On special representations of $p$–adic reductive groups

Elmar Grosse-Klönn
On special representations of $p$-adic reductive groups

ELMAR GROSE-KLÖNNE

Abstract

Let $F$ be a non-Archimedean locally compact field, let $G$ be a split connected reductive group over $F$. For a parabolic subgroup $Q \subset G$ and a ring $L$ we consider the $G$-representation on the $L$-module

$$(\ast) \quad C^\infty(G/Q, L)/ \sum_{Q' \supseteq Q} C^\infty(G/Q', L).$$

Let $I \subset G$ denote a Iwahori subgroup. We define a certain free finite rank $L$-module $\mathfrak{M}$ (depending on $Q$; if $Q$ is a Borel subgroup then $(\ast)$ is the Steinberg representation and $\mathfrak{M}$ is of rank one) and construct an $I$-equivariant embedding of $(\ast)$ into $C^\infty(I, \mathfrak{M})$. This allows the computation of the $I$-invariants in $(\ast)$. We then prove that if $L$ is a field with characteristic equal to the residue characteristic of $F$ and if $G$ is a classical group, then the $G$-representation $(\ast)$ is irreducible. This is the analog of a theorem of Casselman (which says the same for $L = \mathbb{C}$); it had been conjectured by Vignéras.

Introduction

Let $F$ be a non-Archimedean locally compact field with ring of integers $\mathcal{O}_F$ and residue field $k_F$. Let $G$ be a connected split reductive group over $F$. Let $T$ be a split maximal torus, $N \subset G$ its normalizer and $W = N/T$, the corresponding Weyl group. Let $\Phi \subset X^*(T)$ be the set of roots, let $\Phi^+ \subset \Phi$ be the set of positive roots with respect to a Borel subgroup $P$ containing $T$ and let $\Delta \subset \Phi^+$ be the corresponding set of simple roots. For a subset $J \subset \Delta$ let $W_J \subset W$ denote the subgroup generated by the simple reflections associated with the elements of $J$. Let $P_J$ denote the parabolic subgroup generated by $P$ and by representatives (in $N$) of the elements of $W_J$. Any parabolic subgroup of $G$ is conjugate to $P_J$ for some $J$. For a ring $L$ (commutative, with $1 \in L$) we call the $G$-representation

$$\text{Sp}_J(G, L) = \frac{C^\infty(G/P_J, L)}{\sum_{\alpha \in \Delta \setminus J} C^\infty(G/P_{J \cup \{\alpha\}}, L)}$$

the $J$-special representation of $G$ with coefficients in $L$. For $J = \emptyset$ this is the Steinberg representation of $G$ with coefficients in $L$. By an old theorem of Casselman, the representations

MSC Classification 22E50, 11S99

Keywords: $p$-adic reductive group, modular representation, Steinberg representation
Sp\(_J(G, \mathbb{C})\) are irreducible for all \(J\), they form the irreducible constituents, each with multiplicity one, of \(C^\infty(G/P, \mathbb{C})\). Published proofs of this irreducibility use techniques specific for the coefficient field \(L = \mathbb{C}\), see [4] ch. X, Theorem 4.11 or [10] Theorem 8.1.2. For \(L\) a field of characteristic \(\ell \neq p = \text{char}(k_F)\) it is known that the irreducibility of say \(\text{Sp}_\emptyset(G, L)\) depends on \(\ell\). See e.g. [16] III, Theorem 2.8 (b).

In this paper we investigate the representation \(\text{Sp}_J(G, L)\) for arbitrary coefficient rings \(L\) (and on the way obtain results previously unknown even for \(L = \mathbb{C}\)). We need the \(L\)-module

\[
\mathfrak{M}_J(L) = \frac{L[W/W_J]}{\sum_{\alpha \in \Delta - J} L[W/W_{J, \alpha}]}.
\]

Let \(I \subset G\) be an Iwahori subgroup adapted to \(P\), i.e. such that we have an Iwahori decomposition \(G = \cup_{w \in W} IW_P\). Our first main theorem is the following (Theorem 2.3), which even for \(L = \mathbb{C}\) seems to have been unknown before:

**Theorem 1:** There exists an \(I\)-equivariant embedding

\[
\text{Sp}_J(G, L) \hookrightarrow C^\infty(I, \mathfrak{M}_J(L));
\]

its formation commutes with base changes in \(L\).

Using the Iwahori decomposition, the proof of Theorem 1 is reduced to the proof of exactness of a certain natural sequence

\[
\bigoplus_{\alpha \in \Delta - J, w \in W/W_{J, \alpha}} C^\infty(I/I \cap wP_{J, \alpha}w^{-1}, L) \longrightarrow \bigoplus_{w \in W/W_J} C^\infty(I/I \cap wP_Jw^{-1}, L) \longrightarrow C^\infty(I, \mathfrak{M}_J(L))
\]

(Proposition 2.2). This exactness proof proceeds by induction along a certain filtration of (1). The key to defining this filtration is to consider certain subsets of \(\Phi\) which we call \(J\)-quasi-parabolic: a subset \(D \subset \Phi\) is called \(J\)-quasi-parabolic if \(\prod_{\alpha \in D} U_\alpha\) is the intersection of unipotent radicals of parabolic subgroups which are \(W\)-conjugate to \(P_J\). Here \(U_\alpha \subset G\) denotes the root subgroup associated to \(\alpha\). For such \(D\) we define a subset \(W^J(D)\) of \(W/W_J\) as consisting of those classes \(wW_J\) for which \(\prod_{\alpha \in D} U_\alpha\) is contained in the unipotent radical of the parabolic subgroup opposite to \(wP_Jw^{-1}\). Fixing a size-increasing enumeration of all \(J\)-quasi-parabolic subsets \(D\), the corresponding \(W^J(D)\)'s give the said filtration of (1). The exactness of (1) is then reduced to the exactness, for any \(D\), of

\[
\bigoplus_{\alpha \in \Delta - J} L[W^{J[D]}(\alpha)] \longrightarrow L[W^J(D)] \longrightarrow \mathfrak{M}_J(L)
\]

(Proposition 1.2), a purely combinatorial fact on finite crystallographic reflection groups. We mention that if \(L\) is a complete field extension of \(F\), Theorem 1 holds verbatim, with the same proof, for the corresponding representations on spaces of locally analytic (rather than locally constant) functions.
A vigorously emerging subject in current $p$-adic number theory is the smooth representation theory of $p$-adic reductive groups, like $G$, on $\mathbb{F}_p$-vector spaces. So far, the research has focused mostly on the case $G = \text{GL}_2(F)$, for finite extensions $F$ of $\mathbb{Q}_p$, but even for those $G$ the theory turns out to be fairly complicated and is far from being well understood. However, it already becomes quite clear that a good understanding of the theory depends crucially on a good understanding of the functor 'taking invariants under a (pro-$p$)-Iwahori-subgroup'. At present there is literally no general technique available to compute this functor. For example, although Vignéras had proved the irreducibility of the Steinberg representation of our $G$’s in characteristic $p$, the space of its (pro-$p$)-Iwahori invariants was not known (except for $G = \text{GL}_2(F)$); this was the motivating problem for our investigations.

As an immediate consequence of Theorem 1 we obtain that the submodule of $I$-invariants $\text{Sp}_J(G, L)^I$ is free of rank at most the rank of $\mathfrak{M}_J(L)$, i.e. $\text{rk}_L(\text{Sp}_J(G, L)^I) \leq \text{rk}_L(\mathfrak{M}_J(L))$, as was conjectured by Vignéras [15]. The reverse inequality $\text{rk}_L(\text{Sp}_J(G, L)^I) \geq \text{rk}_L(\mathfrak{M}_J(L))$ follows easily by summing over all $J$, using that $\sum_J \text{rk}_L(\mathfrak{M}_J(L)) = |W|$. In particular, the module of $I$-invariants in the Steinberg representation is free of rank one, for any $L$.

The reductive group underlying $G$ can be defined over $\mathcal{O}_F$; as such we denote it by $G_{x_0}$. Its group $G_{x_0}(\mathcal{O}_F)$ of $\mathcal{O}_F$-rational points is a subgroup of $G$, let $G = G_{x_0}(k_F)$ denote the group of $k_F$-rational points of the split reductive group over $k_F$ obtained by reduction. Its root system is the same as for $G$. We may copy the definition of the $G$-representations $\text{Sp}_J(G, L)$ to define $\overline{G}$-representations $\text{Sp}_J(\overline{G}, L)$, for all $J \subset \Delta$ (replace locally constant functions on $G$ by functions on $\overline{G}$). Let $\mathcal{P} \subset \overline{G}$ denote the Borel subgroup obtained by reduction of $I \subset G_{x_0}(\mathcal{O}_F)$. Then using Theorem 1 we find a canonical identification (Proposition 3.2):

$$(2) \quad \text{Sp}_J(G, L)^I = \text{Sp}_J(\overline{G}, L)^{\mathcal{P}}.$$

Our second main theorem is the analog of Casselman’s theorem for a field $L$ with $p = \text{char}(L) = \text{char}(k_F)$ (of course, this analog implies and gives a purely algebraic proof of Casselman’s theorem). Let $I_1 \subset I$ denote the pro-$p$-Iwahori subgroup inside $I$. The $G$-representation $\text{Sp}_J(G, L)$ is generated by $\text{Sp}_J(G, L)^I = \text{Sp}_J(G, L)^{I_1}$ (see [15]). As any smooth representation of a pro-$p$-group on a non-zero vector space in characteristic $p$ admits a non-zero invariant vector, it is enough to show that $\text{Sp}_J(G, L)^I$ is irreducible as a module under the Iwahori Hecke algebra $\mathcal{H}(G, I)$. We may view $\text{Sp}_J(G, L)^I = \text{Sp}_J(\overline{G}, L)^{\mathcal{P}}$ as a module under the Hecke algebra $\mathcal{H}(\overline{G}, \mathcal{P})$. In a first step we show (Proposition 3.4) that each $\mathcal{H}(\overline{G}, \mathcal{P})$-submodule of $\text{Sp}_J(G, L)^I = \text{Sp}_J(\overline{G}, L)^{\mathcal{P}}$ contains the class of the characteristic function $\chi_{Iw_\Delta J}$ of the subset $Iw_\Delta J \subset G$; here $w_\Delta \in W$ denotes the longest element. This follows from explicit formulae for the action on $\text{Sp}_J(\overline{G}, L)^{\mathcal{P}}$ of the Hecke operators associated to simple reflections (these formulae boil down to the Bruhat decomposition of $\overline{G}$ and require our assumption $p = \text{char}(L) = \text{char}(k_F)$), together with a combinatorial lemma (Lemma 1.5) on $W$. In a second step we need to show that the class of $\chi_{Iw_\Delta J}$ generates $\text{Sp}_J(G, L)^I$ as a $\mathcal{H}(G, I)$-module. We can prove this if the root system $\Phi$ belongs to one of the infinite series $(A_l)_l$, $(B_l)_l$, $(G_l)_l$ or $(D_l)_l$. Our argument uses a combinatorial result (Proposition 1.6) on the weak (left)ordering of $W$ (an ordering weaker than the Bruhat ordering)
which we can prove only for such root systems. It may also hold true for the root systems of type $E_6$ or $E_7$ (hence we would get the irreducibility result in these cases, too), but certainly fails for the root systems of the types $E_8$, $F_4$ and $G_2$. Thus in these cases another argument (for the generation of $\text{Sp}_J(G, L)^I$ by $\chi_{Iw_\Delta P_J}$) would be needed. In conclusion, what we prove is (Corollary 4.3, Corollary 4.4):

**Theorem 2:** If $L$ is a field with $\text{char}(L) = \text{char}(k_F)$ and if the root-system $\Phi$ is of type $A_l$, $B_l$, $C_l$ or $D_l$, then the $G$-representation $\text{Sp}_J(G, L)$ is irreducible. The $\text{Sp}_J(G, L)$ for the various $J$ form the irreducible constituents, each one occurring with multiplicity one, of $C^\infty(G/P, L)$.

This theorem had been conjectured by Vignéras [15] (without the restriction on $\Phi$), and, as indicated above, she had proven the irreducibility of the Steinberg representation $\text{Sp}_\emptyset(G, L)$.

In the final section $L$ is arbitrary as before and we consider realizations of $\text{Sp}_J(G, L)$ as modules of harmonic chains on the (semisimple) building $X$ of $G$. It follows from the results of [3] that if $C = S(\emptyset)$ denotes the set of all pointed chambers of $X$, the Steinberg representation $\text{Sp}_\emptyset(G, L)$ is the quotient of the $G$-representation $L[C]$ divided by all sums of pointed chambers which share a common pointed one-codimensional face. For general $J$ it is still easy to see that $\text{Sp}_J(G, L)$ is a quotient of the $G$-representation $L[S(J)]$ for a suitable $G$-stable set $S(J)$ of pointed $|\Delta - J|$-dimensional simplices in $X$. Indeed, using the previous notations, it is not hard to see that the $\overline{G}$-representation $\text{Sp}_J(\overline{G}, L)$ can be realized as a quotient of $L[\overline{S}(J)]$, where $\overline{S}(J)$ denotes a certain $G_{x_0}(O_F)$-stable set of $|\Delta - J|$-dimensional simplices in $X$ containing the unique special vertex $x_0$ of $X$ fixed by $G_{x_0}(O_F)$. Now we simply endow the elements of $\overline{S}(J)$ with the pointing by $x_0$: then $S(J) = G\overline{S}(J)$ works. However, for $J \neq \emptyset$ it is a hard problem to give explicit local generators for the kernel of $L[S(J)] \to \text{Sp}_J(G, L)$, i.e. the needed 'harmonicity' relations. This problem has been solved in the case $G = \text{GL}_n(F)$ for some $J$, namely for $J$ consisting of the first $|J|$ simple roots in the Dynkin digram. (See [7] and [1]. The important definitions, as well as the proof in the case char($F$) = 0, as a byproduct of another investigation, are due to de Shalit. A later proof for general $F$ is due to Aït Amrane. In fact, the definitions of de Shalit for such $J$ realize $\text{Sp}_J(G, L)$ even as a quotient of the free $L$-module on the set of all pointed $|\Delta - J|$-dimensional simplices, instead of just the set $S(J)$ considered above. For general $J$ this may be asking for too much.) Here we give local harmonicity relations for all $J$ if $G = \text{GL}_n(F)$ (Theorem 5.1). Finally we give an explicit description of our embedding from Theorem 1 in terms of this realization of $\text{Sp}_J(G, L)$ (if $G = \text{GL}_n(F)$).

