Quasi-parton distribution functions: 
two-dimensional scalar and spinor QCD  

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We construct the quasi-parton distributions of mesons for two-dimensional QCD with either scalar or spinor quarks using the \(1/N_c\) expansion. We show that in the infinite momentum limit, the parton distribution function is recovered in both leading and sub-leading order in \(1/N_c\).

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I. INTRODUCTION

Light cone distribution amplitudes are central to the description of hard exclusive processes with large momentum transfer. They account for the non-perturbative quark and gluon content of a hadron in the infinite momentum frame. Using factorization, hard cross sections can be split into soft partonic distributions convoluted with perturbatively calculable processes. The partonic distributions are inherently non-perturbative. They are currently estimated using experiments, lattice simulations or models.

Recently one of us [1] has suggested that the light cone hadronic wavefunctions can be recovered from Euclidean correlators in hadronic states using instead quasi-parton distribution functions through pertinent renormalization in the infinite momentum limit. Preliminary lattice simulations have proven very promising [2,3]. The purpose of this letter is to explore this construct in two-dimensional scalar and spinor QCD in the non-perturbative \(1/N_c\) expansion.

Two-dimensional scalar QCD has a smooth large \(N_c\) limit with a confining spectrum [4-6]. In this model the current correlators exhibits many features of four-dimensional QCD in contrast to two-dimensional spinor QCD [7]. In the deep inelastic regime the results exhibit expected scaling laws, and are overall in support of the Feynman partonic picture and the light cone expansion. In this paper, these two models will be used interchangeably to test the concept of the quasi-distributions in a non-perturbative context, as they differ by a minor change in the algebra of the pertinent bosonic operators. Specifically, we construct the quasi-parton distributions for both scalar and spinor QCD in leading and sub-leading order in \(1/N_c\) and show that they merge with the expected light cone distributions in the infinite momentum limit without additional renormalization. Our leading conclusion for two-dimensional spinor QCD is in agreement with a recent study [8].

The organization of the paper is as follows: in section II we discuss a canonical quantization of two-dimensional scalar QCD in the axial gauge. We make explicit the Hamiltonian of the model in leading order in \(1/N_c\) using bosonized fields. Some renormalization issues are also discussed. In section III we explicit the wavefunction for scalar QCD in the light cone limit. In section IV we construct the quasi-parton distribution function in leading order in \(1/N_c\), and show that it reduces to the light cone wavefunction in the infinite momentum limit. We also discuss the leading correction in \(1/P\). In section V, we show how to generalize the bosonization scheme algebraically for both scalar and spinor QCD, and use it for a systematic organization of the operators in \(1/N_c\). This scheme is used in section VI, VII to correct the light cone parton distribution and quasi-distribution in spinor two-dimensional QCD through standard perturbation theory. We show that the subleading corrections to the quasi-parton distribution function merges with the parton distribution function in the infinite momentum limit without renormalization. Our conclusions are in section VIII. In the Appendix we summarized some elements of two-dimensional spinor QCD pertinent for our canonical analysis both in light-cone and axial gauge.

II. QUANTIZATION OF SCALAR QCD IN AXIAL GAUGE

We first discuss the general structure of the Hamiltonian in two dimensions for scalar SU(\(N\)) QCD in the axial gauge \(A_\perp = 0\). The same discussion for two-dimensional spinor QCD in both the light-cone and axial gauge is summarized briefly in the Appendix. The starting Lagrangian is

\[
\mathcal{L} = \frac{1}{2} tr F_{\mu \nu}^2 + (D_\mu \phi)^\dagger D^\mu \phi - m^2 \phi^\dagger \phi
\]  

In terms of the canonical momenta \(\pi^\dagger = \Pi_\phi = (D_0 \phi)^\dagger\) and \(\pi = \Pi_\phi = D_0 \phi\), the corresponding Hamiltonian reads
\[ H = \int dx \left( \pi^1 \pi + |\partial_1 \phi|^2 + m^2 |\phi|^2 + ig \text{Tr} A_0 (\pi \phi^1 - \phi \pi^1) - \frac{1}{2} \text{Tr} (\partial_1 A_0)^2 \right) \] (2)

