Symmetric Graphs and their Quotients

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Introduction

A graph is a combinatorial object that captures abstractly the idea of a relationship amongst the elements in a set. Associated with every combinatorial object is a group of symmetries, or an automorphism group. An automorphism is, loosely speaking, a structure preserving map from the object to itself. A homomorphism maps a complex object onto a simpler one in such a way that certain features of the original object are preserved while others are lost.

In this paper we study a family of highly symmetric graphs using a mixture of group theoretic and combinatorial techniques. The graphs we study have the property that locally they “look the same” at every vertex, while globally they are rich in structure. In particular we look at homomorphic images, or quotients, of symmetric graphs. We would like to understand how the combinatorial structure of a symmetric graph is related to that of its quotients.

When passing from a graph to its quotient, information is lost. For any given symmetric graph there are, in fact, infinitely many larger symmetric graphs which admit the given graph as a quotient. Where is this information being lost to? How is it possible to take a symmetric graph and “unfold” it into a larger symmetric graph which admits the original as a quotient? What extra information is needed?

In Chapter 1, we introduce the basic notions from the theory of permutation groups necessary to give the definition of a symmetric graph, an imprimitive symmetric graph and the quotient of an imprimitive symmetric graph.

In Chapter 2 we introduce the idea of coset spaces and coset graphs. We see that, in some sense, symmetric graphs capture combinatorially the way that a subgroup sits inside a larger group.

In Chapter 3 we look at the “extension problem” for symmetric graphs and describe a “geometric approach” to the problem suggested by Gardiner and Praeger.

In Chapter 4 we look at a number of methods for constructing symmetric graphs with a given quotient.
CHAPTER 1

G-Sets and G-Graphs

1. Permutation Groups

Definition 1 (Permutation group). A permutation group is a triple \((G, \Omega, \rho)\) where \(G\) is a group, \(\Omega\) is a set and \(\rho\) is a homomorphism:

\[ \rho : G \to \text{Aut}(\Omega). \]

We say that \(G\) acts on \(\Omega\) as a group of permutations. The action is said to be faithful if \(\ker(\rho) = \{1\}\).

We will usually neglect to mention the homomorphism \(\rho\) explicitly, and speak of the permutation group \((G, \Omega)\). We write \(\alpha^{\rho(g)}\) or just \(\alpha^g\) to indicate the action of a permutation \(g \in G\) on a point \(\alpha \in \Omega\). This allows us to compose permutations as \(\alpha^{(gh)} = (\alpha^g)^h\) rather than \((gh)(\alpha) = h(g(\alpha))\).

If \(\Delta\) is a subset of \(\Omega\) and \(g\) is an element of \(G\), then we write \(\Delta^g = \{\alpha^g : \alpha \in \Delta\}\) to indicate the image of \(\Delta\) under the action of \(g\).

Sometimes we will call \((G, \Omega)\) a representation of \(G\) as a group of permutations of \(\Omega\), rather than a permutation group. If \(|\Omega| = n\) then we say that \((G, \Omega)\) is a permutation representation of \(G\) of degree \(n\).

Definition 2 (Permutation Equivalence). Two permutation groups \((G_1, \Omega_1)\) and \((G_2, \Omega_2)\) are said to be permutation equivalent if there is an isomorphism \(\varphi : G_1 \to G_2\) and a bijection \(\eta : \Omega_1 \to \Omega_2\) such that for every \(g \in G_1\) the following diagram commutes:

\[
\begin{array}{ccc}
\Omega_1 & \xrightarrow{g} & \Omega_1 \\
\eta \downarrow & & \eta \downarrow \\
\Omega_2 & \xrightarrow{\varphi(g)} & \Omega_2
\end{array}
\]

Permutation equivalence is an equivalence relation on the set of permutation representations of given group. If we relax the condition that \(\eta\) is bijective we obtain a permutation homomorphism.
Definition 3 (Transitive Permutation Group). A permutation group \((G, \Omega)\) is said to be transitive if for any \(\alpha, \beta \in \Omega\) there exist a \(g \in G\) such that \(\alpha^g = \beta\).

Definition 4 (Orbit). If \((G, \Omega)\) is a permutation group and \(\alpha \in \Omega\) then the orbit of \(\alpha\) under the action of \(G\) is:

\[
\alpha^G = \{\alpha^x : x \in G\}
\]

For a transitive permutation group \(\alpha^G = \Omega\). Otherwise \(\Omega\) maybe be partitioned into orbits: \(\Omega = \bigsqcup_{i=1}^{n} \Omega_i\) such that for each \(i\), the permutation group \((G, \Omega_i)\) is transitive.

2. Graphs

The most general definition of a graph is a pair of sets \((V, A)\) with \(A \subseteq V \times V\). The elements of \(V\) are refered to as the vertices of the graph and the elements of \(A\) are refered to as arcs. A simple graph is a graph such that:

\((\alpha, \beta) \in A \iff (\beta, \alpha) \in A\) for all \(\alpha, \beta \in V\).

A graph that is not simple is said to be directed. The term “digraph” is also frequently used. An edge of a graph is an unordered pair \(\{\alpha, \beta\}\) where \((\alpha, \beta)\) is an arc and \(\alpha \neq \beta\). A loop is an arc of the form \((\alpha, \alpha)\). In this paper we will be mostly interested in simple graphs without loops and shall use the term “graph” to mean “simple graph without loops”. We give a formal definition for ease of reference.

Definition 5 (Graph). A graph is a pair of sets \((V, A)\) such that:

\[A \subseteq (V \times V) \setminus \{(\alpha, \alpha) : \alpha \in V\}\]

and:

\[(\alpha, \beta) \in A \iff (\beta, \alpha) \in A\] for all \(\alpha, \beta \in V\).

When more than one graph is being discussed, we shall write \(\Gamma = (V\Gamma, A\Gamma)\) or \(\Sigma = (V\Sigma, A\Sigma)\) so that it is clear to which graph each set of vertices and arcs belongs. If \(\alpha\) is a vertex of some graph \(\Gamma\), then the neighbourhood of \(\alpha\) in \(\Gamma\) is the set \(\Gamma(\alpha) = \{\beta \in V\Gamma : (\alpha, \beta) \in A\Gamma\}\). If \(\alpha\) is a vertex of the graph \(\Sigma\) then the neighbourhood of \(\alpha\) in \(\Sigma\) is denoted by \(\Sigma(\alpha)\). We write \(E\Gamma\) (or \(E\Sigma\)) to denote the set of edge of \(\Gamma\) (or \(\Sigma\)) and \(N\Gamma\) (or \(N\Sigma\)) to denote the set of neighbourhoods.

Definition 6 (Automorphism of a Graph). An automorphism of a graph \(\Gamma\) is a bijection \(\varphi : V\Gamma \rightarrow V\Gamma\) such that:

\[(\alpha, \beta) \in A\Gamma \iff (\varphi(\alpha), \varphi(\beta)) \in A\Gamma\] for all \(\alpha, \beta \in V\Gamma\).
The automorphisms of a graph form a group which we denote by \( \text{Aut}(\Gamma) \). If \( \rho : G \to \text{Aut}(\Gamma) \) is a homomorphism then \( (G, VT, \rho) \) is a permutation group and we say that \( G \) acts on \( \Gamma \) as a group of automorphisms.

**Definition 7 (Vertex-transitive Graph).** A vertex transitive graph is a triple \( (G, \Gamma, \rho) \) where \( \rho : G \to \text{Aut}(\Gamma) \) is a homomorphism and \( (G, \Gamma, \rho) \) is transitive.

As with permutation groups, we will often neglect to mention \( \rho \) explicitly. We shall also sometimes say that \( \Gamma \) is a \( G \)-vertex transitive graph rather than \( (G, \Gamma) \) is a vertex transitive graph. If we say simply that \( \Gamma \) is a vertex transitive graph then we mean that \( (\text{Aut}(\Gamma), \Gamma) \) is an vertex transitive graph.

### 3. Local Properties

**Definition 8 (Stabilizer).** Let \( (G, \Omega) \) be a transitive permutation group, and let \( \alpha \) be any point of \( \Omega \). The **point stabilizer** of \( \alpha \) is:

\[
G_\alpha = \{ x \in G : \alpha^x = \alpha \}
\]

**Lemma 1.** \( G_\alpha g = g^{-1}G_\alpha g \) for any \( g \in G \).

**Proof.** Suppose that \( x \in G \) is such that \( (\alpha^g)^x = \alpha^g \). It follows that \( \alpha^{gxg^{-1}} = \alpha \) and so \( gxg^{-1} \in G_\alpha \). That is \( x \in g^{-1}G_\alpha g \). Conversely suppose that \( x \in g^{-1}G_\alpha g \). Then \( gxg^{-1} \in G_\alpha \) and so \( \alpha^{gxg^{-1}} = \alpha \). That is \( \alpha^g = \alpha \), so \( x \in G_\alpha^g \). \( \square \)

Since \( (G, \Omega) \) is transitive, every element of \( \Omega \) may be expressed in the form \( \alpha^g \) for some \( g \in G \). Thus lemma 1 tells us that the point stabilizers of a transitive representation of \( G \) form a family of conjugate subgroups of \( G \).

If \( G \) acts on \( \Gamma \) as a group of automorphisms and \( G_\alpha \) is the stabilizer of the vertex \( \alpha \) then \( (G_\alpha, \Gamma(\alpha)) \) is a permutation group.

**Proposition 1.** if \( (G, \Gamma) \) is a vertex transitive graph then for all \( \alpha, \beta \in VT \) the permutation groups \( (G_\alpha, \Gamma(\alpha)) \) and \( (G_\beta, \Gamma(\beta)) \) are permutation equivalent.

**Proof.** Suppose that \( \beta = \alpha^g \). Let \( \eta : \Gamma(\alpha) \to \Gamma(\beta) \) be given by: \( \gamma \mapsto \gamma^g \), and let \( \varphi : G_\alpha \to G_\beta \) be given by: \( x \mapsto g^{-1}xg \) then:

\[
\eta(\gamma^x) = \gamma^{gx} = \gamma^{gs^{-1}xg} = (\gamma^g)^{\varphi(x)} = \eta(\gamma)^{\varphi(x)}
\]
for any $\gamma \in \Gamma(\alpha)$ and any $x \in G_\alpha$. Thus the pair of maps $(\eta, \varphi)$
establish the desired permutation equivalence.

If $\Gamma$ is $G$-vertex transitive, then for any permutation group theoretic
property $\mathcal{P}$ we say that $\Gamma$ is $G$-locally $\mathcal{P}$ if $(G_\alpha, \Gamma(\alpha))$ has property $\mathcal{P}$.

4. Symmetric Graphs

Definition 9 (Symmetric Graph). A symmetric graph is a triple $(G, \Gamma, \rho)$ where $G$ is a group, $\Gamma$ is a graph and $\rho$ is a homomorphism:

$$\rho : G \to \text{Aut}(\Gamma)$$

such that $\Gamma$ is $G$-vertex transitive and $G$-locally transitive.

Again, we won’t always mention $\rho$ explicitly and will speak of the
symmetric graph $(G, \Gamma)$, or sometimes the $G$-symmetric graph $\Gamma$.

Definition 10 ($s$-arc). An $s$-arc of a graph $\Gamma$ is a sequence of
vertices $(v_0, v_1, ... v_s)$ such that $v_i$ is adjacent to $v_{i+1}$ for each $i$ and
$v_{i-1} \neq v_{i+1}$. The set of $s$-arcs of $\Gamma$ is denoted by $\text{Arc}_s(\Gamma)$.

If $G$ acts on $\Gamma$ as a group of automorphisms, then $G$ also acts on
$\text{Arc}_s(\Gamma)$ in a natural way:

$$(v_0, v_1, ... v_s)^g = (v_0^g, v_1^g, ... v_s^g)$$

The graph $\Gamma$ is said to be $(G, s)$-arc transitive if $(G, \text{Arc}_s(\Gamma))$ is transitive. Historically there has been much interest in highly arc-transitive graphs, that is graphs that are $s$-arc transitive for large $s$.

One of the oldest results in the area is the theorem by Tutte \[32, 33\], proved using combinatorial arguments, that there are no $s$-arc transitive graphs of valency 3 for $s > 5$. More recently Weiss \[37\] was able to show, with the aid of the classification of finite simple groups
that there are no $s$-arc transitive graphs, of any valency, for $s > 7$.

Symmetric graphs have historically been characterized by their arc transitivity rather than their local transitivity. A $(G, 0)$-arc transitive
graph is just a $G$-vertex transitive graph. The next theorem shows
that $(G, 1)$-arc transitive graphs without isolated vertices are symmet-
tric graphs.

Proposition 2. If $(G, \Gamma)$ is a symmetric graph, the $\Gamma$ is $(G, 1)$-arc
transitive. Conversely, if $\Gamma$ contains no isolated vertices and is $(G, 1)$-
arc transitive, then $(G, \Gamma)$ is a symmetric graph.

Proof. Suppose that $(G, \Gamma)$ is a symmetric graph, then $G$ acts on
$\text{Arc}$ in the obvious way: $(\alpha, \beta)^x = (\alpha^x, \beta^x)$. Let $(\alpha_1, \beta_1)$ and $(\alpha_2, \beta_2)$ be
any two arcs. By the $G$-vertex transitivity of $\Gamma$ we can find a $g \in G$ such
that $\alpha_1^g = \alpha_2$. Since $\beta_1 \in \Gamma(\alpha_1)$ and $g$ is an automorphism, we must have $\beta_1^g \in \Gamma(\alpha_1^g) = \Gamma(\alpha_2)$. By $G$-local transitivity we can find $h \in G_{\alpha_2}$ such that $(\beta_1^g)^h = \beta_2$. It follows that $(\alpha_1, \beta_2)^{gh} = (\alpha_1^{gh}, \beta_1^{gh}) = (\alpha_2, \beta_2)$. Thus $G$ acts transitively on the arcs of $\Gamma$.

Now suppose that $\Gamma$ has no isolated vertices and $(G, A\Gamma)$ is transitive. For any two vertices $\alpha_1$ and $\alpha_2$, there is some arc beginning at $\alpha_1$, say $(\alpha_1, \beta_1)$ and some arc beginning at $\alpha_2$, say $(\alpha_2, \beta_2)$. By arc transitivity we can find some $g \in G$ carrying $(\alpha_1, \beta_1)$ to $(\alpha_2, \beta_2)$, and this $g$ must carry $\alpha_1$ to $\alpha_2$. So $\Gamma$ is $G$-vertex transitive.

Conversely suppose that $\gamma_1$ and $\gamma_2$ are both elements of $\Gamma(\alpha)$. The $(\alpha, \gamma_1), (\alpha, \gamma_2) \in A\Gamma$. By arc transitivity we can find a $g \in G$ such that $(\alpha, \gamma_1)^g = (\alpha, \gamma_2)$ and this $g$ must carry $\gamma_1$ to $\gamma_2$. Thus $\Gamma$ is $G$-locally transitive. The result follows.

5. Orbital Graphs

If $(G, \Omega)$ is a transitive permutation group, then $G$ has a natural action on $\Omega \times \Omega$ given by:

$$(\alpha, \beta)^g = (\alpha^g, \beta^g)$$

Although $(G, \Omega)$ is transitive, $(G, \Omega \times \Omega)$ need not be. The orbits of $G$ on $\Omega \times \Omega$ are called orbitals.

**Lemma 2.** If $(G, \Omega)$ is a transitive permutation group and $\alpha \in \Omega$ then there is a natural correspondence between the orbitals of $(G, \Omega)$ and the orbits of $(G_{\alpha}, \Omega)$.

**Proof.** Let $\Delta$ be any orbital of $(G, \Omega)$ and let $(\gamma, \delta)$ be any element of $\Delta$. Let $g \in G$ be such that $\gamma^g = \alpha$. Since $\Delta$ is an orbital, it follows that $(\alpha, \delta^g) \in \Delta$. Thus every orbital of $(G, \Omega)$ contains an element of the form $(\alpha, \beta)$.

Let $\eta : \Omega \to \Omega \times \Omega$ be the map given by $\beta \mapsto (\alpha, \beta)$. Suppose that $\beta_1$ and $\beta_2$ lie in the same orbit of $(G_{\alpha}, \Omega)$. Then there is some $g \in G_{\alpha}$ such that $\beta_1^g = \beta_2$. It follows that $(\alpha, \beta_1)^g = (\alpha, \beta_2)$ and so $(\alpha, \beta_1)$ and $(\alpha, \beta_2)$ lie in the same orbital of $(G, \Omega)$.

Conversely suppose that $(\alpha, \beta_1)$ and $(\alpha, \beta_2)$ lie in the same orbital of $(G, \Omega)$. Then there is some $g \in G$ such that $(\alpha, \beta_1)^g = (\alpha, \beta_2)$. That is $\alpha^g = \alpha$ and $\beta_1^g = \beta_2$. It follows that $g \in G_{\alpha}$ and $\beta_1$ and $\beta_2$ lie in the same orbit of $(G_{\alpha}, \Omega)$. Thus, the map $\eta$ in fact induces a map from the orbits of $(G_{\alpha}, \Omega)$ to the orbitals of $(G, \Omega)$.

The orbital $\{(\alpha, \alpha) : \alpha \in \Omega\}$ is given a special name. Its called the diagonal orbital. The rank of a permutation group is the number of
orbitals. An orbital $\Delta$ is said to be *self-paired* if

$$(\alpha, \beta) \in \Delta \iff (\beta, \alpha) \in \Delta$$

**Definition 11 (Orbital Graph).** If $(G, \Omega)$ is a transitive permutation group and $\Delta$ is a self-paired orbital then the *orbital graph* $\text{Orb}_\Delta(G, \Omega)$ is the graph with vertex set $\Omega$ and arc-set $\Delta$. Need to say more about the diagonal orbital.

Note that if $\Delta$ is taken to be the diagonal orbit, then the resulting graph has loops, and so is not actually a graph by our definition. Generally we assume that $\Delta$ is not the diagonal orbit.

**Proposition 3.** Orbital graphs are symmetric. Every symmetric graph is an orbital graph.

**Proof.** Suppose that $\Gamma = \text{Orb}_\Delta(G, \Omega)$ for some transitive permutation group $(G, \Omega)$ and some self-paired orbital $\Delta$. Since $V \Gamma = \Omega$ and $(G, \Omega)$ is transitive, $\Gamma$ is $G$-vertex transitive. Since $A \Gamma = \Delta$ and $\Delta$ is an orbital $\Gamma$ is $G$-arc transitive. It follows that $\Gamma$ is symmetric.

Now suppose that $\Gamma$ is a $G$-symmetric graph. Since $\Gamma$ is $G$-vertex transitive $(G, V \Gamma)$ is a transitive permutation group. Since $A \Gamma \subseteq V \Gamma \times V \Gamma$ and $\Gamma$ is $G$-arc transitive, $A \Gamma$ is an orbital of $(G, V \Gamma)$. It follows that $\Gamma = \text{Orb}_{A \Gamma}(G, V \Gamma)$. □

Relaxing the condition that $\Delta$ must be self-paired leads to a symmetric digraph.

6. Imprimitive Symmetric Graphs

Recall that if $\Delta$ is a subset of $\Omega$ and $g$ is an element of $G$, then $\Delta^g$ denotes the image of $\Delta$ under the action of $g$.

**Definition 12 (Block of Imprimitivity).** If $(G, \Omega)$ is a transitive permutation group then a subset $\Delta$ of $\Omega$ is said to be a *block of imprimitivity* if for every $x \in G$ either $\Delta^x = \Delta$ or $\Delta^x \cap \Delta = \emptyset$.