We expect that the methods and results of this paper are indispensable for further investigations on the representations $\text{Sp}_J(G, L)$, for $L$ a field of characteristic $p$. For example, if $L = \mathbb{C}, \mathbb{Q}_\ell$ or if char($L$) = $\ell \neq p$, cohomological results on the representations $\text{Sp}_J(G, L)$ obtained in [5], [11] and [12] have been important for understanding the cohomology of the Drinfel’d symmetric space $\mathcal{X}$ associated with $G = \text{GL}_n(F)$, see [5] and [12]. For $L = k_F$ some of the representations $\text{Sp}_J(G, L)$ occur in the (coherent) cohomology of the natural formal $O_F$-model of $\mathcal{X}$.
It is a great pleasure to express my deep gratitude to Marie-France Vignéras. She invited me to spend June 2006 at Université Paris 7. During that visit she suggested the problem of computing the Iwahori invariants in \( p \)-modular Steinberg representations: this was the origin of the present work. Later she gave helpful comments on a preliminary version of this paper. I am extremely grateful to Peter Schneider. Having explained to him an unnecessarily complicated proof of Theorem 1, valid only in a restricted setting, he insisted on getting a better conceptual understanding. His numerous suggestions were decisive for approaching Theorem 1 in the correct context and for discovering the proof in its full generality. He also outlined some possible further developments. Thanks go to Yacine Aït Amrane for an enlightening discussion on harmonic (co)chains and special representations. Thanks also to Guy Henniart for his interest in this work, and to Alain Genestier for a suggestion concerning the remaining cases of exceptional groups. I thank the universities Paris 7 and Paris 13 for giving me the opportunity to present my results in their seminars. I thank the Deutsche Forschungs Gemeinschaft (DFG) as this work was done while I was supported by the DFG as a Heisenberg fellow.

**Contents**

1 Reflection groups  
2 Functions on the Iwahori subgroup  
3 Special representations of finite reductive groups  
4 Irreducibility in the residual characteristic  
5 Harmonic Chains  

1 Reflection groups

In this section we collect some results on finite crystallographic reflection groups. Proposition 1.2 will be needed for Theorem 2.3, the embedding of \( \text{Sp}_J(G, L) \) into \( C^\infty(I, \mathcal{M}_J(L)) \). Lemma 1.5 will be needed for Proposition 3.4 which concerns the \( \mathcal{H}(G, \overline{P}; L) \)-module structure of \( \text{Sp}_J(G, L)^I \), and Corollary 1.7 will be needed for the proof of Theorem 4.2 on the irreducibility of \( \text{Sp}_J(G, L)^I \) as a \( \mathcal{H}(G, I; L) \)-module.

Consider a reduced crystallographic root system \( \Phi \) and let \( W \) be its corresponding Weyl group. Fix a system \( \Delta \subset \Phi \) of simple roots and denote by \( \Phi^+ \subset \Phi \) the corresponding set of positive roots. Let \( \Phi^- = \Phi - \Phi^+ \). For \( \alpha \in \Phi \) let \( s_\alpha \in W \) denote the associated reflection. Let \( \ell(\cdot) : W \to \mathbb{Z}_{\geq 0} \) be the length function with respect to \( \Delta \). For a subset \( J \subset \Delta \) let \( W_J \subset W \) be the subgroup generated by all \( s_\alpha \) for \( \alpha \in J \). Let

\[
\Phi_J(1) = \Phi^- - (\Phi^- \cap W_J.J)
\]
where \( W_J \cdot J = \{ w\alpha | w \in W_J, \alpha \in J \} \subset \Phi \) is the sub-root system generated by \( J \). For \( w \in W \) we then define the subset
\[
\Phi_J(w) = w\Phi_J(1)
\]
of \( \Phi \). It depends only on the class of \( w \) in \( W/W_J \). Observe \( \Phi_J(w) \subset \Phi_J(w) \) for \( J \subset J' \). We say that a subset \( D \subset \Phi \) is \( J \)-quasi-parabolic if it is the intersection of subsets \( \Phi_J(w) \) for some (at least one) \( w \in W \). Let
\[
W^J = \{ w \in W | w(J) \subset \Phi^+ \}.
\]
It is well known (cf. e.g. [15], remark after definition 6) that this is a set of representatives for \( W/W_J \) and can alternatively be described as
\[
W^J = \{ w \in W | \ell(ws_\alpha) > \ell(w) \text{ for all } \alpha \in J \}.
\]
For a subset \( D \subset \Phi \) let
\[
W^J(D) = \{ w \in W^J | D \subset \Phi_J(w) \}.
\]
Let
\[
V^J = W^J - \bigcup_{\alpha \in \Delta - J} W^{J \cup \{ \alpha \}}.
\]
Then \( W = \cup_{J \subset \Phi} V^J \) (disjoint union). We have
\[
V^J = \{ w \in W^J | w(\Delta - J) \subset \Phi^- \}.
\]

**Lemma 1.1.** For \( J \subset J' \) and \( w \in W^{J'} \) we have \( \Phi_J(w) - \Phi_{J'}(w) \subset \Phi^- \).

**Proof:** Each element in \( \Phi_J(w) - \Phi_{J'}(w) = w(\Phi_J(1) - \Phi_{J'}(1)) \) can be written as \( w(\sum_{\nu} -\alpha_\nu) \) with certain \( \alpha_\nu \in J' \). As \( w \in W^{J'} \) the claim follows. \( \square \)

Let \( L \) be a ring. For a set \( S \) let \( L[S] \) denote the free \( L \)-module with basis \( S \).

**Definition:** We define the \( L \)-module \( \mathfrak{M}_J(L) \) and the \( L \)-linear map \( \nabla \) by the exact sequence of \( L \)-modules
\[
\bigoplus_{\alpha \in \Delta - J} L[W^{J \cup \{ \alpha \}}] \xrightarrow{\partial} L[W^J] \xrightarrow{\nabla} \mathfrak{M}_J(L) \xrightarrow{} 0
\]
where for \( w \in W^{J \cup \{ \alpha \}} \) we set
\[
\partial(w) = \sum_{w' \in W^J, w'wJ \subset W^{J \cup \{ \alpha \}}} w'.
\]

**Proposition 1.2.** (a) \( \nabla \) induces a bijection between \( V^J \) and an \( L \)-basis of \( \mathfrak{M}_J(L) \); in particular, \( \mathfrak{M}_J(L) \) is \( L \)-free of rank \( |V^J| \), and \( \mathfrak{M}_J(L') = \mathfrak{M}_J(L) \otimes_L L' \) for any ring morphism \( L \rightarrow L' \).

(b) Let \( D \subset \Phi \) be a \( J \)-quasi-parabolic subset. We have \( \partial(\bigoplus_{\alpha \in \Delta - J} L[W^{J \cup \{ \alpha \}}(D)]) \subset L[W^J(D)] \), and the sequence
\[
\bigoplus_{\alpha \in \Delta - J} L[W^{J \cup \{ \alpha \}}(D)] \xrightarrow{\partial^D} L[W^J(D)] \xrightarrow{\nabla^D} \mathfrak{M}_J(L)
\]
obtained by restricting (4) is exact.
PROOF: For $w \in W^{J_{\cup} (\alpha)}$ and $w' \in W^J$ with $w' W_J \subset wW_{J_{\cup} (\alpha)}$ we have $\Phi_{J_{\cup} (\alpha)} (w) = \Phi_{J_{\cup} (\alpha)} (w') \subset \Phi_J (w')$. This shows $\partial (\oplus_{\alpha \in \Delta - J} L[W^{J_{\cup} (\alpha)} (D)]) \subset L[W^J (D)]$, for any subset $D$ of $\Phi$.

First Step: Let $D \subset \Phi^+$ be a subset. Define $\mathfrak{M}_{J,D} (L)$ and $\mathfrak{V}^D$ by the exact sequence

$$
\bigoplus_{\alpha \in \Delta - J} L[W^{J_{\cup} (\alpha)} (D)] \xrightarrow{\partial^D} L[W^J (D)] \xrightarrow{\mathfrak{V}^D} \mathfrak{M}_{J,D} (L) \longrightarrow 0.
$$

Let $V^J (D) = V^J \cap W^J (D)$.

Claim: For all $\ell$ and all $w \in V^J (D)$ with $\ell (w) \geq \ell$ we have $\mathfrak{V}^D (w) \in \mathfrak{V}^D (L[V^J (D)])$.

We prove this by descending induction on $\ell$. Suppose we are given such a $w \in V^J (D)$ with $\ell (w) \geq \ell$. If $w \in V^J$ we are done. Otherwise there is some $\alpha \in \Delta - J$ with $w \in W^{J_{\cup} (\alpha)}$. By Lemma 1.1 we have $\Phi_J (w) - \Phi_{J_{\cup} (\alpha)} (w) \subset \Phi^-$, thus our assumption $D \subset \Phi^+$ implies even $w \in W^{J_{\cup} (\alpha)} (D)$. For all $w' \in W^J - \{w\}$ with $w' W_J \subset wW_{J_{\cup} (\alpha)}$ we have $\ell (w') > \ell (w)$ (because $w' W_J \subset wW_{J_{\cup} (\alpha)}$ implies $w' W_{J_{\cup} (\alpha)} = wW_{J_{\cup} (\alpha)}$, but in view of (3) we know that $w$ is the unique element of $wW_{J_{\cup} (\alpha)}$ of minimal length). Moreover we have $w' \in W^J (D)$ (as noted at the beginning of this proof), thus by induction hypothesis we get $\mathfrak{V}^D (w') \in \mathfrak{V}^D (L[V^J (D)])$ for all such $w'$. Now

$$w = \partial^D (w) - \sum_{w' \in W^J - \{w\}, w' W_J \subset wW_{J_{\cup} (\alpha)}} w'$$

(inside $L[W^J (D)]$) which shows $\mathfrak{V}^D (w) \in \mathfrak{V}^D (L[V^J (D)])$, as desired.

The claim is proved. In particular, setting $\ell = 0$, we get $\mathfrak{V}^D (L[V^J (D)]) = \mathfrak{M}_{J,D} (L)$.

Second Step: Here we prove (a). That the image of $V^J$ generates the $L$-module $\mathfrak{M}_J (L)$ follows from the first step (with $D = \emptyset$ there). To see that it remains linearly independent we may assume $L = \mathbb{Z}$ (because the situation for general $L$ arises by base change $\mathbb{Z} \to L$ from the one with $L = \mathbb{Z}$). But then, to prove the linear independence we may just as well assume $L = \mathbb{Q}$ and our task is to show $\dim_{\mathbb{Q}} \mathfrak{M}_J (\mathbb{Q}) = |V^J|$.

By definition, the $\mathbb{Q}$-vector spaces $\mathbb{Q}[W^J]$ and $\mathbb{Q}[W^J_{J_{\cup} (\alpha)}]$ come with the distinguished bases $W^J$ and $W^J_{J_{\cup} (\alpha)}$, hence with isomorphisms with their duals $\mathbb{Q}[W^J] \cong \mathbb{Q}[W^J]^*$ and $\mathbb{Q}[W^J_{J_{\cup} (\alpha)}] \cong \mathbb{Q}[W^J_{J_{\cup} (\alpha)}]^*$. One easily checks that under these identifications, the map

$$L[W^J] \xrightarrow{\partial^*} \bigoplus_{\alpha \in \Delta - J} L[W^J_{J_{\cup} (\alpha)}]$$

dual to $\partial$ is given as follows: for $w' \in W^J$ the $\alpha$-component of $\partial^* (w')$ is the unique $w \in W^J_{J_{\cup} (\alpha)}$ with $w' W_{J_{\cup} (\alpha)} = wW_{J_{\cup} (\alpha)}$. Therefore the kernel of $\partial^*$ is the $\mathbb{Q}$-vector space generated by $\cap_{\alpha} (W^J \cap W_{J_{\cup} (\alpha)}) = \cap_{\alpha} (W^J - W^J_{J_{\cup} (\alpha)}) = W^J - \cup_{\alpha} W^J_{J_{\cup} (\alpha)} = V^J$. Thus $\dim_{\mathbb{Q}} \mathfrak{M}_J (L) = \dim_{\mathbb{Q}} \ker (\partial) = \dim_{\mathbb{Q}} \ker (\partial^*) = |V^J|$.
Third Step: Here we prove (b). As $D$ is $J$-quasi-parabolic we find some $w \in W$ with $wD \subset \Phi^+$. We have a commutative diagram

$$\begin{array}{c}
\bigoplus_{a \in \Delta - J} L[W^J \{a\} (D)] \xrightarrow{\partial^D} L[W^J (D)] \xrightarrow{\nabla^D} \mathfrak{M}_J (L) \\
\bigoplus_{w \in W^J \{a\}} L[W^J \{a\} (wD)] \xrightarrow{\partial^{wD}} L[W^J (wD)] \xrightarrow{\nabla^{wD}} \mathfrak{M}_J (L)
\end{array}$$

where the second and the third (resp. the first) vertical isomorphism is induced by the bijection $W^J \to W^J$, $w' \mapsto (ww')^J$ (resp. $W^J \{a\} \to W^J \{a\}$, $w' \mapsto (ww')^J \{a\}$); here $(v)^J$ for $v \in W$ and $J' \subset \Delta$ denotes the unique representative in $W^{J'}$ of the class of $v$ in $W/W_{J'}$. Therefore we may assume from the beginning that $D \subset \Phi^+$. It suffices to see that the natural map $\mathfrak{M}_{J,D} (L) \to \mathfrak{M}_J (L)$ is injective. By (a) we know that the image of $V^J$, hence in particular the image of $V^J (D)$ in $\mathfrak{M}_J (L)$ is linearly independent. Together with the result of the first step this shows the wanted injectivity of $\mathfrak{M}_{J,D} (L) \to \mathfrak{M}_J (L)$. \qed

For $w \in W$ let $(w)^J$ denote the unique element of $W^J$ with $(w)^J w_J = wW_J$. Thus, $(.)^J$ is the projection from $W$ onto the first factor in the direct product decomposition $W = W^J W_J$. Loosely speaking, applying $(.)^J$ means cutting off $W_J$-factors on the right.

We write $S = \{s_\alpha | \alpha \in \Delta\}$. Consider the following partial ordering $< J$ on $W^J$. For $w, w' \in W^J$ we write $w < J w'$ if there are $s_1, \ldots, s_r \in S$ such that, setting $w^{(i)} = (s_i \ldots s_1 w)^J$ for $0 \leq i \leq r$, we have $\ell(w^{(i)}) > \ell(w^{(i-1)})$ for all $i \geq 1$, and $w^{(r)} = w'$. We denote by $w_\Delta \in W$ resp. $w_J \in W_J$ the respective longest elements.

**Lemma 1.3.** (a) For any $w \in W$ we have $l(w) \geq l((w)^J)$.
(b) For $w_1 \in W_J$ and $w_2 \in W_J$ we have $l(w_1 w_2) = l(w_1) + l(w_2)$.
(c) For any $w \in W$ we have $l(w_\Delta w) = l(w w_\Delta) = l(w_\Delta) - l(w)$.

