Primes Appearing in the Tower Factorization of Integers

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Abstract

In this note, we introduce and discuss a new (and still very much open!) problem in elementary number theory. In particular, every number can be uniquely expressed as a ‘tower factorization’ into primes, and for q prime, we can let M(q) denote the set of all integers whose tower factorization contains the prime q. We then explore the limiting densities of these sets. Although we are able to obtain some results, there is still much more to be done.

1 Introduction

Everybody knows—or at least readily believes—that if you pick a positive integer at random, then the probability that it’s even is 1/2. But what happens if we relax the condition that the randomly selected number must be divisible by 2? For example, what if we choose to also allow numbers of the form 5^2 · 7^3, or 3 · 17^{52 · 12^{11}} · 31, or even 11^{(35^2 · 7^5)} · 23^{19} to be “divisible” by 2? Then in general, these numbers do not contain the prime 2 in their usual prime factorizations (perhaps not even as exponents!), but nonetheless, they all contain a 2 somewhere in a more refined factorization. We call this representation the tower factorization of a number, which can be defined as follows:

Let n ≥ 1 be any arbitrary integer. Then its tower factorization is recursively given by

(0) If n = 1, then its tower factorization is just 1.

(1) If n > 1, let n = p_1^{e_1} · · · p_k^{e_k} be its usual prime factorization. Then the tower factorization of n is given by

\[ n = p_1^{(f_1)} · · · p_k^{(f_k)} = \prod_{i=1}^{k} p_i^{(f_i)} , \]

where f_i is the tower factorization of e_i.

In other words, to obtain the tower factorization of n, we first write n into its prime factorization n = p_1^{e_1} · · · p_k^{e_k}. Then we factor each exponent e_i > 1 into its prime factorization e_i = p_{i,1}^{f_{i,1}} · · · p_{i,k_i}^{f_{i,k_i}}. We continue to factor each subsequent exponent of those prime factorizations until n is written in the form

\[ n = \prod_i p_i^{\left(\prod_{j=1}^{k_i} (\Pi_{e_{ij}})^{-1}\right)} , \]

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\[ \text{1This representation of a number was motivated by computational considerations as discussed in [?].} \]
for some prime numbers $p_i, p_{i,j}, \ldots$. Then by the fundamental theorem of arithmetic, this tower factorization exists and is unique up to reordering.

With this in place, we now pose the main question of this paper:

Let $q$ be any fixed prime number (e.g., $q = 2$). Then what is the probability that a randomly selected positive integer contains the prime $q$ in its tower factorization? To be perfectly precise, let $M(q)$ be the set of all positive integers whose tower factorization contains the prime $q$ at least once. Then we wish to determine the natural density, $d(q)$, of the set $M(q)$. That is to say, what is

$$d(q) := \lim_{N \to \infty} \frac{|M(q) \cap \{1, 2, \ldots, N\}|}{N},$$

and does such a density even exist? (Stated more colorfully, $d(q)$ is the probability that a positive integer ‘chosen at random’ contains the prime $q$ in its tower factorization.)

**Initial thoughts**

Before continuing to discuss the problem in its generality, it is worth-while to first try to develop some intuition for the smallest possible case—namely, $q = 2$. On the one hand, the set $M(2)$ contains every even number, so right away we see that if its density $d(2)$ exists, then it would be at least $1/2$. But as Table 1 illustrates, $M(2)$ also contains many odd numbers as well, and it is not immediately clear how much of a contribution these would have on the density of $M(2)$.

| Number, $n$ | Tower Factorization | Primes $q$ with $n \in M(q)$ |
|-------------|----------------------|--------------------------------|
| 1           | $1$                  | $\emptyset$                   |
| 144         | $2^{(2^2)} \cdot 3^2$ | $\{2, 3\}$                   |
| 625         | $5^{(2^3)}$          | $\{2, 5\}$                   |
| 3,378,7663  | $7 \cdot 13^{(2^3)}$ | $\{2, 3, 7, 13\}$            |
| 37,349 $\cdot$ 11,669,921,475 | $13^3 \cdot 17 \cdot 11^{(7^3 \cdot 5^{(3^2)}}$ | $\{2, 3, 5, 7, 11, 13, 17\}$ |

