More on strongly coupled quenched QED

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March 25, 2022

Abstract

We study the critical region of lattice QED4 in the quenched approximation. The issue of triviality is addressed by contrasting simulation results for \( \langle \bar{\psi}\psi \rangle \) and for the susceptibilities with the predictions of two critical scenarios – powerlaw scaling, and triviality à la Nambu–Jona Lasinio. We discriminate among the two possibilities with reasonable accuracy and we confirm previous results for the critical point and exponents thanks to new analysis strategies and good quality data. The interplay of chiral symmetry breaking with the Goldstone mechanism is studied in detail, and some puzzling features of past results are clarified. Chiral symmetry restoration is observed in the spectrum: the candidate Goldstone boson decouples in the weak coupling phase, while the propagators of the chiral doublets become degenerate. We also present the first measurements of the full mesonic spectrum, relevant for the study of flavour/rotational symmetry restoration. The systematic effects associated with our measurements are discussed in detail.
1 Introduction

One of the most fundamental and difficult questions in high energy theoretical physics is whether theories which are strongly interacting at short distances can exist. It is generally assumed that only asymptotically free theories exist although the evidence for this point of view is based almost wholly on perturbation theory. If a class of theories in which interactions are strong at short distance does in fact exist, then new theoretical approaches to symmetry breaking in the Standard Model (the Higgs sector) would become feasible. In addition, the existence of Quantum Electrodynamics (QED) as a self-contained theory would become a viable theoretical possibility and this fact would have considerable impact on unification schemes [1].

Past studies of lattice QED have discovered a chiral symmetry breaking phase transition at a relatively large value of the coupling constant [2]. This transition is second order in the non-compact version of the theory [3]. By combining accurate measurements of the chiral condensate with spectroscopy calculations, an equation of state and scaling laws were found which are characteristic of an interacting, non-trivial underlying theory.

The results were particularly accurate for the quenched theory, due to the simplicity of the underlying free dynamics [4], [5]. Past work on quenched QED studied the interplay of spectroscopy, scaling laws and critical indices. An equation of state, and scaling laws were derived for the techni-meson masses by exploiting correlation length scaling. The resulting universality relations were confirmed by simulations and the anomalous dimension $\eta$ was measured to be approximatively 0.50 [5] in good agreement with past lattice simulations [4] and hyperscaling relations, as well as with the model’s analytic solution [6], [7].

New analytic insights into chiral transitions were also developed to make the most efficient and compelling use of the simulation data [8].

Despite these successes, the consistency of the data with a logarithmically trivial equation of state is still an open problem. While corrections a’ la $\lambda\phi^4$ were found to be inconsistent with a fermionic theory [9], a trivial theory based on a Nambu–Jona Lasinio model [10] is still possible.

In this work we address the issue of triviality in quenched QED by discussing, and comparing in detail, the two possibile critical behaviors – power law scaling, and triviality a’ la Nambu–Jona–Lasinio. We have performed new, extensive measurements for the chiral condensate and the suscepti-
bility in the scalar channel. A detailed study of finite size effects gave us good control over systematics errors. The good quality of the data available for quenched QED, together with old and new “smart” analysis strategies enabled us to discriminate between the two scenarios with reasonable accuracy. Previous results for the critical point and exponents are confirmed, and cross–checked by making use of the new numerical analysis strategies, which complement the ones discussed in §.

Another central issue is the mechanism of chiral symmetry restoration at the transition. The pion mass, measured in the past, displayed some rather puzzling systematics: it decreased with $\beta$ even as we pass through the chiral restoration transition. In the present work, we study the behaviour of the scalar and pseudoscalar propagator in much more detail than in past work. The numerical results for the pion mass are confirmed. Nevertheless, we observe clear indications of chiral symmetry restoration, both in the scalar and in the vector sector: the propagators of the chiral doublets become degenerate, and the amplitude of the Goldstone mode drops by an order of magnitude across the transition. We speculate that in the thermodynamic limit, the Goldstone mode would decouple completely at the chiral symmetry restoration transition.

We have also obtained estimates for the sigma mass, which evaded us in previous measurements [5]. Unfortunately, they are not accurate enough to be compared with the fermion mass, which would be relevant for the triviality issue. We also present the first measurements of the full mesonic spectrum, which should provide information on the restoration of the rotational/flavor symmetry – i.e. on the symmetries of the continuum theory.

Finally, despite the simplicity of the quenched theory, the spectroscopy computation, and the systematics of finite size effects contain several peculiarities. We collect these issues in the last section for the interested reader.

2 Equation of state and scaling

In our past work we have discussed how the chiral equation of state can be derived, and used to obtain information about the nature of the continuum limit defined at the chiral transition point. Briefly, given that a direct computation of the renormalized charge is plagued with both methodological and computational problems, one resorts to the study of an effective action,
which in turn provides information on the critical exponents. The key point is the expression for the renormalized charge \[ g_R \simeq \xi^{(2\Delta - \gamma - d\nu)} \] (1)

where the critical indices in this expression should be familiar. From the above expression, it is clear that a necessary condition for \( g_R \) to be different from zero in the continuum (\( \xi \to \infty \)) limit is

\[ 2\Delta - \gamma - \nu = 0 \] (2)

i.e. hyperscaling must hold. Our strategy is therefore to identify the proper equation of state, and compute the relevant exponents. (The obvious limitation is that this implies a certain amount of guesswork, whose validity is not always straightforward to verify a posteriori.)

In past work, the critical exponents have been computed with good accuracy in the framework of an equation of state derived in analogy with ferromagnetic systems. The critical exponents are different from mean field ones, and satisfy hyperscaling. That hints, according to the previous discussion, at a non-trivial continuum limit.

However in four dimensions, it is possible that logarithmic corrections to scaling drive \( g_R \) to zero. These corrections would produce "effective" exponents distinct from mean field, and they would still satisfy hyperscaling to good accuracy. So the non--trivial behaviour observed when testing the data against the hypothesis of power-law scaling could be misleading.

This possibility was extensively studied in \[12\], \[13\]. The authors however made use of an EOS based on a \( \lambda \phi^4 \) model, ruled out in \[9\], \[14\] both on numerical and theoretical grounds. We refer the reader to the original literature for discussions of this point.

On the contrary, the theoretically motivated possibility of trivial behavior realized à la Nambu–Jona Lasinio has not been studied in detail. Recall that the continuum model must be the gauged Nambu–Jona Lasinio model, in order for an analytic renormalization group to exist. That is, the theory must be parametrized by two coupling constants, the electrodynamic and the four fermi interaction strengths \[11\], \[12\]. From this point of view, the question is if the gauge fields are enough to 'promote' the trivial Nambu–Jona Lasinio model to an interacting model, characterized by power law scaling with non
mean field exponents, or if, instead, triviality survives the introduction of the gauge interaction.

To our knowledge, the only work which addresses the possibility of trivial logarithms à la Nambu–Jona Lasinio in QED is [15], where this possibility was first put forward, but the numerical study made use of data obtained with very limited resources (small lattices) and was not compelling.

We thus decided to explore this issue in detail, and in the following we will consider in parallel the power–law equation of state and the NJL log-corrected mean field.

To make this short exposition self contained, we recall that the first one can be derived by the chiral Equation of State written in a standard form, under the assumption of scaling

\[ m_e = \langle \bar{\psi}\psi \rangle^\delta f(t/\langle \bar{\psi}\psi \rangle^{1/\beta_{mag}}) \] (3)

Its first order approximation reads

\[ m_e = a^P(\beta_c - \beta) \langle \bar{\psi}\psi \rangle^{\delta-1/\beta} + b^P \langle \bar{\psi}\psi \rangle^\delta \] (4)

Past work has shown that \( \delta - 1/\beta \) (the exponent \( \gamma \)) is, with excellent accuracy, 1. The characteristics of the universal functions \( f \) associated with such a value of the \( \gamma \) exponent were discussed in [8], and verified in [5]. Our candidate equation of state for a possible non-trivial cutoff-free theory will thus be

\[ m_e = a^P(\beta_c - \beta) \langle \bar{\psi}\psi \rangle + b^P \langle \bar{\psi}\psi \rangle^\delta \] (5)

Coming to the log-corrected mean field behaviour, we study the NJL EOS

\[ m_e = a^NJL(\beta_c - \beta) \langle \bar{\psi}\psi \rangle + b^NJL \langle \bar{\psi}\psi \rangle^3 \log \langle \bar{\psi}\psi \rangle \] (6)

which can be derived from the large-N expansion of a four-fermi lagrangian [15], [16].

Once more, we stress that it is important to notice the position of the logarithms, which dictate the sign of the scaling violations: as discussed in [9], [10] their positions are independent of approximations and are, in fact, generic to any fermionic model.

We will comment at the end of this Section on the possible corrections to this leading-log behaviour.
2.1 Equation of state fits

As discussed above, we have compared our data with the two EOS’s

\[ m_e = a^P(\beta_c - \beta) < \bar{\psi}\psi > + b^P < \bar{\psi}\psi >^\delta \]  \hspace{1cm} (7)

and

\[ m_e = a^{NJL}(\beta_c - \beta) < \bar{\psi}\psi > + b^{NJL} < \bar{\psi}\psi >^3 \log < \bar{\psi}\psi > \]  \hspace{1cm} (8)

We have used our old data for the chiral condensate obtained on the $24^4$ lattice, for $\beta$ ranging from .260 to .235, $\Delta \beta = .001$, and five bare electron mass values equi-spaced between .001 and .005. We have fitted the data for several $\beta$ ranges. The inclusion of $\beta < .240$ spoils the power-law fits. Apparently such strong couplings are too far from the critical point to be described by a simplified EOS. The fits, however, are satisfactory and stable once $\beta$ is restricted to the range .240 to .260. We show in Fig. 1 the results of the fit for the power law form for two $\beta$ intervals :[.250,.260], and [.240,.260]. Consider, for instance, the fit over the interval [.250 .260], which includes only 10 $\beta$ values. Its validity extends well beyond the fitted interval, and its results agree with the ones obtained by enlarging the $\beta$ interval.

