Characterization of 3-bridge links with infinitely many
3-bridge spheres

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Abstract
In [9], the author constructed an infinite family of 3-bridge links each of which
admits infinitely many 3-bridge spheres up to isotopy. In this paper, we prove that if a
prime, unsplittable link \( L \) in \( S^3 \) admits infinitely many 3-bridge spheres up to isotopy
then \( L \) belongs to the family.

1 Introduction

An \( n \)-bridge sphere of a link \( L \) in \( S^3 \) is a 2-sphere which meets \( L \) in \( 2n \) points and cuts
\( (S^3, L) \) into \( n \)-string trivial tangles \((B_1, t_1)\) and \((B_2, t_2)\). Here, an \( n \)-string trivial tangle is a
pair \((B_3, t)\) of the 3-ball \( B_3 \) and \( n \) arcs properly embedded in \( B_3 \) parallel to the boundary
of \( B_3 \). We call a link \( L \) an \( n \)-bridge link if \( L \) admits an \( n \)-bridge sphere and does not admit
an \((n-1)\)-bridge sphere. Two \( n \)-bridge spheres \( S_1 \) and \( S_2 \) of \( L \) are said to be pairwise
isotopic (isotopic, in brief) if there exists a homeomorphism \( f: (S^3, L) \to (S^3, L) \) such that
\( f(S_1) = S_2 \) and \( f \) is pairwise isotopic to the identity, i.e., there is a continuous family of
homeomorphisms \( f_t: (S^3, L) \to (S^3, L) \) \((0 \leq t \leq 1)\) such that \( f_0 = f \) and \( f_1 = \text{id} \).

It is known by Otal [20] and [21] that the unknot (resp. any 2-bridge link) admits a
unique \( n \)-bridge sphere up to isotopy for \( n \geq 1 \) (resp. \( n \geq 2 \)). These results were recently
refined by Scharlemann and Tomova [23]. The author constructed an infinite family of
links each of which admits infinitely many 3-bridge spheres up to isotopy in [9], and gave
a classification of 3-bridge spheres of 3-bridge arborescent links in [11]. In this paper, we
prove the following theorem.

Theorem 1.1 Let \( L \) be a prime, unsplittable link in \( S^3 \). Then \( L \) admits infinitely many
3-bridge spheres up to isotopy if and only if \( L \) is equivalent to a link \( L(q/2p; \beta_1/\alpha_1, \beta_2/\alpha_2) \)
(see Figure 1) with \( q \neq 1 \) \((\text{mod } p)\) and \( |\alpha_1| > 1 \) \((\text{or } |\alpha_2| > 1)\).

Here, two links are said to be equivalent if there exists an orientation-preserving homeo-

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morphism of \( S^3 \) which sends one to the other. The link \( L(q/2p; \beta_1/\alpha_1, \beta_2/\alpha_2) \) in Figure
1 is obtained as follows. Let \( V_0 \) be a solid torus standardly embedded in \( S^3 \) and \( L_0 \) a link
in \( V_0 \) obtained by connecting two rational tangles of slopes \( \beta_1/\alpha_1 \) and \( \beta_2/\alpha_2 \) by “trivial
Let $K_1 \cup K_2$ be a 2-bridge link in $S^3$ of type $(2p,q)$ and $V_1$ the regular neighborhood of $K_1$. For $i = 0, 1$, let $l_i$ be the preferred longitude of $V_i$, that is, $l_i$ is an essential loop on $\partial V_i$ which is null-homologous in $S^3 \setminus \text{Int}(V_i)$. Let $h : V_0 \to V_1$ be a homeomorphism which carries $l_0$ to $l_1$. We denote by $L(q/2p; \beta_1/\alpha_1, \beta_2/\alpha_2)$ the union of $h(L_0)$ and $K_2$.

Remark 1.2 We can also see that any 3-bridge sphere of $L(q/2p; \beta_1/\alpha_1, \beta_2/\alpha_2)$ is isotopic to $P_i$ for some integer $i$, where $P_i$ is obtained from $P_0$ by applying the $i$-th power of the “half Dehn twist” along the torus $T$ as illustrated in Figure 1 (see [9] for detailed description of $P_i$). This implies that any prime, unsplittable link admits only finitely many 3-bridge spheres up to homeomorphism.

Theorem 1.1 gives a partial answer to an analogy of the Waldhausen conjecture in terms of knot theory, namely, a prime, unsplittable link with atoroidal complement admits only finitely many $n$-bridge spheres up to isotopy for a given $n \in \mathbb{N}$. The (original) Waldhausen conjecture asserts that a closed orientable atoroidal 3-manifold admits only finitely many Heegaard splittings of given genus $g \in \mathbb{N}$ up to isotopy and was proved to be true by Johannson [13] and Li [17].

## 2 Heegaard splittings of 3-manifolds

Let $M$ be a closed orientable 3-manifold. A genus-$g$ Heegaard splitting of $M$ is a tuple $(V_1, V_2; F)$, where $V_1$ and $V_2$ are genus-$g$ handlebodies in $M$ such that $M = V_1 \cup V_2$ and $F = \partial V_1 = \partial V_2 = V_1 \cap V_2$. Two Heegaard splittings $(V_1, V_2; F)$ and $(W_1, W_2; G)$ of a 3-manifold $M$ are said to be isotopic if there exists a self-homeomorphism $f$ of $M$ such that $f(F) = G$ and $f$ is isotopic to the identity map $\text{id}_M$ on $M$.

For a genus-2 Heegaard splitting $(V_1, V_2; F)$ of $M$, it is known that there is an involution $\tau_F$ on $M$ satisfying the following condition.

(*) $\tau_F(V_i) = V_i$ ($i = 1, 2$) and $\tau_F|_{V_i}$ is equivalent to the standard involution $\mathcal{T}$ on a standard genus-2 handlebody $V$ as illustrated in Figure 2. To be precise, there is a homeomorphism $\psi_i : V_i \to V$ such that $\mathcal{T} = \psi_i(\tau_F|_{V_i})\psi_i^{-1}$ ($i = 1, 2$).
The strong equivalence class of \( \tau_F \) is uniquely determined by the isotopy class of \((V_1, V_2; F)\) (cf. [10] Proposition 5) and we call \( \tau_F \) the hyper-elliptic involution associated with \((V_1, V_2; F)\) (or associated with \( F \), in brief). Here, two involutions \( \tau \) and \( \tau' \) are said to be strongly equivalent if there exists a homeomorphism \( h \) on \( M \) such that \( h\tau h^{-1} = \tau' \) and that \( h \) is isotopic to the identity map \( \text{id}_M \).

Let \( L \) be a prime, unsplittable 3-bridge link. Let \( M \) be the double branched covering of \( S^3 \) branched along \( L \) and \( \tau_L \) the covering transformation. Let \( \Phi_L \) be the natural map from the set of isotopy classes of 3-bridge spheres of \( L \) to the set of isotopy classes of genus-2 Heegaard surfaces of \( M \) whose hyper-elliptic involution is \( \tau_L \). The following proposition is proved in [11].

**Proposition 2.1** \( \Phi_L \) is at most \( 2^{-1} \). Moreover, \( \Phi_L \) is injective if \( L \) is not a non-elliptic Montesinos link.

In the rest of this section, we recall a characterization of genus-2 3-manifolds admitting nontrivial torus decompositions due to Kobayashi [15] (and [16]). We use the following notation.

- \( D[r] \) (resp. \( M\tilde{o}[r], A[r] \)) : the set of all orientable Seifert fibered spaces over a disk \( D \) (resp. a Möbius band \( M\tilde{o} \), an annulus \( A \)) with \( r \) exceptional fibers.
- \( M_K \) : the set of the exteriors of the nontrivial 2-bridge knots.
- \( M_L \) : the set of the exteriors of the nontrivial 2-component 2-bridge links.
- \( L_K \) : the set of the exteriors of the 1-bridge knots in lens spaces each of which admits a complete hyperbolic structure or admits a Seifert fibration whose regular fiber is not a meridian loop.
- \( KI \) : the twisted \( I \)-bundle on the Klein bottle.

If \( K \) is a 2-bridge knot (resp. a 2-bridge link, a 1-bridge knot in a lens space), \( E(K) \) denotes a manifold in \( M_K \) (resp. \( M_L, L_K \)) obtained as the exterior of \( K \).

