THE GOLOD PROPERTY FOR STANLEY-REISNER RINGS IN VARYING CHARACTERISTIC

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Abstract. We give examples of simplicial complexes $\Delta$, such that Golod property of the Stanley-Reisner ring $K[\Delta]$ depends on the characteristic of the field $K$. More precisely, for every finite set $T$ of prime numbers we construct simplicial complexes $\Delta$ and $\Gamma$, such that $K[\Delta]$ is Golod exactly in the characteristics in $T$ and $K[\Gamma]$ is Golod exactly in the characteristics not in $T$.

Along the way, we also show that a one-dimensional simplicial complex is Golod if and only if it is chordal.

1. Introduction

Let $(A, m, K)$ be a standard graded $K$-algebra with maximal ideal $m$. The Poincaré-series of $A$ is the formal power series

$$P_A(t, x) = \sum_{i,j \geq 0} (\dim_K \text{Tor}_i^R(K, K)_j) t^i x^j.$$

The algebra $A$ is called Golod if the following holds:

$$P_A(t, x) = \prod_{i=1}^{n} (1 - tx)^{\frac{1}{1 - \sum_{i\geq 1} \sum_{j\geq 0} \dim_K(\text{Tor}_i^S(A, K)_j) t^{i+1} x^j}},$$

where $S$ is a polynomial ring such that $A = S/I$ for some homogeneous ideal $I \subseteq S$. In general, $P_A(t, x)$ is componentwise bounded above by the right-hand side of (1).

Golod algebras are surprisingly common. For example, it has been proven by Herzog and Huneke [HH13] that if $I \subseteq S$ is a homogeneous ideal, then $S/I^k$ is Golod for every $k > 1$. Further, Seyed Fakhari and Welker showed in [SFW14] that if $I, J \subseteq S$ are two monomial ideals, then $S/IJ$ is Golod. In fact, it has even been conjectured that the same holds without the monomial assumption.

In [Jö10] and [BJ07] Berglund and Jöllenbeck considered the Golod property for Stanley-Reisner rings. They give a combinatorial characterization of Golodness in the class of flag simplicial complexes, which in particular implies that the Golod property of these complexes does not depend on the field $K$ of coefficients. It seems natural to whether this also holds for general simplicial complexes.

The general expectation seems to be that this is not the case, i.e. for sufficiently complicated complexes the Golod property might depend $K$. However, no example of this phenomenon was known. In the present note, we provide a construction for such examples. More precisely, we prove the following:

Theorem (Theorem 3.4 and Theorem 3.5). Let $T$ be a finite set of prime numbers.

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(1) There exists a simplicial complex $\Delta$ such that $K[\Delta]$ is Golod if and only if $\text{char } K \in T$.

(2) Also, there exists a simplicial complex $\Delta$ such that $K[\Delta]$ is Golod if and only if $\text{char } K \not\in T$.

We remark that it is well-known that for the Cohen-Macaulay property only the second case appears.

Let us explain the mechanisms which cause these two cases. By a result of Iriye and Kishimoto [IK14], the Golod property can be characterized by the vanishing of certain maps between (co-)homology groups, see Proposition 2.1. On the one hand, it might happen that these homology groups are torsion groups and thus vanish in all but finitely many characteristics. On the other hand, a map between the free parts of the homology groups might be the multiplication by some number $N$. In this case, the map vanishes exactly for the finitely many divisors of $N$.

Given our main result, one might be tempted to ask if the finiteness assumption is necessary. In other words, one could ask if there exists a simplicial complex which is Golod in infinitely many characteristics, and non-Golod in infinitely many other characteristics. Our second result gives a negative answer to this question.

**Proposition** (Proposition 3.6). For a simplicial complex $\Delta$, the following holds:

(1) If $K$ and $K'$ are two fields with the same characteristic, then $K[\Delta]$ is Golod if and only if $K'[\Delta]$ is Golod.

(2) If $\mathbb{Q}[\Delta]$ is Golod, then $\mathbb{F}_p[\Delta]$ is Golod for all but at most finitely many primes $p$.

(3) If $\mathbb{Q}[\Delta]$ is not Golod, then $\mathbb{F}_p[\Delta]$ is Golod for at most finitely many primes $p$.

Here, $\mathbb{F}_p$ denotes the field with $p$ Elements.

