Microscopic conservation laws for the derivative Nonlinear Schrödinger equation

Xingdong Tang · Guixiang Xu

Abstract

Compared with macroscopic conservation law for the solution of the derivative nonlinear Schrödinger equation (DNLS) with small mass in Klaus and Schippa (A priori estimates for the derivative nonlinear Schrödinger equation. Accepted by Funkcial. Ekvac), we show the corresponding microscopic conservation laws for the Schwartz solutions of DNLS with small mass. The new ingredient is to make use of the logarithmic perturbation determinant \( A(\kappa) \) introduced in Rybkin (in: Topics in Operator Theory, Birkhäuser Verlag, Basel, 2010), Simon (in: Trace ideals and their applications, American Mathematical Society, Providence, 2005) to show one-parameter family of microscopic conservation laws of the \( A(\kappa) \) flow and the DNLS flow, which is motivated by Harrop-Griffiths et al. (Sharp well-posedness for the cubic NLS and MKdV in \( H^s \). arXiv:2003.05011), Killip and Visan (Ann Math 190:249–305, 2019), Killip et al. (Geom Funct Anal 28:1062–1090, 2018).

Keywords

Microscopic conservation law · Derivative nonlinear Schrödinger equation · Diagonal Green’s function · Perturbation determinant

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1 Introduction

We consider the derivative nonlinear Schrödinger equation (DNLS)

\[ i \partial_t q + \partial_x^2 q + i \partial_x(|q|^2 q) = 0, \quad (1.1) \]

where \( q : \mathbb{R} \times \mathbb{R} \to \mathbb{C} \). (1.1) is \( L^2 \)-critical since the dilation

\[ q(t, x) \mapsto q(\lambda t, \lambda x) = \lambda^{1/2} q(\lambda^2 t, \lambda x) \quad (1.2) \]

leaves both (1.1) and the \( L^2 \) norm invariant. The derivative nonlinear Schrödinger equation appears in plasma physics (see [32,33,39], and references therein).

Local well-posedness result for (1.1) in the energy space was worked out by Hayashi and Ozawa [17,34]. They combined the fixed point argument with the \( L^4_t W^{1,\infty}_x \) estimate to construct local-in-time solution with arbitrary data in energy space. For other results, we can refer to [15,16]. Since (1.1) is energy subcritical case, the maximal time interval of existence only depends on \( H^1 \) norm of initial data. Later, local well-posedness result for (1.1) in \( H^s \), \( s \geq 1/2 \) was obtained by Takaoka [40] using Bourgain’s Fourier restriction method. The sharpness was shown in [41] in the sense that nonlinear evolution \( u(0) \mapsto u(t) \) fails to be \( C^3 \) or even uniformly \( C^0 \) in this topology, even when \( t \) is arbitrarily close to zero and \( H^s \) norm of the data is small (see also Biagioni-Linares [4]).

Global well-posedness was shown for (1.1) in the energy space in [34], under the smallness condition

\[ \|u_0\|_{L^2}^2 < 2\pi, \quad (1.3) \]

the argument is based on the sharp Gagliardo–Nirenberg inequality and the energy method (conservation of mass and energy). This result was improved by Takaoka [41] using Bourgain’s restriction method, who proved global well-posedness in \( H^s \) for \( s > 32/33 \) under condition (1.3). In [7,8], Colliander, Keel, Staffilani, Takaoka and Tao made use of almost conservation law [44] to show global well-posedness in \( H^s, s > 1/2 \) under (1.3). Miao et al. [28] combined almost conservation law and the refined resonant decomposition technique to obtain the global well-posedness in \( H^{1/2} \) under (1.3). Later, Wu used the generalized Gagliardo–Nirenberg inequality to improve the global well-posedness of (1.1) in the energy space under the condition

\[ \|u_0\|_{L^2}^2 < 4\pi \quad (1.4) \]

in [45], where \( 4\pi \) is the mass of the solitary waves with critical parameters of (1.1). Miao, Tang and Xu used the structure analysis and classical variational argument to show the existence of solitary waves with two parameters and improved the global result of (1.1) in the energy space in [30], and further used perturbation argument, modulation analysis and Lyapunov stability to show the orbital stability of weak interaction multi-soliton solution with subcritical parameters in the energy space in [29]. We can also refer to [6,10,27,31,43] for the stability analysis of the solitary waves of
the (generalized) derivative nonlinear Schrödinger equation in the energy space and to [11] for lower regularity result of (1.1) by almost conservation law in [44].

Since (1.1) is an integrable system in [1,21], there are lots of results about global well-posedness of (1.1) in the weighted Sobolev space based on the inverse scattering method, please refer to [18–20,35,36] and reference therein.

The conjecture about (1.1) is the following.

**Conjecture 1.1** Let \( s > 0 \). (1.1) is globally well-posed for all initial data in \( H^s(\mathbb{R}) \) in the sense that the solution map \( \Phi \) extends uniquely from Schwartz space to a jointly continuous map \( \Phi : \mathbb{R} \times H^s(\mathbb{R}) \to H^s(\mathbb{R}) \).

According to the well-known local well-posedness result in [40,41], we need to loosen the continuous dependence of the solution on initial data to consider the solution of (1.1) in \( H^s(\mathbb{R}) \) with \( s \in (0, 1/2) \). The basic question is that how to control the uniform estimates of the solution of (1.1).

Motivated by Killip–Visan–Zhang’s argument in [24], Klaus and Schippa combined the integrability of (1.1) with the series expansion of the perturbation determinant [37,38] to obtain a macroscopic conservation law for the Schwartz solution of (1.1) with small mass in [25]. In this paper, we will show the corresponding microscopic form and obtain one-parameter family of microscopic conservation laws for the \( A(\kappa) \) flows [see (4.1)] and the DNLS flow by Harrop-Griffiths–Killip–Visan’s argument in [12,23]. Compared with the macroscopic form, the microscopic conservation law with coercivity helps to show the local smoothing effect for (1.1) in \( H^s(\mathbb{R}) \), which will be addressed in the future, and can be further applied into global well-posedness analysis. We can refer to [12,23,42] and reference therein for related cases.

We now recall some Hamiltonian mechanics background. Equation (1.1) is Hamiltonian equation with respect to the following Poisson structure:

\[
\{F, G\} := \int_{\mathbb{R}} \delta F(\delta G) \partial_x(\delta G) + \delta F(\delta q) \partial_q(\delta q) \, dx,
\]

(1.5)

where the operators \( \delta q \) and \( \delta r \) denote the functional Fréchet derivatives. Any Hamiltonian \( H(q, r, t) \) generates a flow via the equation

\[
\partial_t \begin{bmatrix} q \\ r \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \partial_x \begin{bmatrix} \frac{\delta H}{\delta q} \\ \frac{\delta H}{\delta r} \end{bmatrix}.
\]

(1.6)

Correspondingly, (1.1) is the Hamiltonian flow associated with

\[
H_{\text{DNLS}} := \int_{\mathbb{R}} -iqr' + \frac{1}{2}q^2r^2 \, dx,
\]

(1.7)

where \( r = -\bar{q} \). Two other important Hamiltonian quantities for (1.1) are

\[
M := \int_{\mathbb{R}} qr \, dx, \quad E_{\text{DNLS}} := \int_{\mathbb{R}} q'r' - \frac{3}{2}iq^2rr' + \frac{1}{2}q^3r^3 \, dx.
\]

(1.8)
Conservations of $M$, $H_{DNLS}$ and $E_{DNLS}$ is due to gauge, space translation and time translation invariance of (1.1), the commutativity\(^1\) of $H_{DNLS}$ and $E_{DNLS}$ is based on the fact that they are completely integrable, which means the existence of an infinite family of commuting flows for (1.1). We can refer to [1,25] for more details. Based on the recent breakthrough in KdV, mKdV and NLS by Harrop-Griffiths, Killip and Visan in [12,23], the commuting flow approximation will be the robust method for showing global well-posedness theory of (1.1) in the lower regularity space. We can also refer to [26] and reference therein.

Let us write the Lax operator related to (1.1) and its unperturbed one

$$L(\kappa) := \begin{bmatrix} \partial + i \kappa^2 & -\kappa q \\ -\kappa r & \partial - i \kappa^2 \end{bmatrix}$$

and

$$L_0(\kappa) := \begin{bmatrix} \partial + i \kappa^2 & 0 \\ 0 & \partial - i \kappa^2 \end{bmatrix}.$$ (1.9)

By simple calculations, we know that

$$R_0(\kappa) := L_0(\kappa)^{-1} = \begin{bmatrix} (\partial + i \kappa^2)^{-1} & 0 \\ 0 & (\partial - i \kappa^2)^{-1} \end{bmatrix}$$

admits the integral kernel

$$G_0(x, y; \kappa) = e^{-i \kappa^2 |x-y|} \begin{bmatrix} \mathbb{1}_{\{y < x\}} & 0 \\ 0 & -\mathbb{1}_{\{x < y\}} \end{bmatrix} \text{ for } \kappa^2 > 0.$$ For $\kappa^2 < 0$, we may use $G_0(x, y; \kappa) = -\tilde{G}_0(y, x; -\kappa)$. The resolvent operator $R(\kappa) := L(\kappa)^{-1}$ for Schwartz function $q$ with small mass also has the integral kernel $G(x, y; \kappa)$ (See Proposition 3.1)

$$\begin{bmatrix} G_{11}(x, y, \kappa) & G_{12}(x, y, \kappa) \\ G_{21}(x, y, \kappa) & G_{22}(x, y, \kappa) \end{bmatrix}.$$ Let us define three key functional as follows.

