Chaos in a $Q\bar{Q}$ system at finite temperature and baryon density

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Onset of chaos for the holographic dual of a $Q\bar{Q}$ system at finite temperature and baryon density is studied. We consider a string in the AdS Reissner-Nordstrom background near the black-hole horizon, and investigate small time-dependent perturbations of the static configurations. The proximity to the horizon induces chaos, which is softened increasing the chemical potential. A background geometry including the effect of a dilaton is also examined. The Maldacena, Shenker and Stanford bound on the Lyapunov exponents characterizing the perturbations is satisfied for finite baryon chemical potential and when the dilaton is included in the metric.

I. INTRODUCTION

It has been recently conjectured under general assumptions that, for a thermal quantum system at temperature $T$, some out-of-time-ordered correlation functions involving Hermitian operators, for determined time intervals, have an exponential time dependence characterized by an exponent $\lambda$, and that such exponent obeys the bound

$$\lambda \leq 2\pi T$$

(in units in which $\hbar = 1$ and $k_B = 1$). The correlation functions are related to the thermal expectation values of the (square) commutator of two Hermitian operators at a time separation $t$, which quantify the effect of one operator on later measurements of the other one, a framework for introducing chaos for a quantum system. The conjectured bound, proposed by Maldacena, Shenker and Stanford [1], is remarkable due to its generality. It has been inspired by the observation that black holes (BH) are the fastest ‘scramblers’ in nature: the time needed for a system near a BH horizon to lose information depends logarithmically on the number of degrees of freedom of the system [2,3].

The consequences on the connection between chaotic quantum systems and gravity have been soon investigated [4–7]. A relation between the size of operators on the boundary quantum theory, involved in the temporal evolution of a perturbation, and the momentum of a particle falling in the bulk has been proposed in a holographic framework [8,9].

A generalization of the bound [10] for a thermal quantum system with a global symmetry has been proposed [10]:

$$\lambda \leq \frac{2\pi T}{1 - \frac{|\mu|}{\mu_c}}$$

where $\mu$ is the chemical potential related to the global symmetry, and $\mu_c$ is a critical value above which the thermodynamical ensemble is not defined. The inequality [2] is conjectured for $\mu \ll \mu_c$ and relaxes the bound [1]. Our purpose is to test this generalization.

Several analyses have been devoted to check Eq. [1] using the AdS/CFT correspondence [11–13], adopting a dual geometry with a black hole, identifying $T$ with the Hawking temperature, for example in [14,15]. In particular, the heavy quark-antiquark pair, described holographically by a string hanging in the bulk with end points on the boundary [16–19], has been studied in this context [20]. For this system $\lambda$ is the Lyapunov exponent characterizing the chaotic behavior of time-dependent fluctuations around the static configuration.

To test the generalized bound [2] one has to include the chemical potential in the holographic description. In QCD, a $U(1)$ global symmetry is connected to the conservation of the baryon number. A dual metric has been identified with the AdS Reissner-Nordstrom (RN) metric for a charged black-hole. We can use such a background for testing Eq. [2].

The discussion of the 5d AdS-RN metric as a dual geometry for a thermal system with conserved baryon number can be found, e.g., in [23,24]. The metric is defined by the line element

$$ds^2 = -r^2 f(r) \, dt^2 + r^2 d\bar{x}^2 + \frac{1}{r^2 f(r)} \, dr^2$$

We denote by $r_0$ the position of the tip of the string as shown in Fig. 1 and $l$ the proper distance measured along the string starting from $r_0$. Choosing $\tau = t$ and $\sigma = l \ (l$-gauge), for a static string laying in the $x-r$ plane with $X^M = (t, x(l), 0, 0, r(l))$ the Nambu-Goto action reads:

$$S = -\frac{T}{2\pi\alpha'} \int_{-\infty}^{\infty} dl \sqrt{F^2(r) \dot{x}^2(l) + G^2(r) \dot{r}^2(l)},$$

where $\dot{x} = \frac{\partial x}{\partial r}$ and $\dot{r} = \frac{\partial r}{\partial r}$, $F^2(r) = g_{tt}(r) g_{xx}(r)$ and $G^2(r) = g_{tt}(r) g_{rr}(r)$. For the metric (3), one has $F^2(r) = r^4 f(r)$ and $G(r) = 1$.

