Dirac Operators and the Calculation of the Connes Metric on arbitrary (Infinite) Graphs

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Abstract

As an outgrowth of our investigation of non-regular spaces within the context of quantum gravity and non-commutative geometry, we develop a graph Hilbert space framework on arbitrary (infinite) graphs and use it to study spectral properties of graph-Laplacians and graph-Dirac-operators. We define a spectral triplet sharing most of the properties of what Connes calls a spectral triple. With the help of this scheme we derive an explicit expression for the Connes-distance function on general directed or undirected graphs. We derive a series of apriori estimates and calculate it for a variety of examples of graphs. As a possibly interesting aside, we show that the natural setting of approaching such problems may be the framework of (non-)linear programming or optimization. We compare our results (arrived at within our particular framework) with the results of other authors and show that the seeming differences depend on the use of different graph-geometries and/or Dirac operators.

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1 Introduction

In recent years we started a programme to reconstruct continuum physics and/or mathematics from an underlying more primordial and basically discrete theory living on the Planck-scale (cf. [1] to [4] or [37]). As sort of a “spin-off” various problems of a more mathematical and technical flavor emerged which may have an interest of their own. Discrete differential geometric concepts were dealt with in [1], the theory of random graphs was a central theme of [2], topics of dimension theory and fractal geometry were addressed in [4], lump-spaces and random metrics in [37].

If one wants to recover the usual (differential) operators (or more generally, the concepts of standard functional analysis), being in use in ordinary continuum physics and mathematics by some sort of limiting process from their discrete protoforms, which live, on their part, on a relatively disordered discrete background like, say, a network, it is reasonable to analyse in a first step these discrete counterparts more closely. This will be one of our themes in the following with particular emphasis on discrete Laplace- and Dirac-operators on general graphs. In contrast to [16] we now also include arbitrary directed graphs. The main thrust goes however into an analysis of metrical concepts on discrete (non-commutative) spaces like general graphs, being induced by graph Dirac-operators and the Connes-distance functional.

We note in passing that functional analysis on graphs is both of interest in pure and applied mathematics and also in various fields of (mathematical) physics. For one, discrete systems have an increasing interest of their own or serve as easier to analyse prototypes of their continuum counterparts. To mention a few fields of applications: graph theory in general, analysis on (discrete) manifolds, lattice or discretized versions of physical models in statistical mechanics and quantum field theory, non-commutative geometry, networks, fractal geometry etc. From the widely scattered (mathematical and physical) literature we mention (possibly) very few sources we are aware of. Some of them were of relevance for our own motivation, some others we came across only recently (see e.g. [3], [4], [13], [14] or [29]). Some more literature like e.g. [17] was pointed out to us by Mueller-Hoissen; the possible relevance of references [17] to [19] were brought to our notice by some unknown referee. Last, but not least, there is the vast field of discretized quantum gravity (see e.g. [21] or [22]). All this shows that the sort of discrete functional analysis we are dealing with in the following, is presently a very active field with a lot of different applications.

For the convenience of the reader we begin with compiling some concepts and tools dealing with graph Hilbert spaces which we then use to investigate the spectral properties of graph Laplacians and Dirac operators. In a next step we study and test concepts and ideas, which arose in the context of non-commutative geometry. As we (and others) showed in preceding papers, networks and graphs may (or even should) be understood as examples of non-commutative spaces.
A currently interesting topic in this field is the investigation of certain distance functionals on “nasty” or non-standard spaces and their mathematical or physical “naturalness”. Graphs carry, on the one hand, a natural metric structure given by a distance function \(d(x, y)\), with \(x, y\) two nodes of the graph (see the following sections). This fact was already employed by us in e.g. [4] to develop dimensional concepts on graphs. Having Connes’ concept of distance in noncommutative geometry in mind (cf. chapt. VI of [5]), it is natural to try to compute it in model systems, which means in our context: arbitrary graphs, and compare it with the already existing notion of graph distance mentioned above. (We note in passing that the calculation of the Connes distance for general graphs turns out to be surprisingly complex and leads to perhaps unexpected connections to fields of mathematics like e.g. (non-)linear programming or optimization; see the last section).

Therefore, as one of many possible applications of our formalism, we construct a protoform of what Connes calls a spectral triple, that is, a Hilbert space structure, a corresponding representation of a certain (function) algebra and a (in our framework) natural candidate for a so-called Dirac operator (not to be confused with the ordinary Dirac operator of the Dirac equation), which encodes certain properties of the graph geometry. This will be done in section 3.

In the last (and central) section, which deals with the distance concept deriving from this spectral triplet we will investigate this concept more closely as far as graphs and similar spaces are concerned. In this connection some recent work should be mentioned, in which Connes’ distance function was analyzed in certain simple models like e.g. one-dimensional lattices ([25]-[27]). These papers already show that it is a touchy business to isolate “the” appropriate Dirac operator (after all, different Dirac operators are expected to lead to different geometries!) and that it is perhaps worthwhile to scrutinize the whole topic in a more systematic way. We show in particular that one may choose different Dirac-operators on graphs (or rather, different types of graphs over the same node set) which may lead to different results for e.g. the corresponding Connes-distance.

The problem of finding suitable metrics on “non-standard” spaces is a particularly interesting research topic of its own, presently pursued by quite a few people (see the papers by Rieffel or Weaver [12], [13] and the references mentioned therein). Another, earlier (and important) source is [15]. We recently extended the investigation of metric structures to so-called lump spaces and probabilistic metric spaces (see [37] and [38]) and employed it in the general context of quantum gravity (cf. also [2]).

Remark: When we wrote an earlier draft of this paper we were unaware of the content of [15]. It happened only recently that we realized that various of the results we derived in connection with Dirac-operators and the Connes-distance can already be found in [15] and we try to take care of this fact in the following. Both the technical approach and the motivation are, however, not entirely
identical. The interested reader may consult the electronic version of our earlier draft (16).

The same applies to paper [10] where some of the Hilbert space methods were developed we later rederived in [16] being unaware of the prior results. As a consequence we drop most of the technical steps leading to that part of our previous results overlapping with corresponding parts in the above mentioned papers and refer, for simplicity to [16] where the interested reader can find more details.

2 A brief Survey of Differential and Operator Calculus on Graphs and Graph Hilbert Spaces

We give a brief survey of certain concepts and tools needed in the following analysis. While our framework may deviate at various places from the more traditional one, employed in e.g. algebraic graph theory (see e.g. 8, 9, or 17), this is mainly done for reasons of greater mathematical flexibility and generality and, on the other side, possible physical applications (a case in point being the analysis of non-commutative spaces). Most of the technical tools which are not defined in detail in the following have been introduced in section 3 of 1.

2.1 Simple (Symmetric) Graphs

For simplicity we assume the graph to be connected and locally finite (no elementary loops and multiedges, whereas these could easily be incorporated in the framework), i.e. each node (or vertex) is incident with only a finite number of edges (or bonds). To avoid operator domain problems we usually make the even stronger assumption that the vertex– or node degree, $v_i$, ($i$ labelling the nodes), is globally bounded but this restriction is frequently not really necessary. Furthermore, it has turned out to be algebraically advantageous to identify an (undirected) labelled graph with a directed graph having two oppositely directed edges for each undirected edge, the directed edge, pointing from node $n_i$ to node $n_k$, being denoted by $d_{ik}$, the oppositely directed edge by $d_{ki}$ and the undirected (but orientable) edge by their superposition

$$b_{ik} = d_{ik} - d_{ki} = -b_{ki}$$

(1)

($d_{ik}$ and $d_{ki}$ are treated as independent basis vectors; cf. 1 or 16).

