Dynamics of Distillability

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The time evolution of a maximally entangled state is presented in this paper. The distillability criterion is given in terms of Kraus operators. Using the criterion, we discuss the distillability of $2 \times 2$ and $n \times n (n > 2)$ systems in their evolution process. There are two distinguished processes, dissipation and decoherence, which may destroy the distillability. We discuss the effects of those processes on distillability in details.

PACS number(s): 03.67.-a, 03.65.Bz

Maximally entangled states are an essential ingredient in most applications of quantum information[1,2]. In particular, in quantum communications, by using entangled states several proposals were devised to transmit secret messages between two locally separated parties[3-5]. These proposals have been successfully implemented experimentally by using pairs of photons generated via parametric down conversion[6-9].

On the other hand, quantum entanglement is a fragile feature, which can be destroyed by interaction with the environment. To overcome this difficulty, Bennett et al., Deutsch et al. and Gisin et al. presented several schemes to distill maximally entangled states of two qubits out of a set of pairs in mixed entangled states[10-12]. These processes are called entanglement distillation, which only consists of local actions and classical communication. Unfortunately, it is not known in general, which mixed state are distillable at that time. Later on, the Horodecki family shown that all entangled states of $2 \times 2$ and $2 \times 3$ systems can be distilled into a singlet[13], and they proved that nonpositivity of the partial transposition is a necessary condition for the state of an arbitrary bipartite system to be distillable[14]. But, it does not be the sufficient one. In fact, there are states which have non-positive partial transpose[15] but they are not distillable[16-18]. Recently, a useful sufficient criterion, the so-called reduction criterion, has been derived[18]. It shown that one can explicitly construct a protocol to distill $\rho$ if there exists some vector $|\psi\rangle$ satisfying

$$\langle \psi | T_B \rho \otimes 1 - \rho | \psi \rangle < 0,$$

where $T_B$ stand for the trace over the second subsystem. Moreover, some results of the distillability may be generalized to the case of continuous variable systems[19].

Since the early study on the entanglement and distillability, most works in this area are mainly concentrated on separability and distillability for a concrete class of mixed states. In practice, however, the destruction of a maximally entangled state is closely related to a dynamical process. One of the examples is that the destruction of a maximally entangled state due to the interaction with the environment is a dynamical process. Some authors become aware of the importance of the dynamical properties for quantum entanglement[20,21].

In this paper we investigate the time evolution of the distillability for a initially maximally entangled state. We consider a simple bipartite system which consists of two particles with the same dimensions. Two cases are taken into account in this paper. the first case is that only one of the two particles is subjected to the environment, the other case is two particles in the bipartite system are entirely under the effects of the environment. From the viewpoint of dimensions, this paper may be divided into two parts, i.e., $2 \times 2$ systems and $n \times n (n > 2)$ systems. For the $2 \times 2$ systems, there is an alternative necessary and sufficient condition for distillability. For high dimension, however, we only have necessary or sufficient criterion for distillability, but not a criterion for both. Our results show that the decoherence (caused by the environment considered here) do not change the distillability of the $2 \times 2$ systems, whereas the distillability for a dissipation systems remains unchanged only within a short time scales. For high dimension, however, the distillability condition is more complicated. It does not only depend on the coupling of the systems to the environment, but also on the initial condition.

**Distillability.** We consider two parties, Alice and Bob, who share several pairs of particles. For simplicity, we assume here that the particle has the same dimension $d$. Each pair is initially in a maximally entangled state

$$|\psi^+\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i,i\rangle,$$

Interactions with the environment transform this pure state into mixed state. This process may be described by a linear, trace-preserving, completely positive map $L$ as

$$L(\rho_0) = \rho_f = \sum_{i=1}^{k} A_i \rho_0 A_i^\dagger,$$

where $\rho_0$ stands for the initial state, throughout this paper we assume that $\rho_0 = |\psi^+\rangle\langle \psi^+|$, i.e., the system
is initially in a maximally entangled state. The trace-preserving property implies that the operator $A_i$ obey the constraint

$$\sum_{i=1}^{k} A_i^\dagger A_i = 1, \quad (4)$$

with 1 the identity matrix on the Hilbert space. One of the physical implementations of the Kraus operators $A_i$ is as follows. We consider a system interacting with the environment. The evolution of the total system (system plus the environment) is governed by a unitary operator $U(t)$. The reduced density matrix of the system may be given by tracing the total density operator over the environment,

