Abstract

This paper investigates the discrete-time per-sample model of the zero-dispersion optical fiber. It is shown that the capacity-achieving input distribution is unique, has (continuous) uniform phase and discrete amplitude with a finite number of mass points. The optimality of this multi-ring input holds when the channel is subject to general input cost constraints that include peak power constraint and a joint average and peak power constraint.

Zero-dispersion optical fiber, channel capacity.

I. INTRODUCTION

Signal propagation in optical fibers can be modeled by the stochastic nonlinear Schrödinger (NLS) equation [1], capturing chromatic dispersion, Kerr nonlinearity, and amplified spontaneous emission (ASE) noise in fiber. Finding the capacity and spectral efficiency of such a channel remains a formidable challenge even in the simpler zero dispersion setup. One chief reason for this is that the Fourier spectrum of a signal propagating in an optical fiber expands because of nonlinearity and that this bandwidth expansion is still unquantified (see, e.g., [2, Section VIII], [3]). Therefore, the problem of determining an explicit discrete-time model of the form \( \Pr(y|x) \), with input and output vectors \( x \) and \( y \), that corresponds to the NLS equation is still open. As a consequence, to date the only non-asymptotic capacity result states that the capacity of the NLS channel is always upper bounded by \( \log(1 + \text{SNR}) \), where SNR is signal-to-noise ratio [1], [4].

In this paper we consider the discrete-time per-sample model of the zero-dispersion optical fiber. This model is obtained by sampling the output signal at the input signal bandwidth rate (i.e., bandwidth expansion is ignored and the output signal is potentially sub-optimally sampled). As a consequence, the channel is now given by \( \Pr(y|x) \), with complex input and output scalars \( x \) and \( y \), and we show that the optimal input distribution is unique, has (continuous) uniform phase and discrete amplitude with finitely many mass points. This result holds whenever the input satisfies a peak power constraint or a joint peak and average power constraint. This proves a conjecture made in [2] and, in principle, helps with the design of optimal input constellations by restricting the search space for optimal inputs. Furthermore, it implies that multi-ring modulation formats, potentially with non-uniform ring spacing and probabilities, are optimal for the zero-dispersion fiber.

Related works

In the zero-dispersion case, the conditional probability density function (PDF) of the channel output \( Y \in \mathbb{C} \) given channel input \( X \in \mathbb{C} \) was derived in [2], [5], [6]. It was also established that the asymptotic capacity of the per-sample discrete-time model as a function of the average input power \( P \) is \( \frac{1}{2} \log P + o(1) \) [2], [6]. When combined with the above general capacity upper bound \( \log(1 + \text{SNR}) \) this implies that nonlinearity reduces capacity by at most \( 1/2 \) degree-of-freedom. Finally, the capacity of the discrete-time model of the dispersive optical fiber was shown to be \( \frac{1}{2n} \log \log(P) + o(1) \) as \( P \to \infty \), where \( n \) is the fixed size of the input vector [7, Theorem 1]. As a result, dispersion and nonlinearity substantially reduce capacity.

Capacity achieving input distributions have been investigated for a variety of channels. In particular, since the work by Smith [8] several authors established discreteness of capacity-achieving input distributions for a variety of combinations of channels and input constraints (see, e.g., [8]–[18]).

In the present work the investigation of the capacity achieving input distribution has two parts. In a first part we prove uniqueness of the optimal (complex) input, optimality of the uniform phase, and independence of the optimal phase and the optimal amplitude. These proofs rest on a symmetry argument coupled with a tight discrete approximation of the channel by a cascade of simpler (invertible) channels. In the second part we prove that the optimal amplitude takes a discrete number of values following Smith’s arguments. Although the proof roadmap here is known, implementing it is far from trivial. Previous work includes linear additive noise channels [11], [13], [17], [18], conditionally Gaussian channels [12], [14]–[16] or both [8], [10]. In contrast, the optical channel is non-linear, non-conditionally Gaussian with non-additive noise (see the PDF (3)). Particularly, phase noise is complex and signal-dependent, depending on the input amplitude. This makes the analysis of the conditional entropy more difficult compared to [8]–[18].

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The rest of the paper is organized as follows. In Section II we recall the discrete-time per-sample model of the zero-dispersion optical fiber channel and present the main result in Section III which characterizes the optimal input. In Section IV we first establish uniqueness of the optimal input and the optimality of the uniform input phase and, second, we establish discreteness of the optimal input amplitude. Section V concludes the paper.

Notation

Upper and lower case letters represent scalar random variables and their realizations, respectively. The cumulative distribution function (CDF) and the probability density function (PDF) of a random variable $X$ are denoted by $F_X(x)$ and $f_X(x)$, respectively. The uniform distribution on the interval $[a, b]$ is denoted by $U(a, b)$. Let $f(x) : \mathbb{R} \mapsto \mathbb{R}$ and $g(x) : \mathbb{R} \mapsto \mathbb{R}$ be functions. We use the standard order notation and write $f(x) = o(g(x))$, or equivalently $g(x) = o(f(x))$, if for any $k > 0$ there exists a $c > 0$ such that $|f(x)| > k|g(x)|$ for all $|x| \geq c$; $f(x) = \Omega(g(x))$ if there exists a $k > 0$ and $c > 0$ such that $|f(x)| > k|g(x)|$ for all $|x| \geq c$; and $f(x) \equiv g(x)$ as $x \to x_0$ if $\lim_{x \to x_0} \frac{f(x)}{g(x)} = 1$.

II. ZERO-DISPERSION OPTICAL FIBER: PER SAMPLE MODEL

We consider a zero-dispersion single-mode fiber with length $L$. In fiber-optic communication systems, signal amplification can be performed in a lumped or distributed manner. In the lumped amplification model, erbium-doped fiber amplifiers (EDFAs) are placed at the end of every fiber span, amplifying the signal and introducing noise. In the ideal distributed Raman amplification (DRA), fiber loss is perfectly compensated while noise is added continuously along the fiber. In this paper, we consider an ideal DRA model.

A. The Continuous-time Model

The propagation of the complex envelope of a signal in one polarization of the single-mode zero-dispersion fiber with ideal DRA is described by the ordinary differential equation [2, Eq. 1]

$$j\frac{dQ(t, z)}{dz} = \gamma Q(t, z)|Q(t, z)|^2 + N(t, z), \ 0 \leq z \leq L$$

where $Q(t, z)$ is the signal as a function of time $t$ and distance $z$, $\gamma$ is the nonlinearity coefficient and $N(t, z)$ is a circularly symmetric complex Gaussian noise process satisfying

$$\mathbb{E}(N(t, z)N^*(t', z')) = \sigma^2_0 \delta_{WW}(t - t')\delta(z - z'),$$

where $\delta_{WW}(x) \overset{d}{=} W \text{sinc}(Wx)$, $\text{sinc}(x) \overset{d}{=} \sin(\pi x)/(\pi x)$, $W$ is channel bandwidth, $\delta(x)$ is the Dirac delta function and $\sigma^2_0$ is the noise in-band power spectral density. Equation (1) models Kerr nonlinearity, ASE noise and their interaction in fiber.

B. A Discrete-time Model

Differential equation (1) defines a channel, parametrized by the time $t$, from input $Q(t, 0)$ to random output $Q(t, L)$. Although the signal duration remains constant during propagation, the signal bandwidth might change due to the nonlinearity. Typically, $W(L) \geq W(0)$ where $W(z)$ is the signal bandwidth at distance $z$.

In this paper we consider a potentially sub-optimal discretization where the output signal is sampled at the input rate $1/W(0)$. The resulting discrete-time model is described by a set of identical and independent scalar channels (called per-sample channels) given by

$$Q(L) = |Q(0) + W(L)|e^{j\gamma \int_0^L |Q(t)|^2 dt},$$

(2)

where $Q(0)$, $Q(L) \in \mathbb{C}$ denote the channel input and the corresponding output, respectively, and where $W(\ell)$ is a complex Wiener process with $\mathbb{E}(W(z)W^*(z')) = \sigma^2 \min(z, z')$, and where $\sigma^2 = \sigma^2_0 W(0)$. Note that, because of the potentially sub-optimal discretization, the capacity of (2), measured in bits/2D, may not be equal to the spectral efficiency of (1), measured in bits/s/Hz [2, Section VIII].

