First class models from linear and nonlinear second class constraints

Mehdi Dehghani

Department of Physics, Faculty of Science, Shahrekord University, P. O. Box 115, Shahrekord, Iran.

Maryam Mardaani, Majid Monemzadeh, Salman Abarghooeinejad

Department of Physics, University of Kashan, Kashan, Iran.

Abstract

Two models with linear and nonlinear second class constraints are considered and gauged by embedding in an extended phase space. These models are the free non-relativistic particle on a hyperplane and hyper sphere in configuration space. For the first model we construct its gauged corresponding by the condition of converting second class system to first class one, directly. In contrast the first class system related to the free particle on hyper sphere is derived by the BFT embedding procedure, where its steps are infinite. We give a practical formula for gauging linear and some of the nonlinear second class systems, based on the simplified BFT method. As a result of the gauging two models, we show that in the conversion of second class to the first class constraints the minimum number of phase space degrees of freedom for both systems is a pair of phase space coordinate. This pair for first system is a coordinate and its momentum conjugate, but Poisson structure of embedded non-relativistic particle on hyper sphere is a non-trivial one. We derive infinite correction terms for the Hamiltonian of the nonlinear constraints. The summation is done and constructs an interacting gauged Hamiltonian. We find an open algebra for three first class objects of the embedded nonlinear system.
1. Introduction

It seems that first and second class constrained systems are physically different from each other. Although a first class constrained system is a gauge system, the second class constrained systems must be reduced to physical system with physical degrees of freedoms. First class systems are more handle in contrast to the second class systems in the process of canonical quantization. The first class constraints, in quantized version, acts on all states in Hilbert or Fock space to select a physical ones from many copies. Second class constraints, when they are in linear combinations of original phase space variables, can be removed from phase space to find physical phase space and construct Hilbert space in quantization. To quantize both systems one must be achieve to the reduced phase space. The reduced phase space of a first class system is selection a copy of many similar points in phase space that, all of them satisfy in first class identities. This selection is done by other identities which considered by a gauge fixer. The gauge fixing conditions make the set of first class constraints to second class one. In this way some degrees of the primary model are removed. In contrast, for the second class systems the reduced phase space is constructed by removing non-physical degrees of freedom corresponding to the constraints. For both first and second class systems the removing procedure is done by calculation the Dirac bracket.

The gauge fixing process in theory of constrained systems is the key point for conversion a second class model to a gauge model directly [1] or conceptually [2, 3, 4]. One may imagine a set of second class constraints as a set of first classes and their gauge fixing conditions. The Dirac bracket remove additional degrees of freedom. Thus in reverse way, by adding some degrees of freedom, say embedding the model in a extended phase space, to the second class functions we can convert them to a set of first class constraints. Because the decomposition of a set of second class constraints into the first classes and gauge fixing conditions
can be done in many way, extracting gauge theory from a second class system has more than one solution. In this paper we see this point in the conversion of two sets of constraints. Although for one of the conversions (for linear constraints) we use a direct gauging process, for another one we use the famous method BFT \cite{2, 3, 4} rewritten for Abelian and non-Abelian systems \cite{5}.

In present paper we want to make a comparison between gauging a theory with linear constraints and gauging a theory with nonlinear constraints. In this manner, in section \( \text{(2)} \) we use a direct calculation for gauging the model of a free non-relativistic particle on a hyper plane. Although this model and its results is a partition of our comparison, it is teaching us the concept of gauging and embedding in a simple way. Section \( \text{(3)} \), which is the main part of our paper, is somehow a revising on gauging the Skyrme model or related models such as nonlinear sigma (\( \mathcal{O}(N) \) invariant) model by the BFT embedding formalism. In most of the paper about these models people search for a consistent canonical quantization and quantum spectrum of those. This family of models were considered in several approaches including: the symplectic embedding \cite{6, 7}, the BFT formalism \cite{7, 8, 9, 10, 11, 12, 13}, Stuckelberg field shifting \cite{14, 15}, or mixed approaches based on first principles of the making gauge systems \cite{7, 15, 16, 17, 18}. The problem stems from second class essence of the models, so they try to change it, quantize it and resile to it. We want to know what a kind of gauge theory can be constructed from such models, specially by the BFT method, and what is their classical characteristics. For our aims, we clean the model from mass and limit it to the definite degrees of freedom to see the result, clearly. Extension to the more realistic models isn’t rigorous. In part \( \text{(3.1)} \) we make a brief review on the BFT formalism. We reduce the formula of the BFT for our purposes to apply it on the model of free non-relativistic particle on a hyper sphere. This model and application the BFT method on it introduced in \( \text{(3.2)} \). We find general form of the embedded Hamiltonian of the free particle on hyper sphere in \( \text{(3.2)} \) and make some comments on it in \( \text{(3.3)} \). We give conclusions on our results in section \( \text{(4)} \).
2. Gauging by embedding a system with linear constraints