Table 1: Examples of tower factorizations and to which $M(q)$ each number belongs

On the other hand, it is easy to see that $M(2)$ does not contain any odd square-free integers. Therefore, since the odd square-free integers have density $(2/3) \cdot (6/\pi^2) = 4/\pi^2$, we have that if $d(2)$ exists, then it is bounded by $d(2) \leq 1 - 4/\pi^2 = 0.592716 \ldots$. Thus, if the limiting density of $M(2)$ exists, we intuitively find it would satisfy $0.5 \leq d(2) \leq 0.592716$.

Beyond this first analysis, the intuitive arguments become more difficult. For instance, does the density $d(2)$ exist, and if so is it closer to 0.5 or to 0.592716? Although— or perhaps because—these sort of questions are difficult to answer by ‘blind’ intuition, looking at computer data can be particularly insightful. Slapping together some naïve code, we readily compute the density $|M(2) \cap \{1, 2, \ldots, N\}|/N$ for various values $N$ as plotted in Figure 1.

Now that we can ‘see’ how the density of $M(2)$ in $\{1, 2, \ldots, N\}$ changes as $N$ grows larger and larger, we are able to come up with better guesses. For example, although at this point, we have virtually no rigorous results whatsoever, the experimental data suggests that the limit for $d(2)$ converges and that the first digits of $d(2)$ seem to be about 0.5773 \ldots, which leads to several tempting conjectures. Before continuing, we invite and encourage the reader to spend a moment or so to form an opinion on what they expect about this limiting density.

**Outline of paper**

It is first best to note that the arguments in this paper are all very elementary, and the authors are certainly not number theorists. Therefore, those bored with the presentation can take solace in that they may be able to surpass our results easily (which they are very encouraged to do!).

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2For later notational convenience, for all $q$, we set $1 \notin M(q)$. 
We begin in Section 2 by deriving a few formulas for the densities $d(q)$ [establishing their existence] with the result of Proposition 2 being particularly inviting. Then in Section 3, we use these formulas to get a few bounds on $d(q)$ and establish their asymptotics. We also settle some tempting conjectures about the value of $d(2)$. We conclude with a few obvious open questions, and we provide some numerically computed bounds for $d(q)$ in an appendix.

2 Representations for $d(q)$

As the name of this section might suggest, we will now show that for all primes $q$, the limiting density $d(q)$ is given by any one of several convergent formulas, and therefore it does in fact exist. However, as the existence of subsequent sections might suggest, these formulas are not in a “closed enough” form to just call it a day after their derivation!

The formulas for these limiting densities are found by what has now become a routine ‘probabilistic’ approach, and so for ease of reading, proofs are confined to their most important aspects. The argument has one ‘hand-wavy’ part (which we will be kind enough to point out), but we believe that this presentation is the most intuitive one, and we trust our readers will forgive us.

Theorem 1. Let $q$ be any arbitrary prime, and let $M(q)$ be as previously defined. Then the density $d(q)$ exists and is given by

$$d(q) = 1 - \left(1 - \frac{1}{q}\right) \prod_{p \neq q, \text{prime}} \left[1 - \left(1 - \frac{1}{p}\right) \sum_{m \in M(q)} \frac{1}{p^m}\right].$$

Moreover, if $M^c(q)$ is defined as $M^c(q) := \{z \in \{0, 1, \ldots\} : z \notin M(q)\}$, then we also have

$$d(q) = 1 - \left(1 - \frac{1}{q}\right) \prod_{p \neq q, \text{prime}} \left[1 - \left(1 - \frac{1}{p}\right) \sum_{m \in M^c(q)} \frac{1}{p^m}\right] = 1 - \left(\sum_{m \in M^c(q)} q^{-m}\right)^{-1} \prod_{p, \text{prime}} \left[1 - \left(1 - \frac{1}{p}\right) \sum_{m \in M^c(q)} \frac{1}{p^m}\right].$$