C. Per-Sample Conditional PDF

Let $(R_0, \Phi_0)$ and $(R, \Phi)$ denote the polar coordinates of the channel input $Q(0)$ and output $Q(L)$, respectively, in (2). The conditional PDF of the output given the input is derived in [2, Section III. A, Eq. 18], [5], [6]:

$$p_{R, \Phi|R_0, \Phi_0}(r, \phi|r_0, \phi_0) = \frac{1}{2\pi} p_{R|R_0}(r|r_0)$$

$$+ \frac{1}{\pi} \sum_{m=1}^{+\infty} \mathcal{R}(C_m(r, r_0)e^{-j m(\phi - \phi_0 - \gamma t_0^2 L)}),$$

(3)

where $p_{R|R_0}(r|r_0)$ is the conditional probability density function of $r$ given $r_0$, $\gamma$ is the nonlinearity coefficient, $C_m(r, r_0)$ are the complex-valued Hermite polynomials, and $\mathcal{R}$ is the real part.
where
\[ p_{R_0(r|r_0) \equiv \frac{2r}{\sigma^2 L} e^{-\frac{r^2 + r_0^2}{\sigma^2 L}} I_0 \left( \frac{2rr_0}{\sigma^2 L} \right), \] (4)
and
\[ C_m(r, r_0 \equiv rb_m e^{-a_m (r^2 + r_0^2)} I_m (2b_m r_0 r), \] (5)
with
\[ a_m \equiv \frac{\sqrt{jm \gamma}}{\sigma} \coth \left( \sqrt{jm \gamma} \sigma^2 L \right), \] (6)
\[ b_m \equiv \frac{\sqrt{jm \gamma}}{\sigma} \sinh \left( \sqrt{jm \gamma} \sigma^2 L \right). \] (7)

III. MAIN RESULTS

We consider the per-sample channel model (3) when the input amplitude \( r_0 \) is subject to a peak constraint, an average cost constraint, or both. The input is subject to a peak constraint if its amplitude distribution belongs to the set
\[ \mathcal{P} = \{ F_{R_0(r_0) : \int_0^\rho dF_{R_0(r_0)} = 1} \} \] (8)
for some non-negative constant \( \rho < \infty \). The input is subject to an average cost constraint if its amplitude distribution belongs to the set
\[ \mathcal{A} = \{ F_{R_0(r_0) : \int_0^{+\infty} C(r_0) dF_{R_0(r_0)} \leq A} \} \] (9)
for some \( 0 < A < \infty \) and cost function \( C(r_0) \) satisfying:
C1 (Continuity): \( C(r_0) \) is lower semi-continuous, non-decreasing, \( C(0) = 0 \) and \( \lim_{r_0 \to +\infty} C(r_0) = +\infty \);
C2 (Analyticity): There exists a horizontal strip in the complex plane
\[ S = \{ z \in \mathbb{C} : \Re(z) > 0, |\Im(z)| < \delta, \delta > 0 \} \]
for some \( \delta > 0 \) such that \( C(r_0) \) can be analytically extended from \( (0, \infty) \) to \( S \).

Example 1 The moment cost function \( C(r_0) = r_0^q \) satisfies C1-C2 for \( q > 0 \).

The main result of the paper is the following theorem.

**Theorem 1** Suppose channel (3) is subject to one of the following input constraints:
i) a peak constraint,
ii) an average cost constraint satisfying \( C(r_0) = \omega(r_0^2) \),
iii) a joint peak and average cost constraint.
Then channel capacity is achieved with a unique input distribution \( F_{R_0^*, \Phi^*(r_0, \phi_0)} \). This distribution has a continuous uniform phase \( \Phi_0^* \) and an amplitude \( R_0^* \) that is independent of \( \Phi_0^* \) and that takes a finite number of values.

It follows from Theorem 1 that a peak power constraint is sufficient to guarantee the discreteness of the optimal amplitude distribution. If there is no peak constraint, then the optimal amplitude distribution is discrete if the input is subject to a cost constraint with a cost function that grows faster than \( r_0^2 \). Discreteness of the input amplitude with only a second moment cost function \( C(r_0) = r_0^2 \) remains an open problem.

Remark 1 Theorem 1 part ii) extends to the case where the input is subject to several average cost constraints with \( C_i(r_0) \), \( 1 \leq i \leq M < \infty \), satisfying C1-C2, such that \( C_i(r_0) = \omega(r_0^2) \) for at least one constraint. An example is a joint second and a fourth moment constraint (as in the non-coherent Rician fading channel considered in [15]).
IV. PROOF OF THEOREM 1

Let $T_0 = (R_0, \Phi_0)$ and $T_L = T = (R, \Phi)$. We use the symbol $F$ to represent any of the three input spaces considered in Theorem 1. The channel capacity is then given by

$$C = \sup_{F \in F} I(T_0, T), \quad (10)$$

where $I(T_0, T)$ denotes the mutual information between $T_0$ and $T$.

**Lemma 1** The supremum in (10) is achieved and is finite.

**Proof:** We proceed with the optimization in the weak topology (see for example [8], [12]). It is shown in [8] and [19] that the sets $P$ and $A$ are convex and compact. Consequently $F$ is convex and compact. Now we prove that $I(F_{T_0})$ is continuous in $F_{T_0} \in F$. The output entropy is known to be continuous for channels with additive noise subject to an input peak or average cost constraint [13], [19]. The same proof can be readily extended to $h(T)$. In Appendix I it is shown that the conditional entropy $h(T|T_0)$ is also a continuous function of the input distribution. Therefore $I(F_{T_0})$ is continuous over $F$ which implies that it is bounded and that the supremum in (10) is achieved. \[\blacksquare\]

A. Uniqueness of the optimal input

We prove next the uniqueness of the capacity-achieving input by showing that channel 2 is injective, that is, to different inputs $Q(0)$ correspond different output distributions $Q(L)$.

**Lemma 2** Under the input constraints considered in Theorem 1 the capacity-achieving input of channel 2 is unique.

**Proof:** We start by showing concavity of $I(T_0, T) = h(T) - h(T|T_0)$ in $F_{T_0}$. Indeed, $h(T|T_0)$ is linear in $F_{T_0}$ and $h(T)$ is concave in $F_{T_0}$ since $p_T(r, \phi) = \int p_T(T, r, \phi) dF_{T_0}(r, \phi)$ is linear in $F_{T_0}$ and $h(T)$ is strictly concave in $p_T(r, \phi)$. Strict concavity of $I(T_0, T)$ in $F_{T_0}$ and hence uniqueness of the optimal input follows from the fact that $p_T(r, \phi)$ is injective in $F_{T_0}$ which we show next.

From a direct inspection of equation (3) it is unclear whether the channel is injective. To establish injectivity we show that channel 2 can be seen as the limiting case of a cascade of identical channels

$$Q_0 = X_1 \rightarrow Y_1 = X_2 \rightarrow Y_2 = X_3 \rightarrow Y_3 \ldots X_k \rightarrow Y_k$$

that are all injective. Given integer $i \geq 1$ the channels are given by

$$Y_i = [X_i + N_i]e^{j\varepsilon}$$

where $X_i$ and $Y_i$ denote the input and the output of the $i$th channel, where

$$\varepsilon = L/k,$$

and where $N_i$ is a circularly symmetric Gaussian random variable with variance $\varepsilon$. Hence,

$$Y_1 = [X_1 + N_1]e^{j\varepsilon}$$

and

$$Y_2 = [X_2 + N_2]e^{j\varepsilon}$$

$$= [X_1 + N_1]e^{j\varepsilon} + N_2$$

$$= [Y_1 + N_1]e^{j\varepsilon} + N_2$$

$$d \equiv [X_1 + N_1]e^{j\varepsilon} + N_2$$

$$= [Y_1 + N_1 + N_2]e^{j\varepsilon}$$

Iterating we get

$$Y_k = [Q(0) + \sum_{i=1}^{k} N_i]e^{j\varepsilon} = Q(L)$$

and therefore, uniformly over $Q(0)$, we have

$$Y_k \xrightarrow{d} Q(L)$$
as $\varepsilon \to 0$ (or, equivalently, as $k \to \infty$) where $Q(L)$ is defined in (2). It then follows that the output $Y_k$ of the cascade of channels converges to $Q(L)$ uniformly over $Q(0)$. Finally, to show that channels $X_i \to Y_i$ are concave and weakly (Gateaux) differentiable functions of $\Phi$, it suffices to observe

$$X_i = Y_i e^{-j\varepsilon |Y_i|^2} - N_i.$$  

\[ \square \]

B. Optimal phase: uniformity and independence of optimal amplitude

**Lemma 3** The following properties hold true for the zero-dispersion channel (2):

1. $R$ is independent of $\Phi_0$.

2. Under the input constraints considered in Theorem 1 any capacity-achieving input has uniform phase $\Phi_0^* \sim U(0, 2\pi)$ independent of $R_0^*$, i.e.,

$$dF(R_0^*, \Phi_0^*) (r_0, \phi_0) = \frac{1}{2\pi} d\phi_0 dF_R^r (r_0).$$

(11)

The corresponding (unique) capacity achieving output has a uniform phase that is independent of the amplitude.

**Proof:**

1) From [2, Eq. 30]

$$R = |R_0 e^{j\theta_0} + W(z)|$$

$$= |R_0 + W(z)|$$

$$\overset{\text{d}}{=} |R_0 + W(z)|,$$

(12)

where $W(z) = \int_0^z N(l) dl$ is a standard Wiener process. For the last equation we used the fact that $W(z) = W(z) e^{-j\theta_0}$ and $W(z)$ are identically distributed by the circularly symmetry property. Hence $R$ does not depend on $\Phi_0$.

