DENDRITES AND CHAOS

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Abstract. We answer the two questions left open in [Z. Kočan, Internat. J. Bifur. Chaos Appl. Sci. Engrg. 22, article id: 125025 (2012)] i.e. whether there is a relation between $\omega$-chaos and distributional chaos and whether there is a relation between an infinite LY-scrambled set and distributional chaos for dendrite maps. We construct a continuous self-map of dendrite without any DC3 pairs but containing an uncountable $\omega$-scrambled set. To answer for the second question we construct dendrite $D$ and continuous dendrite map without an infinite LY-scrambled set but with DC1 pairs.

1. Introduction

Kočan in [10] studied relations between a variety of chaotic behavior of continuous maps on compact metric spaces leaving open problems in the case of dendrites. In this paper we focus on the relation between $\omega$-chaos and distributional chaos on dendrites and we answer two of them. To construct appropriate example we use properties of spacing shifts (see [1]). In this paper after reviewing the basic definitions and properties of dynamical systems (Section 2) we recall a variety of definitions of chaos (Section 3). We also state a few useful properties about Gehman dendrite (Section 4). In Section 5 we construct the self-map of dendrite which does not have DC3 pairs, but has uncountable $\omega$-scrambled set and in Section 6 of this article we show how construct extension of the shift without DC3 pairs to the mixing shift with no DC3 pairs. In the last Section 7 we show an example of shift with DC1 pair but without an infinite LY-scrambled set.

2. Definitions and notations

Throughout this paper $\mathbb{N}$ denotes the set $\{1,2,3,\ldots\}$ and $\mathbb{N}_0$ denotes $\mathbb{N} \cup \{0\}$. By a dynamical system we mean a pair $(X,f)$, where $X$ is a compact metric space with a metric $\rho$ and $f$ is a continuous map from $X$ to itself. The orbit of $x \in X$ is the set $O(x) := \{f^k(x) : k \geq 0\}$, where $f^k$ stands for the $k$-fold composition of $f$ with itself. For $x \in X$ the $\omega$-limit set is the set

$$\omega_f(x) := \{y \in X : \exists \{n_k\}_{i=1}^{\infty}, n_k \nearrow \infty f^{n_k}(x) = y \}.$$ 

A set $A \subset X$ is invariant under $f$ if $f(A) \subset A$ and minimal if it is nonempty, closed, invariant under $f$, and it does not contain any proper subset which satisfies these three conditions. We say that the dynamical system $(X,f)$ is minimal if $X$ is a minimal set. It is known that $(X,f)$ is minimal if and only if each $x \in X$ has dense orbit or, equivalently, $\omega_f(x) = X$ for any $x \in X$. A point $x \in X$ is regularly recurrent if for any neighborhood $U$ of $x$ there exists $k \in \mathbb{N}$ such that, for any $i \in \mathbb{N}$
we have $f^k(x) \in U$. It is known (see [3]) that every regular recurrent point is an element of its own $\omega$-limit set which is minimal. By $I$ we denote the interval $[0, 1]$. An arc is any topological space homeomorphic to $I$. A continuum is a nonempty connected compact metric space. A dendrite is a locally connected continuum containing no subset homeomorphic to the circle. A point $x$ of a continuum is its end point if for every neighborhood $U$ of point $x$ there exist a neighborhood $V$ of $x$ such that $V \subseteq U$ and boundary of $V$ is a one-point set.

Now let us present some standard notation related to symbolic dynamics. Let $A$ be any finite set (an alphabet) and let $A^*$ denote the set of all finite words over $A$ including the empty word. For any word $w \in A^*$ we denote by $|w|$ the length of $w$, that is the number of letters which form this word. If $w$ is the empty word then we put $|w| = 0$. An infinite word is a mapping $w : \mathbb{N} \to A$, in other words it is an infinite sequence $w_1w_2w_3\ldots$ where $w_i \in A$ for any $i \in \mathbb{N}$. The set of all infinite words over an alphabet $A$ is denoted by $A^\mathbb{N}$. By $0^\infty$ we will denote the infinite word $0^\infty = 000\cdots \in A^*$. If $x \in A^\mathbb{N}$ and $i, j \in \mathbb{N}$ with $i \leq j$ then we denote $x_{[i,j]} = x_ix_{i+1}\ldots x_{j-1}$ ($x_{[i,i]}$ is empty word) and given $X \subset A^\mathbb{N}$ by $\mathcal{L}(X)$ we denote the language of $X$, that is, the set $\mathcal{L}(X) := \{x_{[1,k]} : x \in X, k > 0\}$. We write $\mathcal{L}_n(X)$ for the set of all $n$-blocks contained in $\mathcal{L}(X)$. If a word $u \in A^*$ appears in $z \in A^\mathbb{N}$ (the same for $z \in A^*)$, then we denote it by $u \subset z$ and say that $u$ is a subword of $z$. If $u_k$ is a sequence of words such that $|u_k| \to \infty$ then we write $z = \lim_{k \to \infty} u_k$ if the limit $z = \lim_{k \to \infty} u_k 0^\infty$ exists in $A^\mathbb{N}$. By $\Sigma_n^+$ we denote the dynamical system $([0, \ldots, n-1]^\mathbb{N}, \sigma)$, where $\sigma$ is a shift map defined by $(\sigma(x))_i = x_{i+1}$.

If $S \subset \Sigma_n^+$ is nonempty, closed and $\sigma$-invariant then the restriction $\sigma|_S : S \to S$ (or even the set $S$) is called a subshift of $\Sigma_n^+$. The set $A$ is endowed with the discrete topology and $A^n$ is endowed with the product topology that is metrizable by the metric $\rho : \Sigma_n^+ \times \Sigma_n^+ \to \mathbb{R}$

$$\rho(x, y) = \begin{cases} 2^{-k}, & \text{if } x \neq y, \\ 0, & \text{otherwise} \end{cases}$$

where $k$ is the length of maximal common prefix of $x$ and $y$, that is $k = \max\{i \geq 1 : x_{[1,i]} = y_{[1,i]}\}$.

Let $X$ be a subshift. We say that $(X, \sigma)$ is

(1) weakly mixing if for any $m > 0$ and any words $u_1, u_2, v_1, v_2$ length of $m$ from $\mathcal{L}(X)$ there is $n > 0$ and a joining word $w$ length of $n$ such that $u_1wv_1, u_2wv_2 \in \mathcal{L}(X)$.

(2) mixing if for any $u, v \in \mathcal{L}(X)$ there is $N > 0$ such that for any $n \geq N$ there exists a word $w$ of length $n$ such that $uwv \in \mathcal{L}(X)$.

(3) exact if for any $u \in \mathcal{L}(X)$ there is $n > 0$ such that for every $v \in \mathcal{L}(X)$ there exists a word $w$ of length $n$ such that $uwv \in \mathcal{L}(X)$.

By $\mathcal{C}[w] = \{x \in X : x_{[0,|w|]} = w\}$ we denote an open set in $X$ (the so-called cylinder set) and by $\mathcal{C}_A[w] = \mathcal{C}[w] \cap A$ we denote trace of cylinder set $\mathcal{C}[w]$ on $X$.

The collection of all cylinder sets form a basis of the topology of $\Sigma_n^+$.

**Definition 2.1** (Number of occurrences). Let $A$ be a finite alphabet and $X$ be a shift space over $A$. For every symbol $a \in A$ and every point $x \in X$ we define number of occurrences $\|x\|_a$ of the symbol $a$ in $x$. Let $x = x_1\ldots x_k$ and let $\|x\|_a$
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denote the number of a’s in x, that is

\|x\|_a = |\{1 \leq j \leq k : x_j = a\}|.

Observe that x ∈ X is a minimal point for σ if for any open and nonempty
set U ⊆ X there exists a positive number m such that for any i > 0 there exists
j ∈ [i, i + m] such that σ^j(x) ∈ U. By N(U, V) we denote set of all positive numbers
i such that σ^i(U) ∩ V ≠ ∅.

For any P ⊆ N define

Σ_P = \{s ∈ Σ^+ : s = s_j = 1 ⇒ |i − j| ∈ P \cup \{0\}\}

which is a subshift. We will call a subshift defined in this way the spacing shift
since it is defined by restricting the spacings between 1’s (see [1]). A subset
P of N is called thick (or replete) if it contains arbitrarily long blocks of consecutive integers.
In other words

(∀ n ∈ N)(∃ m ∈ N){m, m + 1, ..., m + n} ⊆ P.

We recall that Σ_P is weakly mixing iff P is thick (see [13] or [1]).

We restate a version of Mycielski’s theorem ([18], Theorem 1) that will play a
crucial role in the proof of the main results Theorem 5.4.

Definition 2.2. Let X be a complete metric space. We call S ⊆ X a Mycielski
set if it has the form S = \bigcup_{j=1}^{∞} C_j with C_j a Cantor set for every j.

