Three-Dimensional Dynamic Modelling and Validation for Vibration of a Beam-Cable System

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ABSTRACT

In order to understand and to predict cable effects on structures, three-dimensional numerical models for a stranded cable and a beam–cable system consisting of a cantilever beam and two connected cables are presented. The multibond graph formalism is used to model the coupled cable–beam system, with the cable and beam substructures using 3D rigid lumped segments. The stranded cables are modelled considering the bending stiffness, tension and sag due to self-weight. The generally applicable cable-structure modelling approach in this paper is applied to vibration-based non-destructive evaluation of electrical utility poles, where simulated modal testing of the pole-conductor system is required. Experimental parametrisation of a stranded cable is carried out using specially designed apparatus to accurately measure the bending stiffness at different tensions, and to measure the axial stiffness and axial damping. A reduced-scale lab set-up and finite element models are developed for verification of the numerical models. Experimental free and forced vibration testing is performed on individual cantilever beam and stranded cable subsystems, and on the coupled cable–beam system to verify the numerical models in the frequency and time domains. It is concluded that the 3D bond graph models can be used to understand the interaction between cable and structure, allowing prediction of the in-plane and out-of-plane natural frequencies and time response of the connected pole. It is also concluded that by adding the cable to the pole structure, some modes emerge in the eigenvalue solution of the system which may be categorized as cable-dominated modes, pole-dominated or hybrid modes.

1. Introduction

Dynamic interaction between cables and the main structure in cabled structures such as cable-stayed bridges, guyed towers, cable-stayed wind turbines, and power transmission lines has been a topic of interest of researchers for many years. Understanding the interaction of cables and the main structure would be facilitated by computer simulations of coupled cable–beam response. The application area that is the focus of this paper is single-pole transmission line modelling for simulated modal testing in order to develop
a non-destructive evaluation (NDE) method. This literature review consists of two parts: cable-only system modelling followed by cable–beam system analysis.

Models of cables typically treat the cable as a string or beam, or as a series of beam-like segments. Models vary in their treatment of bending stiffness and cable sag due to self-weight. Ricciardi et al. [1] developed a continuous model for vibration analysis of cables with sag, considering the bending stiffness. The conductor was treated as a tensioned Euler-Bernoulli beam under self-weight, and Hamilton’s principle was used to obtain the equations of motion. Natural frequencies and symmetric and anti-symmetric mode shapes were obtained, and the results were validated by finite difference (FD) and finite element (FE) analysis. It was concluded that the bending stiffness and sag of the cable has a significant effect on vibration properties. Sousa et al. [2] used Euler-Bernoulli and Timoshenko beam theories to investigate vibration of transmission line conductors. Neglecting shear and rotational inertia caused a very small error compared to neglecting the bending and geometric stiffness of the cable. Papailiou [3] proposed a model for bending of stranded cables taking into account the slip and friction between the layers, obtaining an experimentally verified bending stiffness model dependent on curvature and tension. Ni et al. [4] analysed the vibration of large diameter sagged cables using a three-node finite element. A case study compared the measured and computed natural frequencies, with and without considering cable bending stiffness, of the main cables of the Tsing Ma Bridge. Ignoring the bending stiffness of sagged cables led to unacceptable errors in predicting the natural frequencies. Jalali et al. [5] developed a 2D numerical model based on the bond graph method for vibration analysis of power lines. Lumped segments were joined with axial and torsional springs and dampers representing the compliance and damping of the cable. Bending stiffness of the cable, without tension, was measured experimentally. Analytical and finite element modelling of stranded cables and wire ropes has been performed by researchers in order to study axial, bending and torsional deformations [6–9]. In general, the literature shows that bending stiffness and cable sag have significant and non-negligible effects on vibration of cable structures and should be considered in the modelling to ensure reliable results.

Dynamic analysis of cable–beam systems has been attempted using various approaches. Gattulli et al. [10] studied the linear and nonlinear dynamic behaviour of cable-stayed bridge structures, neglecting bending stiffness of the cable in solving the equations of in-plane and out-of-plane motion. The existence of global (beam dominated), local (cable dominated) and coupled modes was revealed by linear analysis. Chen et al. [11] reviewed analysis and modelling of transmission towers, transmission lines and transmission tower-line systems subjected to dynamic excitations due to wind, ice effects and seismic motion. The review suggested that future improvements in analytical models of tower-line systems are motivated and expected. Li et al. [12] presented a simplified computational model of a high-voltage transmission tower-line system under out-of-plane and in-plane vibrations due to seismic excitations. The transmission cables and their supporting towers were modelled as a lumped mass system, validated with experiments. McClure et al. [13] presented a modelling approach, using the specialized finite element software ADINA*, for dynamic investigation of a tower-line section subjected to
conductor rupture shock loads in two- and three-dimensional space. A case study of a tower-line system failure during an ice storm was used to validate their model. Cables were modelled using two-node iso-parametric truss elements. Pinto et al. [14] developed a two-dimensional bond graph model for a pole with a cable attached to one side. The cable was modelled as a series of point masses connected by translational springs, and the pole was represented by a modal expansion based on separation of variables. Modal parameters of the cable and pole-cable system were obtained numerically and experimentally. The paper concluded that a more complex model is needed to get accurate time responses.