Thus, the complexes constructed in Theorem 3.4 and Theorem 3.5 are “worst possible”.

This paper is organized as follows. In Section 2 we collect some background information on Golod rings. In the third section, we prove Theorem 3.4 and Theorem 3.5, and also Proposition 3.6. Also, we prove our characterization of one-dimensional Golod complexes. In the fourth section we provide two explicit examples of Stanley-Reisner rings whose Golodness depends on the field. Finally, in the last section shortly discuss a relation to decomposition $k$-chordal complexes [ANS15], and pose a question for an improved criterion for Golodness.

2. Preliminaries about Golod algebras

In this section we recall some facts about Golod algebras. We refer the reader to [Avr98] for a comprehensive treatment.

Let $S = K[X_1, \ldots, X_n]$ be a polynomial ring and let $A = S/I$ for some homogeneous ideal $I$. Let further $K_A$ denote the Koszul complex of $A$ with respect to the sequence $X_1, \ldots, X_n$. By definition, $A$ is called Golod if the equality (1) of power series holds. Golod [Gol62] showed that this is equivalent to the condition that the homology $H_*(K_A)$ of the Koszul complex of $A$ admits trivial Massey products. In the case that $I$ is a monomial ideal, Berghund and Jöllenbeck [BJ07]
Thm. 5.1] extended this by showing that it is in fact sufficient that the product on $H_*(K_A) \cong \text{Tor}_*^S(A, K)$ is trivial.

From now on we assume that $I$ is a squarefree monomial ideal, i.e., the Stanley-Reisner ideal of some simplicial complex $\Delta$. Hochster [Hoc77] gave a topological interpretation of the homogeneous strands in the Koszul complex $K_A$. Building on this, Iriye and Kishimoto [IK14] obtained a topological interpretation of the product on $H_*(K_A)$ which leads to the following characterization of Golod Stanley-Reisner rings:

**Proposition 2.1** (Prop. 6.3, [IK14]). Let $\Delta$ be a simplicial complex with vertex set $V$. Then the Stanley-Reisner ring $K[\Delta]$ is Golod if and only if the inclusion $\Delta_{I,J} \hookrightarrow \Delta_I \ast \Delta_J$ induces the zero map in homology with coefficients in $K$ for all $\emptyset \neq I, J \subset V$ with $I \cap J = \emptyset$.

Here $\Delta_I := \{ F \in \Delta : F \subset I \}$ denotes the restriction of $\Delta$ to $I$, and $\Delta \ast \Gamma := \{ F \cup G : F \in \Delta, G \in \Gamma \}$ denotes the join of two simplicial complexes $\Delta$ and $\Gamma$ on disjoint vertex sets.

**Notation.** Let $i \in \mathbb{N}$ and let $I, J$ be two non-empty disjoint subsets of the vertex set of some simplicial complex $\Delta$. We write $\varphi_{I,J}^i : H_i(\Delta_{I,J}; K) \to H_i(\Delta_I \ast \Delta_J; K)$ for the map induced by the inclusion $\Delta_{I,J} \hookrightarrow \Delta_I \ast \Delta_J$.

The following is just a reformulation of Proposition 2.1:

**Corollary 2.2.** Let $\Delta$ be a simplicial complex with vertex set $V$. Then $K[\Delta]$ is Golod if and only if $\varphi_{I,J}^i = 0$ for all $i \in \mathbb{N}$ and all nonempty disjoint $I, J \subset V$.

### 3. Proof of the main results

#### 3.1. The 1-Skeleton

In this section, let $\Delta$ be a fixed simplicial complex with vertex set $V$. We write $C_i(\Delta; K)$ for the vector space of $i$-chains of $\Delta$. An $i$-cycle $\omega \in C_i(\Delta; K)$ is called complete [ANS15] if it is the boundary of a $(i+1)$-simplex (which does not need to be a face of $\Delta$).

**Lemma 3.1.** Every complete $i$-cycle $c \in C_i(\Delta_{I,J}; K)$ becomes a boundary in $C_{i}(\Delta_I \ast \Delta_J; K)$.

**Proof.** First, assume that some vertices of $c$ are in $I$ and some are in $J$, i.e., there are non-empty faces $\sigma \in \Delta_I, \tau \in \Delta_J$ such that the support of $c$ equals $\sigma \cup \tau$. It holds that $c = \partial(\sigma \cup \tau)$ because $c$ is complete. But $\sigma \cup \tau \in \Delta_I \ast \Delta_J$, so $c$ is a boundary.

