On solving the slice-by-slice three-dimensional 2-tensor tomography problems using the approximate inverse method

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Abstract. A numerical solution of the problem of recovering the solenoidal part of a three-dimensional symmetric 2-tensor field using the incomplete tomographic data is proposed. The initial data of the problem are values of the ray transform for all straight lines, which are parallel to at least one of the planes from a finite set of planes. We consider two sets of planes, the number of planes in which are three and six. The recovery algorithms are based on the approximate inverse method.

1. Introduction

In this paper, we consider the following three-dimensional 2-tensor tomography problem. Let a symmetric 2-tensor field \( w \) in a bounded domain of the space \( \mathbb{R}^3 \) be unknown. It is required to reconstruct the field by its known values of the ray transform \( [I^3 w] \).

The ray transform \( I^3 \) on \( \mathbb{R}^3 \) maps a symmetric 2-tensor field \( w (w_{kl} = w_{lk} \text{ for all } k, l = 1, 2, 3) \) onto the function \( [I^3 w] \) on the manifold of oriented lines

\[
l = \{ x + t \xi \mid x, \xi \in \mathbb{R}^3, |\xi| = 1, x \in \xi \perp, t \in \mathbb{R} \}
\]

by the formula

\[
[I^3 w](\xi, x) = \sum_{k,l=1}^{3} \int_{-\infty}^{\infty} \xi_k \xi_l w_{kl}(x + t \xi) \, dt.
\]

The problem of reconstructing symmetric 2-tensor fields \( w \) by its ray transform \( [I^3 w] \) has no unique solution. Namely, the operator \( I^3 \) possesses not trivial kernel consisting of potential symmetric 2-tensor fields \( dv \) with potentials (vector fields) vanishing on the boundary of the domain (see, for example, [1]). Here \( d \) is the inner derivation operator, which is a composition of the gradient and symmetrization operators. Therefore, we can reconstruct by \( [I^3 w] \) only the solenoidal part \( \ast w \) of the symmetric 2-tensor field \( w \).

In \( \mathbb{R}^3 \), the problem of reconstruction of the solenoidal part of the symmetric 2-tensor field by its known values of the ray transform is overdetermined in terms of its dimension, because we try to recover the functions \( \ast w_{kl}(x) \), where \( x \in \mathbb{R}^3 \), by the values of the function \( [I^3 w] \) on
the four-dimensional manifold of oriented lines. Namely, a pair of the variables \((\xi, x)\) has the dimension six in total, but there are two conditions: \(|\xi| = 1, \ x \in \xi^\perp\). Therefore, it is natural to pose the problem of recovering \(\ast w\) by incomplete data \([I^3 w]_{M^3}\), where \(M^3\) is some three-dimensional submanifold of the manifold of oriented lines. It should be noted that the task of reconstructing a two-dimensional symmetric 2-tensor field \(w\) by the values of the ray transform \([I^2 w]\) is not overdetermined (see, for example, [2]–[5]). Investigation of scalar incomplete data problems is a classical subject of the mathematical tomography (see, for example, [6]–[8]). As far as studying incomplete data problems of the vector and 2-tensor tomography is concerned, this direction is in its very beginning [9]–[12]. We mention the papers [13], [14] devoted to the development and numerical study of the algorithms based on the inversion formulas [12].

In this paper, the problem of reconstructing the solenoidal part \(\ast w\) of the symmetric 2-tensor field \(w\) in the following statement is considered. Let the values of ray transform \([I^3 w]_{M^3}\) be known, where \(M^3\) is a set of all the straight lines parallel to at least one of the planes from a finite set of planes. We propose to use the approximate inverse method, developed by A.K. Louis and his disciples for more than 30 years [15]–[18], to solve the problem. This numerical method was successfully applied, in particular, to solve the vector tomography problems in \(\mathbb{R}^3\) [19]–[21] and the vector [22], [23], 2-tensor [24] and \(m\)-tensor tomography [25] in \(\mathbb{R}^2\).

