On $n$-trivial Extensions of Rings

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Abstract. The notion of trivial extension of a ring by a module has been extensively studied and used in ring theory as well as in various other areas of research like cohomology theory, representation theory, category theory and homological algebra. In this paper we extend this classical ring construction by associating a ring to a ring $R$ and a family $M = (M_i)_{i=1}^n$ of $n$ $R$-modules for a given integer $n \geq 1$. We call this new ring construction an $n$-trivial extension of $R$ by $M$. In particular, the classical trivial extension will be just the 1-trivial extension. Thus we generalize several known results on the classical trivial extension to the setting of $n$-trivial extensions and we give some new ones. Various ring-theoretic constructions and properties of $n$-trivial extensions are studied and a detailed investigation of the graded aspect of $n$-trivial extensions is also given. We end the paper with an investigation of various divisibility properties of $n$-trivial extensions. In this context several open questions arise.

2010 Mathematics Subject Classification. primary 13A02, 13A05, 13A15, 13B99, 13E05, 13F05, 13F30; secondary 16S99, 17A99.

Key Words. trivial extension; $n$-trivial extension; graded rings; homogeneous ideal.
1 Introduction

Except for a brief excursion in Section 2, all rings considered in this paper are assumed to be commutative with an identity; in particular, $R$ denotes such a ring, and all modules are assumed to be unitary left modules. Of course left-modules over a commutative ring $R$ are actually $R$-bimodules with $mr := rm$. Let $\mathbb{Z}$ (resp., $\mathbb{N}$) denote the set of integers (resp., natural numbers). The set $\mathbb{N} \cup \{0\}$ will be denoted by $\mathbb{N}_0$. The ring $\mathbb{Z}/n\mathbb{Z}$ of the residues modulo an integer $n \in \mathbb{N}$ will be noted by $\mathbb{Z}_n$.

Recall that the trivial extension of $R$ by an $R$-module $M$ is the ring denoted by $R \ltimes M$ whose underlying additive group is $R \oplus M$ with multiplication given by $(r, m)(r', m') = (rr', rm' + mr')$. Since its introduction by Nagata in [40], the trivial extension of rings (also called idealization since it reduces questions about modules to ideals) has been used by many authors and in various contexts in order to produce examples of rings satisfying preassigned conditions (see, for instance, [9] and [38]).

It is known that the trivial extension $R \ltimes M$ is related to the following two ring constructions (see for instance [9, Section 2]):

**Generalized triangular matrix ring.** Let $\mathcal{R} := (R_i)_{i=1}^n$ be a family of rings and $\mathcal{M} := (M_{i,j})_{1 \leq i < j \leq n}$ be a family of modules such that for each $1 \leq i < j < k \leq n$, $M_{i,j}$ is an $(R_i, R_j)$-bimodule. Assume for every $1 \leq i < j < k \leq n$, there exists an $(R_i, R_k)$-bimodule homomorphism $M_{i,j} \otimes_{R_j} M_{j,k} \longrightarrow M_{i,k}$ denoted multiplicatively such that $(m_{i,j}m_{j,k})m_{k,l} = m_{i,j}(m_{j,k}m_{k,l})$ for every $(m_{i,j}, m_{j,k}, m_{k,l}) \in M_{i,j} \times M_{j,k} \times M_{k,l}$. Then the set consisting of matrices

$$
\begin{pmatrix}
R_1 & M_{1,2} & \cdots & \cdots & M_{1,n-1} & M_{1,n} \\
0 & R_2 & \cdots & \cdots & M_{2,n-1} & M_{2,n} \\
: & : & \ddots & \ddots & : & : \\
: & : & \ddots & \ddots & : & : \\
0 & 0 & \cdots & 0 & R_{n-1} & M_{n-1,n} \\
0 & 0 & \cdots & 0 & 0 & R_n
\end{pmatrix}
$$

,$$m_{i,i} \in R_i$$ and $m_{i,j} \in M_{i,j}$ ($1 \leq i < j \leq n$),
with the usual matrix addition and multiplication is a ring called a generalized (or formal) triangular matrix ring and denoted also by $T_n(\mathcal{R}, \mathcal{M})$ (see [16] and [17]). Then the trivial extension $R \ltimes M$ is naturally isomorphic to the subring of \[
abla \begin{pmatrix} R & M \\ 0 & R \end{pmatrix}\] consisting of matrices \[
abla \begin{pmatrix} r & m \\ 0 & r \end{pmatrix}\] where $r \in R$ and $m \in M$ (note that, since $R$ is commutative $rm = mr$).

**Symmetric algebra.** Recall that the symmetric algebra associated to $M$ is the graded ring quotient $S_R(M) := T_R(M)/H$ where $T_R(M)$ is the graded tensor $R$-algebra with $T^n_R(M) = M^\otimes n$ and $H$ is the homogeneous ideal of $T_R(M)$ generated by \{m \otimes n - n \otimes m | m, n \in M\}. Note that \[S_R(M) = \bigoplus_{n=0}^\infty S^n_R(M)\] is a graded $R$-algebra with $S^0_R(M) = R$ and $S^1_R(M) = M$ and, in general, $S^n_R(M)$ is the image of $T^n_R(M)$ in $S_R(M)$. Then $R \ltimes M$ and $S_R(M)/\bigoplus_{n \geq 2} S^n_R(M)$ are naturally isomorphic as graded $R$-algebras.

It is also worth recalling that when $M$ is a free $R$-module with a basis $B$, the trivial extension $R \ltimes M$ is also naturally isomorphic to $R[(X_b)_{b \in B}]/[(X_b)_{b \in B}]^2$ where $(X_b)_{b \in B}$ is a set of indeterminates over $R$. In particular, $R \ltimes R \cong R[X]/(X^2)$.

Inspired by the facts above, we introduce an extension of the classical trivial extension of rings to extensions associated to $n$ modules for any integer $n \geq 1$.

In the literature, particular cases of such extensions have been used to solve some open questions. In [10] the authors introduced an extension for $n = 2$ and they used it to give a counterexample of the so-called Faith conjecture. Also, in the case $n = 2$, an extension is introduced in [33] to give an example of a ring which has a non-self-injective injective hull with compatible multiplication. This gave a negative answer of a question posed by Osofsky. In [33] the author introduced and studied a particular extension for the case $n = 3$ to obtain a Galois covering for the enveloping algebras of trivial extension algebras of triangular algebras. Also, there is a master’s thesis [39] which introduced and studied factorization properties of an extension of the trivial extension of a ring by itself (i.e., self-idealization). In this paper, we introduce the following extension ring construction for an arbitrary integer $n \geq 1$.

Let $M = (M_i)_{i=1}^n$ be a family of $R$-modules and $\varphi = \{\varphi_{i,j}\}_{i+j \leq n}^{1 \leq i, j \leq n-1}$ be a family of bilinear maps such that each $\varphi_{i,j}$ is written multiplicatively:

$$\varphi_{i,j} : M_i \times M_j \longrightarrow M_{i+j}$$

$$(m_i, m_j) \mapsto \varphi_{i,j}(m_i, m_j) := m_i m_j.$$ In particular, if all $M_i$ are submodules of the same $R$-algebra $L$, then the bilinear maps, if they are not specified, are just the multiplication of $L$ (see examples in Section 2). The $n$-$\varphi$-trivial extension of $R$ by $M$ is the set denoted by $R \ltimes \varphi M_1 \ltimes \cdots \ltimes M_n$ or simply $R \ltimes \varphi M$ whose
underlying additive group is $R \oplus M_1 \oplus \cdots \oplus M_n$ with multiplication given by

$$(m_0, \ldots, m_n)(m'_0, \ldots, m'_n) = \left( \sum_{j+k=i} m_j m'_k \right)$$

for all $(m_i), (m'_i) \in R \ltimes M$. We could also define the product $\phi_{i,j} : M_i \times M_j \rightarrow M_{i+j}$ as an $R$-bimodule homomorphism $\overline{\phi}_{i,j} : M_i \otimes M_j \rightarrow M_{i+j}$; see Section 2 for details. For the sake of simplicity, it is convenient to set $M_0 = R$. In what follows, if no ambiguity arises, the $n$-$\varphi$-trivial extension of $R$ by $M$ will be simply called an $n$-trivial extension of $R$ by $M$ and denoted by $R \ltimes_n M_1 \ltimes \cdots \ltimes M_n$ or simply $R \ltimes_n M$.

While in general $R \ltimes_n M$ need not to be a commutative ring, in Section 2, we give conditions on the maps $\phi_{i,j}$ that force $R \ltimes_n M$ to be a ring. Unless otherwise stated, we assume the maps $\phi_{i,j}$ have been defined so that $R \ltimes_n M$ is a commutative associative ring with identity. Thus $R \ltimes_n M$ is a commutative ring with identity $(1, 0, \ldots, 0)$. Moreover, $R \ltimes_n M$ is naturally isomorphic to the subring of the generalized triangular matrix ring

$$
\begin{pmatrix}
R & M_1 & M_2 & \cdots & M_n \\
0 & R & M_1 & \cdots & M_{n-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & M_1 \\
0 & 0 & 0 & \cdots & R
\end{pmatrix}
$$

consisting of matrices where $r \in R$ and $m_i \in M_i$ for every $i \in \{1, \ldots, n\}$.

When, for every $k \in \{1, \ldots, n\}$, $M_k = S^k_R(M_1)$, the ring $R \ltimes_n M$ is naturally isomorphic to $S_R(M_1)/\bigoplus_{k \geq n+1} S^k_R(M_1)$. In particular, if $M_1 = F$ is a free $R$-module with a basis $B$, then the $n$-trivial extension $R \ltimes F \ltimes S_R^0(F) \ltimes \cdots \ltimes S_R^n(F)$ is also naturally isomorphic to $R[\{X_b\}_{b \in B}]/(\{X_b\}_{b \in B})^{n+1}$ where $\{X_b\}_{b \in B}$ is a set of indeterminates over $R$. Namely, when $F \cong R$,

$$(R \ltimes_n R \ltimes \cdots \ltimes R) \cong R[X]/(X^{n+1}).$$

\footnote{When $R$ is a field and $M_i = R$ for every $i \in \{1, \ldots, n\}$, these matrices are well-known as upper triangular Toeplitz matrices. In [39], the author used the same terminology for such matrices with entries in a commutative ring.}
Also, in [13], the trivial extension of a ring $R$ by an ideal $I$ is connected to the Rees algebra $\mathcal{R}_+$ associated to $R$ and $I$ which is precisely the following graded subring of $R[t]$ (where $t$ is an indeterminate over $R$):
\[ \mathcal{R}_+ := \bigoplus_{n \geq 0} I^n t^n. \]

Using [13, Lemma 1.2 and Proposition 1.3], we get, similar to [13, Proposition 1.4], the following diagram of extensions and isomorphisms of rings:

\[
\begin{array}{ccc}
R & \longrightarrow & \mathcal{R}_+/(I^{n+1} t^{n+1}) \\
\downarrow & \cong & \downarrow \\
R & \longrightarrow & R \ltimes_n I \ltimes I^2 \ltimes \cdots \ltimes I^n \longrightarrow R \ltimes_n R \ltimes \cdots \ltimes R
\end{array}
\]

In this paper, we study some properties of the ring $R \ltimes_n M$, extending well-known results on the classical trivial extension of rings. The paper is organized as follows.

In Section 2, we carefully define the $n$-trivial extension $R \ltimes_n M$ giving conditions on the maps $\varphi_{i,j}$ so that $R \ltimes_n M$ is actually a commutative ring with identity. Actually, we investigate the situation in greater generality where $R$ is not assumed to be commutative and $M_i$ is an $R$-bimodule for $i = 1, \ldots, n$. We end the section with a number of examples.

In Section 3, we investigate some ring-theoretic constructions of $n$-trivial extensions. We begin by showing that $R \ltimes_n M$ may be considered as a graded ring for three different grading monoids, in particular, $R \ltimes_n M$ may be considered as $\mathbb{N}_0$-graded ring or $\mathbb{Z}_{n+1}$-graded ring. We then show how $R \ltimes_n M$ behaves with respect to polynomials (Corollary 3.4) and power series (Theorem 3.5) extensions and localization (Theorem 3.7). In Theorem 3.9 we show that the $n$-trivial extension of a finite direct product of rings is a finite direct product of $n$-trivial extensions. We end with two results on inverse limits and direct limits of $n$-trivial extensions (Theorems 3.10 and 3.11).

In Section 4, we present some natural ring homomorphisms related to $n$-trivial extensions (see Proposition 4.3). Also, we study some basic properties of $R \ltimes_n M$. Namely, we extend the characterization of prime and maximal ideals of the classical trivial extension to $R \ltimes_n M$ (see Theorem 4.7). As a consequence, the nilradical and the Jacobson radical are determined (see Corollary 4.8). Finally, as an extension of [9, Theorems 3.5 and 3.7], the set of zero divisors, the set of units and the set of idempotents of $R \ltimes_n M$ are also characterized (see Proposition 4.9).

In Section 5, we investigate the graded aspect of $n$-trivial extensions. The motivation behind this study is that, in the classical case (where $n = 1$), the study of trivial extensions as $\mathbb{Z}_2$-graded rings has lead to some interesting properties (see [9]) and has shed more light on the structure of ideals of the trivial extensions. In Section 5, we extend some of results given in [9] and we
give some new ones. Namely, among other results, we characterize the homogeneous ideals of $R \ltimes_n M$ (Theorem 5.1) and we investigate some of their properties (Propositions 5.2 and 5.3). We devote the remainder of Section 5 to investigate the question “When is every ideal of a given class $\mathcal{I}$ of ideals of $R \ltimes_n M$ homogeneous?” (see the discussion after Proposition 5.3). In this context various results and examples are established.

Section 6 is devoted to some classical ring-theoretic properties. Namely, we characterize when $R \ltimes_n M$ is, respectively, Noetherian, Artinian, (Manis) valuation, Prüfer, chained, arithmetical, a $\pi$-ring, a generalized ZPI-ring or a PIR. We end the section with a remark on a question posed in [2] concerning $m$-Boolean rings.

Finally, in Section 7 we study divisibility properties of $n$-trivial extensions. We are mainly interested in showing how one could extend results on the classical trivial extension presented in [9, Section 5] to the context of $n$-trivial extensions.

## 2 The general $n$-trivial extension construction and some examples

The purpose of this section is to formally define the $n$-trivial extension ($n \geq 1$) $R \ltimes_n M_1 \ltimes \cdots \ltimes M_n$ where $R$ is a commutative ring with identity and each $M_i$ is an $R$-module, and to give some interesting examples of $n$-trivial extensions. However, to better understand the construction and the underlying multiplication maps $\varphi_{i,j} : M_i \times M_j \to M_{i+j}$, we begin in the more general context of $R$ being an associative ring (not necessarily commutative) with identity and the $M_i$’s being $R$-bimodules. Also, as there is a significant difference in the cases $n = 1$, $n = 2$ and $n \geq 3$, we handle these three cases separately.

Let $R$ be an associative ring with identity and $M_1, \ldots, M_n$ be unitary $R$-bimodules (in the case where $R$ is commutative we will always assume that $rm = mr$ unless stated otherwise).

**Case $n = 1$.**

$R \ltimes_1 M_1 = R \ltimes M_1 = R \oplus M_1$ is just the trivial extension with multiplication $(r, m)(r', m') = (rr', rm' + mr')$. Here $R \ltimes_1 M_1$ is an associative ring with identity where the associative and distributive laws follow from the ring and $R$-bimodule axioms. For $R$ commutative, we write $(r, m)(r', m') = (rr', rm' + r'm)$ as $r'm = mr'$. Now $R \ltimes_1 M_1$ is an $\mathbb{N}_0$-graded or a $\mathbb{Z}_2$-graded ring isomorphic to $T_R(M_1)/ \bigoplus_{i \geq 2} T^i_R(M_1)$ or $S_R(M_1)/ \bigoplus_{i \geq 2} S^i_R(M_1)$ and to the matrix ring representation mentioned in the introduction. Note that we could drop the assumption that $R$ has an identity and $M_1$ is unitary. We then get that $R \ltimes_1 M_1$ has an identity (namely $(1, 0)$) if and only if $R$ has an identity and $M_1$ is unitary.
Case $n = 2$.
Here $R \rtimes_2 M_1 \times M_2 = R \oplus M_1 \oplus M_2$ with coordinate-wise addition and multiplication

$$(r, m_1, m_2)(r', m_1', m_2') = (rr', rm_1' + m_1 r', rm_2' + m_1 m_1' + m_2 r')$$

where $m_1 m_1' := \varphi_{1,1}(m_1, m_1')$ with the map $\varphi_{1,1} : M_1 \times M_1 \to M_2$. We readily see that $R \rtimes_2 M_1 \times M_2$ satisfying the distributive laws is equivalent to $\varphi_{1,1}$ being additive in each coordinate. Since $R$ is assumed to be associative and $M_1$ and $M_2$ to be $R$-bimodules, $R \rtimes_2 M_1 \times M_2$ is associative precisely when $(rm_1)m_1' = r(m_1 m_1')$, $(m_1 r)m_1' = m_1(rm_1')$, and $(m_1 m_1')r = m_1(m_1'r)$ for $r \in R$ and $m_1, m_1' \in M_1$. This is equivalent to $\varphi_{1,1}(rm_1, m_1') = r\varphi_{1,1}(m_1, m_1')$, $\varphi_{1,1}(m_1 r, m_1') = \varphi_{1,1}(m_1, rm_1')$, and $\varphi_{1,1}(m_1, m_1')r = \varphi_{1,1}(m_1, m_1'r)$. For $R$-bimodules $M$, $N$ and $L$, we call a function $f : M \times N \to L$ a pre-product map if it is additive in each coordinate, is middle linear (i.e., $f(mr, m') = f(m, rm')$) and is left and right homogeneous (i.e., $f(m, m') = rf(m, m')$ and $f(m, mm') = f(m, m')r$). Note that a pre-product map $f : M \times N \to L$ uniquely corresponds to an $R$-bimodule homomorphism $\tilde{f} : M \otimes_R N \to L$ with $f(m, n) = \tilde{f}(m \otimes n)$.

Thus a pre-product map $\varphi_{1,1} : M_1 \times M_1 \to M_2$ corresponds to an $R$-bimodule homomorphism $\varphi_{1,1} : M_1 \otimes_R M_1 \to M_2$. So we could equivalently define $m_1 m_1' := \varphi_{1,1}(m_1 \otimes m_1')$.

So $R \rtimes_2 M_1 \times M_2$ is an (associative) ring with identity precisely when $\varphi_{1,1}$ is a pre-product map or $\varphi_{1,1} : M_1 \otimes_R M_1 \to M_2$ is an $R$-bimodule homomorphism. We can identify $R \rtimes_2 M_1 \times M_2$ with the matrix representation given in the introduction: $(r, m_1, m_2)$ is identified with

$$
\begin{pmatrix}
  r & m_1 & m_2 \\
  0 & r & m_1 \\
  0 & 0 & r
\end{pmatrix}
$$

But the relationship with a tensor algebra or symmetric algebra is more difficult. When $R \rtimes_2 M_1 \times M_2$ is an associative ring, we can define a ring epimorphism

$$
T_R(M_1 \oplus M_2)/ \bigoplus_{i \geq 3} T^i_R(M_1 \oplus M_2) \to R \rtimes_2 M_1 \times M_2
$$

by

$$(r, (m_1, m_2), \sum_{i=1}^{l}(m_{1,i}, m_{2,i}) \otimes (m'_{1,i}, m'_{2,i})) + \bigoplus_{i \geq 3} T^i_R(M_1 \oplus M_2) \mapsto (r, m_1, m_2 + \sum_{i=1}^{l} m_{1,i} m'_{1,i}).$$

For the commutative case, we get a similar ring epimorphism $S_R(M_1 \oplus M_2)/ \bigoplus_{i \geq 3} S^i_R(M_1 \oplus M_2) \to R \rtimes_2 M_1 \times M_2$.

For $R \rtimes_2 M_1 \times M_2$ to be a commutative ring with identity we need $R$ to be commutative with identity and $m_1 m_1' = m_1' m_1$ for $m_1, m_1' \in M_1$, or $\varphi_{1,1}(m_1, m_1') = \varphi_{1,1}(m_1', m_1)$. Thus for $R$
commutative, \( R \ltimes_2 M_1 \times M_2 \) is a commutative ring if and only if \( \varphi_{1,1} \) is a symmetric \( R \)-bilinear map, or equivalently, \( \tilde{\varphi}_{1,1}(m_1 \otimes m_1') = \tilde{\varphi}_{1,1}(m_1' \otimes m_1) \).

Case \( n \geq 3 \).

Here again \( R \) is an associative ring with identity and \( M_1, \ldots, M_n \) (\( n \geq 3 \)) are \( R \)-bimodules. So \( R \ltimes_n M_1 \times \cdots \times M_n = R \oplus M_1 \oplus \cdots \oplus M_n \) with coordinate-wise addition. Assume we have pre-product maps \( \varphi_{i,j} : M_i \times M_j \to M_{i+j} \), or equivalently, the corresponding \( R \)-bimodule homomorphism \( \tilde{\varphi}_{i,j} : M_i \otimes_R M_j \to M_{i+j} \) for \( 1 \leq i, j \leq n-1 \) with \( i + j \leq n \). As usual set

\[
m_i m_j := \varphi_{i,j}(m_i, m_j) = \tilde{\varphi}_{i,j}(m_i \otimes m_j)
\]

for \( m_i \in M_i \) and \( m_j \in M_j \). Setting \( R = M_0 \), we can write the multiplication in \( R \ltimes_n M_1 \times \cdots \times M_n \) as \( (m_0, \ldots, m_n)(m_0', \ldots, m_n') = (m_0'' \ldots, m_n'') \) where \( m_i'' = \sum_j m_j m_k' \). Then \( R \ltimes_n M_1 \times \cdots \times M_n \) satisfies the distributive laws because the maps \( \varphi_{i,j} \) are additive in each coordinate. So \( R \ltimes_n M_1 \times \cdots \times M_n \) is a not necessarily associative ring with identity \((1,0,\ldots,0)\) (see Example 2.2 for a case where \( R \ltimes_n M_1 \times \cdots \times M_n \) is not associative). Note that \( R \ltimes_n M_1 \times \cdots \times M_n \) is associative precisely when \( (m_i m_j m_k) = m_i (m_j m_k) \) for \( m_i \in M_i \), \( m_j \in M_j \) and \( m_k \in M_k \) with \( 1 \leq i, j, k \leq n-2 \) and \( i + j + k \leq n \). In terms of the pre-product maps, this says that \( \varphi_{i+j,k}(\varphi_{i,j}(m_i, m_j), m_k) = \varphi_{i,j+k}(m_i, \varphi_{j,k}(m_j, m_k)) \), or equivalently,

\[
\tilde{\varphi}_{i,j+k} \circ (\tilde{\varphi}_{i,j} \otimes id_{M_k}) = \tilde{\varphi}_{i,j+k} \circ (id_{M_i} \otimes \tilde{\varphi}_{j,k})
\]

where \( id_{M_l} \) is the identity map on \( M_l \) for \( l \in \{1, \ldots, n\} \). In other words, the diagram below commutes:

\[
\begin{array}{ccc}
M_i \otimes M_j \otimes M_k & \xrightarrow{id_{M_i} \otimes \tilde{\varphi}_{j,k}} & M_i \otimes M_{j+k} \\
\tilde{\varphi}_{i,j} \otimes id_{M_k} & \downarrow & \tilde{\varphi}_{i,j+k} \\
M_{i+j} \otimes M_k & \xrightarrow{\tilde{\varphi}_{i+j,k}} & M_{i+j+k}
\end{array}
\]

Let us call a family \( \{\varphi_{i,j}\}_{i+j \leq n} \) (or \( \{\tilde{\varphi}_{i,j}\}_{i+j \leq n} \)) of pre-product maps satisfying the previously stated associativity condition a family of product maps. So, when \( \{\varphi_{i,j}\}_{i+j \leq n} \) (or equivalently \( \{\tilde{\varphi}_{i,j}\}_{i+j \leq n} \)) is a family of product maps, \( R \ltimes_n M_1 \times \cdots \times M_n \) is an associative ring with identity. Further, for \( R \ltimes_n M_1 \times \cdots \times M_n \) to be a commutative ring with identity we need \( R \) to be commutative with identity and \( \varphi_{i,j}(m_i, m_j) = \varphi_{j,i}(m_j, m_i) \) for every \( 1 \leq i, j \leq n-1 \) with \( i + j \leq n \), or equivalently, \( \tilde{\varphi}_{i,j} = \tilde{\varphi}_{j,i} \circ \tau_{i,j} \) where \( \tau_{i,j} : M_i \otimes M_j \to M_j \otimes M_i \) is the ‘flip’ map defined by \( \tau_{i,j}(m_i \otimes m_j) = m_j \otimes m_i \) for every \( m_i \otimes m_j \in M_i \otimes M_j \). In other words, the diagram
below commutes:

\[
\begin{array}{ccc}
M_i \otimes M_j & \xrightarrow{\varphi_{i,j}} & M_{i+j} \\
\downarrow{n_{i,j}} & & \downarrow{\tilde{\varphi}_{i,j}} \\
M_j \otimes M_i & & 
\end{array}
\]

In this case, the family \( \{ \varphi_{i,j} \}_{i+j \leq n} \) (or \( \{ \tilde{\varphi}_{i,j} \}_{i+j \leq n} \)) will be called a family of commutative product maps. So, when \( R \) is commutative and \( \{ \varphi_{i,j} \}_{i+j \leq n} \) (or equivalently \( \{ \tilde{\varphi}_{i,j} \}_{i+j \leq n} \)) is a family of commutative product maps, \( R \ltimes_n M_1 \times \cdots \times M_n \) is a commutative ring with identity.

