A Model-Theoretic Semantics for Defeasible Logic *

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Abstract. Defeasible logic is an efficient logic for defeasible reasoning. It is defined through a proof theory and, until now, has had no model theory. In this paper a model-theoretic semantics is given for defeasible logic. The logic is sound and complete with respect to the semantics. We also briefly outline how this approach extends to a wide range of defeasible logics.

1 Introduction

Defeasible logic is a logic designed for efficient defeasible reasoning. The logic was designed by Nute [23,24] with the intention that it be efficiently implementable. This intention has been realised in systems that can process hundreds of thousands of defeasible rules quickly [21]. Over the years, Nute and others have proposed many variants of defeasible logic [24,1]. In this paper we will address a particular defeasible logic, which we denote by DL [5,1]. However, this work is easily modified to address other defeasible logics.

Defeasible logics are, in general, paraconsistent. The nature of defeasible reasoning, where one chain of reasoning may defeat another, predisposes the logics to avoid inconsistent inferences, and provides for a natural treatment of inconsistencies, when they occur. In the case of DL, which supports only sceptical reasoning, inconsistent inferences are almost completely avoided. They only occur as a result of inconsistencies in the definite knowledge expressed by a theory. When such inconsistencies occur, the inconsistent literals may be used individually – or even together – in further inferences, but no form of ex falso quodlibet reasoning is possible.

DL, and similar logics, have been proposed as the appropriate language for executable regulations [4], negotiations [10], contracts [26], and business rules [14]. The logics are considered to have satisfactory expressiveness and the efficiency of the implementations supports real-time response in applications such

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as electronic commerce [14,11]. Indeed, propositional DL has been shown to have linear time complexity [19].

On the other hand, neither DL nor any other variant of modern defeasible logic has a model theory. DL is defined purely in proof-theoretic terms [24,13]. Furthermore, a model theory based on the idea of extensions (such as is used for default logic) is likely to be inappropriate for defeasible logic, since the kind of scepticism that is developed from intersection of extensions in default logic is different from the kind of “direct” scepticism [15] that occurs in defeasible logic [27].

In early work on semantics for defeasible logics, Nute [23] defined a model theory for LDR, a substantially simpler precursor of DL, in terms of a minimal belief state for each theory. LDR defines defeat only in terms of definite provability; this limitation is the main reason why the approach is successful [23]. Recently, this approach has been extended [8,25] to a defeasible logic that is closer to DL, and more general in some respects. However, the semantics is based on the idea of intersection of extensions and – perhaps consequently – the logic is sound but not complete for this semantics.

Although there is not a model theory for defeasible logic, there has been work on providing a semantics for DL in other styles. In [20] we showed that DL can be defined in terms of a metapogram, defined to reflect the inference rules of the logic, and a semantics for the language of the metapogram. This approach was successful in establishing a relationship between DL and Kunen’s semantics of negation-as-failure. Although it did not directly address model-theoretic reasoning, we will use this semantics as a key intermediate step in verifying the correctness of our model-theoretic semantics.

We have described DL and its variants in argumentation-theoretic terms [12,13]. Such a characterization is useful for the applications of the logic that we have in mind, but the resulting semantics is again a meta-level treatment of the proof theory: proof trees are grouped together as arguments, and conflicting arguments are resolved by notions of argument defeat that reflect defeat in defeasible logic. Thus this work also fails to address model theory.

Finally, a denotational-style semantics [28] has been given to DL [18]. The semantics is compositional, and fully abstract in all but one syntactic class. Although this semantics provides a useful analysis of DL, it does not provide a model theory.

In this paper, we define a model-theoretic semantics for DL. This semantics follows Nute’s semantics for LDR in that models represent a state of mind or “belief state” in which definite knowledge (that which is “known”) is distinguished from defeasible knowledge (that which is “believed”). A major difference from [23] is that it adopts partial models as the basis from which to work. The approach can be adapted easily to a wide range of defeasible logics.

The structure of the paper is as follows. In the next section we introduce the constructs of defeasible logic, and the proof theory of DL. We then introduce the model-theoretic semantics and prove the soundness and completeness of the proof system with respect to this semantics. Finally, we discuss the extension
of this work to other defeasible logics, including logics which admit pre-defined relations – constraints in the sense of constraint logic programming [16].

2 The Defeasible Logic DL

We begin by presenting the basic ingredients of defeasible logic. A defeasible theory contains five different kinds of knowledge: facts, strict rules, defeasible rules, defeaters, and a superiority relation.

