Cellular Automata Based Model for Pedestrian Dynamics

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We construct a two dimensional Cellular Automata based model for the description of pedestrian dynamics. Wide range of complicated pattern formation phenomena in pedestrian dynamics are described in the model, e.g. lane formation, jams in a counterflow and egress behavior. Mean-field solution of the densely populated case and numerical solution of the sparsely populated case are provided. This model has the potential to describe more flow phenomena.

INTRODUCTION

In pedestrian dynamics, several interesting phenomena have been intensively studied: herding effect, lane formation\cite{1} and jams in a counterflow, egress behavior\cite{2}, panic situation\cite{3} etc.. Currently two kinds of models and their extensions\cite{4,5} have been proposed to describe these phenomena: ”social force” model\cite{1} and Cellular Automata based ”floor field” model\cite{6,7}. The first one follows from molecular dynamics model, it involves a group of particles interacting by long range ”social force” induced by behaviors of the individuals. However, the choice of the particular form of long-range ”social force” is somehow arbitrary and generally pedestrians do not decide their behaviour in such a sophisticated way. The second one introduced certain kinds of particles that transfer inter-person forces. The particle density field is often referred to as the ”floor field”. This model is derived from models for insects behavior with chemotaxis. This model also achieved much success in describing the above phenomena, but is somehow a bit far from the reality.

In this paper, we present a Cellular Automata based model focusing more on how the complex patterns and interesting phenomena can emerge from the very simplicity of the person-person and person-environment interaction. (From now on, we abbreviate ”Cellular Automata” to ”CA”. ) This CA model can be regarded as an extension of the Ising Model for magnetic spins. Mean-field solution for the densely populated case of this model is derived from models for insects behavior with chemotaxis. This model also achieved much success in describing the above phenomena, but is somehow a bit far from the reality.

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In section II the basic CA model setting is presented, a model Hamiltonian is given for the system; In section III, the extended Glauber dynamics\cite{8} that governs the time evolution of the system is presented; In section IV, the dense populated case is discussed, we apply mean-field approximation to discuss the equilibrium flow. The results are in consistency with numerical results; In section V, the sparsely populated case is discussed. We numerically simulated lane formation and jams phenomena in counterflow situation as well as egress phenomenon. Good consistency is found with others’ results as well as the reality. Conclusions are given in section VI. In the appendix we present the specific derivation of the mean-field solution.

MODEL

Our model is based on the local decision making process of a single pedestrian, given its surrounding environment and destination direction. The surrounding environment contains the neighbouring pedestrian and their walking directions, i.e. pedestrian tend to move in the same direction with neighbours; The surrounding environment also contains the available empty spaces: pedestrian tend to avoid crowded area so they move to emptier place; they avoid walking towards a wall. The influence of the destination direction is modeled by an external field that guides them to certain directions.

We consider a group of pedestrians in certain plain space with different destination directions. We call our model ”N dimensional” if at most a single pedestrian can have \( q = 2N \) neighbouring pedestrians(see Fig 1). For simplicity we do not consider specific details of a single pedestrian, e.g. his weight, height, age, gender, etc.. Since here the space that one pedestrian occupies is the smallest space unit we are considering, the space is considered to be \( n \) discrete lattices, each cell can contain at most one pedestrian. Then the total carrying capacity of the space is \( n \). The number of pedestrian in this space can vary from zero up to \( n \), characterizing the level of crowdness of the space. In such a N-dimensional model, we characterize a pedestrian by \( q \)-dimensional vectors \( \mathbf{s}^{(i)} \) and \( \mathbf{p}^{(i)} \): \( \mathbf{s}^{(i)} \) denotes the direction of one pedestrian and \( \mathbf{p}^{(i)} \) denotes the condition of neighbors of one pedestrian. Since our model share similarity with Ising spin model, from now on we call a pedestrian a spin. Notice that walls and obstacles are equivalent to pedestrians that never move.

The neighbors of one lattice is coded in an order that neighbor \( i \) and neighbor \( i + N \) are on opposite direction to this lattice, which then forms a \( q \)-dimensional coordinates. This coordinates system distinguishes with N-dimensional conventional coordinates and is designed to describe the conditions of the \( q \) neighbors of each lat-
FIG. 1: schematic drawing of a pedestrian and its surrounding lattices of the system, the surrounding lattices are numbered so that opposite neighbours are \( i \) and \( i + N, i = 1, \ldots, N \).