Theorem 2.3 (Mycielski). Let X be a perfect complete metric space and for each
n ∈ N let R_n ⊂ X^n be a residual subset of X^n. Then there is a dense Mycielski
set S ⊂ X such that (x_1, x_2, ..., x_n) ∈ R_n for any n ∈ N and any pairwise distinct
points x_1, x_2, ..., x_n ∈ S.

3. Definition of chaos

In the following, we provide definitions of some kinds of chaotic behavior of maps.

3.1. Topological chaos. Let ε > 0 and n ∈ N. A set A ⊆ X is an (f, n, ε)-separated if for each x, y ∈ A with x ≠ y there is an integer 0 ≤ i < n_0, such that
ρ(f^i(x), f^i(y)) > ε. Let s(f, n, ε) denote the maximal cardinality of an (f, n, ε)-separated set.

The topological entropy of f is

h(f) = \lim_{ε \to 0^+} \lim_{n \to \infty} \frac{1}{n} \log s(f, n, ε).

We say that f is topologically chaotic (abbreviated PTE) if f has positive topological entropy (see [4] and [6]).

3.2. Li-Yorke chaos. A set S ⊆ X is called LY-scrambled for f, if it contains at
least two points and for any x, y ∈ S with x ≠ y, we have

\lim_{n \to \infty} \rho(f^n(x), f^n(y)) = 0

and

\lim_{n \to \infty} \rho(f^n(x), f^n(y)) > 0.

If there is a two point, or an infinite, or an uncountable LY-scrambled set for
f, we say that f is LY_2, LY_∞ or LY_u chaotic, respectively (see [9]). We say that
Theorem: $X 	o X$ is $\omega$-chaotic for $f$ if it contains at least two points and for any $x,y \in S$ with $x \neq y$, we have:

1. $\omega_f(x) \setminus \omega_f(y)$ is uncountable,
2. $\omega_f(x) \cap \omega_f(y)$ is nonempty,
3. $\omega_f(x)$ is not contained in the set of periodic points.

We say that $f$ is, respectively, $\omega_2$, $\omega_\infty$, or $\omega_n$-chaotic if there is a two-point, an infinite, or an uncountable $\omega$-chaotic set for $f$. (see [9], [14]).

4. **Distributional chaos**. This type of chaos was introduced in [21]. Given $f$, $x,y \in X$ and a positive integer $n$, define a distribution function $F_{xy}^{(n)} : (0, \text{diam}X] \to [0,1]$ by

$$F_{xy}^{(n)}(t) = \frac{1}{n} \# \{ i : 0 \leq i < n \text{ and } \rho(f^i(x), f^i(y)) < t \}.$$

Then $F_{xy}^{(n)}$ is a left-continuous nondecreasing function. We define the lower distribution function $F_{xy}$ and the upper distribution function $F_{xy}^*$ generated by $f$, $x$ and $y$ by

$$F_{xy}(t) = \liminf_{n \to \infty} F_{xy}^{(n)}(t)$$

and

$$F_{xy}^*(t) = \limsup_{n \to \infty} F_{xy}^{(n)}(t).$$

We extend $F_{xy}$ and $F_{xy}^*$ to the whole real line by setting $F_{xy}(t) = F_{xy}^*(t) = 0$ for $t \leq 0$ and $F_{xy}(t) = F_{xy}^*(t) = 1$ for $t$ which is strictly larger than the diameter of $X$. Clearly, $F_{xy}(t) \leq F_{xy}^*(t)$ for every $t \in \mathbb{R}$. We say that a pair $x,y \in X$ is:

DC1: if $F_{xy}^*(t) = 1$ for all $t > 0$ and there is $s > 0$ such that $F_{xy}(s) = 0$,

DC2: if $F_{xy}^*(t) = 1$ for all $t > 0$ and there is $s > 0$ such that $F_{xy}(s) < 1$,

DC3: if there are $a < b$ such that $F_{xy}^*(t) > F_{xy}(t)$ for every $t \in (a,b)$.

A set $S \subseteq X$ is distributionally chaotic for $f$, if it contains at least two points and for any $x,y \in S$ with $x \neq y$ is a DC1 pair

$$F_{xy}^* = 1 \text{ and } F_{xy} = 0 \text{ for some } t.$$ 

If there is a distributionally chaotic set for $f$, then we say that $f$ exhibits distributional chaos, briefly, DC1-chaotic. Note that the weaker notions than DC1 distributional chaos denoted by DC2 and DC3 were introduced by Smítal and Štefánková (see [22]).

4. **Gehman Dendrite**

Let us recall the construction of a continuous dendrite map from [11]. Let $G$ be the Gehman dendrite (see [8]). It is well-known that the Gehman dendrite can be written as the closure of the union of countably many arcs in $\mathbb{R}^2$: $B_0 = [p,p_0]$, $B_1 = [p,p_1]$, and for every $n \in \mathbb{N} \setminus \{0\}$, $B_{i_1i_2\ldots i_n} = [p_{i_1i_2\ldots i_n}p_{i_1i_2\ldots i_{n+1}}]$ where every $i_k$ is either 0 or 1. Let $E$ denote the set of end points of $G$. With every point $x \in E$ we can uniquely associate a sequence of zeros and ones $i_1i_2i_3\ldots$ in such way that the limit of the codes of the arcs converging to the point.

We define a continuous map $g$ on a dendrite $G$ in the following way. Let $g(B_0) = g(B_1) = \{p\}$. For every $i_1,i_2,\ldots,i_n$, let $g|_{B_{i_1i_2\ldots i_n}} : B_{i_1i_2\ldots i_n} \to B_{i_2i_3\ldots i_n}$ be
a homeomorphism such that \( g(p_{i_1,...,i_n}) = p_{i_2,...,i_n} \), and let \( g \) act on \( E \) as the shift map on the space \( \Sigma^+ \). Let \( X \) be a closed \( g \)-invariant subset of \( E \). Denote \( D_X = \bigcup_{x \in X} [x, \xi] \) and \( f = g|D_X \).

**Lemma 4.1.** A set \( D_X \) is subdendrite of \( G \) and \( D_X \) is an \( f \)-invariant subset of \( G \).

**Proof.** Let us notice that \( D_X \) is a union of arcs \([x, \xi] \) so it is a locally connected continuum and from the construction it is obvious that \( D_X \) does not contain subset homeomorphic to the circle. It shows that \( D_X \) is a subdendrite of the Gehman dendrite \( G \). To show that \( D_X \) is a closed \( f \)-invariant subset of \( E \) we get that \( X \) is \( f \)-invariant which completes the proof. \( \square \)

**Lemma 4.2.** Let \( X \) be closed and nonempty subset of \( \Sigma^+ \) without isolated points. Then the set \( D_X = \bigcup_{x \in X} [x, \xi] \subset G \) is the Gehman dendrite.

**Proof.** Let \( X \) be nonempty subset of \( \Sigma^+ \) homeomorphic to the Cantor set. Then the set \( D_X = \bigcup_{x \in X} [x, \xi] \) is subdendrite of \( G \) and has property that every ramification point of \( D \) has order \( 3 \). Now using \( n = 3 \) in Theorem 4.1 in [5] we get that the set \( D_X \) is the Gehman dendrite which completes the proof. \( \square \)

5. **Uncountable \( \omega \)-scrambled set without \( DC3 \) pairs on dendrite**

**Lemma 5.1.** There is a Cantor set \( \Sigma \subset \Sigma^+ \) such that for any \( n \geq 2 \) and any distinct points \( x_1, x_2, ..., x_n \in \Sigma \) and any \( \{i_1, i_2, ..., i_k\} \subset \{1, 2, ..., n\} \) where \( k \in \{1, 2, ..., n\} \), there is \( j > 0 \) such that

\[
x_i(j) = 1 \quad \text{for} \quad i \in \{i_1, i_2, ..., i_k\}
\]

and

\[
x_i(j) = 0 \quad \text{for} \quad i \not\in \{i_1, i_2, ..., i_k\}.
\]

**Proof.** Fix \( n \geq 2, 1 \leq k \leq n \) and indices \( \{i_1, i_2, ..., i_k\} \subset \{1, 2, ..., n\} \). Define

\[R_n^{(i_1, i_2, ..., i_k)} \subset (\Sigma^+)^n\]

by

\[
R_n^{(i_1, i_2, ..., i_k)} = \{(x_1, x_2, ..., x_n) \in (\Sigma^+)^n : x_i(j) = 1 \quad \text{for} \quad i \in \{i_1, i_2, ..., i_k\} \}
\]

and \( x_i(j) = 0 \) for \( i \not\in \{i_1, i_2, ..., i_k\} \).

