We give the simple proof of the star-triangle relation of the chiral Potts model. We also give the constructive way to understand the star-triangle relation of the chiral Potts model, which may give the hint to give the new integrable models.
1 Introduction

The Ising model is the most important integrable model in two-dimension\[1\], but this model has various nice properties so that we cannot guess what kind of properties are inherited in the more general two-dimensional integrable model. In this sense, the eight-vertex model\[2\] and the chiral Potts model\[3, 4\], which include the Ising model as a special case, are key models to understand the mechanism of the integrability in two-dimension. Moreover, the methods for analyzing these models are applicable to another ones and we can obtain various integrable models in two-dimension by modifying these key models.

Especially, the chiral Potts model is related not only to the two-dimensional integrable model but also to the three-dimensional integrable models. The Bazhanov-Baxter model\[5\], which is the generalization of the three-dimensional integrable Zamolodchikov model\[6\], is constructed through the two-dimensional integrable sl\(_n\) chiral Potts model\[7, 8\]. To understand the mathematical structure of the three-dimensional integrable model\[7, 8\], we must clarify more about the integrability, the star-triangle relation, in the chiral Potts model. Au-Yang et. al. have given the proof of the star-triangle relation in the chiral Potts model\[11\], but they used the complicated recursion relation so that the origin of the integrability is not clear in their proof.

In this paper, we give the simple proof of the star-triangle relation of the chiral Potts model. Our method is more clear about the mathematical structure and gives the hint to generalize it. We also give the constructive way of understanding the star-triangle relation of the chiral Potts model, which may give the new integrable models.

2 Rewriting the star-triangle relation of the chiral Potts model

In order to prove the star-triangle relation, we first rewrite the star-triangle relation in a more convenient way to prove it in this section. We also show various formulae, which is used in the proof.

2.1 Preparation

The Boltzmann weight of \(N\)-state chiral Potts model is given by \(W_{pq}(n)\) or \(\overline{W}_{pq}(n)\) depending on the direction. Here \(\{p, q\}\) represents the rapidity, and \(n\) is the spin variable with \(n = 1, 2, \cdots, N\). The Boltzmann weights are periodic with spin, that is, \(W_{pq}(n+N) = W_{pq}(n)\).
and \( \overline{W}_{pq}(n + N) = \overline{W}_{pq}(n) \). The star-triangle relation of the chiral Potts model is

\[
\sum_{d=0}^{N-1} \overline{W}_{qr}(b - d)W_{pr}(a - d)\overline{W}_{pq}(d - c) = R_{pqr}W_{pq}(a - b)\overline{W}_{pr}(b - c)W_{qr}(a - c). \tag{1}
\]

The Boltzmann weights \( W_{pq}(n) \) and \( \overline{W}_{pq}(n) \) in Eq. (1) are

\[
W_{pq}(n) = \rho_1 \prod_{j=1}^{n} \frac{d_p b_q - a_p c_q \omega^j}{b_p d_q - c_p a_q \omega^j}, \quad \overline{W}_{pq}(n) = \rho_2 \prod_{j=1}^{n} \frac{\omega a_p d_q - d_p a_q \omega^j}{c_p b_q - b_p c_q \omega^j}, \tag{2}
\]

where

\[
a_p N + k' b_p N = k q'_p N, \quad k' a_p N + b_p N = k c_p N, \tag{3}\]

with \( \omega = e^{2\pi i/N} \) and \( k^2 + k'^2 = 1 \).

By making the recursion relation, we have the following Fourier transforms of \( W \) and \( \overline{W} \) as \( \tilde{W} \) and \( \overline{W} \) in the form

\[
\tilde{W}_{pq}(m) = \sum_{k=0}^{N-1} \omega^{mk}W_{pq}(k) = \rho'_1 \prod_{j=1}^{m} \frac{b_p d_q - d_p b_q \omega^{j-1}}{c_p a_q - a_p c_q \omega^j}, \tag{4}
\]

\[
\overline{W}_{pq}(m) = \sum_{k=0}^{N-1} \omega^{mk}\overline{W}_{pq}(k) = \rho'_2 \prod_{j=1}^{m} \frac{c_p b_q - b_p c_q \omega^j}{b_p d_q - d_p a_q \omega^j}, \tag{5}
\]

where \( \rho'_1 = \rho_1 \prod_{n=0}^{N-1} \frac{d_p b_q - a_p c_q \omega^j}{b_p d_q - c_p a_q \omega^j} \) and \( \rho'_2 = \rho_2 \prod_{n=0}^{N-1} \frac{\omega a_p d_q - d_p a_q \omega^j}{c_p b_q - b_p c_q \omega^j} \).