The fit à la Nambu–Jona-Lasinio (Fig. 2), on the other hand, are qualitatively poorer, and the resulting parameters are sensitive to the $\beta$ interval chosen. These results are summarized in the Table 1.

The coefficient of the term linear in $< \bar{\psi}\psi >$ is the same for the two models. Note that for both models the coefficient $a$ can be expressed as

\[ a = 1/ < \bar{\psi}\psi > (\partial < \bar{\psi}\psi > / \partial \beta)(\partial m_e/ \partial < \bar{\psi}\psi >) \]  \hspace{1cm} (9)

i.e. it can be shown explicitly that $a$ is independent of the nature of the critical singularities, and it could be read off the data in an ‘effective analysis’ style, for either EOS. Differences in the two parametrizations come from the chiral condensate’s behaviour close to the critical point.

The fits clearly favour power–law behavior, with $\beta_c = .2573(1)$, $\delta = 2.13(1)$, in agreement with previous findings (the errors have been estimated by jack-knifing the results obtained, discarding one point at a time).
Table 1: Results of the EOS fits

| β range | $\alpha^{N\!J\!L}$ | $\beta^{N\!J\!L}$ | $\beta_c^{N\!J\!L}$ | $\alpha^P$ | $\beta^P$ | $\beta_c^P$ | δ |
|---------|-------------------|-----------------|-------------------|---------|---------|---------|---|
| .250 – .260 | -5.43(4) | -2.76(4) | .2542(1) | -5.40(2) | 1.04(14) | .2572(6) | 2.12(8) |
| .245 – .260 | -5.41(3) | -2.59(3) | .2538(1) | -5.42(2) | 1.05(2) | .2572(1) | 2.13(1) |
| .240 – .260 | – | – | – | -5.54(2) | 1.07(2) | .2573(1) | 2.13(1) |

2.2 The exponent triangle

We show here that the results for the critical exponents can be obtained very nicely in graphic form, under the single assumption of the existence of an Equation of State which obeys scaling:

$$t = \langle \bar{\psi}\psi \rangle^{1/\beta_{mag}} F(m/ \langle \bar{\psi}\psi \rangle^\delta) \tag{10}$$

or, equivalently:

$$m = \langle \bar{\psi}\psi \rangle^\delta f(t/ \langle \bar{\psi}\psi \rangle^{1/\beta_{mag}}) \tag{11}$$

It is convenient to notice that these two forms are related by the ‘symmetry’ transformation:

$$m \leftrightarrow t$$

$$\beta_{mag} \leftrightarrow 1/\delta$$

$$f \leftrightarrow F$$

(a simple example of this transformation is of course

$$\langle \bar{\psi}\psi \rangle_{t=0} \propto m^{1/\delta} \leftrightarrow \langle \bar{\psi}\psi \rangle_{m=0} \propto t^{\beta_{mag}}.$$ )

Consider now the two logarithmic derivatives

$$R_t = \left. \frac{t}{\langle \psi \psi \rangle} \frac{\partial \langle \bar{\psi}\psi \rangle}{\partial t} \right|_{\psi \psi \rangle}, \quad R_m = \left. \frac{m}{\langle \psi \psi \rangle} \frac{\partial \langle \bar{\psi}\psi \rangle}{\partial m} \right|_{\psi \psi \rangle} \tag{12}$$

$$R_t$$ can be computed from the EOS, eq. (10)

$$1/R_t = 1/\beta_{mag} - \delta y F'(y)/F(y) \tag{13}$$

where $y = m/ \langle \bar{\psi}\psi \rangle^\delta$. The ‘symmetric’ expression for $R_m$ reads:

$$1/R_m = \delta - 1/\beta_{mag} x f'(x)/f(x) \tag{14}$$

where $x = t/ \langle \bar{\psi}\psi \rangle^{1/\beta_{mag}}$.  

\( R_m \) can also be expressed as a function of the susceptibilities

\[
\chi_{\sigma} = \int_x < \bar{\psi}\psi(x)\bar{\psi}\psi(0) >, \quad \chi_{\pi} = \int_x < \bar{\psi}i\gamma_5\psi(x)\bar{\psi}i\gamma_5\psi(0) >
\]

which are the zero momentum projections of the scalar and pseudoscalar propagators. These two susceptibilities are related to the order parameter via: \( \chi_{\sigma} = \partial < \bar{\psi}\psi > / \partial m \) and the Ward identity \( \chi_{\pi} = < \bar{\psi}\psi > / m \). These relationships were discussed in [8], and we will exploit them in the next Section. Here, we will simply make use of the numerical derivatives of the order parameter.

Using Eq.(10) (11) and the symmetry transformation, it is easy to verify that \( R_t \) and \( R_m \) satisfy:

\[
\frac{R_t}{\beta_{mag}} + \delta R_m = 1
\]

We emphasize that this result depends only on the scaling form of the EOS.

When \( R_m \) is 0, in the chiral limit of the strong coupling phase, \( R_t \) defines the exponent \( \beta_{mag} \). The critical point corresponds to \( R_t = 0 \), and correspondingly we have \( R_m = 1/\delta \). Finally, \( R_t \) is \(-\gamma (= \beta_{mag}(\delta - 1)) \) in the chiral limit of the weak coupling phase (\( R_m = 1 \)).

Thus, it is possible to read off the exponents \( \beta_{mag}, \delta \) and \( \gamma \) from the plot \( R_t \) vs. \( R_m \) which itself only requires \( \beta_c \) as input.

We show the 'triangle plot' in Fig. 3 for the \( 24^4 \) lattice. On this lattice we have a grid of data fine enough to permit the accurate calculation of numerical derivatives of the chiral condensate. Since only the global averages and uncertainties were saved, we cannot make comprehensive error estimates for the derivatives, so we include only the errors on the chiral condensate itself in the plot. Since the results for different values of the parameters are strongly correlated (remember that the same gauge configurations were used for all the parameter values), we are confident that the errors on the derivatives are small.

We see in the plot (Fig. 3) that the data points lie on a straight line, in very good agreement with the fit prediction, and the exponents can be read off the axes, as anticipated.
2.3 Susceptibilities analysis

In ref. [8] we have discussed the properties of the ratio \( R_m(t, m) = \frac{m}{\langle \bar{\psi}\psi \rangle} \times \partial \langle \bar{\psi}\psi \rangle / \partial m \). It was shown how its behaviour is dictated by symmetry arguments, and provides information on the critical point, and critical exponents, with no a priori assumptions. In particular, \( R_m(t, 0) = 0(1) \) in the strong (weak) coupling limit and \( R_m(0, m) = 1/\delta \). For power law scaling \( R_m \) is thus expected to be constant with \( m \) and equal to \( 1/\delta \) at the critical point, while for the NJL model it would follow the ‘effective’ \( \delta, R_m(0, m) = 1/\delta_{eff} = 1/(3 + 1/\log \langle \bar{\psi}\psi \rangle) \). \( R_m \) can thus be used to discriminate among different critical behaviours.

Motivated by these considerations, in this new round of simulations we have measured the scalar susceptibility by making use of a noisy estimator. The details of these measurements can be found in the last Section. As we shall show, the new data for the scalar susceptibility provides an independent check on the results obtained above.

We show in Fig. 4a the ratio of susceptibilities plotted at fixed \( \beta \) as a function of the bare electron mass. We plot our results on the 16\(^4\) lattice, where the sigma susceptibility was “directly” computed and, as a cross check, the results on the 24\(^4\) lattice, where the sigma susceptibility was obtained by numerical differentiation, wherever they overlap. Note the nice agreement in the critical region (see also the zoomed view Fig. 4b where we have included all the available \( \beta \)'s on the big lattice), and the (small) discrepancy at strong coupling which we will discuss at the end. This discrepancy is of course irrelevant for the issue of the critical behaviour. The straight line is drawn giving \( 1/\delta \) as estimated by the fit, and falls, as it should, half way between \( \beta = 0.255 \) and \( \beta = 0.260 \). In Fig. 5 we test the susceptibility data against the prediction of the Equation of State à la Nambu–Jona–Lasinio. To do so, it is more convinient to plot \( 1/R_m - 3 \) versus \( 1/\langle \bar{\psi}\psi \rangle \) since this is predicted to be a straight line with unit slope, which is drawn as a solid line. Clearly, the data does not follow it. (It could be that the inclusion of a scale in the log changes the trend in the right direction. However, we have tried to include a scale in the direct fits to the equation of state discussed above, and the results do not change qualitatively. In particular, the scale and the parameter \( b \) are sensitive to the width of the \( \beta \) interval.) The dashed line, again, is the power law prediction: we learn that the susceptibility data favors powerlaw scaling.
2.4 EOS’s higher order corrections

Finally, a word of criticism is in order. The above results have been obtained by assuming that the theory at the critical point is well described by an effective model written in terms of the chiral condensate. Besides that, we restricted ourselves to the simplest parametrizations of the two universality classes we have considered. It could be that the difference in behaviour we have observed between power law scaling and log-corrected mean field are due to an inadequate parametrization of the trivial EOS. In particular, the derivation of the $NJL$ EOS suggests that the effective theory should be formulated in terms of the renormalized electron mass, as opposed to the chiral condensate. Unfortunately, the quality of the fermion mass data is not adequate for an EOS study, but we can try to get a feeling of the magnitude of the corrections. We expect the dependence of the chiral condensate on the dynamical fermion mass is ‘somewhere between’ the strong coupling limit and pure free field behavior (we stress again that we are only making an ‘order of magnitude’ estimate).