We claim that the set \( R_n^{(i_1, i_2, ..., i_k)} \) is open and dense in the product space \((\Sigma^+)^n\).

We endow \((\Sigma^+)^n\) with the maximum metric \( \rho_n \), i.e.

\[
\rho_n((x_1, ..., x_n), (y_1, ..., y_n)) = \max_{i=1,...,n} \rho(x_i, y_i).
\]

Observe that \( R_n^{(i_1, i_2, ..., i_k)} \) is open, since for every \( j \) there is \( \varepsilon > 0 \) such that if \( \rho_n((x_1, ..., x_n), (y_1, ..., y_n)) < \varepsilon \) then \( x_i(j) = y_i(j) \) for \( i = 1, ..., n \). It remains to prove that \( R_n^{(i_1, i_2, ..., i_k)} \) is also dense. Fix nonempty words \( w_i \) for \( i = 1, ..., n \) and put \( v_i = w_i w_{i+1} ... w_n w_1 ... w_{i-1} \). Note that \( |v_1| = |v_2| = \ldots = |v_n| \). We define

\[
u_i = \begin{cases} v_i 10^\infty, & \text{if } i \in \{i_1, i_2, ..., i_k\}, \\ v_i 0^\infty, & \text{otherwise.}\end{cases}
\]

Clearly,

\[
(u_1, ..., u_n) \in R_n^{(i_1, i_2, ..., i_k)} \cap (C[w_1] \times \ldots \times C[w_n]).
\]
which proves that \( R^{(i_1,\ldots,i_k)}_n \) is dense. It follows that for each \( n \) and \( \{i_1,\ldots,i_k\} \subseteq \{1,\ldots,n\} \) the set \( R^{(i_1,\ldots,i_k)}_n \) is residual in \((\Sigma^2_2)^n\). Given \( n \geq 2 \) let \( \Gamma_n \) be the set of all finite and nonempty subsets of \( \{1,\ldots,n\} \). Then the set
\[
R_n = \bigcap_{A \in \Gamma_n} R^A_n
\]
is also open and dense, hence is also residual in \((\Sigma^2_2)^n\). We constructed a sequence of residual relations, therefore by Mycielski Theorem there is a Cantor set of all finite and nonempty subsets of \( \{1,\ldots,n\} \) which proves that \( 1 \).

**Proof.** For every thick set \( P \) there are thick sets \( P_i \) such that \( P = \bigcup_{i \in \mathbb{N}} P_i \) and \( P_i \cap P_j = \emptyset \) provided that \( i \neq j \).

**Proof.** For any integer \( n \geq 2 \) take \( j_n \) such that \( Q_n = \{j_n,\ldots,j_n+n\} \subseteq P \). We may that \( j_{n+1} > j_n + n \). Put \( Q_1 = P \setminus \bigcup_{n=2}^{\infty} Q_n \). Take any bijection \( F : \mathbb{N} \rightarrow \mathbb{N} \times \mathbb{N} \). Let \( I_j = \{n \in \mathbb{N} : F(n) \in \{j\} \times \mathbb{N} \} \). For each \( j \in \mathbb{N} \) set \( P_j = \bigcup_{i \in I_j} P_i \). By the construction each \( P_j \) is a thick, set \( P_j \)'s are pairwise disjoint and \( P = \bigcup_{i \in \mathbb{N}} P_i \) which completes the proof.

**Lemma 5.2.** For every thick set \( P \) there are thick sets \( P_i \) such that \( P = \bigcup_{i \in \mathbb{N}} P_i \) and \( P_i \cap P_j = \emptyset \) provided that \( i \neq j \).

**Proof.** For any integer \( n \geq 2 \) take \( j_n \) such that \( Q_n = \{j_n,\ldots,j_n+n\} \subseteq P \). We may that \( j_{n+1} > j_n + n \). Put \( Q_1 = P \setminus \bigcup_{n=2}^{\infty} Q_n \). Take any bijection \( F : \mathbb{N} \rightarrow \mathbb{N} \times \mathbb{N} \). Let \( I_j = \{n \in \mathbb{N} : F(n) \in \{j\} \times \mathbb{N} \} \). For each \( j \in \mathbb{N} \) set \( P_j = \bigcup_{i \in I_j} P_i \). By the construction each \( P_j \) is a thick, set \( P_j \)'s are pairwise disjoint and \( P = \bigcup_{i \in \mathbb{N}} P_i \) which completes the proof.

**Theorem 5.3.** For every thick set \( P \) there exists a Cantor set \( \Gamma \subseteq \Sigma_P \) such that for any \( n \geq 2 \) any distinct points \( y_1, y_2, \ldots, y_n \in \Gamma \) and any choice of indexes \( \{i_1,\ldots,i_k\} \subseteq \{1,\ldots,n\} \) the set
\[
\bigcap_{i \in \{i_1,\ldots,i_k\}} \omega(y_i) \setminus \bigcup_{j \notin \{i_1,\ldots,i_k\}} \omega(y_j)
\]
contains an uncountable set \( D \) without minimal points.

**Proof.** Using Lemma 5.2 find a decomposition of \( P \) into pairwise disjoint thick sets \( P = \bigcup_{i \in \mathbb{N}} P_i \). Let \( \Sigma \) be provided by Lemma 5.1. For every \( x \in \Sigma \) denote
\[
Q_x = \bigcup_{x_n = 1} P_n \subseteq P.
\]
Clearly, each \( Q_x \) is thick, since every \( x \in \Sigma \) contains at least one symbol 1. Thus \( Q_x \) defines a weakly mixing spacing shift \( \Sigma_{Q_x} \) (see [13] or [1]) and so we can select a point \( z_x \in \Sigma_{Q_x} \) with a dense orbit in \( \Sigma_{Q_x} \). Denote
\[
\Gamma = \{z_x : x \in \Sigma\}.
\]
Fix any \( n \geq 2 \) and any \( \{i_1, i_2, \ldots, i_k\} \subseteq \{1,2,\ldots,n\} \). Pick any pairwise distinct points \( y_1, \ldots, y_n \in \Gamma \) and let \( x_1, x_2, \ldots, x_n \in \Sigma \) be such that \( y_i = z_{x_i} \). Then there is \( j > 0 \) such that \( x_i(j) = 1 \) for \( i \in \{i_1, i_2, \ldots, i_k\} \) and \( x_i(j) = 0 \) for \( i \notin \{i_1, i_2, \ldots, i_k\} \). This implies that \( P_j \subseteq Q_{x_i} \) for each \( i \in \{i_1, i_2, \ldots, i_k\} \) and \( P_j \cap Q_{x_i} = \emptyset \) for each \( i \notin \{i_1, i_2, \ldots, i_k\} \). Then
\[
\Sigma_{P_j} \subseteq \bigcap_{i \in \{i_1,\ldots,i_k\}} \omega(y_i) = \bigcap_{i \in \{i_1,\ldots,i_k\}} \Sigma_{Q_{x_i}}.
\]
Furthermore, if \( i \notin \{i_1,\ldots,i_k\} \) then in any point \( z \in \Sigma_{P_j} \cap \Sigma_{Q_{x_i}} \), the symbol 1 occurs at most once. There are at most countably many such points, and every weakly mixing spacing shift \( \Sigma_{P_j} \) is uncountable. Furthermore \( P_j \) has thick complement.
(e.g. contains $P_{j+1}$ in its complement), hence $\Sigma_{P_j}$ is proximal by [1]. But the only minimal point in a proximal system is the fixed point, hence

$$\bigcap_{i \in \{1, \ldots, i_k\}} \omega(y_i) \setminus \bigcup_{j \notin \{1, \ldots, i_k\}} \omega(y_j)$$

contains an uncountable set $D$ without minimal points. \hfill \Box

Observe that the set $\Gamma$ in Theorem 5.3 satisfies the following strong version of $\omega$-chaos (in particular is $\omega$-scrambled). In fact, conditions posed in (5.1) are among strongest possible dependences between $n$-tuples of $\omega$-limit sets.

For any integer $n \geq 2$ let take $j_n$ such that $Q_i = \{4^i, 4^i + 1, \ldots, 4^i + i - 1\}$. It is not hard to see that both sets $P$ and $\mathbb{N} \setminus P$ are thick. Then combining Theorem 5.3 with [1] we obtain the following:

**Theorem 5.4.** There exists a spacing shift without DC3 pairs $\Sigma_P$ and a Cantor set $\Gamma \subset \Sigma_P$ such that for any $n \geq 2$, any distinct points $y_1, y_2, \ldots, y_n \in \Gamma$ and any choice of indexes $\{i_1, \ldots, i_k\} \subset \{1, \ldots, n\}$ the set

$$\bigcap_{i \in \{1, \ldots, i_k\}} \omega(y_i) \setminus \bigcup_{j \notin \{1, \ldots, i_k\}} \omega(y_j)$$

contains an uncountable set $D$ without minimal points.