Facts are indisputable statements, for example, “Tweety is an emu”. This might be expressed as \( \text{emu}(\text{tweety}) \).

Strict rules are rules in the traditional sense: whenever the premises are indisputable (e.g. facts) then so is the conclusion. An example of a strict rule is “Emus are birds”. Written formally:

\[ \text{emu}(X) \rightarrow \text{bird}(X). \]

Defeasible rules are rules that can be defeated by contrary evidence. An example of such a rule is “Birds typically fly”; written formally:

\[ \text{bird}(X) \Rightarrow \text{flies}(X) \]

The idea is that if we know that something is a bird, then we may conclude that it flies, unless there is other evidence suggesting that it may not fly.

Defeaters are rules that cannot be used to draw any conclusions. Their only use is to prevent some conclusions. In other words, they are used to defeat some defeasible rules by producing evidence to the contrary. An example is “If an animal is heavy then it might not be able to fly”. Formally:

\[ \text{heavy}(X) \sim \neg \text{flies}(X) \]

The main point is that the information that an animal is heavy is not sufficient evidence to conclude that it doesn’t fly. It is only evidence that the animal may not be able to fly. In other words, we don’t wish to conclude \( \neg \text{flies}(\text{tweety}) \) if \( \text{heavy}(\text{tweety}) \), we simply want to prevent a conclusion \( \text{flies}(\text{tweety}) \).

The superiority relation among rules is used to define priorities among rules, that is, where one rule may override the conclusion of another rule. For example, given the defeasible rules

\[ r : \quad \text{bird}(X) \Rightarrow \text{flies}(X) \]
\[ r' : \quad \text{brokenWing}(X) \Rightarrow \neg \text{flies}(X) \]

which contradict one another, no conclusive decision can be made about whether a bird with a broken wing can fly. But if we introduce a superiority relation \( > \) with \( r' > r \), then we can indeed conclude that the bird cannot fly. We assume that \( > \) is acyclic. It turns out that we only need to define the superiority relation over rules with contradictory conclusions.
We now turn to a formal description of the language of defeasible logic and the inference rules of the defeasible logic DL.

We assume a signature $\Sigma$ containing predefined function and predicate symbols, including $=$, with their arities. Initially, we will assume that it contains only constants and $=$. Later we will consider the full generality of $\Sigma$. We also assume a signature $\Pi$ containing the set of predicate symbols, with their arities, that are defined by a defeasible theory.

An atom has the form $p(t_1, \ldots, t_n)$ where $p \in \Pi$ has arity $n$ and $t_1, \ldots, t_n$ are terms. A literal has the form $a$ or $\neg a$, where $a$ is an atom.

A rule $r$ consists of an optional label, an antecedent (or body) $A(r)$ which is a finite set of literals, an arrow, and its head, which is a literal. Given a set $R$ of rules, we denote the set of all strict rules in $R$ by $R_s$, the set of strict and defeasible rules in $R$ by $R_{sd}$, and the set of defeaters in $R$ by $R_d$. $R[q]$ denotes the set of rules in $R$ with head $q$. If $q$ is a literal, $\sim q$ denotes the complementary literal (if $q$ is a positive literal $p$ then $\sim q$ is $\neg p$; and if $q$ is $\neg p$, then $\sim q$ is $p$).

A defeasible theory $T$ is a triple $(F, R, >)$ where $F$ is a finite set of literals (called facts), $R$ a finite set of rules, and $>$ a superiority relation on the labels of $R$.

**Example 1.** We will use the following defeasible theory to demonstrate some elements of defeasible logic. We assume there are only the constants ethel and tweety in the language. Let $T_{bird} = (F_{bird}, R_{bird}, >_{bird})$ where: $F_{bird}$ is the set of facts

$$emu(ethel),$$
$$bird(tweety).$$

$R_{bird}$ is represented by the set of rule schemas

$$r_1 : \quad emu(X) \rightarrow bird(X).$$
$$r_2 : \quad bird(X) \Rightarrow flies(X).$$
$$r_3 : \quad heavy(X) \sim \neg flies(X).$$
$$r_4 : \quad brokenWing(X) \Rightarrow \neg flies(X).$$
$$r_5 : \quad \Rightarrow heavy(ethel).$$

and the superiority relation $>_{bird}$ contains only $r_4 >_{bird} r_2$.

