Contact Hamiltonian mechanics. An extension of symplectic Hamiltonian mechanics

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Abstract. This contribution gives a possible solution of the major question whether it is possible to construct a classical mechanical theory which not only contains all the advantages of the Hamiltonian formalism, but also takes into account the effects of the environment on the system.

1. Introduction
The Hamiltonian formulation of classical mechanics consists of an even-dimensional phase space with symplectic structure, the dynamics is determined by the Hamiltonian vector field associated with the differential of the Hamiltonian. In addition, we can mention that the Hamiltonian structure is directly connected with other areas of physics such as optics, statistical physics and quantum mechanics. Despite all these advantages, this formulation exclusively describes isolated systems with reversible dynamics, while real systems are constantly in interaction with an environment that introduces the phenomena of dissipation and irreversibility.

Recently, in order to include the dissipation phenomenon in a Hamiltonian theory, the contact Hamiltonian formalism has been proposed in [1]. This formalism consists of enlarging the symplectic phase space of classical mechanics by adding an additional dimension, the new phase space is an odd-dimensional manifold but now with a contact structure. In analogy to the standard theory, the dynamics is determined by the contact vector field induced by the differential of the so called contact Hamiltonian function. There are two major advantage of this formalism, the first is the fact that the symplectic phase space maybe embedded in it and the second is that the phenomenon of irreversible dissipation can be defined straightforward.

This work gives a concise and short survey of the concept of contact Hamiltonian dynamics and its application in classical dissipative system, enhancing the fact that this theory includes all the important properties of the symplectic one. Besides, along this discussion, we provide examples which enable us to show the usefulness of our formalism. Finally, we conclude this contribution introducing a possible quantization route of the contact formalism leading to a nonlinear Schrödinger equation (NLSE) for the description of quantum dissipative systems, which provides a theoretical justification for the introduction of the nonlinear phenomenological Schrödinger equation described in the literature [2, 3, 4]. In particular, we show the solution of the NLSE for the parametric oscillator systems and its corresponding Wigner function.
2. Contact Hamiltonian mechanics

The phase space of a conservative system is the cotangent bundle of the configuration manifold, that is an $2n$-dimensional manifold with local system of coordinates $(p^a, q^a)$, for $a = 1, \ldots, n$. Then any point in the phase space locally is given by a point in the configuration manifold and one in the corresponding fiber in the cotangent bundle. Here we are interested in adding an additional dimension to the phase space to model dissipative phenomena, then we are dealing with a $(2n+1)$-dimensional manifold. However an odd-dimensional manifold cannot admit a symplectic structure. The analogue to the symplectic structure for odd-dimensional manifolds is the contact structure [5].

The contact phase space is defined to be a contact manifold $\mathcal{T}$, a $(2n+1)$-dimensional manifold endowed with a global differential 1-form $\eta$, that satisfies the condition

$$\eta \wedge (d\eta)^n \neq 0,$$

(1)

everywhere on $\mathcal{T}$, where $\wedge$ denotes the exterior product and the exponent denotes the $n$th exterior power. The last property is known as the condition of maximal nonintegrability [5, 6], note that a contact manifold in this sense is orientable. The left hand side in (1) provides a volume form on $\mathcal{T}$. Associated with the definition of the contact 1-form on a contact manifold, there is another fundamental object called the Reeb vector field $\xi$, which is defined intrinsically by the conditions

$$i_\xi \eta = 1, \quad i_\xi d\eta = 0,$$

(2)

and then

$$\mathcal{L}_\xi \eta = i_\xi d\eta + di_\xi \eta = 0.$$

(3)

The Reeb vector field is unique and defines at every point a ‘vertical’ direction with respect to $2n$-dimensional horizontal distribution $\text{ker}(\eta)$. Therefore, the contact structure of the new phase space is completely determined by the triad $(\mathcal{T}, \eta, \xi)$.

Now, assuming that the phase space has a contact structure we can define a dynamics on it. Given the 1-form $\eta$, one can associate to every differentiable function $H : \mathcal{T} \to \mathbb{R}$, a vector field $X_H$, called the contact Hamiltonian vector field generated by $H$, defined through the two (intrinsic) relations

$$\mathcal{L}_X H = f_H \eta \quad \text{and} \quad H = -i_{X_H} \eta,$$

(4)

where $\mathcal{L}_X$ is the Lie derivative along a vector field $X$ and $f_H \in C^\infty(\mathcal{T})$ is a function depending on $H$ defined by the condition $f_H = -\xi(H)$. The function $H$ is called the contact Hamiltonian [6, 7]. Using the Cartan’s identity

$$f_H \eta = \mathcal{L}_{X_H} \eta = i_{X_H} d\eta + di_{X_H} \eta$$

$$= i_{X_H} d\eta - d\mathcal{H},$$

(5)

and the conditions in (4), it follows that

$$i_{X_H} d\eta = d\mathcal{H} - (\mathcal{L}_\xi \mathcal{H}) \eta,$$

(7)

this equation determines uniquely $X_H$ up to a vector field proportional to $\xi$. This additional term is fixed by the requirement $\mathcal{H} = -i_{X_H} \eta$. In addition, from Eq. (7) it is direct that the evolution of the contact Hamiltonian function along its flow is

$$\mathcal{L}_{X_H} \mathcal{H} = f_H \mathcal{H}.$$

(8)