Next, consider the case that all vertices of $c$ are in one of the sets, say $I$. Then $c$ is the boundary of the cone $c \ast \{ w \}$ for any vertex $w \in J$.

It was proven in [BJ07, Proposition 6.4] that the 1-skeleton of a Golod simplicial complex is chordal. The following is a more precise version of this result.

**Proposition 3.2.** For a simplicial complex $\Delta$ on a vertex set $V$, the following are equivalent:

1. The 1-skeleton of $\Delta$ is a chordal graph.
2. $\varphi_{I,J}^1 = 0$ for all non-empty disjoint subsets $I, J \subset V$. 
Lemma 3.1]

2) \( \Rightarrow \) 1) Assume that the 1-skeleton of \( \Delta \) contains a chordless cycle \( C \) of length at least 4. Let \( v \) be any vertex of \( C \) and let \( w_1, w_2 \) denote its two neighbors. Set \( I := \{w_1, w_2\} \) and let \( J \) be the set of all other vertices of \( C \).

Let \( \tilde{c} \in \tilde{H}_1(\Delta_{I,J}; \mathbb{K}) \) denote the homology class corresponding to \( C \). As \( C \) is chordless, the edge \( w_1w_2 \) is not contained in \( \Delta \). Hence the edge \( vw_1 \) is a maximal face of \( \Delta_I \ast \Delta_J \), and thus \( \tilde{c} \) cannot be a boundary in this complex. We conclude that \( \varphi_{I,J}^1(\tilde{c}) \neq 0 \).

\[
\square
\]

In \cite[Thm 6.7]{BJ07} it was shown that a flag simplicial complex is Golod if and only if its one-skeleton is chordal. A one-dimensional simplicial complex does not need to be flag, so the following can be seen as a partial extension of that result.

**Corollary 3.3.** Let \( \Delta \) be a one-dimensional simplicial complex. Then \( \mathbb{K}[\Delta] \) is Golod if and only if \( \Delta \) is chordal.

### 3.2. Construction of the examples.

**Theorem 3.4.** Let \( T \subset \mathbb{N} \) be a finite set of prime numbers. Then there exists a simplicial complex \( \Delta \) such that \( \mathbb{K}[\Delta] \) is Golod if and only if \( \text{char} \mathbb{K} \in T \).

![Figure 1. The complex \( \tilde{\Delta} \) for the proof of Theorem 3.4.](image)

**Proof.** Let \( N \) be the product of the elements of \( T \). Let \( e_1^2, e_2^2 \) be two 2-cells. We glue the boundary of each cell onto a circle \( C \approx S^1 \) such that it winds around it \( N \) times, and call the resulting space \( X \).

Let us compute \( \tilde{H}_2(X; \mathbb{K}) \). The boundary of both \( e_1^2 \) and \( e_2^2 \) is \( N \) times \( C \). So \( e_1^2 \) and \( e_2^2 \) are both cycles if \( \text{char} \mathbb{K} \) divides \( N \). Further, their difference \( \sigma := e_1^2 - e_2^2 \) is a cycle independently of the field.

We will choose a triangulation \( \Delta \) of \( X \) with the following properties:

1. There are two vertices \( a, b \) in the interior of \( e_1^2 \) and \( e_2^2 \), respectively, and
2. the restriction \( \Gamma \) of \( \Delta \) to the other vertices retracts to the circle \( C \).

Let \( I := \{a, b\} \) and let \( J \) be the set of the remaining vertices of \( \Delta \). Then \( \Delta_{I,J} = \Delta \) and \( \Delta_I \ast \Delta_J \) is (homotopic to) a suspension of \( C \), i.e., a 2-sphere.

Consider the case that \( \text{char} \mathbb{K} \) divides \( N \) and let \( \omega \in \tilde{H}_2(X; \mathbb{K}) \) be one of the single cells \( e_1^2 \) and \( e_2^2 \). Its image in \( \Delta_I \ast \Delta_J \) is supported on the contractible subcomplex \( \{a\} \ast \Delta_J \) (respectively \( \{b\} \ast \Delta_J \)), so \( \varphi_{I,J}^2 \) sends it to zero. Thus, if \( \text{char} \mathbb{K} \) divides \( N \) then \( \varphi_{I,J}^2 \) is the zero map for the given choice of \( I \) and \( J \).

