RIGIDITY OF NON-NEGATIVELY CURVED METRICS
ON OPEN FIVE-DIMENSIONAL MANIFOLDS

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Abstract. As the first step in the direction of the Hopf conjecture on the non-existence of metrics with positive sectional curvature on $S^2 \times S^2$ the authors of [GT] suggested the following (Weak Hopf) conjecture (on the rigidity of non-negatively curved metrics on $S^2 \times R^3$): "The boundary $S^2 \times S^2$ of the $S^2 \times B^3 \subset S^2 \times R^3$ with an arbitrary complete metric of non-negative sectional curvature contains a point where a curvature of $S^2 \times S^2$ vanish". In this note we verify this.

More "flats" in $M^5$

Let $(M^n, g)$ be a complete open Riemannian manifold of non-negative sectional curvature. Remind that as follows from [CG] and [P] an arbitrary complete open manifold $M^n$ of non-negative sectional curvature contains a closed absolutely convex and totally geodesic submanifold $\Sigma$ (called a soul) such that the projection $\pi : M \rightarrow \Sigma$ of $M$ onto $\Sigma$ along geodesics normal to $\Sigma$ is well-defined and is a Riemannian submersion.\(^1\) The (vertical) fibers $W_P = \pi^{-1}(P), P \in \Sigma$ of $\pi$ define a metric foliation in $M$ and two distributions: a vertical $V$ distribution of subspaces tangent to fibers and a horizontal distribution $H$ of subspaces normal to $V$. For an arbitrary point $P$ on $\Sigma$, an arbitrary geodesic $\gamma(t)$ on $\Sigma$ and arbitrary vector field $V(t)$ which is parallel along $\gamma$ and normal to $\Sigma$ the following

\[ \Pi(t, s) = exp_{\gamma(t)} sV(t) \]

are totally geodesic surfaces in $M^n$ of zero curvature, i.e., flats.

Sometimes, these are the only directions of zero curvature in open $M^n$ (e.g., when $M^4$ is the tangent bundle to the two-dimensional sphere with the Cheeger-Gromoll metric, see [M2]). The objective of this note is to verify the (Weak Hopf) conjecture from [GT] and to point to more directions of zero curvature in our particular case of a five-dimensional $M$. The following statement is true.

\(^1\) of class $C^{1,1}$. Some additional efforts should be made to verify that $\pi$ is of the same class of smoothness as $M$ in order to have O’Neill’s fundamental equations for Riemannian submersions, see Appendix A below for this and some other results.
Theorem A. There does not exist a metric of nonnegative curvature on $M^5$ for which the boundary of a small metric tube about the soul has positive curvature in the induced metric.

Clearly, the only difficult case in the Theorem A is of two-dimensional $\Sigma$ diffeomorphic to $S^2$. For other cases of $\text{codim}(\Sigma) = 1, 2$ or 4, or two-dimensional and non-orientable $\Sigma$ or torus might be easily treated or by going to the oriented covering, or by applying "the straight line splitting off" theorem by Toponogov. Note also that unlike [GT] we are not assuming that the normal bundle of the soul is topologically trivial.\(^2\)

The proof of Theorem A is based on the consideration of some family of holonomy operators in $M$.\(^3\) More precisely, we consider a disk $\Omega$ in $\Sigma$ bounded by a curve $\omega$, construct a smooth homotopy $\omega_x$ of this curve to a point and consider the family of parallel transports $I_{\omega_x}$ along $\omega_x$ acting on vectors normal to $\Sigma$. Our construction heavily depends on $\text{dim}(\Sigma) = 2$ and $\text{codim}(\Sigma) = 3$ conditions which makes its generalizations to higher dimensions difficult.

The proof of the Theorem A is given in the section 5 after the construction of the family of holonomy operators in the section 1, curvature calculations in the section 2, constructions of the local (and the global) parallel sections in the section 3 (and 4 correspondingly).

1. The Holonomy and the O'Neill's $A$-tensor

Let $\Omega$ be a disk in two-dimensional sphere $\Sigma$ bounded by a closed curve $\omega$. According to the construction given below (see subsection 1.3) $\Omega = \{\omega_x(y)\mid 0 < x < 1\} \cup \{O\}$, where $O$ is some interior point ("center"), the boundary curve $\omega = \partial \Omega$ equals $\omega_1$, $\omega_2(y), 0 \leq y \leq 2\pi$ is the family of closed curves such that $\{x, y\mid 0 < x \leq 1, 0 < y < 2\pi\}$ are ("polar-type") "coordinates" in $\Omega\{O\}$. The point with "coordinates" $\{x, y\}$ we denote by $P(x, y)$, and do not assume that the correspondence $(x, y) \rightarrow P(x, y)$ is one-to-one. We also assume that the parameter $y$ on $\omega_x$ is proportional to the arc-length. Let $X(x, y)$ and $Y(x, y)$ be an orthonormal base of $TP_{(x, y)}\Sigma$ with positive orientation such that unit $Y(x, y)$ has the same directions as $\omega_x(y) = \partial P(x, y)/\partial y$.

Fix some positive $s_0$ smaller than a focal radius of $\Sigma$ in $M$. For some $s < s_0$ denote by $N\Sigma(s)$ the boundary of an $s$-neighborhood of $\Sigma$. Due to our choice it is a smooth manifold. It consists of all points $Q(P, V) = exp_P(sV)$, where $P$ is a point on $\Sigma$ and $V$ is a unit vector normal to $\Sigma$ at $P$. A unit normal $V(Q)$ to hyper-surface $N\Sigma(s)$ at $Q(P, V)$ is the parallel translation of $V$ from $P$ to $Q$ along a vertical geodesic $exp_V(sV), 0 \leq s' \leq s$. By $\tilde{X}(x, y, s, V)$ and $\tilde{Y}(x, y, s, V)$ (or simple $\tilde{X}(x, y)$ and $\tilde{Y}(x, y)$ if there is no confusion) we denote horizontal lifts of $X(x, y)$ and $Y(x, y)$ from $P = P(x, y)$ to $Q(P, V)$.

By a vertical lift of a point $P \in \Sigma$ in direction $V \in \nu_P\Sigma$ we mean a point $Q = exp_P(sV)$ for some $s > 0$. Correspondingly, $\omega(y, V(y)) = exp_{\omega(y)}(sV(y))$ is said to be a vertical lift of $\omega(y)$ along some vertical vector field $V(y)$ along $\omega$. Due to (1) when $V(y)$ is a parallel vertical vector field along $\omega$ its vertical lift $\omega(y, V(y))$ is a horizontal curve (i.e., its speed is a horizontal vector everywhere). In this case we say, as usual, that $\omega(y, V(y))$ is a horizontal lift of $\omega$ (see [O’N]). The map $\pi : \omega(y, V(y)) \rightarrow \omega(y)$ decrease the distance (i.e., is "short") and is an isometry iff $\omega(y, V(y))$ is a horizontal lift of $\omega$.

\(^2\)Which is not really a strong restriction since there are only two non-homotopic vector bundles over $S^2$: trivial and non-trivial for which the corresponding unit-sphere bundle is a ruled surface - the only non-trivial $S^2$-bundle over $S^2$. These bundles correspond to elements of the $\pi_1(SO(3)) = \mathbb{Z}_2$ and both admit a non-zero section. Thus, our main technical result, Theorems 2a and 2b below may be considered as yet another splitting result: the local existence of the parallel sections when the curvature of $M$ is non-negative.

\(^3\)and is a further development of our "prism" construction from [M1.3].
The total vertical lift of $\omega_x$, i.e., the sub-manifold $\Psi_x(y, V) = \exp_{\omega_x(y)} s V$ for all unit $V \in T_{\omega_x(y)} \Sigma$ is a collection over $\omega_x$ of all vertical $s$-spheres. If $X(x, y)$ is a unit vector in $T_{\omega_x(y)} \Sigma$ normal to $\omega_x(y)$ then its parallel transport $\tilde{X}(Q)$ from $\omega_x(y)$ to $Q = Q(\omega_x(y), V)$ (along vertical geodesic) is a normal to $\Psi_x$, so that $\{V(Q), \tilde{X}(Q)\}$ is an orthonormal base of the normal subspace to $\Psi_x$ at $Q$.

1.1 O’Neill’s fundamental equations.

Remind, that according to the fundamental O’Neill’s formula (see [O’N])\(^4\)

\[(2) \quad (R(X, V)Y, W) = ((\nabla_X T)_V W, Y) + ((\nabla_Y A)_X Y, W) - (T_Y X, T_Y W) + (A_X V, A_W Y)\]

where $X, Y$ are horizontal vectors (i.e., belonging to $\mathcal{H}$), $V, W$ are vertical vectors (i.e., belonging to $\mathcal{V}$) and $T$ and $A$ are O’Neills fundamental tensors defined as follows

\[T_E F = \nabla \nabla_{(E)} \mathcal{H}(F) = \mathcal{H}(\nabla (E)) \quad \text{and} \quad A_E F = \nabla \nabla_{(E)} \mathcal{V}(F) = \mathcal{H}(\nabla (E)).\]

Here tensor $T$ is the second fundamental form of vertical fibers, while $A$ measures non-integrability of the horizontal distribution. Therefore,

\[(3) \quad (R(X, W)W, X) = ((\nabla_X T)_V W, X) - \|T_W X\|^2 + \|A_X W\|^2\]

because, as easy to verify,

\[(\nabla_W A)_X X, W = 0\]

due to the fact that $A$ is anti-symmetric and horizontal, see again [O’N]. Vanishing of the curvature term in (3) will imply below Theorem A. Another fundamental formula by O’Neill:

\[(4) \quad (R(X, Y)Y, X)(P) = (R(X, Y)Y, X)(Q(P, V)) + 3\|A_X Y\|^2(Q(P, V)).\]

1.2 Prism construction.

From [M1-3] we have the following.