As in the case \( n = 2 \), when \( R \ltimes_n M_1 \times \cdots \times M_n \) is an (associative) ring with identity, we can identify \( R \ltimes_n M_1 \times \cdots \times M_n \) with the matrix representation given in the introduction: \( (r, m_1, ..., m_n) \) is identified with

\[
\begin{pmatrix}
0 & r & m_1 & m_2 & \cdots & m_n \\
0 & 0 & r & m_1 & \cdots & m_{n-1} \\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \cdots & m_1 \\
0 & 0 & 0 & 0 & \cdots & r
\end{pmatrix}
\]

Also, as in the case \( n = 2 \), when \( R \ltimes_n M_1 \times \cdots \times M_n \) is an associative ring, we can define a ring epimorphism \( T_R(M_1 \oplus \cdots \oplus M_n) / \oplus_{i \geq n+1} T^i_R(M_1 \oplus \cdots \oplus M_n) \rightarrow R \ltimes_n M_1 \times \cdots \times M_n \) and we have a similar result concerning the symmetric algebra when \( R \ltimes_n M_1 \times \cdots \times M_n \) is commutative.

**Remark 2.1**

1. Let \( R_1 \) and \( R_2 \) be two rings and \( H \) an \((R_1, R_2)\)-bimodule. It is well-known that every generalized triangular matrix ring is naturally isomorphic to the trivial extension of \( R_1 \times R_2 \) by \( H \) where the actions of \( R_1 \times R_2 \) on \( H \) are defined as follows: \( (r_1, r_2)h = r_1hr_2 \) and \( h(r_1, r_2) = hr_2 \) for every \((r_1, r_2) \in R_1 \times R_2 \) and \( h \in H \). Below we see that an observation on the product of two matrices of the generalized triangular matrix ring shows that this fact can be extended to \( n \)-trivial extensions.

Consider the generalized triangular matrix ring

\[
T_n(\mathcal{A}, \mathcal{M}) = \begin{pmatrix}
R_1 & M_{1,2} & \cdots & \cdots & M_{1,n-1} & M_{1,n} \\
0 & R_2 & \cdots & \cdots & M_{2,n-1} & M_{2,n} \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & R_{n-1} & M_{n-1,n} \\
0 & 0 & \cdots & 0 & 0 & R_n
\end{pmatrix}
\]
where \((R_i)_{i=1}^n\) is a family of rings and \((M_{i,j})_{1 \leq i < j \leq n}\) is a family of modules such that for each \(1 \leq i < j \leq n\), \(M_{i,j}\) is an \((R_i, R_j)\)-bimodule. Assume, for every \(1 \leq i < j < k \leq n\), there exists an \((R_i, R_k)\)-bimodule homomorphism
\[
M_{i,j} \otimes_{R_j} M_{j,k} \rightarrow M_{i,k}
\]
denoted multiplicatively such that
\[
(m_{i,j}m_{j,k})m_{k,l} = m_{i,j}(m_{j,k}m_{k,l})
\]
for every \((m_{i,j}, m_{j,k}, m_{k,l}) \in M_{i,j} \times M_{j,k} \times M_{k,l}\).
Consider the finite direct product of rings \(R = R_1 \times \cdots \times R_n\) and set, for \(2 \leq i \leq n\), 
\(M_i = M_{1,i} \times M_{2,i+1} \times \cdots \times M_{n-(i-1),n}\) (for \(i = n\), \(M_n = M_{1,n}\)). We need to define an action of \(R\) on each \(M_i\) and a family of product maps so that \(R \rtimes_{n-1} M_2 \rtimes \cdots \rtimes M_n\) is an \(n-1\)-trivial extension isomorphic to \(T_n(\mathcal{R}, \mathcal{M})\).
First, note that, for every matrix \(A = (a_{i,j})\) of \(T_n(\mathcal{R}, \mathcal{M})\) and for every \(2 \leq i \leq n\), the \(i\)-th diagonal above the main diagonal of \(A\) naturally corresponds to the following \((n - i + 1)\)-tuple \((a_{1,i}, a_{2,i+1}, \ldots, a_{n-(i-1),n})\) of \(M_i\). On the other hand, consider two matrices \(A = (a_{i,j})\) and \(B = (b_{i,j})\) of \(T_n(\mathcal{R}, \mathcal{M})\), and denote the product \(AB\) by \(C = (c_{i,j})\). Then using the above correspondence for \(2 \leq i \leq n\), the \(i\)-th diagonal above the main diagonal of \(C\) can be seen as the \((n - i + 1)\)-tuple
\[
c_{j,i+j-1} = \sum_{k=j}^{i+j-1} a_{j,k}b_{k,i+j-1}
\]
Then
\[
c_i = \left( \sum_{k=1}^{i} a_{j,k+j-1}b_{k+j-1,i+j-1} \right) j
\]
Thus the cases \(k = 1\) and \(k = i\) allow us to define the left and right actions of \(R\) on \(M_i\) as follows: For every \((r_i) \in R\) and \((m_{j,i+j-1}) \in M_i,
\[
(r_i)(m_{j,i+j-1}) := (r_jm_{j,i+j-1})_j
\]
and
\[
(m_{j,i+j-1})(r_i) := (m_{j,i+j-1}r_{i+j-1})_j.
\]
The other cases of $k$ can be used to define the product maps $M_k \times M_{i-k} \to M_i$ as follows: Fix $k$, $1 < k < i$, and consider $e_k = (e_{j,k+j-1})_{1 \leq j \leq n-k+1} \in M_k$ and $f_{i-k} = (f_{j,i-k+j-1})_{1 \leq j \leq n-i+k+1} \in M_{i-k}$. Then
\[ e_k f_{i-k} := (e_{j,k+j-1} f_{k+j-1,i+j-1})_{1 \leq j \leq n-i+1}. \]

Therefore, endowed with these products, $R \times_{n-1} M_2 \times \cdots \times M_n$ is an $n-1$-trivial extension naturally isomorphic to the generalized triangular matrix ring $T_n(\mathcal{A}, \mathcal{M})$.

2. It is known that the generalized triangular matrix ring $T_n(\mathcal{A}, \mathcal{M})$ can be seen as a generalized triangular $2 \times 2$ matrix ring. Namely, there is a natural ring isomorphism between $T_n(\mathcal{A}, \mathcal{M})$ and $T_2(S, N)$ where
\[ S = (T_{n-1}((R_i)_{i=1}^{n-1}), (M_{i,j})_{1 \leq i < j \leq n-1}, R_n) \]
and
\[ N = \begin{pmatrix} M_{1,n} \\ M_{2,n} \\ \vdots \\ M_{n-1,n} \end{pmatrix} \]

However, an $n$-trivial extension is not necessarily a 1-trivial extension. For that consider, for instance, the 2-trivial extension $S = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. One can check easily that $S$ cannot be isomorphic to any 1-trivial extension.

We end this section with a number of examples.

**Example 2.2** Suppose that $R$ is a commutative ring and consider $R \times R \times \cdots \times R$ ($n \geq 1$) with a family of product maps $\varphi_{i,j} : R \times R \to R$ where, for $k \in \{1, \ldots, n\}$, $e_k = (0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 in the $k+1$'th place.

For $n = 1$, $R \times_1 R \cong R[X]/(X^2)$.

Suppose, $n = 2$ and $e_1^2 = r_{1,1} e_1$. Then $R \times_2 R \times R \cong R[X,Y]/(X^2 - r_{1,1} Y, XY, Y^2)$ where $X$ and $Y$ are commuting indeterminates. So in the case where $r_{1,1} = 1$, we get $R \times_2 R \times R \cong R[X,Y]/(X^2 - Y, XY, Y^2) \cong R[X]/(X^2)$.

The case $n = 3$ is more interesting. Now, for $1 \leq i,j \leq 2$ with $i + j \leq 3$, $\varphi_{i,j} : R \times R \to R$ with $\varphi_{i,j}(r,s) = r \varphi_{i,j}(1,1)s$. Put $\varphi_{i,j}(1,1) = r_{i,j}$; so $(re_i)(se_j) = rr_{i,j} se_{i+j}$. Now, $R \times_3 R \times R \times R$ is commutative if and only if $e_1 e_2 = e_2 e_1$ or $r_{1,2} = r_{2,1}$. And $R \times_3 R \times R \times R$ is associative if and only if $(e_1 e_1) e_1 = e_1 (e_1 e_1)$ or $r_{1,1} r_{2,1} = r_{1,2} r_{1,1}$. Thus if $R \times_3 R \times R \times R$ is commutative, it is also associative. However, if $R$ is a commutative integral domain and $r_{1,1} \neq 0$, $R \times_3 R \times R \times R$

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\(^2\)We are indebted to J. R. García Rozas (Universidad de Almería, Spain) who pointed out this remark.
is associative if and only if it is commutative. Thus if we take \( R = \mathbb{Z}, r_{1,1} = 1, r_{1,2} = 1 \) and \( r_{2,1} = 2, R \bowtie_3 R \times R \times R \) is a non-commutative and non-associative ring.

For \( n = 4 \), the reader can easily check that \( R \times_4 R \times R \times R \) is commutative if and only if \( r_{i,j} = r_{j,i} \), for \( 1 \leq i, j \leq 3 \) with \( i + j \leq 4 \), and that \( R \times_4 R \times R \times R \) is associative if and only if \( r_{1,1}r_{2,1} = r_{1,2}r_{1,1}, r_{2,1}r_{3,1} = r_{2,2}r_{1,1}, r_{1,2}r_{3,1} = r_{1,3}r_{2,1}, r_{1,3}r_{2,1} = r_{1,3}r_{1,2} \). Thus if \( R \) is a commutative integral domain with \( r_{1,1} \neq 0 \), then \( r_{1,1}r_{2,1} = r_{1,2}r_{1,1} \) if and only if \( r_{2,1} = r_{1,2} \). So if \( r_{1,1} \neq 0 \) and \( r_{1,2} \neq 0 \), then \( r_{1,2}r_{3,1} = r_{1,3}r_{2,1} \) if and only if \( r_{3,1} = r_{1,3} \). Thus if \( r_{1,1} \neq 0 \) and \( r_{1,2} \neq 0 \), then \( R \bowtie_4 R \times R \times R \times R \) is associative and in this case \( R \times_4 R \bowtie_4 R \times R \times R \) is associative if and only if \( r_{1,1}r_{2,2} = r_{1,3}r_{1,2} \). Thus if three of the numbers \( r_{1,1}, r_{2,2}, r_{1,3} \) and \( r_{1,2} \) are given and nonzero, then there is only one possible choice for the remaining \( r_{i,j} \) for \( R \bowtie_4 R \times R \times R \times R \) to be associative. If we take \( R = \mathbb{Z} \) and \( r_{1,1} = 1, r_{2,1} = r_{1,2} = 2, r_{2,2} = 3 \) and \( r_{1,3} = r_{3,1} = 4 \), then the resulting ring is commutative but not associative.

For \( n \geq 5 \), the reader can easily write conditions on the \( r_{i,j} = \varphi_{i,j}(1,1) \) for \( R \bowtie_n R \times \cdots \times R \) to be commutative or associative.

**Example 2.3** Let \( R \) be a commutative ring and \( N_1, \ldots, N_n \) be \( R \)-submodules of an \( R \)-algebra \( T \) with \( N_iN_j \subseteq N_{i+j} \) for \( 1 \leq i, j \leq n-1 \) with \( i + j \leq n \). Then, using the multiplication from \( T \), \( R \bowtie_n N_1 \times \cdots \times N_n \) is a ring which is commutative if \( T \) is commutative. The following are some interesting special cases:

(a) Let \( R \) be a commutative ring and \( I \) an ideal of \( R \). Then \( R \bowtie_n I \times I^2 \times \cdots \times I^n \) is the quotient of the Rees ring \( R[I]/(I^{n+1}I^n) \) mentioned in the introduction.

(b) Let \( R \) be a commutative ring, \( T \) an \( R \)-algebra, and \( J_1 \subseteq \cdots \subseteq J_n \) ideals of \( T \). Then \( R \bowtie_n J_1 \times \cdots \times J_n \) is an example of \( n \)-trivial extension since \( J_iJ_j \subseteq J_i \subseteq J_{i+j} \) for \( i+j \leq n \). For example, we could take \( R \bowtie_2 XR[X] \times R[X] \).

(c) Suppose that \( R_1 \subseteq \cdots \subseteq R_n \) are \( R \)-algebras where \( R \) is a commutative ring. Let \( N \) be an \( R_{n-1} \)-submodule of \( R_n \) (in particular, we could take \( N = R_n \)). Then \( R \bowtie_n R_1 \times \cdots \times R_{n-1} \times N \) with the multiplication induced by \( R_n \) is a ring. For example, we could take \( \mathbb{Z} \bowtie_3 \mathbb{Q} \times \mathbb{R} \times \mathbb{N} \) where \( N \) is the \( \mathbb{R} \)-submodule of \( \mathbb{R}[X] \) of polynomials of degree \( \leq 5 \).

**Example 2.4** Let \( R \) be a commutative ring and \( M \) an \( R \)-module. Let \( S := R \bowtie_n R \times \cdots \times R \times M \) with \( \varphi_{i,j} : R \times R \rightarrow R \) the usual ring product in \( R \) for \( i + j \leq n - 1 \), but, for \( i + j = n \) and \( i, j \geq 1 \), \( \varphi_{i,j} \) is the zero map. So

\[
(r_0, \ldots, r_{n-1}, m_n)(r'_0, \ldots, r'_{n-1}, m'_n) = (r_0r'_0, r_0r'_1 + r_1r'_0, \ldots, r_0r'_{n-1} + \cdots + r_{n-1}r'_0, r_0m'_0 + r'_0m_n).
\]

Then \( S \cong \mathbb{R}[X]/(X^n) \times M \) where \( M \) is considered as an \( \mathbb{R}[X]/(X^n) \)-module with \( f(X)m = f(0)m \).
n-Trivial Extensions of Rings

Example 2.5 Let $R$ be a commutative ring and $T$ an $R$-algebra. Let $J_1 \subseteq \cdots \subseteq J_n$ be ideals of $T$. Then take $R \ltimes_n T/J_1 \ltimes \cdots \ltimes T/J_n$ where the product $T/J_i \times T/J_j \to T/J_{i+j}$ is given by $(t_i + J_i) (t_j + J_j) = t_i t_j + J_{i+j}$ for $i + j \leq n$.

Example 2.6 Let $R$ be a commutative ring, $N_1, \ldots, N_{n-1}$ ideals of $R$ and $N_n = Ra$ a cyclic $R$-module. Then consider $R \ltimes_n N_1 \ltimes \cdots \ltimes N_n$ where the products $N_i \times N_j \to N_{i+j}$ are the usual products for $R$ when $i + j \leq n - 1$, and for $i + j = n$ define $n_i n_j = n_i n_j a$.

In what follows we adopt the following notation.

Notation. Unless specified otherwise, $R$ denotes a non-trivial ring and, for an integer $n \geq 1$, $M = (M_i)_{i=1}^n$ is a family of $R$-modules with bilinear maps as indicated in the definition of the $n$-trivial extension defined so that $R \ltimes_n M$ is a commutative associative ring with identity. So $R \ltimes_n M$ is indeed a commutative ring with identity. Let $S$ be a nonempty subset of $R$ and $N = (N_i)_{i=1}^n$ be a family of ideals such that, for every $i$, $N_i \subseteq M_i$. Then as a subset of $R \ltimes_n M$, $S \times N_1 \times \cdots \times N_n$ will be denoted by $S \ltimes_n N_1 \times \cdots \times N_n$ or simply $S \ltimes_n N$.

3 Some ring-theoretic constructions of $n$-trivial extensions

In this section we investigate some ring-theoretic constructions of $n$-trivial extensions. First we investigate the graded aspect of $n$-trivial extensions.

For the convenience of the reader we recall the definition of graded rings. Let $\Gamma$ be a commutative additive monoid. Recall that a ring $S$ is said to be a $\Gamma$-graded ring, if there is a family of subgroups of $S$, $(S_\alpha)_{\alpha \in \Gamma}$, such that $S = \bigoplus_{\alpha \in \Gamma} S_\alpha$ as an abelian group, with $S_\alpha S_\beta \subseteq S_{\alpha + \beta}$ for all $\alpha, \beta \in \Gamma$. And an $S$-module $N$ is said to be $\Gamma$-graded if $N = \bigoplus_{\alpha \in \Gamma} N_\alpha$ (as an abelian group) and $S_\alpha N_\beta \subseteq N_{\alpha + \beta}$ for all $\alpha, \beta \in \Gamma$. Note that $S_0$ is a subring of $S$ and each $N_\alpha$ is an $S_0$-module. When $\Gamma = N_0$, a $\Gamma$-graded ring (resp., a $\Gamma$-graded module) will simply be called a graded ring (resp., a graded module). See, for instance, [11] and [12] for more details about graded rings although [11] deals with group graded rings.

Now, $R \ltimes_n M_1 \ltimes \cdots \ltimes M_n = R \oplus M_1 \oplus \cdots \oplus M_n$ may be considered as a graded ring for the following three different grading monoids:

**As an $\mathbb{N}_0$-graded ring.** In this case we set $M_k = 0$ for all $k \geq n + 1$ and we extend the definition of $\varphi_{i,j}$ to all $i,j \geq 0$ as follows: For $i$ or $j = 0$,

$$
\varphi_{0,j}: \quad R \times M_j \to M_j \quad (r, m_j) \mapsto \varphi_{0,j}(r, m_j) := rm_j \quad \text{and} \quad \varphi_{i,0}: \quad M_i \times R \to M_i \quad (m_i, r) \mapsto \varphi_{i,0}(m_i, r) := m_i r
$$
are just the multiplication of \( R \) when \( i = j = 0 \) or the \( R \)-actions on \( M_j \) and \( M_i \) respectively when \( j > 0 \) and \( i > 0 \) respectively. For \( i, j \geq 0 \) such that \( i + j \geq n + 1 \), we define \( \varphi_{i,j} : M_i \times M_j \to M_{i+j} \) by \( \varphi_{i,j}(m_i, m_j) = 0 \) for all \((m_i, m_j) \in M_i \times M_j\). Thus \( R \otimes_n M_1 \times \cdots \times M_n \) is an \( \mathbb{N}_0 \)-graded ring \( \bigoplus_i R_i \) where \( R_0 = R \) and \( R_i = M_i \) for \( i \in \mathbb{N} \).

**As a \( \mathbb{Z}_{n+1} \)-graded ring.** In this case we consider, for \( a \in \mathbb{Z} \), the least nonnegative integer \( \hat{a} \) with \( \hat{a} \equiv a \text{mod}(n+1) \), and we set \( M_{\hat{a}} := M_a \). Then for \( a, b \in \mathbb{Z} \), we define maps \( \varphi_{\hat{a},\hat{b}} : M_{\hat{a}} \times M_{\hat{b}} \to M_{\hat{a}+\hat{b}} \) by \( \varphi_{\hat{a},\hat{b}} = \varphi_{a,b} \) when \( \hat{a} + \hat{b} \leq n \) and \( \varphi_{\hat{a},\hat{b}} \) to be the zero map when \( \hat{a} + \hat{b} > n \). Then \( R \otimes_n M_1 \times \cdots \times M_n \) is a \( \mathbb{Z}_{n+1} \)-graded ring \( R_{\hat{0}} \oplus R_{\hat{1}} \oplus \cdots \oplus R_{\hat{2n}} \) where \( R_{\hat{0}} = R \) and \( R_{\hat{1}} = M_1 \) for \( a = 1, \ldots, n \).

Note that each of these gradings have the same set of homogeneous elements.

We have observed that \( R \otimes_n M_1 \times \cdots \times M_n \) is an \( \mathbb{N}_0 \)-graded ring \( \bigoplus_i R_i \) where \( R_0 = R \), \( R_i = M_i \) for \( i = 1, \ldots, n \) and \( R_i = 0 \) for \( i > n \). So \( R \otimes_n M_1 \times \cdots \times M_n \) is a graded ring isomorphic to \( \bigoplus_{i=0}^{\infty} R_i \). The following result presents the converse implication. Namely, it shows that the \( n \)-trivial extensions can be realised as quotients of graded rings.

**Proposition 3.1** Let \( \bigoplus_i S_i \) be an \( \mathbb{N}_0 \)-graded ring and \( m \in \mathbb{N} \). Then \( S_0 \times_m S_1 \times \cdots \times S_m \) with the product induced by \( \bigoplus_i S_i \) is naturally an \( \mathbb{N}_0 \)-graded ring isomorphic to \( \bigoplus_i S_i \).  

**Proof.** Obvious. \( \blacksquare \)

The following result presents a particular case of Proposition 3.1.

**Proposition 3.2** For an \( R \)-module \( N \), we have the following two natural ring isomorphisms:

\[
T_R(N)/\bigoplus_{i \geq n+1} T_R^i(N) \cong R \otimes_n N \times T_R^1(N) \times \cdots \times T_R^n(N), \quad \text{and}
\]

\[
S_R(N)/\bigoplus_{i \geq n+1} S_R^i(N) \cong R \otimes_n N \times S_R^1(N) \times \cdots \times S_R^n(N).
\]

Moreover, suppose that \( N \) is a free \( R \)-module with a basis \( B \), then \( R \otimes_n N \times S_R^1(N) \times \cdots \times S_R^n(N) \) is (graded) isomorphic to \( R[[X_b]_{b \in B}]/(X_b)_{b \in B}^{n+1} \).

In particular, \( R \otimes_n R \times \cdots \times R \) with the natural maps is isomorphic to \( R[X]/(X^{n+1}) \).

**Proof.** Obvious. \( \blacksquare \)
Our next result shows that the n-trivial extension of a graded ring by graded modules has a natural grading. It is an extension of [9 Theorem 4.5].

**Theorem 3.3** Let $\Gamma$ be a commutative additive monoid. Assume that $R = \bigoplus_{\alpha \in \Gamma} R_{\alpha}$ is $\Gamma$-graded and $M_i = \bigoplus_{\alpha \in \Gamma} M_{\alpha}^i$ is $\Gamma$-graded as an $R$-module for every $i \in \{1, \ldots, n\}$, such that $\varphi_{i,j}(M_{\alpha}^i, M_{\beta}^j) \subseteq M_{\alpha+i+\beta}^{i+j}$. Then $R \ltimes_n M_1 \ltimes \cdots \ltimes M_n$ is a $\Gamma$-graded ring with $(R \ltimes_n M_1 \ltimes \cdots \ltimes M_n)_{\alpha} = R_{\alpha} \oplus M_{\alpha}^1 \oplus \cdots \oplus M_{\alpha}^n$.

**Proof.** Similar to the proof of [9 Theorem 4.5].

In the case where $R$ is either a polynomial ring or a Laurent polynomial ring we get the following result in which the first assertion is an extension of [9 Corollary 4.6 (1)].

**Corollary 3.4** The following statements are true.

1. $(R \ltimes_n M_1 \ltimes \cdots \ltimes M_n)[X_\alpha] \cong R[X_\alpha] \ltimes_n M_1[X_\alpha] \ltimes \cdots \ltimes M_n[X_\alpha] \cong R_{\alpha} \oplus M_{\alpha}^1 \oplus \cdots \oplus M_{\alpha}^n$ for any set of indeterminates $\{X_\alpha\}$ over $R$.