Proof. Let $n \in \{1, \ldots, N\}$ be chosen uniformly at random (we will let $N$ go to infinity). Then for all primes $p$ and all nonnegative integers $m$, let $D(p, m)$ be the event that $p^m$ exactly divides

\[3\text{Note that } \{0,1\} \subseteq M^c(q)\]
n (that is $p^m$ divides $n$, but $p^{m+1}$ does not). Then the probability that $n$ is in $M(q)$ is given by:

\[
\mathbb{P}(n \in M(q)) = \mathbb{P}[(q \text{ divides } n) \cup (p^m \text{ exactly divides } n, \text{ where } p \neq q, \text{ and } m \in M(q))] = \mathbb{P}
\left[
(q \text{ divides } n) \cup \left(\bigcup_{p \neq q, \text{ prime}} \bigcup_{m \in M(q)} D(p, m)\right)\right].
\]

These events are not disjoint, which makes this probability difficult to deal with. However, by considering $\mathbb{P}(n \notin M(q)) = 1 - \mathbb{P}(n \in M(q))$, we have the simplification

\[
1 - \mathbb{P}(n \in M(q)) = \mathbb{P}\left[
(q \text{ does not divide } n) \cap \left(\bigcap_{p \neq q, \text{ prime}} \left(\bigcup_{m \in M(q)} D(p, m)\right)\right)\right].
\]

As $N$ goes to infinity, we may treat these events as independent. Thus, as $N \to \infty$ we have

\[
1 - d(q) = \lim_{N \to \infty} \mathbb{P}(q \text{ does not divide } n) \cdot \mathbb{P}\left[
\bigcap_{p \neq q, \text{ prime}} \left(\bigcup_{m \in M(q)} D(p, m)\right)\right].
\]

For any fixed $p$, the events $D(p, m)$ are clearly disjoint. Finally, as $N$ goes to infinity, we have

\[
\lim_{N \to \infty} \mathbb{P}(D(p, m)) = \lim_{N \to \infty} \mathbb{P}(p^m \text{ divides } n) \cdot \mathbb{P}(p^{m+1} \text{ does not divide } n \text{ given } p^m \text{ does})
\]

Thus, our formula simplifies to

\[
d(q) = 1 - \lim_{N \to \infty} \mathbb{P}(q \text{ does not divide } n) \cdot \prod_{p \neq q, \text{ prime}} \left[1 - \sum_{m \in M(q)} \left(\frac{1}{p^m}\right) \cdot \left(1 - \frac{1}{p}\right)\right]
\]

On the other hand, since we have

\[
1 - \mathbb{P}\left(\bigcup_{0 \leq m \in M(q)} D(p, m)\right) = \mathbb{P}\left(\bigcup_{0 \leq m \notin M(q)} D(p, m)\right),
\]

then if we define $M^c(q) = \{z \in \{0, 1, \ldots\} : z \notin M(q)\}$ (with $\{0, 1\} \subseteq M^c(q)$), we have

\[
d(q) = 1 - \lim_{N \to \infty} \mathbb{P}(q \text{ does not divide } n) \cdot \prod_{p \neq q, \text{ prime}} \left[\mathbb{P}\left(\bigcup_{0 \leq m \notin M(q)} D(p, m)\right)\right] \]

\[
= 1 - \lim_{N \to \infty} \mathbb{P}(q \text{ does not divide } n) \cdot \prod_{p \neq q, \text{ prime}} \left[\sum_{m \in M^c(q)} \left(\frac{1}{p^m}\right) \cdot \left(1 - \frac{1}{p}\right)\right]
\]

\[
= 1 - \left(\frac{1}{q}\right) \cdot \prod_{p \neq q, \text{ prime}} \left[\left(1 - \frac{1}{p}\right) \sum_{m \in M^c(q)} \frac{1}{p^m}\right],
\]

as desired.  

\[\square\]

This is the aforementioned ‘hand-wavy’ part, but it is neither particularly difficult nor enlightening to make this step rigorous. In fact, this would be a good exercise for any undergraduate students reading this. Note that in going to the infinite product, attention needs to be given to show that the error terms do not accumulate.
Having obtained this, we are then able to get very concise and beautiful representations for these densities such as the following.

