I. INTRODUCTION

Quantum coherence, stemming from the superposition rule of quantum mechanics, can capture the feature of quantumness in a single system, and play an important role in a variety of applications ranging from thermodynamics [1, 2] to metrology [3]. Recently, following the method in quantum information theory, the resource theory of coherence has been developed [4–10]. Besides quantum coherence, there are other resource theory including quantum entanglement [11], asymmetry [12–18], thermodynamics [19], and steering [20], where all these quantum resource are helpful to quantum information processing tasks.

Any resource theory consists of two basic elements: free states and free operation. The state (operation) outside the sets of free states (operation) is called resource. For example, the free states in the resource theory of coherence is called incoherent states and the corresponding free operations is called incoherent operations [4]. The resource measures are introduced to quantify the amount of resource in a given quantum state. To quantify the coherence in a single system, several operational coherence measures has been proposed, namely, relative entropy of coherence [4], $l_1$ norm of coherence [4], coherence of formation [7], robustness of coherence [21], coherence weight [22] and max-relative entropy of coherence [23, 24], where relative entropy of coherence characterizes the optimal rate to distill maximally coherent state from a given quantum state [7], coherence of formation is equal to minimal cost of maximal coherent state to prepare the given state [7] and max-relative entropy of coherence can be interpreted as the maximal overlap with the maximally coherent state under incoherent operations [23].

However, as the observation and characterization of the properties of quantum systems is often affected by the coupling to the environment, the effect of environment on quantifying coherence has to be taken into account. Thus, incoherent-quantum (IQ) coherence measures defined by max-relative, relative, robustness, and monogamy of coherence [25], where relative entropy of coherence characterizes the property of IQ coherence measures in details and the distribution of coherence in bipartite systems in terms of other coherence measures such as coherence of formation and assistance.

Here, we introduce incoherent-quantum (IQ) coherence measures defined by relative entropy, max-relative entropy and $l_1$ norm on bipartite systems to quantify the coherence in the system with the access to a quantum memory. We also introduce the IQ coherence of formation and assistance on bipartite systems, by which we find the distribution of coherence formation and assistance in bipartite systems: the total coherence of formation is lower bounded by the sum of coherence of formation in each subsystem and the entanglement of formation between the subsystems, while the total coherence of assistance is upper bounded by the sum of coherence of assistance in each subsystem and the entanglement of assistance between subsystems. Besides, we find the relationship between coherence cost (distillable coherence) and entanglement cost (distillable entanglement) in bipartite systems. Moreover, we obtain the monogamy relationship for IQ coherence measures (such as IQ coherence of assistance and formation) in tripartite systems, which illustrates the distribution of coherence in multipartite systems. Furthermore, we discuss the relationship between different IQ coherence measures, such as the equivalence between IQ coherence measures defined by max-relative entropy and $l_1$ norm.
II. PRELIMINARIES

Let $\mathcal{H}$ be a d-dimensional Hilbert space and $\mathcal{D}(\mathcal{H})$ be the set of density operators acting on $\mathcal{H}$. Let us first recall some basic facts about max- and min- relative entropies and the resource theory of coherence.

Max-and min-relative entropy.---Given two operators $\rho$ and $\sigma$ with $\rho \geq 0$, $\Tr \rho \leq 1$ and $\sigma \geq 0$, the max-relative entropy of $\rho$ with respect to $\sigma$ [26, 27] is defined as

$$D_{\text{max}}(\rho||\sigma) := \min \{ \lambda \in \mathbb{R}_+ : \rho \leq 2^\lambda \sigma \},$$

where $D_{\text{max}}(\rho||\sigma)$ is well defined if $\text{supp}[\rho] \subset \text{supp}[\sigma]$ with $\text{supp}[\rho]$ being the support of $\rho$. The min-relative entropy of $\rho$ with respect to $\sigma$ [26, 27] is defined as

$$D_{\text{min}}(\rho||\sigma) := -\log \Tr [\Pi_\rho \sigma],$$

where $\Pi_\rho$ is the projector on the support of $\rho$.

Resource theories of quantum coherence.---Given a fixed reference basis $\{|i\rangle\}_{i=0}^{d-1}$ for some $d$-dimensional Hilbert space, any quantum state which is diagonal in the reference basis is the free state in the resource theory of coherence and the set of incoherent states is denoted by $\mathcal{I}$. However, there is still general consensus on the set of free operations in the resource theory of coherence. Here, we refer incoherent operations (IO) [4] as the free operations, where incoherent operations (IO) is the set of all quantum operations $\Phi$ that admit a set of Kraus operators $\{K_i\}$, such that $\Phi(\cdot) = \sum_i K_i(.) K_i^\dagger$ and $K_i \mathcal{I} K_i^\dagger \subset \mathcal{I}$ for any $i$ [4]. Besides, several operational coherence measures have been proposed, which are listed as follows,

(i) $l_1$ norm of coherence [4],

$$C_{l_1} (\rho) = \sum_{i,j=0}^{d-1} \| |i\rangle \langle j| \rho \|,$$

(ii) relative entropy of coherence [4],

$$C_r (\rho) = S(\Delta(\rho)) - S(\rho),$$

where $S(\rho) = -\Tr [\rho \log \rho]$ is von Neumann entropy,

(iii) max-relative entropy of coherence [23],

$$C_{\text{max}} (\rho) = \min_{\sigma \in \mathcal{I}} D_{\text{max}}(\rho||\sigma),$$

(iv) coherence of formation [7],

$$C_f (\rho) = \min_{\rho = \sum_i p_i |\psi_i\rangle \langle \psi_i|} \sum_i p_i S(\Delta(\psi_i)),$$

where the minimization is taken over all pure state decomposition of $\rho$,

(v) coherence of assistance [9],

$$C_a (\rho) = \max_{\rho = \sum_i p_i |\psi_i\rangle \langle \psi_i|} \sum_i p_i S(\Delta(\psi_i)),$$

where the maximization is taken over all pure state decomposition of $\rho$,

(vi) coherence weight [22],

$$C_w (\rho) = \min \{ s \geq 0 : \rho = (1-s)\sigma + s\tau, \sigma \in \mathcal{I}, \tau \in \mathcal{D}(\mathcal{H}) \}.$$

III. ENTROPIC IQ COHERENCE MEASURE

Given a bipartite system $\mathcal{H}_A \otimes \mathcal{H}_B$ with a fixed basis $\{|i\rangle_A\}_i$ of $\mathcal{H}_A$, we can define the relative entropy of incoherent-quantum (IQ) coherence for any bipartite state $\rho_{AB} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$ as follows [9],

$$C^I_{r}(\rho_{AB}) = \min_{\sigma_{AB} \in \mathcal{I} Q} S(\rho_{AB}||\sigma_{AB}),$$

where the set of incoherent-quantum states $\mathcal{I} Q$ [9, 28] is given by

$$\mathcal{I} Q = \{ \sigma_{AB} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B) | \sigma_{AB} = \sum_i p_i \sigma_i^A \otimes \tau_i^B, \sigma_i^A \text{ is incoherent, } \tau_i^B \in \mathcal{D}(\mathcal{H}_B) \},$$

and $C^I_{r}$ gives an upper bound for assistant distillation of coherence [9, 28]. Max- and min-relative entropies of IQ coherence have also been defined in Ref. [25] as follows,

$$C^I_{\text{max}} (\rho_{AB}) = \min_{\sigma_{AB} \in \mathcal{I} Q} D_{\text{max}}(\rho_{AB}||\sigma_{AB}),$$

$$C^I_{\text{min}} (\rho_{AB}) = \min_{\sigma_{AB} \in \mathcal{I} Q} D_{\text{min}}(\rho_{AB}||\sigma_{AB}),$$

where $C^I_{\text{max}}$ captures the maximal advantage of bipartite states in certain subchannel discrimination problems [25].

For IQ coherence measure $C^I_{r}$, the following properties are considered: (i) positivity, $C^I_{r}(\rho_{AB}) \geq 0$ and $C^I_{r}(\rho_{AB}) = 0$ iff $\rho_{AB} \in \mathcal{I} Q$; (ii) monotonicity under incoherent operation on A side, that is, $C^I_{r}(\rho_{AB}) \leq C^I_{r}(\rho_{AB})$; (iii) strong monotonicity under incoherent operation on A side, that is, for incoherent operation $K_i \mathcal{I} K_i^\dagger \subset \mathcal{I}$, $\sum_i p_i C^I_{r}(\rho_i) \leq C^I_{r}(\rho)$, where $p_i = \Tr [K_i^A \rho_{AB} K_i^A]$ and $p_i = K_i^A \rho_{AB} K_i^A / p_i$; (iv) monotonicity under quantum operation on B side, that is, $C^I_{r}(\rho_{AB}) \leq C^I_{r}(\rho_{AB})$; (v) convexity, that is, for $\rho_{AB} = \sum_i p_i \rho_i$, $C^I_{\text{max}} (\rho_{AB}) \leq \sum_i C^I_{\text{max}} (\rho_i)$.

