The Twisted Heisenberg Algebra $\mathcal{U}_{h,w}(\mathcal{H}(4))$

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Abstract

A two parametric deformation of the enveloping Heisenberg algebra $\mathcal{H}(4)$ which appear as a combination of the standard and a nonstandard quantization given by Ballesteros and Herranz is defined and proved to be Ribbon Hopf algebra. The universal $\mathcal{R}$-matrix and its associated quantum group are constructed. New solution of Braid group are obtained. The contribution of these parameters in invariants of links and WZW model are analyzed. General results for twisted Ribbon Hopf algebra are derived.

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I  Introduction

An enveloping Lie algebra \( \mathcal{U}(\mathcal{G}) \) has many quantizations \((\geq 2)\): The first one is called the Drinfeld-Jimbo quantization \([1, 2]\), whereas the other ones are called the nonstandard quantizations \([3]\). Recently there is much interest in studies relating to various aspects of the nonstandard quantizations.

The standard \(q\)-Heisenberg algebras and their universal \(\mathcal{R}\)-matrices \([8]\) has recently attracted wide attention. The use of \(q\)-Heisenberg algebras to describe composite particles \([9]\), the description of certain classes of exactly solvable potentials in terms of a \(q\)-Heisenberg dynamical symmetry \([10]\), the link between deformed oscillator algebras and superintegrable systems \([11, 12]\) and the relations between these deformed algebras and \(q\)-orthogonal polynomials \([13]\) are the most attractive examples. In reference \([14]\), the quasitriangular \(q\)-oscillator algebra has been found to be related to Yang-Baxter systems and link invariants.

The Heisenberg algebras has also many nonstandard quantizations. Recently, the coboundary Lie bialgebras and their corresponding Poisson-Lie structures has been constructed for the Heisenberg algebra. The quantum nonstandard Heisenberg algebras are derived from these bialgebras by using the Lyakhovsky and Mudrov formalism and for some cases, quantizations at both algebra an group levels have been obtained, including their universal \(\mathcal{R}\)-matrices \([15]\).

Let us mention that the combination of some nonstandard quantizations to the standard ones permits to equip the enveloping algebras with multiparameter quantizations. Following this way, we define a new two parametric quantization of the Heisenberg algebras \(\mathcal{U}_{h,w}(\mathcal{H}(4))\) by combination of the standard and a nonstandard quantization given in \([15]\). This type of deformed bosons can be expected to build up \([h, w]\)-boson realizations of a possible two parametric \(\mathcal{U}_{h,w}(\mathfrak{sl}(2))\) algebra which not exit in the literature at our knowledge.

One of the instances in which both the parameters of the nonstandard Heisenberg algebra become relevant is in the context of reflection equation

\[
R_{12}K_1R_{21}K_2 = K_2R_{12}K_1R_{21}.
\]  \hspace{1cm} (I.1)

It was introduced in \([16]\) in the study of two particle scattering on a half-line, where the matrix \(K\) described the reflection of a particle at the end point. It is also known \([17]\) to have other applications, such as, a generalization of the inverse scattering method to the case of non-ultralocal commutation relations and a lattice regulatized version of Kac-Moody algebras. The reflection algebra is closely connected to the quantum group and the matrix \(K\) may be realized \([18]\) from the knowledge of the Lax operators \(L^\pm\) :

\[
K = S \left(L^-\right) L^+.
\]  \hspace{1cm} (I.2)

\footnote{In the case of \(\mathcal{U}(\mathfrak{sl}(2))\), the nonstandard quantization \([3]\) was obtained as contraction of the Drinfeld-Jimbo one (see \([1]\)). The universal \(\mathcal{R}\)-matrix of the nonstandard algebra \(\mathcal{U}_h(\mathfrak{sl}(2))\) was obtained \([1, 2]\).}
One viewpoint advocated by Majid [19] is the ‘transmutation procedure’ converting the quantum groups to the braided groups, which may be looked simply as a generalization of the supergroups with ± Bose-Fermi statistics being replaced by braid statistics. There are indications [20] that particles of braid statistics arise in low dimensional field theory. As the construction of the above braided algebra can be done for any regular invertible \( R \)-matrix, it is particularly relevant to study the deformed Heisenberg algebras, which are closely connected to the statistics problem.

From the knowledge of the universal \( R \)-matrix of \( U_{h,\omega}(H(4)) \), we can construct the Lax operators \( L^\pm \), which, as a consequence of (1.2), generates the reflection operator \( K \). This necessarily depends on both deformation parameters \((h, \omega)\) and, in the limit \( \omega \to 0 \) limit, yields the \( K \) operator of the \( U_h(H(4)) \) algebra. In the present context we would just like to point out that in the braid statistics problem, which appears for a chain of \((h, \omega)\)-deformed oscillators, both deformation parameters are physically important.

The algebra \( U_{h,w}(H(4)) \) can also be interpreted as two parametric deformation of an extended \((1 + 1)\)-poincaré algebra. Others interesting applications related to braid groups and special functions theory of our two parametric quantization can be found.

The main purpose of this work is fourfold:

(i) To provide the enveloping Heisenberg algebras with a two parametric deformation which appear as a combination of the standard quantization and a nonstandard one [15].

(ii) We derive its universal \( R_{h,w} \)-matrix, we use it to make explicit connection with the formalism of matrix quantum pseudogroups due to Woronowicz [21, 22] and to define the Hopf algebra of the representative elements.

(iii) Using the results obtained in [14], this two parametric deformation is proved to be a ribbon algebra. General results for a twisted ribbon Hopf algebra are also derived.

(iv) We give a new solution of the braid group \( B_m \) and we analyze briefly the contribution of these two parameters in WZW model and in invariants of links.

This paper is organized as follows: In section II, we give the definitions of the standard and nonstandard quantizations of the Heisenberg algebra, their central elements and their universal \( R \)-matrices. In section III, we introduce the two parametric deformation and its quasitriangular Hopf structure. New solution of the braid group \( B_m \) and the contribution of these parameters in the theory of invariants of links is also discussed. In the section IV, we determine the infinitesimal generators of the matrix pseudogroup and we calculate both commutations relations, coproducts, counits and antipodes. The two parametric deformation is proved to be a ribbon Hopf algebra in section V. General results are also established. Some comments concerning the WZW model are also presented. Finally, we conclude with some remarks and perspectives in section VI.
II Standard and Nonstandard Heisenberg Algebras and their Universal $\mathcal{R}$-matrices

In this paper, $h$ and $w$ are arbitrary complex numbers. We denote by $n$, $e$, $a^+$ and $a$ the generators of the Heisenberg algebra $\mathcal{H}(4)$ that satisfy the commutation relations

\[
[a, a^+] = e, \quad [n, a] = -a, \quad [n, a^+] = a^+, \quad [e, .] = 0
\]  

(II.1)

and denote by $\mathcal{U}(\mathcal{H}(4))$ the enveloping algebra of $\mathcal{H}(4)$. The Heisenberg group is denoted by $H(4)$. Let

\[
r = a \otimes a^+ - e \otimes n
\]  

(II.2)

and introduce the notation $r_{ij}$, $1 \leq i < j \leq 3$, where

\[
r_{12} = \sum_i a_i \otimes b_i \otimes 1, \quad r_{23} = \sum_i 1 \otimes a_i \otimes b_i
\]

and

\[
r_{13} = \sum_i a_i \otimes 1 \otimes b_i
\]

if $r = \sum_i a_i \otimes b_i \in \mathcal{U}(\mathcal{H}(4)) \otimes \mathcal{U}(\mathcal{H}(4))$. The element (II.2) solves the classical Yang-Baxter equation (CYBE)

\[
[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0
\]  