As the elementary building blocks of our graph Hilbert spaces we take \{$n_i$\} and \{d_{ik}\} as basis elements of a certain hierarchy of Hilbert spaces over, say, $\mathbb{C}$ with scalar product induced by

$$\langle n_i | n_k \rangle = \delta_{ik} \quad (d_{ik} | d_{lm}) = \delta_{il} \cdot \delta_{km}$$

(2)
This definition implies $(b_{ik}b_{ik}) = 2$.

**Definition 2.1 (Vertex-, Edge Hilbert Space)** The Hilbert spaces $H_0$, $H_1^a$ (a for antisymmetric) and $H_1$ consist of the formal sums

$$f := \sum f_in_i \quad g := \sum g_{ik}d_{ik} \quad \text{with} \quad g_{ik} = -g_{ki} \quad \text{and} \quad g' := \sum g_{ik}d_{ik}$$

(3)

$$\sum |f_i|^2 < \infty \quad \sum |g_{ik}|^2 < \infty$$

(4)

$f_i, g_{ik}$ ranging over a certain given field like e.g. $\mathbb{C}$ (sometimes only rings like e.g. $\mathbb{Z}$ are admitted; then we are dealing only with modules). We evidently have $H_1^a \subset H_1$.

Remark: One could continue this row of vector spaces in ways which are common practice in, say, algebraic topology (see [1] sections 3.1 and 3.2). In this context they are frequently called chain complexes (see also [20]). On the other hand, the above vector spaces could as well be viewed as discrete function spaces over the node-, bond set with $n_i, d_{ik}$ now representing the elementary indicator functions.

Proceeding in this spirit we can now introduce two linear maps between $H_0, H_1$ extending the usual boundary- and coboundary map. On the basis elements they act as follows:

$$\delta : d_{ik} \rightarrow n_k \quad \text{hence} \quad b_{ik} \rightarrow n_k - n_i$$

(5)

$$d : n_i \rightarrow \sum_k (d_{ki} - d_{ik}) = \sum_k b_{ki}$$

(6)

and linearly extended. That is, $\delta$ maps the directed bonds $d_{ik}$ onto the terminal node and $b_{ik}$ onto its (oriented) boundary, while $d$ maps the node $n_i$ onto the sum of the ingoing directed bonds minus the sum of the outgoing directed bonds or on the sum of oriented ingoing bonds $b_{ki}$.

As was shown in [1] (we later realized that the same definition was already employed in [15]), these definitions lead in fact to a kind of discrete differential calculus on $H_0, H_1$, that is we have

$$df = d(\sum f_in_i) = \sum_{k,i} (f_k - f_i)d_{ik}$$

(7)

Combining now the operators $\delta$ and $d$, we can construct the canonical graph Laplacian. On the vertex space it reads:

$$\delta df = -\sum_i (\sum_k f_k - v_i \cdot f_i)n_i = -\sum_i (\sum_k (f_k - f_i))n_i =: -\Delta f$$

(8)
where \( v_i \) denotes the node degree or valency defined above and the \( k \)-sum extends over the nodes adjacent to \( n_i \).

Remark: Note that there exist several variants in the literature (see e.g. [17] or [9]). Furthermore, many mathematicians employ a different sign-convention. We stick in the following to the convention being in use in the mathematical-physics literature where \(-\Delta\) is the positive(!) operator.

This graph Laplacian is intimately connected with another important object, employed in algebraic graph theory, i.e. the adjacency matrix, \( A \), of a graph, its entries, \( a_{ik} \), having the value one if the nodes \( n_i, n_k \) are connected by a bond and are zero elsewhere. If the graph is undirected (but orientable), the relation between \( n_i, n_k \) is symmetric, i.e.

\[
a_{ik} = 1 \implies a_{ki} = 1
\]

This has the obvious consequence that in case the graph is simple and undirected, \( A \) is a symmetric matrix with zero diagonal elements.

Remark: More general \( A \)'s occur if more general graphs are admitted (e.g. general multigraphs).

With our definition of \( \Delta \) it holds:

\[
\Delta = A - V
\]

where \( V \) is the diagonal degree matrix, having \( v_i \) as diagonal entries. (Note that the other sign-convention would lead to \( \Delta = V - A \)).

To avoid domain problems we assume from now on that the node degree, \( v_i \), is uniformly bounded on the graph \( G \), i.e.

\[
v_i \leq v_{\text{max}} < \infty
\]

Defining \( d_{1,2} \) as

\[
d_{1,2} : n_i \to \sum d_{ki}, \sum d_{ik}
\]

respectively and linearly extended, we get

\[
d = d_1 - d_2
\]

Similarly we make the identification \( \delta =: \delta_1 \) with

\[
\delta_{1,2} : d_{ik} \to n_k, n_i
\]

It is noteworthy (but actually not surprising) that \( v_i \leq v_{\text{max}} \) implies that all the above operators are bounded (in contrast to their continuous counterparts, which are typically unbounded). Taking this for granted at the moment, a straightforward analysis yields the following relations:
Lemma 2.2

1. The adjoint \( d^* \) of \( d \) with respect to the spaces \( H_0, H_1^a \) is \( 2\delta \).

2. On the other side we have for the natural extension of \( d, \delta \) to the larger space \( H_1 \):

\[
\delta_1 = (d_1)^* \quad \delta_2 = (d_2)^* \tag{15}
\]

hence

\[
(\delta_1 - \delta_2) = (d_1 - d_2)^* = d^* \neq 2\delta = 2\delta_1 \tag{16}
\]

3. Furthermore it holds

\[
d_1^* \cdot d_1 = \delta_1 \cdot d_1 = d_2^* \cdot d_2 = V : n_i \to v_i n_i \tag{17}
\]

(with \( V \) the vertex degree matrix)

\[
d_1^* \cdot d_2 = \delta_1 \cdot d_2 = d_2^* \cdot d_1 = \delta_2 \cdot d_1 = A : n_i \to \sum_{k-i} n_k \tag{18}
\]

Similar geometric properties of the graph are encoded in the products coming in reversed order.

(Here and in the following \( k - i \) means summation over the first index and runs through the set of labels of nodes directly connected with \( n_i \)).

That and how \( d, d^* \) encode certain geometric information about the graph can be seen from the following domain- and range-properties (cf. [16], for corresponding results in the more traditional approach see [8], p.24ff).

