delta = 2\hbar^2 \partial_z \partial_{\bar{z}}$. Note that $\Phi[B]$ vanishes for magnetic fields with non-trivial magnetic charge; this is the canonical counterpart of the arguments of reference [23]. It is now easy to show that
\[
H_0 \Phi[B] = \frac{1}{2} \exp \left\{ -\frac{k}{4\pi} \int d^2 x \sqrt{\hbar} B \Delta^{-1} B \right\} \left[ \gamma \frac{\delta}{\delta B} \Delta \frac{\delta}{\delta B} + \frac{1}{\gamma} B \Delta^{-1} \left( \Delta - m^2 \right) B \right] \xi[B],
\] (31)
where $m = k\gamma/2\pi$ is the usual topological mass of the photon. We thus obtain the standard Hamiltonian for a free field $(\gamma \Delta)^{-1/2} B$ of mass $m$; in particular, the ground state on the plane,
\[
\xi[B] = \exp \left\{ -\frac{1}{2} \int d^2 x d^2 y B(x) G(x, y) B(y) \right\}, \quad G(x, y) = \int \frac{d^2 p}{(2\pi)^2} e^{ip(x-y)} \sqrt{p^2 + m^2} \frac{\gamma p^2}{\gamma p^2}
\] (32)
is precisely Jackiw’s ground state wave function \[13\] for a massive photon.

For this abelian theory, moreover, the Chern-Simons states \(\chi_{\vec{n}}[A_z]\) can be written explicitly \[24\]. Let us decompose \(A_z\) as in \(6\), and set \(a(z) = i\pi \tilde{a} \cdot (\text{Im } \Omega)^{-1} \cdot \omega(z)\), where \(\Omega\) is the period matrix of \(\Sigma\), the coefficients \(\tilde{a}\) are constant, and the \(\omega(z)\) are a basis of holomorphic differentials. Then

\[
\chi_{\vec{n}}[A_z] = \exp \left\{ \frac{ik}{4\pi} \int d^2z \sqrt{h} (\epsilon^{z\bar{z}} \partial_z (\phi - i\varphi) \partial_{\bar{z}} (\phi - i\varphi)) + \frac{k\pi}{2} \tilde{a} \cdot (\text{Im } \Omega)^{-1} \cdot \tilde{a} \right\} \vartheta \left[ \frac{n}{k} / k \Omega \right].
\]

It is now straightforward to combine the exponential factors from equations \(29\), \(30\), and \(33\); the \(a\)-independent part is

\[
\exp \left\{ -\frac{ik}{4\pi} \int d^2x \sqrt{h} \phi B \right\},
\]

again in agreement with Jackiw’s expression for the planar case \[13\]. As anticipated in the discussion following equation \(6\), the quantum states associated with the harmonic modes \(a_r\), represented by \(\vartheta\)-functions, decouple from the propagating modes described by the functional \(\Phi[B]\). An abelian topologically massive gauge theory is thus degenerate at all energy levels, with a degeneracy given by the number of quantum states of the pure Chern-Simons theory. The degeneracy of the vacuum is the same for all values of the coupling \(\gamma\), although it is only in the limit \(\gamma \to \infty\) that the term \(\Phi[B]\) disappears and the vacuum states become pure holomorphic Chern-Simons states.

Returning to the nonabelian case, on the other hand, we now see that the ansatz of factorization of the wave function into a holomorphic Chern-Simons part and a gauge-invariant one does not yield a complete splitting of the Hamiltonian, and is not useful in analyzing the spectrum of the theory. However, the failure of the Hamiltonian to split with such an ansatz is not in itself sufficient to show a lack of degeneracy at finite \(\gamma\). In order to analyze this issue in more detail, we shall consider the following simple perturbative argument.

As in equation \(14\), we begin by separating out a factor of

\[
\exp \left\{ -\frac{ik}{4\pi} \int d^2x \sqrt{h} \epsilon^{z\bar{z}} A_z A_{\bar{z}} \right\}
\]

from the wave function (although we no longer explicitly include a Chern-Simons wave function \(\chi[A_z]\)). The Hamiltonian density is then

\[
H = \text{Tr} \left\{ -\gamma h^{z\bar{z}} \left( \frac{\delta}{\delta A_z} - \frac{ik}{2\pi} \epsilon^{z\bar{z}} A_{\bar{z}} \right) \frac{\delta}{\delta A_{\bar{z}}} + \frac{1}{2\gamma} (\epsilon^{z\bar{z}} F_{z\bar{z}})^2 \right\}.
\]

Observe that the two terms of the Hamiltonian have different behaviors in the topological limit \(\gamma \to \infty\). Indeed, if we denote

\[
H_\infty = -\text{Tr} \left\{ \gamma h^{z\bar{z}} \left( \frac{\delta}{\delta A_z} - \frac{ik}{2\pi} \epsilon^{z\bar{z}} A_{\bar{z}} \right) \frac{\delta}{\delta A_{\bar{z}}} \right\}
\]

and

\[
V = \frac{1}{2} \text{Tr} (\epsilon^{z\bar{z}} F_{z\bar{z}})^2,
\]

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the Hamiltonian density is \( H = \gamma H_\infty + \gamma^{-1}V \); in the topological limit, \( \gamma H_\infty \) is the leading operator, while \( \gamma^{-1}V \) becomes irrelevant of order \( O(\frac{1}{\gamma}) \). Note that both terms are positive, but they do not commute, so it is not possible to simultaneously diagonalize the two operators.

