Exact dynamics of entanglement and entropy in structured environments

L. Mazzola, S. Maniscalco, J. Piilo, and K. - A. Suominen
Department of Physics and Astronomy, University of Turku, FI-20014 Turun yliopisto, Finland

We study the exact entanglement dynamics of two qubits interacting with a common zero-temperature non-Markovian reservoir. We consider the two qubits initially prepared in Bell-like states or extended Werner-like states. We study the dependence of the entanglement dynamics on both the degree of purity and the amount of entanglement of the initial state. We also explore the relation between the entanglement and the von Neumann entropy dynamics and find that these two quantities are correlated for initial Bell-like states.

I. INTRODUCTION

Entanglement is a key resource in modern quantum information theory and technology. Entangled states play a central role in quantum key distribution, superdense coding, quantum teleportation and quantum error correction [1]. However, realistic quantum systems are never completely isolated from their surroundings. The inevitable interaction between a system and its environment leads to decoherence phenomena and degradation of entanglement [2].

Once entanglement has been lost, it cannot be restored by local operations [3]. It is therefore important to understand the process of disentanglement in order to control the effects of noise and to preserve entanglement. For these reasons the study of entanglement dynamics of quantum systems in realistic situations has become of increasing importance.

Recently much attention has been devoted to the process of finite-time disentanglement, also known as “entanglement sudden death” (ESD), in a bipartite system [4]. This phenomenon consists in the complete disappearance of the bipartite entanglement in a finite time, as opposed to the time-evolution of the coherences of the single parts which vanish only asymptotically.

ESD sets a limit on the life-time and usability of entanglement for practical purposes. Hence, lots of efforts have been done in order to understand the conditions under which ESD occurs [5]. Finite-time disentanglement has been found in different physical systems, such as qubits or harmonic oscillators [6]. ESD appears when the two parts of the system interact with either independent environments [4, 7] or a common one [8]. The first studies within the Markov approximation have been extended to non-Markovian environments, where the memory of the reservoir adds revivals to entanglement dynamics [6, 10].

In Ref. [5] we have studied entanglement sudden death when two-qubits are prepared in a Bell-like state with two excitations, and entanglement sudden birth for qubits prepared in a separable state. Here we focus on the dynamics of a class of states having an “X”-structure density matrix, namely the extended Werner-like states (EWL). This class of states plays a crucial role in many applications of quantum information theory, such as teleportation [11] and quantum key distribution [12]. Moreover, such a choice will give us the chance to study how the entanglement dynamics and its revivals are related to the purity and the amount of entanglement of the initial state. We also focus on the interplay between entanglement and mixedness for the qubits states. We investigate the connection between these two quantities, comparing concurrence and von Neumann entropy dynamics for initial Bell-like states, and finding clear correlations between them.

The paper is organized as follows. In Sec. II we review the exactly solvable model of two qubits interacting with a common Lorentzian structured reservoir. In Sec. III we study the entanglement and von Neumann entropy dynamics for initial Bell-like states and we prove that sudden death of entanglement can never occur if the qubits are initially in a mixed state having at most one excitation. In Sec. IV we focus on the time evolution of EWL states, and compare our results with those obtained in Ref. [13] for independent structured reservoirs. Finally, we summarize our results in Sec. V.

II. THE MODEL

In this section we describe the model we use to study the dynamics of two two-level systems (qubits) interacting with a common zero-temperature bosonic reservoir. Our approach is non-Markovian and non-perturbative, i.e., it does not rely on either the Born or the Markov approximations [3].

The Hamiltonian of the system, in the rotating wave approximation, is given by

$$H = H_0 + H_{\text{int}},$$

$$H_0 = \omega_0 (\sigma^A_+ \sigma^A_- + \sigma^B_+ \sigma^B_-) + \sum_k \omega_k a_k ^\dagger a_k,$$  \hspace{1cm} (1)

$$H_{\text{int}} = (\sigma^A_+ + \sigma^B_+) \sum_k g_k a_k + \text{h.c.},$$  \hspace{1cm} (2)

where $\sigma^A_+$ and $\sigma^B_+$ are the Pauli raising and lowering operators for qubit A and B respectively, $\omega_0$ is the Bohr frequency of the two identical qubits, $a_k$ and $a_k ^\dagger$, $\omega_k$ and
\( g_k \) are the annihilation and creation operators, the frequency and the coupling constant of the field mode \( k \), respectively.

In order to solve the dynamics of the two qubits we need to specify the properties of the environment. In the following we assume that the two qubits interact resonantly with a non-Markovian Lorentzian structured reservoir, such as the electromagnetic field inside a lossy cavity \[14\], having spectral distribution

\[
J(\omega) = \frac{\Omega^2}{2\pi} \frac{\Gamma}{(\omega - \omega_0)^2 + (\Gamma/2)^2},
\]

where \( \Gamma \) is the width of the Lorentzian function and \( \Omega \) the coupling strength.

