On general Caffarelli-Kohn-Nirenberg type inequalities involving non-doubling weights

Toshio Horiuchi

December 19, 2022

Abstract

By using $W(R_+) = P(R_+) \cup Q(R_+)$ as a class of weight functions, we will establish the Caffarelli-Kohn-Nirenberg type inequalities with non-doubling weights being permitted. The classical Caffarelli-Kohn-Nirenberg type inequalities are categorized into non-critical and critical cases, and it is known that there is some kind of mysterious relationship between them. Interestingly the new framework in this treatise allows them to be integrated and reveals the meaning of mysterious relationships.

1 Introduction

The main purpose of the present paper is to study the Caffarelli-Kohn-Nirenberg type inequalities, which are abbreviated as the CKN-type inequalities. We will establish the CKN-type inequalities with non-doubling weights. For this purpose we introduce a class of weight functions denoted by $W(R_+) (R_+ = (0, \infty))$, that is

$$W(R_+) = \{w \in C^1(R_+) : w > 0, \lim_{t \to +0} w(t) = a \text{ for some } a \in [0, \infty]\}.$$ 

Further we define two subclasses of $W(R_+)$, that is

$$P(R_+) = \{w(t) \in W(R_+) : w(t)^{-1} \notin L^1((0, \eta)) \text{ for some } \eta > 0\},$$

$$Q(R_+) = \{w(t) \in W(R_+) : w(t)^{-1} \in L^1((0, \eta)) \text{ for any } \eta > 0\}. \tag{1.1}$$

Clearly $W(R_+) = P(R_+) \cup Q(R_+)$ and $P(R_+) \cap Q(R_+) = \emptyset$ hold. See Section 2 for the precise definition of these classes. A positive continuous function $w(t)$ on $(0, \infty)$ is said to be a doubling weight if there exists a positive number $C$ such that we have $C^{-1}w(t) \leq w(2t) \leq Cw(t)$ $(0 < t < \infty)$, where $C$ is independent of each $t \in (0, \infty)$. If $w(t)$ does not possess this property, then $w(t)$ is said to be a non-doubling weight, and typically $c^{-1/\mu} \in P(R_+)$ and $c^{1/\mu} \in Q(R_+)$ are non-doubling weights. It will be seen that our results on the CKN-type inequalities essentially depend on whether $w$ belongs to $P(R_+)$ or $Q(R_+)$. The classical CKN-type inequalities are categorized into the non-critical inequalities (2.1) and the critical inequalities (2.2), and there is some kind of mysterious relationship between them. For the details see (2.10) in Remark (2.2) and Proposition (7.3) in Section 7. By using a new framework, we will show that they can be treated in a unified manner, and as a result we will make clear the meaning of the relationship. Let us explain the situation in a little more detail. We shall establish the following CKN-type inequalities (1.2) with non-doubling weights that contain the classical CKN-type inequalities.

Let $w \in W(R_+)$, $1 < p \leq q < \infty$, $\eta > 0$, $\mu > 0$ and $0 \leq 1/p - 1/q \leq 1/n$. Then, there exists a positive number $C_n = C_n(p, q, \eta, \mu, w)$ such that we have

$$\int_{B_\eta} |\nabla u|^p w(|x|)^{p-1} |x|^{1-n} \, dx \geq C_n \left( \int_{B_\eta} \frac{|u|^q |x|^{1-n} \, dx}{w(|x|) f_\eta(|x|)^{1+q/p}} \right)^{p/q}, \quad u \in C^c_c(B_\eta \setminus \{0\}), \tag{1.2}$$
where \( p' = p/(p-1) \), \( B_\eta \) is the ball \( \{ x \in \mathbb{R}^n : |x| < \eta \} \) and

\[
 f_\eta(t) = \begin{cases} 
 \frac{1}{t^{p'-p} \gamma}, & \text{if } \gamma > 0 \text{ and } \mu = \frac{1}{p'} \eta^{-p' \gamma}, \\
 \mu + \log \eta/t, & \text{if } \gamma = 0, \\
 -\frac{1}{p'} t^{-p' \gamma}, & \text{if } \gamma < 0,
\end{cases}
\]

(1.3)

The inequalities (1.2) clearly involve non-doubling weights. We remark that when \( p = q \), they are reduced to the Hardy-type inequality and have been established in [6], Theorem 3.1. (cf. [7])

For now, let’s admit the inequalities (1.2) and show how they unify non-critical and critical CKN-type inequalities. For this purpose, we set \( w(t) = t^{p' \gamma + 1}, \gamma \in \mathbb{R} \). By the definitions of \( P(\mathbb{R}_+), Q(\mathbb{R}_+) \) and \( w(t) \), we immediately see that \( w(t) \in P(\mathbb{R}_+) \) if \( \gamma \geq 0 \), and \( w(t) \in Q(\mathbb{R}_+) \) if \( \gamma < 0 \). Since \( w(|x|)^{p'-1}|x|^{-n} = |x|^{p(1+\gamma)\gamma - n} \) holds, we are able to check that

\[
f_\eta(t) = \begin{cases} 
 \frac{1}{t^{p' \gamma}}, & \text{if } \gamma > 0 \text{ and } \mu = \frac{1}{p'} \eta^{-p' \gamma}, \\
 \frac{1}{|x|^{\gamma}(\log(|x|/\eta))^{1+q/p'}}, & \text{if } \gamma = 0 \text{ and } R = e^\mu, \\
 (-p' \gamma)^{1+q/p'} |x|^{q \gamma - n}, & \text{if } \gamma < 0.
\end{cases}
\]

(1.4)

(1.5)

If \( \gamma \neq 0 \), then the inequalities (1.2) coincide with the non-critical CKN-type inequalities (2.1) with \( \mathbb{R}^n \) replaced by \( B_\eta \), and if \( \gamma = 0 \), then they coincide with the critical CKN-type inequality (2.6). Therefore, it follows from (2.1) and (2.6), together with the remarks just after (2.5) and (2.9), that the desired inequalities (1.2) hold, provided that \( C_n \) satisfies \( C_n (p' \gamma)^{p(1+q/p')/(q+1/p')} \leq S^{q/p'} \) if \( \gamma \neq 0 \); \( C_n \leq C^{p,q,R} \) if \( \gamma = 0 \). Thus we see that the inequalities (1.2) unify the non-critical and the critical CKN-type inequalities.

Here we clearly state that the inequalities (1.2) do not hold for \( w \in W(\mathbb{R}_+) \) unconditionally, unless \( n = 1 \) or \( p = q \) (the Hardy-type). In order to study the validity of (1.2) with each \( w(t) \in W(\mathbb{R}_+) \) we introduce the non-degenerate condition (3.2) in Section 3 concerning the behavior of \( w(t) \) near \( t = 0 \), and in Theorem 3.4 we establish the validity of (1.2) under (3.7). Roughly speaking, (3.7) assures that \( w(t) \) does not behave so badly as \( t \to +0 \), and hence the function \( H(\rho) \), given by Definition (3.2), is bounded away from 0. On the contrary if \( \lim_{\rho \to +0} H(\rho) = 0 \) is assumed, then we show in Theorem 3.2 that the inequalities (1.2) fail to hold, and in Theorem 3.3 we characterize a set of weight functions for which (3.7) is violated.

In [3] they established general multiplicative inequalities with weights being powers of distance from the origin. The inequalities they presented include the classical Hardy-Sobolev inequalities (cf. [4] [13] [17] [18]), which later became known as the CKN-type inequalities, are still the subject of many interesting studies. In the study of the CKN-type inequalities, the presence of weight functions in the both sides prevents us from employing effectively the so-called spherically symmetric rearrangement. Further the invariance of \( \mathbb{R}^n \) by the group of dilatations creates some possible loss of compactness. Partly because of these difficulties, it was a very interesting subject of research. In [5] [8] [9] [10] we have also studied these inequalities intensively to show that the existence of extremals, the values of best constants and their asymptotic behaviors essentially depend upon the relations among parameters in the inequalities.
Recently, there are authors who have studied classical inequalities such as Hardy-Sobolev type and CKN-type employing various transformations to obtain fruitful results (cf. [11] [14] [15] [16] [20]). We have also revisited weighted Hardy’s inequalities as follows: In [11] [12] we improved them under one-sided boundary conditions, and in [2] [5] [7] we introduced a new framework with a class of weight functions \( W(R+) \). This work is based on them.

This paper is organized in the following way: The proof Theorem 3.1 is given in Section 4. Theorem 5.2 and Theorem 5.3 are established in Section 5 and in Section 6 respectively. In Appendix we collect useful relations among the best constants of the CKN-type inequalities for the sake of self-containedness.