Note that, $C^I_{r}$ satisfies all these properties, and $C^I_{\text{max}}$ satisfies all these properties except (v). However, $C^I_{\text{max}}$ satisfies the quasi-convexity instead of convexity, that is, for \(\rho_{AB} = \sum_i p_i \rho_i\), $C^I_{\text{max}} (\rho_{AB}) \leq \max_i C^I_{\text{max}} (\rho_i)$ (See [25]).

For bipartite pure state $|\psi_{AB}\rangle$, it can be written as $|\psi_{AB}\rangle = \sum_{i=1}^{d_A} \sqrt{p_i} |i\rangle_A |u_i\rangle_B$ in the local basis $\{|i\rangle\}_i$ of $\mathcal{H}_A$, thus

$$C^I_{r}(\psi_{AB}) = -\sum_i p_i \log p_i = S(\Delta(\rho_{AB})) \leq \log d_A$$

and

$$C^I_{\text{max}} (\psi_{AB}) = 2 \log (\sum_{i=1}^{d_A} \sqrt{p_i}) \leq \log d_A.$$ Due to the convexity of $C^I_{r}$ and quasi-convexity of $C^I_{\text{max}}$, $C^I_{r}(\rho_{AB}) \leq \log d_A$ and $C^I_{\text{max}} (\rho_{AB}) \leq \log d_A$ for any bipartite state $\rho_{AB} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$, which means that the maximal value for IQ coherence measures $C^I_{r}$ and $C^I_{\text{max}}$ does not depend on the subsystem B. Here, we only consider the monotonicity of IQ coherence measures under local incoherent operations on A side and local quantum operations on B side, while the (strong) monotonicity of IQ coherence under the local
incoherent-quantum operations and classical communication (LIQCC) [9, 28] is still unknown as the characterization of LIQCC (such as the Kraus operators of LIQCC) is unclear.

According to the definition, \( C_r^{AB}(\rho_{AB}) \geq C_r(\rho_A) \) with \( \rho_A = \text{Tr}_B[\rho_{AB}] \) being the reduced state. For any pure bipartite state \(|\psi\rangle_{AB} \), the following relation holds,

\[
C_r^{AB}(\rho_{AB}) = C_r(\rho_A) + S(\rho_B),
\]

where \( S(\rho_B) \) is the von Neumann entropy of the reduced state \( \rho_B = \text{Tr}_A[|\psi\rangle\langle\psi|_{AB}] \) on system B. This comes directly from the definition of \( C_r^{AB} \) and the fact that \( S(\rho_A) = S(\rho_B) \) for pure bipartite state. In general, for any bipartite state \( \rho_{AB} \), \( C_r^{AB}(\rho_{AB}) \geq C_r(\rho_A) + \delta_{A\rightarrow B} \) [25], where \( \delta_{A\rightarrow B} \) is the quantum discord between A and B for state \( \rho_{AB} \). For pure tripartite states, we have the following proposition.

**Proposition 1.** Given a pure tripartite state \(|\psi\rangle_{ABC} \in D(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C) \), it holds that

\[
C_r^{AB}(\rho_{AB}) - C_r^{AC}(\rho_{AC}) = S(\rho_B) - S(\rho_C),
\]

where \( \rho_{AB}, \rho_{AC} \) are the corresponding reduced states of \( |\psi\rangle_{ABC} \).

**Proof.** Any tripartite pure state \(|\psi\rangle_{ABC} \) can be written as

\[
|\psi\rangle_{ABC} = \sum_i \sqrt{p_i} |i\rangle_A \otimes |u_i\rangle_{BC},
\]

in the local basis \( \{|i\rangle_A\}_{i} \) of \( \mathcal{H}_A \) with \( p_i \geq 0 \), \( \sum_i p_i = 1 \), and thus the reduced states \( \rho_{AB} \) and \( \rho_{AC} \) can be written as

\[
\rho_{AB} = \sum_{i,j} \sqrt{p_i p_j} |i\rangle\langle j|_A \otimes \text{Tr}_C[|u_i\rangle\langle u_j|_{BC}],
\]

\[
\rho_{AC} = \sum_{i,j} \sqrt{p_i p_j} |i\rangle\langle j|_A \otimes \text{Tr}_B[|u_i\rangle\langle u_j|_{BC}].
\]

Due to the definition,

\[
C_r^{AB}(\rho_{AB}) = S(\Delta_A(\rho_{AB})) - S(\rho_{AB}) = S(\Delta_A(\rho_{AB})) + \sum_i p_i S(\text{Tr}_C[|u_i\rangle\langle u_i|_{BC}]) - S(\rho_{AB}),
\]

\[
C_r^{AC}(\rho_{AC}) = S(\Delta_A(\rho_{AC})) - S(\rho_{AC}) = S(\Delta_A(\rho_{AC})) + \sum_i p_i S(\text{Tr}_B[|u_i\rangle\langle u_i|_{BC}]) - S(\rho_{AC}).
\]

As for pure states, the von Neumann entropy of the reduced states is equal, thus \( S(\text{Tr}_C[|u_i\rangle\langle u_i|_{BC}]) = S(\text{Tr}_B[|u_i\rangle\langle u_i|_{BC}]) \) for any \( i \), and \( S(\rho_{AB}) = S(\rho_C), S(\rho_{AC}) = S(\rho_B) \). Therefore, we obtain the result. \( \square \)

The Proposition 1 illustrates that the difference between \( C_r^{AB}(\rho_{AB}) \) and \( C_r^{AC}(\rho_{AC}) \) for bipartite pure state is equal to the difference between the amount of information encoded in ancillary systems B and C.

In tripartite systems, the monogamy relation for relative entropy of IQ coherence has been proposed as \( C_r^{ABC}(\rho_{ABC}) \geq C_r^{AC}(\rho_{AC}) + C_r^{BC}(\rho_{BC}) \) [25], where \( \rho_{AB} \) and \( \rho_{AC} \) are the corresponding reduced states. However, the relationship between \( C_r^{ABC}(\rho_{ABC}) \) and \( C_r^{AB}(\rho_{AB}) + C_r^{AC}(\rho_{AC}) \) is still unknown, that is, whether the following relation holds for all tripartite states remains to be verified,

\[
C_r^{ABC}(\rho_{ABC}) \geq C_r^{AB}(\rho_{AB}) + C_r^{AC}(\rho_{AC}).
\]

We give an upper bound for the quantity \( C_r^{ABC}(\rho_{ABC}) - C_r^{AB}(\rho_{AB}) - C_r^{AC}(\rho_{AC}) \) in terms of conditional entropy and find that the relation (3) may not hold in general.

**Lemma 2.** Given a tripartite state \( \rho_{ABC} \in D(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C) \), then

\[
C_r^{ABC}(\rho_{ABC}) - C_r^{AB}(\rho_{AB}) - C_r^{AC}(\rho_{AC}) \leq -S(A|BC) + S(A|B) + S(A|C),
\]

where the conditional entropy is defined as \( S(X|Y) = S(\rho_{XY}) - S(\rho_Y) \).

**Proof.** Let us take another system \( \mathcal{H}_A' = \mathcal{H}_A \) and the local basis \( \{|i\rangle_{A'}\}_{i} \). Define an unitary operator on \( \mathcal{H}_A \otimes \mathcal{H}_{A'} \) such that \( \rho_{AA'} = |i\rangle_{A'}\langle i| \otimes \rho_{AA'} \). Hence, for any tripartite state \( \rho_{ABC} = \sum_{i,j} |i\rangle\langle i|_A \otimes |\rho_{iiBC}\rangle \otimes |j\rangle\langle j|_{A'} \),

\[
\sigma_{ABC} = \Delta_A \otimes \Delta_{A'}(U_{AA'}\rho_{ABC} \otimes |0\rangle\langle 0|_{A'}U_{AA'}^\dagger)
\]

\[
= \sum_i |i\rangle\langle i|_A \otimes \rho_{iiBC} \otimes |i\rangle_{A'}. \]

Thus, \( \sigma_{AB} = \text{Tr}_{A'C}[\sigma_{ABC}] = \sum_i |i\rangle\langle i|_A \otimes \rho_i^B \), \( \sigma_{AC} = \text{Tr}_{AB}[\sigma_{ABC}] = \sum_i |i\rangle\langle i|_{A'} \otimes \rho_i^C \), and \( S(\sigma_{AB}) = S(\Delta_A(\rho_{AB})), S(\sigma_{AC}) = S(\Delta_A(\rho_{AC})) \) where \( \rho_i^B = \text{Tr}_C[\rho_{iiBC}^B] \) and \( \rho_i^C = \text{Tr}_B[\rho_{iiBC}^C] \). Then, as \( S(\sigma_{ABC}) = S(\Delta_A(\rho_{ABC})) \), we have

\[
S(\sigma_{ABC}||\sigma_{AB} \otimes \sigma_{AC}) = S(\rho_{AB}) + S(\rho_{AC}) - S(\sigma_{ABCA})
\]

\[
= S(\rho_{AB}) + S(\rho_{AC}) - S(\Delta_A(\rho_{ABC})).
\]

