On algebraic properties of matroid polytopes

Michał Lason* and Mateusz Michałek†

Abstract. A toric variety is constructed from a lattice polytope. It is common in algebraic combinatorics to carry this way a notion of an algebraic property from the variety to the polytope. From the combinatorial point of view, one of the most interesting constructions of toric varieties comes from the base polytope of a matroid.

Matroid base polytopes and independence polytopes are Cohen–Macaulay. We study two natural stronger algebraic properties – Gorenstein and smooth. We provide a full classifications of matroids whose independence polytope or base polytope is smooth or Gorenstein. The latter answers to a question raised by Herzog and Hibi.

1. Introduction

Matroids and lattice polytopes are combinatorial objects that are fundamental for combinatorial algebraic geometry. They belong to a part of mathematics where the interaction of algebra and geometry with combinatorics is particularly strong and significant. We explore this connection.

1.1. Algebraic motivation. Toric varieties are a class of algebraic varieties that on the one hand capture many varieties seen in applications and on the other hand are more amenable to combinatorial techniques than general algebraic varieties. Indeed, the geometry of a toric variety is fully determined by the combinatorics of its associated lattice polytope. When an algebraic variety is constructed using only combinatorial data, one expects to have a combinatorial description of its algebraic properties. An attempt to achieve this description often leads to surprisingly deep combinatorial questions.

Toric variety of a matroid is a particularly interesting example. For a representable matroid it has a nice geometric description – it is the torus orbit closure in a Grassmannian, moreover every orbit closure arises in this way [13]. Affine toric variety of a matroid is recognized mostly due to a famous conjecture of White [37]. The conjecture provides a description of generators of the toric ideal of a matroid.

Key words and phrases. matroid, toric ideal, base polytope, independence polytope, smooth polytope, Gorenstein polytope.

* michalason@gmail.com; Institute of Mathematics of the Polish Academy of Sciences, ul. Śniadeckich 8, 00-656 Warszawa, Poland
† wajcha2@poczta.onet.pl; Max Planck Institute, Mathematics in Sciences, Inselstrasse 22, 04103 Leipzig, Germany
that is the ideal defining the matroid affine variety. In particular, it states that this
ideal is generated in degree two. The conjecture was confirmed for several special
classes of matroids. For general matroids it was proved ‘up to saturation’ \[25\], and
later upgraded for ‘high degrees w.r.t. the rank’ \[27\]. In full generality White’s
conjecture remains open since its formulation in 1980.

In this paper we study algebraic properties of the toric variety of a matroid.
White \[36\] already proved that the affine toric variety of a matroid is normal.
Hence, by a celebrated result of Hochster, it satisfies Cohen–Macaulay property. We
investigate two natural stronger properties – Gorenstein property and smoothness
of the projective variety, as an affine toric variety is smooth only for a unit simplex.
The notion of the Gorenstein property goes back to Grothendieck. It reflects many
symmetries of cohomological properties. It also implies that singularities of the
variety are not ‘too bad’. In particular, there is the following chain of inclusions
(for both affine or projective varieties).

\[
\text{smooth varieties} \subset \text{Gorenstein varieties} \subset \text{Cohen–Macaulay varieties}
\]

Though, not every matroid variety is Gorenstein nor smooth. Thus, there is a
need for a classification. This line of research was pioneered by Herzog and Hibi
\[15\] who classified ‘generic’ discrete polymatroids with Gorenstein property. As
they wrote the whole classification seems to be ‘quite difficult’. For other combina-
torial objects this question was also intensively studied – e.g. for perfect matchings
polytopes of grid graphs \[4\], for cut polytopes \[30\], for symmetric edge polytopes
\[28, 20\], and for order polytopes \[19\]. Next, for two special classes of matroids a
classification of Gorenstein matroids was obtained – for graphic matroids \[18\] (with
an extension to multigraphs \[23\]), and for lattice path matroids \[22\].

We obtain a complete classifications of matroids whose affine toric variety is
Gorenstein, and of those whose projective toric variety is smooth.

1.2. Combinatorial meaning. Matroid base polytope is the convex hull of
the indicator vectors of all bases of the matroid. It is a well-established object
of study in matroid theory. Its edges correspond to a single element exchanges
between bases \[13\]. Thus, the 1-skeleton of the matroid base polytope is the basis
graph of the matroid – a graph on all bases of the matroid and edges between
two bases differing only by one element. Matroid base polytopes are generalized
permutohedra \[1\], and possess integer Carathéodory property \[14\].

Another polytope naturally associated to a matroid is the matroid indepen-
dence polytope. As the name suggests, it is the convex hull of the indicator vectors
of all independent sets.

We consider two properties – smoothness, and Gorenstein property.

A lattice polytope \( P \) is smooth if for every vertex \( v \) of \( P \), the lattice part of
the affine cone starting at vertex \( v \) and generated by \( P \) is lattice isomorphic to the
lattice part of the positive orthant of a linear space.

A lattice polytope \( P \) is Gorenstein if there exists a positive integer \( \delta \) and a
point \( v \) in \( \delta P \) such that \( v \) has the smallest positive distance to every facet of \( \delta P \)
among all lattice points. In other words, a polytope \( P \) is Gorenstein if and only
if for some positive integer \( \delta \) the polytope \( \delta P \) is reflexive. Reflexive polytopes
play a prominent role in algebraic combinatorics with important connections to
mirror symmetry \[2, 3\]. A lot of studies concern their classification and properties,
cf. \[24, 34\].
We classify matroids for which base polytope or independence polytope is smooth or Gorenstein. In order to classify Gorenstein matroid polytopes, in Sections 5 and 7 we introduce two new families of matroids parameterized by families of subsets with certain intersection properties. We believe that our results, apart from algebraic meaning, are also interesting for their own combinatorial sake. As one of possible applications, our results give a construction of a large class of Gorenstein polytopes on which conjectures involving Gorenstein polytopes may be verified.

1.3. Our results. We provide a complete classification of matroids for which the independence polytope or the base polytope is smooth or Gorenstein.

By Lemmas 2.2, 2.6, and Theorem 3.1 we obtain a classification of matroids for which the independence polytope is smooth.

Theorem 1.1. The independence polytope $P(M)$ of a matroid $M$ is smooth if and only if $M$ is a direct sum of loops and uniform matroids of rank one.

By Lemmas 2.4, 2.6, and Theorem 4.1 we obtain a classification of matroids for which the base polytope is smooth.

Theorem 1.2. The base polytope $B(M)$ of a matroid $M$ is smooth if and only if $M$ is a direct sum of loops, uniform matroids of rank one, and uniform matroids of corank one.

By Definition 2.8, Lemmas 2.2, 2.11, and Theorem 6.5 we obtain a classification of matroids for which the independence polytope is Gorenstein. The class of $F_\delta$-matroids (which appear in the classification) is constructed in Section 5.

Theorem 1.3. The independence polytope $P(M)$ of a matroid $M$ is Gorenstein if and only if there exists an integer $\delta \geq 2$ such that $M$ is a direct sum of loops and $(\delta - 1)$-blow ups of connected $F_\delta$-matroids.

By Definition 2.8, Lemmas 2.3, 2.11 and Theorems 8.11 and 8.14 we obtain a classification of matroids for which the base polytope is Gorenstein. The class of $G_\delta$-matroids (which appear in the classification) is constructed in Section 7.

Theorem 1.4. The base polytope $B(M)$ of a matroid $M$ is Gorenstein if and only if there exists an integer $\delta \geq 2$ such that: when $\delta > 2$, $M$ is a direct sum of loops and $G_\delta$-matroids with contracted some $(\delta - 1)$-ears; when $\delta = 2$, $M$ is a direct sum of loops and $G_2'$-matroids.

2. Polytopes – definitions and properties

Throughout the paper by $M$ we denote a matroid, by $E$ its ground set, and by $r$ the rank function $2^E \to \mathbb{N}$. The matroidal closure of a set $A$ we denote by $cl(A)$. We say that a set $F \subset E$ is a flat, or that it is closed, when $cl(F) = F$. A set $A$ is called indecomposable when it can not be decomposed into proper subsets $A = A_1 \sqcup A_2$ (by $\sqcup$ we denote the disjoint union of sets) such that $r(A) = r(A_1) + r(A_2)$. A set $A$ is called connected if every two elements of $A$ belong to some circuit contained in $A$. Recall that a set is connected if and only if it is indecomposable if and only if it is not a direct sum of two or more nontrivial matroids. We will use these notions interchangeably. In Sections 8 and 7 we will intensively use one more notion – a good flat is a flat $G$ such that both: the restriction of the matroid to $G$ and the contraction of $G$ in the matroid are connected. For a general background of matroid theory we refer the reader to [32].
2.1. Matroid independence polytope.

Definition 2.1. The independence polytope of a matroid $M$ on the ground set $E$, denoted by $P(M)$, is the convex hull of points $v_I := \sum_{e \in I} \chi_e \in \mathbb{Z}^E$ over all independent sets $I$ of the matroid $M$.

Since indicator vectors of independent sets are vertices of the hypercube $[0,1]^E$, every indicator vector $e_I$ of an independent set $I$ is a vertex of $P(M)$.

It is straightforward to show the following lemma.

Lemma 2.2. Suppose a matroid $M$ is the direct sum of matroids $M_1, \ldots, M_k$. Then the independence polytope $P(M)$ is the cartesian product of the independence polytopes $P(M_1), \ldots, P(M_k)$. In particular, the independence polytope of a loop is a single vertex, so when $e$ is a loop in $M$, then $P(M)$ and $P(M \setminus \{e\})$ are lattice isomorphic.

In Section 8 we will use the following (not as obvious as the above) lemma describing facets of the independence polytope $P(M)$.

Lemma 2.3. Let $M$ be a loopless matroid. The matroid independence polytope $P(M)$ is full dimensional in $\mathbb{R}^E$, and the following set of inequalities is minimal defining $P(M)$:

\begin{enumerate}[(i)]
  \item $0 \leq x_e$, for every $e \in E$,
  \item $\sum_{e \in F} x_e \leq r(F)$, for every indecomposable flat $F$.
\end{enumerate}

That is, the intersection of $P(M)$ with each of the supporting hyperplanes of the above half spaces is a facet of $P(M)$.

2.2. Matroid base polytope.

Definition 2.3. The base polytope of a matroid $M$ on the ground set $E$, denoted by $B(M)$, is the convex hull of points $v_B := \sum_{e \in B} \chi_e \in \mathbb{Z}^E$ over all bases $B$ of the matroid $M$.

Since indicator vectors of bases of $M$ are vertices of the hypercube $[0,1]^E$, every indicator vector $v_B$ of a basis $B$ is a vertex of $B(M)$. Clearly, $B(M) = P(M) \cap \{v : \sum_{e \in E} v_e = r(M)\}$, thus $B(M)$ is a face of $P(M)$.

It is straightforward to show the following lemma.

Lemma 2.4. Suppose a matroid $M$ is the direct sum of matroids $M_1, \ldots, M_k$. Then the base polytope $B(M)$ of $M$ is the cartesian product of the base polytopes $B(M_1), \ldots, B(M_k)$. In particular, the base polytope of a loop is a single vertex, so when $e$ is a loop in $M$, then $B(M)$ and $B(M \setminus \{e\})$ are lattice isomorphic.

In Section 8 we will use the following (not as obvious as the above) lemma describing facets of the base polytope $B(M)$.

Lemma 2.5. Let $M$ be a connected matroid. The matroid base polytope $B(M)$ is full dimensional in an affine hyperplane $L := \{x \in \mathbb{R}^E : \sum_{e \in E} x_e = r(E)\}$, and the following set of inequalities is minimal defining $B(M)$ in the hyperplane $L$:

\begin{enumerate}[(i)]
  \item $0 \leq x_e$, for every $e \in E$ such that $M \setminus \{e\}$ is connected,
  \item $\sum_{e \in G} x_e \leq r(G)$, for every proper good flat $G$ – a flat $\emptyset \neq G \subseteq E$ such that: restriction of $M$ to $G$ and contraction of $G$ in $M$ are connected.
\end{enumerate}

That is, the intersection of $B(M)$ with each of the supporting hyperplanes of the above half spaces is a facet of $B(M)$.
2.3. Smooth polytope. By a lattice we mean a free abelian group \( \mathbb{Z}^d \). By a lattice polytope we mean a convex polytope in \( \mathbb{R}^d \) with vertices in \( \mathbb{Z}^d \).

**Definition 2.5.** A lattice polytope \( P \subset \mathbb{R}^d \) is smooth (regular) if for every vertex \( v \) of \( P \) the primitive edge directions from \( v \) are a part of a \( \mathbb{Z} \)-basis of \( \mathbb{Z}^d \).

In particular, every vertex of a smooth polytope \( P \) can have at most \( \dim(P) \) incident edges – polytopes with this property are called simple.

The above is a combinatorial definition of a smooth polytope. It is already in use on its own combinatorial sake – see e.g. \([7, 10]\). However, the notion of a smooth polytope originates from algebraic geometry. It means that the projective variety associated to the graded semigroup algebra \( \mathbb{C}[P] \) of lattice points in the cone over \( P \) is smooth.

**Lemma 2.6.** Let \( P_1, P_2 \) be two lattice polytopes. The product polytope \( P_1 \times P_2 \) is smooth if and only if both \( P_1 \) and \( P_2 \) are smooth.

**Proof.** The lemma follows easily from the fact that vertices of \( P_1 \times P_2 \) are exactly pairs \((v_1, v_2)\) where \( v_i \) is a vertex of \( P_i \). \( \square \)

2.4. Gorenstein polytope. Recall that a lattice polytope is reflexive if \( 0 \) is the only lattice point in its interior and the dual (polar) polytope is again a lattice polytope. A lattice polytope \( P \) has integer decomposition property if every vector in \( kP \cap \mathbb{Z}^d \) is a sum of \( k \) vectors from \( P \cap \mathbb{Z}^d \) (cf. \([8]\)).

**Definition 2.7.** A full-dimensional lattice polytope \( P \subset \mathbb{R}^d \) with integer decomposition property is Gorenstein if there exists a positive integer \( \delta \) and a lattice point \( v \in \delta P \) such that \( \delta P - v \) is a reflexive polytope.

The above is a combinatorial definition of the Gorenstein property. It is already in use on its own combinatorial sake – see e.g. \([3, 4, 6, 17, 29, 30, 31, 33]\). However, the notion of a Gorenstein polytope originates from algebra. We will not present here a definition of a Gorenstein algebra. Instead, we list a few equivalent conditions with combinatorial, algebraic, and geometric meaning. Let \( P \subset \mathbb{R}^d \) be a full-dimensional lattice polytope with integer decomposition property, and let \( \mathbb{C}[P] \) be the semigroup algebra of lattice points in the cone over \( P \) – a standard toric construction \([12, 9, 35, 5]\). The following conditions are equivalent \([2, 16]\):

1. \( P \) is a Gorenstein polytope,
2. \( \mathbb{C}[P] \) is a Gorenstein algebra,
3. the affine variety associated to \( \mathbb{C}[P] \) is Gorenstein,
4. the numerator of the Hilbert series of \( \mathbb{C}[P] \) is palindromic,
5. the canonical divisor of \( \text{Spec} \mathbb{C}[P] \) is Cartier.

