ON THE MAGNITUDE AND INTRINSIC VOLUMES OF A CONVEX BODY IN EUCLIDEAN SPACE

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Abstract. Magnitude is an isometric invariant of metric spaces inspired by category theory. Recent work has shown that the asymptotic behavior under rescaling of the magnitude of subsets of Euclidean space is closely related to intrinsic volumes. Here we prove an upper bound for the magnitude of a compact, convex set in Euclidean space in terms of its intrinsic volumes. The result is deduced from an analogous known result for magnitude in $\ell^N_1$, via approximate embeddings of Euclidean space into high-dimensional $\ell^N_1$ spaces. The upper bound is also shown to be sharp to first order for an odd-dimensional Euclidean ball shrinking to a point; this complements recent work investigating the asymptotics of magnitude for large dilatations of sets in Euclidean space.

1. Introduction and main results

Magnitude is an isometric invariant of metric spaces defined by Leinster [12] based on category-theoretic considerations. It is an abstract notion of the size of a metric space, which in some ways serves as an “effective number of points” in the space. Magnitude turns out to encode many classical invariants from integral geometry and geometric measure theory, including volume, capacity, dimension, and surface area. See [15] for a survey of connections between magnitude and geometry. In other directions, magnitude has connections to graph invariants [13], theoretical ecology [22, 14], and homology theory [10, 16, 20, 9].

The purpose of this note is to show that the magnitude of a compact convex set $K$ in the $d$-dimensional Euclidean space $\ell^d_2$ is bounded above by a particular linear combination of the intrinsic volumes of $K$ (Theorem 1). The only such sets whose magnitudes are known explicitly are Euclidean balls for odd $d$, and even in those cases the statement for arbitrary odd $d$ is quite complicated [11, 24] (see Theorem 10 below). The upper bound is sharp to first order for odd-dimensional Euclidean balls with small radius, as shown in Theorem 8. These results can be used to clarify the asymptotic behavior of the magnitude of a convex body in $\ell^d_2$ as it shrinks to a point (Corollaries 2 and 5).

Magnitude can be defined in several equivalent ways (see [15]). For the purposes of this paper the following will suffice. A metric space $(X, d)$ is called positive definite if, for each $n \in \mathbb{N}$ and each collection of distinct $x_1, \ldots, x_n \in X$, the matrix $(e^{-d(x_i, x_j)})_{1 \leq i, j \leq n}$ is positive definite. Every subset of $L_p$ for $1 \leq p \leq 2$ is positive definite; this of course includes subsets of $\ell^d_p$, the space $\mathbb{R}^d$ equipped with the $\ell_p$ metric for $1 \leq p \leq 2$. (See [18, Theorem 3.6] for a broad list of positive definite metric spaces.) If $(X, d)$ is a compact
positive definite metric space, then the \textbf{magnitude} of $X$ is
\begin{equation}
\text{Mag} (X) = \sup \left\{ \frac{\left( \sum_{i=1}^{n} w_i \right)^2}{\sum_{i,j=1}^{n} e^{-d(x_i,x_j)} w_i w_j} \middle| n \in \mathbb{N}, x_1, \ldots, x_n \in X, 0 \neq w \in \mathbb{R}^n \right\}.
\end{equation}

For $0 \leq k \leq d$, the intrinsic volumes of a compact convex set $K \subseteq \ell_2^d$ can be defined by the Kubota formula
\begin{equation}
V_k(K) = \frac{d}{k} \frac{\omega_d}{\omega_k \omega_{d-k}} \int_{\text{Gr}_{d,k}} \text{vol}_k(\pi_P(K)) \, d\mu_{d,k}(P),
\end{equation}
where $\text{Gr}_{d,k}$ is the Grassmann manifold of $k$-dimensional subspaces of $\mathbb{R}^d$, $\mu_{d,k}$ denotes the rotation-invariant probability measure on $\text{Gr}_{d,k}$, $\pi_P$ denotes the orthogonal projection onto $P$, and
\[
\omega_n = \frac{\pi^{n/2}}{\Gamma\left(1 + \frac{n}{2}\right)}
\]
is the volume of the unit ball in $\ell_2^n$; see e.g. [21, p. 222]. The normalization is chosen such that if $T : \ell_2^d \rightarrow \ell_2^N$ is an isometric embedding and $K \subseteq \ell_2^d$ is compact and convex, then $V_k(T(K)) = V_k(K)$ for all $0 \leq k \leq d$. In particular, if $K \subseteq \ell_2^d$ is a $k$-dimensional compact, convex set, then $V_k(K) = \text{vol}_k(K)$.

The first main result of this paper is the following, which will be proved in section 2.

\textbf{Theorem 1.} If $K \subseteq \ell_2^d$ is compact and convex, then
\begin{equation}
\text{Mag} (K) \leq \sum_{k=0}^{d} \frac{\omega_k}{4k} V_k(K),
\end{equation}
with equality if $d = 1$.

