Sequential hyperon decays in the reaction $e^+e^- \rightarrow \Sigma^0\bar{\Sigma}^0$

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Abstract

We report on a study of the sequential hyperon decay $\Sigma^0 \rightarrow \Lambda \gamma; \Lambda \rightarrow p\pi^-$ and its corresponding anti-hyperon decay. We derive a multi-dimensional and model-independent formalism for the case when the hyperons are produced in the reaction $e^+e^- \rightarrow \Sigma^0\bar{\Sigma}^0$. Cross-section distributions are calculated using the folding technique. We also study sequential decays of single-tagged hyperons.

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I. INTRODUCTION

The BESIII experiment \cite{1} has created new opportunities for research into hyperon physics, based on $e^+e^-$ annihilation into hyperon-anti-hyperon pairs. Such possibilities are interesting, and for several reasons:

- They offer the currently only feasible way for investigating the electromagnetic structure of hyperons \cite{2}.
- By measuring in the vicinity of vector-charmonium states, one gains information on the strong baryon-antibaryon decay processes of charmonia.
- They offer a model-independent method for measuring weak-decay-asymmetry parameters, which can probe CP symmetry \cite{3}.

The basic reaction, $e^+e^- \rightarrow Y\bar{Y}$, is graphed in Fig.1. In the continuum region, \textit{i.e.}, in energy regions that do not overlap with energies of vector charmonia like $J/\psi$, $\psi'$ and $\psi(2S)$, the production process is dominated by one-photon exchange, $e^+e^- \rightarrow \gamma^* \rightarrow Y\bar{Y}$. The reaction amplitude is then governed by the electromagnetic form factors $G_E$ and $G_M$. In the vicinity of vector resonances, the electromagnetic form factors are replaced by hadronic form factors $G^\psi_E$ and $G^\psi_M$. However, the shapes of the differential-cross-section distributions are the same in the two cases: all physics of the production mechanism is contained within the form factors, or equivalently, the ratio of form-factor magnitudes, $\alpha_\psi$, and the relative phase of form factors, $\Delta\Phi_\psi$.

Analyses of joint-decay distributions of hyperons, such as $\Lambda(\rightarrow p\pi^-)\bar{\Lambda}(\rightarrow \bar{p}\pi^+)$, enables us to determine some of the weak-interaction-decay parameters, $\alpha\beta\gamma$.

The theoretical description of the annihilation reaction of Fig.1 is described in Ref.\cite{4}, and the corresponding annihilation reaction mediated by $J/\psi$ in Ref.\cite{5}. Accurate experimental results for the form-factor parameters $\alpha_\psi$ and $\Delta\Phi_\psi$ and the weak-interaction parameters $\alpha_\Lambda(\alpha_{\bar{\Lambda}})$ for the latter annihilation process are all reported in Ref.\cite{3}. A precise knowledge of the asymmetry parameters $\alpha_\Lambda(\alpha_{\bar{\Lambda}})$ is needed for studies of spin polarization in $\Omega^-$, $\Xi^-$, and $\Lambda^+_c$ decays, and for tests of the standard model.

The graph of Fig.1 can be generalized in the sense that it can include hyperons that decay sequentially. It can also include cases where the produced hyperon is of a different type than the produced antihyperon, \textit{i.e.}, $e^+e^- \rightarrow Y_1\bar{Y}_2$. 
FIG. 1: Graph describing the electromagnetic annihilation reaction $e^+e^- \rightarrow \bar{\Lambda}\Lambda$. The same reaction can also proceed hadronically via vector charmonium states such as $J/\psi$, $\psi'$, or $\psi(2S)$, in place of the photon.

In this note we shall consider annihilation into $\Sigma^0\bar{\Sigma}^0$ pairs, Ref. [6]. The $\Sigma^0$ decays electromagnetically, $\Sigma^0 \rightarrow \Lambda\gamma$, and subsequently the Lambda hyperon decays weakly, $\Lambda \rightarrow p\pi^-$. The interest of this measurement is many-fold:

- The form factors provide information about the production process. So far, literature has focused on electromagnetic form factors whose interpretation is straightforward [2, 7]. However, recent experimental advances calls for an interpretation also of the hadronic form factors. In particular, it would be interesting to compare the decay of $J/\psi$ into various hyperon-antihyperon pairs with the corresponding decays of other vector charmonia.

- The BESIII collaboration plans to perform a first measurement of the branching fraction of the $\Sigma^0$ Dalitz decay $\Sigma^0 \rightarrow \Lambda\gamma^*, \gamma^* \rightarrow e^+e^-$ using the large data sample available for the $e^+e^- \rightarrow J/\psi \rightarrow \bar{\Sigma}^0\Sigma^0$ process. Then, the most important background will come from $e^+e^- \rightarrow J/\psi \rightarrow \bar{\Sigma}^0\Sigma^0; (\Sigma^0 \rightarrow \Lambda\gamma; \Lambda \rightarrow p\pi^- + c.c)$, where one of the photons undergoes external conversion into an $e^+e^-$ pair. This is because the branching ratio of the $\Sigma^0 \rightarrow \Lambda\gamma$, according to QED, is three orders of magnitude larger than that of the Dalitz decay. In order to properly account for the background, precise knowledge of the joint angular distribution is required.

- It can provide an independent measurement of the Lambda asymmetry parameters $\alpha_\Lambda$ and $\alpha_{\bar{\Lambda}}$.

- It can provide a first test of strong CP symmetry in the $\Sigma^0 \rightarrow \Lambda\gamma$ decay Ref. [8].
Our calculation is performed in steps. First, we review some important facts; the spin structure of the $e^+e^- \to \Sigma^0\bar{\Sigma}^0$ annihilation reaction Ref.[4]; the classical $\alpha\beta\gamma$ description of hyperon decays Ref.[9]; the description of the electromagnetic $\Sigma^0 \to \Lambda\gamma$ decay, both for real and virtual photons Refs.[6][10]. The virtual photons decay into Dalitz lepton pairs. An important element of our calculation is the factorization of the squared amplitudes into a spin-independent fractional decay rate and a spin-density distribution.

Following these reviews we demonstrate how the folding method of Ref.[11] is adapted to sequential decays. Both simple and double decay chains are treated. Finally, we join production and decay steps to give the cross-section distributions.

The information we are hoping to gain resides in the angular distributions, and we are therefore not overly concerned with absolute normalizations, although they may be obtained without too much effort.

II. BARYON FORM FACTORS

The diagram in Fig.1 describes the annihilation reaction $e^-(k_1)e^+(k_2) \to Y(p_1)\bar{Y}(p_2)$ and involves two vertex functions; one of them leptonic, the other one baryonic. The strength of the lepton-vertex function is determined by the electric charge $e$, but two form factors $G_M(s)$ and $G_E(s)$ are needed for describing the baryonic vertex function. Here, $s = (p_1+p_2)^2$ with $p_1$ and $p_2$ as defined in Fig.1.

The strength of the baryon form factors is measured by the function $D(s)$,

$$D(s) = s |G_M|^2 + 4M^2 |G_E|^2,$$

a factor that multiplies all cross-section distributions. The ratio of form factors is measured by $\eta(s)$,

$$\eta(s) = \frac{s|G_M|^2 - 4M^2 |G_E|^2}{s|G_M|^2 + 4M^2 |G_E|^2},$$

with $\eta(s)$ satisfying $-1 \leq \eta(s) \leq 1$. The relative phase of form factors is measured by $\Delta\Phi(s)$,

$$\frac{G_E}{G_M} = e^{i\Delta\Phi(s)} \left| \frac{G_E}{G_M} \right|.$$

In Ref.[3] annihilation in the region of the $J/\psi$ and $\psi(2S)$ masses is considered. The photon propagator of Fig.1 is then replaced by the appropriate vector-meson propagator.
III. CROSS SECTION FOR $e^-e^+ \rightarrow Y(s_1)\bar{Y}(s_2)$

Our first task is to review the calculation of the cross-section distribution for $e^+e^-$ annihilation into baryon-antibaryon pairs, with baryon-four-vector polarizations $s_1$ and $s_2$ [4, 5]. From the squared matrix element of this process, $|\mathcal{M}|^2$, we remove a factor $e^4/e^2s^2$, which is the square of the propagator, and get

$$d\sigma = \frac{1}{2s} \frac{e^4}{s^2} |\mathcal{M}_{\text{red}}(s_1, s_2)|^2 \text{dLips}(k_1 + k_2; p_1, p_2),$$  \hspace{1cm} (III.4)

with $s = (p_1 + p_2)^2$, and dLips the phase-space element of Ref.[12]. For a baryon of momentum $\mathbf{p}$ the four-vector spin $s$ is related to the three-vector spin $\mathbf{n}$, the spin in the rest system, by

$$s(\mathbf{p}, \mathbf{n}) = \frac{n_{||}}{M}(|\mathbf{n}|, E\hat{\mathbf{p}}) + (0, \mathbf{n}_{\perp}).$$  \hspace{1cm} (III.5)

Longitudinal and transverse directions of vectors are relative to the $\hat{\mathbf{p}}$ direction.

In the global c.m. system kinematics simplifies. There, three-momenta $\mathbf{p}$ and $\mathbf{k}$ are defined such that

$$\mathbf{p}_1 = -\mathbf{p}_2 = \mathbf{p},$$  \hspace{1cm} (III.6)
$$\mathbf{k}_1 = -\mathbf{k}_2 = \mathbf{k},$$  \hspace{1cm} (III.7)

and with scattering angle $\theta$ defined by,

$$\cos \theta = \hat{\mathbf{p}} \cdot \hat{\mathbf{k}}.$$  \hspace{1cm} (III.8)

Furthermore, in the global c.m. system the phase-space factor reads

$$\text{dLips}(k_1 + k_2; p_1, p_2) = \frac{p}{32\pi^2k^3} \text{d}\Omega,$$  \hspace{1cm} (III.9)

with $p = |\mathbf{p}|$ and $k = |\mathbf{k}|$.

