EXTENSION FROM LC CENTRES AND THE EXTENSION THEOREM OF DEMAILLY BUT WITH ESTIMATES

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ABSTRACT. This paper generalises the result of Jean-Pierre Demailly on his Ohsawa–Takegoshi-type $L^2$ extension theorem, which guarantees holomorphic extensions for some sections $f$ on analytic subspaces $Y$ defined by multiplier ideal sheaves of plurisubharmonic functions. The result in this paper provides $L^2$ estimates for the extended sections even if those $f$ do not vanish on the singular sets of the reduced loci of the subspaces $Y$, at least for the case when the ambient manifold $X$ is compact and the relevant metrics have only neat analytic singularities. This is achieved by replacing the generalised Ohsawa measure by a measure supported on log-canonical centres.

In this paper, the following notations are used throughout.

Notation 0.0.1. Set $i := \frac{\sqrt{-1}}{2\pi}$.

Notation 0.0.2. Each potential $\varphi$ (of the curvature of a metric) on a holomorphic line bundle $L$ in the following represents a collection of local functions $\{\varphi_\gamma\}_\gamma$ with respect to some fixed local coordinates and trivialisation of $L$ on each open set $V_\gamma$ in a fixed open cover $\{V_\gamma\}_\gamma$ of $X$. The functions are related by the rule $\varphi_\gamma = \varphi_{\gamma'} + 2 \text{Re} h_{\gamma\gamma'}$ on $V_\gamma \cap V_{\gamma'}$ where $e^{h_{\gamma\gamma'}}$ is a (holomorphic) transition function of $L$ on $V_\gamma \cap V_{\gamma'}$ (such that $s_\gamma = s_{\gamma'} e^{h_{\gamma\gamma'}}$, where $s_\gamma$ and $s_{\gamma'}$ are the local representatives of a section $s$ of $L$ under the trivialisations on $V_\gamma$ and $V_{\gamma'}$ respectively).

Notation 0.0.3. For any prime divisor $E$, let

- $\phi_E := \log |s_E|^2$, representing the collection $\{\log |s_{E,\gamma}|^2\}_\gamma$, denote a potential (of the curvature of the metric) on the line bundle associated to $E$ given by the collection of local representations $\{s_{E,\gamma}\}_\gamma$ of some canonical section $s_E$ (thus $\phi_E$ is uniquely defined up to an additive constant);
- $\varphi_{E, sm}^m$ denote a smooth potential on the line bundle associated to $E$;
- $\psi_E := \phi_E - \varphi_{E, sm}^m$, which is a global function on $X$, when both $\phi_E$ and $\varphi_{E, sm}^m$ are fixed.

All the above definitions are extended to any $\mathbb{R}$-divisor $E$ by linearity. For notational convenience, the notations for a $\mathbb{R}$-divisor and its associated $\mathbb{R}$-line bundle are used interchangeably.

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1The notation is chosen by mimicking the reduced Planck constant $\hbar = \frac{h}{2\pi}$. It is typeset with the code \textbackslash{}raisebox{-4.25pt}\{\$\mathchar'26\$\textbackslash{}mkern-7mu \textbackslash{}i\}.
1. Introduction

This work is spun off from the research on the conjectural holomorphic extension theorem on dlt pairs, a collaborative work with Young-Jun Choi from the National Pusan University, which is related to the Abundance Conjecture in algebraic geometry (see, for example, \[13\], [12] and [14]).

1.1. Background. Since the inception of the celebrated Ohsawa–Takegoshi extension theorem in [32], there are numerous improvements and generalisations in various aspects: on the assumptions on the ambient manifold and the subvariety from which holomorphic sections are extended from ([29], [27], [9], [24]); on the method of proof ([35], [2], [7], [3]); and on the weights and constant in the estimate ([28], [5], [17]). The author cannot do better than the survey by Varolin ([37]) which provides an account of the development of the \(L^2\) extension theorems up till the extension theorem with optimal estimates (see [5], [17] and [3]). Interested readers are referred to that.

The present work takes off from the work of Demailly in [11]. Let \(X\) be a weakly pseudoconvex Kähler manifold, \(K_X\) its canonical bundle and \((L,e^{-\varphi_L})\) a hermitian line bundle on \(X\) equipped with a hermitian metric \(e^{-\varphi_L}\). Assume the metric to be smooth for the moment. In [11], Demailly proves a new version of the Ohsawa–Takegoshi \(L^2\) extension theorem applicable to the set-up in which

1. the ambient manifold \(X\) is a weakly pseudoconvex Kähler manifold and need not be Stein on the complement of any subvariety (a feature appeared already in [9]),

2. the relevant curvature form/current associated to the line bundle \(L\) is only required to be weakly positive and need not be strictly positive (a feature appeared already in [36] and [30] with erratum [31] for non-Stein ambient manifolds),

3. the analytic subspace \(Y\), from which any legitimate \((K_X \otimes L)|_Y\)-valued holomorphic section \(f\) is extended, need not be the zero locus of a holomorphic section of some vector bundle, but is more generally defined by the multiplier ideal sheaf \(\mathcal{I}(m_p\psi)\) (or, more precisely, by the structure sheaf \(\mathcal{O}_Y := \mathcal{O}_X/\mathcal{I}(m_p\psi)\)) of a quasi-plurisubharmonic (quasi-psh) function \(\psi\) on \(X\) with jumping numbers \(0 = m_0 < m_1 < \cdots < m_p\) (thus the subspace \(Y\) can possibly be non-reduced and reducible) (a feature which can be considered as a generalisation to the result in the work of Dano Kim, [23] and [24], in which an \(L^2\) extension theorem for extending holomorphic sections on maximal log-canonical centres of some log-canonical pairs \((X, D)\) is proved).

An interesting new input of this version of \(L^2\) extension theorem is that, if the ambient manifold \(X\) is compact (or if it is holomorphically convex, see [3]), via a brilliant use of the Hausdorff-ness of the topology on the relevant cohomology groups, a holomorphic extension of \(f\) can be assured without the need of any \(L^2\) assumption on the section \(f\) (provided that the suitable weak positivity assumption involving \(\varphi_L\) and \(\psi\) still holds true).

Demailly’s result holds true more generally for the cases when \(L\) is a vector bundle of arbitrary rank or when \(L\) is a line bundle but the metric \(e^{-\varphi_L}\) is singular. In the latter case, \(Y = Y^{(m_p)}\) is given by \(\mathcal{I}(\varphi_L + m_p\psi)\) instead and \(f\) is required to have some holomorphic extension locally which is a section of \(\mathcal{I}(\varphi_L)\).

\[2\text{In }[12],\text{ the curvature is even allowed to be slightly negative provided that it is replaced by a sequence such that the negativity dies out in the limit.}\]
Note that the Ohsawa measure in the estimates given by all different versions of the Ohsawa–Takegoshi extension theorem diverges to infinity around the singular points of the subspaces \( Y^{(mp)} \), or more precisely, the singular points of the reduced subvariety \( S^{(mp)} \subset Y^{(mp)} \) which is the schematic difference between \( Y^{(mp)} \) and \( Y^{(mp-1)} \). A section \( f \) on \( Y^{(mp)} \) (and thus on \( S^{(mp)} \)) can then never be \( L^2 \) with respect to the Ohsawa measure if it does not vanish along the singular locus of \( S^{(mp)} \).

Although the extension theorem of Demailly without the \( L^2 \) assumption loses the estimate on the extended section, it was considered advantageous since the sections to be extended would not have to vanish along the singular locus of \( S^{(mp)} \). It was hoped that, with this feature in the new version of the extension theorem, one could follow the arguments as in [12] to construct a suitable psh potential in order to prove the so called “dlt extension” (see [12, Conj. 1.3]). Unfortunately, it was impossible for the author to avoid a convergence argument in the construction of the required psh potential, and the argument required estimates which was provided by the use of the Ohsawa–Takegoshi extension theorem, as in [12].