The equation of motion for \( A_0 \) is a constraint equation that can be solved in terms of \( \phi, \pi \), to yield the canonical Hamiltonian

\[ H = H_0 + H_{\text{int}} \]
\[ H_0 = \int dx (\pi^1 \pi + |\partial_1 \phi|^2 + m^2 |\phi|^2) \]
\[ H_{\text{int}} = \frac{g^2}{2} \int dx \left( J^a - \frac{1}{2} \partial_1^a J^a \right) \]
\[ J^a = i(\phi^1 T^a \pi - \pi^1 T^a \phi) \] (3)

To proceed, we will use a free-like representation for the field and its conjugate

\[ \phi_a = \int \frac{dk}{\sqrt{4\pi E_k}} e^{-ikx} (a_k + b_k^\dagger) \]
\[ (\pi^1)_a = i \int \frac{dk}{\sqrt{4\pi E_k}} e^{ikx} E_k (a_k^\dagger - b_{-k}) \] (4)

However, instead of the free dispersion law \( E_k = \sqrt{k^2 + m^2} \), we will use an arbitrary \( E(k) \) that will be fixed self-consistently below in the planar approximation, with \( E_k \to |k| \) asymptotically.

### A. Hamiltonian to order \( 1/\sqrt{N_c} \)

The Hamiltonian (3) is quartic in \( a_k, a_k^\dagger \). We now choose to bosonize it, by re-writing it in terms of the quadratic operators

\[ M(k_1, k_2) = \frac{1}{\sqrt{N}} \sum_\alpha a_\alpha(k_1) b_\alpha(k_2) \]
\[ N(k_1, k_2) = \sum_\alpha a_\alpha^\dagger(k_1) a_\alpha(k_2) \]
\[ \tilde{N}(k_1, k_2) = \sum_\alpha b_\alpha^\dagger(k_1) b_\alpha(k_2) \] (5)

In leading order of \( 1/\sqrt{N_c} \),

\[ N(k, p) = \int dq M^\dagger(q, k) M(p, q) \]
\[ \tilde{N}(k, p) = \int dq M^\dagger(q, k) M(p, q) \] (6)

Using (5) and the identity \( \sum_\alpha (T^a)_{ij} (T^a)_{kl} = \delta_{ij} \delta_{kl} - \frac{1}{N} \delta_{ij} \delta_{kl} \), the Hamiltonian (3) now reads to order \( 1/\sqrt{N_c} \) as

\[ H = H_2 + H_4 \]
\[ H_2 = \int dk (N(k) + \tilde{N}(k)) \Pi^+(k) \]
\[ + \sqrt{N_c} \int dk (M(k) + M^+(k)) \Pi^-(k) \]
\[ H_4 = \lambda \int dk_1 dk_2 dk_3 dk_4 \delta(k_1 + k_2 + k_3 + k_4) \]
\[ \times \left(-2 f_+(k_1, k_2) f_+(k_3, k_4) M^1(k_1, k_4) M(-k_2, -k_3) \right. \]
\[ + f_-(k_1, k_2) f_-(k_3, k_4) M^1(k_1, k_4) M(k_3, k_2) \]
\[ + f_-(k_1, k_2) f_-(k_3, k_4) M^1(k_1, k_4) M(k_3, k_2) \]
\[ \left. + \mathcal{O} \left( \frac{1}{\sqrt{N_c}} \right) \right) \] (7)

Here \( \lambda = g^2 N_c \) is the standard 't Hooft coupling. We have made use of the notation \( M(k) = M(k, -k) \), \( N(k) = N(k, k) \), and

\[ f_{\pm}(k_1, k_2) = \sqrt{\frac{E_2}{E_1}} \pm \sqrt{\frac{E_1}{E_2}} \]
\[ \Pi^\pm = \frac{1}{2} \left( k^2 + m^2 \pm E_k \right) + \lambda \int \frac{dk_1 E_1}{E_k} \frac{\pm E_1}{(k + k_1)^2} \] (8)

For a consistent expansion in \( 1/N_c \), we can eliminate the \( \sqrt{N_c} \) term in (7) by setting \( \Pi^-(k) = 0 \). The result is a gap equation for \( E(k) \)

\[ \frac{k^2 + m^2}{E_k} - E_k + \lambda \frac{1}{4\pi} \int dk_1 \left( \frac{E_{k_1}}{E_k} - \frac{E_k}{E_{k_1}} \right) \frac{1}{|k_k^1|^2} = 0 \] (9)