The five rule schemas give rise to nine propositional rules by instantiating each variable to ethel and tweety respectively. Those rules, which make up $R_{bird}$ are

$$r_{1,e} : \quad emu(ethel) \rightarrow bird(ethel).$$
$$r_{1,t} : \quad emu(tweety) \rightarrow bird(tweety).$$
$$r_{2,e} : \quad bird(ethel) \Rightarrow flies(ethel).$$
$$r_{2,t} : \quad bird(tweety) \Rightarrow flies(tweety).$$
$$r_{3,e} : \quad heavy(ethel) \sim \neg flies(ethel).$$
$$r_{3,t} : \quad heavy(tweety) \sim \neg flies(tweety).$$
$$r_{4,e} : \quad brokenWing(ethel) \Rightarrow \neg flies(ethel).$$
$$r_{4,t} : \quad brokenWing(tweety) \Rightarrow \neg flies(tweety).$$
$$r_5 : \quad \Rightarrow heavy(ethel).$$
The rules have been re-labelled purely to simplify later reference\footnote{Without the re-labelling we would have different rules with the same label. This is not a problem, formally, but might be confusing.}. As a result, the superiority relation becomes \( \{ r_{4,e} >_{bird} r_{2,e}, r_{4,e} >_{bird} r_{2,t}, r_{4,t} >_{bird} r_{2,e}, r_{4,t} >_{bird} r_{2,t} \} \). As noted in earlier, the two statements \( r_{4,e} >_{bird} r_{2,t} \) and \( r_{4,t} >_{bird} r_{2,e} \) have no effect, since they do not involve rules with conflicting heads.

In this defeasible theory, \( R_s = \{ r_{1,e}, r_{1,t} \} \) and \( R_d[\neg flies(tweety)] = \{ r_{3,t}, r_{4,t} \} \).

A conclusion of \( T \) is a tagged literal and can have one of the following four forms:

\begin{itemize}
  \item \(+\Delta q\), which is intended to mean that the literal \( q \) is definitely provable in \( T \) (i.e., using only facts and strict rules).
  \item \( -\Delta q\), which is intended to mean that we have proved that \( q \) is not definitely provable in \( T \).
  \item \(+\partial q\), which is intended to mean that \( q \) is defeasibly provable in \( T \).
  \item \( -\partial q\) which is intended to mean that we have proved that \( q \) is not defeasibly provable in \( T \).
\end{itemize}

Thus, conclusions are meta-theoretical statements about provability. They do not appear in a defeasible theory. Notice the distinction between \( -\Delta \), which is used only to express failure-to-prove, and \( \neg \), which expresses classical negation. For example, \( -\Delta \neg flies(tweety) \) means that it has been proved that the negated proposition \( \neg flies(tweety) \) cannot be proved definitely in the defeasible theory.

Provability is based on the concept of a derivation (or proof) in \( T = (F,R,>) \). A derivation is a finite sequence \( P = (P(1), \ldots P(n)) \) of conclusions constructed by inference rules.

The preceding descriptions and definitions are common to many variants of defeasible logic, although in some variants the limitation to four tags and four forms of conclusion has been discarded\footnote{Without the re-labelling we would have different rules with the same label. This is not a problem, formally, but might be confusing.}. The inference rules we present below are specific to, and characterize, the defeasible logic \( DL \). We follow the presentation of Billington\footnote{Without the re-labelling we would have different rules with the same label. This is not a problem, formally, but might be confusing.}.

There are four inference rules for \( DL \) (corresponding to the four kinds of conclusion) that specify how a derivation can be extended. The formulation of these inference rules assumes a propositional defeasible theory. \( P(1..i) \) denotes the initial part of the sequence \( P \) of length \( i \):

\[ +\Delta : \text{We may append } P(i + 1) = +\Delta q \text{ if either } \]
\[ q \in F \text{ or } \exists r \in R_s[q] \forall a \in A(r) : +\Delta a \in P(1..i) \]

This means, to prove \(+\Delta q\) we need to establish a proof for \( q \) using facts and strict rules only. This is a deduction in the classical sense. To prove \( -\Delta q \) it is required to show that every attempt to prove \(+\Delta q\) fails in a finite time. Thus the inference rule for \(-\Delta\) is the constructive complement of the inference rule for \(+\Delta\).\footnote{Without the re-labelling we would have different rules with the same label. This is not a problem, formally, but might be confusing.}
We may append $P(i + 1) = -q$ if

$q \notin F$ and

$\forall r \in R_s[q] \exists a \in A(r) : -\Delta a \in P(1..i)$

From $T_{bird}$ in Example 1 we can infer $+\Delta mu(ethel)$ (and $+\Delta bird(tweety)$) immediately, in a proof of length 1. Using $r_{1,e}$ and the second clause of the inference rule, we can infer $+\Delta bird(ethel)$ in a proof of length 2. We can infer $-\Delta heavy(tweety)$ and $-\Delta \neg flies(tweety)$ immediately (among many others), since in these cases $R_s[q]$ is empty.