The models developed in this paper are motivated by ongoing development of a non-destructive evaluation (NDE) method for utility poles based on vibration testing. Existing NDE methods for poles, such as ultrasonic, x-ray and resistograph, are localized damage detection methods that evaluate the strength of the pole at one specific axial location [15]. The authors are developing an NDE method to detect damage at any location, based on vibration response from modal impact testing at a single arbitrary location [16–19]. Since power lines (conductors) are attached to the poles, vibrations of the conductors affect the modal properties of the system. Removing the effect of the conductors to reveal pole properties requires a system model that captures the complex interactions between the cables and the pole [20].

The bond graph method allows simple submodels to be easily combined into a complex model [21]. Bond graphs, based on explicit energy flow paths among storage, dissipative, and source elements, also facilitate derivation of the dynamic equations of the system [22]. Three-dimensional vectorial multibond graph models [22] are developed for the stranded cable and beam substructures individually, and then for a coupled system with one cantilever beam and cables on two sides. These models are easily extended to larger utility pole lines by ‘cutting and pasting’ pole-cable submodels. Since the bending stiffness of the cables has a significant effect on modal properties of the coupled system, the cable is modelled considering the bending stiffness and sag and the beam is modelled as a cantilevered beam. The 3D rigid lumped segments are connected to each other with translational and rotational springs and dampers to model the compliance and damping of the cable and beam. The modal properties of the cable, beam and the cable–beam system are obtained and time response analysis in in-plane and out-of-plane directions is carried out in free and forced vibration simulation. The numerical models are validated with finite element and experimental results in the frequency and time domains. Bending stiffness of the stranded cable under different tensions was measured experimentally, as was the axial stiffness/damping. The measured values were used in the bond graph and FE models. The following section describes the bond graph model and theoretical formulation, Section 3 presents the experimental cable parametrization and modal testing; an elementary verification of the cable and beam submodels is presented in Section 4 and the coupled system results are presented in Section 5.
2. Discrete physical modelling

The cable is assumed to have a small sag $s$ to length $l_c$ ratio (namely $s/l_c < \frac{1}{8}$) based on Irvine's model [23] and is modelled as a tensioned Euler–Bernoulli beam with sag due to self-weight. The cable–beam system is composed of one horizontal suspended cable attached to the tip of a cantilevered beam and the cantilevered beam is considered as an Euler-Bernoulli beam. Figure 1 shows the schematic of the cable–beam system and the cable. The $x$-axis is in the beam in-plane direction and the $z$-axis is in the beam out-of-plane direction. In Figure 1, the bending stiffness of the beam is $EI_b$ and the bending stiffness of the cable is $EI_c$.

2.1 Bond graphs

The bond graph method is used owing to the ease with which submodels of cables and beams can be connected to each other. In bond graphs, a small set of generalized elements is used to represent energy exchange with the environment, energy storage and dissipation, generalized loop and node laws, and power-conserving transformations and gyrator effects [24,25]. The connection of the cable bond graph submodels to the beam bond graph submodel is as simple as drawing a power bond between them.

2.1.1 3D segment submodel

The cables and beam are modelled with 3D rigid lumped segments connected with translational and rotational springs and dampers. The bond graph model with lumped segments has been used for different applications for modelling coupled system dynamics [24,26,27]. The greater the number of segments, the more accurate is the model. Excessive segments, beyond the number required for the numerical model results to converge to analytical results, increase simulation time. In this model, convergence studies in the time and frequency domains were performed and when increasing the number of segments above 30 for cable and beam, there was no significant improvement in the results. Thus, 30 segments were used for modelling of the cable and 30 segments were used to model the cantilevered beam. The lumped segments of the cable and beam
are joined with a translational spring and three rotational springs. Figure 2 shows beam and cable segments connected with springs. The translational springs correspond to the axial compliance of the beam and cable, \( K_{\text{axial, beam}} \) and \( K_{\text{axial, cable}} \), respectively. The shear compliance of the beam and cables are not considered in the analyses as they have negligible effect \([2]\). Rotary springs \( K_{\text{bend, beamy}} \) and \( K_{\text{bend, beamz}} \) are the bending compliance of the beam about the local \( y \) and \( z \) directions, respectively and \( K_{\text{bend, cable}} \) is the bending compliance of the cable about the local body-fixed \( y \) and \( z \) directions. These stiffnesses are the same because of the axial symmetry of the cables. \( K_{\text{tor, beam}} \) is the torsional compliance of the beam about body-fixed \( x \) coordinate and \( K_{\text{tor, cable}} \) is the torsional compliance of the cable about body-fixed \( x \)-direction. In Figure 2, the bending rotary springs (about the \( z \)-direction, not shown for clarity) are oriented along the body \((i+1)\)-fixed frame for the beam and cable. A similar description of the lumped segment model can also be found in \([26]\) for different applications.