Consider a free non-relativistic particle that is described by coordinate \( q_i \) in which we display them by a \( D \)-dimensional array \( \vec{q} \). To have a second class constrained system we assume the particle is confined on a hyperplane. The mass of the particle play no rule in our analysis, so we scale the momenta by the mass of particle. The dynamics of such a system described by the total Hamiltonian.

\[
H_T = \frac{1}{2} \vec{p} \cdot \vec{p} + \lambda \phi_1
\]

(1)

Here \( \vec{a} \) is a constant vector independent from phase space variables which is normal of the hyperplane. By this quantity we calculate the dynamics of \( \phi_1 \) and derive secondary constraint,

\[
\phi_2 = \vec{a} \cdot \vec{p}.
\]

(2)

We see that consistency of hyperplane in configuration space, as a primary constraint, leads to another hyperplane in momenta sub-phase space with the same normal vector. The non-vanishing normal vector condition for the normal vector \( \vec{a} \) makes primary and secondary constraints as second classes and truncates consistency process of the constraints.

At now we transform our constraints to first classes by extending the phase space. By a direct sum we paste the auxiliary variables to the phase space and deduce the new phase space as follows,

\[
(q_i, p_i) \oplus (Q_j, P_j), \quad i = 1, \ldots D \quad j = 1, \ldots d
\]

\[
\{q_i, p_{i'}\} = \delta_{ii'}, \quad \{Q_j, P_{j'}\} = \delta_{jj'}.
\]

(3)

In the extended phase space we require that the constraints are corrected by new variables, linearly.

\[
\tilde{\phi}_1 = \phi_1 + \vec{b} \cdot \vec{P}, \quad \tilde{\phi}_2 = \phi_2 + \vec{c} \cdot \vec{Q},
\]

where \( \vec{b} \) and \( \vec{c} \) are two unknown vectors which is determined by two condition. The first is first class condition,

\[
\{\tilde{\phi}_1, \tilde{\phi}_2\} \approx 0.
\]

(5)
Where weak equality means equality on corrected constraints surface. The second is that \( \tilde{\phi}_2 \) derived from consistency of the \( \tilde{\phi}_1 \). The later also required additional corrections to the Hamiltonian. It means that in the new phase space the system affected by a potential where for simplicity we assume such potential is a function in the new configuration space, say \( V = V(\tilde{Q}) \). So arrive to following equations.

\[
\tilde{b}.\tilde{c} = \tilde{a}\.\tilde{a},
\tilde{c}.\tilde{Q} + \tilde{b}\.\nabla_{\tilde{Q}}V = 0,
\]

where the \( \nabla_{\tilde{Q}} \) is the gradient operator with respect to \( \tilde{Q} \). It is worse to noting that there is a third condition which implies that, after consistency investigation of the \( \tilde{\phi}_2 \) in the new model no other constraints must appear.

\[
\{\phi_2, H_c\} + \tilde{\lambda}\{\tilde{\phi}_1, \tilde{\phi}_2\} \approx 0.
\]

In above equation both terms vanish identically, so no any other equation emerges. The first one comes from the characteristic of the primary model. The second term is due to the first classy of the new model. The set of partial differential equations \([8]\) have many solutions. A category of solutions can be derived by considering \( \tilde{b} \) and \( \tilde{c} \) as constant vectors. In this way the primary free second class system \([1]\) converts to the following interactive gauge system.

\[
\tilde{H}_c = \frac{1}{2}\tilde{p}.\tilde{p} - \frac{\tilde{a}^2}{2\tilde{b}^2}\tilde{Q}.\tilde{Q},
\tilde{\phi}_1 = \tilde{a}.\tilde{q} + \tilde{b}.\tilde{P},
\tilde{\phi}_2 = \tilde{a}.\tilde{p} + \frac{1}{2}\frac{\tilde{a}\tilde{a}}{\tilde{b}\tilde{b}}\tilde{b}.\tilde{Q}.
\]

We see that the chain structure in the gauged model doesn’t change, i.e. if we consider the \( \tilde{\phi}_1 \) as the primary constraint its consistency gives the \( \tilde{\phi}_2 \) and the consistency of the later vanishes, strongly. Moreover, the algebra of the embedded constraints is an Abelian one.