**Theorem 2.2** Let \( M \) be a closed, connected, orientable Haken 3-manifold of Heegaard genus 2 which admits a nontrivial torus decomposition. Let \((V_1, V_2; F)\) be a genus-2 Heegaard splitting of \( M \). Then \( M \) satisfies one of the following four conditions (1), (2), (3) and (4), and \( F \) is isotopic to a Heegaard surface, denoted by the same symbol \( F \), as follows (see Figure [5]).

1. \( M \) is obtained from \( M_1 \in D[2] \) and \( M_2 = E(K) \in L_K \) by identifying their boundaries so that the regular fiber of \( M_1 \) is identified with the meridian loop of \( K \). Moreover,
Figure 3:

- $M_1 \cap F$ is an essential annulus saturated in the Seifert fibration of $M_1$, and
- $M_2 \cap F$ is a 2-holed torus which gives a 1-bridge decomposition of the 1-bridge knot $K$.

Moreover, $V_i \cap T$ ($i = 1, 2$) consists of a single separating essential annulus, where $T = \partial M_1 = \partial M_2$.

(2) $M$ is obtained from $M_1 \in D[2] \cup D[3]$ and $M_2 = E(K) \in M_K$ by identifying their boundaries so that the regular fiber of $M_1$ is identified with the meridian loop of $K$. Moreover,

- $M_1 \cap F$ consists of two disjoint essential saturated annuli in $M_1$ which divide $M_1$ into three solid tori, and
- $M_2 \cap F$ is a 2-bridge sphere of the nontrivial 2-bridge knot $K$.

Moreover, by exchanging $V_1$ and $V_2$ if necessary,

(i) $V_1 \cap T$ consists of two disjoint non-separating essential annuli satisfying the following condition: there exists a complete meridian disk system $(D_1, D_2)$ of $V_1$ such that $D_1 \cap (V_1 \cap T) = \emptyset$ and $D_2 \cap (V_1 \cap T)$ consists of essential arcs properly embedded in each annulus of $V_1 \cap T$, and

(ii) $V_2 \cap T$ consists of disjoint non-parallel separating essential annuli,

where $T = \partial M_1 = \partial M_2$.

(3) $M$ is obtained from

(3-1) $M_1 \in M_{\mathbb{R}}(r)$ ($r = 0, 1, 2$) and $M_2 = E(K) \in M_K$, or
(3-2) $M_1 \in A[r]$ ($r = 0, 1, 2$) and $M_2 = E(K) \in M_L$

by identifying their boundaries so that the regular fiber of $M_1$ is identified with the meridian loop of $K$. Moreover,
• $M_1 \cap F$ consists of two disjoint essential saturated annuli in $M_1$ which divide $M_1$ into two solid tori, and

• $M_2 \cap F$ is a 2-bridge sphere of the 2-bridge link $K$.

Moreover, $V_i \cap T$ ($i = 1, 2$) consists of two disjoint non-separating essential annuli satisfying the condition (i) of (2), where $T = \partial M_1 = \partial M_2$.

(4) $M$ is obtained from $M_1, M_2 \in D[2]$ and $M_3 = E(K_1 \cup K_2) \in M_L$ by identifying their boundaries so that the regular fiber of $M_i$ is identified with the meridian loop of $K_i$ ($i = 1, 2$). Moreover,

• $M_i \cap F$ is an essential saturated annulus in $M_i$ ($i = 1, 2$), and

• $M_3 \cap F$ is a 2-bridge sphere of the 2-bridge link $K_1 \cup K_2$.

Moreover, $V_i \cap T$ ($i = 1, 2$) consists of two disjoint non-parallel separating essential annuli satisfying the condition (ii) of (2), where $T = \partial M_1 \cup \partial M_2 = \partial M_3$.

Proof

Let $\Gamma$ be the union of tori which gives the torus decomposition of $M$. If each component of $\Gamma$ is separating, then we see from the proof of the main theorem of [15] that $M$ and $F$ satisfies one of the conditions (1), (2), (3-1) and (4), where $\Gamma = T$. In the rest of this proof, we assume that $\Gamma$ has a non-separating component and show that $M$ and $F$ satisfy the condition (3-2). By the proof of the main theorem of [15], $M$ satisfies the condition (3-2). In particular, $M$ is obtained by gluing $M_1 \in A[r]$ ($r = 0, 1, 2$) and $M_2 = E(K_1 \cup K_2)$, where $K_1 \cup K_2$ is a nontrivial 2-bridge link. In the rest of this proof, we see that $F$ satisfies the condition (3-2) in this case.

First assume that neither $M_1$ nor $M_2$ is homeomorphic to $S^1 \times S^1 \times I$. Then $\Gamma$ consists of two components. By an argument similar to that for the main theorem of [15], together with Lemmas 3.1, 3.2 and 3.3 of [16], we can see that $F$ satisfies the condition (3-2).

In the remainder of this proof, assume that either $M_1$ or $M_2$ is homeomorphic to $S^1 \times S^1 \times I$. Then we may assume that $\Gamma$ is a component of $T$.

If $M_1$ is homeomorphic to $S^1 \times S^1 \times I$, then $M \setminus \Gamma$ is homeomorphic to the interior of $M_2$. By an argument similar to that for the main theorem of [15], together with Lemmas 3.1, 3.2 and 3.3 of [16], we can see that $\Gamma \cap V_i$ is a non-separating annulus as illustrated in Figure 4. Let $\tau_F$ be the hyper-elliptic involution associated with $F$. Then $\tau_F(\Gamma)$ is an essential torus in $M \setminus \Gamma$ (see Figure 4). Note that $M_2$ is homeomorphic to $A(1/n)$ for some integer $n$ or is hyperbolic (see [15] Lemma 4.4 and see [10] Section 2 for notation). Thus any essential torus in $M_2$ is $\partial$-parallel, and hence, $\tau_F(\Gamma)$ is isotopic to $\Gamma$ in $M$. This implies that $T$ is isotopic to $\Gamma \cup \tau_F(\Gamma)$ since $M_1$ is homeomorphic to $S^1 \times S^1 \times I$, and we see that $F$ satisfies the condition (3).
If $M_2 = E(K_1 \cup K_2)$ is homeomorphic to $S^1 \times S^1 \times I$, then $K_1 \cup K_2$ is a Hopf link and $M \setminus \Gamma$ is homeomorphic to the interior of $M_1$. By an argument similar to that in the previous case, we see that $\Gamma \cap V_i$ is a non-separating annulus as illustrated in Figure 4 and that $\tau_F(\Gamma)$ is also a non-separating essential torus in $M \setminus \Gamma$. If $M \setminus \Gamma(\cong M_1) \in A[r]$ for $r \leq 1$, then any essential torus in $M_1$ is $\partial$-parallel, and hence, we see that $\Gamma$ satisfies the condition (3) by an argument similar to that in the previous case. (We use the same symbol $M \setminus \Gamma$ to denote the manifold obtained by closing-up $M \setminus \Gamma$ with two tori.) If $M \setminus \Gamma(\cong M_1) \in A[2]$, then any essential torus in $M_1$ is either $\partial$-parallel or an essential torus which divides $M_1$ into two Seifert fibered spaces belonging to $A[1]$ whose fibrations are identical on $\tau_1(F)$, this implies that the meridian of a component of $L'$ is a regular fiber of $E(L'(\cong A[1]))$. However, this is impossible (cf. [10, Lemma 1] or [15, Lemma 4.4]). Hence, any essential torus in $M_1$ is $\partial$-parallel, and we see that $\Gamma$ satisfies the condition (3).