Lets us recall open questions formulated by Kočan in [10].

(1) Does the existence of uncountable $\omega$-scrambled set imply distributional chaos?

(2) Does the existence of uncountable $\omega$-scrambled set imply existence of an infinite LY–scrambled set?

(3) Does distributional chaos imply the existence of an infinite LY–scrambled set?

By our construction we immediately obtain a negative answer to the first one i.e. there is an uncountable $\omega$–scrambled set which does not imply distributional chaos.

**Corollary 5.5.** There exists a continuous self-map $f$ of Gehman dendrite such that:

(1) $f$ does not have DC3 pairs,

(2) $f$ has uncountable $\omega$-scrambled set.

**Proof.** Let $\Sigma_P$ be a spacing subshift provided by Theorem 5.4. By Lemma 4.1 we obtain a map $f: D \to D$ on Gehman dendrite $D$ such that we may view $\Sigma_P \subset D$ as an invariant subset (up to conjugacy of these subsystems). Furthermore, there is a fixed point $p \in D \setminus \Sigma_P$ and for every $y \in D \setminus \{p\}$ there is $z \in \Sigma_P$ such that $\lim_{n \to \infty} d(f^n(y), f^n(z)) = 0$.

It is clear that (2) is satisfied, since $(\Sigma_P, \sigma)$ is a subsystem of $(D, f)$ containing an uncountable $\omega$-scrambled set. Furthermore, if pairs $(z_1, y_1)$, $(z_2, y_2)$ are asymptotic then $F_{z_1, z_2} = F_{y_1, y_2}$ and $F_{z_1, z_2}^* = F_{y_1, y_2}^*$ (e.g. see [17, Lemma 5]). This shows that there is no DC3 pair in $D \setminus \{p\}$. But distance in $D$ is given by arclength, hence we may assume that $p(x, x) = 1$ for every $x \in \Sigma_P$. This implies that $(p, x)$ is not DC3 for any $x \in \Sigma_P$ and automatically $(p, y)$ is not DC3 for any $y \in D \setminus \{p\}$. The proof is finished. \hfill \Box
6. Mixing

Remark 6.1. Let $X$ be a shift. Note that $\rho(x, y) < 2^{-k}$ for some $k \geq 0$ implies that $\rho(x, y) \leq 2^{-k-1}$, hence $\rho(x, y) < t$ for every $t \in (2^{-k-1}, 2^{-k})$.

Proof. Indeed, for each $n \geq 1$, $k \geq 0$ and $t \in (2^{-k-1}, 2^{-k})$ we have
\[
\frac{1}{n} |\{0 \leq m < n : d(\sigma^m(x), \sigma^m(y)) < t\}| = \frac{1}{n} |\{0 \leq m < n : d(\sigma^m(x), \sigma^m(y)) \leq 2^{-k-1}\}| = \frac{1}{n} |\{0 \leq m < n : d(\sigma^m(x), \sigma^m(y)) < 2^{-k}\}|.
\]
It follows that $F_x^{(n)}(t) = F_x^{(n)}(2^{-k})$ for each $n \geq 1$, $k \geq 0$ and $t \in (2^{-k-1}, 2^{-k})$. Thus $F_{xy}(t) = F_{xy}(2^{-k})$ and $F_{xy}^*(t) = F_{xy}^*(2^{-k})$ for $t \in (2^{-k-1}, 2^{-k})$. In other words, $F_{xy}$ and $F_{xy}^*$ are piecewise constant functions. \(\square\)

Remark 6.2. Let $X$ be a shift. Then $(x, y) \in \mathcal{X} \times \mathcal{X}$ is a $(DC3)$ pair if and only if there exist an integer $k \geq 0$ such that $F_{xy}(2^{-k}) < F_{xy}^*(2^{-k})$.

Proof. Assume first that $F_{xy}(2^{-k}) < F_{xy}^*(2^{-k})$ for some integer $k \geq 0$. Let $F_{xy}(2^{-k}) = \alpha \in (0, 1)$ and $F_{xy}^*(2^{-k}) = \beta \in (0, 1)$. Note that
\[
\frac{1}{n} |\{0 \leq m < n : d(\sigma^m(x), \sigma^m(y)) < 2^{-k}\}| = \frac{1}{n} |\{0 \leq m < n : d(\sigma^m(x), \sigma^m(y)) \leq 2^{-k-1}\}| = \frac{1}{n} |\{0 \leq m < n : d(\sigma^m(x), \sigma^m(y)) < t\}|.
\]
for $t \in [2^{-k-1}, 2^{-k})$. Now calculating $F_{xy}(t)$ and $F_{xy}^*(t)$ we get that $\alpha = F_{xy}(t) < F_{xy}^*(t) = \beta$, so $(x, y)$ is a $(DC3)$ pair.

On the other hand, if $(x, y)$ is a $(DC3)$ pair then there exists $(a, b) \subset [0, 1]$ such that $F_{xy}(t) < F_{xy}^*(t)$ for all $t \in (a, b)$. Let $k \geq 0$ be such that $2^{-k-1} < b \leq 2^{-k}$.

By Remark 6.1 we have also
\[
F_{xy}(t) = F_{xy}(2^{-k}) < F_{xy}^*(2^{-k}) = F_{xy}^*(t)
\]
for $t \in (2^{-k-1}, 2^{-k})$. Hence $(x, y)$ is a $(DC3)$ pair. \(\square\)

Lemma 6.3. If $X$ is a shift space such that for every $x \in X$ we have $d(\{i : x_i = 0\}) = 1$ then there is no DC3-pair in $X$.

Proof. Let $x, y \in X$. Then
\[
d(\{i : x_i \neq y_i\}) = d(x, y) \leq d(x, 0^\infty) + d(0^\infty, y) = d(\{i : x_i \neq 0\}) + d(\{i : y_i \neq 0\}) = 0.
\]
Thus $d(\{i : x_i = y_i\}) = 1$ which clearly implies that $(x, y)$ is not a DC3 pair. \(\square\)

Remark 6.4. Let $X$ be a shift space. Then $d(\{i : x_i = 0\}) = 1$ for every $x \in X$ if and only if the measure concentrated on $0^\infty$ is the only invariant measure for $X$ (see [12]).

Lemma 6.5. Let $X$ be a shift and $x \in X$ is such that $d(\{i : x_i = 0\}) = 1$. If $(x, y)$ is a $(DC3)$ pair then $(0^\infty, y)$ is a $(DC3)$ pair.
Proof: By Remark 6.2 it is enough to check that \( F_{xy}(2^{-k}) < F_{xy}^*(2^{-k}) \) for some \( k \geq 0 \).

If \((x, y)\) is a \((DC3)\) pair then there exist an integer \( l > 0\), an increasing sequence \( \{s_i\}_{i=1}^\infty \) of positive integers and \( \gamma > 0\) such that

\[
\frac{1}{s_i} \left| \{0 \leq k < s_i : x[k+k+l] \neq y[k+k+l] \} \right| \geq \gamma.
\]

Observe that we can write the set \( \{0 \leq k < s_i : x[k+k+l] \neq y[k+k+l] \} \) as the disjoint union of

\[
\{0 \leq k < s_i : x[k+k+l] \neq y[k+k+l] \} \text{ and } \{0 \leq k < s_i : y[k+k+l] = 0^l\}.
\]

Now, \( d(\{i : x_i = 0\}) = 1 \) implies that for every \( k > 0\) we have

\[
d(\{i : x_{i+k} = 0^k\}) = 1.
\]

We get that for every \( \delta > 0 \) there exist \( N \) such that for all \( i > N \) we have

\[
\frac{1}{s_i} \left| \{0 \leq k < s_i : x[k+k+l] \neq y[k+k+l] \} \right| < \delta.
\]

From that we obtain

\[
\gamma \leq \frac{1}{s_i} \left| \{0 \leq k < s_i : x[k+k+l] \neq y[k+k+l] \} \right| \leq \frac{1}{s_i} \left| \{0 \leq k < s_i : y[k+k+l] = 0^l\} \right| \leq \frac{1}{s_i} \left| \{0 \leq k < s_i : y[k+k+l] = 0^l\} \right| + \delta\]

and finally

\[
\frac{1}{s_i} \left| \{0 \leq k < s_i : y[k+k+l] = 0^l\} \right| \geq \gamma - \delta.
\]

On the other hand there exist a decreasing sequence \( \{t_i\}_{i=1}^\infty \), \( l > 0 \) and \( 0 < \alpha < \gamma \)

\[
\frac{1}{t_i} \left| \{0 \leq k < t_i : x[k+k+l] \neq y[k+k+l] \} \right| \leq \alpha.
\]

A similar calculations as above gives that

\[
\frac{1}{t_i} \left| \{0 \leq k < t_i : y[k+k+l] = 0^l\} \right| \leq \alpha + \varepsilon
\]

where \( \varepsilon > 0 \) is such that

\[
\frac{1}{t_i} \left| \{0 \leq k < t_i : x[k+k+l] = y[k+k+l] \} \right| \leq \varepsilon.
\]

Since \( \varepsilon \) and \( \delta \) can be arbitrarily small this completes the proof that \((0^\infty, y)\) is a \((DC3)\) pair.

\[\square\]

**Theorem 6.6.** Let \( X \) be a shift such that for every \( x \in X \) we have \( d(\{i : x_i = 0\}) = 1 \). Then there exists a mixing shift \( Y \) containing \( X \) such that there is no \((DC3)\) pair in \( Y \).