**Definition 13 (G-invariant partition).** If $(G, \Omega)$ is a transitive permutation group, then a partition $\mathcal{B}$ of $\Omega$ is said to be $G$-*invariant* if for each $\Delta \in \mathcal{B}$ and each $x \in G$ we have $\Delta^x \in \mathcal{B}$. That is, $\mathcal{B}$ admits $G$ as a group of permutations in a natural way.

**Proposition 4.** If $(G, \Omega)$ is a transitive permutation group and $\Delta$ is a block of imprimitivity then $\mathcal{B} = \{\Delta^g : g \in G\}$ is a $G$-invariant partition.
6. IMPRIMITIVE SYMMETRIC GRAPHS

PROOF. Since \((G, \Omega)\) is transitive, every \(\alpha \in \Omega\) is contained in \(\Delta^g\) for some \(g \in G\). Thus \(\bigcup_{g \in G} \Delta^g = \Omega\). If \(\Delta^{g_1} \cap \Delta^{g_2} \neq \emptyset\) for some \(g_1, g_2 \in G\) then \(\Delta \cap \Delta^{g_2g_1^{-1}} \neq \emptyset\). Since \(\Delta\) is a block of imprimitivity, this implies that \(\Delta = \Delta^{g_2g_1^{-1}}\) and so \(\Delta^{g_1} = \Delta^{g_2}\). Thus \(B\) is a partition of \(\Omega\). For any \(\Delta^g \in B\) and any \(x \in G\) \((\Delta^g)^x = \Delta^{gx} \in B\). Thus \(B\) is a \(G\)-invariant partition of \(\Omega\).

PROPOSITION 5. If \((G, \Omega)\) is a transitive permutation group and \(B\) is a \(G\)-invariant partition of \(\Omega\), then each \(B\) in \(B\) is a block of imprimitivity.

PROOF. Since \(B\) is \(G\)-invariant, \(B^g \in B\) for any \(g \in G\). Since \(B\) is a partition, if \(B^g \neq B\) then \(B^g \cap B = \emptyset\). The result follows.

PROPOSITION 6. The map \(\pi : \Omega \to B\) which sends each point of \(\Omega\) to the block of \(B\) containing it induces a surjective permutation homomorphism from \((G, \Omega)\) to \((G, B)\).

PROOF. Need to show that \(\pi(\alpha^x) = \pi(\alpha)^x\) for any \(\alpha \in \Omega\) and any \(x \in G\). By the previous proposition \(\pi(\alpha)\) is a block of imprimitivity. Let \(\Delta = \pi(\alpha)\). Since \(\alpha \in \Delta\) we must have \(\alpha^x \in \Delta^x\) That is \(\pi(\alpha^x) = \Delta^x = \pi(\alpha)^x\).

For any \(B \in B\) we refer to the set \(\pi^{-1}(B)\) as the fiber of the homomorphism \(\pi\) at \(B\). The fibers of a permutation homomorphism are blocks of imprimitivity. Conversely each block of imprimitivity \(\Delta\) gives rise to the \(G\)-invariant partition: \(B = \{\Delta^x : x \in G\}\) and thus also to the homomorphism \((G, \Omega) \mapsto (G, B)\).

For every permutation group both \(\{a\}\) and \(\Omega\) are trivially blocks of imprimitivity. A permutation group is primitive if it admits no nontrivial blocks of imprimitivity.

DEFINITION 14 (Imprimitive Symmetric Graph). A symmetric graph \((G, \Gamma)\) is said to be imprimitive if the induced permutation group, \((G, VT)\), is imprimitive.

DEFINITION 15 (Quotient Graph). Suppose that \((G, \Gamma)\) is an imprimitive symmetric graph with \(B\) a nontrivial \(G\)-invariant partition of the vertex set \(VT\). Define the quotient graph \(\Gamma_B\) of \((G, \Gamma)\) with respect to \(B\) to be the graph with vertex set \(\mathcal{B}\), and an arc \((B, C)\), whenever there is some \(\alpha \in B\) and some \(\beta \in C\) such that \((\alpha, \beta)\) is an arc of \(\Gamma\).

PROPOSITION 7. \((G, \Gamma_B)\) is a symmetric graph.

PROOF. We show that \(G\) acts transitively on the arcs of \(\Gamma_B\). Let \((B, C)\) and \((D, E)\) be any two arcs of \(\Gamma_B\). By the definition of the
quotient graph, there must be some \( \alpha \in \pi^{-1}(B) \) and \( \beta \in \pi^{-1}(C) \) such that \((\alpha, \beta) \in A\Gamma\). Similarly, there must be some \( \gamma \in \pi^{-1}(D) \) and \( \delta \in \pi^{-1}(E) \) such that \((\gamma, \delta) \in A\Gamma\). By the symmetry of \((G, \Gamma)\), we can find an element \( g \in G \) which carries \((\alpha, \beta)\) to \((\gamma, \delta)\). Since \( \pi \) is a \( G \)-homomorphism, it follows that:

\[
\begin{align*}
(B, C)^g &= (\pi(\alpha), \pi(\beta))^g \\
&= (\pi(\alpha)^g, \pi(\beta)^g) \\
&= (\pi(\alpha^g), \pi(\beta^g)) \\
&= (\pi(\gamma), \pi(\delta)) \\
&= (D, E).
\end{align*}
\]

\( \square \)

Suppose that some arc of \( \Gamma \) has both its endpoints in the same fiber of \( B \), that is, there exists some \((\alpha, \beta) \in A\Gamma\) such that \( \pi(\alpha) = \pi(\beta) \). Since the fibers are blocks of imprimitivity, it follows that \( \pi(\alpha^g) = \pi(\beta^g) \) for each \( g \in G \). The arc transitivity of \( \Gamma_B \) then implies every arc of \( \Gamma \) has both its endpoints in the same fiber. In this case \( \Gamma_B \) is the empty graph (no edges) with one vertex per connected component of \( \Gamma \). This case is not very interesting, to exclude it we say that the quotient is nontrivial if it has valency at least one. By the above discussion, \( \Gamma_B \) is nontrivial if and only if each of the fibers of \( B \) is an independent set of \( \Gamma \). For the remainder of this paper we shall always assume that the quotient of a \( G \)-symmetric graph homomorphism is non-trivial, even if we neglect to state this explicitly.
CHAPTER 2

Coset Graphs

In this chapter we first show that for any group \( G \), the transitive representations of \( G \) together with \( G \)-homomorphisms between them form a lattice isomorphic to a quotient of the subgroup lattice of \( G \). Next we describe a construction of Sabidussi’s for vertex transitive graphs, and give a group theoretic characterization of symmetric graphs. Finally we consider quotients of symmetric graphs from a group theoretic perspective.

1. Transitive Permutation Groups

Definition 16 (Core of a subgroup). If \( H \) is a subgroup of \( G \), then the core of \( H \) in \( G \) is:

\[
\text{Core}_G(H) = \bigcap_{x \in G} x^{-1}Hx
\]

Proposition 8. If \( H \) is a subgroup of \( G \), then the core of \( H \) in \( G \) is a normal subgroup of \( G \).

Proof. Since \( gx \) runs over all the elements of \( G \) as \( x \) does, we have:

\[
g^{-1}\text{Core}_G(H)g = \bigcap_{x \in G} (gx)H(gx)^{-1} = \bigcap_{x \in G} xHx^{-1}
\]

for any \( g \in G \). \( \square \)

Definition 17 (Coset Representation). For \( H \) a subgroup of \( G \), let \( \text{Cos}_G(H) \) denote the right cosets of \( H \) in \( G \). We may define an action of \( G \) on \( \text{Cos}_G(H) \) by:

\[
(Ha)^x = Hax
\]

This is indeed an action, since:

\[
(Ha)^{xy} = (Hax)^y = (Haxy) = (Ha)^{xy}
\]
for any “$Ha$” a coset of $H$ in $G$, and any $x, y \in G$. We call a permutation group of the form $(G, \cos_G(H))$ a coset representation of $G$.

**Example 1 (Right Regular Representation).** For any group $G$, the right regular representation of $G$ is the permutation group $(G, G)$ with the action given by:

$$g^h = gh.$$ 

This is a special case of a coset representation where $H$ is the trivial group $\{1\}$.

**Proposition 9.** For any pair of groups $H \leq G$, the action of $G$ on the cosets of $H$ is transitive with kernel $\text{Core}_G(H)$.

**Proof.** Let $Ha_1$ and $Ha_2$ be any two cosets of $H$ in $G$, then:

$$(Ha_1)^{a_1^{-1}a_2} = Ha_2$$

so the action is transitive.

The stabilizer of the point “$H$” is $H$. By Lemma 1, Chapter 1, the stabilizer of the point “$Ha$” is $a^{-1}Ha$. If $g \in G$ is such that $g$ stabilizes every coset of $H$ in $G$, then we must have $g \in a^{-1}Ha$ for each $a \in G$. That is $g \in \text{Core}_G(H)$.

□

The next proposition tells us that every transitive representation of a group $G$ is permutation equivalent to some coset representation.

**Proposition 10.** If $(G, \Omega)$ is a transitive permutation group and $\alpha$ is some point of $\Omega$, then $(G, \Omega)$ is permutation equivalent to $(G, \cos_G(G_\alpha))$.

**Proof.** For each $\beta \in G$ Consider:

$$S_\beta = \{x \in G : \alpha^x = \beta\}$$

If $\beta = \alpha^g$ then it’s not hard to see that $S_\beta$ is a right coset of $G_\alpha$:

$$S_\beta = G_\alpha g$$

Define $\eta : \Omega \to \cos_G(G_\alpha)$ by $\eta(\beta) = S_\beta$ and take $\varphi : G \to G$ to be the identity. We have, for any $\beta \in \Omega$ and any $x \in G$, if $\beta = \alpha^g$ then:

$$\eta(\beta^x) = S_{\beta^x} = S_{\alpha^g x} = G_\alpha g x = G_\alpha g \varphi(x) = \eta(\alpha^g) \varphi(x) = \eta(\beta) \varphi(x)$$

Thus $\eta$ and $\varphi$ establish the desired permutation equivalence. □
2. Imprimitive Permutation Groups

Definition 18. If \((G, \Omega)\) is a transitive permutation group and \(\Delta\) is a subset of \(\Omega\) then the setwise stabilizer of \(\Delta\) is:

\[ G_\Delta = \{ x \in G : \Delta^x = \Delta \} \]

Lemma 3. If \((G, \Omega)\) is a transitive permutation group and \(\Delta\) is a block of imprimitivity, then \(G_\alpha \leq G_\Delta \leq G\) for any \(\alpha \in \Delta\).

Proof. Since \(\alpha \in \Delta\), for any \(g \in G\) we have \(\alpha^g \in \Delta^g\). Since \(\Delta\) is a block of imprimitivity, if \(\alpha^g = \alpha\), then \(\Delta^g \cap \Delta \neq \emptyset\) so \(\Delta^g = \Delta\). That is \(G_\alpha \leq G_\Delta\).

Proposition 11. If \((G, \Omega)\) is a transitive permutation group and \(\Delta\) is a block of imprimitivity, then \((G_\Delta, \Delta)\) is a transitive permutation group.

Proof. Suppose that \(\alpha, \beta \in \Delta\). Since \((G, \Omega)\) is transitive, there is some \(g \in G\) such that \(\alpha^g = \beta\). Since \(\alpha^g \in \Delta^g\) and \(\beta \in \Delta\), we must have \(\Delta^g \cap \Delta \neq \emptyset\). But \(\Delta\) is a block of imprimitivity, so this implies that \(\Delta^g = \Delta\). That is \(g \in G_\Delta\), so \(G_\Delta\) acts transitively on \(\Delta\).

Proposition 12. Suppose that \((G, \Omega)\) is a permutation group and \(\alpha \in \Omega\). For any subgroup \(H\) such that \(G_\alpha \leq H \leq G\), the set \(\alpha^H = \{ \alpha^h : h \in H \}\) is a block of imprimitivity.

Proof. Let \(\Delta = \alpha^H\). If \(\Delta^g \cap \Delta \neq \emptyset\) for some \(g \in G\) then there must be some \(x, y \in H\) such that \(\alpha^{xg} = \alpha^y\). It follows that \(g \in x^{-1}G_\alpha y \leq H\) and \(\Delta^g = \Delta\). That is \(\Delta^g \cap \Delta \neq \emptyset \Rightarrow \Delta^g = \Delta\), so \(\alpha^H\) is a block of imprimitivity.

Lemma 4. If \((G, \Omega)\) is a transitive permutation group and \(\alpha\) is some point of \(\Omega\), then \(G_{\alpha^H} = H\) for any \(H\) such that \(G_\alpha \leq H \leq G\).

Proof. Let \(\Delta = \alpha^H\). Clearly \(H \leq G_\Delta\). To show that \(H = G_\Delta\) it suffices to show that \(G_\alpha\) and \(G_\Delta\) have the same index in \(H\). Since \(\alpha^H\) is an orbit of \(H\) on \(\Omega\), the permutation group \((H, \alpha^H)\) is transitive. Since \(G_\alpha \leq H\), the stabilizer of the point \(\alpha\) in \((H, \alpha^H)\) is \(H \cap G_\alpha = G_\alpha\). Thus by the orbit stabilizer theorem we have \(\Delta = [H : G_\alpha]\).

On the other hand, by proposition 11 and 12, \(\Delta\) is a block of imprimitivity and \((G_\Delta, \Delta)\) is a transitive permutation group. By lemma 3 we have \(G_\alpha \leq G_\Delta\) so the stabilizer of the point \(\alpha\) in \((G_\Delta, \Delta)\) is \(G_\Delta \cap G_\alpha = G_\alpha\). So again by the orbit stabilizer lemma we have \(|\Delta| = [G_\Delta : G_\alpha]|\). The result follows.
Lemma 5. If \((G, \Omega)\) is a transitive permutation group, \(\Delta\) is a block of imprimitivity, and \(\alpha\) is any point of \(\Delta\), then \(\alpha^{G_\Delta} = \Delta\).

Proof. Clearly \(\alpha^{G_\Delta} \leq \Delta\). Let \(H = G_\Delta\). We have \(G_\alpha \leq H \leq G\) and \((H, \alpha^H)\) is a transitive permutation group equivalent to \((H, \cos_H(G_\alpha))\). Since \((G_\Delta, \Delta)\) is also a transitive permutation group equivalent to \((H, \cos_H(G_\alpha))\) we must have \(|\Delta| = |\alpha^H|\). Since \(\alpha^H \leq \Delta\), this implies that \(\alpha^H = \Delta\).

\[\square\]

Let \((S, \leq)\) be the set of subgroups of \(G\) containing \(G_\alpha\), partially ordered by the subgroup relation. Let \((P, \subseteq)\) be the set of blocks of imprimitivity containing \(\alpha\), partially ordered by the subset relation.

Proposition 13. \((S, \leq)\) is order isomorphic to \((P, \subseteq)\)

Proof. Let \(\Phi : S \to P\) be given by \(\Phi(\Delta) = G_\Delta\) and let \(\Psi : P \to S\) be given by \(\Psi(H) = \alpha^H\). Making use of lemma ?? and ??, for any \(H \in S\) we have:

\[\Phi \circ \Psi(H) = \Phi(\alpha^H) = G_{\alpha^H} = H\]

and for any \(\Delta \in P\) we have:

\[\Psi \circ \Phi(\Delta) = \Psi(G_\Delta) = \alpha^{G_\Delta} = \Delta\]

so \(\Phi\) and \(\Psi\) are inverses. To see that \(\Phi\) is order preserving note that:

\[G_{\Delta_1} \leq G_{\Delta_2} \iff \alpha^{G_{\Delta_1}} \leq \alpha^{G_{\Delta_2}} \iff \Delta_1 \subseteq \Delta_2\]

\[\square\]

Proposition 14. Let \((G, \Omega_1)\) and \((G, \Omega_2)\) be two transitive representations of \(G\). For any \(\alpha \in \Omega_1, \beta \in \Omega_2\), the permutation groups \((G, \Omega_1)\) and \((G, \Omega_2)\) are permutation equivalent if and only if there is an automorphism of \(G\) carrying \(G_\alpha\) to \(G_\beta\).

Proof. Suppose that \((G, \Omega_1)\) and \((G, \Omega_2)\) are permutation equivalent. Let \(\varphi : G \to G\) and \(\eta : \Omega_1 \to \Omega_2\) be the maps establishing the equivalence. Let \(\gamma = \eta(\alpha)\). Clearly \(G_\gamma = \varphi(G_\alpha)\). By transitivity there must be some \(g \in G\) such that \(\beta = \gamma^g\). Let \(\psi_g : G \to G\) be the map:

\[x \mapsto g^{-1}xg\]

We have:

\[G_\beta = g^{-1}G_\gamma g = \psi(G_\gamma) = \psi \circ \varphi(G_\alpha)\]

So \(\psi \circ \varphi\) carries \(G_\alpha\) to \(G_\beta\).

For the other direction, let \(\psi\) be the map carrying \(G_\alpha\) to \(G_\beta\). By proposition 1, \((G, \Omega_1)\) is permutation equivalent to \((G, \cos(G_\alpha))\) and
(G, Ω2) is permutation equivalent to (G, cos(Gα)). Take ϕ = ψ and let
η : cos(Gα) → η(Gβ) be given by (Gα)g → (Gβ)ψ(g). For any x ∈ G
we have:

\[ η((Gαg)^x) = η(Gαgx) = (Gβ)ψ(gx) = (Gβ)ψ(g)^ψ(x) = η(Gαg)^ψ(x) \]

thus ϕ and η establish a permutation equivalence between (G, cos(Gα)) and (G, cos(Gβ)) and hence between (G, Ω1) and (G, Ω2).

Define an equivalence relation on the set of subgroups of G as fol-
lows: H1 ∼ H2 if and only if there is an automorphism of G carrying
H1 to H2. Let Z denote the equivalence classes of this relation. De-
fine a partial order on Z by [H1] ≤ [H2] if and only if there is some
H̃1 ∈ [H1] and some H̃2 ∈ [H2] such that H̃1 ≤ H̃2.

The previous result tells us that, upto permutation equivalence,
the essential information about the transitive representations of G is
contained in (Z, ≤). In particular all the transitive representations of
G are to be found as quotients of the right regular representation.

3. Sabidussi’s Construction

We saw in the last section that every permutation group is equiv-
alent to one of the form (G, cos(Gα)). Thus, upto isomorphism, tran-
sitive permutation groups are uniquely determined by pairs of group s
(G, H) with H a subgroup of G. In this section and the next we shall
see that upto isomorphism, a symmetric graph is uniquely determined
by a triple (G, H, a) where H is a subgroup of G and a ∈ G \ H is an
involution.

Given a triple (G, H, a), the idea is to construct a graph whose ver-
tices are the cosets of H in G. We call such a graph a coset graph. The
idea of a coset graph originally goes back to Sabidussi who was study-
ing vertex transitive graphs. Sabidussi’s construction was essentially a
generalization of the Cayley graph construction.

DEFINITION 19. (Cayley Graph) Given a group G and a subset
D ⊆ G the Cayley graph, Cay(G, D) is the directed graph with vertices
the elements of G and arc set \{(x, y) : xy^{-1} ∈ D\}

A Cayley Graph will contain no loops, provided that 1 ∉ D. A
Cayley Graph is simple if and only if the set D is closed under taking
inverses, and connected if and only if D is a generating set for G. The
vertices are the elements of the group $G$, and the action of $G$ on the vertices is permutation equivalent to the right regular representation of $G$.