**Theorem 2.3** Let the graph be connected and finite, \(|V| = n\), then

\[
dim(Rg(d^*)) = n - 1 \tag{19}
\]

\[
dim(Ker(d^*)) = \sum_i v_i - (n - 1) \tag{20}
\]

With \( \dim(H_1) = \sum_i v_i \), \( \dim(H_1^a) = 1/2 \cdot \dim(H_1) \) we have

\[
codim(Ker(d^*)) = \dim(Rg(d)) = n - 1 \tag{21}
\]

We see that both \( Rg(d^*) \) and \( Rg(d) \) have the same dimension \((n - 1)\).
Remark 2.4 In case the graph has, say, \( c \) components, the above results are altered in an obvious way; we have for example

\[
\dim(\text{Rg}(d^*)) = n - c \tag{22}
\]

In the literature \( \text{Ker}(d^*) \) is called (for obvious reasons) the cycle subspace (cf e.g. [8]). On the antisymmetric subspace \( H_1^s \) we have \( d^* = 2\delta \) and \( \delta(b_{ik}) = n_k - n_i \). Choosing now a cycle, given by its sequence of consecutive vertices \( n_{i_1}, \ldots, n_{i_k}; n_{i_{k+1}} := n_{i_1} \), we have

\[
d^*(\sum b_{i_{k+1}i}) = 2 \sum (n_{i_{k+1}} - n_{i_k}) = 0 \tag{23}
\]

that is, vectors of this kind lie in the kernel of \( d^* \).

We will now provide quantitative lower and upper bounds for the respective norms of the occurring operators. We have:

\[
\|df\|^2 = \sum_{ik} |(f_k - f_i)|^2 = \sum_i v_i \cdot |f_i|^2 + \sum_k v_k \cdot |f_k|^2 - \sum_{i \neq k} (\overline{f}_k f_i + \overline{f}_i f_k)
\]

\[
= 2 \cdot \sum_i v_i |f_i|^2 - 2 \cdot \sum_{i \neq k} \overline{f}_k f_i \tag{24}
\]

which can be written as:

\[
\|df\|^2 = 2((f|Vf) - (f|Af)) = (f| - 2\Delta f) \tag{25}
\]

and shows the close relationship of the norm of \( d \) with the expectation values of the adjacency and degree matrix or the graph Laplacian. That is, norm estimates for, say, \( d \), derive in a natural manner from the corresponding estimates for \( A \) or \(-\Delta\). With

\[
\|df\|^2 = (f|d^* df) = (f| - 2\Delta f) \tag{26}
\]

we have

\[
0 \leq d^* d = -2\Delta \quad \text{and} \quad \|d\|^2 = \sup_{\|f\|=1} (f| - 2\Delta f) = \| - 2\Delta \| \tag{27}
\]

Furthermore via

\[
0 < \sup_{\|f\|=1} (f| - 2\Delta f) \leq 2v_{\text{max}} + 2 \sup_{\|f\|=1} |< f|Af >| \tag{28}
\]

we get

\[
\| - \Delta \| \leq v_{\text{max}} + \|A\| \tag{29}
\]
Remark: We want to mention that we are using the usual operator norm also for matrices (in contrast to most of the matrix literature), which is also called the spectral norm. It is unique in so far as it coincides with the so-called spectral radius (cf. e.g. [29] or [30]), that is

$$\|A\| := \sup\{\|\lambda\|; \lambda \in \text{specr}(A)\}$$  \hspace{1cm} (30)

In a first step we give upper and lower bounds for the operator norm of the adjacency matrix, $A$, both in the finite- and infinite-dimensional case. There are various proofs available of a varying degree of generality (see e.g. [10], [17] or [16]) to which we refer the reader. In the following we give only the final results. Note however that the transition from finite to infinite graphs is far from straightforward as in some of the necessary technical steps entirely new methods are needed.

**Theorem 2.5 (Norm of $A$)** With the adjacency matrix $A$ finite or infinite and a finite $v_{\text{max}}$ we have the following result (a certain fixed labelling of the nodes being assumed):

$$\limsup \frac{1}{n} \cdot \sum_{i=1}^{n} v^{(n)}_i \leq \limsup \|A_n\| = \|A\| = \sup\{\|\lambda\|; \lambda \in \text{specr}(A)\} \leq v_{\text{max}}$$  \hspace{1cm} (31)

Here $A_n$ are the adjacency matrices for the induced subgraphs, living over the first $n$ labelled nodes, $v^{(n)}_i$ is the corresponding induced (and $n$-dependent) node degree.

As a byproduct we have the important lemma:

**Lemma 2.6** The adjacency matrices, $A_n$, converge strongly to $A$ and we have in particular $\|A_n\| \not\rightarrow \|A\|$.

(For a proof of the latter result see [10] or [16])

**Remark 2.7** To prove strong convergence of operators is of some relevance for the limit behavior of spectral properties of the operators $A_n, A$. That is (cf. e.g. [7] section VIII.7), we have in that case ($A_n, A$ selfadjoint and uniformly bounded) $A_n \rightarrow A$ in strong resolvent sense, which implies that the spectrum of the limit operator, $A$, cannot suddenly expand, i.e.

$$\lambda \in \text{spec}(A) \Rightarrow \exists \lambda_n \in \text{spec}(A_n) \text{ with } \lambda_n \rightarrow \lambda$$  \hspace{1cm} (32)

and for $a, b \notin \text{spec}_{\text{pp}}(A)$

$$P_{(a,b)}(A_n) \rightarrow P_{(a,b)}(A) \text{ strongly}$$  \hspace{1cm} (33)
To test the effectiveness of the upper and lower bounds given above, we apply them to a non-trivial model recently discussed in [32], i.e. the infinite binary tree with root $n_0$ where $v_0$ is two and $v_i$ equals three for $i \neq 0$. The authors show (among other things) that the spectrum consists of the interval $[-2\sqrt{2},2\sqrt{2}]$, i.e. $\|A\| = 2\sqrt{2}$. $v_{\max}$ is three, we have to calculate $\limsup 1/n \cdot \sum v_i$. For simplicity we choose a subsequence so that $n := n(N)$ with $N$ denoting the $N$-th level (consisting of $2^N$ nodes) of the tree starting from the root $n_0$. Note that in the corresponding induced subgraph $G_N$ the boundary nodes sitting in the $N$-th level have only node degree one with respect to $G_N$ but three viewed as nodes in the full tree.

We then have
\[
n = \sum_{k=1}^{N} 2^k, \quad \sum_{i=0}^{n(N)} v_i = 2 + 3 \cdot \sum_{k=1}^{N-1} 2^k + 2^N = 3 \cdot \sum_{k=0}^{N} 2^k - 2 \cdot 2^N - 1 \tag{34}\]

Hence
\[
\lim_{n(N)} 1/n(n(N)) \sum_{i=0}^{n(N)} v_i = 3 - 2 \lim N \sum_{0}^{N} 2^{k-N} - 1 = 2 \tag{35}\]

That is, our general estimate imply $2 \leq \|A\| \leq 3$, which is not so bad.

