A solution of the Einstein-Maxwell equations describing conformally flat spacetime outside a charged domain wall

Øyvind Grøn* and Steinar Johannesen*

* Oslo University College, Department of Engineering, P.O.Box 4 St.Olavs Plass, N-0130 Oslo, Norway

Abstract We derive and discuss the physical interpretation of a conformally flat, static solution of the Einstein-Maxwell equations. It is argued that it describes a conformally flat, static spacetime outside a charged spherically symmetric domain wall. The acceleration of gravity is directed away from the wall in spite of the positive gravitational mass of the electric field outside the wall, as given by the Tolman-Whittaker expression. The reason for the repulsive gravitation is the strain of the wall which is calculated using the Israel formalism for singular surfaces.

1. Introduction

B. Bertotti [1] and I. Robinson [2] have found a conformally flat solution of the Einstein-Maxwell equations. The solution can be represented by the line element (12) below. N. Tariq and B.O. J. Tupper [3] have proved that this is the only conformally flat, static solution of the Einstein-Maxwell equations in which the energy-momentum tensor has no contribution other than that of a nonnull stationary electromagnetic field. A corresponding deduction for null electromagnetic fields has been given by R. G. McLenaghan, N. Tariq and B. O. J. Tupper [4], and has also been discussed by M. Cahen and J. Leroy [5].

The physical interpretation of the solution has been discussed by D. Lovelock [6,7] and P. Dolan [8]. Lovelock [6] argues that the solution describes spacetime outside a static, massless, charged particle. He also notes that the solution (12) is different from the Reissner-Nordström solution with $m = 0$, although both cause repulsive gravitation. The reason for the repulsive gravitation and the difference between the solution (12) and the Reissner-Nordström solution were, however, not discussed. Ø. Grøn [9] has shown that the reason for the Reissner-Nordström repulsion is Poincaré stresses which are always present in a static, charged object.

It should be noted, however, that Lovelock’s assumption of a massless, charged particle is not admitted. In order to avoid a naked singularity the particle must have a charge satisfying $R_Q < 2R_s$, where $R_Q$ is the length defined in equation (11) and $R_s$ is the Schwarzschild radius of the particle.

Dolan [7] has argued that the Bertotti-Robinson-Lovelock solution does not represent spacetime outside a charged particle. He has imbedded the curved 4-dimensional spacetime in a 6-dimensional pseudo-Euclidean space, and claims that this construction shows that the 3-space of the 4-dimensional spacetime is not spherically symmetric. In
particular, he points out that there is no physical singularity at \( r = 0 \), which it should be if there was a point charge at \( r = 0 \).

In the present paper we shall construct a solution of the Einstein-Maxwell equations where the problem at \( r = 0 \) is avoided, so that spherical symmetry of the 3-space can be maintained. This is obtained by excluding \( r = 0 \) from the conformally flat solution. We introduce a charged, spherical shell with center at \( r = 0 \) with flat spacetime inside it. The mechanical properties of this shell is determined using Israel’s relativistic theory [10] for singular layers. It turns out that this construction will also explain the repulsive gravity of the conformally flat solution, and the difference between this solution and the Reissner-Nordström solution. We will also show that an extension of the Tolman-Whittaker expression [11] of gravitational mass is needed in order to obtain a description which is generally consistent with the acceleration of gravity obtained from the geodesic equation.

2. A conformally flat static spacetime with an electric field

We shall consider a spherically symmetric, conformally flat and static spacetime with a radial electric field. Using conformally flat spacetime (CFS) coordinates \((T, R, \theta, \phi)\) [12 - 14] the line element can be written

\[
ds^2 = e^{2a}(-dT^2 + dR^2 + R^2 d\Omega^2),
\]

where \( a \) is a function of \( R \) alone and \( d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2 \). The field equations take the form

\[
G^T_T = e^{-2a}(2a'' + a'^2 + \frac{4}{R} a') = -e^{-4a} \frac{GQ^2}{4\pi\epsilon_0 R^4},
\]

\[
G^R_R = e^{-2a}(3a'^2 + \frac{4}{R} a') = -e^{-4a} \frac{GQ^2}{4\pi\epsilon_0 R^4},
\]

