COMPONENTS OF $V(\rho) \otimes V(\rho)$

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(WITH AN APPENDIX BY ROCCO CHIRIVÌ AND ANDREA MAFFEI)

1. INTRODUCTION

Let $\mathfrak{g}$ be any simple Lie algebra over $\mathbb{C}$. We fix a Borel subalgebra $\mathfrak{b}$ and a Cartan subalgebra $\mathfrak{t} \subset \mathfrak{b}$ and let $\rho$ be the half sum of positive roots, where the roots of $\mathfrak{b}$ are called the positive roots. For any dominant integral weight $\lambda \in \mathfrak{t}^*$, let $V(\lambda)$ be the corresponding irreducible representation of $\mathfrak{g}$. B. Kostant initiated (and popularized) the study of the irreducible components of the tensor product $V(\rho) \otimes V(\rho)$. In fact, he asked (or possibly even conjectured) if the following is true.

**Question 1. (Kostant)** Let $\lambda$ be a dominant integral weight. Then, $V(\lambda)$ is a component of $V(\rho) \otimes V(\rho)$ if and only if $\lambda \leq 2\rho$ under the usual Bruhat-Chevalley order on the set of weights.

It is, of course, clear that if $V(\lambda)$ is a component of $V(\rho) \otimes V(\rho)$, then $\lambda \leq 2\rho$.

One of the main motivations behind Kostant’s question was his result that the exterior algebra $\wedge \mathfrak{g}$, as a $\mathfrak{g}$-module under the adjoint action, is isomorphic with $2^r$ copies of $V(\rho) \otimes V(\rho)$, where $r$ is the rank of $\mathfrak{g}$ (cf. [Ko]). Recall that $\wedge \mathfrak{g}$ is the underlying space of the standard chain complex computing the homology of the Lie algebra $\mathfrak{g}$, which is, of course, an object of immense interest.

**Definition 2.** An integer $d \geq 1$ is called a saturation factor for $\mathfrak{g}$, if for any $(\lambda, \mu, \nu) \in D^3$ such that $\lambda + \mu + \nu$ is in the root lattice and the space of $\mathfrak{g}$-invariants:

$$[V(N\lambda) \otimes V(N\mu) \otimes V(N\nu)]^0 \neq 0$$

for some integer $N > 0$, then

$$[V(d\lambda) \otimes V(d\mu) \otimes V(d\nu)]^0 \neq 0,$$

where $D \subset \mathfrak{t}^*$ is the set of dominant integral weights of $\mathfrak{g}$. Such a $d$ always exists (cf. [Ku; Corollary 44]).

Recall that 1 is a saturation factor for $\mathfrak{g} = sl_n$, as proved by Knutson-Tao [KT]. By results of Belkale-Kumar [BK] (also obtained by Sam [S]) and Hong-Shen [HS], $d$ can be taken to be 2 for $\mathfrak{g}$ of types $B_r, C_r$ and $d$ can be
taken to be 4 for $\mathfrak{g}$ of type $D$, by a result of Sam [S]. As proved by Kapovich-Millson [KM$_1$, KM$_2$], the saturation factors $d$ of $\mathfrak{g}$ of types $G_2, F_4, E_6, E_7, E_8$ can be taken to be 2 (in fact any $d \geq 2$), 144, 36, 144, 3600 respectively. (For a discussion of saturation factors $d$, see [Ku, §10].)

Now, the following (weaker) result is our main theorem. The proof uses a description of the eigencone of $\mathfrak{g}$ in terms of certain inequalities due to Berenstein-Sjamaar coming from the cohomology of the flag varieties associated to $\mathfrak{g}$, a ‘non-negativity’ result due to Belkale-Kumar and Proposition (9) due to R. Chirivi and A. Maffei given in the Appendix.

An interesting aspect of our work is that we make an essential use of a solution of the eigenvalue problem and saturation results for any $\mathfrak{g}$.

**Theorem 3.** Let $\lambda$ be a dominant integral weight such that $\lambda \leq 2\rho$. Then, $V(d\lambda) \subset V(d\rho) \otimes V(d\rho)$, where $d \geq 1$ is any saturation factor for $\mathfrak{g}$.

In particular, for $\mathfrak{g} = sl_n$, $V(\lambda) \subset V(\rho) \otimes V(\rho)$.

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### 2. Proof of Theorem (3)

We now prove Theorem (3).