The inference rule for defeasible conclusions is complicated by the defeasible nature of DL: opposing chains of reasoning must be taken into account.

We may append $P(i + 1) = +p$ if either

1. $+\Delta q \in P(1..i)$ or
2. (2.1) $\exists r \in R_s[q] \forall a \in A(r) : +\partial a \in P(1..i)$ and
   (2.2) $-\Delta \neg q \in P(1..i)$ and
   (2.3) $\forall s \in R[\neg q]$ either
      (2.3.1) $\exists a \in A(s) : \neg \partial a \in P(1..i)$ or
      (2.3.2) $\exists t \in R_s[q]$ such that $\forall a \in A(t) : +\partial a \in P(1..i)$ and $t > s$

Let us work through this inference rule. To show that $q$ is provable defeasibly we have two choices: (1) We show that $q$ is already definitely provable; or (2) we need to argue using the defeasible part of $T$ as well. In particular, we require that there must be a strict or defeasible rule with head $q$ which can be applied (2.1). But now we need to consider possible “attacks”, that is, reasoning chains in support of $\neg q$. To be more specific: to prove $q$ defeasibly we must show that $\neg q$ is not definitely provable (2.2). Also (2.3) we must consider the set of all rules which are not known to be inapplicable and which have head $\neg q$ (note that here we consider defeaters, too, whereas they could not be used to support the conclusion $q$; this is in line with the motivation of defeaters given earlier). Essentially each such rule $s$ attacks the conclusion $q$. For $q$ to be provable, each such rule $s$ must be counterattacked by a rule $t$ with head $q$ with the following properties: (i) $t$ must be applicable at this point, and (ii) $t$ must be stronger than $s$. Thus each attack on the conclusion $q$ must be counterattacked by a stronger rule.

From $T_{bird}$ in Example 1 we can infer $+\partial bird(ethel)$ in a proof of length 3, using part (1) of the $+\partial$ inference rule. We can infer $+\partial bird(ethel)$ in a proof of length 3, using (1). Other applications of this inference rule first require application of the inference rule $-\partial$. This inference rule completes the definition of the proof system of defeasible logic. It is a strong negation of the inference rule $+\partial$.

We may append $P(i + 1) = -p$ if

1. $-\Delta q \in P(1..i)$ and
2. (2.1) $\forall r \in R_s[q] \exists a \in A(r) : \neg \partial a \in P(1..i)$ or
   (2.2) $+\Delta \neg q \in P(1..i)$ or
It is formulated in a truth-functional manner to simplify later statements, but $D$ is interpreted as identity on $A$. Here $D$ is a set of definite reasoning. It is clear that $DL$ is a paraconsistent logic. If a defeasible theory $T$ contains facts $a$ and $\neg a$ and a rule $b \rightarrow b$, then the inconsistency concerning $a$ has no effect on what is provable about $b$. When considering defeasible reasoning, a main feature of defeasible logics is that potentially inconsistent inferences are avoided either by use of the superiority relation or by simply failing to infer either of the consequences, as in the previous discussion of a bird with a broken wing. Some inconsistent defeasible inferences may be made, as consequences of inconsistent definite inferences, but the logic displays the same kind of paraconsistency as discussed above for definite reasoning.

3 A Model-Theoretic Semantics

A domain is a $\Sigma$-structure $D$ that defines a set $D$ of objects over which formulas may be interpreted, and gives a meaning to all function symbols and pre-defined predicate symbols as functions (respectively, relations) over $D$. $=$ is interpreted as identity on $D$. Given a domain $D$, the $D$-base of a language is $B_D = \{ p(d_1, \ldots, d_n), \neg p(d_1, \ldots, d_n) \mid n \geq 0, p \in \Pi, p \text{ has arity } n, d_i \in D \text{ for } i = 1, 2, \ldots, n \}$. We sometimes refer to elements of $B_D$ as interpreted literals.

