Generalized dynamic scaling for quantum critical relaxation in imaginary time

Shuyi Zhang, Shuai Yin,* and Fan Zhong
State Key Laboratory of Optoelectronic Materials and Technologies, School of Physics and Engineering, Sun Yat-sen University, Guangzhou 510275, People’s Republic of China

We study the imaginary-time relaxation critical dynamics of a quantum system with a vanishing initial correlation length and an arbitrary initial order parameter $M_0$. We find that in quantum critical dynamics, the behavior of $M_0$ under scale transformations deviates from a simple power-law and is described by a universal characteristic function, similar to classical critical dynamics. This characteristic function is shown to be able to describe the quantum critical dynamics in both short- and long-time stages of the evolution. The one-dimensional transverse-field Ising model is employed to numerically determine the specific form of the characteristic function. We demonstrate that it is applicable as long as the system is in the vicinity of the quantum critical point. The universality of the characteristic function is confirmed by numerical simulations of the one-dimensional transverse Ising-ladder model, which belongs to the same universality class.

PACS numbers: 05.30.-d, 64.70.Tg, 64.60.Ht

I. INTRODUCTION

Divergent properties of a system near the critical point are usually characterized by the critical exponents, describing the dependence of macroscopic quantities on the deviation from the critical point. For example, at a distance $g$ from the critical point, the order parameter $M$ behaves as $M \sim g^\beta$, the correlation length $\xi$ as $\xi \sim g^{-\nu}$, and the correlation time $\zeta \sim \xi^z$, where $\beta$ and $\nu$ are the static exponents and $z$ is the dynamic exponent. Although the number of microscopic degrees of freedom is huge, the number of independent critical exponents is small, as these critical exponents satisfy several scaling laws. Whether the power laws with the several exponents are enough to describe universal critical properties is an interesting and fundamental question in both classical and quantum phase transitions.

In this paper, we study the relaxation quantum critical dynamics. Theoretical studies on the quantum critical dynamics have been partly stimulated by developments in experimental technologies, which provide effective platforms to manipulate quantum many-body systems and detect their nonequilibrium quantum phenomena. Although a lot of effort has been devoted to understanding the universal dynamic properties, long-range entanglement and nonequilibrium nature make this issue difficult to tackle in both analytical and numerical aspects.

In classical critical dynamics, the dynamic exponent and the static ones are decoupled. In quantum case, it is well known that there is a mapping between a $d$-dimensional quantum system and a corresponding $(d+1)$-dimensional classical system. The additional dimension comes from the inverse of the temperature, and plays the role of the imaginary time. As the imaginary time has the identical dimension to the real one, parts of the dynamic and static critical properties are intimately intertwined in quantum critical phenomena. For example, hyper-scaling scaling laws in quantum criticality include the dynamic exponent $z$ and the static ones together.

So, besides the usual static critical exponents and the dynamic one, it is believed that no additional exponent is necessary in quantum critical dynamics, compared to the equilibrium critical phenomena. A typical example is the driving quantum critical dynamics, in which the Kibble-Zurek mechanism and the finite-time scaling theory show that nonequilibrium critical properties are well described by the exponents in the equilibrium quantum critical phenomena, even in the finite-temperature region.

However, it has been discovered that analogous to the case of classical short-time dynamics, an additional critical exponent $\theta$ is needed to describe the universal critical initial slip in the imaginary-time evolution near a quantum critical point. Starting with a state with vanishing initial correlation and small initial order parameters $M_0$, after a microscopic stage, the system enters the critical initial slip stage, in which $M$ increases with the imaginary time $\tau$ as $M \propto M_0 \tau^\theta$. This behavior lasts until a crossover time $\tau_{cr}$, characterised by $\tau_{cr}^{-1} \sim M_0^{-\alpha_0/z}$ with $\alpha_0$ being the scaling dimension of $M_0$ and satisfying $\alpha_0 = \theta z + \beta/\nu$. For $\tau > \tau_{cr}$, $M$ decays according to $M \sim \tau^{-\beta/\nu}$. Scaling behaviors in both short and long times are shown to be described by unified scaling forms, which can be used to determine the critical point and critical exponents in the short-time stage. The advantage of this method is that it overcomes the critical slowing down and the divergence of the entanglement entropy.