The inverse Fourier transform is

\[
W_{pq}(m) = \frac{1}{N} \sum_{k=0}^{N-1} \omega^{-mk}\tilde{W}_{pq}(k), \quad \overline{W}_{pq}(m) = \frac{1}{N} \sum_{k=0}^{N-1} \omega^{-mk}\overline{W}_{pq}(k). \tag{6}\]

Using the above Fourier transform, \( R_{pqr} \) is expressed by \( \overline{W} \) and \( \overline{W} \) in the form

\[
R_{pqr} = f_{pq}f_{qr}/f_{pr}, \quad f_{pq} = \left[ \prod_{m=0}^{N-1} \overline{W}_{pq}(m)/W_{pq}(m) \right]^{1/N}. \tag{7}\]

We define

\[
f_{pq} = f_{pq}^{(1)}/f_{pq}^{(2)}, \quad f_{pq}^{(1)} = \left[ \prod_{m=0}^{N-1} \overline{W}_{pq}(m) \right]^{1/N}, \quad f_{pq}^{(2)} = \left[ \prod_{m=0}^{N-1} W_{pq}(m) \right]^{1/N}. \tag{8}\]
which gives

\[ R_{pqr} = f^{(1)}_{pq} f^{(2)}_{pr} f^{(1)}_{pr} f^{(2)}_{qr} \] . Replacing \( b \to b + a, \ c \to c + a \) and \( d \to d + a \) in Eq. (1), we have

\[ N - 1 \sum_{d=0}^{N-1} \overline{W}_{q r} (b - d) W_{p r} (-d) \overline{W}_{p q} (d - c) = R_{p q r} W_{p q} (-b) \overline{W}_{p r} (b - c) W_{q r} (-c). \quad (9) \]

We give the proof of this star-triangle relation in the next chapter.

2.2 Rewriting the star-triangle relation with the cyclic representation

Using the cyclic representation of \( su(2) \), we rewrite the above expression. The basis of the cyclic representation of \( su(2) \) are \( X \) and \( Z \) with the properties

\[ ZX = \omega XZ, \quad X^N = Z^N = 1. \quad (10) \]

The explicit \( N \times N \) matrix representation is

\[ X_{\alpha, \beta} = \delta_{\alpha, \beta+1} + \delta_{\alpha, \beta+1-N}, \quad Z_{\alpha, \beta} = \omega^\alpha \delta_{\alpha, \beta}. \quad (11) \]

We define

\[ T_{pq} = \sum_{m=0}^{N-1} \overline{W}_{pq} (m) Z^m, \quad S_{pq} = \sum_{m=0}^{N-1} \overline{W}_{pq} (m) X^m, \quad (12) \]

and make the following quantities

\[ T_{pq} S_{pr} T_{qr} = \sum_{l,m,n} \overline{W}_{pq} (l) \overline{W}_{pr} (m) \overline{W}_{qr} (n) \omega^{lm} X^m Z^{l+n}, \quad (13) \]

\[ S_{qr} T_{pr} S_{pq} = \sum_{l,m,n} \overline{W}_{qr} (l) \overline{W}_{pr} (m) \overline{W}_{pq} (n) \omega^{mn} X^{l+n} Z^m. \quad (14) \]

We take the component of these operators and obtain

\[ (T_{pq} S_{pr} T_{qr})_{bc} = \sum_{l,m,n} \sum_d \overline{W}_{pq} (l) \overline{W}_{pr} (m) \overline{W}_{qr} (n) \omega^{lm} (X^m)_{bd} (Z^{l+n})_{dc} \]
\[ = \sum_{l,n} \overline{W}_{pq} (l) \overline{W}_{pr} (b - c) \overline{W}_{qr} (n) \omega^{l b + c n} \]
\[ = N^2 \overline{W}_{pq} (-b) \overline{W}_{pr} (b - c) \overline{W}_{qr} (-c), \quad (15) \]
\[(S_{qr}T_{pr}S_{pq})_{bc} = \sum_{l,m,n} \sum_{d} W_{qr}(l)\overline{W}_{pr}(m)\overline{W}_{pq}(n)\omega^{mn}(X^{l+n})_{bd}(Z^{m})_{dc}\]
\[= \sum_{m,n} W_{qr}(b - c - n)\overline{W}_{pr}(m)\overline{W}_{pq}(n)\omega^{m(n+c)}\]
\[= N \sum_{n} W_{qr}(b - c - n)W_{pr}(-n - c)\overline{W}_{pq}(n)\]
\[= N \sum_{d} W_{qr}(b - d)W_{pr}(-d)\overline{W}_{pq}(d - c). \quad (16)\]