Theorem 1 can be compared to the erstwhile conjecture (see [17], [12, Conjecture 3.5.10]) that if $K \subseteq \ell_2^d$ is compact and convex, then
\begin{equation}
\text{Mag} (K) = \sum_{k=0}^{d} \frac{1}{k! \omega_k} V_k(K).
\end{equation}
The explicit computation of magnitude for Euclidean balls in [11] showed that (4) is false for $d \geq 5$ (although it does hold if $K$ is a three-dimensional Euclidean ball). Since that work, attention has turned to weaker versions of this conjecture, in particular the question of whether intrinsic volumes can be recovered from magnitude. We note that the first two terms of the right hand sides of both (3) and (4) are $1 + \frac{1}{2}V_1(K)$; after that the coefficients in the upper bound in (3) are larger.

The \textbf{magnitude function} of a compact set $X \subseteq \ell_2^d$ is the function $t \mapsto \text{Mag} (tX)$ for $t > 0$. Since $V_k$ is homogeneous of degree $k$, (3) is equivalent to the following polynomial upper bound on the magnitude function of a compact, convex set $K \subseteq \ell_2^d$:
\begin{equation}
\text{Mag} (tK) \leq \sum_{k=0}^{d} \frac{\omega_k}{4k} V_k(K) t^k
\end{equation}
for $t \geq 0$. 

A first consequence of Theorem 1 is a new proof of the following surprisingly nontrivial fact about the behavior of the magnitude function of \( X \subseteq \ell_d^2 \) when \( t \to 0 \).

**Corollary 2.** Suppose that \( X \subseteq \ell_d^2 \) is compact. Then

\[
\lim_{t \to 0^+} \text{Mag} (tX) = 1.
\]

**Proof.** If \( Y \) is any compact positive definite metric space and \( \emptyset \neq X \subseteq Y \), then

\[
1 \leq \text{Mag} (X) \leq \text{Mag} (Y) ;
\]

this follows immediately from our definition (1) of magnitude. If we now let \( Y \) be the convex hull of \( X \subseteq \ell_d^2 \), then (5) and (6) imply that

\[
1 \leq \text{Mag} (tX) \leq \sum_{k=0}^{d} \frac{\omega_k V_k(Y)}{4k} t^k = 1 + O(t)
\]

as \( t \to 0 \), which immediately implies the claim. \( \square \)

Corollary 2 was first proved in [1, Theorem 1] using Fourier-analytic techniques and a potential-theoretic characterization of magnitude in \( \ell_d^2 \) from [19]. It was reproved in [24, Corollary 1] using an exact expression for the magnitude of odd-dimensional Euclidean balls (stated as Theorem 10 below). The corresponding result for subsets of \( \ell_1^d \) is much simpler (see [15, Proposition 4.4]). On the other hand, there exists a six-point metric space \( (X,d) \) such that \( tX := (X,td) \) is positive definite for every \( t > 0 \) (such a metric space is known as a space of **negative type**) for which \( \lim_{t \to 0^+} \text{Mag} (tX) = 6/5 \) [12, Example 2.2.8].

For odd-dimensional Euclidean balls, the upper bound in Theorem 1 — and therefore the previously conjectured formula (4) — also captures the correct first-order behavior of the magnitude function as \( t \to 0 \), as the following theorem shows.

**Theorem 3.** Suppose that \( d \) is odd, and let \( B_d^2 \) denote the Euclidean unit ball in \( \ell_d^2 \). Then

\[
\lim_{t \to 0^+} \frac{\text{Mag} (tB_d^2) - 1}{t} = \frac{1}{2} V_1 (B_d^2).
\]

Theorem 3 was conjectured by Simon Willerton in response to a question by the author, on the basis of computer calculations using the results of [24]. The proof appears in section 3. The result suggests the following conjecture.

**Conjecture 4.** If \( K \subseteq \ell_d^2 \) is compact and convex, then

\[
\lim_{t \to 0^+} \frac{\text{Mag} (tK) - 1}{t} = \frac{1}{2} V_1 (K).
\]

Theorems 1 and 3 can be combined to prove a partial result in the direction of Conjecture 4. We denote by \( A_{d,k} \) the set of \( k \)-dimensional affine subspaces of \( \mathbb{R}^d \), and for \( E \in A_{d,k} \) we let \( \text{inrad}(K \cap E) \) be the largest radius of a \( k \)-dimensional Euclidean ball contained in \( K \cap E \).
Corollary 5. There is an absolute constant \( c > 0 \) such that if \( K \subseteq \mathbb{R}^d \) is compact and convex, then

\[
c \max_{1 \leq k \leq d, \ E \in A_{d,k}} \sqrt{k} \text{inrad}(K \cap E) \leq \liminf_{t \to 0^+} \frac{\text{Mag}(tK) - 1}{t} \leq \limsup_{t \to 0^+} \frac{\text{Mag}(tK) - 1}{t} \leq \frac{V_1(K)}{2}.
\]

Proof. The upper bound follows immediately from (1). For the lower bound, for each odd \( k \) and each \( k \)-dimensional affine subspace \( E, K \) contains an isometric copy of \( \text{inrad}(K \cap E) B^k_2 \), and so by Theorem 3 and (6),

\[
\text{Mag}(tK) \geq \text{Mag} \left( t \text{inrad}(K \cap E) B^k_2 \right) = 1 + \frac{V_1(B^k_2)}{2} \text{inrad}(K \cap E) t + o(t)
\]

\[
\geq 1 + c\sqrt{k} \text{inrad}(K \cap E) t + o(t).
\]

The limits inferior and superior in Corollary 5 are necessarily both homogeneous of degree 1 as functions of \( K \), as are the stated upper and lower bounds. It is not a priori obvious, however, that the limits inferior and superior are finite and nonzero.

