Heating the $O(N)$ nonlinear sigma model

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Abstract
The thermodynamics of the $O(N)$ nonlinear sigma model in $1 + 1$ dimensions is studied. We calculate the finite temperature effective potential in leading order in the $1/N$ expansion and show that at this order the effective potential can be made finite by temperature independent renormalization. We will show that this is no longer possible at next-to-leading order in $1/N$. In that case one can only renormalize the minimum of the effective potential in a temperature independent way, which gives us finite physical quantities like the pressure.

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1 Introduction

The nonlinear sigma model is a scalar field theory with an $O(N)$ symmetry. It is described by a Lagrangian density which only consists of a kinetic term,

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi_i \partial^\mu \phi_i ,$$

and a constraint which enforces all the $\phi$ fields to lie on a $N-1$ sphere:

$$\phi_i(x) \phi_i(x) = N/g^2 \quad i = 1 \ldots N .$$

This model has some nice features in 1 + 1 dimensions, which makes it interesting to study as a toy model for QCD. First it is renormalizable. Furthermore it is asymptotically free, such that at very high temperatures it approaches a free field theory. The model also has a dynamically generated mass for the $\phi$ fields. If $N = 3$ the model has instanton solutions. Finally, for $N = 2$ we recover a free field theory, which can be used as a check of the calculations.

In this article we will study the thermodynamical properties of the nonlinear sigma model. In particular we will calculate the pressure. In Sec. 2 we briefly discuss some aspects of thermal field theory. In Sec. 3 we calculate the pressure in the weak-coupling expansion. In Sec. 4 we calculate the effective potential and pressure to leading order in the $1/N$ expansion. The next-to-leading order (NLO) correction is discussed in Sec. 5.

2 The pressure in a field theory

In this section we briefly review how one calculates the pressure in a thermal field theory. For a more complete introduction see Refs. [1, 2].

In classical statistical mechanics one can derive all thermodynamic quantities from the partition function. The partition function $Z$ is given by

$$Z = \sum_n \langle n \mid \exp[-\beta \hat{H}] \mid n \rangle ,$$

where the sum is over all eigenstates of the Hamiltonian $\hat{H}$ and $\beta = 1/T$, the inverse temperature. For example the pressure $P$ is given by

$$P = \frac{1}{\beta} \frac{\partial \log Z}{\partial V} .$$

We next express the partition function in terms of fields. The easiest way to do this is to consider a transition matrix element in ordinary field theory. One can write such a transition element in terms of a path integral in the following way

$$\langle \phi_f \mid \exp[-i(t_f - t_i)\hat{H}] \mid \phi_i \rangle = \int \mathcal{D}\phi \ \exp \left[ i \int_{t_i}^{t_f} dt \int d^d x \mathcal{L}(\phi) \right] ,$$

where $\mathcal{L}$ is a Lagrangian density which has a Minkowskian metric and does not have derivative interactions. Now if one makes the identification $t = -i \tau$ one finds

$$\langle \phi_f \mid \exp[-\beta \hat{H}] \mid \phi_i \rangle = \int \mathcal{D}\phi \ \exp \left[ -\int_0^\beta d\tau \int d^d x \mathcal{L}(\phi) \right] ,$$
where we from now on denote the zero component of a $(d + 1)$-vector by $\tau$ and hence use a Euclidean metric. The last equation enables us to write the partition function in terms of a path integral,

$$
\mathcal{Z} = \int \mathcal{D}\phi \exp \left[ -\int_0^\beta d\tau \int d^d x \mathcal{L}(\phi) \right] \delta(\phi(\tau=0)=\phi(\tau=\beta)) ,
$$

where one implicitly integrates over all states which obey the periodicity condition $\phi(\tau = 0, \vec{x}) = \phi(\tau = \beta, \vec{x})$. So equilibrium thermal field theory is in essence a Euclidean field theory where one dimension ($\tau$) is compactified to a circle. As a consequence, the Fourier transform of a field becomes a sum over modes,

$$
\phi(\tau, \vec{x}) = \frac{1}{\beta} \sum_n \int \frac{d^d k}{(2\pi)^d} e^{i\omega_n \tau + i\vec{k} \cdot \vec{x}} \tilde{\phi}(k) = \sum\int_{\mathcal{K}} e^{i\omega_n \tau + i\vec{k} \cdot \vec{x}} \tilde{\phi}(k) ,
$$

where $\omega_n = \frac{2\pi n}{T}$. This implies that in a loop diagram one should not take the integral over internal momentum but rather the sum-integral $\sum\int_{\mathcal{K}}$.