### 2.2 Arbitrary Directed Graphs

If we deal with general directed graphs we have both ingoing and outgoing edges at each node but in general they do no longer occur in a symmetric way. But nevertheless, most of our concepts and tools, developed in the foregoing subsection, do still exist. The definitions of $d$ and $d^*$ are unaltered. As each edge, $d_{ik}$, is an outgoing edge for node $n_i$ and an ingoing edge for node $n_k$, the same expression holds for $df$, i.e.
\[
df = \sum_{i,k} (f_i - f_k)d_{ik} \quad \text{with} \quad f = \sum_{i} f_i n_i \tag{36}\]

(the sum of course only extends over those directed edges which do exist in the directed graph; in particular each edge, $d_{ik}$, is only counted once in the sum as an outgoing edge with respect to the label $i$.) Furthermore the notions of $d_{1,2}$ and $\delta_{1,2}$ remain the same, mapping nodes or edges on ingoing edges, outgoing nodes and vice versa. We again have
\[
\delta_{1,2} = d^*_{1,2} \quad \text{and} \quad (\delta_1 - \delta_2) = d^* = (d_1 - d_2) \tag{37}\]

We can now calculate $d^*_1 \cdot d_1$ and $d^*_2 \cdot d_2$ and get:
\[
d^*_1 \cdot d_1(n_i) = v_{i}^{in} \cdot n_i \quad , \quad d^*_2 \cdot d_2(n_i) = v_{i}^{out} \cdot n_i \tag{38}\]
with \( v_i^{\text{in, out}} \) the \( \text{in-, out-degree} \) of the node \( n_i \). In the same way we can calculate

\[
\delta_1 \cdot d_2(n_i) = \sum_{k-i, \text{out}} n_k \quad \text{and} \quad \delta_2 \cdot d_1(n_i) = \sum_{k-i, \text{in}} n_k
\] (39)

This yields

\[
d^*d = (\delta_1 - \delta_2) \cdot (d_1 - d_2) = (V^{\text{in}} + V^{\text{out}}) - (A^{\text{in}} + A^{\text{out}})
\] (40)

where the occurring operators on the rhs are the \( \text{in-, out-vertex degree matrices} \), \( \text{in-, out-adjacency matrices} \) respectively.

In general the individual \( \text{in-, out-adjacency matrices} \) are non-symmetric. Their sum, however, is symmetric (and is, in the case of a symmetric graph, twice the adjacency matrix of the undirected graph, i.e. 2\( A \)). One can now again define a (positive, i.e. symmetric) Laplace operator for a non-symmetric (directed) graph, that is:

**Conclusion 2.8**

\[
-\Delta := d^*d = (V^{\text{in}} + V^{\text{out}}) - (A^{\text{in}} + A^{\text{out}}) =: V_d - A_d
\] (41)

(\( \Delta \) Note the now missing factor two in front of \( \Delta! \))

3 \ The Spectral Triplet on a general (directed or undirected) Graph

We begin by making some remarks on various concepts, being in use in the more recent literature. We note that our version of a Dirac operator (defined below) intertwines node-vectors and bond-vectors while in other examples it maps node- to node-functions. Our bond-functions have (in some sense) the character of \( \text{cotangential-vectors} \), while in other approaches derivatives of functions are interpreted as \( \text{tangent-vectors} \). In our view, the latter formalism is only effective in certain classes of highly regular models (like e.g. lattices) where one has kind of global directions and will become cumbersome for general graphs. We developed this latter approach a little bit in section 3.3 of [1] and showed how these cotangent and tangent vectors can be mapped into each other. We think that, on the other side, our framework is more flexible in the general case. This holds in particular for our Dirac operator, which encodes certain geometric properties of the underlying discrete “manifold”.

The Hilbert space we will use in the following is

\[
H = H_0 \oplus H_1
\] (42)
The *natural representation* of the function algebra $\mathcal{F}$ (consisting of the bounded node functions)

$$\{f; f \in C^\infty \text{ i.e. } \sup_i |f_i| < \infty\} \quad (43)$$

on $H$ by bounded operators is given by:

$$H_0 : f \cdot f' = \sum f_i f'_i \cdot n_i \text{ for } f' \in H_0 \quad (44)$$

$$H_1 : f \cdot \sum g_{ik} d_{ik} := \sum f_i g_{ik} d_{ik} \quad (45)$$

From previous work ([1]) we know that $H_1$ carries also a right-module structure, given by:

$$\sum g_{ik} d_{ik} \cdot f := \sum g_{ik} f_k \cdot d_{ik} \quad (46)$$

(For convenience we do not distinguish notationally between elements of $\mathcal{F}$ and their Hilbert space representations).

Remark: The same bi-module structure and the Dirac operator defined below were already employed in [15], p.414ff.

An important object in various areas of modern analysis on manifolds or in Connes’ approach to noncommutative geometry is the so-called *Dirac operator* $D$ (or rather, a certain version or variant of its classical counterpart; for the wider context see e.g. [3] or [33] to [35]). As $D$ we will take in our context the operator:

$$D := \begin{pmatrix} 0 & d^* \\ d & 0 \end{pmatrix} \quad (47)$$

acting on

$$H = \begin{pmatrix} H_0 \\ H_1 \end{pmatrix} \quad (48)$$

with

$$d^* = (\delta_1 - \delta_2) \quad (49)$$

Note however, that there may exist in general several possibilities to choose such an operator. On the other hand, we consider our personal choice to be very natural from a geometrical point of view.
Lemma 3.1 There exists in our scheme a natural chirality- or grading operator, \( \chi \) and an antilinear involution, \( J \), given by
\[
\chi := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\] (50)
with
\[
[\chi, \mathcal{F}] = 0 \quad \chi \cdot D + D \cdot \chi = 0
\] (51)
and
\[
J : \begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} \overline{x} \\ \overline{y} \end{pmatrix}
\] (52)
so that
\[
J \cdot f \cdot J = \overline{f}
\] (53)

These are some of the ingredients which establish what Connes calls a spectral triple (cf. e.g. [23] or [24]). We do not want, however, to introduce the full machinery at the moment as our scheme has an independent geometric meaning of its own. Note in particular what we are saying below about (non) compactness of various operators in observation 3.3.

Definition 3.2 (Spectral Triplets) As spectral triplet on a general graph we take
\[(H, \mathcal{F}, D)\] (54)

In our general framework we got in a relatively straightforward manner a Hilbert space being the direct sum of the node space (a function space) and the bond space (resembling the set of cotangent vectors) and a Dirac operator which emerged naturally as kind of a square root of the Laplacian.

On the other side, if one studies simple models as e.g. in [25] to [27], other choices are possible. In [25], [24], where the one-dimensional lattice was studied, the symmetric difference operator was taken as Dirac operator. In [27] the one-dimensional lattice was assumed to be directed (i.e., in our notation, only \( d_{i,i+1} \) were present) and the Dirac operator was defined (somewhat ad hoc) as a certain self adjoint “doubling” of the (one-sided, i.e. non-symmetric) adjacency matrix.

As we will show below, this latter model fits naturally in our general approach which includes both directed and undirected graphs. All these Dirac operators are different and it is hence no wonder that they lead to different consequences (see below). It is our opinion that, in the end, an appropriate choice has to be dictated by physical intuition. Nevertheless, this apparent non-uniqueness should be studied more carefully.
As can be seen from the above, the connection with the graph Laplacian is relatively close since for e.g. a symmetric graph we have:

\[ D^2 = \begin{pmatrix} d^*d & 0 \\ 0 & dd^* \end{pmatrix} \]  

(55)

and

\[ d^*d = -2\Delta \]  

(56)

\( dd^* \) is the corresponding object on \( H_1 \). (In the vector analysis of the continuum the two entries correspond to \( \text{divgrad} \) \, \( \text{graddiv} \) respectively).

In the original approach of Connes great emphasis was laid on the compactness of operators like the inverse of the Dirac operator and it is part of the general definition of a spectral triple. As discrete spaces of the kind we are studying are non-trivial examples of non-commutative spaces, it is interesting that we can easily test whether this assumption is fulfilled in our particular setting. As may be expected, for graphs with globally bounded node degree, we have the following result:

**Observation 3.3** Note that all our operators are bounded, the Hilbert space is (in general) infinite dimensional, hence there is no chance to have e.g. \( (D - z)^{-1} \) or \( (D^2 - z)^{-1} \) compact. At the moment we are sceptical whether this latter phenomenon dissappears generically if the vertex degree is allowed to become infinite. There are some results on spectra of random graphs which seem to have a certain bearing on this problem (cf. e.g. [28]).