Moreover, by virtue of the work of Batyrev \([2]\) Gorenstein polytopes play an important role in mirror symmetry and for this reason are very intensively studied, see e.g. \([9\text{ Section 8.3]}\) and references therein.

We will use the following ‘working’ description of the Gorenstein property of a polytope. As we easily show in Proposition \([2.9]\) this coincides with Definition \([2.7]\).

**Definition 2.8.** Let \( \delta \) be a positive integer. A full-dimensional lattice polytope \( P \subset \mathbb{R}^d \) with integer decomposition property is \( \delta \)-Gorenstein if there exists a lattice point \( v \in \delta P \) such that for every supporting hyperplane of the cone over \( P \) its reduced equation \( h \) (that is, \( h \) such that \( h(\mathbb{Z}^d) = \mathbb{Z} \)) satisfies \( h(v) = 1 \).
A non necessary full-dimensional lattice polytope \( P \subset \mathbb{R}^d \) with integer decomposition property is \( \delta \)-Gorenstein if \( P \) is \( \delta \)-Gorenstein in the affine lattice it spans.

A lattice polytope \( P \) with integer decomposition property is Gorenstein if \( P \) is \( \delta \)-Gorenstein for some positive integer \( \delta \).

**Proposition 2.9.** Definitions 2.7 and 2.8 agree for full-dimensional polytopes.

**Proof.** When \( P \) is \( \delta \)-Gorenstein, then vertices of the dual polytope \((\delta P - v)^*\) correspond to reduced equations of supporting hyperplanes of the cone over \( P \), and therefore they are integral. Conversely, if \( h(v) > 1 \) for the reduced equation of a supporting hyperplane of the cone over \( P \), then the coordinates of the corresponding vertex of the dual polytope \((\delta P - v)^*\) are fractions with denominator \( h(v) \), hence they are not integer. \( \square \)

**Remark 2.10.** The polytopes \( P(M) \) and \( B(M) \) for a matroid \( M \) do not contain interior lattice points, thus they are not \( 1 \)-Gorenstein unless they are \( 0 \)-dimensional.

**Lemma 2.11.** Let \( P_1, P_2 \) be two lattice polytopes with integer decomposition property. The product \( P_1 \times P_2 \) is \( \delta \)-Gorenstein if and only if \( P_1 \) and \( P_2 \) are \( \delta \)-Gorenstein.

**Proof.** If both \( P_1 \) and \( P_2 \) are \( \delta \)-Gorenstein with lattice points \( v_1 \) and \( v_2 \), as in Definition 2.8, then \( (v_1, v_2) \in \delta(P_1 \times P_2) \) is the point which proves that \( P_1 \times P_2 \) is \( \delta \)-Gorenstein. Conversely, suppose \( P_1 \times P_2 \) is \( \delta \)-Gorenstein with a point \((v_1, v_2)\) with lattice distance one from all facets. Then \( v_1 \) (resp. \( v_2 \)) is a point which proves that \( P_1 \) (resp. \( P_2 \)) is \( \delta \)-Gorenstein. \( \square \)

In particular, when \( M \) is a matroid which is the direct sum of matroids \( M_i \), then \( P(M) \) (resp. \( B(M) \)) is \( \delta \)-Gorenstein if and only if for every \( i \) the polytope \( P(M_i) \) (resp. \( B(M_i) \)) is \( \delta \)-Gorenstein.

**3. Smooth matroid independence polytopes**

**Theorem 3.1.** The independence polytope \( P(M) \) of a connected loopless matroid \( M \) is smooth if and only if \( M \) is a uniform matroid of rank one \( U_{1,n} \).

**Proof.** It is easy to see the independence polytope of \( U_{1,n} \) is a simplex, thus it is simple and smooth.

Suppose now that the polytope \( P(M) \) is smooth. We will use that it is simple.

**Claim 3.2.** Let \( B \) be a basis of \( M \), and let \( C_x \) be the fundamental circuit of an element \( x \notin B \) with respect to \( B \). Then \( C_x = \{ x, b \} \) for some \( b \in B \).

**Proof.** For every \( x \notin B \) and \( b \in C_x \) the set \( B_{x,b} = B \cup x \setminus b \) is a basis of \( M \) and the vertex \( v_{B_{x,b}} \) is a neighbor of \( v_B \) in \( P(M) \) — that is, both vertices of \( P(M) \) are joined by an edge. Additionally, vertices \( v_{B \setminus b} \) for \( b \in B \) are also neighbors of \( v_B \) in \( P(M) \). Thus, there are exactly \( |B| + \sum_{x \notin B} |C_x \cap B| \) neighbors of \( v_B \). Since there are at most \( |E| \) neighbors of a vertex in a simple polytope in \( \mathbb{R}^{|E|} \), we get that \( |C_x \cap B| = 1 \) for every \( x \notin B \). \( \square \)

Every circuit in a matroid is a fundamental circuit of some element with respect to some basis. Thus, every circuit in \( M \) has cardinality 2. Since \( M \) is connected, it has to be a uniform matroid \( U_{1,n} \) of rank one. \( \square \)

**Remark 3.3.** By the proof, the independence polytope \( P(M) \) of a connected loopless matroid \( M \) is simple if and only if \( M \) is a uniform matroid of rank one.
4. Smooth matroid base polytopes

Theorem 4.1. The base polytope $B(M)$ of a connected loopless matroid $M$ is smooth if and only if $M$ is a uniform matroid of rank one $U_{1,n}$ or a uniform matroid of corank one $U_{n-1,n}$.

Proof. The base polytope $B(M^*)$ of the dual matroid to $M$ is isomorphic to $B(M)$ via the map $Z^{|E|} \ni v \to (1, \ldots, 1) - v \in Z^{|E|}$. The base polytope $B(M)$ of a uniform matroid $U_{1,n}$ is smooth, clearly, the polytope of the dual matroid, that is $U_{n-1,n}$, is also smooth.

Let $M$ be a connected loopless matroid such that the base polytope $B(M)$ is smooth. We will show that if $r(M) \leq \frac{1}{2}|E|$, then $M$ is a uniform matroid $U_{1,n}$. This finishes the proof, because when $r(M) > \frac{1}{2}|E|$ then we can apply the following reasoning to the dual matroid $M^*$.

Claim 4.2. Let $B$ be a basis of $M$, and let $C_x, C_y$ be the fundamental circuits of elements $x \notin B$ and $y \notin B$ with respect to $B$. Then $|C_x \cap C_y| \leq 1$.

Proof. Suppose contrary, that there exist two elements $b_1, b_2 \in C_x \cap C_y$. Now $B_{1,x} = (B \cup x) \setminus b_1, B_{2,x} = (B \cup x) \setminus b_2, B_{1,y} = (B \cup y) \setminus b_1, B_{2,y} = (B \cup y) \setminus b_2$ are bases in $M$ whose corresponding vertices are neighbors of $v_B$. We have an equality $v_{B_{1,x}} + v_{B_{2,x}} = v_{B_{1,y}} + v_{B_{2,y}}$, thus the polytope $B(M)$ is not smooth – a contradiction.

Claim 4.3. Let $C_1, C_2$ be circuits with $|C_1| \geq 3$ and $|C_2| = 2$. Then $C_1 \cap C_2 = \emptyset$.

Proof. Suppose contrary, that is there exists $x \in C_1 \cap C_2$. Denote $C_2 = \{x, y\}$. Extend the independent set $C_1 \setminus x$ to a basis $B$. Now the fundamental circuit of $x$ with respect to $B$ is $C_1$. It is easy to see that the fundamental circuit of $y$ with respect to $B$ is $C_2 = (C_1 \cup y) \setminus x$. And, $|C_1 \cap C_2| \geq 2$. This gives a contradiction with Claim 4.2.

Claim 4.4. Let $B$ be a basis of $M$, and let $C_x, C_y$ be the fundamental circuits of $x \notin B$ and $y \notin B$ with respect to $B$. If $|C_x| \geq 3$ and $|C_y| \geq 3$, then $C_x \cap C_y = \emptyset$.

Proof. Suppose contrary, that there exists $b \in C_x \cap C_y$. From the circuit axiom there exists a circuit $C \subseteq C_x \cup C_y \setminus b$. We have that $x, y \in C$, since there is only one fundamental circuit for every element. We have also that $C \cap B \neq \emptyset$, since otherwise $C = \{x, y\}$ contradicting Claim 4.3. Thus there exists $z \in C \cap B$, without loss of generality $z \in C_x$. Now look at a basis $B' = (B \cup x) \setminus b$. The fundamental circuit of $b$ with respect to $B'$ is $C_x$. The fundamental circuit of $y$ with respect to $B'$ is $C_x$. But $x, z \in C_x \cap C$ contradicting Claim 4.2.

Suppose the matroid $M$ does not have a circuit of size 2. Let $B$ be a basis of $M$, and let $C_x$ be the fundamental circuit of $x$ with respect to $B$ for every $x \in E \setminus B$. We have $|C_x| \geq 3$, thus by Claim 4.4 all circuits $C_x$ are disjoint. Hence, $|E \setminus B| = \sum_{x \in E \setminus B} 1 \leq \sum_{x \in E \setminus B} \frac{1}{2}|C_x \setminus x| \leq \frac{1}{2}|B|$. Thus, $r(M) \geq \frac{3}{2}|E|$ which contradicts the assumption that $r(M) \leq \frac{1}{2}|E|$.

Therefore, the matroid $M$ has a circuit $\{x, y\}$ of size 2. Let $S$ be the set of all elements parallel to $x$ (that is, dependent with $x$). Due to Claim 4.3 every circuit of $M$ is either contained in $S$ or disjoint from $S$. Thus, $M$ decomposes into $M|_S \oplus M|_{E \setminus S}$, and so $S = E$. Hence, $M$ is a uniform matroid of rank one. □
5. $F$-matroids

In this section we define a class of matroids that will play a central role in classification of Gorenstein independence polytopes. We first define a special family of subsets of the ground set — a family satisfying the properties of indecomposable flats in a matroid whose independence polytope is Gorenstein. Our goal is to achieve a matroid whose set of indecomposable flats coincides with the original family.

5.1. $F_δ$-families. Fix an integer $δ ≥ 2$.

Definition 5.1. A family $F$ of subsets of a finite ground set $E$ is called an $F_δ$-family if it satisfies the following conditions:

1. For every $e ∈ E$, $\{e\} ∈ F$,
2. For every $F_i, F_j ∈ F$, if $F_i \cap F_j ⊆ \emptyset$, then $F_i \cap F_j$ and $F_i \cup F_j$ belong to $F$,
3. For every $F_i ∈ F$, its cardinality equals $\frac{δf_i - 1}{δ - 1}$ for some integer $f_i ≥ 1$ (mod $δ - 1$).

We call the $F_δ$-family $F$ connected when $E ∈ F$.

Notice that for $F_k = F_1 \cup F_2$ and $F_1 = F_i \cap F_j$ as in (2), by counting their cardinalities we get an equality $f_k + f_1 = f_1 + f_j$ between the corresponding numbers.

$F_δ$-families can be easily generated.

Either globally — by setting an intersection scheme satisfying condition (2) between sets $F_i$, assigning numbers $f_i$ in a strictly monotone way, and finally filling every set with a right number of points (so that condition (3) holds).

Or inductively — every $F_δ$-family is a union of connected $F_δ$-families on disjoint ground sets. We describe two constructions of a connected $F_δ$-family from smaller connected $F_δ$-families. Every connected $F_δ$-family achieved in one of these ways.

Definition 5.2. Suppose $F_1$ and $F_2$ are two connected $F_δ$-families on ground sets $E_1, E_2$ such that $E_1 \cap E_2 = F ∈ F_1 \cap F_2$ and $\{A_1 ∈ F_1 : A_1 ⊆ F\} = \{A_2 ∈ F_2 : A_2 ⊆ F\}$. The fiber sum of families $F_1, F_2$ is a connected $F_δ$-family $F$ on the ground set $E_1 \cup E_2$ defined by $F := F_1 \cup F_2 \cup \{A_1 ∪ A_2 : A_i ∈ F_i, A_1 \cap F = A_2 \cap F = \emptyset\}$. We denote it by $F_1 \oplus F_2$.

Definition 5.3. Suppose $F_1, \ldots, F_k$ are $k$ connected $F_δ$-families on disjoint ground sets $E_1, \ldots, E_k$ and ranks $r_1, \ldots, r_k$ respectively. Suppose $k ≡ 1$ (mod $δ$). The connection of $F_1, \ldots, F_k$ is a connected $F_δ$-family $F$ of rank $r_1 + \cdots + r_k - \frac{k - 1}{δ}$ on the ground set $E = E_1 \cup \cdots \cup E_k$ defined by $F := F_1 \cup \cdots \cup F_k \cup \{E\}$. We denote it by $\text{Conn}(F_1, \ldots, F_k)$.

Proposition 5.4. A connected $F_δ$-family on $|E| > 1$ is equal to either:

- a fiber sum $F_1 \oplus F_2$ of connected $F_δ$-families, or
- a connection $\text{Conn}(F_1, \ldots, F_k)$ of $k ≡ 1$ (mod $δ$) connected $F_δ$-families.

Proof. Suppose $F$ is a connected $F_δ$-family on $|E| > 1$. Consider maximal (w.r.t. inclusion) proper subsets of $E$ that belong to $F$. There are two cases:

- there exist two sets with nonempty intersection, or
- all such sets are disjoint.

In the first case, let $F_1, F_2 ∈ F$ be such that $F_1 \cap F_2 ≠ \emptyset$. Then, by (2) we have that $F_1 ∪ F_2 ∈ F$, hence (from maximality of $F_1$ and $F_2$) $F_1 ∪ F_2 = E$. Consider two connected $F_δ$-families $F_i = \{F ∈ F : F ⊆ F_i\}$ for $i = 1, 2$. Then, it is easy to show that $F = F_1 \oplus F_2$.  

In the second case, let \( F_1, F_2, \ldots, F_k \) be all of them. Since \(|E| > 1, k > 1\). Consider connected \( F_3 \)-families \( \mathcal{F}_i = \{ F \in \mathcal{F} : F \subseteq F_i \} \) for \( i = 1, \ldots, k \). It is easy to verify that in this case \( \mathcal{F} = \text{Conn}(F_1, \ldots, F_k) \). □

5.2. \( F_3 \)-matroids. We present a construction of matroids corresponding to \( F_3 \)-families.

**Definition 5.5.** Let \( \mathcal{F} \) be an \( F_3 \)-family. Let \( \mathcal{C}_\mathcal{F} \) be a family of minimal sets (w.r.t. inclusion) among all \((f_i + 1)\)-element subsets of \( F_i \) over all \( F_i \in \mathcal{F} \). The matroid corresponding to \( \mathcal{F} \) is a matroid on the ground set \( E \) with the set of circuits equal to \( \mathcal{C}_\mathcal{F} \). We denote it by \( M_\mathcal{F} \), and call an \( F_3 \)-matroid.

**Remark 5.6.** A set \( A \subseteq E \) is independent in \( M_\mathcal{F} \) if and only if for every \( F_i \in \mathcal{F} \) an inequality \(|F_i \cap A| \leq f_i \) holds. In particular, the rank of \( F_i \) in \( M_\mathcal{F} \) is less or equal to \( f_i \).