The leading order Hamiltonian simplifies to

\[ H = \int dp dq M^\dagger(p, q) M(p, q) (\Pi^+(p) + \Pi^+(q)) \]
\[ + \lambda \int \frac{dk_1 dk_2 dk_3 dk_4 \delta(k_1 + k_2 + k_3 + k_4)}{16\pi} \left(-2 f_+(k_1, k_2) f_+(k_3, k_4) M^1(k_1, k_4) M(-k_2, -k_3) \right. \]
\[ + f_-(k_1, k_2) f_-(k_3, k_4) M^1(k_1, k_4) M(k_3, k_2) \]
\[ + f_-(k_1, k_2) f_-(k_3, k_4) M^1(k_1, k_4) M(k_3, k_2) \] (10)

### B. Renormalization

The integral in the gap equation (10) and subsequently the Hamiltonian contains a divergence and requires regul-
larization. For that we regularize \( \frac{1}{(x - y)^2} \) using the standard principal value (PV) prescription

\[
\int dx \frac{f(x)}{(x - y)^2} \rightarrow \text{PV} \int dx \frac{f(x) - f(y)}{(x - y)^2} + \frac{2f(y)}{\epsilon}
\]  

(11)

It is readily seen that \( \Pi^- \) is finite but \( \Pi^+ \) diverges as

\[
\Pi^+ = \Pi^+_1 + \frac{\lambda}{2\pi\epsilon} \tag{12}
\]

with \( \Pi_1 \) finite. We have checked that for physical states (on mass shell) the \( \epsilon \)-contributions cancel out (see below).

The solution to the gap equation (9) that asymptotes \( E_k \rightarrow |k| \) still suffers from a logarithmic divergence even after the PV prescription, namely

\[
\frac{\lambda}{8\pi E_k} \int dk_1 \frac{E_{k_1}}{k_1^2} \int dk_2 \frac{E_{k_2}}{k_2^2}
\]

(13)

This is actually related to the mass divergence for the scalar one-loop self energy, and renormalizes the scalar mass

\[
m^2_r = m^2 + \frac{\lambda}{4\pi} \int dk_1 \frac{E_{k_1}}{k_1^2} \tag{14}
\]

From here on, we will refer to \( \Pi^+ \) as the renormalized momentum operator, and \( m \) as the renormalized mass, and omit the r-label for convenience. With this in mind, the renormalized gap equation (10) now reads

\[
\frac{k^2 + m^2}{E_k} - E_k + \frac{\lambda}{4\pi} \int dk_1 \left( \frac{E_{k_1}}{E_k} \left( \frac{E_{k_1}}{k_1 + k_1} \right) \text{PV} \frac{E_{k_1}}{E_k} \frac{1}{k_1^2} \right) = 0
\]

(15)

\[\text{III. WAVE-FUNCTION AND LIGHT CONE LIMIT}\]

To construct the light cone wave-function of the scalar quarks, we define

\[
S_{\pm}(p, k, P) = f_{\pm}(p - P, k - P)f_{\pm}(p, k)
\]

\[A = 2S_+(p, k, P)M_1(p - P, k)M_2(k - P, k)
\]

\[B = S_-(p, k, P)(M_1(p, p - P)M_2(k - P, k) + c.c)
\]

and use them to re-write (10) in the form

\[
H = \int dp dq M_1(p, q)M_2(p, q)(\Pi^+(p) + \Pi^+(q))
\]

\[
- \frac{\lambda}{16\pi} \int dP \int dk dp \frac{A + B}{(p - k)^2}
\]

(17)

The bi-local operator \( M(p, q) \) can be decomposed in modes

\[
M(p - P, p) = \frac{1}{\sqrt{|P|}} \sum_n \left( m_n(P)\phi_n^+(q, P) - m_n^0(-P)\phi_n^-(q - P, -P) \right)
\]

(18)

where the first contribution refers to the light cone wavefunction describing a pair of scalar quarks moving forward in the light front, while the second contribution refers to a pair moving backward in the front form. The pair is characterized by a relative momentum \( p \) and a center of mass momentum \( P \). Here \( m_n, m_n^0 \) are canonical bosonic annihilation and creation operators. The equation of motion follows by commutation

\[
(\Pi^+(p) + \Pi^+(P - p) \mp P^0_n)\phi_n^+(p, P) = \frac{\lambda}{8\pi} \int \frac{dk}{(p - k)^2}
\]

\[
\times (S_+(p, k, P)\phi_n^+(k, P) - S_-(p, k, P)\phi_n^-(k, P))
\]

(19)

We can check that the \( \epsilon \)-dependent divergences noted in the momentum operator cancel out. Indeed, using (12) the LHS in (19) produces \( \frac{1}{8\pi} \phi\dagger \), while the RHS in (19) produces \( \frac{1}{8\pi} S^+(k, k)\phi\dagger = \frac{1}{8\pi} \phi\dagger \), both of which cancel out. This checks the consistency of the renormalization procedure for scalar QCD. No such renormalization is needed for spinor QCD.