In the next sections we introduce and calculate the so-called Connes-distance functional and compare it, among other things, with the ordinary graph distance. In doing this we have to calculate the commutator \([D, f]\) applied to an element \( f' \in H_0 \). We have:

\[ (d \cdot f) f' = \sum_{ik} (f_k f'_k - f_i f'_i) d_{ik} \]  

(57)

\[ (f \cdot d) f' = \sum_{ik} f_i (f'_k - f'_i) d_{ik} \]  

(58)

hence

\[ [D, f] f' = \sum_{ik} (f_k - f_i) f'_k d_{ik} \]  

(59)

On the other side the right-module structure allows us to define \( df \) as an operator on \( H_0 \) via:

\[ df \cdot f' = (\sum_{ik} (f_k - f_i) d_{ik}) \cdot (\sum_k f'_k n_k) = \sum_{ik} (f_k - f_i) f'_k d_{ik} = [D, f] f' \]  

(60)
In a next step we define $df$ as operator on $H_1$ which is not as natural as on $H_0$. We define:

$$df|_{H_1} : d_{ik} \to (f_i - f_k)n_k$$

(61)

and linearly extended. A short calculation shows

$$df|_{H_1} = -(d\bar{f}|_{H_0})^* = [d^*, f]$$

(62)

with

$$[d^*, f]g = d^*(f \cdot g) - fd^*g \quad (g \in H_1)$$

(63)

This then has the following desirable consequence:

**Conclusion 3.4** With the above definitions the representation of $df$ on $H$ is given by

$$df|_{H} = \begin{pmatrix} 0 & df|_{H_1} \\ df|_{H_0} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -(d\bar{f}|_{H_0})^* \\ df|_{H_0} & 0 \end{pmatrix}$$

(64)

and it immediately follows

$$df|_{H} = \begin{pmatrix} 0 & [d^*, f] \\ [d, f] & 0 \end{pmatrix} = [D, f]$$

(65)

### 4 The Connes-Distance Function on Graphs

From the general theory of operators on Hilbert spaces we know that:

$$\|T\| = \|T^*\|$$

(66)

Hence

**Lemma 4.1**

$$\|[d, f]\| = \|[d, \bar{f}]\| = \|[d^*, f]\|$$

(67)

and

$$\|[D, f]\| = \|[d, f]\|$$

(68)

Proof: The left part of (67) is shown below and is a consequence of formula (74); the right identity follows from (66). With

$$X := \begin{pmatrix} x \\ y \end{pmatrix}$$

(69)
and \( T_1 := [d, f], T_2 := [d^*, f] \), the norm of \([D, f]\) is:

\[
||[D, f]||^2 = \text{sup}\{||T_1 x||^2 + ||T_2 y||^2 ; ||x||^2 + ||y||^2 = 1\}
\]

(70)

Normalizing now \( x, y \) to \( ||x|| = ||y|| = 1 \) and representing a general normalized vector \( X \) as:

\[
X = \lambda x + \mu y , \lambda, \mu > 0 \text{ and } \lambda^2 + \mu^2 = 1
\]

(71)

we get:

\[
||[D, f]||^2 = \text{sup}\{\lambda^2||T_1 x||^2 + \mu^2||T_2 y||^2 ; ||x|| = ||y|| = 1, \lambda^2 + \mu^2 = 1\}
\]

(72)

where now \( x, y \) can be varied independently of \( \lambda, \mu \) in their respective admissible sets, hence:

\[
||[D, f]||^2 = \text{sup}\{\lambda^2||T_1||^2 + \mu^2||T_2||^2\} = ||T_1||^2 \quad \Box
\]

(73)

(as a consequence of equation (67)).

We see that in calculating \( ||[D, f]\| \) we can restrict ourselves to the simpler expression \( ||[d, f]\| \). We infer from the above calculations \( x \in H_0 \):

\[
||df \cdot x||^2 = \sum_i \left( \sum_{j=1}^{v_i} |f_i - f_{kj}|^2 \cdot |x_i|^2 \right)
\]

(74)

and the corresponding expression for a directed graph with \( v_i \) replaced by \( v_i^{out} \). Abbreviating

\[
\sum_{j=1}^{v_i} |f_{kj} - f_i|^2 =: a_i \geq 0
\]

(75)

and calling the supremum over \( i \) \( a_s \), it follows:

\[
||df \cdot x||^2 = a_s \cdot \left( \sum_i a_i/a_s \cdot |x_i|^2 \right) \leq a_s
\]

(76)

for \( ||x||^2 = \sum_i |x_i|^2 = 1 \).

On the other side, choosing an appropriate sequence of normalized basis vectors \( e_\nu \) so that the corresponding \( a_\nu \) converge to \( a_s \) we get:

\[
||df \cdot e_\nu||^2 \rightarrow a_s
\]

(77)

We hence have
Theorem 4.2

\[ \| [D, f] \| = \sup_{i} (\sum_{j=1}^{v_i} |f_{kj} - f_i|^2)^{1/2}, \sup_{i} (\sum_{j=1}^{v_{out}} |f_{kj} - f_i|^2)^{1/2} \] respectively \quad (78)

The Connes-distance functional between two nodes, \( n, n' \), is now defined as follows:

**Definition 4.3 (Connes-distance function)**

\[ \text{dist}_C(n, n') := \sup \{ |f_{n'} - f_n| ; \| [D, f] \| = \| df \| \leq 1 \} \] \quad (79)

We would like to note that Davies in [15] introduced several metrics on graphs, which have been motivated, as he remarks, by his study of heat kernels on Riemannian manifolds. What he calls metric \( d_3 \), is related to the rhs of the equation in theorem 4.2 (he uses slightly different Hilbert spaces). He then shows in a longer proof that this metric is identical to another one, \( d_4 \), which corresponds to the lhs in the above theorem. In our approach, on the other side, the content of theorem 4.2 is derived in a relatively transparent and straightforward way.

**Remark 4.4** It is easy to prove that this defines a metric on the graph.

**Corollary 4.5** It is sufficient to vary only over the set \( \{ f ; \| df \| = 1 \} \).

Proof: This follows from

\[ |f_k - f_i| = c \cdot |f_k/c - f_i/c| ; c = \| df \| \] \quad (80)

and

\[ \| d(f/c) \| = c^{-1} \| df \| = 1 \] \quad (81)

with \( c \leq 1 \) in our case. \( \square \)

In general it turns out to be a nontrivial task to calculate this distance on an arbitrary graph as the nature of the above constraint is quite subtle. The underlying reason is that the constraint is, in some sense, inherently *non-local*. As \( f \) is a function, the difference, \( f_{n'} - f_n \), has to be the same independently of the path we follow, connecting \( n' \) and \( n \). On the other side, in a typical optimization process one usually deals with the individual jumps, \( f_k - f_i \), between neighboring points along some path. It is then not at all clear that these special choices of jumps along such a path can be extended to a global function without violating the overall constraint on the expression in theorem 4.2. Nevertheless we think the above closed form is a solid starting point for the calculation of \( \text{dist}_C \) on various classes of graphs or lattices. We illustrate this by proving some apriori estimates concerning this distance function and by evaluating it for some examples.
4.1 Some General Estimates

Having an admissible function $f$ so that $\sup_i (\sum_{k=1}^n |f_k - f_i|^2)^{1/2} \leq 1$, this implies that, taking a minimal path $\gamma$ from, say, $n$ to $n'$, the jumps $|f_{\nu+1} - f_\nu|$ between neighboring nodes along the path have to fulfill:

$$|f_{\nu+1} - f_\nu| \leq 1$$

and, a fortiori, have to be strictly smaller than 1 in the general situation.