The following is the main theorem of our classification of matroids whose independence polytope is Gorenstein. It shows that conditions from Definition 5.1 are not only a necessary conditions for a family of indecomposable flats (see Subsection 6.1), but also sufficient conditions.

**Theorem 5.7.** Suppose \( \mathcal{F} \) is an \( F_3 \)-family for some \( \delta \geq 2 \). Then, \( M_\mathcal{F} \) is a matroid in which the set of indecomposable flats (closed and connected sets) is equal to \( \mathcal{F} \), and every \( F_i \in \mathcal{F} \) has rank equal to \( f_i \). In particular, the matroid \( M_\mathcal{F} \) is connected if and only if \( \mathcal{F} \) is a connected \( F_3 \)-family.

**Proof.** Firstly, we show that \( M_\mathcal{F} \) is indeed a matroid. Let \( C_i \) be a circuit, i.e. a \((f_i + 1)\)-element subset of \( F_i \), and let \( C_j \) be another circuit, i.e. a \((f_j + 1)\)-element subset of \( F_j \). Let \( x \in C_i \cap C_j \). It is enough to show that there exists a circuit \( C_k \subset C_i \cup C_j \setminus x \). Since \( x \in F_i \cap F_j \), by (2) we get that \( F_i = F_j \cap F_j \) and \( F_k = F_i \cup F_j \) belong to \( \mathcal{F} \). Now, \(|C_i \cap C_j| \leq f_i \), as otherwise a \((f_i + 1)\)-element subset of \( F_i \) would be a subset of \( C_i \) and \( C_j \) which is a contradiction (\( C_i \) and \( C_j \) would not be minimal). Thus, \(|C_i \cup C_j| \geq f_i + f_j - f_i + 1 \geq f_k + 1 \). It is a subset of \( F_k \) of cardinality at least \( f_k + 1 \), so it contains a circuit.

Next, we prove that \( F_i \in \mathcal{F} \) has rank equal to \( f_i \) by induction on the size. If the size of \( F_i \) is 1, then clearly its rank equals 1 since there are no loops in \( M_\mathcal{F} \). When the size of \( F_i \) is greater than 1, consider all maximal proper subsets of \( F_i \) that belong to \( \mathcal{F} \). There are two cases:

- there exist two sets with nonempty intersection, or
- all such sets are disjoint.

In the first case, let \( F_j \) be one of them. So, there exists a proper subset \( F_k \in \mathcal{F} \) of \( F_i \) such that \( F_i = F_j \cup F_k \). Let \( F_k \) be the minimum set with this property. Let \( F_j \cap F_k = F_l \). From maximality of \( F_j \) follows that any proper subset \( F_x \in \mathcal{F} \) of \( F_i \) is either contained in \( F_j \), contained in \( F_k \setminus F_j \), or it contains \( F_k \) (from minimality of \( F_k \)). Let \( I_1 \) be a basis of \( F_i \) (by ind. ass. of size \( f_1 \)), and let \( I_j \) (by ind. ass. of size \( f_j \)) and \( I_k \) (by ind. ass. of size \( f_k \)) be its extensions to bases of \( F_j \) and \( F_k \), respectively. Then, the set \( I_i = I_j \cup I_k \) of size \( f_j + f_k - f_i = f_i \) is independent in \( F_i \). Indeed, it is easy to verify that for every set \( F_x \in \mathcal{F} \) we have \(|F_x \cap I_1| \leq f_x \). If \( F_x \) is contained in \( F_j \), then it follows from the fact that \( I_i \cap F_j = I_j \) is independent. If \( F_x \) is contained in \( F_k \setminus F_j \), then it follows from
the fact that \( I_1 \cap F_k = I_k \) is independent. Finally, when \( F_x \) contains \( F_k \), then let
\[ F_y = F_x \cap F_k. \]
Now, \( |F_x \cap I_1| + f_k - f_1 \leq f_y + f_k - f_1 = f_x. \)

In the second case, let \( F_1, F_2, \ldots, F_k \) be all of them. Clearly, \( F_1 \cup \cdots \cup F_k = F_1 \).

Let \( I_1, \ldots, I_k \) be bases of the corresponding sets. Notice that we have an equality
\[ \frac{\delta f_i - 1}{\delta} \cdot |F_i| = |F_i| + \cdots + |F_k| = \frac{\delta f_i - 1}{\delta} + \cdots + \frac{\delta f_k - 1}{\delta}, \]
so \( f_i = f_1 + \cdots + f_k - \frac{(k-1)}{\delta} \).

Thus, the only obstructions for the set \( I_1 \cup \cdots \cup I_k \) of size \( f_1 + \cdots + f_k \) to be independent in \( F_i \) are the \((f_i + 1)\)-element circuits of \( F_i \). So any \( f_1 \)-element subset of \( I_1 \cup \cdots \cup I_k \) is an independent set in \( F_i \), and hence by Remark 5.4 it is a basis.

Now, we argue that sets \( F_i \in \mathcal{F} \) are closed. Suppose that \( C \) is a circuit in \( M_F \) and \( |C \setminus F_i| \leq 1 \). Since circuits belong to the family \( \mathcal{C}_x \), \( C \) is a \((f_j + 1)\)-element subset of a set \( F_j \in \mathcal{F} \). Then \( F_k = F_i \cap F_j \) has rank at least \( |C| - 1 = f_j \). But its upset \( F_j \) has rank \( f_j \). Since both have the same rank, by (3) they have the same cardinality, and hence they coincide. As a consequence \( F_j \subseteq F_i \). Thus, \( C \subseteq F_i \).

Notice that the closure of a circuit \( C \) in \( M_F \) that corresponds to a set \( F_i \in \mathcal{F} \) is equal to \( F_i \). Indeed, the rank of \( cl(C) \) is \(|C| - 1 = f_i \). It is equal to the rank of \( F_i \), which is its upset since \( F_i \) is closed. Hence, \( cl(C) = F_i \).

Finally, let \( A \) be a closed set in \( M_F \). Consider all maximal (not necessarily proper) subsets of \( A \) that belong to \( \mathcal{F} \). Denote them by \( F_1, \ldots, F_k \). They are clearly disjoint (from maximality and (2)). If \( k > 1 \), then the sets \( F_1, \ldots, F_k \) form a decomposition of \( A \) into closed sets in \( M_F \). Indeed, otherwise there exists a circuit \( C \) corresponding to a set \( F \in \mathcal{F} \), which intersects at least two sets \( F_i, F_j \) among \( F_1, \ldots, F_k \). Then \( cl(C) = F \) also intersects \( F_i, F_j \), and it is contained in \( A \) (since \( A \) is closed). This contradicts maximality of \( F_i \), since \( F_i \cup F \cup F_j \in \mathcal{F} \) is a larger subset of \( A \). Therefore, if a closed set \( A \) is indecomposable, then it belongs to \( \mathcal{F} \).

For the opposite implication, let \( F \in \mathcal{F} \). We already know that it is closed. Let \( F = F_1 \cup \cdots \cup F_k \) be its decomposition into closed and indecomposable subsets (we already know that \( F_1, \ldots, F_k \in \mathcal{F} \)). Then we have two equalities, \( f = f_1 + \cdots + f_k \) and \( |F| = |F_1| + \cdots + |F_k| \). Together with (3) they give that \( k = 1 \). Thus, \( F \) is an indecomposable flat.

\[ \Box \]

6. Gorenstein matroid independence polytopes

6.1. Combinatorial reformulation when \( P(M) \) is \( \delta \)-Gorenstein. Elements \( e \) and \( f \) of a matroid are called parallel when \( \{e, f\} \) is a circuit. The relation of being parallel is an equivalence relation in which the equivalence class of \( x \) equals \( cl(x) \). The following operation allows to enlarge these equivalence classes.

**Definition 6.1.** A \( k \)-blow up of an element \( e \in E \) in a matroid \( M \) is the matroid \( M \) enlarged in the following way:

- the ground set \( E \) is enlarged by new elements \( e_2, \ldots, e_k \),
- the set of bases in enlarged by new bases \( (B \setminus e) \cup e_i \) for every \( i = 2, \ldots, k \) and every basis \( B \) of \( M \) containing \( e \).

Then elements \( e, e_2, \ldots, e_k \) are parallel elements.

A \( k \)-blow up of a matroid is the \( k \)-blow up of every element of its ground set.

Notice, that the structure of a matroid in some sense does not change after a \( k \)-blow up – as \( k \)-blow up only makes parallel elements classes \( k \) times larger, and being a basis or an independent set depends only on elements equivalence classes.
THEOREM 6.2. Fix a positive integer \( \delta \). Let \( M \) be a connected loopless matroid. The matroid independence polytope \( P(M) \) is \( \delta \)-Gorenstein if and only if \( M \) is a \((\delta - 1)\)-blow up of a connected loopless matroid \( M' \) which satisfies (\( \clubsuit \)):\n
1. \( \delta r(F) = (\delta - 1)|F| + 1 \) for every indecomposable flat \( F \) in \( M' \).

The above theorem is a corollary of Theorem 7.3 from [15]. However, for sake of completeness we include a proof. First, we present a description of facets of the matroid independence polytope.

LEMMA 6.3. Let \( M \) be a loopless matroid. The matroid independence polytope \( P(M) \) is full dimensional in \( \mathbb{R}^E \), and the following set of inequalities is minimal defining \( P(M) \):

1. \( 0 \leq x_e \), for every \( e \in E \),
2. \( \sum_{e \in F} x_e \leq r(F) \), for every indecomposable flat \( F \).

That is, the intersection of \( P(M) \) with each of the supporting hyperplanes of the above half spaces is a facet of \( P(M) \).

PROOF. Firstly, notice that by the matroid union theorem we have that \( P_M \{ x \in \mathbb{R}^E : 0 \leq x_e \text{ for every } e \in E, \text{ and } \sum_{e \in A} x_e \leq r(A) \text{ for every } A \subset E \} \). Hence, supporting inequalities are contained in the above set of inequalities.

Clearly, the inequality \( 0 \leq x_e \) for every \( e \in E \) is supporting.

If \( A \) is not a flat, then an inequality \( \sum_{e \in \text{cl}(A)} x_e \leq r(\text{cl}(A)) = r(A) \) together with some inequalities \( 0 \leq x_e \) implies \( \sum_{e \in A} x_e \leq r(A) \). When \( F \) is a flat decomposable into \( F_1, \ldots, F_k \), then its inequality follows from inequalities of \( F_i \). Hence, if an inequality \( \sum_{e \in A} x_e \leq r(A) \) is supporting, then \( A \) is an indecomposable flat.

Suppose \( F \) is an indecomposable flat and its inequality is not supporting. Then by a standard linear programming method the inequality \( \sum_{e \in F} x_e \leq r(F) \) has to be a rational convex combination of other inequalities defining \( P(M) \). After simplifying we must have that \( kF = F_1 \cup \cdots \cup F_1 \) as multisets for proper subsets \( F_i \subseteq F \) such that \( kr(F) = r(F_1) + \cdots + r(F_1) \). To show that this is not possible, we apply the following process – if there are two sets \( F_i, F_j \) such that no one is contained in the other, we exchange them into sets \( F_i \cap F_j \) and \( F_i \cup F_j \). After each step the multiset union of all sets remains the same, and the sum of values of \( r \) on these sets weakly decreases, as the rank function \( r \) is submodular. The process ends at \( F_1 = \cdots = F_k = F, F_{k+1} = \cdots = F_1 = \emptyset \). Notice that the sum of ranks is unchanged, hence also all intermediate steps sums of ranks are the same.

Consider one but last step. Then we must have \( F_1 = \cdots = F_{k-1} = F, F_k = A \neq \emptyset, F_{k+1} = B \neq \emptyset, F_{k+2} = \cdots = F_1 = \emptyset, \) and \( A \cup B = F \). Moreover, \( r(F) = r(A) + r(B) \) contradicting indecomposability of \( F \).

PROOF OF THEOREM 6.2. Suppose (\( \clubsuit \))\((1) \) holds for a matroid \( M' \) on the ground set \( E' \). Let \( M \) (on the ground set \( E \)) be the \((\delta - 1)\)-blow up of \( M' \). Notice that an indecomposable flat \( F \) in \( M \) is exactly the \((\delta - 1)\)-blow up of an indecomposable flat \( F' \) in \( M' \). But, their ranks are the same. Hence, by (1) we have \( \delta r(F) = \delta r(F') = (\delta - 1)|F'| + 1 = |F| + 1 \) for every indecomposable flat \( F \) in \( M \). Let \( v = (1, 1, \ldots, 1) \in \mathbb{R}^E \). By Lemma 6.3 vector \( v \) belongs to \( \delta P_M \), and further reduced equations of facets of \( \delta P_M \) of both types (i), (ii) evaluated at \( v \) give 1. Therefore, the polytope \( P(M) \) is \( \delta \)-Gorenstein.
Conversely, suppose the polytope $P(M)$ is $\delta$-Gorenstein. Therefore, there exists a lattice point $v \in \delta P(M)$ such that $\delta P(M) - v$ is a reflexive polytope. Firstly, by Lemma 5.3 (i) we have $v_\emptyset = 1$. By (ii) for every indecomposable flat $F$ in $M$ the equation $\sum_{e \in F} x_e = \delta r(F)$ is the reduced equation of a supporting hyperplane to $\delta P(M)$. Thus, $|F| + 1 = \delta r(F)$. For every element $e \in E$ consider its closure $cl(e)$. It is an indecomposable flat of rank 1. Hence, $|cl(e)| = \delta - 1$. Notice that $cl(e)$ is exactly the set of all elements in $M$ parallel to $e$. Let $M'$ be the matroid $M$ restricted to representatives of all parallel element classes. Then clearly $M$ is the $(\delta - 1)$-blow up of $M'$. Moreover, for every indecomposable flat $F'$ in $M'$ we have $\delta r(F') = \delta r(F) = |F| + 1 = (\delta - 1)|F'| + 1$. That is, $(\spadesuit)_\delta$ (1) holds for $M'$.

For the remaining part of this subsection let $\delta \geq 2$ be a fixed integer, and let $M'$ be a fixed matroid on the ground set $E$ satisfying condition $(\spadesuit)_\delta$ (1).

For every $A \subset E$ define

$$c(A) := \delta r(A) - (\delta - 1)|A|.$$ 

It is straightforward to check that the function $c$ has the following properties:

(2) $c(A) = c(cl(A)) + (\delta - 1)|cl(A) \setminus A|$, 

(3) $c(A) \leq c(cl(A))$, 

(4) $c(A) = c(A_1) + \cdots + c(A_k)$ if $A_1, \ldots, A_k$ are connected components of $A$, 

(5) $c(A \cup B) + c(A \cap B) \leq c(A) + c(B)$ — using submodularity of rank function.

Moreover, using $(\spadesuit)_\delta$ (1) it is easy to check that the function $c$ characterizes indecomposable flats:

(6) $c(A) \geq 0$, 

(7) $c(A) = 0$ if and only if $A = \emptyset$, 

(8) $c(A) = 1$ if and only if $A$ is an indecomposable flat.