**Proposition 2.** If \( I(n) \) is the indicator function for the event \( n \in M^c(q) \), then we have

\[
1 - d(q) = \prod_{p \text{ prime}} \left( 1 - \frac{1}{p} \right) \frac{1}{\sum \frac{I(p^m)}{p^m}} .
\]

**Proof.** Note that \( I(n) \) can be defined as the multiplicative function such that for all \( n \geq 1 \),

\[
I(p^n) = I(n) \cdot I(p),
\]

where \( I(p) \) coincides with the indicator function for \( p \neq q \). Thus the last theorem can be rewritten

\[
1 - d(q) = \left( 1 - \frac{1}{q} \right) \prod_{p \neq q, \text{ prime}} \left( 1 - \frac{1}{p} \right) \frac{1}{\sum \frac{I(p^m)}{p^m}} = \prod_{p \text{ prime}} \left( 1 - \frac{1}{p} \right) \frac{1}{\sum \frac{I(p^m)}{p^m}} ,
\]

as desired. □

Because \( I(m) \) as defined is a multiplicative function that behaves so simply on the primes, this above representation will likely lead the analytically inclined reader to consider the function \( f(s) := \sum_{m \geq 1} I(m) m^{-s} \) [and perhaps \( f(s)/\zeta(s) \)]. Many promising things can be said and done with this function, and the authors believe that this approach should be quite revealing. However, as the authors are neither number theorists nor analysts, we personally were unable to exploit this formula for anything truly useful (though not for want of effort).

### 3 Bounds and asymptotics

Armed with the formulas of Theorem\[3\] we dive into some bounds, which ultimately lead us to a very good asymptotic understanding of \( d(q) \). The first of these results follows so readily from Theorem\[3\] that it needs no proof.

**Theorem 3.** For each prime \( p \neq q \), let \( S(p) \), \( T(p) \), \( A(p) \), and \( B(p) \) be integer subsets such that \( S(p) \subseteq M(q) \subseteq T(p) \) and \( A(p) \subseteq M^c(q) \subseteq B(p) \). Then we have

\[
d(q) \geq 1 - \left( 1 - \frac{1}{q} \right) \prod_{p \neq q, \text{ prime}} \left( 1 - \frac{1}{p} \right) \sum_{s \in S(p)} \frac{1}{p^s} ,
\]

\[
d(q) \leq 1 - \left( 1 - \frac{1}{q} \right) \prod_{p \neq q, \text{ prime}} \left( 1 - \frac{1}{p} \right) \sum_{t \in T(p)} \frac{1}{p^t} ,
\]

\[
d(q) \leq 1 - \left( 1 - \frac{1}{q} \right) \prod_{p \neq q, \text{ prime}} \left( 1 - \frac{1}{p} \right) \sum_{a \in A(p)} \frac{1}{p^a} ,
\]

\[
d(q) \geq 1 - \left( 1 - \frac{1}{q} \right) \prod_{p \neq q, \text{ prime}} \left( 1 - \frac{1}{p} \right) \sum_{b \in B(p)} \frac{1}{p^b} .
\]

After a little thought, we are able to use this result to obtain the following easier bounds. These are more useful because they involve only finite sums and products, and yet in the limit they still squeeze together.

\[5\]In fact, those unhappy with our previous proof could directly prove this result instead and use it to derive our previous formulas.
Proposition 4. Let $P$ be any set of primes with $q \notin P$, let $S \subseteq M(q)$, let $A \subseteq M^c(q) \subseteq B$. Then we have

\[
d(q) \geq 1 - \left(1 - \frac{1}{q}\right) \cdot \prod_{p \in P} \left[1 - \left(1 - \frac{1}{p}\right) \sum_{s \in S} \frac{1}{p^s}\right],
\]

\[
d(q) \geq 1 - \left(1 - \frac{1}{q}\right) \cdot \prod_{p \in P} \left(1 - \frac{1}{p}\right) \sum_{b \in B} \frac{1}{p^b},
\]

\[
d(q) \leq 1 - \frac{q^q - q^{q-1}}{q^q - 1} \cdot \frac{1}{\zeta(q)} \prod_{p \in P} \left[1 - \left(1 - \frac{1}{p}\right) \sum_{s \in S} \frac{1}{p^s}\right].
\]