The matrix element in Eq.(III.4) can be written as a sum of terms that depend on the baryon and antibaryon spin directions in their respective rest systems, $\mathbf{n}_1$ and $\mathbf{n}_2$,

$$|\mathcal{M}_{\text{red}}(e^+e^- \rightarrow Y(s_1)\bar{Y}(s_2))|^2 = sD(s)S(\mathbf{n}_1, \mathbf{n}_2),$$  \hspace{1cm} (III.10)

with the strength function $D(s)$ defined in Eq.(II.1). We call a function such as $S(\mathbf{n}_1, \mathbf{n}_2)$ a spin density. In the present case, the spin density is a sum of seven mutually orthogonal
contributions \cite{4},

\[
S(n_1, n_2) = R + SN \cdot n_1 + SN \cdot n_2 + T_1 n_1 \cdot \hat{p} n_2 \cdot \hat{p} \\
+ T_2 n_{1\perp} \cdot n_{2\perp} + T_3 n_{1\perp} \cdot \hat{k} n_{2\perp} \cdot \hat{k} \\
+ T_4 \left( n_1 \cdot \hat{p} n_{2\perp} \cdot \hat{k} + n_2 \cdot \hat{p} n_{1\perp} \cdot \hat{k} \right),
\] (III.11)

where \( N \) is the normal to the scattering plane,

\[
N = \frac{1}{\sin \theta} \hat{p} \times \hat{k}.
\] (III.12)

The seven structure functions \( R, S, \) and \( T \) of Eq.(III.11) depend on the scattering angle \( \theta \), the ratio function \( \eta(s) \), and the phase function \( \Delta \Phi(s) \). Their detailed expressions are given in Appendix B.

The cross-section distribution for polarized final-state hyperons becomes

\[
\frac{d\sigma}{d\Omega} = \frac{p}{k} \alpha_e^2 D(s) \frac{1}{4s^2} S(n_1, n_2),
\] (III.13)

where \( \alpha_e \) is the fine-structure constant.

If we sum over baryon and antibaryon final-state polarizations we get a well-known result,

\[
\frac{d\sigma}{d\Omega} = \frac{p}{k} \alpha_e^2 D(s) \frac{1}{s^2} R.
\] (III.14)

Summing only over the antibaryon polarizations gives

\[
\frac{d\sigma}{d\Omega} = \frac{p}{k} \alpha_e^2 D(s) \frac{1}{2s^2} (R + SN \cdot n_1).
\] (III.15)

This result tells us that the baryon is polarized and that its polarization is directed along the normal to the scattering plane, \( \hat{p} \times \hat{k} \), and that the value of the polarization is

\[
P_Y(\theta) = \frac{S}{R} = \frac{\sqrt{1 - \eta^2 \cos \theta \sin \theta}}{1 + \eta \cos^2 \theta} \sin(\Delta \Phi)
\] (III.16)

From Eq.(III.11) we conclude that there is a corresponding result for the antibaryon, but it should then be remembered that \( p \) is the momentum of the baryon but \( -p \) that of the antibaryon.

Baryon and antibaryon polarizations in \( e^+e^- \) annihilation were first discussed by Dubničkova et al.\cite{13}, but with results slightly different from ours. For details see Ref.\cite{4}.
IV. WEAK BARYON DECAYS

Weak decays of spin one-half baryons, such as $\Lambda \rightarrow p\pi^-$, involve two amplitudes, one S-wave and one P-wave amplitude, and the decay distribution is commonly parametrized by three parameters, denoted $\alpha\beta\gamma$, and which fulfill a relation

$$\alpha^2 + \beta^2 + \gamma^2 = 1.$$  \hspace{1cm} (IV.17)

Details of this description can be found in Refs.\[14\] or \[4, 9\].

Since we shall encounter several weak baryon decays of the same structure as the $\Lambda \rightarrow p\pi^-$ decay, we shall use a generic notation, $c \rightarrow d\pi$, for those decays.

The matrix element describing the decay of a polarized $c$ baryon into a polarized $d$ baryon is

$$\mathcal{M}(c \rightarrow d\pi) = \bar{u}(p_d, s_d)(A + B\gamma_5)u(p_c, s_c),$$  \hspace{1cm} (IV.18)

with $p$ and $s$ with appropriate indices denoting momenta and spin four-vectors of the baryons. The square of this matrix element we factorize, writing

$$\left|\mathcal{M}(c \rightarrow d\pi)\right|^2 = \text{Tr}\left[\frac{1}{2}(1 + \gamma_5\xi_d)(\hat{p}_d + m_d)(A + B\gamma_5)
\times (\hat{p}_c + m_c)\frac{1}{2}(1 + \gamma_5\xi_c)(A^* - B^*\gamma_5)\right].$$

$$= R(c \rightarrow d\pi) G(n_c, n_d)$$  \hspace{1cm} (IV.19)

where $n_c$ and $n_d$ are the spin vectors of baryons $c$ and $d$ in their rest frames, Eq.(III.5). The $R$-factor is a spin independent factor, defined by

$$R(c \rightarrow d\pi) = 2m_c\Gamma(c \rightarrow d\pi)/\Phi(c \rightarrow d\pi),$$

$$= |A|^2 \left((m_c + m_d)^2 - m_\pi^2\right)$$

$$+ |B|^2 \left((m_c - m_d)^2 - m_\pi^2\right),$$  \hspace{1cm} (IV.20)

where $\Phi (c \rightarrow d\pi) = \Phi(m_c; m_d, m_\pi)$ is the phase-space volume of $C$. We refer to $R(c \rightarrow d\pi)$ as the fractional decay rate, since it is a decay rate per unit phase space. Further inspection of Eq.(IV.19) tells us, that $\Gamma(c \rightarrow d\pi)$ is defined as an average over the spins of both initial and final baryons.

The spin-density-distribution function, $G(n_c, n_d)$ of Eq.(IV.19), is a Lorentz scalar, which we choose to evaluate in the rest system of the mother baryon, $c$,

$$G(c,d) = 1 + \alpha_c n_c \cdot l_d + \alpha_c n_d \cdot l_c + n_c \cdot L_c (n_d, l_d),$$  \hspace{1cm} (IV.21)
with

\[ \mathbf{L}_c(\mathbf{n}_d, \mathbf{l}_d) = \gamma_c \mathbf{n}_d + \left[ (1 - \gamma_c) \mathbf{n}_d \cdot \mathbf{l}_d \right] \mathbf{l}_d + \beta_c \mathbf{n}_d \times \mathbf{l}_d. \]  

(IV.22)

The vector \( \mathbf{l}_d \) is a unit vector in the direction of motion of the daughter baryon, \( d \), in the rest system of mother baryon \( c \). The indices on the \( \alpha \beta \gamma \) parameters remind us they characterize baryon \( c \). A spin density is normalized if the spin-independent term is unity.

We observe an important symmetry,

\[ \mathbf{n}_c \cdot \mathbf{L}_c(\mathbf{n}_d, \mathbf{l}_d) = \mathbf{n}_d \cdot \mathbf{L}_c(\mathbf{n}_c, -\mathbf{l}_d). \]  

(IV.23)

Since the spin of baryon \( d \) is usually not measured, the interesting spin-density is obtained by taking the average over the spin directions \( \mathbf{n}_d \),

\[ W_c(\mathbf{n}_c, \mathbf{l}_d) = \left\langle G_c(c, d) \right\rangle_{\mathbf{n}_d} = U_c + \mathbf{n}_c \cdot \mathbf{V}_c, \]  

(IV.24)

with

\[ U_c = 1, \quad \mathbf{V}_c = \alpha_c \mathbf{l}_d. \]  

(IV.25)

For an initial state polarization \( \mathbf{P}_c \) we put \( \mathbf{n}_c = \mathbf{P}_c \), and get an angular distribution known from the weak hyperon decay \( \Lambda \to p\pi^- \) \cite{4,9}.

The matrix element describing the decay of a polarized \( \bar{c} \) (anti)baryon into a polarized \( \bar{d} \) (anti)baryon is similar to that of Eq.(IV.18),

\[ \mathcal{M}(\bar{c} \to \bar{d}\pi) = \bar{v}(p_{\bar{c}}, s_{\bar{c}})(A' + B'\gamma_5)\nu(p_{\bar{d}}, s_{\bar{d}}). \]  

(IV.26)

The relation between the parameters \( A, B \) and \( A', B' \) is clarified in Refs.\cite{17,18}.

The square of the anti-baryon matrix element of Eq.(IV.26) is factorized exactly as the baryon-matrix element of Eq.(IV.19),

\[ |\mathcal{M}(\bar{c} \to \bar{d}\pi)|^2 = R(\bar{c} \to \bar{d}\pi) G(\mathbf{n}_\bar{c}, \mathbf{n}_\bar{d}), \]  

(IV.27)

where \( \mathbf{n}_\bar{c} \) and \( \mathbf{n}_\bar{d} \) are the spin vectors of baryons \( \bar{c} \) and \( \bar{d} \) in their rest systems.