Stiffness values are computed for a segment lengths of \( l_b = \frac{l_b}{n_b} \) (beam of length \( L_b \) with \( n_b \) segments) and \( l_c = \frac{l_c}{n_c} \) (cable of length \( L_c \) with \( n_c \) segments) \([21,26]\): \[K_{\text{axial, beam}} = \frac{E_b A_b}{l_b}\] (1) \[K_{\text{axial, cable}} = \frac{E_c A_c}{l_c}\] (2) \[K_{\text{bend, beamy}} = \frac{E_b I_{by}}{l_b}, \quad K_{\text{bend, beamz}} = \frac{E_b I_{bz}}{l_b}\] (3) \[K_{\text{bend, cable}} = \frac{E_c I_c}{l_c}\] (4)

**Figure 2.** Successive multibody segments of (a) beam (b) cable.
\[
K_{\text{tor, beam}} = \frac{G_b J_b}{I_b} \\
K_{\text{tor, cable}} = \frac{G_c J_c}{I_c}
\]

where \(E_b\) and \(E_c\) are the elastic moduli of the beam and cable, respectively. \(A_b\) and \(A_c\) are the cross-sectional areas of beam and cable, respectively. \(I_{by}, I_{bz}\) and \(I_c\) are the moments of area of the beam about local \(y\) and \(z\) axes, and cable moment of area, respectively. \(G_b\) and \(G_c\) are the modulus of rigidity of cable and beam, respectively, and \(J_b\) and \(J_c\) are the polar moments of area. The cable axial damping values are measured experimentally (Section 3.2) and other damper values are tuned in the models to give a close time response compared to experimental time responses. The axial \((E_c A_c)\), and bending \((E_c I_c)\) stiffnesses of the cable were measured (Section 3) and the resulting values were used in the models.

### 2.1.2 Bond graph of segments and joints

Mechanics in bond graphs is done from a velocity standpoint, necessitating relative velocity equations between segment centre of gravity and connection points. The following equation relates the velocity of end point A of body i to velocity of the centre of gravity G of body i (Figure 2) [26]:

\[
i\vec{V}_{Ai} = i\vec{V}_{Gi} + i\vec{V}_{Ai/Gi}
\]

where the left superscript \(i\) indicates that the vector is presented in a local body-fixed coordinate system (Figure 2), \(i\vec{V}_{Ai}\) is the absolute velocity of point A on body \(i\) resolved along body-fixed component directions, \(i\vec{a}_i\) is the angular velocity vector, \(i\vec{r}_{Ai/Gi}\) is the position vector of point A with respect to G and \(i\vec{r}_{Ai/Gi}\) is the skew-symmetric matrix containing the relative position vector components. The relative position matrix \(i\vec{r}_{Ai/Gi}\) is used here to express the equations in a matrix multiplication form which is common in multibond graphs [28] and is beneficial in programming the equations. The relative position vector \(i\vec{r}_{Ai/Gi}\) and relative position matrix \(i\vec{r}_{Ai/Gi}\) are the following:

\[
i\vec{r}_{Ai/Gi} = \left( -\frac{l_s}{2}\right) \hat{i} + (0)\hat{j} + (0)\hat{k}
\]

\[
i\vec{r}_{Ai/Gi} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{l_s}{2} \\ 0 & \frac{l_s}{2} & 0 \end{bmatrix}
\]

where \(l_s\) is the segment length (can be \(l_b\) or \(l_c\) in Equations (1)-(6)) and \(\hat{i}, \hat{j}\) and \(\hat{k}\) are along the \(x, y\) and \(z\) body-fixed coordinates of segment \(i\), respectively (Figure 2). All the other vectors (linear and rotational velocity, etc.) are also represented by column vectors in the local coordinate system. The equations below describe Newton’s law in three dimensions for the segment in local coordinates [26].
\[
\sum o\vec{F} = \frac{d}{dt}(m^o\vec{V}_{Gi}) = m^o\vec{V}_{Gi}
\]  
(11)

\[
\sum i\vec{r} = ddt.H = J^i\vec{\omega} + i\vec{\omega}_i \times J_i\vec{\omega}_i
\]  
(12)

where left superscript \( o \) indicates that the vector is defined in the inertial coordinate system, \( m \) is the mass and \( J \) is the rotational inertia. The first term in the right side of Equation (12) is the rotational inertia term and the second term is a gyrational term. Including forces such as gravity and equating vectors in different reference frames requires coordinate transforms from the body fixed frame to the inertial frame. This is accomplished with a rotation matrix. Rotation matrices are made for each of the three rotations \( \psi, \theta, \phi \) about \( z, y \) and \( x \) axes, respectively. These matrices are multiplied together to create the transformation matrix. The final simplified rotation matrix is shown in the following equation that transforms coordinate frames from body fixed to inertial [21]:

\[
\begin{bmatrix}
X \\
Y \\
Z
\end{bmatrix}
= 
\begin{bmatrix}
\cos \theta \cos \psi & -\cos \theta \sin \psi & \sin \theta \\
\cos \theta \sin \psi + \sin \phi \sin \theta \cos \psi & \cos \phi \cos \psi - \sin \phi \sin \theta \sin \psi & -\sin \phi \cos \theta \\
\sin \phi \sin \psi - \cos \phi \sin \theta \cos \psi & \sin \phi \cos \psi + \cos \phi \sin \theta \sin \psi & \cos \phi \cos \theta
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
\]  
(13)

The following equation relates angular velocity components to the rates of change of orientation angles [21]:

\[
\vec{\omega} = 
\begin{bmatrix}
\omega_x \\
\omega_y \\
\omega_z
\end{bmatrix}
= R\vec{\Omega} = 
\begin{bmatrix}
1 & 0 & -\sin \theta \\
0 & \cos \phi & \cos \theta \sin \phi \\
0 & -\sin \phi & \cos \phi \sin \theta
\end{bmatrix}
\begin{bmatrix}
\vec{\phi} \\
\vec{\theta} \\
\vec{\psi}
\end{bmatrix}
\]  
(14)