2 Preliminaries

2.1 A review of the classical CKN-type inequalities

We begin with reviewing the CKN-type inequalities as a background for this paper. We describe fundamental facts on the CKN-type inequalities according to [10], and classify the CKN-type inequalities into two cases, according to the range of the parameter \( \gamma \in \mathbb{R} \).

Definition 2.1. The parameter \( \gamma \) is said to be critical and non-critical if \( \gamma \) satisfies \( \gamma = 0 \) and \( \gamma \neq 0 \) respectively.

Remark 2.1. The non-critical case \( (\gamma \neq 0) \) was further classified in [10].

In the non-critical case, the CKN-type inequalities have the following form:

\[
\int_{\mathbb{R}^n} |\nabla u(x)|^p |x|^{p(1+\gamma)-n} \, dx \geq S^{p,q,\gamma} \left( \int_{\mathbb{R}^n} |u(x)|^q |x|^{q(n-n)} \, dx \right)^{p/q}, \quad u \in C_0^\infty (\mathbb{R}^n \setminus \{0\}),
\]

where \( \gamma \neq 0, n \geq 1, 1 < p \leq q < \infty \) and \( 0 \leq 1/p - 1/q \leq 1/n \). Here \( S^{p,q,\gamma} = S^{p,q,\gamma}(\mathbb{R}^n) \) is called the best constant and given by the following variational problem:

\[
S^{p,q,\gamma} = \inf \{ E^{p,q,\gamma}[u] : u \in C_0^\infty (\mathbb{R}^n \setminus \{0\}) \setminus \{0\} \},
\]

where

\[
E^{p,q,\gamma}[u] = \frac{\int_{\mathbb{R}^n} |\nabla u(x)|^p |x|^{p(1+\gamma)-n} \, dx}{\left( \int_{\mathbb{R}^n} |u(x)|^q |x|^{q(n-n)} \, dx \right)^{p/q}}, \quad u \in C_0^\infty (\mathbb{R}^n \setminus \{0\}) \setminus \{0\}.
\]

We also define the radial best constant \( S^{p,q,\gamma}_{rad} = S^{p,q,\gamma}(R^n) \) as follows.

Definition 2.2. Let \( \Omega \) be a radially symmetric domain. For any function space \( V(\Omega) \) on \( \Omega \), we set

\[
V(\Omega)_{rad} = \{ u \in V(\Omega) : u \text{ is radial} \}.
\]

Then we define

\[
S^{p,q,\gamma}_{rad} = \inf \{ E^{p,q,\gamma}[u] : u \in C_0^\infty (\mathbb{R}^n \setminus \{0\})_{rad} \setminus \{0\} \}.
\]

It follows from Proposition 5.1 that \( S^{p,q,\gamma} = S^{p,q,\gamma}_{rad} \) and \( S^{p,q,\gamma}_{rad} = S^{p,q,\gamma}_{rad} \). By \( S^{p,q,\gamma}(\Omega) \) we denote the best constant with \( \mathbb{R}^n \) replaced by \( \Omega \). Here we remark that the best constants \( S^{p,q,\gamma}(= S^{p,q,\gamma}(\mathbb{R}^n)) \) is invariant if \( \mathbb{R}^n \) is replaced by an arbitrary domain \( \Omega \) containing the origin. \( S^{p,q,\gamma}_{rad}(= S^{p,q,\gamma}_{rad}(\mathbb{R}^n)) \) is also invariant if \( \mathbb{R}^n \) is replaced by a radially symmetric domain \( \Omega \) containing the origin. Further \( S^{p,q,\gamma} \) and \( S^{p,q,\gamma}_{rad} \) are not attained by a function having compact support. (cf. [5] [19]).
In the critical case that $\gamma = 0$, the CKN-type inequalities have the following form: For $\eta > 0$, let $B_\eta$ be the ball $\{x \in \mathbb{R}^n : |x| < \eta\}$.

$$\int_{B_\eta} |\nabla u(x)|^p |x|^{-n} dx \geq C_{\gamma,R}^{p,q,R}(\int_{B_\eta} |u(x)|^q |x|^{-n} (\log(\eta R / |x|))^{1+q/p'} dx)^{p/q'}, \ u \in C_\infty^c(B_\eta \setminus \{0\}), \quad (2.6)$$

where $R$ is a positive number satisfying $R > 1$ and the best constant $C_{\gamma,R}^{p,q,R} = C_{\gamma,R}^{p,q,R}(B_\eta)$ is given by the variational problem:

$$C_{\gamma,R}^{p,q,R} = \inf \{ F_{p,q,R}[u] : u \in C_\infty^c(B_\eta \setminus \{0\}) \setminus \{0\} \}, \quad (2.7)$$

where

$$F_{p,q,R}[u] = \frac{\int_{B_\eta} |\nabla u(x)|^p |x|^{-n} dx}{\left( \int_{B_\eta} |u(x)|^q |x|^{-n} (\log(\eta R / |x|))^{1-q/p'} dx \right)^{p/q'}} \text{ for } u \in C_\infty^c(B_\eta \setminus \{0\}) \setminus \{0\}. \quad (2.8)$$

We also define the radial best constant

$$C_{\gamma,R}^{p,q,R} = \inf \{ F_{p,q,R}[u] : u \in C_\infty^c(B_\eta \setminus \{0\})_{\text{rad}} \setminus \{0\} \}. \quad (2.9)$$

Here we remark that the functional $F_{p,q,R}[u]$ itself depends on each $\eta > 0$, nevertheless the best constants $C_{\gamma,R}^{p,q,R}$ and $C_{\gamma,R}^{p,q,R}$ are independent of $\eta > 0$. To see this it suffices to employ a change of variables given by $x = \eta y$.

**Remark 2.2.** It follows from Proposition 7.1 and Proposition 7.2 we have interesting and mysterious relationships among the best constants as follows:

If $n \geq 2$, $1/p' \leq \gamma_{p,q}$, $R \geq R_{p,q}$ and $1/p - 1/q \leq 1/n$, then it holds that

$$S_{p,q;1/p'} = S_{\gamma,R}^{p,q;1/p'} = C_{\gamma,R}^{p,q,R} = C_{\gamma,R}^{p,q,R}. \quad (2.10)$$

Here $\gamma_{p,q}$ and $R_{p,q}$ are positive numbers defined by (7.1) and (7.2) in Appendix.

If $n = 1$, then we have the following that is proved for the sake of self-containedness.

**Lemma 2.1.** Assume that $n = 1$, $1 < p \leq q < \infty$ and $\gamma \neq 0$. Then we have the followings:

1. $S_{p,q;\gamma}(\mathbb{R}) = S_{p,q;\gamma}((-\infty,0)) = S_{p,q;\gamma}(0,\infty))$.
2. $S_{\gamma,R}^{p,q;\gamma}(\mathbb{R}) = 2^{1-p/q} S_{p,q;\gamma}(\mathbb{R})$.

**Proof of Lemma 2.1.** For any $u \in C_\infty^c(\mathbb{R} \setminus \{0\})$, temporally we set

$$||u||_{1}(\Omega) = \left( \int_{\Omega} |u(t)|^p |t|^{n(1+\gamma)-1} dt \right)^{1/p}, \quad ||u||_{2}(\Omega) = \left( \int_{\Omega} |u(x)|^q |x|^{q-1} dx \right)^{1/q},$$

where $\Omega = (-\infty, 0)$ or $(0, \infty)$. Since $1 < p \leq q < \infty$, we have $(1+t^p)^{1/p} \geq (1+t^q)^{1/q}$ for $t \geq 0$. Then we have

$$E_{p,q;\gamma}[u]^{1/p} \geq \min \left\{ \frac{||u'||_{1}((-\infty,0))}{||u'||_{2}((-\infty,0))}, \frac{||u'||_{1}((0,\infty))}{||u'||_{2}((0,\infty))} \right\} \quad (2.11)$$

Thus we have

$$S_{p,q;\gamma}(\mathbb{R}) \geq \min(S_{p,q;\gamma}((-\infty,0)), S_{p,q;\gamma}((0,\infty))).$$
By the symmetry, we have $S^{p,q;γ}(−∞, 0) = S^{p,q;γ}(0, ∞)$. Since the reverse inequality $S^{p,q;γ}(R) \leq S^{p,q;γ}(0, ∞)$ is obvious by the definition (2.4), the assertion (1) is proved.