Therefore, at difference to the symplectic theory, for the contact mechanics the contact Hamiltonian is not preserved along the evolution.
It is important to observe that from Eq. (7), if we consider $\mathcal{L}_\xi\mathcal{H} = 0$, we recover the usual symplectic result. Therefore this term is responsible to introduce dissipation into the dynamics, because this term introduce a dissipative force in the direction of the Reeb vector field.

Assigning a set of local (Darboux) coordinates $(q^a, p_a, S)$ for $\mathcal{T}$ [5], referred to as contact coordinates, the 1-form $\eta$ and the Reeb vector field $\xi$ can be written as

$$\eta = dS - p_a dq^a, \quad \xi = \frac{\partial}{\partial S},$$

(9)

where here and in the following Einstein’s summation convention over repeated indices is assumed. The second expression in (9) directly implies that in these coordinates

$$f_{\mathcal{H}} = -\frac{\partial \mathcal{H}}{\partial S}.$$  (10)

Therefore, the contact Hamiltonian vector field $X_{\mathcal{H}}$ takes the form

$$X_{\mathcal{H}} = \left( p_a \frac{\partial \mathcal{H}}{\partial p_a} - \mathcal{H} \right) \frac{\partial}{\partial S} - \left( p_a \frac{\partial \mathcal{H}}{\partial S} + \frac{\partial \mathcal{H}}{\partial q^a} \right) \frac{\partial}{\partial p_a} + \left( \frac{\partial \mathcal{H}}{\partial p_a} \right) \frac{\partial}{\partial q^a},$$

(11)

where it is direct that the flow of $X_{\mathcal{H}}$ obtains in contact coordinates by solving

$$\dot{q}^a = \frac{\partial \mathcal{H}}{\partial p_a},$$

(12)

$$\dot{p}_a = -\frac{\partial \mathcal{H}}{\partial q^a} - p_a \frac{\partial \mathcal{H}}{\partial S},$$

(13)

$$\dot{S} = p_a \frac{\partial \mathcal{H}}{\partial p_a} - \mathcal{H}.$$  (14)

The similarity of Eqs. (12)–(14) with Hamilton’s equations of symplectic mechanics is evident. In fact, this set of equations is a generalization of Hamilton’s equations, because, when $\mathcal{H}$ does not depend on the $S$-variable, Eqs. (12) and (13) give exactly Hamilton’s equations in the symplectic phase space and the remaining Eq. (14) in this case is the usual definition of Hamilton’s principal function. Finally, given an arbitrary function in the contact phase space $\mathcal{F} \in C^\infty(\mathcal{T})$, we can define its evolution according to Eqs. (12)–(14) by $X_{\mathcal{H}}[\mathcal{F}]$, i.e.

$$\frac{d\mathcal{F}}{dt} = -\mathcal{H} \frac{\partial \mathcal{F}}{\partial S} + p_a \left[ \frac{\partial \mathcal{F}}{\partial S} \frac{\partial \mathcal{H}}{\partial p_a} - \frac{\partial \mathcal{F}}{\partial p_a} \frac{\partial \mathcal{H}}{\partial S} \right] + \frac{\partial \mathcal{F}}{\partial p_a} \frac{\partial \mathcal{H}}{\partial S} - \frac{\partial \mathcal{F}}{\partial p_a} \frac{\partial \mathcal{H}}{\partial q^a},$$

(15)

where $\{ , \}_{(q^a, p_a)}$ is the standard Poisson bracket and the remaining terms are contact corrections. Therefore, the contact Hamilton’s theory can include a much larger class of models than the symplectic formalism.

