A review of aggregation techniques for agent-based models: understanding the presence of long-term memory

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Abstract A key feature of agent-based modeling is the understanding of the macroscopic behavior based on data at the microscopic level. In this respect, financial market models are requested to replicate, at the aggregate level, the stylized facts of empirical data. Among them, a remarkable role is played by the long term behavior. Indeed, the study of the long-term memory is relevant, in that it describes if and how past events continue to maintain their influence for the future evolution of a system. In economic applications, this is relevant for understanding the reaction of the system to micro- and macro-economic shocks. Moreover, further information on the long-term memory properties of a system can be obtained by analyzing agents heterogeneity and the outcome of their aggregation. The aim of this paper is to review a few techniques—though the most relevant in our opinion—for studying the long-term memory as emergent property of systems composed by heterogeneous agents. Theorems relevant to the present analysis are summarized and their applications in four structural models with long-term memory are shown. This property is assessed through the analysis of the functional relation between model parameters.

Keywords Long-term memory · Agent-based models · Parameters distribution

1 Introduction

The presence of long-term memory property (LTM, hereafter) is a remarkable feature of time series, when the autocorrelation function decays hyperbolically as the time lag increases. The
property is relevant because of its reaction to shocks: systems with high dependence on past events need more time to recover from shocks than systems with a fast decay of the correlation. Models with LTM were introduced in the physical sciences since at least 1950, when some studies in applied statistics detected the presence of LTM within hydrologic and climate data. Classical references on this field are Hurst (1951, 1957), Mandelbrot (1972), Mandelbrot and Wallis (1968) and McLeod and Hipel (1978). Quantitative studies on stock markets have shown persistence properties in financial time series. LTM has been evidenced through the analysis of many different time series, such as speculative returns (Bollerslev and Mikkelsen 1996; Ding and Granger 1996b), foreign exchange rate returns (Ausloos and Ivanova 2000; Ivanova and Ausloos 1999; Reboredo et al. 2013) and their power transformation (Ding and Granger 1996a), and stock prices time series (Ausloos and Ivanova 1999; Reboredo et al. 2013; Vandewalle and Ausloos 1998). For what concerns the persistence of the prices, this property has been tackled in the context of the agricultural futures by Wei and Leuthold (2000), while Cheung and Lai (1993) and Lo (1991) have focussed on the evidence of LTM in certain stock prices and in the gold market returns. Yet, Fung et al. (1994) has shown no consistent pattern of persistence in S&P 500 index futures prices.

The microeconomic explanation of these data is far from being obvious. We are mostly interested on agent-based models for financial time series, and on the composition of possible actions in the market that lead to persistence properties of prices. Specifically, we aim at reviewing some remarkable theoretical results for assessing the presence of LTM. The considered approaches differ from the most part of literature, where the presence of LTM is measured through numerical estimates (Bianchi et al. 2013; Bianchi and Pianese 2008; Kirman and Teyssiére 2002; Lux and Ausloos 2002; Rotundo et al. 2007; Weron 2002).

However, the numerical perspective is important, and some comments are needed. Beyond the mere regression on the autocorrelation function, other methods have been developed for the numerical estimate of LTM, aiming at shortening the confidence intervals and improving the reliability of results: in this perspective, it is worth mentioning the Hurst’s exponent $H$, the Detrended Fluctuation Analysis (DFA), the spectrum (within some boundaries) and multifractals (Diebolt and Guiraud 2005; Hurst 1951, 1957; Mandelbrot 1972; Mandelbrot and Wallis 1968; McLeod and Hipel 1978; Peng et al. 1994; Stanley et al. 2002). Literature reports also studies on the systematic bias in the over- or under-estimate of specific procedures (Battacharya and Waymire 2009; Bianchi 1997, 2004).

As we will see, for what concerns LTM in agent-based models, a key role is played by the heterogeneity property of the system. In this respect, for what concerns the specific context of finance, empirical evidence suggests that the interaction among agents leads to an imitative behavior, that can affect the structure of the asset price dynamics. Several authors focus their research on describing the presence of an imitative behavior in financial markets (see, for instance, Avery and Zemsky 1988; Bischi et al. 2006; Brianzoni et al. 2010; Chiarella et al. 2002).

In general, the traditional viewpoint of agent-based models in economics and finance relies on the existence of representative rational agents. Two different behaviors of agents follow from the property of rationality: firstly, a rational agent analyzes the choices of the other actors and tends to maximize utility and profit or minimize the risk. Secondly, rationality consists in having rational expectations, i.e. the forecast on the future realizations of the variables are assumed to be identical to the mathematical expectations conditioned on the available information set. Thus, the assumption of rationality implies agents’ full knowledge of the market dynamics and equilibrium, and ability to solve the related equilibrium equations.

However, rationality is a rather controversial hypothesis. It is worth mentioning Simon (1957), who argues that assuming a complete knowledge about the economic environment
seems to be unrealistic. Moreover, if the equilibrium model equations are nonlinear or involve a large number of parameters, it can be hard to find a solution.

An heterogeneous agents system is more realistic, since it allows the description of agents’ heterogeneous behaviors evidenced in the financial markets (see Kirman 2006 for a summary of some stylized facts supporting the agents’ heterogeneity assumption). Moreover, heterogeneity allows to avoid the unrealistic assumption of perfect knowledge of agent beliefs, and this leads to the more reasonable condition of bounded rationality (Hommes 2001).

Brock and Hommes (1997, 1998) have much contributed on this field. The authors introduce the learning strategies theory to discuss agents’ heterogeneity in economic and financial models. More precisely, they assume that different types of agents have different beliefs about future variables’ realizations and the forecast rules are commonly observable by all the agents.

In Brock and Hommes (1998), the authors consider an asset in a financial market populated by two typical investor types: fundamentalists and chartists. An agent is a fundamentalist if she/he believes that the price of the aforementioned asset is determined by its fundamental value. In contrast, chartists perform a technical analysis of the market and do not take into account the fundamentals.

More recently, relevant contributions on heterogeneity in agent-based models can be found in Alfarano et al. (2008), Chiarella et al. (2006), Chiarella and He (2002) and Foellmer et al. (2005). For an excellent survey on this topic, see Hommes (2006).

Furthermore, also aggregation and spreading of opinions give an insight of social interactions. Models that allow for opinion formation are mostly based on random interactions among agents, and they are refined considering constraints to the social contact, as an example modeled through scale-free networks. It has already been shown that the relevant number of social contacts in financial markets is very low, being between 3 and 4 (Ausloos et al. 2002; Reka and Barabasi 2002; Rotundo and Ausloos 2007; Vandervalle et al. 1999), opening the way to lattice-based models.

In the context of complex systems, the interpretation of heterogeneity as diversity is worth to be cited. The analysis of the diversity became a remarkable aspect of decision theory on what concerns the selection of multiple elements belonging to different families of candidates (Yager 2010).

The possibility to provide theoretical results on LTM of time series generated from heterogeneous agent-based models overcomes at once the problem of the reliability of numerical methods, the time-consuming computational time, the need to run the algorithms many times, so to confirm the results and to derive the ones related to the mean and variance of the estimated variable, the reliability of random variables generators, and the problem of managing many variables, that often cause numerical instabilities. Therefore, we here aim at focussing on structural models for LTM.

The literature on this specific subject is not wide. Some references are Box-Steffenmaier and Smith (1996), Byers et al. (1997), Diebolt and Guiraud (2005), Tscherning (1995) and Willinger et al. (1998). The keypoint of the quoted references is to assume distributional hypothesis on model parameters, in order to detect the presence of LTM in time series.

Specifically, despite the economic/financial relevance of LTM, there is a substantial lack of theoretical studies grounded on the functional properties of model parameters. This is the rational that led us to select theorems and results aiming at theoretically proving the presence (or absence) of LTM. The present report aims at giving a critical review on the used approaches and shows the application of the main theoretical results to four different agent-based models. The four models follow Foellmer et al. (2005) in developing a chartist/fundamentalist framework, but are radically different in the microeconomic approach.
and in modeling heterogeneity. Specifically, Cerqueti and Rotundo (2003) bases on the model of Kirman and Teyssiére (2002), and generalizes it to the study of the LTM of exchange rates; in Cerqueti and Rotundo (2007) the maximum of expected utility is studied. Heterogeneity among the agents also includes mutual influence and the case of dependence among their decisions; Cerqueti and Rotundo (2010) includes the analysis of returns through the usage of a result proved in Dittman and Granger (2002). Cerqueti and Rotundo (2012) proposes a condition of balance among excess of demand and excess of supply. The presence of spot traders is analyzed alongside the chartists and fundamentalist forecasts. In general, the analysis contained in the models listed above extend some existing results (Zaffaroni 2004, 2007a, b) about LTM arising due to the aggregation of micro units, by enlarging the class of probability densities of agents’ parameters.