Now assume that \( \text{char} \mathbb{K} \) does not divide \( N \). In this case \( \tilde{H}_2(X; \mathbb{K}) \) is generated by \( \sigma := e_1^2 - e_2^2 \). A generator \( \tau \) for \( \tilde{H}_2(\Delta_I \ast \Delta_J; \mathbb{K}) \) is given by the suspension of \( C \), i.e. a signed sum of all triangles in \( \Delta_I \ast \Delta_J \) containing one edge of \( C \) and either
a or b. As σ winds around C for N times, we see that \( \varphi_{I,J}^{1}\sigma = \pm N \cdot \tau \neq 0 \). In conclusion, \( \varphi_{I,J}^{1} \) is zero if and only if \( \text{char} \mathbb{K} \) divides \( N \).

To finish the proof, we have to find a triangulation \( \Delta \) satisfying the conditions. Further, we have to ensure that \( \varphi_{I,J}^{1} = 0 \) and that \( \varphi_{I,J}^{i} = 0 \) independently of the field for any two disjoint vertex sets \( I', J' \) with \( \{I', J'\} \neq \{I, J\} \) and \( i = 1, 2 \).

Let \( \tilde{\Delta} \) be the simplicial complex indicated in Fig. 1 where vertices with the same label are to be identified. Construct a complex \( \Delta' \) from \( \tilde{\Delta} \) by adding two cones with apexes \( a \) and \( b \) along the dashed line. Now \( \Delta \) is constructed from \( \Delta' \) by adding edges between any two vertices except \( a \) and \( b \). We denote the vertex set of \( \Delta \) by \( V \).

The 1-skeleton of \( \Delta \) is complete except for the missing edge from \( a \) to \( b \), so it is clearly chordal. Hence \( \varphi_{1}^{I,J} \) is the zero map for any \( I \) and \( J \). Further, the Künneth formula implies that for \( H_{2}(\Delta_{I'} \ast \Delta_{J'}; \mathbb{K}) \) being nontrivial, it is necessary that either \( \tilde{\Delta} \) or \( \mathbb{K} \) is not a field for any two disjoint vertex sets \( I, J \) such that \( \{I, J\} \neq \emptyset \) and \( \Delta \) is clearly chordal. Hence \( \tilde{\Delta} \) is homotopy equivalent to the space \( H_{2}(\Delta; \mathbb{K}) = \tilde{H}_{2}(\Delta'; \mathbb{K}) = \tilde{H}_{2}(X; \mathbb{K}) \). Here, the first equality uses that attaching edges does not affect \( \pi_{2} \).

Further, our computation of \( \varphi_{2}^{I,J} \) from above stays valid, as the additional edges in \( \Delta_{J} \) only yield additional 2-cells in \( \Delta_{I} \ast \Delta_{J} \) and this does not affect whether some 2-cycle is a boundary or not. \( \square \)

**Theorem 3.5.** Let \( T \subset \mathbb{N} \) be a finite set of prime numbers. Then there exists a simplicial complex \( \Delta \) such that \( \mathbb{K} \langle \Delta \rangle \) is Golod if and only if \( \text{char} \mathbb{K} \notin T \).

\[ \begin{align*}
\varphi_{1}^{I,J} & = 0 \\
\varphi_{2}^{I,J} & = 0
\end{align*} \]

**Figure 2.** The complex \( \tilde{\Delta}_{2} \) for the proof of Theorem 3.5.

**Proof.** As above, we start with a topological consideration. Let \( N \) be the product of the numbers in \( T \). Consider the space \( X \) which is obtained by gluing the boundary of a 2-cell \( e^{2} N \) times around a circle \( C \approx S^{1} \). Clearly,

\[ \tilde{H}_{2}(X; \mathbb{K}) = \begin{cases} 
\mathbb{K} & \text{if } \text{char} \mathbb{K} \in T \\
0 & \text{otherwise}
\end{cases} \]
where in the former case a generator of the homology is given by $e^2$ itself. We will choose a triangulation $\Delta$ of $X$ such that the following holds:

1. There is a vertex $a$ in the interior of $e^2$ and
2. the restriction of $\Delta$ to the neighbors of $a$ is a one-dimensional complex.

