VERTEX OPERATORS ARISING FROM JACOBI-TRUDI IDENTITIES

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Abstract. We give an interpretation of the boson-fermion correspondence as a direct consequence of the Jacobi–Trudi identity. This viewpoint enables us to formulate a unified theory of generalized Schur symmetric functions and obtain a generalized Giambelli identity. It also allows us to construct the action of the Clifford algebra (fermions) on the polynomial algebra from a generalized version of the Jacobi–Trudi identity. As applications, we obtain explicit vertex operators corresponding to characters of the classical Lie algebras, shifted Schur functions, and generalized Schur symmetric functions associated to linear recurrence relations.

1. Introduction

We start by recalling the classical boson-fermion correspondence. Let \( V = \bigoplus_{j \in \mathbb{Z}} \mathbb{C} v_j \) be an infinite-dimensional complex vector space with a linear basis \( \{v_j\}_{j \in \mathbb{Z}} \). Define \( F^{(m)} \ (m \in \mathbb{Z}) \) to be a linear span of semi-infinite wedge products \( v_i^m \wedge v_i^{m-1} \wedge \ldots \) with the properties

1. \( i_m > i_{m-1} > \ldots \),
2. \( i_k = k \) for \( k << 0 \).

The monomial of the form \( |m\rangle = v_m \wedge v_{m-1} \wedge \ldots \) is called the \( m \)th vacuum vector. The elements of \( F^{(m)} \) are linear combinations of monomials \( v_I = v_{i_1} \wedge v_{i_2} \wedge \ldots \) that are different from \( |m\rangle \) only at finitely many places.

We define the Fock space as the graded space \( \mathcal{F} = \bigoplus_{m \in \mathbb{Z}} F^{(m)} \). The algebra of fermions (Clifford algebra) acts on the Fock space \( \mathcal{F} \) by wedge operators \( v_k^+ \) and contraction operators \( v_k^- \ (k \in \mathbb{Z}) \). More precisely, these operators are defined by

\[
    v_k^+ (v_{i_1} \wedge v_{i_2} \wedge \ldots) = v_k \wedge v_{i_1} \wedge v_{i_2} \wedge \ldots,
\]

and

\[
    v_k^- (v_{i_1} \wedge v_{i_2} \wedge \ldots) = \delta_{k,i_1} v_{i_2} \wedge v_{i_3} \wedge \ldots - \delta_{k,i_2} v_{i_1} \wedge v_{i_3} \wedge \ldots + \delta_{k,i_3} v_{i_1} \wedge v_{i_2} \wedge \ldots - \ldots.
\]

Then the following relations are satisfied:

\[
    v_k^+ v_m^- + v_m^+ v_k^- = \delta_{k,m}, \quad v_k^+ v_m^+ + v_m^+ v_k^- = 0, \quad v_k^- v_m^- + v_m^- v_k^- = 0.
\]

Combine the operators \( v_k^\pm \) in generating functions (formal distributions)

\[
    \Gamma^+(u) = \sum_{k \in \mathbb{Z}} v_k^+ u^k \quad \text{and} \quad \Gamma^-(u) = \sum_{k \in \mathbb{Z}} v_k^- u^{-k}.
\]

Using the normal ordered product, we introduce the formal distribution

\[
    \alpha(u) =: \Gamma^+(u) \Gamma^-(u) =: \Gamma^+(u) \Gamma^-(u) - \Gamma^-(u) \Gamma^+(u),
\]

2010 Mathematics Subject Classification. Primary 05E05, Secondary 17B65, 17B69, 11C20.

Key words and phrases. Jacobi-Trudy identity, Giambelli identity, Boson-Fermion correspondence, vertex operators, Schur functions, characters of classical Lie algebras.
where by definition of the normal ordered product,
\[ \Gamma^+(u)_+ = \sum_{k \geq 1} v_k^+ u^k, \quad \Gamma^+(u)_- = \sum_{k \leq 0} v_k^+ u^k. \]

It can be verified that the coefficients \( \alpha_k \) of the formal distribution \( \alpha(u) = \sum \alpha_k u^{-k} \) and the central element 1 satisfy relations of the Heisenberg algebra \( \mathcal{A} \) (see e.g. [6], 16.3):
\[ [1, \alpha_k] = 0, \quad [\alpha_k, \alpha_m] = m \delta_{m,-k} \quad (k, m \in \mathbb{Z}). \]

In this way the Fock space \( \mathcal{F} \) becomes an \( \mathcal{A} \)-module.

There is also a natural action of the Heisenberg algebra \( \mathcal{A} \) on the boson space \( \mathcal{B}(m) = z^m \mathbb{C}[h_1, h_2, \ldots] \):
\[ \alpha_n = \frac{\partial}{\partial h_n}, \quad \alpha_{-n} = n h_n, \quad \alpha_0 = m \quad (m \in \mathbb{Z}). \]

The boson–fermion correspondence [2] identifies the spaces \( \mathcal{B}(m) \) and \( \mathcal{F}(m) \) as equivalent \( \mathcal{A} \)-modules (see e.g. [1], [6]). The exact correspondence relies on the interpretation of \( h_k \)'s as complete symmetric functions. Then each graded component \( \mathcal{B}(m) \) is viewed as the ring of symmetric functions with the linear basis consisting of Schur symmetric functions \( z^m s_\lambda \). The linear basis of elements \( v_\lambda = (v_{\lambda_1+m} \wedge v_{\lambda_2+m-1} \wedge v_{\lambda_3+m-2} \ldots) \) of \( \mathcal{F}(m) \), labeled by partitions \( \lambda = (\lambda_1, \geq \lambda_2, \geq \ldots, \geq \lambda_l \geq 0) \) corresponds to the linear basis \( z^m s_\lambda \) of \( \mathcal{B}(m) \) (see e.g. [6] Theorem 6.1).

By the correspondence the action of operators \( v_k^\pm \) on \( \mathcal{F} \) is carried to the action on the graded space \( \mathcal{B} = \bigoplus \mathcal{B}(m) \). It can be described by the generating functions \( \Gamma^\pm(u) \), written in the so-called vertex operator form. Note that interpretation of \( \mathcal{B}(m) \) as a space of symmetric functions implies these formulas for vertex operators as a direct consequence of famous properties of symmetric functions, and of the Jacobi-Trudi identity in particular. This identity allows us to express Schur functions as determinants of matrices with complete symmetric functions as entries.

On the other hand, there are numerous generalizations of symmetric functions in mathematics, and many of these enjoy analogues of the Jacobi-Trudy formula. The examples include and are not limited to characters of simple classical Lie algebras [3, 7], double symmetric functions [11] and, in particular, quantum immanants and shifted symmetric functions [12], Macdonald polynomials [9], generalized symmetric functions in the sense of [13], etc.

This observation motivates the main idea and the goal of this note to develop a uniformed approach. We consider a general version of the Jacobi-Trudi identity, and illustrate that it immediately implies the action of fermions on the corresponding algebra of polynomials. In many cases it also allows us to write the corresponding vertex operators in a nice closed form.

In the sections below we first recall the corresponding properties of classical symmetric functions and indicate what property we will generalize. We introduce the generalized Schur functions based on the Jacobi-Trudi identities in section two. Then we generalize the theory of symmetric functions and prove several analogous of the well-known results: Newton identities, Jacobi-Trudi identities in terms of the generalized elementary symmetric functions and finally the general Giambelli identities which cover all determinant identities. In section three we illustrate how this general theory of symmetric functions recover several families of known examples such as Schur functions, orthogonal and symplectic Schur functions, factorial Schur functions. As a consequence of our general theory, corresponding identities
are also obtained, for instance a Giambelli identity for the factorial Schur functions and Schur orthogonal/symplectic functions. Finally we develop a vertex operator approach to the generalized Schur functions and show their applications with examples.