The rest of the paper is organized as follows. Section 2 contains the key theoretical results used for the assessment of LTM. Sections 3–6 collect the discussion of the theoretical results presented in Cerqueti and Rotundo (2003, 2007, 2010, 2012), respectively. Section 7 concludes. Section 7 is an Appendix which collects some mathematical material used throughout the paper.

2 Theoretical results for assessing LTM

This section introduces the main techniques for LTM, while base definitions are reported in the Appendix. The presence of LTM can be studied by a direct calculus of the correlation of the considered time series, but under proper hypothesis the proof of LTM can be obtained directly from the properties of the process under examination. Some authors (Ding and Granger 1996a; Ding et al. 1993; Granger 1980, to cite a few) have studied various integrated of order \(d\) (or, briefly, I(\(d\))) time series and proposed techniques for the study of the sum of autoregressive processes. We here focus on a specific aggregation technique (Ding and Granger 1996a). Let us consider the \(N\)-component model\(^1\) defined as

\[
    x_t = \sum_{i=1}^{N} w_i x_{i,t}, \quad \text{with} \quad \sum_{i=1}^{N} w_i = 1, \tag{1}
\]

\[
    x_{i,t} = \bar{x}(1 - \alpha_i - \beta_i) + \alpha_i \epsilon_{t-1}^2 + \beta_i x_{i,t-1}, \quad i = 1, \ldots, N; \quad t > 0, \tag{2}
\]

where

\[
    \epsilon_t = \sqrt{x_t} \epsilon_t, \quad \text{with} \quad \epsilon_t \sim \text{i.i.d. } D(0,1), \tag{3}
\]

\(\bar{x} \in \mathbb{R}, \alpha_i, \beta_i \geq 0, \alpha_i + \beta_i < 1, w_i\) are the weights that the \(i\)-th component has in the sum and \(D(0, 1)\) is the notation used for a distribution with support \((0, 1)\). Equation (2) can be rewritten as:

\[
    x_{i,t} = \bar{x} \frac{1 - \alpha_i - \beta_i}{1 - \beta_i} + \frac{\alpha_i}{1 - \beta_i L} \epsilon_{i,t-1}^2, \quad i = 1, \ldots, N; \quad t > 0, \tag{4}
\]

where \(L\) is the one-step time-lag operator, i.e.: \(Lx_t = x_{t-1}\).

By substituting (4) into (2) one gets

\[
    x_t = \sum_{i=1}^{N} w_i \left[ \bar{x} \frac{1 - \alpha_i - \beta_i}{1 - \beta_i} + \frac{\alpha_i}{1 - \beta_i L} \epsilon_{t-1}^2 \right]. \tag{5}
\]

\(^1\) For the sake of simplicity, we will denote hereafter the entire time series or process as \(x_t\) instead of \([x_t]\).
If the number of components is large, as it is in the most part of agent-based models, we can consider a limit for $N \to +\infty$. In this case Eq. (5) transforms into:

$$x_t = \bar{x} \int_0^1 \int_0^1 \frac{1 - \alpha - \beta}{1 - \beta} dF(\alpha, \beta) + \int_0^1 \int_0^1 \frac{\alpha}{1 - \beta L} \epsilon_{t-1}^2 dF(\alpha, \beta),$$

(6)

where $F$ is the joint distribution of two random variables $\alpha$ and $\beta$ from which the numbers $\{\alpha_i\}_{i=1}^N$ and $\{\beta_i\}_{i=1}^N$ are sampled, respectively.

It is worth remarking that, since the limit $N \to \infty$ leads to an integral formulation of the aggregation, the calculus becomes more easy than the discrete case as soon as a joint distribution $F(\alpha, \beta)$ for $\alpha$ and $\beta$ is given\(^2\). Moreover, an appropriate selection of the joint distribution $F(\alpha, \beta)$ leads to the LTM of the process $x_t$. This is the argument of the following important result.

**Theorem 2.1** Let us consider the process given by Eq. (1) under the hypotheses that $\alpha$ and $\beta$ are drawn from the Beta distribution\(^3\) with parameters $p > 0$ and $q \in (0, 1/2]$. Let us further assume\(^4\) that $\alpha = (1 - \beta)\alpha^*$, where $\alpha^* \sim D(0, 1)$ with mean $\mu > 0$ and $\alpha^*$ independent from $\beta$. Then $x_t$ has LTM.

**Proof** By the independence of $\alpha^*$ and $\beta$, we can write

$$F_*(\alpha^*, \beta) = F_1(\alpha^*) F_2(\beta),$$

where $F_*, F_1, F_2$ are the joint distribution function of $(\alpha^*, \beta)$ and the marginals of $\alpha^*$ and $\beta$, respectively. Then:

$$x_t = \bar{x} \int_0^1 \int_0^1 \frac{1 - \alpha - \beta}{1 - \beta} dF(\alpha, \beta) + \int_0^1 \int_0^1 \frac{\alpha}{1 - \beta L} \epsilon_{t-1}^2 dF(\alpha, \beta)$$

$$= \bar{x} \int_0^1 \int_0^1 (1 - \alpha^*) dF_* (\alpha^*, \beta) + \sum_{k=1}^\infty \epsilon_{t-k}^2 \int_0^1 \alpha^* (1 - \beta)^{k-1} dF_*(\alpha^*, \beta)$$

$$= \bar{x} (1 - \mu) + \sum_{k=1}^\infty \epsilon_{t-k}^2 \int_0^1 \alpha^* dF_1(\alpha^*) \int_0^1 (1 - \beta)^{k-1} dF_2(\beta)$$

$$= \bar{x} (1 - \mu) + \mu \sum_{k=1}^\infty \frac{\beta(p + k - 1, q + 1)}{\beta(p, q)} \epsilon_{t-k}^2.$$

(7)

Standard calculus provides:

$$\frac{\beta(p + k - 1, q + 1)}{\beta(p, q)} = \frac{q \Gamma(p + q) \Gamma(p + k - 1)}{\Gamma(p) \Gamma(p + q + k)} \sim \frac{q \Gamma(p + q)}{\Gamma(q) k^{-1-q}} \text{ as } k \to +\infty.$$ 

Since $q \in (0, 1/2]$, this is a characteristic of a process with LTM (Granger 1980). \(\square\)

\(^2\) The discrete case can be seen as a particular case of the continuous one, where $\alpha_i$ and $\beta_i$ are a particular sample from the same distribution.

\(^3\) see the Appendix for the definition of Beta distribution, generalized beta distribution and the related normalization coefficient $\beta(p, q)$.

\(^4\) This hypothesis is not strictly necessary, but it simplifies the calculus.
**Remark 2.2** It is worth stressing the versatility of the $B(p, q)$ distribution, which is grounded on its statistical properties and on the several different shapes that it can assume depending on its parameters values. In this respect, the particular case $p = 1, q = 1$ models the uniform distribution.

Theorem 2.1 is not the only available tool to state LTM of aggregated processes. Of course, a direct estimate of the decay of the correlation function can always be performed, and this can be theoretically achieved through distributional hypotheses on the parameters of the model. Theorem 2.1 can be generalized and the distributional assumption on the Beta can be removed (see Granger 1980 and the definition of integrated processes in the Appendix). This generalization is formalized in the following result:

**Theorem 2.3** Let $x_t$ be defined as follows:

$$x_t = c_1 + c_2 \sum_{k=1}^{\infty} a_k x_{t-k} e_{t-k},$$  \hfill (8)

where $e_t \sim i.i.d. D(0, 1)$, $c_1, c_2 \in \mathbb{R}$. Assume that $a_k \sim k^{-d-1}$, with $d \in (0, 1/2]$. Then $x_t$ has LTM.

The persistence properties of a time series can be assessed also by considering it as an aggregation of processes with LTM. In this respect, we report here a result due to Granger (1980) and particularly useful in our review:

**Proposition 2.4** If $x_t$ and $y_t$ are independent integrated processes of order, respectively, $d_x$ and $d_y$, then the sum $z_t := x_t + y_t$ is an integrated process of order $d_z$, where

$$d_z = \max \{d_x, d_y\}.$$

In the next sections we review four models and show the usage of the above results for the assessment of LTM.

### 3 First setting

We here outline the model developed in Cerqueti and Rotundo (2003) and show how Theorem 2.1 has been used to detect LTM.

#### 3.1 Outline of the model

The variable of interest is the exchange rate. The market is populated by $N$ agents. The $i$-th agent performs a forecast $\Delta P_{i,t+1} | I_{i,t}$ of the exchange rate increment $\Delta P_{i,t+1}$ conditioned to the information available at time $t$, $I_{i,t}$, for $i = 1, \ldots, N$.