Indeed, suppose $A \neq \emptyset$. By (2) we get $c(A) = c(cl(A)) + (\delta - 1)|cl(A) \setminus A| \geq c(cl(A))$. When the flat $F = cl(A)$ decomposes into $k$ indecomposable flats $F_1, \ldots, F_k$, then $c(F) = c(F_1) + \cdots + c(F_k) = k \geq 1$. Hence, $c(A) \geq 1$. Following these inequalities we get (8).

**Lemma 6.4.** Indecomposable flats in $M'$ satisfy the following property:

(9) if $F_1, F_2$ are indecomposable flats and $F_1 \cap F_2 \neq \emptyset$ then $F_1 \cap F_2$ and $F_1 \cup F_2$ are also indecomposable flats.

**Proof.** By (5) and (6) we have $0 \leq c(F_1 \cup F_2) + c(F_1 \cap F_2) \leq c(F_1) + c(F_2) = 2$. Moreover, $F_1 \cap F_2 \neq \emptyset$ and $F_1 \cup F_2 \neq \emptyset$, thus by (7) we get $c(F_1 \cup F_2) = c(F_1 \cap F_2) = 1$, and by (8) sets $F_1 \cap F_2, F_1 \cup F_2$ are indecomposable flats. \hfill $\square$

### 6.2. Classification when $P(M)$ is $\delta$-Gorenstein.

The following is our classification of matroids whose independence polytope is $\delta$-Gorenstein. The class of $F_3$-matroids (which appear in the classification) is constructed in Section 5.

**Theorem 6.5.** Fix an integer $\delta \geq 2$. The independence polytope $P(M)$ of a connected loopless matroid $M$ is $\delta$-Gorenstein if and only if $M$ is a $(\delta - 1)$-blow up of a connected $F_3$-matroid.

**Proof.** Suppose the independence polytope $P(M)$ of a connected loopless matroid $M$ is $\delta$-Gorenstein. Then by Theorem 6.2 the matroid $M$ is the $(\delta - 1)$-blow up of a connected loopless matroid $M'$ which satisfies $(\spadesuit)_\delta$ (1). Denote the ground set of $M'$ by $E$. Let $F$ be the set of all indecomposable flats in $M'$. Notice that, due
to (9) and (1), \( \mathcal{F} \) is a connected \( F_\delta \)-family on the set \( E \) (from Definition 5.1). Now, consider matroids \( M' \) and \( M_\mathcal{F} \) (from Definition 5.5). Clearly, both are on the same ground set \( E \). Using Theorem 5.7\(, \) we get that both matroids have the same rank, the same set of indecomposable flats \( \mathcal{F} \), and that the ranks of these indecomposable flats coincide. Therefore, \( M' = M_\mathcal{F} \). Indeed, by Lemma 5.3 the independence polytopes of both matroids are cut by the same set of halfspaces \( 0 \leq x_e \) over all \( e \in E \), and \( \sum_{e \in F} x_e \leq r(F) \) over all \( F \in \mathcal{F} \). Thus, \( P(M') = P(M_\mathcal{F}) \) and therefore \( M' = M_\mathcal{F} \) is an \( F_\delta \)-matroid.

Suppose now \( M \) is a \((\delta - 1)\)-blow up of a connected \( F_\delta \)-matroid \( M_\mathcal{F} \) (for some \( F_\delta \)-family \( \mathcal{F} \)). By Theorem 6.2\( it is enough to show that the connected matroid \( M_\mathcal{F} \) satisfies condition \((\spadesuit)_\delta \) (1). It does – by Theorem 5.7\(, \) \( \mathcal{F} \) is the set of indecomposable flats, so \((\spadesuit)_\delta \) (1) follows from (3) in Definition 5.1 and again Theorem 5.7. \( \square \)

**Example 6.6.** Fix an integer \( \delta \geq 2 \). The independence polytope of the graphic matroid of the \((\delta - 1)\)-blow up of the \((\delta + 1)\)-cycle is \( \delta \)-Gorenstein, see [18]. By Theorem 6.5\( the independence polytope of a \((\delta + 1)\)-cycle is a connected \( F_\delta \)-matroid. Indeed, the corresponding \( F_\delta \)-family consists of \( \delta + 1 \) singletons \( \{e_i\} \) and a set \( E = \{e_1, \ldots , e_{\delta + 1}\} \).

**Example 6.7.** Fix an integer \( \delta \geq 2 \). The independence polytope of the graphic matroid of the \((\delta - 1)\)-blow up of two \((\delta + 1)\)-cycles joined by an edge is \( \delta \)-Gorenstein, see [18]. By Theorem 6.5\( the independence polytope of two \((\delta + 1)\)-cycles joined by an edge is a connected \( F_\delta \)-matroid. Indeed, the corresponding \( F_\delta \)-family consists of \( 2\delta + 1 \) singletons \( \{e_i\} \), sets \( F_1 = \{e_1, \ldots , e_{\delta + 1}\} \), \( F_2 = \{e_{\delta + 1}, \ldots , e_{2\delta + 1}\} \), and a set \( E = \{e_1, \ldots , e_{2\delta + 1}\} \).

### 7. G-matroids

In this section we define a class of matroids that will play a central role in classification of \( \delta \)-Gorenstein base polytopes. We first define a special family of subsets of the ground set – a family satisfying the properties of good flats in a matroid whose base polytope is \( \delta \)-Gorenstein. Our goal is to achieve a matroid whose set of good flats coincides with the original family. We distinguish two cases \( \delta > 2 \) and \( \delta = 2 \).

#### 7.1. \( G_\delta \)-families for \( \delta > 2 \)

Fix an integer \( \delta \geq 2 \). The results of this subsection are valid also for \( \delta = 2 \) and will be used later in Subsection 7.3.

**Definition 7.1.** A family \( \mathcal{G} \) of subsets of a finite ground set \( E \) is called a \( G_\delta \)-family if it satisfies the following conditions:

1. for every \( e \in E \), \( \{e\} \in \mathcal{G} \),
2. for every \( G_i, G_j \in \mathcal{G} \), if \( G_i \cap G_j \neq \emptyset \) and \( G_i \cup G_j \neq E \), then \( G_i \cap G_j \) and \( G_i \cup G_j \) belong to \( \mathcal{G} \),
3. the cardinality of \( E \) equals \( \frac{\delta g_0}{\delta - 1} \) for some integer \( g_0 \equiv 0 \) (mod \( \delta - 1 \)),
4. for every \( G_i \in \mathcal{G} \) its cardinality equals \( \frac{\delta g_i - 1}{\delta - 1} \) for some \( g_i \equiv 1 \) (mod \( \delta - 1 \)).

\( G_\delta \)-families can be quite easily generated.

Either globally – by setting an intersection scheme satisfying condition (2) between sets \( G_i \), assigning numbers \( g_i \) in a strictly monotone way, and finally filling every set with a right number of points (so that conditions (3) and (4) hold).

Or locally – from connected \( F_\delta \)-families in some analogy how schemes (resp. manifolds) are constructed from affine schemes (resp. open discs).
DEFINITION 7.2. We say that a $G_δ$-family $\mathcal{G}$ comes from an atlas of connected $F_δ$-families $\mathcal{F}_1, \ldots, \mathcal{F}_k$ on $E_1, \ldots, E_k$ when the following conditions are satisfied:

- $\mathcal{G} = \bigcup_{i=1}^k \mathcal{F}_i$ – covering,
- $\{ A \in \mathcal{F}_i : A \subset E_i \cap E_j \} = \{ A \in \mathcal{F}_j : A \subset E_i \cap E_j \}$ – compatibility.

LEMMA 7.3. Every $G_δ$-family $\mathcal{G}$ comes from an atlas of connected $F_δ$-families.

PROOF. When $G \in \mathcal{G}$, then $\mathcal{F}_G := \{ G_i \in \mathcal{G} : G_i \subset G \}$ is clearly a connected $F_δ$-family – compare Definitions 5.1 and 7.1. Let $G_1, \ldots, G_k$ be inclusion maximal elements of $\mathcal{G}$. Then $\mathcal{G}$ comes from an atlas of connected $F_δ$-families $\mathcal{F}_{G_1}, \ldots, \mathcal{F}_{G_k}$.

Indeed, the covering condition follows from the fact that every element of $\mathcal{G}$ is contained in some $G_i$. The compatibility condition is clear, as $F_δ$-families $\mathcal{F}_{G_i}$ come from a common $G_δ$-family. $\square$

REMARK 7.4. Connected $F_δ$-families $\mathcal{F}_1, \ldots, \mathcal{F}_k$ on $E_1, \ldots, E_k$ satisfying the compatibility condition form (an atlas of) a $G_δ$-family on the ground set $E = \bigcup_{i=1}^k E_i$ by $\mathcal{G} := \bigcup_{i=1}^k \mathcal{F}_i$ if and only if the following conditions are satisfied:

- the cardinality of $E$ equals $\frac{g_0}{\delta}$ for some integer $g_0 \equiv 0 \pmod{\delta - 1}$,
- if $F_i \in \mathcal{F}_i, F_j \in \mathcal{F}_j, F_i \cap F_j \neq \emptyset$, and $F_i \cup F_j \neq E$, then $F_i \cup F_j \in \mathcal{F}_s$ for some $s$.

To define a $G_δ$-matroid we need to build some more structure on a $G_δ$-family.

For a $G_δ$-family $\mathcal{G}$ let $\tilde{\mathcal{G}}$ be the family of nonempty intersections of sets from $\mathcal{G}$.

For $F \in \tilde{\mathcal{G}}$ let $c(F)$ be the least $k$ s.t. $F$ is intersection in $E$ of $k$ sets from $\mathcal{G}$.

For $F \in \tilde{\mathcal{G}}$ let $r(F)$ be a positive number equal to $\frac{1}{\delta}((\delta - 1)|F| + c(F))$. That is, the following equality $\delta r(F) = (\delta - 1)|F| + c(F)$ holds for $F \in \tilde{\mathcal{G}}$.

We say that an intersection of sets $G_1, \ldots, G_k$ is transversal when $G_i \cup G_j = E$ for every $i \neq j$. Equivalently, an intersection of $G_1, \ldots, G_k \in \mathcal{G}$ is transversal if and only if sets $E \setminus G_1, \ldots, E \setminus G_k$ are pairwise disjoint.

Observe that every $F \in \tilde{\mathcal{G}}$ is a transversal intersection of $c(F)$ sets from $\mathcal{G}$. Indeed, if $F$ is an intersection of the least number of sets from $\mathcal{G}$, then $G_i \cup G_j = E$ for every $i \neq j$. Otherwise, if $G_i \cup G_j \neq E$, we could replace $G_i, G_j$ by $G_i \cap G_j$ which by (2) would belong to $\mathcal{G}$, resulting in a fewer number of sets.

When $F \in \tilde{\mathcal{G}}$ is a transversal intersection of $k = c(F)$ sets $G_1, \ldots, G_k \in \mathcal{G}$, then

$$\frac{1}{\delta - 1}(\delta r(F) - k) = |F| = |G_1| + \cdots + |G_k| - (k - 1)|E|$$

$$= \frac{\delta g_1 - 1}{\delta - 1} + \cdots + \frac{\delta g_k - 1}{\delta - 1} - (k - 1)\frac{\delta g_0}{\delta - 1}$$

$$= \frac{1}{\delta - 1}(\delta g_1 + \cdots + g_k - (k - 1)g_0 - k),$$

hence $r(F) = g_1 + \cdots + g_k - (k - 1)g_0$ is an integer.

Of course, $\mathcal{G} \subset \tilde{\mathcal{G}}$. Notice that for $G_i \in \mathcal{G}$ we have $c(G_i) = 1$, $r(G_i) = g_i$, and for $E \in \tilde{\mathcal{G}}$ we have $c(E) = 0$, $r(E) = g_0$. 
7.2. \textit{G}_δ\textit{-matroids for } δ \geq 2. Fix an integer δ \geq 2. The results up to Claim 7.14 are valid also for δ = 2 and will be used later in Subsection 7.3.

We present a construction of matroids corresponding to \textit{G}_δ-families. The definition is by circuits, but they can be also introduced by the rank function – using Claim 7.11.

**Definition 7.5.** Let \( \mathcal{G} \) be a \textit{G}_δ-family. Let \( \mathcal{C}_δ \) be a family of minimal sets (w.r.t. inclusion) among all \((r(F_i) + 1)\)-element subsets of \( F_i \) over all \( F_i \in \mathcal{G} \). The matroid corresponding to \( \mathcal{G} \) is a matroid on the ground set \( E \) with the set of circuits equal to \( \mathcal{C}_δ \). We denote it by \( M_δ \), and call a \textit{G}_δ-matroid.

**Remark 7.6.** A set \( A \subset E \) is independent in \( M_δ \) if and only if for every \( F_i \in \mathcal{G} \) an inequality \(|F_i \cap A| \leq r(F_i)\) holds. In particular, the rank of \( F_i \) in \( M_δ \) is less or equal to \( r(F_i) \).

The following theorem is the cornerstone of our classification of matroids whose base polytope is \( δ \)-Gorenstein for \( δ > 2 \). It shows that conditions from Definition 7.1 are not only a necessary conditions for a family of good flats (see Subsection 7.3), but also sufficient conditions.

**Theorem 7.7.** Suppose \( \mathcal{G} \) is a \textit{G}_δ-family for some \( δ > 2 \). Then, \( M_δ \) is a connected matroid in which the set of good flats (sets that are closed, connected, and their contraction is connected) is equal to \( \mathcal{G} \), every \( G_i \in \mathcal{G} \) has rank equal to \( g_i \), and the ground set \( E \) has rank equal to \( g_0 \).

We prove Theorem 7.7 in a sequence of claims, some of which before proving require some additional lemmas. The whole theorem works for \( δ > 2 \), but claims and lemmas up to Claim 7.14 are valid also for \( δ = 2 \). Starting from Lemma 7.8 an additional assumption is made, which holds always when \( δ > 2 \).

**Lemma 7.8.** For every \( F_i, F_j \in \mathcal{G} \), if \( F_i \cap F_j \neq \emptyset \), then \( F_i \cap F_j, F_i \cup F_j \in \mathcal{G} \). Moreover, the function \( r : \mathcal{G} \to \mathbb{N} \) is submodular, that is if \( F_i, F_j \in \mathcal{G} \), \( F_i \cap F_j \neq \emptyset \), then \( r(F_i \cap F_j) + r(F_i \cup F_j) \leq r(F_i) + r(F_j) \).

**Proof.** Suppose \( F_1, F_2 \in \mathcal{G} \), that is \( E \setminus F_1 = E \setminus G_1^1 \cup \cdots \cup E \setminus G_1^k \) and \( E \setminus F_2 = E \setminus G_2^1 \cup \cdots \cup E \setminus G_2^l \), where \( G_1^i, G_2^j \in \mathcal{G} \), \( c(F_1) = k \), and \( c(F_2) = l \).

Suppose that \( F_1 \cap F_2 \neq \emptyset \). Just from the definition of \( \mathcal{G} \) we have that the intersection \( F_1 \cap F_2 \in \mathcal{G} \). For the union we have an equality \( E \setminus (F_1 \cup F_2) = (E \setminus F_1) \cap (E \setminus F_2) = \bigsqcup_{i,j} (E \setminus G_1^i) \cap (E \setminus G_2^j) = \bigsqcup_{i,j} G_1^i \cup G_2^j \neq E \setminus (G_1^i \cup G_2^j) \). Since \( G_1^i \cap G_2^j \neq \emptyset \) (as \( F_1 \cap F_2 \neq \emptyset \)) and \( G_1^1 \cap G_2^1 \neq E \), we have that \( G_1^1 \cap G_2^1 \in \mathcal{G} \). Hence \( F_1 \cup F_2 \in \mathcal{G} \).