The Hamiltonian described in the equation (1) can be understood directly: herding effect in micro scale is represented by the item \(- \frac{1}{2} \sum_{<i,j>} \vec{v}_i^{(N)} \cdot \vec{v}_j^{(N)}\); the tendency to avoid crowded situation is expressed by the item \( a \sum_{i=1}^n \vec{p}_i^{(q)} \cdot \vec{s}_i^{(q)} \) and an external field plays the role of pedestrian destination direction in the item \(- \sum_{i=1}^n \vec{B}^{(q)} \cdot \vec{s}_i^{(q)}\); The neighboring lattice occupied by wall will result in huge increase of energy, then the spin will turn away from the wall immediately.

**DYNAMICS RULES**

We do not need to calculate the specific microscopic interaction force in the dynamical evolution of the system. The dynamics here is that the system is approaching a energy minimum state. An extended Glauber Dynamics is applied, the detailed dynamical evolution rules of the CA model is as follows:

- Denote \( w(\vec{s}^{(q)}) \) as the probability per unit time of flipping the spin to direction \( \vec{s}^{(q)} = \vec{e}_i^{(q)} (i = 1, 2, \ldots, q) \)

\[
w(\vec{s}^{(q)}) = \frac{e^{-\beta E(\vec{e}_i^{(q)})}}{\sum_{j=0}^q e^{-\beta E(\vec{e}_j^{(q)})}}
\]

\( E(\vec{e}_i^{(q)}) \) is the energy of the system when \( \vec{s}^{(q)} = \vec{e}_i^{(q)} (i = 1, 2, \ldots, q) \). \( \beta = \frac{1}{k_B T} \), \( T \) is the effective "temperature" of the crowd, which reflects the level
of fluctuation in the system. When temperature is high, the spin has higher possibility of changing directions, corresponding to an excited crowd who are willing to change directions now and then; when temperature is low, the spin has lower possibility of changing directions, corresponding to an inert crowd who tend to keep the same direction.

- If one spin has the direction $\vec{s}(q) = \vec{e}(q)$ and neighbor $i$ is unoccupied, then move the spin to neighbor $i$.

- A periodical boundary is applied, i.e. when a spin moves to the edge it will appear on the other side of the space.

We adopt a rule called Q2R Rule [9] developed by Vichniac, basically the rule states that any Cellular Automata that changes the state of cells all at the same time can not reach equilibrium state, which results from energy conservation law. Under this dynamics, We have the number of spins conserved. Numerical realization of the dynamical evolution share resemblance with Monte-Carlo simulation used in various areas of physics as well as math. The random number code is from ref[10].

In the following sections we will focus on two cases of the system. When the lattice is densely populated, i.e. vacancy of the lattice only appears occasionally, then the homogeneous distribution of spins allow us to apply mean-field approximation to probe its dynamical properties. When the lattice is sparsely populated, then mean-field theory no longer applies, we turn to numerical simulation to study counterflow and egress phenomena. We show that even though the dynamical rule is simple, the phenomena that emerge from the CA model are complex and interesting.

**DENSE POPULATED CASE: MEAN-FIELD APPROXIMATION SOLUTION**

When the lattice is densely populated, mean-field approximation is effective to analyze the equilibrium flow of the system (for details see appendix). We consider $M$ of the $n$ lattices are occupied by spins, define parameter $x_l$ to characterize the mean flow,

$$x_l = \begin{cases} <s_{N+l} - s_l>, & l = 1, 2, ..., N \\ <s_l - s_{l-N}>, & l = N + 1, ..., 2N \end{cases}$$

We introduce a mean-field $B_{eff}(q)$, which contains both the influence of the spin-spin interaction and the spin-external field interaction.

$$B_{eff} = fqx_l + B_l - a <p_l>, l = 1, 2, 3, ..., q$$

Then the mean-field free energy of the system can be given as:

$$F \simeq (1/2) fqn \sum_{i=1}^{N} x_i^2 + a <s(q) > \sum_{i=1}^{n} p^{(q)} - na <p^{(q)} > <s(q) >$$

$$- 1/\beta \ln(A_n^M) - 1/\beta M \ln( \sum e^{\beta B_{eff}^{(q)} s^{(q)}_i})$$

**Self-consistent Equations**

If the system is in equilibrium, the average of $\sum_{i=1}^{n} s_i^{(q)}$ can be calculated out: $\sum_{i=1}^{n} s_i^{(q)} = - \frac{\partial F}{\partial B^{(q)}_i}$. Also from the definition we have $\sum_{i=1}^{n} s_i^{(q)} = n <s(q) >$. If the periodic boundary condition is adapted or the system
is large enough, then We make a further approximation $<p_l >\ll <p_0>$, $l = 1, 2, 3, \ldots, q$, consequently

$$n < s_l > = M \sum_{i=1}^{q} e^{\beta (f q x_i + B_i)}$$

For simplicity and without loss of generality, we only consider the external field as follows: $B_1 = B_0, B_{N+1} = -B_0, B_k = 0$, for other $k$. Then the solution is

$$x_l = \{ M_n e^{\beta (f q x_1 + B_0)} - e^{\beta (f q x_{N+1} + B_0)} \}, \quad l = 1 \quad (7)$$

Numerical solutions of the self consistent equations are in Figure 3.

