Computing secular motion under slowly rotating quadratic perturbation

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ABSTRACT
We consider secular perturbations of nearly Keplerian two-body motion under a perturbing potential that can be approximated to sufficient accuracy by expanding it to second order in the coordinates. After averaging over time to obtain the secular Hamiltonian, we use angular momentum and eccentricity vectors as elements. The method of variation of constants then leads to a set of equations of motion that are simple and regular, thus allowing efficient numerical integration. Some possible applications are briefly described.

Key words: celestial mechanics – binaries:general – comets:general

1 INTRODUCTION
If a binary star, or a planet revolving around a star, is perturbed by one or more distant relatively slowly moving masses the perturbation can be approximated by a tidal field. Fast computation of secular perturbations in such a potential is thus desirable. Potential applications include the Kozai resonance, motion of a well isolated binary in a star cluster and cometary motion under the galactic tidal field.

Several authors have considered the computation of cometary motion perturbations due to the galactic tidal field e.g. Heisler and Tremaine (1986), Wiegert and Tremaine (1999), Brasser (2001). More recently Fouchard (2004) suggested the use of the time averaged secular perturbing function and the Lagrangian equations for computing the evolution of orbital elements of a comet. While Fouchard et al. (2005) compared various ways to compute the evolution of orbital elements (using Keplerian or Delaunay elements). Breiter and Ratajczak (2005) used the so called vectorial elements, i.e. the areal velocity vector and the eccentricity vector. However, the perturbing potential considered in all these studies was a simplified one, in most cases just the time term causing the force perpendicular to the galactic plane was included.

In this paper we consider a general quadratic potential. For more mathematical generality we include the possibility of a constant force (although such a term does not appear in a tidal field) and also a slow rotation of the quadratic perturbing field is allowed. We form the simple regular equations of motion using the vectorial elements.

Finally it is necessary to stress that the main purpose of this paper is to introduce the simple secular equations of motion when a two-body system is perturbed by a (at most) quadratic perturbing potential.

2 PERTURBATIONS OF VECTORIAL ELEMENTS
2.1 Perturbing function
Consider a system in which the Keplerian motion of a particle around a central mass $m$ is perturbed by a perturbing potential $U(r, t)$. In addition, a slow rotation (with angular velocity $N$) is assumed. The Hamiltonian may be written

$$H = \frac{1}{2}v^2 - \frac{m}{r} - U(r, t) = -\frac{m}{2a} - U(r, t),$$

where $m$ is the mass of the central body, $r$ is the position vector with components $(x_1, x_2, x_3)$, $v$ is the velocity vector with components $(v_1, v_2, v_3)$ i.e. $\dot{r} = v$ (which also can be identified as the momentum vector) and $a$ is the semi-major axis of the Kepler motion. The function $U$ is the perturbing function, assumed at most quadratic in the coordinates. If the time dependence of $U$ is due to rotation of the potential with a constant angular velocity, then one can consider the system in the rotating coordinate system in which the perturbing function is a constant. Thus, using the suffix notation, one may write in place of $U$

$$\tilde{U} = f_i x_i + \frac{1}{2}x_i G_{ij} x_j + \epsilon_{ijk} N_i x_j x_k,$$

where there is no time dependence, $f_i$ are components of a (possible) constant acceleration vector $f$ and $G_{ij}$ is the second derivative matrix.
\[ f_i = \frac{\partial U}{\partial x_i} \mid _{|r=0}; \quad G_{ij} = \frac{\partial^2 U}{\partial x_i \partial x_j} \mid _{|r=0}. \]  

The last term in (2) is the scalar product of the angular velocity vector \( \mathbf{\Omega} \) of the perturbing potential and the angular momentum vector of the moving particle, here written using the Levi-Civita symbol \( \epsilon_{ijk} \).

Since the matrix \( G_{ij} \) is symmetric, it can be diagonalized by selecting a suitable coordinate system. However, there may be situations in which this is not convenient. Therefore we consider the general non-diagonal case. In the theory presented here that does not add any substantial complications.

### 2.2 Averaging the Hamiltonian

The auxiliary quantities needed in averaging the Hamiltonian are:

(i) The angular momentum vector per unit of mass (actually areal speed) of the particle

\[ \mathbf{J} = \mathbf{r} \times \mathbf{v}. \tag{4} \]

(ii) The eccentricity vector (also known as the Runge-Lenz- or Laplace vector)

\[ \mathbf{e} = \frac{\mathbf{v} \times \mathbf{J}}{m} - \frac{\mathbf{r}}{r}. \tag{5} \]

or, for our purposes, the vector

\[ \mathbf{E} = \sqrt{mae} \]

is more convenient.

(iii) The mean values of the position vector components

\[ <x_i> = -\frac{3}{2}ae_i = -\frac{3}{2} \sqrt{\frac{\mu}{m}} E_i. \tag{7} \]

(iv) The time averaged coordinate products can be written

\[ <x_i x_j> = \frac{a}{2m} J^2 \delta_{ij} - \frac{a}{2m} J_i J_j + \frac{5a}{2m} E_i E_j, \tag{8} \]

where \( \delta_{ij} \) is the Kronecker \( \delta \). This result was easy to derive for \( <x_i^2> \). The generalization was then formed by educated guess and use of computer algebra for confirmation.