2) From (2) and by the circularly symmetry of the complex Wiener process, if output $Q(z)$ corresponds to input $Q(0)$ then output $e^{j\theta} Q(L)$ corresponds to input $e^{j\theta} Q(0)$, for any fixed $\theta \in [0, 2\pi)$. Therefore, the mutual information of the non-dispersive fiber channel is invariant under an input rotation. Hence, if $Q^*(0)$ is capacity-achieving, so is $e^{j\theta} Q^*(0)$. By Lemma 2 the capacity achieving input distribution is unique

$$Q^*(0) \overset{\text{d}}{=} e^{j\theta} Q^*(0)$$

(13)

for any $\theta \in [0, 2\pi)$ which implies that $Q^*(0)$ has a uniform phase $\Phi_0^* \sim U(0, 2\pi)$ that is independent of its norm. Equation (12) together with (2) implies that $Q(L)$ and $e^{j\theta} Q(L)$ have the same distribution irrespectively of $\theta$. Hence the capacity achieving output has a uniform phase that is independent of the amplitude.

From Lemma 1 and Lemma 3 we get:

**Corollary 1** Capacity expression (10) simplifies to

$$C = \max_{F_{R_0}} I \left( F_{R_0} \right),$$

(14)

where

$$I \left( F_{R_0} \right) \overset{\text{def}}{=} I \left( R_0, \Phi_0^* ; R, \Phi \right)$$

$$= h(R, \Phi) - h(R, \Phi | R_0, \Phi_0^*)$$

$$= h(R) + \ln(2\pi) - h(R, \Phi | R_0, \Phi_0^*),$$

with $\Phi_0^* \sim U(0, 2\pi)$.

C. Discreteness of optimal amplitude distribution

The proof of the second part of Theorem 1 pertaining to the discreteness of optimal amplitude distribution is based on the following well-known argument pioneered by Smith in 1971 [8]):

- State the Karush-Kuhn-Tucker (KKT) equations for capacity-achieving input distributions;
- Show, through a Complex Analysis argument, that the KKT conditions imply that the optimal input has a discrete number of mass points.

**Lemma 4** Whenever $F_{R_0} \in F$, functions $I \left( F_{R_0} \right)$ (defined in Corollary 1) and $g \left( F_{R_0} \right) = \int C(r_0) dF_{R_0}$ (where $C(r_0)$ is defined at the beginning of Section 7) are concave and weakly (Gateaux) differentiable functions of $F_{R_0}$.

**Proof:** The concavity of $I \left( F_{R_0} \right)$ in $F_{R_0}$ follows from the fact that $h(R)$ is concave and $h(R, \Phi | R_0, \Phi_0^*)$ is linear in $F_{R_0}$. As for $g_j \left( F_{R_0} \right)$, it is linear in $F_{R_0}$.
The weak-differentiability of the output entropy of certain additive noise channels was established for an amplitude constraint in [13] and for generic cost constraints in [18]. A similar approach can be used to prove the weak differentiability of \( h(R) \) in \( F_{\Phi_0} \) for \( F_{\Phi_0} \in \mathcal{F} \). The weak differentiability of \( h(R, \Phi|\Phi_0) \) follows from its linearity.

We next state the KKT necessary and sufficient conditions for \( F_{\Phi_0}^* = F_0^* \) to be optimal for the optimization problem (14). The reader is referred to [12, Appendix II] for the derivation of the KKT conditions.

1) KKT conditions for a peak power constraint: An input amplitude random variable \( R_0^* \) with CDF \( F_0^* \) in the set (3) achieves the capacity \( C \) of channel (3) if and only if

\[
\text{LHS}_{\rho}(r_0) \overset{\text{def}}{=} C - \ln(2\pi) + \int_0^{+\infty} p(r|\rho) \ln p(r; F_0^*) \, dr + \frac{1}{2\pi} \int_0^{2\pi} h(R, \Phi|\rho, \phi_0) \, d\phi_0 \geq 0, \tag{15}
\]

for all \( 0 \leq r_0 \leq \rho \), with equality if \( r_0 \) is a point of increase of \( F_0^* \).

2) KKT conditions for an average cost constraint: An input amplitude random variable \( R_0^* \) with CDF \( F_0^* \) in the set (9) achieves the capacity \( C \) of channel (3) if and only if there exists \( \nu \geq 0 \) such that

\[
\text{LHS}_A(r_0) \overset{\text{def}}{=} C - \ln(2\pi) + \int_0^{+\infty} p(r|\rho) \ln p(r; F_0^*) \, dr + \nu(C(r_0) - A) + \frac{1}{2\pi} \int_0^{2\pi} h(R, \Phi|\rho, \phi_0) \, d\phi_0 \geq 0, \tag{16}
\]

for all \( r_0 \geq 0 \), with equality if \( r_0 \) is a point of increase of \( F_0^* \). Under a joint peak and average cost constraint, the KKT condition is similar to (16) for all \( 0 \leq r_0 \leq \rho \).

D. Contradictions

The KKT conditions \( \text{LHS}_{\rho}(r_0) \) and \( \text{LHS}_A(r_0) \) in (15)-(16) are analytically extended in \( r_0 \) from the positive real line to an open connected region \( \mathcal{H} \) in the complex plane in Appendix III. We proceed with the proof of the discreteness of the optimal input amplitude for each of the three cases in Theorem [1].

1) Peak power constraint: Suppose that \( F_0^* \) has an infinite number of points of increase in the interval \((0, \rho]\) and consider the analytic extension of \( \text{LHS}_{\rho}(r_0) \) to the complex plane. Since \( \text{LHS}_{\rho}(z) \) is holomorphic on \( \mathcal{H} \) and has infinitely many zeros on the bounded interval \((0, \rho]\) in \( \mathcal{H} \), it is identically zero on \( \mathcal{H} \) by the Complex Analysis Identity Theorem (see, e.g., [20]). Therefore the KKT conditions (15) are satisfied with equality for all \( r_0 > 0 \). Let \( 0 < \epsilon < 1 \). Using the lower bound on \( \text{LHS}_{\rho}(r_0) \) proved in Lemma 7 in Appendix III, there exists \( K > 0 \) such that

\[
0 = \text{LHS}_{\rho}(r_0) \leq C + \ln \left( \frac{1}{K} \right) + \frac{r_0^2}{\sigma^2} \ln(1 - \epsilon r_0) + (\rho - (1 - \epsilon) r_0) \sqrt{\frac{\pi}{\sigma^2}} L_{1/2} \left( \frac{-r_0^2}{\sigma^2} \right), \tag{17}
\]

where \( L_{1/2}(\cdot) \) represents a Laguerre polynomial and \( \xi(r_0) \to 0 \) as \( r \to +\infty \). Since \( L_{1/2}(x) \equiv \frac{|x|^{1/2} e^{-x/2}}{\sqrt{\pi} 1_{x > 0}} \) as \( x \to -\infty \) [21], we have

\[
\lim_{r_0 \to +\infty} \frac{r_0 \sqrt{\frac{\pi}{\sigma^2}} L_{1/2} \left( \frac{-r_0^2}{\sigma^2} \right)}{r_0^2} = \frac{2}{\sigma^2}. \tag{18}
\]

Dividing by \( r_0^2 \) and taking the limit as \( r_0 \to +\infty \), we obtain

\[
0 \leq \frac{1 - 2(1 - \epsilon)}{\sigma^2}, \quad \forall \epsilon \in (0, 1), \tag{19}
\]

where we used the fact that \( \lim_{r_0 \to +\infty} \xi(r_0) = 0 \). Since (19) holds for any \( 0 < \epsilon < 1 \), choosing \( \epsilon = \frac{1}{2} \) yields a contradiction. It follows that \( F_0^* \) is discrete with a finite number of mass points in \((0, \rho]\) and hence in \([0, \rho]\).

2) Average cost constraints: Consider the analytic extension \( \text{LHS}_A(z) \) of \( \text{LHS}_A(r_0) \) to the complex plane and suppose that \( F_0^* \) has an infinite number of points of increase in a bounded interval of \( \mathbb{R}^+ \setminus \{0\} \). Since \( \text{LHS}_A(z) \) is holomorphic on \( \mathcal{H} \) and has infinitely many zeros in a bounded interval in \( \mathcal{H} \), then it is identically zero on \( \mathcal{H} \) by the Identity Theorem. As a result, the KKT conditions are satisfied with equality for all \( r_0 > 0 \) and we have

\[
0 = \text{LHS}_A(r_0) > \nu(C(r_0) - A) + C + \ln \left( \frac{k_1}{2\pi k_0} \right) + \frac{1}{2\pi} \sqrt{\frac{\pi}{\sigma^2}} L_{1/2} \left( \frac{-r_0^2}{\sigma^2} \right), \tag{20}
\]

where the inequality in equation (20) is due to equation (54). Dividing equation (20) by \( r_0^2 \) and taking the limit as \( r_0 \to +\infty \) gives

\[
\nu \lim_{r_0 \to +\infty} \frac{C(r_0)}{r_0^2} \leq -\frac{1}{\sigma^2} < 0, \tag{21}
\]

where we used equation (13) in order to write equation (21). Equation (21) is impossible for \( C(r_0) = 0 \) unless \( \nu = 0 \) which can be formally ruled out (see [18, Lemma 5] for example).