In the large momentum limit \( P \) the equation simplifies. For that we set \( p = xP, k = yP \), and \( P \rightarrow \infty \) on both sides of (19). In this limit the backward wavefunction vanishes \( \phi_- \rightarrow 0 \). Since

\[
\Pi^+(P) + \Pi^+((1 - x)P) - \sqrt{P^2 + M^2} = \frac{1}{2P} \left( \frac{m^2}{x} + \frac{m^2}{1 - x} - M^2 \right) + O \left( \frac{1}{P^2} \right)
\]

(20)

and

\[
S_+(xP, yP, P) = \frac{(2 - x - y)(x + y)}{\sqrt{x(1 - x)y(1 - y)}}
\]

(21)

the equation of motion (19) involves only the forward wavefunction in the form

\[
\frac{m^2}{x} + \frac{m^2}{1 - x} - M^2 \phi_n(x) = \frac{\lambda}{4\pi} \text{PV} \int \frac{dy}{(x - y)^2} \frac{(2 - x - y)(x + y)}{\sqrt{x(1 - x)y(1 - y)}} \phi_n(y)
\]

(22)

where we have defined \( \phi_n^+(xP, P) = \phi_n(x) \), and PV refers to the principal value of the integral. (22) was obtained initially in [5] using different arguments.
IV. QUASI-PARTON DISTRIBUTION FUNCTION

The light cone distribution for scalar quarks is just $|\phi_n(x)|^2$ in leading order in $1/N_c$. We now show that to the same order, the light cone distribution function and the quasi-distribution function as defined in [1] are in agreement without further normalization. For that, we define the quasi-distribution function

$$q(x, P) = +i \int \frac{dz}{4\pi} e^{izP_{xz}} \langle P| (\partial_z \phi(z))^{\dagger} W[z, 0] \phi(0) |P\rangle$$

$$-i \int \frac{dz}{4\pi} e^{izP_{xz}} \langle P| (\phi(z))^{\dagger} W[z, 0] \partial_z \phi(0) |P\rangle$$

where $|P\rangle$ refers to the meson state. In the axial gauge, the Wilson line $W[z, 0] = 1$. Using the mode decomposition [1] and the relations [6] we obtain for the quasi-distribution

$$q(x, P) = \frac{E_n(P)}{P} \frac{xP}{E(xP)}$$

$$\times \left( |\phi_n^+(xP, P)|^2 + |\phi_n^+(-xP, P)|^2 \right)$$

$$+ |\phi_n^-(xP, P)|^2 + |\phi_n^-(xP, P)|^2$$

$$23$$

For $P \to \infty$, we have $E_n = P$ and $xP = E(xP)$ and all $\phi_-$ vanish. The quasi-parton distribution function reduces identically to the parton distribution function $|\phi_n(x)|^2$.

For finite $P$, (23) shows that the backward moving pair in $\phi^-$ contributes. To assess this quantitatively, we now expand in $P$ the contributions $\phi^\pm$ in (24). For that, we go back to (19) and expand in $\frac{1}{P}$, namely

$$\Pi^+ = |P| + \frac{m^2}{2|P|^2} + \frac{\beta_1}{|P|^3} + O\left(\frac{1}{|P|^4}\right)$$

$$E(P) = |P| + \frac{\beta_2}{|P|^2} + O\left(\frac{1}{|P|^3}\right)$$

The coefficients $\beta_1$ is fixed through a straightforward Taylor expansion of $\Pi^+$, while $\beta_2$ is fixed by the gap equation. Their explicit form is not needed for the general arguments to follow. With this in mind, the leading correction to $\phi^-$ is

$$\phi^- = P^2 \phi_n^-(x) = \frac{\lambda}{24\pi \sqrt{x(1-x)}} \int_0^1 \frac{dy}{\sqrt{y(1-y)}}$$

and the subleading correction for $\phi^+$ is $\phi^+(x) + \frac{1}{2} \phi_1^+(x)$ formally solves