As in the symplectic case, the contact phase space is a manifold equipped with a standard volume form: $V = \eta \wedge (d\eta)^n$. However, the Lie derivative of this volume form with respect to the contact flow $X_{\mathcal{H}}$ is different to zero, in fact it contracts (or expands) along the flow according to

$$\mathcal{L}_{X_{\mathcal{H}}} V = -(n + 1) \frac{\partial \mathcal{H}}{\partial S} V,$$

(16)

which means that the contact flow has a non-zero divergence, namely

$$\text{div}(X_{\mathcal{H}}) = -(n + 1) \frac{\partial \mathcal{H}}{\partial S}.$$  (17)
and therefore Liouville’s theorem does not hold. However, it is important to mention that the non-vanishing divergence of the contact vector field depends on the choice of the volume form. For instance, let us consider the volume form \( \tilde{V} = H^{-(n+1)} \) proposed in [8]. Then for such volume form we have that

\[
\mathcal{L}_{X_{\mathcal{H}}} \tilde{V} = \left( \mathcal{L}_{X_{\mathcal{H}}} H^{-(n+1)} \right) V - H^{-(n+1)} \mathcal{L}_{X_{\mathcal{H}}} V \quad (18)
\]

\[
= \left( X_{\mathcal{H}} [H^{-(n+1)}] - (n+1) \mathcal{H}^{-(n+1)} \frac{\partial \mathcal{H}}{\partial S} \right) V, \quad (19)
\]

where the last equality follows from (16). Finally, applying the contact Hamiltonian vector field to \( \mathcal{H}^{-(n+1)} \), namely

\[
X_{\mathcal{H}} [\mathcal{H}^{-(n+1)}] = (n+1) \mathcal{H}^{-(n+1)} \frac{\partial \mathcal{H}}{\partial S}, \quad (20)
\]

one finds that

\[
\mathcal{L}_{X_{\mathcal{H}}} \tilde{V} = 0. \quad (21)
\]

Therefore, for this choice of the volume form the contact flow has zero divergence.

3. Contact mechanics for dissipative systems

To model dissipative system by means of contact mechanics, we can use the fact that this formalism includes a new variable in the dynamics, the \( S \)-variable, which now affects the evolution of the system. Then, it is possible to connect this variable with the degree of freedom of loss of mechanical energy between the system and the environment. Let us consider the case, in which the contact Hamiltonian is described by

\[
\mathcal{H} = H_{\text{mech}}(q^a, p_a) + h(S), \quad (22)
\]

where \( H_{\text{mech}}(q^a, p_a) \) is the mechanical energy of the system and \( h(S) \) characterizes effectively the interaction with the environment. Then, from (15), the evolution of the mechanical energy is

\[
\frac{dH_{\text{mech}}}{dt} = -p_a \frac{\partial H_{\text{mech}}}{\partial p_a} h'(S), \quad (23)
\]

from which it is clear that \( h(S) \) behaves like a potential that generates dissipative forces. For example, we can consider the one-dimensional contact Hamiltonian system

\[
\mathcal{H}_S = \frac{p^2}{2m} + V(q) + \gamma S, \quad (24)
\]

where \( V(q) \) is the mechanical potential and \( \gamma \) is a constant parameter associated with the friction coefficient, also known in the literature as Ohmic damping. Then for this case, the rate of dissipation of the mechanical energy is

\[
\frac{dH_{\text{mech}}}{dt} = -\frac{\gamma}{m} p^2, \quad (25)
\]

which agrees with standard results based on Rayleigh’s dissipation function [9]. Then, the equations of motion (12)-(14) for this system are given by

\[
\dot{q} = \frac{p}{m}, \quad (26)
\]

\[
\dot{p} = -\frac{\partial V(q)}{\partial q} - \gamma p, \quad (27)
\]

\[
\dot{S} = \frac{p^2}{2m} - V(q) - \gamma S. \quad (28)
\]
From Eqs. (26) and (27) one can derive the damped Newtonian equation
\[ \ddot{q} + \gamma \dot{q} + \frac{1}{m} \frac{\partial V(q)}{\partial q} = 0, \tag{29} \]
which describes all one-dimensional systems with a friction force that depends linearly on the velocity. Therefore, the contact phase space is a natural space of evolution for position and momentum quantities for dissipative systems, because of the Eq. (27) cannot be derived from a Hamiltonian function with the canonical symplectic structure of the cotangent bundle.

Finally, as we mention in the previous section the contact Hamiltonian vectorial field with respect to the standard volume form has a non-zero divergence. This property of the field is usually interpreted as a sign of dissipation [10, 11]. However, in this formalism this property does not classify the systems as non-dissipative or dissipative, because, as we already proved, we can always find a volume form such as the vector field has a zero divergence. Therefore, we can argue that being dissipative is not an intrinsic property of the vector field.