The initial magnetization $M_0$ plays an essential role in characterizing the initial condition in classical and quantum short-time critical dynamics. Both $M_0 = 0$ and the saturated $M_0$ are the fixed points of the initial order parameter under scale transformations. When evolution begins with $M_0 = 0$, all the evolution is the initial slip. When evolution begins with the saturated $M_0$, there is no critical initial slip. When $M_0$ is very small and thus close to the fixed point $M_0 = 0$, the rescaled order parameter, $M_0$, under a scale transformation with a rescaling factor $b$, is $M_0 = b^{\alpha_0} M_0$. In between, for an arbitrary $M_0$, in classical critical dynam-
ics, Zheng showed that universal behavior also exists, however, \( b^0 M_0 \) cannot characterize the rescaled order parameter \( M_0 \) for large \( M_0 \). He proposed that \( M_0 \) can be represented by a universal characteristic function, with \( b \) and \( M_0 \) being its arguments. Numerical results have confirmed this proposal. Moreover, the universality of the characteristic function has been verified in other classical systems. A question is then does quantum imaginary-time evolution also need such a universal characteristic function?

In this paper, we show that the power-law scale transformation of \( b^0 M_0 \) is not enough to characterize the rescaled initial magnetization in the imaginary-time relaxation near the critical point with an arbitrary \( M_0 \). We suggest that a universal characteristic function \( U(b, M_0) \) can be introduced, as in the classical case, to characterize the behavior of an arbitrary initial parameter \( M_0 \) under the scale transformation. Both the short- and the long-time critical dynamics are well described by the generalized scale transformation containing this function as the rescaled initial order parameter. The characteristic function is shown to be determined by the scaling factor \( b \) and the original \( M_0 \), and independent of \( y \) and the symmetry-breaking field \( h \). When \( M_0 \) is small, \( U(b, M_0) \) recovers the power-law form \( b^0 M_0 \), while for the saturated \( M_0 \) we have \( U(b, M_0) = M_0 \). This function is universal, since it is identical for different models belonging to one universality class. These conclusions are verified by numerical results of the order parameter and the entanglement entropy in the one-dimensional (1D) transverse-field Ising model and the transverse-field Ising-ladder model.

The rest of the paper is organized as follows. In Sec. II, we illustrate the imaginary-time Schrödinger equation and compare it with the classical master equation. Then, in Sec. III, we discuss the scale transformations containing the universal characteristic function \( U(b, M_0) \) and its properties. In Sec. IV, the theory is confirmed with the models of the quantum Ising chain and Ising ladder. A summary is given in Sec. V.

II. IMAGINARY-TIME QUANTUM EVOLUTION

Studies on the imaginary-time quantum critical dynamics have attracted a lot of attention recently. Some scaling properties of the imaginary-time quantum critical dynamics can be used to predict the behavior of the real-time critical dynamics. Moreover, simulations of the imaginary-time dynamics can be readily realized in both quantum Monte Carlo and density-matrix renormalization group methods. More importantly, the imaginary-time quantum dynamics has its own physical realization. For example, it can be regarded as the evolution controlled by a non-Hermitian Hamiltonian with strong dissipation.

The imaginary-time evolution of a quantum state \( |\psi(\tau)\rangle \) is governed by the Schrödinger equation with a Hamiltonian \( H \) and the time \( t \) replaced by \( -\tau \),

\[
\frac{\partial}{\partial \tau} |\psi(\tau)\rangle = -H |\psi(\tau)\rangle,
\]

in which \( |\psi(\tau)\rangle \) satisfies the normalization condition \( \langle \psi(\tau) | \psi(\tau) \rangle = 1 \). In the real-time evolution, the normalization condition is naturally satisfied because of the unitarity of the evolution operator. For the imaginary-time evolution, the normalization condition has to be imposed on \( \psi(\tau) \). The formal solution to Eq. (1) is

\[
|\psi(\tau)\rangle = Z^{-1} \exp(-H \tau) |\psi_0\rangle,
\]

where \( |\psi_0\rangle = |\psi(0)\rangle \) is the initial state, and

\[
Z = \| \exp(-H \tau) |\psi_0\rangle \|,
\]

is the normalization factor with \( \| \cdot \| \) denoting a modulo operation.