(II.3)

and its called a classical $r$-matrix. The quantum Hopf algebra $\mathcal{U}_h(\mathcal{H}(4))$ which quantizes the standard bialgebra generated by the classical $r$-matrix (II.2) is defined as:

**Definition 1** The quantum standard algebra $\mathcal{U}_h(\mathcal{H}(4))$ is the unital associative algebra with the generators $N$, $E$, $A^+$, $A$ and the relations

\[
[A, A^+] = \frac{\sinh(hE)}{h}, \quad [N, A^+] = A^+, \quad [N, A] = -A, \quad [E, .] = 0
\]  

(II.4)

The algebra $\mathcal{U}_h(\mathcal{H}(4))$ admits a Hopf structure with coproducts, counits and antipodes determined by

\[
\begin{align*}
\Delta_h(N) &= N \otimes 1 + 1 \otimes N, & S_h(N) &= -N, & \varepsilon_h(N) &= 0, \\
\Delta_h(E) &= E \otimes 1 + 1 \otimes E, & S_h(E) &= -E, & \varepsilon_h(E) &= 0, \\
\Delta_h(A^+) &= A^+ \otimes 1 + 1 \otimes A^+, & S_h(A^+) &= -A^+, & \varepsilon_h(A^+) &= 0, \\
\Delta_h(A) &= A \otimes e^{hE} + e^{-hE} \otimes A, & S_h(A) &= -A, & \varepsilon_h(A) &= 0.
\end{align*}
\]  

(II.5)

In this point of view, the generators $N$, $E$ and $A^+$ are primitive elements.

There exist another element belonging to the center of the algebra $\mathcal{U}_h(\mathcal{H}(4))$ given by

\[
C_h = N \frac{\sinh(hE)}{h} - \frac{1}{2}(A^+A + A A^+).
\]  

(II.6)

From the relations (II.5), we have $S_h^2 = \text{id}$. The quantum Hopf algebra (II.4) and (II.5) is equivalent to the structure defined by Celeghini et al \[3\]. The coproduct $\Delta_h$, counit $\varepsilon_h$ and antipode $S_h$ (II.5) are related to Celeghini et al coproduct $\Delta_c$, counit $\varepsilon_c$ and antipode $S_c$ by

\[
\begin{align*}
\Delta_h &= A^{-1} \Delta_c A, & \varepsilon_h &= \varepsilon_c, & S_h &= S_c,
\end{align*}
\]  

(II.7)
where, \( A = e^{-\frac{h}{2}(E \otimes N - N \otimes E)} \). The classical \( r \)-matrix associated to the quantum structure defined in \( \mathbb{R} \) is given by

\[
 r = a \otimes a^+ - \frac{1}{2}(e \otimes n + n \otimes e). \tag{II.8}
\]

Recall that a quasitriangular Hopf algebra is a pair \((U, \mathcal{R})\) where \( U \) is a Hopf algebra and \( \mathcal{R} \) is invertible element obeying to the following axioms

\[
 \mathcal{R} \triangle (u)\mathcal{R}^{-1} = \sigma \circ \triangle(u), \quad u \in U, \tag{II.9}
\]

\[
 (\triangle \otimes \text{id})(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{23}, \quad (\text{id} \otimes \triangle)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{12}, \tag{II.10}
\]

where, if \( \mathcal{R} = \sum_i a_i \otimes b_i \in U \otimes U \), we denote \( \mathcal{R}_{12} = \sum_i a_i \otimes b_i \otimes 1 \in U \otimes U \otimes U \), \( \mathcal{R}_{13} = \sum_i 1 \otimes a_i \otimes b_i \) and \( \sigma \) is the flip operator \( \sigma(a \otimes b) = b \otimes a \). The relation (II.9) indicates that \( \mathcal{R} \) being a intertwining operator on the coproduct \( \triangle \). \( \mathcal{R} \) is called an universal \( \mathcal{R} \)-matrix and satisfies the quantum Yang-Baxter equation (QYBE)

\[
 \mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}. \tag{II.11}
\]

Now, given a quasitriangular Hopf algebra \((U, \mathcal{R})\) and an invertible element \( F \in U \otimes U \) satisfying the following conditions:

\[
 (\triangle_h \otimes \text{id})(F)\mathcal{F}_{12} = (\text{id} \otimes \triangle_h)(F)\mathcal{F}_{23}, \tag{II.12}
\]

\[
 \mathcal{F}_{12}^{-1}(\triangle_h \otimes \text{id})(F^{-1}) = \mathcal{F}_{23}^{-1}(\text{id} \otimes \triangle_h)(F^{-1}), \tag{II.13}
\]

\[
 (\varepsilon_h \otimes \text{id})(F) = (\text{id} \otimes \varepsilon_h)(F), \tag{II.14}
\]

\[
 (\varepsilon_h \otimes \text{id})(F^{-1}) = (\text{id} \otimes \varepsilon_h)(F^{-1}) \tag{II.15}
\]

one can form a new quasitriangular Hopf algebra \( U_F \) by twisting \( U \): \( U_F \) retains the vector space and multiplication of \( U \) while its quasitriangular Hopf structure is given by

\[
 \triangle_F \equiv F^{-1} \triangle F, \\
 \varepsilon_F = \varepsilon, \\
 S_F = v^{-1}Sv, \\
 \mathcal{R}_F = \mathcal{F}_{21}^{-1}\mathcal{R}\mathcal{F} \tag{II.16}
\]

where, the element \( v \) and its inverse are obtained from the invertible element \( F \in U \otimes U \) as follows:

\[
 v = m(S \otimes \text{id})(F) \quad v^{-1} = m(\text{id} \otimes S)(F^{-1}). \tag{II.17}
\]

If \( S^2 = \text{id} \), the twisted quasitriangular Hopf algebra \( U_F \) thus not conserve the same property, namely,

\[
 S_F^2(a) = v^{-1}S(v)aS(v^{-1})v. \tag{II.18}
\]
Proposition 1 Let \((U, R)\) a quasitriangular Hopf algebra and an element \(F \in U \otimes U\) satisfying the conditions (II.12)-(II.15). The element \(v\) defined in (II.17) has the following properties:

\[
\varepsilon(v) = 1, \quad \Delta(v) = (S \otimes S)(F_{21}^{-1})(v \otimes v) \cdot F^{-1},
\]

(II.19)

and, for the element \(v^{-1}S(v)\), we have

\[
\Delta(v^{-1}S(v)) = F(v^{-1}S(v) \otimes v^{-1}S(v))(S^2 \otimes S^2)(F^{-1}).
\]

(II.20)

If \(S^2 = \text{id}\), the relation (II.20) reads

\[
\Delta_F(v^{-1}S(v)) = v^{-1}S(v) \otimes v^{-1}S(v).
\]

(II.21)

Proof: The equations (II.19) arise by direct calculations. The relations (II.20) and (II.21) are derived using (II.19).

Proposition 2 The Hopf algebra \(U_h(H(4))\) is quasitriangular. The universal \(R\)-matrix has the following form

\[
R^h = \exp(-2h E \otimes N) \exp(2h e^h E A \otimes A^+) \quad \text{(II.22)}
\]

and satisfies the Quantum Yang-Baxter equation (II.11).