Not every vertex transitive graph is a Cayley Graph. Sabidussi’s idea was to generalize Cayley’s construction by considering the action of $G$ on the cosets of some non-trivial subgroup $H$.

**Definition 20.** (Sabidussi Graph) Given a group $G$, a subgroup $H$, and a set $D \subseteq G$, the Sabidussi graph, $\text{Sab}(G, H, D)$, is the directed graph with vertex set $\text{cos}_G(H)$ and arc set $\{(Hx, Hy) : xy^{-1} \in D\}$

A Sabidussi graph will contain no loops provided that $D \cap H = \emptyset$. A Sabidussi Graph is simple if and only if $D$ is closed under taking inverses and connected if and only if $D \cup H$ generates $G$. The vertices of the Sabidussi graph are the cosets of the subgroup $H$ in $G$, and the action of $G$ on the vertices is $(G, \text{cos}_G(H))$.

**Proposition 15 (Sabidussi).** Sabidussi Graphs are vertex transitive. Every vertex transitive graph is isomorphic to a Sabidussi Graph.

**Proof.** Let $\Gamma = \text{Sab}(G, H, D)$ be any sabidussi graph (simple and without loops). The group $G$ has a natural transitive action on the vertices of $\Gamma$ given by $Hx^g = Hxg$. To prove that $\Gamma$ is $G$-vertex transitive, we must check that this action preserves the adjacency structure of $\Gamma$.

Suppose that $(Hx, Hy)$ is an arc of $\Gamma$, so $xy^{-1} \in D$. Since $xg(yg)^{-1} = xgg^{-1}y^{-1} = xy^{-1} \in D$ for any $g$, it follows immediately that $(Hx, Hy)^g = (Hxg, Hyg)$ is also an arc of $\Gamma$. This proves the first part.

Now, let $\Gamma$ be any $G$-vertex transitive graph. It follows that $(G, V\Gamma)$ is permutation equivalent to $(G, \text{cos}_G(H))$ for some $H < G$. Let $\eta : \Omega \to \text{cos}_G(H)$ be the map establishing the permutation equivalence, and let $\mu = \eta^{-1}$

Let $N$ be the neighbourhood of the vertex $\mu(H)$, and let

$$D = \left( \bigcup_{\alpha \in N} \eta(\alpha) \right) \setminus H$$

We will show that $\Gamma \cong \text{Sab}(G, H, \eta(D))$.

The map $\eta : V\Gamma \to \text{cos}_G(H)$ establishes a permutation isomorphism between the vertices of $\Gamma$ and the vertices of $\text{Sab}(G, H, D)$. Let $(\alpha, \beta)$ be any arc of $\Gamma$. By $G$-vertex transitivity, we can find some $g \in G$ such that $\alpha^g = \mu(H)$ and $\beta^g \in E$. Suppose that $\eta(\alpha) = Hx$ and $\eta(\beta) = Hy$. We must check that $xy^{-1} \in D$
Since $\eta$ is a permutation isomorphism, it follows that $\eta(\alpha^g) = Hxg$ and $\eta(\beta^g) = Hyg$. Since $\beta^g \in E$ we must have $xg(yg)^{-1} = xgg^{-1}y = xy^{-1} \in D$. The result follows.

4. A Group Theoretic Characterization of Symmetric Graphs

Since symmetric graphs are vertex transitive, every symmetric graph is a Sabidussi graph. However, not every vertex transitive graph is symmetric, so we expect there to be some extra conditions on the subset $D$ in the symmetric case.

Recall from Chapter 1 that every symmetric graph is an orbital graph. In the first section of this chapter, we saw that if $H < G$ is the stabilizer of some point in $\Omega$, then $(G, \Omega)$ is permutation equivalent to $(G, \cos_G(H))$. We shall see in this section, the orbitals of $(G, \Omega)$ actually correspond to double cosets of $H$ in $G$.

**Definition 21.** If $H$ is a subgroup of $G$, then a double coset of $H$ in $G$ is a subset of $G$ of the form $HxH = \{h_1xh_2 : h_1, h_2 \in H\}$ for some $x \in G$.

**Lemma 6.** Each double coset of $H$ in $G$ is a union of right cosets of $H$. The double cosets of $H$ in $G$ form a partition of $G$.

**Proof.** The first part is obvious since $HxH = \bigcup_{h \in H} Hxh$ for any $x \in G$. For each element $g \in G$ it is clear that $g \in HgH$ so $\bigcup_{x \in G} HxH = G$. If $HxH \cap HyH \neq \emptyset$ then $h_1xh_2 = h_3yh_4$ for some $h_1, h_2, h_3, h_4 \in H$. It follows that $y = h_1^{-1}h_3xh_4h_2^{-1}$ and $HyH = Hh_1^{-1}h_3xh_4h_2^{-1}H = HxH$. This shows that any two double cosets are equal or disjoint, so the double cosets partition $G$.

**Proposition 16 (Lorimer).** For any permutation group $(G, \Omega)$ with point stabilizer $H$, the orbitals are in natural bijection with the double cosets of $H$ in $G$. The self-paired orbitals correspond to those double cosets which contain an involution.

**Proof.** From the first section in this chapter we know that $(G, \Omega)$ is permutation equivalent to $(G, \cos_G(H))$. Recall from chapter 1 that the orbitals of a permutation group are in bijection with the orbits of the point stabilizer. We must show that orbit of $(H, \cos_G(H))$ corresponds to a double coset of $H$ in $G$.

Let $\kappa$ be the map $Hx \mapsto HxH$. Suppose that $Hx$ and $Hy$ are in the same orbit of $(H, \cos_G(H))$. Then there is some $h \in H$ such that
$Hxh = Hy$. It follows that $y \in Hxh \subset HxH$. That is $HyH = HxH$ and so $\kappa(Hx) = \kappa(Hy)$.

Conversely, suppose that $\kappa(Hx) = \kappa(Hy)$ for some pair of cosets $Hx$ and $Hy$. Then $x \in HyH$ and so $x = h_1y_2$ for some $h_1, h_2 \in H$. It follows that $Hx = Hyh_2$, that is $Hx$ and $Hy$ lie in the same orbit of $(H, \cos_G(H))$. Since $\kappa$ is surjective, this proves the first part.

Suppose that $a \in G$ is such that $a^2 = 1$ and $a \not\in H$. Since $(H, Ha)^a = (Ha, Ha^2) = (Ha, H)$, it follows that the orbital $\Delta$ of $(G, \cos_G(H))$ containing $(H, Ha)$ is self-paired. Clearly $a \in HaH = \kappa(Ha)$, so the double coset associated with $\Delta$ contains an involution.

Conversely, if the double coset $HxH$ contains an involution $a$, then $HxH = HaH = \kappa(Ha)$. Thus $HxH$ is associated with the orbital containing $(H, Ha)$ which is self-paired. This proves the second part.

\begin{proof}
Let $\Gamma = \text{Sab}(G, H, HaH)$. By proposition ?? $\Gamma$ is $G$-vertex transitive. Since $HaH$ is an orbit of the action of $H$ on $\cos_G(H)$, the graph $\Gamma$ is also $G$-locally transitive.

Suppose that $\Gamma$ is a $G$-symmetric graph. Let $\alpha$ be any vertex, and let $H = G_\alpha$. By proposition ??, the permutation group $(G, V\Gamma)$ is equivalent to $(G, \cos_G(H))$. Let $\eta : V\Gamma \to \cos_G(H)$ be any map inducing this equivalence. Let $\beta$ be any neighbour of $\alpha$ in $\Gamma$ and let $a \in G$ be such that $(\alpha, \beta)^a = (\beta, \alpha)$. Clearly $a$ is an involution and $\eta(\beta) = Ha$.

Let $\Gamma' = \text{Sab}(G, H, HaH)$. The map $\eta$ induces a permutation equivalence between the vertices of $\Gamma$ and the vertices of $\Gamma'$. To show that $\Gamma$ is isomorphic to $\Gamma'$ we must show that $\eta$ preserves adjacency.

Let $(\omega, \delta)$ be any arc of $\Gamma$. Let $g \in G$ be such that $\omega^g = \alpha$. Then $\delta^g$ is some neighbour of $\alpha$. Thus there exists $h \in H$ such that $\delta^gh = \beta$. Therefore $\eta(\delta^gh) = \eta(\beta) = Ha$. Since $\eta$ is a permutation equivalence, $\eta(\delta^gh) = \eta(\delta)^gh$, so $\eta(\delta) = Ha^{(\delta^gh)^{-1}} = Ha^{-1}g^{-1}$. Similarly, $\eta(\omega^g) = \eta(\omega)^g$ so $\eta(\omega) = Hg^{-1} = Hg^{-1}$. Now $ah^{-1}g^{-1}g = ah^{-1} \in HaH$ so $(\eta(\omega), \eta(\delta))$ is an arc of $\Gamma'$. The result follows.
\end{proof}

\begin{lemma}
$|HxH| = |H||H|/|x^{-1}Hx \cap H|$. Fix me.
\end{lemma}

\begin{proof}
The mapping $HxH \to x^{-1}HxH$ given by $h_1xh_2 \mapsto x^{-1}h_1xh_2$ is bijective, so $|HxH| = |x^{-1}HxH|$. But for the rule for the product
of two subgroups: \(|x^{-1}HxH| = |x^{-1}Hx||H|/|x^{-1}Hx \cap H|\), of course, \(|x^{-1}Hx| = |H|\). and the result follows. □

**Proposition 17.** If \(\Gamma\) is a \(G\)-symmetric graph isomorphic to the Sabidussi graph \(\text{Sab}(G,H,HaH)\) the the stabilizer of an arc of \(\Gamma\) is isomorphic to \(a^{-1}Ha \cap H\) and the valency of \(\Gamma\) is \(|H|/|x^{-1}Hx \cap H|\).

**Proof.** Consider the arc \((H,Ha)\). The stabilizer of this arc is the intersection between the stabilizer of the vertex “\(H\)” and the stabilizer of the vertex “\(Ha\)”. But the stabilizer of the vertex “\(H\)” is just \(H\) and the stabilizer of the vertex “\(Ha\)” is \(a^{-1}Ha\). So the stabilizer of the arc is \(a^{-1}Ha \cap H\).

Consider the action of \(H\) on the neighbours of the vertex “\(H\)”.
Since \(\Gamma\) is symmetric, this action is transitive. If “\(Ha\)” is any neighbour of “\(H\)” , then the action of \(H\) on the neighbours of “\(H\)” is permutation equivalent to the action of \(H\) on the cosets of the stabilizer of the arc \((H,Ha)\). Since the stabilizer of the arc \((H,Ha)\) is \(a^{-1}Ha \cap H\) it follows that the valency of \(\Gamma\) is \(|H : a^{-1}Ha \cap H| = |H|/|x^{-1}Hx \cap H|\)

□

In some sense the valancy of a symmetric graph is measuring the extent to which a subgroup fails to be normal.

### 5. Quotient Graphs

In the first section of this Chapter, we determined, for any two transitive permutation groups \((G,\Omega_1)\) and \((G,\Omega_2)\), the conditions under which there exists a \(G\)-homomorphism from \((G,\Omega_1)\) to \((G,\Omega_2)\).

We saw that if \((G,\Omega_1)\) is permutation equivalent to \((G,\cos_G(H))\) for some \(H < G\), and \((G,\Omega_2)\) is permutation equivalent to \((G,\cos_G(K))\) for some \(K < G\), then there is a \(G\)-homomorphism from \((G,\Omega_1)\) to \((G,\Omega_2)\) if and only if \(H < K < G\).

In this section we give analogous conditions under which there exists a \(G\)-homomorphism from a \(G\)-symmetric graph \(\Gamma\) to a \(G\)-symmetric graph \(\Sigma\).

**Theorem 2 (Lorimer).** If \(\Gamma\) is a \(G\)-symmetric graph isomorphic to \(\text{Sab}(G,H,HaH)\) and \(\Sigma\) is a quotient of \(\Gamma\), then \(\Sigma\) is isomorphic to \(\text{Sab}(G,K,KaK)\) for some \(H < K < G\).

**Proof.** Let \(\eta : V\Gamma \to \text{Cos}_G(H)\) be the map inducing the isomorphism between \(\Gamma\) and \(\text{Sab}(G,H,HaH)\). Let \(\pi : V\Gamma \to V\Sigma\) be the map inducing the homomorphism between \(\Gamma\) and \(\Sigma\). Let \(\alpha = \eta^{-1}(H)\), let \(B = \pi(\alpha)\) and let \(\Delta = \pi^{-1}(B)\).
By proposition ?? $\Delta$ is a block of imprimitivity. If $K = G_\Delta$ then the permutation group $(G, V\Sigma)$ is equivalent to $(G, \cos_G(K))$. Let $\mu : V\Sigma \rightarrow \cos_G(K)$ be any map inducing the equivalence. If $\Sigma' = \Sab(G, K, KaK)$ then $\mu$ induces a permutation equivalence between the vertices of $\Sigma$ and the vertices of $\Sigma'$. We must check that it preserves adjacency. Blah blah blah...

If $a \in K$ then the valency of the quotient is one.
CHAPTER 3

The Extension Problem

We would like to understand how a symmetric graph \((G, \Gamma)\) can be “unfolded” into a larger imprimitive symmetric graph \((\tilde{G}, \tilde{\Gamma})\) admitting the original graph as a quotient. In particular we’d like to understand how pairs of graphs \((\Gamma, \tilde{\Gamma})\) where \(\tilde{\Gamma}\) is an “extension” of \(\Gamma\) are related combinatorially. That is, we’d like to be able to describe the the structure of the graph \(\tilde{\Gamma}\) in terms of the structure of the graph \(\Gamma\).

The quotient of a symmetric graph contains considerably less information than the original graph. Gardiner and Praeger [1] observed that some of the information that is lost may be recovered from the induced bipartite graph between adjacent blocks of the partition, and from a combinatorial design induced on the blocks themselves.

In this chapter we describe Gardiner and Praeger’s observations as well as some of the questions is raises.

1. Induced Bipartite graph

Let \(\Gamma\) be an imprimitive \(G\)-symmetric graph, with \(B\) a non-trivial \(G\)-invariant partition of the vertices. As always, we assume that the quotient \(\Gamma_B\) has valency at least one, so the blocks of \(B\) are independent sets. For any arc \((B, C)\) of the quotient, the subgraph of \(\Gamma\) induced by \(B \cup C\) must be bipartite – possibly containing some isolated vertices. If we restrict ourselves to the subgraph induced by \((\Gamma(C) \cap B) \cup (\Gamma(B) \cap C)\) then we obtain a bipartite graph with no isolated vertices. We call this graph the induced bipartite graph of \((B, C)\), and denote it by \(\Gamma[B, C]\).

**Proposition 18.** The induced bipartite graph \(\Gamma[B, C]\) is \(G_{B \cup C}\) symmetric.

**Proof.** Let \((\alpha, \beta)\) and \((\gamma, \delta)\) be any two arcs of \(\Gamma[B, C]\). Without loss of generality we may assume that \(\alpha \in B\) and \(\beta \in C\). By the \(G\)-symmetry of \(\Gamma\) we can find \(g \in G\) such that \((\alpha, \beta)^g = (\gamma, \delta)\). Since \(\Gamma[B, C]\) is bipartite, either \(\gamma \in B\) and \(\delta \in C\) in which case \(B^g = B\) and \(C^g = C\), or \(\gamma \in C\), \(\delta \in B\), in which case \(B^g = C\) and \(C^g = B\). Either way, \(g \in G_{B \cup C}\) and the result follows. 

[25]
Proposition 19. For any two arcs in the quotient, the induced bipartite graphs are isomorphic.

Proof. Let \((B, C)\) and \((D, E)\) be any two arcs of \(\Gamma_B\). Since \(\Gamma_B\) is \(G\)-symmetric we can find some \(g \in G\) such that \((B, C)^g = (D, E)\). Clearly
\[
(\Gamma(C) \cap B)^g = (\Gamma(C^g) \cap B^g) = (\Gamma(E) \cap D),
\]
and
\[
(\Gamma(B) \cap C)^g = (\Gamma(B^g) \cap C^g) = (\Gamma(D) \cap E).
\]
So \(g\) induces a bijection
\[
(\Gamma(C) \cap B) \cup (\Gamma(B) \cap C) \rightarrow (\Gamma(E) \cap D) \cup (\Gamma(D) \cap E).
\]
Since \(g\) is an automorphism, adjacency is preserved and we have an isomorphism from \(\Gamma[B, C]\) to \(\Gamma[D, E]\).

When passing from a symmetric graph to its quotient we preserve the adjacency structure of the blocks, and discard the exact details of which vertices are connected to which others. Some of the information that has been lost is recoverable from the induced bipartite graph. But the induced bipartite graph reveals only the local connectivity. To reconstruct the original graph from its quotient we need to know how these induced bipartite graphs are “glued together”. Gardiner and Praeger observed that some of this global information about the way the induced bipartite graphs fit together is captured in a combinatorial design induced on each of the blocks. Before we can describe this design we need a little background.

2. Combinatorial Designs

Definition 22. An incidence structure is a triple \((P, B, I)\) where \(P\) and \(B\) are sets, usually referred to as points and blocks respectively, and \(I\) is an incidence relation \(I \subseteq P \times B\).

It is often convenient to visualize an incidence structure as a bipartite graph. Take the points as vertices of one side of the bipartition and the blocks as vertices of the other. Draw an edge between the vertex corresponding to a point \(p\) and the vertex corresponding to a block \(b\) if and only if \((p, b) \in I\).

We will be interested in incidence structures satisfying strong regularity and symmetry conditions.

Definition 23. A \((v, k, \lambda)\)-design is an incidence structure such that:

\(1\) There are \(v\) points in total
(2) Each block is incident with exactly \( k \) points.
(3) Each point is incident with exactly \( \lambda \) blocks.

When visualizing an incidence structure as a bipartite graph, the extra regularity conditions of a design correspond to the condition that any two vertices in the same bipartite half of the graph must have the same valency.

Note that our definition here of a “design” corresponds to what is usually referred to as a 1-design or “tactical configuration” in the literature. There is a more general definition of a \( t \)-design of which the definition of a 1-design is a special case. We will not be needing this more general definition.

The incident point-block pairs of a design are usually referred to as flags. For a design \( \mathcal{D} \) we will use the notation \( P_\mathcal{D}, B_\mathcal{D} \) and \( F_\mathcal{D} \) to denote the points, blocks and flags of \( \mathcal{D} \) respectively.

Define the trace of a block \( b \) to be the set \( T(b) = \{ p \in P : (p, b) \in I \} \). Similarly define the trace of a point \( p \) to be the set \( T(p) = \{ b \in B : (p, b) \in I \} \).

**Proposition 20.** In a design, the number of blocks with the same trace is a constant that is independent of the choice of block.

We denote this constant by \( m \) and call it the multiplicity of the design. If distinct blocks have distinct traces, then we may identify the blocks with their traces, and take \( B \) to be a subset of the power set of \( P \). Otherwise we say that the design contains repeated blocks.

**Proposition 21.** If a design contains repeated blocks, then the incidence structure obtained by identifying blocks with the same trace is again a design.

**Definition 24.** If \( \mathcal{D} = (P_\mathcal{D}, B_\mathcal{D}, I) \) is a 1-design then the dual of \( \mathcal{D} \) is the design \( \mathcal{D}^* = (B_\mathcal{D}, P_\mathcal{D}, I^*) \) where \( (b, p) \in I^* \) if and only if \( (p, b) \in I \).