$x$ is a cyclic coordinate, hence:

$$\dot{x}(l) = \pm \frac{\dot{r}(l)}{\sqrt{r_0^4 f(r_0) (r^4 f(r) - r_0^4 f(r_0))}}.$$  \hspace{1cm} (9)

The solution of equation (9) is obtained considering that

$$dl^2 = g_{xx}(r) dx^2 + g_{rr}(r) dr^2.$$  \hspace{1cm} (10)

For the unit vector $t^M = (0, \dot{x}(l), 0, 0, \dot{r}(l))$ tangent to the string at the point with coordinate $l$ the relation holds:

$$g_{MN} t^M t^N = g_{xx}(r) \dot{x}^2(l) + g_{rr}(r) \dot{r}^2(l) = r^2 \dot{x}^2(l) + \frac{1}{r^2 f(r)} \dot{r}^2(l) = 1.$$  \hspace{1cm} (11)

Including this constraint in Eq. (9) gives

$$\dot{r} = \frac{\sqrt{r^4 f(r) - r_0^4 f(r_0)}}{r} \hspace{1cm} (12)$$

$$\dot{x} = \frac{\sqrt{r_0^4 f(r_0)}}{r^3 \sqrt{f(r)}}.$$  \hspace{1cm} (13)

The function $r(l)$ for the static string can be computed integrating Eq. (12).

The dependence of $L$, the distance between the string endpoints on the boundary, on $r_0$ is obtained:

$$L(r_0) = 2 \int_{r_0}^{\infty} dr \frac{r^2 \sqrt{f(r)}}{r^2 \sqrt{f(r)} \sqrt{r^4 f(r) - r_0^4 f(r_0)}}.$$  \hspace{1cm} (14)

The energy of the string configuration

$$E(r_0) = \frac{1}{\pi\alpha'} \int_{r_0}^{\infty} dr \frac{r^2 \sqrt{f(r)}}{\sqrt{r^4 f(r) - r_0^4 f(r_0)}}.$$  \hspace{1cm} (15)
diverges and needs to be regularized. A possible prescription is to subtract the bare quark masses, interpreted as the energy of the string consisting in two straight lines from the boundary to the horizon,

\[ m_Q = \frac{1}{2\pi\alpha'} \int_{r_H}^{\infty} dr, \]

obtaining

\[ E_{QQ}(r_0) = \frac{1}{\pi\alpha'} \left( \int_{r_0}^{\infty} dr \frac{r^2 f(r)}{\sqrt{r^4 f(r) - r_0^4 f(r_0)}} - \int_{r_H}^{\infty} dr \right). \]

The function \( E_{QQ} \) can be expressed vs \( L \). For the metric in Eq. (4), the distance \( L(r_0) \) has a maximum \( L_{\text{max}} \), and all values \( L \in [0, L_{\text{max}}] \) are obtained for two positions \( r_0 \). Also the function \( E_{QQ}(r_0) \) has a maximum, which decreases and is reached earlier as \( \mu \) increases. For each value of the chemical potential there is a value of \( r_0 \) above which there is one energy value indicating a stable string configuration. Below such \( r_0 \), as shown in Fig. 2, the \( E_{QQ}(L) \) is not single valued: for each \( L \) there are profiles identified by different \( r_0 \), with different energies, corresponding to stable and unstable configurations.

III. SQUARE STRING

As suggested in [20], a simple model suitable for an analytical treatment of the time-dependent perturbations is a square string in the AdS-RN background geometry [3], depicted in Fig. 3. The model describes quite well a string near the horizon, as shown in Fig. 4, where the profile of the string approaching the horizon is drawn.

It is convenient to work in the \( r \)-gauge (\( \tau = t \) and \( \sigma = r \)). The embedding functions for a string in the \( x-r \) plane are \( X^M = (t, x(t, r), 0, 0, r) \), and the NG action reads

\[ S = -\frac{1}{2\pi\alpha'} \int dt dr \sqrt{1 + \dot{x}^2 \left( r^4 f(r) - \frac{1}{f(r)} \dot{r}^2 \right)}. \]

For a static string \( X^M = (t, x(r), 0, 0, r) \) this reduces to

\[ S = -\frac{T}{2\pi\alpha'} \int dr \sqrt{r^4 f(r) \dot{x}^2 + 1}. \]