2. General Jacobi–Trudi identity

Classical case – symmetric functions. For more details please refer e.g. to [10, 8]. Recall that the ring of classical symmetric functions in the variables \((x_1, x_2, \ldots)\) is a polynomial ring in the coherent variables \(\{h_k\}_{k \in \mathbb{Z}_{\geq 0}}\), which are complete symmetric functions defined by

\[
h_k = \sum_{i_1 \leq i_2 \leq \cdots \leq i_k} x_{i_1} x_{i_2} \cdots x_{i_k}, \quad h_0 = 1.
\]

(2.1)

The ring of symmetric functions possesses a linear basis of Schur functions \(s_\lambda\), labeled by partitions \(\lambda = (\lambda_1, \geq \lambda_2, \geq \cdots, \geq \lambda_l \geq 0)\). It is known that Schur symmetric functions can be expressed as polynomials in the \(h_k\) by the Jacobi-Trudi formula

\[
s_\lambda = \det [h_{\lambda_i - j}]_{1 \leq i,j \leq l}.
\]

(2.2)

Generalization. We generalize the classical setting to combine all the examples of interest in one picture. We start with a set of independent variables \(\{h_k^{(0)}\}_{k \in \mathbb{Z}_{> 0}}\). Let \(B = \mathbb{C}[h_1^{(0)}, h_2^{(0)}, \ldots]\) be a polynomial ring in these variables. Let

\[
h_0^{(0)} = 1, \quad \text{and} \quad h_k^{(0)} = 0 \quad \text{for} \quad k < 0.
\]

The algebra \(B\) is graded by \(\text{deg}(h_n^{(0)}) = n\), so

\[
B = \bigoplus_{n=0}^{\infty} B_n,
\]

(2.3)

where \(B_n\) is the homogeneous subspace of degree \(n\) and \(B_0 = \mathbb{C}\). Let \(B^{\leq n} = \bigoplus_{i=0}^{n} B_{n-i}\). Then \(B = \bigcup_{n=0}^{\infty} B^{\leq n}\) and clearly \(\text{gr} B \simeq B\). Suppose that \(\{h_k^{(r)}\} (k, r \in \mathbb{Z})\) is a set of elements of \(B\) such that \(h_k^{(r)} \in B^{\leq k+r}\). We require that

\[
h_{-k}^{(r)} = 1, \quad \text{and} \quad h_k^{(r)} = 0 \quad \text{for} \quad k + r < 0.
\]

(2.4)

For any partition \(\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l \geq 0)\) the polynomial \(s_\lambda\) in \(\{h_k^{(0)}\}_{k \in \mathbb{Z}}\) is defined by the formula

\[
s_\lambda = \det \left[ h_{\lambda_i - i+1}^{(r-1)} \right]_{1 \leq i,j \leq l}.
\]

We also require that polynomials \(\{s_\lambda\}\) form a linear basis of \(B = \mathbb{C}[h_1^{(0)}, h_2^{(0)}, \ldots]\) when \(\lambda\) goes over the set of all partitions. In the following we will frequently refer to \(s_\lambda\) as the generalized Schur function associated to the partition \(\lambda\).

Moreover, we can extend the definition of polynomials \(s_\lambda \in B\) and define such a polynomial for any integer vector \(\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)\) – it does not have to be a partition or even a composition, parts \(\lambda_i\) can be negative, and can be listed in any order. We represent the integer vector \(\lambda\) as a sequence with finitely many non-zero terms: \(\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l, 0, 0, 0 \ldots)\).
Given such \( \lambda \), we associate the infinite matrix

\[ S_\lambda = \begin{pmatrix}
    h^{(0)}_{\lambda_1} & h^{(1)}_{\lambda_1} & h^{(2)}_{\lambda_1} & \ldots & h^{(l-1)}_{\lambda_1} & \ldots \\
    h^{(0)}_{\lambda_2} & h^{(1)}_{\lambda_2} & h^{(2)}_{\lambda_2} & \ldots & h^{(l-1)}_{\lambda_2} & \ldots \\
    \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \\
    h^{(0)}_{\lambda_{l-1}} & h^{(1)}_{\lambda_{l-1}} & h^{(2)}_{\lambda_{l-1}} & \ldots & h^{(l-1)}_{\lambda_{l-1}} & \ldots \\
    h^{(0)}_{-l} & h^{(1)}_{-l} & h^{(2)}_{-l} & \ldots & h^{(l-1)}_{-l} & \ldots \\
    \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \\
    \end{pmatrix}
\]

Note that by property (3.1), \( S_\lambda \) has a block form

\[
\begin{pmatrix}
    A & B \\
    0 & D
\end{pmatrix}
\]

with \( A \) being an \( l \times l \) matrix and \( D \) – an infinite upper-triangular matrix with one’s on the diagonal. Therefore, if we consider the sequence of determinants of the the \( N \times N \) parts of the matrix \( S_\lambda \)

\[
\det \left[ h^{(j-1)}_{\lambda_i-1} \right]_{i,j=1,\ldots,N}, \quad N = 1, 2, 3, \ldots,
\]

for large enough \( N \) it will stabilize to a polynomial in variables \( h_1^{(0)}, h_2^{(0)}, \ldots \), which we will still denote as \( s_\lambda \) or as \( \det S_\lambda \).

**Lemma 2.1.** For any integer vector \( \lambda \) the following properties hold:

A) \[ s(\lambda_1, \lambda_2, \ldots, \lambda_k, \ldots, \lambda_l) = -s(\lambda_1, \lambda_2, \ldots, \lambda_{k-1}, \lambda_{k+1}, \ldots, \lambda_l) \] \hspace{1cm} (2.5)

B) If \( \lambda_k - \lambda_m = k - m \), for some \( k, m \), then \( s_\lambda = 0 \). \hspace{1cm} (2.6)

**Proof.** Change of the order of rows in the determinant \( \det S_\lambda \). \( \square \)

**Remark 2.1.** The case of classical symmetric functions correspond to \( h_1^{(0)} = h_k \) the ordinary complete symmetric functions, and \( h_k^{(p)} = h_{k+p} \).

**Elementary symmetric functions.** There are several ways to define elementary symmetric functions. Traditionally, classical elementary symmetric functions are defined by the formula

\[ e_k = \sum_{i_1 < i_2 < \cdots < i_k} x_{i_1} x_{i_2} \ldots x_{i_k}, \quad e_0 = 1. \]

Elementary symmetric functions enjoy several important properties that are equivalent to the traditional definition. For our purposes it is convenient to consider the so-called Newton’s identity as the primary defining property of elementary symmetric functions. The classical Newton’s formula relates classical elementary symmetric functions \( e_i \) to complete symmetric functions \( h_k \):

\[ \sum_{i=0}^n (-1)^i e_i h_{n-i} = 0 \quad \text{for} \quad n \geq 1, \quad e_0 h_0 = 1. \] \hspace{1cm} (2.7)
This identity is also equivalent to the condition that for any \( n \in \mathbb{Z}_{\geq 0} \), the matrix 
\( E = ((-1)^{i-j} e_{i-j})_{\{i,j=1,\ldots,n\}} \) is the inverse of the matrix \( H = (h_{i-j})_{\{i,j=1,\ldots,n\}} \). The formula (2.7) can be considered as one of alternative definitions of elementary symmetric functions, since (2.7) implies that \( e_k \)’s are uniquely defined polynomials in \( h_i \)’s given by the formulas

\[
e_k = \det[h_{1-i+j}]_{1 \leq i,j \leq k}.
\]

(2.8)

In our case we generalize the formula (2.8) to become the definition of elementary symmetric functions.

**Definition 2.1.** For \( a, p \in \mathbb{Z} \) define the *generalized elementary symmetric function* \( e_a^{(p)} \) as follows:

\[
e_a^{(p)} = \begin{cases} 
0, & \text{for } p < a, \\
1, & \text{for } p = a, \\
\det \left[ h_{p+1-i}^{(p-j)} \right]_{1 \leq i,j \leq p-a}, & \text{for } p > a.
\end{cases}
\]

Thus, for \( p > a \), the polynomial \( e_a^{(p)} \) is a \((p-a) \times (p-a)\) determinant

\[
e_a^{(p)} = \det \left( \begin{array}{cccc}
     h_{p}^{(-p+1)} & h_{p}^{(-p+2)} & \ldots & h_{p}^{(-a-1)} & h_{p}^{(-a)} \\
     1 & h_{p}^{(-p+2)} & \ldots & h_{p}^{(-a-1)} & h_{p}^{(-a)} \\
     \ldots & \ldots & \ldots & \ldots & \ldots \\
     0 & 0 & \ldots & 1 & h_{a+1}^{(-a)}
\end{array} \right).
\]

Note that \( e_a^{(a+1)} = h_{a+1}^{(-a)} \), and that, comparing the definition with formula (2.4), \( e_{-a}^{(1)} = s_{(1^{a+1})} \) if \( a \geq -1 \).

