ON CLOSED LIE IDEALS AND CENTER OF GENERALIZED GROUP ALGEBRAS

VED PRAKASH GUPTA, RANJANA JAIN AND BHARAT TALWAR

Abstract. For any locally compact group \( G \) and any Banach algebra \( A \), a characterization of the closed Lie ideals of the generalized group algebra \( L^1(G, A) \) is obtained in terms of left and right actions by \( G \) and \( A \). In addition, when \( A \) is unital and \( G \) is an \([\text{SIN}]\) group, we show that the center of \( L^1(G, A) \) is precisely the collection of all center valued functions which are constant on the conjugacy classes of \( G \). As an application, we establish that \( Z(L^1(G) \otimes A) = Z(L^1(G)) \otimes A \), for a class of groups and Banach algebras. And, prior to these, for any finite group \( G \), the Lie ideals of the group algebra \( \mathbb{C}[G] \) are identified in terms of some canonical spaces determined by the irreducible characters of \( G \).

1. Introduction

An associative algebra \( A \) inherits a canonical Lie algebra structure with respect to the Lie bracket given by the commutator \( [x, y] := xy - yx \). And, a subspace \( L \) of \( A \) is said to be a Lie ideal if \( [L, A] \subseteq L \), where

\[
[L, A] := \text{span}\{[x, a] : x \in L, a \in A\}.
\]

The project of analysis of ideals of various tensor products of operator algebras has attracted some well known operator algebraists and a substantial amount of work has been done in this direction - see, for instance, \([9, 1, 3, 21, 22, 11, 38]\) and the references therein.

Analysis of Lie ideals of associative rings was initiated as early as in 1955 by Herstein \((14, 15)\), for whom the motivation was purely algebraic, and was followed up enthusiastically by Herstein himself and many other algebraists. On the other hand, the study of closed Lie ideals of operator algebras is primarily motivated by the well understood relationship that exists between commutators, projections and closed Lie ideals in \( C^* \)-algebras - see \([33, 26]\). Given its relevance, a good amount of work has also been done to examine the Lie ideals of pure as well as Banach algebras - see \([28, 7, 25, 27, 17, 5, 26, 35]\).

However, unlike the ideals of tensor products of operator algebras, not much was known about the closed Lie ideals of various tensor products of operator algebras. The pioneering works in this direction appear mainly in the works of Marcoux \((25)\) and Brešar et al. \((5)\). Marcoux, basically, identified all the closed Lie ideals of \( A \otimes^\text{min} C(X) \) for a UHF \( C^* \)-algebra \( A \) and a compact space \( X \). And, in 2008, relying heavily on the Lie ideal structure of tensor products of pure algebras, Brešar et al. proved that for a unital Banach algebra (resp., \( C^* \)-algebra) \( A \), the closed Lie ideals of the Banach space projective tensor product \( A \otimes^\gamma K(H) \) and of the Banach space injective tensor product \( A \otimes^\Lambda K(H) \) (if it is a Banach algebra) (resp., of \( A \otimes^\text{min} K(H) \)) are precisely the closed ideals.

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Motivated by these works of Marcoux and Brešar et al., we analyzed the structure of closed Lie ideals of certain ($C^*$ and other) tensor products of $C^*$-algebras in \cite{10, 12, 25, 38}, which also includes a generalization of Marcoux’s result. This article is basically a continuation of this project to the analysis of closed Lie ideals of the projective tensor product of certain Banach algebras. More specifically, in this article, we mainly analyze the closed Lie ideals of the projective tensor product $L^1(G) \otimes \gamma A$ (which is also identified with the generalized group algebra $L^1(G, A)$), for any locally compact group $G$ and any Banach algebra $A$. The main result in this direction is the following characterization:

**Theorem 3.4.** Let $G$ be a locally compact group and $A$ be a Banach algebra. Then, a closed subspace $L$ of $L^1(G, A)$ is a Lie ideal if and only if

$$
\Delta(x^{-1})(f \cdot x^{-1})a - a(x \cdot f) \in L
$$

for every $f \in L$, $x \in G$ and $a \in A$, where $\Delta$ denotes the modular function of group $G$.

It would be appropriate to mention here that there is a subtle difference between the perspectives of some earlier works in this direction and of this article. In a nutshell, the focus of the articles \cite{5, 10, 12, 25, 38} was on characterizing and identifying closed Lie ideals of some Banach algebras and various tensor products of $C^*$-algebras by exploiting, among others, the notion of the Lie normalizer of an ideal, whereas the (above) characterization that we obtain in this article is essentially motivated by a technique that was considered by Laursen (\cite{23}, Theorem 2.2) to characterize the closed (left and right) ideals of the generalized group algebra. Section 3 is devoted to this characterization, few generalities and some applications.

As the title suggests, the other important aspect of this article is the analysis of the center of a generalized group algebra and Section 4 is completely devoted to this theme. There already exist some satisfying results related to the center of a group algebra in literature - see \cite{29, 30, 24}. For instance, it is known that for a locally compact group $G$, the center $\mathcal{Z}(L^1(G))$ is non-trivial if and only if $G$ is an $\text{IN}$ group (\cite{31}). Its analogue for generalized group algebras holds as is shown in Lemma \ref{lem:in-group-center}. Also, for an $\text{IN}$ group $G$, $f \in L^1(G)$ is central if and only if $f$ is constant on the conjugacy classes of $G$ (\cite{24}). Taking cue from this, we prove the following:

**Theorem 4.7.** Let $G$ be a locally compact $\text{IN}$ group and $A$ be a unital Banach algebra. If either $G$ is an $\text{SIN}$ group or $\mathcal{Z}(A) = C_1$, then

$$\mathcal{Z}(L^1(G, A)) = \{f \in L^1(G, \mathcal{Z}(A)) : f \text{ is constant on the conjugacy classes of } G\}.$$ 

On the other hand, for any two $C^*$-algebras $A$ and $B$, Haydon and Wassermann (\cite{13}) showed that $\mathcal{Z}(A \otimes \text{min} B) = \mathcal{Z}(A) \otimes \text{min} \mathcal{Z}(B)$, which was later extended to any $C^*$-tensor product by Archbold (\cite{2}). Similar isometric identification was obtained for the Haagerup tensor product of unital $C^*$-algebras by Smith (\cite{37}) and then for non-unital $C^*$-algebras by Allen, Sinclair and Smith (\cite{11}). Then, the second named author along with Kumar (\cite{20}) obtained a continuous (not necessarily isometric) isomorphism between $\mathcal{Z}(A \hat{\otimes} B)$ and $\mathcal{Z}(A) \hat{\otimes} \mathcal{Z}(B)$, where $\hat{\otimes}$ denotes the operator space projective tensor product; and, very recently, the first and the second named authors established an isometric isomorphism for the Banach space projective tensor product in \cite{11}. However, such an identification is out of reach for Banach algebras in full generality. For instance, if we take the pathological example of a Banach algebra $A$ with the trivial multiplication, i.e., $ab = 0$ for all $a, b \in A$, then $\mathcal{Z}(L^1(G) \otimes^\gamma A) = L^1(G) \otimes^\gamma A$ but $\mathcal{Z}(L^1(G) \otimes^\gamma \mathcal{Z}(A) = (0) \otimes^\gamma A)$ is trivial whenever $G$ is not an $\text{IN}$ group. Interestingly when $A$ is unital, as an application of above mentioned results, we obtain the following:
Theorem 4.13. Let $G$ be a locally compact group and $A$ be a unital Banach algebra. If either
(1) $\mathcal{Z}(L^1(G))$ is complemented in $L^1(G)$ by a projection of norm one with either $G$ discrete or $\mathcal{Z}(A) = \mathbb{C}1_A$, or,
(2) $G$ is an FIA group, then 
$$\mathcal{Z}(L^1(G)) \otimes^\gamma \mathcal{Z}(A) = \mathcal{Z}(L^1(G) \otimes^\gamma A).$$

Prior to obtaining the above results, we identify all the Lie ideals of the group algebra of a finite group in terms of some canonical spaces determined by the irreducible characters of the group, in Section 2.

2. Lie ideals of finite group algebras

To begin with, in this section, we identify the Lie ideals of the (complex) group algebra of a finite group. Since the group algebra of any finite group is a finite direct sum of matrix algebras, we first establish a basic result concerning Lie ideals of such algebras.

It is tempting to assume that for a direct sum of Banach algebras $A = \bigoplus_i A_i$, a closed subspace $L$ of $A$ is a Lie ideal if and only if $L = \bigoplus_i L_i$ for some closed Lie ideals $L_i$ in $A_i$. However, this is not true as we shall see below.

For $n \in \mathbb{N}$, let $M_n$ denote the set of all $n \times n$ complex matrices; $I_n$ and $0_n$ denote its identity and zero matrices, respectively; $O_n := \{0_n\}$ and $sl_n$ denote the set of all trace zero matrices in $M_n$. It is known that $M_n$ possesses precisely $4$ Lie ideals, namely, $(0)$, $\mathbb{C}I_n$, $sl_n$ and $M_n$ (see [14, Theorem 5]). However, there are uncountably many Lie ideals in the direct sums of matrix algebras, as we see from the following observation.

Proposition 2.1. A subspace $L$ of $A := \oplus_{i=1}^k M_n$ is a Lie ideal if and only if it satisfies
$$\oplus_{i=1}^k \delta_i \cdot s_{n_i} \subseteq L \subseteq \oplus_{i=1}^k \delta_i \cdot M_{n_i} + \oplus_{i=1}^k \mathbb{C}I_{n_i},$$
for $\delta_i = \delta_i \in \{0, 1\}$, $1 \leq i \leq k$.

Proof. It is known that a closed subspace $L$ of a $C^*$-algebra $A$ is a Lie ideal if and only if there exists a closed ideal $J$ in $A$ such that $[J, A] \subseteq L \subseteq N(J)$, where $N(J) := \{x \in A : [x, a] \in J \text{ for all } a \in A\}$ (see [5, Corollary 5.26, Theorem 5.27]). Also, $J = \oplus_{i=1}^k J_{n_i}$, where $J_{n_i} = O_{n_i}$ or $M_{n_i}$ for $1 \leq i \leq k$, so that $[J, A] = \oplus_{i=1}^k [J_{n_i}, M_{n_i}]$. Then, note that $[J_{n_i}, M_{n_i}] = sl_{n_i}$ when $J_{n_i} = M_{n_i}$, and $[J_{n_i}, M_{n_i}] = O_{n_i}$ otherwise [14, Theorem 5]. The result now follows from the fact that $N(J) = J + \oplus_{i=1}^k \mathbb{C}I_{n_i}$, which can be verified easily.

Recall that for a finite group $G$, the vector space $\mathbb{C}[G] := \{\sum_{x \in G} a_x x : a_x \in \mathbb{C}\}$ is a unital $*$-algebra with multiplication induced by group multiplication and involution obtained as conjugate linear extension of the map $G \ni x \mapsto x^* := x^{-1} \in G$, and is known as the group algebra of $G$.

A unitary representation $\pi : G \to U(V)$ of $G$ (on an inner product space $V$ - also called a $G$-module and - is written simply as $(V, \pi)$) can be linearly extended to a unital $*$-representation $\pi : \mathbb{C}[G] \to B(V)$ of $\mathbb{C}[G]$, and, conversely, a unital $*$-representation of $\mathbb{C}[G]$ restricts to a unitary representation of $G$ in a bijective way. Note that an arbitrary finite dimensional (complex) representation $\rho : G \to GL(V)$ can be treated as a unitary representation with respect to the inner product $\langle v, w \rangle := \sum_x \langle \rho_x(v), \rho_x(w) \rangle$, where $\langle \cdot, \cdot \rangle_0$ is any inner product on $V$.

