BASS NUMBERS OVER LOCAL RINGS
VIA STABLE COHOMOLOGY

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To Jürgen Herzog on his 70th birthday.

ABSTRACT. For any non-zero finite module $M$ of finite projective dimension over a Noetherian local ring $R$ with maximal ideal $m$ and residue field $k$, it is proved that the natural map $\text{Ext}_R(k, M) \to \text{Ext}_R(k, M/mM)$ is non-zero when $R$ is regular and is zero otherwise. A noteworthy aspect of the proof is the use of stable cohomology. Applications include computations of Bass series over certain local rings.

1. Introduction. Let $(R, m, k)$ denote a commutative Noetherian local ring with maximal ideal $m$ and residue field $k$; when $R$ is not regular we say that it is singular.

This article revolves around the following result:

**Theorem.** If $(R, m, k)$ is a singular local ring and $M$ an $R$-module of finite projective dimension, then $\text{Ext}_R(k, \pi^M) = 0$ for the canonical map $\pi^M : M \to M/mM$.

Special cases, known for a long time, are surveyed at the end of Section 2. Even in those cases our proof is new. It utilizes a result of Martsinkovsky [11] through properties of Vogel’s stable cohomology functors [3, 6] recalled in Section 1. It also suggests extensions to DG modules over certain commutative DG algebras; these will be discussed in [2]. Applications of the theorem include new criteria for regularity of local rings (in Section 2) and explicit computations of Bass numbers of modules (in Section 3).

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1. Stable cohomology. In this section we recall the construction of stable cohomology and basic results required in the sequel. The approach we adopt is based on a construction by Vogel and described in Goichot [6]; see also [3].

Let $R$ be an associative ring, and let $R^e$ denote its center. Given left $R$-modules $L$ and $M$, choose projective resolutions $P$ and $Q$ of $L$ and $M$, respectively. Recall that a homomorphism $P \to Q$ of degree $n$ is a family $\beta = (\beta_i)_{i \in \mathbb{Z}}$ of $R$-linear maps $\beta_i : P_i \to Q_{i+n}$, that is, an element of the $R^e$-module

$$\operatorname{Hom}_R(P,Q)_n = \prod_{i \in \mathbb{Z}} \operatorname{Hom}_R(P_i,Q_{i+n}).$$

This module is the $n$th component of a complex $\operatorname{Hom}_R(P,Q)$, with differential

$$\partial_n(\beta)_i = \partial^Q_{i+n} \beta_i - (-1)^n \beta_{i-1} \partial^P_i.$$

The maps $\beta : P \to Q$ with $\beta_i = 0$ for $i \gg 0$ form a subcomplex with component

$$\widehat{\operatorname{Hom}}_R(P,Q)_n = \prod_{i \in \mathbb{Z}} \operatorname{Hom}_R(P_i,Q_{i+n}) \quad \text{for } n \in \mathbb{Z}.$$

We write $\widehat{\operatorname{Hom}}_R(P,Q)$ for the quotient complex. It is independent of the choices of $P$ and $Q$ up to $R$-linear homotopy, and so is the exact sequence of complexes

$$(1.0.1) \quad 0 \longrightarrow \widehat{\operatorname{Hom}}_R(P,Q) \longrightarrow \operatorname{Hom}_R(P,Q) \xrightarrow{\theta} \widehat{\operatorname{Hom}}_R(P,Q) \longrightarrow 0.$$

The stable cohomology of the pair $(L,M)$ is the graded $R^e$-module $\widehat{\operatorname{Ext}}_R(L,M)$ with

$$\widehat{\operatorname{Ext}}^n_R(L,M) = H^n(\widehat{\operatorname{Hom}}_R(P,Q)) \quad \text{for each } n \in \mathbb{Z}.$$

It is equipped with functorial homomorphisms of graded $R^e$-modules

$$(1.0.2) \quad \operatorname{Ext}^n_R(L,M) \xrightarrow{\eta^n(L,M)} \widehat{\operatorname{Ext}}^n_R(L,M) \quad \text{for all } n \in \mathbb{Z}.$$
1.1. If \( \text{pd}_R L \) or \( \text{pd}_R M \) is finite, then \( \widehat{\text{Ext}}_R^n(L, M) = 0 \) for all \( n \in \mathbb{Z} \).