When external field $B$ approaches zero there is still non-zero solution when $T$ is below certain critical point; Certain low temperature phase transition exists when $B \to 0^+$. Approximation of $B \to 0^+$ situation of this phase transition is done using Taylor expansion, which is shown as the black line in Figure 3. Define the critical point as $T_c = \frac{M_f}{k_B n} = \frac{2 M f}{k_B n}$, Taylor expansion gives:

$$x_1 = \frac{M}{n} \frac{T}{T_c} \sqrt{3\left(1 - \frac{T}{T_c}\right)}$$

Since the approximation only holds when temperature is close to the critical point. We can see that the black line fits well with the numerical solution around the critical point and deviates from the numerical solution when $T$ continues to decrease.

Two dimensional simulations are carried out to check this mean field approximation results(See Figure 3a,b,c). For each parameters we perform 10 separate simulations and calculate the average. In order to show the variance we plot the maximum and minimum and result as well. Generally the results fits well, critical temperature turns out to be $T_c^* \simeq 0.4 T_c$. High temperature and low temperature limits of the solution is discussed in the appendix.

**Spontaneous Self-organization**

Spontaneous self-organization exists when $B = 0$, then the effective field do not contain external field. We require the free energy(equation 5) of the system approaches the minimum, so $\frac{dF}{d(x_l^2)} = 0$. If we do Taylor expansion on the equation(see appendix), then the approximate solution to $\frac{dF}{d(x_l^2)} = 0$ is:

$$x_l = \pm \frac{M}{N n} \frac{T}{T_c} \sqrt{3\left(1 - \frac{T}{T_c}\right)}, \quad l = 1, 2, 3, \ldots, q \quad (9)$$

The definition of $T_c$ is the same with equation 8. The uncertainty in the sign as well as in $l$ of $x_l$ is caused by fluctuations. This is essentially the symmetry breaking phenomena in our CA model. The solution is sensitive to tiny asymmetry of the initial condition. Physically it means: when temperature of the system is below $T_c$, the system will form constant flow spontaneously. Note that this spontaneous self-organization is different from the solution of the self consistent equations when $B \to 0^+$: compared with the none external field condition, a tiny external field will result in great change in the system.

![Figure 3: (a) Numerical solution of $x_1$ in Equation 7, $N = 1$, $T_c = \frac{2 M f}{k_B n}$ (b) Two dimensional($N = 2$) Simulation results compared with theory, critical temperature turns out to be $T_c^* \sim 0.4 T_c$. (c) Two dimensional($N = 2$) Simulation results compared with theory, critical temperature turns out to be $T_c^* \sim 0.4 T_c$. All data are averaged from over 10 separate simulations, with maximum and minimum data also plotted.](image)
in consistence with the terms used by others. In normal
phase is discovered. Block phase, normal phase and lane forma-
tion do not have a clear boundary, lane forma-
tion is shown in Figure 5g. Lane formation phase and

\[ M \text{total number of spins} \]

states keep changing in specific shapes, but remain the
form lanes and move efficiently. Metastable means the
"lane formation" phase (Figure 5e, f), i.e. the pedestrians
be observed. We call the three states different phases
of the system: "block phase" (Figure 4d), i.e. pedestrian
are labeled into two kinds, either tends
to be parallel or anti-parallel with the external field. The
initial positions of pedestrian are randomly chosen.

Choose the parameters as follows: \( a = 100, f = 1 \), \( B = 1, n = 10000 \). This corresponds to a
very active way of walking, pedestrians cooperate and
also they avoid crowded areas. Figure 4a is a snap-
shot of the initial condition, color indicates the direc-
tion of pedestrian: black indicates \( s = (1, 0, 0, 0) \)(upside),
green indicates \( s = (0, 0, 1, 0) \)(downside), blue indicates
\( s = (0, 0, 0, 1) \)(right), red indicates \( s = (1, 0, 0, 0) \)(left).

After encounter process (See Figure 4b) and merge pro-
cess (See Figure 4c), three kinds of metastable states can be
observed. We call the three states different phases of the system: "block phase" (Figure 4d), i.e. pedestrian
different directions block each other’s road and can-
not move to the other side; "normal phase" (Figure 5a),
i.e. the system is not in order and fluctuates greatly;
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same kind of pattern. The final state is dominated by the
total number of spins \( M \) and temperature \( T \), the phase
diagram is shown in Figure 5g. Lane formation phase and
normal phase do not have a clear boundary, lane forma-
tion phase has a rectangle shape in the diagram approxi-
ately. A linear boundary between the block phase and
normal phase is discovered.