2. Definitions and statement of problems

In this paper, we deal with the tensor fields in \(\mathbb{R}^n, n = 2, 3\). Therefore, we give some definitions for \(\mathbb{R}^n\) for arbitrary \(n\). Let \(\xi \in \mathbb{S}^{n-1}\) be a directional line vector and \(w\) be a symmetric 2-tensor field in \(\mathbb{R}^n\), then the ray transform of \(w\) is a linear operator, which acts according the formula

\[
[I^n w](\xi, x) = \sum_{k,l=1}^{n} \int_{\mathbb{R}} \xi_k \xi_l w_{kl}(x + t\xi) \, dt,
\]

where \(x = (x_1, \ldots, x_n) \in \xi^\perp\).

It is known [1] that every symmetric 2-tensor field \(w\) may be uniquely presented as the sum

\[
w = \ast w + dv,
\]

where \(\ast w\) satisfies the equality

\[
(\delta \ast w)_k = \sum_{l=1}^{n} \frac{\partial (\ast w_{kl})}{\partial x_l} = 0,
\]

for all \(k = 1, \ldots, n\), and the vector field \(v\) is such as \(v(x) \to 0\) at \(|x| \to \infty\). The field \(\ast w\) is called the solenoidal part of the 2-tensor field \(w\), and the second term in (1)

\[
(dv)_{kl} = \frac{1}{2} \left( \frac{\partial v_k}{\partial x_l} + \frac{\partial v_l}{\partial x_k} \right), \quad k, l = 1, \ldots, n,
\]

is called the potential part of \(w\). The ray transform of the potential field is identically zero, i.e. we have \([I^n w] = [I^n \ast w]\). Thus, we can reconstruct only the solenoidal part \(\ast w\) of the field \(w\) by its known values of \([I^n w]\).

Let \(\pi_j = \{x_j = 0\} \subset \mathbb{R}^3, j = 1, 2, 3\), be the coordinate planes. We define the manifolds

\[
M^3(\pi_j) = \{ (\xi, x) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid |\xi| = 1, \ \xi_j = 0, \ \langle \xi, x \rangle = 0 \}, \quad j = 1, 2, 3.
\]

We introduce the coordinates \((\theta, s, z)\) on \(M^3(\pi_j)\) so that

\[
\xi = \cos \theta e_{j+1} + \sin \theta e_{j+2}, \quad x = s(- \sin \theta e_{j+1} + \cos \theta e_{j+2}) + ze_j,
\]

(2)
where \( \{e_1, e_2, e_3\} \) is the standard basis. The lower indices take values 1, 2, 3, so, for example, if \( j = 3 \), we have: \( e_{j+1} = e_1, e_{j+2} = e_2 \). For the three-dimensional symmetric 2-tensor field \( w \) the function \( [T^3_j]w = [T^3 w]_{M^3(\pi_j)} \) is defined in the coordinates (2) on \( M^3(\pi_j) \) by the following formula

\[
[T^3_j]w(\theta, s, z) = \sum_{k,l=1}^{2} \int_{\mathbb{R}} \xi_{j+k} \xi_{j+l} w_{j+k,j+l}(x + t\xi) dt.
\]  

(3)

**The 3P-problem.** It needs to reconstruct the solenoidal part \( ^wS \) of the symmetric 2-tensor field \( w \) by the three given functions \( [T^3_j]w, j = 1, 2, 3 \).

Let \( \pi_{j+1} \subset \mathbb{R}^3, j = 1, 2, 3 \) be the planes with the normal vectors \( (e_j + e_{j+1})/\sqrt{2} \) and passing through the origin. We define the manifolds

\[
M^3(\pi_{j+1}) = \{(\xi, x) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid |\xi| = 1, \xi_j + \xi_{j+1} = 0, \langle \xi, x \rangle = 0 \}.
\]

We introduce the coordinates \( (\theta, s, z) \) on \( M^3(\pi_{j+1}) \) so that

\[
\xi = \cos \frac{\theta}{\sqrt{2}} e_j - \sin \frac{\theta}{\sqrt{2}} e_{j+1} + \sin \theta e_{j+2},
\]

\[
x = z - s \sin \frac{\theta}{\sqrt{2}} e_j + z + s \sin \theta e_{j+1} + s \cos \theta e_{j+2}.
\]

(4)

(5)

For the three-dimensional symmetric 2-tensor field \( w \), the function \( [T^3_{j+1}]w = [T^3 w]_{M^3(\pi_{j+1})} \) is defined in coordinates (4), (5) on \( M^3(\pi_{j+1}) \) by the following formula

\[
[T^3_{j+1}]w(\theta, s, z) = \sum_{k,l=0}^{2} \int_{\mathbb{R}} \xi_{j+k} \xi_{j+l} w_{j+k,j+l}(x + t\xi) dt.
\]

(6)

**The 6P-problem.** It is required to reconstruct the solenoidal part \( ^wS \) of the symmetric 2-tensor field \( w \) by the six given functions \( [T^3_j]w, [T^3_{j+1}]w, j = 1, 2, 3 \).

