On spanning maximum $k$-edge-colorable subgraphs

Vahan V. Mkrtchyan$^{ab}$, and Gagik N. Vardanyan$^a$

$^a$Department of Informatics and Applied Mathematics, Yerevan State University, Yerevan, 0025, Armenia

$^b$Institute for Informatics and Automation Problems, National Academy of Sciences of Republic of Armenia, 0014, Armenia

A subgraph $H$ of a graph $G$ is called spanning, if any vertex of $G$ is not isolated in $H$, while it is called maximum $k$-edge-colorable, if $H$ is $k$-edge-colorable and contains as many edges as possible. We show that any connected graph containing a matching that misses at most one vertex, has a spanning maximum 2-edge-colorable subgraph. We also show that any graph whose minimum degree is at least two and maximum degree is $r$, $r \geq 3$, has a spanning maximum $(r - 1)$-edge-colorable subgraph. This particularly, implies that any graph whose vertices are of degree two or three, has a spanning maximum 2-edge-colorable subgraph. In the end of the paper we present a conjecture, which claims that any almost regular graph has a spanning maximum 2-edge-colorable subgraph.

1. Introduction

Let $N$ denote the set of positive integers. In this paper graphs are assumed to be finite, undirected and without loops, though they may contain multiple edges. Graphs having no multiple edges will be called simple. If $G$ is a graph, then for a vertex $x \in V(G)$ $d_G(x)$ denotes the degree of $x$ in $G$. Moreover, let $\Delta(G)$ and $\delta(G)$ denote the maximum and minimum degrees of vertices in $G$, respectively. A vertex is defined to be isolated in $G$, if its degree is zero. If $G'$ is a subgraph of $G$, then we say that $G'$ covers (misses) a vertex $x$ of $G$, if $d_{G'}(x) \geq 1$ ($d_{G'}(x) = 0$). A subgraph is spanning, if it covers all the vertices of the graph. A point that should be made clear here, is that if a vertex $x$ of $G$ is not a vertex of a subgraph $G'$, then we assume that $d_{G'}(x) = 0$.

The length of a path $P$ of a graph $G$ is the number of edges lying on $P$. If $G$ is a connected graph, then for any two vertices $u$ and $v$ let $\rho(u, v)$ denote the length of a shortest path connecting these two vertices. Usually $\rho(u, v)$ is called the distance between the vertices $u$ and $v$ in the graph $G$.

A subset of edges of a graph is called matching, if it contains no adjacent edges. Usually, a vertex that is (not) incident to an edge from a matching, is said to be covered (missed).
by the matching. A matching is maximum, if it contains as many edges as possible, while it is perfect, if any vertex is incident to an edge from the matching.

A k-edge-coloring of a graph G is a partition of E(G) into k matchings. Usually, these matchings into which E(G) is partitioned, are called color-classes of the edge-coloring. The least integer k for which G has a k-edge-coloring is called the chromatic index of G and is denoted by χ′(G). Clearly, χ′(G) ≥ Δ(G) for any graph G, and the following classical theorems of Shannon and Vizing give non-trivial upper bounds for χ′(G):

**Theorem 1 (Shannon [19])**. For every graph G
\[ \Delta(G) \leq \chi'(G) \leq \left\lceil \frac{3\Delta(G)}{2} \right\rceil. \]  

**Theorem 2 (Vizing, [22])**: \( \Delta(G) \leq \chi'(G) \leq \Delta(G) + \mu(G) \), where \( \mu(G) \) denotes the maximum multiplicity of an edge in G.

Note that Shannon’s theorem implies that if we consider a cubic graph G, then \( 3 \leq \chi'(G) \leq 4 \), thus \( \chi'(G) \) can take only two values. In 1981 Holyer proved that the problem of deciding whether \( \chi'(G) = 3 \) or not for cubic graphs G is NP-complete [9], thus the calculation of \( \chi'(G) \) is already hard for cubic graphs.

For a graph G and \( k \in N \) define
\[ B_k(G) \equiv \{(H_1, ..., H_k) : H_1, ..., H_k \text{ are pairwise edge-disjoint matchings of } G\}, \]

and let
\[ \nu_k(G) = \max\{|H_1| + ... + |H_k| : (H_1, ..., H_k) \in B_k(G)\}. \]

Define:
\[ \alpha_k(G) = \max\{|H_1|, ..., |H_k| : (H_1, ..., H_k) \in B_k(G) \text{ and } |H_1| + ... + |H_k| = \nu_k(G)\}, \]

and let
\[ M_2(G) = \{(H, H') \in B_2(G) : |H| + |H'| = \nu_2(G), |H| = \alpha_2(G)\}. \]

Though, we have introduced the parameters \( \nu_k(G) \) and \( \alpha_k(G) \) very formally, the reader can easily see that \( \nu_k(G) \) is just the number of edges in a maximum k-edge-colorable subgraph, while \( \alpha_k(G) \) is the maximum number of edges that can appear in a color-class of a maximum k-edge-colorable subgraph of a graph G. Recall that a subgraph of a graph is called to be maximum k-edge-colorable, if it is k-edge-colorable and contains maximum possible number of edges.