$$\phi \equiv (K_0 - H_0)\phi_1^+ = -K_1 \phi + H_1 \phi - H_0 \phi^-$$

Here we have defined

$$K_0(x) = \frac{m^2}{x} + \frac{m^2}{x} - M_n^2$$

$$H_0(x, y) = \frac{\lambda}{4\pi} \frac{(x + y)(x + y)}{\sqrt{xyxy}} \left(\frac{1}{x} - \frac{1}{y}\right)^2$$

$$K_1(x) = \frac{\beta_1}{x^3} + \frac{\beta_1}{x^3}$$

$$H_0(x, y) = -\frac{\lambda}{4\pi\sqrt{xyxy}}$$

$$H_1(x, y) = \frac{1}{(x-y)^2\sqrt{xxyy}}$$

$$+ \beta_2(x^2 - y^2) \left(\frac{1}{y^2} - \frac{1}{x^2}\right)$$

$$+ \beta_2(x^2 - y^2) \left(\frac{1}{y^2} - \frac{1}{x^2}\right)$$

$$28$$

with $\bar{x} = 1 - x$ and $\bar{y} = 1 - y$. In general, this equation is solved in the same Hilbert space that defines $K_0 - H_0$, if we note that $K_0 - H_0$ is hermitian in the space defined with the measure $\int \phi^\dagger \phi$ where the set of $\phi_n$ forms a complete basis set. The formal solution to (28) is

$$\phi_1^+(x) = \sum_{m \neq n} \phi_m(x) \int_0^1 \frac{dy}{\sqrt{M_n^2 - M_m^2}}$$

$$29$$

The $\frac{1}{P}$ expansion now clearly shows that the rate at which the quasi-distribution (24) approaches the asymptotic light-cone distribution $|\phi_n(x)|^2$ is smooth for all $x \neq 0, 1$. It is singular for $x = 0, 1$ through the contribution of the backward moving pair $\phi^-$ in (24). So the large $P$ limit should be taken before the $x \to 0, 1$ limits at the edges.

V. ALGEBRAIC STRUCTURE

The algebraic framework we have developed allows us to go beyond the leading order in $1/N_c$, and therefore check the proposal in [1] beyond the leading order we have so far established. For that, we note that the bi-local operators [5] obey a closed algebra

$$[M_{12}, M_{34}]$$

$$\delta_{13}\delta_{24} + \frac{s}{N_c} (\delta_{13} N_{42} + \delta_{42} N_{31})$$

$$[M_{12}, N_{34}] = \delta_{13} M_{42}$$

$$[M_{12}, \bar{N}_{34}] = \delta_{23} M_{14}$$

$$[M_{12}, M_{34}] = [N_{12}, \bar{N}_{34}] = 0$$

$$[N_{12}, N_{34}] = \delta_{23} N_{14} - \delta_{14} N_{32}$$

$$30$$
with $N_{12}^4 = N_{21}$. The sign assignment for the bosonization of scalar QCD is $s = +1$ as all underlying operators are bosonic.

A solution to this algebraically closed set can be found by organizing the bi-local operator in $1/N_c$,

$$M = M^0 + \frac{1}{N_c} M^1 + O\left(\frac{1}{N_c^2}\right)$$

$$N = N^0 + \frac{1}{N_c} N^1 + O\left(\frac{1}{N_c^2}\right)$$

(31)

where $M^0$ satisfies the commutation relation

$$[M^0(k_1, k_2), M^0(k_3, k_4)] = \delta(k_1 - k_3)\delta(k_2 - k_4)$$

(32)

in the instant $N_c$ limit. In terms of $M^0$ the solution to (30) can be found by inspection in leading and next to leading order

$$N^0_{12} = \int d3M^0_{13} M^0_{23}$$

$$N^0_{12} = \int d3M^0_{31} M^0_{32}$$

$$M^1_{12} = \mp \frac{1}{2} \int d3d4M^0_{34} M^0_{14} M^0_{32}$$

$$N^1 = 0$$

(33)

It is important to note that the expantion of the $N$’s starts at the second order! From now on to avoid cluttering, we omit the $O$ for the large $N_c$ asymptotic operator.