**Proposition 2.1.** The elements \( e_a^{(p)} \) are solutions of the following Newton’s identity:

\[
\sum_{p=-\infty}^{\infty} (-1)^{a-p} h_b^{(p)} e_a^{(-p)} = \delta_{a,b} \quad \text{for any } a, b \in \mathbb{Z}.
\]

(2.9)

**Proof.** For \( b > a \) expand the determinant in the definition of \( e_a^{(b)} \) by the first row:

\[
e_a^{(b)} = h_{b}^{(-b+1)} e_a^{(b-1)} - h_{b}^{(-b+2)} e_a^{(b-2)} + \ldots + (-1)^{b-a-1} h_{b}^{(-a)} e_a^{(a)},
\]

which gives \( \sum_{s=0}^{b-a} (-1)^s h_{b}^{(-b+s)} e_a^{(b-s)} = 0 \). Since \( h_{b}^{(-b+s)} = 0 \) for \( s < 0 \) and \( e_a^{(b-s)} = 0 \) for \( s > b - a \), we can rewrite the last equality

\[
\sum_{p=-\infty}^{\infty} (-1)^{a-p} h_b^{(p)} e_a^{(-p)} = 0 \quad \text{for } b > a.
\]

It is easy to get the same equality for \( b < a \), since in this case (2.9) contains only zero terms. Finally, for \( a = b \),

\[
\sum_{p=-\infty}^{\infty} (-1)^{a-p} h_a^{(p)} e_a^{(-p)} = h_a^{(-a)} e_a^{(a)} = 1.
\]

\[\square\]
Let us introduce the infinite matrices $H$ and $E$ with entries

$$H_{bp} = h^{(p)}_{-b}, \quad E_{pa} = (-1)^{a-p}e^{-p}_{-a}, \quad (a, b, p \in \mathbb{Z}).$$

(2.10)

They can be displayed as follows.

$$H = \begin{pmatrix}
\ldots & h_2^{(-1)} & h_2^{(0)} & h_2^{(1)} & h_2^{(a-1)} & h_2^{(a)} & \ldots \\
\ldots & 1 & h_1^{(0)} & h_1^{(1)} & h_1^{(a-1)} & h_1^{(a)} & \ldots \\
\ldots & 0 & 1 & h_0^{(1)} & h_0^{(a-1)} & h_0^{(a)} & \ldots \\
\ldots & 0 & 0 & 0 & 1 & h_{a+1}^{(a)} & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\end{pmatrix},$$

$$E = \begin{pmatrix}
\ldots & -e_0^{(1)} & e^{(1)}_{-1} & -e_{-1}^{(1)} & \ldots & \pm e_{-a}^{(1)} & \ldots \\
\ldots & 0 & -e_0^{(0)} & e^{(0)}_{-1} & \ldots & \pm e_{-a}^{(0)} & \ldots \\
\ldots & 0 & 0 & -e_{-2}^{(-1)} & \ldots & \pm e_{-a}^{(-1)} & \ldots \\
\ldots & 0 & 0 & 0 & 1 & e_{-a}^{(-2)} & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\end{pmatrix},$$

where the columns are numerated to grow from the left to the right and rows are indexed to increase from the top to the bottom. Both matrices are upper-triangular with one’s on the diagonal. Then (2.9) can be interpreted as $HE = Id$ for infinite matrices, or, more carefully, for any $M < N$, for finite upper-triangular submatrices $H(M, N) = (H_{bp})_{(M \leq b, p \leq N)}$, and $E(M, N) = (E_{qa})_{(M \leq q, a \leq N)}$:

$$\sum_{k=M}^{N} H_{kk}E_{ka} = \sum_{s=0}^{a-b} H_{b,b+s}E_{b+s,a} = \sum_{s=0}^{a-b} (-1)^{-s+a-b}h_{-b}^{(b+s)}e_{-a}^{(-b-s)} = (-1)^{a-b} \delta_{-a,-b} = \delta_{a,b}.$$

Hence for any $M < N$ one has $H(M, N)E(M, N) = Id$.

The following lemma is proved in [3] (Lemma A.42).

**Lemma 2.2.** Let $A$ and $B$ be $r \times r$ matrices whose product is a scalar matrix $c \cdot Id$. Let $(S, S')$ and $(T, T')$ be permutations of the sequence $(1, \ldots, r)$, where $S$ and $T$ each consists of $k$ integers, $S', T'$ of $r - k$. Denote as $A_{S,T}$ the corresponding minor (it is the determinant of the $k \times k$ matrix whose $i, j$ entry is $a_{s_i,t_j}$ with $s_i \in S$, $t_j \in T$). Then

$$c^{r-k}A_{S,T} = \varepsilon \det(A)B_{T',S'},$$

where $\varepsilon$ is the product of the signs of the two permutations.

**Corollary 2.1.** Let $\lambda = (\lambda_1, \ldots, \lambda_l)$ be a partition, and let $\mu = (\mu_1, \ldots, \mu_k)$ be the conjugate partition. Then

$$s_{\lambda} = \det [e_{j-\mu_j}^{(i)}]_{1 \leq i, j \leq k}.$$

**Proof.** From (2.4), $s_{\lambda}$ is the minor $H(-k, l-1)_{S,T}$ of the matrix $H(-k, l-1)$ with $S = \{-\lambda_1, \ldots, -\lambda_l + l - 1\}$ and $T = \{0, 1, \ldots, l - 1\}$. Then $S' = \{\mu_1 - 1, \ldots, \mu_k - k\}$, $T'$ =
\{-k, \ldots, -1\}, and \( S \cup S' = T \cup T' = \{-k, \ldots, l-1\} \), with \( \varepsilon = (-1)^{\sum \mu_j} = (-1)^{\sum \lambda_j} \). By Lemma 2.2 it follows that

\[ s_\lambda = \det \mathcal{H}(-k, l-1)_{S,T} = \varepsilon \det \mathcal{E}(-k, l-1)_{T',S'} = \varepsilon \det \left[ (-1)^{\mu_j - j - \mu_j} e^{(i)}_{j-\mu_j} \right]_{1 \leq i, j \leq k} = \varepsilon (-1)^{\sum (\mu_j - j) - \sum i} \det [e^{(i)}_{j-\mu_j}]_{1 \leq i, j \leq k} = \det [e^{(i)}_{j-\mu_j}]_{1 \leq i, j \leq k}. \]

\[ \square \]

**General Giambelli identities.** For any \( m, n \in \mathbb{Z} \), we define the hook Schur function \( s_{(m|n)} \) by

\[ s_{(m|n)} = \sum_{p=0}^{n} (-1)^{p} h_{m+1}^{(p)} e^{(-p)}_{-n}. \]  \hspace{1cm} (2.11)

If \( m \geq 0 \) and \( n \geq 0 \), \( s_{(m|n)} \) is exactly the generalized Schur function \( s_{(m+1,1^n)} \) of the hook \((m+1,1^n)\) according to (2.4). This can be easily seen by expanding the Jacobi-Trudi determinant along the first row and observe that the \((1,p+1)\)-minor is exactly the elementary symmetric function \( e^{(-p-1)}_{-n} \) by Definition 2.1 for \( p = 0, 1, \ldots, n \).

Clearly by Newton identity (2.1), \( s_{(m|n)} = 0 \) if either \( m < 0 \) or \( n < 0 \) except that \( s_{(m|n)} = (-1)^n \) when \( m + n = -1 \).

Recall that the Frobenius notation \((\alpha_1 \cdots \alpha_r | \beta_1 \cdots \beta_r) = (\alpha|\beta)\) of the partition \( \lambda \) is defined by

\[ \alpha_i = \lambda_i - i, \]
\[ \beta_i = \lambda'_i - i, \]

for \( i = 1, \ldots, r \), where \( r \) is the length of the main diagonal in \( \lambda \). Clearly the conjugate of \((\alpha|\beta)\) is \((\beta|\alpha)\), and the hook \((m+1,1^n)\) is \((m|n)\) in Frobenius notation. Sometimes it is convenient to allow \( i > r \) and still use the formula to extend the Frobenius notation. For example, \( \lambda = (3 2^3) = (2 0 3 2) \) in Frobenius notation and \( \lambda = (2,0,-1,-2,\cdots|3,2,1,0,-1,\cdots) \) or any cut-off beyond \( r \) in the extended Frobenius notation.