**Proof:** Any $v \in W^J$ is the unique element of minimal length in the set of representatives for the coset $vW_J$; this gives (a). For the easy statements (b) and (c) see [6] 1.8 and 1.10. \qed

**Lemma 1.4.** Let $w \in W^J$ and $s \in S$.

(a) $w < J (sw)^J$ implies $\ell(w) < \ell(sw)$.
(b) $\ell(w) < \ell(sw)$ and $w \neq (sw)^J$ together imply $sw \in W^J$, hence $w < J (sw)^J = sw$.
(c) $(sw)^J < J w$ if and only if $\ell(sw) < \ell(w)$.
(d) There exists a unique maximal element $z^J \in W^J$ for the ordering $< J$; it lies in $V^J$. We have $z^J = w_\Delta w_J$. For any $u \in W$ such that $z^J \leq_\emptyset u$ and for any $s \in S$ with $\ell(sz^J) < \ell(z^J)$ we have $\ell(sw) < \ell(w)$.

**Proof:** (a) We have $l(w) < l((sw)^J) \leq l(sw)$ where the first inequality follows from the definition of $< J$ and the second one from Lemma 1.3 (a) (applied to $sw$).
To prove (b) assume $\ell(w) < \ell(sw)$ and $sw \notin W^J$. Then we find some $\alpha \in J$ with $\ell(sws_\alpha) =

\[8\]
$\ell(sw) - 1 = \ell(w)$. Take a reduced expression $w = \sigma_1 \ldots \sigma_r$ with $\sigma_i \in S$. By the deletion condition for Weyl groups we get a reduced expression for $sws_\alpha$ by deleting some factors in the string $s_1 \ldots \sigma_r s_\alpha$. Namely, as $\ell(sw) = \ell(w)$, exactly two factors must be deleted. If $s$ remained this would mean $\ell(sw) < \ell(w)$, contradicting $w \in W^J$. If $s_\alpha$ remained this would mean $\ell(sw) < \ell(w)$, contradicting our hypothesis. Thus $sws_\alpha = w$, i.e. $w = (sw)^J$.

(c) First assume $\ell(sw) < \ell(w)$. Then we get $l((sw)^J) < l(w)$ from Lemma 1.3 (a) (applied to $sw$). As $(sw)^J = \ell(w)$ we get $(sw)^J < w$ from the definition of $< J$. Now assume $\ell(sw) > \ell(w)$ and $(sw)^J < w$. Then there are $\alpha_1, \alpha_2 \in J$ such that $\ell(sw_{\alpha_1 \alpha_2}) < \ell(w)$. On the other hand, $w \in W^J$ implies $\ell(sw_{\alpha_1}) > \ell(w)$ and $\ell(sw_{\alpha_2}) > \ell(w)$. From $\ell(sw) > \ell(w)$ (or from $\ell(sw_{\alpha_1}) > \ell(w)$) together with $\ell(sw_{\alpha_1 \alpha_2}) < \ell(w)$ it follows that $\ell(sw_{\alpha_1}) = \ell(w)$. As $\ell(sw) > \ell(w)$ and $\ell(sw_{\alpha_1 \alpha_2}) > \ell(w)$ this implies $w = sws_\alpha$ as in the proof of (b). But then $\ell(sw_{\alpha_2}) > \ell(w)$ contradicts $\ell(sw_{\alpha_1 \alpha_2}) < \ell(w)$.

(d) From Lemma 1.3 (c) it follows that $(sw)_J = \Delta w_J$. We claim that $z^J = (sw)_J = \Delta w_J$ is maximal in $W^J$ with respect to $< J$, and is uniquely determined by this property. To see this we need to show, by (b), that for any $w \in W^J - \{z^J\}$ there is some $s \in S$ with $\ell(sw) > \ell(w)$ and $w \neq (sw)^J$. As $w \neq z^J = \Delta w_J$ we find $s \in S$ with $\ell(sw) = \ell(w) + 1$, hence

$$\ell(sw) \geq \ell(sw \cdot w_J) = \ell(w_J) + 1 - \ell(w_J) > \ell(w)$$

where we used $\ell(w_J) = \ell(w) + \ell(w_J)$ as recorded in Lemma 1.3 (b). If we had $w = (sw)^J$ this would mean $sw = uw$ for some $u \in W_J$, hence $\ell(sw_J) = \ell(uw) \leq \ell(w_J)$ by Lemma 1.3 (b): contradiction!

Finally, we have $z^J = \Delta w_J = w_J \Delta$ for

$$\tilde{J} = \{ 1 \in \Delta | s_1 = w_\Delta s_{\alpha_1} w_\Delta \text{ for some } \alpha \in J \}.$$

For $u \in W$ such that $z^J = w_J w_\Delta < u = (uw_\Delta)w_\Delta$ we get $uw_\Delta \in W_J$. Similarly, $z^J = w_J w_\Delta$ means that $\ell(sz^J) < \ell(z^J)$ for $s \in S$ can only happen if $s = s_\alpha$ for some $\alpha \in \Delta - J$. Therefore $\ell(sw_\Delta) > \ell(uw_\Delta)$ since $uw_\Delta \in W_J$. By Lemma 1.3 (c) this means $\ell(sw) < \ell(u)$. □

**Lemma 1.5.** For each $w \in V^J - \{z^J\}$ there is some $w' \in V^J$ and some $s \in S$ with $w < J w'$, with $\ell((sw)^J) < \ell(w)$ and with $\ell((sw)^J) \geq \ell(w')$.

**Proof:** Consider the set

$$J' = \{ \alpha \in \Delta \ | \ \ell(s_\alpha w) > \ell(w) \}$$

and let $w_{J'}$ denote the longest element of $W_{J'}$. For any given $\alpha \in \Delta$ we have $\alpha \notin J'$ if and only if $\ell((s_\alpha w)^J) < \ell(w)$, by Lemma 1.4.

**Case (i):** $z^J w^{-1} \notin W_{J'}$. Take a reduced expression $z^J w^{-1} = \sigma_1 \ldots \sigma_r$ with $\sigma_i \in S$. Let $1 \leq i \leq r$ be maximal such that $\sigma_r = s_\alpha$ for some $\alpha \in \Delta - J'$ (such an $i$ exists since $z^J w^{-1} \notin W_{J'}$). By Lemma 1.4(c) we have $\ell(z^J) = r + \ell(w)$, by Lemma 1.4(b) we then see
$w' \in V^J$ for $w' = \sigma_{i+1} \ldots \sigma_rw$. This $w'$ together with $s = s_\alpha$ is fine. 

Case (ii): $z^Jw^{-1} \in W_{J'}$. Here we claim that $w' = z^J$ satisfies the wanted conclusion. Assume on the contrary that $\ell(s_\alpha z^J) < \ell(z^J)$ for all $\alpha \in \Delta - J'$. Then we also have $\ell(s_\alpha w_{J'}w) < \ell(w_{J'}w)$ for all $\alpha \in \Delta - J'$. This follows from Lemma 1.4(d) since $z^Jw^{-1} \in W_{J'}$ implies $z^J \leq w_{J'}w$. On the other hand $\ell(s_\alpha w_{J'}w) < \ell(w_{J'}w)$ for all $\alpha \in J'$, too (because $\ell(w_{J'}w) = \ell(w_{J'}) + \ell(w)$ as follows from the definition of $J'$), hence for all $\alpha \in \Delta$. This means $w_{J'}w = w_\Delta$. But then $w = w_\Delta w_J$ for some $J \subset \Delta$ (as in the proof of Lemma 1.4(d)). As $V^J \cap V^J = \emptyset$ for $J \neq J'$ this shows $J = J'$ and $w = z^J$, contradicting our hypothesis $w \neq z^J$. □

The next result concerns the partial ordering $<_\emptyset$ of $W$ (i.e. $<_J$ for $J = \emptyset$), called the weak ordering of $W$ in [2].

Consider the following subgroup $W_\Omega$ of $W$. We write our set of simple roots as $\Delta = \{\alpha_1, \ldots, \alpha_l\}$ and denote by $\alpha_0 \in \Phi$ the unique highest root. Then we define the elements $\epsilon_1, \ldots, \epsilon_l$ in the $\mathbb{R}$-vector space dual to the one spanned by $\Phi$ by requiring $(\epsilon_i, \alpha_j) = \delta_{ij}$ for $1 \leq i, j \leq l$. For $1 \leq i \leq l$ we let $w_{\Delta(i)} \in W$ denote the longest element of the subgroup of $W$ generated by the set $\{s_{\alpha_j} | j \neq i\}$. Then

$$W_\Omega - \{1\} = \{w_{\Delta(i)}w_\Delta | 1 \leq i \leq l, (\epsilon_i, \alpha_0) = 1\}.$$ 

The conjugation action of $W_\Omega$ on $\{s_{\alpha_0}, s_{\alpha_1}, \ldots, s_{\alpha_l}\}$ identifies $W_\Omega$ with the automorphism group of the Dynkin diagram of the affine root system (see [8] pp. 18-20).

**Proposition 1.6.** Suppose that the underlying root-system is of type $A_l$, $B_l$, $C_l$ or $D_l$. There exists a sequence $w_\Delta = w_0, w_1, \ldots, w_r = 1$ in $W$ such that for all $i \geq 1$ we have $w_{i-1} <_\emptyset w_i$, or $w_i = uw_{i-1}$ for some $u \in W_\Omega$.

**Proof:** We use the respective descriptions of $W_\Omega$ given in [8] pp. 18-20. We write $s_i = s_{\alpha_i}$.

Case $A_l$: Then $W$ can be identified with the symmetric group in $\{1, \ldots, l+1\}$. We write an element $w \in W$ as the tuple $[w(1), \ldots, w(l+1)]$. As simple reflections we take the transpositions $s_i = [1, \ldots, i-1, i+1, i, i+2, \ldots, l] \in W$ for $i = 1, \ldots, l$. Then $W_\Omega$ consists of the elements

$$w_{\Delta(i)}w_\Delta = [i+1, \ldots, l+1, 1, \ldots, i] \quad (0 \leq i \leq l).$$

The length $\ell(w)$ of $w \in W$ is the number of all pairs $(i, j)$ with $i < j$ and $w(i) > w(j)$. We pass from $w_\Delta$ to 1 via the sequence

$$w_\Delta = [l+1, \ldots, 1] \xrightarrow{(\ast)} [1, l+1, \ldots, 2] <_\emptyset [l, l+1, l-1, \ldots, 1]$$

$$\xrightarrow{(\ast)} [1, 2, l+1, l, \ldots, 3] <_\emptyset [l-1, l, l+1, l-2, \ldots, 1]$$

$$\xrightarrow{(\ast)} \ldots <_\emptyset [2, \ldots, l+1, 1] \xrightarrow{(\ast)} [1, \ldots, l+1] = 1.$$

Here each step of type $(\ast)$ is obtained by left-multiplication with an element of $W_\Omega$.

Case $B_l$: Here $W$ can be identified with the group of signed permutations of $\{\pm 1, \ldots, \pm l\}$, i.e.
with all bijections \( w : \{ \pm 1, \ldots, \pm l \} \to \{ \pm 1, \ldots, \pm l \} \) satisfying \(-w(a) = w(-a)\) for all \(1 \leq a \leq l\).

We write an element \( w \in W \) as the tuple \( [w(1), \ldots, w(l)] \). As simple reflections we take the elements \( s_i = [1, \ldots, l - i - 1, l - i + 1, l - i, l - i + 2, \ldots, l] \) for \( 1 \leq i \leq l - 1 \), together with \( s_l = [-1, 2, \ldots, l] \). Then the length of \( w \in W \) can be computed as

\[
\ell(w) = |\{(ij) : i < j, w(i) > w(j)\}| - \sum_{w(j) < 0} w(j)
\]

(for all this see [2] chapter 8.1). The group \( W_\Omega \) consists of two elements, its non-trivial element is

\[
w_\Delta \Delta = [1, \ldots, l - 1, -l].
\]

For \( 1 \leq i \leq l \) let

\[
a_i = [-i, \ldots, -l, l - 1, \ldots, 1],
\]

\[
b_i = [-i, \ldots, 1 - l, l, i - 1, \ldots, 1].
\]

We pass from \( w_\Delta \) to 1 via the sequence

\[
w_\Delta = [-1, \ldots, -l] = a_1 \rightarrow a_1 < \emptyset a_2 \rightarrow a_2 < \emptyset a_3 \rightarrow \ldots
\]

\[
\ldots < \emptyset a_l \rightarrow a_l b_l = [l, 1, \ldots, 1] \rightarrow [1, \ldots, l] = 1.
\]

Here the relations \(< \emptyset \) result from left-multiplications with \( s_{l-1} \ldots s_1 \), increasing the length by \( l - 1 \), as one easily checks. Each step of type \((*)\) is obtained by left-multiplication with \( w_\Delta \Delta \).

It remains to justify the step \((**)\). Observe that

\[
w_\Delta \Delta s_1 \ldots s_l = [l, 1, \ldots, l - 1].
\]

Moreover, for each \( w \in W \) satisfying \( w(i) > 0 \) for all \( 1 \leq i \leq l \) we have \( w < \emptyset s_1 \ldots s_l w \). Together it follows that, to prove that the step \((**)\) is permissible, it suffices to show that \((**)\) decomposes into left-multiplications with (powers of) \([l, 1, \ldots, l - 1]\), and transpositions \( s_1, \ldots, s_{l-1} \). But this was shown in our analysis of case \( A_l \).

**Case \( C_l \):** Here \( W \) is the same as in case \( B_l \) and we take the same simple reflections. Again \( W_\Omega \) consists of two elements, but this time its non-trivial element is

\[
w_\Delta \Delta = [-l, \ldots, -1].
\]

We pass from \( w_\Delta \) to 1 via the sequence

\[
w_\Delta = [-1, \ldots, -l] \rightarrow [l, 1, \ldots, l] \rightarrow [1, \ldots, l] = 1.
\]

Here \((*)\) is obtained by left-multiplication with \( w_\Delta \Delta \). To justify the step \((**)\) observe that

\[
w_\Delta \Delta s_l w_\Delta \Delta s_1 \ldots s_l = [l, 1, \ldots, l - 1].
\]
Moreover, for each \( w \in W \) satisfying \( w(i) > 0 \) for all \( 1 \leq i \leq l \) we have \( w <_{B} s_{1} \ldots s_{l} w \) (as already noted above), and
\[
w_{\Delta(1)} w_{\Delta} s_{1} \ldots s_{l} w <_{B} s_{l} w_{\Delta(1)} w_{\Delta} s_{1} \ldots s_{l} w.
\]
Thus left-multiplication of \([l, 1, \ldots, l - 1] \) to such \( w \in W \) is a permissible operation for our purposes. Therefore we may conclude as in the case \( B_{l} \).

