GLOBAL WEAK SOLUTIONS FOR AN INCOMPRESSIBLE, GENERALIZED NEWTONIAN FLUID INTERACTING WITH A LINEARLY ELASTIC KOITER SHELL

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ABSTRACT. In this paper we analyze the interaction of an incompressible, generalized Newtonian fluid with a linearly elastic Koiter shell whose motion is restricted to transverse displacements. The middle surface of the shell constitutes the mathematical boundary of the three-dimensional fluid domain. We show that weak solutions exist as long as the magnitude of the displacement stays below some (possibly large) bound which is determined by the geometry of the undeformed shell.

1. INTRODUCTION

Fluid-solid interaction problems involving moving interfaces have been studied intensively during the last two decades. The interaction with elastic solids has proven to be particularly difficult, due to apparent regularity incompatibilities between the parabolic fluid phase and the hyperbolic or dispersive solid phase, see, e.g., [3, 4, 13, 14, 8, 7, 20, 30, 29] and the references therein. In [7, 20] the global-in-time existence of weak solutions for the interaction of an incompressible, Newtonian fluid with a Kirchhoff-Love plate is shown. In [29] we generalized this result to the case of a linearly elastic Koiter shell. The aim of the present paper is to extend the result in [29] to generalized Newtonian fluids, i.e., to fluids with a shear-dependent viscosity. A common model for the viscous (extra) stress tensor $S$ of such fluids is given by

$$S = \mu_0 (\delta + |D|) p^{-2} D$$

for constants $\mu_0 > 0$, $\delta \geq 0$, $1 < p < \infty$. Here, $D$ is the shear rate tensor. The mathematical analysis of such fluids in fixed spatial domains was initiated by Ladyzhenskaya [24, 25, 26] and Lions [31] in the late sixties. For $p \geq 11/5$ (in three space dimensions) the global existence of weak solutions follows from a combination of monotone operator theory and a compactness argument which is quite standard today. In [18] the Lipschitz truncation technique was used for the first time to study the existence of stationary weak solutions in the case of smaller exponents. This technique was improved in [16] and transferred to the nonstationary case in [17]. In the latter paper the existence of global weak solutions is shown for arbitrary $p$ strictly greater than the natural bound $6/5$. This result is based on a parabolic Lipschitz truncation and a deep understanding of the pressure. However, in [6] the existence proof was considerably simplified by the introduction of solenoidal parabolic Lipschitz truncations which are considerably more flexible. We shall employ these in the present paper. It seems that [21] is the only analytical result so far dealing with the interaction of generalized Newtonian fluids with elastic solids. In this paper the existence of global weak solutions for shear-thickening fluids, i.e., $p \geq 2$, is shown under the assumptions of cylindrical symmetry, resulting in a two-dimensional problem, and a very strong mathematical damping of the elastic solid.
In the present paper we extend [29] to generalized Newtonian fluids. In doing so we have to deal with three new substantial difficulties. The first one is the well-known problem of identifying the limit of the extra stress tensor. Here, we have to apply the techniques developed in [6]. The second difficulty is due to the fact that the proof of relative $L^2$-compactness of bounded sequences of weak solutions developed in [29] needs substantial modification if $p$ is not larger than $3/2$. Finally, due to the additional nonlinearity in the system we cannot proceed as in [29] and apply the Kakutani-Glicksberg-Fan theorem. Instead, we have to construct an approximate decoupled system that is uniquely solvable on the one hand and that gives rise to an approximate coupled system accessible to the Lipschitz-truncation technique on the other hand. In order to deal with the approximate system we have to transfer monotone operator theory techniques to the present “non-cylindrical” situation.

The present paper is partly based on the author’s Ph.D. thesis [28]. It is organized as follows. In Subsection 1.1 we introduce Koiter’s energy for elastic shells, in Subsection 1.2 we introduce the coupled fluid-shell system, and in Subsection 1.3 we derive formal a-priori estimates for this system. In Section 2 we give some results concerning domains with non-Lipschitz boundaries. Then, in Section 3 we state the main result of the paper. The rest of the section is devoted to the proof of this result. In Subsection 3.1 we give the proof of compactness of sequences of weak solutions. Subsequently, in Subsection 3.2 we analyse a decoupled variant of our original system, while in Subsection 3.3 we apply a fixed-point argument to this decoupled system. In Subsection 3.4 we conclude the proof by letting the regularisation parameter, which we introduced earlier, tend to zero. Finally, some further results and technical computations can be found in the appendix.

1.1. Koiter’s energy. Throughout the paper, let $\Omega \subset \mathbb{R}^3$ be a bounded, non-empty domain of class $C^4$ with outer unit normal $\nu$. We denote by $g$ and $h$ the first and the second fundamental form of $\partial \Omega$, induced by the ambient Euclidean space, and by $dA$ the surface measure of $\partial \Omega$. Furthermore, let $\Gamma \subset \partial \Omega$ be a union of domains of class $C^{1,1}$ having non-trivial intersection with all connected components of $\partial \Omega$. We set $M := \partial \Omega \setminus \Gamma$; note that $M$ is compact. Let $\partial \Omega$ represent the middle surface of an elastic shell of thickness $2\varepsilon_0 > 0$ in its rest state where $\varepsilon_0$ is taken to be small compared to the reciprocal of the principal curvatures. Furthermore, we assume that the elastic shell consists of a homogeneous, isotropic material whose linear elastic behavior may be characterized by the Lamé constants $\lambda$ and $\mu$. We restrict the deformation of the middle surface to displacements along the unit normal field $\nu$, and we assume the part $\Gamma$ of the middle surface to be fixed. Hence, we can describe the deformation by a scalar field $\eta : M \to \mathbb{R}$ vanishing at the boundary $\partial M$. We model the elastic energy of the deformation by Koiter’s energy for linearly elastic shells and transverse displacements

$$K(\eta) = K(\eta, \eta) = \frac{1}{2} \int_M \varepsilon_0 (C, \sigma(\eta) \otimes \sigma(\eta)) + \frac{\varepsilon_0^3}{3} (C, \xi(\eta) \otimes \xi(\eta)) \, dA.$$

Here,

$$C_{\alpha\beta\gamma\delta} := \frac{4\lambda\mu}{\lambda + 2\mu} g_{\alpha\beta} g_{\gamma\delta} + 2\mu (g_{\alpha\gamma} g_{\beta\delta} + g_{\alpha\delta} g_{\beta\gamma}).$$
is the elasticity tensor of the shell, and
\[ \sigma(\eta) = -h \eta, \quad \zeta(\eta) = \nabla^2 \eta - k \eta, \]
are the linearized strain tensors, where \( k_{\alpha\beta} := h_{\alpha\beta}^0 \). See [22], [23], [11], [12] for Koiter’s energy for nonlinearly elastic shells, and [12] for the derivation of the linearization; cf. also [29]. \( K \) is a quadratic form in \( \eta \) which is coercive on \( H^2_0(M) \), i.e., there exists a constant \( c_0 \) such that
\[ K(\eta) \geq c_0 \| \eta \|^2_{H^2_0(M)}, \quad (1.1) \]
see the proof of Theorem 4.4-2 in [12]. Using integration by parts and taking into account some facts from Riemannian geometry one can show that the \( L^2 \)-gradient of this energy has the form
\[ \text{grad}_2 K(\eta) = \varepsilon_0 \frac{8\mu(\lambda + \mu)}{3(\lambda + 2\mu)} \Delta^2 \eta + B \eta \]
where \( B \) is a second order differential operator which vanishes on flat parts of \( M \), i.e., where \( h = 0 \). The details can be found in [28]. Thus, we obtain a generalization of the linear Kirchhoff-Love plate equation for transverse displacements, and [12] for the derivation of the linearization; cf. for instance [10].

By Hamilton’s principle, the displacement \( \eta \) of the shell must be a stationary point of the action functional
\[ \mathcal{A}(\eta) = \int_{t_0}^{t_f} \int_M \left( \frac{\partial \eta(t, \cdot)}{\partial t} \right)^2 dA - K(\eta(t, \cdot)) dt \]
where \( I := (0, T) \), \( T > 0 \). Here we assume that the mass density of \( M \) may be described by a constant \( \varepsilon_0 \rho_0 \). Hence, the integrand with respect to time is the difference of the kinetic and the potential energy of the shell. The corresponding Euler-Lagrange equation is
\[ \varepsilon_0 \rho_0 \delta_t^2 \eta + \text{grad}_2 K(\eta) = 0 \text{ in } I \times M. \]

1.2. Statement of the problem. We denote by \( \Omega_{\eta(t)}, t \in I \), the deformed domain (cf. (2.1)) and by
\[ \Omega^t_\eta := \bigcup_{t \in I} \{ t \} \times \Omega_{\eta(t)} \]
the deformed spacetime cylinder. Let us suppose that the variable domain \( \Omega_{\eta} \) is filled by a homogeneous, incompressible, generalized Newtonian fluid whose isothermal motion is governed by the system
\[ \begin{align*}
\rho_F (\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u}) - \text{div} \left( S(D \mathbf{u}) - \pi \text{id} \right) &= \rho_F \mathbf{f} & \text{in } \Omega^t_\eta, \\
\text{div} \mathbf{u} &= 0 & \text{in } \Omega^t_\eta, \\
\mathbf{u}(\cdot, \cdot + \eta \mathbf{v}) &= \partial_t \eta \mathbf{v} & \text{on } I \times M, \\
\mathbf{u} &= 0 & \text{on } I \times \Gamma.
\end{align*} \quad (1.2) \]

Here, \( \mathbf{u} \) is the velocity field, \( \pi \) is the pressure field, \( D \mathbf{u} \) is the symmetric part of the gradient of \( \mathbf{u} \), \( S \) is the extra stress tensor, \( \text{id} \) denotes the \( 3 \times 3 \) unit matrix, and \( \mathbf{f} \) is an external body force. We assume that \( S \) possesses a \( p \)-structure, i.e., for some \( 6/5 < p < \infty \) and \( \delta \geq 0 \) we have

- \( S : M_{\text{sym}} \to M_{\text{sym}} \) continuous,
- Growth: \( |S(D)| \leq c_0 (\delta + |D|)^{p-2} |D| \) for all \( D \in M_{\text{sym}} \) and some \( c_0 > 0 \),
- Coercivity: \( S(D) : D \geq c_1 (\delta + |D|)^{p-2} |D|^2 \) for all \( D \in M_{\text{sym}} \) and some \( c_1 > 0 \),
- Strict monotonicity: \( (S(D) - S(E)) : (D - E) > 0 \) for all \( D, E \in M_{\text{sym}}, D \neq E \).
Here, $M_{\text{sym}}$ denotes the space of real, symmetric $3 \times 3$ matrices. In the following, we divide equation (1.2), by the constant fluid density $\rho_F$, denoting $S/\rho_F$ and $\pi/\rho_F$ again by $S$ and $\pi$. (1.2) is the no-slip condition in the case of a moving boundary, i.e., the velocity of the fluid at the boundary equals the velocity of the boundary. The force exerted by the fluid on the boundary is given by the evaluation of the stress tensor at the deformed boundary in the direction of the inner normal $-\nu \eta(t)$, i.e., by

$$
\rho_F \left( -S(Du(t, \cdot)) \nu \eta(t) + \pi(t, \cdot) \nu \eta(t) \right).
$$

(1.3)

Thus, the equation for the displacement of the shell takes the form

$$
\varepsilon_0 \rho_S \frac{d^2}{dt^2} \eta + \text{grad}_L^2 K(\eta) = \varepsilon_0 \rho_S g + \rho_F F \cdot \nu \quad \text{in } I \times M,
$$

$$
\eta = 0, \ \nabla \eta = 0 \quad \text{on } I \times \partial M.
$$

(1.4)

where $g$ is a given body force and

$$
F(t, \cdot) := \left( -S(Du(t, \cdot)) \nu \eta(t) + \pi(t, \cdot) \nu \eta(t) \right) \circ \Phi \eta(t) \left| \det d\Phi \eta(t) \right|
$$

with $\Phi \eta(t) : \partial \Omega \to \partial \Omega \eta(t)$, $q \mapsto (t, q) v(q)$. In the following, we divide (1.4) by $\varepsilon_0 \rho_S$, denote $K/\varepsilon_0 \rho_S$ again by $K$, and assume, for the sake of a simple notation, that $\rho_F/\varepsilon_0 \rho_S = 1$.

Finally, we specify initial values

$$
\eta(0, \cdot) = \eta_0, \ \partial_t \eta(0, \cdot) = \eta_1 \quad \text{in } M \quad \text{and} \quad u(0, \cdot) = u_0 \quad \text{in } \Omega_0.
$$

(1.5)

In the following, we will analyze the system (1.2), (1.4), (1.5).

### 1.3. Formal a-priori estimates.

Let us now formally derive energy estimates for this parabolic-dispersive system. To this end, we multiply (1.2) by $u$, integrate the resulting identity over $\Omega \eta(t)$, and obtain after integrating by parts the stress tensor

$$
\int_{\Omega \eta(t)} \partial_t u \cdot u \ dx + \int_{\Omega \eta(t)} (u \cdot \nabla) u \cdot u \ dx
$$

(1.6)

\[
= - \int_{\Omega \eta(t)} S(Du) : Du \ dx + \int_{\Omega \eta(t)} f \cdot u \ dx + \int_{\partial \Omega \eta(t) \setminus \Gamma} (S(Du) \nu \eta(t) - \pi \nu \eta(t)) \cdot u \ dA \eta(t).
\]

Here, $dA \eta(t)$ denotes the surface measure of the deformed boundary $\partial \Omega \eta(t)$. Taking into account that

$$
\int_{\Omega \eta(t)} (u \cdot \nabla) u \cdot u \ dx = - \int_{\Omega \eta(t)} (u \cdot \nabla) u \cdot u \ dx + \int_{\partial \Omega \eta(t)} u \cdot \nu \eta(t) \ |u|^2 \ dA \eta(t),
$$

(1.7)

we may apply Reynold’s transport theorem $\text{A.1}$ to the first two integrals in (1.6) to obtain

$$
\frac{1}{2} \frac{d}{dt} \int_{\Omega \eta(t)} |u|^2 \ dx = - \int_{\Omega \eta(t)} S(Du) : Du \ dx + \int_{\Omega \eta(t)} f \cdot u \ dx
$$

(1.8)

\[
+ \int_{\partial \Omega \eta(t) \setminus \Gamma} (S(Du) \nu \eta(t) - \pi \nu \eta(t)) \cdot u \ dA \eta(t).
\]

Multiplying (1.4) by $\partial_t \eta$, integrating the resulting identity over $M$, integrating by parts, and using the fact that $(\text{grad}_L^2 K(\eta), \partial \eta)_{L^2} = 2K(\eta, \partial \eta)$, we obtain

$$
\frac{1}{2} \frac{d}{dt} \int_{M} |\partial_t \eta|^2 \ dA + \frac{d}{dt} K(\eta) = \int_{M} g \partial_t \eta \ dA + \int_{M} F \cdot \nu \partial_t \eta \ dA.
$$

(1.9)

\footnote{For the sake of a better readability we suppress the dependence of the unknown on the independent variables, e.g. we write $u = u(t, \cdot)$.}
Adding (1.8) and (1.9), taking into account the definition of $F$, (1.2), and applying a change of variables to the boundary integral, we obtain the energy identity
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega(t)} |\mathbf{u}|^2 \, dx + \frac{1}{2} \frac{d}{dt} \int_{\partial M} |\partial_t \eta|^2 \, dA + \frac{d}{dt} K(\eta) = -\int_{\Omega(t)} S(D\mathbf{u}) : D\mathbf{u} \, dx + \int_{\Omega(t)} f \cdot \mathbf{u} \, dx + \int_{\partial M} g \partial_t \mathbf{u} \, dA.
\] (1.10)

In view of (1.1) and the coercivity of $S$, an application of Gronwall’s lemma gives
\[
\|\mathbf{u}(t, \cdot)\|_{L^2(\Omega(t))}^2 + \int_0^t \|D\mathbf{u}(s, \cdot)\|_{L^p(\Omega(t))}^p \, ds + \|\partial_t \eta(t, \cdot)\|_{L^2(M)}^2 + \|\eta(t, \cdot)\|_{H^2(M)}^2 \leq c e^{\int_0^t L_1(s)}(\|\mathbf{u}_0\|_{L^2(\Omega_0)}^2 + \|\mathbf{u}_0\|_{L^\infty(\Omega_0)}^\infty + \|D\mathbf{u}_0\|_{L^p(\Omega_0)}^p) \leq c(T, \text{data}).
\] (1.11)

Hence, we have
\[
\|\eta\|_{W^{1,-(L^2(\Omega(t)))'}}(t) + \|\mathbf{u}\|_{L^2(\Omega_0)} + \|D\mathbf{u}\|_{L^p(\Omega_0)} \leq c(T, \text{data}).
\]

We shall construct weak solutions in this regularity class. In view of the embedding $H^1(\partial \Omega) \hookrightarrow C^{0,\theta}(\partial \Omega)$ for $\theta < 1$, this implies that the boundary of our variable domain will be the graph of a Hölder continuous function which, in general, is not Lipschitz continuous. Since, in general, Korn’s inequality is false in non-Lipschitz domains, c.f. [2], we cannot expect an estimate of $\mathbf{u}$ in $L^p(I, W^{1,p}(\Omega(t)))$. In the next section, we collect some facts about a class of non-Lipschitz domains.