Now, suppose that \( F_0^* \) have an infinite number of points of increase with only a finite number of them on a bounded interval of \( \mathbb{R}^+ \setminus \{0\} \). Then, the points of increase tend to infinity. We establish in Lemma 8 in Appendix III a lower bound on \( \text{LHS}_A(r_0) \).
which diverges to $+\infty$ as $r_0 \to +\infty$. This implies that $\text{LHS}_A(r_0) \neq 0$ at large values of $r_0$ which contradicts the possibility of having arbitrarily large mass points. It follows that $R^*_0$ is discrete with a finite number of mass points in $\mathbb{R}^+ \setminus \{0\}$ and hence in $\mathbb{R}^+$.

The above proof immediately generalize to the case with multiple input cost constraints satisfying properties C1 and C2 and where at least one of them is $\omega(r_0^2)$.

3) A joint average cost and peak power constraints: Assuming that $F^*_0$ has an infinite number of points of increase in $(0, \rho]$, equation (16) implies that

$$\text{LHS}_A(r_0) = 0, \quad \forall r_0 > 0.$$  

Thus (21) holds true and

$$\nu \lim_{r_0 \to +\infty} C(r_0) - A < \frac{1}{\sigma^2 L}.$$  

On the other hand, since the support of $F^*_0$ is restricted to $[0, \rho]$, we use the established upper bound in Lemma 7 by which equation (16) implies

$$\nu (C(r_0) - A) + C + \ln \left( \frac{1}{K} \right) + \frac{r_0^2}{\sigma^2 L} - \ln (1 - \xi(r_0)) + (\rho - (1 - \epsilon)r_0) \sqrt{\frac{\pi}{\sigma^2 L} 1 - \frac{1}{2} \left( - \frac{r_0^2}{\sigma^2 L} \right)} \geq 0.$$  

Dividing by $r_0 > 0$ and letting $r_0 \to +\infty$ gives

$$\nu \lim_{r_0 \to +\infty} \frac{C(r_0)}{r_0^2} \geq \frac{2(1 - \epsilon) - 1}{\epsilon \sigma^2 L}, \quad \forall \epsilon \in (0, 1).$$  

Taking the limit $\epsilon \to 0$

$$\nu \lim_{r_0 \to +\infty} \frac{C(r_0)}{r_0^2} \geq \frac{1}{\sigma^2 L}.$$  

Equations (23) and (24) establish a contradiction. It follows that $F^*_0$ is discrete with a finite number of points of increase in $[0, \rho]$.

V. CONCLUSIONS

How to efficiently communicate over an optical fiber is a question whose answer has important practical consequences. In this paper we studied the capacity-achieving input distribution in the per-sample discrete-time model of the zero-dispersion optical fiber. We showed that, for a variety of practical input constraints such as a peak-power constraint, the capacity-achieving input is unique, has a uniform phase distribution that is independent of the amplitude, and the distribution of the amplitude is discrete with a finite number of mass points. In other words, multi-ring modulation formats commonly used in optical communication can achieve channel capacity—potentially with non-uniform ring spacing and probabilities. Whether such constellations are optimal for the dispersive optical fiber is an interesting and challenging open problem.

APPENDIX I

CONTINUITY OF THE CONDITIONAL ENTROPY

We establish the weak continuity of $h(T|T_0)$ in $F_{T_0} \in \mathcal{F}$. To this end, let $F_0$ be a distribution function of $T_0$ and let $\{F_m\} \subset \mathcal{F}$, $m \geq 1$, be a sequence of distribution functions converging weakly to $F_0$. Also, let

$$i(r, \phi, r_0, \phi_0) = -p(r, \phi|r_0, \phi_0) \ln p(r, \phi|r_0, \phi_0),$$

where $p(r, \phi|r_0, \phi_0) \overset{\text{def}}{=} p_{R, \Phi|R_0, \Phi_0}(r, \phi|r_0, \phi_0)$ is given by equation (3). We note that $h(T|T_0)$ is finite for any $F_{T_0} \in \mathcal{F}$. The continuity of $h(T|T_0)$ in $F_0 \in \mathcal{F}$ is guaranteed by the following set of equations:

$$h(T|T_0) = \int h(T|r_0, \phi_0) dF_0(r_0, \phi_0)$$

$$= \lim_{r, \phi, m \to +\infty} \int i(r, \phi, r_0, \phi_0) dF_m(r_0, \phi_0) dr d\phi$$

$$= \lim_{m \to +\infty} \int i(r, \phi, r_0, \phi_0) dF_m(r_0, \phi_0) dr d\phi$$

$$= \lim_{m \to +\infty} \int i(r, \phi, r_0, \phi_0) dr d\phi dF_m(r_0, \phi_0)$$

$$= \lim_{m \to +\infty} h(T|T_m).$$
The interchange in the order of integration in equations (25) and (28) is justified by Fubini’s theorem. This is due to the fact that the integrals in these equations have finite values and that \( p(r, \phi | r_0, \phi_0) \) is bounded (see Lemma 5). Equation (26) is due to the definition of weak convergence by virtue of the fact that \(-p(r, \phi | r_0, \phi_0) \ln p(r, \phi | r_0, \phi_0)\) is continuous and bounded in \((r_0, \phi_0)\) where the boundedness follows from Lemma 5 below. Finally, equation (27) holds by the Dominated Convergence Theorem. In fact, we find next a function \( d(r, \phi) \) such that:

\[
\left| \int_{r, \phi} i(r, \phi, r_0, \phi_0) \, dF_m(r_0, \phi_0) \right| \leq d(r, \phi),
\]

and

\[
\int_{r, \phi} d(r, \phi) \, dr \, d\phi < \infty.
\]

To this end we start by writing

\[
\left| \int_{r, \phi} i(r, \phi, r_0, \phi_0) \, dF_m(r_0, \phi_0) \right| \leq I^+(r, \phi) + I^-(r, \phi),
\]

where

\[
I^+(r, \phi) \equiv \int_{p(r, \phi | r_0, \phi_0) \leq 1} i(r, \phi, r_0, \phi_0) \, dF_m(r_0, \phi_0)
\]

\[
I^-(r, \phi) \equiv - \int_{p(r, \phi | r_0, \phi_0) > 1} i(r, \phi, r_0, \phi_0) \, dF_m(r_0, \phi_0).
\]

We consider first the case when \( F_m(r_0, \phi_0) \) is subjected to a peak power constraint. Using the lower bound proved in Lemma 5 there exists a \( c > 0 \) for which \( \xi(c) < \frac{1}{2} \) and such that for \( 0 \leq r_0 \leq \rho \) we have

\[
\ln \frac{1}{p(r, \phi | r_0, \phi_0)} \leq \ln \frac{1}{2 \pi p_{R|R}(r | c) \left(1 - \xi(c) \right)} \leq \ln \frac{4 \pi}{p_{R|R}(r | c)}.
\]

This is always possible since \( p(r, \phi | r_0, \phi_0) \to 0 \) as \( r_0 \to +\infty \) by virtue of equation (3). Therefore,

\[
I^+(r, \phi) \leq \int_{0 < p(r, \phi | r_0, \phi_0) \leq 1} p(r, \phi | r_0, \phi_0) \ln \frac{4 \pi}{p_{R|R}(r | c)} \, dF_m(r_0, \phi_0)
\]

\[
\leq (\ln(4 \pi) - \ln p_{R|R}(r | c)) p(r, \phi)
\]

\[
\leq \left( \ln(4 \pi) - \ln \frac{2r}{\sigma^2 L} + \frac{r^2 + c^2}{\sigma^2 L} \right) p(r, \phi),
\]

(29)

where we used the fact that

\[
p_{R|R}(r | c) \geq \frac{2r}{\sigma^2 L} e^{-\frac{r^2 + c^2}{2 \sigma^2 L}}.
\]

Since the output of the channel given by equation (3) has a finite second moment whenever \( F \in \mathcal{F} \) by Lemma 6 below, the Left-Hand Side (LHS) of inequality (29) is integrable. As for \( I^-(r, \phi) \) we have

\[
I^-(r, \phi) \leq \int_{p(r, \phi | r_0, \phi_0) > 1} p(r, \phi | r_0, \phi_0) \ln k_u p_{R|R}(r | r_0) \, dF_m(r_0, \phi_0)
\]

(30)

\[
\leq \left( \ln k_u + \ln \frac{2r}{\sigma^2 L} \right) p(r, \phi),
\]