In view of this, the goal of the present work is to resume the estimate of the Ohsawa–Takegoshi \( L^2 \) extension theorem under Demailly’s setting by replacing the (generalised) Ohsawa measure by the “measure on log-canonical centres”, or the “lc-measure” for short, which is defined in Section 1.3. The latter measure, instead of diverging to infinity around the singular locus of \( S^{(mp)} \), is indeed supported in the singular locus of \( S^{(mp)} \) (or on some lc centres of \((X, S^{(mp)})\) if \( S^{(mp)} \) is a divisor). This provides the means to get some sort of control over the \( L^2 \) norm of the holomorphic extensions, and eventually can be useful in proving the “dlt extension”.

For the purpose of introducing the lc-measure and stating the main theorems, the basic set-up of this paper is first given here.

1.2. Set-up. Let \((X, \omega)\) be a compact Kähler manifold of complex dimension \( n \), and let \( \mathcal{I}(\varphi) := \mathcal{I}_X(\varphi) \) be the multiplier ideal sheaf of the potential \( \varphi \) on \( X \) given at each \( x \in X \) by

\[
\mathcal{I}(\varphi)_x := \mathcal{I}_X(\varphi)_x := \left\{ f \in \mathcal{O}_{X,x} \left| f \right. \text{ is defined on a coord. neighbourhood } V_x \ni x \right. \text{ and } \int_{V_x} |f|^2 e^{-\varphi} d\lambda_{V_x} < +\infty \right\},
\]

where \( d\lambda_{V_x} \) is the Lebesgue measure on \( V_x \). Throughout this paper, the following are assumed on \( X \):

1. \((L, e^{-\varphi_L})\) is a hermitian line bundle with an analytically singular metrics \( e^{-\varphi_L} \), where \( \varphi_L \) is locally equal to \( \varphi_1 - \varphi_2 \), where each of the \( \varphi_i \)'s is a quasi-psh local function with neat analytic singularities, i.e. locally

\[
\varphi_i \equiv c_i \log \left( \sum_j |g_{ij}|^2 \right) \mod \mathcal{C}^\infty,
\]

where \( c_i \in \mathbb{R}_{\geq 0} \) and \( g_{ij} \in \mathcal{O}_X \);

2. \( \psi \) is a global function on \( X \) such that it can also be expressed locally as a difference of two quasi-psh functions with neat analytic singularities;

3. \( \sup_X \psi \leq 0 \) (which implies that \( \psi \) is quasi-psh after some blow-ups as it has only neat analytic singularities);
(4) there exist numbers $m_0, m_1 \in \mathbb{R}_{\geq 0}$ with $m_0 < m_1$ such that
\[ \mathcal{I}(\varphi_L + m_0 \psi) = \mathcal{I}(\varphi_L + m_1 \psi) \subseteq \mathcal{I}(\varphi_L + m_1 \psi) \quad \text{for all } m \in [m_0, m_1), \]
i.e. $m_1$ is a jumping number of the family $\{\mathcal{I}(\varphi_L + m \psi)\}_{m \in \mathbb{R}_{\geq 0}}$ (such numbers exist on compact $X$ as $\psi$ is quasi-psh after suitable blow-ups and thus it follows from the openness property of multiplier ideal sheaves and \( \text{(eq 1.3.1)} \));

(5) $S := S^{(m_1)}$ is a reduced subvariety defined by the annihilator
\[ \operatorname{Ann}_{\mathcal{O}_X}\left( \frac{\mathcal{I}(\varphi_L + m_0 \psi)}{\mathcal{I}(\varphi_L + m_1 \psi)} \right); \]
in particular, $S \subset (\psi)^{-1}(-\infty)$.

**Remark 1.2.1.** If $\varphi_L + (m_1 + \beta)\psi$ is a quasi-psh potential for all $\beta \in [0, \delta]$ for some $\delta > 0$ (which holds true in all the main theorems in this paper), openness property guarantees that, on every compact subset $K \subset X$, there exists $\varepsilon > 0$ such that $\mathcal{I}(\varphi_L + (m_1 + \varepsilon)\psi)|_K = \mathcal{I}(\varphi_L + m_1 \psi)|_K$ (see \[20\] Main Thm. (iii) or \[26\] Thm. 1.1; see also \[18\]). This then implies that the analytic subspace defined by $\operatorname{Ann}_{\mathcal{O}_X}\left( \frac{\mathcal{I}(\varphi_L + m_0 \psi)}{\mathcal{I}(\varphi_L + m_1 \psi)} \right)$ is automatically reduced, following the arguments in \[11\] Lemma 4.2.

**Remark 1.2.2.** Most of the arguments in this paper can be adapted to the case when $X$ is just a weakly pseudoconvex (non-compact) Kähler manifold (after passing to a relatively compact exhaustion) provided that the upper-boundedness on $\psi$ in \( (3) \) still holds true on $X$. As this is not automatic on a non-compact manifold, and the most interesting applications which the author concerns about are on compact manifolds, the background manifold is assumed to be compact in this paper for the sake of clarity.

**Definition 1.2.3.** Suppose that $\varphi$ is a potential or a global function on $X$ such that it is locally a difference $\varphi_1 - \varphi_2$ of quasi-psh local functions with neat analytic singularities as in \([1]\) above. The polar ideal sheaf of $\varphi$ is defined to be the ideal sheaf generated by the local holomorphic functions $g_{i,j}$ for all $j$’s and $i = 1, 2$.

**Notation 1.2.4.** Given a set $V \subset X$, a section $f$ of $\frac{\mathcal{I}(\varphi_L + m_0 \psi)}{\mathcal{I}(\varphi_L + m_1 \psi)}$ on $V$ (which is supported in $S$), and a section $F$ of $\mathcal{I}(\varphi_L + m_0 \psi)$ on $V$, the notation
\[ F \equiv f \mod \mathcal{I}(\varphi_L + m_1 \psi) \quad \text{on } V \]
is set to mean that, for all $x \in V$, if $(F)_x$ and $(f)_x$ denote the germs of $F$ and $f$ at $x$ respectively, one has
\[ ((F)_x \mod \mathcal{I}(\varphi_L + m_1 \psi))_x = (f)_x. \]
If such a relation between $F$ and $f$ holds, $F$ is said to be an extension of $f$ on $V$. If the set $V$ is not specified, it is assumed to be the whole space $X$. Such notation is also applied to cases with a slight variation of the sheaf $\mathcal{I}(\varphi_L + m_1 \psi)$ (for example, with $\mathcal{I}(\varphi_L + m_1 \psi)$ replaced by $\mathcal{I}(\varphi_L + m_1 \psi) \otimes \mathcal{I}(\varphi_L + m_1 \psi) \otimes \mathcal{I}(\varphi_L + m_1 \psi)$ (see Definition 2.3.1)).

1.3. **lc-measure and main theorems.** As explained above, the goal of this paper is to replace the generalised Ohsawa measure $\left| J^{m_1} \int_{\omega}^2 d \operatorname{vol}_{\omega, \varphi_L}[\psi] \right|$ in the previous versions of $L^2$ extension theorem (as in \[11\]) by the measure on log-canonical (lc) centres given by (eq 1.3.1)
\[ \mathcal{C}^\infty(S) \ni g \mapsto \int_{\mathcal{I}(\varphi_L + m_1 \psi)} g \left| f \right|^2 \omega d \text{lc vol}_{\omega, \varphi_L}[\psi] := \lim_{\varepsilon \to 0} \int_{\mathcal{I}(\varphi_L + m_1 \psi)} g \left| f \right|^2 e^{-\varphi_L - m_1 \psi} d \text{vol}_{\omega, \varphi_L}[\psi] \].
where \((X)\text{is assumed to be compact here, and}\)

- \(f \in H^0\left( S, K_X \otimes L \otimes \mathcal{O}_{\varphi_L+m_0\psi} \right)\) and \(\tilde{f}\) is a smooth extension of \(f\) to a section on \(X\) such that \(\tilde{f} \in C^\infty \otimes \mathcal{I}(\varphi_L + m_0\psi)\);
- \(\tilde{g}\) is any smooth extension of \(g\) to a function on \(X\);
- \(\sigma\) is a non-negative integer and the measure \(|f|_{\omega}^2 \text{d}lcv_{\omega,\varphi_L}^{\sigma}(m_1) [\psi]|\) is supported (if finite) on a (reduced) subvariety \(lcv_X^\sigma(S)\) of \(S\), which is the union of lc centres of \((X,S)\) of codimension \(\sigma\) in \(X\) if \(S\) is a divisor (see Definition 2.3.1), and is the image of such kind of union under some log-resolution (see Section 2.1).