If \( \nu(G) \) denotes the cardinality of the largest matching of G, then it is clear that \( \alpha_k(G) \leq \nu(G) \) for all G and k. Let us also note that \( \nu_1(G) \) and \( \alpha_1(G) \) coincide with \( \nu(G) \).

In contrast with the theory of 2-matchings, where every graph G admits a maximum 2-matching that includes a maximum matching [10], there are graphs that do not have a “maximum” pair of disjoint matchings (a pair \( (H, H') \in B_2(G) \) with \( |H| + |H'| = \nu_2(G) \)) that includes a maximum matching.
On spanning maximum $k$-edge-colorable subgraphs

The following is the best result that can be stated about the ratio $\nu(G)/\alpha_2(G)$ for any graph $G$ (see [13]):

$$1 \leq \nu(G)/\alpha_2(G) \leq 5/4.$$ 

Very deep characterization of graphs $G$ satisfying $\nu(G)/\alpha_2(G) = 5/4$ is given in [21]. Let us also note that by Mkrtchyan’s result [11], reformulated as in [6], if $G$ is a matching covered tree, then $\alpha_2(G) = \nu(G)$. Note that a graph is said to be matching covered (see [12]), if its every edge belongs to a maximum matching (not necessarily a perfect matching as it is usually defined, see e.g. [10]).

The quantitative aspect of the investigation of maximum $k$-edge-colorable subgraphs of graphs and particularly, $r$-regular graphs has attracted a lot of attention, previously. The basic problem that researchers were interested was the following: what is the proportion of edges of a graph (or an $r$-regular graph, and particularly, cubic graph), that we can cover by its $k$ matchings?

For the case $k = 1$ in [8] an investigation is carried out in the class of simple cubic graphs, and in [4, 7, 16, 17, 23] for the general case. Let us also note that the relation between $\nu_1(G)$ and $|V|$ has also been investigated in the regular graphs of high girth [5].

The same is true for the case $k = 2, 3$. Albertson and Haas investigate these ratios in the class of simple cubic and 4-regular graphs in [11, 2], and Steffen investigates the problem in the class of bridgeless cubic graphs in [20]. Similar investigations are done in [18] for subcubic graphs. In [14] the problem is addressed in the class of cubic graphs. Finally, a best-possible bound is proved in [15] for the case $k = \Delta(G)$ in the class of all graphs.

However, it worths to be mentioned that the quantitative line of the research was not the only one. Previously, a special attention was also paid to structural properties of maximum $k$-edge-colorable subgraphs, and sometimes this kind of results have helped researchers to get quantitative results. A typical example of a structural result is the one proved in [2], which states that in any cubic graph $G$ there is a maximum 2-edge-colorable subgraph $H$, such that the graph $G \setminus E(H)$ is 2-edge-colorable. Recently, in [15] new such results are presented for maximum $\Delta(G)$-edge-colorable subgraphs of graphs $G$. Particularly, there it is shown that any set of vertex-disjoint cycles of a graph $G$ (particularly, any 2-factor) can be extended to a maximum $\Delta(G)$-edge-colorable subgraph of $G$ if $\Delta(G) \geq 3$. Also, there it is shown that for any maximum $\Delta(G)$-edge-colorable subgraph $H$ of $G$ $|\partial_H(X)| \geq \lceil \frac{\Delta_G(X)}{2} \rceil$ for each $X \subseteq V(G)$, where $\partial_K(X)$ is the set of edges of a graph $K$ with exactly one end-vertex in $X$. Note that this result implies theorem 5 formulated below. Finally, in [3] it is shown that the edges of a cubic graph lying outside a maximum 3-edge-colorable subgraph form a matching. Though this result does not have a direct generalization, using the ideas of the proof of Vizing theorem for simple graphs from [24], in [15] it is shown that a simple graph $G$ has a maximum $\Delta(G)$-edge-colorable subgraph $H$, such that the edges of $G$ that do not belong to $H$ form a matching.