On the other side, for any compact \( X \subseteq \ell^d_2 \), \( \text{Mag}(X) \geq \frac{\text{vol}_d(X)}{d! \omega_d} \) [12, Theorem 3.5.6] and (7) \( \lim_{t \to \infty} \frac{\text{Mag}(tX)}{t^d} = \frac{\text{vol}_d(X)}{d! \omega_d} \) [1, Theorem 1] (which was consistent with the formerly conjectured formula (4)). Thus (5) captures the correct order of growth of \( \text{Mag}(tK) \) as \( t \to \infty \) when \( K \) has nonempty interior, but with the wrong constant if \( K \) is greater than one-dimensional.

When \( X \subseteq \ell^d_2 \) is the closure of a bounded, open set with smooth boundary and \( d \geq 3 \) is odd, there is the finer asymptotic expansion

\[
\text{Mag}(tX) = \frac{1}{d! \omega_d} \left( \text{vol}_d(X)t^d + \frac{d+1}{2} \text{vol}_{d-1}(\partial X)t^{d-1} + \frac{(d-1)(d+1)^2}{8} \left( \int_{\partial X} H \ dS \right) t^{d-2} \right) + O(t^{d-3})
\]

as \( t \to \infty \) [3]. Here \( H \) is the mean curvature on \( \partial X \) and \( S \) is the surface area measure. When \( K \subseteq \ell^d_2 \) is a compact, convex set with nonempty interior and smooth boundary, (8) becomes

\[
\text{Mag}(tK) = \frac{1}{d! \omega_d} \left( V_d(K)t^d + (d+1)V_{d-1}(K)t^{d-1} + \frac{\pi}{4}(d+1)^2 V_{d-2}(K)t^{d-2} \right) + O(t^{d-3}).
\]

This implies that \( V_{d-1}(K) \) and \( V_{d-2}(K) \) can also be recovered from the magnitude function of \( K \). It also shows that, although the upper bound in (5) only matches the \( t \to \infty \) asymptotics of the magnitude function of \( K \) in a rough sense, the dependence of the three top-order terms on \( K \) is, intriguingly, correct up to scalar multiples. However, the next term in the asymptotic expansion (8) turns out not to be a multiple of an intrinsic volume [5].
2. Proof of Theorem 1

Theorem 1 follows from a similar result for magnitude of convex sets in $\ell_1^N$. For $0 \leq k \leq N$, the $\ell_1$ intrinsic volumes of a compact convex set $K \subseteq \ell_1^N$ are defined by

$$V'_k(K) = \sum_{P \in \text{Gr}'_{N,k}} \text{vol}_k(\pi_P(K)),$$

where $\text{Gr}'_{N,k}$ denotes the set of $k$-dimensional coordinate subspaces of $\mathbb{R}^N$ and $\pi_P$ denotes the coordinate projection onto $P$ [11]. (In fact, the natural class of sets to consider is somewhat larger than convex sets, but this point will not be used here.) Note that if $K$ lies in a $d$-dimensional subspace of $\ell_1^N$, then $V'_k(K) = 0$ for $k > d$.

**Theorem 6 ([15, Theorem 4.6]).** If $K \subseteq \ell_1^N$ is compact and convex, then

$$\text{Mag}(K) \leq \sum_{k=0}^{N} \frac{1}{2^k} V'_k(K),$$

with equality if $K$ has nonempty interior, or if $N = 2$.

We note that, by the $\ell_1$ analogue of Steiner’s formula [11, Theorem 6.2], the right hand side of (9) is equal to $\text{vol}_N\left(\frac{1}{2}K + [0,1]^N\right)$. There does not appear to be such a simple interpretation of the upper bound in (3).

The idea of the proof of Theorem 1 is to approximate the Euclidean space $\ell_2^d$ by subspaces of $\ell_1^N$ for large $N$, and show that the $\ell_1$ intrinsic volumes approximate scalar multiples of the classical intrinsic volumes in those subspaces.

Let $\Omega_{d,n} = \{-1,1\}^d$, equipped with the uniform probability measure $\mathbb{P}_{d,n}$. We will consider $L_1(\Omega_{d,n}) = L_1(\Omega, \mathbb{P}_{d,n})$ and $\ell_1(\Omega_{d,n}) \cong \ell_1^{2nd}$, which are both the space of functions $f : \Omega_{d,n} \to \mathbb{R}$ but with different norms:

$$\|f\|_{L_1} = \mathbb{E}_{d,n}|f| = \frac{1}{2^{nd}} \sum_{x \in \Omega_{d,n}} |f(x)| = \frac{1}{2^{nd}} \|f\|_{\ell_1}.$$

For $1 \leq i \leq d$ and $1 \leq j \leq n$, define $X_{i,j} = X_{i,j}^{(d,n)} : \Omega_{d,n} \to \mathbb{R}$ by $X_{i,j}(x) = x_{i,j}$. Then, with respect to $\mathbb{P}_{d,n}$, $\{X_{i,j} | 1 \leq i \leq d, 1 \leq j \leq n\}$ are independent, identically distributed random variables with $\mathbb{P}_{d,n}[X_{i,j} = 1] = \mathbb{P}_{d,n}[X_{i,j} = -1] = 1/2$.