We need to study the manifolds satisfying one of the following conditions in the proof of Theorem 1.1 (cf. Section 4).

\begin{enumerate}
  \item [(M1)] $M$ is obtained by gluing $M_1 \in D[2]$ and $M_2 = L(p, q) \setminus N(K)$ as in Theorem 2.2 (1), where $M_2$ is hyperbolic,
  \item [(M2)] $M$ is obtained by gluing $M_1 \in D[2] \cup D[3]$ and $M_2 = E(K) \in M_K$ as in Theorem 2.2 (2), where $M_2$ is hyperbolic,
  \item [(M3-1-1)] $M$ is obtained by gluing $M_1 \in M[r]$ \((r = 1, 2)\) and $M_2 = E(K) \in M_K$ as in Theorem 2.2 (3), where $K$ is a torus knot of type \((2, n)\),
  \item [(M3-1-2)] $M$ is obtained by gluing $M_1 \in M[r]$ \((r = 0, 1, 2)\) and $M_2 = E(K) \in M_K$ as in Theorem 2.2 (3), where $M_2$ is hyperbolic,
  \item [(M3-2-1)] $M$ is obtained by gluing $M_1 \in A[r]$ \((r = 0, 1, 2)\) and $M_2 = E(K) \in M_L$ as in Theorem 2.2 (3), where $K$ is a torus link of type \((2, n)\),
  \item [(M3-2-2)] $M$ is obtained by gluing $M_1 \in A[r]$ \((r = 0, 1, 2)\) and $M_2 = E(K) \in M_L$ as in Theorem 2.2 (3), where $M_2$ is hyperbolic,
  \item [(M4)] $M$ is obtained by gluing $M_1, M_2 \in D[2]$ and $M_3 = E(K) \in M_K$ as in Theorem 2.2 (4), where $M_3$ is hyperbolic.
\end{enumerate}

Remark 2.3 The double branched covering of $S^3$ branched over $L(q/2p; \beta_1/\alpha_1, \beta_2/\alpha_2)$ satisfies the condition (M3-1-2) or (M3-2-2), where $r \geq 1$ (cf. [9]).

3 Mapping class groups

In this section, we calculate certain subgroups of the mapping class groups of the Seifert fibered spaces and the manifolds which arose in Theorem 2.2. This enables us to compare the hyper-elliptic involutions of genus-2 Heegaard surfaces of 3-manifolds. For a hyperbolic
3-manifold $N$, let $\mathcal{M}(N)$ be the (orientation-preserving) mapping class group of $N$. For a Seifert fibered space $N$, let $\mathcal{M}(N)$ be the subgroup of the (orientation-preserving) mapping class group of $N$ which consists of elements preserving each singular fiber of $N$. (See [10] for more details.) When $N$ is a Seifert fibered space over a surface $F$, let $\mathcal{M}^0(N)$ be the subgroup of $\mathcal{M}(N)$ which consists of the elements inducing the identity map on $F$, $\mathcal{M}^*(F)$ the mapping class group of $(F, \text{exceptional points})$. For a 3-manifold $M$ in Theorem 2.2, let $\mathcal{M}(M)$ be the subgroup of the (orientation-preserving) mapping class group of $M$ which consists of the elements preserving each piece of the torus decomposition of $M$ and each singular fiber of the Seifert pieces.

We describe some elements of the mapping class groups of certain Seifert fibered spaces. Let $N_1$ and $N_2$ be the Seifert fibered spaces $M\tilde{o}(\beta_1/\alpha_1, \beta_2/\alpha_2) \in M\tilde{o}[2]$ and $A(\beta_1/\alpha_1, \beta_2/\alpha_2) \in A[2]$, respectively (see [10] Section 2 for notation). We define $g_i, h_i, b \in \mathcal{M}(N_1)$ and $a, D_j \in \mathcal{M}(N_2)$ ($i, j \in \{1, 2\}$) as follows. We denote by $g_i$ and $h_i$ the involutions as illustrated in Figure 5. The symbols $a$ and $b$ denote the Dehn twist along saturated annuli $A_a$ and $A_b$, respectively, in the direction of a fiber, and $D_j$ are the Dehn twists along saturated tori $T_{D_j}$, respectively, in the direction of loops intersecting regular fibers in one point, as illustrated in Figure 5. For more precise description of the above elements, see [10] Section 5 and Remark 2.

**Lemma 3.1**

(1) If $N$ is a Seifert fibered space $M\tilde{o}(\beta_1/\alpha_1, \beta_2/\alpha_2) \in M\tilde{o}[2]$, then $\mathcal{M}(N) = \langle b \rangle \times \langle D_1, D_2 \rangle \times \langle g_1, g_2 \rangle$ and has a group presentation

$$\mathcal{M}(N) = \langle D_1, D_2, g_1, g_2, b \mid g_1^2, [g_1, g_2], g_1D_1g_1 = D_1^{-1}, g_2D_2g_2 = D_2^{-1}, b^2, [g_1, b], [D_j, b] \ (i, j \in \{1, 2\}) \rangle.$$ 

In particular, the subgroup $\langle D_1, D_2 \rangle$ of $\mathcal{M}(N)$ is a free group of rank 2.

(2) If $N$ is a Seifert fibered space $A(\beta_1/\alpha_1, \beta_2/\alpha_2) \in A[2]$, then $\mathcal{M}(N) = \langle a \rangle \times \langle D_1', D_2' \rangle \times \langle g_1, g_2 \rangle$.
(h₁, h₂) and has a group presentation

\[ \mathcal{M}(N) = \langle D'_1, D'_2, h_1, h_2, a \mid h_1^2[h_1, h_2], h_1D'_1h_1 = D'_j^{-1}, h_2D'_1h_2 = D'_2^{-1}, h_1a = h_2ah_2 = a^{-1}, D'_jaD'_j^{-1} = a \ (i, j \in \{1, 2\}) \rangle. \]

In particular, the subgroup \( \langle D'_1, D'_2 \rangle \) of \( \mathcal{M}(N) \) is a free group of rank 2.

**Proof** By [12] Proposition 25.3, we have a split exact sequence

\[ 1 \to \mathcal{M}^0(N) \to \mathcal{M}(N) \to \mathcal{M}^*(F) \to 1. \]

(1) By [12] Lemma 25.2, \( \mathcal{M}^0(N) \) is an order-2 group generated by \( b \). On the other hand, by [11] Section 4.1, we have the following exact sequence, called the “Birman exact sequence”.

\[ 1 \to \pi_1(F', x_0) \to \mathcal{M}^*(F)(\cong M_2) \to \mathcal{M}_1 \to 1, \]

where \( \pi_1(F', x_0) \) denotes the fundamental group of a once-punctured Möbius band and \( \mathcal{M}_n \) denotes the mapping class group of a Möbius band fixing \( n \) specified points. Recall that \( \pi_1(F', x_0) \) is a free group of rank 2 and that \( \mathcal{M}_1 = \langle g_1, g_2 \rangle \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \) (cf. [10] Lemma 4). Moreover, we may take the images of the projection map as the generators of \( \mathcal{M}_1 \). Then their images in \( \mathcal{M}^*(F) \), by the second map in the above exact sequence, are \( D_1 \) and \( D_2 \). Moreover, the conjugation of \( D_j \) by \( g_i \) \((i, j \in \{1, 2\})\) is as follows:

\[ g_1D_jg_1 = D_j^{-1}, \quad g_2D_1g_2 = D_2. \]

Hence, by using an argument in [14] p.136–139, we obtain the following group presentation of \( \mathcal{M}^*(F) \).

\[ \mathcal{M}^*(F) = \langle D_1, D_2, g_1, g_2 \mid g_1^2, [g_1, g_2], g_1D_jg_1 = D_j^{-1}, g_2D_1g_2 = D_2 \ (i, j \in \{1, 2\}) \rangle. \]

Since the conjugation of \( b \) by \( D_j \) or \( g_1 \) is \( b \), we obtain the desired result by using an argument in [14] p.136–139 again.

(2) can be proved similarly. \( \square \)

Let \( M \) be a manifold in Theorem 2.2 and \( T \) the union of tori in the theorem. Let \( D \) be the subgroup of \( \mathcal{M}(M) \) generated by the all possible Dehn twists along \( T \). Then we obtain the following, which can be proved by an argument similar to that for [5] Proposition 15.2 or [10] Lemma 3.

**Lemma 3.2** Let \( M \) be a manifold in Theorem 2.2.

(1) If \( M \) satisfies the condition (M1) or (M2), then \( D \) is an infinite cyclic group generated by \( D_1 \), where \( D_1 \) is the Dehn twist along (a component of) \( T \) in the direction of a longitude of \( K \).

(2) If \( M \) satisfies the condition (M3-1-1), then \( D \cong \langle D_1 \rangle \cong \mathbb{Z} \), where \( D_1 \) is the Dehn twist along (a component of) \( T \) in the direction of a longitude of \( K \).