In this paper, we concentrate on spanning maximum $k$-edge-colorable subgraphs of graphs. Our starting point is a statement mentioned in [20], which claims that any bridgeless cubic graph has a spanning maximum 2-edge-colorable subgraph. We offer two generalizations of this result. The first of them states that in a graph with a perfect matching there is a spanning maximum 2-edge-colorable subgraph with an additional
property. The other generalization states that in a graph of maximum degree $r$ ($r \geq 3$) without an isolated or a pendant vertex, there is a spanning maximum $(r - 1)$-edge-colorable subgraph.

Non-defined terms and concepts can be found in [10, 24].

2. The main results

Before we proceed to the formulation and proof of our results, we would like to state the following result which we will need later.

**Theorem 3** [13] Over all pairs $(H, H') \in M_2(G)$ and all maximum matchings $M$ of a graph $G$, consider the pairs $((H, H'), M)$ for which $|M \cap H|$ is maximized. Among these, choose a pair $((H, H'), M)$ such that $|M \cap H'|$ is maximized. Then:

1. there are no $M - H$ alternating cycles and even paths (lemma 1 of [13]);
2. the lengths of $M - H$ alternating odd paths is at least five; moreover the first and last edges of those paths belong to $H'$ (lemma 5 of [13]);
3. any vertex lying on a $M - H$ alternating odd path is incident to an edge from $H'$ (corollary 3 of [13]);
4. any vertex incident to an edge from $M \setminus (H \cup H')$, is incident to one edge from $H$ and one edge from $H'$ (lemma 2 of [13]).

**Remark 1** An attentive reader might have noted that the graphs considered in [13] are simple, while in this paper we allow graphs to possess a multiple edge. However, we would like to point out that theorem 3 can be proved exactly by the same methods of [13]. To avoid repeating the same claims and proofs, we have decided to omit the proof of theorem 3.

We are ready to state the first result of the paper:

**Theorem 4** Suppose that a connected graph $G$ has a matching that misses at most one vertex. Then $G$ has a spanning maximum 2-edge-colorable subgraph, one of whose color-classes contains $\alpha_2(G)$ edges.

**Proof.** First let us assume that $G$ has a perfect matching. Consider the pair $((H, H'), M)$ satisfying the conditions of the theorem 3 and let us show that any vertex $v \in V(G)$ is adjacent to an edge from $H \cup H'$.

Since $M$ is a perfect matching, there is an edge $e \in M$ incident to $v$. Note that if $e \in H \cup H'$, then we are done. On the other hand, if $e \not\in H \cup H'$, then the statement follows from (4) of theorem 3.

We are left with case when $G$ has a matching that misses exactly one vertex. Clearly such a matching is maximum. Again, we consider the pair $((H, H'), M)$ satisfying the conditions of the theorem 3. Similar to the case considered above, the reader can easily verify that if a vertex is covered $M$, then it is incident to an edge from $H \cup H'$. Thus, we
only need to consider the case of the vertex $v$ that is missed by $M$, and we can assume that it is not covered by an edge from $H \cup H'$.

Now, since $G$ is a connected graph, $v$ is adjacent to a vertex $w$. Note that $w$ is covered by $M$, thus it is also covered by an edge from $H \cup H'$. This implies that $w$ lies on an $H - H'$ alternating path or cycle. We will differ two cases.

Case A: $w$ lies on an $H - H'$ cycle $C$. Define:

$$H'_1 = (H' \setminus \{f\}) \cup \{e\},$$

where $f$ is the edge from $E(C) \cap H'$ that is incident to $w$. Note that the pair $(H, H'_1)$ corresponds to a spanning maximum 2-edge-colorable subgraph that we were looking for.

Case B: $w$ lies on an $H - H'$ path $P$. Note that $w$ cannot be an end-point of $P$, since $(H, H') \in M_2(G)$. Thus there are $h \in H$ and $h' \in H'$ that lie on $P$ and are incident to $w$. Suppose that $h = (w, w_1)$ and $h' = (w, w_2)$. Clearly, one of $h$ or $h'$ does not belong to $M$. Without loss of generality, we can assume that $h \not\in M$. Since $M$ covers all vertices but $v$, $w_1$ is incident to an edge $h_0 \in M$.

Note that $w_1$ is incident to an edge from $H'$, since if $h_0 \in H'$ then this is trivial. On the other hand, if $h_0 \not\in H'$ then, as $h \in H$, we have $h_0 \in M \setminus (H \cup H')$, and therefore we are done by (4) of theorem 3.

