$B^0_q$-$\bar{B}^0_q$ Mixing and Matching with Fermilab Heavy Quarks

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We discuss the matching procedure for heavy-light 4-quark operators using the Fermilab method for heavy quarks and staggered fermions for light quarks. These ingredients enable us to construct the continuum-limit operator needed to determine the oscillation frequency of neutral $B$ mesons. The matching is then carried out at the one-loop level. We also present an updated preliminary result for the SU(3)-breaking ratio $\xi$, based on calculations using the MILC Collaboration’s ensembles of lattice gauge fields.

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1. Introduction

All neutral mesons—$K^0$, $B^0$, $B_s$, $D^0$—have been observed to oscillate from particle to antiparticle. The oscillation frequency $\Delta M$ tests the Standard Model’s pattern of flavor violation. The phenomenology is especially simple for neutral $B$ mesons (normal and strange), because the flavor-changing dynamics play out predominantly at distances much shorter than the scale of QCD. In the case of the $B$ mesons, the width difference $\Delta \Gamma$ of the two propagating eigenstates also arises predominantly at short distances. It is especially intriguing (at least for now), because measurements of $\Delta \Gamma_s$ and the $CP$ phase $\phi_s$ of the $B_s$ are in imperfect agreement with the Standard Model\cite{1,2}.

Neutral $B$ mixing stems from $\Delta B = 2$ flavor-changing transitions. In the Standard Model these arise first at the one-loop level, so non-Standard contributions are conceivably of comparable size. The observables are then (approximately) $\Delta M = 2|M_{12}|$, $\Delta \Gamma = 2|\Gamma_{12}| \cos \phi$, and $\phi = \arg (\Delta M_{12}/\Gamma_{12})$, where $M_{12}$ and $\Gamma_{12}$ are the off-diagonal elements of the mass and width matrices of the two-state systems:

$$M_{12} = \frac{G_F^2 M_W^2}{8\pi^2 M_{b_s}} (V_{ts}^* V_{tb})^2 S_0 (m_t^2/M_W^2) \eta_b (\mu) \langle B| \bar{q}_L \gamma_\mu b \gamma^\mu b | \bar{B} \rangle + \text{BSM}, \quad (1.1)$$

$$\Gamma_{12} = -\frac{G_F^2 m_b^2}{6\pi M_{b_s}} [G(V, \mu) \langle B| \bar{q}_L \gamma_\mu b \gamma^\mu b | \bar{B} \rangle + G_S(V, \mu) \langle B| \bar{q}_L b \bar{q}_L b | \bar{B} \rangle] + \text{BSM}, \quad (1.2)$$

where $V$ is the CKM matrix, and $S_0$, $\eta_b$, $G$, and $G_S$ are short-distance effects, computed in electroweak and QCD perturbation theory. Contributions beyond the Standard Model ("BSM") are not written out explicitly. Because of the $V-A$ structure of the electroweak interaction, only the left-handed (light) quark field $\bar{q}_L = q_L (1 + \gamma_5)$ appears.

The remainder of this paper is organized as follows. Section 2 constructs lattice operators with staggered light quarks and Fermilab heavy quarks, corresponding to the 4-quark operators in Eqs. (1.1) and (1.2). (The construction suffices for any light quark with chiral symmetry and heavy quark with heavy-quark symmetry.) We give a status report of our numerical results in Sec. 3. Section 4 summarizes and presents some of our plans for the future.

2. Short-Distance Matching

To compute the hadronic matrix elements in Eqs. (1.1) and (1.2), one has to derive an expression in lattice gauge theory that approximates well $\bar{q}_L \gamma_\mu b \bar{q}_L \gamma^\mu b$ and $\bar{q}_L b \bar{q}_L b$. The lattice operators can then be computed, and the numerical and other uncertainties estimated, to determine $M_{12}$ and $\Gamma_{12}$. Similar operators appear BSM, for which the following derivation serves as a template.