The functions \( R(\bar{c} \to \bar{d}\pi) \) and \( G(\mathbf{n}_\bar{c}, \mathbf{n}_\bar{d}) \) are tied to hyperons \( \bar{c} \) and \( \bar{d} \) in exactly the same way as those tied to hyperons \( c \) and \( d \), Eqs.(IV.20) and (IV.21), or to be specific,

\[ G(\bar{c}, \bar{d}) = 1 + \alpha_c \mathbf{n}_\bar{c} \cdot \mathbf{l}_\bar{d} + \alpha_c \mathbf{n}_\bar{d} \cdot \mathbf{l}_\bar{c} + \mathbf{n}_\bar{c} \cdot \mathbf{L}_c(\mathbf{n}_\bar{d}, \mathbf{l}_d). \]  

(IV.28)
For CP conserving interactions the asymmetry parameters of the hyperon pair \( c, d \) are related to those of anti-hyperon pair \( \bar{c}, \bar{d} \) by

\[
\alpha_c = -\alpha_{\bar{c}}, \quad \beta_c = -\beta_{\bar{c}}, \quad \gamma_c = \gamma_{\bar{c}}.
\] (IV.29)

V. ELECTROMAGNETIC HYPERON DECAYS: REAL PHOTONS

Electromagnetic transitions such as \( \Sigma^0 \to \Lambda \gamma \) and \( \Xi^0 \to \Sigma^0 \gamma \) are readily investigated in \( e^+e^- \) annihilation. The electromagnetic \( \Sigma^0 \to \Lambda \) transition is caused by the four-current \([12]\)

\[
J_\mu(c \to d) = \frac{1}{m_c + m_d} \left[ F_1 \left\{ \frac{k^2}{m_d - m_c} \gamma_\mu + k_\mu \right\} 
+ F_2 i \sigma_\mu \nu k^\nu \right],
\] (V.30)

with \( k = p_c - p_d \). This transition current is gauge invariant, meaning \( k \cdot J = 0 \). The \( F_1(k^2) \) and \( F_2(k^2) \) contributions are each, by themselves, gauge invariant. We shall ignore the \( F_1 \) term, which vanishes for real photons, \( k^2 = 0 \), and stay with the \( F_2 \) term. We denote by \( \mu_{cd} \),

\[
\mu_{cd} = e F_2 / (m_c + m_d),
\] (V.31)

the strength of the magnetic-moment transition. As a consequence, the expression for the matrix element for any electromagnetic \( \Sigma^0 \to \Lambda \gamma \) like decay, becomes

\[
\mathcal{M}_\gamma(c \to d\gamma) = \mu_{cd} \bar{u}_d(p_d, s_d) \left( \sigma^{\mu\nu} e_\mu^* (-i k_\nu) \right) u_c(p_c, s_c)
= \mu_{cd} \bar{u}_d(p_d, s_d) \left( \phi^* k \right) u_c(p_c, s_c),
\] (V.32)

where \( s_c \) and \( s_d \) are the spin four-vectors of the two baryons.

It is convenient to write the square of this matrix element on the form

\[
|\mathcal{M}_\gamma(c \to d\gamma)|^2 = \mu^2_{cd} \text{Tr} \left[ \frac{1}{2} (1 + \gamma_5 \phi_d)(\phi_d + m_d) \phi^* k \right.
\times (\phi_c + m_c) \frac{1}{2} (1 + \gamma_5 \phi_c) \phi^* k \left. \right]
= H_{\gamma}^{\mu\nu}(k) e_\mu(k) e_\nu^*(k),
\] (V.33)

with \( H_{\gamma}^{\mu\nu}(k) \) referred to as the hadron tensor. We have also made use of the simplifying identity

\[
e^\mu i \sigma_{\mu\nu} k^\nu = -\phi k,
\] (V.34)
valid for real photons.

Summation over the two photon-spin directions entails replacing $\epsilon_\mu(k)\epsilon_\nu^*(k)$ by $-g_{\mu\nu}$. This leads to

$$\sum_{\epsilon_\gamma} |M_\gamma(c \rightarrow d\gamma)|^2 = R(c \rightarrow d\gamma) G_\gamma(n_c, n_d), \quad (V.35)$$

and again $n_c$ and $n_d$ are the spin vectors of baryons $c$ and $d$ in their rest systems. Photon polarizations are summed over. There are also electromagnetic transitions between charged baryons, but in this section we limit ourselves to electromagnetic transitions between neutral baryons.

The factorization of Eq.(V.35) is chosen so that the fractional decay rate $R(c \rightarrow d\gamma)$ is the unpolarized part of Eq.(V.35) and its $G_\gamma(n_c, n_d)$ factor the normalized spin-density-distribution function. Here, unpolarized means averaged over the spin directions of both initial and final baryons.

The fractional decay rate, $R(c \rightarrow d\gamma)$ of Eq.(V.36), has the same structure as the corresponding one for weak baryon decays, Eq.(IV.20),

$$R(c \rightarrow d\gamma) = \frac{2m_c\Gamma(c \rightarrow d\gamma)}{\Phi(c \rightarrow d\gamma)},$$

$$= \frac{\mu_{cd}^2(m_c^2 - m_d^2)^2}{m_c^4}, \quad (V.36)$$

where $\Phi(c \rightarrow d\gamma) = \Phi(m_c; m_d, m_\gamma)$ is the phase-space volume.

The electromagnetic decay width is

$$\Gamma(c \rightarrow d\gamma) = \frac{1}{2\pi}\mu_{cd}^2\omega^3, \quad (V.37)$$

where $\omega$ is the photon energy. Remember, that this width is obtained after averaging over both initial and final baryon spin states.

The spin-density-distribution function of Eq.(V.35) involves an implicit summation over photon polarizations. For such a case

$$G_\gamma(n_c, n_d) = 1 - n_c \cdot l_\gamma l_\gamma \cdot n_d, \quad (V.38)$$

where $l_\gamma$ is a unit vector in the direction of motion of the photon, and $l_d = -l_\gamma$ a unit vector in the direction of motion of baryon $d$, both in the rest system of baryon $c$.

We notice that when both hadron spins are parallel or anti-parallel to the photon momentum, then the decay probability vanishes, a property of angular-momentum conservation. We also notice that expression (V.38) cannot be written in the $\alpha\beta\gamma$ representation of Eqs.(IV.21) and (IV.22).
When the spin of the final-state baryon \( d \) is not measured, the relevant spin-density is obtained by forming the average over the spin directions \( n_d \),

\[
W_\gamma(n_c; l_d) = \left\langle G_\gamma(c, d) \right\rangle_{n_d} = U_c + n_c \cdot V_c,
\]

with

\[
U_c = 1, \quad V_c = 0.
\]

Thus, the decay-distribution function is independent of the initial-state baryon spin vector \( n_c \).

The anti-particle matrix element corresponding to the particle matrix element of Eq.(V.32), is simply

\[
\mathcal{M}_\gamma(\bar{c} \rightarrow \bar{d} \gamma) = \mu_{cd} \bar{v}(p_{\bar{c}}, s_{\bar{c}})(\mathbf{\phi}^* \mathbf{\kappa}) v_{\bar{d}}(p_{\bar{d}}, s_{\bar{d}}).
\]

We assume the parameter \( \mu \) is the same for particle transitions \( c \rightarrow d \) as for anti-particle transitions \( \bar{c} \rightarrow \bar{d} \).

The normalized spin density corresponding to the antiparticle matrix element of Eq.(V.41) is the same as that corresponding to the particle matrix element of Eq.(V.32), as given in Eq.(V.38), provided we replace the particle spin vectors \( n_c \) and \( n_d \) by the anti-particle spin vectors \( n_{\bar{c}} \) and \( n_{\bar{d}} \).

The possibility to search for P-violating admixtures in the electromagnetic decay \( \Sigma^0 \rightarrow \Lambda \gamma \) was suggested in Ref.[8]. Such contributions are created by making the substitution

\[
\mathbf{\phi}^* \mathbf{\kappa} \rightarrow (1 - b_{\gamma_5})\mathbf{\phi}^* \mathbf{\kappa},
\]

in the decay amplitude. Moreover, if one can measure hyperon and anti-hyperon sequential decays simultaneously tests for CP violation become possible.

The substitution (V.42) changes the normalized spin density (V.38) into

\[
G_\gamma(n_c, n_d) = 1 - n_c \cdot \mathbf{l}_\gamma n_d + \rho_c [n_c \cdot \mathbf{l}_\gamma - n_d \cdot \mathbf{l}_\gamma],
\]

with asymmetry parameter

\[
\rho_c = \frac{2\mathfrak{R}(b)}{1 + |b|^2},
\]
Similarly, the decay width of Eq. (V.37) is changed into
\[
\Gamma(c \rightarrow d\gamma) = \frac{1}{2\pi} (1 + |b|^2)\mu_{cd}^2\omega^3. \tag{V.45}
\]
Parity violating admixtures in the anti-particle decay \(\bar{\Sigma}^0 \rightarrow \bar{\Lambda}\gamma\) can also be simulated by the substitution of Eq. (V.42). For generality we replace \(b\) by \(\bar{b}\), and simultaneously remark that hemiticity requires \(\bar{b} = -b^*\). The spin density for the anti-particle decay becomes
\[
G_\gamma(n_c, n_d) = 1 - n_c \cdot l_\gamma \otimes n_d - \rho_e [n_c \cdot l_\gamma - n_d \cdot l_\gamma], \tag{V.46}
\]
and
\[
\rho_e = \frac{2\Re(\bar{b})}{1 + |b|^2}. \tag{V.47}
\]
The P-violating interference term now enters with the opposite sign. If CP is conserved then \(\bar{b} = -b\). For a full discussion of P and CP conservation in this context we refer to Ref. [8].

