COVARIANT PHASE SPACE, CONSTRAINTS, GAUGE AND THE PEIERLS FORMULA

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It is well known that both the symplectic structure and the Poisson brackets of classical field theory can be constructed directly from the Lagrangian in a covariant way, without passing through the non-covariant canonical Hamiltonian formalism. This is true even in the presence of constraints and gauge symmetries. These constructions go under the names of the covariant phase space formalism and the Peierls bracket. We review both of them, paying more careful attention, than usual, to the precise mathematical hypotheses that they require, illustrating them in examples. Also an extensive historical overview of the development of these constructions is provided. The novel aspect of our presentation is a significant expansion and generalization of an elegant and quite recent argument by Forger & Romero showing the equivalence between the resulting symplectic and Poisson structures without passing through the canonical Hamiltonian formalism as an intermediary. We generalize it to cover theories with constraints and gauge symmetries and formulate precise sufficient conditions under which the argument holds. These conditions include a local condition on the equations of motion that we call hyperbolizability, and some global conditions of cohomological nature. The details of our presentation may shed some light on subtle questions related to the Poisson structure of gauge theories and their quantization.

Keywords: Classical field theory; Covariant phase space; Peierls bracket; Symplectic structure; Poisson structure.

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1. Introduction

A classical field theory is essentially defined by a local variational principle for a given set of dynamical fields on a given spacetime manifold. The variational principle determines a set of partial differential equations (PDEs), the equations of motion, to be imposed on the dynamical fields. The equations of motion are typically hyperbolic (or can be made so with gauge fixing if gauge invariance is present). What distinguishes variational PDEs among the more general class of hyperbolic PDEs is that their solution spaces can be naturally endowed with symplectic and hence Poisson structure, making it into a phase space. The algebra of smooth functions on the phase space with the corresponding Poisson bracket then constitutes the Poisson algebra of observables. The existence of this algebraic formulation is what allows for quantization.

It is clear that the symplectic and Poisson structure on its phase space is a crucial ingredient in the description of a classical field theory. The most common way
of building these structures is via the canonical formalism\textsuperscript{1,2} (sometimes known as
the 3+1 Hamiltonian formalism, when spacetime is 4-dimensional), which requires
an explicit choice of a time function, even when no one such choice is natural, and
the application of a Legendre transform, which may be only ambiguously defined,
for instance, in the presence of gauge invariance. However, it is also well known
that both can be defined completely \textit{covariantly} (that is, without choosing an ex-
licit time function or applying a Legendre transform) directly from the Lagrangian,
without going through the canonical formalism. These methods are known, respec-
tively, as the \textit{covariant phase space formalism}\textsuperscript{3–5} and the \textit{Peierls bracket}\textsuperscript{6,7}.
They are clearly preferable when the canonical formalism explicitly breaks some of the
natural symmetries of the theory (any relativistic theory is an example). The sym-
plectic 2-form and the Poisson bivector constructed in this way are equivalent (they
are mutual inverses). Despite the covariance of both constructions, until recently,
their equivalence was only known via the intermediary of the non-covariant canoni-
cal formalism\textsuperscript{8,9}. That is, until Forger & Romero\textsuperscript{10} provided an elegant, completely
covariant proof of the equivalence in the case of a scalar field.

In this review, we describe in depth the constructions of symplectic and Poisson
structures of classical field theory, as well as their equivalence, all in a covariant
way. A novel contribution of our exposition is an extension of the Forger-Romero
argument to field theories where constraints and gauge invariance are present. An-
other is that we do not restrict ourselves to the class of wave-like equations defined
on Lorentzian manifolds (though that will be the main source of our examples).
We also pay special attention to several aspects that are often omitted or left im-
plcit in the existing literature. Neither the covariant phase space nor the Peierls
constructions are automatic. That is, besides a given Lagrangian density, several
conditions must be fulfilled for the corresponding formulas to make sense. We make
these conditions explicit in both cases: existence of a Cauchy surface and spacelike
compact support for solutions, in the construction of the symplectic structure, and
existence of hyperbolic PDE system, with retarded and advanced Green functions,
closely related to the equations of motions, in the construction of the Poisson struc-
ture. Furthermore, the statement of equivalence also requires a certain sufficient
condition on the cohomological properties of the constraint and gauge generator
differential operators.

Note that we restrict our attention to the geometric and algebraic aspects of
the constructions and systematically avoid analytical details. In particular, the con-
struction of the space of solutions of a PDE system on a spacetime of dimension
greater than one (which corresponds to an ordinary mechanical system) requires a
theory of infinite dimensional differential geometry. We treat the minimal amount
of the needed infinite dimensional geometry in a formal way. On the other hand, an
honest attempt to present the relevant functional analytical details can be found in
Refs. 11–14. Our results could then form the core identities of a future investigation
along similar lines that could extend them beyond the formal level.
In Sec. 2, we present some background material on Green functions and hyperbolic PDEs. There we define the notion of Green hyperbolicity that will be crucial in the later construction of the Peierls formula and the study of its properties. The main body of this review is contained in Sec. 3. There, we review the formal differential geometry of the (possibly infinite dimensional) solution spaces (Sec. 3.2). Also, we review the covariant phase space formalism (Secs. 3.1, 3.3) and the Peierls formula (Sec. 3.3). Finally, Sec. 3.3.5 shows the equivalence between the corresponding symplectic and Poisson structures by a generalization of the Forger-Romero argument, under precise sufficient conditions (Secs. 3.2.2, 3.2.3 and 3.3.1). We conclude with some examples in Sec. 4 and a discussion in Sec. 5. In addition, some further needed background information is given in the appendices, which includes the jet bundle approach to PDEs, conservation laws and variational forms, as well as a generalization of causal structure on smooth manifolds beyond Lorentzian geometry. In particular, Appendix B sets up some notation that is used throughout the paper to describe PDE systems.

Finally, we conclude this introductory section with an extensive (though still incomplete) historical overview of the literature on the covariant phase space formalism and the Peierls formula. This historical material may be safely skipped at first reading.

1.1. Historical overview

The canonical formalism in mechanics (what we would now call the construction of the symplectic and Poisson structures on the phase space of a mechanical system) has a long history and is most closely associated with the names of Hamilton and Jacobi. Though, undoubtedly, its roots go back even to Lagrange. Its main features are (a) the identification of the phase space with the space of initial data and (b) the use of the Legendre transform to determine special coordinates on the phase space in which the symplectic form takes a certain canonical form (hence naming the formalism). Because of the relation of Poisson brackets to quantization, the problem of the quantization of fields in the early 20th century required the translation (see for instance Refs. 1, 20) of the canonical formalism from mechanics to field theories (multiple independent variables instead of just one).

As already mentioned earlier, the main features of the canonical formalism quickly began to clash with the relativistic nature and spacetime covariance of field theories. That was already clear in the works of Rosenfeld and Dirac. The possibility that both of these unpleasant features could be avoided became realized very slowly and is still not in mainstream use by theoretical physicists. It appears that parts of it were rediscovered multiple times and it is hard to trace them to any one source. Below, we discuss some key references that made it clear that it is possible to construct the phase space itself, as well as its symplectic and Poisson structures in a fully covariant way, avoiding both features (a) and (b) of the canonical formalism.
An important figure in some of the developments described below is that of Souriau, despite lack of many explicit references to his work. Perhaps his importance is not surprising, since he was one of the people responsible for abstracting (in the 1960s) the modern notion of a symplectic manifold as the appropriate arena for mechanics.\textsuperscript{19} In particular, the name of Souriau is closely associated to identifying the classical phase space with the set of solutions of the equations of motion, rather than the set of initial data. Also, in Souriau’s book\textsuperscript{19} can be found a construction of the symplectic structure on the phase space directly from the Lagrangian, which he attributes to Lagrange himself.\textsuperscript{21} Even though his book treated only mechanical systems and not field theories, these ideas seem to have been rather influential.

1.1.1. Peierls formula

The covariant construction of Poisson brackets in field theory can in fact be traced to a single source: the seminal 1952 paper of Peierls.\textsuperscript{6} In that paper, he introduced what is now known as the \textit{Peierls bracket}, which we prefer to call the \textit{Peierls formula}, as reviewed in Sec. 3.3.4. The formula for the causal Green function as the difference of the retarded and advanced Green functions, $G = G^+ - G^-$, appeared there, though in a somewhat implicit form. It is likely that Peierls was guided by experience. Having seen the unequal time Poisson bracket (or rather the quantum commutator) of point fields in many examples, computed using the canonical method, but expressed in relativistically invariant form, he was probably lead to a guess for its general formula. He showed that this formula is in fact antisymmetric and is equivalent to the canonical bracket for equal time fields, in the non-singular case (without gauge invariance). He, however, did not give an independent proof of its non-degeneracy or the Jacobi identity. Peierls showed that gauge invariance was not an obstacle to defining the Poisson bracket by his method, as long as one restricted oneself to gauge invariant observables. He also showed how the formula extends to fermionic fields: each fermionic field can be reduced to a bosonic one after multiplication by a formal anticommuting parameter.

A somewhat later 1957 paper of Glaser, Lehmann & Zimmermann\textsuperscript{22} treated the perturbative expansion of interacting fields in terms of retarded products of incoming free fields, in contrast to the usual expansion in terms of time ordered products of asymptotic fields. As their name suggests, retarded fields are defined using retarded Green functions. They did not explicitly discuss Poisson structures, but their work folds into the thread of ideas we are discussing in a slightly different way. Their formula for the commutator of interacting fields involved differences of of retarded and advanced Green functions, what we would now call causal Green functions, which also occur in the Peierls formula. This is not so surprising given the intimate relationship between quantum commutators and Poisson brackets.

In 1960 came a paper of Segal\textsuperscript{23} where he discussed the canonical quantization of field theories with non-linear hyperbolic equations of motion by identifying their phase space with the space of solutions and endowing it with a Poisson structure
in a covariant way. His formula for the Poisson bracket also involved the causal Green function G. His construction appears to have been independent of Peierls, but motivated much in the same way. In particular, he constructs G not as the difference of retarded and advanced Green functions, but as a distributional solution of the linearized equations with specific initial conditions designed to reproduce the equal time commutation relations. Further, though minor, developments of these ideas appeared in a monograph\cite{24} and in some conference proceedings, including Ref. 25.

Unfortunately, not many people paid attention to Peierls’ paper. Notable exceptions were Bergmann and DeWitt. In fact, DeWitt\cite{7,26} quickly became an early adopter and proponent. He can be said to be responsible for clarifying the role of the causal Green function of the Jacobi equation (the linearized Euler-Lagrange equation) in Peierls’ construction and showing that it can be consistently used with gauge fixing. Incidentally, he also clarified the role of classical fermi fields in terms of anticommuting Grassmann variables and thus seeded the germs of supergeometry.

In 1971, Steinmann\cite{27} published a monograph where he adapted the retarded products of Glaser, Lehmann & Zimmermann as a way of formalizing renormalized perturbation theory within the context of Axiomatic Quantum Field Theory.

DeWitt’s formulation saw relatively few improvements until the early ‘90s when Marolf\cite{28-30} (DeWitt’s PhD student at the time) realized that the Peierls formula can be taken off-shell and define a Poisson bracket for off-shell observables. He essentially showed that, by using the Peierls formula directly, the (off-shell) field configuration space can be given the structure of a degenerate (though regular) Poisson manifold. The symplectic leaves of this Poisson structure are copies of the (on-shell) solution space with the standard canonical symplectic structure. The following interpretation is present though implicit in Marolf’s papers: given a Lagrangian density $L[\phi] + \phi \cdot J$, where $\phi$ denotes the dynamical fields and $J$ the corresponding external sources, the solution space corresponding to a fixed external source profile $J$ is one leaf of the Poisson structure on the field configuration space constructed from the source-less Lagrangian density $L[\phi]$. Unfortunately, most of Marolf’s discussion is non-covariant, as for simplicity it introduces a fixed time coordinate.

More recently, the Peierls formula became an important ingredient in the construction of perturbative Algebraic Quantum Field Theory (pAQFT). Its use came to prominence in the 2000s with Refs. 31, 32. In these papers, cues were taken partially from the perturbative QFT tradition of Refs. 22, 27, with retarded, advanced and time-ordered products adapted to an off-shell setting. At the same time, they realized that very similar formulas come about at the classical level from the use of the off-shell Peierls formula (as defined by Marolf) in perturbative classical field theory. It was there that the use of the off-shell Peierls formula was formulated in a systematic and covariant way. The first direct demonstration of the Jacobi identity for the Peierls formula was given in Ref. 32. More recently, this formulation of the off-shell Peierls formula has been extended to classical fermi fields\cite{33} which paved the way for its inclusion in a BV-BRST treatment of gauge theories in pAQFT\cite{12,13}.

Another recent development is the incorporation of the off-shell Peierls formula
in a serious functional analytical effort to describe the infinite dimensional spaces of field configurations and algebras of observables using infinite dimensional differential geometry.\textsuperscript{12–14}

Finally, we should mention another recent paper\textsuperscript{34} that has a non-trivial overlap with this review. Its goals also include clarifying the specific conditions needed to be satisfied by a linear gauge theory that guarantee that the Peierls construction works. On the other hand, their geometric set up is somewhat less general than ours and they do not consider the relation with the covariant phase space formalism in detail. Though, unlike here, they also treat fermi fields and further consider quantization.

1.1.2. Covariant phase space formalism

Much of the impetus for the development of a covariant Poisson bracket came from covariant quantization, or more precisely the need for a solid classical analog of the unequal time commutation relations in QFT. On the other hand, the development of the covariant phase space formalism was spurred by the desire to understand conservation laws in field theory, as well as trying to improve upon the canonical quantization program of Dirac and Bergmann.

In the physics literature, the roots of this formalism, though in a rather obscure form, can be found in the 1953 paper of Bergmann & Schiller.\textsuperscript{35} This paper was part of Bergmann’s program to study the implications of general covariance in General Relativity (or any other second order covariant theory) for the structure of its conservation laws, its stress energy tensor, and its canonical quantization. There already appear formulas for what we call the presymplectic current density and its potential.

This possibility of obtaining the symplectic structure of a theory directly from the Lagrangian by the methods of Bergmann & Schiller remained somewhat unknown, except possibly to a small group of experts. For example, in 1962, Komar\textsuperscript{36} (a student of Bergmann) used these methods to define the symplectic structure on the space of initial data on a null surface. Few other papers using this formalism appeared until its apparently independent resurgence in the ’80s.

The 1975 article by Ashtekar & Magnon\textsuperscript{37} used the integrated symplectic potential current density as a symplectic form for the Klein-Gordon field on curved spacetime, citing the work of Segal\textsuperscript{25} as a similar previous treatment in Minkowski space, which goes back to the aforementioned Ref. 23. However, it appears that the formalism of Ashtekar & Magnon was privately inspired\textsuperscript{38} by the ideas of Souriau, which they generalized to field theories. Later Ashtekar also applied the same formalism to general relativity.\textsuperscript{39} Starting around this time, perhaps again due to the influence of the ideas of Souriau, the identification of the phase space with the space of solutions rather than with that of initial data starts to become more prevalent.

Another independent appearance of the (pre)symplectic form as an integral of the covariant (pre)symplectic current density is found in the 1978 article of Friedman\textsuperscript{40} and a later article of Friedman & Schutz.\textsuperscript{41} The examples considered there
consisted of a scalar field and of the combined system of linearized gravitational and hydrodynamic modes describing a relativistic star. The latter presymplectic form is degenerate due to general covariance (gravitational gauge symmetry) and, in fact, its kernel was used in the analysis of linear stability to discard possibly unstable unphysical modes. Friedman did not cite any preceding sources, but apparently was inspired by some lectures on the covariant treatment of conservation laws in variational theories (obviously connected to symplectic structure by Noether’s theorem) that were delivered by Trautman at Chicago in 1971. The relevant content from these lectures later appeared as Ref. 43. Trautman’s main influence seems to have been the gradual refinement of the original ideas of Noether in connecting symmetries (including gauge symmetries) with conservation laws in the calculus of variations with multiple independent variables.\textsuperscript{45–47} In the latter formalism, the geometric structure that is closely related (but not identical to) the covariant presymplectic (potential) current density is the Poincaré-Cartan form.

Another independent instance of the covariant symplectic formalism, though only for mechanical systems and not field theories, appeared in the 1982 article of Henneaux\textsuperscript{48} on the inverse problem of the calculus of variations. No sources were cited in the article, but apparently the main inspiration were the ideas of Souriau.\textsuperscript{19}

The breakthrough point for a more widespread appreciation of the covariant phase space formalism was the 1987 paper of Crnković & Witten,\textsuperscript{4} which gave explicit covariant constructions for the symplectic forms of scalar fields, Yang-Mills theory and General Relativity. This paper was in fact an expansion of one section, where this formalism was laid out in some generality, of an earlier paper of Witten\textsuperscript{50} on open string field theory. Witten did not cite any preceding sources and it is not clear what his main influences were.

Almost simultaneously, very similar ideas were laid out by Zuckerman.\textsuperscript{5} It is not completely clear what was Zuckerman’s main influence. It is likely to have been similar to Trautman’s, as the main preceding reference that he cites is the original article by Noether.\textsuperscript{44} He does, however cite\textsuperscript{50,51} independent contemporary presentations of very similar ideas, including a private letter of Deligne. Deligne’s presentation of these ideas was later recorded in the Ref. 52, where only Zuckerman is cited explicitly.

Soon thereafter, several reviews appear presenting the formalism in its general and essentially modern form. Crnković\textsuperscript{53} follows Ref. 4, Lee & Wald follow Refs. 4, 40 and Ashtekar, Bombelli & Reula\textsuperscript{54} follow Refs. 19, 37, 39. The Introduction in Ref. 54 contains further contemporary references. The paper by Lee & Wald has to be singled out for its exceptional clarity of presentation. The generality of the presentation was lacking only the treatment of classical fermi fields, which appeared even in a cursory way only in Ref. 52. That was rectified only much later in a paper of Hollands & Marolf.\textsuperscript{55}

In the mean time, some of the formal manipulations involved in deriving the covariant presymplectic current density were formalized in the framework of the
variational bicomplex, as can be seen for instance in Sec. 8.3 of Ref. 56. It is essentially this presentation that we review later in Sec. 3.1.

1.1.3. **Equivalence**

While symplectic and Poisson structures are obviously related, as can be seen from the above references, the two covariant formalisms that we have described naturally appear in somewhat different problems. Thus, it is not surprising that most researchers would prefer to use just one or the other, without attempting to relate the two. Some have actually done so, though, until recently only using the fact that they are both equivalent (under appropriate assumptions of course) to the corresponding canonical constructions.

Recall that in his original paper Peierls\(^6\) showed, by explicit calculation, the equivalence of the Peierls formula with the canonical Poisson bracket for non-singular field theories. DeWitt\(^26\) extended that to the case of gauge fixed singular theories, provided the canonical formalism was applied after gauge fixing.

Even before that, having taken note of Peierls’ paper, Bergmann’s group showed\(^57\) that both the Bergmann-Schiller and Peierls formulations of Poisson brackets are essentially equivalent for non-singular field theories (those without gauge invariance). They also compare the Peierls bracket with the Dirac bracket in singular theories. Unfortunately, by modern standards their discussion is rather obscure.

Much later, Barnich, Henneaux & Schomblond\(^8\) showed that the Peierls bracket coincides with the Dirac bracket in the canonical formalism even if both first and second class constraints are present (hence including gauge theories). Their presentation is very clear. It is also very insightful in the way that they included the covariant phase space formalism. They showed that the canonical formalism is in fact a special case of the covariant one, provided one uses Hamilton’s least action principle as the variational functional. Further, they noted that the canonical variables can be introduced by adjoining some auxiliary fields\(^9\) (the canonical momenta) to the Lagrangian formulation, while showing that the symplectic structure is invariant under the adjunction or elimination of auxiliary fields.

Finally, rather recently, a breakthrough appeared in a paper of Forger & Romero.\(^10\) They managed to prove the equivalence of the covariant constructions of the symplectic and Poisson structures in an elegant and fully covariant way, thus without using the canonical formalism as an intermediary. Their proof was restricted to the case of a non-singular scalar field theory. It is their argument whose generalization we present in expanded detail in the bulk of this review, Sec. 3. It should be mentioned that Forger & Romero also very clearly compared the geometric structure of the covariant phase space formalism to the related but non-identical geometric structure of the **multisymplectic** formalism.
2. Linear PDE theory

In this section, we describe a number of important technical results about linear hyperbolic equations that will be crucial for the later discussion of the Peierls bracket in Sec. 3. Here we are mostly concerned with the linear algebra of the inhomogeneous system of partial differential equation (PDE system)

\[ f[\phi] = \tilde{\alpha}^*, \quad (1) \]
on an \( n \)-dimensional spacetime manifold \( M \), where \( f : \Gamma(F) \to \Gamma(\tilde{F}^*) \) is a linear partial differential operator acting on smooth sections \( \phi \in \Gamma(F) \) of a field (vector) bundle \( F \to M \) and taking values in the densitized dual bundle \( \tilde{F}^* \to M \), where \( \tilde{F}^* \cong F^* \otimes_M \Lambda^n M \) is the linear dual bundle \( F^* \) tensored with the bundle of volume forms on \( M \). Typically, \( \tilde{\alpha}^* \in \Gamma(F) \) is a compactly supported dual density. We choose to always have the PDE valued in dual densities out of convenience, as will be evidenced later in Sec. 2.5. Keeping with the terminology of Appendix B, we use such an \((f, \tilde{F}^*)\) as our preferred equation form for linear PDE systems.

We will consider the case where \( f \) is hyperbolic and hence possesses Green functions (Sec. 2.1). An important idea that is often necessary to relate physical equations of motion to hyperbolic equations is that of compatible constraints (Sec. 2.2). Solution spaces can be conveniently parametrized using special, causal Green functions (Sec. 2.3 and 2.4). Both the differential operator \( f \) and its Green functions have adjoints, which are important in the definitions of various natural bilinear pairings (Sec. 2.5).

Sometimes we will refer to basic background information on jet bundles, the interpretation of differential operators as maps between jet bundles and the interpretation of PDEs as submanifolds of jet bundles. The relevant information is summarized in Appendix A and Appendix B. Also, hyperbolic PDEs naturally define a generalized kind of causal structure on \( M \). The necessary ideas and definitions are summarized in Appendix C, with the notation similar to the standard one used in Lorentzian geometry. This causal structure can be used to restrict the supports of field and dual density sections.

**Definition 2.1.** Consider a vector bundle \( V \to M \). We define the following subspaces of the space of sections \( \Gamma(V) \):

\[ \Gamma_0(V) = \{ \phi \in \Gamma(V) \mid \text{supp } \phi \text{ is compact} \}, \]
\[ \Gamma_+(V) = \{ \phi \in \Gamma(V) \mid \text{supp } \phi \text{ is retarded} \}, \]
\[ \Gamma_-(V) = \{ \phi \in \Gamma(V) \mid \text{supp } \phi \text{ is advanced} \}, \]
\[ \Gamma_{SC}(V) = \{ \phi \in \Gamma(V) \mid \text{supp } \phi \text{ is spacelike compact} \}, \]
where retarded support, advanced support, or spacelike compact support means, respectively, that \( \text{supp } \phi \subset \overline{I^+(K)} \), \( \text{supp } \phi \subset \overline{I^-(K)} \), or \( \text{supp } \phi \subset \overline{I(K)} \) for some compact \( K \subset M \). The corresponding subspaces of the solution space \( \mathcal{S}(F) \) of \( f[\phi] = 0 \)
are denoted by
\[ S_{0,\pm SC}(F) = \mathcal{S}(F) \cap \Gamma_{0,\pm SC}(F). \] (6)

2.1. Green hyperbolicity

Below, we define the notion of a Green hyperbolic PDE system as one that possesses unique advanced and retarded Green functions. We will rely heavily on the existence and properties of these Green functions in later sections. The class of Green hyperbolic systems is quite large, including for instance wave-like equations on globally hyperbolic Lorentzian spacetimes\(^58\) as well as symmetric (or even symmetrizable) hyperbolic systems (cf. Refs. 59 and [60, Sec.4]) that satisfy a similar global causal condition. Note that the second class of examples does not require a background Lorentzian metric to be defined.

Before proceeding, we need the notion of a causal structure (a priori independent of any Lorentzian metric), with respect to which the notions of advanced and retarded support will be defined. In the literature on relativity, the two are usually introduced together. However, a deeper investigation of hyperbolic PDE systems shows that the notion of causality can be defined independently and intrinsically from a given PDE. It so happens that, for equations with a d’Alambert-like principal symbol, the causal relations deduced directly from the PDE, on the one hand, and from the background Lorentzian metric, on the other, actually coincide. The basic relevant notions and definitions are summarized in Appendix C. See Ref. [60, Secs.3,4] for a more in depth discussion.

Definition 2.2. The PDE system \( f[\phi] = 0 \) is said to be Green hyperbolic if there exists a globally hyperbolic conal structure on \( M \) such that (a) the inhomogeneous equation \( f[\phi] = \hat{\alpha}^* \in \Gamma_{\pm}(\tilde{F}^*) \) and (b) a solution \( \phi_{\pm} \in \Gamma_{\pm}(F) \) exists, is unique and satisfies the support condition supp \( \phi_{\pm} \subseteq \overline{T^+(\text{supp} \hat{\alpha}^*)} \). We denote the unique two-sided inverses by \( G_{\pm} : \Gamma_{\pm}(\tilde{F}^*) \rightarrow \Gamma_{\pm}(F) \) and refer to them as the retarded (+) and advanced (−) Green functions. In adapted local coordinates \( (x^i, u^a) \) on \( F \) and \( (x^i, u_b) \) on \( \tilde{F}^* \), where \( u^a(\phi(x)) = \phi^a(x), u_b(\hat{\alpha}^*(x)) = \alpha_b(x) \) and \( d\tilde{x} = dx^1 \wedge \cdots \wedge dx^n \), the Green functions can be represented as integral kernels
\[ \phi_{\pm}^a(x) = (G_{\pm}[\hat{\alpha}^*])^a(x) = \int_M G_{\pm}^{ab}(x;y)\alpha_b(y)\,dy. \] (7)

It is sufficient that \( G_{\pm} \) be defined on \( \Gamma_0(\tilde{F}^*) \). It can then be extended to \( \Gamma_{\pm}(\tilde{F}^*) \) by an exhaustion argument (Cor.5 in Ref. 61). Ideally, the Green functions should be well-defined distributions (be continuous in the appropriate function space topology), but we will not discuss this issue here and instead concentrate on their algebraic and geometric properties.

Remark 2.1. In this paper, we rely heavily on the notion of Green hyperbolicity. On the other hand, many PDE references only treat well-posedness of the Cauchy
problem of homogeneous equations and do not address the inhomogeneous problem. For instance, Cauchy well-posedness of linear symmetric hyperbolic systems is established in [62, Ch.7], but the existence of Green functions is not addressed. Fortunately, there is an argument in the classic PDE literature, known as Duhamel’s principle, that essentially establishes the equivalence between hyperbolic systems with a well-posed Cauchy problem and Green hyperbolic systems. Usually, this argument is discussed only for specific examples, but in principle it works quite generally. Thus, we can appeal to this very large class of PDE systems when considering examples, rather than restricting ourselves only to wave-like equations on Lorentzian manifolds for which Green hyperbolicity is well established.58,59,63

2.2. Compatible constraints

Consider a linear PDE system on the field bundle $F \to M$ that consists of

$$f[\phi] = 0, \quad c[\phi] = 0,$$

where $f: \Gamma(F) \to \Gamma(\tilde{F}^*)$ is a hyperbolic partial differential operator and $c: \Gamma(F) \to \Gamma(E)$ is another partial differential operator valued in a vector bundle $E \to M$. We refer to $f[\phi] = 0$ as the hyperbolic subsystem and to $c[\phi] = 0$ as the constraints subsystem, with $E \to M$ the constraints bundle. The equation form of the total system is $(f \oplus c, \tilde{F}^* \oplus E)$.