For the light valence quark we take naive asqtad propagators

$$\langle \Upsilon(x) \bar{\Upsilon}(y) \rangle_U = \Omega(x) \Omega^{-1}(y) \langle \chi(x) \bar{\chi}(y) \rangle_U, \quad (2.1)$$

where $\chi$ is the one-component staggered fermion field; $\Upsilon$ is a 4-component naive field, and $\langle \cdot \cdot \cdot \rangle_U$ denotes the fermion average in a fixed gauge field $U$. For the heavy quark we use

$$\Psi = |1 + d_L \langle m_0 a \rangle \gamma \cdot D| \psi, \quad (2.2)$$

where $\psi$ is the fermion field appearing in the Fermilab action [3] or an improved action with the same design features [3].
We aim to construct lattice operators $Q$ and $Q_S$ such that
\begin{align}
Q &\doteq \bar{q}_L \gamma^\mu b \bar{q}_L \gamma^\mu b + O(a^2), \\
Q_S &\doteq \bar{q}_L b \bar{q}_L b + O(a^2),
\end{align}
where $\doteq$ means “has the same matrix elements as.” Here the $O(a^2)$ term depends on $m_b a$. As long as one retains small corrections to heavy-quark symmetry, it remains bounded even as $m_b a \to \infty$; as long as certain Dirac off-diagonal improvements are consistently introduced $[3, 4]$, they vanish as $a \to 0$. These two elements are the essence of the Fermilab method.

Our construction starts with the lattice operators $\bar{Y}_L \gamma^\mu \gamma^\nu \Psi$ and $\bar{Y}_L \gamma^\nu \Psi$. According to the HQET theory of cutoff effects $[5, 6, 7]$, these lattice operators can be described by
\begin{align}
\bar{Y}_L \gamma^\mu \gamma^\nu \Psi &\doteq 2C_{\text{lat}} \bar{q}_L \gamma^\mu (q^+ h^-) \bar{q}_L \gamma^\nu h^- + 2\delta C_{\text{lat}} \bar{q}_L h^+(q^-) \bar{q}_L h^- + \sum_{i=1}^5 B_{i\text{lat}} \mathcal{O}_i + \cdots, \\
\bar{Y}_L \gamma^\nu \Psi &\doteq 2\delta C_{\text{lat}} \bar{q}_L \gamma^\nu h^+(q^-) \bar{q}_L h^- + 2C_{\text{lat}} \bar{q}_L h^+(q^-) \bar{q}_L h^- + \sum_{i=1}^5 B_{i\text{lat}} \mathcal{O}_i + \cdots,
\end{align}
where $h^{(\pm)}$ are the heavy-quark fields of the heavy-quark effective theory (HQET), satisfying $h^{(\pm)} = \frac{1}{2}(1 \pm \gamma_4) h^{(\pm)}$. The sums are over five dimension-7, $\Delta B = 2$, four-quark operators, similar to those written out, but with an extra derivative. The series continues with operators of dimension 8 and higher. On the right-hand side of Eqs. (2.5) and (2.6) the operators are to be understood as those written out, but with an extra derivative. The series continues with operators of dimension 8 and higher. On the right-hand side of Eqs. (2.5) and (2.6) the operators are to be understood with some continuum regulator and renormalization scheme. Discretization effects are lumped into the short-distance coefficients $C_{\text{lat}}$, $\delta C_{\text{lat}}$, and $B_{i\text{lat}}$, which depend on the couplings of the lattice action, as well as the lattice spacing $a$ and the (renormalized) gauge coupling and quark masses.

The next step is to note that the target operators have a completely parallel description in HQET, namely
\begin{align}
\bar{q}_L \gamma^\mu b \bar{q}_L \gamma^\nu b &\doteq 2C \bar{q}_L \gamma^\mu (q^+ h^-) \bar{q}_L \gamma^\nu h^- + 2\delta C \bar{q}_L h^+(q^-) \bar{q}_L h^- + \sum_{i=1}^5 B_i \mathcal{O}_i + \cdots, \\
\bar{q}_L b \bar{q}_L b &\doteq 2\delta C \bar{q}_L \gamma^\nu h^+(q^-) \bar{q}_L h^- + 2C \bar{q}_L h^+(q^-) \bar{q}_L h^- + \sum_{i=1}^5 B_{i\text{lat}} \mathcal{O}_i + \cdots,
\end{align}
where the (continuum HQET) operators on the right-hand sides of Eqs. (2.7) and (2.8) are precisely the same as those on the right-hand sides of Eqs. (2.5) and (2.6). The coefficients differ, however, because the lattice does not appear on the left-hand side of Eqs. (2.7) and (2.8).