(3) If \( M \) satisfies the condition (M3-1-2), then \( D \) is generated by \( D_m \) and \( D_1 \), where \( D_m \) and \( D_1 \) are the Dehn twists along \( T \) in the direction of a meridian and a longitude of \( K \), respectively. Moreover, \( D \cong \langle D_m, D_1 \rangle \cong \mathbb{Z} \oplus \mathbb{Z} \) otherwise.

(4) If \( M \) satisfies the condition (M3-2-1), namely, \( M \) is obtained by gluing \( M_1 \in A[r] \) \((r = 0, 1, 2)\) and \( M_2 = E(K) = A(1/n) \) so that the regular fibers of \( M_1 \) are identified with
the meridians of $K$, then $\mathcal{D}$ is an abelian group generated by $D_m$ and $D_l$, where $D_m$ and $D_l$ are the Dehn twists along (a component of) $T$ in the direction of a meridian and a longitude of $K$, respectively.

(5) If $M$ satisfies the condition (M3-2-2), then $\mathcal{D}$ is generated by $D_{m_1}, D_{l_1}$ and $D_{l_2}$, where $D_{m_1}$ and $D_{l_1}$ are the Dehn twists along a component $T_i$ ($i = 1, 2$) of $T$ in the direction of a meridian and a longitude of $K$, respectively. Moreover, $\mathcal{D} \cong \langle D_{m_1}, D_{l_1} \rangle \cong \mathbb{Z} \oplus \mathbb{Z}$ if $M_1 \in \mathbb{A}[0]$, and $\mathcal{D} \cong \langle D_{m_1}, D_{l_1}, D_{l_2} \rangle \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ otherwise.

(6) If $M$ satisfies the condition (M4), then $\mathcal{D} \cong \langle D_{l_1}, D_{l_2} \rangle \cong \mathbb{Z} \oplus \mathbb{Z}$, where $D_{l_i}$ is the Dehn twist along a component $T_i$ ($i = 1, 2$) of $T$ in the direction of a longitude of $K$.

We define some self-homeomorphisms of $M$ when $M$ satisfies the condition (M3-1-1) or (M3-2-1) as follows.

**Definition 3.3** Let $M$ be a manifold which satisfies the condition (M3-1-1) or (M3-2-1) with $r = 2$.

(1) When $M$ satisfies the condition (M3-1-1), we define self-homeomorphisms $G_1$, $G_2$ and $B$ of $M$ as follows.

$$
G_1|_{M_1} = g_1, \quad G_1|_{M_2} = f, \quad G_1|_{T \times [1, 2]} = R, \\
G_2|_{M_1} = g_2, \quad G_2|_{M_2} = id, \quad G_2|_{T \times [1, 2]} = R_lD_1^{1/2}, \\
B|_{M_1} = b, \quad B|_{M_2} = id, \quad B|_{T \times [1, 2]} = R_mD_3^{1/2}, \\
D_j|_{M_1} = D_j, \quad D_j|_{M_2} = id, \quad D_j|_{T \times [1, 2]} = id \quad (j = 1, 2).
$$

Here, $g_1$, $g_2$ and $b$ are involutions of $M_1$ as described in Lemma 3.1 $f$ is an involution of $M_2 = E(K)$ which gives a strong inversion of the torus knot $K$ (see [10] Remark 7), and $R$ and $R_\alpha$ ($\alpha = m$ or $l$) are the self-homeomorphisms of $T \times [1, 2]$ defined by $R([\bar{x}], t) = ([\bar{x}], t)$ and $R_\alpha([\bar{x}], t) = ([\bar{x} + \frac{1}{2} \bar{d}], t)$, respectively. Here, we identify $T$ with $\mathbb{R}^2/\mathbb{Z}^2$ and $[\bar{x}]$ denotes the point of $\mathbb{R}^2/\mathbb{Z}^2$ determined by $\bar{x} \in \mathbb{R}^2$. In the identity $D_j|_{M_1} = D_j$, the right-hand side represents the homeomorphisms in Lemma 3.1 (1). The symbols $D_l$ and $D_m$ denote the Dehn twists given in Lemma 3.2 (2) and (4), and $D_l^{1/2}$ and $D_m^{1/2}$ denote the half Dehn twists in the direction of $l$ and $m$, respectively (see [10] Section 5) for definition of half Dehn twists.

(2) When $M$ satisfies the condition (M3-2-1), we define self-homeomorphisms $H_1$ and $H_2$ of $M$ as follows.

$$
H_1|_{M_1} = h_1, \quad H_1|_{M_2} = h_1, \quad H_1|_{T \times [1, 2]} = R, \\
H_2|_{M_1} = h_2, \quad H_2|_{M_2} = h_2, \quad H_2|_{T \times [1, 2]} = h_2|_{T \times [1, 2]}, \\
D_j'|_{M_1} = D_j', \quad D_j'|_{M_2} = id, \quad D_j'|_{T \times [1, 2]} = id.
$$

Here, $h_1$ is an involution of $M_1$ as described in Lemma 3.1 $h_2$ is an involution of $M_2 = E(K)$ which gives a strong inversion of the torus link $K$ (see [10] Lemma 4 (3)), and $R$ is the self-homeomorphism of $T \times [1, 2]$ defined in (1). In the identity $D_j'|_{M_1} = D_j'$, the right-hand side represents the homeomorphisms in Lemma 3.1 (2).

In [10] Proposition 6], the author calculated $\mathcal{M}(M)$ for certain manifolds in Theorem 2.2 by using [3] Theorem 15.1 and [22]. The following theorem can be obtained by a similar argument together with Lemmas 3.1 and 3.2.
**Theorem 3.4** Let $M$ be a manifold which satisfies the condition (M3-1-1) or (M3-2-1) with $r = 2$.

1. If $M$ satisfies the condition (M3-1-1), then the subgroup $\langle D_1, D_2 \rangle$ of $\mathcal{M}(M)$ generated by $D_1$ and $D_2$ is a free group of rank 2.

2. If $M$ satisfies the condition (M3-2-1), then the subgroup $\langle D_1', D_2', D_1 \rangle$ of $\mathcal{M}(M)$ is the direct product of the infinite cyclic group generated by $D_1$ and the free group of rank 2 generated by $D_1'$ and $D_2'$.

**Proof** Recall from [5 Theorem 15.1] and [22 (cf. [10, Section 5])] that there is an exact sequence

$$1 \to D \to \mathcal{M}(M) \to \Delta \to 1,$$

(1)

where $\Delta$ is the subgroup of $\mathcal{M}(M_1) \times \mathcal{M}(M_2)$ consisting of all elements $(f_1, f_2)$ such that $f_1|_\tau$ is isotopic to $f_2|_\tau$.

1. Let $M$ be a manifold which satisfies the condition (M3-1-1). Then we have $\mathcal{M}(M_1) = \langle b \rangle \times \langle (D_1, D_2) \rangle$ by Lemma 3.11 (1), and $\mathcal{M}(M_2) = \langle f \rangle \cong \mathbb{Z}_2$ by [10] Lemma 4 (1). Note that the subgroup of $\mathcal{M}(M_1)$ generated by $D_1$ and $D_2$ is a free group of rank 2. Hence, we see that

$$\Delta = \langle (b, id) \rangle \times \langle (D_1, id), (D_2, id) \rangle \times \langle (g_1, f), (g_2, id) \rangle$$

and the subgroup of $\Delta$ generated by $(D_1, id)$ and $(D_2, id)$ is also a free group of rank 2. Hence, we see from the exact sequence (1) that the subgroup of $\mathcal{M}(M)$ generated by $D_1$ and $D_2$ is a free group of rank 2.