Lemma 1.

\[\|A_X Y\|^2(Q(P, V)) = \frac{s^2}{4}\|R(X, Y)V\|^2(P).\]

The sketch of the proof of Lemma 1 is (see [M1-3] for calculations): take a small triangle $\triangle PP_1 P_2$ with sides parallel to $X$ and $Y$, translate parallel $V$ along these sides to vectors $V_1$ and $V_2$ at $P_1$ and $P_2$ correspondingly and lift-up the vertices of the triangle in obtained directions: $\triangle(s') = \triangle P(P_1) P_1 P_2$ (we have a "prism") where $P(s) = \exp s V$, $P_1(s) = \exp s V_1$, $P_2(s) = \exp s V_2$. From (1) it follows that the angle $\angle P(s)$ and sides of $\triangle(s)$ have zero first and second derivatives. Hence, the second derivative of the length of the third side $P_1(s) P_2(s)$ is proportional to the second derivative of the curvature of $M$ in two-dimensional direction $\{X, Y\}$. The same second derivative of the length of the third side $P_1(s) P_2(s)$ can be computed in a different way: by comparing $V_2$ with the parallel translation $V_2'$ of $V_1$ from $P_1$ to $P_2$ along $P_1 P_2$. By Ambrose-Singer theorem $V_2 - V_2'$ translated

\(^4\)and also [M1-3] for an exposition adapted to our case.
from \( P_2 \) to \( P \) equals \( R(X,Y)V \) times the area of the triangle \( \triangle PP_1 P_2 \) up to higher order terms. Then the second variation formula due to (1) implies the claim of the Lemma 1.

Before going further remark, that \( A_X Y(Q) \) does not depend on the particular choice of the orthonormal base \( X,Y \) with a positive orientation of a horizontal subspace \( \mathcal{H}_Q \). Indeed, due to \( A_H H \equiv 0 \) for another orthonormal base with a positive orientation \( \tilde{X} = \cos(\alpha)X + \sin(\alpha)Y, \tilde{Y} = -\sin(\alpha)X + \cos(\alpha)Y \) we have

\[
A_{\tilde{X}} \tilde{Y} = (\cos^2(\alpha) + \sin^2(\alpha))A_X Y = A_X Y.
\]

Therefore, in what follows we denote \( A_X Y(Q(P,V)) \) simply \( A(Q) \) for \( Q = Q(P,V) \).

Vanishing of \( A \) implies

\[
(5) \quad A_X W = 0
\]

for all horizontal \( X \) and vertical \( W \), i.e., that the vertical subspace is parallel in horizontal direction. Indeed,

\[
A_X W = (A_X W, X)X + (A_X W, Y)Y = (A_X W, Y)Y = (H(\nabla_{H(X)}(W(F))), Y) = -(W, A_X Y).
\]

Because \( A(Q) \) is orthogonal to the normal \( V(Q) \) of \( N\Sigma(s) \) and vertical, it defines a vector field tangent to the vertical two-dimensional sphere \( S^2(P) = N\Sigma(s) \cap W_P \). Therefore, \( A(Q) \) vanish at some \( Q^* = Q(P,V^*(P,s)) \) for every \( P \). Note that from the Lemma 1 we deduce:

**Lemma 2.** For a given \( P \) the vector \( V^*(P,s) \) does not depend on \( s \) and satisfies \( R(X,Y)V^* = 0 \). For a fixed \( s \) the set of all \( Q = Q(P,V,s) = \exp_{sV}V \) such that \( A(Q) = 0 \) is in one-to-one correspondence with the set of \( V \in \nu_P \Sigma \) such that \( R(X,Y)V = 0 \).

Now we employ \( \text{codim}\Sigma = 3 \).

As we saw \( A(Q) \) is proportional to the generator \( R(X,Y)V \) of the holonomy group of the normal bundle \( \nu \Sigma \), and therefore \( A \) or is identically zero on a given vertical sphere \( S^2(P) \), or vanish for two opposite to each other normals \( V_1^* \) and \( V_2^* = -V_1^* \), which by the Lemma 2 does not depend on the radius \( s \) of vertical spheres; while parallel translations of the space \( \nu_P \Sigma \) normal to \( \Sigma \) around small closed contours around \( P \) in positive direction are rotations about \( V_1^* \) in positive direction with a speed equals to the area bounded by the contour times \( \|R(X,Y)W,U\| \) where \( W,U \) from \( \nu_P \Sigma \) are orthonormal and orthogonal to \( V_1^* \). If we denote by \( Hol(P) \) a rotation of \( \nu_P \Sigma \) about an axis \( V_1^* \) in positive direction and speed \( \|R(X,Y)W,U\| \) (a density of the holonomy operator according to Ambrose-Singer theorem) we will have a continuous map \( Hol : \Sigma \rightarrow SO(3) \), which we call an infinitesimal holonomy map - a nice geometric representation for the holonomy of the normal bundle \( \nu \Sigma \).

1.3 Construction of the homotopy \( \omega_x \).

In [M1] (see also [M3,4]) we proved that if the holonomy of the normal bundle \( \nu \Sigma \) of the simply connected soul in an open manifold \( M^n \) of non-negative curvature is trivial then the manifold \( M^n \) is isometric to the direct product. In this case the Theorem A is obviously true. Thus, we may assume that at some point \( O \in \Omega^2 \) the generator of the holonomy operator is not zero, i.e., \( R(X,Y)V(O) \neq 0 \) so that the vector \( V^*(O) \) as above is uniquely defined. Our construction we start with some initial homotopy \( \omega_x, 0 \leq x \leq 1 \) of \( \omega = \partial \Omega \) to a point \( O \), i.e., such that \( \Omega = \{ \omega_x(y) | 0 < x \leq 1 \} \cup \{ O \}; \) and then will change it if necessary.

\(^5\text{change } \Omega \text{ to } \Sigma \setminus \Omega \text{ if necessary}\)
Consider the parallel translation \( I_x \) of \( \nu_{P(x)} \Sigma \), where \( P(x) = \omega_x(0) \), into itself along \( \omega_x \) - we call it the holonomy along \( \omega_x \). Due to our choice of \( O \) it is not the identity map for small \( x \), and because \( \text{codim} \Sigma = 3 \) this holonomy is a rotation about some uniquely defined axis generated by a vector \( V(x) \in \nu_{P(x)} \Sigma \) such that \( V(x) \to V^*(O) \) as \( x \to 0 \). For definiteness we choose \( \omega_x \) equals a circle of radius \( x \) around \( P \) for small \( x \). Then \( I_x \) depends smoothly on \( x \), and because \( V(x) \) is uniquely defined - it also depends smoothly on \( x \) for sufficiently small \( x \). Then the image \( V(x,y) \) of the vector \( V(x) \) under the parallel transport \( I_x(y) \) along \( \omega_x \) from \( P(x) \) to \( \omega_x(y) \) is also a smooth vector field. This will imply that the surfaces \( \Omega \) along \( \omega_x \) will be smooth.

Note that it always holds

\[
\nabla_Y V(x,y) = 0,
\]

and it is not difficult to see that all first covariant derivatives of \( V(x,y) \) actually vanish at \( O \).

Consider how \( I_x \) varies for bigger \( x \). If for all \( 0 < x \leq 1 \) it is a rotation on non-zero angle about some uniquely defined vector \( V(x) \) we have our homotopy defined. Otherwise for some \( x \neq x^* \) the family of holonomies \( I_x \) converges (in a natural sense) to \( I_{x^*} \) which is the identity map. In other words, if \( H : (0,1] \to SO(3) \) is the action of \( I_x \) on \( \nu_{O} \Sigma \) as follows:

\[
H(x)(V) = J^{-1}(x) \circ I_x \circ J(x)(V),
\]

where \( J(x) \) is a parallel translation from \( O \) to \( P(x) \), then \( H(x) \in SO(\nu_{P} \Sigma) = SO(3) \) and \( H(x^*) = id \). Having this trouble, i.e., \( H(x^*) = id \) we may try to "take off" the curve \( H(x), x^* - \delta < x < x^* + \delta \) of orthogonal transformations from an identity point \( id \) in \( SO(3) \) by varying "the curve of curves" - the family \( \omega_x \), i.e., taking some variation \( \tilde{\omega}_{x,\epsilon} \), where \( \tilde{\omega}_{x,0} = \omega_x \) such that the new holonomy curve \( H(\epsilon, x) = J^{-1}(x)\circ I_{x,\epsilon} \circ J(x) \) where \( I_{x,\epsilon} \) is a parallel translation along \( \tilde{\omega}_{x,\epsilon} \), does not go through \( id \) in \( SO(3) \).