Block phase (Figure 5h) can also be called jam phase in
consistence with the terms used by others. In normal
phase (Figure 5h) the system is not in order and fluctu-
ates greatly, though lane formation phase can be seen,
but the lanes stay just for several time steps and are not stable (In each step of time a pedestrian can take no
more than one step). Transient block phenomenon (See
Figure 5e, c, d) is observed before lane formation. In lane
formation phase (Figure 5e, f), long lanes are formed and
the system remain relatively stable. Pedestrian can move
fluently and the transport efficiency is maximum.

### Egress phenomenon

Egress phenomenon [2] refers to the process of a group of pedestrian leaving a room in a certain period of time. An external field acts as the aim of the crowd that guides the crowd to the door. The external field is set so that pedestrian are moving towards the up side of the space from the exit in the middle (See Figure 6a). Then the crowd clogs around the door until the whole egress is over (See Figure 6b, c, d). During the egress, asymmetric behavior of the clogging crowd around the exit is ob-
served (see Figure 6b). This phenomenon formed in the
following process: a pedestrian either from the left side or
the right side of the clogging crowd enter the exit, then a
lane is formed since the pedestrian of his side follows his
movement while the pedestrian from the other side are
expelled by the lane and await until all the pedestrian
from the left side enter the exit.

Figure 7 shows the relationship between egress time
and door width. Certain critique width is discovered, the result is consistent with Ref.[2], the critical width of a
door is 3 ~ 4 lattice width. According to Ref.[7], each
lattice is approximately 40cm × 40cm, which should be
narrowly able to contain one standing pedestrian, thus the
width of a door is about 1.2m ~ 1.6m, which is
approximately consistent with our daily life, e.g. typi-
cal door in Peking University dorm building.

### CONCLUSION

We construct a Cellular Automata based model to
describe pedestrian dynamics. We focus on the emer-
gence of complex phenomena resulting from simple local
interaction between pedestrian. Analytical mean-field
solution of the model is given to describe dense popu-
lated case. Simulations are carried out to verify the re-
sults. Numerical simulations are carried out to probe the
sparsely populated case, which is beyond the mean-field
approximation. Typical phenomena like lane formation
[1] and jams in a counterflow and egress behavior[2] are
found. Block phase, normal phase and lane formation
phase are distinguished and the phase diagram shows
that only in very limited situation lane formation can be
observed. A linear boundary between block phase and

### SPARSELY POPULATED CASE: COUNTERFLOW AND EGRESS PHENOMENON

In a sparsely populated case, the total spins in the lat-
tice \( M \) is far smaller than the total number of lattices
\( n \). Then mean-field approximation does not make sense
since the system is generally heterogeneous in sparsely
populated case. So we turn to numerical simulation to
probe the various phenomena in this case, e.g. lane for-
mation and jams in counterflow as well as egress phe-
omena.

### Counterflow

Study on counterflow can be found in Refs.[1] [4] [6] [7]. Basically, the counterflow situation refers to the en-
counter of two crowds of pedestrian coming from two op-
posite directions. The phenomena can take place in sys-
tem with a periodical boundary or an open boundary and
also in a narrow corridor. Several distinctive phenomena
will happen depending on the density of the crowd, the
average speed of the pedestrian and the crowd’s level of
anxiety. We conducted several simulations of our model
and showed the different phases of the pedestrian under
different parameters. Since it’s counterflow, in our simu-
lation pedestrian are labeled into two kinds, either tends
to be parallel or anti-parallel with the external field. The
initial positions of pedestrian are randomly chosen.

Choose the parameters as follows: \( a = 100, f = 1, B = 1, n = 10000 \). This corresponds to a
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FIG. 4: color indicates the direction of pedestrian: black indicates \( s=(1,0,0,0) \) (upside), green indicates \( s=(0,0,1,0) \) (downside), blue indicates \( s=(0,0,0,1) \) (right), red indicates \( s=(1,0,0,0) \) (left). (a) Initial state of counterflow, two crowd of pedestrian with different directions. (b) Encounter. (c) Merge. (d) Jam or block phase, pedestrian tangle together and cannot form efficient flow.

FIG. 5: color indicates the direction of pedestrian: black indicates \( s=(1,0,0,0) \) (upside), green indicates \( s=(0,0,1,0) \) (downside), blue indicates \( s=(0,0,0,1) \) (right), red indicates \( s=(1,0,0,0) \) (left). Figure (b)-(f) show the whole process of lane formation. (a) Example of normal phase. (b) Lane formation, transient block phenomenon before lane formation happens. (c) Lane formation, transient block is starting to approach lane formation phase. (d) Lane formation, transient block is disappearing. (e) Lane formation. (f) Lane formation. (g) Phase diagram, \( T_0 \) is the numerical unit of temperature.

