HAMILTON CYCLES IN CAYLEY GRAPHS ON GENERALIZED DIHEDRAL GROUPS

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Abstract. We prove that every connected Cayley graph on a generalized dihedral group has a Hamilton cycle. In particular, this confirms a long-standing conjecture of Holsztyński and Strube in 1978 stating that every connected Cayley graph on a dihedral group has a Hamilton cycle.

Key words: Hamilton cycle; Cayley graph; generalized dihedral group.

MSC: 05C45, 05C25.

1. Introduction

In this paper, all graphs considered are finite, simple and undirected. For a group $G$ and an inverse-closed subset $S$ of $G\setminus \{1\}$, the Cayley graph $Cay(G, S)$ with connection set $S$ is the graph with vertex set $G$ and edge set $\\{(g, sg) \mid g \in G, s \in S\}$. It is easily seen that $Cay(G, S)$ is a regular graph of valency $|S|$, and it is connected if and only if $G = \langle S \rangle$. Lovász's 1969 question [10], whether every connected vertex-transitive graph has a Hamilton path, inspired the following:

Question 1.1. Does every connected Cayley graph with more than two vertices have a Hamilton cycle?

Despite the fact that Question 1.1 has attracted considerable attention for more than forty years, an answer to this question, or more generally, a determination of which Cayley graphs have a Hamilton cycle, still seems quite far away. A graph is said to be Hamilton-connected if for each pair of distinct vertices there is a Hamilton path connecting them, and is said to be Hamilton-laceable if it is bipartite and for each pair of vertices from different parts there is a Hamilton path connecting them. Clearly, Hamilton-connected graphs and Hamilton-laceable graphs with more than two vertices all have Hamilton cycles. Following a remarkable theorem of Chen and Quimpo [3] stating that every connected Cayley graph of valency at least 3 on an abelian group is either Hamilton-connected or Hamilton-laceable, it is known that the answer to Question 1.1 is affirmative for Cayley graphs on abelian groups (see also [11]).

There has been effort by many authors to extend the positive answer to Question 1.1 for Cayley graphs on abelian groups to other families. For example, it was answered in affirmative for Cayley graphs on semidirect products of a cyclic group of prime order by an abelian group by Durnberger [4], on nilpotent groups with a cyclic commutator subgroup by Ghaderpour and Morris [6], and on groups with a cyclic commutator subgroup of prime-power order by Keating and Witte [8]. See survey papers [4, 9, 12] for more results. However, for Cayley graphs on dihedral groups, a family of groups whose structure is simple enough, Question 1.1 has been open, although attempts to prove the existence of Hamilton cycles in this family of Cayley graphs date back to
at least forty years ago. In fact, Holsztyński and Strube [7, page 270] conjectured in 1978:

**Conjecture 1.2.** Every connected Cayley graph on a dihedral group has a Hamilton cycle.

In the same paper Holsztyński and Strube verified this conjecture for dihedral groups of order twice a prime. For cubic Cayley graphs on dihedral groups, the conjecture was proved to be true by Alspach and Zhang [2] in 1989. Moreover, in 2010 Alspach, Chen and Dean investigated Question 1.1 for Cayley graphs on generalized dihedral groups [1], where they proved:

**Theorem 1.3.** [1, Theorem 1.8] Every connected Cayley graph of valency at least 3 on a generalized dihedral group of order divisible by 4 is either Hamilton-connected or Hamilton-laceable.

Recall that for a nontrivial abelian group $A$ the generalized dihedral group $\text{Dih}(A)$ of $A$ is the semidirect product of $A$ by $Z_2$ with $Z_2$ acting on $A$ by inverting elements. Thus dihedral groups are precisely generalized dihedral groups of cyclic groups. As a consequence of Theorem 1.3, every connected Cayley graph on a generalized dihedral group of order divisible by 4 has a Hamilton cycle. In this paper we prove that every connected Cayley graph on a generalized dihedral group has a Hamilton cycle. This gives a positive answer to Question 1.1 and, in particular, verifies Conjecture 1.2. Our main result is as follows.

**Theorem 1.4.** Every connected Cayley graph on a generalized dihedral group has a Hamilton cycle.

The proof of Theorem 1.4 will be given in the next section. For a graph $\Gamma$ denote the vertex set and edge set of $\Gamma$ by $V(\Gamma)$ and $E(\Gamma)$, respectively. Denote the cycle of length $n$ by $C_n$ for $n \geq 3$, and denote the path of length $m - 1$ by $P_m$ for $m \geq 2$.