In order to compare the imaginary-time quantum dynamics with the classical thermal dynamics, we now derive an equation describing the evolution of the normalized wave function, \( |\phi(\tau)\rangle \equiv Z^{-1} |\psi(\tau)\rangle \). Substituting Eqs. (2) and (3) into Eq. (1), we obtain

\[
\frac{\partial}{\partial \tau} |\phi(\tau)\rangle = -H |\phi(\tau)\rangle + \bar{E}(\tau) |\phi(\tau)\rangle,
\]

where \( \bar{E}(\tau) = \langle \phi(\tau) | H |\phi(\tau)\rangle \) is the averaged energy. Expanding \( |\phi(\tau)\rangle \) in the eigenstates of the Hamiltonian,

\[
|\phi(\tau)\rangle = \sum_i C_i(\tau) |E_i\rangle,
\]

we arrive at the evolution of the probability \( P_i(\tau) \equiv |C_i(\tau)|^2 \) of finding the \( i \)th eigenstate \( |E_i\rangle \) with the eigenvalue \( E_i \),

\[
\frac{\partial P_i(\tau)}{\partial \tau} = - [E_i - \bar{E}(\tau)] P_i(\tau).
\]

Two remarks are in order here. (a) Equation (6) is a typical dissipation equation. If \( E_i > \bar{E}(\tau) \), \( P_i \) will decay; if \( E_i < \bar{E}(\tau) \), on the other hand, \( P_i \) will increase. The steady solution of this equation corresponds to \( \bar{E} = E_i \). (b) For a system with a first exciting gap \( \Delta = E_1 - E_0 \), the ground-state energy \( E_0 \) is always smaller than \( \bar{E}(\tau) \). Thus the system will tend to its ground state after a typical time scale \( \zeta_r \sim \Delta^{-1} \). Hence the imaginary-time evolution is a commonly used method to find the ground state.

The reason for the similarity between the imaginary-time quantum critical dynamics and the classical critical dynamics can be inspected by comparing Eq. (6) with the classical master equation,

\[
\frac{\partial P_i(t)}{\partial t} = \sum_j [W_{j-i} P_j(t) - W_{i-j} P_i(t)],
\]
in which \( W_{j \rightarrow i} \) is the transition probability from the \( j \)th to the \( i \)th state. \( W_{j \rightarrow i} \) must fulfill the detailed balance condition, which is \( W_{j \rightarrow i} / W_{i \rightarrow j} = \exp \left[ -(E_i - E_j) / T \right] \) with \( T \) being the temperature. Both equations describe a dissipation process. The probability of the high-energy excitations decays fast with the time evolution, whereas the low-energy modes, controlling the critical phenomena, are left over. As a result, Eqs. (6) and (7) exhibit similar evolution properties, especially the critical initial slip behavior in the short-time stage. However, the critical dynamics described by Eq. (6) is essentially different from that described by the classical master equation (7), even for two models in the same static universality class. For example, the dynamic exponent corresponding to Eq. (6) is \( z = 1 \) for the 1D quantum Ising model, while to Eq. (7) is \( z \approx 2.1667(5) \) for the 2D classical Ising model. The difference between them also explains the fact that the initial-slip exponent \( \theta \) is different from the classical one.

III. GENERALIZED SHORT-TIME CRITICAL DYNAMICS

In classical short-time critical dynamics, a universal characteristic function has been proposed to describe the rescaled initial order parameter in the scale transformation characterizing evolution with an arbitrary initial magnetization. In this section, we suggest a similar scale transformation which contains a universal characteristic function to describe the imaginary-time evolution starting with an arbitrary initial magnetization in quantum critical dynamics.

Universal behavior near the quantum critical point is controlled by the low-lying energy levels. Because of the dissipative nature of Eq. (1), when a quantum system is quenched to the vicinity of its critical point, contributions from the high-energy levels decay fast in the microscopic initial stage. After this stage, the critical system enters the universal short-time stage. For both very small \( M_0 \) and the saturated \( M_0 \), previous studies have shown that a scaling form connects the scaling behavior in this stage to that in the long-time stage. Thus, it is justified to expect that criticality should also exist for an arbitrary \( M_0 \). However, as \( M_0 \) is not in the vicinity of its fixed point, a simple power-law relation is not enough to describe the universal behavior. For example, if the relation \( M_0 = b^z M_0 \) were right for all \( M_0 \), \( M_0 \) would be larger than the saturated value, which is clearly not physical. Inspired by the idea of the universal characteristic function applied to the classical case, we suggest that the universal characteristic function \( U(b, M_0) \) is the rescaled initial order parameter \( M_0 \) for the rescaling factor \( b \).