Such a forecast relies on technical analysis (chartist approach) $\Delta P_{i,t+1}^c | I_{i,t}$ and on fundamentals evaluation (fundamentalist approach) $\Delta P_{i,t+1}^f | I_{i,t}$, conditioned to her/his information at time $t$, as follows:

$$(\Delta P_{i,t+1} | I_{i,t}) = \psi_i (\Delta P_{i,t+1}^f | I_{i,t}) + \xi_i (\Delta P_{i,t+1}^c | I_{i,t}),$$

where $\psi_i + \xi_i = 1$ and $\psi_i$ and $\xi_i$ represent the proportions of fundamentalist and chartist approaches in making forecast, respectively.

 başlayan konu, belirli bir çevre ve değerler üzerindeki varyans ve diğer çeşitli şekillerin olduğu eko"{s}ikliklere dayanıyor. Bu türde, $p = 1, q = 1$ durumunda da uniform dağılımı modeli.
The exchange rate of the market is given by the average of the exchange rates associated to the agents, i.e.

\[ P_t = \sum_{i=1}^{N} \frac{1}{N} P_{i,t}. \] (9)

The chartist forecast is given by:

\[ \Delta P_{i,t+1}^c | I_{t,i} = h_{t-1} (P_t - P_{t-1}) + \tilde{\alpha}_i (P_{t-1} - P_{i,t-1}), \] (10)

where \( h_t \) is a deterministic function of time and \( \tilde{\alpha}_i \in D(0, 1), \forall i. \)

The fundamentalist forecast is given by:

\[ \Delta P_{i,t+1}^f | I_{i,t} = v \tilde{P}_{i,t} + v (\beta_{i,t} - 1) P_t, \] (11)

where \( v \neq 0, \beta_{i,t} \in D(0, 1) \) and

\[ \tilde{P}_{i,t} = \tilde{P}_{i,t-1} + \epsilon_t, \quad \epsilon_t \sim N(0, \sigma^2). \]

Each agent may invest in foreign risky value with stochastic interest rate \( \rho_t \sim N(\rho, \sigma^2) \) and in riskless bonds with the constant interest rate \( r, \) with \( \rho > r. \)

A mean–variance utility maximization procedure leads to the equation of interest:

\[ P_{i,t} = \left( \frac{1}{c} \right) \frac{\nu \psi_i}{c - 1} \tilde{P}_{i,t} + \frac{1}{c} \left[ (\beta_{i,t} - 1) \nu \psi_i + \xi_i h_{t-1} \right] P_t \]

\[ + \frac{1}{c} \cdot \xi_i (\tilde{\alpha}_i - h_{t-1}) P_{t-1} - \frac{1}{c} \cdot \xi_i \tilde{\alpha}_i P_{i,t-1}, \] (12)

with \( c = \frac{1+\rho}{r-\rho}. \)

3.2 LTM assessment

Theorem 2.1 has been used to prove LTM of \( P_t \) in (9):

**Proposition 3.1** Suppose that the following conditions hold:

1. \( \beta_{i,t} = -\frac{h_{t-1}}{\nu \psi_i} + \frac{h_{t-1}}{\nu} + 1; \)
2. \( \tilde{\alpha}_i = \xi_i^\delta; \)
3. \( \xi_i \sim B(p, p); \)
4. \( p \in (0, 1/2). \)

Then, for \( N \to +\infty, \) we have that \( P_t \) has LTM with Hurst’s exponent given by \( H = p + \frac{1}{2}. \)

The proof follows the same trace as in the proof of Theorem 2.1.

**Proof** Define \( \hat{\beta}_i = -\frac{1}{\xi_i} \tilde{\alpha}_i, \hat{\alpha}_i \hat{\epsilon}_{t-1} = \frac{1}{\xi_i} (\tilde{\alpha}_i - h_{t-1}). \)

By hypothesis 3. and by applying Proposition 8.15 (in the Appendix), we get \( \xi_i \left( -\frac{1}{\xi_i} \right) \sim b \left( p, p, -\frac{1}{c}, 1 \right). \) It is worth noting that, for particular choices of \( \tilde{\alpha}_i, \hat{\beta}_i \) obeys still to a beta distribution. In particular, hypothesis 2. assures that \( \hat{\beta} \sim b(p, p, -\frac{1}{c}, \delta). \)

By hypothesis 1., then, (12) becomes:

\[ P_{i,t} = \frac{(1/c) \nu \psi_i + 1}{1 - \hat{\beta}_i L} \tilde{P}_{i,t} + \frac{\hat{\alpha}_i \hat{\epsilon}_{t-1}}{1 - \hat{\beta}_i L} P_{t-1}. \] (13)
By formulas (9) and (13), we can write

\[ P_t = \sum_{i=1}^{N} \frac{1}{N} \left[ \frac{(1/c)\psi_i + \hat{\alpha}_i \hat{\epsilon}_{t-1}}{1 - \hat{\beta}_i L} \tilde{P}_{i,t} + \frac{\hat{\beta}_i}{1 - \hat{\beta}_i L} P_{t-1} \right]. \quad (14) \]

In the limit for \( N \to \infty \) and by the definition of \( \tilde{P} \) we have

\[ P_t = \mathbb{E} \left[ \frac{(1/c)\psi_i + \hat{\alpha}_i \hat{\epsilon}_{t-1}}{1 - \hat{\beta}_i L} \tilde{P}_{i,t} \right] + \mathbb{E} \left[ \frac{\hat{\beta}_i}{1 - \hat{\beta}_i L} P_{t-1} \right] = \sum_{k=1}^{\infty} P_{t-k} \hat{\epsilon}_{t-k} \int_{0}^{1} \int_{0}^{1} \frac{\hat{\alpha}}{(1 - \hat{\beta} L)} dF(\hat{\alpha}, \hat{\beta}), \]

where \( F \) is the joint distribution function of the random variables \( \hat{\alpha} \) and \( \hat{\beta} \), from which the samples \( \{\hat{\alpha}_i\}_{i=1}^{N} \) and \( \{\hat{\beta}_i\}_{i=1}^{N} \) are extracted and \( \mathbb{E} \) is the expectation operator.

Now, consider the random variable \( \alpha^* \sim D(0, 1) \) with mean \( \mu \) such that \( \hat{\alpha} = (1 - \hat{\beta})\alpha^* \), and \( \alpha^* \) is independent from \( \hat{\beta} \). Thus

\[ P_t = \sum_{k=1}^{\infty} P_{t-k} \hat{\epsilon}_{t-k} \int_{0}^{1} \int_{0}^{1} \alpha^*(1 - \hat{\beta})\hat{\beta}^{k-1} dF_*(\alpha^*, \hat{\beta}) \]

\[ = \sum_{k=1}^{\infty} P_{t-k} \hat{\epsilon}_{t-k} \int_{0}^{1} \alpha^* dF_1(\alpha^*) \int_{0}^{1} (1 - \hat{\beta})\hat{\beta}^{k-1} dF_2(\hat{\beta}) \]

\[ =: \sum_{k=1}^{\infty} a_k P_{t-k} \hat{\epsilon}_{t-k}, \quad (15) \]

where \( F_*, F_1, F_2 \) are the joint distribution function of \( (\alpha^*, \hat{\beta}) \) and the marginals of \( \alpha^* \) and \( \hat{\beta} \), respectively.

Standard calculus gives:

\[ a_k = c_1 \frac{\beta(p + k - 1, p - 1)}{\beta(p, p)} \sim c_2 k^{-1-p}. \quad (16) \]

Hence, \( P_t \) is an I(\( d \)) process, with \( d = p \). Hence, it has LTM, with Hurst’s exponent \( H = p + \frac{1}{2} \) (Ding and Granger 1996a,b; Granger 1980; Granger and Joyeux 1980; Hosking 1981; Jonas 1981; Lo 1991).

4 Second setting

The financial market model proposed in Cerqueti and Rotundo (2007) is now presented. Such a model is able to capture LTM of prices as it arises from the market microstructure.

4.1 Outline of the model

The variable of interest is the price of an asset.

The market is populated by \( N \) agents. The \( i \)-th agent performs a forecast of the market price at time \( t \) as follows:

\[ P_{i,t} = \psi_i P_{i,t}^f + \xi_i P_{i,t}^c, \quad (17) \]
where $\psi_i = 1 - \xi_i$, $\{\xi_i\}_{i=1}^N$ are sampled from a random variable $\xi \in D(0, 1)$, and $P_{i,t}^f$ and $P_{i,t}^c$ represent the fundamentalist and chartist contribution to the forecast, respectively.

Market price is given by a combination of trading prices weighted with respect to the size of the order placed by the agents:

$$P_t = \sum_{i=1}^N \omega_{i,t} P_{i,t},$$

(18)

where $\omega_{i,t}$ is a proxy of the order placed by the $i$-th agent at time $t$, and there exists a function $\tilde{\omega}_t$ such that:

$$\tilde{\omega}_t = \sum_{i=1}^N \omega_{i,t},$$

with $\omega < \tilde{\omega}_t < \bar{\omega}$, for suitable thresholds $\omega$ and $\bar{\omega}$.