2. Let $M$ be a manifold which satisfies the condition (M3-2-1). Then we have $\mathcal{M}(M_1) = \langle a \rangle \times \langle (D_1', D_2') \times \langle h_1, h_2 \rangle \rangle$ by Lemma 3.11 (2). On the other hand, we have $\mathcal{M}(M_2) = \langle a \rangle \times \langle h_1, h_2 \rangle$ (cf. [10] Lemma 4 (2)). Note that the subgroup of $\mathcal{M}(M_1)$ generated by $D_1'$ and $D_2'$ is a free group of rank 2. We see that

$$\Delta = \langle (D_1', id), (D_2', id) \rangle \times \langle (h_1, h_1), (h_2, h_2) \rangle$$

and the subgroup of $\Delta$ generated by $(D_1', id)$ and $(D_2', id)$ is also a free group of rank 2. Hence, we see from the exact sequence (1) that the subgroup of $\mathcal{M}(M)$ generated by $D_1'$ and $D_2'$ is a free group of rank 2. On the other hand, the subgroup of $\mathcal{M}(M)$ generated by $D_i$ is an infinite cyclic group by Lemma 3.2 (3). Since we can easily see that $D_i$ commutes with $D_1'$ and $D_2'$, we obtain the desired result. □

**Remark 3.5** Let $M$ be a manifold in Theorem 3.4.

1. If $M$ satisfies the condition (M3-1-1), then $\mathcal{M}(M) = \langle B \rangle \times \langle (D_1, D_2) \rangle \times \langle (G_1, G_2) \rangle$, and has a group presentation

$$\mathcal{M}(M) = \langle D_1, D_2, G_1, G_2, B \ | \ G_1^2, [G_1, G_2], G_1D_1G_1 = D_1^{-1}, G_2D_1G_2 = D_2^{-1}, B^2 = D_m, [G_1, B] = D_m^{-1}, [G_2, B], [D_j, B] \ (i, j \in \{1, 2\}) \rangle.$$

2. If $M$ satisfies the condition (M3-2-1), then $\mathcal{M}(M) = \langle D_m, D_1 \rangle \times \langle (D_1', D_2') \times \langle (H_1, H_2) \rangle \rangle$, and has a group presentation

$$\mathcal{M}(M) = \langle D_1, D_2, H_1, H_2, D_m, D_l \ | \ H_1^2, [H_1, H_2], H_1D_1' H_1 = D_1'^{-1}, H_2D_1' H_2 = D_2'^{-1}, D_1'D_m D_1'^{-1} = D_m, D_1'D_l D_1'^{-1} = D_l, H_1D_m H_1 = D_m^{-1}, H_2D_m H_2 = D_m, H_1D_1H_1 = D_l^{-1} \ (i, j \in \{1, 2\}) \rangle.$$
4 Proof of Theorem 1.1

Since the if part is already proved in [9], we prove the only if part. Namely, we show that any prime, unsplittable 3-bridge link which admits infinitely many 3-bridge spheres up to isotopy is equivalent to a link $L(q/2p; \beta_1/\alpha_1, \beta_2/\alpha_2)$ in Figure 1 with $q \not\equiv 1 \pmod{p}$ and $|\alpha_1| > 1$ (or $|\alpha_2| > 1$).

Let $L$ be a prime, unsplittable 3-bridge link in $S^3$, and assume that $L$ admits infinitely many 3-bridge spheres up to isotopy. Let $M = M_2(L)$ be the double branched cover of $S^3$ branched along $L$ and $\tau_L$ the covering transformation. By Proposition 2.1, $M$ admits infinitely many genus-2 Heegaard surfaces, up to isotopy, whose hyper-elliptic involutions are $\tau_L$. By [17, Theorem 1.1], $M$ is toroidal, and hence, either $M$ is a Seifert fibered space or $M$ admits a nontrivial torus decomposition.

Case 1 $M$ is a Seifert fibered space.

By the orbifold theorem [3, 7] together with [8, Section 5], $L$ is a generalized Montesinos link or a Seifert link, that is, either $L$ is equivalent to a link in Figure 6 or $S^3 \setminus L$ admits a Seifert fibration.

Assume first that $L$ is a generalized Montesinos link. By [4, Theorem 2.1], $L$ is equivalent to one of the links in Figure 7 since $L$ is a 3-bridge link. By [10, 11], $L$ admits at most six 3-bridge spheres, at most two 3-bridge spheres or a unique 3-bridge sphere up to isotopy according as $L$ is equivalent to the link in Figure 7 (1), (2) or (3). This contradicts the assumption that $L$ admits infinitely many 3-bridge spheres up to isotopy.

Next, assume that $L$ is a Seifert link. By [6] and by the assumption that $L$ is a 3-bridge link, we see that $L$ is equivalent to a (nontrivial) $(3, n)$-torus link or the union of a $(2, n)$-torus knot and its core of index 2.

If $L$ is equivalent to a $(3, n)$-torus knot or the union of a $(2, n)$-torus knot and its core of index 2, then $M$ is a small Seifert fibered space and admits at most four genus-2 Heegaard
surfaces up to isotopy by [2], which is a contradiction. We also prove the following proposition in Section 5.

**Proposition 4.1** If $L$ is a $(3, 3n')$-torus link for some nonzero integer $n'$, then $L$ admits a unique 3-bridge sphere up to isotopy.

Hence, $M$ cannot be a Seifert fibered space.

**Case 2** $M$ admits a nontrivial torus decomposition.

If $L$ is an arborescent link, then $L$ admits at most four 3-bridge spheres by [11]. Hence, we assume that $L$ is not an arborescent link. Then, by Theorem 2.2 and [10, Proof of Theorem 1], $M$ satisfies one of the conditions (M1), (M2), (M3-1-1), (M3-1-2), (M3-2-1), (M3-2-2) and (M4) introduced at the end of Section 2. Let $T$ be the union of tori as in Theorem 2.2.

**Case 2.1** $M$ satisfies the condition (M1).

Note that $M$ is obtained by gluing $M_1 \in D[2]$ and $M_2 = L(p,q) \setminus N(K)$, where $K$ is a 1-bridge knot in a lens space $L(p,q)$, and that $M_2$ is hyperbolic. Since $M_1$ is also simple, we can see that $T = \partial M_1 = \partial M_2$ is the only essential torus in $M$ up to isotopy. By [13, Theorem 4], there exist genus-2 Heegaard surfaces $F_1, F_2, \ldots, F_n$ of $M$ such that any genus-2 Heegaard surface $F$ can be obtained from some $F_i$ by applying Dehn twists along $T$. Recall from Theorem 2.2 that $F \cap M_1$ is an essential saturated annulus of $M_1$ and $F \cap M_2$ is a 2-fold torus which gives a 1-bridge presentation of $K$. Let $\mu$ and $\lambda$ be the meridian and a longitude of $K$, and denote the Dehn twist along $T$ in the direction of $\mu$ and $\lambda$ by $D_\mu$ and $D_\lambda$, respectively. Then $F$ is isotopic to $D_\mu^{n_1}D_\lambda^{n_2}(F_i)$ for some integers $n_1$ and $n_2$. Note that any genus-2 Heegaard surface meets $T$ in the union of two meridians of $K$. Hence, $D_\mu^{n_1}D_\lambda^{n_2}(F_i) = D_\lambda^{n_2}(F_i)$. Note also that $\tau_{F_j}D_\lambda \tau_{F_j} = D_\lambda^{-1}$ because $\tau_{F_j}$ reverses the orientation of $\lambda$ (see Figure 8). Thus,

$$\tau_F = \tau_{D_\lambda^{n_2}(F_j)} = D_\lambda^{n_2}\tau_{F_j}D_\lambda^{-n_2} = D_\lambda^{2n_2}\tau_{F_j}.$$  

Since $D$ is an infinite cyclic group generated by $D_\lambda$ by Lemma 5.2 (1), $\{D_\lambda^{2n_2}\tau_{F_j}\}_{n_2 \in \mathbb{Z}}$ are mutually distinct, and hence, there is at most one $n_2 \in \mathbb{Z}$ such that $D_\lambda^{2n_2}\tau_{F_j} = \tau_L$. So, for each Heegaard surface $F_j$, the hyper-elliptic involution associated with $D_\lambda^{n_2}(F_j)$ is strongly equivalent to $\tau_L$ for at most one $n_2 \in \mathbb{Z}$. Hence, the number of genus-2 Heegaard surfaces whose hyper-elliptic involutions are $\tau_L$ is finite. This contradicts the assumption.