The left regular representation $\lambda : G \to U(\ell^2(G))$ of $G$ given by $\lambda_x(\xi)(y) = \xi(x^{-1}y)$ for $x, y \in G, \xi \in \ell^2(G)$, extends to an injective $*$-representation $\lambda : \mathbb{C}[G] \to B(\ell^2(G))$. As a consequence, $\mathbb{C}[G]$ inherits a $C^*$-norm given by $\|x\| = \|\lambda_x\|$ for $x \in \mathbb{C}[G]$. Being finite dimensional,
$\mathbb{C}[G]$ is a $C^*$-algebra with this norm and this is the unique $C^*$-norm on $\mathbb{C}[G]$. If $G$ has $k$ conjugacy classes, say, \{$G_j : 1 \leq j \leq k$\}, then the center of $\mathbb{C}[G]$ is a $k$-dimensional $*$-subalgebra with a basis given by \{$z_j := \sum_{x \in C_j} x : 1 \leq j \leq k$\}. Thus, from the structure theorem of (unital) finite dimensional $C^*$-algebras, there exists an algebra $*$-isomorphism $\psi : \mathbb{C}[G] \to \oplus_{j=1}^k M_{n_j}$, where each $n_j$ is the dimension of an irreducible unital $*$-representation of $\mathbb{C}[G]$ (and hence of $G$).

**Remark 2.2.** An algebra isomorphism from $\mathbb{C}[G]$ onto $\oplus_{j=1}^k M_{n_j}$ can also be obtained from purely algebraic methods. For instance, by Artin-Molien-Wedderburn Theorem, there exists an algebra isomorphism $\psi : \mathbb{C}[G] \to \oplus_{j=1}^k M_{n_j}$, where each $n_j$ is the dimension of an irreducible $G$-module.

For any finite dimensional complex representation $(V, \pi)$ of $G$, its character $\chi : G \to \mathbb{C}$ is given by $\chi(x) := \text{trace}(\pi(x))$, where trace is not normalized and is defined with respect to one (equivalently, any) basis of $V$. Each such character $\chi$ extends linearly to a map $\tilde{\chi} : \mathbb{C}[G] \to \mathbb{C}$ and, by linearity of the trace map, we have $\tilde{\chi}(z) = \text{trace}(\pi(z))$ for all $z \in \mathbb{C}[G]$. Two representations of $G$ are equivalent if and only if they have same characters (see [18, Theorem 14.21]). The character of an irreducible representation (equivalently, all equivalent irreducible representations) of $G$ is, in short, also called an irreducible character of $G$. If $G$ has $k$ conjugacy classes, then $G$ has precisely $k$ distinct irreducible characters. We assert the following identification of Lie ideals of group algebras.

**Theorem 2.3.** Let $G$ be a finite group with $k$ conjugacy classes. Let \{${\chi}_j : 1 \leq j \leq k$\} denote the set of irreducible characters of $G$ and $n_j$ denote the dimension of any representation corresponding to $\chi_j$. Let $\omega_j := \frac{1}{|G|} \sum_{x \in G} \chi_j(x^{-1})x \in \mathbb{C}[G]$ and $K_j$ denote the ideal generated by $\omega_j$ for $1 \leq j \leq k$. Then,

1. \{$\omega_j : 1 \leq j \leq k$\} is the set of minimal central projections of $\mathbb{C}[G]$, and
2. \{$K_j : 1 \leq j \leq k$\} is the set of distinct non-zero minimal ideals of $\mathbb{C}[G]$.

In particular, a subspace $L$ of $\mathbb{C}[G]$ is a Lie ideal if and only if

$$\sum_{j=1}^k \delta_j \ker(\tilde{\chi}_j) \omega_j \subseteq L \subseteq \sum_{j=1}^k \delta_j K_j + \text{span}\{\omega_j : 1 \leq j \leq k\},$$

for $\delta_j = \tilde{\delta}_j \in \{0, 1\}$, $1 \leq j \leq k$.

In order to prove Theorem 2.3 in view of Proposition 2.1 and the isomorphism $\psi : \mathbb{C}[G] \to \oplus_{j=1}^k M_{n_j}$, it suffices to make the following observations, which are most probably folklore. We include the details just for the sake of completeness.

**Proposition 2.4.** Let $G$, $\chi_j$, $\omega_j$ and $\tilde{\chi}_j$ for $1 \leq j \leq k$ be as in Theorem 2.3 and $\psi : \mathbb{C}[G] \to \oplus_{j=1}^k M_{n_j}$ be a $*$-isomorphism. Then,

1. $\omega_j = \psi^{-1}((0_{n_1}, \ldots, 0_{n_{j-1}}, I_{n_j}, 0_{n_{j+1}}, \ldots, 0_{n_k}))$; so that $\psi^{-1}(M_{n_j}) = K_j$, and
2. $\ker(\tilde{\chi}_j) = \psi^{-1}(M_{n_1} \oplus \cdots \oplus M_{n_{j-1}} \oplus s_{n_j} \oplus M_{n_{j+1}} \oplus \cdots \oplus M_{n_k})$; so that $\ker(\tilde{\chi}_j) \omega_j = \psi^{-1}(O_{n_1} \oplus \cdots \oplus O_{n_{j-1}} \oplus s_{n_j} \oplus O_{n_{j+1}} \oplus \cdots \oplus O_{n_k})$

for all $1 \leq j \leq k$.

**Proof.** Note that the $C^*$-algebra $\oplus_{j=1}^k M_{n_j}$ possesses a natural inner product given by

$$\langle (X_1, \ldots, X_k), (Y_1, \ldots, Y_k) \rangle = \sum_{j=1}^k \text{trace}_{M_{n_j}}(Y_j^* X_j),$$
where trace is not normalized. Let $\widetilde{M}_{n_j} := O_{n_1} \oplus \cdots \oplus O_{n_j-1} \oplus M_{n_j} \oplus O_{n_{j+1}} \oplus \cdots \oplus O_{n_k}$. Then, $\widetilde{M}^\perp_{n_j} = M_{n_1} \oplus \cdots \oplus M_{n_{j-1}} \oplus O_{n_j} \oplus M_{n_{j+1}} \oplus \cdots \oplus M_{n_k}$. Also, let $U(n_j)$ denote the set of all unitaries in $M_{n_j}$.

The proof now is mainly a book keeping exercise. Let $A := \bigoplus_{j=1}^k M_{n_j}$. For each $1 \leq j \leq k$, consider the $*$-representation $\theta_j : A \rightarrow M_{n_j} = B(\mathbb{C}^{n_j})$ given by $\theta_j((X_1, \ldots, X_k)) = X_j$. Then, $\{\theta_j : 1 \leq j \leq k\}$ constitutes a complete set of inequivalent irreducible $*$-representations of the $C^*$-algebra $A$.

Taking $\pi_j = (\theta_j \circ \psi)|_{C} : G \rightarrow U(n_j)$, we observe that $\{\pi_j : 1 \leq j \leq k\}$ forms a complete set of inequivalent irreducible representations of $G$; so that $\chi_j = \chi_{\pi_j}$ for all $1 \leq j \leq k$.

Next, consider the canonical $*$-representation $\Gamma : A \rightarrow B(A)$ of $A$ given by $\Gamma(a)(b) = ab$ for $a, b \in A$, with respect to the canonical inner product on $A$ described in the preceding paragraph. Then, $\widetilde{M}_{n_j}$ is an $A$-submodule of $A$. To make it explicit, let this $A$-module be denoted by $(\widetilde{M}_{n_j}, \rho_j)$. Then, $\rho_j \equiv (\theta_j \oplus \cdots \oplus \theta_j)$ ($n_j$-fold direct sum) as $A$-modules for all $1 \leq j \leq k$.

(1): Let $e_j := \psi^{-1}((0, n_1, \ldots, 0_{n_{j-1}}, I_{n_j}, 0_{n_{j+1}}, \ldots, 0_{n_k}))$ for $1 \leq j \leq k$. Consider $\mathbb{C}[G]$ with the inner product induced by the isomorphism $\psi$. Then, again via $\psi$ and $\Gamma$, $\mathbb{C}[G]$ becomes a $\mathbb{C}[G]$-module (equivalently, $G$-module) and the action is again the left multiplication on $\mathbb{C}[G]$. As a result, $\psi^{-1}(M_{n_j})$ and $\psi^{-1}(\widetilde{M}^\perp_{n_j})$ are $\mathbb{C}[G]$-submodules of $\mathbb{C}[G]$.

Now, let $\varphi_j := (\psi^{-1}|_{\widetilde{M}_{n_j}} : \widetilde{M}_{n_j} \rightarrow \psi^{-1}(\widetilde{M}_{n_j})$ for $1 \leq j \leq k$. Then, $\varphi_j$ is a $G$-module isomorphism between $\widetilde{M}_{n_j}$ and $\psi^{-1}(\widetilde{M}_{n_j})$: thus, the $G$-module $\psi^{-1}(\widetilde{M}_{n_j})$ is also isomorphic to $n_j$-copies of $\pi_j := (\theta_j \circ \psi)|_{C}$, so its character is given by $n_j \chi_j$ (see, for instance, [13] Page 141]) and $\psi^{-1}(\widetilde{M}_{n_j})$ and $\psi^{-1}(\widetilde{M}^\perp_{n_j})$ contain no isomorphic irreducible $\mathbb{C}[G]$-submodules. Also, $\mathbb{C}[G] = \psi^{-1}(\widetilde{M}_{n_j}) \oplus \psi^{-1}(\widetilde{M}^\perp_{n_j})$ and $1 = e_j + \tilde{e}_j$, where $\tilde{e}_j := \psi^{-1}((I_{n_1}, \ldots, I_{n_{j-1}}, 0_{n_j}, I_{n_{j+1}}, \ldots, I_{n_k})) \in \widetilde{M}^\perp_{n_j}$. So, by [13] Proposition 14.10, we obtain $e_j = \sum_{x \in G} n_j \chi_j(x^{-1}) x = \omega_j$.

(2): Let $(X_1, \ldots, X_k) \in M_{n_1} \oplus \cdots \oplus M_{n_{j-1}} \oplus sl_{n_j} \oplus M_{n_{j+1}} \oplus \cdots \oplus M_{n_k}$. Then,
\[
\tilde{\chi}_j(\psi^{-1}((X_1, \ldots, X_k))) = \text{trace}(\theta_j \circ \psi \circ \psi^{-1}((X_1, \ldots, X_k))) = \text{trace}(\chi_j) = 0.
\]

So, $\psi^{-1}(M_{n_1} \oplus \cdots \oplus M_{n_{j-1}} \oplus sl_{n_j} \oplus M_{n_{j+1}} \oplus \cdots \oplus M_{n_k}) \subseteq \ker(\tilde{\chi}_j)$. And, both subspaces, being of co-dimension 1 in $\mathbb{C}[G]$, are therefore equal.

As an application, we provide a non-central Lie ideal in $\mathbb{C}[D_6]$ which is not an ideal.

**Example 2.5.** Let $D_6 = \langle r, s : r^3 = 1, s^2 = 1, srs = r^{-1} \rangle$ denote the Dihedral group consisting of symmetries of an equilateral triangle. $D_6$ has three conjugacy classes and its irreducible characters are well understood - see [13] Page 121]. If $\{\chi_1, \chi_2, \chi_3\}$ forms a complete set of characters of irreducible representations of $D_6$, then $\chi_3$ can be taken to be the one that satisfies $\chi_3(e) = 2$, $\chi_3(r) = \chi_3(r^2) = -1$ and $\chi_3(s) = \chi_3(rs) = \chi_3(r^2s) = 0$.

Let $L := \ker(\chi_3) \omega_4$. Then, by Theorem 2.3, $L$ is a Lie ideal in $\mathbb{C}[D_6]$, and it is non-central. Also, it is routine to check that $L = \{c_1(r - r^2) + c_2(s - r^2s) + c_3(rs - r^2s) : c_i \in \mathbb{C}\}$, so, being a 3-dimensional subspace, it can not be an ideal of $\mathbb{C}[D_6] \cong \mathbb{C} \oplus \mathbb{C} \oplus M_2$.

### 3. Closed Lie ideals of generalized group algebras

First, recall that for a general measure space $(\Omega, \mathcal{M}, \mu)$ and a Banach space $X$, a function $f : \Omega \rightarrow X$ is said to be
(1) simple \((\mathcal{M}, \mu)\)-measurable if \(f\) takes only finitely many values and \(f^{-1}(x) \in \mathcal{M}\) for every \(x \in X\).

(2) \((\mathcal{M}, \mu)\)-measurable if there exists a sequence \(\{s_n\}\) of \(X\)-valued simple \((\mathcal{M}, \mu)\)-measurable functions on \(\Omega\) such that \(s_n \to f\) a.e. \(|\mu|\).