The leading term of the strong coupling expansion $^{[17]}$ lagrangean is given by

$$S_{QEDS,C} = 1/4(\bar{\psi}\psi)(\bar{\psi}\psi)_{x+\mu} \tag{17}$$

which gives,

$$2 < \bar{\psi}\psi > + m_e = M_F \tag{18}$$

The analogous relationship in the free field limit was discussed in $^{[12]}$ and reads:

$$< \bar{\psi}\psi > = .62M_f \tag{19}$$

We then plot (Fig. 6) the fermion mass versus the chiral condensate. We see that the numerical results lie between the two limiting behaviors (we can ignore $m_e$ in the plot at strong coupling), and that, in our region of masses, the corrections induced by the replacement of the chiral condensate by the dynamical fermion mass in the Equation of State are less than the statistical errors.
3 Spectroscopy

We discuss the results obtained on a $16^3 \times 32$ lattice, at $\beta = (.250, .255, .260, .270, .280)$, for bare quark masses ranging from .003 to .025. For a very limited subsample of parameters we also took data on a $16^3 \times 64$ lattice, for checking purposes. We have used a variety of sources, which allows us to gain better control and understanding of our results. We have used a standard point source and a wall-noisy source. We have also implemented the sources for the measurements of the full mesonic spectroscopy, which we discuss in the next Section. This allows us to cross-check the results for the pion, which couples to all the sources we have used, and to understand the systematics affecting each measurement technique. For the $\sigma$ particle, the use of a point/wall-noisy source was mandatory, since as we have discussed in [5] this particle does not couple to a rigid wall in the quenched case. Here we were able to obtain reasonable sigma propagators, even if the extraction of the sigma mass was problematic, as we discuss below. We will give the details of the analysis in the last Section, where we will also collect all the Tables.

3.1 Chiral symmetry restoration and level ordering

The analysis of the scalar and pseudoscalar propagators provides information on the mechanism of chiral symmetry breaking and the appearance of a Goldstone mode in the meson spectrum.

The main qualitative feature we observe in the (pseudo)scalar sector of the spectrum is the following: the mass of the lowest pseudoscalar state decreases with $\beta$ at fixed quark mass, while the mass of the first excited state in the pseudoscalar propagator is always comparable with the sigma mass. Also, in past work we have studied the property of the pion mass, and found that the deviations from Goldstone behaviour were not dramatic in the weak coupling phase.

At first sight, this seems puzzling and at odds with chiral symmetry restoration at the transition, where one expects that the sigma and the pion masses become degenerate, and that the pion is no longer a Goldstone particle. One possible solution of this apparent paradox, and a key to gaining better understanding of the mechanism of chiral symmetry restoration, relies on a more detailed analysis of the propagator’s behaviour. As we now
show, in the weak coupling phase the amplitude of the excited states in the pseudoscalar channel becomes dominant, while the fundamental one is suppressed. This explains how the low-lying state in the pseudoscalar channel maintains the property of a Goldstone boson beyond the transition point, as noticed in past work, but there is no contradiction with the Goldstone Theorem. The candidate Goldstone boson simply decouples in the weak coupling phase.

The Tables 12, 13, and the figures 7-8, show the amplitudes of the fundamental and first excited states. In the tables the propagators were normalized to $G_\pi(t = 0)$, so the quoted amplitudes represent the fraction that each state (Goldstone, 1st excited) contributes to the complete propagator. The Goldstone contribution $A_G$ is about 70\% at $m = .003$ and $\beta = .250$, while it is only 10\% at the same mass value, and $\beta = .280$. Around the critical point the Goldstone amplitude become dominant, in the small ($\leq .01$) mass limit, and does not depend much on the mass itself. For higher masses, the two amplitudes are rather insensitive to $\beta$, as expected. Note that 1 is an upper bound for $A_G + A_{1_{\text{st excited}}}$, because of positivity, while $1 - (A_G + A_{1_{\text{st excited}}})$ represents the contribution of higher excitations, which is small. The amplitude itself (i.e., without normalization) $G_\pi(t = 0)$ is a rather smooth function of $\beta$ and mass. For instance, at $\beta = .250(.280)$ it goes from $3.61(1.71)$ at $m = .003$ to $2.61(1.68)$ at $m = .009$. In Fig. 7 we plot the amplitude in the pion channel, and the total amplitude minus the amplitude in the pion channel, for $m = .003$ and $m = .009$. The change in the trend around the transition is very clear, the effect being more pronounced at small masses, as it should be.

Apparently, in the chiral symmetric phase the Goldstone mode simply disappears, and the mass of the pseudoscalar meson (the first excited state) becomes degenerate with the sigma mass, consistent with chiral symmetry restoration. This is certainly what ultimately happens in the perturbative limit. The amplitude of the Goldstone channel apparently is not related to the lattice geometry (at least, it does not change in a significant way on the $16^3 \times 64$ lattice), but it might be controlled by the lattice size. It is very possible that on a larger lattice the suppression of the Goldstone amplitude is abrupt. It would be interesting to check this point.

Another point to be considered is that on the lattice chiral symmetry should be realized at the level of propagators – i.e. the sigma and pion propagators are related by a lattice chiral symmetry transformation. From
this point of view, the mass degeneracy is a by-product of a more general symmetry, and it is not surprising that the correct behaviour at long distance is the last one to be recovered. To study the short-distance behaviour of the restoration of chiral symmetry, we can consider the zero distance propagator. For instance, we show in Fig. 8 the ratio of the zero distance propagators $G(0)$ in the scalar and pseudoscalar channel plotted versus $\beta$ at fixed electron mass: the behaviour is the one expected on the grounds of chiral symmetry, and it is analogous to the trend of the susceptibility ratio.

In a completely analogous way we can study the (pseudo)vector sector. The pseudovector propagators are rather intractable, and we do not have a good estimate for the $A_1$ mass. But again, in order to study chiral symmetry breaking/restoration, we can use, as in the scalar/pseudoscalar case, the propagators themselves. Fig. 9 shows the results for the ratio of the $\rho$ to the $A_1$ propagators at zero distance. Again, chiral symmetry restoration is quite visible for $\beta > 0.260$.

Another important issue is the relationship between the $\sigma$ mass and the fermion. From the data we quote (Table 14, Table 16) the $\sigma$ mass is definitively heavier than twice the fermion mass. Naive Nambu-Jona-Lasinio behavior predicts $\sigma = 2m_f$, while $\sigma < 2m_f$ could be interpreted as a signature of non-triviality (there is nonzero binding energy in the sigma channel). Our simulation results do not fit into either scenario. However, our estimate of the $\sigma$ mass is effected by the uncertainties discussed in the last Section: basically, the long distance behaviour was not clear enough, and our results come from the analysis of intermediate times. The ordering $m_\sigma > 2m_f$ suggested by our data should not be taken as conclusive.

3.2 Full mesonic spectrum and the restoration of the flavor/rotational symmetry

All the symmetries of the continuum have to be realized at the chiral transition, if there is a bona fide continuum limit at that point. Golterman [18] has constructed the rest-frame meson operators, classified them according to the lattice symmetry group, and found relationships with the continuum operators. We borrowed these operators for our QED computations. They have been used first in QCD by the High Energy Monte Carlo Grand Challenge group. We refer the reader to the original works for more detail, and
just recall here that the Dirac matrix $\gamma$ and $\xi$ act in spinor and flavor space, respectively. In this notation the Goldstone meson is excited by the operator $\gamma_4\gamma_5\xi_4\xi_5$.

We have analyzed the data for these operators along with the spectrum data discussed above, on the same $16^3 \times 32$ lattices. The extraction of the direct component of the propagators was necessary to obtain a satisfactory plateau for the effective masses.

All the (pseudo)vector particles turned out to be degenerate for the full set of parameters we have explored, and we will not comment further on them.

The scalar sector is more interesting. We display a representative sample of plots for the effective masses in Fig. 10. The curves level off for relatively small $t$. We chose $t=5$ as a safe starting point for the weighted average of the results. The quality of the data for the pseudoscalar mesons is operator dependent as can be seen from the magnitude of the errors in the Tables 18, 19. We show only a subset of the results, to illustrate their essential features.

Basically, all the non-Goldstone mesons turn out to be mutually (almost) degenerate. A certain trend towards the restoration of continuum symmetry can be observed, especially for the meson excited by $\gamma_5\xi_k\xi_5$, the most significant for the study of flavor symmetry restoration. This trend is shown in Fig. 11. However, it is not clear if this trend toward degeneracy, apparent from the data and from the figures, is induced by the chiral transition, or by the perturbative limit.