The energy of any stationary state \( \psi[A_z, A_\bar{z}] \) will grow as \( \gamma \) in the large \( \gamma \) regime unless it is annihilated by \( H_\infty \),

\[
H_\infty \psi[A_z, A_\bar{z}] = 0. \tag{37}
\]

Therefore as the system approaches the topological regime, its physical quantum states are reduced to the null eigenfunctions of \( H \) that satisfy the Gauss law, i.e., holomorphic functionals \( \chi[A_z] \) satisfying the Gauss law constraint \( \Box \). We thus recover the Chern-Simons states and the degeneracy of the Chern-Simons theory in the topological limit \( \gamma \to \infty \).

For finite values of \( \gamma \), on the other hand, since \( V \) does not commute with \( H_\infty \), the holomorphic form of the eigenfunctions is not preserved, and the degeneracy may be removed \( \Box \). This can be seen at leading order in perturbation theory. The leading correction to the ground state energy generated by the Yang-Mills potential term \( \gamma^{-1}V \) is given by the eigenvalues of the matrix

\[
M_{mn} = \frac{1}{\gamma} \langle \chi_m | V[A] | \chi_n \rangle, \tag{38}
\]

where \( \chi_m, \chi_n \) denote arbitrary states of an orthonormal basis of the Chern-Simons theory. As usual in field theory, the (finite-dimensional) matrix \( M \) is divergent, indicating that the perturbative corrections to the vacuum energy require a renormalization. The ultraviolet divergences can be partially regularized by introducing a point splitting operator \( K_\epsilon(z, \bar{z}; w, \bar{w}) \) between the two curvature terms of the potential interaction,

\[
V_\epsilon[A_z, A_\bar{z}] = \frac{1}{2} \int d^2 z \sqrt{h} \int d^2 w \sqrt{h} \text{Tr} \left\{ \epsilon^{z\bar{z}} (D_z A_\bar{z} - \partial_z A_\bar{z}) K_\epsilon(z, \bar{z}; w, \bar{w}) e^{w\bar{w}} (D_w A_\bar{w} - \partial_w A_\bar{w}) \right\} . \tag{39}
\]

The regulating operator \( K_\epsilon \) may be chosen in a gauge-invariant way — for instance, it can be given by parallel transport along a geodesic of length \( \epsilon \) connecting the points \( z \) and \( w \), or as the inverse of an elliptic gauge-invariant differential operator such as \( K_\epsilon = (I + \epsilon \partial_z \partial_{\bar{z}})^{-3} \). For simplicity, however, we shall use the noninvariant regulator \( K_\epsilon = (I + \epsilon \partial_z \partial_{\bar{z}})^{-3} \); gauge invariance will be recovered after renormalization when the regulator is removed \( (\epsilon \to 0) \).

The matrix element \( M_{mn} \) is then given by

\[
M_{mn} = \frac{1}{\gamma} \int [\delta A] \exp \left\{ \frac{-ik}{2\pi} \int d^2 u \sqrt{h} \epsilon^{u\bar{u}} A_u A_{\bar{u}} \right\} \hat{\chi}_m[A_z] V_\epsilon[A_z, A_\bar{z}] \chi_n[A_\bar{z}] . \tag{40}
\]

We can eliminate the dependence of \( V_\epsilon \) on \( A_\bar{z} \) by functional integration by parts, using the exponential prefactor in \( \Box \) to replace \( A_\bar{z} \) by a functional derivative with respect to \( A_z \):

\[
M_{mn} = -\frac{1}{2\gamma} \int [\delta A] \exp \left\{ \frac{-ik}{2\pi} \int d^2 u \sqrt{h} \epsilon^{u\bar{u}} A_u A_{\bar{u}} \right\} \hat{\chi}_m[A_z] \int d^2 z \sqrt{h} \int d^2 w \sqrt{h} K_\epsilon(z, \bar{z}, w, \bar{w}) \left[ \text{Tr} \left\{ \epsilon^{z\bar{z}} (D_z A_\bar{z} - \partial_z A_\bar{z}) \left( \frac{2\pi i v}{k} D_w \frac{\delta}{\delta A_\bar{w}} + v w \bar{w} \partial_w A_\bar{w} \right) - \frac{2\pi i c v}{k} \epsilon^{w\bar{w}} A_w \delta^2(z - w) A_\bar{z} \right\} \right]
\]

\[
-\frac{2\pi i}{k} \text{dim} G \epsilon^{w\bar{z}} \partial_w \partial_{\bar{z}} \delta^2(z - w) \chi_n[A_\bar{z}] , \tag{41}
\]

\[k \]
where $c_v$ denotes the dual Coxeter number of $G$ ($c_v = N$ for $SU(N)$). The first term vanishes because the Chern-Simons state $\chi_n$ satisfies the Gauss law, but the remainder still depends on $A_z, A_\bar{z}$, and does not vanish in general.