2. VARIABLE DOMAINS

We denote by $S_{\alpha}$, $\alpha > 0$, the open set of points in $\mathbb{R}^3$ whose distance from $\partial \Omega$ is less than $\alpha$. It’s a well known fact from elementary differential geometry, see for instance [27], that there exists a maximal $\kappa > 0$ such that the mapping
\[
\Lambda : \partial \Omega \times (-\kappa, \kappa) \to S_{\kappa}, \ (q, s) \mapsto q + s \nu(q)
\]
is a $C^3$-diffeomorphism. For the inverse $\Lambda^{-1}$ we shall write $x \mapsto (q(x), s(x))$. Note that $\kappa$ is not necessarily small; if $\Omega$ is the ball of radius $R$, then $\kappa = R$. Let $B_{\alpha} := \Omega \cup S_{\alpha}$ for $0 < \alpha < \kappa$. The mapping $\Lambda(\cdot, \alpha) : \partial \Omega \to \partial B_{\alpha}$ is a $C^3$-diffeomorphism as well. Hence, $B_{\alpha}$ is a bounded domain with $C^3$-boundary.\footnote{In fact, it’s even $C^4$.} For a continuous function $\eta : \partial \Omega \to (-\kappa, \kappa)$ we set
\[
\Omega_\eta := \Omega \setminus S_{\kappa} \cup \{x \in S_{\kappa} \mid s(x) < \eta(q(x))\}. \quad (2.1)
\]
$\Omega_\eta$ is an open set. For $\eta \in C^k(\partial \Omega)$, $k \in \{1, 2, 3\}$ we denote by $\nu_\eta$ and $dA_\eta$ the outer unit normal and the surface measure of $\partial \Omega_\eta$, respectively. In [29] we showed that the mapping $\Psi_\eta : \Omega \to \Omega_\eta$, defined to be the identity in $\Omega \setminus S_{\kappa}$ and defined in $S_{\kappa} \cap \Omega$ by
\[
x \mapsto (q(x) + \nu(q(x))(s(x) + \eta(q(x)))\beta(s(x)/\kappa)) \quad (2.2)
\]
for a suitable function $\beta : \mathbb{R} \to \mathbb{R}$, is a homeomorphism, and even a $C^k$-diffeomorphism provided that $\eta \in C^k(\partial \Omega)$ with $k \in \{1, 2, 3\}$. Furthermore, we showed that the homeomorphism
\[
\Phi_\eta := \Psi_\eta |_{\partial \Omega} : \partial \Omega \to \partial \Omega_\eta, \ q \mapsto q + \eta(q) \nu(q)
\]
is a $C^k$-diffeomorphism provided that $\eta \in C^k(\partial \Omega)$, $k \in \{1, 2, 3\}$. Finally, we argued that $\Psi_\eta$ and $\Phi_\eta$ become singular as $\tau(\eta) \to \infty$ where

$$\tau(\eta) := \begin{cases} (1 - \|\eta\|_{L^\infty(\partial \Omega)}/\kappa)^{-1} & \text{if } \|\eta\|_{L^\infty(\partial \Omega)} < \kappa, \\ \infty & \text{else.} \end{cases} \quad (2.3)$$

Remark 2.4. For $\eta \in C^2(\partial \Omega)$ with $\|\eta\|_{L^\infty(\partial \Omega)} < \kappa$ and $\varphi : \Omega \to \mathbb{R}^3$ we denote by $\mathcal{T}_\eta \varphi$ the pushforward of $(\det d\Psi_\eta)^{-1} \varphi$ under $\Psi_\eta$, i.e.,

$$\mathcal{T}_\eta \varphi := (d\Psi_\eta (\det d\Psi_\eta)^{-1} \varphi) \circ \Psi_\eta^{-1}.$$  

The mapping $\mathcal{T}_\eta$ with the inverse $\mathcal{T}_\eta^{-1} \varphi := (d\Psi_\eta^{-1} (\det d\Psi_\eta) \varphi) \circ \Psi_\eta$ obviously defines isomorphisms between the Lebesgue and Sobolev spaces on $\Omega$ and $\Omega_\eta$, respectively, as long as the order of differentiability is not larger than 1. Moreover, the mapping preserves vanishing boundary values. We saw in [29] that it also preserves the divergence-free constraint and hence defines isomorphisms between corresponding spaces of solenoidal functions on $\Omega$ and $\Omega_\eta$, respectively.

A bi-Lipschitz mapping of domains induces isomorphisms of the corresponding $L^p$ and $W^{1,p}$ spaces. For $\eta \in H^2(\partial \Omega)$ the mapping $\Psi_\eta$ is barely not bi-Lipschitz, due to the embedding $H^2(\partial \Omega) \hookrightarrow C^{0,\theta} (\partial \Omega)$ for $\theta < 1$. Hence a small loss, made quantitative in the next lemma, will occur.

Lemma 2.5. Let $1 < p \leq \infty$ and $\eta \in H^2(\partial \Omega)$ with $\|\eta\|_{L^\infty(\partial \Omega)} < \kappa$. Then the linear mapping $v \mapsto v \circ \Psi_\eta$ is continuous from $L^p(\Omega_\eta)$ to $L^p(\Omega)$ and from $W^{1,p}(\Omega_\eta)$ to $W^{1,p}(\Omega)$ for all $1 \leq r < p$. The analogous claim with $\Psi_\eta$ replaced by $\Psi_\eta^{-1}$ is true. The continuity constants depend only on $\Omega$, $p$, $r$, and a bound for $\|\eta\|_{H^2(\partial \Omega)}$ and $\tau(\eta)$.

Proof. See [29].

In the following we denote by $\cdot |_{\partial \Omega}$ the usual trace operator for Lipschitz domains. From the continuity properties of this trace operator and Lemma 2.5 we deduce the following assertion.

Corollary 2.6. Let $1 < p \leq \infty$ and $\eta \in H^2(\partial \Omega)$ with $\|\eta\|_{L^\infty(\partial \Omega)} < \kappa$. Then the linear mapping $\operatorname{tr}_\eta : v \mapsto (v \circ \Psi_\eta)|_{\partial \Omega}$ is well defined and continuous from $W^{1,p}(\Omega_\eta)$ to $W^{1,1/r'}(\partial \Omega)$ for all $1 < r < p$. The continuity constant depends only on $\Omega$, $p$, $r$, and a bound for $\|\eta\|_{H^2(\partial \Omega)}$ and $\tau(\eta)$.

From Lemma 2.5 and the Sobolev embeddings for regular domains we deduce Sobolev embeddings for our special domains.

Corollary 2.7. Let $1 < p < 3$ and $\eta \in H^2(\partial \Omega)$ with $\|\eta\|_{L^\infty(\partial \Omega)} < \kappa$. Then

$$W^{1,p}(\Omega_\eta) \hookrightarrow L^s(\Omega_\eta)$$

for $1 \leq s < p^* = 3p/(3-p)$. The embedding constant depends only on $\Omega$, $p$, $s$, and a bound for $\|\eta\|_{H^2(\partial \Omega)}$ and $\tau(\eta)$.

We denote by $H$ the mean curvature (with respect to the outer normal) and by $G$ the Gauss curvature of $\partial \Omega$.  

\[^3\]The symbol $\hookrightarrow\hookrightarrow$ indicates that the embedding is compact.
Proposition 2.8. Let $1 < p \leq \infty$ and $\eta \in H^2(\partial \Omega)$ with $||\eta||_{L^\infty(\partial \Omega)} < \kappa$. Then, for $\varphi \in W^{1,p}(\Omega)$ with $\text{tr}_\eta \varphi = b \eta$, $b$ a scalar function, and $\psi \in C^1(\overline{\Omega})$ we have
\[
\int_{\Omega} \varphi \cdot \nabla \psi \, dx = -\int_{\Omega} \text{div} \varphi \, \psi \, dx + \int_{\partial \Omega} b (1 - 2H\eta + G\eta^2) \text{tr}_\eta \psi \, dA.
\]
Proof. See [29].

We showed in [29] that the function $\gamma(\eta) := 1 - 2H\eta + G\eta^2$ is positive as long as $|\eta| < \kappa$. Now, consider the space
\[
E^p(\Omega) := \{ \varphi \in L^p(\Omega) \mid \text{div} \varphi \in L^p(\Omega) \}
\]
for $1 \leq p \leq \infty$, endowed with the canonical norm.

Proposition 2.9. Let $1 < p < \infty$ and $\eta \in H^2(\partial \Omega)$ with $||\eta||_{L^\infty(\partial \Omega)} < \kappa$. Then there exists a continuous, linear operator
\[
\text{tr}_\eta^* : E^p(\Omega) \to (W^{1,p'}(\partial \Omega))'
\]
such that for $\varphi \in E^p(\Omega)$ and $\psi \in C^1(\overline{\Omega})$
\[
\int_{\Omega} \varphi \cdot \nabla \psi \, dx = -\int_{\Omega} \text{div} \varphi \, \psi \, dx + \langle \text{tr}_\eta \varphi, \text{tr}_\eta \psi \rangle_{W^{1,p'}(\partial \Omega)}.
\]
The continuity constant depends only on $\Omega$, $p$, and a bound for $\tau(\eta)$.

Proof. See [29].

Proposition 2.10. Let $1 < p < \infty$, $\eta \in H^2(\partial \Omega)$ with $||\eta||_{L^\infty(\partial \Omega)} < \kappa$, and $\alpha$ such that $||\eta||_{L^\infty(\partial \Omega)} < \alpha < \kappa$. Then there exists a bounded, linear extension operator
\[
\mathcal{F}_\eta : \{ b \in W^{1,p}(\partial \Omega) \mid \int_{\partial \Omega} b \gamma(\eta) \, dA = 0 \} \to W^{1,p}_{\text{div}}(B\alpha),
\]
in particular $\text{tr}_\eta \mathcal{F}_\eta b = b \eta$. We also have
\[
\mathcal{F}_\eta : \{ b \in L^p(\partial \Omega) \mid \int_{\partial \Omega} b \gamma(\eta) \, dA = 0 \} \to \{ \varphi \in L^p(B\alpha) \mid \text{div} \varphi = 0 \}
\]
as a bounded, linear operator with $\text{tr}_\eta^* \mathcal{F}_\eta b = b \gamma(\eta)$. The continuity constants depend only on $\Omega$, $p$, and a bound for $||\eta||_{H^2(\partial \Omega)}$ and $\tau(\alpha)$.

Proof. See [29].

Of course, these extension operators are not optimal in the sense that they don’t produce any regularity.

Proposition 2.11. Let $6/5 < p < \infty$ and $\eta \in H^2(\partial \Omega)$ with $||\eta||_{L^\infty(\partial \Omega)} < \kappa$. Then extension by 0 defines a bounded, linear operator from $W^{1,p}(\Omega)$ to $H^s(\mathbb{R}^3)$ for some $s > 0$. The continuity constant depends only on $\Omega$, $p$, and a bound for $||\eta||_{H^2(\partial \Omega)}$ and $\tau(\eta)$.

Proof. Let $6/5 < r < \infty$. By standard embedding theorems we have $W^{1,r}(\Omega) \to H^s(\Omega)$ for some $s > 0$. In order to prove the claim we can proceed exactly like in the proof of [29].
Hence, the application of Hölder’s inequality below is not optimal.

But the identity

\[
\int_{\Omega} |v(x) - v(y)| \, dy = \int_{\Omega} \int_{|x-y|^{3+2s}} |v(x) - v(y)|^2 \, dx \, dy + 2 \int_{\Omega} \int_{\Omega \setminus \Omega} |v(x)|^2 \, dx \, dy \]

for \( v \in H^s(\Omega) \). While the first term on the right-hand side is dominated by \( c\|v\|_{H^s(\Omega)} \) for all \( s \leq \bar{s} \), we can estimate the interior integral of the second term by

\[
\int_{|x-y| > d(x)} \frac{1}{|x-y|^{3+2s}} \, dx = \frac{c(s)}{d(x)^{3s}}
\]

where \( d(x) \) denotes the distance from \( x \) to \( \partial \Omega \). Again by standard embedding results, we have \( H^s(\Omega) \hookrightarrow L^r(\Omega) \) for some \( r > 2 \). An application of Hölder’s inequality now shows that the second term on the right-hand side of (2.12) is dominated by

\[
c(s) \|v\|_{L^r(\Omega)}^2 \|d(\cdot)^{-2s}\|_{L^r(\Omega)}
\]

But the identity

\[
\int_{\mathbb{R}^3} |d(x)|^{-2s(2r/2)'} \, dx = \int_{\mathbb{R}^3} \int_{|\beta/2|}^0 |\det d^\alpha| \alpha^{-2s(2r/2)'} \, d\alpha \, dA,
\]

a consequence of a change of variables, proves that the last factor in this expression is finite for sufficiently small \( s \).

Let us now prove a suitable variant of Korn’s inequality for non-Lipschitz domains.

**Proposition 2.13.** Let \( 1 < p < \infty \) and \( \eta \in H^2(\partial \Omega) \) with \( \|\eta\|_{L^r(\partial \Omega)} < \kappa \). Then, for all \( \varphi \in C^1(\overline{\Omega}) \) and all \( 1 \leq r < p \), we have

\[
\|
\varphi
\|_{W^{1,r}(\Omega)} \leq c \left( \|D\varphi\|_{L^p(\Omega)} + \|\varphi\|_{L^p(\Omega)} \right).
\]

The constant \( c \) depends only on \( \Omega, p, r, \) and a bound for \( \|\eta\|_{H^2(\partial \Omega)} \) and \( \tau(\eta) \).

**Proof.** For \( 1/r = 1 - 1/p \) and \( 1 - 1/(2r) < \beta < 1 \), we have

\[
\|\nabla \varphi\|_{L^r(\Omega)} \leq \|\nabla \varphi \|_{L^p(\Omega)} \|d^{\beta-1}\|_{L^r(\Omega)}.
\]

Here, \( d(x) \) denotes the distance from \( x \in \Omega \) to \( \partial \Omega \). Since \( \Omega \) is a \( \beta \)-Hölder domain, by \( \parallel \) Theorem 3.1], the term \( \|\nabla \varphi \|_{L^p(\Omega)} \|d^{\beta-1}\|_{L^r(\Omega)} \) is dominated by the right-hand side of (2.14). Hence, all we need to do is to bound the \( L^r(\Omega) \)-norm of \( d^{\beta-1} \). To this end, we note that, since \( \eta \) is 1/2-Hölder continuous, for \( |s| < \kappa \) and \( q, \bar{q} \in \partial \Omega \), we have

\[
|\eta(q) - s| \leq |\eta(q) - \eta(\bar{q})| + |\eta(\bar{q}) - s| \leq c |q - \bar{q}|^{1/2} + |\eta(\bar{q}) - s| \leq c (|q - \bar{q}| + |\eta(\bar{q}) - s|) \leq c |q - \bar{q} + s v| \geq c |\eta(q) - s|^2.
\]

For the second inequality, we used the fact that the geodesic distance on \( \partial \Omega \) and the Euclidean distance in \( \mathbb{R}^3 \) of \( q \) and \( \bar{q} \) are comparable. We deduce that \( d(q + s v) \geq c |\eta(q) - s|^2 \).

\[\text{In fact, it is possible to show that extension by 0 is continuous from } H^s(\Omega) \text{ to } H^s(\mathbb{R}^3) \text{ for all } 0 < s < 1/2. \]

Hence, the application of Hölder’s inequality below is not optimal.
Thus, using a change of variables, for \( \| \eta \|_{H^1(\partial \Omega)} < \alpha < \kappa \), we obtain
\[
\int_{\partial \Omega} |\det d\Lambda| d(q + s \mathbf{v})^{(\beta - 1)\gamma} \, d\sigma(q) \leq c \int_{\partial \Omega} |\eta(q)| - s^{(\beta - 1)\gamma} \, d\sigma(q).
\]
By the assumption on \( \beta \) the last integral is bounded. \( \square \)

The usual Bochner spaces are not the right objects to deal with functions defined on time-dependent domains. For this reason we now define an (obvious) substitute for these spaces. For \( I := (0, T) \), \( T > 0 \), and \( \eta \in C(\tilde{I} \times \partial \Omega) \) with \( \| \eta \|_{L^1(I \times \partial \Omega)} < \kappa \) we set \( \Omega^\eta_I := \bigcup_{t \in I} \{ t \} \times \Omega^\eta(t) \). Note that \( \Omega^\eta_I \) is a domain in \( \mathbb{R}^d \). For \( 1 \leq p, r \leq \infty \) we set
\[
L^p(I, L^r(\Omega^\eta_I)) := \{ v \in L^1(\Omega^\eta_I) \mid \| v(t, \cdot) \|_{L^r(\Omega^\eta(t))} \in L^p(I) \},
\]
\[
L^p(I, W_{\text{div}}^{1,r}(\Omega^\eta_I)) := \{ v \in L^p(I, W_{\text{div}}^{1,r}(\Omega^\eta_I)) \mid \text{div} v = 0 \},
\]
\[
W^{1,r}(I, W_{\text{div}}^{1,r}(\Omega^\eta_I)) := \{ v \in L^p(I, W_{\text{div}}^{1,r}(\Omega^\eta_I)) \mid \text{div} v = 0 \}.
\]
Here \( \nabla \) and \( \text{div} \) are acting with respect to the space variables. Furthermore, we set
\[
\Psi^\eta_I : I \times \tilde{\Omega} \rightarrow \Omega^\eta_I, \quad (t, x) \mapsto (t, \Psi^\eta_I(t, x))
\]
and
\[
\Phi^\eta_I : I \times \partial \Omega \rightarrow \bigcup_{t \in I} \{ t \} \times \partial \Omega^\eta(t), \quad (t, x) \mapsto (t, \Phi^\eta_I(t, x)).
\]
If \( \eta \in L^\infty(I, H^2(\partial \Omega)) \) we obtain “instationary” versions of the claims made so far by applying these at (almost) every \( t \in I \). For instance, from Corollary 27 we deduce that
\[
L^2(I, H^1(\Omega^\eta(t))) \hookrightarrow L^2(I, L^2(\Omega^\eta_I))
\]
for \( 1 \leq s < 2^* \). Note that the construction given above does not provide a substitute for Bochner spaces of functions with values in negative spaces. Furthermore, note that for all \( 1/2 < \theta < 1 \) we have
\[
W^{1,\infty}(I, L^2(\partial \Omega)) \cap L^\infty(I, H^2(\partial \Omega)) \hookrightarrow C^{0,1-\theta}(\tilde{I}, H^{2+\theta}(\partial \Omega)) \rightarrow C^{0,1-\theta}(\tilde{I}, C^{0,2\theta-1}(\partial \Omega)),
\]
(2.15)

**Proposition 2.16.** Let \( \eta \in W^{1,\infty}(I, L^2(\partial \Omega)) \cap L^\infty(I, H^2(\partial \Omega)) \) be given with \( \| \eta \|_{L^1(I \times \partial \Omega)} < \kappa \) and \( \alpha \) a real number such that \( \| \eta \|_{L^1(I \times \partial \Omega)} < \alpha < \kappa \). The application of the extension operators from Proposition 2.10 at (almost) all times defines a bounded, linear extension operator \( \mathcal{F}_\eta \) from
\[
\left\{ b \in H^1(I, L^2(\partial \Omega)) \cap L^2(I, H^2(\partial \Omega)) \mid \int_{\partial \Omega} b(t, \cdot) \gamma(\eta(t, \cdot)) \, dA = 0 \text{ for all } t \in I \right\}
\]

\[
\{ \varphi \in H^1(I, L^2(B_\alpha)) \cap C(\tilde{I}, H^1(B_\alpha)) \mid \text{div} \varphi = 0 \}
\]
as well as a bounded, linear extension operator \( \mathcal{F}_\eta \) from
\[
\left\{ b \in C(\tilde{I}, L^2(\partial \Omega)) \mid \int_{\partial \Omega} b(t, \cdot) \gamma(\eta(t, \cdot)) \, dA = 0 \text{ for almost all } t \in I \right\}
\]
to
\[ \{ \varphi \in C(I, L^2(B_\alpha)) \mid \text{div} \varphi = 0 \}. \]
The continuity constants depend only on \( \Omega \) and a bound for \( \| \eta \|_{W^{1,\infty}(I, L^2(\partial \Omega))} \) and \( \tau(\alpha) \).  

Proof. See [29]. \( \square \)

Remark 2.17. For \( \eta \in C^2(I \times \partial \Omega) \) with \( \| \eta \|_{L^\infty(I \times \partial \Omega)} < \kappa \) an application of \( \mathcal{T}_{\eta(t)} \) for each \( t \in I \) defines isomorphisms between appropriate function spaces on \( I \times \Omega \) and \( \Omega_{\eta(t)} \), respectively, as long as the order of differentiability is not larger than 1.