To simplify forthcoming computations we consider variations given by

\[
(6) \quad \tilde{\omega}_x(\epsilon, y) = \exp_{\omega_x(y)}(\epsilon \phi_x(y)X(x,y)), \quad x^* - \delta < x < x^* + \delta
\]

where \( X(x,y) \), as above, is a unit vector normal to \( Y(x,y) \); and a smooth function \( \phi_x(y) \) satisfying restrictions:

\[
\phi_x(y) \equiv 0 \quad \text{for} \quad x < x^* - \delta, x^* + \delta < x.
\]

The varied family of holonomies along \( \tilde{\omega}_x(\epsilon, y) \) defines the map \( H(x, \epsilon) \) on two variables (and depending on the "profile function" \( \phi \)) into three-dimensional \( SO(3) \) as follows:

\[
H(x, \epsilon)(V) = J^{-1}(x, \epsilon) \circ I_x(\epsilon) \circ J(x, \epsilon)(V),
\]

where \( J(x, \epsilon) = \tilde{J}(\epsilon) \circ J(x) \) and \( \tilde{J}(\epsilon) \) is a parallel translation from \( P(x) \) to \( P(x, \epsilon) = \tilde{\omega}_x(\epsilon, 0) \) along \( \tilde{\omega}_x(\epsilon, 0) \), and \( I_x(\epsilon) \) is a parallel translation along \( \tilde{\omega}_x(\epsilon, y) \) of vectors from \( \nu_{P(x, \epsilon)} \Sigma \). If partial derivatives of \( H(x, \epsilon) \) on \( x \) and on \( \epsilon \), i.e., two vectors \( \partial H(x, \epsilon)/\partial x \) and \( \partial H(x, \epsilon)/\partial \epsilon \) are linearly independent at \( (x^*, 0) \) for a given \( \phi \) then, obviously, there exists a variation \( \tilde{\omega}_{x,\epsilon} \) for which the curve \( H(x, \epsilon) \) does not go through the point \( id \in SO(3) \) for sufficiently small \( \epsilon \). By Ambrose-Singer theorem the action of the derivative \( \partial H(x, \epsilon)/\partial \epsilon \) on a vector \( W \) from \( \nu_O \Sigma \) is

\[
(\partial H(x, \epsilon)/\partial \epsilon)(W) = J^{-1}(x^*)(\int_{\tilde{\omega}_x} \phi_x(y)I_x^{-1}(y)R(X, \partial/\partial y)W_x(y)dy) =
\]
This coordinate description might be useful, e.g., to verify (12) below. Then without loss of generality we may assume that

\[ \partial H(x, \epsilon) = \partial H(x, \epsilon) / \partial x \]

where \( \psi_x(y) = (X, \partial / \partial x) \).

We consider \( R(X, Y) \) as an operator \( R(X, Y) : V \to R(X, Y)V \) from the Lie-algebra of \( SO(3) \) which generates the holonomy group and obtain some conditions on these generators in the case when both the first and the second differentials of \( H(x, \epsilon) \) are degenerated at \((x^*, 0)\) which do not allow to "take off" the curve \( H(x) \) of the point \( id \) in \( SO(3) \). For short we denote below \( d_x H(x) = \partial H(x, \epsilon) / \partial x \) and \( d_\epsilon H = \partial H(x, \epsilon) / \partial \epsilon \) and consider two possibilities for the vector \( R = d_x H(x^*, 0)^6 \)

1) it does not equal zero,

6The same consideration might be done in coordinates: choose an orthonormal base \( \{ E_i, i = 1, 2, 3 \} \) in \( \nu_0 \Sigma \) and the corresponding standard basis of \( so(3) \) consisting of three generators \( R_{12}, R_{13}, R_{23} \) of \( so(3) \) which are unit tangents to rotations in \( \nu_0 \Sigma \) with axes \( E_3, E_2, E_1 \) correspondingly. Define vector fields \( E_i(x, y) = I_x(y)J(x)E_i \) along \( \omega_x \). Because \( H(x^*, 0) = id \) these are (continuous) parallel vector fields along \( \omega_x \). If \( H(x, \epsilon) \) is given by the matrix \( (H_{ij}(x, \epsilon); i, j = 1, 2, 3) \) then its derivatives have the following components:

\[ \frac{\partial H_{ij}(x^*, \epsilon)}{\partial \epsilon} \bigg|_{\epsilon=0} = \frac{L(x)}{2\pi} \int_{\omega_x} \phi_x(y)(R(X, Y)E_i(y), E_j(y))dy. \]

where \( E_i(y) = E_i(x^*, y) \) and

\[ \frac{\partial H_{ij}(x, 0)}{\partial x} = \frac{L(x)}{2\pi} \int_{\omega_x} \psi_x(y)(R(X, Y)E_i(y), E_j(y))dy. \]

These components are coordinates of derivatives of \( H(x, \epsilon) \) in the basis \( R_{12}, R_{13}, R_{23} \) of \( so(3) \):

\[ \partial H(x, \epsilon) / \partial \epsilon = \sum_{ij=12,13,23} \partial H_{ij}(x, \epsilon) / \partial \epsilon R_{ij}. \]

This coordinate description might be useful, e.g., to verify (12) below. Then without loss of generality we may assume that \( R \) in \( so(3) \) is proportional to (say) the coordinate vectors \( R_{23} \), or

\[ d_x H_{12} = d_x H_{13} = 0 \]

and

\[ d_x H(x^*, 0) = R_{23} \int (R(X, Y)E_2(y), E_3(y))dy \neq 0. \]
2) it equals zero.

Below we consider in details the first (principal) case when the rank of the differential of \( H(x, \epsilon) \) is at least one at \((x^*, 0)\). After that it will be easy to see that our main technical results, see the Theorems 1a and 1b below, can be obtained in the same line of arguments also in the second case.

By the same arguments as in the fundamental lemma of the calculus of variations we see that if the rank of the map \( H(x, \epsilon) \) is one for all possible variations \( \phi \) then all the vectors \( R(X, Y) \) and both \( \nabla_Y R(X, Y) \) and \( \nabla_X R(X, Y) \) are proportional to \( R \) along \( \omega_x \). Indeed, take two different points \( P_i = \omega_x(y_i), i = 1, 2 \) and assume that \( R_1 = I_x(y_1)R(X, Y)(P_1) \) not proportional to \( R_2 = I_x(y_2)R(X, Y)(P_2) \). Then choosing two \( \delta \)-like functions \( \phi^i_\omega(x) \) concentrated near these points \( P_i \) we define two variations (6) for which according to (7) derivatives of the holonomy \( H(x, \epsilon) \) will be close to \( R_1 \) and \( R_2 \) correspondingly and linearly independent, making our ”taking off” possible. Hence, we come to the following conclusion.

**Lemma 3.** If for all variations \( \omega_x(y) \) given by (6) the holonomy curve \( H(x, \epsilon) \) goes through \( id \) in \( SO(3) \), then all

\[
I_x(y)R(X, Y)(y)
\]

are proportional to the vector \( R \).

Remark that from the Lemma 3 and the formula (7) it follows that for an arbitrary vector field \( W_x(y) \) which is parallel along \( \omega_x \) we have

\[
\nabla_X W_x(y) = J^{-1}(x^*)(L(x)\frac{I_x}{2\pi} \int_0^{2\pi} \psi_x(y)I_x^{-1}(y)R(X, Y)W_x(y)dy,
\]

or that

\[
(11) \quad \nabla_X W_x(y) \text{ is parallel to } R(X, Y)W_x(y),
\]

where here \( R(X, Y) \) is understood as the vector from \( so(3) \) at \( \omega_x(y) \), i.e., an anti-symmetric operator on \( \nu_{\omega_x,y} \Sigma \).

Note, that vector fields \( V(x,y) \) are parallel along \( \omega_x \) for \( x < x^* \) under rotations \( H(x) \) which are approximately \( id - (x^* - x)R \). Because by assumption \( R \neq 0 \) they are close to \( V^* \) such that \( R(V^*) \equiv 0 \), i.e., we arrive at the following statement.

**Lemma 4.** The vector fields \( V(x,y) \) parallel along \( \omega_x \) and invariant under \( H(x) \) converge to the vector field \( V^* \) along \( \omega_x(y) \) such that

\[
R(X, Y)V^*(y) \equiv 0.
\]

The vector field \( V^*(y) \) is parallel along \( \omega_x \) as the limit of parallel vector fields.

Next we note that taking the covariant derivative \( \nabla_Y R(X, Y) \) of \( R(X, Y) \) along \( \omega_x \), we should have according to the Lemma 3 the vector field which is also parallel to \( R \). The same is true for the covariant derivative of the field of transformations \( R(X, Y) \) in the direction \( X \) normal to \( Y \). Indeed, take an arbitrary \( P_i = \omega_x(y), y \neq 0 \) different from \( P \), and consider again a variation with a \( \delta \)-like function \( \phi^i_\omega(y) \) concentrated near \( P_i \) and zero at
P. Let we know that all first variations of $H(x^*, \epsilon)$ at $\epsilon = 0$ are proportional to $R$. Then, as easy computations show that the second variation of $H(x, \epsilon)$ acts on the vector $W$ as follows

$$\delta_2^2 H(x^*, \epsilon)|_{\epsilon=0}(W) = J^{-1}(x^*) \frac{L(x)}{2\pi} \int_{\omega_x} \phi_x(y) I_x^{-1}(y) (\nabla_X R(X, Y) W(y)) dy +$$

$$J^{-1}(x^*) \int_{\omega_x} \phi_x(y) I_x^{-1}(y) R(X, Y) (\nabla_X W_x(y)|_{x=x^*}) dy +$$

$$J^{-1}(x^*) \int_{\omega_x} (\frac{L'(x)}{2\pi} + \phi'_x(y)) I_x^{-1}(y) R(X, Y) W(y) dy,$$

where $\phi'_x(y)$ stands for the partial derivative of $\phi_x(y)$ on $x$, $W(y) = W(x^*, y)$ and $W_x(y)$ as above is parallel along $\omega_x$. Note that here this derivative is not a skew-adjoint map on $W$ due to the fact that $W$ also depends on the variation, i.e., on $\epsilon$. This is given by the second right term in the previous equality which by (11) is

$$J^{-1}(x^*) \int_{\omega_x} (\frac{L'(x)}{2\pi} + \phi'_x(y)) I_x^{-1}(y) R(X, Y) W(y) dy,$$

Operators $R$ and $R^2$ are correspondingly the first (tangent) and the second derivatives in $GL(3) \supset SO(3)$ to the one-parameter group of rotations in $SO(3)$ (a circle) issuing from $id$ in the same direction $R$ as the family of holonomies $H(x)$. The third term in the right-hand side of (12) is proportional to $R(W)$. If the first vector in the right-hand side of (12) is not proportional to $R(W)$ we again, as above in the case of the first variation non-proportional to $R$, would have the variation which deform the holonomy curve $H(x)$ to $H(x, \epsilon)$ which for sufficiently small but positive $\epsilon$ avoids $id$ in $SO(3)$. Hence, the following is true.