Proof: Similar to the proof given in [8] (see proposition II.9). The \(R\)-matrix (II.22) is related to \(R_c\)-matrix given in [8] by the following twist

\[
R^h = A_{21}^{-1} R_c A.
\]

(II.23)

Now, let us consider the nonstandard classical \(r\)-matrix

\[
r = n \otimes a^+ - a^+ \otimes n,
\]

(II.24)

which solves the classical Yang-Baxter equation (II.3). The quantum Hopf algebra which quantizes the nonstandard bialgebra generated by the classical \(r\)-matrix (II.24) was given by Ballesteros and Herranz [15]:

Definition 2 The quantum nonstandard algebra \(U_w(H(4))\) is the unital associative algebra generated by \(N, E, A^+\) and \(A\), satisfying the commutations relations

\[
[A, A^+] = E e^{w A^+}, \quad [N, A^+] = \frac{e^{w A^+} - 1}{w}, \quad [N, A] = -A, \quad [E, \cdot \ ] = 0. \quad \text{(II.25)}
\]
The algebra $\mathcal{U}_w(\mathcal{H}(4))$ has the following Hopf structure

$$
\begin{align*}
\Delta_w(N) &= N \otimes e^{wA^+} + 1 \otimes N, & S_w(N) &= -Ne^{-wA^+}, & \varepsilon_w(N) &= 0, \\
\Delta_w(E) &= E \otimes 1 + 1 \otimes E, & S_w(E) &= -E, & \varepsilon_w(E) &= 0, \\
\Delta_w(A^+) &= A^+ \otimes 1 + 1 \otimes A^+, & S_w(A^+) &= -A^+, & \varepsilon_w(A^+) &= 0
\end{align*}
$$

and

$$
\begin{align*}
\Delta_w(A) &= A \otimes e^{wA^+} + 1 \otimes A + wN \otimes E e^{wA^+}, & S_w(A) &= -A e^{-wA^+} + wNE e^{-wA^+}, \\
\varepsilon_w(A) &= 0.
\end{align*}
$$

The generators $E$ and $A_+$ are primitives.

We will not consider here the others nonstandard deformations. The Casimir element of $\mathcal{U}_w(\mathcal{H}(4))$ is given by [15]

$$
C_w = N E + \frac{e^{-wA^+} - 1}{2w} A + A e^{-wA^+} - \frac{1}{2w}.
$$

**Proposition 3** The Hopf algebra $\mathcal{U}_w(\mathcal{H}(4))$ is quasitriangular. The universal $\mathcal{R}$-matrix has the following form [13]:

$$
\mathcal{R}^w = \exp(-w A^+ \otimes N) \exp(w N \otimes A^+) \exp(wN \otimes A^+)
$$

and satisfies the quantum Yang-Baxter equation (II.11).

In passing, let us mention that $\mathcal{R}^w = \mathcal{F}^{-1}_{21} \mathcal{F}$, with $\mathcal{F} = \exp(wN \otimes A^+)$ and $\Delta_w = \mathcal{F}^{-1} \Delta_0 \mathcal{F}$, where $\Delta_0$ is the coproduct of the enveloping algebra $\mathcal{U}(\mathcal{H}(4))$, extended to $\mathcal{U}(\mathcal{H}(4))[w]$, namely

$$
\begin{align*}
\Delta_0(N) &= N \otimes 1 + 1 \otimes N, \\
\Delta_0(E) &= E \otimes 1 + 1 \otimes E, \\
\Delta_0\left(\frac{1-e^{-wA^+}}{w}\right) &= \left(\frac{1-e^{-wA^+}}{w}\right) \otimes 1 + 1 \otimes \left(\frac{1-e^{-wA^+}}{w}\right), \\
\Delta_0(A) &= A \otimes 1 + 1 \otimes A.
\end{align*}
$$

$\mathcal{F}$ appear as an element which deforms the coproduct $\Delta_0$ of $\mathcal{U}(\mathcal{H}(4))$ to the coproduct $\Delta_w$ of $\mathcal{U}_w(\mathcal{H}(4))$. Furthermore, the antipodes and counits are

$$
\begin{align*}
S_0(N) &= -N, & S_0(A) &= -A, & S_0\left(\frac{1-e^{-wA^+}}{w}\right) &= -\left(\frac{1-e^{-wA^+}}{w}\right), \\
\varepsilon_0(N) &= \varepsilon_0(A) = \varepsilon_0\left(\frac{1-e^{-wA^+}}{w}\right) = 0.
\end{align*}
$$
Let us just mention, that there exist other solutions of the classical Yang-Baxter equation given by
\[ r(\mu, \nu) = \mu(a \otimes e - e \otimes a) + \nu(a^+ \otimes e - e \otimes a^+), \quad \mu, \nu \in \mathbb{C}. \quad (\text{II.32}) \]
The quantum Hopf algebra which quantizes the nonstandard bialgebra generated by the classical \( r \)-matrix \( r(\mu, 0) \) can be obtained as contraction limit from \( U_h(sl(2)) \otimes u(1) \). Another interesting solution will be the subject of the following section.

### III A Two Parametric Deformation of \( \mathcal{U}(\mathcal{H}(4)) \) and Links Invariants

In this section, we analyse the algebra which corresponds to a combination of the standard deformation (II.4) and the nonstandard one (II.25). Let us consider the element
\[ r(h, w) = 2h(a \otimes a^+ - e \otimes n) + w(n \otimes a^+ - a^+ \otimes n), \quad h, w \in \mathbb{C} \quad (\text{III.1}) \]
which satisfies the classical Yang-Baxter equation (II.3). In this case, the quantum Hopf algebra \( \mathcal{U}_{h,w}(\mathcal{H}(4)) \) which quantizes the bialgebra generated by the \( r(h, w) \)-matrix (III.1) will be characterized by two parameters \( h \) and \( w \) associated respectively to the standard \( r \)-matrix \( r(h, 0) = 2h(a \otimes a^+ - e \otimes n) \) and to the nonstandard \( r \)-matrix \( r(0, w) = w(n \otimes a^+ - a^+ \otimes n) \):

**Proposition 4** The two parametric deformed algebra \( \mathcal{U}_{h,w}(\mathcal{H}(4)) \) is an associative algebra over \( \mathbb{C} \) generated by \( N, E, A^+ \) and \( A \), satisfying the commutations relations
\[ [A, A^+] = \frac{\sinh(hE)}{h}e^{wA^+}, \quad [N, A^+] = \frac{e^{wA^+} - 1}{w}, \quad [N, A] = -A \quad (\text{III.2}) \]

where, \( E \) is still central. The algebra (III.2) admit the following Hopf structure
\[ \Delta_{h,w}(N) = N \otimes e^{wA^+} + 1 \otimes N, \quad S_{h,w}(N) = -Ne^{-wA^+}, \quad \varepsilon_{h,w}(N) = 0, \]
\[ \Delta_{h,w}(E) = E \otimes 1 + 1 \otimes E, \quad S_{h,w}(E) = -E, \quad \varepsilon_{h,w}(E) = 0, \quad (\text{III.3}) \]
\[ \Delta_{h,w}(A^+) = A^+ \otimes 1 + 1 \otimes A^+, \quad S_{h,w}(A^+) = -A^+, \quad \varepsilon_{h,w}(A^+) = 0, \]

and
\[ \Delta_{h,w}(A) = A \otimes e^{hE}e^{wA^+} + e^{-hE} \otimes A + we^{-hE} N \otimes \frac{\sinh(hE)}{h}e^{wA^+}, \quad \varepsilon_{h,w}(A) = 0, \]
\[ S_{h,w}(A) = -Ae^{-wA^+} + wN \frac{\sinh(hE)}{h}e^{-wA^+}. \quad (\text{III.4}) \]

**Proof:** All the Hopf algebra axioms can be verified by direct calculations. The elements \( E \) and \( A^+ \) are primitives.  