4 Numerical details, discussion of finite size effects and of the spectroscopy analysis

As anticipated, this Section is devoted to the discussion of the details of the new simulations, and contains all the relevant Tables. In the new simulations we have used the same algorithm as in the past (see [20]). It begins in momentum space, and produces the appropriate Gaussian distribution of photons. Then, using a Fast Fourier Transform it generates a set of dimensionless gauge fields in coordinate space. The coupling of the gauge fields to the electrons is then implemented by an appropriate rescaling of the gauge fields.
The data for the chiral condensate used in this paper are from our old simulation on the $24^4$ lattice, and from new simulations on the $16^3 \times 32$, $16^4$ and $8^4$ lattices.

The chiral condensate was measured by inverting the Dirac operator with a noisy source defined on the even sites of the lattice. We have also measured $\frac{\partial \langle \bar{\psi} \psi \rangle}{\partial m}$ (the scalar susceptibility, $\chi_\sigma$), by using the same noisy background as for the chiral condensate. Several Tables (2–7) collect our results.

As is well known, the finite size effects on the order parameter (and consequently its derivative, the sigma susceptibility) are not dramatic. However, they are not negligible, and have some peculiar characteristics which should be discussed. First, the qualitative behaviour of the finite size/geometry effects depend on the phase—whether chiral symmetry is broken or restored.

We discuss first the behaviour in the strong coupling phase. We plot in Fig. 12 the data for the chiral condensate on the three symmetric lattices. The results on the small lattice are effected by finite size effects of the order of several percent, but on all the lattices the chiral condensate behaves in a qualitative correct way. The only perplexing point is that the chiral con-

|     | .250  | .255  | .260  | .265  | .270  | .275  | .280  |
|-----|-------|-------|-------|-------|-------|-------|-------|
| 0.003 | 0.1036(64) | 0.0786(53) | 0.0630(38) | 0.0453(27) | 0.0349(25) | 0.0294(15) | 0.0236(10) |
| 0.007 | 0.1331(52) | 0.1132(45) | 0.0989(37) | 0.0781(26) | 0.0645(20) | 0.0585(20) | 0.0512(20) |
| 0.011 | 0.1501(41) | 0.1338(33) | 0.1209(35) | 0.1034(27) | 0.0883(20) | 0.0805(19) | 0.0710(17) |
| 0.015 | 0.1606(30) | 0.1473(30) | 0.1340(26) | 0.1201(26) | 0.1094(21) | 0.0989(20) | 0.0887(17) |
| 0.019 | 0.1792(30) | 0.1665(28) | 0.1511(26) | 0.1395(25) | 0.1267(21) | 0.1172(20) | 0.1042(17) |
| 0.023 | 0.1915(30) | 0.1779(27) | 0.1673(26) | 0.1508(23) | 0.1360(19) | 0.1303(21) | 0.1173(18) |
| 0.027 | 0.2031(27) | 0.1866(25) | 0.1730(23) | 0.1614(22) | 0.1543(19) | 0.1440(21) | 0.1316(17) |
| 0.031 | 0.2121(28) | 0.1984(27) | 0.1891(26) | 0.1722(21) | 0.1609(19) | 0.1542(19) | 0.1417(18) |
| 0.035 | 0.2238(27) | 0.2074(24) | 0.1958(23) | 0.1837(22) | 0.1709(21) | 0.1634(20) | 0.1552(21) |
| 0.039 | 0.2256(27) | 0.2171(25) | 0.2064(27) | 0.1939(23) | 0.1806(19) | 0.1751(20) | 0.1634(18) |
| 0.043 | 0.2351(25) | 0.2205(23) | 0.2100(22) | 0.1997(21) | 0.1931(20) | 0.1814(20) | 0.1714(18) |
| 0.047 | 0.2426(23) | 0.2343(23) | 0.2209(21) | 0.2113(22) | 0.2023(21) | 0.1916(19) | 0.1803(17) |
| 0.051 | 0.2495(24) | 0.2403(23) | 0.2299(23) | 0.2168(20) | 0.2056(20) | 0.1981(20) | 0.1877(18) |
| 0.055 | 0.2592(23) | 0.2447(23) | 0.2346(21) | 0.2239(20) | 0.2164(18) | 0.2079(19) | 0.1974(17) |

### 4.1 Chiral condensate and sigma susceptibilities

The data for the chiral condensate used in this paper are from our old simulation on the $24^4$ lattice, and from new simulations on the $16^3 \times 32$, $16^4$ and $8^4$ lattices.

The chiral condensate was measured by inverting the Dirac operator with a noisy source defined on the even sites of the lattice. We have also measured $\frac{\partial \langle \bar{\psi} \psi \rangle}{\partial m}$ (the scalar susceptibility, $\chi_\sigma$), by using the same noisy background as for the chiral condensate. Several Tables (4–7) collect our results.

As is well known, the finite size effects on the order parameter (and consequently its derivative, the sigma susceptibility) are not dramatic. However, they are not negligible, and have some peculiar characteristics which should be discussed. First, the qualitative behaviour of the finite size/geometry effects depend on the phase—whether chiral symmetry is broken or restored.

We discuss first the behaviour in the strong coupling phase. We plot in Fig. 12 the data for the chiral condensate on the three symmetric lattices. The results on the small lattice are effected by finite size effects of the order of several percent, but on all the lattices the chiral condensate behaves in a qualitative correct way. The only perplexing point is that the chiral con-
Table 3: $m_x \times \chi_{\sigma}$ on the $8^4$ lattice

|    | .250  | .255  | .260  | .265  | .270  | .275  | .280  |
|----|-------|-------|-------|-------|-------|-------|-------|
| 0.003 | 0.0359(43) | 0.0381(31) | 0.0371(21) | 0.0324(13) | 0.0259(12) | 0.0244(8) | 0.0212(6) |
| 0.007 | 0.0364(37) | 0.0417(33) | 0.0454(22) | 0.0483(14) | 0.0457(12) | 0.0435(14) | 0.0410(9) |
| 0.011 | 0.0421(29) | 0.0507(22) | 0.0530(21) | 0.0564(15) | 0.0565(13) | 0.0572(11) | 0.0537(10) |
| 0.015 | 0.0586(19) | 0.0562(21) | 0.0599(16) | 0.0630(16) | 0.0655(11) | 0.0645(12) | 0.0623(9) |
| 0.019 | 0.0594(22) | 0.0632(19) | 0.0656(17) | 0.0676(15) | 0.0710(13) | 0.0701(11) | 0.0702(11) |
| 0.023 | 0.0629(19) | 0.0683(16) | 0.0686(18) | 0.0723(14) | 0.0759(11) | 0.0742(11) | 0.0757(10) |
| 0.027 | 0.0688(20) | 0.0726(17) | 0.0769(15) | 0.0779(14) | 0.0777(13) | 0.0802(13) | 0.0812(11) |
| 0.031 | 0.0692(19) | 0.0742(18) | 0.0772(17) | 0.0809(14) | 0.0837(12) | 0.0837(12) | 0.0861(10) |
| 0.035 | 0.0760(20) | 0.0780(17) | 0.0806(16) | 0.0852(15) | 0.0871(13) | 0.0871(12) | 0.0884(11) |
| 0.039 | 0.0779(18) | 0.0819(16) | 0.0840(16) | 0.0877(14) | 0.0887(12) | 0.0917(12) | 0.0917(11) |
| 0.043 | 0.0822(16) | 0.0831(16) | 0.0862(15) | 0.0902(14) | 0.0920(12) | 0.0927(12) | 0.0954(11) |
| 0.047 | 0.0831(16) | 0.0864(16) | 0.0888(14) | 0.0906(14) | 0.0950(13) | 0.0946(12) | 0.0985(11) |
| 0.051 | 0.0852(16) | 0.0884(16) | 0.0894(16) | 0.0941(13) | 0.0981(12) | 0.0973(13) | 1.003(11) |
| 0.055 | 0.0870(16) | 0.0905(16) | 0.0955(13) | 0.0978(13) | 0.0972(13) | 0.0997(14) | 1.034(12) |

densate increases with decreasing volume. One should expect the opposite trend, since chiral symmetry breaking, which produces a non-zero value of the chiral condensate in the chiral limit, is an infinite volume effect which should be obscured on a finite lattice. However, from Fig. 12 we observe that the derivative of the chiral condensate with respect to the quark mass is increasing as the volume decreases at smaller masses. This effect can be read off also from the susceptibility data. (This is consistent with the ‘early’ transition observed for small volume: finite volume increases the slope of the chiral condensate as a function of the bare quark mass, thus mimicking the approach to the critical point.) If this trend is maintained, the curves for the chiral condensate are going to ‘cross’ at some point, thus recovering the expected pattern for finite size effects in the chiral limit— that the chiral condensate decreases with volume. Indeed, qualitative arguments leading to the conclusion that the chiral condensate should decrease with volume, require that the dominant contribution to the chiral condensate is from spontaneous (as opposed to the explicit) symmetry breaking. On small lattices, and large masses, the main contributions to the chiral condensate/susceptibility come from the excited states, so the standard arguments do not necessarily apply.
Another possibility is that finite volume effects push up the physical pion mass, in such a way that the pion on a $8^4$ lattice is even heavier than the one on a $16^4$. Then PCAC would tell us that the chiral condensate increases with volume, if $f_\pi$ is (almost) constant with volume.

So, we can find several qualitative explanations for this seemingly puzzling behaviour. However, it should be noticed that in the unquenched model, the behaviour is the conventional one, so it is possible that these results point out some characteristic/pathology of the quenched model not yet explored.

