Stabilization of the $m = 1$ mode in a long-thin mirror trap with high-beta anisotropic plasmas

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Stability of a “rigid” ballooning mode $m = 1$ is studied in application to a mirror axisymmetric trap designed to confine anisotropic plasma with a large beta ($\beta$, the ratio of plasma pressure to magnetic field pressure). It was found that for effective stabilization by lateral perfectly conducting wall, the beta parameter must exceed some critical value $\beta_{\text{crit}}$. The dependence of $\beta_{\text{crit}}$ on the plasma anisotropy, mirror ratio and width of vacuum gap between plasma and the wall was studied.

Unlike the works of other authors focused on the plasma model with a sharp boundary, we calculated the boundaries of the stability zone for a number of diffuse radial pressure profiles and several axial magnetic field profiles.

With a combination of a conducting lateral wall and conducting end plates imitating the attachment of end MHD stabilizers to the central cell of an open trap, there are two critical values of beta and two stability zones, $\beta < \beta_{\text{crit1}}$ and $\beta > \beta_{\text{crit2}}$, which can merge, making the entire range of allowable beta values $0 < \beta < 1$ stable.

Keywords: plasma, MHD stability, ballooning modes, mirror trap, Gas-Dynamic Trap

I. INTRODUCTION

Continuing the study of ballooning instabilities in isotropic plasma started in our article [1], in this paper we present the results of the study of the so-called wall stabilization of rigid flute and ballooning perturbations with azimuthal mode number $m = 1$ in a mirror trap (also called open trap) with anisotropic plasma. According to the physics of the case, wall stabilization of plasma with a sufficiently high pressure is achieved by exciting image currents in the conducting lateral walls of the vacuum chamber surrounding the plasma column. These currents are directed opposite by the diamagnetic current in the plasma, and the opposite currents, as is known, are pushed apart. Such pushing returns the pop-up “tongues” of plasma back to the axis of the trap.

In order to avoid a possible misunderstanding of the significance of the stabilization of some mode $m = 1$, we should probably recall what constitutes a ballooning instability in an open axially symmetric trap. It is generally accepted to think of ballooning perturbations as small-scale deformations of the plasma equilibrium with a large azimuthal mode number $m \gg 1$. It has long been known that the finite Larmor radius (FLR) effects can stabilize small-scale flute-type perturbations if $m > 1$ [2]. Ballooning perturbations are, in fact, related to flute perturbations; therefore, it is commonly believed that they are also stabilized by the FLR effects.

In its pure form, flute perturbations can be detected in a low-pressure plasma (that is, in the limit of $\beta \to 0$, where $\beta$ is the ratio of plasma pressure to the magnetic pressure field). They represent a magnetic tube of field lines, which, together with the plasma captured in it, floats to the periphery of the plasma column without changing its shape and without distorting the magnetic field. The classical method for stabilizing flute disturbances is to attach end MHD stabilizers (for example, of the cusp type) to the central section of the open trap, which “clamp” the ends of the force tube. The same “clamping” effect is achieved by placing conductive plates at the ends of the plasma column. If the ends of the magnetic tube are “frozen” into the end plates, the tube cannot float without bending and deforming the magnetic field. Thus, the flute perturbation is transformed into a balloon type perturbation. Deformation of the magnetic field requires energy, which must be withdrawn from the thermal energy of the plasma. Therefore ballooning instability is possible if plasma beta exceeds a certain threshold value; examples of calculating the threshold can be found in the articles [3–5].

The FLR effects are not capable of stabilizing flute-balloon perturbations with $m = 1$. However, they impose rigidity on such perturbations, so that the plasma density distribution does not change in each cross section of the plasma column. The plasma column bends, shifting from the trap axis to different distances in different sections. Stabilization of such a “rigid” mode $m = 1$ would mean stabilization of all flute-balloon perturbations, provided that the modes $m > 1$ are stabilized by the FLR effects.

Unlike many other works on the stability of a rigid ballooning mode $m = 1$ [6–16], which were focused exclusively on the plasma model with a stepwise pressure profile along the radius, in our article [1] we studied the stability of the $m = 1$ mode in a plasma with a diffuse pressure profile, but then limited ourselves to the case of an isotropic plasma. Specifically, we calculated the so-called critical beta $\beta_{\text{crit}}$, such that the rigid flute and

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ballooning modes \( m = 1 \) can be stabilized by the lateral conducting wall without using any other MHD stabilization methods if \( \beta > \beta_{\text{crit}} \). A second case was also analyzed when the lateral conductive wall was supplemented with conducting end plates installed in magnetic mirror throats, which should have simulated the installation of end MHD stabilizers, such as cusp. In this case, two stability zones were found: the first at small beta, \( \beta < \beta_{\text{crit1}} \), the second at large beta, \( \beta > \beta_{\text{crit2}} \), and these two zones can merge.

In this article, we study wall stabilization on one particular model of anisotropic plasma, which allows us to perform a significant part of the calculations in an analytical form. The key equation for solving the problem was derived by Lynda LoDestro [17] in 1986, but, in fact, neither she nor anyone else has ever used this equation. We found few papers [18–21] that refer to LoDestro’s paper but do not use her equation. Probably oblivion for many years of LoDestro’s work is due to the early termination of the TMX (Tandem Mirror Experiment) and MFTF-B (Mirror Fusion Test Facility B) projects in the USA in the same 1986 [22]. However, the achievement of high electron temperature and high beta in the gas dynamic trap (GDT) at the Institute of Budker Institute of Nuclear Physics in Novosibirsk [23–31], emergence of new ideas [32] and new projects [33–35] makes us rethink old results.

To avoid unnecessary repetitions, we will not analyze the content of the articles cited above, since their detailed review was previously done in our article [1], and we will immediately proceed to the description of the new calculations. In section II, LoDestro’s equation will be written and the necessary notation will be introduced. In section III, the anisotropic pressure model used below will be formulated. The results of calculating some coefficients in the LoDestro equation for this model are presented in Appendices A and B. Section IV presents the results of calculating the critical beta in the limit when the conducting wall surrounding the plasma column is almost conforms to the lateral boundary of the plasma column, but does not touch it. In this limit, LoDestro’s ordinary differential equation of second order reduces to an integral along the \( z \) coordinate on the trap axis; the integral turns to zero at the critical beta value. The section V describes solution of the LoDestro equation by the shooting method and presents the results of calculations for several model profiles of pressure and magnetic field. In section VI, the shooting method is again used to solve the LoDestro equation with other boundary conditions that imitate the effect of conducting end plates or MHD anchors installed in magnetic plugs. The final section VII summarizes our results and conclusions.

II. LODESTRO EQUATION

The LoDestro equation is a second-order ordinary differential equation for the function

\[
\phi(z) = a(z)B_v(z)\xi_n(z),
\]

which depends on one coordinate \( z \) along the trap axis and is expressed in terms of the variable radius of the plasma column boundary \( a = a(z) \), the vacuum magnetic field \( B_v = B_v(z) \) and the virtual small displacement \( \xi_n(z) \) of the plasma column from the axis. It is obtained on the assumption that

- FLR effects are strong enough, \( m = 1 \);
- the paraxial (long-thin) approximation applies;
- the pressure tensor depends on only two functions: \( p_\perp(\psi, B), p_\parallel(\psi, B) \);
- plasma beta is not necessarily small.

The LoDestro equation does not take into account the resistance of plasma and conductive walls. It is also possible that the assumption about the dominance of the FLR effects is violated near the boundary of the plasma column, which may result in an incorrect account of the plasma drifts and rotation. Plasma rotation is taken into account by A. Beklemishev’s theory of vortex confinement [30], but it is limited by the zero beta limit. The resistance was taken into account by Kang, Lichtenberg, and Nevins [37] in an isotropic plasma model with a sharp boundary.

In its final form, the LoDestro equation reads

\[
0 = \frac{d}{dz} \left[ \Lambda + 1 - \left( \frac{p_\perp + p_\parallel}{B_v^2} \right) \frac{d\phi}{dz} \right. \\
\left. + \phi \left( - \frac{d}{dz} \left( \frac{B_v'}{B_v} + \frac{2\alpha'}{a} \right) \left( 1 - \frac{\langle p_\perp + p_\parallel \rangle}{2B_v^2} \right) \right. \\
\left. + \frac{\omega^2 \langle r \rangle}{B_v^2} \frac{\langle p_\perp + p_\parallel \rangle}{a_v} \\
\left. - \frac{1}{2} \left( \frac{B_v'}{B_v} + \frac{2\alpha'}{a} \right)^2 \left( 1 - \frac{\langle p_\perp + p_\parallel \rangle}{2B_v^2} \right) \right) \right, \tag{2}
\]

where the derivative \( d/dz \) in the first two lines acts on all factors to the right of it, and the prime (’’) is a shortcut for \( d/dz \). The parameter \( \psi_\alpha \), used below, has the meaning of the reduced (i.e. divided by \( 2\pi \)) magnetic flux through the plasma cross section \( \pi a^2 \). It is related to the plasma radius \( a = a(z) \) by the equation

\[
\frac{a^2}{2} = \int_0^{\psi_\alpha} \frac{d\psi}{B} . \tag{3}
\]

The magnetic flux \( \psi \) through an arbitrary circular section of the plasma and the radial coordinate \( r \) relates the
The magnetic field $B = B(\psi, z)$ in the paraxial (longitudinal) approximation (i.e., with a small curvature of field lines) is related to the vacuum magnetic field $B_v = B_v(z)$ by the transverse equilibrium equation

$$B^2 = B_v^2 - 2p_\perp. \quad (5)$$

Following LoDestro and some other authors we use rationalized electromagnetic units (also known as Heaviside—Lorentz units) where the factor $[4\pi]$ before $p_\perp$ and $p_\parallel$ is dropped in Eqs. (2) and (5). This factor occurs in the Gaussian system of units, but we omit it throughout the paper.

The kinetic theory predicts (see, for example, [38]) that the longitudinal and transverse plasma pressures can be considered as functions of $B$ and $\psi$, i.e. $p_\perp = p_\perp(B, \psi)$, $p_\parallel = p_\parallel(B, \psi)$. In Eq. (2), one must assume that the magnetic field $B$ is already expressed in terms of $\psi$ and $z$, and therefore $p_\perp = p_\perp(\psi, z)$, $p_\parallel = p_\parallel(\psi, z)$. In what follows, we will also use the notation

$$p = \frac{p_\perp + p_\parallel}{2}. \quad (6)$$

One should distinguish between the actual plasma radius $a = r(\psi, a, z)$ and the vacuum plasma radius

$$a_v(z) = \sqrt{\frac{2\psi_a}{B_0(z)}}. \quad (7)$$

It enters Eq. (2) as the ratio $a''_v/a_v$. LoDestro draws the reader’s attention to the fact that only the vacuum field line curvature $a''_v$ enters into the equation, but in fact the curvature $a''$ of the plasma boundary arises when calculating the derivative in the second line of the equation. We also point out that $\rho$ is the plasma mass density, and $\omega$ is the oscillation frequency.