VI. ELECTROMAGNETIC HYPERON DECAYS: VIRTUAL PHOTONS

The leptonic decay \(\Sigma^0 \rightarrow \Lambda e^+e^-\) is a small fraction of the electromagnetic decay \(\Sigma^0 \rightarrow \Lambda\gamma\) [15, 16]. The lepton pair of the leptonic decay is interpreted as the decay product of a virtual, massive photon. This pair is often called a Dalitz lepton pair.

The steps to follow in order to find the cross-section distribution for virtual photons are well known. The square of the reduced matrix element is written as
\[
|\mathcal{M}_e(c \rightarrow de^+e^-)|^2 = \frac{1}{m_e^4} H^\mu_\nu L_{\mu\nu}, \tag{VI.48}
\]
where \(H^\mu_\nu\) is the hadron tensor and \(L_{\mu\nu}\) the lepton tensor.

The hadron tensor can be extracted from Eq. (V.33),
\[
H^\mu_\nu(c \rightarrow de^+e^-) = \mu_{cd}^2 \text{Tr} \left[ \frac{1}{2} (1 + \gamma_5 \gamma_\nu)(\not{p}_d + m_d) \sigma^{\mu\tau} k_\tau \right. \\
\left. \times (\not{p}_e + m_e) \frac{1}{2} (1 + \gamma_5 \gamma_\mu) \sigma^{\nu\lambda} k_\lambda \right]. \tag{VI.49}
\]

We need the square of \(\mathcal{M}_e\) averaged over baryon spins but summed over lepton spins. The summation over lepton spins leads to a lepton tensor,
\[
L_{\mu\nu}(k_1, k_2) = e^2 \sum_{l_{\text{spin}}} \bar{v}(k_2)\gamma_\mu u(k_1)\bar{u}(k_1)\gamma_\nu v(k_2) \\
= 4e^2 \left[ k_\mu k_\nu - k_{1\mu}k_{1\nu} - k_{2\mu}k_{2\nu} - \frac{1}{2} g_{\mu\nu} k^2 \right]. \tag{VI.50}
\]
Next, we integrate over the lepton momenta. For this purpose we rewrite the phase-space element as
\[ \text{dLips}(p_c; p_d, k_1, k_2) = \frac{1}{2\pi} dm^2 \text{dLips}(p_c; p_d, k) \, \text{dLips}(k; k_1, k_2) \] (VI.51)
with \( k^2 = m^2_\gamma \) and \( \text{dLips}(k; k_1, k_2) \) the phase-space element for the lepton pair, as in Appendix C. However, care should be exercised since in many experiments the efficiency is very sensitive to the lepton momentum.

The integration over the lepton phase space affects only the lepton tensor. Thus, we note that
\[ \langle k_1 \mu k_1 \nu \rangle = \left[ \frac{1}{3} (1 - \frac{m^2_\gamma}{k^2}) k_\mu k_\nu - \frac{1}{12} k^2 (1 - \frac{4m^2_\gamma}{k^2}) g_{\mu \nu} \right] \langle 1 \rangle , \] (VI.52)
and similarly for \( \langle k_2 \mu k_2 \nu \rangle \), with brackets denoting integration over lepton phase space, \( \text{dLips}(k; k_1, k_2) \), and \( \langle 1 \rangle \) denoting the phase-space volume itself. The term proportional to \( k_\mu k_\nu \) in Eq.(VI.52) vanishes due to gauge invariance. As a consequence, we get as average of the lepton tensor,
\[ \langle L_{\mu \nu} \rangle = L(k^2)(-g_{\mu \nu}) \] (VI.53)
\[ L(k^2) = \alpha_e k^2 \sqrt{1 - \frac{4m^2_\gamma}{k^2}} \left[ 1 - \frac{1}{3} \left( 1 - \frac{4m^2_\gamma}{k^2} \right) \right]. \] (VI.54)

The lepton tensor \( L_{\mu \nu} \) of Eq.(VI.53) comes with a factor \((-g_{\mu \nu})\). Contracting it with the hadron tensor \( H_{\mu \nu}(c \rightarrow dg) \), with \( g \) representing the virtual photon, is equivalent to summing over photon polarizations. We write
\[ |\mathcal{M}_e(c \rightarrow dg)|^2 = -H^\mu_\mu(c \rightarrow dg), \]
\[ = R(c \rightarrow dg) G(n_c, n_d). \] (VI.55)

The factorization is chosen so that \( R(c \rightarrow dg) \) is spin independent, and so that the spin-independent term of \( G(n_c, n_d) \) is unity.

The functions \( R \) and \( G \) are easily calculated. Neglecting terms unimportant for the \( \Sigma^0 \rightarrow \Lambda \gamma \) transition, we get for the fractional decay rate of Eq.(VI.55),
\[ R(c \rightarrow dg) = 2m_\Sigma \Gamma(c \rightarrow dg)/\Phi(c \rightarrow dg), \]
\[ = \mu^2_{cd} \left[ (m_c - m_d)^2 - m^2_\gamma \right] (m_c + m_d)^2. \] (VI.56)
where \( \Phi(c \rightarrow dg) = \Phi(m_c, m_d, m_\gamma) \) is the phase-space volume. For \( m_\gamma = 0 \) we recover \( R(c \rightarrow d\gamma) \) for real photons, Eq.(V.36).
Again neglecting terms unimportant for the $\Sigma^0 \to \Lambda\gamma$ transition, the properly normalized spin density reads,

$$G(n_c, n_d) = 1 - n_c \cdot l_\gamma \cdot n_d. \quad (VI.57)$$

Thus, it is in this approximation also equal to the normalized spin density for real photons, Eq.(V.38). The exact expressions for $R$ and $G$ are given in Appendix E.

Next, we combine the matrix elements for the transitions $c \to dg$ and $g \to e^+e^-$, $g$ representing a virtual photon of mass $m_\gamma$.

Since the lepton tensor of Eq.(VI.54) lacks spin dependence, $G(g \to e^+e^-) = 1$, we have the spin-density relation

$$G(c \to de^+e^-) = G(c \to dg)G(g \to e^+e^-), \quad (VI.58)$$

and a corresponding $R$-factor relation

$$R(c \to de^+e^-) = R(c \to dg)R(g \to e^+e^-). \quad (VI.59)$$

The function $R(g \to e^+e^-)$ collects the remains, the lepton tensor of Eq.(VI.54) multiplied by the propagator $1/k^4$ of Eq.(VI.48),

$$R(g \to e^+e^-) = \frac{\alpha e}{k^2} \sqrt{1 - \frac{4m_e^2}{k^2}} \left[ 1 - \frac{1}{3} \left( 1 - \frac{4m_e^2}{k^2} \right) \right]. \quad (VI.60)$$

This expression comes with the phase-space element

$$dLips = \frac{1}{2\pi} \, dm_\gamma^2 \, dLips(p_c; p_d; k), \quad (VI.61)$$

where $k$ is the four-momentum of the virtual photon and $k^2 = m_\gamma^2$. Remember that $m_\gamma^2 \geq 4m_e^2$ so there is no singularity in $R(g \to e^+e^-)$.

Deviations from the Dalitz-distribution function of Eq.(VI.60) signals the importance form factors in the virtual photon exchange.

**VII. FOLDING**

Our general aim is to calculate the cross-section distributions for $e^+e^-$ annihilation into $\Sigma^0\overline{\Sigma}^0$ pairs that subsequently decay, as $\Sigma^0 \to \Lambda \to p$ or $\overline{\Sigma}^0 \to \overline{\Lambda} \to \overline{p}$, and as illustrated in Fig. 2. The first step in this endeavour is to perform the folding of a product of spin densities, a technique especially adapted to spin one-half baryons.
FIG. 2: Graph describing the reaction $e^+e^- \rightarrow \Sigma^0\Sigma^0$, and the subsequent decays, $\Sigma^0 \rightarrow \Lambda\gamma; \Lambda \rightarrow p\pi^-$ and $\Sigma^0 \rightarrow \bar{\Lambda}\gamma; \bar{\Lambda} \rightarrow \bar{p}\pi^+$. The reaction graphed can, in addition to photons, be mediated by vector charmonia, such as $J/\psi$, $\psi'$ and $\psi(2S)$. Solid lines refer to baryons, dashed to mesons, and wavy to photons.

A folding procedure implies forming an average over intermediate spin directions $\mathbf{n}$ according to the prescription

$$\langle 1 \rangle_n = 1, \quad \langle \mathbf{n} \rangle_n = 0, \quad \langle \mathbf{n} \cdot \mathbf{k} \cdot \mathbf{l} \rangle_n = \mathbf{k} \cdot \mathbf{l}. \quad \text{(VII.62)}$$

For more details see Ref.[11].

In the present case there are five spin densities; the annihilation spin density $S(\mathbf{n}_\Sigma, \mathbf{n}_\Sigma)$ of Eq.(III.11); the spin densities of the electromagnetic and weak decays, Eqs.(V.38) and (IV.21),

$$G(\Sigma^0 \rightarrow \Lambda\gamma) = 1 - \mathbf{n}_\Sigma \cdot \mathbf{l}_\gamma \cdot \mathbf{n}_\Lambda, \quad \text{(VII.63)}$$

$$G(\Lambda \rightarrow p\pi^-) = 1 + \alpha_\Lambda \mathbf{n}_\Lambda \cdot \mathbf{l}_p + \alpha_\Lambda \mathbf{n}_p \cdot \mathbf{l}_p$$

$$+ \mathbf{n}_\Lambda \cdot \mathbf{L}_\Lambda(\mathbf{n}_p, \mathbf{l}_p), \quad \text{(VII.64)}$$

with $\mathbf{L}_\Lambda(\mathbf{n}_p, \mathbf{l}_p)$ defined in Eq.(IV.22); and the anti-hyperon versions of the last two spin densities. Remember that the symbol $\mathbf{l}$ represents a unit vector.