Then the star-triangle relation is rewritten as
\[S_{qr}T_{pr}S_{pq} = R_{pqr}T_{pq}S_{pr}T_{qr}/N, \quad (17)\]
which becomes
\[NS_{qr}T_{pr}S_{pq}/f_{qr}^{(1)}f_{pr}^{(2)}f_{pq}^{(1)} = T_{pq}S_{pr}T_{qr}/f_{pq}^{(2)}f_{pr}^{(1)}f_{qr}^{(2)}, \quad (18)\]
by using Eq.(7) and Eq.(8). Then if we define the following quantities \(\hat{T}_{pq}\) and \(\hat{S}_{pq}\)
\[\hat{T}_{pq} = \frac{T_{pq}}{N \left(\Pi_{m=0}^{N-1} W_{pq}(m)\right)^{1/N}} = \frac{T_{pq}}{N f_{pq}^{(2)}}, \quad (19)\]
\[\hat{S}_{pq} = \frac{S_{pq}}{\left(\Pi_{m=0}^{N-1} \overline{W}_{pq}(m)\right)^{1/N}} = \frac{S_{pq}}{f_{pq}^{(1)}}, \quad (20)\]
we can rewrite the original star-triangle relation into the following form
\[\hat{S}_{qr}\hat{T}_{pr}\hat{S}_{pq} = \hat{T}_{pq}\hat{S}_{pr}\hat{T}_{qr}. \quad (21)\]

### 2.3 Properties of \(\hat{T}_{pq}\) and \(\hat{S}_{pq}\)

Here we show various properties of these \(\hat{T}_{pq}\) and \(\hat{S}_{pq}\). Date et. al. [14] have given the similar formula of ours through the help of the recursion relation. We use the explicit representation of \(Z, X\), and this method is more apparent than that of Date et. al.

We first rewrite \(T_{pq}\) in the following form
\[T_{pq} = \sum_{l=0}^{N-1} \overline{W}_{pq}(l)Z^{l} = \begin{pmatrix}
\sum_{l=0}^{N-1} \overline{W}_{pq}(l) & 0 & 0 & \cdots \\
0 & \sum_{l=0}^{N-1} \overline{W}_{pq}(l)\omega^{l} & 0 & \cdots \\
0 & 0 & \sum_{l=0}^{N-1} \overline{W}_{pq}(l)\omega^{2l} & \cdots \\
\cdots & \cdots & \cdots & \cdots
\end{pmatrix}\]
If we use the relation $W_{pq}(n)W_{qp}(n) = \rho_1^2$ (n-independent), we have $T_{pq}T_{qp} = N^2 \rho_1^2 \times 1$, and the determinant is

$$\det(T_{pq}) = N^N \prod_{l=0}^{N-1} W_{pq}(-l) = N^N \prod_{l=0}^{N-1} W_{pq}(l).$$

(23)

Then we define the normalized quantity

$$\hat{T}_{pq} = \frac{T_{pq}}{N \left| \prod_{m=0}^{N-1} W_{pq}(m) \right|^{1/N}} = \frac{T_{pq}}{|\det(T_{pq})|^{1/N}},$$

(24)

which satisfies $\det(\hat{T}_{pq}) = 1$. Using this normalized quantity, we have $\hat{T}_{pq}\hat{T}_{qp} = 1$.

Similarly, we can prove $\hat{S}_{pq}\hat{S}_{qp} = 1$ in the following way. By diagonalizing $X$ by the unitary matrix $U$ in the form $U^{-1}XU = Z$, we have

$$S_{pq} = \sum_{l=0}^{N-1} \tilde{W}_{pq}(l)X^l = U \begin{pmatrix} \sum_{l=0}^{N-1} \tilde{W}_{pq}(l) & 0 & \cdots \\ 0 & \sum_{l=0}^{N-1} \tilde{W}_{pq}(l)\omega^l & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} U^{-1}$$

$$= U \begin{pmatrix} \tilde{W}_{pq}(0) & 0 & \cdots \\ 0 & \tilde{W}_{pq}(-1) & \cdots \\ 0 & 0 & \tilde{W}_{pq}(-2) & \cdots \end{pmatrix} U^{-1}.$$ \hspace{1cm} (25)

If we use the relation $\tilde{W}_{pq}(n)\tilde{W}_{qp}(n) = \rho_2^2$ (n-independent), we have $S_{pq}S_{qp} = \rho_2^2 \times 1$, and the determinant is

$$\det(S_{pq}) = \prod_{l=0}^{N-1} \tilde{W}_{pq}(-l) = \prod_{l=0}^{N-1} \tilde{W}_{pq}(l).$$

(26)

Then we define the normalized quantities

$$\tilde{S}_{pq} = \frac{S_{pq}}{\left| \prod_{m=0}^{N-1} \tilde{W}_{pq}(m) \right|^{1/N}} = \frac{S_{pq}}{|\det(S_{pq})|^{1/N}},$$

(27)
which satisfies det(\(\hat{S}_{pq}\)) = 1. Using this normalized quantity, we have \(\hat{S}_{pq}\hat{S}_{qp} = 1\).