3. Main result

For the rest of the paper we shall fix some \( 6/5 < p < \infty \). We define
\[ Y^I := W^{1,\infty}(I, L^2(M)) \cap L^\infty(I, H^2(M)), \]
and for \( \eta \in Y^I \) with \( \| \eta \|_{L^\infty(I \times M)} < \kappa \) we set
\[ X^I_{\eta,p} := L^\infty(I, L^2(\Omega_{\eta(t)})) \cap L^p(I, W^{1,p}_{\text{div}}(\Omega_{\eta(t)})). \]

Here and throughout the rest of the paper, we tacitly extend functions defined in \( M \) by 0 to \( \partial \Omega \). Note that, by Proposition 2.13, the space \( X^I_{\eta,p} \) embeds into \( L^r(I, W^{1,r}_{\text{div}}(\Omega_{\eta(t)})) \) for all \( 1 \leq r < p \). We define the space of test functions \( T^I_{\eta,p} \) to consist of all couples
\[ (b, \varphi) \in (H^1(I, L^2(M)) \cap L^p(I, H^2_0(M))) \times (H^1(I, L^2(\Omega_{\eta(t)})) \cap L^p(I, W^{1,p}_{\text{div}}(\Omega_{\eta(t)}))) \]
such that \( b(T, \cdot) = 0 \), \( \varphi(T, \cdot) = 0 \)\(^5\), and \( \varphi - \mathcal{T}_{\eta} b \in H_0^0 \). Here, \( H_0^0 \) denotes the closure in \( H^1(I, L^2(\Omega_{\eta(t)})) \cap L^p(I, W^{1,p}_{\text{div}}(\Omega_{\eta(t)})) \) of the elements of this space that vanish at \( t = T \) and whose supports are contained in \( \Omega^I_{\eta(t)} \). From the last requirement we infer that \( \text{tr}_{\eta} \varphi = \text{tr}_{\eta} \mathcal{T}_{\eta} b = \mathbf{b} \). In particular, \( \varphi \) vanishes on \( \Gamma \). Furthermore, the finite exponent \( \bar{p} \) needs to be larger than \( (5p/6)' \) and not smaller than \( p \), so let us choose \( \bar{p} := \max((5p/6)' + 3, p) \).

We call the data \( (f, g, u_0, \eta_0, \eta_1) \) admissible if \( f \in L^2_{\text{loc}}([0, \infty) \times \mathbb{R}^3), g \in L^2_{\text{loc}}([0, \infty) \times M), \eta_0 \in H^2_0(M) \) with \( \| \eta_0 \|_{L^\infty(M)} < \kappa, \eta_1 \in L^2(M) \), and \( u_0 \in L^2(\Omega_{\eta_0}) \) with \( \text{div} u_0 = 0, \text{tr}_{\eta_0} u_0 = \eta_1 \gamma(\eta_0) \).

Definition 3.1. A couple \( (\eta, u) \) is a weak solution of (1.2), (1.4), and (1.5) for the admissible data \( (f, g, u_0, \eta_0, \eta_1) \) in the interval \( I \) if \( \eta \in Y^I \) with \( \| \eta \|_{L^\infty(I \times M)} < \kappa, \eta(0, \cdot) = \eta_0, u \in X^I_{\eta,p} \) with \( \text{tr}_{\eta} u = \partial_t \eta \mathbf{v} \), and
\[ -\int_I \int_{\Omega_{\eta(t)}} u \cdot \partial_t \varphi \, dx \, dt + \int_I \int_{\Omega_{\eta(t)}} u \otimes u : D\varphi \, dx \, dt + \int_I \int_{\Omega_{\eta(t)}} S(Du) : D\varphi \, dx \, dt - \int_I \int_M \partial_t \eta \partial_t b \, dA \, dt + 2 \int_k K(\eta, b) \, dt = \int_I \int_{\Omega_{\eta(t)}} f \cdot \varphi \, dx \, dt + \int_I \int_M g b \, dA \, dt + \int_{\Omega_{\eta_0}} u_0 \cdot \varphi(0, \cdot) \, dx + \int_M \eta_1 b(0, \cdot) \, dA \]
for all test functions \( (b, \varphi) \in T^I_{\eta,p} \).

\(^5\)We saw in [29] that it makes sense to evaluate \( \varphi \) at a fixed point \( t \) in time and that \( \varphi(t, \cdot) \in L^2(\Omega_{\eta(t)}) \).
Like in [29] the weak formulation (3.2) arises formally by multiplication of (1.2) with a test function \( \varphi \), integration over space and time, integration by parts, and taking into account (1.4). Here, the boundary integrals resulting from integrating by parts the time-derivative of \( u \) and the convective term cancel. By Corollary 2.7 and interpolation (with a weight of \( \theta = 2/5 \) on the bound for the kinetic energy), we have \( u \in L'(\Omega^I_t) \) for all \( 1 \leq r < 10p/6 \). Hence, in view of the assumption on \( p \), the second term in (3.2) is well-defined and finite.

**Theorem 3.3.** For arbitrary admissible data \( (f, g, u_0, \eta_0, \eta_1) \) there exist a time \( T^* \in (0, \infty) \) and a couple \( (\eta, u) \) such that for all \( T < T^* \) \( (\eta, u) \) is a weak solution of (1.2), (1.4), and (1.5) in the interval \( I = (0, T) \). Furthermore, we have

\[
\|\eta\|_{\dot{Y}^2}^2 + \|u\|_{L^2(I;L^2(\Omega^I_0)))}^2 + \|Du\|_{L^p(\Omega^I_0)}^p \leq c e^{T} \left( \|u_0\|_{L^2(\Omega^I_0)}^2 + \|\eta_1\|_{L^2(M)}^2 + \|\eta_0\|_{H^1(M)}^2 + \int_0^T \|f(s,\cdot)\|_{L^2(\Omega^I_0)}^2 + \|g(s,\cdot)\|_{L^2(M)}^2 ds \right). 
\]

Either \( T^* = \infty \) or \( \lim_{t \to T^*} \|\eta(t,\cdot)\|_{L^p(\Omega^I_0)} = \kappa \).

In the following we will denote the right-hand side of (3.4) as a function of \( T, \Omega^I_t, f, g, u_0, \eta_0, \eta_1 \).

### 3.1. Compactness.

Similarly to [29] we can show strong \( L^2 \)-compactness of the shell and the fluid velocities for bounded sequences of weak solutions. However, for the compactness of the shell velocities we need to assume that \( p > 3/2 \). The reason is that we need the shell velocities to be uniformly bounded in a spatial regularity class that embeds compactly into \( L^2(M) \). By taking the trace of the fluid velocities, we obtain the boundedness of the shell velocities in \( L^p(I;W^{1-1/r,r}(M)) \) for all \( 1 \leq r < p \). But \( W^{1-1/r,r}(M) \) embeds compactly into \( L^2(M) \) if and only if \( r > 3/2 \). While the weak formulation (3.2) of our original system is linear in the shell velocity (and compactness of the shell velocities is therefore not needed), this is not the case in our regularized system. On the other hand, since the extra stress tensor of our regularized system will possess a \( p \)-structure for some large \( p \) (partly in order to make the problem accessible to monotone operator theory), in the end, we can deal with arbitrary \( p > 6/5 \).

**Proposition 3.5.** Let \( (f, g, u_0^n, \eta_0^n, \eta_1^n) \) a sequence of admissible data with

\[
\sup_n \left( \tau(\eta_0^n) + \|\eta_0^n\|_{H^1_0(M)} + \|\eta_1^n\|_{L^2(M)} + \|u_0^n\|_{L^2(\Omega^I_0)} \right) < \infty. 
\]

Furthermore, let \( (\eta_n, u_n) \) be a sequence of weak solutions of (1.2), (1.4), and (1.5) for the above data in the interval \( I = (0, T) \) such that

\[
\sup_n \left( \tau(\eta_n) + \|\eta_n\|_{Y'} + \|u_n\|_{Y^p_{\theta,p}} \right) < \infty. 
\]

Then the sequence \( (u_n) \) is relatively compact in \( L^2(I \times \mathbb{R}^3) \).

If \( p > 3/2 \), the sequence \( (\partial_t \eta_n) \) is relatively compact in \( L^2(I \times \mathbb{R}^3) \).

---

\[\text{[6] Here and throughout the rest of the paper, if not explicitly stated otherwise, we (tacitly) extend functions defined in a domain of } \mathbb{R}^3 \text{ by } 0 \text{ to the whole space.}\]
Here, we extend the functions $\nabla u_n$ and $\nabla u$, which a-priori are defined only in $\Omega^b_n$ and $\Omega^b_1$, respectively, by 0 to $I \times \mathbb{R}^3$. Let us deal with the case $p > 3/2$ first. The proof of this case is a rather simple modification of the proof of [29, Proposition 3.8]. Therefore, we only give a sketch. We saw in the proof of [29, Proposition 3.8] that it is enough to show that it is a rather simple modification of the proof of [29, Proposition 3.8]. Therefore, we only give a sketch. We saw in the proof of [29, Proposition 3.8] that it is enough to show that

$$
\eta_n \to \eta \quad \text{weakly}^* \text{ in } L^\infty(I, H^2_0(M)) \text{ and uniformly},
\partial_t \eta_n \to \partial_t \eta \quad \text{weakly}^* \text{ in } L^\infty(I, (H^1_0(M))
\nu u_n \to \nu u \quad \text{weakly}^* \text{ in } L^\infty(I, L^2(\mathbb{R}^3)),
\nu u_n \to \nu u \quad \text{weakly} \text{ in } L^2(I \times \mathbb{R}^3).
$$

Here, we extend the functions $\nabla u_n$ and $\nabla u$, which a-priori are defined only in $\Omega^b_n$ and $\Omega^b_1$, respectively, by 0 to $I \times \mathbb{R}^3$. Let us deal with the case $p > 3/2$ first. The proof of this case is a rather simple modification of the proof of [29, Proposition 3.8]. Therefore, we only give a sketch. We saw in the proof of [29, Proposition 3.8] that it is enough to show that

$$
\int_I \int_{\Omega^b(t)} u_n \cdot \mathcal{F}_\eta \partial_t \eta_n \, dx \, dt + \int_I \int_M |\partial_t \eta_n|^2 \, dA
\to \int_I \int_{\Omega^b(t)} u \cdot \mathcal{F}_\eta \partial_t \eta \, dx \, dt + \int_I \int_M |\partial_t \eta|^2 \, dA,
$$

$$
\int_I \int_{\Omega^b(t)} (u_n - \mathcal{F}_\eta \partial_t \eta_n) \, dx \, dt \to \int_I \int_{\Omega^b(t)} (u - \mathcal{F}_\eta \partial_t \eta) \, dx \, dt.
$$

Here, we assume that the number $\alpha$ in the definition of $\mathcal{F}$, see Proposition [2.16] satisfies the inequality $\sup_{\alpha_i} \|\eta_i\|_{L^{1/(1-\lambda)}(M)} < \alpha < \kappa$. Let us start with the demonstration of (3.9). For $b \in H^2_0(M)$ we employ the special test functions $(\mathcal{M}_b, \mathcal{F}_b, \mathcal{M}_b)$, see Lemma [A.4] for the definition of the operators $\mathcal{M}_b$. From this lemma, Proposition [2.16 Proposition 2.10] and (3.7) we deduce the estimate

$$
\|\mathcal{M}_b\|_{H^1(I, L^2(\mathcal{B}))} + \|\mathcal{F}_b, \mathcal{M}_b\|_{H^1(I, L^2(\mathcal{B}))} \leq c \|b\|_{H^2_0(M)}.
$$

As in the proof of [29, Proposition 3.8] we use equation (3.2) to show that the functions

$$
c_{b_n}(t) := \int_{\Omega^b(t)} u_n(t, \cdot) \cdot (\mathcal{F}_\eta \cdot \mathcal{M}_b)(t, \cdot) \, dx + \int_M \partial_t \eta_n(t, \cdot) (\mathcal{M}_b)(t, \cdot) \, dA
$$

are bounded in $C^{0, \beta}(\overline{I})$ for some $\beta \in (0, 1)$ independently of $\|b\|_{H^2_0(M)} \leq 1$. Here, the convective term has to be estimated in the form

$$
\|\int_{\Omega^b(t)} u_n \otimes \nabla u_n : D\mathcal{F}_\eta \cdot \mathcal{M}_b \, dx\|_{L^{1/(1-\beta)}(I, L^p(\mathcal{B}))} \leq \|u_n\|_{L^p(\mathcal{B})}^2 \|D\mathcal{F}_\eta \cdot \mathcal{M}_b\|_{L^p(\mathcal{B})}.
$$

Note that $2p' < 10p/6$. From this fact and (3.7) we deduce as before by the Arzela-Ascoli argument that the functions

$$
h_n(t) := \sup_{\|b\|_{L^2_0(M)} \leq 1} \left(c_{b_n}(t) - c_b(t)\right),
$$

where $c_b$ is defined as $c_{b_n}$ with $(\eta_n, u_n)$ replaced by $(\eta, u)$, converge to zero in $C(\overline{I})$. By [29, Lemma A.13], for the functions

$$
g_n(t) := \sup_{\|b\|_{L^2_0(M)} \leq 1} \left(c_{b_n}(t) - c_b(t)\right)
$$
and all $3/2 < r < p$ we have
\[ \int g_n(t) \, dt \leq c \epsilon \left( \|u_n\|_{L^p(I; W^{1,1}(\Omega_{\eta(t)}))} + \|u\|_{L^p(I; W^{1,1}(\Omega_{\eta(t)}))} \right) + c(\epsilon) \int h_n(t) \, dt, \]  
(3.11)

proving that $(g_n)$ tends to zero in $L^1(I)$. As in the proof of \cite[Proposition 3.8]{29} we can infer (3.9). Let us proceed with the proof of (3.9). We fix a sufficiently small $\sigma > 0$ and $\delta_\sigma \in C^1(I \times \partial \Omega)$ such that $\|\delta_\sigma - \eta\|_{L^\infty(I \times \partial \Omega)} < \sigma$ and $\delta_\sigma < \eta$ in $I \times \partial \Omega$. For $\sigma \in H(\Omega)$ and $t \in I$ we set
\[ c^\sigma_{\phi,n}(t) := \int_{\Omega_{\eta(t)}} u_n(t, \cdot) \cdot \mathcal{T}_{\delta_\sigma(t)} \phi \, dx, \]
see Remark 2.4 for the definition of $\mathcal{T}_{\delta_\sigma(t)}$, and we define the functions $c^\sigma_\phi$ analogously. From Remark 2.4 and Remark 2.17 we deduce that
\[ \| \mathcal{T}_{\delta_\sigma} \phi \|_{H^1(I; L^2(B_\delta)) \cap C(I; W^{1,p}(B_\delta))} \leq c \| \phi \|_{W^{1,p}_{0,\text{div}}(\Omega)}, \]
As before, using equation (3.2), we infer that the functions
\[ h^\sigma_n(t) := \sup_{\|\phi\|_{W^{1,p}_{0,\text{div}}(\Omega)} \leq 1} (c^\sigma_{\phi,n}(t) - c^\sigma_\phi(t)) \]
are bounded in some Hölder space, independently of $\|b\|_{H^s_0(M)}$. Again by the Arzela-Ascoli argument, we obtain that $(h^\sigma_n)$ tends to zero in $C(I)$, and, by an application of \cite[Lemma A.13]{29}, that the functions
\[ g^\sigma_n(t) := \sup_{\|\phi\|_{H^1(A)} \leq 1} (c^\sigma_{\phi,n}(t) - c^\sigma_\phi(t)) \]
converge to zero in $L^1(I)$. Finally, \cite[Lemma A.16]{29} yields the existence of functions $\Psi_{t,n}$ as in the proof of \cite[Proposition 3.8]{29} satisfying the estimate
\[ \|u_n(t, \cdot) - (\mathcal{T}_{\eta(t)} \partial_\eta(t) - \Psi_{t,n}) \|_{H^1(\mathbb{R}^3)} < \epsilon, \]
for arbitrary, but fixed $s > 0$. On the other hand, by Lemma 2.11 the functions $u_n$ and $u$, extended by 0 to $I \times \mathbb{R}^3$, are uniformly bounded in $L^p(I; H^s(\mathbb{R}^3))$ for sufficiently small $s$. Thus, we can infer (3.9) as in the proof of \cite[Proposition 3.8]{29}.

Now, let us consider the case $6/5 < p \leq 3/2$. In view of (3.8), we have
\[ \limsup_{n \to \infty} \left( \int_{\Omega_{\eta(t)}} |u|^2 \, dx dt - \int_{\Omega_{\eta(t)}} |u_n|^2 \, dx dt \right) \leq 0. \]
Thus, it suffices to show that
\[ \limsup_{n \to \infty} \left( \int_{\Omega_{\eta(t)}} |u_n|^2 \, dx dt - \int_{\Omega_{\eta(t)}} |u|^2 \, dx dt \right) \leq 0. \]  
(3.12)

While we can prove (3.9) exactly as before we are not able to show (3.9). This is due to the fact that the first part of \cite[Lemma A.13]{29} is not applicable anymore. Nevertheless, defining
\[ c_{b,n}(t) := \int_{\Omega_{\eta(t)}} u_n(t, \cdot) \cdot (\mathcal{T}_{\eta(t)} \mathcal{M}_{\eta(t)} b)(t, \cdot) \, dx + \int_M \partial_\eta(t, \cdot) \cdot (\mathcal{M}_{\eta(t)} b)(t, \cdot) \, dA, \]

\[ \text{See Lemma A.14 for the definition of the operators } \mathcal{M}_{\eta(t)}. \]
for \( t \in I \), defining \( c_h \) analogously with \((\eta_n, u_n)\) replaced by \((\eta, u)\), and defining \( h_n \) as in Section 3.10, we can make use of Lemma A.6 to show as before that \((h_n)\) tends to zero in \( C(I) \).