On the other side the Connes distance can only become identical to the ordinary distance $d(n, n')$ if there exist a sequence of admissible node functions with all these jumps approaching the value 1 along such a path, which is however impossible in general as can be seen from the structure of the constraint on the expression in theorem 4.2. Only in this case one gets:

$$\sum_\gamma (f_{\nu+1} - f_\nu) \to \sum_\gamma 1 = \text{length}(\gamma)$$

We formulate this observation as follows:

**Lemma 4.6 (Connes-distance)** Within our general scheme one has the following inequality

$$\text{dist}_C(n, n') \leq d(n, n')$$

By the same token one can prove that $\text{dist}_C$ between two nodes is bounded by the corresponding Connes-distance calculated for the (one-dimensional) sub-graph formed by a minimal path connecting these nodes, i.e.

$$\text{dist}_C(n, n'; G) \leq \text{dist}_C(\text{min.path})(n, n')$$

The reason is that one has more admissible functions at ones disposal for a subgraph. With $G'$ a connected subgraph of $G$, the set of admissible function, $S_{G'}$, on $G'$ contains the restrictions of the functions of the corresponding set, $S_G$, belonging to $G$, as each restriction to $G'$ of a member belonging to $S_G$ lies in $S_{G'}$. Hence the supremum is in general larger on $S_{G'}$. The distance along such a path, on the other side, can be rigorously calculated (see the discussion of some examples below) and is for non-neighboring nodes markedly smaller than the ordinary graph distance. From what we have said we can also infer the following corollary.

**Corollary 4.7** With $G'$ a connected subgraph of $G$ it holds (with $n, n' \in V' \subset V$)

$$\text{dist}_C(n, n'; G) \leq \text{dist}_C(n, n'; G')$$

One can also give sufficient criteria for $\text{dist}_C(n, n') < d(n, n')$. The cases of undirected, directed graphs, respectively, have to be treated a little bit differently.
Lemma 4.8  Let $G$ be an undirected graph and $\gamma$ a minimal path of length, $l > 1$, connecting $n, n'$. There is at least one node, $n^*$, belonging to $\gamma$, having node degree $\geq 2$ (as there are at least two consecutive edges, belonging to $\gamma$). If

$$\text{dist}_C(n, n') = l = d(n, n')$$

all the individual jumps along $\gamma$ have to be one. But then the corresponding function cannot be admissible at $n^*$. Hence

$$\text{dist}_C(n, n') < d(n, n')$$

Let $G$ now be a directed graph and let there exist two different paths, $\gamma, \gamma'$, of equal length, $l > 1$, connecting $n, n'$. Again there exists a node, $n^*$, on $\gamma$ so that it is incident with two edges, the one belonging to $\gamma$, the other to $\gamma'$. Along both edges the jumps have to be one and again the admissibility of the corresponding function is violated. We again can conclude

$$\text{dist}_C(n, n') < d(n, n')$$

Remark: The latter situation will be discussed below in the example of the directed $\mathbb{Z}^2$-lattice.

We remarked above that the calculation of the Connes distance on graphs is to a large part a continuation problem for admissible functions, defined on subgraphs. Then the following question poses itself. For what classes of graphs and/or subgraphs do we have an equality in the above corollary? We start from a given graph, $G_0 = (V_0, E_0)$, and then add new nodes and bonds, yielding a new graph, $G' = (V', E')$. We consider two fixed nodes, $n_0, n'_0$ in $G_0$.

Assumption 4.9  We assume that the above process does not create new paths between nodes belonging to $G_0$. In other words, the paths, connecting $n_0$ and $n'_0$ are contained in $G_0$.

Lemma 4.10  Under this assumption each admissible function on $G_0$ can be extended to an admissible function on $G'$.

Proof: In a first step we construct the set of nearest neighbors, $V_1 \setminus V_0$ in $V' \setminus V_0$ relative to $V_0$. Each new node in $V_1 \setminus V_0$ has a unique nearest neighbor in $V_0$ since otherwise there would exist a new path between these two nodes lying in $G_0$. With $n \in V_1 \setminus V_0$ we extend an admissible function on $G_0$ as follows:

$$f_n := f_{n_0} \quad n_0 \text{ the unique nearest neighbor in } V_0$$

This extended function is an admissible function on $G_1 := (V_1, E_1)$. Note however that, by assumption, there do not exist bonds in $E_1$, connecting nodes in $V_1 \setminus V_0$.\[18]
We can now continue this process until we arrive at the graph $G'$.

By the same token we see that

$$dist_C(n_0, n'_0; G_1) = dist_C(n_0, n'_0; G_0)$$

This holds at every intermediate step and we get:

**Lemma 4.11** Under the above assumption we have

$$dist_C(n_0, n'_0; G_0) = dist_C(n_0, n'_0; G')$$

**Corollary 4.12** If $G$ is a tree, it holds

$$dist_C(n; n'; \text{minimal path})$$

Proof: In a tree there exists, by definition, at most one path, connecting two nodes. We can take this path as connected subgraph, $G_0$, and make the above extension, since $G$ and $G_0$ fulfill the assumption.

The graphs, so constructed are however rather special, consisting, so to speak, of a start graph plus some added hair.

Above we have given sufficient conditions for

$$dist_C(n, n'; G_0) = dist_C(n, n'; G)$$

with $G$ an extension of $G_0$ and $n, n' \in V_0$. We show now that the emergence of too short new paths is representing the obstruction for such a result to hold in general.

So let again $G_0$ be a graph and assume the existence of two nodes, $n, n'$, in $V_0$ with

$$dist_C(n, n'; G_0) > l \in \mathbb{N}$$

We extend $G_0$ to some $G$ by adding new nodes and edges. We know that for admissible functions the elementary jumps along an edge have to fulfill $|f_i - f_k| \leq 1$. If there exists a new path, $\gamma$, in $G$, connecting $n, n'$, with

$$\text{length}(\gamma) \leq l$$

we can conclude that for each admissible function on $G$ it must hold

$$|f(n) - f(n')| \leq l$$

We hence have
Lemma 4.13 If two nodes in $G_0$ have $\text{dist}_C(n,n';G_0) > l$ and if there exists a path, $\gamma$, in $G$, connecting $n, n'$ and having length $l \leq l_1$, it necessarily holds that

$$\text{dist}_C(n,n';G) \leq l < \text{dist}_C(n,n';G_0)$$ (98)

Up to now we have derived upper bounds on $\text{dist}_C$ relative to $\text{dist}_C$ on subgraphs or the canonical graph distance $d(n,n')$. In the following we will derive a quite efficient lower bound. This is done by defining a particular admissible function, depending on an arbitrary base node, $n_0$. That is, we fix an arbitrary node, dubbed $n_0$, and take as admissible function the canonical distance, divided by the local vertex degree:

$$f_{n_0}(n) := (v_n)^{-1/2} \cdot d(n_0,n) \ , \ f_{n_0}(n) := (v_n^{(\text{out})})^{-1/2} \cdot d(n_0,n) \ , \ f_{n_0}(n_0) = 0$$ (99)

for undirected, directed graphs, respectively.