The sigma susceptibility, again in the broken phase, is very sensitive to finite volume effects: at $\beta = .250$, mass $\approx .003$, the difference between the results on the $8^4$ and on the $16^4$ lattice is around 30%, as opposed to the 10% difference for the chiral condensate. This can be understood by recalling that in the strong coupling, infinite volume limit, the sigma susceptibility is 0 for every value of the electron mass, while it is necessarily finite on a finite volume. So, roughly speaking, we can expect $\chi_\sigma \approx 1/V$, which justifies the considerable

| Table 4: Chiral condensate on the $16^4$ lattice |
|-----------------------------------------------|
|                                | .250  | .255  | .260  | .265  | .270  | .275  | .280  |
|                                | .003  | .0977(17) | .0711(17) | .0558(14) | .0412(10) | .0316(6) | .0258(4) | .0214(3) |
|                                | .007  | .1189(17) | .1016(13) | .0866(12) | .0725(10) | .0620(8) | .0529(7) | .0463(5) |
|                                | .011  | .1407(14) | .1250(12) | .1091(10) | .0971(9) | .0827(7) | .0745(7) | .0663(6) |
|                                | .015  | .1606(12) | .1418(11) | .1286(11) | .1146(10) | .1032(9) | .0929(7) | .0842(6) |
|                                | .019  | .1726(14) | .1577(12) | .1430(10) | .1313(10) | .1192(8) | .1097(7) | .0990(6) |
|                                | .023  | .1843(12) | .1687(11) | .1555(9) | .1453(10) | .1325(8) | .1236(8) | .1131(7) |
|                                | .027  | .1955(11) | .1833(11) | .1688(10) | .1554(9) | .1448(8) | .1351(8) | .1266(7) |

| Table 5: $m_e \times \chi_\sigma$ on the $16^4$ lattice |
|-----------------------------------------------|
|                                | .250  | .255  | .260  | .265  | .270  | .275  | .280  |
|                                | .003  | .0238(16) | .0304(10) | .0297(9) | .0291(5) | .0266(3) | .0233(2) | .0203(2) |
|                                | .007  | .0414(11) | .0434(8) | .0445(7) | .0455(5) | .0440(4) | .0419(3) | .0391(3) |
|                                | .011  | .0492(9) | .0527(8) | .0546(7) | .0563(5) | .0552(4) | .0541(3) | .0520(3) |
|                                | .015  | .0564(9) | .0603(7) | .0621(6) | .0640(5) | .0634(5) | .0627(5) | .0617(3) |
|                                | .019  | .0633(8) | .0664(7) | .0694(6) | .0706(6) | .0710(5) | .0703(4) | .0695(4) |
|                                | .023  | .0685(8) | .0714(7) | .0745(6) | .0761(6) | .0765(5) | .0774(5) | .0760(4) |
|                                | .027  | .0735(7) | .0756(7) | .0782(6) | .0806(5) | .0814(5) | .0818(5) | .0817(4) |
Table 6: Chiral condensate on the $16^3 \times 32$ lattice

|       | .250 | .255 | .260 | .270 | .280 |
|-------|------|------|------|------|------|
| .003  | .0980(18) | .0783(26) | .0633(14) | .0435(9) | .0312(10) |
| .005  | .1116(17) | – | – | .0560(9) | .0418(8) |
| .007  | .1244(16) | – | – | .0664(9) | .0524(9) |
| .009  | .1323(15) | .1143(18) | .1002(12) | .0795(10) | .0606(8) |
| .011  | – | – | – | .0895(20) | – |
| .013  | – | – | – | .0959(16) | – |
| .015  | – | .1446(21) | .1280(12) | .1046(18) | – |
| .017  | – | – | – | .1141(17) | – |
| .019  | .1739(28) | – | – | .1237(18) | – |
| .021  | .1774(22) | .1666(18) | .1507(11) | .1276(18) | – |
| .023  | .1840(25) | – | – | .1337(15) | – |
| .025  | .1904(21) | – | – | .1407(17) | – |

Table 7: $m \times \chi$ on the $16^3 \times 32$ lattice

|       | .250 | .255 | .260 | .270 | .280 |
|-------|------|------|------|------|------|
| .003  | .0237(9) | .0219(13) | .0235(7) | .0205(5) | .0167(6) |
| .005  | .0324(8) | – | – | .0302(5) | .0254(6) |
| .007  | .0375(9) | – | – | .0381(5) | .0339(5) |
| .009  | .0435(8) | .0465(10) | .0477(6) | .0452(4) | .0409(5) |
| .011  | – | – | – | .0503(8) | – |
| .013  | – | – | – | .0575(8) | – |
| .015  | – | .0599(10) | .0615(6) | .0612(8) | – |
| .017  | – | – | – | .0650(10) | – |
| .019  | .0614(13) | – | – | .0688(9) | – |
| .021  | .0652(12) | .0704(9) | .0714(5) | .0710(8) | – |
| .023  | .0671(12) | – | – | .0750(7) | – |
| .025  | .0708(11) | – | – | .0785(9) | – |

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sensitivity of the sigma susceptibility to the volume.

The geometry effects, as deduced from the comparison of the results on the $16^4$ and $16^3 \times 32$ lattice are instead very small, thus showing that the long-distance behaviour in the strong coupling phase does not have any peculiarity.

Consider now the weak coupling phase. The pattern of finite size effects in the chiral condensate is analogous to the one observed at strong coupling, so apparently the finite size effects do not 'see' the transition. The finite size effects on the sigma susceptibility are instead different, and not as strong in the weak coupling phase. Indeed, here the chiral condensate should extrapolate to 0, and its derivative is supposed to have a weaker volume dependence in this phase.

In fact, finite volume corrections can be written as

\[
\langle \bar{\psi} \psi \rangle = \langle \bar{\psi} \psi \rangle (V = \infty, m_c) (1 - A/V^\alpha). \]

When \( \langle \bar{\psi} \psi \rangle = m_c^2 \), finite volume effects on its derivative at finite \( m_c \) are still \( O(1 - A/V^\alpha) \). The finite volume effects on the chiral condensate and on its derivative are comparable, and thus smaller, for the derivative, than the ones observed in the strong coupling phase.

The most peculiar systematics we have observed in the weak coupling phase come from the small mass results on the asymmetric lattice (Fig. 13). They look at odds with chiral symmetry restoration, which is instead apparent even on the smaller lattice. We attribute them to a lack of complete suppression of the Goldstone mode in the weak coupling phase, as we have discussed above.

Summarizing, the finite volume effects, as studied on symmetric lattices, are most important for the sigma susceptibilities in the strong coupling phase, and can be traced back to the difficulties of a finite lattice breaking chiral symmetry. They are not very strong, but somehow puzzling, for the chiral condensate itself.

The geometry effects are instead most significant in the weak coupling phase, at small masses, thus demonstrating a rather complicated, and maybe not completely understood, long distance behaviour.

Anyway, the data for the chiral condensate obtained on the $16^4$ lattice are reasonably free of systematics in the entire range of parameters we have explored, as can be inferred from their comparison with the $24^4$ lattice results.

The same can be said for the sigma susceptibility, which deserves, however, a special remark. A rather delicate point about the susceptibility computation is the subtraction of the disconnected part. In principle, this is not performed by our noisy estimator. The long distance behaviour of the sigma
Table 8: Results for from the effective mass analysis of the wall propagators in the pseudoscalar channel (t=14).

|   | .250  | .255  | .260  | .270  | .280  |
|---|-------|-------|-------|-------|-------|
| .003 | .1798(60) | .1727(88) | .1473(83) | .1040(111) | .0609(251) |
| .005 | .2288(44) | – | – | .1521(70) | .1133(12) |
| .007 | .2676(36) | – | – | .1927(52) | .1537(82) |
| .009 | .3003(31) | .2925(41) | .2761(38) | .2283(42) | .1884(63) |
| .011 | – | – | – | .2620(45) | – |
| .013 | – | – | – | .2907(51) | – |
| .015 | – | .3708(29) | .3602(27) | .3173(46) | – |
| .017 | – | – | – | .3421(43) | – |
| .019 | .4242(35) | – | – | .3665(48) | – |
| .021 | .4423(34) | .4317(24) | .4247(21) | .3888(45) | – |
| .023 | .4593(33) | – | – | .4096(43) | – |
| .025 | .4754(32) | – | – | .4293(41) | – |

For the meson operators the standard parametrization of the propagators reads:

4.2 Spectroscopy analysis

The spectroscopy data are from a $16^3 \times 32$ lattice, where a point, a wall source, and suitable point split wall sources were used for the inversion of the Dirac propagator. We took some measurements also on a $16^3 \times 64$ lattice for checking purposes. The computation of the spectrum with the (rigid) point split wall requires gauge fixing. The gauge is not fixed in the noisy wall inversion, where our aim is to obtain the point propagator with better statistics. Thus, in this case all the gauge dependent contributions have to cancel out.

