Fractional spin - a property of particles described with a fractional Schrödinger equation

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Abstract. It is shown, that the requirement of invariance under spatial rotations reveals an intrinsic fractional extended translation-rotation-like property for particles described with the fractional Schrödinger equation, which we call fractional spin.

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1. Introduction

The fractional calculus [1,4] provides a set of axioms and methods to extend the coordinate and corresponding derivative definitions in a reasonable way from integer order \( n \) to arbitrary order \( \alpha \):

\[
\{ x^n, \frac{\partial^n}{\partial x^n} \} \rightarrow \{ x^\alpha, \frac{\partial^\alpha}{\partial x^\alpha} \}
\]  

(1)

The definition of the fractional order derivative is not unique, several definitions e.g. the Riemann, Caputo, Weyl, Riesz, Grünwald fractional derivative definition coexist [6]-[13]. To keep this paper as general as possible, we do not apply a specific representation of the fractional derivative operator.

We will only assume, that an appropriate mapping on real numbers of coordinates \( x \) and fractional coordinates \( x^\alpha \) and functions \( f \) and fractional derivatives \( g \) exists and that a Leibniz product rule is defined properly. Therefore we use \( x^\alpha \) as a short hand notation for e.g. \( \text{sign}(x)|x|^\alpha \) as demonstrated in [5] and \( \frac{\partial^\alpha}{\partial x^\alpha} = \frac{\partial^\alpha}{\partial x^\alpha} \) as a short hand notation for e.g. the fractional left and right Liouville derivative \( (D_+^\alpha, D_-^\alpha) \):

\[
(D_+^\alpha f)(x) = \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial x} \int_{-\infty}^{x} d\xi (x-\xi)^{\alpha} f(\xi)
\]  

(2)

\[
(D_-^\alpha f)(x) = \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial x} \int_{x}^{\infty} d\xi (\xi-x)^{\alpha} f(\xi)
\]  

(3)

which may be combined via

\[
\frac{\partial^\alpha}{\partial x^\alpha} f(x) = \frac{D_+^\alpha - D_-^\alpha}{2 \sin(\alpha \pi/2)} f(x)
\]  

(4)

\[
= \Gamma(1+\alpha) \frac{\cos(\alpha \pi/2)}{\pi} \int_{0}^{\infty} \frac{f(x+\xi)-f(x-\xi)}{\xi^{\alpha+1}} d\xi
\]  

(5)

\[ 0 \leq \alpha < 1 \]
For this derivative definition, the invariance of the scalar product follows:

$$\int_{-\infty}^{\infty} \left( \frac{\partial^\alpha}{\partial x^\alpha} f(x) \right)^* g(x) \, dx = - \int_{-\infty}^{\infty} f(x)^* \left( \frac{\partial^\alpha}{\partial x^\alpha} g(x) \right) \, dx$$

(6)

where * denotes the complex conjugate.

The Leibniz product rule is used in the following form [1], [2]:

$$\frac{\partial^\alpha}{\partial x^\alpha} (f \psi) = \sum_{k=0}^{\infty} \binom{\alpha}{k} \left( \frac{\partial^k}{\partial x^k} f \right) \left( \frac{\partial^{\alpha-k}}{\partial x^{\alpha-k}} \psi \right)$$

(7)

where the fractional binomial is given by

$$\binom{\alpha}{k} = \frac{\Gamma(1 + \alpha)}{\Gamma(1 + k) \Gamma(1 + \alpha - k)}$$

(8)

and $\Gamma(z)$ is the Euler $\Gamma$-function.

We define the following set of conjugated operators on an euclidean space for $N$ particles in space coordinate representation:

$$\hat{P}_\mu = \{\hat{P}_0, \hat{P}_i\} = \{i\hbar \partial_t, -i (\frac{\hbar}{mc})^\alpha mc \partial^\alpha_i \}$$

(9)

$$\hat{X}_\mu = \{\hat{X}_0, \hat{X}_i\} = \{t, (\frac{\hbar}{mc})^{1-\alpha} (x^o_i)\}$$

(10)

Due to (6), these operators are hermitean.