**Case \( D_{l} \):** Here \( W \) can be identified with the group of signed permutations of \( \{\pm 1, \ldots, \pm l\} \) having an even number of negative entries, i.e. with all bijections \( w : \{\pm 1, \ldots, \pm l\} \to \{\pm 1, \ldots, \pm l\} \) satisfying \( -w(a) = w(-a) \) for all \( 1 \leq a \leq l \), and such that the number \( |\{i \mid w(i) < 0\}| \) is even. We write an element \( w \in W \) as the tuple \([w(1), \ldots, w(l)]\). As simple reflections we take the elements \( s_{j} \) for \( 1 \leq i \leq l - 1 \) used in cases \( B_{l} \) and \( C_{l} \), together with
\[
s_{l} = [-2, -1, 3, \ldots, l].
\]
The length of \( w \in W \) can be computed (see [2] chapter 8.2) as
\[
\ell(w) = |\{(ij) \mid i < j, w(i) > w(j)\}| + |\{(ij) \mid w(i) + w(j) < 0\}|.
\]
\( W_{\Omega} \) consists of the four elements \( 1, w_{\Delta(1)} w_{\Delta}, w_{\Delta(l-1)} w_{\Delta} \) and \( w_{\Delta(l)} w_{\Delta} \). We have
\[
w_{\Delta(1)} w_{\Delta} = [-1, 2, \ldots, l - 1, -l]
\]
and, according to the parity of \( l \),
\[
w_{\Delta(1)} w_{\Delta} = [-l, \ldots, -1] \quad (l \text{ even})
\]
\[
w_{\Delta(l)} w_{\Delta} = [l, 1 - l, \ldots, -1] \quad (l \text{ odd})
\]
(and \( w_{\Delta(l-1)} w_{\Delta} = [l, 1 - l, \ldots, -2, 1] \) if \( l \) is even, \( w_{\Delta(l-1)} w_{\Delta} = [-l, \ldots, -2, 1] \) is \( l \) is odd). We pass from \( w_{\Delta} \) to 1 via the sequence
\[
w_{\Delta} = [-1, \ldots, -l] \xrightarrow{(s)} [l, \ldots, 1] \xrightarrow{(**) \ 1, \ldots, l] = 1 \quad (l \text{ even})
\]
\[
w_{\Delta} = [1, -2, \ldots, -l] \xrightarrow{(s)} [l, \ldots, 1] \xrightarrow{(**) \ 1, \ldots, l] = 1 \quad (l \text{ odd}).
\]
Here \( (s) \) is obtained by left-multiplication with \( w_{\Delta(1)} w_{\Delta} \). To justify the step \( (**) \) observe that
\[
w_{\Delta(1)} s_{1} \ldots s_{l-2} = [l, 1, \ldots, l - 1].
\]
For each \( w \in W \) with \( w(i) > 0 \) for all \( 1 \leq i \leq l - 2 \) we have \( w <_{B} s_{1} \ldots s_{l-2} w \). Thus left-multiplication of \([l, 1, \ldots, l - 1] \) to such \( w \in W \) is a permissible operation for our purposes and we may conclude as before. \( \square \)

**Corollary 1.7.** For each \( w \in V^{J} \) there is a sequence \( z^{J} = w_{0}, w_{1}, \ldots, w_{r} = w \) in \( W \) such that for all \( i \geq 1 \) we have \( w_{i}^{J} = uw_{i-1}^{J} \) for some \( u \in W_{\Omega} \), or \( \ell(w_{i-1}^{J}) < \ell(w_{i}^{J}) \) and \( w_{i}^{J} = s(w_{i-1}^{J}) \) for some \( s \in S \). |
Proof: Recall that $z^J = (w_{\Delta})^J$. Furthermore observe that $\ell(w') < \ell(w)$ and $w = sw'$ for some $s \in S$ implies that $[w^J = s(w')^J$ and $\ell((w')^J) < \ell(w^J)]$ or $w^J = (w')^J$. Thus the corollary follows from Proposition 1.6. \qed

Remark: For the irreducible reduced root systems of type $E_8$, $F_4$ and $G_2$ we have $W_\Omega = \{1\}$ by [8]. Therefore the statement of Proposition 1.6 cannot hold true in these cases. We do not discuss the cases $E_6$, $E_7$.

2 Functions on the Iwahori subgroup

Let $F$ be a non-Archimedean locally compact field, $\mathcal{O}_F$ its ring of integers, $p_F \in \mathcal{O}_F$ a fixed prime element and $k_F$ its residue field. Let $G$ be a split reductive group over $F$, connected and different from its center. (Here we commit the usual abuse of notation: what we really mean is that $G$ is the group of $F$-rational points of such an algebraic $F$-group scheme, similarly for the subgroups considered below.) Let $T$ be a split maximal torus, $N \subset G$ its normalizer in $G$ and let $W = N/T$, the corresponding Weyl group. For any $w \in W$ we choose a representative (with the same name) $w \in N$. Let $P = TU$ be a Borel subgroup with unipotent radical $U$. Let $\Phi \subset X^*(T) = \text{Hom}_{alg}(T, G_m)$ be the set of roots, let $\Phi^+ \subset \Phi$ be the set of $P$-positive roots, let $\Delta \subset \Phi^+$ be the set of simple roots. Since $T$ is split this root system is reduced.

For $\alpha \in \Phi$ let $U_\alpha \subset G$ be the associated root subgroup. Then $U = \prod_{\alpha \in \Phi^+} U_\alpha$ (direct product, for any ordering of $\Phi^+$). We need the parabolic subgroups $P_J = PW_JP$ of $G$; each parabolic subgroup of $G$ containing $P$ is of this form (for a suitable $J$). For $w \in W$ let $P_{J,w} = wP_Jw^{-1}$ and let $P_{J,w}^-$ be the parabolic subgroup of $G$ opposite to $P_{J,w}$. We then find

$$\Phi - \Phi_J(w) = \{\alpha \in \Phi \mid U_\alpha \subset P_{J,w}\}$$

or equivalently: $\prod_{\alpha \in \Phi_J(w)} U_\alpha$ is the unipotent radical of $P_{J,w}^-$. Note that $P_{J,w} = P_{J,w'}$ for any $w' \in wW_J$.

We choose an Iwahori subgroup $I$ in $G$ compatible with $P$, in the sense that we have the Iwahori decomposition

$$G = \bigcup_{w \in W} IwP$$

(disjoint union). For any subgroup $H$ in $G$ we write $H^0 = H \cap I$.

Lemma 2.1. Let $D \subset \Phi$ be a $J$-quasi-parabolic subset. Then $\prod_{\alpha \in D} U_\alpha^0$ is a subgroup of $G$ and is independent of the ordering of $D$. We denote it by $U_D^0$.

Proof: Take any ordering of $D$. Then choose an ordering of $\Phi$ which restricts to this ordering on $D$ and such that the product map

$$\prod_{\alpha \in \Phi} U_\alpha \rightarrow G$$
is injective. Write \( D = \cap_{w \in T} \Phi_J(w) \) (some \( T \subset W \)). Then of course
\[
\prod_{\alpha \in D} U^0_{\alpha} = \cap_{w \in T} \prod_{\alpha \in \Phi_J(w)} U^0_{\alpha}
\]
(all products w.r.t. the fixed ordering of \( \Phi \), and the intersection is taken inside \( G \)). Hence it is enough to see that \( \prod_{\alpha \in \Phi_J(w)} U^0_{\alpha} \) is independent of the ordering of \( \Phi_J(w) \) — but this is clear: \( \prod_{\alpha \in \Phi_J(w)} U^0_{\alpha} \) is the intersection of \( I \) with the unipotent radical of \( P^+_{J,w} \).

For a topological space \( T \) and an \( L \)-module \( M \) let \( C^\infty(T,M) \) denote the \( L \)-module of locally constant \( M \)-valued functions on \( T \).

Applying the functor \( C^\infty(\cdot, \cdot) \) to the exact sequence (4) we obtain an exact sequence
\[
(5) \quad C^\infty(I, \bigoplus_{w \in W_J} L[W^J, w]) \rightarrow C^\infty(I, L[W^J]) \rightarrow C^\infty(I, \mathfrak{M}_J(L)) \rightarrow 0.
\]

Observe that we have natural embeddings, which we view as inclusions,
\[
\bigoplus_{w \in W_J} C^\infty(I/P^0_{J,w}, L) \subset C^\infty(I, \bigoplus_{w \in W_J} L[W^J, w]),
\]
\[
\bigoplus_{w \in W^J} C^\infty(I/P^0_{J,w}, L) \subset C^\infty(I, L[W^J]),
\]
by summing over the respective direct summands.

**Proposition 2.2.** The sequence

\[
\bigoplus_{w \in W_J} C^\infty(I/P^0_{J,w}, L) \xrightarrow{\partial_C} \bigoplus_{w \in W^J} C^\infty(I/P^0_{J,w}, L) \xrightarrow{\nabla_C} C^\infty(I, \mathfrak{M}_J(L))
\]

obtained by restricting (5) is exact.

**Proof:** Choose an enumeration \( D_0, D_1, D_2, \ldots \) of all \( J \)-quasi-parabolic subsets of \( \Phi \) such that \( n < m \) implies \( |D_n| \leq |D_m| \). Let \( (f_w)_{w \in W^J} \in \text{Ker}(\nabla_C) \). By induction on \( m \) we show: adding to \( f \) an element in the image of \( \partial_C \) if necessary, we may assume \( f_w|_{U^0_{D_n}} = 0 \) for all \( w \in W^J \), all \( n \leq m \).

Assume we have \( f_w|_{U^0_{D_n}} = 0 \) for all \( w \in W^J \), all \( n < m \). Let us write \( D = D_m \).

(i) We first claim \( f_w|_{U^0_{D}} = 0 \) for all \( w \in W^J \setminus W^J(D) \). Indeed, for such \( w \) we have \( |D \cap \Phi_J(w)| < |D| \), hence \( D \cap \Phi_J(w) = D_n \) for some \( n < m \). Thus
\[
f_w(U^0_D) = f_w(U^0_{D_n} \prod_{\alpha \in D \setminus D_n} U^0_{\alpha}) = f_w(U^0_{D_n}) = 0
\]
where in the first equation we used that we may form \( U^0_D \) with respect to any ordering of \( D \), where the second equation follows from \( U^0_{\alpha} \subset P^0_{J,w} \) for \( \alpha \notin \Phi_J(w) \) (and the invariance property...
of $f_w$), and where the last equation holds true by induction hypothesis.

(ii) Our sequence in question restricts to a sequence

$$\bigoplus_{\alpha \in \Delta - J} C^\infty(I/P^0_{J \cup \{\alpha\}}, w, L) \xrightarrow{\partial C^0} \bigoplus_{w \in W^J(D)} C^\infty(I/P^0_{J,w}, L) \xrightarrow{\nabla C^0} C^\infty(I, \mathfrak{M}_f(L)).$$

For any $x \in U^0_D$, evaluating functions at $x$ transforms (6) into a sequence isomorphic with the one from Proposition 1.2 (b). Let us denote by $(\partial C^0)_x$ resp. by $(\nabla C^0)_x$ the differentials of this sequence, which by Proposition 1.2 (b) is exact. From (i) it follows that

$$f^D(x) = (f_w(x))_{w \in W^J(D)} \in \ker((\nabla C^0)_x),$$

hence this lies in the image of $(\partial C^0)_x$. For all $x \in U^0_D$ choose preimages of $f^D(x)$ under $(\partial C^0)_x$. Since the $f_w$ are locally constant, these preimages can be arranged to vary locally constantly on $U^0_D$, and moreover, in view of (i) we may assume that for all $x \in U^0_D \cap \cup_{n \leq m} U^0_{D_n}$ these preimages are zero.

For any $\alpha \in \Delta - J$ and $w \in W^{J \cup \{\alpha\}}(D)$ the natural map $U^0_D \rightarrow I/P^0_{J \cup \{\alpha\}, w}$ is injective. Thus we find an element

$$g^D = (g_{\alpha,w})_{\alpha,w} \in \bigoplus_{w \in W^{J \cup \{\alpha\}}(D)} C^\infty(I/P^0_{J \cup \{\alpha\}}, w, L),$$

which on $U^0_D$ assumes the preimages of the $f^D(x)$ just chosen, and which vanishes at all $x \in \cup_{n \leq m} U^0_{D_n}$ with $x \notin U^0_D$. We obtain

$$f^D(x) - \partial C^0(g^D)(x) = 0$$

for all $x \in \cup_{n \leq m} U^0_{D_n}$; for $x \in U^0_D$ this follows from our definition of $g^D|_{U^0_D}$, for $x \in \cup_{n \leq m} U^0_{D_n}$ with $x \notin U^0_D$ this follows from the vanishing of $g^D$ at such $x$ together with the induction hypothesis. Now set $g_{\alpha,w} = 0$ for all $\alpha \in \Delta - J$ and $w \in W^{J \cup \{\alpha\}} - W^{J \cup \{\alpha\}}(D)$. By (i) and what we just saw we find

$$((f_w)_w - \partial C((g_{\alpha,w})_{\alpha,w}))(x) = 0$$

for all $x \in \cup_{n \leq m} U^0_{D_n}$. The induction is complete. In other words, we have shown that, adding to $(f_w)_w$ an element in the image of $\partial C$ if necessary, we may assume $f_w|_{U^0_D} = 0$ for all $w \in W^J$, all $J$-quasi-parabolic subsets $D$. In particular we find $f_w|_{U^0_{\Phi J(w)}}$ for all $w \in W^J$. But $U^0_{\Phi J(w)}$ is a set of representatives for $I/P^0_{J,w}$, hence $f_w = 0$. We are done. \hfill $\square$

**Definition:** Let $J$ be a subset of $\Delta$. We define the $G$-representation $\text{Sp}_f(G, L)$ by the exact sequence of $G$-representations

$$\bigoplus_{\alpha \in \Delta - J} C^\infty(G/P_{J \cup \{\alpha\}}, L) \xrightarrow{\partial} C^\infty(G/P_J, L) \rightarrow \text{Sp}_f(G, L) \rightarrow 0,$$

where $\partial$ is the sum of the canonical inclusions, and the $G$-action is by translation of functions on $G$. We call $\text{Sp}_f(G, L)$ the $J$-special $G$-representation with coefficients in $L$. 

15
Theorem 2.3. \( \text{Sp}_J(G, L) \) is \( L \)-free. There exists an \( I \)-equivariant embedding
\[
\text{Sp}_J(G, L) \overset{\lambda_L}{\to} C^\infty(I, \mathcal{M}_J(L)).
\]
Its formation commutes with base changes: for a ring morphism \( L \to L' \) the composite
\[
\text{Sp}_J(G, L) \otimes_L L' \cong \text{Sp}_J(G, L') \overset{\lambda_L'}{\to} C^\infty(I, \mathcal{M}_J(L')) \cong C^\infty(I, \mathcal{M}_J(L)) \otimes_L L'
\]
is \( \lambda_L \otimes_L L' \).

Proof: Recall that for \( w \in W \) we defined \( P^0_{\lambda_J, w} = I \cap wP_Jw^{-1} \). Note that \( P^0_{\lambda_J, w} \) and \( wP_J \) depend only on the coset \( wW_J \), not on the specific representative \( w \in wW_J \). The same is true for the isomorphism
\[
I/P^0_{\lambda_J, w} \cong IwP_J/P_J,
\]
\( i \mapsto iw \).