In this way, we summarize the properties of \(\hat{T}_{pq}\) and \(\hat{S}_{qp}\) in the following way

\[
\text{det}(\hat{T}_{pq}) = 1, \quad \text{det}(\hat{S}_{pq}) = 1, \\
\hat{T}_{qp} = \hat{T}_{pq}^{-1}, \quad \hat{S}_{qp} = \hat{S}_{pq}^{-1}.
\]  

(28)

2.4 Special case: the Ising model

The Ising model is the \(N = 2\) case of the chiral Potts model\[11\]. The star-triangle relation is \[12\]

\[
\sum_{d=\pm 1} \exp\{d(L_1a + K_2b + L_3c)\} = R \exp\{K_1bc + L_2ca + K_3ab\},
\]  

(29)

where \(a, b, c\) takes \(\pm 1\). We exchange the symbol of \(L_2\) and \(K_2\) from the standard notation in order to compare the Ising model with the chiral Potts model. \{\(K_1, L_2, K_3, R\}\} are expressed by \{\(L_1, K_2, L_3\}\) in the following way

\[
\exp(4K_1) = \frac{\cosh(L_1 + K_2 + L_3) \cosh(-L_1 + K_2 + L_3)}{\cosh(L_1 - K_2 + L_3) \cosh(L_1 + K_2 - L_3)},
\]  

(30)

\[
\exp(4L_2) = \frac{\cosh(L_1 + K_2 + L_3) \cosh(L_1 - K_2 + L_3)}{\cosh(-L_1 + K_2 + L_3) \cosh(L_1 + K_2 - L_3)},
\]  

(31)

\[
\exp(4K_3) = \frac{\cosh(L_1 + K_2 + L_3) \cosh(L_1 + K_2 - L_3)}{\cosh(-L_1 + K_2 + L_3) \cosh(L_1 - K_2 + L_3)},
\]  

(32)

\[
R = \sqrt{2 \sinh(2L_1) \sinh(2L_3) / \sinh(2L_2)},
\]  

(33)

where \(L_2\) in Eq.(33) must be expressed by \{\(L_1, K_2, L_3\)\} by using Eq.(31). We rewrite Eq.(29) by using the formula

\[
\exp(L_1ab) = \sqrt{2 \sinh(2L_1)} \left(\exp(L_1^*\sigma_x)\right)_{ab},
\]  

(34)

\[
\delta_{dd'} \exp(K_1db) = \left(\exp(K_1b\sigma_z)\right)_{dd'},
\]  

(35)

where the dual variables \(L_1^*\) are defined by \(\tanh L_1^* = \exp\{-2L_1\}\). Then the star-triangle relation is expressed by

\[
\sum_{dd'} \sqrt{2 \sinh(2L_3)} \left(\exp(L_3^*\sigma_x)\right)_{cd} \left(\exp(K_2b\sigma_z)\right)_{dd'} \sqrt{2 \sinh(2L_1)} \left(\exp(L_1^*\sigma_x)\right)_{d'a}
\]

\[
= R \sum_{dd'} \left(\exp(K_1b\sigma_z)\right)_{cd} \sqrt{2 \sinh(2L_2)} \left(\exp(L_2^*\sigma_x)\right)_{dd'} \left(\exp(K_3b\sigma_z)\right)_{d'a}.
\]  

(36)
Using Eq.(33), this relation can be rewritten in the following form
\[ \exp\{L^*_3 \sigma_x\} \exp\{bK_2 \sigma_z\} \exp\{L^*_1 \sigma_x\}_{ca} = \exp\{bK_1 \sigma_z\} \exp\{L^*_2 \sigma_x\} \exp\{bK_3 \sigma_z\}_{ca}. \tag{37} \]

If we notice that \( b = \pm 1 \), we have the star-triangle relation in the following simple form
\[ \exp\{L^*_3 \sigma_x\} \exp\{\pm K_2 \sigma_z\} \exp\{L^*_1 \sigma_x\} = \exp\{\pm K_1 \sigma_z\} \exp\{L^*_2 \sigma_x\} \exp\{\pm K_3 \sigma_z\}, \tag{38} \]

which is the Euclidean version of the spherical trigonometry relation.