The following definition introduces the notion of a defeasible interpretation. It is formulated in a truth-functional manner to simplify later statements, but
it might equally well be formulated without introducing truth, at the expense of clusier definitions later.

**Definition 1.** A defeasible interpretation \( \mathcal{I} \) consists of a domain \( \mathcal{D}_I \), and two partial functions \( \mathcal{I}_\Delta \) and \( \mathcal{I}_\emptyset \) which map \( B_{\mathcal{D}_I} \) to \{True, False\}, which satisfy the following conditions:

\[
\forall q \in B_{\mathcal{D}_I} \quad \mathcal{I}_\Delta(q) = \text{True} \rightarrow \mathcal{I}_\emptyset(q) = \text{True}
\]

and

\[
\forall q \in B_{\mathcal{D}_I} \quad \mathcal{I}_\emptyset(q) = \text{False} \rightarrow \mathcal{I}_\Delta(q) = \text{False}
\]

A \( \mathcal{D} \)-interpretation is a defeasible interpretation over the domain \( \mathcal{D} \).

Intuitively, \( \mathcal{I}_\Delta \) specifies those evaluated literals that are known definitely (map to True) or known not to be known definitely (map to False). Similarly, \( \mathcal{I}_\emptyset \) specifies those evaluated literals that are believed defeasibly (map to True) or known not to be defeasibly believed (map to False). Thus a defeasible interpretation represents a state of mind in which some evaluated literals are known or believed, while for others there is awareness that they are not known or believed, and still others have undefined status – neither known, nor unknown; neither believed, nor unbelieved. We refer to the values of \( \mathcal{I}_\Delta \) and \( \mathcal{I}_\emptyset \) on an evaluated literal \( q \) as the epistemic status of \( q \) in \( \mathcal{I} \).

The two conditions on defeasible interpretations enforce that that which is known is believed, and that which is not believed is not known. In other words, they enforce the expected relationship between knowledge and belief.

The functions \( \mathcal{I}_\Delta \) and \( \mathcal{I}_\emptyset \) are easily extended to conjunctions of evaluated literals. The extension follows the same pattern for both functions, so we show only \( \mathcal{I}_\Delta \).

\[
\mathcal{I}_\Delta(q_1, \ldots, q_n) = \bigwedge_{i=1}^n \mathcal{I}_\Delta(q_i)
\]

where the truth table for \( \land \) is

|       | True | False | undefined |
|-------|------|-------|-----------|
| True  | True | False | undefined |
| False | False| False | False     |
| undefined | undefined | False | undefined |

Among the \( \mathcal{D} \)-interpretations, we distinguish the \( \mathcal{D} \)-models of a defeasible theory \( T \): those \( \mathcal{D} \)-interpretations which are consistent with the rules of reasoning specified by \( T \) and \( DL \).

\(^2\) Actually these conditions need not be imposed: even if we did not impose them on interpretations, all models of a defeasible theory satisfy these conditions. But we prefer to explicitly require that interpretations respect this fundamental relationship between definite and defeasible knowledge.
Definition 2. A $\mathcal{D}$-interpretation $I$ is a $\mathcal{D}$-model if it satisfies the following four conditions. Let $q$ range over elements of $\mathcal{B}_D$.

$I_\Delta(q) = \text{True}$ iff
$q \in F \text{ or } \exists r \in R_s[q] \ I_\Delta(A(r)) = \text{True}$

$I_\Delta(q) = \text{False}$ iff
$q \notin F \text{ and } \forall r \in R_s[q] \ I_\Delta(A(r)) = \text{False}$

$I_\partial(q) = \text{True}$ iff
(1) $I_\Delta(q) = \text{True}$ or
(2) (2.1) $\exists r \in R_{sd}[q] \ I_\partial(A(r)) = \text{True}$ and
(2.2) $I_\partial(q) = \text{False}$ and
(2.3) $\forall s \in R[q] \text{ such that}$
\hspace{0.5cm} (2.3.1) $I_\partial(A(s)) = \text{False}$ or
\hspace{0.5cm} (2.3.2) $\exists t \in R_{sd}[q] \text{ such that}$
\hspace{1cm} $I_\partial(A(t)) = \text{True and } t \succ s$