We are mainly interested in the dynamics of entanglement and in the effects that the non-Markovian reservoir induces on the correlation between the two qubits. To quantify entanglement we use the Wootters concurrence \[15\], defined as

\[
C(t) = \max \{0, C_1(t), C_2(t)\},
\]

where

\[
C_1(t) = 2|w(t)| - 2\sqrt{b(t)c(t)},
\]

\[
C_2(t) = 2|z(t)| - 2\sqrt{a(t)d(t)}.
\]

We notice that coherences give a positive contribution to \( C_1(t) \) and \( C_2(t) \) and so to concurrence, while the negative parts involve populations only.

In the next section we will also consider the evolution of the mixedness of the two qubit state, which we quantify through the von Neumann entropy, defined as

\[
S(\rho) = -\text{Tr}\{\rho \ln(\rho(t))\}.
\]

Von Neumann entropy is equal to zero for pure states, and attains its maximum value (equal to \( \ln N \) with \( N \) the dimension of the Hilbert space) for a maximally mixed state.

III. ENTANGLEMENT AND VON NEUMANN ENTROPY DYNAMICS FOR BELL-LIKE STATES

Here, we seek for the connection between the dynamics of entanglement and von Neumann entropy of two qubits prepared in Bell-like states

\[
|\Phi\rangle = \alpha|0\rangle + e^{i\theta}(1 - \alpha^2)^{1/2}|01\rangle,
\]

and

\[
|\Psi\rangle = \alpha|00\rangle + e^{i\theta}(1 - \alpha^2)^{1/2}|11\rangle.
\]

Our aim is to understand the interplay between these two different physical quantities, in particular when peculiar phenomena such as ESD or “entanglement sudden birth”(ESB) \[9, 11, 17\], and revivals of entanglement occur.

The entanglement dynamics of two qubits in a Lorentzian structured reservoir, prepared in a Bell-like state with two excitations as in Eq. (8), has been presented in Ref. \[9\]. There we have studied in detail the difference with the common Markovian reservoir case and the independent reservoirs non-Markovian case. The evolution of entanglement for a Bell-like state as in Eq. (9) has been studied also in Ref. \[18\].

For both the Bell-like states in Eqs. \[8\] and \[9\] the entanglement dynamics is the result of two combined effects: the backaction of the non-Markovian reservoir and the reservoir-mediated interaction between the qubits.

The memory effects due to the non-Markovianity of the reservoir causes oscillations in entanglement dynamics. These oscillations are typical also of the independent reservoirs case \[10\], but they disappear completely for Markovian reservoirs \[8\]. The sharing of the reservoir plays also a special role. Indeed, the common reservoir provides an effective coupling between the qubits, and so consequently creates quantum correlations between them. As a result, qubits prepared in a factorized state can become entangled due to the interaction with the common reservoir. This is in contrast with the independent reservoirs case in which a factorized state of the qubits can never evolve into an entangled state.

The results in Ref. \[18\] show that ESD does not occur for a Bell-like state with one excitation as in Eq. \[8\] for any value of \( \alpha^2 \). ESD does not appear for the same Bell-like state even if the dipolar interaction between the qubits is included \[19\]. Actually, a straightforward calculation shows that for every pure or mixed state of the qubits containing at most one excitation, ESD and ESB cannot take place. In fact, the density matrix describing a generic mixed state with maximum one excitation,
written in the same basis of Eq. (3), has the form
\[
\rho(t) = \begin{pmatrix}
    a(t) & j(t) & k(t) & 0 \\
    j^*(t) & b(t) & z(t) & 0 \\
    k^*(t) & z^*(t) & c(t) & 0 \\
    0 & 0 & 0 & 0
\end{pmatrix}.
\] (10)

The expression for the concurrence for any value of the parameters is
\[
C(t) = \max\{0, 2|z(t)|\}.
\] (11)

Here concurrence is directly given by the coherence between the $|10\rangle$ and $|01\rangle$ states. Since the coherence vanishes in asymptotic way, there cannot be ESD for any generic state with maximum one excitation. Analogously, entanglement can be smoothly generated but it cannot suddenly appear. This result does not depend on the degree of purity of the state. This is true as long as the form of the density matrix in Eq. (10) is maintained. On the other hand, if some population is transferred to the two excitations state then ESD can appear. This is the case of two qubits in a Bell state interacting with a non-RWA common reservoir [20].

One problem we want to address here is to understand how the amount of entanglement changes with the purity of the state. In the next section we will explore this aspect by preparing the qubits in a particular class of mixed states. Furthermore, it is useful to see how the mixedness of the initial state changes with time because of the interaction with the environment, and if there exists some connection between the von Neumann entropy and entanglement dynamics.

