THE PROBLEM OF MISSING TERMS IN TERM BY TERM INTEGRATION INVOLVING DIVERGENT INTEGRALS

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Abstract. Term by term integration may lead to divergent integrals, and naive evaluation of them by means of, say, analytic continuation or by regularization or by the finite part integral may lead to missing terms. Here, under certain analyticity condition, the problem of missing terms for the incomplete Stieltjes transform, \( \int_0^\infty f(x)(\omega + x)^{-1} \, dx \), and the Stieltjes transform itself, \( \int_0^\infty f(x)(\omega + x)^{-1} \, dx \), is resolved by lifting the integration in the complex plane. It is shown that the missing terms arise from the singularities of the complex valued function \( f(z)(\omega + z)^{-1} \), with the divergent integrals arising from term by term integration interpreted as finite part integrals.

1. Introduction

Term by term integration is one of the frequently used methods in evaluating integrals and in constructing asymptotic expansions of integrals. It typically involves expanding the integrand or a factor of it and then interchanging the order of summation and integration. Of course the order cannot be arbitrarily interchanged without satisfying certain uniformity conditions. But we, nevertheless, often chivalrously perform the interchange at the cost of introducing divergent integrals in an otherwise well defined integral. We often see this happen in asymptotic evaluation of integrals where the first step in constructing an asymptotic expansion is by deriving a formal expansion, say, by means of an appropriate expansion of the integrand followed by a term by term integration \cite{1}. The emergence of divergent integrals in such formal manipulation calls for their proper interpretation and treatment. The interpretation of divergent integral is made complicated by the fact that a divergent integral may be assigned different values corresponding to different ways of interpreting the divergent integral. A divergent integral can be assigned meaningful value by means of analytic continuation \cite{2}, Caesaro limits \cite{3}, regularization methods \cite{4}, distribution theory \cite{5}, or finite part integration \cite{6} or still by some other means \cite{7, 8, 9}. These generally assign different values to the divergent integral. A naive application of any of these interpretations of a divergent integral in term by term integration can lead to misleading results, sometimes yielding the correct value but more often correctly reproducing only some of the terms of the correct results but completely missing out some group of terms.

The problem with naive term by term integration is exemplified by the asymptotic evaluation of the Stieltjes transform \( \int_0^\infty h(x)(\omega + x)^{-1} \, dx \). The canonical example is given by the integral \cite{1}

\[
(1) \quad \int_0^\infty \frac{1}{\sqrt{1 + x (\omega + x)}} \, dx.
\]

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Inserting the binomial expansion of \((1 + x)^{-1/3}\) at infinity and performing term by term integration lead to the infinite series
\[
\int_0^\infty \frac{1}{\sqrt[3]{1+x}(\omega + x)} \, dx \sim \sum_{s=0}^\infty \left(-\frac{1}{3}\right)^s \int_0^\infty \frac{x^{-s-1/3}}{((\omega + x))} \, dx.
\]

Only the \(j = 0\) term exists and the rest of the integrals do not exist due to the non-integrable singularity at the origin. The question now arises as to how the divergent integrals should be interpreted.

The divergent integrals can be interpreted in several ways. One way is to interpret them via analytic continuation. The idea behind “analytic continuation is to match the divergent function of interest to an analytic function in a region where the original function is well behaved” \[2\]. For instance, we have the integral
\[
\int_0^\infty x^{-\mu} \frac{\pi \csc \pi \mu}{(\omega + x)} \, dx = \frac{\pi}{\omega^{\mu}}, \quad 0 < \mu < 1.
\]

We can assign the divergent integral the value equal to the value of the analytic continuation of the right hand side of equation \[3\] for \(\mu = s + 1/3\). Applying this interpretation, term by term integration leads to the asymptotic expansion
\[
\sum_{s=0}^\infty (-1)^s \frac{1}{\omega^{s+1/3}}.
\]

The same result is obtained when the divergent integral is interpreted as a finite part integral. However, the given integral \[1\] can be evaluated explicitly and the result expanded for \(\omega > 1\). The result is given by
\[
\sum_{s=0}^\infty (-1)^s \frac{1}{\omega^{s+1/3}} - \sum_{s=1}^\infty \frac{3^s(s-1)!}{2 \cdot 5 \cdot \ldots \cdot (3s-1)} \frac{1}{\omega^s}, \quad \omega > 1.
\]

Comparing expressions \[4\] and \[5\] we find that expression \[4\] correctly reproduces the first group of terms of \[5\] but completely misses out the second group. This demonstrates how naive term by term integration involving divergent integrals can lead to missing terms. This problem for the Stieltjes transform was resolved by McClure and Wong by interpreting the singular factors in the integrand as distributions and the divergent integral as a linear functional \[5, 10\].

In this paper we give term by term integration another look and the interpretation of divergent integrals that accompanies in such operation without appealing to distribution theory. Here we will consider the incomplete Stieltjes transform
\[
S_a(\omega) = \int_0^a \frac{h(x)}{\omega + x} \, dx, \quad \omega > 0, \quad a > 0,
\]
from which the well-known Stieltjes transform can be recovered in the limit as \(a \to \infty\), provided the limit exists. We will consider two expansions of the integrand leading to divergent integrals upon term by term integration, which are
\[
\frac{1}{\omega + x} = \sum_{j=0}^\infty (-1)^j \frac{\omega^j}{x^{j+1}}
\]
\[
h(x) = \sum_{j=0}^\infty \frac{c_j}{x^{j+\alpha}}
\]
Expansion 7 is appropriate for term by term integration for small $\omega$ or at the origin; and expansion 8 is for term by term integration for large $\omega$ or at infinity. Both expansions lead to the family of divergent integrals

$$\int_0^a f(x) \frac{1}{x^{m+\nu}} \, dx, \quad m = 1, 2, \ldots, \quad 0 \leq \nu < 1, \quad a > 0.$$  

(9)

for some function $f(x)$ with $f(0) \neq 0$. The later expansion 8 has been the area of asymptotics and the area of application of the distributional approach due to McClure and Wong [5, 10]. On the other hand, it is for the first time that term by term integration involving divergent integrals arising from expansion 7 is dealt with.