3 The star-triangle relation in the chiral Potts model

In this chapter, we first give the simple proof of the star-triangle relation of the chiral Potts model. Starting from the basis \( \{X, XZ, Z\} \) of the cyclic representation of \( su(2) \), we next give the constructive proof of the star-triangle relation and give the same expression of \( T_{pq} \) and \( S_{pq} \) as in Eq.(24) and Eq.(27). We also demonstrate our method to give some generalization of the chiral Potts model.

3.1 The Proof of the star-triangle relation

From the expression of \( T_{pq} \) and \( S_{pq} \) in Eq.(12), we can show the following formula
\[ T_{pq} (d_p b_q Z/\omega - a_p c_q) X = (b_p d_q Z/\omega - c_p a_q) XT_{pq}, \tag{39} \]
\[ S_{pq} (\omega a_p d_q X - c_p b_q) Z = (\omega d_p a_q X - b_p c_q) ZS_{pq}. \tag{40} \]

Using the above relation, we first operate \( T_{pq} S_{pr} T_{qr} \) to the following quantity \( I_{pqr} \)
\[ I_{pqr} = -a_p a_q c_r X + a_p d_q b_r X Z - c_p b_q b_r Z/\omega. \tag{41} \]

Then we can show
\[ T_{qr} I_{pqr} = I_{pq} T_{qr}, \quad S_{pr} I_{pqr} = I_{rqp} S_{pr}, \quad T_{pq} I_{rqp} = I_{qrp} T_{pq}, \tag{42} \]
by using Eq.(39) and Eq.(40). This gives
\[ T_{pq} S_{pr} T_{qr} I_{pqr} = I_{rqp} T_{pq} S_{pr} T_{qr}. \tag{43} \]

Similarly we operate \( S_{qr} T_{pr} S_{pq} \) to the same quantity \( I_{pqr} \) and we can show
\[ S_{pq} I_{pqr} = I_{qpr} S_{pq}, \quad T_{pr} I_{qpr} = I_{qrp} T_{pr}, \quad S_{qr} I_{qrp} = I_{rqp} S_{qr}. \tag{44} \]
which gives
\[ S_{qr} T_{pr} S_{pq} I_{pqr} = I_{rqp} S_{qr} T_{pr} S_{pq}. \] (45)

From Eq.\((43)\) and Eq.\((45)\), we have
\[ \left[ (S_{qr} T_{pr} S_{pq})^{-1} T_{pq} S_{pr} T_{qr}, I_{pqr} \right] = 0. \] (46)

In the same way, using the formula
\[ X^{-1}(d_p b_q - \omega a_p c_q Z^{-1}) T_{pq} = T_{pq}(b_p d_q - c_p a_q Z^{-1}) X^{-1}, \] (47)
\[ Z^{-1}(a_p d_q - c_p b_q X^{-1}/\omega) S_{pq} = S_{pq}(d_p a_q - b_p c_q X^{-1}) Z^{-1}, \] (48)
and operate \(T_{pq} S_{pr} T_{qr}\) and \(S_{qr} T_{pr} S_{pq}\) on the quantity \(J_{pqr}\)
\[ J_{pqr} = -b_p b_q d_r X^{-1}/\omega + b_p c_q a_r X^{-1} Z^{-1} - d_p a_q a_r Z^{-1}, \] (49)
we have
\[ T_{qr} J_{pqr} = J_{prq} T_{qr}, \quad S_{pr} J_{prq} = J_{rpq} S_{pr}, \quad T_{pq} J_{rqp} = J_{rqp} T_{pq}, \] (50)
and
\[ S_{pq} J_{pqr} = J_{qpr} S_{pq}, \quad T_{pr} J_{qpr} = J_{qrp} T_{pr}, \quad S_{qr} J_{qrp} = J_{rqp} S_{qr}. \] (51)

From Eq.\((50)\) and Eq.\((51)\), we have
\[ \left[ (S_{qr} T_{pr} S_{pq})^{-1} T_{pq} S_{pr} T_{qr}, J_{pqr} \right] = 0. \] (52)