(31)

where we used Lemma 5 for inequality (30). Inequality (31) holds true since \( p_{R|R} \leq \frac{2r}{\sigma^2 L} \) and is integrable. This completes the justification of equation (27) for the case of a peak power constraint and \( h(T | T_0) \) is continuous in \( F_0 \). Now, consider \( F_m(r_0, \phi_0) \) under an average cost constraint. The integrable bound on \( I^-(r, \phi) \) given by (31) holds exactly the same in this case. It remains to find such one for \( I^+(r, \phi) \). To this end we let \( B > 0 \) be large enough and we write:

\[
I^+(r, \phi) \leq \int_{0 < p(r, \phi | r_0, \phi_0) \leq 1, r_0 \leq B} i(r, \phi, r_0, \phi_0) \, dF_m(r_0, \phi_0) + \int_{0 < p(r, \phi | r_0, \phi_0) \leq 1, r_0 > B} i(r, \phi, r_0, \phi_0) \, dF_m(r_0, \phi_0).
\]

(32)
An integrable bound to the first integral in equation (32) is identical to the one given by equation (29). The second integral in equation (32) is upper bounded as follows:

\[ \int_{0 < p(r, \phi | r_0, \phi_0) \leq 1, r_0 > B} i(r, \phi, r_0, \phi_0) \, dF_m(r_0, \phi_0) \]

\[ \leq \int_{0 < p(r, \phi | r_0, \phi_0) \leq 1, r_0 > B} p(r, \phi | r_0, \phi_0) \ln \frac{4\pi}{p_{R | R_0}(r | r_0)} \, dF_m(r_0, \phi_0) \]

\[ \leq \left( \ln(4\pi) - \frac{2r}{\sigma^2 L} + \frac{r^2}{\sigma^2 L} \right) p(r, \phi) + \frac{1}{\sigma^2 L} \int_{0 < p(r, \phi | r_0, \phi_0) \leq 1, r_0 > B} r_0^2 p(r, \phi | r_0, \phi_0) \, dF_m(r_0, \phi_0) \]

\[ \leq \left( \ln(4\pi) - \frac{2r}{\sigma^2 L} + \frac{r^2}{\sigma^2 L} \right) p(r, \phi) + \int r_0^2 p(r, \phi | r_0, \phi_0) \, dF_m(r_0, \phi_0) \] (33)

The first term in equation (33) is clearly integrable. The second term is also integrable by Fubini. In fact

\[ \int_{r, \phi} \int_{r_0} r_0^2 p(r, \phi | r_0, \phi_0) \, dF_m(r_0, \phi_0) \, dr \, d\phi \]

\[ = \int r_0^2 \int_{r, \phi} p(r, \phi | r_0, \phi_0) \, dr \, d\phi \, dF_m(r_0, \phi_0) \]

\[ = \int r_0^2 \, dF_m(r_0, \phi_0) \]

\[ \leq A, \]

where the last inequality is justified by virtue of the second moment constraint. This completes the proof of equation (27) for the case of an average cost constraint. Finally, \( h(T | T_0) \) is weak continuous on \( F \).

**Lemma 5 (Bounds on the Conditional PDF)** The conditional PDF \( p_{R, \phi | R_0, \phi_0}(r, \phi | r_0, \phi_0) \) satisfies the following properties.

1) **Upper bound:**

\[ p_{R, \phi | R_0, \phi_0}(r, \phi | r_0, \phi_0) \leq k_u p_{R | R_0}(r | r_0), \]

where \( k_u \) \( \defeq \frac{1}{2\pi} \left( 1 + \sqrt{2} \sum_{m=1}^{+\infty} \frac{\beta_m}{\sinh(\beta_m)} \right) < \infty \) and \( \beta_m = \sqrt{\frac{2}{\sigma^2 L}}. \)

2) **Lower bound**

\[ p_{R, \phi | R_0, \phi_0}(r, \phi | r_0, \phi_0) \geq \frac{1}{2\pi} p_{R | R_0}(r | r_0) \left( 1 - \xi(r_0) \right), \]

where \( \xi(r_0) \) \( \defeq \sqrt{2} e^{-\left( \mathcal{R}(a_m) - \frac{1}{\sigma^2 L} \right)^2} \sum_{m=1}^{+\infty} \frac{\beta_m}{\sinh(\beta_m)} \to 0 \) as \( r_0 \to +\infty. \)

**Proof:**

**Upper Bound.** Considering equation (35), a direct upper bound is:

\[ p_{R, \phi | R_0, \phi_0}(r, \phi | r_0, \phi_0) \]

\[ \leq \frac{1}{2\pi} p_{R | R_0}(r | r_0) + \frac{1}{\pi} \sum_{m=1}^{+\infty} |C_m(r, r_0)| \]

\[ \leq \frac{1}{2\pi} p_{R | R_0}(r | r_0) + \frac{1}{\pi} \sum_{m=1}^{+\infty} r b_m |e^{-\mathcal{R}(a_m)(r^2 + r_0^2)} I_0(2\mathcal{R}(b_m) r_0 r) |, \] (34)

where in order to write the last equation we used equation (35) and the fact that \( |I_m(z)| \leq I_0(\mathcal{R}(z)) \) for \( z \in \mathbb{C} \) and \( m \in \mathbb{N}. \)

Let \( \beta_m = \sqrt{\frac{2}{\sigma^2 L}}, m \geq 1. \) After straightforward manipulations we obtain for \( m \geq 1:\)

\[ \mathcal{R}(a_m) = \frac{\beta_m \sinh(2\beta_m) + \sin(2\beta_m)}{2\sigma^2 L \sinh^2(\beta_m) + \sin^2(\beta_m)} > \frac{1}{\sigma^2 L}, \] (35)

where the inequality is due to the fact that the function \( t(x) = \frac{x \sinh(2x) + \sin(2x)}{2x \sinh^2(x) + \sin^2(x)} \) is increasing on \( x > 0 \) and that \( \lim_{x \to 0} t(x) = 1. \) Also,

\[ \mathcal{R}(b_m) = \frac{\beta_m \sinh(\beta_m) \cos(\beta_m) + \cosh(\beta_m) \sin(\beta_m)}{\sinh^2(\beta_m) + \sin^2(\beta_m)} \leq \frac{1}{\sigma^2 L}, \] (36)

and

\[ |b_m| = \frac{\sqrt{2} \beta_m}{\sigma^2 L \sinh^2(\beta_m) + \sin^2(\beta_m)} \leq \frac{\sqrt{2} \beta_m}{\sigma^2 L \sinh(\beta_m)}, \] (37)
Back to equation (34), we obtain
\[
p_{R,\Phi|R_0,\Phi_0}(r, \phi|r_0, \phi_0) \leq \frac{1}{2\pi} p_{R|R_0}(r|r_0) + \frac{1}{\pi} \frac{\sqrt{2}r}{\sigma^2 L} e^{-\frac{\sigma^2 m^2}{|\sinh(\beta_m)|}} I_0 \left( \frac{2r_0 r}{\sigma^2 L} \right) \sum_{m=1}^{+\infty} \frac{\beta_m}{\sinh(\beta_m)}
\]
where \( k_n p_{R|R_0}(r|r_0) \),

where the last inequality is justified by virtue of (36) and (37). As for the sequence of coefficients \( \{ \mathcal{R}(a_m) \}_{m \geq 1} \), we use a tighter bound than the one given by inequality (35). In fact, since \( \tilde{t}(x) \) is increasing in \( x > 0 \):
\[
\mathcal{R}(a_1) = \frac{1}{\sigma^2 L} t(\beta_1) > \frac{1}{\sigma^2 L} t(\beta_1) > \frac{1}{\sigma^2 L},
\]

where \( \frac{1}{\sigma^2 L} t(\beta_1) = \mathcal{R}(a_1) = \frac{\beta_1}{\sigma^2 L} \frac{\sinh(2\beta_1)+\sin^2(\beta_1)}{\sinh^2(\beta_1)+\sin^2(\beta_1)} \). Hence, inequality (39) implies
\[
p_{R,\Phi|R_0,\Phi_0}(r, \phi|r_0, \phi_0) \geq \frac{1}{2\pi} p_{R|R_0}(r|r_0) - \frac{1}{\pi} \frac{\sqrt{2}r}{\sigma^2 L} e^{-\mathcal{R}(a_1)(r^2+r_0^2)} I_0 \left( \frac{2r_0 r}{\sigma^2 L} \right) \sum_{m=1}^{+\infty} \frac{\beta_m}{\sinh(\beta_m)}
\]
\[
= \frac{1}{2\pi} p_{R|R_0}(r|r_0) \left( 1 - \xi(r_0) \right).
\]

Since \( \mathcal{R}(a_1) > \frac{1}{\sigma^2 L} \), inequality (40) holds true and \( \xi(r_0) = \sqrt{2} e^{-\frac{\mathcal{R}(a_1)}{\sqrt{2}}} r_0^2 \sum_{m=1}^{+\infty} \frac{\beta_m}{\sinh(\beta_m)} \to 0 \) as \( r_0 \to +\infty \).