**Proof.** First, note that from Lemma 6.3 we have that there in no DC3-pair in \( X \). Now we inductively construct an increasing sequence of shift spaces \( X_0 \subset X_1 \subset \ldots \) and then define space \( Y \) as the closure of the union of all \( X_n \)'s i.e.

\[
Y = \bigcup_{n=0}^{\infty} X_n.
\]
Let \( X_0 = X \cup W \), where \( W \) is any weak mixing shift such that the only invariant measure for \( W \) is concentrated on \( 0^\infty \). We define the set \( X_1 \) adding to \( X_0 \) orbits of points of the form 
\[
0^a10^b10^\infty, \quad \text{where} \quad \alpha \geq 0, \quad \beta \geq 2.
\]
Note that every block added at first step has at most two occurrences of the symbol 1. Inductively, for given \( X_n \) and \( n \geq 0 \) we construct a shift space \( J_{n+1} \) and set \( X_{n+1} = X_n \cup J_{n+1} \), where 
\[
J_{n+1} = \bigcup_{m=0}^{\infty} \sigma^m \left\{ 0^a u 0^b v 0^\infty : \alpha \geq 0, \ u, v \in \mathcal{L}(X_n), \ \beta \text{ such that } \phi_{\beta} > \phi_{2n} \right\}
\]
and \( \phi_n, \phi_n' \) denote the maximum number of occurrences of the symbol 1 among all blocks of length \( n \) in \( X \) and \( X_i \), respectively, that is,
\[
\phi_n = \max\{ \|w\|_1 : w \in \mathcal{L}(X) \}
\]
and
\[
\phi_n' = \max\{ \|w\|_1 : w \in \mathcal{L}(X_i) \}.
\]
Note that each \( X_n \) is a subshift and let us first notice that the sequence \( \{\phi_n\}_{n=1}^\infty \) is non-negative and subadditive (i.e. \( 0 \leq \phi_{m+n} \leq \phi_m + \phi_n \)).

We claim that \( \lim_{n \to \infty} \phi_n = \lim_{n \to \infty} \phi_n' \) for all \( l \geq 0 \) and \( n > 0 \). Indeed, from property that \( X \subset X_i \) we have that \( \phi_n \leq \phi_n' \). By the other hand note that if \( u \in X_{n+1} \) and \( |u| \geq \beta \) then either
\[
u \in \mathcal{L}(X_n)
\]
or
\[
\|u\|_1 \leq 2 \max\{ \|v\|_1 : v \in \mathcal{L}(X) \} \leq 2 \phi_n \leq \phi_n' \leq \phi_{|u|}.
\]
Therefore
\[
\phi_{n+1} \leq \phi_{|u|}.
\]
If \( |u| \leq \beta \) we have that \( u = 0^k w 0^s \) where \( k, s \geq 0 \). Since \( w \) is a word with length at most \( n \) from \( X_n \) so we get
\[
\|u\|_1 \leq \phi_n.
\]

Let us notice that the condition \( \lim_{n \to \infty} \phi_n = 0 \) is equivalent to \( d(\{i : x_i = 0\}) = 1 \) (see \cite{7}, Theorem 3).

Now if we denote
\[
Y_{n+1} = J_{n+1}
\]
we get equality
\[
X_{n+1} = X_n \cup Y_{n+1}.
\]
The set \( Y_{n+1} \) is shift invariant and points from \( Y_{n+1} \) guarantee that the mixing condition holds for pairs of blocks \( u, v \) from \( \mathcal{L}(X_n) \).

We claim that
\[
Y_{n+1} = J_{n+1} \cup \{ 0^a u 0^\infty : u \in \mathcal{L}(X_n) \} \cup \{ 0^\infty \}
\]
for all \( s \geq 0 \).

Indeed, let \( \{x_k\}_{k=1}^\infty \subset J_{n+1} \) for some sequence \( \{n_k\}_{k=1}^\infty \subset \mathbb{N} \) and fix \( x = \lim_{k \to \infty} x_k \). Without loss of generality we can assume that for positive integer \( k \) there exist \( m_k, \alpha_k, \beta_{nk} \geq 0 \) such that
\[
x_k = \sigma^{m_k} \left( 0^{\alpha_k} u 0^{\beta_{nk}} v 0^\infty \right) \in \{ 0^{j_k} u 0^{\beta_{nk}} v 0^\infty, 0^{j_k} v 0^\infty, 0^\infty \}
\]
for \( j_k \leq \alpha_k \), \( s_k \leq \beta_n \) and where \( u, v \in \mathcal{L}_{\alpha_k}(X_{n_k}) \), \( \alpha_k \geq 0, \beta_n \geq 0 \) and such that 
\[ \varphi_B > \varphi_{2n_k}. \]

Let choose first \( x_k = 0^j u^0 \beta_k v^0 \). If \( j_k \) tends to infinity as \( k \to \infty \) then we get that 
\[ x = \lim_{k \to \infty} x^{(k)} = 0^\infty. \] If \( j_k \to j \neq \infty \) for some \( j \geq 0 \) then without loss of generality we can assume that 
\[ x_k = 0^j u^0 \beta_k v^0 \]
and now if \( \beta_n \to \infty \) we get \( x = 0^a v^0 \), but if \( \beta_n \to \beta \neq \infty \) we get that 
\[ x_k \to 0^j u^0 v^0 \infty \] as \( k \to \infty \) so it means that \( x \in J_{n_k + 1}. \)

Now, if we choose \( x_k = 0^a v^0 \infty \) and if \( s_k \to \infty \) we get that \( x = 0^\infty \), but on the other hand if \( s_k \to s \neq \infty \) we get \( x = 0^a v^0 \infty \). Finally if we choose \( x_k = 0^\infty \) then we get \( x = 0^\infty \).

Now we will prove that \( d(\{i : y_i = 1\}) = 0 \) for every \( y \in Y \). If \( y \in \bigcup_{n=0}^\infty X_n \) then \( y \in X_k \) for some \( k \) and then 
\[ \frac{1}{n} |\{0 \leq i < n : y_i = 1\}| \leq \frac{1}{n} \varphi_n \leq \frac{\varphi_n}{n} \to 0 \]
Now fix \( y \in Y \setminus \bigcup_{n=0}^\infty X_n \). Then there exists a sequence 
\[ x_k \in \bigcup_{n=0}^\infty X_n \]
such that 
\[ \lim_{k \to \infty} x_k = y \in Y \setminus \bigcup_{n=0}^\infty X_n. \]
For each \( n \) there exist \( N \geq 0 \) such that for all \( k \geq N \)
\[ y_{(0,n)} = (x_k)_{(0,n)}. \]
Since \( x_k \in X_k \) we get that 
\[ |\{0 \leq i < n : (x_{(0,n)})_i = 1\}| \leq \varphi_n \leq \varphi_n. \]

The shift space \( Y \) is topologically mixing. Indeed, taking two nonempty blocks \( u, v \in \mathcal{L}(Y) \) there is \( k \geq 0 \) such that \( u, v \in \mathcal{L}(X_k) \). Therefore \( u^0 \in \mathcal{L}(X_k) \), \( u^1 \in \mathcal{L}(X_k) \) for all \( \alpha \geq 0 \) and it ends the proof that \( Y \) is topologically mixing. 

\[ \square \]

7. DISTRIBUTIONAL CHAOS WITHOUT AN INFINITE LY-SCRAMBLED SET

Denote \( I = [0, 1] \). We perform an inductive construction. In the initial step set \( m_0 = 1 \) and \( z_1^{(0)} = \frac{1}{2} \). Define \( Z^{(0)} = \{ z_1^{(0)} \} \), \( x_0^{(0)} = 0 \), \( x_1^{(0)} = z_1^{(0)} \), \( x_2^{(0)} = 1 \), and 
\[ l_0 = 1, l_{-1} = 0. \] Note that the sequence \( (x_i^{(0)})_{i=0}^2 \) contains \( Z^{(0)} \), constructed so far, and endpoints of \([0, 1]\). Furthermore \( x_0^{(0)} < x_1^{(0)} < x_2^{(0)} \). For the inductive step denote for \( i \in \mathbb{N} \)
\[ L_i = (l_{i-1} + 1)m_i \]
with 
\[ m_{i+1} \geq 2^{i}l_i \]
and assume that we have just constructed sets 
\[ Z^{(0)}, Z^{(1)}, \ldots, Z^{(n)} \subset I \]
where
\[ |Z^{(i)}| = L_i \]
and
\[ l_i = \sum_{j=0}^{i} |Z^{(j)}|. \]