In this paper we will approach the problem of missing terms by lifting the integration in the complex plane. Our solution to the problem is motivated by our recent work that saw the representation of the Cauchy principal value and the Hadamard finite part of the generalized principal value by Fox [11] as complex contour integrals [12]. Our results there hint that the emergent divergent integrals 9 can be interpreted as a finite part integral provided an appropriate contour integral representation of the finite part is obtained, and provided the original integral is likewise represented as a contour integral consistent with the contour integral representation of the finite part integral. We will pursue that hint here. To do so we will obtain the contour integral representation of the finite part of the divergent integrals 9 from which we read off the required consistent contour integral representation of the incomplete Stieltjes transform. Replacing the integral 9 with its equivalent contour integral representation in the complex plane and performing the term by term integration over this representation will lead us to uniquely identify the divergent integrals that arise as finite part integrals. Moreover, we will find that the missing terms emerge as contributions coming from the complex singularities of the function $h(z)(\omega + z)^{-1}$ (which is the extension of the integrand $h(x)(\omega + x)^{-1}$ in integral 6 in the complex plane).

Here we will primarily treat term by term integration arising from expansion 7 and show that the finite part integral in complex contour integral representation leads to an analytic (exact) evaluation of the incomplete Stieltjes transform and the Stieltjes transform in positive powers of $\omega$. We will find, in particular, that the missing term arise from the simple pole of $(\omega + z)^{-1}$. On the other hand, we will cursorily consider term by term integration arising from expansion 8 contenting ourselves with an application of our method to the specific example of a generalization of the canonical example, and demonstrating that our approach reproduces a known result of the distributional approach to asymptotics. We will find that, for the given example, the missing terms arise from the branch point of $h(z)$. We will treat elsewhere an in depth comparison of our approach to that of the distributional approach of McClure and Wong. Overall, lifting the problem of term by term integration in the complex plane by way of the finite part integral offers a novel method in evaluating analytically and asymptotically integral transforms.

The paper is organized as follows. In Section 2 we obtain the contour integral representation of the finite part of the divergent integrals 9 for pole ($\nu = 0$) and branch point ($\nu \neq 0$) singularities. In Section 3 we perform term by term integration at the origin corresponding to the expansion 7 for both cases of singularities involving entire integrands; we derive the known series expansions of the exponential function and the incomplete gamma function as examples. In Section 4 we
consider expansion at the origin for holomorphic integrands. In Section 5 we perform term by term integration at infinity corresponding to the expansion 8 for a specific class of integrals, the canonical example 1 is a special case of which.

2. Contour integral representation of finite part integrals of divergent integrals with singularities at the origin

We now obtain the contour integral representation of the finite part of the divergent integral 9 under certain conditions to be detailed below. Assuming that the function $f(x)$ is locally integrable in the interval $[0,a]$, the finite part integral is obtained by replacing the lower limit of integration by some positive $\epsilon < a$. The resulting integral is then grouped in two terms,

$$\int_{\epsilon}^{a} \frac{f(x)}{x^{m+\nu}} \, dx = C_\epsilon + D_\epsilon$$

where $C_\epsilon$ is the group of terms that possess a finite value in the limit as $\epsilon \to 0$, and $D_\epsilon$ is the group of terms that diverge in the same limit. The finite part of the divergent integral is obtained by dropping the diverging term $D_\epsilon$ and assigning the limit of $C_\epsilon$ as the value of the divergent integral, otherwise known as the finite part integral (FPI) [13, 14], that is

$$\text{FPI} \int_{0}^{a} \frac{f(x)}{x^{m+\nu}} \, dx = \lim_{\epsilon \to 0} C_\epsilon.$$

Equivalently the finite part integral can expressed in the form

$$\text{FPI} \int_{0}^{a} \frac{f(x)}{x^{m+\nu}} \, dx = \lim_{\epsilon \to 0} \left[ \int_{\epsilon}^{a} \frac{f(x)}{x^{m+\nu}} \, dx - D_\epsilon \right].$$

These different representations of the finite part integral will serve us separate purposes. We will use equation 11 to compute explicitly the finite part integral later; while we will use equation 12 to establish the desired contour integral representation of the finite part integral.

Now given the function $f(x)$, the complex extension of $f(x)$ is the function, $f(z)$, obtained from $f(x)$ itself by replacing the real variable $x$ with the complex variable $z$. Here, as in [12], we assume that the function $f(x)$ has a complex extension, $f(z)$, that is analytic in some region of the complex plane containing the strip $[0,a]$. We will denote the finite part of the divergent integral 9 under this assumption by $\int_{0}^{a}$.

We distinguish this with the more familiar notation $\text{FPI} \int_{0}^{a}$ because the function $f(x)$ does not necessarily have an analytic extension in the complex plane for the finite part integral to exist. However, this is not much of a restriction because the functions $f(x)$ involved in practice have analytic extensions.

The contour integral representation of the divergent integral 9 will depend on the value of $\nu$. The cases $\nu = 0$ (pole singularity) and $\nu \neq 0$ (branch point singularity) are distinct and have to be considered separately. The representation of the finite part integral given by equation 11 or 12 by a contour integral in the complex plane means that the FPI is the value of an absolutely convergent integral. This lifts the somewhat vague meaning of the finite part, and endows it with an unambiguous interpretation as a legitimate integral itself.
2.1. Pole singularity.

**Proposition 1.** Let \( f(x) \) be \((n+1)\) continuously differentiable in the interval \([0,a]\) and \( M = \sup\{|f^{(n+1)}(x)|, x \in [0,a]\} < \infty \). If \( f(0) \neq 0 \), then we have the finite part integral

\[
\text{FPI} \int_0^a \frac{f(x)}{x^{n+1}} \, dx = \lim_{\epsilon \to 0^+} \left[ \int_\epsilon^a \frac{f(x)}{x^{n+1}} - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!(n-k)} \frac{1}{\epsilon^{n-k}} + \frac{f^{(n)}(0)}{n!} \ln \epsilon \right],
\]

for all \( n = 0, 1, 2, \ldots \).

**Proof.** For some \( \epsilon \in (0, a) \) we consider the convergent integral \( \int_\epsilon^a f(x)x^{-(n+1)} \, dx \). By the differentiability of \( f(x) \) up to the \((n+1)\)-th order we have the expansion about \( x = 0 \),

\[
f(x) = \sum_{k=0}^n \frac{f^{(k)}(0)}{k!} x^k + R_{n+1}(x).
\]

Inserting this expansion and performing the integral yields

\[
\int_\epsilon^a f(x)x^{-(n+1)} \, dx = -\sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!(n-k)} \left( \frac{1}{\epsilon^{n-k}} - \frac{1}{\epsilon^{n-k}} \right) + \frac{f^{(n)}(0)}{n!} \ln \epsilon + \int_\epsilon^a \frac{R_{n+1}(x)}{x^{n+1}} \, dx.
\]