FIG. 6: (a) Schematic drawing of the external field \( B \) and the situation of egress. (b) Initial condition. (c) Egress behavior. (d) Asymmetric egress, a lane is formed on the left side of the crowd around the door so that the left side go through the door first while the right side pedestrian remain blocked.
normal phase is discovered. Asymmetric egress behavior is discovered. The critical width for the door is qualitatively measured, results are consistent with Ref. [2] and real life.

Further application of this Cellular Automata based model can be expected in qualitative descriptions of other flow phenomena. It shows that Cellular Automata is a powerful tool to describe non-linear and complex phenomena.

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APPENDIX DERIVATION

Details of Mean-field approximation Solution

Now we study the hamiltonian of the system

\[ H = \frac{1}{2} \sum_{<i,j>} f \cdot \vec{v}_i^{(N)} \cdot \vec{v}_j^{(N)} + \sum_{i=1}^{n} a \cdot \vec{p}_i^{(q)} \cdot \vec{p}_i^{(q)} - \sum_{i=1}^{n} \vec{B}_i^{(q)} \cdot \vec{s}_i^{(q)} \]

we have:

\[ \vec{v}_i^{(N)} \cdot \vec{v}_j^{(N)} = (s_{i1} - s_{iN+1}, s_{i2} - s_{iN+2}, \ldots, s_{iN} - s_{i2N}) \cdot (s_{j1} - s_{jN+1}, s_{j2} - s_{jN+2}, \ldots, s_{jN} - s_{j2N}) \]

\[ = \sum_{k=1}^{N} s_{ik} \cdot s_{jk} + \sum_{k=1}^{N} s_{ik+N} \cdot s_{jk+N} - \sum_{k=1}^{N} s_{ik} \cdot s_{jk} + \sum_{k=1}^{N} s_{ik+N} \cdot s_{jk} \]

Define

\[ s_i^{(q)} \bigoplus s_j^{(q)} = \sum_{k=1}^{N} s_{ik} \cdot s_{jk+N} + \sum_{k=1}^{N} s_{ik+N} \cdot s_{jk} \]

It is easy to see that:

\[ s_i^{(q)} \bigoplus s_j^{(q)} = s_j^{(q)} \bigoplus s_i^{(q)} \]
and

$$(\vec{A}(q) + \vec{B}(q)) \otimes \vec{C}(q) = \vec{A}(q) \otimes \vec{C}(q) + \vec{B}(q) \otimes \vec{C}(q)$$

then

$$\vec{v}_{i_j}^{(N)} \vec{v}_{j_i}^{(N)} = \vec{s}_i^{(q)} \cdot \vec{s}_j^{(q)} - \vec{s}_i^{(q)} \otimes \vec{s}_j^{(q)}$$

Apply mean-field approximation to it, we have

$$\vec{s}_i^{(q)} \cdot \vec{s}_j^{(q)} = (\vec{s}_i^{(q)} - <\vec{s}_i^{(q)}> + \vec{s}_i^{(q)}) \cdot (\vec{s}_j^{(q)} - <\vec{s}_j^{(q)}> + <\vec{s}_j^{(q)}>$$

$$\simeq (\vec{s}_i^{(q)} + \vec{s}_j^{(q)}) \cdot (<\vec{s}_i^{(q)}> - <\vec{s}_j^{(q)}) \cdot (<\vec{s}_j^{(q)}>$$

since $<\vec{s}_i^{(q)}> = <\vec{s}_j^{(q)}>$, denote them as $<\vec{s}^{(q)}>$, then

$$\vec{s}_i^{(q)} \cdot \vec{s}_j^{(q)} \simeq (\vec{s}_i^{(q)} + \vec{s}_j^{(q)}) \cdot (<\vec{s}^{(q)}>) - (<\vec{s}^{(q)}) \cdot (<\vec{s}^{(q)}>$$

Similarly,

$$\vec{s}_i^{(q)} \otimes \vec{s}_j^{(q)} \simeq (\vec{s}_i^{(q)} + \vec{s}_j^{(q)}) \otimes <\vec{s}^{(q)}> - <\vec{s}^{(q)}> \otimes <\vec{s}^{(q)}>$$

$$\vec{p}_i^{(q)} \vec{s}_i^{(q)} \simeq \vec{p}_i^{(q)} \cdot <\vec{s}^{(q)}> + <\vec{p}^{(q)}> \cdot <\vec{s}^{(q)}> - <\vec{p}^{(q)}> \otimes <\vec{s}^{(q)}>$$