In particular \( Z^{(i)} \cap Z^{(j)} = \emptyset \) for \( i \neq j \). Note that \( l_i \) and \( L_i \) are constructed in such a way that \( l_i + 1 \geq 2^i \) and \( L_i \geq 2^i \) for all \( i \in \mathbb{N} \cup \{0\} \). We also assume that all elements of set
\[ \bigcup_{j=0}^{n} Z^{(j)} = \left( x^{(n)}_j \right)_{j=1}^{l_n} \]
were enumerated in such a way that
\[ 0 = x^{(n)}_0 < x^{(n)}_1 < \cdots < x^{(n)}_{l_n+1} = 1. \]

Additionally we also assume that sets \( Z^{(i)} = \{ z_1^{(i)}, \ldots, z_{L_i}^{(i)} \} \) are such that if we put \( z_0^{(i)} = 0 \) and \( z_{L_i+1}^{(i)} = 1 \) then \( |Z^{(i+1)} \cap \left( z_j^{(i)}, z_{j+1}^{(i)} \right)| = m_{i+1} \) for all \( i \in \mathbb{N}_0 \).

Fix any
\[ m_{n+1} \geq 2^n l_n \]
and define
\[ z^{(n+1)}_{i+m_{n+1}+k} = x^{(n)}_i + \frac{k}{m_{n+1}+1} \left( x^{(n)}_{i+1} - x^{(n)}_i \right) \]
for \( i = 0, 1, \ldots, l_n, k = 1, 2, \ldots, m_{n+1} \). Then put
\[ Z^{(n+1)} = \{ z^{(n+1)}_j : j = 1, 2, \ldots, L_{n+1} \}. \]

Finally enumerate elements of set
\[ \bigcup_{j=0}^{n+1} Z^{(j)} = \left( x^{(n+1)}_j \right)_{j=1}^{l_{n+1}} \]
in such a way that
\[ 0 = x^{(n+1)}_0 < x^{(n+1)}_1 < \cdots < x^{(n+1)}_{l_{n+1}+1} = 1. \]

Now we define the sequence of points
\[ A_{n,k} = \left( z^{(n)}_k, \frac{1}{2^n} \right) \subset I \times I \]
for \( n \in \mathbb{N}_0 \) and \( k \in \{1, 2, \ldots, L_n\} \).

Let us notice that by our construction
\[ \text{diam} \left( z^{(k+1)}_j, z^{(k+1)}_{j+1} \right) \leq \frac{1}{m_{k+1}+1} \text{diam} \left( z^{(k)}_j, z^{(k)}_{j+1} \right) \]
for any \( k \) and \( j \) (see 7.1).

If \( \pi \) denotes projection on the first coordinate, i.e. \( \pi(x, y) = (x, 0) \), we get that \( \pi(A_{n,i}) \neq \pi(A_{m,j}) \) for every \( n \neq m \) and every \( i \in \{1, 2, \ldots, L_n\}, j \in \{1, 2, \ldots, L_m\} \).
Lemma 7.1. The set
\[ \mathcal{D} = (I \times \{0\}) \cup \bigcup_{n=0}^{\infty} \bigcup_{k=1}^{L_n} \left\{ \left( z_k^{(n)}, y \right) : y \in \left[ 0, \frac{1}{2^n} \right] \right\} \]
is a dendrite.

Proof. It is enough to show that \( \mathcal{D} \) is locally connected. Let \( x \in \mathcal{D} \setminus (I \times \{0\}) \) and let \( U \) be a non-empty open set such that \( x \in U \). There exist some \( t \geq 0 \) such that \( x = (a, b) \in I \times \left[ \frac{1}{2^n}, \frac{1}{2^n} \right] \). Let \( \hat{\pi} \) be the projection such that \( \hat{\pi}(u, v) = u \) and let denote
\[ \epsilon_1 = \min \left\{ \left| b - \frac{1}{2^n-1} \right|, \left| b - \frac{1}{2^n+2} \right| \right\} \]
and
\[ \delta_t = \min_{n \leq t+1, \frac{1}{2^n} \leq \epsilon_1} \left\{ \left| \hat{\pi}(x) - \hat{\pi} \left( z_l^{(n)} \right) \right| : \hat{\pi}(x) \neq \hat{\pi} \left( z_l^{(n)} \right) \right\}. \]
Now taking \( \epsilon = \frac{1}{2} \min \{ \epsilon_1, \delta_t \} \) the set
\[ V = \left( (a - \epsilon, a + \epsilon) \times (b - \epsilon, b + \epsilon) \right) \cap \mathcal{D} \]
is connected because it is a segment \( \{a\} \times (b - \epsilon, b + \epsilon) \). Finally if \( x = (a, 0) \in I \times \{0\} \) then taking neighbourhood \( (x - \epsilon_0, x + \epsilon_0) \times [0, \epsilon_0) \) for some \( \epsilon_0 > 0 \) we get connected set \( V \subset U \). It ends the proof. \( \square \)

Now we define map \( f : \mathcal{D} \to \mathcal{D} \) in the following way
(i) if \( n \) is even
\[ f \left( \left( z_k^{(n)}, y \right) \right) = \begin{cases} 
\left( z_{k+1}^{(n)}, \phi_n(y) \right) & \text{if } k = 1, 2, \ldots, L_n - 1, \\
\left( z_{L_n+1}^{(n+1)}, \frac{1}{2} \phi_n(y) \right) & \text{if } k = L_n.
\end{cases} \]
for \( y \in \left[ \frac{3}{2^n+2}, \frac{1}{2^n} \right] \),
(ii) \[ f \left( (z_k^{(n)}, y) \right) = (\psi_n(y), 0) \quad \text{for } y \in \left[ \frac{1}{2n+1}, \frac{3}{2n+1} \right], \]

(2) if \( n \) is odd

(i) \[ f \left( (z_{L_n+1-k}^{(n)}, y) \right) = \begin{cases} (z_{L_n-k}^{(n)}, \phi_n(y)) & \text{if } k = 1, 2, \ldots, L_n - 1, \\ (z_1^{(n+1)}, \frac{1}{2} \phi_n(y)) & \text{if } k = L_n. \end{cases} \]

for \( y \in \left[ \frac{3}{2n+2}, \frac{1}{2n} \right] \),

(ii) \[ f \left( (z_k^{(n)}, y) \right) = \left( \psi_n(y), 0 \right) \quad \text{for } y \in \left[ \frac{1}{2n+1}, \frac{3}{2n+1} \right], \]

(3) \[ f \left( (z_k^{(n)}, y) \right) = \pi(A_n, k) \]

for all \( y \in [0, 0] \) and any \( n \),

(4) \[ f((x, 0)) = (x, 0) \quad \text{for all } x \in I, \]

where functions \( \phi_n, \psi_n \) and \( \hat{\psi}_n \) are increasing linear functions such that

\[
\phi_n \left( \left[ \frac{3}{2n+2}, \frac{1}{2n} \right] \right) = \left[ 0, \frac{1}{2n} \right],
\]

\[
\psi_n \left( \left[ \frac{1}{2n+1}, \frac{3}{2n+2} \right] \right) = \begin{cases} [z_k^{(n)}, z_{k+1}^{(n)}] & \text{if } k = 1, 2, \ldots, L_n - 1, \\ [z_{L_n+1}^{(n+1)}, z_{L_n+1}^{(n)}] & \text{if } k = L_n, \end{cases}
\]

\[
\hat{\psi}_n \left( \left[ \frac{1}{2n+1}, \frac{3}{2n+2} \right] \right) = \begin{cases} [z_{L_n-k}^{(n)}, z_{L_n+1-k}^{(n)}] & \text{if } k = 1, 2, \ldots, L_n - 1, \\ [z_1^{(n+1)}, z_2^{(n+1)}] & \text{if } k = L_n. \end{cases}
\]

By \( I_k^n \) we denote the segment connecting points \( (z_k^{(n)}, 0) \) and \( A_n, k \). Note that \( \text{diam } I_k^n = \frac{1}{2n} \) for all \( k \in \{1, 2, \ldots, L_n\} \).

**Lemma 7.2.** The map \( f \) is a continuous map on \( D \).

**Proof.** Let \( y \in D \) and \( (y_n)_{n=1}^{\infty} \subset D \) be the sequence such that \( \lim_{n \to \infty} y_n = y \).