We next define

$$S^n_i = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} X_{i,j}$$

for $1 \leq i \leq d$, and let $T^n_d : \ell_2^d \to L_1(\Omega_{d,n})$ be given by $T^n_d(e_i) = S^n_i$. We also write $\tilde{T}^n_d = \sqrt{\frac{\pi}{2}} 2^{-nd} T^n_d$, so that

$$\|\tilde{T}^n_d(y)\|_{\ell_1} = \sqrt{\frac{\pi}{2}} \|T^n_d(y)\|_{L_1}.$$

To deduce Theorem 1 from Theorem 6, we will use two technical results, both of which are applications of the central limit theorem.
Lemma 7. For every $d$, $n$, and nonzero $y \in \mathbb{R}^d$, 
\[1 - \frac{4}{\sqrt{n}} \leq \left\| \tilde{T}_n^d(y) \right\|_{\ell_1} \leq 1 + \frac{4}{\sqrt{n}}.\]

Proof. Without loss of generality we may assume that $\|y\|_2 = 1$. We have
\[T_n^d(y) = \sum_{i=1}^d \sum_{j=1}^n \frac{y_i}{\sqrt{n}} X_{i,j}.\]
By a version of the Berry–Esseen theorem for Lipschitz test functions,
\[\left| \mathbb{E}_{d,n} f(T_n^d(y)) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-t^2/2} \, dt \right| \leq 3 \sum_{i=1}^d \sum_{j=1}^n \left| \frac{y_i}{\sqrt{n}} \right|^3 \leq \frac{3}{\sqrt{n}} \]
for any 1-Lipschitz function $f : \mathbb{R} \to \mathbb{R}$. (This is essentially contained in the work of Esseen [2]; see [6, Proposition 2.2] for an explicit statement which includes the precise constant given here.) In particular, letting $f(t) = |t|$, this implies that
\[\left| \left\| T_n^d(y) \right\|_{L_1} - \sqrt{\frac{2}{\pi}} \right| \leq \frac{3}{\sqrt{n}},\]
from which the lemma follows. (The stated constant 4 is not sharp.) □

Proposition 8. If $K \subseteq \ell_2^d$ is compact and convex, then for each $0 \leq k \leq d$,
\[\lim_{n \to \infty} V_k'(\tilde{T}_n^d(K)) = \frac{\omega_k}{2^k} V_k(K).\]

Proof. The case $k = 0$ is trivial, since $V_0' = V_0 = 1$ always. Now given distinct $x_1, \ldots, x_k \in \Omega_{d,n}$, we denote $\pi_{x_1,\ldots,x_k}(f) = (f(x_1), \ldots, f(x_k))$. Then the $\ell_1$ intrinsic volumes of $X \subseteq \ell_1(\Omega_{d,n})$ can be equivalently expressed as
\[V_k'(X) = \frac{1}{k!} \sum_{x_1,\ldots,x_k \text{ distinct}} \text{vol}_k(\pi_{x_1,\ldots,x_k}(X)).\]

Now
\[\pi_{x_1,\ldots,x_k}(\tilde{T}_n^d(y)) = \sqrt{\frac{\pi}{2}} 2^{-nd} (\langle y, S^n(x_1) \rangle, \ldots, \langle y, S^n(x_k) \rangle),\]
where $S^n(x) = (S^n_{x_1}(x), \ldots, S^n_{x_k}(x)) \in \mathbb{R}^d$. Equivalently,
\[\pi_{x_1,\ldots,x_k}(\tilde{T}_n^d(y)) = \sqrt{\frac{\pi}{2}} 2^{-nd} M(x_1, \ldots, x_k)^t y,\]
where $M(x_1, \ldots, x_k)$ is the $d \times k$ matrix with entries $(S^n_{i,j}(x))_{1 \leq i \leq d, 1 \leq j \leq k}$ and $M^t y$ is given by matrix multiplication. It follows that
\[\text{vol}_k(\pi_{x_1,\ldots,x_k}(\tilde{T}_n^d(K))) = \left( \frac{\pi}{2} \right)^{k/2} 2^{-ndk} \sqrt{\det(M^t M)} \text{vol}_k(\pi_C(M)(K)),\]
where \( C(M) \) is the subspace of \( \ell^d_k \) spanned by the columns of \( M = M(x_1, \ldots, x_k) \). Note that if any two \( x_j \) are equal, then \( M^tM \) has rank smaller than \( k \), and so \( \det M^tM = 0 \). This implies that

\[
V_k'(\widetilde{T}_d^n(K)) = \frac{1}{k!} \left( \frac{\pi}{2} \right)^{k/2} 2^{-ndk} \sum_{x_1, \ldots, x_k} \sqrt{\det(M^tM) \vol_k(\pi_{C(M)}(K))}
\]

without the restriction to distinct summands, and so

\[
V_k'(\widetilde{T}_d^n(K)) = \frac{1}{k!} \left( \frac{\pi}{2} \right)^{k/2} \mathbb{E} \sqrt{\det(M^tM) \vol_k(\pi_{C(M)}(K))},
\]

where \( M \) is a \( d \times k \) random matrix with independent entries each distributed as \( \frac{1}{\sqrt{n}} \sum_{j=1}^n X_{1,j} \).