In the case of the square profile, Eq. (19) is determined integrating along the three sides of the string. The result can be regularized:

\[ S_{\text{reg}} = -\frac{T}{2\pi\alpha'} \left( L^2_0 \sqrt{f(r_0)} - 2 (r_0 - r_H) \right), \]

where \( L \) still denotes the distance between the endpoints on the boundary. For \( r_0 \) near the horizon the energy

\[ E = -\frac{S_{\text{reg}}}{T} \]

has a local maximum, hence upon small perturbations the string departs toward an equilibrium configuration. The stationary point for \( E \) is determined solving

\[ 2Lr_0\sqrt{f(r_0)} + \frac{r_0^2 L}{2\sqrt{f(r_0)}} \frac{\partial f(r_0)}{\partial r_0} - 2 = 0. \]

For the metric function \( f(r) \) in (4), expanding the l.h.s. of Eq. (22) for \( r_0 \to r_H \) gives:

\[ r_{0,\text{sol}} = \frac{r_H \left( L^2 \left( 2r_H^2 + 11\mu^2 \right) - 8 \right)}{2L^2 \left( 2r_H^2 + 5\mu^2 \right) - 8}. \]

Moreover, expanding for \( L \to 0 \) at \( \mathcal{O}(L^2) \) gives

\[ r_{0,\text{sol}} = r_H \left( 1 + \frac{L^2}{8} \left( 2r_H^2 - \mu^2 \right) \right). \]
We now consider a fluctuating string described by the action \[ \mathcal{L} \text{,} \] and introduce a small time-dependent perturbation \( \delta r(t) \) to the static solution, \( r_0(t) = r_{0,\text{sol}} + \delta r(t) \); indeed, for the square string a perturbation makes time-dependent the position \( r_0 \) of the bottom side. The regularized action is given by

\[
S^{\text{reg}} = -\frac{1}{2\pi} \int dt \left\{ L \sqrt{r_0^4 f(r_0) - \frac{1}{f(r_0)} r_0^2} - 2(r_0 - r_H) \right\}.
\]  

The Lagrangian

\[
\mathcal{L} = L \sqrt{r_0^4 f(r_0) - \frac{1}{f(r_0)} r_0^2} - 2(r_0 - r_H)
\]  

can be expanded around \( r_{0,\text{sol}} \) to second order in \( \delta r(t) \):

\[
\mathcal{L} \approx -2r_{0,\text{sol}} + 2r_H + L r_{0,\text{sol}}^2 \sqrt{f(r_{0,\text{sol}})} + \delta r(t) \left( -2 + 2L r_{0,\text{sol}} \sqrt{f(r_{0,\text{sol}})} + \frac{L r_{0,\text{sol}}^2 f'(r_{0,\text{sol}})}{2 \sqrt{f(r_{0,\text{sol}})}} \right) \\
+ L \delta r(t)^2 \left( \frac{1}{\sqrt{f(r_{0,\text{sol}})}} \right) - \frac{r_{0,\text{sol}}^2 f'(r_{0,\text{sol}})^2}{8 f(r_{0,\text{sol}})^{3/2}} + \frac{r_{0,\text{sol}}^2 f''(r_{0,\text{sol}})}{4 \sqrt{f(r_{0,\text{sol}})}} \right) - \frac{L}{2r_{0,\text{sol}}^3 f(r_{0,\text{sol}})^{3/2}},
\]

and the equation of motion for \( \delta r(t) \) reads:

\[
\ddot{\delta r}(t) - \frac{L}{r_{0,\text{sol}}^3 f(r_{0,\text{sol}})^{3/2}} + L \delta r(t) \left( 2 \frac{r_{0,\text{sol}} f'(r_{0,\text{sol}})}{\sqrt{f(r_{0,\text{sol}})}} + \frac{2r_{0,\text{sol}}^2 f'(r_{0,\text{sol}})^2}{4 f(r_{0,\text{sol}})^{3/2}} + \frac{r_{0,\text{sol}}^2 f''(r_{0,\text{sol}})}{2 \sqrt{f(r_{0,\text{sol}})}} \right) - 2 + 2L r_{0,\text{sol}} \sqrt{f(r_{0,\text{sol}})} + \frac{L r_{0,\text{sol}}^2 f'(r_{0,\text{sol}})}{2 \sqrt{f(r_{0,\text{sol}})}} = 0.
\]  