Since relative entropy is monotone under partial trace, then

\[
S(\sigma_{ABCA}||\sigma_{AB} \otimes \sigma_{AC}) \geq S(\sigma_{BC}||\sigma_{B} \otimes \sigma_{C}),
\]

where \( \sigma_{BC} = \rho_{BC}, \sigma_B = \rho_B \) and \( \sigma_C = \rho_C \), that is,

\[
S(\Delta_A(\rho_{AB})) + S(\Delta_A(\rho_{AC})) - S(\Delta_A(\rho_{ABC})) \geq S(\rho_B) + S(\rho_C) - S(\rho_{BC}).
\]

Therefore,

\[
C_r^{ABC}(\rho_{ABC}) - C_r^{AB}(\rho_{AB}) - C_r^{AC}(\rho_{AC})
\]

\[
= S(\rho_{AB}) + S(\rho_{AC}) - S(\rho_{ABC})
\]

\[
- [S(\Delta_A(\rho_{AB})) + S(\Delta_A(\rho_{AC})) - S(\Delta_A(\rho_{ABC}))]
\]

\[
\leq S(\rho_{AB}) + S(\rho_{AC}) - S(\rho_{ABC})
\]

\[
- [S(\rho_B) + S(\rho_C) - S(\rho_{BC})]
\]

\[
= -S(A|BC) + S(A|B) + S(A|C).
\]

\( \square \)
The negative conditional entropy quantifies the amount of entanglement as $S(A|B) < 0$ indicates the entanglement between A and B [30]. Thus, the following relation

$$-S(A|BC) \geq -S(A|B) - S(A|C),$$

(5)

can be viewed as a monogamy relation of entanglement, which holds for any pure tripartite state. Besides, Lemma 2 illustrates that the violation of the relation (5) will lead to the violation of the relation (3).

**Proposition 3.** There exists some tripartite state $\rho_{ABC} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C)$ such that

$$C^{ABC}_r(\rho_{ABC}) \leq C^{AB}_r(\rho_{ABC}) + C^{AC}_r(\rho_{AC}).$$

**Proof.** It is easy to verify that the tripartite state with the form $\rho_{ABC} = \rho_{AB} \otimes \rho_{AC}$ violates the relation (5) where $\mathcal{H}_A = \mathcal{H}_A_1 \otimes \mathcal{H}_A_2$. Thus, the relation (3) does not hold in general.

In view of the discussion in Ref. [25], the relation (3) cannot hold in general as the term $C^{AB}_r(\rho_{ABC}) + C^{AC}_r(\rho_{AC})$ contains two copies of local coherence $C_r(\rho_A)$, whereas the term $C^{ABC}_r(\rho_{ABC})$ only contains one copy of $C_r(\rho_A)$. The relation (3) will be violated for the tripartite state $\rho_{ABC}$ with weak correlation between B and C, e.g., $\rho_{ABC} = \rho_{AB} \otimes \rho_{AC}$ where $\mathcal{H}_A = \mathcal{H}_A_1 \otimes \mathcal{H}_A_2$.

By introducing smooth max and min-relative entropies of IQ coherence, the distribution of coherence quantified by relative entropy in multipartite systems has been obtained in Ref. [25]. Besides relative entropy of coherence, we find the distribution of coherence of formation $C_f$ and assistance $C_a$ in bipartite systems by introducing the corresponding IQ coherence measures. The IQ coherence of formation on bipartite systems is defined as follows,

$$C^{AB}_f(\rho_{AB}) := \min_i \sum_i p_i C^{AB}_r(\psi_i^A|\psi_i^B)$$

$$= \min_i \sum_i p_i S(\Delta_A(\psi_i^A|\psi_i^B)), \quad (6)$$

where the minimization is taken over all pure state decomposition $\rho_{AB} = \sum_i p_i|\psi_i^A|\psi_i^B)$. Since $S(\Delta_A(\psi_i^A|\psi_i^B)) = S(\Delta_A(Tr_B[|\psi_i^A|\psi_i^B]))$ (see Lemma 20 in Appendix A) and von Neumann entropy is concave, then we have

$$C^{AB}_f(\rho_{AB}) = \min_i \sum_i p_i S(\Delta_A(Tr_B[|\psi_i^A]))$$

$$\geq C^{AB}_r(\rho_{AB}) + C_f(\rho_{AB}), \quad (7)$$

where the minimization is taken over all pure state decomposition of $\rho_{AB} = \sum_i p_i|\psi_i^A$ without the restriction of $p_i^A$ to be pure state. $C^{AB}_f$ satisfy the properties (i)-(v), where (i) and (v) are obvious, (iii) and (iv) are presented in Appendix A, and (ii) comes directly from (iii) and (v).

Here, we consider the distribution of coherence of formation in bipartite systems in terms of the IQ coherence of formation $C^{AB}_f$, where $C^{AB}_f$ contains not only the local coherence in subsystem but also the entanglement of formation $E_f$ [31] between A and B, for which we have the following relation.

**Lemma 4.** Given a bipartite state $\rho_{AB} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$, then

$$C^{AB}_r(\rho_{AB}) \geq C_r(\rho_A) + E_f(\rho_{AB}), \quad (8)$$

where $\rho_A$ is the reduced state on subsystem A, and $E_f(\rho_{AB}) = \min \sum_i p_i S(Tr_A[|\psi_i^A|\psi_i^B])$ with the minimization being taken over all pure state decomposition of $\rho_{AB} = \sum_i p_i|\psi_i^A|\psi_i^B$.

**Proof.** For any pure state decomposition of $\rho_{AB} = \sum_i p_i|\psi_i^A|\psi_i^B$ with $\rho_i^A = Tr_B[|\psi_i^A|\psi_i^B]$ and $\rho_i^B = Tr_A[|\psi_i^A|\psi_i^B]$, we have

$$\sum_i p_i S(\Delta_A(\psi_i^A|\psi_i^B)) = \sum_i p_i [C_r(\rho_i^A) + S(\rho_i^B)]$$

$$\geq C_r(\rho_A) + E_f(\rho_{AB}),$$

where the first line comes from (1) and the second line comes from the convexity of $C_r$ and definition of $E_f$. Thus, we get the result.

**Lemma 5.** Given a bipartite state $\rho_{AB} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$, then

$$C_f(\rho_{AB}) \geq C^{AB}_f(\rho_{AB}) + C_f(\rho_B), \quad (9)$$

where $\rho_B$ is the reduced state of $\rho_{AB}$ on subsystem B.

**Proof.** There exists an optimal pure state decomposition of $\rho_{AB} = \sum_i p_i|\psi_i^A|\psi_i^B$ such that $C_f(\rho_{AB}) = \sum_i p_i S(\Delta_A(\psi_i^A|\psi_i^B)) = \sum_i p_i S(\Delta_A(Tr_B[|\psi_i^A|\psi_i^B]))$ (see Lemma 20 in Appendix A) and von Neumann entropy is concave, then we have

$$C_f(\rho_{AB}) = \sum_i p_i S(\Delta_A(\psi_i^A|\psi_i^B))$$

$$\geq C^{AB}_f(\rho_{AB}) + C_f(\rho_B),$$

where the inequality results from the definitions of $C^{AB}_f$ and $C_f$, and the fact that $\rho_B = \sum_i p_i |u_{i,j}|(u_{i,j}|B)$.}

Combining the above two lemmas, we can obtain the distribution of coherence of formation in bipartite systems, where the total coherence of formation is lower bounded by the sum of local coherence of formation in subsystems A and B and the entanglement of formation between the subsystems.

**Theorem 6.** Given a bipartite state $\rho_{AB} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$, it holds that,

$$C_f(\rho_{AB}) \geq \max \{ C_r(\rho_A) + C_f(\rho_B), C_r(\rho_B) + C_f(\rho_A) \} + E_f(\rho_{AB}), \quad (10)$$

where $\rho_A$ and $\rho_B$ are the corresponding reduced states of $\rho_{AB}$. 

\[ \blacksquare \]
Proof. Based on Lemmas 4 and 5, we have
\[
C_f(\rho_{AB}) \geq C_f^{AB}(\rho_{AB}) + C_f(\rho_B) \\
\geq C_f(\rho_A) + E_f(\rho_{AB}) + C_f(\rho_B).
\]
Similarly, \(C_f(\rho_{AB}) \geq C_f(\rho_B) + C_f(\rho_A) + E_f(\rho_{AB})\) can be obtained.