Repeating the integration by parts to eliminate the remaining dependence on $A_z$, we find

$$M_{mn} = \frac{c_v}{2\gamma} \left( \frac{2\pi}{k} \right)^2 \int [\delta A] \exp \left\{ -\frac{ik}{2\pi} \int d^2 u \sqrt{\hbar} \epsilon^{u\bar{u}} A_u A_{\bar{u}} \right\} \hat{\chi}_m[A_{\bar{z}}] \int d^2 z \sqrt{\hbar} \int d^2 w \sqrt{\hbar} K_\epsilon(z, \bar{z}; w, \bar{w}) \left[ \text{Tr} \left\{ A_w \delta^2(z - w) \frac{\delta}{\delta A_w} \right\} + \text{dim} G (\delta^2(z - w))^2 + \frac{ik \text{dim} G}{2\pi c_v} \epsilon^{w\bar{z}} \partial_z \partial_w \delta^2(z - w) \right] \chi_n[A_{\bar{z}}].$$

(42)

The degeneracy of the ground state will be preserved only if $M_{mn}$ is a diagonal matrix of the form $M_{nn} = \lambda \delta_{mn}$; if $M$ is not proportional to the identity operator, the degeneracy of the topologically massive theory will be lower than that of the corresponding Chern-Simons theory.

In the abelian case, equation (42) reduces to

$$M_{mn} = \frac{\pi i}{k\gamma} \text{dim} G \int [\delta A] \exp \left\{ -\frac{ik}{2\pi} \int d^2 u \sqrt{\hbar} \epsilon^{u\bar{u}} A_u A_{\bar{u}} \right\} \hat{\chi}_m[A_{\bar{z}}] \int d^2 z \sqrt{\hbar} \int d^2 w \sqrt{\hbar} \epsilon^{w\bar{z}} \partial_z \partial_w \delta^2(z - w) K_\epsilon(z, \bar{z}; w, \bar{w}) \chi_n[A_{\bar{z}}] \quad (43)$$

which is proportional to the identity. Hence the degeneracy is preserved at least at first order in perturbation theory. Higher order computations will show that wave functionals become nonholomorphic and differ from Chern-Simons wave functionals, but we know from our previous analysis that the degeneracy is preserved at all orders of perturbation theory.

In the nonabelian case, on the other hand, the extra term

$$\left( \frac{2\pi}{k} \right)^2 \frac{c_v}{2\gamma} \int [\delta A] \exp \left\{ -\frac{ik}{2\pi} \int d^2 u \sqrt{\hbar} \epsilon^{u\bar{u}} A_u A_{\bar{u}} \right\} \hat{\chi}_m[A_{\bar{z}}] \int d^2 z \sqrt{\hbar} \int d^2 w \sqrt{\hbar} \text{Tr} \left\{ A_w \delta^2(z - w) K_\epsilon(z, \bar{z}; w, \bar{w}) \frac{\delta}{\delta A_w} \right\} \chi_n[A_{\bar{z}}]$$

depends on $A_z$. Hence $M_{mn}$ is not proportional to the identity, implying that the degeneracy is broken. The extra term (44) of $M_{mn}$ can be expressed in terms of the expectation values of the Kač-Moody currents $J^z$ and $J^\bar{z}$. Because of the identity

$$\frac{\delta}{\delta A_z} \chi_n[A_{\bar{z}}] = J^z \chi_n[A_{\bar{z}}],$$

we have

$$M_{mn} = \left( \frac{2\pi}{k} \right)^3 \frac{c_v}{2\gamma} \int d^2 z \sqrt{\hbar} \int d^2 w \sqrt{\hbar} \chi_m[A_{\bar{z}}] J^z \chi_n + \lambda \delta_{mn} \quad (45)$$

where $\lambda$ denotes the constant coefficients of the last two terms of expression (12). This connection between topologically massive gauge theory and conformal field theory might be exploited to explicitly evaluate the matrix elements of $M$, thus clarifying the pattern of topological
symmetry breaking. Although such a connection may not be useful for higher orders in perturbation theory, its presence at the lowest order opens up new perspectives in the application of conformal field theory techniques to three-dimensional systems. This topic deserves further study.

In summary, the existence of a coupling between topological and propagating modes in non-abelian topologically massive gauge theories yields an observable physical effect, the breaking of the degeneracy of the energy levels. This fact is not unrelated to the existence of a shift in the effective coupling constant of the pure Chern-Simons theory when it is viewed as the infinite mass limit of the topologically massive theory \[25, 26, 14\]: in a sense, this shift is also due to the coupling of the topological and propagating modes of the massive theory.

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