With Eqs. (2.5)-(2.8) the desired construction of $Q$ and $Q_S$ is immediate:
\begin{align}
Q &= Z \bar{Y}_L \gamma^\mu \gamma^\nu \Psi + \delta Z \bar{Y}_L \gamma^\nu \Psi + \sum_i b_i Q_i, \\
Q_S &= Z_S \bar{Y}_L \gamma^\mu \gamma^\nu \Psi + \sum_i b_{i\text{lat}} Q_i,
\end{align}
where the $Q_i$ are lattice discretizations of the $\mathcal{O}_i$, such that $Q_i \doteq C_{ij} \mathcal{O}_j + \text{dimension 8}$. Simple algebra then shows that if
\begin{align}
Z &= [CC_{\text{lat}} - \delta C \delta C_{\text{lat}}] / [C_{\text{lat}} C_{\text{lat}} - \delta C_{\text{lat}} \delta C_{\text{lat}}], \\
\delta Z &= [\delta C - Z \delta C_{\text{lat}}] / C_{\text{lat}}, \\
b_i &= [B_j - Z B_j] \delta Z B_{i\text{lat}} C_{\text{lat}}^{-1} j_i,
\end{align}
3. Long-Distance Matrix Elements

To compute the matrix elements we use a data-object called the open-meson propagator \cite{10}. Valence quark propagators are started at an origin \((x_0, t_0)\), where the 4-quark operator sits, out to all \((x, t)\). Since, for this problem, we are interested only in zero-momentum pseudoscalars, at each \(t\) the Dirac indices are contracted with \(\gamma_5\), and this contraction is summed over all \(x\). On the other hand, \(M_{12}\) and \(\Gamma_{12}\) require two (several) Dirac structures in (beyond) the Standard Model. Therefore we leave the Dirac and color indices free at \((x_0, t_0)\), writing out one \(12 \times 12 \times N_4\) data-object per configuration, where \(N_4\) is the total number of time slices. Three-point functions are formed by contracting open-meson propagators at times \(t_i\) and \(t_f\) with the Dirac structure of each 4-quark operator. Two-point functions from \(t_0\) to \(t\) are used to normalize the matrix elements and to provide a cross-check with our separate calculations of \(B\)-meson decay constants \cite{3,4}.
Our calculations are carried out on several ensembles of lattice gauge fields with a realistic sea of 2+1 flavors, made available by the MILC Collaboration [12, 13]. The ensembles used here are listed in Table 1 together with the valence quark masses. The sea quarks are simulated with the asqtad action for staggered quarks, and with the fourth-root procedure to reduce the number of species from 4 to 1.

To discuss the analysis, it is helpful to introduce some notation. The four-quark matrix elements are written

\[ \langle B_q^0 | \bar{\Psi} L \gamma_\mu \Psi | B_q^0 \rangle = \frac{2}{3} M_{B_q} \beta_q^2, \]

(3.1)

where the quantity \( \beta_q \) is well-behaved in the heavy-quark limit. We extract \( \beta_s \) and \( \beta_d \) from 2- and 3-point functions. With staggered valence quarks these correlators have contributions from wrong-parity states with time dependence \( (-1)^{t/a} \). We are careful to disentangle these states. To isolate the ground state we use Bayesian fits, varying the number of states.

We then carry out a partially-quenched (i.e., \( m_q \) and \( m_l \) varying independently) chiral extrapolation of \( \beta_q/\beta_s \) to obtain \( \beta_d/\beta_s \), using rooted staggered chiral perturbation theory for \( \beta_q \) [14, 15]. With more valence masses than sea masses, the effects of partial quenching constrain the parameters of \( \chi \)PT more stringently than would unitary \( (m_q = m_l) \) data alone. Fitting the ratio \( \beta_d/\beta_s \) yields smaller statistical errors than fitting \( r_1^{3/2} \beta_q \) directly. We also carry out a chiral extrapolation of \( r_1^{3/2} \beta_s \), which is mild, because it depends only on the sea masses \( (am_l, am_h) \).

