CONTINUOUS PROJECTIONS ONTO IDEAL CONVERGENT SEQUENCES

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Abstract. Let \( I \subseteq \mathcal{P}(\omega) \) be a meager ideal. Then there are no continuous projections from \( \ell_\infty \) onto the set of bounded sequences which are \( I \)-convergent to 0. In particular, it follows that the set of bounded sequences statistically convergent to 0 is not isomorphic to \( \ell_\infty \).

1. Introduction

A closed subspace \( X \) of a Banach space \( B \) is said to be complemented in \( B \) if there exists a continuous projection from \( B \) onto \( X \). It is known that \( c_0 \), the space of real sequences convergent to 0, is not complemented in \( \ell_\infty \), cf. [10, 12].

The aim of this note is to show the ideal analogue of this result.

Let \( I \subseteq \mathcal{P}(\omega) \) be an ideal, that is, a family closed under subsets and finite unions. It is also assumed that \( \text{Fin} := \{\omega \}^{<\omega} \subseteq I \) and \( \omega \notin I \). Set \( I^+ := \mathcal{P}(\omega) \setminus I \).

In particular, each \( I \) can be regarded as a subset of the Cantor space \( 2^\omega \) with the product topology, so we can speak of Borel ideals, \( F_\sigma \) ideals, etc. An ideal \( I \) is said to be a P-ideal if it is \( \sigma \)-directed modulo finite sets, i.e., for each sequence \( (A_n) \) in \( I \) there exists \( A \in I \) such that \( A_n \setminus A \) is finite for all \( n \in \omega \). We refer to [7] for a recent survey on ideals and filters.

A real sequence \( (x_n) \) is said to be \( I \)-convergent to \( y \) if \( \{n : x_n \notin U\} \in I \) for all neighborhoods \( U \) of \( y \). We denote by \( c(I) \) [resp. \( c_0(I) \)] the space of real sequences which are \( I \)-convergent [resp. \( I \)-convergent to 0]. The set of bounded real \( I \)-convergent sequences has been studied, e.g., in [2, 6, 8]. By an easy modification of [8, Theorem 2.3], \( c_0(I) \cap \ell_\infty \) is a closed linear subspace of \( \ell_\infty \) (with the sup norm).

The question addressed here, posed at the open problem session of the 45th Winter School in Abstract Analysis (Czech Republic, 2017), follows:

**Question 1.** Is \( c_0(I) \cap \ell_\infty \) complemented in \( \ell_\infty \)?

Before proving our main result, we recall the following:

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Lemma 1.1. An infinite dimensional subspace $X$ of $\ell_\infty$ is complemented in $\ell_\infty$ if and only if it is isomorphic to $\ell_\infty$.

Proof. See [1, Proposition 2.5.2 and Theorem 5.6.5].

Hence, Question 1 can be reformulated as:

Question 2. Is $c_0(\mathcal{I}) \cap \ell_\infty$ isomorphic to $\ell_\infty$?

We will prove that the answer is negative for a large class of ideals. To state our result, we recall that a family $\mathcal{A} \subseteq \mathcal{I}^+$ is said to be $\mathcal{I}$-maximal-almost-disjoint (in short, $\mathcal{I}$-mad) if $\mathcal{A}$ is a maximal family (with respect to inclusion) such that $A \cap B \in \mathcal{I}$ for all distinct $A, B \in \mathcal{A}$, so that for each $X \in \mathcal{I}^+$ there exists $A \in \mathcal{A}$ such that $X \cap A \in \mathcal{I}^+$. (The minimal cardinality $a(\mathcal{I})$ of an $\mathcal{I}$-mad has been studied in the literature: e.g., it is known that, if $\mathcal{I}$ is an analytic P-ideal, $a(\mathcal{I}) > \omega$ if and only if $\mathcal{I}$ is $F_\sigma$, cf. [4, 5].)

Our main result follows:

Theorem 1.2. Let $\mathcal{I}$ be an ideal for which there exists an uncountable $\mathcal{I}$-mad family. Then $c_0(\mathcal{I}) \cap \ell_\infty$ is not complemented in $\ell_\infty$.

It can be shown that, if $\mathcal{I}$ is a meager ideal, there is an $\mathcal{I}$-mad family of cardinality $c$, see Lemma 2.3 below. In particular

Corollary 1.3. $c_0(\mathcal{I}) \cap \ell_\infty$ is not complemented in (and not isomorphic to) $\ell_\infty$ whenever $\mathcal{I}$ is meager.

As an important example, the family of asymptotic density zero sets $\mathcal{Z} := \{S \subseteq \omega : |S \cap [1, n]|/n \to 0\}$ is an analytic P-ideal, hence meager. Therefore:

Corollary 1.4. The set of bounded real sequences statistically convergent to 0 (i.e., $c_0(\mathcal{Z})$) is not isomorphic to $\ell_\infty$.

Lastly, we obtain an analogue of the main result in [9] (for summability matrices):

Corollary 1.5. $c$ is complemented in $c(\mathcal{I}) \cap \ell_\infty$ if and only if $\mathcal{I} = \operatorname{Fin}$.

It is worth noting that Theorem 1.2 cannot be extended to all ideals $\mathcal{I}$. Indeed, if $\mathcal{I}$ is maximal, then the set of bounded $\mathcal{I}$-convergent sequences, which is isomorphic to $c_0(\mathcal{I}) \cap \ell_\infty$, is exactly $\ell_\infty$.