This equation is solved by

\[
\delta r(t) = A \exp(\lambda t) + B \exp(-\lambda t).
\]  

The coefficient \( \lambda \), our Lyapunov exponent, determines the time growth of the perturbation. It is given by:

\[
\lambda = \frac{r_{0,\text{sol}}}{2} \left( -8 f(r_{0,\text{sol}})^2 + r_{0,\text{sol}}^2 f'(r_{0,\text{sol}})^2 \\
- 2 r_{0,\text{sol}} f(r_{0,\text{sol}}) \left( 4 f'(r_{0,\text{sol}}) + r_{0,\text{sol}} f''(r_{0,\text{sol}}) \right) \right)^{1/2}.
\]

(30)

Expanding \( f(r_{0,\text{sol}}) \), \( f'(r_{0,\text{sol}}) \) and \( f''(r_{0,\text{sol}}) \) at second order in \( L \) we have:

\[
\lambda = 2r_H \left( 1 - \frac{\mu^2}{2r_H^2} \right) \left( 1 - \frac{L^2}{4} \left( 2r_H^2 - \mu^2 \right) \right)
\]

(31)

and, using Eq. [5]:

\[
\lambda = 2\pi T_H \left( 1 - \frac{L^2}{2} \pi T_H r_H \right).
\]

(32)

The exponent \( \lambda \) saturates the bound [1] at the lowest order in \( L \). The \( O(L^2) \) gives a negative correction. No effect of the chemical potential is found, but for the one encoded in \( T_H \) through Eq. [5].
IV. PERTURBED STRING

To study the onset of chaos in a more realistic configuration, we perturb the static solution of a string near the black-hole horizon by a small time-dependent effect.

There are different ways to introduce a small time-dependent perturbation. We follow [20], and perturb the string along the orthogonal direction at each point with coordinate \( l \) in the \( r-x \) plane, as in Fig. 5. For the unit vector \( n^M = (0, n^x, 0, 0, n^r) \) orthogonal to \( t^M \) we have:

\[
g_{rr}(r) (n^r)^2 + g_{xx}(r) (n^x)^2 = 1 \tag{33}
\]
\[
\dot{r}(l) g_{rr}(r) n^r + \dot{x}(l) g_{xx}(r) n^x = 0 \tag{34}
\]

The solution for the components \( n^x \) and \( n^r \) is

\[
n^x(l) = \sqrt{\frac{g_{rr}}{g_{xx}}} \dot{x}(l), \quad n^r(l) = -\sqrt{\frac{g_{xx}}{g_{rr}}} \dot{r}(l) \tag{35}
\]

for an outward perturbation, as in Fig. 5. Introducing a time-dependent perturbation \( \xi(t,l) \) along \( n \) one has:

\[
r(t,l) = r_{BG}(l) + \xi(t,l) n^r(l); \quad x(t,l) = x_{BG}(l) + \xi(t,l) n^x(l), \tag{36}
\]

with \( r_{BG}(l) \) and \( x_{BG}(l) \) the static solutions obtained integrating Eqs. (12) and (13).

To describe the dynamics of the perturbation (assuming it is small), we expand the metric function around the static solution \( r_{BG}(l) \) to the third order in \( \xi(t,l) \).

To the third order in \( \xi \) the NG action involves a quadratic and a cubic term. The quadratic term has the form:

\[
S^{(2)} = \frac{1}{2\pi\alpha'} \int dt \int_{-\infty}^{\infty} dl \left( C_{tt}\dot{\xi}^2 + C_{ll}\xi^2 + C_{00}\xi^2 \right), \tag{37}
\]

with \( C_{tt}, C_{ll} \) and \( C_{00} \) depending on \( l \). For the metric in Eq. (3) with a generic metric function \( f(r) \) the coefficients \( C_{tt}, C_{ll} \) and \( C_{00} \) read:

\[
C_{tt}(l) = \frac{1}{2r_{BG}\sqrt{f(r_{BG})}},
\]
\[
C_{ll}(l) = -\frac{1}{4C_{tt}(l)},
\]
\[
C_{00}(l) = \frac{1}{4r_{BG}^3 f(r_{BG})^{3/2}} \left\{ -2r_{BG}^4 f(r_{BG})^2 \left( 2f(r_{BG}) + r_{BG} f'(r_{BG}) \right) \right.
\]
\[
+ r_{0}^4 f(r_0) \left( 4f(r_{BG})^2 + r_{BG}^2 f'(r_{BG})^2 + r_{BG} f(r_{BG}) \left( f'(r_{BG}) - r_{BG} f''(r_{BG}) \right) \right) \right\}. \tag{38}
\]