In the phenomenology of \( B-B \) mixing it is conventional to write the matrix element as

\[ \langle B_q^0 | \bar{q}_L \gamma_\mu b_L e^{i \alpha} | B_q^0 \rangle = \frac{2}{3} f_{B_q}^2 M_{B_q}^2 B_{B_q}. \]

(3.2)

Neglecting \( Z - 1 \) and \( \delta Z \) in Eq. (2.9) one sees that

\[ \beta_q = f_{B_q} \sqrt{M_{B_q} B_{B_q}}. \]

Of special importance is

\[ \xi = f_{B_q} B_{B_q}^{1/2} / f_{B_q} B_{B_q}^{1/2} = (M_{B_q}/M_{B_q})^{1/2} (\beta_s/\beta_d), \]

(3.3)

where, again, the right-most expression neglects \( Z - 1 \) and \( \delta Z \). We use the experimentally measured meson masses and our chirally extrapolated \( \beta_s \) and \( \beta_d/\beta_s \) to obtain \( f_{B_q} B_{B_q}^{1/2} \) and \( \xi \). The light-quark-mass dependence is shown in Fig. 1. Further plots can be found in Ref. [16].

A preliminary, but comprehensive, error budget is given in Table 2. The \( B^*-B-\pi \) coupling \( g_{B^*B\pi} \) enters the expressions for the chiral extrapolation. The data are not precise enough to determine

| \( a \) (fm) | Lattice | \( N_{\text{conf}} \) | Sea \( (am_l, am_h) \) | Valence \( am_q \) |
|---|---|---|---|---|
| 0.12 | \( 24^3 \times 64 \) | 529 | (0.005, 0.005) | 0.005, 0.007, 0.01, 0.02, 0.03, 0.0415 |
| “coarse” | \( 20^3 \times 64 \) | 833 | (0.007, 0.005) | 0.005, 0.007, 0.01, 0.02, 0.03, 0.0415 |
| | \( 20^3 \times 64 \) | 592 | (0.01, 0.05) | 0.005, 0.007, 0.01, 0.02, 0.03, 0.0415 |
| | \( 20^3 \times 64 \) | 460 | (0.02, 0.05) | 0.005, 0.007, 0.01, 0.02, 0.03, 0.0415 |
| 0.09 | \( 28^3 \times 96 \) | 557 | (0.0062, 0.031) | 0.0031, 0.0044, 0.062, 0.0124, 0.0272, 0.031 |
| “fine” | \( 28^3 \times 96 \) | 534 | (0.0124, 0.031) | 0.0031, 0.0042, 0.062, 0.0124, 0.0272, 0.031 |

Table 1: Input parameters for the numerical calculations. The lattice spacings listed are approximate mnemonics. The heavier sea mass \( m_h \) is close to the strange mass, which then is subject to retuning \( a \) posteriori, yielding the last value of \( am_q \) for the coarse ensembles.
$g_{B^*B\pi}$, so it must be set with a prior distribution in the chiral fits. A range that encompasses phenomenological and quenched lattice estimates is $g_{B^*B\pi} = 0.35 \pm 0.14$. The error in Table 2 corresponds to this range, while the prior width in the fits is $\pm 0.28$.

Until the perturbation theory has been checked, we prefer not to report a value for $f_{B_1/2}$. The matching corrections nearly cancel in the ratio $\beta_q/\beta_s$; the results with and without $Z - 1$ and $\delta Z$ are nearly the same, as shown in Fig. 1b. With the error budget discussed above we find

$$\xi = 1.205 \pm 0.037_{\text{stat}} \pm 0.034_{\text{syst}}, \quad (3.4)$$

unchanged since *Lattice 2008* [15].

### 4. Future Prospects

When the perturbative matching has been completely checked, we will be in a position to present final results. We can also compare different strategies, in particular, whether the perturbative expansion seems to work better for $\rho_{(S)}$ or $Z_{(S)}$ (cf. Eq. (2.14)).

In the longer term, we plan to obtain results for 4-quark operators that enter beyond the Standard Model. Furthermore, the MILC ensembles now not only have much higher statistics than the...
current project at $a = 0.12$ and 0.09 fm, but also extend to smaller lattice spacings, $a = 0.06$ and 0.045 fm. New runs with higher statistics and five lattice spacings (also 0.15 fm) are underway.