The central limit theorem now implies that

\[
\lim_{n \to \infty} V_k'(\widetilde{T}_d^n(K)) = \frac{1}{k!} \left( \frac{\pi}{2} \right)^{k/2} \mathbb{E} \sqrt{\det(G^tG) \vol_k(\pi_{C(G)}(K))},
\]

where \( G \) is a \( d \times k \) random matrix with i.i.d. standard Gaussian entries, and \( C(G) \) is the span of the columns of \( G \). (The unboundedness of \( \det(M^tM) \) can be handled with a standard truncation argument, using for example the fact that \( \det(M^tM) \leq \prod_{j=1}^k \|S^n(x_j)\|_2 \) by Hadamard’s inequality, combined with Hoeffding’s inequality for sums of independent bounded random variables.)

As is well known, the rotation invariance of the standard Gaussian distribution on \( \ell^d_2 \) implies that \( C(G) \) is uniformly distributed in \( \Gr_{d,k} \), independently of the singular values of \( G \). Hence

\[
\mathbb{E} \sqrt{\det(G^tG) \vol_k(\pi_{C(V)}(K))} = \left( \mathbb{E} \sqrt{\det(G^tG)} \right) \int_{\Gr_{d,k}} \vol_k(\pi_{C(V)}(K)) \, d\mu_{d,k}
\]

(11)

\[
= \left( \mathbb{E} \sqrt{\det(G^tG)} \right) \frac{\omega_d \omega_{d-k}}{\binom{d}{k}} \omega_k V_k(K).
\]

Now \( \det(G^tG) \) is distributed as a product of \( k \) independent \( \chi^2 \) random variables with \( d, d-1, \ldots, d-k+1 \) degrees of freedom respectively [23] (cf. [7]), which implies that

\[
\mathbb{E} \sqrt{\det(G^tG)} = 2^{k/2} \prod_{i=1}^k \frac{\Gamma\left(\frac{d-k+i+1}{2}\right)}{\Gamma\left(\frac{d-k+i}{2}\right)} = 2^{k/2} \frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d-k+1}{2}\right)}.
\]

The Legendre duplication formula

\[
\Gamma(x)\Gamma\left(x + \frac{1}{2}\right) = \frac{2\sqrt{\pi}\Gamma(2x)}{2^{2x}}
\]

now implies that

\[
\left(\mathbb{E} \sqrt{\det(G^tG)}\right) \frac{\omega_{d-k}}{\binom{d}{k} \omega_d} = \left( \frac{2}{\pi} \right)^{k/2} \frac{k!}{2^k}.
\]

(12)

The proposition now follows by combining (10), (11), and (12). \( \square \)

**Proof of Theorem 4** Recall that the Lipschitz distance between two homeomorphic metric spaces \((X, d_X)\) and \((Y, d_Y)\) is defined to be

\[
\inf \{ \| \log \text{dil}(f) \| + \| \log \text{dil}(f^{-1}) \| \mid f : X \to Y \text{ bi-Lipschitz} \},
\]
where
\[
d\mathcal{I}(f) = \sup_{x_1 \neq x_2} \frac{d_Y(f(x_1), f(x_2))}{d_X(x_1, x_2)}
\]
and \(d\mathcal{I}(f^{-1})\) is defined similarly.

If \(X \subseteq \ell^d_2\) is a fixed compact set (equipped with the \(\ell^d_2\) metric), then Lemma 7 implies that the metric spaces \(\widetilde{T}_n^d(X) \subseteq \ell_1(\Omega_{d,n})\) (equipped with the \(\ell_1(\Omega_{d,n})\) metric) converge to \(X\) in the Lipschitz distance when \(n \to \infty\). This implies that \(\widetilde{T}_n^d(X) \xrightarrow{n \to \infty} X\) also in the Gromov–Hausdorff distance (see [8, Section 3.A]).

Magnitude is lower semicontinuous with respect to the Gromov–Hausdorff topology on the collection of positive definite metric spaces [18, Theorem 2.6]. It follows that
\[
\text{Mag}(X) \leq \liminf_{n \to \infty} \text{Mag}\left(\widetilde{T}_n^d(X)\right).
\]

If \(K \subseteq \ell^d_2\) is compact and convex, Theorem 6 then implies that
\[
\text{Mag}(K) \leq \liminf_{n \to \infty} \sum_{k=0}^{\text{dim}^d_n} \frac{1}{2^{kd}} V'_k(\widetilde{T}_n^d(K)) = \liminf_{n \to \infty} \sum_{k=0}^{d} \frac{1}{2^{kd}} V'_k(\widetilde{T}_n^d(K)).
\]
The upper bound in (3) now follows from Proposition 8.

Equality for \(d = 1\) follows from the known formula \(\text{Mag}([0, \ell]) = 1 + \frac{1}{2} \ell\) [17, Theorem 7].