In order to provide a clear interpretation of the dynamics, it is useful to recall the four-state effective description [9]. Indeed, the state of the two qubits is equivalent to a four-state system, in which three states \{\langle 00 \rvert, \langle 0 \rvert + \langle 1 \rvert \rvert \rangle / \sqrt{2}, \langle 11 \rvert \rangle \} interact in ladder configuration with the electromagnetic field, and the fourth state \{\langle - \rangle = \langle 10 \rangle - \langle 01 \rangle / \sqrt{2} \} is completely decoupled from the other states and the electromagnetic field. The states $\langle + \rangle$ and $\langle - \rangle$ are known as super-radiant and sub-radiant states, respectively.

We look at the evolution of the von Neumann entropy of the qubit pair when the state is initially prepared in a Bell-like state of the form (8) and (9). It is particularly interesting to see how the degree of purity of the state evolves when the qubits undergo ESD. Since ESD occurs when $\alpha^2 \lesssim 1/4$ [9], we choose $\alpha^2 = 1/20$ and $\theta = 0$ for the sake of convenience. In Fig. 1 we notice that ESD appears for high value of mixedness of the system (entropy peaks coincide with minima of concurrence). Moreover, the revivals of entanglement appear roughly in correspondence of the minimum of the von Neumann entropy, when the state becomes purer. A careful analysis shows that the dynamics of entropy follows the population of the super-radiant state $\rho_{++}(t)$, (in the Appendix we provide the analytic expression of the von Neumann entropy for this initial state). As Fig. 1 shows, they attain their relative maxima and minima at the same times. Moreover, we can deduce from Eqs. (5) and (6) that entanglement dynamics is a function of the population of the super-radiant state, which is $\rho_{++}(t) = 2\sqrt{\rho(t)c(t)}$ for the particular initial state of Eq. (9). Therefore whenever the population $\rho_{++}(t)$ reaches its relative maxima, the state attains a maximum value of mixedness, the time-dependent part of the concurrence $\langle 2|w(t)| - \rho_{++}(t) \rangle$ be-
comes negative and entanglement disappears. On the other hand, whenever the population of the super-radiant state reaches a minimum, the population of the $|1\rangle$ excited state and the $|00\rangle$ ground state have their maxima (see inset in Fig. 1), and the system goes toward a Bell-like state $|\Psi\rangle$. As a consequence the system becomes purer and entanglement is partially recovered.

When the system is prepared in a Bell-like state as the one of Eq. (8) there is no ESD, however it is still of interest to understand how the revivals of entanglement are related to the degree of purity of the state. For the Bell state $|+\rangle$ entanglement has exactly the same dynamics of the population of the super-radiant state, as shown in Fig. 2. Moreover the zeroes of entanglement and entropy coincide. For those times, in fact, the system goes into the ground state which is pure and factorized. When some population returns in the super-radiant-state entanglement is recovered, and the state is again mixed.

When $\alpha^2 \neq 1/2$ and/or $\theta \neq 0$ the initial state is a superposition of super-radiant and sub-radiant states. Although the sub-radiant state does not evolve in time, being decoupled from the super-radiant and ground states, it affects the entanglement and entropy dynamics. For the sake of convenience we choose $\alpha^2 = 1/5$ and $\theta = 0$. In Fig. 3 we see that entropy still follows closely the time evolution of the super-radiant state population, having its relative minima in the same positions of the zeroes of $\rho_{+-}(t)$, (the analytic expression of the von Neumann entropy is in the Appendix). On the contrary, entanglement has new relative maxima when the population of the super-radiant state is zero. This is due to the presence of the sub-radiant state. In fact, in this case both the super-radiant and the sub-radiant states contribute to the total entanglement. Thus there are two different sets of entanglement maxima, those associated with the maxima of the super-radiant state population, and those associated to the sub-radiant state. Entanglement is zero whenever the population of the super-radiant state $\rho_{++}(t)$, the population of the sub-radiant state $\rho_{--}(t)$, and the absolute value of the coherence between super-radiant and sub-radiant states $\rho_{+-}(t)$, are equal. This can be explained in the light of the expressions in Eqs. (A.1) in the Appendix, where $b(t)$, $c(t)$ and $z(t)$ are written as a function of those quantities. In fact, when $\rho_{++}(t)$, $\rho_{--}(t)$ and $|\rho_{+-}(t)|$ have equal value, then $z(t)$ is equal to zero, irrespective of the sign of $\rho_{+-}(t)$. Specifically, this happens in correspondence to the 1st, 4th and 5th zeroes of entanglement, where $b(t) = 0$ and $c(t) = 2\rho_{--} = k$. In this case the state of the system becomes $|1-k\rangle|00\rangle|00\rangle + k|01\rangle|01\rangle|1\rangle$, which is clearly not entangled. Similarly, for the 2nd and 3rd zeroes, $b(t) = k$, $c(t) = 0$ and the non-entangled state is $|1-k\rangle|00\rangle|00\rangle + k|10\rangle|10\rangle$.