When the operators in (33) are inserted back into the Hamiltonian, we obtain a complete expression for the first three terms of the $1/N_c$ expanded Hamiltonian in terms of the large $N_c$ asymptotic operators that define the Hilbert space. Specifically, to order $1/N_c^2$ we have

$$H = K_{MM} M^1 M + \frac{1}{N_c} K_{MM} (M^{11} M + M^1 M^1) + \frac{1}{N_c^2} K_{MM} M^{11} M^1$$

$$+ \frac{K_{NM}}{\sqrt{N_c}} N M^1 + \frac{K_{NM}}{\sqrt{N_c}} N^1 M + \frac{K_{NN}}{N_c} N N$$

(34)

Thus, up to order $1/N_c^2$ we encounter six $M$ interactions, but up to order $1/N_c \sqrt{N_c}$ we are still dealing with more tractable quartic and cubic terms. Our algebraic treatment differs notably from the one presented in [9], in that in ours the algebra is corrected which is required for a consistent expansion. The resulting effective hadronic Hamiltonian is different.

VI. CORRECTION TO THE PDF IN SPINOR QCD

In so far our discussion has concentrated on two-dimensional scalar QCD where we have established that the quasi-parton distribution function reduces to the parton distribution function in leading order in $1/N_c$. We have checked that this is also the case for two-dimensional spinor QCD, in agreement with a recent study [9]. In the Appendix we have briefly summarized the key changes from scalar to spinor in the light cone and axial gauge.

Since in the spinor version, the underlying fields are fermionic and not bosonic, the algebraic structure differs from scalar to spinor QCD only in the sign switch $s = +1 \to -1$, with exactly the same bosonized Hamiltonian [34]. Also, to avoid unnecessary long formula we will only discuss the $1/N_c$ corrections to the parton distribution function in two-dimensional spinor instead of scalar QCD. The arguments for both models are similar, but the formula for scalar QCD are laboriously long as we have checked, with exactly the same conclusion.

Using the definitions for spinor QCD in the Appendix, we use for the bi-local mesonic operator $M$ in the light cone gauge the decomposition

$$M(xP, (1-x)P) = \frac{1}{\sqrt{P}} \sum_n m_n(P) \phi_n(x)$$

(35)

which satisfies (30) with $s = -1$. To order $1/N_c$, the Hamiltonian for two-dimensional spinor QCD is the same as in (34), which after inserting (35) yields the first two leading contributions to the interaction of the form

$$\frac{\lambda}{4\pi\sqrt{N_c}} \int dP dP_1$$

$$\times \left( m_1^l(P_1) m_1^l(P - P_1) m_k(P) f_{ijk}(\frac{P_1}{P}) + c.c. \right)$$

$$+ \frac{1}{N_c} m^l m^{lm}$$

(36)

The quartic contribution in (30) is only shown schematically. It is of order $1/N_c$, and apparently relevant for the $1/N_c$ correction to the parton distribution function. However, by simple inspection it gives zero contribution when acting on a free and leading meson contribution to the state, i.e.

$$\left( \frac{1}{N_c} m^l m^{lm} m^l \right) m^l |0\rangle = 0$$

(37)

It will be dropped. Therefore the leading correction to the parton distribution function is given by

$$\sum_{kl} \int \frac{dk dq}{2\pi} \phi_k \left( \frac{xP + q}{xP + q} \right) \phi_l \left( \frac{k}{k + q} \right)$$

$$\times \langle P_1 | \left( m_1^l(xP + q)m_1^l(k + q) \right) \frac{1}{\sqrt{(xP + q)(k + q)}} | P_1 \rangle$$

(38)

Here $| P_1 \rangle$ is the first order perturbation of the meson state $m_1^l(P) |0\rangle$, which by standard perturbation theory reads...
\[ |P\rangle^1 = \frac{\lambda}{2\sqrt{2\pi N_c}} \int dP_1 \times \sum_{kl} \left( \frac{f_{kl}(P)}{m^2_x + m^2_{1-x} - m^2_i} \right) |0\rangle \] (39)

Inserting (39) into (38) and carrying out the contractions as a correction to the leading parton distribution function and axial gauge using the changes in the Appendix. To the description of two-dimensional spinor QCD in the large momentum limit. For that, we switch is still of the form (34). We now note that the contribution limit these terms drop out as we have shown earlier, so they will be ignored. The only surviving terms in the Hamiltonian to first order in \( \lambda \) c. The structure of the Hamiltonian are of the form

\[ H_1 = \frac{1}{\sqrt{N_c}} \sum_{123} f_{123} m^1_1 m^2_1 m^3_3 \]
\[ H_2 = \frac{1}{N_c} \sum_{1234} f_{1234} m^1_1 m^2_1 m^3_3 m^4_4 \] (43)

