HARARY ENERGY OF COMPLEMENT OF LINE GRAPHS OF REGULAR GRAPHS

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Abstract. The Harary matrix of a graph $G$ is defined as $H(G) = [h_{ij}]$, where $h_{ij} = \frac{1}{d(v_i, v_j)}$, if $i \neq j$ and $h_{ij} = 0$, otherwise, where $d(v_i, v_j)$ is the distance between the vertices $v_i$ and $v_j$ in $G$. The $H$-energy of $G$ is defined as the sum of the absolute values of the eigenvalues of Harary matrix. Two graphs are said to be $H$-equienergetic if they have same $H$-energy. In this paper we obtain the $H$-energy of the complement of line graphs of certain regular graphs in terms of the order and regularity of a graph and thus constructs pairs of $H$-equienergetic graphs of same order and having different $H$-eigenvalues.

1. Introduction

Let $G$ be a simple, undirected, connected graph with $n$ vertices and $m$ edges. Let the vertices of $G$ be labeled as $v_1, v_2, \ldots, v_n$. The adjacency matrix of a graph $G$ is the square matrix $A(G) = [a_{ij}]$, in which $a_{ij} = 1$ if $v_i$ is adjacent to $v_j$ and $a_{ij} = 0$, otherwise. The eigenvalues of $A(G)$ are the adjacency eigenvalues of $G$, and they are labeled as $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. These form the adjacency spectrum of $G$ [4]. Two graphs are said to be cospectral if they have same spectra.

The distance between the vertices $v_i$ and $v_j$, denoted by $d(v_i, v_j)$, is the length of the shortest path joining $v_i$ and $v_j$. The diameter of a graph $G$, denoted by $\text{diam}(G)$, is the maximum distance between any pair of vertices of $G$. A graph $G$ is said to be $r$-regular graph if all of its vertices have same degree equal to $r$.

The Harary matrix [9] of a graph $G$ is a square matrix $H(G) = [h_{ij}]$ of order $n$, where

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The Harary matrix was used in the study of molecules in the quantitative structure-property relationship (QSPR) models \[9\].

The **Harary index** defined as the sum of the reciprocal of the distances between all pairs of vertices and it can be derived from the Harary matrix. It has interesting properties in structure-property correlations \[11,16\].

The eigenvalues of \(H(G)\) labeled as \(\xi_1 \geq \xi_2 \geq \cdots \geq \xi_n\) are said to be the **Harary eigenvalues** or **H-eigenvalues** of \(G\) and their collection is called **Harary spectrum** or **H-spectrum** of \(G\). Two non-isomorphic graphs are said to be **H-cospectral** if they have same \(H\)-spectra.

The **Harary energy** or **H-energy** of a graph \(G\), denoted by \(HE(G)\), is defined as \[5\]

\[
HE(G) = \sum_{i=1}^{n} |\xi_i| .
\]

The Harary energy is defined in full analogy with the **ordinary graph energy** \(E(G)\), defined as \[6\]

\[
E(G) = \sum_{i=1}^{n} |\lambda_i| .
\]

The ordinary graph energy has a relation with the total \(\pi\)-electron energy of a molecule in quantum chemistry \[10\]. Bounds for the Harary energy of a graph are reported in \[3,5\].

Two connected graphs \(G_1\) and \(G_2\) are said to be **Harary equienergetic** or **H-equienergetic** if \(HE(G_1) = HE(G_2)\). The \(H\)-equienergetic graphs are reported in \[12,13\]. The distance energy of complements of iterated line graphs of regular graphs has been obtained in \[8\]. In this paper we use similar technique of \[8\] to obtain the \(H\)-energy of the complement of line graphs of certain regular graphs and thus construct \(H\)-equienergetic graphs having different \(H\)-spectra.

The **complement** of a graph \(G\) is a graph \(\overline{G}\), with vertex set same as of \(G\) and two vertices in \(\overline{G}\) are adjacent if and only if they are not adjacent in \(G\). The **line graph** of \(G\), denoted by \(L(G)\) is the graph whose vertices corresponds to the edges of \(G\) and two vertices of \(L(G)\) are adjacent if and only if the corresponding edges are adjacent in \(G\). For \(k = 1, 2, \ldots\) the \(k\)-th iterated line graph of \(G\) is defined as \(L^k(G) = L(L^{k-1}(G))\), where \(L^0(G) = G\) and \(L^1(G) = L(G)\) \[7\].