The angle brackets in Eq. (2) denote the average

$$\langle g \rangle = \frac{\int_0^{\psi_a} \frac{d\psi}{B} g}{\int_0^{\psi_a} \frac{d\psi}{B}} = \frac{2}{a^2} \int_0^{\psi_a} \frac{d\psi}{B} \langle g \rangle \quad (8)$$

of an arbitrary function $g(\psi, z)$ over the plasma cross section. For example,

$$\langle p \rangle = \frac{\int_0^{\psi_a} \frac{d\psi}{B} p}{\int_0^{\psi_a} \frac{d\psi}{B}} = \frac{2}{a^2} \int_0^{\psi_a} \frac{d\psi}{B} \langle p \rangle \quad (9)$$

Finally, parameter

$$\Lambda = \frac{r_w^2 + a^2}{r_w^2 - a^2} \quad (10)$$

is expressed in terms of the actual radius of the plasma/vacuum boundary $a = a(z)$ and the radius of the conducting cylinder $r_w = r_w(z)$, which surrounds the plasma column.

The parameter $\Lambda = \Lambda(z)$ is generally a variable function of the $z$ coordinate, but in the remainder of the paper we assume that $\Lambda$ is a constant. This means that the shape of the lateral conducting wall on an enlarged scale reproduces the shape of the plasma column. We will call such a chamber proportional. It is hardly possible and necessary to make a proportional chamber, but the assumption $\Lambda(z) = \text{const}$ greatly simplifies the calculation. The larger the $\Lambda$ value, the closer the conducting cylinder is located to the plasma boundary. The $\Lambda \rightarrow \infty$ limit corresponds to the case when the conducting side wall approaches the plasma boundary, repeating its shape, but does not touch the plasma. The limit $\Lambda \rightarrow 1$ means that the lateral conducting wall is removed to infinity.

We have repeated the derivation of the LoDestro equation and now we are sure that it is correct, although there are typos in a pair of intermediate formulas in Ref. [17]. It should also be clarified that Lodestro’s work builds on the “shoulders” of predecessors, in particular [12, 38–40].

The boundary conditions for Eq. (2) and similar equations in the study of ballooning instability are traditionally set at the ends of the plasma column at the magnetic field maxima $B_v = B_{\text{max}}$, where $B_v' = 0$ and $p_\perp = p_\parallel = 0$. In accordance with the geometry of actually existing open traps, it is usually assumed that the magnetic field is symmetrical with respect to the median $z = 0$ plane, and the magnetic mirrors (i.e., field maxima) are located at $z = \pm L$.

Traditionally, two types of boundary conditions are considered. In the presence of conducting end plates directly in magnetic mirrors, it is required that the boundary condition

$$\phi = 0 \quad (11)$$

be satisfied at $z = \pm L$. A similar boundary condition is usually used in studying the stability of small-scale ballooning disturbances, thereby modeling the presence of a stabilizing cell behind a magnetic mirror (see, for example, [5]).

If the plasma ends are isolated, the boundary condition

$$\phi' = 0 \quad (12)$$

is applied. As a rule, it implies that other methods of MHD stabilization in addition to stabilization by a conducting lateral wall are not used. It is this boundary condition (12) that was used earlier in the works on the stability of the $m = 1$ ballooning mode.

### III. ANISOTROPIC PRESSURE

In publications on the stability of the rigid ballooning mode, two models of anisotropic pressure have previously been used. Kesner in his paper [13] indicates that in
the first model the transverse pressure in a nonuniform magnetic field \( B \leq B_{\max} \) varies according to the law
\[
p_{\perp} \propto B_{\max}^2 - B^2,
\]
while in the second model
\[
p_{\perp} \propto (B/B_{\max})^2(1 - B/B_{\max})^{n-1}.
\]
The first model approximately describes the pressure distribution in an open trap, which arises when beams of neutral atoms (NB) are injected into plasmas at a right angle to the trap axis into the minimum of the magnetic field. This is the so-called normal injection. The second model corresponds to oblique injection, which forms a population of so-called sloshing ions.

In the present paper, we will restrict ourselves to the first model for three reasons. Firstly, it roughly describes the current state of a real experiment on the Gas Dynamic Trap (GDT) at the Budker INP, and even more suitable for Compact Axisymmetric Toroid (CAT) also constructed at BINP \([28]\). Secondly, in this model, the coefficients in the LoDestro equation can be calculated analytically. Lastly, in the second model, the ultimate beta can actually be limited by either mirror or the firehose instabilities. The second model, in a certain sense, gives quite different results and will be analyzed sometime in the future.

In kinetic theory \([38]\) it is proved that if one of the two pressures \( p_{\perp} \) and \( p_{\parallel} \) is given as a function of \( B \), then the other is uniquely determined using the parallel equilibrium equation. The latter can be rewritten in terms of the partial derivative with respect to \( B \) for a constant magnetic flux \( \psi \) as
\[
p_{\perp} = -B^2 \frac{\partial}{\partial B} \frac{p_{\parallel}}{B}.
\]
Another key result of the kinetic theory is the assertion that the function \( p_{\perp}/B^2 \) always decreases as \( B \) increases, i.e.
\[
\frac{\partial}{\partial B} \frac{p_{\perp}}{B^2} \leq 0.
\]
By an application of (15), one can rewrite this last result as
\[
\frac{\partial}{\partial B} \left( p_{\perp} + p_{\parallel} \right) \leq 0.
\]
First, for stability against the so-called “firehose mode”, it is required that
\[
p_{\parallel} - \frac{B^2}{2} \leq p_{\perp} + \frac{B^2}{2}.
\]
Secondly, stability against the so-called “mirror mode” implies that
\[
\frac{\partial}{\partial B} \left( p_{\perp} + \frac{B^2}{2} \right) > 0.
\]
All these equations are satisfied by the functions\(^1\)
\[
p_{\perp}(B, \psi) = p(\psi) \left( 1 - \frac{B^2}{B_{\max}^2} \right),
\]
\[
p_{\parallel}(B, \psi) = p(\psi) \left( 1 - \frac{B}{B_{\max}} \right)^2,
\]
\[
f(B, \psi) = p(\psi) \left( 1 - \frac{B}{B_{\max}} \right).
\]
They describe the plasma pressure profile with a peak near the magnetic field minimum in the median plane of an open trap. Both functions \( p_{\perp} \) and \( p_{\parallel} \) simultaneously vanish at \( B = B_{\max} \). We assume that \( B_{\max} \) does not exceed the magnetic field \( B_{\max} \) in magnetic mirrors, bearing in mind that there is some cold plasma in the region \( B_{\max} < B < B_{\max} \), but its pressure is negligible. Such a profile approximately describes the real pressure distribution in the current experiments at the GDT facility, where the field \( B_{\max} \) approximately corresponds to the so-called stop point of sloshing ions.

In the case (20) Eq. (5) can be solved and the magnetic field \( B \) weakened by the diamagnetic effect can be explicitly expressed in terms of the vacuum field \( B_{\psi} \):
\[
B(\psi, z) = B_{\psi} \sqrt{\frac{B_{\psi}^2(z) - 2p(\psi)}{B_{\max}^2 - 2p(\psi)}}.
\]
In the second variant (14), Eq. (5) can be solved in analytical form only if \( n = 2 \) or \( n = 3 \), but in any case, numerical integration is necessary when calculating such functions as \( a \) and \( \langle \bar{p} \rangle \), while for the functions (18)–(20) these integrals can be calculated analytically.

The pressure functions expressed in terms of \( B_{\psi} \) will be denoted by the capital letter \( P \),
\[
P_{\perp}(B_{\psi}, \psi) = p(\psi) \frac{B_{\psi}^2 - B_{\max}^2}{B_{\max}^2 - 2p(\psi)},
\]
\[
P_{\parallel}(B_{\psi}, \psi) = p(\psi) \left( 1 - \frac{B_{\psi}^2 - 2p(\psi)}{B_{\max}^2 - 2p(\psi)} \right)^2,
\]
\[
P(\psi, \psi) = p(\psi) \left( 1 - \sqrt{\frac{B_{\max}^2 - 2p(\psi)}{B_{\max}^2 - 2p(\psi)}} \right).
\]
Turning to dimensionless variables, we take the minimum vacuum magnetic field in the median plane \( \min(B_{\psi}) \equiv B_{\psi0} \) as a unit of measurement, and denote the mirror ratio \( B_{s}/B_{\psi0} \) at the stopping point by the letter \( R \). Then, keeping the same notation for dimensionless quantities as for dimensional quantities, we have
\[
B_{\psi0} = 1, \quad B_{s} = R.
\]
The unit of length along the radius \( r \) is further fixed by the fact that we take the magnetic flux \( \psi_{\alpha} \) through

\(^{1}\) The inequalities (16), (17) and (18) are satisfied if \( p_0 > 0 \). The inequality (19) formally leads to the condition \( 2p_0 < B_{\max}^2 \), while the transverse equilibrium equation (5) has a solution in the entire region \( B_{\psi} < B_{\max} \) under the obviously more stringent condition \( 2p_0 < \min(B_{\psi}^2) < B_{\max}^2 \).
the plasma section as a unit of measurement, assuming further that $\psi_0 = 1$. As for the coordinate $z$ along the trap axis, it will be further normalized to the distance $L$ between the median plane $z = 0$ and the magnetic mirror. With this normalization, it turns out that the plugs are located in the planes $z = \pm 1$.

Below we represent the dependence of pressure on the magnetic flux $\psi$ as

$$p(\psi) = p_0 f_k(\psi),$$  \hspace{1cm} (27)

where the dimensionless function $f_k(\psi)$ is defined as

$$f_k(\psi) = \begin{cases} 1 - \psi^k, & \text{if } 0 \leq \psi \leq 1 \\ 0, & \text{otherwise} \end{cases}$$ \hspace{1cm} (28)

for integer values of index $k$, and for $k = \infty$ is expressed in terms of a $\theta$-function such that $\theta(x) = 0$ for $x < 0$ and $\theta(x) = 1$ for $x > 0$:

$$f_\infty(\psi) = \theta(1 - \psi).$$ \hspace{1cm} (29)

The parameter beta $\beta$ is defined as the maximum of the ratio $2p_{\perp}/B_c^2$. The maximum is reached on the trap axis (where $\psi = 0$) at the minimum of the vacuum field (where $\min(B_v) = 1$), so that in dimensionless notation

$$p_0 = \frac{\beta R^2}{2(\beta + R^2 - 1)}. \hspace{1cm} (30)$$

The parameter $p_0$ can vary within the range $0 < p_0 < 1/2$, and $p_0 \to 1/2$ at $\beta \to 1$. At $\beta > 1$, plasma equilibrium is impossible, since Eq. (5) does not have a smooth solution.