**Theorem 2.1.** For any partition \( \lambda = (\alpha|\beta) = (\alpha_1 \cdots \alpha_r | \beta_1 \cdots \beta_r) \) and any \( n \geq l(\lambda) \), the generalized Schur function \( s_{(\alpha|\beta)} \) satisfies

\[ s_{(\alpha|\beta)} = \det [s_{(\lambda_i-i|n-j)}]_{1 \leq i, j \leq n} = \det [s_{(\alpha_i|\beta_j)}]_{1 \leq i, j \leq r}. \] \hspace{1cm} (2.12)

**Proof.** Let \( (\alpha|\beta) = \lambda = (\lambda_1, \cdots, \lambda_l) \). For any \( n \geq l(\lambda) \), consider the matrix

\[ \left[ ((-1)^{i-1} e^{(-i+1)}_{-n-j}) \right]_{1 \leq i, j \leq n}. \]

As \( e^{(-i+1)}_{-n+j} = 0 \) for \( i + j > n + 1 \), the matrix has determinant 1 regardless of the parity of \( n \). By the definition of the hook Schur function (2.11) it follows that

\[ s_{(\lambda_i-i|n-j)} = \sum_{p=0}^{n-1} (-1)^{p} h_{\lambda_i-i+1}^{(p)} e^{(-p)}_{-n+j}. \] \hspace{1cm} (2.13)

This implies the following matrix identity in \( \operatorname{Mat}_n(B) \):

\[ [s_{(\lambda_i-i|n-j)}] = [h_{\lambda_i-i+1}^{(j-1)}]((-1)^{i-1} e^{(-i+1)}_{-n+j}). \] \hspace{1cm} (2.14)

Taking the determinant of (2.14) we get that

\[ s_\lambda = \det [s_{(\lambda_i-i|n-j)}]_{1 \leq i, j \leq n}. \]
For each $i > r$, the $i$th row has only one non-zero entry $s_{(\lambda_i-i|n-j)} = (-1)^{n-j}$ at the column $j = n+1+\alpha_i$. It is well-known that $\{n+1+\alpha_i\}_{1 \leq i \leq n} \cup \{n-\beta_j\}_{1 \leq j \leq m} = \{1, 2, \ldots, m+n\}$ for any $m \geq \lambda_1$ (cf. [10], Ch. 1, (1.7)), i.e. $\{-1-\alpha_i\}_{1 \leq i \leq n} \cup \{\beta_j\}_{1 \leq j \leq m} = \{-m, \ldots, n-1\}$. So when we remove the last $n-r$ rows and the $n-r$ columns numbered by $n+1+\alpha_{r+1}, \ldots, n+1+\alpha_n$, the supplement $r \times r$-minor of $[s_{(\lambda_i-i|n-j)}]$ is exactly $\det[s_{(\alpha_i|\beta_j)}]_{r \times r}$. Therefore $s_{\lambda} = \pm \det[s_{(\alpha_i|\beta_j)}]$. The overall sign factor is given by

$$(-1)^{\sum_{t=r+1}^{n}(n-j_t)+(0+j_t)-(i-r-1)} = (-1)^{\sum_{t=r+1}^{n}(n+r+1)} = (-1)^{(n-r)(n+r+1)} = 1,$$

which shows the Giambelli identity.

**Remark 2.2.** It is clear that the Jacobi-Trudi identities for both $h_n^{(i)}$ and $v_n^{(i)}$ are special cases of the Giambelli identity.

Ex. $\lambda = (3 \ 2 \ 3 \ 2) = (2 \ 0 \ 3 \ 2)$, so $r = 2$. Take $n = 4$ and the extended Giambelli determinant is

$$\det[s_{(\alpha_i|4-j)}] = \det \begin{vmatrix}
s_{(2|3)} & s_{(2|2)} & s_{(2|1)} & s_{(2|0)} \\
s_{(0|3)} & s_{(0|2)} & s_{(0|1)} & s_{(0|0)} \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0
\end{vmatrix} = \det[s_{(\alpha_i|\beta_j)}]_{2 \times 2}.$$

### 3. Fermions and Vertex Operator Presentation.

In accordance with the classical case, to construct the action of fermions on the boson space, we need to take multiple copies of the polynomial ring $B$. Namely, let $z$ be a variable, set $B^{(m)} = z^m B$ and $B = \oplus B^{(m)} = \mathbb{C}[z, z^{-1}, h_1^{(0)}, h_2^{(0)}, \ldots]$. The elements $\{s_\lambda z^m\}$ form a linear basis of $B$, where $s_\lambda$ are defined by (2.4) and are labeled by partitions $\lambda$, and $m \in \mathbb{Z}$.

Define the operators $v_k^+$ and $v_k^- (k \in \mathbb{Z})$ acting on this basis by the following rules:

$$v_k^+(s_\lambda z^m) = s_{(k-m-1,\lambda)} z^{m+1}, \quad (3.1)$$

$$v_k^-(s_\lambda z^m) = \sum_{t=1}^{\infty} (-1)^{t+1} \delta_{k-m-1,1-t} s_{(\lambda_1+1,\ldots,\lambda_t-1,1,\lambda_{t+1},\ldots)} z^{m-1}. \quad (3.2)$$

Note that in the sum (3.2), only one term survives by the property (2.6) of $s_\lambda$. A direct check on the basis elements $\{s_\lambda z^m\}$ immediately proves the following proposition:

**Proposition 3.1.** (1) The action of $v_k^\pm$ satisfies the commutation relations of the Clifford algebra of fermions:

$$v_k^+ v_l^+ + v_l^+ v_k^+ = 0, \quad v_k^- v_l^- + v_l^- v_k^- = 0, \quad v_k^+ v_l^- + v_l^- v_k^+ = \delta_{k,l}.$$

(2) Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l)$ be a partition $\lambda \vdash |\lambda|$, and let $\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k)$ be the conjugate of the partition $\lambda$. Then

$$v_{\lambda_1+1}^+ \cdots v_{\lambda_2+2}^+ v_{\lambda_1+1}^+ (1) = s_\lambda z^l,$$

$$v_{-\lambda_1-1}^- \cdots v_{-\lambda_{l-1}-1}^- v_{-\lambda_l}^- (1) = (-1)^{|\lambda|} s_\mu z^{-l}.$$

**Remark 3.1.** Following [15], where the first formula in (2) was stated for classical symmetric functions, presentations of this kind are called sometimes “Bernstein (vertex) operators presentations”. The second formula in (2) can be found in [4].
Our next goal is to combine the operators \( v_k^\pm \) into generating functions and write them in the form of “vertex operators”. Let
\[
\Gamma^+(u, m) = \sum_{k \in \mathbb{Z}} v_k^+ |_{B^m u^k}, \quad \Gamma^-(u, m) = \sum_{k \in \mathbb{Z}} v_k^- |_{B^m u^{-k}}.
\] (3.3)

**Vertex operators for classical symmetric functions.** First, let us review the combinatorial description of operators (3.3) in the classical case. Recall (see e.g. [10]) that the ring of symmetric functions possess a scalar product with classical Schur functions \( s_\lambda \) as an orthonormal basis: \( \langle s_\lambda, s_\mu \rangle = \delta_{\lambda, \mu} \). Then for any symmetric function \( f \) one can define an adjoint operator acting on the ring of symmetric functions by the standard rule:
\[
\langle Df g, w \rangle = \langle g, fw \rangle
\]
for any symmetric functions \( g, f, w \). Using notation \( D\lambda = Ds_\lambda \), one gets
\[
\langle D\lambda s_\mu, s_\nu \rangle = \langle s_\mu, s_\lambda s_\nu \rangle,
\]
hence
\[
D\lambda s_\mu = s_{\mu/\lambda},
\]
where \( s_{\mu/\lambda} \) is the skew-Schur function of shape \( \mu/\lambda \). This symmetric function can be expressed as a determinant
\[
s_{\mu/\lambda} = \det[h_{\mu_i - \lambda_j - i + j}]_{1 \leq i, j \leq n}.
\]
Introduce generating functions for \( e_k, h_k \) and corresponding adjoint operators:
\[
E(u) = \sum_{k \geq 0} e_k u^k, \quad H(u) = \sum_{k \geq 0} h_k u^k,
\]
\[
DE(u) = \sum_{k \geq 0} D_e u^k, \quad DH(u) = \sum_{k \geq 0} D_h u^k.
\]
Then it is known that for classical symmetric functions,
\[
\Gamma^+(u, m) = u^{m+1} z H(u) DE \left( \frac{-1}{u} \right),
\] (3.4)
\[
\Gamma^-(u, m) = u^{-m} z^{-1} E(-u) DH \left( \frac{1}{u} \right).
\] (3.5)

Note that very often these formulas are written through power sums (see e.g. [6], Lecture 5). Namely, introduce the (normalized) classical power sums, which are symmetric functions of the form
\[
p_k = \frac{1}{k} \sum_i x_i^k.
\] (3.6)
Generating function \( P(u) = \sum_{k \geq 1} p_k u^{k-1} \) satisfies the relation \( H(u) = \exp(\sum_{k \geq 1} p_k u^k) \). We also get \( E(u) = \exp(-\sum_{k \geq 1} p_k (-u)^k) \), since \( H(u) E(-u) = 1 \).