The cardinality \(|F|\) is a modular function. From \( r(F) = \frac{1}{δ}(1 + (δ - 1)|F| + c(F)) \) we get that the function \( r(F) \) is submodular if and only if the function \( c(F) \) is. We will show that \( c \) is a submodular function.

Let \( H \) be a bipartite graph with two classes of vertices \( A = \{a_1, \ldots, a_k\} \) and \( B = \{b_1, \ldots, b_l\} \). An edge joins vertices \( a_i \) and \( b_j \) if the corresponding sets \( E \setminus G_1^i \) and \( E \setminus G_2^j \) have nonempty intersection. In other words, \( H \) is the intersection graph of sets \( E \setminus G_1^1, \ldots, E \setminus G_1^k \cup E \setminus G_2^1, \ldots, E \setminus G_2^l \), as sets \( E \setminus G_1^i \) over \( i \) (and also sets \( E \setminus G_2^j \) over \( j \)) are pairwise disjoint.

Observe that \( c(F_1 \cap F_2) \) is at most the number of connected components of \( H \). Indeed, \( E \setminus (F_1 \cap F_2) = E \setminus F_1 \cup E \setminus F_2 = (E \setminus G_1^1 \cup \cdots \cup E \setminus G_1^k) \cup (E \setminus G_2^1 \cup \cdots \cup E \setminus G_2^l) \).
Whenever in this union there are two sets $E \setminus G_i, E \setminus G_j$ with nonempty intersection we replace them by their union $E \setminus (G_i \cap G_j)$ (which is also of a form $E \setminus G$ for some $G \in \mathcal{G}$, as $\emptyset \neq F_1 \cap F_2 \subset G_1 \cap G_2$). This way sets corresponding to vertices of every connected component of $H$ will become their union as edges of $H$ indicate which pairs of sets have nonempty intersection. In general, the union of sets corresponding to vertices of a connected subgraph $H'$ of $H$ is of a form $E \setminus G_{H'}$ for some $G_{H'} \in \mathcal{G}$.

Our previous bound on $c(F_1 \cup F_2)$ was $k \cdot l -$ the number of sets $(E \setminus G_1^i) \cap (E \setminus G_2^j)$. In order to get a better bound we need to cover all these sets with a fewer number of sets of a form $E \setminus G$ for $G \in \mathcal{G}$, contained in $E \setminus (F_1 \cup F_2) = (E \setminus F_1) \cap (E \setminus F_2)$. Notice that when $H'$ is a connected subgraph of $H$, $a_i \notin H'$, and $H' \cap B = \{b_{j_1}, \ldots, b_{j_s}\}$, then $E \setminus G_1^i \cap E \setminus G_{H'} = (E \setminus G_1^i) \cap (E \setminus G_{H'}^j) \cup \cdots \cup (E \setminus G_1^i) \cap (E \setminus G_{H'}^j)$. Moreover, this set is either empty, or of a form $E \setminus G$ for some $G \in \mathcal{G}$. We say that a strange pair $(a_i, H')$ (where $H'$ is a connected subgraph, $a_i \notin H'$, and $H' \cap B = \{b_{j_1}, \ldots, b_{j_s}\}$) covers $(a_i, b_{j_1}), \ldots, (a_i, b_{j_s})$. Our task is to cover all intersections $E \setminus G_1^i \cap E \setminus G_{H'}^j$. But when $a_i$ and $b_j$ are not joined by an edge, this set is empty. Hence, it is enough to cover edges of $H$ by strange pairs of types $(a_i, H')$ and $(H', b_j)$ (analogously). The number of strange pairs in a covering of the edge set of $H$ (which depends purely on the graph $H$) gives an upper bound on $c(F_1 \cup F_2)$.

We show by induction on $n$ that in every graph $G$ on $n$ vertices the sum of the number of connected components and the number of strange pairs in some covering (in the above sense) of all its edges is at most $n$. First, notice that it is enough to consider connected graphs. Second, let $v$ be a vertex which does not disconnect the graph $G$, i.e. $G \setminus v$ is connected (there always exists such a vertex). Then, a strange pair $(v, G \setminus v)$ covers all edges incident to $v$. Moreover, from the inductive assumption the connected graph $G \setminus v$ possesses a covering with at most $n - 2$ pairs. Hence, $G$ has a covering with at most $n - 1$ pairs and the inductive assertion follows.

Concluding, $c(F_1 \cup F_2) + c(F_1 \cup F_2) \leq |H| = k + l = c(F_1) + c(F_2)$. \hfill \qed

**Claim 7.9.** $M_G$ is a matroid.

**Proof.** Let $C_i$ be a circuit, i.e. a $(r(F_i) + 1)$-element subset of $F_i$, and let $C_j$ be another circuit, i.e. a $(r(F_j) + 1)$-element subset of $F_j$. Let $x \in C_i \cap C_j$. From the circuit axioms (see 3.2), it is enough to show that there exists a circuit $C_k \subset C_i \cup C_j \setminus x$. Since $x \in F_i \cap F_j$, by Lemma 7.8 we get that $F_i = F_i \cap F_j$ and $F_k = F_i \cup F_j$ belong to $\mathcal{G}$. Now, $|C_i \cap C_j| \leq r(F_i)$, as otherwise a $(r(F_i) + 1)$-element subset of $F_i$ would be a subset of $C_i$ and $C_j$ which is a contradiction ($C_i$ and $C_j$ would not be minimal). Thus, $|C_i \cup C_j| \geq r(F_i) + 1 + r(F_j) \geq r(F_i) + r(F_j) - r(F_i) + 1 \geq r(F_k) + 1$ by Lemma 7.8. It is a subset of $F_k$ of cardinality at least $r(F_k) + 1$, so it contains a circuit. \hfill \qed

Using the submodular function $r : \mathcal{G} \to \mathbb{N}$ we define a new function $\tilde{r}$ defined on all subsets of $E$ in a following way – for $A \subset E$ let

$$\tilde{r}(A) = \min\{r(F_1) + \cdots + r(F_k) : F_1, \ldots, F_k \in \mathcal{G} \text{ and } A \subset F_1 \cup \cdots \cup F_k\}.$$  

**Lemma 7.10.** The function $\tilde{r} : 2^E \to \mathbb{N}$ is

1. proper – for $X \subset E$, $0 \leq \tilde{r}(X) \leq |X|$,
2. weakly increasing – for $Y \subset X$, $\tilde{r}(Y) \leq \tilde{r}(X)$, and
3. submodular – for $X, Y \subset E$, $\tilde{r}(X \cap Y) + \tilde{r}(X \cup Y) \leq \tilde{r}(X) + \tilde{r}(Y)$.  

PROOF. Clearly, \( \tilde{r} \) is proper (by \([7,1]\)(1)) and weakly increasing. Let \( A_1, A_2 \subseteq E \). Suppose that \( \tilde{r}(A_1) = r(F^1_1) + \cdots + r(F^1_l) \) and \( \tilde{r}(A_2) = r(F^2_1) + \cdots + r(F^2_l) \), where \( F^1_1, \ldots , F^1_k, F^2_1, \ldots , F^2_l \in \mathcal{G} \), \( A_1 \subseteq F^1_1 \cup \cdots \cup F^1_k \), and \( A_2 \subseteq F^2_1 \cup \cdots \cup F^2_l \). Moreover, we can assume that sets \( F^1_1, \ldots , F^1_k \) (and also sets \( F^2_1, \ldots , F^2_l \)) are pairwise disjoint (if two sets have nonempty intersection we can replace them by their union). For submodularity of \( \tilde{r} \) we need to show that

\[
\tilde{r}(A_1 \cap A_2) + \tilde{r}(A_1 \cup A_2) \leq \tilde{r}(A_1) + \tilde{r}(A_2) = r(F^1_1) + \cdots + r(F^1_l) + r(F^2_1) + \cdots + r(F^2_l).
\]

We begin on the right side of the above formula and apply the following process – if there are two sets \( C_i, C_j \) intersecting properly (their intersection is nonempty and one is not contained in the other) we exchange them into \( C_i \cap C_j \) and \( C_i \cup C_j \). After each step all sets belong to \( \mathcal{G} \), the multiset union of all sets remains the same, and the sum of values of \( r \) on these sets weakly decreases, as by Lemma \([7,8]\) an inequality \( r(C_i \cap C_j) + r(C_i \cup C_j) \leq r(C_i) + r(C_j) \) holds. The process clearly ends, as for e.g. the sum of squares of cardinalities of the sets grows, and at the same time it is bounded. We obtain

\[
r(C_1) + \cdots + r(C_s) \leq r(F^1_1) + \cdots + r(F^1_l) + r(F^2_1) + \cdots + r(F^2_l),
\]

where \( C_1 \cup \cdots \cup C_s = F^1_1 \cup \cdots \cup F^1_k \cup F^2_1 \cup \cdots \cup F^2_l \) as multisets (notice that the multiplicity of every element is either 1, or 2), and no \( C_i, C_j \) intersect properly. Observe that the last property implies that \( C_1, \ldots , C_s \) can be split into (without loss of generality) \( C_1, \ldots , C_t \) and \( C_{t+1}, \ldots , C_s \) such that as sets

\[
C_1 \cup \cdots \cup C_t = (F^1_1 \cup \cdots \cup F^1_k) \cap (F^2_1 \cup \cdots \cup F^2_l),
\]

\[
C_{t+1} \cup \cdots \cup C_s = (F^1_1 \cup \cdots \cup F^1_k) \cup (F^2_1 \cup \cdots \cup F^2_l).
\]

Then, we get inequalities

\[
\tilde{r}(A_1 \cap A_2) \leq \tilde{r}((F^1_1 \cup \cdots \cup F^1_k) \cap (F^2_1 \cup \cdots \cup F^2_l)) \leq r(C_1) + \cdots + r(C_t),
\]

\[
\tilde{r}(A_1 \cup A_2) \leq \tilde{r}((F^1_1 \cup \cdots \cup F^1_k) \cup (F^2_1 \cup \cdots \cup F^2_l)) \leq r(C_{t+1}) + \cdots + r(C_s),
\]

and finally

\[
\tilde{r}(A_1 \cap A_2) + \tilde{r}(A_1 \cup A_2) \leq \tilde{r}(A_1) + \tilde{r}(A_2).
\]

\( \square \)

**Claim 7.11.** The rank in \( M_{\mathcal{G}} \) is given by the function \( \tilde{r} \).

**Proof.** By \([32]\) Corollary 1.3.4 a proper, weakly increasing, and submodular function is the rank function of a matroid. Thus, Lemma \([7,10]\) guarantees that \( \tilde{r} \) is the rank function of a matroid. Denote this matroid by \( M_{\mathcal{G}} \). We will show the set of circuits of \( M_{\mathcal{G}} \) coincides with the set of circuits of \( M_{\mathcal{F}} \), and therefore both matroids are the same.

Suppose \( C \) is a circuit of \( M_{\mathcal{G}} \). Then, \( C \) is a \( (r(F_i) + 1) \)-element subset of \( F_i \in \mathcal{G} \). We have \( \tilde{r}(C) \leq r(F_i) = |C| - 1 < |C| \) since \( C \subseteq F_i \). Every proper subset \( C' \subseteq C \) is independent in \( M_{\mathcal{G}} \), so if \( C' \subseteq F'_1 \cup \cdots \cup F'_k \), then \( |C'| \leq |C' \cap F'_1| + \cdots + |C' \cap F'_k| \leq r(F'_1) + \cdots + r(F'_k) \) by Remark \([7,6]\). Hence, \( |C'| \leq \tilde{r}(C') \). As a consequence, \( C \) is a circuit in \( M_{\mathcal{F}} \).

Suppose now that \( C \) is a circuit in \( M_{\mathcal{F}} \). Then, \( \tilde{r}(C) < |C| \). Suppose \( \tilde{r}(C) = r(F_1) + \cdots + r(F_k) \) for \( F_1, \ldots , F_k \in \mathcal{G} \) and \( C \subseteq F_1 \cup \cdots \cup F_k \). For every \( i \) we have \( C \cap F_i \subseteq F_i \). Thus either \( k = 1 \), or for every \( i \) the set \( C \cap F_i \subseteq C \) is not a circuit in \( M_{\mathcal{F}} \), and so \( r(F_i) \geq \tilde{r}(C \cap F_i) \geq |C \cap F_i| \). Altogether it gives
\[ r(F_1) + \cdots + r(F_k) \geq |C|, \] which is a contradiction. Therefore, \( k = 1 \). Since \( C \) is a circuit in \( M_F \), \( r(F_1) = \tilde{r}(C) = |C| - 1 \), so \( |C| = r(F_1) + 1 \). Now, if \( C \) was not minimal in the set of \( (r(F_1) + 1) \)-element subsets of \( F_1 \in \tilde{G} \), it would also be not minimal in the set \( \{ C \in E : \tilde{r}(C) < |C| \} \). Hence, \( C \) is a circuit in \( M_G \).

**Remark 7.12.** Notice that it may happen for \( F \in \tilde{G} \) that \( \tilde{r}(F) < r(F) \). It may even happen that \( |F| < r(F) \). In particular, \( F \) does not have to be the closure of a \((r(F) + 1)\)-element circuit in \( M_G \).

**Claim 7.13.** Every \( G_i \in G \) has rank equal to \( g_i \), and \( E \) has rank equal to \( g_0 \).

Moreover, for every \( \emptyset \neq X \subset E \) the rank of \( E \) is greater of equal to \( \frac{1}{\delta}((\delta - 1)|X| + 1) \).

**Proof.** For \( \emptyset \neq X \subset E \), by Claim 7.11 we have that
\[
\tilde{r}(X) = \min \{ r(F_1) + \cdots + r(F_k) : F_i \in \tilde{G}, X \subset \bigcup_i F_i \}
\]
\[
= \min \left\{ \sum_i \frac{1}{\delta}((\delta - 1)|F_i| + c(F_i)) : F_i \in \tilde{G}, X \subset \bigcup_i F_i \right\}
\]
\[
\geq \frac{1}{\delta}((\delta - 1)|X| + 1).
\]

For \( G_i \in G \) we get that \( \tilde{r}(G_i) \geq \frac{1}{\delta}((\delta - 1)|G_i| + 1) = r(G_i) = g_i \). On the other hand, by Remark 7.6 \( \tilde{r}(G_i) \leq g_i \). Hence, \( \tilde{r}(G_i) = g_i \).

A similar calculation for \( E \) gives \( \tilde{r}(E) = \frac{1}{\delta}((\delta - 1)|E| = r(E) = g_0 \).