If we notice the relation
\[ [I_{pqr}, J_{pqr}] = \frac{\omega - 1}{\omega^2} \left( a_p b_p b_q d_q b_r d_r Z - \omega a_p d_q a_q d_q a_r b_r X - \omega a_p b_q a_q c_q a_r c_r Z^{-1} \right. \]
\[ + \omega a_p d_q a_q^2 a_r c_r X Z^{-1} + b_p c_p b_q c_q a_r b_r X^{-1} - b_p c_p b_q^2 b_r d_r X^{-1} Z/\omega \right) \neq 0, \]
and the cyclic representation is generated by \(\{X, Z\}\), we can conclude from Eq.\((52)\) that the operator
\[ (S_{qr} T_{pr} S_{pq})^{-1} T_{pq} S_{pr} T_{qr}, \]
commute with the non-commutative quantities \( \{ I_{pqr}, J_{pqr} \} \), which means

\[
(S_{qr} T_{pr} S_{pq})^{-1} T_{pq} S_{pr} T_{qr} = \rho_0 \times 1.
\]  

(53)

Rewriting the above relation with the normalized quantities \( \hat{T} \) and \( \hat{S} \), we have

\[
\hat{T}_{pq} \hat{S}_{pr} \hat{T}_{qr} = \rho'_0 \hat{S}_{qr} \hat{T}_{pr} \hat{S}_{pq}.
\]  

(54)

Taking the determinant of the both side of Eq.(54), we have

\[
\rho'_0 N = 1,
\]

which gives

\[
\rho'_0 = \omega^m,
\]

where \( m = \text{(integer)} \). If we take the special limit \( q \to p \), we have

\[
T_{pp} = 1 \quad \text{and} \quad S_{pp} = 1,
\]

which gives \( \rho'_0 |_{q \to p} = 1 \), but the integer \( m \) does not change in the limit \( q \to p \), which gives \( \rho'_0 = 1 \) in general.

In this way, we have proved the star-triangle relation of the chiral Potts model in the form

\[
\hat{T}_{pq} \hat{S}_{pr} \hat{T}_{qr} = \hat{S}_{qr} \hat{T}_{pr} \hat{S}_{pq}.
\]  

(55)

### 3.2 Constructive understanding of the star-triangle relation

We start from the quantity \( I_{pqr} \) in the general form \( I_{pqr} = \alpha_{pqr} X + \beta_{pqr} XZ + \gamma_{pqr} Z \) and construct \( T_{pq}, S_{pq} \) in such a way as it satisfies the relation

\[
T_{qr} I_{pqr} = I_{prq} T_{qr}, \quad S_{pq} I_{pqr} = I_{qpr} S_{pq}.
\]  

(56)

Imposing the \( Z_N \) periodicity, we will show that the constructed \( T_{pq}, S_{pq} \) becomes in the form of Eq.(12) and \( I_{pqr} \) becomes in the form of Eq.(41).

We first construct \( T_{pq} \) from Eq.(56), that is,

\[
T_{qr}(Z) (\beta_{pqr} Z/\omega + \alpha_{pqr}) X = (\beta_{qpr} Z/\omega + \alpha_{qpr}) XT_{qr}(Z), \quad \gamma_{pqr} = \gamma_{qpr}.
\]  

(57)

This give the expression in the form

\[
T_{qr} = \sum_{k=0}^{N-1} \hat{W}_{qr}(k) Z^k,
\]  

\[
\hat{W}_{qr}(k) = \prod_{l=1}^{k} \frac{\beta_{prq} - \beta_{pqr} \omega^{l-1}}{-\alpha_{prq} + \alpha_{pqr} \omega^l} = (p - \text{independent}).
\]  

(59)

Similarly we construct \( S_{pq} \) from Eq.(56), that is,

\[
S_{pq}(X) (\beta_{pqr} X + \gamma_{pqr}) Z = (\beta_{qpr} X + \gamma_{qpr}) Z S_{pq}(X), \quad \alpha_{pqr} = \alpha_{qpr}.
\]  

(60)
This gives the expression of the form

\[ S_{pq} = \sum_{k=0}^{N-1} \mathbf{W}_{pq}(k) X^k, \quad (61) \]

\[ \mathbf{W}_{pq}(k) = \prod_{l=1}^{k} \frac{-\beta_{pqr} + \beta_{qpr} \omega^{l-1}}{\gamma_{pqr} - \gamma_{qpr} \omega^l} = (r - \text{independent}). \quad (62) \]