Let us consider the cases.

**Case 1.** Let \( y \in D \setminus (I \times \{0\}) \) i.e. \( y \in I_k^m \) for some \( m \in \mathbb{N} \) and \( k \in \{1, 2, \ldots, L_m\} \).

(1a) If we have that \( \rho(y_n, \pi(y_n)) > \frac{1}{2n+1} \) for all \( n \) then \( y_n \in \bigcup_{j \leq m} \bigcup_{i \leq L_j} I_j^n \). Since there are finitely many \( I_j^n \) we may assume that \( y_n \in I_j^n \) for some fixed \( j, r \) and all \( n \). But then \( r = m \) and \( j = k \) and by continuity of functions \( \phi_m, \psi_m \) and \( \hat{\psi}_m \) we get that \( f \) is continuous.
Figure 2. Sketch how the map $f$ works

(1b) If $\rho(y, \pi(y)) \leq \frac{1}{2m+1}$. It means that there exists $t > m$ such that $y = (y_1, y_2) \in I^m_k \cap (I \times \left(\frac{1}{2^t}, \frac{1}{2^{t-1}}\right))$. Let us denote

$$\varepsilon_t = \min_{n \leq t+2} \min_{1 \leq l \leq L} \left\{ \left| \hat{\pi}(y) - \hat{\pi}(z_{j}^{(n)}) \right| : \hat{\pi}(y) \neq \hat{\pi}(z_{j}^{(n)}) \right\}$$

and

$$\delta_j = \left| y_2 - \frac{1}{2^j} \right|.$$

Now take

$$\varepsilon = \min \left\{ \frac{\varepsilon_t}{2}, \frac{\delta_{t-1}}{2}, \frac{\delta_{t+2}}{2} \right\}$$

and assume that $\rho(y_n, y) < \varepsilon$. We get that there exists $N \in \mathbb{N}$ such that for all $n > N$ we have $y_n \in I^m_k$ and

$$\text{diam}[y_n, \pi(y_n)] < \frac{1}{2m+1}$$

and therefore

$$\rho(f(y_n), f(y)) < \rho(y_n, y) \rightarrow 0$$

which proves that $f$ is continuous.

Case 2. Let $y \in I \times \{0\}$ and $\rho(y_n, \pi(y_n)) < \frac{1}{2m+1}$ for all $n$.

(2a) If $f(y_n) = \pi(y_n)$ for all $n$ then

$$\rho(f(y_n), f(y)) \leq \rho(\pi(y_n), y) < \rho(y_n, y) \rightarrow 0$$

so the $f$ is continuous.

(2b) If $f(y_n) \neq \pi(y_n)$ for all $n$ then $y_n \in I^r_j$ for some $r > 0$ and $j \in \{1, 2, \ldots, L_r\}$ and $\text{diam} I^r_j < \frac{1}{2^m}$. It means that for even $n$ we have

$$f(y_n) \in I^r_{j+1}$$

or

$$f(y_n) \in \left[z_{j}^{(r)}, z_{j+1}^{(r)}\right]$$

or

$$f(y_n) \in I^r_{L_r+1}$$

for $j = L_r$. 
or
\[ f(y_n) \in \left[ z_{r+1}^{(r)}, z_{r+2}^{(r)} \right] \]
and by the other hand if \( r \) is odd we have
\[ f(y_n) \in I_{r-1} \]
or
\[ f(y_n) \in \left[ z_{j-1}^{(r)}, z_{j}^{(r)} \right] \]
or
\[ f(y_n) \in I_{r+1}^{n+1} \text{ for } j = 1 \]
or
\[ f(y_n) \in \left[ z_{1}^{(r+1)}, z_{1}^{(r)} \right] . \]

Now we get
\[ \rho(f(y_n), f(y)) \leq \rho(f(y_n), y_n) + \rho(y_n, y). \]
If \( f(y_n) \in I_{j+1} \) then
\[ \rho(f(y_n), y_n) \leq \frac{2}{r} + \frac{1}{l_r+1} \leq \frac{3}{2r} < \frac{1}{2r-2}. \]
and if \( f(y_n) \in I_{L+1}^{r+1} \) or \( f(y_n) \in I_{1}^{r+1} \) we get that
\[ \rho(f(y_n), y_n) \leq \frac{1}{2r} + \frac{1}{l_r+1} + \frac{1}{2r+1} \leq \frac{3}{2r} < \frac{1}{2r-2}. \]
If \( f(y_n) \in I \times \{0\} \) we get
\[ \rho(f(y_n), y_n) \leq \frac{1}{2r} + \frac{1}{l_r+1} \leq \frac{1}{2r-1} < \frac{1}{2r-2}. \]

Now we have that
\[ \rho(f(y_n), f(y)) \leq \rho(f(y_n), y_n) + \rho(y_n, y) < \frac{1}{2r-2} + \rho(y_n, y) \to 0 \]
because if \( n \) grows then \( r \) also grows. Indeed, by contradiction suppose that there exists \( r(n) \) and subsequence \( \{y_{n_k}\}_{k=1}^{\infty} \) such that \( y_{n_k} \in I_j \) for all \( k \). We get that \( y \in I_j \) and
\[ \rho(f(y_{n_k}), \pi(y_{n_k})) < \frac{1}{2r+1} \]
and it implies that \( f(y_{n_k}) = \pi(y_{n_k}) \) which is contradiction with assumptions that \( f(y_n) \neq \pi y_n \) in this case. So, the proof of continuity of the map \( f \) is complete.

Lemma 7.3. Let \( x, y \in D \). If for every positive integer \( n \) points \( f^n(x), f^n(y) \) are not fixed points of \( D \) then
\[ \lim_{n \to \infty} \rho(f^n(x), f^n(y)) = 0. \]
Proof. Without loss of generality we may assume that there are \( m, i \geq 0 \) and \( k \geq 0 \) such that \( y \in I_i^m \) and \( f^k(x) \in I_i^m \). For all \( N \) there exist \( s \) and \( m' > N \) such that

\[
f^s(y) \in I_{i+k}^{m'},
\]

and

\[
\text{dist} \left( I_{i+k}^{m'}, I_i^{m'} \right) < \frac{k}{l_{m'} + 1},
\]

for \( i, i + k \in \{1, 2, \ldots, L_{m'}\} \).

Now we have

\[
\rho(f^s(x), f^s(y)) \leq \text{diam} I_{i+k}^{m'} + \text{dist} \left( I_{i+k}^{m'}, I_i^{m'} \right) + \text{diam} I_i^{m'} \leq \frac{1}{2^{m'-1}} + \frac{k}{l_{m'} + 1}.
\]

Let choose \( r > s \) and assume that \( f^r(x) \in I_j^{m''} \) for some \( j \) and \( m'' > m' \). Let us consider the cases.

(a) If \( r \) is even and \( f^r(y) \in I_j^{m''} \) for \( j + k \leq L_{m''} \) or if \( r \) is odd and \( f^r(y) \in I_j^{m''} \) for \( j - k \geq 1 \) we get that

\[
\rho(f^r(x), f^r(y)) \leq \text{diam} I_j^{m''} + \text{dist} \left( I_j^{m''}, I_i^{m''} \right) + \text{diam} I_i^{m''} \leq \frac{1}{2^{m''-1}} + \frac{k}{l_{m''} + 1} \leq \frac{1}{2^{m'''-1}} + \frac{k}{l_{m'} + 1}.
\]

(b) If \( j + k > L_{m''} \) and \( r \) is even then \( f^r(y) \in I_{L_{m''}+1}^{m''+1} + L_{m''} - j - k \) and we have

\[
\rho(f^r(x), f^r(y)) \leq \text{diam} I_{L_{m''}+1}^{m''+1} + L_{m''} - j - k + \text{dist} \left( I_{L_{m''}+1}^{m''+1}, I_{L_{m''}+1}^{m''} \right) + \text{dist} \left( I_{L_{m''}+1}^{m''}, I_j^{m''} \right) + \frac{1}{2^{m''}} \leq \frac{1}{2^{m''}} + \frac{j + k - L_{m''}}{l_{m''} + 1} + \frac{L_{m''} - j}{l_{m''} + 1} + \frac{1}{2^{m''}} \leq \frac{1}{2^{m''-1}} + \frac{k}{l_{m''} + 1} \leq \frac{1}{2^{m'''-1}} + \frac{k}{l_{m'} + 1}.
\]