As in our first article [1], the subsequent calculations were performed in the Wolfram Mathematica® for four values of the index $k = \{1, 2, 4, \infty\}$. The $f_1$ function describes the smoothest pressure profile. For $\beta/B_c^2 \ll 1$ it approximately gives the parabolic dependence of the pressure $p$ on the coordinate $r$. The larger the index $k$, the more table-like distribution $p$ near the axis of the plasma column and the steeper it is near the boundary of the column. Index $k = \infty$ corresponds to a pressure profile in the form of a step with a sharp boundary.

For the above radial pressure profiles, Wolfram Mathematica® was able to calculate integrals (3) and (9) analytically. The results of the calculations are summarized in Appendixes A and B.

An important characteristic of the plasma is the degree of its anisotropy. If the latter is formally defined as

$$A = \frac{P_{\perp}(B_v, 0) - P_{\parallel}(B_v, 0)}{P_{\perp}(B_v, 0) + P_{\parallel}(B_v, 0)}, \hspace{1cm} (31)$$

relating it to the plasma pressure at the minimum of the magnetic field on the trap axis, we obtain

$$A = \frac{\sqrt{1 - \beta}}{R}. \hspace{1cm} (32)$$

This shows that the anisotropy is the greater, the smaller $R$, reaching a maximum at $R \to 1$. In this connection, we will sometimes refer to the parameter $R$ as the anisotropy parameter. It is useful to note that the limit $B_v = R \to \infty$ corresponds to the case of an isotropic plasma. Taking this limit in the formulas collected in Appendixes A and B, it is possible to re-derive the formulas for isotropic plasma that were previously published in Ref. [1]. However in the present paper we restrict our consideration to the case $R \leq K$ since the plasma pressure should tend to zero at the magnetic mirror throat where $B_v = K$, and $K$ is the mirror ratio.

IV. LIMIT OF ZERO VACUUM GAP

As the lateral conducting wall approaches the plasma/vacuum boundary, where it produces its maximum stabilizing effect, parameter $\Lambda$ tends to infinity, $\Lambda \to \infty$. In this limit, it is possible to make analytic progress in solving Eq. (2), and, hence, in assessing the effects of a diffuse profile on the rigid ballooning mode stability.

Stabilization of the $m = 1$ mode by a conducting wall in the $\Lambda \to \infty$ limit was previously studied by Kesner [13], Li, Kesner and Lane [14] in the case of a plasma with a sharp-boundary radial profile. In particular, apart of isotropic plasma Kesner analyzed anisotropic pressure model (14) and concluded the critical beta becomes smaller as the degree of anisotropy increases.

For $\Lambda \to \infty$ the first term in Eq. (2) is formally greater than all the others, so the derivative $d\phi/dz$ must tend to zero in proportion to $1/\Lambda$, i.e.

$$\phi(z) = \phi_0 + \delta\phi(z), \hspace{1cm} (33)$$

moreover, $\delta\phi(z) = O(1/\Lambda)$, and the constant $\phi_0$ can be considered equal to 1 due to the linearity of Eq. (2) with respect to the function $\phi$. Substituting $\phi = \phi_0 = 1$ into the second term in Eq. (2) (that’s the whole square bracket on three lines) and integrating the whole equation from $z = -1$ to $z = 1$ drops out the first term (with large $\Lambda$) provided that the boundary condition $\phi' = \delta\phi' = 0$ is used at $z = \pm 1$. Performing the indicated procedure yields the integral equation

$$\omega^2 \int_{-1}^{1} \left( \begin{array}{l} \langle \rho \rangle \\ B_v', 2a' \\ B_v, 2a' \end{array} \right) \left( 1 - \frac{\langle \rho \rangle}{B_v^2} \right) \left( 1 - \frac{a'}{B_v^2} \right) \left( 1 - \frac{\langle \rho \rangle}{B_v^2} \right) \left( 1 - \frac{a'}{B_v^2} \right) \left( 1 - \frac{\langle \rho \rangle}{B_v^2} \right) \left( 1 - \frac{a'}{B_v^2} \right) dz, \hspace{1cm} (34)$$

It allows one to calculate the squared oscillation frequency $\omega^2$ if the radial profile of pressure $p = (p_{\perp} + p_{\parallel})/2$, density $\rho$, and vacuum magnetic field $B_v$ are given. At the margins of the stable regime, the oscillation frequency is equal to zero, $\omega^2 = 0$. This fact is proved in the theory.
of ideal magnetohydrodynamics (see, for example, [41–43]). In the stability region $\omega^2 > 0$, and instability takes place if $\omega^2 < 0$.

The bracket $\langle B' v \rangle / B_v + 2a'/a$ is proportional to $B'_v$, so the first term on the right-hand side of Eq. (34) is zero if (as we assume) the boundary conditions at $z = \pm 1$ are set in the throats of magnetic mirrors, where $B'_v = 0$.

In the case of a plasma with a sharp boundary, we have

$$\frac{B'_v}{B_v} + \frac{2a'}{a} = \frac{B'_v}{B_v} - \frac{B' v}{B} = \frac{(B/B_v)'}{B/B_v} = -\frac{1}{2} \frac{(1 - \beta_{L})'}{1 - \beta_{L}} = \frac{1}{2} \frac{\beta_{L}'}{1 - \beta_{L}}, \tag{35}$$

where $B = B_v\sqrt{1 - \beta_{L}}$, $a^2 = 2/B$, $\beta_{L} = 2p_{\perp}/B_v^2$, so Eq. (34) becomes

$$\omega^2 \int_{-1}^{1} \frac{\rho}{B_v^2} \, dz = \int_{-1}^{1} \left[ \frac{2\beta}{B_v^2} \frac{\alpha''}{a_v} + \frac{1}{8} \left( \frac{\beta_{L}'}{1 - \beta_{L}} \right)^2 \left( 1 - \frac{\bar{p}}{B_v^2} \right) \right] dz. \tag{36}$$

This result is identical to equation (19) in Kaiser and Pearlstein’s paper [12]. The first term in square brackets on the right-hand side is a generalization of the Rosenbluth–Longmire criteria [44] for the rigid mode. In the case of isotropic plasma, it is always negative (see, for example, [45]). The second term is certainly positive, but for $\beta \ll 1$ it is less than the first one. This means that $\omega^2 < 0$ in the limit of $\beta \to 0$. Therefore, if stability is possible, then only if beta exceeds some limit value, $\beta > \beta_{\text{crit}} > 0$.

In an anisotropic plasma, the integral of the first term can theoretically become positive with sufficiently strong anisotropy and a special choice of the axial profile of the magnetic field. This is the method of flute instability stabilization by sloshing ions [46, 47]. However, for our chosen pressure model (13) stabilization by sloshing ions is impossible.

In a plasma with anisotropic pressure, condition (18) together with Eq. (3) guarantees that $\bar{p} < B_v^2/2$. Hence, the multiplier $(1 - \bar{p}/B_v^2) > 0$ and the second term in the square brackets are always positive. Therefore, the rigid ballooning mode in a plasma with anisotropic wall pressure can also be stabilized by a conducting wall located close enough to the lateral surface of the plasma column.

Further in this section, we present the results of calculating the critical value of beta in an anisotropic plasma in the limit $\Lambda = \infty$. The calculation method is described in detail in our previous article using the example of isotropic plasma [1]. Here, we restrict ourselves to indicating that the critical value $\beta_{\text{crit}}$ of the parameter $\beta$, corresponding to the marginal stability $\omega^2 = 0$, is defined as a root of the equation

$$W(\beta_{\text{crit}}) = 0, \tag{37}$$

where (in dimensionless variables)

$$W(\beta) = \int_{-1}^{1} \left[ \frac{2(\bar{p})}{B_v^2} \frac{\alpha''}{a_v} + \frac{1}{2} \left( \frac{B'_v}{B_v} + \frac{2a'}{a} \right)^2 \left( 1 - \frac{\bar{p}}{B_v^2} \right) \right] dz. \tag{38}$$

Some peculiarities of searching for roots of this strongly non-linear equation in the Wolfram Mathematica® are described in Appendix D.

As in Ref. [1], we used two models of the vacuum magnetic field. In the first model, the axial profile of the field was given by the family of functions $B_v(z) = \left[ 1 - \left( 1 - K^{-\nu/2} \right) |z|^\nu \right]^{-2/\nu}$, (39), which depended on three parameters: the mirror ratio $K$ and two indices $\mu$ and $\nu$. The calculations were performed for the following combinations of these parameters: $K = \{16, 8, 4\}$, $\mu = \{1, 2, 4, 6\}$ and $\nu = \{0.5, 2, 6\}$. Graphs of functions (39) are given in Ref. [1]. Parameter $K$ is numerically equal to the ratio $B_v(\pm 1)/B_v(0)$. Parameters $\mu$ and $\nu$ determine the width and “steepness” of magnetic mirrors. As both $\mu$ and $\nu$ increase, the profile steepens near the magnetic mirrors while the quasi-homogeneous region at the center of the trap expands. The combination $\mu = 1$, $\nu = 2$ is remarkable in that in the case of an isotropic plasma, the function corresponding to it minimizes the absolute value of the integral in the Rosenbluth–Longmire criterion [44], which determines the stability condition for flute oscillations in mirror traps (see [45]).

As we explained in Ref. [1], in the case of the model field (39), the integrand in Eq. (38) can be singular due to the presence of delta functions $\delta(z)$ and $\delta(z \pm 1)$ in the vacuum curvature $\alpha''$. In the same article, we formulated a method for dealing with these singularities. Here, we just clarify that in anisotropic plasma model the singularity in the magnetic mirror at $z = \pm 1$ makes a zero contribution, since the pressure $\bar{p}$ there is zero. As for the singularity at the point $z = 0$, it gives a nonzero contribution only for $\mu = 1$.