In order that \( \bar{W}_{qr} \) is independent of \( p \) in the right-hand side of Eq. (59) and \( \mathbf{W}_{pq} \) is independent of \( r \) in the right-hand side of Eq. (62), we can parametrize \( \alpha_{pqr}, \beta_{pqr}, \gamma_{pqr} \) in the form

\[ \alpha_{pqr} = A \alpha_{p}^{(1)} \alpha_{q}^{(1)} \alpha_{r}^{(2)}, \quad \beta_{pqr} = B \alpha_{p}^{(1)} \beta_{q}^{(1)} \gamma_{r}^{(1)}, \quad \gamma_{pqr} = C \gamma_{p}^{(2)} \gamma_{q}^{(1)} \gamma_{r}^{(1)}, \quad (63) \]

where we used the symmetry relations Eq. (57) and Eq. (60). Denoting \( \alpha_{p}^{(1)} = a_{p}, \gamma_{p}^{(1)} = b_{p}, \alpha_{p}^{(2)} = c_{p}, \beta_{p}^{(1)} = d_{p}, \gamma_{p}^{(2)} = e_{p}, \) the periodic condition of \( \bar{W}_{pq} \) and \( \mathbf{W}_{pq} \) give the conditions

\[ \frac{b_{p} N d_{q} N - d_{p} N b_{q} N}{c_{p} N a_{q} N - a_{p} N c_{q} N} = 1, \quad \frac{a_{p} N d_{q} N - d_{p} N a_{q} N}{c_{p} N b_{q} N - b_{p} N e_{q} N} = 1, \quad (64) \]

where we impose \((-A)^N = B^N = (-C)^N\), and we choose \( A = -1, B = 1, C = -1/\omega\). Up to this level, it is not necessary to take \( e_{p} = c_{p} \), and we have the following expression

\[ I_{pqr} = -a_{p} d_{q} c_{r} X + a_{p} d_{q} b_{r} X Z - e_{p} b_{q} b_{r} Z/\omega, \quad (65) \]

\[ T_{qr} = \sum_{k=0}^{N-1} \bar{W}_{qr}(k) Z^k, \quad \bar{W}_{qr}(k) = \rho_{1}^{1} \prod_{j=1}^{k} b_{q} a_{r} - d_{q} b_{r} \omega^{j-1}, \quad (66) \]

\[ S_{pq} = \sum_{k=0}^{N-1} \mathbf{W}_{pq}(k) X^k, \quad \mathbf{W}_{pq}(k) = \rho_{2}^{1} \prod_{j=1}^{k} \omega a_{p} d_{q} - d_{p} a_{q} \omega^{j}, \quad (67) \]

Similarly, we start from the general form \( J_{pqr} = \alpha_{pqr}' X^{-1} + \beta_{pqr}' X^{-1} Z^{-1} + \gamma_{pqr}' Z^{-1} \), which satisfies the properties

\[ T_{qr} J_{pqr} = J_{prq} T_{qr}, \quad S_{pq} J_{pqr} = J_{qpr} S_{pr}, \quad (68) \]

which gives

\[ T_{qr} = \sum_{k=0}^{N-1} \bar{W}_{qr}(k) Z^k, \quad (69) \]
\[
\tilde{W}_{qr}(k) = \rho_1 \prod_{l=1}^{k-1} \frac{-\omega \alpha'_{prq} + \alpha'_{prq}\omega^l}{\beta'_{prq} - \beta'_{prq}\omega^l} = (p \text{ - independent}), \quad (70)
\]
\[
S_{pq} = \sum_{k=0}^{N-1} \tilde{W}_{pq}(k)X^k, \quad (71)
\]
\[
\tilde{W}_{pq}(k) = \rho_2 \prod_{l=1}^{k} \frac{\omega \gamma'_{apq} - \gamma'_{apq}\omega^l}{\beta'_{apq} + \beta'_{apq}\omega^l} = (r \text{ - independent}), \quad (72)
\]

with \(\gamma'_{apq} = \gamma'_{prq}\) and \(\alpha'_{prq} = \alpha'_{apq}\). Then we can parametrize
\[
\alpha'_{prq} = A' \alpha_p^{(1)} \alpha_q^{(1)} \alpha_r^{(2)}, \quad \beta'_{apq} = B' \alpha_p^{(1)} \beta_q^{(1)} \gamma_r^{(1)}, \quad \gamma'_{apq} = C' \gamma_p^{(2)} \gamma_q^{(1)} \gamma_r^{(1)}. \quad (73)
\]