The constraints are said to be hyperbolically integrable if there exists a pair of linear differential operators $h: \Gamma(E) \to \Gamma(\tilde{E}^*)$ and $q: \Gamma(\tilde{F}^*) \to \Gamma(\tilde{E}^*)$ that satisfy the identity $h \circ c = q \circ f$, or

$$h[c[\phi]] = q[f[\phi]],$$

for any $\phi \in \Gamma(F)$, where the operator $h$ itself is hyperbolic. The existence of this identity implies that the vanishing of the constraints $c[\phi] = 0$ at some initial time implies that $c[\phi] = 0$ everywhere on $M$, provided $f[\phi] = 0$. We call the PDE $h[\psi] = 0$ on $E \to M$ the consistency subsystem and the joint system $f[\phi] = 0, h[\psi] = 0$ on $F \oplus_M E \to M$ the compound system. We write the corresponding equation form as $(f \oplus h, \tilde{F}^* \oplus_M \tilde{E}^*)$.

If the above conditions are satisfied, the PDE system $f[\phi] = 0, c[\phi] = 0$ is said to be hyperbolic with constraints. Whenever we refer to a causal structure induced by a hyperbolic system with constraints, we actually mean the one induced by the corresponding hyperbolic compound system.

2.3. Causal Green function (without constraints)

Now that we are sure to have access to the retarded/advanced green functions $G_{\pm}$ for the Green hyperbolic system $f[\phi] = 0$, we can define the so-called causal Green function

$$G = G_+ - G_-.$$
This new Green function helps to conveniently parametrize the space of solutions $S_{SC}(F) \cong \ker f \subset \Gamma_{SC}(F)$ by featuring in the following

**Proposition 2.1.** The sequence

$$
0 \longrightarrow \Gamma_0(F) \xrightarrow{f} \Gamma_0(\tilde{F}^*) \xrightarrow{G} \Gamma_{SC}(F) \xrightarrow{f} \Gamma_{SC}(\tilde{F}^*) \longrightarrow 0,
$$

(11)

is exact (in the sense of linear algebra).

That is, the image of each map coincides with the kernel of the next map. The proof for wave-like equations, which is given in [58, Thm.3.4.7] and [64, Lem.3.2.1] and unfortunately excludes the final surjection, directly carries through to the Green hyperbolic case. The final surjection is covered by the proof of Cor.5 in Ref. 61. A complete proof actually follows from the identities given in Lem. 2.1 below and the fact that $f$ is invertible on $\Gamma_{\pm}(F)$, and hence a fortiori injective on $\Gamma_0(F)$.

We can interpret the above proposition in the following way. Since $S_{SC}(F) \cong \text{im } G$, we can express any solution to the homogeneous problem as $\phi = G[\tilde{\alpha}^*]$, where $\alpha \in \Gamma_0(\tilde{F}^*)$ is some smooth dual density of compact support. Equivalently, by exactness, $S_{SC}(F) \cong \text{coker } f = \Gamma_0(\tilde{F}^*)/\text{im } f$. Also, since $\Gamma_{SC}(F) \cong \text{im } f$, for any dual density $\tilde{\alpha}^*$ with spacelike compact support, there exists a solution $\phi$ with spacelike compact support of the inhomogeneous problem $f[\phi] = \tilde{\alpha}^*$.

**Definition 2.3.** Consider one Cauchy surface $\Sigma \subset M$ and two more Cauchy surfaces $\Sigma^\pm \subset M$ to the past and future of $\Sigma$, where $\Sigma^\pm \subset I^\pm(\Sigma)$, and let $S^\pm = I^\pm(\Sigma^\pm)$. Let $\{\chi_+, \chi_-, \}$ be a partition of unity adapted to the open cover $\{S^+, S^-\}$ of $M$, that is, $\chi_+ + \chi_- = 1$ and $\text{supp } \chi^\pm \subset S^\pm$. We call $\{\chi_+, \chi_- \}$ a partition of unity adapted to the Cauchy surface $\Sigma$.

**Lemma 2.1.** The exact sequence of Prp. 2.1 splits at

$$
\Gamma_0(\tilde{F}^*) \cong \Gamma_0(F) \oplus S_{SC}(F) \quad \text{and} \quad \Gamma_{SC}(F) \cong S_{SC}(F) \oplus \Gamma_{SC}(\tilde{F}^*). 
$$

(12)

Given a partition of unity $\{\chi_+, \chi_- \}$ adapted to a Cauchy surface $\Sigma$, there exist (noncanonical) splitting maps

$$
f_\chi : \text{im } G \rightarrow \Gamma_0(\tilde{F}^*), \quad f_\chi[\phi] = \pm f_\chi^+[\phi] = \pm f[\chi_+ \phi],
$$

(13)

$$
G_\chi : \Gamma_{SC}(\tilde{F}^*) \rightarrow \Gamma_{SC}(F), \quad G_\chi[\tilde{\alpha}^*] = G_+[\chi_+ \tilde{\alpha}^*] + G_-[\chi_- \tilde{\alpha}^*].
$$

(14)

**Proof.** Note that these splitting maps are not canonical, as they depend on the choice of a Cauchy surface and a partition of unity adapted to it.

When $\phi \in S_{SC}(F) \cong \text{im } G$, it is clear that $f_\chi^+[\phi] = f[\chi_+ \phi]$ does in fact have compact support, as supp $\phi$ is spacelike compact while $\text{supp } d\chi_+ \subset S^+ \cap S^-$ is timelike compact,$^a$ and $f[\chi_+ \phi] \neq 0$ only on supp $\phi$ \cap $\text{supp } d\chi_+$, which is by definition compact. Also, since $d(\chi_+ + \chi_-) = 0$, we have $f_\chi^+[\phi] + f_\chi^-[\phi] = 0$,

---

$^a$A set is **timelike compact** if its intersection with every spacelike compact set is compact.$^{55}$
which means that the map \( f_\chi = \pm f_\chi^\pm \) is well defined. On the other hand, we have \( G_\pm [f(\chi \pm \phi)] = \chi \pm \phi \) from the uniqueness of solutions to the inhomogeneous problem with retarded/advanced support. The definition of the causal Green function then immediately implies that \( G \circ f_\chi = \pm \text{id} \) on \( \Gamma_{SC}( \tilde{F}^* ) \). Also, a direct calculation shows that \( f \circ G_\chi = \text{id} \) on \( \Gamma_{SC}( \tilde{F}^* ) \):

\[
\begin{align*}
\Phi_+ \phi^* + \chi_- \Phi_-^* = \phi^*.
\end{align*}
\]

(15)

This concludes the proof.

\[ \square \]

2.4. Causal Green function (with constraints)

We will not discuss the most general kind of constraints and restrict our attention only to parametrizable ones. By the term parametrizable, we mean that there exist an additional vector bundle \( E' \to M \) and additional differential operators \( h', c' \) and \( q' \), which fit into the following commutative diagram

\[
\begin{array}{ccc}
\Gamma(E') & \xrightarrow{c'} & \Gamma(F) & \xrightarrow{c} & \Gamma(E) \\
\downarrow{h'} & & \downarrow{f} & & \downarrow{h} \\
\Gamma(E'^*) & \xrightarrow{q'} & \Gamma(F^*) & \xrightarrow{q} & \Gamma(E^*)
\end{array}
\]

(16)

such that \( h' \) is hyperbolic, and that the horizontal rows form complexes of differential operators \( (c \circ c' = 0 \text{ and } q \circ q' = 0) \) that are formally exact (Appendix B.2). Since both \( h \) and \( h' \) are hyperbolic, we can define their retarded/advanced Green functions, \( H_+ \) and \( H'_+ \), as well as their causal Green functions, \( H = H_+ - H_- \) and \( H' = H'_+ - H'_- \). All these operators then fit into the following commutative diagram:

\[
\begin{array}{ccc}
0 & \xrightarrow{c'} & \Gamma_0(E') & \xrightarrow{c'} & \Gamma_0(E'^*) & \xrightarrow{c'} & \Gamma_0(\tilde{E}^*) & \xrightarrow{q'} & 0 \\
\downarrow{h'} & & \downarrow{H'} & & \downarrow{H_+} & & \downarrow{H} & & \downarrow{q'} \\
0 & \xrightarrow{c} & \Gamma_0(F) & \xrightarrow{c} & \Gamma_0(F^*) & \xrightarrow{c} & \Gamma_0(\tilde{F}^*) & \xrightarrow{q} & 0 \\
\downarrow{h} & & \downarrow{H} & & \downarrow{H_+} & & \downarrow{H} & & \downarrow{q} \\
0 & \xrightarrow{c} & \Gamma_0(E) & \xrightarrow{c} & \Gamma_0(E^*) & \xrightarrow{c} & \Gamma_0(\tilde{E}^*) & \xrightarrow{q} & 0
\end{array}
\]

(17)

Note that the causally restricted supports in the above diagram should be defined with respect to a causal structure that is defined by the total compound system with equation form \( (h' \oplus f \oplus h, \tilde{E}^* \oplus_M \tilde{F}^* \oplus_M \tilde{E}^*) \).

Lemma 2.2. The retarded/advanced inhomogeneous problem

\[
\begin{align*}
\Phi(\phi) &= \beta^*, \\
c(\phi) &= \gamma,
\end{align*}
\]

(18)

(19)

with \( \beta^* \in \Gamma_0(\tilde{F}^*) \) and \( \gamma \in \Gamma_{SC}(E) \), is solvable for \( \phi \in \Gamma_{SC}(F) \) iff \( h[\gamma] = q[\beta^*] \).
Proof. In one direction, if $\phi$ is the desired solution, then $h[\gamma] = h[c[\phi]] = q[f[\phi]] = q[\tilde{\beta}^*]$. In the other direction, let $\phi = G_+[\tilde{\beta}^*]$. We obviously have $f[\phi] = \tilde{\beta}^*$. It remains to check

$$c[\phi] = c[G_+[\tilde{\beta}^*]] = H_+[q[\tilde{\beta}^*]] = H_+[h[\gamma]] = \gamma.$$ (20)

This concludes the proof.

It is convenient to state here a lemma concerning formally exact complexes of differential operators, which shall be referred to in later sections.

Lemma 2.3. Suppose that linear differential operators $c': \Gamma(E') \rightarrow \Gamma(F)$ and $c: \Gamma(F) \rightarrow \Gamma(E)$ form a formally exact complex. Then any linear differential operator $l: \Gamma(F) \rightarrow \Gamma(L)$ such that $l \circ c' = 0$ factors as $l = l_c \circ c$, for some linear differential operator $l_c: \Gamma(F) \rightarrow \Gamma(E')$. Similarly, any linear differential operator $r: \Gamma(R) \rightarrow \Gamma(F)$ such that $c \circ r = 0$ factors as $r = c' \circ r_c$, for some linear differential operator $r_c: \Gamma(R) \rightarrow \Gamma(E')$.

Proof. We represent all differential operators as maps from appropriate jet bundles (see Sec. Appendix B). The fact that the differential operators $c': J^\infty E' \rightarrow J^\infty F$ and $c: J^\infty F \rightarrow E$ form a formally exact complex shows that the prolongations $p^\infty c': J^\infty E' \rightarrow J^\infty F$ and $p^\infty c: J^\infty F \rightarrow J^\infty E$ compose into an exact sequence of vector bundle maps. Hence, the image of $p^\infty c'$ coincides with the kernel of $p^\infty c$. By hypothesis, the kernel of the linear bundle map $l: J^\infty F \rightarrow L$ contains $\text{im} p^\infty c'$, while the image of the linear bundle map $p^\infty r: J^\infty R \rightarrow J^\infty F$ is contained in the image of $p^\infty c'$. Therefore, desired factorization formulas follow straightforwardly from linear algebra.

Such arguments are common in the formal theory of PDEs and can even be generalized to the nonlinear setting.\textsuperscript{66–68}

2.5. Pairings and adjoints

We conclude this section by remarking the identities

$$(G_\pm)^* = G_\mp^*,$$ (21)

where on the left hand side $(G_\pm)^*$ denotes the adjoint of the retarded/advanced Green function $G_\pm$ of the equation $f[\phi] = 0$, and on the right hand side $G_\mp^*$ denotes the advanced/retarded Green function of the adjoint equation $f^*[\phi] = 0$. Note that taking the adjoint flips the support between retarded and advanced.

Definition 2.4. Given two differential operators $f, f^*: \Gamma(F) \rightarrow \Gamma(\tilde{F}^*)$ are said to be mutually \textit{adjoint} if there exists a bilinear differential operator $G: \Gamma(F) \times \Gamma(F) \rightarrow \Omega^{n-1}(M)$ such that

$$f[\phi] \cdot \psi - \phi \cdot f^*[\psi] = dG[\phi, \psi]$$ (22)
Lemma 2.4. Following the notation of Def. 2.5, the Green pairing \( \langle -, - \rangle_G \) between a solution \( \phi \in \Gamma_{SC}(F) \) of \( f[\phi] = 0 \) and a solution \( \psi \in \Gamma_{SC}(F) \) of \( f^*[\psi] = 0 \) is given by

\[
\langle \phi, \psi \rangle_G = \int_{\Sigma} \iota^* G[\phi, \psi]
\]

The pairing is only partially defined. That is, there exist arguments for which the integral does not converge. For simplicity, we will consider it only for those pairs of sections for which the integrand \( \phi \cdot \tilde{\alpha}^* \) has compact support. This pairing is non-degenerate in either argument, as follows from the standard argument of the fundamental lemma of the calculus of variations \([70, \S IV.3.1]\). It will play an important role in Secs. 3.2.4, 3.2.5. It is easy to show that the formal adjoint \( f^* \) coincides with the adjoint of \( f \) with respect to this natural pairing: \( \langle f[\phi], \psi \rangle = \langle \phi, f^*[\psi] \rangle \). This natural pairing allows us to define adjoints for integral operators like Green functions, namely \( \langle G_{\pm} [\tilde{\alpha}^*], \tilde{\beta}^* \rangle = \langle \tilde{\alpha}^*, (G_{\pm})^*[\tilde{\beta}^*] \rangle \).

It is straightforward that that the natural pairing \( \langle -, - \rangle \) is non-degenerate on the spaces \( \Gamma_{\pm}(F) \times \Gamma_{\mp}(F^*) \). And since, the identities \( f \circ G_{\pm} = G_{\pm} \circ f = \text{id} \) hold on \( \Gamma_{\pm}(F) \), it is now easy to verify the adjoint identities (21) since

\[
\begin{align*}
\int_M \phi_{\mp} \cdot f \circ G_{\pm} [\tilde{\alpha}_+^*] &= \langle \phi_{\mp}, \tilde{\alpha}_+^* \rangle = \int_M (G_{\pm})^* \circ f^*[\phi_{\mp}] \cdot \tilde{\alpha}_+^* \tag{24} \\
\int_M G_{\mp} \circ f [\phi_{\pm}] \cdot \tilde{\alpha}_-^* &= \langle \phi_{\pm}, \tilde{\alpha}_-^* \rangle = \int_M \phi_{\mp} \cdot f^* \circ (G_{\mp})^*[\tilde{\alpha}_-^*], \tag{25}
\end{align*}
\]

for any \( \phi_{\pm} \in \Gamma_{\pm}(F) \) and \( \tilde{\alpha}_{\pm}^* \in \Gamma_{\pm}(F) \). The causal Green functions then satisfy \( (G)^* = -G^* \), where \( G^* \) is the causal Green function for \( f^* \).

Definition 2.5. Let \( \iota : \Sigma \subset M \) be a future oriented, Cauchy surface. The Green pairing \( \langle -, - \rangle_G \) between a solution \( \phi \in \Gamma_{SC}(F) \) of \( f[\phi] = 0 \) and a solution \( \psi \in \Gamma_{SC}(F) \) of \( f^*[\psi] = 0 \) is given by

\[
\langle \phi, \psi \rangle_G = \int_{\Sigma} \iota^* G[\phi, \psi]
\]

Lemma 2.4. Following the notation of Def. 2.4, the Green pairing \( \langle -, - \rangle_G \) depends only on the equivalence class \( [G] \) of \( G \) and is independent of \( \Sigma \).

Proof. Any two representatives \( G_1 \) and \( G_2 \) of \([G]\) will differ by an exact term \( dH \), with \( H[-,-] \) a bilinear bidifferential operator. Therefore, the integrands \( \iota^* G_1[\varphi_1, \varphi_2] \) will differ by the exact term \( d\varphi^* H[\varphi_1, \varphi_2] \), with necessarily compact support. Therefore, since \( \Sigma \) has no boundary, we can use any representative of \([G]\) to evaluate the pairing.
Now, let \( \iota ': \Sigma' \subset M \) be another Cauchy surface and let \( S \subseteq M \) be such that \( \partial S = \Sigma' - \Sigma \). Then an application of Stokes’ theorem shows the following:

\[
\int_{\Sigma'} \iota'^* G[\phi, \psi] - \int_{\Sigma} \iota^* G[\phi, \psi] = \int_S (f[\phi] \cdot \psi - \phi \cdot f^*[\psi]) = 0,
\]

where we used the solution properties \( f[\phi] = 0 \) and \( f^*[\psi] = 0 \). This shows the independence of the Green pairing from the choice of a Cauchy surface.

**Lemma 2.5.** The Green pairing has the following alternative forms:

\[
\langle \psi, \xi \rangle_G = \int_M \hat{\alpha}^* \cdot \xi = \int_M \hat{\alpha}^* \cdot G^*[\hat{\beta}^*] = - \int_M G[\hat{\alpha}^*] \cdot \hat{\beta}^* = - \int_M \psi \cdot \hat{\beta}^*,
\]

where \( \phi = G[\hat{\alpha}^*] \) and \( \xi = G^*[\hat{\beta}^*] \), \( \hat{\alpha}^*, \hat{\beta}^* \in \Gamma_0(\tilde{F}^*) \). Moreover, it is non-degenerate.

Note that the following proof was partly inspired by Sec. 3.3 of Ref. 10 and Lem. 3.2.1 of Ref. 64. It has the same skeletal structure as one of the main technical lemmas (Lem. 3.12) of Sec. 3.

**Proof.** Consider a future oriented Cauchy surface \( \iota': \Sigma \subset M \) (see Appendix C) and a partition of unity \( \{ \chi_{\pm} \} \) adapted to it. Then, recalling the notation for the splitting maps in Lem. 2.1, direct calculation gives

\[
\langle \psi, \xi \rangle_G = \int_{\Sigma} \iota^* G[\psi, \xi] = \sum_{\pm} \int_{\Sigma} \iota^* G[\psi, \chi_{\pm} \xi]
\]

\[
= \sum_{\pm} \pm \int_{I^+(\Sigma)} dG[\psi, \chi_{\pm} \xi]
\]

\[
= \sum_{\pm} \pm \int_{I^+(\Sigma)} (f[\psi] \cdot (\chi_{\pm} \xi) - \psi \cdot f^*[\chi_{\pm} \xi])
\]

\[
= - \int_{I^-(\Sigma)} \psi \cdot f^*_X[\xi] - \int_{I^+(\Sigma)} \psi \cdot f^*_X[\xi] = - \int_M \psi \cdot f^*_X[\xi]
\]

\[
= - \int_M G[\hat{\alpha}^*] \cdot f^*_X[\xi] = \int_M \hat{\alpha}^* \cdot (G^* \circ f^*_X)[\xi]
\]

\[
= \int_M \hat{\alpha}^* \cdot \xi = \int_M \hat{\alpha}^* \cdot G^*[\hat{\beta}^*]
\]

\[
= - \int_M G[\hat{\alpha}^*] \cdot \hat{\beta}^* = - \int_M \psi \cdot \hat{\beta}^*.
\]

Appealing to the exact sequence of Prp. 2.1, the dual densities \( \hat{\alpha}^* \) and \( \hat{\beta}^* \) are only defined up to an element of \( \text{im} f \) and \( \text{im} f^* \) respectively. However, the formulas show that this freedom does not affect the result.

Now, based on the formula \( \langle \psi, \xi \rangle_G = - \int_M \psi \cdot \hat{\beta}^* \), the fact that \( \hat{\beta}^* \) could be arbitrary and the non-degeneracy of the natural pairing \( \langle - , - \rangle \) we can see that \( \langle - , - \rangle_G \) must be non-degenerate in its second argument. The same reasoning establishes non-degeneracy in the first argument as well. \( \square \)
3. Covariant phase space formalism, Peierls formula

This section constitutes the main body of this review. At this point it is helpful to at least skim the contents of Appendix A and Appendix B, as they summarize relevant concepts and notation. Its culmination is a precise set of conditions (Secs. 3.2.2, 3.2.3 and 3.3.1) that are sufficient to establish the validity of the covariant phase space formalism and the Peierls formula constructions, respectively, of the symplectic (Sec. 3.3.3) and Poisson (Sec. 3.3.4) structures on the phase space of a classical field theory, and their equivalence via a generalized Forger-Romero\textsuperscript{10} argument (Sec. 3.3).

Below, we study their argument in depth and generalize it to include field theories with constraints and gauge invariance. Sec. 3.1 defines variational PDE systems and shows how the covariant symplectic formalism arises from their geometry. Sec. 3.2 discusses the differential geometry of the space of solutions of the PDE system. Since the focus of this work is more geometrical than analytical, we avoid a detailed discussion of the subtleties of infinite dimensional manifolds. Instead, we define so-called formal tangent and cotangent spaces to the solution space and then fix a particular background solution, so that we need only consider a single formal tangent and cotangent fiber at that point of the phase space. This is sufficient for defining the formal symplectic form and the Poisson bivector and proving their equivalence. An in-depth discussion of the functional analytical details that go into defining the necessary infinite dimensional differential geometry can be found in Refs. 11–14.

3.1. Variational systems

Consider a field vector bundle $F \to M$ over an $n$-dimensional manifold $M$. A local action functional of order $k$ on $F \to M$ is a function $S[\phi]$ of sections $\phi: M \to F$, $S[\phi] = \int_M (j^k \phi)^* \mathcal{L}$, \hspace{1cm} (36)

where $\mathcal{L}$, the Lagrangian density, is a section of the bundle $(\Lambda^n M)^k \to J^k F$ densities, which could depend on jet coordinates of order up to $k$ (see Appendix A). The Lagrangian density is called local because, given a section $\phi$ and local coordinates $(x^i, u_I^a)$ on $J^k F$, the pullback at $x \in M$ can be written as $(j^k \phi)^* \mathcal{L}(x) = \mathcal{L}(x^i, \partial I^a(\phi(x)))$, \hspace{1cm} (37)

which depends only on $x$ and on the derivatives of $\phi$ at $x$ up to order $k$. For the most part, the integral over $M$ can be considered formal, since all the necessary properties will be derived from $\mathcal{L}$. On the other hand, the finiteness of $S[\phi]$ or related quantities may be important while discussing boundary conditions in spacetimes with non-compact spatial extent. However, we will not discuss these issues below.

Recall that Appendix A introduces the variational bicomplex $\Omega^{h,v}(F)$ of vertically and horizontally graded differential forms on $J^{\infty} F$. Below, we use the notation introduced there. A Lagrangian density is then an element $\mathcal{L} \in \Omega^{n,0}(F)$ that can
be projected to \(J^kF\). Incidentally the usual variational derivative of variational calculus can be put into direct correspondence with the vertical differential \(d_v\) on this complex, which is how the name \textit{variational bicomplex} was established.\(^{71,73}\)

Let \((x^i, u_I^a)\) be a set of adapted coordinates on the infinite-jet bundle \(J^\infty F\), where all the following calculations can be lifted. Any result that depends only on jets of finite order can then be projected onto the appropriate finite dimensional jet bundle. Using the integration by parts identity \((A.19)\) if necessary, we can always write the first vertical variation of the Lagrangian density as

\[
\begin{align*}
\delta v L &= E_L a \wedge dv u^a - dh \theta. \\
&(38)
\end{align*}
\]

All terms proportional to \(dv u^a_I, |I| > 0\), have been absorbed into \(dh \theta\). In the course of the performing the integrations by parts, \(E_L a\) can acquire dependence on jets up to order \(2k\) (see Appendix B), and \(\theta\) on jets up to order \(2k - 1\). Note that \(E_L a = 0\) are the \textit{Euler-Lagrange equations} associated with the action functional \(S[\phi]\) or the Lagrangian density \(L\). We can identify the form \(E_L a \wedge dv u^a\) with a possibly nonlinear differential operator \(E_L: \Gamma(F) \to \Gamma(\tilde{F}^*)\), or equivalently a bundle morphism \(E_L: J^{2k}F \to \tilde{F}^*\). A PDE system with an equation form given by Euler-Lagrange equations of a Lagrangian density is said to be \textit{variational}. Also, the form \(\theta\) is an element of \(\Omega^{n-1,1}(F)\), projectable to \(J^{2k-1}F\). It is referred to as the \textit{presymplectic potential current density}. Applying the vertical exterior differential to \(\theta\) we obtain the \textit{presymplectic current density} (or the \textit{presymplectic current density} defined by \(L\) if the extra precision is necessary):

\[
\omega = dv \theta, \\
&(39)
\]

with \(\omega \in \Omega^{n-1,2}(F)\). This terminology implies that \(\omega\) can be integrated over a codim-1 spacetime surface to construct a presymplectic form (Sec. 3.3.3). Such a form on the solution space is referred to as \textit{local}. This method of constructing a symplectic form on the phase space of classical field theory is sometimes referred to as the \textit{covariant phase space method}.\(^{3-5,54}\)

The following lemma is an easy consequence of the definition of \(\omega\).

\textbf{Lemma 3.1.} The form \(\omega \in \Omega^{n-1,2}(F)\) defined in Eq. \((39)\) is both horizontally and vertically closed when pulled back to \(i_\infty: \mathcal{E}_{EL}^\infty \subseteq J^\infty F:\)

\[
\begin{align*}
&d_h i^*_\infty \omega = 0, \\
&d_v i^*_\infty \omega = 0.
\end{align*}
\]

\textbf{Proof.} The horizontal and vertical differentials on \(\mathcal{E}_{EL}^\infty\) are defined by pullback along \(i_\infty\), that is, \(d_h i^*_\infty = i^*_\infty d_h\) and \(d_v i^*_\infty = i^*_\infty d_v\). Since \(\omega = dv \theta\) is already vertically closed on \(J^\infty F\), it is a fortiori vertically closed on \(\mathcal{E}_{EL}^\infty\). The rest is a consequence
of the nilpotence and anti-commutativity of $d_h$ and $d_v$:

$$0 = d_h^2 = d_h d_v u^a - d_v d_h \theta,$$

$$d_h \omega = d_h d_v \theta = -d_v \text{EL}_a \wedge d_v u^a,$$

$$d_h \iota^* \omega = \iota^* d_h \omega = -\iota^* d_v \text{EL}_a \wedge d_v u^a = 0,$$

since $\text{EL}_a$ and $d_v \text{EL}_a$ generate the differential ideal in $\Omega(J^\infty F)$ that is annihilated by the pullback $\iota^*$.

In fact, we will promote the name *presymplectic current density* to any form satisfying these properties.

**Definition 3.1.** It is interesting to note that the existence of a presymplectic current density compatible with a PDE $\mathcal{E}$ is almost equivalent to $\mathcal{E}$ being variational.\textsuperscript{74}

Given a PDE system $\iota: \mathcal{E} \subset J^k F$ we call a form $\omega$ a *presymplectic current density compatible with $\mathcal{E}$* if $\omega \in \Omega^{n-1,2}(F)$ and it is both horizontally and vertically closed on solutions:

$$d_h \iota^* \omega = 0,$$

$$d_v \iota^* \omega = 0.$$

The particular form $\omega$ defined by Eq. (39) will be referred to as the presymplectic current density associated to or obtained from the Lagrangian density $\mathcal{L}$, if there is any potential confusion.

### 3.2. Formal differential geometry of solution spaces

Before describing the symplectic and Poisson structures on the space of solutions, we should say something about the differential geometry of the manifold of solutions of a PDE system as well as its tangent and cotangent spaces. As usual for infinite dimensional manifolds, there are some subtleties.