Thus, in both formulations of the 2-tensor tomography problem, the data space has dimension 3. It is proved [12] that the 3P-problem has an unique solution, and the solution of the 6P-problem is not only unique but stable, as well.

3. The theoretical background of algorithms

Let us present the theoretical basis of algorithms proposed for a solving of the posed problems.

3.1. The two-dimensional slices of the three-dimensional 2-tensor field

Let us know values of the functions \( [T^3_j]w, j = 1, 2, 3 \), and \( S_{(j)}, j = 1, 2, 3 \), be an orthogonal projector of the symmetric 2-tensor field \( w \) on the plane \( \pi_j \). Namely,

\[
S_{(1)}w = \begin{pmatrix}
0 & 0 & 0 \\
0 & w_{22} & w_{23} \\
0 & w_{23} & w_{33}
\end{pmatrix}, \quad S_{(2)}w = \begin{pmatrix}
w_{11} & 0 & w_{13} \\
0 & 0 & 0 \\
w_{13} & 0 & w_{33}
\end{pmatrix}, \quad S_{(3)}w = \begin{pmatrix}
w_{11} & w_{12} & 0 \\
w_{12} & w_{22} & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]

We can consider the field \( (S_{(j)}w)(x) \) for the given \( w(x) \) as the three-dimensional symmetric 2-tensor field satisfying the equalities \( (S_{(j)}w)_j = (S_{(j)}w)_j = (S_{(j)}w)_j = 0 \), or as the two-dimensional 2-tensor field

\[
S_{(j)}w = \begin{pmatrix}
w_{j+1,j+1} & w_{j+1,j+2} \\
w_{j+1,j+2} & w_{j+2,j+2}
\end{pmatrix}.
\]
on the plane \((x_{j+1}, x_{j+2})\) with a smooth dependence on \(x_j\). The two-dimensional 2-tensor field \((S_{(j)} w)(x_{j+1}, x_{j+2})(z)\) is called the slice of the three-dimensional 2-tensor field \(w(x)\) on the plane \(\{x_j = z\}\).

Note that \([\mathcal{I}^3_{(j)} w](\theta, s, x_j) = [\mathcal{I}^2(S_{(j)} w)(x_j)](\theta, s)\), where \((S_{(j)} w)(x_j)\) is the \(j\)-th slice of \(w\) for fixed \(x_j\). In this way only the solenoidal part \(s(S_{(j)} w)\) of the two-dimensional symmetric 2-tensor field \(S_{(j)} w, j = 1, 2, 3\) may be reconstructed by its known values of \([\mathcal{I}^3_{(j)} w], j = 1, 2, 3\).

There is a connection between components of the solenoidal part of the three-dimensional symmetric 2-tensor field and components of the solenoidal parts of its slices \([12]\). Namely, for \(j = 1, 2, 3\), there are equalities

\[
y_{j+1}^2 F[^s w](x_{j+1}, x_{j+2}) - 2y_{j+1}y_{j+2} F[^s w](x_{j+1}, x_{j+2}) + y_{j+2}^2 F[^s w](x_{j+1}, x_{j+2}) = y_{j+2}^2 F[^s w](x_{j+1}, x_{j+2}) + 2y_{j+1}y_{j+2} F[^s w](x_{j+1}, x_{j+2}) + y_{j+2}^2 F[^s w](x_{j+1}, x_{j+2}),
\]

where \(F[\cdot]\) is the three-dimensional Fourier transform defined by the formula

\[
F[g](y) = (2\pi)^{-3/2} \iiint_{\mathbb{R}^3} \exp(-i\langle y, x \rangle) g(x) \, dx.
\]