Theorem 1 and its proof highlight some open questions about continuity properties of magnitude. As noted in the statement of Theorem 6, the upper bound in (3) is actually equal to \(\text{Mag}(K)\) if \(K \subseteq \ell^1_N\) is \(N\)-dimensional; the upper bound for lower-dimensional sets in \(\ell^1_N\) follows by approximation by \(N\)-dimensional sets. As we have seen, Theorem 1 is similarly deduced by approximating \(K \subseteq \ell^d_2\) by subsets of \(\ell^N_1\) which are homeomorphic to \(K\).

The \(t \to \infty\) asymptotics of the magnitude function in (7) show that if \(K\) is greater than one-dimensional, then the upper bound on \(\text{Mag}(tK)\) in (5) must be strict for large enough \(t\). This implies that somewhere in the string of approximations leading from Theorem 6 for \(N\)-dimensional sets in \(\ell^N_1\) to Theorem 1 for compact, convex sets in \(\ell^d_2\), continuity must fail. In particular, at least one of the two following statements must be false:

- For each \(N\), if \(K \subseteq \ell^N_1\) is a compact, convex set then \(\text{Mag}(K) = \sum_{k=0}^{N} 2^{-k}V'_k(K)\) ([12, Conjecture 3.4.10], [15, Conjecture 4.5]). Equivalently, magnitude is continuous with respect to the Hausdorff distance on the collection of compact, convex sets in \(\ell^N_1\).
- For each \(d\), magnitude is continuous with respect to the Hausdorff distance on the collection of \(d\)-dimensional compact, convex sets in \(L_1\).

Magnitude is known to be continuous on the collection of convex bodies in any fixed finite-dimensional subspace of \(L_1\) [15, Theorem 4.15]. Moreover, the known examples of discontinuity of magnitude all involve change of topology. This includes the six-point space from [12, Example 2.2.8] discussed above shrinking to a one-point space, as well as the approximation of a sphere in Euclidean space by spherical shells [14, 25]. Available evidence
is thus in favor of the second statement above (although it is possible that both statements are false). In fact, we conjecture the following stronger statement:

**Conjecture 9.** Let $(X, d_X)$ be a compact metric space of negative type. Then magnitude is continuous with respect to the Lipschitz distance on the family of metric spaces $(Y, d_Y)$ of negative type which are bi-Lipschitz equivalent to $(X, d_X)$.

As noted above, Conjecture 9 and known results would show that [12, Conjecture 3.4.10] and [15, Conjecture 4.5] are false for compact, convex sets in $\ell^N_1$ without interior.

### 3. Proof of Theorem 3

Theorem 3 depends on an exact combinatorial formula for the magnitude of a Euclidean ball in odd dimensions due to Willerton [24]. To state it, we first need some terminology and notation.

A **Schröder path** is a finite directed path in $\mathbb{Z}^2$ in which each step with starting point $(x, y) \in \mathbb{Z}^2$ is either an **ascent** to $(x + 1, y + 1)$, a **descent** to $(x + 1, y - 1)$, or a **flat step** to $(x + 2, y)$. For $k \geq 0$, a **disjoint $k$-collection** is a family of Schröder paths from $(-i, i)$ to $(i, i)$ for each $0 \leq i \leq k$, such that no node in $\mathbb{Z}^2$ is contained in two of the paths. (Since all nodes of the paths have an even sum of coordinates, it follows that the paths do not cross.) We denote by $X_k$ the set of all disjoint $k$-collections, and by $X^j_k$ the set of disjoint $k$-collections with exactly $j$ flat steps. The set $X^0_k$ consists of a single collection, denoted $\sigma^0_{\text{roof}}$ in [24], in which for each $i$, the $i$th path consists of $i$ ascents followed by $i$ descents.

For a collection $\sigma \in X_k$ we write $\tau \in \sigma$ if $\tau$ is a step in one of the paths in $\sigma$. For an indeterminate $t$ define

$$w_j(\tau) = \begin{cases} 
1 & \text{if } \tau \text{ is an ascent}, \\
t & \text{if } \tau \text{ is a flat step}, \\
y + 1 - j & \text{if } \tau \text{ is a descent from height } y \text{ to height } y - 1.
\end{cases}$$

**Theorem 10 ([24, Corollary 27]).** Let $d = 2m + 1$ be odd. Then

$$\text{Mag}(tB^d_2) = \frac{\sum_{\sigma \in X_{m+1}} \prod_{\tau \in \sigma} w_2(\tau)}{d! \sum_{\sigma \in X_{m-1}} \prod_{\tau \in \sigma} w_0(\tau)}$$

for all $t > 0$.

**Proof of Theorem**

First note that by the Kubota formula [2],

$$V_1(B^d_2) = \frac{(2m + 1)\sqrt{\pi} \Gamma(m + 1)}{\Gamma\left(m + \frac{3}{2}\right)} = 2 \left(m - \frac{1}{2}\right)^{-1},$$

where $\binom{x}{k} = \frac{x(x-1)\cdots(x-k+1)}{k!}$ denotes the generalized binomial coefficient for $x \in \mathbb{R}$ and $k$ a nonnegative integer (with the convention that $\binom{x}{0} = 1$).