The main goal of this section is to describe the *formal* tangent and cotangent spaces of the manifold of arbitrary field sections and the manifold of solution sections. The adjective formal, in the last sentence, alludes to the fact that we avoid most technical issues of infinite dimensional analysis and concentrate on what would be dense subspaces of the true tangent and cotangent spaces with a reasonable choice for their topologies. Results are algebraic and (finite dimensional) geometric identities that would form the core of an earnest functional analytical formulation of their non-formal versions. The formal tangent and cotangent spaces have a natural dual pairing, which we prove to be non-degenerate, as a substitute for the absence of true topological duality between them. In the presence of constraints, the proof is carried out under some additional sufficient conditions.

We start with Sec. 3.2.1, which explains how linearizing the linearized equations of motion are related to the formal cotangent space of the space of solutions. Secs. 3.2.2 and Sec. 3.2.3 discuss sufficient conditions on the constraints and gauge
transformations needed for later results. Secs. 3.2.4 and 3.2.5 define the formal tangent and cotangent spaces in the progressively more complicated cases of the space of field configurations, the space of solutions (without constraints), and the space of solutions (with constraints).

3.2.1. Non-linear equations and linearization

In the preceding section (Sec. 3.1) we have discussed general variational systems, without regard for either linearity or hyperbolicity. Note that the notion of Green hyperbolicity that we discussed earlier in Sec. 2 is only applicable to linear systems. The way that we shall restrict our discussion to linear systems is by linearization, which is justified below.

Let us denote by $S(F)$ the set of solutions of the equations of motion of a given non-linear classical field theory on a field bundle $F \rightarrow M$. For instance, for General Relativity $S(F)$ would include metrics of all possible signatures, not just Lorentzian ones. So, obviously, we shall not be interested in all possible solutions, but those that have good causal behavior. We shall not delve here into the question of what constitutes good causal behavior in a non-linear field theory, but refer the reader to Sec. 4.2 of Ref. 60. We shall simply postulate that there is a subset $S_H(F) \subseteq S(F)$ that consists of all solutions with good causal behavior, with the subscript $H$ nominally standing for globally hyperbolic. For us, the most important property of any background solution $\phi \in S_H(F)$ is that the linearized equations of motion about $\phi$ are Green hyperbolic (Sec. 2). Sometimes, we shall also refer to a background solution $\phi$ as a dynamical linearization point.

We shall also assume the hypothesis that $S_H(F)$ can be seen as a possibly infinite dimensional manifold (see Refs. 11–14 for attempts to make that precise). When dealing with either symplectic or Poisson structure, we need also the notions of the tangent $TS_H(F)$ and cotangent $T^*S_H(F)$ bundles, since these structures are needed to define 2-form or bivector tensors on $S_H(F)$. The de Rham closedness and Jacobi identities that respectively identify symplectic and Poisson structures require a notion of differentiation, that is, a differential structure on $TS_H(F)$ and $T^*S_H(F)$ as well. However, if we concentrate on the mutual inverse relationship between a symplectic form $\Omega$ and a Poisson bivector $\Pi$, we are allowed to work with a single tangent space $T_\phi S_H(F)$ and a single cotangent space $T^*_\phi S_H(F)$ at a time, with $\phi \in S_H(F)$, verifying this property for each pair $\Omega_\phi$ and $\Pi_\phi$ individually. This is precisely what we do below.

From now on, we fix $\phi \in S_H(F)$ to be a particular dynamical linearization point. Given any, possibly non-linear, differential operator, we denote its linearization by the same symbol but with a dot, e.g., $\hat{f} : \Gamma(F) \rightarrow \Gamma(\tilde{F})$ is the linearization of $f : \Gamma(F) \rightarrow \Gamma(\tilde{F})$ about $\phi$. Since solution space $S_H(F)$ is embedded in the total field configuration space $\Gamma(F)$, the tangent space at $\phi$ is defined by the space of linearized solutions, that is, solutions of the linearized equations. The following sections, Secs. 3.2.2 and 3.2.3, introduce the linear differential operators that we
expect to obtain after linearizing the equations of motion with constraints and gauge invariance. Later, in Sec. 3.3.1, we consider the Euler-Lagrange equations of a possibly non-linear classical field theory and linearize them, together with the corresponding hyperbolic, constraint and gauge generator differential operators.

3.2.2. Constraints

Earlier, in Sec. 2.2, we discussed linear constrained hyperbolic systems. This notion can actually be extended to non-linear systems, with a very similar structure of identities satisfied by the differential operators involved. See Refs. 75 and 60 for details. At this point, we will presume that we are dealing with a linearization of a possibly non-linear constrained hyperbolic system, whose linearization is itself a linear constrained hyperbolic system of the kind described in Sec. 2.2. To keep the linearization in mind, we put a dot on all the differential operators. Thus, we have a linear constrained hyperbolic system defined by the operators $\dot{f}: \Gamma(F) \rightarrow \Gamma(\tilde{E}^*)$, $\dot{c}: \Gamma(F) \rightarrow \Gamma(E)$, $\dot{h}: \Gamma(E) \rightarrow \Gamma(\tilde{E}^*)$, $\dot{\tilde{q}}: \Gamma(\tilde{E}^*) \rightarrow \Gamma(\tilde{E}^*)$, satisfying the identity $\dot{h} \circ \dot{c} = \dot{\tilde{q}} \circ \dot{f}$. An important thing to note is that their formal adjoints will satisfy the related identity $\dot{c}^* \circ \dot{h}^* = f^* \circ \dot{q}^*$, which is exploited below.

When dealing with constrained systems, some results covered later will require the further sufficient condition that the constraints be parametrizable (see Sec. 2.4) so that we can extend both the linearized system and its adjoint to the following commutative diagrams:

\[
\begin{align*}
0 & \xrightarrow{\dot{c}^*} \Gamma_0(E^*) \xrightarrow{\dot{h}^*} \Gamma_0(\tilde{E}^*) \xrightarrow{\dot{q}^*} \Gamma_0(E') \xrightarrow{\dot{h}^*} \Gamma_0(\tilde{E}^*) \xrightarrow{\dot{q}^*} 0 \\
0 & \xrightarrow{\dot{f}^*} \Gamma_0(F) \xrightarrow{\dot{G}} \Gamma_0(\tilde{E}^*) \xrightarrow{\dot{f}^*} \Gamma_0(\tilde{E}^*) \xrightarrow{\dot{q}^*} 0 \\
0 & \xrightarrow{\dot{h}^*} \Gamma_0(E) \xrightarrow{\dot{H}} \Gamma_0(\tilde{E}^*) \xrightarrow{\dot{h}^*} \Gamma_0(\tilde{E}^*) \xrightarrow{\dot{q}^*} 0
\end{align*}
\] (47)

and

\[
\begin{align*}
0 & \xrightarrow{\dot{c}^*} \Gamma_{SC}(E^*) \xrightarrow{\dot{h}^*} \Gamma_{SC}(E') \xrightarrow{\dot{q}^*} \Gamma_0(E^*) \xrightarrow{\dot{h}^*} \Gamma_0(E') \xrightarrow{\dot{q}^*} 0 \\
0 & \xrightarrow{\dot{f}^*} \Gamma_{SC}(F) \xrightarrow{\dot{G}^*} \Gamma_0(\tilde{E}^*) \xrightarrow{\dot{f}^*} \Gamma_0(\tilde{E}^*) \xrightarrow{\dot{q}^*} 0 \\
0 & \xrightarrow{\dot{h}^*} \Gamma_{SC}(E^*) \xrightarrow{\dot{h}^*} \Gamma_{SC}(E) \xrightarrow{\dot{h}^*} \Gamma_{SC}(E) \xrightarrow{\dot{q}^*} 0
\end{align*}
\] (48)

The rows form exact sequences, while the columns form formally exact complexes, as described in Sec. 2.4. Note that the adjoint diagram also describes a hyperbolic system with hyperbolically integrable constraints, except that the role of the
constraint subsystem is now played by \((q^*, F^*)\) and the consistency subsystem is \((\bar{h}^*, \bar{E}^*)\), which satisfies the consistency identity \(\bar{h}^* \circ q^* = \bar{e}^* \circ \bar{f}^*\).

It is convenient to introduce here a cohomological condition, to be applied in later sections, on the columns of the above commutative diagrams. First, let us introduce some notation for the respective cohomologies. Because the columns are so short, the cohomologies can only be defined at the middle nodes. Each cohomology can be identified by the vector bundle where it is defined, \(F\) or \(\bar{F}^*\), the support restriction, 0 or \(SC\), and the diagram used to define it, (47) or (48). Thus, we denote the cohomologies defined by the columns of diagram (47) by \(H^c_0(F), H^c_0(\bar{F}^*), H^c_{SC}(F)\) and \(H^c_{SC}(\bar{F}^*)\), while those defined by the columns of diagram (48) by \(H^c_0(F), H^c_0(\bar{F}^*), H^c_{SC}(F)\) and \(H^c_{SC}(\bar{F}^*)\). A cocycle section \(\psi \in \Gamma(F)\) (which is annihilated by \(\bar{c}\) or \(q^*\)), with appropriately restricted support, represents a cohomology class denoted by \([\psi]_c\) or \([\psi]_{c^*}\), and similarly for cocycle sections in \(\Gamma(\bar{F}^*)\) (which are annihilated by \(\hat{q}\) or \(\hat{c}^*\)).

Second, recall that, given \(\psi \in \Gamma(F)\) and \(\bar{a}^* \in \Gamma(\bar{F}^*)\), there is a natural pairing \(\langle \psi, \bar{a}^* \rangle = \int_M \psi \cdot \bar{a}^*\), provided the integral is finite. This pairing is indeed well defined between the corresponding nodes of the diagrams (47) and (48). Moreover, when restricted to cocycle sections, this pairing descends to cohomology classes, say \(\langle [\psi]_c, [\bar{a}^*]_{c^*} \rangle = \langle \psi, \bar{a}^* \rangle\). Thus, we have well defined natural pairings on \(H^c_0(F) \times H^c_{SC}(\bar{F}^*), H^c_0(\bar{F}^*) \times H^c_{SC}(F)\), \(H^c_{SC}(\bar{F}^*) \times H^c_{SC}(F)\), \(H^c_{SC}(\bar{F}^*) \times H^c_{SC}(F)\), \(H^c_{SC}(\bar{F}^*) \times H^c_{SC}(F)\).

Third, we must recall that the commutativity of the diagrams (50) and (51) allows us to consider the respective cohomologies at \(\Gamma_0(\bar{F}^*)\) modulo \(\text{im} \bar{f}\) and at \(\Gamma_{SC}(F)\) restricted to \(\text{im} G^*\). In both cases, by the exactness of the rows of these diagrams, we are simply describing the vertical cohomologies in the space of solutions of \(\bar{f}\) and \(\bar{f}^*\), respectively. We shall denote them by \(H^c_{SC}(F, \bar{f})\) and \(H^c_{SC}(F, \bar{f}^*)\), respectively. The natural pairing also descends to the cohomologies in solutions as follows, with say \(\psi = G[\bar{a}^*]\) and \(\xi = G^*[\bar{\beta}^*]\).

\[
\langle [\psi]_c, [\xi]_{c^*} \rangle_G = \langle \psi, \xi \rangle_G, \tag{49}
\]

where on the right-hand-side \(\langle -, - \rangle_G\) is the Green pairing from Def. 2.5.

**Definition 3.2.** The constraints are said to be *globally parametrizable* if the natural pairing between the vertical cohomologies in solutions, \(H^c_{SC}(F, \bar{f})\) and \(H^c_{SC}(F, \bar{f}^*)\), defined using the commutative diagrams (47) and (48), is non-degenerate.

**Remark 3.1.** Note that, as also mentioned in Sec. 2.4, when dealing with parametrizable constraints, the causal structure that is in use is that of the total compound system, whose equation form is \((h^* \oplus f \oplus \bar{h}, E^{**} \oplus_M E^* \oplus_M \bar{E}^*)\). It is easy to show that the adjoint system \((h^* \oplus f \oplus \bar{h}^*, E^* \oplus_M F^* \oplus_M \bar{E}^*)\) defines the same causal structure.
3.2.3. Gauge transformations

Many important classical field theories exhibit gauge invariance, like Maxwell theory, Yang-Mills theory, and General Relativity. A gauge transformation is a family of maps $g_\varepsilon : \Gamma(F) \to \Gamma(F)$, parametrized by sections $\varepsilon \in \Gamma(P)$ of the gauge parameter bundle $P \to M$, that take solutions to solutions, while not modifying a field section outside the support of $\delta \in \Gamma(P)$. $g_\delta[\phi](x) = \phi(x)$ if $x \not\in \text{supp} \delta$, which may be compact and arbitrarily small. If we linearize about some pair of background section $\delta \to \delta + \varepsilon$, we obtain a linearized gauge transformation $g_\delta[\phi] \to g_\delta[\phi] + \dot{g}[\varepsilon]$.

It is another requirement on gauge transformations that the generator of linearized gauge transformations $\dot{g} : \Gamma(P) \to \Gamma(F)$ is a differential operator, which may depend on the background sections $\delta$ and $\phi$.

Equivalence classes of sections under gauge transformations are considered physically equivalent. Therefore, physical observables will consist only of those functions on phase space that are gauge invariant (constant on orbits of gauge transformations). Equivalently, observables are annihilated by the action of linearized gauge transformations. Another way to look at it, is to consider observables as functions on the space of gauge orbits. We denote the space of solutions of the possibly nonlinear equations of motion (with good causal behavior, cf. Sec. 3.2.1) modulo gauge transformations, $\bar{S}_H(F) = S_H(F)/\sim$ and call it the physical phase space.

Often it is convenient to impose subsidiary conditions on field sections, called gauge fixing, that restrict the choice of representatives of gauge equivalence classes. The gauge fixing is called full if they only allow a unique representative from each equivalence class, and otherwise called partial. The gauge transformations that are compatible with a partial gauge fixing are called residual.

Unfortunately, PDE systems with gauge invariance cannot have a well-posed initial value problem, and hence cannot be hyperbolic. In particular, their linearizations cannot be Green hyperbolic. However, the addition of subsidiary conditions on field sections can make the new PDE system equivalent to a hyperbolic one, usually with constraints. In practice, many hyperbolic systems with constraints arise after adding such gauge fixing conditions to a non-hyperbolic system with gauge invariance. Interestingly, after many convenient gauge fixings, there may remain non-trivial residual gauge freedom. For later convenience, as we did with constraints, we restrict our attention to what we call recognizable gauge transformations. That is, given linearized gauge transformations of the form $\dot{g}[\varepsilon]$ and a partially gauge fixed hyperbolic system with equation form $(\dot{f}, \dot{F}^*)$, we can fit them into the following...
the natural pairing between the cohomologies $H$ and $\tilde{H}$.

Commutative diagrams (50) and (51) are required to be hyperbolic and $P' \to M$ is called the gauge invariant field bundle, while $\tilde{g}'$ is called the operator of gauge invariant field combinations.

The above commutative diagrams are very similar to those used to define parametrizable constraints in Sec. 3.2.2. Thus, we can define all the same cohomologies: $H^\vartheta_0(F), H^\vartheta_0(\tilde{F}^\star), H^\vartheta_{SC}(F), H^\vartheta_{SC}(\tilde{F}^\star), H^\vartheta_{SC}(F, \tilde{f})$ defined by diagram (50), and $H^\vartheta_{SC}(\tilde{F}^\star), H^\vartheta_{SC}(\tilde{F}^\star), H^\vartheta_0(\tilde{F}^\star), H^\vartheta_0(\tilde{F}^\star), H^\vartheta_{SC}(F, \tilde{f}^\star)$ defined by diagram (51).

The systems with equation forms $(\hat{k}, \tilde{P}^\star)$ and $(\hat{k}'^\star, \tilde{P}'^\star)$ are required to be hyperbolic and $P' \to M$ is called the gauge invariant field bundle, while $\tilde{g}'$ is called the operator of gauge invariant field combinations.

The above commutative diagrams are very similar to those used to define parametrizable constraints in Sec. 3.2.2. Thus, we can define all the same cohomologies: $H^\vartheta_0(F), H^\vartheta_0(\tilde{F}^\star), H^\vartheta_{SC}(F), H^\vartheta_{SC}(\tilde{F}^\star), H^\vartheta_{SC}(F, \tilde{f})$ defined by diagram (50), and $H^\vartheta_{SC}(\tilde{F}^\star), H^\vartheta_{SC}(\tilde{F}^\star), H^\vartheta_0(\tilde{F}^\star), H^\vartheta_0(\tilde{F}^\star), H^\vartheta_{SC}(F, \tilde{f}^\star)$ defined by diagram (51).

Also, in exactly the same way, there are bilinear pairings $\langle -, - \rangle$ and $\langle -, - \rangle_G$ defined on respective pairs of these cohomologies. On the other hand, the following definition is not quite analogous, reflecting how this hypothesis is used in later sections.

**Definition 3.3.** The gauge transformations are said to be globally recognizable if the natural pairing between the cohomologies $H^\vartheta_{SC}(F)$ and $H^\vartheta_0(\tilde{F}^\star)$, defined using the commutative diagrams (50) and (51), is non-degenerate.

### 3.2.4. Formal $T$ and $T^*$ for configurations

Here we consider a section $\phi \in \mathcal{S}_H(F) \subset \Gamma(F)$ and examine the formal tangent and cotangent spaces at $T_\vartheta \Gamma = T_\vartheta \Gamma(F)$ and $T^*_\vartheta \Gamma = T^*_\vartheta \Gamma(F)$ at $\phi$.

**Definition 3.4.** We define the formal full tangent space at $\phi$ as the set of spacelike compactly supported sections and we define the formal full cotangent space at $\phi$ as
the set
\[ T_\phi \Gamma \cong \Gamma_{SC}(F) \quad \text{and} \quad T^*_\phi \Gamma \cong \Gamma_0(\tilde{F}^*). \] (52)
The natural pairing \( \langle -, - \rangle : T_\phi \Gamma \times T^*_\phi \Gamma \to \mathbb{R} \) is
\[ \langle \psi, \tilde{\alpha}^* \rangle = \int_M \psi \cdot \tilde{\alpha}^*. \] (53)

**Lemma 3.2.** The natural pairing between \( T_\phi \Gamma \) and \( T^*_\phi \Gamma \) is non-degenerate.

This is essentially the fundamental lemma of the calculus of variations and the proof is standard [70, §IV.3.1].

Since the physical phase space will be identified with the space of gauge orbits \( \bar{S}_H(F) \) in the solution space \( S_H(F) \), given a solution section \( \phi \in S_H(F) \), the formal tangent space \( T_\phi \bar{S} = T_\phi \bar{S}_H(F) \) at the corresponding equivalence class \( [\phi] \in \bar{S}_H(F) \) in the space of gauge orbits consists of equivalence classes of linearized solutions up to linearized gauge transformations. Dually, the formal cotangent space \( T^*_\phi \bar{S} = T^*_\phi \bar{S}_H(F) \) will consist of dual densities annihilated by the adjoint of infinitesimal gauge transformation generator.

Since gauge transformations act on field configurations and not just solutions, it makes sense to consider all field configurations related by gauge transformations as physically equivalent. The tangent space \( T_\phi \Gamma \) will be reduced to the quotient (or physical) tangent space \( T_\phi \bar{\Gamma} \) and the cotangent space \( T^* \Gamma \) to the subset \( T^*_\phi \bar{\Gamma} \) of gauge invariant elements. The natural pairing between them is shown to be non-degenerate under the condition of global recognizability, that is, the vertical formally exact complexes in diagrams (50) and (51) are exact. We deal with field configurations first and delay the discussion of solutions to the next section.

The exactness of the composition \( \dot{\gamma}^* \circ \dot{\gamma} = 0 \) ensures that we can recognize pure gauge field configurations, which are of the form \( \psi = \dot{\gamma}[\varepsilon] \) for some spacelike compactly supported section \( \varepsilon : M \to \tilde{P} \), precisely as those spacelike compact field sections \( \psi : M \to F \) that give vanishing gauge invariant field combinations \( \dot{\gamma}^*[\psi] = 0 \). On the other hand, the exactness of the dual composition \( \dot{\gamma}^* \circ \dot{\gamma}^* = 0 \) ensures that we can parametrize gauge invariant, compactly supported dual densities \( \tilde{\alpha}^* : M \to \tilde{F}^* \), those satisfying \( \dot{\gamma}^*[\alpha] = 0 \), precisely as the image of the differential operator \( \dot{\gamma}^* \) acting on compactly supported sections of \( \tilde{P}^{**} \to M \).

**Definition 3.5.** The formal gauge invariant full tangent space at \( \phi \) is the set of gauge equivalence classes of \( \phi \)-spacelike compactly supported sections,
\[ T_\phi \bar{\Gamma} = \{[\psi] \mid \psi \in \Gamma_{SC}(F)\}, \] (54)
\[ [\psi] \sim \psi + \dot{\gamma}[\varepsilon], \quad \text{with} \quad \varepsilon \in \Gamma_{SC}(P). \] (55)

The formal gauge invariant full cotangent space at \( \phi \) is the set of compactly supported gauge invariant dual densities,
\[ T^*_\phi \bar{\Gamma} = \{\tilde{\alpha}^* \in \Gamma_0(\tilde{F}^*) \mid \dot{\gamma}^*[\tilde{\alpha}^*] = 0\}. \] (56)
The natural pairing \( \langle - , - \rangle : T_\phi \Gamma \times T^*_\phi \Gamma \to \mathbb{R} \) is

\[
\langle [\psi], \tilde{\alpha}^* \rangle = \int_M \psi \cdot \tilde{\alpha}^*.
\] (57)

**Lemma 3.3.** If the gauge transformations are globally recognizable (Def. 3.3), the natural pairing between the gauge invariant spaces \( T_\phi \Gamma \) and \( T^*_\phi \Gamma \) is non-degenerate.

**Proof.** Non-degeneracy in the second argument follows once again from the fundamental lemma of the calculus of variations: \( \langle [\psi], \tilde{\alpha}^* \rangle = \langle \psi, \tilde{\alpha}^* \rangle = 0 \) for all \( \psi \in T_\phi \Gamma \), implies that \( \tilde{\alpha}^* = 0 \).

Non-degeneracy in the first argument is more complicated, since we can no longer use arbitrary \( \tilde{\alpha}^* \) in the second argument. It now requires an appeal to the global recognizability of the gauge transformations. Suppose that \( \langle [\psi], \tilde{\alpha}^* \rangle = 0 \) for all \( \tilde{\alpha}^* \in T^*_\phi \Gamma \). We need to show that this implies \( \psi = \hat{g}[\epsilon] \) is pure gauge, for some spacelike compactly supported \( \epsilon \in \Gamma_{SC}(P) \).

By definition, a gauge invariant dual density represents a cohomology class \( [\tilde{\alpha}^*]_{g^*} \in H^0_{\Gamma} (\hat{F}^*), \) defined in Sec. 3.2.3. In fact, any such class could be represented. Considering \( \tilde{\alpha}^* = \hat{g}'[\tilde{\epsilon}^*] \) with arbitrary \( \tilde{\epsilon}^* \in \Gamma_0(\hat{P}^*) \), we find

\[
\langle [\psi], \tilde{\alpha}^* \rangle = \langle \psi, \tilde{\alpha}^* \rangle = \langle \psi, \hat{g}'[\tilde{\epsilon}^*] \rangle = \langle \hat{g}'[\psi], \tilde{\epsilon}^* \rangle.
\] (58)

Since \( \tilde{\epsilon}^* \) could be arbitrary, the vanishing of \( \langle \hat{g}'[\psi], \tilde{\epsilon}^* \rangle \) implies that \( \hat{g}'[\psi] = 0 \). That is, \( \psi \) necessarily represents a cohomology class \( [\psi]_g \in H^0_{SC}(F) \), also defined in Sec. 3.2.3. Therefore we find that non-degeneracy in the first argument implies that

\[
\langle \psi, \tilde{\alpha}^* \rangle = \langle [\psi]_g, [\tilde{\alpha}^*]_{g^*} \rangle = 0,
\] (59)

where the last pairing is in the respective cohomologies and the class \( [\tilde{\alpha}^*]_{g^*} \) allowed to be arbitrary. But the global recognizability hypothesis specifies precisely that the above pairing in cohomology is non-degenerate and implies that \( [\psi]_g = [0] \) and hence that \( \psi = \hat{g}[\epsilon] \), with \( \epsilon \in \Gamma_{SC}(P) \), is pure gauge. \( \square \)

**3.2.5. Formal \( T \) and \( T^* \) for solutions**

The formal tangent space \( T_\phi \mathcal{S} \) will consist of linearized solutions, that is solutions of the linearized constrained hyperbolic system \( \hat{f}[\psi] = 0 \) and \( \hat{c}[\psi] = 0 \). The formal cotangent space will naturally consist of equivalence classes of dual densities up to the images of the adjoints of \( \hat{f} \) and \( \hat{c} \). After giving the precise definitions below, we prove that that the natural pairing between these formal tangent and cotangent spaces is non-degenerate.

**Definition 3.6.** We define the **formal solutions tangent space** at \( \phi \) as the set of spacelike compactly supported linearized solution sections,

\[
T_\phi \mathcal{S} = T_\phi \mathcal{S}_H(F) = \{ \psi \in \Gamma_{SC}(F) \mid \hat{f}[\psi] = \hat{c}[\psi] = 0 \}.
\] (60)
We define the formal solutions cotangent space at $\phi$ as the set of equivalence classes of compactly supported dual densities,

$$T^*_\phi S = T^*_\phi S_H(F) = \{[\tilde{\alpha}^*] \mid \tilde{\alpha}^* \in \Gamma_0(\tilde{F}^*)\},$$  \hspace{1cm} (61)

$$[\tilde{\alpha}^*] \sim \tilde{\alpha}^* + \dot{f}^*[\xi] + \dot{c}^*[\tilde{\varepsilon}^*], \text{ with } \xi \in \Gamma_0(F), \ \tilde{\varepsilon}^* \in \Gamma_0(\tilde{E}^*).$$  \hspace{1cm} (62)

The natural pairing $\langle -, - \rangle : T_\phi S \times T^*_\phi S \to \mathbb{R}$ is

$$\langle \psi, [\tilde{\alpha}^*] \rangle = \int_M \psi \cdot \tilde{\alpha}^*. \hspace{1cm} (63)$$

As a warm-up before the main result of this section, we fist handle the case where the constraints and gauge transformations are trivial.

**Lemma 3.4.** If the constraints $\dot{c}^*[\phi] = 0$ and the gauge transformations are trivial, then the natural pairing between $T_\phi S$ and $T^*_\phi S$ is non-degenerate.

**Proof.** Non-degeneracy in the first argument follows again from the fundamental lemma of the calculus of variations: $\langle \psi, [\tilde{\alpha}^*] \rangle = 0$ for all $\tilde{\alpha}^* \in \Gamma_0(\tilde{F}^*)$ implies that $\psi = 0$.

Non-degeneracy in the second argument is more tricky, since now $\psi$ can no longer be arbitrary. Suppose we have $\langle \psi, [\tilde{\alpha}^*] \rangle = 0$ for all spacelike compactly supported linearized solutions $\psi \in T_\phi S$. From this, we need to deduce that $[\tilde{\alpha}^*] = [0]$, which means $\tilde{\alpha}^* = \dot{f}^*[\xi]$ for some compactly supported $\xi \in \Gamma_0(F)$.

By Prp. 2.1, we can parametrize all solutions as $\psi = G[\tilde{\beta}^*]$, using unrestricted $\tilde{\beta}^* \in \Gamma_0(\tilde{F}^*)$. The following simple calculation

$$\langle G[\tilde{\beta}^*], [\tilde{\alpha}^*] \rangle = \langle G[\tilde{\beta}^*], \tilde{\alpha}^* \rangle = -\langle \tilde{\beta}^*, G^*[\tilde{\alpha}^*] \rangle$$  \hspace{1cm} (64)

and an application of the fundamental lemma of the calculus of variations shows that $G^*[\tilde{\alpha}^*] = 0$. But, once again appealing to Prp. 2.1, this implies that $\tilde{\alpha}^* = \dot{f}^*[\xi]$ with $\xi \in \Gamma_0(F)$, which concludes the proof.