2. $(R \ltimes_n M_1 \ltimes \cdots \ltimes M_n)[X_\alpha^\pm] \cong R[X_\alpha^\pm] \ltimes_n M_1[X_\alpha^\pm] \ltimes \cdots \ltimes M_n[X_\alpha^\pm] \cong R_{\alpha} \oplus M_{\alpha}^1 \oplus \cdots \oplus M_{\alpha}^n$ for any set of indeterminates $\{X_\alpha\}$ over $R$.

Also, as in the classical case, we get the related (but not graded) power series case. It is a generalization of [9 Corollary 4.6 (2)]. First recall that, for a given set of analytic indeterminates $\{X_\alpha\}_{\alpha \in \Lambda}$ over $R$, we can consider three types of power series rings (see [44] for further details about generalized power series rings):

$$R[[\{X_\alpha\}_{\alpha \in \Lambda}]]_1 \subseteq R[[\{X_\alpha\}_{\alpha \in \Lambda}]]_2 \subseteq R[[\{X_\alpha\}_{\alpha \in \Lambda}]]_3.$$

Here

$$R[[\{X_\alpha\}_{\alpha \in \Lambda}]]_1 = \bigcup \{R[[\{X_{\alpha_1}, \ldots, X_{\alpha_\ell}\}]](\alpha_1, \ldots, \alpha_\ell) \subseteq \Lambda\},$$

$$R[[\{X_\alpha\}_{\alpha \in \Lambda}]]_2 = \{\sum_{i=0}^{\infty} f_i | f_i \in R[[\{X_\alpha\}_{\alpha \in \Lambda}]] is homogeneous of degree i\} \quad \text{and}$$

$$R[[\{X_\alpha\}_{\alpha \in \Lambda}]]_3 = \{\sum_{i=0}^{\infty} f_i | f_i is a possibly infinite sum of monomials of degree i with at most one monomial of the form r_{\alpha_1, \ldots, \alpha_n} X_{\alpha_1}^{i_1} \cdots X_{\alpha_n}^{i_n} for each set \{\alpha_1, \ldots, \alpha_n\} with i_1 + \cdots + i_n = i\}.$$

More generally, given a partially ordered additive monoid $(S, +, \leq)$, the generalized power series ring $R[[X, S^{\leq}]]$ consists of all formal sums $f = \sum_{s \in S} a_s X^s$ where $\text{supp}(f) = \{s \in S | a_s \neq 0\}$ is Artinian and narrow (i.e., has no infinite family of incomparable elements) where addition and multiplication are carried out in the usual way. If $\Lambda$ is a well-ordered set, $S = \bigoplus_{\lambda \in \Lambda} N_0$ and $\leq$ is the reverse lexicographic order on $S$, then $R[[X, S^{\leq}]] \cong R[[\{X_\alpha\}]]_3$.

Note, that in a similar manner we can define three types of power series over a module. The routine proof of the following theorem is left to the reader.
**Theorem 3.5** 1. Let \( \{X_\alpha\}_{\alpha \in \Lambda} \) be a set of analytic indeterminates over \( R \). Then, for \( i = 1, 2, 3 \),

\[
(R \times_n \times M_1 \times \cdots \times M_n)[\{(X_\alpha)_{\alpha \in \Lambda}\}]_i \cong R[[\{(X_\alpha)_{\alpha \in \Lambda}\}]_i \times_n M_1[[\{(X_\alpha)_{\alpha \in \Lambda}\}]_i \times \cdots \times M_n[[\{(X_\alpha)_{\alpha \in \Lambda}\}]_i].
\]

2. Let \((S, +, \preceq)\) be a partially ordered additive monoid. Then

\[
(R \times_n \times M_1 \times \cdots \times M_n)[[X, S \preceq]] \cong R[[X, S \preceq]] \times_n M_1[[X, S \preceq]] \times \cdots \times M_n[[X, S \preceq]].
\]

Now, we give, as an extension of [9, Theorem 4.1], the following result which investigates the localization of an \( n \)-trivial extension. For this we need the following technical lemma.

**Lemma 3.6** For every \((m_i) \in R \times_n M\) and every \( k \in \{1, \ldots, n\} \),

\[
(m_0, 0, \ldots, 0, m_k, m_{k+1}, \ldots, m_n)(m_0, 0, \ldots, 0, -m_k, 0, \ldots, 0) = (m_0^2, 0, \ldots, 0, e_{k+1}, \ldots, e_n)
\]

where \( e_l = m_0 m_l - m_k m_{l-k} \) for every \( l \in \{k + 1, \ldots, n\} \). Consequently, there is an element \((f_i)\) of \( R \times_n M\), such that

\[
(m_i)(f_i) = (m_0^{2^n}, 0, \ldots, 0).
\]

We will denote the element \((f_i)\) in Lemma 3.6 by \((\tilde{m}_i)\) so \((m_i)(\tilde{m}_i) = (m_0^{2^n}, 0, \ldots, 0)\).

**Theorem 3.7** Let \( S \) be a multiplicatively closed subset of \( R \) and \( N = (N_i) \) be a family of \( R \)-modules where \( N_i \) is a submodule of \( M_i \) for each \( i \in \{1, \ldots, n\} \) and \( N_i N_j \subseteq N_{i+j} \) for every \( 1 \leq i, j \leq n-1 \) and \( i + j \leq n \). Then the set \( S \times_n N \) is a multiplicatively closed subset of \( R \times_n M \) and we have a ring isomorphism

\[
(R \times_n M)_{S \times_n N} \cong R_S \times_n M_S
\]

where \( M_S = (M_i S) \).

**Proof.** It is trivial to show that \( S \times_n N \) is a multiplicatively closed subset of \( R \times_n M \). Now in order to show the desired isomorphism, we need to make, as done in the proof of [9, Theorem 4.1 (1)], the following observation: Let \((m_i) \in R \times_n M\) and \((s_i) \in S \times_n N\). Then using the notation of Lemma 3.6,

\[
\frac{(m_i)}{(s_i)} = \frac{(m_i)(\tilde{s}_i)}{(S_0, 0, \ldots, 0)} = \frac{(m'_i)}{(S_0, 0, \ldots, 0)}
\]

where \((m'_i) = (m_i)(\tilde{s}_i)\) and \( S_0 = S_0^{2^n} \). Then the map

\[
f : \frac{(m_i)}{(s_i)} \mapsto \frac{(m'_i)}{(S_0, m'_i, \ldots, m'_i)}
\]

is the desired isomorphism.
As a simple but important particular case of Theorem 3.7, we get the following result which extends [9, Theorem 4.1 and Corollary 4.7]. In Theorem 4.7, we will show that if $P$ is a prime ideal of $R$, then $P \triangleleft_n M$ is a prime ideal of $R \triangleleft_n M$. This fact is used in the next result to show that the localization of an $n$-trivial extension at a prime ideal is isomorphic to an $n$-trivial extension. In what follows, we use $T(A)$ to denote the total quotient ring of a ring $A$. In Proposition 4.9, we will prove that $S \triangleleft_n M$, where $S = R - (Z(R) \cup Z(M_1) \cup \cdots \cup Z(M_n))$, is the set of all regular elements of $R \triangleleft_n M$. Thus $T(R \triangleleft_n M) = (R \triangleleft_n M)_{S \triangleleft_n M}$.

**Corollary 3.8** The following assertions are true.

1. Let $P$ be a prime ideal of $R$. Then we have a ring isomorphism

   $$(R \triangleleft_n M)_{P \triangleleft_n M} \cong R_P \triangleleft_n M_P$$

   where $M_P = (M_{iP})$.

2. We have a ring isomorphism

   $$T(R \triangleleft_n M) \cong R_S \triangleleft_n M_S$$

   where $S = R - (Z(R) \cup Z(M_1) \cup \cdots \cup Z(M_n))$.

3. For an indeterminate $X$ over $R$, we have a ring isomorphism

   $$(R \triangleleft_n M_1 \times \cdots \times M_n)(X) \cong R(X) \triangleleft_n M_1(X) \times \cdots \times M_n(X).$$

**Proof.** All the proofs are similar to the corresponding ones for the classical case. ■

Our next result generalizes [9, Theorem 4.4]. It shows that the $n$-trivial extension of a finite direct product of rings is a finite direct product of $n$-trivial extensions. For the reader’s convenience we recall here some known facts on the structure of modules over a finite direct product of rings. Let $R = \prod_{i=1}^s R_i$ be a finite direct product of rings where $s \in \mathbb{N}$. For $j \in \{1, ..., s\}$, we set $\bar{R}_j := 0 \times \cdots \times 0 \times R_j \times 0 \times \cdots \times 0$ and, for an $R$-module $N$, $N_j := \bar{R}_j N$. Then $N_j$ is a submodule of $N$ and we have $N = N_1 \oplus \cdots \oplus N_s$. Namely, every element $x$ in $N$ can be written in the form $x = x_1 + \cdots + x_s$ where $x_j = e_j x \in N_j$ for every $j \in \{1, ..., s\}$ (here $e_j = (0, ..., 0, 1, 0, ..., 0)$ with 1 in the $j$’th place). Note that each $N_j$ is also an $\bar{R}_j$-module and $N_1 \times \cdots \times N_s$ is an $R$-module isomorphic to $N$ via the following $R$-isomorphism:

$$\sum_{i=1}^s e_j x = x \quad \mapsto \quad (e_1 x_1, \ldots, e_s x_s) \quad \text{and} \quad (y_1, \ldots, y_s) \quad \mapsto \quad \sum y_j$$
Now, consider the family of commutative product maps \( \varphi = \{ \varphi_{i,j} \}_{i+j \leq n} \) and define the following maps:

\[
\varphi_{j,i,k} : M_{j,i} \times M_{j,k} \rightarrow M_{j,i+k} \\
(m_{j,i}, m_{j,k}) \mapsto \varphi_{j,i,k}(m_{j,i}, m_{j,k}) = e_j \varphi_{i,k}(m_{j,i}, m_{j,k})
\]

where \( M_{j,i} := \overline{R_j} M_i \) for \( j \in \{1, ..., s\} \) and \( i \in \{1, ..., n\} \). It is easily checked that, for every \( j \in \{1, ..., s\} \), \( \varphi_j = \{ \varphi_{j,i,k} \}_{i+k \leq n} \) is a family of commutative product maps and \( R_j \ltimes \varphi_j M_{j,1} \ltimes \cdots \ltimes M_{j,n} \) is a \( n \)-\( \varphi_j \)-trivial extension. Furthermore,

\[
\varphi_{i,k} : M_i \times M_k \rightarrow M_{i+k} \\
(m_i, m_k) \mapsto \varphi_{i,k}(m_i, m_k) = \sum_{j=1}^{s} \varphi_{j,i,k}(m_{j,i}, m_{j,k}).
\]

With this notation in mind, we are ready to give the desired result.

**Theorem 3.9** Let \( R = \prod_{i=1}^{s} R_i \) be a finite direct product of rings where \( s \in \mathbb{N} \). Then

\[
R \ltimes \varphi M_1 \ltimes \cdots \ltimes M_n \cong (R_1 \ltimes \varphi_1 M_{1,1} \ltimes \cdots \ltimes M_{1,n}) \times \cdots \times (R_s \ltimes \varphi_s M_{s,1} \ltimes \cdots \ltimes M_{s,n}).
\]

**Proof.** It is easily checked that the map \( (r, m_1, ..., m_n) \mapsto ((r_j, m_{j,1}, ..., m_{j,n}))_{1 \leq j \leq s} \) is an isomorphism. \( \square \)

We end this section with two results which investigate the inverse limit and direct limit of a system of \( n \)-trivial extensions. Namely, we show that, under some conditions, the inverse limit or direct limit of a system of \( n \)-trivial extensions is isomorphic to an \( n \)-trivial extension. The inverse limit case is a generalization of [9, Theorem 4.11].

Let \( \Gamma \) be a directed set and \( \{ M_{\alpha}; f_{\alpha\beta} \} \) be an inverse system of abelian groups over \( \Gamma \) (so for \( \alpha \leq \beta \), \( f_{\alpha\beta} : M_{\beta} \rightarrow M_{\alpha} \)). We know that the inverse limit \( \overleftarrow{\lim}_{\alpha} M_{\alpha} \) is isomorphic to the following subset of the direct product \( \prod_{\alpha} M_{\alpha} \):

\[
M_{\infty} := \{(x_{\alpha})_{\alpha \in \Gamma} | \lambda \leq \mu \Rightarrow x_\lambda = f_{\lambda\mu}(x_\mu)\}.
\]

In the next result, by \( \overleftarrow{\lim}_{\alpha} M_{\alpha} \) we mean exactly the set \( M_{\infty} \).

**Theorem 3.10** Let \( \Gamma \) be a directed set and \( n \geq 1 \) be an integer. Consider a family of inverse systems \( \{ M_{i,\alpha}; f_{i,\alpha,\beta} \} \) over \( \Gamma \) (for \( i \in \{0, ..., n\} \)) which satisfy the following conditions:

1. For every \( \alpha \in \Gamma \), \( M_{0,\alpha} = R_\alpha \) is a ring,
2. For every $\alpha \in \Gamma$ and every $i \in \{1, \ldots, n\}$, $M_{i,\alpha}$ is an $R_\alpha$-module, and

3. For every $\alpha \in \Gamma$, $R_\alpha \ltimes_n M_{1,\alpha} \ltimes \cdots \ltimes M_{n,\alpha}$ is an $n$-trivial extension with a family of commutative product maps:

$$\varphi_{i,j,\alpha} : M_{i,\alpha} \times M_{j,\alpha} \to M_{i+j,\alpha}$$

which satisfy, for every $\alpha \leq \beta$,

$$\varphi_{i,j,\alpha}(f_{i,\beta,\alpha}(m_{i,\beta}), f_{j,\beta,\alpha}(m_{j,\beta})) = f_{i+j,\beta,\alpha}(\varphi_{i,j,\beta}(m_{i,\beta}, m_{j,\beta})).$$

Then $\lim \leftarrow R_\alpha \ltimes_n \lim \leftarrow M_{1,\alpha} \ltimes \cdots \ltimes \lim \leftarrow M_{n,\alpha}$ is an $n$-trivial extension with the following family of well-defined commutative product maps:

$$\varphi_{i,j,\alpha} : \lim \leftarrow M_{i,\alpha} \times \lim \leftarrow M_{j,\alpha} \to \lim \leftarrow M_{i+j,\alpha}$$

$$((m_{i,\alpha}), (m_{j,\alpha})) \mapsto (\varphi_{i,j,\alpha}(m_{i,\alpha}, m_{j,\alpha})).$$

Moreover, there is a natural ring isomorphism:

$$\lim \leftarrow (R_\alpha \ltimes_n M_{1,\alpha} \ltimes \cdots \ltimes M_{n,\alpha}) \cong \lim \leftarrow R_\alpha \ltimes \lim \leftarrow M_{1,\alpha} \ltimes \cdots \ltimes \lim \leftarrow M_{n,\alpha}.$$ 

**Proof.** The result follows using a standard argument. \qed

Let $\Gamma$ be a directed set and $\{M_\gamma; f_\gamma\}$ a direct system of abelian groups over $\Gamma$ (so for $\gamma \leq \lambda$, $f_\gamma : M_\gamma \to M_\lambda$). We know that the direct limit $\lim \leftarrow M_\gamma$ is isomorphic to $\bigoplus M_\gamma / S$ where $S$ is generated by all elements $\lambda_\gamma(f_\alpha(a)) - \lambda_\alpha(a)$ where $\alpha \leq \beta$ and $\lambda_\alpha : M_\lambda \to \bigoplus M_\gamma$ is the natural inclusion map for $\lambda \in \Gamma$. Since $\Gamma$ is directed, every element of $\bigoplus M_\gamma / S$ has the form $\lambda_\alpha(a) + S$ for some $\alpha \in \Gamma$ and $a_\alpha \in M_\alpha$.

**Theorem 3.11** Let $\Gamma$ be a directed set and $n \geq 1$ be an integer. Consider a family of direct systems $\{M_{i,\gamma}; f_{i,\gamma,\beta}\}$ over $\Gamma$ (for $i \in \{0, \ldots, n\}$) which satisfy the following conditions:

1. For every $\alpha \in \Gamma$, $M_{0,\alpha} = R_\alpha$ is a ring,

2. For every $\alpha \in \Gamma$ and every $i \in \{1, \ldots, n\}$, $M_{i,\alpha}$ is an $R_\alpha$-module, and

3. For every $\alpha \in \Gamma$, $R_\alpha \ltimes_n M_{1,\alpha} \ltimes \cdots \ltimes M_{n,\alpha}$ is an $n$-trivial extension with a family of commutative product maps:

$$\varphi_{i,j,\alpha} : M_{i,\alpha} \times M_{j,\alpha} \to M_{i+j,\alpha}$$

which satisfy, for every $\beta \leq \alpha$,

$$\varphi_{i,j,\alpha}(f_{i,\beta,\alpha}(m_{i,\beta}), f_{j,\beta,\alpha}(m_{j,\beta})) = f_{i+j,\beta,\alpha}(\varphi_{i,j,\beta}(m_{i,\beta}, m_{j,\beta})).$$
Then \( \lim_{\longrightarrow} R_\alpha \ltimes_n \lim_{\longrightarrow} M_{1,\alpha} \times \cdots \times \lim_{\longrightarrow} M_{n,\alpha} \) is an \( n \)-trivial extension with the following family of well-defined commutative product maps:

\[
\varphi_{i,j,\alpha} : \lim_{\longrightarrow} M_{i,\alpha} \times \lim_{\longrightarrow} M_{j,\alpha} \rightarrow \lim_{\longrightarrow} M_{i+j,\alpha}
\]

\[
((m_{i,\alpha}), (m_{j,\alpha}))_{\alpha} \mapsto (\varphi_{i,j,\alpha}(m_{i,\alpha}, m_{j,\alpha}))_{\alpha}.
\]

Moreover, there is a natural ring ismorphism:

\[
\lim_{\longrightarrow} (R_\alpha \ltimes_n M_{1,\alpha} \ltimes \cdots \ltimes M_{n,\alpha}) \cong \lim_{\longrightarrow} R_\alpha \ltimes_n \lim_{\longrightarrow} M_{1,\alpha} \ltimes \cdots \ltimes \lim_{\longrightarrow} M_{n,\alpha}.
\]

**Proof.** The result follows using a standard argument. \( \square \)

### 4 Some basic algebraic properties of \( R \ltimes_n M \)

In this section we give some basic properties of \( n \)-trivial extensions. Before giving the first result, we make the following observations on situations where a subfamily of \( M \) is trivial.

**Observation 4.1** 1. If there is an integer \( i \in \{1, \ldots, n-1\} \) such that \( M_j = 0 \) for every \( j \in \{i+1, \ldots, n\} \), then there is a natural ring isomorphism

\[
R \ltimes_n M_1 \ltimes \cdots \ltimes M_i \ltimes 0 \ltimes \cdots \ltimes 0 \cong R \ltimes_i M_1 \ltimes \cdots \ltimes M_i.
\]

If \( M_1 = \cdots = M_{n-1} = 0 \), then \( R \ltimes_n M \) can be represented as \( R \ltimes_1 M_n \). However, if \( n \geq 3 \) and there is an integer \( i \in \{1, \ldots, n-2\} \) such that, for \( j \in \{1, \ldots, n\} \), \( M_j = 0 \) if and only if \( j \in \{1, \ldots, i\} \), then in general \( R \ltimes_n 0 \ltimes \cdots \ltimes 0 \ltimes M_{i+1} \ltimes \cdots \ltimes M_n \) cannot be represented as an \( n-i \)-trivial extension as above. Indeed, if for example, \( i \) satisfies \( 2i + 2 \leq n \), then \( R \ltimes_{n-i} M_{i+1} \ltimes \cdots \ltimes M_n \) makes no sense (since \( \varphi_{i+1,i+1}(M_{i+1}, M_{i+1}) \) is a subset of \( M_{2i+2} \) not of \( M_{i+2} \)).

2. If \( M_{2k} = 0 \) for every \( k \in \mathbb{N} \) with \( 1 \leq 2k \leq n \), then \( R \ltimes_n M \) can be represented as the trivial extension of \( R \) by the \( R \)-module \( M_1 \times M_3 \times \cdots \times M_{2n'+1} \) where \( 2n' + 1 \) is the biggest odd integer in \( \{1, \ldots, n\} \). Namely, there is a natural ring isomorphism

\[
R \ltimes_n M \cong R \ltimes_1 (M_1 \times M_3 \times \cdots \times M_{2n'+1}).
\]

3. If \( M_{2k+1} = 0 \) for every \( k \in \mathbb{N} \) with \( 1 \leq 2k+1 \leq n \), then there is a natural ring isomorphism

\[
R \ltimes_n M \cong R \ltimes_{n''} M_2 \ltimes M_4 \ltimes \cdots \ltimes M_{2n''}
\]
Proposition 4.3

The following assertions are true.

1. Let \( G \) be a submonoid of \( \Gamma_{n+1} \) and consider the family of \( R \)-modules \( M' = (M'_i)_{i=1}^n \) such that \( M'_i = M_i \) if \( i \in G \) and \( M'_i = 0 \) if \( i \notin G \). Then we have the following (natural) ring extensions:

\[
R \hookrightarrow R \triangleleft M' \hookrightarrow R \triangleleft n M.
\]

In particular, for every \( m \in \{1, ..., n\} \), we have the following (natural) ring extensions:

\[
R \hookrightarrow R \triangleleft n 0 \triangleleft \cdots \triangleleft 0 \triangleleft M_m \triangleleft \cdots \triangleleft M_n \hookrightarrow R \triangleleft n M_1 \triangleleft \cdots \triangleleft M_n.
\]

The extension \( R \hookrightarrow R \triangleleft M_1 \triangleleft \cdots \triangleleft M_n \) will be denoted by \( i_n \).

2. For every \( m \in \{1, ..., n\} \), \( 0 \triangleleft n 0 \triangleleft \cdots \triangleleft 0 \triangleleft M_0 \triangleleft \cdots \triangleleft M_n \) is an ideal of \( R \triangleleft n M \) and an \( R \triangleleft j M_1 \triangleleft \cdots \triangleleft M_j \)-module for every \( j \in \{n - m, ..., n\} \) via the action

\[
(x_0, x_1, ..., x_j)(0, ..., 0, y_m, ..., y_n) := (x_0, x_1, ..., x_j, 0, ..., 0, y_m, ..., y_n) = (x_0, x_1, ..., x_{n-m}, 0, ..., 0)(0, ..., 0, y_m, ..., y_n).
\]

Moreover, the structure of \( 0 \triangleleft n 0 \triangleleft \cdots \triangleleft 0 \triangleleft M_0 \triangleleft \cdots \triangleleft M_n \) as an ideal of \( R \triangleleft n M \) is the same as the \( R \triangleleft j M_1 \triangleleft \cdots \triangleleft M_j \)-module structure for every \( j \in \{n - m, ..., n\} \). In particular, the structure of the ideal \( 0 \triangleleft n 0 \triangleleft \cdots \triangleleft 0 \triangleleft M_n \) is the same as the one of the \( R \)-module \( M_n \).

where \( 2n'' \) is the biggest even integer in \( \{1, ..., n\} \). In general, for every cyclic submonoid \( G \) of \( \Gamma_{n+1} \) generated by an element \( g \in \{1, ..., n\} \), if \( M_i = 0 \) if and only if \( i \notin G \), then there is a natural ring isomorphism

\[
R \triangleleft n M \cong R \triangleleft s M_g \triangleleft M_{2g} \triangleleft \cdots \triangleleft M_{sg}
\]

where \( sg \) is the biggest integer in \( G \cap \{1, ..., n\} \).

As observed above, if one would discuss according to whether a subfamily of \( M \) is trivial or not, then various situations may occur. Thus, for the sake of simplicity, we make the following convention.

Convention 4.2

Unless explicitly stated otherwise, when we consider an \( n \)-trivial extension for a given \( n \), then we implicitly suppose that \( M_i \neq 0 \) for every \( i \in \{1, ..., n\} \). This will be used in the sequel without explicit mention.