**Lemma 6 (Finite Second Moment)** Whenever the distribution \( F_{R_0} \) satisfies any of the input constraints considered in Theorem 7 the channel output has a finite second moment.

**Proof:** Whenever \( F_{R_0} \) belongs to any of the input constraints considered in this paper, the channel input has a finite second moment. The fact that the second moment of the output is also finite can be verified using, for example, the monotonicity of power in the dispersive optical channel [4] which also holds in the special case of zero dispersion.

**APPENDIX II**

**ANALYTICITY PROPERTIES OF LHS(z), z \in \mathbb{C}**

1) \( p_{R,\Phi|R_0,\Phi_0}(r, \phi|z, \phi_0) \): Using equation (6), one can find after some manipulations:
\[
\mathcal{R}(a_m) \overset{\text{def}}{=} \frac{1}{2\sigma^2 L} t(\beta_1) = \frac{\beta_m}{\sigma^2 L} \frac{\sinh(2\beta_m) + \sin(2\beta_m)}{\sinh^2(\beta_m) + \sin^2(\beta_m)}
\]
\[
\mathcal{I}(a_m) \overset{\text{def}}{=} \frac{1}{2\sigma^2 L} \tau(\beta_m) = \frac{\beta_m}{\sigma^2 L} \frac{\sinh(2\beta_m) - \sin(2\beta_m)}{\sinh^2(\beta_m) + \sin^2(\beta_m)}
\]
\[
\geq 0,
\]

where \( \beta_m = \sqrt{\frac{m^2}{2\pi^2} - \frac{n^2}{2\pi^2}} \), \( m > 0 \) and \( n \). Consider the function \( D(x) = \frac{\pi(x)}{4|\pi(x)|} \) for \( x > 0 \). Noticing that \( D(x) \) is positive, continuous, \( \lim_{x \to 0} D(x) = 0 \) and \( \lim_{x \to +\infty} D(x) = 1 \), then \( D(x) \) is upper bounded on \( x \geq 0 \). This in turn implies that
\[
M = \sup_{m \geq 0} \frac{\mathcal{I}(a_m)}{\mathcal{R}(a_m)} = \text{finite where } \phi_0 = \frac{1}{\sigma^2 L}. \]

Now, define the open subset \( \mathcal{E} = \{ z \in \mathbb{C} : R(z^2) - M |\mathcal{I}(z^2)| > 0 \} \) and denote by \( \mathbb{R}^{++} = \{ z \in \mathbb{R} : z > 0 \} \). We note that \( \mathcal{E} \supset \mathbb{R}^{++} \) and that for any \( z \in \mathcal{E} \) we have that \( \mathcal{R}(a_m z^2) > 0 \). We extend equation (6) to the complex plane in \( r_0 \geq 0 \). In fact, let
\[
s(z; r, \phi, \phi_0) = \frac{1}{2\pi} \sum_{m=-\infty}^{+\infty} C_m(r, z) e^{-j m (\phi - \phi_0 - \gamma z^2)}
\]
be an extension of \( p_{R, \Phi \mid R_0, \phi_0} (r, \phi \mid r_0, \phi_0) \) to the complex plane where \( C_0 (r, z) \) is given by the following:

\[
\lim_{z \to z_0} s(z; r, \phi, \phi_0) = \lim_{z \to z_0} \frac{1}{2\pi} \sum_{m=-\infty}^{+\infty} C_m (r, z) e^{-jm(\phi - \phi_0 - \gamma z^2 \mathcal{L})} = \frac{1}{2\pi} \sum_{m=-\infty}^{+\infty} \lim_{z \to z_0} C_m (r, z) e^{-jm(\phi - \phi_0 - \gamma z^2 \mathcal{L})} = \frac{1}{2\pi} \sum_{m=-\infty}^{+\infty} C_m (r, z_0) e^{-jm(\phi - \phi_0 - \gamma z_0^2 \mathcal{L})},
\]

where the interchange in the order of the limit and the summation signs holds true since \( |C_m (r, z)| \) decays fast enough in a neighbourhood of \( z_0 \) by virtue of the fact that \( \mathcal{R}(a_m z^2) > 0 \). Next we integrate \( s(z; r, \phi, \phi_0) \) on the boundary \( \partial \Delta \) of a compact triangle \( \Delta \subset \mathcal{E} \):

\[
\int_{\partial \Delta} s(z; r, \phi, \phi_0) dz = \frac{1}{2\pi} \sum_{m=-\infty}^{+\infty} \int_{\partial \Delta} C_m (r, z) e^{-jm(\phi - \phi_0 - \gamma z^2 \mathcal{L})} dz = 0,
\]

where the interchange in the order between the integral and the summation signs holds true since \( \mathcal{R}(a_m z^2) > 0 \). The fact that the end result evaluates to zero is justified since

\[
C_m (r, z) e^{-jm(\phi - \phi_0 - \gamma z^2 \mathcal{L})} = rb_m e^{-a_m (r^2 + z^2)} I_m (2b_m z r) e^{-jm(\phi - \phi_0 - \gamma z^2 \mathcal{L})},
\]

is analytic in \( z \in \mathbb{C} \). Equations (42) and (43) implies that \( s(z; r, \phi, \phi_0) \) is analytic in \( z \) whenever \( z \in \mathcal{E} \) from Morera’s theorem. 2) \( \int_{0}^{2\pi} h(R, \Phi \mid z, \phi_0) d\phi_0 \). We show next that there exists an open covering \( \mathcal{O} \) of the positive real line such that \( u(z) = \int_{0}^{2\pi} h(R, \Phi \mid z, \phi_0) d\phi_0 \) is analytic in \( z \in \mathcal{O} \). Explicitly, we write

\[
u(z) = \int_{0}^{2\pi} \int_{0}^{2\pi} \int_{0}^{+\infty} s(z; r, \phi, \phi_0) \ln (s(z; r, \phi, \phi_0)) dr d\phi d\phi_0.
\]

It has been proved in the previous section that \( s(z; r, \phi, \phi_0) \) is analytic in \( z \in \mathcal{E} \). When it comes to \( \ln (s(z; r, \phi, \phi_0)) \) we have the following:

- For any positive real \( z_1 > 0 \), there exits an open ball \( B_i(z_1, \xi_i) \subset \mathcal{E} \) centered at \( z_1 \) such that \( \mathcal{R}(s(z; r, \phi, \phi_0)) > 0 \), \( z \in B_i \). This is due to the fact that \( s(r_0; r, \phi, \phi_0) = p_{R, \Phi \mid r_0, \phi_0}(r, \phi \mid r_0, \phi_0) \) is positive in \( r_0 > 0 \) and continuous on \( \mathcal{E} \). Though it is not clear from equation [3] that \( p_{R, \Phi \mid r_0, \phi_0}(r, \phi \mid r_0, \phi_0) > 0 \), the positiveness of \( p_{R, \Phi \mid r_0, \phi_0}(r, \phi \mid r_0, \phi_0) \) would be trivial when considering the incremental model presented in [2, Equation 3].

- Therefore, there exists a sequence \( \{ z_i \}_{i \geq 1} \) of real positive numbers and an open covering \( \mathcal{O} = \bigcup_{i \geq 1} B_i \subset \mathcal{E} \) of the positive real line such that \( \mathcal{R}(s(z; r, \phi, \phi_0)) > 0 \).

Using the principal branch of the logarithm, we obtain that \( \ln (s(z; r, \phi, \phi_0)) \) is analytic in \( z \in \mathcal{O} \).

Hence, \( s(z; r, \phi, \phi_0) \ln (s(z; r, \phi, \phi_0)) \) is analytic in \( z \in \mathcal{O} \). The analyticity of \( u(z) \) in \( z \in \mathcal{O} \) follows by adopting Morera’s theorem in a similar manner to what was done in the previous section where the various interchange arguments are justified since \( \mathcal{R}(a_m z^2) > 0 \).

3) \( \int_{0}^{+\infty} p_{R \mid R_0}(r \mid z) \ln (p_{R}(r)) dr \): By similar arguments as above, it can be shown that:

\[
w(z) = \int_{0}^{+\infty} p_{R \mid R_0}(r \mid z) \ln (p_{R}(r)) dr,
\]

is analytic on \( \mathcal{O} \).