The Heisenberg subalgebra generated by $E$, $A^+$ and $A$ is not a Hopf subalgebra. The Casimir element of the quantum algebra $\mathcal{U}_{h,w}(\mathcal{H}(4))$ is given by

$$C_{h,w} = N \frac{\sinh hE}{h} + \frac{e^{-wA_+} - 1}{2w} A + A \frac{e^{-wA_+} - 1}{2w}. \quad (\text{III.5})$$

When $h$ is equal to zero, $C_{h,w}$ correspond to the central element given in [13]. The explicit expression of $N$ in terms of $A$, $A^+$ and $E$ is given by the series

$$N \equiv \left( \frac{1 - e^{-wA_+}}{w} \right) A \left[ \frac{\sinh(hE)}{h} \right]^{-1} = 2h \left( \frac{1 - e^{-wA_+}}{w} \right) A \sum_{k=0}^{\infty} e^{-(2k+1)hE}. \quad (\text{III.6})$$

The elements $E^i(A^+)^j N^k A^l$ where $(i, j, k, l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ build a Poincaré-Birkhoff-Witt basis of $\mathcal{U}_{h,w}(\mathcal{H}(4))$:

**Proposition 5** Each infinite dimensional irreducible representation $\pi_{e,n}$ of $\mathcal{U}_{h,w}(\mathcal{H}(4))$ is labeled by two parameters $e$ and $n$. A generic representation $\pi_{e,n}$ is defined as follows:

$$A |r\rangle = \left( \frac{\sinh(he)}{h} \right)^{1/2} \sqrt{r} |r-1\rangle,$$

$$A^+ |r\rangle = \sum_{k=0}^{\infty} \frac{w^k}{(k+1)} \left( \frac{\sinh(he)}{h} \right)^{(k+1)/2} \sqrt{\frac{(r+k+1)!}{r!}} |r+k+1\rangle,$$

$$\left( \frac{1 - e^{-wA_+}}{w} \right) |r\rangle = \left( \frac{\sinh(he)}{h} \right)^{1/2} \sqrt{r+1} |r+1\rangle,$$

$$E |r\rangle = e |r\rangle,$$

$$N |r\rangle = (r+n) |r\rangle \quad (\text{III.7})$$

where $\{|r\rangle\}_{r=0}^{\infty}$ is an orthonormal basis of the module $V_{e,n}$.

The invertible element $\mathcal{F} = e^{wN \otimes A^+}$ satisfies the relations (II.12)-(II.15) on the enveloping algebra $\mathcal{U}_h(\mathcal{H}(4))$. Now, let us turn to the universal $\mathcal{R}$-matrix of $\mathcal{U}_{h,w}(\mathcal{H}(4))$:

**Proposition 6** The Hopf algebra $\mathcal{U}_{h,w}(\mathcal{H}(4))$ is quasitriangular. The universal $\mathcal{R}$-matrix has the following form

$$\mathcal{R}^{h,w} = (\sigma \circ \mathcal{F})^{-1} \mathcal{R}^{h} \mathcal{F} \quad (\text{III.8})$$

where

$$\mathcal{F} = e^{wN \otimes A^+}, \quad (\text{III.9})$$

$$\mathcal{R}^{h} = e^{-2h E \otimes N} \exp \left( 2h e^{hE} A \otimes \left( \frac{1 - e^{-wA_+}}{w} \right) \right). \quad (\text{III.10})$$

The universal $\mathcal{R}$-matrix (III.8) solves the quantum Yang-Baxter equation (II.11).
Proof: The relations (II.9) are verified using the Campbell-Baker-Hausdorff formula

\[ e^\phi \triangle(.) e^{-\phi} = \triangle(.) + \sum_{n=1}^{\infty} \frac{1}{n!} [\phi, \ldots [\phi, \triangle(.)] \ldots] . \]  

(III.11)

Recall that \( \mathcal{F} \) satisfies the relations (II.12)-(II.15). Let us check that \( (\triangle_{h,w} \otimes \text{id})(\mathcal{R}^{h,w}) = \mathcal{R}_{13}^{h,w} \mathcal{R}_{23}^{h,w} \). This equation reduces to

\[
\mathcal{F}_{31}^{-1} \mathcal{R}_{13}^{h} \mathcal{F}_{13} \mathcal{F}_{23}^{-1} \mathcal{R}_{23}^{h} \mathcal{F}_{23} = \mathcal{F}_{12}^{-1} (\triangle_{h} \otimes \text{id})(\mathcal{F}_{21}^{-1})(\triangle_{h} \otimes \text{id})(\mathcal{R}^{h})(\triangle_{h} \otimes \text{id})(\mathcal{F}) \mathcal{F}_{12},
\]

\[
= \mathcal{F}_{12}^{-1} (\triangle_{h} \otimes \text{id})(\mathcal{F}_{21}^{-1}) \mathcal{R}_{13}^{h} \mathcal{R}_{23}^{h} (\triangle_{h} \otimes \text{id})(\mathcal{F}) \mathcal{F}_{12},
\]

\[
= \mathcal{F}_{12}^{-1} (\triangle_{h} \otimes \text{id})(\mathcal{F}_{21}^{-1}) \mathcal{R}_{13}^{h} \mathcal{R}_{23}^{h} (\text{id} \otimes \triangle_{h})(\mathcal{F}) \mathcal{F}_{23},
\]

namely,

\[
\mathcal{F}_{12}^{-1} (\triangle_{h} \otimes \text{id})(\mathcal{F}_{21}^{-1}) \mathcal{R}_{13}^{h} = \mathcal{F}_{31} \mathcal{R}_{13}^{h} \mathcal{F}_{13} \mathcal{F}_{23}^{-1} (\text{id} \otimes \triangle'_{h})(\mathcal{F}^{-1})
\]

(III.12)

where, \( \triangle'_{h} = \sigma \circ \triangle_{h} \). Applying the flip \( \sigma_{23} \) to both sides, the preceding equation is equivalent to

\[
\mathcal{F}_{13}^{-1} \sigma_{23}(\triangle_{h} \otimes \text{id})(\mathcal{F}_{21}^{-1}) \mathcal{R}_{12}^{h} = \mathcal{F}_{21} \mathcal{R}_{12}^{h} \mathcal{F}_{12} \mathcal{F}_{23}^{-1} (\text{id} \otimes \triangle_{h})(\mathcal{F}^{-1})
\]

\[
= \mathcal{F}_{21}^{-1} \mathcal{R}_{12}^{h} (\text{id} \otimes \triangle_{h})(\mathcal{F}^{-1})
\]

namely,

\[
\mathcal{F}_{13}^{-1} \sigma_{23}(\triangle_{h} \otimes \text{id})(\mathcal{F}_{21}^{-1}) = \mathcal{F}_{21}(\triangle'_{h} \otimes \text{id})(\mathcal{F}^{-1}).
\]

(III.13)

It is easy to see that applying \( \sigma_{12} \) to the left-hand side of the preceding equation gives \( \mathcal{F}_{23}^{-1} (\text{id} \otimes \triangle_{h})(\mathcal{F}^{-1}) \), and applying it to the right-hand side gives \( \mathcal{F}_{12}^{-1} (\triangle_{h} \otimes \text{id})(\mathcal{F}^{-1}) \). So the result follows on using (II.13) once again. The relation \( (\text{id} \otimes \triangle_{h,w})(\mathcal{R}^{h,w}) = \mathcal{R}_{13}^{h,w} \mathcal{R}_{23}^{h,w} \) is verified using the same method.