An application of Lemma A.7 yields that for

\[
g_n(t) := \sup_{\|b\|_{L^k(M)} \leq 1} (c_{b,n}(t) - c_b(t))
\]

estimate (3.11) holds for all \( 6/5 < r < p \), thus proving that \((g_n)\) tends to zero in \( L^1(I) \). Of course, we can not proceed as in the case \( p > 3/2 \) by setting \( b = \partial_t \eta_n(t, \cdot) \) since we have no bound of \( \partial_t \eta_n(t, \cdot) \) in \( L^4(M) \). Instead, we replace \( b \) by suitable spatial-high-frequency cut-offs of the shell velocities. To this end, we fix some orthonormal basis of \( N \), and denote by \( \mathcal{P}_k \) the orthogonal projection onto the first \( k \) basis functions. By adding a zero sum, for fixed \( k \in \mathbb{N} \) we obtain the identity

\[
\int \int_{\Omega_{\eta_n}(I)} u_n \cdot \mathcal{F}_{\eta_n} \mathcal{M}_{\eta_n}^\perp \mathcal{P}_k \eta_n \partial_t \eta_n \, dx \, dt + \int_{I} \int_{M} \partial_t \eta_n \mathcal{M}_{\eta_n}^\perp \mathcal{P}_k \partial_t \eta_n \, dAdt
\]

\[
- \int \int_{\Omega_{\eta_n}(I)} u_n \cdot \mathcal{F}_{\eta_n} \mathcal{M}_{\eta_n}^\perp \mathcal{P}_k \partial_t \eta \, dx \, dt + \int_{I} \int_{M} \partial_t \eta_n \mathcal{M}_{\eta_n}^\perp \mathcal{P}_k \partial_t \eta \, dAdt
\]

\[
= \int \int_{\Omega_{\eta_n}(I)} u_n \cdot \mathcal{F}_{\eta_n} \mathcal{M}_{\eta_n}^\perp \mathcal{P}_k \partial_t \eta_n \, dx \, dt + \int_{I} \int_{M} \partial_t \eta_n \mathcal{M}_{\eta_n}^\perp \mathcal{P}_k \partial_t \eta_n \, dAdt
\]

\[
- \int \int_{\Omega_{\eta_n}(I)} u_n \cdot \mathcal{F}_{\eta_n} \mathcal{M}_{\eta_n}^\perp \mathcal{P}_k \partial_t \eta_n \, dx \, dt + \int_{I} \int_{M} \partial_t \eta_n \mathcal{M}_{\eta_n}^\perp \mathcal{P}_k \partial_t \eta_n \, dAdt
\]

\[
+ \int \int_{\Omega_{\eta_n}(I)} u_n \cdot \mathcal{F}_{\eta_n} \mathcal{M}_{\eta_n}^\perp \mathcal{P}_k (\partial_t \eta_n - \partial \eta) \, dx \, dt + \int_{I} \int_{M} \partial_t \eta_n \mathcal{M}_{\eta_n}^\perp \mathcal{P}_k (\partial_t \eta_n - \partial \eta) \, dAdt.
\]

Of course, it’s not a restriction to assume that the basis functions lie in \( L^4(M) \). Thus, by (3.7), for fixed \( k \) the first two lines of the right-hand side of (3.13) are bounded by \( c\|g_n\|_{L^1(I)} \) for some constant \( c > 0 \). Since the sequences \((\mathcal{M}_{\eta_n}^\perp \mathcal{P}_k (\partial_t \eta_n - \partial \eta))_n\) and \((\mathcal{F}_{\eta_n} \mathcal{M}_{\eta_n}^\perp \mathcal{P}_k (\partial_t \eta_n - \partial \eta))_n\) converge to zero weakly in \( L^2(I \times M) \) and \( L^2(I \times B_{\alpha}) \), respectively, for fixed \( k \) the right-hand side of (3.13) vanishes in the limit \( n \to \infty \). Moreover, by adding a zero sum, we obtain

\[
\int \int_{\Omega_{\eta_n}(I)} |u_n|^2 \, dx \, dt + \int_{I} \int_{M} |\mathcal{P}_k \partial_t \eta_n|^2 \, dAdt - \int \int_{\Omega_{\eta_n}(I)} |u|^2 \, dx \, dt - \int_{I} \int_{M} |\mathcal{P}_k \partial_t \eta|^2 \, dAdt
\]

\[
= \int \int_{\Omega_{\eta_n}(I)} u_n \cdot (\mathcal{F}_{\eta_n} \mathcal{M}_{\eta_n}^\perp \mathcal{P}_k \partial_t \eta_n - \partial \eta) \, dx \, dt + \int_{I} \int_{M} \partial_t \eta_n \mathcal{M}_{\eta_n}^\perp \mathcal{P}_k \partial_t \eta_n \, dAdt
\]

\[
- \int \int_{\Omega_{\eta_n}(I)} u_n \cdot (\mathcal{F}_{\eta_n} \mathcal{M}_{\eta_n}^\perp \mathcal{P}_k \partial_t \eta_n - \partial \eta) \, dx \, dt - \int_{I} \int_{M} \partial_t \eta_n \mathcal{M}_{\eta_n}^\perp \mathcal{P}_k \partial_t \eta \, dAdt
\]

\[
+ \int \int_{\Omega_{\eta_n}(I)} u_n \cdot (\partial_t \eta_n - \partial \eta) \, dx \, dt + \int_{I} \int_{M} (\mathcal{P}_k \partial_t \eta - \partial \eta) \, dAdt.
\]

Here, we used the orthogonality of the projections \( M_{\eta_n}^\perp \), \( M_{\eta_n}^\perp \), and \( \mathcal{P}_k \). In view of (3.9) and the convergence of (3.13), for fixed \( k \) the first two lines of the right-hand side of (3.14)
vanish in the limit \( n \to \infty \). Furthermore, by the definition of \( \mathcal{F}_\eta \), see [29], we have

\[
\int_I \int_{\Omega_{\eta(t)}} \mathbf{u} \cdot \mathcal{F}_\eta \mathcal{M}^\perp_{\eta} (\mathcal{P}_k \partial_t \eta - \partial_t \eta) \, dx \, dt = \int_I \int_M \int_{-\alpha}^\eta \exp \left( \int_\eta^\tau \beta(q + \tau \nu) \right) \mathbf{v} \cdot \mathbf{u}(q + s \nu) |\det \Delta| \, ds \, \mathcal{M}^\perp_{\eta} (\mathcal{P}_k \partial_t \eta - \partial_t \eta) \, dA(q) \, dt
\]

\[
= \int_I \int_M \psi_0 \mathcal{F}_\eta \mathcal{M}^\perp_{\eta} (\mathcal{P}_k \partial_t \eta - \partial_t \eta) \, dA \, dt \leq c \| \psi_0 \|_{L^p(I,W^{1,r}(M))} \| \mathcal{M}^\perp_{\eta} (\mathcal{P}_k \partial_t \eta - \partial_t \eta) \|_{L^p(I,W^{1,r}(M)')} \| \mathcal{M}^\perp_{\eta} (\mathcal{P}_k \partial_t \eta - \partial_t \eta) \|_{L^p(I,W^{1,r}(M)')}
\]

for all \( 6/5 \leq r < p \). A simple calculation using Corollary [26] shows that we can bound the \( L^p(I,W^{1,r}(M)) \)-norm of \( \psi_0 \) by the \( L^p(I,W^{1,r}(\Omega_{\eta(t)})) \)-norm of \( \mathbf{u} \). Moreover, we have

\[
\int_I \int_{\Omega_{\eta(t)}} \mathbf{u} \cdot \mathcal{F}_\eta \mathcal{M}^\perp_{\eta} (\mathcal{P}_k \partial_t \eta - \partial_t \eta) \, dx \, dt \leq \| \mathbf{u} \|_{L^\infty(I,L^2(\Omega_{\eta(t)}))} \| \mathcal{F}_\eta \mathcal{M}^\perp_{\eta} (\mathcal{P}_k \partial_t \eta - \partial_t \eta) \|_{L^\infty(I,L^2(\Omega_{\eta(t)})^\perp)}.
\]

Remember that in \( \Omega \setminus \Sigma_{\alpha} \) the extension \( \mathcal{F}_\eta (\mathcal{P}_k \partial_t \eta - \partial_t \eta) \) is given by the solution of the Stokes system with vanishing right-hand side and boundary values on \( \partial (\Omega \setminus \Sigma_{\alpha}) \) given by

\[
\exp \left( \int_{\eta = q}^{\eta = q + \tau \nu} d\tau \right) (\mathcal{M}^\perp_{\eta} (\mathcal{P}_k \partial_t \eta - \partial_t \eta) \mathbf{v}) = \psi_1 \mathcal{M}^\perp_{\eta} (\mathcal{P}_k \partial_t \eta - \partial_t \eta) \mathbf{v}.
\]

By a change of variables and the regularity of \( \psi_1 \), it’s easy to see that the \( (H^{1/2}(\partial (\Omega \setminus \Sigma_{\alpha}))^\perp) \)-norm of this function can be bounded by the \( (H^{1/2}(M))' \)-norm of \( \mathcal{M}^\perp_{\eta} (\mathcal{P}_k \partial_t \eta - \partial_t \eta) \). On the other hand, Theorem 3 in [19] shows that the solution operator of the Stokes system is bounded from the space of functionals \( \mathbf{g} \in (H^{1/2}(\partial (\Omega \setminus \Sigma_{\alpha})))^\perp \) with \( \langle \mathbf{g}, \nu \rangle = 0 \) to \( L^2(\Omega \setminus \Sigma_{\alpha}) \). Combining these estimates we obtain that

\[
\int_I \int_{\Omega_{\eta(t)}} \mathbf{u} \cdot \mathcal{F}_\eta \mathcal{M}^\perp_{\eta} (\mathcal{P}_k \partial_t \eta - \partial_t \eta) \, dx \, dt \leq c \| \mathcal{M}^\perp_{\eta} (\mathcal{P}_k \partial_t \eta - \partial_t \eta) \|_{L^\infty(I,H^{1/3}(M)'\perp)},
\]

and, similarly, we have

\[
\int_I \int_{\Omega_{\eta(t)}} \mathbf{u}_n \cdot \mathcal{F}_{\eta_n} \mathcal{M}^\perp_{\eta_n} (\mathcal{P}_k \partial_t \eta_n - \partial_t \eta_n) \, dx \, dt \leq c \| \mathcal{M}^\perp_{\eta_n} (\mathcal{P}_k \partial_t \eta_n - \partial_t \eta_n) \|_{L^\infty(I,H^{1/3}(M)'\perp)}.
\]

Using Lemma [A.6] Lemma [A.8] and duality, we can make the right-hand sides small by choosing \( k \) large, independently of \( n \). Thus, for each \( \varepsilon > 0 \) we can find some fixed large \( k \) such that the lim sup in \( n \) of the left-hand side of (3.14) is bounded by \( \varepsilon \). This proves (3.12) since for fixed \( k \) we have

\[
\limsup_{n \to \infty} \left( \int_I \int_M |\mathcal{P}_k \partial_t \eta|^2 \, dA \, dt - \int_I \int_M |\mathcal{P}_k \partial_t \eta_n|^2 \, dA \, dt \right) \leq 0.
\]

\[\text{ Here, } \nu \text{ denotes the (outer) unit normal of } \partial (\Omega \setminus \Sigma_{\alpha}).\]

\[\text{At first sight, it might seem awkward that we need spatial regularity of } \mathbf{u} \text{ to control the integral over } \Omega_{\eta(t)} \cap S_{\alpha} \text{ while this is not the case for the integral over } \Omega \setminus \Sigma_{\alpha}. \text{ Obviously, this is due to the fact that the extension operator } \mathcal{F}_\eta \text{ is not optimal in the sense that it produces no spatial regularity in } S_{\alpha}.\]
3.2. **The regularized and decoupled system.** We have to regularize (and decouple) our system. As discussed in [29] it is essential to regularize the motion of the boundary. Furthermore, for technical reasons, we want to avoid to apply the proof of strong $L^2$-compactness to the Galerkin system, i.e., to the finite-dimensional approximations. For this reason, we (slightly) regularize the explicit nonlinearities in the system. Furthermore, since we want to apply monotone operator theory to the regularized system, we have to make sure that a weak solution $(\partial_t \eta, u)$ possesses a (formal) time-derivative in the dual of the energy class. This is achieved by perturbing the extra stress tensor $S$ into an operator $S_{\varepsilon}$ with a $p_0$-structure for $p_0 \geq 11/5$ and by adding the term $\text{grad}_x K(\partial_t \eta)$ to the shell equation, resulting in a “parabolization” of the whole system. Finally, we need the weak solutions of our regularized (and decoupled) system to be unique which is most easy to prove for $p_0 \geq 4$. Thus, we set $S_{\varepsilon}(D) := S(D) + \varepsilon |D|^2 D$ and $p_0 := \max(p, 4)$.

We shall use the regularization operators $\mathcal{R}_\varepsilon$ constructed in [29 Subsection 3.2]. Remember that $\mathcal{R}_\varepsilon \eta_0$ approximates $\eta_0$ uniformly from above. Furthermore, we note that $\text{tr}_{\varepsilon_0} (u_0 - \mathcal{F}_{\varepsilon_0} \eta_1) = 0$. Thus, extending $u_0 - \mathcal{F}_{\varepsilon_0} \eta_1 \in L^2(\Omega_{\varepsilon_0})$ by 0 to $\mathbb{R}^3$ yields a divergence-free vector field in $L^2(\mathbb{R}^3)$ whose support is contained in $\Omega_{\varepsilon_0, \eta_0}$. Let $u^\varepsilon_0$ denote a smooth divergence-free approximation of this field whose support is contained in $\Omega_{\varepsilon_0, \eta_0}$ as well. Moreover, let $\eta^\varepsilon_1$ be a smooth, $C^4$ say, approximation of $\eta_1$ satisfying\(^{13}\)

\[
\int_{\partial \Omega} \eta^\varepsilon_1 \gamma(\mathcal{R}_\varepsilon \eta_0) \, dA = 0,
\]

and let $u^\varepsilon_0 := u^\varepsilon_0 + \mathcal{F}_{\varepsilon} \eta_0, \eta^\varepsilon_1$. Then we have $u^\varepsilon_0 \in C^1(\Omega_{\varepsilon_0, \eta_0}) \cap \Omega_{\varepsilon_0, \eta_0}) \cap \Omega_{\varepsilon_0, \eta_0}$, $\text{div} u^\varepsilon_0 = 0$, $\text{tr}_{\varepsilon_0} u^\varepsilon_0, u^\varepsilon_0 = \eta^\varepsilon_1 \gamma(\mathcal{R}_\varepsilon \eta_0)$. From the definition of the operator $\mathcal{F}$ it is not hard to see that $\chi_{\Omega_{\varepsilon_0, \eta_0}} \mathcal{F}_{\varepsilon_0} \eta_1$ converges to $\chi_{\Omega_{\varepsilon_0, \eta_0}} \mathcal{F}_{\varepsilon_0} \eta_1$ in $L^2(\mathbb{R}^3)$ and thus

\[
\eta^\varepsilon_1 \to \eta_1 \quad \text{in} \quad L^2(\mathbb{R}^3),
\]

\[
\chi_{\Omega_{\varepsilon_0, \eta_0}} u^\varepsilon_0 \to \chi_{\eta_0} u_0 \quad \text{in} \quad L^2(\mathbb{R}^3).
\]

In the following, let $I = (0, T)$, $T > 0$, be a fixed time interval and $\delta \in C(\bar{I} \times \partial \Omega)$ be an arbitrary, but fixed function such that $\|\delta\|_{L^\infty(\bar{I} \times \partial \Omega)} < \kappa$ and $\delta(0, \cdot) = \eta_0$. Let $\varphi$ be a vector field defined in $\Omega \setminus \delta$ and $b$ a function defined in $I \times M$. For $t \in I$ we define

\[
(\mathcal{R}^0_{\varepsilon} \varphi)(t, \cdot) := \mathcal{F}_{\varepsilon} \delta(t), \quad \frac{1}{\varepsilon} \int_{t_1 - \varepsilon}^{t_1} \mathcal{F}^{-1}_{\varepsilon} \delta(t) \varphi(s, \cdot) \, ds,
\]

\[
(\mathcal{R}^1_{\varepsilon} b)(t, \cdot) := \frac{1}{\varepsilon} \int_{t_1 - \varepsilon}^{t_1} \det(d\mathcal{F}_{\varepsilon} \delta(t))^{-1} b(s, \cdot) \, ds
\]

where we extend the integrands by 0 to the whole time axis. We have $\text{tr}_{\varepsilon, \delta} \mathcal{R}^0_{\varepsilon} \varphi = \mathcal{R}^1_{\varepsilon} b$ provided that $\text{tr}_{\varepsilon, \delta} \varphi = b v$. Furthermore, we note that $\mathcal{R}^0_{\varepsilon}$ preserves the divergence-free constraint. Let us now define our decoupled and regularized problem.

**Definition 3.16.** Let $\varepsilon, \tilde{\varepsilon} > 0$. A couple $(\eta, u)$ is called a weak solution of the decoupled and regularized system with datum $\delta$ in the interval $I$ if $\eta \in Y^I \cap H^1(I, H^2_0(M))$ with

\(^{12}\)Note that the classical limit exponent $11/5$ is, in fact, not the limit exponent in our case, due to the weak regularization of the convective term announced above.

\(^{13}\)given, e.g., by convolution with a mollifier kernel

\(^{14}\)given, e.g., by applying a regularization operator similar to $\mathcal{R}_{\varepsilon}$ followed by an application of $\mathcal{R}_{\varepsilon, \eta_0}$
\[ \eta(0, \cdot) = \eta_0, \ u \in X_{\varepsilon, \delta, p_0} \] with \( w_{\varepsilon, \delta} u = \partial_t \eta \cdot v \), and

\[ -\int_{\Omega_{\varepsilon, \delta}(t)} u \cdot \partial_t \varphi \ dx \ dt - \int_{\Omega_{\varepsilon, \delta}(t)} (R_{\varepsilon}^0 u) \otimes u : D\varphi \ dx \ dt \]

\[ -\frac{1}{2} \int_M \partial_t \eta \partial_t \varepsilon b \gamma(\varepsilon \delta) \ dAdt + \frac{1}{2} \int_M (R_{\varepsilon}^1 \partial_t \eta) \partial_t \eta b \gamma(\varepsilon \delta) \ dAdt \]

(3.17)

\[ + \int_M S(\partial_t \varphi) : D\varphi \ dx \ dt - \int_M \partial_t \eta \partial_t b \ dAdt + 2 \int M K(\eta + \varepsilon \partial_t \eta, b) \ dt \]

\[ = \int M \int_t M f \cdot \varphi \ dx \ dt + \int M g b \ dAdt + \int_{\Omega_{\varepsilon, \delta}(0)} \eta^0 \cdot \varphi(0, \cdot) \ dx + \int M \eta^0 b(0, \cdot) \ dA \]

for all test functions \( (b, \varphi) \in T^I_{\varepsilon, \delta} \).

Concerning the space of test functions, we note that \( \tilde{\rho} \geq p_0 \). Thus, the term involving the modified extra stress tensor is well-defined and finite. Furthermore, note that we introduced two regularization parameters, \( \varepsilon \) for the extra stress tensor and \( \tilde{\varepsilon} \) for the rest. The reason is that, if we let first \( \varepsilon \) tend to zero, then the explicit nonlinearity in \( \partial_t \eta \) will vanish. This way, for the second limit \( \varepsilon \searrow 0 \), the restriction \( p > 3/2 \) in Proposition 3.18 is irrelevant.