Define:

$$H_1 = (H \setminus \{h\}) \cup \{e\}.$$

Again note that the pair $(H_1, H')$ corresponds to a spanning maximum 2-edge-colorable subgraph that we were looking for. □

The reader may wonder whether the existence of a spanning maximum 2-edge-colorable subgraph can be proved under the weaker assumption that the maximum matching of a graph misses two vertices. The example of a claw (a graph with four vertices and three edges where one vertex is adjacent to the other three vertices) demonstrates that this fails. Rather surprisingly, the presence of a claw as an induced subgraph is a necessary condition for the absence of a spanning maximum 2-edge-colorable subgraph in a graph. More precisely, the following is true:

**Corollary 1** Let $G$ be a claw-free graph, that is, let $G$ be a graph, which contains no induced subgraph that isomorphic to claw. Then $G$ contains a spanning maximum 2-edge-colorable subgraph.

**Proof.** Let $G_1$ be any connected component of $G$. By the well-known results of Sumner, Jünger, Pulleyblank and Reinelt (see theorem 3.3.20 and the sentence after its proof in [10]), $G_1$ contains a matching missing at most one vertex. Apply theorem 4. Note that the union of spanning maximum 2-edge-colorable subgraphs of connected components of a graph is a spanning maximum 2-edge-colorable subgraph of the original graph. □
We would like also to mention the following corollary of theorem 4 and the well-known Petersen theorem (see, for example, theorem 3.4.1 of [10]), whose statement was already mentioned in the proof of theorem 4.1 from [20]. We will come back to this corollary in the end of the paper.

**Corollary 2** Let $G$ be a bridgeless cubic graph. Then $G$ contains a spanning maximum 2-edge-colorable subgraph.

We now turn to the problem of existence of spanning maximum $k$-edge-colorable subgraphs in arbitrary graphs for arbitrary values of $k$. In [15] the following result is proved:

**Theorem 5** Let $G$ be any graph, and let $G'$ be any maximum $\Delta(G)$-edge-colorable subgraph of $G$. Then for any vertex $v \in V(G)$ we have $d_{G'}(v) \geq \left\lceil \frac{d_G(v)}{2} \right\rceil$.

Though this theorem in [15] is proved for the case $k = \Delta(G)$, let us note that it can be easily verified that its proof works for any $k \geq \Delta(G)$. Observe that this implies that any maximum $k$-edge-colorable subgraph of $G$ is spanning provided that $k \geq \Delta(G)$ and $G$ contains no isolated vertex.

Thus it is natural to concentrate our attention on the case $k < \Delta(G)$. The example of the claw shows that in general a graph $G$ may not possess a spanning maximum $k$-edge-colorable subgraph for any $k < \Delta(G)$. Therefore we will restrict the class of graphs in order to be able to prove the existence of such subgraphs.

Our second result states the existence of a spanning maximum $(\Delta(G)-1)$-edge-colorable subgraph in graphs $G$ with $\delta(G) \geq 2$.

**Theorem 6** Let $G$ be a graph with $2 \leq \delta(G) \leq \Delta(G) = r$, where $r \geq 3$. Then $G$ contains a spanning maximum $(r - 1)$-edge-colorable subgraph.

**Proof.** Clearly, we can assume that $G$ is a connected graph. Let $S(G)$ denote the set of all maximum $(r - 1)$-edge-colorable subgraphs of $G$ that cover maximum possible number of vertices. The proof of the theorem will be completed if we show that any member of $S(G)$ covers all isolated vertex.

Under the opposite assumption, we present some properties of subgraphs from $S(G)$ that later will enable us to derive a contradiction.

Let $G'$ be a member of $S(G)$, and let $u$ be a vertex of $G$ missed by $G'$. Consider the vertices $u_1, \ldots, u_k \ (2 \leq k \leq r)$ that are adjacent to $u$. Since $G'$ is a maximum $(r - 1)$-edge-colorable subgraph of $G$, we have:

(a) $d_G(u_i) = r$ for $i = 1, \ldots, k$;

(b) $d_{G'}(u_i) = r - 1$ for $i = 1, \ldots, k$.

Let $v_i$ be any neighbour of the vertex $u_i \ (1 \leq i \leq k)$ that is different from $u$. Let us show that

(c) $d_{G'}(v_i) = 1$. 

(b) implies that $d_{G'}(v_i) \geq 1$. Now if $d_{G'}(v_i) \geq 2$, then define a subgraph $G''$ of $G$ as follows:

$$G'' = (G' \setminus \{(u_i, v_i)\}) \cup \{(u, u_i)\}.$$

Clearly $G''$ is a maximum $(r-1)$-edge-colorable subgraph of $G$. Moreover, $G''$ covers more vertices of $G$ than $G'$ does, which contradicts the choice of $G'$. Thus (c) must hold.

Taking into account that $r - 1 \geq 2$, the statement (c) implies:

1. there is no edge in $G$ that connects two vertices from $\{u_1, ..., u_k\}$;
2. any two vertices from $\{u_1, ..., u_k\}$ have exactly one common neighbour, which is the vertex $u$.