4) \( \text{LHS}(z) \): Let \( \text{LHS}_A(z) \) be an extension of the LHS of equation (16) in \( r_0 \) to the complex plane:

\[
\text{LHS}(z) = \nu(C(z) - A) + C + \int_{0}^{+\infty} p(r \mid z) \ln p(r; F_0) dr + \frac{1}{2\pi} \int_{0}^{2\pi} h(R, \Phi \mid z, \phi_0) d\phi_0.
\]

According to the results of paragraphs 1, 2, 3 and Property C3, \( \text{LHS}_A(z) \) is analytic whenever \( z \in \mathcal{O} \cap \mathcal{S}_8 \). As for \( \text{LHS}_p(z) \) the complex extension of the LHS of equation (15), it is analytic on \( \mathcal{O} \). We denote the region where both \( \text{LHS}_A(z) \) and \( \text{LHS}_p(z) \) are analytic by \( \mathcal{H} = \mathcal{O} \cap \mathcal{S}_8 \). We note that \( \mathcal{H} \) is open and connected and that \( \mathcal{H} \supset \{ \mathbb{R}^+ \setminus \{0\} \} \).
Lemma 7 (Upper Bound on LHS\(_\rho(r_0)\)) For large values of \(r_0\), the LHS of equation (46) LHS\(_\rho(r_0)\) satisfies: for any 0 < \(\epsilon < 1\), there exists \(K > 0\) such that

\[
\text{LHS}_{\rho}(r_0) \leq C + \ln \left( \frac{1}{K} \right) + \frac{r_0^2}{\sigma^2 L} - \ln (1 - \xi(r_0)) + (\rho - (1 - \epsilon)r_0) \sqrt{\frac{\pi}{\sigma^2 L}} L_{\frac{1}{2}} \left( -\frac{r_0^2}{\sigma^2 L} \right),
\]

where \(\xi(r_0) = \sqrt{2} e^{-\left(R(\cdot) - \frac{r^2}{2\sigma^2}\right)} \sum_{m=1}^{\infty} \frac{\beta_m}{\text{min}(\beta_m)} \to 0\), \(\beta_m = \sqrt{2} \sigma L\) and where \(L_{\frac{1}{2}}(\cdot)\) denotes a Laguerre polynomial.

**Proof:** We start by finding an upper bound on \(p(r, F_0^*)\). In fact:

\[
p(r, F_0^*) = \int_0^{+\infty} p_{R|F_0}(r|r_0) \, dF_0^*(r_0)
= \frac{2r}{\sigma^2 L} \int_0^{\infty} e^{-\frac{r^2 + r_0^2}{\sigma^2 L}} I_0 \left( \frac{2r r_0}{\sigma^2 L} \right) \, dF_0^*(r_0)
\leq \frac{2r}{\sigma^2 L} e^{-\frac{2r r_0}{\sigma^2 L}}
\]

(44)

where we used the fact that \(I_0 \left( \frac{2r r_0}{\sigma^2 L} \right) \leq e^{-\frac{2r r_0}{\sigma^2 L}}\). Equation (44) implies:

\[
\int_0^{+\infty} p_{R|F_0}(r|r_0) \ln p(r; F_0^*) \, dr
\leq \ln \left( \frac{2}{\sigma^2 L} \right) + \int_0^{+\infty} \ln(r) p_{R|F_0}(r|r_0) \, dr - \frac{1}{\sigma^2 L} \int_0^{+\infty} r^2 p_{R|F_0}(r|r_0) \, dr + \frac{2\rho}{\sigma^2 L} \int_0^{+\infty} r p_{R|F_0}(r|r_0) \, dr
\]

(45)

where we used the fact that \(p_{R|F_0}(r|r_0)\) is a Rician PDF with parameters \(\left(r_0, \frac{2r}{\sigma^2 L}\right)\). The first two moments of such a PDF are respectively

\[
\int_0^{+\infty} r p_{R|F_0}(r|r_0) \, dr = \frac{\sigma \sqrt{\pi L}}{2} L_{\frac{1}{2}} \left( -\frac{r_0^2}{\sigma^2 L} \right)
\]

(46)

\[
\int_0^{+\infty} r^2 p_{R|F_0}(r|r_0) \, dr = \sigma^2 L + \frac{r_0^2}{\sigma^2 L}
\]

(47)

Next, we find an upper bound on \(\frac{1}{2\pi} \int_0^{2\pi} h(R, \Phi|r_0, \phi_0) \, d\phi_0\) for large values of \(r_0\) as follows:

\[
\frac{1}{2\pi} \int_0^{2\pi} h(R, \Phi|r_0, \phi_0) \, d\phi_0
= -\int_0^{2\pi} \int_0^{+\infty} \frac{p(r, \phi|r_0, \phi_0) \ln p(r, \phi|r_0, \phi_0)}{2\pi} \, dr \, d\phi_0
\]

(\(a\)) \(\leq\ln(2\pi) - \ln(1 - \xi(r_0)) - \int_0^{+\infty} p_{R|F_0}(r|r_0) \ln p_{R|F_0}(r|r_0) \, dr
\]

\[
= -\ln \left( \frac{1}{\pi \sigma^2 L} \right) - \int_0^{+\infty} \ln(r) p_{R|F_0}(r|r_0) \, dr + \frac{1}{\sigma^2 L} \left( r_0^2 + \int_0^{+\infty} r^2 p_{R|F_0}(r|r_0) \, dr \right)
\]

\[
- \int_0^{+\infty} \ln \left( I_0 \left( \frac{2r r_0}{\sigma^2 L} \right) \right) p_{R|F_0}(r|r_0) \, dr - \ln(1 - \xi(r_0))
\]

\[
\leq \ln \left( \frac{\pi \sigma^2 L}{K} \right) - \int_0^{+\infty} \ln(r) p_{R|F_0}(r|r_0) \, dr - \ln(1 - \xi(r_0)) + 2\frac{r_0^2 + \sigma^2 L}{\sigma^2 L} - \int_0^{+\infty} \ln \left( K e^{(1 - \epsilon) \frac{2r r_0}{\pi \sigma^2 L}} \right) p_{R|F_0}(r|r_0) \, dr
\]

\[
= -\ln \left( \frac{\pi \sigma^2 L}{K} \right) - \int_0^{+\infty} \ln(r) p_{R|F_0}(r|r_0) \, dr - \ln(1 - \xi(r_0)) + 2\frac{2r_0^2 + \sigma^2 L}{\sigma^2 L} - (1 - \epsilon) \frac{2r_0}{\sigma^2 L} \int_0^{+\infty} r p_{R|F_0}(r|r_0) \, dr
\]

(48)

where \(K > 0\) and \(0 < \epsilon < 1\) are such that \(I_0(x) \geq K e^{(1 - \epsilon)x}, x \geq 0\). This is true since for any 0 < \(\epsilon < 1\), the positive and continuous function \(e^{(1 - \epsilon)x} I_0(x)\) is lower bounded by a positive \(K\) since \(e^{(1 - \epsilon)x} I_0(x) \equiv \frac{e^{x^2}}{\sqrt{2\pi} x} \) as \(x \to +\infty\) [21].
Lemma 8 (Lower Bound on LHS$_A(r_0)$) The LHS of equation (15) LHS$_A(r_0)$ satisfies:

$$\text{LHS}_A(r_0) > \nu(C(r_0) - A) + C + \ln\left(\frac{k_1}{2\pi k_u}\right) + \frac{1}{\sigma^2 L} r_0 - r_0 \sqrt{\frac{\pi}{\sigma^2 L}} \left(\frac{r_0^2}{\sigma^2 L}\right),$$

where $k_1 = \int_0^{\infty} e^{-\frac{r^2}{2\sigma^2}} dF_{R_0'}(r_0)$, $k_u = \frac{1}{2\pi} \left(1 + \sqrt{2} \sum_{\nu=1}^{\infty} \frac{\beta_{\nu}}{\sinh(\beta_{\nu})}\right) < \infty$, $\beta_{\nu} = \sqrt{\frac{\pi}{2}} \sigma L$ and where $L_{\nu/2}(\cdot)$ denotes a Laguerre polynomial.

Proof: We start by obtaining a lower bound on the output PDF $p(r; F_0^*)$:

$$p(r, F_0^*) = \int_0^{\infty} p_{R|R_0}(r|r_0) dF_{R_0}(r_0)$$

$$= \int_0^{\infty} \frac{2r}{\sigma^2 L} e^{-\frac{r^2}{2\sigma^2}} I_0\left(\frac{2r r_0}{\sigma^2 L}\right) dF_{R_0}(r_0)$$

$$\geq (a) \frac{2k_1 r}{\sigma^2 L} e^{-\frac{r^2}{2\sigma^2}},$$

where $k_1 = \int_0^{\infty} e^{-\frac{r^2}{2\sigma^2}} dF_{R_0'}(r_0)$ and step $(a)$ follows because $I_0(x) \geq 1$. The first integral in LHS$_A(r_0)$ is hence lower bounded as

$$\int_0^{\infty} p_{R|R_0}(r|r_0) \ln p(r; F_0^*) \, dr \geq \int_0^{\infty} \ln(r) p_{R|R_0}(r|r_0) \, dr + \ln\left(\frac{2k_1}{\sigma^2 L}\right) - \frac{\sigma^2 L + r_0^2}{\sigma^2 L},$$

where we used equation (46) in order to write inequality (50). The second integral in LHS$_A(r_0)$ is lower bounded as