Now write

$$N(t) = \sum_{\sigma \in X_{m+1}} \prod_{\tau \in \sigma} w_2(\tau) \quad \text{and} \quad D(t) = \sum_{\sigma \in X_{m-1}} \prod_{\tau \in \sigma} w_0(\tau).$$
Willerton showed in [24, Theorem 28] that $N(0) = d!D(0)$. We wish to compute
\begin{equation}
\frac{d}{dt} \text{Mag} \left( tB_2^d \right) \bigg|_{t=0} = \frac{N'(0)D(0) - N(0)D'}{d!D(0)^2} = \frac{N'(0) - d!D'(0)}{N(0)}.
\end{equation}

We have that
\begin{align}
N(0) &= \sum_{\sigma \in X_{m-1}} \prod_{\tau \in \sigma} w_2(\tau) = \prod_{\tau \in \sigma_{\text{root}}} w_2(\tau), \\
N'(0) &= t^{-1} \sum_{\sigma \in X_{m-1}} \prod_{\tau \in \sigma} w_2(\tau), \\
D'(0) &= t^{-1} \sum_{\sigma \in X_1} \prod_{\tau \in \sigma} w_0(\tau).
\end{align}

It is easy to give an explicit expression for $N(0)$, but it is more convenient here to leave it in the form above.

We instead begin by simplifying the right hand side of [14] via the same trick used in [24] to show $N(0) = d!D(0)$. Namely, each $\sigma \in X_{m-1}$ gives rise to a $\mu(\sigma) \in X_{m+1}$ by shifting all paths up two units, adding ascents from $(-i, i)$ to $(-i+1, i+1)$ and descents from $(i-1, i+1)$ to $(i, i)$ for $1 \leq i \leq m$, and finally adding a path from $(-m+1, m+1)$ to $(m+1, m+1)$ consisting of $m+1$ ascents followed by $m+1$ descents (see [24, Figure 4]). Then $\mu(\sigma)$ has the same number of flat steps as $\sigma$, and
\begin{equation}
\prod_{\tau \in \mu(\sigma)} w_2(\tau) = d! \prod_{\tau \in \sigma} w_0(\tau).
\end{equation}

It therefore follows from (14) and (15) that
\begin{equation}
\frac{d}{dt} \text{Mag} \left( tB_2^d \right) \bigg|_{t=0} = \sum_{\sigma \in X_{m+1} \setminus \mu(X_{m-1})} \frac{t^{-1} \prod_{\tau \in \sigma} w_2(\tau)}{\prod_{\tau \in \sigma_{\text{root}}} w_2(\tau)}.
\end{equation}

For $1 \leq p \leq k$ and $0 \leq q \leq k - p$, let $\sigma_{p,q}^k$ denote the disjoint $k$-collection described as follows: the $p^{th}$ path consists of $p - 1$ ascents, one flat step, and $p - 1$ descents. For $p + 1 \leq i \leq p + q$, the $i^{th}$ path consists of $i - 1$ ascents, one descent, one ascent, and $i - 1$ descents. For $i < p$ and $i > p + q$, the $i^{th}$ path consists of $i$ ascents followed by $i$ descents. (See Figure 1.) It is not hard to show that
\begin{equation}
X_1^k = \left\{ \sigma_{p,q}^k \mid 1 \leq p \leq k, \ 0 \leq q \leq k - p \right\}.
\end{equation}

Moreover,
\begin{equation}
X_{m+1} \setminus \mu(X_{m-1}) = \left\{ \sigma_{1,q}^{m+1} \mid 0 \leq q \leq m - 1 \right\} \cup \left\{ \sigma_{p,m+1-p}^{m+1} \mid 1 \leq p \leq m + 1 \right\},
\end{equation}
where the parameter ranges are chosen so that this is a disjoint union. We therefore have that
\begin{equation}
\frac{d}{dt} \text{Mag} \left( tB_2^d \right) \bigg|_{t=0} = \sum_{q=0}^{m-1} \frac{t^{-1} \prod_{\tau \in \sigma_{1,q}^{m+1}} w_2(\tau)}{\prod_{\tau \in \sigma_{\text{root}}} w_2(\tau)} + \sum_{p=1}^{m+1} \frac{t^{-1} \prod_{\tau \in \sigma_{p,m+1-p}^{m+1}} w_2(\tau)}{\prod_{\tau \in \sigma_{\text{root}}} w_2(\tau)}.
\end{equation}
Figure 1. The disjoint 4-collection $\sigma^4_{2,1}$.

By considering only which descents in $\sigma^{m+1}_{p,q}$ are not in $\sigma^{m+1}_{\text{root}}$, and vice versa, we find that

$$t^{-1} \frac{\prod_{\tau \in \sigma^{m+1}_{p,q}} w_2(\tau)}{\prod_{\tau \in \sigma^{m+1}_{\text{root}}} w_2(\tau)} = \frac{\prod_{j=1}^q [2(p+j) - 2]}{\prod_{j=0}^q [2(p+j) - 1]}.$$

With some algebraic manipulation, the right hand side of (16) becomes

$$\sum_{q=0}^{m-1} \binom{q + \frac{1}{2}}{q}^{-1} + \binom{m + \frac{1}{2}}{m}^{-1} \sum_{k=0}^m \binom{k - \frac{1}{2}}{k}.$$

We claim that this sum is equal to $\binom{m - \frac{1}{2}}{m}^{-1}$, which would complete the proof by (13).