$$\gamma(x; \kappa) := \text{sgn}(i \kappa^2)[G_{11}(x, x; \kappa) - G_{22}(x, x; \kappa)] - 1,$$

$$g_{12}(x; \kappa) := \text{sgn}(i \kappa^2)G_{12}(x, x; \kappa),$$

$$g_{21}(x; \kappa) := \text{sgn}(i \kappa^2)G_{21}(x, x; \kappa).$$

With these preparations, the main result in this paper is

**Theorem 1.2** Let $s \in (0, 1/2)$, $i \kappa^2 \in \mathbb{R} \setminus (-1, 1)$, $q(0) \in H^s(\mathbb{R}) \cap \mathcal{S}$ with small mass. Suppose that $q$ is a solution to (1.1), then we have

$$\partial_t \rho(\kappa) + \partial_x j_{DNLS}(\kappa) = 0,$$

\(^1\) Which means $\{H_{DNLS}, E_{DNLS}\} = 0$, where $\{\cdot, \cdot\}$ is defined by (1.5).
where the density $\rho$ and the flux $j_{\text{DNLS}}$ are defined as follows

$$
\rho(\kappa) : = -\kappa \frac{q_{21}(\kappa) + rg_{12}(\kappa)}{2 + \gamma(\kappa)},
$$

$$
j_{\text{DNLS}}(\kappa) : = i \kappa \frac{q'_{21}(\kappa) - r'g_{12}(\kappa)}{2 + \gamma(\kappa)} - i \kappa^2 qr - 2 \kappa^2 \rho(\kappa) - qr \rho(\kappa).
$$

**Remark 1.3** We give some remarks as follows.

1. In this paper, we consider microscopic conservation law for the solution of (1.1). That is the reason why we consider the Schwartz solution. At the same time, Harrop-Griffiths, Killip and Visan prove the microscopic conservation laws for integrable lattice models in [14]. On the other hand, mass threshold is another interesting problem. Please refer to [17,30,45] and reference therein. Recently, it is very interesting that Bahouri and Perelman obtain global well-posedness for (1.1) in $H^{1/2}(\mathbb{R})$ without mass restriction by combining the profile decomposition techniques with the integrability structure in [2]. We can also refer to [3].

2. The above microscopic conservation law corresponds to macroscopic conservation law of (1.1) in [25]. In fact, we have

$$
A(\kappa) = \text{sgn}(i \kappa^2) \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j} \text{tr} \left\{ \left( \sqrt{R_0} \left( L - L_0 \right) \sqrt{R_0} \right)^j \right\}
$$

$$
= \int_{\mathbb{R}} \rho(\kappa) \, dx,
$$

see (4.13) for instance. To the best of our knowledge, this identity is firstly discovered in this paper. Compared to macroscopic conservation law, microscopic conservation law with coercivity can be used to show the local smoothing estimate for the solution of (1.1) with small mass in $H^s \cap \mathcal{S}$, $s \in (0, 1/2)$ and others. Please refer to [23,24].

3. The leading order term $^{2}$ in $\rho$ has the following form

$$
\left| \text{Re} \int_{\mathbb{R}} \rho^{[2]}(\kappa) \, dx \right| \approx \left| \kappa \right|^2 \left\| q \right\|^2_{H^{1/2}(\mathbb{R})},
$$

which captures $L^2$ norm of the part of $q$ living at frequencies $|\xi| \lesssim |\kappa|^2$ [See also (2.8) and (4.2)]. Combining the above conservation law and the similar argument on Besov norm estimate in [24,25] for any Hamiltonian flow preserving $A(\kappa)$ for all $|\kappa| \geq 1$, we can obtain a uniform bound of $H^s(\mathbb{R})$ norm of the Schwartz solution of (1.1) with small mass

$$
\left\| q(t) \right\|_{H^s} \lesssim \left\| q(0) \right\|_{H^s}, \quad \text{for } s \in (0, 1/2).
$$

$^{2}$ In fact, we have $\rho^{[2]}(\kappa) = -\kappa^2 \left( \frac{q r}{2 i \kappa^2 - \delta} + r \frac{q}{2 i \kappa^2 + \delta} \right)$ by (3.24), (3.25) and (3.26).
4. After we complete this work, there are some important progresses about the global well-posedness of the derivative nonlinear Schrödinger equation in $H^s(\mathbb{R})$, $s \geq 1/6$ based on the $L^2$-equicontinuity property of (DNLS) flow and related Hamiltonian commuting flows without/with the mass threshold. We can refer to [13,22].

Lastly, the paper is organized as follows. In Sect. 2, we recall some notations and preliminary estimates. In Sect. 3, we show the existence and some properties of the Green’s function related to the Lax operator $L(\kappa)$. In Sect. 4, we introduce the invariant quantity $A(\kappa)$ from the logarithmic perturbation determinant and show its microscopic conservation laws for the $A(\kappa)$ flow and the DNLS flow.

2 Some notation and preliminary estimates

In this paper, we take $r = -\bar{q}$ and choose $s \in (0, \frac{1}{2})$, and all implicit constants can depend on $s$. We denote by $\mathcal{S}$ the space of Schwartz class functions on $\mathbb{R}$, and introduce the notation

$$B_\delta := \{ q \in H^s : \|q\|_{H^s} \leq \delta \}. \quad (2.1)$$

which can be ensured by scaling argument under the assumption that the mass $\|q\|_{L^2}$ is small enough.

We use the inner product on $L^2(\mathbb{R})$ as follows

$$\langle f, g \rangle = \int f(x)g(x)\,dx,$$

which also gives the dual product between $H^s(\mathbb{R})$ and $H^{-s}(\mathbb{R})$. In addition, If $F : \mathcal{S} \to \mathbb{C}$ is $C^1$, we have

$$\left. \frac{d}{d\theta} \right|_{\theta = 0} F(q + \theta f) = \{ \tilde{f}, \frac{\delta F}{\delta q} \} - \{ f, \frac{\delta F}{\delta r} \}, \quad (2.2)$$

where $r = -\bar{q}$. The Fourier transform is defined by

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\xi x} f(x)\,dx, \quad \text{whence} \quad \widehat{fg}(\xi) = \frac{1}{\sqrt{2\pi}} [\hat{f} * \hat{g}](\xi).$$

2.1 Sobolev spaces

For complex $\kappa$ with $|\kappa| \geq 1$ and $\sigma \in \mathbb{R}$ we define the norm

$$\|q\|_{H^s_{\kappa}}^2 := \int_{\mathbb{R}} \left(4|\kappa|^4 + \xi^2\right)^\sigma |\hat{q}(\xi)|^2\,d\xi$$

and write $H^\sigma := H^\sigma_1$. 

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For $0 < s < \frac{1}{2}$, simple calculation yields the Sobolev inequality
\[
\| f \|_{L^\infty} \lesssim \| \hat{f} \|_{L^1} \leq \| f \|_{H^{s+\frac{1}{2}}} \left( |\xi|^2 + 4|\kappa|^4 \right)^{-\frac{2s+1}{2}} \lesssim |\kappa|^{-2s} \| f \|_{H^{s+\frac{1}{2}}}. \tag{2.3}
\]
Consequently, we have the following algebra property of $H^{s+\frac{1}{2}}$ space:
\[
\| fg \|_{H^{s+\frac{1}{2}}} \lesssim |\kappa|^{-2s} \| f \|_{H^{s+\frac{1}{2}}} \| g \|_{H^{s+\frac{1}{2}}}. \tag{2.4}
\]
By duality and the fractional product rule used as [5, Proposition 3.3], Sobolev embedding, and (2.3), we obtain
\[
\| qf \|_{H^{s-\frac{1}{2}}} \lesssim |\kappa|^{-2s} \| q \|_{H^{s-\frac{1}{2}}} \| f \|_{H^{s+\frac{1}{2}}}. \tag{2.5}
\]

### 2.2 Operator estimates and Trace

For $0 < \sigma < 1$ and $i\kappa^2 \in \mathbb{R}$, $|\kappa| \geq 1$, we define the operator $(i\kappa^2 \mp \partial)^{-\sigma}$ using the Fourier multiplier $(i\kappa^2 \mp i\xi)^{-\sigma}$, where, for $\arg z \in (-\pi, \pi]$, we define
\[
z^{-\sigma} = |z|^{-\sigma} e^{-i\sigma \arg z}. \tag{2.6}
\]
Therefore, for all $i\kappa^2 \in \mathbb{R}$ with $|\kappa| \geq 1$, we have
\[
\left((i\kappa^2 \mp \partial)^{-\sigma}\right)^* = (i\kappa^2 \pm \partial)^{-\sigma},
\]
and
\[
\left\| (i\kappa^2 \mp \partial)^{-\sigma} \right\|_{op} \leq |\kappa|^{-2\sigma}.
\]
We denote $\mathcal{I}_p$ the Schatten class of compact operators on $L^2(\mathbb{R})$ whose singular numbers are $l^p$ summable. $\mathcal{I}_p$ is complete and is an embedded subalgebra of bounded operators on $L^2(\mathbb{R})$. Moreover, we have
\[
\mathcal{I}_p \subset \mathcal{I}_q, \quad \text{for} \quad p \leq q. \tag{2.7}
\]
Let us recall some facts about the class $\mathcal{I}_p$ that we will use repeatedly in Sect. 3: an operator $A$ on $L^2(\mathbb{R})$ is Hilbert–Schmidt class ($\mathcal{I}_2$) if and only if it admits an integral kernel $a(x, y) \in L^2(\mathbb{R} \times \mathbb{R})$, and
\[
\| A \|_{op}^2 \leq \| A \|_{\mathcal{I}_2}^2 = \iint_{\mathbb{R} \times \mathbb{R}} |a(x, y)|^2 \, dx \, dy.
\]
The product of two Hilbert–Schmidt operators is trace class. Moreover, we have
\[
\text{tr}(AB) := \int_{\mathbb{R} \times \mathbb{R}} a(x, y)b(y, x) \, dy \, dx = \text{tr}(BA),
\]
\[
|\text{tr}(AB)| \leq \|AB\|_{\mathcal{J}_1} \leq \|A\|_{\mathcal{J}_2} \|B\|_{\mathcal{J}_2}.
\]

The class $\mathcal{J}_p$ forms a two-sided ideal in the algebra of bounded operators on $L^2$; indeed, for any bounded operators $B, C$ on $L^2(\mathbb{R})$, we have
\[
\|BAC\|_{\mathcal{J}_p} \leq \|B\|_{\text{op}} \|A\|_{\mathcal{J}_p} \|C\|_{\text{op}}.
\]

We can refer to [9,38] for more details.