**Claim 7.14.** The matroid \( M_G \) is connected.

**Proof.** Let \( F_1, \ldots, F_k \) be the decomposition of the ground set \( E \) into indecomposable flats. Suppose \( k > 1 \). Then, \( \emptyset \neq F_i \subset E \) and by Claim 7.13
\[
\frac{1}{\delta}((\delta - 1)|E| = \tilde{r}(E) = \tilde{r}(F_1) + \cdots + \tilde{r}(F_k)
\]
\[
\geq \frac{1}{\delta}((\delta - 1)|F_1| + 1) + \cdots + \frac{1}{\delta}((\delta - 1)|F_k| + 1) > \frac{1}{\delta}((\delta - 1)|E|).
\]

This is a contradiction. Hence, \( k = 1 \) and therefore the matroid is connected.

The remaining part of this subsection holds for \( G_\delta \)-families \( G \) satisfying an additional condition:

(5) for every \( e \in E \), \( E \setminus e \notin G \).

Notice that when \( \delta > 2 \), then (5) already follows from conditions (3) and (4). Indeed, if \( G_i = E \setminus e \in G \), then \( (\delta - 1)|E| - (\delta - 1) = (\delta - 1)|E \setminus e| = \delta g_i - 1 \) and \( (\delta - 1)|E| = \delta g_0 \). Hence, \( \delta g_0 = \delta g_i + (\delta - 1) - 1 = \delta(g_i + 1) - 2 \), which is possible for integers \( g_0, g_i \) only when \( \delta = 2 \).

Suppose now that \( \delta > 2 \), or \( \delta = 2 \) and \( E \setminus e \notin G \) for every \( e \in E \).

**Lemma 7.15.** The function \( r \) is strictly increasing, that is if \( F_i, F_j \in \tilde{G} \), \( F_i \subset F_j \), then \( r(F_i) < r(F_j) \).

**Proof.** Suppose \( F_1, F_2 \in \tilde{G} \), that is \( E \setminus F_1 = E \setminus G_1^1 \sqcup \cdots \sqcup E \setminus G_k^1 \) and \( E \setminus F_2 = E \setminus G_1^2 \sqcup \cdots \sqcup E \setminus G_k^2 \), where \( G_1^1, G_2^2 \in G \), \( r(F_1) = k \), and \( r(F_2) = l \).

Suppose \( F_1 \not\subset F_2 \). To show that \( r \) is strictly increasing we need to prove that \( (\delta - 1)|F_1| + c(F_1) < (\delta - 1)|F_2| + c(F_2) \) holds. It is equivalent to an inequality \( c(F_1) - c(F_2) < (\delta - 1)|F_2 \setminus F_1| \). Observe that the number of sets \( E \setminus G_1^1 \) intersecting
nonempty $E \setminus F$ is at most $c(F)$. Indeed, no more than one of them can intersect nonempty $E \setminus G_j$ as otherwise the union of sets which intersect nonempty $E \setminus G_j$ together with this set would be of a form $E \setminus G$ for $G \in \mathcal{G}$ allowing a disjoint decomposition of $E \setminus F$ into a fewer number of sets. The number of sets $E \setminus G_j$ contained in $F_2 \setminus F_1$ is less than $|F_2 \setminus F_1|$ as every such set has more than one element (by condition (5)). Therefore, the function $r$ is strictly increasing. \hfill \Box

Claim 7.16. In the matroid $M_\mathcal{G}$ there is the following chain of inclusions:

indecomposable flats (closed and connected sets) $\subset \tilde{\mathcal{G}} \subset$ flats (closed sets).

Proof. First, we argue that a set $F_i \in \tilde{\mathcal{G}}$ is closed. Suppose that $C$ is a circuit in $M_\mathcal{G}$ and $|C \setminus F_i| \leq 1$. Since circuits belong to the family $\mathcal{C}_\mathcal{G}$, $C$ is a $(r(F_j) + 1)$-element subset of some set $F_j \in \tilde{\mathcal{G}}$. Then, $F_k = F_i \cap F_j \in \tilde{\mathcal{G}}$ has rank at least $|C| - 1 = r(F_j)$. On the other hand, by Remark 7.15 it has rank at most $r(F_k)$. Hence, since by Lemma 7.15 the function $r$ is strictly increasing, both sets have to coincide. As a consequence $F_j \subset F_i$. Thus $C \subset F_i$.

Notice that the closure of a circuit $C$ in $M_\mathcal{G}$ that corresponds to a set $F_i \in \tilde{\mathcal{G}}$ is equal to $F_i$. Indeed, the rank of $cl(C)$ is $|C| - 1 = r(F_i)$. It is greater than or equal to the rank of $F_i$, which is its upset since $F_i$ is closed. Hence, $cl(C) = F_i$.

Finally, let $A$ be a closed set in $M_\mathcal{G}$. Consider all maximal subsets of $A$ that belong to $\tilde{\mathcal{G}}$. Denote them by $F_1, \ldots, F_k$. They are clearly disjoint (from maximality and Definition 7.1 (2)). Moreover, when $k > 1$, then sets $F_1, \ldots, F_k$ form a decomposition of $A$ into closed sets in $M_\mathcal{G}$. That is, $\tilde{r}(A) = \tilde{r}(F_1) + \cdots + \tilde{r}(F_k)$. Indeed, otherwise there exists a circuit $C \subset A$ corresponding to a set $F \in \tilde{\mathcal{G}}$, which intersects at least two sets $F_j, F_j$ among $F_1, \ldots, F_k$. Then $cl(C) = F \in \tilde{\mathcal{G}}$ also intersects $F_j, F_j$, and it is contained in $A$ (since $A$ is closed). This contradicts maximality of $F_i$, since $F_1 \cup F \cup F_j \in \tilde{\mathcal{G}}$ is a larger subset of $A$. Therefore, if a closed set $A$ is indecomposable, then $k = 1$ and $A \in \tilde{\mathcal{G}}$. \hfill \Box

Claim 7.17. The set of good flats in $M_\mathcal{G}$ (sets that are closed, connected, and their contraction is connected) is equal to $\tilde{\mathcal{G}}$.

Proof. Suppose $F$ is a good flat in $M_\mathcal{G}$. Since $F$ is an indecomposable flat, from Claim 7.16 we know that $F \in \tilde{\mathcal{G}}$. Suppose $\tilde{r}(F) = r(F_1) + \cdots + r(F_k)$ for $F_i \in \tilde{\mathcal{G}}$, with $F \subseteq \bigcup_i F_i$. Then, clearly $F = \bigcup_i F_i$. Indeed, if $F \not\subseteq F$, then it can be replaced by $F_1 \cap F$ and $r(F_1 \cap F) < r(F_i)$ by Lemma 7.16. If $F_1, F_2$ intersect nonempty, then they can be replaced by $F_1 \cup F_j$ and $r(F_1 \cup F_j) < r(F_i) + r(F_j)$. Now, $\tilde{r}(F) = \tilde{r}(F_1) + \cdots + \tilde{r}(F_k)$, and therefore $F$ decomposes into $F_1, \ldots, F_k$. Thus, $k = 1$. That is, $\tilde{r}(F) = r(F)$.

Suppose now that $c(F) = k$. That is, $F$ is a transversal intersection of $G_1, \ldots, G_k \in \mathcal{G}$. Define $F_i = \bigcap_{j \neq i} G_j$. Clearly, $\bigcup_i F_i = E \cup (k - 1)F$ as multisets. For every $i$ we have that $c(F_i) = k - 1$. Hence,

$$\tilde{r}(F_i) - \tilde{r}(F) \leq r(F_i) - r(F) = \frac{1}{\delta}((\delta - 1)|F_i| + k - 1) - \frac{1}{\delta}((\delta - 1)|F| + k) = \frac{1}{\delta}((\delta - 1)|F_i \setminus F| - 1).$$
Thus, we get that the sum of ranks of sets $F_i \setminus F$ in $M_G / F$ is less or equal to
\[
(\hat{r}(F_1) - \hat{r}(F)) + \cdots + (\hat{r}(F_k) - \hat{r}(F))
\leq \frac{1}{\delta}((\delta - 1)|F_1 \setminus F| - 1) + \cdots + \frac{1}{\delta}((\delta - 1)|F_k \setminus F| - 1)
\leq \frac{1}{\delta}((\delta - 1)|E \setminus F| - k) = r(E) - r(F) = \hat{r}(E) - \hat{r}(F).
\]

Hence, $M_G / F$ decomposes into $F_1, \ldots, F_k$. Since $F$ is a good flat, we get that $k = 1$. Thus, $F \in \mathcal{G}$.

Other way round, let $G \in \mathcal{G}$. Suppose that $G = F_1 \cup \cdots \cup F_k$ is the decomposition of $G$ into indecomposable flats (we already know that $F_1, \ldots, F_k \in \mathcal{G}$). Then, by Claim 7.13
\[
\frac{1}{\delta}((\delta - 1)|G| + 1) = \hat{r}(G) = \hat{r}(F_1) + \cdots + \hat{r}(F_k)
\geq \frac{1}{\delta}((\delta - 1)|F_1| + 1) + \cdots + \frac{1}{\delta}((\delta - 1)|F_k| + 1)
\geq \frac{1}{\delta}((\delta - 1)|G| + k).
\]

Thus, $k = 1$ and so $G$ is an indecomposable flat.

Let $A_1, \ldots, A_k$ be connected components of $M_G / G$. Suppose $k > 1$. Then,
\[
\frac{1}{\delta}((\delta - 1)|E \setminus G| - 1) = r(E) - r(G) = \hat{r}(E) - \hat{r}(G)
\geq \frac{1}{\delta}((\delta - 1)|A_1| + \cdots + \frac{1}{\delta}((\delta - 1)|A_k|)
\geq \frac{1}{\delta}((\delta - 1)|E \setminus G|),
\]
which is a contradiction. Hence, $k = 1$ and so $G$ is a good flat. \hfill \Box

**Remark 7.18.** A set $F \in \tilde{\mathcal{G}}$ may be decomposable even though $\hat{r}(F) = r(F)$.

### 7.3. $G'_2$-families and $G'_2$-matroids.

**Definition 7.19.** A family $\mathcal{G}'$ of subsets of a finite ground set $E$ is called a $G'_2$-family if it satisfies the following conditions:

1. for every $e \in E$, $\{e\} \in \mathcal{G}'$ and $E \setminus e \notin \mathcal{G}'$,
2. for every $G_i, G_j \in \mathcal{G}'$, if $G_i \cap G_j \neq \emptyset$ and $G_i \cup G_j \neq E$, then $G_i \cap G_j \in \mathcal{G}'$,
3. the cardinality of $E$ equals $2g_0$ for some integer $g_0$,
4. for every $G_i \in \mathcal{G}'$, its cardinality equals $2g_i - 1$ for some integer $g_i$.

Notice that conditions (3) and (4) are the same as in Definition 7.1 for $\delta = 2$, while in conditions (1') and (2') there is a slight difference – sets $E \setminus e$ for $e \in E$ are treated differently.

For a $G'_2$-family $\mathcal{G}'$ let $\tilde{\mathcal{G}}'$ be the family of nonempty intersections of sets from $\mathcal{G}'$, and let $\hat{r}'(F), r'(F)$ for $F \in \tilde{\mathcal{G}}'$ be defined as in Subsection 7.1.
We present a construction of matroids corresponding to a $G'_2$-family $G'$. The matroid base polytope $B$ about the union of sets in $G$ has rank equal to lemmas, remarks, and claims up to Claim 7.14 are valid also for the matroid $G$ not only the necessary conditions for a family of good flats (see Subsection 8.1), base polytope is 2-Gorenstein. It shows that conditions from Definition 7.19 are Proposition 7.21 and using Claim 7.11.

**Definition 7.20.** Let $G'$ be a $G'_2$-family. Let $C_{G'}$ be a family of minimal sets (w.r.t. inclusion) among all $(r'(F_i) + 1)$-element subsets of $F_i$ over all $F_i \in \mathcal{G}'$. The matroid corresponding to $G'$ is a matroid on the ground set $E$ with the set of circuits equal to $C_{G'}$. We denote it by $M_{G'}$, and call a $G'_2$-matroid.

**Proposition 7.21.** Suppose $G'$ is a $G'_2$-family. Then, $G = G' \cup \{E \setminus e : e \in E\}$ is a $G'_2$-family (from Definition 7.1), and the matroid $M_{G'}$ coincides with the matroid $M_{G}$ (from Definition 7.3).

**Proof.** It is straightforward that $G = G' \cup \{E \setminus e : e \in E\}$ is a $G'_2$-family, that is $G$ satisfies conditions (1) – (4) from Definition 7.1. Let $G$ be a family of nonempty intersections of sets from $G$, and let $c(F)$, $r(F)$ for $F \in G$ be defined as in Subsection 7.2.

Clearly, $G' \subset G$. Notice that on $F \in G'$ functions $c'$ and $r$, and therefore also functions $r'$ and $c$, coincide. Indeed, if $E \setminus F = E \setminus (G_1 \cup \cdots \cup G_k)$ with $G_i \in G'$, then allowing to use also sets $E \setminus (E \setminus e) = \{e\}$ cannot make $k$ smaller. Moreover, if for $F \in G$ we have $E \setminus F = E \setminus (G_1 \cup \cdots \cup G_i \cup \{e_{i+1}\} \cup \cdots \cup \{e_k\})$ with $G_i \in G'$ and $k = c(F)$, then $F' \subset F = G_1 \cap \cdots \cap G_i \in G'$ and $c(F') = l$. Therefore,

$$r(F') = \frac{1}{2}(|F'| + c(F')) = \frac{1}{2}(|F| + (k-l) + l) = \frac{1}{2}(|F| + k) = \frac{1}{2}(|F| + c(F)) = r(F).$$

Suppose $C$ is a $(r(F_i) + 1)$-element subset of $F_i$ for $F_i \in G$. Then, by the above, $C$ is also a $(r'(F'_i) + 1) = (r(F_i) + 1)$-element subset of $F'_i \supset F_i$ for $F'_i \in G'$.

The opposite follows from $G' \subset G$. Hence, $\mathcal{E}_{G'} = \mathcal{E}_G$, and finally $M_{G'} = M_G$. □

The following theorem is the cornerstone of our classification of matroids whose base polytope is 2-Gorenstein. It shows that conditions from Definition 7.19 are not only the necessary conditions for a family of good flats (see Subsection 8.1), but also sufficient conditions.

**Theorem 7.22.** Suppose $G'$ is a $G'_2$-family. Then, $M_{G'}$ is a connected matroid in which the set of good flats (sets that are closed, connected, and their contraction is connected) is equal to $G'$, every $G_i \in G'$ has rank equal to $g_i$, and the ground set $E$ has rank equal to $g_0$.

The proof of Theorem 7.22 follows the lines of the proof of Theorem 7.17. All lemmas, remarks, and claims up to Claim 7.14 are valid also for the matroid $M_G$ corresponding to the $G_2$-family $G = G' \cup \{E \setminus e : e \in E\}$. By Proposition 7.21, $M_G$ is equal to $M_{G'}$. Lemma 7.15 and Claims 7.16, 7.17 hold for $M_{G'}$ as for every $e \in E$, $E \setminus e \notin G'$ by (1') (and proofs of these lemma and claims do not use the condition about the union of sets in $G'$).

