Pinned Distance Sets Using Effective Dimension

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Let $E \subseteq \mathbb{R}^n$. The distance set of $E$ is

$$\Delta E = \{|x - y| \mid x, y \in E\}.$$

More generally, if $x \in \mathbb{R}^n$, the pinned distance of $E$ w.r.t. $x$ is

$$\Delta_x E = \{|x - y| \mid y \in E\}.$$
When $E$ is a finite set, Erdős conjectured that $|\Delta E|$ is at least (almost) linear in terms of $|E|$.

- In a breakthrough paper, Guth and Katz proved this in the plane.
- Still an important open problem for $\mathbb{R}^n$ with $n \geq 3$.

Falconer posed an analogous question for the case that $E$ is infinite, known as Falconer’s distance set problem.

- If $E \subseteq \mathbb{R}^n$ has $\dim_H(E) > n/2$, then $\Delta E$ has positive measure.
- Still open in all dimensions.
- Guth, Iosevich, Ou and Wang, proved that if $E \subseteq \mathbb{R}^2$ and $\dim_H(E) > 5/4$, then $\mu(\Delta E) > 0$. 
Substantial progress has been made in a slightly different direction, on the Hausdorff dimension of *pinned distance sets* in the plane for “many” \( x \in E \).

- Shmerkin proved that, if \( \dim_H(E) > 1 \) and \( \dim_H(E) = \dim_P(E) \), then \( \dim_H(\Delta_x E) = 1 \).

- Liu showed that, if \( \dim_H(E) = s \in (1, 5/4) \), then \( \dim_H(\Delta_x E) \geq \frac{4}{3}s - \frac{2}{3} \).

- Shmerkin improved this bound when \( \dim_H(E) = s \in (1, 1.04) \), by proving that
  \[
  \dim_H(\Delta_x E) \geq 2/3 + 1/42 \approx 0.6904
  \]

- S. proved that, for any \( E \) with \( \dim_H(E) > 1 \),
  \[
  \dim_H(\Delta_x E) \geq \frac{\dim_H(E)}{4} + \frac{1}{2}
  \]
Our results

Let $E \subseteq \mathbb{R}^2$ be analytic and $1 < d < \text{dim}_H(E)$.

- There is a subset $F$ of full dimension such that, for all $x \in F$,
  $$\text{dim}_H(\Delta_x E) \geq \frac{d(d-4)}{d-5}$$

  - This improves the best known bounds when $\text{dim}_H(E) \in (1, 1.127)$.

- For all $x$ outside a set of dimension 1
  $$\text{dim}_H(\Delta_x E) \geq \frac{\text{dim}_P(E)+1}{2\text{dim}_P(E)}.$$

- If $\text{dim}_P(E) < \frac{d(3+\sqrt{5})-1-\sqrt{5}}{2}$, then for all $x$ in a subset of full dimension $\text{dim}_H(\Delta_x E) = 1$.

- There is a point $x \in E$ such that
  $$\text{dim}_P(\Delta_x E) \geq \frac{12-\sqrt{2}}{8\sqrt{2}} \approx 0.9356$$
  - Improves (slightly) the Keleti-Shermkin bound for packing dimension.
Regularity results

We can generalize the problem, by considering a pinned set $X$, and a test set $Y$, and investigating

$$\sup_{x \in X} \dim_H(\Delta_x Y).$$

We proved that, under some regularity assumptions, the distance sets achieve maximal dimension.

1. If $Y$ is analytic, with $\dim_H(Y) > 1$ and $\dim_P(Y) < 2 \dim_H(Y) - 1$. Then, for any subset $X$ with $\dim_H(X) > 1$,

$$\dim_H(\Delta_x Y) = 1.$$

for all $x \in X$ outside a set of (Hausdorff) dimension one.

2. If $Y$ is analytic with $\dim_H(Y) > 1$, and $X$ satisfies $\dim_H(X) = \dim_P(X) > 1$, then there is a subset $F \subseteq X$ such that,

$$\dim_H(\Delta_x Y) = 1,$$

for all $x \in F$. 
Fix a universal Turing machine $U$. Let $x \in \mathbb{R}$ and $r \in \mathbb{N}$. The **Kolmogorov complexity of $x$ at precision $r$** is

$$K_r(x) = \text{minimum length input } \pi \in \{0, 1\}^\ast \text{ such that } U(\pi) = x \upharpoonright r,$$

where $x \upharpoonright r$ is the first $r$ bits in the binary representation of $x$.

- Can think of $U$ as a computer.
- Can think of $\pi$ as a program written in, e.g., Python.