4. Time-dependent contact Hamiltonian systems
To include time-dependence in the system we extend the contact phase space by adding the time variable to it. Then we have an extended phase space \( T^E = T \times \mathbb{R} \) with coordinates as \( (q^a, p_a, S, t) \) and extended 1-form
\[ \eta^E = dS - p_a dq^a + \mathcal{H} dt, \tag{30} \]
where \( \mathcal{H} \) is the contact Hamiltonian. To define the dynamics on \( T^E \) under the assumption that \( d\eta^E \) is not degenerate, we set the two (intrinsic) simultaneous conditions
\[ i_{X^E_{\mathcal{H}}} d\eta^E + d\eta^E i_{X^E_{\mathcal{H}}} \eta^E = g_{\mathcal{H}} \eta^E \quad \text{and} \quad i_{X^E_{\mathcal{H}}} \eta^E = 0, \tag{31} \]
with \( g_{\mathcal{H}} \in C^\infty(T^E) \) a function depending on \( \mathcal{H} \), where this factor in coordinates corresponds to
\[ g_{\mathcal{H}} = \frac{\partial \mathcal{H}}{\partial S}. \tag{32} \]
Notice that (31) is the natural extension of (4) to \( T^E \). These two conditions define a vector field \( X^E_{\mathcal{H}} \) on \( T^E \) which is completely equivalent to the contact Hamiltonian flow (11) with general form
\[ X^E_{\mathcal{H}} = X_{\mathcal{H}} + \frac{\partial}{\partial t}, \tag{33} \]
with \( X_{\mathcal{H}} \) given by (11), the proof of this result can be found in [1]. From this, it can be recognised immediately that the equations of motion given by such a field on \( T^E \) are the same as those of the contact Hamiltonian vector field (11), with the addition of the trivial equation \( \dot{t} = 1 \). We call a system defined by a contact Hamiltonian \( \mathcal{H}(q^a, p_a, S, t) \) and by the vector field \( X^E_{\mathcal{H}} \) of the form (33) a time-dependent contact Hamiltonian system. Besides, from (33) and (15) it follows that the evolution of any function \( \mathcal{F} \in C^\infty(T^E) \) under the dynamics given by a time-dependent contact Hamiltonian system is
\[ \frac{d\mathcal{F}}{dt} = -\mathcal{H} \frac{\partial \mathcal{F}}{\partial S} + p_a \{ \mathcal{F}, \mathcal{H} \}(S,p_a) + \{ \mathcal{F}, \mathcal{H} \}(q^a,p_a) + \frac{\partial \mathcal{F}}{\partial t}. \tag{34} \]
Therefore, in general, we say that a function \( \mathcal{F} \in C^\infty(T^E) \) is a first integral (or invariant) of the contact dynamics given by \( X^E_{\mathcal{H}} \) if \( \mathcal{F} \) is constant along the flow of \( X^E_{\mathcal{H}} \), that is if \( X^E_{\mathcal{H}}[\mathcal{F}] = 0 \).
5. **Hamilton-Jacobi formulation**

In the symplectic theory, the Hamilton–Jacobi formulation is a powerful tool which enables to re-express Hamilton’s equations in terms of a single partial differential equation whose solution, a function of the configuration space, has all the necessary information to obtain the trajectories of the mechanical system. The same formulation can be obtained from the contact theory. Therefore, in this section we introduce a Hamilton–Jacobi formulation of contact Hamiltonian systems, which establishes a connection with the configuration space, where the phenomenological equations are defined.

As we mentioned before, the Hamilton–Jacobi equation is a re-formulation of the dynamical equations in terms of a single partial differential equation (PDE) for the function $S(q^a,t)$. Thus, we are looking for a PDE of the form

$$
\mathcal{F}(q^a, \frac{\partial S}{\partial q^a}, S, t, \frac{\partial S}{\partial t}) = 0,
$$

(35)

and whose characteristic curves are equivalent to the contact Hamiltonian dynamics (12)–(14). To construct such PDE, let us define the function

$$
\mathcal{F}(q^a, p_a, S, t, E) \equiv E - \mathcal{H}(q^a, p_a, S, t).
$$

(36)

It turns out that the solution of the Eq. $\mathcal{F} = 0$ is defined by

$$
\eta^E = dS - p_a dq^a + \mathcal{H} dt = 0,
$$

(37)

that is by the two conditions

$$
p_a = \frac{\partial S}{\partial q^a} \quad \text{and} \quad \mathcal{H}\left(q^a, \frac{\partial S}{\partial q^a}, S, t\right) = -\frac{\partial S}{\partial t}
$$

(38)

gives exactly the contact Hamiltonian Eqs. (12)–(14). Therefore we call the second equation in (38) the contact Hamilton-Jacobi equation. Additionally, we found that the $S$-variable is the generalization of Hamilton’s principal function which satisfies the contact version of the Hamilton–Jacobi equation.