Results of calculation are presented in tables I and in figures 1. Calculations were made for discrete values of parameter $R$ ranging from $R = 1.1$ to $R = K$. Each of the four tables I is calculated for one fixed value of the parameter $\mu$ from the full set $\{1, 2, 4, 6\}$, while the parameters $k$ and $\nu$ run through all the values for which the calculations were made. All four tables are constructed for the mirror ratio $K = 16$. Comparison with tables for other mirror ratios (which we omit here) led us to the startling conclusion that critical beta is independent of $K$. Figure 2 explains this fact. It shows that the graphs of the integral $W(\beta)$ depend on the mirror ratio $K$, but intersect the axis $W(\beta) = 0$ at the point $\beta = \beta_{\text{crit}}$, common to all graphs. Looking ahead, we note that for the

\[ \text{We use double brackets to denote numbers of equations in cited papers.} \]
second model of the magnetic field, which we will describe below, the critical beta still depends on the mirror ratio although very weakly. Figure 2 also proves that there is only one critical beta value $\beta = \beta_{\text{crit}}$ for each set of parameters $k, \mu, \nu, K, R$, and the stable region lays at $\beta > \beta_{\text{crit}}$, where $W(\beta) > 0$.

As we noted above, the parameter $R$ can serve as a characteristic of plasma anisotropy. According to the formula (32), the greater the degree of anisotropy, the closer the $R$ parameter is to unity. As can be seen from table 1, the critical beta rapidly approaches zero at $R \to 1$, but in our calculations we did not take $R$ values less than 1.1, believing that they are unlikely can be achieved experimentally. At $R = 1.1$, the critical beta ranged from 0.5 for the smoothest radial profile $k = 1$ to 0.17 for a plasma with a sharp boundary at $k = \infty$. The minimum values of $\beta_{\text{crit}}$ in each of the tables I are in blue bold.

Within each individual table, it is not difficult to detect a trend towards a decrease in the critical value of beta with an increase in the steepness of the radial pressure profile as the index $k$ increases from $k = 1$ to $k = \infty$ for a fixed pair of indices $\nu$ and $\mu$. The abbreviation N/F instead of a number says that the root was not found. This can mean both that the root does not exist, or that it exists, but less than 1 by about $10^{-6}$. From a practical point of view, it’s all the same: it’s hard to imagine that in a real mirror trap experiment one can get so close to the theoretical limit $\beta = 1$. We will also see later that the paraxial approximation implied in the derivation of the LoDestro equation breaks down in the limit as beta tends to unity.

In a more detailed analysis of the tables I and figures 1, we see that for a fixed value of the $R$ and $k$ parameters, the critical beta slightly increases as the index $\nu$ increases (in tables from top to bottom from 1 to 6), and when index $\nu$ increases (vertically inside a separate table from top to bottom from 0.5 to 6). In other words, stabilization of the rigid ballooning mode is more problematic in traps with short and steep magnetic mirrors. For sufficiently large values of $R \geq 3/5$, the critical beta approaches unity, and the closer the smaller the parameter $k$. It is significant that with an increase in $\nu$, the stability zone disappears for smooth radial profiles (primarily for $k = 1$).

In the second model, the vacuum magnetic field was given by a two-parameter family of functions

$$B_v(z) = 1 + (K - 1) \sin^2(\pi z/2)$$

(40)

with three values of the index $q = \{2, 4, 8\}$ and four values of the mirror ratio $K = \{16, 8, 4, 2\}$. The results are summarized in table II and figure 3. The minimum value of critical beta is again reached at maximum anisotropy (at $R = 1.1$), and it varies approximately within the same limits as in the first model: from 0.5 at $k = 1$ to 0.18 at $k = \infty$. The dependence on the parameter $q$, which characterizes the width and steepness of the magnetic mirrors, does not seem to be very significant, except that
profiles with an increased $q$ index are more prone to the disappearance of the stability zone as the parameter $R$ increases. In contrast to the field of the form (39), for the field of the form (40) we noticed a slight dependence of the critical beta on the mirror ratio $K$. As $K$ decreases from 16 to 2, the value of $\beta_{\text{crit}}$ decreases only in the third decimal place after the decimal point.

As stated in Ref. [1], unlike the magnetic field of the first model (39), function (40) is everywhere smooth and has no breaks. In this sense, it is more realistic. But even in the absence of such a kink on the profile of the vacuum field $B_z(z)$, on the profile of the plasma boundary $a(z)$ near the median plane $z = 0$, a “swell” is formed in the form of a “thorn” with a large curvature on point. An example of such a “spike” for $q = 2$ is shown in Fig. 4(a). At $q = 8$, the “thorn” expands, forming a diamagnetic “bubble” named after Beklemishev, as in Fig. 4(c). The graphs of the plasma boundary $a(z)$ in Fig. 4 are plotted for different values of $k$ and different values of $\beta_{\text{crit}}$ corresponding to them, but with the same value of the magnetic flux $\psi = \psi_a = 1$ captured into the plasma. Interestingly, such plots $a(z)$ almost coincide, although the values of $\beta_{\text{crit}}$ for radial pressure profiles with different $k$ differ quite significantly. Similar effect was reported in Ref. [1] for isotropic plasma.

The formation of a region of large curvature at $\beta \rightarrow 1$ leads to a violation of the paraxial approximation, which was used in the derivation of the LoDestro equation. The formal condition for the applicability of the paraxial approximation can be written as

$$\psi_a \frac{d^2 a_k}{dz^2} \ll 1,$$

where the functions $a_k^2$ for pressure profiles with different
An example of the dependence of the integral (38) on $\beta$ for a field of the form (39) with different mirror ratios $K$. All plots cross the x-axis at the same point $\beta = \beta_{crit}$.

Table II: Critical beta for an anisotropic plasma in the second model of magnetic field (40) at $\Lambda = \infty$ and $K = 16$.

indices $k$ are defined by the formulas (A1) in Appendix A. According to their definition (3) the functions $a_k^2$ are proportional to the magnetic flux $\psi_a$ trapped in the plasma, but when writing formulas (A1) parameter $\psi_a$ was set to one, so it was added explicitly to the condition (41). Formulating the paraxiality condition, we assume that $\psi_a = a_{v0}^2/2$, where $a_{v0}$ is the radius of the plasma column in the median plane of a vacuum magnetic field at $\beta = 0$.

An analysis of the graphs of the left side of the condition (41) for a magnetic field of the form (40) shows that it is most difficult to fulfill this condition in the case of $q = 2$, when the maximum curvature is reached in the median plane $z = 0$. In the other two variants $q = 4$ and $q = 8$, the first two derivatives of the vacuum field $B_v(z)$ are equal to zero at $z = 0$, so the curvature peak is formed at some distance from the median plane, where the vacuum field is larger and the local beta value is smaller. For a field of type $q = 2$ near the median plane, we approximately have

$$B_v = 1 + z^2/2l^2,$$

where $l$ determines the scale of the change in the vacuum field in this region. Substituting (42) into the formulas (A1) and calculating the right side of the condition (41) at the point $z = 0$, we obtain the desired constraint on the paraxiality parameter. For the smoothest pressure profile $k = 1$, this method yields

$$\frac{2\sqrt{R^2 - 1} a_{v0}^2}{\sqrt{1 - \beta R}} \ll 1.$$  

(43a)

For the other three cases ($k = \{2, 4, \infty\}$), respectively, we have

$$\frac{\sqrt{R^2 - 1} a_{v0}^2}{(1 - \beta R)^{1/3}} \ll 1,$$

(43b)

$$\frac{2\sqrt{R^2 - 1} (5/4)^2 a_{v0}^2}{\sqrt{R} (1 - \beta R)^{3/4}} \ll 1,$$

(43c)

$$\frac{\sqrt{R^2 - 1}}{(1 - \beta)^{3/2}} \ll 1.$$  

(43d)

This shows that the most severe restriction on the value of beta takes place for the steepest pressure profile ($k = \infty$). From Eq. (43d) we obtain the condition

$$1 - \beta \gg (a_{v0}/l)^{4/3}.$$  

(44)

From the formal point of view of a refined mathematician, it can be true even for $\beta \to 1$ if the parameter $a_{v0}/l$ tends to zero even faster, but in real open traps the parameter $a_{v0}/l$, although small, has a finite value. Accordingly, our results will be unreliable if $\beta$ is too close to one.

The displacement profiles of the plasma column $\xi_a(z)$ are shown in Fig. 5 for all radial pressure profiles $k$ at the same value $\beta = 0.9$. We emphasize that the displacement is not constant, although, as mentioned above, $\phi(z) = const$ for $\Lambda \to \infty$. At the critical values of beta indicated in Table II and in Fig. 4, the displacement profiles would be practically the same for all $k$, since the profiles of the plasma boundary $a(z)$ in Fig. 4 practically coincide.

The next section describes the method and results of solving the LoDestro equation (2) with a finite value of the parameter $\Lambda$. We used the tables I and II to check convergence of that method.

V. CASE OF FINITE VACUUM GAP

We used the ParametricNDSolveValue built-in utility to find solution to the ordinary differential equation (2) in the Wolfram Mathematica® system. The utility returns a reference $pf$ to an interpolation function of the $z$ coordinate, which also depends on the free parameters $\beta$.
Figure 3: Critical beta in a magnetic field of the form \((40)\) depending on the anisotropy parameter \(R\) at \(K = 8\) and different values of the index \(q\) in the limit \(\Lambda \to \infty\). The stability zone for the radial profile with index \(k\) is located above the corresponding curve.

Figure 4: Axial profile of the plasma boundary in the magnetic field in the form \((40)\) for \(\Lambda \to \infty\), \(R = 2\), \(K = 16\), different parameters \(q\) (shown in the figures) and critical values of beta for different pressure profiles \(k\) (also shown in the figures). The area occupied by plasma at \(\beta = 0\) is shaded.

Figure 5: Axial profile of the displacement \(\xi_n(z)\) of the plasma column in the magnetic field \((40)\) for \(\beta = 0.9\), \(\Lambda \to \infty\), \(R = 4\), \(K = 16\) and various values of the parameters \(q\) and \(k\) (indicated in the figures).

and \(\Lambda\). Other parameters \((K, R, \mu, \nu\) in the first magnetic field model and \(K, R, q\) in the second model) were given by numbers.

Taking into account the symmetry of the magnetic field with respect to the median plane \(z = 0\), it suffices to find a solution to the equation at half the distance between the magnetic mirrors, for example, on the interval \(0 < z < 1\), excluding the end points \(z = 0\) and \(z = 1\). As we explained in our previous article \([1]\), at these points the coefficients of the equation can be singular in the case of the model field \((39)\) due to the presence of delta functions \(\delta(z)\) and \(\delta(z - 1)\). Excluding the ends of the interval \(z = 0\) and \(z = 1\) from the domain of the LoDestro equation eliminates singularities, but the boundary conditions for the regularized equation must be set at the point \(z = 0+\) just to the right of the point \(z = 0\) and at the point \(z = 1-\) slightly to the left of the point \(z = 1\). The method of transferring the boundary conditions to these points is described in the Appendix to Ref. \([1]\).