(3) Bochner integrable if there exists a sequence \(\{s_n\}\) of \(X\)-valued simple \((\mathcal{M}, \mu)\)-measurable functions on \(\Omega\) with \(\mu(\text{supp}(s_n)) < \infty\) for all \(n\) such that \(s_n \to f\) a.e. \(|\mu|\) and 
\[
\int_{\Omega} \|f - s_n\|d\mu \to 0, \text{ where } \|g\| \text{ denotes the scalar valued function } x \mapsto \|g(x)\|.
\]

It is known that \(f\) is Bochner integrable if and only if \(f\) is \((\mathcal{M}, \mu)\)-measurable and \(\|f\| \in L^1(\Omega, \mu)\). For such an \(f\), its Bochner integral is defined as 
\[
\int_{\Omega} f(x)d\mu = \lim_n \int_{\Omega} s_n(x)d\mu.
\]

The space of \(X\)-valued Bochner integrable functions on \(\Omega\) is denoted by \(L^1(\Omega, X)\). See [30] for further details.

We now come to the objects of present interest, namely, generalized group algebras. As is standard, by a locally compact group we shall mean a topological group which is both Hausdorff and locally compact.

Let \(G\) be a locally compact group with a left Haar measure \(m\). Then, there exists a continuous group homomorphism \(\Delta : G \to \mathbb{R}_{>0}\) such that \(m(Sx) = \Delta(x)m(S)\) for all \(S \in B_G\) and \(x \in G\), where \(B_G\) denotes the Borel \(\sigma\)-algebra of \(G\). This group homomorphism is known as the modular function of \(G\). It is a well known fact that when \(A\) is a Banach algebra, the space \(L^1(G, A)\), known as a generalized group algebra, is a Banach algebra with respect to the multiplication given by the convolution \((f \ast g)(x) := \int_G f(xs)g(s^{-1})ds\) and norm given by 
\[
\|f\|_1 = \int_G \|f(x)\|dx.
\]

For every \(a \in A\) and \(f \in L^1(G)\), define \(fa : G \to A\) by \((fa)(x) = f(x)a\) for all \(x \in G\). It is easily seen that \(fa \in L^1(G, A)\) for all \(a \in A\) and \(f \in L^1(G)\). We have the following well known identification.

**Theorem 3.1.** The map \(L^1(G) \otimes A \ni \sum f_i \otimes a_i \mapsto \sum f_i a_i \in L^1(G, A)\) extends to an isometric isomorphism from \(L^1(G) \otimes A\) onto \(L^1(G, A)\).

In particular, the subspace \(\text{span}\{fa : f \in L^1(G), a \in A\}\) is dense in \(L^1(G, A)\).

As per the requirements of this article, we discuss some useful properties, whose analogues are well known for group algebras (see, for instance, [19], [31]).

For each \(x \in G\) and \(f \in L^1(G, A)\), one defines \(x \cdot f, f \cdot x : G \to A\) by 
\[
(x \cdot f)(y) := f(x^{-1}y) \text{ and } (f \cdot x)(y) := f(yx) \text{ for all } y \in G;
\]

It can be easily verified that \(x \cdot f, f \cdot x \in L^1(G, A)\) for all \(x \in G\) and \(f \in L^1(G, A)\).

**Lemma 3.2.** For any \(f \in L^1(G, A)\), the mapping \(G \ni x \mapsto \Delta(x^{-1})f(x^{-1}) \in A\) is Bochner integrable and 
\[
(3.1) \quad \int_G f(x)dx = \int_G \Delta(x^{-1})f(x^{-1})dx.
\]

Also, for any \(y \in G\) and \(f \in L^1(G, A)\), we have 
\[
(3.2) \quad \int_G (f \cdot y)(x)dx = \Delta(y^{-1}) \int_G f(x)dx.
\]

**Proof.** Fix a sequence \(\{s_n\} \subset L^1(G, A)\) of simple \((B_G, m)\)-measurable functions converging almost everywhere to \(f\) and satisfying \(\lim_{n \to \infty} \int_G \|f - s_n(x)\|dx = 0\). Since the sequence of scalar functions \(\{\|f - s_n\|\}\) is contained in \(L^1(G)\), from [31] page 1484, we see that the map
where the second equality holds because each $s_n$ functions such that it converges almost everywhere to $f$. In order to prove (3.2), fix a sequence \( \{x_n\} \subset L^1(G,A) \) of simple \((B_G, m)\)-measurable functions such that it converges almost everywhere to $f$ and $\lim_{n \to \infty} \|(f \cdot y)(x) - t_n(x)\|dx = 0$. From [31] page 1484 again, we obtain

\[
\int_G (\|f \cdot y - t_n\| \cdot y^{-1})(x)dx = \Delta(y) \int_G \|f \cdot y - t_n\|dx
\]

for all $n$. In particular,

\[
(3.3) \int_G \|f - t_n \cdot y^{-1}\|dx = \int_G (\|f \cdot y - t_n\| \cdot y^{-1})(x)dx = \Delta(y) \int_G \|f \cdot y - t_n\|dx \to 0.
\]

Thus,

\[
\int_G (f \cdot y)(x)dx = \lim_n \int_G t_n(x)dx = \lim_n \int_G t_n(xy^{-1}y)dx = \Delta(y^{-1}) \lim_n \int_G t_n(xy^{-1}y)dx = \Delta(y^{-1}) \int_G f(x)dx.
\]

This completes the proof.

\[\square\]

**Lemma 3.3.** For each $f \in L^1(G,A)$, the maps $G \ni x \mapsto x \cdot f, f \cdot x \in L^1(G,A)$ are continuous.

**Proof.** Let $\epsilon > 0$. Fix an $f_0 = \sum_{i=1}^r g_i a_i \in \text{span}\{g \cdot a \in L^1(G) : a \in A\}$ such that $\|f - f_0\| < \epsilon/3$. For continuity of left multiplication, note that for any $x \in G$, we have

\[
\|x \cdot f - f\|_1 = \|x \cdot (\sum_{i=1}^r g_i a_i) - \sum_{i=1}^r g_i a_i\|_1 \leq \sum_{i=1}^r \|x \cdot g_i - g_i\|_1 \|a_i\|.
\]

Using continuity of left multiplication by elements of $G$ in $L^1(G)$ (see [19] §A.4]), there exists a neighbourhood $V$ of $e$ such that $\sup_{1 \leq i \leq r} \|x \cdot g_i - g_i\|_1 \leq sup\{\|a_i\| : 1 \leq i \leq r\}^{-1} \epsilon/3r$ for all $x \in V$. Then, using left invariance of Haar measure, we obtain $\|x \cdot f - x \cdot f_0\|_1 = \|f - f_0\|_1 \leq \epsilon/3$ for all $x \in G$. Thus, we have $\|x \cdot f - f\|_1 < \epsilon$ for all $x \in V$. This proves continuity at $e$.

Now, let $x \in G$ and $\{x_n\}$ be a net converging to $x$ in $G$. Then, $x^{-1}x_n \to e$ and hence,

\[
\|x_n \cdot f - x \cdot f\|_1 = \|x_n \cdot (x^{-1}x_n \cdot f - f)\|_1 = \|(x^{-1}x_n \cdot f - f)\|_1 \to 0,
\]

where the second equality follows by the left invariance of Haar measure.
And, for continuity of the right multiplication at \( e \), following above steps, it suffices to show that \( \| f \cdot x - f_0 \cdot x \|_1 \leq \epsilon/3 \) in a neighbourhood of \( e \), which follows readily from the continuity of \( \Delta \) and the relation

\[
\| f \cdot x - f_0 \cdot x \|_1 = \int_G \| ((f - f_0) \cdot x)(y) \| dy = \Delta(x^{-1}) \int_G \| (f - f_0)(y) \| dy,
\]

where the last equality follows from the preceding lemma. Once continuity at \( e \) is established, then continuity at an arbitrary \( x \in G \) is obtained on similar lines as above. \( \square \)

That was all that we will require from the basic theory. We now proceed to characterize the closed Lie ideals of generalized group algebras.

For every \( a \in A \) and \( f \in L^1(G, A) \), consider \( af, fa : G \to A \) given by \((af)(x) = af(x)\) and \((fa)(x) = f(x)a\) for all \( x \in G \). It is easily seen that \( af, fa \in L^1(G, A) \) for all \( a \in A \) and \( f \in L^1(G, A) \). In 1969, Laursen [23] had shown if \( A \) is a Banach algebra with approximate identity, then a closed subspace \( I \) of \( L^1(G, A) \) is an ideal if and only if \( I \) is \( A \)-translation invariant, i.e.,

\[
f \cdot x, x \cdot f, af, fa \in I \text{ for all } f \in I, a \in A, x \in G.
\]

It is easily seen that, for a countable discrete group \( G \) with the (unimodular) counting Haar measure and any Banach algebra \( A \), a closed subspace \( L \) of \( L^1(G, A) \) is a Lie ideal if and only if

\[
(f \cdot x^{-1})a - a(x \cdot f) \in L \text{ for all } f \in L, a \in A, x \in G.
\]

Indeed, setting \( \chi_x := \chi_x \) for \( x \in G \), since \( \text{span}\{a \chi_x : a \in A, x \in G\} \) is dense in \( L(G, A) \), a closed subspace \( L \) of \( L(G, A) \) is a Lie ideal if and only if

\[
f \ast (a \chi_x) - (a \chi_x) \ast f \in L \text{ for all } f \in L, a \in A, x \in G.
\]

Then, note that

\[
(f \ast (a \chi_x))(t) = \sum_{s \in G} f(ts)a \chi_x(s^{-1}) = f(t x^{-1})a = ((f \cdot x^{-1})a)(t), \text{ i.e., } f \ast (a \chi_x) = (f \cdot x^{-1})a;
\]

and, likewise, we have \((a \chi_x) \ast f = a(x \cdot f)\).

Unlike the collection \( \{ \chi_x : x \in G \} \) for a discrete group \( G \), there is no canonical way of identifying a copy of \( G \) in \( L^1(G) \) when \( G \) is not discrete. Thus, for an arbitrary locally compact group \( G \), we need a more analytical approach. We obtain the following general form of the characterization observed in [3,4].

**Theorem 3.4.** Let \( G \) be a locally compact group and \( A \) be a Banach algebra. Then, a closed subspace \( L \) of \( L^1(G, A) \) is a Lie ideal if and only if

\[
\Delta(x^{-1})(f \cdot x^{-1})a - a(x \cdot f) \in L
\]

for every \( f \in L, x \in G \) and \( a \in A \).

**Proof.** Let \( L \) be a closed Lie ideal in \( L^1(G, A) \), \( f \in L, x \in G \) and \( a \in A \). Since \( L \) is closed, it is enough to show that \( \Delta(x^{-1})(f \cdot x^{-1})a - a(x \cdot f) \in L \). Let \( \epsilon > 0 \).