**Proposition 3.18.** Let \( \varepsilon, \tilde{\varepsilon} > 0 \). There exists a unique weak solution \((\eta, u)\) of the decoupled and regularized system with datum \( \delta \) in the interval \( I \) which satisfies the estimate

\[ \|\eta\|_{L^2(I,H_0^2(\Omega_{\varepsilon, \delta}(I)))} + \|u\|_{L^2(I,L^2(\Omega_{\varepsilon, \delta}(I)))} + \|D\varphi\|_{L^p(\Omega') \leq c_1(T, \Omega(\varepsilon, \delta), f, g, u_0, \eta_0, \eta_0')}. \]

(3.19)

In particular, the left-hand side is bounded independently of \( \varepsilon, \tilde{\varepsilon}, \) and \( \delta \). Furthermore, for some constant \( c > 0 \), we have

\[ \varepsilon \|\partial_t \eta\|_{L^2(I,H_0^2(M))} + \tilde{\varepsilon} \|D\varphi\|_{L^p(\Omega')} \leq c. \]

(3.20)

For the sake of a better readability, for the moment, we will suppress the parameters \( \varepsilon, \tilde{\varepsilon} \) in the notation. In particular, \( u_0 \) and \( \eta_1 \) denote the regularized initial values \( u_0^\varepsilon \) and \( \eta_1^\varepsilon \), respectively, and \( S \) denotes \( S_{\varepsilon} \). For the proof of this proposition we will need the following lemma. Let

\[ E_{\eta, u}(t) = \frac{1}{2} \int_{\Omega_{\varepsilon, \delta}(t)} |u(t, \cdot)|^2 \ dx + \frac{1}{2} \int_M |\partial_t \eta(t, \cdot)|^2 \ dA + K(\eta(t, \cdot)). \]

Note that, a-priori, this function is only defined almost everywhere.

**Lemma 3.21.** Let \((\eta, u)\) be a weak solution of the decoupled and regularized system with datum \( \delta \) in the interval \( I \) where the field \( S(D\varphi) \) in (3.17) may be replaced by an arbitrary field \( \xi \in L^p(\Omega_{\varepsilon, \delta}) \). Then, \( \partial_t \eta \in C(I, L^2(M)) \) with \( \partial_t \eta(0, \cdot) = \eta_1, \ u \in C(I, L^2(\mathbb{R}^3)) \) with \( u(0, \cdot) = u_0 \), and for all \( t \in I \) we have the energy identity

\[ E_{\eta, u}(t) - E_{\eta, u}(0) = -\int_0^t \int_{\Omega_{\varepsilon, \delta}(r)} \xi : Du \ dx \ ds - 2 \int_0^t K(\partial_t \eta) \ ds \]

(3.22)

\[ + \int_0^t \int_{\Omega_{\varepsilon, \delta}(r)} f \cdot u \ dx \ ds + \int_0^t \int_M g \partial_t \eta \ dAdt. \]

**Proof.** Let

\[ V := \{ (b, \varphi) \in L^2(I,H_0^2(M)) \times L^p(I, W^{1,p}_0(\Omega_{\varepsilon, \delta}(I))) | w_{\varepsilon, \delta} \varphi = b \cdot v \}. \]

Using Remark 2.17 and the fact that \( D\delta \) is smooth, we see that classical Korn’s inequality holds uniformly.
and define $u_k := \mathcal{R}_L \psi u + u_{0,k}, \partial_t \eta_k := \mathcal{R}_L \psi \partial_t \eta + \eta_{1,k}$, and $\eta_k(t, \cdot) := \eta_0 + \int_0^t \partial_t \eta_k(s, \cdot) \, ds$

where

$$u_{0,k}(t, \cdot) := (1-kt) \chi(0,1/k)(t) \mathcal{F}_{\delta \mathcal{R}_L}(t) \mathcal{F}_{\delta \mathcal{R}_L}^{-1} u_0,$$

$$\eta_{1,k}(t, \cdot) := \text{tr}_{\delta \mathcal{R}_L}(t) u_0^1(t, \cdot) = (1-kt) \chi(0,1/k)(t) \frac{\det(d\Psi_{\delta \mathcal{R}_L})}{\det(d\Psi_{\delta \mathcal{R}_L}(t))} \eta_1.$$

Remember that $u_0$ and $\eta_1$ are smooth and note that $\text{tr}_{\delta \mathcal{R}_L}(u_k) = \partial_t \eta_k \nu$. We claim that $\eta_k \in Y^I \cap H^2(I, H^2(M)), \partial_t \eta_k \nu \in W^{1,p_0}(\Omega, \mathcal{R}_L(\Omega, \mathcal{R}_L))$, that

$$(\partial_t \eta_k, u_k) \to (\partial_t \eta, u) \quad \text{in } V$$

for $k \to \infty$, and that the functionals

$$\frac{d}{dt}(\partial_t \eta_k, u_k)(b, \varphi) := \int_I \int_{\Omega \setminus \delta(t)} \partial_t u_k \cdot \varphi \, dx \, dt + \frac{1}{2} \int_M \partial_t \eta_k \partial_t \mathcal{R}_L \nu(t, \cdot) \, dA(t)$$

$$+ \int_I \int_{\Omega \setminus \delta(t)} \partial_t^2 \eta_k b \, dA(t) \quad (3.23)$$

are bounded in $V'$. Except for the inclusion $(\partial_t \eta_k) \subset L^{p_0}(\Omega^I)$ and the boundedness of the functionals, these claims are obvious if we remember that $\mathcal{F}_{\delta \mathcal{R}_L}$ is an isomorphism between the involved function spaces on $\Omega^I$ and the corresponding function spaces on $I \times \Omega$, see Remark 2.17. Before proving the remaining assertions, let us draw the relevant conclusions. We can proceed as in [29] Remark 1.17 to show that the extension of $u_k$ by $(\partial_t \eta_k \nu) \circ q$ lies in $C(\bar{I}, L^{p_0}(B_\alpha))$ for $\|\mathcal{R}_L\|_{L^\infty(I \times M)} < \alpha < \kappa$. Thus, the extension of $u_k$ by $0$ lies in $C(\bar{I}, L^2(\mathbb{R}^3))$. For all $s, t \in [0, T]$ we have

$$\left\langle \frac{d}{dt}(\partial_t \eta_k, u_k)(b, \varphi) \right\rangle = \frac{1}{2} \int_{\Omega \setminus \delta(t)} |u_k(t, \cdot)|^2 \, dx + \frac{1}{2} \int_M |\partial_t \eta_k(t, \cdot)|^2 \, dA$$

$$- \frac{1}{2} \int_{\Omega \setminus \delta(t)} |u_k(s, \cdot)|^2 \, dx + \frac{1}{2} \int_M |\partial_t \eta_k(s, \cdot)|^2 \, dA \quad (3.24).$$

Replacing $\eta_k$ by $\eta_k - \eta_i$ and $u_k$ by $u_k - u_i$ and integrating the resulting identity over $I$ with respect to $s$, we obtain

$$\int_{\Omega \setminus \delta(t)} (|u_k - u_i|(t, \cdot)|^2 \, dx + \int_M |\partial_t (\eta_k - \eta_i)(t, \cdot)|^2 \, dA$$

$$\leq c \left( \left\| \frac{d}{dt}(\partial_t (\eta_k - \eta_i), (u_k - u_i))(b, \varphi) \right\|_{V'} \right) \left( \|\partial_t (\eta_k - \eta_i)(b, \varphi)\| + \|u_k - u_i\|_{L^2(\Omega^I)}^2 \right.$$

$$+ \|\partial_t \eta_k - \partial_t \eta_i\|_{L^2(I \times M)}^2 \right).$$

Extending the functions $u_k$ and $u$ by $0$ to $I \times \mathbb{R}^3$, we deduce from this estimate and the properties of the approximations that the sequences $(u_k)$ and $(\partial_t \eta_k)$ converge to $u$ in $C(\bar{I}, L^2(\mathbb{R}^3))$ and to $(\partial_t \eta)$ in $C(\bar{I}, L^2(M))$, respectively. By an argument analogous to the one given in [29] Remark 3.1, using the $L^2$-continuity of $\partial_t \eta$ and $u$, we can show that (3.17) holds with $u_0 (0, \cdot)$ and $\partial_t \eta (0, \cdot)$ in place of $u_0$ and $\eta_1$, respectively, proving that $\partial_t \eta (0, \cdot) = \eta_1, u(0, \cdot) = u_0$.

Choosing $s = 0$ in (3.24), the right-hand side converges to

$$\frac{1}{2} \int_{\Omega \setminus \delta(t)} |u(t, \cdot)|^2 \, dx + \frac{1}{2} \int_M |\partial_t \eta(t, \cdot)|^2 \, dA$$

$$- \frac{1}{2} \int_{\Omega \setminus \delta(0)} |u_0|^2 \, dx - \frac{1}{2} \int_M |\eta_1|^2 \, dA.$$
By the uniform boundedness of \( (d/dt(\partial_t \eta_k, u_k)) \) in \( V' \), the left-hand side converges to 
\( \langle \Sigma, (\partial_t \eta, u) \rangle_{\chi(0,t)} \) where \( \Sigma \in V' \) is given by

\[
\langle \Sigma, (b, \varphi) \rangle_{V'} = \int_I \int_{\Omega_{\delta(t)}} (\mathcal{R}^0 b) \otimes u : D\varphi \, dx \, dt - \frac{1}{2} \int_I \int_M (\mathcal{R}^1 \partial_t \eta) \partial_t \eta \, b \, \gamma(\mathcal{R} \delta) \, d\text{Adt} \\
- \int_I \int_{\Omega_{\delta(t)}} \chi : D\varphi \, dx \, dt - 2 \int_I K(\eta + \partial_t \eta, b) \, dt + \int_I \int_{\Omega_{\delta(t)}} \mathbf{f} \cdot \varphi \, dx \, dt \\
+ \int_I \int_M g \, b \, d\text{Adt}
\]  
(3.25)

for \( (b, \varphi) \in V \). This can be seen as follows. For \( (b, \varphi) \in \mathcal{T}_{\mathcal{R} \delta, p} \) with \( b(0, \cdot) = 0, \varphi(0, \cdot) = 0 \), an application of Reynold’s transport theorem shows that

\[
\langle \frac{d}{dt}(\partial_t \eta_k, u_k), (b, \varphi) \rangle = -\int_I \int_{\Omega_{\delta(t)}} u_k \cdot \partial_t \varphi \, dx \, dt - \frac{1}{2} \int_I \int_M \partial_t \eta_k \partial_t \mathcal{R} \delta \, b \, \gamma(\mathcal{R} \delta) \, d\text{Adt} \\
- \int_I \int_M \partial_t \eta_k \partial_t b \, d\text{Adt}.
\]

Now, if we let \( k \to \infty \), use (3.17), and note that these test functions are dense in \( V \)\(^{16}\) we obtain (3.25). Noting that

\[
(\mathcal{R}^0 b) \otimes u : D\varphi = (\mathcal{R}^0 u) \cdot \nabla \left| \frac{u}{2} \right|
\]

we see that for \( (b, \varphi) = (\partial_t \eta, u) \chi(0,t) \) the first two terms on the right-hand side of (3.25) cancel. Hence, (3.22) follows.

Now, let us prove the inclusion \( (\partial_t u_k) \subset L^0(\Omega_{\mathcal{R} \delta}, \partial_t) \) and the boundedness of the functionals (3.23). Note that

\[
\partial_t((\mathcal{R}^0_{1/k} u_k)(t, \cdot)) = \left( \frac{d}{dt} \left( \frac{T^{-1}_{\mathcal{R} \delta(t)}}{\text{det} d\Psi_{\mathcal{R} \delta(t)}} k^t_{s} T^{-1}_{\mathcal{R} \delta(s)} u(s, \cdot) \, ds \right) \right) \circ \Psi^{-1}_{\mathcal{R} \delta(t)} \partial_t \Psi^{-1}_{\mathcal{R} \delta(t)} \\
+ \left( \frac{d}{dt} \left( \frac{T^{-1}_{\mathcal{R} \delta(t)}}{\text{det} d\Psi_{\mathcal{R} \delta(t)}} k^t_{s} T^{-1}_{\mathcal{R} \delta(s)} u(s, \cdot) \, ds \right) \right) \circ \Psi^{-1}_{\mathcal{R} \delta(t)}.
\]

(3.26)

While the second term on the right-hand side obviously lies in \( L^0(\Omega_{\mathcal{R} \delta}, \partial_t) \), the same is true for the first term because the spatial derivatives of \( u \) lie in \( L^\infty(\Omega_{\mathcal{R} \delta}) \). We can similarly show that \( \partial_t u_{0,k} \in L^0(\Omega_{\mathcal{R} \delta}, \partial_t) \). Let us proceed with the boundedness of the functionals (3.23). For \( (b, \varphi) \in \mathcal{T}_{\mathcal{R} \delta, p} \), we have

\[
\langle \frac{d}{dt}(\partial_t \eta_k, u_k), (b, \varphi) \rangle = \int_I \int_{\Omega_{\delta(t)}} \partial_t u_{0,k} \cdot \varphi \, dx \, dt + \int_I \int_M \partial_t^2 \eta_{1,k} \, b \, d\text{Adt} \\
- \int_I \int_{\Omega_{\delta(t)}} \mathcal{R}^0 u \cdot \partial_t \varphi \, dx \, dt - \frac{1}{2} \int_I \int_M \partial_t \eta_k \partial_t \mathcal{R} \delta \, b \, \gamma(\mathcal{R} \delta) \, d\text{Adt} \\
- \int_I \int_M (\mathcal{R}^1_{1/k} \partial_t \eta_k) \partial_t b \, d\text{Adt}.
\]

(3.27)

In order to deal with the third and the fifth term on the right hand side, let us denote the \( L^2(\Omega_{\mathcal{R} \delta}) \)-adjoints of \( \mathcal{R}^0_{1/k} \) and \( \mathcal{R}^1_{1/k} \) by \( (\mathcal{R}^0_{1/k})' \) and \( (\mathcal{R}^1_{1/k})' \), respectively. We have

\[\text{In order to prove the denseness, we can proceed analogously to the proof of denseness employed in \[20\] just before the proof of Proposition 3.15.}\]
\((\mathcal{D}^i_{1/k})' = \mathcal{D}^{1/k}_{-1/k}\) and
\[
((\mathcal{D}^0_{1/k})' \varphi)(t, \cdot) = (\mathcal{D}^{-1}_{\mathcal{G}(t)})' k \int_t^{t+\frac{1}{k}} (\mathcal{D}^0_{\mathcal{G}(s)})' \varphi(s, \cdot) \, ds
\]
where we extend the integrands by 0 to the whole time axis and
\[
(\mathcal{D}^0_{\mathcal{G}(t)})' \varphi(t, \cdot) = (d\Psi_{\mathcal{G}(t)})' \varphi(t, \cdot) \circ \Psi_{\mathcal{G}(t)}^{-1},
\]
\[
(\mathcal{D}^{-1}_{\mathcal{G}(t)})' \varphi(t, \cdot) = (d\Psi_{\mathcal{G}(t)})' T \varphi(t, \cdot) \circ \Psi_{\mathcal{G}(t)}^{-1}.
\]
We compute
\[
- \int_I \int_{\Omega_{\mathcal{G}(t)}} \mathcal{R}^0_{1/k} \mathbf{u} \cdot \partial_t \varphi \, dA dt = - \int_I \int_{\Omega_{\mathcal{G}(t)}} \mathbf{u} \cdot (\mathcal{D}^0_{1/k})' \partial_t \varphi \, dA dt
\]
\[
= - \int_I \int_{\Omega_{\mathcal{G}(t)}} \mathbf{u} \cdot (\mathcal{D}^0_{1/k})' \partial_t \varphi \, dA dt + \int_I \int_{\Omega_{\mathcal{G}(t)}} \mathbf{u} \cdot [\partial_t, (\mathcal{D}^0_{1/k})'] \varphi \, dA dt.
\]
Here, the commutator
\[
([\partial_t, (\mathcal{D}^0_{1/k})'] \varphi)(t, \cdot) = [\partial_t, (\mathcal{D}^{-1}_{\mathcal{G}(t)})'] k \int_t^{t+\frac{1}{k}} (\mathcal{D}^0_{\mathcal{G}(s)})' \varphi(s, \cdot) \, ds
\]
\[
+ (\mathcal{D}^{-1}_{\mathcal{G}(t)})' k \int_t^{t+\frac{1}{k}} [\partial_t, (\mathcal{D}^{1/k}_{\mathcal{G}(s)})'] \varphi(s, \cdot) \, ds.
\]
is acting derivatively only on the spatial variable of \(\varphi\), cf. (3.26). Thus, the \(L^{p_0}(\Omega_{\mathcal{G}(t)})\)-norm of \([\partial_t, (\mathcal{D}^0_{1/k})'] \varphi\) is bounded by the \(L^{p_0}(I, W^{1,p_0}(\Omega_{\mathcal{G}(t)}))\)-norm of \(\varphi\). Analogously, we have
\[
- \int_I \int_{M} (\mathcal{D}^1_{1/k} \partial_t \eta) b \, dA dt = - \int_I \int_{M} \partial_t \eta (\mathcal{D}^1_{1/k})' b \, dA dt
\]
\[
= - \int_I \int_{M} \partial_t \eta \partial_t ((\mathcal{D}^1_{1/k})' b) \, dA dt + \int_I \int_{M} \partial_t \eta \partial_t (\mathcal{D}^1_{1/k})' b \, dA dt,
\]
and, here, the \(L^2(I \times M)\)-norm of \([\partial_t, (\mathcal{D}^1_{1/k})'] b\) is even bounded by the \(L^2(I \times M)\)-norm of \(b\). Unfortunately, in general, \((\mathcal{D}^1_{1/k})' \varphi, (\mathcal{D}^1_{1/k})' b \notin T^I_{\Omega_{\mathcal{G}(t)}}\) since the adjoint operator \((\mathcal{D}^0_{1/k})'\) preserves neither the divergence-free constraint nor the structure of the boundary values.