The secular perturbing function \( R = \langle \tilde{U} \rangle \) takes the form

\[ R = f_i <x_i> + \frac{a}{4m} G_{ij} <x_i x_j> + N_i J_i. \tag{9} \]

If one introduces the matrices \( A_{ij} \) and \( B_{ij} \) with the elements

\[ A_{ij} = \frac{a}{2m} \left( \sum_k G_{kk} \delta_{ij} - G_{ij} \right) \tag{10} \]

\[ B_{ij} = \frac{5a}{2m} G_{ij}, \tag{11} \]

and the vector \( \mathbf{F} \)

\[ \mathbf{F} = -\frac{3}{2} \sqrt{\frac{a}{m}} \mathbf{f}, \tag{12} \]

then the averaged perturbing function can be written in the simple form

\[ R = \mathbf{F} \mathbf{E} + \frac{1}{2} \mathbf{J} A_{ij} J_j + \frac{1}{2} \mathbf{E} B_{ij} E_j + N_i J_i, \tag{13} \]

where the vectors \( \mathbf{F}, \mathbf{N} \), and matrices \( \mathbf{A}, \mathbf{B} \) are constants in any particular orbit.

### 2.3 Equations of motion

The reason for the choice of the elements \( \mathbf{J} \) and \( \mathbf{E} \) was that one obtains for their components the simple Poisson bracket relations

\[ \{ J_i, J_j \} = \epsilon_{ijk} J_k, \quad \{ E_i, E_j \} = \epsilon_{ijk} J_k, \quad \{ J_i, E_j \} = \epsilon_{ijk} E_k. \tag{14} \]

With these formulae one gets the equations of motion (Breiter and Rataczak, 2005) and references therein

\[ \dot{\mathbf{J}} = -\{ \mathbf{J}, \mathbf{R} \} = \mathbf{J} \times \frac{\partial \mathbf{R}}{\partial \mathbf{J}} + \mathbf{E} \times \frac{\partial \mathbf{R}}{\partial \mathbf{E}}, \tag{15} \]

\[ \dot{\mathbf{E}} = -\{ \mathbf{E}, \mathbf{R} \} = \mathbf{E} \times \frac{\partial \mathbf{R}}{\partial \mathbf{J}} + \mathbf{J} \times \frac{\partial \mathbf{R}}{\partial \mathbf{E}}, \tag{16} \]

and by \( \mathbf{E} \) the partial derivatives of \( \mathbf{R} \) with respect to the vectorial elements are

\[ \frac{\partial \mathbf{R}}{\partial \mathbf{J}} = \mathbf{N} + \mathbf{A} \mathbf{J}, \tag{17} \]

\[ \frac{\partial \mathbf{R}}{\partial \mathbf{E}} = \mathbf{F} + \mathbf{B} \mathbf{E}, \tag{18} \]

where we have used the vector-matrix notation. In deriving the above equations one should note that \( \mathbf{J} \) and \( \mathbf{E} \) are constants of motion under the two-body Hamiltonian \( H = -m_i/(2a_i) \). Thus \( a \) can be considered a (numerical) constant since \( \{ \mathbf{J}, a \} = \{ \mathbf{E}, a \} = 0 \). The equations of motion have the integrals

\[ \mathbf{J} \cdot \mathbf{E} = 0, \quad J^2 + E^2 = ma, \quad R = \text{constant}. \tag{19} \]

Here we must remark that the form of the perturbing function is not unique. This is because of the larger-than-necessary number of variables and because one may e.g. add terms proportional to \( J^2 + E^2 \) and/or to \( \mathbf{J} \cdot \mathbf{E} \) which would not affect the final expressions for the derivatives of the elements in (15) and (16).

Note that when the rotation term \( \mathbf{N} \cdot \mathbf{J} \) is included, the result for the evolution of the vectorial elements will be obtained in the rotating coordinate system.

### 3 APPLICATIONS

Possible applications of the vectorial element equations include:

(i) Constant force, or rotating constant force. This may be restricted to toy applications, but was included in our treatment for the sake of generality and because it is mathematically simple.

(ii) Isolated binary in a star cluster. Assuming the perturbing stars are distant and move relatively slowly, one may approximate the perturbing potential by

\[ U = \sum_k \frac{m_k}{2|R_k|^3} \left( r^2 - 3 \frac{R_k \cdot r}{|R_k|^2} \right)^2, \tag{20} \]

where \( R_k \) are the positions of the perturbers relative to the centre-of-mass of the binary. Writing \( R_k = (X_{k1}, X_{k2}, X_{k3}) \) we get for the components of the matrix \( \mathbf{G} \)

\[ G_{ij} = \sum_k \frac{m_k}{|R_k|^3} (\delta_{ij} - 3X_{ki}X_{kj}), \tag{21} \]

where the matrix elements may be time dependent, but this does not invalidate the equations presented above.
(iii) Kozai resonance. This term is often used in connection of triple systems in which averaging over the outer orbit gives the perturbing potential of the form

\[ U = \frac{m_3}{8b_3^3} (r^2 - 3(z \cdot r)^2), \]

where \( m_3 \) is the mass of the third body, \( b_3 \) is the semi-minor axis of the outer ellipse and \( z \) is the unit normal vector of the orbital plane of the distant body. In this case we have

\[ G_{ij} = \frac{m_3}{40b_3^5} (\delta_{ij} - 3z_i z_j), \]

where \( z_i \) are the components of the vector \( z \).