Given that \(|R_{n+1}(x)| \leq M|x|^{n+1}/(n+1)!\), for some constant \( M > 0 \), we obtain the following bound for the integral over the remainder term,

\[
\left| \int_\epsilon^a \frac{R_{n+1}(x)}{x^{n+1}} \, dx \right| \leq \int_\epsilon^a \frac{|R_{n+1}(x)|}{x^{n+1}} \, dx \leq \frac{M}{(n+1)!} (a - \epsilon) \leq \frac{Ma}{(n+1)!}.
\]

Thus the remainder integral exists in the limit as \( \epsilon \to 0 \). Collecting all the terms that diverge individually as \( \epsilon \to 0 \) in one side, we have

\[
\int_\epsilon^a f(x)x^{-(n+1)} \, dx - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!(n-k)} \frac{1}{\epsilon^{n-k}} + \frac{f^{(n)}(0)}{n!} \ln \epsilon
\]

\[
= -\sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!(n-k)} \frac{1}{\epsilon^{n-k}} + \frac{f^{(n)}(0)}{n!} \ln \epsilon + \int_\epsilon^a \frac{R_{n+1}(x)}{x^{n+1}} \, dx.
\]

The limit of the right hand side of the equation [17] exists as \( \epsilon \to 0 \), and so must be the left hand side in the same limit. Finally taking the limit in equation [17] as \( \epsilon \to 0 \) yields the finite part integral [13]. \( \square \)

The second and third terms in the left hand side of equation [17] are exactly the diverging terms that have to be removed from the diverging first term to obtain a finite limit, or the diverging terms that have to be dropped to obtain the Hadamard finite of the integral.

**Theorem 1.** Let the complex extension, \( f(z) \), of \( f(x) \) be analytic in some neighborhood of the interval \([0,a]\). If \( f(0) \neq 0 \), then

\[
\int_0^a \frac{f(x)}{x^{n+1}} \, dx = \frac{1}{2\pi i} \int_C \frac{f(z)}{z^{n+1}} (\log z - \pi i) \, dz, \ n = 0, 1, \ldots
\]
where log $z$ is the complex logarithm whose branch cut is the positive real axis and $C$ is the contour straddling the branch cut of $\log z$ starting from $a$ and ending at $a$ itself, as depicted in Figure-1.

**Proof.** Deform the contour $C$ into the contour $C'$ shown in Figure-1 and evaluate the integral along that contour. Then

$$
\int_C f(z) \frac{\log z}{z^{n+1}} \, dz = \int_a^a f(x) \frac{\log x}{x^{n+1}} \, dx + \int_\epsilon^\epsilon f(z) \frac{\log (x e^{2\pi i})}{x^{n+1}} \, dx
$$

where the first term comes from the upper part of the contour, the second term from the small circle about the origin, and the third term from the lower part of the contour. With $\log(x e^{2\pi i}) = \ln x + 2\pi i$, equation 19 simplifies to

$$
\int_C f(z) \frac{\log z}{z^{n+1}} \, dz = 2\pi i \int_a^a f(x) \frac{\log x}{x^{n+1}} \, dx + \int_\epsilon^\epsilon f(z) \frac{\log z}{z^{n+1}} \, dz.
$$

We now evaluate the integral along the small circle, the second term in equation 20. Since, by assumption, $f(z)$ is analytic at $z = 0$, we have the expansion

$$
f(z) = \sum_{k=0}^n \frac{f^{(k)}(0)}{k!} z^k + O(z^{n+1}), \quad n = 0, 1, \ldots.
$$

Substituting this back into the integral and using the parametrization $z = e^{i\theta}$, $0 \leq \theta < 2\pi$, we obtain

$$
\int_\epsilon^\epsilon f(z) \frac{\log z}{z^{n+1}} \, dz = \sum_{k=0}^n \frac{f^{(k)}(0)}{k!} \frac{1}{\epsilon^{n-k}} \left[ i \ln \epsilon \int_0^{2\pi} e^{-i(n-k)\theta} d\theta - \int_0^{2\pi} e^{-i(n-k)\theta} d\theta \right] + O(\epsilon),
$$

where we have used $\log(e^{i\theta}) = \ln \epsilon + i\theta$. The integrals are evaluated with the following integrals

$$
\int_0^{2\pi} e^{-i(n-k)\theta} d\theta = \begin{cases} 0 & , n \neq k \\ 2\pi & , n = k \end{cases},
$$

Figure 1. The contour of integration. The contour $C$ does not enclose any pole of $f(z)$. 
Substituting the above integrals in equation 20, the integral along the entire contour $C$ assume the form

$$\int_C f(z) \log z z^{m+1} \, dz = 2\pi i \left[ \int_\epsilon^{a-\epsilon} \frac{f(x)}{x^{n+1}} \, dx - \sum_{j=0}^{m-1} j! \frac{f^{(j)}(0)}{(n+j)!} \frac{1}{\epsilon^{n+j-1}} + \frac{f^{(m)}(0)}{m!} \right] - 2\pi^2 \frac{f^{(n)}(0)}{n!} + O(\epsilon)$$

(25)

We can already discern the Hadamard finite part from the group of first terms in the limit as $\epsilon \to 0$. To arrive at the final form, we use the Cauchy integral formula to rewrite the derivative term. Since the contour $C$ is closed, by the Cauchy integral formula we have

$$f^{(n)}(0) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{z^{n+1}} \, dz$$

(26)

We substitute equation 26 back into equation 25, divide the entire expression by $2\pi i$ and then take the limit as $\epsilon \to 0$ leads us to the result [18].

2.2. Branch Point Singularity.

**Proposition 2.** Let $f(x)$ be $m$ continuously differentiable in the interval $[0, a]$ and $M = \sup\{ |f^{(m)}(x)|, x \in [0, a] \} < \infty$. If $f(0) \neq 0$, then we have the finite part integral

$$\int_0^a \frac{f(x)}{x^{m+\nu}} \, dx = \lim_{\epsilon \to 0^+} \left[ \int_{\epsilon}^{a-\epsilon} \frac{f(x)}{x^{m+\nu}} \, dx - \sum_{j=0}^{m-1} j! \frac{f^{(j)}(0)}{(n+j)!} \frac{1}{\epsilon^{n+j-1}} \right]$$

(27)

for all $m = 1, 2, \ldots$ and $\nu \in (0, 1)$.

**Proof.** The term proportional to $x^m$ in the Taylor expansion of $f(x)$ leads to the integral $\int_\epsilon^{a-\epsilon} x^{-\nu} g(x) \, dx$ with some $g(x)$ integrable in $[0, a]$. This term has a finite limit as $\epsilon \to 0$. We then expand $f(x)$ about $x = 0$ up to the order $(m-1)$ and then proceed in the same manner as we have done in Theorem 1 to establish the finite part integral [27].