Now for example the partition function of the nonlinear sigma model is given by

$$
\mathcal{Z} = \int \prod_{i=1}^N \mathcal{D}\phi_i \prod_x \delta(\phi_i(x)\phi_i(x) - N/g^2) \exp \left[ -\int_0^\beta d\tau \int dx \mathcal{L}(\phi) \right] ,
$$

where from now on we work in one spatial dimension. To obtain the pressure we have to calculate $\mathcal{Z}$. We will follow two paths. The first one is making an expansion around $g^2 = 0$. This will only give us the leading term of the pressure. The second way is an expansion in $1/N$ which will generates additional contributions which are non-analytical in $g^2$.

### 3 The pressure in the weak-coupling expansion

One can get rid of the constraint by integrating out one of the $\phi$ fields, which results in

$$
\mathcal{Z} = \int \prod_{i=1}^{N-1} \mathcal{D}\pi_i \prod_x \theta(N/g^2 - \pi_i\pi_i) \exp \left[ -\int_0^\beta d\tau \int dx \mathcal{L}_{\text{eff}}(\pi) \right] ,
$$

where $\theta(x)$ is the step function and the effective Lagrangian density $\mathcal{L}_{\text{eff}}$ is given by

$$
\mathcal{L}_{\text{eff}}(\pi) = \frac{1}{2} \partial_\mu \pi_i \partial^\mu \pi_i + \frac{g^2}{2} \frac{(\pi_i\partial_\mu \pi_i)^2}{N - g^2 \pi_i\pi_i} - \frac{1}{2} \beta V \log \left( N/g^2 - \pi_i\pi_i \right) .
$$

For small values of $g^2$ the $\theta(x)$ function is only vanishing when $\pi(x)$ is large. Since large values of $\pi$ give a small contribution to the path integral we approximate $\theta(N/g^2 - \pi_i\pi_i) \approx 1$ which gives

$$
\mathcal{Z} = \int \prod_{i=1}^{N-1} \mathcal{D}\pi_i \exp \left[ -\int_0^\beta d\tau \int d^d x \mathcal{L}_{\text{eff}}(\pi) \right] .
$$

We will not calculate $\mathcal{Z}$ but rather $\frac{1}{\beta V} \log \mathcal{Z}$, where $V$ is the volume of our 1 dimensional space. Because $\log \mathcal{Z}$ is an extensive quantity, i.e. it is linear in $V$, the pressure is equal to $\frac{1}{\beta V} \log \mathcal{Z}$. Since in general $\frac{1}{\beta V} \log \mathcal{Z}$ does not vanish at zero temperature, we subtract the zero temperature contribution to normalize the pressure to zero at zero temperature.
If $g^2 = 0$ it can be seen from $\mathcal{L}_{\text{eff}}$ that one has $N - 1$ noninteracting $\pi$ fields. Hence it is easy to show that leading term is equal to the pressure of $N - 1$ free fields

$$
P = - \frac{N-1}{2} \left[ \sum_K \log(K^2) - \int_K \log(K^2) \right] = (N-1) \frac{\pi}{6} T^2 ,
$$

(13)

where $K = (\omega_n, k)$ is a Euclidean two vector and we defined

$$
\int_K \equiv \int \frac{d^2k}{(2\pi)^2}.
$$

(14)

By calculating the loop diagrams, one can show that up to and including order $g^4$ one only finds the pressure of a free gas in $d = 1 + 1$ [3, 4]. However one finds corrections to the free pressure in a $1/N$ expansion. This may indicate that the pressure is completely non-analytical in $g^2$.

4 The effective potential in leading order in $1/N$

Another way to implement the constraint on the $\phi$ fields is by using a Lagrange multiplier field which we will denote by $\alpha$. This gives the following expression for the partition function,

$$
\mathcal{Z} = \int \prod_{i=1}^N D\phi_i D\alpha \exp \left\{ -\frac{1}{2} \int_0^\beta d\tau \int dx \partial^\mu \phi_i \partial_\mu \phi_i \\
- \frac{1}{2} \int_0^\beta d\tau \int dx (\phi_i(x)\phi_i(x) - N/g^2) \right\} .
$$

(15)

In this way the action still is quadratic in the $\phi$ fields, so one can easily integrate them out. This gives

$$
\mathcal{Z} = \int D\alpha \exp \left\{ -S[\alpha] + \frac{N}{2g^2} \int_0^\beta d\tau \int dx \alpha(x) \right\} ,
$$