Such measure vanishes when \(\sigma\) is large and diverges when \(\sigma\) is small (see Section 3), and can be finite and non-zero only at one particular value of \(\sigma\). Here an ad hoc definition of such special value of \(\sigma\) is given so that the main theorems can be stated properly.

**Definition 1.3.1.** Given the setting above, the codimension of minimal lc centres (mlc) of \((X,S)\) (or of \((X,\varphi_L,\psi,m_1)\)) with respect to \(f\), denoted by \(\sigma_f\), is the smallest integer \(\sigma\) such that

\[
\int_{lcv_X^\sigma(S)} |f|_{\omega}^2 \text{d}lcv_{\omega,\varphi_L}^{\sigma}(m_1) [\psi] < \infty.
\]

The precise behaviour of such measure is discussed in Section 3. It can be seen from the calculation there that this definition coincides with the one in Definition 3.0.5 when \(S\) is a divisor with \(\text{simple normal crossings (snc)}\). Indeed, \(\sigma_f\) is ranging between 0 and the codimension of mlc of \((X,S)\) when \(S\) is an snc divisor. If \(\mathcal{I}(\varphi_L + m_0\psi) = \emptyset_X\) and if \(\sigma_f \geq 1\), then \(f\) vanishes on all lc centres of \((X,S)\) with codimension \(\sigma_f\) in \(X\) but is non-trivial on at least one lc centre of codimension \(\sigma_f\). Moreover, from the discussion in Section 2.1, if \(\pi: \bar{X} \to X\) is a log-resolution of \((X,\varphi_L,\psi)\), the codimension \(\sigma_f\) coincides with \(\sigma_{\pi \circ \varphi_L \circ \pi}^{\sigma_f}\) (see Section 2.1 for the meaning of \(s_E\) and log-resolution of \((X,\varphi_L,\psi)\)).

The author would like to mention that the use of such lc-measure was inspired by the study of residue currents in [4], [33] and [1]. In their works, the kind of current-valued function (in 1-variable case)

\[
\mathbb{R}_{>0} \ni \varepsilon \mapsto \varepsilon \frac{idz \wedge d\bar{z}}{|z|^{2(1-\varepsilon)}}
\]

is studied. Such function gives a holomorphic family of currents for \(\varepsilon > 0\) and can be analytically continued across \(\varepsilon = 0\). Its value at \(\varepsilon = 0\) is a residue measure on \(\{z = 0\}\).

The lc-measure considered in this paper is essentially given by the value at \(\varepsilon = 0\) of the current-valued function

\[
\mathbb{R}_{>0} \ni \varepsilon \mapsto \varepsilon \frac{\bigwedge_{j=1}^g (idz_j \wedge d\bar{z}_j)}{\prod_{j=1}^g |z_j|^2 \cdot \sum_{j=1}^g \sigma \cdot \log |z_j|^2}^{\sigma + \varepsilon}
\]

after analytically continued across \(\varepsilon = 0\).

It happens that the lc-measure above can be fitted into the Ohsawa–Takegoshi-type \(L^2\) extension theorem. The main result of this paper can be stated as follows.

**Theorem 1.3.2.** Let \(\sigma_S := \max \sigma_f\) (see Definition 1.3.1 for the definition of \(\sigma_f\), where the maximum is taking over all \(f \in H^0\left( S, K_X \otimes L \otimes \mathcal{O}_{\varphi_L+m_0\psi} \right)\)) (which is at most \(\text{“vol”}\)).
some snc divisors (and thus $S$ for all $\sigma$ with estimates the lines of arguments in [11].

Then, for any holomorphic section $f \in H^0(S, K_X \otimes L \otimes \mathcal{J}(\varphi_L + m_0 \psi))$, there exists a sequence $\{F_\sigma\}_{\sigma=1}^\infty \subset H^0(X, K_X \otimes L \otimes \mathcal{J}(\varphi_L + m_0 \psi))$ such that

$$\sum_{\sigma=1}^{\sigma_f} F_\sigma \equiv f \mod \mathcal{J}(\varphi_L + m_1 \psi)$$

with estimates

$$\int_X \frac{|F_\sigma|^2 e^{-\varphi_L - m_1 \psi}}{|\psi|^{\sigma}(\sigma \log |\psi|)^2 + 1} \leq \frac{5}{|\psi|} \int_{\mathcal{L}_{\psi}^\sigma(S)} f - \sum_{i=\sigma+1}^{\sigma_f} F_i |d(\mathcal{L}_{\psi}^\sigma [\psi]) < \infty$$

for all $\sigma = 1, \ldots, \sigma_f$. ($\sum_{i=\sigma+1}^{\sigma_f} F_i$ is defined to be 0 when $\sigma = \sigma_f$.)

See Remark 2.2.6 for the purpose of the number $\ell$ in the estimate. Notice that, even in the case $\sigma_f = 1$, the weight in the estimate of $F_\sigma$ above is pointwisely dominating (up to a multiple constant) the weight in the estimate in [11] (which is in the magnitude of $e^{\varphi_L + m_1 \psi}$). Therefore, the above estimate includes the estimate in [11] up to a constant multiple.

For the proof, it is first argued in Section 2.1 that it suffices to consider the case where the polar ideal sheaves of $\varphi_L$ and $\psi$ (see Definition 1.2.3) are the defining ideal sheaves of some snc divisors (and thus $S$ is an snc divisor in particular). The proof then goes along the lines of arguments in [11].

For the sake of simplicity, the proofs in Section 2 are given for the case where $m_0 = 0$ and $m_1 = 1$. The result for the general $m_0$ and $m_1$ can be obtained by replacing $\varphi_L$ by $\varphi_L + m_0 \psi$ and $\varphi_L + \psi$ by $\varphi_L + m_1 \psi$ in the arguments.

As in the classical cases, the problem is reduced to solve for a weak solution of a $\overline{\partial}$-equation (derived from the smooth extension of $f$, and depending on the $\varphi$ in the definition of the lc-measure) with “error” using the twisted Bochner–Kodaira inequality (eq 2.2.2) with suitably chosen auxiliary functions (see Theorems 2.2.1 and 2.3.3), at least on the compliment of the polar sets of $\varphi_L$ and $\psi$. Part of the reasons that the “error” term arises, especially when $\sigma_f > 1$, is that the curvature term in the twisted Bochner–Kodaira inequality has an unavoidable negative summand from the choice of the auxiliary functions. Extra treatment has to be made in order to control the negative term so that it goes to 0 when $\varepsilon \to 0^+$. That is why the proof for the cases $\sigma_f = 1$ and $\sigma_f > 1$ are handled separately (see Sections 2.2 and 2.3 respectively) for the clarity of arguments.

The weak solution with “error” can be continued across the polar sets of $\varphi_L$ and $\psi$ via a generalised version of [8, Lemme 6.9], which may be considered as a “weighted-$L^2$ version.
of the Riemann continuation theorem 4 (see Lemmas 2.2.3 and 2.3.1). The required holomorphic extension of \( f \) is then constructed from the above solution with the “error” and letting \( \varepsilon \to 0^+ \). The required estimate is also obtained after taking the necessary limits.

The regularity of the limit is assured by following a similar argument as in [11]. Extra technical difficulty comes from the fact that the “error” term goes to 0 in a weighted \( L^2 \) norm instead of the unweighted one when \( \sigma_f > 1 \). Lemma 2.3.6 is there for handling that situation.

Theorem 1.3.2 essentially provides a “quantitative” \( L^2 \) extension theorem of Demailly, answering [11] Question 2.15, at least for the case where \( L \) is a line bundle (instead of a vector bundle) and \( \varphi_L \) has only neat analytic singularities (instead of general singularities). For the sake of completeness, the result is stated below.