The spin density for the $\Sigma^0 \rightarrow p$ transition is obtained by folding a product of spin densities. Averaging over the Lambda and final-state proton spins, according to the folding
prescription Eq. (VII.62), gives us

\[
G(\Sigma^0 \to p) = \left\langle G(\Sigma^0 \to \Lambda\gamma) G(\Lambda \to p\pi^-) \right\rangle_{n_\Lambda, n_p} = 1 - \alpha_\Lambda n_\Sigma \cdot l_\gamma l_\gamma \cdot l_p.
\]  
(VII.65)

We notice that this spin density does not depend on the asymmetry parameters \(\beta_\Lambda\) and \(\gamma_\Lambda\), a consequence of the average over the final-state-proton-spin directions.

To the baryon decay chain \(\Sigma^0 \to \Lambda \to p\) there is a corresponding anti-baryon decay chain \(\bar{\Sigma}^0 \to \bar{\Lambda} \to \bar{p}\), and a corresponding transition-spin density.

To go from the baryon to the anti-baryon case, we simply replace the baryon variables by their anti-baryon counterparts, \(n_\Sigma \to n_{\bar{\Sigma}}\), \(\alpha_\Lambda \to \alpha_{\bar{\Lambda}}\), etc.

The inclusion of parity violation in the \(\Sigma^0 \to \Lambda\gamma\) decay is straightforward. To this end we replace \(G(\Sigma^0 \to \Lambda\gamma)\) of Eq. (VII.63) by

\[
G(\Sigma^0 \to \Lambda\gamma) = 1 - n_\Sigma \cdot l_\gamma l_\gamma \cdot n_\Lambda + \rho_\Sigma [n_\Sigma \cdot l_\gamma - n_\Lambda \cdot l_\gamma],
\]  
(VII.66)

of Eq. (V.43), and get

\[
G(\Sigma^0 \to p) = \left\langle G(\Sigma^0 \to \Lambda\gamma) G(\Lambda \to p\pi^-) \right\rangle_{n_\Lambda, n_p} = \left(1 - \rho_\Sigma \alpha_\Lambda l_\gamma \cdot l_p\right) - \alpha_\Lambda n_\Sigma \cdot l_\gamma l_\gamma \cdot l_p.
\]  
(VII.67)

Therefore, as a consequence of parity violation the normalization of the cross-section distribution acquires a small angular dependent term.

VIII. SINGLE CHAIN DECAYS

Single-chain decays of \(\Sigma^0\) hyperons can be studied in the \(e^+e^-\) annihilation into \(\Sigma^0\bar{\Sigma}^0\) pairs, provided the \(\bar{\Sigma}^0\) is somehow identified, e.g., as a missing hyperon Ref.\([6]\). The spin-density state of the \(\Sigma^0\) will then be obtained from Eq. (III.11) as,

\[
\left\langle S(n_\Sigma, n_{\bar{\Sigma}}) \right\rangle_{n_{\bar{\Sigma}}} = \mathcal{R} + SN \cdot n_\Sigma.
\]  
(VIII.68)

A \(\Sigma^0\) hyperon in a state of polarization \(P_\Sigma\), subject to the condition \(|P_\Sigma| \leq 1\), is characterized by a normalized spin-density function,

\[
S_\Sigma(n_\Sigma) = 1 + P_\Sigma \cdot n_\Sigma.
\]  
(VIII.69)
Therefore, by Eq. (VIII.68) the hyperon polarization is in the present case equal to \( P_\Sigma = SN/R \).

If a \( \Sigma^0 \) hyperon of polarization \( P_\Sigma \) undergoes an electromagnetic decay, \( \Sigma^0 \to \Lambda \gamma \), we can determine the spin-density distribution of the \( \Lambda \) hyperon by folding the initial state \( \Sigma^0 \) spin density of Eq. (VIII.69) with the \( \Sigma^0 \) decay distribution of Eq. (V.38), to get

\[
W_\Lambda(n_\Lambda; l_\Lambda) = \left\langle S_\Sigma(n_\Sigma) G_\gamma(n_\Sigma, n_\Lambda) \right\rangle_{n_\Sigma} \\
= 1 - P_\Sigma \cdot l_\Lambda l_\Lambda \cdot n_\Lambda, \tag{VIII.70}
\]

with \( l_\Lambda = -l_\gamma \) and a \( \Lambda \) polarization

\[
P_\Lambda = -P_\Sigma \cdot l_\Lambda l_\Lambda. \tag{VIII.71}
\]

Consequently, the \( \Lambda \) polarization is directed along the \( \Lambda \) momentum \( l_\Lambda \), a fact which is independent of the initial \( \Sigma^0 \) hyperon spin.

Let us now consider also the weak decay of the \( \Lambda \)-hyperon, \( \Lambda \to p\pi^- \), which is described by the spin density \( G_\Lambda(n_\Lambda, n_p) \) of Eq. (IV.21). Since the spin of the final-state proton is usually not measured, we form the average over the proton spin directions. The spin-density-distribution function of Eq. (VIII.70) is now expanded to

\[
W_p(l_\Lambda, l_p) = \left\langle S_\Sigma(n_\Sigma) G_\gamma(n_\Sigma, n_\Lambda) G_p(n_\Lambda, n_p) \right\rangle_{n_\Sigma, n_\Lambda, n_p} \\
= 1 - \alpha_\Lambda P_\Sigma \cdot l_\Lambda l_\Lambda \cdot l_p, \\
= 1 + \alpha_\Lambda P_\Lambda \cdot l_p. \tag{VIII.72}
\]

The decay chain \( \Sigma^0 \to \Lambda \gamma \to p\pi^- \) makes part of our annihilation process and it is therefore of interest to investigate what additional information may be obtained by measuring the spin of the final-state proton. Thus, instead of the spin density of Eq. (VII.67) we investigate the spin density

\[
G(\Sigma^0 \to p) = \left\langle G(\Sigma^0 \to \Lambda \gamma) G(\Lambda \to p\pi^-) \right\rangle_{n_\Lambda}. \tag{VIII.73}
\]

Invoking the vector-function identity of Eq. (IV.22) we get

\[
G(\Sigma^0 \to p) = 1 + \alpha_\Lambda n_p \cdot l_p \\
- n_\Sigma \cdot l_\gamma \left[ \alpha_\Lambda l_\gamma \cdot l_p + n_p \cdot L_\Lambda(l_\gamma, -l_p) \right]. \tag{VIII.74}
\]
Finally, the spin-density-distribution function for the final state proton is obtained as

\[
S(n_p) = \langle S(n_\Sigma) S(n_\Sigma, n_p) \rangle_{n_\Lambda, n_\Sigma} = U_p + V_p \cdot n_p,
\]

(VIII.75)

\[
U_p = 1 - \alpha \Lambda n_p \cdot l_\gamma,
\]

(VIII.76)

\[
V_p = \alpha \Lambda l_p - P_\Sigma \cdot l_\gamma L_\Lambda (l_\gamma, -l_p).
\]

(VIII.77)

This result describes a proton polarization which is \(v_p/u_p\). It is explicitly dependent on \(\alpha \Lambda\), but there is a hidden dependence on \(\beta \Lambda\) and \(\gamma \Lambda\) in the vector function \(L_\Lambda\).

**IX. PRODUCTION AND DECAY OF \(\Sigma^0\bar{\Sigma}^0\) PAIRS**

Now, we come to the main task of our investigation; production and decay of \(\Sigma^0\bar{\Sigma}^0\) pairs. The starting point is the reaction \(e^+ e^- \rightarrow \Sigma^0\bar{\Sigma}^0\), the spin-density distribution of which was calculated in Sect.3. We name it \(S(n_\Sigma, n_\Sigma)\). The explicit expression is given by Eq.(III.11), with \(n_1, n_2\) replaced by \(n_\Sigma, n_\Sigma\).

The spin-density distribution \(W_\Sigma(n_\Sigma, n_p)\) for the decay chain \(\Sigma^0 \rightarrow \Lambda \gamma; \Lambda \rightarrow p \pi^-\) is given in Eq.(VIII.74). We write

\[
W_\Sigma(n_\Sigma, n_p) = U_\Sigma + n_\Sigma \cdot V_\Sigma,
\]

IX.78

\[
U_\Sigma = 1 + \alpha \Lambda n_p \cdot l_\gamma
\]

IX.79

\[
V_\Sigma = -l_\gamma [\alpha \Lambda l_\gamma \cdot n_p + n_p \cdot L_\Lambda (l_\gamma, -l_p)],
\]

IX.80

and ditto for \(W_\bar{\Sigma}(n_\bar{\Sigma}, n_\bar{p})\). We are only interested in decay chains of \(\Sigma^0\) and \(\bar{\Sigma}^0\) which are each others anti chains.