Further, we will not be distracted by such “trifles”, formulating boundary conditions at the end points \(z = 0\) and \(z = 1\), especially since below we present the results of calculations only for the second, everywhere smooth, magnetic field model, in which the LoDestro equation \((2)\) does not have singularities.

Due to the symmetry of the magnetic field noted above, the desired function \(\phi(z)\) must be even, hence

\[\phi'(0) = 0.\]  (45)

As for the point \(z = 1\), then according to Eq. \((12)\) there must also be the equality

\[\phi'(1) = 0,\]  (46)

which corresponds to the case when the plasma ends are isolated from the metal elements of the structure of the mirror trap. An obvious fit to the LoDestro equation with boundary conditions \((45)\) and \((46)\) is the trivial solution
The critical value of beta, \( \beta_{\text{crit}} \), the application of the shooting method for solving such a problem in the Wolfram Mathematica\textsuperscript{©} system was described earlier in Ref. [1] using the example of an isotropic plasma. Let us dwell on the features of the calculation for the anisotropic plasma.

In what follows, we denote by \( z_R \) the coordinate of the stop point on the \( z \) axis, where \( B_v = R \), and \( P_\perp = p_\parallel = 0 \).

For the second magnetic field model

\[
z_R = \frac{2}{\pi} \arcsin \left( \frac{R-1}{K-1} \right)^{1/4}. \tag{48}\]

Its solution is the equality

\[
[\Lambda(z) + 1] \phi'(z) = \text{const}, \tag{50}\]

where we assume for a moment that \( \Lambda \) may be a function of \( z \), bearing in mind a future analysis of a conducting wall with different shape. The constant on its right-hand side can be found from the boundary condition at \( z = 1 \), where the derivative \( \phi'(1) \) is equal to zero. It follows that the constant on the right-hand side of Eq. (50) is also zero. Since the factor \( \Lambda(z) + 1 \) is greater than zero everywhere, we conclude that \( \phi'(z) = 0 \) in the entire region \( z_R < z < 1 \). Thus, it suffices to find the numerical solution of the original equation (2) in the region \( 0 < z < z_R \).

It should be taken into account that the derivative \( \phi'(z) \) undergoes a jump at \( z = z_R \). Indeed, integrating Eq. (2) over an infinitesimal neighborhood of the point \( z_R \) from \( z_R^- \) to \( z_R^+ \), we obtain the equation

\[
[\Lambda(z_R) + 1] \left[ \phi'(z_R^-) - \phi'(z_R^+) \right] = [Q(z_R^+) - Q(z_R^-)] \phi(z_R), \tag{51}\]

in which we took into account that \( \Lambda, \phi \) and \( (p) \) are continuous at the point \( z = z_R \), in contrast to the derivative \( \phi'(z) \) and the coefficient

\[
Q(z) = B_v' + \frac{2a'}{a} = \frac{2a}{a} - \frac{2a}{a}. \tag{52}\]

The jump in the value of the coefficient is due to the fact that for \( B_v = R \) the derivative of the function (22) has a jump. Since \( \phi'(z_R^+) = 0 \) and \( Q(z_R^+) = 0 \), from Eq. (51) we find the value that the derivative of \( \phi'(z) \) must have at the point \( z_R^- \) on the right boundary of the interval \( 0 < z < z_R \) from its inner side:

\[
\phi'(z_R^-) = \frac{Q(z_R^-)}{\Lambda(z_R^-) + 1} \phi(z_R). \tag{53}\]

When solving Eq. (2) on the interval \( 0 < z < z_R^- \), the boundary condition (53) should be used instead of (46).

The calculations were done in the second magnetic field model (40) for mirror ratios \( K = \{16, 8, 4\} \) and those combinations of parameters \( k = \{1, 2, 4, \infty\} \), \( q = \{2, 4, 8\} \), which are listed in section IV in table II, for discrete constant values of \( \Lambda = \{1, 1.001, 1.002, 1.003, \ldots, 400, 450, 500\} \) in the interval from \( \Lambda = 1 \) to \( \Lambda = 500 \). For \( \Lambda = 500 \), the critical beta value we calculated differed from the value found in previous section for \( \Lambda = \infty \) only in the third decimal place, while in the isotropic plasma model the difference was observed only in the fifth decimal place [1]. Parameter \( R \) varied from \( R = 1.1 \) to \( R = K \).

Critical beta values for the \( \Lambda = 1 \) case, when the conducting side wall is removed to infinity, have not been found. However, such values were found for \( \Lambda \to 1^+ \). The discussion of the stability zone at values of \( \Lambda \) close to unity is relegated to Appendix C, since it is of academic rather than practical interest.

Figures 6, 7, and 8 show plots of \( \beta_{\text{crit}} \) versus ratio \( r_w/a = \sqrt{(\Lambda + 1)/(\Lambda - 1)} \) for mirror ratios \( K = \{4, 8, 16\} \) at various combinations of \( k, q, \) and \( R \). Comparison of subfigures in consecutive rows demonstrates strong dependence of critical betas on parameter \( R \), which characterizes the degree of anisotropy (the smaller the \( R \), the stronger the anisotropy, the lower the \( \beta_{\text{crit}} \)). Comparison of subfigures within each row confirms a weak tendency noted in section IV to increase critical beta as the magnetic mirrors steepen with increasing parameter \( q \). Comparison of figures 6–8 for different \( K \) demonstrates some decrease in the stability zone as the mirror ratio increases.

We also see that parabolic radial pressure profile (51) is unstable for all ratios \( r_w/a \) if \( R \geq 2 \) and \( q = 4 \) or \( q = 8 \). For the case of isotropic plasma [1], the stability zone disappeared also for the next steepest profile (52) and sufficiently steep axial profiles of the magnetic field, but in the case of anisotropic plasma this profile can always be made stable by increasing \( \Lambda \).

At the minimum anisotropy that was allowed in our calculations, that is, at \( R = K \), the model of normal
injection of neutral beams (13) and the model of isotropic plasma give very close results. This fact is confirmed by Fig. 9.

The main trends identified by our calculations are listed below:

- If the stability zone can in principle exist for a given set of parameters $K$, $R$, $q$ (i.e. if in table II for this set, the numerical value $\beta_{\text{crit}}$ is specified), then it occurs if the parameter $\Lambda$ exceeds some minimum value $\Lambda_{\text{min}}$.
- This value is the smaller, the steeper the radial pressure profile (the larger the parameter $k$), the smoother the axial profile of the vacuum magnetic field (the smaller $q$), the smaller the mirror ratio $K$, and the greater the anisotropy (the smaller $R$). The stability zone narrows with increasing $\Lambda_{\text{min}}$ and may disappear altogether for smooth pressure profiles ($k = 1$).
- As expected, critical betas for the case $\Lambda = 1$, when conducting lateral wall is removed ($r_w/a = \infty$), have not been found.

Figure 6: Critical beta for a magnetic field of the form (40) in a perfectly conducting cylindrical chamber with variable radius $r_w(z)$ proportional to the plasma radius $a(z)$; anisotropic plasma pressure model (13) simulating normal NB injection is assumed, $K = 4, R = \{1.1, 1.5, 2, 4\}$. 

- (a) $q=2, K=4, R=1.1$
- (b) $q=2, K=4, R=1.2$
- (c) $q=2, K=4, R=1.5$
- (d) $q=2, K=4, R=2$
- (e) $q=2, K=4, R=1.1$
- (f) $q=4, K=4, R=4$
- (g) $q=4, K=4, R=1.5$
- (h) $q=4, K=4, R=2$
- (i) $q=4, K=4, R=1.1$
- (j) $q=8, K=4, R=4$
- (k) $q=8, K=4, R=1.5$
- (l) $q=8, K=4, R=2$
- (m) $q=8, K=4, R=1.1$


Figure 7: Critical beta for a magnetic field of the form (40) in a perfectly conducting cylindrical chamber with variable radius \( r_w(z) \) proportional to the plasma radius \( a(z) \); anisotropic plasma pressure model (13) simulating normal NB injection is assumed, \( K = 8, R = \{1.1, 1.5, 2, 4, 8\} \).
Figure 8: Critical beta for a magnetic field of the form (40) in a perfectly conducting cylindrical chamber with variable radius $r_w(z)$ proportional to the plasma radius $a(z)$; anisotropic plasma pressure model (13) simulating normal NB injection is assumed, $K = 16$, $R = \{1.1, 1.5, 2, 4, 8, 16\}$. 

$\beta = \frac{q}{K}$

$\frac{r_w}{a}$

$\frac{r_w}{a}$

$\frac{r_w}{a}$

$\frac{r_w}{a}$

$\frac{r_w}{a}$

$\frac{r_w}{a}$

$\frac{r_w}{a}$

$\frac{r_w}{a}$
Figure 9: Comparison of the isotropic plasma model [1] with the anisotropic plasma model (13) with minimum anisotropy $R = K$. Plots in odd and odd series show the critical beta for isotropic plasma and anisotropic plasma, respectively.
• As expected, the smaller the vacuum gap between the plasma and the conducting wall (the larger $\Lambda$), the wider the stability zone (the smaller $\beta_{\text{crit}}$).

• The minimum $\beta_{\text{crit}}$ found for the studied set of radial and axial profiles and parameter of anisotropy $R = 1.1$ is about 17%, which is much lower than the value 70% reported for isotropic plasmas. We expect $\beta_{\text{crit}} \to 0$ at $R \to 1$, but such a limit does not seem achievable in a real experiment.

VI. COMBINED WALL STABILIZATION

Finally, we repeat calculations of previous section with a minor replacement. We take boundary condition (11), which means that the plasma is frozen into the conducting end plates, instead of boundary condition (12) describing insulating ends of a mirror trap.

In recent paper [1] we reported two critical beta values for the case of isotropic plasmas, $\beta_{\text{crit1}}$ and $\beta_{\text{crit2}}$: one at low beta due to the balancing of the ponderomotive force with the curvature drive, and one at high beta due to the proximity of the conducting wall which enables magnetic line bending to balance the curvature drive. Corresponding to two critical values of beta, there are two zones of stability. The first zone exists at low plasma pressure, at $0 < \beta < \beta_{\text{crit1}}$, and the second one exists at high pressure, at $\beta_{\text{crit1}} < \beta < 1$. These two zones merge at larger $\Lambda$ providing overall stability at any beta. In this section we extend these results to the case of anisotropic plasma produced by normal NB injection. The main difference from the isotropic model is that the degree of anisotropy provides an additional “knob” for controlling the experimental parameters.