We proceed to the assertion (2). If $u \in C_c^\infty(R \setminus \{0\} \setminus \{0\}$ is radial, then

$$
\|u\|_1(R) = 2^{1/p}\|u\|_1((−∞, 0)) = 2^{1/p}\|u\|_1((0, ∞))
$$
$$
\|u\|_2(R) = 2^{1/q}\|u\|_2((−∞, 0)) = 2^{1/q}\|u\|_2((0, ∞)).
$$

Then we have

$$
S^{p,q;γ}_\text{rad}(R) = 2^{1/p/q} S^{p,q;γ}((−∞, 0)) = 2^{1/p/q} S^{p,q;γ}((0, ∞)).
$$

Therefore the assertion (2) follows from the assertion (1).

\[\square\]

### 2.2 A class of weight functions

First we introduce a class of weight functions which is crucial in this paper.

**Definition 2.3.** Let us set $R_+ = (0, ∞)$ and

$$
W(R_+) = \{w \in C^1(R_+) : w > 0, \lim_{t \to +0} w(t) = a \text{ for some } a \in [0, ∞]\}. \tag{2.12}
$$

In the next we define two subclasses of this large class.

**Definition 2.4.** Let us set

$$
P(R_+) = \{w \in W(R_+) : w^{-1} \notin L^1((0, η)) \text{ for some } η > 0\} \tag{2.13}
$$

$$
Q(R_+) = \{w \in W(R_+) : w^{-1} \notin L^1((0, η)) \text{ for any } η > 0\} \tag{2.14}
$$

**Example 2.1.**

1. $t^α \in P(R_+)$ if $α ≥ 1$ and $t^α \in Q(R_+)$ if $α < 1$.
2. $e^{-1/η} \in P(R_+)$ and $e^{1/η} \in Q(R_+)$.
3. For $α ∈ R$, $t^{α}e^{-1/η} \in P(R_+)$ and $t^{α}e^{1/η} \in Q(R_+)$.

**Remark 2.3.**

1. From Definition 2.3 and Definition 2.4 it follows that $W(R_+) = P(R_+) ∪ Q(R_+)$ and $P(R_+) \cap Q(R_+) = ∅$.
2. If $w^{-1} \notin L^1((0, η))$ for some $η > 0$, then $w^{-1} \notin L^1((0, η))$ for any $η > 0$. Similarly if $w^{-1} \in L^1((0, η))$ for some $η > 0$, then $w^{-1} \in L^1((0, η))$ for any $η > 0$.
3. If $w \in P(R_+)$, then $\lim_{t \to +0} w(t) = 0$. Hence by setting $w(0) = 0$, $w$ is uniquely extended to a continuous function on $[0, ∞)$. On the other hand if $w \in Q(R_+)$, then possibly $\lim_{t \to +0} w(t) = +∞$.

Lastly we define a function $f_η(t)$ in order to introduce variants of the Hardy potentials:

**Definition 2.5.**

$$
f_η(t) = \begin{cases} 
\mu + \int_{r}^{η} \frac{1}{w(s)} ds, & \text{if } w \in P(R_+), \\
\int_{0}^{η} \frac{1}{w(s)} ds, & \text{if } w \in Q(R_+). \tag{2.15}
\end{cases}
$$
3 Main results

3.1 The \(n\)-dimensional CKN-type inequalities

Let \(B_\eta\) be the ball \(\{x \in \mathbb{R}^n : |x| < \eta\}\). Define a strictly monotone function \(\varphi(\rho) \in C^1(0, \varphi^{-1}(\eta))\) as follows:

**Definition 3.1.** 1. For \(w \in P(\mathbb{R}_+), \) by \(\varphi(\rho)\) we denote the unique solution of the integral equation

\[
\rho^{-1} = \mu + \int_{\rho}^{\eta} \frac{1}{w(s)} \, ds.
\]

(3.1)

2. For \(w \in Q(\mathbb{R}_+), \) by \(\varphi(\rho)\) we denote the unique solution of the integral equation

\[
\rho = \int_{0}^{\varphi(\rho)} \frac{1}{w(s)} \, ds.
\]

(3.2)

First we assume that \(w \in P(\mathbb{R}_+), \) then by differentiating (3.1) we have

\[
\varphi'(\rho) = w(\varphi(\rho)) \frac{\rho \varphi'(\rho)}{\rho^2}, \quad \lim_{\rho \to +0} \varphi(\rho) = 0, \quad \varphi(1/\mu) = \eta.
\]

(3.3)

We see that \(\varphi\) is strictly monotone and satisfies \(\rho^{-1}(t) = 1/\left(\mu + \int_{\rho}^{\eta} \frac{1}{w(s)} \, ds\right).\)

Secondly we assume that \(w \in Q(\mathbb{R}_+), \) then by differentiating (3.2) we have

\[
\varphi'(\rho) = w(\varphi(\rho)), \quad \lim_{\rho \to +0} \varphi(\rho) = 0.
\]

(3.4)

Again \(\varphi\) is strictly monotone and satisfies \(\varphi^{-1}(t) = \int_{0}^{t} \frac{1}{w(s)} \, ds.\)

Then we define

**Definition 3.2.** 1. For \(w \in P(\mathbb{R}_+), \) we set for \(\rho \in (0, \varphi^{-1}(\eta))\)

\[
H(\rho) = \rho \varphi'(\rho) \frac{w(\varphi(\rho))}{\varphi(\rho)} = \frac{w(\varphi(\rho))}{\varphi(\rho)} \left(\mu + \int_{\rho}^{\eta} w(s)^{-1} \, ds\right).
\]

(3.5)

2. For \(w \in Q(\mathbb{R}_+)\) we set for \(\rho \in (0, \varphi^{-1}(\eta))\)

\[
H(\rho) = \rho \varphi'(\rho) \frac{w(\varphi(\rho))}{\varphi(\rho)} = \frac{w(\varphi(\rho))}{\varphi(\rho)} \int_{0}^{\varphi(\rho)} w(s)^{-1} \, ds.
\]

(3.6)

Now we introduce the non-degenerate condition (NDC) on \(H(\rho)\) which assures that \(H(\rho)\) is bounded away from 0 as \(\rho \to +0:\)

**Definition 3.3.** (the non-degenerate condition) Let \(\eta > 0\) and \(w \in W(\mathbb{R}_+).\) A weight function \(w\) is said to satisfy the non-degenerate condition (3.7) if

\[
C_0 := \inf_{0 < \rho < \varphi^{-1}(\eta)} H(\rho) > 0.
\]

(3.7)

Here \(H(\rho)\) is defined by Definition 3.2.

Here we state the \(n\)-dimensional CKN-type inequalities, which is a natural extension of the classical CKN-type inequalities \((2.1)\) and \((2.6)\) in Section 2.
Theorem 3.1. Let \( n \geq 1, 1 < p \leq q < \infty, \mu > 0, \eta > 0 \) and \( 0 \leq 1/p - 1/q \leq 1/n \). Assume that \( w \in W(R_+) \). Moreover assume that if \( n > 1 \) and \( p < q, H(\rho) \) satisfies the non-degenerate condition (3.7). Then, we have the followings:

1. There exists a positive number \( C_n = C_n(p,q,\eta,\mu,w) \) such that for any \( u \in C^\infty_c(B_\eta \setminus \{0\}) \) we have
   \[
   \int_{B_\eta} |\nabla w|^p w^{(p-1)}|x|^{1-n} dx \geq C_n \left( \int_{B_\eta} \frac{|u|^{q}|x|^{1-n} dx}{w(|x|)} f_\eta(|x|)^{1+q/p'} \right)^{p/q} \tag{3.8}
   \]
where \( f_\eta(t) \) is defined by (2.7).\( ^{\text{11}} \)

2. If \( C_n \) is the best constant, then \( C_n \) satisfies the followings:
   
   (a) If \( n = 1 \), then \( C_1 = S_{p,q;1/p'}(R) = 2^{p/q-1}S_{p,q;1/p'}(R) \).
   
   (b) If \( n > 1 \), then \( C_n \geq \min(C_0,1)S_{p,q;1/p'}(\mathbb{R}^n) \) \( (p < q) \); \( C_n = (1/p')^p \) \( (p = q) \).
   
   (c) If \( C_0 \geq 1 \) and \( 1/p' \leq \gamma_{p,q}, \) then
   \[
   C_n = S_{p,q;1/p'}(\mathbb{R}^n) = S_{\text{rad}}^{p,q;1/p'}(\mathbb{R}^n).
   \]

Here \( C_0 \) is given by (3.7) and \( \gamma_{p,q} \) is given by
\[
\gamma_{p,q} = \frac{n-1}{1+q/p'} \tag{3.9}
\]

Theorem 3.1 will be established in Section 4 by virtue of the variational principle and the non-critical CKN-type inequalities. We also remark that if \( p = q \), then this has been partially treated in [6] as the Hardy-type inequality.