The terms $P_{i,t}^f$ and $P_{i,t}^c$ are given by:

$$P_{i,t}^f = v \tilde{P}_{i,t-1} + v(\xi_i - 1)P_{t-1},$$

(19)

where $v \in \mathbb{R}$, $\xi_i$ are sampled by a real random variable $\zeta$ with finite expected value $\bar{\zeta}$ and independent on $\xi$ and

$$\tilde{P}_{i,t} = \tilde{P}_{i,t-1} + \epsilon_t, \quad \epsilon_t \sim N(0, \sigma_\epsilon^2);$$

$$P_{i,t}^c = \gamma_i \tilde{P}_{i,t-1},$$

(20)

where the adjustment factor $\gamma_i$ are i.i.d, with support in the interval $(1 - \delta, 1 + \delta)$, with $\delta \in [0, 1]$, such that

$$\mathbb{E}[\gamma_i] = \bar{\gamma}, \quad i = 1, \ldots, N,$$

and $\gamma_i$ are independent on $\xi_i$ and $\xi_i$.

By (17), (19) and (18), the market price is:

$$P_t = \sum_{i=1}^N \omega_{i,t} \left[ v\psi_i \tilde{P}_{i,t-1} + v\psi_i(\xi_i - 1)P_{t-1} + \gamma_i \xi_i P_{i,t-1} \right].$$

(21)

4.2 LTM assessment

Two cases are presented: absence and presence of imitative behaviors between the agents. These conditions have been captured by imposing absence/presence of dependence between the parameter $\xi_i$ and the weight $\omega_{i,t}$ of agent $i$ at time $t$.

4.2.1 First case: absence of imitative behaviors

The following result holds.

**Theorem 4.1** Given $i = 1, \ldots, N$, let $\xi_i$ be a sampling drawn from a random variable $\xi$ such that

$$\mathbb{E}[\xi^k] \sim O(c)k^{-1-p} + o(k^{-1-p}) \quad \text{as} \quad k \to +\infty.$$  

(22)

Moreover, given $i = 1, \ldots, N$, let $\xi_i$ be a sampling drawn from a random variable $\zeta$. 
Let us assume that \( \xi \) and \( \zeta \) are mutually independent. Furthermore, suppose that there exists \( q > 0 \) such that
\[
(\mathbb{E}[\gamma_i])^{k-1} = \tilde{\gamma}^{k-1} \sim k^{-q}, \quad \text{as } k \to +\infty.
\] (23)

Then, for \( N \to +\infty \) and \( p + q \in [-\frac{1}{2}, \frac{1}{2}] \), we have that \( P_t \) has LTM with Hurst’s exponent given by \( H = p + q + \frac{1}{2} \).

Proof By definition of \( P_{t,i} \), we have
\[
(1 - \gamma_i \xi_i L) P_{t,i} = v \psi_i \tilde{P}_{t,i-1} + v \psi_i (\xi_i - 1) P_{t-1},
\]
and then
\[
P_{t,i} = \frac{v \psi_i}{1 - \gamma_i \xi_i L} \tilde{P}_{t,i-1} + \frac{v \psi_i (\xi_i - 1)}{1 - \gamma_i \xi_i L} P_{t-1}.
\]
(25)

By the definition of \( P_t \) and (25), we have
\[
P_t = \sum_{i=1}^{N} \omega_{i,t} \left[ \frac{v \psi_i}{1 - \gamma_i \xi_i L} \tilde{P}_{t,i-1} + \frac{v \psi_i (\xi_i - 1)}{1 - \gamma_i \xi_i L} P_{t-1} \right].
\]
(26)

Setting the limit as \( N \to \infty \), by replacing \( \psi_i = 1 - \xi_i \) and by the definition of \( \tilde{P} \), a series expansion gives
\[
P_t = v \sum_{k=1}^{\infty} \tilde{\omega}_t P_{t-k} \int_{\mathbb{R}} \int_{0}^{1} (\zeta - 1)(1 - \xi) \xi^{k-1} \tilde{\gamma}^{k-1} dF(\zeta, \xi),
\]
where \( F \) is the joint distribution function of \( \zeta \) and \( \xi \).

Since, by hypothesis, \( \zeta \) and \( \xi \) are mutually independent, with marginal distributions \( F_1 \) and \( F_2 \) respectively, we have
\[
P_t = v \sum_{k=1}^{\infty} \tilde{\omega}_t P_{t-k} \tilde{\gamma}^{k-1} \int_{\mathbb{R}} \int_{0}^{1} (\zeta - 1)(1 - \xi) \xi^{k-1} dF_1(\zeta) dF_2(\xi)
\]
\[
= v \sum_{k=1}^{\infty} \tilde{\omega}_t P_{t-k} \tilde{\gamma}^{k-1} \int_{\mathbb{R}} (\zeta - 1) dF_1(\zeta) \int_{0}^{1} (1 - \xi) \xi^{k-1} dF_2(\xi)
\]
\[
= v(\zeta - 1) \sum_{k=1}^{\infty} \tilde{\omega}_t P_{t-k} \tilde{\gamma}^{k-1}(M_{k-1} - M_k),
\]
where \( M_k \) is the \( k \)-th moment of \( \xi \) and satisfies condition (22). In accord to Theorem 2.3, the proof of the decay of the coefficient of \( P_{t-k} \) gives a sufficient condition for stating the presence of LTM. By the hypotheses on \( \omega, \xi \) and \( \gamma \), we have
\[
\omega \sum_{k=1}^{\infty} P_{t-k}(M_{k-1} - M_k) < \sum_{k=1}^{\infty} \tilde{\omega}_t P_{t-k}(M_{k-1} - M_k) \leq \frac{\omega}{\tilde{\omega}} \sum_{k=1}^{\infty} P_{t-k}(M_{k-1} - M_k)
\]
and
\[
M_{k-1} - M_k \sim k^{-p-1}.
\]
(28)
Hence, we obtain
\[ \tilde{\gamma}^{k-1}(M_{k-1} - M_k) \sim k^{-q-p-1}. \] (29)
Therefore, \( P_t \) is an I(\( d \)) process with \( d = p + q + 1 \). Thus, \( P_t \) has LTM, with Hurst’s exponent \( H = p + q + \frac{1}{2} \) (Ding and Granger 1996a, b; Granger 1980; Granger and Joyeux 1980; Lo 1991).

Remark 4.2 We can use the Beta distribution \( B(p, q) \) for defining the random variable \( \xi \). In fact, a Beta-distributed random variable satisfies the relation stated in (22).

Remark 4.3 In the particular case \( \gamma_i = 1 \), for each \( i = 1, \ldots, N \), then LTM is allowed uniquely for persistence processes. In this case it results \( q = 0 \) and, since \( p > 0 \) by definition, Theorem 4.1 assures that \( H \in (1/2, 1) \).

Remark 4.4 Structural changes drive a modification of the Hurst’s parameter of the time series, and thus the degree of memory of the process. In fact, if the chartist calibrating parameter \( \gamma_i \) or the proportionality factor between chartist and fundamentalist, \( \xi_i \), varies structurally, then the distribution parameters \( p \) and \( q \) of the related random variables change as well. Therefore \( H \) varies, since it depends on \( q \) and \( p \). Furthermore, a drastic change can destroy the stationarity property of the time series. In fact, in order to obtain such stationarity property for \( P_t \), we need that \( p + q \in [-1/2, 1/2] \), and modifications of \( q \) and/or \( p \) must not exceed such range.

Remark 4.5 The parameters \( q \) and \( p \) could be calibrated in order to obtain a persistent, antipersistent or uncorrelated time series.

4.2.2 Second case: presence of imitative behaviors

The dependence structure allowing the size of the order to change the proportion between fundamentalist and chartist forecasts is modeled through a function
\[ f_{\omega_i,t} : D(0, 1) \rightarrow D(0, 1) \] such that \( f_{\omega_i,t}(\xi) = \tilde{\xi}, \forall i, t, \) (30)
for each weight \( \omega_{i,t} \).

In this case, LTM of the price process can be assessed as follows:

Theorem 4.6 Given \( i = 1, \ldots, N \), let \( \xi_i \) be a sampling drawn from a random variable \( \xi \in D(0, 1) \).

Fixed \( \omega_{i,t} \), let \( f_{\omega_{i,t}} \) be a random variable transformation defined as in (30) such that
\[ \mathbb{E}[(f_{\omega_{i,t}}(\xi))^k] = \mathbb{E}[\tilde{\xi}^k] \sim O(c)k^{-1-\tilde{p}} + o(k^{-1-\tilde{p}}) \text{ as } k \rightarrow +\infty. \] (31)
Moreover, given \( i = 1, \ldots, N \), let \( \xi_i \) be a sampling drawn from a random variable \( \xi \), where \( \tilde{\xi} \) and \( \xi \) are mutually independent.