The element (III.10) correspond to the \( \mathcal{R}_{h} \)-matrix (II.22) of the algebra (II.4) generated by \( N, E, A \) and \( (1 - e^{-wA^+})/w \) (the generator \( A^+ \) is replaced by \( (1 - e^{-wA^+})/w \)), whereas \( \mathcal{F}^{-1} \) is the twist which deforms the coproducts (II.5) to the coproducts (III.3)-(III.4), namely

\[
\triangle_{h,w} = \mathcal{F}^{-1} \triangle_{h} \mathcal{F}.
\]

(III.14)

Furthermore,

\[
\varepsilon_{h,w}(\cdot) = \varepsilon_{h}(\cdot), \quad S_{h,w}(\cdot) = v^{-1}S_{h}(\cdot)v
\]

(III.15)

where, \( v \) is the invertible element given by

\[
v = \sum_{k=0}^{\infty} \frac{(-1)^{k}w^{k}}{k!}N^{k}(A^{+})^{k}
\]

\[
v^{-1} = \sum_{k=0}^{\infty} \frac{1}{k!}N^{k}(\ln(2 - e^{-wA^{+}}))^{k}
\]

(III.16)
which satisfies the following relation
\[ v^{-1}S_h(v) = e^{wA^+}. \] (III.17)

From the universal \( R \)-matrix, by standard techniques, we can readily deduce a new \( R \)-matrix depending on a continuous parameter \( u \). In fact, defining the operator \( T_u \) by its action:
\[ T_uA^+ = e^{-u}A^+, \quad T_uw = e^uw, \quad T_uh = h, \quad T_uN = N, \] (III.18)

we can define
\[ R^{h,w}(u) = (T_u \otimes 1)R^{h,w} = e^{-wA^+ \otimes N}e^{-2hE \otimes N}\exp\left(2he^ue^{hE}A \otimes \left(\frac{1 - e^{-wA^+}}{w}\right)\right)e^{wN \otimes A^+} \] (III.19)

and again by direct calculations we have: The matrix \( R^{h,w}(u) \) defined in (III.19) satisfies the Yang-Baxter equation
\[ R^{h,w}_{12}(u)R^{h,w}_{13}(u + v)R^{h,w}_{23}(v) = R^{h,w}_{23}(v)R^{h,w}_{13}(u + v)R^{h,w}_{12}(u). \] (III.20)

The classical \( r(u) \)-matrix corresponding to the universal \( R^{h,w}(u) \)-matrix (III.19), depending on the parameter \( u \), is the following
\[ r(u) = w(n \otimes a^+ - a^+ \otimes n) + 2h(e^u a \otimes a^+ - e \otimes n) \] (III.21)
and solves the parameter-dependent classical Yang-Baxter equation
\[ [r_{12}(u), r_{13}(u + v)] + [r_{12}(u), r_{23}(v)] + [r_{13}(u + v), r_{23}(v)] = 0. \] (III.22)

Now, consider the infinite dimensional irreducible representations \( \pi_{e,n} \) with the condition \( e^{2he} \neq 1 \). Evaluates the interwiner matrices \( R(e_1, e_2) \equiv e^{2he_1n_2}e^{(\pi_{e_1,n_1} \otimes \pi_{e_2,n_2})}R^{h,w} : V_{e_1,n_1} \otimes V_{e_2,n_2} \rightarrow V_{e_2,n_2} \otimes V_{e_1,n_1} \) and \( R^{-1}(e_1, e_2) \equiv e^{-2he_2n_1}e^{(\pi_{e_1,n_1} \otimes \pi_{e_2,n_2})}[R^{h,w}]^{-1} \sigma : V_{e_1,n_1} \otimes V_{e_2,n_2} \rightarrow V_{e_2,n_2} \otimes V_{e_1,n_1} \), we obtain
\[
[R]_{r_1',r_2'}^{r_1,r_2}(e_1, e_2) = w^{r_1' + r_2' - r_1 - r_2}(\frac{e^{he_1} - e^{-he_1}}{2h})(r_2' - r_1)/2 \cdot \frac{e^{he_2} - e^{-he_2}}{2h})(r_1' - r_2)/2 \cdot \left(\frac{r_1'r_2'}{r_1'r_2}\right)^{1/2} \sum_{s = \sup(r_1' - r_2', 0)}^\infty \phi_{r_1' - r_2}^{-s} \phi_{r_2' - r_1 + s}(e^{he_1} - e^{-he_1})^s e^{h(s - 2r_1')e_1}, \] (III.23)
\[
[R^{-1}]_{r_1',r_2'}^{r_1,r_2}(e_1, e_2) = w^{r_1' + r_2' - r_1 - r_2}(\frac{e^{he_1} - e^{-he_1}}{2h})(r_2' - r_1)/2 \cdot \frac{e^{he_2} - e^{-he_2}}{2h})(r_1' - r_2)/2 \cdot \left(\frac{r_1'r_2'}{r_1'r_2}\right)^{1/2} \sum_{s = \sup(r_2' - r_1, 0)}^\infty (-1)^s \frac{(r_1' + r_2')}{s} \phi_{r_1' - r_2 + s} \phi_{r_2' - r_1 - s}(e^{he_2} - e^{-he_2})^s e^{h(s + 2r_1')e_2} \] (III.24)
with
\[ \phi_{r'_1-r_2+s} = \sum_{k=0}^{r'_1-r_2+s} \frac{f^r}{k!} (r_1 + n_1)^k, \quad \phi_{r'_2-r_1+s} = \sum_{k=0}^{r'_2-r_1+s} (-1)^k \frac{f^r}{k!} (r'_1 + n_2)^k, \]

where
\[ f^r = \begin{cases} 1 & \text{if } k = 0 \\ \sum_{i_1 + i_2 + \cdots + i_k = s} \frac{1}{(i_1 + 1)(i_2 + 1) \cdots (i_k + 1)} & \text{if } k \geq 1 \end{cases} \quad (\text{III.25}) \]

Taking \( w = 0 \), the relations (III.23) and (III.24) reduces to the results obtained by Gomez and Sierra in [14], namely

\[
[R_g]_{r_1,r_2}^{r'_1,r'_2}(e_1, e_2) = \delta_{r'_1+r'_2, r_1+r_2} \left( r'_1 \right)^{1/2} \left( r'_2 \right)^{1/2} e^{-h(r'_1+r_2)e_1} \\
\left[ (e^{he_1} - e^{-he_1})(e^{he_2} - e^{-he_2}) \right]^{(r'_1-r_2)/2} \\
[R_g^{-1}]_{r_1,r_2}^{r'_1,r'_2}(e_1, e_2) = \delta_{r'_1+r'_2, r_1+r_2} \left( r'_1 \right)^{1/2} \left( r'_2 \right)^{1/2} (-1)^{r_2-r'_1} e^{h(r'_1+r_1)e_2} \\
\left[ (e^{he_1} - e^{-he_1})(e^{he_2} - e^{-he_2}) \right]^{(r_2-r'_1)/2}. \quad (\text{III.26})
\]

Recall that, a braid group \( \mathcal{B}_m \) is an abstract group generated by the elements \( \sigma_i, \ 1 \leq i \leq m-1 \) satisfying the relations \( \sigma_i \sigma_j = \sigma_j \sigma_i \) if \(|i-j| \geq 2\), \( \sigma_i^{-1} = \sigma_i^{-1} \) and \( \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \) if \( 1 \leq i \leq m-2 \). In colored braid groups cases, \( \sigma_i \) contain different nontrivial parameters called the string variables. The \( R \) matrix (III.23) build a new infinite dimensional representation \( \rho_m \) of the colored braided group \( \mathcal{B}_m \) via

\[ \rho_m : \mathcal{B}_m \longrightarrow \text{End}(\otimes_{i=1}^m V_{e_i,n_i}) \]
\[ \sigma_i \longrightarrow \text{id} \otimes R^{e_i,e_{i+1}} \otimes \text{id} \otimes (m-i-1) \quad (\text{III.27}) \]

which satisfies

\[ R_i^{e_2,e_3} R_{i+1}^{e_1,e_2} R_{i+1}^{e_1,e_3} = R_{i+1}^{e_1,e_2} R_i^{e_1,e_3} R_{i+1}^{e_2,e_3}. \quad (\text{III.28}) \]

The noncolored version correspond to take \( e_1 = e_2 = \cdots = e_m = e \).