Let $y \in \mathbb{R}$. The **Kolmogorov complexity of $x$ at precision $r$ given $y$ at precision $s$** is

$$K_{r,s}(x \mid y) = \text{minimum length input } \pi \in \{0, 1\}^\ast \text{ such that } U(\pi, y \upharpoonright s) = x \upharpoonright r.$$

For every $x \in \mathbb{R}^n$ and $r \in \mathbb{N}$, $0 \leq K_r(x) \leq nr + O(\log r)$.

- If $x$ is rational, then $K_r(x) = O(\log r)$
- Almost every point satisfies $K_r(x) = r - O(\log r)$ for every $r \in \mathbb{N}$. We call these points random.

Symmetry of information: For every $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, and $r, t \in \mathbb{N}$,

$$K_{r,t}(x, y) = K_t(y) + K_{r,t}(x \mid y) + O(\log r + \log t).$$

We can relativize the definitions in the natural way to get $K_r^A(x)$, $K_{r,t}^A(x \mid y)$, ... for any oracle $A \subseteq \mathbb{N}$. 
Let $x \in \mathbb{R}^n$. The (effective Hausdorff) dimension of $x$ is

$$\dim(x) = \liminf_{r \to \infty} \frac{K_r(x)}{r}.$$ 

The (effective) packing dimension of $x$ is

$$\text{Dim}(x) = \limsup_{r \to \infty} \frac{K_r(x)}{r}.$$ 

- $0 \leq \dim(x) \leq \text{Dim}(x) \leq n$.
- Almost every point satisfies $\dim(x) = \text{Dim}(x) = n$.
- If $x$ is rational, then $\dim(x) = \text{Dim}(x) = 0$. 

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Theorem (J. Lutz and N. Lutz)

For every set $E \subseteq \mathbb{R}^n$,

$$\dim_H(E) = \min_{A \subseteq \mathbb{N}} \sup_{x \in E} \dim^A(x) \quad \text{and} \quad \dim_P(E) = \min_{A \subseteq \mathbb{N}} \sup_{x \in E} \Dim^A(x).$$

- The Hausdorff dimension of a set is characterized by the (effective) dimension of the points in the set.
- Allows us to use computability to attack problems in geometric measure theory.
Pinned distance sets using effective dimension

Theorem (Fiedler, S.)

Let \( X, Y \subseteq \mathbb{R}^2 \) such that \( Y \) is analytic, with \( 1 < \dim_H(Y), \dim_H(X) \). Then there is a subset \( F \subseteq X \) of full dimension, such that, for all \( x \in F \),

\[
\dim_H(\Delta_x Y) \geq d \left( 1 - \frac{(D-1)(D-d)}{2(D^2+D-1)-2d(2D-1)} \right),
\]

where \( d = \min\{\dim_H(X), \dim_H(Y)\} \) and \( D = \max\{\dim_P(X), \dim_P(Y)\} \).

We reduce our main theorem on the Hausdorff dimension of pinned distance sets to this pointwise analog.

1. Orponen’s theorem on radial projections.
2. Point-to-set principle.
Pinned distance sets using effective dimension

Theorem (Fiedler, S.)

Suppose that \( x, y \in \mathbb{R}^2, e_1 = \frac{y-x}{|y-x|} \) satisfy the following.

(C1) \( \dim(x), \dim(y) > 1 \)

(C2) \( K_r^x(e_1) \approx r \) for all \( r \).

(C3) \( K_r^x(y) \approx K_r(y) \) for all sufficiently large \( r \).

(C4) \( K_r(e_1 \mid y) \approx r \) for all \( r \).

Then

\[
\dim^x(|x - y|) \geq d \left( 1 - \frac{(D-1)(D-d)}{2(D^2+D-1)-2d(2D-1)} \right),
\]

where \( d = \min\{\dim(x), \dim(y)\} \) and \( D = \max\{\text{Dim}(x), \text{Dim}(y)\} \).
We fix $x, y \in \mathbb{R}^2$ satisfying the conditions. To get a lower bound on $\dim^x(|x - y|)$, we show the analogous bound at each precision (scale). Fix a precision $r \in \mathbb{N}$.

1. Symmetry of information: proving a lower bound on $K_r^x(|x - y|)$ is equivalent to establishing an upper bound on $K_r^x(y \mid |x - y|)$.
   - How many bits are needed to specify (a $2^{-r}$-approximation of) $y$ if you know ($2^{-r}$-approximations of) $x$ and $|x - y|$.