The final property of $G'$ that we will need is the following:

3. no two vertices $v$ and $w$ lying on a distance two from the vertex $u$, are adjacent.

For the proof of this property, let us assume that $v$ and $w$ are adjacent. Moreover, let us also assume that $v$ and $w$ are adjacent to the vertices $u_i$ and $u_j$, respectively. Note that it is possible that $i = j$.

Let $H_1, ..., H_{r-1}$ be the color-classes of the subgraph $G'$, and suppose that the color-classes are enumerated in the way, that $(u_i, v) \in H_i$. Since $r > 2$, (c) implies that there is $l$ $(1 \leq l \leq r - 1)$ such that $w$ is not covered by $H_l$. Again note that it is possible that $i = l$.

Let us do the following two steps:

Step 1: Remove the edge $(u_i, v)$ from $H_i$, and add $(u, u_i)$ to it;
Step 2: Add the edge $(v, w)$ to $H_l$.

Note that after these steps $H_1, ..., H_{r-1}$ remain matchings, moreover, $H_1 \cup ... \cup H_{r-1}$ now contains more edges than in the beginning, which contradicts the maximality of $G'$. Thus (3) must be true, too.

We are ready to derive the required contradiction. First of all, let us show that any subgraph from $\mathcal{S}(G)$ misses at most one vertex. On the opposite assumption, let us consider a mapping $w : \mathcal{S}(G) \to N$ defined as follows:

$$w(G') = \min\{\rho(u^0, v^0) : u^0 \text{ and } v^0 \text{ are different vertices missed by the subgraph } G'\}.$$  

Note that the parameter is well-defined since by assumption any maximum $(r-1)$-edge-colorable subgraph of $G$ misses at least two vertices.

Now choose $G' \in \mathcal{S}(G)$ minimizing $w$, and let $u, v$ be two vertices of $G$ that are not covered by $G'$ and $w(G') = \rho(u, v)$. Note that (b) implies that $v \notin \{u_1, ..., u_k\}$ and $w(G') = \rho(u, v) \geq 3$, where $u_1, ..., u_k$ are the neighbours of the vertex $u$.

Consider a path $P_{u,v}$ of $G$ having length $\rho(u, v)$ and connecting the vertices $u$ and $v$. Let $u_j$ $(1 \leq j \leq k)$ be the vertex of $P_{u,v}$ next to $u$, and let $v_j$ be the vertex next to $u_j$. (b) implies that $(u_j, v_j) \in E(G')$. Consider a subgraph $G''$ of $G$ defined as:

$$G'' = (G' \setminus \{(u_j, v_j)\}) \cup \{(u, u_j)\}.$$
Clearly $G''$ is a maximum $(r-1)$-edge-colorable subgraph of $G$. Moreover, (c) implies that $v_j$ is not covered in $G''$, therefore $G'' \in S(G)$. However,

$$w(G') = \rho(u, v) > \rho(v_j, v) \geq w(G''),$$

which contradicts the choice of $G'$.

This implies that any member of $S(G)$ misses at most one vertex. Thus to complete the proof of the theorem, it remains to rule out the case when a member of $S(G)$ misses exactly one vertex.

By induction on $s \in N$, let us show that the following properties are true for any $G' \in S(G)$, where we assume that $u$ is the only vertex of $G$ missed by $G'$, $u_1^{1s-1}, ... , u^{k_2s-1}_{2s-1}$ ($k_2s-1 \geq 0$) and $u_2^1, ... , u^{k_2s}_{2s}$ ($k_2s \geq 0$) are the vertices of $G$ placed on the distance $2s - 1$ and $2s$ from $u$, respectively:

(A) $d_G(u^1_{2s-1}) = r$ and $d_G'(u^1_{2s-1}) = r - 1$ for $i = 1, ..., k_2s-1$;

(B) no two vertices from $\{u^1_{2s-1}, ..., u^{k_2s-1}_{2s-1}\}$ are adjacent;

(C) no two vertex from $\{u^1_{2s-1}, ..., u^{k_2s-1}_{2s-1}\}$ have a common neighbour from $\{u^1_{2s}, ..., u^{k_2s}_{2s}\}$;

(D) $d_G'(u^1_{2s}) = 1$ for $i = 1, ..., k_2s$;

(E) no two vertex from $\{u^1_{2s}, ..., u^{k_2s}_{2s}\}$ are adjacent, and any neighbour of a vertex from $\{u^1_{2s}, ..., u^{k_2s}_{2s}\}$ that is not placed on a distance $2s - 1$ from $u$, is placed on distance $2s + 1$ from $u$.