Now, we give an example such that the equality in Theorem 6 holds. For any quantum state \(\rho_B \in D(\mathcal{H}_B)\), there exists an optimal pure state decomposition of \(\rho_B = \sum_i p_i |u_i\rangle|u_i\rangle_B\) such that \(C_f(\rho_B) = \sum_i p_i S(\Delta_B(|u_i\rangle|u_i\rangle_i))\). Let us take the pure bipartite state \(|\psi_{AB}\rangle = \sum_i \sqrt{p_i} |i\rangle_A |u_i\rangle_B\), then \(C_f(\psi_{AB}) = C_f(\rho_A) + C_f(\rho_B) + E_f(\psi_{AB})\).

Besides, due to the equivalence between coherence of formation \(C_f\) and coherence cost \(C_c\) [7], we can obtain the relationship between coherence cost \(C_c\) and entanglement cost \(E_c\) in bipartite systems from Theorem 6.

**Corollary 7.** Given a bipartite state \(\rho_{AB} \in D(\mathcal{H}_A \otimes \mathcal{H}_B)\), it holds that,
\[
C_c(\rho_{AB}) \geq \max \{ C_c(\rho_A) + C_c(\rho_B), C_f(\rho_B) + C_c(\rho_A) \} + E_c(\rho_{AB}),
\]
where \(\rho_A\) and \(\rho_B\) are the corresponding reduced states of \(\rho_{AB}\), and the entanglement cost \(E_c\) [32] is defined as
\[
E_c(\rho) = \inf \{ t : \lim_{n \to \infty} \| \rho^{\otimes n} - \Lambda_{\text{LOCC}}(\phi_+^{\otimes n}) \|_{\text{tr}} = 0 \},
\]
with \(\phi_+ = \frac{1}{\sqrt{2}} (|01\rangle - |10\rangle)\), \(\Lambda_{\text{LOCC}}\) being the local operation and classical communication (LOCC) and trace norm \(\| A \|_{\text{tr}} = \text{Tr} [\sqrt{A^\dagger A}]\).

Proof. In view of Theorem 6, we have the following relationship for the bipartite state \(\rho_{AB}^{\otimes n}\),
\[
C_f(\rho_{AB}^{\otimes n}) \geq C_f(\rho_A^{\otimes n}) + C_f(\rho_B^{\otimes n}) + E_f(\rho_{AB}^{\otimes n}).
\]
Since both \(C_r\) and \(C_f\) are additive [7] and \(C_c\) is equivalent to the regularized entanglement of formation \(E_f\) [32], we have
\[
C_f(\rho_{AB}) \geq C_f(\rho_A) + C_f(\rho_B) + E_c(\rho_{AB}).
\]
Similarly, we can also obtain the following relation,
\[
C_f(\rho_{AB}) \geq C_r(\rho_B) + C_f(\rho_A) + E_c(\rho_{AB}).
\]
Therefore, we obtain the result.

It has been proved that relative entropy of coherence \(C_r\) is equivalent to distillable coherence \(C_d\) [7]. Thus we can obtain the relationship between the distillable coherence and distillable entanglement in bipartite systems as follows.

**Corollary 8.** Given a bipartite state \(\rho_{AB} \in D(\mathcal{H}_A \otimes \mathcal{H}_B)\), \(C_d(\rho_{AB})\) and \(E_d(\rho_{AB})\) has the following relationship,
\[
C_d(\rho_{AB}) \geq C_d(\rho_A) + C_d(\rho_B) + E_d(\rho_{AB}),
\]
where \(\rho_A\) and \(\rho_B\) are the corresponding reduced states of \(\rho_{AB}\) and the distillable entanglement \(E_d\) [33] is defined as
\[
E_d(\rho) = \inf \{ t : \lim_{n \to \infty} \| \Lambda_{\text{LOCC}}(\rho^{\otimes n}) - \phi_+^{\otimes n} \|_{\text{tr}} = 0 \}.
\]

Proof. It has been proved in Ref. [25] that
\[
C_r(\rho_{AB}) \geq C_r(\rho_A) + C_r(\rho_B) + E_r^{\infty}(\rho_{AB}),
\]
where \(E_r^{\infty}\) is the regularized relative entropy of entanglement [34–36]. Due to the equivalence between \(C_r\) and \(C_d\) [7] and the fact that \(E_r^{\infty} \geq E_d\) [11], we obtain the result.

In tripartite systems, the monogamy relation of coherence has been considered for relative entropy of coherence \(C_r\) and it has been shown in Refs. [37, 38] that it does not hold in general for \(C_r\). However, the monogamy relation for IQ coherence measure \(C_r^{AB}\) has been established in Ref. [25]. Here, we obtain the monogamy relation for \(C_r^{AB}\) in tripartite systems as follows.

**Proposition 9.** Given a bipartite state \(\rho_{ABC} \in D(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C)\), then
\[
C_r^{AB|C}(\rho_{ABC}) \geq C_r^{AB|C}(\rho_{ABC}) + C_r^{BC|C}(\rho_{ABC}).
\]
which implies the following monogamy relation,
\[
C_r^{AB|C}(\rho_{ABC}) \geq C_r^{AB|C}(\rho_{AC}) + C_r^{BC|C}(\rho_{BC}).
\]

Proof. For any pure tripartite state \(|\psi\rangle_{ABC} = \sum_i \sqrt{p_i} |i\rangle_A \otimes |u_i\rangle_B \otimes |v_i\rangle_C\),
\[
C_r^{AB|C}(|\psi\rangle_{ABC})
= C_r^{AB|C}(|\psi\rangle_{ABC})
= S(\Delta_A \otimes \Delta_B(\rho_{AB}))
= S(\Delta_A(\rho_A)) + \sum_i p_i S(\Delta_B(\rho_i^B))
= S(\Delta_A(\rho_A)) + \sum_i p_i S(\Delta_B(\text{Tr}_C|u_i\rangle|u_i\rangle_{BC}))
\geq C_r^{AB|C}(|\psi\rangle_{ABC}) + C_r^{BC|C}(\rho_{BC}),
\]
where the last inequality results from the fact that \(S(\Delta_A(\rho_A)) = C_r^{AB|C}(\psi_{ABC})\) for pure state \(\psi_{ABC}\) and \(\sum_i p_i S(\Delta_B(\text{Tr}_C|u_i\rangle|u_i\rangle_{BC})) \leq C_r^{BC|C}(\rho_{BC})\) due to the definition of \(C_r^{BC|C}\). For any tripartite states \(\rho_{ABC}\), there exists an optimal pure state decomposition of \(\rho_{ABC} = \sum_i \lambda_i |\psi_i\rangle_{ABC}\) such that \(C_r^{AB|C}(\rho_{ABC}) = \sum_i \lambda_i C_r^{AB|C}(|\psi_i\rangle_{ABC})\). Thus,
\[
C_r^{AB|C}(\rho_{ABC}) = \sum_i \lambda_i C_r^{AB|C}(|\psi_i\rangle_{ABC})
\geq \sum_i \lambda_i [C_r^{AB|C}(|\psi_i\rangle_{ABC}) + C_r^{BC|C}(\rho_i^{BC})]
\geq C_r^{AB|C}(\rho_{ABC}) + C_r^{BC|C}(\rho_{BC}).
\]
Similar to coherence of formation $C_f$, coherence of assistance $C_a$ is also defined by taking the pure state decompositions of the given state [9]. Here, we introduce the IQ coherence of assistance $C_{a}^{AB}$ on bipartite systems, which is defined as follows

$$C_{a}^{AB}(\rho_{AB}) := \max \sum_i p_i C_r^{AB}(|\psi_i\rangle\langle\psi_i|_{AB}) = \max \sum_i p_i S(\Delta_A(|\psi_i\rangle\langle\psi_i|_{AB})), \quad (15)$$

where the maximization is taken over all pure state decomposition $\rho_{AB} = \sum_i p_i |\psi_i\rangle\langle\psi_i|_{AB}$. Following the similar method, we can obtain the relationship between coherence of assistance $C_a$ and entanglement of assistance $E_a$ [39] in bipartite systems as follows.

**Theorem 10.** Given a bipartite state $\rho_{AB} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$, it holds that,

$$C_{a}^{AB}(\rho_{AB}) \leq C_a(\rho_A) + E_a(\rho_B),$$
$$C_a(\rho_{AB}) \leq C_{a}^{AB}(\rho_{AB}) + C_a(\rho_B),$$
$$C_{a}(\rho_{AB}) \leq C_a(\rho_A) + C_a(\rho_B) + E_a(\rho_{AB}),$$

where $E_a(\rho_{AB}) = \max \sum_i p_i S(\Tr_A [\rho_{AB}])$ with the maximization being taken over all pure state decomposition of $\rho_{AB} = \sum_i p_i |\psi_i\rangle\langle\psi_i|_{AB}$ and $\rho_A, \rho_B$ are the reduced states of $\rho_{AB}$ on subsystems $A$ and $B$, respectively.