Equation (14) transforms the orientation angle rates into the body fixed angular velocity components. Taking the inverse of matrix \( R \) will allow us to transform in the other direction. The inverse transformation matrix is shown below.

\[
R^{-1} = 
\begin{bmatrix}
1 & \sin \phi \tan \theta & \cos \phi \tan \theta \\
0 & \cos \phi & -\sin \phi \\
0 & \frac{\sin \phi}{\cos \theta} & \frac{-\sin \phi}{\cos \theta}
\end{bmatrix}
\]  
(15)

Figure 3 shows the symbols and constitutive laws of sources, storage and dissipative elements, and power-conserving elements in the scalar form [29]. Figure 4 shows the vector bond graph submodel representing the body segment \( i \) of the cable and beam (developed in 20sim* software). The dynamic equations for the cable and beam segments are the same; thus, the bond graph formulation is the same for cable and beam. In Figure 4, the one-junctions are labelled with the corresponding translational or rotational velocities. Translational velocity Equation (7) for defining the velocity of left end point \( A \) of the segment and the similar equation for the velocity of the right end point \( B \) of the segment with respect to centre of mass in the body-fixed coordinates are represented by two modulated transformers (MTF) and two 0-junctions in Figure 4. A 0-junction is used to add \( i\vec{V}_{Gi} \) to \( i\vec{r}_{Ai/Gi}i\vec{\omega}_i \) to obtain \( i\vec{V}_{Ai} \) and a 0-junction is used to add \( i\vec{V}_{Gi} \) to \( i\vec{r}_{Bi/Gi}i\vec{\omega}_i \) to obtain \( i\vec{V}_{Bi} \). The two MTFs representing relative velocity cross products use the relative position vector as an input signal. The coordinate transformation between
| Symbol | Constitutive Law |
|--------|-----------------|
| Flow   | $f = f(t)$      |
| Effort | $e = e(t)$      |
| Inertia| $f = \frac{1}{I} \int e \, dt$ |
| Capacitor | $e = \frac{1}{C} \int f \, dt$ |
| Resistor | $e = Rf$ |
| Transformer | $e_2 = ne_1$ |
| Modulated transformer | $f_1 = nf_2$ |
| Gyrator | $e_2 = n(\theta)f_1$ |
| Modulated Gyrator | $e_1 = n(\theta)f_2$ |
| 1-junction | $f_1 = f_2$ |
| 0-junction | $e_1 = e_2$ |

Figure 3. Bond graph elements [29].

Body-fixed and inertial frames is represented by an MTF that multiplies the transformation matrix and $\dot{\mathbf{V}}_{Gi}$ to obtain inertial centre of gravity velocity $\mathbf{V}_{Gi}$. Translational force Equation (11) is represented by a mass matrix multiport $I$ element and the gravity effort source connected to the $\mathbf{V}_{Gi}$ 1-junction. Orientation angle rates $[\dot{\theta} \dot{\psi} \dot{\phi}]^T$ are obtained by multiplication of the $\dot{i}\bar{\omega}_i$ body-fixed angular velocity by the matrix in Equation (15), followed by integration to obtain the transformation angles $[\theta \psi \phi]^T$. The right-hand side of Equation (12) is represented by a rotational inertia $I$ element and a modulated gyrator MGY. Other elements bonded to the $i\bar{\omega}_i$ 1-junction create the external moments from joints i and i-1 and moments from forces at end points A and B. An MGY is used to represent the cross product $i\bar{\omega}_i \times \mathbf{J}\bar{\omega}_i$.

Figure 5 shows the bond graph submodel of joint i between successive segments (developed in 20sim software). The 0-junctions represent the translational and angular
velocity constraints that are caused by parallel spring/dampers between the left end point A of body $i + 1$ and right end point B of body $i$ (Figure 2). The velocity of B on body $i$, $\dot{V}_{Bi}$, and angular velocity $\omega_i$ are first transformed to the inertial coordinate system using a pair of MTFs and then transformed to the body $i + 1$ local coordinate system using another pair of MTFs. The C multiport elements model the stiffness described in Equations (1)-(6) using a diagonal stiffness matrix. The R elements define the damping using a damping matrix. The cable axial damping is measured and the other damping
values are tuned to reduce discrepancies between numerical and experimental time response and natural frequencies.

2.1.3 **Cable–beam connection bond graph**

Figure 6 illustrates the segments of the beam and cable at the connection point. The connection of the cable to the tip of the beam is realized by bonding the 1-junctions associated with the beam tip velocities and the cable endpoint velocities. Figure 7 shows the bond graph model at the connection point of beam tip and the cable. The angles that
are calculated in the \( n_c \)th joint submodel of the left cable are input as a signal to the beam tip segment and then the calculated angles in the beam tip segment are input signal for the first joint of the cable attached to the right of the beam. Beam tip velocity at point A (Figure 2), \( \vec{V}_{A_{n_b}} \) is connected to the lower joint submodel of beam (joint \( n_b \) in Figure 7). Velocity of point B of the highest beam segment, \( \vec{V}_{B_{n_c}} \) is connected to the joint \( n_c \) of left cable and the first joint of right cable in the beam \( n_b \)th coordinate system. Rotational velocity 1-junctions are also bonded in the cable endpoints and the beam tip to enforce angular velocity continuity at the fixed connection.