Now, let us concentrate in noncolored case. The noncolored braid group representation (NCBG) admits an extension à la Turaev [24], if there exists an isomorphism \( \mu : V_{e,n} \rightarrow V_{e,n} \) which transforms the basis \( \{ |r \rangle \}_{r=0}^\infty \) into \( \{ \mu_r |r \rangle \}_{r=0}^\infty \) satisfying the following three conditions:

\[ (\mu_i \mu_j - \mu_k \mu_l) R^{k,l}_{i,j}(e,e) = 0, \quad (\text{III.29}) \]
\[ \sum_j R_{i,j}^{k,j}(e, e) \mu_j = \delta_i^k a b, \quad (\text{III.30}) \]
\[ \sum_j [R_{i,j}^{-1}]^{k,j}(e, e) \mu_j = \delta_i^k a^{-1} b \quad (\text{III.31}) \]

where \( a, b \) are constants. The Turaev conditions (III.29)-(III.31) hold if

\[ \mu = \text{id}, \quad a = e^{he}, \quad b = e^{-he}. \quad (\text{III.32}) \]

Let \( \zeta : \mathcal{B}_m \to \mathbb{Z} \) such that \( \zeta(\sigma_{i}^{\pm1}) = \pm 1 \), \( \zeta(xy) = \zeta(x) + \zeta(y) \). Then, the link invariant \( P : \prod_{m \geq 2} \mathcal{B}_m \to \mathbb{C} \) associated to the enhanced Yang-Baxter operator (EYB-operator) \((R, \text{id}, e^{he}, e^{-he})\) is

\[ P(x) = q^{e(-\zeta(x)+m)} Tr[\rho_m(x)], \quad \forall x \in \mathcal{B}_m \quad (\text{III.33}) \]

which is invariant under the two Markov moves \( P(xy) = P(yx), x, y \in \mathcal{B}_m \) (type I) and \( P(x\sigma_{m}^{\pm1}) = P(x), x \in \mathcal{B}_m, \sigma_{m} \in \mathcal{B}_{m+1} \) (type II). Let us remark that:

(i) The combination of the standard and nonstandard quantizations permit to build a new infinite dimensional representation of the colored braided group \( \mathcal{B}_m \).

(ii) The link invariant (III.33) is equal to the one obtained in reference [14]. The link invariants obtained for a structure defined as combination of the standard and nonstandard quantizations are exactly equal to those calculated for the standard quantization separately. The invariants (III.33) are polynomials only in the variable \( q^{he} \). It's easy to see that the nonstandard parameter \( w \) have no contribution in (III.33) (see the results (III.23) and (III.24)).

(iii) These comments are true for the colored braiding version. (See reference [14] for more details).

**IV The Two Parametric Heisenberg Group \( H_{h,w}(4) \)**

A 3 \( \times \) 3 dimensional representation \( \pi_3 \) of the two parametric deformed Heisenberg algebra \( \mathcal{U}_{h,w}(\mathcal{H}(4)) \) is given by

\[ \pi_3(A) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \pi_3(A^+) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \]
\[ \pi_3(E) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \pi_3(N) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (\text{IV.1}) \]
This representation remains undeformed. Correspondingly, the $R$-matrix (III.8) is represented by the $9 \times 9$ matrix

$$R_{h,w}^{\pi_3} = (\pi_3 \otimes \pi_3)(R_{h,w}^{\pi_3}) = \begin{pmatrix} I_3 & 2h\pi_3(A^+) & -2h\pi_3(N) \\
0 & I_3 + \omega\pi_3(A^+) & -w\pi_3(N) \\
0 & 0 & I_3 \end{pmatrix}, \quad (IV.2)$$

$I_3$ being the $3 \times 3$ identity matrix. We shall now present the quantum group $H(4)$ as a matrix quantum group à la Woronowicz [21, 22]. Let us consider a $3 \times 3$ matrix of the following form

$$T = e^{\beta \otimes \pi_3(E)}e^{\gamma \otimes \pi_3(N)}e^{\alpha \otimes \pi_3(A)} = \begin{pmatrix} 1 & \alpha & \beta \\
0 & e^\gamma & \delta \\
0 & 0 & 1 \end{pmatrix}, \quad (IV.3)$$

where the matrix elements $\alpha$, $\beta$, $\gamma$ and $\delta$ generate the algebra of functions on the quantum group $\mathcal{F}_{h,w}(H(4))$. The matrix elements $\alpha$, $\beta$, $\gamma$ and $\delta$ of $T$ satisfies the relation:

$$R_{h,w}^{\pi_3} T_1 T_2 = T_2 T_1 R_{h,w}^{\pi_3} \quad (IV.4)$$

where, $T_1 = T \otimes 1$ and $T_2 = 1 \otimes T$. The algebra $\mathcal{F}_{h,w}(H(4))$ can be endowed with a Hopf algebra by defining a comultiplication $\Delta$ and counit $\varepsilon$ as

$$\Delta (T) = T \hat{\otimes} T, \quad \varepsilon(T) = 1, \quad (IV.5)$$

where $\hat{\otimes}$ denotes matrix multiplication and tensor product of the $\mathbb{C}^*$-algebras of noncommutative representative functions. The inverse matrix then defines the antipode, namely

$$S(T) = T^{-1}. \quad (IV.6)$$

**Proposition 7**

(i) The relations between the generators $\alpha$, $\beta$, $\gamma$ and $\delta$ are the following:

$$[\alpha, \beta] = 2h\alpha + w\alpha^2, \quad [\gamma, \delta] = w(e^\gamma - 1),$$

$$[\alpha, \delta] = wae^\gamma, \quad [\beta, \gamma] = -w\alpha,$$

$$[\beta, \delta] = [\alpha, \gamma] = 0. \quad (IV.7)$$

(ii) The generators $\alpha$, $\beta$, $\gamma$ and $\delta$ with the relations specified in (IV.7) have coproducts, counits and antipodes given by

$$\Delta(\alpha) = \alpha \otimes e^\gamma + 1 \otimes \alpha, \quad S(\alpha) = -e^{-\gamma} \alpha, \quad \varepsilon(\alpha) = 0,$$

$$\Delta(\beta) = \beta \otimes 1 + 1 \otimes \beta + \alpha \otimes \delta, \quad S(\beta) = -\beta + e^{-\gamma}\alpha\delta, \quad \varepsilon(\beta) = 0,$$

$$\Delta(e^\gamma) = e^\gamma \otimes e^\gamma, \quad S(e^\gamma) = e^{-\gamma}, \quad \varepsilon(e^\gamma) = 0,$$

$$\Delta(\delta) = \delta \otimes 1 + e^\gamma \otimes \delta, \quad S(\delta) = -e^{-\gamma}\delta, \quad \varepsilon(\delta) = 0. \quad (IV.8)$$
Proof: The relations (IV.7) arises from the relation (IV.4). The Hopf structure of $F_{h,w}(H(4))$ reads from the equations (IV.5) and (IV.6), namely

$$T^{-1} = \begin{pmatrix} 1 & -e^{-\gamma}\alpha & -\beta + e^{-\gamma}\alpha\delta \\ 0 & e^{-\gamma} & -e^{-\gamma}\delta \\ 0 & 0 & 1 \end{pmatrix}$$