Accordingly, in case of the order parameter, we have

\[
M(\tau, g, h, M_0) = b^{-3/2} M(b^{-3/2} \tau, b^{1/2} g, b^{3/2} h, U(b, M_0)),
\]

where \( U(b, M_0) \) is the universal characteristic function, and \( \delta \) is another static critical exponent defined as \( M \propto b^{3/\delta} \) at \( g = 0 \).

It should be noted that \( U \) is to be included in the scale transformations for all macroscopic physical quantities. Besides the order parameter, we can also measure the evolution of the entanglement. As a unique physical quantity of a quantum system, the entanglement is usually measured as \( S = -\text{Tr}(\rho \log \rho) \) if we apply the definition of the von-Neumann entropy, where \( \rho \) is the reduced density matrix of half of the system and the base of logarithm is 2 throughout. For a 1D system near its critical point \( S = (c/6) \ln \xi^{21-24} \), with \( c \) being the central charge. Using the characteristic function, we can write down the generalized scale transformation of the correlation length,

\[
\xi(\tau, g, h, M_0) = b^{3} \xi(b^{3-\tau} \tau, b^{1/2} g, b^{3/2} h, U(b, M_0)).
\]

Therefore, the entanglement entropy \( S \) satisfies

\[
S(\tau, g, h, M_0) = \frac{c}{6} \ln b^{3} S(b^{3-\tau} \tau, b^{1/2} g, b^{3/2} h, U(b, M_0)).
\]

Here are some properties of the function \( U(b, M_0) \). (a) At both fixed points of \( M_0 \), i.e., \( M_0 = 0 \) or the saturated one, we have \( U(b, M_0) = M_0 \) for any given \( b \). (b) The value \( U(b, M_0) \) depends only on \( M_0 \) and the rescaling factor \( b \), but not on any other parameters, \( g \) and \( h \), for example. In other words, Eq. (8) always holds as long as the system is near its critical point. (c) When the initial order parameter is small, \( M_0 \to 0 \), the characteristic function returns to a simple power-law relation \( U(b, M_0) \to b^{\tau_{00}} M_0 \). (d) Exactly at the critical point \( g = 0 \) and \( h = 0 \), by setting \( b = \tau^{1/2} \), Eq. (8) becomes

\[
M(\tau, M_0) = \tau^{-\beta/\nu} f_M(U(\tau^{-1/2}, M_0)),
\]

where \( f_M \) is the scaling function related to \( M \). In the long-time stage, any information of the starting \( M_0 \) should be “forgotten”, except for the sign of \( M_0 \). Hence we have \( f_M(U(\tau^{-1/2}, M_0)) \sim \text{sgn}(M_0) \) for any \( M_0 \) when \( \tau \to \infty \), where \( \text{sgn} \) is the sign function. In the short-time stage, \( \tau \ll M_0^{-1/2z_0} \), the scaling behavior for very small \( M_0 \) can be easily restored. As \( U(b, M_0) \to b^{\tau_{00}} M_0 \), we have \( f_M(U(\tau^{-1/2}, M_0)) \sim M_0^{\tau_{00}/z} \) and \( M_0 \) increases with \( \tau \) as \( M \sim M_0^{\tau/\nu} \), as in the classical situation. Similar to the case of \( M \), let \( b = \tau^{1/2} \), and the generalized scaling form for \( S \) reads

\[
S(\tau, M_0) = \frac{c}{6z} \ln \tau + f_S(U(\tau^{1/2}, M_0)).
\]

We define an entanglement entropy-difference, \( \Delta S \), by

\[
\Delta S(\tau, M_0) = f_S(U(\tau^{1/2}, M_0)) - f_S(U(\tau^{1/2}, 0)).
\]

For long times, if \( M_0 \neq 0 \), \( f_S(U(\tau^{1/2}, M_0)) \) tends to a constant, \( \Delta S(\infty, 1) \), independent of \( M_0 \), whereas in the short times and for a small \( M_0 \), we restore

\[
\Delta S(\tau, M_0) \propto M_0^{\tau_{00}^{-1/2z_0}/z}.
\]
IV. VERIFICATION OF THE GENERALIZED SCALE TRANSFORMATIONS

In this section, we verify the scale transformation proposed for the universal imaginary-time quantum critical dynamics with an arbitrary initial magnetization $M_0$ and determine the universal characteristic function $U(b, M_0)$. The 1D transverse-field Ising model is taken as an example. First, we shall confirm Eq. (8) at $g = 0$ and $h = 0$ and determine $U(b, M_0)$. Second, we show that $U(b, M_0)$ is independent of $g$ and $h$. Third, the universal properties are confirmed by determining $U$ for the quantum Ising-ladder and comparing it with the Ising chain.