$$\rho_f(t) = Tr_B[U(t) \rho_0(0) \otimes |0_B\rangle \langle 0_B| U^\dagger(t)], \quad (5)$$

where $|0_B\rangle$ stands for the initial state of the environment. For a set of complete and orthonormal bases $\{\mu^i_B\}$ of the environment, one obtains

$$\rho_f(t) = \sum_{i=1}^{k} A_i \rho_0 A_i^\dagger, \quad (6)$$

with $A_i = \langle \mu^i_B | U(t) |0_B\rangle$ is the so-called Kraus operator. It is evident that $A_i$ satisfy $\sum_{i=1}^{k} A_i^\dagger A_i = 1$, for $U(t)$ is unitary. So $L(\rho_0) \rightarrow \rho_f(t)$ is a trace preserving completely positive map.

For $2 \times 2$ systems, the sufficient and necessary condition for distillability is

$$F_f \equiv Tr(\rho_0 \rho_f) > \frac{1}{2}, \quad (7)$$

In terms of the Kraus operators, this condition is

$$F_f = \sum_{i=1}^{k} |\langle \psi^+| A_i |\psi^+\rangle|^2 > \frac{1}{2}. \quad (8)$$

In derivation of eq.(8), we used the initial condition $\rho_0 = |\psi^+\rangle \langle \psi^+|$. For high dimension ($>2$), there is a reduction criterion, it states that if there exists some vector $|\psi\rangle$ such that

$$G_f \equiv \langle \psi| Tr_B(1 - \rho |\psi\rangle < 0, \quad (9)$$

then the final state $\rho_f$ is distillable. An important aspect of this criterion is that if one finds a state $|\psi\rangle$ satisfying eq.(9), then one can explicitly construct a protocol to distill $\rho_f$[18]. For a initially maximally entangled state $|\psi^+\rangle$, we suppose that the most promising state, by which the distillation proposal is constructed, is $|\psi^+\rangle$ itself. In practice, we may always perform a distillation before the maximally entangled state decoheres far away from its initial state. In this sense the sufficient condition for distillability is

$$G_f = \sum_{i=1}^{k} \langle \psi^+| Tr_B(A_i |\psi^+\rangle \langle \psi^+| A_i^\dagger) \otimes 1 |\psi^+\rangle < \sum_{i=1}^{k} |\langle \psi^+| A_i |\psi^+\rangle|^2. \quad (10)$$

Therefore if we get the Kraus operators, we may know exactly the distillability of a state. Some words of caution are now in order. The Kraus operators are not unique in general. For example, in eq.(6) we may choose the other bases $\{\nu^i_B\}$ instead of $\{\mu^i_B\}$ to compute the Kraus operators. However, different sets of Kraus operators, which describe the same dynamical process, may be transformed each to other by a unitary transformation. In this sense, the distillability criterion do not depend on the choice of Kraus operators.

$2 \times 2$ system with decoherence. Decoherence occurs due to unwanted interactions between our quantum system and its environment. These interactions cause only information leak out of the system. Typically, this process may be described by the following two Kraus operators[22].

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & e^{-\gamma t} \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{\sqrt{1-e^{-2\gamma t}}} \end{pmatrix}. \quad (11)$$

We consider a pair of entangled particles $a$ and $b$. If only one of them (say $a$) is subjected to environment, the time evolution of a maximally entangled state

$$|\psi^+\rangle = \frac{1}{\sqrt{2}}(|0_a, 1_b\rangle - |1_a, 0_b\rangle)$$

is then given by

$$\rho_f(t) = \begin{pmatrix} 0 & 0 & \frac{1}{2} e^{-\gamma a t} & 0 \\ 0 & \frac{1}{2} e^{-\gamma a t} & 0 & 0 \\ \frac{1}{2} e^{-\gamma a t} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (12)$$

where $\gamma_a$ is the decay rate for particle $a$. This process may occur when a photon from a entangled pair is transmitted through a fiber whose length is randomly modulated by acoustic waves, or an atom from the pair is exposed to interactions with a environment that consists of a set of harmonic oscillators[24]. Substituting eq.(11) into eq.(8), we obtain

$$F_f = \frac{1}{2} + \frac{1}{2} e^{-\gamma a t}. \quad (13)$$

It is evident that the final state is always distillable. If the two particles are entirely under the effect of the environment, it is easy to show that the new Kraus operator for the whole system are

$$A_1 = A_{1a} \otimes A_{1b}, \quad A_2 = A_{2a} \otimes A_{2b}, \quad (14)$$
So, $F_f$ in this case is

$$F_f = \frac{1}{2} + \frac{1}{2} e^{-\gamma t}, \quad (15)$$

the final state is always distillable, too.