Let’s move on to solving Eq. (2) with the boundary conditions (11) for $z = 1$, and (45) and (47) for $z = 0$. As shown in section V, in the region $z_R < z < 1$ the desired solution satisfies Eq. (50). Noticing that the constant in this equation is now $\text{const} = \{\Lambda(z_R) + 1\} \phi' (z_R^+)$, rather than zero, we find that the derivative $\phi' (z)$ at $z = z_R^+$ is equal to

$$\phi' (z_R^+) = \frac{-\phi (z_R)}{[\Lambda (z_R) + 1] \int_{z_R^+}^{z_R} \frac{dz}{\Lambda (z_R) + 1}}.$$  \hspace{1cm} (54)

Substituting $\phi (z_R^+)$ into Eq. (51) and taking into account that $Q(z_R^+) = 0$, we find the derivative $\phi' (z_R^+)$ on the right boundary of the interval $0 < z < z_R$ from its inner side:

$$\phi' (z_R^+) = \left[ \frac{Q(z_R^+)}{\Lambda (z_R) + 1} + \frac{1}{[\Lambda (z_R) + 1] \int_{z_R^+}^{z_R} \frac{dz}{\Lambda (z_R) + 1}} \right] \phi (z_R).$$  \hspace{1cm} (55)

This is the boundary condition that should be used instead of (53) in the problem of ballooning instability with a combination of wall stabilization and stabilization by conducting end plates. In case of $\Lambda (z) = \text{const}$ under consideration here it reduces to

$$\phi' (z_R^+) = \left[ \frac{Q(z_R^+)}{\Lambda + 1} - \frac{1}{1 - z_R} \right] \phi (z_R).$$  \hspace{1cm} (56)

Following same scheme as in Ref. [1], we performed a series of calculations for the vacuum magnetic field (40) with three values of the index $q = \{2, 4, 8\}$ and a mirror ratio from a limited set $K = \{24, 16, 8, 4\}$. Parameter of anisotropy $R \leq K$ was taken from the list $R = \{1, 1.2, 1.5, 2, 4, 8, 16, 24\}$.

A series of figures 10 illustrates the results of calculations at the minimum degree of anisotropy, i.e. at $R = K$, when the instability zone has the maximum dimensions. The zone of instability in this and subsequent figures for each radial pressure profile with a given index $k = \{1, 2, 4, \infty\}$ lies between the lower and upper curves of the corresponding color. They represent $\beta_{\text{crit1}}$ and $\beta_{\text{crit2}}$, respectively. If there is only one curve, as in the figure 10(a) for the profile $k = 1$, it should be interpreted as $\beta_{\text{crit1}}$. If there are no curves of a given color on the graph, it must be understood that the corresponding pressure profile is stable in the entire range of $0 < \beta < 1$ and in the entire range of values of $r_w/a$, which is shown in the figure. In the range $r_w/a$ to the left of the left edge of the displayed interval, all profiles are stable in the entire interval $0 < \beta < 1$. It is easy to see that smooth pressure profiles ($k = 1, k = 2$) are more stable than steep profiles ($k = 4, k = \infty$). The same trend takes place for small-scale ballooning disturbances when end MHD stabilizers are used [3].

For comparison, Fig. 11 shows the results of calculations for isotropic plasma with the same combinations of indices $q$ and mirror ratios $K$. As can be seen from these two figures, 10 and 11, the dimensions of the instability zones for the same mirror ratio are very close, but in the case of an anisotropic plasma they are still smaller, which becomes more noticeable as the mirror ratio decreases. Both in the case of anisotropic and isotropic plasma models, the dimensions of the instability zone $\beta_{\text{crit1}} < \beta < \beta_{\text{crit2}}$ (which is not shaded) are maximum for the steepest magnetic field profile with index $q = 2$. They are minimal at $q = 8$.

On the contrary, as can be seen from the analysis of figures 6, 7 and 8, with only wall stabilization implemented without end plates installed, the instability zone at $\beta < \beta_{\text{crit}}$ gets slightly larger as $q$ increases. In case of combined stabilization, for a fixed set of parameters $q, K, R$ the instability zone is maximum for the steepest radial pressure profile ($k = \infty$) and may be completely absent for smooth profiles ($k = 1, k = 2$). Without stabilization by the conducting end plates, the opposite situation takes place: the stability zone is smaller and may be completely absent for smooth profiles. It means that the effect of end plates dominates. Smooth radial profiles can be totally stabilized without lateral walls.

Increasing the anisotropy only enhances the effect of the end plates. A series of figures 12 illustrates the depen-
Summing up everything said in this section, we can state the following:

- Two stability zones are found for moderate values of parameter $\Lambda$ and a sufficiently large mirror ratio $K$. The lower zone $\beta < \beta_{\text{crit}}$ exists even for $\Lambda = 1$.
- With other things being equal, the instability zone is maximum for the steepest pressure profile ($k = \infty$) and might not exist at all for smooth radial profiles ($k = 1$ or $k = 2$). Recall that in section V it is for the stepwise profile that the instability zone had the

Figure 10: Stability zone for normal injection of neutral atoms in the case of combined stabilization by end MHD stabilizers in a proportional chamber at different indices $q = \{2, 4, 8\}$ of the axial magnetic field profile (40), different mirror ratio $K = \{24, 16, 8, 4\}$ and minimal anisotropy $R = K$. The ratio $r_w/a = r_w(z)/a(z)$ of the proportional chamber radius $r_w(z)$ to the variable plasma column radius $a(z)$ is constant for each chamber implementation, and anisotropic plasma pressure model (13) is assumed. Instability zone is located between the lower curve $\beta_{\text{crit}}(r_w/a)$ and the upper curve $\beta_{\text{crit}}(r_w/a)$ of that same colors; the zone of instability is not shaded for a plasma with a sharp boundary ($k = \infty$), for which it has the maximum dimensions. Compare with Fig. 11.
Figure 11: Stability zone for an isotropic plasma in the case of combined stabilization by end MHD stabilizers in a proportional chamber at different indices $q = \{2, 4, 8\}$ of the axial magnetic field profile (40), different mirror ratio $K = \{24, 16, 8, 4\}$. Instability zone is located between $\beta_{\text{crit}1}(r_w/a)$ (lower curve) and $\beta_{\text{crit}2}(r_w/a)$ (upper curve of the same color). The instability zone is not shaded for a plasma with a sharp boundary ($k = \infty$), for which it has maximum dimensions. Compare with Fig. 10.

- For a fixed value of the parameter $\Lambda$, the stability zones expand and can merge with a decrease in the mirror ratio $K$ and/or a smoothing of the radial pressure profile (a decrease in $k$).

- The zone of instability decreases and may even disappear as the plasma anisotropy increases (as the parameter $R$ decreases). The dimensions of the unstable zone are maximum at the minimum studied degree of anisotropy $R = K$, considered for a given mirror ratio $K$. In this limit, the dimensions of the unstable zone are close to those found for an isotropic plasma.

- If an instability zone exists between two stability zones for some combinations of the parameters $k$, $q$, $K$, and $R$, then it disappears if $\Lambda > \Lambda_{\text{crit}} > 1$. The value of $\Lambda_{\text{crit}}$ is the smaller, the smaller $k$, $K$ and the larger $q$. The largest value $\Lambda_{\text{crit}} = 1.52$ ($r_w/a = 2.20140$) in our calculations was found at $q = 2$, $K = R = 24$, $k = \infty$. 
VII. CONCLUSIONS

In the present work, we have studies the possibility of stabilizing the \( m = 1 \) rigid flute and ballooning modes in an axially symmetric mirror trap using a conducting cylindrical wall surrounding a plasma column with anisotropic pressure model (13) simulating normal injection of fast neutral beams. In contrast to many predecessors, who studied the not quite realistic sharp-boundary or staircase models of radial plasma pressure profile, we considered four variants of a diffuse pressure profile (28) with different degrees of steepness, specified by the index \( k \), as well as several variants of the axial profile of the vacuum magnetic field given by the functions (39) and (40) with different values of the indices \( \mu, \nu, q \) at different mirror ratios \( K \).

Stabilization by a conducting wall without any additional means of MHD stabilisation is achieved at a sufficiently high plasma pressure, when the parameter \( \beta \) (the dimensionless ratio of the plasma pressure to the magnetic field pressure) exceeds a certain critical value \( \beta_{\text{crit}} \). Therefore, our goal was to calculate this critical value and study its dependence on the degree of plasma anisotropy, the shape of the radial pressure profile, the axial profile of the magnetic field, the mirror ratio, and the size of the vacuum gap between the plasma and the conducting wall. For calculations, we developed a numerical code in the Wolfram Mathematica\textsuperscript®, which solved Eq. (2), previously derived by Lynda LoDestro.

Our calculations showed that the stability zone expands significantly due to a decrease in the critical beta \( \beta_{\text{crit}} \) as the degree of plasma anisotropy increases. The mirror ratio and axial profile of the magnetic field have relatively smaller effect on the value of \( \beta_{\text{crit}} \). The dependence of \( \beta_{\text{crit}} \) on the radial profile and the gap width between the plasma column and the lateral conducting wall are more significant.

From a practical point of view, a noticeable decrease in the value of \( \beta_{\text{crit}} \) is achieved if the radius of the conducting wall \( r_w \) exceeds the plasma radius \( a \) by no more than 2 times, \( r_w/a \leq 2 \). The influence of the conducting wall practically disappears (in the sense that \( \beta_{\text{crit}} \to 1 \)) if \( r_w/a \geq 4 \). For effective wall stabilization, the ratio \( r_w/a \) must be less than the indicated limits. On the other hand, too small a vacuum gap between the plasma and the wall worsens the vacuum conditions due to the recy-
clinging of neutrals into the plasma from the wall surface, which leads to degradation of the plasma parameters. For this reason, plasma stabilization using only the side wall might seem difficult to implement, but the parameter range near \( r_w/a = 2 \) seems quite comfortable from the point of view of many experimenters.

Even more promising is the stabilization of the rigid ballooning mode with a combination of a conducting lateral wall and conducting end plates, which imitate the attachment of end MHD stabilizers to the central cell of a mirror trap. In contrast to pure wall stabilization, mirror ratio and magnetic field profile have strong effect of the combined stabilization.

Our calculations have shown the great efficiency of this method of stabilization. We found the existence of two stability zones. One at low beta due to the balancing of the lateral wall effect with the curvature drive, and one at high beta due to the proximity of the conducting wall which enables magnetic line bending to balance the curvature drive. Corresponding to two critical values of beta, there are two zones of stability. The first zone exists at low plasma pressure, at \( 0 < \beta < \beta_{\text{crit}1} \), and the second one exists at high pressure, at \( \beta_{\text{crit}2} < \beta < 1 \). These two zones merge, making the entire range of allowable beta values \( 0 < \beta < 1 \) stable, as the mirror ratio decreases, as the vacuum gap between the plasma and the lateral wall decreases, or as the plasma pressure anisotropy increases.