Indeed, in this case we may choose \( P \) or \( Q \) to be a bounded complex. The definitions then yield \( \text{Hom}_R(P, Q) = \text{Hom}_R(P, Q) \), and hence \( \text{Hom}_R(P, Q) = 0 \).

1.2. For a family \( \{M_j\}_{j \in J} \) of \( R \)-modules and every integer \( n \), the canonical inclusions \( M_j \to \bigoplus_{j \in J} M_j \) induce, by functoriality, a commutative diagram of \( R \)-modules

\[
\begin{array}{ccc}
\text{Ext}_R^n(L, \bigoplus_{j \in J} M_j) & \xrightarrow{\eta^n(L, \bigoplus_{j \in J} M_j)} & \text{Ext}_R^n(L, \bigoplus_{j \in J} M_j) \\
\bigoplus_{j \in J} \text{Ext}_R^n(L, M_j) & \xrightarrow{\bigoplus_{j \in J} \eta^n(L, M_j)} & \bigoplus_{j \in J} \text{Ext}_R^n(L, M_j)
\end{array}
\]

(1.2.1)

**Proposition 1.3.** Suppose \( L \) admits a resolution by finite projective \( R \)-modules.

For every \( n \), the vertical maps in (1.2.1) are bijective. In particular, the map \( \eta^n(L, \bigoplus_{j \in J} M_j) \) is injective or surjective for some \( n \) if and only if \( \eta^n(L, M_j) \) has the corresponding property for every \( j \in J \).

**Proof.** Let \( P \) be a resolution of \( L \) by finite projective \( R \)-modules and \( Q_j \) a projective resolution of \( M_j \). The complex \( \bigoplus_{j \in J} Q_j \) is a projective resolution of \( \bigoplus_{j \in J} M_j \), and we have a commutative diagram of morphisms of complexes of \( R \)-modules

\[
\begin{array}{cccccccccccc}
0 & \xrightarrow{} & \text{Hom}_R(P, \bigoplus_{j \in J} Q_j) & \xrightarrow{} & \text{Hom}_R(P, \bigoplus_{j \in J} Q_j) & \xrightarrow{} & \text{Hom}_R(P, \bigoplus_{j \in J} Q_j) & \xrightarrow{} & 0 \\
0 & \xrightarrow{} & \bigoplus_{j \in J} \text{Hom}_R(P, Q_j) & \xrightarrow{} & \bigoplus_{j \in J} \text{Hom}_R(P, Q_j) & \xrightarrow{} & \bigoplus_{j \in J} \text{Hom}_R(P, Q_j) & \xrightarrow{} & 0
\end{array}
\]
with natural vertical maps. The map $H^n(\mathcal{S})$ is bijective, as it represents

$$\prod_{j \in J} \text{Ext}^n_R(L, M_j) \longrightarrow \text{Ext}^n_R\left(L, \prod_{j \in J} M_j\right),$$

which is bijective due to the hypothesis on $L$. As $\mathcal{S}$ is evidently bijective, $H^n(\mathcal{S})$ is an isomorphism. The right-hand square of the diagram above induces (1.2.1). ■

2. Local rings. The next theorem is the main result of the paper. It concerns the maps

$$\text{Ext}^n_R(k, \beta) : \text{Ext}^n_R(k, M) \longrightarrow \text{Ext}^n_R(k, V)$$

induced by some homomorphism $\beta : M \rightarrow V$, and is derived from a result of Martsinkovsky [11] by using properties of stable cohomology, described in Section 1.

Theorem 2.1. Let $(R, m, k)$ be a local ring and $V$ an $R$-module such that $mV = 0$.