Theorem 10 illustrates that the total coherence of assistance in bipartite systems is upper bounded by the sum of coherence of assistance in each subsystem and the entanglement of formation between subsystems. The proof of Theorem 10 is almost the same as that of Theorem 6, thus we omit it here. The regularized version of coherence of assistance $C_a^{\infty}$ has also been proposed in Ref. [9], which is defined as $C_a^{\infty}(\rho) := \lim_{n \to \infty} \frac{1}{n} C_a^{\rho^\otimes n}(\rho) = S(\Delta(\rho))$. Moreover, for any state extension $\rho_{AB}$ of a given state $\rho_A$, i.e., $\Tr_A[\rho_{AB}] = \rho_A$, $C_a^{\infty}(\rho_A)$ is an upper bound of $C_{a}^{AB}(\rho_{AB})$. In fact, $C_a^{\infty}(\rho_A)$ is the maximum value of $C_{a}^{AB}(\rho_{AB})$ for the state extension $\rho_{AB}$ of $\rho_A$.

**Proposition 11.** Given a quantum state $\rho_A \in \mathcal{D}(\mathcal{H}_A)$, then

$$C_{a}^{\infty}(\rho_A) = \max_{\rho_{AB}, \Tr_B[\rho_{AB}] = \rho_A} C_{a}^{AB}(\rho_{AB}). \quad (16)$$

*Proof.* First, for pure bipartite state $\psi_{AB}$ with $\Tr_B[\psi_{AB}] = \rho_A$, then $C_{a}^{AB}(\psi_{AB}) = S(\Delta_A(\rho_A)) = C_{a}^{\infty}(\rho_A)$. Besides, for mixed bipartite state $\rho_{AB}$ with $\Tr_B[\rho_{AB}] = \rho_A$, there exists a purification $\psi_{ABC}$ of $\rho_{AB}$ such that $\Tr_C[\psi_{ABC}] = \rho_{AB}$. Since $C_{a}^{AB}$ is monotone under completely positive and trace preserving (CPTP) maps on $B$ side, then $C_{a}^{AB}(\rho_{AB}) \leq C_{a}^{ABC}(\psi_{ABC}) = S(\Delta_A(\rho_A)) = C_{a}^{\infty}(\rho_A)$.

\[ \Box \]

**IV. $l_1$ NORM OF IQ COHERENCE**

In order to introduce $l_1$ norm of IQ coherence on bipartite systems, let us first introduce a new norm $\|Q\|_{l_1}^{\tr}$ on $B(\mathcal{H}_A \otimes \mathcal{H}_B)$ with a fixed basis $\{|i\}_A \otimes \{|j\}_B$ of $\mathcal{H}_A \otimes \mathcal{H}_B$. For any operator $Q \in B(\mathcal{H}_A \otimes \mathcal{H}_B)$ written as $Q = \sum_{i,j=1}^{d_A} |i\rangle\langle j|_A \otimes Q_{ij}$, the norm $\|Q\|_{l_1}^{\tr}$ is defined as follows,

$$\|Q\|_{l_1}^{\tr} := \sum_{i,j=1}^{d_A} \|Q_{ij}\|_{l_1}^{\tr}, \quad (17)$$

where $\|A\|_{l_1}^{\tr} = \Tr \sqrt{A^\dagger A}$. It is easy to show that $\|\cdot\|_{l_1}^{\tr}$ is a norm, that is, it satisfies the following properties: (i) Positivity, $\|Q\|_{l_1}^{\tr} \geq 0$ and $\|Q\|_{l_1}^{\tr} = 0 \iff Q = 0$; (ii) $\|\alpha Q\|_{l_1}^{\tr} = |\alpha| \|Q\|_{l_1}^{\tr}$ for any $\alpha \in \mathbb{C}$; (iii) Triangle inequality, $\|Q + P\|_{l_1}^{\tr} \leq \|Q\|_{l_1}^{\tr} + \|P\|_{l_1}^{\tr}$ for any operators $Q, P \in B(\mathcal{H}_A \otimes \mathcal{H}_B)$.

Based on this new norm, we define $l_1$ norm of IQ coherence on bipartite systems as follows,

$$C_{l_1}^{AB}(\rho_{AB}) := \min_{\sigma_{AB} \in \mathcal{E}(\mathcal{Q})} \|\rho_{AB} - \sigma_{AB}\|_{l_1}^{\tr}$$
$$= \sum_{i \neq j} \|\rho_{ij}\|_{l_1}^{\tr}, \quad (18)$$

where $\rho_{ij} = \langle i | \rho_{AB} | j \rangle_A$. Note that, $C_{l_1}^{AB}$ satisfies the properties (i)-(v), where the positivity of $C_{l_1}^{AB}$ (i.e., property (i)) comes from the positivity of the norm $\|\cdot\|_{l_1}^{\tr}$, (iv) results from the contractivity of $\|\cdot\|_{l_1}^{\tr}$ under CPTP maps, (v) comes from the triangle inequality of the norm $\|\cdot\|_{l_1}^{\tr}$, and (v) lead to the property (ii). Thus, we only need to prove (iii), which is presented in the Appendix B.

Due to the definition, $C_{l_1}^{AB}(\rho_{AB}) \geq C_{l_1}(\rho_A)$ with $\rho_B$ being the reduced state of $\rho_{AB}$, which comes from the fact that $\|\rho_{ij}\|_{l_1}^{\tr} \geq |\Tr [\rho_{ij}]|$. If the subsystem $B$ is a trivial system, i.e., $\dim \mathcal{H}_B = 1$, then $C_{l_1}^{AB}(\rho_{AB})$ reduces to $C_{l_1}(\rho_A)$. Besides, for bipartite pure state $|\psi_{AB}\rangle$, which can be written as $|\psi_{AB}\rangle = \sum_{i=1}^{d_A} \sqrt{p_i} |i\rangle_A |u_i\rangle_B$, $C_{l_1}^{AB}(\psi_{AB}) = (\sum_{i=1}^{d_A} \sqrt{p_i})^2 - 1 \leq d_A - 1$. Thus, the maximum value for $C_{l_1}^{AB}$ is $d_A - 1$ which does not depend on the subsystem $B$.

**Proposition 12.** Given an bipartite state $\rho_{AB} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$, then

$$C_{l_1}(\rho_{AB}) \geq C_{l_1}^{AB}(\rho_{AB}) + C_{l_1}(\rho_B), \quad (19)$$

where $\rho_B$ is the reduced state of $\rho_{AB}$.

*Proof.* For any bipartite state $\rho_{AB} = \sum_{i,j} |i\rangle\langle j|_A \otimes \rho_{ij}$, the reduced state $\rho_B$ can be written as $\rho_B = \sum_i \rho_{ii}$. Thus

$$C_{l_1}^{AB}(\rho_{AB}) = \sum_{i \neq j} \|\rho_{ij}\|_{l_1}^{\tr},$$
$$C_{l_1}(\rho_B) = \sum_{i} \|\rho_{ii}\|_{l_1}^{\tr} - 1 = \sum_{i \neq j} \|\rho_{ij}\|_{l_1}^{\tr} - \sum_{i \neq j} \|\rho_{ij}\|_{l_1}^{\tr},$$
$$C_{l_1}(\rho_{AB}) = \sum_{i \neq j} \|\rho_{ij}\|_{l_1}^{\tr} + \sum_{i \neq j} \|\rho_{ij}\|_{l_1}^{\tr},$$
Since $\sum_{j \neq k} |\sum_i \langle j | p_{ij}^B | k \rangle_B| \leq \sum_{i \neq j} |\sum_i \langle j | p_{ij}^B | k \rangle_B|$, and $\sum_{i \neq j} \|p_{ij}^B\|_{tr} \leq \sum_{i \neq j} \|p_{ij}^B\|_{l_1}$ (see Lemma 21 in Appendix C), we get the result.

This relation (19) is stronger than the known result $C_{l_1}(\rho_{AB}) \geq C_{l_1}(\rho_A) + C_{l_1}(\rho_B)$ as $C_{l_1}^{A|B}(\rho_{AB}) \geq C_{l_1}(\rho_A)$. Besides, $C_{l_1}^{A|B}$ contains not only the local coherence in subsystem A but also the nonlocal correlation between A and B from the following proposition.