**Theorem 2.** Let the complex extension, $f(z)$, of $f(x)$ be analytic in some neighborhood of the interval $[0, a]$. If $f(0) \neq 0$, then

$$\int_0^a \frac{f(x)}{x^{m+\nu}} \, dx = \frac{1}{(e^{2\pi \nu i} - 1)} \int_C \frac{f(z)}{z^{m+\nu}} \, dz, \quad m = 1, 2, \ldots, \quad 0 < \nu < 1,$$

(28)

where the branch of $z^{-\nu}$ is such that it is positive on top of the positive real axis and the contour $C$ is the contour straddling the branch cut of $z^{-\nu}$ starting from $a$ and ending at $a$ itself, as depicted in Figure 1.

**Proof.** We proceed in the same manner as we did in Theorem 1 to establish the equality [28].
2.3. Example. Let us compute the finite part integral of the divergent integral
\[ \int_0^a x^{-n} \, dx \] for \( n = 1, 2, \ldots \), \( 0 < \nu < 1 \) and \( a > 0 \). For \( \epsilon > 0 \), we have
\[ \int_0^a \frac{1}{x^{n+\nu}} \, dx = -\frac{1}{(n+\nu-1)} a^{n+\nu-1} \frac{1}{(n+\nu-1} \epsilon^{n+\nu-1}. \]

The second term diverges as \( \epsilon \to 0 \) so we drop it, leaving only the first term to give the finite part integral
\[ \text{FPI} \int_0^a \frac{1}{x^{n+\nu}} \, dx = -\frac{1}{(n+\nu-1)} a^{n+\nu-1}. \]

This exactly the same result obtained using equation 27. To use the integral representation, we choose the unit circle centered at the origin as our contour of integration. With the parametrization \( z = a e^{i\theta} \), we obtain
\[ \text{FPI} \int_0^a \frac{1}{x^{n+\nu}} \, dx = \frac{1}{(e^{-2\pi i\nu} - 1)} \int_0^{2\pi} \frac{1}{a^{n+\nu} e^{(n+\nu)i\theta}} i a e^{i\theta} d\theta = -\frac{1}{(n+\nu-1)} a^{n+\nu-1}. \]

Any contour continuously deformable to the circle will yield the same value of the finite part integral.

2.4. Remark. It is imposed above that the function \( f(x) \) does not vanish at the origin so that the origin is necessarily non-integrable. However, it may happen that \( f(x) \) vanishes there, in particular, \( f(x) = x^m g(x) \), where \( g(0) \neq 0 \), for some positive integer \( m \). Then the integral \( \int_0^a f(x) x^{-m-\nu} \, dx \) exists for all \( 0 \leq \nu < 1 \), and its value is given by \( \int_0^a g(x) x^{-\nu} \, dx \). Provided the complex extension function \( g(z) \) has the desired analytic properties, it is straightforward to show that equations 18 and 28 yield the respective values \( \int_0^a g(x) \, dx \) (for \( \nu = 0 \)) and \( \int_0^a g(x) x^{-\nu} \, dx \) (for \( 0 < \nu < 1 \)). That is the finite part integral and the regular integral (e.g. Riemann integral) coincide when the given integral \( \int_0^a f(x) x^{-m-\nu} \, dx \) converges absolutely. This is expected as the finite part integral of an absolutely convergent integral is just the value of the integral itself, i.e. the integral has no diverging component. This observation allows us to remove the restriction \( f(0) \neq 0 \) and interpret all integrals that appear as finite part integral wherever appropriate.

3. Term by term integration at the origin for functions with entire extensions

Here and in the succeeding section, we will consider the term by term integration of the incomplete Stieltjes transform
\[ \int_0^a \frac{x^{-\nu} f(x)}{\omega + x} \, dx, \quad 0 \leq \nu < 1, \]
in the neighborhood of \( \omega = 0 \). The overarching idea behind our approach to the missing terms is to obtain a contour integral representation of the integral 31 that is consistent with the contour integral representation of the finite part integral. By consistent we mean that the same contour used in the contour integral representation of the finite part integral is the same contour used in the contour integral representation of the incomplete Stieltjes transform. The representation will be dictated by the value of \( \nu \), either \( \nu = 0 \) or \( \nu \neq 0 \), and the analytic behavior...
of $f(z)$ in the complex plane. In this section, we will consider the case in which $f(x)$ has an entire complex extension $f(z)$.

3.1. Case 1: $\nu = 0$

**Lemma 1.** Let $g(x)$ be integrable in the interval $[0,a]$ and that its complex extension $g(z)$ is analytic in a region containing the interval $[0,a]$ in its interior and holomorphic elsewhere. Then

$$\int_0^a g(x) \, dx = \frac{1}{2\pi i} \int_C g(z) \log z \, dz - \sum_k \text{Res} \{ \log z \, g(z) \}_{z_k},$$

where the branch cut of $\log z$ is chosen to be the positive real axis and $C$ is the contour straddling the branch cut of $\log z$ starting from $a$ and ending at $a$ itself, as depicted in Figure-2, and the $z_k$’s are the poles of $g(z)$ enclosed by $C$, with no pole of $g(z)$ lying along $C$.

**Proof.** Given the contour integral $\int_C g(z) \log z \, dz$, deform the contour $C$ into the branch cut by means of the contour $C'$ in Figure-2, eventually taking the limit $\epsilon \to 0$. \qed

We will find that the first term of equation 32 yields the result of naive term by term integration; and the second term, the contribution coming from the poles, is the origin of the missing terms. In general, the integral representation of the incomplete Stieltjes transform relevant to us is of the form similar to equation 32. In particular, the representation will have a contour integral term which accounts for the term by term integration, and a term arising from contributions of the poles or branch points of the function $g(z)$, which are the missing terms. Here and in the following section, we will consider missing terms coming pole singularities; later we will give an example of missing terms arising from a branch point.
Theorem 3. Let the complex extension, \( f(z) \), of the function \( f(x) \) be entire, then

\[
\int_0^a \frac{f(x)}{\omega + x} \, dx = \sum_{j=0}^{\infty} (-1)^j \omega^j \int_0^a \frac{f(x)}{x^{j+1}} \, dx - f(-\omega) \ln \omega, \quad \omega < a.
\]  

Proof. With \( f(z) \) entire, the complex valued function \( f(z)(\omega + z)^{-1} \) has a simple pole at \( z = -\omega \). Then equation (32) leads to the following contour integral representation of the incomplete Stieltjes transform,

\[
\int_0^a \frac{f(x)}{\omega + x} \, dx = -f(-\omega) (\ln \omega + i\pi) + \frac{1}{2\pi i} \int_C \frac{f(z)}{\omega + z} \, dz.
\]