**Remark 3.2.** At this point, it is worth mentioning that the natural pairing between $T_\phi S$ and $T^*_\phi S$ (again, in the absence of constraints or gauge transformations) is essentially equivalent, via Lem. 2.5, to the Green pairing (Def. 2.5), which is also non-degenerate.

In the presence of gauge symmetries, the formal tangent space consists of equivalence classes of linearized solutions up to gauge transformations. On the other hand, the formal cotangent space is restricted to equivalence represented by gauge invariant dual densities. After giving the precise definitions below, we prove that the natural pairing between these formal tangent and cotangent spaces is non-degenerate, provided the constraints $\dot{c}[\phi] = 0$ are globally parametrizable and the gauge transformation are globally recognizable.

The following technical definition is motivated by following steps: we first construct the solution space $T_\phi S$ and then quotient by its purge gauge subspace.
Definition 3.7. We define the formal gauge invariant solutions tangent space at φ as the set of gauge equivalence classes of φ-spacelike compactly supported linearized solution sections,

$$T_φ\overline{S} = T_φ\overline{S}_H(F) = \{[\psi] \mid \psi \in \Gamma_{SC}(F), \tilde{f}[\psi] = 0, \tilde{c}[\psi] = 0\},$$

$$[\psi] \sim \psi + \tilde{g}[\varepsilon],$$

with $\varepsilon \in \Gamma_{SC}(P)$

and $\tilde{f}[\tilde{g}[\varepsilon]] = 0, \tilde{c}[\tilde{g}[\varepsilon]] = 0.$

The formal gauge invariant solutions cotangent space at φ is the set of equivalence classes of compactly supported gauge invariant dual densities,

$$T_φ^*\overline{S} = T_φ^*\overline{S}_H(F) = \{[\tilde{\alpha}^*] \mid \tilde{\alpha}^* \in \Gamma_0(\tilde{F}^*), \tilde{g}^*[\tilde{\alpha}^*] = \tilde{g}^*[\tilde{f}^*[\xi] + \tilde{c}^*[\tilde{\varepsilon}^*]],$$

with $\xi \in \Gamma_0(F), \tilde{\varepsilon}^* \in \Gamma_0(\tilde{E}^*), [\tilde{\alpha}^*] \sim \tilde{\alpha}^* + \tilde{f}^*[\xi] + \tilde{c}^*[\tilde{\varepsilon}^*],$

with $\xi \in \Gamma_0(F), \tilde{\varepsilon}^* \in \Gamma_0(\tilde{E}^*).$

The natural pairing $(-,-): T_φ\overline{S} \times T_φ^*\overline{S} \to \mathbb{R}$ is

$$\langle [\psi], [\tilde{\alpha}^*] \rangle = \int_M \psi \cdot \tilde{\alpha}^*. \quad (67)$$

We now prove the main result of this section.

Lemma 3.5. If the constraints $\tilde{c}[\phi] = 0$ are globally parametrizable (Def. 3.2) and the gauge transformations are globally recognizable (Def. 3.3), then the natural pairing between $T_φ\overline{S}$ and $T_φ^*\overline{S}$ is non-degenerate.

Proof. Unfortunately, we now cannot directly rely on the fundamental lemma of the calculus of variations to prove non-degeneracy in either argument. Instead, we proceed roughly as in the proof of Lem. 3.3.

To prove non-degeneracy in the first argument, suppose we have $\langle [\psi], [\tilde{\alpha}^*] \rangle = 0$ for arbitrary $[\tilde{\alpha}^*] \in T_φ^*\overline{S}$. It is easy to see from the definition that we can restrict ourselves to representatives that satisfy $\tilde{g}^*[\tilde{\alpha}^*] = 0$. Then still, $\tilde{\alpha}^*$ may represent an arbitrary cohomology class $[\tilde{\alpha}^*]_{\tilde{g}^*} \in H^0_0(\tilde{F}^*)$, defined in Sec. 3.2.3. We now need to show that $[\psi] = [0]$, or equivalently that $\psi = \tilde{g}[\varepsilon]$ for some $\varepsilon \in \Gamma_{SC}(P)$. Considering $\tilde{\alpha}^* = \tilde{g}^*[\tilde{\varepsilon}^*]$ with arbitrary $\tilde{\varepsilon}^* \in \Gamma_0(\tilde{P}^*)$, we have

$$\langle [\psi], [\tilde{\alpha}^*] \rangle = \langle \psi, \tilde{g}^*[\tilde{\varepsilon}^*] \rangle = \langle \tilde{g}^*[\psi], \tilde{\varepsilon}^* \rangle. \quad (68)$$

Since $\tilde{\varepsilon}^*$ could be arbitrary, the vanishing of $\langle \tilde{g}^*[\psi], \tilde{\varepsilon}^* \rangle$ implies that $\tilde{g}^*[\psi] = 0$. That is, $\psi$ necessarily represents a cohomology class $[\psi]_{\tilde{g}} \in H^0_{SC}(F)$, also defined in Sec. 3.2.3. Therefore, for any $[\tilde{\alpha}^*] \in T_φ^*\overline{S}$, we find

$$0 = \langle \psi, \tilde{\alpha}^* \rangle = \langle [\psi], [\tilde{\alpha}^*] \rangle_{\tilde{g}}, \quad (69)$$

where the last pairing is in the respective cohomologies and the class $[\tilde{\alpha}^*]_{\tilde{g}}$ is allowed to be arbitrary. But the global recognizability hypothesis specifies precisely...
that the above pairing in cohomology is non-degenerate and implies that \([\psi]_g = [0]\) and hence that \(\psi = \hat{g}[\varepsilon]\), with \(\varepsilon \in \Gamma_{SC}(P)\), is pure gauge.

To prove non-degeneracy in the second argument, suppose we have \(\langle [\psi], [\tilde{\alpha}^*] \rangle = 0\) for arbitrary \([\psi] \in T_0\hat{S}\), which is represented by a solution of \(\hat{f}[\psi] = 0\), \(\hat{c}[\psi] = 0\). Then, from the definition, it is clear that a solution \(\psi\) may also represent an arbitrary cohomology class \([\psi]_g \in H^n_{SC}(F, \hat{f})\), defined in Sec. 3.2.2. We now need to show that \([\tilde{\alpha}^*] = [0]\), or equivalently that \(\tilde{\alpha}^* = \hat{f}^*[\xi] + \hat{c}^*[\tilde{\varepsilon}^*]\) for some \(\xi \in \Gamma_0(F)\) and \(\tilde{\varepsilon}^* \in \Gamma_0(\hat{E}^*)\). We can always choose \(\psi = \hat{c}^*[\zeta^*]\) with \(\zeta^* \in \Gamma_{SC}(E^*)\) such that \(\hat{h}'[\zeta^*] = 0\), or equivalently \(\zeta^* = \hat{H}'[\tilde{\varepsilon}^*]\) and \(\psi = \hat{c}^* \circ \hat{H}'[\tilde{\varepsilon}^*] = G \circ \hat{q}'[\tilde{\varepsilon}^*]\), with \(\tilde{\varepsilon}^* \in \Gamma_0(\hat{E}^*)\) arbitrary. We then have
\[
\langle [\psi], [\tilde{\alpha}^*] \rangle = \langle G \circ \hat{q}'[\tilde{\varepsilon}^*], \tilde{\alpha}^* \rangle = -\langle \tilde{\varepsilon}^*, \hat{q}'[G^*[\tilde{\alpha}^*]] \rangle. \tag{70}
\]
Since \(\tilde{\varepsilon}^*\) could be arbitrary, the vanishing of \(\langle \tilde{\varepsilon}^*, \hat{q}'[G^*[\tilde{\alpha}^*]] \rangle\) implies that \(\hat{q}'[G^*[\tilde{\alpha}^*]]\). That is, \(G^*[\tilde{\alpha}^*]\) represents a cohomology class \([G^*[\tilde{\alpha}^*]]_{c^*} \in H^0_{c^*}(\hat{F}^*, \hat{f}^*)\), also defined in Sec. 3.2.2. Therefore, for any \([\psi] \in T_0\hat{S}\), we find
\[
0 = \langle [\psi], [\tilde{\alpha}^*] \rangle = \langle \psi, \tilde{\alpha}^* \rangle = -\langle \psi, G^*[\tilde{\alpha}^*] \rangle_G = -\langle [\psi], [G^*[\tilde{\alpha}^*]]_{c^*} \rangle_G, \tag{71}
\]
where the last two pairings are the Green pairing (Def. 2.5, and Lem. 2.5) and its descent to the respective cohomologies. But the global parametrizability hypothesis specifies precisely that the above pairing is non-degenerate and implies that \([G^*[\tilde{\alpha}^*]]_{c^*} = [0]\), or equivalently that there exists a \(\tilde{\varepsilon}^* \in \Gamma_0(\hat{E}^*)\) such that
\[
G^*[\tilde{\alpha}^*] = \hat{q}' \circ \hat{H}^*[\tilde{\varepsilon}^*] = G^* \circ \hat{c}^*[\tilde{\varepsilon}^*]. \tag{72}
\]
This, in turn, implies that \(G^*[\tilde{\alpha}^* - \hat{c}^*[\tilde{\varepsilon}^*]] = 0\). Hence, by the exact sequence of Lem. 2.1, we have that \(\tilde{\alpha}^* = \hat{f}^*[\xi] + \hat{c}^*[\tilde{\varepsilon}^*]\) for some \(\xi \in \Gamma_0(F)\).

### 3.3. Symplectic and Poisson structure

In this section, we endow the space of solutions \(S_H(F)\) of a variational PDE system with the structures of both a symplectic and a Poisson manifold (or rather formal versions of these structures), really turning it into the phase space of classical field theory.

In general, a variational system may have gauge symmetries. These must be gauge fixed. The resulting system should then be put into the form of a constrained hyperbolic system. Or, rather, what is most important for us is that these steps can be carried out for the linearization of our variational system (Sec. 3.3.1). If the constraints are (a) globally parametrizable, (b) the gauge transformations globally recognizable and (c) the gauge fixing satisfies an extra compatibility condition, then we can apply a generalized Forger–Romero argument (Sec. 3.3.2). Next, we use the covariant phase space formalism to build the formal symplectic form (Sec. 3.3.3) and the Peierls formula to build the formal Poisson bivector (Sec. 3.3.4). Finally, we prove that the two structures are equivalent (Sec. 3.3.5).
For the remainder of this section, let us fix a Lagrangian density $\mathcal{L} \in \Omega^{n,0}(F)$. Following Sec. 3.1, it defines a presymplectic current density $\omega \in \Omega^{n-1,2}(F)$ and its Euler-Lagrange equations define a PDE system with equation form $(\text{EL}, \tilde{F}^*)$.

### 3.3.1. Variational systems, gauge fixing and constraints

There are many reasons why the equation form $(\text{EL}, \tilde{F}^*)$ of the equations of motion of the classical field theory is not optimal for our analysis. As we shall see later on, the Peierls formula calls for a Green function of the linearized equations of motion. However, the particular form of the differential operator EL may not be one that directly falls into one of the classes of differential operators that are easily recognized as hyperbolic, so that its linearizations possess Green functions.

For instance, in the presence of gauge invariance, we must first add a gauge fixing condition, say $c_g[\phi] = 0$ valued in a bundle $E_g \to M$. Also, in many cases, either due to the extra gauge fixing equations or due to internal integrability conditions (Sec. Appendix B.1), the equations can only be cast in hyperbolic form with constraints. Of course, let us not forget that, a priori, we haven’t yet restricted the choice of $\mathcal{L}$ in any way that would guarantee that its Euler-Lagrange system is not elliptic or of some other hyperbolic type. So, we call the Euler-Lagrange system hyperbolizable if, after a possible gauge fixing, it can be shown to be equivalent to a constrained hyperbolic system in a way that we make precise below. From now on, we require that for a classical field theory the Lagrangian $\mathcal{L}$ is chosen such that its Euler-Lagrange equations are hyperbolizable. We shall see later in Sec. 4, that many relativistic field theories of physical interest are in fact hyperbolizable.

Consider the gauge fixed Euler-Lagrange system, whose equation form is $(\text{EL} \oplus c_g, \tilde{F}^* \oplus_M E_g)$. It is hyperbolizable if it equivalent (in the sense of Sec. Appendix B.1) to a constrained hyperbolic system $(f \oplus c, \tilde{F}^* \oplus_M E)$. Again, we are not going into the details of what constitutes a non-linear constrained hyperbolic system but defer instead to Refs. 75 and 60. The equivalence must have the following form:

$$\begin{align*}
\begin{cases}
\text{EL} = R \circ (f \oplus c) \\
c_g = R_g \circ c
\end{cases} & \iff \\
\begin{cases}
f = \tilde{R} \circ (\text{EL} \oplus c_g) \\
c = \tilde{R}_g \circ (\text{EL} \oplus c_g)
\end{cases},
\end{align*}$$

(73)

where the $R$, $\tilde{R}$, $R_g$ and $\tilde{R}_g$ are possibly non-linear differential operators.

As discussed earlier, in Sec. 3.2.1, for the purposes of our discussion, it is sufficient to pick a single dynamical linearization point $\phi \in S_H(F)$ and linearize the above PDE systems about it. In particular, the linearized equations will be sufficient to define the formal tangent and cotangent spaces $T_\phi \mathcal{S}, T^*_\phi \mathcal{S}$ and their gauge invariant analogs $T_\phi \bar{\mathcal{S}}, T^*_\phi \bar{\mathcal{S}}$. In other words, we need to work with the linearized versions of each of the hyperbolic system, the constraints, the Euler-Lagrange system, the gauge fixing conditions, the gauge transformations, as well as the hyperbolization. As before, we denote the equation form of the linearized hyperbolic system $(\dot{f}, \dot{\tilde{F}}^*)$. The linearized constraints are presumed to be globally parametrizable and fit into the commutative diagrams (47) and (48). The linearized gauge transformations are
presumed to be globally recognizable and fit into the commutative diagrams (50) and (51). The linearized EL equations are denoted \((J, \tilde{F}^s)\) and are also called the Jacobi system,

\[ J[\psi]_a(x) = J_{ab}^i \partial_f \phi^b(x) = 0, \tag{74} \]

with \(J\) the Jacobi operator, while the linearized gauge fixing conditions are denoted by the equation form \((\hat{c}_g, E_g)\). In local coordinates \((x^i, u^a)\) on \(F\), the components of the Jacobi operator satisfy the identity

\[ J_{ab}^i \wedge d_v u^b = d_v \text{EL}_a. \tag{75} \]

The equivalence of the linearized systems takes the following form:

\[
\begin{align*}
J &= r \circ \dot{f} + r_c \circ \dot{c} \\
\dot{c}_g &= r_g \circ \dot{c}
\end{align*} \iff \begin{align*}
\dot{f} &= \tilde{r} \circ J + \tilde{r}_c \circ \dot{c}_g \\
\dot{c} &= \tilde{r}_1 \circ J + \tilde{r}_g \circ \dot{c}_g
\end{align*} \tag{76}
\]

If the operator \(\tilde{r}_1\) is non-vanishing, it means that part of the constraints consist of integrability conditions of the Jacobi system.

Note that, strictly speaking, the \(r\)- and \(\tilde{r}\)-differential operators effecting the equivalence are not inverses of each other. Their compositions may differ from the identity by some differential operator that factors through a differential identity, that is, \(\dot{q} \circ \dot{f} - \dot{h} \circ \dot{c} = 0\) or \(\dot{g}^* \circ J = 0\). In other words, we must have

\[
\begin{align*}
0 &= r \circ \tilde{r} + r_c \circ \tilde{r}_j = \text{id} + p_3 \circ \tilde{g}^* \\
r \circ r_c + r_g \circ \tilde{r}_g = 0 &\iff \tilde{r} \circ r = \text{id} + p_f \circ \dot{q}, \tag{77} \\
r_g \circ \tilde{r}_j &= p_g \circ \tilde{g}^* &\iff \tilde{r}_j \circ r = p_c \circ \dot{q}, \tag{78} \\
r_g \circ \tilde{r}_g &= \text{id} &\iff \tilde{r}_g \circ r_g + \tilde{r}_1 \circ r_c = \text{id} - p_c \circ \dot{h}. \tag{79}
\end{align*}
\]

for some differential operators \(p_3, p_f, p_g\) and \(p_c\). Also, the identity \(\dot{q} \circ \dot{f} - \dot{h} \circ \dot{c} = 0\), when expressed in terms of the \(J\) and \(\dot{c}_g\) operators, is identically satisfied when

\[
\begin{align*}
\dot{q} \circ \tilde{r} - \dot{h} \circ \tilde{r}_j &= q_1 \circ \tilde{g}^*, \tag{80} \\
\dot{q} \circ \tilde{r}_c - \dot{h} \circ \tilde{r}_g &= 0. \tag{81}
\end{align*}
\]

It is worth noting that the above relations involving the \(r\)- and \(\tilde{r}\)-operators follow from the equivalence (76) only when \(J\) and \(\dot{c}_g\) satisfy no additional differential identities. However, we will simply presume that they hold as needed sufficient conditions for the derivation of the Peierls formulas later in Sec. 3.3.4.

Finally, to make sure that the condition \(\dot{c}_g[\psi] = 0\) in fact constitutes a gauge fixing condition, we require the following compatibility between the gauge transformation operator and the constraints that we shall refer to as the gauge fixing compatibility condition:

\[ \dot{s}' \circ \tilde{r}_c \circ \dot{c}_g = 0. \tag{83} \]

This condition connects constraints (represented by \(\dot{c}_g\)) and gauge transformations (represented by \(\dot{s}'\)). Roughly speaking, this condition says that the part of the constraints \(\dot{c}[\psi] = 0\) that comes from \(\dot{c}_g[\psi] = 0\) is sufficient, when adjoined to \(J[\psi] = 0\)
to make the gauge fixed system hyperbolizable. This compatibility condition becomes important later on, in Lem. 3.9, to show that the gauge invariant formal cotangent space $T^*_{\phi \bar{S}}$ can be equivalently defined in two ways, involving either the J operator or the $\dot{f}$, $\dot{c}$ operators. Also, it helps prove that the hyperbolic differential operator $\dot{k}$, that acts on gauge invariant field combinations in the presence of recognizable gauge transformations, is actually independent of the choice of gauge fixing operator $\dot{c}_g$ as long as the compatibility condition is satisfied (see Cor. 3.2).

For future reference, it is convenient to state here the following

**Lemma 3.6.** The gauge fixing compatibility condition Eq. (83) is equivalent to the existence of a differential operator $\bar{r}_s : \Gamma(F) \to \Gamma(\tilde{P}^*)$ such that

$$\bar{r}_c \circ \dot{c}_g = \dot{s} \circ \bar{r}_s.$$  \hspace{1cm} (84)

That is, $\bar{r}_c \circ \dot{c}_g$ factors through $\dot{s}$.

**Proof.** This follows directly from the gauge compatibility condition (83) and Lem. 2.3.

We summarize the conditions listed in this section in the following

**Definition 3.8.** The Euler-Lagrange system (EL, $\tilde{F}^*$) or just the Jacobi system (J, $\tilde{F}^*$) is said to be hyperbolizable if the following conditions are met: (a) there exists a gauge fixing and an equivalence with a constrained hyperbolic system of the form (73) or (76), (b) the constraints are parametrizable and the gauge transformations are recognizable with respect to the resulting hyperbolic subsystem, and (c) the gauge fixing compatibility condition (83) is satisfied.

The consequences of hyperbolizability are explored in the following section. We stress that these conditions are sufficient for our purposes and can in fact be satisfied by many relativistic field theories of physical interest, but some of the same results could also hold under weaker conditions.

### 3.3.2. Causal Green functions

The goal of this section is to use the gauge fixed equivalence (76) with a constrained hyperbolic system to construct a causal Green function for the Jacobi system.

First, we show that the residual gauge transformations (those that are still allowed by the gauge fixing condition $\dot{c}_g[\psi] = 0$) essentially come from gauge parameters that satisfy the symmetric hyperbolic equation $\dot{k}[\varepsilon] = 0$. The main purpose of this lemma is to serve as a reference argument for one of the sub-results of Thm. 3.1.

**Lemma 3.7.** Given a $\psi \in \Gamma_{SC}(F)$ such that $\dot{c}_g[\psi] = 0$ and $\psi \in \text{im} \dot{y}$, there exists $\varepsilon \in \Gamma_{SC}(P)$ such that $\dot{k}[\varepsilon] = 0$ and $\psi = \dot{y}[\varepsilon]$ precisely when the image of a map (to be defined in the proof) $\circ K$: $\ker \dot{s} \subseteq \Gamma_{SC}(\tilde{P}^*) \to H^2_{\text{SC}}(F, \dot{f})$ is trivial (see Sec. 3.2.3).
Proof. Recall that the Jacobi system, due to its variational character is easily shown to be self-adjoint:

\[ J^* = J. \]  \hspace{1cm} (85)

Also, gauge invariance and Noether’s second theorem imply the identities

\[ J \circ \hat{g} = 0 \quad \text{and} \quad \hat{g}^* \circ J = 0. \]  \hspace{1cm} (86)

The equivalence of the gauge fixed Jacobi system with the constrained hyperbolic system postulated in (76) then gives

\[ s \circ \dot{k} = \dot{f} \circ \dot{g} = \check{r}_c \circ \dot{c}_g \circ \dot{g}. \]  \hspace{1cm} (87)

Suppose that \( \psi \in \Gamma_{SC}(F) \) such that \( \dot{c}_g[\psi] = 0 \) and \( \psi = \hat{g}[\varepsilon'] \) for some \( \varepsilon' \in \Gamma_{SC}(P) \). Let \( \hat{\beta}^* = \hat{k}[\varepsilon'] \in \Gamma_{SC}(\hat{P}^*) \) and note that

\[ \hat{s}[\hat{\beta}^*] = \check{s} \circ \dot{k}[\varepsilon'] = \check{r}_c \circ \dot{c}_g \circ \dot{g}[\varepsilon'] = \check{r}_c \circ \dot{c}_g[\psi] = 0. \]  \hspace{1cm} (88)

Then also, of course, \( \dot{f}[\psi] = \dot{f} \circ \hat{g}[\varepsilon'] = \check{s} \circ \dot{k}[\varepsilon'] = \hat{s}[\hat{\beta}^*] = 0 \). Hence, \( \psi \) represents a cohomology class \( [\psi]_g \in H^2_{SC}(F, \hat{f}) \) (Sec. 3.2.3). The conclusion of this lemma holds precisely when this cohomology class is trivial, \( [\psi]_g \).

Let \( \{ \chi_{\pm} \} \) be a partition of unity adapted to a Cauchy surface (Def. 2.3) and recall the associated splitting map (Lem. 2.1) \( K_\chi : \Gamma_{SC}(\hat{P}^*) \rightarrow \Gamma_{SC}(P) \) that inverts \( \dot{k} \) from the right. We can make two observations: (a) the difference \( \varepsilon' - K_\chi[\hat{\beta}^*] \) is in \( \mathcal{S}_{SC}(P) \), and (b) \( \eta_\chi = \hat{g}[K_\chi[\hat{\beta}^*]] = G[\check{s}_\chi[\hat{\beta}^*]] \) is in \( \mathcal{S}_{SC}(F) \), where \( \check{s}_\chi[\hat{\beta}^*] = \pm \hat{s}[\chi_{\pm} \hat{\beta}^*] \) has compact support since \( \hat{s}[\hat{\beta}^*] = 0 \). Hence \( \eta \) and \( \psi \) define the same cohomology class, \( [\eta_\chi]_g = [\psi]_g \). Moreover, the choice of the adapted partition of unity \( \{ \chi_{\pm} \} \) doesn’t matter, since for any other choice \( \{ \chi'_{\pm} \} \) the differences \( \chi'_{\pm} - \chi_{\pm} \hat{\beta}^* \) have compact support, so that \( [\eta_{\chi'}]_g = [\eta_\chi]_g \).

Thus, the composition of maps

\[ gK : \ker \hat{s} \subset \Gamma_{SC}(\hat{P}^*) \xrightarrow{\hat{g}K} \mathcal{S}_{SC}(F) \rightarrow H^2_{SC}(F, \hat{f}) \]  \hspace{1cm} (89)

is independent of the choice of the adapted partition of unity and, as desired, its image coincides with the image of the subset of \( \mathcal{S}_{SC}(F) \) consisting of elements of the form \( \psi = \hat{g}[\varepsilon'] \) with \( \varepsilon' \in \Gamma_{SC}(P) \). Thus, each such \( \psi = \hat{g}[\varepsilon] \) where \( \dot{k}[\varepsilon] = 0 \) precisely when the image of \( gK \) is trivial.

It was remarked in the above proof that \( J \circ \hat{g} = 0 \) and \( \hat{g}^* \circ J = 0 \). This is actually enough information to prove that the Jacobi operator must factor through \( \hat{g}' \) on the right and through \( g'^* \) on the left.

Lemma 3.8. There exists a differential operator \( J_g \) such that \( J = J_g \circ \hat{g}' = \hat{g}'^* \circ J_g^* \).

Proof. This is a simple consequence of Eq. (86), Lem. 2.3, and the self-adjointness of \( J \).
Next, we prove a lemma that will be used to establish an alternative characterization of the formal gauge invariant solutions cotangent space (Def. 3.7) in the main theorem of this section.

Lemma 3.9. Provided the sufficient conditions listed in Sec. 3.3.1 hold, any compactly supported dual density \( \tilde{\alpha}^* \in \Gamma_0(\tilde{F}^*) \) that satisfies
\[
\hat{g}^*[\tilde{\alpha}^*] = \hat{g}^*[\hat{f}^*[\psi] + \hat{c}^*[\tilde{\beta}]],
\]
for some \( \psi \in \Gamma_0(F) \) and \( \tilde{\beta}^* \in \Gamma_0(\tilde{E}^*) \), can be written as \( \tilde{\alpha}^* = J[\xi] \), for some \( \xi \in \Gamma_0(F) \).

Proof. We start by defining a field section \( \psi_+ \) with retarded support such that
\[
\hat{f}^*[\psi_+] = \hat{f}^*[\psi] + \hat{c}^*[\tilde{\beta}^*].
\]
If it is simple to check that we can define \( \psi_+ = \psi + \hat{q}^* \circ H^*_+[\tilde{\beta}^*] \). (Using advanced support would have also been possible.) Then,
\[
\hat{k}^* \circ \hat{s}^*[\psi_+] = \hat{g}^* \circ \hat{f}^*[\psi_+] = \hat{g}^*[\tilde{\alpha}^*] = 0.
\]

Since \( \hat{k}^* \) is invertible on sections of retarded support, we must have \( \hat{s}^*[\psi_+] = 0 \). On the other hand,
\[
\tilde{\alpha}^* = \hat{f}^*[\psi_+] = J^* \circ \hat{r}^*[\psi_+] + \hat{c}^*_g \circ \hat{r}_c[\psi_+]
\]
\[
= J[p^*[\psi] + \hat{r}^* \circ \hat{g}^* \circ H^*_+[\tilde{\beta}^*] + \hat{r}_c^*[\hat{s}^*[\psi_+]]]
\]
\[
= J[p^*[\psi] + \hat{g} \circ q_3^* \circ H^*_+[\tilde{\beta}^*] + \hat{r}_c^* \circ (\hat{h}^* \circ H^*_+[\tilde{\beta}^*])]
\]
\[
= J[p^*[\psi] + \hat{r}_c^*[\tilde{\beta}^*]],
\]
where we have used the equivalence (76), the formal self-adjointness of \( J \), Lem. 3.6, and the identities (81) and \( J \circ \hat{g} = 0 \). Therefore, the desired conclusion holds with \( \xi = \hat{r}^*[\psi] + \hat{r}_c^*[\tilde{\beta}^*] \).