Note also that the nature of the maps \( \varphi_{i,j} \) can affect the structure of the \( n \)-trivial extension. For example, in case where \( n = 2 \), if \( \varphi_{1,1} = 0 \), then \( R \triangleleft 2 M_1 \triangleleft M_2 \cong R \triangleleft (M_1 \triangleleft M_2) \). For example, if \( I \subseteq J \) is an extension of ideals of \( R \), then \( R \triangleleft 2 I \triangleleft R/J \cong R \triangleleft (I \triangleleft R/J) \).

Let us start with the following result which presents some relations (easily established) between \( n \)-trivial extensions.

Proposition 4.3

The following assertions are true.

1. Let \( G \) be a submonoid of \( \Gamma_{n+1} \) and consider the family of \( R \)-modules \( M' = (M'_i)_{i=1}^n \) such that \( M'_i = M_i \) if \( i \in G \) and \( M'_i = 0 \) if \( i \notin G \). Then we have the following (natural) ring extensions:

\[
R \hookrightarrow R \triangleleft M' \hookrightarrow R \triangleleft n M.
\]

In particular, for every \( m \in \{1, ..., n\} \), we have the following (natural) ring extensions:

\[
R \hookrightarrow R \triangleleft n 0 \triangleleft \cdots \triangleleft 0 \triangleleft M_m \triangleleft \cdots \triangleleft M_n \hookrightarrow R \triangleleft n M_1 \triangleleft \cdots \triangleleft M_n.
\]

The extension \( R \hookrightarrow R \triangleleft M_1 \triangleleft \cdots \triangleleft M_n \) will be denoted by \( i_n \).

2. For every \( m \in \{1, ..., n\} \), \( 0 \triangleleft n 0 \triangleleft \cdots \triangleleft 0 \triangleleft M_0 \triangleleft \cdots \triangleleft M_n \) is an ideal of \( R \triangleleft n M \) and an \( R \triangleleft j M_1 \triangleleft \cdots \triangleleft M_j \)-module for every \( j \in \{n - m, ..., n\} \) via the action

\[
(x_0, x_1, ..., x_j)(0, ..., 0, y_m, ..., y_n) := (x_0, x_1, ..., x_j, 0, ..., 0, y_m, ..., y_n) = (x_0, x_1, ..., x_{n-m}, 0, ..., 0)(0, ..., 0, y_m, ..., y_n).
\]

Moreover, the structure of \( 0 \triangleleft n 0 \triangleleft \cdots \triangleleft 0 \triangleleft M_0 \triangleleft \cdots \triangleleft M_n \) as an ideal of \( R \triangleleft n M \) is the same as the \( R \triangleleft j M_1 \triangleleft \cdots \triangleleft M_j \)-module structure for every \( j \in \{n - m, ..., n\} \). In particular, the structure of the ideal \( 0 \triangleleft n 0 \triangleleft \cdots \triangleleft 0 \triangleleft M_n \) is the same as the one of the \( R \)-module \( M_n \).
3. For every \( m \in \{1, \ldots, n\} \), we have the following natural ring isomorphisms:

\[ R \times_n M_1 \times \cdots \times M_n/0 \times_n 0 \times \cdots \times 0 \times M_m \times \cdots \times M_n \cong R \times_{m-1} M_1 \times \cdots \times M_{m-1} \]

obtained from the natural ring homomorphism:

\[
\pi_{m-1} : \quad R \times_n M_1 \times \cdots \times M_n \quad \mapsto \quad R \times_{m-1} M_1 \times \cdots \times M_{m-1}
\]

\((r, x_1, \ldots, x_n) \mapsto (r, x_1, \ldots, x_{m-1})\)

where for \( m = 1 \), \( R \times_{m-1} M_1 \times \cdots \times M_{m-1} = R \).

To give another example for the assertion (1), one can show that, for \( n = 3 \), \( \{0, 2\} \) is a submonoid of \( \Gamma_4 \). Then we have the following (natural) ring extensions:

\[ R \hookrightarrow R \times M_2 \hookrightarrow R \times_3 M_1 \times M_2 \times M_3. \]

**Remark 4.4** We have seen that, in the case of \( n = 1 \), the ideal structure of \( 0 \times_1 M_1 \) is the same as the \( R \)-module structure of \( 0 \times_1 M_1 \). Actually, Nagata [40] used this to reduce proofs of module-theoretic results to the ideal case. However, for \( n \geq 2 \), the \( R \)-module structure of \( 0 \times_n M_1 \times \cdots \times M_n \) need not be the same as the ideal structure. For instance, consider the 2-trivial extension \( \mathbb{Z} \times_2 \mathbb{Z} \times \mathbb{Z} \) (with the maps induced by the multiplication in \( \mathbb{Z} \)). Then \( \mathbb{Z}(0, 1, 1) = \{(0, m, m) \mid m \in \mathbb{Z}\} \) while the ideal of \( \mathbb{Z} \times_2 \mathbb{Z} \times \mathbb{Z} \) generated by \( (0, 1, 1) \) is \( 0 \times_2 \mathbb{Z} \times \mathbb{Z} \). However, according to Proposition 4.3 (2), \( (\mathbb{Z} \times_1 \mathbb{Z})(0, 1, 1) = (\mathbb{Z} \times_2 \mathbb{Z} \times \mathbb{Z})(0, 1, 1) \).

The notion of extensions of ideals under ring homomorphisms is a natural way to construct examples of ideals. In this context, we use the ring homomorphism \( i_m \) (indicated in Proposition 4.3) to give such examples.

**Proposition 4.5** For an ideal \( I \) of \( R \), we have the following assertions:

1. The ideal \( I \times_n IM_1 \times \cdots \times IM_n \) of \( R \times_n M \) is the extension of \( I \) under the ring homomorphism \( i_n \), and we have the following natural ring isomorphism:

\[ (R \times_n M)/(I \times_n IM_1 \times \cdots \times IM_n) \cong (R/I) \times_n (M_1/IM_1) \times \cdots \times (M_n/IM_n) \]

where the multiplications are well-defined as follows:

\[ \overline{\phi}_{i,j} : M_i/IM_i \times M_j/IM_j \quad \mapsto \quad M_{i+j}/IM_{i+j} \]

\[ (\overline{m_i}, \overline{m_j}) \quad \mapsto \quad \overline{m_i m_j} := \overline{\varphi}_{i,j}(\overline{m_i}, \overline{m_j}) := \overline{\varphi}_{i,j}(m_i, m_j) = m_i m_j. \]

2. The ideal \( I \times_n IM_1 \times \cdots \times IM_n \) is finitely generated if and only if \( I \) is finitely generated.

**Proof.** 1. The proof is straightforward.

2. Using \( \pi_0 \) it is clear that if \( I \times_n IM_1 \times \cdots \times IM_n \) is generated by elements \((r_j, m_{j,1}, \ldots, m_{j,n})\) with \( j \in E \) for some set \( E \), then \( I \) is generated by the \( r_j \)'s. Conversely, if \( I \) is generated by elements \( r_j \) with \( j \in E \) for some set \( E \), then \( I \times_n IM_1 \times \cdots \times IM_n \) is generated by the \((r_j, 0, \ldots, 0)'s. \)

\[ Q.E.D. \]
Now, we determine the radical, prime and maximal ideals of $R \ltimes_n M$. As in the classical case, we show that these ideals are particular cases of the homogeneous ones, which are characterized in the next section. However, we give these particular cases here because of their simplicity which is reflected, using the following lemma, on the fact that they contain the nilpotent ideal $0 \ltimes_n M$ (of index $n + 1$).

**Lemma 4.6** Every ideal of $R \ltimes_n M$ which contains $0 \ltimes_n M$ has the form $I \ltimes_n M$ for some ideal $I$ of $R$. In this case, we have the following natural ring isomorphism:

$$R \ltimes_n M/I \ltimes_n M \cong R/I.$$  

**Proof.** Let $J$ be an ideal of $R \ltimes_n M$ which contains $0 \ltimes_n M$ and consider the ideal $I = \pi_0(J)$ of $R$ where $\pi_0$ is the surjective ring homomorphism used in Proposition 4.3. Then $J \subseteq I \ltimes_n M$ and by the fact that $0 \ltimes_n M \subseteq J$, we deduce that $J = I \ltimes_n M$. Finally, using $\pi_0$ and the fact that $\pi_0^{-1}(I) = J$, we get the desired isomorphism. 

The following result is an extension of [9, Theorem 3.2].

**Theorem 4.7** Radical ideals of $R \ltimes_n M$ have the form $I \ltimes_n M$ where $I$ is a radical ideal of $R$. In particular, the maximal (resp., the prime) ideals of $R \ltimes_n M$ have the form $M \ltimes_n M$ (resp, $P \ltimes_n M$) where $M$ (resp., $P$) is a maximal (resp., a prime) ideal of $R$.

**Proof.** Using Lemma 4.6, it is sufficient to note that every radical ideal contains $0 \ltimes_n M$ since $(0 \ltimes_n M)^{n+1} = 0$. 

Theorem 4.7 allows us to easily determine both the Jacobson radical and the nilradical of $R \ltimes_n M$.

**Corollary 4.8** The Jacobson radical $J(R \ltimes_n M)$ (resp., the nilradical $\text{Nil}(R \ltimes_n M)$) of $R \ltimes_n M$ is $J(R) \ltimes_n M$ (resp., $\text{Nil}(R) \ltimes_n M$) and the Krull dimension of $R \ltimes_n M$ is equal to that of $R$.

We end this section with an extension of [9] Theorems 3.5 and 3.7] which determines, respectively, the set of zero divisors $Z(R \ltimes_n M)$, the set of units $U(R \ltimes_n M)$ and the set of idempotents $Id(R \ltimes_n M)$ of $R \ltimes_n M$. It is worth noting that trivial extensions have been used to construct examples of rings with zero divisors that satisfies certain properties. As mentioned in the introduction, particular 2-trivial extensions are used to settle some questions. Recently, in [14], a 2-trivial extension is used in the context of zero-divisor graphs to give an appropriate example.

**Proposition 4.9** The following assertions are true.

1. The set of zero divisors of $R \ltimes_n M$ is

$$Z(R \ltimes_n M) = \{(r, m_1, ..., m_n) | r \in Z(R) \cup Z(M_1) \cup ... \cup Z(M_n), m_i \in M_i \text{ for } i \in \{1, ..., n\} \}.$$
Hence $S \ltimes_n M$ where $S = R - (Z(R) \cup Z(M_1) \cup \cdots \cup Z(M_n))$ is the set of regular elements of $R \ltimes_n M$.

2. The set of units of $R \ltimes_n M$ is $U(R \ltimes_n M) = U(R) \ltimes_n M$.

3. The set of idempotents of $R \ltimes_n M$ is $\text{Id}(R \ltimes_n M) = \text{Id}(R) \ltimes_n 0$.

**Proof.** All the proofs are similar to the corresponding ones for the classical case. For completeness, we give a proof of the first assertion.

Let $(r, m_1, ..., m_n) \in R \ltimes_n M$ such that $r \in Z(R) \cup Z(M_1) \cup \cdots \cup Z(M_n)$. If $r = 0$, then $(0, m_1, ..., m_n)(0, ..., m_n') = (0, ..., 0)$ for every $m_n' \in M_n$. Hence $(r, m_1, ..., m_n) \in Z(R \ltimes_n M)$.

Suppose $r \neq 0$. If $r \in Z(R)$, there exists a nonzero element $s \in R$ such that $rs = 0$, so $(r, 0, ..., 0)(s, 0, ..., 0) = (0, ..., 0)$ and hence $(r, 0, ..., 0) \in Z(R \ltimes_n M)$. If $r \in Z(M_i)$, for some $i \in \{1, ..., n\}$, there exists a nonzero element $m_i''$ of $M_i$ such that $rm_i'' = 0$, so

$$(r, 0, ..., 0)(0, ..., 0, m_i'', 0, ..., 0) = (0, ..., 0).$$

Hence $(r, 0, ..., 0) \in Z(R \ltimes_n M)$. Now, since $Z(R \ltimes_n M)$ is a union of prime ideals and $\text{Nil}(R \ltimes_n M)$ is contained in each prime ideal and using the fact that $(0, m_1, ..., m_n) \in \text{Nil}(R \ltimes_n M)$, we conclude that $(r, m_1, ..., m_n) = (r, 0, ..., 0) + (0, m_1, ..., m_n) \in Z(R \ltimes_n M)$. This gives the first inclusion.

Conversely, let $(r, m_1, ..., m_n) \in Z(R \ltimes_n M)$. Then there is $(s, m_1', ..., m_n') \in R \ltimes_n M - \{(0, ..., 0)\}$ such that $(0, ..., 0) = (r, m_1, ..., m_n)(s, m_1', ..., m_n') = (rs, rm_1'sm_1, rm_2'sm_2, ..., rm_n'sm_n)$. If $s \neq 0$, then $r \in Z(R)$, and if $s = 0$, we get $r \in Z(M_1)$ if $m_1' \neq 0$, otherwise we pass to $m_2'$ and so on we continue until we arrive at $s = 0$ and $m_i' = 0$ for all $i \in \{1, ..., n-1\}$. Then $rm_n = 0$ and $m_i' \neq 0$, so $r \in Z(M_n)$. This gives the desired inclusion. 

\[\blacksquare\]

5 Homogeneous ideals of $n$-trivial extensions

The study of the classical trivial extension as a graded ring established some interesting properties (see, for instance, [9, Section 3]). Namely, in [9], studying homogeneous ideals of the trivial extension shed more light on the structure of their ideals. Then naturally one would like to extend this study to the context of $n$-trivial extensions. In this section we extend this study to the context of $n$-trivial extensions, where here $R \ltimes_n M$ is a ($\mathbb{N}_0$-)graded ring with, as indicated in Section 3, $(R \ltimes_n M)_0 = R$, $(R \ltimes_n M)_i = M_i$, for every $i \in \{1, ..., n\}$, and $(R \ltimes_n M)_i = 0$ for every $i \geq n + 1$. Note that we could also consider $R \ltimes_n M$ as a $\mathbb{Z}_{n+1}$-graded ring or $\Gamma_{n+1}$-graded ring as mentioned in Section 3.

For that, it is convenient to recall the following definitions: Let $\Gamma$ be a commutative additive monoid and $S = \bigoplus_{\alpha \in \Gamma} S_\alpha$ be a $\Gamma$-graded ring. Let $N = \bigoplus_{\alpha \in \Gamma} N_\alpha$ be a $\Gamma$-graded $S$-module. For every
Theorem 5.1 The following assertions are true. A submodule $N'$ of $N$ is said to be homogeneous if one of the following equivalent assertions is true.

1. $N'$ is generated by homogeneous elements,

2. If $\sum_{\alpha \in G'} n_{\alpha} \in N'$, where $G'$ is a finite subset of $\Gamma$ and each $n_{\alpha}$ is homogeneous of degree $\alpha$, then $n_{\alpha} \in N'$ for every $\alpha \in G'$, or

3. $N' = \bigoplus_{\alpha \in \Gamma} (N' \cap N_{\alpha})$.

In particular, an ideal $J$ of $R \ltimes_{\text{n}} M$ is homogeneous if and only if $J = (J \cap R) \oplus (J \cap M_1) \oplus \cdots \oplus (J \cap M_n)$. Note that $I := J \cap R$ is an ideal of $R$ and, for $i \in \{1, \ldots, n\}$, $N_i := J \cap M_i$ is an $R$-submodule of $M_i$ which satisfies $IM_i \subseteq N_i$ and $N_iM_j \subseteq N_{i+j}$ for every $i, j \in \{1, \ldots, n\}$.

The next result extends \cite[Theorem 3.3 (1)]{9}. Namely, it determines the structure of the homogeneous ideals of $n$-trivial extensions.

In what follows, we use the ring homomorphism $\Pi_0 := \pi_0$ (used in Proposition \ref{proposition}) and, for $i \in \{1, \ldots, n\}$, the following homomorphism of $R$-modules:

$$\Pi_i: \quad R \ltimes_{\text{n}} M_1 \times \cdots \times M_n \to M_i,$$

where the multiplications are well-defined as follows:

$$\varphi_{i,j}: \quad M_i/C_i \times M_j/C_j \to M_{i+j}/C_{i+j}, \quad (\overline{m_i}, \overline{m_j}) \mapsto \overline{m_i} \overline{m_j}.$$

In particular, $(R \ltimes_{\text{n}} M_1 \times \cdots \times M_n)\langle 0 \ltimes_{\text{n}} C_1 \times \cdots \times C_n \rangle \cong R \ltimes_{\text{n}} (M_1/C_1) \times \cdots \times (M_n/C_n)$.

1. Let $I$ be an ideal of $R$ and let $C = (C_i)_{i \in \{1, \ldots, n\}}$ be a family of $R$-modules such that $C_i \subseteq M_i$ for every $i \in \{1, \ldots, n\}$. Then $I \ltimes_{\text{n}} C$ is a (homogeneous) ideal of $R \ltimes_{\text{n}} M$ if and only if $IM_i \subseteq C_i$ and $C_iM_j \subseteq C_{i+j}$ for all $i, j \in \{1, \ldots, n\}$ with $i + j \leq n$. Thus if $I \ltimes_{\text{n}} C$ is an ideal of $R \ltimes_{\text{n}} M$, then $M_i/C_i$ is an $R/I$-module for every $i \in \{1, \ldots, n\}$, and we have a natural ring isomorphism

$$(R \ltimes_{\text{n}} M_1 \times \cdots \times M_n)/(I \ltimes_{\text{n}} C_1 \times \cdots \times C_n) \cong (R/I) \ltimes_{\text{n}} (M_1/C_1) \times \cdots \times (M_n/C_n)$$

where the multiplications are well-defined as follows:

$$\varphi_{i,j}: \quad M_i/C_i \times M_j/C_j \to M_{i+j}/C_{i+j}, \quad (\overline{m_i}, \overline{m_j}) \mapsto \overline{m_i} \overline{m_j}.$$
(a) $K$ is an ideal of $R$ and $N_i$ is a submodule of $M_i$ for every $i \in \{1, \ldots, n\}$ such that $K M_i \subseteq N_i$ and $N_j M_j \subseteq N_{i+j}$ for every $j \in \{1, \ldots, n\}$ with $i + j \leq n$. Thus $K \times_n N_1 \times \cdots \times N_n$ is a homogeneous ideal of $R \times_n M_1 \times \cdots \times M_n$.

(b) $J \subseteq K \times_n N_1 \times \cdots \times N_n$.

(c) The ideal $J$ is homogeneous if and only if $J = K \times_n N_1 \times \cdots \times N_n$.

**Proof.** 1. If $I \times_n C_1 \times \cdots \times C_n$ is an ideal of $R \times_n M$, then $(R \times_n M_1 \times \cdots \times M_n)(I \times_n C_1 \times \cdots \times C_n) = I \times_n (IM_1 + C_1) \times (IM_2 + C_2 + M_1) \times \cdots \times (IM_n + C_n + \sum_{i+j=n} C_i M_j)$. Thus $I M_i \subseteq C_i$ and $C_i M_j \subseteq C_{i+j}$ for every $i, j \in \{1, \ldots, n\}$.

Conversely, suppose that we have $I M_i \subseteq C_i$ and $C_i M_j \subseteq C_{i+j}$ for all $i, j \in \{1, \ldots, n\}$ with $i + j \leq n$. Then $M_i/C_i$ is an $R/I$-module for every $i \in \{1, \ldots, n\}$ and the map

$$f: \quad R \times_n M_1 \times \cdots \times M_n \to (R/I) \times_n (M_1/C_1) \times \cdots \times (M_n/C_n)$$

$$(r, m_1, \ldots, m_n) \mapsto (r + I, m_1 + C_1, \ldots, m_n + C_n)$$

is a well-defined surjective homomorphism with $\ker f = I \times_n C_1 \times \cdots \times C_n$, so $I \times_n C_1 \times \cdots \times C_n$ is an ideal of $R \times_n M_1 \times \cdots \times M_n$ and

$$(R \times_n M_1 \times \cdots \times M_n)/(I \times_n C_1 \times \cdots \times C_n) \cong (R/I) \times_n (M_1/C_1) \times \cdots \times (M_n/C_n).$$

In particular, $(R \times_n M_1 \times \cdots \times M_n)/(0 \times_n C_1 \times \cdots \times C_n) \cong R \times_n (M_1/C_1) \times \cdots \times (M_n/C_n)$.

2. All of the three statements are easily checked.

The following result presents some properties of homogeneous ideals of $R \times_n M$. It is an extension of both [9, Theorem 3.2 (3)] and [9, Theorem 3.3 (2) and (3)]. In particular, we determine, as an extension of [9, Theorem 3.3 (3)], the form of homogeneous principal ideals. In fact, the characterization of homogeneous principal ideals plays a key role in studying homogeneous ideals. This is due to (the easily checked) fact that an ideal $I$ of a graded ring is homogeneous if every principal ideal generated by an element of $I$ is homogeneous.

**Proposition 5.2** The following assertions are true.

1. Let $I \times_n N_1 \times \cdots \times N_n$ and $I' \times_n N'_1 \times \cdots \times N'_n$ be two homogeneous ideals of $R \times_n M$. Then we have the following homogeneous ideals of $R \times_n M$:

   (a) $(I \times_n N_1 \times \cdots \times N_n) + (I' \times_n N'_1 \times \cdots \times N'_n) = (I + I') \times_n (N_1 + N'_1) \times \cdots \times (N_n + N'_n)$,

   (b) $(I \times_n N_1 \times \cdots \times N_n) \cap (I' \times_n N'_1 \times \cdots \times N'_n) = (I \cap I') \times_n (N_1 \cap N'_1) \times \cdots \times (N_n \cap N'_n)$,

   (c) $(I \times_n N_1 \times \cdots \times N_n)(I' \times_n N'_1 \times \cdots \times N'_n) = I I' \times_n (\sum_{i+j=n} \times (IN'_n + I' N_n + \sum_{i+j=n} N_i N'_j))$, and
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For a given ring \( R \) generated, the following assertions are true.

1. The proof for each of the first three statements is similar to the corresponding one of [8, Theorem 25.1 (2)]. The last statement easily follows from the fact that the residual of two homogeneous ideals is again homogeneous.

2. A principal ideal \( (a, m_1, ..., m_n) \) of \( R \times_n M \) is homogeneous if and only if \( (a, m_1, ..., m_n) = aR \times_n (Rm_1 + aM_1) \times (Rm_2 + aM_2 + m_1M_1) \times \cdots \times (Rm_n + aM_n + \sum_{i+j=n} m_iM_j) \).

3. For an ideal \( J \) of \( R \times_n M \), \( \sqrt{J} = \sqrt{\Pi_0(J)} \times_n M \). In particular, if \( I \times_n C_1 \times \cdots \times C_n \) is a homogeneous ideal of \( R \times_n M \), then \( \sqrt{I} \times_n C_1 \times \cdots \times C_n = \sqrt{I} \times_n M \).

Proof. 1. The proof for each of the first three statements is similar to the corresponding one of [8, Theorem 25.1 (2)]. The last statement easily follows from the fact that the residual of two homogeneous ideals is again homogeneous.

2. Apply assertion (1) and Theorem 5.1 (1).

3. The proof is similar to the one of [9, Theorem 3.2 (3)].

It is a known fact that, in case where \( n = 1 \), even if a homogeneous ideal \( I \times C \) is finitely generated, the \( R \)-module \( C \) is not necessarily finitely generated (you can consider \( \mathbb{Z} \times \mathbb{Q} \) and the principal ideal \( (2, 0) = 2\mathbb{Z} \times \mathbb{Q} \) as an example). The following result presents, in this context, some particular cases obtained using standard arguments.

Proposition 5.3 The following assertions are true.

1. The ideal \( 0 \times_n M \) of \( R \times_n M \) is finitely generated if and only if each \( R \)-module \( M_i \) is finitely generated.

2. If a homogeneous ideal \( I \times_n C_1 \times \cdots \times C_n \) of \( R \times_n M \) is finitely generated, then \( I \) is a finitely generated ideal of \( R \).

The converse implication is true when \( C_i \) is a finitely generated \( R \)-module for every \( i \in \{1, ..., n\} \).

From the previous section, we note that every radical (hence prime) ideal of \( R \times_n M \) is homogeneous. However, it is well-known that the ideals of the classical trivial extensions are not in general homogeneous (see [9]). Then natural questions arise:

Question 1: When every ideal of a given class \( \mathcal{I} \) of ideals of \( R \times_n M \) is homogeneous?

Question 2: For a given ring \( R \) and a family of \( R \)-modules \( M = (M_i)_{i=1}^n \), what is the class of all homogeneous ideals of \( R \times_n M \)?