For fixed \( \omega \) and \( z \neq -\omega \), we have the expansion

\[
\frac{1}{\omega + z} = \sum_{j=0}^{n-1} \frac{(-1)^j \omega^j}{z^{j+1}} + \frac{(-1)^n \omega^n}{z^n(\omega + z)}, \quad n = 1, 2, \ldots
\]

Substituting this expansion back into the integral in equation (34) yields

\[
\int_0^a \frac{f(x)}{\omega + x} \, dx = -f(-\omega) (\ln \omega + i\pi) + \sum_{j=0}^{n-1} (-1)^j \omega^j \frac{1}{2\pi i} \int_C \frac{f(z) \log z}{z^{j+1}} \, dz
\]

\[
+ (-1)^n \omega^n \frac{1}{2\pi i} \int_C \frac{f(z) \log z}{z^n(\omega + z)} \, dz.
\]  

(36)

We now make the trivial replacement \( \log z = \log z - i\pi + i\pi \) in the second term of equation (36)

\[
\int_0^a \frac{f(x)}{\omega + x} \, dx = -f(-\omega) (\ln \omega + i\pi) + \sum_{j=0}^{n-1} (-1)^n \omega^j \frac{1}{2\pi i} \int_C \frac{f(z) \log z}{z^{j+1}} \, dz
\]

\[
+ i\pi \frac{1}{2\pi i} \int_C f(z) \left[ \sum_{j=0}^{n-1} (-1)^j \frac{\omega^j}{z^{j+1}} \right] \, dz + (-1)^n \omega^n \frac{1}{2\pi i} \int_C \frac{f(z) \log z}{z^n(\omega + z)} \, dz.
\]

We recognize that the contour integral in the second term is just the finite part integral of the divergent integral \( \int_0^a f(x)x^{-j-1} \, dx \). We replace the summation in the third term with equation (35). A residue term will emerge in the process which will cancel out the imaginary term in the first term. We arrive at the expression

\[
\int_0^a \frac{f(x)}{\omega + x} \, dx = -f(-\omega) \ln \omega + \sum_{j=0}^{n-1} (-1)^n \omega^j \int_0^a \frac{f(x)}{x^{j+1}} \, dx + R_n(\omega)
\]

where the remainder term is given by

\[
R_n(\omega) = (-1)^n \omega^n \frac{1}{2\pi i} \int_C \frac{f(z)}{z^n(\omega + z)} \, dz.
\]

Despite appearances, the remainder is not a finite part integral because of the presence of the pole enclosed by the contour. We estimate the remainder by deforming the contour into a circle centered at the origin with radius \( a \). Then we have the bound

\[
|R_n(\omega)| \leq \omega^n \int_0^{2\pi} \frac{|f(ae^{i\theta})|}{|\omega + ae^{i\theta}|} (\ln a + |\theta - \pi|) \, d\theta
\]

\[
\leq \left( \frac{\omega}{a} \right)^n \frac{M(f,a)(\ln a + \pi^2)}{|a - \omega|}
\]

(40)
in which $M(f, a)$ is the maximum modulo of $f(z)$ along the contour of integration. When $\omega < a$, the remainder term vanishes in the limit as $n \to \infty$. Then equation \ref{33} follows.

3.1.1. Example. Let us reproduce a known series expansion of the exponential integral function, $E_1(z)$, which assumes the integral representation \cite{15} p150,#6.2.2

$$E_1(z) = e^{-z} \int_0^\infty \frac{e^{-x}}{z + x} \, dx, \quad |\operatorname{Arg}z| < \pi.$$  

By inspection, the integral involved in the representation is a Stieltjes transform of the function $f(x) = e^{-x}$. We restrict ourselves to $z = \omega > 0$ and just effect an analytic continuation in the complex plane if it is desired. Since the complex extension, $e^{-z}$, of the real valued function $e^{-x}$ is entire, it admits an expansion given by equation \ref{33} which we now derive.

To proceed we cut the upper limit of integration to some finite $a > 0$ and then eventually take the limit $a \to \infty$. The relevant finite part integrals to compute are $\int_0^a e^{-x} x^{-j-1} \, dx$. For some $\epsilon > 0$, with $0 < \epsilon < a$, we obtain, after term by term integration through the series expansion of $e^{-x}$, the integral

$$\int_0^a e^{-x} x^{-j+1} \, dx = -\sum_{k=0}^{n-1} \frac{(-1)^k}{k!} \left( \frac{1}{a^{j-k}} - \frac{1}{\epsilon^{j-k}} \right) + \frac{(-1)^j}{j!} (\ln a - \ln \epsilon)$$

$$+ \sum_{k=j+1}^{\infty} \frac{(-1)^k}{k!} \frac{1}{(k-j)} \left( (a^{k-j} - \epsilon^{k-j}) \right).$$

The finite part of the divergent integral $\int_0^a e^{-x} x^{-j-1} \, dx$ is obtained by dropping all terms that diverge as $\epsilon \to 0$ and keeping only the terms with a well defined limit in the said limit in equation \ref{42}. Then we obtain the finite part integral

$$\int_0^a e^{-x} x^{-j+1} \, dx = -\sum_{k=0}^{j-1} \frac{(-1)^k}{k!} \frac{1}{a^{j-k}}$$

$$+ \frac{(-1)^j}{j!} \ln a + \frac{(-1)^j+1}{(j+1)!} a \cdot 2F_2(1, 1; j + 2; -a),$$

where the contributing infinite series in equation \ref{42} has been summed in terms of the hypergeometric function $2F_2(a_1, a_2; b_1, b_2; z)$.

We now take the limit as $a$ approaches infinity. Only the last two terms contribute in the limit. The hypergeometric function involved has a double pole, so that its relevant asymptotic expansion needed to compute the limit is given by \cite{16}

$$2F_2(a_1, a_1; b_1, b_2; z) = \frac{\Gamma(b_1) \Gamma(b_2)}{\Gamma(a_1)^2} e^z (1 + O(z^{-1})) z^{a_1-b_1-b_2}$$

$$+ \frac{\Gamma(b_1) \Gamma(b_2)}{\Gamma(a_1) \Gamma(b_1-a_2) \Gamma(b_2-a_1)} \left[ \log(-z) (1 + O(z^{-1})) \right.$$

$$\left. - (\psi(b_1-a_1) + \psi(b_2-a_1) + \psi(a_1) + 2\gamma) (1 + O(z^{-1})) \right], \quad |z| \to \infty,$$}

where $\psi(z)$ is the logarithmic derivative of the gamma function and $\gamma$ is the Euler constant, with $\psi(1) = -\gamma$. Only the second term contributes in the limit.
Substituting the appropriate values of the parameters of the hypergeometric function and implementing the limit, we obtain the desired finite part integral

\[
\int_{0}^{\infty} \frac{e^{-x}}{x^{j+1}} \, dx = \frac{(-1)^j}{j!} \psi(j+1), \quad j = 0, 1, 2, \ldots.
\]