(16)

where

$$
S[\alpha] = \frac{N}{2} \text{Tr} \log[-\partial^2 + \alpha(x)] .
$$

(17)

The pressure is equal to the minimum of the effective potential, which one can calculate by expanding the $\alpha$ field around its vacuum expectation value $m^2$. By considering the propagator of the $\phi$ fields, one can show that to leading order in $1/N$, $m$ is equal to the physical mass of the $\phi$ fields. This is, however, not longer the case at NLO, [5, 6]. The effective potential can be obtained from the effective action by division by $\beta V$. To calculate the effective potential we write $\alpha = m^2 + \tilde{\alpha}/\sqrt{N}$ and expand the action around $m^2$ [7],

$$
S[\alpha] = \frac{N}{2} \text{Tr} \log[-\partial^2 + m^2] + \frac{\sqrt{N}}{2} \text{Tr} \left( \frac{1}{-\partial^2 + m^2} \tilde{\alpha} \right) + \frac{1}{4} \text{Tr} \left( \frac{1}{-\partial^2 + m^2} \tilde{\alpha} \right)^2 + \mathcal{O}(1/\sqrt{N}) .
$$

(18)

From this equation it can easily be seen that the effective potential can be calculated in a $1/N$ expansion. The leading order effective potential is given by the classical action. The corrections are obtained by integrating over the $\tilde{\alpha}$ field.

To calculate the leading order effective potential we introduce a momentum cutoff $\Lambda$ and subtract $m$ and $T$-independent constants from the effective potential. This subtraction will not
change the physics, since it only shifts the whole effective potential by a constant. One finds for the effective potential at leading order in $1/N$

$$V(m^2) = \frac{Nm^2}{2g_b^2} - \frac{N}{2} \left[ \sum_P \log(P^2 + m^2) - \int_P \log(P^2) \right]$$

$$= \frac{N}{2} \left[ \frac{m^2}{g_b^2} - \frac{m^2}{4\pi} \left( 1 + \log \frac{\Lambda^2}{m^2} \right) + \frac{T^2}{4\pi} J_0(\beta m) \right], \quad (19)$$

where $g_b$ is the bare coupling constant. $J_0(\beta m)$ is given by

$$J_0(\beta m) = \frac{8}{T^2} \int_0^\infty dp \frac{p^2 n(\omega_p)}{\omega_p}, \quad (21)$$

where $n(x) = 1/(e^x - 1)$ and $\omega_p^2 = p^2 + m^2$. One is able to renormalize the leading order effective potential in a temperature independent way by replacing $g_b^2 \rightarrow Z g_b^2(\mu)$ where

$$\frac{1}{Z g^2} = 1 + \frac{g^2}{4\pi} \log \frac{\Lambda^2 g^2(\mu)}{\mu^2}. \quad (22)$$

and $g^2 = g^2(\mu)$. From this equation it follows that the $\beta$-function of $g^2$ is given by

$$\beta(g^2) \equiv \mu \frac{dg^2(\mu)}{d\mu} = -\frac{g^4(\mu)}{2\pi}. \quad (23)$$

The leading order $\beta$-function is exact in $g^2$. Since the $\beta$-function is negative, $g^2$ approaches zero for large values of $\mu$. This shows that the theory is asymptotically free.

With use of the renormalization of the coupling constant one finds the following finite expression for the effective potential

$$V(m^2) = \frac{N}{2} \left[ \frac{m^2}{g^2} - \frac{m^2}{4\pi} \left( 1 + \log \frac{\mu^2}{m^2} \right) + \frac{1}{4\pi} T^2 J_0(\beta m) \right]. \quad (24)$$

One can easily show that the effective potential is independent of the renormalization scale $\mu$. This is expected since the choice of $\mu$ is completely arbitrary.

To obtain the pressure, one has to minimize the effective potential with respect to $m^2$. Minimization gives the so-called gap equation

$$\frac{1}{g^2} = \sum_P \frac{1}{P^2 + m^2} = \frac{1}{4\pi} \log \left( \frac{\mu^2}{m^2} \right) + \frac{1}{4\pi} J_1(\beta m) \equiv \frac{1}{4\pi} \log \left( \frac{\mu^2}{m^2} \right), \quad (25)$$

where $J_1(\beta m)$ is defined by

$$J_1(\beta m) = 4 \int_0^\infty dp \frac{n(\omega_p)}{\omega_p}. \quad (26)$$

The solution of the gap equation determines the leading order physical mass of the $\phi$ fields as a function of temperature. At $T = 0$ one can solve this equation to show that the mass is completely non-analytical in $g^2$,