It is clear that these questions depend on the structure of both \( R \) and each \( M_i \). For instance, for \( n = 1 \), if \( R \) is a quasi-local ring with maximal \( m \), then a proper homogeneous ideal of \( R \times R/m \).
has either the form $I \ltimes R/m$ or $I \ltimes 0$ where $I$ is a proper ideal of $R$. And a proper homogeneous principal ideal of $R \ltimes R/m$ has either the form $0 \ltimes R/m$ or $I \ltimes 0$ where $I$ is a principal ideal of $R$. Then, for instance, a principal ideal of $R \ltimes R/m$ generated by an element $(a, e)$ where $a$ and $e$ are both nonzero with $a \in m$, is not homogeneous.

Question 1 was investigated in [9] for the case where $I$ is the class of regular ideals of $R \ltimes 1 M$ [9, Theorem 3.9]. Also, under the condition that $R$ is an integral domain, a characterization of trivial extension rings over which every ideal is homogeneous is given (see [9, Theorem 3.3 and Corollary 3.4]). Our aim in the remainder of this section is to extend this study to $n$-trivial extensions. It is worth noting that in the classical case (where $n = 1$), ideals $J$ with $\Pi_0(J) = 0$ are homogeneous. This shows that the condition that all ideals $J$ with $\Pi_0(J) \neq 0$ are homogeneous implies that all ideals of $R \ltimes 1 M$ are homogeneous. In the context of $R \ltimes_n M$ for $n \geq 2$ we show that more situations can occur.

Let us begin with the class of ideals $J$ of $R \ltimes_n M$ with $\Pi_0(J) \cap S \neq \emptyset$ for a given subset $S$ of regular elements of $R$.

Recall that a ring $S$ is said to be présimplifiable if, for every $a$ and $b$ in $S$: $ab = a$ implies $a = 0$ or $b \in U(S)$. Présimplifiable rings were introduced and studied by Bouvier in a series of papers (see references) and they have also been investigated in [7, 8]. In [9], the notion of a présimplifiable ring is used when homogeneous ideals of the classical trivial extensions were studied. For example, we have that if $R$ is présimplifiable but not an integral domain, then every ideal of $R \ltimes 1 M$ is homogeneous if and only if $M_1 = 0$ (see [9, Theorems 3.3 (4)]). This is why we first consider just subsets of regular elements.

**Theorem 5.4** Let $S$ be a nonempty subset of $R - Z(R)$ and let $I$ be the class of ideals $J$ of $R \ltimes_n M$ with $\Pi_0(J) \cap S \neq \emptyset$. Then the following assertions are equivalent.

1. Every ideal in $I$ is homogeneous.
2. Every principal ideal in $I$ is homogeneous.
3. For every $s \in S$ and $i \in \{1, ..., n\}$, $sM_i = M_i$.
4. Every principal ideal $(s, m_1, ..., m_n)$ with $s \in S$ has the form $I \ltimes_n M$ where $I$ is a principal ideal of $R$ with $I \cap S \neq \emptyset$.
5. Every ideal in $I$ has the form $I \ltimes_n M$ where $I$ is an ideal of $R$ with $I \cap S \neq \emptyset$.

**Proof.** (1) $\Rightarrow$ (2). Obvious.

(2) $\Rightarrow$ (3). Let $s \in S$ and $i \in \{1, ..., n\}$. We only need to prove that $M_i \subseteq sM_i$. Consider an element $m_i$ of $M_i$. Since $s \in S$, $\langle(s,0,\ldots,0,m_i,0,\ldots,0)\rangle$ is homogeneous. Then $(s,0,\ldots,0) \in \langle(s,0,\ldots,0,m_i,0,\ldots,0)\rangle$, so there is $(x,e_1,\ldots,e_n) \in R \ltimes_n M_1 \ltimes \cdots \ltimes M_n$ such that

$$(s,0,\ldots,0,m_i,0,\ldots,0)(x,e_1,\ldots,e_n) = (s,0,\ldots,0).$$
Since \( s \) is regular, \( x = 1 \). Then \( m_i = (-s)e_i \), as desired.

(3) \( \Rightarrow \) (4). Let \( \langle (s, m_1, ..., m_n) \rangle \) be a principal ideal of \( R \times_n M \) with \( s \in S \). By (3),

\[
(s, m_1, ..., m_n)(0 \times_n 0 \times \cdots \times 0 \times M_n) = 0 \times_n 0 \times \cdots \times 0 \times M_n.
\]

This implies that \( 0 \times_n 0 \times \cdots \times 0 \times M_n \subset \langle (s, m_1, ..., m_n) \rangle \). Using this inclusion and (3), we get \( 0 \times_n 0 \times \cdots \times 0 \times M_{n-1} \times 0 \subset \langle (s, m_1, ..., m_n) \rangle \). Then inductively we get

\[
0 \times_n 0 \times \cdots \times 0 \times M_i \times 0 \times \cdots \times 0 \subset \langle (s, m_1, ..., m_n) \rangle
\]

for every \( i \in \{1, ..., n\} \). Thus \( 0 \times_n M_1 \times \cdots \times M_n \subset \langle (s, m_1, ..., m_n) \rangle \). Therefore by Lemma 4.6 and Proposition 5.2 (2), \( \langle (s, m_1, ..., m_n) \rangle \) has the form \( I \times_n M \) where \( I = sR \).

(4) \( \Rightarrow \) (5). Consider an ideal \( J \) in \( \mathcal{S} \). Then there is an element \( (s, m_1, ..., m_n) \in J \) such that \( s \in \Pi_0(J) \cap S \). Therefore using (4) and Lemma 4.6 we get the desired result.

(5) \( \Rightarrow \) (1). Obvious. \( \blacksquare \)

As an example, we can consider the trivial extension \( S := \mathbb{Z} \times_2 \mathbb{Z}_W \times \mathbb{Q} \) where \( \mathbb{Z}_W \) is the ring of fractions of \( \mathbb{Z} \) with respect to the multiplicatively closed subset \( W = \{2^k | k \in \mathbb{N}\} \) of \( \mathbb{Z} \). Then the principal ideal \( \langle (3, 1, 0) \rangle \) of \( S \) is not homogeneous. Deny, we must have \( (3, 0, 0) \in \langle (3, 1, 0) \rangle \).

Thus there is \( (a, e, f) \in S \) such that \( (3, 0, 0) = (3, 1, 0)(a, e, f) \). But this implies that \( a = 1 \) and then \( e = -\frac{1}{3} \), which is absurd.

The following result is an extension of [9, Theorem 3.9]. Recall that an ideal is said to be regular if it contains a regular element. Here, from Proposition 4.9 an ideal of \( R \times_n M \) is regular if and only if it contains an element \( (s, m_1, ..., m_n) \) with \( s \in R \setminus (Z(R) \cup Z(M_1) \cup \cdots \cup Z(M_n)) \).

**Corollary 5.5** Let \( S = R - (Z(R) \cup Z(M_1) \cup \cdots \cup Z(M_n)) \). Then the following assertions are equivalent.

1. Every regular ideal of \( R \times_n M \) is homogeneous.
2. Every principal regular ideal of \( R \times_n M \) is homogeneous.
3. For every \( s \in S \) and \( i \in \{1, ..., n\} \), \( sM_i = M_i \) (or equivalently, \( M_iS = M_i \)).
4. Every principal ideal \( \langle (s, m_1, ..., m_n) \rangle \) with \( s \in S \) has the form \( I \times_n M \) where \( I \) is a principal ideal of \( R \) with \( I \cap S \neq \emptyset \).
5. Every regular ideal of \( R \times_n M \) has the form \( I \times_n M \) where \( I \) is an ideal of \( R \) with \( I \cap S \neq \emptyset \).

Consequently, if \( R \times_n M \) is root closed (in particular, integrally closed), then every regular ideal of \( R \times_n M \) has the form given in (5).

**Proof.** The proof is similar to the one of [9, Theorem 3.9]. \( \blacksquare \)
Compare the following result with [9, Corollary 3.4].

**Corollary 5.6** Assume that $R$ is an integral domain. Then the following assertions are equivalent.

1. Every ideal $J$ of $R \ltimes_n M$ with $\Pi_0(J) \neq 0$ is homogeneous.
2. Every principal ideal $J$ of $R \ltimes_n M$ with $\Pi_0(J) \neq 0$ is homogeneous.
3. For every $i \in \{1, \ldots, n\}$, $M_i$ is divisible.
4. Every principal ideal $\langle (s, m_1, \ldots, m_n) \rangle$ of $R \ltimes_n M$ with $s \neq 0$ has the form $I \ltimes_n M$ where $I$ is a nonzero principal ideal of $R$.
5. Every ideal $J$ of $R \ltimes_n M$ with $\Pi_0(J) \neq 0$ has the form $I \ltimes_n M$ where $I$ is a nonzero ideal of $R$.
6. Every ideal of $R \ltimes_n M$ is comparable to $0 \ltimes_n M$.

**Proof.** The equivalence (5) $\Leftrightarrow$ (6) is a simple consequence of Lemma [10].

The proof of Theorem 5.4 shows that another situation can be considered. This is given in the following result. We use $\text{Ann}_R(H)$ to denote the annihilator of an $R$-module $H$.

**Theorem 5.7** Let $\mathcal{I}$ be the class of ideals $J$ of $R \ltimes_n M$ with $\Pi_0(J) \cap S \neq \emptyset$ where $S$ is a nonempty subset of $R - \{0\}$ such that, for every $s \in S$, $\text{Ann}_R(s) \subseteq \text{Ann}_R(M_i)$. Then the following assertions are equivalent.

1. Every ideal in $\mathcal{I}$ is homogeneous.
2. Every principal ideal in $\mathcal{I}$ is homogeneous.
3. For every $s \in S$ and $i \in \{1, \ldots, n\}$, $sM_i = M_i$.
4. Every principal ideal $\langle (s, m_1, \ldots, m_n) \rangle$ with $s \in S$ has the form $I \ltimes_n M$ where $I$ is a principal ideal of $R$ with $I \cap S \neq \emptyset$.
5. Every ideal in $\mathcal{I}$ has the form $I \ltimes_n M$ where $I$ is an ideal of $R$ with $I \cap S \neq \emptyset$.

**Proof.** We only need to prove the implication (2) $\Rightarrow$ (3). Let $s \in S$ and $i \in \{1, \ldots, n\}$ and consider an element $m_i$ of $M_i - \{0\}$. Since $s \in S$, $\langle (s, 0, \ldots, 0, m_i, 0, \ldots, 0) \rangle$ is homogeneous. Then $(s, 0, \ldots, 0) \in \langle (s, 0, \ldots, 0, m_i, 0, \ldots, 0) \rangle$, so there is $(x, e_1, \ldots, e_n) \in R \ltimes_n M_1 \times \cdots \times M_n$ such that $(s, 0, \ldots, 0, m_i, 0, \ldots, 0)(x, e_1, \ldots, e_n) = (s, 0, \ldots, 0)$. Then $sx = s$ and, by the hypothesis on $S$, $(x - 1)m_i = 0$. Therefore $m_i = xm_i = (-s)e_i$, as desired. ■
For an example of a ring that satisfies the condition of the previous result, consider a ring $R$ with an idempotent $e \in R - \{1, 0\}$ and set $S = \{e\}$ and $M_i = Re$ for every $i \in \{1, \ldots, n\}$. Thus, since $eM_i = M_i$ for every $i \in \{1, \ldots, n\}$, every ideal $J$ of $R \times_n M$ with $e \in \Pi_0(J)$ is homogeneous.

Unlike the classical case (where $n = 1$), the fact that, for every $i \in \{1, \ldots, n\}$, $M_i$ is divisible does not necessarily imply that every ideal is homogeneous. For that, we consider the 2-trivial extension $S := k \times_2 (k \times k) \times (k \times k)$ where $k$ is a field. Then the principal ideal $\langle (0, (1, 0), (0, 1)) \rangle$ of $S$ is not homogeneous. Indeed, if it were homogeneous, we must have $(0, (1, 0), (0, 0)) \in \langle (0, (1, 0), (0, 1)) \rangle$. Thus there is $(a, (e, f), (e', f')) \in S$ such that $(0, (1, 0), (0, 0)) = (0, (1, 0), (0, 1))(a, (e, f), (e', f'))$. But this implies that $(a, 0) = (1, 0)$ and $(0, 0) = (e, a)$, which is absurd.

This example naturally leads us to investigate when every ideal $J$ of $R \times_n M$ with $\Pi_0(J) = 0$ is homogeneous. In this context, the notion of a présimplifiable module is used. For that, recall that an $R$-module $H$ is called $R$-présimplifiable if, for every $r \in R$ and $h \in H$, $rh = h$ implies $h = 0$ or $r \in U(R)$. For example, over an integral domain, every torsion-free module is présimplifiable (see [7] and also [8]).

In studying the question when every ideal $J$ of $R \times_n M$ with $\Pi_0(J) = 0$ is homogeneous, several different cases occur. For this we use the following lemma.

**Lemma 5.8** Let $J$ be an ideal of $R \times_n M$ such that, for $i \in \{1, \ldots, n\}$, $\Pi_0(J) = 0, \ldots, \Pi_{i-1}(J) = 0, \Pi_i(J) \neq 0$. Then the following assertions are true.

1. For $i = n$, the ideal $J$ is homogeneous and it has the form $0 \times_n 0 \times \cdots \times 0 \times \Pi_n(J)$.

2. For $i \neq n$, if $0 \times_n 0 \times \cdots \times 0 \times M_{i+1} \times \cdots \times M_n \subset J$, then $J$ is homogeneous and it has the form $0 \times_n 0 \times \cdots \times 0 \times \Pi_i(J) \times M_{i+1} \times \cdots \times M_n$.

**Proof.** Straightforward. 

**Theorem 5.9** Assume that $n \geq 2$ and $M_j$ is présimplifiable for a given $j \in \{1, \ldots, n-1\}$. Let $\mathcal{J}$ be the class of ideals $J$ of $R \times_n M$ with $\Pi_i(J) = 0$ for every $i \in \{0, \ldots, j-1\}$ and $\Pi_j(J) \neq 0$. Then the following assertions are equivalent.

1. Every ideal in $\mathcal{J}$ is homogeneous.

2. Every principal ideal in $\mathcal{J}$ is homogeneous.

3. For every $k \in \{j + 1, \ldots, n\}$ and every $m_j \in M_j - \{0\}$, $M_k = m_jM_{k-j}$.

4. Every principal ideal $\langle (0, 0, \ldots, 0, m_j, \ldots, m_n) \rangle$ with $m_j \neq 0$ has the form $0 \times_n 0 \times \cdots \times 0 \times N \times M_{j+1} \times \cdots \times M_n$ where $N$ is a nonzero cyclic submodule of $M_j$. 


5. Every ideal in $\mathcal{I}$ has the form $0 \times_n 0 \times \cdots \times 0 \times N \times M_{j+1} \times \cdots \times M_n$ where $N$ is a nonzero submodule of $M_j$.

6. Every ideal in $\mathcal{I}$ contains $0 \times_n 0 \times \cdots \times 0 \times M_{j+1} \times \cdots \times M_n$.

**Proof.** The implication $(3) \implies (4)$ is proved similarly to the implication $(3) \implies (4)$ of Theorem 5.4. The implication $(6) \implies (1)$ is a simple consequence of Lemma 5.8. Then only the implication $(2) \implies (3)$ needs a proof. Let $k \in \{j + 1, \ldots, n\}$, $m_j \in M_j - \{0\}$ and $m_k \in M_k - \{0\}$. Then the principal ideal $p = \langle (0, \ldots, 0, m_j, 0, \ldots, 0, m_k, 0, \ldots, 0) \rangle$ is homogeneous. This implies that $(0, \ldots, 0, m_j, 0, \ldots, 0) \in p$ and so there exists $(r, e_1, \ldots, e_n) \in R \times_n M$ such that

$$(0, \ldots, 0, m_j, 0, \ldots, 0) = (r, e_1, \ldots, e_n)(0, \ldots, 0, m_j, 0, \ldots, 0, m_k, 0, \ldots, 0).$$

Then $rm_j = m_j$ and $rm_k + e_{k-j}m_j = 0$. Since $M_j$ is pr´esimplifiable, $r$ is invertible and then $m_k = -r^{-1}e_{k-j}m_j$, as desired.

For examples of rings that satisfy the conditions of the previous result, we can consider the following two 2-trivial extensions: $Z \times_2 Z_W \times Q$ and $Z \times_2 Z_W \times Z_W$ where $Z_W$ is the ring of fractions of $Z$ with respect to the multiplicatively closed subset $W = \{2^k | k \in \mathbb{N}\}$ of $Z$.

The following particular cases are of interest.

**Corollary 5.10** Assume that $n \geq 2$ and $M_{n-1}$ is pr´esimplifiable. Let $\mathcal{I}$ be the class of ideals $J$ of $R \times_n M$ with $\Pi_i(J) = 0$ for every $i \in \{0, \ldots, n-2\}$. Then the following assertions are equivalent.

1. Every ideal in $\mathcal{I}$ is homogeneous.

2. For every $m_{n-1} \in M_{n-1} - \{0\}$, $M_n = m_{n-1}M_1$.

3. Every ideal in $\mathcal{I}$ is comparable to $0 \times_n 0 \times \cdots \times 0 \times M_n$.

**Proof.** $(1) \implies (2)$. This is a particular case of the corresponding one in Theorem 5.9.

$(2) \implies (3)$. Let $I$ be an ideal of $R \times_n M$ in $\mathcal{I}$. If $\Pi_{n-1}(I) \neq 0$, then Theorem 5.9 shows that $I$ contains $0 \times_n 0 \times \cdots \times 0 \times M_n$. Otherwise, $\Pi_{n-1}(I) = 0$ which means that $0 \times_n 0 \times \cdots \times 0 \times M_n$ contains $I$.

$(3) \implies (1)$. Let $I$ be a nonzero ideal of $R \times_n M$ in $\mathcal{I}$. If $\Pi_{n-1}(I) \neq 0$, then Theorem 5.9 shows that $I$ is homogeneous. The other case is a consequence of the assertion $(1)$ of Lemma 5.8.

When $n = 2$, we get the following particular case of Corollary 5.10.

**Corollary 5.11** Assume that $M_1$ is pr´esimplifiable and $n = 2$. Let $\mathcal{I}$ be the class of ideals $J$ of $R \times_2 M$ with $\Pi_0(J) = 0$. Then the following assertions are equivalent.
1. Every ideal in \( \mathcal{I} \) is homogeneous.
2. For every \( m_1 \in M_1 - \{0\} \), \( M_2 = m_1 M_1 \).
3. Every ideal in \( \mathcal{I} \) is comparable to \( 0 \times_2 0 \times M_2 \).

When \( j = 1 \) in Theorem 5.9, there are additional conditions equivalent to (1)-(6). The study of this case leads us to introduce the following notion in order to avoid trivial situations.

**Definition 5.12** Assume that \( n \geq 2 \). For \( i \in \{1, \ldots, n - 1\} \) and \( j \in \{2, \ldots, n\} \) with product \( ij \leq n \), \( M_i \) is said to be \( \phi-j \)-integral (where \( \phi = \{\phi_{i,j}\}_{1 \leq i,j \leq n-1} \) is the family of multiplications) if, for any \( j \) elements \( m_{i_1}, \ldots, m_{i_j} \) of \( M_i \), if the product \( m_{i_1} \cdots m_{i_j} = 0 \), then at least one of the \( m_{i_k} \)'s is zero. If no ambiguity arises, \( M_i \) is simply called \( j \)-integral.

**Corollary 5.13** Assume that \( n \geq 2 \), \( M_1 \) is pr\'esimplifiable and \( k \)-integral for every \( k \in \{2, \ldots, n-1\} \). Let \( \mathcal{I} \) be the class of ideals \( J \) of \( R \times_n M \) with \( \Pi_0(J) = 0 \) and \( \Pi_1(J) \neq 0 \). Then the following assertions are equivalent.

1. Every ideal in \( \mathcal{I} \) is homogeneous.
2. Every principal ideal in \( \mathcal{I} \) is homogeneous.
3. For every \( k \in \{2, \ldots, n\} \) and every \( m_1 \in M_1 - \{0\} \), \( M_k = m_1 M_{k-1} \).
4. For every \( k \in \{2, \ldots, n\} \) and every nonzero elements \( m_{i_1}, \ldots, m_{i_{k-1}} \in M_1 - \{0\} \), \( M_k = m_{i_1} \cdots m_{i_{k-1}} M_1 \).
5. For every \( k \in \{2, \ldots, n\} \) and every nonzero element \( m \in M_1 - \{0\} \), \( M_k = m^{k-1} M_1 \).
6. Every principal ideal \( \langle(0, m_1, \ldots, m_n)\rangle \) with \( m_1 \neq 0 \) has the form \( 0 \times_n N \times M_2 \cdots \times M_n \) where \( N \) is a nonzero cyclic submodule of \( M_1 \).
7. Every ideal in \( \mathcal{I} \) has the form \( 0 \times_n N \times M_2 \cdots \times M_n \) where \( N \) is a nonzero submodule of \( M_1 \).
8. Every ideal in \( \mathcal{I} \) contains \( 0 \times_n 0 \times M_2 \cdots \times M_n \).

**Proof.** The equivalences (3) \( \iff \) (4) \( \iff \) (5) are easily proved. \( \blacksquare \)

The following result shows that, in fact, the conditions of Corollary 5.13 above are necessary and sufficient to show that every ideal \( J \) of \( R \times_n M \) with \( \Pi_0(J) = 0 \) is homogeneous. Note that Corollary 5.11 presents the case \( n = 2 \). Thus in the following result we may assume that \( n \geq 3 \).

**Corollary 5.14** Assume that \( n \geq 3 \) and \( M_1 \) is pr\'esimplifiable and \( k \)-integral for every \( k \in \{2, \ldots, n-1\} \). Then the following assertions are equivalent.
1. Every ideal $J$ of $R \times_n M$ with $\Pi_0(J) = 0$ and $\Pi_1(J) \neq 0$ is homogeneous.

2. For every $j \in \{1, ..., n - 1\}$, $M_j$ is présimplifiable and every ideal $J$ of $R \times_n M$ with $\Pi_0(J) = 0$ is homogeneous.

**Proof.** We only need to prove that (1) $\Rightarrow$ (2). Let $j \in \{1, ..., n - 1\}$ and consider $m_j \in M_j - \{0\}$. Let $r \in R$ such that $rm_j = m_j$. By Corollary 5.13 (4), there are $m_{1j}, ..., m_{nj} \in M_1 - \{0\}$ such that $m_j = m_{1j} \cdots m_{nj}$. Then $rm_{1j} \cdots m_{nj} = m_{1j} \cdots m_{nj}$ which implies that $(rm_{1j} - m_{1j})m_{12} \cdots m_{nj} = 0$. Now, since $M_1$ is $k$-integral for every $k \in \{2, ..., n - 1\}$, $rm_{1j} - m_{1j} = 0$. Therefore $r$ is invertible since $M_1$ is présimplifiable. So $M_j$ is présimplifiable.

Now, to prove that every ideal $J$ of $R \times_n M$ with $\Pi_0(J) = 0$ is homogeneous, it suffices to prove that $M_k = m_jM_{k-j}$ for every $k \in \{2, ..., n\}$, every $j \in \{1, ..., k-1\}$ and every $m_j \in M_j - \{0\}$ (by Theorem 5.9). The case where $k = 2$ is trivial. Thus fix $k \in \{3, ..., n\}$ and $j \in \{1, ..., k-1\}$. Consider $m_j \in M_j - \{0\}$ and $m_k \in M_k - \{0\}$. We prove that $m_k = m_jm_{k-j}$ for some $m_{k-j} \in M_{k-j} - \{0\}$. By Corollary 5.13 (5), $m_j = m^{j-1}_1m_1$ for some $m, m_1 \in M_1 - \{0\}$. And, by Corollary 5.13 (3), $m_k = m_1m_{k-1}$ for some $m_{k-1} \in M_{k-1} - \{0\}$. Also, by Corollary 5.13 (5), $m_{k-1} = m^{k-2}_1m'_1$ for some $m'_1 \in M_1 - \{0\}$. Then $m_k = m^{k-2}_1m_1m'_1 = (m^{j-1}_1m_1)(m^{k-j-1}_1m'_1) = m_jm_{k-j}$ where $m_{k-j} = m^{k-j-1}_1m'_1 \in M_{k-j} - \{0\}$, as desired.