**Lemma 5.** If for all variations $\tilde{\omega}_x(y)$ given by (6) the holonomy curve $H(x, \epsilon)$ goes through $id \in SO(3)$, then

$$I_{x^*}(y)\nabla_X R(X, Y)(y)$$

and

$$I_{x^*}(y)\nabla_Y R(X, Y)(y)$$

are proportional to the vector $R$.

We say that the holonomy $I_{x^*}$ is vanishing along $\omega_{x^*}$ if $H(x^*) = id$ and the claims of the Lemmas 3-5 are true. If this happens $V^*(x, y)$ belongs to the kernel of the actions of all first and second variations of $H(x^*, \epsilon)$ on $\epsilon = 0$, the vector field $V(x, y)$ converges to $V^*(y)$ as $x \to x^*$ and the following is true

$$\nabla_X V(x, y)|_{x=x^*} \equiv 0.$$

Note that the same arguments work also when the vector $d_x H$ is zero. Indeed, because all $H_x$ for $x < x^*$ are non-trivial rotations the vector field $V(x, y)$ is correctly defined, and for $x \neq x^*$ this vector field converges to $V^*(y)$ on $\omega_{x^*}$. If $R(X, Y) \neq 0$ then for some variation given by some profile $\phi_x(y)$ the vector $R = (\partial H(x^*, \epsilon)/\partial \epsilon)|_{\epsilon=0}$ is not zero. Thus after a small variation given by $\phi_x(y)$ we are in the case $R \neq 0$ and may apply arguments above to obtain the claim of the Lemma 5. Similarly, if $R(X, Y) \equiv 0$ along $\omega_{x^*}$ but the claim of the Lemma 5 is not true, then (12) shows that after some deformation $\tilde{\omega}_x(y)$ we arrive at the first case when $R \neq 0$.

We summarize the obtained results in the following theorem.
Theorem 1a. For a domain $\Omega$ bounded by a closed curve $\omega$ there exists a smooth homotopy $\omega_x$, $0 \leq x \leq 1$ of $\omega = \omega_1$ to a point $O$ such that

1) or holonomies $I_x$ along $\omega_x$ are non-trivial rotations of $\nu_{\partial \Sigma}$ about axis $V(x)$ for all $0 < x \leq 1$;

2) or for some $x \not\geq x^* < 1$ these holonomies converge to the identity map. Then

$$\nabla_x V(x,y)|_{x = x^*} = 0,$$

where $V(x,y) = I_x(y)V(x)$ are parallel translations of $V(x)$ along $\omega_x$.

It might be useful to note that we may choose variations above (constructed in order to deform the initial homotopy $\omega_x$) not diminishing domains bounded by $\omega_x$, i.e., such that we have $\Omega_x \subset \tilde{\Omega}_x$ for domains $\Omega_x$, $\tilde{\Omega}_x$ bounded by $\omega_x$ and $\tilde{\omega}_x$ correspondingly. Hence, varying an initial center $O$ in $\Omega$ or we can construct for a given point $O'$ a homotopy $\omega_x^{O'}$ between $\omega$ and the constant curve $O'$ with non-vanishing holonomies along all curves in the homotopy family, or $O' \in \text{int}(\Omega^{O'})$ for some domain with vanishing holonomy along $\partial \Omega^{O'}$.

Finally, note that for the given homotopy $\omega_x$, $0 \leq x \leq 1$ we may start the construction of the vector field $V(x,y)$ first by defining $V(1,y)$ and then considering the family of holonomy operators $I_x$ along $\omega_x$ for $x$ close to 1. In the same way as before we conclude that if $I_1$ is not the identity map then $V(x,y)$ is defined for all $x > 1 - \epsilon$ close enough to 1. When the holonomy $I_1$ is trivial, i.e., $I_1 = \text{id}$ but the operators $R(X,Y) : V \to R(X,Y)V$ of the infinitesimal holonomies do not vanish along $\omega_1$ we still will be able, deforming the initial homotopy $\omega_x$ as above if necessary, to define the unit normal vector field $V(x,y)$ parallel along curves $\omega_x$ for small $x$ close enough to 1. Therefore, as in the Theorem 1a above we see that the following is true.

Theorem 1b. For a domain $\Omega$ bounded by a closed curve $\omega$ there exists a smooth homotopy $\omega_x$, $0 \leq x \leq 1$ of $\omega = \omega_1$ to a point $O$ such that

1) or holonomies $I_x$ along $\omega_x$ are non-trivial rotations of $\nu_{\partial \Sigma}$ about axis $V(x)$ for all $0 < x \leq 1$;

2) or $I_1 \neq \text{id}$ but for some $1 > x > x^* > 0$ these holonomies converge to the identity map. Then

$$\nabla_x V(x,y)|_{x = x^*} = 0,$$

where $V(x,y) = I_x(y)V(x)$ are parallel translations of $V(x)$ along $\omega_x$.

3) Or $I_1 = \text{id}$ and all operators $R(X,Y) : V \to R(X,Y)V$ of the infinitesimal holonomies vanish along $\omega_1$.

Below we estimate curvature of a vertical lift $\Omega_\Sigma$ of $\Omega$ in a direction of the vector field $V(x,y)$ using some coordinates which might be different from our ”coordinates” $\{x,y\}$ above.

2. Curvature of a local vertical lift

Consider the vertical lift of $\Omega$ along the given vector field

$$\Omega_\Sigma(x,y) = \exp_{p(x,y)} s V(x,y).$$

During this section the local coordinates $\{x,y\}$ in $\Omega$ will be chosen in a process of our calculations in order to simplify them. In particular, they are not assumed to coincide with those from the previous section.
Denote by $\tilde{X}(x, y; s)$ and $\tilde{Y}(x, y; s)$ the $x$- and $y$-coordinate vectors on $\Omega_s$. By $\tilde{X}(x, y; s)$ and $\tilde{Y}(x, y; s)$ we denote the horizontal lifts of $X(x, y; 0)$ and $Y(x, y; 0)$ (basic horizontal vector fields): vertical $V(x, y; s)$ - the parallel transport of $V(x, y)$ along vertical geodesic from $P(x, y)$ to $P_s(x, y) = \Omega_s(x, y)$, and by $\tilde{X}(x, y; s)$ and $\tilde{Y}(x, y; s)$ the unit vectors of the same directions as $\tilde{X}(x, y; s)$ and $\tilde{Y}(x, y; s)$. We usually assume that at the given point of consideration (only) $\tilde{X}(x, y; s)$ and $\tilde{Y}(x, y; s)$ are unit and normal to each other (i.e., coincide with $\tilde{X}(x, y; s)$ and $\tilde{Y}(x, y; s)$) and their first covariant derivatives vanish at this point. It holds

\begin{equation}
\mathcal{H}(\tilde{X}(x, y; s)) = \tilde{X}(x, y; s), \quad \mathcal{V}(\tilde{X}(x, y; s)) = s\nabla_X V(x, y) + o(s^2)
\end{equation}

and

\begin{equation}
\mathcal{H}(\tilde{Y}(x, y; s)) = \tilde{Y}(x, y; s), \quad \mathcal{V}(\tilde{Y}(x, y; s)) = s\nabla_Y V(x, y) + o(s^2).
\end{equation}

Next we do calculations of some curvature tensor terms\footnote{We work first with curvature tensor terms to simplify calculations. The obtained in Lemmas 7-9 formulas then will provide estimates for the sectional curvature of $\Omega_s$.} with $o(s^2)$ precision, i.e., up to $O(s^2)$-terms.