The following estimates are the elementary estimates in this paper.

**Lemma 2.1** Let $s \in (0, 1/2)$, $i \kappa^2 \in \mathbb{R}$, we have
\[
\| (\partial + i \kappa^2)^{-s - \frac{1}{2}} q (\partial - i \kappa^2)^{-\frac{1}{2}} \|_{\mathcal{J}_2} \lesssim \|q\|_{L^2},
\]
(2.8)
\[
\| (\partial + i \kappa^2)^{s - \frac{1}{2}} q (\partial - i \kappa^2)^{-\frac{1}{2}} \|_{\mathcal{J}_2} \lesssim |\kappa| \|q\|_{H^{-s - \frac{1}{2}}},
\]
(2.9)

and
\[
\| (\partial + i \kappa^2)^{-s - \frac{1}{2}} f (\partial - i \kappa^2)^{-s - \frac{1}{2}} \|_{\text{op}} \lesssim |\kappa|^{-2s} \|f\|_{H^{-s - \frac{1}{2}}}. \tag{2.10}
\]

**Proof** Estimate (2.8) can be proved by [24, Lemma 4.1]. For (2.9), it suffices to consider the case $i \kappa^2 = 1$ by a scaling argument. By Plancherel’s Theorem, we have
\[
\| (1 - \partial)^{-s - \frac{1}{2}} q (1 + \partial)^{-\frac{1}{2}} \|_{\mathcal{J}_2}^2 = \text{tr} \left\{ (1 - \partial)^{s - \frac{1}{2}} q (1 - \partial)^{-\frac{1}{2}} \hat{q} \right\} = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{|\hat{q}(\xi - \eta)|^2}{(1 + \xi^2)^{\frac{1}{2} - s} (1 + \eta^2)^{\frac{1}{2}}} \, d\eta \, d\xi,
\]

Note that
\[
\int_{\mathbb{R}} \frac{1}{(1 + (\xi + \eta)^2)^{\frac{1}{2} - s} (1 + \eta^2)^{\frac{1}{2}}} \, d\eta \lesssim (4 + \xi^2)^{s - \frac{1}{2}},
\]
we can obtain estimate (2.9).

Lastly, we can obtain (2.10) by duality and (2.4) as follows
\[
\left| \int_{\mathbb{R}} fgh \, dx \right| \lesssim \|f\|_{H^{-s - \frac{1}{2}}_{\kappa}} \|gh\|_{H^{s + \frac{1}{2}}_{\kappa}} \lesssim |\kappa|^{-2s} \|f\|_{H^{-s - \frac{1}{2}}_{\kappa}} \|g\|_{H^{s + \frac{1}{2}}_{\kappa}} \|h\|_{H^{s + \frac{1}{2}}_{\kappa}}.
\]
This completes the proof. \qed

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3 The diagonal Green’s functions

In this Section, motivated by the ideas in [12, 23], we introduce three key quantities $g_{12}$, $g_{21}$, and $\gamma$ from the diagonal Green’s function related to the Lax operator $L(\kappa)$ for (1.1), and establish some elementary estimates about them. Recall that

$$L(\kappa) = L_0(\kappa) + \begin{bmatrix} 0 & -\kappa q \\ -\kappa r & 0 \end{bmatrix} \text{ where } L_0(\kappa) := \begin{bmatrix} \partial + i\kappa^2 & 0 \\ 0 & \partial - i\kappa^2 \end{bmatrix}. \quad (3.1)$$

Since we only consider the case $i\kappa^2 \in \mathbb{R}$ with $|\kappa| \geq 1$, we have

$$L(\kappa)^* = \begin{bmatrix} -\partial - i\bar{\kappa}^2 & -\bar{\kappa}\bar{r} \\ -\bar{\kappa}\bar{q} & -\partial + i\bar{\kappa}^2 \end{bmatrix} = -\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} L(-\bar{\kappa}) \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \quad (3.2)$$

We now construct the Green’s function associated with $L_0(\kappa)$ and $L(\kappa)$, respectively. By the Fourier transformation, the resolvent operator $R_0(\kappa) := L_0(\kappa)^{-1}$ admits the integral kernel

$$G_0(x, y; \kappa) = e^{-i\kappa^2|x-y|} \begin{bmatrix} \mathbb{1}_{\{y<x\}} & 0 \\ 0 & -\mathbb{1}_{\{x<y\}} \end{bmatrix} \text{ for } i\kappa^2 > 0. \quad (3.3)$$

For $i\kappa^2 < 0$, we may use $G_0(x, y; \kappa) = -G_0(y, x; -\bar{\kappa})$ by (3.2).

By the perturbation theory and the resolvent identity, the resolvent operator $R(\kappa) := L(\kappa)^{-1}$ can be formally expressed as

$$R = R_0 + \sum_{\ell=1}^{\infty} (-1)^\ell \sqrt{R_0} \left( \sqrt{R_0(L - L_0)} \sqrt{R_0} \right)^\ell \sqrt{R_0}, \quad (3.4)$$

where

$$\sqrt{R_0(L - L_0)} \sqrt{R_0} = -\begin{bmatrix} 0 & \Lambda \\ \Gamma & 0 \end{bmatrix}, \quad (3.5)$$

$$\Lambda := (\partial + i\kappa^2)^{-1/2} \kappa q (\partial - i\kappa^2)^{-1/2} \text{ and } \Gamma := (\partial - i\kappa^2)^{-1/2} \kappa r (\partial + i\kappa^2)^{-1/2}. \quad (3.6)$$

and fractional powers of $R_0$ are defined via (2.6). By (2.8), we have

$$\|\Lambda\|_{\mathcal{B}_2} = \|\Gamma\|_{\mathcal{B}_2} \lesssim \|q\|_{L^2} \lesssim \delta. \quad (3.7)$$

Now we have the convergence of series (3.4) as part of the following result.
Proposition 3.1 (Existence of the Green’s function) There exists \( \delta > 0 \) such that \( L(\kappa) \) is invertible as an operator on \( L^2(\mathbb{R}) \), for all \( q \in B_\delta \) and all \( i \kappa^2 \in \mathbb{R} \) with \( |\kappa| \geq 1 \). The resolvent operator \( R(\kappa) := L(\kappa)^{-1} \) admits an integral kernel \( G(x, y; \kappa) \) satisfying

\[
\begin{bmatrix}
G_{11}(x, y, \kappa) G_{12}(x, y, \kappa) \\
G_{21}(x, y, \kappa) G_{22}(x, y, \kappa)
\end{bmatrix} = - \begin{bmatrix}
\tilde{G}_{11}(y, x, -\kappa) \tilde{G}_{21}(y, x, -\kappa) \\
\tilde{G}_{12}(y, x, -\kappa) \tilde{G}_{22}(y, x, -\kappa)
\end{bmatrix}
\]

(3.8)
such that the mapping

\[
H_\kappa^s(\mathbb{R}) \ni q \mapsto G - G_0 \in H_\kappa^{s+\frac{1}{2}} \otimes H_\kappa^{s+\frac{1}{2}}
\]

(3.9)
is continuous. Moreover, \( G - G_0 \) is continuous as a function of \( (x, y) \in \mathbb{R} \times \mathbb{R} \). Lastly, we have

\[
\partial_x G(x, y; \kappa) = \begin{bmatrix}
-i \kappa^2 & \kappa q(x) \\
\kappa r(x) & i \kappa^2
\end{bmatrix} G(x, y; \kappa) + \begin{bmatrix}
\delta(x - y) & 0 \\
0 & \delta(x - y)
\end{bmatrix},
\]

(3.10)

\[
\partial_y G(x, y; \kappa) = G(x, y; \kappa) \begin{bmatrix}
i \kappa^2 & -\kappa q(y) \\
-\kappa r(y) & -i \kappa^2
\end{bmatrix} - \begin{bmatrix}
\delta(x - y) & 0 \\
0 & \delta(x - y)
\end{bmatrix}
\]

(3.11)
in the sense of distributions.

Proof The proof is similar to those in [12, Proposition 3.1]. We sketch the proof for completeness. From (3.7), we have\(^3\)

\[
\| \sqrt{R_0(L - L_0)} \sqrt{R_0} \|_{\mathcal{J}_2} \leq \sqrt{2} \| A \|_{\mathcal{J}_2} \lesssim \| q \|_{L^2} \lesssim \delta
\]

uniformly for \( |\kappa| \geq 1 \). Thus, for \( \delta > 0 \) sufficiently small, series (3.4) converges in operator norm uniformly for \( |\kappa| \geq 1 \). It is easy to verify that the sum acts as an inverse to \( L(\kappa) \).

From (3.7), we also have \( R - R_0 \in \mathcal{J}_2 \). In particular, the operator \( R - R_0 \) admits an integral kernel \( G - G_0 \) in \( L^2(\mathbb{R} \times \mathbb{R}) \). Moreover, by (2.9), we know that \( q := R - R_0 \) is continuous in the sense of Hilbert–Schmidt operators from \( H_\kappa^{-s-\frac{1}{2}} \) to \( H_\kappa^{s+\frac{1}{2}} \), which implies (3.9).