2. Preliminaries and Proofs

Thanks to Lemma 1.1, a negative question to Question 1 would follow if $c_0(\mathcal{I}) \cap \ell_\infty$ was separable (indeed $\ell_\infty$ is nonseparable, hence they cannot be isomorphic). However, this works only if $\mathcal{I} = \operatorname{Fin}$:

Lemma 2.1. $c_0(\mathcal{I})$ is separable if and only if $\mathcal{I} = \operatorname{Fin}$. 
Proof. The if part is known. Conversely, let us suppose that there exists \( A \in \mathcal{I} \cap [\omega]^\omega \). For each \( X \subseteq \omega \) and \( \varepsilon > 0 \), let \( B(1_X, \varepsilon) \) be the open ball with center \( 1_X \) and radius \( \varepsilon \). The collection \( \mathcal{B} := \{B(1_X, 1/2) : X \in [A]^\omega\} \) is an uncountable family of nonempty open sets which are pairwise disjoint, hence \( c_0(\mathcal{I}) \) is not separable.

At this point, recall the following characterization, see [11] and [3, Theorem 4.1.2]:

**Lemma 2.2.** \( \mathcal{I} \) is a meager ideal if and only if there exists a finite-to-one function \( f : \omega \to \omega \) such that \( f^{-1}(A) \in \mathcal{I} \) if and only if \( A \) is finite.

In other words, the second condition is \( \text{Fin} \leq_{\text{RB}} \mathcal{I} \), where \( \leq_{\text{RB}} \) is the Rudin–Blass ordering. This is sufficient to prove the existence of an uncountable \( \mathcal{I} \)-mad family:

**Lemma 2.3.** There exists an \( \mathcal{I} \)-mad family of cardinality \( \mathfrak{c} \), provided \( \mathcal{I} \) is meager.

**Proof.** It is known that there is a \( \text{Fin-mad family} \ \mathcal{A} \) of cardinality \( \mathfrak{c} \), cf. [12]. Then, thanks to Lemma 2.2, there exists a finite-to-one function \( f : \omega \to \omega \) such that \( f^{-1}(A) \in \mathcal{I} \) if and only if \( A \) is finite, hence \( \{f^{-1}(A) : A \in \mathcal{A}\} \) is the claimed \( \mathcal{I} \)-mad family. \( \square \)

Let us prove our main result:

**Proof of Theorem 1.2.** Let us suppose for the sake of contradiction that \( c_0(\mathcal{I}) \cap \ell_\infty \) is complemented in \( \ell_\infty \) and denote by
\[
\pi : \ell_\infty \to c_0(\mathcal{I}) \cap \ell_\infty
\]
the canonical projection. Define \( T := I - \pi \), hence \( T \) is bounded linear operator such that \( T(x) = 0 \) for each \( x \in c_0(\mathcal{I}) \cap \ell_\infty \). Note also that, if \( B \notin \mathcal{I} \), then \( 1_B \) is a bounded sequence which is not \( \mathcal{I} \)-convergent to 0, hence \( \pi(1_B) \neq 1_B \) and \( T(1_B) \neq 0 \).

At this point, let \( \{A_j : j \in J\} \) be an uncountable \( \mathcal{I} \)-mad family, which exists by hypothesis. We are going to show that there exists \( j \in J \) such that \( T(1_{A_j}) = 0 \), which is impossible since \( A_j \in \mathcal{I}^+ \). Indeed, let us suppose that, for each \( j \in J \), there exists \( x_j = (x_{j,n}) \in \ell_\infty \) supported on \( A_j \) with \( T(x_j) \neq 0 \) and, without loss of generality, \( \|x_j\|_\infty = 1 \). It follows that there exists \( m, k \in \omega \) such that \( J := \{j \in J : |x_{j,m}| \geq 2^{-k}\} \) is uncountable. Also, by possibly replacing \( x_j \) with \( -x_j \), let us suppose without loss of generality that \( x_{j,m} > 0 \) for all \( j \in J \).

For each nonempty finite set \( F \subseteq J \), define \( s_F = (s_{F,n}) := \sum_{j \in F} x_j \). In particular,
\[
\|T(s_F)\|_\infty \geq s_{F,m} \geq |F|2^{-k}.
\] (1)
Note also that \( I := \bigcup(A_i \cap A_j) \), where the sum is extended over all distinct \( i, j \in F \), belongs to \( \mathcal{I} \). This implies that the sequence \( s_F \upharpoonright I \) is \( \mathcal{I} \)-convergent to 0,
hence \( T(s_F) = T(s_F \upharpoonright I^c) \). Therefore
\[
\|T(s_F)\|_{\infty} = \|T(s_F \upharpoonright I^c)\|_{\infty} \leq \|T\| \cdot \|s_F \upharpoonright I^c\|_{\infty} \leq \|T\|,
\]
which, together with (1), implies \(|F| \leq 2^k\|T\|\). This contradicts the fact the \( \tilde{J} \) is infinite. \( \square \)

**Proof of Corollary 1.5.** There is nothing to prove if \( \mathcal{I} = \text{Fin} \). Conversely, fix \( I \in \mathcal{I} \setminus \text{Fin} \) and define \( X := \{ x \in \ell_\infty : x_i \neq 0 \text{ only if } i \in I \} \) and \( Y := X \cap c_0 \). It is clear that
\[
c \subseteq Y \subseteq X \subseteq c(\mathcal{I}) \cap \ell_\infty
\]
and that \( X \) and \( Y \) are isometric to \( \ell_\infty \) and \( c_0 \), respectively. Hence, it is known that \( c \) can be projected continuously onto \( Y \), let us say through \( T \), see [10]. To conclude the proof, let us suppose that there exists a continuous projection \( H : c(\mathcal{I}) \cap \ell_\infty \to c \). Then the restriction \( T \circ H \upharpoonright X \) is a continuous projection \( \ell_\infty \to c_0 \). This contradicts Theorem 1.2 (in the case \( \mathcal{I} = \text{Fin} \)). \( \square \)

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