6. **Damped parametric oscillator**

For its wide range of applications, the study of the dynamics of the parametric oscillator, i.e. an oscillator with time-dependent frequency, has been the subject of major interest, leading to the search of exact invariants in the Hamiltonian theory [12, 13]. However, the study of a parametric oscillator that includes the effects of the environment on the Hamiltonian theory has not been entirely satisfactory. The most traditional route to describe dissipative systems in the standard theory is to consider systems with explicit time dependence. The idea is to introduce a convenient time-dependence into the Hamiltonian such that it reproduces the phenomenological equations of motion with energy dissipation. As an example, we have the Caldirola [14] and Kanai [15] approach. But the major problem in such models is that the canonical and physical momenta and positions do not coincide [16, 17]. Here we study the damped parametric oscillator in the contact formalism by means of invariants in the contact phase space, which enables us to show the usefulness of our formalism and also show that the contact formalism does not have the ambiguity existing in the traditional approaches to dissipative systems.

The contact Hamiltonian for the system under consideration is given by

$$
\mathcal{H} = \frac{p^2}{2m} + \frac{1}{2}m\omega^2(t)q^2 + \gamma S,
$$

(39)
and the equations of motion can be easily deduced by the contact Hamiltonian Eqs. (26)–(28). Now, by direct substitution into result (34) it has been proved in [1] that the damped parametric oscillator possesses the quadratic invariant

$$ I(q,p,t) = m e^{\gamma t} \left[ \left( \alpha(t) \frac{p}{m} - \dot{\alpha}(t) - \frac{\gamma}{2} \alpha(t) \right) q \right]^2 + \left( \frac{q}{\alpha(t)} \right)^2, \quad (40) $$

where the time-dependent function $\alpha(t)$ satisfies the Ermakov equation [18]

$$ \ddot{\alpha} + \left( \omega^2(t) - \frac{\gamma^2}{4} \right) \alpha = \frac{1}{\alpha}, \quad (41) $$

and the $S$-dependent invariant

$$ \mathcal{I}(q,p,S,t) = e^{\gamma t} \left[ S - \frac{q p}{2} \right]. \quad (42) $$

The invariant $\mathcal{I}(q,p,t)$ is a generalization of the canonical invariant found by H. R. Lewis Jr. for the parametric oscillator [12], which is recovered when $\gamma \to 0$. Now, by means of these invariants it is possible to find the general solution of the system, by means of the time-dependent contact transformation, see Appendix A,

$$ \tilde{Q} = \arctan \left( \alpha \left[ \frac{\dot{\alpha}}{2} \left( \frac{\dot{\alpha}}{\alpha} - \frac{\gamma}{2} \right) - \frac{p}{m q} \right] \right), \quad (43) $$

$$ \tilde{P} = \mathcal{I}(q,p,t), \quad (44) $$

$$ \tilde{S} = \mathcal{I}(q,p,S,t). \quad (45) $$

with conformal factor $f = e^{\gamma t}$. The contact Hamiltonian expressed in this new variables takes the simple form

$$ \mathcal{K} = \mathcal{I}/\alpha^2. \quad (46) $$

Therefore, as $\mathcal{K}$ does not depends on the variables $\tilde{Q}$ and $\tilde{S}$, the new contact Hamiltonian equations have the trivial form

$$ \dot{\tilde{Q}} = \frac{1}{\alpha^2}, \quad \dot{\tilde{P}} = 0, \quad \dot{\tilde{S}} = 0, \quad (47) $$

with solutions

$$ \tilde{Q}(t) = \int_t^\tau \frac{d\tau}{\alpha^2(\tau)}, \quad \tilde{P}(t) = \mathcal{I} \quad \text{and} \quad \tilde{S}(t) = \mathcal{I}. \quad (48) $$

Finally, inverting the transformation (43)–(45), one obtains the solutions in the original (physical) coordinates, i.e.

$$ q(t) = \sqrt{\frac{2}{m}} e^{\gamma t} \alpha(t) \cos \phi(t), \quad \frac{d}{dt}(q(t)) = \mathcal{I}, \quad \frac{d}{dt}(\tilde{Q}(t)) = \tilde{Q}(t), \quad \frac{d}{dt}(p(t)) = \mathcal{I}, \quad \frac{d}{dt}(S(t)) = \mathcal{I}, \quad (49) $$

$$ p(t) = \sqrt{2 m \mathcal{I}} e^{\gamma t} \left[ \left( \frac{\dot{\alpha}}{2} \right) - \frac{\gamma}{2} \alpha(t) \right] \cos \phi(t) - \frac{1}{\alpha} \sin \phi(t), \quad (50) $$

$$ S(t) = e^{-\gamma t} \mathcal{I} + \frac{q(t)p(t)}{2}, \quad (51) $$

where $\phi(t) = \tilde{Q}(t)$ and the values of the constants $\mathcal{I}$ and $\mathcal{I}$ are determined by the initial conditions. Therefore, we have derived here the solutions of the equations of motion of the damped parametric oscillator using the invariants of the contact Hamiltonian system, where all the dynamics of the system is encoded in the Ermakov equation (41).
7. Schrödinger quantization of contact Hamiltonian systems