As the normal vector of the constrained surfaces \((\tilde{a})\) was characteristic of the linear second class constrained model the constant vector \( \tilde{b} \) incorporate to \( \tilde{a} \) are for the gauged system. In addition, we see surfaces that is described by the \((\tilde{\phi}_1, \tilde{\phi}_2)\) are also another hyperplanes. But despite the primary model which its constraint surfaces were in the coordinate and momenta sub-phase space,
in the new model the \((\tilde{\phi}_1, \tilde{\phi}_2)\) are hyperplanes on the \(\vec{q} \oplus \vec{P}\) and \(\vec{p} \oplus \vec{Q}\) sub-phase spaces. Also, for primary model normal vectors of the hyperplanes were the same, but for the new model they are different. There is a comment about correction term adds to the hamiltonian. A physical interpretation for this term is that, despite to the free model its gauged model is an interactive one. This describes an oscillator with incorrect sign in the potential. One may imagine that oscillator gives its energy from other part, according to the minus sign. This fact is understood better if one, by a canonical transformation, transforms the \(\vec{Q}\) coordinates to momenta \(\vec{P}'\).

Conclusively, we can select a minimal solution by considering only a pair of coordinate-momentum conjugate for gauging the free particle on a hyper surface as:

\[
\tilde{H}_c = \frac{1}{2}\vec{p}.\vec{p} - \frac{1}{2}\vec{a}.\vec{a}Q^2, \quad \tilde{\phi}_1 = \vec{a}.\vec{q} + P, \quad \tilde{\phi}_2 = \vec{a}.\vec{p} + \vec{a}.\vec{a}Q.
\]

For other second class constrained models the extension of phase space and adding correction terms to the constraints and the Hamiltonian isn’t as simple as models with linear constraints. There are some algorithm for conversion to gauge systems. The BFT method is one of them we use it in next section to convert the simplest model contains nonlinear second class constraints.

### 3. Gauging by embedding a system with nonlinear constraints

In this section we simplify the general form of the BFT algorithm to apply it on a model with nonlinear constraints. The present problem in its general form related to the problem of the quantization of the free particle on sphere that is an introduction to the fundamental problem of the quantization in curved space-times. We focus only on the process of the conversion a classical second class system to a classical first class system, yet. In other words we work in the realm of pre-quantization of the systems with finite degrees of freedom and nonlinear second class constraints. Extension of the BFT formalism to infinite degrees (field) models is trivial but to the models with arbitrary nonlinearity of the constraints isn’t.
3.1. BFT method

The BFT algorithm, when it is applied on the systems with bosonic degrees of freedom, essentially comes from the conditions \((3,5)\). Poisson structure of the glued phase space to primary phase space isn’t arbitrary but depends to the algebra of second class constraints. In this approach to the gauge theory making we have two sets of generators for generating correction terms to the constraints and the Hamiltonian. They are denoted by the square matrix \(B\) and the vector \(G\) for the constraints and the Hamiltonian in the following relations, respectively. The correction terms are computed by,

\[
\begin{align*}
\phi_a^{(1)} &= \chi_{ab} \eta_b, \\
B_{ab}^{(1)} &= \{\phi_a^{(0)}, \phi_b^{(1)}\} - \{\phi_b^{(0)}, \phi_a^{(1)}\}, \\
B_{ab}^{(n)} &= \sum_{m=0}^{n} \{\phi_a^{(n-m)}, \phi_b^{(m)}\} + \sum_{m=0}^{n-2} \{\phi_a^{(n-m)}, \phi_b^{(m+2)}\}_{\eta}, \\
\phi_a^{(n+1)} &= -\frac{1}{n + 2} \eta_{a\omega} \chi_{cb}^{-1} \chi_{cb} B_{ba}^{(n)}, \quad n \geq 1,
\end{align*}
\]

for embedding the constraints and by

\[
\begin{align*}
G_a^{(0)} &= \{\phi_a^{(0)}, H^{(0)}\}, \\
G_a^{(1)} &= \{\phi_a^{(1)}, H^{(0)}\} + \{\phi_a^{(0)}, H^{(1)}\} + \{\phi_a^{(2)}, H^{(1)}\}_{\eta}, \\
G_a^{(n)} &= \sum_{m=0}^{n} \{\phi_a^{(n-m)}, H^{(m)}\} \\
&\quad + \sum_{m=0}^{n-2} \{\phi_a^{(n-m)}, H^{(m+2)}\}_{\eta} + \{\phi_a^{(n+1)}, H^{(1)}\}_{\eta}, \\
H^{(n+1)} &= -\frac{1}{n + 1} \eta_{a\omega} \chi_{bc}^{-1} \chi_{bc} G_c^{(n)}, \quad n \geq 0
\end{align*}
\]

for gauging the Hamiltonian. The embedded Hamiltonian \((\tilde{H})\) and the first class constraints \((\tilde{\phi})\) are derived by summation on corresponding correction terms.