**Lemma 7.**

\[
(R(\tilde{X}(x, y; s), \tilde{Y}(x, y; s))\tilde{Y}(x, y; s), \tilde{X}(x, y; s)) - (R(\tilde{X}(x, y; s), \tilde{Y}(x, y; s))\tilde{Y}(x, y; s), \tilde{X}(x, y; s)) = 
\]

\[
s^2((\nabla_Y R(\tilde{X}(x, y; 0), \tilde{Y}(x, y; 0))V(x, y; 0))\nabla_X V(x, y; 0)) - 
\]

\[
(R(\tilde{X}(x, y; 0), \tilde{Y}(x, y; 0))V(x, y; 0))\nabla_Y V(x, y; 0))
\]

\[
+ s^2 (R(\tilde{X}(x, y; 0), \tilde{Y}(x, y; 0))\nabla_X V(x, y; 0), \nabla_Y V(x, y; 0)) + o(s^2).
\]

**Proof.** We fix a point $P = P(x, y; 0)$ in $\Omega$ and to simplify notations drop below $(x, y)$-arguments. By the same reason we also drop an $s$-argument if it equals zero. By $\tilde{V}$ we denote a vector field on $\Omega$ which is parallel at $P$, i.e., $\nabla_X \tilde{V}(P) = \nabla_Y \tilde{V}(P) = 0$ and equals $V$ at the point $P$. From (15) we deduce

\[
(R(\tilde{X}(s), \tilde{Y}(s))\tilde{Y}(s), \tilde{X}(s)) - (R(\tilde{X}(s), \tilde{Y}(s))\tilde{Y}(s), \tilde{X}(s)) = 
\]

\[
2s((R(\tilde{X}(s), \tilde{Y}(s))\tilde{Y}(s), \nabla_X V(s)) + (R(\tilde{X}(s), \tilde{Y}(s))\nabla_Y V(s), \tilde{X}(s)) + 
\]

\[
2s^2((R(\tilde{X}, \tilde{Y})\nabla_Y V, \nabla_X V)(\tilde{X}, \tilde{Y}) + (R(\nabla_X V, \tilde{Y})\nabla_Y V, \tilde{X}))(\tilde{X}, \tilde{Y}) + o(s^2)
\]

Because

\begin{equation}
(R(\tilde{X}, \tilde{Y})\tilde{Y}, W) = 0 \quad \text{and} \quad (R(\tilde{X}, \tilde{Y})U, \tilde{X}) = 0
\end{equation}

for any vertical $W, U$ we have

\[
(R(\tilde{X}(s), \tilde{Y}(s))\tilde{Y}(s), \nabla_X V(s)) + (R(\tilde{X}(s), \tilde{Y}(s))\nabla_Y V(s), \tilde{X}(s)) = 
\]
By the second Bianchi identity

\[ \nabla_V R(X, Y)Y + \nabla_X R(Y, \bar{V})Y + \nabla_Y R(\bar{V}, X)Y = 0, \]

and (17)

\[ \nabla_V R(X, Y)Y = \nabla_Y R(X, \bar{V})Y. \]

Or, using

\[ R(X, \bar{V})Y - R(\bar{V}, Y)X = R(X, \bar{V})Y + R(Y, \bar{V})X = R(X + Y, \bar{V})(X + Y) - R(X, \bar{V})X - R(Y, \bar{V})Y = 0 \]

and the first Bianchi identity

\[ R(X, \bar{V})Y + R(\bar{V}, Y)X + R(Y, X)\bar{V} = 0 \]

we conclude

\[ \nabla_V R(X, Y)Y = \nabla_Y R(X, \bar{V})Y = (\nabla_Y R)(X, Y)\bar{V}/2. \]

In the same way

\[ \nabla_V R(X, Y)X = \nabla_X R(Y, \bar{V})X = (\nabla_X R)(X, Y)\bar{V}/2. \]

and

\[ 2s((R(\bar{X}(s), \bar{Y}(s))\bar{Y}(s), \nabla_X V(s)) + (R(\bar{X}(s), \bar{Y}(s))\nabla_Y V(s), \bar{X}(s))) = \]

\[ s^2((\nabla_Y R)(X, Y)V, \nabla_X V) - ((\nabla_X R)(X, Y)V, \nabla_Y V)) \]

in (16).

Next we note that for an arbitrary operator the derivative \( \nabla_X (R(V)) \) equals \( (\nabla_X R)(V) + R(\nabla_X V), \) or

\[ ((\nabla_Y R)(X, Y)V, \nabla_X V) - ((\nabla_X R)(X, Y)V, \nabla_Y V) + 2(R(X, Y)\nabla_X V, \nabla_Y V) = \]

\[ (((\nabla_Y R)(X, Y)V, \nabla_X V) + (R(X, Y)\nabla_Y V, \nabla_X V)) - (((\nabla_X R)(X, Y)V, \nabla_Y V) + (R(X, Y)\nabla_X V, \nabla_Y V)) \]

\[ = (\nabla_Y (R(X, Y)V), \nabla_X V) - (\nabla_X (R(X, Y)V), \nabla_Y V). \]

which implies from (16) that

\[ (R(\bar{X}(s), \bar{Y}(s))\bar{Y}(s), \bar{X}(s)) = (R(\bar{X}(s), \bar{Y}(s))\bar{Y}(s), \bar{X}(s)) = \]
\[ s^2((\nabla_Y (R(X,Y)V), \nabla_X V) - (\nabla_X (R(X,Y)V), \nabla_Y V)) + 2s^2(R(\nabla_X V, \tilde{Y})\nabla_Y V, \tilde{X}) + o(s^2). \]

Again, from the first Bianchi identity

\[ (R(W,Y)U, X) + (R(Y,U)W, X) + (R(U,W)Y, X) = 0 \]

and

\[ (R(W,Y)U, X) - (R(Y,U)W, X) = (R(W,Y)U, X) + (R(U,Y)W, X) = \]

\[ (R(W + U, Y)W + U, X) - (R(W,Y)W, X) - (R(U,Y)U, X) = 0 \]

following from (17), we have

\[ (R(\nabla_X V, \tilde{Y})\nabla_Y V, \tilde{X})) = (R(X,Y)\nabla_Y V, \nabla_X V) / 2 \]

which finely implies through (16*)

\[ (R(\tilde{X}(s), \tilde{Y}(s))\tilde{Y}(s), \tilde{X}(s)) = (R(\tilde{X}(s), \tilde{Y}(s))\tilde{Y}(s), \tilde{X}(s)) = \]

\[ s^2((\nabla_Y (R(X,Y)V), \nabla_X V) - (\nabla_X (R(X,Y)V), \nabla_Y V)) + s^2(R(X,Y)\nabla_Y V, \nabla_X V) + o(s^2). \]

The Lemma 7 is proved.

Note that the Lemma 7 formula can be re-written as follows.

**Lemma 8.**

\[ (R(\tilde{X}(s), \tilde{Y}(s))\tilde{Y}(s), \tilde{X}(s)) = \]

\[ s^2(||R(X,Y)V||^2 - det \begin{vmatrix} \nabla_X \nabla_X V & \nabla_X \nabla_Y V \\ \nabla_Y \nabla_X V & \nabla_Y \nabla_Y V \end{vmatrix}) + s^2D + o(s^2), \]

where multiplications in a formal determinant above are scalar products and \( D \) is:

\[ D = \frac{1}{2}(-Y(\nabla_Y \nabla_X V, \nabla_X V) - X(\nabla_X \nabla_Y V, \nabla_Y V) + X(\nabla_Y \nabla_X V, \nabla_Y V) + X(\nabla_Y \nabla_Y V, \nabla_X V)). \]

**Proof.** Note that, e.g.,

\[ (\nabla_Y (R(X,Y)V), \nabla_X V) = Y(R(X,Y)V, \nabla_X V) - (R(X,Y)V, \nabla_Y \nabla_X V)) \]

Repeating similar transformations for all terms in (28) and in

\[ (R(X,Y)\nabla_Y V, \nabla_X V) = (\nabla_X \nabla_Y \nabla_Y V, \nabla_X V) - (\nabla_Y \nabla_X \nabla_Y V, \nabla_X V) \]
we obtain for the right-hand term of (28)
\[
(\nabla_Y(R(X,Y)V), \nabla_X V) - (\nabla_X(R(X,Y)V), \nabla_Y V)) + (R(X,Y)\nabla_Y V, \nabla_X V) = \\
D - ((R(X,Y)V, \nabla_Y \nabla_X V) - (R(X,Y)V, \nabla_X \nabla_Y V) + (\nabla_Y \nabla_Y V, \nabla_X \nabla_X V) - (\nabla_X \nabla_Y V, \nabla_Y \nabla_X V))
\]
\[
D + \|R(X,Y)V\|^2 - \det \begin{vmatrix} \nabla_X \nabla_X V & \nabla_X \nabla_Y V \\ \nabla_Y \nabla_X V & \nabla_Y \nabla_Y V \end{vmatrix},
\]

since \( R(X,Y)V = \nabla_X \nabla_Y V - \nabla_Y \nabla_X V \) and hence
\[
(R(X,Y)V, \nabla_Y \nabla_X V) - (R(X,Y)V, \nabla_X \nabla_Y V) = -\|R(X,Y)V\|^2;
\]
while for the "D-part" (which contains derivatives of scalar products) in the formula above we obtain
\[
D = \frac{1}{2}(Y(\nabla_X \nabla_Y V, \nabla_X V) - Y(\nabla_Y \nabla_X V, \nabla_X V) - X(\nabla_X \nabla_Y V, \nabla_Y V) + \\
X(\nabla_Y \nabla_X V, \nabla_Y V) + X(\nabla_Y \nabla_Y V, \nabla_X V) - Y(\nabla_X \nabla_Y V, \nabla_Y V)),
\]
which implies the claim of the Lemma 8. The Lemma 8 is proved.