**Proposition 13.** Given a bipartite state $\rho_{AB} \in D(H_A \otimes H_B)$, then we have the following relationship,

$$C_{l_1}^{A|B}(\rho_{AB})^2 - C_{l_1}(\rho_A)^2 \geq 2(\text{Tr} \left[ \rho_{AB}^2 \right] - \text{Tr} \left[ \rho_A^2 \right]) \tag{20}$$

**Proof.** Any bipartite state $\rho_{AB}$ can be written as $\rho_{AB} = \sum_{i,j=0}^{d_A} |i\rangle \langle i|_A \otimes \rho_{ij}^B$ and thus

$$C_{l_1}^{A|B}(\rho_{AB}) = 2 \sum_{i<j} |\text{Tr} [\rho_{ij}^B]|,$$

$$C_{l_1}(\rho_A) = 2 \sum_i |\text{Tr} [\rho_{i}^B]|.$$

Moreover, the term $\text{Tr} \left[ \rho_{AB}^2 \right] - \text{Tr} \left[ \rho_A^2 \right]$ has the following upper bound,

$$\text{Tr} \left[ \rho_{AB}^2 \right] - \text{Tr} \left[ \rho_A^2 \right]$$

$$= \left( \sum_i \text{Tr} \left[ |\rho_{ii}^B|^2 \right] + 2 \sum_{i<j} |\text{Tr} [\rho_{ij}^B]| \right)$$

$$- \left( \sum_i \text{Tr} \left[ |\rho_{ii}^B|^2 \right] + 2 \sum_{i<j} |\text{Tr} [\rho_{ij}^B]| \right)$$

$$\leq 2 \sum_{i<j} (\text{Tr} \left[ |\rho_{ij}^B|^2 \right] - |\text{Tr} [\rho_{ij}^B]|^2)$$

$$\leq 2 \sum_{i<j} (\text{Tr} \left[ |\rho_{ij}^B|^2 \right] - |\text{Tr} [\rho_{ij}^B]|^2),$$

where the first and second inequalities come from the fact that $\text{Tr} \left[ |\rho_{ij}^B|^2 \right] \leq \text{Tr} \left[ |\rho_{ij}^B| \right]$. Therefore,

$$C_{l_1}^{A|B}(\rho_{AB})^2 - C_{l_1}(\rho_A)^2$$

$$= 4 \left( \sum_{i<j} \text{Tr} \left[ |\rho_{ij}^B| \right] - \left( \sum_i \text{Tr} [\rho_{ij}^B] \right)^2 \right)$$

$$= 4 \left( \sum_{i<j} \left( \text{Tr} \left[ |\rho_{ij}^B| \right] - |\text{Tr} [\rho_{ij}^B]| \right) \right) \left( \sum_{i<j} \left( \text{Tr} \left[ |\rho_{ij}^B| \right] + |\text{Tr} [\rho_{ij}^B]| \right) \right)$$

$$\geq 4 \left( \sum_{i<j} \text{Tr} \left[ |\rho_{ij}^B| \right] - |\text{Tr} [\rho_{ij}^B]| \right)^2$$

$$\geq 2 \left( \text{Tr} \left[ \rho_{AB}^2 \right] - \text{Tr} \left[ \rho_A^2 \right] \right),$$

where the first inequality comes directly from the fact that $\text{Tr} \left[ |\rho_{ij}^B| \right] \leq \text{Tr} \left[ |\rho_{ij}^B| \right]$ and the second inequality comes from the upper bound of $\text{Tr} \left[ \rho_{AB}^2 \right] - \text{Tr} \left[ \rho_A^2 \right]$.

The term $\text{Tr} \left[ \rho_{AB}^2 \right] - \text{Tr} \left[ \rho_A^2 \right]$ quantifies the entanglement between A and B as

$$\text{Tr} \left[ \rho_{AB}^2 \right] - \text{Tr} \left[ \rho_A^2 \right] > 0 \tag{21}$$

only if $\rho_{AB}$ is entangled [40] and the inequality (21) provides a powerful tool in the detection of entanglement in experiments [41, 42]. Thus, the above proposition implies that the total coherence in bipartite system quantified by $l_1$ norm consists of the nonlocal correlation between A and B and the local coherence $C_{l_1}(\rho_A)$ and $C_{l_1}(\rho_B)$. Furthermore, we obtain the monogamy relation of $C_{l_1}^{A|B}$ in tripartite systems, which clarifies the distribution of coherence by $l_1$ norm in multipartite systems.

**Proposition 14.** Given a tripartite state $\rho_{ABC} \in D(H_A \otimes H_B \otimes H_C)$, then

$$C_{l_1}^{A|B|C}(\rho_{ABC}) \geq C_{l_1}^{A|C}(\rho_{ABC}) + C_{l_1}^{B|C}(\rho_{ABC}), \tag{22}$$

which implies the following monogamy relation,

$$C_{l_1}^{A|B|C}(\rho_{ABC}) \geq C_{l_1}^{A|C}(\rho_{ABC}) + C_{l_1}^{B|C}(\rho_{ABC}), \tag{23}$$

where $\rho_{AC}, \rho_{BC}$ are the corresponding reduced states of $\rho_{ABC}$.

**Proof.** Any tripartite state $\rho_{ABC}$ can be written as $\rho_{ABC} = \sum_{i,j} \sum_{m,n} |i\rangle \langle j|_A \otimes |m\rangle \langle n|_B \otimes \rho_{ij,mn}^{ABC}$ with the local basis $\{ |i\rangle_1 \}_i$ and $\{ |m\rangle_1 \}_m$ of $H_A$ and $H_B$. Then the reduced state $\rho_{BC} = \sum_i \sum_{m,n} |m\rangle \langle n|_B \otimes \rho_{i,mn}^{BC}$. Thus

$$C_{l_1}^{ABC}(\rho_{ABC}) = \sum_{(i,m) \neq (j,n)} \|\rho_{ij,mn}^{ABC}\|_{tr},$$

$$C_{l_1}^{A|BC}(\rho_{ABC}) = \sum_{i \neq j} \sum_{m,n} \|\rho_{i,mn}^{BC}\|_{tr} \leq \sum_{i \neq j} \sum_{m,n} \|\rho_{i,mn}^{BC}\|_{tr},$$

$$= \sum_{i \neq j} \sum_{m,n} \|\rho_{ij,mn}^{ABC}\|_{tr},$$

$$C_{l_1}^{B|C}(\rho_{BC}) = \sum_{m \neq n} \sum_i \|\rho_{i,mn}^{BC}\|_{tr} \leq \sum_{m \neq n} \|\rho_{i,mn}^{BC}\|_{tr},$$

where $(i,m) \neq (j,n)$ means $i \neq j$ or $m \neq n$. Therefore, we get the result.

Now, let us consider the relationship between $C_{l_1}^{A|B}$ and $C_{l_1}^{A|B}$, where we find that $C_{l_1}^{A|B}$ is closely related to $C_{l_1}^{A|B}$ and they are equal for certain type of bipartite states.

**Proposition 15.** Given a bipartite state $\rho_{AB} \in D(H_A \otimes H_B)$, then

$$1 + \frac{1}{d_A - 1} C_{l_1}^{A|B}(\rho_{AB}) \leq 2 C_{l_1}^{A|B}(\rho_{AB}) \leq 1 + C_{l_1}^{A|B}(\rho_{AB}). \tag{24}$$

where $d_A$ is the dimension of $H_A$. 
Proof. Any bipartite state $\rho_{AB}$ can be written as $\rho_{AB} = \sum_{i,j} |j\rangle_A \otimes \rho_{ij}^B$, then $C_{i_1}^{AB}(\rho_{AB}) = \sum_{i,j} \|\rho_{ij}^B\|_{tr} = \sum_{i,j} \|\rho_{ij}^B\|_{tr} - 1$. For $\rho_{ij}^B$ with $i < j$, there exists a unitary $U_{ij}^B$ such that $U_{ij}^B \rho_{ij}^B = |\rho_{ij}^B|^B$, and thus $\rho_{ij}^B U_{ij}^B \rho_{ij}^B = |\rho_{ij}^B|^B$ as $\langle \rho_{ij}^B \rangle = \rho_{ij}^B$. Now, let us take the positive operator $M$ as follows,

$$M = \frac{1}{d_A - 1} \sum_{i<j} [|i\rangle_i |A \otimes \|B + |j\rangle_i |A \otimes B_B + |i\rangle_i |A \otimes U_{ij}^B + |j\rangle_i |A \otimes U_{ij}^B] = I_{AB} + \frac{1}{d_A - 1} \sum_{i<j} [|i\rangle_i |A \otimes U_{ij}^B + |j\rangle_j |A \otimes U_{ij}^B],$$

where the positivity of $M$ comes from the fact that $|i\rangle_i |A \otimes \|B + |j\rangle_j |A \otimes B_B = |i\rangle_i |A \otimes U_{ij}^B + |j\rangle_j |A \otimes U_{ij}^B$ and the fact that $|X| + X \geq 0$ for any Hermitian operator $X$.