Before we proceed with the proof, let us make the following

**Remark 2** In the the statements (A)-(E), we do not assume or prove that for each $s \in N$ there exist vertices that are placed on a distance $2s - 1$ or $2s$ from $u$. We simply claim that if they exist, then they must satisfy (A)-(E).

Assume that $s = 1$. Then (a) and (b) imply (A), (1) implies (B), (2) implies (C), (c) implies (D), (3) implies (E). Now, by induction, let us assume that the statements (A)-(E) are true for $s - 1$, and let us show that they remain valid for $s$. Below, we assume that $H_1, ..., H_{r-1}$ are the color-classes of $G'$.

Suppose that (A) is not true for $s$. Then, there exists a vertex $u^i_{2s-1}$ and $H_j$ ($1 \leq j \leq r - 1$), such that $H_j$ misses $u^i_{2s-1}$. Assume that $u^i_{2s-1}$ is adjacent to the vertex $u^i_{2s-2}$. By induction hypothesis, $d_G'(u^i_{2s-2}) = 1$. Moreover, due to maximality of $G'$, we have that $H_j$ covers $u^i_{2s-2}$. Define:

$$H_j' = (H_j \setminus \{e\}) \cup \{(u^i_{2s-2}, u^i_{2s-1})\},$$

where $e$ is the edge of $H_j$ incident to $u^i_{2s-2}$. Note that since $r - 1 \geq 2$, $H_1, ..., H'_j, ..., H_{r-1}$ are color-classes of a subgraph $G'' \in S(G)$. However, (A) fails for $G''$ with $s - 1$ contradicting the induction hypothesis. Thus, (A) must be true for $s$, as well.

Suppose that (B) is not true for $s$. Then there are adjacent vertices $u^i_{2s-1}$ and $u^j_{2s-1}$. Assume that $u^i_{2s-1}$ is adjacent to $u^i_{2s-2}$. Since (A) holds for $s$, we have $(u^i_{2s-2}, u^i_{2s-1}) \in E(G')$, thus there is $j$ ($1 \leq j \leq r - 1$), such that $(u^i_{2s-1}, u^j_{2s-1}) \in H_j$. 

Due to induction hypothesis, we have $d_{G'}(u_2^i) = 1$. Thus, there is $i$ (1 ≤ $i$ ≤ $r - 1$), such that $H_i$ covers $u_2^i$. Let us show that without loss of generality, we can assume that $i \neq j$.

If $i = j$, then $H_i = H_j$. Let us assume that $e = (u_2^i, u_2^j) \in H_i$. Since $r - 1 \geq 2$, the induction hypothesis implies that there is $l \neq i$ (1 ≤ $l$ ≤ $r - 1$), such that the vertex $u_2^l$ is covered by an edge $f \in H_l$. Define:

$$H_i' = (H_i \setminus \{e\}) \cup \{f\}, H_i' = (H_i \setminus \{f\}) \cup \{e\}.$$  

Note that $H_1, ..., H_i', ..., H_{r-1}$ are the color-classes of another coloring of $G'$. However, for this new coloring of $G'$, the only edge of $G'$ that covers $u_2^i$ does not belong to $H_j$.

Thus initially, we can assume that $i \neq j$. Define:

$$H_j' = (H_j \setminus \{(u_2^i, u_2^j)\}) \cup \{(u_2^i, u_2^j)\}.$$  

Note that since $r - 1 \geq 2$, $H_1, ..., H_j', ..., H_{r-1}$ are color-classes of a subgraph $G'' \in S(G)$. However, $d_{G''}(u_2^i) = 2$ contradicting (D) of the induction hypothesis. Thus, (B) must be true for $s$, as well.

As (A) is already shown to be true for $s$, we have that in order to prove (C) and (D), it suffices only to show (D). Suppose that (D) fails to be true for $s$, then there is a vertex $u_2^i$ such that $d_{G'}(u_2^i) \geq 2$ (recall that $u$ is the only vertex of $G$ that is missed by $G'$). Thus, there are $H_j$ and $H_l$ covering $u_2^j$ (1 ≤ $j$ ≤ $r - 1$). Suppose that $(u_2^i, u_2^j) \in H_j$. Also assume that the vertex $u_2^l$ is adjacent to $u_2^j$, which in its turn is adjacent to $u_2^j$. Similar to the proof of (B), it can be easily seen, that without loss of generality, we can assume that $(u_2^i, u_2^j) \in H_k$ and $k \neq j$. Define:

$$H_j' = (H_j \setminus \{(u_2^i, u_2^j)\}) \cup \{(u_2^i, u_2^j)\}.$$  

Note that $H_1, ..., H_j', ..., H_{r-1}$ are color-classes of a subgraph $G'' \in S(G)$. However, $d_{G''}(u_2^i) = 2$ contradicting (D) of the induction hypothesis. Thus, (D) (and therefore (C)) must be true for $s$, as well.