The expansion of the Stieltjes transform of \(e^{-t}\) given by equation (33) can now obtained and substituted back into equation (41) to yield the following known expansion of the exponential integral function \([15, \text{p}151, \# 6.6.3]\)

\[
E_1(\omega) = -\ln \omega + e^{-\omega} \sum_{j=0}^{\infty} \frac{\psi(j+1)}{j!} \omega^j.
\]

It is not immediately apparent whether the series converges or not. But we are assured of the convergence because in the limit as \(a \to \infty\) the upper limit becomes arbitrarily large so that the condition \(\omega < a\) is satisfied for any \(\omega\).

3.2. Case 2: \(\nu \neq 0\)

**Lemma 2.** Let \(g(x)\) be integrable in the interval \([0, a]\) and that its complex extension \(g(z)\) is analytic in a region containing the interval \([0, a]\) in its interior. Let \(z^{-\nu}\), for \(0 < \nu < 1\), be such that its branch cut coincides with the real positive axis. Then

\[
\int_{0}^{a} x^{-\nu} g(x) \, dx = \frac{1}{(e^{-2\pi i \nu} - 1)} \int_{C} z^{-\nu} g(z) \, dz - \frac{2\pi i}{(e^{-2\pi i \nu} - 1)} \sum_{k} \text{Res} \left[z^{-\nu} g(z)\right]_{z=z_k},
\]

where \(C\) is the contour straddling the branch cut of \(z^{-\nu}\) starting from \(a\) and ending at \(a\) itself, and the \(z_k\)'s are the poles enclosed by \(C\), with no pole of \(g(z)\) lying on the contour \(C\).

**Proof.** Given the contour integral \(\int_{C} g(z) z^{-\nu} \, dz\), deform the contour \(C\) into the branch cut by means of the contour \(C'\) in Figure-2, again eventually taking the limit \(\epsilon \to 0\). \(\square\)

**Theorem 4.** Let the complex extension, \(f(z)\), of \(f(x)\) be entire. Then

\[
\int_{0}^{a} \frac{x^{-\nu} f(x)}{\omega + x} \, dx = \sum_{j=0}^{\infty} (-1)^j \omega^j \int_{0}^{a} \frac{f(x)}{x^{j+\nu+1}} \, dx + \frac{\pi f(-\omega)}{\omega^\nu \sin \pi \nu}, \quad \omega < a, \quad 0 < \nu < 1.
\]

**Proof.** From Lemma 2, we have the following contour integral representation of the incomplete Stieltjes transform

\[
\int_{0}^{a} \frac{x^{-\nu} f(x)}{\omega + x} \, dx = \frac{\pi f(-\omega)}{\omega^\nu \sin \pi \nu} + \frac{1}{(e^{-2\pi i \nu} - 1)} \int_{C} \frac{f(z)}{z^\nu (\omega + z)} \, dz
\]

We insert the same expansion for \((\omega + z)^{-1}\) in the integral and use the contour integral representation of the finite part integral to obtain

\[
\int_{0}^{a} \frac{x^{-\nu} f(x)}{\omega + x} \, dx = \frac{\pi f(-\omega)}{\omega^\nu \sin \pi \nu} + \sum_{j=0}^{\nu - 1} (-1)^j \int_{0}^{a} \frac{f(x)}{x^{j+1}} \, dx + R_\nu(\omega)
\]

where the remainder term is given by

\[
R_\nu(\omega) = \frac{(-1)^n \omega^n}{(e^{-2\pi i \nu} - 1)} \int_{C} \frac{f(z)}{z^{n+\nu} (\omega + z)} \, dz.
\]
Again deforming the contour to be the circular path with radius $a$ and centered at the origin, we can establish the bound

$$|R_n(\omega)| \leq \left( \frac{\omega}{a} \right)^n \frac{M(f, a)\pi}{a^n|\omega - \nu||\sin \pi \nu|}.$$  

For $\omega < a$ the remainder vanishes as $n$ approaches infinity. This establishes the expansion given by equation 48

\[\square\]

3.2.1. Example. This time let us reproduce a known series expansion of the incomplete gamma function, $\Gamma(a, z)$, which assumes the integral representation [15 p177, # 8.6.4]

\begin{equation}
\Gamma(\nu, z) = \frac{z^\nu e^{-z}}{\Gamma(1 - \nu)} \int_0^\infty \frac{x^\nu e^{-x}}{z + x} \, dx, \quad |\text{Arg}z| < \pi, \quad \Re \nu < 1.
\end{equation}

We recognize that the integral in the representation is the Stieltjes transform of the function $x^{-a}e^{-x}$, which falls under equation 48 with $f(z) = e^{-z}$. In this example, we again make the restriction $z = \omega > 0$ and further restrict ourselves to the case $0 < \Re \nu < 1$. Then the expansion given by equation 49 applies.

We proceed as above in obtaining the required finite part integrals. For some $\epsilon$, with $0 < \epsilon < a$, we have the following integral

$$\int_\epsilon^a \frac{e^{-x}}{x^{j+1+\nu}} \, dx = - \sum_{k=0}^j \frac{(-1)^k}{k!(k - j - \nu)} \left( \frac{1}{a^{j+\nu-k}} - \frac{1}{a^{j+\nu-k}} \right)$$

\begin{equation}
+ \sum_{k=j+1}^\infty \frac{(-1)^k}{k!(k - j - \nu)} \left( a^{k-j-\nu} - \epsilon^{k-j-\nu} \right)
\end{equation}

The finite part of the integral is again just the terms that have a finite limit as $\epsilon \to 0$. Collecting all terms with finite limits, the finite part integral, for finite $a$, is

$$\int_0^a \frac{e^{-x}}{x^{j+1+\nu}} \, dx = - \sum_{k=0}^j \frac{(-1)^k}{k!(k - j - \nu)} \frac{1}{a^{j+\nu-k}}$$

\begin{equation}
+ (-1)^{j+1} \frac{a^{1-\nu}}{(1 - \nu)(j + 1)!} \, 2F_2(1, 1 - \nu; 2 + j, 2 - \nu; -a),
\end{equation}

where the sum in the second term has been expressed as a hypergeometric function.