For any Borel set \( U \) in \( G \) and \( a \in A \), let \( a_U := a \chi_U \). If \( U \) is compact, then \( a_U \in L^1(G, A) \) for all \( a \in A \). So, it is enough to find a compact neighbourhood \( V \) of \( e \in G \) such that

\[
\| \Delta(x^{-1})(f \cdot x^{-1})a - a(x \cdot f) - \frac{1}{m(V)}(f \ast (a_U \chi_V) - (a_U \chi_V) \ast f) \|_1 < \epsilon.
\]

...
Note that for an arbitrary compact symmetric neighbourhood $V$ of $e$, $a \in A$ and $x \in G$, we have

$$m(Vx^{-1}) \| \Delta(x^{-1})(f \cdot x^{-1})a - a(x \cdot f) - \frac{1}{m(V)}(f \ast (a_{xV}) - (a_{xV}) \ast f) \|_1 \leq \| m(Vx^{-1}) \Delta(x^{-1})(f \cdot x^{-1})a - \frac{m(Vx^{-1})}{m(V)} f \ast (a_{xV}) \|_1$$

$$+ \| m(Vx^{-1})a(x \cdot f) - \frac{m(Vx^{-1})}{m(V)}(a_{xV}) \ast f \|_1. \quad (*)$$

Let $E := \{ y \in G : \| (m(Vx^{-1})(f \cdot x^{-1})a - f \ast (a_{xV}))(y) \| > 0 \}$. It is easily seen that $E$ is a $\sigma$-finite Borel subset of $G$. We have

$$\| m(Vx^{-1}) \Delta(x^{-1})(f \cdot x^{-1})a - \frac{m(Vx^{-1})}{m(V)} f \ast (a_{xV}) \|_1$$

$$= \Delta(x^{-1}) \int_G \| (m(Vx^{-1})(f \cdot x^{-1})a - f \ast (a_{xV}))(y) \| dy \quad \text{(since $m(Vx^{-1}) = \Delta(x^{-1})m(V)$)}$$

$$= \Delta(x^{-1}) \int_E \| (f \cdot x^{-1})(y)a \int_{Vx^{-1}} ds - \int_G f(y)s_{xV}(s^{-1})ds \| dy$$

$$= \Delta(x^{-1}) \int_E \| \int_{Vx^{-1}} (f \cdot x^{-1})(y)ads - \int_{Vx^{-1}} f(y)ads \| dy \quad \text{(since $s^{-1} \in Vx^{-1} \iff s \in Vx^{-1}$)}$$

$$= \Delta(x^{-1}) \int_E \| \int_{Vx^{-1}} ((f \cdot x^{-1})a - (f \cdot s)a)(y)ds \| dy$$

$$\leq \Delta(x^{-1}) \int_{Vx^{-1}} \| ((f \cdot x^{-1})a - (f \cdot s)a)(y) \| dsdy$$

$$= \Delta(x^{-1}) \int_{Vx^{-1}} \int_E \| ((f \cdot x^{-1})a - (f \cdot s)a)(y) \| dyds \quad \text{(by Tonelli’s Theorem)}$$

$$\leq \Delta(x^{-1}) \int_{Vx^{-1}} \| (f \cdot x^{-1})a - (f \cdot s)a \|_1 ds$$

$$\leq \Delta(x^{-1})m(Vx^{-1}) \sup_{s \in Vx^{-1}} \| (f \cdot x^{-1} - f \cdot s)a \|_1$$

$$= \Delta(x^{-1})m(Vx^{-1}) \sup_{t \in V} \| (f \cdot x^{-1} - f \cdot (tx^{-1}))a \|_1$$

$$= \Delta(x^{-1})m(Vx^{-1}) \sup_{t \in V} \| (f \cdot x^{-1} - f \cdot (t^{-1}x^{-1}))a \|_1 \quad \text{(since $V$ is symmetric)}$$

$$\leq \Delta(x^{-1})m(Vx^{-1}) \sup_{t \in V} \| f \cdot x^{-1} - (f \cdot x^{-1}) \cdot t^{-1} \|_1 \| a \|;$$

and, on the other hand, considering the $\sigma$-finite Borel set $E' := \{ y \in G : \| (m(V)a(x \cdot f) - a_{xV} \ast f)(y) \| > 0 \}$, we obtain

$$\| m(Vx^{-1})a(x \cdot f) - \frac{m(Vx^{-1})}{m(V)}(a_{xV}) \ast f \|_1$$

$$= \Delta(x^{-1}) \int_G \| (m(V)a(x \cdot f) - a_{xV} \ast f)(y) \| dy$$

$$= \Delta(x^{-1}) \int_{E'} \| af(x^{-1}y) \int_V ds - \int_G a_{xV}(ys)f(s^{-1})ds \| dy$$

$$\leq \Delta(x^{-1})\| a \| \int_V \int_{E'} \| (x \cdot f)(y) - (xs \cdot f)(y) \| dyds$$
\[ \leq \Delta(x^{-1})m(V) \sup_{s \in V} \|a\| \|f - s \cdot f\|_1, \quad \text{(since } \|x \cdot \varphi\|_1 = \|\varphi\|_1 \forall \varphi \in L^1(G, A)) \]

where the second last inequality follows by left invariance and Tonelli’s Theorem.

Note that, by Lemma 4.3 for each \( \varphi \in L^1(G, A) \), the maps \( G \ni s \mapsto s \cdot \varphi, \varphi \cdot s \in L^1(G, A) \) are continuous. So, we can choose a compact symmetric neighbourhood \( V_1 \) of \( e \) such that

\[ \sup_{s \in V_1} \|f - s \cdot f\|_1 < \frac{\epsilon}{2\|a\|}, \]

and another compact symmetric neighbourhood \( V_2 \) of \( e \) such that

\[ \sup_{s \in V_2} \|f \cdot x^{-1} - f \cdot (xs)^{-1}\|_1 < \frac{\epsilon}{2\|a\|\Delta(x^{-1})}. \]

Using a compact neighbourhood \( e \) of \( e \) satisfying \( V \subseteq V_1 \cap V_2 \) in (*), we readily obtain

\[ \|\Delta(x^{-1})((f \cdot x^{-1})a) - (a(x \cdot f)) - \frac{1}{m(V)}(f \cdot (a_{x,V}) - (a_{x,V} \cdot f))\|_1 \leq \epsilon \]

for all \( x \in V \), as was desired.

Conversely, suppose that \( L \) is a closed subspace of \( L^1(G, A) \) such that

\[ \Delta(x^{-1})(f \cdot x^{-1})a - a(x \cdot f) \in L \quad \text{for all } f \in L, x \in G \text{ and } a \in A. \]

We show that \( L \) is a Lie ideal. Since \( C_c(G, A) \) is dense in \( L^1(G, A) \), it is enough show that \( \varphi \ast f - f \ast \varphi \in \overline{L} \) for every \( f \in L \) and \( \varphi \in C_c(G, A) \). Let \( f \in L, \varphi \in C_c(G, A) \) and \( \epsilon > 0 \). Let \( F \coloneqq \operatorname{supp}(\varphi) \). Then, \( K \coloneqq F \cup F^{-1} \) is a symmetric compact set containing the support of \( \varphi \).

As left multiplication, right multiplication, inverse function, the function \( \varphi \) and the modular function are all continuous, there exists a mutually disjoint covering (not necessarily open) \( K_1, \ldots, K_r \) of \( K \) such that

\[ \|\varphi(y)(y \cdot f) - \Delta(y^{-1})(f \cdot y^{-1})\varphi(y) - \varphi(z)(z \cdot f) + \Delta(z^{-1})(f \cdot z^{-1})\varphi(z)\|_1 \leq \epsilon / m(K) \]

for all \( y, z \in K_i, 1 \leq i \leq r \). Fix \( y_i \in K_i \) for every \( 1 \leq i \leq r \). Then,

\[ \|\varphi \ast f - f \ast \varphi - \sum_i m(K_i)(\varphi(y_i)(y_i \cdot f) - \Delta(y_i^{-1})(f \cdot y_i^{-1})\varphi(y_i))\|_1 \]

\[ = \int_G \|\varphi \ast f(x) - \sum_i m(K_i)\varphi(y_i)(y_i \cdot f)(x) - (f \ast \varphi)(x) \]

\[ + \sum_i m(K_i)\Delta(y_i^{-1})(f \cdot y_i^{-1})(x)\varphi(y_i)\| dx \]

\[ = \int_G \int_G \varphi(xy)f(y^{-1})dy - \sum_i \int_{K_i} \varphi(y_i)(y_i \cdot f)(x)dy - \int_G f(xy)\varphi(y^{-1})dy \]

\[ + \sum_i \int_{K_i} \Delta(y_i^{-1})(f \cdot y_i^{-1})(x)\varphi(y_i)dy \]

\[ = \int_G \int_G \varphi(y)f(y^{-1}x)dy - \sum_i \int_{K_i} \varphi(y_i)(y_i \cdot f)(x)dy - \int_K \Delta(y^{-1})(f \cdot y^{-1})(x)\varphi(y)dy \]

\[ + \sum_i \int_{K_i} \Delta(y_i^{-1})(f \cdot y_i^{-1})(x)\varphi(y_i)dy \] (by 3.4)

\[ = \sum_i \int_{K_i} \int_G \|\varphi(y)(y \cdot f)(x) - \varphi(y_i)(y_i \cdot f)(x) - \Delta(y^{-1})(f \cdot y^{-1})(x)\varphi(y) \]

\[ + \Delta(y_i^{-1})(f \cdot y_i^{-1})(x)\varphi(y_i)\| dx \]

\[ = \sum_i \int_{K_i} \int_G \|\varphi(y)(y \cdot f)(x) - \varphi(y_i)(y_i \cdot f) - \Delta(y^{-1})(f \cdot y^{-1})(x)\varphi(y) + \Delta(y_i^{-1})(f \cdot y_i^{-1})\varphi(y_i)\|_1 dy \]
is an ideal if and only if \( f \) as we illustrate in the following example.

**Corollary 3.8.** Let \( G \) be a locally compact group. Then, a closed subspace \( L \) of \( L^1(G) \) is a Lie ideal if and only if 

\[
\Delta(x^{-1})f \cdot x^{-1} - x \cdot f \in L
\]

for all \( f \in L \) and \( x \in G \).

**Remark 3.6.** When the Haar measure is \( \sigma \)-finite (equivalently, \( G \) is \( \sigma \)-compact), \( L^\infty(G) \) is the dual space of \( L^1(G) \); so, sufficiency in Corollary 3.5 follows easily when \( A = \mathbb{C} \). Indeed, for \( f \in L, h \in L^1(G) \) and \( \phi \in L^1 \), we have

\[
\phi(f \ast h - h \ast f) = \int_G \phi(x)(f \ast h - h \ast f)(x)dx
\]

\[
= \int_G \phi(x) \left( \int_G f(xy)h(y^{-1})dy - \int_G h(xy)f(y^{-1})dy \right)dx
\]

\[
= \int_G \phi(x) \left( \int_G h(y^{-1})f(xy)dy - \int_G h(y)f(y^{-1}x)dy \right)dx
\]

\[
= \int_G \phi(x) \left( \int_G h(y)f(xy^{-1})\Delta(y^{-1}) - h(y)f(y^{-1}x) \right)dydx
\]

\[
= \int_G h(y) \left( \int_G \phi(x)(\Delta(y^{-1})f \cdot y^{-1} - y \cdot f)(x)dx \right)dy = 0.
\]

So, by Hahn-Banach Theorem, we deduce that \( L \) is a closed Lie ideal in \( L^1(G) \).

Since \( (G \times \{e_H\}) \cdot \{(e_G) \times H\} = G \times H \), it is easily seen that a subspace \( I \) of \( L^1(G \times H) \) is an ideal if and only if \( f \cdot x \) and \( x \cdot f \in I \) for every \( f \in I \) and \( x \in (G \times \{e_H\}) \cup \{(e_G) \times H\} \). However, these elements are not enough to characterize the closed Lie ideals of \( L^1(G \times H) \), as we illustrate in the following example.

**Example 3.7.** Consider \( G = H = D_6 \). Consider the subspace \( L \) of \( \mathbb{C}[G \times H] \) given by

\[
L = \{c_1(r,e) - c_1(r^2,e) + c_2(s,e) - c_2(r^2s,e) + c_3(rs,e) - c_3(r^2s,e) : c_1, c_2, c_3 \in \mathbb{C} \}.
\]

Since the modular function of a finite group is identically 1, it is readily seen that \( \Delta(x^{-1})f \cdot x^{-1} - x \cdot f \in L \) for every \( f \in L \) and \( x \in (D_6 \times \{e\}) \cup \{(e) \times D_6 \} \). However, \( L \) is not a Lie ideal in \( \mathbb{C}[G \times H] \), as after some routine calculation, we observe that

\[
\begin{align*}
(r,e) - (r^2,e) + (s,e) - (r^2s,e) + (rs,e) - (r^2s,e) \in (r,r) \\
- (r,e) - (r^2,e) + (s,e) - (r^2s,e) + (rs,e) - (r^2s,e) \in (e) \\
= -(r^3, r) + (s, r) - (r^2, r) + (r^2, r) + (s, r) + (s, r) \\
= -3(r^3, r) + 3(s, r) \notin L.
\end{align*}
\]

However, when \( H \) is abelian, then a smaller collection of \( G \times H \) with an extra condition provides sufficiency for a subspace to be a Lie ideal.

**Corollary 3.8.** Let \( G \) and \( H \) be locally compact groups and \( H \) be abelian. Let \( A \) be a Banach algebra and \( L \) be a closed subspace of \( L^1(G \times H, A) \). Then, \( L \) is a Lie ideal if

\[
w \cdot f \in L \quad \text{and} \quad \Delta(z^{-1})(f \cdot z^{-1})a - a(z \cdot f) \in L
\]
for all \( w \in \{ e_G \} \times H, f \in L, a \in A \) and \( z \in G \times \{ e_H \} \).

