Bounds on Codes with Locality and Availability

S. B. Balaji and P. Vijay Kumar, Fellow, IEEE
Department of Electrical Communication Engineering, Indian Institute of Science, Bangalore.
Email: balaji.profess@gmail.com, pvk1729@gmail.com

Abstract

In this paper we investigate bounds on rate and minimum distance of codes with $t$ availability. We present bounds on minimum distance of code with $t$ availability that are tighter than existing bounds. For bounds on rate of a code with $t$ availability, we restrict ourself to a sub class of codes with $t$ availability and derive a tighter rate bound. For $t = 3, 4$, we also present a high-rate construction.

Index Terms

Distributed storage, codes with locality, availability, multiple erasures.

I. INTRODUCTION

Let $C$ denote a linear $[n, k]$ code. The code is said to have locality $r$ if each of the $n$ code symbols of $C$ can be recovered by accessing at most $r$ other code symbols. Equivalently, there exist $n$ codewords $h_1 \cdots h_n$ in the dual code $C^\perp$ such that $c_i \in \text{supp}(h_i)$ and $|\text{supp}(h_i)| \leq r + 1$ for $1 \leq i \leq n$ where $c_i$ denote the $i$th code symbol of $C$ and supp$(h_i)$ denote the support of the codeword $h_i$.

a) Codes with Availability: For every code symbol $c_i$ in $C$, there exists $t$ codewords $h_1^t, \ldots, h_i^t$ in the dual of the code, each of Hamming weight $\leq r + 1$, such that $\text{supp}(h_j) \cap \text{supp}(h_j') = \{i\}$, $\forall 1 \leq g \neq j \leq t$. We denote the matrix with these $h_j^t, 1 \leq j \leq n, 1 \leq i \leq t$ as rows as $H_{des}(C)$.

The parameter $r$ is called the locality parameter and we will formally refer to this class of codes as $(n, k, r, t)$ codes. Again, when the parameters $n, k, r, t$ are clear from the context, we will simply term the code as a code with $t$ availability.

b) Codes with Strict Availability: Codes in this class can be defined as the nullspace of an $(n \times n)$ parity-check matrix $H$, where each row has weight $(r + 1)$ and each column has weight $t$, with $nt = m(r + 1)$. Additionally, if the support sets of the rows in $H$ having a non-zero entry in the $i$th column are given respectively by $S_j^{(i)}, j = 1, 2, \cdots, t$, then we must have that

$$ S_j^{(i)} \cap S_l^{(i)} = \{i\}, 1 \leq j \neq l \leq t. $$

Thus each code symbol $c_i$ is protected by a collection of $t$ orthogonal parity checks (opc) each of weight $(r + 1)$. The parameter $r$ is called the locality parameter and we will formally refer to this class of codes as $(n, k, r, t)$ codes. Again, when the parameters $n, k, r, t$ are clear from the context, we will simply term the code as a code with strict $t$ availability.

A. Background

In [1] P. Gopalan et al. introduced the concept of codes with locality (see also [2], [3]), where an erased code symbol is recovered by accessing a small subset of other code symbols. The size of this subset denoted by $r$ is typically much smaller than the dimension of the code, making the repair process more efficient compared to MDS codes. The authors of [1] considered codes that can locally recover from single erasures (see also [4], [5], [6]).

Approaches for local recovery from multiple erasures can be found in [7], [8], [9], [10], [11], [12], [13], [6], [14], [15]. In this paper we concentrate only codes with $t$ availability and codes with strict $t$ availability.

B. Our Contributions

1) We derive a bound on rate of an $(n, k, r, 3)_{a_2}$ code using a greedy algorithm and we also derive a bound on rate of an $(n, k, r, t)_{a_2}$ code for general $t$ using a simple transpose trick. The resulting bounds are tighter than the bound given in [10].

2) We derive field size dependent and field size independent bounds on minimum distance of an $(n, k, r, t)_{a_1}$ code. The resulting bounds are tighter than the bounds given in [10] and [20].

3) We present a high rate construction with $t = 3, r = 2^g - 1, n = (r + 1)^2$ over $F_2$ and $t = 4, r = 3^g - 1, n = (r + 1)^2$ over $F_3$. The construction can be extended to larger values of $t$ also.

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II. BOUNDS ON THE RATE OF AN \((n, k, r, t)_{a_2}\) CODE

A. A Rate bound for an \((n, k, r, 3)_{a_2}\) code

We present a greedy algorithm for calculating a bound on the rate of an \((n, k, r, 3)_{a_2}\) code and analyse the algorithm to get the bound.

Let \(C\) be an \((n, k, r, 3)_{a_2}\) code. W.l.o.g. the tanner graph of \(C\) is assumed to be connected. If not, we can puncture the code and take a subset of code symbols and form a code \(C'\) with rate \(\geq\) rate of \(C\) such that \(C'\) is also an \((n, k, r, 3)_{a_2}\) code with a connected tanner graph. By definition, \(C\) is the null space of an \(m \times n\) matrix \(H\) with \(m(r + 1) = 3n\) which contains all the orthogonal parities protecting all the symbols.

**Theorem 1.** The code \(C\) has rate upper bounded by the following expression:

\[
\frac{k}{n} \leq 1 - \frac{3(1 + L_1 + L_2)}{(r + 1)(3 + L_1 + 2L_2)}
\]

where:

\[
m = \frac{3n}{r + 1}
\]

\[
L_1' = \left\lfloor \frac{(2r - 1)m}{3(r + 2)} \right\rfloor - \frac{1}{r + 2} - 1
\]

\[
L_2 = \left\lfloor \frac{m - 3 - L_1'}{2} \right\rfloor
\]

\[
L_1 = m - 3 - 2L_2.
\]

**Proof:** We present a greedy algorithm and analyse it to get the bound.

**GreedyAlgorithm:**

1) Let \(S = \emptyset\), \(P = \emptyset\).
2) Pick an arbitrary number \(\sigma_1\) from \([n]\) and add it to the set \(S\).
3) Pick the 3 codewords in the rows of the matrix \(H\) with support containing \(\sigma_1\) and add it the set \(P\).
4) At Step \(i, i \geq 2\):
   - Choose a number \(\sigma_i\) from \([n] - S\) which is in the support of maximum number of distinct codewords in \(P\) but not in the support of 3 distinct codewords in \(P\) and add it to the set \(S\).
   - Take the codewords in the rows of \(H\) containing \(\sigma_i\) in its support which are not in \(P\) and add it to \(P\).
5) Continue the step 3 until \(P\) contains all the rows of \(H\).
6) It is clear that, \(k \leq n - |S|\) at the end of the algorithm.