It follows that for any inclusion of cosets \( wW_J \subset wW_J \cup \{ \alpha \} \) we have a commutative diagram
\[
\begin{array}{ccc}
I/P^0_{\lambda_J, w} & \cong & IwP_J/P_J \\
\downarrow & & \downarrow \\
IwP_J/P_J & \cong & IwP_{J \cup \{ \alpha \}}/P_{J \cup \{ \alpha \}}
\end{array}
\]
where the horizontal arrows are the obvious projections and the vertical arrows are the above isomorphisms. Now recall the Iwahori decompositions
\[
G/P_J = \bigcup_{w \in W_J} IwP_J/P_J, \quad G/P_{J \cup \{ \alpha \}} = \bigcup_{w \in W_{J \cup \{ \alpha \}}} IwP_{J \cup \{ \alpha \}}/P_{J \cup \{ \alpha \}}
\]
(disjoint unions). They give
\[
C^\infty(G/P_J, L) = \bigoplus_{w \in W_J} C^\infty(IwP_J/P_J, L),
\]
\[
C^\infty(G/P_{J \cup \{ \alpha \}}, L) = \bigoplus_{w \in W_{J \cup \{ \alpha \}}} C^\infty(IwP_{J \cup \{ \alpha \}}/P_{J \cup \{ \alpha \}}, L).
\]

With these identifications, the above commutative diagrams (for all \( \alpha \in \Delta - J \)) induce a commutative diagram
\[
\begin{array}{ccc}
\bigoplus_{\alpha \in \Delta - J} C^\infty(G/P_{J \cup \{ \alpha \}}, L) & \cong & C^\infty(G/P_J, L) \\
\downarrow & & \downarrow \\
\bigoplus_{w \in W_{J \cup \{ \alpha \}}} C^\infty(I/P^0_{\lambda_J, w}, L) & \cong & \bigoplus_{w \in W_J} C^\infty(I/P^0_{\lambda_J, w}, L)
\end{array}
\]
where the vertical arrows are isomorphisms. The top row is exact by the definition of \( \text{Sp}_J(G, L) \), the bottom row is exact by Proposition 2.2, and clearly all arrows are \( I \)-equivariant. Hence we get the wanted injection \( \lambda_L : \text{Sp}_J(G, L) \hookrightarrow C^\infty(I, \mathcal{M}_J(L)) \). We then derive the freeness of \( \text{Sp}_J(G, L) \): first for \( L = \mathbb{Z} \) since \( C^\infty(I, \mathcal{M}_J(\mathbb{Z})) \) is \( \mathbb{Z} \)-free, then by base change \( \mathbb{Z} \to L \) for any \( L \). Similarly we get the stated base change property. □
Corollary 2.4. (Conjectured by Vignéras [15]) The submodule $\text{Sp}_J(G,L)^I$ of $I$-invariants in $\text{Sp}_J(G,L)$ is free of rank

$$\text{rk}_L(\text{Sp}_J(G,L)^I) = \text{rk}_L(\mathcal{M}_J(L)) = |V_J|^I.$$ 

Proof: The inequality $\text{rk}_L(\text{Sp}_J(G,L)^I) \leq \text{rk}_L(\mathcal{M}_J(L)) = |V_J|^I$ follows from Theorem 2.3. On the other hand, by the Iwahori decomposition again, $C^\infty(G/P_J,L)$ is free of rank $|W_J|$. ([15] Proposition 9). Now $W_J$ is the disjoint union of all $V_{J'}$ with $J' \supset J$. Since $C^\infty(G/P_J,L)$ admits a $G$-equivariant filtration whose graded pieces are the $\text{Sp}_{J'}(G,L)$, the inequalities $\text{rk}_L(\text{Sp}_{J'}(G,L)^I) \leq \text{rk}_L(\mathcal{M}_{J'}(L)) = |V_{J'}|^I$ for all $J' \supset J$ imply the inequality $\text{rk}_L(\text{Sp}_J(G,L)^I) \geq \text{rk}_L(\mathcal{M}_J(L)) = |V_J|^I$.

Alternatively, the bijectivity of $\text{Sp}_J(G,L)^I \longrightarrow C^\infty(I,\mathcal{M}_J(L))^I \cong \mathcal{M}_J(L)$ follows immediately from the proof of Theorem 2.3, namely from the surjectivity of

$$\bigoplus_{w \in W_J} C^\infty(I/P^0_{J,w},L)^I \longrightarrow C^\infty(I,\mathcal{M}_J(L))^I$$

which we get from the very definition of $\mathcal{M}_J(L)$. \hfill $\square$

Corollary 2.5. Let $\pi$ be a smooth irreducible (hence finite dimensional) representation of $I$ on a $\mathbb{C}$-vector space. Then $\pi$ occurs in $\text{Sp}_J(G,L)$ with multiplicity at most $|V_J| \dim_{\mathbb{C}}(\pi)$.

Proof: $\pi$ occurs in $C^\infty(I,\mathcal{M}_J(\mathbb{C}))$ with multiplicity $|V_J| \dim_{\mathbb{C}}(\pi)$. \hfill $\square$

Remark: If $L$ is a complete field extension of $F$ we may replace all spaces of locally constant functions occurring here by the corresponding spaces of locally $F$-analytic functions. In particular we may define locally analytic $G$-representations $\text{Sp}^{an}_J(G,L)$ and $C^{an}(I,\mathcal{M}_J(L))$. Then Theorem 2.3 and Corollary 2.4 carry over, with the same proofs: there exists an $I$-equivariant embedding

$$\text{Sp}^{an}_J(G,L) \hookrightarrow C^{an}(I,\mathcal{M}_J(L))$$

and we have $\text{rk}_L(\text{Sp}^{an}_J(G,L)^I) = \text{rk}_L(\mathcal{M}_J(L)) = |V_J|^I$.

3 Special representations of finite reductive groups

Now we assume in addition that $G$ is semisimple and that the root system $\Phi$ is irreducible. There is a unique chamber $C$ in the standard apartment associated to $T$ in the Bruhat-Tits-building of $G$ which is fixed by our Iwahori subgroup $I$. Let $x_0$ be the special vertex of $C$ corresponding to our Borel subgroup $P$ (see below for what this means). Let $G_{x_0}/\mathcal{O}_F$ denote the $\mathcal{O}_F$-group
scheme with generic fibre the underlying $F$-group scheme $G$ of $G = G(F)$ and such that for each unramified Galois extension $F'$ of $F$ with ring of integers $\mathcal{O}_{F'}$ we have

$$G_{x_0}(\mathcal{O}_{F'}) = \{ g \in G(F') \mid gx_0 = x_0 \}$$

(see [13] 3.4). This $G_{x_0}$ is a group scheme as constructed by Chevalley ([13] 3.4.1). Its special fibre $G_{x_0} \otimes_{\mathcal{O}_F} k_F$ is a split connected reductive group over $k_F$ with the same root datum as $G$ ([13] 3.8.1; compare also [9], part II, 1.17, and for adjoint $G$ see [8] p.30/31 where the Bruhat decomposition of $G = (G_{x_0} \otimes_{\mathcal{O}_F} k_F)(k_F)$ is discussed similarly to how we are going to use it here). Let $K_{x_0} = G_{x_0}(\mathcal{O}_F)$ and

$$U_{x_0} = \text{Ker} \ [ \ K_{x_0} \rightarrow G_{x_0}(k_F) \ ] .$$

For $H$ any of the groups $G, P, T, N, U, U_\alpha$ let

$$\overline{H} = \frac{(H \cap K_{x_0}, U_{x_0})}{H \cap U_{x_0}} = \frac{H \cap K_{x_0}}{H \cap U_{x_0}} .$$

Our choice of $x_0$ above is characterized by the fact $I$ is the preimage of $P$ under the homomorphism $K_{x_0} \rightarrow \overline{G}$. On groups of $k_F$-rational points we have: $\overline{P}_J$ is a parabolic subgroup in $\overline{G}$, containing the Borel subgroup $\overline{P}$. This $\overline{P}$ has $\overline{U}$ as its unipotent radical and contains the maximal split torus $\overline{T}$, whose normalizer in $\overline{G}$ is $\overline{N}$. The quotient $\overline{N}/\overline{T}$ is canonically identified with the Weyl group $W = N/T$, and similarly as before we choose for any $w \in W$ a representative (with the same name) $w \in \overline{N}$. Let $\overline{P}^w = \overline{T} U^w$ denote the Borel subgroup opposite to $P$, with unipotent radical $\overline{U}^w$. For $w \in W$ let $\overline{U}^w = \overline{U} \cap w\overline{U}^{-1}$. Then

$$\overline{U}^w = \prod_{\alpha \in \Phi^+} \overline{U}_\alpha$$

and $\overline{U}^1 = \{1\}$. By transposition of [15] par. 4.2, Prop. 4 (b) we have

$$(7) \quad \overline{U}^w w \overline{P}_J = \overline{P}_w \overline{P}_J$$

for any $w \in W_J$, and the left hand side product is direct.

**Lemma 3.1.** Let $w \in W_J$ and $s \in S$.

(a) If $(sw)^J = w$ then

$$us \overline{U}^w w \overline{P}_J = \overline{U}^w w \overline{P}_J$$

for each $u \in \overline{U}^s$, and these are direct products.

(b) If $\ell((sw)^J) > \ell(w)$ then

$$\overline{U}^s U^w w \overline{P}_J = \overline{U}^{sw} w \overline{P}_J$$

and these are direct products.

(c) If $\ell((sw)^J) < \ell(w)$, then $w^{-1}(\beta) \in \Phi^-$, where $s = s_\beta$. The product

$$\overline{U}' = \prod_{\alpha \in \Phi^+ \setminus \{\beta\}} \overline{U}_\alpha$$

18
(any ordering of the factors) is a subgroup of $\mathcal{U}^w$. We have
\[
\mathcal{U}^w u \mathcal{U}^w w \mathcal{P}_J = \mathcal{U}^w w \mathcal{P}_J \quad \text{for } u \in \mathcal{U}^s - \{1\},
\]
\[
usw' u \mathcal{P}_J = \mathcal{U}^sw' sw \mathcal{P}_J \quad \text{for } u \in \mathcal{U}^s
\]
and all these are direct products.

**Proof:** We use general facts on Bruhat decompositions.

(a) We have
\[
s\mathcal{U}^w w \mathcal{P}_J = s\mathcal{P} w \mathcal{P}_J \subset \mathcal{P} w \mathcal{P}_J \cup \mathcal{P} sw \mathcal{P}_J = \mathcal{P} w \mathcal{P}_J = \mathcal{U}^w w \mathcal{P}_J
\]
where at the inclusion sign we use $s\mathcal{P} w \subset \mathcal{P} w \cup \mathcal{P} sw$, and where in the equality following it we use the hypothesis $(sw)^J = w$, i.e. $swW_J = wW_J$. Applying $s$ we see that this inclusion is an equality. Since $u \in \mathcal{P}$ and $\mathcal{U}^w w \mathcal{P}_J = \mathcal{P} w \mathcal{P}_J$ we get (a).

(b) $\ell(sw) > \ell(w)$ implies $\ell(sw) > \ell(w)$ and again by general properties of Bruhat decompositions we find
\[
\mathcal{U}^s s\mathcal{U}^w w \mathcal{P}_J = \mathcal{U}^s s\mathcal{P} w \mathcal{P}_J = \mathcal{P} s\mathcal{P} w \mathcal{P}_J = \cup_{u \in W_J} \mathcal{P} s\mathcal{P} w v \mathcal{P}
\]
\[
= \cup_{J} \mathcal{P} sw \mathcal{P}_J = \mathcal{P} sw \mathcal{P}_J = \mathcal{U}^sw' sw \mathcal{P}_J
\]
where the assumption $\ell(sw) > \ell(w)$ implied $\mathcal{P} sw \mathcal{P}_J = \mathcal{P} sw \mathcal{P}$, and where we made repeated use of (7) (in the first and last equation with this $J$, and in the second equation by setting $J = \emptyset$ in (7)).

(c) $\ell(sw) < \ell(w)$ implies $\ell(sw) < \ell(w)$, hence $w^{-1}(\beta) \in \Phi^-$. One checks that $\mathcal{U}' = s\mathcal{U}^sw$, hence this is a subgroup. Moreover, $s\mathcal{U}' = \mathcal{U}'$ and since $\mathcal{U}^s \subset \mathcal{P}$ and $\mathcal{U}^sw sub \mathcal{P}_J = \mathcal{P} sw \mathcal{P}_J$ the last equality follows. Finally, again by general facts on Bruhat decompositions we have
\[
s\mathcal{U}^w w \mathcal{P}_J \subset \mathcal{U}^w w \mathcal{P}_J \cup \mathcal{U}^sw' sw \mathcal{P}_J
\]
and the union on the right hand side is disjoint (since $swW_J \neq wW_J$). We just saw that
\[
s\mathcal{U}' = \mathcal{U}'sw' \mathcal{P}_J, \text{ hence } s(\mathcal{U}^w - \mathcal{U}')w \mathcal{P}_J \subset \mathcal{U}^w w \mathcal{P}_J.
\]
It follows that
\[
\mathcal{U}^s su \mathcal{U}' w \mathcal{P}_J \subset \mathcal{U}^w w \mathcal{P}_J
\]
for $u \in \mathcal{U}^s - \{1\}$. To see the reverse inclusion it is enough to show $\mathcal{U}' w \mathcal{P}_J \subset \mathcal{U}^s su \mathcal{U}' w \mathcal{P}_J$. Since $\mathcal{U}' = s\mathcal{U}^sw$ this boils down to showing $\mathcal{U}^sw \subset s\mathcal{U}^s su \mathcal{U}^sw \mathcal{P}_J$, i.e. (by (7)) to $\mathcal{U}^sw \subset s\mathcal{U}^s subs \mathcal{P} sw \mathcal{P}_J$. A small computation in $\text{SL}_2(k_F)$ shows that, because of $u \neq 1$, there is some $\tilde{u} \in \mathcal{U}^s$ with $\tilde{u} su \subset \mathcal{U}^s$. This implies the wanted inclusion. \qed

**Definition:** Similarly as before, we define the $J$-special $\mathcal{G}$-representation $Sp_J(G, L)$ with coefficients in $L$ by the exact sequence of $\mathcal{G}$-representations
\[
\bigoplus_{\alpha \in \Delta - J} \mathcal{C}(\mathcal{G}/\mathcal{P}_{J, J}(\alpha), L) \xrightarrow{\partial} \mathcal{C}(\mathcal{G}/\mathcal{P}_J, L) \longrightarrow Sp_J(G, L) \longrightarrow 0.
\]
Consider the natural map
\[ C(\overline{G}/P_J, L) \rightarrow C^\infty(G/P_J, L), \]
\[ f \mapsto [g = k y \mapsto f(k)] \]
where we decompose a general element \( g \in G \) as \( g = k y \) with \( k \in K_{x_0} \) and \( y \in P_J \) (using the Iwasawa decomposition \( G = K_{x_0} P_J \)), and where \( k \) denotes the class of \( k \) in \( \overline{G} = K_{x_0}/U_{x_0} \). We have similar maps for the various \( P_J \cup \{ \alpha \} \), hence an embedding
\[ \text{Sp}_J(\overline{G}, L) \hookrightarrow \text{Sp}_J(G, L). \]
(8)

For \( w \in W^J \) we write
\[ g_w = \chi P_w = \chi U_w P_J, \]
the characteristic function of \( P_w = U_w P_J \) on \( G \). We also write \( g_w \) for the class of \( g_w \) in \( \text{Sp}_J(\overline{G}, L) \).