For the meson operators the standard parametrization of the propagators reads:
Table 9: Results from the effective mass analysis of the point source propagators in the pseudoscalar channel (t=14)

|       | .250   | .255   | .260   | .270   | .280   |
|-------|--------|--------|--------|--------|--------|
| .003  | 1.819(73) | 1.764(108) | 1.598(12) | 1.267(102) | 0.665(468) |
| .005  | 2.346(59)  | –      | –      | 1.713(68)  | 1.174(202) |
| .007  | 2.758(51)  | –      | –      | 2.112(53)  | 1.560(130) |
| .009  | 3.098(43)  | 3.019(51)  | 2.932(52)  | 2.470(45)  | 1.906(98) |
| .011  | –      | –      | –      | 2.791(69)  | –      |
| .013  | –      | –      | –      | 3.088(60)  | –      |
| .015  | –      | 3.827(39)  | 3.768(39)  | 3.362(54)  | –      |
| .017  | –      | –      | –      | 3.615(49)  | –      |
| .019  | 4.239(49)  | –      | –      | 3.909(46)  | –      |
| .021  | 4.414(48)  | 4.430(33)  | 4.396(31)  | 4.132(44)  | –      |
| .023  | 4.578(46)  | –      | –      | 4.340(42)  | –      |
| .025  | 4.734(45)  | –      | –      | 4.536(40)  | –      |

Table 10: Results for the pion mass from two particle fits for $t > 2$ of the point/noisy propagators

|       | .250   | .255   | .260   | .270   | .280   |
|-------|--------|--------|--------|--------|--------|
| .003  | 1.843(44)  | 1.500(323)  | 1.484(87)  | 1.145(64)  | 0.638(120) |
| .005  | 2.349(36)  | –      | –      | 1.591(55)  | 1.149(74) |
| .007  | 2.752(27)  | –      | –      | 2.045(38)  | 1.609(198) |
| .009  | 3.054(209)  | 2.923(112)  | 2.899(39)  | 2.419(38)  | 1.930(100) |
| .011  | –      | –      | –      | 2.748(67)  | –      |
| .013  | –      | –      | –      | 3.099(68)  | –      |
| .015  | –      | 3.679(228)  | 3.738(61)  | 3.372(66)  | –      |
| .017  | –      | –      | –      | 3.615(69)  | –      |
| .019  | 3.856(302)  | –      | –      | 3.853(71)  | –      |
| .021  | 4.037(280)  | 4.127(342)  | 4.191(512)  | 4.000(58)  | –      |
| .023  | 4.183(443)  | –      | –      | 4.195(99)  | –      |
| .025  | 4.281(975)  | –      | –      | 4.375(3165) | –      |
Table 11: Results for the first excited state in the pseudoscalar channel, from two particle fits for $t > 2$ of the noisy propagators

|       | .250          | .255          | .260          | .270          | .280          |
|-------|---------------|---------------|---------------|---------------|---------------|
| .003  | .5022 (1179)  | .3364(79)     | .4584(55)     | ...           | ...           |
| .005  | .5380 (920)   | –             | –             | .5642(232)    | .5495(18)     |
| .007  | .5982 (870)   | –             | –             | .5798(208)    | .5653(17)     |
| .009  | .5420 (44)    | .4980(722)    | .6289(481)    | .5962(199)    | .5773(18)     |
| .011  | –             | –             | –             | .6730(486)    | –             |
| .013  | –             | –             | –             | .6958(488)    | –             |
| .015  | –             | .5246(1467)   | .7364(349)    | .7179(499)    | –             |
| .017  | –             | –             | –             | .7405(510)    | –             |
| .019  | .4630 (283)   | –             | –             | .7057(433)    | –             |
| .021  | .4786 (196)   | .5292(593)    | .5764(247)    | .7261(.4521)  | –             |
| .023  | .4917 (244)   | –             | –             | .7467(.47)    | –             |
| .025  | .4962 (461)   | –             | –             | .7613(1.17)   | –             |

Table 12: Amplitude of the fundamental state from two particle fits for $t > 2$ of $G_\pi(t)/G_\pi(0)$

|       | .250          | .255          | .260          | .270          | .280          |
|-------|---------------|---------------|---------------|---------------|---------------|
| .003  | .711(63)      | .413(135)     | .376(48)      | .234(24)      | .104(21)      |
| .005  | .708(54)      | –             | –             | .225(19)      | .115(15)      |
| .007  | .726(47)      | –             | –             | .231(18)      | .121(15)      |
| .009  | .701(270)     | .481(101)     | .492(33)      | .240(18)      | .126(15)      |
| .011  | –             | –             | –             | .268(40)      | –             |
| .013  | –             | –             | –             | .288(40)      | –             |
| .015  | –             | .474(254)     | .534(54)      | .309(44)      | –             |
| .017  | –             | –             | –             | .333(41)      | –             |
| .019  | .272(180)     | –             | –             | .349(34)      | –             |
| .021  | .34(31)       | .343(296)     | .431(490)     | .324(30)      | –             |
| .023  | .30(17)       | –             | –             | .329(55)      | –             |
| .025  | .21(42)       | –             | –             | .34(93)       | –             |
Table 13: Amplitude of the 1st excited state from two particle fits for $t > 2$ of $G_\pi(t)/G_\pi(0)$

|       | .250  | .255  | .260  | .270  | .280  |
|-------|-------|-------|-------|-------|-------|
| .003  | .264 (41) | .549 (192) | .539 (57) | .671 (40) | .710 (46) |
| .005  | .253 (39) | –     | –     | .674 (37) | .719 (46) |
| .007  | .231 (35) | –     | –     | .667 (35) | .719 (46) |
| .009  | .227 (163) | .412 (90) | .453 (40) | .656 (33) | .716 (46) |
| .011  | –   | –   | –   | .705 (88) | –   |
| .013  | –   | –   | –   | .689 (85) | –   |
| .015  | –   | .397 (197) | .386 (37) | .671 (83) | –   |
| .017  | –   | –   | –   | .651 (80) | –   |
| .019  | .643 (256) | –   | –   | .622 (52) | –   |
| .021  | .649 (190) | .552 (286) | .425 (460) | .600 (300) | –   |
| .023  | .675 (248) | –   | –   | .578 (359) | –   |
| .025  | .829 (517) | –   | –   | .562 (880) | –   |

Table 14: Sigma masses from fits for $t > 3$ to $A e^{\rho - m_\rho} + B$ of the noisy scalar propagators

|       | .250  | .255  | .260  | .270  | .280  |
|-------|-------|-------|-------|-------|-------|
| .003  | .4566 (245) | .5289 (339) | .4706 (183) | .5663 (180) | .6152 (274) |
| .005  | .5095 (189) | –   | –   | .5567 (131) | .6006 (167) |
| .007  | .5926 (197) | –   | –   | .5697 (95) | .6019 (116) |
| .009  | .5888 (148) | .5817 (161) | .5490 (96) | .5951 (98) | .6138 (111) |
| .011  | –   | –   | –   | .6034 (138) | –   |
| .013  | –   | –   | –   | .6144 (194) | –   |
| .015  | –   | .6551 (114) | .6452 (85) | .6344 (112) | –   |
| .017  | –   | –   | –   | .6462 (130) | –   |
| .019  | .7488 (316) | –   | –   | .6658 (120) | –   |
| .021  | .7264 (209) | .7367 (127) | .7076 (87) | .7055 (85) | –   |
| .023  | .7969 (250) | –   | –   | .6994 (84) | –   |
| .025  | .8094 (188) | –   | –   | .7337 (108) | –   |
Table 15: Background B from fits for $t > 3$ to $A e^{\sigma m - \sigma_\sigma} + B$ of the noisy propagators in the scalar channel

|       | .250       | .255       | .260       | .270       | .280       |
|-------|------------|------------|------------|------------|------------|
| .003  | 9.2(20.6)E-04 | -2(189)E-05 | 7.00(1.76)E-03 | 1.33(18)E-02 | 1.40(27)E-02 |
| .005  | 12.3(5.7)E-04 | -          | -          | 4.76(68)E-03 | 7.07(1.03)E-03 |
| .007  | 1.4(2.5)E-04 | -          | -          | 2.18(31)E-03 | 3.21(48)E-03 |
| .009  | -1.8(14.8)E-05 | 8(1430)E-07 | 2.21(1.22)E-04 | 1.18(13)E-03 | 2.38(24)E-03 |
| .011  | -          | -          | -          | 5.38(1.66)E-04 | -          |
| .013  | -          | -          | -          | 3.96(93)E-04 | -          |
| .015  | -          | 2.00(3.42)E-05 | 4.31(2.56)E-05 | 1.53(60)E-04 | -          |
| .017  | -          | -          | -          | 7.82(3.37)E-05 | -          |
| .019  | -2.5(2.2)E-05 | -          | -          | 1.25(21)E-04 | -          |
| .021  | 2.17(1.49)E-05 | 5.82(9.16)E-05 | 2.97(0.88)E-05 | 6.58(1.68)E-05 | -          |
| .023  | 1.07(0.99)E-06 | -          | -          | 3.13(1.16)E-05 | -          |
| .025  | -7.58(7.74)E-06 | -          | -          | 1.11(86)E-05 | -          |

Table 16: Electron masses from point source propagators.

|       | .250       | .255       | .260       | .270       | .280       |
|-------|------------|------------|------------|------------|------------|
| .003  | .187(30)   | .157(45)   | .113(38)   | .094(11)   | .063(14)   |
| .005  | .215(27)   | -          | -          | .115(11)   | .080(12)   |
| .007  | .232(26)   | -          | -          | .133(12)   | .097(12)   |
| .009  | .246(24)   | .204(23)   | .181(28)   | .150(12)   | .113(13)   |
| .011  | -          | -          | -          | .118(18)   | -          |
| .013  | -          | -          | -          | .134(17)   | -          |
| .015  | -          | .236(19)   | .235(27)   | .147(17)   | -          |
| .017  | -          | -          | -          | .160(17)   | -          |
| .019  | .293(69)   | -          | -          | .198(16)   | -          |
| .021  | .310(70)   | .265(17)   | .279(27)   | .210(16)   | -          |
| .023  | .325(70)   | -          | -          | .223(17)   | -          |
| .025  | .341(71)   | -          | -          | .235(18)   | -          |
Table 17: Results for the $\rho$ mass from two particle fit of the wall propagators.