Furthermore, suppose that there exists \( q > 0 \) such that
\[ (\mathbb{E}[\gamma_i])^{k-1} = \tilde{\gamma}^{k-1} \sim k^{-q}, \text{ as } k \rightarrow +\infty. \]
Then, for \( N \rightarrow +\infty \) and \( \tilde{p} + q \in \left[-\frac{1}{2}, \frac{1}{2}\right] \), we have that \( P_t \) has LTM with Hurst’s exponent given by \( H = \tilde{p} + q + \frac{1}{2} \).

Proof The proof is similar to the one given for Theorem 4.1. \( \square \)

Remark 4.7 Remark 4.2 guarantees, that the \( f_{\omega_{i,t}} \) can transform any random variable \( X \sim B(p, q) \) in \( f_{\omega_{i,t}}(X) \sim B(\tilde{p}, \tilde{q}) \). Therefore, the changing of the strategy used by the investors, driven by the weights \( \omega \)’s, can be attained by calibrating the parameters of a Beta distribution.
5 Third setting

We now present the study developed in Cerqueti and Rotundo (2010). The approach used for the estimate of LTM differs from the ones used in the previous settings, since it proceeds through a direct calculus of the decay of the autocorrelation function and the application of Proposition 2.4. Some relevant consequences—in terms of LTM assessment—of a result in Dittman and Granger (2002) on non-linear transformations of \( I(d) \) processes are also shown.

5.1 Outline of the model

The variables of interest are two: the price of an asset and its return. The model is outlined for the price, while the asset return is treated separately as a transformation of price.

The market is populated by \( N \) agents. The \( i \)-th agent performs a forecast \( \Delta P_{i,t+1} | I_i, t \) of the price increment \( \Delta P_{i,t+1} \) conditioned to the information available at time \( t, I_i, t \), for \( i = 1, \ldots, N \).

Such a forecast relies on technical analysis (chartist approach) \( \Delta P_{c}^{i} | I_i, t \) and on fundamentals evaluation (fundamentalist approach) \( \Delta P_{f}^{i} | I_i, t \), conditioned to her/his information at time \( t \), as follows:

\[
(\Delta P_{i,t+1} | I_i, t) = \Delta P_{f}^{i} | I_i, t + \Delta P_{c}^{i} | I_i, t + u_t
\]

where \( u_t \) is a stochastic term representing an error in the forecast at time \( t \).

The market clearing condition is

\[
P_t = \sum_{i=1}^{N} \frac{1}{N} P_{i,t}. \tag{33}
\]

The chartist forecast is given by:

\[
\Delta P_{c}^{i} | I_i, t = \alpha_i^{(1)} (P_i - P_{i,t-1}) + \alpha_i^{(2)} (P_t - P_{t-1}), \tag{34}
\]

with \( \alpha_i^{(1)}, \alpha_i^{(2)} \in \mathbb{R}, \forall i \).

The fundamentalist forecast is given by:

\[
\Delta P_{f}^{i} | I_i, t = \nu \bar{P}_{i,t} + \nu (\beta_i - 1) P_t, \tag{35}
\]

where \( \nu \in \mathbb{R}, \beta_i \) are parameters drawn by sampling from the cartesian product \( (1 - \bar{\beta}, 1 + \bar{\beta})^N, \bar{\beta} > 0 \), equipped with the relative product probability measure and

\[
\bar{P}_{i,t} = \bar{P}_{i,t-1} + \epsilon_t, \quad \epsilon_t \sim N(0, \sigma_\epsilon^2), \tag{36}
\]

for each \( i = 1, \ldots, N \).

Each agent may invest in risky asset with stochastic interest rate \( \rho_t \sim N(\rho, \sigma_\rho^2) \) and in riskless bonds with the constant interest rate \( r \), with \( \rho > r \).

The utility function maximized by the agents is of mean–variance type, with risk aversion term \( \mu \). Utility maximization procedure leads to the equation of interest:

\[
P_t = \frac{1}{N} \sum_{i=1}^{N} \left\{ \frac{1}{1 + c} \cdot \frac{1 - \lambda_i}{1 - \lambda_i L} \left( \gamma_{i,t} - c v \bar{P}_{i,t} \right) - \frac{c}{1 + c} \cdot \frac{1 - \lambda_i}{1 - \lambda_i L} u_t - \frac{c}{1 + c} \cdot \frac{1 - \lambda_i}{1 - \lambda_i L} \left[ \nu (\beta_i - 1) - \alpha_i \right] P_t - \frac{\lambda_i}{1 - \lambda_i L} P_{t-1} \right\}. \tag{37}
\]
where:

$$\gamma_{i,t} = \frac{X_{i,t}}{b_{i,t}}, \quad c = \frac{1+r}{r - \rho}, \quad \lambda_i := \frac{-ca_i^{(2)}}{1 + ca_i^{(1)}}.$$ 

with $X_{i,t}$ supply function at time $t$ of agent $i$ and

$$b_{i,t} = \frac{\rho - r}{2\mu V((P_{i,t+1}|I_{i,t})(1 + \rho_{t+1}))},$$

denoting $V$ the usual variance operator.

5.2 LTM assessment

We here present how LTM has been detected. Two cases are separately treated: asset price and asset return.

5.2.1 First case: asset price

A preliminary technical assumption is stated:

$$\alpha_i = \nu(\beta_i - 1) < -\frac{1}{c}. \quad (38)$$

By (37) and (38), market price can rewritten as follows:

$$P_t = \frac{1}{N} \cdot \frac{1}{1+c} \sum_{i=1}^{N} \frac{1 - \lambda_i}{1 - \lambda_i L} \gamma_{i,t} - \frac{1}{N} \cdot \frac{1}{1+c} \sum_{i=1}^{N} \frac{1 - \lambda_i}{1 - \lambda_i L} \mu_t$$

$$- \frac{1}{N} \cdot \frac{cV}{1+c} \sum_{i=1}^{N} \frac{1 - \lambda_i}{1 - \lambda_i L} \bar{P}_{i,t} - \frac{1}{N} \sum_{i=1}^{N} \frac{\lambda_i}{1 - \lambda_i L} P_{t-1} =: A_1^3 + A_2^3 + A_3^3 + A_4^3, \quad (39)$$

and $\lambda_i \in (0, 1)$, for each $i = 1, \ldots, N$.

The specific term $A_3^3$ is properly rewritten as

$$\bar{P}_{i,t} = \bar{P}_{i,t-1} + \epsilon_t, \quad \epsilon_t \sim N(0, \sigma_t^2) \quad (40)$$

so giving rise to

$$A_3^3 = \frac{1}{N} \sum_{i=1}^{N} \frac{-c}{1+c} \frac{1 - \lambda_i}{1 - \lambda_i L} \Gamma_t, \quad (41)$$

with

$$\Gamma_t = \sum_{j=0}^{t-1} \epsilon_{t-j} + \bar{P}_{t,0} \sim N(0, \sigma_t^2). \quad (42)$$

LTM of $P_t$ is the content of the following result:

**Theorem 5.1** Let us assume that there exists $p$, $q \in (0, +\infty)$ such that $\lambda_i \in [0, 1]$ and $\lambda_i$ are sampled by a $B(p, q)$ distribution.

For each fixed $i = 1, \ldots, N$, let $\gamma_{i,t}$ be a stationary stochastic process such that$^5$

$^5$ $\delta_{i,j}$ is the usual Kronecker symbol, e.g. $\delta_{i,j} = 1$ for $i = j$; $\delta_{i,j} = 0$ for $i \neq j$.  

\[ Springer \]
Moreover, let us assume that \( u_t \) is a stationary stochastic process, with
\[
\mathbb{E}[u_t] = 0; \
\mathbb{E}[u_t u_{t+i}] = \delta_{i,0} \sigma_u^2.
\]

Fix \( r = 1, 2, 3, 4 \). Then, as \( N \to +\infty \), LTM for \( A^r_t \) holds, with Hurst’s exponent \( H_r \), in the following cases:

- \( q > 1 \) implies \( H_r = 1/2 \);
- \( q \in (0, 1) \) and the following equation holds:
\[
\sum_{h=-\infty}^{+\infty} \mathbb{E}[A^r_t A^r_{t-h}] = 0,
\]

imply \( H_r = (1-q)/2 \). In this case it results \( H_r < 1/2 \), and the process is mean reverting.

Moreover, \( P_t \) has LTM with Hurst’s exponent \( H \) given by
\[
H = \max \left\{ H_1, H_2, H_3, H_4 \right\},
\]

**Proof** By Proposition 2.4 and by the relationship between the order \( d \in [-1/2, 1/2] \) of an I(\(d\)) process and Hurst’s exponent \( H = d + 1/2 \), it is sufficient to show LTM for each component of \( P_t \). We proceed by considering \( A_1^r \).