We can overcome this problem by replacing \((\mathcal{D}^0_{1/k})' \varphi, (\mathcal{D}^1_{1/k})' b\) by \((\mathcal{D}^0_{1/k} \varphi, (\mathcal{D}^{1/k}_{-1/k})' b) \in T^I_{\Omega_{\mathcal{G}(t)}}\). In order to do so, remembering that \((\mathcal{D}^1_{1/k})' = \mathcal{D}^{1/k}_{-1/k}\), we need to bound the functionals
\[
(b, \varphi) \mapsto \int_I \int_{\Omega_{\mathcal{G}(t)}} \mathbf{u} \cdot \partial_t ((\mathcal{D}^0_{1/k})' - (\mathcal{D}^0_{-1/k})') \varphi \, dA dt
\]
in \(V'\). As we saw above the commutators \([\partial_t, (\mathcal{D}^0_{1/k})]\) and \([\partial_t, (\mathcal{D}^{-1}_{\mathcal{G}(t)})']\) are acting derivatively only on the spatial variable so that the corresponding terms in (3.29) can be estimated by a constant multiple of the \(L^{p_0}(I, W^{1,p_0}(\Omega_{\mathcal{G}(t)}))\)-norm of \(\varphi\). Hence, we need to deal with the terms resulting from the time-derivative acting on the Steklov means. These terms evaluated at \(t \in I\) give
\[
k (\mathcal{D}^{-1}_{\mathcal{G}(t)})' (\mathcal{D}^{1/2}_{\mathcal{G}(t+\frac{1}{k})})' \varphi(t + 1/k, \cdot) - (\mathcal{D}^{1/2}_{\mathcal{G}(t)})' (\mathcal{D}^{-1}_{\mathcal{G}(t)})' \varphi(t, \cdot)
\]
\[
- k (\mathcal{D}^{1/2}_{\mathcal{G}(t+\frac{1}{k})})' (\mathcal{D}^{-1}_{\mathcal{G}(t+\frac{1}{k})})' \varphi(t + 1/k, \cdot) - (\mathcal{D}^{1/2}_{\mathcal{G}(t+\frac{1}{k})})' (\mathcal{D}^{-1}_{\mathcal{G}(t)})' \varphi(t, \cdot)
\]
\[
= k (\mathcal{D}^{-1}_{\mathcal{G}(t)}) (\mathcal{D}^{1/2}_{\mathcal{G}(t+\frac{1}{k})})' \varphi(t + 1/k, \cdot).
\]
But from the smoothness of $\mathcal{B}$ we can deduce that
\[
\int \int_{\Omega_{\mathcal{B}(t)}} \left| \left( \mathcal{F}_{\mathcal{B}(t)}^{-1}(t+\frac{1}{2}) - \left( \mathcal{F}_{\mathcal{B}(t)}^{-1}(t+\frac{1}{2}) \right)' \right) \varphi(t+1/k, \cdot) \right|^{p_0} \, dx dt \\
= \int \int_{\Omega} |d\Psi_{\mathcal{B}(t)}(\text{det} d\Psi_{\mathcal{B}(t)})(\text{det} d\Psi_{\mathcal{B}(t+\frac{1}{2})}^{-1}) \circ \Psi_{\mathcal{B}(t+\frac{1}{2})} - (d\Psi_{\mathcal{B}(t)}^{-1}(t+\frac{1}{2}))^T \circ \Psi_{\mathcal{B}(t)} d\Psi_{\mathcal{B}(t+\frac{1}{2})}^{1/p_0} \, | \varphi(t+1/k, \cdot) \circ \Psi_{\mathcal{B}(t+\frac{1}{2})} |^{p_0} \, dx dt \\
\leq \frac{c}{k} \int_{t} \int_{\Omega} |\varphi(t+1/k, \cdot) \circ \Psi_{\mathcal{B}(t+\frac{1}{2})}|^{p_0} \, dx dt \leq \frac{c}{k}
\]
proving the boundedness of $(3.29)$ in $V'$. In view of $(3.27)$ and from what we showed so far it remains to bound the functionals
\[
(b, \varphi) \mapsto \int_{t} \int_{\Omega_{\mathcal{B}(t)}} \partial_{1} u_{0, k} \cdot \varphi \, dx dt + \int_{t} \int_{M} \partial_{1} \eta_{1, k} b \, dA dt - \int_{t} \int_{\Omega_{\mathcal{B}(t)}} u \cdot \partial_{1} (\mathcal{A}_{0,1/k} \varphi) \, dx dt \\
- \frac{1}{2} \int_{t} \int_{M} \partial_{1} \eta \partial_{1} \mathcal{A}_{0,1/k} \varphi(\mathcal{A}_{0,1/k} \varphi) \, dA dt - \int_{t} \int_{M} \partial_{1} \eta \partial_{1} (\mathcal{A}_{0,1/k} \varphi) \, dA dt \quad (3.31)
\]
in $V'$. Using $(3.17)$ we can replace the last three terms in $(3.31)$ by
\[
\int_{t} \int_{\Omega_{\mathcal{B}(t)}} u_{0} \cdot (\mathcal{A}_{0,1/k} \varphi)(0, \cdot) \, dx + \int_{t} \int_{M} \eta_{1} (\mathcal{A}_{0,1/k} \varphi)(0, \cdot) \, dA
\]
since the remaining terms in $(3.17)$ can be bounded by a constant multiple of $\| (b, \varphi) \|_{V}$, reflecting the fact that the formal time-derivative of $(\partial_{1} \eta, u)$ lies in $V'$. Here, the convective term is the crucial one. By interpolation, we have
\[
\int_{t} \int_{\Omega_{\mathcal{B}(t)}} (\mathcal{A}_{0,1/k} \varphi) \cdot u : D(\mathcal{A}_{0,1/k} \varphi) \, dx dt \\
\leq \| \mathcal{A}_{0,1/k} \varphi \|_{L^{1/3}(\Omega_{\mathcal{B}(t)})} \| u \|_{L^{11/3}(\Omega_{\mathcal{B}(t)})} \| D(\mathcal{A}_{0,1/k} \varphi) \|_{L^{11/3}(\Omega_{\mathcal{B}(t)})} \\
\leq c \| u \|_{L^{11/5}(\Omega_{\mathcal{B}(t)})} \| u \|_{H^{11/5}(\Omega_{\mathcal{B}(t)})} \| \varphi \|_{L^{11/5}(\Omega_{\mathcal{B}(t)})} \| D(\mathcal{A}_{0,1/k} \varphi) \|_{L^{11/5}(\Omega_{\mathcal{B}(t)})}.
\]
Thus, the lemma is proved if we can bound the functionals
\[
(b, \varphi) \mapsto \int_{t} \int_{\Omega_{\mathcal{B}(t)}} \partial_{1} u_{0, k} \cdot \varphi \, dx dt + \int_{t} \int_{M} \partial_{1} \eta_{1, k} b \, dA dt \\
+ \int_{t} \int_{\Omega_{\mathcal{B}(t)}} u_{0} \cdot (\mathcal{A}_{0,1/k} \varphi)(0, \cdot) \, dx + \int_{t} \int_{M} \eta_{1} (\mathcal{A}_{0,1/k} \varphi)(0, \cdot) \, dA \quad (3.32)
\]
in $V'$. A simple computation shows that the sum of the first and the third term equals
\[
k \int_{t}^{x} \int_{\Omega_{\mathcal{B}(t)}} \left( (\mathcal{F}_{\mathcal{B}(t)}^{-1}(t) - \mathcal{F}_{\mathcal{B}(t)}^{-1}(t)) \mathcal{A}_{0,1/k} \varphi \right) \cdot \varphi \, dx dt \\
+ \int_{t}^{x} \int_{\Omega_{\mathcal{B}(t)}} \left( (1 - k t) \partial_{1} (\mathcal{F}_{\mathcal{B}(t)}^{-1}(t)) \mathcal{A}_{0,1/k} \varphi \right) \cdot \varphi \, dx dt.
\]
This expression can be bounded by a constant multiple of the $L^{p_0}(\Omega_{\mathcal{B}(t)})$-norm of $\varphi$ analogously to $(3.30)$ and $(3.26)$. The sum of the second and the fourth term in $(3.32)$ can be handled similarly. This completes the proof.
Proof. (of Proposition 3.18) We use the Galerkin method. We proceed exactly as in [29] for the construction of time-dependent basis functions \((W_k)\) and \(W_k\) such that

\[
\text{span}\{\{\varphi W_k, \varphi W_k\} | \varphi \in C^1_0([0, T]), k \in \mathbb{N}\}
\]

is dense in \(T^I_{\mathbb{R}^d, \text{pyc}}\). We seek functions \(\alpha_k^n : [0, T] \rightarrow \mathbb{R}, k, n \in \mathbb{N}\), such that \(u_n := \alpha_k^n W_k\) and \(\eta_n(t, \cdot) := \int_0^t \alpha_k^n W_k \, ds + \eta_0\) (summation with respect to \(k\) from 1 to \(n\)) solve the equation

\[
\begin{aligned}
\int_{\Omega_{\mathbb{R}^d(t)}} \partial_t u_n \cdot W_j \, dx - \int_{\Omega_{\mathbb{R}^d(t)}} (\mathcal{R} \vartheta u_n) \otimes u_n : D W_j \, dx + \frac{1}{2} \int_M \partial_t \eta_n \partial \mathcal{R} b \gamma(\mathcal{R} \delta) \, dA \\
+ \frac{1}{2} \int_M \mathcal{R} (\partial_\eta \eta_n) \partial \eta_n W_j \gamma(\mathcal{R} \delta) \, dA + \int_{\Omega_{\mathbb{R}^d(t)}} S(D u_n) : D W_j \, dx + \int_M \partial^2_t \eta_n W_j \, dA \\
+ 2K(\eta_n + \partial_\eta \eta_n, W_j) = \int_{\Omega_{\mathbb{R}^d(t)}} f_n \cdot W_j \, dx + \int_M g_n W_j \, dA \quad (3.33)
\end{aligned}
\]

for all \(1 \leq j \leq n\). Here, \(f_n\) and \(g_n\) are smooth functions which converge to \(f\) and \(g\) in \(L^2_{\text{loc}}([0, T] \times \mathbb{R}^3)\) and \(L^2_{\text{loc}}([0, T] \times M)\), respectively. As in [29], we construct initial conditions \(\alpha_0^n(0)\) such that

\[
\begin{aligned}
\partial_\eta \eta_n(0, \cdot) &\rightarrow \eta_1 \quad \text{in } L^2(M), \\
u_n(0, \cdot) &\rightarrow u_0 \quad \text{in } L^2(\Omega_{\mathbb{R}^d(0)}).
\end{aligned}
\]

With these initial conditions, (3.33) is a Cauchy problem for a system of ordinary integro-differential equations of the form (1 \(\leq j \leq n\), summation with respect to \(k\) from 1 to \(n\))

\[
A_{jk}(t) \dot{\alpha}^k(t) = B_j(t, \alpha(t)) + \int_0^t C_j(\alpha(t), \alpha(s), t, s) \, ds + D_j(t).
\]

Here, the functions \(A_{jk}, D_j : [0, T] \rightarrow \mathbb{R}\) and \(B_j : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}\), given by

\[
\begin{aligned}
A_{jk}(t) &= \int_{\Omega_{\mathbb{R}^d(t)}} W_k \cdot W_j \, dx + \int_M W_k W_j \, dA, \\
B_j(t, \alpha(t)) &= \left( -\int_{\Omega_{\mathbb{R}^d(t)}} \partial_t W_k \cdot W_j \, dx - \int_M \partial_t W_k W_j \, dA - 2K(W_k, W_j) \\
&\quad - \frac{1}{2} \int_M W_k W_j \gamma(\mathcal{R} \delta) \, dA \right) \alpha^k(t) - \int_{\Omega_{\mathbb{R}^d(t)}} S(\alpha^k(t) D W_k) : D W_j \, dx,
\end{aligned}
\]

are continuous, while the functions \(C_j : \mathbb{R}^d \times \mathbb{R}^d \times [0, T] \times [0, T] \rightarrow \mathbb{R}\), given by

\[
C_j(\alpha(t), \alpha(s), t, s) = -2K(W_k(s), W_j(t)) \alpha^k(s) + \frac{1}{\varepsilon} \chi_{(t-\varepsilon,t)}(s) E_{jk}(t) \alpha^k(s) \alpha^l(t)
\]

with

\[
E_{jk}(t) = \int_{\Omega_{\mathbb{R}^d(t)}} W_j \otimes W_k : D W_j \, dx - \frac{1}{2} \int_M W_j W_k W_j \gamma(\mathcal{R} \delta) \, dA,
\]

are measurable and bounded on compact subsets of their domain. Furthermore, we saw in [29] that the matrices \(A(t)\) are invertible. Now, one can easily adapt the proof of Peano’s existence theorem to show that there exists a unique, local \(C^1\)-solution \(\alpha\) which exists as long as \(|\alpha(t)|\) stays bounded, cf. [28] Appendix A.3. Let us now test (3.33) with
We saw in the proof of Lemma \[3.2\] that the second and the fourth term on the left-hand side cancel, while the first and the third term yield

\[
\frac{d}{dt} \frac{1}{2} \int_{\Omega_{\delta}(t)} |u_n|^2 \, dx.
\]

Thus, we can proceed as in Subsection 1.3 to obtain

\[
\|\eta_n\|_{L^2(I,L^2(\Omega_{\delta}(t)))}^2 + \|u_n\|_{L^p(\Omega_{\delta}(t))}^p + \|D\eta_n\|_{L^p(\Omega_{\delta}(t))}^p \leq c_0(T, \Omega_{\delta}, f_n, g_n, u_n(0, \cdot), \eta_0, \partial_t \eta_n(0, \cdot)),
\]

as well as

\[
\epsilon \|\partial_t \eta_n\|_{L^2(I,H^2_0(M))} + \tilde{c} \|D\eta_n\|_{L^p(\Omega_{\delta}(t))} \leq c
\]

for some constant \(c > 0\). In particular, the solutions exist on the whole time interval \([0, T]\).

From these bounds and \(3.26\) we deduce that

\[
\|S(D\eta_n)\|_{L^p(I_H^0(\Omega_{\delta}(t)))} + \|\partial_t \eta_n\|_{H^1(I; L^2_0(M))} + \|\partial_t^2 \eta_n\|_{H^1(I; L^2_0(M))} \leq c' \tag{3.34}
\]

for another constant \(c' > 0\). Hence, for a subsequence (again denoted by the index \(n\)) we have

\[
\eta_n \to \eta \quad \text{weakly in } H^1(I, H^2_0(M)),
\]

\[
\partial_t \eta_n \to \partial_t \eta \quad \text{weakly}^* \text{ in } L^\infty(I, L^2(M)),
\]

\[
\partial_t^2 \eta_n \to \partial_t^2 \eta \quad \text{in } L^2(I \times M),
\]

\[
u_n \to \nu \quad \text{weakly}^* \text{ in } X_{\delta,p}^I,
\]

\[
u_n \to \nu \quad \text{in } L^p_0(\Omega_{\delta}(t)),
\]

\[
S(D\eta_n) \to \xi \quad \text{weakly in } L^p_0(\Omega_{\delta}(t)).
\]

The above convergences and \(\text{tr}_{\delta} u_n = \partial_t \eta_n \nu\) imply the identity \(\text{tr}_{\delta} u = \partial_t \eta \nu\). Furthermore, by the lower semi-continuity of the norms with respect to weak and weak* convergence, we deduce \(3.19\) and \(3.20\). Multiplying \(3.33\) by \(\varphi(t)\), where \(\varphi \in C^1_0([0, T])\), integrating over \(I\), integrating by parts in time, letting \(n \to \infty\), and using the denseness of the test functions in \(T_{\delta,p}^I\) we see that the couple \((\eta, u)\) satisfies \(3.17\) with \(\xi\) in place of \(S(Du)\). Thus, it remains to identify \(\xi\). In view of the \(L^2\)-continuity of \(\partial_t \eta\) and \(u\), we can proceed analogously to the proof of \([29, \text{Remark 3.3}]\) to show that, for \((b, \varphi) \in T_{\delta,p}^I\), with the constraint \(b(T, \cdot) = 0\), \(\varphi(T, \cdot) = 0\) replaced by \(b(0, \cdot) = 0\), \(\varphi(0, \cdot) = 0\), \((\eta, u)\) satisfies \(3.17\) with \(\xi\) in place of \(S(Du)\) and with the right-hand side replaced by

\[
\int_I \int_{\Omega_{\delta}(t)} f \cdot \varphi \, dx dt + \int_I \int_M g b \, dA dt - \int_{\Omega_{\delta}} u(T, \cdot) \cdot \varphi(T, \cdot) \, dx - \int_M \eta(T, \cdot) b(T, \cdot) \, dA.
\]

On the other hand, we note that subsequences of \((u_n(T, \cdot))\) and \((\partial_t \eta_n(T, \cdot))\) converge weakly to functions \(u^*\) and \(\eta^*\) in \(L^2(\Omega_{\delta}(T))\) and \(L^2(M)\), respectively. Thus, multiplying \(3.33\) by \(\varphi(t)\), \(\varphi \in C^1_0([0, T])\), and taking the limit as before, we see that, for \((b, \varphi)\) as above, \((\eta, u)\) satisfies \(3.17\) with \(\xi\) in place of \(S(Du)\) and with the right-hand side replaced by

\[
\int_I \int_{\Omega_{\delta}(t)} f \cdot \varphi \, dx dt + \int_I \int_M g b \, dA dt - \int_{\Omega_{\delta}} u^* \cdot \varphi(T, \cdot) \, dx - \int_M \eta^* b(T, \cdot) \, dA.
\]

The denseness can be shown by exactly the same argument used in \([29]\) just before the proof of Proposition 3.15.
This yields \( u^* = u(T, \cdot), \eta^* = \partial_t \eta(T, \cdot) \). Furthermore, a subsequence of \((\eta_n(T, \cdot))\) converges to \( \eta(T, \cdot) \) weakly in \( H_0^1(M) \). We have already seen that the Galerkin solutions satisfy the energy identity
\[
\int_I \int_{\Omega \times (0, T)} S(Du_n) : Du_n \ dx dt + 2 \int_I K(\partial_t \eta_n) \ dt 
= -E_{\eta_n, u_n}(T) + E_{\eta_n, u_n}(0) + \int_I \int_{\Omega \times (0, T)} f_n : u_n \ dx dt + \int_I \int_M g_n \partial_t \eta_n \ dA dt.
\]
Taking the lim sup of this equation, exploiting the weak lower semi-continuity of the energy \( E \) and noting that \( \eta_n(0, \cdot) = \eta_0 \) for all \( n \in \mathbb{N} \), we obtain
\[
\limsup_n \int_I \int_{\Omega \times (0, T)} S(Du_n) : Du_n \ dx dt + 2 \int_I K(\partial_t \eta_n) \ dt 
\leq -E_{\eta, u}(T) + E_{\eta, u}(0) + \int_I \int_{\Omega \times (0, T)} f : u \ dx dt + \int_I \int_M g \partial_t \eta \ dA dt.
\]
From the energy identity for the weak solution \((\eta, u)\) (with \( \xi \) in place of \( S(Du) \)) we deduce that
\[
\limsup_n \int_I \int_{\Omega \times (0, T)} S(Du_n) : Du_n \ dx dt + 2 \int_I K(\partial_t \eta_n) \ dt 
\leq \int_I \int_{\Omega \times (0, T)} \xi : Du \ dx dt + 2 \int_I K(\partial_t \eta) \ dt.
\]
Using this estimate and the weak convergences, we obtain
\[
0 \leq \limsup_n \left( \int_I \int_{\Omega \times (0, T)} (S(Du_n) - S(Du)) : (Du_n - Du) \ dx dt 
+ 2 \int_I K(\partial_t \eta_n - \partial_t \eta, \partial_t \eta_n - \partial_t \eta) \ dt \right) 
= \limsup_n \left( \int_I \int_{\Omega \times (0, T)} S(Du_n) : Du_n + S(Du) : Du 
- S(Du_n) : Du - S(Du) : Du_n \ dx dt 
+ 2 \int_I K(\partial_t \eta_n, \partial_t \eta_n) + K(\partial_t \eta, \partial_t \eta) - 2K(\partial_t \eta_n, \partial_t \eta) \ dt \right) 
\leq \int_I \int_{\Omega \times (0, T)} \xi : Du + S(Du) : Du - \xi : Du - S(Du) : Du \ dx dt 
+ 2 \int_I K(\partial_t \eta, \partial_t \eta) + K(\partial_t \eta, \partial_t \eta) - 2K(\partial_t \eta, \partial_t \eta) \ dt 
= 0.
\]

Hence, for a subsequence, we have
\[
(S(Du_n) - S(Du)) : (Du_n - Du) \to 0
\]

\(^{18}\)Note that each continuous, non-negative quadratic form, e.g. \( K \), is weakly lower semi-continuous. This follows by taking the lim inf of the inequality
\[
0 \leq K(\eta_n - \eta, \eta_n - \eta) = K(\eta_n) - 2K(\eta, \eta_n) + K(\eta).
\]
a.e. in $Ω_{\bar{\delta} \tilde{\delta}}$. By Proposition A.3 we infer that $D u_n \to D u$ and hence $S(D u_n) \to S(D u)$ a.e. in $Ω_{\bar{\delta} \tilde{\delta}}$. Finally, Vitali’s convergence theorem yields $\bar{\xi} = S(D u)$. This proves the existence of weak solutions.

