ON EXPLICIT PROBABILITY DENSITIES
ASSOCIATED WITH FUSS-CATALAN NUMBERS

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Abstract. In this paper we give explicitly a family of probability densities, the moments of which are Fuss-Catalan numbers. The densities appear naturally in random matrices, free probability and other contexts.

In this paper we study a family of probability densities \( \pi_s \), \( s \in \mathbb{N} \), which are uniquely determined by the moment sequences \( \{m_0, m_1, \ldots, m_k, \ldots\} \) [1]. Here

\[
m_k = \frac{1}{sk + 1} \binom{sk + k}{k}
\]

are known as Fuss-Catalan numbers in free probability theory [7]. The densities \( \pi_s \) belong to the class of free Bessel laws [3] and are known to appear in several different contexts, for instance, random matrices [1, 3, 7], random quantum states [4], free probability and quantum groups [3, 6]. More precisely, \( \pi_s \) is the limit spectral distribution of random matrices in the forms such as \( X_s^1 X_s^* \) and \( X_1 \cdots X_s X_s^* \cdots X_1^* \), where \( X_1, \ldots, X_s \) are independent \( N \times N \) random matrices (the matrix elements of different matrices are independent); in free probability we have the free convolution relation: \( \pi_s = \pi_1 \boxplus \pi_s^1 \) [7, 3].

More generally, T. Banica et al. [3] introduce a remarkable family of probability distributions \( \pi_{s,t} \) with \( (s, t) \in (0, \infty) \times (0, \infty) - (0, 1) \times (1, \infty) \), called free Bessel laws. \( \pi_s \) is the special case where \( t = 1 \), i.e., \( \pi_{s,1} = \pi_s \). The moments of \( \pi_{s,t} \) are called the Fuss-Catalan polynomials (Theorem 5.2, [3]):

\[
m_k(t) = \sum_{j=1}^{k} \frac{1}{j} \binom{k - 1}{j - 1} \binom{sk}{j - 1} t^j.
\]

Indeed, the following relation holds [3]:

\[
\pi_{s,1} = \pi_s^{\boxplus s}, \quad \pi_{1,t} = \pi_1^{\boxplus t}.
\]

The distribution \( \pi_{1,t} \), the famous Marchenko-Pastur law of parameter \( t \), also called free Poisson law, has an explicit formula:

\[
\pi_{1,t} = \max(1 - t, 0) \delta_0 + \frac{\sqrt{4t - (x - 1 - t)^2}}{2\pi x}.
\]
In the special case,
\[(0.5) \quad \pi_1 = \pi_{1,1} = \frac{1}{2\pi} \sqrt{4x^{-1} - 1}.
\]

Another special case of \(\pi_{s,t}\) where an explicit density formula is available is due to Penson and Solomon [8]:
\[(0.6) \quad \pi_2 = \pi_{2,1} = \frac{\sqrt{2}\sqrt{3} \sqrt{2}(27 + 3\sqrt{81 - 12s})^{2/3} - 6 \sqrt{x}}{12\pi x^{2/3}(27 + 3\sqrt{81 - 12s})^{1/3}} 1_{(0,27/4)}(x).
\]

To the best of our knowledge, except for the special cases above there are no explicit formulae available for the other \(\pi_{s,t}\). The aim of this work is to give explicit densities of \(\pi_s = \pi_{s,1}, s \in \mathbb{N}\). The proof of the following result is based on the method to find an explicit density from a given moment sequence used in [9, 5].

**Theorem 0.1.** Let \(\pi_s, s \in \mathbb{N}\), be the unique densities determined by the Fuss-Catalan numbers in Eq. (0.1). Then we have the following formulae:

\[(0.7) \quad \pi_s(x) = \frac{1_{[0,K]}(x)}{B\left(\frac{s}{2}, \frac{1}{s} + \frac{1}{s}\right)} \int_{[0,1]^s} \frac{\tau K - x}{\sqrt{x}} \left(\tau K\right)^{1/s} F(t_1, \ldots, t_s) 1_{\{\tau K \geq x\}} d^s t,
\]

where \(K = (s + 1)^{s+1}/s^s\), \(\tau = \sum_{j=1}^s t_j\) and \(F(t_1) = \delta(t_1 - 1),\) while for \(s > 1\)

\[(0.8) \quad F(t_1, \ldots, t_s) = \frac{1}{B\left(\frac{s}{2}, \frac{1}{s} + \frac{1}{s}\right) \prod_{j=2}^s B\left(\frac{j}{s+1}, \frac{j}{s+1}\right)} \times t_1^{s+1-1}(1-t_1) \prod_{j=2}^s t_j^{s+1-1}(1-t_j) \prod_{j=1}^s \frac{1}{s_j^{s_j+1}}.
\]

**Proof.** First, we derive a family of symmetric distributions \(\sigma_s\), the 2\(k\)-moments of which are \(m_k\) in Eq. (0.1).