Two end points of the right and left cables are pinned to the wall and therefore have zero translational velocity. Thus, a zero-flow source is used at each end point to define the pin connection. The fixed boundary condition of the beam is applied by zero flow sources for translational and rotational velocities at the bottom-most point of the beam. In order to create the desired tension in the cable, the model is first given zero initial displacement and then stretched until the desired tension is achieved. Five values of tension are considered for the analysis. For applying the stretching displacement to the cable in the bond graph model, a flow source was temporarily applied to the 1-junction associated with the velocities of the end points of the left and right cables in the local \( x \) direction. After creating the desired tension, the zero flow source was reinstated at the right end.

### 2.2 Finite element model

The three-dimensional finite element models of the cable and beam–cable system were developed in Ansys* using solid elements. The beam-cable model (Figure 8) consists of approximately 36,000 solid elements with each element having 48 degrees of freedom. The displacement of the free end of the beam is defined to be equal to the displacements of cross sections of each of the left and right cables where they connect to the beam. The cables are pinned at the other end points. A pre-tension is first applied to the cable and then Eigen Value analysis is performed and the natural frequencies and mode shapes of the system are obtained in a pre-stressed modal analysis. The cables are modelled as cylinders but the geometric and mechanical properties of the cables are defined such that they match the measured bending stiffness and axial stiffness (Section 3) and mass of the

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**Figure 8.** Finite element model of the cable–beam system.
cable at each value of tension. Young’s modulus, volumetric density and diameter of the cylinder model are adjusted at each value of tension so that the following equations are enforced:

\[ E_c I_c = a \Rightarrow E_c \frac{\pi d_c^4}{64} = a \]

\[ E_c A_c = b \Rightarrow E_c \frac{\pi d_c^2}{4} = b \]

\[ \rho_c A_c = c \Rightarrow \rho_c \frac{\pi d_c^2}{4} = c \]  \hspace{1cm} (16)

where \(a\), \(b\) are the measured bending stiffness and axial stiffness of the cable (Section 3) and \(c\) is the linear density of the cable, known from the cable manufacturer catalogue. Equation (16) has three unknowns that can be solved so that the values of \(E_c\), \(d_c\) and \(\rho_c\) can be calculated for each tension. The measured bending stiffness is different in each cable tension but the axial stiffness and linear density are the same; thus, all the three values of \(E_c\), \(d_c\) and \(\rho_c\) are different in each tension.

3. Experimental cable parameterization and vibration testing

In order to have a reliable numerical model for the stranded cable, the cable mechanical properties must be measured accurately. The stranded cable used in this study is a steel \(7 \times 19\) cable with a nominal diameter of a quarter inch and linear density of 0.16667 kg/m. An apparatus was designed to measure the bending stiffness \((E_c I_c)\) of the cable under different tensions, and a second apparatus was designed to measure the axial stiffness \((E_c A_c)\) and axial damping ratio \((\xi)\). In Section 3.1, the bending stiffness measurement is described, the axial stiffness test is explained in section 3.2 and the experimental vibration tests on the cable and cable–beam system are detailed in Section 3.3.

3.1 Bending stiffness measurement

3.1.1 Theory

Figure 9 shows the schematic of the test set-up used to measure the bending stiffness of the cable under different tensions. The load \(Q\) is applied at the midpoint of the span length \(l\) of the cable causing the deflection \(\delta\) at the mid-span and the cable is under tension \(P\).

![Figure 9. Schematic of bending stiffness tests.](image-url)
By solving the partial differential equation for the static deflection of an Euler beam under tension, deflection of the mid-span of the beam is [30]:

\[
\delta = \frac{Q l^3}{48 E I_c} \left[ \frac{3\lambda \sinh \lambda + 6(1 - \cosh \lambda)}{\lambda^3 \sinh \lambda} \right]
\]

(17)

where \( l \) is the span length, \( E I_c \) is bending stiffness of the cable and \( \lambda \) is the following:

\[
\lambda = \sqrt{\frac{p l^2}{4 E I_c}}
\]

(18)

Equation (17) can be rewritten as [30]:

\[
\frac{4P\delta}{Ql} = 1 + \frac{2(1 - \cosh \lambda)}{\lambda \sinh \lambda}
\]

(19)

In order to obtain the bending stiffness \( EI \) of the cable, the ratio of \( \frac{4P\delta}{Ql} \) was calculated for each value of tension \( P \) since the deflection-force diagram was considered linear. Equation (19) was then solved numerically for each value of \( P \) to find \( \lambda \) and then \( E I_c \) was calculated using Equation (18). The bending stiffness of stranded cables is bounded by two flexural stiffnesses \( EI_{\text{max}} \) and \( EI_{\text{min}} \) [18]. The former corresponds to a solid cross section with no slippage of the strands and wires relative to each other, and the latter corresponds to unrestricted wires and strand slip [18]:

\[
EI_{\text{max}} = E \sum_{i=1}^{N_s} \frac{\pi d_{i}^4}{64} \left( \frac{\pi d_{i}^2}{4} y_{i}^2 \right) \approx E \frac{\pi d_{c}^4}{64}
\]

(20)

\[
EI_{\text{min}} = E \sum_{i=1}^{N_s} \frac{\pi d_{i}^4}{64}
\]

(21)

where \( d_{i} \) is the diameter of the \( i \)th strand, \( y_{i} \) is the distance from the centre of gravity of the \( i \)th strand to the centre of gravity of the conductor, \( d_{c} \) is the diameter of the cable and \( N_s \) is the number of strands in the cable. The bending stiffness values that were measured in the present study fall within this range for all tensions, and are closer to the minimum bending stiffness limit.