2 × 2 system with dissipation.– Different from the case of decoherence, dissipation leads not only to the decay of off-diagonal elements of the density matrix, but also the energy loss (decay of the diagonal element of the density matrix). The effect of energy loss to the environment is usually described by a master equation[25]. which in the Born-Markov approximation may be represented in terms of Kraus operators,

$$A_1 = \begin{pmatrix} 0 & \sqrt{e^{-\gamma t}} \\ 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & -\sqrt{1-e^{-\gamma t}} \\ 0 & 0 \end{pmatrix}. \quad (16)$$

If we only transmit one of the entangled particles through a noisy channel, the $F_f$ is then given by

$$F_f = \sum_{i=1}^{2} |\langle \psi | A_i | \psi \rangle|^2 = e^{-\frac{1}{\gamma} f(t)}. \quad (17)$$

It is greater than $\frac{1}{2}$ only for $t < 2ln2/\gamma$. This indicates that we must perform distillation within $2ln2/\gamma$ in order to distill a maximally entangled particles. Similarly, we obtain $F_f = e^{-\gamma t}$ for the case that the two entangled particles are both under the effects of the environment. It is well known that the character time of decoherence is much shorter than the dissipation one, so we can ignore the effects of dissipation in general. From the viewpoint of distillation, however, the dissipation is more destructive, for the information loss due to decoherence may be reconstructed by distillation, but the information loss caused by dissipation does not. Figure 1 shows a results of $F_f(t)$ defined in eq.(7) by solving master equations

$$\dot{\rho} = -i[H_0, \rho] + \frac{\gamma}{2} \sum_{i=a, b} (2\sigma_i^- \rho \sigma_i^+ - \sigma_i^+ \sigma_i^- \rho - \rho \sigma_i^+ \sigma_i^-),$$

and

$$\dot{\rho} = -i[H_0, \rho] + \frac{\gamma}{2} \sum_{i=a, b} (2\sigma_i^+ \rho \sigma_i^- - \sigma_i^- \sigma_i^+ \rho - \rho \sigma_i^- \sigma_i^+)$$

umerically, where $H_0$ represents the free Hamiltonian and is defined as $H_0 = \omega(\sigma_3^+ + \sigma_3^-)$. It is well known that the first master equation describes the dynamics for a dissipation system, while the second one governs a decoherence process. Both master equations can be derived by using Markov approximation and assuming system-bath interaction $\sum_i (\sigma_i^+ a_i + h.c.)$ and $\sum_i (\sigma_i^- a_i + h.c.)$, respectively. Here we use a notation $a_i$ to denote the bath mode annihilation operator. It is clear that $F_f(t)$ corresponding a dissipation process goes down below 0.5 earlier than a decoherence process. In this sense, we say dissipation is more destructive.

FIG. 1. $F_f(t)$ versus $t$, the dashed line is drawn for a decoherence system, while the solid line is for a dissipation system. Parameters chosen are $\gamma = 0.6\omega$, $t$ is chosen in units of $1/\omega$.

$n \times n$ system with decoherence.– Without loss of generality, we present the following model to describe the effect of decoherence, the model Hamiltonian is

$$H = H_s + H_B + H_i,$$

$$H_i = \sum_j f_j(\{a_s\}) \otimes g_j(\{b_B\}), \quad (18)$$

where $f_j$ and $g_j$ are functions of system operators and bath operators, respectively. In order to ensure that only the decoherence occur in the system, we make a constraint $[a_s, H_s] = 0$ on the operator $a_s$. This condition implies that we may factorize the time evolution operator of the composite system in the following way

$$U(t) = e^{-i(H_s + H_B)t} \cdot \prod_{j=1}^{N} e^{F_j(t) \cdot f_j(\{a_s\}) \otimes G_j(\{b_B\})}. \quad (19)$$