Any symmetric function \( f \) can be expressed as a polynomial \( f = \varphi(p_1, 2p_2, 3p_3, \ldots) \) in (normalized) power sums. Then one has \( Df = \varphi(\partial_{p_1}, \partial_{p_2}, \partial_{p_3}, \ldots) \). (See e.g. [10], Example 1.3.) Hence,
\[
DH(u) = \exp\left(\sum_k \frac{\partial p_k}{k} u^k\right), \quad DE(u) = \exp\left(-\sum_k \frac{\partial p_k}{k} (-u)^k\right),
\]
and we can write

\[ \Gamma^+(u, m) = u^{m+1} z \exp \left( \sum_{j \geq 1} p_j u^j \right) \exp \left( - \sum_{j \geq 1} \frac{\partial_p u}{j} u^{-j} \right), \]

(3.7)

\[ \Gamma^-(u, m) = u^{-m} z^{-1} \exp \left( - \sum_{j \geq 1} p_j u^j \right) \exp \left( \sum_{j \geq 1} \frac{\partial_p u}{j} u^{-j} \right). \]

(3.8)

**Operators** $v_k^\pm$ **expressed through** $D_p$ **and** $D^{(p)}$. Our next goal is to express operators $v_k^\pm$ through analogues of $e_k^{(p)}$, $h_k^{(p)}$, $D_{ek}$, $D_{hk}$. In some examples we will be able to go further and write nice formulas in the spirit of (3.4) for generating functions $\Gamma^\pm(u, m)$.

We introduce the following two operators on $B$. Define

\[ D_p(s_{\lambda_1, \ldots, \lambda_l}) := \begin{cases} 
\det[h_{\lambda_i-1}^{(j)}(j=0,1,\ldots,l, i=1,\ldots,l)], & \text{if } 0 \leq p \leq l, \\
0, & \text{otherwise}. 
\end{cases} \]

(3.9)

The following interpretation of this definition will be useful. For $0 \leq p \leq l$, one can write

\[ D_p(s_{\lambda_1, \ldots, \lambda_l}) = \det \begin{pmatrix}
\begin{array}{cccc}
  h_{\lambda_1-1}^{(0)} & \cdots & h_{\lambda_1-1}^{(p)} & \cdots & h_{\lambda_1-1}^{(l)} \\
  h_{\lambda_2-2}^{(0)} & \cdots & h_{\lambda_2-2}^{(p)} & \cdots & h_{\lambda_2-2}^{(l)} \\
  \vdots & \cdots & \vdots & \cdots & \vdots \\
  h_{\lambda_i-1}^{(0)} & \cdots & h_{\lambda_i-1}^{(p)} & \cdots & h_{\lambda_i-1}^{(l)} \\
  h_{\lambda_{l-1}}^{(0)} & \cdots & h_{\lambda_{l-1}}^{(p)} & \cdots & h_{\lambda_{l-1}}^{(l)} \\
\end{array}
\end{pmatrix} .
\]

(3.10)

The matrix above is obtained from the matrix $S_{\lambda}$ by subtracting 1 from all the lower indices in the entries of $S_{\lambda}$ and deleting the $p$-th column. Note that the resulting matrix has the form

\[ \begin{pmatrix} 
A & B \\
0 & D
\end{pmatrix}, \]

where $A$ is $(l - 1) \times (l - 1)$ matrix, and $D$ is upper-triangular matrix with 1’s on the diagonal. Note that this interpretation naturally extends to the case $p > l$, where $D$ block would have several zero’s on the diagonal. We can define a sequence of polynomials in $h_k^{(a)}$ – the determinants of $N \times N$ left upper part of that matrix. This sequence stabilizes for big enough $N$, and we conclude that presented by (3.10) determinant $D_p(s_{\lambda_1, \ldots, \lambda_l})$ is a well-defined polynomial in $h_k^{(a)}$. Note that $D_0(1) = 1$ and $D_p(1) = 0$ for $p \neq 0$. Operator $D_p$ is the combinatorial analogue of the classical operator $D_{ep}$.

Define

\[ D^{(p)}(s_{\lambda_1, \ldots, \lambda_l}) = \sum_{l=1}^{\infty} (-1)^{l+1} h_{\lambda_l-1}^{(p)} s^{(\lambda_1, \ldots, \lambda_{l-1}+1, \lambda_{l+1}, \ldots, \lambda_l, 0, \ldots)}. \]

(3.11)
This also can be written as

\[
D^{(p)}(s_{\lambda_1, \ldots, \lambda_l}) := \det\begin{pmatrix}
  h_{\lambda_1+1}^{(p)} & h_{\lambda_1+1}^{(0)} & \ldots & h_{\lambda_1+1}^{(l-1)} & \ldots \\
  h_{\lambda_2}^{(p)} & h_{\lambda_2}^{(0)} & \ldots & h_{\lambda_2}^{(l-1)} & \ldots \\
  \vdots & \vdots & \ddots & \vdots & \ddots \\
  h_{\lambda_{l-i+2}}^{(p)} & h_{\lambda_{l-i+2}}^{(0)} & \ldots & h_{\lambda_{l-i+2}}^{(l-1)} & \ldots \\
  h_{-l}^{(p)} & h_{-l}^{(0)} & \ldots & h_{-l}^{(l-1)} & \ldots \\
  \vdots & \vdots & \ddots & \vdots & \ddots 
\end{pmatrix}.
\]

The matrix above is obtained from the matrix \( S_{\lambda} \) by raising all the lower indices of its elements by 1 and by adding the first column with entries \((h_{\lambda_{l-i+2}}^{(p)})_{i=1,2,\ldots} \). Then this matrix has the form

\[
\begin{pmatrix}
  A & B \\
  C & D
\end{pmatrix},
\]

where \( A \) is \((l+1)\) by \((l+1)\) matrix, \( C \) is a matrix with all zero-entires except (may be) the first column, and \( D \) is upper-triangular matrix with 1’s on the diagonal. Note that \( h_{p+2}^{(p)} = 0 \) for \(-i+2+p < 0\), and therefore, the sequence of determinants of \( N \times N \) upper left corner submatrices of this matrix stabilizes to a well-defined polynomial in variables \( h_{\lambda i}^{(k)} \). Hence \( D^{(p)} \) is a well-defined operator on \( B \).

Note that \( D^{(p)} \equiv 0 \) for \( p \geq 0 \), since in this case the defining matrix in \( D^{(p)}(s_{\lambda_1, \ldots, \lambda_l}) \) will have two identical columns. Also note that \( D^{(p)}(s_{\lambda_1, \ldots, \lambda_l}) = 0 \) if \(-p > \lambda_1 + 1\), \( D^{(-1)}(1) = 1 \), and \( D^{(p)}(1) = 0 \) for \( p \neq -1 \). Thus, \( D^{(-p)} \) is the analogue of \( D_{h_{p-1}} \).