The ensuing shifts caused by (43) on the mesonic state to first order in \( \lambda \) are respectively of the form

\[ |i\rangle^1 = \frac{1}{\sqrt{N_c}} \sum_{12} |12\rangle \alpha_{12i} \]
\[ |i\rangle^2 = \frac{1}{N_c} \sum_{123} |123\rangle \alpha_{123i} \] (44)

with

\[ \alpha_{12i} = \frac{f_{12i}}{E_1 + E_2 - E_i} \]
\[ \alpha_{123i} = \frac{f_{123i}}{E_1 + E_2 + E_3 - E_i} \]
\[ + \sum_4 \frac{f_{1234i}}{(E_1 + E_2 + E_3 - E_i)(E_3 + E_4 - E_i)} \] (45)

and the coefficients \( f_{ijk} \) and \( f_{ijki} \) are

\[ f_{ijk}(P_1, P_2, P_3) = \frac{\lambda}{4\pi} \int dk_1 dk_2 dk_3 dk_4 \]
\[ \times \delta(k_1 + k_2 + k_3 - k_4) \]
\[ \times \delta(k_1 + k_2 - P_1) \delta(k_3 + q - P_2) \delta(k_4 + q - P_3) \]
\[ \times \left( \phi^+_{j}(k_1, P_1) \phi^+_{j}(k_3, P_2) \phi^+_{j}(k_4, P_3) S(k_1, k_2, k_3, k_4) \right) \]
\[ \times (k_1 - k_4)^2 \]
\[ \times \phi^+_{j}(k_1, P_1) \phi^+_{j}(q, P_2) \phi^+_{j}(q, P_3) S(k_2, k_1, k_3, k_4) \]
\[ \times (k_1 + k_3)^2 \]
\[ + f_{ijk} \] (46)

where we have set

\[ S(k_1, k_2, k_3, k_4) = \cos \left( \frac{\theta(k_1) - \theta(k_3)}{2} \right) \sin \left( \frac{\theta(k_2) + \theta(k_3)}{2} \right) \] (47)

The last contribution \( f_{ijk} \) involves at least one \( \phi^- \) and therefore drops out in the large momentum limit, so it will not be quoted.

All contributions of the form \( f_{ijki} \) involve at least one \( \phi^- \) and also drop out in the large momentum limit. More specifically, in the large momentum limit, we set \( P_3 = P \rightarrow +\infty \), and we change our variables to \( P_1 = xP, P_2 = yP, \) and \( P_3 = zP \), then any term which contains \( \phi^- (x_1 P, x_2 P) \) vanishes in this limit, an example is the \( f_{1234} \) term.

VII. CORRECTION TO THE QUASI-PDF IN SPINOR QCD

In this section we derive the \( 1/N_c \) correction to the quasi-parton distribution function for two-dimensional spinor QCD and show that it is in agreement with the \( 1/N_c \) correction to the parton distribution we just established in the large momentum limit. For that, we switch to the description of two-dimensional spinor QCD in the axial gauge using the changes in the Appendix.

In the axial gauge, the Hamiltonian is written in terms of \( m_a(P) \) and \( \phi^\pm \). The structure of the Hamiltonian is still of the form (34). We now note that the contributions to the first order shift of the state \( |P\rangle^1 \) of the form \( m^1_1 m^1_1 m^1_1 \) always carries \( \phi^- \). In the large momentum limit these terms drop out as we have shown earlier, so they will be ignored. The only surviving terms in the Hamiltonian at large momentum are also of the form \( m^1_1 m^1_1 m^1_1 + c.c. \).

With the above in mind and to be more specific, the parts of the Hamiltonian (34) that will contribute to the quasi-parton distribution function in leading order in perturbation theory are of the form

\[ H_1 = \frac{1}{\sqrt{N_c}} \sum_{123} f_{123} m^1_1 m^2_1 m^3_3 \]
\[ H_2 = \frac{1}{N_c} \sum_{1234} f_{1234} m^1_1 m^2_1 m^3_3 m^4_4 \] (43)
The parton fractions are constrained kinematically. For instance, the energy denominator

$$\frac{1}{E_x P + E_y P + E_z - E_P}$$

implies $0 < x, y, z < 1$ in leading order in $1/P$, otherwise the contribution is subleading. In this case, the only term in $H^1$ which contains only $\phi^+$ (first contribution in (49)) will reduce to the light cone gauge term if one identifies the creation operators in both cases using

$$\begin{equation}
\phi_n^+ (xP, P) \rightarrow \phi_n (x) \frac{1}{E_x P + E_y P + E_z - E_P} \rightarrow \frac{2P}{m_1^2 + m_2^2 - m_1^2} \tag{48}
\end{equation}$$