By (3.9), we obtain that the kernel function \( G - G_0 \) is continuous in \( (x, y) \) since \( s + \frac{1}{2} > \frac{1}{2} \).

For regular \( q \), identities (3.10) and (3.11) precisely express the fact that \( G \) is an integral kernel for \( R(\kappa) \). They also hold for irregular \( q \) by (3.9).

Let us define \( \gamma, g_{12} \) and \( g_{21} \) as follows:

\[
\gamma(x; \kappa) := \text{sgn}(i \kappa^2)[G_{11}(x, x; \kappa) - G_{22}(x, x; \kappa)] - 1,
\]

(3.12)

\(^3\) Since the convergence of the tails of series (3.4) is key to the existence of the Green’s function, we can pay some regularity \( s > s_0 > 0 \) with \( s_0 \in (0, 1/2) \), use the \( \mathcal{J}_p \) estimate instead of the \( \mathcal{J}_2 \) estimate to remove small mass assumption. Here, we pay attention to the regularity problem of the solution for (1.1) in this paper.

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where \( G_{ij}(x, y, \kappa), 1 \leq i, j \leq 2, \) are the entries of the integral kernel \( G(x, y, \kappa). \) If \( q \in B_\delta \cap \mathcal{S}, \) we may use (3.10) and (3.11) to obtain

\[
\gamma' = 2\kappa (q g_{21} - r g_{12}),
\]

(3.15)

\[
g_{12}' = -2i\kappa^2 g_{12} - \kappa q [\gamma + 1],
\]

(3.16)

\[
g_{21}' = 2i\kappa^2 g_{21} + \kappa r [\gamma + 1].
\]

(3.17)

By (3.8), we have

\[
\gamma(\kappa) = \bar{\gamma}(-\bar{\kappa}) \quad \text{and} \quad g_{12}(\kappa) = -\bar{g}_{21}(-\bar{\kappa}).
\]

(3.18)

Moreover, by (3.15), (3.16), and (3.17), we have the following identity

\[
\frac{\kappa^2 - \kappa^2}{\kappa^2} [g_{12}'(\kappa) g_{21}(\kappa) + g_{21}'(\kappa) g_{12}(\kappa)]
= [g_{12}(\kappa) g_{21}(\kappa) + g_{21}(\kappa) g_{12}(\kappa)]' + \frac{\kappa}{2\kappa} [((\gamma(\kappa) + 1)(\gamma(\kappa) + 1))]',
\]

(3.19)

which is closely connected to the commutativity of the \( A(\kappa)'s \) flows under the Poisson bracket (1.5). From series representation (3.4) of \( R(\kappa) \) in \( q \) and \( r, \) we can deduce the corresponding series representations of \( g_{12}, g_{21}, \) and \( \gamma \) in \( q \) and \( r. \) We use the square brackets notation as follows

\[
g_{12}^{[2m+1]}(\kappa) := \text{sgn}(i \kappa^2) \left\{ \delta_x, (\partial + i\kappa^2)^{-\frac{1}{2}} \Lambda (\Gamma \Lambda)^m (\partial - i\kappa^2)^{-\frac{1}{2}} \delta_x \right\},
\]

(3.20)

\[
g_{21}^{[2m+1]}(\kappa) := \text{sgn}(i \kappa^2) \left\{ \delta_x, (\partial - i\kappa^2)^{-\frac{1}{2}} \Gamma (\Lambda \Gamma)^m (\partial + i\kappa^2)^{-\frac{1}{2}} \delta_x \right\},
\]

(3.21)

with \( g_{12}^{[2m]}(\kappa) = g_{21}^{[2m]}(\kappa) := 0, \) and similarly, \( \gamma^{[2m+1]}(\kappa) := 0 \) and

\[
\gamma^{[2m]}(\kappa) := \text{sgn}(i \kappa^2) \left\{ \delta_x, (\partial + i\kappa^2)^{-\frac{1}{2}} (\Lambda \Gamma)^m (\partial + i\kappa^2)^{-\frac{1}{2}} \delta_x \right\}
- \text{sgn}(i \kappa^2) \left\{ \delta_x, (\partial - i\kappa^2)^{-\frac{1}{2}} (\Lambda \Gamma)^m (\partial - i\kappa^2)^{-\frac{1}{2}} \delta_x \right\}.
\]

(3.22)

then by (3.4), we have

\[
g_{12}(\kappa) = \sum_{\ell=1}^{\infty} g_{12}^{[\ell]}(\kappa), \quad g_{21}(\kappa) = \sum_{\ell=1}^{\infty} g_{21}^{[\ell]}(\kappa), \quad \text{and} \quad \gamma(\kappa) = \sum_{\ell=2}^{\infty} \gamma^{[\ell]}(\kappa).
\]

(3.23)
We also write the tails of these series as

\[ g_{12}^{[\geq m]}(\kappa) := \sum_{\ell=m}^{\infty} g_{12}^{[\ell]}(\kappa), \quad g_{21}^{[\geq m]}(\kappa) := \sum_{\ell=m}^{\infty} g_{21}^{[\ell]}(\kappa), \]

\[ \gamma^{[\geq m]}(\kappa) := \sum_{\ell=m}^{\infty} \gamma^{[\ell]}(\kappa). \]

By (3.16), (3.17) and (3.36), we obtain the identities

\[ g_{12} = -(2i\kappa^2 + \partial)^{-1}[\kappa q + \kappa \gamma q], \quad g_{21} = -(2i\kappa^2 - \partial)^{-1}[\kappa r + \kappa \gamma r], \]

and

\[ \gamma = -2g_{12}g_{21} - \frac{1}{2}\gamma^2, \]

from which we have the explicit expressions for the leading order terms

\[ g_{12}^{[1]}(\kappa) = -\frac{kq}{2i\kappa^2 + \partial}, \quad g_{12}^{[3]}(\kappa) = \frac{2}{2i\kappa^2 + \partial} (\kappa q \cdot \frac{kq}{2i\kappa^2 + \partial} \cdot \frac{kq}{2i\kappa^2 + \partial}), \]

\[ g_{21}^{[1]}(\kappa) = -\frac{kr}{2i\kappa^2 - \partial}, \quad g_{21}^{[3]}(\kappa) = \frac{2}{2i\kappa^2 - \partial} (\kappa r \cdot \frac{kr}{2i\kappa^2 - \partial} \cdot \frac{kr}{2i\kappa^2 - \partial}), \]

and

\[ \gamma^{[2]}(\kappa) = -2 \frac{kq}{2i\kappa^2 + \partial} \cdot \frac{kr}{2i\kappa^2 - \partial}, \]

\[ \gamma^{[4]}(\kappa) = \frac{4}{2i\kappa^2 + \partial} (\kappa q \cdot \frac{kq}{2i\kappa^2 - \partial} \cdot \frac{kq}{2i\kappa^2 - \partial}) + \frac{4}{2i\kappa^2 + \partial} (\kappa r \cdot \frac{kr}{2i\kappa^2 - \partial} \cdot \frac{kr}{2i\kappa^2 - \partial}) \]

\[ - 2 \frac{kq}{2i\kappa^2 + \partial} \cdot \frac{kr}{2i\kappa^2 - \partial} \cdot \frac{kq}{2i\kappa^2 - \partial} \cdot \frac{kr}{2i\kappa^2 - \partial}. \]

We are now ready to obtain some basic estimates on \( g_{12}, \ g_{21} \).

**Proposition 3.2** (Properties of \( g_{12} \) and \( g_{21} \)) There exists \( \delta > 0 \) such that for all \( i \kappa^2 \in \mathbb{R} \) with \( |\kappa| \geq 1 \) the maps \( q \mapsto g_{12}(\kappa) \) and \( q \mapsto g_{21}(\kappa) \) are (real analytic) diffeomorphisms of \( B_{\delta} \) into \( H^{s+\frac{1}{2}} \) satisfying the estimates

\[ \|g_{12}(\kappa)\|_{H^{s+\frac{1}{2}}} + \|g_{21}(\kappa)\|_{H^{s+\frac{1}{2}}} \lesssim |\kappa| \|q\|_{H^{s-\frac{1}{2}}}. \]

Further, the remainders satisfy the estimate

\[ \|g_{12}^{[\geq 3]}(\kappa)\|_{H^{s+\frac{1}{2}}} + \|g_{21}^{[\geq 3]}(\kappa)\|_{H^{s+\frac{1}{2}}} \lesssim |\kappa| \|q\|_{H^{s-\frac{1}{2}}} \|q\|_{L^2}, \]

uniformly in \( \kappa \). Finally, if \( q \) is Schwartz then so are \( g_{12}(\kappa) \) and \( g_{21}(\kappa) \).
Proof It suffices to consider the case $i \kappa^2 \geq 1$ as the case $i \kappa^2 \leq -1$ is similar, and by (3.18), it suffices to consider $g_{12}(\kappa)$. Recalling (3.24), we obtain
\[
\|g_{12}^{[1]}(\kappa)\|_{H_{\kappa}^{s+\frac{1}{2}}} = |\kappa| \|q\|_{H_{\kappa}^{s-\frac{1}{2}}}.
\] (3.30)

To bound the remainder terms in the series, we employ duality and Lemma 2.1:
\[
\left|\langle f, g_{12}^{[geq 3]}(\kappa) \rangle \right| \leq \|\partial - i \kappa^2\|^{-s-\frac{1}{2}} \left\langle \partial + i \kappa^2 \right\rangle^{-s-\frac{1}{2}} \left\langle 2\delta + 1 \right\rangle |\kappa|^{-2s(2\ell-1)}
\times \sum_{\ell=1}^{\infty} \|\partial + i \kappa^2\|^{-s-\frac{1}{2}} \|kq\|^{-s-\frac{1}{2}} \|q\|_{H_{\kappa}^{s-\frac{1}{2}}}^{2\ell+1} |\kappa|^{-2s(2\ell-1)}
\lesssim |\kappa|^{-2s} \|f\|_{H_{\kappa}^{s-\frac{1}{2}}} \sum_{\ell=1}^{\infty} \|q\|_{H_{\kappa}^{s-\frac{1}{2}}} \left( |\kappa|^{-1-2s} \|q\|_{H_{\kappa}^{s-\frac{1}{2}}} \right)^{2\ell}
\lesssim \|f\|_{H_{\kappa}^{s-\frac{1}{2}}} |\kappa| \|q\|_{H_{\kappa}^{s-\frac{1}{2}}} \|q\|_{L^2}^2
\] (3.31)
provided $\delta > 0$ is sufficiently small. This proves (3.29) and completes the proof of (3.28).