(IV.9)

and

$$T \otimes T = \begin{pmatrix} 1 \otimes 1 & \alpha \otimes e^\gamma + 1 \otimes \alpha & \beta \otimes 1 + 1 \otimes \beta + \alpha \otimes \delta \\ 0 & e^\gamma \otimes e^\gamma & \delta \otimes 1 + e^\gamma \otimes \delta \\ 0 & 0 & 1 \otimes 1 \end{pmatrix}.$$  

(IV.10)

When the parameter $h$ is equal to zero, $F_{h,w}(H(4))$ is reduced to the algebra $F_{w}(H(4))$ obtained by Ballesteros and Herranz [15]:

$$[\alpha, \beta] = w\alpha^2, \quad [\gamma, \delta] = w(e^\gamma - 1),$$

$$[\alpha, \delta] = wae^\gamma, \quad [\beta, \gamma] = -w\alpha,$$

$$[\alpha, \gamma] = [\beta, \delta] = 0.$$  

(IV.11)

Whereas, if $w = 0$, the algebra $F_{h}(H(4))$ being the standard ones (according to the algebra obtained in [8], we take $\gamma = 0$), i.e.

$$[\alpha, \beta] = 2h\alpha, \quad [\alpha, \delta] = [\beta, \delta] = 0$$  

(IV.12)

and

$$\Delta(\alpha) = \alpha \otimes 1 + 1 \otimes \alpha, \quad S(\alpha) = -\alpha, \quad \epsilon(\alpha) = 0,$$

$$\Delta(\beta) = \beta \otimes 1 + 1 \otimes \beta + \alpha \otimes \delta, \quad S(\beta) = -\beta + \alpha\delta, \quad \epsilon(\beta) = 0,$$

$$\Delta(\delta) = \delta \otimes 1 + 1 \otimes \delta, \quad S(\delta) = -\delta, \quad \epsilon(\delta) = 0.$$  

(IV.13)

The relations (IV.13) are obtained in the reference [8]. The algebra (IV.12) is lightly different from the algebra $F_{h}(H(4))$ obtained in [8].

V The Ribbon Hopf Algebra $U_{h,w}(\mathcal{H}(4))$

Any quasitriangular Hopf algebra $(U, \mathcal{R})$ has an invertible element, usually called $u$, with the property that

$$S^2(a) = uau^{-1}, \quad \forall a \in U.$$  

(V.1)

The element $u$ and its inverse $u^{-1}$ can be obtained from the universal $\mathcal{R}$-matrix as follows:

$$u = m(S \otimes \text{id})(\mathcal{R}_{21}),$$

$$u^{-1} = m(S^{-1} \otimes \text{id})(\mathcal{R}_{21}^{-1}).$$  

(V.2)
The elements $u$ and $uS(u)$ satisfies the relations \[ \varepsilon(u) = 1, \quad \triangle(u) = (\mathcal{R}_{21}\mathcal{R})^{-1}(u\otimes u) = (u\otimes u)(\mathcal{R}_{21}\mathcal{R})^{-1}, \]
\[ \triangle(uS(u)) = (\mathcal{R}_{21}\mathcal{R})^{-2}(u\otimes u). \]

(V.3)

The element $uS(u) = S(u)u$ is central in $\mathcal{U}$. The elements $u$ and $S(u)$ of $\mathcal{U}$ does not in general commute with the others elements of $\mathcal{U}$, however $u$ and $S(u)$ are central if $S^2 = id$.

Recall that, a Ribbon Hopf algebra $(\mathcal{U}, \mathcal{R}, \theta)$ is a quasitriangular Hopf algebra with a central element $\theta$ satisfying
\[ \theta^2 = uS(u), \quad S(\theta) = \theta, \]
\[ \triangle(\theta) = (\mathcal{R}_{21}\mathcal{R})^{-1}(\theta \otimes \theta), \quad \varepsilon(\theta) = 1. \]

(V.4)

Proposition 8 Let $(\mathcal{U}, \mathcal{R})$ be a quasitriangular Hopf algebra and $\mathcal{F}$ an invertible element of $\mathcal{U}\otimes\mathcal{U}$ satisfying (II.12)-(II.15). The $u\mathcal{F}$ element of the twisted quasitriangular Hopf algebra $\mathcal{U}_\mathcal{F}$ is
\[ u\mathcal{F} = v^{-1}S(v)u = v^{-1}uS^{-1}(v), \]
where $v$ and its inverse are given by (II.17). The element $u\mathcal{F}$ has the property that
\[ S_\mathcal{F}^2(a) = u\mathcal{F}a u\mathcal{F}^{-1}, \quad a \in \mathcal{U}_\mathcal{F}, \]
\[ \varepsilon_\mathcal{F}(u\mathcal{F}) = 1, \]
\[ \triangle_\mathcal{F}(u\mathcal{F}) = (\mathcal{R}_{21}\mathcal{R}_\mathcal{F})^{-1}(u\mathcal{F} \otimes u\mathcal{F}) = (u\mathcal{F} \otimes u\mathcal{F})(\mathcal{R}_{21}\mathcal{R}_\mathcal{F})^{-1}, \]

(V.5)

Proof: For the proof, we choose $\mathcal{F} = \sum(f) f_1 \otimes f_2$, $\mathcal{F}^{-1} = \sum(g) g_1 \otimes g_2$, $\mathcal{R} = \sum(r) r_1 \otimes r_2$, $u = \sum(r) S(r_2)r_1$, $v = \sum(f) S(f_1)f_2$ and $v^{-1} = \sum(g) g_1S(g_2)$. The element $u\mathcal{F}$ is obtained from the universal $\mathcal{R}_\mathcal{F}$-matrix as follows:
\[ u\mathcal{F} = m(S_\mathcal{F} \otimes id)(\mathcal{R}_{21}_\mathcal{F}), \]
\[ = m(v^{-1} \otimes id)(S \otimes id)(\mathcal{F}^{-1}\mathcal{R}_{21}\mathcal{F})(1 \otimes v), \]
\[ = \sum(f) \sum(g) \sum(r) v^{-1}S(f_2)S(r_2)S(g_1)vg_2r_1f_1. \]

From the relation $m(S \otimes id)(\mathcal{F}\mathcal{F}^{-1}) = 1$, we obtain that $\sum(g) S(g_1)vg_2 = 1$. Now, the preceding relations reads
\[ u\mathcal{F} = \sum(f) \sum(r) v^{-1}S(f_2)uf_1 \]
\[ = \sum(f) \sum(g) v^{-1}S(f_2)S^2(f_1)u. \]