**Proposition 22.** The dual of a 1-design is a 1-design.

### 3. Flag Transitive Designs

An automorphism of a design \( \mathcal{D} \) is a pair \( (\eta, \mu) \) of bijections \( \eta : P_\mathcal{D} \to P_\mathcal{D} \) and \( \mu : B_\mathcal{D} \to B_\mathcal{D} \) with the property that \( (p, b) \) is a flag of \( \mathcal{D} \) if and only if \( (\eta(p), \mu(b)) \) is a flag. The automorphisms of a design form a group.

As with sets and graphs, we say that a group \( G \) acts on a design \( \mathcal{D} \) as a group of automorphisms if there is some homomorphism \( \rho : G \to \)
Aut(\(D\)). We don’t require this homomorphism to be injective, that is we don’t require that \(G\) acts \textit{faithfully} on \(D\).

If the group \(G\) acts on the design \(D\) as a group of automorphisms then we get three induced permutation groups, namely:

1. \((G, P_D)\) – The induced permutation group on the points.
2. \((G, B_D)\) – The induced permutation group on the blocks.
3. \((G, F_D)\) – The induced permutation group on the flags.

We will be interested in highly symmetric designs. In particular, we will be interested in pairs \((G, D)\) where \(G\) acts on \(D\) in such a way that the induced permutation group on the flags \((G, F_D)\) is transitive.

We will either call the pair \((G, D)\) a flag-transitive design, or we call the design \(D\) a \(G\)-flag transitive design. Many interesting groups arise naturally as flag-transitive automorphism groups of designs, including a number of the sporadic simple groups.

**Proposition 23.** If \((G, D)\) is a flag-transitive design then the induced permutation groups \((G, P_D)\) and \((G, B_D)\) are both transitive.

**Proposition 24.** If \((G, D)\) is a flag-transitive design and \(H_1 < G\) is the stabilizer of some point \(p\) then the induced permutation group \((H_1, T(p))\) is transitive. Similarly if \(H_2 < G\) is the stabilizer of some block \(b\) then the induced permutation group \((H_2, T(b))\) is transitive.

If \(D_1\) and \(D_2\) are two \(G\)-flag transitive designs, then a \(G\)-\textit{design homomorphism} from \(D_1\) to \(D_2\) is a pair of maps \(\rho = (\rho_P, \rho_B)\) such that:

1. \(\rho_P : P_{D_1} \to P_{D_2}\) induces a permutation homomorphism between \((G, P_{D_1})\) and \((G, P_{D_2})\).
2. \(\rho_B : B_{D_1} \to B_{D_2}\) induces a permutation homomorphism between \((G, B_{D_1})\) and \((G, B_{D_2})\).
3. \((p, b) \in I_1\) if and only if \((\rho_P(p), \rho_B(b)) \in I_2\).

When both \(\rho_P\) and \(\rho_B\) are bijections we have a \(G\)-\textit{design isomorphisms}. When confusion is unlikely to result, we will sometimes write \(\rho(p)\) instead of \(\rho_P(p)\) when \(p\) is a point and \(\rho(b)\) instead of \(\rho_B(b)\) when \(b\) is a block.

**Proposition 25.** If \(\rho\) is a \(G\)-design homomorphism from \(D_1\) to \(D_2\) then \(\rho\) induces a \(G\)-permutation homomorphism from \(F_{D_1}\) to \(F_{D_2}\).

### 4. \(G\)-symmetric designs

In this section we show how \(G\)-symmetric designs may be viewed as a special kind of flag-transitive design. I’m calling them symmetric designs, but I think they should probably be called square designs.
Definition 25. A $G$-flag transitive design $\mathcal{D}$ is self-dual if there exists a $G$-isomorphism $\rho = (\rho_P, \rho_B)$ between $\mathcal{D}$ and $\mathcal{D}^*$. The $G$-isomorphism $\rho$ is called a duality of $\mathcal{D}$.

Definition 26. A $G$-symmetric design is self-dual $G$-flag transitive design $\mathcal{D}$ which admits a duality $\rho = (\rho_P, \rho_B)$ with the property that $\rho_B \circ \rho_P = id_P$ and $\rho_P \circ \rho_B = id_B$. In this case the $G$-isomorphism $\rho$ is called a polarity of $\mathcal{D}$.

Observe that if $\mathcal{D}$ is a $G$-symmetric design with the property that $(p, \rho(p))$ is a flag for some $p \in P_\mathcal{D}$, then since $\rho$ is a $G$-isomorphism it follows that the point-stabilizer of $\mathcal{D}$ is isomorphic to the flag-stabilizer of $\mathcal{D}$. Thus each point is incident with exactly one block and vice versa. We consider such $G$-symmetric designs to be “degenerate”, and unless an explicit statement to the contrary is given shall take the expression “$G$-symmetric design” to mean “non-degenerate $G$-symmetric design”.

In a sense that will become clear a little later, these “degenerate” $G$-symmetric designs correspond to the “degenerate” orbital graph $\text{Orb}_\Delta(G, \Omega)$ which could be formed from the permutation group $(G, \Omega)$ by taking $\Delta$ to be the diagonal orbit, and also to the “degenerate” Sabidussi graph $\text{Sab}(G, H, a)$ which could be formed by taking $a \in H$.

We shall see that each $G$-symmetric graph gives rise in a natural way to a $G$-symmetric design and conversely a $G$-symmetric design together with a “marked” polarity give rise to a $G$-symmetric graph. We shall also see that in some cases, by choosing a different polarity it is possible to construct two non-isomorphic $G$-symmetric graphs from the same $G$-symmetric design. For any graph $\Gamma$, let $\mathcal{N}_\Gamma = \{\Gamma(v) : v \in V_\Gamma\}$ denote the set of neighbourhoods of $\Gamma$.

Proposition 26. If $\Gamma$ is a $G$-symmetric graph, then the incidence structure $\mathcal{D}(\Gamma) = (V_\Gamma, N_\Gamma, I)$ where $(v, n) \in I$ if and only if $v \in n$ is a $G$-symmetric design.

Proof. The $G$-arc transitivity of $\Gamma$ is sufficient to ensure that $\mathcal{D}(\Gamma)$ is a $G$-flag transitive design. To see that it is in fact a $G$-symmetric design we must exhibit a polarity.

Let $\rho_P : P \to B$ be given by $v \mapsto \Gamma(v)$. The map $\rho_P$ is clearly bijective, and since $G$ acts on $\Gamma$ as a group of automorphisms we have:

\[
\rho_P(v^g) = \Gamma(v^g) = \Gamma(v)^g = \rho_P(v)^g.
\]
So \( \rho_P \) induces a permutation isomorphism. Now, take \( \rho_B = \rho_P^{-1} \). We
must check that the pair \( \rho = (\rho_P, \rho_B) \) preserve the incidence structure
of the design.

Suppose that \((v, \Gamma(w))\) is a flag of \( \mathcal{D} \), so \( v \in \Gamma(w) \). Since \( \Gamma \) is a
simple graph it follows immediately that \( w \in \Gamma(v) \) and so \((w, \Gamma(v))\) is
also a flag of \( \mathcal{D} \). That is \((\Gamma(v), w)\) is a flag of \( \mathcal{D}^* \). But
\[
\rho((v, \Gamma(w))) = (\rho_P(v), \rho_B(\Gamma(w))) = (\Gamma(v), w).
\]

So we are done.

**Proposition 27.** If \( \mathcal{D} \) is a (non-degenerate) \( G \)-symmetric design
with “marked” polarity \( \rho \) then the graph \( \Gamma(\mathcal{D}, \rho) \) with vertex set \( P_\mathcal{D} \) and
arc set \( \{(p, q) : (q, \rho_P(p)) \in I_\mathcal{D}\} \) is \( G \)-symmetric.

**Proof.** We must first check that \( \Gamma(\mathcal{D}, \rho) \) is well-defined. The non-
degenerateness of \( \mathcal{D} \) ensures that there are no loops. If \((p, q)\) is an arc
of \( \Gamma(\mathcal{D}, \rho) \) then \((q, \rho_P(p))\) is a flag of \( \mathcal{D} \). Since \( \rho \) is an isomorphism, if
\((q, \rho_P(p))\) is a flag of \( \mathcal{D} \) then \((\rho_P(q), \rho_B \circ \rho_P(p)) = (\rho(q), p)\) is a flag of
\( \mathcal{D}^* \). It follows that \((p, \rho(q))\) is a flag of \( \mathcal{D} \), and thus \((q, p)\) is an arc of
\( \Gamma(\mathcal{D}, \rho) \). So \( \Gamma(\mathcal{D}, \rho) \) is simple.

By proposition [?] \( G \) acts transitively on the points \( \mathcal{D} \) so \( \Gamma(\mathcal{D}, \rho) \)
is \( G \)-vertex transitive. Suppose that \((p, q_1)\) and \((p, q_2)\) are two distinct
arcs of \( \Gamma(\mathcal{D}, \rho) \). Then we have \( q_1, q_2 \in T(\rho(p)) \). By proposition [?] the
stabilizer of \( \rho(p) \) acts transitively on \( T(\rho(p)) \). Thus we can find some
\( g \in G \) which fixes \( \rho(p) \) and carries \( q_1 \) to \( q_2 \). Since \( \rho \) is a \( G \)-isomorphism,
if \( g \) fixes \( \rho(p) \) it also fixes \( p \), thus \( g \) carries the arc \((p, q_1)\) to the arc
\((p, q_2)\). So \( \Gamma(\mathcal{D}, \rho) \) is \( G \)-locally transitive. The result follows.

**Proposition 28.** For any design \( \mathcal{D} \) and any polarity \( \rho \) the design
\( \mathcal{D}(\Gamma(\mathcal{D}, \rho)) \) is isomorphic to \( \mathcal{D} \)

**Proof.** later

5. “cross-section” of a graph homomorphism

Suppose that \( \Gamma \) is an imprimitive \( G \)-symmetric graph with \( \mathcal{B} \) a
non-trivial \( G \)-invariant partition of the vertices. Assume further that
the quotient \( \Gamma_\mathcal{B} \) has valency at least one so that the blocks of \( \mathcal{B} \) are
independent sets.

For any vertex \( \alpha \) of \( \Gamma \), let \( B(\alpha) \) denote the block of \( \mathcal{B} \) containing
\( \alpha \). Let \( \Gamma(\alpha) = \{ \beta \in V\Gamma : (\alpha, \beta) \in A\Gamma \} \) denote the neighbours of \( \alpha \) in
\( \Gamma \). Let \( \Gamma_\mathcal{B}(B) = \{ C \in \mathcal{B} : (B, C) \in A\Gamma_\mathcal{B} \} \) denote the neighbours of \( B \)
6. RECONSTRUCTION PROBLEM

in the quotient. For any $\alpha \in B$, let $\Gamma_B(\alpha) = \{ B(\beta) : \beta \in \Gamma(\alpha) \}$ denote the blocks of $B$ containing the neighbours of $\alpha$ in $\Gamma$.

Construct a design with point set $B$, block set $\Gamma_B(B)$ and an incidence relation $I \subseteq B \times \Gamma_B(B)$ defined by $(\alpha, C) \in I$ if and only if $C \in \Gamma_B(\alpha)$.

**Proposition 29.** The incidence structure $D(B) = (B, \Gamma_B(B), I)$ is a $H$-flag transitive $(v, k, \lambda)$ design, where:

$$
\begin{align*}
v & := |B| \\
k & := |\Gamma(C) \cap B|, \\
\lambda & := |\Gamma_B(\alpha)|
\end{align*}
$$

and $H$ is the stabilizer of the block $B \in B$.

The design $D(B)$ gives, in a sense, a “cross-section” of the homomorphism $\Gamma \mapsto \tilde{\Gamma}$.

6. Reconstruction problem

Gardiner and Praeger observed that if $\Gamma$ is a $G$-symmetric graph and $B$ is a non-trivial $G$-invariant partition of the vertices, then the graph $\Gamma$ “decomposes” into the triple $(\Gamma_B, \Gamma[B, C], D(B))$.

We may ask now, to what extent the combinatorial structure of $\Gamma$ is determined by the triple $(\Gamma_B, \Gamma[B, C], D(B))$? Do these three components contain sufficient information to reconstruct $\Gamma$ uniquely? If not, what extra information is required?

Suppose we are given a symmetric graph $\Lambda$, a symmetric bipartite graph $\Sigma$ and a flag transitive design $D$ without any particular groups acting upon them. Let us say that a symmetric graph $\Gamma$ is “product” of $(\Lambda, \Sigma, D)$ if there is a group $G$ acting $\Gamma$ and a $G$-invariant partition $B$ of the vertices of $\Gamma$ such that the triple $\Gamma$ decomposes into the triple $(\Lambda, \Sigma, D)$.

What are the necessary and sufficient conditions under which these three components $(\Lambda, \Sigma, D)$ can be “glued together” into some “product” $\Gamma$?

If the necessary conditions are satisfied, could there be more than one way to “glue together” a given triple $(\Lambda, \Sigma, D)$ into an imprimitive symmetric graph $\Gamma$?
Some Constructions

The problem of relating the structure of $\Gamma$ to that of the triple $(\Gamma_B, \Gamma[B, C], \mathcal{D}(B))$ is very difficult when $\Gamma$ is taken to be an arbitrary imprimitive symmetric graph. The approach which has been taken by researchers in the area is to first impose additional assumptions on one or more of the components of the decomposition, then study the subfamily of imprimitive symmetric graphs admitting a decompositions which satisfies these additional assumptions.

The special case where the parameters of the “kernel design” $\mathcal{D}(B)$ satisfy:

$$v = k - 1 \geq 2.$$  

was studied by Li, Praeger and Zhou [7]. It was found that for this special case, if the design $\mathcal{D}(B)$ contains no repeated blocks, then there exists an elegant combinatorial method for constructing $\Gamma$ from $\Gamma_B$. This is the three-arc graph construction which will be described in the next section.

In a later paper Zhou [41] showed that the three-arc graph construction actually applies to a wider family of triples $(G, \Gamma, B)$ than those originally studied. In particular the construction may be used for any triple $(G, \Gamma, B)$ satisfying the following condition:

CONDITION (PE). The induced actions of $G_B$ on $B$ and $\Gamma_B(B)$ are permutationally equivalent with respect to some bijection $\rho : B \to \Gamma_B(B)$.

On a group theoretic level this is quite a natural condition to impose. Suppose that the triple $(G, \Gamma, \mathcal{B})$ satisfies the above condition and that $\Gamma \cong \text{Sab}(G, K, a)$ and $\Gamma_B \cong \text{Sab}(G, H, a)$. If $B$ is a block of $\mathcal{B}$ then by proposition [?] the action of $G_B$ on $B$ is permutation equivalent to $(H, \cos(K))$. The action of $G_B$ on $\Gamma_B(B)$ is just the “local action” of $\Gamma_B$ and so the permutation group $(G_B, \Gamma_B(B))$ is permutation equivalent to $(H, \cos(a^{-1}Ha \cap H))$.

If the condition above holds, then we must have $K \cong a^{-1}Ha \cap H$. This means that the stabilizer of a point of $\Gamma$ is isomorphic to the stabilizer of an arc of $\Gamma_B$. This strong “coupling” between the points of
The three-arc graph construction was first introduced by Li, Praeger and Zhou in [7]. Let $\Sigma$ be any $G$-symmetric graph. Recall that an $s$-arc of $\Sigma$ is a sequence $(\alpha_0, \alpha_1, \ldots, \alpha_s)$ of vertices in $\Sigma$ such that $\alpha_i, \alpha_{i+1}$ are adjacent in $\Sigma$ and $\alpha_{i-1} \neq \alpha_{i+1}$ for each $i$. The set of $s$-arcs of $\Sigma$ is denoted by $\text{Arc}_s(\Sigma)$. Consider the induced action of $G$ on $\text{Arc}_3(\Sigma)$ given by:

$$(\sigma_0, \sigma_1, \sigma_2, \sigma_3)g = (\sigma_0^g, \sigma_1^g, \sigma_2^g, \sigma_3^g).$$

This action is, in general, intransitive. We may however partition the set $\text{Arc}_3(\Sigma)$ into orbits on which $G$ acts transitively. For such an orbit $\Delta$, let $\Delta^o = \{(\sigma_4, \sigma_3, \sigma_2, \sigma_1) : (\sigma_1, \sigma_2, \sigma_3, \sigma_4) \in \Delta\}$ denote the pair of $\Delta$. It is not hard to check that $\Delta^o$ is again an orbit of $G$ on $\text{Arc}_3(\Sigma)$. If $\Delta = \Delta^o$ then $\Delta$ is said to be self paired.

**Definition 27.** Given a $G$-symmetric graph $\Sigma$ and a self-paired orbit $\Delta$ on $\text{Arc}_3(\Sigma)$, the three-arc graph $\Gamma = \text{Arc}_\Delta(\Sigma)$ is the graph with vertex set $A\Sigma$ and arc set $\{((\sigma, \tau), (\sigma', \tau') : (\tau, \sigma, \sigma', \tau') \in \Delta\}$.

Note that the requirement that $\Delta$ is self-paired ensures that the resulting graph is simple.

**Proposition 30.** With $G$, $\Sigma$ and $\Delta$ as in definition 27, the three-arc graph $\Gamma = \text{Arc}_\Delta(\Sigma)$ is $G$-symmetric.

**Proof.** Immediate from the construction. $\square$

For each vertex $\sigma$ of $\Sigma$, let $B(\sigma) = \{(\sigma, \tau) : \tau \text{ is a neighbour of } \sigma\}$ be the set of arcs of $\Sigma$ with initial vertex $\sigma$. Clearly $B = \{B(\sigma) : \sigma \text{ is a vertex of } \Sigma\}$ is a partition of $A\Sigma$. For any $g \in G$ we have

$$B(\sigma)^g = \{(\sigma, \tau)^g : \tau \text{ is a neighbour of } \sigma\} = \left\{(\sigma^g, \tau^g) : \tau \text{ is a neighbour of } \sigma^g\right\} = \left\{(\sigma^g, \tau) : \tau \text{ is a neighbour of } \sigma^g\right\} = B(\sigma^g).$$

Thus $B = \{B(\sigma) : \sigma \in V\Sigma\}$ is in fact a $G$-invariant partition of $V\Gamma$.

**Proposition 31.** With $\Gamma = \text{Arc}_\Delta(\Sigma)$ and $B$ as defined above, The quotient graph $\Gamma_B$ is isomorphic to $\Sigma$.

**Proof.** The map $\sigma \mapsto B(\sigma)$ identifies the vertices of $\Sigma$ with the vertices of $\Gamma_B$. We must show that $(\sigma, \sigma')$ is an arc of $\Sigma$ if and only if $(B(\sigma), B(\sigma'))$ is an arc of $\Gamma_B$. 

$\Gamma$ and the arcs $\Gamma_B$ allows us to construct $\Gamma$ from $\Gamma_B$ in a straightforeward manner.
If \((B(\sigma), B(\sigma'))\) is an arc of \(\Gamma_B\) then for some \(\tau, \tau' \in V\Sigma\) the arcs \((\sigma, \tau)\) and \((\sigma', \tau')\) are adjacent in \(\Gamma\). It follows that \((\tau, \sigma, \sigma', \tau')\) is a three-arc of \(\Sigma\) and so \(\sigma\) is adjacent to \(\sigma'\) in \(\Sigma\).