Here, a neighbor of $a$ is any vertex sharing an edge with $a$. Let $V$ denote the vertex set of $\Delta$, let $J \subset V$ be the set of neighbors of $a$ and let $I := V \setminus J$. We claim that the image of $e^2$ in $\Delta_I \ast \Delta_J$ is not a boundary. Indeed, as $\Delta_J$ is one-dimensional, there is no 3-simplex containing $a$ in $\Delta_I \ast \Delta_J$. Thus the triangles containing $a$ cannot be obtained as boundaries. Thus, the map $\varphi^{I,J}_2$ is nonzero if and only if char $K \notin T$.

Again as above, we finish the proof by picking an actual simplicial complex and showing that all other maps $\varphi^{I',J'}_2$ vanish if char $K / \notin T$. Consider the complex $\tilde{\Delta}_2$ in Fig. 2. Let $\Delta'$ be the complex obtained by gluing a cone with apex $a$ onto $\tilde{\Delta}_2$ along the dashed line. Clearly, $\Delta'$ satisfies our assumptions. We construct $\Delta$ from $\Delta'$ by adding all edges which do not contain $a$. This ensures that the 1-skeleton of $\Delta$ is chordal without affecting $\tilde{H}_2(\Delta; K)$. So by the argument above, $\varphi^{I,J}_2$ is nonzero if and only if char $K \in T$. If $I', J'$ are other disjoint vertex sets with $I' \cup J' = V$, then clearly $\varphi^{I',J'}_2 = 0$ for char $K \notin T$, because in this case $\tilde{H}_2(\Delta; K) = 0$.

Finally, consider two disjoint vertex sets $I', J'$ with $I' \cup J' \subset V$. In this case, one shows similarly to the argument in the proof of Theorem 3.4 that $\tilde{H}_2(\Delta_{I' \cup J'}; K) = 0$, so the corresponding map vanishes.

3.3. A general finiteness result. Given the two constructions in the previous section, one might ask if the finiteness of the set $T$ is really necessary. In other words, could there be a simplicial complex $\Delta$ such that $K[\Delta]$ is Golod in infinitely many characteristics and non-Golod in infinitely many other characteristics? Indeed, such a phenomenon is excluded by the following result. For completeness, we also show that the Golod property only depends on the characteristic.

Proposition 3.6. For a simplicial complex $\Delta$, the following holds:

1. The Golod property of $K[\Delta]$ depends only on the characteristic of $K$. More precisely, if $K$ and $K'$ are two fields with the same characteristic, then $K[\Delta]$ is Golod if and only if $K'[\Delta]$ is Golod.
2. If $Q[\Delta]$ is Golod, then $F_p[\Delta]$ is Golod for all but at most finitely many primes $p$.
3. If $Q[\Delta]$ is not Golod, then $F_p[\Delta]$ is Golod for at most finitely many primes $p$.

Here $F_p$ denotes the field with $p$ Elements.

Proof. The claim follows from Lemma 3.7 below, applied to the maps $\varphi^{I,J}_i$ for all $i$ and all non-empty disjoint subsets $I, J$ of the vertex set of $\Delta$.

We doubt that this Lemma is a new result, but as we could not locate it in the literature we provide a proof.

Lemma 3.7. Let $(C, \partial), (C', \partial')$ be two bounded complexes of finitely generated free abelian groups. Let further $\varphi : C \to C'$ be a map of complexes, i.e., a $\mathbb{Z}$-linear map such that $\partial' \circ \varphi = \varphi \circ \partial$. Then the following holds:
(1) If \(H(\varphi \otimes \mathbb{Q}) : H(C \otimes \mathbb{Q}) \to H(C' \otimes \mathbb{Q})\) is the zero map, then \(H(\varphi \otimes \mathbb{F}_p) : H(C \otimes \mathbb{F}_p) \to H(C' \otimes \mathbb{F}_p)\) is also the zero map for all but finitely many primes \(p\).

(2) On the other hand, if \(H(\varphi \otimes \mathbb{Q}) : H(C \otimes \mathbb{Q}) \to H(C' \otimes \mathbb{Q})\) is not the zero map, then \(H(\varphi \otimes \mathbb{F}_p) : H(C \otimes \mathbb{F}_p) \to H(C' \otimes \mathbb{F}_p)\) is also nonzero for all but finitely many primes \(p\).