**Case 2.2** $M$ satisfies the condition (M2).
be the Heegaard surface $D$, elements whose restrictions to $M$ (cf. [10, Section 5]) that there is an exact sequence

Some $f \in M$ and for some $P$ is isotopic to $F$. Then we have $M_n$ and for some $G$ involutions are $\tau$. In Case 2.1, we can see that the number of genus-2 Heegaard surfaces such that $\tau$ is fixed. Then $T$ consists of all elements whose restrictions to $M$. Let $\lambda$ denote the Dehn twist along $\lambda$. First assume that $M$ contains a unique essential annulus up to isotopy. By [10, Lemma 6], and that there exist genus-2 Heegaard surfaces $F_0$, $F_1$, $F_2$ and $F_3$ of $M$ such that any genus-2 Heegaard surface of $M$ is isotopic to $D^{n/4}_\lambda(F_i)$ for some integer $n$ and for some $i = 0, 1, 2, 3$. (We remark that $F_i = D^{n/4}_\lambda(F_0)$.) Recall that $\tau g, D^{n/4}_\lambda = D^{-1}_\lambda$. By an argument similar to that in Case 2.1, we can see that the number of genus-2 Heegaard surfaces whose hyper-elliptic involutions are $\tau_L$ is finite, a contradiction.

Next, assume that $M_1 \in D[3]$. Note that $F \cap M_1$ is homeomorphic to one of $G_1, G_2$ and $G_3$ in Figure 9. To be precise, $F \cap M_1$ is isotopic to $f_1(G_i)$ for some $f_1 \in M(M_1)$ and for some $i = 1, 2, 3$. (We may assume that $f_1|_{\partial M_1} = \text{id.}$.) For each $i = 1, 2, 3$, let $F_i$ be a genus-2 Heegaard surface such that $F_i \cap M_1 = G_i$ and $F_i \cap M_2$ is the 2-bridge sphere of $K$. By [10, Lemma 6 (1)], any genus-2 Heegaard surface $F$ is isotopic to $D^{n/4}_\lambda f(F_i)$ for some integer $n$ and for some $i = 1, 2, 3$ and for some homeomorphism $f$ of $M$ which is obtained from some $f_1 \in M(M_1)$ by the rule $f|_{\partial M_1} = f_1 \in M(M_1)$ and $f|_{\partial M_2} = \text{id.}$ Let $F_i^j$ ($j = 0, 1, 2, 3$) be the Heegaard surface $D^{n/4}_\lambda(F_i)$. Let $M_0(M)$ be the subgroup of $M(M)$ consisting of all elements whose restrictions to $M_2$ are the identity. Then the above argument implies that $F$ is isotopic to $g(F_i^j)$ for some $g \in M_0(M)$ and for some $F_i^j$.

Claim 1 For each $F_i^j$, at most one of $\{g(F_i^j)\}_{g \in M_0(M)}$ can have $\tau_L$ as hyper-elliptic involution.

Proof We show this only for $F_i^0$. (The other cases can be treated similarly.) Put $\tau := \tau_{F_i^0}$. Then $\tau g(F_i^0) = g \tau g^{-1}$. Recall by [10, Proof of Theorem 2 (3)] that $M(M_1) \cong (P_3/(\langle xy \rangle^3)) \rtimes \langle \tau \rangle < (B_3/(\langle xy \rangle^3)) \rtimes \langle \tau \rangle$,

where $P_3$ and $B_3$ are the pure 3-braid group and the 3-braid group, respectively. Let $M_0(M_1)$ be the subgroup of $M(M_1)$ consisting of all elements whose restrictions to $T$ are the identity. Then we have $M_0(M_1) \cong P_3/(\langle xy \rangle^3)$. Recall from [5, Theorem 15.1] and [22] (cf. [10, Section 5]) that there is an exact sequence

$1 \to D \to M(M) \to \Delta \to 1,$
where $\Delta$ is the subgroup of $\mathcal{M}(M_1) \times \mathcal{M}(M_2)$ consisting of all elements $(f_1, f_2)$ such that $f_1|_\tau$ is isotopic to $f_2|_\tau$. Since $\mathcal{M}_0(M)$ is the subgroup of $\mathcal{M}(M)$ consisting of all elements whose restrictions to $M_2$ are the identity, we obtain an exact sequence

$$1 \to \mathcal{D} \to \mathcal{M}_0(M) \to \mathcal{M}_0(M_1) \to 1.$$ 

Recall from [10, Claim 1 (2)] that the “centralizer”

$$Z(\tau, \mathcal{M}_0(M_1)) = \{ f \in \mathcal{M}_0(M_1) \mid f\tau = \tau f \}$$

of $\tau$ in $\mathcal{M}_0(M_1)$ is $\{1\}$. By using this fact, the identity $D_\lambda \tau D_\lambda^{-1} = D_{\lambda^{-1}} \tau$ and Lemma 3.2 (1), we can see that the “centralizer” $Z(\tau, \mathcal{M}_0(M)) = \{ f \in \mathcal{M}_0(M) \mid f\tau = \tau f \}$ of $\tau$ in $\mathcal{M}_0(M)$ is $\{1\}$. This implies that the hyper-elliptic involution associated with $g(F_\lambda^0)$ is strongly equivalent to $\tau_L$ for at most one $g \in \mathcal{M}_0(M)$. \hfill $\square$

Hence, the number of genus-2 Heegaard surfaces of $M$ whose hyper-elliptic involutions are $\tau_L$ is at most twelve, a contradiction.

**Case 2.3** $M$ satisfies the condition (M3-1-1).

Recall that $M$ is obtained by gluing $M_1 \in \mathcal{M}_0[r]$ ($r = 1, 2$) and $M_2 = E(K)$, where $K$ is a $(2, n)$-torus knot, so that the regular fiber of $M_1$ is identified with the meridian loop of $K$. By Theorem [22] for any genus-2 Heegaard surface $F$, $F \cap M_1$ is the union of two essential saturated annuli which cuts $M_1$ into two solid tori and $F \cap M_2$ is a 2-bridge sphere of $K$.

Assume first that $r = 1$. Note that $M_1$ contains a unique essential saturated annulus up to isotopy and that $K$ admits a unique 2-bridge sphere up to isotopy preserving $K$ (see [19, Theorem 4]). Let $\mu$ and $\lambda$ be the meridian and a longitude of $K$, respectively. Let $F_0$ be a genus-2 Heegaard surface of $M$. Then, by [10, Lemma 6 (2)], any genus-2 Heegaard surface $F$ of $M$ is isotopic to $D_\lambda^n/F_0$ for some integer $n$. Note that $\mathcal{D}$ is the infinite cyclic group generated by $D_\lambda$ (see [10, Lemma 3]). Hence, by an argument similar to that in the previous cases, we see that the number of genus-2 Heegaard surfaces whose hyper-elliptic involutions are $\tau_L$ is finite, a contradiction.

Assume first that $r = 2$. Pick a “standard” genus-2 Heegaard surface $F_0$ of $M$, such that $F_0 \cap M_1$ is preserved by the homeomorphisms $g_1, g_2$ and $b$ in Lemma 3.1 (1). Then we may assume that $\tau_{F_0} = \tau_L = G_2$. By Theorem 3.4 (1) and [10, Lemma 6 (2)], any genus-2 Heegaard surface $F$ of $M$ is isotopic to $D_1^{n_1} D_2^{n_2} \cdots D_1^{n_1} D_2^{n_2} (F_0)$ for some integers $n_1, n_2, \ldots, n_{2m-1}$ and $n_{2m}$, where $n_2, \ldots, n_{2m-1}$ are nonzero. Then

$$\tau_F = D_1^{n_1} D_2^{n_2} \cdots D_1^{n_{2m-1}} D_2^{n_{2m}} G_2 D_2^{-n_{2m}} D_1^{n_{2m-1}} \cdots D_2^{-n_2} D_1^{-n_1}.$$ 

Since $G_2 D_1 G_2 = D_2^{-1}$ by Remark 3.3, we have

$$\tau_F = D_1^{n_1} D_2^{n_2} \cdots D_1^{n_{2m-1}} D_2^{n_{2m}} D_1^{n_{2m-1}} D_2^{n_{2m}} \cdots D_1^{n_2} D_2^{n_1} G_2.$$ 

By Theorem 3.4 (1), we can see that $\tau_F = \tau_{F_0}$ implies $n_1 = n_2 = 0$, which means $F$ is isotopic to $F_0$. This contradicts the assumption.