Here and further, we denote the points in the main space by \(x\), and the points in the space of the Fourier transform image by \(y\). For the Fourier images of components of the solenoidal part the following equalities are also fulfilled

\[
y_1 F[^s w]_{k1}(y) + y_2 F[^s w]_{k2}(y) + y_3 F[^s w]_{k3}(y) = 0, \quad k = 1, 2, 3.
\]

Let us know values of the functions \([\mathcal{I}^3_{(j+1)} w], j = 1, 2, 3\). For each \(j\), we choose a new orthogonal basis in \(\mathbb{R}^3\):

\[
\tilde{e}_j = \frac{1}{\sqrt{2}}(e_j + e_{j+1}), \quad \tilde{e}_{j+1} = \frac{1}{\sqrt{2}}(e_j - e_{j+1}), \quad \tilde{e}_{j+2} = e_{j+2}.
\]

In the new coordinate system, the planes, in which \([\mathcal{I}^3_{(j+1)} w](\theta, s, z)\) are calculated for fixed \(z\), are parallel to the coordinate planes. Let \(\tilde{w}\) be the field \(w\) in the new coordinate system, then we have \([\mathcal{I}^3_{(j+1)} w](\theta, s, z) = [\mathcal{I}^3_{(j)} \tilde{w}](\tilde{\theta}, \tilde{s}, \tilde{z}), j = 1, 2, 3\), where

\[
\tilde{z} = z + s(\cos \theta - \sin \theta)/2, \quad \quad \tilde{s} = 2s \sqrt{4 \cos^2 \theta + 1 + 2 \sin \theta \cos \theta},
\]

\[
\cos \tilde{\theta} = \frac{\cos \theta}{2\sqrt{4 \cos^2 \theta + 1 + 2 \sin \theta \cos \theta}}, \quad \quad \sin \tilde{\theta} = \frac{\sin \theta + \cos \theta}{4\sqrt{4 \cos^2 \theta + 1 + 2 \sin \theta \cos \theta}}.
\]

Thus, the problem is reduced to the problem with the initial data \([\mathcal{I}^3_{(j)} \tilde{w}](\tilde{\theta}, \tilde{s}, \tilde{z}), j = 1, 2, 3\), and we have the following connection between components of the solenoidal part of the field \(w\) and components of the solenoidal parts of slices of the field \(\tilde{w}\)

\[
(y_1 - y_{j+1})^2 \tilde{u}_{j+2,j+2}(y) - 2(y_1 - y_{j+1})y_{j+2} \tilde{u}_{j+1,j+2}(y) + y_{j+2}^2 \tilde{u}_{j+1,j+1}(y)
\]

\[
= (y_1 - y_{j+1})^2 F[^s w]_{j+2,j+2}(y) - 2(y_1 - y_{j+1})y_{j+2} F[^s w]_{j+1,j+2}(y) + y_{j+2}^2 F[^s w]_{j+1,j+1}(y),
\]

\[
+ y_{j+1}^2 F[^s w]_{j,j}(y) - 2F[^s w]_{j,j+1} + F[^s w]_{j+1,j+1}.
\]

Here we use notation \(\tilde{u} = F[^s(S_{(j)} \tilde{w})]\).
3.2. The approximate inverse method for the recovery of the 2-tensor fields in $\mathbb{R}^2$

We propose to use the approximate inverse method for reconstructing the solenoidal parts of two-dimensional slices of the three-dimensional symmetric 2-tensor field. This approach was successfully applied in [24].

We use a connection between the ray transform $I^2$ of the two-dimensional symmetric 2-tensor field and the Radon transform $R$ of the function. The Radon transform acts on the function $f(x), x \in \mathbb{R}^2$ by the formula

$$[Rf](\theta, s) = \int_{\mathbb{R}} f(s\xi^\perp + t\xi) dt,$$

where $\xi = (\cos \theta, \sin \theta), \xi^\perp = (-\sin \theta, \cos \theta)$. Namely, there are the following equalities

$$[R(s^2 v)]_{kl}(\theta, s) = \xi_k \xi_l [I^2 v](\theta, s),$$

$$k, l = 1, 2.$$

It is necessary to remind the scheme of the approximate inverse method for the function reconstruction by its known values of the Radon transform (see the details in [18]). Let $e \in L_2(\mathbb{R}^2)$ be a function with the feature $\|e\|_{L_1(\mathbb{R}^2)} = 1$. Using the operator of translating and dilating $T_{p, \gamma}$, we form the mollifier $e_{\gamma}^p$ from the function $e$ by the formula