## 8. Gorenstein matroid base polytopes

### 8.1. Combinatorial reformulation when $B(M)$ is $\delta$-Gorenstein.

**Theorem 8.1.** Fix a positive integer $\delta$. Let $M$ be a connected loopless matroid. The matroid base polytope $B(M)$ is $\delta$-Gorenstein if and only if $M$ satisfies (♠)$_{\delta}$.
exists. Let an affine hyperplane $L_{\delta B}$ have the matroid base polytope.

$(B$ and the following set of inequalities is minimal defining $r$ inequalities ($\star$)

connected in $B$ show that a weight function defined by $w$ gives that $(\star)$ holds. That is, the intersection of $M$ to $G$ and contraction of $G$ in $M$ are connected.

Before we proceed to the proof of Theorem 8.1 recall a description of facets of the matroid base polytope.

Lemma 8.2 (21). Let $M$ be a connected matroid. The matroid base polytope $B(M)$ is full dimensional in the hyperplane $L := \{ x \in \mathbb{R}^E : \sum_{e \in E} x_e = r(E) \}$, and the following set of inequalities is minimal defining $B(M)$ in the hyperplane $L$:

(i) $0 \leq x_e$, for every $e \in E$ such that $M \setminus \{ e \}$ is connected,

(ii) $\sum_{e \in G} x_e \leq r(G)$, for every proper good flat $G - a$ flat $\emptyset \neq G \subseteq E$ such that: restriction of $M$ to $G$ and contraction of $G$ in $M$ are connected.

That is, the intersection of $B(M)$ with each of the supporting hyperplanes of the above half spaces is a facet of $B(M)$.

Proof of Theorem 8.1. Suppose $(\dagger)_\delta$ (0) holds, i.e. the weight function $w$ exists. Let $v$ be a lattice point given by $v_e = w(e)$. The affine hyperplane $L$ and inequalities (i), (ii) define $B(M)$, thus by multiplying by $\delta$ their constants we get an affine hyperplane $L_{\delta}$ and inequalities $(\dagger)_\delta$ defining the dilated polytope $\delta B(M)$. Now, conditions $(\dagger)_\delta$ give that $v \in \delta B(M)$. Further, we claim that both inequalities $(\dagger)_\delta$, $(\star)_\delta$ provide reduced equations of the facets of $\delta B(M)$. Condition $(\dagger)_\delta$ (0) gives that $(\dagger)_\delta$ evaluated at $v$ is equal to 1, and condition $(\dagger)_\delta$ (2) gives that $(\star)_\delta$ evaluated at $v$ is equal to 1. Therefore, the polytope $B(M)$ is $\delta$-Gorenstein.

Conversely, suppose the polytope $B(M)$ is $\delta$-Gorenstein. Therefore, there exists a lattice point $v \in \delta B(M)$ such that $\delta B(M) - v$ is a reflexive polytope. We will show that a weight function defined by $w(e) = v_e$ satisfies conditions $(\dagger)_\delta$. First, $v \in \delta B(M)$, so $\sum_{e \in E} v_e = \delta r(E)$ (((1) holds). By $(\star)_\delta$ for every good flat $G$ the equation $\sum_{e \in G} x_e = \delta r(G)$ is a reduced equation of a supporting hyperplane to $\delta B(M)$. Thus, $\sum_{e \in G} v_e + 1 = \delta r(G)$ (((2) holds). Now, if $M \setminus e$ is connected, then by $(\star)_\delta$ the equation $0 = x_e$ is a reduced equation of a supporting hyperplane to $\delta B(M)$. Hence $v_e = 1$ (the first part of (0) holds). Otherwise, if $M \setminus e$ is not connected, then by Lemma 8.3 $M/e$ is connected and so $\{ e \}$ is a good flat. The corresponding supporting hyperplane to $\delta B(M)$ is $x_e \leq \delta r(e) = \delta$. Thus, $v_e + 1 = \delta$ (the second part of (0) holds).

We show that the weight function $w : E \to \{ 1, \delta - 1 \}$ from Theorem 8.3 is already defined by $(\dagger)_\delta$ (0) for every $e \in E$.

Lemma 8.3. Suppose $M$ is a connected matroid. Then for every element $e$ of the ground set, its deletion $M \setminus e$ is connected or its contraction $M/e$ is connected.

Proof. Suppose that $M \setminus e$ is not connected and decomposes into connected components $A_1 \cup \cdots \cup A_k$ for $k \geq 2$. Since $M$ is connected $e$ is not a coloop, hence $r(M) = r(A_1) + \cdots + r(A_k)$. Moreover, for every proper subset of the set of $A_i$’s we have $r(e \cup A_1 \cup \cdots \cup A_j) = r(e) + r(A_i \cup \cdots \cup A_k)$ as otherwise $e$ would be in the closure of a proper subset of $A_i$’s and $M$ would be not connected. In particular, $r(e \cup A_i) = r(e) + r(A_i)$ so there is no circuit between $e$ and $A_i$, thus every $A_i$ is connected in $M/e$ as it was in $M$.
Suppose now that $M/e$ decomposes. This forms a decomposition of the set of $A_i$'s into proper subsets $A_i, \ldots, A_k$. We have a contradiction

$$r_M(E) - 1 = r_{M/e}(E \setminus e) = r_{M/e}(A_{i_1} \cup \cdots \cup A_{i_l}) + r_{M/e}(A_{j_1} \cup \cdots \cup A_{j_k})$$

$$= r_M(A_{i_1} \cup \cdots \cup A_{i_l}) + r_M(A_{j_1} \cup \cdots \cup A_{j_k}) = r_M(E).$$

**Lemma 8.4.** Let $M$ be a connected matroid, let $\emptyset \neq F \subseteq E$ be an indecomposable flat, and let $F_1, \ldots, F_k$ be connected components of $M/F$. Then, $E \setminus F_1, \ldots, E \setminus F_k$ are good flats.

**Proof.** From the closure properties $cl_M(E \setminus F_i) = cl_{M/F}((E \setminus F_i) \setminus F) \cup F$. The set $(E \setminus F_i) \setminus F$ is a union of connected components in $M/F$, so it is closed. Hence, $E \setminus F_i$ is closed in $M$, i.e. it is a flat.

The set $F_i \cup F$ is connected in $M$. Indeed, suppose contrary $F_i \cup F = C \cup D$. The set $F$ is indecomposable, so without loss of generality we have $F \subset C$. Now, $F_i = C \setminus F \cup D$ in $M/F$, so it is a decomposition of a connected component (which is not possible), unless $C = F$. But if $C = F$, then $M$ decomposes into $D$ and $E \setminus D$ contradicting connectivity of $M$. Now, $E \setminus F_i$ is a union of connected sets $F_j \cup F$ (for $j \neq i$) with nonempty intersection $F$, hence it is also connected.

The matroid $M/(E \setminus F_i)$ is isomorphic to $M/F|_{E_i}$, hence it is connected.

Concluding, $E \setminus F_1, \ldots, E \setminus F_k$ are good flats in $M$. □

**Lemma 8.5.** The matroid base polytope of $M$ is $\delta$-Gorenstein if and only if the matroid base polytope of $M^*$ is $\delta$-Gorenstein (with weights 1 and $\delta - 1$ reversed).

**Proof.** Recall that $B(M^*) = (1, \ldots, 1) - B(M)$. Hence, $B(M)$ and $B(M^*)$ are isomorphic as lattice polytopes. Moreover, $M \setminus e$ is connected if and only if $(M \setminus e)^* = M^*/e$ is connected. □

For the remaining part of this subsection let $\delta \geq 2$ be a fixed integer, and let $M$ be a fixed matroid satisfying conditions (\(\bullet\))$_{\delta}$ (0), (1), (2).

For every $A \subset E$ define

$$c(A) := \delta r(A) - w(A).$$

It is straightforward to check that the function $c$ has the following properties:

1. $c(A) = c(cl(A)) + w(cl(A) \setminus A)$,
2. $c(A) \geq c(cl(A))$,
3. $c(A) = c(A_1) + \cdots + c(A_k)$ if $A_1, \ldots, A_k$ are connected components of $A$,
4. $c(A) = \frac{1}{2}c(A_1) + \cdots + c(A_k)$ if $A_1 \setminus A, \ldots, A_k \setminus A$ are connected components of $M/A$ (we assume here that $A_i$ contains $A$) – using (1),
5. $c(A) = c(E \setminus A_1) + \cdots + c(E \setminus A_k)$ if $A_1, \ldots, A_k$ are connected components of $M/A$ (we assume here that every $A_i$ is disjoint from $A$) – using (1),
6. $c(A \cup B) + c(A \cap B) \leq c(A) + c(B)$ – using submodularity of rank function.

Moreover, using (\(\bullet\))$_{\delta}$ we prove that the function $c$ characterizes good flats:

1. $c(A) \geq 0$,
2. $c(A) = 0$ if and only if $A = E$ or $A = \emptyset$,
3. $c(A) = 1$ if and only if $A$ is a good flat or $A = E \setminus e$ and $w(e) = 1$. 
Indeed, suppose \( A \neq \emptyset, E \). By (3) \( c(A) = c(d(A)) + w(c(A) \setminus A) \). If \( F = c(A) \neq E \), then let \( F_1, \ldots, F_k \) be its decomposition. By (5) we have \( c(F) = c(F_1) + \cdots + c(F_k) \). Now, let \( F_i^1, \ldots, F_i^l_i \) be connected components of \( M/F_i \). Then, by Lemma 8.4 every \( E \setminus F_i^j \) is a good flat, and therefore by (2) we have \( c(E \setminus F_i^j) = 1 \). By (7) \( c(F_i) = c(E \setminus F_i^j) + \cdots + c(E \setminus F_i^{1}) = l_i \geq 1 \). Hence, \( c(A) \geq 1 \), and following these inequalities we get also (11).

\textbf{Lemma 8.6.} Good flats satisfy the following properties:

\begin{align*}
(12) & \text{ if } G_1, G_2 \text{ are good flats and } \begin{cases} G_1 \cap G_2 \neq \emptyset \\ G_1 \cup G_2 \neq E \end{cases} \text{ then } G_1 \cap G_2 \text{ is a good flat,} \\
(13) & \text{ if } G_1, G_2 \text{ are good flats and } \begin{cases} G_1 \cap G_2 \neq \emptyset \\ G_1 \cup G_2 \neq E \end{cases} \text{ then } G_1 \cup G_2 \text{ is a good flat.}
\end{align*}

\textbf{Proof.} From (8), (9), and (11) we get that

\[ 0 \leq c(G_1 \cup G_2) + c(G_1 \cap G_2) \leq c(G_1) + c(G_2) = 2. \]

If \( G_1 \cap G_2 \neq \emptyset \) and \( G_1 \cup G_2 \neq E \), then from (10) we know that \( 1 \leq c(G_1 \cap G_2) \) and \( 1 \leq c(G_1 \cup G_2) \). Hence \( c(G_1 \cap G_2) = c(G_1 \cup G_2) = 1 \). Thus from (11) the set \( G_1 \cap G_2 \) is a good flat. Also from (11) if \( G_1 \cup G_2 \neq E \) \( e \) or \( w(e) \neq 1 \) (for every \( e \)), then \( G_1 \cup G_2 \) is a good flat. \( \square \)

\textbf{8.2. Classification when } \( B(M) \text{ is } \delta\text{-Gorenstein for } \delta > 2, \)

We call a set of \( s \) elements \( \{e_1, \ldots, e_s\} \) of the ground set of a matroid an \( s\text{-ear} \) if every circuit of the matroid contains either none of these elements, or all of them.

\textbf{Definition 8.7.} A matroid \( M \) with an element \( e \in E \) replaced by an \( s\text{-ear} \) is the matroid \( M \) modified in the following way:

- the ground set \( E \) is enlarged by new elements \( e_2, \ldots, e_s \),
- in the set of circuits, every circuit containing \( e \) is replaced by a circuit containing \( e = e_1, e_2, \ldots, e_s \).

Clearly, \( \{e_1, \ldots, e_s\} \) is an \( s\text{-ear} \) in the above matroid.

This operation is a composition of \( s - 1 \) operations known as \textit{series extension}, see \[32\]. We can also define a matroid with an element replaced by an \( s\text{-ear} \) as a composition of better known operations. Let \( M^* \) be a matroid \( M^* \) with added \( (s - 1) \) elements \( e_2, \ldots, e_s \) parallel to \( e = e_1 \). Then, \( M' = (M^*)^* \) is the matroid \( M \) with \( e \) replaced by an \( s\text{-ear} \). Notice that when the matroid \( M \) is graphic, then it is just the replacement of an edge \( e \) by a path of \( s \) edges \( e = e_1, \ldots, e_s \) in \[18\], we called it an \( s\text{-ear} \) because it looks like an ear.

\textbf{Definition 8.8.} Let \( M \) be a matroid with an \( s\text{-ear} \) \( \{e_1, \ldots, e_s\} \). A matroid \( M \) with contracted \( s\text{-ear} \) \( \{e_1, \ldots, e_s\} \) is the matroid \( M/\{e_2, \ldots, e_s\} \).

In other words, a matroid with a contracted \( s\text{-ear} \) is the matroid with contracted all but one elements of that ear. Notice that operations from Definitions \[8.7\] and \[8.8\] are inverse to each other.

\textbf{Proposition 8.9.} Fix an integer \( \delta > 2 \). Let \( M \) be a connected matroid satisfying conditions (\( \bullet \))\( \delta \). Suppose \( w_M(e) = 1 \) for an element \( e \) of the ground set. Then the matroid \( M \) with an element \( e \) replaced by a \((\delta - 1)\text{-ear} \) \( \{e = e_1, \ldots, e_{\delta-1}\} \) is
connected, satisfies conditions \((\clubsuit)_\delta\), and the weight of every \(e_i\) equals \(\delta - 1\). Moreover, when \(M\) is a connected matroid satisfying conditions \((\clubsuit)_\delta\), then the matroid \(M\) with a contracted \((\delta - 1)\)-ear is also connected and satisfies conditions \((\clubsuit)_\delta\).

**Proof.** Suppose \(w_M(e) = 1\) for some \(e\) (so \(M \setminus e\) is connected and \(M/e\) is not connected). Then \(w_{M^*}(e) = \delta - 1\) (so \(M^* \setminus e\) is not connected and \(M^*/e\) is connected). By Lemma 8.5 the dual matroid \(M^*\) also satisfies \((\clubsuit)_\delta\). Let \(M''\) be the matroid \(M^*\) with added \((\delta - 2)\) parallel elements \(e_2, \ldots, e_{\delta-1}\) to \(e = e_1\) so that the set consisting of \(e\) and all its parallel elements have cardinality \(\delta - 1\). Now, since \(\delta - 1 > 1\), \(M'' \setminus e_i\) is connected and \(M''/e_i\) is not connected. Hence, we set \(w_{M''}(e_i) = 1\). It is easy to check that good flats \(G'\) in \(M''\) correspond to good flats \(G\) in \(M^*\) via the rules that if \(e \in G\) then \(e_1, \ldots, e_{\delta-1}\) \(\in G',\) and if \(e_i \in G'\) then \(e \in G\). Taking into account weights, \(M''\) satisfies \((\clubsuit)_\delta\). Let \(M'\) be the dual matroid to \(M''\), which by Lemma 8.5 satisfies \((\clubsuit)_\delta\). Clearly, \(M'\) is equal to the matroid \(M\) with an element \(e\) replaced by a \((\delta - 1)\)-ear \(\{e = e_1, \ldots, e_{\delta-1}\}\), and \(w_{M'}(e_i) = \delta - 1\) for every \(e_i\). The opposite implication goes analogously. \(\square\)

**Proposition 8.10.** Fix an integer \(\delta > 2\). Let \(M\) be a connected loopless matroid satisfying conditions \((\clubsuit)_\delta\). Then there exists a connected loopless matroid \(M'\) satisfying conditions \((\clubsuit)_\delta\) with all weights equal to \(\delta - 1\), such that \(M\) is equal to \(M'\) with contracted some \((\delta - 1)\)-ears.