In addition, we have
\[
\frac{\delta g_{12}}{\delta q}(\kappa)|_{q=0} = -\frac{\kappa}{2i\kappa^2 + \partial} \quad \text{and} \quad \frac{\delta g_{12}}{\delta r}(\kappa)|_{q=0} = 0
\]
which is an isomorphism, as noted already in (3.30). Furthermore, for any $f \in \mathcal{S}$, we have
\[
\lim_{\varepsilon \to 0} \frac{d}{d\varepsilon} \bigg|_{\varepsilon = 0} G(x, z; q + \varepsilon f) = -\int G(x, y; q) \begin{bmatrix} 0 & f(y) \\ -\bar{f}(y) & 0 \end{bmatrix} G(y, z; q) \, dy.
\]
By an analysis similar to that used to prove (3.29), we have
\[
\|\frac{\delta g_{12}}{\delta r}(\kappa)\|_{H_{\kappa}^{s-\frac{1}{2}} \to H_{\kappa}^{s+\frac{1}{2}}} + \|\frac{\delta g_{12}}{\delta q}(\kappa)\|_{H_{\kappa}^{s-\frac{1}{2}} \to H_{\kappa}^{s+\frac{1}{2}}} \lesssim \|q\|_{L^2}^2 \lesssim \delta^2,
\]
and so the inverse function theorem implies the diffeomorphism property for sufficiently small $\delta$.

The regularity result can be easily obtained in an argument similar to those in [23, Proposition 2.2] and [12, Proposition 3.2].

We also have some estimates on $\gamma$ as follows.

---

4 If we use $I_p$ with $p > 2$ instead of $I_2$ once again, we may pay some regularity on $s > s_0 > 0$ to remove the small mass assumption.
Proposition 3.3 (Properties of $\gamma$) There exists $\delta > 0$ such that for all $i \kappa^2 \in \mathbb{R}$ with $|\kappa| \geq 1$ the map $q \mapsto \gamma(\kappa)$ is bounded from $B_\delta$ to $L^1 \cap H^{s+\frac{1}{2}}$ and we have the estimates

\[
\|\gamma(\kappa)\|_{H^{s+\frac{1}{2}}} \lesssim |\kappa|^{2-2s}\|q\|^2_{H^{s-\frac{1}{2}}},
\]

(3.32)

\[
\|\gamma(\kappa)\|_{L^\infty} \lesssim |\kappa|^{2-4s}\|q\|^2_{H^{s-\frac{1}{2}}},
\]

(3.33)

\[
\|\gamma(\kappa)\|_{L^1} \lesssim |\kappa|^2\|q\|^2_{H^{s-1}} + |\kappa|^{-2(4s-1)}\|q\|_4^4 H^{s-\frac{1}{2}}.
\]

(3.34)

\[
\|\gamma^{[\geq 4]}(\kappa)\|_{L^1} \lesssim |\kappa|^{-2(4s-1)}\|q\|_4^4 H^{s-\frac{1}{2}},
\]

(3.35)

uniformly in $\kappa$. Further, we have the quadratic identity

\[
\gamma + \frac{1}{2} \gamma^2 = -2g_{12}g_{21}.
\]

(3.36)

and if $q$ is Schwartz, then so is $\gamma(\kappa)$.

Proof Once again it suffices to consider the case $i \kappa^2 \geq 1$. Using (3.26) and (2.4), we obtain

\[
\|\gamma^{[2]}\|_{H^{s+\frac{1}{2}}} \lesssim \|\frac{\kappa q}{2i\kappa^2 + \hat{a}} \cdot \frac{\kappa r}{2i\kappa^2 - \hat{a}}\|_{H^{s+\frac{1}{2}}} \lesssim |\kappa|^{2-2s}\|q\|^2_{H^{s-\frac{1}{2}}}.\]

To handle $\gamma^{[\geq 4]}$ we use series representation (3.23) and the same dual argument used to prove (3.29). Estimate (3.33) then follows from (3.32) via (2.3).

Choosing $\sigma = \kappa$ in (3.19), we obtain

\[
\partial_x \left\{ 2g_{12}(x; \kappa)g_{21}(x; \kappa) + \frac{1}{2} \gamma(x; \kappa)^2 + \gamma(x; \kappa) \right\} = 0.
\]

By (3.28) and (3.32), we know that the term in the braces vanishes as $|x| \to \infty$. Thus, identity (3.36) follows by integration.

By using this quadratic identity, we may write

\[
\gamma^{[\geq 4]} = -\frac{1}{2} \gamma^2 - 2g_{12}^{[\geq 3]} : g_{21}^{[\geq 3]} = -2g_{12}^{[1]} : g_{21}^{[\geq 3]} - 2g_{12}^{[2]} : g_{21}^{[\geq 3]}.
\]

(3.37)

By Proposition 3.2 and (3.32), we have

\[
\|g_{12}^{[\geq 3]}\|_{L^2} + \|g_{21}^{[\geq 3]}\|_{L^2} \lesssim |\kappa|^{-2(s+\frac{1}{2})} \left[ \|g_{12}^{[\geq 3]}\|_{H^{s+\frac{1}{2}}} + \|g_{21}^{[\geq 3]}\|_{H^{s+\frac{1}{2}}} \right]
\]

\[
\lesssim |\kappa|^{-6s+2}\|q\|^3_{H^{s-\frac{1}{2}}},
\]

\[
\|g_{12}\|_{L^2} + \|g_{21}\|_{L^2} \lesssim |\kappa|^{-2(s+\frac{1}{2})} \left( \|g_{12}\|_{H^{s+\frac{1}{2}}} + \|g_{21}\|_{H^{s+\frac{1}{2}}} \right) \lesssim |\kappa|^{-2s}\|q\|_{H^{s-\frac{1}{2}}},
\]

\[
\implies \text{Springer}
\]
and
\[ \| \gamma \|_{L^2} \lesssim |\kappa|^{-2(s+\frac{1}{2})} \| \gamma \|_{H^{s+\frac{1}{2}}} \lesssim |\kappa|^{-4s+1} \| q \|^2_{H^{s-\frac{1}{2}}}, \]
which together with Hölder's inequality imply that
\[ \| \gamma^{[\geq 4]} \|_{L^1} \lesssim \| \gamma \|_{L^2}^2 + \| g_{12}^{[\geq 3]} \|_{L^2} \| g_{21} \|_{L^2} + \| g_{12}^{[1]} \|_{L^2} \| g_{21}^{[\geq 3]} \|_{L^2} \lesssim |\kappa|^{-2(4s-1)} \| q \|^{4}_{H^{s-\frac{1}{2}}}, \]
thus we obtain (3.35). Estimate (3.34) then follows from applying the Cauchy–Schwarz inequality to (3.26).

The regularity result can also be deduced by an argument similar to those in [23, Proposition 2.2] and [12, Proposition 3.3]. \( \square \)

Due to the structure of microscopic conservation law in (4.13), the combination function \( \frac{g_{12}(k)}{2+\gamma(k)} \) will be also used later. We now give the analogue estimates. Firstly, we denote:
\[ g_{12}(\frac{k}{2+\gamma}) = \left( g_{12}(\frac{k}{2+\gamma}) \right)^{[1]} + \left( g_{12}(\frac{k}{2+\gamma}) \right)^{[3]} + \left( g_{12}(\frac{k}{2+\gamma}) \right)^{[\geq 5]}, \]
where the leading order terms are given by
\[ \left( g_{12}(\frac{k}{2+\gamma}) \right)^{[1]} = \frac{1}{2} g_{12}^{[1]} \quad \text{and} \quad \left( g_{12}(\frac{k}{2+\gamma}) \right)^{[3]} = \frac{1}{2} g_{12}^{[3]} - \frac{1}{4} g_{12}^{[1]} \gamma^{[2]}, \quad (3.38) \]
and the remainder term is given by
\[ \left( g_{12}(\frac{k}{2+\gamma}) \right)^{[\geq 3]} = \frac{1}{2} g_{12}^{[3]} - g_{12}(\frac{k}{2+\gamma}). \quad (3.39) \]

We can now show the following estimates about \( \frac{g_{12}(k)}{2+\gamma(k)} \).