**Theorem 1.3.3.** Under the setting given in Section 1.2, suppose that \( 0 = m_0 < m_1 < m_2 < \cdots < m_p < \ldots \) are the jumping numbers of the family \( \{ F(\varphi_L + m\psi) \}_{m \in \mathbb{R}_{\geq 0}} \). Suppose further that

1. for some given positive integer \( p \), there exists \( \delta > 0 \) such that
   \[
   i\partial\bar{\partial}\varphi_L + \bar{\beta}i\partial\bar{\partial}\psi \geq 0 \quad \text{on } X \quad \text{for all } \beta \in [m_1, m_p + \delta],
   \]
2. for any given constant \( \ell > 0 \) and using the notation \( e_\sigma := \sqrt[\sigma]{\ell} \), the function \( \psi \) is normalised (by adding to it a suitable constant) such that
   \[
   \psi < -\frac{e_{m_\sigma}}{\ell} \quad \text{and} \quad \frac{5}{|\psi| \log(e_{m_\sigma}/|\psi|)} + \frac{n}{|\psi|} \leq \delta.
   \]

Let \( Y^{(m_\sigma)} \) be the analytic space defined by the ideal sheaf \( F(\varphi_L + m\psi) \). Then, for any holomorphic section \( f \in H^0(Y^{(m_\sigma)}, K_X \otimes L \otimes \frac{\mathcal{I}(\varphi_L)}{\mathcal{I}(\varphi_L + m\psi)}) \), there exists a sequence \( \{ F_{k,\sigma} \}_{\sigma = 1}^{\sigma_f(m_\sigma)} \subset H^0(X, K_X \otimes L \otimes \mathcal{I}(\varphi_L + m_{\sigma-1}\psi)) \) for each \( \kappa = 1, 2, \ldots, p \) such that

\[
\sum_{k=1}^{\kappa} \sum_{\sigma=1}^{\sigma_f(m_\sigma)} F_{k,\sigma} \equiv f \mod \mathcal{I}(\varphi_L + m_\kappa\psi)
\]

with estimates

\[
\int_X \frac{|F_{k,\sigma}|^2 e^{-\varphi_L - m_\kappa\psi}}{|\psi|^{\sigma_f((\sigma \log |\psi|)^2 + 1)}} \leq \frac{1}{\sigma} \int_{|\kappa-1(S(m_\sigma))|} |f^{(m_\kappa)}(\sigma) - \sum_{i=1}^{\sigma_f(m_\kappa)} F_{k,i}|^2 \omega_{\omega,\varphi_L}^{\sigma_f(m_\kappa)}[\psi] < \infty
\]

for all \( \sigma = 1, \ldots, \sigma_f(m_\kappa) \) and \( \kappa = 1, \ldots, p \), where

\[
f^{(m_\kappa)} := f - \sum_{k=1}^{\sigma_f-1} \sum_{i=1}^{\sigma_f(m_\kappa)} F_{k,i} \mod \mathcal{I}(\varphi_L + m_\kappa\psi)
\]

\[
\in H^0(S^{(m_\kappa)}, K_X \otimes L \otimes \frac{\mathcal{I}(\varphi_L + m_{\kappa-1}\psi)}{\mathcal{I}(\varphi_L + m_\kappa\psi)}).
\]

---

4 This is usually named as the “Riemann extension theorem”. The current naming “Riemann continuation theorem” is used just to distinguish this theorem from the Ohsawa–Takegoshi-type extension theorem which is studied in this paper. The use of “continuation” is found in Grauert–Remmert’s book [10] Section A.3.8 (English translation by Huckleberry), but “extension” is used in [15] Section 7.1, a later publication of the same authors.
1.4. Further questions. There are several questions that the author would still like to understand.

(1) As stated in Remark 2.2.8, it is not yet clear to the author whether the current result (if allowing $X$ to be non-compact) includes the results on optimal constant in [5] and [17]. Moreover, is it possible to determine the “optimal constant” in this Ohsawa–Takegoshi-type extension theorem with lc-measure?

(2) The lc-measure is inspired by the residue currents studied in [4], [33] and [1], obtained by replacing their residue currents (i.e. limit of $\varepsilon e^{-\varphi_L-(1-\varepsilon)m_1\psi} \omega_X$ in the notation of this paper) by a currents with “Poincaré-growth” singularities (i.e. limit of $\varepsilon e^{-\varphi_L-m_1\psi} \omega_X$). Is it possible to replace the lc-measure in the main theorem by some measure which is defined by currents which diverge to infinity even faster, like the limits of

$$
\varepsilon e^{-\varphi_L-m_1\psi} \omega_X, \varepsilon e^{-\varphi_L-m_1\psi} \omega_X, \varepsilon e^{-\varphi_L-m_1\psi} \omega_X, \varepsilon \frac{\varepsilon e^{-\varphi_L-m_1\psi} \omega_X}{|\psi|^\sigma (\log |\psi|)^{1+\varepsilon}}, \varepsilon \frac{\varepsilon e^{-\varphi_L-m_1\psi} \omega_X}{|\psi|^\sigma \log |\psi| (\log \log |\psi|)^{1+\varepsilon}}, \varepsilon \frac{\varepsilon e^{-\varphi_L-m_1\psi} \omega_X}{|\psi|^\sigma \log |\psi| \log^2 |\psi| (\log^3 |\psi|)^{1+\varepsilon}}, \ldots
$$

and so on (where $\log^j$ denotes the composition of $j$ copies of log functions)? It seems to the author that this could be related to the question stated in Remark 2.2.7, which is asking for the estimates with some better weights given in [28].

(3) The author started to consider the lc-measure during the study of analytic adjoint ideal sheaves with Chen-Yu Chi from National Taiwan University. The lc-measure on various lc centres can possibly be the means to generalise the works of Guenancia ([19]) and Dano Kim ([25]) on this subject. Furthermore, these lc-measures provide a way to characterise the lc centres which can be defined by the multiplier ideal sheaves of quasi-psh functions. Considering such linkage, it would be of interest to see more of their applications in analytic and algebraic geometry.

Theorem 1.3.2 would be more useful if $\varphi_L$ and $\psi$ are allowed to be replaced by a sequence and the curvature assumption is allowed to have a slight negativity which converges to 0 in the limit, as in [12]. This will be discussed in the collaborative work with Young-Jun Choi, in which Theorem 1.3.2 is applied to deal with the “dlt extension”.

2. Proof of the main theorem

Let $(X, \omega)$ be a compact Kähler manifold of dimension $n$ equipped with a (smooth) Kähler metric $\omega$. All notations follow the ones given in Section 1.2.

To start the proof, the problem is first reduced to the case where $S$ is a divisor with simple normal crossings (snc).

For simplicity, it is assumed in Sections 2.2 and 2.3 that $m_0 = 0$ and $m_1 = 1$. The arguments remain the same for the case of general jumping numbers.

2.1. Effects of log-resolutions. As $\varphi_L$ and $\psi$ are locally difference of quasi-psh functions with neat analytic singularities, by the result of [21], there is a log-resolution $\pi: \tilde{X} \to X$ of $(X, \varphi_L, \psi)$, which is the composition of a sequence of blow-ups at smooth centres such that the polar ideal sheaves of $\varphi_L$ and $\psi$ (see Definition 1.2.3) are principal ideal sheaves given by some divisors, and the sum of these divisors together with the exceptional divisors of $\pi$ (i.e. components of the relative canonical divisor $K_{\tilde{X}/X} := K_{\tilde{X}}/\pi^*K_X$) has only snc.