For the other direction, let \((\alpha, \beta, \gamma, \delta)\) be any three-arc of \(\Sigma\) contained in \(\Delta\). If \((\sigma, \sigma')\) is any arc of \(\Sigma\), then by the \(G\)-arc-transitivity of \(\Sigma\) there exists a \(g \in G\) such that \((\beta, \gamma)^g = (\sigma, \sigma')\). Since \(\Delta\) is an orbit of \(G\) on the three-arcs of \(\Sigma\), it follows that \((\alpha, \beta, \gamma, \delta)^g = (\alpha^g, \sigma, \sigma', \delta^g)\) is contained in \(\Delta\). Thus the arc \((\sigma, \alpha^g)\) is adjacent to the arc \((\sigma', \delta^g)\) in \(\Gamma\) and so \(B(\sigma)\) is adjacent to \(B(\sigma')\) in \(\Gamma_B\).

\[\square\]

**Proposition 32.** If \((G, \Gamma, B)\) is such that \(\Gamma\) is a three-arc graph of \(\Gamma_B\), then for any \(B \in \mathcal{B}\) the permutation groups \((G_B, B)\) and \((G, \Gamma_B(B))\) are permutation equivalent.

**Proof.** Each \(B \in \mathcal{B}\) is of the form \(B(\sigma)\) for some \(\sigma\) a vertex of \(\Gamma_B\). The vertices of \(\Gamma\) contained in \(B(\sigma)\) are precisely the arcs of \(\Gamma_B\) of the form \((\sigma, \tau)\).

Let \(\rho : B(\sigma) \to \Gamma_B(B(\sigma))\) be given by \((\sigma, \tau) \mapsto B(\tau)\). Clearly \(\eta\) is a bijection. If \(g\) is any element of the stabilizer of \(B(\sigma)\) then we have:

\[
\rho((\sigma, \tau)^g) = \rho((\sigma^g, \tau^g)) \\
= \rho((\sigma, \tau^g)) \\
= B(\tau^g) \\
= B(\tau)^g.
\]

Thus \(\rho\) induces a permutation equivalence between the permutation groups \((G_{B(\sigma)}, B(\sigma))\) and \((G_{B(\sigma)}, \Gamma_B(B(\sigma)))\).

\[\square\]

**Proposition 33.** If \((G, \Gamma, B)\) is such that \((G, B)\) is permutation equivalent to \((G, \Gamma_B(B))\) for some \(B \in \mathcal{B}\), then the action of \(G\) on the vertices of \(\Gamma\) is permutation equivalent to the action of \(G\) on the arcs of \(\Gamma_B\).

**Proof.** Let \(\rho : B \to \Gamma_B(B)\) be any bijection inducing a permutation equivalence between \((G, B)\) and \((G, \Gamma_B(B))\) and let \(\pi : VT \to \mathcal{B}\) be the map which sends each vertex of \(\Gamma\) to the block of \(\mathcal{B}\) containing it. Fix a vertex \(\alpha\) of \(\Gamma\). Since \(\Gamma\) is \(G\)-vertex transitive, every vertex of \(\Gamma\) may be written in the form \(\alpha^g\) for some \(g \in G\). Let \(\mu : VT \to A\Gamma_B\) be given by \(\alpha^g \mapsto (\pi(\alpha)^g, \rho(\alpha)^g)\). It is not too hard to see that \(\mu\) is a
bijection. For any $\beta = \alpha^g$ and any $x \in G$ we have:

$$
\mu(\beta^x) = \mu(\alpha^gx) = (\pi(\alpha)^{gx}, \eta(\alpha)^{gx}) = (\pi(\alpha)^x, \eta(\alpha)^x) = (\pi(\beta)^x, \eta(\beta)^x) = (\pi(\beta), \eta(\beta))^x = \mu(\beta)^x
$$

Thus $\mu$ establishes a permutation isomorphism between $(G, V\Gamma)$ and $(G, A\Gamma_B)$.

□

We may make use the map $\mu$ to “label” the vertices of $\Gamma$ by the arcs of $\Gamma_B$. For any arc $(B, C)$ of $\Gamma_B$, let $v_{BC} = \mu^{-1}((B, C))$ denote the vertex of $\Gamma$ which is mapped to $(B, C)$ by $\mu$.

**Proposition 34.** Some stuff about how $G$ acts on labelled vertices and how the initial block contains the vertex.

**Proposition 35.** Provided that $\Gamma$ has valency at least two, for each arc $(v_{BC}, v_{DE})$ of $\Gamma$, $(C, B, D, E)$ is a 3-arc of $\Gamma_B$.

**Proof.** Suppose that $v_{BC}$ were adjacent to $v_{CB}$. Since $\text{val}(\Gamma) \geq 2$, the vertex $v_{CB}$ is adjacent to some other vertex $v_{B_1C_1}$ distinct from $v_{BC}$. By the $G$-symmetry of $\Gamma$ there exists a $g \in G$ such that $(v_{BC}, v_{CB})^g = (v_{B_1C_1}, v_{CB})$. By the previous proposition, this implies that $B = B^x = B_1$ and $C = C^x = C_1$, so $v_{BC} = v_{B_1C_1}$. A contradiction. Thus $v_{BC}$ is not adjacent to $v_{CB}$.

Suppose now that $v_{BC}$ were adjacent to $v_{CE}$ for some $E \neq B$. Since $\text{val}(\Gamma) \geq 2$ we can find another vertex $v_{C_1E_1}$ distinct from $v_{CE}$ and $v_{BC}$. By the $G$-symmetry of $\Gamma$ we can find some $g_1 \in G$ such that $(v_{BC}, v_{CE})^{g_1} = (v_{B_1C_1}, v_{CB})$. By the previous proposition, this implies that $C = C^{g_1} = C_1$. We can also find a $g_2 \in G$ such that $(v_{BC}, v_{CE})^{g_2} = (v_{C_1E_1}, v_{BC})$ and so $B = C^{g_2} = E_1$. Combining these gives $v_{C_1E_1} = v_{CB}$, but $v_{BC}$ is not adjacent to $v_{CB}$ so again we have a contradiction.

We know now that if $(v_{BC}, v_{DE})$ is an arc of $\Gamma$, then $D \neq C$. A similar argument shows that $B \neq E$, so $B, C, D, E$ are distinct vertices with $B$ adjacent to $C$ and $D$ adjacent to $E$. Since, by the previous proposition $B(v_{BC}) = B$ and $B(v_{DE}) = D$, we have $B$ adjacent to $D$ and so $C, B, D, E$ is a 3-arc of $\Gamma_B$ as claimed.

□

Concluding remarks.
2. Locally imprimitive quotient

Suppose that $\Gamma = \text{Sab}(G, H, a)$. Let $\overline{H} = a^{-1}Ha \cap H$, so the local permutation group induced at each vertex of $\Gamma$ is equivalent to $(H, \cos(\overline{H}))$. Suppose further that there is some $K$ such that $\overline{H} < K < H$ and $a \notin K$. Then the graph $\tilde{\Gamma} = \text{Sab}(G, K, a)$ is an extension of $\Gamma$. Let $\overline{K} = a^{-1}Ka \cap K$. Since $K < H$ we must also have $a^{-1}Ka < a^{-1}Ha$ so $\overline{K} \leq \overline{H}$. Since $\overline{H} < K$ we must also have $a^{-1}\overline{H}a < a^{-1}Ka$ but since $a$ is an involution $a^{-1}\overline{H}a = \overline{H}$ so we have $\overline{H} = \overline{K}$. That is $\overline{H} = \overline{K}$ and the local permutation group induced at each vertex of $\tilde{\Gamma}$ is $(K, \cos(\overline{H}))$.

The pair $(\Gamma, \tilde{\Gamma})$ satisfy the property that globally $\tilde{\Gamma}$ is imprimitive, admitting $\Gamma$ as quotient, but locally $\Gamma$ is imprimitive, and the local permutation group of $\tilde{\Gamma}$ is a quotient of the local permutation group of $\Gamma$. We wish to understand how $\Gamma$ and $\tilde{\Gamma}$ are related structurally, and ideally find some method of constructing $\tilde{\Gamma}$ from $\Gamma$.

As a preliminary observation, let $n = [G : H]$ be the number of vertices of $\Gamma$, let $v = [K : H]$ be the valency of $\tilde{\Gamma}$ and let $r = [H : K]$. The graph $\tilde{\Gamma}$ has $[G : K] = [G : H][H : K] = nr$ vertices. That is, $r$ times as many vertices as $\Gamma$. The valency of $\tilde{\Gamma}$ is $[H : \overline{H}] = [H : K][K : \overline{H}] = rv$. That is $r$ times the valency of $\Gamma$. Since the number of edges in a graph is equal to half the valency times the number of vertices, it follows that both $\Gamma$ and $\tilde{\Gamma}$ have the same number of edges.

The action of $G$ on the arcs of $\Gamma$ is permutation equivalent to $(G, \cos(\overline{H}))$. Since $\Gamma$ is simple, each arc $(\alpha, \beta)$ has a pair namely $(\beta, \alpha)$. The function $\varphi : A\Gamma \to A\Gamma$ which sends each arc to its pair is an involution It also preserves the action of $G$.

Now, since $\overline{H} < H < G$ the action of $G$ on the arcs of $\Gamma$ is imprimitive. That is, there is some $G$-invariant partition $B$ of $A\Gamma$ such that the action of $G$ on $B$ is permutation equivalent to $(G, \cos(H))$. In fact, this is the partition $B = \{B(\alpha) : \alpha \in V\Gamma\}$ where $B(\alpha)$ is the set of all arcs who’s initial vertex is $\alpha$.

Let $\pi : A\Gamma \to B$ be the map which sends each arc to the block of $B$ containing it. We may define a new graph $\Sigma$ we vertex set $B$ where $p$ is adjacent to $b$ if and only if there is some $x \in \pi^{-1}(p)$ and some $y \in \pi^{-1}(b)$ such that $y = \varphi(x)$. This new graph $\Sigma$ is in fact isomorphic to $\Gamma$.

The action of $G$ on the arcs of $\Gamma$ is permutation isomorphic to the action of $G$ on the arcs of $\tilde{\Gamma}$. Both actions are of the form $(G, \cos(\overline{H}))$.

Since $\overline{H} < K < G$ there exists some $G$-invariant partition $B$ of $A\Gamma$ such
that the action of $G$ on $\mathcal{B}$ is permutation equivalent to $(G, \cos(H))$. The partition $\mathcal{B}$ is a refinement of the partition $\mathcal{B}$.

Let $\lambda : A\Gamma \to \mathcal{B}$ be the map which sends each arc to the block of $\mathcal{B}$ containing it. We may define a new graph $\tilde{\Gamma}$ we vertex set $\mathcal{B}$ where $p$ is adjacent to $b$ if and only if there is some $x \in \lambda^{-1}(p)$ and some $y \in \lambda^{-1}(b)$ such that $y = \varphi(x)$. This new graph $\tilde{\Gamma}$ is in fact isomorphic to $\tilde{\Gamma}$. Thus we have a method of construction $\tilde{\Gamma}$ from $\Gamma$.

3. Covering Graphs

The three-arc graph construction allows us, under certain conditions, to “unfold” a $G$-symmetric graph into a larger, imprimitive $G$-symmetric graph, admitting the original graph as a quotient. The covering graph construction \cite{5} is similar in this respect. It also, under certain conditions, allows us to “unfold” a $G$-symmetric graph into a larger one. It differs from the three-arc graph construction, however, in that it requires a simultaneous “unfolding” of the group $G$.

Recall that given two groups $N$ and $G$ and a homomorphism $$\rho : G \to \text{Aut}(N)$$

we may form the semidirect product of $N$ by $G$. This is the group $\tilde{G} = N \rtimes_{\rho} G$, whose elements are the ordered pairs:

$$\{(n, g) : n \in N, g \in G\}$$

with multiplication given by:

$$(n_1, g_1)(n_2, g_2) = (n_1^{\rho(g_1)}n_2, g_1g_2).$$

The functions $i_1 : N \to \tilde{G}$ and $i_2 : G \to \tilde{G}$ given by $n \mapsto (n, 1)$ and $g \mapsto (1, g)$ respectively give natural embeddings of $N$ and $G$ into $\tilde{G}$. Identifying $N$ with its image under $i_1$, the semidirect product has the property that $N$ is normal in $\tilde{G}$ and $\tilde{G}/N \cong G$.

**Definition 28** ($N$-chain). Suppose that $\Gamma$ is a $G$-symmetric graph and $N$ is a group. An $N$-chain is a function $\phi : A\Gamma \to N$ satisfying $\phi((v, u)) = \phi((u, v))^{-1}$ for all $(u, v) \in A\Gamma$.

**Definition 29** (Compatible $N$-chain). Suppose that $\Gamma$ is a $G$-symmetric graph, $N$ is a group and $\rho : G \to \text{Aut}(N)$ is a homomorphism. An $N$-chain $\phi$ is said to be compatible with $\rho$ if for every $g \in G$ the following diagram commutes:
Definition 30 (Biggs Cover). Suppose that $\Gamma$ is a $G$-symmetric graph, $N$ is a group, $\rho : G \rightarrow \text{Aut}(N)$ is a homomorphism and $\phi$ is a compatible $N$-chain. The Biggs Cover $\tilde{\Gamma}(N, \rho, \phi)$ of $\Gamma$ with respect to $\phi$ is the graph with vertex set: $\{(g, v) : k \in G, v \in V\}$ and arc set: $\{((g_1, v_1), (g_2, v_2)) : (v_1, v_2) \in A\Gamma, g_2 = g_1\phi((v_1, v_2))\}$

Note that the condition $\phi((v, u)) = \phi((u, v))^{-1}$ ensures that the resulting graph is simple.

Proposition 36. The covering graph $\tilde{\Gamma}(N, \rho, \phi)$ is $(N \rtimes \rho G)$-symmetric.

Proof. Define an action of $N \rtimes \rho G$ on $\tilde{\Gamma}$ by:

$$((n, v))^{(\eta, g)} = (n^{\rho(g)}, \eta, v^g).$$

This action is well-defined since:

$$(n, v)^{(n_1, g_1, n_2, g_2)} = (n^{\rho(g_1)}n_1^{g_1}, v^{g_1})^{(n_2, g_2)} = (n^{\rho(g_1)}\rho(g_2), n_1^{\rho(g_2)}n_2^{g_1g_2}) = (n, v)^{(n_2^{\rho(g_2)}, n_2^{g_1g_2})}.$$

We must check that it preserves the adjacency structure of $\tilde{\Gamma}$. Observe that, by the compatibility of $\rho$ and $\phi$, if $n_2 = n_1\phi(v_1, v_2)$ then:

$$n_2^{\rho(g)}\phi(v_1^g, v_2^g) = n_2^{\rho(g)}\phi(v_1, v_2)^{\phi(g)} = (n_2\phi(v_1, v_2))^{\phi(g)} = n_1^{\phi(g)}$$

for any $(\eta, g) \in N \rtimes \rho G$. Also, since $G$ is a group of automorphisms of $\Gamma$, we know that $(v_1, v_2) \in A\Gamma$ if and only if $(v_1^g, v_2^g) \in A\Gamma$. Thus the action defined above does indeed preserves adjacency.

For any $(n_1, v_1), (n_2, v_2) \in V\tilde{\Gamma}$, by the $G$-symmetry of $\Gamma$ we can find a $g \in G$ such that $v_1^g = v_2$. Let $\eta = n_1^{\rho(g)^{-1}}n_2$. We have:

$$(n_1, v_1)^{(n, g)} = (n_1^{\rho(g)}\eta, v_1^g) = (n_1^{\rho(g)}, n_1^{\rho(g)^{-1}}n_2, v_2) = (n_2, v_2).$$

Thus the action is vertex transitive.
Suppose that \((n_1, v_1)\) and \((n_2, v_2)\) are both adjacent to \((n, g)\), so \(n_1 = n\phi(v, v_1)\) and \(n_2 = n\phi(v, v_2)\). By the \(G\)-symmetry of \(\Gamma\) we can find a \(g \in G_v\) such that \(v_1^g = v_2\). Let \(\eta = (n^{-1})^{\rho(g)} n\). We have:

\[
(n, v)^{(\eta, g)} = (n^{\rho(g)} \eta, v^g) \\
= (n^{\rho(g)} (n^{-1})^{\rho(g)} n, v) \\
= (n, v)
\]

and:

\[
(n_1, v_1)^{(\eta, g)} = (n_1^{\rho(g)} \eta, v_1^g) \\
= (n_1^{\rho(g)} (n^{-1})^{\rho(g)} n, v_2) \\
= ((n_1 n^{-1})^{\rho(g)} n, v_2) \\
= (\phi(v, v_1)^{\rho(g)} n, v_2) \\
= (\phi(v, v_2) n, v_2) \\
= (n_2 n^{-1} n, v_2) \\
= (n_2, v_2)
\]

Thus the action is locally transitive. The result follows.

In fact, a stronger result than this holds. If \(\Gamma\) is \((G, s)\)-arc transitive then \(\tilde{\Gamma}\) is \((\tilde{G}, s)\)-arc transitive, where \(\tilde{G} = N \rtimes_\rho G\). The proof is by induction and may be found in [5]. This construction was originally used by Conway to produce an infinitely family of 5-arc transitive graphs.

For each \(v \in V\Gamma\), let \(B(v) = \{(n, v) : n \in N\}\). For any \((\eta, g) \in \tilde{G}\) we have:

\[
B(v)^{(\eta, g)} = \{(n, v)^{(\eta, g)} : n \in N\} \\
= \{((\eta n^{\rho(g)}), v^g) : n \in N\} \\
= \{(n, v^g) : n \in N\} \\
= B(v^g).
\]

Thus the partition: \(B = \{B(v) : v \in V\Gamma\}\) is \(\tilde{G}\)-invariant.

**Proposition 37.** The quotient of the Biggs cover \(\tilde{\Gamma}\) with respect to the partition \(B\) is isomorphic to the original graph \(\Gamma\).

**Proof.** The map \(\eta : V\Gamma \to V\tilde{\Gamma}_B\) given by \(v \mapsto B(v)\) establishes a bijection between the vertices of \(\tilde{\Gamma}\) and the vertices of \(\tilde{\Gamma}_B\). If \((u, v)\) is an arc of \(\Gamma\), then \(((1, u), (\phi(((u, v)), v)))\) is an arc of \(\tilde{\Gamma}\), and so \((B(u), B(v))\)
is an arc of \( \Gamma_B \). If \((u,v)\) is not an arc of \( \Gamma \), then for all \( n_1, n_2 \in N \) \((n_1, u), (n_2, v)\) is not an arc of \( \Gamma \) and so \((B(u), B(v))\) is not an arc of \( \Gamma_B \). Thus \( \eta \) establishes an isomorphism between \( \Gamma \) and \( \Gamma_B \). \( \square \)

Let \((B, C)\) be any arc of \( \Gamma \). By the above \( B = B(u) \) and \( C = B(v) \) for some \((u, v)\) \( \in A\Gamma \). Its not too hard to see that each \((n, u)\) \( \in B \) has a unique neighbour in \( C \), namely \((n\phi(u, v), v)\). Thus the induced bipartite graph \( \Gamma[B, C] \) is a matching. It follows immediately that the valency of \( \Gamma \) is the same as the valency of \( \Gamma \).

4. **Group Theoretic analysis of covering graph construction**

The Bigg’s covering graph construction encompasses all pairs of graphs \( \Gamma(G, H, a) \) and \( \Gamma(G, H, \tilde{a}) \) where \( \tilde{G} \) is a semidirect product of \( N \) by \( G \) for some \( N \). The local permutation groups of \( \Gamma \) and \( \tilde{\Gamma} \) are the same, and \( N \) acts regularly on the fibers.