Finally, we give a case when we can characterize rings in which every ideal is homogeneous. Note that, when $R$ is a ring with $aM_i = M_i$ for every $i \in \{1, ..., n - 1\}$ and every $a \in R - \{0\}$, and $M_i = m^{i-1}_iM_i$ for every $i \in \{2, ..., n\}$ and every nonzero element $m \in M_1 - \{0\}$, then $R$ is an integral domain and $M_i$ must be torsion-free for every $i \in \{1, ..., n - 1\}$.

**Corollary 5.15** Suppose that $n \geq 2$ and $R$ is an integral domain. Assume that $M_i$ is torsion-free, for every $i \in \{1, ..., n - 1\}$, and that $M_1$ is $k$-integral for every $k \in \{2, ..., n - 1\}$. Then the following assertions are equivalent.

1. Every ideal of $R \times_n M$ is homogeneous.

2. The following two conditions are satisfied:

   i. For every $i \in \{1, ..., n\}$, $M_i$ is divisible, and

   ii. For every $i \in \{2, ..., n\}$ and every $m_1 \in M_1 - \{0\}$, $M_i = m_1M_{i-1}$.

**Proof.** Simply use Corollaries 5.6 and 5.13 and Theorem 5.9.

It is easy to show that the two $n$-trivial extensions $\mathbb{Z} \times_n \mathbb{Q} \times \cdots \times \mathbb{Q}$ and $\mathbb{Z} \times_n \mathbb{Q} \times \cdots \times \mathbb{Q} \times \mathbb{Q}/\mathbb{Z}$ satisfy the conditions of the above result and so every ideal of these rings is homogeneous.

We end this section with the following particular case. 
Corollary 5.16 Suppose that $n \geq 2$. Consider the $n$-trivial extension $S := k \ltimes E_1 \ltimes \cdots \ltimes E_n$ where $k$ is a field and, for $i \in \{1, \ldots, n\}$, $E_i$ is a $k$-vector space. Suppose that $E_1$ is $k$-integral for every $k \in \{2, \ldots, n-1\}$. Then the following assertions are equivalent.

1. Every ideal of $S$ is homogeneous.
2. For every $k \in \{2, \ldots, n\}$, every $j \in \{1, \ldots, k-1\}$ and every $e_j \in E_j - \{0\}$, $E_k = e_j E_{k-j}$.

As a particular case, we can consider a field extension $K \subseteq F$, then every ideal of $S := K \ltimes n F \ltimes \cdots \ltimes F$ is homogeneous. Namely, every proper ideal of $S$ has the form $0 \ltimes n 0 \ltimes \cdots \ltimes 0 \ltimes N \ltimes F \ltimes \cdots \ltimes F$ where $N$ is a $K$-subspace of $F$.

6 Some ring-theoretic properties of $R \ltimes n M$

In this section, we determine when $R \ltimes n M$ has certain ring properties such as being Noetherian, Artinian, Manis valuation, Prüfer, chained, arithmetical, a $\pi$-ring, a generalized ZPI-ring or a PIR. We end the section with a remark on a question posed in [2] concerning $m$-Boolean rings.

We begin by characterizing when the $n$-trivial extensions are Noetherian (resp., Artinian). The following result extends [9, Theorem 4.8].

**Theorem 6.1** The ring $R \ltimes n M$ is Noetherian (resp., Artinian) if and only if $R$ is Noetherian (resp., Artinian) and, for every $i \in \{1, \ldots, n\}$, $M_i$ is finitely generated.

**Proof.** Similar to the proof of [9, Theorem 4.8].

The following result is an extension of [9, Theorem 4.2 and Corollary 4.3]. It investigates the integral closure of $R \ltimes n M$ in the total quotient ring $T(R \ltimes n M)$ of $R \ltimes n M$.

**Theorem 6.2** Let $S = R - (Z(R) \cup Z(M_1) \cup \cdots \cup Z(M_n))$. If $R'$ is the integral closure of $R$ in $T(R)$, then $(R' \cap R_S) \ltimes n M_1 S \ltimes \cdots \ltimes M_n S$ is the integral closure of $R \ltimes n M$ in $T(R \ltimes n M)$. In particular,

1. If $R$ is an integrally closed ring, then $R \ltimes n M_1 S \ltimes \cdots \ltimes M_n S$ is the integral closure of $R \ltimes n M_1 \ltimes \cdots \ltimes M_n$ in $T(R \ltimes n M_1 \ltimes \cdots \ltimes M_n)$, and

2. If $Z(M_i) \subseteq Z(R)$ for all $i \in \{1, \ldots, n\}$, then $R \ltimes n M_1 S \ltimes \cdots \ltimes M_n S$ is integrally closed if and only if $R$ is integrally closed.

**Proof.** All statements can be proved similarly to the corresponding ones of [9, Theorem 4.2 and Corollary 4.3].
It is worth noting as in the classical case that $R \otimes_n M$ can be integrally closed without $R$ being integrally closed (see the example given after [9 Corollary 4.3]).

Similar to the classical case [9 Theorem 4.16 (1) and (2)], as a consequence of Theorem 6.2 and Corollary 5.5 we give the following result which characterizes when $R \otimes_n M$ is (Manis) valuation and when it is Prüfer. First, recall these two notions.

Let $S$ be a subring of a ring $T$, and let $P$ be a prime ideal of $S$. Then $(S, P)$ is called a valuation pair on $T$ (or just $S$ is a valuation ring on $T$) if there is a surjective valuation $v : T \rightarrow G \cup \{\infty\}$ $(v(xy) = v(x) + v(y), v(x + y) \geq \min\{v(x), v(y)\}$, $v(1) = 0$ and $v(0) = \infty$) where $G$ is a totally ordered abelian group, with $S = \{x \in T|v(x) \geq 0\}$ and $P = \{x \in T|v(x) > 0\}$. This is equivalent to if $x \in T - S$, then there exists $x' \in P$ with $xx' \in S - P$. A valuation ring $S$ is called a (Manis) valuation ring if $T = T(S)$. Also, $S$ is called a Prüfer ring if every finitely generated regular ideal of $S$ is invertible. This is equivalent to every overring of $S$ being integrally closed (see [38] for more details).

**Corollary 6.3** Let $S = R - (Z(R) \cup Z(M_1) \cup \cdots \cup Z(M_n))$.

1. $R \otimes_n M$ is a Manis valuation ring if and only if $R$ is a valuation ring on $R_S$ and $M_i = M_iS$ for every $i \in \{1,...,n\}$.

2. $R \otimes_n M$ is a Prüfer ring if and only if, for every finitely generated ideal $I$ of $R$ with $I \cap S \neq \emptyset$, $I$ is invertible and $M_i = M_iS$ for every $i \in \{1,...,n\}$.

Now, as an extension of [9 Theorem 4.16 (3)], we characterize when $R \otimes_n M$ is a chained ring. Recall that a ring $S$ is said to be chained if the set of ideals of $S$ is totally ordered by inclusion.

As an exception to Convention 4.2 in the following results (Lemma 6.4, Theorem 6.5, Corollary 6.6, Lemma 6.7, and Theorem 6.8), a module in the family associated to an $n$-trivial extension can be zero.

The proof of the desired result uses the following lemma which gives another characterization of a particular $n$-trivial extension with the property that every ideal is homogeneous.

**Lemma 6.4** Assume that $R$ is quasi-local with maximal ideal $m$. Suppose also that at least one of the modules of the family $M$ is nonzero. Then every ideal of $R \otimes_n M$ is homogeneous if and only if the following three conditions are satisfied:

1. $R$ is an integral domain.

2. For every $i \in \{1,...,n\}$, $M_i$ is divisible.

3. For every $1 \leq i \leq j \leq n$ (when $n \geq 2$), if $M_i \neq 0$ and $M_j \neq 0$, then $M_{j-i} \neq 0$ and $eM_i = M_j$ for every $e \in M_{j-i}$.

In this case, each ideal has one of the forms $I \otimes_n M$, for some ideal $I$ of $R$, or $0 \otimes_n 0 \otimes \cdots \otimes 0 \otimes N \otimes M_{j+1} \otimes \cdots \otimes M_n$ where $N$ is a nonzero submodule of $M_j$ for some $j \in \{1,...,n\}$.
Proof. \(\Rightarrow\) Clearly the first assertion is a simple consequence of the second one. Then we only need to prove the second and the third assertions.

(2). Let \(r \in R - \{0\}\) and \(i \in \{1, \ldots, n\}\). Consider an element \(m_i \in M_i\). If \(r \not\in m_i\), the maximal ideal of \(R\), then \(r\) is not divisible and trivially we get the result. Next assume \(r \in m_i\). By hypothesis, the ideal \(\langle (r, 0, \ldots, 0, m_i, 0, \ldots, 0) \rangle\) is homogeneous, so there is \((r', m'_1, \ldots, m'_n)\) such that

\[
(r, 0, \ldots, 0) = (r, 0, \ldots, 0, m_i, 0, \ldots, 0)(r', m'_1, \ldots, m'_n).
\]

Then \(rr' = r\) and \(0 = rm' + r'm_i\). Thus \(r'\) cannot be in \(m_i\), so \(r'\) is invertible and thus \(m_i = -(r')^{-1}rm'_i\), as desired.

(3). Let \(1 \leq i \leq j \leq n\) such that \(M_i \neq 0\) and \(M_j \neq 0\). Consider \(m_i \in M_i - \{0\}\) and \(m_j \in M_j - \{0\}\). By hypothesis, \(\langle (0, \ldots, 0, m_i, 0, \ldots, 0, m_j, 0, \ldots, 0) \rangle\) is homogeneous. Then

\[
(0, \ldots, 0, m_j, 0, \ldots, 0) = (0, \ldots, 0, m_i, 0, \ldots, 0, m_j, 0, \ldots, 0)(r', m'_1, \ldots, m'_n)
\]

for some \((r', m'_1, \ldots, m'_n) \in R \otimes M\). This implies that

\[
r'm_i = 0 \quad \text{and} \quad r'm_j + m_i m'_{j-i} = m_j.
\]

If \(M_{j-i} = 0\), we get \(r'm_i = 0\) and \((r' - 1)m_j = 0\). This is impossible since either \(r'\) or \(r' - 1\) is invertible. Then \(M_{j-i} \neq 0\). Now, suppose that \(r' \neq 0\). By (2), there exists \(m''_{j-i} \in M_{j-i}\) such that \(m''_{j-i} = r'm'_{j-i}\). Hence using the fact that \(r'm_i = 0\), the equality \(r'm_j + m_i m'_{j-i} = m_j\) becomes \(r'm_j = m_j\). As in the previous case, this is impossible. Therefore \(r' = 0\) and this gives the desired result.

\(\Leftarrow\) We only need to prove that every principal ideal \(\langle (s, m_1, \ldots, m_n) \rangle\) of \(R \otimes M\) is homogeneous. For this, distinguish two cases \(s \neq 0\) and \(s = 0\) and follow an argument similar to that of (3) \(\Rightarrow\) (4) of Theorem 5.4 \(\blacksquare\)

**Theorem 6.5** Assume that \(n \geq 2\) and that at least one of the modules of the family \(M\) is nonzero. Then the ring \(R \otimes M\) is chained if and only if the following conditions are satisfied:

1. \(R\) is a valuation domain,

2. For every \(i \in \{1, \ldots, n\}\), \(M_i\) is divisible,

3. For every \(1 \leq i \leq j \leq n\), if \(M_i \neq 0\) and \(M_j \neq 0\), then \(M_{j-i} \neq 0\) and \(eM_i = M_j\) for every \(e \in M_{j-i}\), and

4. For every \(i \in \{1, \ldots, n\}\), the set of all (cyclic) submodules of \(M_i\) is totally ordered by inclusion.
Proof. \(\implies\) First, we prove that \(R\) is a chained ring. Consider two ideals \(I\) and \(J\) of \(R\). Then \(I \leq_n M\) and \(J \leq_n M\) are two ideals of \(R \leq_n M\). Then they are comparable and so are \(I\) and \(J\) as desired. A similar argument can be used to prove the last assertion.

Now, we prove that every ideal of \(R \leq_n M\) is homogeneous. Then by Lemma 5.8 we get the other assertions. Consider a nonzero ideal \(K\) of \(R \leq_n M\). If \(\Pi_0(K) \neq 0\), then necessarily \(0 \leq_n M \subseteq K\). Then by Lemma 4.6 \(K\) is homogeneous. Now, let \(i \geq 1\) be the smallest integer such that \(\Pi_i(K) \neq 0\). If \(i = n\), then by the first assertion of Lemma 5.8 \(K\) is homogeneous. If \(i \neq n\), then necessarily \(0 \leq_n \cdots \leq 0 \times M_{i+1} \times \cdots \times M_n \subseteq K\). Thus by the second assertion of Lemma 5.8 \(K\) is homogeneous, as desired.

\(\Longleftarrow\) Using Lemma 5.8 we deduce that any two ideals \(I\) and \(J\) of \(R \leq_n M\) have the forms \(I = 0 \leq_n \cdots \leq 0 \times I_i \times M_{i+1} \times \cdots \times M_n\) and \(J = 0 \leq_n \cdots \leq 0 \times J_j \times M_{j+1} \times \cdots \times M_n\) for some \(i, j \in \{0, \ldots, n\}\) where \(I_i\) and \(J_j\) are submodules of \(M_i\) and \(M_j\) respectively (here \(M_0 = R\)). If \(i \neq j\), then obviously \(I\) and \(J\) are comparable. If \(i = j\), then using the first and the last assertion, we can show that \(I_i\) and \(J_j\) are comparable and so are \(I\) and \(J\), as desired.

Using Theorem 5.5 and Corollary 3.8, we get an extension of [9, Theorem 4.16 (4)] which characterizes when \(R \leq_n M\) is arithmetical. Recall that a ring \(S\) is arithmetical if and only if \(S_P\) is chained for each prime (maximal) ideal \(P\) of \(S\). Also, recall that, for a ring \(S\), an \(S\)-module \(H\) is called arithmetical if, for each prime (maximal) ideal \(P\) of \(S\), the set of submodules of \(H_P\) is totally ordered by inclusion. Finally, recall the support of an \(S\)-module \(H\), \(\text{supp}(H)\), over a ring \(S\) is the set of all prime ideals \(P\) of \(S\) such that \(H_P \neq 0\).

**Corollary 6.6** The ring \(R \leq_n M\) is arithmetical if and only if the following conditions are satisfied:

1. \(R\) is arithmetical,
2. For every \(i \in \{1, \ldots, n\}\), \(M_i\) is an arithmetical \(R\)-module,
3. For every \(P \in \bigcup_i \text{supp}(M_i)\), \(R_P\) is a valuation domain,
4. For every \(i \in \{1, \ldots, n\}\) and every \(P \in \text{supp}(M_i)\), \(M_{iP}\) is a divisible \(R_P\)-module, and
5. For every \(1 \leq i \leq j \leq n\), if \(P \in \text{supp}(M_i) \cap \text{supp}(M_j)\), then \(P \in \text{supp}(M_{j-i})\) and \(eM_{iP} = M_{jP}\) for every \(e \in M_{(j-i)P}\).

Recall that a ring \(S\) is called a generalized ZPI-ring (resp., a \(\pi\)-ring) if every proper ideal (resp., proper principal ideal) of \(S\) is a product of prime ideals. An integral domain which is a \(\pi\)-ring is called a \(\pi\)-domain. Clearly, a generalized ZPI-domain is nothing but a Dedekind domain. It is well known (for example, see [28, Sections 39 and 46]) that \(S\) is a \(\pi\)-ring (resp., a generalized ZPI-ring, a principal ideal ring (PIR)) if and only if \(S\) is a finite direct product.
of the following types of rings: (1) $\pi$-domains (resp., Dedekind domains, PIDs) which are not fields, (2) special principal ideal rings (SPIRs) and (3) fields.

Our next results extend [9] Lemma 4.9 and Theorem 4.10. They characterize when $R \bowtie_n M$ is a $\pi$-ring, a generalized ZPI-ring or a PIR.

**Lemma 6.7** If $R \bowtie_n M$ is a $\pi$-ring (resp., a generalized ZPI-ring, a PIR), then $R$ is a $\pi$-ring (resp., a generalized ZPI-ring, a PIR). Hence $R = R_1 \times \cdots \times R_s$ where $R_i$ is either (1) a $\pi$-domain (resp., a Dedekind domain, a PID) but not a field, (2) an SPIR, or (3) a field. Let $M_{j,i} = (0 \times \cdots 0 \times R_j \times 0 \times \cdots 0)M_i$ where $1 \leq i \leq n$ and $1 \leq j \leq s$. If $R_i$ is a domain or SPIR, but not a field, then $M_{j,i} = 0$ while if $R_i$ is a field, $M_{j,i} = 0$ or $M_{j,i} \cong R_i$.

Conversely, if $R = R_1 \times \cdots \times R_s$ and $M_i = M_{1,i} \times \cdots \times M_{s,i}$ are as above and $R$ is a $\pi$-ring (resp., a generalized ZPI-ring, a PIR), then $R \bowtie_n M$ is a $\pi$-ring (resp., a generalized ZPI-ring, a PIR).

**Proof.** Using Theorem 3.9, the proof is similar to that of [9] Lemma 4.9.

**Theorem 6.8** $R \bowtie_n M$ is a $\pi$-ring (resp., a generalized ZPI-ring, a PIR) if and only if $R$ is a $\pi$-ring (resp., a generalized ZPI-ring, a PIR) and $M_i$ is cyclic with annihilator $P_{i_1} \cdots P_{i_s}$ where $P_{i_1}, \ldots, P_{i_s}$ are some idempotent maximal ideals of $R$ (if $i_s = 0$, $\text{Ann}(M_i) = R$, that is, $M_i = 0$).

**Proof.** Similar to the proof of [9] Theorem 4.10.

We end the section with a remark on a question posed in [2]. Recall that a ring $R$ is called $m$-Boolean for some $m \in \mathbb{N}$, if $\text{char } R = 2$ and $x_1x_2 \cdots x_m(1 + x_1) \cdots (1 + x_m) = 0$ for all $x_1, \ldots, x_m \in R$. Thus Boolean rings are just 1-Boolean rings. It is shown in [2, Theorem 10] that 2-Boolean rings can be represented as trivial extensions. Namely, it is proved that if $R$ is 2-Boolean, then $R \cong B \bowtie \text{Nil}(R)$ where $B = \{b \in R | b^2 = b\}$ ([2, Theorem 10]). Based on this result the following natural question is posed (see [2] page 74]): Whether [2, Theorem 10] can be extended to $m$-Boolean rings for $m \geq 2$?

One can ask whether the $n$-trivial extension is the suitable construction to solve this question. Using [2, Theorem 6], one can show that the amalgamated algebras along an ideal (introduced in [34]) resolve partially this question. Recall that, given a ring homomorphism $f : A \rightarrow B$ and an ideal $J$ of $B$, the amalgamation of $A$ with $B$ along $J$ with respect to $f$ is the following subring of $A \times B$:

$$A \bowtie f B = \{(a, f(a) + j)|a \in A, j \in J\}.$$ 

Note that $A \bowtie f B \cong A \hat{+} J$ where $A \hat{+} J \subseteq A \times B$ is the ring whose underlying group is $A \hat{+} J$ with multiplication given by $(a, x) (a', x') = (aa', ax' + a'x + xx')$ for all $a, a' \in A$ and $x, x' \in J$. Here $J$ is an $A$-module via $f$ and then $ax' := f(a)x'$ and $a'x := f(a')x$ (see [34] for more details).

Now, if $R$ is $m$-Boolean for $m \geq 2$, then from [2, Theorems 6 and 7], $R = B \bowtie \text{Nil}(R)$ where $B = \{b \in R | b^2 = b\}$. Then

$$R \cong B \hat{+} \text{Nil}(R) \cong B \bowtie f \text{Nil}(R)$$
where $i : B \hookrightarrow R$ is the canonical injection.

Actually any $n$-trivial extension $R \ltimes_n M$ can be seen as the amalgamation of $R$ with $R \ltimes_n M$ along $0 \ltimes_n M$ with respect to the canonical injection. This leads to pose the following question for every $m \geq 2$: Is any $m$-Boolean ring an $m$-trivial extension?

## 7 Divisibility properties of $R \ltimes_n M$

Factorization in commutative rings with zero divisors was first investigated in a series of papers by Bouvier, Fletcher and Billis (see References), where the focus had been on the unicity property. The papers [1], [4], [8] marked the start of a systematic study of factorization in commutative rings with zero divisors. Since then, this theory has attracted the interest of a number of authors. The study of divisibility properties of the classical trivial extension has lead to some interesting examples and then to answering several questions (see [9], Section 5). In this section we are interested in extending a part of this study to the context of $n$-trivial extensions.

First, we recall the following definitions. Let $S$ be a commutative ring and $H$ an $S$-module. Two elements $e, f \in H$ are said to be associates (written $e \sim f$) (resp., strong associates (written $e \approx f$), very strong associates (written $e \cong f$)) if $Se = Sf$ (resp., $e = uf$ for some $u \in U(S)$, $e \sim f$ and either $e = f = 0$ or $e = rf$ implies $r \in U(S)$). Taking $H = S$ gives the notions of “associates” in $S$. We say that $H$ is strongly associate if for every $e, f \in H$, $e \sim f \Rightarrow e \approx f$. When $S$ is strongly associate as an $S$-module, we also say that $S$ is strongly associate. Finally, recall that $H$ is said to be $S$-présimplifiable if for $r \in S$ and $e \in H$, $re = e \Rightarrow r \in U(S)$ or $e = 0$. If $S$ is $S$-présimplifiable we only say that $S$ is présimplifiable.

We begin with an extension of [9, Theorem 5.1].

**Proposition 7.1** Let $R \subseteq S$ be a ring extension such that $U(S) \cap R = U(R)$.

1. If $S$ is présimplifiable, then every $R$-submodule of $S$ is présimplifiable. In particular, $R$ is présimplifiable.

2. Suppose that $S = R \oplus N$ as an $R$-module where $N$ is a nilpotent ideal of $S$ which satisfies either $N^2 = 0$ or $N = \bigoplus_{i \in \mathbb{N}} N_i$ as an $R$-module where $S = R \oplus N_1 \oplus N_2 \oplus \cdots$ is a graded ring. Then $S$ is présimplifiable if $R$ is présimplifiable and $N$ is $R$-présimplifiable.

**Proof.** 1. Let $H$ be an $R$-submodule of $S$. Consider $e = xe$ with $e \in H - \{0\}$ and $x \in R - \{0\}$. Since $S$ is présimplifiable, $x \in U(S)$ and so $x \in U(S) \cap R = U(R)$.

2. Let $x = r_x + n_x \neq 0$ and $y = r_y + n_y$ be two elements of $R \oplus N = S$ where $r_x, r_y \in R$ and $n_x, n_y \in N$, such that $x = yx$. Assume that $r_x \neq 0$. Then $r_x = r_y r_x$ implies that $r_y \in U(R) \subseteq U(S)$, and, since $N$ is nilpotent, $y = r_y + n_y$ is invertible in $S$, as desired. Next, assume now that $r_x = 0$. Then $n_x \neq 0$ and so $n_x = r_y n_x + n_y n_x$. In the case $N^2 = 0$, we have
Proposition 7.2 Let $R = \bigoplus_{i \in \mathbb{N}_0} R_i$ be a graded ring.

1. If $R$ is strongly associate, then $R_0$ is a strongly associate ring and $R_i$ is a strongly associate $R_0$-module for every $i \in \mathbb{N}$.

2. Suppose there exists $n \in \mathbb{N}$ such that $R_i = 0$ for every $i \geq n + 1$, that is, $R = R_0 \ltimes_n R_1 \ltimes \cdots \ltimes R_n$, and assume that $R_0$ is a présimplifiable ring and $R_1, \ldots, R_{n-1}$ are présimplifiable $R_0$-modules. Then $R$ is strongly associate if and only if $R_n$ is strongly associate.

Proof. 1. Let $x_i, y_i \in R_i - \{0\}$ for $i \in \mathbb{N}_0$ such that $R_0x_i = R_0y_i$. Then $Rx_i = Ry_i$. Hence there is $u = u_0 + u_1 + \cdots \in U(R)$ such that $x_i = uy_i$. Then $u_0 \in U(R_0)$ and $x_i = u_0y_i$, as desired.