**Proof.** Let $\Gamma_3(\mathfrak{g})$ be the saturated tensor semigroup defined by

$$\Gamma_3(\mathfrak{g}) = \{ (\lambda, \mu, \nu) \in D^3 : [V(N\lambda) \otimes V(N\mu) \otimes V(N\nu)]^G \neq 0 \text{ for some } N > 0 \}.$$

To prove the theorem, it suffices to prove that $(\rho, \rho, \lambda^*) \in \Gamma_3(G)$, where $\lambda^*$ is the dual weight $-w_o\lambda$, $w_o$ being the longest element of the Weyl group of $\mathfrak{g}$. Let $G$ be the connected, simply-connected complex algebraic group with Lie algebra $\mathfrak{g}$. Let $B$ (resp. $T$) be the Borel subgroup (resp. maximal torus) of $G$ with Lie algebra $\mathfrak{b}$ (resp. $\mathfrak{t}$). Let $W$ be the Weyl group of $G$. For any standard parabolic subgroup $P \supset B$ with Levi subgroup $L$ containing $T$, let $W^P$ be the set of smallest length coset representatives in $W/W_L$, $W_L$ being the Weyl group of $L$. Then, we have the Bruhat decomposition:

$$G/P = \sqcup_{w \in W^P} \Lambda_w^P, \text{ where } \Lambda_w^P := BwP/P.$$

Let $\bar{\Lambda}_w$ denote the closure of $\Lambda_w$ in $G/P$. We denote by $[\bar{\Lambda}_w]$ the Poincaré dual of its fundamental class. Thus, $[\bar{\Lambda}_w]$ belongs to the singular cohomology:

$$[\bar{\Lambda}_w] \in H^{2\dim G/P - \ell(w)}(G/P, \mathbb{Z}),$$

where $\ell(w)$ is the length of $w$.

Let $\{x_j\}_i \subseteq \mathfrak{t}$ be the dual to the simple roots $\{\alpha_i\}_i \subseteq \mathfrak{t}$, i.e.,

$$\alpha_i(x_j) = \delta_{i,j}.$$
In view of [BS] (or [Ku; Theorem 10]), it suffices to prove that for any standard maximal parabolic subgroup $P$ of $G$ and triple $(u, v, w) \in (W^P)^3$ such that the cup product of the corresponding Schubert classes in $G/P$:

(1) \[ [\overline{\Lambda}_u^P] \cdot [\overline{\Lambda}_v^P] \cdot [\overline{\Lambda}_w^P] = k[\overline{\Lambda}_e^P] \in H^*(G/P, \mathbb{Z}), \text{ for some } k \neq 0, \]

the following inequality is satisfied:

(2) \[ \rho(ux_P) + \rho(vx_P) + \lambda'(wx_P) \leq 0. \]

Here, $x_P := x_{\alpha_P}$, where $\alpha_P$ is the unique simple root not in the Levi of $P$.

Now, by [BK1; Proposition 17(a)] (or [Ku; Corollary 22 and Identity (9)]), for any $u, v, w \in (W^P)^3$ such that the equation (1) is satisfied,

(3) \[ (\chi_{w_ow_0^P} - \chi_u - \chi_v)(x_P) \geq 0, \]

where $w_0^P$ is the longest element in the Weyl group of $L$ and

$\chi_w := \rho - 2\rho^L + w^{-1}\rho$

($\rho^L$ being the half sum of positive roots in the Levi of $P$).

Now,

(4) \[ (\chi_{w_ow_0^P} - \chi_u - \chi_v)(x_P) = (\rho - w_0^P w^{-1}\rho - \rho - u^{-1}\rho - v^{-1}\rho)(x_P), \text{ since } \rho^L(x_P) = 0 \]

Combining (3) and (4), we get

(5) \[ (\rho + u^{-1}\rho + v^{-1}\rho + w^{-1}\rho)(x_P) \leq 0, \text{ if (1) is satisfied.} \]

We next claim that for any dominant integral weight $\lambda \leq 2\rho$ and any $u, v, w \in (W^P)^3$,

(6) \[ \rho(ux_P) + \rho(vx_P) + \lambda'(wx_P) \leq (\rho + u^{-1}\rho + v^{-1}\rho + w^{-1}\rho)(x_P), \]

which is equivalent to

(7) \[ \lambda'(wx_P) \leq (\rho + w^{-1}\rho)(x_P). \]

Of course (5) and (6) together give (2). So, to prove the theorem, it suffices to prove (7). Since the assumption on $\lambda$ in the theorem is invariant under the transformation $\lambda \mapsto \lambda'$, we can replace $\lambda'$ by $\lambda$ in (7). By Proposition (9) in the appendix, $\lambda = \rho + \beta$, where $\beta$ is a weight of $V(\rho)$ (i.e., the weight space of $V(\rho)$ corresponding to the weight $\beta$ is nonzero). Thus,

\[ \lambda(wx_P) = \rho(wx_P) + \beta(x_P), \text{ for some weight } \beta \text{ of } V(\rho). \]

Hence,

\[ \lambda(wx_P) = \rho(wx_P) + \beta(x_P) \leq (w^{-1}\rho + \rho)(x_P). \]

This establishes (7) and hence the theorem is proved. \qed
We recall the following conjecture due to Kapovich-Millson [KM$_1$] (or [Ku; Conjecture 47]).