**Corollary 3.4** Let \( s \in (0, 1/2) \) and \( q \in B_{\delta} \). there exists \( \delta > 0 \) such that for all \( i \kappa^2 \in \mathbb{R} \) with \( |\kappa| \geq 1 \), we have the estimates
\[ |\kappa|^2 \left\| \frac{g_{12}(k)}{2+\gamma(k)} \right\|_{H^{s-1/2}} + \left\| \frac{g_{12}(k)}{2+\gamma(k)} \right\|_{H^{s+1/2}} \lesssim |\kappa| \| q \|_{H^{s-1/2}}, \quad (3.40) \]
\[ |\kappa|^2 \left\| \left( \frac{g_{12}(k)}{2+\gamma(k)} \right)^{[\geq 3]} \right\|_{H^{s-1/2}} + \left\| \left( \frac{g_{12}(k)}{2+\gamma(k)} \right)^{[\geq 3]} \right\|_{H^{s+1/2}} \lesssim |\kappa| \| q \|_{H^{s+1/2}} \| q \|_{L^2}^2. \quad (3.41) \]

**Proof** From (3.38) and (3.24), we see that
\[ |\kappa|^2 \left\| \left( \frac{g_{12}(k)}{2+\gamma(k)} \right)^{[1]} \right\|_{H^{s-1/2}} + \left\| \left( \frac{g_{12}(k)}{2+\gamma(k)} \right)^{[1]} \right\|_{H^{s+1/2}} \]
\[ \approx \left\| (2i\kappa^2 + \partial) \left( \frac{g_{12}(k)}{2+\gamma(k)} \right)^{[1]} \right\|_{H^s} \approx |\kappa| \| q \|_{H^{s-1/2}}. \]
Thus, it suffices for (3.40) to show (3.41). Moreover, by (3.16), we have
\[ (2i\kappa^2 + \partial) \left( \frac{g_{12}(k)}{2+\gamma(k)} \right)^{[\geq 3]} = -\kappa \frac{\gamma'}{2(2+\gamma)} q + \frac{g_{12}(k)}{(2+\gamma)^2} \gamma'. \]
and therefore, we have

\[ \text{LHS of (3.41)} \lesssim |\kappa| \left\| \frac{\gamma}{2+\gamma} q \right\|_{H^{s-1/2}} + \left\| \frac{g_{12}}{(2+\gamma)^2} \gamma' \right\|_{H^{s-1/2}} \]

\[ \lesssim |\kappa| |^{1-2s} \left\| q \right\|_{H^{s-1/2}} + \left\| \frac{g_{12}}{(2+\gamma)^2} \right\|_{H^{s+1/2}} \]

\[ \lesssim |\kappa| |^{1-2s} \left\| q \right\|_{H^{s-1/2}} + |\kappa| |^{2s} \left\| \gamma' \right\|_{H^{s-1/2}} + \left\| g_{12} \right\|_{H^{s+1/2}} \]

\[ \lesssim |\kappa| |\| q \|_{H^{s-1/2}} \| q \|^2_{L^2}, \]

where the second step we use (2.5) and (3.32), and the third step we expand as series and employ the algebra property (2.4), together with (3.28) and (3.32). This yields (3.40) for sufficiently small \( \delta \). □

### 4 Conservation laws and dynamics

In this section, we will firstly introduce the invariant quantity \( A(\kappa) \) from the logarithmic perturbation determinant, which is related to the integrability and spectral invariance of (1.1), then we show the dynamics and microscopic conservation laws of the \( A(\kappa) \)'s flow and the DNLS (i.e., \( H_{\text{DNLS}} \)) flow, respectively.

#### 4.1 Conservation Laws

Inspired by [12,23,24,37], we formally define the logarithmic perturbation determinant \( \text{sgn}(i\kappa^2) \log \det(L_0^{-1}L) \) as follows:

\[ A(\kappa; q, r) := \text{sgn}(i\kappa^2) \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j} \text{tr} \left\{ (\sqrt{R_0} (L - L_0) \sqrt{R_0})^j \right\}. \tag{4.1} \]

By (3.4), simple calculations deduce that for \( q, r \in \mathcal{S} \), we have

\[ A(\kappa; q, r) = -\text{sgn}(i\kappa^2) \sum_{m=1}^{\infty} \frac{1}{m} \text{tr} \left\{ (\Lambda \Gamma)^m \right\}. \tag{4.2} \]

In the following, we will use (4.2) as the definition of \( A(\kappa; q, r) \). For the sake of simplicity, we write

\[ A(\kappa; q, r) = \sum_{m=1}^{\infty} A_m(\kappa; q, r), \quad A_m(\kappa; q, r) := -\text{sgn}(i\kappa^2) \frac{1}{m} \text{tr} \left\{ (\Lambda \Gamma)^m \right\}. \tag{4.3} \]

Firstly, we have
Lemma 4.1  Firstly, we have A) There exists $\delta > 0$ such that for all $q \in B_\delta \cap \mathcal{S}$ and $i\kappa^2 \in \mathbb{R}$ with $|\kappa| \geq 1$, series (4.2) converges absolutely. Moreover, we have

$$\frac{\delta}{\delta q} A(\kappa) = -\kappa g_{21}(\kappa), \quad \frac{\delta}{\delta r} A(\kappa) = -\kappa g_{12}(\kappa),$$

(4.4)

and

$$\gamma'(\kappa) = 2 \left( -q \frac{\delta}{\delta q} A(\kappa) + r \frac{\delta}{\delta r} A(\kappa) \right).$$

(4.5)

Proof  By (3.7), we know that $A(\kappa; q, r)$ converges absolutely for all $q \in B_\delta \cap \mathcal{S}$. By simple calculations, we have

$$\frac{\delta}{\delta q} A_m = -\kappa g_{21}^{[2m-1]} \quad \text{and} \quad \frac{\delta}{\delta r} A_m = -\kappa g_{12}^{[2m-1]},$$

(4.6)

which together with (3.15), (3.23) implies (4.5).

Next, we show that the mass, the Hamiltonian and the energy defined by (1.7) and (1.8), respectively, arise as the coefficients in the asymptotic expansion of $A(\kappa)$ as $|\kappa| \to \infty$. More precisely, we have.

Lemma 4.2  (Asymptotic expansion of $A(\kappa)$) For $q \in B_\delta \cap \mathcal{S}$, we have, as $|\kappa| \to \infty$,

$$A(\kappa) = (-i) \frac{M}{2} - \frac{(-i)^2}{2i\kappa^2} H^{DNLS}_{\frac{1}{2}} + \frac{(-i)^3}{(2i\kappa^2)^2} E^{DNLS}_{\frac{1}{2}} + O(|\kappa|^{-6}).$$

(4.7)

Proof  By (3.16) and (3.17), and (4.5), we have

$$2i \kappa^2 \frac{\delta A}{\delta q} = \partial \frac{\delta A}{\delta q} + 2\kappa^2 r \cdot \partial^{-1} \left( -q \frac{\delta A}{\delta q} + r \frac{\delta A}{\delta r} \right) + \kappa^2 r,$$

$$2i \kappa^2 \frac{\delta A}{\delta r} = \partial \frac{\delta A}{\delta r} + 2\kappa^2 q \cdot \partial^{-1} \left( -q \frac{\delta A}{\delta q} + r \frac{\delta A}{\delta r} \right) + \kappa^2 q.$$

By the asymptotic analysis, we have

$$\frac{\delta A}{\delta q} = -\kappa g_{21}$$

$$= -i \frac{1}{2} r + \frac{1}{2i\kappa^2} \left( -i \frac{1}{2} r' + \frac{3}{2} q r^2 \right) + O(|\kappa|^{-6}),$$

$$\frac{\delta A}{\delta r} = -\kappa g_{12}$$

$$= -i \frac{1}{2} q + \frac{1}{2i\kappa^2} \left( -i \frac{3}{2} q' + \frac{1}{2} q^2 r \right) + O(|\kappa|^{-6}),$$

(4.8)

which together with the fact that

$$A(q, r) = \int_0^1 \partial_\theta A(\theta q, \theta r) \, d\theta$$

(4.9)
Remark 4.3 By computing $\gamma = 2\partial^{-1}\left(-q\frac{\delta A}{\delta q} + r\frac{\delta A}{\delta r}\right)$, we can obtain asymptotic expansion for $\gamma$,

$$\gamma = \frac{1}{2\kappa^2}qr - \frac{1}{4\kappa^4}(i \, q'r - i \, qr' + \frac{3}{2} q^2 r^2) + O(|\kappa|^{-6}). \quad (4.10)$$

Lemma 4.4 (Density function of $A(\kappa)$) For all $q \in B_\delta \cap \mathcal{S}$ and $i\kappa^2 \in \mathbb{R}$ with $|\kappa| \geq 1$, we have

$$A(\kappa) = -\tilde{A}(-\tilde{\kappa}), \quad (4.11)$$

$$\frac{\partial}{\partial \kappa} A(\kappa) = \int_{\mathbb{R}} \left[2i\kappa \cdot \gamma(\kappa) - (qg_{21}(\kappa) + rg_{12}(\kappa))\right] dx, \quad (4.12)$$

$$A(\kappa) = \int_{\mathbb{R}} \rho(\kappa) \, dx, \quad \text{where} \quad \rho(\kappa) = -\kappa\frac{qg_{21}(\kappa) + rg_{12}(\kappa)}{2 + \gamma(\kappa)}. \quad (4.13)$$

Proof Firstly, we show that (4.12) holds. In fact, simple calculations imply that

$$\frac{\partial}{\partial \kappa} A_m = \int_{\mathbb{R}} \left[2i \, \kappa \gamma^{[2m]} - (qg_{21}^{[2m-1]} + rg_{12}^{[2m-1]})\right] dx. \quad (4.14)$$

By summation with respect to $m$, we can obtain (4.12).