**Analysis of the greedy algorithm:**

Let \(g_i\) be the number of new codewords added to \(P\) at step \(i\). Since the tanner graph of \(C\) is connected, \(g_i \in \{1, 2\} \) for \(2 \leq i \leq |S|\). We define \(g_{|S|+1} = 0\). Let the codewords in \(P\) after the step \(i\) and write it as a partial parity check matrix. Let \(s_1^i, s_2^i, s_3^i\) be the number of weight 1, 2, 3 columns in the partial parity check matrix respectively. Lets introduce four collection of variables \(I_i, \phi_i, J_i, \psi_i\) for each \(1 \leq i \leq |S| - 1\).

We have the following recursive update for \(i \geq 2\):

\[
\text{if } g_{i+1} = 2, g_{i+2} = 2 \text{ then}
\]

\[
s_1^{i+1} = s_1^i + 2r - 1
\]

\[
s_2^{i+1} = s_2^i + 0
\]

\[
s_3^{i+1} = s_3^i + 1
\]

\[
I_i = J_i = \psi_i = \phi_i = 0.
\]
if \( g_{i+1} = 1, g_{i+2} = 2 \) then
\[
\begin{align*}
    s_1^{i+1} &= s_1^i - \phi_i + r + 1 \\
    s_2^{i+1} &= s_2^i - \phi_i \\
    s_3^{i+1} &= s_3^i + \phi_i
\end{align*}
\]
for some \( 0 \leq \phi_i \leq r + 1 \)
\[
I_i = J_i = \psi_i = 0.
\]

if \( g_{i+1} = 2, g_{i+2} = 1 \) then
\[
\begin{align*}
    s_1^{i+1} &= s_1^i + 2r - 1 - 2I_i \\
    s_2^{i+1} &= s_2^i + I_i \\
    s_3^{i+1} &= s_3^i + 1
\end{align*}
\]
for some \( 0 < I_i \leq 2r \)
\[
\phi_i = J_i = \psi_i = 0.
\]

if \( g_{i+1} = 1, g_{i+2} = 1 \) then
\[
\begin{align*}
    s_1^{i+1} &= s_1^i + r + 1 - J_i - 2\psi_i \\
    s_2^{i+1} &= s_2^i - J_i + \psi_i \\
    s_3^{i+1} &= s_3^i + J_i
\end{align*}
\]
for some \( s_2^i \geq J_i \geq r + 1 \geq J_i + \psi_i \geq 0 \)
\[
I_i = \phi_i = 0.
\]

if \( g_{i+1} = 1, g_{i+2} = 0 \) then
\[
\begin{align*}
    s_1^{i+1} &= s_1^i + 0 \\
    s_2^{i+1} &= s_2^i - (r + 1) \\
    s_3^{i+1} &= s_3^i + r + 1
\end{align*}
\]
\[
I_i = J_i = \psi_i = \phi_i = 0.
\]

Writing the first two steps of the update explicitly:
\[
\begin{align*}
    s_1^1 &= 3r \\
    s_2^1 &= 0 \\
    s_3^1 &= 1 \\
    s_1^2 &= 3r + 2r + 2 - 3 - 2\gamma_1 \\
    s_2^2 &= \gamma_1 \\
    s_3^2 &= 2
\end{align*}
\]
for some \( 0 \leq \gamma_1 \leq 4 \)
\[
I_1 = J_1 = \psi_1 = \phi_1 = 0.
\]

Let \( l_{kj} \) be the number of times \( g_i = k, g_{i+1} = j \) for \( i \geq 3 \).

Now using the Global constriants (i.e., after the final step all the columns in the partial parity check matrix must have weight 3 (Hence partial parity check matrix after the final step is the full matrix H with rows permuted)): 
\[ s_3 = 2l_{22} + l_{23} + \sum J_i + \sum \phi_i + r + 1 = n \]  \tag{2}

\[ s_2 = \gamma_1 - \sum \phi_i + \sum I_i - \sum J_i + \sum \psi_i - (r + 1) = 0 \]  \tag{3}

\[ s_1 = 5r - 1 - 2\gamma_1 + l_{22}(2r - 1) + l_{21}(2r - 1) - 2\sum J_i + l_{11}(r + 1) - \sum \psi_i + l_{12}(r + 1) - \sum \phi_i = 0 \]  \tag{4}

\[ l_{11} \geq \sum_{i : g_{i+1} = 1, g_{i+2} = 1} \frac{I_i}{2} \]  \tag{5}

\[ l_{21} - 1 \leq l_{12} \leq l_{21} \]  \tag{6}

\[ m = \frac{3n}{r + 1} = 5 + g_3 + 2(l_{22} + l_{12}) + l_{21} + l_{11} \]  \tag{7}