Finally, turning to the proof of (E), let us note that it suffices to show that no two vertices $u_2^i$ and $u_2^j$ are adjacent. On the opposite assumption, suppose that $u_2^i$ is adjacent to $u_2^j$ and $(u_2^i, u_2^j) \in H_i$ (see (A)) and $u_2^j$ is adjacent to $u_2^j$ and $(u_2^j, u_2^j) \in H_l$. Let us note that it is possible that $u_2^j = u_2^j$ or $H_i = H_l$.

Also assume that the vertex $u_2^j$ is adjacent to $u_2^j$, which in its turn is adjacent to $u_2^j$. Again, similar to the proof of (B), it can be easily seen, that without loss of generality, we can assume that $(u_2^i, u_2^j) \in H_k$ and $k \neq i$.

Let us do the following two steps:

Step 1: Remove the edge $(u_2^i, u_2^j)$ from $H_i$, and add $(u_2^i, u_2^j)$ to it;

Step 2: Add the edge $(u_2^i, u_2^j)$ to $H_p$, where $p \neq l$.

Note that after these steps $H_1, ..., H_{r-1}$ remain matchings, moreover, $H_1 \cup ... \cup H_{r-1}$ now contains more edges than in the beginning, which contradicts the maximality of $G'$.
(the edge \((v_2^{i_1}, u_2^{i_2})\) cannot belong to \(G'\), this follows from (D), which we already know to be true for \(s\)). Thus (E) must be true, too.

We are ready to derive the contradiction that we were looking for. We achieve this by showing that our conditions and the already established properties imply that for any even number \(2q\) \((q \in N)\), \(G' \in \mathcal{S}(G)\) and the vertex \(u\) missed by \(G'\), there is a vertex of \(G\) such that the distance between this vertex and \(u\) is \(2q\), which clearly will contradict the finiteness of \(G\).

We will prove this statement by induction on \(q\). Let \(G' \in \mathcal{S}(G)\) be fixed and let \(u\) be the vertex missed by \(G'\). Since \(\delta(G) \geq 2\), there is a vertex \(v\) adjacent to \(u\). Let this edge be \(e = (u, v)\). Again, \(\delta(G) \geq 2\) implies that \(v\) is incident to an edge \(f \neq e\). Since \(u\) is not covered by \(G'\), (A) implies that \(u\) is not incident to \(f\). Let \(w\) be the other end-vertex of \(f\). (B) implies that \(u\) and \(w\) are not adjacent, and hence \(\rho(u, w) = 2\) which proves our statement for \(q = 1\).

Now, by induction assume that we have already proved that there is a vertex \(v\) such that \(\rho(u, v) = 2(q - 1)\). Since \(\delta(G) \geq 2\), there is an edge \(e\) incident to \(v\) that does not lie on a path \(P\) that connects \(u\) and \(v\) and has length \(\rho(u, v) = 2(q - 1)\). (A) implies that the vertex preceding \(v\) on \(P\) cannot be incident to \(e\). \(e\) cannot also be incident to other vertices of \(P\) since \(P\) is a shortest path connecting \(u\) and \(v\). Let \(w\) be the other end-vertex of \(e\). (C) and (E) imply that \(\rho(u, w) > 2(q - 1)\), thus \(\rho(u, w) = 2(q - 1) + 1\). Since \(\delta(G) \geq 2\), \(w\) is incident to an edge \(e = (w, z) \neq e\). (A) implies that \(z \neq v\). (A), (E) and \(\rho(u, w) = 2(q - 1) + 1\) imply that \(\rho(u, z) > 2(q - 1) + 1\). Thus \(\rho(u, z) = 2q\) and \(z\) is the vertex that we were looking for. The proof of the theorem is completed.

**Corollary 3** Let \(G\) be a graph, whose vertices are of degree two or three. Then \(G\) has a spanning maximum 2-edge-colorable subgraph.

Note that this corollary in its turn implies the already stated corollary that we deduced from theorem [1]. As we have mentioned the statement of this corollary first appeared in the proof of theorem 4.1 from [20]. However, an attentive reader probably has already realized that the proof given in [20] is wrong.

Retaining the notations of [20], let us, first explain, what is wrong there. The gap is that when the author removes the edges \(e_1\) and \(e_2\) from a maximum 2-edge-colorable subgraph \(H\) and adds the edges \((v, u_1)\) and \((v, u_2)\) to it to get a new maximum 2-edge-colorable subgraph \(H'\), he may leave the other \((\neq u_1\) and \(\neq u_2\), respectively) end-vertices isolated, so after this operation one can not conclude that \(V(H') = V(H) \cup \{v\}\) as it is done there.