More specifically, the first order correction to the quasi-parton distribution function is proportional to

$$\begin{align*}
\langle P | \int dp dq & \sin \left[ \frac{\theta(xP) + \theta(p)}{2} \right] M^\dagger (xP, q) M(p, q) | P \rangle + \\
\langle P | \int dp dq & \sin \left[ \frac{\theta(xP) + \theta(p)}{2} \right] M^\dagger (q, -p) M(q, -xP) | P \rangle
\end{align*} \tag{49}$$

with $| P \rangle$ corrected to first order. There are two type of contributions in (49) as we now discuss.

First, the $m_m m_m$ term. For this only the $| i \rangle^1$ in the shift of the state contributes, and the specific contribution with only $\phi^+$ is

$$\begin{align*}
2 \sin \frac{\theta(xP)}{2} & \sum_{kk'ii} \gamma_{kli} \phi^+_k (xP, p_k) \phi^+_k (xP, p_k) \\
+ 2 \sin \frac{\theta(xP)}{2} & \sum_{kk'ii} \gamma_{kli} \phi^+_k (xP, p_k) \phi^+_k (xP, p_k)
\end{align*} \tag{50}$$

In the large momentum limit, we have $p_k = yP$, and $p_l = (1 - y)P$ as discussed above. The first term is non-zero if $0 < x < y$, and the second term is always zero for $0 < x < 1$ since $(x + y) > y$. Thus by shifting $y \rightarrow y + x$ with $0 < y < 1 - x$, and taking care of factors of $P$, this contribution matches the correction to the parton distribution function in the light cone gauge (10).

Second, the $m_m m_m m^{1}$ term comes with at least one $\phi^-$, and is always zero in the large $P$ limit as discussed above. It follows, that the order $1/N_c$ contribution to the quasi-parton distribution matches the parton distribution in the large momentum limit without renormalization in two-dimensional spinor QCD. We have explicitly checked that the same holds for two-dimensional scalar QCD.

VIII. CONCLUSIONS

Using a bosonized form of two-dimensional scalar and spinor QCD, we have analyzed the quasi-parton distribution of a meson state. In the infinite momentum limit, the quasi-distribution matches the parton distribution on the light cone both in leading and sub-leading order without further renormalization, but the limit is subtle at the parton fractions $x = 0, 1$. This provides a non-perturbative check on the proposal put forth by one of us [1] for extracting the QCD light cone partonic distributions from their quasi-distribution counterparts using pertinent equal-time Euclidean correlators through suitably matching at large momentum.

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X. APPENDIX : TWO-DIMENSIONAL SPINOR QCD IN THE LIGHT-CONE AND AXIAL GAUGE

Here and for convenience we briefly summarize some of the changes needed to recover spinor QCD from scalar QCD as developed in the main text. Both in the light cone and axial gauge the mesonic operators $M$ and $N$ are defined as in section V with $s = -1$. The fermionic fields in terms of creation-annihilation operators are defined as

$$\psi = \int_0^\infty \frac{dp^+}{2\pi} (a(p^+)e^{-ip^+x^-} + b^+(p^+)e^{-ip^+x^-})$$

$$\psi = \int \frac{dp}{2\pi} e^{ipx} (a(p)u(p) + b^l(-p)v(-p)) \tag{51}$$

in the light cone and axial gauge respectively, with

$$u(p) = e^{-\frac{i}{2} \theta(p) \gamma^3} (1, 0)^T$$

$$v(-p) = e^{-\frac{i}{2} \theta(p) \gamma^3} (0, 1)^T \tag{52}$$

The mode decomposition in the light cone gauge is given in (45), and in the axial gauge as

$$M(k_1, P - k_1) = \frac{1}{|P|} \times \sum_n \phi_n^+ (k_1, P)m_n (P) - \phi_n (-k_2, -P)m_n^\dagger (-P) \tag{53}$$

The bosonized Hamiltonian is still of the form (54), with the relevant $M^\dagger MM$ term given in the main text.
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