$I_\partial(q) = \text{False}$ iff
(1) $I_\Delta(q) = \text{False}$ and
(2) (2.1) $\forall r \in R_{sd}[q] \ I_\partial(A(r)) = \text{False}$ or
(2.2) $I_\Delta(q) = \text{True}$ or
(2.3) $\exists s \in R[q] \text{ such that}$
\hspace{0.5cm} (2.3.1) $I_\partial(A(s)) = \text{True}$ and
\hspace{1cm} (2.3.2) $\forall t \in R_{sd}[q] \text{ such that}$
\hspace{1.5cm} $I_\partial(A(t)) = \text{False or } t \prec s$

Clearly there is a close correspondence between these conditions and the inference rules of $DL$. The four conditions are if-and-only-if statements. The “if” part ensures that all models are deductively closed. The “only-if” part ensures that models are abductively closed, in the following sense: if an evaluated literal has some epistemic status in a model, then there is a reason, in the model, why that status is given.

We must also represent the meaning of the symbols of $\Sigma$. We might simply use a $\Sigma$-structure $\mathcal{D}$ for this purpose, but it turns out that, in general, models based on the intended $\Sigma$-structure $\mathcal{D}$ do not characterize provability in $DL$.

Instead we will use the first-order theory of $\mathcal{D}$, denoted $Th(\mathcal{D})$. Thus we will consider all domains that agree with $\mathcal{D}$ on first-order sentences. We write $Th(\mathcal{D}), T \models_{DL} c$ to denote that the conclusion $c$ holds in all defeasible models $M$ of $T$ such that $\mathcal{D}_M$ is a model of $Th(\mathcal{D})$ (in the conventional sense of first-order logic). That is, if $c$ is $+\Delta q$ then $M_\Delta(q) = \text{True}$, if $c$ is $-\Delta q$ then $M_\Delta(q) = \text{False}$, etc. We say that $c$ is a defeasible logical consequence of $T$ and $Th(\mathcal{D})$.

Taking the defeasible models of $Th(\mathcal{D}), T$ as the semantics of the defeasible theory $T$ over $\mathcal{D}$, it is straightforward to prove that the proof system is sound.
Theorem 1. Let $T$ be a defeasible theory over a domain $D$. Let $c$ be a conclusion.

If $T \vdash c$ then $Th(D), T \models_{DL} c$

The proof is by induction on the length of proofs.

4 Completeness

We now establish the completeness of the inference rules with respect to these models. We do this by establishing an intimate connection between $D$-models of a defeasible theory $T$ and partial models of a corresponding metaprogram. In this section we only outline the proof, since there is not enough space to present the requisite background material.

In [20] we developed a logic metaprogram $M_T$ for $DL$. If $T$ is a defeasible theory, then $M_T$ is $M$ augmented with a representation of $T$. It was shown in [20] that the consequences of $T$ are characterized by $M_T$ under Kunen’s semantics for logic programs [17].

Theorem 2 ([20]). Let $T$ be a defeasible theory and let $M_T$ denote its metaprogram counterpart. Let $H$ be the Herbrand domain over which $M_T$ is defined.

For each literal $p$,

1. $T \models +\Delta p$ iff $M_T, Th(H) \models_K \text{definitely}(p)$;
2. $T \models -\Delta p$ iff $M_T, Th(H) \models_K \neg\text{definitely}(p)$;
3. $T \models +\partial p$ iff $M_T, Th(H) \models_K \text{ defeasibly}(p)$;
4. $T \models -\partial p$ iff $M_T, Th(H) \models_K \neg\text{ defeasibly}(p)$;

Kunen also characterized his semantics of logic programs in terms of 3-valued models of the logic program. For any logic program $P$ and literal $q$, Kunen showed that $P, Th(H) \models_K q$ iff $P, Th(H) \models_K ^* q$ [17], where $\models_K$ denotes the 3-valued logical consequence relation.

The remaining step is to relate defeasible models of a defeasible theory $T$ with 3-valued models of $M_T$. There is a technical difficulty: defeasible interpretations of $T$ involve domains with signature $\Sigma$, whereas the signature of domains of $M_T$ extends $\Sigma$ with function symbols corresponding to $\Pi$ (and other, auxiliary function symbols). The problem arises because the metaprogram represents predicates as functions.

We address this problem by defining, for every domain $D$ over $\Sigma$, a domain extension $D^*$ over the extended signature. We are then able to relate $D$-interpretations of $T$ with 3-valued interpretations of $M_T$ over $D^*$. Thus we obtain

Proposition 1. Let $T$ be a defeasible theory over signature $\Sigma$ and let $D$ be a domain over this signature.