## 2. Proof of Theorem 1.4

The Cartesian product $\Gamma \Box \Sigma$ of graphs $\Gamma$ and $\Sigma$ is the graph with vertex set $V(\Gamma) \times V(\Sigma)$ such that two vertices $(u, x)$ and $(v, y)$ are adjacent in $\Gamma \Box \Sigma$ if and only if either $u$ is adjacent to $v$ in $\Gamma$ and $x = y$ or $u = v$ and $x$ is adjacent to $y$ in $\Sigma$.

**Lemma 2.1.** Let $n \geq 3$ and $m \geq 2$. Then $C_n \Box P_m$ has a Hamilton cycle.

**Proof.** Let $\Gamma = C_n \Box P_m$ with $V(\Gamma) = \{1, 2, \ldots, n\} \times \{1, 2, \ldots, m\}$ and

$$E(\Gamma) = \{ \{(i, j), ((i \mod n) + 1, j)\} \mid (i, j) \in \{1, 2, \ldots, n\} \times \{1, 2, \ldots, m\} \}$$

$$\cup \{ \{(i, j), (i, j + 1)\} \mid (i, j) \in \{1, 2, \ldots, n\} \times \{1, 2, \ldots, m - 1\} \} .$$

Put

$$S_1 = \{ \{(i, 1), (i + 1, 1)\} \mid i \in \{1, 2, \ldots, n - 1\} \}$$

to be a subset of $E(\Gamma)$,

$$S_j = \{ \{(i, j), (i + 1, j)\} \mid i \in \{2, 3, \ldots, n - 1\} \}$$

for $j = 2, 3, \ldots, m - 1$, and

$$S_m = \begin{cases} \{\{(i, m), (i + 1, m)\} \mid i \in \{1, 2, \ldots, n - 1\}\} & \text{if } m \text{ is even} \\ \{\{(i, m), ((i \mod n) + 1, m)\} \mid i \in \{2, 3, \ldots, n\}\} & \text{if } m \text{ is odd}. \end{cases}$$
Moreover, put
\[ T_1 = \{(1, j), (1, j + 1) \mid j \in \{1, 2, \ldots, m - 1\}\}, \]
\[ T_2 = \{(2, j), (2, j + 1) \mid j \in \{1, 2, \ldots, m - 1\}, \ j \text{ is even}\}, \]
\[ T_n = \{(n, j), (n, j + 1) \mid j \in \{1, 2, \ldots, m - 1\}, \ j \text{ is odd}\}. \]
Then the spanning subgraph of \( \Gamma \) with edge set \( S_1 \cup S_2 \cup \cdots \cup S_m \cup T_1 \cup T_2 \cup T_n \) is a Hamilton cycle of \( \Gamma \).

For an even integer \( n \geq 4 \) and an integer \( m \geq 2 \), define the \textit{brick chimney graph} \( B_{n,m} \) to be the graph with vertex set \( \{1, 2, \ldots, n\} \times \{1, 2, \ldots, m\} \) and edge set
\[ \{(i, j), ((i \mod n) + 1, j) \mid (i, j) \in \{1, 2, \ldots, n\} \times \{1, 2, \ldots, m\}\} \]
\[ \cup \{(i, j), (i, j + 1) \mid (i, j) \in \{1, 2, \ldots, n\} \times \{1, 2, \ldots, m - 1\}, \ i \equiv j \pmod{2}\} \]
and call
\[ \{(i, 1) \mid i \in \{1, 2, \ldots, n\}, \ i \equiv 0 \pmod{2}\} \]
and
\[ \{(i, m) \mid i \in \{1, 2, \ldots, n\}, \ i \equiv m \pmod{2}\} \]
the \textit{gluing sets} of \( B_{n,m} \). The following result of Alspach and Zhang \cite{AlspachZhang} will play an important role in our proof of Theorem \ref{thm:main}.

**Lemma 2.2.** \cite{AlspachZhang} Corollary 2.3 Let \( n \geq 4 \) be an even number, \( m \geq 2 \) be an integer and \( \Gamma \) be the graph obtained by adding to \( B_{n,m} \) an arbitrary perfect matching between its two gluing sets. Then \( \Gamma \) has a Hamilton cycle.