|       | .250     | .255     | .260     | .270     | .280     |
|-------|----------|----------|----------|----------|----------|
| .003  | .3394(94)| .2816(99)| .2293(58)| .1661(44)| .1206(47)|
| .005  | .3387(181)| –        | –        | .2097(34)| .1630(35)|
| .007  | .4226(61)| –        | –        | .2476(29)| .1981(30)|
| .009  | .4573(302)| .4107(509)| .3539(66)| .2821(26)| .2295(28)|
| .011  | –        | –        | –        | .3126(44)| –        |
| .013  | –        | –        | –        | .3414(37)| –        |
| .015  | –        | .5016(131)| .4446(67)| .3681(116)| –        |
| .017  | –        | –        | –        | .4000(403)| –        |
| .019  | .6276(122)| –        | –        | .4202(286)| –        |
| .021  | .6501(139)| .5757(173)| .5188(69)| .4442(234)| –        |
| .023  | .6717(138)| –        | –        | .4672(372)| –        |
| .025  | .6937(148)| –        | –        | .4895(330)| –        |

Table 18: Psudoscalar meson masses for $m_e = .003$. Weighted results of effective masses, $t > 5$

|       | .250     | .255     | .260     | .270     | .280     |
|-------|----------|----------|----------|----------|----------|
| $\gamma_5\xi_5$ | .3353(477)| .2684(411)| .2431(263)| .1698(159)| .1265(157)|
| $\gamma_5\xi_5\xi_5$ | .3266(226)| .2803(195)| .2336(125)| .1499(70)| .0979(77)|
| $\gamma_4\gamma_5\xi_5\xi_4$ | .3469(338)| .2677(268)| .2394(170)| .1678(103)| .1324(111)|
| $\gamma_4\gamma_5\xi_5\xi_m$ | .3238(208)| .2951(201)| .2233(108)| .1535(65)| .1011(75)|
| $\gamma_5\xi_5\xi_4$ | .3457(293)| .2884(260)| .2461(149)| .1636(87)| .1346(91)|
| $\gamma_5\xi_4$ | .3274(194)| .2735(185)| .2305(105)| .1537(63)| .1072(74)|

Table 19: Psudoscalar meson masses for $m_e = .009$. Weighted results of effective masses, $t > 5$

|       | .250     | .255     | .260     | .270     | .280     |
|-------|----------|----------|----------|----------|----------|
| $\gamma_5\xi_5$ | .4544(287)| .4103(256)| .3628(156)| .2813(92)| .2319(83)|
| $\gamma_5\xi_5\xi_5$ | .4522(141)| .4071(128)| .3605(85)| .2744(48)| .2168(48)|
| $\gamma_4\gamma_5\xi_5\xi_m$ | .4629(181)| .4022(193)| .3645(110)| .2815(67)| .2330(69)|
| $\gamma_4\gamma_5\xi_m$ | .4568(135)| .4143(138)| .3583(82)| .2762(46)| .2216(44)|
| $\gamma_5\xi_m\xi_4$ | .4644(167)| .4073(164)| .3656(99)| .2822(58)| .2369(58)|
| $\gamma_5\xi_4$ | .4521(118)| .4086(134)| .3581(74)| .2763(47)| .2334(49)|
\[ G(\tau) = a [\exp(-M\tau) + \exp(-M(Nr - \tau))] + a1 [\exp(-M1\tau) + \exp(-M1(Nr - \tau))] + \\
(-1)^\tau \tilde{a} [\exp(-\tilde{M}\tau) + \exp(-\tilde{M}(Nr - \tau))] \]  

(20)

In many cases, the oscillating channel, and/or the radial excitation proved to be irrelevant. For the sigma particle, the inclusion of a constant (background) term was necessary, as discussed below.

For the pion, the contribution of the oscillating channel is very small in the strong coupling phase. At weak coupling, especially with the wall operator, there is a modest oscillation which can be dealt with by extracting the direct channel contribution in the usual way. We have thus performed the effective masses analysis of the direct channel, computed as

\[ G_{\text{direct}}(2t) = 2G(2t) - G(2t - 1) - G(2t + 1) \]  

(21)

(note that according to [19] this should give us the exact propagator in the direct channel).

Many of the results are displayed in Fig.14–16. We observed some discrepancies, at the smaller values of the quark masses, between the results with the point and the wall source. This occurred even for the apparently asymptotic parts of the propagators. In these cases we preferred the results of two particle fits (M and M1 according to eq. (18)) (the inclusion of an oscillating state \( \tilde{M} \) proved to be irrelevant). The mass of the fundamental particle, the Goldstone boson in the strong coupling region, is in between the results for the effective masses from point and wall sources, as it should be, since these provide lower and upper bounds, respectively. We show a sample of the results of the fits in Fig. 17, whose \( \chi^2 \)'s are in the range of a few percent, very small as they should be for correlated data.

Rather than discussing the systematics further, we prefer to show the full collection of the results in Tables 8, 9, 10. The errors come from jack-knife estimates, performed by discarding one propagator at a time. Wherever they agree, the effective mass result, either from a wall or point source measurement, is more accurate. In case of (mild) disagreement, the fit result is safe. We show also the results of the first excited state, again from the two particle fits, in Table 11.
The behaviour of the $\sigma$ particle is rather peculiar. Its main characteristic is a plateau in the propagators which shows up in the weak coupling region. In the strong coupling region, one particle fits were compelling, and in agreement with effective mass analyses. (in this case the oscillating component is small, so we could use the standard even-even, odd-odd combinations for the extraction of the mass values). In the weak coupling region, a one-particle fit badly fails.

Two particle (M and M1) fits give results for the lower mass in agreement with one-particle fits in the strong coupling region, although noisier, while in the weak coupling region, at small quark masses, the fundamental mass is extremely low, often compatible with zero.

A very reasonable ansatz which works in the whole range of parameters was thus to fit the propagators with a form $A e^{-mt} + B$, with symmetrization (Figs. 18 and 19). $B$ is compatible with zero in the strong coupling region, with masses in agreement with the ones estimated via the one particle fit. On the other hand, in the weak coupling region, and at small masses, $B$ is distinctly different from zero, and the mass is consistent with the one determined via a two particle fit. Not surprisingly, the situation at $\beta = .260$ is the most ambiguous, and it is not clear if there is a background, a real pole mass, or what. For instance, we show in Fig. 20 the results from a fit with background for $t > 3$ (dot), and two particle fit for $t > 2$ (dash) at $\beta = .260$ and $m = .003$. Both $\chi^2$ are a few percent, and the fits are practically indistinguishable. The mass for the fit with background is $47(2)$ and the background itself is $7(1)E-03$. The two particle fit gives a mass of $51$ and a smaller mass $13$ with an amplitude of $6E-02$.

It is unlikely that a low mass particle really exists here since the background, or, equivalently, its amplitude, decreases when the bare quark mass increases. The power law behaviour, discussed in [21], could be appropriate in the immediate vicinity of the critical region, but a detailed test of this hypothesis would require much more statistics, insight etc. Also, recall that the naive symmetrization used here is not compelling. Multiple images can alter the behavior on a torus from that expected on an infinite volume. This is true even for the naive exponential behavior, and probably even more for some slow-decaying function. Apparently, a fixed background is enough to summarize what is going on, thus for the time being, we take our results from fits in the form discussed above. We quote the results, with the extra-caveats discussed above for $\beta = .260$, in Tables 14, 15.
A rather convincing argument in favour of the chosen parametrization comes from the study of effective masses once the background $B$ was subtracted ‘by hand’. $B$ can be estimated from the long distance behaviour of the sigma propagator, once the plateau has set in ($t = 13$ proved to be a reasonable starting point for the computation of the background). We show the results in Fig. 21: the effective masses computed after the subtraction of the background flatten, giving estimates in agreement with the ones obtained with the background fits. We also show what we would have obtained without subtraction, a decreasing plot suggesting an unnaturally low mass.

To learn more about the nature of the background, we measured the propagators on a bigger lattice, for the four smaller masses at $\beta = .270$. We see (Fig. 22) that the background is clearly reduced, and we tentatively conclude that it becomes less and less significant while increasing the lattice length. Our parametrization explains the odd behaviour of the $\sigma$ susceptibility on ‘small’ cylindrical lattices, and it explains why the symmetric lattices are apparently satisfactory (they do not take information from the unsafe long distance behavior, dominated by finite size effects), and suggests that the proper behaviour is recovered in the infinite volume limit. We conclude that we have safe estimates of the mass in the sigma channel which is dominating in the time distance 4–13, in lattice units. As is evident from the plots, the signals are lost at larger distance, and we cannot exclude that lower mass excitations are present in the scalar spectrum. This, as discussed above, would be especially relevant when comparing the sigma mass with twice the mass of the fermion.

The fermion propagators is well described by the usual form
\[ G(t) = a(e^{-mt} - (-1)^t e^{-m(T-t)}) \]
for all euclidean times. We have checked that the results do not change, within rather large statistical errors, while changing the fitting interval. Wherever the comparison is possible, our new results obtained with a point source agree with the previous ones, coming from wall source measurements. The results of the fits (a subset of the plots are shown in Fig. 23) are given in Table 16.