We now prove (4.13). On one hand, by differentiating (3.16), (3.17), and (3.36) with respect to $\kappa$, we obtain that,

$$\partial_x \left(g_{12} \frac{\partial g_{21}}{\partial \kappa} - \frac{\partial g_{12}}{\partial \kappa} \cdot g_{21}\right) = \kappa(qg_{21} + rg_{12}) \frac{\partial}{\partial \kappa}(\gamma + 1) - \kappa(\gamma + 1) \frac{\partial}{\partial \kappa}(qg_{21} + rg_{12})$$

$$- 2i \kappa \gamma(\gamma + 2) + (\gamma + 1)(qg_{21} + rg_{12}).$$

Using (3.15), we have

$$-(g_{12} \frac{\partial g_{21}}{\partial \kappa} - \frac{\partial g_{12}}{\partial \kappa} \cdot g_{21}) \gamma' = -\gamma(2 + \gamma) \frac{\partial}{\partial \kappa}(qg_{21} - rg_{12}) + (qg_{21} - rg_{12})(1 + \gamma) \frac{\partial \gamma}{\partial \kappa}.$$ 

Combining the above two identities, we get

$$\partial_x \frac{g_{12} \frac{\partial g_{21}}{\partial \kappa} - \frac{\partial g_{12}}{\partial \kappa} \cdot g_{21}}{2 + \gamma} = (qg_{21} + rg_{12} - 2i \, \kappa \gamma) - \frac{\partial}{\partial \kappa} \left(\kappa \frac{qg_{21} + rg_{12}}{2 + \gamma}\right),$$

which can be integrated in $x$ to yield

$$\frac{\partial}{\partial \kappa} \int_{\mathbb{R}} \kappa \frac{qg_{21} + rg_{12}}{2 + \gamma} \, dx = \int_{\mathbb{R}} qg_{21} + rg_{12} - 2i \, \kappa \gamma \, dx = -\frac{\partial A}{\partial \kappa}. \quad (4.15)$$

On the other hand, by (4.7), we have

$$A(\kappa) = -i \frac{1}{2} M + O(|\kappa|^{-2}), \quad \text{as} \quad \kappa \to \infty. \quad (4.16)$$
and by (4.8) and (4.10), we obtain

\[ \rho(\kappa) = -i \frac{1}{2} qr + O(|\kappa|^{-2}), \quad \text{as} \quad \kappa \to \infty. \]  

(4.17)

Combining (4.15), (4.16) and (4.17), we can obtain (4.13).

Lastly, (4.11) is obvious from (3.18) and (4.13).

Next, we show the commutation of \( A(\kappa)'s \) under the Poisson bracket (1.5), which implies that \( A(\varkappa) \) is an invariant quantity under the \( A(\kappa)'s \) flows.

Lemma 4.5 (Poisson brackets) There exists \( \delta > 0 \) such that for any \( \kappa \) and \( \kappa' \) with \( i\kappa^2, i\kappa'^2 \in \mathbb{R} \setminus (-1, 1) \) with \( \kappa^2 \neq \varkappa^2 \) and any \( q \in B_{\delta} \cap \mathcal{I} \) we have

\[ \{ A(\kappa), A(\varkappa) \} = 0. \]  

(4.18)

Proof By (4.5) and (1.5), we have

\[ \{ A(\kappa), A(\varkappa) \} = \kappa \varkappa \int g_{12}(\kappa) g'_{21}(\varkappa) + g_{21}(\kappa) g'_{12}(\varkappa) \, dx, \]

which together with (3.19) implies that (4.18) holds.

Next we will exhibit the dynamics of \( g_{12}, g_{21} \) and \( \gamma \) defined by (3.12), (3.13) and (3.14) along the \( A(\kappa) \) flow and DNLS flow, respectively.

4.2 Dynamics I: the \( A(\kappa) \) flow.

Firstly, we have

Lemma 4.6 (Dynamics of \( A(\kappa) \) flow) Let \( i\kappa^2 \in \mathbb{R} \) with \( |\kappa| > 1 \). Under the \( A(\kappa) \) flow, we have

\[ \frac{d}{dt} q = -\kappa g'_{12}(\kappa) \quad \text{and} \quad \frac{d}{dt} r = -\kappa g'_{21}(\kappa). \]  

(4.19)

Proof It is obvious from (4.4) and (1.6).

Lemma 4.7 (Lax pair for the \( A(\kappa) \) flow) Let \( i\kappa^2, i\varkappa^2 \in \mathbb{R} \setminus (-1, 1) \) with \( \kappa^2 \neq \varkappa^2 \), and \( L(\varkappa) \) be defined by (1.9). Under the \( A(\kappa) \) flow, we have

\[ \frac{d}{dt} L(\varkappa) = [ P_{A(\kappa)}, L(\varkappa) ], \]  

(4.20)

where

\[ P_{A(\kappa)} = \begin{bmatrix} -\frac{1}{2} \mathcal{E}(\gamma(\kappa) + 1) & -\Theta g_{12}(\kappa) \\ -\Theta g_{21}(\kappa) & \frac{1}{2} \mathcal{E}(\gamma(\kappa) + 1) \end{bmatrix} \quad \text{with} \quad \Theta = \frac{\sqrt{\kappa} g^3}{\kappa^2 - \varkappa^2}, \quad \mathcal{E} = \frac{\sqrt{\kappa}^2 \kappa^2}{\kappa^2 - \varkappa^2}. \]  

(4.21)
Proof} Firstly, by (3.15) and the fact that
\[ \gamma'(\kappa) = (\partial + i \kappa^2)(\gamma(\kappa) + 1) - (\gamma(\kappa) + 1)(\partial + i \kappa^2) = (\partial - i \kappa^2)(\gamma(\kappa) + 1) - (\gamma(\kappa) + 1)(\partial - i \kappa^2), \]
we have
\[ (\partial + i \kappa^2)(\gamma(\kappa) + 1) - (\gamma(\kappa) + 1)(\partial + i \kappa^2) - 2\kappa(qg_{21}(\kappa) - rg_{12}(\kappa)) = 0. \]  \( (4.22) \)

(\partial - i \kappa^2)(\gamma(\kappa) + 1) - (\gamma(\kappa) + 1)(\partial - i \kappa^2) - 2\kappa(qg_{21}(\kappa) - rg_{12}(\kappa)) = 0. \)  \( (4.23) \)

Next, by (3.16) and the fact that
\[ g'_{12}(\kappa) + 2i \kappa^2 g_{12}(\kappa) = (\partial + i \kappa^2)g_{12}(\kappa) - g_{12}(\kappa)(\partial - i \kappa^2), \]
we get
\[ (\kappa^2 - \kappa^2)g'_{12}(\kappa) = \kappa^2[(\partial + i \kappa^2)g_{12}(\kappa) - g_{12}(\kappa)(\partial - i \kappa^2)] + \kappa \kappa^2 q(\gamma(\kappa) + 1). \]  \( (4.24) \)

Finally, by (3.17) and the fact that
\[ g'_{21}(\kappa) + 2i \kappa^2 g_{21}(\kappa) = (\partial - i \kappa^2)g_{21}(\kappa) - g_{21}(\kappa)(\partial + i \kappa^2), \]
we obtain
\[ (\kappa^2 - \kappa^2)g'_{21}(\kappa) = \kappa^2[(\partial - i \kappa^2)g_{21}(\kappa) - g_{21}(\kappa)(\partial + i \kappa^2)] - \kappa \kappa^2 r(\gamma(\kappa) + 1). \]  \( (4.25) \)

Combining (1.9), (4.19) with (4.22), (4.23), (4.24), (4.25), we can obtain (4.20). \( \square \)

**Proposition 4.8** *(Microscopic conservation law for the A(\kappa) flow)* Let \( i \kappa^2, i \kappa^2 \in \mathbb{R} \setminus (-1, 1) \) with \( \kappa^2 \neq \kappa^2 \). Under the A(\kappa) flow, we have
\[ \frac{d}{dt} g_{12}(\kappa) = -\mathcal{E}g_{12}(\kappa)[\gamma(\kappa) + 1] + \Theta g_{12}(\kappa)[\gamma(\kappa) + 1], \]  \( (4.26) \)
\[ \frac{d}{dt} g_{21}(\kappa) = \mathcal{E}g_{21}(\kappa)[\gamma(\kappa) + 1] - \Theta g_{21}(\kappa)[\gamma(\kappa) + 1], \]  \( (4.27) \)
\[ \frac{d}{dt} \gamma(\kappa) = -2\Theta [g_{12}(\kappa)g_{21}(\kappa) - g_{21}(\kappa)g_{12}(\kappa)]. \]  \( (4.28) \)

and the following microscopic conservation laws
\[ \partial_t \left[ 2i \kappa \cdot \gamma(\kappa) - (qg_{21}(\kappa) + rg_{12}(\kappa)) \right] + \partial_x j_{\gamma}(\kappa, \kappa) = 0, \]  \( (4.29) \)
\[ \partial_t \rho(\kappa) + \partial_x j_{A(\kappa)}(\kappa, \kappa) = 0, \]  \( (4.30) \)
where the flux functions \( j_\gamma \) and \( j_{A(\kappa)} \) are determined by

\[
j_\gamma(\kappa, \kappa) := -\frac{k^3(\kappa^2 + \kappa z^2)}{(\kappa^2 - \kappa z^2)^2} \left[ g_{12}(\kappa)g_{21}(\kappa) + g_{21}(\kappa)g_{12}(\kappa) \right]
\]
\[
- \frac{k^3 z}{(\kappa^2 - \kappa z^2)^2} \left[ (\gamma(\kappa) + 1)(\gamma(\kappa) + 1) \right].
\] (4.31)

\[
j_{A(\kappa)}(\kappa, \kappa) := -\Theta \frac{g_{12}(\kappa)g_{21}(\kappa) + g_{12}(\kappa)g_{21}(\kappa)}{2 + \gamma(\kappa)} - \frac{\Theta z}{2k} \gamma(\kappa).
\] (4.32)

**Remark 4.9** Due to the coercivity of the quadratic term of the flux, the conservation law (4.30) is more useful than (4.29).