\[
G^\theta_\theta = G^\phi_\phi = e^{-2a}(2a'' + a'^2 + \frac{2}{R} a') = e^{-4a} \frac{GQ^2}{4\pi\epsilon_0 R^4},
\]

where \( G^\mu_\nu \) are the components of the Einstein tensor, \( \epsilon_0 \) is the permittivity in vacuum, and \( Q = Q(R) \) is the charge inside a spherical surface with radius \( R \) [15]. Subtracting equation (4) from equation (2) we obtain

\[
e^{2a} a' = -\frac{GQ^2}{4\pi\epsilon_0 R^3}. \tag{5}
\]

Inserting this into equation (3) leads to two different solutions, either

\[
Q(R) = 0 \tag{6}
\]

or

\[
Q(R) = \pm \sqrt{\frac{4\pi\epsilon_0}{G} Re^a}. \tag{7}
\]

From equations (5) and (6) we find that \( a = a_0 \) where \( a_0 \) is a constant. We may adjust the rate of the coordinate clocks and the length of the measuring rods so that
$a_0 = 0$. This means that the solution $Q(R) = 0$ represents the Minkowski spacetime.

Combining equations (5) and (7) leads to

$$a' = -\frac{1}{R}.$$  \hfill (8)

Integration gives

$$e^a = \frac{R_Q}{R},$$

where $R_Q$ is a positive constant of integration. From equation (7) we now get

$$Q(R) = \pm \sqrt{\frac{4\pi \epsilon_0}{G}} R_Q.$$  \hfill (9)

Hence the integration constant

$$R_Q = \sqrt{\frac{G}{(4\pi \epsilon_0 c^4)}} Q$$  \hfill (10)

represents the length corresponding to the charge $Q$ (where we have included the velocity of light $c$ in this conversion formula between charge and length). Equation (10) implies that the charge density vanishes in the considered region. Thus the assumption of a conformally flat, static and spherically symmetric spacetime implies that there is no charge in this region, although there is a radial electric field there. Hence there must be a charge inside the conformally flat region.

The line element describing spacetime outside the charge distribution takes the form

$$ds^2 = \frac{R_Q^2}{R^2} (-dT^2 + dR^2 + R^2 d\Omega^2) = \frac{R_Q^2}{R^2} (-dT^2 + dR^2) + R_Q^2 d\Omega^2,$$  \hfill (11)

Since the metric is static, the coordinate clocks go at the same rate everywhere, equal to the rate of the standard clocks at $R = R_Q$. We see that the rate of time as shown by standard clocks at rest is slower for increasing $R$. This indicates that the field of gravity points in the direction of increasing $R$. The Kretschmann curvature scalar is constant in this spacetime,

$$R^{\mu \nu \alpha \beta} R_{\mu \nu \alpha \beta} = \frac{8}{R_Q^4}.$$  \hfill (12)

Hence there is no physical singularity in the spacetime represented by the line element (11).

We shall assume that the charge $Q$ is distributed uniformly on a singular spherical shell with a spacetime described by the line element (11) outside the shell and Minkowski spacetime inside it. Furthermore, we choose as a coordinate condition that the radial coordinate is continuous at the shell. From the line element (11) it then follows that the CFS radius of the shell is $R = R_Q$.

The geometry of the spacetime outside the charged shell is rather strange. From the line element (11) we see that the area of a spherical surface is independent of the radius. One may wonder if this means that there is no usual kind of spherical symmetry as suggested by Doland [7]. We shall here keep to the interpretation that the line element (11) represents a spherically symmetric space with a rather unusual geometry.