In this section we sketch briefly a possible quantization of the contact formalism. Thus using the fact that the additional contact variable is a generalization of Hamilton’s principal function, and that the canonical momenta and positions in our formalism coincide with the physical ones, then we suggest based on the Schrödinger quantization introduced in the early days of the quantum mechanics [19], a canonical quantization of the contact Hamiltonian based on the standard rules of canonical quantization in the position representation, namely

\[ p_a \rightarrow \frac{i}{\hbar} \frac{\partial}{\partial q^a}, \quad q^a \rightarrow q^a, \quad S(q^a, t) \rightarrow \frac{i}{\hbar} \ln \Psi(q^a, t), \]  

(52)

where the last relation is Schrödinger’s original definition of the wave function in terms of the Hamilton’s principal function \( S \) [19]. Using such rules to quantize the contact Hamiltonian \( \mathcal{H} \) and obtain the operator \( \hat{\mathcal{H}} \), one can define the contact Schrödinger equation

\[ i\hbar \frac{\partial \Psi}{\partial t} = \hat{\mathcal{H}} \Psi. \]  

(53)

This equation has a fundamental property: in the case in which the contact Hamiltonian reduces to a symplectic Hamiltonian (i.e. when \( \mathcal{H} \) does not depend on \( S \) explicitly) the dynamics reduces to the standard quantum dynamics. However, this equation has the disadvantage that in general it does not conserve the norm of the wave function. But for systems with contact Hamiltonian of the form \( \mathcal{H} = H_{\text{mec}} + h(S) \), normalization is achieved following the procedure of Gisin [20], which consists in subtracting the mean value of \( \hat{h} \), that is \( \hat{h} \rightarrow \hat{h} - \langle \hat{h} \rangle \) where

\[ \langle \hat{h} \rangle = \int dq^a \Psi^* \hat{h}(S) \Psi. \]  

(54)

This leads to the nonlinear Schrödinger equation

\[ i\hbar \frac{\partial \Psi}{\partial t} = \hat{H}_{\text{mec}} \Psi + (\hat{h} - \langle \hat{h} \rangle) \Psi. \]  

(55)

We can mention that at variance with Gisin’s proposal [20] we have specified the choice of \( \hat{h} \) associated with the dependence of the contact Hamiltonian on the \( S \)-variable that is responsible for dissipation.

As an example we can apply this nonlinear Schrödinger equation (NLSE) to the contact Hamiltonian with arbitrary potential and linear dependence on \( S \) given in Eq. (24), leading to the nonlinear evolution equation

\[ i\hbar \frac{\partial \Psi}{\partial t} = \left[ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial q^2} + V(q) + \frac{\gamma}{i} \frac{\hbar}{\ln \Psi - \langle \ln \Psi \rangle} \right] \Psi, \]  

(56)

which is exactly the phenomenological NLSE introduced in [2, 3, 4] for the description of dissipative systems and has been a matter of recent research [21, 17]. Therefore, we have provided a theoretical justification for the introduction of the nonlinear phenomenological Schrödinger equation that displays consistency between the classical and quantum descriptions in terms of the contact formalism.

This derivation of the NLSE has important implications. It is well-known, in the standard theory, that the classical canonical evolution is associated in quantum mechanics with a unitary evolution. Then, because the NLSE has been obtained from the contact formalism, whose evolution is governed by contact transformations the evolution of the wave function \( \Psi \) is no
longer unitary. On the other hand, from Eq. (56) it is possible to deduce the Smoluchowski equation
\[
\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial q_j} j(q, t) - D(t) \frac{\partial^2 \rho}{\partial q^2} = 0
\]
(57)
for the probability density \( \rho(q, t) = \Psi(q, t)\Psi^*(q, t) \), where the current of probability has the well-known form
\[
j(q, t) = \rho(q, t) v(q, t) = \frac{\hbar^2}{2m} \left( \Psi^* \frac{\partial \Psi}{\partial q} - \Psi \frac{\partial \Psi^*}{\partial q} \right)
\]
(58)
and the diffusion term is defined by the relation
\[
-D(t) \frac{1}{\rho} \frac{\partial^2 \rho}{\partial q^2} = \gamma [\ln \rho - \langle \ln \rho \rangle].
\]
(59)
Therefore, the continuity equation for this case is substituted by a Smoluchowski equation with an extra (possible time-dependent) diffusion coefficient \( D(t) \).