In July of 2009, during a discussion with the first author over this issue, Eckhard Steffen has proposed a strategy for repairing the proof of this statement. Rather surprisingly, his approach was very similar to our proof of theorem though for the sake of fairness, we should point out, that he was deriving the necessary contradiction not on the infiniteness of the counter-example like we did above.

We finish the paper with the following:

**Conjecture 1** Any graph \(G\) with \(\Delta(G) - \delta(G) \leq 1\) has a spanning maximum 2-edge-colorable subgraph.
REFERENCES

1. M.O. Albertson, R. Haas, Parsimonious edge coloring, Discrete Math. 148 (1996) 1–7.
2. M.O. Albertson, R. Haas, The edge chromatic difference sequence of a cubic graph, Discrete Math. 177, (1997) 1–8.
3. D. Aslanyan, V. V. Mkrtchyan, S. S. Petrosyan, G. N. Vardanyan, On disjoint matchings in cubic graphs: maximum 2- and 3-edge-colorable subgraphs, under review, (available at: http://arxiv.org/abs/0909.2767)
4. B. Bollobás, Extremal graph theory, Academic Press, London-New York-San Francisco, 1978.
5. A. D. Flaxman, S. Hoory, Maximum matchings in regular graphs of high girth, The Electronic Journal of Combinatorics, 14, N 1, 2007, pp. 1-4.
6. F. Harary, M.D. Plummer, On the core of a graph, Proc. London Math. Soc. 17 (1967), pp. 305–314.
7. M.A. Henning, A. Yeo, Tight Lower Bounds on the Size of a Maximum Matching in a Regular Graph, Graphs and Combinatorics, vol. 23, N 6, 2007, pp. 647-657
8. A. M. Hobbs, E. Schmeichel, On the maximum number of independent edges in cubic graphs, Discrete Mathematics 42, 1982, pp. 317-320.
9. I. Holyer, The NP-completeness of edge coloring, SIAM J. Comput. 10, N4, 718-720, 1981 (available at: http://cs.bris.ac.uk/ian/graphs)
10. L. Lovász, M.D. Plummer, Matching theory, Ann. Discrete Math. 29 (1986).
11. V. V. Mkrtchyan, On trees with a maximum proper partial 0-1 coloring containing a maximum matching, Discrete Mathematics 306, (2006), pp. 456-459.
12. V. V. Mkrtchyan, A note on minimal matching covered graphs, Discrete Mathematics 306, (2006), pp. 452-455.
13. V. V. Mkrtchyan, V. L. Musoyan, A. V. Tserunyan, On edge-disjoint pairs of matchings, Discrete Mathematics 308, (2008), pp. 5823-5828 (available at: http://arxiv.org/abs/0708.1903).
14. V. V. Mkrtchyan, S. S. Petrosyan, G. N. Vardanyan, On disjoint matchings in cubic graphs, Disc. Math. 310 (2010) 1588-1613
15. V. V. Mkrtchyan, E. Steffen, Maximum ∆-edge-colorable subgraphs of class II graphs, J. Graph Theory 2011, to appear (available at: http://arxiv.org/abs/1002.0783)
16. T. Nishizeki and I. Baybars, Lower bounds on the cardinality of the maximum matchings of planar graphs, Discrete Math., 28, 255-267, 1979.
17. T. Nishizeki, On the maximum matchings of regular multigraphs, Discrete Mathematics 37, 1981, pp. 105-114.
18. R. Rizzi, Approximating the maximum 3-edge-colorable subgraph problem, Disc. Math. 309 (2009) 4166-4170
19. C. E. Shannon, A theorem on coloring the lines of a network, J. Math. And Phys., 28 (1949), pp. 148-151
20. E. Steffen, Measurements of edge-uncolorability, Discrete Mathematics 280 (2004), pp. 191 – 214.
21. A. V. Tserunyan, Characterization of a class of graphs related to pairs of disjoint matchings, Discrete Mathematics 309, (2009), pp. 693-713, (available at: http://arxiv.org/abs/0712.1014)
22. V. G. Vizing, The chromatic class of a multigraph, Kibernetika No. 3, (1965) Kiev, pp. 29-39 (in Russian)

23. J. Weinstein, Large matchings in graphs, Canadian J. Math., 26, 6,(1974), pp. 1498-1508.

24. D. B. West, Introduction to Graph Theory, Prentice-Hall, Englewood Cliffs, 1996.