Let \( A \) be an abelian group. Then for each \( t \in \text{Dih}(A) \setminus A \), we have
\[ \text{Dih}(A) = A \sqcup tA, \]
where \( \sqcup \) denotes the disjoint union. Moreover, elements of \( \text{Dih}(A) \setminus A \) are involutions and the product of two elements in \( \text{Dih}(A) \setminus A \) are in \( A \). With this in mind, we are ready to prove our main result.

**Proof of Theorem \ref{thm:main}**. We prove the theorem by induction on the valency of the graph. If the valency is two then the Cayley graph itself is a Hamilton cycle. Suppose that every connected Cayley graph on a generalized dihedral group of valency less than \( k \) has a Hamilton cycle, where \( k \geq 3 \). Consider a connected Cayley graph \( \Gamma = \text{Cay}(G, S) \) on a generalized dihedral group \( G \) of valency \( |S| = k \), where \( G = \text{Dih}(A) \) for some abelian group \( A \). We aim to show that \( \Gamma \) has a Hamilton cycle. For a subgraph \( \Delta \) of \( \Gamma \) and \( g \in G \), let \( \Delta g \) denote the image of \( \Delta \) under the automorphism of \( \Gamma = \text{Cay}(G, S) \) induced by the right multiplication of \( g \).

First assume that \( S \cap A \neq \emptyset \). We may assume that \( |A| \) is odd as the case that \( |A| \) is even is treated in Theorem \ref{thm:even}. Take an arbitrary \( s \in S \cap A \). Then \( s \) is not an involution. If \( |S| = 3 \), then \( S = \{s, s^{-1}, g\} \) for some involution \( g \) and so \( \Gamma \) consists of two cycles \( \{1, s, \ldots, s^{|A|-1}\}, \ (g, gs, \ldots, gs^{|A|-1}\) and the perfect matching \( \{\{s^i, gs^i\} \mid i \in \{0, 1, \ldots, |A| - 1\}\} \) between them, whence \( \Gamma \cong C_{|A|} \square P_2 \) has a Hamilton cycle by Lemma \ref{lem:3cycle}. Now assume that \( |S| \geq 4 \). Let \( T = S \setminus \{s, s^{-1}\}, \ H = \langle T \rangle, \ B = H \cap A \) and \( \Sigma = \text{Cay}(H, T) \). Since \( G = \langle S \rangle = \langle T, s \rangle \) and \( s \in A \), we know that \( T \not\subseteq A \). Hence \( |H| \geq 4 \) and \( H \not\subseteq A \), which implies that \( |H:B| = 2 \). Since the conjugation of each element of \( G \setminus A \) inverses elements of \( A \), every subgroup of \( A \) is
normal in $G$. Take an arbitrary $t \in T \setminus A$. Then $H = B \rtimes \langle t \rangle = \text{Dih}(B)$. By the inductive hypothesis, $\Sigma = \text{Cay}(H, T)$ has a Hamilton cycle $C \cong C_{|H|}$. If $G = H$, then $C$ is a Hamilton cycle of $\Gamma$. In the following we assume that $G \neq H$. Then $A \neq B$. It follows that
\[
G = \langle S \rangle = \langle T, s \rangle = \langle H, s \rangle = \langle (B, t), s \rangle = \langle B, s, t \rangle = \langle B, s \rangle \langle t \rangle
\]
and
\[
A = A \cap G = A \cap \langle B, s \rangle \langle t \rangle = \langle B, s \rangle = B \langle s \rangle.
\]
Let $m$ be the smallest positive integer such that $s^m \in B$. Then $m > 1$ and
\[
A = B \uplus Bs \uplus \cdots \uplus Bs^{m-1}.
\]
Viewing $G = A \uplus tA$ and $H = B \uplus tB$, we then deduce that
\[
G = (B \uplus tB) \uplus (Bs \uplus tBs) \uplus \cdots \uplus (Bs^{m-1} \uplus tBs^{m-1}) = H \uplus Hs \uplus \cdots \uplus Hs^{m-1}.
\]
Since $A$ is abelian, we have $s \cdot bs^{i-1} = bs^i$ and $s^{-1} \cdot tbs^{i-1} = tsbs^i = tbs^i$ for all $b \in B$ and $i \in \{1, 2, \ldots, m-1\}$. Hence
\[
M_i := \{\{bs^{i-1}, bs^i\} \mid b \in B\} \uplus \{\{tsbs^{i-1}, tbs^i\} \mid b \in B\}
\]
is a perfect matching between $Hs^{i-1}$ and $Hs^i$ in $\Gamma$ for each $i \in \{1, 2, \ldots, m-1\}$. Let $\Gamma_1$ be the spanning subgraph of $\Gamma$ with edge set
\[
E(C) \uplus E(Cs) \uplus E(Cs^2) \uplus \cdots \uplus E(Cs^{m-1}) \uplus M_1 \uplus M_2 \uplus \cdots \uplus M_{m-1}.
\]
Then $\Gamma_1 \cong C_{|H|} \uplus P_m$, and thus Lemma 2.1 shows that $\Gamma_1$ has a Hamilton cycle, which is also a Hamilton cycle of $\Gamma$.