$$m_{T=0} = \mu \exp \left( -\frac{2\pi}{g^2} \right). \quad (27)$$

We can use Eq. (27) to normalize the minimum of the effective potential at $T = 0$ to be zero which gives

$$V(m^2) = \frac{N}{2} \left[ \frac{m^2}{g^2} - \frac{m^2}{4\pi} \left( 1 + \log \frac{\mu^2}{m^2} \right) + \frac{1}{4\pi} T^2 J_0(\beta m) + \frac{m_{T=0}^2}{4\pi} \right]. \quad (28)$$
The effective potential as a function of $m$ for different temperatures is shown in Fig. 1, for the arbitrary choice $g^2(\mu = 500) = 10$. The quantities $T$, $m$, $\mu$, $V/T$ and $P/T$ are all in the same arbitrary units. The solid curve which is the minimum of the effective potential is equal to the pressure.

Figure 1: The leading order effective potential as function of $m$ for different temperatures with $g^2(\mu = 500) = 10$.

5 Next-to-leading order correction in $1/N$

The term linear in $\alpha$ in Eq. (18) gives no contribution to the effective potential since it gives rise to a tadpole [8]. The first $1/N$ correction to the effective potential stems from the last term of Eq. (18). By going to momentum space one can show that the correct ion is given by

$$V_1(m^2) = -\frac{1}{2} \sum P \log \left[ \frac{1}{Q} \frac{1}{Q^2 + m^2} \right] .$$  \hspace{1cm} (29)

We calculated this correction in Ref. [6]. In the limit $\Lambda \to \infty$, one obtains

$$V_1(m^2) = -\frac{1}{8\pi} \left( \Lambda^2 \ln \frac{\Lambda^2}{m^2} - \bar{m}^2 \ln \frac{\Lambda^2}{\bar{m}^2} \right) - \frac{m^2}{4\pi} \left( \ln \ln \frac{\Lambda^2}{\bar{m}^2} - \ln \frac{\Lambda^2}{4\bar{m}^2} \right) + F(m,T) ,$$ \hspace{1cm} (30)

where $\bar{m}$ is defined in Eq. (25). In (30), we have subtracted $m$ and $T$-independent constants and dropped terms that vanish in the limit $\Lambda \to \infty$. $F(m,T)$ is a finite term and the logarithmic integral $\text{li}$ is defined by

$$\text{li}(x) = \mathcal{P} \int_0^x \frac{dt}{\log t} ,$$ \hspace{1cm} (31)

where $\mathcal{P}$ stands for principal value. The first two terms of Eq. (30) are problematic. It is impossible to remove these divergences by renormalizing $g^2$ in a temperature independent way or by subtracting $m$ and $T$-independent constants. However this is possible at the minimum of the effective potential. At the minimum, one can use the leading order gap equation, Eq. (25), to show that $\bar{m}$ is independent of $T$. So one could add

$$\frac{\Lambda^2}{8\pi} \left\{ \ln \frac{4\pi}{g_b^2} - \exp \left( -\frac{4\pi}{g_b^2} \right) \ln \left[ \exp \left( \frac{4\pi}{g_b^2} \right) \right] \right\}$$ \hspace{1cm} (32)
to the effective potential which yields an effective potential that can be renormalized at the
minimum. Using this renormalization at the minimum we have calculated the pressure \( \mathcal{P} \) as a
function of \( N \). The result is depicted in Fig. 2. One clearly sees a crossover which is not a phase
transition. This is in accordance with the Mermin-Wagner-Coleman theorem \([9, 10]\) which forbids
spontaneous breakdown of a continuous symmetry in \( 1 + 1 \) dimensions. The figure furthermore
shows that the \( 1/N \) expansion is relatively good since the corrections are really of order \( 1/N \).
Finally it can be seen from the figure that the theory is asymptotically free, because in the limit
\( T \to \infty \) the pressure approaches the pressure of a free gas, Eq. (13).

Figure 2: Pressure \( \mathcal{P} \) normalized to \( NT^2 \) as a function of temperature for different values of \( N \)
with \( g^2(\mu = 500) = 10 \) \([6]\).

6 Summary and Conclusions

We find that the pressure of the nonlinear sigma model in the weak-coupling expansion through
order \( g^4 \) only consist of the free term. Furthermore, we showed that in a \( 1/N \) expansion we
can renormalize the leading order effective potential in a temperature independent way. This is,
however, impossible for the effective potential at next-to-leading order in \( 1/N \). In that case one
can only renormalize in a temperature-independent way a physical quantity, like the pressure.

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