To conclude this section we consider the following mixed and factorized initial state

$$
\rho(0) = (\alpha^2 |0_A\rangle\langle0_A| + (1-\alpha^2)|1_A\rangle\langle1_A|) \otimes (\alpha^2 |0_B\rangle\langle0_B| + (1-\alpha^2)|1_B\rangle\langle1_B|).
$$

(12)

The dynamics of this state in a common non-Markovian reservoir is characterized by ESB and revivals of disentanglement, as we have shown in Ref. [5]. In Fig. 4 we see how these interesting features are related to the degree of purity of the state. As we have seen before, the positions of maxima and minima of the entropy and of the super-radiant state population match. The sudden creation of entanglement happens roughly when the entropy hits
IV. WERNER STATE

In this section we study the two qubits entanglement dynamics of extended Werner-like states in a common zero-temperature Lorentzian structured reservoir. Our goal here is to study how, starting from an initial state that is not perfectly pure, as in realistic experimental conditions, affects the entanglement dynamics, and in particular the occurrence of ESD and ESB phenomena.

A. Terminology and previous works

The standard two-qubit Werner state, introduced in 1989 by Werner [21], is defined as

$$\rho_W = r|\Psi\rangle\langle\Psi| + \frac{1-r}{4}I,$$  \hspace{1cm} (13)

where $|\Psi\rangle$ is the singlet state. In Ref. [21], Werner demonstrated that while pure entangled states always violate the Bell inequality, mixed entangled states might not. The Werner state is the first entangled state to be proven not violating any Bell inequalities [21]. The generalized or Werner-like states are defined as

$$\rho_{WL} = r|M\rangle\langle M| + \frac{1-r}{4}I$$  \hspace{1cm} (14)

with $|M\rangle$ one of the four maximally entangled Bell states. For a given $r$, Werner and Werner-like states exhibit the same entanglement. A further generalization is the extended Werner-like states (EWL), containing a non-maximally entangled state part, which are defined as

$$\rho_{EWL}^\Phi = r|\Phi\rangle\langle\Phi| + \frac{1-r}{4}I$$  \hspace{1cm} (15)

where $|\Phi\rangle = \alpha|00\rangle + e^{i\theta}(1-\alpha^2)^{1/2}|01\rangle$, and

$$\rho_{EWL}^\Psi = r|\Psi\rangle\langle\Psi| + \frac{1-r}{4}I$$  \hspace{1cm} (16)

with $|\Psi\rangle = \alpha|00\rangle + e^{i\theta}(1-\alpha^2)^{1/2}|11\rangle$.

The dynamics of entanglement of two qubits prepared in Werner, Werner-like or extended Werner-like states has attracted a lot of attention. The appearance of ESD has been demonstrated for two qubits prepared in a Werner-like state and interacting with Markovian independent reservoirs [22, 23] or with independent noisy channels [24]. Violation of Bell inequality has been examined in independent thermal reservoirs [25, 26]. Entanglement has been studied in a Markovian common thermal reservoir [27], in the presence of collective dephasing [28] and in the maximal noise limit [29]. Finally, the dynamics of an EWL in two independent Lorentzian structured reservoirs has been investigated in Ref. [13].

Moreover, Werner and Werner-like states have been used so far in many applications in quantum information processes such as teleportation [11] and entanglement teleportation [30]. The experimental preparation and characterization of the Werner states have also been widely investigated. Werner states are prepared via spontaneous parametric down-conversion [31] or using a universal source of entanglement [32], and used in ancilla-assisted process tomography [33] and secure quantum key distribution [12].

B. Entanglement dynamics

We evaluate the dynamics of the entanglement of EWL states as a function of the initial amount of mixedness, controlled by the purity parameter $r$, and as a function of the initial degree of entanglement measured by $\alpha^2$.

Looking at the evolution of this kind of states gives the possibility to study how the degree of purity of the initial state influences the entanglement dynamics. We also compare our results with those obtained in Ref. [13] for two qubits in two independent Lorentzian structured reservoirs.