Let us consider the case where (NDC) is violated, that is, \( \inf_{0 < \rho < \Phi^{-1}(\eta)} H(\rho) = 0 \) holds. From Definition 3.2, we see that \( H(\rho) > 0 \) for \( \rho \in (0,\Phi^{-1}(\eta)) \), hence \( \lim_{\rho \to +0} H(\rho) = 0 \). If we assume \( \lim_{\rho \to +0} H(\rho) = 0 \) in stead, then the inequality (3.3) fails to hold. Namely:

Theorem 3.2. Let \( n > 1, 1 < p < q < \infty, \mu > 0, \) and \( 0 \leq 1/p - 1/q \leq 1/n \) and \( R > 0 \). Assume that \( w \in W(R_+) \). If \( \lim_{\rho \to +0} H(\rho) = 0 \), then the inequality (3.3) is not valid.

Intuitively speaking, if \( w \in P(R_+) \) and \( w \) vanishes infinitely at the origin or if \( w \in Q(R_+) \) and \( w \) blows up infinitely at the origin, then (NDC) is violated. To explain more accurately, we introduce the following notion.

Definition 3.4. 1. For \( w(t) \in P(R_+) \), \( w(t) \) is said to vanish at the origin in the infinite order, if and only if for an arbitrary positive integer \( m \) there exist a positive number \( t_m \) such that \( t_m \to 0 \) as \( m \to \infty \) and
\[
w(t) \leq t^m, \quad t \in (0,t_m). \tag{3.10}
\]

2. For \( w(t) \in Q(R_+) \) \( w(t) \) is said to blow up at the origin in the infinite order, if and only if for an arbitrary positive integer \( m \) there exists a positive \( t_m \) such that \( t_m \to 0 \) as \( m \to \infty \) and
\[
w(t) \geq t^{-m}, \quad t \in (0,t_m). \tag{3.11}
\]

Then we have the following:
**Theorem 3.3.** If \( w \in P(\mathbb{R}_+) \) and \( w \) vanishes at the origin in the infinite order, or if \( w \in Q(\mathbb{R}_+) \) and \( w \) blows up at the origin in the infinite order, then (NDC) is not satisfied.

To help understand the theorems, we give typical examples:

**Example 3.1.**

1. When either \( w(t) = e^{-t/|t|} \in P(\mathbb{R}_+) \) or \( w(t) = e^{t/|t|} \in Q(\mathbb{R}_+) \),
   
   \( H(\varphi^{-1}(t)) = O(t) \) as \( t \to +0 \), or equivalently \( H(\rho) = O(\varphi(\rho)) \) as \( \rho = \varphi^{-1}(t) \to +0 \).

2. Moreover, if \( w(t) = \exp(\pm t^{-\alpha}) \) with \( \alpha > 0 \), then \( H(\varphi^{-1}(t)) = O(t^\alpha) \) as \( t \to +0 \). In fact, it holds that \( \lim_{t \to +0} H(\varphi^{-1}(t))/t^\alpha = 1/\alpha \).

**Example 3.2.** Let \( n \geq 1, \ 1 < p \leq q < \infty, \ 0 \leq 1/p - 1/q \leq 1/n \) and \( 0 < \eta \).

Let \( w(t) = t^{n/p}r^{1} \).

If \( \gamma > 0 \), then \( w(t) \in P(\mathbb{R}_+) \) and we have for \( \mu = \frac{1}{p'} \eta^{-p'} \gamma \),

\[
 f_\eta(t) = \frac{1}{p'} t^{-p'} \gamma, \quad \frac{1}{p} = \frac{1}{p'} \eta^{-p'} \gamma \quad \text{and} \quad H(\rho) = \frac{\rho \varphi'(\rho)}{\varphi(\rho)} = \frac{1}{p'} \gamma.
\]

If \( \gamma = 0 \), then \( w(t) \in P(\mathbb{R}_+) \) and we have for \( \mu > 0 \),

\[
 f_\eta(t) = \mu + \log \eta / t, \quad \frac{1}{p} = \mu + \log \eta / \varphi(\rho) \quad \text{and} \quad H(\rho) = \frac{1}{p} (\geq \mu).
\]

If \( \gamma < 0 \), then \( w(t) \in Q(\mathbb{R}_+) \)

\[
 f_\eta(t) = -\frac{1}{p'} t^{-p'} \gamma, \quad \rho = -\frac{1}{p'} \eta^{-p'} \gamma \quad \text{and} \quad H(\rho) = \frac{\rho \varphi'(\rho)}{\varphi(\rho)} = -\frac{1}{p'} \gamma.
\]

Thus we have

\[
 C_0 = \begin{cases} 
 (p'|\gamma|)^{-1}, & \text{if } \gamma \neq 0, \\
 \mu, & \text{if } \gamma = 0.
\end{cases}
\]

Assume that either \( 0 < \lvert \gamma \rvert \leq 1/p' \) or \( \gamma = 0, \mu \geq 1 \). Then we have \( H(\rho) \geq 1 \) in \((0, \varphi^{-1}(\eta))\).

From Theorem 3.7 noting that \( w(|x|)^{p-1}|x|^{1-n} = |x|^{p(1+\gamma)-n} \), we have

\[
 \int_{B_\eta} |\nabla u|^{p} |x|^{p(1+\gamma)-n} \, dx \geq C \left( \int_{B_{\eta}} \frac{|u|^q |x|^{1-n}}{w(|x|) f_\eta(|x|)^{1+q/p'}} \right)^{p/q} 
\]

(3.12)

where \( C = \min(C_0^p, 1)^{Sp:q;1/p'} = Sp:q;1/p' \).

We give a non-doubling weight function \( w(t) \in W(\mathbb{R}_+) \) for which the condition (NDC) is satisfied.

**Example 3.3.** Let \( 1 < \eta \) and for simplicity we make examples with \((0, \eta]\) instead of \( \mathbb{R}_+ \). First we give an example in \( P((0, \eta]) \). (For the definition of \( P((0, \eta]) \) and \( Q((0, \eta]) \), see Remark 3.7.) Define

\[
 z(t) = \frac{1}{\log(\eta^t/t)}, \quad 0 < t \leq \eta.
\]

We take a function \( z_1(t) \in C^1((0, \eta]) \) such that \( z_1(t) \) satisfies the estimate

\[
 z(t) \leq z_1(t) \leq 1
\]

and the condition

\[
 \begin{cases} 
 z_1(t_{2m}) = z(t_{2m}), \\
 z_1(t_{2m+1}) = 1, \quad (m = 1, 2, \cdots),
\end{cases} 
\]

(3.13)
where \( t_m = 1/m \) \((m = 1, 2, \cdots)\). Now we set
\[
w(t) = tz_1(t) \in C^1((0, \eta]).
\]

Since \( w(t) = tz_1(t) \leq t \) holds, we have \( \int_0^\eta w(s)^{-1} \, ds \geq \int_0^\eta 1/s \, ds = \log(\eta / t) \to \infty \) \((t \to 0)\). Therefore \( w(t) \in \mathcal{P}(0, \eta]) \). We show that \( w(t) \) becomes a non-doubling weight function. In fact, if \( m \) is an odd number, then we have
\[
\frac{w(tz_2m)}{w(tz_2m)} = \frac{w(tz_2m)}{w(tz_2m)} = \frac{1}{2 \log(2m\eta)} \to 0 \text{ as } m \to \infty.
\]

On the other hand, for \( p = \varphi^{-1}(t) \) \((0 < t \leq \eta)\),
\[
H(\varphi^{-1}(t)) = \frac{w(t)}{t} \left( \mu + \int_t^\eta \frac{1}{w(s)} \, ds \right) = z_1(t) \left( \mu + \int_t^\eta \frac{1}{sz_1(s)} \, ds \right)
\]
\[
\geq z(t) \left( \mu + \int_t^0 \frac{1}{s} \, ds \right) = \frac{1}{\log(e\eta / t)} \cdot \left( \mu + \log(\eta / t) \right) \geq 1 \quad \text{for } \mu \geq 1.
\]

Hence the condition \((\text{NDC})\) is satisfied.