In the above equations the upper indexes in the parentheses denote to the order of correction. Also the \(\eta_a\) is a vector represents the new phase space variables, so the suffix \(\eta\) under the brackets means Poisson bracket in the glued sub-phase space. The quantities \(\phi_a^{(0)}\) and \(H^{(0)}\) are no things more than uncorrected
constraints and canonical Hamiltonian of the uncorrected model. The roman indexes $a, b, \cdots$ take their values from $\{1, 2, \cdots, \#\}$ of the second class constraints and everywhere in this paper the summation convention is considered for repeated indexes. The square matrices $\chi$ and $\omega$ determine how the elements of vector $\eta$ appear in the correction terms. They satisfy in the master equation of the BFT,

$$\Delta + \chi^T \omega \chi = 0. \quad (12)$$

By determining $\omega$ also Poisson structure of the new sub-phase space will be determined because we consider that there is no interaction between two parts of phase space, off-shell.

As authors [19, 20] stated there are more than one solution for equation (12), thus for a given second class system there are so many corresponding first class systems which divert to it after gauge fixing. In some cases the elements of the matrix of Poisson brackets of the second class constraints, say $\Delta_{ab}$, on constraints surface are constant; for such cases there is a simple solution for (12) as one could assume the unknown matrixes have elements independent from the phase space variables even or $\chi = 1$ and $\omega = -\Delta$. Such a solution reduces the recursion relations (10,11) to a simple form. In this regime the generating functions $B^{(n)}$ vanish as same as Poisson brackets on new sub-phase space. Conclusively, the constraints are corrected by only one term, say $\tilde{\phi}_a = \phi_a^{(0)} + \eta_a$. The recursion relation for the $n$th order of the Hamiltonian reduces to,

$$G_a^{(n)} = \{\phi_a^{(0)}, H^{(n)}\},$$

$$H^{(n+1)} = \frac{1}{n+1} \eta_a \Delta_{ab}^{-1} G_b^{(n)}, \quad n \geq 0. \quad (13)$$

The fractional factor can be absorbed in generating vector and the matrix $\Delta^{-1}$ rearranges the elements of $\tilde{\eta}$, i.e. we can order the recursion relations as,

$$G_a^{(n)} = \frac{1}{n+1} \{\phi_a^{(0)}, H^{(n)}\},$$

$$\eta_b' = \eta_a \Delta_{ab}^{-1},$$

$$H^{(n+1)} = \eta_b' G_b^{(n)}, \quad n \geq 0. \quad (14)$$
Although for the problem that we faced it the Δ matrix hasn’t constant element, we use the above simplified equations in an appropriate manner for our goal.

3.2. BFT embedding a non-relativistic particle on hyper sphere

The full dynamics of the free non-relativistic particle confined on a D-dimensional sphere is given by

\[ H_T = \frac{1}{2} \vec{p} \cdot \vec{p} + \lambda (\vec{q} \cdot \vec{q} - 1), \]  

(15)

where we assume the confinement condition as a primary constraint. The \( \lambda \) is Lagrange multiplier, adds the primary constraints \( \phi_1 \) to the canonical Hamiltonian. This is the simplest model with nonlinear constraint in the configuration space which produces its second class partner in the phase space as \( \phi_2 = \vec{q} \cdot \vec{p} \). Due to the nonlinear nature of the constraints the extension of phase space to convert the constraints as gauge symmetries of a new model isn’t trivial. But in the BFT formalism we have sufficient equations to add a linear term to the constraints making it first class. This assumption determines the symplectic structure of the extended phase space. So, our consideration is that deformation of constraints by new phase space variables \( (\eta_1, \eta_2) \) as \( \tilde{\phi}_1 = \phi_1 + \eta_1 \) and \( \tilde{\phi}_2 = \phi_2 + \eta_2 \) make them first class constraints. Before starting the BFT embedding the matrix elements of Poisson bracket of the constraints off the constraints surfaces is \( \Delta_{ab} = 2\vec{q}^2 \epsilon_{ab} \). In the process of the BFT the constraints surface changes, so we can’t compute the Δ matrix on the constraints surface unless at the end of our calculations when it corrected. This subtle point make the using of the (14) problematic. We eliminate this problem by choosing a suitable ansatz, in continuance. In this way, according to the corrected constraints we reach to the following nontrivial and nonconstant symplectic structure for the new part of the phase space,

\[ \{ \eta_a, \eta_b \} = -2(1 - \eta_1) \epsilon_{ab}. \]  