Figures 5, 6 and 7 show the entanglement dynamics as a function of the dimensionless quantity $\gamma_0 t$ (with $\gamma_0 = 4\Omega^2/\Gamma$ the Markovian decay rate of the atoms) and of the purity parameter $r$. In Fig. 5, the qubits are initially in the state given by Eq. (15) with $\alpha^2 = 1/2$ and $\theta = 0$. When $r = 1$ the qubits are prepared in the super-radiant state, entanglement exhibits oscillations and ESD is never present. However, whenever a little amount of mixedness is added, periods of finite-time disentanglement immediately occur. This is due to the appearance of some
population in the excited state $|11\rangle$. As a consequence the oscillating part of the concurrence $\{2|z(t)| − 2\sqrt{a(t)d(t)}\}$ can become negative. Due to the non-Markovianity of the system and to the effective coupling provided by the common reservoir, ESD regions are followed by revivals, and for long times entanglement reaches a stationary value. When $r < 1/3$ the state is initially factorized, but as time passes, because of the reservoir-mediated interaction, entanglement between the qubits is suddenly created. The non-Markovianity of the reservoir enriches the dynamics causing eventually revivals of disentanglement. Eventually the entanglement reaches a non-zero stationary value. The amount of entanglement that has been created depends on the population of the sub-radiant state, which carries the entanglement.

If the qubits are prepared in the state given by Eq. (15) with $\alpha^2 = 1/2$ and $\theta = \pi$, the dynamics is completely different compared to the previous $\theta = 0$ case, as shown in Fig. 6. In fact when $r = 1$ the qubits are prepared in the sub-radiant state and concurrence does not evolve, being at any time equal to 1. When the mixed part is present little oscillations appear, but eventually the concurrence attains the stationary value $(1 + 3r)/4$. When $r < 1/3$ the initial state is factorized, entanglement is suddenly created, a revival of disentanglement is present, and again after some oscillations concurrence reaches its stationary value.

When $\alpha^2 \neq 1/2$ and/or $\theta \neq 0$, $\pi$ the non-mixed part of the initial state is a superposition of super-radiant and sub-radiant states. Thus the entanglement dynamics depends on the weights of those two states. As a result there is a wide variety of entanglement dynamics in between the two asymptotic behaviors described above.

In Fig. 7 the qubits are prepared in the state given by Eq. (16) with $\alpha^2 = 1/2$ and $\theta = 0$. Note however that anyway the results are independent of the choice of the relative phase $\theta$. For $r = 1$ the initial state is the Bell state $(|00\rangle + |11\rangle)/\sqrt{2}$. For this initial condition not only there is no ESD, but also concurrence vanishes only for infinite time. When an increasing amount of mixedness is present in the initial state, finite-time disentanglement appear. ESD is then followed by revivals and, as expected, a certain amount of entanglement is preserved. Analogously to the other EWL state, for $r < 1/3$ the initial state is factorized. For the same reasons previously mentioned, entanglement is suddenly created, momentarily deteriorated, and it finally goes to the stationary value $r/4$, coinciding with the population of the sub-radiant state. For this type of EWL state different choices of $\alpha$ lead to the same qualitative behavior of the entanglement dynamics.

The results we just described for two qubits in a common structured reservoir are quite different from those presented in Ref. [13] for two qubits in two independent Lorentzian structured reservoirs. First of all, if two uncoupled qubits interact with two independent reservoirs entanglement cannot be created from a factorized state. Hence, the ESB region, characterizing the dynamics in a common reservoir, is not present in the results in Ref. [12], for any of the EWL states. Moreover, for qubits prepared in the initial state of Eq. (16), we notice that the reservoir-mediated interaction between the qubits keeps the value of concurrence higher compared to the two independent reservoirs case.

The crucial difference between the common and independent reservoirs cases is that for qubits in two independent reservoirs the decoherence-free sub-radiant state does not exist. When the qubits are in two independent reservoirs, the entanglement dynamics of the $|+\rangle$ and $|−\rangle$ states are the same. As a consequence the relative phase $\theta$ in Eqs. (8) and (15) does not affect the results. This is definitely in contrast with the results we present in Figs. 5 and 6 showing two completely different asymptotic behavior when changing the relative phase $\theta$ of a $\pi$. 
factor.

V. CONCLUSIVE REMARKS

In this paper we have investigated the connection between entanglement and entropy dynamics in a system of two qubits interacting with a common zero temperature non-Markovian reservoir. We have used the exactly solvable model presented in Ref. [9], in the Appendix we attach the analytical solution in the Laplace transform space in the case of EWL states.

We have compared the entanglement and von-Neumann entropy time-evolution for two qubits prepared in a Bell-like state. We have noticed that ESD seems to appear when the state becomes highly mixed, whereas revivals of entanglement are associated to minima of the entropy, where the state becomes purer. On the other hand, when starting from a factorized state, sudden birth of entanglement occurs for lower values of mixedness, while revivals of disentanglement are accompanied by peaks of the entropy.

For a Bell-like state with one excitation the picture seems to be more complex. However, by studying the dynamics of the population of the super-radiant and sub-radiant states, we have realized that two sets of maxima of entanglement can be identified. The super-radiant state mainly carries the entanglement when it is maximally populated; when its population is zero, the entanglement is associated to the sub-radiant state. When the populations of the states are equal, the entanglement vanishes.