Now, let us show uniqueness. For weak solutions $(\eta_0, u_0)$ and $(\eta_1, u_1)$ of the regularized and decoupled system with datum $\delta$ in the interval $I$ their difference $(\eta := \eta_1 - \eta_0, u := u_1 - u_0)$ is a weak solution, too, with $S(D u)$ replaced by $\bar{\xi} = S(D u_1) - S(D u_0)$ and $f, g$ replaced by

$$\bar{f} := -(R^0(0) \cdot \nabla) u_0 - (R^0(0) \cdot \nabla) u, \quad \bar{g} := \frac{1}{2} (R^1(0) \partial_t \eta) - \frac{1}{2} (R^1(0) \partial_t \eta_0) \partial_t \eta,$$

respectively. Thus, by Lemma 3.21 we have

$$E_{\eta, u}(t) \leq - \int_0^t \int_{\Omega_{\bar{\delta} \tilde{\delta}(\xi)}} (R^0(0) \cdot \nabla) u_0 \cdot u \, dx \, dt + \frac{1}{2} \int_0^t \int_{\Omega_{\bar{\delta} \tilde{\delta}(\xi)}} (\partial_t \eta_0) \partial_t \eta_0 \, dx \, dt$$

for some nonnegative function $c \in L^1(I)$. In the first inequality we used the monotonicity of $S$ and the fact that the second terms in $\bar{f}$ and $\bar{g}$ cancel when tested against $(\partial_t \eta, u)$. By Gronwall’s inequality, we have $E_{\eta, u} \equiv 0$, proving that the solutions coincide. □

3.3. Fixed-point argument. Let us now define solutions of our regularized problem.

**Definition 3.35.** Let $\varepsilon, \bar{\varepsilon} > 0$. A couple $(\eta, u)$ is a weak solution of the $(\varepsilon, \bar{\varepsilon})$-regularized system in the interval $I$ if $\eta \in Y^I \cap H^1(I, H_0^2(M))$ with $\|\eta\|_{L^\infty(I \times M)} < \kappa$, $\eta(0, \cdot) = \eta_0$, and $u \in X^{\varepsilon, \bar{\varepsilon}}_{\eta, 0}$ with $tr_{\varepsilon, \bar{\varepsilon}} u = \partial_t \eta \, \nu$, and

$$- \int_I \int_{\Omega_{\varepsilon, \bar{\varepsilon}(\xi)}} u \cdot \partial_t \varphi \, dx \, dt - \int_I \int_{\Omega_{\varepsilon, \bar{\varepsilon}(\xi)}} (R^0(0) \cdot \nabla) u \cdot D \varphi \, dx \, dt$$

$$- \frac{1}{2} \int_I \int_M \partial_t \eta \, \partial_t \eta \, b(\gamma(\mathcal{R}_\varepsilon)) \, dx \, dt + \frac{1}{2} \int_I \int_M R^1(0) \partial_t \eta \, b(\gamma(\mathcal{R}_\varepsilon)) \, dx \, dt$$

$$+ \int_I \int_{\Omega_{\varepsilon, \bar{\varepsilon}(\xi)}} S_\varepsilon(D u) : D \varphi \, dx \, dt - \int_I \int_M \partial_t \eta \, b \, dx \, dt + \frac{1}{2} \int_I K(\eta + \varepsilon \partial_t \eta, b) \, dt$$

for all test functions $(b, \varphi) \in T^1_{\varepsilon, \bar{\varepsilon}}(\eta, \nu)$.

**Proposition 3.37.** There exists a $T > 0$ such that for all sufficiently small $\varepsilon, \bar{\varepsilon} > 0$ there exists a weak solution $(\eta, u)$ of the $(\varepsilon, \bar{\varepsilon})$-regularized system in the interval $I = (0, T)$. Furthermore, we have

$$\|\eta\|_{L^\infty(I \times M)} + \|u\|_{L^2(I \times M)} + \|D u\|_{L^p(I \times M)} \leq c_0(T, \Omega_{\varepsilon, \bar{\varepsilon}}(\eta_0), f, g, u_0, \eta_0, \eta_0^f)$$

and $\sup_{\varepsilon} \tau(\eta_0) < \infty$. The time $T$ can be chosen to depend only on $\tau(\eta_0)$ and the bound (3.38) for the $Y^I$-norm of $\eta$. Finally, for some constant $c > 0$, we have

$$\varepsilon \|\partial_t \eta\|_{L^2(I \times M)}^2 + \bar{\varepsilon} \|D u\|_{L^p(I \times M)} \leq c.$$
Proof. We set }\alpha := (\|\eta_0\|_{L^\infty(M)} + \kappa)/2\text{ and fix arbitrary but sufficiently small }\varepsilon, \bar{\varepsilon} > 0.\text{ For a better readability, in the following, we will omit the symbols }\varepsilon, \bar{\varepsilon}.\text{ We want to use Schauder’s fixed point theorem. To this end, we define the space }Z := C(I \times \partial \Omega)\text{ with the closed, convex subset}

\[ D := \{ (\delta, v) \in Z \mid \delta(0, \cdot) = \eta_0, \|\delta\|_{L^\infty(I \times \partial \Omega)} \leq \alpha \}. \]

Let }F : D \to Z\text{ map each }\delta \in D\text{ to the component }\eta\text{ of the unique weak solution }\eta, u\text{ of the decoupled and regularized system with datum }\delta.\text{ From (3.19) we deduce that the norm of }\eta\text{ in}

\[ Y^I \hookrightarrow C^{0,1-\theta}(I, C^{0,2\theta - 1}(\partial \Omega)) \quad (1/2 < \theta < 1) \quad (3.40) \]

is bounded. Since }\eta(0, \cdot) = \eta_0,\text{ we can choose the time interval }I = (0, T)\text{ so small that }\|\eta\|_{L^\infty(I \times MC)} \leq \alpha,\text{ independently of the parameter }\varepsilon; \text{ in particular, }\tau(\eta) \leq c(\alpha).\text{ Thus, }F\text{ maps }D\text{ into itself. Furthermore, from (3.40) and the compact embedding of the Hölder space into }Z,\text{ we see that }F(D)\text{ is relatively compact in }Z. \text{ It remains to show that }F\text{ is continuous. To this end, we let }\delta_n \subset D\text{ be a sequence converging to }\delta\text{ in }Z\text{ and }\eta_n, u_n\text{ be the corresponding weak solutions given by Proposition 3.21 in view of (3.19), (3.20), and (3.34). We can deduce that for a subsequence we have}

\[ \eta_n \to \eta \quad \text{weakly in } H^1(I, H^2(M)) \text{ and, thus, uniformly}, \]

\[ \partial_t \eta_n \to \partial_t \eta \quad \text{weakly}^* \text{ in } L^\infty(I, L^2(M)), \]

\[ \mathcal{R}^1 \partial_t \eta_n \to \mathcal{R}^1 \partial_t \eta \quad \text{in } L^2(I \times M), \]

\[ u_n \to u \quad \text{weakly}^* \text{ in } L^\infty(I, L^2(\mathbb{R}^3)), \]

\[ \nabla u_n \to \nabla u \quad \text{weakly}^* \text{ in } L^{p_0}(I \times \mathbb{R}^3), \]

\[ \mathcal{R}^0 u_n \to \mathcal{R}^0 u \quad \text{in } L^{p_0}(I \times \mathbb{R}^3), \]

\[ S(Du_n) \to \xi \quad \text{weakly} \text{ in } L^{p_0}(\mathcal{Q}^I_{\mathcal{R}^0 \delta}). \]

Here, we extend }\nabla u_n, S(Du_n),\text{ and }\nabla u,\text{ which are a-priori defined on }\Omega^I_{\mathcal{R}^0 \delta}\text{ and }\Omega^I_{\mathcal{R}^0 \delta},\text{ respectively, by 0 to the whole of }I \times \mathbb{R}^3.\text{ We have to show that }\eta = F(\delta).\text{ The property }\eta(0, \cdot) = \eta_0\text{ follows immediately from the uniform convergence of }\eta_n.\text{ Moreover, we can show exactly like in the proof of }\delta_0.\text{ Proposition 3.35 that }\partial_t \eta = \text{tr } \mathcal{R} \delta u. \text{ It remains to prove that (3.17) is satisfied. For all }n\text{ and all test functions }\langle b_n, \varphi_n \rangle \in T^I_{\mathcal{R} \delta_0, p},\text{ we have}

\[- \int_I \int_{\Omega_{\mathcal{R} \delta_0}} \mathcal{D} \varphi_n dx dt + \int_I \int_{\Omega_{\mathcal{R} \delta_0}} \mathcal{R}^0 u_n \otimes D \varphi_n dx dt \]

\[- \frac{1}{2} \int_I \int_M \partial_t \eta_n \partial_t \mathcal{R} \delta_0 b_n \gamma(\mathcal{R} \delta_0) dA dt + \frac{1}{2} \int_I \int_M \mathcal{R}^1 \partial_t \eta_n \partial_t \mathcal{R} \delta_0 b_n \gamma(\mathcal{R} \delta_0) dA dt \]

\[ + \int_I \int_{\Omega_{\mathcal{R} \delta_0}} S(Du_n) : D \varphi_n dx dt - \int_I \int_M \partial_t \eta_n \partial_t b_n dA dt + 2 \int_I K(\eta_n + \partial_t \mathcal{R} \delta_0 b_n) dt \]

\[ = \int_I \int_{\Omega_{\mathcal{R} \delta_0}} \mathcal{F} \cdot \varphi_n dA dt + \int_I \int_M b \cdot \varphi_n dA dt + \int_I \int_{\Omega_{\mathcal{R} \delta_0}} \mathcal{D} \varphi_n(0, \cdot) dx + \int_I \int_{\mathcal{R} \delta_0} \eta_1 b_n(0, \cdot) dA. \]

As in }\delta_0.\text{ Proposition 3.35, we can pass to the limit in this equation (with the exception of the extra stress tensor) by using, for given }\langle b, \varphi \rangle \in T^I_{\mathcal{R} \delta_0, p},\text{ the special test functions }\langle b_n, \varphi_n \rangle := (\mathcal{M}_{\mathcal{R} \delta_0 b, \mathcal{R} \delta_0, \mathcal{M}_{\mathcal{R} \delta_0 b}) \in T^I_{\mathcal{R} \delta_0, p}\text{ on the one hand, and the test functions }\langle 0, \varphi \rangle \in T^I_{\mathcal{R} \delta_0}\text{ with }\varphi(T, \cdot) = 0\text{ and supp }\varphi \subset \Omega^I_{\mathcal{R} \delta_0}\text{ on the other hand. Here, additionally to (3.41), we need to take into account the assertions (1.2), (2.2) in Lemma A.5. Finally, in order to identify the}
function $\xi$, we can proceed almost literally as in the proof of Proposition 3.21. Essentially, we only have to replace the integrals over $\Omega^I_{\varepsilon,b}$ by integrals over $I \times \mathbb{R}^3$, extending the corresponding functions by 0 to $I \times \mathbb{R}^3$. This shows that $(\eta, u)$ is the unique weak solution of the decoupled and regularized problem with datum $\delta$. Thus, $F$ is continuous and, by Schauder’s fixed point theorem, possesses a fixed point. This concludes the proof. \hfill \Box

3.4. Limiting process. Now, we can prove our main result by letting the regularizing parameters in Definition 3.3 tend to zero.

Proof: (of Theorem 3.3) First, we fix $\varepsilon > 0$ and let $\varepsilon \downarrow 0$. We have shown that there exists a $T > 0$ such that for all $\varepsilon = 1/n, n \in \mathbb{N}$ sufficiently large, there exists a weak solution $(\eta_{\varepsilon}, u_{\varepsilon})$ of the $(\varepsilon, \bar{\varepsilon})$-regularized problem in the interval $I = (0, T)$. For a subsequence, the estimates (3.38), (3.39), Proposition 2.13, and the compact embedding $Y^I \hookrightarrow C(I \times \partial \Omega)$ yield the following convergences

$$
\eta_{\varepsilon}, \mathcal{R}_{\varepsilon} \eta_{\varepsilon} \rightarrow \eta \quad \text{weakly}^{*} \text{ in } L^{\infty}(I, H^2(\mathcal{M})) \quad \text{and uniformly,}
$$

$$
\partial_t \eta_{\varepsilon}, \partial_t \mathcal{R}_{\varepsilon} \eta_{\varepsilon} \rightarrow \partial_t \eta \quad \text{weakly}^{*} \text{ in } L^{\infty}(I, L^2(\mathcal{M})),
$$

$$
u_{\varepsilon} \rightarrow u \quad \text{weakly}^{*} \text{ in } L^{\infty}(I, L^2(\mathbb{R}^3)), \quad (3.43)
$$

$$
S(Du_{\varepsilon}) \rightarrow \xi \quad \text{weakly in } L^p(I \times \mathbb{R}^3).
$$

Here, we extend $Du_{\varepsilon}, \nu_{\varepsilon}, S(Du_{\varepsilon})$, and $Du$ which are a-priori defined only on $\Omega^I_{\varepsilon, b}$ and $\Omega^I_{\eta}$, respectively, by 0 to the whole of $I \times \mathbb{R}^3$. The uniform convergence of $(\mathcal{R}_{\varepsilon} \eta_{\varepsilon})$ follows from the estimate

$$
||\mathcal{R}_{\varepsilon} \eta_{\varepsilon} - \eta||_{L^\infty(I \times \partial \Omega)} \leq ||\mathcal{R}_{\varepsilon} (\eta_{\varepsilon} - \eta)||_{L^\infty(I \times \partial \Omega)} + ||\mathcal{R}_{\varepsilon} \eta - \eta||_{L^\infty(I \times \partial \Omega)}.
$$

Now, we can repeat the proof of Proposition 3.3 almost literally to show that

$$
\partial_t \eta_{\varepsilon} \rightarrow \partial_t \eta \quad \text{in } L^2(I \times \mathcal{M}),
$$

$$
u_{\varepsilon}, \mathcal{R}_{\varepsilon}^0 \nu_{\varepsilon} \rightarrow u \quad \text{in } L^2(I \times \mathbb{R}^3). \quad (3.44)
$$

From these convergences and the definition of $\mathcal{R}_{\varepsilon}^0, \mathcal{R}_{\varepsilon}^1$ it is not hard to see that

$$
\mathcal{R}_{\varepsilon}^1 \partial_t \eta_{\varepsilon} \rightarrow \partial_t \eta \quad \text{in } L^2(I \times \mathcal{M}),
$$

$$
\mathcal{R}_{\varepsilon}^0 \nu_{\varepsilon} \rightarrow u \quad \text{in } L^2(I \times \mathbb{R}^3). \quad (3.45)
$$

As in the proof [29] Proposition 3.35), we obtain the identity $\text{tr}_n u = \partial_t \eta v$. By (3.38), Proposition 2.13, and Corollary 2.7, the sequence $(u_{\varepsilon})$ is bounded in $L^\infty(I, L^2(\mathbb{R}^3)) \cap L^p(I, L^p(\mathbb{R}^3))$ for all $r < p^{19}$ From this bound, (3.44), and (3.45), by interpolation, we obtain that

$$
u_{\varepsilon}, \mathcal{R}_{\varepsilon}^0 \nu_{\varepsilon} \rightarrow u \quad \text{in } L^r(I \times \mathbb{R}^3) \quad (3.46)
$$

for all $1 \leq r < 10p/6$. Similarly, we deduce from (3.38), (3.44), and (3.45) that

$$
\partial_t \eta_{\varepsilon}, \mathcal{R}_{\varepsilon}^0 \partial_t \eta_{\varepsilon}, \partial_t \mathcal{R}_{\varepsilon} \eta_{\varepsilon} \rightarrow \partial_t \eta \quad \text{in } L^4(I, L^2(\mathcal{M})). \quad (3.47)
$$

The lower semi-continuity of the norms yields the estimate (3.4), while the uniform convergence of $(\eta_{\varepsilon})$ gives $\eta(0, \cdot) = \eta_0$. For all $\varepsilon = 1/n, n$ sufficiently large, and all $(b_{\varepsilon}, \varphi_{\varepsilon}) 

\text{In fact, this is true for } p_0 \text{ in place of } p. \text{ But since we will have to repeat this argument when taking the limit } \bar{\varepsilon} \downarrow 0, \text{ we don’t want to make use of this fact.}
\[ T^l_{\varepsilon \eta_{\varepsilon, b}} \text{ we have} \]

\[ - \int_J \int_{\Omega_{\varepsilon \eta_{\varepsilon}}} \mathbf{u}_\varepsilon \cdot \partial_t \varphi \, dxdt - \int_J \int_{\Omega_{\varepsilon \eta_{\varepsilon}}} (\mathcal{A}_0^0 \mathbf{u}_\varepsilon) \otimes \mathbf{u}_\varepsilon : \nabla \varphi \, dxdt \quad (3.48) \]

\[ - \frac{1}{2} \int_J \int_M \partial_t \eta_{\varepsilon} \partial_t \mathcal{A}_0 \eta_{\varepsilon} b_{\varepsilon} \gamma(\mathcal{A}_0 \eta_{\varepsilon}) \, dAdt + \frac{1}{2} \int_J \int_M \mathcal{A}_0^1 (\partial_t \eta_{\varepsilon}) \partial_t \eta_{\varepsilon} b_{\varepsilon} \gamma(\mathcal{A}_0 \eta_{\varepsilon}) \, dAdt \]

\[ + \int_J \int_{\Omega_{\varepsilon \eta_{\varepsilon}}} S^e(D\mathbf{u}_\varepsilon) : D\varphi - \int_J \int_M \partial_t \eta_{\varepsilon} \partial_t b_{\varepsilon} \, dAdt + 2 \int_J K(\eta_{\varepsilon} + \varepsilon \partial_t \eta_{\varepsilon}, b_{\varepsilon}) \, dt \]

\[ = \int_J \int_{\Omega_{\varepsilon \eta_{\varepsilon}}} f \cdot \varphi \, dxdt + \int_J \int_M g b_{\varepsilon} \, dAdt + \int_{\Omega_{\varepsilon \eta_{\varepsilon}}} \mathbf{u}_0^0 \cdot \varphi(0, \cdot) \, dx + \int_M \eta_{\varepsilon} b_{\varepsilon}(0, \cdot) \, dA. \]