References

[1] A. Lenz and U. Nierste, Theoretical update of $B_s$-$\bar{B}_s$ mixing, *JHEP* **0706** (2007) 072 [arXiv:hep-ph/0612167].

[2] M. Bona et al. [UTfit Collaboration], First evidence of new physics in $b \leftrightarrow s$ transitions, arXiv:0803.0659 [hep-ph].

[3] A. X. El-Khadra, A. S. Kronfeld, and P. B. Mackenzie, Massive fermions in lattice gauge theory, *Phys. Rev. D* **55** (1997) 3933 [arXiv:hep-lat/9604004].

[4] M. B. Oktay and A. S. Kronfeld, New lattice action for heavy quarks, *Phys. Rev. D* **78** (2008) 014504 [arXiv:0803.0523 [hep-lat]].

[5] A. S. Kronfeld, Application of heavy-quark effective theory to lattice QCD I: power corrections, *Phys. Rev. D* **62** (2000) 014505 [arXiv:hep-lat/0002008].

[6] J. Harada, S. Hashimoto, K. I. Ishikawa, A. S. Kronfeld, T. Onogi, and N. Yamada, Application of heavy-quark effective theory to lattice QCD II: radiative corrections to heavy-light currents, *Phys. Rev. D* **65** (2002) 094513 [arXiv:hep-lat/0112044]; Erratum *ibid.* **71** (2005) 019903.

[7] J. Harada, S. Hashimoto, A. S. Kronfeld, and T. Onogi, Application of heavy-quark effective theory to lattice QCD III: radiative corrections to heavy-heavy currents, *Phys. Rev. D* **65** (2002) 094514 [arXiv:hep-lat/0112045].

[8] H. W. Lin and N. Christ, Non-perturbatively determined relativistic heavy quark action, *Phys. Rev. D* **76** (2007) 074506 [arXiv:hep-lat/0608005].

[9] J. A. Bailey et al. [Fermilab Lattice and MILC Collaborations], The $B \to \pi \ell \nu$ semileptonic form factor from three-flavor lattice QCD: a model-independent determination of $|V_{ub}|$, *Phys. Rev. D* **79** (2009) 054507 [arXiv:0811.3640 [hep-lat]].

[10] R. T. Evans, A. X. El-Khadra, and M. Di Pierro [Fermilab Lattice and MILC Collaborations], A study of the $B_s$-$\bar{B}_s$ mass and width difference in 2+1 flavor lattice QCD, in proceedings of *Lattice 2006*, PoS (LATTICE 2006) 081.

[11] C. Bernard et al. [Fermilab Lattice and MILC Collaborations], $B$ and $D$ Meson Decay Constants, in proceedings of *Lattice 2008*, PoS (LATTICE 2008) 278 [arXiv:0904.1895 [hep-lat]].

[12] C. W. Bernard et al. [MILC Collaboration], The QCD spectrum with three quark flavors, *Phys. Rev. D* **64** (2001) 054506 [arXiv:hep-lat/0104002].

[13] C. Aubin et al. [MILC Collaboration], Light hadrons with improved staggered quarks: approaching the continuum limit, *Phys. Rev. D* **70** (2004) 094505 [arXiv:hep-lat/0402030].

[14] C. Bernard, J. Laiho, and R. S. Van de Water, private communication. The most salient formulae can be found in Ref. [15].

[15] R. T. Evans, A. X. El-Khadra, and E. Gámiz [Fermilab Lattice and MILC Collaborations], A determination of the $B_s^0$ and $B_d^0$ mixing matrix elements in 2+1 lattice QCD, in proceedings of *Lattice 2008*, PoS (LATTICE 2008) 052.

[16] R. T. Evans, E. Gámiz, A. X. El-Khadra, and M. Di Pierro [Fermilab Lattice and MILC Collaborations], A determination of the $B_s^0$ and $B_d^0$ mixing parameters in 2+1 lattice QCD, in proceedings of *Lattice 2007*, PoS (LATTICE 2007) 354 [arXiv:0710.2880 [hep-lat]].