(3) If \(K\) and \(K'\) are two fields with the same characteristic, then \(H(\varphi \otimes K) = 0\) if and only if \(H(\varphi \otimes K') = 0\).

Proof. (1) If \(H(\varphi \otimes \mathbb{Q}) = 0\), then the image of every cycle \(c \in \ker \partial\) is a boundary over \(\mathbb{Q}\). By the Universal Coefficient Theorem, there are only finitely many primes \(p\) such that \(C \otimes \mathbb{F}_p\) contains a cycle \(c'\) which is not a boundary and which is not defined over \(\mathbb{Z}\). Clearly, \(\varphi\) induces the zero map in all other characteristics.

(2) If \(H(\varphi \otimes \mathbb{Q})\) is not the zero map, then there exists a cycle \(c \in \ker(\partial \otimes \mathbb{Q})\) such that \(c' := \varphi(c)\) is not a boundary. By clearing denominators we may assume that \(c\) is defined over \(\mathbb{Z}\). Moreover, \(c\) is clearly a cycle over any field. Now \(c'\) is a boundary if and only if the system of linear equations \(\partial' y = c'\) is not solvable over \(\mathbb{Q}\), equivalently, if and only if the rank of the extended matrix \((\partial', c')\) is strictly greater than the rank of \(\partial'\). The rank of a matrix with integer entries is the same in all but finitely many characteristics, as it can be expressed by the non-vanishing of certain minors. Hence \(c'\) is not a boundary in all but finitely many characteristics.

(3) We may assume that \(K\) is the prime field of \(K'\). Then \(H(\varphi \otimes K') = H(\varphi \otimes K) \otimes_K K'\), so one map vanishes if and if only the other one vanishes.

4. Two explicit examples

We give two explicit examples of simplicial complexes which show the phenomena observed in the last section.

Example 4.1. Let \(\Delta\) be the simplicial complex with the facets

\[
\begin{array}{cccccccc}
124 & 235 & 341 & 452 & 513 \\
12a & 23a & 34a & 45a & 51a \\
12b & 23b & 34b & 45b & 51b
\end{array}
\]

on the vertex set \(V = \{1, 2, 3, 4, 5, a, b\}\). The generators of the corresponding Stanley-Reisner ideal are

\[
\begin{align*}
& x_1 x_2 x_3 \quad x_2 x_3 x_4 \quad x_3 x_4 x_5 \quad x_4 x_5 x_1 \quad x_5 x_1 x_2 \\
& x_1 x_3 x_a \quad x_1 x_4 x_a \quad x_2 x_4 x_a \quad x_2 x_5 x_a \quad x_5 x_3 x_a \\
& x_1 x_3 x_b \quad x_1 x_4 x_b \quad x_2 x_4 x_b \quad x_2 x_5 x_b \quad x_5 x_3 x_b \\
& x_a x_b
\end{align*}
\]

Geometrically, \(\Delta\) is a Möbius strip with two 2-balls glued along its boundary. By the same argument as in the proof of Theorem 3.4, one shows that \(K[\Delta]\) is Golod if and only if \(\text{char } K = 2\), where the "critical" sets are \(I = \{a, b\}\) and \(J = \{1, \ldots, 5\}\). Note that the 1-skeleton of \(\Delta\) is already chordal, so we do not need to add additional edges as in the proof of Theorem 3.4.

Example 4.2. Let \(\Delta\) be the simplicial complex with the facets
on the vertex set \( V = \{1, 2, 3, 4, 5, a, b\} \). The generators of the corresponding Stanley-Reisner ideal are

\[
x_1x_2x_3 \quad x_2x_3x_4 \quad x_3x_4x_5 \quad x_4x_5x_1 \quad x_5x_1x_2 \\
x_1x_3x_a \quad x_1x_4x_a \quad x_2x_4x_a \quad x_2x_5x_a \quad x_5x_3x_a \\
x_5x_1x_a \quad x_2x_b \quad x_3x_b \quad x_4x_b
\]

This is a triangulation of the real projective plane, which is obtained from the usual 6-vertex triangulation by subdividing the 2-cell 51a. It can be shown as in the proof of Theorem 3.5 that \( \mathbb{K}[\Delta] \) is Golod if and only if \( \text{char } \mathbb{K} \neq 2 \). Here, the ”critical“ sets are \( I = \{2, 3, 4, b\} \) and \( J = \{5, 1, a\} \).