(16)

The two dimensional antisymmetric tensor \( \epsilon_{ab} \) is characterized by \( \epsilon_{12} = 1 \).
We define the following objects to employ the relation (14).

\[ \phi_0 = H_c \quad \phi_1 = \phi_1 + 1 \quad \phi_2 = \phi_2. \] (17)

One can shows, these quantities form a closed algebra with following structure constants.

\[
\{ \phi_\alpha, \phi_\beta \} = f_{\alpha\beta\gamma} \phi_\gamma,
\]

\[
f_{\alpha\beta\gamma} = 2(\epsilon_{\alpha\beta0}\delta_{\gamma1} + \epsilon_{\alpha\beta1}\delta_{\gamma0} - \epsilon_{\alpha\beta2}\delta_{\gamma2}),
\]

where Greek indexes take their values from the set \{0, 1, 2\}. The \(\delta_{\alpha\beta}\) and the \(\epsilon_{\alpha\beta\gamma}\) are the conventional, 3-dimensional Kronecker discrete delta function and the full antisymmetric tensor with \(\epsilon_{012} = 1\), respectively. In this version, the three functions are first class in terminology of the constrained systems. But they aren’t a full chain of a Hamiltonian and some constraints, specially \(\phi_1\) isn’t a constraint. In other words the BFT process intends to induce a chain structure of Hamiltonian and constraints on these objects by adding corrections on them.

By running the machinery of the simplified BFT (14) for systems with constant \(\Delta\) matrix (not only weakly but also off the constraints surface) one can deduces a general formula for \(n\)th order correction term of the embedded Hamiltonian, inductively.

\[
H^{(n)} = \frac{1}{n!} \phi_{\gamma_n} \prod_{m=1}^{n} q'_{a_m} f_{a_m \gamma_m - 1 \gamma_m}, \quad n \geq 1,
\] (19)

where summation convention as before is considered for indexes in their domain. The set \{1, 2\} for Roman indexes and the set \{0, 1, 2\} for Greek indexes. Also the \(\gamma_0\) takes only the value 0. Besides the at hand problem, we give an alternative approach for the solving of the linear problem in appendix (5) by the manipulation of the above instruction in the presence of the central charges. This formula and its twin, that is given in (5), is one of the main results of this paper for gauging linear and nonlinear second class systems. Those are applicable and practical whenever, one can to perform on second class functions.
linear operations to create an algebra with the constant structure functions and central charges.

As we see in (19) the $H^{(n)}$ can be expanded in terms of three elements of the closed algebra (18). In each level of iteration, due to the $\Delta^{-1}$ that appears in the second equation of the (14), also a factor $(\tilde{\phi}_1)^{-1}$ enters in $H^{(n)}$. Thus, the solution isn’t exactly linear in $\tilde{\phi}_1$ as (19). Via afore thoughts we guess the ansatz,

$$H^{(n)} = \frac{1}{(\tilde{\phi}_1)^n} \tilde{\phi}_\mu F_\mu(n; \bar{\eta}),$$

(20)

for the $n$th level of the correction to the Hamiltonian. After some calculation which appears in appendix (6) we arrive to the solutions,

$$F_0(n + 1; \bar{\eta}) = (-\eta_1)^{n+1},$$

$$F_1(1; \bar{\eta}) = 0$$

$$F_1(n + 1; \bar{\eta}) = \frac{1}{2} (\eta_2)^2 (-\eta_1)^{n-1},$$

$$F_2(n + 1; \bar{\eta}) = \eta_2 (-\eta_1)^n.$$  \hspace{1cm} (21)

Conclusively, the embedded Hamiltonian is a summation on all corrected terms plus the primary canonical Hamiltonian.

$$\tilde{H} = H_c + \sum_{n=1}^{\infty} H^{(n)}.$$  \hspace{1cm} (22)

According to the equations (21) and the ansatz (20), the output is decomposed to three parts,

$$\tilde{H} = \frac{1}{\phi_1 + \eta_1} (\tilde{\phi}_1 \tilde{\phi}_0 + \frac{1}{2} \eta_2 \tilde{\phi}_2 + \eta_2 \tilde{\phi}_2).$$  \hspace{1cm} (23)

The convergence condition for the summations on the series appears in the corrected Hamiltonian forces the value of the new phase space variable to the intervals $\eta_1 < \frac{1}{2}$.