In the next we give an example in \( \mathcal{Q}(0, \eta]) \). We set
\[
z(t) = \log(e\eta / t) \quad 0 < t \leq \eta.
\]

Take a function \( z_1(t) \in C^1((0, \eta]) \) satisfying the condition \((3.13)\) and the estimate
\[
1 \leq z_1(t) \leq z(t).
\]

We define
\[
w(t) = tz(t)^2z_1(t) \in C^1((0, \eta]).
\]

Since \( w(t) = tz(t)^2z_1(t) \geq tz(t)^2 \) holds, we have \( \int_0^t w(s)^{-1} \, ds \leq \int_0^t 1/(s(\log(e\eta / s)^2) \, ds = 1 \). Therefore \( w(t) \in \mathcal{Q}(0, \eta]) \). We show that \( w(t) \) becomes a non-doubling weight function. If \( m \) is an odd number, then we have
\[
\frac{w(tz_2m)}{w(tz_2m)} = \frac{w(tz_2m)}{w(tz_2m)} = \frac{1}{2 \log(2m\eta)} \to \infty \text{ as } m \to \infty.
\]

On the other hand, for \( p = \varphi^{-1}(t) \), \( 0 < t \leq \eta \),
\[
H(\varphi^{-1}(t)) = \frac{w(t)}{t} \int_0^t \frac{1}{w(s)} \, ds = z(t)^2z_1(t) \int_0^t \frac{1}{sz_1(s)} \, ds
\]
\[
\geq z(t)^2 \int_0^t \frac{1}{sz(s)} \, ds = \frac{(\log(e\eta / t)^2 \cdot (\log(e\eta / t))^2}{2} = \frac{1}{2}.
\]

Hence the condition \((\text{NDC})\) is satisfied.

Remark 3.1.  
1. In Example \((3.3)\) \( \mathcal{P}(0, \eta]) \) and \( \mathcal{Q}(0, \eta]) \) are defined in a similar way by \((1.1)\) with \( \mathbb{R}_+ \) replaced by \((0, \eta]\).

2. Theorem \((3.1)\), Theorem \((3.2)\) and Theorem \((3.3)\) will be established in Section 4, Section 5 and Section 6 respectively.

3. When \( p = 1 \), one can obtain similar results as in this section under a suitable modification. See \((7)\).
4 Proof of Theorem 3.1

Proof of Theorem 3.1. We employ a polar coordinate system $x = r\omega$ for $r = |x|$ and $\omega \in S^{n-1}$. By $\Delta_{S^{n-1}}$ we denote the Laplace-Beltrami operator on a unit sphere $S^{n-1}$. Then a gradient operator $\Lambda$ on $S^{n-1}$ is defined by

$$\int_{S^{n-1}} (-\Delta_{S^{n-1}} \xi_1)\xi_2\,dS = \int_{S^{n-1}} \Lambda \xi_1 \Lambda \xi_2\,dS \quad \text{for} \quad \xi_1, \xi_2 \in C^2(S^{n-1}).$$

(4.1)

Here we have

$$\Delta \xi_1 = \frac{1}{r^{n-1}} \partial_r (r^{n-1} \partial_r \xi_1) + \frac{1}{r^2} \Delta_{S^{n-1}} \xi_1, \quad |\nabla \xi_1|^2 = |\partial_r \xi_1|^2 + \frac{1}{r^2}|\Lambda \xi_1|^2,$$

(4.2)

where

$$\partial_r \xi_1(x) = \frac{x}{|x|} \nabla \xi_1(x).$$

(4.3)

Then, the inequality (3.8) is transformed to the following:

$$\int_{S^{n-1}} dS \int_0^{\bar{\eta}} \left( \frac{(\partial_r \xi_1)^2 + (\Lambda \xi_1)^2}{r^2} \right) w(r)^{p-1} dr \geq C_n \left( \int_{S^{n-1}} dS \int_0^{\bar{\eta}} \frac{|u|^q dr}{w(r)f_q (r)1/q} \right)^{p/q}, \quad (u(x) = u(r\omega) \in C^\infty_c(B_{\bar{\eta}} \setminus \{0\})).$$

(4.4)

Here $dS$ denotes surface elements on $S^{n-1}$ if $n > 1$ and a counting measure on $S^0 = \{-1,1\}$ if $n = 1$. Here we remark that if $n = 1$, then $\Delta_{S^{n-1}} = \Delta = 0$.

Proof of (1). First we consider the case where $w \in \mathcal{P}(\mathbb{R}_+)$, and we set $y = \rho \omega \rho = |y|$, $r = \varphi(\rho)$ and $U(y) = U(\rho \omega) = u(\varphi(\rho) \omega)$, then $U(y) \in C^\infty_c(B_{\bar{\eta}} \setminus \{0\})$ with $\bar{\eta} = \varphi^{-1}(\eta) (= 1/\mu)$. Then, the inequality (4.4) is transformed to the following:

$$\int_{S^{n-1}} dS \int_0^{\bar{\eta}} \left( (\partial_r U)^2 + H(\rho) \frac{(\Lambda U)^2}{\rho^2} \right) \rho^{2(p-1)} d\rho \geq C_n \left( \int_{S^{n-1}} dS \int_0^{\bar{\eta}} |u|^q \rho^{-1+q/p} d\rho \right)^{p/q},$$

(4.5)

or equivalently

$$\int_{B_{\bar{\eta}}} \left( (\partial_\rho U)^2 + H(\rho) \frac{(\Lambda U)^2}{\rho^2} \right) \rho^{2p-2-n} d\rho \geq C_n \left( \int_{B_{\bar{\eta}}} |U|^q \rho^{q/p'-n} d\rho \right)^{p/q}.$$

(4.6)

In order to show the inequality (3.8), it suffices to establish (4.6) for some positive number $C_n$. If $p = q$ holds, then (4.6) follows direct from the classical Hardy inequality, hence we assume that $p < q$. From (3.7), the left hand side of (4.6) is estimated from below in the following way.

$$\int_{B_{\bar{\eta}}} \left( (\partial_\rho U)^2 + H(\rho) \frac{(\Lambda U)^2}{\rho^2} \right) \rho^{2p-2-n} d\rho \geq \int_{B_{\bar{\eta}}} \left( (\partial_\rho U)^2 + C_0^2 \frac{(\Lambda U)^2}{\rho^2} \right) \rho^{2p-2-n} d\rho$$

$$\geq \min(C_0^2, 1) \int_{B_{\bar{\eta}}} \left( (\partial_\rho U)^2 + \frac{(\Lambda U)^2}{\rho^2} \right) \rho^{2p-2-n} d\rho$$

$$= \min(C_0^2, 1) \int_{\mathbb{R}^n} \frac{1}{|x|} \nabla U^p |\nabla U|^{p-1} \rho^{-n} d\rho$$

$$\geq \min(C_0^2, 1) S^p |x|^{p-1} \left( \int_{B_{\bar{\eta}}} |U|^q \rho^{q/p' - n} d\rho \right)^{p/q}.\]
In the last step we used the CKN type inequality \(2.1\) with \(\gamma = 1/p'\). This proves the assertion \(4.6\) with \(C_\gamma \geq \min(C_0^p, 1)S^{p,q,1/p'}\).

Secondly we assume that \(w \in Q(R_+^+)\). By putting \(r = \phi(p)\), in a similar way the inequality \(4.4\) is transformed to the following:

\[
\int_{S^{p-1}} dS \int_0^{\hat{\eta}} \left( (\partial_\rho U)^2 + H(\rho)^2 \frac{(\Lambda U)^2}{\rho^2} \right) \rho^{p/2} d\rho \geq C_\gamma \left( \int_{S^{p-1}} dS \int_0^{\hat{\eta}} |U|^q \rho^{-1-q/p'} d\rho \right)^{p/q},
\]

or equivalently

\[
\int_{B_{\hat{\eta}}} \left( (\partial_\rho U)^2 + H(\rho)^2 \frac{(\Lambda U)^2}{\rho^2} \right) \rho^{p/2} d\rho \geq C_\gamma \left( \int_{B_{\hat{\eta}}} |U|^q \rho^{-1-q/p'} d\rho \right)^{p/q}.
\]

We assume that \(p < q\) by the same reason in the previous step. Then we have

\[
\int_{B_{\hat{\eta}}} \left( (\partial_\rho U)^2 + H(\rho)^2 \frac{(\Lambda U)^2}{\rho^2} \right) \rho^{p/2} d\rho \geq \min(C_0^p, 1) \int_{B_{\hat{\eta}}} \left( (\partial_\rho U)^2 + \frac{(\Lambda U)^2}{\rho^2} \right) \rho^{p/2} d\rho
\]

\[
= \min(C_0^p, 1) \int_{B_{\hat{\eta}}} |\nabla U|^p \rho^{p(1-1/p')-n} d\rho
\]

\[
\geq \min(C_0^p, 1)S^{p,q,1/p'} \left( \int_{B_{\hat{\eta}}} |U|^q \rho^{-q/p'-1} d\rho \right)^{p/q}.
\]

In the last step we used the CKN type inequality \(2.1\) with \(\gamma = -1/p'\). This proves the assertion \(4.8\) with \(C_\gamma \geq \min(C_0^p, 1)S^{p,q,1/p'} = \min(C_0^p, 1)S^{p,q,1/p'}\).