Assume that $\varphi_L + m_1\psi$ is psh. Note that $K_{\tilde{X}/X}$ can be decomposed into two effective $\mathbb{Z}$-divisors $E$ and $R$ (with the corresponding canonical holomorphic sections denoted by
Suppose that \( \pi \) is restricted to \( \pi^{-1}(V) \) where \((V, z)\) is a coordinate neighbourhood in \( X \), and let \( \{U_\gamma\}_\gamma \) be a covering of \( \pi^{-1}(V) = \bigcup_\gamma U_\gamma \) by coordinate neighbourhoods \((U_\gamma, w_\gamma)\) in \( \tilde{X} \). Let also \( \{\varrho_\gamma\}_\gamma \) be a partition of unity subordinated to \( \{U_\gamma\}_\gamma \). Then, with \( s_E, s_R, z \) and \( w_\gamma \)'s suitably chosen, one has, for any \( f \in H^0(V, K_X \otimes L) \) which is viewed as an \((n, 0)\)-form \( f = f_V dz^1 \wedge \cdots \wedge dz^n \),

\[
\int_V |f|^2 e^{-\varphi_L - m\psi} = \int_V |f_V|^2 e^{-\varphi_L - m\psi} c_n dz^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^1 \wedge \cdots \wedge d\bar{z}^n
\]

\[
= \sum_\gamma \int_{U_\gamma} \varrho_\gamma |f_V|^2 e^{-\varphi_L - m\psi} |s_{E,\gamma} \otimes s_{R,\gamma}|^2 c_n dw_\gamma^1 \wedge \cdots \wedge dw_\gamma^n \wedge d\bar{w}_\gamma^1 \wedge \cdots \wedge d\bar{w}_\gamma^n
\]

\[
= \sum_\gamma \int_{U_\gamma} \varrho_\gamma |\pi^* f_V \otimes s_E|^2 e^{-\varphi_L - m\psi + \phi_R} c_n dw_\gamma^1 \wedge \cdots \wedge dw_\gamma^n \wedge d\bar{w}_\gamma^1 \wedge \cdots \wedge d\bar{w}_\gamma^n,
\]

where \( c_n := (-1)^{n(n-1)/2} (\sqrt{1})^n \). Since \( \pi \) is bimeromorphic (birational) and irreducible components of \( K_{\tilde{X}/X} = E + R \) are exceptional under \( \pi \), i.e. their images under \( \pi \) are of codimension at least 2 in \( X \), it can be seen from the Riemann continuation theorem\(^5\) and the identity theorem for holomorphic functions that any local \( \pi^* L \otimes E \)-valued holomorphic section \( \tilde{f}_U \) on an open set \( U \subset \tilde{X} \) can be expressed as \( \pi^* f_{\pi(U)} \otimes s_E \) for some local \( L \)-valued holomorphic section \( f_{\pi(U)} \) on \( \pi(U) \). It follows that, for any \( m \in [m_0, m_1] \),

\[
(2.1.1) \quad K_{\tilde{X}} \otimes \pi^* L \otimes R^{-1} \otimes \mathcal{I}_{\tilde{X}}(\pi^* \varphi_L - \phi_R + m\pi^* \psi) = \pi^*(K_X \otimes L) \otimes E \otimes \mathcal{I}_X(\varphi_L + m\psi) \cdot \mathcal{O}_{\tilde{X}}.
\]

This shows, in particular, that \( m_1 \) is a jumping number of the family \( \{\mathcal{I}_X(\varphi_L + m\psi)\}_{m \in \mathbb{R}_{\geq 0}} \) if and only if it is a jumping number of \( \{\mathcal{I}_X(\pi^* \varphi_L - \phi_R + m\pi^* \psi)\}_{m \in \mathbb{R}_{\geq 0}} \). Furthermore, suppose that \( \varphi_L + (m_1 + \beta) \psi \) is quasi-psh for all \( \beta \in [0, \delta] \) for some \( \delta > 0 \), and if \( \tilde{S} \) is the reduced divisor defined by \( \text{Ann}_{\mathcal{I}_{\tilde{S}}}(\mathcal{I}_{\tilde{X}}(\pi^* \varphi_L - \phi_R + m_0 \pi^* \psi)] \), one then has \( \pi(\tilde{S}) = S \).

As a result, by replacing \( \pi^* \psi \) by \( \psi \), \( \pi^* \varphi_L - \phi_R \) by \( \varphi_L \), and \( \pi^* f_V \otimes s_E \) by \( f_V \) (thus replacing \( \pi^* L \otimes R^{-1} \) by \( L \), and therefore \( \pi^* K_X \otimes \pi^* L \otimes E = K_{\tilde{X}} \otimes \pi^* L \otimes R^{-1} \) by \( K_X \otimes L \)), it can be assumed in the following that

- \( S \) is a reduced divisor, and
- the polar ideal sheaves of \( \varphi_L \) and \( \psi \) are principal and the corresponding divisors have only nc with each other.

Moreover, the estimates on the holomorphic extension obtained in the main theorems in the following sections are valid even before blowing up.

### 2.2. Extension from lc centres of codimension 1

For simplicity, suppose that \( m_0 = 0 \) and \( m_1 = 1 \). As discussed in Section 2.1 one can assume that \( S \) is a reduced divisor in \( X \) and that \((X, S)\) is a log-smooth and log-canonical (lc) pair.

---

\(^5\)See footnote\(^4\)
The goal of the following is to replace the generalised Ohsawa measure in the Ohsawa–Takegoshi $L^2$ extension theorem by the lc-measure given by

\[
|f|^2 \omega \sigma \psi L [\omega] := \lim_{\varepsilon \to 0^+} \int f^2 e^{-\frac{\varphi_L - \psi}{\varepsilon}} \sigma_{\varphi_L} \sigma \omega X, \omega \psi \, d \text{vol}_{X, \omega},
\]

where $\sigma \geq 0$ and $f$ is any smooth extension of $f$ on $X$ such that $|f|^2 e^{-\varphi_L}$ is locally integrable. The behaviour of such measure is discussed in Section 3.

Set

\[
P_{\varphi_L} := \varphi_L^{-1}(-\infty) \text{ and } P_{\psi} := \psi^{-1}(-\infty)
\]

only negative poles), which are closed analytic subsets of $X$ by the assumptions on $\varphi_L$ and $\psi$ (Section 2.2). As discussed in Section 2.1, it is assumed in what follows that the polar divisors of $\varphi_L$ and $\psi$ have only snc with each other. In particular, $P_{\varphi_L} \cup P_{\psi}$ has only snc.

The key tool for proving this version of extension theorem is still the twisted Bochner–Kodaira inequality

\[
\| \sqrt{\eta} \partial \xi \|_{X^\omega}^2 + \| \sqrt{\eta + \lambda} \partial \xi \|_{X^\omega}^2 \geq \int_{X^\omega} \Theta^\omega(\xi, \xi) \omega
\]

for any compactly supported $K_X \otimes L$-valued smooth $(0, q)$-forms $\xi \in \mathcal{A}^0\omega_{\delta}(X^\omega; K_X \otimes L)$ on $X^\omega$ as in [11, §3.C, (3.10)] (see also [10, Ch. VIII]) or [28, Lemma 2.1, (9)], where

- $X^\omega := X \setminus (P_{\varphi_L} \cup P_{\psi})$, which has the structure of a complete Kähler manifold;
- $\varphi := \varphi_L + \psi + \nu$ is a potential (of the curvature of a hermitian metric) on $L$, where $\nu$ is a real-valued smooth function on $X^\omega$;
- $\Theta$ is the curvature form in \eqref{eq:2.2.4} and $\Theta^\omega(\xi, \xi)_\varphi$ is the trace of the contraction between $\partial \varphi$ and $e^{-\varphi} \xi \wedge \bar{\xi}$ with respect to the hermitian metric on $X$ given by $\omega$ (in the convention such that $\Theta^\omega(\xi, \xi)_\varphi \geq 0$ whenever $\Theta \geq 0$);
- $\varphi$ is the formal adjoint of $\partial$ with respect to the inner product corresponding to the global $L^2$-norm $\|\cdot\|_{X^\omega}^2 \omega \varphi$ on $X^\omega$.

The completeness of $X^\omega$ guarantees that $\omega$ can be modified to a complete metric, and, in that case, the inequality \eqref{eq:2.2.2} holds true also for all (weighted) $L^2$ $(0, q)$-forms $\xi$ in both of the domains of $\partial$ and its Hilbert space adjoint $\partial^*$ (see, for example, [10, Ch. VIII, §3]), and thus Riesz Representation Theorem can be invoked.

Let $\theta: [0, \infty) \to [0, 1]$ be a smooth non-increasing function such that $\theta \equiv 1$ on $[0, 1/2]$ and $\equiv 0$ on $[1/2, \infty)$, and $|\theta'| \leq \frac{AB}{\lambda B} + \varepsilon_0$ on $[0, \infty)$ for some positive constant $\varepsilon_0$. Define also that $\theta_\varepsilon := \theta \circ |\psi|^{-\varepsilon}$ and $\theta_\varepsilon := \theta' \circ |\psi|^{-\varepsilon}$ for convenience.

It is shown below (Theorem 2.2.3) that the Ohsawa measure in the Ohsawa–Takegoshi extension theorem can be replaced by the lc-measure \eqref{eq:2.2.4} in the classical case, i.e. when minimal lc centres (mlc) of $(X, S)$ are of codimension 1 and $S$ is smooth as $(X, S)$ is log-smooth), or when the holomorphic section to be extended vanishes on the singular locus of $S$.