Moreover, if $q \in S$, then we also have

\[
d(q) \geq 1 - \frac{q^{q+1} - q^q}{q^{q+1} - 1} \cdot \frac{1}{\zeta(q + 1)} \prod_{p \in P} \left[1 - \left(1 - \frac{1}{p}\right) \sum_{s \in S} \frac{1}{p^s}\right].
\]

Proof. The first two inequalities follow immediately from Theorem 3 by using the set families

\[
S(p) := \begin{cases} S, & \text{for } p \in P, \\ \emptyset, & \text{for } p \notin P, \end{cases} \quad \text{and} \quad B(p) := \begin{cases} B, & \text{for } p \in P, \\ \{0, 1, 2, \ldots\}, & \text{for } p \notin P. \end{cases}
\]

The third inequality follows again from Theorem 3 by using the set family

\[
A(p) := \begin{cases} A, & \text{for } p \in P, \\ \{0, 1, 2, \ldots, q - 1\}, & \text{for } p \notin P. \end{cases}
\]

To prove the final inequality, note from Theorem 4 we have

\[
d(q) = 1 - \frac{1 - \frac{1}{q}}{1 - \frac{1}{q^{q+1}}} \cdot \prod_{p \neq q, \text{prime}} \left[1 - \left(1 - \frac{1}{p}\right) \sum_{m \in M(q)} \frac{1}{p^m}\right],
\]

\[
= 1 - \frac{q^{q+1} - q^q}{q^{q+1} - 1} \cdot \frac{1}{\zeta(q + 1)} \prod_{p \neq q, \text{prime}} \left[1 - \left(1 - \frac{1}{p}\right) \sum_{m \in M(q)} \frac{1}{p^m}\right].
\]

Now for all primes $p \neq q$, since $q \in S$ and $p \geq 2$, we have

\[
\frac{1 - \left(1 - \frac{1}{p}\right) \sum_{m \in M} \frac{1}{p^m}}{1 - \frac{1}{p^{q+1}}} \leq \frac{1 - \left(1 - \frac{1}{p}\right) \sum_{s \in S} \frac{1}{p^s}}{1 - \frac{1}{p^{q+1}}} \leq \frac{1 - \left(1 - \frac{1}{p}\right) \frac{1}{p^q}}{1 - \frac{1}{p^{q+1}}} \leq \frac{1 - \frac{1}{p} \cdot \frac{1}{p^q}}{1 - \frac{1}{p^{q+1}}} = 1,
\]

which we are then able to use to truncate the infinite product as desired.

For fixed values of $q$, these allow us to use the computer to rigorously calculate digits of $d(q)$, and they also allow the following bounds.

Proposition 5. For all $q$, we have

\[
1 - \frac{q^{q+1} - q^q}{q^{q+1} - 1} \cdot \frac{1}{\zeta(q + 1)} \leq d(q) \leq 1 - \frac{q^q - q^{q-1}}{q^q - 1} \cdot \frac{1}{\zeta(q)}.
\]

Therefore, we have

\[
\frac{1}{2q+1} (1 - q^{-1}) - \frac{1}{q^q} \leq d(q) - \frac{1}{q} \leq \frac{1}{2q} (1 + q^{-1}).
\]
Proof. The first inequalities follow immediately from the last proposition by taking \( P = \emptyset \). The second inequalities then follow from routine computations after using the elementary bounds

\[
1 + \frac{1}{2^s} \leq \zeta(s) \leq 1 + \frac{1}{2^s} + \int_2^\infty \frac{dx}{x^s} = 1 + \frac{2 + s - 1}{2^s(s - 1)},
\]

which are valid for all real values of \( s \) greater than 1. \( \square \)