**Conjecture 4.** Let \( \mathfrak{g} \) be a simple, simply-laced Lie algebra over \( \mathbb{C} \). Then, \( d = 1 \) is a saturation factor for \( \mathfrak{g} \).

The following theorem follows immediately by combining Theorem (3) and Conjecture (4).

**Theorem 5.** For any simple, simply-laced Lie algebra \( \mathfrak{g} \) over \( \mathbb{C} \), assuming the validity of Conjecture (4), Question (1) has an affirmative answer for \( \mathfrak{g} \), i.e., for any dominant integral weight \( \lambda \leq 2\rho \), \( V(\lambda) \) is a component of \( V(\rho) \otimes V(\rho) \).

Thus, assuming the validity of Conjecture (4), Question (1) has an affirmative answer for any simple \( \mathfrak{g} \) of type \( D_r \) (\( r \geq 4 \); \( E_6 \); \( E_7 \); and \( E_8 \) as well (apart from \( \mathfrak{g} \) of type \( A_r \) as in Theorem (3)).

**Remark 6.** By an explicit calculation using the program LIE, it is easy to see that Question (1) has an affirmative answer for simple \( \mathfrak{g} \) of types \( G_2 \) and \( F_4 \) as well.
3. APPENDIX (DUE TO R. CHIRIVÌ AND A. MAFFEI)

We follow the notation and assumptions from the Introduction. In particular, \(\mathfrak{g}\) is a simple Lie algebra over \(\mathbb{C}\). Let \(\{\omega_i\}_{i \in I}\) be the fundamental weights, \(\{\alpha_i\}_{i \in I}\) the simple roots, and \(\{s_i\}_{i \in I}\) the simple reflections, where \(I := \{1 \leq i \leq r\}\). For any \(J \subset I\), let \(W_J\) be the parabolic subgroup of the Weyl group \(W\) generated by \(s_j\) with \(j \in J\) and let \(\Phi_J\) be the root system generated by the simple roots \(\alpha_j\) with \(j \in J\). Set

\[\Omega := \bigoplus_{i \in I} \mathbb{R}\omega_i; \quad \Omega_J := \bigoplus_{j \in J} \mathbb{R}\omega_j,\]

and let \(\pi_J : \Omega \to \Omega_J\) be the projection with kernel \(\Omega_{I \setminus J}\). The projection \(\pi_J(\Phi_J)\) of the roots in \(\Phi_J\) gives a root system whose fundamental weights are given by \(\{\omega_j : j \in J\}\).

Let \(A \subset \mathfrak{t}^*\) be the dominant cone, \(B \subset \mathfrak{t}^*\) the cone generated by \(\{-\alpha_i : i \in I\}\) and \(C := 2\rho + B\). We want to describe the vertices of the polytope \(A \cap C\). For \(J \subset I\) define

\[A_J := \mathbb{R}_{\geq 0}[\omega_j : j \in J], \quad B_J := \mathbb{R}_{\geq 0}[-\alpha_j : j \in J] \quad \text{and} \quad C_J := 2\rho + B_J.\]

The sets \(A_J\) and \(B_J\) are the faces of \(A\) and \(B\). The vertices of the polytope \(A \cap C\) are given by the zero dimensional nonempty intersections of the form \(A_J \cap C_H\).

For any \(J \subset I\), let \(b_J := \sum_{\alpha \in \Phi_J^+} \alpha\) and \(c_J := 2\rho - b_J\). All these points are different. Moreover, \(c_I = 0\) and \(c_0 = 2\rho\).

**Lemma 7.** For each \(J \subset I\), we have

\[A_{I \setminus J} \cap C_J = \{c_J\}.\]

Moreover, none of the other intersections \(A_H \cap C_K\) give a single point.

**Proof.** Observe that

\[b_J = 2 \sum_{j \in J} \omega_j + \sum_{\ell \not\in J} a_\ell \omega_\ell, \quad \text{where} \quad a_\ell \leq 0.\]

Hence, \(c_J \in A_{I \setminus J} \cap C_J\).