(c) If \( f^r(y) \in I_{k-j}^{m''-1} \) for \( j - k < 1 \) and \( r \) is odd then in the similar way as above we get

\[
\rho(f^r(x), f^r(y)) \leq \text{diam} I_{k-j}^{m''-1} + \text{dist} \left( I_{k-j}^{m''-1}, I_j^{m''} \right) + \text{diam} I_j^{m''} \leq \frac{1}{2^{m''-1}} + \text{dist} \left( I_{k-j}^{m''-1}, I_j^{m''} \right) + \frac{1}{2^{m''}} \leq \frac{1}{2^{m''-1}} + \frac{k - j}{l_{m''-1} + 1} + \frac{j}{l_{m''-1} + 1} + \frac{1}{2^{m''}} \leq \frac{1}{2^{m'''-2}} + \frac{k}{l_{m'''-1} + 1} \leq \frac{1}{2^{m'''-1}} + \frac{k}{l_{m'} + 1}.
\]

Finally we get that for all \( r \geq s \)

\[
\rho(f^r(x), f^r(y)) \leq \frac{1}{2^{m'''-1}} + \frac{k}{l_{m'} + 1}.
\]
but \( m' \) can be arbitrarily large, thus
\[
\lim_{r \to \infty} \rho(f^r(x), f^r(y)) = 0
\]
and we get that \((x, y) \in \mathcal{D} \times \mathcal{D}\) is asymptotic. \( \square \)

**Lemma 7.4.** Let \( f : \mathcal{D} \to \mathcal{D} \) then there exists the sequence \((w_n)_{n=0}^{\infty}\) such that
\[
\lim_{n \to \infty} w_n = 0
\]
and for each even \( n \) and for \( j \in \{1, 2, \ldots, m_n + 1\} \)
\[
\rho\left(f^{l_n + j}\left(\frac{1}{2}, 1\right), (1, 0)\right) \leq w_n
\]
and for odd \( n \) we have
\[
\rho\left(f^{l_n + j}\left(\frac{1}{2}, 1\right), (0, 0)\right) \leq w_n.
\]

**Proof.** We claim that for even \( k \) we have
\[
\text{diam}\left(z^{(k)}_L, 1\right) \leq \frac{1}{l_k + 1}
\]
and for odd \( k \)
\[
\text{diam}\left(0, z^{(k)}_1\right) \leq \frac{1}{l_k + 1}.
\]
We will prove the claim by induction on \( k \). It is easy check that for \( k = 0 \) the inequality holds. If \( k = 1 \) then by (7.2) we have
\[
\text{diam}\left(0, z^{(1)}_1\right) \leq \frac{1}{m_0 + 1} \cdot \text{diam}\left(0, z^{(0)}_1\right) \leq \frac{1}{m_0 + 1} \cdot \frac{1}{l_0 + 1} = \frac{1}{l_1 + 1} \leq \frac{1}{4}.
\]
Now, assume that the claim holds for some even \( k > 0 \). By (7.1) we get that
\[
\text{diam}\left(z^{(k+2)}_{L_{k+2}}, 1\right) \leq \frac{1}{m_{k+2} + 1} \cdot \text{diam}\left(z^{(k+1)}_{L_{k+1}}, 1\right) \leq \frac{1}{m_{k+2} + 1} \cdot \frac{1}{m_{k+1} + 1} \cdot \text{diam}\left(z^{(k)}_L, 1\right)
\]
Now by induction assumptions on \( \text{diam}\left(z^{(k)}_L, 1\right)\) and property that \( L_{k+1} + l_k = l_{k+1} \) we have
\[
\text{diam}\left(z^{(k+2)}_{L_{k+2}}, 1\right) \leq \frac{1}{(m_{k+2} + 1)(m_{k+1} + 1)} \cdot \frac{1}{l_k + 1} = \frac{1}{m_{k+2} + 1} \cdot \frac{1}{m_{k+1} + 1} \cdot \frac{1}{l_k + 1} \leq \frac{1}{m_{k+2} + 1} \cdot \frac{1}{m_{k+1} + 1} \cdot \frac{1}{l_k + 1} = \frac{1}{l_{k+2} + 1}.
\]
So this case is proved. The case that \( k \) is odd is proved in the same way.

Note that
\[
\pi\left(f^{l_n + j}\left(\frac{1}{2}, 1\right)\right) \in \left(z^{(n)}_{L_n}, 1\right) \times \{0\} \subset I \times \{0\}
\]
for \( j = 1, \ldots, m_{n+1} \) and \( n \) even. Now we have
\[
\rho \left( f^{l_n+j} \left( \frac{1}{2}, 1 \right), (1,0) \right) \leq \rho \left( f^{l_n+j} \left( \frac{1}{2}, 1 \right), \pi \left( f^{l_n+j} \left( \frac{1}{2}, 1 \right) \right) \right) + \rho \left( \pi \left( f^{l_n+j} \left( \frac{1}{2}, 1 \right) \right), (1,0) \right) \leq \frac{1}{2^n} + \text{diam} \left( z^{(n)}_{1_n}, 1 \right) \leq \frac{1}{2^n} + \frac{1}{l_n + 1}.
\]
When \( n \) is odd we have that
\[
\pi \left( 0, f^{l_n+j} \left( \frac{1}{2} \right) \right) \in \left( 0, z_1^{(n)} \right) \times \{0\} \subset I \times \{0\}
\]
for all \( j \in \{1, 2, \ldots, m_{n+1} \} \) and
\[
\rho \left( f^{l_n+j} \left( \frac{1}{2}, 1 \right), (0,0) \right) \leq \frac{1}{2^n} + \frac{1}{l_n + 1}.
\]
It is enough to take
\[
w_n = \frac{1}{l_n + 1} + \frac{1}{2^n}
\]
for all \( n \in \mathbb{N} \cup \{0\} \).

**Lemma 7.5.** The map \( f : \mathbb{D} \to \mathbb{D} \) has DC1 pair.

**Proof.** We will show that points \( \left( \frac{1}{2}, 1 \right) \) and \( (1,0) \) form a (DC1) pair i.e.
\[
\liminf_{n \to \infty} \frac{1}{n} \left| \left\{ 0 \leq m \leq n - 1 : \rho \left( f^m \left( \frac{1}{2}, 1 \right), (1,0) \right) < s \right\} \right| = 0
\]
for some \( s > 0 \) and
\[
\limsup_{n \to \infty} \frac{1}{n} \left| \left\{ 0 \leq m \leq n - 1 : \rho \left( f^m \left( \frac{1}{2}, 1 \right), (1,0) \right) < t \right\} \right| = 1
\]
for all \( t > 0 \).

From lemma 7.4 we have for each even \( n \) and for \( j \in \{1, 2, \ldots, m_{n+1} \} \)
\[
\rho \left( f^{l_n+j} \left( \frac{1}{2}, 1 \right), (1,0) \right) \leq w_n
\]
and for all sufficiently large odd \( n \) we have
\[
\rho \left( f^{l_n+j} \left( \frac{1}{2}, 1 \right), (1,0) \right) > \frac{1}{2}.
\]

Now if we fix sufficiently large \( N = N(t) \in \mathbb{N} \) then for all odd \( n \geq N \) inequality holds
\[
\left| \left\{ 0 \leq j \leq l_n + m_{n+1} - 1 : \rho \left( f^j \left( \frac{1}{2}, 1 \right), (1,0) \right) < w_N < t \right\} \right| \geq \frac{l_n + m_{n+1}}{t_n + m_{n+1}} \geq 1 - \frac{l_n}{m_{n+1}} \geq 1 - \frac{l_n}{2^n l_n} = 1 - \frac{1}{2^n} \to 1.
\]
To show the second property of (DC1) let \( s = \frac{1}{2} \) and then for all even \( n \geq N \) we get
\[ \left| \{ 0 \leq j \leq l_n + m_{n+1} - 1 : \rho \left( f^j \left( \frac{1}{n}, 1 \right), (1, 0) \right) < \frac{1}{2n} \} \right| \leq \]
\[ \leq 1 - \frac{m_{n+1}}{l_n + m_{n+1}} \leq \frac{l_n}{l_n + m_{n+1}} \leq \frac{l_n}{2^nl_n} = \frac{1}{2^n} \to 0. \]

It completes the proof. \qed

**Theorem 7.6.** There exists a continuous self-map \( f \) of dendrite such that:

1. \( f \) has DC1 pair,
2. \( f \) does not have an infinite LY-scrambled set.

**Proof.** From lemma 7.5 we have that \( f \) has (DC1) pair and from Lemma 7.3 we get that there is no \( (x, y, z) \) scrambled set. Indeed, let assume that \( f^n(x) \notin I \) for all \( n \in \mathbb{N} \). Now, if \( (x, y) \) and \( (x, z) \) are a LY-pairs then exist \( n, m \in \mathbb{N} \) such that \( f^n(y) \in I \) and \( f^m(z) \in I \). We get that \( (y, z) \) is not LY-pair because if \( f^n(y) \neq f^m(z) \) then the distance between them is always positive and by the other hand if \( f^n(y) = f^m(z) \) then the distance is always zero. \qed

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