Note that, by \((3.39)\), we have

\[ \left| \int_J K(\varepsilon \partial_t \eta_{\varepsilon}, b_{\varepsilon}) \, dt \right| \leq c \varepsilon \|\nabla \eta_{\varepsilon}\|_{L^2(\Omega_{\varepsilon, b}(M))} \|b_{\varepsilon}\|_{L^2(\Omega_{\varepsilon, b}(M))} \leq \sqrt{c} \|b_{\varepsilon}\|_{L^2(\Omega_{\varepsilon, b}(M))}, \]

and thus, this term vanishes in the limit. Just like in \((29)\), we make use of the special test functions \((b_{\varepsilon}, \varphi_{\varepsilon}) := (\mathcal{A}_0 \eta_{\varepsilon}, \mathcal{A}_0 \eta_{\varepsilon} b_{\varepsilon}) \in T^l_{\varepsilon \eta_{\varepsilon}, b}\) for given \((b, \varphi) \in T^l_{\eta, b}\) on the one hand, and the test functions \((0, \varphi) \in T^l_{\eta, b}\) with \(\varphi(T_{\varepsilon, b}) = 0\) and \(\text{supp} \varphi \subset \Omega_{\varepsilon, b}\) on the other hand. Using \((3.43), (3.46), (3.47), \) and \((3.15), \) as well as the assertions \((1.2), (2.1)\) in Lemma \((A.5)\) we can pass to the limit in \((3.48)\) with the exception of the extra stress tensor. The convergences \((3.46)\) are needed for the second term, while the convergences \((3.47)\) are needed for the third and the fourth term. This implies the validity of \((3.2)\) with \(\varepsilon = 0\) and \(S(D\mathbf{u})\) replaced by \(\tilde{\varepsilon}\) for \((b, \varphi) = (b, \mathcal{A}_0 b) \in T^l_{\eta, b}\); note that the third and the fourth term in \((3.36)\) cancel for \(\varepsilon = 0.\)

Now, let us identify \(\tilde{\varepsilon}\). Let us fix some open, bounded interval \(J\) and some open ball \(B\) such that for the cylinder \(Q := J \times B\) we have \(\partial Q \subset \Omega_{\varepsilon, b}\). By the uniform convergence of \(\mathcal{A}_0 \eta_{\varepsilon}\), we have \(\Omega_{\varepsilon} \subset \Omega_{\varepsilon, \eta_{\varepsilon}}\) for sufficiently small \(\varepsilon\). Thus, setting \(v_{\varepsilon} := \mathbf{u}_\varepsilon - \mathbf{u}\) and \(G_{\varepsilon} := G_{\varepsilon}^0 + G_{\varepsilon}^1\) with

\[ G_{\varepsilon}^0 := S^e(D\mathbf{u}_\varepsilon) - \tilde{\varepsilon}, \]

\[ G_{\varepsilon}^1 := \mathcal{A}_0^0 \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon - \mathbf{u} \otimes \mathbf{u}, \]

for a subsequence we have\(^{20}\)

\[ v_{\varepsilon} \rightarrow 0 \] weakly in \(L^{p_0}(J, W^{1, p_0}(Q))\),

\[ G_{\varepsilon}^0 \rightarrow 0 \] weakly in \(L^{p_0}(Q)\),

\[ G_{\varepsilon}^1 \rightarrow 0 \] in \(L^r(Q)\) for all \(1 \leq r < 5p/6\)

and

\[ - \int_Q v_{\varepsilon} \cdot \partial_t \varphi + G_{\varepsilon} : \nabla \varphi \, dxdt = 0 \]

for all vector fields \(\varphi \in C_0^\infty(J \times B)\) with \(\text{div} \varphi = 0\) and sufficiently small \(\varepsilon\). By Hölder’s inequality and Theorem \((A.2)\)(a), we deduce that for \(\theta \in (0, 1)\) and \(\zeta \in C_0^\infty(Q)\) with \(\chi_{\varepsilon} \leq \)

\(^{20}\)Note that classical Korn’s inequality holds in \(B\).
\[ \zeta \leq \chi_{\frac{1}{2}} \] we have

\[
\limsup_{\epsilon \to 0} \int_{Q} \left( (S_{\epsilon}(Du_{e}) - S_{\tilde{\epsilon}}(Du)) : D(u_{e} - u) \right)^{\theta} \zeta \, dx \, dt \\
\leq c 2^{(\theta - 1)k} + \limsup_{\epsilon \to 0} \int_{Q} (S_{\epsilon}(Du_{e}) - S_{\tilde{\epsilon}}(Du)) : \nabla v_{\epsilon} \zeta O_{n,k} \, dx \, dt.
\]

The second term on the right-hand side is bounded by

\[
\limsup_{\epsilon \to 0} \int_{Q} G_{\epsilon} : \nabla v_{\epsilon} \zeta O_{n,k} \, dx \, dt + \limsup_{\epsilon \to 0} \int_{Q} (\xi - S_{\tilde{\epsilon}}(Du)) : \nabla v_{\epsilon} \zeta O_{n,k} \, dx \, dt.
\]

Here, by Theorem A.2(b), the first term is bounded by \(c 2^{-k/p}\), while, by Theorem A.2(a) and the weak convergence of \((\nabla v_{\epsilon})\), the second term can be estimated by

\[
\limsup_{\epsilon \to 0} \int_{Q} (\xi - S_{\tilde{\epsilon}}(Du)) : \nabla v_{\epsilon} \zeta \, dx \, dt + \limsup_{\epsilon \to 0} \int_{Q} (\xi - S_{\tilde{\epsilon}}(Du)) : \nabla v_{\epsilon} \zeta O_{n,k} \, dx \, dt \leq c 2^{-k}.
\]

Thus, we showed that

\[
\limsup_{\epsilon \to 0} \int_{Q} \left( (S_{\epsilon}(Du_{e}) - S_{\tilde{\epsilon}}(Du)) : D(u_{e} - u) \right)^{\theta} \zeta \, dx \, dt = 0
\]

and, hence, for a subsequence

\[
(S_{\epsilon}(Du_{e}) - S_{\tilde{\epsilon}}(Du)) : D(u_{e} - u) \to 0
\]
a.e. in \(\frac{1}{2}Q\). As in the proof of Proposition 3.18 we can infer that \(\bar{\xi} = S_{\tilde{\epsilon}}(Du)\) in \(\frac{1}{2}Q\). Since \(Q\) was arbitrary, we have \(\bar{\xi} = S_{\tilde{\epsilon}}(Du)\) in \(\Omega_{\tilde{\eta}}\).

We can repeat these arguments almost literally for the limit \(\tilde{\epsilon} \to 0\). The main difference is that, in general, we won’t have strong \(L^{2}\)-compactness of \((\partial_{\eta} \eta_{\tilde{\epsilon}})\) if \(p \leq 3/2\). But since equation (3.2) (with \(S\) replaced by \(S_{\tilde{\epsilon}}\)) contains no (explicit) nonlinearities in \((\partial_{\eta} \eta_{\tilde{\epsilon}})\), strong compactness is not needed. Furthermore, in this limit process we have to use that

\[
\int_{\Omega_{\tilde{\eta}}^{(i)}} |Du_{e}|^{p-2}Du_{e} : D\Phi_{\tilde{\epsilon}} \, dx \leq c \|Du_{e}\|^{p-1}_{L^{p}(\Omega_{\tilde{\eta}}^{(i)})} \|D\Phi_{\tilde{\epsilon}}\|_{L^{p}(\Omega_{\tilde{\eta}}^{(i)})} \leq c \tilde{\epsilon}^{1/p}.
\]

Finally, we can proceed as in [29] to show that the solution exists as long as the inequality \(\|\eta(t, \cdot)\|_{L^{p}(\Omega)} < \kappa\) holds.

**APPENDIX A. APPENDIX**

The following classical result can be found in [3].

**Proposition A.1. (Reynolds transport theorem)** Let \(\Omega \subset \mathbb{R}^{3}\) be a bounded domain with \(C^{1}\)-boundary, let \(I \subset \mathbb{R}\) be an interval, and let \(\Psi \in C^{1}(I \times \mathbb{R}^{3}, \mathbb{R}^{3})\) such that

\[
\Psi_{t} := \Psi(t, \cdot) : \mathbb{R}^{3} \to \Psi_{t}(\Omega)
\]

is a diffeomorphism for all \(t \in I\). We set \(\Omega_{t} := \Psi_{t}(\Omega)\) and \(\nu := (\partial_{\nu} \Psi) \circ \Psi_{t}^{-1}\). Then we have for all \(\bar{\xi} \in C^{1}(\cup_{t \in I} \{t\} \times \overline{\Omega_{t}})\) and \(t \in I\)

\[
\frac{d}{dt} \int_{\Omega_{t}} \bar{\xi}(t, x) \, dx = \int_{\Omega_{t}} \partial_{t} \bar{\xi}(t, x) \, dx + \int_{\partial \Omega_{t}} \nu \cdot v_{t} \bar{\xi}(t, \cdot) \, dA_{t}.
\]

Here, \(dA_{t}\) denotes the surface measure and \(v_{t}\) denotes the outer unit normal of \(\partial \Omega_{t}\).
Theorem A.2. Let $1 < p, r < \infty$, $d \in \mathbb{N}$, $B \subset \mathbb{R}^d$ an open ball, and $J$ an open, bounded interval. We set $Q := J \times B$ and assume that the vector fields $\mathbf{v}_n$ and the $\mathbb{R}^{d \times d}$-valued fields $G_n^0$ and $G_n^1$, $n \in \mathbb{N}$, satisfy

\[
\begin{align*}
\mathbf{v}_n &\to 0 \quad \text{weakly in } L^p(J, W^{1,p}(B)), \\
G_n^0 &\to 0 \quad \text{weakly in } L^p(Q), \\
G_n^1 &\to 0 \quad \text{strongly in } L^r(Q).
\end{align*}
\]

Furthermore, we assume that the sequence $(\mathbf{v}_n)$ is bounded in $L^\infty(I, L^r(B))$ and that for $G_n = G_n^0 + G_n^1$ and all vector fields $\varphi \in C_0^\infty(I \times B)$ with $\nabla \varphi = 0$ we have

\[
\int_Q \mathbf{v}_n \cdot \partial_t \varphi + G_n : \nabla \varphi \, dx \, dt = 0.
\]

Then\[\footnote{For $\alpha > 0$ we denote by $\alpha Q$ the cylinder $Q$ scaled by $\alpha$ with respect to its center.}^2\]

for $\zeta \in C_0^\infty(\partial_+ Q)$ with $\chi_{\frac{1}{2}Q} \leq \zeta \leq \chi_Q$ and all $n, k \in \mathbb{N}$, $k$ sufficiently large, there exist open sets $O_{n,k} \subset Q$ such that

\[
\begin{align*}
(a) \quad &\limsup_{n \to \infty} |O_{n,k}| \leq c 2^{-k}, \\
(b) \quad &\limsup_{n \to \infty} \int_Q G_n^0 \cdot \nabla \mathbf{v}_n \zeta \chi_{O_{n,k}} \, dx \, dt \leq c 2^{-k/p}.
\end{align*}
\]

Proof. See [6].

Proposition A.3. Let $S : M_\text{sym} \to M_\text{sym}$ be continuous and strictly monotone, i.e.,

\[
(S(A) - S(B)) : (A - B) > 0
\]

for $A, B \in M_\text{sym}$, $A \neq B$. Furthermore, let $(A_n)_{n \in \mathbb{N}} \subset M_\text{sym}$ be a sequence such that

\[
\lim_{n \to \infty} (S(A_n) - S(A)) : (A_n - A) = 0
\]

for some $A \in M_\text{sym}$. Then $\lim_{n \to \infty} A_n = A$.

Proof. See [15].

Lemma A.4. Let $\eta \in Y^1$ with $\|\eta\|_{L^\infty(I \times \mathcal{M})} < \kappa$. There exists a linear operator $\mathcal{M}_\eta$ such that

\[
\begin{align*}
\|\mathcal{M}_\eta b\|_{L^r(I \times \mathcal{M})} &\leq c \|b\|_{L^r(I \times \mathcal{M})}, \\
\|\mathcal{M}_\eta b\|_{C(I \times \mathcal{L}^r(M))} &\leq c \|b\|_{C(I \times \mathcal{L}^r(M))}, \\
\|\mathcal{M}_\eta b\|_{L^1(I \mathcal{H}^2_0(M))} &\leq c \|b\|_{L^1(I \mathcal{H}^2_0(M))}, \\
\|\mathcal{M}_\eta b\|_{H^1(I \mathcal{L}^2_0(M))} &\leq c \|b\|_{H^1(I \mathcal{L}^2_0(M))}
\end{align*}
\]

for all $1 \leq r \leq \infty$ and

\[
\int_M (\mathcal{M}_\eta b)(\cdot, t) \gamma(\eta(\cdot, t)) \, dA = 0
\]

for almost all $t \in I$. The constant $c$ depends only on $\Omega$, $\|\eta\|_{Y^1}$ and $\tau(\eta)$; it stays bounded as long as $\|\eta\|_{Y^1}$ and $\tau(\eta)$ stay bounded.

Proof. We only give the definition of the operator, for a proof of the claims see [29]. For fixed, but arbitrary $\psi \in C_0^\infty(\text{int}\mathcal{M})$ with $\psi \geq 0$, $\psi \not\equiv 0$ and

\[
a(b(t, \cdot), \eta(t, \cdot)) := \int_M b(t, \cdot) \gamma(\eta(t, \cdot)) \, dA
\]

for $\alpha > 0$ we denote by $\alpha Q$ the cylinder $Q$ scaled by $\alpha$ with respect to its center.
For all \( N \)

Let the sequence \((2.a)\)

Proof.

Lemma A.6.

Proof.

Finally, for \((2.b)\) the sequence \((\mathcal{F}_{\eta_n}, \mathcal{M}_n b)\) converges to \(\mathcal{F}_{\eta_n}, \mathcal{M}_n b\) in \(H^1(I, L^2(B_\alpha)) \cap L'((I, W^{1,1}(B_\alpha))).\)

(2.c) Provided that \((b_n)\) converges to \(b\) weakly in \(L^2(I \times M)\) the sequence \((\mathcal{F}_{\eta_n}, b_n)\) converges to \(\mathcal{F}_{\eta_n} b\) weakly in \(L^2(I \times B_\alpha)\).

Proof. Comparing with [29, Lemma A.11] only assertions (1.b) and (2.b) have changed. The proof of (1.b) proceeds exactly as before. The same is true for assertion (2.b) with the exception that, here, for the convergence in \(L'(I, W^{1,1}(B_\alpha))\) we have to exploit the fact that \(Y\) embeds compactly into \(L^\infty(I, W^{1,1}(M))\) which is a consequence of the classical Aubin-Lions lemma.

In the proof of Proposition 3.5 it comes in handy to have an \(L^2\)-orthogonal variant \(\mathcal{M}_\eta^\perp\) of the operator \(\mathcal{M}_\eta\) which is defined by

\[(\mathcal{M}_\eta^\perp b)(t, \cdot) := b(t, \cdot) - \gamma(\eta(t, \cdot), a(\eta(t, \cdot), \eta(t, \cdot)))\]

Lemma A.6. For \(\eta \in Y\) with \(\|\eta\|_{L^\infty(I \times M)} < \kappa\) the assertions of Lemma 3.5 with \(\mathcal{M}_\eta^\perp\) in place of \(\mathcal{M}_\eta\) hold. Furthermore, for all \(1 \leq r \leq \infty\) and \(0 \leq s \leq 2\) we have

\[\|\mathcal{M}_\eta^\perp b\|_{L^r(I, H^s(M))} \leq c \|b\|_{L^r(I, H^s(M))} \cdot\]

Finally, for \((\eta_n) \subset Y\) with \(\sup_n \|\eta_n\|_{L^\infty(I \times M)} < \alpha \leq \kappa\) and (3.8) the claims (1.a) and (2.a) of Lemma 3.5 are true with \(\mathcal{M}_\eta^\perp\) in place of \(\mathcal{M}_\eta\).

Proof. The proofs proceed almost exactly as before. Note that \(H^2(M)\) is an algebra.

Lemma A.7. For all \(N \in \mathbb{N}, 6/5 < p \leq \infty\) and \(\varepsilon > 0\) there exists a constant \(c\) such that for all \(\eta, \tilde{\eta} \in H_0^2(M)\) with \(\|\eta\|_{H_0^2(M)} + \|\tilde{\eta}\|_{H_0^2(M)} + \tau(\eta) + \tau(\tilde{\eta}) \leq N\) and all \(v \in W^{1,p}(\Omega,\),
\( \tilde{v} \in W^{1,p}(\Omega_{\tilde{\eta}}) \) we have
\[
\sup_{\|b\|_{L^4(M)}} \left( \int_{\Omega_{\tilde{\eta}}} v \cdot \mathcal{T}_{\eta} \mathcal{M}_{\eta} b \, dx - \int_{\Omega_{\tilde{\eta}}} \tilde{v} \cdot \mathcal{T}_{\eta} \mathcal{M}_{\eta} b \, dx + \int_{M} \mathcal{T}_{\eta} \nu \cdot \mathcal{M}_{\eta} b - \mathcal{T}_{\eta} \tilde{v} \cdot \mathcal{M}_{\eta} b \, dA \right)
\leq c \sup_{\|b\|_{L^4(M)}} \left( \int_{\Omega_{\tilde{\eta}}} v \cdot \mathcal{T}_{\eta} \mathcal{M}_{\eta} b \, dx - \int_{\Omega_{\tilde{\eta}}} \tilde{v} \cdot \mathcal{T}_{\eta} \mathcal{M}_{\eta} b \, dx + \int_{M} \mathcal{T}_{\eta} \nu \cdot \mathcal{M}_{\eta} b - \mathcal{T}_{\eta} \tilde{v} \cdot \mathcal{M}_{\eta} b \, dA \right) + \varepsilon \left( \|v\|_{W^{1,p}(\Omega_{\tilde{\eta}})} + \|	ilde{v}\|_{W^{1,p}(\Omega_{\tilde{\eta}})} \right).
\]

**Proof.** The proof is a very simple modification of the proof of [29, Lemma A.13] if we note that \( W^{1-1/r}(M) \) embeds compactly into \( L^{4/3}(M) \) for all \( 6/5 < r < \infty \). \( \Box \)

**Lemma A.8.** Let \( X \) be a function space that embeds compactly into \( L^2(M) \), and let \( \mathcal{P}_k \), \( k \in \mathbb{N} \), be the projection operators from the proof of Proposition 3.5. Then for each \( \varepsilon > 0 \) we have
\[
\| \mathcal{P}_k - \text{id} \|_{L^2(L^2(M))} \leq \varepsilon
\]
provided that \( k \) is sufficiently large.

**Proof.** By a simple compactness argument it suffices to show that for fixed \( b \in L^2(M) \) we have
\[
\| \mathcal{P}_k b - b \|_{L^2(M)} \leq \varepsilon
\]
provided that \( k \) is sufficiently large. But this is an elementary consequence of the definition of the projection operators \( \mathcal{P}_k \). \( \Box \)

**Acknowledgements**

The author would like to thank Lars Diening, Philipp Nägele, and Michael Růžička for their valuable and helpful comments and the fruitful discussions on the topic. The research of the author was partly supported by the project C2 of the SFB/TR 71 “Geometric Partial Differential Equations”.

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