The final-state-angular distributions are obtained by folding the spin distributions for production and decay, according to prescription (VII.62). Invoking Eq.(III.11) for the production step and Eqs.(IX.78) and its anti-distribution for the decay steps, we get the angular
distribution

\[
W_{\Sigma \Sigma}(l_a) = \left\langle S(n_{\Sigma}, n_{\Sigma}) W_{\Sigma}(n_{\Sigma}, n_p) W_{\Sigma}(n_{\Sigma}, n_{\bar{\Sigma}}) \right\rangle_{n_{\Sigma}, n_{\bar{\Sigma}}}
\]

\[
= RU_{\Sigma} U_{\bar{\Sigma}} + SU_{\Sigma} N \cdot V_{\Sigma} + SU_{\bar{\Sigma}} N \cdot V_{\bar{\Sigma}}
+ \mathcal{T}_1 V_{\Sigma} \cdot \hat{p} V_{\Sigma} \cdot \hat{p} + \mathcal{T}_2 V_{\Sigma \perp} \cdot V_{\Sigma \perp}
+ \mathcal{T}_3 V_{\Sigma \perp} \cdot \hat{k} V_{\Sigma \perp} \cdot \hat{k}
+ \mathcal{T}_4 \left( V_{\Sigma} \cdot \hat{p} V_{\Sigma \perp} \cdot \hat{k} + V_{\bar{\Sigma}} \cdot \hat{p} V_{\bar{\Sigma} \perp} \cdot \hat{k} \right),
\]

(IX.81)

where \(l_a\) denotes the ensemble of \(l\) values in the decays.

The angular distributions of Eq.(IX.81) still depend on the spin vectors \(n_p\) and \(n_{\bar{p}}\) which are difficult to measure. If we are willing to consider the spin averages, then variables \(U\) and \(V\) simplify,

\[
U_{\Sigma} = 1, \quad V_{\Sigma} = -\alpha_\Lambda l_\Lambda \cdot l_p l_\Lambda
\]

\[
U_{\bar{\Sigma}} = 1, \quad V_{\bar{\Sigma}} = -\alpha_{\bar{\Lambda}} l_{\bar{\Lambda}} \cdot l_\bar{p} l_{\bar{\Lambda}}. \quad (IX.82)
\]

Since \(U_{\Sigma} = U_{\bar{\Sigma}} = 1\) the effect of the folding is to make, in the spin-density function \(S(n_{\Sigma}, n_{\Sigma})\) of Eq.(III.11), the replacements \(n_{\Sigma} \rightarrow V_{\Sigma}\) and \(n_{\bar{\Sigma}} \rightarrow V_{\bar{\Sigma}}\). We notice that the \(U\) and \(V\) variables are independent of the weak-asymmetry parameters \(\beta_\Lambda\) and \(\gamma_\Lambda\). Their dependence is hidden in the vector function \(L_\Lambda(l_\gamma, -l_p)\) of Eq.(IX.80), and which is absent in Eq.(IX.81).

Inserting the expressions of Eq.(IX.82) into the spin-density function of Eq.(IX.81) we get

\[
W_{\Sigma \Sigma}(l_a) = \mathcal{R} - \alpha_\Lambda S N \cdot l_\Lambda l_\Lambda \cdot l_p - \alpha_{\bar{\Lambda}} S N \cdot l_{\bar{\Lambda}} l_\Lambda \cdot l_{\bar{\Lambda}}
+ \alpha_\Lambda \alpha_{\bar{\Lambda}} l_\Lambda \cdot l_p l_\Lambda \cdot l_{\bar{\Lambda}} \left[ \mathcal{T}_1 l_\Lambda \cdot \hat{p} l_\Lambda \cdot \hat{p} \right.
+ \mathcal{T}_2 l_{\Lambda \perp} \cdot l_{\Lambda \perp} + \mathcal{T}_3 l_{\Lambda \perp} \cdot \hat{k} l_{\Lambda \perp} \cdot \hat{k}
+ \mathcal{T}_4 \left( l_\Lambda \cdot \hat{p} l_{\Lambda \perp} \cdot \hat{k} + l_{\bar{\Lambda}} \cdot \hat{p} l_{\Lambda \perp} \cdot \hat{k} \right).
\]

(IX.83)

Thus, this is the angular distribution obtained when folding the product of spin densities for production and decay.
X. DIFFERENTIAL DISTRIBUTIONS

Explicit expressions for the structure functions $R$, $S$, and $T$ are given in Appendix B. With their help we can rewrite the differential distribution function of Eq. (IX.83) as

$$W(\xi) = \left[F_0 + \eta F_1\right]$$

$$- \sqrt{1 - \eta^2} \sin(\Delta \Phi) \sin \theta \cos \theta \left[\alpha_L F_2 F_5 + \alpha_L F_3 F_6\right]$$

$$+ \alpha_L \alpha_L F_2 F_3 \left[(\eta + \cos^2 \theta) F_4 - \eta \sin^2 \theta F_7\right]$$

$$+ (1 + \eta) \sin^2 \theta F_8$$

$$+ \sqrt{1 - \eta^2} \cos(\Delta \Phi) \sin \theta \cos \theta F_9,$$

where the argument $\xi$ of the angular functions is a nine-dimensional vector $\xi = (\theta, \Omega_L, \Omega_p, \Omega_{\bar{L}}, \Omega_{\bar{p}})$.

The ten angular functions $F_k(\xi)$ are defined as:

- $F_0(\xi) = 1$
- $F_1(\xi) = \cos^2 \theta$
- $F_2(\xi) = l_L \cdot l_p$
- $F_3(\xi) = l_{\bar{L}} \cdot l_p$
- $F_4(\xi) = l_L \cdot \hat{p} l_{\bar{L}} \cdot \hat{p}$
- $F_5(\xi) = N \cdot l_L$
- $F_6(\xi) = N \cdot l_{\bar{L}}$
- $F_7(\xi) = l_{L\perp} \cdot l_{\bar{L}\perp}$
- $F_8(\xi) = l_{L\perp} \cdot k l_{\bar{L}\perp} \cdot \hat{k} / \sin^2 \theta$
- $F_9(\xi) = \left(l_L \cdot \hat{p} l_{\bar{L}\perp} \cdot \hat{k} + l_{\bar{L}} \cdot \hat{p} l_{L\perp} \cdot \hat{k}\right) / \sin \theta.$

The cross-section distribution (IX.83), and also the ten angular functions above, depend on a number of unit vectors; $\hat{p}$ and $-\hat{p}$ are unit vectors along the directions of motion of the $\Sigma^0$ and the $\Sigma^0$ in the c.m. system; $\hat{k}$ and $-\hat{k}$ are unit vectors along the directions of motion of the incident electron and positron in the c.m. system; $l_L$ and $l_{\bar{L}}$ are unit vectors along the directions of motion of the $\Lambda$ and $\bar{\Lambda}$ in the rest systems of the $\Sigma^0$ and the $\bar{\Sigma}^0$; $l_p$ and $l_{\bar{p}}$
are unit vectors along the directions of motion of the $p$ and the $\bar{p}$ in the rest systems of the $\Lambda$ and the $\bar{\Lambda}$. Longitudinal and transverse components of vectors are defined with respect to the $\hat{p}$ direction.

The differential distribution function $W(\xi)$ of Eq.(X.84) involve two parameters related to the $e^+e^- \rightarrow \Sigma^0\Sigma^0$ reaction that can be determined by data: the ratio of form factors $\eta$, and the relative phase of form factors $\Delta \Phi$. In addition, the distribution function $W(\xi)$ depends on the weak-asymmetry parameters $\alpha_\Lambda$ and $\alpha_{\bar{\Lambda}}$ of the two Lambda-hyperon decays. The dependence on the weak-asymmetry parameters $\beta$ and $\gamma$ drops out, since final-state-proton and anti-proton spins are not measured.

An important conclusion to be drawn from the differential distribution of Eq.(X.84) is that when the phase $\Delta \Phi$ is small, the parameters $\alpha_\Lambda$ and $\alpha_{\bar{\Lambda}}$ are strongly correlated and therefore difficult to separate. In order to contribute to the experimental precision of $\alpha_\Lambda$ and $\alpha_{\bar{\Lambda}}$ a non-zero value of $\Delta \Phi$ is required.

The sequential differential decay distribution of a single-tagged $\Sigma^0$ produced in $e^+e^-$ annihilation can be obtained form Eq.(X.84) by suitably integrating over the angular variable $\Omega_{\bar{\Lambda}}$ and $\Omega_\Lambda$. As a result we get the differential distribution for $\Sigma^0$ production and decay,

$$d\sigma \propto \left[ R - \alpha_\Lambda S \mathbf{N} \cdot \mathbf{l}_\Lambda \mathbf{l}_\Lambda \cdot \mathbf{l}_\bar{p} \right] d\Omega_{\bar{\Lambda}} d\Omega_{\Lambda} d\Omega_\bar{p},$$

$$= \left[ 1 + \eta \cos^2 \theta - \alpha_\Lambda \sqrt{1 - \eta^2} \sin(\Delta \Phi) \sin \theta \cos \theta \times \cos \theta_{\Lambda p} \sin \theta_\Lambda \sin \phi_\Lambda \right] d\Omega_\Lambda d\Omega_{\bar{\Lambda}} d\Omega_\bar{p}. \quad (X.86)$$

Here, $\theta$ is the $\Sigma^0$ production angle, $\theta_{\Lambda p}$ the relative angle between the vectors $\mathbf{l}_\Lambda$ and $\mathbf{l}_\bar{p}$, and $\theta_\Lambda$ and $\phi_\Lambda$ the directional angles of $\mathbf{l}_\Lambda$ in the global coordinate system of $E$. From the angular distribution of Eq.(X.86) we can determine the product $\alpha_\Lambda \sin(\Delta \Phi)$, and from the corresponding $\Sigma^0$ distribution the product $\alpha_{\bar{\Lambda}} \sin(\Delta \Phi)$.