5. **Subgraph Extension**

I have been thinking about the idea of a “combinatorial” description of the extension of a graph in terms of its quotient. That is, a description of the adjacency structure of \( \tilde{\Gamma} \) directly in terms of the adjacency structure of \( \Gamma \). This section is a sketch of an idea I had, that perhaps there is a connection between the subgraph structures of different \( G \)-symmetric graphs for the same \( G \).

**Definition 31 (Subgraph).** Suppose that \( \Gamma = (V, A) \) is a graph. If \( W \) is a subset of \( V \) and \( B \) is a subset of \( (W \times W) \cap A \) then \( \Upsilon = (W, B) \) is a subgraph of \( \Gamma \). We write \( \Upsilon < \Gamma \) and allow for the possibility that \( \Upsilon \) is a directed graph.

**Definition 32 (Stabilizer of a Subgraph).** Suppose that \((G, \Gamma)\) is a symmetric graph and \( \Upsilon \) is a directed subgraph. The stabilizer of \( \Upsilon \) in \((G, \Gamma)\) is \( \text{Aut}(\Upsilon) \cap G \). It is denoted by \( G_\Upsilon \).

**Definition 33 (Subgraph Graph).** Suppose that \((G, \Gamma)\) is a symmetric graph, \( \Upsilon \) is a directed subgraph of \( \Gamma \) and \( a \) is an involution of \( \Gamma \) which fixes an arc. The subgraph graph \( \text{Sub}(\Gamma, \Upsilon, a) \) of \( \Gamma \) with respect to \( \Upsilon \) and \( a \) is the graph with vertices \( V = \{\Upsilon^g : g \in G\} \) and arcs \( A = \{((\Upsilon \cup \Upsilon^a)^g) : g \in G\} \).

**Proposition 38.** Subgraph graph’s are symmetric, with point stabilizer \( G_\Upsilon \).

**Proof.** Obvious. \( \square \)
Example 2 (cube from tetrahedron). Let $\Gamma$ be the tetrahedron with $S4$ acting on it. Let $\Upsilon$ be any directed three-cycle. Let $a$ be any involution fixing an arc of $\Upsilon$, then the subgraph graph $Sub(\Gamma, \Upsilon, a)$ is isomorphic to the cube.

Definition 34 (Symmetric subgraph). Suppose that $(G, \Gamma)$ is a symmetric graph. The digraph $\Upsilon < \Gamma$ is a symmetric subgraph of $\Gamma$ if $(G_\Upsilon, \Upsilon)$ is a symmetric graph.

Question 1. If $(G, \Gamma)$ is a symmetric graph, for which subgroups $K$ of $G$ do their exist a subgraphs of $\Upsilon$ of $\Gamma$ such that $G_\Upsilon = K$? For which subgroups $K$ of $G$ do their exist a symmetric subgraph $\Upsilon$ of $\Gamma$ such that $G_\Upsilon = K$?

Question 2. Which extensions of a given $G$-symmetric graph $\Gamma$ are isomorphic to subgraph graphs of $\Gamma$? Which are isomorphic to subgraph graphs of $\Gamma$ with respect to some symmetric subgraph (digraph) of $\Gamma$?
Bibliography

[1] C. E. Praeger A. Gardiner. A geometrical approach to imprimitive symmetric graphs. *Proc. London. Math. Soc.*, 1993.
[2] C.E. Praeger A. Gardiner. Topological covers of complete graphs. *Math. Proc. Camb. Phil. Soc.*, 1994.
[3] S. Zhou A. Gardiner, C.E. Praeger. Cross ratio graphs.
[4] H.A Priestley B.A Davey. *Introduction to Lattices and order*. Cambridge University Press, London, 2002.
[5] N. Biggs. *Algebraic Graph Theory*. Cambridge University Press, London, 1974.
[6] G. Royle C. Godsil. *Algebraic Graph Theory*. Springer, New York, 2001.
[7] S. Zhou C. H. Li, C. E. Praeger. A class of finite symmetric graphs with 2-arc transitive quotients. *Math. Proc. Camb. Phil. Soc.*, 2000.
[8] P.J Cameron. Finite permutation groups and finite simple groups. *Bull. London Math. Soc.*, 1981.
[9] Z. Venkatesh S. Zhou C.H. Li, C.E. Praeger. Finite locally-quasiprimitive graphs. *Discrete Mathematics*, 2002.
[10] R. Diestel. *Graph Theory*. Springer, New York, 2002.
[11] A. Gardiner. Arc-transitivity in graphs. *Quart. J. Math. Oxford*, 1973.
[12] A. Gardiner. Arc-transitivity in graphs ii. *Quart. J. Math. Oxford*, 1974.
[13] A. Gardiner. Arc-transitivity in graphs ii. *Quart. J. Math. Oxford*, 1976.
[14] R. L. Griess. *Twelve Sporadic Groups*. Springer-Verlag, Berlin, 1998.
[15] L. G. Grove. *Classical Groups and Geometric Algebra*. American Mathematical Society, 2002.
[16] B. Mortimer J. D. Dixon. *Permutation Groups*. Springer, New York, 1996.
[17] E. S. Landau. *Symmetric Designs: An Algebraic Approach*. Cambridge University Press, London, 1983.
[18] P. Lorimer. Vertex-transitive graphs: Symmetric graphs of prime valency. *Journal of Graph Theory*, 1984.
[19] S. Zhou M.A.Iranmanesh, C.E. Praeger. Finite symmetric graphs with two-arc transitive quotients. *Journal of Combinatorial Theory*, 2002.
[20] B. Polster. *A Geometrical Picture Book*. Springer, New York, 1998.
[21] A. Gardiner. C.E. Praeger. Symmetric graphs with complete quotients. *draft*, 1999.
[22] C.E. Praeger. Finite transitive permutation groups and finite vertex-transitive graphs. *University of Western Australia internal publication*.
[23] C.E. Praeger. Imprimitive symmetric graphs. *Ars Combinatoria*.
[24] C.E Praeger. On automorphism groups of imprimitive symmetric graphs. *Ars. Combinatoria*, 1987.
[25] C.E Praeger. Finite symmetric graphs. *University of Western Australia internal publication*, 2000.
[26] R. J. Wilson R. C. Read. *An Atlas of Graphs*. Clarendon Press, Oxford, 1998.
[27] D. J. S. Robinson. *A Course in the Theory of Groups*. Springer, New York, 1996.
[28] G. Sabidussi. Vertex-transitive graphs. *Monatsch. Math*, 1964.
[29] C.G. Sims. Graphs and finite permutation groups. *Math. Z*, 1967.
[30] C.G. Sims. Graphs and finite permutation groups ii. *Math. Z*, 1968.
[31] D.H. Smith. Primitive and imprimitive graphs. *Quart. J. Math.*, 1971.
[32] W.T. Tutte. A family of cubical graphs. *Proc. Cambridge Philos. Soc.*, 1947.
[33] W.T. Tutte. On the symmetry of cubic graphs. *Canad. J. Math.*, 1959.
[34] S. Klavzar W. Imrich. *Product Graphs*. John Wiley and Sons, New York, 2000.
[35] W. D. Wallis. *Combinatorial Designs*. Marcel Dekker, New York, 1988.
[36] Z. Wan. *Geometry of Classical Groups over Finite Fields*. Studentlitteratur, Sweden, 1993.
[37] Richard Weiss. The nonexistence of 8-transitive graphs. *Combinatorica*, 1981.
[38] S. Zhou. Almost covers of 2-arc transitive graphs. *Math. Proc. Camb. Phil. Soc.*, 2000.
[39] S. Zhou. Imprimitive symmetric graphs. *PhD. Thesis. University of Western Australia*, 2000.
[40] S. Zhou. Constructing a class of symmetric graphs. *Europ. J. Combinatorics*, 2002.
[41] S. Zhou. Imprimitive symmetric graphs, 3-arc graphs and 1-designs. *Discrete Mathematics*, 2002.
[42] S. Zhou. A local analysis of imprimitive symmetric graphs. *J. Algebraic Combinatorics*, 2005.
1. Description of the problem

We would like to understand the ways in which a symmetric graph \((G, \Gamma)\) can be “unfolded” into a larger imprimitive symmetric graph \((\tilde{G}, \tilde{\Gamma})\) which admits the original graph as a quotient. On a group theoretic level, if we knew the subgroup structure of \(G\) and we knew all the groups \(\tilde{G}\) which have \(G\) as a composition factor then in a sense we would “know” all the possible unfoldings. What would still be lacking is a combinatorial understanding of how the structure of the new graph is related to that of the original.

The three-arc graph construction gives a nice description of the graph \(\tilde{\Gamma}\) in terms of the graph \(\Gamma\) in the special case where \(\Gamma \cong \text{Sab}(G, H, a)\), \(\tilde{\Gamma} \cong \text{Sab}(G, K, a)\) and \(K \cong a^{-1}Ha \cap H\). That is, the special case where the stabilizer of an arc of \(\Gamma\) is isomorphic to the stabilizer of a vertex of \(\tilde{\Gamma}\). One may ask whether there other group theoretic assumptions which may be imposed on a pair of symmetric graphs \((\Gamma, \tilde{\Gamma})\) which will yield a nice description of \(\tilde{\Gamma}\) in terms of \(\Gamma\).

I have considered the case where \(\Gamma \cong \text{Sab}(G, H, a)\), \(\tilde{\Gamma} \cong \text{Sab}(G, K, a)\) and there exists a normal subgroup \(N\) of \(G\) such that \(G\) is a semidirect product of \(N\) by \(H\). Here are some rough notes.

2. Labelling technique

Every \(g \in G\) has a unique expression of the form \(g = (h, n)\) where \(h \in H\) and \(n \in N\). Since the elements of \(N\) form a traversal of the cosets of \(H\) in \(G\), the vertices of \(\Gamma\) may labelled by elements of \(N\) and in fact form a group. The action of \(G\) on the vertices of \(\Gamma\) is given by \(m^{(h,n)} = h^{-1}mhn\). Let us take \(D\) to be the the design induced on the fiber of the vertex of \(\Gamma\) which is labelled by the identity of \(N\).

The points of \(\tilde{\Gamma}\) may be labelled by ordered pairs of the form \((x, m)\) where \(x\) is a coset of \(K\) and \(m \in N\). The action of \(G\) on the points of \(\tilde{\Gamma}\) is then given by: \((x, m)^{(h,n)} = (x^h, h^{-1}mhn)\). This is indeed an action, since:

\[
(x, m)^{(h_1,n_1)(h_2,n_2)} = (x^{h_1}, h_1^{-1}mh_1n_1)^{(h_2,n_2)} \\
= (x^{h_1h_2}, h_2^{-1}h_1^{-1}mh_1n_1h_2n_2) \\
= (x^{h_1h_2}, h_2^{-1}h_1^{-1}mh_1h_2h_2^{-1}n_1h_2n_2) \\
= (x, m)^{(h_1h_2h_2^{-1}n_1h_2n_2)} \\
= (x, m)^{(h_1,n_1)(h_2,n_2)}
\]
3. Self-paired orbital on the flags $\mathcal{D}$

I will show that there is a self-paired orbital on the flags of $\mathcal{D}$ which can be used to reconstruct $\tilde{\Gamma}$ from $\Gamma$.

Suppose that $((x, p), (y, b))$ is an arc $\tilde{\Gamma}$. Then $(p, q)$ is an arc of the quotient $\Gamma$. Making use of the automorphism $(1, p^{-1})$ of $\tilde{\Gamma}$ we have that $((x, p), (y, b))^{(1, p^{-1})} = ((x, 1), (y, bp^{-1}))$ is also an arc of $\tilde{\Gamma}$, and so $(x, bp^{-1})$ is a flag of $\mathcal{D}$.

Since $\tilde{\Gamma}$ is assumed to be a simple graph, if $((x, p), (y, b))$ is an arc of $\tilde{\Gamma}$ then so is $((y, b), (x, p))$. Making use of the automorphism $(1, b^{-1})$ we have that and $((y, b), (x, p))^{(1, b^{-1})} = ((y, 1), (x, pb^{-1}))$ is an arc of $\tilde{\Gamma}$, and so $(y, pb^{-1})$ is a flag of $\mathcal{D}$.

What we have done is “decomposed” the arc $((x, p), (y, b))$ of $\tilde{\Gamma}$ into the arc $(p, q)$ of $\Gamma$ together with the pair of flags $((x, bp^{-1}), (y, pb^{-1}))$ of $\mathcal{D}$. Let $\Delta$ be the orbital on the flags of $\mathcal{D}$ which contains the pair $((x, bp^{-1}), (y, pb^{-1}))$. That is $\Delta = \{((x, bp^{-1}), (y, pb^{-1}))^h : h \in H\}$.

Suppose we choose a different arc $((w, q), (z, d))$ of $\tilde{\Gamma}$. Then by arguments identical to those above $(q, d)$ is an arc of $\Gamma$ and $(w, dq^{-1})$ and $(z, qd^{-1})$ are flags of $\mathcal{D}$. Since $\tilde{\Gamma}$ is $G$-arc transitive, we can find some $g = (h, n)$ such that $((x, p), (y, b))^{(h, n)} = ((w, q), (z, d))$. That is:

$$
\begin{align*}
w &= x^h \\
z &= y^h \\
q &= h^{-1}phn \\
d &= h^{-1}bhn
\end{align*}
$$

So:

$$
(w, dq^{-1}) = (x^h, h^{-1}bhn^{-1}h^{-1}p^{-1}h) = (x^h, h^{-1}bp^{-1}h) = (x, bp^{-1})^h.
$$

Similarly:

$$
(z, qd^{-1}) = (y^h, h^{-1}phnn^{-1}h^{-1}b^{-1}h) = (y^h, h^{-1}pb^{-1}h) = (y, pb^{-1})^h.
$$
It follows immediately that \(((w, dq^{-1}), (z, qd^{-1})) \in \Delta\). Thus, any arc of \(\widetilde{\Gamma}\) can be “decomposed” into an arc of \(\Gamma\) together with a pair of flags of \(\mathcal{D}\) contained in the orbital \(\Delta\).

I will now show that this orbital \(\Delta\) on the flags of \(\mathcal{D}\) is self-paired. Since \(\widetilde{\Gamma}\) is simple. There must be some \(g = (h, n)\) which “flips” the arc \(((x, p), (y, b))\). That is, there must be some \(g = (h, n)\) such that \(((x, p), (y, b))^{(h, n)} = ((y, b), (x, p))\). By an argument identical to the one on the previous page, we must have that

\[
(y, pb^{-1}) = (x, bp^{-1})^h \\
(x, bp^{-1}) = (y, pb^{-1})^h
\]

This shows that the orbital \(\Delta\) is self-paired.

4. RECONSTRUCTION

Given \(\Gamma\) and \(\mathcal{D}\) together with a self paired orbital \(\Delta\) on the flags of \(\mathcal{D}\) it should be possible to reconstruct \(\widetilde{\Gamma}\). Let \(N(1)\) denote the neighbourhood of the vertex “1” of the \(\Gamma\) and let \(P\) and \(B\) denote the points and blocks respectively of the design \(\mathcal{D}\). Let \(\eta : N(1) \to B\) be some map establishing a permutation isomorphism between the permutation groups \((H, N(1))\) and \((H, B)\).

The graph \(\widetilde{\Gamma}\) can be described as the graph with vertex set \(P \times V\) and arc set \(((x, v), (y, w))\) such that

1. \((v, w)\) is an arc of \(\Gamma\)
2. \((x, \eta(vw^{-1}))\) is a flag of \(\mathcal{D}\)
3. \((y, \eta(wv^{-1}))\) is a flag of \(\mathcal{D}\)
4. \(((x, \eta(vw^{-1})), (y, \eta(wv^{-1}))) \in \Delta\)

The action of \(G\) on \(P \times V\) is that given in section 2. Since \(\mathcal{D}\) is \(H\)-flag transitive, for any pair of vertices \((x, p)\) and \((y, q)\), we can find some \(h \in H\) such that \(x^h = y\). Now:

\[
(x, p)^{(h, h^{-1}p^{-1}hq)} = (x^h, h^{-1}phh^{-1}p^{-1}hq) = (y, q).
\]

So this action is transitive.
We need to check that this action is well defined on the arcs. Suppose that \([(x,p), (y,q)]\) is any arc and \((h,n)\) is any element of \(G\). We need to check that \([(x,p), (y,q)]^{(h,n)}\) is also an arc.

Since \([(x,p), (y,q)]\) is an arc of \(\tilde{\Gamma}\) we know that \((p,q)\) is an arc of \(\Gamma\) and that:
\[
((x, \eta(pq^{-1})), (y, \eta(qp^{-1})) \in \Delta
\]

The arc \([(x,p), (y,q)]^{(h,n)}\) of \(\tilde{\Gamma}\) decomposes into the arc \((p,q)^{(h,n)}\) of \(\Gamma\) and the pair of flags:
\[
((x, \eta(pq^{-1})), (y, \eta(qp^{-1}))^{h} \in \Delta
\]
So the action is well-defined on arcs.

Finally we need to check that this action is transitive on the arcs. If \([(x,p), (y,b)]\) and \([(w,q), (z,d)]\) are any two arcs of \(\tilde{\Gamma}\) then we must have:
\[
((x, \eta(pb^{-1})), (y, \eta(bp^{-1})) \in \Delta
\]
and:
\[
((w, \eta(qd^{-1})), (z, \eta(dq^{-1})) \in \Delta.
\]
So there must be some \(h \in H\) such that:
\[
((x, \eta(pb^{-1})), (y, \eta(bp^{-1}))^{h} = ((w, \eta(qd^{-1})), (z, \eta(dq^{-1})).
\]
Since \(\eta\) is a permutation homomorphism, \(\eta(bp^{-1})^{h} = \eta(dq^{-1})\) implies that \(\eta(h^{-1}bp^{-1}h) = \eta(dq)\) which implies that \(h^{-1}bp^{-1}h = dq^{-1}\). Now we have:
\[
(x,p)^{(h,h^{-1}p^{-1}hq)} = (x^h, h^{-1}phh^{-1}p^{-1}hq) = (w, q)
\]
And:
\[
(y,b)^{(h,h^{-1}p^{-1}hq)} = (y^h, h^{-1}bhh^{-1}p^{-1}hq) = (z, h^{-1}bp^{-1}hq) = (z, dq^{-1}q) = (z, d)
\]
Thus the action is transitive on the arcs as claimed.
5. Another Special Case

Suppose that \( \Gamma = \text{Sab}(G, H, a) \). Let \( \overline{H} = a^{-1}Ha \cap H \), so the local permutation group induced at each vertex of \( \Gamma \) is equivalent to \( (H, \cos(\overline{H})) \). Suppose further that there is some \( K \) such that \( \overline{H} < K < H \) and \( a \not\in K \). Then the graph \( \tilde{\Gamma} = \text{Sab}(G, K, a) \) is an extension of \( \Gamma \).

Let \( \overline{K} = a^{-1}Ka \cap K \). Since \( \overline{H} < K \) we must also have \( a^{-1}\overline{H}a < a^{-1}Ka \) so \( \overline{K} \leq \overline{H} \). Since \( \overline{H} < K \) we must also have \( a^{-1}\overline{H}a \) \( \neq a^{-1}Ka \) but since \( a \) is an involution \( a^{-1}\overline{H}a = \overline{H} \) so we have \( \overline{H} \leq \overline{K} \). That is \( \overline{H} = \overline{K} \) and the local permutation group induced at each vertex of \( \tilde{\Gamma} \) is \( (K, \cos(\overline{H})) \).