Proposition 3.2.

\[
v_k^+ |_{B^m} = \sum_{p \in \mathbb{Z}} (-1)^p h_{k-m-1}^{(p)} D_p z, \quad (3.12)
\]

\[
v_k^- |_{B^m} = (-1)^{k-m+1} \sum_{p \in \mathbb{Z}} (-1)^p h_{k-m+1}^{(p)} D^{(-p)} z^{p-1}. \quad (3.13)
\]

Proof. We compare the action of operators on both sides on the basis elements \( s_\lambda z^m \). For the proof of \( (3.12) \) use the expansion of the determinant \( s_{(k-m-1, \lambda)} \) by the first row:

\[
v_k^+(s_\lambda z^m) = s_{(k-m-1, \lambda)} z^{m+1} = \sum_{p=0}^l (-1)^p h_{k-m-1}^{(p)} \det[h_{\lambda_{i-1}}^{(j)}]_{j=0,1,\ldots,l,i=1,\ldots,l} z^{m+1}
\]

\[
= \left( \sum_{p \in \mathbb{Z}} (-1)^p h_{k-m-1}^{(p)} D_p z \right) s_\lambda z^m.
\]

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For the proof of (3.13) we apply Newton’s formula:

\[
\left(\sum_{p \in \mathbb{Z}} (-1)^p k^{m+1} e_k^{(p)} D^{(-p)} z \right) s_\lambda z^m
\]

\[
= \sum_{p \in \mathbb{Z}} (-1)^{p+k-m+1} e_k^{(p)} \sum_{t=1}^{\infty} (-1)^{t+1} h_{(t+1)}^{(p)} \frac{s(\lambda_1, \lambda_2, \ldots, \lambda_m)}{z^{m-1}}
\]

\[
= \sum_{t=1}^{\infty} (-1)^{t+1} \frac{\sum_{p \in \mathbb{Z}} (-1)^{p+k-m+1} e_k^{(p)} \frac{s(\lambda_1, \lambda_2, \ldots, \lambda_m)}{z^{m-1}}}{z^{m-1}}
\]

\[
= \sum_{t=1}^{\infty} (-1)^{t+1} \delta_{k-m+1, \lambda_t+1} s(\lambda_1, \lambda_2, \ldots, \lambda_m) z^{m-1} = v_k^{-}(s_\lambda z^m).
\]

\[
\square
\]

4. Applications - Explicit formulas for vertex operators

Characters of classical Lie algebras. Let \( g \) be a symplectic or orthogonal Lie algebra. In [7], it is shown that, similarly to Schur functions, universal characters \( \chi_\lambda(a) \) of irreducible \( g \)-representations form linear bases of the ring of ordinary symmetric functions. For \( a > 0 \) denote as \( J_a \) the universal character of irreducible \( g \)-representation with the highest weight \( a \omega_1 \), where \( \omega_1 \) is the fundamental weight. Set \( J_0 = 1 \), and \( J_a = 0 \) if \( a < 0 \). Several forms of Jacobi-Trudi identities for characters of irreducible representations of classical Lie algebras are known (e.g. [3], [7]). Here we would like to use the identities of [3] (Propositions 24.22, 24.33, 24.44):

**Proposition 4.1.** Let \( \lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 0) \) be a partition. Then the universal character \( \chi_\lambda(a) \) of irreducible \( g \)-representation with the highest weight \( \lambda \) is given by

\[
\chi_\lambda(a) = \det [h_{(i)}^{(j)}]_{i,j=1 \ldots r},
\]

\[
h_a^{(r)} = \begin{cases} J_{a+r} + J_{a-r}, & \text{if } r > 0, \\ J_{a+r}, & \text{if } r \leq 0. \end{cases} \tag{4.1}
\]

Hence we consider the boson space with \( \{h_a^{(r)}\} \) as in (4.1), and the linear basis consisting of elements \( s_\lambda = \chi_\lambda(\lambda) \).

**Proposition 4.2.** Generating functions (3.13) are given by

\[
\Gamma^+(u, m) = u^{m+1} z J(u)(DE(-u) + DE(-1/u) - D_0),
\]

\[
\Gamma^-(u, m) = u^{-m+1} z^{-1} J(u)^{-1}(DH(u) - DH(1/u)),
\]

where \( J(u) = \sum_{s \in \mathbb{Z}} J_s u^s \), and \( DE(u) = \sum_{p \in \mathbb{Z}} D_p u^p \), \( DH(u) = \sum_{p \in \mathbb{Z}} D(p) u^p \), with operators \( D_p \) and \( D^{(p)} \) defined by (3.12), (3.11).
Proof. Using that $D_p \equiv 0$ for $p < 0$, we derive
\[
\Gamma^+(u, m) = u^{m+1} z \sum_{s \in \mathbb{Z}} \sum_{p \geq 0} (-1)^p h_s^{(p)} D_p u^s
\]
\[
= u^{m+1} z \sum_{s \in \mathbb{Z}} \left( \sum_{p \geq 0} (-1)^p (J_{s+p} + J_{s-p})D_p - J_s D_0 \right) u^s
\]
\[
= u^{m+1} z J(u)(DE(-1/u) + DE(-u) - D_0).
\]
Let $K(-u) = \sum_{p \in \mathbb{Z}} K_p(-u)^p$ be a formal series defined by the property
\[
J(u)K(-u) = 1. \tag{4.2}
\]
As usual, (4.2) is equivalent to the matrix form relation $JK = Id$, where $J = \sum_{i,j \in \mathbb{Z}} J_{i+j} E_{ij}$, and $K = \sum_{i,j \in \mathbb{Z}} (-1)^{i-j} K_{j-i} E_{ij}$. Define the the matrix $A = \sum_{i \in \mathbb{Z}} E_{ii} + \sum_{i > 0} E_{-i}$ and the inverse $A^{-1} = \sum_{i \in \mathbb{Z}} E_{ii} - \sum_{i > 0} E_{-i}$. Then the matrix $H$ in (2.10) for $h_a^{(r)}$ from (1.1) is given by $H = JA$, and, since $HE = Id$, we have $E = A^{-1}K$ and
\[
e_{j}^{(i)} = \begin{cases} \frac{K_{i-j}}{E_{ij}} & \text{for } i \leq 0, \\ K_{i-j} - K_{-i-j} & \text{for } i > 0. \end{cases}
\]
Using that $D^{(-p)} \equiv 0$ for $p \leq 0$, we derive
\[
\Gamma^-(u, m) = u^{-m-1} z^{-1} \sum_{s \in \mathbb{Z}} \sum_{p > 0} (-1)^p e_s^{(p)} (-u)^{-s} D^{(-p)}
\]
\[
= u^{-m-1} z^{-1} \sum_{s \in \mathbb{Z}} \sum_{p > 0} (-1)^p (K_{p-s} - K_{-p-s}) (-u)^{-s} D^{(-p)}
\]
\[
= u^{-m-1} z^{-1} K(-u) \sum_{p > 0} (-1)^p ((-u)^{-p} - (-u)^p) D^{(-p)}
\]
\[
= u^{-m-1} z^{-1} K(-u)(DH(u) - DH(1/u)).
\]
\[\square\]

Remark 4.1. Relation (4.2) implies that, together with $K_0 = 1$, and $K_p = 0$ for $p < 0$, $K_p = \det [J_{1+i+j}]_{1 \leq i,j \leq p}$, for $p > 0$, and $\chi_{g}((1^p)) = K_p - K_{p-2}$.

Remark 4.2. It is known that characters of the classical Lie algebras can be expressed through evaluations of classical symmetric functions, where the following identifications take place:
\[
J_a = h_a(x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}) - h_{a-2}(x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}) \quad \text{for } g = \mathfrak{o}_{2n},
\]
\[
J_a = h_a(x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}, 1) - h_{a-2}(x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}, 1) \quad \text{for } g = \mathfrak{o}_{2n+1},
\]
\[
J_a = h_a(x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}) \quad \text{for } g = \mathfrak{sp}_{2n},
\]
where $h_k$ are the ordinary complete symmetric functions (2.1). Accordingly, formulas of Proposition 4.2 can be rewritten in terms of the power sums (3.6), and then one can compare Proposition 4.2 with the vertex operators introduced in [5]. In [5] the opposite direction is undertaken: vertex operators that give realization of characters of classical Lie algebras are introduced through their action on the boson space. The operators satisfy generalized fermions relations, this allows the authors of [5] to deduce several versions of Jacobi-Trudi identities for characters of the orthogonal and symplectic Lie algebras.
**Shifted Schur functions.** We follow the notation and definitions in [12], to which we refer the reader for details. Let \( s^*_\lambda(x_1, \ldots, x_n) \) denote a shifted Schur polynomial:

\[
s^*_\lambda(x_1, \ldots, x_n) = \frac{\det(x_i + n - i|\lambda_j + n - j)}{\det(x_i + n - i|n - j)},
\]

(4.3)

where

\[
(x|k) = \begin{cases} 
  x(x-1)\ldots(x-k+1) & \text{for } k = 1, 2, \ldots, \\
  1 & \text{for } k = 0, \\
  \frac{1}{(x+1)\ldots(x+(-k))} & \text{for } k = -1, -2, \ldots.
\end{cases}
\]