Recall that we do calculations with \( o(s^2) \)-precision, i.e., up to \( O(s^2) \)-terms. Next we estimate the "external curvature" term\(^8\) of \( \Omega_s \) at some point \( P(x,y; s) \). To make calculations simpler we may assume that covariant derivatives of coordinate vector fields vanish at this point, i.e.,
\[
\nabla_X X(P) = \nabla_Y Y(P) = 0.
\]
Rotating if necessary the orthonormal base \( \{X,Y\} \) we may also assume that vertical vectors \( \nabla_X V \) and \( \nabla_Y V \) are normal at the point \( P(x,y; 0) \). We denote them by \( dW \) and \( eU \) where \( \{W,U\} \) unit and normal to each other. Then the normal space of \( \Omega_s \) at the considered point \( P_s(x,y) \) is generated by \( \{V, M, N\}(x,y; s) \) where
\[
M = W - sd\bar{X} \quad \text{and} \quad N = U - se\bar{Y}
\]
Correspondingly, the unit normals to \( \Omega_s \) which are normal to each other are
\[
M = \bar{M}/\|\bar{M}\| = \frac{W - sd\bar{X}}{\sqrt{1 + (sd)^2}} + o(s^2), \quad \text{and} \quad N = \bar{N}/\|\bar{N}\| = \frac{U - se\bar{Y}}{\sqrt{1 + (se)^2}} + o(s^2).
\]
From the Gauss equation we see that the external curvature term \( R_s^{ext}(x,y) \) of \( \Omega_s \) (i.e., the difference between the curvature term \( \tilde{R}(x,y; s) \) of the surface \( \Omega_s(\epsilon) \) and the curvature tensor term \( R(x,y; s) \) of the ambient manifold \( M \) in the same two-dimensional direction) equals\(^9\)
\[
(31) \quad R_s^{ext}(x,y) = \sum_{Z \in \{M,N\}} (\nabla_X \tilde{X}, Z)(\nabla_Y \tilde{Y}, Z) - (\nabla_X \bar{X}, \bar{Z})^2,
\]
\(^8\)recall the footnote before Lemma 7
\(^9\)See last two footnotes above. To compute the Gauss curvature we should divide these curvature terms by the area of the element \( d\bar{X} \wedge d\bar{Y} \).
since the normal $V$ does not contribute to the Gauss formula. Because the second fundamental form of vertical fibers vanish along $\Sigma$ for every vertical $W$ and horizontal $X$ we have

$$
\|\nabla_W X\| = O(s),
$$

and routine calculations give

$$
\nabla_{\tilde{X}} \tilde{X} - H_1 = \nabla_{X+sdW} X + sdW = sd_x W + sd\nabla_X W + O(s^2)
$$

where $H_1$ tangent to $\Omega_s$. In the same way

$$
\nabla_{\tilde{Y}} \tilde{Y} - H_2 = \nabla_{X+sdW} Y + seU = \nabla_X Y + se_x U + se\nabla_X U + O(s^2)
$$

and

$$
\nabla_{\tilde{Y}} \tilde{Y} - H_3 = \nabla_{Y+seU} Y + seU = se_y U + se\nabla_Y U + O(s^2)
$$

for some $H_2, H_3$ tangent to $\Omega_s$. Which after substitution into (30) leads to the following formulas up to $O(s^2)$-terms

$$(\nabla_{\tilde{X}} \tilde{X}, M) = (sd_x W + sd\nabla_X W, W - sd\tilde{X}) = sd_x'$$

$$(\nabla_{\tilde{X}} \tilde{Y}, M) = (\nabla_X Y + se_x' U + se\nabla_X U, W - sd\tilde{X}) = (\nabla_X Y, W) + se(\nabla_X U, W),$$

$$(\nabla_{\tilde{Y}} \tilde{Y}, M) = (se_y' U + se\nabla_Y U, W - sd\tilde{X}) = se(\nabla_Y U, W);$$

and

$$(\nabla_{\tilde{X}} \tilde{X}, N) = (sd_x' W + sd\nabla_X W, U - se\tilde{Y}) = sd(\nabla_X W, U),$$

$$(\nabla_{\tilde{X}} \tilde{Y}, N) = (\nabla_X Y' + se_x' U + se\nabla_X U, U - se\tilde{Y}) = (\nabla_X Y, U) + se_x'',$nabla_X Y = -s R(X, Y)V/2

implies for the external curvature term

$$s^{-2}R_{ext}^e(x, y) = d_x' e(\nabla_Y U, W) - (- (R(X, Y)V, W))/2 + e(\nabla_X U, W)^2 +$$

$$e_y' d(\nabla_X W, U) - (- (R(X, Y)V, U))/2 + e_x')^2.$$

Because

$$d_x' = (\|\nabla_X V\|)'_x = (\nabla_X V, \nabla_X V)/\|\nabla_X V\| = (\nabla_X V, W)$$

$$e_x' = (\|\nabla_Y V\|)'_x = (\nabla_Y V, \nabla_Y V)/\|\nabla_Y V\| = (\nabla_Y V, U)$$

$$e_y' = (\|\nabla_X V\|)'_y = (\nabla_Y V, \nabla_Y V)/\|\nabla_Y V\| = (\nabla_Y V, U)$$

which with the help of

$$\nabla_X Y = -s R(X, Y)V/2$$

implies for the external curvature term

$$s^{-2}R_{ext}^e(x, y) = d_x' e(\nabla_Y U, W) - (- (R(X, Y)V, W))/2 + e(\nabla_X U, W)^2 +$$

$$e_y' d(\nabla_X W, U) - (- (R(X, Y)V, U))/2 + e_x')^2.$$
and, e.g.,
\[
(\nabla_Y U, W) = \|\nabla_Y V\|(\nabla_Y \frac{\nabla_Y V}{\|\nabla_Y V\|}, W) = (\nabla_Y \nabla_Y V, W)
\]
due to the fact that \((W, U) = 0\); by direct calculations we conclude
\[
s^{-2}R^\text{ext}_s(x, y) = \\
(\nabla_X \nabla_X V, W)(\nabla_Y \nabla_Y V, W) - ((R(X, Y)V)W)^2/4 + 2(R(X, Y)V)(\nabla_X \nabla_Y V, W) - (\nabla_X \nabla_Y V, W)^2 + (\nabla_X \nabla_X V, U)(\nabla_Y \nabla_Y V, U) - R(X, Y)V, U)^2/4 + (R(X, Y)V, U)(\nabla_X \nabla_Y V, U) - (\nabla_X \nabla_Y V, U)^2) =
\]
(36)
\[
(\nabla_X \nabla_X V, \nabla_Y \nabla_Y V) - \|R(X, Y)V\|^2/4 + (R(X, Y)V, \nabla_X \nabla_Y V) - \|\nabla_X \nabla_Y V\|^2.
\]
Because
\[
(R(X, Y)V, \nabla_X \nabla_Y V) = (\nabla_X \nabla_Y V - \nabla_Y \nabla_X V, \nabla_X \nabla_Y V)
\]
we obtain the following statement.

**Lemma 9.** The external curvature term \(R^\text{ext}_s(x, y)\) of \(\Omega_s\) is given by the formula
\[
R^\text{ext}_s(x, y) = s^2(-\|R(X, Y)V\|^2/4 + \det \begin{vmatrix} \nabla_X \nabla_X V & \nabla_X \nabla_Y V \\ \nabla_Y \nabla_X V & \nabla_Y \nabla_Y V \end{vmatrix})
\]
where, as before, multiplications in a formal determinant above are scalar products.\(^{10}\)

Now we put formulas above together and draw some conclusions. Because by the fundamental O’Neill’s formula (4)
\[
(\nabla_X \nabla_Y V(0, 0), \nabla_X \nabla_Y V(0, 0), \nabla_Y \nabla_Y V(0, 0) = 0
\]
for all \(0 \leq \alpha \leq 2\pi\), which after taking derivative on \(\alpha\) gives
\[
\nabla_X \nabla_X V(0, 0), \nabla_X \nabla_Y V(0, 0), \nabla_Y \nabla_Y V(0, 0) = 0,
\]
which in turn as in (15) implies
(15') \[
Y'(0, 0; s) = \hat{Y}(0, 0; s), \quad X'(0, 0; s) = \hat{X}(0, 0; s) \quad \text{and} \quad \nabla_X, X'(x', y'; s), \nabla_Y, Y'(x', y; s) = 0
\]
for new coordinate vectors \(X'(x', y'; s), Y'(x', y'; s)\) on \(\Omega_s\), and by (31) implies for the external curvature of \(\Omega_s\) at \(P(0, 0; s)\) the claim of the Lemma. The proof of the Lemma 9 is complete.

\(^{10}\)We may complete the proof of the Lemma 9 with the following analysis of the external curvature term of \(\Omega_s\) at the point \(O(s) = P(x, y, 0)\) in the coordinates from the Theorem 1, where our \(y\)-coordinate curves \(\omega_\xi\) degenerate to a point and \(kg(x, y) \to \infty\) as \(x \to 0\). Easy to see this is singularity of the coordinate system, which does not yield the singularity of \(\Omega_s\). Indeed, as we already noted, at this point the vector field \(V\) is smooth with all first-order covariant derivatives vanishing, i.e., \(d(0, y) = 0\). Take other than our “polar”-type coordinates: let, for instance, the “new” coordinate \(\{x', y'\}\) be the normal coordinates on \(\Sigma\) with the center at \(O\). Because \(\omega_\xi\) for small \(x\) are circles with radius \(x\) around \(O\) we have in new coordinates:
\[
\nabla_{Y \cos(\alpha) - x \sin(\alpha)} V(x \cos(\alpha), x \sin(\alpha)) \equiv 0
\]
for all \(0 \leq \alpha \leq 2\pi\), which after taking derivative on \(\alpha\) gives
\[
\nabla_X \nabla_X V(0, 0), \nabla_X \nabla_Y V(0, 0), \nabla_Y \nabla_Y V(0, 0) = 0,
\]
which in turn as in (15) implies
(15') \[
Y'(0, 0; s) = \hat{Y}(0, 0; s), \quad X'(0, 0; s) = \hat{X}(0, 0; s) \quad \text{and} \quad \nabla_X, X'(x', y'; s), \nabla_Y, Y'(x', y; s) = 0
\]
for new coordinate vectors \(X'(x', y'; s), Y'(x', y; s)\) on \(\Omega_s\), and by (31) implies for the external curvature of \(\Omega_s\) at \(P(0, 0; s)\) the claim of the Lemma. The proof of the Lemma 9 is complete.
\[ (R(\tilde{X}, \tilde{Y})\tilde{Y}, \tilde{X}) - \frac{3}{4}\|R(X, Y)V\|^2 \]