**Theorem 2.2.1.** Suppose that

1. there exists $\delta > 0$ such that

\[
i \partial \overline{\partial} (\varphi_L + \psi) + \beta i \partial \overline{\partial} \psi \geq 0 \quad \text{on } X \text{ for all } \beta \in [0, \delta], \text{ and}
\]
(2) for any given constant $\ell > 0$, the function $\psi$ is normalised (by adding to it a suitable constant) such that
\[
\psi < -\frac{e}{\ell} \quad \text{and} \quad \frac{2}{|\psi| \log \left| \frac{\ell \psi}{e} \right|} + \frac{1}{|\psi|} \leq \delta.
\]
(See Remark 2.2.6 for the use of the constant $\ell$.)

Let $\tilde{f}$ be an $K_X \otimes L$-valued smooth section on $X$ such that $|\bar{\Omega} f|^2_\omega e^{-\varphi L - \psi} \log \left| \frac{\ell \psi}{e} \right|$ is integrable over $X$. Then, for any numbers $\varepsilon, \varepsilon' > 0$, the $\bar{\Omega}$-equation
\[
\bar{\Omega} u_\varepsilon = v_\varepsilon := \bar{\Omega} \left( \theta \left( \frac{1}{|\psi|} \right) \tilde{f} \right) = \varepsilon \theta' \bar{\partial} \psi \wedge \tilde{f} + \theta \bar{\partial} f
\]
can be solved with an $\varepsilon'$-error, in the sense that there are a smooth $K_X \otimes L$-valued $(0,1)$-form $w^{(0,1)}_{\varepsilon', \varepsilon}$ and a smooth section $u_{\varepsilon', \varepsilon}$ on $X^0$ such that
\[
(eq\ 2.2.3) \quad \bar{\Omega} u_{\varepsilon', \varepsilon} + w^{(0,1)}_{\varepsilon', \varepsilon} = v_\varepsilon \quad \text{on} \quad X^0,
\]
with the estimates
\[
\int_{X^0} \frac{|u_{\varepsilon', \varepsilon}|^2 e^{-\varphi L - \psi}}{|\psi|^{1-\varepsilon} ((\log |\ell \psi|)^2 + 1)} + \frac{1}{\varepsilon'} \int_{X^0} \frac{|w^{(0,1)}_{\varepsilon', \varepsilon}|^2 e^{-\varphi L - \psi} \log \left| \frac{\ell \psi}{e} \right|}{e} \leq \frac{1}{\varepsilon} \int_X \left| \theta \bar{\partial} f \right|^2_\omega e^{-\varphi L - \psi} \log \left| \frac{\ell \psi}{e} \right| + \frac{\varepsilon}{1 - \varepsilon} \int_X \frac{|\theta'|^2 f^2_\omega e^{-\varphi L - \psi}}{|\psi|^{1+\varepsilon}}.
\]

Remark 2.2.2. It is well known that a locally $L^1$ function $f$, which satisfies
\[
i \theta \bar{\partial} f \geq 0 \quad \text{as a current},
\]
coincides with a uniquely determined psh function almost everywhere (see, for example, [22, Thm. 1.6.10, Thm. 1.6.11]). Since $\varphi_L$ and $\psi$ are locally differences of quasi-psh functions, a simple argument shows that $(\varphi_L + \psi) + \beta \psi$ is a psh potential on $X$ for every $\beta \in [0, \delta]$ by the assumption [11].

Proof. Let $L$ be endowed with a metric with potential $\varphi_L + \psi + \nu$, where $\nu := \nu(\psi)$ and $\nu$ is a smooth real-valued function on $(-\infty, -\frac{e}{\ell})$. Put $\eta_\varepsilon := \tilde{\eta}(\psi)$ and $\lambda_\varepsilon := \tilde{\lambda}(\psi)$, where $\tilde{\eta}$ and $\tilde{\lambda}$ are smooth positive functions on $(-\infty, -\frac{e}{\ell})$. The curvature form in concern (i.e. the curvature form in (eq 2.2.2); see [11, §3.C, (3.10)] or [28, Lemma 2.1, (9)]) is then
\[
\Theta := \eta_\varepsilon i \theta \bar{\partial}(\varphi_L + \psi + \nu) - i \theta \bar{\partial} \eta_\varepsilon - \frac{1}{\lambda_\varepsilon} i \theta \bar{\partial} \eta_\varepsilon \wedge \bar{\theta} \eta_\varepsilon
\]
\[
(eq\ 2.2.4) \quad \Theta = \eta_\varepsilon i \theta \bar{\partial}(\varphi_L + \psi) + \eta_\varepsilon (\nu'' - (\log \tilde{\eta}_\varepsilon')') \circ \psi i \theta \bar{\partial} \psi + \left( \frac{\eta_\varepsilon' - \tilde{\eta}_\varepsilon' - (\tilde{\eta}_\varepsilon')^2}{\lambda_\varepsilon} \right) \circ \psi i \theta \bar{\partial} \psi \wedge \bar{\theta} \psi.
\]

The objective now is to find a function $\tilde{\nu}$ and positive functions $\tilde{\eta}$ and $\tilde{\lambda}$ such that
\[
(eq\ 2.2.5a) \quad \Theta \geq \Gamma(\psi) i \theta \bar{\partial} \psi \wedge \bar{\theta} \psi \quad \text{on} \quad X, \quad \text{and}
\]
To see how the required solution and estimate are obtained from these inequalities, write $\langle \cdot, \cdot \rangle := \langle \cdot, \cdot \rangle_{X^0, \omega, \varphi}$ as the global inner product on $X^0$ induced by the potential $\varphi := \varphi_L + \psi + \nu$ and $\| \cdot \| := \| \cdot \|_{X^0, \omega, \varphi}$ the corresponding norm. Although $\omega$ is not assumed to be complete in the statement, the standard argument (see, for example, [10, Ch. VIII, §6]) reduces the problem to the case where $\omega$ is complete on $X^0$, which is assumed to be the case in what follows.

When all inequalities in (eq 2.2.5) hold, and **assuming that** $v^2(2) = 0$ on $X$, the usual argument with the Cauchy–Schwarz inequality and the twisted Bochner–Kodaira inequality (eq 2.2.2) yields, for any compactly supported smooth $K_X \otimes L$-valued $(0, 1)$-form $\zeta$ on $X^0$, that

$$\langle \zeta, v^1 \rangle = \left\langle \langle \zeta \rangle_{ker \overline{\partial}} , v^1 \right\rangle = \left\langle (\partial \psi)^\omega \cdot j(\zeta)_{ker \overline{\partial}} , \frac{\varepsilon \theta'_\varepsilon f}{|\psi|^{1+\varepsilon}} \right\rangle$$

*given $\Gamma \geq 0$*

$$\leq \frac{\varepsilon}{1 - \varepsilon} \int_{\text{supp} \theta'_\varepsilon} \left| \varepsilon \theta'_\varepsilon f \right|^2 e^{-\varphi_L - \psi - \nu} \left( \frac{\varepsilon}{1 - \varepsilon} \int_{\text{supp} \theta'_\varepsilon} |\theta'_\varepsilon|^2 \left| \frac{\tilde{f}}{e^{\varphi_L - \psi}} \right|^2 \right)^{1/2}$$

*by (eq 2.2.3)*

$$\leq \frac{\varepsilon}{1 - \varepsilon} \int_{\text{supp} \theta'_\varepsilon} \left| \varepsilon \theta'_\varepsilon f \right|^2 e^{-\varphi_L - \psi} \left( \frac{\varepsilon}{1 - \varepsilon} \int_{\text{supp} \theta'_\varepsilon} |\theta'_\varepsilon|^2 \left| \frac{\tilde{f}}{e^{\varphi_L - \psi}} \right|^2 \right)^{1/2}$$

*by (eq 2.2.2)*

where $(\cdot)_{ker \overline{\partial}}$ denotes the orthogonal projection to the closed subspace $\text{ker} \overline{\partial}$ with respect to $\langle \cdot, \cdot \rangle$, and $(\partial \psi)^\omega \cdot j \cdot$ denotes the adjoint of $\overline{\partial} \psi \wedge \cdot$ with respect to $\langle \cdot, \cdot \rangle$. The completeness of $X^0$ and the Riesz Representation Theorem then assure the existence of the solution $u^1$ to the equation $\overline{\partial} u^1 = v^1$ with the estimate

$$\int_{X^0} |u^1|^2 e^{-\varphi_L - \psi - \nu} \leq \frac{\varepsilon}{1 - \varepsilon} \int_{\text{supp} \theta'_\varepsilon} \left| \varepsilon \theta'_\varepsilon f \right|^2 e^{-\varphi_L - \psi}$$

One then obtains the required estimate provided that $(\eta_L + \lambda) e^{\nu} \leq |\psi|^{1-\varepsilon}((\log |\ell \psi|)^2 + 1)$.