A. Model, numerical method, and initial state

In the following, we mainly use the 1D transverse field Ising model. The Hamiltonian reads

$$H_I = -\sum_n \sigma_n^x \sigma_{n+1}^x - h_x \sum_n \sigma_n^z - h \sum_n \sigma_n^z,$$

where $\sigma_n^x$ and $\sigma_n^z$ are the Pauli matrices in $x$ and $z$ direction, respectively, at site $n$, $h_x$ is the transverse field and $h$ is the symmetry-breaking field. We have set the Ising coupling to unity as our energy unit. The initial-slip exponent $\theta$ and determine dynamics with an arbitrary initial magnetization $g$.

where $\sigma_n^x$ and $\sigma_n^z$ are the Pauli matrices in $x$ and $z$ direction, respectively, and $\sigma_n^y$ is the symmetry-breaking field. We have set the Ising coupling to unity as our energy unit. The initial-slip exponent $\theta$ and determine $U$ for the quantum Ising-ladder and comparing it with the Ising chain.

where $\sigma_n^x$ and $\sigma_n^z$ are the Pauli matrices in $x$ and $z$ direction, respectively, at site $n$. The critical point of model (15) is $h_{xc} = 1$ and $h = 0$. The exact critical exponents are $\beta = 1/8$, $\nu = 1$, $\delta = 15$ and $z = 1^x$, and the central charge $c = 1/2$. The initial-slip exponent $\theta$ is estimated to be $\theta = 0.373$. This model is realized in CoNb$_2$O$_6$ experimentally.

In order to show the universality of the characteristic function, we also employ the quantum Ising ladder with the Hamiltonian

$$H_L = -\sum_n \sum_{\alpha=1,2} \sigma_{n,\alpha}^x \sigma_{n+1,\alpha}^x - \sum_n \sum_{\alpha=1,2} \sigma_{n,\alpha}^z \sigma_{n+1,\alpha}^z - h_L \sum_n \sum_{\alpha=1,2} \sigma_{n,\alpha}^y - h \sum_n \sigma_{n,\alpha}^z,$$

where the first and the second terms are the interactions along the ladder and on the rung, respectively, and $\alpha$ denotes the legs of the ladder. The critical point of this model was determined by finite-time scaling method to be $h_L = 1.8323$ and the static critical exponents and the dynamic exponent $z$ determined by the same method show that it belongs to the same universality class as model (15). It has also been shown that the initial-slip exponent, $\theta_L$, is very close for the two models.

The infinite time-evolving block decimation (ITEBD) algorithm is used to calculate the imaginary-time evolution. A quantum state in 1D can be represented in a matrix product form via Vidal’s decomposition and each site is attached with such a matrix. By taking the translational invariance of an infinite homogeneous condition into account, the ITEBD algorithm represents the matrix product form with repeated matrices in one primitive cell. The evolution of a state then is represented by the updating of these matrices according to the local evolution operators, which are obtained by the Suzuki-Trotter decomposition of $\exp(-H\tau)$. The time interval is chosen as 0.01 and 100 states are kept. These values are identical with the previous study. Three decimal places are kept in our results, as the increment of $M_0$, which we choose to determine $U$, is 0.002.

The initial state with an order parameter $M_0$ is prepared in a direct product state. It has already been confirmed that the universal critical behavior in the short times is not sensitive to the microscopic details of the initial state. So we choose the homogeneous direct product state for simplicity. For the transverse-field Ising model (15), this is

$$|\psi_0\rangle = \bigotimes_n [(a_n^\uparrow + b_n^\downarrow)],$$

where $a_n$ and $b_n$ are coefficients of the local state at site $n$, $|\uparrow\rangle$ and $|\downarrow\rangle$ are eigenvectors of $\sigma^z$, and $a_n = \sqrt{(1 + M_0)/2}$ and $b_n = \sqrt{(1 - M_0)/2}$ for a given $M_0$.