Then we have the summation as follows:

$$\sum_{<i,j>} \vec{v}_{i_j}^{(N)} \vec{v}_{j_i}^{(N)} \simeq 2q <\vec{s}^{(q)}> \sum_{i=1}^{n} \vec{s}_i^{(q)} - nq <\vec{s}^{(q)}> <\vec{s}^{(q)}> -$$

$$(2q <\vec{s}^{(q)}> \otimes \sum_{i=1}^{n} \vec{s}_i^{(q)} - nq <\vec{s}^{(q)}> \otimes <\vec{s}^{(q)}>$$

$$\sum_{i=1}^{n} \vec{p}_i^{(q)} \vec{s}_i^{(q)} \simeq <\vec{s}^{(q)}> \sum_{i=1}^{n} \vec{p}_i^{(q)} + <\vec{p}^{(q)}> \sum_{i=1}^{n} \vec{s}_i^{(q)} - n <\vec{p}^{(q)}> <\vec{s}^{(q)}>$$

and it is easy to see

$$<\vec{s}^{(q)}> \otimes <\vec{s}^{(q)}> - <\vec{s}^{(q)}> \cdot <\vec{s}^{(q)}> = - \sum_{l=1}^{N} x_l^2$$

letting $x_l = (<s_l> - <s_{l+N}>)$,

Also we define $x_{N+1} = -x_1$ for later use.

So the Hamiltonian can be approximated as:

$$H \simeq -\vec{B}_{eff}^{(q)} \sum_{i=1}^{n} \vec{s}_i^{(q)}$$

$$+ \frac{1}{2} fqn \sum_{i=1}^{N} x_i^2 + a <\vec{s}^{(q)}> \sum_{i=1}^{n} \vec{p}_i^{(q)} - na <\vec{p}^{(q)}> <\vec{s}^{(q)}>$$

Here $\vec{B}_{eff}^{(q)}$ is the mean-field, the projection of $\vec{B}_{eff}^{(q)}$ in direction $l$ suffices:

$$B_{eff} = f qx_l + B_l - a <p_l>, l = 1, 2, 3, \ldots, q$$
The partition function of the system is:

\[ Z = \sum_{\vec{s}} \cdots \sum_{\vec{s}_n} e^{-\beta H} \]

\[ \simeq e^{-\beta \left( \frac{1}{2} f q_n \sum_{i=1}^{N} x_i^2 + a \langle \vec{s}^{(q)} \rangle > \sum_{i=1}^{n} \vec{p}^{(q)} - n a \langle \vec{p}^{(q)} \rangle > \langle \vec{s}^{(q)} \rangle > \right) \sum_{\vec{s}_1} \cdots \sum_{\vec{s}_n} e^{\beta \vec{B}_{e}^{(q)} \vec{\xi}^{(q)}}} \]

since M of the n lattices is occupied by spins,

\[ Z \simeq e^{-\beta \left( \frac{1}{2} f q_n \sum_{i=1}^{N} x_i^2 + a \langle \vec{s}^{(q)} \rangle > \sum_{i=1}^{n} \vec{p}^{(q)} - n a \langle \vec{p}^{(q)} \rangle > \langle \vec{s}^{(q)} \rangle > \right) A_n^M \left( \sum_{\vec{s}_1} \cdots \sum_{\vec{s}_n} e^{\beta \vec{B}_{e}^{(q)} \vec{\xi}^{(q)}} \right)^M} \]

**Free energy**

Free energy of the system is;

\[ F = -\frac{1}{\beta} \ln Z \]

\[ \simeq \left( \frac{1}{2} f q_n \sum_{i=1}^{N} x_i^2 + a \langle \vec{s}^{(q)} \rangle > \sum_{i=1}^{n} \vec{p}^{(q)} - n a \langle \vec{p}^{(q)} \rangle > \langle \vec{s}^{(q)} \rangle > \right) - \frac{1}{\beta} \ln \left( A_n^M \right) - \frac{1}{\beta} M \ln \left( \sum_{\vec{s}_1} \cdots \sum_{\vec{s}_n} e^{\beta \vec{B}_{e}^{(q)} \vec{\xi}^{(q)}} \right) \]

**Details of Self-consistent Equations**

1. **High Temperature Limit Situation**

\( \beta (f q x_l + B_l) \ll 1 \), so we can do Taylor expansion to the item \( e^{\beta (f q x_l + B_l)} \)

remember \( x_l = \langle s_i \rangle > \langle x_l + N \rangle > \rangle = -x_{N+l} \) and \( (B_{N+1}, B_{N+2}, B_{N+3}, \ldots, B_{2N}) = -(B_1, B_2, B_3, \ldots, B_N) \), so

\[ n < s_i > \simeq M \left( 1 + \beta (f q x_l + B_l) \right) \frac{1}{2N + \sum_{i=1}^{N} \frac{1}{2} \beta^2 (f q x_i + B_i)^2} \]