First of all, we need to show that
\[
\mathbb{E}\left[A_1^r A_{1-h}^r\right] \sim h^{-1-q}, \text{ as } N \to +\infty.
\]

Let us examine \( A_1^r A_{1-h}^r \).

\[
A_1^r A_{1-h}^r = \frac{1}{N^2(1+c)^2} \sum_{i=1}^{N} \frac{1 - \lambda_i}{1 - \lambda_i L} \gamma_{i,t} \sum_{j=1}^{N} \frac{1 - \lambda_j}{1 - \lambda_j L} \gamma_{j,t-h}
\]
\[
= \frac{1}{N^2(1+c)^2} \sum_{i=1}^{N} (1 - \lambda_i) \left[ \sum_{l=0}^{\infty} (\lambda_i L)^l \right] \gamma_{i,t} \sum_{j=1}^{N} (1 - \lambda_j) \left[ \sum_{m=0}^{\infty} (\lambda_j L)^m \right] \gamma_{j,t-h}.
\]

The terms of the series are positive, and so it is possible to exchange the order of the sums:

\[
A_1^r A_{1-h}^r = \frac{1}{(1+c)^2} \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} \frac{1}{N^2} \sum_{i=1}^{N} (1 - \lambda_i) \lambda_i^l (1 - \lambda_j) \lambda_j^m \gamma_{i,t-m} \gamma_{j,t-h-l}.
\]

In the limit as \( N \to +\infty \) and setting \( x := \lambda_i, y := \lambda_j \), (49) becomes:

\[
A_1^r A_{1-h}^r = \frac{1}{(1+c)^2} \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} \int_0^1 (1-x)^l (1-y)^m F(x, y) dF(x, y),
\]

where \( F \) is the joint distribution over \( x \) and \( y \).
Taking the mean w.r.t. the time and by using the hypothesis (44), we get
\[
E[A_1^t A_{t-h}^1] = \frac{1}{(1+c)^2} \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} \int_0^1 \int_0^1 (1-x)^l (1-y)^m \delta_{x,y} \delta_{m,l+h} \sigma_y \sigma_x^2 dF(x, y)
\]
(51)
\[
= \frac{1}{\beta(p, q)} \cdot \frac{\sigma_y^2}{(1+c)^2} \sum_{l=0}^{\infty} \int_0^1 (1-x)^{1+q} x^{2l+h+p-1} dx.
\]
(52)

By using the distributional hypothesis on \( \lambda_i \), for each \( i \), we get
\[
E[A_1^t A_{t-h}^1] = \frac{1}{\beta(p, q)} \cdot \frac{\sigma_y^2}{(1+c)^2} \sum_{l=0}^{\infty} \Gamma(h+p+2l) \Gamma(q+2) \Gamma(h+p+q+2l+2) \Gamma(h+1-q).
\]
(53)

Now, the rate of decay of the autocorrelation function related to \( A_1^1 \) is given by (53). By using the results in Rangarajan and Ding (2000) on such rate of decay and the Hurst’s exponent of the time series, we obtain the thesis.

The proof is similar for \( A_2^1, A_3^1, \) and \( A_4^1 \).

5.2.2 Second case: asset return

The assessment of LTM for asset return moves from Dittman and Granger (2002), where the LTM of nonlinear transformation of \( I(d) \) processes is theoretically proven for the transformations that can be written as a finite sum of Hermite polynomials. This is specifically the case of log-returns.

**Theorem 5.2** If \( \log(P_t) \) is \( I(d) \), then returns are \( I(d') \), where

1. if \( -1/2 < d \leq 0 \), then \( d' = -1 \);
2. if \( 0 < d < 1/2 \), then \( d' = d - 1 \).

**Proof** Differencing theory gives that if \( \log(P_t) \) is \( I(d) \) then -applying the result proved in Dittman and Granger (2002)—the log-returns time series \( r_t = \log(P_t) - \log(P_{t-1}) \) is \( d' = d - 1 \).

Theorem 5.2 leads to two simple important consequences.

**Corollary 5.3** Uncorrelated returns \( (d' = 0) \) are obtained if \( d = 1 \).

**Corollary 5.4** LTM in returns \( (d' > 0) \) is obtained if \( d > 1 \).

6 Fourth setting

The last model we present is Cerqueti and Rotundo (2012). LTM is detected by employing all the techniques shown in Sect. 2. Moreover, it adds a further remark on the relative weights of different groups of agents. Each group contributes to the LTM of the aggregate in accord to Proposition 2.4.
6.1 Outline of the model

The hypothesis on the market recall the ones of the previous model for what concerns the price of a risky asset as variable of interest. Henceforth, the chartist/fundamentalist approach and the market clearing condition are still (32) and (33). This model differs from the chartist and fundamentalist evaluation of price, here got as a correction over the price at the previous time. The chartist forecast is given by:

\[ \Delta P_{i,t+1}^c | I_{i,t} = \alpha_i^{(1)}(P_{i,t} - P_{i,t-1}) + \alpha_i^{(2)} P_t \]  

(54)

with \( \alpha_i^{(1)}, \alpha_i^{(2)} \in [0, +\infty), \forall i \).

The fundamentalist forecast is linked to the chartist approach as follows:

\[ \Delta P_{i,t+1}^f | I_{i,t} = v \tilde{P}_{i,t} + \alpha_i^{(2)} P_t , \]  

(55)

where \( v \in \mathbb{R} \) and \( \tilde{P}_{i,t} \) is defined as in (40) for each \( i = 1, \ldots, N \).

Each investor face the decision of investing in a riskless bond that has the constant interest rate \( r \). A technical assumption is required, and it is assumed that \( r > \alpha_i^{(2)} \), for each \( i \).

The utility function maximized by the agents is of mean–variance type, with risk aversion term \( \mu \). Utility maximization procedure leads to the equation of the market price:

\[ P_t = \frac{1}{N} \sum_{i=1}^{N} \left[ \frac{1}{1 + c} \cdot \frac{1 - \lambda_i}{1 - \lambda_i L} \left\{ \gamma_{i,t} - cv \tilde{P}_{i,t} - u_t \right\} \right] , \]  

(56)

where:

\[ \gamma_{i,t} = \frac{X_{i,t}}{b_{i,t}} , \quad c = -\frac{1}{r} , \quad \lambda_i := \frac{-c\alpha_i^{(2)}}{1 + c\alpha_i^{(1)}} , \]

with \( X_{i,t} \) supply function at time \( t \) of agent \( i \) and

\[ b_{i,t} = \frac{-r}{2\mu \sqrt{((P_{i,t+1} | I_{i,t}))}} , \]

denoting \( \sqrt{\cdot} \) the usual variance operator.

6.2 LTM assessment

The detection of LTM for asset prices moves from the statement of some technical assumptions, which play also a role in the economic context of Cerqueti and Rotundo (2012):

Assumption 6.1 There exist \( N \) random variables \( w_1, \ldots, w_N \) and a stochastic process \( z_t \), independent on \( u_t \), such that:

- \( \mathbb{E}[w_j] = \bar{w} \in \mathbb{R} \), for each \( j = 1, \ldots, N \);
- \( w_i \) is independent on \( \lambda_j \), for each \( i = 1, \ldots, N \);
- \( z_t \) are i.i.d., with mean 0 and variance \( \sigma^2 \);
- for each \( i = 1, \ldots, N \) and \( t > 0 \), it results \( \gamma_{i,t} = z_t \cdot w_i \).

Two cases are presented: homogeneous and heterogeneous agents.
6.2.1 First case: homogeneous agents

Homogeneity is captured through an identical distribution for the parameters $\lambda$’s. In this respect, a very general case is presented:

**Theorem 6.2** Let us assume that there exists $p, q \in (0, +\infty)$ such that $\lambda_i$ are sampled by a $B(p, q)$ distribution, for each $i = 1, \ldots, N$.

Then, as $N \to +\infty$, LTM for $P_t$ holds, with Hurst’s exponent $H_B \leq 1/2$.

**Proof** To prove the result, we need to rewrite the process $P_t$ as the sum of three components:

$$P_t = \Gamma_t^1 + \Gamma_t^2 + \Gamma_t^3,$$

where

$$
\begin{align*}
\Gamma_t^1 &= \frac{1}{N(1+c)} \sum_{i=1}^{N} \frac{1 - \lambda_i}{1 - \lambda_i} \gamma_{i,t}; \\
\Gamma_t^2 &= -\frac{c}{N(1+c)} \sum_{i=1}^{N} \frac{1 - \lambda_i}{1 - \lambda_i} \bar{P}_{i,t}; \\
\Gamma_t^3 &= -\frac{1}{N(1+c)} \sum_{i=1}^{N} \frac{1 - \lambda_i}{1 - \lambda_i} u_t.
\end{align*}
$$

(58)

By definition of the model, the processes $\Gamma_t^i$’s are independent. Hence, we can analyze separately LTM of the $\Gamma_t^i$’s.

Denote as $\lambda$ and $w$ the random identically distributed random variables $\lambda_i$ and $w_j$. Furthermore, denote as $F$ the joint cumulative distribution function of $(\lambda, w)$ and $F_{\Lambda}$ be the marginal distribution of $\lambda$.