Inequality (5) is true because whenever \( g_{i+1} = 2, g_{i+2} = 1 \), it is followed by at least \( z = \frac{L - 2}{2} \), (1,1) transitions i.e., \( g_{i+2+j} = 1, g_{i+2+j+1} = 1, \forall 0 \leq j \leq z - 1 \). This is because at least \( z + 1 \) new 2 weight columns appear in the partial parity check matrix obtained from codewords in \( P \) after the step \( i + 1 \), which is a subset of support of one of the new codewords added in step \( i + 1 \) to the set \( P \) and hence these \( z + 1 \) new 2 weight columns can become 3 weight columns only one by one at each of the following steps (since there will be 2 weight codewords in each of these steps, we will be adding only one codeword to \( P \) in each of these steps). Let,

\[ \sum J_i = \sum_{i : g_{i+1} = 1, g_{i+2} = 1} J_i = l_{11} \sum_{i : g_{i+1} = 1, g_{i+2} = 1} \frac{J_i}{l_{11}} = l_{11}J \]

\[ \sum \psi_i = \sum_{i : g_{i+1} = 1, g_{i+2} = 1} \psi_i = l_{11} \sum_{i : g_{i+1} = 1, g_{i+2} = 1} \frac{\psi_i}{l_{11}} = l_{11}\psi \]

\[ \sum I_i = \sum_{i : g_{i+1} = 1, g_{i+2} = 1} I_i = l_{21} \sum_{i : g_{i+1} = 1, g_{i+2} = 1} \frac{I_i}{l_{21}} = l_{21}I \]

\[ \sum \phi_i = \sum_{i : g_{i+1} = 1, g_{i+2} = 1} \phi_i = l_{12} \sum_{i : g_{i+1} = 1, g_{i+2} = 1} \frac{\phi_i}{l_{12}} = l_{12}\phi \]

Now \( r + 1 \geq J + \psi \geq 0 \)

\[ 0 \leq \phi \leq r + 1 \]

\[ 0 < I \leq 2r. \]

Now (5) becomes:

\[ 2l_{11} + 2l_{21} \geq l_{21}I \]  \tag{8}

1) Manipulating the equations and inequalities given above:

By equation (3) : \[ l_{12}\phi + l_{11}J + (r + 1) - \gamma_1 = l_{21}I + l_{11}\psi \]

Using \( r + 1 \geq J + \psi \)

\[ l_{12}\phi + l_{11}(r + 1 - \psi) + (r + 1) - \gamma_1 \geq l_{21}I + l_{11}\psi \]

\[ l_{12}\phi - l_{21}I + l_{11}(r + 1) + (r + 1) - \gamma_1 \geq l_{11}\psi \]  \tag{9}
Using (2) and (3) :
\[
0 = \gamma_1 + l_{21}l + l_{11}\psi - (r + 1) + 2 + l_{22} + l_{21} + r + 1 - n
\]
\[
l_{22} = n - \gamma_1 - 2 - l_{21} - l_{21}l - l_{11}\psi
\]
Substituting (9) :
\[
l_{22} \geq n - \frac{\gamma_1}{2} - 2 - l_{21} - \frac{l_{21}l}{2} - \frac{l_{12}\phi}{2} - \frac{l_{11}(r + 1)}{2} - \frac{(r + 1)}{2}
\]
Substituting \( \frac{3n}{r + 1} \):
\[
m = \frac{3n}{r + 1} = 5 + g_3 + 2(l_{22} + l_{12}) + l_{21} + l_{11}
\]
\[
m \geq 5 + g_3 + 2n - \gamma_1 - (r + 1) - 4 - 2l_{21} - 2l_{11} - 2l_{21} - l_{12}\phi
\]
\[-l_{11}(r + 1) + 2l_{12} + l_{21} + l_{11}
\]
\[0 \geq 5 + g_3 + 2n - m - \gamma_1 - (r + 1) - 4 - 3l_{21}
\]
\[-l_{12}(\phi - 2) - l_{11}(r + 2)
\]
Now first substituting \( \phi \leq r + 1 \) and then substituting \( l_{12} \leq l_{21} \) in the above inequality :
\[0 \geq 5 + g_3 + 2n - m - \gamma_1 - (r + 1) - 4 - l_{21}(r + 2) - l_{11}(r + 2)
\]
\[l_{21} + l_{11} \geq \frac{5 + g_3 + 2n - m - \gamma_1 - (r + 1) - 4}{r + 2}
\]
\[l_{21} + l_{11} \geq \frac{(2r - 1)m}{3(r + 2)} + \frac{5 + g_3 - \gamma_1 - (r + 1) - 4}{r + 2}
\]
Substituting \( g_3 \geq 1, \gamma_1 \leq 4 \):
\[l_{21} + l_{11} \geq \frac{(2r - 1)m}{3(r + 2)} - \frac{(r + 3)}{r + 2}
\]
which gives
\[l_{11} + l_{21} \geq \frac{(2r - 1)m}{3(r + 2)} - \frac{1}{r + 2} - 1
\](10)

2) Let \( L_1, L_2 \) be the number of times \( g_i = 1, 2 \) respectively. Using \( |S| = L_1 + L_2 + 1 \) (at the end of the algorithm) and \( m = L_1 + 2L_2 + 3 \) we get:
\[\frac{k}{n} \leq 1 - \frac{|S|}{n} = 1 - \frac{1 + L_1 + L_2}{n}
\]
\[\frac{k}{n} \leq 1 - \frac{m(1 + L_1 + L_2)}{nm} = 1 - \frac{3(1 + L_1 + L_2)}{(r + 1)(3 + L_1 + 2L_2)}
\]
(11)

3) Now using (10) we get :
\[L_1 \geq l_{11} + l_{21} \geq \left\lceil \frac{(2r - 1)m}{3(r + 2)} - \frac{1}{r + 2} - 1 \right\rceil
\]

Using the above inequality on \( L_1 \) and \( m = L_1 + 2L_2 + 3 \), we have:
\[L_1' = \left\lceil \frac{(2r - 1)m}{3(r + 2)} - \frac{1}{r + 2} - 1 \right\rceil
\]
\[L_2 \leq \left\lceil \frac{m - 3 - L_1'}{2} \right\rceil
\]
(12)
\[L_1 \geq m - 3 - 2\left\lceil \frac{m - 3 - L_1'}{2} \right\rceil.
\]
(13)
Substituting the bounds (12), (13) in (11) we get the bound given in the theorem.

Fig. 1 shows the plot of the new rate bound (1) along with the rate bounds given in [10] \( \left( \frac{k}{n} \leq \frac{1}{\prod_{j=1}^{t} (1 + \frac{1}{j^t})} \right) \) and [17] \( \left( \frac{k}{n} \leq \frac{r}{r+1} \right) \) for \( t = 3, n = t^{+3} \). It can be seen that the new bound given in (1) is tighter. Even if we plot for different \( n \), the new bound remains tighter barring some small values of \( r \) as low as \( r \leq 3 \). The fact that the new rate bound depends on \( n \) is by itself interesting and such bounds might throw insight on optimal value of \( n \) for \((n, k, r, 3)_{a_2}\) codes with connected tanner graph.