We impose that \(\tilde{W}_{qr}(k)\) in Eq.(55) and Eq.(70) gives the same expression and \(\tilde{W}_{pq}(k)\) in Eq.(77) and Eq.(72) gives the same expression, we obtain \(\gamma_p^{(1)} = a_p, \alpha_q^{(1)} = b_p, \beta_q^{(1)} = c_p = e_p, \alpha_p^{(2)} = \gamma_q^{(2)} = d_p\) by choosing \(A' = -1/\omega, B' = 1, C' = -1\). In this way, \(e_p = c_p\) become necessary to exist \(J_{prq}\), we have the desired expression
\[
I_{pqr} = -a_p a_q c_r X + a_p d_q b_r X Z - c_p b_q b_r Z/\omega, \quad (74)
\]
\[
J_{pqr} = -b_p b_q d_r X^{-1}/\omega + b_p c_q a_r X^{-1} Z^{-1} - d_p a_q a_r Z^{-1}, \quad (75)
\]
\[
T_{qr} = \sum_{k=0}^{N-1} \tilde{W}_{qr}(k)Z^k, \quad \tilde{W}_{qr}(k) = \rho_1 \prod_{j=1}^{k} \frac{b_q d_r - d_q b_r \omega^j - 1}{c_q a_r - a_q c_r \omega^j}, \quad (76)
\]
\[
S_{pq} = \sum_{k=0}^{N-1} \tilde{W}_{pq}(k)X^k, \quad \tilde{W}_{pq}(k) = \rho_2 \prod_{j=1}^{k} \frac{\omega a_p d_q - d_p a_q \omega^j}{c_p b_q - b_p c_q \omega^j}, \quad (77)
\]

and the periodic condition becomes
\[
\frac{b_p N d_q N - d_p N b_q N}{c_p N a_q N - a_p N c_q N} = 1, \quad \frac{a_p N d_q N - d_p N a_q N}{c_p N b_q N - b_p N c_q N} = 1, \quad (78)
\]

which is satisfied by the condition
\[
a_p N + k'b_p N = kd_p N, \quad k'a_p N + b_p N = kc_p N, \quad (79)
\]

where \(\{k, k'\}\) are constants, which are not always necessary to satisfy \(k^2 + k'^2 = 1\) in general.

The rest of the proof of the star-triangle relation is the same as that in the previous section.
3.3 Some generalization

From the construction of the previous section, we can construct the some generalized model for $N = uv$ ($u, v$ : relatively prime integers). For this case, we consider $X' = X^u$, $Z' = Z^u$ and these satisfy $Z'X' = \omega'X'Z'$ with $\omega' = \omega u^2$. Then we replace $X \rightarrow X'$, $Z \rightarrow Z'$, $\omega \rightarrow \omega'$ in $T_{qr}$ and $S_{pq}$, which gives

$$T_{qr} = \sum_{k=0}^{v-1} \tilde{W}_{qr}(k) Z^{uk}, \quad \tilde{W}_{qr}(k) = \rho'_1 \prod_{j=1}^{k} \frac{b_q d_r - d_q b_r \omega'^2(j-1)}{c_q a_r - a_q c_r \omega'^2 j},$$

$$S_{pq} = \sum_{k=0}^{v-1} \tilde{W}_{pq}(k) X^{uk}, \quad \tilde{W}_{pq}(k) = \rho_2 \prod_{j=1}^{k} \frac{\omega'^2 a_p d_q - d_p a_q \omega'^2 j}{c_p b_q - b_p c_q \omega'^2 j}.$$  

If we notice that $u$ and $v$ are relatively prime, the periodic condition is

$$a_p^v + k' b_p^v = kd_p^v, \quad k'a_p^v + b_p^v = kc_p^v.$$  

We define the normalized quantity as before

$$\hat{T}_{pq} = \frac{T_{pq}}{|\det(T_{pq})|^{1/N}}, \quad \hat{S}_{pq} = \frac{S_{pq}}{|\det(S_{pq})|^{1/N}},$$

we can show the star-triangle relation of the form

$$\hat{S}_{qr} \hat{T}_{pr} \hat{S}_{pq} = \hat{T}_{pq} \hat{S}_{pr} \hat{T}_{qr},$$

for this generalized model. In the special case of $u = 1, v = N$, this reduce the original chiral Potts model.

4 Summary and discussion

We give the simple proof of the star-triangle relation of the chiral Potts model. We also give the constructive way to understand the star-triangle relation for the chiral Potts model, which may give the hint to give the new integrable model. We can rewrite the star-triangle relation with the group element of the cyclic representation of $su(2)$. Our approach may give the hint to give the new two dimensional integrable model by considering the cyclic representation of more general Lie algebra. We can prove the Yang-Baxter relation of the vertex type chiral Potts model from the star-triangle relation[15], which suggests that the
star-triangle relation of the chiral Potts model is more fundamental than the Yang-Baxter relation of the vertex type chiral Potts model. As the Bazhanov-Baxter model is constructed from the vertex type chiral Potts model of $sl(n)$, we expect that the mathematical structure of Bazhanov-Baxter model is understood from the star-triangle relation of the chiral Potts model.

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