From our general results we have

$$\|df\| = \sup_i \left( \sum_{j=1}^{v_i} |f_{k_j} - f_i|^2 \right)^{1/2}$$ (100)

or $v_i$ replaced by $v_i^{(\text{out})}$. Inserting the above particular function we get

$$\|df\| \leq 1$$ (101)

as each term, $|f_{k_j} - f_i|$, is either zero or one (depending of whether the distance to the base point remains constant or changes by $\pm 1$).

Lemma 4.14 The functions, $f_{n_0}(n)$, $n_0$ an arbitrary node in $G$, are admissible.

With $n, n'$ two arbitrary nodes in $G$ we take $n$ as base point, $n_0$, and have

$$f_{n_0}(n') - f_{n_0}(n) = f_{n_0}(n') = (v_{n'})^{-1/2} \cdot d(n',n)$$ (102)

(as $f_{n_0}(n) = f_{n_0}(n_0) = 0$). As $\text{dist}_C$ is the supremum over admissible functions, we get:

Theorem 4.15

$$\text{dist}_C(n,n') \geq (v_{(n,n')})^{-1/2} \cdot d(n,n')$$ (103)

with $v_{(n,n')}$ the minimum of the (out-) vertex degrees at $n, n'$ respectively.

Note that one can of course either choose $n$ or $n'$ as base point in the definition of the above admissible function.
4.2 Examples

The general results derived above should be compared with the results in e.g. [23] to [27]. Choosing the symmetric difference operator as “Dirac operator” in the case of the one-dimensional lattice the authors in [25, 26] got a distance which is strictly greater than the ordinary distance but their choice does not fulfill the above natural constraint given in Theorem 4.2. Note in particular that our operator \( d \) is a map from node- to bond-functions which is not the case in these examples. In [27] the authors employed a symmetric doubling of the non-symmetric adjacency matrix of the one-dimensional directed lattice, \( \mathbb{Z}^1 \) as Dirac operator. With \( v^{\text{(out)}}_i = 1 \) in this example, our above general estimate yields

\[
d(n, n') \leq \text{dist}_C(n, n') \leq d(n, n') \Rightarrow \text{dist}_C(n, n') = d(n, n')
\]

that is, we get the same result for our Dirac operator as for the choice made in [27].

We want to close this paper with the discussion of several examples which show that, in general, it is quite a non-trivial task to calculate \( \text{dist}_C \). The first one is a simple warm-up exercise, the second one is the one-dimensional non-directed lattice, \( \mathbb{Z}^1 \), discussed also by some of the authors mentioned above (treated however within their own schemes) and is not so simple. The last one is the directed \( \mathbb{Z}^2 \)-lattice, which we do not solve in closed form, but we provide several estimates. The technique used in approaching some of the problems may be interesting in general. It turns out that the proper mathematical context, to which our strategy does belong, is the field of (non-)linear programming or optimization (see e.g. [36] or any other related textbook). This can be inferred from the structure of the constraints we get. This means that the techniques developed in this field may perhaps be of use in solving such intricate problems.

Example 1: The square with vertices and edges:

\[
x_1 - x_2 - x_3 - x_4 - x_1
\]

Let us calculate the Connes-distance between \( x_1 \) and \( x_3 \). As the sup is taken over functions, the summation over elementary jumps is (or rather: has to be) path-independent (this represents a subtle constraint for practical calculations). It is an easy exercise to see that the \textit{sup} can be found in the class where the two paths between \( x_1, x_3 \) have the \textit{valuations} \( 1 \geq a \geq 0 \):

\[
x_1 - x_2 : a, \quad x_2 - x_3 : (1 - a^2)^{1/2}
\]

\[
x_1 - x_4 : (1 - a^2)^{1/2}, \quad x_4 - x_3 : a
\]

Hence one has to find \( \sup_{0 \leq a \leq 1}(a + \sqrt{1 - a^2}) \). Setting the derivative with respect to \( a \) to zero one gets \( a = \sqrt{1/2} \). That is:
Example 4.16 (Connes-distance on a square)

\[ \text{dist}_C(x_1, x_3) = \sqrt{2} < 2 = d(x_1, x_3) \] (108)

Remark 4.17 As \( v_i = 2 \), our apriori estimate in theorem 4.15 is saturated as

\[ \text{dist}_C(x_1, x_2) \geq (2)^{-1/2} \cdot 2 = \sqrt{2} \] (109)

The next example is considerably more complicated.

Example 2: The undirected one-dimensional lattice:

The nodes are numbered by \( \mathbb{Z} \). We want to calculate \( \text{dist}_C(0, n) \) within our general framework. The calculation will be done in two main steps. In the first part we make the (in principle quite complicated) optimization process more accessible. For the sake of brevity we state without proof that it is sufficient to discuss real monotonely increasing functions with

\[ f(k) = \begin{cases} f(0) & \text{for } k \leq 0 \\ f(n) & \text{for } k \geq n \end{cases} \] (110)

and we write

\[ f(k) = f(0) + \sum_{i=1}^{k} h_i \] for \( 0 \leq k \leq n \) \( h_i \geq 0 \) (111)

The above optimization process then reads:

Observation 4.18 Find \( \sup \sum_{i=1}^{n} h_i \) under the constraint

\[ h_1^2 \leq 1, h_2^2 + h_1^2 \leq 1, \ldots, h_n^2 + h_{n-1}^2 \leq 1, h_n^2 \leq 1 \] (112)

The simplifying idea is now the following. Let \( h := (h_i)_{i=1}^{n} \) be an admissible sequence with all \( h_{i+1}^2 + h_i^2 < 1 \). We can then find another admissible sequence \( h' \) with

\[ \sum h'_i > \sum h_i \] (113)

Hence the supremum cannot be taken on the interior. We conclude that at least some \( h_{i+1}^2 + h_i^2 \) have to be one. There is then a minimal \( i \) for which this holds. We can convince ourselves that the process can now be repeated for the substring ending at \( i + 1 \). Repeating the argument we can fill up all the entries up to place \( i + 1 \) with the condition \( h_{i+1}^2 + h_i^2 = 1 \) and proceeding now upwards we end up with
Lemma 4.19  The above supremum is assumed within the subset

\[ h_1^2 \leq 1, h_1^2 + h_2^2 = 1, \ldots, h_{n-1}^2 + h_n^2 = 1, h_n^2 \leq 1 \]  \hspace{1cm} (114)

This concludes the first step.