For the Ising-ladder, the local basis vectors are $|\uparrow\downarrow\rangle$, $|\uparrow\downarrow\rangle$, $|\downarrow\uparrow\rangle$ and $|\downarrow\uparrow\rangle$, where $l$ and $r$ label the two spins on the same rung. The initial state for the Ising ladder is

$$|\psi_0\rangle = \bigotimes_n [(a_n^l + b_n^r)(a_n^r + b_n^l)],$$

where $a_n^l = a_n^r = \sqrt{(1 + M_0)/2}$ and $b_n^l = b_n^r = \sqrt{(1 - M_0)/2}$ for a given $M_0$.

B. Numerical results

Figure 1(a) shows the imaginary-time evolution of the order parameter $M$ with different initial magnetization $M_0$ for the quantum Ising model (15). The initial-slip stage, in which $M$ increases, shrinks as $M_0$ increases. Then, $M$ enters the power-law decay stage, in which $M \sim \tau^{-\beta/\nu z}$, independent of the value of $M_0$. These features are in consistent with those with very small $M_0$. However, in the initial slip stage, the shapes of the curves of $M$ versus $\tau$ change with $M_0$. This is different from the situation for very small $M_0$, for which the curves in the initial-slip stage are straight lines with the identical slope on a double-logarithmic scale. This indicates the necessity of the presence of the universal characteristic function $U$. The evolution of entanglement entropy difference $\Delta S$ is shown in Fig. 1 (b). $\Delta S$ saturates at a constant independent of $M_0$ for long times. In the early stage, the slopes for larger $M_0$ are different from those for small $M_0$. This also indicates the necessity of $U$. 

shows the determination of Eq. (rescaled curves as can be seen in the insets. Then we run the system at \( b = 5 \) with the \( U(5, M_0) \), one finds that they fit well when \( M_0 \) is small and deviates from each other when \( M_0 \) grows larger. \( U \) also increases as \( b \) increases. For larger \( b \), the curve deviates from the power-law form for smaller \( M_0 \), and tends to saturate at smaller \( M_0 \).

We can apply the scale transformation of the entropy \( S \), Eq. (10), to determine \( U \) as well. However, we apply this to check the foregoing determined \( U \). According to Eq. (10), the scale-transformed curve \( S(b^{2/\nu}g, b^{3\delta/\nu}h, U(b, M_0)) \) coincides with the curve of \( S(\tau, b^{1/\nu}g, b^{3\delta/\nu}h, U(b, M_0)) \). From Fig. 4, it can be seen that, after a microscopic scale, the curve from \( M_0 = U(b, M_0) \), determined in Fig. 2, and the corresponding rescaled curve perfectly collapse onto each other. As the entanglement entropy \( S \) is closely related with the correlation length \( \xi \), which is responsible to the universal behavior in the critical region, we expect that the universal characteristic function \( U \) must be considered for all the macroscopic quantities.

1. Determination of \( U(b, M_0) \)

To determine numerically the form of \( U \), according to Eq. (8), we first let the system evolve at a certain \( g \) and \( h \) starting from an initial magnetization \( M_0 \), and rescale \( M \) and \( \tau \) by \( M \rightarrow b^{3/\nu}M \) and \( \tau \rightarrow b^{2/\nu} \), respectively. Then we run the system at \( b^{1/\nu} \) and \( b^{3\delta/\nu} \) starting from a series of \( M_0' \), and compare each of these curves with the rescaled curve to find the one that most fit it. The corresponding initial magnetization thus satisfies \( M_0' = U(b, M_0) \).

The generalized scale transformation (8) show that \( U \) is independent of the choice of \( g \) and \( h \). So, we work at the critical point, \( g = 0 \) and \( h = 0 \), for simplicity. Figure 2 shows the determination of \( U \) for two sets of values. It is clear that after the non-universal microscopic time, the curves starting with \( U(b, M_0) \) fit perfectly with the rescaled curves, in both the short- and long-time stages. In contrast, if the initial magnetization is not \( U(b, M_0) \), the curves of the subsequent evolution deviate from the rescaled curves as can be seen in the insets.