(V.9)
So the result follows on using the element \( v \) (II.17). The equations (V.6) and (V.7) are verified by direct calculations. The relations (V.8) arise as follows:

\[
\Delta_F(u_F) = F^{-1} \Delta (v^{-1}S(v)u) F,
= F^{-1} \Delta (v^{-1}S(v)) \Delta (u) F,
= (v^{-1}S(v) \otimes v^{-1}S(v))(S^2 \otimes S^2)F^{-1}(\mathcal{R}_{21}\mathcal{R})^{-1}F,
\]

\[
\Delta_F(u_F) = (u_F \otimes u_F)(\mathcal{R}^{F}_{21}\mathcal{R}^{F})^{-1}.
\] (V.10)

**Proposition 9** Let \( (U, \mathcal{R}, \theta) \) be a ribbon Hopf algebra and \( F \) an invertible element of \( U \otimes U \) satisfying (II.12)-(II.15). The twisted algebra \( U_{\mathcal{F}} \) is a ribbon Hopf algebra with \( \theta_{\mathcal{F}} = \theta \) and the relations

\[
S_{\mathcal{F}}(\theta_{\mathcal{F}}) = 1, \quad \varepsilon_{\mathcal{F}}(\theta_{\mathcal{F}}) = 1,
\]

\[
\Delta_{\mathcal{F}}(\theta_{\mathcal{F}}) = (\mathcal{R}^{\mathcal{F}}_{21}\mathcal{R}^{\mathcal{F}})^{-1}(\theta_{\mathcal{F}} \otimes \theta_{\mathcal{F}}).
\] (V.11)

**Proof:** The results (V.11) arises from \( \theta^{2}_{\mathcal{F}} = u_{\mathcal{F}}S_{\mathcal{F}}(u_{\mathcal{F}}) = uS(u) \), namely, \( \theta_{\mathcal{F}} = \theta \).

In the particular case of the universal \( \mathcal{R}_{h} \)-matrix (II.22) of the Heisenberg algebra \( U_{h}(\mathcal{H}(4)) \) we obtain from proposition 9 and [14]

\[
u_{h} = \sum_{l \geq 0} \frac{(-1)^{l}(2h)^{l}}{l!} e^{-hlE}(A^{+})^{l}A^{2hEN},
\]

\[
u^{-1}_{h} = \sum_{l \geq 0} \frac{(2h)^{l}}{l!} e^{hlE}(A^{+})^{l}A^{2hEN},
\] (V.12)

For the standard Heisenberg algebra \( U_{h}(\mathcal{H}(4)) \), we have from (II.5) that \( S^{2} = \text{id} \) so that \( u_{h} \) is central. Similarly, \( S_{h}(u_{h}) \) is also central and we have

\[
S_{h}(u_{h}) = e^{-2hE}u_{h}.
\] (V.13)

Gomez and Sierra have proved that \( U_{h}(\mathcal{H}(4)) \) is a Ribbon Hopf algebra with \( \theta_{h} = e^{-hE}u_{h} \) [14]. For the two parametric Heisenberg algebra \( U_{h,w}(\mathcal{H}(4)) \), the \( u_{h,w} \) element and its inverse reads

\[
u_{h,w} = e^{wA^{+}} \sum_{l \geq 0} \frac{(-1)^{l}(2h)^{l}}{l!} e^{-hlE} \left( \frac{1 - e^{-wA^{+}}}{w} \right)^{l} A^{2hEN},
\]

\[
u^{-1}_{h,w} = \sum_{l \geq 0} \frac{(2h)^{l}}{l!} e^{hlE} \left( \frac{1 - e^{-wA^{+}}}{w} \right)^{l} A^{2hEN} e^{-wA^{+}},
\] (V.14)
and

\[ S_{h,w}(u_{h,w}) = e^{-2hE}e^{-wA^+}u_{h,w}. \]  \hfill (V.15)

For the representation \( \pi_{e,n} \), we obtain

\[ u_{h,w}|r\rangle = e^{2nhe} \sum_{l \geq 0} \frac{w^l}{l!} \left( \frac{t - t^{-1}}{2h} \right)^{1/2} \left( \frac{r + l}{r!} \right)^{1/2} |r + l\rangle, \]

\[ u_{h,w}^{-1}|r\rangle = e^{-2nhe}|r\rangle - wt^{-2n} \left( \frac{\sinh(he)}{h} \right)^{1/2} \sqrt{r + 1} |r + 1\rangle. \]  \hfill (V.16)

**Proposition 10**  

The quasitriangular Hopf algebra \((U_{h,w}(\mathcal{H}(4)), R_{h,w})\) is a Ribbon Hopf algebra with

\[ \theta_{h,w} = e^{-hE} \sum_{l \geq 0} \frac{(-1)^l(2h)^l}{l!} e^{-hlE} \left( \frac{1 - e^{-wA^+}}{w} \right)^l A^l e^{2hEN} \]

\[ \theta_{h,w}^{-1} = e^{hE} \sum_{l \geq 0} \frac{(2h)^l}{l!} e^{hlE} \left( \frac{1 - e^{-wA^+}}{w} \right)^l A^l e^{-2hEN} \]  \hfill (V.17)

which in the irreducible representation \( \pi_{e,n} \) takes the value

\[ \theta_{h,w}|r\rangle = e^{(2n-1)he}|r\rangle, \quad \theta_{h,w}^{-1}|r\rangle = e^{-(2n-1)he}|r\rangle. \]  \hfill (V.18)

Finally, let us remark that:

(i) The eigenvalue of the central element \( \theta_{h,w} \) and its inverse depend only on the standard parameter \( h \).

(ii) The value of \( \theta_{h,w} \) in a given irreducible representation \( \pi_{e,n} \) contains some interesting information of the corresponding conformal field theory (CFT) associated to the quantum algebra \( U_{h,w}(\mathcal{H}(4)) \). In the cases of a semisimple algebra \( G \) it was shown that the conformal weight \( \Delta_\alpha \) of a primary field of the WZW model \( \hat{\mathcal{G}}_k \) is related to the value \( \theta_\alpha \) by

\[ \theta_\alpha = e^{2\pi i \Delta_\alpha} \]  \hfill (V.19)

where \( \theta_\alpha \) is the value of \( \theta \) on the irreducible representation \( \alpha \) of \( U_{q_1, \ldots, q_\xi}(G) \) associated to the primary field \( \alpha \). If the formula (V.19) holds true for the quantum Heisenberg algebra \( U_{h,w}(\mathcal{H}(4)) \) it would imply that

\[ e^{(2n-1)he} = e^{2\pi i \Delta_{e,n}}. \]  \hfill (V.20)

The conformal weight \( \Delta_{e,n} \) depend only on the standard parameter \( h \). The nonstandard parameter \( w \) has no contribution in WZW model.
VI Conclusions and Perspectives

Let us remark that the combination of the standard and nonstandard deformations in the case of the enveloping Heisenberg algebra is possible because the invertible element $\mathcal{F}$ satisfies the axioms (II.12)-(II.15) on the standard algebra $\mathcal{U}_h(\mathcal{H}(4))$. The parameter arising from the nonstandard quantization not appear in the conformal weight $\Delta_{e,n}$ of the primary field of the WZW model $\hat{G}_k$ and in the link invariants. The parameters which plays a relevant roles arises from the standard quantization. In the case of the quantum algebra $\mathcal{U}_q(sl(2))$, the analogs two parametric deformation cannot be defined because the twist $D$ do not obeys to the axioms (II.12)-(II.15).

Using the annihilator and creator operators of the infinite Heisenberg algebra $\mathcal{U}_{h,w}(\mathcal{H}(\infty))$ and the Sugawara construction, the two parametric deformation can be extended to Virasoro algebra. The two parametric deformation of the Galilei group obtained by contraction can be also used to study the magnetic chain following the approach developed in [26]. These problems will be studied else where.

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