When $v^2(2) \neq 0$, one can handle the situation using the argument as in [11] after (5.20) or the following slight variation of that. For any compactly supported smooth $K_X \otimes L$-valued $(0, 1)$-form $\zeta$ on $X^0$, one can apply the Cauchy–Schwarz inequality directly to

\[ \int_{X^0} |u^1|^2 e^{-\varphi_L - \psi - \nu} \leq \frac{\varepsilon}{1 - \varepsilon} \int_{\text{supp} \theta'_\varepsilon} \left| \varepsilon \theta'_\varepsilon f \right|^2 e^{-\varphi_L - \psi}. \]

\[ \text{Note that } \varphi_L + \psi \text{ is, being psh by Remark 2.2.2 locally bounded from above, so the weight in the norm } \| \cdot \|_{X^0, \omega, \varphi} \text{ is everywhere positive on } X^0 \text{ even though } \varphi_L \text{ itself may go to } +\infty. \]
yields
\[
|\langle \xi, v_e \rangle| \leq |\langle (\xi)_{ker \theta}, v_e^{(1)} \rangle| + |\langle (\xi)_{ker \theta}, v_e^{(2)} \rangle| \\
\leq \mathcal{N}_1(\partial \xi) \mathcal{N}_2(f) + \|\xi\| \left\| \theta_e \overline{f} \right\| \\
\leq (\mathcal{N}_1(\partial \xi) + \varepsilon') \|\xi\|^2 \left( \left( \mathcal{N}_2(f) \right)^2 + \frac{1}{\varepsilon'} \left\| \theta_e \overline{f} \right\|^2 \right)^{\frac{1}{2}}
\]

for any \( \varepsilon' > 0 \). Note that the norm-square \( \left\| \theta_e \overline{f} \right\|^2 = \int_X \left| \theta_e \overline{f} \right|^2 e^{-\varphi_L - \psi - \nu} \) converges on \( X \) by assumption (given the choice of \( \tilde{\nu} \) below). The twisted Bochner–Kodaira inequality and the Riesz Representation Theorem then assure the acclaimed existence of solution \( (u_{e',e}, w_{e',e}(0,1)) \) and estimate, assuming that \( (\eta_e + \lambda_e) e^\nu \leq |\psi|^{1-\varepsilon} (|\log |\psi||^2 + 1) \).

Note also that the smoothness of \( (u_{e',e}, w_{e',e}(0,1)) \) follows from the smoothness of \( v_e \) and the regularity of the \( \overline{\theta} \) operator.

To find \( \tilde{\nu}, \tilde{\eta}_e \) and \( \tilde{\lambda}_e \) such that \( \text{eq 2.2.5a} \) and \( \text{eq 2.2.5b} \) hold, note that \( \Gamma \), as defined in \( \text{eq 2.2.4} \), can be expressed as

\[
(\text{eq 2.2.6}) \quad \Gamma = \tilde{\eta}_e \left( \tilde{\nu}'' - (\log \tilde{\eta}_e)'' - \left( 1 + \frac{\tilde{\eta}_e}{\lambda_e} \right) \left( \log \tilde{\eta}_e \right)^2 \right) .
\]

Thus, letting \( t \) be the variable of the functions \( \tilde{\nu}, \tilde{\eta}_e, \tilde{\lambda}_e \) and \( \Gamma \), \( \text{eq 2.2.5b} \) is equivalent to

\[
\frac{\Gamma}{\tilde{\eta}_e} = \tilde{\nu}'' - (\log \tilde{\eta}_e)'' - \left( 1 + \frac{\tilde{\eta}_e}{\lambda_e} \right) \left( \log \tilde{\eta}_e \right)^2 \geq \begin{cases} 0 & \text{for } t \leq -\frac{\varepsilon}{\ell} , \\
\frac{(1-\varepsilon)\varepsilon}{\tilde{\eta}_e} \left| \frac{\varphi}{|\varphi|} \right| |t|^{1+\varepsilon} & \text{for } B \leq |t| \leq A . \end{cases}
\]

Now putting

\[
\tilde{\eta}_e := |t|^{1-\varepsilon} \log \left| \frac{\ell t}{\varepsilon} \right| = |t|^{1-\varepsilon} e^{-\tilde{\nu}}
\]

\[
(\log \tilde{\eta}_e)' = -\frac{1-\varepsilon}{|t|} - \frac{1}{|t| \log \left| \frac{\varphi}{|\varphi|} \right|}
\]

\[
(\log \tilde{\eta}_e)'' = -\frac{1-\varepsilon}{|t|^2} - \frac{1}{|t|^2 \log \left| \frac{\varphi}{|\varphi|} \right|} + \frac{1}{|t|^2 (\log \left| \frac{\varphi}{|\varphi|} \right|)^2}
\]

into the above inequalities and setting

\[
\Gamma = \frac{(1-\varepsilon)\varepsilon}{\varphi} |t|^{1+\varepsilon} \quad \text{for all } t < -\frac{\varepsilon}{\ell} ,
\]

the function \( \tilde{\lambda}_e \) is then defined. To see that \( \tilde{\lambda}_e \) so defined is positive, notice that

\[
\frac{\Gamma}{\tilde{\eta}_e} = \frac{1-\varepsilon}{|t|^2} + \frac{2}{|t|^2 \log \left| \frac{\varphi}{|\varphi|} \right|} + \frac{2}{|t|^2 (\log \left| \frac{\varphi}{|\varphi|} \right|)^2} - \left( 1 + \frac{\tilde{\eta}_e}{\lambda_e} \right) \left( \frac{1-\varepsilon}{|t|} + \frac{1}{|t| \log \left| \frac{\varphi}{|\varphi|} \right|} \right)^2 = \frac{(1-\varepsilon)\varepsilon}{|t|^2} ,
\]
which yields
\[
1 + \tilde{\eta}_e = 1 + \left(\frac{1}{1 - \varepsilon + \frac{1}{\log |z_1|}}\right)^2 \left(\frac{2\varepsilon}{\log \frac{\ell t}{e}} + \frac{1}{(\log \frac{\ell t}{e})^2}\right)
\]
\[
\Rightarrow \tilde{\lambda}_e = \tilde{\eta}_e \left(\frac{1 - \varepsilon + \frac{1}{\log |z_1|}}{2\varepsilon} + \frac{1}{(\log |z_1|)^2}\right) > 0.
\]
Moreover, since
\[
\tilde{\lambda}_e \leq \tilde{\eta}_e \left(\frac{1 + \frac{1}{\log |z_1|}}{1}\right)^2 = |t|^{1-\varepsilon} (\log |\ell t|)^2 \log \frac{|\ell t|}{e} = t^{1-\varepsilon} (\log |\ell t|)^2 e^{-\tilde{\nu}},
\]
it follows that \((\eta_e + \lambda_e) e^{\nu} \leq |\psi|^{1-\varepsilon} (\log |\ell \psi|)^2 + 1\).

Finally, with the above choices of \(\nu\) and \(\tilde{\eta}_e\), one has
\[
0 \leq (\tilde{\nu}' - (\log \tilde{\eta}_e)) \circ \psi = \frac{2}{|\psi| \log \frac{\ell t}{e}} + \frac{1 - \varepsilon}{|\psi|} \leq \delta
\]
on \(X\) by the normalisation assumption [2]. Therefore, it can be seen from [eq 2.2.4] that [eq 2.2.5a] holds due to the curvature assumption [1] in the hypothesis.

\[\square\]

Theorem 2.2.1 holds true irrespective of the codimension of mlc of \((X, S)\). The required extension of \(f\) with estimate given in terms of the measure in [eq 2.2.1] can be obtained by letting \(\varepsilon \to 0^+\) (after estimating \(|\theta'|^2\) by a constant and followed by \(\varepsilon' \to 0^+\)), provided that the right-hand-side of the estimate converges. However, before starting the limit process, the solutions of the equation [eq 2.2.3] should be continued to the whole of \(X\).