Posing $\hat{c} = -\frac{1}{1+c} \omega$, in the limit for $N \to \infty$ we have

$$
\begin{align*}
\Gamma_t^1 &= \lim_{N \to +\infty} \frac{1}{N(1+c)} \sum_{i=1}^{N} \frac{1 - \lambda_i}{1 - \lambda_i} \gamma_{i,t} = -\frac{1}{1+c} \int_0^1 \int_0^1 \frac{1 - \lambda}{1 - \lambda L} w z_t dF(\lambda, w) \\
&= \hat{c} \int_0^1 \frac{1 - \lambda}{1 - \lambda L} z_t dF_{\Lambda}(\lambda) = \hat{c} \int_0^1 (1 - \lambda) \sum_{k=0}^{\infty} (\lambda L)^k z_t dF_{\Lambda}(\lambda) \\
&= \hat{c} \sum_{k=0}^{\infty} \int_0^1 (1 - \lambda)\lambda^k z_{t-k} dF_{\Lambda}(\lambda) = \hat{c} \sum_{k=0}^{\infty} \left[ \int_0^1 (1 - \lambda)\lambda^k dF_{\Lambda}(\lambda) \right] z_{t-k} =: \hat{c} \sum_{k=0}^{\infty} a_k z_{t-k},
\end{align*}
$$

(59)

where

$$a_k \sim \int_0^1 (1 - \lambda)\lambda^{k-1} dF_{\Lambda}(\lambda) = \mathbb{E}[\lambda^k] - \mathbb{E}[\lambda^{k+1}].$$

Since $\lambda \sim B(p, q)$, we have:

$$a_k \sim k^{-q-2}.$$

(60)

Therefore, $\Gamma_t^1$ faces the same asymptotic behavior of an I($d$) process, with $d = -q - 1$. Since $q > 0$, we have that $\Gamma_t^1$ can be represented as an integrated process of order $d < -1$. Hence, $\Gamma_t^1$ does not exhibit LTM.
For what regards the process $\Gamma^3_t$, fixed $h > 0$, we have

$$
\mathbb{E}\left[\Gamma^3_t \Gamma^3_{t-h}\right] = \mathbb{E}\left[\frac{1}{N} \cdot \frac{-c}{1+c} \sum_{i=1}^{N} \frac{1-\lambda_i}{1-\lambda_i L} \mu_t \cdot \frac{1}{N} \cdot \frac{-c}{1+c} \sum_{j=1}^{N} \frac{1-\lambda_j}{1-\lambda_j L} \mu_{t-h}\right]
$$

$$
= \mathbb{E}\left[\frac{c^2}{(1+c)^2} \sum_{m=0}^{\infty} \int_0^1 (1-\lambda)^m u_{t-m} dF(\lambda) \sum_{l=0}^{\infty} \int_0^1 (1-\mu)^l u_{t-h-l} dF(\mu)\right]
$$

$$
= \frac{1}{\beta(p,q)} \cdot \frac{c^2 \sigma_u^2}{(1+c)^2} \sum_{l=0}^{\infty} \int_0^1 (1-\lambda)^{1+q} \lambda^{2l+h+p-1} d\lambda
$$

$$
= \frac{1}{\beta(p,q)} \cdot \frac{\sigma_u^2}{(1+c)^2} \sum_{l=0}^{\infty} \frac{\Gamma(h+p+2l)\Gamma(q+2)}{\Gamma(h+p+q+2l+2)} \sim \frac{1}{\beta(p,q)} \cdot \frac{\sigma_u^2}{(1+c)^2} h^{-1-q}.
$$

(61)

Then, Rangarajan and Ding (2000) assures that: as $N \to +\infty$, LTM for $\Gamma^3_t$ holds, with Hurst’s exponent $H_3$ as follows:

- $q > 1$ implies $H_3 = 1/2$:
- $q \in (0, 1)$ and the following equation holds:

$$
\sum_{h=-\infty}^{+\infty} \mathbb{E}[\Gamma^3_t \Gamma^3_{t-h}] = 0,
$$

(62)

imply $H_3 = (1 - q)/2$. In this case it results $H_3 < 1/2$, and the process $\Gamma^3_t$ is mean reverting.

Since

$$
\tilde{P}_{i,t} = \sum_{j=0}^{t-1} \epsilon_{t-j} + \tilde{P}_{i,0},
$$

then $\tilde{P}_{i,t}$ is a stationary process, and the arguments carried out for $\Gamma^3_t$ can be replicated to state that LTM holds for $\Gamma^2_t$ as $N \to +\infty$. The Hurst’s exponent is $H_2$.

By Granger (1980), we have that

$$
H_B = \max\{H_2, H_3\}.
$$

(63)

The proof of Theorem 6.2 suggests the possibility of relaxing the distributional hypotheses on the $\lambda$’s. The following Corollary states immediately:

**Corollary 6.3** Fix $i = 1, \ldots, N$. Assume that:

$$
\mathbb{E}[\lambda^k_i] \sim O(c)k^{-1-q} + o(k^{-1-q}) \text{ as } k \to +\infty,
$$

(64)

with $q > 0$. Then, as $N \to +\infty$, LTM for $P_t$ holds, with Hurst’s exponent $H_B \leq 1/2$. 

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6.2.2 Second case: heterogeneous case

Heterogeneity is modeled through the employment of mixtures of absolute continuous distributions for the parameters $\lambda$’s.

More precisely, we include in the homogeneous framework a set of spot traders, whose behavior is captured by a measure of Dirac-type $\delta_x(y)$, defined as follows:

$$
\delta_x(y) = \begin{cases} 
1, & \text{for } x = y, \\
0, & \text{for } x \neq y.
\end{cases}
$$

Next result formalizes LTM detection in the heterogeneous case.

**Theorem 6.4** Consider $p, q_1, \ldots, q_k \in (0, +\infty)$ and $A_1(N), \ldots, A_k(N) \subseteq \{1, \ldots, N\}$ such that $\lambda_i$ are sampled by $B(p, q_j)$ distribution, for each $i \in A_j(N)$, $j = 1, \ldots, k$.

Moreover, consider $d_{k+1}, \ldots, d_n \in (0, 1)$ and $A_{k+1}(N), \ldots, A_n(N) \subseteq \{1, \ldots, N\}$ such that $\lambda_i \sim \delta_{d_i}$, for each $i \in A_j(N)$, $j = k + 1, \ldots, n$.

Assume that there exists $\theta_j \in (0, 1)$ such that

$$
\lim_{N \to +\infty} \frac{\text{card} A_j(N)}{N} = \theta_j, \quad \forall j = 1, \ldots, n.
$$

Furthermore, assume that $\lambda_i$ are sampled by independent random variables.

Then, as $N \to +\infty$, $P_t$ has LTM, with Hurst’s exponent $H_D \leq 1/2$.

**Proof** The process $P_t$ can be disaggregated as follows:

$$
P_t = \sum_{j=1}^{k} \Phi^j_t + \sum_{j=k+1}^{n} \Psi^j_t, \quad (65)
$$

where

$$
\begin{align*}
\Phi^j_t &= \frac{1}{N} \sum_{i \in A_j(N)} \left[ \frac{1}{1 + c} \cdot \frac{1 - \lambda_i}{1 - \lambda_i L} \left\{ \gamma_{t,i} - cv\bar{P}_{t,i} - u_t \right\} \right], \quad j = 1, \ldots, k; \\
\Psi^j_t &= \frac{1}{N} \sum_{i \in A_j(N)} \left[ \frac{1}{1 + c} \cdot \frac{1 - \lambda_i}{1 - \lambda_i L} \left\{ \gamma_{t,i} - cv\bar{P}_{t,i} - u_t \right\} \right], \quad j = k + 1, \ldots, n.
\end{align*}
$$

In order to proceed, we need to study the behavior of the $k$-th moments of the Dirac distribution, with $k \in \mathbb{N}$.

A direct computation gives:

$$
E\left[ (\delta_x)^k \right] = \int_{-\infty}^{+\infty} \xi^k \delta_x(\xi) d\xi = x^k.
$$

Therefore, the terms related to the processes $\Psi$’s do not contribute to LTM of the process $P_t$.

By Theorem 6.2, we have that the process $\Phi^j_t$ has an Hurst’s exponent $H_j \leq 1/2$. Since the $\lambda$’s are independent and by Granger (1980), we obtain that

$$
H_D = \max\{H_1, \ldots, H_k\} \leq 1/2, \quad (67)
$$

and this completes the proof. \hfill \Box
7 Conclusions

We have shown how to use the aggregation technique proposed in Granger (1980) for the theoretical assessment of LTM in heterogeneous agent-based models. In the reviewed models, agents are supposed to drive actively the formation of specific financial variables. Heterogeneity mirrors in the way to make forecasts (chartist and fundamentalist) and to technically analyze the market (distribution of the parameters). The discussed models extend some results already known in the literature about the arise of LTM of aggregations of independent micro units. The most relevant results, along with the differences between the models, are discussed. The remarkable role played by the diversity between units is specifically acknowledged, mainly in the context of decision theory for what concerns multiple selections in different families of elements. In some other frameworks, it is worth noting how diversity rules the connection among heterogeneous agents to share information and collaborate or compete. In this respect, the diversity might be an indicator of the performance of the strategies in dynamic optimization models.