To prove this claim, note first that by the generalized binomial theorem, the sequence $\left(\binom{k - \frac{1}{2}}{k}\right)_{k \geq 0}$ has the generating function

$$g(x) = \sum_{k=0}^{\infty} \binom{k - \frac{1}{2}}{k} x^k = (1 - x)^{-3/2}.$$

Therefore the sequence $\left(\sum_{k=0}^{m} \binom{k - \frac{1}{2}}{k}\right)_{m \geq 0}$ has generating function

$$\sum_{m=0}^{\infty} \left(\sum_{k=0}^{m} \binom{k - \frac{1}{2}}{k}\right) x^m = \frac{g(x)}{1 - x} = (1 - x)^{-3/2} = \sum_{m=0}^{\infty} \binom{m + \frac{1}{2}}{m} x^m,$$

and so

$$\sum_{k=0}^{m} \binom{k - \frac{1}{2}}{k} = \binom{m + \frac{1}{2}}{m}$$

for each $m \geq 0$.

The claim thus reduces to showing that

$$\sum_{q=0}^{m-1} \binom{q + \frac{1}{2}}{q}^{-1} = \binom{m - \frac{1}{2}}{m}^{-1} - 1 \quad (17)$$
for $m \geq 0$. This follows by observing that both sides of (17) are 0 when $m = 0$, and that

$$\left(\frac{m + \frac{1}{2}}{m + 1}\right)^{-1} - \left(\frac{m - \frac{1}{2}}{m}\right)^{-1} = \left(\frac{m + \frac{1}{2}}{m}\right)^{-1}.$$

□

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REFERENCES

[1] J. A. Barceló and A. Carbery. On the magnitudes of compact sets in Euclidean spaces. Amer. J. Math., 140:449–494, 2018.
[2] C.-G. Esseen. On mean central limit theorems. Kungl. Tekn. Högsk. Handl. Stockholm, 121:31 pp, 1958.
[3] H. Gimperlein and M. Goffeng. On the magnitude function of domains in Euclidean space. Preprint, available at https://arxiv.org/abs/1706.06839, 2017.
[4] H. Gimperlein and M. Goffeng. On the magnitude function of domains in Euclidean space, III: Questions and examples from a geometric analyst’s perspective (a). Blog post at https://golem.ph.utexas.edu/category/2019/01/on_the_magnitude_function_of_d_2.html, 23 January 2019.
[5] M. Goffeng. Personal communication. 2018.
[6] L. Goldstein. $L^1$ bounds in normal approximation. Ann. Probab., 35(5):1888–1930, 2007.
[7] N. R. Goodman. The distribution of the determinant of a complex Wishart distributed matrix. Ann. Math. Statist., 34:178–180, 1963.
[8] M. Gromov. Metric Structures for Riemannian and Non-Riemannian Spaces. Birkhäuser, Boston, 2001.
[9] R. Hepworth. Magnitude cohomology. Preprint, available at https://arxiv.org/abs/1807.06832, 2018.
[10] R. Hepworth and S. Willerton. Categorifying the magnitude of a graph. Homology, Homotopy, and Applications, 19(2):31–60, 2017.
[11] T. Leinster. Integral geometry for the 1-norm. Advances in Applied Mathematics, 49:81–96, 2012.
[12] T. Leinster. The magnitude of metric spaces. Doc. Math., 18:857–905, 2013.
[13] T. Leinster. The magnitude of a graph. Math. Proc. Camb. Phil. Soc., 166:247–264, 2019.
[14] T. Leinster and M. W. Meckes. Maximizing diversity in biology and beyond. Entropy, 18(3):88, 2016.
[15] T. Leinster and M. W. Meckes. The magnitude of a metric space: from category theory to geometric measure theory. In Measure Theory in Non-Smooth Spaces, Partial Differ. Equ. Meas. Theory, pages 156–193. De Gruyter Open, Warsaw, 2017.
[16] T. Leinster and M. Shulman. Magnitude homology of enriched categories and metric spaces. Preprint, available at https://arxiv.org/abs/1711.00802, 2017.
[17] T. Leinster and S. Willerton. On the asymptotic magnitude of subsets of Euclidean space. Geom. Dedicata, 164:287–310, 2013.
[18] M. W. Meckes. Positive definite metric spaces. Positivity, 17(3):733–757, 2013.
[19] M. W. Meckes. Magnitude, diversity, capacities, and dimensions of metric spaces. Potential Anal., 42(2):549–572, 2015.
[20] N. Otter. Magnitude meets persistence. Homology theories for filtered simplicial sets. Preprint, available at https://arxiv.org/abs/1807.01540, 2018.
[21] R. Schneider and W. Weil. Stochastic and Integral Geometry. Probability and its Applications (New York). Springer-Verlag, Berlin, 2008.
[22] A. R. Solow and S. Polasky. Measuring biological diversity. Environmental and Ecological Statistics, 1:95–107, 1994.
[23] S. S. Wilks. Moment-generating operators for determinants of product moments in samples from a normal system. Ann. of Math. (2), 35(2):312–340, 1934.
[24] S. Willerton. The magnitude of odd balls via Hankel determinants of reverse Bessel polynomials. Preprint, available at https://arxiv.org/abs/1708.03227, 2017.
[25] S. Willerton. Personal communication. 2018.

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