Finally, we motivate the Peierls formula and then state and prove the main theorem of this section. Equivalence with a constrained hyperbolic system now allows us to solve the inhomogeneous problem
\[
J[\psi] = \tilde{\alpha}^*, \quad \hat{c}_\nu[\psi] = 0,
\]
where the source must necessarily satisfy the gauge invariance condition \( \hat{g}^*[\tilde{\alpha}^*] = 0 \).

The equivalent inhomogeneous problem in symmetric hyperbolic form is
\[
\hat{f}[\psi] = \hat{r}[^*\tilde{\alpha}], \quad \hat{c}[\psi] = \hat{r}_J[^*\tilde{\alpha}].
\]
Recall from Lem. 2.2 that this system is solvable iff the sources satisfy the consistency identity:
\[
\hat{q}[\hat{r}[^*\tilde{\alpha}]] - \hat{h}[\hat{r}_J[^*\tilde{\alpha}]] = p_1 \circ \hat{g}^*[\tilde{\alpha}^*] = 0,
\]
which is obviously satisfied, after using identity (81), for any gauge invariant source. The retarded and advanced solutions to this inhomogeneous problem are then \( \psi_\pm = G_\pm[\hat{r}[^*\tilde{\alpha}]] \). This means that \( \hat{r}_J[^*\tilde{\alpha}] = \hat{c}[G_\pm[\hat{r}[^*\tilde{\alpha}]]] \) and, in particular, \( \hat{c}_g[\psi] = 0 \).
Motivated by this formula, we introduce the following retarded, advanced and causal Green functions for the gauge fixed Jacobi system.

**Definition 3.9.** Let $E_\pm = G_\pm \circ \tilde{r}$. The *Peierls formula* is

$$E = E_+ - E_- = G \circ \tilde{r}.$$  \hfill(99)

We also call $E$ the *Peierls* or *Jacobi causal Green function*.

One can immediately check that $\psi = E[\tilde{\alpha}^*]$ satisfies both $\tilde{f}[\psi] = 0$ and $\tilde{c}[\psi] = 0$, whenever $\tilde{\alpha}^*$ is a gauge invariant dual density. By the equivalence (76), the same solution also satisfies $J[\psi] = 0$ and $\tilde{c}_g[\psi] = 0$.

**Theorem 3.1.** Provided the gauge fixed Jacobi system $J[\psi] = 0$, $\tilde{c}_g[\psi] = 0$ is hyperbolizable (Def. 3.8), the Jacobi causal Green function $E$ defined in Eq. (99) fits into the following commutative diagram

$$\begin{array}{ccccccccc}
\Gamma_0(P) & \longrightarrow & 0 & \longrightarrow & \Gamma_{SC}(P) & \longrightarrow & 0 \\
\downarrow \hat{\delta} & & \downarrow \hat{\delta} & & \downarrow \hat{\delta} \\
0 & \longrightarrow & \Gamma_0(F) & \stackrel{\hat{J}}{\longrightarrow} & \Gamma_0(\tilde{F}^*) & \longrightarrow & E & \longrightarrow & \Gamma_{SC}(\tilde{F}^*) & \longrightarrow & 0, \\
\downarrow \hat{\delta} & & \downarrow \hat{\delta} & & \downarrow \hat{\delta} & & \downarrow \hat{\delta} & & \downarrow \hat{\delta} \\
0 & \longrightarrow & \Gamma_0(\tilde{F}^*) & \longrightarrow & \Gamma_{SC}(\tilde{F}^*) \\
\end{array}$$  \hfill(100)

which becomes a complex (successive arrows compose to 0) after taking the vertical cohomologies. That complex is exact at $\Gamma_0(\tilde{F}^*)$ and $\Gamma_{SC}(F)$, while at $\Gamma_0(F)$ and $\Gamma_{SC}(\tilde{F}^*)$ its cohomologies coincide respectively with $H_0^2(\tilde{F}^*)$ and $H_{SC}^2(\tilde{F}^*)$.

Moreover, with reference to Def. 3.7, we have the isomorphisms $T_0 \tilde{S} \cong \ker J / \text{im} \hat{g}$ at $\Gamma_{SC}(F)$, and $T_0 \tilde{S} \cong \ker \hat{g}^* / \text{im} J$ at $\Gamma_0(\tilde{F}^*)$.

Finally, a Cauchy surface $\Sigma \subset M$ and a partition of unity $\{\chi_{\pm}\}$ adapted to it define the following splittings at $\Gamma_0(\tilde{F}^*)$ and $\Gamma_{SC}(F)$:

$$\ker \hat{g}^* \cong \text{im} J \oplus T_0 \tilde{S} \quad \text{and} \quad \text{coker} \hat{g} \cong T_0 \tilde{S} \oplus \text{im} \hat{g}^*,$$  \hfill(101)

where (cf. Def. 2.3, Lem. 2.1, and Def. 3.6)

$$J_\chi: T_0 \tilde{S} \rightarrow \Gamma_0(\tilde{F}^*), \quad J_\chi[\psi] = \pm J[\chi_{\pm}\psi],$$  \hfill(102)

$$E_\chi: \hat{\text{im} \hat{g}^*} \rightarrow \Gamma_{SC}(F), \quad E_\chi[\tilde{\alpha}^*] = \hat{r}^* \circ G^*_\chi[\tilde{\alpha}^*].$$  \hfill(103)

The conclusion of the theorem is rather dense with information, so its proof is somewhat lengthy. However it simply consists of checking the properties of the horizontal sequence in the above diagram at each of its objects.

**Proof.** The fact that successive maps compose to zero, after taking the vertical cohomologies, is established in items (2) and (3) below, which also prove exactness of the resulting complex at $\Gamma_0(\tilde{F}^*)$ and $\Gamma_{SC}(F)$. On the other hand, cohomologies at $\Gamma_0(F)$ and $\Gamma_{SC}(\tilde{F}^*)$ are computed in items (1) and (4).
The isomorphism $T_\phi \tilde S \cong \ker J / \im \hat g$ is established as follows: on the one hand, it is obvious that $\hat f[\psi] = 0$, $\hat c[\psi] = 0$ implies $J[\psi] = 0$; on the other hand, item (3) shows that both $\hat f \circ E = 0$ and $\hat c \circ E = 0$, while $\im E = \ker J$ (mod $\im \hat g$). The isomorphism $T_\phi \tilde S \cong \ker \hat g^* / \im J$ is established as follows: on the one hand, by the equivalence (76), it is obvious that $\tilde\alpha^* + J[\xi]$, with $\hat g^* [\tilde\alpha^*] = 0$ and $\xi$ arbitrary, represents a unique element of $T_\phi \tilde S$; on the other hand, if $\hat g^* [\tilde\alpha^*] = \hat g^* [\hat f[\psi] + \hat c^* [\hat \beta^*]]$, then $[\tilde\alpha^*] = [\tilde\alpha^* - \hat f[\psi] - \hat c^* [\hat \beta^*]]$ in $T_\phi \tilde S$ and Lem. 3.9 shows that any representative of $[0] \in T_\phi \tilde S$ represents only $[0] \in \ker \hat g^* / \im J$.

Finally, the splittings (101), with the corresponding splitting map identities, are established in items (3) and (4).

Note that below we make liberal use of various maps defined using the adapted partition of unity $\{\chi_\pm\}$ introduced in the hypothesis of the theorem (cf. Lem. 2.1).

1. If $\psi \in \Gamma_0(F)$, then $J[\psi] = 0$ is equivalent to $\hat g'[\psi] = 0$.
   If $\psi \in \ker \hat g'$, then by Lem. 3.8 ($J = \hat J \circ \hat g'$) we certainly have $\psi \in \ker J$. On the other hand, if $\psi \in \ker J$, then
   \[
   \hat k'[\hat g'[\psi]] = s'[\hat f[\psi]] = s'[\hat r_e \circ J[\psi] + \hat r_c \circ \hat c_g[\psi]] = s' \circ \hat r_c \circ \hat c_g[\psi] = 0,
   \]
   where the last equality holds due to the gauge fixing compatibility condition (83) and we have also used the equivalence (76). But, since $\hat k'$ is injective on $\Gamma_0(P')$, this can only be if $\psi \in \ker \hat g'$. Therefore, after taking vertical cohomologies, the cohomology of (100) at $\Gamma_0(F)$ is isomorphic to $H^{0}_0(F)$.

2. At $\Gamma_0(F^*)$, we have $E \circ J = 0$ (mod $\im \hat g'$). Any $\tilde\alpha^* \in \Gamma_0(F^*)$ such that $\hat g^* [\tilde\alpha^*] = 0$ and $E[\tilde\alpha^*] = \hat g[\epsilon]$, with $\epsilon \in \Gamma_{SC}(P)$, can be written as $\tilde\alpha^* = J[\xi]$, with $\xi \in \Gamma_0(F)$. For the first part, write the equivalent form (84) of the gauge fixing compatibility condition. Direct calculation, with $\xi \in \Gamma_0(F)$, then gives
   \[
   E[\xi] = G[\hat r \circ J[\xi]] = G[\hat f[\xi] - \hat r_c \circ \hat c_g[\xi]]
   \]
   \[
   = G[\hat s \circ \hat r_s[\xi]]
   \]
   \[
   = \hat g[\hat K[\hat r_s[\xi]]].
   \]
   For the second part, let $\psi = E[\tilde\alpha] = \hat g[\epsilon] \in \Gamma_{SC}(F)$ and $\tilde\beta^* = \hat k[\epsilon] \in \Gamma_{SC}(\hat F^*)$. Note that $\hat f[\psi] = \hat f \circ G[\hat r[\tilde\alpha^*]] = 0$ and also $\hat s[\tilde\beta^*] = \hat s \circ \hat k[\epsilon] = \hat f \circ \hat g[\epsilon] = 0$.
   Using the same logic and notation as in the proof of Lem. 3.7, we can write
   \[
   \psi = G[\hat s[\tilde\beta^*] + \hat s[\tilde\gamma^*]] = G \circ \hat s[\chi_+ \tilde\beta^* + \tilde\gamma^*],
   \]
   for some $\tilde\gamma^* \in \Gamma_0(\hat F^*)$. Note that the argument of $G$ has compact support. Recalling the definition $\psi = G[\hat r[\tilde\alpha^*]]$ and the fact that $\ker G = \im \hat f$, there must exist a $\xi \in \Gamma_0(F)$ such that
   \[
   \hat f[\xi] = \hat r[\tilde\alpha^*] - \hat s[\chi_+ \tilde\beta^* + \tilde\gamma^*].
   \]
   By uniqueness of solutions with retarded support, we must have $\xi = G_+[\hat r[\tilde\alpha^*] - \hat s[\chi_+ \tilde\beta^* + \tilde\gamma^*]] = G_+[\hat r[\tilde\alpha^*]] - \hat g \circ K_+ [\chi_+\tilde\beta^* + \tilde\gamma^*]$ (we could have also used $G_-$.
Therefore, the desired conclusion holds, \( \psi' = E \circ J_\chi[\psi] = \hat{g}[\varepsilon], \) with \( \varepsilon = K_+ \circ \tilde{r}_s[\chi_+] + K_- \circ \tilde{r}_s[\chi_-] \in \Gamma_{SC}(P). \)
(4) If $\tilde{\alpha}^* \in \Gamma_{\text{SC}}(\tilde{F}^*)$, then $\tilde{\alpha}^* = J[\psi]$, for some $\psi \in \Gamma_{\text{SC}}(F)$, is equivalent to $\tilde{\alpha}^* = \hat{g}^*[\tilde{\beta}^*]$, for some $\tilde{\beta}^* \in \Gamma_{\text{SC}}(\tilde{P}^*)$.

If $\tilde{\alpha}^* = J[\psi]$, then by Lem. 3.8 ($J = \hat{g}^* \circ J_\gamma^*\gamma$) we certainly have $\tilde{\alpha}^* \in \text{im} \hat{g}^*$. On the other hand, if $\tilde{\alpha}^* = \hat{g}^*[\tilde{\beta}^*]$, let $\xi = K_{\alpha}^*[\tilde{\beta}^*] \in \Gamma_{\text{SC}}(P')$. Direct calculation then shows
\[
J[\tilde{\alpha}^* \circ \hat{s}^*[\xi]] = (J^* \circ \tilde{r}^*) \circ \hat{s}^*[\xi] = \hat{J}^* \circ \tilde{r}^*[\xi] = \tilde{\alpha}^*,
\]
where we have used the formal adjoint versions of Eqs. (76) and the gauge fixing compatibility condition (83). Therefore, the desired conclusion holds, with $\psi = \tilde{r}^* \circ \hat{s}^*[\xi] \in \Gamma_{\text{SC}}(F)$. Simplifying the last expression, we get $\psi = \tilde{r}^* \circ (\hat{s}^* \circ K_{\alpha}^*)[\tilde{\beta}^*] = \tilde{r}^* \circ G_{\alpha}^*[\hat{g}^*[\tilde{\beta}^*]] = \tilde{r}^* \circ G_{\alpha}^*[\hat{\alpha}^*]$ and hence
\[
J \circ E_\lambda[\tilde{\alpha}^*] = J \circ \tilde{r}^* \circ G_{\alpha}^*[\hat{\alpha}^*] = \tilde{\alpha}^*.
\]
Therefore, we have established that, after taking vertical cohomologies, the cohomology of (100) at $\Gamma_{\text{SC}}(\tilde{F}^*)$ is isomorphic to $H^g_{\text{SC}}(\tilde{F}^*)$.

Remark 3.3. The hypotheses of Thm. 3.1 make use of the notion of hyperbolizability (Def. 3.8), which in turn requires the corresponding constraints to be parametrizable (Sec. 3.2.2) and the gauge transformations to be recognizable (Sec. 3.2.3). But only the local versions of these were used. That is, Thm. 3.1 holds even if global parametrizability and recognizability fail. The global conditions instead will appear in the representation (161) of the formal presymplectic form with respect to the natural pairing between the formal tangent and cotangent spaces.

As mentioned before, it is easy to see from its variational nature that the Jacobi operator is self-adjoint $J^* = J$. If it were directly invertible, the Green functions $E_\pm$ would satisfy the same relation with their adjoints as shown in Sec. (2.5), making the causal Green function anti-self-adjoint, $(E)^* = -E$. However, due to gauge invariance the relation of the gauge fixed Green functions to their adjoints is more complicated.

Lemma 3.10. When restricted to act on gauge invariant dual densities, the causal Green function of the gauge fixed Jacobi system is anti-self-adjoint up to gauge:
\[
(E)^* = -E \quad \text{(mod im} \hat{g}).
\]

Proof. First, note that from identities (76) and (77) we have
\[
J \circ E_\pm[\tilde{\alpha}^*] = \sum_{\pm} (r \circ \tilde{f} + r_c \circ \tilde{c}) \circ G_{\pm} \circ \tilde{r}[\tilde{\alpha}^*] = r \circ \tilde{r}[\tilde{\alpha}^*] + r_c \circ \tilde{r}_J[\tilde{\alpha}^*] = (\text{id} + p_J \circ \hat{g}^* \circ \tilde{r}_J)[\tilde{\alpha}^*].
\]
It then follows that
\[(E \mp) \star \circ J \circ E = (E \mp) \star \circ (J \circ E) = (E \mp) \star \circ p_1 \circ \dot{g}^*,\]  
(137)
\[(E \mp) \star \circ J \circ E \pm = ((E \mp) \star \circ J \star) \circ E \pm = E \pm + \dot{g} \circ p_j \circ E \pm,\]  
(138)
and hence
\[E_{\pm} = (E \mp) \star \circ (E \mp) \star \circ p_1 \circ \dot{g}^* - \dot{g} \circ p_j \circ E \pm.\]  
(139)

Given that \(E = E_+ - E_-\), we then have
\[E = -(E)^* - \dot{g} \circ p_j^* \circ E - (E)^* \circ p_j^* \circ \dot{g}^*,\]  
(140)
which gives the desired conclusion.

We conclude this section by drawing attention to the fact that the kind of gauge fixing that features in a hyperbolization, as discussed in Sec. 3.3.1 is a special kind of partial gauge fixing. We refer to it as purely hyperbolic. Any further gauge fixing conditions are then called residual. We leave the consideration of residual gauge fixing to future work. A principal difficulty in dealing with residual gauge fixing conditions is that the resulting constraints are no longer parametrizable (such as operators that are elliptic on a family of spatial slices). Thus, the kernel of the gauge fixing conditions may contain very few, if any solutions with spacelike compact support, which would be difficult to fit into the current formal framework for tangent and cotangent spaces to the space of solutions.

3.3.3. Formal symplectic structure

Below, we construct a formal symplectic form \(\bar{\Omega}\) using the covariant phase space formalism. That is, we will integrate the presymplectic current density \(\omega\), derived in Sec. 3.1, over a Cauchy surface. Any Cauchy surface would do, giving the same result. The resulting form can in general be degenerate, though, and only becomes symplectic once projected to the gauge invariant formal tangent space \(T_{\phi} \bar{S}\).

**Definition 3.10.** Consider a variational system (Sec. 3.1) with presymplectic form \(\omega \in \Omega^{n-1,2}(F)\) (Appendix A). Suppose that it is hyperbolizable (Sec. 3.3.1) and \(\phi \in \mathcal{S}_H(F)\) is a background solution with good causal behavior (Sec. 3.2.1), so that the linearized equations of motion endow \(M\) with a globally hyperbolic causal structure (Appendix C). Then, given a Cauchy surface \(\Sigma \subset M\), we define the formal *presymplectic* 2-form \(\Omega\) on the formal tangent space \(T_{\phi} \bar{S}\) by the formula
\[\Omega(\psi, \xi) = \int_{\Sigma} \omega[\psi, \xi] = \int_{\Sigma} (J^\infty \phi)^* [j^\xi \dot{\psi} \circ \omega].\]  
(141)

Recall that for any section \(\psi, \xi \in \Gamma(F)\) we can define the prolonged evolutionary vector fields \(\dot{\psi}, \dot{\xi}\) on \(J^\infty F\) (Appendix A), which can be then be contracted with \(\omega \in \Omega^*(J^\infty F)\).

Ideally, we would now show that \(\Omega\) defines a smooth, closed differential form on the possibly infinite dimensional space of solutions \(\mathcal{S}_H(F)\). However, we would then need to make explicit use of the infinite dimensional differential structure...
on $S_H(F)$ and $T S_H(F)$, which we have consistently avoided doing in this review, preferring a formal approach, with minimal analytical details. So instead, we will settle for showing that it is formally smooth and closed. These names are simply place holders for the identities demonstrated in the proof of the following

**Lemma 3.11.** Under the hypotheses of Def. 3.10, the definition of $\Omega$ is independent of the choice of Cauchy surface $\Sigma \subset M$. Moreover, $\Omega$ is formally closed.

**Proof.** First, we note that if $\chi, \xi \in T_\phi \mathcal{S}$ then the integral defining $\Omega$ is necessarily finite, since the integrand $\omega[\chi, \xi] = (j^\infty \phi)^*[\iota^*_\xi \iota^*_\chi \omega]$ has spacelike compact support as both $\chi$ and $\xi$ do. Independence of the choice of $\Sigma$ follows if we can show that $\omega[\chi, \xi]$ is de Rham closed on $M$. This follows directly from the horizontal, on-shell closedness of $\omega$ in the variational bicomplex (Lem. 3.1):

$$d[(j^\infty \phi)^*[\iota^*_\xi \iota^*_\chi \omega]] = (j^\infty \phi)^*[d_h [\iota^*_\xi \iota^*_\chi \omega]] = 0.$$ (142)

For a fixed background solution $\phi \in S_H(F)$, the formal 2-form $\Omega(\chi, \xi)$ is defined as a Cauchy surface integral of a bidifferential operator $\omega[\chi, \xi]$, which is defined by a form $\omega \in \Omega^*(J^\infty F)$. Hence we are happy to declare $\Omega$ to be formally smooth in its dependence on $\phi$, as long as $\omega$ itself is smooth, which it is by construction. Also, in this simple case, we are justified in declaring the formal de Rham differential $\delta$ on $S_H(F)$ to act on $\Omega$ as the vertical differential $d_v$ under the integral sign. Therefore, in this context, it is straightforward to check that $\Omega$ is formally closed since $\omega$ is on-shell, vertically closed (Lem. 3.1):

$$d_v[(j^\infty \phi)^*[\iota^*_\xi \iota^*_\chi \omega]] = (j^\infty \phi)^*[d_v [\iota^*_\xi \iota^*_\chi \omega]] = 0.$$ (143)

This concludes the proof. \hfill $\Box$

Though this was not attempted in Refs. 12–14, their rigorous setting for infinite dimensional geometry can be used to remove the formal character of the above lemma. Also, it is quite clear from the proof that the integration surface $\Sigma$ in the definition of $\Omega$ need not actually be a Cauchy surface. It need only be in the same homology class as a Cauchy surface. In particular, it is enough that $\Sigma$ coincides with some Cauchy surface outside a compact set.

A bilinear form defines a linear map from a vector space to its algebraic dual. A similar statement holds for a continuous bilinear form and the topological dual space. However, our formal cotangent spaces $T^*_\phi \mathcal{S}$ and $T^*_\overline{\phi} \mathcal{S}$ are neither the algebraic

\[\text{The appropriate homology theory here should correspond to a variant of locally finite Borel-Moore homology, where one considers only chains whose intersection with every spacelike compact set is compact. This variant does not appear to have gotten any attention in the literature and thus deserves further study.}\]
nor the topological duals of the formal tangent spaces $T_\phi S$ and $T_\phi S$. Thus we have to check this property for $\Omega$ by hand. This will be accomplished using one of the splitting maps for the Jacobi system from Thm. 3.1, which is analogous Lem. 2.1 for hyperbolic systems. The argument in the proof was inspired by Sec. 3.3 of Ref. 10 and Lem. 3.2.1 of Ref. 64.

**Lemma 3.12.** Provided the constraints are globally parametrizable (Sec. 3.2.2) and the gauge transformations are globally recognizable (Sec. 3.2.3), the presymplectic form $\Omega$ defines the following map from the formal tangent space to the formal cotangent space:

$$\Omega: T_\phi S \to T^*_\phi S$$

$$\psi \mapsto [\tilde{\alpha}^*],$$

with $\tilde{\alpha}^* = J[\chi_\psi] = \pm J[\chi_{\psi}].$

(145)

(146)

(147)

where $J: \Gamma(F) \to \Gamma(\tilde{F}^*)$ the Jacobi differential operator and $\{\chi_{\pm}\}$ is a partition of unity adapted to a Cauchy surface $\Sigma$.

**Proof.** Using the adapted partition of unity, we can write any spacelike compactly supported solution $\psi$ of $\dot{f}[\psi] = 0$ as $\psi = \psi_+ + \psi_-$, with $\psi_{\pm} = \chi_{\pm}\psi$ now being of retarded and advanced supports. If $\psi$ also satisfies the constraints $\dot{c}[f] = 0$, then by Thm. 3.1 it also satisfies the Jacobi equation $J[\psi] = 0$. Hence $J[\psi_+ + \psi_-] = 0$ or $J[\psi_+] = -J[\psi_-] = J[\chi_\psi]$. Note that the support of $J[\psi_{\pm}]$ is compact, since $\psi_{\pm}$ satisfy the Jacobi equation away from the intersection $S^+ \cap S^- \cap \text{supp } \psi$, which is by hypothesis compact.

Next, we want to find a compactly supported dual density $\tilde{\alpha}^*$ that satisfies $\Omega(\xi, \psi) = \langle \xi, \tilde{\alpha}^* \rangle$ for any $\xi \in T_\phi S$, which in particular satisfies $J[\xi] = 0$. Recall that an adapted partition of unity also depends on two additional Cauchy surfaces $\Sigma^\pm \subset I^\pm(\Sigma)$ and the supports of the partition are contained in $\text{supp } \chi_\pm \subseteq S^\pm = I^\pm(\Sigma^\pm)$. 
The following direct calculation helps us identify \( \tilde{\alpha}^* \).

\[
\Omega(\xi, \psi) = \int_\Sigma \omega(\xi, \psi) \alpha = \sum_{\pm} \int_{\Sigma^\pm} (j^\infty \phi)^* \omega(\xi, \psi_{\pm}) \tag{148}
\]

\[
= \sum_{\pm} \int_{\Sigma^\pm} (j^\infty \phi)^* \omega(\xi, \psi_{\pm}) + \sum_{\pm} \int_{S^\pm \cap I^\pm(\Sigma)} d(j^\infty \phi)^* \omega(\xi, \psi_{\pm}) \tag{149}
\]

\[
= \sum_{\pm} \int_{I^\pm(\Sigma)} (j^\infty \phi)^* (d_h \omega)(\xi, \psi_{\pm}) \tag{150}
\]

\[
= \sum_{\pm} \int_{I^\pm(\Sigma)} (j^\infty \phi)^* (-d_v EL_a \wedge d_v u^a)(\xi, \psi_{\pm}) \tag{151}
\]

\[
= \sum_{\pm} \int_{I^\pm(\Sigma)} [(J^I_a b \partial_I \xi^b)_{\pm} - (J^I_a b \partial_I \psi^b)_{\pm}] \tag{152}
\]

\[
= \int_{I^-(\Sigma)} \xi \cdot J[\psi_{\pm}] - \int_{I^+(\Sigma)} \xi \cdot J[\psi_{\pm}] \tag{153}
\]

\[
= \int_{I^-(\Sigma)} \xi \cdot J[\psi] - \int_{I^+(\Sigma)} \xi \cdot J[\psi] \tag{154}
\]

\[
= \int_M \xi \cdot J[\psi] \tag{155}
\]

Note that after the integration by parts, the boundary integrals over \( \Sigma^\pm \) were dropped since they did not intersect the support of their integrands. Then, since \( \text{supp} \psi_{\pm} \subseteq S^\pm \), the integration over \( S^\pm \cap I^\pm(\Sigma) \) was extended to all of \( I^\pm(\Sigma) \). Finally, the term \( \psi_{\pm} \cdot J[\xi] \) was dropped since \( \xi \) is a linearized solution.

To complete the proof, we use the non-degeneracy of the natural pairing between \( T_\phi S \) and \( T^*_\phi S \) (Lem. 3.5, which we can invoke because of the global parametrizability and recognizability hypotheses) to define the operator \( \Omega \) by the formula

\[
\langle \xi, \Omega \psi \rangle = \Omega(\xi, \psi) = \langle \xi, \tilde{\alpha}^* \rangle = (\xi, [\tilde{\alpha}^*]), \tag{156}
\]

so that \( \Omega \psi = [\tilde{\alpha}^*] \in T_\phi^* S \), with \( \tilde{\alpha}^* = J[\psi] \).

**Corollary 3.1.** Provided the constraints are globally parametrizable (Sec. 3.2.2) and the gauge transformations are globally recognizable (Sec. 3.2.3), the 2-form \( \Omega \) on \( T_\phi S \) projects to a 2-form \( \bar{\Omega} \) on \( T^*_\phi \bar{S} \) and hence defines a map

\[
\bar{\Omega}: T_\phi \bar{S} \to T^*_\phi \bar{S} \tag{157}
\]

\[
[\psi] \mapsto [\tilde{\alpha}^*], \tag{158}
\]

with \( \tilde{\alpha}^* = J[\psi] \). \( \Box \)

**Proof.** Notice that in the presence of gauge symmetries (residual gauge freedom is present after a purely hyperbolic gauge fixing) the form \( \Omega \) is degenerate, since
every pure gauge solution lies in its kernel:
\[
\Omega(\dot{g}[\varepsilon], \psi) = \langle \dot{g}[\varepsilon], J[\chi_{\pm} \psi] \rangle = \pm \langle \varepsilon, \dot{g}^* \circ J[\chi_{\pm} \psi] \rangle = 0
\]  
for any \( \psi \), since Noether’s second theorem implies\(^3\) that \( \dot{g}^* \circ J = 0 \). So, the first part is established.