### 3.1.2 Experimental set-up for bending stiffness measurement

Figure 10 shows the set-up for bending stiffness measurement. A hydraulic cylinder was used to apply the load \( Q \) at the middle of the cable and a high accuracy linear variable displacement transducer OMEGA LD621-150 (LVDT) was attached to the cylinder to measure the deflection of the cable at mid-span (\( \delta \)). A threaded rod was used to adjust the tension in the cable and an inline tension/compression load cell STI 1,608,966 was used to measure cable tension \( P \). An S-type load cell INTERFACE SSMF was attached to the cylinder to measure the applied load.

The bending tests were carried out with cable tension values of 25, 100, 200 and 300 N. Table 1 presents the measured bending stiffness values. The theoretical limits of bending stiffness using Equations (20) and (21) are \( EI_{\text{max}} = 15.5632 \) and \( EI_{\text{min}} = 0.0306 \), respectively. As can be seen from Table 1, the bending stiffness values increase with tension.
3.2 Axial stiffness-damping measurement

3.2.1 Theory
Experimental characterization was carried out to measure the axial stiffness and damping of the cable. Axial stiffness of the cable is \( \frac{E_A}{l} \) (Equation (2)) and deflection is assumed to be in the linear elastic range governed by Hooke’s Law [31]:

\[
F = kx
\]

where \( F \) is the applied tension, \( k \) is the spring constant of the cable and \( x \) is the elongation caused by the tension \( F \). For a homogeneous solid cylinder, the spring constant can be calculated using the Young’s modulus of the material and the geometrical properties.
However, the $E_cA_c$ value for a stranded cable cannot be calculated because of the complicated axial deformation behaviour arising from the multiple wires and strands [29]. Previous attempts to obtain analytical axial and/or torsional stiffness theories for wire ropes and stranded cables can be found in [8, 31, 32, 33, 34], but alternatively the spring constant of the cable can be measured using single degree of freedom (SDOF) mass-spring-damper oscillator theory. By initiating free vibration of a known mass hanging from the cable, the stiffness and damping properties can be identified [35, 36]. Figure 11 shows the schematic of a SDOF system with cable as the spring.

The damped frequency $\omega_d = \frac{2\pi}{\tau_d}$ can be calculated by measuring the time period of the oscillation ($\tau_d$) and, after measuring damping ratio using logarithmic decrement, the following equation can be used to determine stiffness:

$$\sqrt{\frac{k}{M + \frac{m_c}{3}}} \sqrt{1 - \xi^2} = \omega_d$$  \hspace{1cm} (23)

where $M$ is the hung mass (Figure 11) and $m_c$ is the mass of the cable. The equivalent damping constant $c$ can be calculated using the damping ratio:

![Figure 11. Cable and mass as SDOF oscillator.](image-url)
The calculated $E_c A_c$ value and the damping constant were used in the bond graph model for axial stiffness and damping, respectively.

### 3.2.2 Experimental set-up

Figure 12 shows the schematic of the axial stiffness-damping measurement set-up.

A known mass (2.2 kg) was suspended from a measured length of the cable and an in-line load cell (Figure 12) was used to monitor the tension during the oscillation. In order to hang the mass from the end of the cable, a cylindrical adapter was used. Three sample lengths of 70 cm, 1 m and 1.3 m were used, and for each sample length, five impulses were applied to the sprung mass. Figure 13 shows the set-up used for measurement. The tension was recorded with 500 Hz sampling frequency.

The difference of the results obtained from the three different sample lengths was negligible and the average results obtained from three sample lengths were considered for the final result. Figure 14 shows a recorded tension obtained from a 1 m sample length after curve fitting. Tension time response in Figure 14 was used to obtain the parameters.

The axial stiffness of the cylindrical adapter used for hanging the cable had a small but non-negligible effect that was accounted for by treating it as a spring in series with the cable.

\[
\frac{1}{k_c} = \frac{1}{k} - \frac{1}{k_m}
\]

where $k_m$ is the cylindrical adapter axial stiffness $\frac{E_m A_m}{l_m}$ that can be calculated from material and geometric properties, $k$ is the calculated spring constant from Equation (23) and $k_c$ is the cable spring constant. The value of $k_c l_c$ is a constant for a given cable.
and independent of cable length. The measured values for axial stiffness and damping ratio are \((E_c A_c) = 221.78 \text{kN} \text{ and } \xi = 0.017\).

### 3.2 Vibration testing

In order to validate the numerical models for cable, beam and cable–beam systems, free and forced vibration tests were performed and the modal properties and time responses were obtained. The following sections present the procedure for vibration testing on the cable and cable–beam systems. Modal testing was also carried out on the cantilever beam, but this was sufficiently straightforward that only the results are presented in Section 4.
3.2.1 Cable vibration tests
Figure 15 presents the schematic of the vibration tests on the cable. Two rigid frames were secured to the floor. The right end of the cable was attached to a threaded rod to adjust the tension. An in-line load cell was used at the left end of the cable to measure the longitudinal tension. The test set-up, in-line load cell and the threaded rod are shown in Figure 16. Two types of vibration tests were performed on the cable: Experiment A and Experiment B. Experiment A (Figure 15(a)) is the modal testing for obtaining the natural frequencies of the cable in in-plane and out-of-plane directions. The modal tests were carried out with 25 N, 100 N, 200 N and 300 N cable tensions. The hammer hits were applied one quarter of the way along the cable in in-plane and out-of-plane directions in two separate tests and the accelerometers were placed at the mid-span. In addition to the in-plane accelerometer shown in Figure 15(a), another accelerometer that is not shown in the figure was also used (perpendicular to the plane of Figure 15) to measure the out-of-plane vibration due to a hammer impact load in that direction. Each test was performed five times and the average result was considered for the analysis. The cable was supported with a pin connection at both ends and the span length between the rigid frames was 2.4 m.