Finally we take the limit as $a \to \infty$. The first term vanishes in the limit. The hypergeometric function has two simple poles, so its relevant asymptotic expansion in the computation of the limit is given by [17]

$$2F_2(a_1, a_2; b_1, b_2; z) = \frac{\Gamma(b_1)\Gamma(b_2)\Gamma(a_2 - a_1)}{\Gamma(a_2)\Gamma(b_1 - a_1)\Gamma(b_2 - a_1)} (-z)^{-a_1} (1 + O(z^{-1}))$$

$$+ \frac{\Gamma(b_1)\Gamma(b_2)\Gamma(a_1 - a_2)}{\Gamma(a_1)\Gamma(b_1 - a_2)\Gamma(b_2 - a_2)} (-z)^{-a_2} (1 + O(z^{-1}))$$

\begin{equation}
+ \frac{\Gamma(b_1)\Gamma(b_2)}{\Gamma(a_1)\Gamma(a_2)} e^{z} z^{a_1+a_2-b_1-b_2} (1 + O(z^{-1})), \quad |z| \to \infty
\end{equation}

for $a_1 \neq a_2$. Only the second term contributes in the limit under the condition that $\Re \nu > 0$. 

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Substituting the appropriate values of the parameters of the hypergeometric function and implementing the limit gives the desired finite part integral,

\[
\int_0^\infty \frac{e^{-x}}{x^{j+1+\nu}} \, dx = (-1)^{j+1} \frac{\Gamma(1-\nu)\Gamma(\nu)}{\Gamma(j+\nu+1)}.
\]

We substitute this back into equation 48 to obtain the expansion of the Stieltjes transform and use the well-known identity \(\Gamma(z)\Gamma(1-z) = \pi \csc \pi \nu\) to finally obtain the following known expansion for the incomplete gamma function [15, p178, #8.7.3]

\[
\Gamma(a,z) = \Gamma(a) \left[ 1 - e^{-z} \sum_{j=0}^{\infty} \frac{z^{j+\nu}}{\Gamma(j+\nu+1)} \right].
\]

Notice that the series is clearly convergent everywhere. This is expected as in the limit \(a \to \infty\) the upper limit \(a\) becomes arbitrarily large so that the condition \(\omega < a\) is satisfied for any finite \(\omega\).

3.3. Remark. What happens if we choose a contour \(\tilde{C}\) that excludes the lone pole at \(z = -\omega\)? It appears that under this condition the pole contribution will drop out and only the term by term integrated terms will remain in the expansion, apparently leading to a contradiction with our results above. The apparent contradiction is resolved by recognizing that the equality in equations 33 and 48 only hold under the condition that the remainder vanishes as the number of terms in the partial sum increases indefinitely. When the pole is excluded, the remainder will not vanish in the limit; and, when the remainder is appropriately dealt with, we will arrive at the same result.

We can demonstrate this with the later case of \(\nu \neq 0\). Using Lemma 2 and following the same steps above, we obtain the finite expansion of the integral as
follows
\[ \int_0^a \frac{x^{-\nu} f(x)}{\omega + x} \, dx = \sum_{j=0}^{n-1} (-1)^j \int_0^a \frac{f(x)}{x^{j+1+\nu}} \, dx + \tilde{R}_n(\omega), \]
where the remainder term is given by
\[ \tilde{R}_n(\omega) = \frac{(-1)^n \omega^n}{(e^{-2\pi i \nu} - 1)} \int_{\tilde{C}} \frac{f(z)}{z^{n+\nu}(\omega + z)} \, dz. \]

Contradiction with the result 48 arises if the remainder \( \tilde{R}_n(\omega) \) vanishes as \( n \to \infty \). But it does not vanish. We can see that by deforming the contour \( \tilde{C} \) into the circular contour \( \tilde{C}' \) depicted in Figure 3. Along this contour, the remainder term assumes the form
\[ \tilde{R}_n(\omega) = \frac{(-1)^n \omega^n}{(e^{-2\pi i \nu} - 1)} \int_{\tilde{C}'} \frac{f(z)}{z^{n+\nu}(\omega + z)} \, dz - 2\pi i \text{Res}\left[ \frac{(-1)^n \omega^n}{(e^{-2\pi i \nu} - 1)} \frac{f(z)}{z^{n+\nu}(\omega + z)} \right] \bigg|_{z=-\omega}. \]

With the bound given by equation 52, the first term vanishes in the limit as \( n \to \infty \). On the other hand, evaluating the residue on the second term yields a term independent of \( n \) and in fact equal to the contribution of the simple pole at \( z = -\omega \) in equation 48. That is the remainder term carries the pole contribution excluded by the contour \( \tilde{C} \), so that the result given by equation 48 is recovered in the limit as \( n \to \infty \). This example emphasizes the need to consider the remainder term in the application of our approach.

4. Term by term integration at the origin for functions with holomorphic extensions

We now consider the case in which the complex extension function \( f(z) \) is holomorphic in the complex plane. Let \( z_1, z_2, z_3, \ldots \) be the poles of \( f(z) \), and \( \zeta_0 \) the greatest lower bound of the modulus of the poles, i.e. \( \zeta_0 = \inf\{|z_1|, |z_2|, \ldots\} \). Then we have the following results.
Theorem 5. If $\zeta_0 \neq 0$, then for every $\omega < a, \zeta_0$ the following expansion holds

$$
\int_0^a \frac{f(x)}{\omega + x} \, dx = \sum_{j=0}^{\infty} (-1)^j \int_0^a \frac{f(x)}{\omega + x + j} \, dx - f(-\omega) \ln \omega.
$$

Proof. Let the contour $C$ be sufficiently large to accommodate the simple pole at $z = -\omega$ but not any of the poles of $f(z)$ (see Figure 4). Then by the representation Lemma 4 and by following the same steps we have used in the entire case, we arrive at the expansion

$$
\int_0^a \frac{f(x)}{\omega + x} \, dx = -f(-\omega) \ln \omega + \sum_{j=0}^{n-1} (-1)^j \int_0^a \frac{f(x)}{\omega + x + j} \, dx + R_n(\omega),
$$

where the remainder term is given by

$$
R_n(\omega) = (-1)^n \frac{1}{2\pi i} \int_{C'} \frac{f(z)(\log z - i\pi)}{z^n(\omega + z)} \, dz.
$$

Under the condition that $\omega < a$, there always exists some $\rho$ such that $\omega < \rho < a$. We then deform the contour of integration into the contour $C'$ shown in Figure 4. Along this contour, the remainder assumes the form

$$
R_n(\omega) = (-1)^n \frac{1}{2\pi i} \int_{C'} \frac{f(z)(\log z - i\pi)}{z^n(\omega + z)} \, dz + (-1)^n \frac{1}{2\pi i} \int_{C'} \frac{f(z)}{z^n(\omega + z)} \, dz.
$$

We can use the bound given by inequality 40 for the first term of equation 63 to obtain the bound for the remainder term

$$
|R_n(\omega)| \leq \left(\frac{\omega}{\rho}\right)^n \left[\frac{M(f, \rho)(\ln \rho + \pi^2)}{\rho - \omega} + \int_\rho^a \frac{|f(x)|}{x(\omega + x)} \, dx\right].
$$

Since $\omega < \rho$ the bound vanishes as $n \to \infty$. Then equation 62 follows. \qed

Using similar steps, we can establish the following results.