The $\rho$ particle does not couple appreciably to the point source, so we quote here the results obtained with the wall source. Wherever a cross check was possible, the two approaches gave consistent results. We note that the fits
look good (a subset of them are shown in Fig. 24) but their \( \chi^2 \)'s are not very low (they are typically order 1, too high for correlated data). The effective mass analysis, especially at strong coupling, is not quite flat. Anyway, it oscillates around the results of the fit. The new data deep in the symmetric phase confirm that an oscillation with a very low (\( \simeq 0 \)) mass is needed in order to obtain good fits. As discussed for the sigma propagator, this very low mass can be traded for a constant (in this case purely oscillating) term. However, in this case the problem is less severe, since it does not affect our estimate in the fundamental channel.

5 Conclusions

The results presented here include a detailed comparison of the data with an equation of state à la Nambu Jona Lasinio, together with a further study of the powerlaw scaling predicted by analytic computations. Our analysis favours power law scaling with non-mean field exponents in agreement with the ones found in past work. We have performed detailed fits of old data for the two Equations of State, and we have confirmed their results by a new numerical analysis. We have analyzed our new data for the susceptibilities, and contrasted them with the predictions of the two scenarios. From any point of view, the power–law hypothesis provides an accurate description of the data, while the NJL analysis gives poorer results (in the case of direct fits of the order parameter), or fails qualitatively (in the case of the analysis of susceptibility data). We stress that the finite size effects and other sources of systematic errors were kept under control, as discussed in detail in the text. We have also commented on possible corrections to the equation of state, and on their impact on the results. We found these corrections to be small, at least in the case we were considering. This point deserves however further investigation.

We have gained a good understanding of the Goldstone mechanism, and of chiral symmetry breaking in the spectrum. We have found that the candidate Goldstone boson decouples in the weak coupling phase. The behaviour of the amplitudes in the (would be) Goldstone and non-Goldstone mode is clear, and picks up the critical point with good accuracy. This result also explains how the low-lying state in the pseudoscalar channel maintains the property of a Goldstone boson – a puzzling result of previous works. We
have analyzed in detail the behaviour of the propagators in the scalar and vector sector, and found clear indications of chiral symmetry restoration at the transition point. A point left open is the level ordering $\sigma$ – fermion. The fermion mass was measured again, and found in good agreement with previous work. We have also obtained good estimates for the scalar propagator in the intermediate time distance region. However, the long distance region still evades us, and we do not have firm conclusions for the physically relevant low-lying pole of the scalar propagator. We mention that the source used for the inversion of the Dirac operator is crucial in this case: in previous work with a wall source we did not get any signal for the sigma. A computation of the sigma mass probably requires a dedicated source (i.e. with the appropriate quantum numbers).

Finally, we have presented the first computation in QED of the full mesonic spectrum local in time. We have observed indications of flavor/rotational symmetry restoration. However, it is not clear if such behaviour is induced by the chiral transition, or by the perturbative limit.

We would like to thank the National Science Foundation for support under grant NSF-PHY92-00148. The computer simulations reported here were done on the C90-CRAYs at the Pittsburgh Supercomputer Center (PSC) and the National Energy Research Supercomputer Center (NERSC), and on the CM–2 at the PSC. We wish to thank Jean–Francois Lagaë for conversations and Donald K. Sinclair for some of the meson spectroscopy codes.

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Figure captions

1. Results of power–law fits for various $\beta$ intervals. Circles, $0.240 \leq \beta \leq 0.260$, crosses, $0.245 \leq \beta \leq 0.260$, squares, $0.250 \leq \beta \leq 0.260$.

2. Results of the fits à la Nambu–Jona-Lasinio for various $\beta$ intervals. Crosses, $0.245 \leq \beta \leq 0.260$, squares, $0.250 \leq \beta \leq 0.260$.

3. $R_t$ vs $R_m$. The data are from the $24^4$ lattice, the logarithmic derivatives are evaluated numerically. The input value for the critical $\beta$ was 0.2572. The straight line is the prediction of the EOS fit, $R_t = (1 - \delta R_m)/(\delta - 1)$.

4. (a) Susceptibility ratio, $16^4$ lattice, circles. $\beta = 0.250, 0.255, 0.260, 0.265, 0.270, 0.275, 0.280$, from bottom to top. The crosses are for the same ratio, evaluated by taking the logarithmic derivative numerically on the $24^4$ lattice, at $\beta = 0.250, 0.255, 0.260$. The straight line is drawn at $1/\delta$, with $\delta = 2.12$, and shows that $0.255 < \beta_c < 0.260$ in agreement with the results of the EOS fit. (b) Zoomed view around the critical region. Also, the $24^4$ lattice data are shown in steps of $\Delta \beta = 0.001$, beginning from $\beta = 0.260$ from top to bottom.

5. $\chi_\pi/\chi_\sigma - 3$ vs $1/\log < \bar{\psi}\psi >$. Data in the $\beta$ interval $0.250 - 0.260$, $\Delta \beta(0.005, 0.001)$, (circles, crosses), ($16^4, 24^4$). The solid line is the NJL prediction at criticality, the dashed one is the power law prediction, consistent with the data at $\beta = 0.257$.

6. Fermion mass as a function of $< \bar{\psi}\psi >$. The data for the chiral condensate are from $24^4$ lattice, the data for the fermion mass are taken on a $16^3 \times 32$ lattice, ref. [5]. The solid line is the free fermion prediction, the dashed one the strong coupling limit. $m_e = (0.002, 0.003, 0.004, 0.005)$, (diamonds, crosses, squares, circles).

7. Amplitude of the Goldstone mode (solid), and sum of the amplitudes of the excited states (dash) for $m_e = 0.003$ (top) and $m_e = 0.009$ (bottom), as a function of $\beta$. The Goldstone mode becomes dominant around the critical $\beta$, the effect being more pronounced at small mass.

8. Ratio of the sigma to the pion propagator at zero distance. Squares, circles, crosses, fancy crosses are for $m_e = 0.003, 0.009, 0.015, 0.021$. The
short distance behaviour shows clear indications of chiral symmetry breaking/restoration at small masses.

9. Ratio of the A1 to the $\rho$ propagator at zero distance. Squares, circles, crosses, fancy crosses are for $m_e = .003, .009, .015, .021$. Also in this case, the short distance behaviour shows clear indications of chiral symmetry breaking/restoration at small masses.

10. Effective mass plot for the pseudoscalar mesons at $\beta = .260, m_e = .003$. The Goldstone is the lowest, all the others are degenerate.

11. Mass of the Goldstone (diamonds) and of the $\gamma_5 \xi_5 \xi_5$ (crosses) mesons as a function of $\beta$. $m_e = .003$, upper portion of the figure, $m_e = .009$ lower.

12. Finite size effects on the chiral condensate in the strong coupling region at $\beta = ( .250, .255 )$ (solid, dashed). Squares are for the $24^4$ lattice, diamonds for the $16^4$, crosses for $8^4$.

13. Pion (dash) and sigma (solid) susceptibilities at $\beta = .280$. Crosses, diamonds, squares are for the $8^4, 16^4, 16^3 \times 32$ lattices.

14. Effective mass for the pion at quark masses .003 (crosses) and .009 (circles) at $\beta = .250$, point and wall source.

15. Effective mass plot for the pion at $\beta = .250, m_e = .019$ (crosses). The results from an high-statistics run on a $16^2 \times 64$ lattice are shown for comparison (circles).

16. Effective mass for the pion at quark masses .003 (crosses) and .009 (circles) at $\beta = .280$, point and wall source.

17. Two particle fits for the pseudoscalar propagator (point source) at $\beta = .260$, masses .003, .009, .051, .021, $t > 2$. The $\chi^2$ ranges from 2/1000 to 6/100.

18. Scalar propagator (noisy source) fits to $A e^{-m_\sigma t} + B$, $m_e = .003$, $t > 3$. Crosses, diamonds, squares are for $\beta = .250, .260, .280$. The $\chi^2$'s are 5,7,9 /100.
19. Scalar propagator (noisy source) fits to $Ae^{-m\sigma t} + B$, $m_e = .007$, $t > 3$. Crosses, diamonds, squares are for $\beta = .250, .270, .280$. The $\chi^2$'s are 3,40,30/100.

20. Scalar propagator (noisy source) fits at $\beta = .260$ and $m_e = .003$. The results of a fit with background for $t > 3$ (dot) and a two particle fit for $t > 2$ (dash) are practically coincident.

21. Effective mass (even-even, odd-odd) for the sigma particle. From top to bottom, $(\beta = .250, m_e = .007), (\beta = .260, m_e = .003), (\beta = .280, m_e = .007)$. The squares are the naive results, the crosses with the subtracted background ($t > 13$).

22. Scalar propagator at $\beta = .270, m_e = .003$ on the $16^3 \times 32$ and $16^3 \times 64$ lattices. Note the agreement at short distance, and the clear size-dependent plateau showing up at large time separations.

23. Electron propagator fits in the interval $0 < t < 17$, $m_e = .009$, $\beta = .250, .255, .270, .280$ from bottom to top.

24. Two particle fits (direct + oscillating) for $t > 4$ to the vector propagator at $m_e = .003$, $\beta$ ranges from .250 (bottom) to .280 (top).
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