**Proposition 3.2.** (a) The embedding (8) induces an isomorphism
\[ \text{Sp}_J(\overline{G}, L) \cong \text{Sp}_J(G, L)^I. \]
(b) The set \( \{ g_w \mid w \in V^J \} \) is an \( L \)-basis of \( \text{Sp}_J(\overline{G}, L)^P \).

**Proof:** For \( \overline{G} = \text{GL}_n(k_F) \) (some \( n \)) a proof of (b) is given in [12] par.6. For general \( \overline{G} \) the proof carries over (this is then similar to [15] par.4). But of course, to compute \( \text{Sp}_J(\overline{G}, L)^P \) (i.e. proving (b)) one may also proceed as in the proof of Corollary 2.4 above, and then (a) follows by comparing with the very statement of Corollary 2.4. □

We define the Hecke-Algebra
\[ \mathcal{H}(\overline{G}, P; L) = \text{End}_{L[\overline{G}]} L[\overline{P}\backslash \overline{G}]. \]

For a \( \overline{G} \)-representation on a \( L \)-vector space \( V \) with subspace \( V^P \) of \( P \)-invariants, Frobenius reciprocity tells us that there is an isomorphism
\[ \text{Hom}_{L[\overline{G}]}(L[\overline{P}\backslash \overline{G}], V) \cong \text{Hom}_{L[\overline{P}]}(L, V) \cong V^P \]
which sends \( \psi \in \text{Hom}_{L[\overline{G}]}(L[\overline{P}\backslash \overline{G}], V) \) to \( \psi(\overline{P}) \in V^P \). Hence \( V^P \) becomes a right \( \mathcal{H}(\overline{G}, P; L) \)-module. For \( g \in \overline{G} \) we define the Hecke operator \( T_g \in \mathcal{H}(\overline{G}, P; L) \) by setting
\[ (T_g f)(\overline{P}h) = \sum_{\overline{P}h' \subset g^{-1} \overline{P}h} f(\overline{P}h') \]
for \( f \in L[\overline{P}\backslash \overline{G}] \), where for the moment we identify \( L[\overline{P}\backslash \overline{G}] \) with the \( L \)-module of functions \( \overline{P}\backslash \overline{G} \rightarrow L \). For \( n \in \mathbb{N} \) the Hecke operator \( T_n \) only depends on the class of \( n \) in \( W = \mathbb{N}/T \). It acts on \( v \in V^P \) as
\[ v T_n = \sum_{u \in \overline{P}/(\overline{P} n^{-1} \overline{P} n)} u n^{-1} v. \]
(9)
Notice that for $s \in S$ we may identify $U^s \cong \mathcal{P}/(\mathcal{P} \cap s\mathcal{P})$. Thus formula (9) for the Hecke operator $T_s$ acting on $g_w \in \text{Sp}_J(G,L)^\mathcal{P}$ becomes

$$g_w T_s = \sum_{u \in U^s} \text{(the class of } \chi_{usu'wP_J})$$

in $\text{Sp}_J(G,L)^\mathcal{P}$.

For the rest of this section we assume that $L$ is a field with $\text{char}(L) = \text{char}(k_F)$.

**Lemma 3.3.** Let $w \in W^J$ and $s \in S$.

(a) If $(sw)^J = w$ then

$$g_w T_s = 0.$$

(b) If $\ell((sw)^J) > \ell(w)$ then

$$g_w T_s = g_{sw}.$$

(c) If $\ell((sw)^J) < \ell(w)$ then

$$g_w T_s = -g_w.$$

**Proof:** This follows from Lemma 3.1 and from $|U^s| = 0$ in $L$. For example, for (c) we compute, using the notations of Lemma 3.1 (c), in particular the direct product decomposition $U^w = U^sU'$:

$$g_w T_s = \sum_{u \in U'} [\chi_{usu'wP_J}] = \sum_{u \in U'} \sum_{u' \in U'} [\chi_{usu'wP_J}]$$

$$= \sum_{u \in U'} \sum_{u' \in U'} [\chi_{usu'wP_J}] + \sum_{u \in U'} [\chi_{usw'wP_J}] - \sum_{u \in U'} [\chi_{uswP_J}].$$

Lemma 3.1 (c) together with $|U^s| = 0$ in $L$ shows that the second term vanishes and that the first term is $-|\chi_{uswP_J}|$. □

**Proposition 3.4.** Each non-zero $\mathcal{H}(G,P;L)$-submodule $E$ of $\text{Sp}_J(G,L)^\mathcal{P}$ contains the element $g_{z^J}$. In particular, the $\mathcal{H}(G,P;L)$-module $\text{Sp}_J(G,L)^\mathcal{P}$ is indecomposable.

**Proof:** By Proposition 3.2 we find an element

$$h = \sum_{w \in V^J} \beta_w g_w$$

in $E$, with certain $\beta_w \in L$, not all of them zero. Choose an enumeration $z^J = w_0, w_1, w_2, \ldots$ of $V^J$ such that $w_j <_J w_i$ implies $i < j$. For $t \geq 0$ consider the property

$$\Psi(t) = [\beta_{w_i} = 0 \text{ for all } i > t].$$

By descending induction it is enough to show the following: If $\Psi(t)$ holds true for some $t > 0$, then passing to another $h \neq 0$ if necessary, $\Psi(t')$ holds true for some $t > t' \geq 0$. Notice that in view of the decreasing nature of our enumeration, Lemma 3.3 shows that the property $\Psi(t)$ is
preserved under application of $T_s$ to $h$, for any $s \in S$.

Let $t$ be minimal such that $\mathcal{P}(t)$ holds true (i.e. such that in addition $\beta_{w_t} \neq 0$), and assume $t > 0$ (otherwise we are done). By Lemma 1.5 we find $s_1, \ldots, s_r \in S$ such that, setting $w^{(i)} = (s_i \ldots s_1 w_t)^j$ for $0 \leq i \leq r$, we have the following: $\ell(w^{(i+1)}) > \ell(w^{(i)})$ for all $i \geq 0$, and $\ell((sw^{(i)})^j) < \ell(w^{(i)})$ for all $r > i \geq 0$, and $\ell((sw^{(i)})^j) \geq \ell(w^{(i)})$. By Lemma 3.3 we may replace $h$ by $hT_s$ to assume $\beta_{w(s)} = 0$ [while keeping the other hypotheses on $h$; in particular, $\beta_{w_1} \neq 0$ also for the new $h$ — this follows from our induction hypothesis which tells us that for the old $h$ we have $\beta_{(sw_t)^j} = 0$ (if $(sw_t)^j \in V^I$), therefore this old $\beta_{(sw_t)^j}$ (if $(sw_t)^j \in V^I$) does not, by an instance of Lemma 3.3 (b), contribute to the new $\beta_{w_1} = \beta_{(sw_t)^j}$.] By descending subinduction on $0 \leq g \leq r$ we show that, passing to another $h \neq 0$ if necessary, we may assume $\beta_{w_i} = 0$ for all $i > t$, and $\beta_{w_0} = 0$. For $g = 0$ this is what we want. For $g = r$ this was just shown. Now if for $0 \leq g < r$ we have $\beta_{w(g)} \neq 0$ and $\beta_{w(g+1)} = 0$, we replace $h$ by $h + hT_{s_{g+1}}$: then, inspecting once more the formulae of Lemma 3.3, we find $\beta_{w(g)} = 0$ for this new $h$, but $\beta_{w(g+1)} \neq 0$, ensuring $h \neq 0$. \hfill $\square$

4 Irreducibility in the residual characteristic

Following our conventions we put $T^0 = I \cap T$ and then let $\widetilde{W} = N/T^0$ (sometimes referred to as the extended affine Weyl group). $\widetilde{W}$ acts on the apartment $A$ and can be canonically identified with the semidirect product $(T/T^0).W$. It contains the affine Weyl-group $W^a$, the subgroup of $\widetilde{W}$ generated by the reflections in the walls of $A$. On the other hand, let $\Omega$ be the subgroup of $\widetilde{W}$ stabilizing the standard chamber in $A$ (i.e. the one fixed by $I$). Then $\tilde{W}$ is canonically identified with the semidirect product $\Omega.W^a$. If $G$ is of adjoint type the canonical projection $\varphi : \widetilde{W} \to W$ is injective on $\Omega$ and its image $W_\Omega = \varphi(\Omega) \subset W$ coincides with the one defined in section 1.

We define the Iwahori Hecke algebra

$$\mathcal{H}(G, I; L) = \text{End}_{L[G]}L[I \backslash G].$$

For a smooth $G$-representation on a $L$-vector space $V$ with subspace $V^I$ of $I$-invariants, Frobenius reciprocity tells us that there is an isomorphism

$$\text{Hom}_{L[G]}(L[I \backslash G], V) \cong \text{Hom}_{L[I]}(L, V) \cong V^I$$

which sends $\psi \in \text{Hom}_{L[G]}(L[I \backslash G], V)$ to $\psi(I) \in V^I$. Hence $V^I$ becomes a right $\mathcal{H}(G, I; L)$-module. For $g \in G$ we define the Hecke operator $T_g$ in $\mathcal{H}(G, I; L)$ by setting

$$(T_g f)(Ih) = \sum_{Ih' \subset I g^{-1}Ih} f(Ih')$$

for $f \in L[I \backslash G]$, where for the moment we identify $L[I \backslash G]$ with the $L$-module of compactly supported functions $I \backslash G \to L$. The Hecke operator $T_n$ for $n \in N$ depends only on the class of
If the underlying root-system is of type $\text{G}$, and the $T_n$ for $n$ running through a system of representatives for $\tilde{W}$ form an $L$-basis of $\mathcal{H}(G, I; L)$ ([14] section 1.3, example 1). They act on $v \in V^I$ as

$$vT_n = \sum_{u \in I/(I \cap n^{-1}I_n)} uu^{-1}v.$$ 

By Proposition 3.2 we have an isomorphism

$$(11) \quad \text{Sp}_J(G, L)^P \cong \text{Sp}_J(G, L)^I.$$ 

For $w \in W$ we had defined a Hecke operator $T_w$ acting on the $\mathcal{H}(G, P, L)$-module $\text{Sp}_J(G, L)^P$. On the other hand, if we denote again by $w$ a representative in $N$ of the image of $w$ in $\tilde{W}$ (under the embedding $W \hookrightarrow (T/T^0)W \cong \tilde{W}$), we get a Hecke operator $T_w$ acting on the $\mathcal{H}(G, I; L)$-module $\text{Sp}_J(G, L)^I$. (Note however that, for fixed Iwahori subgroup $I$, the embedding $W \to \tilde{W}$ depends on the choice of $x_0$ (or equivalently, of $P$). Hence the $\mathcal{H}(G, I; L)$-elements $T_w$ for $w \in W$ depend on this choice.) It is clear from our constructions that these actions coincide under our isomorphism (11). Recall that for $w \in W^I$ we wrote $g_w$ for the class in $\text{Sp}_J(G, L)^P$ of the characteristic function of $\overline{P}w\overline{P}_J$ on $G$. Now we also write $g_w$ for its image in $\text{Sp}_J(G, L)^I$ under (11), i.e. for the class in $\text{Sp}_J(G, L)^I$ of the characteristic function of $IwP_J$ on $G$.

For the rest of this section we assume that $L$ is a field with $\text{char}(L) = \text{char}(k_F)$.

**Lemma 4.1.** Assume that $G$ is of adjoint type. For each $u \in W_\Omega$ there exists a lifting $\bar{u} \in N$ (under the canonical projections $N \to \tilde{W} \to W$) which normalizes $I$ and such that for all $w \in W^I$ we have $g_wT_{\bar{u}}^{-1} = g_{(uw)^{-1}}$ in $\text{Sp}_J(G, L)^I$.

**Proof:** By [8] Proposition 2.10 we can lift $u \in W_\Omega$ to an element $\bar{u} \in N$ which normalizes $I$. Therefore $T_{\bar{u}}^{-1}$ acts on $\text{Sp}_J(G, L)^I$ simply through the action of $\bar{u} \in N \subset G$ and for $w \in W^I$ we compute $\bar{u}wP_J = I\bar{u}wP_J = I(wu)^{-1}P_J$. The Lemma follows. (The hypothesis that $G$ be of adjoint type should be superfluous here, but [8] assumes this.) $\square$

**Theorem 4.2.** If the underlying root-system is of type $A_l, B_l, C_l$ or $D_l$ then the $\mathcal{H}(G, I; L)$-module $\text{Sp}_J(G, L)^I$ is irreducible.

**Proof:** By Proposition 3.4 we know that each non-zero $\mathcal{H}(G, I; L)$-submodule of $\text{Sp}_J(G, L)^I$ contains the element $g_{x^I}$. Therefore it is enough to show that $\text{Sp}_J(G, L)^I$ is generated as a $\mathcal{H}(G, I; L)$-module by the element $g_{x^I}$.

(a) We first assume that $G$ is of adjoint type. We claim that for each subspace $E$ of $\text{Sp}_J(G, L)^I$ containing $g_{x^I}$ and stable under all $T_w$ for $w \in W$, and stable under all $T_{\bar{u}}^{-1}$ for $\bar{u} \in N$ normalizing $I$ as in Lemma 4.1, we have $E = \text{Sp}_J(G, L)^I$. Indeed, we know that $\text{Sp}_J(G, L)^I$ is generated as an $L$-vector space by all $g_w$ for $w \in V^I$. By Lemmata 4.1 and 3.3 it is therefore enough to find for each $w \in V^I$ a sequence $z^I = w_0, w_1, \ldots, w_r = w$ in $W$ such that for all $i \geq 1$ we have $w_i = w_{i-1}$ for some $u \in W_\Omega$, or $[\ell(w_{i-1}^I) < \ell(w_i^I)$ and $w_i^I = s(w_{i-1}^I)$ for some $s \in S]$. But this is the content of Corollary 1.7 which is available since we assume that $G$ be of adjoint type.
(b) In the general case we find a central isogeny \( \pi : G \to G' \) with \( G' \) split, connected, semisimple and of adjoint type, and with the same root system. We find a split maximal torus \( T' \) with normalizer \( N' \), a Borel subgroup \( P' \) and an Iwahori subgroup \( I' \) in \( G' \) such that \( \pi^{-1}(T') = T, \; \pi^{-1}(P') = P, \; \pi^{-1}(I') = I \) and such that \( W \cong N'/T' \). As \( \ker(\pi) \subset T \) it is clear that \( \pi \) induces a \( G \)-equivariant isomorphism \( \Sp_{\mathfrak{g}}(G', L) \cong \Sp_{\mathfrak{g}}(G, L) \) which restricts to an isomorphism of Iwahori invariant spaces \( \Sp_{\mathfrak{g}}(G', L)_I' \cong \Sp_{\mathfrak{g}}(G, L)_I \) (both of dimension \( |V'J| \), by Corollary 2.4).