Experiment B was a free vibration test (Figure 15(b)). In this test, a 1 kg weight was hung from the quarter span of the cable and by cutting the string attached to the weight, an initial displacement was created to cause free vibration of the cable. The in-plane vibration response at mid-span was measured by the in-plane accelerometer.

3.2.2 Pole-line vibration tests
In order to validate the developed numerical model for the cable–beam system, an experimental lab-scale pole-line set-up was built. Figure 17 shows the schematic of experimental tests. Two U-bolts were used in the connection point of the cable and beam. The pole-line and U-bolts are shown in Figure 18. Similar to tests for the cable-only system, two types of vibration tests were performed: Experiment A and Experiment B. Experiment A is shown in the top part of Figure 17. Experiment A was impact testing (forced vibration). Two accelerometers positioned 0.45 m and 0.65 m from the clamped

![Figure 14](image-url) Tension record for the mass-spring-damper system using 1 m sample length.
end of the beam and one accelerometer positioned at mid-span of the left cable were used to measure the response. The position of accelerometers was kept fixed and hammer impacts were made at four different locations. Hit location 1 is 60 cm from the clamped end of the beam, hit location 2 is at the quarter span length of the right cable, hit location 3 is at the mid-span of the right cable and hit location 4 is at the quarter span length of the left cable. Tests were performed with a Brul & Kjaer 8205–002 impact hammer, 4507 B 004 70-g accelerometers, and a National Instruments NI USB-4432 power supply and signal conditioner. The software ModalView [37] was used to analyse the data.

An out-of-plane impulse (perpendicular to the plane of Figure 17) was also applied to excite the beam and an out-of-plane accelerometer in place of Accelerometer 2 was used to measure the response.

In Experiment B (Figure 17(b)), a mass (200 g) was hung at mid-span of the left section of the cable through a string to cause a free vibration (similar to Experiment B for cable-only test). An accelerometer positioned 0.6 m from the clamped end was used to measure the time response.

4. Elementary verification

Simulation models of cable, beam and the coupled cable–beam system were verified with experiment and finite element results in the frequency and time domains. The 20sim® software was used to implement the bond graph models. In addition to a bond graph graphical user interface, the software has a frequency domain toolbox to numerically generate transfer functions and eigenvalues.
4.1 Beam model verification

The hammer impulse was simulated in the 20th segment of the beam bond graph model corresponding to 50 cm from the clamped end and the vibration response was obtained in the 25th segment corresponding to 62.5 cm from the clamp end. Hammer impulse was simulated by applying a pulse signal using a modulated effort source, MSe, to the centre of gravity inertial velocity 1-junction, $\dot{V}_{Gi}$ (Figure 4). Natural frequencies of the
cantilever beam were obtained using 20sim® and the results are compared with finite element analysis and experiments in Table 2.
Table 2. Natural frequency comparison between numerical and experimental results: beam only.

| Mode Number | 1      | 2      | 3      | 4      | 5      |
|-------------|--------|--------|--------|--------|--------|
| Exp (Hz)    | 2.24   | 14.70  | 41.57  | 84.22  | 138.2  |
| BG (Hz)     | 2.34   | 15.79  | 45.40  | 90.56  | 151.65 |
| FE (Hz)     | 2.41   | 15.38  | 43.57  | 86.14  | 143.36 |

4.2 Cable model verification

Before modal testing, the cable model was simulated in the 20sim® software environment. An impulse force was applied at the quarter-span location, and the time response was obtained at mid-span as in the experimental test procedure (Figure 4). The Vode–Adams explicit integrator [38] was used for integration and simulation time was around 15 seconds for 10 seconds of time response simulation. Table 3 presents the comparison of in-plane natural frequencies of the cable with different tensions and Table 4 compares the out-of-plane natural frequencies of the cable.

A free vibration test of the cable was simulated in the cable numerical model. Experimental and simulated free vibration results (Figure 15(b)) for mid-span in-plane acceleration response are compared in Figure 19 for a cable with 200 N tension. While agreement is generally good, there is some discrepancy in decay rate which could be due to the lumped segment modelling of damping in the bond graph model and difficulty in replicating the applied initial displacement caused by the hung mass. It is also possible that there are unmodelled lightly damped dynamics related to the support structure.

5. Coupled system simulation results and validation

The numerical (FE and bond graph) and experimental results for the coupled cable–beam system are presented in this section. In Section 5.1, the natural frequency results from different methods are presented and compared with each other, Section 5.2 presents the mode shapes of the coupled system obtained from FEM and Section 5.3 is the time-domain response analysis.