$$e_{\gamma}^p(x) = T_{p, \gamma} e(x) = \gamma^{-2} e((x - p)/\gamma),$$

$$x, p \in \mathbb{R}^2.$$

Namely, for the function $e_{\gamma}^p(x)$ and an arbitrary function $f \in L_2(\mathbb{R}^2)$, the equality

$$\lim_{\gamma \to 0} \langle f, e_{\gamma}^p \rangle_{L_2(\mathbb{R}^2)} = f(p)$$

holds.

The operator of translating and dilating $T_{p, \gamma}$ at fixed $\gamma > 0, p \in \mathbb{R}^2$ acts on the function $g(\theta, s)$ according to the formula

$$T_{p, \gamma} g(\theta, s) = \gamma^{-2} g(\theta, (s - (p, \xi^\perp))/\gamma)$$

and is connected with $T_{p, \gamma}$ by the equality $R^* T_{p, \gamma} = T_{p, \gamma} R^*$. Here $R^*$ is the adjoined operator for the Radon transform $R$ acting on the function $g(\theta, s)$ by the following formula

$$(R^* g)(x) = \int_{0}^{2\pi} g(\theta, (x, \xi^\perp)) d\theta.$$

The operator $R^*$ is called the back-projection operator.

Let $e$ belong to the range of the back-projection operator and $\psi$ be a solution of the equation $R^* \psi = e$, then for fixed $\gamma$ and $p$, the function

$$\psi_{\gamma}^p = T_{2\gamma}^p \psi$$

is the solution of the equation $R^* \psi_{\gamma}^p = e_{\gamma}^p$. At small $\gamma$ we obtain

$$f(p) \approx \langle f, e_{\gamma}^p \rangle_{L_2(\mathbb{R}^2)} = \langle f, R^* \psi_{\gamma}^p \rangle_{L_2(\mathbb{R}^2)} = \langle Rf, \psi_{\gamma}^p \rangle_{L_2(Z)},$$

where $Z = \{ (\theta, s) \in \mathbb{R}^2 \mid \theta \in [0, 2\pi), s \in \mathbb{R} \}$. Thus, formulas for the approximate inverse of the ray transform operator $I^2$ have the form (at small $\gamma$)

$$s^k \psi_{k}^l(p) \approx \langle \xi_k \xi_l [I^2 v], \psi_{\gamma}^p \rangle_{L_2(Z)},$$

$$k, l = 1, 2. \tag{12}$$
3.3. About the solution of the 3P- and 6P-problems

We introduce the notations for the left hand sides of equalities (7) and (10)

\[ \nu_{(j)}(y) = y^2_{j+1}F^*\left(S_{(j)}w\right)_{j+2,j+2}(y) - 2y_{j+1}y_{j+2}F^*\left(S_{(j)}w\right)_{j+1,j+2}(y) + y^2_{j+2}F^*\left(S_{(j)}w\right)_{j+1,j+1}(y), \]

\[ \nu_{(j,j+1)}(y) = (y_j - y_{j+1})^2F^*\left(S_{(j)}\tilde{w}\right)_{j+2,j+2}(y) - 2(y_j - y_{j+1})y_{j+2}F^*\left(S_{(j)}\tilde{w}\right)_{j+1,j+2}(y) + y^2_{j+2}F^*\left(S_{(j)}\tilde{w}\right)_{j+1,j+1}(y). \]

Finally we obtain

\[ u_{j,j+1}(y) = \frac{1}{2y_jy_{j+1}}\left( y^2_j\nu_{(j)} - y^2_{j+2}\nu_{(j,j+1)} - 2y_jy_{j+1}\nu_{(j+1)} + 2\nu_{(j+1,j+1)} \right). \]