The electric field strength is $E^R = F^{TR}$, where $F^{TR}$ is a contravariant component of
the electromagnetic field tensor. In an orthonormal basis the electric field outside the spherical charge distribution \((10)\) decreases as \(R^{-2}\),

\[
F^{\hat{\tau}\hat{r}} = \frac{Q}{4\pi\varepsilon_0 R^2},
\]

(14)

The coordinate component of the electric field strength in the CFS coordinate system is

\[
E^R = \frac{F^{\hat{\tau}\hat{r}}}{\sqrt{|g_{TT} g_{RR}|}} = e^{-2a} F^{\hat{\tau}\hat{r}} = \frac{R^2}{R_Q^2} F^{\hat{\tau}\hat{r}}.
\]

(15)

Using equations (14) and (15) we obtain

\[
E^R = \frac{Q}{4\pi\varepsilon_0 R_Q^2}.
\]

(16)

We then have the surprising result that the electric field strength outside the charge distribution does not decrease with the distance from the charge in this spherically symmetric space. The reason is that the flux of the electric field lines through a spherical surface about the charge is generally independent of the radius \(R\) of the surface. Since the area of the surface is independent of \(R\), it follows that the electric field strength does not decrease with increasing \(R\). This means that the energy momentum tensor of the electric field is constant, which is the reason why the right hand sides of the field equations \((2) - (4)\) are constant.

We shall now introduce a new radial coordinate \(\hat{r}\) equal to the physical radial distance and rename the time coordinate so that \(t = T\). From the line element (12) it then follows that

\[
d\hat{r} = \frac{R_Q}{R} dR.
\]

(17)

We assume that there is Minkowski spacetime inside the shell and choose as a coordinate condition that the radial coordinate is continuous at the shell. This means that \(\hat{r} = R_Q\) corresponds to \(R = R_Q\). Hence the physical radius of the shell is equal to \(R_Q\). Integration of (17) with this condition gives

\[
\hat{r} = R_Q \left(1 + \ln \frac{R}{R_Q}\right).
\]

(18)

The inverse transformation is

\[
R = R_Q e^{(\hat{r} - R_Q)/R_Q}.
\]

(19)

In terms of the physical radial coordinate the line element takes the form

\[
ds^2 = -e^{-2(\hat{r} - R_Q)/R_Q} dt^2 + d\hat{r}^2 + R_Q^2 d\Omega^2.
\]

(20)

We see that the coordinate clocks go at same rate as a standard clock at the shell where \(\hat{r} = R_Q\). The line element (20) reduces to the Minkowski line element at the shell, showing that the metric is continuous at the shell.
3. A general expression for the acceleration of gravity

We shall calculate an expression for the acceleration of gravity valid in a static spherically symmetric space, modifying the formula given in [11].

Consider a free particle instantaneously at rest in a static gravitational field. It follows a timelike geodesic curve in spacetime, with equation

$$\ddot{x}^\mu + \Gamma_{\alpha\beta}^\mu \dot{x}^\alpha \dot{x}^\beta = 0 \,,$$  \hspace{1cm} (21)

which in the present case reduces to

$$\ddot{R} = -\Gamma_{TT}^R \dot{T}^2 \,,$$  \hspace{1cm} (22)

where a dot indicates differentiation with respect to the proper time of the particle. With the line element (1) we have

$$\Gamma_{TT}^R = a' \,.$$  \hspace{1cm} (23)

The four velocity identity here leads to

$$\dot{T} = |g_{TT}|^{-1/2} = e^{-a} \,,$$  \hspace{1cm} (24)

giving

$$\ddot{R} = -e^{-2a} a' \,.$$  \hspace{1cm} (25)

At the horizon of a black hole we have that $g_{TT} = 0$, making $\ddot{R}$ diverge at the horizon. Hence the acceleration of gravity at the surface of a black hole, the surface gravity, is defined as

$$g = |g_{TT}|^{1/2} \dot{R} \,,$$  \hspace{1cm} (26)

which is finite at the surface of a black hole. This gives

$$g = -e^{-a} a' \,.$$  \hspace{1cm} (27)

Note that $g < 0$ means attractive gravity.