With these operators, the classical, non relativistic Hamilton function $H_c$, which depends on the classical momenta and coordinates $\{p_i, x^i\}$

$$H_c = \sum_{i=1}^{3N} \frac{p_i^2}{2m} + V(x^1, ..., x^i, ..., x^{3N})$$

(11)

is quantized. This yields the Hamiltonian $H^\alpha$

$$H^\alpha = \frac{1}{2} mc^2 \left( \frac{\hbar}{mc} \right)^{2\alpha} \sum_{i=1}^{3N} \partial^\alpha_i \partial^\alpha_i + V(\hat{X}^1, ..., \hat{X}^i, ..., \hat{X}^{3N})$$

(12)

Thus, a time dependent Schrödinger type equation for fractional derivative operators results

$$H^\alpha \Psi = \left( -\frac{1}{2} mc^2 \left( \frac{\hbar}{mc} \right)^{2\alpha} \sum_{i=1}^{3N} \partial^\alpha_i \partial^\alpha_i + V(\hat{X}^1, ..., \hat{X}^i, ..., \hat{X}^{3N}) \right) \Psi = i\hbar \partial_t \Psi$$

(13)

For $\alpha = 1$ this reduces to the classical Schrödinger equation.

2. The internal structure of fractional particles

Properties of particles, which are described by wave equations, may be investigated using the commutation relations of fundamental symmetry operations.

Let's call a particle elementary, if it is described by a potential- and field-free wave equation. If in addition there is an internal structure, which is determined by additional quantum numbers, it may be revealed e.g. considering the behaviour under rotations.
We define a generalized fractional angular momentum operator $K^\beta$ for a single particle (\(V = 0, N = 1\)) under rotations in $R^2$ about the z-axis.

Using the Leibniz product rule (14), the fractional derivative of the product $xf(x)$ is given by

$$\partial_\alpha^\beta (xf(x)) = \sum_{j=0}^{\infty} \left( \begin{array}{c} \alpha \\ j \end{array} \right) (\partial_\alpha^j x)\partial_\alpha^{\beta-j} f(x)$$

(14)

$$= (x\partial_\alpha^\beta + \alpha x\partial_\alpha^{-1}) f(x)$$

(15)

We define a generalized fractional angular momentum operator $K^\beta$ with z-component $K_z^\beta$

$$K_z^\beta = i \left( \frac{\hbar}{mc} \right)^\beta mc (x\partial^\beta_y - y\partial^\beta_x)$$

(16)

The components $K_x^\beta, K_y^\beta$ are given by cyclic permutation of the spatial indices in (16).

Using (14) the commutation relation with the Hamiltonian $H^\alpha$ of the free Schrödinger equation (13) results as (with units $\beta = 1$ we obtain for the z-component of the standard angular momentum

$$[i2K_z^\beta, H^\alpha] = [K_z^\beta, \partial_x^{2\alpha} + \partial_y^{2\alpha} + \partial_z^{2\alpha}]$$

(17)

$$= [K_z^\beta, \partial_x^{2\alpha} + \partial_y^{2\alpha}]$$

$$= K_z^\beta ((\partial_x^{2\alpha} + \partial_y^{2\alpha})) - (\partial_x^{2\alpha} + \partial_y^{2\alpha})K_z^\beta$$

$$= (x\partial_y^\beta - y\partial_x^\beta)(\partial_x^{2\alpha} + \partial_y^{2\alpha}) - (\partial_x^{2\alpha} + \partial_y^{2\alpha})(x\partial_y^\beta - y\partial_x^\beta)$$

$$= x\partial_x^{2\alpha}\partial_y^\beta + x\partial_y^{2\alpha}\partial_x^\beta - y\partial_x^{2\alpha+\beta} - y\partial_y^{2\alpha}\partial_x^\beta$$

$$- (\partial_x^{2\alpha}\partial_y^\beta - y\partial_x^{2\alpha+\beta} + x\partial_y^{2\alpha+\beta} - \partial_y^{2\alpha}\partial_x^\beta)$$

$$= x\partial_x^{2\alpha}\partial_y^\beta - y\partial_x^{2\alpha}\partial_y^\beta$$

$$- (x\partial_x^{2\alpha}\partial_y^\beta + 2\alpha\partial_x^{2\alpha-1}\partial_y^\beta - y\partial_x^{2\alpha}\partial_y^\beta - 2\alpha\partial_x^{2\alpha}\partial_y^{2\alpha-1})$$

$$= -2\alpha (\partial_x^{2\alpha-1}\partial_y^\beta - \partial_x^{2\alpha}\partial_y^{2\alpha-1})$$