**Proof.** By Theorem 3.4, it suffices to show that \( \Delta((x, y^-1)) f \cdot ((x, y^-1) - a((x, y) \cdot f) \in L \)
for all \( f \in L, a \in A \) and \( (x, y) \in G \times H \). Note that, \( \Delta_{(e_G) \times H} \) is the modular function of \( H \) and hence it is trivial on \( H \). Thus, we have

\[
\Delta((x, y^-1)) (f \cdot ((x, y^-1) - a((x, y) \cdot f) = \Delta((x, e_H)) \Delta((e_G, y^-1)(f \cdot (x, e_H)(e_G, y^-1)a - a((x, e_H)(e_G, y) \cdot f) = \Delta((x, e_H)) (f \cdot (x, e_H)(e_G, y^-1)a - a((x, e_H) \cdot f) \cdot (e_G, y^-1) = (e_G, y) \cdot \Delta((x, e_H)) (f \cdot (x, e_H)) a - a((x, e_H) \cdot f),
\]
which, by hypothesis, belongs to \( L \). \( \square \)

**Remark 3.9.** The first part of the hypothesis of Corollary 3.3 is not necessary. For instance, taking \( G \) to be the trivial group, \( H \) abelian and \( A = \mathbb{C} \), we observe that every subspace of \( L^1(G \times H, A) \) is a Lie ideal as \( L^1(G \times H, A) \) is a commutative Banach algebra and if \( L \) is a subspace of \( L^1(G \times H, A) \) which is not a left ideal, then \( (e, y) \cdot f \notin L \) for some \( y \in H \) and \( f \in L \).

### 3.1. A partial dictionary between the Lie ideals of \( L^1(G) \) and those of \( L^1(G, A) \)

We now look for Lie ideals in \( L^1(G, A) \) that can be obtained from those in \( A \). Towards this end, for each subspace \( F \) of \( A \), we consider the subspace

\[
\tilde{F} := \{ f \in L^1(G, A) : f(x) \in F \text{ for almost every } x \in G \}
\]

contained in \( L^1(G, A) \). Clearly, when \( F \) is closed so is \( \tilde{F} \).

**Remark 3.10.** Let \( I \) be a closed subspace of \( A \). Then, it is easily seen that \( I \) is an ideal in \( A \) if and only if \( \tilde{I} \) is an ideal in \( L^1(G, A) \).

However, not every closed ideal in \( L^1(G, A) \) is of the form \( \tilde{I} \). If \( A \) is a simple Banach algebra and every closed ideal of \( L^1(G, A) \) is of the form \( \tilde{I} \), then \( L^1(G, A) \) will be simple as \( (0) \) and \( A \) are the only choices for \( I \) which yield \( (0) = (0) \) and \( \tilde{A} = L^1(G, A) \). But, in general, \( L^1(G, A) \) need not be simple even if \( A \) is so. For instance, whenever \( L^1(G) \) has a proper and non-trivial closed ideal, say, \( J \), then \( J \otimes \tilde{A} \) is a proper (see [2] page 2) and non-trivial closed ideal of \( L^1(G, A) \).

In view of Remark 3.10, one could ask similar questions for closed Lie ideals. It turns out that the characterization obtained in Theorem 3.4 helps us to answer these questions appreciably well.

**Proposition 3.11.** Let \( G \) be a locally compact group, \( A \) be a Banach algebra and \( L \) be a closed subspace of \( A \). If \( \tilde{L} \) is a Lie ideal in \( L^1(G, A) \), then \( L \) is a Lie ideal in \( A \). The converse holds when \( G \) is abelian.

**Proof.** Let \( l \in L \) and \( a \in A \). Then, for any fixed Borel set \( E \) with finite positive measure, \( l \chi_E \in \tilde{L} \). Taking \( x = e \) in Equation (3.5), we see that \( l \chi_E a = a \chi_E \in \tilde{L} \); so that \( la \chi_E (y) - al \chi_E (y) \in \tilde{L} \) a.e. Hence, \( la - al \in L \), which implies that \( L \) is a Lie ideal in \( L \).

When \( G \) is abelian, then the converse follows easily using Theorem 3.4. \( \square \)

The converse of the preceding proposition is not true in general as is clear from the following proposition, which also shows that the commutativity of \( G \) has a significant say in telling whether \( \tilde{L} \) is a Lie ideal in \( L^1(G, A) \) or not.
Proposition 3.12. Let $G$ be a non-abelian locally compact group, $A$ be a Banach algebra and $L$ be a closed subspace of $A$. Then, $\bar{L}$ is a Lie ideal in $L^1(G, A)$ if and only if $L$ is an ideal in $A$ if and only if $\bar{L}$ is an ideal in $L^1(G, A)$.

Proof. In view of Remark 3.10 we only need to prove necessity in the first equivalence. Suppose that $\bar{L}$ is a Lie ideal in $L^1(G, A)$. By Proposition 3.11 $L$ must be a Lie ideal in $A$. Now, suppose on contrary that $L$ is not an ideal in $A$. Then, there exist $a \in A$ and $l \in L$ such that $al \notin L$ (or, $la \notin L$). Also, $G$ being non-abelian, there exist $x, y \in G$ such that $xy \neq yx$. Let $V_1$ and $V_2$ be disjoint compact neighbourhoods of $xy$ and $yx$ respectively. Since $x^{-1}V_1$ and $V_2x^{-1}$ are neighbourhoods of $y$, so is $V = x^{-1}V_1 \cap V_2x^{-1}$. Then, taking $f = l\chi_{V_1} \in \bar{L}$ (or, $f = l\chi_{V_2}$), we observe that for every $z \in V$,

\[
(\Delta(x^{-1})a(x^{-1} \cdot f) - (f \cdot x)a)(z) = \Delta(x^{-1})af(xz) - f(zx)a \\
= \Delta(x^{-1})al\chi_{V_1}(xz) - la\chi_{V_1}(zx) \\
= \Delta(x^{-1})al (or, -la) \notin L,
\]

which gives a contradiction by Theorem 3.4 as $m(V) > 0$.

It now remains only to answer the question whether every closed Lie ideal in $L^1(G, A)$ is of the form $\bar{L}$ or not. It turns out that it is not the case. Recall that a topological group $G$ is said to be an [IN] group if $G$ contains a compact neighbourhood of identity invariant under inner automorphisms. Clearly, every locally compact abelian group is an [IN] group and every [IN] group is unimodular.

Proposition 3.13. Let $G$ be a non-trivial [IN] group and $A$ be a Banach algebra with non-trivial center. Then, there exists a closed Lie ideal in $L^1(G, A)$ which is not of the form $\bar{L}$ for any closed Lie ideal $\bar{L}$ in $A$.

Proof. Let $0 \neq a \in Z(A)$, $V$ be a compact invariant neighbourhood of $e \in G$ and $f := a\chi_V \in L^1(G, A)$. Then, $(f \cdot x)a' - a'(x^{-1} \cdot f) = 0$ for all $x \in G$ and $a' \in A$. Thus, by Theorem 3.4 span$\{f\}$ is a closed Lie ideal in $L^1(G, A)$. We assert that span$\{f\}$ is not of the form $\bar{L}$ for any closed Lie ideal $\bar{L}$ in $A$.

Suppose, on contrary, that there exists a closed Lie ideal $L$ in $A$ such that span$\{f\} = \bar{L}$. Clearly $a \in L$. If we can choose a Borel set $U$ of finite measure such that $m(U\Delta V) \neq 0$, then for $h := a\chi_U \in L^1(G, A)$, we easily see that $h \in \bar{L}$ whereas $h \notin$ span$\{f\}$, which gives a contradiction.

So, it just remains to show that such a choice of $U$ is possible. If $V = \{e\}$ (equivalently, $G$ is discrete), then for any $e \neq y \in G$, $U$ can be taken as any compact neighbourhood of $y$ not intersecting $V$. On the other hand if $V \neq \{e\}$, fix an $e \neq x \in V$, and then take any two disjoint neighbourhoods $V_1$ and $V_2$ of $e$ and $x$, respectively, inside $V$. One may take $U$ as $V_1$ or $V_2$. \hfill $\square$

4. Center of a generalized group algebra

We first recall certain classes of topological groups that we will come across while discussing center of generalized group algebras (in fact, we already met one such class in Proposition 3.13). Recall that, for any topological group $G$,

$$\text{Aut}(G) := \{\beta : G \to G : \beta \text{ is a bi-continuous group isomorphism}\}$$
is a Hausdorff topological group (not necessarily locally compact) with respect to the so-called Birkhoff topology. A basis for the neighbourhoods of the identity automorphism $I$ in this topology is given by the collection

$$\{ N(K, V) : K \subseteq G \text{ is compact and } V \subseteq G \text{ is a neighbourhood of } e \},$$

where $N(K, V) := \{ \beta \in \text{Aut}(G) : \beta(x), \beta^{-1}(x) \in Vx \text{ for all } x \in K \}$. See [4] and [34] for details. A topological group $G$ is said to be an

1. [IN] group if $G$ contains a compact neighbourhood $U$ of the identity of $G$ which is invariant under inner automorphisms, i.e., $gUg^{-1} = U$ for all $g \in G$.
2. [SIN] group if every neighbourhood of identity contains a compact neighbourhood of identity which is invariant under inner automorphisms.
3. [FIA]$^-$ group if $\text{Inn}(G)$, the group of inner automorphisms of $G$, is relatively compact in $\text{Aut}(G)$ with respect to the Birkhoff topology.
4. [FC]$^-$ group if the conjugacy classes of $G$ are all relatively compact.

A detailed analysis of above classes of topological groups is available in [8] (also see [4, 34, 32]).

Remark 4.1. (1) A topological group is an [FIA]$^-$ group if and only if it is an [FC]$^-$ as well as an [SIN] group - see [32].

(2) Birkhoff topology is known to be finer than the topology of uniform convergence on compacta. In particular, this implies that the evaluation map $\text{Aut}(G) \times G \to G$ is continuous - see [4, 34].

We now proceed to examine the elements of the center of a generalized group algebra.

In 1972, Mosak ([30, Proposition 1.2]) proved that for a locally compact group $G$, the center of its group algebra is given by

$$Z(L^1(G)) = \{ f \in L^1(G) : \Delta(x^{-1})(f \cdot x^{-1}) = (x \cdot f), \forall x \in G \}.$$  

Motivated by this, we obtain a similar realization of the center of a generalized group algebra by employing some of the techniques used in Theorem 3.4. Note that $Z(L^1(G, A))$ is a closed Lie ideal in $L^1(G, A)$.

Proposition 4.2. Let $G$ be a locally compact group and $A$ be a Banach algebra. Then,

$$Z(L^1(G, A)) = \{ f \in L^1(G, A) : \Delta(x^{-1})(f \cdot x^{-1})a = a(x \cdot f), \forall x \in G, a \in A \}.$$  

Proof. Let $f \in Z(L^1(G, A))$, $x \in G$ and $a \in A$, then as proved in Theorem 3.4 for $\epsilon > 0$ there exists a compact neighbourhood $V$ of $e$ such that

$$\|\Delta(x^{-1})(f \cdot x^{-1})a - a(x \cdot f) - \frac{1}{m(V)}(f \ast (a_{xV}) - (a \ast V) \ast f)\|_1 \leq \epsilon.$$  

The required equality follows easily from the fact that $f \ast (a_{xV}) = (a_{xV}) \ast f$. For the reverse inclusion, let $f \in L^1(G, A)$ be such that $\Delta(x^{-1})(f \cdot x^{-1})a = a(x \cdot f)$ for every $x \in G$, $a \in A$. It is enough to show that $f \ast \varphi = \varphi \ast f$ for every $\varphi \in C_c(G, A)$. Let $\varphi \in C_c(G, A)$. Then, for given $\epsilon > 0$, as proved in Theorem 3.4 we have

$$\|\varphi \ast f - f \ast \varphi - \sum_{i} m(K_i)(\varphi(y_i)(y_i \cdot f) - \Delta(y_i^{-1})(f \cdot y_i^{-1}\varphi(y_i)))\|_1 \leq \epsilon,$$

where $K_i$ and $y_i$‘s are as mentioned in the proof of Theorem 3.4. Hence the result.  