**Proof of (2).** (a) Assume that \(n = 1\). Then \(3.8\) is equivalent to the following: For \(U \in C_c^\infty(B_{\hat{\eta}} \setminus \{0\})\) with \(\hat{\eta} = \varphi^{-1}(\eta)\),

\[
\begin{align*}
\int_{B_{\hat{\eta}}} |\partial U| \rho^{p(2(p-1)-n)} d\rho & \geq C_1 \left( \int_{B_{\hat{\eta}}} |U|^q \rho^{q/p'-1} d\rho \right)^{p/q}, \quad w \in P(R_+), \\
\int_{B_{\hat{\eta}}} |\partial U| \rho^{p} d\rho & \geq C_1 \left( \int_{B_{\hat{\eta}}} |U|^q \rho^{-q/p'-1} d\rho \right)^{p/q}, \quad w \in Q(R_+).
\end{align*}
\]

Then it follows from the non-critical CKN-type inequalities \(2.1\) with \(n = 1\) that we have \(C_1 = S^{p,q,1/p'}(B_{\hat{\eta}}) = S^{p,q,1/p'}(R^n)\). As for the last equality, see the remark just after Definition 2.2.

(b) Assume that \(n > 1\). If \(p > q\), then from the argument in (1) \(C_\gamma \geq \min(C_0^p, 1)S^{p,q,1/p'}(R^n)\) holds. If \(p = q\), then the inequalities \(4.5\) and \(4.7\) are reduced to the Hardy-type inequalities, hence we have \(C_\gamma = S^{p,r,1/p'} = (1/p')^p\). (See Proposition 7.3 (1.).

(c) Assume that \(C_0 \geq 1\) and \(1/p' \leq \gamma_{p,q} = (n-1)/(1+q/p')\). By Proposition 7.3 together with Proposition 7.3 \(S^{p,q,1/p'} = S^{p,q,1/p'} = S^{p,r,1/p'} = S^{p,r,1/p'}\) holds. Then one can assume \(U \in C_c^\infty(B_{\hat{\eta}} \setminus \{0\})\) so that we have \(\Lambda U \equiv 0\). Therefore the assertion is now clear.

**Remark 4.1.** We note that if \(1 < p \leq (n+1)/2\), then \(1/p' \leq \gamma_{p,q}\) for any \(q \in (p, np/(n-p)]\).
5 Proof of Theorem 3.2

Proof. Step 1. We assume that \( H(\rho) \equiv 0 \) in \((0, \tilde{\eta})\) with \( \varphi(\tilde{\eta}) = \eta \). First we assume that \( w \in P(\mathbb{R}+) \). Then it follows from \((5.5)\) that the desired inequality \((5.12)\) has the form

\[
\int_{S^{n-1}} dS \int_{0}^{\tilde{\eta}} |\partial_{\rho} U|^p \rho^{2(p-1)} d\rho \geq C \left( \int_{S^{n-1}} dS \int_{0}^{\tilde{\eta}} |U|^q \rho^{-1+q/p'} d\rho \right)^{p/q}. \tag{5.1}
\]

By \( M(S^{n-1}) \) we denote a set of all measurable functions on \( S^{n-1} \). For \( A(\rho) \in C^{+}((0, \tilde{\eta})) \setminus \{0\} \) and \( B(\omega) \in M(S^{n-1}) \), we define a test function \( U(\rho) = A(\rho) \cdot B(\omega) \) such that

\[
0 \leq B(\omega) \in L^p(S^{n-1}), \quad \text{but} \quad B(\omega) \notin L^q(S^{n-1}).
\]

Then, in the inequality \((5.1)\), we see that

\[
\begin{cases}
\text{(LHS)} = \int_{S^{n-1}} |B(\omega)|^p dS \int_{0}^{\tilde{\eta}} |\partial_{\rho} A(\rho)|^p \rho^{2(p-1)} d\rho < \infty, \\
\text{(RHS)} = \left( \int_{S^{n-1}} |B(\omega)|^q dS \int_{0}^{\tilde{\eta}} \left| A(\rho) \right|^q \rho^{q/p'-1} d\rho \right)^{p/q} = \infty.
\end{cases} \tag{5.2}
\]

Here (LHS) and (RHS) are abbreviations of the left-hand side and the right-hand side respectively. Therefore the inequality \((5.1)\) does not hold, and this proves the assertion provided that \( H \equiv 0 \). Secondly we assume that \( w \in Q(\mathbb{R}+) \). Then it follows from \((5.7)\) that the desired inequality \((5.12)\) has the form

\[
\int_{S^{n-1}} dS \int_{0}^{\tilde{\eta}} |\partial_{\rho} U|^p \rho^{2(p-1)} d\rho \geq C \left( \int_{S^{n-1}} dS \int_{0}^{\tilde{\eta}} |U|^q \rho^{-1+q/p'} d\rho \right)^{p/q}. \tag{5.3}
\]

Again using the same test function \( U = A(\rho)B(\omega) \), we see that

\[
\begin{cases}
\text{(LHS)} = \int_{S^{n-1}} |B(\omega)|^p dS \int_{0}^{\tilde{\eta}} |\partial_{\rho} A(\rho)|^p d\rho < \infty, \\
\text{(RHS)} = \left( \int_{S^{n-1}} |B(\omega)|^q dS \int_{0}^{\tilde{\eta}} \left| A(\rho) \right|^q \rho^{-q/p'-1} d\rho \right)^{p/q} = \infty.
\end{cases} \tag{5.4}
\]

Therefore the inequality \((5.3)\) does not hold, and we have the desired assertion.

Remark 5.1. From Theorem 3.7 Lemma 2.7 and \((4.9)\) we have the followings:

1. For \( w(\rho) = \rho^2 \), we have

\[
\int_{0}^{\tilde{\eta}} |\partial_{\rho} A(\rho)|^p \rho^{2(p-1)} d\rho \geq C \left( \int_{0}^{\tilde{\eta}} |A(\rho)|^q \rho^{-1+q/p'} d\rho \right)^{p/q}, \tag{5.5}
\]

2. For \( w(\rho) = 1 \) we have

\[
\int_{0}^{\tilde{\eta}} |\partial_{\rho} A(\rho)|^p d\rho \geq C \left( \int_{0}^{\tilde{\eta}} |A(\rho)|^q \rho^{-q/p'-1} d\rho \right)^{p/q}, \tag{5.6}
\]

where \( C \) is a positive number independent of each function \( A(\rho) \in C^{+}((0, \tilde{\eta})) \).

Step 2. We assume that \( \lim_{\rho \to 0} H(\rho) = 0 \) and \( w \in P(\mathbb{R}^+) \). As in the previous step, we take \( B(\omega) \in L^p(S^{n-1}) \) with \( B(\omega) \notin L^q(S^{n-1}) \). Let \( B_j(\omega) \) be a mollification of \( B \) such that \( B_j(\omega) \in C^0(S^{n-1}) \), \( B_j \to B \) in \( L^p(S^{n-1}) \) but

\[
\int_{S^{n-1}} |B_j(\omega)|^q dS \to \infty \quad (j \to \infty). \tag{5.7}
\]
Let \( \{ \varepsilon_j \} \) be a sequence of numbers such that \( 0 < \varepsilon_j < 1, \varepsilon_j \to 0 \) as \( j \to \infty \) and
\[
H(\rho)^p \cdot \int_{S^{n-1}} |\Lambda B_j(\omega)|^p \, dS \leq 1 \quad (0 < \rho \leq \varepsilon_j \eta, j = 1, 2, 3, \ldots).
\] (5.8)

We take and fix an \( A(\rho) \in C_c^\infty(\mathbb{R}_+, 0, \eta) \cap \{ 0 \} \) satisfying
\[
\int_0^{\eta} |\partial_\rho A(\rho)|^p \rho^{2(\rho-1)} \, d\rho = 1.
\] (5.9)

Define
\[
A_j(\rho) = \varepsilon_j^{-1/p'} A(\rho/\varepsilon_j) \quad (j = 1, 2, 3, \ldots).
\] (5.10)

Then, for \( j = 1, 2, 3, \ldots \) we see that \( A_j(\rho) \in C_c^\infty(0, \varepsilon_j \eta) \subset C_c^\infty(0, \eta) \) and
\[
\int_0^{\eta} |\partial_\rho A_j(\rho)|^p \rho^{2(\rho-1)} \, d\rho = \int_0^{\eta} |\partial_\rho A(\rho)|^p \rho^{2(\rho-1)} \, d\rho = 1,
\] (5.11)
\[
\int_0^{\eta} |A_j(\rho)|^q \rho^{-1+q/p'} \, d\rho = \int_0^{\eta} |A(\rho)|^q \rho^{-1+q/p'} \, d\rho.
\] (5.12)