In particular, we can consider the one dimensional contact Hamiltonian for the parametric oscillator (39). The NLSE associated with this contact Hamiltonian possesses analytic wave packet solution of the form [16]
\[
\Psi_{NL}(q, t) = \left[ \frac{m}{\pi \hbar \alpha(t)^2} \right]^{1/4} \exp \left\{ \frac{i m}{2 \hbar} \left[ \frac{\dot{\alpha}(t)}{\alpha(t)} - \frac{\gamma}{2} \frac{1}{\alpha^2(t)} \right] (q - \langle q \rangle)^2 + \frac{i}{\hbar} \langle p \rangle (q - \langle q \rangle) + \frac{i}{2 \hbar} \langle q \rangle \langle p \rangle - i \phi_{NL}(t) \right\}
\]
(60)
where the time-dependent functions \( (\langle q \rangle(t), \langle p \rangle(t)) \) satisfy the contact Hamilton’s equation:
\[
\frac{d\langle q \rangle}{dt} = \frac{\langle p \rangle}{m},
\]
(61)
\[
\frac{d\langle p \rangle}{dt} = -\omega^2(t)\langle q \rangle - \gamma \langle p \rangle,
\]
(62)
the function \( \alpha(t) \) obeys the nonlinear Ermakov equation (41) and the purely time-dependent phase
\[
\phi_{NL}(t) = \frac{1}{2} \int_0^t dt' \left[ \frac{1}{\alpha^2(t')} - \frac{\gamma}{2} \alpha(t') \left( \frac{\dot{\alpha}(t')}{\alpha(t')} - \frac{\gamma}{2} \alpha(t') \right) \right].
\]
(63)
Then, from the form of the wave packet we can see that the maximum of the density distribution follows the classical trajectories according to Ehrenfest’s theorem. Additionally, it is possible to prove that the position and the momentum uncertainties as well as their correlation obey the relations
\[
\sigma_x^2(t) = \frac{\hbar}{2m} \alpha^2(t),
\]
(64)
\[
\sigma_p^2(t) = \frac{m \hbar}{2} \left[ \left( \frac{\dot{\alpha}(t)}{\alpha(t)} - \frac{\gamma}{2} \alpha(t) \right)^2 + \frac{1}{\alpha^2(t)} \right],
\]
(65)
\[
\sigma_{xp}(t) = \frac{\hbar}{2} \alpha(t) \left[ \frac{\dot{\alpha}(t)}{\alpha(t)} - \frac{\gamma}{2} \alpha(t) \right].
\]
(66)
where one can deduce that the diffusion term in this case correspond to \( D(t) = \frac{\gamma}{2} \sigma_x^2(t) \).

Because the Wigner phase space representation of quantum mechanics provides physical insight of the system as well as a natural comparison between quantum and classical dynamics, in the sense of correspondence and non-correspondence concepts [22, 23], we want to conclude this section presenting the Wigner function associated with the Gaussian wave packet (60).
Then, by means of the Wigner transformation [22], the Wigner function corresponding to the wave packet (60) corresponds to

\[ W(x, p; t) = \frac{1}{\pi \hbar} \exp \left\{ -\frac{2}{\hbar} \mathcal{J}(q - \langle q \rangle, p - \langle p \rangle, t) \right\}. \tag{67} \]

where \( \mathcal{J}(q, p, t) \) is identical to the contact invariant (40), only replacing

\[ q \rightarrow q - \langle q \rangle(t) \quad \text{and} \quad p \rightarrow p - \langle p \rangle(t). \tag{68} \]

Therefore, the Wigner function for the damped parametric oscillator is completely determined by a contact invariant of the contact phase space.

8. Conclusion

Throughout this work we have introduced the contact structure on the phase space for the study of classical systems, by means of the contact Hamiltonian vectorial field generated by a contact Hamiltonian. This geometrical description not only contains all the properties of the standard symplectic theory (to describe isolated systems), but also allows to include classical dissipative systems. Then, taking advantage of the fact that we have added an extra dimension in the contact formalism, we connect this new variable with the degree of freedom for the exchange of energy between the system and the environment. This connection allowed us to include the dynamics of basic dissipative systems.

This formalism can also include time-dependent and time-independent systems. Additionally, in both cases the contact transformations, see Appendix A, which are transformation on the contact manifold that preserve the contact structure: \((\mathcal{T}, \eta, \xi)\), and also leave the equations of motion invariant. The usefulness of such transformation, as well as all the concepts developed in the work, were shown in detail for a specific important example, the damped parametric oscillator, for which we have solved the dynamics using the invariants of the system.

To close the study and the comparison of the contact formalism with the symplectic one, the contact Hamilton–Jacobi formalism of contact theory has been introduced, which encloses all the dynamics of the system in a single partial differential equation, called contact Hamilton–Jacobi equation. In addition, this formulation allowed us to show that the additional contact variable is the generalization of the standard Hamilton’s principal function.