For the second part, recall that we are not interested in the dual density \( \tilde{\alpha}^* = J[\chi] \psi \) specifically, which explicitly depends on the adapted partition of unit \( y \), but rather the equivalence class \( \{ \tilde{\alpha}^* \} \in T^*_\phi \tilde{S} \), which is defined modulo \( \text{im} \ 0^* \) and \( \text{im} \ 0^* \). Equivalently, following a conclusion of Thm. 3.1, since we actually want \( \{ \tilde{\alpha}^* \} \in T^*_\phi \tilde{S} \cong T^*_0 S / \text{im} \ 0, \) it is enough to consider \( \tilde{\alpha}^* \) modulo \( \text{im} J \). Consider another adapted partition of unity \( \{ \chi'_{\pm} \} \). Because each partition of unity provides a splitting map (Thm. 3.1), if we consider equivalence classes of solutions modulo \( \text{im} \ 0^* \), we have \( \{ \psi \} = [E \circ J[\chi] \psi] = [E \circ J[\chi'] \psi] \). Then, \( [E[\chi'] \psi - J[\chi] \psi] = [\psi] - [\psi] = [0] \). So, by exactness of the sequence in Thm. 3.1, \( J[\chi] \psi \) and \( J[\chi'] \psi \) must differ by an element of \( \text{im} J \); in other words, they represent the same equivalence class in \( T^*_\phi \tilde{S} \).

Finally, the projected map \( \bar{\Omega}: T_0 \tilde{S} \to T^*_0 \tilde{S} \) is defined by the formula
\[
\langle \{ \xi \}, \bar{\Omega}[\psi] \rangle = \langle \{ \xi \}, \tilde{\alpha}^* \rangle = \langle \{ \xi \}, [\tilde{\alpha}^*] \rangle,
\]  
with \( \tilde{\alpha}^* = J[\chi] \psi \), which is sufficient because the natural pairing between \( T_0 \tilde{S} \) and \( T^*_0 \tilde{S} \) is non-degenerate (Lem. 3.5, again which we can invoke by the global parametrizability and recognizability hypotheses).

So, formally, the quotient projection to the physical phase space effects a presymplectic reduction \( (\tilde{S}_H(F), \Omega) \to (\bar{S}_H(F), \bar{\Omega}) \). We shall see later on that \( \bar{\Omega} \) is non-degenerate and hence symplectic.

**Remark 3.4.** Note that the relation between \( \bar{\Omega} \) as a bilinear form on the formal tangent space \( T_0 \tilde{S} \) and the linear map \( J[\chi]: T_0 \tilde{S} \to T^*_0 \tilde{S} \) relies on the non-degeneracy of the natural pairing between the formal tangent and cotangent spaces. This non-degeneracy, as proven in Sec. 3.2.5, relies on the cohomological conditions that we call *global parametrizability* and *global recognizability* (Secs. 3.2.2 and 3.2.3). It is clear that, if these conditions fail and hence the natural pairing is degenerate, the form \( \bar{\Omega}(\psi, \xi) = \langle \psi, J[\chi] \xi \rangle \) may be degenerate, even if the operator \( J[\chi] \) is not. This is bound to happen, because, as one of the conclusions of Thm. 3.1, \( J[\chi] \) is invertible under the weaker hypotheses of *local parametrizability* and *local recognizability*. Such a degeneracy has already been noted, for instance, in Refs. 34 and 76.

### 3.3.4. Formal Poisson bivector, Peierls formula

Below, we construct a formal *Poisson bivector* \( \Pi \), using the Peierls formula
\[
\Pi = E,
\]  
where \( E \) is again the causal Green function of the Jacobi operator \( J \) as defined in Sec. 3.3.1. To show that \( \Pi \) is indeed a Poisson bivector, it suffices to show that (a)
it is an antisymmetric bilinear form on the formal cotangent space, (b) it defines a map from the formal cotangent space to the formal tangent space and (c) it is a two-sided inverse of \( \bar{\Omega} \) defined in Cor. 3.1. We actually postpone part (c) to Sec. 3.3. The fact that \( \Pi \) defines a Poisson bracket, with its Leibniz and Jacobi identities, then formally follows from standard arguments.

**Lemma 3.13.** The Peierls formula specifies a map from the formal cotangent space to the formal tangent space:

\[
\Pi : T^*_{\bar{\phi}} \bar{\mathcal{S}} \to T_{\phi} \mathcal{S}
\]

\[
[\tilde{\alpha}^*] \mapsto [\psi], \quad \text{with} \quad \dot{\psi}^*[\tilde{\alpha}^*] = 0
\]

and \( \psi = E[\tilde{\alpha}^*] \).

**Proof.** The challenge is to show that \( \Pi \) maps equivalence classes to equivalence classes (Def. 3.7). That is, that any representative \( \tilde{\alpha}^* + \dot{f}^*[\xi] + \dot{c}_g^*[\gamma^*] \) of an equivalence class \( [\tilde{\alpha}^*] \in T^*_{\bar{\phi}} \bar{\mathcal{S}} \), with \( \dot{\psi}^*[\tilde{\alpha}^*] = 0 \), gets mapped to the same equivalence class in \( T_{\phi} \mathcal{S} \).

By linearity, it suffices to check that \( [0] \in T^*_{\bar{\phi}} \bar{\mathcal{S}} \) is mapped to \( [0] \in T_{\phi} \mathcal{S} \). Recall that any solution representing \( [0] \in T_{\phi} \mathcal{S} \) is pure gauge. We have used the identity that \( E \circ \dot{\psi}^*[\tilde{\alpha}^*] = 0 \), gets mapped to the same equivalence class in \( T_{\phi} \mathcal{S} \). Note that the equivalence (76) of the \( (\dot{f} + \dot{c}, \bar{F}^* \oplus E) \) and \( (J \oplus \dot{c}_g, \bar{F}^* \oplus E_g) \) equation forms, together with the self-adjointness of the Jacobi operator \( J^* = J \), allows us to rewrite any representative of \( [0] \in T^*_{\bar{\phi}} \bar{\mathcal{S}} \) as \( J[\xi] + \dot{c}_g^*[\gamma^*] \), for some \( \xi \in \Gamma_0(F) \) and \( \gamma^* \in \Gamma_0(\bar{E}_g^*) \). This representative will also satisfy the identity

\[
\dot{\psi}^* \circ \dot{c}_g^*[\gamma^*] = \dot{\psi}^*[J[\xi] + \dot{c}_g^*[\gamma^*]] = 0.
\]

Direct calculation then shows that

\[
\Pi[J[\xi] + \dot{c}_g^*[\gamma^*]] = E \circ J[\xi] + E \circ \dot{c}_g^*[\gamma^*]
\]

\[
= \dot{\psi}^*[\varepsilon] - (\dot{c}_g \circ \dot{c}_g^*)([\gamma^*]) - \dot{\psi} \circ p_g^* \circ E \circ \dot{c}_g^*[\gamma^*]
\]

\[
- (E)^* \circ p_g^*[\dot{\psi}^* \circ \dot{c}_g^*[\gamma^*]]
\]

\[
= \dot{\psi}^*[\varepsilon] - q_g^* \circ (H)^* \circ r_g^*([\gamma^*]) - p_g^* \circ E \circ \dot{c}_g^*[\gamma^*]
\]

is pure gauge. We have used the identity that \( E \circ J[\xi] = \dot{\psi}^*[\varepsilon] \) for some \( \varepsilon \in \Gamma_0(P) \) (Thm. 3.1), the anti-self-adjointness identity (140), that (Eqs. (76) and (81))

\[
\dot{c}_g \circ E = r_g \circ (\dot{c} \circ G) \circ \bar{r} = r_g \circ (H \circ \dot{q} \circ \bar{r})
\]

\[
= (r_g \circ H \circ q_1) \circ \dot{\psi}^*
\]

and the identity (166).

Therefore, we can conclude that if \( [\tilde{\alpha}^*] = [0] \), then \( [E[\tilde{\alpha}^*]] = [0] \).

**Lemma 3.14.** The Peierls formula defines an antisymmetric bilinear form on the formal cotangent space:

\[
\Pi([\tilde{\alpha}^*], [\tilde{\beta}^*]) = \langle \Pi[\tilde{\alpha}^*], [\tilde{\beta}^*] \rangle = -\Pi([\tilde{\beta}^*], [\tilde{\alpha}^*]),
\]

for any \( [\tilde{\alpha}^*], [\tilde{\beta}^*] \in T^*_{\bar{\phi}} \bar{\mathcal{S}} \).
Proof. Recall that the representatives always satisfy $\dot{g}^\ast [\tilde{\alpha}^\ast] = \dot{g}^\ast [\tilde{\beta}^\ast] = 0$. Appealing directly to the anti-self-adjointness identity (140) we have

$$\Pi([\tilde{\alpha}^\ast], [\tilde{\beta}^\ast]) = \langle \Pi[\tilde{\alpha}^\ast], [\tilde{\beta}^\ast] \rangle = \langle E[\tilde{\alpha}^\ast], \tilde{\beta}^\ast \rangle = \langle (E)^\ast [\tilde{\beta}^\ast], \tilde{\alpha}^\ast \rangle$$

(173)

$$= -\langle (E[\tilde{\beta}^\ast] + \dot{g} \circ p_J \circ E[\tilde{\beta}^\ast] + (E)^\ast \circ p_J \circ \dot{g}^\ast [\tilde{\beta}^\ast]), \tilde{\alpha}^\ast \rangle$$

(174)

$$= -\langle [E[\tilde{\beta}^\ast]], [\tilde{\alpha}^\ast] \rangle = -(\Pi[\tilde{\beta}^\ast], [\tilde{\alpha}^\ast])$$

(175)

$$= -\Pi([\tilde{\beta}^\ast], [\tilde{\alpha}^\ast]).$$

(176)

3.3.5. The Peierls formula inverts the covariant symplectic form

Below, in Thm. 3.2, we state and prove the main result of this section, that $\Pi = \bar{\Omega}^{-1}$. It is worth pausing here and recalling the various hypotheses, assumptions, and intermediate results that have lead up to it.

First of all, the result itself is not completely new. On the one hand, Peierls’ original paper\(^6\) already outlined an argument for the equivalence of his proposed bracket and the standard Poisson bracket of the Hamiltonian formalism, defined with respect to a preferred time foliation. On the other hand, when the covariant phase space formalism was introduced, the use of the symplectic current density\(^3\)–\(^5\) was justified by its agreement with the standard symplectic structure of the Hamiltonian formalism. These two observations were joined into a detailed argument by Barnich, Henneaux and Schomblond,\(^8\) which covered the case when the Hamiltonian formalism includes first class (gauge) and second class constraints.

Note that both the covariant phase space and Peierls bracket formalisms are fully covariant, but their equivalence had only been demonstrated using a non-covariant Hamiltonian formalism as an intermediate step. So, one reason to look for improvements is the desire to make the argument covariant throughout and bypass the Hamiltonian formalism all together. Another reason is to make clear all mathematical assumptions necessary to make the intermediate constructions well defined. In particular, the existence of advanced and retarded Green functions, needed by the Peierls formula, is guaranteed by standard mathematical results in PDE theory only if the field theory equations of motion (the Euler-Lagrange equations) satisfy some local and global hyperbolicity\(^c\) requirements. We have exhibited these assumptions bundled within the notions of hyperbolizability (Def. 3.8), a global causal condition generalizing global hyperbolicity (Appendix C), as well as global parametrizability and recognizability (Secs. 3.2.2 and 3.2.3). Also, the formal presymplectic form (Def. 3.10) is defined only when the integral over the presymplectic current converges. Again, a sufficient condition for this integral to converge is to restrict the support of linearized solutions plugged into the presymplectic form to be spacelike

\(^c\)In the spirit of being inclusive, we have equated our basic notion of hyperbolicity precisely with the existence of retarded and advanced Green functions (Green hyperbolicity). However, as pointed out earlier, there are large classes of PDEs easily identifiable by their principal symbols (including wave-like and symmetric hyperbolic systems) that are well known to be Green hyperbolic.
compact. This restriction is the main reason for defining the formal tangent spaces to consist of field sections of spacelike compact support (Secs. 3.2.4, 3.2.5).

The main technical result leading up to the theorem below is of course Thm. 3.1, which reduces to Prp. 2.1 when the Euler-Lagrange equations are directly in hyperbolic form (gauge invariance and constraints are absent). The exactness of parts of the horizontal sequence (100) (after taking the vertical cohomologies) can be seen as a precise characterization of the kernel and cokernel of the causal Green function $E$, defined in Eq. (99). It is this characterization that is the main motivation behind defining the formal cotangent spaces to consist of dual densities of compact support (Secs. 3.2.4, 3.2.5). If these support restrictions were relaxed, for instance, to time-like compact support for dual densities, then the causal Green function $E$ need not be invertible due to global Aharonov-Bohm type effects. In that case, the relation of the Peierls formula to the presymplectic form must be more subtle. The final technical result that is used in the proof below is the non-degeneracy of the natural pairing between the formal tangent and cotangent spaces (Sec. 3.2.5), which rely on rather technical sufficient conditions that we have dubbed global parametrizability of constraints (Sec. 3.2.2) and global recognizability of gauge transformations (Sec. 3.2.3). If they fail, then the symplectic form $\Omega$ is in fact degenerate. However, as can be seen from its representation in Cor. 3.1, that is not because $J_{\chi} : T_{\phi}\tilde{S} \to T_{\phi}\tilde{S}$ becomes non-invertible, but because the natural pairing $\langle - , - \rangle$ becomes degenerate. Recall that, according to Thm. 3.1, $J_{\chi}$ remains invertible even when only local parametrizability and recognizability hold.

The proof given below was inspired by Sec. 3.3 of Ref. 10 as well as the exact sequence of Prp. 2.1 (see references near its statement), though similar ideas can already be found in Lem. 3.2.1 of Ref. 64. The main limitation of the argument given by Forger & Romero is that it only treats the case when Euler-Lagrange equations are already in hyperbolic form. Our argument is generalized to the case where a hyperbolization may be required, and constraints and gauge may be present. Due to the more complicated hypothesis, the argument itself has been fine grained and split into multiple steps. Also, we show that $\Omega$ and $\Pi$ are two-sided (as opposed to one-sided) inverses of each other. The technical content of the proof of the main Thm. 3 of Ref. 10 is split between our Lem. 3.12 and Cor. 3.1 (rewriting the formal symplectic form), Thm. 3.1 (two-sided inversion), and Lem. 3.5 (natural pairing non-degeneracy).

**Theorem 3.2.** Global parametrizability (Sec. 3.2.2) and global recognizability (Sec. 3.2.3) conditions hold, the Peierls formula gives a two-sided inverse to the formal symplectic form, $\Omega \Pi = \text{id}$ on $T_{\phi}\tilde{S}$ and $\Pi \Omega = \text{id}$ on $T_{\phi}\tilde{S}$.

**Proof.** The proof uses in an essential way the splitting identities of Thm. 3.1. Consider any $[\psi] \in T_{\phi}\tilde{S}$ and $[\alpha] \in T_{\phi}\tilde{S}$. To use these splitting identities, we introduce a Cauchy surface $\Sigma \subset M$ and a partition of unity $\{\chi_{\pm}\}$ adapted to it.
Then
\[
\langle \Pi \bar{\Omega} [\psi], [\tilde{\alpha}^*] \rangle = \langle E \circ J_\chi [\psi], \tilde{\alpha}^* \rangle \quad \text{(using Eq. (102))}
\]
\[
= \langle \psi + \dot{g}[\varepsilon], \tilde{\alpha}^* \rangle \quad \text{(for some } \varepsilon \in \Gamma_{SC}(P))
\]
\[
= \langle [\psi], [\tilde{\alpha}^*] \rangle.
\]
Therefore, from the non-degeneracy of the natural pairing between \(T_\phi \bar{S}\) and \(T_{\phi^*} \bar{S}\) (Lem. 3.5), we concluded that \(\Pi \bar{\Omega} = \text{id}\). Similarly, we have
\[
\langle [\psi], \bar{\Omega} \Pi [\tilde{\alpha}^*] \rangle = \langle \psi, J_\chi \circ E [\tilde{\alpha}^*] \rangle.
\]
But then
\[
E [J_\chi \circ E [\tilde{\alpha}^*] - \tilde{\alpha}^*] = (E \circ J_\chi) \circ E [\tilde{\alpha}^*] - E [\tilde{\alpha}^*] = \dot{g}[\varepsilon],
\]
for some \(\varepsilon \in \Gamma_{SC}(P)\). But, by the exactness (after taking vertical cohomologies) of the horizontal sequence (100) of Thm. 3.1 at \(\Gamma_0(F)\), this means that \(J_\chi \circ E [\tilde{\alpha}^*] - \tilde{\alpha}^* = J[\xi]\) for some \(\xi \in \Gamma_0(F)\). In other words,
\[
\langle [\psi], \bar{\Omega} \Pi [\tilde{\alpha}^*] \rangle = \langle \psi, \tilde{\alpha}^* + J[\xi] \rangle = \langle [\psi], [\tilde{\alpha}^*] \rangle.
\]
Therefore, from the non-degeneracy of the natural pairing between \(T_\phi \bar{S}\) and \(T_{\phi^*} \bar{S}\), we concluded that \(\bar{\Omega} \Pi = \text{id}\).

It is interesting to note that the construction of the Poisson bivector \(\Pi\) via the Peierls formula requires gauge fixing the equations of motion. On the other hand, the construction of the symplectic form \(\bar{\Omega}\) does not. Since, after gauge reduction, the two are mutual inverses, the Poisson bivector on the gauge invariant solutions space ultimately does not depend on gauge fixing. There is another way to see that result. For recognizable gauge transformations, the gauge invariant field combination \(\xi = \dot{g}'[\psi]\) of a solution \(\psi \in \Gamma(F)\) of \(\dot{f} [\psi] = 0\) itself satisfies the hyperbolic PDE system \(\dot{k}'[\xi] = \dot{s}' \circ \dot{f} [\psi] = 0\). On the other hand, the equivalence formulas (76) and the gauge fixing compatibility condition (83) imply that the same is true even if only \(J[\psi] = 0\). In other words, the system \(\dot{k}'[\xi] = 0\) depends on \(J\) and \(\dot{g}'\) but not on the choice of gauge fixing. This is the case, for example, for Maxwell electrodynamics, where Maxwell’s equations for the gauge invariant field strength \(F = F[A]\), where \(A\) is the gauge variant vector potential, by themselves constitute a (constrained) hyperbolic system. It is then not surprising that we can express the Poisson bivector acting on gauge invariant observables formed with respect to the gauge invariant field combinations directly in terms of the causal Green for the \(\dot{k}'\) PDE system. This observation was known already to Peierls and this example of Maxwell electrodynamics appeared in his original paper. Of course, if appropriate cohomologies in diagram (51) do not vanish, there may be gauge invariant observables not of that form, for which the gauge fixed Peierls Green function \(E\) would be necessary.
Corollary 3.2. Given two gauge invariant dual densities of the form \( \tilde{\alpha}^* = \dot{g}'^* [\tilde{\alpha}^*] \) and \( \dot{\tilde{\alpha}}^* = \dot{g}'^* [\dot{\tilde{\alpha}}^*] \), with \( \tilde{\alpha}^*, \dot{\tilde{\alpha}}^* \in \Gamma_0(\tilde{F}^*) \) and \( \tilde{\beta}^*, \dot{\tilde{\beta}}^* \in \Gamma_0(\dot{F}^*) \), we have the following identity
\[
\langle E[\tilde{\alpha}^*], \dot{\tilde{\beta}}^* \rangle = \langle K' \circ \dot{r}' [\tilde{\alpha}^*], \dot{\tilde{\beta}}^* \rangle,
\]
where \( \dot{r}' = s' \circ \dot{r} \circ \dot{g}^* \).

Proof. Direct calculation shows
\[
\langle E[\tilde{\alpha}^*], \dot{\tilde{\beta}}^* \rangle = \langle G \circ \dot{r} \circ \dot{g}^* [\tilde{\alpha}^*], \dot{g}'^* [\dot{\tilde{\beta}}^*] \rangle
= \langle (\dot{g}' \circ G) \circ \dot{r} \circ \dot{g}^* [\tilde{\alpha}^*], \dot{\tilde{\beta}}^* \rangle
= \langle K' \circ (s' \circ \dot{r} \circ \dot{g}^*) [\tilde{\alpha}^*], \dot{\tilde{\beta}}^* \rangle,
\]
which concludes the proof.

We conclude with a simple corollary that is sometimes known as classical microcausality.

Corollary 3.3. Consider two on-shell gauge invariant dual density classes \( [\tilde{\alpha}^*], [\dot{\tilde{\beta}}^*] \in T^*_\phi \bar{S} \) whose supports are spacelike separated, then
\[
\Pi([\tilde{\alpha}^*], [\dot{\tilde{\beta}}^*]) = 0.
\]

Proof. Picking representatives \( \tilde{\alpha}^*, \dot{\tilde{\beta}}^* \in \Gamma_0(\tilde{F}^*) \) with genuinely spacelike separated supports and using the definition of the Poisson bivector, we have \( \Pi([\tilde{\alpha}^*], [\dot{\tilde{\beta}}^*]) = \langle \tilde{\alpha}^*, G[\dot{r} [\dot{\tilde{\beta}}^*]] \rangle = 0 \). This is obvious because \( \text{supp } G[\dot{r} [\dot{\tilde{\beta}}^*]] \subseteq I(\text{supp } \dot{\tilde{\beta}}^*) \), by the properties of the causal Green function \( G \). Hence, the arguments in the natural pairing have non-overlapping supports and give zero.

4. Examples

In this section, we give a few examples of common relativistic field theories and show how they fit into the framework presented in this review. In particular, we make explicit the various identities needed to show that they are hyperbolizable according to Def. 3.8. We freely use the notation introduced in Sec. 3.3.1. In all the examples, we will concentrate on linear theories or the linearizations of non-linear ones, as discussed in Sec. 3.2.1.

4.1. Scalar field

The field bundle \( F = M \times \mathbb{R} \) is the trivial \( \mathbb{R} \)-bundle. The Lagrangian density is
\[
\mathcal{L}[\phi] = -\frac{1}{2} \sqrt{-g} [g^{ij} (\partial_i \phi)(\partial_j \phi) + V(\phi)] d\tilde{x},
\]
where we used coordinates \( (x^i) \) on \( M \),

\[\text{supp } \tilde{\alpha}^* \cap I(\text{supp } \dot{\tilde{\beta}}^*) = \emptyset \text{ and supp } \dot{\tilde{\beta}}^* \cap I(\text{supp } \tilde{\alpha}^*) = \emptyset.\]
$g$ is a globally hyperbolic Lorentzian metric on $M$, and $|g| = \det g_{ij}$. The Jacobi equations
\begin{equation}
J[\psi] = \left( \partial_i (\sqrt{-|g|} g^{ij} \partial_j \psi) - 2 \sqrt{-|g|} V'(\phi) \psi \right) \, d\tilde{x} \tag{188}
\end{equation}
have a wave-like principal symbol and, given the global hyperbolicity of the metric, are well known to be Green hyperbolic,\textsuperscript{58, 63} so $\dot{f} = J$. The constraints and the gauge transformations are trivial, $\dot{c} = 0$ and $\dot{g} = 0$. Note that, for the existence of Green functions, no constraints need to be imposed on $V'(\phi)$ beyond smoothness, so tachyonic theories and theories with variable mass are hyperbolizable as well.

A more detailed treatment can be found for instance in Ref. 10.

### 4.2. Maxwell $p$-form

The field bundle $F = \Lambda^p M$ is the bundle of differential $p$-forms ($p > 0$). The Lagrangian is the generalization of the Maxwell Lagrangian density $L[\phi] = -\frac{1}{4} d\phi \wedge * d\phi$, where $*$ is the Hodge star with respect to a globally hyperbolic metric $g$ on $M$. Below, we identify the densitized dual bundle of the bundle of $p$-forms with $(n-p)$-forms via the pairing formula $\psi \cdot \tilde{\alpha} = \psi \wedge \tilde{\alpha}^*$. The Jacobi equations are
\begin{equation}
J[\psi] = \frac{1}{2} * d \delta \psi, \tag{189}
\end{equation}
where $\delta = * d *$ is the de Rham co-differential. They are invariant under gauge transformations with generator $\dot{g}[\varepsilon] = d \varepsilon$, where the gauge parameter bundle is $P = \Lambda^{p-1} M$. The Lorenz gauge plays the role of a purely hyperbolic gauge fixing, $c_g[\psi] = \delta \psi$. The equivalent constrained hyperbolic system is $\dot{f}[\psi] = * \Box \psi$, $\dot{c} = \delta \psi$; the equivalence is effected by the operators $\bar{r} = 2 \text{id}$, $\bar{r}_c = * d$, $\bar{r}_g = 0$ and $\bar{r}_q = \text{id}$. The operator $\Box = (\delta d + d \delta)$ is the Laplace-Beltrami operator (also known as the $p$-form d’Alambertian) and is well known, again when the metric $g$ is globally hyperbolic,\textsuperscript{58, 63} to be Green hyperbolic.\textsuperscript{58, 63} The parametrizability (diagram (47)) and recognizability (diagram (50)) identities are generated by the following commutative diagrams (which hold possibly up to sign factors):

\begin{equation}
\begin{array}{c}
\Gamma(\Lambda^{p+1} M) \xrightarrow{\delta} \Gamma(\Lambda^p M) \xrightarrow{\dot{c} = \delta} \Gamma(\Lambda^{p-1} M) \\
\downarrow s \Box \downarrow s \Box \downarrow s \Box,
\end{array} \tag{190}
\end{equation}

\begin{equation}
\begin{array}{c}
\Gamma(\Lambda^{n-p-1} M) \xrightarrow{d} \Gamma(\Lambda^{n-p} M) \xrightarrow{d} \Gamma(\Lambda^{n-p+1} M) \\
\Gamma(\Lambda^{p-1} M) \xrightarrow{\dot{g} = d} \Gamma(\Lambda^{p} M) \xrightarrow{d} \Gamma(\Lambda^{p-1} M) \\
\Gamma(\Lambda^{n-p} M) \xrightarrow{\delta} \Gamma(\Lambda^{n-p+1} M) \xrightarrow{\delta} \Gamma(\Lambda^{n-p} M)
\end{array} \tag{191}
\end{equation}

The gauge fixing compatibility condition (83) is obviously satisfied as $\delta \circ * d \circ \delta = * d^2 \delta = 0$, up to sign.

A more detailed treatment can be found for instance in Refs. 76 and 34.
4.3. Proca field

The field bundle $F = T^* M$ is the bundle of 1-forms and the Lagrangian density differs from the Maxwell one by a mass term, $\mathcal{L}[\phi] = -\frac{1}{4}d\phi \wedge *d\phi - \frac{1}{2}m^2 \phi \wedge *\phi$, where $*$ is the Hodge star with respect to a globally hyperbolic metric $g$ on $M$. The Jacobi equations are

$$J[\psi] = \frac{1}{2} * (\delta d\psi - m^2 \psi),$$

where $\delta = *d*$ is the de Rham co-differential. There is no gauge invariance, $\dot{g} = 0$, but there are integrability conditions. The equivalent constrained hyperbolic system is $\ddot{f}[\psi] = * (\Box - m^2) \psi$, $\dot{c}[\psi] = \delta \psi$; the equivalence is effected by the operators $\bar{r} = 2 \text{id}$, $\bar{r}_c = -\frac{2}{m^2} \delta d$, $\bar{r}_J = -\frac{2}{m^2} *d$, and $\bar{r}_g = 0$. Again, $\Box = (\delta d + d\delta)$ is the Laplace-Beltrami operator, which is known to be Green hyperbolic when the metric $g$ is globally hyperbolic. The parametrizability diagram (193) identities are generated by the following commutative diagram (which holds possibly up to sign factors):

$$\Gamma(\Lambda^{p+1}M) \xrightarrow{\delta} \Gamma(\Lambda^p M) \xrightarrow{\varepsilon = \delta} \Gamma(\Lambda^{p-1} M)$$

$$\Gamma(\Lambda^{n-p} M) \xrightarrow{d} \Gamma(\Lambda^{n-p} M) \xrightarrow{\delta} \Gamma(\Lambda^{n-p+1} M).$$

A more detailed treatment can be found for instance in Ref. 77.