This shows that for \( q \) large, the value of \( d(q) \) is very close to \( \frac{1}{q} \). That is, the probability that a number contains \( q \) in its tower factorization is very close to the probability that it is divisible by \( q \). But also note that the value is bounded away from \( \frac{1}{q} \) by an additive term on the order of \( 2^{-q} \). This can be explained by noting that \( 2^{-q} \) is essentially the probability that the term \( 2^q \) appears in the prime factorization of \( n \), and since \( 2 \) is the smallest prime, it makes sense that the contribution due to terms of this type be fundamentally larger than the rest.

These bounds are sufficiently tight to give another very believable result:

**Corollary 6.** The sequence \( d(q) \) is strictly decreasing, and it decreases to 0.

**Numerics for \( d(2) \)**

Let us now return to address the original question: “what is the value of \( d(2) \)?” Using Proposition 5 provides the bounds \( 0.5246243585 \leq d(2) \leq 0.5947152656 \), which is not yet refined enough to rule out tempting conjectures like \( d(2) = \gamma \) or \( d(2) = 1/\sqrt{3} \) that our initial data from Section 1 may have suggested.

Nonetheless, we can use the bounds of Proposition 4 to write a program that (eventually) calculates \( d(2) \) to within arbitrarily precision. More specifically, using these bounds with \( A = M(2) \cap \{1,2,\ldots,20\} \), with \( S = M(2) \cap \{1,2,\ldots,20\} \), and with \( P \) being the set of the first 25,000 primes yields

\[
0.577350376 < d(2) < 0.577350486,
\]

and since \( 1/\sqrt{3} = 0.57735026\ldots \) and \( \gamma = 0.577215\ldots \), this definitively (and perhaps anticlimactically) shows that \( \gamma < 1/\sqrt{3} < d(2) \), which disproves any such conjecture. It is curious to note though how very close \( d(2) \) is to \( 1/\sqrt{3} \), and the authors have no explanation for this.

Using this same technique, we are able to compute numeric bounds on various other values of \( d(q) \), which we present in the appendix.

**4 Conclusion**

So what is the value of \( d(2) \)? Apparently it’s just slightly larger than \( 1/\sqrt{3} \), but what is an exact answer? Is \( d(2) \) algebraic? Is it expressible in terms of elementary functions or more satisfying limits, or is it possible that perhaps \( d(2) \) is in some sense its own transcendental mathematical constant? Unfortunately, after many attempts, the authors were unable to make headway on any of these questions let alone the corresponding questions for \( d(q) \) in general.

Nonetheless, the authors believe this problem is very interesting—especially because the representation of \( d(q) \) in Proposition 2 is so tempting. The problem has a certain fractal-like self-similarity, and it feels like some beautiful idea is just waiting to be applied. The authors hope for progress on the problem, and we wish our readers the best with these loose ends.
5 Appendix

Here we tabulate numeric bounds found on $d(q)$ for various values of $q$. These were found by using Proposition 4 and a simple Maple script. The floating point values in the fifth column are a rigorous lower bound for $d(q)$ (on top) and a rigorous upper bound for $d(q)$ (on bottom). More complete values for these bounds have been computed, but they are truncated to just 35 digits here.

The columns $p$, $a$, and $s$ are parameters for the algorithms used. These parameters correspond to the size of $P$, (roughly) the size of $A$, and (roughly) the size of $S$ as in Proposition 4. Notably, $p$ is the number of primes used in the estimation, which seems to matter much more than the size of $A$ [affecting the upper bound] or $S$ [affecting the lower bound]. Data and code are available on request.