**Case 2.4** $M$ satisfies the condition (M3-1-2).
Note that $M$ is obtained by gluing $M_1 \in M_{\partial[r]}$ ($r = 0, 1, 2$) and $M_2 = E(K)$, where $K = S(p, q)$ is a hyperbolic 2-bridge knot, so that the regular fiber of $M_1$ is identified with the meridian loop of $K$. We may assume that $q$ is odd. Let $F$ be a genus-2 Heegaard surface of $M$. By Theorem 2.2, $F \cap M_1$ is the union of two essential saturated annuli which cuts $M_1$ into two solid tori and $F \cap M_2$ is a 2-bridge sphere of $K$. Note that $M_1/\langle \tau_F \rangle$ is a solid torus and that the image of $\text{Fix} \tau_F \cap M_1$ forms a link in it as illustrated in Figure 10. On the other hand, $M_2/\langle \tau_F \rangle$ is also a solid torus and the image of $\text{Fix} \tau_F \cap M_2$ forms a knot in it such that its exterior in $M_2$ is the exterior of a 2-bridge link $S(2p, q)$ in $S^3$ (cf. [9, Lemma 3.2]). Since the meridian and the longitude of the solid torus $M_1/\langle \tau_F \rangle$ are identified with the longitude and the meridian of the solid torus $M_2/\langle \tau_F \rangle$, respectively, we see that $L$ is equivalent to $L(q/2p; \beta_1/\alpha_1, \beta_2/\alpha_2)$. Moreover, since $K = S(p, q)$ is a hyperbolic 2-bridge knot, we have $q \not\equiv \pm 1 \pmod{p}$ by [18].

First assume that $r = 0$. Note that $M_1$ has a unique essential saturated annulus up to isotopy and that $K$ admits a unique 2-bridge sphere up to isotopy. Let $F_0$ be the pre-image of $P^0$ given in [9] (cf. Figure 4) by the covering map $M \to S^3$ (branched over $L$). By an argument similar to that in the previous cases, $F$ is isotopic to $D^{3/4}_\lambda(F_0)$ for some integer $n$, where $D_\lambda$ is the Dehn twist along a component of $T = \partial M_1 = \partial M_2$ in the direction of a longitude of $K$. However, $F = D^{3/4}_\lambda(F_0)$ is isotopic to $F_0$ since $D^{3/4}_\lambda(F_0) \cap M_1$ can be isotoped to $F_0 \cap M_1$ by an isotopy fixing the boundary of $M_1$ as illustrated in Figure 11. Thus $M$ admits a unique genus-2 Heegaard surface up to isotopy, a contradiction.

Assume that $r = 1$ or $r = 2$. Then, we see by [9] that $L$ is equivalent to a link $L(q/2p; \beta_1/\alpha_1, \beta_2/\alpha_2)$ in Figure 4 and that $L$ admits infinitely many 3-bridge spheres $\{P^i\}_{i \in \mathbb{Z}}$ up to isotopy. (Moreover, we can see that any 3-bridge sphere is isotopic to $P^i$ for some $i \in \mathbb{Z}$ by using an argument similar to that in the previous cases and by Lemma 3.1.)

**Case 2.5** $M$ satisfies the condition (M3-2-1) or (M3-2-2).

Note that $M$ is obtained by gluing $M_1 \in A[r]$ ($r = 0, 1, 2$) and $M_2 = E(K)$, where $K$
is a 2-bridge link, so that the regular fiber of $M_1$ is identified with the meridian loop of $K$. Let $F$ be a genus-2 Heegaard surface of $M$. By Theorem 2.2, $F \cap M_1$ is the union of two essential saturated annuli which cuts $M_1$ into two solid tori and $F \cap M_2$ is a 2-bridge sphere of $K$. If $M_1$ is homeomorphic to a 2-bridge knot exterior, then $F$ can intersect each $M_i$ so that $F \cap M_1$ is a 2-bridge sphere and $F \cap M_2$ is the union of two essential saturated annuli.)

Pick a “standard” genus-2 Heegaard surface $F_0$ of $M$ and assume that $\tau_{F_0} = \tau_L$. By using an argument similar to that in the previous cases, we see the following hold.

- If $M$ satisfies the condition (M3-2-1), where $r = 0$ or 1, then $F$ is isotopic to $D_1^{n/2}(F_0)$ for some integer $n$, where $\lambda$ is a longitude or a meridian of $K$ according as $F_0$ meets $T := \partial M_1 = \partial M_2$ in a meridian or a longitude of $K$. Note that the subgroup of $D$ generated by $D_{\lambda}$ is finite or an infinite cyclic group. If the subgroup is finite, then $M$ admits only finitely many genus-2 Heegaard surfaces up to isotopy, a contradiction. If the subgroup is an infinite cyclic group, then, by an argument similar to that in Cases 2.1 together with Lemma 3.2 (4), we see that the number of genus-2 Heegaard surfaces of $M$ whose hyper-elliptic involutions are $\tau_L$ is finite, a contradiction.

- If $M$ satisfies the condition (M3-2-1), where $r = 2$, then we see that $F$ is isotopic to $D_1^{n_1/2} D_2^{n_2} \cdots D_1^{n_m/2}(F_0)$ for some integers $n_i$ ($i = 0, 1, \ldots, 2m$) by using Theorem 2.4 (2). By an argument similar to that in Case 2.4 together with Theorem 3.4 (2), we see that $\tau_F = \tau_{F_0}(= \tau_L)$ implies $n_i = 0$ for all $i = 0, 1, \ldots, 2m$. Hence, $F$ is isotopic to $F_0$, a contradiction.

- If $M$ satisfies the condition (M3-2-2), where $r = 0$, then $F$ is isotopic to $D^{n/2}_{\lambda}(F_0)$ for some integer $n$, where $\lambda$ is a longitude of $K$. By an an argument similar to that in Cases 2.1 together with Lemma 3.2 (5), we see that the number of genus-2 Heegaard surfaces of $M$ whose hyper-elliptic involutions are $\tau_L$ is finite, a contradiction.

- If $M$ satisfies the condition (M3-2-2), where $r = 1$ or 2, then we see by 9 that the link $L$ is equivalent to a link $L(q/2p; \beta_1/\alpha_1, \beta_2/\alpha_2)$ in Figure 1 and that $L$ admits infinitely many 3-bridge spheres $\{P_i\}_{i \in \mathbb{Z}}$ up to isotopy.

**Case 2.6 M satisfies the condition (M4).**

Note that $M$ is obtained by gluing $M_1, M_2 \in D[2]$ and $M_3 = E(K_1 \cup K_2)$, where $K_1 \cup K_2$ is a hyperbolic 2-bridge link with components $K_1$ and $K_2$, so that the regular fiber of $M_i$ is identified with the meridian loop of $K_i$ ($i = 1, 2$). Recall from Theorem 2.2 that, for any genus-2 Heegaard surface $F$ of $M$, $F \cap M_i$ ($i = 1, 2$) is an essential saturated annulus in $M_1$ and $F \cap M_3$ is the 2-bridge sphere of $K_1 \cup K_2$. Let $D_{\mu_i}$ and $D_{\lambda_i}$ ($i = 1, 2$) denote the Dehn twists along $T = \partial M_1 = \partial M_2$ in the direction of the meridian and a longitude of $K_i$, respectively. Note that $K_1 \cup K_2$ admits a unique 2-bridge sphere up to isotopy by 23, and that $M_i$ contains a unique essential annulus up to isotopy. Let $F_0$ be a “standard” genus-2 Heegaard surface of $M$. Then, by 10 Lemma 6 (1)], any genus-2 Heegaard surface $F$ of $M$ is isotopic to $D^{n_1}_{\lambda_1} D^{n_2}_{\lambda_2}(F_0)$ for some integers $n_1$ and $n_2$ by 10 Lemma 6 (1)). Since $\tau_{F_0} D_{\lambda_i} \tau_{F_0} = D_{\lambda_i}^{-1}$, we have, by 10 Lemma 5,

$$\tau_{D^{n_1}_{\lambda_1} D^{n_2}_{\lambda_2}(F_0)} = D^{n_1}_{\lambda_1} D^{n_2}_{\lambda_2} \tau_{F_0}.$$
Since $D = \langle D_{\lambda_1}, D_{\lambda_2} \rangle \cong \mathbb{Z}^2$ (see Lemma 3.2 (6)), we have $D_{\lambda_1}^{n_1} D_{\lambda_2}^{n_2} \tau_{F_0} = \tau_{F_0}$ if and only if $n_1 = n_2 = 0$. Hence, $M$ admits a unique genus-2 Heegaard surface whose hyper-elliptic involution is $\tau_L$, a contradiction.

This completes the proof of Theorem 1.1.