In this paper, we limited ourselves to considering only one anisotropy model, which approximately describes the pressure distribution in an open trap upon injection of beams of fast neutral atoms into the plasma at a right angle to the trap axis (normal injection model). Last but not least, the choice of this model was due to considerations of ease of development of the numerical code. Also, for reasons of simplicity, the shape of the conductive side wall was chosen, which, on an enlarged scale, repeats the shape of the plasma column (proportional chamber model). In the near future, we plan to expand the set of computational models by supplementing them with a model of oblique injection and a model of a conducting chamber in the form of a straight cylinder.

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Appendix A: Computing \( a_i^k \)

Indicating the values of \( k \) as a subscript, we write down the result of calculating the integral in Eq. (3). For the radial profile \( k = 1 \) Wolfram Mathematica\textsuperscript{©} yields

\[
a_1^2 = \frac{1}{B_s p_0} \left\{ B_s B_v - \sqrt{(B_s^2 - 2p_0) (B_s^2 - 2p_0)} + (B_v^2 - B_s^2) \left[ \tanh^{-1} \left( \frac{B_v}{B_s} \right) - \tanh^{-1} \left( \frac{B_s^2 - 2p_0}{B_v^2 - 2p_0} \right) \right] \right\}. \tag{A1a}
\]

Both terms \( \tanh^{-1}(\ldots) \) inside the square brackets can be converted to a logarithm and combined into a single logarithmic function. Further, in the Appendix B when calculating \( P_1 \) we get a similar difference of functions \( \tanh^{-1}(\ldots) \) whose argument is greater than one and, hence, a separate such function has a complex value, but their difference is real in exact calculations. However, in practice the result sometimes turns out to be complex with a small imaginary part. To exclude imaginary quantities, we convert the difference of hyperbolic arc tangents to a purely real logarithm using the substitution

\[
[\ldots] = \ln \left( \frac{B_s + B_v}{\sqrt{B_s^2 - 2p_0} + \sqrt{B_v^2 - 2p_0}} \right).
\]

For the \( a_1^2 \) function, such a transformation is not necessary, but we do it “for the company”.

For the radial profile \( k = 2 \) Wolfram Mathematica\textsuperscript{©} yields

\[
a_2^2 = -\frac{i}{B_s} \sqrt{\frac{2B_v^2}{p_0} - 4E} \left( \sin^{-1} \left( \frac{2p_0}{\sqrt{B_v^2 - 2p_0}} \right) \right) \frac{B_v^2 - 2p_0}{B_v^2 - 2p_0}. \tag{A1b}
\]
This is equivalent to

\[ a_2^2 = -\frac{i}{B_s} \sqrt{\frac{2B_s^2}{p_0} - 4} E \left( i \sinh^{-1} \left( \frac{\sqrt{2p_0}}{B_s^2 - 2p_0} \right) \right) \]

Here \( E(\phi|m) \) gives the elliptic integral of the second kind. Using Abramowitz & Stegun 17.4.8 and 17.4.9 \cite{Abramowitz1972} we perform the following transform

\[
F \left( i \sinh^{-1}(t) \mid m \right) \rightarrow iF \left( \tan^{-1}(t) \mid 1 - m \right), \\
E \left( i \sinh^{-1}(t) \mid m \right) \rightarrow it \sqrt{\frac{m^2 + 1}{t^2 + 1}} + iF \left( \tan^{-1}(t) \mid 1 - m \right) - iE \left( \tan^{-1}(t) \mid 1 - m \right),
\]
to express incomplete elliptic functions of an imaginary argument in terms of elliptic functions of a real argument. The purpose of such a transformation is again the same—to get rid of imaginary numbers in intermediate calculations.

\[ a_2^2 = \sqrt{\frac{2B_s^2}{p_0} - 4} \left[ F \left( \tan^{-1} \left( \frac{2p_0}{B_s^2 - 2p_0} \right) \right) - E \left( \tan^{-1} \left( \frac{2p_0}{B_s^2 - 2p_0} \right) \right) \right] \]

For the radial profile \( k = 4 \) Wolfram Mathematica\textsuperscript{©} yields

\[
a_4^2 = 10 \left( B_s^2 - 2p_0 \right) \left( B_v^2 - 2p_0 \right) F_1 \left( \frac{1}{4}; \frac{1}{2}; \frac{1}{2}; \frac{5}{4}; \frac{2p_0}{B_s^2 - 2p_0}; \frac{2p_0}{B_v^2 - 2p_0} \right) / B_v \left[ 5 \left( B_s^2 - 2p_0 \right) \left( B_v^2 - 2p_0 \right) F_1 \left( \frac{1}{4}; \frac{1}{2}; \frac{1}{2}; \frac{5}{4}; \frac{2p_0}{B_s^2 - 2p_0}; \frac{2p_0}{B_v^2 - 2p_0} \right) - \right. \\
- 4p_0 \left( B_s^2 - 2p_0 \right) F_1 \left( \frac{5}{4}; \frac{1}{2}; \frac{3}{2}; \frac{9}{4}; \frac{2p_0}{B_s^2 - 2p_0}; \frac{2p_0}{B_v^2 - 2p_0} \right) + \left. \\
+ 4p_0 \left( B_v^2 - 2p_0 \right) F_1 \left( \frac{5}{4}; \frac{1}{2}; \frac{3}{2}; \frac{9}{4}; \frac{2p_0}{B_s^2 - 2p_0}; \frac{2p_0}{B_v^2 - 2p_0} \right) \right]
\]

Here \( F_1(a;b_1,b_2;c;x,y) \) is the Appell hypergeometric function of two variables. After substitutions

\[ B_s \rightarrow \sqrt{2p_0 \left( \frac{1}{x} + 1 \right)}, \quad B_v \rightarrow \sqrt{2p_0 \left( \frac{1}{y} + 1 \right)}, \]

built-in utility FullSimplify managed to significantly simplify this formula by reducing the number of Appell hypergeometric functions from four to one:

\[
a_4^2 = \frac{2}{B_s} \sqrt{\frac{B_s^2 - 2p_0}{B_v^2 - 2p_0}} \left( \frac{1}{4}; \frac{1}{2}; \frac{1}{2}; \frac{5}{4}; \frac{2p_0}{B_s^2 - 2p_0}; \frac{2p_0}{B_v^2 - 2p_0} \right). \quad (A1c)
\]

Similarly, for the case \( k = 8 \) we obtained

\[
a_8^2 = \frac{2}{B_s} \sqrt{\frac{B_s^2 - 2p_0}{B_v^2 - 2p_0}} F_1 \left( \frac{1}{8}; \frac{1}{2}; \frac{1}{2}; \frac{9}{8}; \frac{2p_0}{B_s^2 - 2p_0}; \frac{2p_0}{B_v^2 - 2p_0} \right), \quad (A1d)
\]

however we did not used this formula in the present paper.

For the radial profile \( k = \infty \) Wolfram Mathematica\textsuperscript{©} yields simple result, which can easily be checked without system of computer integration:

\[
a_\infty^2 = \frac{2}{B_s} \sqrt{\frac{B_s^2 - 2p_0}{B_v^2 - 2p_0}}. \quad (A1e)
\]

We found that the duration of calculations for the \( k = 4 \) variant is about 200 ÷ 300 times longer than the computation time for all other variants combined. Apparently, this fact is connected with the complexity of the built-in algorithm for calculating the Appell function in the Wolfram Mathematica\textsuperscript{©} system. However, for a set of Appell function arguments that are of interest to our problem, the calculation time for the \( k = 4 \) variant is reduced by 2 ÷ 2.5 times if the built-in Appell function is replaced by its integral representation in accordance with the Picard formula

\[
F_1(a;b_1,b_2;c;x,y) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{-a+c-1} \frac{1}{(1-xt)^{b_1}(1-yt)^{b_2}} \, dt. \quad (A2)
\]
Appendix B: Integrals $P_k$

For the radial profile $k = 1$ Wolfram Mathematica© yields

$$P_1 = \frac{a^2}{2} \langle P_\perp \rangle_1 = -\frac{1}{16B_s p_0} \left\{ B_s^3 B_v - 3B_s B_v^3 + 8p_0^2 + (-B_s^2 + 4p_0 + 3B_v^2) \sqrt{(B_s^2 - 2p_0)} (B_v^2 - 2p_0) \right.$$

$$+ (B_s^2 + 2B_s B_v^2 - 3B_v^4) \left[ \tanh^{-1} \left( \frac{B_s^2 - 2p_0}{\sqrt{B_s^2 - 2p_0}} \right) - \tanh^{-1} \left( \frac{B_s}{B_v} \right) \right] \right\}. \quad (B1a)$$

Arguments of hyperbolic arc tangents $\tanh^{-1}(\ldots)$ in this formula are greater than one. Moreover, such a separate function of the hyperbolic arc tangent has a complex value, but their difference in square brackets is real. In order not to struggle with the imaginary parts of these functions, which do not always cancel out with sufficient accuracy in the end, we convert the expression in square brackets to a logarithm, which has a purely real value:

$$[\ldots] = \ln \left( \frac{\sqrt{B_s^2 - 2p_0} + \sqrt{B_v^2 - 2p_0}}{B_s + B_v} \right).$$

For the radial profile $k = 2$ Wolfram Mathematica© yields

$$P_2 = \frac{a^2}{2} \langle P_\perp \rangle_2 = \frac{1}{12B_s \sqrt{B_s^2 - 2p_0}} \left\{ -2i\sqrt{2} \left( \frac{B_s^2 - B_v^2}{B_s^2 - 2p_0} \right) \left( B_v^2 + p_0 \right) F \left( i \sinh^{-1} \left( \frac{\sqrt{2p_0}}{B_s^2 - 2p_0} \right) \frac{B_s^2 - 2p_0}{B_v^2 - 2p_0} \right) \right.$$

$$+ i\sqrt{2} \left( \frac{B_s^2 - 2p_0}{B_s^2 - 2p_0} \right) (B_s^2 - 2 (B_v^2 + 2p_0)) \left( \sinh^{-1} \left( \frac{\sqrt{2p_0}}{B_s^2 - 2p_0} \right) \frac{B_s^2 - 2p_0}{B_v^2 - 2p_0} - 2 \sqrt{\frac{p_0}{B_s^2 - 2p_0}} (B_s B_v + 4p_0) \right) \right\}. \quad (B1b)$$

Here $F(\phi|m)$ and $E(\phi|m)$ are the elliptic integral of the first and the second kinds, respectively. Using Abramowitz & Stegun 17.4.8 and 17.4.9 we perform the transform specified in Appendix A to express incomplete elliptic functions of an imaginary argument in terms of elliptic functions of a real argument. The purpose of such a transformation is again the same—to get rid of imaginary numbers in intermediate calculations.