Next assume that $S \subseteq G \setminus A$. Take arbitrary $s \in S$ and $t \in T \setminus \{s\}$. Let $T = S \setminus \{s\}$, $H = \langle T \rangle$, $B = H \cap A$ and $\Sigma = \text{Cay}(H, T)$. Since $T \subseteq G \setminus A$, we have $H \nparallel A$ and so $|H:B| = 2$. Note that $B$ is normal in $G$ and $H = B \rtimes \langle t \rangle = \text{Dih}(B)$. Then by our inductive hypothesis, $\Sigma = \text{Cay}(H, T)$ has a Hamilton cycle $C \cong C_{|H|}$. If $G = H$, then $C$ is a Hamilton cycle of $\Gamma$. Now assume that $G \neq H$. Then $A \neq B$. As $s$ and $t$ are elements of $G \setminus A$, they are both involutions, and $A \cap \langle s, t \rangle = \langle st \rangle$. It follows that
\[
G = \langle S \rangle = \langle T, s \rangle = \langle H, s \rangle = \langle B \langle t \rangle, s \rangle = \langle B, \langle s, t \rangle \rangle = B \langle s, t \rangle
\]
and
\[
A = A \cap G = A \cap B \langle s, t \rangle = B(A \cap \langle s, t \rangle) = B \langle s, t \rangle.
\]
Let $m$ be the smallest positive integer such that $(st)^m \in B$. Then $m > 1$ and
\[
A = B \uplus B(st) \uplus \cdots \uplus B(st)^{m-1}.
\]
This yields that
\[
G = (B \uplus tB) \uplus (B(st) \uplus tB(st)) \uplus \cdots \uplus (B(st)^{m-1} \uplus tB(st)^{m-1}) = H \uplus H(st) \uplus \cdots \uplus H(st)^{m-1}.
\]
Since $T \subseteq S \subseteq G \setminus A$, the graph $\Sigma = \text{Cay}(H, T)$ is bipartite with parts $B$ and $tB$. This implies that $\Sigma(st)^i$ is bipartite with parts $B(st)^i$ and $tB(st)^i$, and so is its Hamilton cycle $C(st)^i$. Moreover, since $A$ is abelian, we have $s \cdot tB(st)^{i-1} = b(st)^i$ for all $b \in B$ and $i \in \{1, 2, \ldots, m\}$. Hence
\[
M_i := \{\{tb(st)^{i-1}, b(st)^i\} \mid b \in B\} \uplus \{\{tsbs^{i-1}, tbs^i\} \mid b \in B\}.
\]
is a perfect matching between $tB(st)^i$ and $B(st)^i$ in $\Gamma$ for each $i \in \{1, 2, \ldots, m\}$. Note that $B(st)^m = B$. Let $\Gamma_1$ be the spanning subgraph of $\Gamma$ with edge set
\[ E(C) \sqcup E(C(st)) \sqcup E(C(st)^2) \sqcup \cdots \sqcup E(C(st)^{m-1}) \sqcup M_1 \sqcup M_2 \sqcup \cdots \sqcup M_{m-1} \]
and $\Gamma_2$ be the spanning subgraph of $\Gamma$ with edge set $E(\Gamma_1) \sqcup M_m$. Then $\Gamma_1 \cong B_{|H|,m}$ is a brick chimney graph with gluing sets $B$ and $tB(st)^{m-1}$. By Lemma 2.2, $\Gamma_2$ has a Hamilton cycle, which is also a Hamilton cycle of $\Gamma$. This completes the proof. □

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