4.4. Graviton

The field bundle $F = S^2 T^* M$ is the bundle of symmetric, rank-2 covariant tensors and the Lagrangian density is the usual Einstein-Hilbert action $\mathcal{L}[\phi] = *\phi(R[\phi] - 2\Lambda)$, where we of course interpret $\phi$ as a Lorentzian metric, $*\phi$ is the corresponding Hodge star operator, $R[\phi]$ the corresponding Ricci scalar and $\Lambda$ the cosmological constant. 78 We will of course use $\nabla$ to denote the covariant derivative compatible with $\phi$. Consider a background solution $\phi \in S_H(F)$, whose good causal behavior property we take to coincide with the usual notion of Lorentzian global hyperbolicity. Let us denote the corresponding volume form as $*\phi_1 = *\phi \phi_1$. Here it convenient to identify any tensor bundle with its densitized dual, with the natural pairing $(\langle - , - \rangle)$ constructed by contracting corresponding indices using the metric $\phi$ or its inverse, multiplying by the volume form $*\phi_1$ and integrating over $M$.

The Jacobi equations are $J[\psi] = L[\psi]$, where $L$ also known as the Lichnerowicz operator. In local coordinates $(x^i)$ on $M$, the components of the Lichnerowicz operator are

$$L_{ij}[\psi] = -\frac{1}{2} \phi_{ij} (\nabla^k \nabla^i \psi_{kl} - \Box \psi - \Lambda \psi) - \Box \psi_{ij} - \Lambda \psi_{ij} - \frac{1}{2} \nabla_i \nabla_j \psi + \nabla^k \nabla (\psi_{ij})_k,$$

where indices are raised and lowered using $\phi$, $\psi = \phi^{kl} \psi_{kl}$ and $\Box = \nabla^k \nabla_k$ is the tensor d’Alambertian.
Before proceeding, we introduce some key linear differential operators and their adjoints. To start, the trace reversal operator $\rho: \Gamma(S^2T^*M) \to \Gamma(S^2T^*M)$ does not actually involve any derivatives and in components is given by $\rho_{ij}[\psi] = \psi_{ij} - \frac{1}{2} \phi_{ij} \psi$. With our conventions, it is self-adjoint, $\rho^* = \rho$, and also idempotent, $\rho \circ \rho = \text{id}$. Given, a 1-form $\nu$, we define $K_{ij}[\nu] = \nabla_i v_j$ and call $K: \Gamma(T^*M) \to \Gamma(S^2T^*M)$ the Killing operator. Its adjoint, $K^*_\gamma[\psi] = -\nabla^i \psi_{ij}$ (recall our identification of each tensor bundle with its own densitized dual) is the divergence operator on symmetric covariant 2-tensors, $K^*: \Gamma(S^2T^*M) \to \Gamma(T^*M)$. Another important operator is the linearized Riemann curvature (cf. Sec. 7.5 of Ref. 78). If $\phi' = \phi + \lambda \psi$, the components of the Riemann tensor of $\phi'$ are given by $R_{ijkl}[\phi'] = R_{ijkl} + \lambda \hat{R}_{ijkl}[\psi] + O(\lambda^2)$, where $R_{ijkl}$ is the Riemann tensor of $\phi$ and

$$
\hat{R}_{ijkl}[\psi] = -2\nabla_i[\psi_{jk}][l,k] + R_{ij[k}^m \psi_{j]m},
$$

(195)

with the usual notation $(-)_i = \nabla_i(-)$, is the linearized Riemann curvature operator $\Gamma(S^2T^*M) \to \Gamma(RM)$ and $RM \to M$ is the sub-bundle of $(T^*) \otimes 4M$ that satisfies the algebraic symmetries of the Riemann tensor. Its adjoint operator $\hat{R}^*: \Gamma(RM) \to \Gamma(S^2T^*M)$ is then

$$
\hat{R}^*_{ij}[\xi] = 2\nabla^l \xi_{k(ij)} l - \hat{R}^k l m (i\xi_j) m k l.
$$

(196)

Finally, we define the following self-adjoint hyperbolic differential operator

$$
W_{ij}[\psi] = \square \psi_{ij} - 2R^k_{ij} \psi_{kl},
$$

(197)

with $W: \Gamma(S^2T^*M) \to \Gamma(S^2T^*M)$. Note that $W$ has a wave-like principal symbol so it is known to be Green hyperbolic.58, 63, 79

The Jacobi equations are invariant under gauge transformations (linearized diffeomorphisms) with generator $\delta [v] = K [v]$, the Killing operator, where the gauge parameter bundle is $P = T^*M$. The de Donder gauge plays the role of a purely hyperbolic gauge fixing $\dot{c}_g = K^* \circ \rho$, or $(c_g)_{ij}[\psi] = \nabla^l \rho [\psi]_{ij}$ in coordinate components. The equivalent constrained hyperbolic system is $\hat{f}[\psi] = W[\psi], \hat{c}[\psi] = K^* \circ \rho [\psi]$. The equivalence is effected by the operators $\bar{\rho} = -2\rho, \bar{r}_c = 2K, \bar{r}_f = 0$ and $\bar{r}_g = \text{id}$. The parametrizability (diagram 47) and recognizability (diagram 50) identities are generated by the following commutative diagrams:

$$
\begin{array}{ccc}
\Gamma(RM) & \xrightarrow{\rho \circ K^{\ast \rho}} & \Gamma(S^2T^*M) \\
\text{W} & \downarrow & \text{W} \downarrow \square + \Lambda, \\
\Gamma(RM) & \xrightarrow{\rho \circ K^{\ast \rho}} & \Gamma(S^2T^*M)
\end{array}
$$

(198)

$$
\begin{array}{ccc}
\Gamma(T^*M) & \xrightarrow{K \circ \rho} & \Gamma(S^2T^*M) \\
\text{W} & \downarrow & \text{W} \\
\Gamma(T^*M) & \xrightarrow{K \circ \rho} & \Gamma(RM)
\end{array}
$$

(199)
We do not give explicit general expressions for the operators $K'$ and $W'$, simply because they do not seem to be available in the literature. On the other hand, they must exist for abstract reasons. Namely, if we define $K'$ as differential operator extending $K$ to a formally exact sequence, then it always exists, as mentioned in Appendix B.2. Further, the composition of operators

$$K' \circ W = K' \circ (-2\rho \circ L + 2K \circ \rho \circ \rho) = -2K' \circ \rho \circ L$$  \hspace{1cm} (200)$$

clearly annihilates the image of the Killing operator $K$, due to the gauge invariance of $L$. Therefore, by Lem. 2.3, there must exist a factorization $K' \circ W = W' \circ K'$, which we will conjecture to have a wave-like principal symbol (like $W$ and $\Box + \Lambda$ do) and hence be Green hyperbolic.

If we assume that the background metric tensor $\phi$ to have constant curvature, we can say much more. In particular we know that $K' = \dot{R}$ and that the composition $\dot{R} \circ K = 0$ forms part of a larger elliptic complex, in many ways analogous to the de Rham complex. In the even more special case of zero curvature (and hence also $\Lambda = 0$), all operators can be expressed with constant coefficients in local inertial coordinate systems formal, which makes it easier to check formal exactness directly. Also, in that case all the hyperbolic operators become equal to $\Box$, the wave operator.

A more detailed treatment of the quantization of the graviton field on arbitrary cosmological vacuum backgrounds can be found in Ref. 79, though without introducing operators analogous to $K'$ and $W'$.

5. Discussion

We have reviewed in detail the covariant phase space formalism and the Peierls formula, which endow the space of solutions of a classical field theory, respectively, with symplectic and Poisson structures, thus giving it the structure of a phase space, well known to be equivalent to the canonical phase space. Each of these constructions is covariant and does not require the non-covariant, canonical Hamiltonian formalism as an intermediary. In distinction with much of the existing literature, where the following aspects have often been left implicit, we have spelled out precise conditions under which these constructions succeed without mathematical ambiguities or difficulties. While it has long been known that the resulting symplectic and Poisson structures are equivalent (the symplectic form and the Poisson bivector are mutual inverses), despite the covariant construction, existing proofs still required the canonical Hamiltonian formalism as an intermediary. The main result in our presentation, which also happens to be novel, is a detailed and completely covariant proof of the equivalence under a precise set of sufficient conditions. The proof follows the ideas of the previous work of Forger & Romero, but is generalized to field theories more general than scalar fields. Our argument holds for theories that also include constraints and that may have gauge symmetries. The list of examples to which the argument is applicable includes essentially all relativistic field theories of physical interest.
Despite the fact that the phase space of a field theory in more than one spacetime dimension (which corresponds to ordinary mechanical systems) is infinite dimensional, we have systematically avoided a discussion of functional analytical details needed in a theory of infinite dimensional geometry. Instead, we have treated formally the minimal geometric details needed in our presentation. Essentially, we have restricted our discussion to linear PDEs (or rather, linearizations of non-linear ones) and their solution spaces by appealing to the fact that the inversion of a symplectic form or a Poisson bivector requires only the tangent or cotangent space at a single point of the phase space (a background solution). However, the precise algebraic and differential geometric identities given here can be used as a core in a future investigation that would fill in the missing functional analytic details. In fact, some attempts along these lines have already been made elsewhere. For instance, Ref. 14 has done precisely that but only for the more restrictive class of scalar field theories. On the other hand, Refs. 12, 13 have considered more general theories, including those with gauge theories. Incidentally, these references have concentrated on the so-called off-shell formalism and, while heavily relying on the Peierls formula, did not consider its relation to the corresponding covariant symplectic structure, which requires restriction to solutions to be well defined.

The sufficient conditions we have introduced for the Peierls inversion formula to hold, the (global) parametrizability of constraints and the (global) recognizability or gauge transformations, have two aspects. See Remarks 3.3 and 3.4 regarding the subtle interplay between these conditions and the hypotheses that are sufficient to establish non-degeneracy of symplectic and Poisson structures described in this review. The local version is expected to hold generically for relativistic field theories of physical interest, as illustrated by the examples of Sec. 4. The global version, on the other hand has a cohomological character and it is actually known to fail in spacetimes with certain topological properties.34,76 The main examples of these problematic cases have come from studying Maxwell electrodynamics on spacetimes with non-trivial spatial topology.34,76 It would be nice to identify more key examples and study their properties. This would require the computation of cohomologies of the de Rham and other formally exact complexes with causally restricted supports (e.g., advanced, retarded, spacelike compact, timelike compact). The techniques needed for such computations go a bit beyond the standard treatments of de Rham cohomology with unrestricted or compact supports, as presented in standard differential geometry and differential topology texts. They will be addressed elsewhere.81

More generally, compact or spacelike compact supports, featuring in the sufficient conditions discussed above, may be too restrictive for physical purposes, for example when dealing with infrared issues on spatially non-compact spacetimes. In those cases, the solution, of course, is to introduce boundary conditions at infinity. However, as is well known, there may not always be a uniquely preferred set of boundary conditions. In fact, boundary conditions are expected to be dictated by detailed physical considerations, which may vary from problem to problem. The
main difficulty in relaxing the spacelike compact support condition on linearized solutions is the divergence of the integral in the definition of the covariant symplectic form, Def. 3.10. This situation is reminiscent of the problem of extending unbounded, symmetric operators on a Hilbert space to larger domains, while maintaining their self-adjointness. Perhaps a similar approach can be applied to the symplectic form, where its anti-symmetry would replace the self-adjointness condition, can be used to study the space of possible boundary conditions at infinity. Notably, an attempt in a direction implicitly similar to this suggestion can be found in Sec. 5.1 of Ref. 56. These ideas will be explored further in future work.

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Appendix A. Jet bundles and the variational bicomplex

In this appendix, we briefly introduce jet bundles and fix the relevant notation. For simplicity, we restrict ourselves to fields taking values in vector bundles. However, the discussion could be straightforwardly generalized to general smooth bundles. More details, as well as a coordinate independent definition, can be found in the standard literature.83–85

Fix a vector bundle $F \rightarrow M$, with $\dim M = n$, with fibers modeled on a vector space $U$, and consider an adapted coordinate patch $\mathbb{R}^n \times U$, with coordinates $(x^i, u^a)$. Extend this patch to a $k$-jet patch $\mathbb{R}^n \times U \times U^m$ by adding extra copies of $U$, with new coordinates $(x^i, u^a, u^a_{i_1}, \ldots, u^a_{i_1 \cdots i_k})$, which formally denote the derivatives of $\partial_{i_1 i_2 \cdots} \phi^a(x)$ of a section $\phi$ at $x$. To keep track of all the derivatives, we introduce multi-index notation. A multi-index $I = i_1 i_2 \cdots i_k$ replaces the corresponding set of symmetric covariant coordinate indices (the multi-index does not change when the defining $i$’s are permuted). The order of this multi-index is given by $|I| = k$, with $|\emptyset| = 0$. To augment a multi-index by adding another index, we use the notation $Ij = jI = i_1 \cdots i_k j$. Thus we can write higher order derivatives as $\partial_{i_1 \cdots i_k} \phi(x) = \partial_I \phi(x)$, the higher order jet coordinates as $u^a_{i_1 \cdots i_k} = u^a_I$ and the total set of coordinates on a $k$-jet patch as $(x^i, u^a_I)$, $|I| \leq k$. In particular the empty multi-index $I = \emptyset$ corresponds to $u^a_{\emptyset} = u^a$.

Since the higher derivatives are symmetric in all indices, the number of extra coordinates is given by $n_k = \sum_{i=1}^k \dim S^k \mathbb{R}^n$, with $S^k$ denoting the symmetric tensor product. Given two different coordinate patches on $F$, we define the transition maps between the corresponding $k$-jet patches according to the usual calculus chain rule applied to higher order derivatives. These $k$-jet patches can be glued together into the total space of the $k$-jet bundle $J^k F \rightarrow M$, which includes $J^0 F \cong F$. 
Since $F \to M$ is a vector bundle, so is $J^kF \to M$. It is isomorphic to $F \oplus_M (F \otimes_M S^1T^*M) \oplus_M \cdots \oplus_M (F \otimes_M S^kT^*M)$, but not naturally. Jet bundles come with natural projections $J^kF \to J^{k-1}F$, which simply discard all derivatives of order $k$. This projection gives $J^kF$ the structure of an affine bundle over the base $J^{k-1}F$, with fibers modeled on the vector bundle $(F \otimes_M S^kT^*M)^{k-1} \to J^{k-1}F$ (see Def. Appendix A.1 next). The bundle $J^kF \to J^{k-1}F$ is affine because, in general, bundle morphisms of $J^kF \to J^kF$ induced by vector bundle automorphisms of $F$ are not linear but affine.

Given a vector bundle $E \to M$ it can be pulled back to the $k$-jet bundle along the projection $J^kF \to M$. We introduce a convenient notation for this pullback.

**Definition Appendix A.1.** We denote by $(E)^k \to J^kF$ the pullback of $E \to M$ to $J^kF$, which then fits into the pullback commutative square

$$(E)^k \to E \quad \downarrow \quad J^kF \to M.$$  

Any smooth section $\phi: M \to F$ automatically gives rise to its $k$-jet prolongation or $k$-prolongation $j^k\phi: M \to J^kF$. Namely $j^k\phi$ is a section of the bundle $J^kF \to M$ that is defined in a local adapted coordinate patch as

$$j^k\phi(x) = (x^i, \phi^a(x), \partial_1\phi^a(x), \ldots, \partial_{i_1\ldots i_k}\phi^a(x)) = (x^i, \partial_I\phi^a(x)), \ |I| \leq k.$$  

One can think of the $k$-prolongation symbol as a differential operator

$$j^k: \Gamma(F) \to \Gamma(J^kF)$$

of order $k$. In fact, any (not necessarily linear) differential operator of order $k$,

$$f: \Gamma(F) \to \Gamma(E), \quad f: \phi \mapsto f[\phi],$$

can be written as a composition of $j^k$ with an order 0 (not necessarily linear) operator $f: J^kF \to E$, such that $f[\phi] = f(j^k\phi)$. Note that we are slightly abusing notation by denoting both the differential operator and the bundle morphism by the same symbol $f$.

Further, we can define an $l$-prolongation of a differential operator $f$ of order $k$,

$$p^l f: J^{k+l}F \to J^lE,$$

which is then a differential operator of order $k + l$, by composing with $j^l$:

$p^l f[\phi] = j^l f[\phi]$. Prolongation is discussed briefly using coordinate-wise operations in Sec. Appendix B.1. The $k$-jet prolongation $j^k\phi$ can now be thought of as a special case of bundle morphisms, that is, $j^k\phi = p^k\phi$, where on the right hand side we interpret $\phi$ as the base fixing bundle morphism to $F \to M$ from the trivial
Given the sequence of projections of $k$-jet bundles over $M$,

\[ \cdots \to J^2F \to J^1F \to J^0F \cong F, \]  

it is convenient to introduce the infinite order jet (or $\infty$-jet) bundle $J^\infty F$ defined as the projective limit over the jet order $k$

\[ J^\infty F = \lim_{\leftarrow} J^kF. \]

This limit implicitly defines $J^\infty F$ as an infinite dimensional smooth manifold. The main advantage of working with $\infty$-jets is that any function or tensor on $J^kF$ for finite $k$ can be pulled back to $J^\infty F$. Conversely, any smooth function or tensor on $J^\infty F$ depends only on jets up to some finite order, say $k$, and can be faithfully projected to $J^kF$. Another major convenience of working on $J^\infty F$ is the ability to decompose the usual de Rham differential into its horizontal and vertical parts

\[ d = d_h + d_v. \]

The defining property of $d_h$ is the following. Given a section $\phi: M \to F$, we must have the identity

\[ (j^\infty \phi)^* d_h \alpha = d((j^\infty \phi)^* \alpha), \]

where $\alpha$ is any differential form on $J^\infty F$ and $d$ is the usual de Rham differential on $M$. On the other hand, $d_v$ is characterized by the fact that its image is annihilated by the pullback to $M$ along any section $\phi$,

\[ (j^\infty \phi)^* d_v \alpha = 0. \]

It can be checked that the horizontal and vertical differentials anti-commute and are separately nilpotent,

\[ d_h d_v + d_v d_h = 0, \quad d_v^2 = 0 = d_h^2. \]

Note that, to apply $d_v$ or $d_h$ to forms defined on a finite order jet bundle $J^kF$, the pullback and projection operations mentioned above will often be applied implicitly. Thus the application of say $d_h$ to a differential form on $J^kF$ may yield that a differential form that projects to $J^{k+1}F$ but not to $J^kF$. In local coordinates $(x^i, u^a)$ on $F$, and the induced coordinates $(x^i, u^a_I)$ on $J^\infty F$, a convenient basis for differential forms is

\[ d_h x^i = dx^i, \quad d_v u^a_I = du^a_I - d_h u^a_I = du^a_I - u^a_{Ii} dx^i. \]
We can also define two special kinds of vector fields. A vector field \( \xi \) is horizontal if its action in local coordinates is
\[
\dot{\xi}(x^i) = \xi_i, \quad \dot{\xi}(u^a_j) = \xi^i u^a_i i.
\] (A.14)
for some \( \xi_i = \xi^i(x, u^a_j) \). In particular, the vector field \( \partial_j \) with \( \xi_i = \delta_j^i \) is horizontal. Note that \([\partial_i, \partial_j] = 0\). A vector field \( \psi \) is evolutionary if its action in local coordinates is
\[
\dot{\psi}(x^i) = 0, \quad \dot{\psi}(u^a_j) = \partial_I(\psi^a),
\] (A.15)
for some \( \psi^a = \psi^a(x, u^a_j) \), where \( \partial_I(f) = \partial_{i_1}(\partial_{i_2}(\ldots \partial_{i_k}(f) \ldots)) \) for multi-index
\( I = i_1 i_2 \ldots i_k \) (the order of application of these vector fields does not matter since they commute). Note that the \( \psi^a \) can be seen as the fiber coordinate components of a section of the bundle \((F)^\infty \rightarrow J^\infty F\). These definitions can be checked to be coordinate independent.

One can show that for a horizontal vector field \( \dot{\xi} \) on \( J^\infty F \) there exists a vector field \( \xi_\phi \) on \( M \) such that their actions on scalar functions are intertwined by the pullback along the jet prolongation \( j^\infty \phi \) of a section \( \phi : M \rightarrow F \),
\[
\dot{\xi}(f(j^\infty \phi)) = \xi_\phi(f(j^\infty \phi)),
\] (A.16)
for any scalar function \( f \) on \( J^\infty F \). Namely, in local coordinates, \( \xi_\phi = \xi^i \partial_i \) with
\[
\xi^i_\phi = (t_\xi dx^i)(j^\infty \phi) = \dot{\xi}(x^i)(j^\infty \phi) = \xi^i(j^\infty \phi).
\] On the other hand, evolutionary vector fields \( \psi \) satisfy the identities
\[
t_\psi(d_h \alpha) + d_h(t_\psi \alpha) = 0, \quad (A.17)
\]
\[
\mathcal{L}_\psi(j^\infty \phi)^* \alpha = \frac{d}{dx} \bigg|_{x=0} [j^\infty(\phi + \varepsilon \psi)]^* \alpha = (j^\infty \phi)^* \mathcal{L}_\phi \alpha,
\]
for any form \( \alpha \in \Omega^*(J^\infty F) \) and section \( \psi : M \rightarrow F \). Actually, \( \psi \) could be a section of \((F)^k \rightarrow J^k F\), that is, it could depend on \( \phi^a(x) \) and its derivatives and not only on \( x \in M \). The only corresponding change in the above formula would be to replace \( \varepsilon \psi \) by \( \varepsilon (j^k \phi)^* \psi \). Ostensibly, \( \mathcal{L}_\psi \) should stand for the Lie derivative on the infinite dimensional manifold of sections of \( F \rightarrow M \), where the section \( \psi \) is identified with the vector field whose action on local coordinates is \( \mathcal{L}_\psi \phi^a(x) = \psi^a(x) \). However, since we do not delve into the differential geometry of infinite dimensional manifolds here, we keep the symbol \( \mathcal{L}_\psi(j^\infty \phi)^* \) primitive and defined as above.

Integrations or differentiations by parts are carried out using the following basic identity
\[
d_\psi u^a_i \wedge dx^i \wedge \alpha = d_\psi(u^a_i dx^i) \wedge \alpha = (d_\psi d_h u^a_i) \wedge \alpha
\] (A.19)
\[
= - (d_h d_\psi u^a_i) \wedge \alpha
\] (A.20)
\[
= -d_\psi u^a_i \wedge d_h \alpha - d_h (d_\psi u^a_i \wedge \alpha).
\]
(A.21)
This split of the de Rham differential into horizontal and vertical differentials also splits the de Rham complex $\Omega^*(J^\infty F)$ of differential forms on $J^\infty F$ into a bicomplex.\cite{71,73} Since the horizontal and vertical 1-forms generate the graded commutative algebra of differential forms, any form $\lambda \in \Omega^*(J^\infty F)$ can be uniquely written as

$$\lambda = \sum_{h,v} \lambda_{h,v},$$

where $0 \leq h \leq n$ and $0 \leq v$ are respectively the horizontal and vertical form degrees. We have thus turned the differential forms into a bigraded complex $\Omega^*(J^\infty F) = \bigoplus_{h,v} \Omega^{h,v}(F)$, with the $d_h$ differential increasing $h$ by 1 and the $d_v$ differential increasing $v$ by 1. This complex is called the variational bicomplex.\cite{71,73} As with any bicomplex, we can consider its cohomology with respect to either or any combination of the two differentials. The horizontal cohomology is $H_{h,v}(d_h) = H(\Omega^*(J^\infty F), d_h)$ in degrees $(h,v)$. The vertical cohomology is $H_{h,v}(d_v) = H(\Omega^*(J^\infty F), d_v)$ in degrees $(h,v)$. Both $(H^{h,*}/d_hH^{h-1,*}, d_v)$ and $(H^{*,v}/d_vH^{*,v-1}, d_h)$ still form complexes, therefore we can also consider their cohomologies. The relative cohomologies are $H_{h,v}(d_h|d_v) = H(H^{h,*}/d_hH^{h-1,*}, d_v)$ and $H^{*,v}(d_h|d_v) = H(H^{*,v}/d_vH^{*,v-1}, d_h)$.

Appendix B. Jet bundles and systems of PDEs

This appendix outlines the description of PDEs as submanifolds of the jet bundle. Jet bundles are briefly introduced in Appendix A, where also notation is fixed (not all of it being completely standard) and standard literature references are given. Such a description of PDEs is more intrinsic than the usual one in terms of equations, but is essentially equivalent. This approach is well known in the geometric and formal theory of differential systems.\cite{86-88}

From now on, fix $M$ to be finite dimensional manifold and let $n = \dim M$. Also fix a vector bundle $F \to M$. We refer to $M$ as the spacetime manifold and to $F$ as the field bundle.

We restrict our attention to regular PDEs in the following sense.

**Definition Appendix B.1.** A PDE system $\mathcal{E}$ of order $k$ is a smooth, closed sub-bundle of $J^k F \to M$, $\mathcal{E} \subset J^k F$.

Note that $\mathcal{E}$ need not be a vector sub-bundle of $J^k F$. The above definition may seem unfamiliar to some, but can be cast in more recognizable form using the following

**Proposition Appendix B.1.** Given a PDE system $\mathcal{E}$ of order $k$, there exists (up to a global obstruction) a vector bundle $E \to M$, a smooth sub-bundle $E' \subset E$ containing the zero section of $E \to M$, and a smooth base fixing smooth bundle morphism $f: J^k F \to E$ such that the image of $f$ is contained in $E'$, the image of $f$ is transverse in $E'$ to the zero section of $E$ and $\mathcal{E}$ is precisely the preimage of the zero section, that is, $\mathcal{E}$ satisfies $f = 0$. 
The proof follows from basic differential topology. The obstruction is of a global topological nature \cite[§7]{66} and is related to the fact that not every embedded submanifold can be represented as the zero-set of a section of a vector bundle. Clearly, the equation form is not unique. For instance, applying any invertible transformation to the equations $f = 0$ gives another equation form $f' = 0$, which describes exactly the same PDE system.

We refer to $E \to M$ as the equation bundle and to $f$ or the pair $(f, E)$ as the equation form of the PDE system $\mathcal{E}$. A section $\phi: M \to F$, also referred to as a field configuration, is said to satisfy the PDE system $\mathcal{E}$ if the $k$-jet prolongation of $\phi$ is contained in $\mathcal{E}$, $j^k\phi(x) \in \mathcal{E}_x \subset J^k_x(F, M)$. Then, equivalently, $j^k\phi$ is a section of $\mathcal{E} \to M$. We denote the space of all solution sections by $\mathcal{S}(F) \subset \Gamma(F)$ or $\mathcal{S}_E(F)$ when the PDE system needs to be mentioned explicitly. Using the above proposition, we can equivalently say that $\phi$ is a solution of the PDE system $\mathcal{E}$ if

$$ f[\phi] = f(j^k\phi) = 0. $$

Expressing the $k$-jet in local coordinates, $j^k\phi(x) = (x, \phi^a(x), \partial_i\phi^a(x), \ldots)$, it is clear that $f(x, \phi^a(x), \partial_i\phi^a(x), \ldots) = 0$ is a system of partial differential equations in the usual sense of the term. Starting with a PDE system in the usual sense, its geometric form as a sub-bundle of the jet bundle can be obtained by a converse of the above lemma. At this point, the regularity assumptions on both $\mathcal{E}$ and $f$ become important. Namely, the transversality properties of $f$ ensure that the zero set of $f = 0$ is a submanifold of $J^kF$ and vice versa.

The linear and affine structures on $J^kF$ give us the possibility of defining the notion of linear and quasilinear PDE systems.