5.1 Frequency analysis

Table 5 presents the comparison of bond graph, finite element and experimental natural frequencies of the beam–cable system. There is a good overall agreement between the numerical and experimental results, suggesting that the bond graph model has enough accuracy to capture the dominant beam–cable system dynamics. Some of the lower modes were not revealed in the bond graph software’s numerical calculation of natural frequencies. Some discrepancy in the frequency values is attributed to the difficulty in exactly replicating the boundary conditions during the experiments.

5.2 Mode shapes

The mode shapes of the coupled cable–beam system are obtained from the FE model. Figure 20 shows the first eight in-plane mode shapes of the coupled cable–beam system.
with 200 N tension in the cable. As can be seen, some of the modes are beam dominated modes (BD), some are cable-dominated modes (CD) and some are hybrid (H) modes (both beam and cable dynamically involved).
Table 5. Natural frequency comparison between numerical and experimental results: beam–cable system.

| Mode number | 1    | 2    | 3    | 4    | 5    | 6    |
|-------------|------|------|------|------|------|------|
| 25 N Test (Hz) | 5.299 | 6.730 | 11.660 | 12.958 | 14.897 | –    |
| 25 N BG (Hz)   | 6.199 | 6.430 | 10.159 | –     | 14.092 | –    |
| 25 N FE (Hz)   | 5.622 | 6.202 | 11.652 | 13.030 | 14.185 | –    |
| 100 N Test (Hz)| 8.610 | 12.670 | 16.890 | 21.850 | 24.320 | 36.087|
| 100 N BG (Hz)  | 6.299 | 12.190 | 15.124 | –     | 24.932 | 37.884|
| 100 N FE (Hz)  | 10.521 | 11.439 | 14.283 | 22.651 | 24.171 | 35.542|
| 200 N Test (Hz)| 13.490 | 14.537 | 16.069 | 26.111 | 32.404 | 39.604|
| 200 N BG (Hz)  | –     | 15.788 | 16.740 | 27.410 | 33.887 | 41.185|
| 200 N FE (Hz)  | 13.730 | 15.417 | 16.136 | 30.561 | 32.362 | 41.347|
| 300 N Test (Hz)| 14.043 | 15.263 | 18.524 | 36.715 | 40.654 | 44.576|
| 300 N BG (Hz)  | –     | 17.094 | 20.705 | 37.719 | 40.825 | 47.188|
| 300 N FE (Hz)  | 14.765 | 18.395 | 19.260 | 36.716 | 39.172 | 43.260|

Figure 20. First eight in-plane mode shapes of the coupled cable–beam system. (a) Mode 1 (BD), (b) Mode 2 (H), (c) Mode 3 (H), (d) Mode 4 (CD), (e) Mode 5 (H), (f) Mode 6 (H), (g) Mode 7 (H), (h) Mode 8 (CD).
In Experiment A and B of the cable–beam system experiments (Figure 17), time response of the beam was measured using accelerometers in forced and free vibration, respectively. To simulate a hammer strike (Experiment A), an impulse signal was applied to a modulated effort source (MSe) element in one segment of the beam model. The amplitude and period of the impulse was chosen to match the energy of the modal testing hammer impact. This value was determined by estimating the area under the force versus time plot from the impact hammer load cell. Experiment B was replicated by applying a step input force equal to the hanging mass weight at mid-span of the left cable. Removing the force simulated cutting the cable. Figure 21 and Figure 22 present the

5.3 Time-domain analysis

Figure 21. Acceleration time response due to impact with 300 N tension –simulation and test. (a) In-plane, (b) out-of-plane.
The acceleration time response of the beam due to impact load, obtained from simulation and test, when the cable has 300 N tension and 200 N tension, respectively. The beginning of the time responses is zoomed to better show the two curves during the initial decay transient. As can be seen, numerical and experimental time responses are in good agreement with each other and show the same frequency content and amplitude and decay rate. Some discrepancies in frequency content are attributed to the difficulty of precisely simulating the hammer strike time series, and to the presence of unmodelled dynamics in the laboratory support structure and load cell connection joint.

Experiment B in Figure 17 is also simulated using the bond graph model and the time responses are presented in Figure 23. The results show that the numerical model can give accurate time responses under free and forced vibration. In Figure 23, in-plane
acceleration response obtained from simulation and test is shown with tensions of 100 and 200 N in the cable.

**Conclusions**

A three-dimensional numerical model based on the bond graph method was developed for vibration analysis of a coupled cable–beam system, which can be extended to a utility pole-conductor system. The cables were modelled considering the bending stiffness and sag due to self-weight, and the beam was modelled as a cantilevered beam. Experimental parametrization for a stranded cable was carried out, measuring the bending stiffness under different tensions and the axial stiffness and damping using newly designed apparatus. The bending stiffness was shown to increase with an increase in tension. Experimental free and forced vibration tests were performed on the substructures of cable and beam and on the lab-scale cable–beam coupled system. The tests were simulated using the numerical models and comparison of the results showed good agreement. The natural frequencies of the system were obtained, and the bond graph results were validated by experimental and finite element results. It is also concluded that by connecting the cable to the beam in a cable–beam system, there emerge 'beam-
dominated', 'cable-dominated', and 'hybrid' modes. It is concluded that the numerical models can be used to simulate the modal testing of any simple structure connected to cables. The authors will apply the modelling method to electricity transmission line systems to facilitate research into non-destructive evaluation of wood poles.

**Disclosure statement**

No potential conflict of interest was reported by the authors.

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