5. Concluding remarks

5.1. The \( i \)-skeleton and higher chordality. In this section, we give two consequences of Lemma 3.1 which we consider to be of independent interest. The first one is the following corollary.

Corollary 5.1. The map \( \varphi_{i,J}^{I,J} \) depends only on the \( i \)-skeleton of \( \Delta \).

Proof. It is clear that the map depends only on the \((i+1)\)-skeleton of \( \Delta \). Adding \((i+1)\)-dimensional simplices to \( \Delta \) only turns complete \( i \)-cycles into boundaries. By Lemma 3.1, all complete \( i \)-cycles lie in the kernel of \( \varphi_{i,J}^{I,J} \), so this does not affect the map. \( \square \)

Remark 5.2. Based on the preceding corollary, one might be tempted to conjecture that the map \( \varphi_{i,J}^{I,J} \) depends only in the pure \( i \)-skeleton of \( \Delta \). But this is false by the following counterexample.

Consider the join of an empty triangle with an \( S^0 \). This complex is not Golod, as it is a join (or Gorenstein*). But the complex \( \Delta \) obtained by adding an edge between the two ”poles“ is Golod: If the two poles are \( I \) and the triangle is \( J \), then \( \Delta_I \) is contractible, so \( \Delta_I \ast \Delta_J \) is as contractible as well. All proper restrictions of \( \Delta \) have no second homology, and the 1-skeleton is chordal. Hence \( \Delta \) is Golod.

Adiprasito, Nevo and Samper define in [ANS15] several high-dimensional extensions of the notion of a chordal graph. In particular, they define a simplicial complex to be decomposition \( k \)-chordal, if every \( k \)-cycle \( z \) can be written as a sum of complete \( k \)-cycles \( (z_i) \), such the vertices of each \( (z_i) \) are also vertices of \( z \). The sufficiency of Proposition 3.2 extends to this setting:

Proposition 5.3. If \( \Delta \) is a decomposition \( k \)-chordal simplicial complex, then \( \varphi_k^{I,J} = 0 \) for all non-empty disjoint subsets \( I, J \subset V \). In particular, if \( \Delta \) is a decomposition \( k \)-chordal for all \( k \), then \( \mathbb{K}[\Delta] \) is Golod.

Proof. This is immediate from Lemma 3.1. \( \square \)

5.2. Degree bounds. Let \( \Delta \) be a simplicial complex with vertex set \( V \). By Corollary 2.2, \( \mathbb{K}[\Delta] \) is Golod if the maps \( \varphi_{i,J}^{I,J} \) vanish for all \( i \in \mathbb{N} \). It is clear that one only has to consider \( i \leq \text{dim } \Delta \). Moreover, it is in fact sufficient to consider
\[ i \leq \text{reg}\ K[\Delta] - 1, \] where \( \text{reg}\ K[\Delta] \) denotes the Castelnuovo-Mumford regularity. This is immediate from Hochster’s formula, which implies that
\[
\text{reg}\ K[\Delta] = \max\{ j : H_{j-1}(\Delta_I; K) \neq 0 \text{ for some } I \subseteq V \}.
\]
It follows from [BJ07, Theorem 6.5] that if \( \Delta \) is flag, then the vanishing of \( \varphi_{I,J}^{i} \) (for all \( I, J \)) is sufficient for \( K[\Delta] \) to be Golod. An optimistic generalization of this would be the following assertion: If \( \Delta \) has no minimal non-faces of dimension \( \geq k \) and \( \varphi_{I,J}^{i} = 0 \) for all \( i \leq k \), then \( K[\Delta] \) is Golod. However, this can easily seen to be false. An easy example is the join of two boundaries of \((k-1)\)-simplices. Instead we ask we following question, which is motivated by an analogous result in [ANS15]:

**Question 5.4.** Let \( \Delta \) be a simplicial complex with vertex set \( V \). Assume that \( \Delta \) has no minimal non-faces of dimension \( \geq k \) and assume further that \( \varphi_{I,J}^{i} = 0 \) for all \( i \leq 2k - 1 \) and all non-empty disjoint subsets \( I, J \subseteq V \). Is \( K[\Delta] \) Golod?

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