and from the Gauss fundamental equation for the curvature term \( \tilde{R}(x, y; s) \) of the surface \( \Omega_s \) follows

\[ \tilde{R}(x, y; s) = (R(\tilde{X}(x, y; s), \tilde{Y}(x, y; s))\tilde{Y}(x, y; s), \tilde{X}(x, y; s)) + R_{s=0}^e(x, y), \]

from Lemmas 7-9 we conclude our main formula of this section

\[ \tilde{R}(x, y; s) = \tilde{R}(x, y; 0) + s^2 D + o(s^2), \]

where

\[ D = \frac{1}{2}(-Y(\nabla_Y \nabla_X V, \nabla_X V) - X(\nabla_X \nabla_Y V, \nabla_Y V) + X(\nabla_Y \nabla_X V, \nabla_Y V) + X(\nabla_Y \nabla_Y V, \nabla_X V)), \]

and the curvature term \( \tilde{R}(x, y; 0) \) of \( \Omega_0 \) equals \( (R(X, Y)Y, X)(x, y; 0) \).\(^{11}\) The sectional curvature \( \tilde{K}(x, y; s) \) of \( \Omega_s \) at the point \( P(x, y; s) \) is given through the curvature term as

\[ \tilde{K}(x, y; s) = \frac{(R(\tilde{X}(x, y; s), \tilde{Y}(x, y; s))\tilde{Y}(x, y; s), \tilde{X}(x, y; s))}{\|X(x, y; s) \wedge Y(x, y; s)\|^2}. \]

Because

\[ ||\tilde{X}(x, y; s) \wedge \tilde{Y}(x, y; s)||^2 = 1 + s^2(d^2 + e^2) \]

the equality (37) gives for the curvature forms

\[ \tilde{K}(x, y; s)d\sigma_s = \tilde{K}(x, y; 0)(1 - \frac{s^2}{2}(d^2 + e^2))d\sigma_0 + s^2 Dd\sigma_0 + o(s^2), \]

where \( d\sigma_s \) denotes the area form \( d\tilde{X}(x, y; s) \wedge d\tilde{Y}(x, y; s) \) of \( \Omega_s \).

3. Local parallel section. Vanishing holonomy case

At this point we should repeat again that to simplify Lemmas 7-9 calculations we have used local coordinates in \( \Omega \) satisfying some assumptions such as: coordinate vectors \( X(x, y) \) and \( Y(x, y) \) at the given point \( P \) were orthonormal and such that derivatives of the vector field \( V(x, y) \) given by the Theorem 1 \( \nabla_X V \) and \( \nabla_Y V \) were orthogonal. We also assumed above that

\[ \nabla_X X(P) = \nabla_X Y(P) = \nabla_Y Y(P) = 0 \]

\(^{11}\)Similar but simpler calculations give

\[ ((R(\tilde{X}, \tilde{Y})\tilde{Y}, \tilde{X})|_{s=0}(x, y)) = ((R(\tilde{X}, \tilde{Y})\tilde{Y}, \tilde{X})(x, y; s)_{s=0} = 0. \]
at the point $P$ where we calculated terms of (40). However in the obtained formula (40) not only curvature terms do not depend on this particular choice of coordinates, but also the term
\begin{equation}
    d^2 + e^2 = (\nabla_X V, \nabla_X V) + (\nabla_Y V, \nabla_Y V)
\end{equation}
can be rewritten as an invariant, known as the "vertical" part of the energy of our vertical lift $V : P(x, y) \rightarrow V(x, y)$, as follows:
\begin{equation}
    E^V(V(x, y)) = g^{ij}(x, y)(\nabla_i V, \nabla_j V)
\end{equation}
for an arbitrary coordinate system $\{x^1, x^2\}$ in the neighborhood of $P$ in $\Sigma$, where $\nabla_i V = \nabla_{\partial_i} V$ and $g^{ij}(x^i, x^j)$ and $g^{ij}(x^i, x^j)$ are metric tensor and its inverse correspondingly. Hence, the same is true also for the $D$-term in (40): it can be expressed in a form which is invariant under coordinate changes. The exact formula easily follows from its origin from Lemmas 7-9’s calculations and is left until the next paper where we study its properties in more details. In this paper the following property of $D$ is crucial.

**Lemma 10.** The two-form $D(x, y)dX(x, y) \wedge dY(x, y)$ is exact
\begin{equation}
    D(x, y)dX(x, y) \wedge dY(x, y) = d\eta(x, y),
\end{equation}
where the one-form $\eta$ has the type:
\begin{equation}
    \eta(x, y) = A(x, y)dX + B(x, y)dY,
\end{equation}
with coefficients $A(x, y), B(x, y)$ of the form
\begin{equation}
    A(x, y) = (A_1(x, y), \nabla_X V(x, y)) + (A_2(x, y), \nabla_Y V(x, y))
\end{equation}
and
\begin{equation}
    B(x, y) = (B_1(x, y), \nabla_X V(x, y)) + (B_2(x, y), \nabla_Y V(x, y))
\end{equation}

**Proof.** The proof is immediate by the definition of the differential:
\begin{align*}
    D(x, y)dX(x, y) \wedge dY(x, y) &= (Y(\nabla_X \nabla_Y V, \nabla_X V) - Y(\nabla_Y \nabla_X V, \nabla_X V) - X(\nabla_X \nabla_Y V, \nabla_Y V) + \\
    X(\nabla_Y \nabla_X V, \nabla_Y V) + Y(\nabla_Y \nabla_Y V, \nabla_X V) - Y(\nabla_X \nabla_Y V, \nabla_X V))dX(x, y) \wedge dY(x, y) = \\
    d((\nabla_X \nabla_Y V, \nabla_X V)dX) - d((\nabla_X \nabla_Y V, \nabla_Y V)dY) + d((\nabla_Y \nabla_X V, \nabla_Y V)dY) + d((\nabla_Y \nabla_Y V, \nabla_X V)dY) = \\
    \frac{1}{2}d((d^2)_g dX - (e^2)'_g dY) - d((f^2)_g dX - (f^2)'_g dY),
\end{align*}
where $f^2 = (\nabla_X V, \nabla_Y V)$; or
\begin{equation}
    \eta = \frac{1}{2}((d^2)^g dX - (e^2)'_g dY) - ((f^2)'^g dX - (f^2)_g dY).
\end{equation}

Now we can prove our main technical results. The first one is about the vector field $V(x, y)$ from the Theorem 1a. Denote for short by $\Omega^* = \{P(x, y)|x \leq x^*\} \subset \Omega$ the domain where the vector field $V(x, y)$ is defined and by $\Omega^*$ the vertical lift of $\Omega$ in direction of this vector field.
\textbf{Theorem 2a.} When the holonomy vanishes along the boundary $\omega_{x^*}$ of $\Omega^*$ then the vector field $V(x, y)$ constructed in the Theorem 1a is parallel on $\Omega^*$:

\[ \nabla_X V(x, y) = \nabla_Y V(x, y) \equiv 0. \]

\textit{Proof.} First we note that all the curves $\omega_s^*(y) = \partial \Omega_s^*$ which are vertical lifts of $\omega_{x^*}(y)$ have the same geodesic curvature. Indeed, from (1) it follows $\nabla Y [V, Y] \equiv 0$ and $R(V, Y)Y \equiv 0$. Hence, from

\[ \nabla V \nabla Y = R(V, Y) + \nabla Y \nabla V + \nabla [V, Y] Y = R(V, Y)Y \]

we have

(45) \[ \nabla Y (\omega_s^*(y)) \equiv k g(y) \bar{X}(\omega_s^*(y)), \]

where $k g(y)$ stands for the geodesic curvature of $\omega_{x^*}(y)$. In the case of the vanishing holonomy by the Theorem 1 and (15) the tangent subspace to $\Omega_s^*$ along $\omega_s^*(y)$ coincides with the horizontal subspace, i.e., contains the vector $\nabla Y (\omega_s^*(y))$ of the geodesic curvature of this vertical lift of $\omega_{x^*}$, which implies that the geodesic curvature of $\omega_s^*(y)$ in $\Omega_s^*$ is the same as the geodesic curvature of $\omega_{x^*}(y)$ in $\Omega^*$. Hence, by the Gauss-Bonnet theorem

(46) \[ \int_{\Omega_s^*} \tilde{K}(x, y; s)d\sigma_s = \int_{\Omega^*} \tilde{K}(x, y; 0)d\sigma_0. \]

If we compare this with (40) we get

(47) \[ \int_{\Omega^*} \tilde{K}(x, y; 0)(d^2 + e^2)d\sigma_0 = 2 \int_{\Omega^*} D d\sigma_0. \]

By the Lemma 10 and Stokes theorem

(48) \[ \int_{\Omega^*} D d\sigma_0 = \int_{\omega_{x^*}} \eta, \]

which in turn equals zero since by the Theorem 1 the one-form $\eta$ vanishes identically along $\omega_{x^*}$. I.e., we have

\[ \int_{\Omega^*} \tilde{K}(x, y; 0)(d^2 + e^2)d\sigma_0 = 0, \]

or

(49) \[ \tilde{K}(x, y; 0)(d^2 + e^2) \equiv 0 \]

from the non-negativity of the curvature. By the "prism"-construction the holonomy $I_c$ along a closed curve $c$ vanishes if $c$ is inside some open domain in $\Sigma$ with zero curvature and is contractible in this domain, see the
Lemma 3.6 [M3]. Therefore, \( d \) and \( e \) vanish in the interior of the closure of the set in \( \Omega \) where \( K(x, y; 0) = 0 \) equals zero. Which leads to

\[
d(x, y) = e(x, y) = 0 \quad \text{if} \quad K(x, y; 0) = 0
\]

because \( d(x, y), e(x, y) \) are smooth functions, and

\[
d(x, y) = e(x, y) = 0 \quad \text{for all} \quad P(x, y) \in \Omega^*
\]

with the help of (50). The Theorem 2a is proved.