**Proof.** Applying Proposition 8.9 to all elements of weight 1 in \(M\) we get a matroid \(M'\) satisfying conditions \((\clubsuit)_\delta\) with all weights equal to \(\delta - 1\). The opposite procedure (to get back from \(M'\) to \(M\)) is by contractions from Definition 8.5. \(\square\)

The following is our classification of matroids whose base polytope is \(\delta\)-Gorenstein, for \(\delta > 2\). The class of \(G_\delta\)-matroids (which appear in the classification) is constructed in Section 7.

**Theorem 8.11.** Fix an integer \(\delta > 2\). The base polytope \(B(M)\) of a connected loopless matroid \(M\) is \(\delta\)-Gorenstein if and only if \(M\) is a \(G_\delta\)-matroid with contracted some \((\delta - 1)\)-ears.

**Proof.** By Theorem 8.1 the base polytope \(B(M)\) of a connected loopless matroid \(M\) is \(\delta\)-Gorenstein if and only if \(M\) satisfies conditions \((\clubsuit)_\delta\).

Suppose a connected loopless matroid \(M\) satisfies conditions \((\clubsuit)_\delta\). Due to Proposition 8.10 there exists a connected loopless matroid \(M'\) satisfying conditions \((\clubsuit)_\delta\) with all weights equal to \(\delta - 1\), such that \(M\) is equal to \(M'\) with contracted some \((\delta - 1)\)-ears. It is enough to prove that \(M'\) is a \(G_\delta\)-matroid. Denote the ground set of \(M'\) by \(E\). Notice that the set \(G\) of good flats in \(M'\) is a \(G_\delta\)-family on the set \(E\) (from Definition 7.1). Indeed, since all weights in \(M'\) are equal to \(\delta - 1\), for every \(e \in E\) the contraction \(M'/e\) is connected and therefore \(\{e\}\) is a good flat in \(M'\) — condition (1) holds. Condition (2) follows from Lemma 8.6 and the fact that all weights are equal to \(\delta - 1 > 1\). Condition (3) follows from \((\clubsuit)_\delta\) (1) and weight \(\equiv \delta - 1\). And, finally condition (4) follows from \((\clubsuit)_\delta\) (2) and weight \(\equiv \delta - 1\). Now, consider matroids \(M'\) and \(M_G\) (from Definition 7.3). Clearly, both are on the same ground set \(E\). Using Theorem 7.7 we get that both matroids have the same rank, the same set of good flats \(G\), and that the ranks of these good flats coincide. Therefore, \(M' = M_G\). Indeed, base polytopes of both matroids are contained in the same affine hyperplane (defined by the rank), and by Lemma 8.2 both are cut
by the same set of halfspaces \( 0 \leq x_e \) over all \( e \in E \), and \( \sum_{e \in G} x_e \leq r(G) \) over all \( G \in \mathcal{G} \). Thus, \( B(M') = B(M_G) \) and therefore \( M' = M_G \).

Suppose now \( M \) is a \( G_\delta \)-matroid \( M_G \) (for some \( G_\delta \)-family \( \mathcal{G} \)) with contracted some \( (\delta - 1) \)-ears. By the second part of Proposition 8.9 it is enough to show that the connected loopless matroid \( M_G \) satisfies conditions \((\dagger)_\delta\). It does – by Theorem 7.7 \( \mathcal{G} \) is the set of good flats, so by Definition 7.1 (1), for every \( e \in E \) the set \( \{ e \} \) is a good flat, and so all weights are equal to \( \delta - 1 \). Now, equations \((\dagger)_\delta\) (1) and (2) follow from conditions (3) and (4) of the \( G_\delta \)-family \( \mathcal{G} \).

\[
\text{Example 8.12. Fix an integer } \delta > 2. \text{ The base polytope of the graphic matroid of the } \delta \text{-cycle is } \delta \text{-Gorenstein, see } 18. \text{ By Theorem } 8.7 \text{ it is a } G_\delta \text{-matroid with contracted some } (\delta - 1) \text{-ears. Indeed, it is a } G_\delta \text{-matroid corresponding to a } G_\delta \text{-family consisting of } \delta \text{ singletons } \{ e_i \} \text{ on a set } E = \{ e_1, \ldots, e_\delta \}.
\]

\[
\text{Example 8.13. Fix an integer } \delta > 2. \text{ The base polytope of the graphic matroid of the } \delta - 1 \text{ disjoint } \delta \text{-cycles joined by an edge is } \delta \text{-Gorenstein, see } 18. \text{ By Theorem } 8.11 \text{ it is a } G_\delta \text{-matroid with contracted some } (\delta - 1) \text{-ears. Indeed, it is a } G_\delta \text{-matroid corresponding to a } G_\delta \text{-family on a set } E = \{ e_{i,1}, \ldots, e_{i,\delta - 1} \} \text{ consisting of } \delta (\delta - 1) \text{-ears } \{ e_{i,1}, \ldots, e_{i,\delta - 1} \} \text{ with contracted one } (\delta - 1) \text{-ear } \{ e_{1,1}, \ldots, e_{1,\delta - 1} \}.
\]

8.3. Classification when \( B(M) \) is 2-Gorenstein. The following is our classification of matroids whose base polytope is 2-Gorenstein. The class of \( G_2 \)-matroids (which appear in the classification) is constructed in Section 7.

\[
\text{Theorem 8.14. The base polytope } B(M) \text{ of a connected loopless matroid } M \text{ is } 2\text{-Gorenstein if and only if } M \text{ is a } G_2 \text{-matroid.}
\]

\[
\text{Proof. By Theorem 8.1 the base polytope } B(M) \text{ of a connected loopless matroid } M \text{ is } 2\text{-Gorenstein if and only if } M \text{ satisfies conditions } (\dagger)_2.
\]

Suppose a connected loopless matroid \( M \) satisfies conditions \((\dagger)_2\). Denote the ground set of \( M \) by \( E \). Let \( \mathcal{G} \) be the set of all good flats in \( M \). Notice that the set \( \mathcal{G}' = \mathcal{G} \cup \{ \{ e \} : e \in E \} \) is a \( G_2 \)-family on the set \( E \) (from Definition 7.19). Indeed, since \( M \) is connected no set \( E \setminus e \) is a flat in \( M \) – condition (1') holds. Condition (2') follows from Lemma 8.1 Conditions (3) and (4) follow from \((\dagger)_2\) (1) and (2) and the fact that all weights are equal to 1. Now, consider matroids \( M \) and \( M_{\mathcal{G}'} \) (from Definition 7.20). Clearly, both are on the same ground set \( E \). Using Theorem 7.22 we get that both matroids have the same rank, the same set of good flats larger than singletons, namely \( \mathcal{G} \setminus \{ \{ e \} : e \in E \} = \mathcal{G}' \setminus \{ \{ e \} : e \in E \} \), and that the ranks of these good flats coincide. Therefore, \( M = M_{\mathcal{G}} \). Indeed, base polytopes of both matroids are contained in the same affine hyperplane (defined by the rank), and by Lemma 8.2 both are cut by the same set of halfspaces \( 0 \leq x_e \) and \( x_e \leq 1 \) over all \( e \in E \), and \( \sum_{e \in G} x_e \leq r(G) \) over all \( G \in \mathcal{G} \setminus \{ \{ e \} : e \in E \} \). Recall that when a good flat is a singleton, then the corresponding supporting hyperplane is \( x_e \leq r(e) = 1 \). Thus, \( B(M) = B(M_{\mathcal{G}'}) \) and therefore \( M = M_{\mathcal{G}'} \).

Suppose now \( M \) is a \( G_2 \)-matroid \( M_G \) (for some \( G_2 \)-family \( \mathcal{G} \)). By Theorem 7.22 the set of good flats of the matroid \( M_G \) is equal to \( \mathcal{G} \). It is easy to verify that the connected loopless matroid \( M_G \) satisfies conditions \((\dagger)_2\). Indeed, equations \((\dagger)_2\) (1) and (2) follow from conditions (3) and (4) of the \( G_\delta \)-family \( \mathcal{G} \).
Example 8.15. The base polytope of the graphic matroid of the clique $K_4$ is 2-Gorenstein, see [18]. By Theorem 8.14 it is a $G'_2$-matroid. Indeed, it is a $G'_2$-matroid corresponding to a $G'_2$-family on the set \{$e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}$\} of edges of $K_4$, consisting of six singletons, and four 3-element sets corresponding to triangles in the clique $K_4$.

Example 8.16. The base polytope of the graphic matroid of two cliques $K_4$ joined by an edge which is removed is 2-Gorenstein, see [18]. By Theorem 8.14 it is a $G'_2$-matroid. Indeed, it is a $G'_2$-matroid corresponding to a $G'_2$-family on the set \{$e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}$\}, consisting of ten singletons, four 3-element sets \{$e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}$\}, two 5-element sets \{$e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}$\}, and four 7-element sets \{$e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}$\}, \{$e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}$\}, \{$e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}$\}, \{$e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}$\}.

References
[1] F. Ardila, C. Benedetti, J. Doker, Matroid Polytopes and their Volumes, Discrete Comput. Geom. 43 (2010), 841-854.
[2] V. Batyrev, Dual polyhedra and mirror symmetry for Calabi-Yau hypersurfaces in toric varieties, J. Alg. Geom. 3 (1994), 493-535.
[3] V. Batyrev, B. Nill, Combinatorial aspects of mirror symmetry, Integer points in polyhedra–geometry, number theory, representation theory, algebra, optimization, statistics, 35-66, Contemp. Math., 452, Amer. Math. Soc., Providence, RI, 2008.
[4] M. Beck, C. Haase, S.V. Sam, Grid Graphs, Gorenstein Polytopes, and Domino Stackings, Graphs Combin. 25 (2009), 409-426.
[5] W. Bruns, J. Gubeladze, Polytopes, rings, and K-theory. Vol. 27, Springer, Dordrecht, 2009.
[6] W. Bruns, T. Römer, h-vectors of Gorenstein polytopes, J. Combin. Theory, Series A 114 (2005), 437-468.
[7] F. Castillo, F. Liu, B. Nill, A. Paffenholz, Smooth polytopes with negative Ehrhart coefficients, J. Combin. Theory, Series A 160 (2014), no. 4, Paper 4.28.
[8] D. Cox, C. Haase, T. Hibi, A. Higashitani, Integer decomposition property of dilated polytopes, Electron J. Combin. 21 (2014), no. 4, Paper 4.28.
[9] D. Cox, J. Little, H. Schenck, Toric Varieties, Grad. Stud. Math., Vol. 124, American Mathematical Society, Providence, 2011.
[10] A. Dickenstein, S. Di Rocco, R. Piene, Classifying smooth lattice polytopes via toric fibrations, Adv. Math. 222 (2009), 240-254.
[11] E.M. Feichtner, B. Sturmfels, Matroid polytopes, nested sets and Bergman fans, Port. Math. 62 (2005), 437-468.
[12] W. Fulton, Introduction to Toric Varieties, Ann. of Math. Stud., Vol. 131, Princeton University Press, Princeton, 1993.
[13] I. Gelfand, R. Goresky, R. MacPherson, V. Serganova, Combinatorial geometries, convex polyhedra, and Schubert cells, Adv. Math. 63 (1987), 301-316.
[14] D. Gijswijt, G. Regts, Polychedra with the Integer Carathéodory Property, J. Combin. Theory Ser. B 102 (2012), 62-70.
[15] J. Herzog, T. Hibi, Discrete Polymatroids, J. Algebraic Combin. 16 (2002), 239-268.
[16] T. Hibi, Dual polytopes of rational convex polytopes, Combinatorica 12 (1992), 237-240.
[17] T. Hibi, A. Higashitani, H. Ohsugi, Roots of Ehrhart polynomials of Gorenstein Fano polytopes, Proc. Amer. Math. Soc. 139 (2011), 3727-3734.
[18] T. Hibi, M. Lasoń, K. Matsuda, M. Michałek, M. Vodicka, Gorenstein graphic matroids, to appear in Israel J. Math., [arXiv:19_05_0418]
[19] T. Hibi, K. Matsuda, H. Ohsugi, K. Shibata, Centrally symmetric configurations of order polytopes, J. Algebra 443 (2015), 469-478.
[20] A. Higashitani, M. Kummer, M. Michałek, Interlacing Ehrhart polynomials of reflexive polytopes, Sel. Math. New Ser. 23 (2017), 2977-2998.
[21] S. Kim, Flag enumerations of matroid base polytopes, J. Combin. Theory, Series A 117 (2010), 928-942.
[22] K. Knauer, private communication.
[23] M. Kölbl, Gorenstein graphic matroids from multigraphs, [arXiv:1912.03862](https://arxiv.org/abs/1912.03862).
[24] M. Kreuzer, H. Skarke, Complete classification of reflexive polyhedra in four dimensions, Adv. Theor. Math. Phys. 4 (2000), 1209-1231.
[25] M. Lasoń, M. Michałek, On the toric ideal of a matroid, Adv. Math. 259 (2014), 1-12.
[26] M. Lasoń, List coloring of matroids and base exchange properties, European J. Combin. 49 (2015), 265-268.
[27] M. Lasoń, On the toric ideals of matroids of a fixed rank, [arXiv:1601.08199](https://arxiv.org/abs/1601.08199).
[28] T. Matsui, A. Higashitani, Y. Nagazawa, H. Ohsugi, T. Hibi, Roots of Ehrhart polynomials arising from graphs, J. Algebr. Comb. 34 (2011), 721-749.
[29] B. Nill, J. Schepers, Gorenstein polytopes and their stringy E-functions, Math. Ann. 355 (2013), 457-480.
[30] H. Ohsugi, Gorenstein cut polytopes, European J. Combin. 38 (2014) 122-129.
[31] H. Ohsugi, T. Hibi, Special simplices and Gorenstein toric rings, J. Combin. Theory, Series A 113 (2006), 718-725.
[32] J.G. Oxley, Matroid Theory, Oxford Science Publications, Oxford University Press, Oxford, 1992.
[33] R. Sanyal, C. Stump, Lipschitz polytopes of posets and permutation statistics, J. Combin. Theory, Series A 158 (2018), 605-620.
[34] H. Sato, Toward the classification of higher-dimensional toric Fano varieties, Tohoku Math. J. 52 (2000), 383-413.
[35] B. Sturmfels, Gröbner Bases and Convex Polytopes, Univ. Lecture Series 8, American Mathematical Society, Providence, 1995.
[36] N. White, The basis monomial ring of a matroid, Adv. Math. 24 (1977), 292-297.
[37] N. White, A unique exchange property for bases, Linear Algebra and its App. 31 (1980), 81-91.