**B. A simple rate bound for an \((n, k, r, t)_{a_2}\) code:**

Here we derive a bound on rate of an \((n, k, r, t)_{a_2}\) code using a very simple transpose trick.

**Theorem 2.**

\[
R(r, t) = \sup \{ (k, n): (n, k, r, t)_{a_2} \text{ code exists} \} \frac{k}{n}
\]

Then
\[
R(r, t) = 1 - \frac{t}{r+1} + \frac{t}{r+1} R(t-1, r+1) \\
R(r, t) \leq 1 - \frac{t}{r+1} + \frac{t}{r+1} \prod_{j=1}^{t} \left\{ 1 + \frac{1}{j(t-1)} \right\}
\]

**Proof:** Let \( R^n(r, t) \) be the maximum achievable rate of a code with strict \( t \) availability for fixed \( n, r, t \). Let \( C \) be an \((n, k, r, t)_{a_2}\) code with rate \( R^n(r, t) \). By definition \( C \) is the null space of an \( m \times n \) matrix \( H \) with column weight \( t \) and row weight \( r+1 \). This matrix \( H \) contains all \( t \) orthogonal parity checks protecting any given symbol. Now the null space of \( H^t \) (transpose of \( H \)) corresponds to an \((n, k, t-1, r+1)_{a_2}\) code. Hence we have the following inequality:

\[
\text{rank}(H) = n(1 - R^n(r, t)) \\
\text{rank}(H) = \text{rank}(H^t) \geq m(1 - R^n(t-1, r+1))
\]
Hence we have
\[
m(1 - R^m(t - 1, r + 1)) \leq n(1 - R^n(r, t))
\]
Using \(m(r + 1) = nt\):
\[
\frac{t}{r + 1}(1 - R^m(t - 1, r + 1)) \leq (1 - R^n(r, t))
\]
\[
R^n(r, t) \leq 1 - \frac{t}{r + 1} + \frac{t}{r + 1} R^m(t - 1, r + 1)
\]
\[
R^n(r, t) \leq 1 - \frac{t}{r + 1} + \frac{t}{r + 1} (\sup_{m \geq 0} R^m(t - 1, r + 1))
\]
\[
\sup_{n \geq 0} R^n(r, t) \leq 1 - \frac{t}{r + 1} + \frac{t}{r + 1} (\sup_{m \geq 0} R^m(t - 1, r + 1))
\]
\[
R(r, t) \leq 1 - \frac{t}{r + 1} + \frac{t}{r + 1} (R(t - 1, r + 1))
\]

Now swapping the roles of \(H\) and \(H^t\) in the above derivation i.e., we take an \((m, k', r, t)_{a_2}\) code \(C\) with rate \(R^m(t - 1, r + 1)\) and repeat the above argument in exactly the same way. By doing so we get:
\[
R(r, t) \geq 1 - \frac{t}{r + 1} + \frac{t}{r + 1} R(t - 1, r + 1)
\]
Hence we get:
\[
R(r, t) = 1 - \frac{t}{r + 1} + \frac{t}{r + 1} R(t - 1, r + 1)
\] (16)

Now substituting the rate bound \(R(t - 1, r + 1) \leq \frac{1}{\prod_{j=1}^{r+1}(1 + \frac{r}{j})}\) given in [10] into (16), we get:
\[
R(r, t) \leq 1 - \frac{t}{r + 1} + \frac{t}{r + 1} \frac{1}{\prod_{j=1}^{r+1}(1 + \frac{1}{j(r-1)})}
\] (17)

**Tightness of the bound:**
The bound on \(R(r, t)\) given in (15) becomes tighter than the bound \(R(r, t) \leq \frac{1}{\prod_{j=1}^{r+1}(1 + \frac{r}{j})}\) given in [10] as \(r\) increases for a fixed \(t\). As an example lets calculate \(R(r, 2)\). From (15):
\[
R(r, 2) \leq 1 - \frac{2}{r + 1} + \frac{2}{r + 1} \frac{1}{\prod_{j=1}^{r+1}(1 + \frac{1}{j})}
\]
\[
R(r, 2) \leq 1 - \frac{2}{r + 1} + \frac{2}{r + 1} \frac{1}{r + 2}
\]
\[
R(r, 2) \leq \frac{r}{r + 2}
\]

The above bound on \(R(r, 2)\) obtained from (14) and (15) is a tight bound as it is known that rate \(\frac{r^2}{r+2}\) is achievable for \(t = 2\) and any \(r\) using a complete graph code (19) and hence clearly tighter than \(R(r, 2) \leq \frac{r^2}{(r+1)(r+2)}\) given in [10]. We show a plot of the bound given in (15) for \(t = 4\) in Fig. 2. The plot shows that the bound given by (15) is tighter than the bound given in [10] for \(t = 4, r > 2\).

Even though our bound becomes tighter as \(r\) increases for a fixed \(t\) than the bound in [10], the bound given in [10] is for \((n, k, r, t)_{a_1}\) codes but the bound in (14) and (15) is applicable only for \((n, k, r, t)_{a_2}\) codes. But we also would like to draw attention to the fact that most of the high rate constructions known in literature for \((n, k, r, t)_{a_1}\) codes are also \((n, k, r, t)_{a_2}\) codes. In fact the most general high rate construction for \((n, k, r, t)_{a_1}\) is given in [18] and it is also an \((n, k, r, t)_{a_2}\) code. Hence there is a very good reason to think that when it comes to rate-optimality \((n, k, r, t)_{a_2}\) codes will be good enough.

### III. Bounds on minimum distance of codes with \(t\) availability

In this section we first present a field size dependent bound on minimum distance of \((n, k, r, t)_{a_1}\) codes. Later as corollaries we present field size independent bounds on minimum distance of \((n, k, r, t)_{a_1}\) codes. Two field size independent bounds are available in literature:

In [10] the following bound on minimum distance of an \((n, k, r, t)_{a_1}\) code was presented:
Fig. 2: Plotting locality vs rate for \( t = 4 \). Here we compare the bound (15) with the bound given in [10] and the achievable rate \( \frac{r}{r+4} \) given by the construction in [18].