The coefficients depend on \( l \) through \( r_{BG}(l) \). Their expressions for the AdS-RN metric are:
The equation of motion from (41) is

\[ C_{ll} \ddot{\xi} + \partial_t \left( C_{lt} \dot{\xi} \right) - C_{00} \dot{\xi} = 0. \tag{40} \]

For \( \xi (t, l) = \xi (l) e^{i\omega t} \) this corresponds to

\[ \partial_t \left( C_{ll} \dot{\xi} \right) - C_{00} \dot{\xi} = \omega^2 C_{ll} \xi, \tag{41} \]

a Sturm-Liouville equation with weight function \( W(l) = -C_{ll}(l) \). We solve Eq. (41) for different values of \( r_0 \) and \( \mu \), imposing the boundary conditions \( \xi (l) \to 0 \) as \( l \to -\infty \). The two lowest eigenvalues \( \omega_0^2 \) and \( \omega_1^2 \), varying \( r_0 \) and \( \mu \), are collected in Table I and in one case the eigenfunctions \( e_0 (l) \) and \( e_1 (l) \) are depicted in Fig. 6.

![FIG. 6. Eigenfunctions \( e_0 (l) \) (black line) and \( e_1 (l) \) (red line) of Eq. (41) for \( r_0 = 1.172 \) and \( \mu = 0.6 \).](image)

There are negative values of \( \omega_0^2 \), corresponding to an unstable sector. For \( \mu = 0 \) the system is stabilized as \( r_0 \) increases, with the tip of the string departing from the BH horizon: \( \omega_0^2 \) becomes positive for \( r_0 \geq 1.177 \). Fixing \( r_0 = 1.1 \), the lowest lying state is stabilized increasing the chemical potential \( \mu \), and \( \omega_0^2 \) is positive for \( \mu \geq 1.2 \). The dependence of \( \omega_0^2 \) and \( \omega_1^2 \) on \( r_0 \) and \( \mu \) is shown in Fig. 7, together with the line demarcating the regions of negative and positive values of \( \omega_0^2 \).

The perturbation can be expanded in terms of the first two eigenfunctions \( e_0 (l) \) and \( e_1 (l) \),

\[ \xi (t, l) = c_0 (t) e_0 (l) + c_1 (t) e_1 (l), \tag{42} \]

with the time dependence dictated by the coefficient functions \( c_0 (t) \) and \( c_1 (t) \). Up to a surface term, the cubic action has the expression:

| \( r_0 \) | \( \mu \) | \( \omega_0^2 \) | \( \omega_0^2 \) |
|---|---|---|---|
| \( r_0 = 1.1 \) | 1.172 | 0 | -0.064 | 10.458 |
| 0.6 | -0.703 | 7.082 | 0.6 | 0.148 | 9.574 |
| 0.9 | -0.388 | 5.605 | 0.9 | 0.324 | 8.428 |
| 1.2 | 0.006 | 3.938 | 1.2 | 0.397 | 6.735 |

| \( r_0 \) | \( \mu \) | \( \omega_0^2 \) | \( \omega_0^2 \) |
|---|---|---|---|
| \( r_0 = 5 \) | 1.172 | 0 | 81.726 | 275.477 |
| 0.3 | 0.124 | 10.537 | 0.3 | 81.706 | 275.458 |
| 0.6 | 0.258 | 9.874 | 0.6 | 81.648 | 275.400 |
| 0.9 | 0.406 | 8.733 | 0.9 | 81.551 | 275.303 |
| 1.2 | 0.449 | 7.046 | 1.2 | 81.415 | 275.168 |