(17)

Setting $\beta = 1$ we obtain for the z-component of the standard angular momentum operator $L_z$ the commutation relation

$$[L_z, H^\alpha] = [K_z^{\beta = 1}, H^\alpha] = -i\alpha (\partial_x^{2\alpha-1}\partial_y - \partial_x\partial_y^{2\alpha-1})$$

(18)

which obviously is not vanishing. Therefore particles described with the fractional Schrödinger equation (13) contain an internal structure for $\alpha \neq 1$.

We now define the fractional total angular momentum $J^\beta$ with z-component $J_z^\beta$.

Setting

$$\beta = 2\alpha - 1$$

(19)

we obtain with $J_z^{2\alpha-1} = K_z^{2\alpha-1}$, and with (17)

$$[J_z^{2\alpha-1}, H^\alpha] = 0$$

(20)

this operator commutes with the Hamilton operator. Therefore $J_z^{2\alpha-1}$ indeed is the fractional analogue of the standard z-component of the total angular momentum.

Now we define the z-component of a fractional intrinsic angular momentum $S_z^\beta$ with

$$J_z^{2\alpha-1} = L_z + S_z^{2\alpha-1}$$

(21)
The explicit form is given by
\[ S^2_{\alpha - 1}z = ix \left( \frac{h}{mc} \right)^{2\alpha - 1} mc \partial_y^{2\alpha - 1} - h\partial_y \]
\[ - iy \left( \frac{h}{mc} \right)^{2\alpha - 1} mc \partial_x^{2\alpha - 1} - h\partial_x \]  
(22)
This operator vanishes for \( \alpha = 1 \), whereas for \( \alpha \neq 1 \) it gives the z-component of a fractional spin.

Let's call the difference between fractional and ordinary derivative \( \delta p \), or more precisely the components
\[ \delta p_i = i \left( \frac{h}{mc} \right)^{2\alpha - 1} mc \partial_i^{2\alpha - 1} - h\partial_i \]  
(23)
for \( S^2_{\alpha - 1} \) we can write
\[ S^2_{\alpha - 1}z = x\delta p_y - y\delta p_x \]  
(24)
The components of a fractional spin vector are then given by the cross product
\[ S^{2\alpha - 1} = \vec{r} \times \delta \vec{p} \]  
(25)
Therefore fractional spin describes an internal fractional rotation, which is proportional to the momentum difference between fractional and ordinary momentum. For a given \( \alpha \) it has exactly one component.

With \( J^z_{\alpha - 1} = K^z_{\alpha - 1} \) and \( J^y_{\alpha - 1} = K^y_{\alpha - 1} \), the commutation relations for the total fractional angular momentum are given by
\[ [J^z_{\alpha - 1}, J^y_{\alpha - 1}] = (2\alpha - 1) \frac{h}{mc} J^z_{\alpha - 1} p^2_{\alpha - 1} \]  
(26)
\[ [J^y_{\alpha - 1}, J^z_{\alpha - 1}] = (2\alpha - 1) \frac{h}{mc} J^y_{\alpha - 1} p^2_{\alpha - 1} \]  
(27)
\[ [J^z_{\alpha - 1}, J^x_{\alpha - 1}] = (2\alpha - 1) \frac{h}{mc} J^z_{\alpha - 1} p^2_{\alpha - 1} \]  
(28)
with components of the momentum operator \( p \) or generators of fractional translations respectively given as
\[ p_i^\beta = i \left( \frac{h}{mc} \right)^\beta mc \partial_i^\beta \]  
(29)
Therefore in the general case \( \alpha \neq 1 \) an extended fractional rotation group is generated, which contains an additional fractional translation factor.

**3. Conclusion**

We conclude, that fractional elementary particles which are described with the fractional Schrödinger equation, carry an internal structure, which we call fractional spin, because analogies to e.g. electron spin are close.

Consequently, the transformation properties of the fractional Schrödinger equation are more related to the ordinary Pauli-equation than to the ordinary (\( \alpha = 1 \)) Schrödinger-equation.
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