If \(G\) is a regular graph of order \(n_0\) and of degree \(r_0\) then the line graph \(L(G)\) is a regular graph of order \(n_1 = (n_0r_0)/2\) and of degree \(r_1 = 2r_0 - 2\). Consequently the order and degree of \(L^k(G)\) are \[1,2\]

\[
n_k = \frac{r_{k-1}n_{k-1}}{2}
\]
and
\[ r_k = 2r_{k-1} - 2, \quad (4) \]
where \( n_i \) and \( r_i \) stands for order and degree of \( L^i(G) \), \( i = 0, 1, \ldots \). Therefore
\[ r_k = 2^k r_0 - 2^{k+1} + 2 \quad (5) \]
and
\[ n_k = \frac{n_0}{2^k} \prod_{i=0}^{k-1} r_i = \frac{n_0}{2^k} \prod_{i=0}^{k-1} (2^i r_0 - 2^{i+1} + 2) \quad (6) \]

We need following results.

**Theorem 1.** [4] If \( G \) is an \( r \)-regular graph, then its maximum adjacency eigenvalue is equal to \( r \).

**Theorem 2.** [15] If \( \lambda_1, \lambda_2, \ldots, \lambda_n \) are the adjacency eigenvalues of a regular graph \( G \) of order \( n \) and of degree \( r \), then the adjacency eigenvalues of \( L(G) \) are
\[ \lambda_i + r - 2, \quad i = 1, 2, \ldots, n, \quad \text{and} \]
\[ -2, \quad n(r-2)/2 \ \text{times}. \]

**Theorem 3.** [14] Let \( G \) be an \( r \)-regular graph of order \( n \). If \( r, \lambda_2, \ldots, \lambda_n \) are the adjacency eigenvalues of \( G \), then the adjacency eigenvalues of \( \overline{G} \), the complement of \( G \), are \( n - \lambda_i - 1, i = 2, 3, \ldots, n \).

**Theorem 4.** [3] Let \( G \) be an \( r \)-regular graph of order \( n \) and let \( \text{diam}(G) \leq 2 \). If \( r, \lambda_2, \ldots, \lambda_n \) are the adjacency eigenvalues of \( G \), then its \( H \)-eigenvalues are \( \frac{1}{2}(n + r - 1) \) and \( \frac{1}{2}(\lambda_i - 1), i = 2, 3, \ldots, n \).

**Lemma 5.** [8] Let \( G \) be an \( r \)-regular graph of order \( n \). If \( r \leq \frac{n-1}{2} \), then \( \text{diam}\left(\overline{L^k(G)}\right) = 2, k \geq 1 \).

### 2. Results

**Theorem 6.** Let \( G \) be an \( r \)-regular graph of order \( n \). If \( r \leq \frac{n-1}{2} \), then
\[ HE\left(L(G)\right) = r(n-2). \]

**Proof.** Let the adjacency eigenvalues of \( G \) be \( r, \lambda_2, \ldots, \lambda_n \). From Theorem 2, the adjacency eigenvalues of \( L(G) \) are
\[
\begin{align*}
2r - 2, & \quad \text{and} \\
\lambda_i + r - 2, & \quad i = 2, 3, \ldots, n, \quad \text{and} \\
-2, & \quad n(r-2)/2 \ \text{times}. 
\end{align*}
\]

\[ \{ \} \quad (7) \]
From Theorem 3 and Eq. (7), the adjacency eigenvalues of $L(G)$ are

$$\begin{align*}
(nr/2) - 2r + 1, & \quad \text{and} \\
-\lambda_i - r + 1, & \quad i = 2, 3, \ldots, n, \quad \text{and} \\
1, & \quad n(r - 2)/2 \text{ times.}
\end{align*}$$

The graph $\overline{L(G)}$ is a regular graph of order $nr/2$ and of degree $(nr/2) - 2r + 1$. Since $r \leq \frac{n-1}{2}$, by Lemma 5, $\text{diam}(\overline{L(G)}) = 2$. Therefore by Theorem 4 and Eq. (8), the $H$-eigenvalues of $\overline{L(G)}$ are

$$\begin{align*}
(nr - 2r)/2, & \quad \text{and} \\
-(\lambda_i + r)/2, & \quad i = 2, 3, \ldots, n, \quad \text{and} \\
0, & \quad n(r - 2)/2 \text{ times.}
\end{align*}$$