If $R$ is singular and $\beta : M \rightarrow V$ is an $R$-linear map that factors through some module of finite projective dimension, then

$$\text{Ext}^n_R(k, \beta) = 0 \text{ for all } n \in \mathbb{Z}.$$

Proof. By hypothesis, there exists an $R$-module $N$ of finite projective dimension such that $\beta$ factors as $M \xrightarrow{\gamma} N \xrightarrow{\delta} V$. The following diagram

$$\begin{array}{ccc}
\text{Ext}^n_R(k, M) & \xrightarrow{\text{Ext}^n_R(k, \beta)} & \text{Ext}^n_R(k, V) \\
\text{Ext}^n_R(k, \gamma) & & \text{Ext}^n_R(k, \delta)
\end{array}$$

$$\begin{array}{ccc}
\text{Ext}^n_R(k, V) & \xrightarrow{\eta^n(k, V)} & \text{Ext}^n_R(k, V) \\
\text{Ext}^n_R(k, \delta) & & \text{Ext}^n_R(k, \delta)
\end{array}$$

$$\begin{array}{ccc}
\text{Ext}^n_R(k, N) & \xrightarrow{\eta^n(k, N)} & \text{Ext}^n_R(k, N) = 0 \\
\text{Ext}^n_R(k, \gamma) & & \text{Ext}^n_R(k, \delta)
\end{array}$$

commutes due to the naturality of the maps involved; the equality comes from 1.1. The map $\eta^n(k, k)$ is injective by [11, Theorem 6]. Proposition 1.3 shows that $\eta^n(k, V)$ is injective as well, so the diagram yields $\text{Ext}^n_R(k, \beta) = 0$. ■
Note that no finiteness condition on $M$ is imposed in the theorem. This remark is used in the proof of the following corollary, which deals with the maps

$$\text{Tor}_R^n(k, \alpha) : \text{Tor}_R^n(k, V) \longrightarrow \text{Tor}_R^n(k, M)$$

induced by some homomorphism $\alpha : V \rightarrow M$.

**Corollary 2.2.** If $R$ is singular and $\alpha : V \rightarrow M$ is an $R$-linear map that factors through some module of finite injective dimension, then $\text{Tor}_R^n(k, \alpha) = 0$ for all $n \in \mathbb{Z}$.

**Proof.** Set $(-)^\vee = \text{Hom}_R(-, E)$, where $E$ is an injective envelope of the $R$-module $k$. Let $V \rightarrow L \rightarrow M$ be a factorization of $\alpha$ with $L$ of finite injective dimension. By Ishikawa [7, 1.5], the module $L^\vee$ has finite flat dimension, so it has finite projective dimension by Jensen [9, 5.8]. As $m(V^\vee) = 0$ and $\alpha^\vee$ factors through $L^\vee$, Theorem 2.1 gives $\text{Ext}_R^n(k, \alpha^\vee) = 0$. The natural isomorphism $\text{Ext}_R^n(k, -^\vee) \cong \text{Tor}_R^n(k, -)^\vee$ now yields $\text{Tor}_R^n(k, \alpha)^\vee = 0$, whence we get $\text{Tor}_R^n(k, \alpha) = 0$, as desired. □

Next we record an elementary observation, where $(-)^* = \text{Hom}_R(-, R)$.

**Lemma 2.3.** Let $(R, m, k)$ be a local ring and $\chi : X \rightarrow Y$ an $R$-linear map. If $\text{Coker}(\chi)$ has a non-zero free summand, then $\text{Ker}(\chi^*) \nsubseteq mY^*$ holds. When $Y$ is free of finite rank the converse holds as well.

**Proof.** The condition on $\text{Coker}(\chi)$ holds if and only if there is an epimorphism $\text{Coker}(\chi) \rightarrow R$, that is, an $R$-linear map $v : Y \rightarrow R$ with $v\chi = 0$ and $v(Y) \nsubseteq m$. When such an $v$ exists, it is in $\text{Ker}(\chi^*)$, but not in $mY^*$, for otherwise $v(Y) \subseteq m$.

When $Y$ is finite free and $\text{Ker}(\chi^*) \nsubseteq mY^*$ holds, pick $v$ in $\text{Ker}(\chi^*)$, but not in $mY^*$. Since $Y^*$ is finite free, $v$ can be extended to a basis of $Y^*$; hence, $v(Y) = R$. □

The theorem in the introduction is the crucial implication in the next result:
Theorem 2.4. Let \((R, \mathfrak{m}, k)\) be a local ring, \(M\) an \(R\)-module, and 
\[ \varepsilon^n_M = \text{Ext}^n_R(k, \pi^M) : \text{Ext}^n_R(k, M) \to \text{Ext}^n_R(k, M/\mathfrak{m}M) \]
the homomorphism induced by the natural map \(\pi^M : M \to M/\mathfrak{m}M\).