Theorem 6. If $\zeta_0 \neq 0$, then for every $\omega < a, \zeta_0$ the following expansion holds

$$
\int_0^a \frac{x^{-\nu} f(x)}{\omega + x} \, dx = \sum_{j=0}^{\infty} (-1)^j \omega^j \int_0^a \frac{f(x)}{\omega + x + j} \, dx + \pi f(-\omega) \frac{\omega^{-\nu} \sin \pi \nu}{\omega^\nu}, \quad 0 < \nu < 1.
$$

5. TERM BY TERM INTEGRATION AT INFINITY

Here we apply our method to a problem that has been treated with the distributional approach [1, 5]. We will only demonstrate how the method can be applied when expansions are made at infinity. A detailed exposition of the relationship between our approach and the distributional approach to asymptotics will be considered elsewhere. Let us consider the integral

$$
I(\omega) = \int_0^\infty \frac{1}{(1 + x)^\nu(\omega + x)} \, dx, \quad 0 < \nu < 1, \quad 0 < \omega.
$$

We wish to obtain the asymptotic expansion for $\omega \to \infty$. The problem discussed in the introduction is a special case of this integral for $\nu = 1/3$. We leave the remainder analysis to the reader.

The first step is to obtain a contour integral representation of the integral given by equation 68. The complex extension function $(1 + z)^{-\nu}(\omega + z)^{-1}$ has a branch point at $z = -1$. We choose the branch of $(1 + z)^{-\nu}$ such that its branch cut is
the line $[-1, \infty)$. Furthermore, we choose the contour given in Figure 5. Since $\omega$ is arbitrarily large, the pole at $z = -\omega$ is outside this contour.

By deforming the contour into the branch cut of $(1+z)^{-\nu}$, we obtain the contour integral representation of the given integral,

$$
\int_0^\infty \frac{1}{(1+x)^\nu(\omega + x)} \, dx = -\int_0^1 \frac{1}{(1-x)^\nu(\omega - x)} \, dx + \frac{1}{(e^{-2\pi \nu i} - 1)} \int_C \frac{1}{(1+z)^\nu(\omega + z)} \, dz
$$

(69)

The first integral exists because $\omega$ is arbitrarily large, i.e. $\omega >> 1$. This time we use the expansion

$$
\frac{1}{(1+z)^\nu} = \sum_{s=0}^\infty (-1)^s \binom{-\nu}{s} \frac{1}{z^{s+\nu}}
$$

(70)

We substitute this expansion back into the second term of equation (69) and make the necessary identification for the finite part integral to obtain the expansion

$$
\int_0^\infty \frac{1}{(1+x)^\nu(\omega + x)} \, dx = -\int_0^1 \frac{1}{(1-x)^\nu(\omega - x)} \, dx + \sum_{s=0}^\infty \binom{-\nu}{s} \int_0^\infty \frac{1}{(\omega + x)^{s+\nu}} \, dx.
$$

(71)

Observe that the second term is just the term by term integration of the expansion, with the integration interpreted as a finite part integral. The first term represents the contribution coming from the branch point at $z = -1$. The expansion in the second term does not have an accompanying pole term because the pole at $z = -\omega$ is outside of the contour of integration.
Let us calculate the indicated finite part integral. For $0 < \epsilon < a < \infty$ we have the integral
\[
\int_{\epsilon}^{a} \frac{1}{(\omega + x)x^{s+\nu}} \, dx = \int_{\epsilon}^{a} \frac{1}{x^{s+\nu}} \left[ \sum_{j=0}^{s-1} (-1)^j \frac{x^j}{\omega^{j+1}} + \frac{(-1)^s x^s}{\omega^s (\omega + x)} \right] \, dx
\]
\[
= -\sum_{j=0}^{s-1} \frac{1}{(s + \nu - j - 1)\omega^{j+1}} \left[ \frac{1}{a^{s+\nu-j-1}} - \frac{1}{\epsilon^{s+\nu-j-1}} \right] + \frac{(-1)^s}{\omega^s} \int_{\epsilon}^{a} \frac{1}{x^{\nu}(\omega + x)} \, dx.
\]
(72)
The finite part for $a < \infty$ is obtained by dropping the terms that diverge as $\epsilon \to 0$, and keeping only those terms with a well defined limit in the said limit. Finally taking the limit as $a \to \infty$ yields the desired finite part integral
\[
\int_{0}^{\infty} \frac{1}{(\omega + x)x^{s+\nu}} \, dx = \frac{(-1)^s}{\omega^s} \int_{0}^{\infty} \frac{1}{x^{\nu}(\omega + x)} \, dx = \frac{(-1)^s \pi}{\omega^{s+\nu} \sin \pi \nu}.
\]
(73)
Finally the first integral can be expanded by expanding $(\omega - x)^{-1}$ at $x = 0$, and then integrating the infinite series term by term. Here no special treatment is needed for the term by term integration because no divergent integral is involved, and the infinite series for the expansion is absolutely convergent in the entire interval of integration for $\omega >> 1$. By substitution of the expansion of the first term and the finite part integrals back in equation 71, we obtain the desired expansion,
\[
\int_{0}^{\infty} \frac{1}{(1 + x)^{\nu}(\omega + x)} \, dx = -\sum_{s=0}^{\infty} \frac{\Gamma(s + 1)\Gamma(1 - \nu)}{\Gamma(s - \nu + 2)} \frac{1}{\omega^{s+1}} \left[ \frac{\pi}{\sin \pi \nu} \sum_{s=0}^{\infty} (-1)^s \left( \frac{-\nu}{s} \right) \frac{1}{\omega^{s+\nu}}, \ \omega \to \infty. \right.
\]
(74)
Both infinite series in equation 74 happen to converge for sufficiently large $\omega$. We recover from this the result for $\nu = 1/3$ given earlier in equation 5.

This example demonstrates that the result of the distributional approach, at least for the example given here, can be obtained by term by term integration via contour integration representation and the finite part integral.

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