Then we define a sequence of test functions \( U_j = A_j(\rho) \cdot B_j(\omega) \in C_c^\infty((0, \varepsilon_j \eta)) \times C_c^\infty(S^{n-1}). \) If we show the following properties, then the assertion clearly follows:

1. \[
\left( \int_{S^{n-1}} dS \int_0^{\eta} \left( (\partial_\rho U_j)^2 + H(\rho)^2 \frac{(A_j)^2}{\rho^2} \right)^{p/2} \rho^{2(\rho-1)} \, d\rho \right)^{p/2} < \infty,
\] (5.13)

2. \[
\left( \int_{S^{n-1}} dS \int_0^{\eta} |U_j|^q \rho^{-1+q/p'} \, d\rho \right)^{p/q} \to \infty \quad \text{as} \quad j \to \infty.
\] (5.14)

It follows from (5.11) and (5.12) we have (5.14), and hence it suffices to show (5.13). By the definition we immediately see that
\[
\int_0^{\eta} |\partial_\rho A_j(\rho)|^p \rho^{2(\rho-1)} \, d\rho \int_{S^{n-1}} |B_j(\omega)|^p \, dS = \int_{S^{n-1}} |B_j(\omega)|^p \, dS < \infty.
\] (5.15)

By Hardy’s inequality, for some positive number \( C \) we have
\[
\int_0^{\eta} |\partial_\rho A_j(\rho)|^p \rho^{2(\rho-1)} \, d\rho \geq C \int_0^{\eta} |A_j(\rho)|^p \rho^{p-2} \, d\rho, \quad (j = 1, 2, 3, \ldots).
\] (5.16)

Then we have
\[
\int_0^{\eta} |A_j(\rho)|^p \frac{H(\rho)^p}{\rho^p} \rho^{2(\rho-1)} \, d\rho \int_{S^{n-1}} |\Lambda B_j(\omega)|^p \, dS
\]
\[
= \int_0^{\eta} |A_j(\rho)|^p \rho^{p-2} H(\rho)^p \, d\rho \int_{S^{n-1}} |\Lambda B_j(\omega)|^p \, dS
\]
\[
\leq C^{-1} \int_0^{\eta} |\partial_\rho A_j(\rho)|^p \rho^{2(\rho-1)} \, d\rho < \infty \quad (5.17, 5.16)
\]

Since \((a^2 + b^2)^{p/2} \leq 2^{p/2}(a^p + b^p), (a, b \geq 0),\) we have (5.9), hence the assertion is proved.

Secondly we assume that \( w \in \mathcal{O}(\mathbb{R}_+). \) Let \( B_j(\omega) \in C_c^\infty(S^{n-1}) (j = 1, 2, 3, \ldots) \) be the same function as before. We take an \( A(\rho) \in C_c^\infty((0, \eta)) \cap \{ 0 \} \) satisfying
\[
\int_0^{\eta} |\partial_\rho A(\rho)|^p \, d\rho = 1.
\] (5.18)
Define
\[ A_j(\rho) = \varepsilon_j^{1/p'} A(\rho/\varepsilon_j) \quad (j = 1, 2, 3, \ldots). \quad (5.19) \]

Then, for \( j = 1, 2, 3, \ldots \) we see that \( A_j(\rho) \in C_c^\infty(0, \varepsilon_j \tilde{\eta}) \subset C_c^\infty((0, \tilde{\eta})) \) and
\[
\int_0^{\tilde{\eta}\varepsilon_j} |\partial_\rho A_j(\rho)|^p d\rho = \int_0^{\tilde{\eta}} |\partial_\rho A(\rho)|^p d\rho = 1, \quad (5.20)
\]
\[
\int_0^{\tilde{\eta}\varepsilon_j} |A_j(\rho)|^q \rho^{-1-\alpha/q'} d\rho = \int_0^{\tilde{\eta}} |A(\rho)|^q \rho^{-1-\alpha/q'} d\rho. \quad (5.21)
\]

Now we define a sequence of test functions \( U_j = A_j(\rho) \cdot B_j(\omega) \in C_c^\infty((0, \varepsilon_j \tilde{\eta})) \times C_c^\infty(S^{n-1}) \). If we can show the following properties, then the assertion follows in a similar way:

1. \[
\int_{S^{n-1}} dS \int_0^{\tilde{\eta}\varepsilon_j} \left( (\partial_\rho U_j)^2 + H(\rho)^2 \frac{(\Lambda U_j)^2}{\rho^2} \right)^{p/2} d\rho < \infty, \quad (5.22)
\]
2. \[
\left( \int_{S^{n-1}} dS \int_0^{\tilde{\eta}\varepsilon_j} |U_j|^q \rho^{-1-\alpha/q'} d\rho \right)^{p/q} \to \infty \text{ as } j \to \infty. \quad (5.23)
\]

Since (5.22) follows directly from (5.7) and (5.21), it suffices to show (5.22). By the definition we immediately see that
\[
\int_0^{\tilde{\eta}\varepsilon_j} |\partial_\rho A_j(\rho)|^p d\rho \int_{S^{n-1}} |B_j(\omega)|^p dS = \int_{S^{n-1}} |B_j(\omega)|^p dS < \infty. \quad (5.24)
\]

By Hardy’s inequality, for some positive number \( C \) we have
\[
\int_0^{\tilde{\eta}\varepsilon_j} |\partial_\rho A_j(\rho)|^p d\rho \geq C \int_0^{\tilde{\eta}\varepsilon_j} |A_j(\rho)|^p \rho^{-p} d\rho, \quad (j = 1, 2, 3, \ldots). \quad (5.25)
\]

Then we have
\[
\int_0^{\tilde{\eta}\varepsilon_j} |A_j(\rho)|^p \frac{H(\rho)^p}{\rho^p} d\rho \int_{S^{n-1}} |AB_j(\omega)|^p dS \leq C^{-1} \int_0^{\tilde{\eta}\varepsilon_j} |\partial_\rho A_j(\rho)|^p d\rho < \infty \quad (5.8, 5.25) \quad (5.26)
\]

Then we have (5.22) as before, hence the assertion is proved. \( \square \)

6 Proof of Theorem 3.3

Proof. First we treat the case that \( w \in P(\mathbb{R}_+) \) and \( w \) vanishes in infinite order at the origin. Namely we assume that for an arbitrary positive number \( m \) there exists a positive \( t_m \) such that \( t_m \to 0 \) as \( m \to \infty \) and
\[
w(t) \leq t^m, \quad t \in (0, t_m). \quad (6.1)
\]

Since \( \varphi(\rho) (\varphi(0) = 0) \) is increasing, for \( \eta > 0 \) we set \( \tilde{\eta} = \varphi^{-1}(\eta) \). Now we assume on the contrary that for some positive number \( C_0 \),
\[
H(\rho) \geq C_0, \quad 0 < \rho \leq \tilde{\eta}.
\]

Then \( C_0/\rho \leq \varphi'(\rho)/\varphi(\rho) \) holds for \( \rho \in (0, \tilde{\eta}] \), hence by integrating both sides over an interval \( [\rho, \tilde{\eta}] \) we have
\[
\varphi(\rho) \leq \frac{\varphi(\tilde{\eta})}{\tilde{\eta} C_0} \rho, \quad \rho \in (0, \tilde{\eta}). \quad (6.2)
\]
Since \( H(\rho) = \rho \varphi'(\rho)/\varphi(\rho) = w(\varphi(\rho))/\varphi(\rho) \) holds, by (6.1) we have \( C_0 \leq \varphi(\rho)^{m-1}/\rho \) for sufficiently small \( \rho > 0 \), more precisely
\[
(C_0 \rho)^{1/(m-1)} \leq \varphi(\rho), \quad \rho \in (0, \varphi^{-1}(t_m)).
\] (6.3)

Then we have
\[
1 \leq \frac{\varphi(\bar{\eta})}{C_{0}(1/(m-1)) \bar{\eta} C_{0}^{-1/(m-1)}} \rho^{m-1} \quad \rho \in (0, \varphi^{-1}(t_m)).
\]

If \( m \) is sufficiently large, then this does not hold, hence the assertion is proved by a contradiction.