It is also known that for any unimodular locally compact group $G$, $f \in L^1(G)$ is central if and only if $f$ is constant on the conjugacy classes of $G$ (see [24]). However, such a characterization does not hold for $L^1(G, A)$, in general. For example, if $G = D_6 := \{ r, s : r^3 = 1, s^2 = 1, srs = r^{-1} \}$ and $A$ is a Banach algebra endowed with the trivial multiplication, i.e., $ab = 0$ for every $a, b \in A$, then $Z(L^1(D_6, A)) = L^1(D_6, A)$ as $f \ast g = 0$ for every $f, g \in L^1(D_6, A)$.  

And, one may define \( f \in L^1(D_6, A) \) such that \( f(s) = c \) and is zero otherwise, for some \( 0 \neq c \in A \); then, \( f(rsr^{-1}) = f(rsr^2) = f(r^2s) = 0 \neq c = f(s) \). Thus, \( f \) is central but not constant on conjugacy classes.

Even the converse is not true in general, i.e., functions which are constant on the conjugacy classes need not be in \( \mathcal{Z}(L^1(G, A)) \). For example, every function in \( L^1(\mathbb{R}, M_2) \) is constant on conjugacy classes. Define \( f(x) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \) when \( x \in [0, 1] \) and zero otherwise. Then, for \( x = 1, a = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \) and for any \( y \in [1, 2] \), we have

\[
(f \cdot x^{-1})a(y) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = a(x \cdot f)(y);
\]

thus, \( f \notin \mathcal{Z}(L^1(\mathbb{R}, M_2)) \), by Proposition 4.2.

Interestingly, when \( G \) is \([\text{SIN}]\) and \( A \) is unital, we can have a satisfying identification of the elements of the \( \mathcal{Z}(L^1(G, A)) \). We need some preparation for this. Our first step in this direction is to prove that the elements of \( \mathcal{Z}(L^1(G, A)) \) are all \( \mathcal{Z}(A) \)-valued.

**Lemma 4.3.** Let \( G \) be a locally compact group and \( A \) be a Banach algebra. Then,

\[
\mathcal{Z}(L^1(G, A)) \subseteq L^1(G, \mathcal{Z}(A)).
\]

**Proof.** Consider a non-zero element \( f \) in \( \mathcal{Z}(L^1(G, A)) \). Suppose there exists a positive measure \( Borel \) set \( E' \) in \( G \) such that \( f(x) \notin \mathcal{Z}(A) \) for every \( x \in E' \). Choose a measurable subset \( E \) of \( E' \) of finite and positive measure.

By Proposition 4.2 we obtain \( fa = af \) for every \( a \in A \). Let \( B_E \) denote the \( \sigma \)-algebra consisting of all Borel sets contained in \( E \). Then, for any \( F \in B_E \), we find that

\[
\left( \int_F f(x) \, dx \right) a = \int_F f(x) a \, dx = \int_F (fa)(x) \, dx = \int_F (af)(x) \, dx = a \left( \int_F f(x) \, dx \right)
\]

for all \( a \in A \), i.e., \( \int_F f \, dx \notin \mathcal{Z}(A) \) for all \( F \in B_E \). Define \( H : B_E \to \mathcal{Z}(A) \) by \( H(F) = \int_F f \, dx \). Then, \( H \) is an \( m \)-continuous (i.e., \( \lim_{m(F) \to 0} H(F) = 0 \)) vector measure of bounded variation ([6 Theorem II.2.4]). Thus, by [6 Corollary III.2.5], there exists a \( g \in L^1(E, \mathcal{Z}(A)) \) such that \( H(F) = \int_F g \, dx \) for all \( F \in B_E \). This shows that \( \int_F (f(x) - g(x)) \, dx = 0 \) for every \( F \in B_E \); so that, \( f = g \) a.e. on \( E \), by [6 Corollary II.2.5]. Since \( g(E) \subseteq \mathcal{Z}(A) \), this is a contradiction to the existence of \( E' \). Hence, \( f(x) \in \mathcal{Z}(A) \) for almost every \( x \in G \).

Before looking at the elements of \( \mathcal{Z}(L^1(G, A)) \), one needs to worry when is it non-trivial. It is known that \( \mathcal{Z}(L^1(G)) \) is non-trivial if and only if \( G \) is an \([\text{IN}]\) group ([29]). Adapting the techniques of [29], we now prove its analogue for generalized group algebras.

**Lemma 4.4.** Let \( G \) be a locally compact group and \( A \) be a unital Banach algebra. Then, \( \mathcal{Z}(L^1(G, A)) \neq (0) \) if and only if \( G \) is an \([\text{IN}]\) group.

**Proof.** For \( 0 \neq f \in \mathcal{Z}(L^1(G, A)) \), the function \( h(x) = \|f(x)\|^{1/2} \in L^2(G) \) is positive. So, the function \( p : G \to \mathbb{C} \) defined as \( p(s) = \int_G h(sy)h(y) \, dy \) belongs to \( C_0(G) \) (see [16 Theorem 20.16]). Also, taking \( a = 1_A \) (the unit of \( A \)) in Proposition 4.2, one can easily verify that for every \( t \in G \), \( h(txs^{-1}) = \Delta(t)^{1/2}h(x) \), for almost every \( x \in G \) (where the null set depends on \( t \)). Thus, \( p \) is invariant under inner automorphisms and \( p(e) = \|f\|_1 > 0 \). Then, for any \( \epsilon \) such that \( p(e) > \epsilon > 0 \), the set \( \{x \in G : p(x) \geq \epsilon\} \) is a compact neighbourhood of \( e \) in \( G \), which is invariant under inner automorphisms. Hence, \( G \) is an \([\text{IN}]\)-group.

Conversely, if \( G \) is an \([\text{IN}]\) group, then \( \mathcal{Z}(L^1(G)) \) is non-trivial and \( (0) \neq \mathcal{Z}(L^1(G)) \otimes \mathbb{C}1 \subseteq \mathcal{Z}(L^1(G) \otimes^\gamma A) \).
Before identifying the elements of $Z(L^1(G, A))$, for the sake of clarity, in addition to the first paragraph of Section 2, we recall few more terminologies and facts related to Bochner integrability of vector valued functions. Consider a measure space $(\Omega, \mathcal{M}, \mu)$, a Banach space $X$ and a function $f : \Omega \to X$. Then, $f$ is said to be

1. Borel $(\mathcal{M}, \mu)$-measurable if $f^{-1}(U) \in \mathcal{M}$ for every open subset $U$ of $X$.
2. $(\mathcal{M}, \mu)$-essentially separably valued if there exists an $E \in \mathcal{M}$ such that $\mu(E^c) = 0$ and $f(E)$ is contained in a separable (closed) subspace of $X$.

A proof of the following useful equivalence can be found, for instance, in [36, Proposition 2.15].

**Proposition 4.5.** Let $(\Omega, \mathcal{M}, \mu)$ be a measure space, $X$ be a Banach space and $f : \Omega \to X$ be a function supported on a $\sigma$-finite set. Then, $f$ is $(\mathcal{M}, \mu)$-measurable if and only if $f$ is Borel $(\mathcal{M}, \mu)$-measurable and $(\mathcal{M}, \mu)$-essentially separably valued.

Let $B_{\text{inv}} := \{B \in B_G : xBx^{-1} = B \text{ for all } x \in G\}$. Clearly, $B_{\text{inv}}$ is a $\sigma$-subalgebra of $B_G$. Let $L^1_{\text{inv}}(G, A)$ denote the corresponding generalized group algebra with respect to the left Haar measure $m$ (as above).

**Lemma 4.6.** Let $G$ be a locally compact group and $A$ be a Banach algebra.

1. If $f : G \to A$ is a function with $\sigma$-finite support (with respect to $B_{\text{inv}}$), then $f$ is $(B_{\text{inv}}, m)$-measurable if and only if $f$ is $(B_G, m)$-measurable and is constant on the conjugacy classes of $G$.
2. $L^1_{\text{inv}}(G, A) \subseteq L^1(G, A)$ as a closed subspace.

In particular,

$$L^1_{\text{inv}}(G, A) = \{f \in L^1(G, A) : f \text{ is constant on the conjugacy classes of } G\}.$$

**Proof.** (1) Let $f$ be $(B_{\text{inv}}, m)$-measurable. Since $B_{\text{inv}} \subseteq B_G$, $f$ is Borel $(B_G, m)$-measurable, by Proposition 4.5. Also, since $f$ is $(B_{\text{inv}}, m)$-essentially separably valued, there exists $E \in B_{\text{inv}} \subseteq B_G$ such that $m(E^c) = 0$ and $f(E)$ is contained in a separable subspace. Hence, $f$ is $(B_G, m)$-essentially separably valued. To see that $f$ is constant on conjugacy classes, consider $x, t \in G$. Since $f^{-1}(\{f(t)\}) \in B_{\text{inv}}$, we have $xf^{-1}(\{f(t)\})x^{-1} = f^{-1}(\{f(t)\})$ so that $f(xtx^{-1}) = f(t)$. Since $x$ and $t$ were arbitrary, it follows that $f$ is constant on the conjugacy classes of $G$.

Conversely, suppose $f$ is $(B_G, m)$-measurable and is constant on conjugacy classes. Then, by Proposition 4.5 again, $f$ is Borel $(B_G, m)$-measurable and $(B_G, m)$-essentially separably valued. We first show that $f$ is Borel $(B_{\text{inv}}, m)$-measurable. Let $x \in G$ and $U \subset G$ be open. Then,

$$x^{-1}f^{-1}(U)x = \{x^{-1}gx : f(g) \in U\} = \{x^{-1}xgx^{-1}x : f(g) \in U\} = f^{-1}(U)$$

because $f(g) = f(xgx^{-1})$ for every $g \in G$. This proves that $f^{-1}(U) \in B_{\text{inv}}$ and hence $f$ is Borel $(B_{\text{inv}}, m)$-measurable.

Further, since $f$ is $(B_G, m)$-essentially separably valued, there exists an $E \in B_G$ such that $m(E^c) = 0$ and $f(E)$ is contained in a separable closed subspace $Y$ of $A$. Let $\tilde{E} = f^{-1}(Y)$. Since $Y^c$ is open and $f$ is Borel $(B_{\text{inv}}, m)$-measurable, $\tilde{E} \in B_{\text{inv}}$. Clearly, $f(\tilde{E}) \subset Y$ and since $f^{-1}(Y^c) \subseteq E^c$, we have $m(\tilde{E}^c) = m(f^{-1}(Y^c)) = 0$. Hence, $f$ is $(B_{\text{inv}}, m)$-essentially separably valued.

In particular, by Proposition 4.5 again, $f$ is $(B_{\text{inv}}, m)$-measurable.

(2) From the definition of Bochner integrability, it is clear that $L^1_{\text{inv}}(G, A) \subseteq L^1(G, A)$. And, since

$$\|f\|_{L^1_{\text{inv}}(G, A)} = \|\|f\|\|_{L^1_{\text{inv}}(G)} = \|\|f\|\|_{L^1(G)} = \|f\|_{L^1(G, A)}$$
for every $f \in L^1_{\text{inv}}(G, A)$, it follows that $L^1_{\text{inv}}(G, A)$ is closed in $L^1(G, A)$.

**Theorem 4.7.** Let $G$ be a locally compact [IN] group and $A$ be a unital Banach algebra. If either $G$ is an [SIN] group or $\mathcal{Z}(A) = C_1 A$, then

$$\mathcal{Z}(L^1(G, A)) = \{f \in L^1(G, \mathcal{Z}(A)) : f \text{ is constant on the conjugacy classes of } G\}.$$  

**Proof.** Suppose $f \in L^1(G, \mathcal{Z}(A))$ and is constant on conjugacy classes. Then, for every $x \in G$ and $a \in A$, we have

$$\left(x^{-1} \cdot ((f \cdot x^{-1})a) - x^{-1} \cdot (a(x \cdot f))\right)(y) = f(xy^{-1})a - af(y) = f(y)a - f(y)a = 0,$$

for every $y \in G$. Thus, $(f \cdot x^{-1})a = a(x \cdot f)$ for all $x \in G$ and $a \in A$, which by Proposition 4.2 shows that $f \in \mathcal{Z}(L^1(G, A))$, since $G$ is unimodular.