Appendix: Mathematical definitions

This sections summarizes the main definition and theorems that we used for the proofs in the models here reviewed.

Long-term memory

The memory is defined “long-term” or, simply, “long” if the decay of the correlation is slow. In details:

**Definition 8.1** A stationary process \((X_t)\) is called stationary process with long memory if its autocorrelation function \(\rho(k)\) has asymptotically the following hyperbolic rate of decay:

\[
\rho(k) \sim L(k)k^{2d-1}, \quad \text{as } k \to \infty,
\]

where \(d \in (-1/2, 1/2)\) and \(L(k)\) is a slowly varying function, i.e. \(L(\lambda k)/L(k) \to 1\) as \(k \to \infty, \forall \lambda > 0\).

The parameter \(d\) summarizes the degree of long range dependence of the series. If \(-0.5 < d < 0\) the series is mean reverting; if \(d = 0\) there is no correlation between the data and \(X_t\) is a short memory process. If \(0 < d < 0.5\), the correlation function decays slowly with the lag \(k\) and the time series has a long range correlation, or long memory property (Cerqueti and Rotundo 2003).

The term slow, referred to the decay of the autocorrelation function, must be intended as compared to the autocorrelation function of a short memory process, that decays to zero at an exponential rate.

The definition is extended to the time series \(x_t\) generated by \(X_t\).

The parameter \(d\) is related to the Hurst’s exponent \(H\), and this provides methods for its estimate.

---

\(^6\) For the sake of simplicity, we will denote hereafter the entire time series (or process) as \(x_t\) (or \(X_t\)) instead of \(\{x_t\}\) (or \(\{X_t\}\)).
Definition 8.2  Given a time series $x_t$ Hurst’s exponent $H$ describes the degree of dependence among the increments of the analyzed process. It can be defined as follows:

$$E(x_{t+T} - x_t)^2 \sim cT^{2H}$$

Several methods are available for its estimate (Kirman and Teyssiére 2002; Lo 1991) and $H = d + \frac{1}{2}$.

Spectral analysis can provide an estimate for $H$. The spectral density of a covariance stationary time series $x_t$ is given by

$$f(\lambda) = \gamma(0) + 2 \sum_{h=1}^{\infty} \gamma(h)\cos(\lambda h),$$

where $\gamma(h) = \text{Cov}(x_t, x_{t-h})$ is the autocovariance function.

The spectrum of stationary processes with long range memory can be approximated in the neighborhood of the zero frequency as

$$S(f) \propto f^{-\alpha}, \quad 1 < \alpha < 3, \quad f \to 0^+$$

The following relation holds: $H = \frac{\alpha - 1}{2}$ (Menna et al. 2002; Osborne and Provenzale 1989).

Fractionally integrated processes

Definition 8.3  [Integrated process of order $d$] Suppose that $x_t$ is a zero-mean time series generated from a zero-mean, variance $\sigma^2$ white noise series $\epsilon_t$ by use of the linear filter $a(L)$, where $L$ is the backward operator, so that $x_t = a(L)\epsilon_t$, $L^k \epsilon_t = \epsilon_{t-k}$, and that $a(L)$ may be written $a(L) = (1 - L)^{-d}a'(L)$, where $a'(z)$ has no poles or roots at $z = 0$. Then $x_t$ will be said to be integrated of order $d$ and denoted $x_t \sim I(d)$.

To avoid a cumbersome notation, we will refer briefly to $x_t \sim I(d)$ as for an integrated process $x_t$ of order $d$.

Note that $d$ need not be an integer (Granger 1980); $d$ is also called the fractional degree of integration of the process.

Granger (1980) gives the following remark.

Remark 8.4  If $x_t$ is an integrated process of order $d$ and $x_t = \sum_{j=0}^{\infty} b_j \epsilon_{t-j}$ then $b_j = \frac{\Gamma(j+d)}{\Gamma(d)\Gamma(j+1)}$, $j \geq 1$.

Therefore, this leads to the equivalent definition:

Definition 8.5  (Integrated process of order $d$) A time series $x_t$ is called fractionally integrated with differencing parameter $d$ ($x_t \sim I(d)$), if

$$x_t = \sum_{j=0}^{\infty} b_j \epsilon_{t-j}, \quad \text{with} \quad b_j = \frac{\Gamma(j+d)}{\Gamma(j+1)\Gamma(d)}$$

and $\epsilon_t \sim i.i.d.(0, \sigma^2)$

The following result has been proved in Granger and Joyeux (1980).

Theorem 8.6  If $x_t = (1 - L)^{-d} \epsilon_t$, then $\text{cov}(x_t, x_{t-k}) = \frac{\sigma^2}{2\pi} \sin(\pi d) \frac{\Gamma(k+d)}{\Gamma(k+1-d)} \Gamma(1 - 2d)$, provided $d < \frac{1}{2}$. The variance of $x_t$ increases as $d$ increases and is infinite for $d \geq \frac{1}{2}$. 

\[ Springer \]
By Theorem 8.6 and Granger (1980) we have:

**Corollary 8.7** \( \rho_k = corr(x_t, x_{t-k}) = \frac{\Gamma(1-d)}{\Gamma(d) \Gamma(k+1-d)}, \) for \( d < \frac{1}{2} \) and \( d \neq 0 \).

**Remark 8.8** Of course \( \rho_k = 0 \) if \( k > 0 \) and \( d = 0 \), which is the white noise case.

**Remark 8.9** Using the fact, derived from Sterling’s theorem, that \( \frac{\Gamma(j+a)}{\Gamma(j+b)} \) is well approximated by \( j^{a-b} \), it follows that \( \rho_j \approx A_1 j^{2d-1}, b_j \approx A_2 j^{d-1} \), where \( A_1 \) and \( A_2 \) are appropriate constraints. Hence, if \( d > 0 \), then the series \( x_t \) possesses the long-memory property.

The algebra of integrated series is quite simple. By continuing to follow Granger (1980):

**Proposition 8.10** If \( x_t \) is an integrated process of order \( d_x \), and an integrating filter is applied to it, to form \( y_t = (1-L)^{-d} x_t \), then \( y_t \) is an integrated process of order \( d_y = d_x + d' \).

Beta distribution and its properties

**Definition 8.11** If \( z \) is an ordinary beta-distributed random variable with support \([0, 1]\), the probability density function of \( z \) is

\[
p(z) = \frac{1}{\beta(p, q)} z^{p-1} (1-z)^{q-1}, \quad 0 \leq z \leq 1,
\]

where \( p, q > 0 \) and

\[
\beta(p, q) = \int_0^1 z^{p-1} (1-z)^{q-1} dz.
\]

We refer to this distribution as \( B(p, q) \).

**Proposition 8.12** If \( x \sim B(p, q) \), then the random variable \( y = 1 - x \) is a beta random variable with law \( B(q, p) \).

Let us now consider \( C > 0, h \in \mathbb{R} \) and a new random variable \( x \) which is related to \( z \) through the power transformation

\[
z = \left( \frac{x}{C} \right)^h \quad \text{or} \quad x = C z^\frac{1}{h}
\]

By the transformation in (70) we can define a generalization of the beta distribution.

**Definition 8.13** The random variable \( x \) defined by (70) has a beta generalized distribution \( b(p, q, C, h) \) if its probability density function is defined by

\[
f(x) = \frac{|h|}{\beta(p, q) C} \left( \frac{x}{C} \right)^{ph-1} \left[ 1 - \left( \frac{x}{C} \right)^h \right]^{q-1}
\]

where \( 0 \leq x \leq C \).

The moment \( M_n \) of order \( n \) for \( x \) is given by

\[
M_n = C^n \frac{\beta\left(p + \frac{n}{h}, q\right)}{\beta(p, q)} = C^n \frac{\Gamma(p + q) \Gamma\left(p + \frac{n}{h}\right)}{\Gamma\left(p + q + \frac{n}{h}\right) \Gamma\left(p\right)}.
\]
Remark 8.14 A standard beta random variable is also a generalized beta random variable with parameters $h = C = 1$. Thus the properties of the beta standard random variable can be extended to the beta generalized random variable.

The beta generalized distribution is close with respect to the class of power transformations.

Proposition 8.15 Let $x \sim b(p, q, C, h)$ and consider

$$y = rx^s,$$

where $r, s \in \mathbb{R}$. Then $y \sim b(p, q, rC^s, \frac{h}{s})$.

Remark 8.16 Given $x \sim b(p, q, C, h)$, Proposition 8.15 implies that,

$$\lambda x \sim b(p, q, \lambda C, h)$$

and

$$x^n \sim b\left(p, q, C^n, \frac{h}{n}\right).$$

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