Given a \((n,k,r,t)_{a_1}\) code over the field \( F_q \) with minimum distance \( d_{\text{min}}(n,k,r,t) \), its co-ordinates are calculated using the Generalized Hamming Weights (GHW) of the dual of the \((n,k,r,t)_{a_1}\) code.

Let \( C \) be an \((n,k,r,t)_{a_1}\) code over the field \( F_q \) with minimum distance \( d_{\text{min}}^q(n,k,r,t) \). Let \( \{d_i^\perp : 1 \leq i \leq n-k\} \) be the GHWs \( C^\perp \). Let \( e_i \) for \( 1 \leq i \leq n-k \) be such that \( d_i^\perp \leq e_i \). Let \( S' = \{s_1,...,s_{d_i^\perp}\} \) be the co-ordinates corresponding to the support of \( i \) dimensional subpace whose support has cardinality exactly \( d_i^\perp \). Add \( e_i - d_i^\perp \) arbitrary extra co-ordinates to \( S' \) and let the resulting set be \( S \). Now shorten the code \( C \) in the co-ordinates given by \( S \). The resulting code has length \( n-e_i \) and dimension \( \geq n-e_i-(n-k-i) = k+i-e_i \) and minimum distance \( \geq d_{\text{min}}^q(n,k,r,t) \) (if \( k+i-e_i > 0 \)) and locality \( r \) and availability \( t \). Hence:

\[
d_{\text{min}}^q(n,k,r,t) \leq \min\{e_i : i \leq k\} \cdot d_{\text{min}}^q(n-e_i,k+i-e_i,r,t)
\]

Theorem 3. Let \( C \) be an \((n,k,r,t)_{a_1}\) code over the field \( F_q \) with minimum distance \( d_{\text{min}}^q(n,k,r,t) \). Then

\[
d_{\text{min}}^q(n,k,r,t) \leq \min\{e_i : i \leq k\} \cdot d_{\text{min}}^q(n-e_i,k+i-e_i,r,t)
\] (18)

Proof: The proof follows by the argument presented above.

Example 1. It can be easily seen that \( d_i^\perp \leq ir + 1 \) for \( r \geq 2, t \geq 2 \). Hence using this upper bound, the bound in [18] can be
written as:

\[ d_{\text{min}}^a(n, k, r, t) \leq \min_{\{i; i(r-1)+1 < k\}} d_{\text{min}}^b(n - ir - 1, k - (i(r - 1) + 1), r, t) \]

which is tighter than the bound given in [19]. The bound given in [19] is for information symbol availability. But here we consider all symbol availability. Hence the tightening may be natural. But we would like to draw the attention to the fact that our bound can be further tightened by using more accurate upper bound \( e_i \) on GHWs of dual.

By using the expression for \( e_i \) given in the following example, we will get even more tighter bound on minimum distance of code with \( t \) availability over \( F_q \).

**Example 2.** We can calculate \( e_i \) using the recursion given in [19] for calculating the upper bounds on GHWs with trivial or slight modification and using the rate bound given in [10] for \( t > 3 \) and rate bound given in [17] for \( t = 3 \) and rate bound given in [19] for \( t = 2 \):

\[
R'(r, t) = \begin{cases} 
\frac{r}{t+1} & \text{if } t = 2 \\
\frac{r^2}{(r+1)^2} & \text{if } t = 3 \\
\frac{1}{\prod_{j=1}^{r} (1 + \frac{1}{r})} & \text{if } t > 3 
\end{cases}
\]  

(19)

\[ b = \lceil n (1 - R'(r, t)) \rceil \]
\[ e_b = n \]
\[ e_{i-1} = \min(e_i, e_i - \left\lfloor \frac{2e_i}{i} \right\rfloor + r + 1) \]

**Corollary 4.** The field size dependency on the bound (18) can be removed and written as (using the same reasoning as before):

\[
d_{\text{min}}(n, k, r, t) \leq \min_{\{i; e_i < k\}} d_{\text{min}}(n - e_i, k + i - e_i, r, t) \\
\leq \min_{\{i; e_i < k\}} n - k - i + 1 - \sum_{j=1}^{t} \left\lfloor \frac{k + i - e_i - 1}{r_j} \right\rfloor
\]

(20)

(21)

where \( e_i, 1 \leq i \leq b \) are calculated using example 2.

In the above we have substituted the bound \( d_{\text{min}}(n - e_i, k + i - e_i, r, t) \leq n - k - i + 1 - \sum_{j=1}^{t} \left\lfloor \frac{k + i - e_i - 1}{r_j} \right\rfloor \) from [10]. We can get other bounds by substituting other bounds on minimum distance of codes with \( t \) availability in literature for \( d_{\text{min}}(n - e_i, k + i - e_i, r, t) \). Which bound on \( d_{\text{min}}(n - e_i, k + i - e_i, r, t) \) we use may matter only if we are able to calculate only small number \( e_i \)'s. Hence we stick to equation (21).

**Tightness of the Bound:** We use the expression in (21) along with expression for \( e_i \) given in example 2 to calculate bound on minimum distance of an \((n, k, r, t)\) code. The resulting bound is tighter than the bounds given in [10] and [20]. We plot the bound for \( t = 3 \) in Fig. 3. It can be seen from the plot that our bound is tighter than the bounds in [10] and [20].

**Corollary 5.** Let \( C \) be an \((n, k, r, t \geq 2)\) code. Let,

\[ B_0 = \text{RowSpan}(H_{\text{des}}(C)) \]

Let the dimension of \( B_0 \) be \( M \). Let the code \( B_0 \) has a generator matrix with full rank such that the Hamming weight of all the columns of the matrix is \( \leq t \) and all the rows have Hamming weight \( \leq r + 1 \). Let the minimum distance of \( C \) be \( d_{\text{min}}(M)(n, k, r, t) \).