Conversely, consider any $f \in \mathcal{Z}(L^1(G, A))$. By Lemma 4.3, $f$ is a locally compact [SIN] group. Then, there exists an approximate identity $\{u_\alpha\}$ for every $x \in L^1(G, \mathcal{Z}(A))$, determined by a family of characteristic functions of compact invariant neighborhoods of identity. Then, $\{u_\alpha \otimes 1_A\}$ is an approximate identity for $L^1(G, A)$ contained in $\mathcal{Z}(L^1(G) \otimes A)$, and $\{f_\alpha = 1_A u_\alpha \}$ is an approximate identity for $\mathcal{Z}(L^1(G))$ contained in $L^\infty(G, A)$.

Then, $f_\alpha \ast f \in \mathcal{Z}(L^1(G, A))$ because $\mathcal{Z}(L^1(G, A))$ is a subalgebra of $L^1(G, A)$. Note that, for any $s, t \in G$, we have

$$\|f_\alpha \ast f(s) - f_\alpha \ast f(t)\| = \left\|\int_G f_\alpha(sx)f(x^{-1})dx - \int_G f_\alpha(tx)f(x^{-1})dx\right\|$$

$$= \left\|\int_G f_\alpha(x^{-1})(f \cdot s)(x)dx - \int_G f_\alpha(x^{-1})(f \cdot t)(x)dx\right\|$$

$$\leq \int_G \|f_\alpha(x^{-1})(f \cdot s)(x) - f_\alpha(x^{-1})(f \cdot t)(x)\|dx$$

$$\leq \int_G \|((f \cdot s) - (f \cdot t))(x)\|dx \quad (\text{since } \|f_\alpha\|_\infty = 1 \forall \alpha)$$

$$= \|f \cdot s - f \cdot t\|_1.$$  

From Lemma 4.3 it then follows that $f_\alpha \ast f$ is measurable.

Since $f_\alpha \ast f \in \mathcal{Z}(L^1(G, A))$ and $A$ is unital, it follows from Proposition 4.2 that $x^{-1} \cdot (f_\alpha \ast f) = (f_\alpha \ast f) \cdot x$ for every $x \in G$. Then, by continuity of $f_\alpha \ast f$, we get $f_\alpha \ast f(xy) = f_\alpha \ast f(yx)$ for every $y \in G$. Replacing $y$ by $x^{-1}y$ we see that $f_\alpha \ast f \in L^1_{\text{inv}}(G, A)$ for all $\alpha$. By Lemma 4.6, $L^1_{\text{inv}}(G, A)$ is closed in $L^1(G, A)$; hence, $f$, being the limit of the net $\{f_\alpha \ast f\}$, belongs to $L^1_{\text{inv}}(G, A)$. This proves that $f$ is constant on the conjugacy classes of $G$. □

It was recently shown (in [11, Theorems 1 & 2]) that for $C^*$-subalgebras $A_0 \subseteq A$ and $B_0 \subseteq B$, $A_0 \otimes B_0$ can be identified with the Banach subalgebra $A_0 \otimes B_0$ of $A \otimes B$; and also that $\mathcal{Z}(A \otimes B) = \mathcal{Z}(A) \otimes \mathcal{Z}(B)$. However, in general, such identifications are not known for Banach algebras. For generalized group algebras, we obtain such an identification of the center for certain classes of Banach algebras. We need some auxiliary results to prove this.

First, two lemmas that will be needed ahead, the first of which is borrowed from [4] (also see [8, Theorem 2.3 and Page 8] and [50, §1.0]).
Lemma 4.8. [30] Let $G$ be a locally compact group. Then, there exists a continuous homomorphism $\Delta : \text{Aut}(G) \to \mathbb{R}_{>0}$ such that

$$\int_G (f \circ \beta^{-1})(x)dx = \Delta(\beta) \int_G f(x)dx$$

for every $f \in L^1(G)$ and $\beta \in \text{Aut}(G)$.

Moreover, $G$ is unimodular if and only if $\Delta(\beta) = 1$ for all $\beta \in \text{Inn}(G)$.

Based on this, we deduce the following:

Lemma 4.9. Let $G$ be a unimodular locally compact group and $A$ be a Banach algebra. Then,

1. $\beta$ is measure preserving for every $\beta \in \text{Inn}(G)$;
2. $\beta : f := f \circ \beta^{-1} \in L^1(G, A)$ and $\|\beta \cdot f\|_1 = \|f\|_1$ for every $\beta \in \text{Inn}(G)$ and $f \in L^1(G, A)$;
3. the mapping $\text{Inn}(G) \ni \beta \mapsto \beta \cdot f \in L^1(G, A)$

is continuous for every $f \in L^1(G, A)$.

Proof. (1): Let $\beta \in \text{Inn}(G)$. Since $G$ is unimodular, for any $S \in B_G$, by Lemma 4.8 we have

$$m(S) = \Delta(\beta)m(S) = \Delta(\beta)\|\chi_S\|_1 = \|\chi_S \circ \beta^{-1}\|_1 = \|\chi_{\beta(S)}\|_1 = m(\beta(S)).$$

(2): Let $\beta \in \text{Inn}(G)$ and $f \in L^1(G, A)$. We first show that $\beta \cdot f$ is $(B_G, m)$-measurable.

Since $f$ is $(B_G, m)$-measurable, there exists a sequence $\{s_n\}$ of $A$-valued simple measurable functions and a null set $V$ such that $s_n \to f$ on $V^c$. Then, by (1), $\beta(V)$ is a null set; and, clearly, $\{\beta \cdot s_n\}$ is a sequence of $A$-valued simple $(B_G, m)$-measurable functions converging to $\beta \cdot f$ on $\beta(V)^c$. This proves that $\beta \cdot f$ is $(B_G, m)$-measurable. For integrability, note that $\|\beta \cdot f\| = \|f\| \circ \beta^{-1} \in L^1(G)$, by Lemma 4.8. Hence $\beta \cdot f \in L^1(G, A)$.

Next, by Lemma 4.8 again, we have

$$\|\gamma \cdot f\|_1 = \int_G \|\gamma \cdot f\| = \int_G \|f\| \circ \gamma^{-1} = \Delta(\gamma) \int_G \|f\| = \Delta(\gamma)\|f\|_1$$

for all $\gamma \in \text{Aut}(G)$. Note that, $G$ being unimodular, we have $\Delta(\gamma) = 1$ for every $\gamma \in \text{Inn}(G)$ (by Lemma 4.8); so that $\|\gamma \cdot f\|_1 = \|f\|_1$ for all $\gamma \in \text{Inn}(G)$.

(3): Let $f \in L^1(G, A)$, $\{\beta_n\}$ be a net converging to some $\beta$ in $\text{Inn}(G)$ and $\epsilon > 0$. Fix an $h = \sum_{i=1}^n f_i \otimes a_i \in L^1(G) \otimes A$ such that $\|f - h\|_1 < \epsilon$. For each $\alpha$, we have

$$\|\beta_\alpha \cdot f - \beta \cdot f\|_1 \leq \|\beta_\alpha \cdot f - \beta_\alpha \cdot h\|_1 + \|\beta_\alpha \cdot h - \beta \cdot h\|_1 + \|\beta \cdot h - \beta \cdot f\|_1.$$ 

On the right hand side of this inequality, the first and the third terms are less than $\epsilon$ for every $\alpha$, because (2); and the middle term simplifies to

$$\|\beta_\alpha \cdot \sum_{i=1}^n f_i a_i - \beta \cdot \sum_{i=1}^n f_i a_i\|_1 \leq \sum_{i=1}^n \|\beta_\alpha \cdot f_i - \beta \cdot f_i\|_1 \|a_i\|$$

for all $\alpha$.

Now, using the fact that for each $g \in L^1(G)$, the mapping $\text{Inn}(G) \ni \beta \mapsto \beta \cdot g \in L^1(G)$ is continuous (see [30], Page no. 281], we can choose an $\alpha_0$ such that the last inequality is less than $\epsilon$ for all $\alpha \geq \alpha_0$, and we are done. \[\square\]

Prior to proving the main result, we also introduce a $\sharp$-operator on $L^1(G, A)$ with some useful properties. Note that such a $\sharp$-operator on $L^1(G)$ has been studied and used by various authors in the past (see [30] and the references therein).
Proposition 4.10. Let $G$ be a locally compact $[\text{FIA}]^-$ group and $A$ be a unital Banach algebra. Then, there exists a projection $\sharp: L^1(G, A) \rightarrow L^1_{\text{inv}}(G, A)$ which maps $L^1(G, Z(A))$ onto $Z(L^1(G, A))$.

Proof. Since $G$ is an $[\text{FIA}]^-$ group, $\text{Inn}(G)$ is a compact subgroup of $\text{Aut}(G)$ with respect to the Birkhoff topology; so, $\text{Inn}(G)$ has a unique normalized Haar measure, say, $d\beta$.

Step I: We find a linear contraction $\sharp: L^1(G, A) \rightarrow L^1_{\text{inv}}(G, A)$.

Note that, for $f \in C_c(G, A)$ and $x \in G$, since the evaluation map $\text{Aut}(G) \times G \rightarrow G$ is continuous (Remark 4.1), we observe that the mapping

$$\beta \mapsto ([\beta: f](x)) \in A$$

is continuous. In particular, the map $\beta \mapsto \|([\beta: f](x))\|$, being continuous on a compact set, is integrable. This allows us to define a function $f^\sharp: G \rightarrow A$ by

$$f^\sharp(x) = \int_{\text{Inn}(G)} ([\beta: f](x))d\beta$$

for $x \in G$.

We assert that $f^\sharp \in C_c(G, A) \cap L^1_{\text{inv}}(G, A)$. Note that if $F$ is the compact support of $f$, then for any $y \notin \text{Inn}(G)(F)$, we have $f^\sharp(y) = 0$, since $\beta^{-1}(y) \notin F$ for any $\beta \in \text{Inn}(G)$; thus, $\text{supp}(f^\sharp) \subseteq \text{Inn}(G)(F)$ which is compact (being the image of the compact set $\text{Inn}(G) \times F$ under the continuous evaluation map $(\beta, x) \mapsto \beta(x)$). Thus, $f^\sharp$ is compactly supported.

We now show that $f^\sharp$ is continuous. Note that, being continuous and supported on a compact set, $f$ is left and right uniformly continuous; so, there exists a neighbourhood $V$ of $e$ such that $\|f(x) - f(y)\| < \epsilon$ whenever $x^{-1}y, yx^{-1} \in V$, by [16, Theorem 4.15]. Further, since $G$ is an $[\text{SIN}]$ group (Remark 4.1), we can choose $V$ to be invariant. Then, for any $x, y \in G$ such that $x^{-1}y \in V$, it is easily seen that $(\beta^{-1}(x))^{-1} \beta^{-1}(y) \in V$ for every $\beta \in \text{Inn}(G)$; so that $\|f(\beta^{-1}(x)) - f(\beta^{-1}(y))\| \leq \epsilon$ for every $\beta \in \text{Inn}(G)$ and, hence,

$$\|f^\sharp(x) - f^\sharp(y)\| \leq \int_{\text{Inn}(G)} \|f(\beta^{-1}x) - f(\beta^{-1}y)\|d\beta \leq \epsilon.$$

This proves that $f^\sharp$ is (uniformly) continuous. Also, note that, for any $x, y \in G$,

$$f^\sharp(xy^{-1}) = f^\sharp(\text{Ad}_{y^{-1}}(x))$$

$$= \int_{\text{Inn}(G)} ([\beta: f](\text{Ad}_{y^{-1}}(x)))d\beta$$

$$= \int_{\text{Inn}(G)} ([\text{Ad}_y \cdot (\beta: f)](x))d\beta$$

$$= \int_{\text{Inn}(G)} ([\beta: f](x))d\beta$$

(by left invariance of the Haar measure $d\beta$)

$$= f^\sharp(x).$$

Hence, $f^\sharp \in L^1_{\text{inv}}(G, A)$, by Lemma 4.6.

We next prove that the map $\sharp: C_c(G, A) \rightarrow L^1_{\text{inv}}(G, A)$ is a linear contraction. Clearly, $\sharp$ is a linear map. And, for any $f \in C_c(G, A)$, by Lemma 4.9 and an appropriate application of
Thus, \( \hat{\zeta} : C_c(G, A) \to L^1_{\text{inv}}(G, A) \) extends to a linear contraction \( \hat{\zeta} : L^1(G, A) \to L^1_{\text{inv}}(G, A) \).