TABLE I. Eigenvalues \( \omega_0^2 \) and \( \omega_1^2 \) of Eq. (41) changing the values of \( r_0 \) and \( \mu \).
\[ S^{(3)} = \frac{1}{2\pi\alpha'} \int dt \int_{-\infty}^{\infty} dl \left\{ D_0 \xi^3 + D_1 \xi \dot{\xi}^2 + D_2 \xi^2 \dot{\xi}^2 \right\}, \] (43)

with \( D_{0,1,2} \) functions of \( l \). This reads, expanding the perturbation \( \xi(t, l) \) as in Eq. (42):

\[
S^{(3)} = \frac{1}{2\pi\alpha'} \int dt \int_{-\infty}^{\infty} dl \left\{ \left( D_0 c_0^3 + D_1 c_0 c_1^2 \right) c_0^2(t) + \left( 3D_0 c_0 c_1^2 + D_1 \left( 2c_0 c_1 + c_0 c_1^2 \right) \right) c_0(t) c_1^2(t) + D_2 \left( e_0 c_1^3 c_0^2 + e_0^3 c_1 c_0 c_1^2 + 2c_0 c_1^2 c_0 c_1 \right) \right\}. \] (44)

Upon integration on \( l \), an action for \( c_0(t) \) and \( c_1(t) \) is obtained summing \( S^{(2)} \) and \( S^{(3)} \):

\[
S^{(2)} + S^{(3)} = \frac{1}{2\pi\alpha'} \int dt \left[ \sum_{n=0,1} \left( \dot{c}_n^2 - \omega_n^2 c_n^2 \right) + K_1 c_0^3 + K_2 c_0 c_1^2 + K_3 c_0 c_1^2 + K_4 c_0 c_1^2 + K_5 \dot{c}_0 c_1 \right]. \] (45)

FIG. 7. Eigenvalues \( \omega_0^2 \) and \( \omega_1^2 \) vs \( r_0 \) and \( \mu \). The green surface corresponds to \( \omega_1^2 \), the red and blue surface to \( \omega_0^2 \). The dark blue line demarcates the (blue) region of negative \( \omega_0^2 \) from the (red) region of positive \( \omega_0^2 \).

The coefficients \( K_{1,2,3,4,5} \) depend on \( r_0 \) and \( \mu \), and are collected in Tab. II choosing a set of values for such quantities.

As one can numerically test, in cases corresponding to negative values of \( \omega_0^2 \) the action describes the motion of \( c_0 \) and \( c_1 \) in a trap, and in some regions within the potential the kinetic term is negative. As suggested in [20], it is useful to replace \( c_{0,1} \rightarrow \tilde{c}_{0,1} \) in the action, with \( c_0 = \tilde{c}_0 + \alpha_1 \tilde{c}_0^2 + \alpha_2 \tilde{c}_1^2 \) and \( c_1 = \tilde{c}_1 + \alpha_3 \tilde{c}_0 \tilde{c}_1 \), neglecting \( O(\tilde{c}_1^4) \) terms, setting the constants \( \alpha \) to ensure the positivity of the kinetic term. We set the constants \( \alpha_1 = -2, \alpha_2 = -0.5 \) and \( \alpha_3 = -1 \), slightly different from [20]. The replacement stretches the potential stabilizing the time evolution: the dynamics is not affected, and a chaotic behaviour shows up also in the transformed system.

The onset of chaos can be investigated constructing Poincaré sections. We show the sections defined by \( \tilde{c}_1(t) = 0 \) and \( \tilde{c}_1(t) > 0 \) for bounded orbits within the trap. In the case \( r_H = 1, r_0 = 1.1 \) and increasing \( \mu \), such sections are collected in Fig. 8. For \( \tilde{c}_0 \) near zero the orbits are scattered points depending on the initial conditions. On the other hand, increasing \( \mu \) the points in the plot form more regular paths: the effect of switching on the chemical potential is to mitigate the chaotic behavior.

For \( \mu = 1.2 \) and \( r_0 = 1.1 \) the eigenvalue \( \omega_0^2 \) becomes positive and the orbits form tori, as one can see in Fig. 9. Moving further away from the horizon, the Poincaré plots for the string dynamics show regular