so for \( l = 1 \)

\[ n < s_1 > \simeq M \left( 1 + \beta (f q x_1 + B_0) \right) \frac{1}{2N + \sum_{i=1}^{N} \frac{1}{2} \beta^2 (f q x_i + B_i)^2} \]

\[ n < s_{N+1} > \simeq M \left( 1 + \beta (f q x_{N+1} - B_0) \right) \frac{1}{2N + \sum_{i=1}^{N} \frac{1}{2} \beta^2 (f q x_i + B_i)^2} \]

for other \( l \)

\[ n < s_l > \simeq M \left( 1 + \beta f q x_l \right) \frac{1}{2N + \sum_{i=1}^{N} \frac{1}{2} \beta^2 (f q x_i + B_i)^2} \]

\[ n < s_{N+l} > \simeq M \left( 1 + \beta f q x_{N+l} \right) \frac{1}{2N + \sum_{i=1}^{N} \frac{1}{2} \beta^2 (f q x_i + B_i)^2} \]
then

\[ n(<s_1 > - < s_{N+1}>) \approx M \frac{\beta(f_q(x_1 - x_{N+1}) + 2B_0)}{2N + \sum_{i=1}^{q} \frac{1}{2} \beta^2(f_q x_i + B_i)^2} \]

\[ n(<s_l > - < s_{N+l}>) \approx M \frac{\beta f_q(x_l - x_{N+l})}{2N + \sum_{i=1}^{q} \frac{1}{2} \beta^2(f_q x_i + B_i)^2} \]

\[ x_l = -x_{N+l} = <s_l > - < s_{l+N} > \]

\[ nx_1 \approx M \frac{2\beta(f_q x_1 + B_0)}{2N + \sum_{i=1}^{q} \frac{1}{2} \beta^2(f_q x_i + B_i)^2} \]

\[ nx_l \approx M \frac{2\beta f_q x_l}{2N + \sum_{i=1}^{q} \frac{1}{2} \beta^2(f_q x_i + B_i)^2} \]

then

\[ x_1 \approx \frac{\beta B_0}{N \frac{\beta}{M} - \beta f_q} \]

\[ x_l \approx 0, l = 2, 3, \ldots, N \]

2. Low Temperature Limit Situation

\[ \beta(f_q x_l + B_l) \gg 1, \text{ if } (f_q x_l + B_l) \neq 0 \]

\[ n < s_l >= M \frac{e^{\beta(f_q x_l + B_l)}}{\sum_{i=1}^{q} e^{\beta(f_q x_i + B_i)}} \]

Guess the solution satisfies

\[ x_1 > 0, \ x_l = 0, l = 2, 3, \ldots, N \]

\[ n < s_l > = M \frac{e^{\beta(f_q x_l + B_l)}}{2N - 2 + e^{\beta(f_q x_1 + B_1)} + e^{-\beta(f_q x_1 + B_1)}} \]

for \( l = 1 \)

\[ n < s_1 > = M \frac{e^{\beta(f_q x_1 + B_0)}}{2N - 2 + e^{\beta(f_q x_1 + B_0)} + e^{-\beta(f_q x_1 + B_0)}} = M, \ \beta(f_q x_1 + B_1) \gg 1 \]

the solution is

\[ < s_1 > = \frac{M}{n} \]

Easy to see that for others

\[ s_l = 0, \ l = 2, 3, \ldots, 2N \]

so

\[ x_1 = \frac{M}{n}, \ x_l = 0, l = 2, 3, \ldots, N \]

which satisfies our guess.
Details of self-organization

\[ \sum_{x^{(q)} \in \mathbf{S}(\{0\})} e^{-\beta B_{x^{(q)}}} = e^{\beta a < p_0 >} \cdot \sum_{i=1}^{N} (e^{-\beta f_{qx}} + e^{\beta f_{qx}}) \]

\[ \simeq 2 e^{\beta a < p_0 >} \cdot \sum_{i=1}^{N} \left( 1 + \frac{1}{2} (\beta f_{qx})^2 + \frac{1}{24} (\beta f_{qx})^4 \right) \]

\[ \simeq 2 N e^{\beta a < p_0 >} \cdot \left( 1 + \frac{N}{2N} (\beta f_{qx})^2 + \sum_{i=1}^{N} \frac{1}{24N} (\beta f_{qx})^4 \right) \]

\[ \ln(\sum_{x^{(q)} \in \mathbf{S}(\{0\})} e^{-\beta B_{x^{(q)}}}) \simeq \ln(2N) + \beta a < p_0 > + \sum_{i=1}^{N} \frac{1}{2N} (\beta f_{qx})^2 + \sum_{i=1}^{N} \frac{1}{24N} (\beta f_{qx})^4 \]