We have also considered the entanglement dynamics of a particular class of mixed states, the extended Werner-like states. We have demonstrated that the amount of purity of the initial state plays a key role in the entanglement dynamics, controlling the appearance of ESD and ESB phenomena. For stronger non-Markovian conditions the dynamics exhibits stronger and longer lasting entanglement oscillations and an increasing number of dark periods and revivals as well. On the other hand, in the Markovian regime, no oscillations are present, however the basic features of the dynamics, and so the ESD and ESB regions, are still present.

In general, entanglement and mixedness are two different physical quantities characterizing the degree of non-classicality of a quantum state. We think that it is important also from fundamental point of view to understand the interplay between entanglement and entropy, and that the present results shed new light on their dynamical relation.

Acknowledgments

We thank Barry Garraway for stimulating discussions. This work has been financially supported by M. Ehrnrooth Foundation, Väisälä Foundation, Turku University Foundation, Turun Collegium of Science and Medicine, and the Academy of Finland (projects 108699, 115682, 115982).

APPENDIX

Here we present the exact analytic solution for two qubits interacting with a Lorentzian structured reservoir, when the qubits are prepared in an EWL state as in Eqs. (15) and (16). We provide the expressions for the density matrix element in Eq. (4) as a function of the solution \( \tilde{\rho}_{ij} \) of the pseudomode master equation [34] in Eqs. (5) and (6) of Ref. [9]

\[
\begin{align*}
 a(t) &= \tilde{\rho}_{aa}(t) + \tilde{\rho}_{bb}(t) + \tilde{\rho}_{cc}(t), \\
 b(t) &= \frac{\tilde{\rho}_{++}(t) + \tilde{\rho}_{--}(t) + \tilde{\rho}_{+-}(t) + \tilde{\rho}_{-+}(t)}{2}, \\
 c(t) &= \frac{\tilde{\rho}_{++}(t) - \tilde{\rho}_{--}(t) - \tilde{\rho}_{+-}(t) - \tilde{\rho}_{-+}(t)}{2}, \\
 z(t) &= \frac{\tilde{\rho}_{++}(t) - \tilde{\rho}_{--}(t) - \tilde{\rho}_{+-}(t) + \tilde{\rho}_{-+}(t)}{2}, \\
 d(t) &= 1 - a(t) - b(t) - c(t) = \tilde{\rho}_{ff}(t), \\
 w(t) &= \tilde{\rho}_{af}(t), \\
 \tilde{\rho}_{++}(t) &= \tilde{\rho}_{dd}(t) + \tilde{\rho}_{ee}(t). 
\end{align*}
\]

The subscripts \( a, b \) and \( c \) refer to the states in which both the qubits are in the ground state, and the pseudomode has 0, 1 or 2 excitations respectively. In \( d \) and \( e \) the atomic system is in the super-radiant state, and the pseudomode has 0 or 1 excitations. \( f \) is the state having 0 excitations in the pseudomode and both the qubits in their excited states.

The analytic expressions are given in the Laplace transform space. In the following we provide the solutions for a EWL state of the form [15]:

\[
\begin{align*}
 \hat{\rho}_{aa}(t) &= L^{-1}\left\{ \frac{1}{4s}(1 + r + \frac{64\Gamma^2\Omega^4(-1 + r)l(s)}{j(s)k(s)} - \frac{8\Gamma\Omega^2(1 + r + 4a\sqrt{1 - \alpha^2r\cos\theta})}{j(s)}\right\}, \\
 \hat{\rho}_{bb}(t) &= L^{-1}\left\{ \frac{2\Omega^2}{j(s)}(1 + r - \frac{8\Gamma\Omega^2(-1 + r)l(s)}{k(s)} + 4a\sqrt{1 - \alpha^2r\cos\theta})\right\}. 
\end{align*}
\]
\[ \rho_{ee}(t) = \mathcal{L}^{-1}\left\{ \frac{-48\Omega^4(-1+r)(\Gamma + s)}{k(s)} \right\}, \] (A.4)

\[ \rho_{dd}(t) = \mathcal{L}^{-1}\left\{ \frac{1}{4j(s)} \left( \frac{1}{k(s)}(8\Gamma\Omega^2(-1+r)(6\Gamma^3 + 31\Gamma^4 s + 20\Gamma^3(-2\Omega^2 + 3s^2) + 5\Gamma^2 s(-20\Omega^2 + 11s^2) + 8\Gamma(56\Omega^4 - \frac{17}{2}\Omega^2 s^2 + 3s^4) + 4(120\Omega^4 s - 2\Omega^2 s^3 + 3s^5))) + (8\Omega^2 + (\Gamma + s)(\Gamma + 2s))(1 + r + 4\alpha\sqrt{1 - \alpha^2 r \cos \theta}) \right) \right\}, \] (A.5)