**XI. CROSS-SECTION DISTRIBUTIONS**

We shall now consider the phase-space imbedding of the differential-distribution function of Eq.(X.84). We start with the cross-section-distribution function for creation of a pair of baryons, $e^+e^- \rightarrow B\bar{B}$. Combining Eqs.(III.4), (III.9), and (III.10), we get

$$d\sigma(e^+e^- \rightarrow B\bar{B}) = \frac{p \alpha_\gamma^2 D(s)}{k} \frac{S(n_b, n_{\bar{b}})}{4s^2} d\Omega,$$

$$= \frac{\alpha_\gamma^2 D(s)}{k} \frac{S(n_b, n_{\bar{b}})}{4s^2} d\Omega. \quad (XI.87)$$
where $\Omega$ are the baryon scattering angles in the c.m. system.

Next we consider the propagator factors associated with the sequential decays of the baryons $\Sigma^0$ and $\bar{\Sigma}^0$ produced in the $e^+e^-$ annihilation process. These sequential decays are illustrated in Fig.2. There are three factors associated with the square of each propagator. Let us consider the decay $c \rightarrow dg$, where $g$ can represent a pion or a photon. Other decay modes are also possible to incorporate. Then, we have

$$P_c = \left[ \frac{\pi}{m_c \Gamma_c} \delta(s_c - m_c^2) \right] \left[ \frac{ds_c}{2\pi} \text{dLips}(p_c; p_d, p_g) \right] \left[ R_c G_c \right].$$

Here, the first factor comes from squaring the propagator in the Feynman diagram; the second factor from dividing the phase-space element into a product of two-body phase-space elements; and the third factor is the reduced matrix element squared for the decay $c \rightarrow dg$, and the product of the normalized spin density $G_c$ and the fractional decay rate $R_c$.

The fractional decay rate $R_c$ is defined in Eq.(V.36) as

$$R(c \rightarrow dg) = \frac{2m_c \Gamma(c \rightarrow dg)}{\Phi(c \rightarrow dg)},$$

where $\Phi$ is the two-body phase-space volume, and $\Gamma(c \rightarrow dg)$ the channel width for the decay $c \rightarrow dg$. It was defined to be spin averaged for both initial and final baryon states. However, in a sequential decay both final spin-state contribution must be included. This is achieved by multiplying $R(c \rightarrow d\gamma)$ by a factor of two. This factor can be incorporated in the channel width $\Gamma(c \rightarrow dg)$, reinterpreting it to include the sum over final baryon spin states. Finally, we observe that

$$\text{dLips}(p_c; p_d, p_g) = \Phi_c(c \rightarrow dg) \frac{d\Omega_c}{4\pi},$$

giving as a consequence a $P$ factor

$$P_c = G_c \frac{\Gamma(c \rightarrow dg)}{\Gamma(c \rightarrow all)} \frac{d\Omega_c}{4\pi},$$

with $\Omega_c$ the angular variable in the rest system of baryon $c$. In our application the indices $c$ and $\bar{c}$ are representatives for $c = \Sigma^0, \Lambda$ and $\bar{c} = \bar{\Sigma}^0, \bar{\Lambda}$.

The differential-distribution function $W(\xi)$ of Eq.(X.84) is obtained by folding a product of spin densities

$$W(\xi) = \left\langle S(n_b, n_{\bar{b}}) \prod_{c, \bar{c}} G_c(n_c, n_d) \right\rangle_n.$$
Folding involves averages over spin directions, but as remarked, cross-section distributions requires summing over the spin directions. Thus, an average over the spin density $S(n_b,n_{\bar{b}})$ is accompanied by an extra factor of four, and it is not normalized to unity either but to $R$.

The folding formula Eq.(XI.92) combined with Eqs.(XI.87) and (III.14) gives the master equation

$$d\sigma = d\sigma(e^+e^- \rightarrow \Sigma^0\Sigma^0) \frac{W(\xi)}{R} \prod_{c,\bar{c}} \left[ \frac{\Gamma(c \rightarrow dg)}{\Gamma(c \rightarrow all)} \frac{d\Omega_c}{4\pi} \right].$$

This readily understood structure agrees with that found in Ref.[11].

Since spin densities are normalized, except for the annihilation density $S(n_b,n_{\bar{b}})$, the overall normalization condition reads

$$\int W(\xi) \prod_{c,\bar{c}} \frac{d\Omega_c}{4\pi} = R.$$ 

This normalization is checked explicitly in single-chain-sequential decay in Ref.[6].

When the $\Sigma^0 \rightarrow \Lambda\gamma$ reaction is involved, and the photon is real, then the channel width in Eq.(XI.93) is for all practical purposes equal to the total width, $\Gamma(\Sigma^0 \rightarrow \Lambda\gamma) = \Gamma(\Sigma^0 \rightarrow all)$.

We also point out that for virtual photons the cross-section distribution of Eq.(XI.93) receives an additional lepton factor,

$$\frac{1}{2\pi} dm^2_\gamma \ R(g \rightarrow e^+e^-),$$

where $m_\gamma$ is the virtual photon mass, and $R$ the Dalitz function

$$R(g \rightarrow e^+e^-) = \frac{\alpha_e}{k^2} \sqrt{1 - \frac{4m^2_e}{k^2}} \left[ 1 - \frac{1}{3} \left( 1 - \frac{4m^2_e}{k^2} \right) \right].$$

The $e^+e^-$ annihilation reactions described above are all concerned with annihilation through ordinary photons, as illustrated in Fig.1. However, the same reactions can be initiated by other vector mesons as well. Of special interest is the $J/\psi$ case, which is treated in Ref.[5], and which is accessible to the BESIII experiment. By making the replacement

$$\frac{\alpha^2_\psi}{s^2} \rightarrow \frac{\alpha_\psi\alpha_g}{(s - m^2_\psi)^2 + m^2_\psi \Gamma(m_\psi)}$$

in the photon-induced reaction, Eq.(XI.87), we get the cross-section-distribution formula for annihilation through the $J/\psi$ meson. The meaning of the parameters $\alpha_\psi$ and $\alpha_g$ is explained in Ref.[5]. This is equivalent to replacing in the master formula of Eq.(XI.93), the photon-induced $e^+e^- \rightarrow \Sigma^0\Sigma^0$ annihilation cross section by the $J/\psi$ induced cross section.
Appendix A: Graph calculation

In this Appendix we shall work out the phase-space density for the two-step case. Our notation follows Pilkuhn [12]. The cross-section distribution can be written as

\[ d\sigma = \frac{1}{2\sqrt{\lambda(s, m_e^2, m_e^2)}} \left| \mathbf{M} \right|^2 \text{dLips}(k_1 + k_2; \{l_i\}, \{l_i'\}), \]  

(A.1)

where \( \{l_i\} \) are the final-state momenta in the hyperon decay chain and \( \{l_i'\} \) are the final-state momenta in the anti-hyperon decay chain. The average over the squared matrix element indicates summation over final-state spins and average over initial-state lepton. The definitions of the particle momenta are explained in Fig.1.

Since \( \Gamma \ll M \) for the intermediate propagators, their squares may be approximated as

\[ \frac{1}{(s - M^2)^2 + M^2 \Gamma^2} = \frac{\pi}{M\Gamma(M)} \delta(s - M^2). \]  

(A.2)

This makes it convenient to pull out a factor \( K \) from the squared matrix element,

\[ K = \prod_i \frac{1}{(s_i - M_i^2)^2 + M_i^2 \Gamma_i^2(M_i)}, \]  

(A.3)

and plug it into the phase-space density. In Eq.(A.3) the product runs over the four intermediate-state hyperons.

After some manipulations we can write the modified phase-space density as

\[ K \text{dLips}(k_1 + k_2; \{l_i\}, \{l_i'\}) = \frac{p}{(4\pi)^2 \sqrt{s}} \text{d}\Omega \bigg|_{\text{CM}} \times \prod_i \left[ \frac{q_i}{8\pi M_i^2 \Gamma_i(M_i)} \text{d}\Omega_i \right] Y, \]  

(A.4)

where index CM refers to the two-body reaction \( e^+e^- \rightarrow YY \), and index Y to each of the four intermediate-state hyperon decays, in their respective hyperon rest systems.

Appendix B: Structure functions

The six structure functions \( R, S, \) and \( T \) of Eq.[III.11] depend on the scattering angle \( \theta \), in the c.m. system, the ratio function \( \eta(s) \), and the phase function \( \Delta \Phi(s) \). To be specific
\[ R = 1 + \eta \cos^2 \theta, \quad (B.1) \]
\[ S = \sqrt{1 - \eta^2} \sin \theta \cos \theta \sin(\Delta \Phi), \quad (B.2) \]
\[ T_1 = \eta + \cos^2 \theta, \quad (B.3) \]
\[ T_2 = -\eta \sin^2 \theta, \quad (B.4) \]
\[ T_3 = 1 + \eta, \quad (B.5) \]
\[ T_4 = \sqrt{1 - \eta^2} \cos \theta \cos(\Delta \Phi). \quad (B.6) \]

The parameters \( \eta \) and \( \Delta \Phi \) are defined in Eqs. (II.2) and (II.3).