For the metric (1) we get for the following combination of the mixed components of the Einstein tensor

$$(G^T_T - G^R_R - G^\theta_\theta - G^\phi_\phi) \dot{R}^2 e^{4a} = -(2 R^2 e^{2a} a')' = (2 R^2 e^{3a} a)'' \,.$$  \hspace{1cm} (28)

Integration from $R_Q$ to $R$ gives

$$2 R^2 e^{3a(R)} g(R) = -2 R_Q^2 e^{2a(R_0)} a'(R_0) + \int_{R_Q}^R (G^T_T - G^R_R - G^\theta_\theta - G^\phi_\phi) \dot{R}^2 e^{4a(R)} d\dot{R} \,.$$  \hspace{1cm} (29)

Using the field equations, we get

$$R^2 e^{3a(R)} g(R) = -R_Q^2 e^{2a(R_0)} a'(R_0) + \frac{\kappa}{2} \int_{R_Q}^R (T^T_T - T^R_R - T^\theta_\theta - T^\phi_\phi) \dot{R}^2 e^{4a(R)} d\dot{R} \,.$$  \hspace{1cm} (30)

This is essentially a modified expression for the Tolman-Whittaker formula of active gravitational mass. If the integral is negative, it contributes with attractive gravity.
4. Repulsive gravitation outside a charged domain wall

For the solution given in section 2 the formula (30) reduces to

\[
\frac{R_Q}{R} g = \frac{1}{R_Q} - \kappa \int_{R_Q}^{R} Q(\tilde{R})^2 \frac{1}{R^2} d\tilde{R} = \frac{1}{R_Q} - \int_{R_Q}^{R} \frac{1}{R^2} d\tilde{R} ,
\]

(31)

giving

\[
g = \frac{1}{R_Q} > 0 .
\]

(32)

This means that a free particle accelerates in the positive \(R\)-direction. Hence there is repulsive gravitation in spite of the fact that the contribution from the Tolman-Whittaker expression is attractive. Note the rather surprising result that the acceleration of gravity is constant and does not diminish with increasing distance from the singular shell. This is similar to the behaviour of the electric field strength which was noted after equation (16).

In order to investigate the reason for the repulsive gravitation the Israel formalism for singular shells [10] will be used to find the mechanical properties of the shell.

The energy momentum tensor of the shell is given by

\[
\kappa S^i_j = [K^i_j] - \delta^i_j [K] ,
\]

(33)

where \(K^i_j\) are the components of the extrinsic curvature tensor of the shell, \(K = K^i_i\) and \([T] = T_+ - T_-\), where \(T_+\) and \(T_-\) are the values of \(T\) outside and inside the surface respectively. We consider the line element

\[
ds^2 = -e^\alpha dt^2 + e^\beta dr^2 + e^\gamma (d\theta^2 + \sin^2(\theta)d\phi^2) ,
\]

(34)

where \(\alpha, \beta\) and \(\gamma\) are functions of \(r\). The unit normal vector to a spherical surface about the origin is given by

\[
n = e^{-\frac{\beta}{2}} e_r .
\]

(35)

The covariant components of the extrinsic curvature tensor are given by

\[
K^i_{ij} = -n_i \Gamma^i_{ij} ,
\]

(36)

where Latin indices run through the surface coordinates \(t, \theta\) and \(\phi\), and Greek indices run through the four spacetime coordinates. The first term \(n_i \Gamma^i_{ij}\) vanishes on the surface, giving

\[
K^i_{ij} = -\frac{1}{2} n^r g_{ij,r} = -\frac{1}{2} e^{-\frac{\beta}{2}} g_{ij,r} .
\]

(37)

This gives

\[
K^t_t = -\frac{1}{2} e^{-\frac{\beta}{2}} \alpha_r,
\]

(38)

and

\[
K^\theta_\theta = K^\phi_\phi = -\frac{1}{2} e^{-\frac{\beta}{2}} \gamma_r .
\]

(39)

Equations (33), (38) and (39) give

\[
\kappa S^t_t = -2[K^\theta_\theta] = [e^{-\frac{\beta}{2}} \gamma_r] .
\]

(40)
and
\[ \kappa S^\theta_\theta = \kappa S^\phi_\phi = -[K^t_t] - [K^\theta_\theta] = \frac{1}{2}e^{-\frac{2}{3} (\alpha_r + \gamma_r)}. \tag{41} \]