\[ \rho_{ff}(t) = \mathcal{L}^{-1}\left\{ \frac{1}{4k(s)} \left( (-1+r)(6\Gamma^3 + 31\Gamma^4 s + 4\Gamma^3(8\Omega^2 + 15s^2) + \Gamma^2 s(412\Omega^2 + 55s^2) + 8(40\Omega^4 + \frac{91}{2}\Omega^2 s^2 + 3s^4) + 4(72\Omega^4 s + 26\Omega^2 s^3 + s^5)) \right) \right\}, \] (A.7)

\[ \rho_{+-}(t) = \mathcal{L}^{-1}\left\{ \frac{r(\Gamma + 2s)(-1 + 2\alpha^2 + 2i\alpha\sqrt{1 - \alpha^2 \sin \theta})}{2(4\Omega^2 + s(\Gamma + 2s))} \right\}, \] (A.8)

\[ \rho_{--}(t) = \rho_{+-}^*(t), \] (A.9)

\[ j(s) = (\Gamma + 2s)(8\Omega^2 + s(\Gamma + s)), \] (A.13)

\[ \rho_{u}(t) = 0. \] (A.11)

The solutions for both the EWL states are written as functions of \(k(s), j(s), l(s)\) defined as

\[ k(s) = (16\Omega^2 + 2\Gamma^2 s + 24\Omega^2 s + 3\Gamma s^2 + s^3) \] (A.12)

\[ (3\Gamma^3 + 28\Omega^2 + 11\Gamma^2 s + 24\Omega^2 s + 12\Gamma s^2 + 4s^3), \]

Here we present the solutions for a EWL state of the form [10]:

\[ \rho_{a\bar{a}}(t) = \mathcal{L}^{-1}\left\{ \frac{-1}{4s} (-1 + r - 4\alpha^2 r + \frac{8\Gamma\Omega^2(-1+r)}{j(s)} + \frac{64\Gamma^2\Omega^4(-1 + (-3 + 4\alpha r)(-1 + (-3 + 4\alpha r)l(s))}{j(s)k(s)} \right\}, \] (A.15)

\[ \rho_{a\bar{a}}(t) = \mathcal{L}^{-1}\left\{ \frac{2\Omega^2}{j(s)} \left( 1 - r - \frac{8\Gamma\Omega^2(-1 + (-3 + 4\alpha^2 r)l(s))}{k(s)} \right) \right\}, \] (A.16)

\[ \rho_{ee}(t) = \mathcal{L}^{-1}\left\{ \frac{48\Omega^4(-1 + (-3 + 4\alpha^2 r)(\Gamma + s))}{k(s)} \right\}, \] (A.17)

\[ \rho_{dd}(t) = \mathcal{L}^{-1}\left\{ \frac{-1}{4j(s)} \left( (8\Omega^2 + (\Gamma + s)(\Gamma + 2s))(1 + r) + \frac{1}{k(s)}(8\Gamma\Omega^2(-1 + (-3 + 4\alpha^2 r)(6\Gamma^5 + 31\Gamma^4 s + 20\Gamma^3(-2\Omega^2 + 3s^2) + 5\Gamma^2 s(-20\Omega^2 + 11s^2) + 8\Gamma(56\Omega^4 - \frac{17}{2}\Omega^2 s^2 + 3s^4) + 4(120\Omega^4 s - 2\Omega^2 s^3 + s^5))) \right) \right\}, \] (A.18)
\[
\hat{\rho}_{ce}(t) = \mathcal{L}^{-1}\left\{ -\frac{2\Omega^2(-1 + (-3 + 4\alpha^2)r)(6\Gamma^3 + 8\Gamma^2s + 13\Gamma^2s + 12\Omega^2s + 9\Gamma s^2 + 2s^3)}{k(s)} \right\},
\]
(A.19)

\[
\hat{\rho}_{ff}(t) = \mathcal{L}^{-1}\left\{ -\frac{1}{4k(s)}\left( ((-1 + (-3 + 4\alpha^2)r)(6\Gamma^5 + 31\Gamma^4s + 4\Gamma^3(38\Omega^2 + 15s^2) + 12s(412\Omega^2 + 55s^2) + 
8\Gamma(40\Omega^4 + \frac{91}{2} \Omega^2 s^2 + 3s^4) + 4(72\Omega^4 s + 26\Omega^2 s^3 + 3s^3)) \right) \right\},
\]
(A.20)

\[
\hat{\rho}_{af}(t) = \mathcal{L}^{-1}\left\{ \frac{\alpha r \sqrt{1 - \alpha^2}}{4\Omega^2 + 12\Gamma^2s^2 + 12\Omega^2s + 3\Gamma s^2 + 2s^3} \right\},
\]
(A.21)

\[
\hat{\rho}_{-}(t) = \hat{\rho}_{+}(t) = \hat{\rho}_{-}(t) = 0.
\]
(A.22)