Let \( e_i, \forall 1 \leq i \leq M \) be calculated as follows. Since the calculation is a function of \( M \), we refer to them as \( e_i(M) \).
Fig. 3: Plotting locality vs minimum distance for $t = 3$ with $n = \left(\frac{r+3}{3}\right)$, $k = \frac{nr}{r+3}$. Here we comparing the bound given in Corollary 4 with the bounds given in [10], [20].

$e_0(M) = 0$

$e_1(M) = r + 1$

$J_1 = 0$

$J'_1 = r + 1 - \left\lfloor \frac{t(n-e_{i-1})}{M-i+1} \right\rfloor$

$J'_2 = \left\lceil \frac{2e_{i-1} - (i-1) - (i-1)(r+1)}{M-i+1} \right\rceil$

$J_i = \begin{cases} 
\max (J'_1, J'_2, 1) & \text{if } n - e_j \geq M, \forall 1 \leq j \leq i - 1, r + 1 - J_{i-1} \geq 2 \\
\max (J'_1, J'_2, 0) & \text{if } n - e_j \geq M, \forall 1 \leq j \leq i - 1, r + 1 - J_{i-1} < 2 \\
\max (J'_1, 1) & \text{if } n - e_j < M, \text{ for some } 1 \leq j \leq i - 1, r + 1 - J_{i-1} \geq 2 \\
\max (J'_1, 0) & \text{if } n - e_j < M, \text{ for some } 1 \leq j \leq i - 1, r + 1 - J_{i-1} < 2 
\end{cases}$

$e_i(M) = e_{i-1}(M) + r + 1 - J_i \quad (22)$

Then,

\[ d_{\min}(M)(n,k,r,t) \leq \min_{\{i:e_{i}(M)-i<k\}} d_{\min}(n-e_{i}(M), k + i - e_{i}(M), r, t) \]

\[ \leq \min_{\{i:e_{i}(M)-i<k\}} n - k - i + 1 - \sum_{j=1}^{t} \left\lfloor \frac{k + i - e_{i}(M) - 1}{r^j} \right\rfloor \quad (23) \]

If we want a bound on minimum distance which is independent of $M$ we take:

\[ d_{\min}(n,k,r,t) \leq \max_{\{n(1-R'(r,t))\leq M\leq n-k\}} d_{\min}(M)(n,k,r,t) \quad (24) \]

Proof: The only thing we have to prove are the expressions given for $e_i(M)$. Proof of equation (22) for calculating $e_i(M)$.

To avoid cumbersome notation we refer to $e_i(M)$ as $e_i$ in the proof. Let $\{h_1, \ldots, h_M\}$ denote a set of $M$ codewords such that $|\text{Supp}(h_i)| \leq r + 1, \forall 1 \leq i \leq M$ which are basis of $B_0$ such that the following matrix $H_1$ has all column weights $\leq t$:

\[
H_1 = \begin{bmatrix}
    h_1 \\
    \vdots \\
    h_M
\end{bmatrix}
\]
Let's take the codeword $h_1$ and set $e_1 = r + 1$. Let $A_1 = \text{Supp}(h_1)$. Now assume that we have a set of co-ordinates $A_{i-1}$ such that $|A_{i-1}| = e_{i-1}$ and $\bigcup_{j=1}^{i-1} \text{Supp}(h_j) \subseteq A_{i-1}$ for some $s_1, ..., s_{i-1}$. Now we are going to select $e_i - e_{i-1}$ co-ordinates from $[n] - A_{i-1}$ and add these co-ordinates to $A_{i-1}$ to form $A_i$ such that $A_i$ contains the support of at least $i$ codewords from \{h_1, ..., h_M\}. Now write the matrix $H_1$ after permuting rows and columns such that the co-ordinates represented by $A_{i-1}$ are the first $e_{i-1}$ columns of the matrix and the first $i - 1$ rows of the matrix are the rows $h_{s_1}, ..., h_{s_{i-1}}$ (upto permutation of columns). Hence the resulting matrix can be written as:

$$
H_1' = \begin{bmatrix}
H_1 & 0 \\
H_2 & H_3
\end{bmatrix}
$$

The matrix $H_2$ is the $i - 1 \times e_{i-1}$ matrix with columns corresponding to the coordinates in $A_{i-1}$ and $h_{s_1}, ..., h_{s_{i-1}}$ as its rows (upto permutation of columns and after throwing away some zero weight columns). Now take the matrix $H_2$. Let's write the matrix $H_2$ as $H_2 = [A|B]$ after permuting its columns such that the matrix $A$ has exactly the columns corresponding to the co-ordinates $F_1 - 1 = \bigcup_{j=1}^{i-1} \text{Supp}(h_j)$. Since the sum of weight of the columns corresponding to $F_1 - 1 = \bigcup_{j=1}^{i-1} \text{Supp}(h_j)$ is $\geq 2|F_1| - (i - 1)$ (since $t \geq 2$ this is the case) the sum of weight of columns of $A$ is $\geq 2|F_1| - (i - 1)(r + 1)$. Now the columns of the matrix $H_4$ corresponding to the columns in $B$ has 0 weight. Hence sum weight of the columns in $B$ is $\geq 2|F_1| - (i - 1)$. If not, we can remove these extra $e_i - e_{i-1} - |F_1| - 1$ co-ordinates corresponding to the columns on $B$ from $A_{i-1}$ and add to $A_{i-1}$ arbitrarily chosen $e_{i-1} - |F_1| - 1$ co-ordinates from the rest of $n - |F_1| - 1$ co-ordinates such that the columns corresponding to the chosen $e_i - 1 - |F_1| - 1$ co-ordinates have hamming weight atleast 2 in each of the column. Such a choice of the columns is always possible at each step $i - 1$ as long as $n - |F_1| - 1 \geq M + e_{i-1} - |F_1| - 1$ as the number of columns having hamming weight one in $H_1$ is atmost $M$. Hence under the condition $n - |F_1| - 1 \geq M + e_{i-1} - |F_1| - 1$, the sum of weights of all columns in $H_2$ is $\geq 2|F_1| - (i - 1) - (i - 1)(r + 1) + 2(e_{i-1} - |F_1| - 1) = 2e_{i-1} - (i - 1)(r + 1)$. The condition boils down to $n - e_{i-1} \geq M$. Now using this condition the average row weight of the matrix $H_2$ is atleast $J_2 = 2e_{i-1} - (i - 1)(r + 1)$. Hence there will be a row $h_i'$ in $H_2$ with weight $\geq |J_2|$. The corresponding row in $H_1'$ will have weight $\leq r + 1 - J_2$ outside the coordinates in $A_{i-1}$. Similarly the sum of column weights in $H_2$ is atmost $t(n - e_{i-1})$. Hence the average row weight of rows in $H_3$ is atmost $J_3 = \frac{t(n - e_{i-1})}{M + 1}$. Hence there is a row $h_i''$ in $H_3$ with weight $\leq |J_3|$. If $r + 1 - J_2 \geq 2$, the number of new co-ordinates that are added to $A_{i-1}$ to form $A_{i-1}$ is atleast 2. Hence the sum weight of all rows in $H_2$ is atleast 1. Hence there is a row $h_i'''$ in $H_2$ with weight $\leq r$.