5 Proof of Proposition 4.1

Let $L$ be a torus link $T(3, 3n')$ for some nonzero integer $n'$, and let $K_1$, $K_2$ and $K_3$ be the three components of $L$. Let $S$ be a 3-bridge sphere of $L$. Let $T$ be the standard torus in $S^3$ containing $L$, and let $A_i$ ($i = 1, 2, 3$) be the closure of a component of $T \setminus L$ bounded by two components of $L$ different from $K_i$. Let $V_1$ and $V_2$ be the two solid tori in $S^3$ bounded by $T$ such that the meridians of $V_1$ and $V_2$ meet $L$ in three points and $|3n'|$ points, respectively. Since $S \cap K_i$ consists of two points for each $i = 1, 2, 3$, $S \cap A_i$ satisfies one of the following conditions (see Figure 12).

(i) $S \cap A_i$ contains two non-separating arcs $\gamma_1^i$ and $\gamma_2^i$.

(ii) $S \cap A_i$ contains two separating arcs $\gamma_1^i$ and $\gamma_2^i$.

Thus one of the following holds.

(S1) $S \cap A_i$ satisfies the condition (i) for every $i = 1, 2, 3$.

(S2) Two of $S \cap A_1$, $S \cap A_2$ and $S \cap A_3$ satisfies the condition (i) and the other satisfies the condition (ii).

(S3) Two of $S \cap A_1$, $S \cap A_2$ and $S \cap A_3$ satisfies the condition (ii) and the other satisfies the condition (i).

(S4) $S \cap A_i$ satisfies the condition (ii) for every $i = 1, 2, 3$.

Assume that $S$ satisfies the condition (S1). Note that $\gamma := \gamma_1^1 \cup \gamma_2^1 \cup \gamma_1^2 \cup \gamma_2^2 \cup \gamma_1^3 \cup \gamma_2^3$ consists of two loops each of which contains one of the two points $S \cap K_i$ for every $i = 1, 2, 3$. Suppose there is a loop component, $\delta$, of $S \cap T$ other than $\gamma$. Then $\delta$ bounds a disk in the interior of $A_i$ disjoint from $\gamma$ for some $i = 1, 2, 3$, and hence, $\delta \cup K_i$ is a trivial 2-component link for each $i = 1, 2, 3$. On the other hand, $\delta$ either bounds a disk in $S \setminus \gamma$ or is isotopic to the core of the annulus component of $S \setminus \gamma$. In the latter case, the linking number of $\delta$ and a component of $L$ is 1 (see Figure 13), a contradiction. Hence, $\delta$ bounds a disk in $S \setminus \gamma$. If we
assume that $\delta$ is innermost (in $S \setminus \gamma$), then we can eliminate it from $S \cap T$ since the union of the two disks described above is a 2-sphere bounding a 3-ball in $S^3 \setminus L$. Hence, we may assume that $S \cap T = \gamma$. Since $\gamma$ cuts $S$ into two disks and an annulus, it bounds two disks in $V_1$ and bounds an annulus in $V_2$. Since such a disk is unique up to isotopy in $(V_1, L)$ and an annulus is unique up to isotopy in $(V_2, L)$, $L$ admits a unique 3-bridge sphere satisfying the condition (S1). (We obtain two 3-bridge spheres when $n' = \pm 1$, but it can be easily seen that they are isotopic.)

Assume that $S$ satisfies the condition (S2). Note that $\gamma := \gamma_1^1 \cup \gamma_1^2 \cup \gamma_2^1 \cup \gamma_2^2 \cup \gamma_3^1 \cup \gamma_3^2$ is a loop containing the six points $S \cap L$. Thus any loop component of $S \cap A_i$ except $\gamma$ bounds a disk in $S \setminus \gamma \subset S^3 \setminus L$. This implies that any loop component of $S \cap A_i$ cannot be a core of the annulus $A_i$ for each $i = 1, 2, 3$, since the linking number of the core of $A_i$ and a component of $L$ is $n'(\neq 0)$. Hence, any loop component of $S \cap A_i$ also bounds a disk in the interior of $A_i$ disjoint from $\gamma$, and hence, we can remove all such components by an isotopy. Thus we may assume that $S \cap T$ consists of only one loop component $\gamma$. Since $\gamma$ itself bounds disks in $S$ on both sides, it must be an inessential loop on $T$.

We show that this case can be reduced to the case where $S$ satisfies the condition (S1). To this end, let $h : S^2 \to [-2, 2]$ be the height function such that $S_i := h^{-1}(t)$ is a 3-bridge sphere of $L$ when $-1 < t < 1$, that $S_1$ is a 2-sphere which meets each $K_i$ in one point when $t = \pm 1$, that $S_2$ is a single point when $t = \pm 2$ and that $S_i$ is a 2-sphere in $S^3 \setminus L$ otherwise. Moreover, we may assume that $S_0 = S$ and that the restriction $g := h|T$ of $h$ to $T$ has at most one non-degenerate singular point at every level. Thus, for every singular value $t_0$, $g^{-1}(t_0)$ contains a maximal point, a minimal point or a saddle point. We represent each saddle point in $g^{-1}(t_0)$ by an arc on $T$ with endpoints on $g^{-1}(t_0 - \varepsilon)$ for sufficiently small $\varepsilon > 0$, as in Figure 13. Such an arc, $\alpha$, is of one of the following three types (see Figure 13):

- $\alpha$ is of type 1 if its endpoints are on the same component of $g^{-1}(t_0 - \varepsilon)$, and $g^{-1}(t_0 + \varepsilon)$ contains a pair of parallel essential loops on $T$,
- $\alpha$ is of type 2 if its endpoints are on the same component of $g^{-1}(t_0 - \varepsilon)$, and $g^{-1}(t_0 + \varepsilon)$ does not contain a pair of parallel essential loops on $T$, and
- $\alpha$ is of type 3 if its endpoints are on different components of $g^{-1}(t_0 - \varepsilon)$.

Put $X_s := g^{-1}([-2, s])$ for any $s \in [-2, 2]$. Since $S(= S_0)$ cuts $T$ into a disk and a 1-holed torus, we may assume that $X_0$ is the disk. Since $L \subset X_1$, we see that $X_1$ contains

![Figure 13:](image-url)
Figure 14:

Figure 15: The dashed and dotted lines give all possible types of an arc $\alpha$ representing a saddle point of $g$.

an essential loop on $T$. Thus, there exists a singular value $s_0 > 0$ and a sufficiently small $\varepsilon > 0$ such that $X_{s_0-\varepsilon}$ does not contain an essential loop on $T$ and $X_{s_0+\varepsilon}$ contains an essential loop on $T$. Note that, if $X_{s_0-\varepsilon}$ does not contain an essential loop on $T$ and the arc representing the singular point at $s_0$ is of type 2 or of type 3, then $X_{s_0+\varepsilon}$ cannot contain an essential loop on $T$. Thus, the arc representing the singular point at $s_0$ must be of type 1, and hence, $g^{-1}(s_0 + \varepsilon)$ contains a pair of parallel essential loops, say $c$ and $c'$, on $T$. Note that $c$ bounds a $k$-holed disk in $S' := S_{s_0+\varepsilon}$ disjoint from $g^{-1}(s_0 + \varepsilon) \setminus (c \cup (\bigcup_{i=1}^{k} c_i))$ together with $k$ components $c_1, c_2, \ldots, c_k$ of $g^{-1}(s_0 + \varepsilon) \setminus (c \cup c')$ for some non-negative integer $k$. We see that $c$ is null-homologous in $V_i$ for some $i = 1, 2$ since $c_1, c_2, \ldots, c_k$ are inessential loops on $T$. (By the choice of $s_0$, all components of $g^{-1}(s_0 + \varepsilon)$ except $c$ and $c'$ are inessential on $T$.) Hence, $c$ and $c'$ must be meridian loops of one of the solid tori $V_1$ and $V_2$. Since each of $c$ and $c'$ meets each component of $L$ in a single point, it intersects each of the annuli $A_1$, $A_2$ and $A_3$ in a non-separating arc. Hence, the 3-bridge sphere $S'$, which is isotopic to $S$, satisfies the condition (S1).

Similarly, the cases where $S$ satisfies the condition (S3) or (S4) is reduced to the first case. This implies that $L$ admits a unique 3-bridge sphere up to isotopy.

This completes the proof of Proposition 4.1.

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