We identify the Bruhat-Tits buildings of \( G \) and \( G' \); then \( C \) is fixed by \( I' \), and \( x_0 \) corresponds to \( P' \subset G' \) (just as it corresponds to \( P \subset G \)). Let \( \tilde{u} \in N' \) as in Lemma 4.1, in particular normalizing \( I' \). For \( n' \in N' \) we have

\[
T_{n'}T_{\tilde{u}}^{-1} = T_{n'\tilde{u}}^{-1} = T_{\tilde{u}^{-1}n'}T_{\tilde{u}}^{-1} \quad \text{in } \mathcal{H}(G', I'; L)
\]

by general facts on \( \mathcal{H}(G', I'; L) \) (the 'braid relations'), or just by the definition of the \( T_g \)'s. Now \( \tilde{u}\pi(N)\tilde{u}^{-1} = \pi(N) \) and this is contained in \( N' \). Since \( \mathcal{H}(G, I; L) \) is generated by the \( T_n \) with \( n \in N \) (see, e.g. [14] section 1.3, example 1), the relations (12) imply

\[
\mathcal{H}(G, I; L)T_{\tilde{u}}^{-1} = T_{\tilde{u}^{-1}}\mathcal{H}(G, I; L)
\]

inside \( \End_L\Sp_{\mathfrak{g}}(G, L)_I \) (here we keep the names of \( \mathcal{H}(G, I; L) \) and \( T_{\tilde{u}}^{-1} \) also for their images in \( \End_L\Sp_{\mathfrak{g}}(G, L)_I \)). We get

\[
(g_{z,j}\mathcal{H}(G, I; L))T_{\tilde{u}}^{-1} \subset (\tilde{u}g_{z,j})\mathcal{H}(G, I; L)
\]

inside \( \Sp_{\mathfrak{g}}(G, L)_I \) (recall that \( T_{\tilde{u}}^{-1} \) acts from the right on \( \Sp_{\mathfrak{g}}(G, L)_I \) by left multiplication with \( \tilde{u} \)). By Proposition 3.4 we have \( g_{z,j} \in (\tilde{u}^{-1}g_{z,j})\mathcal{H}(G, I; L) \). We apply \( T_{\tilde{u}}^{-1} \), by equation (13) again this gives \( \tilde{u}g_{z,j} \in g_{z,j}\mathcal{H}(G, I; L) \), and together with (14) we get

\[
(g_{z,j}\mathcal{H}(G, I; L))T_{\tilde{u}}^{-1} \subset g_{z,j}\mathcal{H}(G, I; L).
\]

By what we have seen in (a) this proves the Theorem. \( \square \)

**Remark:** In conclusion, it turns out that, in case the root system is \( A_t, B_t, C_t \) or \( D_t \) (possibly also in case it is \( E_6, E_7 \)), to prove the irreducibility of the \( \mathcal{H}(G, I; L) \)-module \( \Sp_{\mathfrak{g}}(G, L)_I \) it is enough to use the action of \( \mathcal{H}(\mathcal{G}, \mathcal{P}; L) \) together with the Hecke operators \( T_{\tilde{u}}^{-1} \) of Lemma 4.1. To deal with the remaining exceptional groups where the operators \( T_{\tilde{u}}^{-1} \) are not available one has to work out the action of sufficiently many other Hecke operators (besides those in \( \mathcal{H}(\mathcal{G}, \mathcal{P}; L) \)). We remark that Corollary 2.4 together with [15] Proposition 10 provides us with an isomorphism of \( \mathcal{H}(G, I; L) \)-modules

\[
\Sp_{\mathfrak{g}}(G, L)_I \cong \frac{C^\infty(G/P_J, L)_I}{\sum_{\alpha \in \Delta\setminus J} C^\infty(G/P_{J \cup \{\alpha\}}, L)_I}
\]

In the case \( G = \SL_n(F) \) (or \( G = (P)\GL_n(F) \)) Rachel Ollivier found an independent proof of the irreducibility of the right hand side of (15).
Corollary 4.3. Suppose that the underlying root-system is of type $A_t$, $B_t$, $C_t$ or $D_t$. The $G$-representation $\text{Sp}_J(G,L)$ is irreducible.

Proof: Let $I_1 \subset I$ denote the pro-$p$-Iwahori subgroup in $I$, where $p = \text{char}(k_F)$. Then $I$ is generated by $I_1$ and $T^0 = T \cap I$. As $T$ acts trivially on $\text{Sp}_J(G,L)$, the spaces of invariants under $I$ and $I_1$ are the same:

$$\text{Sp}_J(G,L)^I = \text{Sp}_J(G,L)^{I_1}.$$ 

Replacing $I$ by $I_1$ in our definition of the Iwahori Hecke Algebra $\mathcal{H}(G,I;L)$ we obtain the algebra $\mathcal{H}(G,I_1;L)$. Similarly as before, $\text{Sp}_J(G,L)^{I_1}$ is an $\mathcal{H}(G,I_1;L)$, and the irreducibility of $\text{Sp}_J(G,L)^I$ as an $\mathcal{H}(G,I;L)$-module (Theorem 4.2) immediately implies the irreducibility of $\text{Sp}_J(G,L)^{I_1} = \text{Sp}_J(G,L)^I$ as an $\mathcal{H}(G,I_1;L)$ module. Now recall the well known fact that for every smooth representation of a pro-$p$-group — like $I_1$ — on a non-zero $L$-vector space $E$ the subspace $E^I$ of $I_1$-invariants is non-zero (since $\text{char}(L) = p$). Applied to a non-zero $G$-subrepresentation $E$ of $\text{Sp}_J(G,L)$, the irreducibility of $\text{Sp}_J(G,L)^I$ as an $\mathcal{H}(G,I;L)$ module implies $E^I = \text{Sp}_J(G,L)^I$. But $\text{Sp}_J(G,L)$ is generated as a $L[G]$-module by $\text{Sp}_J(G,L)^I$; this follows from [15], Proposition 9, where it is shown that even the $L[G]$-module $C_G^{\infty}(G/P_J,L)$ is generated by its $I_1$-fixed vectors. Thus $E = \text{Sp}_J(G,L)$ and we are done. \qed

Remark: For any $J$ with $|V^J| = 1$, like $J = \emptyset$, we get the irreducibility of $\text{Sp}_J(G,L)$ for any $G$ (not necessarily of type $A_t$, $B_t$, $C_t$ or $D_t$). The irreducibility of the Steinberg representation $\text{Sp}_\emptyset(G,L)$ had been obtained earlier by Vignéras [15]. In fact she conjectures [15] the irreducibility of $\text{Sp}_J(G,L)$ for any $J$, without any restrictions on $\Phi$ (like those imposed in Corollary 4.3).

Corollary 4.4. (a) (Vignéras) The $G$-representations $\text{Sp}_J(G,L)$ for the various subsets $J \subset \Delta$ are pairwise non-isomorphic.

(b) Suppose that the underlying root-system is of type $A_t$, $B_t$, $C_t$ or $D_t$. The $G$-representations $\text{Sp}_J(G,L)$ with $J$ running through all subsets $J \subset \Delta$ form the irreducible constituents of the $G$-representation $C_G^{\infty}(G/P,L)$, each one occurring with multiplicity one. 

Proof: The irreducibility of the $\text{Sp}_J(G,L)$ in (b) is Theorem 4.3, everything else can be found in the paper [15]. Namely, there it is shown that each $\text{Sp}_J(G,L)$ admits a $P$-equivariant filtration, with factors the natural $P$-representations $C_G^{\infty}(PwP/P,L)$ for $w \in V^J$. These factors are shown to be irreducible ([15] Proposition 1, Theorem 5). They are non-isomorphic for different $w \in W$. Indeed, let $R(w) = \{\alpha \in \Phi^+ \mid w^{-1}(\alpha) \in \Phi^+\}$. Let $U^-$ denote the unipotent radical of the Borel subgroup $P^-$ opposite to $P$. For $w \in W$ let

$$U^w = U \cap wU^-w^{-1} = \prod_{\alpha \in \Phi^+ - R(w)} U_\alpha.$$ 

Similarly to (7) we have $U^w = PwP/P$. Therefore $R(w)$ is the set of all $\alpha \in \Phi^+$ for which $U_\alpha$ acts trivially on $C_G^{\infty}(PwP/P,L)$, but $R(w)$ uniquely determines $w$. \qed
Question: Is the theory of extensions between the various $G$-representations $\text{Sp}_J(G, L)$ (for $L$ a field with $\text{char}(L) = \text{char}(k_F)$) parallel to the theory of extensions between the various $G$-representations $\text{Sp}_J(G, \mathbb{C})$ (as worked out in [11], [12])?

**Corollary 4.5.** Suppose that the underlying root-system is of type $A_l$, $B_l$, $C_l$ or $D_l$. Let $\mathcal{O}_K$ be a complete discrete valuation ring with fraction field $K$ and residue field $k_K$. Suppose $\text{char}(k_K) = \text{char}(k_F)$. Up to $K^\times$-homothety, $\text{Sp}_J(G, \mathcal{O}_K)$ is the unique $G$-stable $\mathcal{O}_K$-lattice inside $\text{Sp}_J(G, K)$.

**Proof:** (I thank Marie-France Vignéras for completing my (originally incomplete) argument here.) Let $N$ be another $G$-stable $\mathcal{O}_K$-lattice inside $\text{Sp}_J(G, K)$. Let $p_K \in \mathcal{O}_K$ be a uniformizer. Since $\text{Sp}_J(G, k_K)$ is irreducible by Corollary 4.3, the image of $p_K^n N \cap \text{Sp}_J(G, \mathcal{O}_K)$ in $\text{Sp}_J(G, \mathcal{O}_K) \otimes K k_K = \text{Sp}_J(G, k_K)$ for $n \in \mathbb{Z}$ must be either (a) zero, or (b) all of $\text{Sp}_J(G, k_K)$. Case (a) implies $p_K^{n+1} N \subset \text{Sp}_J(G, \mathcal{O}_K)$. Case (b) implies

\[(16) \quad \text{Sp}_J(G, \mathcal{O}_K) \subset p_K \text{Sp}_J(G, \mathcal{O}_K) + p_K^n N.\]

Now $\text{Sp}_J(G, \mathcal{O}_K)$ is finitely generated as an $\mathcal{O}_K[G]$-module (e.g. by $\mathcal{O}_K$-generators of $\text{Sp}_J(G, \mathcal{O}_K)^I$, as was already used in the proof of Corollary 4.3), therefore there exists some $m >> 0$ with $p_K^m \text{Sp}_J(G, \mathcal{O}_K) \subset N$. This means that (16) simplifies as $\text{Sp}_J(G, \mathcal{O}_K) \subset p_K^m N$. In view of this dichotomy (a)/(b) for any $n \in \mathbb{Z}$ we get $p_K^n N = \text{Sp}_J(G, \mathcal{O}_K)$ for some $n \in \mathbb{Z}$ since $\bigcap_n p_K^n N = 0$ and $\bigcup_n p_K^n N = \text{Sp}_J(G, K)$. \hfill $\square$

### 5 Harmonic Chains

Here $L$ is an arbitrary ring again and $G = \text{GL}_{d+1}(F)$ (some $d \geq 1$). Let $X$ denote the semisimple Bruhat-Tits building of $G$. Let $X^0$ denote the set of vertices of $X$. For $x \in X^0$ let

\[ K_x = \{ g \in G \mid gx = x \text{ and } \det(g) \in \mathcal{O}_F^\times \} \]

and let $U_x$ be the unique maximal normal open subgroup of $K_x$. Let $P_{J,x} = K_x \cap P_J$. The group $K_x$ acts on the set of simplices of $X$ containing $x$. Let $\sigma_x = \sigma_x(J)$ denote the unique maximal such simplex which is fixed by $P_{J,x}$. It is $k$-dimensional, where $k = |\Delta - J| = d - |J|$. Inside the set of all $k$-dimensional simplices of $X$ we define

\[ X_x(J) = \{ g \sigma_x \mid g \in K_x \}. \]

In each $\sigma \in X_x(J)$ we distinguish the vertex $x \in \sigma$, its pointing. $K_x$ acts on $X_x(J)$. We let

\[ X(J) = \prod_{x \in X^0} X_x(J) \]

and call this the set of pointed $J$-simplices (so by definition this is a disjoint union, i.e. each element of $\sigma \in X(J)$ comes with a distinguished vertex $x \in \sigma$, its pointing). $G$ acts on $X(J)$. Let $x \in X$ and $\alpha \in \Delta - \sigma$. For $\sigma \in X_x(J)$ and $\tau \in X_x(J \cup \{\alpha\})$ (i.e. pointed at the same vertex
we write \( \tau < \sigma \) if \( \tau \subseteq \sigma \). Now let \( x, x' \in X^0 \) such that \( \{x, x'\} \in X^1 \) (i.e. is a 1-simplex in \( X \); we identify simplices in \( X \) with their sets of vertices). Let

\[
U_{x, x'} = U_{x' x} = (U_x, U_{x'}),
\]

the subgroup of \( G \) generated by \( U_x \) and \( U_{x'} \). Then \( U_{x, x'} \subseteq K_x \) and \( U_{x, x'} \subseteq K_{x'} \). For \( k \in K_x \) and \( k' \in K_{x'} \) such that \( k^{-1}k' \in P_j \) we say that the families of pointed \( J \)-simplices

\[
\mathfrak{F} = U_{x, x'} k \sigma_x = \{ \sigma \in X_x(J) \mid \sigma = u k \sigma_x \text{ for some } u \in U_{x, x'} \} \subseteq X_x(J)
\]

and \( \mathfrak{F}' = U_{x, x'} k' \sigma_{x'} \subseteq X_{x'}(J) \) are adjacent in \( X(J) \).

**Definition:** \( \mathfrak{har}_J(1) \) and \( \mathfrak{har}_J(2) \) are the minimal \( L \)-submodules of \( L[X(J)] \) satisfying:

1. For each \( \alpha \in \Delta - J \), each \( \tau \in X(J \cup \{\alpha\}) \), if we let \( \mathcal{B}(\tau) = \{ \sigma \in X(J) \mid \tau < \sigma \} \), then

\[
\sum_{\sigma \in \mathcal{B}(\tau)} \sigma \in \mathfrak{har}_J(1).
\]

2. If \( \mathfrak{F} \) and \( \mathfrak{F}' \) are adjacent families in \( X(J) \), then

\[
\sum_{\sigma \in \mathfrak{F}} \sigma - \sum_{\sigma' \in \mathfrak{F}'} \sigma' \in \mathfrak{har}_J(2).
\]

We let \( \mathfrak{har}_J = \mathfrak{har}_J(1) + \mathfrak{har}_J(2) \) and define

\[
\mathfrak{f}_J(L) = \frac{L[X(J)]}{\mathfrak{har}_J}.
\]

We call \( \mathfrak{f}_J(L) \) the \( L \)-module of \( L \)-valued \( J \)-chains on \( X \). It carries an obvious \( G \)-action.