**Proof** Firstly, we show dynamics (4.26), (4.27) and (4.28). By (1.6), Proposition 3.1 and Lemma 4.7, we have

\[
\frac{d}{dt} G(x, z; \kappa) = -\int G(x, y; \kappa) \frac{d}{dt} L(z) G(y, z; \kappa) dy
\]
\[
= \int G(x, y; \kappa) \left[ \begin{array}{cc} 0 & -\kappa^2 g_{12}'(\kappa) \\ -\kappa^2 g_{21}'(\kappa) & 0 \end{array} \right] G(y, z; \kappa) dy
\]
\[
= P_{A(\kappa)}(x, \kappa, z) G(x, z; \kappa) - G(x, z; \kappa) P_{A(\kappa)}(z; \kappa, \kappa).
\]

By choosing \( x = z \) and (4.20), we have

\[
\frac{d}{dt} G(x, x; \kappa) = -\Theta \left[ \begin{array}{c} -\kappa^2 g_{12}'(\kappa) \\ -\kappa^2 g_{21}'(\kappa) \end{array} \right]
\]
\[
- \Theta \left[ \begin{array}{c} g_{12}(\kappa)g_{21}(\kappa) - g_{21}(\kappa)g_{12}(\kappa) - g_{12}(\kappa)(\gamma(\kappa) + 1) \\ g_{21}(\kappa)(\gamma(\kappa) + 1) - g_{12}(\kappa)g_{21}(\kappa) + g_{21}(\kappa)g_{12}(\kappa) \end{array} \right],
\]

which implies (4.26), (4.27) and (4.28).

Next, we prove (4.29). A direct computation implies that

\[
\partial_t \left[ 2i \kappa \gamma(\zeta) - q g_{21}(\zeta) - r g_{12}(\zeta) \right]
\]
\[
= -g_{21}(\zeta) \frac{d}{dt} q - g_{12}(\zeta) \frac{d}{dt} r + 2i \kappa \frac{d}{dt} \gamma(\zeta) - q \frac{d}{dt} g_{21}(\zeta) - r \frac{d}{dt} g_{12}(\zeta).
\] (4.33)

On the one hand, by Lemma 4.6, we have

\[
- g_{21}(\zeta) \frac{d}{dt} q - g_{12}(\zeta) \frac{d}{dt} r = \kappa \left[ g_{21}(\zeta)g_{12}'(\kappa) + g_{12}(\zeta)g_{21}'(\kappa) \right].
\] (4.34)

On the other hand, by (4.26), (4.27), and (4.28), we obtain that

\[
2i \kappa \frac{d}{dt} \gamma(\zeta) - q \frac{d}{dt} g_{21}(\zeta) - r \frac{d}{dt} g_{12}(\zeta)
\]
\[
= -4i \kappa \Theta g_{12}(\kappa)g_{21}(\zeta) - \Theta g_{21}(\zeta)q \left[ \gamma(\kappa) + 1 \right] + \Theta g_{21}(\kappa)q \left[ \gamma(\kappa) + 1 \right].
\]
\[ + 4i \varphi \Theta g_{12}(\kappa)g_{21}(\varphi) + \mathcal{E} g_{12}(\varphi)r [\gamma(\kappa) + 1] - \Theta g_{12}(\kappa)r [\gamma(\kappa) + 1] \]
\[ = \frac{\varphi \Theta}{\kappa^2} [g_{21}(\varphi)g_{12}'(\kappa) + g_{12}(\varphi)g_{21}'(\kappa)] - \frac{\Theta}{\varphi} [g_{21}(\kappa)g_{12}'(\varphi) + g_{12}(\kappa)g_{21}'(\varphi)] . \]  

(4.35)

Combining (4.33), (4.34), (4.35) with (3.19), we can obtain (4.31).

Finally, let us show that (4.32) holds. A direct calculation implies that

\[
(\gamma(\kappa) + 2)^2 \partial_t \rho(\varphi)
= -\varphi \left[ g_{21}(\varphi) \frac{d}{dt} q + g_{12}(\varphi) \frac{d}{dt} r \right] (\gamma(\kappa) + 2) \]
\[ - \varphi \left[ q \frac{d}{dt} g_{21}(\varphi) + r \frac{d}{dt} g_{12}(\varphi) \right] (\gamma(\kappa) + 2) \]
\[ + \varphi \left[ q g_{21}(\varphi) + r g_{12}(\varphi) \right] \partial_t \gamma(\varphi) . \]

(4.36)

By Lemma 4.6 and (4.26), (4.27), we have

\[
(4.36) + (4.37)
= \varphi \kappa \left[ g_{21}(\varphi)g_{12}'(\kappa) + g_{12}(\varphi)g_{21}'(\kappa) \right] (\gamma(\kappa) + 2)
+ \frac{\varphi \Theta}{2 \kappa} \gamma'(\kappa) (\gamma(\kappa) + 1) (\gamma(\kappa) + 2) - \frac{\mathcal{E}}{2} \gamma'(\kappa) (\gamma(\kappa) + 1) (\gamma(\kappa) + 2)
= \varphi \kappa \left[ g_{21}(\varphi)g_{12}'(\kappa) + g_{12}(\varphi)g_{21}'(\kappa) \right] (\gamma(\kappa) + 2)
- \frac{\mathcal{E}}{2} \left[ (\gamma(\kappa) + 1) (\gamma(\kappa) + 1) \right] (\gamma(\kappa) + 2) + \mathcal{E} \gamma'(\kappa) (\gamma(\kappa) + 1) (\gamma(\kappa) + 2). \]

(4.39)

By (4.28), we have

\[
(4.38) = -2 \varphi \Theta \left[ q g_{21}(\varphi) + r g_{12}(\varphi) \right] g_{12}(\kappa)g_{21}(\varphi) - g_{21}(\kappa)g_{12}(\varphi) \]
\[ = -2 \varphi \Theta \left[ q g_{21}(\varphi) - r g_{12}(\varphi) \right] g_{12}(\kappa)g_{21}(\varphi) + g_{21}(\kappa)g_{12}(\varphi) \]
\[ + 4 \Theta \varphi g_{12}(\varphi)g_{21}(\varphi) [q g_{21}(\kappa) - r g_{12}(\kappa)] \]
\[ = -\Theta \gamma'(\varphi) \left[ g_{12}(\kappa)g_{21}(\varphi) + g_{21}(\kappa)g_{12}(\varphi) \right] - \frac{\varphi \Theta}{2 \kappa} \gamma'(\kappa) \gamma(\varphi) [\gamma(\varphi) + 2] . \]

(4.40)

Combining (4.39), (4.40) and (3.19), we can deduce (4.32).

\[ \square \]

4.3 Dynamics II: the DNLS flow:

Now we turn to the $H_{\text{DNLS}}$ flow. We firstly recall the Lax representation for (1.1) as follows.
Lemma 4.10 (Lax Pair for the DNLS flow, [1,21]) Let $i \kappa^2 \in \mathbb{R} \setminus (-1, 1)$, $L(\kappa)$ be defined by (1.9). Under the DNLS flow, we have

$$
\frac{d}{dt} L(\kappa) = [P_{H_{\text{DNLS}}}, L(\kappa)],
$$

where $P_{H_{\text{DNLS}}} = \begin{bmatrix} -2i \kappa^4 - i \kappa^2 qr & 2\kappa^3 q + i \kappa q' + \kappa q^2 r \\ 2\kappa^3 r - i \kappa r' + \kappa qr^2 & 2i \kappa^4 + i \kappa^2 qr \end{bmatrix}$.

Following the analogue argument as those in Proposition 4.8, we can obtain main result in this paper.

Theorem 4.11 (Microscopic conservation law for the DNLS flow) Let $i \kappa^2 \in \mathbb{R} \setminus (-1, 1)$. Under the DNLS flow, we have

$$
\frac{d}{dt} q = iq'' + (q^2 r)' \quad \text{and} \quad \frac{d}{dt} r = -ir'' + (qr^2)'.
$$

and

$$
\frac{d}{dt} g_{12}(\kappa) = -2 \left( 2i \kappa^4 + i \kappa^2 qr \right) g_{12}(\kappa) - \left( 2\kappa^3 q + i \kappa q' + \kappa q^2 r \right) (\gamma(\kappa) + 1),
$$

$$
\frac{d}{dt} g_{21}(\kappa) = 2 \left( 2i \kappa^4 + i \kappa^2 qr \right) g_{21}(\kappa) + \left( 2\kappa^3 r - i \kappa r' + \kappa qr^2 \right) (\gamma(\kappa) + 1),
$$

$$
\frac{d}{dt} \gamma(\kappa) = 2\kappa^2 \gamma'(\kappa) + 2i \kappa (q' g_{21}(\kappa) + r' g_{12}(\kappa)) + qr \gamma'(\kappa).
$$

Moreover, we have the following microscopic conservation law

$$
\partial_t \rho(\kappa) + \partial_x j_{\text{DNLS}}(\kappa) = 0,
$$

where the density $\rho$ and the flux $j_{\text{DNLS}}$ are defined by (4.13) and

$$
j_{\text{DNLS}}(\kappa) := i \kappa \frac{q' g_{21}(\kappa) - r' g_{12}(\kappa)}{2 + \gamma(\kappa)} - i \kappa^2 qr - 2\kappa^2 \rho(\kappa) - qr \rho(\kappa).
$$

Proof By (1.6), Proposition 3.1 and Lemma 4.10, it is easy to obtain the dynamics of $q$, $r$ and $g_{12}$, $g_{21}$ and $\gamma$ under the $H_{\text{DNLS}}$ flow. Combining the dynamics of $q$, $r$, $g_{12}$, $g_{21}$, $\gamma$ and (3.19), we can complete the proof of the microscopic conservation law (4.44). \qed

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