Note that for generic value of \( x \) we can write \((x|k) = \frac{\Gamma(x+1)}{\Gamma(x+1-k)}\), where \( \Gamma(x) \) is the special gamma-function. Also the following relations are useful in further calculations:

\[
(x|k) (-x - 1| - k) = (-1)^k, \quad (x|a) = (x|b)(x - b|a - b).
\]

(4.4)

The stability property of shifted Schur polynomials allows us to introduce the shifted Schur functions, which we denote as \( s^*_\lambda = s^*_\lambda(x_1, x_2, \ldots) \). In particular, the complete shifted Schur functions \( h^*_r = s^*_1(x) \) are

\[
h^*_r(x_1, x_2, \ldots) = \sum_{1 \leq i_1 < \cdots < i_r < \infty} (x_{i_1} - r + 1)(x_{i_2} - r + 2)\ldots x_{i_r},
\]

and the elementary shifted Schur functions \( e^*_r = s^*_{(1^r)} \) are

\[
e^*_r(x_1, x_2, \ldots) = \sum_{1 \leq i_1 < \cdots < i_r < \infty} (x_{i_1} + r - 1)(x_{i_2} + r - 2)\ldots x_{i_r}.
\]

Theorem 13.1 in [12] states that

\[
s^*_\lambda = \det[\phi^{j-1}h^*_\lambda_{i-j}|1 \leq i, j \leq t],
\]

where \( \phi \) is the automorphism of the algebra of shifted Schur functions, defined by the formula

\[
\phi(h^*_k) = h^*_k + (k - 1)h^*_k-1.
\]

By Corollay 1.6. in [12], shifted Schur functions \( s^*_\lambda \) form a linear basis in the ring of shifted symmetric functions, which is also a polynomial ring in variables \( h^*_1, h^*_2, \ldots \). Thus, we are exactly in the setting of the Section 2 with \( s^*_\lambda = \det[h^*_{\lambda_{i-j}}]_{1 \leq i, j \leq t} \) and

\[
h^{(r)}_k = \phi^r h^*_{k+r}.
\]

In this case our Definition 2.1 gives for \( p > a \),

\[
e_{(p)} = \det[\phi^{-p+j}h^*_{1-i+j}|1 \leq i, j \leq p-a].
\]

(4.5)

Note that \( e_k = e_{1-k}^{(1)} \). Moreover,

\[
\phi^{-p+1}(e^*_k) = \det[\phi^{-p+1}h^*_{1-i+j}|1 \leq i, j \leq k] = \det[\phi^{-p+j}h^*_{1-i+j}|1 \leq i, j \leq k] = e^{(p)}_{p-k}.
\]

Traditionally, for shifted Schur functions generating functions are written not in powers \( u^k \), but in shifted powers \( (u|k) \), which makes some formulas easier to work with (see e.g.
Using that \( \sum_{k \in \mathbb{Z}} v_k^+ |_{B_m} u^k \). Let us consider both cases: as before, we define \( \Gamma^\pm(u, m) \) by (3.3): \( \Gamma^\pm(u, m) = \sum_{k \in \mathbb{Z}} v_k^+ |_{B_m} u^k \). Alternatively, set

\[
\Gamma^+_s(u, m) = \sum_{k \in \mathbb{Z}} v_k^+ |_{B_m} \frac{1}{(u|k - 1)}, \quad \Gamma^-_s(u, m) = \sum_{k \in \mathbb{Z}} v_k^- |_{B_m} (-u|k + 1).
\]

**Proposition 4.3.** Generating functions have the form

\[
\Gamma^+(1/u, m) = zu^{-m-1} \sum_{p \geq 0} (\partial_u - 1)^p (H(1/u)) D_p u^p,
\]

\[
\Gamma^-(1/u, m) = z^{-1} u^m \sum_{p \geq 0} (\partial_u + 1)^p (E(-1/u)) D^{(-p-1)} u^p,
\]

\[
\Gamma^+_s(u, m) = z \frac{1}{(u|m)} H^*(u - m) DE^*(u - m),
\]

\[
\Gamma^-_s(u, m) = z^{-1} (-u|m) E^*(u + m) DH^*(u + m - 1),
\]

where

\[
E(u) = \sum_{r \in \mathbb{Z}} e_r u^r, \quad H(u) = \sum_{r \in \mathbb{Z}} h_r u^r, \quad E^*(u) = \sum_{r \in \mathbb{Z}} e_r^* u^r, \quad H^*(u) = \sum_{r \in \mathbb{Z}} h_r^* u^r,
\]

\[
DE^*(u) = \sum_{p \in \mathbb{Z}} D_p \frac{(-1)^p}{(u| - p)}, \quad DH^*(u) = \sum_{p \in \mathbb{Z}} D(p) \frac{(-1)^p}{(u|p)}.
\]

**Proof.** For the proof of the first equality write:

\[
\Gamma^+(1/u, m) = u^{-m-1} z \sum_{s \in \mathbb{Z}} \left( \sum_{p \geq 0} (-1)^p \phi^p h^*_s D_p \right) u^s
\]

\[
= u^{-m-1} z \sum_{p \geq 0} \left( \sum_{i=0}^{p} \sum_{s \in \mathbb{Z}} \left( \begin{array}{c} p \\ i \end{array} \right) (s + p - 1|i) h^*_{s+p-i} u^{-s} \right) (-1)^p D_p
\]

\[
= u^{-m-1} z \sum_{p \geq 0} \left( \sum_{r \in \mathbb{Z}} \sum_{i=0}^{p} \left( \begin{array}{c} p \\ i \end{array} \right) (r + i - 1|i) h^r u^{-r+i} \right) (-1)^p D_p
\]

\[
= u^{-m-1} z \sum_{r \in \mathbb{Z}} h^r \sum_{p \geq 0} \left( \sum_{i=0}^{p} \left( \begin{array}{c} p \\ i \end{array} \right) (r + i - 1|i) u^{-r-i} \right) (-1)^p D_p u^p.
\]

Using that \( (-\partial_u)^i(u^{-r}) = (r + i - 1|i) u^{-r-i} \), we obtain

\[
\sum_{i=0}^{p} \left( \begin{array}{c} p \\ i \end{array} \right) (r + i - 1|i) u^{-r-i} = \left( \sum_{i=0}^{p} \left( \begin{array}{c} p \\ i \end{array} \right) (-\partial_u)^i \right) (u^{-r}) = (1 - \partial_u)^p(u^{-r}),
\]

and

\[
\Gamma^+(1/u, m) = u^{-m-1} z \left( 1 - \partial_u \right)^p \left( \sum_{r \in \mathbb{Z}} h^r u^{-r} \right) (-1)^p D_p u^p = u^{-m-1} z \sum_{p \geq 0} (\partial_u - 1)^p (H(1/u)) D_p u^p.
\]
Similarly,

\[
\Gamma^{-}(1/u, m) = u^{-1} \sum_{s \in \mathbb{Z}} \left( \sum_{p > 0} (-1)^p \phi^{-p+1} e_{p-s} (-u)^s D(-p) \right) \\
= u^{-1} \sum_{r \in \mathbb{Z}} e_r \sum_{p > 0} \left( \sum_{i=0}^{p-1} \binom{p-1}{i} (r+i-1|i)(-u)^{-r-i} \right) D(-p) u^p \\
= u^{-1} \sum_{p \geq 0} (1 + \partial_u)^{p-1} \left( \sum_{r \in \mathbb{Z}} e_r (-u)^{-r} \right) D(-p) u^p
\]

For the proof of the third equality write, using (4.4),

\[
\Gamma^{+}(u, m) = z \frac{1}{(u|m)} \sum_{k \in \mathbb{Z}} \left( \sum_{p \geq 0} (-1)^p h_{k-m-1}^{(p)} D_p \right) \frac{1}{(u-m|k-m-1)} \\
= z \frac{1}{(u|m)} \sum_{s \in \mathbb{Z}} \left( \sum_{p \geq 0} (-1)^p \phi^p h_{s+p}^* D_p \right) \frac{1}{(u-m|s)} \\
= z \frac{1}{(u|m)} \sum_{p \geq 0} \sum_{s \in \mathbb{Z}} \sum_{i=0}^{p} \binom{p}{i} (s+p-1|i) h_{s+p-i}^* \frac{1}{(u-m|s)} (-1)^p D_p \\
= z \frac{1}{(u|m)} \sum_{r \in \mathbb{Z}} \sum_{p \geq 0} h_r^* \sum_{i=0}^{p} \binom{p}{i} \frac{(r+i-1|i)}{(u-m|r-p+i)} (-1)^p D_p.
\]