\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
\mu & K_1 & K_2 & K_3 & K_4 & K_5 \\
\hline
0 & 11.36 & 21.72 & 10.58 & 3.37 & 6.73 \\
0.6 & 7.22 & 16.76 & 9.98 & 3.44 & 6.88 \\
1.2 & 0.81 & 5.84 & 8.29 & 3.64 & 7.28 \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
\mu & K_1 & K_2 & K_3 & K_4 & K_5 \\
\hline
0 & 7.63 & 20.61 & 8.17 & 2.69 & 5.39 \\
0.6 & 5.13 & 17.30 & 8.04 & 2.81 & 5.62 \\
1.2 & 0.86 & 9.30 & 7.81 & 3.22 & 6.44 \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
\mu & K_1 & K_2 & K_3 & K_4 & K_5 \\
\hline
0 & 7.36 & 20.64 & 8.00 & 2.65 & 5.29 \\
0.6 & 4.97 & 17.45 & 7.90 & 2.76 & 5.53 \\
1.2 & 0.88 & 9.69 & 7.76 & 3.18 & 6.36 \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
\mu & K_1 & K_2 & K_3 & K_4 & K_5 \\
\hline
0 & -15.01 & 560.52 & 7.44 & 2.84 & 5.67 \\
0.6 & -14.88 & 560.57 & 7.44 & 2.84 & 5.67 \\
1.2 & -14.49 & 560.73 & 7.46 & 2.84 & 5.69 \\
\hline
\end{array}
\]
FIG. 8. Poincaré sections for a time-dependent perturbed string, obtained changing the initial conditions, with \( r_0 = 1.1 \) and increasing the chemical potential: \( \mu = 0 \) (top row), \( \mu = 0.03 \) (second row), \( \mu = 0.06 \) (third row) and \( \mu = 0.09 \) (bottom row), for \( \dot{c}_1 = 0 \) and \( \dot{c}_1 \geq 0 \). The plots in the right column enlarge the corresponding ones in the left column in the range of small \( \tilde{c}_0 \), \( \dot{\tilde{c}}_0 \).

orbits regardless of \( \mu \).

The Lyapunov exponents in the four dimensional \( c_0, c_1 \) phase space can be computed for the different values of \( r_0 \) and \( \mu \) using the numerical method in [25]. The results are shown in Figs. 10 and 11. Focusing on the system with \( r_0 = 1.1 \), we have evaluated the convergency plots of the four Lyapunov coefficients, one for each direction of the phase space, varying \( \mu \) from \( \mu = 0 \) to \( \mu = 1.2 \). The cases \( \mu = 0 \) and \( \mu = 0.6 \) are displayed in Fig. 10, the other cases are similar. The largest Lyapunov exponent behaves as an exponentially decreasing oscillating function, which can be extrapolated to large number of time steps as shown in Fig. 11. The values resulting from the fit decrease as \( \mu \) increases: the effect of the chemical potential is to soften the dependence on the initial conditions, making the string less chaotic.

To investigate the behaviour for different \( r_0 \), we have computed the Lyapunov coefficients for \( r_0 = 5 \), away from the horizon, and \( \mu \) up to \( \mu = 1.2 \). The convergency plots show a rapid convergence of all Lyapunov coefficients towards zero. The result of the fit for large time steps, for different values of \( \mu \) is in the same Fig. 11.

To summarize, the Poincaré plots show that chaos is produced in the proximity of the BH horizon, and
FIG. 9. Poincaré section in the case \( r_0 = 1.1, \mu = 1.2, \) energy \( E = 1 \times 10^{-5} \) with \( 8 \times 10^3 \) time steps (top panel), \( r_0 = 5, \mu = 0 \) and energy \( E = 1 \times 10^{-3} \) (bottom panel).

FIG. 10. Convergency plots of the four Lyapunov exponents (LCEs) in the case of a string with \( r_0 = 1.1, \mu = 0 \) (top panel) and \( \mu = 0.6 \) (bottom panel), and \( 2 \times 10^3 \) time steps.

FIG. 11. Fit of the largest Lyapunov coefficient \( \lambda_{MAX} \) for \( r_0 = 1.1 \) (top) and \( r_0 = 5 \) (bottom), varying \( \mu \). The local maxima of plots as in Fig. 10 are fitted.