Due to the definition of $C_{i_1}^{\max}$, there exists an incoherent-quantum state $\tau_{AB}$ such that

$$\rho_{AB} \leq 2C_{i_1}^{\max}(\rho_{AB}) \tau_{AB}.$$ 

Thus

$$\text{Tr}[M \rho_{AB}] \leq 2C_{i_1}^{\max}(\rho_{AB}) \text{Tr}[M \tau_{AB}],$$

which leads to

$$1 + \frac{1}{d_A - 1} C_{i_1}^{AB}(\rho_{AB}) \leq 2C_{i_1}^{\max}(\rho_{AB}).$$

Besides, let us take the incoherent-quantum state $\sigma_{AB}$ to be

$$\sigma_{AB} = \frac{1}{1 + C_{i_1}^{AB}(\rho_{AB})} \sum_i |i\rangle_i |A \otimes \rho_{ii}^B + \sum_{i<j} |i\rangle_i |A \otimes \rho_{ij}^B + |j\rangle_j |A \otimes \rho_{ji}^B|.$$ 

Then $\rho_{AB} \leq (1 + C_{i_1}^{AB}(\rho_{AB})) \sigma_{AB}$, as

$$C_{i_1}^{AB}(\rho_{AB}) \sigma_{AB} - \rho_{AB} = \sum_{i<j} [|i\rangle_i |A \otimes \rho_{ij}^B + |j\rangle_j |A \otimes \rho_{ji}^B| - |i\rangle_i |A \otimes \rho_{ij}^B - |j\rangle_j |A \otimes \rho_{ji}^B| \geq 0,$$

where the inequality comes from the fact that $|i\rangle_i |A \otimes \rho_{ij}^B + |j\rangle_j |A \otimes \rho_{ji}^B| = |i\rangle_i |A \otimes \rho_{ij}^B + |j\rangle_j |A \otimes \rho_{ji}^B|$ for $i \neq j$.

\[\square\]

Proposition 16. Given a bipartite state $\rho_{AB} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$, if there exists a unitary operator $U_{AB} = \sum_i |i\rangle_i |A \otimes U_{ij}^B$ such that $U_{ij}^B U_{ij}^{B,1} = |\rho_{ij}^B|^B$, for any $i,j$, then

$$C_{i_1}^{\max}(\rho_{AB}) = \log(1 + C_{i_1}^{AB}(\rho_{AB})),$$

where $|P|$ is defined as $|P| = \sqrt{P^\dagger P}$.

\[\square\]

V. ADDITIVITY OF IQ COHERENCE MEASURES

The above sections show that IQ coherence measures can capture the nonlocal correlation between subsystems. However, the measure of nonlocal correlation may not be additive, such as the local entropy of entanglement. Thus we
discuss the additivity of IQ coherence measures in this section. Let us begin with the simplest case, relative entropy and $l_1$ norm. In view of the definition, it is easy to see the additivity of $C_A^{1|B}$ and $C_{1i}^{A|B}$: for any two bipartite states $\rho_{A_1B_1} \in D(\mathcal{H}_{A_1} \otimes \mathcal{H}_{B_1})$ and $\rho_{A_2B_2} \in D(\mathcal{H}_{A_2} \otimes \mathcal{H}_{B_2})$, then

$$C_r^{A_1A_2|B_1B_2}(\rho_{A_1B_1} \otimes \rho_{A_2B_2}) = C_r^{A_1|B_1}(\rho_{A_1B_1}) + C_r^{A_2|B_2}(\rho_{A_2B_2}),$$

and

$$1 + C_{1i}^{A_1A_2|B_1B_2}(\rho_{A_1B_1} \otimes \rho_{A_2B_2}) = [1 + C_{1i}^{A_1|B_1}(\rho_{A_1B_1}) \cdot [1 + C_{1i}^{A_2|B_2}(\rho_{A_2B_2})].$$

Now, we consider the additivity of IQ coherence measures $C_{\text{max}}^{A|B}$ and $C_{\text{tr}}^{A|B}$, for which we have the following propositions.

**Proposition 17.** For any two bipartite states $\rho_{A_1B_1} \in D(\mathcal{H}_{A_1} \otimes \mathcal{H}_{B_1})$ and $\rho_{A_2B_2} \in D(\mathcal{H}_{A_2} \otimes \mathcal{H}_{B_2})$, then

$$C_{\text{max}}^{A_1A_2|B_1B_2}(\rho_{A_1B_1} \otimes \rho_{A_2B_2}) = C_{\text{max}}^{A_1|B_1}(\rho_{A_1B_1}) + C_{\text{max}}^{A_2|B_2}(\rho_{A_2B_2}).$$

**Proof.** Due to definition of max-relative entropy of IQ coherence measure, there exists optimal IQ states $\sigma_{A_1|B_1}$ and $\sigma_{A_2|B_2}$ such that $\rho_{A_1B_1} \leq 2C_{\text{max}}^{A_1|B_1}(\rho_{A_1B_1})\sigma_{A_1|B_1}$. Hence, we have the following inequality,

$$C_{\text{max}}^{A_1A_2|B_1B_2}(\rho_{A_1B_1} \otimes \rho_{A_2B_2}) \leq C_{\text{max}}^{A_1|B_1}(\rho_{A_1B_1}) + C_{\text{max}}^{A_2|B_2}(\rho_{A_2B_2}).$$

Now, we prove the converse. It has been proved in Ref. [25] that

$$2C_{\text{max}}^{A|B}(\rho_{AB}) = \max_{\tau_{AB} \geq 0, \Delta \in \otimes \Omega_{AB} = 1} \text{Tr} [\rho_{AB} T_{AB}].$$

Hence, there exist operators $\tau_{A_iB_i}$ such that $\tau_{A_iB_i} \geq 0$, $\Delta_{A_i} \otimes \Omega_{B_i}(\tau_{A_iB_i}) = 1$, and $\Delta_{A_i} \otimes \Omega_{B_i}(\rho_{A_iB_i}) = \text{Tr} [\rho_{AB} T_{AB} B_i]$, for $i = 1, 2$. Then the operator $\tau_{A_1A_2B_1B_2} := \tau_{A_1B_1} \otimes \tau_{A_2B_2}$ satisfies the conditions $\tau_{A_1A_2B_1B_2} \geq 0$ and $\Delta_{A_1} \otimes \Delta_{A_2} \otimes \Omega_{B_1B_2}(\tau_{A_1A_2B_1B_2}) = \Omega_{A_1A_2B_1B_2}$, which implies that $2C_{\text{max}}^{A_1A_2|B_1B_2}(\rho_{A_1B_1} \otimes \rho_{A_2B_2}) \geq \text{Tr} [\rho_{A_1B_1} \otimes \rho_{A_2B_2} T_{A_1A_2B_1B_2}], i.e.,

$$C_{\text{max}}^{A_1A_2|B_1B_2}(\rho_{A_1B_1} \otimes \rho_{A_2B_2}) \geq C_{\text{max}}^{A_1|B_1}(\rho_{A_1B_1}) + C_{\text{max}}^{A_2|B_2}(\rho_{A_2B_2}).$$

Therefore, we obtain the result. \qed

**Proposition 18.** For any two bipartite states $\rho_{A_1B_1} \in D(\mathcal{H}_{A_1} \otimes \mathcal{H}_{B_1})$ and $\rho_{A_2B_2} \in D(\mathcal{H}_{A_2} \otimes \mathcal{H}_{B_2})$, then

$$C_f^{A_1A_2|B_1B_2}(\rho_{A_1B_1} \otimes \rho_{A_2B_2}) = C_f^{A_1|B_1}(\rho_{A_1B_1}) + C_f^{A_2|B_2}(\rho_{A_2B_2}).$$

**Proof.** Due to the definition of $C_f^{A|B}$, it is easy to get the inequality

$$C_f^{A_1A_2|B_1B_2}(\rho_{A_1B_1} \otimes \rho_{A_2B_2}) \leq C_f^{A_1|B_1}(\rho_{A_1B_1}) + C_f^{A_2|B_2}(\rho_{A_2B_2}).$$

Thus, we only need to prove the converse. First, we prove that for any pure state $|\psi\rangle_{A_1A_2B_1B_2}$, the following inequality holds,

$$C_f^{A_1A_2|B_1B_2}(\rho_{A_1A_2B_1B_2}) \geq C_f^{A_1|B_1}(\sigma_{A_1A_2B_1B_2}) + C_f^{A_2|B_2}(\sigma_{A_2B_2}),$$

where $\sigma_{A_1A_2B_1B_2}$ are the corresponding reduced states of $|\psi\rangle_{A_1A_2B_1B_2}$. Since the pure state $|\psi\rangle_{A_1A_2B_1B_2}$ can be written as $|\psi\rangle_{A_1A_2B_1B_2} = \sum_i \sqrt{p_i} |i\rangle_{A_1} |u_i\rangle_{A_2B_1B_2}$, then

$$C_f^{A_1A_2|B_1B_2}(\rho_{A_1B_1}) = S(\Delta_{A_1} \otimes \Delta_{A_2}(\sigma_{A_1A_2B_1B_2})) = S(\Delta_{A_1}(\sigma_{A_1B_1})) + \sum_i p_i S(\Delta_{A_2}(\sigma_{iA_2B_1B_2})) \geq C_f^{A_1|B_1}(\sigma_{A_1B_1}) + C_f^{A_2|B_2}(\sigma_{A_2B_2}),$$

Note that the additivity of $C_A^{1|B}$ is still unclear as the method used in the proof of the additivity of $C_f^{A|B}$ does not work for $C_A^{1|B}$. Nevertheless, if the subsystems $B_i$ ($i = 1, 2$) are trivial, i.e., the dimension is 1, then one has the additivity of the coherence measures. For example, the additivity of IQ
coherence measures $C_{\text{max}}^{A|B}$ will lead to the additivity of $C_{\text{max}}$ if the subsystems $B_i$ ($i = 1, 2$) are trivial.