- $M_T, Th(D^*) \models_K \text{definitely}(q)$ iff $T, Th(D) \models_{DL} +\Delta q$
- $M_T, Th(D^*) \models_K \neg\text{definitely}(q)$ iff $T, Th(D) \models_{DL} -\Delta q$
• $\mathcal{M}_T, Th(D^*) \models_K \text{ defeasibly}(q)$ iff $T, Th(D) \models_{DL + \partial q}$
• $\mathcal{M}_T, Th(D^*) \models_K \neg \text{ defeasibly}(q)$ iff $T, Th(D) \models_{DL - \partial q}$

Combining these three results together, the completeness (and soundness) of the proof system for the model theory is established.

**Theorem 3.** Let $T$ be a defeasible theory over signature $\Sigma$ and let $\mathcal{H}$ be the Herbrand domain over this signature. Let $c$ be any conclusion.

$T \vdash c$ iff $Th(\mathcal{H}), T \models_{DL} c$

5 Other Defeasible Logics

The approach of this paper is more widely applicable than simply to $DL$. In the following, we first look at the extension of this approach to a form of $DL$ that is not essentially propositional. We then discuss its application to other defeasible logics that can be formulated within the flexible framework of $\Sigma$. We only have space to outline this work.

5.1 Non-Propositional Defeasible Logic

The development of the model theory in terms of $D$ and $Th(D)$ is an overkill, technically, for an essentially propositional logic, such as the presentation of $DL$ in Section 2. However, this development makes it easy to extend $DL$ to a first-order logic, and to extend it further with pre-defined functions and predicates – constraints in the sense of constraint logic programming.

The only change to the model theory is that the restriction on $\Sigma$ (to constants and $=$) is removed, with consequent flow-on affects on $D$.

The proof system, as presented in Section 2, is inadequate to handle infinite domains, if rules are regarded as schemas for propositional rules. However [20] gives a bottom-up reformulation of the proof system that can handle infinite domains, and is easily extended to address constraints.

With this reformulated proof system, the soundness and completeness continues to hold in the presence of constraints over infinite domains. There are some elements of the proofs that must change. In particular, the metaprogram $\mathcal{M}$ must now accommodate constraints and Theorem 2 must be extended. But such changes are not difficult. The extension of Kunen’s result, equating $\vdash_K$ and $\models_K$, to arbitrary domains is due to Stuckey [29].

5.2 Variants of $DL$

As proposed in [2], the metaprogram presentation of $DL$ can be generalized to a framework for the definition of many different defeasible logics. [2] presents some variants of $DL$ with different proof-theoretic properties. Some of these variants involve extra levels of belief, beyond definite and defeasible knowledge.

Since failure-to-prove in these variants is still characterized by Kunen’s semantics, we can directly apply the techniques of this paper to establish a model
theory and prove soundness and completeness of these variants. Technically, all
that is needed is a definition of model appropriate to the inference rules of the
logic and a corresponding metaprogram.

The framework of [2] admits other forms of failure-to-prove than Kunen’s se-
manetics. Any semantics for logic programs provides an alternate form of failure-
to-prove. Indeed, [2] showed that Well-Founded Defeasible Logic is character-
ized by metaprogram of Section 4 and the well-founded semantics of logic pro-
grams [30].

In the well-founded semantics – and many other semantics of logic programs
– the semantics rests on a single domain of values $D$. Thus, in these cases, the
introduction of $Th(D)$ is unnecessary. In the case of Well-Founded Defeasible
Logic there is, essentially, a single model of a defeasible theory $T$ over $D$, which
is the reduct of the well-founded (partial) model of $M_T$. The defeasible logic
investigated in [2][3] performs loop-checking to detect failure-to-prove, which
makes failure-to-prove in that logic similar to well-founded semantics. This sug-
gests that a model-theoretic semantics for which that logic is sound and com-
plete might be established by following the pattern of this paper, but using the
well-founded semantics in place of Kunen’s semantics and using a substantially
different metaprogram.

Other semantics, such as the stable model semantics [9], give rise to several
$D$-models for a defeasible theory in the corresponding defeasible logic.

6 Conclusion

We have introduced an approach to defining an appropriate model theory for
defeasible logics. The approach was demonstrated in detail for the logic $DL$, and
we outlined how it can be applied to a wide range of defeasible logics.

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