In the next we treat the case that \( w \in \mathcal{Q}(\mathbb{R}_+) \) and \( w \) blows up at the origin in infinite order. Then we assume for an arbitrary positive number \( m \) there exists a positive \( t_m \) such that \( t_m \to 0 \) as \( m \to \infty \) and
\[
w(t) \geq t^{-m}, \quad t \in (0, t_m).
\] (6.4)

As in the previous step we assume \( \tau_{p, q} \) as well. By the definition of \( \varphi(\rho) \) and (6.4), we have
\[
\varphi'(\rho) = w(\varphi(\rho)) \geq \varphi(\rho)^{-m}, \quad \rho \in (0, \varphi^{-1}(t_m)).
\]

By integrating this over an interval \((0, \rho)\) we have
\[
\varphi(\rho) \geq (m+1)^{1/(m+1)} \rho^{1/(m+1)}, \quad \rho \in (0, \varphi^{-1}(t_m)).
\] (6.5)

Combining this with (6.2) we have
\[
(m+1)^{1/(m+1)} \leq \frac{\varphi(\bar{\eta})}{\bar{\eta} C_{0}^{-1/(m+1)}} \rho^{1/(m+1)} \quad \rho \in (0, \varphi^{-1}(t_m)).
\]

But this does not hold if \( m \) is sufficiently large, and hence the assertion is proved. \( \square \)

7 Appendix; Some relations among the best constants

In this section we review fundamental properties of the best constants \( S^{p,q}_{\text{rad}} \), \( S^{p,q}_{\text{rad}} \), \( C_{p,q;R} \) and \( C_{p,q;R} \). Most of the contents are borrowed from [10] (See Section 2.2 and Section 2.3). Let us introduce some notations.

**Definition 7.1.** For \( 1 < p \leq q < \infty \), we set
\[
\gamma_{p,q} = \frac{n-1}{1+q/p}, \quad S_{p,q} = \left\{ \right.
\begin{array}{ll}
(p')^{p-2+p/q} q^{p/q} & \left(\frac{\omega_n}{\tau_{p,q}} B\left(\frac{1}{p \tau_{p,q}}, \frac{1}{p' \tau_{p,q}}\right)\right)^{1-p/q} \quad \text{if } p < q, \quad (7.1)
\end{array}
\]

Here \( \tau_{p,q} = 1/p - 1/q \) and \( B(\cdot, \cdot) \) is the beta function.

**Proposition 7.1.** (Non-critical CKN-type inequalities)
Assume that \( 1 < p \leq q < \infty \), \( \tau_{p,q} \leq 1/n \) and \( \gamma \neq 0 \). Then we have the following:

1. \( S^{p,q}_{\text{rad}} \geq S^{p,q}_{\text{rad}} > 0 \).
2. \( S^{p,q}_{\text{rad}} = S^{p,q;\gamma}_{\text{rad}} \) and \( S^{p,q;\gamma}_{\text{rad}} = S^{p,q;\gamma}_{\text{rad}} \).
3. \( S^{p,q}_{\text{rad}} = S_{p,q;\gamma | Y |^{p(1-\gamma_{p,q})}} \).
4. \( S^{p,q}_{\text{rad}} = S_{p,q;\gamma | Y |^{p(1-\gamma_{p,q})}} \) for \( 0 < |\gamma| \leq \gamma_{p,q} \).
Definition 7.2. For $1 < p \leq q < \infty$ we set

$$R_{p,q} = \exp \frac{1 + q/p'}{(n-1)p'} \text{ if } n \geq 2, \quad C_{p,q} = S_{p,q}(p')^{p(\tau_{p,q}-1)}. \tag{7.2}$$

Proposition 7.2. (Critical CKN-type inequalities) Assume that $1 < p \leq q < \infty$, $\tau_{p,q} \leq 1/n$ and $R > 1$. Then, we have:

1. $C_{\text{rad}}^{p,q,R} \geq C_{p,q} > 0$
2. $C_{\text{rad}}^{p,q,R} = C_{p,q}$ for $R \geq 1$.
3. $C_{\text{rad}}^{p,q,R} = C_{\text{rad}}^{p_{\text{rad}},q_{\text{rad}}} = C_{p,q}$ for $R \geq R_{p,q}$ if $p \geq n \geq 2$.

From Proposition 7.1 and Proposition 7.2, noting that $S_{p,q}(p')^{p(\tau_{p,q}-1)} = C_{p,q}$ by Definition 7.2, we have an interesting relation among the best constants:

Proposition 7.3. 1. Assume that $n \geq 1$, $1 < p \leq q < \infty$, $\tau_{p,q} \leq 1/n$ and $R \geq 1$. Then

$$S_{\text{rad}}^{p,q,1/p'} = S_{p,q}^{1/p'(1-\tau_{p,q})} = C_{p,q} = C_{\text{rad}}^{p,q,R}. \tag{7.3}$$

2. If $n \geq 2$, $1 < p \leq q < \infty$, $\tau_{p,q} \leq 1/n$, $1/p' \leq \gamma_{p,q}$ and $R \geq R_{p,q}$, then we have the relation

$$S_{\text{rad}}^{p,q,1/p'} = S_{\text{rad}}^{p,q,1/p'} = C_{\text{rad}}^{p,q,R} = C_{p,q,R}. \tag{7.4}$$

References

[1] H. Ando, T. Horiuchi, Weighted Hardy’s inequalities and the variational problem with compact perturbations, Mathematical Journal of Ibaraki University, 52 (2020), pp. 15-26.

[2] H. Ando, T. Horiuchi, Generalized weighted Hardy’s inequalities with compact perturbations. (preprint)

[3] L. Caffarelli, R. Kohn, L. Nirenberg, First order interpolation inequalities with weights, Compositio Math., Vol. 53 (1984), No. 3, pp259-275.

[4] T. Horiuchi, The imbedding theorems for weighted Sobolev spaces, Journal of Mathematics of Kyoto University, Vol. 29 (1989), pp365-403.

[5] T. Horiuchi, Best constant in weighted Sobolev inequality with weights being powers of distance from the origin, Journal of Inequality and Application, Vol. 1 (1997), pp275-292.

[6] T. Horiuchi, Hardy’s inequalities with non-doubling weights and sharp remainders, arXiv:2012.08766 [math.AP], Sci. Math. Jpn. (in Edition Electronica), e-2022-2.

[7] T. Horiuchi, On general Caffarelli-Kohn-Nirenberg type inequalities involving non-doubling weights ($p = 1$). (preprint)

[8] N. Chiba, T. Horiuchi, On radial symmetry and its breaking in the Caffarelli-Kohn-Nirenberg type inequalities for $p = 1$, Math. J. Ibaraki Univ., Vol. 47 (2015), pp49–63.

[9] N. Chiba, T. Horiuchi, Radial symmetry and its breaking in the Caffarelli-Kohn-Nirenberg type inequalities for $p = 1$, Proc. Japan Acad., Ser. A, Math. Sci., Vol. 92 (2016), No. 4, pp51-55.
[10] T. Horiuchi, P. Kumlin, On the Caffarelli-Kohn-Nirenberg type inequalities involving Critical and Supercritical Weights, *Kyoto journal of Mathematics*, **Vol. 52**, No.4 (2012), pp661-742.

[11] N. Ioku, Attainability of the best Sobolev constant in a ball, Math. Ann. 375 (2019), no. 1-2, pp1-16.

[12] X. Liu, T. Horiuchi, H. Ando, One dimensional weighted Hardy’s inequalities and application. *Journal Mathematical Inequalities*, **Vol. 14** (2020), No. 4, pp1203-1222.

[13] V.G. Maz’ja, Sobolev spaces, *Springer*, 1985.

[14] Sano, Megumi and Takahashi, Futoshi, Scale invariance structures of the critical and the sub-critical Hardy inequalities and their improvements, Calc. Var. Partial Differential Equations 56 (2017), no. 3, Art. 69, 14 pp.

[15] Sano, Megumi, Extremal functions of generalized critical Hardy inequalities, J. Differential Equations 267 (2019), no. 4, pp2594-2615.

[16] Sano Megumi, Minimization problem associated with an improved Hardy Sobolev type inequality, Nonlinear Analysis, 200, (2020).

[17] G. Talenti, Best constant in Sobolev inequality, *Ann. Mat. Pura Appl.*, **Vol 110**, 1976, pp 353-372.

[18] G. Talenti, Nonlinear elliptic equations, rearrangements of functions and Orlicz spaces, *Ann. Mat. Pura Appl.*, **Vol 120**(4) (1979), pp160-184.

[19] S.I. Pohozaev, Eigenfunctions of the equation, \(\Delta u + f(u) = 0\), *Soviet Math. Doklady*, 6 (1965), pp1408–1411.

[20] N. B. Zographopoulos, Existence of extremal functions for a Hardy-Sobolev inequality, J. Funct. Anal. 259 (2010), no. 1, 308-314.

**Toshio Horiuchi**
**Department of Mathematics**
**Faculty of Science**
**Ibaraki University**
**Mito, Ibaraki, 310, Japan**
e-mail: toshio.horiuchi.math@vc.ibaraki.ac.jp