3.3. Verification of the results

In the last stage of the gauging the model (15), we verify whether our results are compatible? In this way, we check: being Abelian or non-Abelian nature of the new first class constraints and their chain structure between themselves
and the Hamiltonian. We purify the Hamiltonian in the form of kinetic and potential terms, by reduction of it on the new constraints surface.

In former subsection we see that due the new phase space Poisson structure \[ \{ \tilde{\phi}_1, \tilde{\phi}_2 \} = 2\tilde{\phi}_1. \] (24)

Although in the finite order BFT the first class constraints become Abelian, here we take out non-Abelian ones because of the non-constancy of the \( \Delta \) matrix and infinite order of BFT project on this special problem.

In the next step, we consider that the \( \tilde{\phi}_1 \) as like as its uncorrected partner is a primary constraint for the system which is described by \( \tilde{H} \). Afterwards, we simplify the Hamiltonian by it to obtain

\[ \tilde{H}' = \kappa H_c + \frac{1}{2} \varrho^2 + \varrho \phi_2, \] (25)

where we improve the new variables by redefinitions,

\[ \kappa = 1 - \eta_1, \quad \varrho = \eta_2. \] (26)

Then, the consistency of the \( \tilde{\phi}_1 \) in the new system gives,

\[ \{ \tilde{\phi}_1, \tilde{H}' \} = 2\varrho \tilde{\phi}_1. \] (27)

Where vanish weakly and no another constraint emerges. Consequently the second constraints situates at the primary level, inevitably. So, we project the Hamiltonian on the surface of both constraints, named it \( \tilde{H}_{on} \), and then do the consistency. For the \( \tilde{\phi}_2 \) it terminates to an identity due to the,

\[ \{ \tilde{\phi}_2, \tilde{H}_{on} \} = 0. \] (28)

The curious reality that happen in this stage is that, if we project the \( \tilde{H} \) on the surface of both constraints (assume both of them as primary constraints), then

---

\( ^1 \)Another possibility is that we encounter to a bifurcation in the process of consistency. But, by a direct calculation one can shows that the vanishing of the \( \varrho \) eliminates the pair \( (\kappa, \varrho) \) as a second class pair, which isn’t of our favorite.
the chain structure remains according to the (24), (28) and
\[
\{ \tilde{\phi}_1, \tilde{H}_{on} \} = 2\kappa \tilde{\phi}_2.
\] (29)

The three first class object make an open Lie algebra. Conclusively, after becoming first class the chain structure of the constraints remains only on whole constraints surface. The completely on-shell Hamiltonian and its Poisson structure reduces to the absolute desired:
\[
\tilde{H}_{on} = \frac{1}{2} \kappa \vec{q}^2 - \frac{1}{2} \vec{\kappa}^2, \quad \kappa > \frac{1}{2}
\]
\[
\{ \kappa, q \} = 2\kappa, \quad \{ q_i, p_j \} = \delta_{ij}
\] (30)
\[
\tilde{\phi}_1 = \vec{q}^2 - \kappa \quad \tilde{\phi}_2 = \vec{q} \cdot \vec{p} + \rho.
\]

At first sight we see that the embedded hyper sphere in the configuration space doesn’t transform to another hyper-sphere but to a hyper surface with sections, at \( \kappa = \text{constants} \), in the form of hyper-sphere. It is part of a spherical paraboloid. The minimum number of auxiliary variables are go in the primary model by construction of the BFT formalism, spontaneously.

The minus sign in front of the second term obligate us to interpret both the \( \kappa \) and \( \rho \) as coordinates, unless same as the first example of this paper we assume the primary system exchanges the energy with an external system in the form of an oscillator. An extra evidence that guide us to consider the \((\kappa, \rho)\) as a coordinate pair is a non-canonical transformation
\[
\kappa \rightarrow \ln \sqrt{\kappa}, \quad \rho \rightarrow \rho,
\] (31)

which transforms this pair to usual canonical variables in the sense of usual symplectic structure on phase spaces [21]. So our consideration is reasonable. If we get the \((\kappa, \rho)\) as a pure coordinate pair, in quantization the new part of phase space is a noncommutative plane with position-dependent noncommutativity parameter. After the quantization, it can be investigated in the context of Lie algebra noncommutativity [22, 23] or \( \kappa \)-Minkowski noncommutativity [24, 25, 26].
4. Conclusion