Now we pick a row in $H_1'$ from $[H_2|H_3]$ part of the matrix from among the rows in $H_1$ corresponding to $h_i', h_i'', h_i'''$ such that the row has least weight in the co-ordinates $[n] - A_{i-1}$. Adding the support of the picked row to $A_{i-1}$ we form $A_i$ such that it contains the support of at least $i$ codewords in \{h_1, ..., h_M\} and has cardinality $\leq e_i$. Now add co-ordinates arbitrarily to $A_i$ to form $A_i'$ of cardinality exactly $e_i$.

Since $A_i$ contains the support of $i$ linearly independent codewords in the dual, it is clear the $d_1^i \leq |A_i| = e_i$. 

### Tightness of the bound:

In the Fig. 2 and Fig. 5 we plot the bound given by (23) and (24) respectively for $t = 3$. It can be seen from Fig. 2 that the new bound given in (23) is tighter when plotted for a specific value of $M$ as it optimizes the bound on minimum distance for the given value of $M$. It can be seen from Fig. 5 that the bound given in (24) is still tighter than the bounds given in [10], [20] even after maximizing over all possible $M$. Such bounds depending on $M$ might throw insight as to what $M$ is optimal for minimum distance for a given $n, k, r, t$. It may be expected that the least possible value of $M$ for a given $n, k, r, t$ to be optimal for minimum distance but we are far away from a proof of such statement.

### IV. Constructions

#### A. Construction for $(n, k, r, t)_{a_1}$ code

1. The following is a construction of availability code with $t = 3, r = 2^g - 1, n = (r + 1)^2$. We don’t know the expression for the dimension of the construction. We find the dimension using MATLAB.

2. Form a collection of partitions of $[m] = [2^g], \{P_1, ..., P_{2^{g-1}}\}$ such that each set in a partition contains just 2 elements and if $(a, b)$ appears as a set in partition $P_i$ then the set must not appear in any other partition $P_j, i \neq j$.

It can be recursively shown that such a collection of partition can be formed if $T(m)$ denote the maximum number of such collection of partitions of $m = 2^g$ elements, then it can be seen that $T(m) \geq 2T(m/2) + 1$. From this by recursively solving, we get $T(m) \geq 2^g - 1$. The recursive bound $T(m) \geq 2T(m/2) + 1$ and also the construction can be seen as follows:

a) Pick a partition $P_i = \{A_1 = \{a_1, a_2\}, ..., A_{2^{g-1}} = \{a_{2^{g-1}}, a_{2^g}\}\}$ of $[m]$ arbitrarily.

b) Let $W_{T(m/2)} = \{Q_1, ..., Q_{T(m/2)}\}$ be a collection of $T(m/2)$ partitions of $[2^g]$ satisfying the required conditions.
c) Now take a partition $Q_1$ from $W_{T(m/2)}$. Let the partition be: $Q_1= \{B_1 = \{b_1, b_2\}, ..., B_{2^{g-2}} = \{b_{2^{g-2}-1}, b_{2^{g-2}}\}\}$. 

\[ \text{d) Now using the above partition $Q_1$ we form 2 more partitions } P_3, P_4 \text{ of } [m] \text{ as follows:} \\
\text{Let } P_3 = P_4 = \emptyset \]

For each set $B_j = \{b_{2^{j-1}}, b_{2^j}\}$ from $Q_1$, do the following:

\[ P_3 = P_3 \cup \{\{a_2b_{2^{j-1}} - 1, a_2b_{2^j} - 1\}, \{a_2b_{2^{j-1}}, a_2b_{2^j}\}\} \]
\[ P_4 = P_4 \cup \{\{a_2b_{2^{j-1}} - 1, a_2b_{2^j} - 1\}, \{a_2b_{2^{j-1}}, a_2b_{2^j}\}\} \]

The resulting collection of sets $P_3, P_4$ can be easily seen to be partitions of $[m]$. Similarly we can form 2 more partitions of $[m]$ for each of the partition in $W_{T(m/2)}$. It is easy to see that the resulting collection of partitions of
[m] satisfy the required conditions. Hence \( T(m) \geq 2T(m/2) + 1 \).

4) Now pick a sub collection of \( 2^{n-1} \) partitions from \( \{P_1, ..., P_{2^{n-1}}\} \). Let them be \( \{P_1, ..., P_{2^n-1}\} \). Let order the sets in each partition from 1 to \( 2^{n-1} \).