The pair \((\Gamma, \tilde{\Gamma})\) satisfy the property that globally \( \tilde{\Gamma} \) is imprimitive, admitting \( \Gamma \) as quotient, but locally \( \Gamma \) is imprimitive, and the local permutation group of \( \tilde{\Gamma} \) is a quotient of the local permutation group of \( \Gamma \). We wish to understand how \( \Gamma \) and \( \tilde{\Gamma} \) are related structurally, and ideally find some method of constructing \( \tilde{\Gamma} \) from \( \Gamma \).

As a preliminary observation, let \( n = [G : H] \) be the number of vertices of \( \Gamma \), let \( v = [K : \overline{H}] \) be the valency of \( \tilde{\Gamma} \) and let \( r = [H : K] \). The graph \( \tilde{\Gamma} \) has \([G : K] = [G : H][H : K] = nr \) vertices. That is, \( r \) times as many vertices as \( \Gamma \). The valency of \( \Gamma \) is \([H : \overline{H}] = [H : K][K : \overline{H}] = rv \). That is \( r \) times the valency of \( \tilde{\Gamma} \). Since the number of edges in a graph is equal to half the valency times the number of vertices, it follows that both \( \Gamma \) and \( \tilde{\Gamma} \) have the same number of edges.

The action of \( G \) on the arcs of \( \Gamma \) is permutation equivalent to \((G, \cos(\overline{H}))\). Since \( \Gamma \) is simple, each arc \((\alpha, \beta)\) has a pair namely \((\beta, \alpha)\). The function \( \varphi : A\Gamma \rightarrow A\Gamma \) which sends each arc to its pair is an involution It also preserves the action of \( G \).

Now, since \( \overline{H} < H < G \) the action of \( G \) on the arcs of \( \Gamma \) is imprimitive. That is, there is some \( G \)-invariant partition \( \mathcal{B} \) of \( A\Gamma \) such that the action of \( G \) on \( \mathcal{B} \) is permutation equivalent to \((G, \cos(\overline{H}))\). In fact, this is the partition \( \mathcal{B} = \{B(\alpha) : \alpha \in V\Gamma\} \) where \( B(\alpha) \) is the set of all arcs who’s initial vertex is \( \alpha \).

Let \( \pi : A\Gamma \rightarrow \mathcal{B} \) be the map which sends each arc to the block of \( \mathcal{B} \) containing it. We may define a new graph \( \Sigma \) vertex set \( \mathcal{B} \) where \( p \) is adjacent to \( b \) if and only if there is some \( x \in \pi^{-1}(p) \) and some \( y \in \pi^{-1}(b) \) such that \( y = \varphi(x) \). This new graph \( \Sigma \) is in fact isomorphic to \( \Gamma \).

The action of \( G \) on the arcs of \( \Gamma \) is permutation isomorphic to the action of \( G \) on the arcs of \( \tilde{\Gamma} \). Both actions are of the form \((G, \cos(\overline{H}))\). Since \( \overline{H} < K < G \) there exists some \( G \)-invariant partition \( \mathcal{B} \) of \( A\Gamma \) such
that the action of $G$ on $\mathfrak{B}$ is permutation equivalent to $(G, \cos(H))$. The partition $\mathfrak{B}$ is a refinement of the partition $\mathfrak{B}$.

Let $\lambda : A \Gamma \to \mathfrak{B}$ be the map which sends each arc to the block of $\mathfrak{B}$ containing it. We may define a new graph $\tilde{\Sigma}$ with vertex set $\mathfrak{B}$ where $p$ is adjacent to $b$ if and only if there is some $x \in \lambda^{-1}(p)$ and some $y \in \lambda^{-1}(b)$ such that $y = \varphi(x)$. This new graph $\Sigma$ is in fact isomorphic to $\tilde{\Gamma}$. Thus we have a method of construction $\tilde{\Gamma}$ from $\Gamma$. 


1. \textit{G}-symmetric designs

In this Chapter we shall define a new category \( G \)-Design. The objects of this category are \( G \)-symmetric designs and the morphisms are \( G \)-design homomorphisms. We shall exhibit a “forgetful functor” from \( G \)-graph to \( G \)-design and rephrase some of the questions we have been asking about \( G \)-symmetric graphs into questions about \( G \)-symmetric designs. In particular we shall describe a “decomposition” for \( G \)-symmetric designs that is analogous to the decomposition given by Gardiner and Praeger for \( G \)-symmetric graphs.

\textbf{Definition 1.} A \( G \)-flag transitive design \( D \) is self-dual if there exists a \( G \)-isomorphism \( \rho = (\rho_P, \rho_B) \) between \( D \) and \( D^* \). The \( G \)-isomorphism \( \rho \) is called a duality of \( D \).

\textbf{Definition 2.} A \( G \)-symmetric design is self-dual \( G \)-flag transitive design \( D \) which admits a duality \( \rho = (\rho_P, \rho_B) \) with the property that \( \rho_B \circ \rho_P = \text{id}_P \) and \( \rho_P \circ \rho_B = \text{id}_B \). In this case the \( G \)-isomorphism \( \rho \) is called a polarity of \( D \).

Observe that if \( D \) is a \( G \)-symmetric design with the property that \( (p, \rho(p)) \) is a a flag for some \( p \in P_D \), then since \( \rho \) is a \( G \)-isomorphism it follows that the point-stabilizer of \( D \) is isomorphic to the flag-stabilizer of \( D \). Thus each point is incident with exactly one block and vice versa. We consider such \( G \)-symmetric designs to be “degenerate”, and unless an explicit statement to the contrary is given shall take the expression “\( G \)-symmetric design” to mean “non-degenerate \( G \)-symmetric design”.

In a sense that will become clear a little later, these “degenerate” \( G \)-symmetric designs correspond to the “degenerate” orbital graph \( \text{Orb}_\Delta(G, \Omega) \) which could be formed from the permutation group \( (G, \Omega) \) by taking \( \Delta \) to be the diagonal orbit, and also to the “degenerate” Sabidussi graph \( \text{Sab}(G, H, a) \) which could be formed by taking \( a \in H \).

We shall see that each \( G \)-symmetric graph gives rise in a natural way to a \( G \)-symmetric design and conversely a \( G \)-symmetric design together with a “marked” polarity give rise to a \( G \)-symmetric graph. We shall also see that in some cases, by choosing a different polarity it is possible to construct two non-isomorphic \( G \)-symmetric graphs from the same \( G \)-symmetric design. For any graph \( \Gamma \), let \( \mathcal{N}_\Gamma = \{ \Gamma(v) : v \in V\Gamma \} \) denote the set of \textit{neighbourhoods} of \( \Gamma \).

\textbf{Proposition 1.} If \( \Gamma \) is a \( G \)-symmetric graph, then the incidence structure \( D(\Gamma) = (V\Gamma, \mathcal{N}_\Gamma, I) \) where \( (v, n) \in I \) if and only if \( v \in n \) is a \( G \)-symmetric design.
**Proof.** The $G$-arc transitivity of $\Gamma$ is sufficient to ensure that $\mathcal{D}(\Gamma)$ is a $G$-flag transitive design. To see that it is in fact a $G$-symmetric design we must exhibit a polarity.

Let $\rho_P : P \to B$ be given by $v \mapsto \Gamma(v)$. The map $\rho_P$ is clearly bijective, and since $G$ acts on $\Gamma$ as a group of automorphisms we have:

$$
\rho_P(v^g) = \Gamma(v^g) = \Gamma(v)^g = \rho_P(v)^g.
$$

So $\rho_P$ induces a permutation isomorphism. Now, take $\rho_B = \rho_P^{-1}$. We must check that the pair $\rho = (\rho_P, \rho_B)$ preserve the incidence structure of the design.

Suppose that $(v, \Gamma(w))$ is a flag of $\mathcal{D}$, so $v \in \Gamma(w)$. Since $\Gamma$ is a simple graph it follows immediately that $w \in \Gamma(v)$ and so $(w, \Gamma(v))$ is also a flag of $\mathcal{D}$. That is $(\Gamma(v), w)$ is a flag of $\mathcal{D}^\star$. But

$$
\rho((v, \Gamma(w))) = (\rho_P(v), \rho_B(\Gamma(w))) = (\Gamma(v), w).
$$

So we are done. 

\[\square\]

**Proposition 2.** If $\mathcal{D}$ is a (non-degenerate) $G$-symmetric design with “marked” polarity $\rho$ then the graph $\Gamma(\mathcal{D}, \rho)$ with vertex set $P_\mathcal{D}$ and arc set $\{(p, q) : (q, \rho_P(p)) \in I_\mathcal{D}\}$ is $G$-symmetric.

**Proof.** We must first check that $\Gamma(\mathcal{D}, \rho)$ is well-defined. The non-degenerateness of $\mathcal{D}$ ensures that there are no loops. If $(p, q)$ is an arc of $\Gamma(\mathcal{D}, \rho)$ then $(q, \rho_P(p))$ is a flag of $\mathcal{D}$. Since $\rho$ is an isomorphism, if $(q, \rho_P(p))$ is a flag of $\mathcal{D}$ then $(\rho_P(q), \rho_B(\rho_P(p))) = (\rho(q), p)$ is a flag of $\mathcal{D}^\star$. It follows that $(p, \rho(q))$ is a flag of $\mathcal{D}$, and thus $(q, p)$ is an arc of $\Gamma(\mathcal{D}, \rho)$. So $\Gamma(\mathcal{D}, \rho)$ is simple.

By proposition [?] $G$ acts transitively on the points $\mathcal{D}$ so $\Gamma(\mathcal{D}, \rho)$ is $G$-vertex transitive. Suppose that $(p, q_1)$ and $(p, q_2)$ are two distinct arcs of $\Gamma(\mathcal{D}, \rho)$. Then we have $q_1, q_2 \in T(\rho(p))$. By proposition [?] the stabilizer of $\rho(p)$ acts transitively on $T(\rho(p))$. Thus we can find some $g \in G$ which fixes $\rho(p)$ and carries $q_1$ to $q_2$. Since $\rho$ is a $G$-isomorphism, if $g$ fixes $\rho(p)$ it also fixes $p$, thus $g$ carries the arc $(p, q_1)$ to the arc $(p, q_2)$. So $\Gamma(\mathcal{D}, \rho)$ is $G$-locally transitive. The result follows. 

\[\square\]

**Proposition 3.** For any design $\mathcal{D}$ and any polarity $\rho$ the design $\mathcal{D}(\Gamma(\mathcal{D}, \rho))$ is isomorphic to $\mathcal{D}$

**Proof.** later

\[\square\]
Definition 3. If $D_1$ and $D_2$ are two $G$-symmetric designs then a $G$-design homomorphism from $D_1$ to $D_2$ is a pair of permutation homomorphisms $\eta_P : P_1 \to P_2$ and $\eta_B : B_1 \to B_2$ satisfying $(p, b) \in I_1$ implies $(\eta_P(p), \eta_B(b)) \in I_2$.

Proposition 4. If $D_1$ and $D_2$ are two $G$-symmetric designs then a $G$-design homomorphism from $D_1$ to $D_2$ induces a $G$-permutation homomorphism from the flags of $D_1$ to the flags of $D_2$.

What I want to say is that $G$-symmetric designs together with $G$-design homomorphisms form a category (well, kind of more of a poset really). I also want to say that the category $G$-Graph “projects down” onto the category $G$-Design. $F : G$-Graph $\to G$-Design. If there’s a $G$-graph homomorphism from $x$ to $y$ then there’s a $G$-design homomorphism from $F(x)$ to $F(y)$

2. Kernel of a design homomorphism

Gardiner and Praeger showed that whenever we have a $G$-symmetric graph homomorphism $\Gamma \mapsto \Sigma$ we get an induced “cross-sectional” design on each of the fibers of the kernel. Here I’ll show that you get exactly the same thing whenever you have $G$-symmetric design homomorphism.

So, with an abuse of notation, let $\pi : D \to Q$ be a $G$-symmetric design homomorphism. For any point $q \in P_Q$ consider the fiber over $q$, that is the set $F(q) = \pi^{-1}(q) = \{p \in P_D : \pi(p) = q\}$. The “induced kernel design” on the fiber of $q$ is $K(q) = (F(q), T(q), I_q)$ where $(p, d) \in I_q$ if and only if there is some $b \in \pi^{-1}(d)$ such that $(p, b) \in I_D$.

I think that if $\Gamma \mapsto \Sigma$ is a $G$-symmetric graph homomorphism and $D$ is the “GP-cross-section” of the map then if $F(\Gamma) \mapsto F(\Sigma)$ is the $G$-symmetric design homomorphism then the
1. Quotient Graphs

In the first section of this Chapter, we determined, for any two transitive permutation groups \((G, \Omega_1)\) and \((G, \Omega_2)\), the conditions under which there exists a \(G\)-homomorphism from \((G, \Omega_1)\) to \((G, \Omega_2)\).

We saw that if \((G, \Omega_1)\) is permutation equivalent to \((G, \cos_G(H))\) for some \(H < G\), and \((G, \Omega_2)\) is permutation equivalent to \((G, \cos_G(K))\) for some \(K < G\), then there is a \(G\)-homomorphism from \((G, \Omega_1)\) to \((G, \Omega_2)\) if and only if \(H < K < G\).

In this section we give analogous conditions under which there exists a \(G\)-homomorphism from a \(G\)-symmetric graph \(\Gamma\) to a \(G\)-symmetric graph \(\Sigma\).

**Theorem 1** (Lorimer). If \(\Gamma\) is a \(G\)-symmetric graph isomorphic to \(\text{Sab}(G, H, HaH)\) and \(\Sigma\) is a quotient of \(\Gamma\), then \(\Sigma\) is isomorphic to \(\text{Sab}(G, K, KaK)\) for some \(H < K < G\) with \(a \notin K\).

**Proof.** Suppose that \(\pi : \Gamma \to \Sigma\) is a \(G\)-graph homomorphism. Since \(\Gamma\) is \(G\)-isomorphic to \(\text{Sab}(G, H, HaH)\), the action of \(G\) on the vertices of \(\Gamma\) is permutation equivalent to \((G, \cos_G(H))\). Since \(\pi\) induces a homomorphism from \(V\Gamma\) to \(V\Sigma\), by [where?] \((G, V\Sigma)\) must be permutation equivalent to \((G, \cos_G(K))\) for some \(K\) with \(H < K < G\). What about the \(a\)? \(\square\)

2. Extension Problem

For a given symmetric graph \(\Gamma\), let \(\text{Grp}(\Gamma)\) denote the set of subgroups of \(\text{Aut}(\Gamma)\) which act symmetrically on \(\Gamma\). For a given group \(G\), let \(\text{Ext}(G)\) denote the set of groups \(\tilde{G}\) such that \(\tilde{G}/N \cong G\) for some \(N\) normal in \(\tilde{G}\). Clearly \(\Gamma\) is \(\tilde{G}\)-symmetric if and only if \(\tilde{G} \in \text{Ext}(G)\) for some \(G \in \text{Grp}(\Gamma)\).

**Definition 1.** The symmetric graph \((\tilde{G}, \tilde{\Gamma})\) is said to be an *extension* of the symmetric graph \((G, \Gamma)\) if \(\tilde{G} \in \text{Ext}(G)\) and there exists a \(\tilde{G}\)-homomorphism from \(\tilde{\Gamma}\) to \(\Gamma\).

The “extension problem” for symmetric graph is the problem of finding all the extensions of a given symmetric graph \((G, \Gamma)\).
On a group theoretic level, if we knew the subgroup structure of $G$ and we knew all the groups $\tilde{G}$ which admit $G$ as a quotient then in a sense we would “know” all the possible extensions of $(G, \Gamma)$.

**Proposition 1.** If $\Gamma$ is $G$-isomorphic to $\text{Sab}(G, H, HaH)$ then the faithful extensions of $\Gamma$ are the graphs of the form $\text{Sab}(G, K, KaK)$ where $K < H$ and $KaK \subset HaH$.

**Proposition 2.** For any $\tilde{G} \in \text{Ext}(G)$, let $N \trianglelefteq G$ be such that $\tilde{G}/N \cong G$ and let $\pi : \tilde{G} \to G$ be the natural projection. The $\tilde{G}$-symmetric extensions of $\Gamma$ are the graphs of the form $\text{Sab}(\tilde{G}, R, R\tilde{a}R)$ where $\pi(R) < H$ and $\pi(a) \in HaH$.

From a combinatorial perspective, if $(\tilde{G}, \tilde{\Gamma})$ is an extension of $(G, \Gamma)$ then we would like to understand how the structure of $\tilde{\Gamma}$ is related to the structure of $\Gamma$. Ideally, we’d like to be able to describe the adjacency structure of $\tilde{\Gamma}$ in terms of the adjacency structure of $\Gamma$.

### 3. Unfaithful Extensions

Recall from Chapter 1, that in our definition of a symmetric graph, we do not require the action of the group to be faithful on the vertices of the graph. Suppose that $\Gamma$ is a $G$-symmetric graph isomorphic to $\text{Sab}(G, H, HaH)$. The vertices of $\Gamma$ are the cosets of $H$ in $G$ and the action of $G$ on the vertices of $\Gamma$ is the permutation group $(G, \cos_G(H))$. By proposition [?] the kernel of this action is:

$$\text{Core}_G(H) = \bigcap_{g \in G} g^{-1}Hg.$$  

If this kernel is non-trivial then the action of $G$ on the vertices of $\Gamma$ is unfaithful.

**Proposition 3.** For any pair of groups $\tilde{G}$ and $\tilde{H}$ with $\tilde{H} < \tilde{G}$, and any $a \in \tilde{G}$ with $\tilde{a}^2 = 1$ and $\tilde{a} \notin \tilde{H}$, the graphs $\text{Sab}(\tilde{G}, \tilde{H}, \tilde{H}a\tilde{H})$ and $\text{Sab}(G, H, HaH)$ are isomorphic where:

$$N = \text{Core}_{\tilde{G}}(\tilde{H}),$$

$$G = \tilde{G}/N,$$

$$H = \tilde{H}/N$$

and $a = \pi(\tilde{a})$ where $\pi : \tilde{G} \to G$ is the natural projection.

**Proof.** Let $\tilde{\Gamma} = \text{Sab}(\tilde{G}, \tilde{H}, \tilde{H}a\tilde{H})$ and let $\Gamma = \text{Sab}(G, H, HaH)$. The vertices of $\tilde{\Gamma}$ are the right cosets of $H$ in $G$. The vertices of $\Gamma$ are the
right cosets of $H/N$ in $G/N$. Now,

$$\left[\tilde{G}/N : \tilde{H}/N\right] = \frac{|\tilde{G}|/|N|}{|\tilde{H}|/|N|} = \frac{|\tilde{G}|}{|\tilde{H}|} = \left[\tilde{G} : \tilde{H}\right]$$

so both $\tilde{\Gamma}$ and $\Gamma$ have the same number of vertices. Let $\pi : \tilde{G} \to G$ be the natural projection. Since $N$ is normal in $\tilde{H}$, we have $\pi(\tilde{H}) = H$. Now,

$$H \pi(x) = H \pi(y) \iff \pi(xy^{-1}) \in H \iff xy^{-1} \in \tilde{H} \iff \tilde{H}x = \tilde{H}y$$

so $\pi$ actually induces a bijection between the cosets of $H$ in $G$ and the cosets of $H'$ in $G'$. To see that it preserves adjacency, suppose that $(\tilde{H}x, \tilde{H}y)$ is any arc of $\tilde{\Gamma}$, and so $xy^{-1} \in \tilde{H}\tilde{a}\tilde{H}$. It follows that $\pi(xy^{-1}) \in H\alpha H$ and so $(Hx, Hy)$ is an arc of $\Gamma$. The result follows. □

Suppose that $\tilde{\Gamma}$ is a symmetric graph isomorphic to $\text{Sab}(\tilde{G}, \tilde{H}, \tilde{H}\tilde{a}\tilde{H})$. Suppose further that $\tilde{G}$ acts faithfully on the vertices of $\tilde{\Gamma}$. Let $\Gamma$ be any quotient of $\tilde{\Gamma}$, so that, by proposition [??], $\Gamma$ is isomorphic to $\text{Sab}(\tilde{G}, \tilde{K}, \tilde{K}\tilde{a}\tilde{K})$ for some $\tilde{K}$ such that $\tilde{H} < \tilde{K} < \tilde{G}$ and $\tilde{a} \not\in \tilde{K}$. Although $\tilde{G}$ acts faithfully on the vertices of $\tilde{\Gamma}$ it does not follow that $\tilde{G}$ acts faithfully on the vertices of the quotient $\Gamma$.