In its turn, in the 3P-problem, the values of the three functions \([T^3_{(j)}w], [T^3_{(j,j+1)}w], j = 1, 2, 3\) are known. In this way, there are three functions \(\nu_{(j)}, j = 1, 2, 3\), and it is necessary to solve system (15), (17) for finding the symmetric 2-tensor field \(u\). The functions \(\nu_{(j,j+1)}\) can be found from system (15)–(17)

\[ \nu_{(j,j+1)} = \frac{1}{2} \left( \nu_{(j)} + \nu_{(j+1)} + \frac{1}{y_jy_{j+1}} \left( y^2_j\nu_{(j+1)} - y^2_{j+2}\nu_{(j,j+1)} - 2y_jy_{j+1}\nu_{(j+1)} + 2\nu_{(j+1,j+1)} \right) \right). \]

Finally we obtain

\[ u_{j,j+1}(y) = \frac{1}{2y_jy_{j+1}|y|^2} \left( y^2_j\nu_{(j)} - y^2_{j+2}\nu_{(j,j+1)} - 2y_jy_{j+1}\nu_{(j+1)} + 2\nu_{(j+1,j+1)} \right). \]

Therefore, the solution of system (15), (17) has the form of (18), (20) for each \(y_j \neq 0\) for all \(j = 1, 2, 3\).

4. The schemes of algorithms for solving the 3P- and 6P-problems

4.1. Solving the 6P-problem

Let six functions \([T^3_{(j)}w], [T^3_{(j,j+1)}w], j = 1, 2, 3\) be given for a symmetric 2-tensor field \(w\). It is necessary to realize the following steps for recovering the solenoidal part \(^*w\) of the field \(w\) by these data.

The 1st step. Reconstruction of the components \(^*(S_{(j)}w)_{j+i}, i = 1, 2\) of solenoidal parts of the slices \(S_{(j)}w, j = 1, 2, 3\) of the field \(w\) at fixed \(z\) and the components \(^*(S_{(j)}\tilde{w})_{j+i}, i = 1, 2\) of solenoidal parts of the slices \(S_{(j)}\tilde{w}, j = 1, 2, 3\) of the field \(\tilde{w}\) (in the new coordinate system (9)) at
fixed $\tilde{z}$ using formulas (12) of the approximate inverse method. For the mollifiers construction, we use the Gauss function
\[ e_G(x) = (2\pi)^{-1} \exp(-|x|^2/2). \]
The solution of the equation $R^*\psi_G = e_G$ has the form
\[ \psi_G(s) = (2\pi^2)^{-1} \left(1 - \sqrt{2} s D(s/\sqrt{2})\right), \]
where
\[ D(t) = \exp(-t^2) \int_0^t \exp(r^2) dr \]
is the Dawson integral. The values of the functions $\psi_G^\theta, \psi_G^s$ are calculated using (11).

**The 2nd step.** Finding the functions $\nu_{(j)(y)}(y), j = 1, 2, 3$ using (13) and (14).

**The 3rd step.** Calculating $u(y) = F[\ast w](y)$ by formulas (18) and (19).

**The 4th step.** Finding $\ast w(x)$ by applying the inverse Fourier transform to $u(y)$.

### 4.2. Solving the 3P-problem

Let three functions $[I^3_{(j)}, w], j = 1, 2, 3$ be given for a symmetric 2-tensor field $w$. It is necessary to reconstruct the solenoidal part $\ast w$ of the field $w$.

**The 1st step.** Reconstruction of the components $\ast(S_{(j)}w)_{j+i}, i = 1, 2$ of solenoidal parts of the slices $S_{(j)}w, j = 1, 2, 3$ of the field $w$ at fixed $z$ using formulas (12) of the approximate inverse method. For the mollifiers construction we use the Gauss function.

**The 2nd step.** Finding of functions $\nu_{(j)}(y), j = 1, 2, 3$ using (13).

**The 3rd step.** Calculating $u(y) = F[\ast w](y)$ by formulas (18) and (20).

**The 4th step.** Finding $\ast w(x)$ by applying the inverse Fourier transform to $u(y)$.

### 5. Simulations

We demonstrate the results of the numerical experiment aimed at finding the value of $\gamma$ being optimal (the notation is $\gamma_{opt}$) for the chosen discretization of the input data with respect to the parameters $\theta, s$. Discretization of the ray transform $[I^3_{(j)}, w]$ and $[I^3_{(j+3)}, w]$ with respect to $z$ is 64, discretization with respect to $\theta, s$ is changing and takes values 250, 500, 1000. Values of the parameter $\gamma$ of the approximate inverse method are equal to 0.005, 0.01, 0.02, 0.04. The graphical demonstration of the recovery result is shown for the plane $\{x_3 = 0\}$.