FIG. 12. Largest Lyapunov exponent \( \lambda_{MAX} \) vs \( \mu \), for the tip position \( r_0 = 1.1 \) (top) and \( r_0 = 5 \) (bottom panel).

that the dynamics of the string is less chaotic as the chemical potential increases. This is confirmed by the behavior of the largest Lyapunov coefficient. In all cases the bound Eq. (1) is satisfied: for example, for a system with \( r_0 = 1.1 \) and \( \mu = 0.6 \) we have \( \lambda \simeq (2.7 \times 10^{-2}) \times 2\pi T_{H} \), close to the value computed for \( \mu = 0 \) in [20]. There are no indications of a relaxed bound as foreseen by Eq. (2).
V. GEOMETRY WITH A DILATON

It is interesting to study a different background, a modification of the AdS-RN with the introduction of a warp factor, used to implement a confining mechanism in holographic models of QCD breaking the conformal invariance [20]. The line element is defined as

\[ ds^2 = e^{-\frac{2}{r}} \left( -r^2 f(r) dt^2 + r^2 d\vec{x}^2 + \frac{1}{r^2 f(r)} dr^2 \right), \]

(46)

with the same metric function \( f(r) \) in Eq. (1). The Hawking temperature is in Eq. (5) and does not depend on the dilaton parameter \( c \). The warp factor mainly affects the IR small \( r \) region, and the geometry becomes asymptotically \( AdS_5 \) in the UV \( r \to \infty \) region. Introducing a dilaton factor has been used, in a bottom-up approach, to study features of the QCD phenomenology at finite temperature and baryon density, namely the behaviour of the quark and gluon condensates increasing \( T \) and \( \mu \), the phase diagram, the in-medium broadening of the spectral functions of two-point correlators [24, 27–29].

The analysis for a time-dependent perturbation of the static string in this background can be carried out following the previously adopted procedure. For the square string in the background (46), the Lyapunov exponent computed at \( O(L^2) \) reads:

\[ \lambda = 2\pi T_H \left( 1 - \frac{L^2}{2} \pi T_H r_H \left( 1 + \frac{c^2}{r_H^2} \right) \right). \]

(47)

This expression fulfills the bound (1).

To study the dependence of chaos on the dilaton parameter \( c \), we inspect the Poincarè plots and compute the Lyapunov exponents. The Poincarè section for \( r_H = 1 \), \( r_0 = 1.1 \), \( \mu = 0 \) and \( c = 1 \) is shown in Fig. 13. For small values of \( \tilde{c}_0 \), \( \tilde{c}_0 \) the section shows patterns hinting for a less chaotic system as the constant \( c \) increases. This is confirmed by the Poincarè plot for \( c = 2 \), which shows regular orbits also in the phase space region of small \( \tilde{c}_0 \) and \( \tilde{c}_0 \). That increasing \( c \) the string is less chaotic can also be inferred from Fig. 14 where the Lyapunov coefficient for the string with \( r_H = 1 \), \( r_0 = 1.1 \), \( \mu = 0 \) and a few values of \( c \) is drawn: the exponent monotonically decreases vs \( c \).

VI. CONCLUSIONS

The investigation of a holographic dual of the heavy quark-antiquark system confirms the bounds (1) also in the case of finite baryonic chemical potential. Increasing \( \mu \) the system is less chaotic. This agrees with the conclusion obtained considering the charged particle motion in the RN AdS background, for which a reduction of the chaotic behaviour is observed increasing the chemical potential [30]. Decrease in chaoticity is also observed for a thermal background involving a dilaton warp factor. Even though our analysis is limited to small perturbations around the static string configuration, such conclusions are robust; it seems unlikely that a numerical study of large fluctuations around the static profile, analogous to the one carried out in [20], would lead to different results. The conclusion is that the bound (1) continues to hold in the case of finite chemical potential.

A possible extension of our analysis concerns the interplay between chaos and time-dependent background geometry, namely the hydrodynamic metric worked out in [31–34]. It would be interesting to establish the existence of a bound analogous to Eq. (1) also in these cases.

Acknowledgements.

We thank F. Giannuzzi, A. Mirizzi and S. Nicotri for discussions. This study has been carried out within the INFN project (Iniziativa Specifica) QFT-HEP.

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FIG. 13. Top panels: Poincaré section for the perturbed string in the background geometry with warp factor (46), for $r_0 = 1.1$, $\mu = 0$ and parameter of the dilaton $c = 1$, energy $E = 1 \times 10^{-5}$ and $8 \times 10^{3}$ time steps (left plot). The right plot enlarges the left one in the small $\tilde{c}_0$, $\dot{\tilde{c}}_0$ region. The bottom panels correspond to $c = 2$.

FIG. 14. Largest Lyapunov exponent for $r_0 = 1.1$ and $\mu = 0$, increasing the dilaton constant $c$.

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