The following conditions are equivalent.

(i) \(R\) is regular.

(ii) \(\varepsilon^n_R \neq 0\) for some \(n\).

(iii) \(\varepsilon^n_M \neq 0\) for an \(R\)-module \(M\) with \(\text{pd}_R M < \infty\) and some \(n\).

(iv) \(\varepsilon^n_M \neq 0\) for every finite \(R\)-module \(M \neq 0\) and for \(d = \dim R\).

(v) \(\text{Coker}(\partial_n^F)\), where \(F\) is a minimal free resolution of \(k\) over \(R\), has a non-zero free direct summand for some \(n\).

Proof. Set \(G = \text{Hom}_R(F, R)\) with \(F\) as in (v).

From \(\text{Hom}_R(F, M) \cong G \otimes_R M\) and \(\partial(G \otimes M) \subseteq \mathfrak{m}(G \otimes M)\) (by the minimality of \(F\)) we get a commutative diagram

\[
\begin{array}{ccc}
\text{Ext}^n_R(k, M) & \xrightarrow{\varepsilon^n_M} & \text{Ext}^n_R(k, M/\mathfrak{m}M) \\
\cong & & \cong \\
\text{H}_{-n}(G \otimes_R M) & \xrightarrow{\text{H}_{-n}(G \otimes_R \pi^M)} & \text{H}_{-n}(G \otimes_R (M/\mathfrak{m}M)) = \text{G}_{-n} \otimes (M/\mathfrak{m}M)
\end{array}
\]

(i) \(\implies\) (iv). As \(R\) is regular, \(F\) is the Koszul complex on a minimal generating set of \(\mathfrak{m}\). This gives \(G_d = R\), an isomorphism \(\text{H}_{-d}(G \otimes_R \pi^M)\), and an inequality \(M/\mathfrak{m}M \neq 0\) by Nakayama’s lemma; now the diagram yields \(\varepsilon^d_M \neq 0\).

(iv) \(\implies\) (ii) \(\implies\) (iii). These implications are tautologies.

(iii) \(\implies\) (i). This implication is a special case of Theorem 2.1.

(ii) \(\iff\) (v). The preceding diagram shows that the condition \(\varepsilon^n_R \neq 0\) is equivalent to \(\text{Ker}(\partial_{-n}^G) \not\subseteq \mathfrak{m}G_{-n}\). Thus, the desired assertion follows from Lemma 2.3.

Notes 2.5. The equivalence of conditions (i) and (ii) in Theorem 2.4 was proved by Ivanov [8, Theorem 2] when \(R\) is Gorenstein and by Lescot [10, 1.4] in general.
The equivalence of (i) and (v) is due to Dutta [4, 1.3]. As shown above, it follows from Lescot’s theorem via the elementary Lemma 2.3. Martsinkovsky deduced Dutta’s theorem from [11, Theorem 6] and used the latter to prove regularity criteria different from (ii), (iii) and (iv) in Theorem 2.4, see [11, page 11].

3. Bass numbers of modules. The $n$th Bass number of a module $M$ over a local ring $(R, \mathfrak{m}, k)$ is the integer

$$\mu^n_R(M) = \text{rank}_k \text{Ext}^n_R(k, M).$$

Given a homomorphism $\beta : M \to N$ and an $R$-submodule $N' \subseteq N$, we let $M \cap N'$ denote the submodule $\beta^{-1}(N')$ of $M$.