**Theorem 5.1.** There exists a \( G \)-equivariant isomorphism

\[
\mathfrak{f}_J(L) \cong \text{Sp}_J(G, L).
\]

**Proof:** For \( x \in X^0 \) we have the \( K_x \)-equivariant isomorphism

\[
L[X_x(J)] \cong C(U_x \backslash K_{x/P_{Jx}}, L),
\]

\[
g \sigma_x \mapsto \chi_{U_x g P_{Jx}}
\]

\((g \in K_x)\) where \( \chi_{U_x g P_{Jx}} \) denotes the characteristic function of \( U_x g P_{Jx} \). The Iwasawa decomposition \( G = K_x P_J \) (which holds since \( x \), like all vertices in \( X \), is a special vertex) provides a natural isomorphism

\[
C(U_x \backslash K_{x/P_{Jx}}, L) \cong C(U_x \backslash G/P_J, L).
\]

Together we obtain an isomorphism

\[
\frac{L[X_x(J)]}{L[X_x(J)] \cap \mathfrak{har}_J(1)} \cong \frac{C(U_x \backslash G/P_J, L)}{\sum_{\alpha \in \Delta - J} C(U_x \backslash G/P_{J \cup \{\alpha\}}, L)}.
\]

27
Furthermore, for $\{x, x'\} \in X^1$ the isomorphisms (17) for $x$ and $x'$ induce an isomorphism

$$\mathfrak{har}_J(2) \cap (L[X_J(J)] \oplus L[X_{J'}(J)]) \cong C(U_{x,x'} \setminus G/P_J, L).$$

Together we deduce a $G$-equivariant exact sequence

$$\bigoplus_{\{x,x'\} \in X^1} C(U_{x,x'} \setminus G/P_J, L) \rightarrow \bigoplus_{x \in X_0} \frac{C(U_x \setminus G/P_J, L)}{\sum_{\alpha \in \Delta^{-} J} C(U_{x \cup \omega_x(\alpha)} \setminus G/P_J, L)} \rightarrow \mathfrak{h}_J(L) \rightarrow 0. \tag{18}$$

On the other hand we have according to [12] section 6, Theorem 8 a $G$-equivariant exact sequence

$$\bigoplus_{\{x,x'\} \in X^1} \text{Sp}_J(G, \mathbb{Z})^{U_{x,x'}} \rightarrow \bigoplus_{x \in X_0} \text{Sp}_J(G, \mathbb{Z})^{U_x} \rightarrow \text{Sp}_J(G, \mathbb{Z}) \rightarrow 0. \tag{19}$$

Using [12] section 6 Proposition 15 we see that by base extension $\mathbb{Z} \rightarrow L$ we derive an exact sequence

$$\bigoplus_{\{x,x'\} \in X^1} C(U_{x,x'} \setminus G/P_J, L) \rightarrow \bigoplus_{x \in X_0} \frac{C(U_x \setminus G/P_J, L)}{\sum_{\alpha \in \Delta^{-} J} C(U_{x \cup \omega_x(\alpha)} \setminus G/P_J, L)} \rightarrow \text{Sp}_J(G, L) \rightarrow 0. \tag{19}$$

Comparing the exact sequences (18) and (19) we conclude. \hfill \square

**Remarks:**
(a) We may identify $X^0$ with the set of homothety-classes $[\Lambda] = \{\lambda \Lambda \mid \lambda \in F^x\}$ of free $O_F$-submodules $\Lambda$ of rank $d + 1$ in a fixed $(d + 1)$-dimensional $F$-vector space. A $k$-dimensional simplex in $X$ is then given by the set of its $k + 1$ vertices. This set carries a canonical cyclic ordering, namely the cyclic ordering $\ldots, [\Lambda_0], \ldots, [\Lambda_k], [\Lambda_0], \ldots$ if we can choose the representatives $\Lambda_j$ such that

$$\Lambda_0 \supseteq \Lambda_1 \supseteq \ldots \supseteq \Lambda_k \supseteq p_F \Lambda_0.$$ 

Giving a pointing of the simplex amounts to fixing this cyclic ordering into a true total ordering $([\Lambda_0], \ldots, [\Lambda_k])$ (here $[\Lambda_0]$ is the pointing). For $\{x_0, x'_0\} \in X^1$ and pointed $k$-simplices $(x_0, \ldots, x_k)$ and $(x'_0, \ldots, x'_k)$ (represented as indicated) the families $U_{x_0,x'_0}(x_0, \ldots, x_k)$ and $U_{x_0,x'_0}(x'_0, \ldots, x'_k)$ are adjacent if and only if $x_i, x'_i \in X^1$ for all $0 \leq i \leq k$.

(b) Let $\hat{X}^k$ denote the set of all pointed $k$-dimensional simplices in $X$. One may define a $G$-stable submodule $\mathfrak{har}_J$ of $L[\hat{X}^k]$ as the minimal submodule of $L[\hat{X}^k]$ containing $\mathfrak{har}_J$ and all relations of the following kind. Let $\sigma = (\Lambda \supseteq \Lambda_1 \supseteq \ldots \supseteq \Lambda_k \supseteq p_F \Lambda) \in \hat{X}^k$ (pointed at $[\Lambda]$) and set

$$\mathcal{C}(\sigma) = \{\sigma' = (\Lambda' \supseteq \Lambda'_1 \supseteq \ldots \supseteq \Lambda'_k \supseteq p_F \Lambda) \in X(J) \mid \text{for all } 1 \leq j \leq k \text{ we have } \Lambda'_j \subset \Lambda_j \text{ or } \Lambda_j \subset \Lambda'_j\}.$$ 

Then

$$\sigma - \sum_{\sigma' \in \mathcal{C}(\sigma)} \sigma' \in \mathfrak{har}_J.$$ 

28
One may ask for which \( J \) the inclusion \( L[X(J)] \subset L[\hat{X}^k] \) induces an isomorphism

\[
\mathfrak{F}_J(L) \cong \frac{L[\hat{X}^k]}{\hat{\text{har}}_J}.
\]

In the case where \( J \) consists of the first \( d - k \) simple roots (in the Dynkin diagram) this holds true: this follows from work of de Shalit [7] (he works with a different but equivalent definition of \( \hat{\text{har}}_J \) in this case). For these \( J \) Theorem 5.1 has been obtained by de Shalit in the case \( \text{char}(F) = 0 \), and by Aït Amrane (as the main result of [1]) for \( F \) of arbitrary characteristic.

**Formula:** Let \( J \) be arbitrary again (and \( G = \text{GL}_{d+1}(F) \)). We conclude with an explicit description of the embedding \( \lambda_L : \text{Sp}_J(G, L) \hookrightarrow C^\infty(I, \mathfrak{M}_J(L)) \) of Theorem 2.3 in terms of the isomorphism \( \mathfrak{F}_J(L) \cong \text{Sp}_J(G, L) \) of Theorem 5.1, without giving proofs. We identify \( W \) with the automorphism group of the set \{0, \ldots, d\} and \( \Delta \) with the set of transpositions \((s - 1, s)\) for \( 1 \leq s \leq d \). For \( 0 \leq i \leq d \) let \( e_i \in X_*(T) \) denote the cocharacter \( e_i : \mathbb{G}_m \rightarrow T \) sending \( y \in \mathbb{G}_m \) to the diagonal matrix \( e_i(y) = y^{e_{ii}} = y \) and \( e_i(y) = 1 \) for \( j \neq i \). Let \( \{s_1 < \ldots < s_k\} \) denote the set, in increasing enumeration, of all \( s \in \{1, \ldots, d\} \) such that the transposition \((s - 1, s)\) does not belong to \( J \). In particular, \( k = d - |J| \). For \( 1 \leq i \leq k \) let

\[
\xi_i^J = \sum_{0 \leq j \leq s_i - 1} e_j \in X_*(T).
\]

For \( w \in W^J \) let

\[
\tilde{Y}_A^0(J, w) = \{ \sum_{i=1}^k m_i w(\xi_i^J) | m_i \in \mathbb{Z}_{\geq 0} \} \subset X_*(T).
\]

Under the natural projection

\[
X_*(T) \otimes \mathbb{R} \xrightarrow{\pi} X_*(T) \otimes \mathbb{R}/(e_0 + \ldots + e_d) = A
\]

the set \( \tilde{Y}_A^0(J, w) \) projects to a set \( Y_A^0(J, w) \) of vertices in the standard apartment \( A \) of \( X \). This \( Y_A^0(J, w) \) is the set of vertices of a connected full simplicial subcomplex \( Y_{J,w} \) of \( X \) all of whose maximal simplices are \( k \)-dimensional. We let \( Y_A(J, w) \) denote the subset of \( X(J) \) consisting of all pointed \( J \)-simplices in \( X \) having all their vertices in \( Y_A^0(J, w) \). Thus the simplex underlying an element of \( Y_A(J, w) \) is a chamber in \( Y_{J,w} \). We may assume that \( I \) fixes the chamber in \( X \) whose set of vertices is \( \{\pi(\xi_0^0), \pi(\xi_1^0), \ldots, \pi(\xi_d^0)\} \), where we set \( \xi_0^0 = 0 \in X_*(T) \). Let \( I.Y_A(J, w) \subset X(J) \) denote the union of all \( I \)-orbits of elements of \( Y_A(J, w) \) and then put

\[
Y(J) = \bigcup_{w \in W^J} I.Y_A(J, w)
\]

(this is a disjoint union inside \( X(J) \)). For \( \sigma \in Y(J) \) there exists a unique \( w \in W^J \), a unique \( \sigma' \in Y_A(J, w) \) and some \( g \in I \) such that \( \sigma = g \sigma' \). Here \( g \) is not uniquely determined, but the coset \( gV_{\sigma'} \) in \( I \) is independent of the choice of \( g \), where \( V_{\sigma'} \subset I \) denotes the stabilizer of \( \sigma' \) in \( I \). There is a unique element \( \sigma(w) \in Y_A(J, w) \) which is pointed at the central vertex (i.e. at
\[ \pi(0) \in A \). Let \( m(\sigma(w), \sigma') \in \mathbb{Z}_{\geq 0} \) denote the gallery distance between \( \sigma(w) \) and \( \sigma' \) (i.e between their underlying chambers in \( Y_{J,w} \)). Let

\[ \tilde{\lambda}_L(\sigma) = (-1)^{m(\sigma(w), \sigma')} \chi_{\sigma'} \otimes \nabla(w) \]

(with \( \nabla \) as in the exact sequence (4)), an element of \( C^\infty(I, L) \otimes \mathcal{M}_J(L) = C^\infty(I, \mathcal{M}_J(L)) \). By \( L \)-linearity we obtain a map \( \tilde{\lambda}_L : L[Y(J)] \to C^\infty(I, \mathcal{M}_J(L)) \). One can show:

(i) The canonical map \( L[Y(J)] \to \mathfrak{H}_J(L) \), induced by the inclusion \( L[Y(J)] \subset L[X(J)] \), is surjective. (More precisely, for \( w \in W^J \) the image of \( L[I,Y_A(J, w)] \) in \( \mathfrak{H}_J(L) \) corresponds, under the isomorphism \( \mathfrak{H}_J(L) \cong \text{Sp}_J(G, L) \), to the image of \( C^\infty(I/P^I_{J,w}, L) \) in \( \text{Sp}_J(G, L) \), cf. the proof of Theorem 2.3.)

(ii) The composition

\[ L[Y(J)] \longrightarrow \mathfrak{H}_J(L) \cong \text{Sp}_J(G, L) \overset{\lambda_L}{\longrightarrow} C^\infty(I, \mathcal{M}_J(L)) \]

is the map \( \tilde{\lambda}_L \) just described.

References

[1] Y. Ait Amrane, Cohomology of Drinfeld symmetric spaces and Harmonic cochains, Annales de l’institut Fourier, 56 (2006), no. 3, p. 561-597

[2] A. Björner, F. Brenti, Combinatorics of Coxeter groups, Graduate Texts in Mathematics 231, Springer Verlag (2005)

[3] A. Borel, J.-P. Serre, Cohomologie d’immeubles et de groupes S-arithmétiques, Topology 15 (1976), no. 3, 211–232.

[4] A. Borel and N. Wallach, Continuous cohomology, discrete subgroups, and representations of reductive groups. Second edition. Mathematical Surveys and Monographs, 67. American Mathematical Society, Providence, RI

[5] Jean-François Dat, Espaces symétriques de Drinfeld et correspondance de Langlands locale. Ann. Scient. Éc. Norm. Sup. 39 (1), 1-74 (2006)

[6] J. E. Humphreys, Reflection groups and Coxeter groups. Cambridge Studies in Advanced Mathematics 29. Cambridge University Press, Cambridge (1990).

[7] E. de Shalit, Residues on buildings and de Rham cohomology of \( p \)-adic symmetric domains, Duke Math. J. 106 (2001), no.1, 123–19

[8] N. Iwahori and H. Matsumoto, On some Bruhat decomposition and the structure of the Hecke rings of \( p \)-adic Chevalley groups, Publ. Math. IHES 25 (1965), 3 – 48
[9] J. C. Jantzen, Representations of algebraic groups, Boston: Academic Press (1987)

[10] G. Laumon, Cohomology of Drinfeld Modular varieties, Part I, Cambridge University Press (1996)

[11] S. Orlik, On extensions of generalized Steinberg representations, J. Algebra 293 (2005), no. 2, 611–630.

[12] P. Schneider and U. Stuhler, The cohomology of $p$-adic symmetric spaces, Inventiones Math. 105 (1991), no.1, 47 – 122

[13] J. Tits, Reductive groups over local fields, Proc. Symp. Pure Math.,33 (1979), part 1, 29 – 69

[14] M.F. Vigneras, Pro-$p$-Iwahori Hecke algebra and supersingular $\overline{\mathbb{F}}_p$-representations, Math. Ann. 331 (2005), no. 3, 523–556 and Math. Ann. 333 (2005), no. 3, 699–701

[15] M.F. Vigneras, Série principale modulo $p$ de groupes réductifs $p$-adiques, GAFA vol. in the honour of J. Bernstein (2008)

[16] M. F. Vigneras, Représentations $l$-modulaires d’un groupe réductif $p$-adique avec $l \neq p$. Progress in Math. 131, Birkhauser (1996)

Humboldt-Universität zu Berlin
Institut für Mathematik
Rudower Chaussee 25
12489 Berlin, Germany
E-mail address: gkloenne@math.hu-berlin.de