Here,

$$\{H_s, H_B, f_j(\{a_s\}) \cdot G_j(\{b_B\})(j = 1, 2, ..., N)\}$$

are elements of Lie algebra enlarged by $H_s, H_B, H_i$, while function $F_j(t)$ is determined by

$$\frac{d}{dt} e^{F_j(t) \cdot f_j(\{a_s\}) \otimes G_j(\{b_B\})} = F_j(t) F_j(t) f_j(\{a_s\}) \cdot G_j(\{b_B\}) e^{F_j(t) \cdot f_j(\{a_s\}) \otimes G_j(\{b_B\})}. \quad (20)$$

Eq.(20) indicates that in bases spanned by the eigenstates of $H_s$, the diagonal elements of the system density matrix remain unchanged in the evolution process, while the off-diagonal parts gain a complex phase with the evolution. A simple calculation give the element of reduced density matrix for the system
where \( \bar{f}_{jm} \) satisfies \( f_j(\{\omega\})|m\rangle = \bar{f}_{jm}|m\rangle \). The quantities \( \gamma_{mn}(t) \) have the following properties,

\[
\gamma_{mn}(t) = 0, \text{ for } m = n, \text{ and } \gamma_{mn}(t) > 0
\]

for otherwise. Physically, no energy transfer between the system and the environment require \( \gamma_{mn} = 0 \) for \( m = n \), while \( \gamma_{mn}(t) > 0 \) for \( m \neq n \). For \( \Gamma_{mn}(t) \), we have \( \Gamma_{mn}(t) = -\Gamma_{nm}(t) \), this property is directly from the hermity of \( \rho_f \). For a initially maximally entangled state

\[
\rho_0 = \frac{1}{d} \sum_{m,n} |m_a,m_b\rangle\langle n_a,n_b|,
\]

It is easy to check that

\[
G_f(t) = \text{Tr}(\text{Tr}_b \rho_f \otimes 1 - \rho_f) = -\sum_{m,n} \frac{d}{d\tau} e^{-\gamma_{mn}^a(t) - \gamma_{mn}^b(t)} \cdot \cos(\Gamma_{mn}^a(t) + \Gamma_{nm}^b(t)).
\]

(23)

Here, we assume the two entangled particles are both under the effect of the environment, \( \gamma_{mn}^a(t) \) and \( \gamma_{mn}^b(t) \) (\( \Gamma_{mn}^a(t) \) and \( \Gamma_{mn}^b(t) \)) are defined by eq.(21) for a-particle(b-particle). Eq.(23) is a damping-oscillation function of time. If \( \cos(\Gamma_{mn}^a(t) + \Gamma_{mn}^b(t)) > 0 \), the final state of \( \rho_f \) is distillable. If \( \rho_f \) does not violate the Peres separability criterion, the final state \( \rho_f \) can not be distilled, this can be done for a given state \( \rho_f \). For a model presented in [23], \( \Gamma_{mn}^x(t) \), \( (x = a, b) \) may be rewritten explicitly as

\[
\Gamma_{mn}^x(t) = \int \frac{g_{\omega,x}}{\omega_x^2}\left(m^2 - n^2\right)(\omega t - \sin \omega t) \rho(\omega) d\omega,
\]

(24)

where \( g_{\omega,x} \) is the coupling constant of particle \( x \) to the environment, \( \rho(\omega) \) stands for the spectrum distribution of the environment. Eq.(23) shows that the final state \( \rho_f \) is not always distillable, its distillability would depend on the detailed information of the system even in the case of decoherence.

**n x n system with dissipation.-** We consider a spin-n \((d = 2n + 1)\) particle interacting with its environment. Under the Born-Markov approximation, the dissipation process may be described by a master equation in the Lindblad form[25]

\[
\dot{\rho} = -i[H_0, \rho] + \gamma(2S_-\rho S_+ - S_+ S_- \rho - \rho S_+ S_-),
\]

(25)

where \( S_+ \) and \( S_- \) are defined by \( S_+ \) \((S_-) \) is the spin operator defined by \( S_+ = (S_x + iS_y) \) \((S_-) \) \((S_-) \). \( H_0 = \Omega S_z \) is the free Hamiltonian of the spin-n particle. \( \gamma \) is the decay rate. We would like to note that eq.(25) only described the time evolution of one particle in the entangled pair. In general, the particles in the pair experience different environment, for them are transmitted trough different noisy channels. By using the method presented in[26], we obtain the element of the density matrix up to the first order of \( \gamma \)

\[
\rho_{mn}(t) = \frac{1}{d} \sum_{m,n} 2Re\left[1 + \gamma t \sqrt{(f + m + 1)(f - m + 1)}\right]
\]

\[
\cdot \left( \Omega m + \Omega n - i\gamma(f - m + 1) - i\gamma(f + n) \right)
\]

\[
- \frac{1}{d} \sum_{m,n} (f + m + 1)(f - m + 1) |m\rangle\langle n|)
\]