**Theorem 3.1.** Let $(R, \mathfrak{m}, k)$ be a local ring, $M \to N$ an $R$-linear map, and set

$$r = \text{rank}_k(M/M \cap \mathfrak{m}N).$$

If $R$ is singular and $\text{pd}_R N$ is finite, then there is an equality

$$\mu^n_R(M \cap \mathfrak{m}N) = \mu^n_R(M) + r\mu^{n-1}_R(k), \quad \text{for each } n \in \mathbb{Z}.$$

**Proof.** Set $\overline{M} = M/(M \cap \mathfrak{m}N)$ and $\overline{N} = N/\mathfrak{m}N$, and let $\pi : M \to \overline{M}$ and $\iota : \overline{M} \to \overline{N}$ be the induced maps. They appear in a commutative diagram with exact rows

$$
\begin{array}{ccc}
M & \to & N \\
\downarrow \pi & & \downarrow \\
0 & \to & \overline{M} & \to & \overline{N}
\end{array}
$$

Since $\iota$ is $k$-linear, it is split, so we get a commutative diagram with exact rows

$$
\begin{array}{ccc}
\text{Ext}_R(k, M) & \to & \text{Ext}_R(k, N) \\
\downarrow \text{Ext}_R(k, \pi) & & \downarrow 0 \\
0 & \to & \text{Ext}_R(k, \overline{M}) & \to & \text{Ext}_R(k, \overline{N})
\end{array}
$$
and zero map due to Theorem 2.1. It implies $\text{Ext}_R(k, \pi) = 0$.

By definition, there exists an exact sequence of $R$-modules

$$0 \longrightarrow (M \cap mN) \longrightarrow M \overset{\pi}{\longrightarrow} \overline{M} \longrightarrow 0.$$  

As $\text{Ext}_R(k, \pi) = 0$, its cohomology sequence yields an exact sequence

$$0 \longrightarrow \text{Ext}^{n-1}_R(k, \overline{M}) \longrightarrow \text{Ext}^n_R(k, M \cap mN) \longrightarrow \text{Ext}^n_R(k, M) \longrightarrow 0$$

of $k$-vector spaces for each integer $n$. Computing ranks over $k$ and using the isomorphism $\text{Ext}_R(k, \overline{M}) \cong \text{Ext}_R(k, k) \otimes_k \overline{M}$, we obtain the desired equality. 

Recall that the $n$th Betti number of $M$ is the integer

$$\beta^n_R(M) = \text{rank}_k \text{Ext}^n_R(M, k).$$

**Corollary 3.2.** Assume $R$ is singular, $N$ is a finite $R$-module, $N \supseteq M \supseteq mN$ holds, and set

$$s = \text{rank}_k (N/M).$$

If $\text{pd}_R N = p < \infty$ holds, then for each $n \in \mathbb{Z}$ there is an equality

$$(3.2.1) \quad \mu^n_R(M) = \sum_{i=0}^p \mu^{n+i}_R(R) \beta^R_i(N) + s \beta^R_{n-1}(k).$$

In particular, $\mu^n_R(m) = \mu^n_R(R) + \beta^R_{n-1}(k)$ for each $n \geq 0$.

**Proof.** The hypotheses give $M \cap mN = mN$ and $r = \text{rank}_k (M/mN)$. Apply Theorem 3.1 to the inclusions $M \subseteq N$ and $mN \subseteq N$ to get

$$\mu^n_R(M) = \mu^n_R(mN) - r \beta^R_{n-1}(k)$$

$$= \mu^n_R(N) + \text{rank}_k (N/mN) \beta^R_{n-1}(k) - r \beta^R_{n-1}(k)$$

$$= \mu^n_R(N) + s \beta^R_{n-1}(k).$$
As pd$_R N$ is finite, Foxby [5, 4.3(2)] yields

$$\mu^n_R(N) = \sum_{i=0}^p \mu^{n+i}_R(R) \beta_i^R(N).$$

**Remark 3.3.** The hypothesis $p < \infty$ in the corollary is needed, as otherwise the sum in (3.2.1) is not defined. On the other hand, when $R$ is regular, and so $p$ is necessarily finite, formula (3.2.1) may fail. For instance, with $d = \text{dim } R$, one has

$$\mu^n_R(m) = \begin{cases} \binom{d}{n-1} & \text{for } n = 1, \ldots, d, \\ 0 & \text{otherwise.} \end{cases}$$

Indeed, $0 \to m \to R \to k \to 0$ induces a cohomology long exact sequence where $\mu^n_R(R) = 0$ for $n \neq d$, the map $\varepsilon^d_R$ is bijective by Theorem 2.4, and $\mu^n_R(k) = \binom{d}{n}.$

3.4. Bass numbers are often described in terms of the generating formal power series $I^M_R(t) = \sum_{n \in \mathbb{Z}} \mu^n_R(M) t^n$. We also use the series $P^M_R(t) = \sum_{n \in \mathbb{Z}} \beta^n_R(M) t^n$.