| $q$ | $p$   | $a$ | $s$ | Bounds (First 35 Digits [more available]) | Digits Known |
|-----|-------|-----|-----|------------------------------------------|--------------|
| 2   | 25000 | 20  | 20  | 0.577 350 376 656 807 813 001 171 222 749 099 03 | 6            |
| 3   | 6000  | 100 | 100 | 0.388 807 379 263 994 405 608 000 000 000 000 00 | 10           |
| 5   | 5000  | 100 | 100 | 0.215 118 984 695 585 620 327 888 615 736 044 88 | 19           |
| 7   | 2500  | 100 | 100 | 0.146 500 891 228 438 042 819 116 915 103 810 81 | 29           |
| 11  | 2000  | 200 | 200 | 0.091 134 581 055 674 121 650 272 316 314 808 81 | 44           |
| 13  | 2000  | 200 | 200 | 0.076 979 810 520 294 777 519 659 200 891 501 69 | 52           |
| 17  | 2000  | 200 | 200 | 0.058 827 124 602 119 403 676 736 708 884 910 96 | 60           |
| 19  | 2000  | 200 | 200 | 0.052 632 482 973 467 517 964 355 534 025 063 33 | 60           |
| 23  | 2000  | 200 | 200 | 0.043 478 317 889 484 083 344 293 667 695 938 90 | 60           |
| 29  | 2000  | 200 | 200 | 0.034 482 759 519 907 038 838 884 044 979 223 96 | 60           |
| 31  | 2000  | 200 | 200 | 0.022 588 064 741 450 054 595 016 325 700 965 71 | 60           |
| 37  | 2000  | 200 | 200 | 0.027 027 027 030 566 683 523 134 092 350 343 62 | 60           |
| 41  | 2000  | 200 | 200 | 0.026 362 482 973 467 517 964 355 534 025 063 33 | 60           |
| 43  | 2000  | 200 | 200 | 0.024 390 243 902 660 852 384 118 730 197 418 93 | 60           |
| 47  | 2000  | 200 | 200 | 0.021 276 595 744 684 328 187 880 387 051 387 64 | 60           |
| $q$ | $p$ | $a$ | $s$ | Bounds (First 35 Digits [More available]) | Digits Known |
|-----|-----|-----|-----|---------------------------------|--------------|
| 53  | 200 | 200 | 200 | 0.018 867 924 528 301 941 256 223 883 281 009 21 0.018 867 924 528 301 941 256 223 883 281 009 21 | 60            |
| 59  | 200 | 200 | 200 | 0.016 949 152 542 372 882 208 592 895 018 030 94 0.016 949 152 542 372 882 208 592 895 018 030 94 | 60            |
| 61  | 200 | 200 | 200 | 0.016 393 442 622 950 819 885 416 820 821 976 79 0.016 393 442 622 950 819 885 416 820 821 976 79 | 60            |
| 67  | 200 | 200 | 200 | 0.014 925 373 134 328 358 212 292 786 538 441 94 0.014 925 373 134 328 358 212 292 786 538 441 94 | 60            |
| 71  | 200 | 200 | 200 | 0.014 084 507 042 253 521 126 969 339 106 717 70 0.014 084 507 042 253 521 126 969 339 106 717 70 | 60            |
| 73  | 200 | 200 | 200 | 0.013 698 630 136 986 301 369 915 228 058 392 40 0.013 698 630 136 986 301 369 915 228 058 392 40 | 60            |
| 79  | 200 | 200 | 200 | 0.012 658 227 848 101 265 822 785 626 836 554 17 0.012 658 227 848 101 265 822 785 626 836 554 17 | 60            |
| 83  | 200 | 200 | 200 | 0.012 048 192 771 084 337 349 397 641 437 357 10 0.012 048 192 771 084 337 349 397 641 437 357 10 | 60            |
| 89  | 200 | 200 | 200 | 0.011 235 955 056 179 775 280 898 877 203 211 62 0.011 235 955 056 179 775 280 898 877 203 211 62 | 60            |
| 97  | 200 | 200 | 200 | 0.010 309 278 350 515 463 917 525 773 198 999 20 0.010 309 278 350 515 463 917 525 773 198 999 20 | 60            |
| 101 | 200 | 200 | 200 | 0.009 900 990 099 009 900 990 099 009 901 185 36 0.009 900 990 099 009 900 990 099 009 901 185 36 | 60            |