In the second step we calculate \( \sup |f(0) - f(n)| \) on this restricted set. From the above we now have the constraint:

\[ h_1^2 \leq 1, h_2^2 = 1 - h_1^2, h_3^2 = h_1^2, h_4^2 = 1 - h_1^2, \ldots, h_n^2 = 1 - h_1^2 \]  \hspace{1cm} (115)

depending on \( n \) being even or uneven. This yields

\[
\sup |f(0) - f(n)| = \begin{cases} 
1 & \text{for } n = 1 \\
(n/2) \cdot \sup(h_1 + \sqrt{1 - h_1^2}) = (n/2) \cdot \sqrt{2} & \text{for } n \text{ even} \\
\sup([n/2] \cdot (h_1 + \sqrt{1 - h_1^2} + h_1)) & \text{for } n \text{ uneven}
\end{cases}
\]  \hspace{1cm} (116)

In the even case the rhs can be written as \( \sqrt{n^2/2} = \sqrt{|n^2/2|} \). In the uneven case we get by differentiating the rhs and setting it to zero:

\[
h_1^{\text{max}} = A_n / \sqrt{1 + A_n^2}, \quad \sqrt{1 - (h_1^{\text{max}})^2} = 1 / \sqrt{1 + A_n^2}
\]  \hspace{1cm} (117)

with \( A_n = 1 + 1/[n/2] \). We see that for increasing \( n \) both terms approach \( 1/\sqrt{2} \), the result in the even case. Furthermore we see that the distance is monotonely increasing with \( n \) as should be the case for a distance. This yields in the uneven case

\[
dist_C(0, n) = \frac{([n/2] + 1)A_n + [n/2]}{\sqrt{1 + A_n^2}}
\]  \hspace{1cm} (118)

which is a little bit nasty. Both expressions can however be written in a more elegant and unified way (this was a conjecture by W.Kunhardt, inferred from numerical examples). For \( n \) uneven a short calculation yields

\[
[n^2/2] = (n^2 - 1)/2 = 1/2 \cdot (n - 1)(n + 1) = 2[n/2]([n/2] + 1)
\]  \hspace{1cm} (119)

(with the floor-, ceiling-notation the expressions would become even more elegant). With the help of the latter formula the rhs in (118) can be transformed into

\[
\frac{([n/2] + 1)A_n + [n/2]}{\sqrt{1 + A_n^2}} = \sqrt{[n^2/2] + 1}
\]  \hspace{1cm} (120)

Conclusion 4.20  For the one-dimensional undirected lattice we have

\[
dist_C(0, n) = \begin{cases} 
\sqrt{[n^2/2]} & \text{for } n \text{ even} \\
\sqrt{[n^2/2] + 1} & \text{for } n \text{ uneven}
\end{cases}
\]  \hspace{1cm} (121)
Remark 4.21 Again comparing the exact result with our lower bound, we find for $n$ even:

$$\text{dist}_C(0, n) \geq (2)^{-1/2} \cdot n = (2)^{1/2} \cdot n/2$$

(122)

that is, the lower bound is again saturated. For $n$ uneven we have instead:

$$\text{dist}_C(0, n) = (2)^{-1/2} \cdot (n^2 + 1)^{1/2} > (2)^{-1/2} \cdot n$$

(123)

Example 3: The directed lattice $\mathbb{Z}^2$

The vertices in $\mathbb{Z}^2_d$ are denoted by $(i, j)$ or $(x, y)$. The edges point from $(i, j)$ to $(i + 1, j)$ and $(i, j + 1)$; hence, $v_i^{\text{out}} = 2$. As the system is translation invariant, it suffices to calculate the Connes distance between nodes $(0, 0)$ and $(x, y)$ with $x, y > 0$.

For nodes lying on the same parallel to the $x$, $y$-axis, respectively, we have

$$\text{dist}_C(n, n') = d(n, n')$$

(124)

For $x$ or $y = 0$, there is only one minimal path, connecting $(0, 0)$ and $(x, y)$. Therefore, lemma 4.8 does not apply. For, say, $y = 0$, we choose the following admissible function:

$$f(x, y) := x \quad \text{for all } y$$

(125)

We have

$$|f(x, 0) - f(0, 0)| = |x| = d((0, 0), (x, 0))$$

(126)

and can conclude

$$\text{dist}_C((0, 0), (x, 0)) = d((0, 0), (x, 0))$$

(127)

on $\mathbb{Z}^2_d$. The same holds for the $y$-axis.

For nodes with both $x, y \neq 0$, we have more than one minimal path connecting $(0, 0)$ and $(x, y)$. Our lemma then shows that, necessarily,

$$\text{dist}_C(n, n') < d(n, n')$$

(128)

More detailed estimates will be given below.

If we try to really calculate the Connes distance on $\mathbb{Z}^2_d$ for points in general position, the optimization problem becomes quite involved and we will only provide some estimates. The reason is that the constraint equations are of a quite non-local nature (compared to the simpler undirected $\mathbb{Z}^1$-lattice) and that in general several minimal paths do exist which make the continuation problem quite intricate.
It is easy to see that the canonical graph distance between the points \((0, 0)\) and \((x, y)\) is \(|x| + |y|\) and that all minimal paths have the same length. With \(x, y > 0\), we conjecture (without giving a proof) that it suffices to restrict the variation to admissible functions with positive jumps in the positive \(x-, y-\)directions and that we can set \(f(0, 0) = 0\). A particular minimal path consists of \(x\) steps in the \(x\)-direction followed by \(y\) steps in the \(y\)-direction. We denote (for convenience) the jumps along the \(x-, y-\)axis, respectively, by
\[
h_{i0} := f(i, 0) - f(i - 1, 0) \geq 0 \quad h_{0j} := f(0, j) - f(0, j - 1) \geq 0 \quad (129)
\]

The optimization problem now reads:

**Problem 4.22** Find \(\sup \left( \sum_{i=1}^{x} h_{i0} + \sum_{j=1}^{y} h_{0j} \right)\) under the constraints imposed by the admissibility of the corresponding function, \(f\). Note however that the constraints must hold on the full lattice.

From our general result in theorem 4.15 we know that
\[
\text{dist}_C((0, 0), (x, y)) \geq (2)^{-1/2} \cdot (x + y) \quad (130)
\]

We can construct an admissible function, \(f\), which fulfills
\[
|f(0, 0) - f(x, y)| = (2)^{-1/2} \cdot (x + y) \quad (131)
\]

This can be achieved by setting \(f(0, 0) = 0\) and by choosing all \(x-, y-\)jumps equal to \(a\) with \(2a^2 = 1 \Rightarrow a = (2)^{-1/2}\). This yields the above result.

The question is, whether this is already the supremum over the set of admissible functions. We will show, that this is not the case by providing another admissible function yielding a bigger value. We choose an admissible function with \(x\)-jumps equal to \(a\) and \(y\)-jumps equal to \(b\) with \(a^2 + b^2 = 1\). The admissible function reads
\[
f(x, y) = ax + by \quad (132)
\]

The function \(f\) takes a stationary value at
\[
a = (x^2/(x^2 + y^2))^{1/2} \quad b = (y^2/(x^2 + y^2))^{1/2} \quad (133)
\]

yielding the value
\[
f(x, y) = (x^2 + y^2)^{1/2} \quad (134)
\]

Assuming, for example, that \(x \neq y\), it follows that
\[
f(x, y)^2 - ((2)^{-1/2} \cdot (x + y))^2 = (1/2) \cdot (x^2 + y^2) - x \cdot y = (1/2) \cdot (x - y)^2 > 0 \quad (135)
\]

In other words, we see

**Observation 4.23** For the directed \(\mathbb{Z}^2\)-lattice and \(x \neq y\) \((x, y > 0)\) we have the estimate
\[
d((0, 0), (x, y)) > \text{dist}_C((0, 0), (x, y)) \geq (x^2 + y^2)^{1/2} > (2)^{-1/2}(x + y) \quad (136)
\]
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