$$P_2 = \frac{a^2}{2} \langle P_\perp \rangle_2 = \frac{1}{12B_s^2} \left\{ \frac{1}{\sqrt{p_0 (B_v^2 - 2p_0)}} \right. \left\lfloor \sqrt{2B_s} \left( \frac{B_s^2 - 2p_0}{B_v^2 - 2p_0} \right) (B_v^2 + 4p_0) F \left( \tan^{-1} \left( \frac{\sqrt{2p_0}}{B_s^2 - 2p_0} \right) \frac{B_s^2 - 2p_0}{B_v^2 - 2p_0} \right) \right.$$

$$+ (B_s^2 - 2p_0) \left( B_s^2 - 2 (B_v^2 + 2p_0) \right) \left( \tan^{-1} \left( \frac{\sqrt{2p_0}}{B_s^2 - 2p_0} \right) \frac{B_s^2 - 2p_0}{B_v^2 - 2p_0} - 2 \frac{p_0}{B_s^2 - 2p_0} (B_s B_v + 4p_0) \right) \right\}.$$
The intimidating formula above has been simplified by substituting first
\[
\Lambda = 1
\]
for the case of isotropic plasma \cite{1}, and then caused surprise. \(\beta\) of the parameter profile (the larger the parameter found, was the smaller, the steeper the radial pressure minimum times larger than the plasma radius and reapplying the
\[
\text{FullSimplify}
\]
and then substitution
\[
B_v \rightarrow \sqrt{2p_0 \left(1 + \frac{1}{x}\right)}, \quad B_s \rightarrow \sqrt{2p_0 \left(1 + \frac{1}{y}\right)}
\]
and reapplying the \text{FullSimplify} utility. The result was a reduction in the number of Appel hypergeometric functions from 6 to 2:
\[
P_4 = \frac{a_2^2}{2} \langle P_\perp \rangle_4 = \left\{ 5 \left( B_s^2 - 2p_0 \right) \left( B_v^2 + 8p_0 \right) F_1 \left( \frac{1}{4} : \frac{1}{2} : \frac{1}{2} : \frac{5}{4}; -\frac{2p_0}{B_s^2 - 2p_0}, -\frac{2p_0}{B_v^2 - 2p_0} \right) + \right.
+ 2p_0 \left( -2B_s^2 + 3B_v^2 + 8p_0 \right) F_1 \left( \frac{5}{4} : \frac{1}{2} : \frac{1}{2} : \frac{9}{4}; -\frac{2p_0}{B_s^2 - 2p_0}, -\frac{2p_0}{B_v^2 - 2p_0} \right) -
- 5 \sqrt{(B_s^2 - 2p_0) (B_v^2 - 2p_0)} (B_s B_v + 8p_0) \right\} / \left[ 50 B_s \sqrt{(B_s^2 - 2p_0) (B_v^2 - 2p_0)} \right]. \quad (B1c)
\]
Despite such significant simplifications, the calculation time for the \(k = 4\) options was several hundred times longer than for all other options combined.

Similarly, for the case \(k = 8\) we obtained
\[
P_8 = \frac{a_8^2}{2} \langle P_\perp \rangle_8 = \left\{ 9 \left( B_s^2 - 2p_0 \right) \left( B_v^2 + 16p_0 \right) F_1 \left( \frac{1}{8} : \frac{1}{2} : \frac{1}{2} : \frac{9}{8}; -\frac{2p_0}{B_s^2 - 2p_0}, -\frac{2p_0}{B_v^2 - 2p_0} \right) + \right.
+ 2p_0 \left( -4B_s^2 + 5B_v^2 + 16p_0 \right) F_1 \left( \frac{9}{8} : \frac{1}{2} : \frac{1}{2} : \frac{17}{8}; -\frac{2p_0}{B_s^2 - 2p_0}, -\frac{2p_0}{B_v^2 - 2p_0} \right) -
- 9 \sqrt{(B_s^2 - 2p_0) (B_v^2 - 2p_0)} (B_s B_v + 16p_0) \right\} / \left[ 162 B_s \sqrt{(B_s^2 - 2p_0) (B_v^2 - 2p_0)} \right], \quad (B1d)
\]
however we did not use this formula in the present paper.

For the radial profile \(k = \infty\) Wolfram \textit{Mathematica}® yields very simple result, which can easily be checked without system of computer integration:
\[
P_\infty = \frac{a_\infty^2}{2} \langle P_\perp \rangle_\infty = \frac{p_0}{B_s} \left( \sqrt{\frac{B_s^2 - 2p_0}{B_v^2 - 2p_0}} - 1 \right). \quad (B1e)
\]

**Appendix C: Minimal \(\Lambda\)**

The existence of a stability zone, even if in the limit of \(\beta \to 1\), at a very large gap between the conducting wall and the plasma, was previously discovered for the case of isotropic plasma \cite{1}, and then caused surprise. For \(\Lambda = 1.01\), the radius of the conducting wall \(r_w\) is 14 times larger than the plasma radius \(a\). In general, the minimum \(\Lambda_{\text{min}}\), for which the critical values of beta were found, was the smaller, the steeper the radial pressure profile (the larger the parameter \(k\)), the smoother the axial profile of the vacuum magnetic field (the smaller the parameter \(q\)), and the smaller the mirror ratio \(K\). In the previous paper \cite{1} we did not aim to calculate \(\Lambda_{\text{min}}\) exactly, but simply chose the minimum values of \(\Lambda\) from the available list of discrete values for which we calculated \(\beta_{\text{crit}}\).

In the present work, we tried to calculate \(\Lambda_{\text{min}}\) more accurately. The difficulty of such calculations is that the coefficients of the LoDestro equation are singular at the point \(z = 0\) in the limit \(\beta \to 1\). On the other hand, we showed above that the LoDestro equation is inapplicable in this limit, since the paraxial approximation is violated at \(\beta \to 1\). Nevertheless, we searched for the roots of the dispersion equation (53) with respect to \(\Lambda\) for four sufficiently large beta val-
Appendix D: Calculation of the roots of nonlinear equations

The search for the roots of Eq. (37) was initially carried out using the FindRoot utility built into the Wolfram Mathematica® library. FindRoot searches for the first root near an initial guess β_{start} passed to it. Success or failure in finding the root with this utility depends very much on luck in choosing β_{start}. Therefore, as an additional means of searching for roots, the RootSearch package was included, which was developed by Ted Ersek [49]. This package contains a utility of the same name that searches for all roots within a given interval.

In some intermediate version of our numerical code, a root found by the RootSearch utility was passed as β_{start} to the FindRoot utility to recheck the result of calculation of β_{crit}. In rare cases, when only one of the two utilities found a solution to Eq. (38), the code was analyzed in order to improve it. In particular, the formulas that Wolfram Mathematica® obtained when calculating the integrals a^2 and /b/ were improved. For example, in the formula (A1c) for a^2, the number of Appell hypergeometric functions was reduced from four to one, and in the formula (B1e) for /b/, from six to two. A side result of such code optimization was the “expulsion” of complex numbers in the intermediate calculations of the integral (38). Due to rounding errors for such numbers, it sometimes happened that the integral (38) took on a complex value with a small but finite imaginary part. In such cases, FindRoot skipped the root of Eq. (37). In cases where both FindRoot and RootSearch did not find a solution to Eq. (37), it was considered that the solution did not exist.

Unfortunately, after all this effort, the FindRoot and RootSearch bundle of utilities kept missing roots of Eqs. (37), (53), (56). The problem of losing roots was

| $K = 16$, $\beta = 0.999$ | $K = 8$, $\beta = 0.999$ | $K = 4$, $\beta = 0.999$ |
|-------------------------|-------------------------|-------------------------|
| $k$ | $q/R$ | 1.1 | 1.2 | 1.5 | 2 | 3 | 4 | 6 | 8 | $k$ | $q/R$ | 1.1 | 1.2 | 1.5 | 2 | 3 | 4 | $k$ | $q/R$ | 1.1 | 1.2 | 1.5 | 2 | 3 | 4 |
| 1 | 2 | 2.13 | 2.62 | 4.01 | 6.45 | 11.8 | 17.7 | 30.3 | 44.0 | 75.4 | 113. | 2 | 2.13 | 2.62 | 4.01 | 6.45 | 11.8 | 17.7 | 30.3 | 44.0 | 75.4 | 113. |
| 1 | 4 | 2.55 | 3.61 | 10.5 | N/F | N/F | N/F | N/F | N/F | N/F | N/F | 2 | 2.55 | 3.61 | 10.5 | N/F | N/F | N/F | N/F | N/F | N/F | N/F |
| 8 | 2.74 | 4.16 | 23.9 | N/F | N/F | N/F | N/F | N/F | N/F | N/F | N/F | 8 | 2.74 | 4.16 | 23.9 | N/F | N/F | N/F | N/F | N/F | N/F | N/F |
| 2 | 2 | 1.069 | 1.10 | 1.17 | 1.22 | 1.26 | 1.27 | 1.29 | 1.30 | 1.30 | 2 | 1.069 | 1.10 | 1.17 | 1.22 | 1.26 | 1.27 | 1.29 | 1.30 | 1.30 | 2 |
| 4 | 2 | 1.15 | 1.23 | 1.41 | 1.59 | 1.77 | 1.86 | 1.94 | 1.98 | 2.02 | 2.04 | 8 | 1.19 | 1.31 | 1.59 | 1.92 | 2.34 | 2.58 | 2.83 | 2.96 | 3.08 | 3.15 |
| 4 | 4 | 1.030 | 1.052 | 1.095 | 1.13 | 1.17 | 1.18 | 1.19 | 1.20 | 1.20 | 1.20 | 4 | 1.044 | 1.076 | 1.14 | 1.21 | 1.27 | 1.30 | 1.32 | 1.33 | 1.34 | 1.35 |
| $\infty$ | 4 | 1.0073 | 1.013 | 1.025 | 1.034 | 1.042 | 1.045 | 1.048 | 1.049 | 1.049 | 1.050 | 8 | 1.011 | 1.019 | 1.037 | 1.053 | 1.066 | 1.071 | 1.075 | 1.077 | 1.079 | 1.079 |

Table III: Minimal $\Lambda_{\text{min}}$ for the magnetic field (40) and anisotropic pressure model (13).
solved by applying the ideas formulated in the post of user with the nickname matheorem in the mathematica forum on the stackexchange.com portal. Using his code sample [50] and some of the code from Ted Yersek’s RootSearch [49] package, we wrote a XRootSearch package that replaced the FindRoot and RootSearch utilities.

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