Consider now an intersection of the form \(A_{I \setminus H} \cap C_K\). Assume it is not empty and that \(y = 2\rho - x \in A_{I \setminus H} \cap C_K\). Then, \(x = 2 \sum_{h \in H} \omega_h + \sum_{\ell \in H} d_\ell \omega_\ell\). Now, notice that if \(h \notin K\), the coefficient of \(\omega_h\) in \(x\) can not be positive. So, we must have \(K \supset H\). If \(K \supset H\) and \(K \neq H\), then

\[A_{I \setminus H} \cap C_K \supset (A_{I \setminus H} \cap C_H) \cup (A_{I \setminus K} \cap C_K) \supset \{c_H, c_K\}.\]

Hence, it is not a single point.
It remains to prove that $A_{I-J} \cap C_J \subset \{c_J\}$. Let $y = 2\rho - x$ as before. Notice that $\pi_J(x) = 2 \sum_{j \in J} \omega_j$ and $\pi_J(x) = \sum_{\alpha \in \Phi^+} \pi_J(\alpha)$. Since $\pi_J$ is injective on $B_J$, we must have $x = b_J$ and the claim follows.

We have the following Corollary.

**Corollary 8.** The intersection $A \cap C$ is the convex hull of the points $\{c_J : J \subset I\}$.

We now prove the following main result of this Appendix.

**Proposition 9.** Let $\lambda \leq 2\rho$ be a dominant integral weight. Then,

$$\lambda = \rho + \beta,$$

for some weight $\beta$ of $V(\rho)$.

**Proof.** Let $Q \subset \mathfrak{t}^*$ be the root lattice (generated by the simple roots) and let $H_\rho$ be the convex hull of the weights $\{w(\rho) : w \in W\}$. Recall that the weights of the module $V(\rho)$ are precisely the elements of the intersection

$$(\rho + Q) \cap H_\rho.$$ 

If $\lambda$ is as in the proposition, then it is clear that $\lambda - \rho \in \rho + Q$. So, we need to prove that it belongs to $H_\rho$. To check this, it is enough to check that $(A \cap C) - \rho \subset H_\rho$ or equivalently that

$$c_J - \rho \in H_\rho, \text{ for all } J \subset I.$$

We have

$$c_J - \rho = \rho - b_J = w_o^J(\rho) \in H_\rho,$$

where $w_o^J$ is the longest element in the parabolic subgroup $W_J$. Indeed, to prove the last equality, observe that $\rho - w_o^J(\rho)$ is a sum of roots $\alpha_j$ with $j \in J$. So, since $\pi_J$ is injective on $B_J$, it is enough to check that $\pi_J(\rho - w_o^J(\rho)) = \pi_J(b_J)$. Hence, we are reduced to study the case in which $J = I$, for which we have $w_o^I(\rho) = -\rho$ and $\rho - w_o^I(\rho) = 2\rho = b_I$. \qed

**REFERENCES**

[BK1] P. Belkale and S. Kumar, Eigenvalue problem and a new product in cohomology of flag varieties, *Invent. Math.* 166 (2006), 185-228.

[BK2] P. Belkale and S. Kumar, Eigencone, saturation and Horn problems for symplectic and odd orthogonal groups, *J. Alg. Geom.* 19 (2010), 199–242.

[BS] A. Berenstein and R. Sjamaar, Coadjoint orbits, moment polytopes, and the Hilbert-Mumford criterion, *J. Amer. Math. Soc.* 13 (2000), 433–466.

[HS] J. Hong and L. Shen, Tensor invariants, saturation problems, and Dynkin automorphisms, Preprint (2015).

[KM1] M. Kapovich and J. J. Millson, Structure of the tensor product semigroup, *Asian J. of Math.* 10 (2006), 492–540.
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[KM$_2$] M. Kapovich and J. J. Millson, A path model for geodesics in Euclidean buildings and its applications to representation theory, *Groups, Geometry and Dynamics* 2 (2008), 405–480.

[KT] A. Knutson and T. Tao, The honeycomb model of $\text{GL}_n(\mathbb{C})$ tensor products I: Proof of the saturation conjecture, *J. Amer. Math. Soc.* 12 (1999), 1055–1090.

[Ko] B. Kostant, Clifford algebra analogue of the Hopf-Koszul-Samelson theorem, the $\rho$-decomposition, $C(\mathfrak{g}) = \text{End} V_{\rho} \otimes C(P)$, and the $\mathfrak{g}$-module structure of $\wedge \mathfrak{g}$, *Adv. Math.* 125 (1997), 275–350.

[Ku] S. Kumar, A survey of the additive eigenvalue problem (with appendix by M. Kapovich), *Transformation Groups* 19 (2014), 1051–1148.

[S] S. Sam, Symmetric quivers, invariant theory, and saturation theorems for the classical groups, *Adv. Math.* 229 (2012), 1104–1135.

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