\[ = -\frac{1}{2} \left[ \sum_{i=1}^{N} \frac{1}{2N} (\beta f_{qx})^2 + \sum_{i=1}^{N} \frac{1}{24N} (\beta f_{qx})^4 \right] + \frac{1}{3} \sum_{i=1}^{N} \frac{1}{2N} (\beta f_{qx})^2 + \sum_{i=1}^{N} \frac{1}{24N} (\beta f_{qx})^4 \]

\[ \simeq \ln(2N) + \beta a < p_0 > + \sum_{i=1}^{N} \frac{1}{2N} (\beta f_{qx})^2 + \sum_{i=1}^{N} \frac{1}{24N} (\beta f_{qx})^4 - \sum_{i=1}^{N} \sum_{k=1}^{N} \frac{1}{8N^2} (\beta f)^4 x_i^2 x_k^2 \]

\[ F \simeq -\frac{1}{\beta} \ln(A_n^M) - \frac{M}{\beta} \{ \ln(2N) + \beta a < p_0 > \} \]

\[ + \left( \frac{1}{2} f_{qn} - \frac{\beta M f^2 q^2}{2N} \right) \sum_{l=1}^{N} x_l^2 - \frac{M}{\beta} \sum_{l=1}^{N} \left( \frac{1}{24N} \right)(\beta f_{qx})^4 + \frac{M}{\beta} \sum_{l=1}^{N} \sum_{k=1}^{N} \frac{1}{8N^2} (\beta f)^4 x_l^2 x_k^2 \]

\[ + \frac{M}{\beta} \sum_{l=1}^{N} \sum_{k=1}^{N} \frac{1}{96N^2} (\beta f)^6 x_l^2 x_k^4 - \frac{M}{\beta} \sum_{l=1}^{N} \sum_{k=1}^{N} \sum_{m=1}^{N} \left( \frac{1}{24N^3} \right)(\beta f)^6 x_l^2 x_k^2 x_m^2 \]

\[ \frac{dF}{dx_l^2} = \left( \frac{1}{2} f_{qn} - \frac{\beta M f^2 q^2}{2N} \right) + \frac{M \beta f^4 q^4}{4N} \left( 1 - \frac{1}{3} x_l^2 \right) + \frac{M \beta^3 f^4 q^4}{4N^2} \sum_{k=1, k \neq l}^{N} x_k^2 + \frac{M \beta^5 f^6 q^6}{96N^2} \sum_{k=1}^{N} x_k^4 \]

\[ + \frac{M \beta^5 f^6 q^6}{8N^2} \left( \frac{1}{6} - \frac{1}{N} \right) \sum_{k=1, m=1}^{N} x_k^2 x_m^2 \]
\[ \frac{d^2 F}{d(x_l^2)^2} = \frac{M \beta^3 f^4 q^4}{4N} \left( \frac{1}{N} \right) - \frac{1}{3} + \frac{M \beta^5 f^6 q^6}{48N^2} x_l^2 + \frac{M \beta^5 f^6 q^6}{4N^2} (1 - \frac{1}{N}) \sum_{k=1}^{N} x_k^2 \]

\[ \frac{d^2 F}{d x_l^2 d x_k^2} = \frac{M \beta^3 f^4 q^4}{4N^2} + \frac{M \beta^5 f^6 q^6}{48N^2} x_k^2 + \frac{M \beta^5 f^6 q^6}{4N^2} (1 - \frac{1}{N}) \sum_{b=1}^{N} x_b^2 \]

**When \( \beta > \frac{nN}{\mathcal{F}q} \)**

From \( \frac{dF}{dx_l^2} = 0 \) we have

\[
\begin{pmatrix}
1 & \cdots & 1 & 1 & \cdots & 1 & 1 \\
1 & \cdots & 1 & 1 & \cdots & 1 & 1 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & \cdots & 1 & 1 & \cdots & 1 & 1 \\
1 & \cdots & 1 & 1 & \cdots & 1 & 1 \\
\end{pmatrix}
\begin{pmatrix}
x_1^2 \\
x_2^2 \\
\vdots \\
x_{N-1}^2 \\
x_N^2 \\
\end{pmatrix}
\]

\[= \frac{4N^2}{M \beta^3 f^4 q^4} \left( \frac{M \beta f^2 q^2}{2N} - \frac{f q n}{2} \right) \begin{pmatrix}
1 \\
1 \\
\ldots \\
1 \\
1 \\
\end{pmatrix} \]

so

\[ x_l^2 = \frac{3}{\beta^2 f^2 q^2} (1 - \frac{Nn}{M \beta f q}) \]

or

\[ x_l = \pm \sqrt{\frac{3}{\beta^2 f^2 q^2} (1 - \frac{Nn}{M \beta f q})} \]