**Appendix C: Phase-space volume**

The Lorentz invariant two-body phase-space element is by definition
\[
\text{dLips}(k; k_1, k_2) = \frac{d^3 k_1}{(2\pi)^3 2\omega_1} \frac{d^3 k_2}{(2\pi)^3 2\omega_2} (2\pi)^4 \delta(k - k_1 - k_2). \quad (C.1)
\]
Integration exploiting the delta functions leads to
\[
\int \text{dLips}(k; k_1, k_2) = \frac{k_c}{4\pi \sqrt{s}} \frac{d\Omega_c}{4\pi} \quad (C.2)
\]
where \( \sqrt{s} = M \), \( k_c \) the momentum, and \( \Omega_c \) the angular variable, both in the c.m. system. In terms of the mass variables
\[
k_c^2 = \frac{1}{4M^2} \left[ (M^2 + m_1^2 - m_2^2)^2 - 4M^2 m_1^2 \right]. \quad (C.3)
\]
The phase-space volume \( \Phi \) is obtained from Eq. (C.2) by integration over \( d\Omega_c \),
\[
\Phi(M; m_1, m_2) = \frac{k_c}{4\pi \sqrt{s}}. \quad (C.4)
\]
For equal masses \( m_1 = m_2 = m \) the value of the phase-space volume becomes
\[
\Phi(M; m, m) = \langle 1 \rangle = \frac{1}{8\pi} \sqrt{1 - \frac{4m^2}{M^2}}. \quad (C.5)
\]

**Appendix D: Decay into virtual gamma**

The squared matrix element \( |M(c \rightarrow dg)|^2 \) for the decay of a baryon \( c \) into a baryon \( d \) and a virtual gamma \( g \) of mass \( m_\gamma \) is given in Eq. (VI.55). It can be factorized into factors
$R(c \rightarrow dg)$ and $G_g(n_c, n_d)$. The exact expression for the fractional width is,

$$R(c \rightarrow dg) = \mu_{cd}^2 \left[ (m_c - m_d)^2 - m_{\gamma}^2 \right] \left[ (m_c + m_d)^2 + \frac{1}{2} m_{\gamma}^2 \right],$$

(D.1)

with $2m_c \leq m_{\gamma} \leq (m_c - m_d)$. In the limit $m_{\gamma} = 0$ we recover $R(c \rightarrow d\gamma)$ for real photons, Eq.(V.36). The exact expression for the normalized spin density is,

$$G_g(n_c, n_d) = 1 + B n_c \cdot l_{\gamma} l_{\gamma} \cdot n_d + C n_c \cdot n_d,$$

(D.2)

$$A = (m_c + m_d)^2 + \frac{1}{2} m_{\gamma}^2,$$

$$B = - \frac{(m_c + m_d)^2}{A},$$

$$C = \frac{1}{2} m_{\gamma}^2 / A.$$  

(D.3)

Here, we can without qualm put $B = -1$ and $C = 0$. In this limit we recover the normalized spin density for real photons, Eq.(V.38).

**Appendix E: Angular functions**

The cross-section distribution (IX.83) is a function of two hyperon unit vectors: $l_\Lambda$, the direction of motion of the Lambda hyperon in the rest system of the Sigma hyperon, and $l_p$ the direction of motion of the proton in the rest system of the Lambda hyperon. Plus the corresponding vectors for the anti-hyperon chain. In order to handle these vectors we introduce a common global coordinate system, which we define as follows.

The scattering plane of the reaction $e^+ e^- \rightarrow \Sigma^0 \bar{\Sigma}^0$ is spanned by the unit vectors $\hat{p} = l_\Sigma$ and $\hat{k} = l_e$, as measured in the c.m. system. The scattering plane makes up the $xz$-plane, with the $y$-axis along the normal to the scattering plane. We choose a right-handed coordinate system with basis vectors

$$e_z = \hat{p},$$

$$e_y = \frac{1}{\sin \theta} (\hat{p} \times \hat{k}),$$

$$e_x = \frac{1}{\sin \theta} (\hat{p} \times \hat{k}) \times \hat{p}. (E.1)$$

Expressed in terms of them the initial-state lepton momentum becomes

$$\hat{k} = \sin \theta e_x + \cos \theta e_z.$$  

(E.2)
This coordinate system is used for defining the directional angles of the Lambda and the proton. The directional angles of the Lambda hyperon in the Sigma hyperon rest system are,

\[ \mathbf{l}_\Lambda = (\cos \phi_\Lambda, \sin \phi_\Lambda \sin \theta_\Lambda, \cos \theta_\Lambda) \]  
(E.3)

whereas the directional angles of the proton in the Lambda hyperon rest system are

\[ \mathbf{l}_p = (\cos \phi_p, \sin \phi_p \sin \theta_p, \cos \theta_p) \]  
(E.4)

And so for the anti-hyperons.

An event of the reaction \( e^+ e^- \rightarrow \Sigma^0 \bar{\Sigma}^0; \Sigma^0 \rightarrow \Lambda \rightarrow p; \bar{\Sigma}^0 \rightarrow \bar{\Lambda} \rightarrow \bar{p} \); is specified by a nine-dimensional vector \( \mathbf{\xi} = (\theta, \Omega_\Lambda, \Omega_p, \Omega_{\bar{\Lambda}}, \Omega_{\bar{p}}) \). The differential-cross-section distribution is proportional to a function \( \mathcal{W}(\mathbf{\xi}) \), which according to Eq.(X.84) can be decomposed as

\[
\mathcal{W}(\mathbf{\xi}) = \left[ \mathcal{F}_0(\mathbf{\xi}) + \eta \mathcal{F}_1(\mathbf{\xi}) \right] \\
- \sqrt{1 - \eta^2} \sin(\Delta \Phi) \sin \theta \cos \theta \left[ \alpha_\Lambda \mathcal{F}_2(\mathbf{\xi}) \mathcal{F}_5(\mathbf{\xi}) + \alpha_{\bar{\Lambda}} \mathcal{F}_3(\mathbf{\xi}) \mathcal{F}_6(\mathbf{\xi}) \right] \\
+ \alpha_\Lambda \alpha_{\bar{\Lambda}} \mathcal{F}_2(\mathbf{\xi}) \mathcal{F}_3(\mathbf{\xi}) \left[ (\eta + \cos^2 \theta) \mathcal{F}_4(\mathbf{\xi}) \\
- \eta \sin^2 \theta \mathcal{F}_7(\mathbf{\xi}) + (1 + \eta) \sin^2 \theta \mathcal{F}_8(\mathbf{\xi}) \right] \\
+ \sqrt{1 - \eta^2} \cos(\Delta \Phi) \sin \theta \cos \theta \mathcal{F}_9(\mathbf{\xi}),
\]
(E.5)

The set of ten angular functions, \( \mathcal{F}_0(\mathbf{\xi}) - \mathcal{F}_9(\mathbf{\xi}) \), are defined in Eq.(X.85). The scalar products needed for their determination are as follows:

\[
\mathbf{N} \cdot \mathbf{l}_\Lambda = \sin \theta_\Lambda \sin \phi_\Lambda, \\
\mathbf{l}_\Lambda \cdot \mathbf{l}_p = \sin \theta_\Lambda \sin \theta_p \cos(\phi_\Lambda - \phi_p) + \cos \theta_\Lambda \cos \theta_p \\
\mathbf{l}_\Lambda \cdot \hat{\mathbf{p}} = \cos \theta_\Lambda, \\
\mathbf{l}_\Lambda \perp \cdot \hat{\mathbf{k}} = \sin \theta \sin \theta_\Lambda \cos \phi_\Lambda, \\
\mathbf{l}_\Lambda \perp \cdot \hat{\mathbf{p}} = 0, \\
\mathbf{l}_\Lambda \perp \cdot \mathbf{l}_\Lambda \perp = \sin \theta_\Lambda \sin \theta_\Lambda \cos(\phi_\Lambda - \phi_\Lambda).
\]
(E.6)

We understand that the remaining scalar products are obtained from those above by the substitution \((\Lambda; p) \rightarrow (\bar{\Lambda}; \bar{p})\). With the scalar products of Eq.(E.6) in hand one quickly
determines the ten angular functions $F_k(\xi)$ of Eq. (X.85),

\begin{align}
F_0(\xi) &= 1, \\
F_1(\xi) &= \cos^2 \theta, \\
F_2(\xi) &= \sin \theta \sin \theta_p \cos (\phi_\Lambda - \phi_p) + \cos \theta \cos \theta_p, \\
F_3(\xi) &= \sin \theta \sin \theta_p \cos (\phi_\Lambda - \phi_p) + \cos \theta \cos \theta_p, \\
F_4(\xi) &= \cos \theta \Lambda \cos \theta \bar{\Lambda}, \\
F_5(\xi) &= \sin \theta \Lambda \sin \phi \Lambda, \\
F_6(\xi) &= \sin \theta \Lambda \sin \phi \bar{\Lambda}, \\
F_7(\xi) &= \sin \theta \Lambda \sin \theta \Lambda \cos (\phi_\Lambda - \phi \bar{\Lambda}), \\
F_8(\xi) &= \sin \theta \Lambda \sin \phi \Lambda \sin \theta \Lambda \cos \phi \bar{\Lambda}, \\
F_9(\xi) &= \cos \theta \Lambda \sin \theta \Lambda \cos \phi \Lambda + \sin \theta \Lambda \cos \phi \Lambda \cos \theta \bar{\Lambda}.
\end{align}

(E.7)

The differential distribution of Eq. (E.5) involves two parameters related to the $e^+e^- \rightarrow \Sigma^0\bar{\Sigma}^0$ reaction that can be determined by data: the ratio of form factors $\eta$, and the relative phase of form factors $\Delta \Phi$. In addition, the distribution function $W(\xi)$ depends on the weak-decay parameters $\alpha_\Lambda$ and $\alpha_{\bar{\Lambda}}$ of the two $\Lambda$ hyperon decays. The dependence on the weak decay parameters $\beta$ and $\gamma$ drops out, since final-state proton and anti-proton spins are not measured.

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