2. Let $x = x_m + \cdots + x_n$ and $y = y_m + \cdots + y_n$ be two associate elements of $R$ where $m \in \{0, \ldots, n\}$ and $x_i, y_i \in R_i$ for $i \in \{m, \ldots, n\}$ such that $x_m$ and $y_m$ are nonzero. Then $x_m \sim y_m$. In particular, there is $\alpha = \alpha_0 + \cdots + \alpha_n$ such that $x = \alpha y$. Then $x_m = \alpha_0y_m$. Hence two cases occur. Case $m \neq n$. Since $R_m$ is présimplifiable, $\alpha_0 \in U(R_0)$. Then $\alpha \in U(R)$, as desired. Case $m = n$ (i.e., $x = x_m$ and $y = y_m$). Here, the result follows since $R_n$ is strongly associate.

Now we can give the extension of [9, Theorem 5.1] to the context of $n$-trivial extensions.

Corollary 7.3 The following assertions are true.

1. $R \ltimes_n M_1 \ltimes \cdots \ltimes M_n$ is présimplifiable if and only if $R, M_1, \ldots, M_n$ are présimplifiable.

2. If $R \ltimes_n M_1 \ltimes \cdots \ltimes M_n$ is strongly associate, then $R, M_1, \ldots, M_n$ are strongly associate.

3. Suppose that $R, M_1, \ldots, M_n$ are présimplifiable. Then $R \ltimes_n M_1 \ltimes \cdots \ltimes M_n$ is strongly associate if and only if $M_n$ is strongly associate.

Now we investigate the extension of [9, Theorem 5.4]. It is convenient to recall the following definitions. Let $S$ be a commutative ring. A nonunit $a \in S$ is said to be irreducible or an atom (resp., strongly irreducible, very strongly irreducible) if $a = bc$ implies $a \sim b$ or $a \sim c$ (resp., $a \approx b$ or $a \approx c$, $a \cong b$ or $a \cong c$) and $a$ is said to be $m$-irreducible if $Sa$ is a maximal element of the set of proper principal ideals of $S$. Note that, for a nonzero nonunit $a \in S$, a very strongly irreducible $\Rightarrow a$ is $m$-irreducible $\Rightarrow a$ is strongly irreducible $\Rightarrow a$ is irreducible, but none of these implications can be reversed. In the case of an $S$-module $H$, we say that $e \in H$ is $S$-primitive (resp., strongly $S$-primitive, very strongly $S$-primitive) if for $a \in S$ and $f \in H$, $e = af \Rightarrow e \sim f$ (resp., $e \approx f$, $e \cong f$). And $e$ is $S$-superprimitive if $be = af$ for $a, b \in S$ and
f ∈ H, implies a | b. Note that (1) e is S-primitive ⇔ Se is a maximal cyclic S-submodule of H; (2) e is S-superprimitive ⇒ e is very strongly S-primitive ⇒ e is strongly S-primitive ⇒ e is S-primitive, (3) if \( \text{Ann}(e) = 0 \), e is S-primitive ⇒ e is very strongly S-primitive, and (4) e is S-superprimitive ⇒ \( \text{Ann}(e) = 0 \).

In the following results the homogeneous element \( (0, ..., 0, m_i, 0, ..., 0) \) ∈ \( R \ltimes_n M \) where \( i \in \{1, ..., n\} \) and \( m_i \in M_i \), is denoted by \( \overline{m_i} \). The following result extends [9, Theorem 5.4 (1)].

**Proposition 7.4** Let \( i \in \{1, ..., n\} \) and \( m_i, n_i \in M_i \) \( \setminus \{0\} \). Then \( m_i \sim n_i \) (resp., \( m_i \approx n_i \), \( m_i \cong n_i \)) in \( M_i \) if and only if \( m_i \sim n_i \) (resp., \( m_i \approx n_i \), \( m_i \cong n_i \)) in \( R \ltimes_n M \).

**Proof.** The assertion is proved similarly to the corresponding classical one.

It is worth noting that the analogue of the assertion (4) of [9, Theorem 5.4] does not hold in the context of n-trivial extensions with \( n \geq 2 \). Indeed, consider the 2-trivial extension \( S = \mathbb{Z}_4 \ltimes_2 \mathbb{Z}_4 \ltimes \mathbb{Z}_4 \). It is easy to show that \( \overline{1} \) is superprimitive in the \( \mathbb{Z}_4 \)-module \( \mathbb{Z}_4 \). However, \( (0, 0, 1) \) is not very strongly irreducible in \( S \) (since \( (\overline{2}, \overline{1}, \overline{2})^2 = (0, 0, 1) \)). Moreover, even if we assume that \( R \) is an integral domain, we still don’t have the desired analogue. For this, take \( S = \mathbb{Z} \ltimes_2 \mathbb{Z} \ltimes \mathbb{Z} \). We have \( 1 \) is superprimitive in the \( \mathbb{Z} \)-module \( \mathbb{Z} \). However, \( (0, 0, 1) \) is not very strongly irreducible in \( S \) (since \( (0, 1, 0)^2 = (0, 0, 1) \)). The last example also shows that the assertion (2) of [9, Theorem 5.4] does not hold in the context of n-trivial extensions with \( n \geq 2 \). Namely, if \( 0 \neq m_i = m_j m_k \) where \( (m_i, m_j, m_k) \in M_i \times M_j \times M_k \), \( i \geq 2 \) and \( j, k \in \{1, ..., i - 1\} \) with \( j + k = i \), then \( m_i \) cannot be irreducible. Indeed, \( m_i = m_j m_k \) but neither \( m_j \) nor \( m_k \) are in \( \langle m_i \rangle \subseteq 0 \ltimes_n \mathbb{Z} \ltimes \mathbb{Z} \ltimes M_i \ltimes \mathbb{Z} \ltimes M_i \ltimes \mathbb{Z} \ltimes \mathbb{Z} \ltimes M_i \).

To extend [9, Theorem 5.1 (2)], we need to introduce the following definitions.

**Definition 7.5** Assume \( n \geq 2 \) and each multiplication in the family \( \varphi = \{ \varphi_{i,j} \}_{\substack{1 \leq i,j \leq n \atop i \neq j}} \) is not trivial. Let \( i \in \{2, ..., n\} \). An element \( m_i \in M_i \setminus \{0\} \) is said to be \( \varphi \)-indecomposable (or indecomposable relative to the family of multiplications \( \varphi \)) if, for every \( (m_j, m_k) \in M_j \times M_k \) (where \( j, k \in \{1, ..., i - 1\} \) with \( j + k = i \)), \( m_i \neq m_j m_k \). If no ambiguity can arise, \( \varphi \)-indecomposable elements are simply called indecomposables.

For example, in \( \mathbb{Z} \ltimes_2 \mathbb{Z} \ltimes \mathbb{Q} \), every element in \( \mathbb{Q} - \mathbb{Z} \) is indecomposable. However, every element \( x \in \mathbb{Z} \) (\( \mathbb{Z} \) as a submodule of \( \mathbb{Q} \)) is decomposable (since \( (0, 1, 0)(0, x, 0) = (0, 0, x) \)).

**Definition 7.6** Let \( i \in \{2, ..., n\} \). The \( R \)-module \( M_i \) is said to be \( \varphi \)-integral (or integral relative to the family of multiplications \( \varphi \)) if, for every \( (m_j, m_k) \in M_j \times M_k \) (where \( j, k \in \{1, ..., i - 1\} \) with \( j + k = i \)), \( m_j m_k = 0 \) implies that \( m_j = 0 \) or \( m_k = 0 \). If no ambiguity can arise, a \( \varphi \)-integral \( R \)-module is simply called integral.
For example, for $\mathbb{Z} \times 2 \mathbb{Z} \times \mathbb{Z}$, $M_2 = \mathbb{Z}$ is integral. And, for $\mathbb{Z} \times 2 \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z}$ is not integral since, for instance, $\varphi_{1,1}(1,2) = \overline{1} \overline{2} = \overline{0}$.

**Proposition 7.7** Assume $n \geq 2$. Let $i \in \{1, \ldots, n\}$ and $m_i \in M_i - \{0\}$. If $m_i$ is irreducible (resp., strongly irreducible, very strongly irreducible) in $R \leftrightharpoons n \ M$, then $m_i$ is primitive (resp., strongly primitive, very strongly primitive) in $M_i$.

Conversely, three cases occur:

**Case** $i = 1$. The reverse implication holds if $R$ is an integral domain and $M_j$ is torsion-free for every $j \in \{2, \ldots, n\}$.

**Case** $i = 2$ (here $n \geq 2$). The reverse implication holds if $R$ is an integral domain, $M_j$ is torsion-free for every $j \in \{1, \ldots, n\} - \{2\}$ and $m_2$ is indecomposable.

**Case** $i \geq 3$ (here $n \geq 3$). The reverse implication holds if $R$ is an integral domain, $M_j$ is torsion-free for every $j \in \{1, \ldots, n\} - \{i\}$, $M_j$ is integral for every $j \in \{2, \ldots, i-1\}$, and $m_i$ is indecomposable.

**Proof.** We only prove the primitive (irreducible) case. The other two cases are proved similarly.

$\Rightarrow$ Suppose that $m_i$ is irreducible and let $m_i = an_i$ for some $a \in R$ and $n_i \in M_i$. Then $m_i = (a,0,\ldots,0)n_i$ and so $(R \leftrightharpoons n \ M)m_i = (R \leftrightharpoons n \ M)n_i$. This implies that $Rm_i = Rn_i$, as desired.

$\Leftarrow$ Let $m_i = (a_j)(n_j)$ for some $(a_j), (n_j) \in R \leftrightharpoons n \ M$. Then $a_0n_0 = 0$. First, we show that the case $a_0 = 0$ and $n_0 = 0$ is impossible. Cases $i = 1, 2$ are easy and are left to the reader. So assume $i \geq 3$. Suppose that $a_0 = 0$ and $n_0 = 0$. Then we have the following equalities:

For $j \in \{2, \ldots, i-1\}$, $a_1n_{j-1} + a_2n_{j-2} + \cdots + a_{j-1}n_1 = 0$ and $a_1n_{i-1} + a_2n_{i-2} + \cdots + a_{i-1}n_1 = m_i$.

A recursive argument on these equalities shows that, for $l \in \{2, \ldots, i\}$, there is $k \in \{0, \ldots, l-1\}$ such that $(a_0, \ldots, a_k) = (0, \ldots, 0)$ and $(n_0, \ldots, n_{l-(k+1)}) = (0, \ldots, 0)$. Indeed, it is clear this is true for $l = 2$. Then suppose this is true for a given $l \in \{2, \ldots, i-1\}$. So the equality $a_1n_{l-1} + a_2n_{l-2} + \cdots + a_{l-1}n_1 = 0$ becomes $a_{k+1}n_{l-(k+1)} = 0$. Then since $M_i$ is integral, we get the desired result for $l + 1$. Thus for $l = i$, we get $a_{k+1}n_{l-(k+1)} = m_i$ which is absurd since $m_i$ is indecomposable.

Now, we may assume that $a_0 \neq 0$ and $n_0 = 0$, so $aqn_1 = 0$. Since $M_1$ is torsion-free, $n_1 = 0$. Recursively we get $n_j = 0$ for $j \in \{1, \ldots, i-1\}$. Then $a_0n_i = m_i$, and since $m_i$ is primitive, there exists $b_0 \in R$ such that $n_i = b_0m_i$. It remains to show that there is $b_j \in M_j$ for $j \in \{1, \ldots, n-i\}$ such that $n_{i+j} = b_j m_i$ and this implies that $(n_i) = (0, \ldots, 0, n_i, n_{i+1}, \ldots) = (b_0, \ldots, b_{n-i}, 0, \ldots, 0)m_i$, as desired. We have $aqn_{i+1} + a_1n_i = 0$. Then using both $aqn_i = m_i$ and $n_i = b_0m_i$, we get $a_0n_{i+1} + a_1b_0n_i = 0$. Then $a_0(n_{i+1} + a_1b_0n_i) = 0$, so $n_{i+1} + a_1b_0n_i = 0$ (since $M_{i+1}$ is torsion-free). Then $n_{i+1} = -a_1b_0^2m_i$. Then we set $b_1 = -a_1b_0^2$ and so $n_{i+1} = b_1m_i$. Similarly, using the equality $aqn_{i+2} + a_1n_{i+1} + a_2n_i = 0$ with the equalities $aqn_i = m_i$ and $n_i = b_0m_i$, we get $a_0n_{i+2} + a_1b_1aqn_i + a_2b_0rn_i = 0$. So $n_{i+2} + a_1b_1n_i = 0$. Finally, a recursive argument gives the desired result. ■
The following result extends [9, Theorem 5.4 (3)].

**Proposition 7.8** Suppose that $R$ has a nontrivial idempotent. Then for every $i \in \{1, ..., n\}$ and $m_i \in M_i \setminus \{0\}$, $m_i$ is not irreducible in $R \bowtie_n M$.

**Proof.** The assertion is proved similarly to the corresponding classical one. $\blacksquare$

Now we are interested in some factorization properties. Recall that a ring $S$ is called *atomic* if every (nonzero) nonunit of $S$ is a product of irreducible elements (atoms) of $S$. Note that, as in the domain case, the ascending chain condition on principal ideals (ACCP) implies atomic.

We begin with an extension of [9, Theorem 5.5 (2)] which characterizes when a trivial extension of a ring satisfies ACCP. For this, we need the following lemma.

**Lemma 7.9** Let $i \in \{0, ..., n\}$ and consider two elements $a = (0, ..., 0, a_i, a_{i+1}, ..., a_n)$ and $b = (0, ..., 0, b_i, b_{i+1}, ..., b_n)$ of $R \bowtie_n M$ with $a_i \neq 0$. Then the implication “$(a) \subsetneq (b) \Rightarrow b_i \neq 0$ and $(a_i) \subsetneq (b_i)$” is true if either (1) $0 \leq i \leq n - 1$ and $M_i$ is présimplifiable (here $M_0 = R$) or (2) $i = n$.

**Proof.** Since $(a) \subsetneq (b)$, there is $c = (c_0, ..., c_n) \in R \bowtie_n M - U(R \bowtie_n M)$ such that $a = cb$. Then $a_i = c_0 b_i$ and $c_0 \notin U(R)$. This shows that $(a_i) \subsetneq (b_i)$ in both cases. $\blacksquare$

**Theorem 7.10** Assume $n \geq 2$. Suppose that $M_i$ is présimplifiable for every $i \in \{0, ..., n - 1\}$ (here $M_0 = R$). Then $R \bowtie_n M$ satisfies ACCP if and only if $R$ satisfies ACCP and, for every $i \in \{1, ..., n\}$, $M_i$ satisfies ACC on cyclic submodules.

**Proof.** The proof of the direct implication is easy. Let us prove the converse. Suppose that $R \bowtie_n M$ admits a strictly ascending chain of principal ideals $$(a_1, i) \subsetneq (a_2, i) \subsetneq \cdots$$

If there exists $j_0 \in \mathbb{N}$ such that $a_{j_0, 0} \neq 0$. Then for every $k \geq j_0$, $a_{k, 0} \neq 0$. Then by Lemma 7.9, we get the following strictly ascending chain of principal ideals of $R$: $$\langle a_{j_0, 0} \rangle \subsetneq \langle a_{j_0+1, 0} \rangle \subsetneq \cdots$$

This is absurd since $R$ satisfies ACCP. Now, suppose that $a_{j, 0} = 0$ for every $j \in \mathbb{N}$ and that there exists $j_1 \in \mathbb{N}$ such that $a_{j_1, 1} \neq 0$. Also, by Lemma 7.9, we obtain the following strictly ascending chain of cyclic submodules of $M_1$: $$\langle a_{j_1, 1} \rangle \subsetneq \langle a_{j_1+1, 1} \rangle \subsetneq \cdots$$

This is absurd since $M_1$ satisfies ACC on cyclic submodules. We continue in this way until the case where we may suppose that $a_{j, i} = 0$ for every $i \in \{1, ..., n - 1\}$ and every $j \in \mathbb{N}$. Therefore, by Lemma 7.9, we get the desired result. $\blacksquare$
Now, we investigate when $R \times_n M$ is atomic. Namely, we give an extension of Theorem 5.5 (4). Recall that an $R$-module $N$ is said to satisfy MCC if every cyclic submodule of $N$ is contained in a maximal (not necessarily proper) cyclic submodule of $N$.

**Theorem 7.11** Assume $n \geq 2$. Suppose that $M_i$ is présimplifiable for every $i \in \{0, \ldots, n-1\}$ (here $M_0 = R$). Then $R \times_n M$ is atomic if $R$ satisfies ACCP, $M_i$ satisfies ACC on cyclic submodules, for every $i \in \{1, \ldots, n-1\}$, and $M_n$ satisfies MCC.

**Proof.** The proof is slightly more technical than the one of Theorem 5.5 (4). Here, we need to break the proof into the following $n + 1$ steps such that in the step number $k \in \{1, \ldots, n+1\}$ we prove that every nonunit element $(m_i) \in R \times_n M$ with $m_0 = 0, \ldots, m_k = 0$ and $m_k \neq 0$ is a product of irreducibles.

We use an inductive argument for the first $n$ steps.

**Step 1.** Suppose there is a nonunit element $(m_i)$ of $R \times_n M$ with $m_0 \neq 0$ and such that $(m_i)$ cannot be factored into irreducibles. Then there exist $(a_{1,i}), (b_{1,i}) \in R \times_n M - U(R \times_n M)$ such that $(m_i) = (a_{1,i}) (b_{1,i})$ and neither $(m_i)$ nor $(a_{1,i})$ nor $(b_{1,i})$ are associate. Since $0 \neq m_0 = a_{1,0} b_{1,0}, a_{1,0} \neq 0$ and $b_{1,0} \neq 0$. Clearly $(a_{1,i})$ or $(b_{1,i})$ must be reducible, say $(a_{1,i})$. Also, for $(a_{1,i})$ there are $(a_{2,i}), (b_{2,i}) \in R \times_n M - U(R \times_n M)$ such that $(a_{1,i}) = (a_{2,i}) (b_{2,i})$ and neither $(a_{1,i})$ nor $(a_{2,i})$ nor $(b_{2,i})$ are associate. As above, $a_{2,0} \neq 0$ and $b_{2,0} \neq 0$ and say $(a_{2,i})$ is reducible. So we continue and then we obtain a strictly ascending chain

$$
\langle (m_i) \rangle \subsetneq \langle (a_{1,i}) \rangle \subsetneq \langle (a_{2,i}) \rangle \subsetneq \cdots
$$

Using Lemma [7.9] we get a strictly ascending chain of principal ideals of $R$

$$
\langle m_0 \rangle \subsetneq \langle a_{1,0} \rangle \subsetneq \langle a_{2,0} \rangle \subsetneq \cdots
$$

This is absurd since $R$ satisfies ACCP.

**Step $j$ ($1 \leq j \leq n$).** Suppose there is a nonunit element $(m_i) \in R \times_n M$ with $m_0 = 0, \ldots, m_{j-2} = 0$ and $m_{j-1} \neq 0$ that is not a product of irreducibles. Then there are $(a_{1,i}), (b_{1,i}) \in R \times_n M - U(R \times_n M)$ such that $(m_i) = (a_{1,i}) (b_{1,i})$ and neither $(m_i)$ nor $(a_{1,i})$ nor $(b_{1,i})$ are associate. Then $a_{1,0} b_{1,j-1} + a_{1,1} b_{1,j-2} + \cdots + a_{1,j-2} b_{1,1} + a_{1,j-1} b_{1,0} = m_{j-1} \neq 0$. If $a_{1,k} = 0$ for every $k \in \{0, \ldots, j-2\}$, then necessarily $b_{1,0} \neq 0$. Hence by the preceding steps, $(b_{1,i})$ is a product of irreducibles and then by hypothesis on $(m_i), (a_{1,i})$ is reducible. If $a_{1,k} \neq 0$ for some $k \in \{0, \ldots, j-2\}$, then $(a_{1,i})$ is a product of irreducibles and $(b_{1,i})$ is reducible. Thus, by symmetry, we may assume that $(a_{1,i})$ is reducible and it is not a product of irreducibles. So, necessarily $a_{1,0} = 0, \ldots, a_{1,j-2} = 0$ and $a_{1,j-1} \neq 0$. We repeat the last argument so that we obtain a strictly ascending chain of principal ideals of $R \times_n M$

$$
\langle (m_i) \rangle \subsetneq \langle (a_{1,i}) \rangle \subsetneq \langle (a_{2,i}) \rangle \subsetneq \cdots
$$
such that, for every $k \in \mathbb{N}$, $a_{k,0} = 0$, ..., $a_{k,j-2} = 0$ and $a_{k,j-1} \neq 0$. Then, by Lemma \[7.9\], we get a strictly ascending chain of cyclic submodules of $M_{j-1}$

$$\langle m_{j-1} \rangle \subset \langle a_{1,j-1} \rangle \subset \langle a_{2,j-1} \rangle \subset \cdots$$

which is absurd by hypothesis on $M_{j-1}$, as desired.

**Step n + 1.** It remains to prove that every element of $R \times_n M$ of the form $(0, ..., 0, m_n)$ with $m_n \neq 0$ is a product of irreducibles. Since $M_n$ satisfies MCC, $Rm_n \subseteq Rm$ where $Rm$ is a maximal cyclic submodule of $M_n$. Then $m_n = am$ for some $a \in R - \{0\}$ and so $(0, ..., 0, m_n) = (a, 0, ..., 0)(0, ..., 0, m)$. Now, $a \neq 0$ shows that $(a, 0, ..., 0)$ is a product of irreducibles (by Step 1) and $Rm$ is maximal shows that either $(0, ..., 0, m)$ is irreducible or $(0, ..., 0, m) = (a_i)(b_i)$ where $a_k \neq 0$ and $b_l \neq 0$ for some $k, l \in \{0, ..., n - 1\}$. Then by the preceding steps, $(a_i)$ and $(b_i)$ are products of irreducibles and hence so is $(0, ..., 0, m)$. This concludes the proof. ■

A ring $S$ is said to be a bounded factorial ring (BFR) if, for each nonzero nonunit $x \in S$, there is a natural number $N(x)$ so that for any factorization $x = x_1 \cdots x_s$ where each $x_i$ is a nonunit, we have $s \leq N(x)$. For domains we say BFD instead of BFR. Recall that an $S$-module $H$ is said to be a BF-module if, for each nonzero nonunit $h \in H$, there exists a natural number $N(h)$ so that $h = a_1 \cdots a_{s-1}h_s$ (each $a_i$ a nonunit) $\Rightarrow$ $s \leq N(h)$.

Our next theorem, which is a generalization of [9] Theorem 5.5 (4), investigates when $R \times_n M$ is BFR. It is based on the following lemma.

**Lemma 7.12** For $j \in \mathbb{N} - \{1\}$, a product of $j$ elements of $R \times_n M$ of the form $(0, x_1, ..., x_n)$ is of the form $(0, 0, y_j, ..., y_n)$ (where, if $j \geq n + 1$ the product is zero).

**Theorem 7.13** Assume that $n \geq 2$, $R$ is an integral domain and $M_i$ is torsion-free for every $i \in \{1, ..., n - 1\}$. Then $R \times_n M$ is a BFR if and only if $R$ is a BFD and $M_i$ is a BF-module for every $i \in \{1, ..., n\}$.

**Proof.** --- Clear.

$\Leftarrow$ Let $(m_i)$ be a nonzero nonunit element of $R \times_n M$ and suppose we have a factorization into nonunits $(m_i) = (a_{1,i}) \cdots (a_{s,i})$ for some $s \in \mathbb{N}$. If $m_0 \neq 0$, $m_0 = a_{1,0} \cdots a_{s,0}$ implies that $s \leq N(m_0)$. Otherwise, there is $j \in \{1, ..., n\}$ such that $m_0 = 0, ..., m_{j-1} = 0$ and $m_j \neq 0$. We may assume that $s \geq j + 1$. Since $R$ is an integral domain and by Lemma \[7.12\], we may assume there is $k \in \{1, ..., j\}$ such that $a_{l,0} = 0$ for every $l \in \{1, ..., k\}$ and $a_{l,0} \neq 0$ for every $l \in \{k + 1, ..., s\}$. Let $(0, 0, b_k, ..., b_s) = \prod_{l=1}^{k}(a_{l,i})$ and $(c_0, ..., c_n) = \prod_{l=k+1}^{s}(a_{l,i})$. Since $M_i$ is torsion-free for every $i \in \{1, ..., j - 1\}$ and $c_0 = \prod_{l=k+1}^{s}a_{l,0} \neq 0$, $b_k = 0, ..., b_{j-1} = 0$ and $b_j \neq 0$. Then $m_j = c_0b_j = \prod_{l=k+1}^{s}a_{l,0}b_j$. Therefore $s \leq N(m_j) + k - 1$ (since $M_j$ is a BF-module). ■
Now we investigate the notion of a $U$-factorization. It was introduced by Fletcher \cite{35,36} and developed by Axtell et al. in \cite{11} and \cite{12}. Let $S$ be a ring and consider a nonunit $a \in S$. By a factorization of $a$ we mean $a = a_1 \cdots a_s$ where each $a_i$ is a nonunit. Recall from \cite{35} that, for $a \in S$, $U(a) = \{ r \in S \mid \exists s \in S \text{ with rsa = a} \} = \{ r \in S \mid r(a) = (a) \}$. A $U$-factorization of $a$ is a factorization $a = a_1 \cdots a_s b_1 \cdots b_t$ where, for every $1 \leq i \leq s$, $a_i \in U(b_1 \cdots b_t)$ and, for every $1 \leq i \leq t$, $b_i \notin U(b_1 \cdots \hat{b}_i \cdots b_t)$. We denote this $U$-factorization by $a = a_1 \cdots a_s [b_1 \cdots b_t]$ and call $a_1, ..., a_s$ (resp., $b_1, ..., b_t$) the irrelevant (resp., the relevant) factors.