In this paper we embed a general system with linear second class constraints directly and by the BFT embedding. We show that the embedded constraints also remain linear but the Hamiltonian becomes interactive. For a free particle on a hyper sphere which can be considered as a model with nonlinear constraints we do the embedding procedure by the finite order BFT formalism. But, because of the non-constancy of the $\Delta$ matrix, we derive an infinite series for correction terms for the Hamiltonian. We find, in both systems the correction terms to primary constraints are linear with respect to the new coordinates. It changes hyper sphere to part of a spherical paraboloid. In conversions to first class systems we encounter to non-unique solutions. We find that minimal solutions add additional degrees of freedom in the number of the second class constraints, brings to mind Wess-Zumino variables in gauging a system [15]. Our novelty of work is a prescription for computing the Hamiltonian correction terms, based on the BFT formalism. Our general formula is derived for the fixed $\Delta$ matrix. We generalize it for nonconstant $\Delta$ matrix, i.e. off the constraints surface. We find a nontrivial Poisson structure for the gauged nonlinear system, that in the quantization produces a noncommutative plane. Ultimately, we show that, after the gauging nonlinear system the Hamiltonian and the constraints take place in a chain structure with non-Abelian open algebra.

5. appendix A

For the linear problem we name the three objects of closed algebra as bellow.

$$\overline{\phi}_0 = H_c, \quad \overline{\phi}_1 = \phi_1, \quad \overline{\phi}_2 = \overline{\phi}_2.$$ (32)

All of the conventions for indexes are as before. This algebra has an essential difference with respect to the algebra of the nonlinear problem. In addition to structure constants it has a central charge in the form,

$$\{\overline{\phi}_\alpha, \overline{\phi}_\beta\} = f_{\alpha\beta\gamma} \overline{\phi}_\gamma + c_{\alpha\beta},$$ (33)
\[
f_{\alpha\beta\gamma} = \epsilon_{\alpha\beta\gamma} \delta_{27}
\]
\[
e_{\alpha\beta} = a^2 \epsilon_{0\alpha\beta}.
\]
So, we are in the situation that we can generalize the \((19)\) in the presence of the central charges. In the same way which arrives us to the \((19)\) we can show that the compact term of the \(H^{(n)}\) decomposes to two parts,
\[
H^{(n)} = \frac{1}{n!} (\phi_{n}) \prod_{m=1}^{n} \eta'_{\alpha_{m}} f_{\alpha_{m} \gamma_{m-1}} \gamma_{m} \\
+ e_{\alpha_{m} \gamma_{m-1}} \eta'_{\alpha_{m}} \prod_{m=1}^{n-1} \eta'_{\alpha_{m}} f_{\alpha_{m} \gamma_{m-1}} \gamma_{m}). \tag{34}
\]
For \(n = 1\) the ambiguity in the second term of the parenthesis disappears, because according to the conventions and \((33)\) the \(c_{\alpha_{0} \gamma_{0}} = c_{\alpha_{0} 0}\) vanishes.

In conclusion, we can establish the relations \((33, 34)\) to deduce the minimal solution \((9)\) after truncation at \(n = 2\). It means that we have the finite order BFT which is due to the constant \(\Delta\) matrix.

6. appendix B

We begin with the \((14)\) and the expression for the matrix elements of Poisson brackets of the second class constraints which can be written in the form \(\Delta_{ab} = -\frac{1}{2\phi_{1}} \epsilon_{ab}\). In continue we find the following recursion relation between consecutive corrections of the Hamiltonian,
\[
H^{(n+1)} = -\frac{1}{2\phi_{1}} \eta_{\alpha} \epsilon_{\alpha b} \frac{1}{n+1} (\phi_{b}, H^{(n)}).
\tag{35}
\]
Afterward, we set the ansatz \((20)\) in the above equation to reach
\[
\phi_{\mu} F_{\mu} (n+1; \eta) = -\frac{1}{2} \eta_{\alpha} \epsilon_{ab} \frac{1}{n+1} (F_{\mu} (n; \eta) f_{b \mu \gamma} \phi_{\gamma}) \\
- n F_{\mu} (n; \eta) \phi_{\mu} \phi_{\gamma} f_{b \mu \gamma} \phi_{\gamma}). \tag{36}
\]
The factor \(\frac{1}{\phi_{1}}\) in the second term is destroyer, but due to the special form of its coefficient \(f_{b \mu \gamma}\) such a factor is removed. It confirms the suggested ansatz and
leads to the coupled recursion relations,