\( U(b, M_0) \) measured for model (15) is shown in Fig. 3. Two fixed points \( U(b, 0) = 0 \) and \( U(b, 1) = 1 \) are manifest for different rescaling factors \( b \). Between the two fixed points, \( U \) increases as \( M_0 \) increases. Comparing the line of \( b^{x_0}M_0 \) for \( b = 5 \) with the \( U(5, M_0) \), one finds that they fit well when \( M_0 \) is small and deviates from each other when \( M_0 \) grows larger. \( U \) also increases as \( b \) increases. For larger \( b \), the curve deviates from the power-law form for smaller \( M_0 \), and tends to saturate at smaller \( M_0 \).

We can apply the scale transformation of the entropy \( S \), Eq. (10), to determine \( U \) as well. However, we apply this to check the foregoing determined \( U \). According to Eq. (10), the scale-transformed curve \( S(b^{\tau}, g, h, M_0) - (c/6) \log b \) which starts with \( M_0 \) should coincide with the curve of \( S(\tau, b^{1/\nu}g, b^{3\delta/\nu}h, U(b, M_0)) \). From Fig. 4, it can be seen that, after a microscopic time scale, the curve from \( M_0 = U(b, M_0) \), determined in Fig. 2, and the corresponding rescaled curve perfectly collapse onto each other.
1. Critical-point situation

When the system deviates a little from the critical point, i.e., \( g \neq 0 \) or \( h \neq 0 \) (or both), the generalized scale transformation (8) still holds, and the characteristic function is not dependent on the value of \( g \) or \( h \) as long as the system is close to the critical point.

To show this independency, first we study a system slightly deviated from the critical point with \( g \equiv h_x - h_{xc} \neq 0 \). When \( g > 0 \), the system is in the disordered phase and \( M \) tends to 0 as the evolution continues; when \( g < 0 \), the system is in the ordered phase and \( M \) tends to \( M \propto (−g)β \) in equilibrium. As can be seen from Fig. 5, in both situations, the value \( M_0 \) from which the evolution curve and the rescaled curve from \( M_0 \) collapse, is almost the same as \( U(b, M_0) \) determined at the critical point.

Then we examine the situation with a small symmetry-breaking field \( h \). We will restrict it to be in the same direction of the initial order parameter in order to prevent \( M \) from dropping below zero in the evolution. In equilibrium \( M \) should tend to \( M \propto h^{1/β} \). Figure 6 shows that \( U(b, M_0) \) is identical with the corresponding one in the absence of \( h \).

We measured the dependence of \( U \) on \( b \) and \( M_0 \) in presence of \( g \) and \( h \). \( b \) ranges from 2 to 5 with an increment 1 and \( M_0 \) ranges from 0.1 to 0.9 with an increment 0.1. Part of the results is listed in Table I. From this table, we find that the characteristic function \( U(b, M_0) \) is really independent of \( g \) and \( h \).

2. Off-critical-point situation

When the system deviates a little from the critical point, i.e., \( g \neq 0 \) or \( h \neq 0 \) (or both), the generalized scale transformation (8) still holds, and the characteristic function is not dependent on the value of \( g \) or \( h \) as long as the system is close to the critical point.

To show this independency, first we study a system slightly deviated from the critical point with \( g = h_x - h_{xc} \neq 0 \). When \( g > 0 \), the system is in the disordered phase and \( M \) tends to 0 as the evolution continues; when \( g < 0 \), the system is in the ordered phase and \( M \) tends to \( M \propto (−g)β \) in equilibrium. As can be seen from Fig. 5, in both situations, the value \( M_0 \) from which the evolution curve and the rescaled curve from \( M_0 \) collapse, is almost the same as \( U(b, M_0) \) determined at the critical point.

Then we examine the situation with a small symmetry-breaking field \( h \). We will restrict it to be in the same direction of the initial order parameter in order to prevent \( M \) from dropping below zero in the evolution. In equilibrium \( M \) should tend to \( M \propto h^{1/β} \). Figure 6 shows that \( U(b, M_0) \) is identical with the corresponding one in the absence of \( h \).

We measured the dependence of \( U \) on \( b \) and \( M_0 \) in presence of \( g \) and \( h \). \( b \) ranges from 2 to 5 with an increment 1 and \( M_0 \) ranges from 0.1 to 0.9 with an increment 0.1. Part of the results is listed in Table I. From this table, we find that the characteristic function \( U(b, M_0) \) is really independent of \( g \) and \( h \).