Due to the importance of the symplectic Hamiltonian theory in quantum mechanics, we have given a possible quantum version of the contact formalism. So based on the Schrödinger quantization, introduced in the early days of the quantum mechanics, this quantization leads to a nonlinear Schrödinger equation that has been already proposed in the literature for the description of quantum dissipative system. There are important consequences in the NLSE derived from the contact theory compared to the standard one, like the fact that the evolution of the wave packet solution is no longer a unitary one, besides the density probability obeys a Smoluchowski equation instead of the standard continuity equation. In particular, we showed an analytical solution of the NLSE for the damped parametric oscillator, where the maximum of the probability distribution follows the classical equation of motion according to the Ehrenfest’s theorem. In addition, given the Gaussian wave packet solution we presented the Wigner function associated with it and we enhance that it is completely determined by a contact invariant, which displays an intriguing consistency between the classical and quantum descriptions in the contact formalism.

Finally, it is important mention that if we are not able to find a contact structure we cannot proceed with the analysis carried on in the work and we cannot describe dissipation. To be able to introduce the Hamilton-Jacobi equation we need to identify a “configuration space”. This would not be possible on a compact contact manifold, for instance like the three-dimensional...
sphere. Therefore, the contact manifolds which are being considered in the work are of a very special kind. In this sense, in physical terms we can say that the contact formalism describes a subset of all dissipative systems.

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Appendix A. Contact transformations

A contact transformation is a transformation that leaves the contact form invariant up to multiplication by conformal factor $f$ \cite{24, 25}, namely $\tilde{\eta} = f\eta$. In coordinates the transformation from $(q^a, p_a, S)$ to $(\tilde{Q}^a, \tilde{P}_a, \tilde{S})$ is

$$d\tilde{S} - \tilde{P}_a d\tilde{Q}^a = f(dS - p_a dq^a). \tag{A.1}$$

which is equivalent to

$$f = \frac{\partial \tilde{S}}{\partial S} - \tilde{P}_a \frac{\partial \tilde{Q}^a}{\partial S}, \tag{A.2}$$

$$-fp_i = \frac{\partial \tilde{S}}{\partial q^i} - \tilde{P}_a \frac{\partial \tilde{Q}^a}{\partial q^i}, \tag{A.3}$$

$$0 = \frac{\partial \tilde{S}}{\partial p_i} - \tilde{P}_a \frac{\partial \tilde{Q}^a}{\partial p_i}. \tag{A.4}$$

In addition, we can obtain the generating function of a contact transformation. Assuming that the coordinates $(q^a, \tilde{Q}^a, S)$ are independent, we compute the differential of the generating function $\tilde{S}(q^a, \tilde{Q}^a, S)$, namely

$$d\tilde{S} = \frac{\partial \tilde{S}}{\partial S} dS + \frac{\partial \tilde{S}}{\partial q^a} dq^a + \frac{\partial \tilde{S}}{\partial \tilde{Q}^a} d\tilde{Q}^a. \tag{A.5}$$

Substituting (A.5) into (A.1) we obtain the following conditions for $\tilde{S}$

$$f = \frac{\partial \tilde{S}}{\partial S}, \quad f p_a = -\frac{\partial \tilde{S}}{\partial q^a}, \quad \tilde{P}_a = \frac{\partial \tilde{S}}{\partial \tilde{Q}^a}. \tag{A.6}$$

In particular, for contact transformations with $f = 1$ the conditions in (A.6) imply that the generating function has the form

$$\tilde{S} = S - F_1(q^a, \tilde{Q}^a), \tag{A.7}$$

where $F_1(q^a, \tilde{Q}^a)$ is the generating function of a symplectic canonical transformation. This result implies that all canonical transformations are a special case of contact transformations corresponding to $f = 1$.

For the time-dependent case time-dependent contact transformations are transformation of coordinates

$$(q^a, p_a, S, t) \rightarrow (\tilde{Q}^a, \tilde{P}_a, \tilde{S}, t), \tag{A.8}$$

that preserve the contact structure of the extended phase space (30) up to multiplication by a non-zero conformal function $f$, that is

$$f (dS - p_a dq^a + \mathcal{H} dt) = d\tilde{S} - \tilde{P}_a d\tilde{Q}^a + \mathcal{H} dt, \tag{A.9}$$
where $\mathcal{H}$ is a function on $T^*E$ which is going to be the new contact Hamiltonian in the transformed coordinates. From this conditions one obtains the standard conditions (A.2)–(A.4) for a time-independent contact transformation, together with the following rule for the transformation of the Hamiltonians

$$f\mathcal{H} = \frac{\partial \tilde{S}}{\partial t} - \tilde{P}_a \frac{\partial \tilde{Q}^a}{\partial t} + \mathcal{H}. \quad (A.10)$$

that relates the Hamiltonians.

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