In these terms, the formulas (3.2.1) can be restated as an equality

$$(3.4.1)\quad I^M_R(t) = I^R_R(t) P^R_N(t^{-1}) + stP^R_k(t).$$

3.5. Let $(S, m_S, k)$ and $(T, m_T, k)$ be local rings, and let

$$\varepsilon_S : S \to k \leftarrow T : \varepsilon_T$$

be the canonical maps. The fiber product is defined by the formula

$$(S \times_k T) := \{(s, t) \in S \times T \mid \varepsilon_S(s) = \varepsilon_T(t)\}.$$ 

This is a subring of $S \times T$, which is local with maximal ideal $m = m_S \oplus m_T$ and residue field $k$. Set $R = S \times_k T$.

Let $N$ and $P$ be finite modules over $S$ and $T$, respectively. The canonical maps $S \leftarrow R \to T$ turn $N$ and $P$ into $R$-modules, and for them Lescot [10, 2.4] proved

$$(3.5.1)\quad \frac{I^N_R(t)}{P^R_k(t)} = \frac{I^S_N(t)}{P^S_k(t)} \quad \text{and} \quad \frac{I^P_R(t)}{P^R_k(t)} = \frac{I^P_T(t)}{P^T_k(t)}.$$
If \( N/n_S N = V = P/m_T P \) holds for some \( k \)-module \( V \), then
\[
N \times_V P := \{(n, p) \in N \times P \mid \pi^N(n) = \pi^P(p)\}
\]
has a natural structure of finite \( R \)-module.

**Corollary 3.6.** With notation as in 3.5, set \( v = \text{rank}_k V \) and \( M = N \times_V P \).

If \( S \) and \( T \) are singular and \( \text{pd} \_S N \) and \( \text{pd} \_T P \) are finite, then
\[
\frac{I^M \_R(t)}{P^R \_k(t)} = \frac{I^S \_S(t)P^S \_N(t^{-1})}{P^S \_k(t)} + \frac{I^T \_T(t)P^T \_P(t^{-1})}{P^T \_k(t)} + 2vt.
\]

**Proof.** We have \( mM \cong m_S N \oplus m_T P \) as \( R \)-modules, whence the first equality below:
\[
\frac{I^M \_R(t)}{P^R \_k(t)} = \frac{I^S \_m \_S \_M(t)}{P^R \_k(t)} + \frac{I^T \_m \_T \_M(t)}{P^R \_k(t)}
= \frac{I^S \_S(t)P^S \_N(t^{-1})}{P^S \_k(t)} + vt + \frac{I^T \_T(t)P^T \_P(t^{-1})}{P^T \_k(t)} + vt.
\]
The second one comes from (3.5.1) and (3.4.1), in this order.

**Notes 3.7.** For all finite \( R \)-modules \( N \supseteq M \supseteq mN \), it is proved in [1, Theorem 4] that the Bass numbers of \( M \) and \( k \) asymptotically have the same size, measured on appropriate polynomial or exponential scales. The closed formula in Corollary 3.2 is a much more precise statement, but as noted in Remark 3.3 that formula may not hold when \( \text{pd} \_R N \) is infinite or when \( R \) is regular.

The last conclusion in Corollary 3.2 is Lescot’s result [10, 1.8(2)]. Combining it with the expression for \( I^R \_R(t) \) obtained from Corollary 3.6 by setting \( N = S \), \( P = T \) and \( V = k \), one recovers [10, 3.2(1)]. The proof of Corollary 3.6 faithfully transposes Lescot’s derivation of [10, 3.2(1)] from [10, 1.8(2)].
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