(26)

In derivation of eq.(26), the initial condition \( \rho(0) = \frac{1}{d} \sum_{m,n} |m\rangle\langle n| \) was used. If the two particles are both exposed to the interactions of the environment, following the procedure presented in case A of this section, we arrive at

\[
G_f(t) = -\frac{1}{d} \sum_{m,n} dRe\left[1 + \gamma t \sqrt{(f + m + 1)(f - m + 1)}\right]
\]

\[
\cdot \left( \Omega m + \Omega n - i\gamma(f - m + 1) - i\gamma(f + n) \right)
\]

\[
- \frac{1}{d} \sum_{m,n} (f + m + 1)(f - m + 1) |m\rangle\langle n|)
\]

(27)

According to the sufficient condition(8), the final state is distillable if \( G_f(t) < 0 \). For \( \gamma = 0.2\Omega \), the dependence of the critical time \( t_c \) defined by \( G_f(t_c) = 0 \) on dimension of the system is shown in Fig.2. As Figure 2 shows, the larger the dimension of the system, the shorter the character time of distillability destruction.
Figure 3 is plotted for a numerical simulation of $G_f(t)$ for a decoherence system and a dissipation system, the dynamics of the dissipation system is govern by master equation (25), while the time evolution of the decoherence system are described by

$$\dot{\rho} = -i[H_0, \rho] + \frac{\gamma}{2}(2S_z\rho S_z - S_z S_z\rho - \rho S_z S_z), \quad (28)$$

![Figure 3](image_url)

**FIG. 3.** $G_f(t)$ as a function of time. Solid line is for a decoherence system with dimension 5, dotted line for a dissipation one with dimension 7, dashed line for decoherence with dimension 7 and dotted and dashed line for dissipation with dimension 5. In this plot, we choose $\gamma = 0.6\Omega$ and time $t$ is plotted in units of $1/\Omega$

From figure 3 we can see that in the case of dissipation $G_f(t)$ go up to zero after a short time evolution from our initial state $|\psi(0)\rangle = \frac{1}{d} \sum_m |m, -m\rangle$, whereas $G_f(t)$ for a decoherence process is always below zero. We would like to address that this conclusion is not a general one, which would depend on the initial state of the system and the bath information, for example, if we choose $m = 0$, and $\rho(\omega) = constant$ in eq.(24), we may find a time when $\cos(\Gamma_{mn}(t) + \Gamma_{mb}(t)) < 0$.

In summary, the dynamics of distillability for a bipartite system are investigated in this paper. The destruction of the maximal entanglement for the system is closely related to the interactions with the environment. There are two kinds of interactions which lead to the destruction of maximal entanglement. One is the quantum decoherence, the another is the dissipation. From the viewpoint of distillability, the dissipation is more harmful in quantum communication. For example, the decoherence do not change the distillability of a $2 \times 2$ systems, whereas the dissipation does. For high dimension, both dissipation and decoherence do destroy the quantum entanglement, but in different ways.

We would like to note that the initial state considered here is a maximally entangled state. If the initial state belongs to a special class of entangled states, the local environment can enhance the quantum entangle-

ment[27] from the viewpoint of quantum teleportation. This increases the efficiency of distillation and makes the undistillable state to be distillable. In the framework of quantum information theory, the state change allowed by quantum mechanics may be classified into three types. The first one is the unitary evolution, the second is the interaction with an environment, and the last one is a measurement performed on the quantum system. The unitary evolution is the cause to change the state of the bipartite system, for the two particles in the bipartite systems may interact each other. Although we do not discuss the dynamic of distillability of such a system in this paper, the method of this paper can easily generalized to this case. As to the measurement, the method presented here is also available, because the most general type of measurement can be understood within the framework of unitary evolution[28]. In fact most generalized measurement can be realized through many dynamical processes[29].

**ACKNOWLEDGEMENT:**

This work is supported by the NSF of China.

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