(iv) Cometary motion under the galactic tide. In this case the tidal potential is usually written

\[ \tilde{U} = \frac{1}{2} \sum G_{kk} x_k^2 + N \cdot J, \]

which is possible by choosing the coordinate system such that \( x_3 \) axis is perpendicular to the galactic plane and the \( x_1 \)-axis points to the galactic centre.

(v)\ The perturbing potential due to a small disk of mass can also be expressed easily in terms of the vectorial elements, although not in quadratic form. The secular perturbing function for this case is often written in terms of the Delaunay elements in the form \( R = \epsilon L^{-3} G^{-3} (1 - 3G^{-2} H^2) \), where \( \epsilon \) is a (small) constant and \( L, G, H \) are the Delaunay elements. Here \( L = \sqrt{ma} \) is a constant, \( G = |J| \) and \( H = z \cdot J \), where \( z \) is the unit direction vector of the symmetry axis of the potential. Thus

\[ R = \epsilon L^{-3} |J|^{-3} (1 - 3(z \cdot J)^2 |J|^2). \]

Consequently it is easy to include the effect of any such potential into the equations of motion for the vectorial elements. Note that this kind of a term could be used to approximate the effect of the planets (averaged over orbits) to comets that come close, but not too close, to the Solar System.

4 NUMERICAL ASPECTS

We ran some numerical test computations and compared results against direct numerical integration of the equations of motion for the coordinates. The tests used the tidal field of the Galaxy in the form \( \tilde{U} = \frac{1}{2} \sum G_{kk} x_k^2 + N \cdot J \) (parameters from Fouchard et al. (2005)) and a value for the semi-major-axis of an Oort Cloud comet of \( a \geq 10,000\text{AU} \).

The experiments suggest:

If one needs results for a long time and infrequent output is allowed, then one of the best numerical integrators available is the Burlirsh-Stoer method (actually the code known as DIFSY1, originally written by Burlirsh). However, this is efficient only when run with optimally long steps, which may be too long for some applications.

If one needs frequent output, then the method of choice is the implicit midpoint method. It is very simple to implement and gives quite high precision, especially it conserves the quadratic integrals of motion (Huang and Leimkuhler 1997; Sanz-Serna and Calvo 1994).

In this case one finds that the stepsize equal to one period (i.e. \( 10^6 \) years) gives a clearly satisfactory accuracy: the plots of the vectorial elements from both computations (secular and coordinate integration) agree well, except for a small phase error.

However, for \( a >> 20,000\text{AU} \) the results differ and the reliability of a secular theory becomes questionable.

5 CONCLUSIONS

We have generalized the equations of Breiter and Ratajczak (2004) who considered the simple case of perturbing function of the form \( g_{33} z^3 \).

The equations for vectorial elements, even in the case of a general quadratic potential are simple and regular, contrary to what one obtains when using Keplerian or Delaunay elements (Fouchard et al. (2003)).

The integration of the secular theory equations is typically faster than direct coordinate integration by two orders of magnitude.

The implicit midpoint method may be even faster than Burlirsh-Stoer, but there is no simple way of knowing accuracy and optimal stepsize.

The most reliable integration method is Burlirsh-Stoer extrapolation.

For comets the secular theory results are not accurate for \( a >> 20,000\text{AU} \), as comparison with direct integration shows (simply: perturbation can be too large).

For an integration over only one period at a time the midpoint method and Burlirsh-Stoer are nearly equally fast. Thus in this case the midpoint method may be preferable due to its simplicity.

REFERENCES

Brasser, R. 2001. Monthly Notices of the Royal Astronomical Society 324, 1109-1116.
Breiter, S., Ratajczak, R. 2005. Monthly Notices of the Royal Astronomical Society 364, 1222-1228.
Fouchard, M. 2004. Monthly Notices of the Royal Astronomical Society 349, 347-356.
Fouchard, M., Froeschlé, C., Matese, J. J., Valsecchi, G. 2005. Celestial Mechanics and Dynamical Astronomy 93, 229-262.
Heisler, J., Tremaine, S. 1986. Icarus 65, 13-26.
Huang, W. and Leimkuhler, B. 1997. SIAM J. Sci. Comp., vol., 18, # 1.
Sanz-Serna, J.M. and Calvo, M.P. 1994. Numerical Hamiltonian Problems, Chapman and Hall, London.
Wiepert, P., Tremaine, S. 1999. Icarus 137, 84-121.