3. Universality of \( U(b, M_0) \)

Finally, we confirm the universality of the characteristic function. Figure 7 shows the imaginary-time evolution for the quantum Ising ladder (16). From \( M_0 = U(b, M_0) \), where \( U(b, M_0) \) is chosen to be identical with the value of model (15), the curve and the corresponding rescaled curve from \( M_0 \) collapse onto each other after the early microscopic time stage. Although the shape of the evolution curve in this model is slightly different from that in the 1D Ising chain, the values of \( U \) are identical for both models, thus confirming the the universality of the characteristic function.
FIG. 7: (Color online) (a) $U(2, 0.5) = 0.650$ and (b) $U(4, 0.2) = 0.382$ checked in the quantum Ising-ladder at its critical point. Double-logarithmic scales are used.

V. SUMMARY

We have studied in this paper the relaxation quantum critical dynamics in imaginary time with an arbitrary initial order parameter. We have shown that a universal characteristic function $U$ must be introduced to describe the universal properties in both short and long times for an arbitrary initial magnetization $M_0$ similar to the classical case. This characteristic function contains the rescaling factor $b$ and $M_0$ as its arguments. It is identical for the models belonging to one universality class. According to the scale transformation including $U$, the form of this function has been determined for the 1D transverse-field Ising model. Its universality has been confirmed by the quantum Ising-ladder model.

Acknowledgements

We wish to thank Zhibing Li for his discussions. This project was supported by NNSFC (10625420).
Simulations in Statistical Physics, 2nd edition (Cambridge University Press, Cambridge, 2009).

34 U. Schollwöck, Rev. Mod. Phys. 77, 259 (2005).
35 G. Vidal, Phys. Rev. Lett. 91, 147902 (2003); 93, 040502 (2004); S. R. White and A. E. Feiguin, ibid. 93, 076401 (2004); A. J. Daley, C. Kollath, U. Schollwöck, and G. Vidal, J. Stat. Mech.: Theory Exp. (2004) P04005; H. C. Jiang, Z. Y. Weng, and T. Xiang, Phys. Rev. Lett. 101, 090603 (2008); H. H. Zhao, Z. Y. Xie, Q. N. Chen, Z. C. Wei, J. W. Cai, and T. Xiang, Phys. Rev. B 81, 174411 (2010).
36 N. Moiseyev, Non-Hermitian quantum mechanics, (Cambridge University Press, 2011).
37 A. I. Nesterov and S. G. Ovchinnikov, Phys. Rev. E 78, 015202(R) (2008); S.-D. Liang and G.-Y. Huang, Phys. Rev. A 87, 012118 (2013).
38 J. Zinn-Justin, Quantum field theory and critical phenomena, 3rd Ed., (Clarendon Press, 1996).
39 A. Altland and B. Simons, Condensed Matter Field Theory, (Cambridge University Press, 2006).
40 M. P. Nightingale and H. W. J. Blöte, Phys. Rev. B 62, 1089 (2000).
41 E. V. Albano, M. A. Bab, G. Baglietto, R. A. Borzi, T. S. Grigera, E. S. Loscar, D. E. Rodriguez, M. L. Rubio Puzzo, and G. P. Saracco, Rep. Prog. Phys. 74, 026501 (2011).
42 J. Eisert, M. Cramer, and M. Plenio, Rev. Mod. Phys. 82, 277 (2010).
43 L. Amico, R. Fazio, A. Osterloh, and V. Vedral, Rev. Mod. Phys. 80, 517 (2008).
44 A. Osterloh, L. Amico, G. Falci, and R. Fazio, Nature 416, 608 (2002).
45 R. Coldea, D. A. Tennant, E. M. Wheeler, E. Wawrzynska, D. Prabhakaran, M. Telling, K. Habicht, P. Smeibidl, and K. Kiefer, Science 327, 177 (2010).
46 H-J. Mikeska and A. K. Kolezhuk in Quantum Magnetism, ed. U. Schollwöck, et al., (Springer Press, 2004).
47 Z. Chen, S. Yin, and F. Zhong, unpublished (2013).
48 G. Vidal, Phys. Rev. Lett. 98, 070201 (2007).