5) Form a matrix \( A = [a_{ij}] \) as follows:
   For \( 1 \leq i \leq m \)
   \[ a_{i,1} = 1, a_{i,2+2(i-1)} = 1, a_{i,2+2(i-1)+1} = 1 \]

6) For the \( j^{th} \) set (say \( \{a, b\} \)) in partition \( P_i \) for \( 1 \leq i \leq 2^{n-1} - 1 \):

   \[
   a_{m+i-1,2^{n-1}+1+4(j-1)+1,2+2(a-1)} = 1 \\
   a_{m+i-1,2^{n-1}+1+4(j-1)+1,2+2(b-1)} = 1 \\
   a_{m+i-1,2^{n-1}+1+4(j-1)+1,2+2(b-1)+1} = 1 \\
   a_{m+i-1,2^{n-1}+1+4(j-1)+1,2+2(a-1)+1} = 1 \\
   a_{m+i-1,2^{n-1}+1+4(j-1)+1,2+2(a-1)+2} = 1 \\
   a_{m+i-1,2^{n-1}+1+4(j-1)+1,2+2(a-1)+1} = 1 \\
   a_{m+i-1,2^{n-1}+1+4(j-1)+1,2+2(b-1)+1} = 1 \\
   a_{m+i-1,2^{n-1}+1+4(j-1)+1,2+2(b-1)+2} = 1 \\
   a_{m+i-1,2^{n-1}+1+4(j-1)+1,2+2(b-1)+1} = 1 \\
   
   7) For the \( j^{th} \) set (say \( \{a, b\} \)) in partition \( P_i \) for \( 1 \leq i \leq 2^{n-1} - 1 \):

   \[
   a_{m+i-1,2^{n-1}+1+4(j-1)+1,3+2+2(a-1)} = 1 \\
   a_{m+i-1,2^{n-1}+1+4(j-1)+1,3+2+2(b-1)+1} = 1 \\
   a_{m+i-1,2^{n-1}+1+4(j-1)+1,3+2+2(b-1)+1} = 1 \\
   a_{m+i-1,2^{n-1}+1+4(j-1)+1,3+2+2(a-1)+1} = 1 \\
   a_{m+i-1,2^{n-1}+1+4(j-1)+1,3+2+2(a-1)+2} = 1 \\
   a_{m+i-1,2^{n-1}+1+4(j-1)+1,3+2+2(a-1)+1} = 1 \\
   a_{m+i-1,2^{n-1}+1+4(j-1)+1,3+2+2(b-1)+1} = 1 \\
   a_{m+i-1,2^{n-1}+1+4(j-1)+1,3+2+2(b-1)+2} = 1 \\
   a_{m+i-1,2^{n-1}+1+4(j-1)+1,3+2+2(b-1)+1} = 1 \\
   
   8) For the \( j^{th} \) set (say \( \{a, b\} \)) in partition \( P_{2^n-1} \):

   \[
   a_{m+(2^{n-1}-1)2^{n-1}+1+2(j-1)+1,2+2(a-1)} = 1 \\
   a_{m+(2^{n-1}-1)2^{n-1}+1+2(j-1)+1,2+2(b-1)} = 1 \\
   a_{m+(2^{n-1}-1)2^{n-1}+1+2(j-1)+1,2+2(b-1)+1} = 1 \\
   a_{m+(2^{n-1}-1)2^{n-1}+1+2(j-1)+1,2+2(a-1)+1} = 1 \\
   a_{m+(2^{n-1}-1)2^{n-1}+1+2(j-1)+1,2+2(a-1)+2} = 1 \\
   a_{m+(2^{n-1}-1)2^{n-1}+1+2(j-1)+1,2+2(a-1)+1} = 1 \\
   a_{m+(2^{n-1}-1)2^{n-1}+1+2(j-1)+1,2+2(b-1)+1} = 1 \\
   a_{m+(2^{n-1}-1)2^{n-1}+1+2(j-1)+1,2+2(b-1)+2} = 1 \\
   a_{m+(2^{n-1}-1)2^{n-1}+1+2(j-1)+1,2+2(b-1)+1} = 1 \\
   
   Apart from the \( a_{ij} \) assigned to 1 above, rest of the \( a_{ij} = 0 \).

9) The desired parity check matrix is \( H = A^T \).

10) It is easy to see that the resulting code corresponding to null space of \( H \) has parameters \( t = 3, r = 2^n - 1, n = (r + 1)^2 \) because of the conditions put on the partitions \( \{P_1, ..., P_{2^n-1}\} \).

11) An Example construction for \( t = 3, r = 3 \) (the matrix \( A \) below is written after permutation of rows). In the following matrix, rows 1 to 4 correspond the step 5. Rows 5 to 8 and rows 13 to 16 correspond to the step 6 with partitions
\{\{1, 2\}, \{3, 4\}\} and \{\{1, 3\}, \{2, 4\}\} and rows 9 to 12 correspond to step 7 with partition \{\{1, 2\}, \{3, 4\}\}:

\[
RowPermutation(A) = \begin{bmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

1) The following table gives the rate of the construction mentioned above for a certain choice of partitions:

| t=3 | r=3 | r=7 | r=15 |
|-----|-----|-----|------|
| Our Construction over \(F_2\) | 0.5 (\(n = 16\)) | 0.7031 (\(n = 64\)) | 0.8359 (\(n = 256\)) |
| \(\frac{r}{t+1}\) Construction by [18] | 0.5 (\(n = 20\)) | 0.7 (\(n = 120\)) | 0.8333 (\(n = 816\)) |
| Construction by [21] | 0.45 (\(n = 20\)) | 0.6857 (\(n = 35\)) | 0.8323 (\(n = 155\)) |

2) The following table gives the rate of the construction for a certain choice of partitions:

| t=4 | r=2 | r=8 | r=26 |
|-----|-----|-----|------|
| Our Construction over \(F_3\) | 0.3333 (\(n = 9\)) | 0.6790 (\(n = 81\)) | 0.8724 (\(n = 729\)) |
| \(\frac{r}{t+1}\) construction by [18] | 0.3333 (\(n = 15\)) | 0.6667 (\(n = 495\)) | 0.8667 (\(n = 27405\)) |

The required partitions can be formed using recursive building up of partitions as described in the construction. Simulation shows that as long as certain level of symmetry is maintained in the partition the code yields good rate. The optimal choice of partitions is currently not clear to us. Hence we are not mentioning the choice of partitions used in generating the above table.

It can be seen that our construction has slightly higher rate than the codes given in [18] and [21]. Our construction also has much shorter block length compared to constructions in [18]. The construction given above can be extended further for larger \(t\). We are not listing them as the idea is exactly similar.

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