Note that the proven result does not mean that the holonomy on \( \Omega^* \) is trivial. We have proved only that the vector field \( V(x, y) \) constructed in the Theorem 1a is parallel on \( \Omega^* \), which does not imply that the infinitesimal holonomy operators \( R(X, Y)(x, y) \) vanishes identically. Note also that under condition: \( R(X, Y)(x, y) \neq 0 \) the vector field \( V(x, y) \) coincides with \( V^*(x, y) \) (which is not defined otherwise).

Next we note that the form \( \eta \) also vanishes along an arbitrary geodesic: if some curve \( \omega_x(y) \) is a geodesic and \( V(x, y) \) is a vector field which is parallel along \( \omega_x \) then in a local half-geodesic coordinate system with \( \omega_x \) as an axe (such system of coordinates obviously satisfies our restrictions on coordinate systems where the form \( \eta \) is given by the formula (44) above) it holds:

\[
\eta(\dot{\omega}_x(y)) = -\eta(\partial/\partial y) = -(e^2)'_x = (\nabla_X \nabla_Y V(x, y)Y, \nabla_Y V(x, y)) = 0
\]

since \( \nabla_Y V(x, y) = 0 \) by the definition of \( V(x, y) \). This implies our second main technical result.

**Theorem 2b.** If \( \Omega \) is bounded by the closed geodesic \( \omega(y) \), and we have 1) or 2) case in the Theorem 1b then the vector field \( V(x, y) \), which existence is stated in the Theorem 1b in \( \Omega \) or \( \Omega \backslash \Omega^*_x \), is parallel in the corresponding domain.

**Proof.** The proof is immediate by the same arguments as in the proof of the Theorem 2a. If the vector field \( V(x, y) \) is defined on the whole \( \Omega \) we can define the vertical lift \( \Omega_x \). Because the boundary \( \omega \) of \( \Omega \) is a geodesic its vertical lifts \( \omega_x \) are also geodesics in \( M \) by (45). Then as in the Theorem 2a by the Gauss-Bonnet theorem it holds (46) which with the help of the Stokes formula and (52) implies (49) and the claim (51) of our theorem as above. If the vector field \( V(x, y) \) is defined only on some sub-domain \( \Omega^*(x^*) = \Omega \backslash \Omega^*_x \) for \( 0 < x^* < 1 \) (i.e., we have the second case in the Theorem 1b), then we apply (45) and the Gauss-Bonnet theorem to the vertical lift \( \Omega_x(x^*) \) of this sub-domain. Then the Lemma 10 together with (52) infer (51).

**4. Global section. The case of non-vanishing holonomy.**

If the holonomy never vanishes we may, actually, construct a global parallel section \( V: \Sigma \to \nu \Sigma \) of the unit normal bundle of \( \Sigma \). The proof is easy by going to contradiction. Indeed, assume that at some point \( O_+ \in \Sigma \) the holonomy operator \( R(X, Y) \) is not zero. Then, as we have seen already, in the neighborhood of this point \( O_+ \) the smooth vector field \( V^* \) is correctly defined. Assume that it is not parallel, i.e.,

\[
\nabla_X V^* \quad \text{or} \quad \nabla_X V^* \quad \text{not zero}.
\]
Take another point $O_-$, a disk $\Omega^r$ with a center $O_-$ of a small radius $r$. Next consider the homotopy $\omega_x$ of the boundary of this disk $\omega^r$ to a point $O_+$ inside $\Omega = \Sigma \setminus \Omega^r$. Then the family of parallel transports $I_x$ along $\omega_x$ never vanishes for otherwise we would not have (52) by the Theorem 2. Thus, taking $r \to 0$ we can define the vector field $V$ as in the Theorem 1 on $\Sigma$.

Now, applying Lemmas 7-9 computations to this global section $V$ instead of (48) we have

\[
\int_{\Sigma} Dd\sigma_0 = \int_{\partial \Sigma} \eta = \int_{\emptyset} \eta = 0,
\]

implying (51) as before, i.e., that the constructed section is parallel. Therefore, the following is true.

**Lemma 11.** If the holonomy does not vanish for any $(O_+, O_-)$-homotopy then there exists a global parallel section $V : \Sigma \to \nu \Sigma$ of a unit normal bundle such that the family of corresponding lifts

$$\Sigma_s = \{ \text{exp}_P s V(P) \mid P \in \Sigma, \quad 0 \leq s \leq s_0 \}$$

is isometric to the direct product $\Sigma \times [0, s_0]$.

When the global parallel section $V : P \in \Sigma \to V(P) \in \nu \Sigma$ exists the proof of the Theorem A is easy. Indeed, then all horizontal lifts $\Sigma_s$ are totally geodesic sub-manifolds in $M$ isometric copies of $\Sigma$, or pseudo-souls. Thus, arguing in the same way as in the original paper by Cheeger and Gromoll, see [CG] or [Y], we can prove that the sectional curvature of $M$ vanish in all two-dimensional "vertizontal" directions along $\Sigma_s$, i.e., generated by one vector tangent to $\Sigma_s$ and another - normal to it.\footnote{First on $\Sigma \setminus \{O_-\}$, but then arguments as above in footnote 10 shows that the vector field $V$ can be smoothly continued to the point $O_-$ as well.}

It would be interesting to understand when the global section exists. Note that, as we will prove in an instant (see the next section) along every closed geodesic $\gamma$ on $\Sigma$ their exists a parallel normal vector field with vanishing covariant derivatives. Thus, it would be natural to conjecture the existence of the global parallel section in a case when for every point of $\Sigma$ goes some closed geodesic.

### 5. The proof of the Theorem A

There exists at least one closed geodesic $\gamma$ in $\Sigma$ which is contractible since $\Sigma$ is diffeomorphic to the sphere $S^2$. Consider the homotopy $\omega_x, 0 \leq x \leq 1$ between some point $O$ and $\gamma$. According to the Theorems 1b and 2b there exists a vector field $V^*(y)$ parallel along $\gamma(y)$ such that

\[
R(X, Y)V^*(y) \equiv 0.
\]

Then by the Lemmas 1 and 2 we conclude

\[
A(Q(V^*(y), s)) \equiv 0,
\]

\footnote{or even, using Perelman's arguments, that (1) is fulfilled for $\Sigma_s$ too. For the proof consider "up-and-down" construction from [M1] and proceed as in [P].}
i.e., along all geodesics $\gamma_s(y) = \exp_{\gamma(s)} sV^*(y)$ which are horizontal lifts of $\gamma$ the fundamental $A$-tensor vanishes. This implies that the family of vertical spaces $V(Q(V^*(y), s))$ is parallel along $\gamma_s$. Fix some $s > 0$, two unit and vertical parallel vector fields $W(y)$ and $U(y)$ along $\gamma_s(y), 0 \leq y \leq 2\pi$, and consider the mean curvature vector $H(y)$ of the vertical fiber at $\gamma_s(y)$:

$$H(y) = T_{W(y)}W(y) + T_{U(y)}U(y).$$

It does not depend on the particular choice the orthonormal base $\{W(y), U(y)\}$ and therefore is a smooth vector field along $\gamma_s$. Thus the scalar product $(H(y), Y(y))$, where $Y(y) = \dot{\gamma}_s(y)$ is a periodic function along $\gamma_s$. For its derivative we have

$$(H(y), \dot{\gamma}_s(y))_y = ((\nabla_Y T)_{W(y)}W(y) + (\nabla_Y T)_{U(y)}U(y))$$

since $\gamma_s$ is a geodesic and $\{W(y), U(y)\}$ are parallel along $\gamma_s$. An integral of (57) over closed $\gamma_s$ equals zero, which with the help of (56) and (3) implies

$$\int_{\gamma_s} (R[W(y), Y(y)] + R[U(y), Y(y)])dy = -\int_{\gamma_s} (\|T_W Y(y)\|^2 + \|T_U Y(y)\|^2)dy.$$

Because the curvature is non-negative we conclude that curvatures $R[W(y), Y(y)]$ and $R[U(y), Y(y)]$ vanish along the geodesic $\gamma_s$ together with the second fundamental form of vertical fibers relative to the normal $Y(y)$:

$$R[W(y), Y(y)] = R[U(y), Y(y)] = 0$$

and

$$T_W(y)Y(y) = T_U(y)Y(y) = 0,$$

which due to the Gauss fundamental equation implies that not only the sectional curvature of $M$ vanish along $\gamma_s$ in two-dimensional directions $\{W(y), Y(y)\}$ but also that the sectional curvature of the hypersurface $N\Sigma$ in the same direction equals zero.

Theorem A is proved.

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