We shall now apply these formulae to the spherical shell defined above. Then \( \alpha_+ = \beta_+ = 2a \) and \( \gamma_+ = 2(a + \ln R) \) outside the shell, where \( a \) is given by equation (9), and \( \alpha_- = \beta_- = 0 \) and \( \gamma_- = R^2 \) inside it. This gives
\[ \kappa S^T_T = \kappa S^\theta_\theta = \kappa S^\phi_\phi = -\frac{2}{R_Q}. \tag{42} \]

Surprisingly, the mass density and strain of the shell are independent of its radius. We see that the components of the energy momentum tensor of the singular shell may be written
\[ S^i_j = -\sigma \delta^i_j, \tag{43} \]
which characterizes a domain wall [16]. Hence the energy momentum tensor in equation (42) describes a charged, spherical domain wall with mass density
\[ \sigma = \frac{2}{\kappa R_Q}, \tag{44} \]
showing that the shell has a positive mass \( M = 4\pi \sigma R^2_Q = R_Q/G \), i.e. the mass of the shell is proportional to its charge. This means that it is not possible to change the charge of the shell without simultaneously changing its mass. In particular, it is not possible to neutralize it, which is the reason why the line element of the spacetime outside the shell does not contain independent parameters for the mass and the charge, as in the Reissner-Nordström spacetime. In fact the relationship \( R_Q = G M \) corresponds to a maximally charged Reissner-Nordström black hole. Equation (43) shows that
\[ S^\theta_\theta = S^\phi_\phi = -\sigma. \tag{45} \]
This means that there is a strain in the shell equal to minus its mass density. We can now understand the reason for the repulsive gravitation which was noted after equation (32). As applied to the domain wall the Tolman-Whittaker formula gives
\[ g(R_Q) = \frac{\kappa}{2} (S^T_T - S^\theta_\theta - S^\phi_\phi) = \frac{1}{R_Q}, \tag{46} \]
in accordance with equation (32), showing that the domain wall repels gravitationally the surrounding matter [17]. The energy of the electric field outside the charged domain wall has positive gravitational mass which causes attractive gravitation. But the strain of the domain wall causes repulsive gravitation which dominates over the attractive gravitation of the electric field at all distances from the domain wall.

5. Comparison with the Reissner-Nordström metric

The Reissner-Nordström spacetime is usually represented by the line element
\[ ds^2 = -\left(1 - \frac{R_\Sigma}{r} + \frac{R^2_Q}{r^2}\right)dt^2 + \left(1 - \frac{R_\Sigma}{r} + \frac{R^2_Q}{r^2}\right)^{-1}dr^2 + r^2d\Omega^2, \tag{47} \]
where \( R_s = 2GM \) is the Schwarzschild radius of the mass \( M \) in the Reissner-Nordström metric. A maximally charged Reissner-Nordström spacetime has \( R_s = 2R_Q \), giving

\[
ds^2 = -\left(1 - \frac{R_Q}{r}\right)^2 dt^2 + \left(1 - \frac{R_Q}{r}\right)^{-2} dr^2 + r^2 d\Omega^2.
\]  

(48)

In order to compare the solution (12) with the maximally charged Reissner-Nordström spacetime, we shall express the line element (12) in coordinates \((r, t)\) where \( g_{tt} g_{rr} = -1 \). From the line element (12) we then have

\[
dr = \frac{R_Q^2}{R^2} dR.
\]

(49)

Integration with the condition that \( r = R_Q \) corresponds to \( R = R_Q \) gives

\[
r = R_Q \left(2 - \frac{R_Q}{R}\right),
\]

(50)

mapping the region \( R \geq R_Q \) onto the bounded region \( R_Q \leq r < 2R_Q \). The inverse transformation is

\[
R = R_Q \left(2 - \frac{r}{R_Q}\right)^{-1}.
\]

(51)

In terms of the radial coordinate \( r \) the line element (12) takes the form

\[
ds^2 = -\left(2 - \frac{r}{R_Q}\right)^2 dt^2 + \left(2 - \frac{r}{R_Q}\right)^{-2} dr^2 + R_Q^2 d\Omega^2.
\]

(52)

There is a certain similarity between the line elements (52) and (48), but it is actually more remarkable that they represent different spacetimes that cannot be related to each other by a coordinate transformation. This was already noted by Lovelock [7] in connection with the Jebsen-Birkhoff theorem [18,19] and the line element (12).