The components of the test solenoidal symmetric 2-tensor field $w(x)$ is given by the equalities
\[ w_{jj}(x) = \left(\frac{\partial}{\partial x_{j+2}} - \frac{\partial}{\partial x_{j+1}}\right)^2 f(x), \]
\[ w_{j,j+1}(x) = \left(\frac{\partial}{\partial x_{j+2}} - \frac{\partial}{\partial x_{j+1}}\right) \left(\frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_{j+2}}\right) f(x), \]
for $j = 1, 2, 3$, where
\[ f(x) = \begin{cases} 
(0.81 - x_1^2 - 2x_2^2 - x_3^2)^3, & x_1^2 + 2x_2^2 + x_3^2 < 0.81, \\
0, & \text{otherwise}.
\end{cases} \]

In Table 1 we show values of the best relative error of the reconstruction of $w$ (in percents) and corresponding $\gamma_{opt}$ in solving the 3P- and 6P-problems for different discretization of the ray transform with respect to the parameters $\theta, s$. It can be seen from Table 1 that with an increase
of the number of input data, the accuracy of recovery when solving the \(3P\)-problem remains practically unchanged. In turn, when solving the \(6P\)-problem, increasing the number of input data gives a noticeable gain in the accuracy of recovery. As is expected, the recovery accuracy when solving the \(6P\)-problem significantly exceeds the recovery accuracy for the \(3P\)-problem.

Table 1. The relative error and \(\gamma_{opt}\) for different discretizations of the ray transform

| discretization \ result | 250   | 500   | 1000  |
|-------------------------|-------|-------|-------|
| the \(3P\)-problem      | 37.02 | 0.04  | 33.39 | 0.01  | 32.78 | 0.005 |
| the \(6P\)-problem      | 39.58 | 0.02  | 18.12 | 0.01  | 13.3  | 0.005 |

Figure 1. The components of the test field (the 1st column) and the components of its approximations for solving the \(3P\)-problem (the 2nd column) and for solving the \(6P\)-problem (the 3rd column) for discretization 1000 with respect to \(\theta, s\).
In Figure 1 (in the columns) we demonstrate the components of the test solenoidal symmetric 2-tensor field \( w \) (the 1st column), and its approximations for the 3P-problem (the 2nd column) and for the 6P-problem (the 3rd column), the data discretization is 1000 with respect to \( \theta, s \).

In the columns, the components are shown in the following order (from top to bottom): \( w_{11}, w_{12}, w_{13}, w_{22}, w_{23}, w_{33} \).

6. Conclusion

We consider the problem of reconstructing the solenoidal part \( s w \) of the symmetric 2-tensor field \( w \) in \( \mathbb{R}^3 \) by incomplete data. Namely, it is required to recover \( s w \) by the values of the ray transform \( [I^3 w]_{|M^3} \), where \( M^3 \) is the set of all lines parallel at least to one of the planes from a finite set of planes. We use the notations:

- \( [I^3_{(j)} w] = [I^3 w]_{|M^3(\pi_j)} \), \( j = 1, 2, 3 \) for the values of ray transform known for all lines parallel to the coordinate plane \( \pi_j = \{ x_j = 0 \} \),
- \( [I^3_{(j,j+1)} w] = [I^3 w]_{|M^3(\pi_{j,j+1})} \), where \( \pi_{j,j+1} = \{ e_j + e_{j+1} \} / \sqrt{2} \) and passing through the origin.

Two statements of the problem are considered:

- **The 3P-problem.** It is required to recover the solenoidal part \( s w \) of the symmetric 2-tensor field \( w \) by the three given functions \( [I^3_{(j)} w], j = 1, 2, 3 \).
- **The 6P-problem.** It needs to reconstruct the solenoidal part \( s w \) of the 2-tensor field \( w \) by the six given functions \( [I^3_{(j)} w], [I^3_{(j,j+1)} w], j = 1, 2, 3 \).

For a numerical solution of the problems, we have developed the algorithms based on the approximate inverse method.

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