2. A $2^{-r}$-approximation of $|x - y|$ gives an annulus of thickness $2^{-r}$. We need to specify where $y$ is on this annulus (trivial bound - give $r$ bits).

3. Use an induction on scales approach - find a “nice” sequence of precisions $r_1 < r_2 < \ldots r_k = r$. To specify $y$ to precision $r$, first specify to precision $r_1$, using this to specify $y$ to precision $r_2$, and so on.

4. We choose our precisions based on the behavior of the complexity function $K_s(y)$.
   - When the complexity is growing very quickly (slope at least 1), we are able to show that $K(|x - y|)$ is growing at a rate of 1 (best possible).
Complexity of $y$ increases slowly

Suppose that between precisions $r_i$ and $r_{i+1}$, the complexity of $y$ is increasing slowly (much less than slope 1)

- Want to specify $y$ up to precision $r_{i+1}$, given $x$ and $|x - y|$ and given $y$ up to precision $r_i$.
- Want to show that $y$ is essentially the only point in the annulus whose complexity is growing slowly.
- That is, if $z$ is in the annulus, then either $z$ is very close to $y$, or the complexity of $z$ is growing quickly.
Goal: Prove the complexity of other points on annulus are growing more quickly than that of $y$.

Reduce this to *projections*.

Main idea: we can compute $x$ if we know $y$, $z$ and the position of $x$ along the line with direction $e_2^\perp$ containing $x$.

$$K(x | y) \lesssim K(z | y) + K(x | p_{e_2}x, e_2)$$

Thus, if complexities of $z$ and $y$ are growing very slowly, then the complexity of $x$ is growing slowly.

Goal is to prove that $K(x | p_{e_2}x, e_2)$ is small.
Reducing to projections

Suppose that $|x - z| = |x - y|$, and let $s := -\log \|y - z\|$.

$$K_{r-s}(x \mid y) \lesssim K_r(z \mid y) + K_{r-s,r-r}(w, e_2 \mid y, z) + K_{r-s,r}(x \mid w, e_2)$$

$$\lesssim K_r(z \mid y) + K_{r-s}(x \mid p_{e_2}x, e_2).$$

We know that $K_{r-s}(x \mid y) \gtrsim d(r - s)$. So, if

- the complexity of $z$ increases at least as slowly as that of $y$, and
- we can get a strong enough upper bound on $K_{r-s}(x \mid p_{e_2}x, e_2)$

we have a contradiction - i.e., no such $z$ exists on the annulus.
Projection theorem

- We want to bound $K_r(x \mid p_e x, e)$ - the complexity of computing (an approximation of) $x$ given (approximations of) $p_e x$ and $e$.

- When the direction $e$ is random relative to $x$, i.e., $K_r^x(e) \approx r$, we know that $K_r(x \mid p_e x, e) \approx K_r(x) - r$.
  - This is the pointwise analog of Marstrand’s projection theorem.

- Unfortunately we don’t have enough control over the direction to directly apply this result.

- However, we do have enough control to ensure that $e$ is random up to some initial precision:

$$K_s^x(e) \approx s,$$

where $s = -\log |z - y|$. 

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Pinned Distance Sets
## Projection theorem

Theorem (Fiedler, S.)

Let $x \in \mathbb{R}^2$, $e \in S^1$, $\varepsilon \in \mathbb{Q}^+$, $C \in \mathbb{N}$, $A \subseteq \mathbb{N}$, and $t, r \in \mathbb{N}$. Suppose that $r$ is sufficiently large, and that the following hold.

(P1) $1 < d \leq \text{dim}^A(x) \leq \text{Dim}^A(x) \leq D$.

(P2) $t \geq \frac{d(2-D)}{2}r$.

(P3) $K_{s,x}^A(e) \geq s - C \log s$, for all $s \leq t$.

Then

$$K_r^A(x \mid p_e x, e) \leq \max\{\frac{D-1}{D}(dr - t) + K_r^A(x) - dr, K_r^A(x) - r\} + \varepsilon r.$$
Complexity of $y$ increases slowly

**Goal:** Prove the complexity of other points on annulus are growing more quickly than that of $y$.

For any $e \in S^1$ and $x \in \mathbb{R}^2$, $p_e x = e \cdot x$.

- We can compute $x$ if we know $y$, $z$ and the position of $x$ along the line with direction $e_2^\perp$ containing $x$.
  $$K(x \mid y) \lesssim K(z \mid y) + K(x \mid p_{e_2} x, e_2)$$

- Thus, if complexities of $z$ and $y$ are growing sufficiently slowly, we have a contradiction.
Thank you!