**Definition Appendix B.2.** A PDE system $\mathcal{E} \subset J^kF$ is called linear if $\mathcal{E} \to M$ is a vector sub-bundle of the vector bundle $J^kF \to M$. The PDE system is called quasilinear if $\mathcal{E} \to J^{k-1}F$ is an affine sub-bundle of the affine bundle $J^kF \to J^{k-1}F$.

The connection to the usual meanings of these terms can be seen through adapted equation forms.

**Lemma Appendix B.1.** The PDE system $\mathcal{E} \subset J^kF$ is linear iff it has an equation form $(f, E)$, where $f: J^kF \to E$ is a morphism of vector bundles over $M$.

The PDE system $\mathcal{E} \subset J^kF$ is quasilinear iff it has an equation form $(f, E)$, where $f: J^kF \to E$ is a morphism of affine bundles, which fits into the commutative diagram

$$
\begin{array}{ccc}
J^kF & \xrightarrow{f} & E \\
\downarrow & & \downarrow \\
J^{k-1}F & \xrightarrow{} & M
\end{array}
$$

where the vertical maps define the affine bundles, with the vector bundle $E \to M$ naturally considered an affine one.
The proof is immediate. Alternatively, the quasilinear case can be cast into the form of a base fixing affine bundle morphism \( f : J^k F \to (E)^{k-1} \), where both bundles are over \( J^{k-1} F \). Such equation forms are called adapted.

In the more common language of adapted local coordinates, the conditions of linearity and quasilinearity are expressed as follows. Consider adapted local coordinates \((x^i, v_A)\) on the equation bundle \( E \), \((x^i, u^a)\) on the field bundle \( F \), and the corresponding \((x^i, u^a_I)\) on the \( k \)-jet bundle \( J^k F \). Let \( \phi : M \to F \) be a field configuration, then its \( k \)-jet in local coordinates is \( j^k \phi(x) = (x^i, \partial_I \phi^a(x)) \). The above lemma asserts the existence of an equation form that looks like

\[
f^I_{Aa}(x) \partial_I \phi^a(x) = 0. \tag{B.3}
\]

Note that this equation is linear in \( \phi(x) \) and its derivatives and that the coefficients \( f^I_{Aa}(x) \), with multi-indices \( I \), depend only on the base space coordinates \( x \). On the other hand, for a quasilinear equation, the lemma asserts the existence of an equation form that looks like (with \(|I| = k\))

\[
f^I_{Aa}(x, j^{k-1} \phi(x)) \partial_I \phi^a(x) + f_A(x, j^{k-1} \phi(x)) = 0. \tag{B.4}
\]

Note that, in the linear case, the fact that the coefficients of \( f : J^k (F, M) \to E \) only depend on the base space coordinates \( x \) is captured by the requirement that it is a morphism of vector bundles over \( M \). In the quasilinear case, the coefficients of \( f \) can obviously depend on both \( x, \phi(x) \) as well as all derivatives \( \partial_I \phi(x) \) up to order \(|I| = k - 1\), which is captured by allowing \( f : J^k F \to (E)^{k-1} \) to be a (base fixing) bundle morphism over \( J^{k-1} F \). It is worth remarking that any linear PDE system is also naturally quasilinear.

Recall that the affine bundle \( J^k F \to J^{k-1} F \) is modeled on the vector bundle \((S^k T^* M \otimes_M F)^{k-1} \to J^{k-1} F \). Therefore, an adapted equation form \((f, E)\) of a quasilinear PDE system \( \mathcal{E} \subset J^k F \) naturally singles out a section

\[
\tilde{f} : J^{k-1} F \to (E \otimes_M F^* \otimes_M S^k TM)^{k-1}. \tag{B.5}
\]

In local coordinates, \( \tilde{f} \) corresponds to the coefficient \( f^I_{Aa} \) of the highest derivative term \( \partial_I \phi^a(x) \) with \(|I| = k\) in Eq. (B.4). This section \( \tilde{f} \) is called the principal symbol of the given equation form of \( \mathcal{E} \). If the equation is linear, rather than quasilinear, \( \tilde{f} \) can be projected from a section on \( J^{k-1} F \) to a section on \( M \). Moreover, if we fix \( x \in M \) and \( p \in T_x^* M \), we can define the linear map \( \tilde{f}_{x,p} = \tilde{f}_x : p^\otimes k \to E_x \).

\[
\tilde{f}_{x,p} : E_x \to F_x. \tag{B.6}
\]

which we also refer to as the principal symbol.

**Appendix B.1. Prolongation, integrability, equivalence**

One reason to discuss PDE systems as submanifolds of a jet bundle is independence of a particular equation form. Any two equation forms are equivalent if they define the same PDE system manifold. We should specify our notion of equivalence.
Definition Appendix B.3. Consider two field bundles $F_i \to M$, $i = 1, 2$, and two PDE systems $E_i \subseteq J^k F_i$. Denote the corresponding spaces of smooth solution sections by $S_i(F_i)$. The PDE systems $E_1$ and $E_2$ are said to be equivalent if there exist bundle morphisms $e_{ij} : J^{l_i} F_i \to F_j$, $i \neq j$, such that

$$\phi_i \in S_i(F_i) \text{ and } \phi_j = e_{ij} \circ j^{l_i} \phi_i \implies \phi_j \in S_j(F_j), \quad (B.7)$$

as well as that $e_{12} \circ j^{l_1}$ and $e_{21} \circ j^{l_2}$ are mutual inverses when restricted to the solution spaces $S_1(F_1)$ and $S_2(F_2)$.

We can easily extend the notion of equivalence to equation forms of PDE systems. In that case two different equations forms that define the same PDE system manifold are trivially equivalent. Note that neither the field bundles nor the orders of the PDE systems need to be same for equivalence to hold.

Let us restrict to the case that is of importance elsewhere in this review, namely of $F_1 = F_2 = F$ and $e_{12}$ and $e_{21}$ respectively equal to the canonical projections $J^{l_1} F \to F$ and $J^{l_2} F \to F$, which are in a sense trivial. In this case, it is certainly sufficient that $E_1 = E_2$ for equivalence to hold, but it is not necessary. In fact $E_1$ and $E_2$ could be of different orders. To obtain necessary conditions for equivalence, we need to consider prolongation of PDE systems and the possible resulting integrability conditions.

A discussion of these notions in the setting of the jet bundle description of PDE systems can be rather technical. On the other hand, the theory of equivalence of PDE systems formulated in these terms has become quite mature and has yielded some important results. The technical details of this theory can be found for example in Refs. 86, 87. Below we give a brief non-technical introduction to this theory and state some simplified results relevant for hyperbolic systems.

The step by step derivation and inclusion of integrability conditions into a PDE is called prolongation. It is easiest to define prolongation in equation form and in local coordinates. Consider an equation form $(f, E)$ of a PDE system $E \subseteq J^k F$, as well as local coordinates $(x^i, u^a)$ on $F$ and $(x^i, v_A)$ on $E$. If the section $\phi : M \to F$ satisfies the PDE system, we have the following system of equations holding in local coordinates

$$f_A(x^i, \partial J^k \phi^a) = 0. \quad (B.8)$$

These equations hold for each point $x \in M$, therefore when both sides are differentiated with respect to the coordinates on $M$, the resulting equations are still satisfied,

$$\partial_i f_A(x^i, \partial J^k \phi^a) = (\hat{\partial}_i f_A)(x^i, \partial J^k \phi^a) = 0, \quad (B.9)$$

where $\hat{\partial}_i f_A$ are functions on $J^{k+1} F$ obtained by pulling back the functions $f_A$ from $J^k F$ to $J^{k+1} F$ and applying the horizontal vector field $\hat{\partial}_i$. These new functions $f_{iA} = \hat{\partial}_i f_A$, together with the old $f_A$ ones, constitute the local coordinate expression for the equation form $(p^1 f, J^1 E)$, where $p^1$ is the 1-prolongation defined in
Sec. Appendix A. We call the corresponding PDE system $\mathcal{E}^1 = \mathcal{E}_{p^c} \subset J^{k+1} F$ the first prolongation of $\mathcal{E}$ or also its prolongation to order $k + 1$. Prolongations to any higher order, $(p^c f, J^l E)$ and $\mathcal{E}^l \subset J^{k+l} F$, are defined iteratively.

Let $p_l : J^{k+l} F \rightarrow J^k F$ be the canonical jet projection, which restricts to $p_l : \mathcal{E}^l \rightarrow \mathcal{E}$. Notice that we necessarily have $p_l (\mathcal{E}^l) \subseteq \mathcal{E}$, since the prolonged system contains the original one as a subsystem. We have just shown that sections satisfying $\mathcal{E}$ automatically satisfy $\mathcal{E}^l$, and vice versa. In other words, $\mathcal{S}_\mathcal{E} (F) = \mathcal{S}_{\mathcal{E}^l} (F)$ and the two PDE systems are equivalent. However, the inclusion $p_l (\mathcal{E}^l) \subseteq \mathcal{E}$ may be strict, which would mean that there exist non-trivial integrability conditions. There exists an equation form $(f \oplus g, E \oplus G)$ for $p_l (\mathcal{E}^l) \subset J^k F$, where $(g, G)$ is an equation form for the integrability conditions. These observations provide another sufficient condition for the equivalence of two PDE systems, namely that there exists an order $l \geq k_1, k_2$ such that $\mathcal{E}^{l-k_1} = \mathcal{E}_{l-k_2}^l$ as subsets of $J^l F$.

Prolongation can be iterated indefinitely. Taking this process to its limit, we obtain the infinite order prolongation $\mathcal{E}^\infty \subset J^\infty F$ from the equation form $(p^\infty f, J^\infty E)$, which takes all possible integrability conditions into account. One can then show that the equality $\mathcal{E}^{\infty}_1 = \mathcal{E}^{\infty}_2$, as subsets of $J^\infty F$, is both a necessary and a sufficient condition for the equivalence of two PDE systems. It is a deep theorem of the geometric theory of PDE systems\cite{56, 86, 87} that, for any given PDE system, there exists a finite order $l$ such that prolongations above that order introduce no new integrability conditions. Therefore, this restricted version of the equivalence problem can be decided in finitely many steps.

Remark Appendix B.1. In mathematical physics, PDEs systems are often obtained in variational form (as Euler-Lagrange equations of some Lagrangian). However, this form need not be one for which the existence of Green functions are readily available. Thus, it becomes important to formalize, as is done above, how these systems can be brought into an equivalent form that can be shown to be Green hyperbolic (cf. Sec. 2.1) using standard methods. For example, the Klein-Gordon equation is normal hyperbolic,\cite{58} but not symmetric hyperbolic.\cite{59, 75} The Dirac and Proca equations are neither. However, each of these equations can be shown to be equivalent to either a symmetric or normal hyperbolic system with constraints.\cite{58, 59, 63, 75} The inclusion of constraints into the description of a hyperbolic system is discussed in Sec. 2.4.

Appendix B.2. Formal exactness

An important concept used in this work is that of a formally exact complex (or sequence) of differential operators. There are several related concepts, which we discuss briefly below. More details can be found in the Refs. 67, 68, 89.

A complex of differential operators or differential complex consists of vector bundles $E, F, G \rightarrow M$ and differential operators $f : E \rightarrow F$ and $g : F \rightarrow G$ such
that \( g \circ f = 0 \); it is often written as
\[
\Gamma(E) \xrightarrow{f} \Gamma(F) \xrightarrow{g} \Gamma(G) \tag{B.10}
\]
or
\[
J^\infty E \xrightarrow{p^\infty f} J^\infty F \xrightarrow{p^\infty g} J^\infty G . \tag{B.11}
\]

Of course, the sequence of operators in a differential complex could have any length, not just two. Given a complex, we are of course free to define its cohomology, \( \ker g / \text{im } f \). If this cohomology vanishes, then the complex is said to be exact or to form an exact sequence. Different kinds of cohomologies can be defined by considering different spaces on which the differential operators are defined. There are of course different kinds of exactness associated to them.

The complex is locally exact if for every \( x \in M \) there exists a neighborhood \( U \subseteq M \) of \( x \) such that the sequence
\[
\Gamma(E|_U) \xrightarrow{f} \Gamma(F|_U) \xrightarrow{g} \Gamma(G|_U) \tag{B.12}
\]
is exact. The complex is globally exact if the sequence (B.10) is exact. Of course there are different versions of global exactness if we replace arbitrary smooth sections by say sections with compact support, or other restriction. In practical applications, it is global exactness or the knowledge of the global cohomology that is important. Local exactness is important because it allows the use of sheaf-theoretic methods to compute the global cohomology.

Local exactness by itself is a difficult property to check, because it is essentially a functional analytical condition. A simpler geometric condition is formal exactness. The complex is formally exact if the sequence (B.11) is exact as a sequence of (infinite dimensional) vector bundles, which is the same as exactness each of the sequences
\[
J^{s+k+l} E \xrightarrow{p^s t^l f} J^{s+l} F \xrightarrow{p^s g} J^s G \tag{B.13}
\]
of (finite dimensional) vector bundles, with \( k \) and \( l \) being the respective orders of the operators \( f \) and \( g \). Formal exactness is not a sufficient condition for local exactness, but it is necessary and is a first step in trying to establish the stronger condition. For a given differential operator \( f : \Gamma(E) \to \Gamma(F) \), the existence of a differential operator \( g : \Gamma(F) \to \Gamma(G) \) extending \( f \) to a formally exact sequence is assured for abstract reasons,\(^67,89\) provided the usual regularity conditions hold.

Yet another related and simpler condition is ellipticity\(^90, \S XIX.4\). The complex is said to be elliptic if for every \( x \in M \) and non-zero \( p \in T^*_x M \) the sequence of principal symbols (cf. Eq. (B.6))
\[
E_x \xrightarrow{\tilde{f}_x p} F_x \xrightarrow{\tilde{g}_x p} G_x \tag{B.14}
\]
is exact. The de Rham complex is a prototypical example of both an elliptic and a formally exact complex. However, these conditions are in general independent,\(^91\) though in many cases ellipticity has proven helpful in checking formal exactness.
Appendix C. Causal structure on conal manifolds

In the literature on relativity, causal structure is most often studied as a subset of Lorentzian geometry. The term Lorentzian geometry refers to the study of structures induced on spacetime manifolds by the presence of a Lorentzian metric. One of these structures consists of the cones of null vectors. In particular, it is these cones that determine causal relationships between points in Lorentzian spacetimes. While causal relationships themselves can be defined solely in terms of the null cones, the reason they deserve the name causal is that they also describe the maximal speed at which disturbances can travel in solutions of hyperbolic PDEs with wave-like principal symbols. However, a similar property holds for more general classes of hyperbolic equations, even those that have no relation to a Lorentzian metric. It stands to reason then that causal relationships should be definable in terms of the intrinsic geometry of such PDEs. Indeed, if we consider cones of so-called characteristic covectors and define causal relationships in terms of them, surprisingly few changes are necessary, with Lorentzian null cones appearing as special cases for the class of wave-like PDEs mentioned above. It stands to reason to give the study of such cones the name characteristic geometry. On the other hand, we find it convenient to generalize even further and consider simply a priori given cones (in the tangent and cotangent space), thus abstracting and clarifying the geometric notions that go into the definitions and basic properties of causal relations. Thus, we shall actually be studying conal geometry or what have sometimes been called conal manifolds. This abstraction highlights the fact that a basic tool in the study of causal structures should be differential topology, rather than pseudo-Riemannian geometry.

Characteristic geometry and its abstraction to conal geometry are discussed in a fuller and more integrated way in Ref. Some cues have been taken from previous attempts to abstract the notion of causal structure or causal order in Lorentzian geometry. The generalization from Lorentzian cones to more general ones, for the purposes of describing causality in quantum field theory has been considered before, but not in a concrete way.

Each point of a conal manifold, referred to here as a cone bundle, is smoothly assigned an open cone (a set invariant under multiplication by positive scalars) of tangent or cotangent vectors.

**Definition Appendix C.1.** A smooth bundle $C \to M$ of finite dimensional manifolds is termed a cone bundle if there exists an enveloping vector bundle $E \to M$ and an inclusion bundle morphism $\nu: C \subset E$, such that each fiber $C_x$, $x \in M$, is an open convex cone in the corresponding fiber $E_x$ (where we have implicitly identified $C$ with its image $\nu(C) \subset E$). A bundle map $\chi: C \to C'$ is a cone bundle morphism if, given corresponding enveloping vector bundles $E \to M$ and $E' \to M'$, there exists a vector bundle morphism $\psi: E \to E'$ such that $\chi = \psi|_C$. Namely, the
following diagrams exist and commute:

\[
\begin{array}{ccc}
E & \xrightarrow{\psi} & E' \\
\downarrow & & \downarrow \\
M & \xleftarrow{\subset} & M'
\end{array},
\quad
\begin{array}{ccc}
C & \xrightarrow{\chi} & C' \\
\downarrow & & \downarrow \\
E & \xrightarrow{\psi} & E'
\end{array}.
\]

Before proceeding, we need some terminology concerning cones and operations on them. These notions are often used in convex geometry.\textsuperscript{102}

**Definition Appendix C.2.** Given a finite dimensional vector space \( V \) and a convex cone \( C \subset V \), denote its closure by \( \bar{C} \) and its open interior by \( \mathring{C} \). We define the convex dual (a.k.a. polar dual) \( C^* \subset V^* \) as the set

\[
C^* = \{ u \in V^* \mid u \cdot v \geq 0 \text{ for all } v \in C \}.
\]

We define the strict convex dual \( C^{\circ} \subset V^* \) as the set

\[
C^{\circ} = \{ u \in V^* \mid u \cdot v > 0 \text{ for all } v \in \bar{C} \setminus \{0\} \}.
\]

The attribute strict may be dropped from the description of \( C^{\circ} \) when it is clear from context. It is easy to check the following

**Proposition Appendix C.1.** Consider a convex cone \( C \).

(i) The convex dual \( C^* \) is always closed and also convex. In addition, \( C^{**} = \bar{C} \).

The strict convex dual \( C^{\circ} \) is always open and convex.

(ii) \( C^* \setminus \{0\} \) is non-empty iff \( C \) is contained in a closed half space. \( C^{\circ} \) is non-empty iff \( C \) contains no affine line (it is salient).

(iii) If \( C \) is open and salient, then \( C^{**} = C \).

(iv) The inclusion of cones \( C_1 \subseteq C_2 \) implies the reverse inclusion of their duals, \( C_1^* \supseteq C_2^* \) and \( C_1^{\circ} \supseteq C_2^{\circ} \).

(v) The convex dual of the intersection of closures of cones \( C_1 \) and \( C_2 \) is the convex union (convex hull of the union) of their duals, \((\bar{C}_1 \cap \bar{C}_2)^* = C_1^* + C_2^* \), where the right hand side is written as a Minkowski sum, which for cones coincides with the convex hull of the union. The converse identity holds as well, \((\bar{C}_1 + \bar{C}_2)^* = C_1^* \cap C_2^* \). Similarly, if \( C_1 \) and \( C_2 \) are open and salient, then \((C_1 \cap C_2)^{\circ} = C_1^{\circ} + C_2^{\circ}\) and \((C_1 + C_2)^{\circ} = C_1^{\circ} \cap C_2^{\circ}\).

We extend these operations to cone bundles by acting fiberwise. So, if \( C \to M \) is a cone sub-bundle of a vector bundle \( E \to M \), then the convex dual cone bundle \( C^* \to M \) is the cone sub-bundle of \( E^* \to M \) such that each fiber \( C_x^* \) is the convex dual of the corresponding fiber \( C_x \), for \( x \in M \). Similarly, we can define the strict convex dual cone bundle \( C^{\circ} \to M \). The operations of intersection (\( \cap \)) and convex union (\( + \)) are extended to cone bundles in the same way. Clearly if \( C^{\circ} \to M \) is a smooth bundle, it is also a cone bundle. On the other hand, \( C^* \to M \) is usually not (since it is usually not open), though for brevity we shall sometimes refer to it as a cone bundle anyway.
In the next definition, we make a slight break with the usual terminology concerning covectors in Lorentzian geometry. A covector naturally defines a codimension-1 subspace, its annihilator, of the tangent space. In the presence of a metric, a covector can be canonically identified with a vector. If this vector is timelike, the corresponding codimension-1 subspace is called spacelike. However, a direct means of identifying covectors with vectors is missing in general. On the other hand, the link between covectors and codimension-1 tangent subspaces is metric independent and since the term *spacelike* still makes sense for tangent subspaces, we transfer it to corresponding covectors naming them *spacelike* as well.

**Definition Appendix C.3.** Given a manifold $M$, a **chronal cone bundle** on $M$ is a cone bundle $C \to M$ enveloped by the tangent bundle $TM \to M$ such that each fiber $C_x, x \in M$, is a *proper cone* (non-empty, open, convex, salient). The elements of $C$ are **future directed, timelike vectors**. The strict convex dual $C^\circ \to M$, enveloped by the cotangent bundle $T^*M \to M$, is the corresponding **spacelike cone bundle**. The elements of $C^\circ$ are **future oriented, spacelike covectors**. The corresponding cone bundles $\bar{C} \to M$ and $\bar{C}^\circ \to M$ are referred to, respectively, as the **causal** and **cocausal** cone bundles. A morphism of two chronal or spacelike cone bundles $C \to M$ and $C' \to M'$ is induced by pushforward $\chi^*: TM \to TM'$ (or $T^*M \to T^*M'$) of an open embedding $\chi: M \to M'$.

Note that the open convex dual of a proper cone is again a proper cone. Prototypical examples of chronal and spacelike cone bundles are the cone bundles of timelike vectors and spacelike covectors on a Lorentzian manifold.

At this point, one may recall some standard notions of Lorentzian geometry, as long as they are defined only in terms of timelike cones, and apply them to the geometry of cone bundles. Many of the standard theorems translate as well, some directly and others with some extra effort. We restrict ourselves to those that are relevant to the issues at hand.

Fix a chronal cone bundle $C \to M$ on a spacetime manifold $M$. Note that any chronal cone bundle is **time oriented**, since by definition the fibers consist of single cones rather than double cones like in the Lorentzian case. By assumptions the cones are directed into the future.

**Definition Appendix C.4.** A smooth curve $\gamma$ in $M$ is called **future directed, timelike** if the tangent to $\gamma$ is everywhere contained in $C$. The **chronal precedence relation** $I^+ \subseteq M \times M$ (also $I^+_x$) is defined as

$$I^+ = \{(x, y) \in M \times M \mid \exists \gamma, \text{ future directed, timelike curve from } x \text{ to } y\}. \quad (C.4)$$

When $(x, y) \in I^+$, we say that $x$ **chronologically precedes** $y$ and also write $x \ll y$.

If we replace $C$ by $\bar{C}$ in the above definitions, we obtain **causal curves** and the **causal precedence relation** $J^+ \subseteq M \times M$, denoted $x < y$. The inverse relations are written $I^-$ and $J^-$. 
It is clear that $I^+$ is an open, transitive relation ($x \ll y$ and $y \ll z$ implies $x \ll z$). Using this relation, we can define the usual causal hierarchy.

**Definition Appendix C.5.**

(i) If $I^+$ is irreflexive ($x \not\ll x$ or, equivalently, no closed timelike curves exist), then $C$ is chronological.

(ii) Given an open $N \subseteq M$, $I^+_{C|N}$ and $I^+_C \cap (N \times N)$ are both relations on $N \times N$. We say that $N$ is *chronologically compatible* if both of these relations coincide. More conventionally, this means that any two points of $N$ that can be joined by a timelike curve in $M$ can also be joined by a timelike curve in $N$.

(iii) An open $N \subseteq M$ is called *chronologically convex* if any timelike curve that joins any two points of $N$ must also lie in $N$, which is a stronger condition than chronological compatibility.

(iv) The chronal cone bundle $C$ is said to be *strongly chronological* if it is chronological and for every $x \in M$ and every neighborhood $N \subseteq M$ of $x$, there exists a smaller open neighborhood $L \subseteq N$ that is chronologically convex.

(v) The chronal cone bundle $C$ is said to be *stably chronological* if there exists another chronal cone bundle $C'$ that is itself chronological and an open neighborhood of the closure of $C$, that is, $\bar{C} \setminus \{0\} \subset C'$. For spacelike cone bundles, stable chronology is equivalent to the reverse inclusion $\bar{C}' \setminus \{0\} \subset C$.

Each of these definitions has an obvious analog when the adjectives *chronological* or *timelike* are replaced by *causal*.

Note that *stably chronological* is equivalent to *stably causal*, so these terms will be used interchangeably. Moreover, the chronological chronal cone bundle $C'$ such that it contains a stably chronological chronal cone bundle $C$ can itself be chosen to be stably chronological. Next we turn from curves to surfaces.

**Definition Appendix C.6.** Each of the following concepts may be prefaced with $C$- or $C^\oplus$- to be more specific.

(i) An oriented codim-1 surface $S \subset M$ is called *future oriented, spacelike* if its oriented conormals are everywhere contained in $C^\oplus$.

(ii) A codim-1 surface $S \subset M$ is called *achronal* if it $(S \times S) \cap I^+ = \emptyset$, that is, no two points of $S$ are connected by a timelike curve. Similarly, $S$ is *acausal* when $(S \times S) \cap J^+ = \emptyset$.

(iii) A codim-1 surface $S \subset M$ is called *Cauchy* if it is acausal and every inextensible causal curve intersects $S$ exactly once.

(iv) A chronal cone bundle $C \to M$ is called *globally hyperbolic* if there exists a Cauchy surface $\bar{S} \subseteq M$. A spacelike cone bundle $C^\oplus \to M$ is called *globally hyperbolic* if $C \to M$ is.

Next we define some commonly used domains. Fix $S \subset M$ to be a $C$-acausal codim-1 submanifold, such that either $S$ is closed or $\bar{S} \subset M$ is a submanifold with boundary.
Definition Appendix C.7.

(i) The future/past domain of influence $I^\pm(N)$ of a subset $N \subseteq M$ is the set of points $y \in M$ such that there exists $x \in N$ with either $x \ll y$ (+) or $y \ll x$ (−). Let $I(N) = I^+(N) \cup I^-(N) \cup N$.

(ii) The domain of dependence $D(S)$ is the largest open subset of $M$ for which $S$ is a Cauchy surface. More commonly, $D(S)$ is the set of points $y \in M$ such that every inextensible timelike curve through $y$ intersects $S$. Let $D^\pm(S) = D(S) \cap I^\pm(S)$.

(iii) An open subset $L \subseteq M$ is lens-shaped with respect to $S \subseteq D$ if it can be smoothly factored as $L \cong (-1, 1) \times S$, with $t: L \to (-1, 1)$ denoting the projection onto the first factor (the temporal function), such that the level set $t = 0$ is $S$ and all other level sets are spacelike as well as share the same boundary as $S$ in $M$ (which may be empty).

The literature in relativity and Lorentzian geometry mostly makes use of the notion of global hyperbolicity as given above, but specialized to Lorentzian cone bundles. On the other hand, the literature on hyperbolic PDE systems (including symmetric and regular hyperbolic ones) mostly makes use of lens-shaped domains. It is a non-trivial fact that these two notions coincide. The argument is essentially that the temporal function of a lens-shaped domain foliates it with Cauchy surfaces. Conversely, a globally hyperbolic cone bundle admits a temporal function and a smooth factorization that turns it into a lens-shaped domain (glossing over some details related to spatial compactness). The original argument establishing the converse link in Lorentzian geometry is due to Geroch. However, his argument only established the existence of a continuous temporal function. The details necessary to establish the smooth version of the result are due to more recent work of Bernal and Sanchez. The very recent result by Fathi and Siconolfi, using completely different methods, established the existence of a smooth temporal function and factorization for a more general class of cone bundles that is sufficient for our purposes.

Proposition Appendix C.2. If a cone bundle $C \to M$ is globally hyperbolic, then there exists a smooth temporal function $t: M \to \mathbb{R}$, whose level sets are all diffeomorphic and are $C$-Cauchy surfaces. Hence, an open subset $D \subseteq M$ is $C$-lens-shaped with respect to $S \subseteq D$ iff it is globally $C|_D$-hyperbolic, with $S$ being $C|_D$-Cauchy.

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