\textbf{Step II:} We now show that the operator \( \hat{\zeta} \) is identity on \( L^1_{\text{inv}}(G, A) \).

Let \( f \in L^1_{\text{inv}}(G, A) \) and \( \epsilon > 0 \). It suffices to show that \( \| \hat{\zeta} f - f \|_1 \leq \epsilon \). For this, we first assert that \( \beta \cdot f = f \) for every \( \beta \in \text{Inn}(G) \). Note that, for \( \beta = \text{Ad}_y \in \text{Inn}(G) \) for some \( y \in G \), by Lemma 4.6 we have

\[
(\beta \cdot f)(x) = f(yxy^{-1}) = f(x) \quad \text{for all } x \in G.
\]

So, \( \beta \cdot f = f \) for every \( \beta \in \text{Inn}(G) \); and, by the continuity of the map \( \beta \mapsto \beta \cdot f \) (Lemma 4.9), we deduce that

\[
\beta \cdot f = f \quad \text{for every } \beta \in \text{Inn}(G).
\]

Now, fix an \( h \in C_c(G, A) \) such that \( \| f - h \|_1 < \epsilon/3 \). Then, imitating the proof [30, Lemma 1.4] verbatim, we see that \( h^\sharp \) belongs to the closed convex hull of \( \{ \beta \cdot h : \beta \in \text{Inn}(G) \} \). So, there exist non-negative scalars \( c_i, 1 \leq i \leq r \) with \( \sum_{i=1}^r c_i = 1 \) and automorphisms \( \{ \beta_i : 1 \leq i \leq r \} \subset \text{Inn}(G) \) such that

\[
\| h^\sharp - \sum_{i=1}^r c_i (\beta_i \cdot h) \|_1 \leq \epsilon/3.
\]

Thus,

\[
\| f^\sharp - f \|_1 \leq \| f^\sharp - h^\sharp \|_1 + \| h^\sharp - \sum_{i=1}^r c_i (\beta_i \cdot h) \|_1 + \sum_{i=1}^r c_i \| (\beta_i \cdot h) - (\beta_i \cdot f) \|_1,
\]

\[
\leq \| f - h \|_1 + \| h^\sharp - \sum_{i=1}^r c_i (\beta_i \cdot h) \|_1 + \sum_{i=1}^r c_i \| h - f \|_1 \quad \text{(by Lemma 4.9)}
\]

\[
\leq \epsilon.
\]

\textbf{Step III:} Finally, we show that \( \hat{\zeta} \) maps \( L^1(G, Z(A)) \) onto \( Z(L^1(G, A)) \).

Let \( f \in L^1(G, Z(A)) \). Consider a sequence \( \{ f_n \} \subset C_c(G, Z(A)) \) such that \( f_n \to f \), then \( f_n^\sharp \to f^\sharp \). By the definition of the \( \hat{\zeta} \) operator, \( f_n^\sharp \in C_c(G, Z(A)) \subset L^1(G, Z(A)) \), so that \( f_n^\sharp \in L^1(G, Z(A)) \). The result now follows from the fact that \( Z(L^1(G, A)) = L^1(G, Z(A)) \cap L^1_{\text{inv}}(G, A) \) (by Theorem 4.7).

\textbf{Lemma 4.11.} Let \( A \) be a Banach algebra and \( G \) be a locally compact group such that \( Z(L^1(G)) \) is complemented in \( L^1(G) \) by a projection of norm one. Then,

\[
Z(L^1(G)) \otimes Z(A) \subseteq Z(L^1(G) \otimes A).
\]

\textbf{Proof.} Since \( Z(L^1(G)) \) is complemented in \( L^1(G) \) by a projection of norm one, appealing to [36, Proposition 2.4] and [36, Page 30], the algebraic embeddings

\[
Z(L^1(G)) \otimes Z(A) \subseteq L^1(G) \otimes Z(A) \subseteq L^1(G) \otimes A
\]

extend to isometric embeddings

\[
Z(L^1(G)) \otimes Z(A) \subseteq L^1(G) \otimes A \subseteq L^1(G) \otimes A.
\]

Thus, it suffices to show that \( Z(L^1(G)) \otimes Z(A) \subseteq Z(L^1(G) \otimes A) \). This follows easily by observing that \( (f \otimes a)(g \otimes b) = fg \otimes ab = gf \otimes ba = (g \otimes b)(f \otimes a) \) for every \( f \in Z(L^1(G)) \), \( a \in Z(A) \), \( g \in L^1(G) \) and \( b \in A \).
Remark 4.12. For every $[\text{FC}]^-$ group $G$, $Z(L^1(G))$ is complemented in $L^1(G)$ by a projection of norm one (see [39]). So, the preceding result holds for a large class of groups.

We have equality in Lemma 4.11 for some classes of groups as we demonstrate below. For instance, when $G$ is abelian, then $G$ is an $[\text{FC}]^-$ group and $Z(L^1(G)) = L^1(G)$. This gives

$$L^1(G) \otimes \gamma Z(A) = Z(L^1(G)) \otimes \gamma Z(A) \subseteq Z(L^1(G) \otimes \gamma A) \subseteq L^1(G) \otimes \gamma Z(A),$$

where the last inclusion follows from Lemma 4.3. Hence, $Z(L^1(G) \otimes \gamma A) = Z(L^1(G)) \otimes \gamma Z(A)$. And, more generally, we have the following:

Theorem 4.13. Let $G$ be a locally compact group and $A$ be a unital Banach algebra. If either

1. $Z(L^1(G))$ is complemented in $L^1(G)$ by a projection of norm one with either $G$ discrete or $Z(A) = \mathbb{C}1_A$, or,
2. $G$ is an $[\text{FIA}]^-$ group, then

$$Z(L^1(G)) \otimes \gamma Z(A) = Z(L^1(G)) \otimes \gamma A).$$

Proof. The direct inclusion follows from Lemma 4.11. We prove the reverse inclusion in both the cases. Consider a non-zero element $f \in Z(L^1(G) \otimes \gamma A)$. Note that, by Lemma 4.3,

$$f \in L^1(G, Z(A)).$$

(1) Let $G$ be such that $Z(L^1(G))$ is complemented in $L^1(G)$ by a projection of norm one.

First suppose that $Z(A) = \mathbb{C}1_A$. Note that, from Theorem 4.7, $f$ is constant on the conjugacy classes of $G$ and takes values in $Z(A) = \mathbb{C}1_A$. So, there exists a $g \in L^1(G)$ such that $f = g1_A$. Note that $g$ will also be constant on the conjugacy classes of $G$; so that $g \in Z(L^1(G))$ (by [24]). This proves that $f = g \otimes 1_A \in Z(L^1(G)) \otimes \gamma \mathbb{C}1_A$ which gives the result.

Next suppose that $G$ is discrete. Then, $G$ is an $[\text{SIN}]$ group and from Theorem 4.7 it follows that $f$ is constant on the conjugacy classes. Let $\{a_j : j \in \Gamma \} \subseteq Z(A)$ denote the set of all possible values taken by $f$ on the conjugacy classes of $G$, for some indexing set $\Gamma$. For each $j \in \Gamma$, let $C_j$ denote the union of all those conjugacy classes on which $f$ takes the value $a_j$, and let $C := G \setminus \bigcup\{C_j : a_j \neq 0\}$; so that $f(x) = 0$ for all $x \in C$ if $C \neq \emptyset$. We assert that $\Gamma$ is countable.

For each $n \in \mathbb{N}$, let $\Gamma_n = \{j \in \Gamma : m(C_j) \|a_j\| > 1/n\}$. Since $G$ is discrete, it is easily seen that $\Gamma \setminus \bigcup_{n=1}^{\infty} \Gamma_n$ is at most a singleton; so, it suffices to show that $\Gamma_n$ is finite for all $n$. Suppose, on contrary, that for some $n \in \mathbb{N}$, $\Gamma_n$ contains countably infinite members, say, $\{j_k\}_{k \in \mathbb{N}}$. Then,

$$\int_G \|f\| \, dx \geq \int_{\bigcup_{k=1}^{\infty} C_{j_k}} \|f\| \, dx = \sum_{j=1}^{\infty} \int_{C_{j_k}} \|f\| \, dx = \sum_{k=1}^{\infty} m(C_{j_k}) \|a_{j_k}\| \geq \sum_{k=1}^{\infty} \frac{1}{n} = \infty,$$

which is a contradiction to the fact that $f$ is integrable, thereby establishing our assertion.

Two possibilities arise, namely, either $\Gamma$ is finite or infinite. If $\Gamma$ is finite, then $f = \sum_{j \in \Gamma} \chi_{C_j} \otimes a_j \in Z(L^1(G)) \otimes Z(A)$. And, if $\Gamma$ is infinite, we can consider $\Gamma \setminus \{j : a_j = 0\}$ to be same as $\mathbb{N}$. Note that, $\|f - \sum_{j=1}^{\infty} \chi_{C_j} a_j\| \leq \|f\|$ (as scalar functions) for all $n \geq 1$ and $\|f - \sum_{j=1}^{\infty} \chi_{C_j} a_j\| \to 0$ pointwise on $G$. Thus, $f = \sum_{j=1}^{\infty} \chi_{C_j} a_j$ in $L^1(G, A)$, by Lebesgue Dominated Convergence Theorem. Also, $\chi_{C_j} a_j$ corresponds to $\chi_{C_j} \otimes a_j$ which is in $Z(L^1(G)) \otimes \gamma Z(A) \subseteq Z(L^1(G) \otimes \gamma A)$ for all $j \in \Gamma$. Hence the result.

(2) Let $G$ be an $[\text{FIA}]^-$ group. Since $f \in L^1(G) \otimes \gamma Z(A)$ and $C_c(G) \otimes Z(A)$ is dense in $L^1(G) \otimes \gamma Z(A)$, there exists a sequence $\{f_n\}$ in $C_c(G) \otimes Z(A)$ such that $\lim_{n \to \infty} \|f_n - f\|_1 = 0$. 

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By Proposition 4.10 we have
\[ \| f_n^2 - f \|_1 = \|(f_n - f)^2 \|_1 \leq \| f_n - f \|_1 \]
for all \( n \in \mathbb{N} \); so that \( \lim_{n \to \infty} \| f_n^2 - f \|_1 = 0 \). From Lemma 4.11 we know that the norm of \( L^1(G) \otimes^\gamma Z(A) \) restricted to \( \mathcal{Z}(L^1(G)) \otimes Z(A) \) coincides with the norm coming from \( \mathcal{Z}(L^1(G)) \otimes^\gamma Z(A) \). So, we shall be done if we can show that \( f_n^2 \in \mathcal{Z}(L^1(G)) \otimes Z(A) \) for all \( n \). It actually suffices to show that \( (h \otimes a)^2 = h^2 \otimes a \) for every \( h \in C_c(G) \) and \( a \in A \), where the function \( \xi \) on the right hand side represents the projection from \( L^1(G) \) onto \( \mathcal{Z}(L^1(G)) \) as in Proposition 4.10 (by taking \( A = \mathbb{C} \)). This follows rather easily as
\[
(h \otimes a)^2(x) = \int_{\text{Inn}(G)} (\beta \cdot (ha))(x) d\beta = \left( \int_{\text{Inn}(G)} h(\beta^{-1}(x)) d\beta \right) a = (h^2 \otimes a)(x)
\]
for all \( x \in G \).

Since every compact group is an \([\text{FIA}]^-\) group (see [32, Diagram 1]), we deduce the following:

**Corollary 4.14.** Let \( G \) be a compact group and \( A \) be a unital Banach algebra. Then,
\[
\mathcal{Z}(L^1(G) \otimes^\gamma A) = \mathcal{Z}(L^1(G)) \otimes^\gamma Z(A).
\]

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Ved Prakash Gupta, School of Physical Sciences, Jawaharlal Nehru University, New Delhi
E-mail address: vedgupta@mail.jnu.ac.in, ved.math@gmail.com

Ranjana Jain, Department of Mathematics, University of Delhi, Delhi
E-mail address: rjain@maths.du.ac.in

Bharat Talwar, Department of Mathematics, University of Delhi, Delhi
E-mail address: btalwar.math@gmail.com