\[
F_0(n+1; \vec{\eta}) = (-\eta_1)F_0(n; \vec{\eta}),
\]
\[
F_1(n+1; \vec{\eta}) = \frac{1}{n+1}(- (n-1)\eta_1 F_1(n; \vec{\eta}) + \eta_2 F_2(n; \vec{\eta})),
\]
\[
F_2(n+1; \vec{\eta}) = \frac{1}{n+1}(\eta_2 F_0(n; \vec{\eta}) - n\eta_1 F_2(n; \vec{\eta})),
\]

for three types of unknown functions $F_\mu(n; \vec{\eta})$. The first and third equation of the above equations can be solved immediately up to initial conditions $F_0(1; \vec{\eta})$ and $F_2(1; \vec{\eta})$. Then, by these solutions we solve the second equation for $n \geq 1$.

Eventually, we read off the initial conditions and the $F_1(1; \vec{\eta})$ from $H^{(1)}$, directly. Putting them in the solutions,

\[
F_0(n+1; \vec{\eta}) = (-\eta_1)^n F_0(1; \vec{\eta}),
\]
\[
F_1(n+1; \vec{\eta}) = \frac{1}{2(n+1)}(-\eta_1)^n - 2(n-1)\eta_2^2 F_0(1; \vec{\eta})
\]
\[
- 2\eta_1 \eta_2 F_2(0; \vec{\eta}),
\]
\[
F_2(n+1; \vec{\eta}) = \frac{1}{n+1}(-\eta_1)^n - 1(n\eta_2 F_0(1; \vec{\eta}) - \eta_1 F_2(1; \vec{\eta})),
\]

build the answer upstanding.

References

[1] P. Mitra, R. Rajaraman, Ann. Phys. (N.Y.) 203 (1990) 157.
[2] I. A. Batalin, E. S. Fradkin, Phys. Lett. B 180 (1986) 157.
[3] I. A. Batalin, E. S. Fradkin, Nucl. Phys. B 279 (1987) 514.
[4] I. A. Batalin, I. V. Tyutin, Int. J. Mod. Phys. A 6 (1991) 3255.
[5] N. Banerjee, R. Banerjee, S. Ghosh, Ann. Phys. (N.Y.) 241 (1994) 237.
[6] J. Ananias Neto, C. Neves, W. Oliveira, Phys. Rev. D 63 (2001) 085018.
[7] S.-T. Hong, Y.-W. Kim, Y.-J. Park, K. Rothe, J. Phys. A 36 (2003) 1643.

[8] W. Oliveira, J. Ananias Neto, Int. J. Mod. Phys. A 12 (1997) 4895.

[9] W. Oliveira, J. Ananias Neto, Nucl. Phys. B 533 (1998) 611.

[10] S.-T. Hong, W.-T. Kim, Y.-J. Park, Phys. Rev. D 59 (1999) 114026.

[11] N. Banerjee, S. Ghosh, R. Banerjee, Nucl. Phys. B 417 (1994) 257.

[12] J. Barcelos-Neto, Phys. Rev. D 55 (1997) 2265.

[13] J. Barcelos-Neto, W. Oliveira, Phys. Rev. D 56 (1997) 2257.

[14] C. Neves, C. Wotzasek, J. Phys. A 33 (2000) 6447.

[15] J. Ananias Neto, Phys. Lett. B 571 (2003) 105.

[16] E. M. C. Abreu, J. Ananias Neto, A. C. R. Mendes, C. Neves, W. Oliveira, Ann. Phys 524 (2012) 434.

[17] S.-T. Hong, W.-T. Kim, Y.-J. Park, Phys. Rev. D 60 (1999) 125005.

[18] A. A. Deriglazov, B. F. Rizzuti, Phys. Rev. D 83 (2011) 125011.

[19] M. Monemzadeh, A. Shirzad, Phys. Rev. D 72 (2005) 045004.

[20] M. Monemzadeh, A. Shirzad, Int. J. Mod. Phys. A 18 (2003) 5613.

[21] M. Chaichian, M. M. Sheikh-Jabbari, A. Tureanu, Phys. Rev. Lett 86 (2001) 2716.

[22] V. P. Nair, Phys. Lett. B 505 (2001) 249.

[23] A. A. Deriglazov, Phys. Lett. B 530 (2002) 235.

[24] J. Lukierski, H. Ruegg, W. J. Zakrzewski, Ann. Phys 243 (1995) 90.

[25] G. Amelino-Camelia, Phys. Lett. B 510 (2001) 255.

[26] G. Amelino-Camelia, Int. J. Mod. Phys. Lett. D 11 (2002) 35.