**Definition 2.** If $(G, \Gamma)$ is a symmetric graph with $G$ acting faithfully on the vertices of $\Gamma$, then an *unfaithful* extension of $(G, \Gamma)$ is an extension of the form $(\tilde{G}, \tilde{\Gamma})$ where $\tilde{G} \neq G$.

**Proposition 4.** Suppose that $(G, \Gamma)$ is a symmetric graph with $\Gamma \cong \text{Sab}(G, H, HaH)$ and that $(\tilde{G}, \tilde{\Gamma})$ is an unfaithful extension of $(G, \Gamma)$ with $\tilde{\Gamma} \cong \text{Sab}(\tilde{G}, \tilde{H}, \tilde{H}\tilde{a}\tilde{H})$. If $\pi : \tilde{G} \to G$ is the natural projection, then $\pi(\tilde{H}) < H$ and $\pi(\tilde{a}) \in HaH$.

**Proof.** Let $N = \ker(\pi)$ Since $\Gamma$ is a quotient of $\text{Gamma}$ we must have $\Gamma \cong \text{Sab}(\tilde{G}, \tilde{K}, \tilde{K}\tilde{a}\tilde{K})$ for some $\tilde{H} < \tilde{K} < \tilde{G}$. Since $\Gamma \cong \text{Sab}(G, H, HaH)$ we must have $\text{Core}_{\tilde{G}}(K) = N$ and $\pi(K) = H$. Since $\tilde{H} < K$ it follows that $\pi(\tilde{H}) < \pi(K) < H$. What about the $a$? □
4. Covering Graphs

**Definition 3.** The graph $\tilde{\Gamma}$ is said to be a *cover* of the graph $\Gamma$ if there exists a graph homomorphism $\pi : \tilde{\Gamma} \to \Gamma$ with the property that, for any arc $(B, C)$ of $\Gamma$, and any $v \in \pi^{-1}(B)$ there is a unique $w \in \pi^{-1}(C)$ such that $(v, w)$ is an arc of $\tilde{\Gamma}$.

**Definition 4.** The graph $\tilde{\Gamma}$ is said to be a *cover* of the graph $\Gamma$ if there exists a graph homomorphism $\pi : \tilde{\Gamma} \to \Gamma$ with the property that, for any arc $(B, C)$ of $\Gamma$, and any $v \in \pi^{-1}(B)$ there exists some $w \in \pi^{-1}(C)$ (not necessarily unique) such that $(v, w)$

Clearly a multicover is a special case of a cover. We shall see that in this section that if $(\tilde{G}, \tilde{\Gamma})$ is an unfaithful extension of $(G, \Gamma)$ then $\tilde{\Gamma}$ is a multicover of $\Gamma$.

**Proposition 5.** Let $(\tilde{G}, \tilde{\Gamma})$ be an unfaithful extension of $(G, \Gamma)$ and let $N$ be a normal subgroup of $\tilde{G}$ such that $\tilde{G}/N \cong G$. If $B$ is any vertex of $\Gamma$ then $N$ acts transitively on the fiber $\pi^{-1}(B)$.

*Proof.* Let $\Delta = \pi^{-1}(B)$. By proposition [?], $\Delta$ is a block of imprimitivity of $V\tilde{\Gamma}$. Suppose that $\Gamma \cong \text{Sab}(G, H, HaH)$ and $\tilde{\Gamma} \cong \text{Sab}(\tilde{G}, \tilde{H}, \tilde{H}a\tilde{H})$. Then the stabilizer of $\Delta$ is $HN$ and the action of $HN$ on the fiber is permutation equivalent to $(HN, \cos(H))$.

$\square$
5. Combinatorial Perspective

In this chapter we look at a number of well-known “combinatorial” methods for constructing extensions of symmetric graphs. We first describe the constructions, and then analyze them group theoretically. In the next chapter we describe a “geometrical” approach to the extension problem for symmetric graphs which was proposed by Gardiner and Praeger.

6. Direct Sum and Lexicographic Product

There are a number of ways of forming “products” of graphs. See for example [?].

Definition 5. For any two graphs Γ and Σ, then the direct sum of Γ and Σ is the graph with vertex set \( V \Gamma \times V \Sigma \) where \((v, x)\) is adjacent to \((w, y)\) if and only if \(v\) is adjacent to \(w\) in \(\Gamma\) and \(x\) is adjacent to \(y\) in \(\Sigma\).

Proposition 6. If \(\Gamma\) and \(\Sigma\) are both symmetric graphs then the direct product of \(\Gamma\) and \(\Sigma\) is symmetric.

Definition 6. For any two graphs \(\Gamma\) and \(\Sigma\), the lexicographic product of \(\Gamma\) by \(\Sigma\) is the graph with vertex set \(V \Gamma \times V \Sigma\) where \((v, x)\) is adjacent to \((w, y)\) if and only if \(v\) is adjacent to \(w\) in \(\Gamma\) or \(v = w\) and \(x\) is adjacent to \(y\) in \(\Sigma\).

Proposition 7. If \(\Gamma\) is any symmetric graph and \(\Sigma\) is the empty graph on \(n\) vertices, then the lexicographic product of \(\Gamma\) by \(\Sigma\) is symmetric.

7. Bigg’s Covering Graph Construction

The three-arc graph construction allows us, under certain conditions, to “unfold” a \(G\)-symmetric graph into a larger, imprimitive \(G\)-symmetric graph, admitting the original graph as a quotient. The covering graph construction [?] is similar in this respect. It also, under certain conditions, allows us to “unfold” a \(G\)-symmetric graph into a larger one. It differs from the three-arc graph construction, however, in that it requires a simultaneous “unfolding” of the group \(G\).

Recall that given two groups \(N\) and \(G\) and a homomorphism

\[
\rho : G \rightarrow \text{Aut}(N)
\]

we may form the semidirect product of \(N\) by \(G\). This is the group \(\tilde{G} = N \rtimes_{\rho} G\), whose elements are the ordered pairs:

\[
\{(n, g) : n \in N, g \in G\}
\]
with multiplication given by:

\[(n_1, g_1)(n_2, g_2) = (n_1^{\rho(g_1)} n_2, g_1 g_2).\]

The functions \(i_1 : N \to \tilde{G}\) and \(i_2 : G \to \tilde{G}\) given by \(n \mapsto (n, 1)\) and \(g \mapsto (1, g)\) respectively give natural embeddings of \(N\) and \(G\) into \(\tilde{G}\). Identifying \(N\) with its image under \(i_1\), the semidirect product has the property that \(N\) is normal in \(\tilde{G}\) and \(\tilde{G}/N \cong G\).

**Definition 7 (N-chain).** Suppose that \(\Gamma\) is a \(G\)-symmetric graph and \(N\) is a group. An \(N\)-chain is a function \(\phi : A\Gamma \to N\) satisfying \(\phi((v, u)) = \phi((u, v))^{-1}\) for all \((u, v) \in A\Gamma\).

**Definition 8 (Compatible N-chain).** Suppose that \(\Gamma\) is a \(G\)-symmetric graph, \(N\) is a group and \(\rho : G \to \text{Aut}(N)\) is a homomorphism. An \(N\)-chain \(\phi\) is said to be *compatible* with \(\rho\) if for every \(g \in G\) the following diagram commutes:

\[
\begin{array}{ccc}
A\Sigma & \xrightarrow{\phi} & K \\
\downarrow{g} & & \downarrow{\rho(g)} \\
A\Sigma & \xrightarrow{\phi} & K
\end{array}
\]

**Definition 9 (Biggs Cover).** Suppose that \(\Gamma\) is a \(G\)-symmetric graph, \(N\) is a group, \(\rho : G \to \text{Aut}(N)\) is a homomorphism and \(\phi\) is a compatible \(N\)-chain. The Biggs Cover \(\tilde{\Gamma}(N, \rho, \phi)\) of \(\Gamma\) with respect to \(\phi\) is the graph with vertex set: \(\{(g, v) : k \in G, v \in V\Gamma\}\) and arc set: \(\{((g_1, v_1), (g_2, v_2)) : (v_1, v_2) \in A\Gamma, g_2 = g_1 \phi((v_1, v_2))\}\)

Note that the condition \(\phi((v, u)) = \phi((u, v))^{-1}\) ensures that the resulting graph is simple.

**Proposition 8.** The covering graph \(\tilde{\Gamma}(N, \rho, \phi)\) is \((N \rtimes_\rho G)\)-symmetric.

*Proof.* Define an action of \(N \rtimes_\rho G\) on \(\tilde{\Gamma}\) by:

\[(n, v)^{(\eta, g)} = (n^{\rho(\eta)} \eta, v^g).\]

This action is well-defined since:

\[
(n, v)^{(\eta_1, g_1)(\eta_2, g_2)} = (n^{\rho(g_1)} \eta_1, v^{g_1})(\eta_2, v^{g_2}) = (n^{\rho(g_1)^{\rho(g_2)} \eta_1^{\rho(g_2)}}, v^{g_1 g_2}) = (n, v)^{(\eta_1^{\rho(g_2)} \eta_2, g_1 g_2)}
\]
We must check that it preserves the adjacency structure of \( \tilde{\Gamma} \). Observe that, by the compatibility of \( \rho \) and \( \phi \), if \( n_2 = n_1 \phi(v_1, v_2) \) then:

\[
\begin{align*}
    n_2^{\rho(g)} \phi(v_1^g, v_2^g) &= n_2^{\rho(g)} \phi(v_1, v_2)^{\phi(g)} \\
    &= (n_2 \phi(v_1, v_2))^{\phi(g)} \\
    &= n_1 \phi(g)
\end{align*}
\]

for any \((\eta, g) \in N \rtimes \rho G\). Also, since \( G \) is a group of automorphisms of \( \Gamma \), we know that \((v_1, v_2) \in A\Gamma\) if and only if \((v_1^g, v_2^g) \in A\Gamma\). Thus the action defined above does indeed preserves adjacency.

For any \((n_1, v_1), (n_2, v_2) \in V\tilde{\Gamma}\), by the \( G \)-symmetry of \( \Gamma \) we can find a \( g \in G \) such that \( v_1^g = v_2 \). Let \( \eta = n_1^{\rho(g)^{-1}} n_2 \). We have:

\[
\begin{align*}
    (n_1, v_1)^{(\eta, g)} &= (n_1^{\rho(g)} \eta, v_1^g) \\
    &= (n_1^{\rho(g)}, n_1^{\rho(g)^{-1}} n_2, v_2) \\
    &= (n_2, v_2).
\end{align*}
\]

Thus the action is vertex transitive.

Suppose that \((n_1, v_1)\) and \((n_2, v_2)\) are both adjacent to \((n, g)\), so \( n_1 = n \phi(v, v_1) \) and \( n_2 = n \phi(v, v_2) \). By the \( G \)-symmetry of \( \Gamma \) we can find a \( g \in G_v \) such that \( v_1^g = v_2 \). Let \( \eta = (n^{-1})^{\rho(g)} n \). We have:

\[
\begin{align*}
    (n, v)^{(\eta, g)} &= (n^{\rho(g)} \eta, v^g) \\
    &= (n^{\rho(g)} (n^{-1})^{\rho(g)} n, v) \\
    &= (n, v)
\end{align*}
\]

and:

\[
\begin{align*}
    (n_1, v_1)^{(\eta, g)} &= (n_1^{\rho(g)} \eta, v_1^g) \\
    &= (n_1^{\rho(g)} (n^{-1})^{\rho(g)} n, v_2) \\
    &= ((n_1 n^{-1})^{\rho(g)} n, v_2) \\
    &= (\phi(v, v_1)^{\rho(g)} n, v_2) \\
    &= (\phi(v, v_2) n, v_2) \\
    &= (n_2^{n^{-1}} n, v_2) \\
    &= (n_2, v_2)
\end{align*}
\]

Thus the action is locally transitive. The result follows. \( \square \)
In fact, a stronger result than this holds. If $\Gamma$ is $(G, s)$-arc transitive then $\tilde{\Gamma}$ is $(\tilde{G}, s)$-arc transitive, where $\tilde{G} = N \rtimes_{\rho} G$. The proof is by induction and may be found in [?]. This construction was originally used by Conway to produce an infinitely family of 5-arc transitive graphs.

For each $v \in V\Gamma$, let $B(v) = \{(n, v) : n \in N\}$. For any $(\eta, g) \in \tilde{G}$ we have:

$$
B(v)^{(\eta, g)} = \{(n, v)^{(\eta, g)} : n \in N\} = \{((\eta n)^{(\rho(g)}, v^g) : n \in N\} = \{(n, v^g) : n \in N\} = B(v^g).
$$

Thus the partition: $B = \{B(v) : v \in V\Gamma\}$ is $\tilde{G}$-invariant.

**Proposition 9.** The quotient of the Biggs cover $\tilde{\Gamma}$ with respect to the partition $B$ is isomorphic to the original graph $\Gamma$.

**Proof.** The map $\eta : V\Gamma \to V\tilde{\Gamma}_B$ given by $v \mapsto B(v)$ establishes a bijection between the vertices of $\Gamma$ and the vertices of $\tilde{\Gamma}_B$. If $(u, v)$ is an arc of $\Gamma$, then $((1, u), (\phi((u, v)), v))$ is an arc of $\tilde{\Gamma}$, and so $(B(u), B(v))$ is an arc of $\tilde{\Gamma}_B$. If $(u, v)$ is not an arc of $\Gamma$, then for all $n_1, n_2 \in N$ $((n_1, u), (n_2, v))$ is not an arc of $\tilde{\Gamma}$ and so $(B(u), B(v))$ is not an arc of $\tilde{\Gamma}_B$. Thus $\eta$ establishes an isomorphism between $\Gamma$ and $\tilde{\Gamma}_B$. □
Let $(B, C)$ be any arc of $\tilde{\Gamma}$. By the above $B = B(u)$ and $C = B(v)$ for some $(u, v) \in A\Gamma$. It's not too hard to see that each $(n, u) \in B$ has a unique neighbour in $C$, namely $(n\phi(u, v), v)$. Thus the induced bipartite graph $\Gamma[B, C]$ is a matching. It follows immediately that the valency of $\tilde{\Gamma}$ is the same as the valency of $\Gamma$.

8. **Group Theoretic Analysis of Bigg’s Covering Graph Construction**

The group $\tilde{G}$ acts unfaithfully on $B$ with kernel

$$\tilde{N} = \{(n, 1) : n \in N\} \cong N.$$  

If $|N| = n$ then $|\tilde{G}| = n|G|$, and since $\tilde{\Gamma}$ has exactly $n$ times as many vertices as $\Gamma$, we have $[\tilde{G} : G_{B(\alpha)}] = n[G : G_\alpha]$. It follows that $|G_{B(\alpha)}| = |G_\alpha|$.

Let $\pi : \tilde{G} \to G$ be the natural projection. Clearly $|\pi(\tilde{G}_{B(\alpha)})| \leq |\tilde{G}_{B(\alpha)}|$. Since $G_\alpha \leq \pi(\tilde{G}_{B(\alpha)})$, we must have $G_\alpha = \pi(\tilde{G}_{B(\alpha)})$ and $|\pi(\tilde{G}_{B(\alpha)})| = |\tilde{G}_{B(\alpha)}|$, so $\tilde{G}_{B(\alpha)} \cong \pi(\tilde{G}_{B(\alpha)}) \cong G_\alpha$. In fact, the permutation groups $(G_\alpha, \Gamma(\alpha))$ and $(\tilde{G}_{B(\alpha)}, \tilde{\Gamma}(B(\alpha)))$ are equivalent via the bijection $\eta : \Gamma(\alpha) \to \tilde{\Gamma}(B(\alpha))$ which sends $\beta$ to $B(\beta)$. Probably a much easier way to say this.

If $\Gamma$ is isomorphic to the coset graph $\Gamma(G, H, a)$ then since the local actions of $\Gamma$ and $\tilde{\Gamma}$ are the same, $\tilde{\Gamma}$ must be isomorphic to the coset graph $\Gamma(\tilde{G}, H, \tilde{a})$ where $\tilde{a}$ is such that $\pi(\tilde{a}) = a$. That is $\tilde{a} \in \{(n, a) : n \in N\}$. Thus for any $N$ there are exactly $n$ possible Biggs covers of $\Gamma$.

Looking closely at the compatibility condition between $\rho$ and $\phi$ we see that, since $G$ acts transitively on the arcs of $\Gamma$, the map $\phi$ is completely determined by where it sends the arc $(v_H, v_{Ha})$. Choosing where to send this arc is essentially equivalent to choosing which preimage of $a$ we want. Requiring that $\phi((v_H, v_{Ha})) = n$ is essentially the same as determining that $\tilde{a} = (n, a)$.

I think that Bigg’s covering graph construction encompasses all pairs of graphs $\Gamma(G, H, a)$ and $\Gamma(\tilde{G}, H, \tilde{a})$ where $\tilde{G}$ is a semidirect product of $N$ by $G$ for some $N$. In the more general case where $\tilde{G}$ is an extension of $N$ by $G$ but not a split extension $\Gamma(\tilde{G}, H, \tilde{a})$ should be a cover of $\Gamma(G, H, a)$ in the sense that the induced bipartite graph is a matching, but Bigg’s covering graph construction cannot be used to construct $\Gamma(\tilde{G}, H, \tilde{a})$ from $\Gamma(G, H, a)$.
9. Generalization of Bigg’s Covering Graph Construction

The Bigg’s covering graph construction applies to the case where $\tilde{G}$ is a semidirect product of $N$ by $G$ and $H < G$ is such that $H \cap N = \{1\}$. The graph $\tilde{\Gamma}$ is isomorphic to $\text{Sab}(\tilde{G}, H, HaH)$ and the graph $\Gamma$ is isomorphic to $\text{Sab}(\tilde{G}, \tilde{H}, \tilde{Ha}\tilde{H})$ where $\tilde{H} = HN$.

In this section we consider a generalization of the Bigg’s covering graph construction in which the condition that $H \cap N = \{1\}$ is relaxed. We find that in this case $\tilde{\Gamma}$ is a multicover of $\Gamma$.

**Lemma 1.** If $H$ is a subgroup of $G$ and $N$ is a normal subgroup of $G$ then $H \cap N$ is a normal subgroup of $H$.

Let $M = N/H \cap N$ and let $\lambda : N \to M$ be the natural projection.