All adjacency eigenvalues of a regular graph of degree $r$ satisfy the condition $-r \leq \lambda_i \leq r$ [4]. Therefore $\lambda_i + r \geq 0$, $i = 1, 2, \ldots, n$. Therefore by (9),

$$HE(\overline{L(G)}) = \frac{nr - 2r}{2} + \sum_{i=2}^{n} \frac{(\lambda_i + r)}{2} + |0| \times \frac{n(r - 2)}{2}$$

$$= r(n - 2) \quad \text{since} \quad \sum_{i=2}^{n} \lambda_i = -r.$$

\[\square\]

Figure 1. Cycle $C_6$ and $\overline{L(C_6)}$.

**Example 7.** Consider the cycle $C_6$. It satisfies the conditions of Theorem 6. Complement of $L(C_6)$ is shown in the Figure 1. The $H$-eigenvalues of $\overline{L(C_6)}$ are $4, 0, -0.5, -0.5, -1.5, -1.5$. Hence $HE(L(C_6)) = 8$ and by Theorem 6 also, $HE(\overline{L(C_6)}) = 8$. 
Corollary 8. Let $G$ be a regular graph of order $n_0$ and of degree $r_0$. Let $n_k$ and $r_k$ be the order and degree respectively of the $k$-th iterated line graph $L^k(G)$, $k \geq 1$. If $r_0 \leq \frac{n_0-1}{2}$, then

$$HE(L^k(G)) = r_{k-1}(n_{k-1} - 2).$$

Proof. If $r_0 \leq \frac{n_0-1}{2}$, then by Eqs. (3) and (4), we have

$$r_1 = 2r_0 - 2 \leq n_0 - 3 \leq \frac{n_0r_0}{2} - 1 = \frac{n_1 - 1}{2}.$$

Hence

$$r_{k-1} \leq \frac{n_{k-1} - 1}{2}.$$

Therefore by Theorem 6,

$$HE(L^k(G)) = HE(L(L^{k-1}(G))) = r_{k-1}(n_{k-1} - 2).$$

Corollary 9. Let $G$ be a regular graph of order $n_0$ and of degree $r_0$. Let $n_k$ and $r_k$ be the order and degree respectively of the $k$-th iterated line graph $L^k(G)$, $k \geq 1$. If $r_0 \leq \frac{n_0-1}{2}$, then

$$HE(L^k(G)) = \left[ \frac{n_0}{2^{k-1}} \prod_{i=0}^{k-1} (2^i r_0 - 2^{i+1} + 2) \right] - 2(2^{k-1} r_0 - 2^k + 2).$$

### 3. $H$-equienergetic graphs

If $G_1$ and $G_2$ are the regular graphs of same order and of same degree. Then $L(G_1)$ and $L(G_2)$ are of the same order and of same degree. Further their complements are also of same order and of same degree.

Lemma 10. Let $G_1$ and $G_2$ be regular graphs of the same order $n$ and of the same degree $r$. If $r \leq \frac{n-1}{2}$, then $L(G_1)$ and $L(G_2)$ are $H$-cospectral if and only if $G_1$ and $G_2$ are cospectral.

Proof. Follows from Eqs. (7), (8) and (9).

Lemma 11. Let $G_1$ and $G_2$ be regular graphs of the same order $n$ and of the same degree $r$. If $r \leq \frac{n-1}{2}$, then for $k \geq 1$, $L^k(G_1)$ and $L^k(G_2)$ are $H$-cospectral if and only if $G_1$ and $G_2$ are cospectral.

Theorem 12. Let $G_1$ and $G_2$ be regular, non $H$-cospectral graphs of the same order $n$ and of the same degree $r$. If $r \leq \frac{n-1}{2}$, then for $k \geq 1$, $L^k(G_1)$ and $L^k(G_2)$ form a pair of non $H$-cospectral, $H$-equienergetic graphs of equal order and of equal number of edges.

Proof. Follows from Lemma 11 and Corollary 9.
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