Consider the characteristic function of \(\sigma_s\) as follows:

\[(0.9) \quad \int_{-\infty}^{+\infty} e^{ix\sigma_s(x)} dx = \sum_{k=0}^{\infty} \frac{(-\xi^2)^k}{(2k)!} m_k = \sum_{k=0}^{\infty} \beta_k \frac{(-\xi^2)^k}{k!},
\]

where

\[(0.10) \quad \beta_k = \frac{1}{sk + 1} \binom{sk + k}{k} \frac{k!}{(2k)!} = \frac{1}{sk + 1} \frac{(sk + k)!}{(2k)!}.
\]

A direct computation shows that the ratio

\[
\frac{\beta_{k+1}}{\beta_k} = \frac{sk + 1}{(sk + k + 1)(2k + 1)(2k + 2)} \times \frac{(sk + k + 1)(sk + k + 2) \cdots (sk + k + s + 1)}{(sk + 1)(sk + 2) \cdots (sk + s)}
\]

\[= K \frac{(k + \frac{1}{s+1})(k + \frac{2}{s+1}) \cdots (k + \frac{s}{s+1})}{(k + \frac{1}{s})(k + \frac{2}{s}) \cdots (k + \frac{s}{s})} \frac{1}{k + 1 + \frac{1}{s}}
\]

\[= K \frac{(k + a_1)(k + a_2) \cdots (k + a_s)}{(k + b_1)(k + b_2) \cdots (k + b_s)} \frac{1}{k + b_{s+1}}.
\]
Here

\[(0.11) \quad b_1 = \frac{1}{2}, \quad a_i = \frac{i}{s+1} \quad \text{and} \quad b_{i+1} = \frac{i+1}{s} \quad \text{for} \quad i = 1, 2, \ldots, s.\]

Therefore, using the generalized hypergeometric function, we rewrite

\[(0.12) \quad \int_{-\infty}^{+\infty} e^{i\xi x} \sigma_s(x) dx = {}_sF_{s+1}(a_1, \ldots, a_s; b_1, \ldots, b_s, b_{s+1}; -\frac{K}{4} \xi^2)\]

\[(0.13) \quad = \int_{[0,1]} F(t_1, \ldots, t_s) \, \text{d}^s t.\]

Note that \(a_1 = b_1\) for \(s = 1\) but \(b_j > a_j > 0, \quad j = 1, 2, \ldots, s, \) when \(s > 1.\) In Eq. \((0.13)\) we have made use of Euler’s integral representation of the generalized hypergeometric function \([2]\).

Next, with the help of the integral representation of the Bessel function of the first kind \([2]\), that is, for \(\alpha > -\frac{1}{2},\)

\[(0.14) \quad J_{\alpha}(z) = \frac{(z/2)^{\alpha}}{\Gamma(\alpha + 1)} \, \text{d} \int_{-1}^{1} e^{i\xi x} (1 - x^2)^{\alpha - \frac{1}{2}} dx,\]

we get

\[(0.15) \quad \text{d} \Gamma(\alpha + 1) = \frac{1}{B\left(\frac{1}{2}, \alpha + \frac{1}{2}\right)} \, \text{d} \int_{-1}^{1} e^{i\xi x} (1 - x^2)^{\alpha - \frac{1}{2}} dx.\]

Set \(\alpha = 1/s, \quad z = \sqrt{\tau K} \xi.\) We have

\[(0.16) \quad \text{d} \Gamma(\alpha + 1) = \frac{1}{B\left(\frac{1}{2}, \alpha + \frac{1}{2}\right)} \, \text{d} \int_{-1}^{1} e^{i\xi x} (1 - x^2)^{\alpha - \frac{1}{2}} dx\]

Combining \((0.16)\) and \((0.13)\) after interchanging the order of integration, we obtain

\[(0.17) \quad \sigma_s(x) = \frac{1}{B\left(\frac{1}{2}, \alpha + \frac{1}{2}\right)} \int_{[0,1]} \frac{\left(\tau K - x^2\right)^{1/s - 1/2}}{(\tau K)^{1/s}} F(t_1, \ldots, t_s) \, 1_{\{\tau K \geq x^2\}} \, \text{d}^s t.\]

Noting the fact that

\[(0.18) \quad \pi_s(x) = \frac{\sigma_s(\sqrt{x})}{\sqrt{x}} 1_{(0, \infty)},\]

the proof is then complete. \(\square\)

We remark that some properties of the density \(\pi_s(x)\) follow easily from Theorem \((0.11)\); for instance, there are no atoms, the support is \([0, K],\) and the density is analytic on \((0, K)\) (these properties were obtained earlier by implicit complex-analytic methods; see Theorem 2.1, \([3]\)).
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