$$\int_0^{2\pi} h(R, \Phi|r_0, \phi_0) \, d\phi_0 \geq \frac{2\pi}{\sigma^2 L} \int_0^{2\pi} \int_0^{2\pi} p(r, \phi|r_0, \phi_0) \ln \left(k_u p_{R|R_0}(r|r_0)\right) \, d\phi \, d\phi_0$$

$$= \int_0^{2\pi} p_{R|R_0}(r|r_0) \, dr \int_0^{2\pi} p(r, \phi|r_0, \phi_0) \, d\phi \, d\phi_0$$

$$= -2\pi \int_0^{2\pi} \ln \left(k_u p_{R|R_0}(r|r_0)\right) p_{R|R_0}(r|r_0) \, dr$$

$$= 2\pi \left(h(R|r_0) - \ln(k_u)\right),$$

(a) is due to Lemma 5. Finally, using inequalities (45) and (48), the LHS of equation (15) can be upper bounded when $r_0$ is sufficiently large as follows:

$$\text{LHS}_p(r_0) = C - \ln(2\pi) + \int_0^{\infty} p(r|r_0) \ln p(r; F_0^*) \, dr + \frac{1}{2\pi} \int_0^{2\pi} h(R, \Phi|r_0, \phi_0) \, d\phi_0$$

$$\leq C - \ln(2\pi) + \ln\left(\frac{2}{\sigma^2 L}\right) + \int_0^{\infty} \ln(r)p_{R|R_0}(r|r_0) \, dr - \frac{1}{\sigma^2 L} \left(\sigma^2 L + r_0^2\right)$$

$$\geq C + \frac{1}{\sigma^2 L} \left(\sigma^2 L + r_0^2\right) + \int_0^{\infty} \ln(r)p_{R|R_0}(r|r_0) \, dr - \ln(1 - \xi(r_0))$$

$$\geq C + \frac{1}{\sigma^2 L} - \ln(1 - \xi(r_0)) + (R - 1 - \xi(r_0)) \sqrt{\frac{\pi}{\sigma^2 L}} \left(-\frac{r_0^2}{\sigma^2 L}\right).$$

(49)
where step (b) is due to Lemma 5. From (4), 
\[ h(R|r_0) = -\int_0^{+\infty} p_{R|R_0}(r|r_0) \ln \left( p_{R|R_0}(r|r_0) \right) \, dr \]

From (4), where we used equation (46) in order to write the last equation. Inequality (52) is true since  
\[ \int_0^\infty \ln(r)p_{R|R_0}(r|r_0) \, dr + \frac{1}{\sqrt{2\pi L}} e^{\sigma^2 L \frac{r_0^2}{2}} - r_0 \sqrt{\frac{\pi}{\sigma^2 L}} L^{\frac{1}{2}} \left( -\frac{r_0^2}{2\sigma^2 L} \right) \]

Therefore inequality (51) gives
\[ \frac{1}{2\pi} \int_0^{2\pi} h(R, \Phi|r_0, \phi_0) \, d\phi_0 > \ln \left( \frac{e \sigma^2 L}{2k_u} \right) + \frac{2}{\sigma^2 L} r_0^2 - \int_0^{+\infty} \ln(r)p_{R|R_0}(r|r_0) \, dr - r_0 \sqrt{\frac{\pi}{\sigma^2 L}} L^{\frac{1}{2}} \left( -\frac{r_0^2}{2\sigma^2 L} \right) \]

and inequalities (50) and (53) imply
\[ \text{LHS}_A(r_0) > \nu(C(r_0) - A) + C - \ln(2\pi) + \frac{2k_1}{\sigma^2 L} + \ln \left( \frac{e \sigma^2 L}{2k_u} \right) \]

\[ + \int_0^{+\infty} \ln(r)p_{R|R_0}(r|r_0) \, dr - \int_0^{+\infty} \ln(r)p_{R|R_0}(r|r_0) \, dr \]

\[ - \frac{1}{\sigma^2 L} \left( \sigma^2 L + r_0^2 \right) + \frac{2}{\sigma^2 L} r_0^2 - r_0 \sqrt{\frac{\pi}{\sigma^2 L}} L^{\frac{1}{2}} \left( -\frac{r_0^2}{2\sigma^2 L} \right) \]

\[ = \nu(C(r_0) - A) + C + \ln \left( \frac{k_1}{2\pi k_u} \right) + \frac{1}{\sigma^2 L} r_0^2 - r_0 \sqrt{\frac{\pi}{\sigma^2 L}} L^{\frac{1}{2}} \left( -\frac{r_0^2}{2\sigma^2 L} \right) \]

REFERENCES

1. G. Kramer, M. I. Yousefi, and F. Kschischang, “Upper bound on the capacity of a cascade of nonlinear and noisy channels,” in IEEE Info. Theory Workshop, Jerusalem, Israel, Apr. 2015, pp. 1–4.
2. M. I. Yousefi and F. R. Kschischang, “On the per-sample capacity of nondispersive optical fibers,” IEEE Trans. Inf. Theory, vol. 57, no. 11, pp. 7522–7541, Nov. 2011.
3. G. Kramer, “Autocorrelation function for dispersion-free fiber channels with distributed amplification,” 2017, http://arxiv.org/abs/1705.00454.
4. M. I. Yousefi, G. Kramer, and F. R. Kschischang, “Upper bound on the capacity of the nonlinear Schrödinger channel,” in Canadian Workshop on Inf. Theory, St. John’s, Newfoundland, Canada, July 2013, pp. 1–5.
5. A. Mecozzi, “Limits to long-haul coherent transmission set by the Kerr nonlinearity and noise of the in-line amplifiers,” IEEE J. Lightw. Technol., vol. 12, no. 11, pp. 1993–2000, Nov. 1994.
6. K. S. Turitsyn, S. A. Derevyanko, I. V. Yurkevich, and S. K. Turitsyn, “Information capacity of optical fiber channels with zero average dispersion,” Phys. Rev. Lett., vol. 91, no. 20, p. 203901, Nov. 2003.
7. M. I. Yousefi, “The asymptotic capacity of the optical fiber,” arXiv:1610.06458, pp. 1–12, Nov. 2016.
8. J. G. Smith, “The information capacity of amplitude- and variance-constrained scalar Gaussian channels,” Inf. Contr., vol. 18, no. 3, pp. 203–219, Apr. 1971.
9. W. Hirt and J. Massey, “Capacity of the discrete-time Gaussian channel with intersymbol interference,” IEEE Trans. Inf. Theory, vol. 34, no. 3, pp. 380–388, May 1988.
10. S. Shamai and I. Bar-David, “The capacity of average and peak-power-limited quadrature Gaussian channels,” IEEE Trans. Inf. Theory, vol. 41, no. 4, pp. 1060–1071, July 1995.
11. A. Das, “Capacity-achieving distributions for non-Gaussian additive noise channels,” in IEEE International Symposium on Information Theory, Sorrento, Italy, June 2000, p. 432.
12. I. Abou-Faycal, M. D. Trott, and S. Shamai, “The capacity of discrete-time memoryless Rayleigh-fading channels,” IEEE Trans. Inf. Theory, vol. 47, no. 4, pp. 1290–1301, May 2001.
13. A. Tchamkerten, “On the discreteness of capacity-achieving distributions,” IEEE Trans. Inf. Theory, vol. 50, no. 11, pp. 2773–2778, Nov. 2004.
14. T. H. Chan, S. Hranilovic, and F. Kschischang, “Capacity-achieving probability measure for conditionally Gaussian channels with bounded inputs,” IEEE Trans. Inf. Theory, vol. 51, no. 6, pp. 2073–2088, June 2005.
15. M. G. Gursoy, H. V. Poor, and S. Verdú, “The noncoherent Rician fading channel–Part I: structure of the capacity-achieving input,” IEEE Trans. Wireless Commun., vol. 4, no. 5, pp. 2193–2206, Sept. 2005.
16. J. F. Fahs and I. Abou-Faycal, “Using Hermite bases in studying capacity-achieving distributions over AWGN channels,” IEEE Trans. Inf. Theory, vol. 58, no. 8, pp. 5302–5322, Aug. 2012.
17. J. F. Fahs, N. Ajeeb, and I. Abou-Faycal, “The capacity of average power constrained additive non-Gaussian noise channels,” in IEEE Int. Conf. Telecommun., Beirut, Lebanon, Apr. 2012.
18. J. F. Fahs and I. Abou-Faycal, “Input constraints and noise density functions: a simple relation for bounded-support and discrete capacity-achieving inputs,” arXiv:1602.00878, Sept. 2016.
19. ———, “On the finiteness of the capacity of continuous channels,” IEEE Trans. Commun., vol. 64, no. 1, pp. 166–173, Jan. 2016.
20. H. Silverman, Complex Variables. Houghton Mifflin Company, 1975.
21. M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 10th ed., ser. Appl. Math. Washington, D.C., USA: National Bureau of Standards, Dec. 1972.