**Corollary 19.** Given two quantum states $\rho_1 \in \mathcal{D}(\mathcal{H}_{A_1})$ and $\rho_2 \in \mathcal{D}(\mathcal{H}_{A_2})$, it holds that

$$C_{\text{max}}(\rho_1 \otimes \rho_2) = C_{\text{max}}(\rho_1) + C_{\text{max}}(\rho_2).$$

(29)

Due to the additivity of $C_{\text{max}}$, we can obtain the additivity of robustness of coherence $\text{ROC}$ [21] as follows,

$$1 + \text{ROC}(\rho_1 \otimes \rho_2) = [1 + \text{ROC}(\rho_1)] \cdot [1 + \text{ROC}(\rho_2)],$$

(30)

which comes directly from the fact that $C_{\text{max}}(\rho) = \log(1 + \text{ROC}(\rho))$ [23]. Following the same method, it is easy to obtain the additivity of coherence weight $C_w$ as following,

$$1 - C_w(\rho_1 \otimes \rho_2) = [1 - C_w(\rho_1)] \cdot [1 - C_w(\rho_2)].$$

(31)

Thus, the additivity of robustness of coherence and coherence weight are proved here, which will be useful to the further study on the distribution of coherence in multipartite systems quantified by robustness of coherence and coherence weight.

**VI. CONCLUSION**

In this work, we have investigated the properties of the incoherent-quantum coherence measures defined by relative entropy, max-relative entropy and $l_1$ norm on bipartite systems. We also introduce the IQ coherence of formation and assistance on bipartite systems. And we have found the distribution of coherence of formation $C_f$ and assistance $C_a$ in bipartite systems: the total coherence of formation is lower (upper) bounded by the sum of coherence of formation (assistance) in each local subsystem and entanglement of formation (assistance) between subsystems. Besides, we have obtained the tradeoff relation between coherence cost and entanglement cost, distillable coherence and distillable entanglement in bipartite systems. Moreover, we have obtained the monogamy relationship of the IQ coherence of formation and assistance in tripartite systems. Furthermore, the additivity of IQ coherence measures have been discussed. These results substantially advance the understanding of the physical laws that governs the distribution of quantum coherence in bipartite systems and pave the way for the further researches in this direction.

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where the first inequality comes from the fact that $C_f^{A|B}$ is strong monotonicity under IO on A side for $|\psi\rangle_{AB}$ and the second inequality comes from the fact the concavity of von Neumann entropy.

**Lemma 20.** Given a bipartite pure state $|\psi\rangle_{AB}$, it holds that

$$S(\Delta_A(\psi_{AB})) = S(\Delta_A(\text{Tr}_B [\psi_{AB}])).$$

**Proof.** Since the pure state $|\psi\rangle_{AB}$ can be expressed in the given basis $\{|i\rangle_A\}_i$ of $\mathcal{H}_A$ as follows,

$$|\psi\rangle_{AB} = \sum_i \sqrt{p_i} |i\rangle_A |u_i\rangle_B,$$

with $p_i \geq 0$ and $\sum_i p_i = 1$. Then

$$\Delta_A(\psi_{AB}) = \sum_i p_i |i\rangle_A \otimes |u_i\rangle_B |u_i\rangle_B,$$

$$\Delta_A(\text{Tr}_B [\psi_{AB}]) = \sum_i p_i |i\rangle_A |u_i\rangle_B |u_i\rangle_B,$$

which implies that $S(\Delta_A(\psi_{AB})) = S(\Delta_A(\text{Tr}_B [\psi_{AB}]))) = -\sum_i p_i \log p_i.$
\[ K^A_{\mu} \rho_{AB} K^{A,1}_{\mu}/p_{\mu} \text{ and } p_{\mu} = \text{Tr} \left[ K^A_{\mu} \rho_{AB} K^{A,1}_{\mu} \right]. \] Thus

\[
\sum_{\mu} p_{\mu} C_i^{A|B}(\rho^B_{\mu}) = \sum_{\mu} p_{\mu} \sum_{i \neq j} \| \rho^B_{\mu} \|_{tr}
\]

\[
= \sum_{\mu} \sum_{i \neq j} \| \langle i | K^A_{\mu} \rho_{AB} K^{A,1}_{\mu} | j \rangle \|_{tr}
\]

\[
= \sum_{\mu} \sum_{i \neq j} \left\| \sum_{r,s} [K^A_{\mu}]_{ir} [K^{A,1}_{\mu}]_{sj} \rho^B_{rs} \right\|_{tr}
\]

where \( \rho^B_{rs} = \langle r | \rho^B \| s \rangle \in B(\mathcal{H}_B) \).

Since \( A^A \) is incoherent, then \([K^A_{\mu}]_{ir} [K^{A,1}_{\mu}]_{rj} = \delta_{ij} \) for any \( r \), where \( \delta_{ij} = 1 \) if \( i = j \), otherwise \( \delta_{ij} = 0 \). Therefore

\[
\sum_{\mu} p_{\mu} C_i^{A|B}(\rho^B_{\mu}) = \sum_{\mu} \sum_{i \neq j} \sum_{r,s} \left\| [K^A_{\mu}]_{ir} [K^{A,1}_{\mu}]_{sj} \rho^B_{rs} \right\|_{tr}
\]

\[
\leq \sum_{\mu} \sum_{i \neq j} \left\| [K^A_{\mu}]_{ir} [K^{A,1}_{\mu}]_{sj} \rho^B_{rs} \right\|_{tr}
\]

\[
= \sum_{r \neq s} \left\| \rho^B_{rs} \right\|_{tr}
\]

where the last inequality comes from the fact that \( \sum_{\mu} \sum_{i \neq j} \left\| [K^A_{\mu}]_{ir} [K^{A,1}_{\mu}]_{sj} \right\| \leq 1 \) given in [4]. Thus, we obtain the strong monotonicity of \( C_i^{A|B} \) under IO on A side.

### Appendix C: Relation between \( l_1 \) norm and trace norm

The trace norm \( \| \cdot \|_{tr} \) is closely related to the \( l_1 \) norm \( \| \cdot \|_{l_1} \), for which we have the following relationship.

**Lemma 21.** Given an operator \( P \in B(\mathcal{H}) \) and a fixed reference basis \( \{ | i \rangle \}_i \) of \( \mathcal{H} \). Then

\[
\| P \|_{tr} = \min_{U,V} \| U P V \|_{l_1},
\]

where the minimization is taken over all the unitaries \( U,V \) acting on \( \mathcal{H} \) and \( \| \cdot \|_{l_1} \) is defined by the the given basis.

**Proof.** First, let us prove that for operator \( P \in B(\mathcal{H}) \), \( \| P \|_{tr} \leq \| P \|_{l_1} \). Due to single value decomposition of \( P \), there exists two orthonormal basis \( \{ | x_i \rangle \}_i \) and \( \{ | y_i \rangle \}_i \) such that

\[
\| P \|_{tr} = \sum_i \langle x_i | P | y_i \rangle
\]

\[
= \sum_i \sum_{j,k} \langle x_i | j \rangle \langle j | P | k \rangle \langle k | y_i \rangle
\]

\[
\leq \sum_{j,k} \| j \| P \| k \| \sum_i \| \langle x_i | j \rangle \langle k | y_i \rangle \|
\]

\[
\leq \sum_{j,k} \| j \| P \| k \|,
\]

where the last inequality comes from the fact that \( \sum_i \| \langle x_i | j \rangle \langle k | y_i \rangle \| \leq \left( \sum_i \| \langle x_i | j \rangle \|^2 \right)^{\frac{1}{2}} \left( \sum_i \| \langle k | y_i \rangle \|^2 \right)^{\frac{1}{2}} = 1 \) with \( \{ | x_i \rangle \}_i \) and \( \{ | y_i \rangle \}_i \) being the orthonormal basis. Thus, \( \| P \|_{tr} = \| U P V \|_{l_1} \leq \| U P V \|_{l_1} \) for any two unitaries.

Besides, there exist unitaries \( U \) and \( V \) such that \( U P V = \sum_i s_i | i \rangle \langle i | \) with \( \{ s_i \}_i \) being the single value of \( P \), and thus

\[
\| U P V \|_{l_1} = \| P \|_{tr}.
\]

\( \square \)