We also provide the expressions of the von Neumann entropy for the Bell-like states in Eqs. (8) and (9).

\[
S_\Phi = \frac{1}{2}\left( (a(t) + d(t))\log\frac{4}{2\hat{\rho}_{++}(t)\log\hat{\rho}_{++}(t)} - (a(t) + d(t) - f(t))(\log[a(t) + d(t) - f(t)])
- (a(t) + d(t) + f(t))\log[a(t) + d(t) + f(t)] \right),
\]
(A.23)

\[
S_\Phi = \frac{1}{2}\left( -2a(t)\log[a(t)] - (\hat{\rho}_{++}(t) + \hat{\rho}_{-}(t) - j(t))\log[\hat{\rho}_{++}(t) + \hat{\rho}_{-}(t) - j(t)]
- (\hat{\rho}_{++}(t) + \hat{\rho}_{-}(t) + j(t))\log[\hat{\rho}_{++}(t) + \hat{\rho}_{-}(t) + j(t)] \right).
\]
(A.24)

[1] M. A. Nielsen and I. L. Chuang, Quantum Computation and Quantum Information, (Cambridge University Press, Cambridge, England, 2000); S. Stenholm and K.-A. Suominen, Quantum Approach to Informatics, (Wiley, Hoboken, NJ, 2005).
[2] H.-P. Breuer and F. Petruccione, The Theory of Open Quantum systems (OUP, Oxford, 2002).
[3] M. B. Plenio, S. Virmani, Quantum Inf. Comput. 7, 1 (2007).
[4] T. Yu and J. H. Eberly, Phys. Rev. Lett. 93, 140404 (2004).
[5] Ting Yu and J. H. Eberly, Science 323, 598 (2009).
[6] Juan Pablo Paz and Augusto J. Roncaglia, Phys. Rev. Lett. 100, 220401 (2008).
[7] Sumanta Das, G. S. Agarwal, arXiv:0901.2114, M. Scala, R. Migliore, A. Messina, J. Phys. A: Math. Theor. 41, 435304 (2008).
[8] Z. Ficek and R. Tanaś, Phys. Rev. A 74, 024304 (2006).
[9] L. Mazzola, S. Maniscalco, J. Piilo, K.-A. Suominen, B. M. Garraway, Phys. Rev. A 74, 042302 (2009).
[10] B. Bellomo et al., Phys. Rev. Lett. 99, 160502 (2007).
[11] S. Adhikari et al., arXiv:0812.3772v1; M. Horodecki, P. Horodecki, R. Horodecki, Phys. Rev. A 60, 1888 (1999).
[12] A. Acín, N. Gisin and L. Masanes, Phys. Rev. Lett. 97, 120405 (2006).
[13] B. Bellomo, R. Lo Franco, and G. Compagno, Phys. Rev. A 77, 032342 (2008).
[14] S. Haroche and J.-M. Raimond, Exploring the Quantum: Atoms, Cavities, and Photons, (OUP, Oxford, 2006).
[15] W. K. Wootters, Phys. Rev. Lett. 80, 2245 (1998).
[16] Z. Ficek and R. Tanaś, Phys. Rev. A 77, 054301 (2008).
[17] C. E. López et al., Phys. Rev. Lett. 101, 080503 (2008).
[18] S. Maniscalco et al., Phys. Rev. Lett. 100, 090503 (2008).
[19] Yang Li, Jiang Zhou and Hong Guo, Phys. Rev. A 79, 012309 (2009).
[20] Jun Jing, Zhi-Guo Lü, Zbigniew Ficek, arXiv:0811.4633v1, accepted by PRA.
[21] R. F. Werner, Phys. Rev. A 40, 4277 (1989).
[22] Ting Yu and Eberly, J. Mod. Opt. 54, 2289-2296 (2007).
[23] Ting Yu and J. H. Eberly, Quantum Information and Computation 7, 459-468 (2007).
[24] C. J. Shan et al., arXiv:0808.3690
[25] A. Miranowicz, Phys. Letters A 327 272-283 (2004).
[26] C. F. Wildfeuer and J. P. Dowling, Phys. Rev. A 78, 032113 (2008).
[27] Shang-Bin Li and Jing-Bo Xu, The European Physical Journal D 41 2 (2007) 377-383.
[29] Lech Jakóbczyk, Ann Jamróz, Phys. Letters A 347 (2005) 180-190.
[30] J. Lee and M. S. Kim, Phys. Rev. Lett. 84, 4236 (2000).
[31] Yong-Sheng Zhang et al., Phys. Rev. A 66, 062315 (2002).
[32] M. Barbieri et al., Phys. Rev. Lett. 92, 177901 (2004).
[33] J. B. Altepeter et al., Phys. Rev. Lett. 90, 193601 (2003).
[34] B. M. Garraway, Phys. Rev. A 55, 2290 (1997).