Our next result investigates when an $n$-trivial extension is a $U$-FFR. First, recall the following definitions.

A ring $S$ is called a finite factorization ring (FFR) (resp., a $U$-finite factorization ring ($U$-FFR)) if every nonzero nonunit of $S$ has only a finite number of factorizations (resp., $U$-factorizations) up to order and associates (resp., associates on the relevant factors). A ring $S$ is called a weak finite factorization ring (WFR) (resp., a $U$-weak finite factorization ring ($U$-WFR)) if every nonzero nonunit of $R$ has only a finite number of nonassociate divisors (resp., nonassociate relevant factors). We have $FFR \Rightarrow WFR$ and the converse holds in the domain case. But $\mathbb{Z}_2 \times \mathbb{Z}_2$ is a WFR that is not an FFR. However, from \cite{11} Theorem 2.9], $U$-FFR $\Leftrightarrow$ $U$-WFR.

The study of the notions above on the classical trivial extensions has lead to consider the following notion (see \cite{11}). Let $N$ be an $S$-module. For a nonzero element $x \in N$, we say that $Sd_1d_2 \cdots d_s x$ is a reduced submodule factorization if, for every $j \in \{1, ..., s\}$, $d_j \notin U(S)$ and for no cancelling and reordering of the $d_j$’s is it the case that $Sd_1d_2 \cdots d_s x = Sd_1d_2 \cdots d_s x$ where $t < s$. The module $N$ is said to be a $U$-FF module if for every nonzero element $x \in N$, there exist only finitely many reduced submodule factorizations $Sx = Sd_1d_2 \cdots d_s x$, up to order and associates on the $d_i$, as well as up to associates on the $x_k$. In our context, we introduce the following definition.

**Definition 7.14** Assume $n \geq 2$ and consider $i \in \{1, ..., n\}$.

1. Let $m_i \in M_i - \{0\}$, $s \in \mathbb{N}$ and $(d_{i_1}, ..., d_{i_s}) \in M_{i_1} \times \cdots \times M_{i_s}$ where $\{i_1, i_2, ..., i_s\} \subseteq \{0, ..., n\}$ with $i_1 + \cdots + i_s = i$. We say that $Rd_{i_1}d_{i_2} \cdots d_{i_s} m_i \subseteq M_i$ is a $\varphi$-reduced submodule factorization if, for every $j \in \{1, ..., s\}$ such that $i_j = 0$, $d_j \notin U(R)$ and for no cancelling and reordering of the $d_j$’s is it the case that $Rd_{i_1}d_{i_2} \cdots d_{i_s} = Rd_{i_1}d_{i_2} \cdots d_{i_t}$ where $t < s$. If no ambiguity can arise, a $\varphi$-reduced submodule factorization is simply called a reduced submodule factorization.

2. The $R$-module $M_i$ is said to be a $\varphi$-$U$-FF module (or simply $U$-FF module) if, for every nonzero element $x \in M_i$, there exist only finitely many reduced submodule factorizations $Rx = Rd_{i_1}d_{i_2} \cdots d_{i_s}$, up to order and associates on the $d_i$.

It is clear that, for $i = 1$, the notion of $U$-FF module defined here is the same as the Axtell’s one.
Based on the proof of [11, Theorem 4.2], it is asserted in [12, Theorem 3.6] that, if \( R \bowtie 1 M_1 \) is a \( U \)-FFR, then for every nonzero nonunit \( d \in R \), there are only finitely many distinct principal ideals \( \langle (d, m) \rangle \) in \( R \bowtie 1 M_1 \). However, a careful reading of this proof shows that the case of ideals \( \langle (d, m) \rangle \) with \( dM_1 = 0 \) should be also treated. One can confirm the validity of this assertion for reduced rings. However, the context of \( n \)-trivial extensions seems to be more complicated. Nevertheless, under some certain conditions, we next investigate when \( R \bowtie n M \) is a \( U \)-FFR.

**Lemma 7.15** Assume \( n \geq 2 \) and \( M_n \) is integral. Then for every nonzero nonunit \( d \in R \), the following assertions are true.

1. For every \( i \in \{1, \ldots, n-1\} \), the following assertions are equivalent:
   - 1.a. \( dM_i = 0 \).
   - 1.b. \( dm_i = 0 \) for some \( m_i \in M_i - \{0\} \).
   - 1.c. \( dM_{n-i} = 0 \).
   - 1.d. \( dm_{n-i} = 0 \) for some \( m_{n-i} \in M_{n-i} - \{0\} \).

2. The following assertions are equivalent:
   - 2.a. \( dM_i = 0 \) for some \( i \in \{1, \ldots, n-1\} \).
   - 2.b. \( dM_i = 0 \) for every \( i \in \{1, \ldots, n-1\} \).

3. If \( dM_n = 0 \), then \( dM_i = 0 \) for every \( i \in \{1, \ldots, n-1\} \).

4. If \( M_n \) is torsion-free, then \( M_i \) is torsion-free for every \( i \in \{1, \ldots, n-1\} \).

**Proof.** (1). For the implications (1.a) \( \Rightarrow \) (1.b) and (1.c) \( \Rightarrow \) (1.d) there is nothing to prove.

(1.b) \( \Rightarrow \) (1.c). Let \( m \in M_{n-i} \). Then \( dm_i m = 0 \in M_n \). Therefore \( dm = 0 \) (since \( M_n \) is integral and \( m_i \neq 0 \)).

(1.d) \( \Rightarrow \) (1.a). Similar to the previous proof.

(2). For the implication (2.b) \( \Rightarrow \) (2.a) there is nothing to prove.

(2.a) \( \Rightarrow \) (2.b). First, we prove that \( dM_1 = 0 \). For every \( m_1 \in M_1 - \{0\} \), \( dm_1 = 0 \in M_i \) and so \( dm^n_1 = 0 \in M_n \). Therefore \( dm_1 = 0 \) (since \( M_n \) is integral and \( m_1 \neq 0 \)). Now, consider any \( j \in \{1, \ldots, n-1\} \) and any \( m_j \in M_j - \{0\} \). Then for every \( m_1 \in M_1 - \{0\} \), \( dm_j m^n_{1-j} = 0 \in M_n \) which shows that \( dm_j = 0 \).

(3). This is proved as above.

(4). If there is \( m_1 \in M_1 - \{0\} \) and \( r \in R - \{0\} \) such that \( rm_1 = 0 \), then \( rm^n_1 = 0 \in M_n \). Since \( M_n \) is torsion-free and \( r \neq 0 \), \( m^n_1 = 0 \in M_n \) so \( m_1 = 0 \) (since \( M_n \) is integral). This is absurd since \( m_1 \neq 0 \). Finally, by assertions (1) and (2), we conclude that \( M_i \) is torsion-free for every \( i \in \{1, \ldots, n-1\} \). ■
Theorem 7.16 Assume \( n \geq 2 \). If \( R \varpropto_n M \) is a U-FFR (equivalently, a U-WFFR), then the following conditions are satisfied:

1. \( R \) is an FFR.

2. \( M_i \) is a U-FF module for every \( i \in \{1, \ldots, n\} \).

Moreover, if \( R \) is an integral domain and \( M_n \) is integral and torsion-free, then

3. For every nonzero nonunit \( d \in R \), there are only finitely many distinct principal ideals \( \langle (d,m_1,\ldots,m_n) \rangle \) in \( R \varpropto_n M \).

4. For every \( i \in \{1, \ldots, n-1\} \) and every \( m \in M_i - \{0\} \), there are only finitely many distinct principal ideals \( \langle (0,\ldots,0,m,m_{i+1},\ldots,m_n) \rangle \) in \( R \varpropto_n M \).

Conversely, if \( R \) is an integral domain and \( M_n \) is integral and torsion-free, then the assertions (1)-(4) imply that \( R \varpropto_n M \) is a U-FFR.

Proof. The proof of the “converse” part is similar to the corresponding one of [11] Theorem 4.2.\( \Rightarrow \) The proof of each (1) and (2) is similar to that given in [11] Theorem 4.2.

3. Suppose, by contradiction, there exists a nonzero nonunit \( d \in R \) for which there is a family of distinct principal ideals of the form \( \langle (d,m_1,\ldots,m_{j,n}) \rangle \) where \( j \) is in an infinite indexing set \( \Gamma \). We prove this is impossible by showing that, for every \( j \neq k \) in \( \Gamma \), there exists \( (1,x_1,\ldots,x_n) \in R \varpropto_n M \) such that \( (d,m_{j,1},\ldots,m_{j,n}) = (1,x_1,\ldots,x_n)(d,m_{k,1},\ldots,m_{k,n}) \).

A recursive argument shows that the fact that every equation \( dX = b_i \), with \( b_i \in M_i \) admits a solution \( X \in M_i \) implies the existence of the desired \( (1,x_1,\ldots,x_n) \). Note that, from Lemma 7.15, \( M_i \) is torsion-free for every \( i \in \{1, \ldots, n\} \). First, consider an element \( b_n \in M_n - \{0\} \). For every \( j \in \Gamma \), \( (d,m_{j,1},\ldots,m_{j,n})(0,\ldots,0,b_n) = (0,\ldots,0,db_n) \). Then \( (0,\ldots,0,db_n) = (d,m_{j,1},\ldots,m_{j,n})[(0,\ldots,0,b_n)] \) is the only possible corresponding U-factorization of \( (0,\ldots,0,db_n) \) (since \( R \varpropto_n M \) is a U-FFR), so there exists \( r \in R \) such that \( b_n = db_n \). This shows that the above equation admits a solution for \( i = n \). Now consider \( k \in \{1, \ldots, n-1\} \) and any \( b_k \in M_k \). For every \( b_{n-k} \in M_{n-k} - \{0\} \), \( b_kb_{n-k} \in M_n - \{0\} \) and so there is \( r \in R \) such that \( b_kb_{n-k} = db_kb_{n-k} \). Then \( (b_k - db_k)b_{n-k} = 0 \). Therefore \( b_k = db_k \) (since \( M_n \) is integral).

4. Let \( i \in \{1, \ldots, n-1\} \). Suppose, by contradiction, there exists \( m \in M_i - \{0\} \), for which there is a family of distinct principal ideals of the form \( \langle (0,\ldots,0,m,m_{i+1},\ldots,m_{j,n}) \rangle \) where \( j \) is in an infinite indexing set \( \Gamma \). Let \( m_{n-i} \in M_{n-i} - \{0\} \). Necessarily, \( mm_{n-i} \neq 0 \). Then

\[
(0,\ldots,0,m,m_{j,i+1},\ldots,m_{j,n})(0,\ldots,0,m_{n-i},0,\ldots,0) = (0,\ldots,0,mm_{n-i}).
\]

Then \( (0,\ldots,0,mm_{n-i}) = (0,\ldots,0,m,m_{j,i+1},\ldots,m_{j,n})[(0,\ldots,0,m_{n-i},0,\ldots,0)] \) is the only possible corresponding U-factorization of \( (0,\ldots,0,mm_{n-i}) \) (since \( R \varpropto_n M \) is a U-FFR), so there exists \( (r_0,r_1,\ldots,r_n) \) such that

\[
(0,\ldots,0,m,m_{j,i+1},\ldots,m_{j,n})(r_0,r_1,\ldots,r_n)(0,\ldots,0,m_{n-i},0,\ldots,0) = (0,\ldots,0,m_{n-i},0,\ldots,0),
\]
equivalently \((0, \ldots, 0, r_0m_{m-1}) = (0, \ldots, 0, m_{n-1}, 0, \ldots, 0)\), which is absurd. 

A ring \(S\) is called a \textit{\(U\)-bounded factorization} ring (\(U\)-BFR) if, for each nonzero nonunit \(x \in S\), there is a natural number \(N(x)\) so that, for any factorization \(x = a[b_1 \cdots b_t]\), we have \(t \leq N(x)\). An \(S\)-module \(H\) is said to be a \(U\)-BF module if for every \(h \in H - \{0\}\) there exists a natural number \(N(h)\) so that if \(Sh = Sd_1 \cdots d_t h'\) where \(d_j \notin U(S)\), \(t > N(h)\) and \(h' \in H\), then, after cancellation and reordering of some of the \(d_j\)'s we have \(Sh = Sd_1 \cdots d_s h'\) for some \(s \leq N(h)\).

The question of when the classical trivial extension is a \(U\)-BFR is still open. However, there is an answer to this question for an integral domain \(D\) [11, Theorem 4.4]: For a \(D\)-module \(N\), \(D \times N\) is a \(U\)-BFR if and only if \(D\) is a BFD and \(N\) is a \(U\)-BF \(R\)-module. Two more general results for the direct implication were established in [12, Theorem 3.7 and Lemma 3.8]. Here, we extend these results to our context. For this we need to introduce the following definition.

**Definition 7.17** Assume \(n \geq 2\) and consider \(i \in \{1, \ldots, n\}\). The \(R\)-module \(M_i\) is said to be a \(\varphi\)-\(U\)-BF module (or simply a \(U\)-BF module) if, for every nonzero element \(x \in M_i\), there exists a natural number \(N(x)\) so that if \(Rx = Rd_{i_1}d_{i_2} \cdots d_{i_t}\) where \(t \in N\), \((d_{i_1}, \ldots, d_{i_t}) \in M_{i_1} \times \cdots \times M_{i_t}\) for some \(\{i_1, i_2, \ldots, i_t\} \subseteq \{0, \ldots, n\}\) with \(i_1 + \cdots + i_t = i\), \(d_{i_j} \notin U(R)\) when \(i_j = 0\), and \(t > N(x)\), then, after cancellation and reordering of some of the \(d_{i_j}\)'s in \(R\), we have \(Rx = Rd_{i_1}d_{i_2} \cdots d_{i_s}\) for some \(s \leq N(h)\).

**Theorem 7.18** If \(R \ltimes_n M\) is a \(U\)-BFR, then \(R\) is a \(U\)-BFR and \(M_i\) is a \(U\)-BF module for every \(i \in \{1, \ldots, n\}\). Moreover, if \(R\) is présimplifiable, then \(R\) is a BFR.

Conversely, assume \(R\) to be an integral domain. If \(R\) is a BFD and for every \(i \in \{1, \ldots, n\}\), \(M_i\) is a \(U\)-BF module, then \(R \ltimes_n M\) is a \(U\)-BFR.

**Proof.** Similar to the classical case.

A ring \(S\) is called \(\textit{\(U\)-atomic}\) if every nonzero nonunit element of \(S\) has a \(U\)-factorization in which all the relevant factors are irreducibles. The question of when the classical trivial extension is \(U\)-atomic is still unsolved. In [11, Theorem 4.6], Axtell gave an answer to this question for an integral domain \(D\) with ACCP: For a \(D\)-module \(N\), \(D \times N\) is atomic if and only if \(D \times N\) is \(U\)-atomic. In [12, Theorem 3.15], it is shown that the condition that the ring is an integral domain could be replaced by the ring is présimplifiable. The following result gives an extension of [12, Theorem 3.15] to the context of \(n\)-trivial extensions.

**Theorem 7.19** Assume \(n \geq 2\). Suppose that \(M_i\) is présimplifiable for every \(i \in \{0, \ldots, n-1\}\) (here \(M_0 = R\)), \(R\) satisfies ACCP and \(M_i\) satisfies ACC on cyclic submodules for every \(i \in \{1, \ldots, n-1\}\). Then \(R \ltimes_n M\) is atomic if and only if \(R \ltimes_n M\) is \(\textit{\(U\)-atomic}\).

**Proof.** \(\implies\) Clear.

\(\iff\) Suppose that \(R \ltimes_n M\) is not atomic. Then by the proof of Theorem 7.11 there exists \(m_n := (0, \ldots, 0, m_n) \in R \ltimes_n M\) with \(m_n \neq 0\) which is not a product of irreducibles. Since \(R \ltimes_n M\)
is $U$-atomic, $m_n$ admits a $U$-factorization of the form $m_n = a_1 \cdots a_s [b_1 \cdots b_t]$ such that the $b_i$'s are irreducibles. Since $m_n$ cannot be a product of irreducibles and by the proof of Theorem 7.11 necessarily $s = 1$ and $a_1$ has the form $x_n := (0, \ldots, 0, x_n)$. But $x_n \langle b_1 \cdots b_t \rangle = \langle b_1 \cdots b_t \rangle$ and so $b_1 \cdots b_t$ has the form $y_n := (0, \ldots, 0, y_n)$. This is impossible since $m_n = x_n y_n = 0$. 

References

[1] A. G. Ağargüm, D. D. Anderson and S. Valdes-Leon, Factorization in commutative rings with zero divisors, III, Rocky Mountain J. Math. 31 (2001), 1–21.

[2] D. D. Anderson, Generalizations of Boolean rings, Boolean-like rings and von Neumann regular rings, Comment. Math. Univ. St. Paul. 35 (1986), 69–76.

[3] D. D. Anderson, M. Axtell, S. J. Forman and J. Stickles, When are associates unit multiples?, Rocky Mountain J. Math. 34 (2004), 811–828.

[4] D. D. Anderson and R. Markanda, Unique factorization rings with zero divisors, Houston J. Math. 11 (1985), 15–30.

[5] D. D. Anderson and R. Markanda, Unique factorization rings with zero divisors: Corrigendum, Houston J. Math. 11 (1985), 423–426.

[6] D. D. Anderson and J. Pascual, Regular ideals in commutative rings, sublattices of regular ideals, and Prüfer rings, J. Algebra 111 (1987), 404–426.

[7] D. D. Anderson and S. Valdes-Leon, Factorization in commutative rings with zero divisors, Rocky Mountain J. Math. 26 (1996), 439–480.

[8] D. D. Anderson and S. Valdes-Leon, Factorization in commutative rings with zero divisors, II, Factorization in integral domains, Lecture Notes in Pure and Appl. Math., 189, Marcel Dekker, New York, (1997), 197–219.

[9] D. D. Anderson and M. Winders, Idealization of a module, J. Commut. Algebra 1 (2009), 3–56.

[10] P. Ara, W. K. Nicholson and M. F. Yousif, A look at the Faith-Conjecture, Glasgow Math. J. 42 (2001), 391–404.

[11] M. Axtell, U-factorizations in commutative rings with zero divisors, Comm. Algebra 30 (2002), 1241–1255.

[12] M. Axtell, S. Forman, N. Roersma, and J. Stickles, Properties of U-factorizations, Int. J. Commut. Rings 2 (2003), 83–99.
[13] V. Barucci, M. D’Anna and F. Strazzanti, *A family of quotients of the Rees algebra*, Comm. Algebra 43 (2015), 130–142.

[14] D. Bennis, J. Mikram and F. Taraza, *On the extended zero divisor graph of commutative rings*, Turk. J. Math. 40 (2016), 376–388.

[15] M. Billis, Unique factorization in the integers modulo $n$, Amer. Math. Monthly 75 (1968), 527.

[16] G. F. Birkenmeier, H. E. Heatherly, J. Y. Kim and J. K. Park, *Triangular matrix representations*, J. Algebra 230 (2000), 558–595.

[17] G. F. Birkenmeier, J. K. Park and S. T. Rizvi, *Extensions of Rings and Modules*, Springer Science+Business Media, New York, 2013.

[18] A. Bouvier, *Demi-groupes de type (R). Demi-groupes commutatifs à factorisation unique*, C. R. Acad. Sci. Paris 268 (1969), 372–375.

[19] A. Bouvier, *Demi-groupes de type (R)*, C. R. Acad. Sci. Paris 270 (1969), 561–563.

[20] A. Bouvier, *Factorisation dans les demi-groupes de fractions*, C. R. Acad. Sci. Paris 271 (1970), 924–925.

[21] A. Bouvier, *Factorisation dans les demi-groupes*, C. R. Acad. Sci. Paris 271 (1970), 533–535.

[22] A. Bouvier, *Anneaux présimplifiables et anneaux atomiques*, C. R. Acad. Sci. Paris Sér. A-B 272 (1971), 992–994.

[23] A. Bouvier, *Sur les anneaux de fractions des anneaux atomiques présimplifiables*, Bull. Sci. Math. 95 (1971), 371–377.

[24] A. Bouvier, *Sur les anneaux de fractions des anneaux atomiques présimplifiables*, Bull. Sci. Math. 95 (1971), 371–377.

[25] A. Bouvier, *Anneaux de Gauss*, C. R. Acad. Sci. Paris 273 (1971), 443–445.

[26] A. Bouvier, *Remarques sur la factorisation dans les anneaux commutatifs*, Pub. Dépt. Math. Lyon 8 (1971), 1–18.

[27] A. Bouvier, *Résultats nouveaux sur les anneaux présimplifiables*, C. R. Acad. Sci. Paris 275 (1972), 955–957.

[28] A. Bouvier, *Anneaux présimplifiables*, C. R. Acad. Sci. Paris Sér. A-B 274 (1972), 1605–1607.
[29] A. Bouvier, *Résultats nouveaux sur les anneaux présimplifiables*, C. R. Acad. Sci. Paris Sér. A-B 275 (1972), 955–957.

[30] A. Bouvier, *Anneaux présimplifiables*, Rev. Roumaine Math. Pures Appl. 19 (1974), 713–724.

[31] A. Bouvier, *Structure des anneaux à factorisation unique*, Pub. Dépt. Math. Lyon 11 (1974), 39–49.

[32] A. Bouvier, *Sur les anneaux principaux*, Acta. Math. Sci. Hung. 27 (1976), 231–242.

[33] V. Camillo, I. Herzog and P. P. Nielsen, *Non-self-injective injective hulls with compatible multiplication*, J. Algebra 314 (2007), 471–478.

[34] M. D’Anna, C. A. Finocchiaro, M. Fontana, *Amalgamated algebras along an ideal*, in Commutative Algebra and Applications, Proceedings of the Fifth International Fez Conference on Commutative Algebra and Applications, Fez, Morocco, 2008, W. de Gruyter Publisher, Berlin 2009, 155–172.

[35] C. R. Fletcher, *Unique factorization rings*, Proc. Camb. Phil. Soc. 65 (1969), 579–583.

[36] C. R. Fletcher, *The structure of unique factorization rings*, Proc. Camb. Phil. Soc. 67 (1970), 535–540.

[37] C. R. Fletcher, *Euclidean rings*, J. London Math. Soc. 41 (1971), 79–82.

[38] J. A. Huckaba, *Commutative Rings with Zero Divisors*, Monographs and Textbooks in Pure and Applied Mathematics 117, Marcel Dekker, Inc., New York, 1988.

[39] A. McQueen, *Factorization in rings of upper-triangular Toeplitz matrices*, Master’s thesis, University of Central Missouri, 2014.

[40] M. Nagata, *Local Rings*, Interscience Publishers, New York-London-Sydney, 1962.

[41] C. Nastasescu and F. Van Oystaeyen, *Methods of Graded Rings*, Lecture Notes in Math. 1836, Springer-Verlag, Berlin, 2004.

[42] D. G. Northcott, *Lessons on Rings, Modules, and Multiplicities*, Cambridge Univ. Press, Cambridge, 1968.

[43] Z. Pogorzaly, *A generalization of trivial extension algebras*, J. Pure Appl. Algebra 203 (2005), 145–165.

[44] P. Ribenboim, *Rings of generalized power series. II. Units and zero-divisors*, J. Algebra 168 (1994), 71–89.