Using that for \( v \notin \mathbb{Z}_{\leq 0} \) and \( r, t \in \mathbb{Z}, p \in \mathbb{Z}_{>0} \),

\[
\sum_{k=0}^{p} \binom{p}{k} \frac{(r+k-1|k)}{(v|t-p+k)} = \frac{\Gamma(v-t+1)\Gamma(v-t+p+r+1)}{\Gamma(v-t+r+1)\Gamma(v+1)} = \frac{1}{(v-t+r|r)(v|t-p-r)},
\]

we derive the third statement:

\[
\Gamma^{+}(u, m) = z \frac{1}{(u|m)} \sum_{r \in \mathbb{Z}} \frac{h_r^*}{(u-m|r)} \sum_{p \geq 0} \frac{(-1)^p D_p}{(u-m|p)}.
\]
Similarly,
\[ \Gamma_+^-(u, m) = z^{-1}(-u|m) \sum_{k \in \mathbb{Z}} \sum_{p > 0} e_{k-m+1}^{(p)} (-1)^{k-m+1} (-u - m | k - m + 1) (-1)^p D^{(-p)} \]
\[ = z^{-1}(-u|m) \sum_{s \in \mathbb{Z}} \sum_{p > 0} \phi^{-p+1}(e_{s}^{*}) (-1)^s (-u - m | s) (-1)^p D^{(-p)} \]
\[ = z^{-1}(-u|m) \sum_{s \in \mathbb{Z}} \left( \sum_{p > 0} \sum_{i=0}^{p-1} \left( \frac{p - 1}{i} \right) \frac{(p - s - 1|i)}{(u + m - 1 - s)} e_{s-i}^{*}(-1)^p D^{(-p)} \right) \]
\[ = z^{-1}(-u|m) \sum_{r \in \mathbb{Z}} \left( \sum_{p > 0} \sum_{i=0}^{p-1} \left( \frac{r + i - 1|i}{u + m - 1 | r + i - p} \right) (-1)^p D^{(-p)} \right) \]
\[ = z^{-1}(-u|m) \sum_{r \in \mathbb{Z}} \frac{e_{r}^{*}(-1)^p D^{(-p)}}{(u + m | r)(u + m - 1 - p)} \]
\[ = z^{-1}(-u|m) \sum_{r \in \mathbb{Z}} E^*(u + m) DH^*(u + m - 1). \]

and the fourth statement is proved. \( \square \)

**Linear recurrence.** Assume that elements \( h_k^{(p)} \) of \( B \) are given by a linear recurrence relation
\[ h_k^{(p)} = \sum_i t_{ki} h_i^{(p-1)}, \quad t_{ki} \in \mathbb{C}, \quad k, i \in \mathbb{Z}. \quad (4.6) \]

Note that condition (3.1) implies that \( t_{k,k+1} = 1 \) and \( t_{km} = 0 \) for \( m > k + 1 \). We combine the coefficients into an infinite matrix \( T = \sum_{kl} t_{-k-l} E_{kl} \).

In general, a recursion of type (4.6) may not lead to a decomposition of generating functions \( \Gamma^\pm(u, m) \) as a product of generating functions \( H(u), DH(u), E(u), DE(u), \) etc. One can try to substitute the relation on generating functions by a relation on generating matrices and vectors.

Namely, let \( \mathcal{H} \) and \( \mathcal{E} \) be infinite generating matrices, defined as in (2.10). Let \( \mathcal{H}^{(p)} \) be the \( p \)-th column of \( \mathcal{H} \), and let \( \mathcal{E}^{(p)} \) be the the \((p)\)-th row of \( \mathcal{E} \). Note that \( \mathcal{H}^{(p)} = T \mathcal{H}^{(p-1)} \) and the property \( \mathcal{H} \mathcal{E} = \mathcal{E} \mathcal{H} = Id \) implies that \( \mathcal{E}^{(p)} T = \mathcal{E}^{(p-1)} \), and
\[ e_k^{(p)} = \sum_i (-1)^{k+i} e_i^{(p-1)} t_{ik}. \]

Define the generating vectors \( \Gamma^\pm(m) \) for \( v_k^\pm \) acting on \( B \) as follows:
\[ \Gamma^+(m) = \left( \ldots, v_{k+1}^+, v_k^+, v_{k-1}^+, \ldots \right), \quad \Gamma^-(m) = \left( \ldots, v_{k+1}, v_k^-, v_{k-1}^-, \ldots \right), \]

both vectors have \( v_k^\pm \big|_{B_m} \) entry on the \((k)\)th place. Denote as \( L \) the shift operator \( \sum_k E_{k,k+1} \). Then (3.13) and (3.12) can be expressed as a relation on vectors and matrices those entire
are operators on $\mathcal{B}$:

$$\Gamma^+(m) = zL^{m+1} \sum_{p \geq 0} (-T)^p \mathcal{H}^{(0)} D_p, \quad \Gamma^-(m) = z^{-1} L^{m-1} \sum_{p \geq 0} \mathcal{E}^{(0)} T^p D^{-p}.$$  

**Example 4.1.** For ordinary symmetric functions we have in this formulation $T = L^{-1}$.

**Example 4.2.** In [13] a Jacobi–Trudi identity is proved for a large class of generalized symmetric polynomials, where generalized Schur polynomials are given by the formula (2.4) with recurrence relation

$$h_k^{(p+1)} = h_k^{(p)} + a(k)h_k^{(p)} + b(k)h_{k-1}^{(p)},$$

which corresponds to the tridiagonal operator $T = \sum_k E_{k,k-1} + a(-k)E_{k,k} + b(-k)E_{k,k+1}$.

**Example 4.3.** Let the elements $h_k^{(p)}$ be defined as

$$h_k^{(p)} = \sum_{i=1}^l a_i h_k^{(p-1)}, \quad (4.7)$$

where $\{a_{-1} = 1, a_0, \ldots, a_l\}$ are fixed complex constants. This recursion corresponds to the sum of powers of a shift operator: $T = \sum_{i=1}^l a_i L^i$. Set $f(u) = \sum_{i=1}^l a_i u^i$.

**Proposition 4.4.** The recurrence relation (4.7) with the constants $\{a_{-1} = 1, a_0, \ldots, a_l\}$ gives the generating functions $\Gamma^+(u, m)$

$$\Gamma^+(u, m) = u^{m+1} z H(u) D E(-f(u)), \quad (4.8)$$

$$\Gamma^-(u, m) = u^{-m+1} z^{-1} E(-u^{-1}) D H(f(u)^{-1}), \quad (4.9)$$

where $H(u) = \sum_{k \in \mathbb{Z}} h_k^{(0)} u^k$, $E(u) = \sum_{k \in \mathbb{Z}} e_k^{(0)} u^k$, $D E(u) = \sum_{p \in \mathbb{Z}} D_p w^p$, $D H(u) = \sum_p D^{(p)} w^p$.  

**Proof.**

$$\Gamma^+(u, m) = u^{m+1} z \sum_{p \geq 0} (-1)^p \sum_{s \in \mathbb{Z}, 1 \leq i_1 \ldots i_p \leq l} a_{i_1} \ldots a_{i_p} h_{s-i_1 \ldots i_p}^{(0)} D^p u^s$$

$$= u^{m+1} z \sum_{t \in \mathbb{Z}} h_t^{(0)} u^t \sum_{p \geq 0} (-1)^p D^p$$

$$= u^{m+1} z \sum_{t \in \mathbb{Z}} h_t^{(0)} u^t \sum_{p \geq 0} (-1)^p D^p,$$

and (4.8) follows. Using that $e_k^{(p)} = \sum_{i=-1}^l (-1)^{i-1} a_i e_{k+i}^{(p-1)}$, the proof of the second equality follows the same lines. \hfill \Box

**Remark 4.3.** Recall that for ordinary symmetric functions, $h_k^{(p)} = h_{p+k}$, $e_k^{(p)} = e_{p-k}$, $D^{(-p)} = D_{h_{p-1}}$, $D_p = D_{e_p}$, and all together it is a particular case of Proposition 4.4 that corresponds to the sequence $\{a_{-1} = 1, a_1 = 0, \ldots, a_l = 0\}$. Thus, Proposition 4.4 is a simple alternative combinatorial way to deduce the formulas of classical vertex operators [3.4] [3.5] [3.7] [3.8].
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