One may wonder why the spacetime outside the charged spherical domain wall is different from the Reissner-Nordström spacetime. After all, both are spaces outside a charged spherical body. The difference is in the mechanical properties of the bodies containing the mass and charge appearing in the metrics.

Applying the Israel formalism to a singular spherical shell having radius \( r = r_0 \), with Reissner-Nordström spacetime outside and Minkowski spacetime inside, we find that the components of the energy momentum tensor of the shell are

\[
S^t_t = -\frac{2}{\kappa r_0} \left[1 - \left(1 - \frac{R_s}{r_0} + \frac{R_Q^2}{r_0^2}\right)^{1/2}\right]
\]

and

\[
S^\theta_\theta = S^\phi_\phi = -\frac{1}{\kappa r_0} \left[1 - \left(1 - \frac{R_s}{2r_0}\right)\left(1 - \frac{R_s}{r_0} + \frac{R_Q^2}{r_0^2}\right)^{-1/2}\right].
\]

Hence the shell has a positive mass density and is strained. In the Newtonian limit

\[
S^t_t = -\frac{M}{4\pi r_0^2}.
\]

(53)

(54)

(55)
Since $S^t_t$ is interpreted as minus the mass density $\sigma$, we get

$$M = 4\pi r_0^2 \sigma,$$

(56)

showing that the mass of the Reissner-Nordström metric comes from the spherical shell.

This shall now be compared with the solution above, where the singular shell is a spherical domain wall with a charge corresponding to a maximally charged source of the Reissner-Nordström spacetime $R_s = 2R_Q$. In the case where this source has the same radius $r_0 = R_Q$ as the domain wall, the components of its energy momentum reduce to

$$S^t_t = -\frac{2}{\kappa R_Q}, \quad S^\theta_\theta = S^\phi_\phi = 0.$$

(57)

Hence the spherical shell inside the Reissner-Nordström spacetime has the same mass density as the domain wall given in equation (44), but it has a vanishing strain. This difference in the mechanical properties of the shells is the reason for the different geometries of the spacetimes outside the shells.

6. Conclusion

About fifty years ago Bertotti and Robinson found a solution of the Einstein-Maxwell equations representing a static, conformally flat spacetime with a uniform electromagnetic field. There has been no general agreement as to the physical interpretation of the solution, which is most simply represented by the line element (12) above.

It was presented by Robinson as a spherically symmetric solution of the field equations, while Bertotti writes that the main qualitative physical feature of the solution is the anisotropy of space. Furthermore, while the Reissner-Nordström solution contains two physical parameters, one representing mass and the other charge, this solution contains only one parameter which represents the charge producing the electric field.

Lovelock [6] shows from the line element (12) that some of the components of the Riemann curvature tensor are proportional to $1/r^2$ and hence claims that spacetime has a physical singularity at $r = 0$. Because of this and the fact that the line element contains only one physical parameter he concludes [7] that the line element (12) corresponds to the spacetime outside a massless point charge at rest at $r = 0$. Lovelock also notes that there is repulsive gravity in this spacetime, i.e. a free neutral particle is accelerated away from the charge at $r = 0$. This is said to be an unphysical property of the solution, and Lovelock points out that the solution shares this property with the Reissner-Nordström solution with $m = 0$.

We have given a new physical interpretation of the solution which explains its seemingly unphysical properties. Firstly we have shown that the Kretschmann curvature scalar is constant, which means that there is indeed only a coordinate singularity at $r = 0$. Using Israel’s relativistic theory of singular shells we have constructed a physical system where this coordinate singularity does not appear. It consists of a charged domain wall with Minkowski spacetime inside the wall and Robinson-Bertotti-Lovelock spacetime outside it. The charge of the wall corresponds to that of a maximally charged Reissner-Nordström spacetime, which explains that there is only one physical parameter in the solution. The Tolman-Whittaker expression for the gravitational mass density of the wall is negative,
which means that the repulsive character of the conformally flat spacetime is due to the
strain of the domain wall.

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