The following generalisation to [8, Lemme 6.9] is needed.

Lemma 2.2.3. On an open set \(\Omega \subset \mathbb{C}^n\), let \(z_1\) be a complex coordinate on \(\Omega\) such that \(\sup_{\Omega} |z_1|^2 < 1\). If there are

- a \((p, q)\)-form \(v\) with coefficients in \(L^1_{\text{loc}}(\Omega)\) (locally \(L^1\) with respect to the unweighted Lebesgue measure \(d\lambda\)) and
- a \((p, q - 1)\)-form \(u\) with coefficients in \(L^2_{\text{loc}}(\Omega; |\log |z_1|^2|^{-1-\delta})\) (locally \(L^2\) with respect to the weighted Lebesgue measure \(\frac{d\lambda}{|\log |z_1|^2|^{1-\delta}}\)), where \(\delta \in [0, 1]\),

satisfying
\[
\widetilde{\partial} u = v \quad \text{on} \quad \Omega \setminus \{z_1 = 0\}
\]
in the sense of currents, then the same equation holds in the sense of currents on the whole of \(\Omega\).

Proof. The cases when \(\delta = 1\) is already included in [8, Lemme 6.9], so it suffices to consider only the cases when \(\delta \in [0, 1)\) in what follows.

The aim is to show that, for all compactly supported smooth \((n - p, n - q)\)-form \(\zeta\) on \(\Omega\), one has
\[
\int_{\Omega} \zeta \wedge v = (-1)^{p+q+1} \int_{\Omega} \widetilde{\partial} \zeta \wedge u.
\]
Let \( \chi : \mathbb{R} \to [0, 1] \) be a smooth non-decreasing function with \( \text{supp } \chi = [\frac{1}{2}, +\infty) \) and \( \chi(t) = 1 \) for all \( t \geq 1 \). Set \( \chi_\varepsilon := \chi \circ \frac{1}{\log |z|} \) and \( \chi'_\varepsilon := \chi' \circ \frac{1}{\log |z|^2} \) for any \( \varepsilon > 0 \) for convenience. Apparently, \( \chi_\varepsilon \zeta \) is compactly supported in \( \Omega \setminus \{ z_1 = 0 \} \) and thus
\[
\int_\Omega \chi_\varepsilon \zeta \wedge v = (-1)^{p+q+1} \int_\Omega \partial \chi_\varepsilon \wedge \zeta \wedge u + (-1)^{p+q+1} \int_\Omega \chi_\varepsilon \partial \zeta \wedge u
\]
by assumption. It can be seen that
\[
\int_\Omega \chi_\varepsilon \zeta \wedge v \to \int_\Omega \zeta \wedge v \quad \text{and} \quad \int_\Omega \chi_\varepsilon \partial \zeta \wedge u \to \int_\Omega \partial \zeta \wedge u \quad \text{as } \varepsilon \to 0^+
\]
by the dominated convergence theorem thanks to the regularity on \( v \) and \( u \) (note that
\[
\left| \int_\Omega \partial \zeta \wedge u \right|^2 \leq \int_\Omega | \log |z_1|^2 |^s |\partial \zeta|^2 d\lambda \int_{\text{supp } \zeta} \frac{|u|^2}{| \log |z_1|^2 |^\delta} d\lambda
\]
by the Cauchy–Schwarz inequality and \( | \log |z_1|^2 |^s \) is integrable with respect to \( d\lambda \), thus \( \partial \zeta \wedge u \) is integrable).

It remains to show that \( \int_\Omega \partial \chi_\varepsilon \wedge \zeta \wedge u \to 0 \) as \( \varepsilon \to 0^+ \). As \( \partial \chi_\varepsilon = \frac{\varepsilon \chi'_\varepsilon |z|^2}{|z|^2 + \varepsilon} \), one has
\[
\left| \int_\Omega \partial \chi_\varepsilon \wedge \zeta \wedge u \right|^2 \leq \varepsilon^2 \int_\Omega \frac{|\chi'_\varepsilon |z|^2 |\zeta|^2}{|z|^2 + \varepsilon} d\lambda \int_{\text{supp } \zeta} \frac{|u|^2}{| \log |z_1|^2 |^{1+\delta}} d\lambda.
\]

Since
\[
\varepsilon \int_\Omega \frac{d\lambda}{|z_1|^2 | \log |z_1|^2 |^{1+\delta}} \quad \text{converges when } \varepsilon \to 0^+,
\]
together with the fact that the integral involving \( u \) converges by assumption, the integral \( \int_\Omega \partial \chi_\varepsilon \wedge \zeta \wedge u \) converges to 0 as \( \varepsilon \to 0^+ \). This concludes the proof. \( \square \)

**Proposition 2.2.4.** Under the assumptions (1) and (2) in Theorem 2.2.1, the equation (eq 2.2.3), namely \( \partial u_{\varepsilon, \psi} + w^{(0,1)}_{\varepsilon, \psi} = v_{\varepsilon} \), which comes with the estimate given in Theorem 2.2.1, holds true on the whole of \( X \).

**Proof.** Notice that \( v_{\varepsilon} \) is smooth on \( X \). In view of [8, Lemme 6.9] and Lemma 2.2.3, it suffices to show that both \( u_{\varepsilon, \psi} \) and \( w^{(0,1)}_{\varepsilon, \psi} \) are in \( L^2_{\text{loc}}(X) \), or at least locally in \( L^2 \) under some suitable weights.

The curvature assumption (1) in Theorem 2.2.1 infers that \( \varphi_L + \psi \) is psh on \( X \) (see Remark 2.2.2), thus locally bounded above by some constant. Since \( \psi \) is also bounded above by assumption, it follows that \( e^{-\varphi_L - \psi} \log |c_\psi| \) is bounded from below by some positive constant. From the estimate provided by Theorem 2.2.1 \( w^{(0,1)}_{\varepsilon, \psi} \) is in \( L^2_{\text{loc}}(X) \).

The weight \( \frac{1}{|\psi|^{1+\varepsilon}(|\log |\ell \psi||)^2 + 1} \) is locally bounded from below by some positive constant on \( X \setminus P_\psi \). A similar argument thus shows that \( u_{\varepsilon, \psi} \) is in \( L^2_{\text{loc}}(X \setminus P_\psi) \). It follows immediately from [8, Lemme 6.9] that the equation (eq 2.2.3) holds on \( X \setminus P_\psi \).

From the fact that
\[
(\text{eq 2.2.7}) \quad x^s |\log x|^s \leq \frac{s^s}{e^s e^s^s}
\]
for all \( x \in [0, 1] \), \( s > 0 \) and \( \delta \geq 0 \) (if \( 0^0 \) is treated as 1), it can be seen easily that
\[
\frac{1}{|\psi|^{1+\varepsilon}(|\log |\ell \psi||)^2 + 1} \geq \frac{1}{|\psi| \ell^s (2^s e^{-2} e^{-2} + 1)}.
\]
Together with the fact that \( \varphi_L + \psi \) being locally bounded from above, it yields \( u_{\varepsilon', \varepsilon} \in L^2_{\text{loc}}(X; |\psi|^{-1}) \). Recall that the polar ideal sheaf of \( \psi \) is assumed to be generated by an snc divisor. Therefore, for any point \( p \in P_\psi \setminus \text{Sing}(P_\psi) \), there is a neighbourhood \( V_p \) of \( p \) lying within a coordinate chart of \( X \) such that

\[
P_\psi \cap V_p = \{ z_1 = 0 \} ,
\]

where \( z_1 \) is a complex coordinate with \( \sup_{V_p} |z_1| < 1 \), and

\[
|\psi| \big|_{V_p} \leq \alpha \log |z_1|^2 + C ,
\]

where \( \alpha \) and \( C \) are positive constants. It follows that \( u_{\varepsilon', \varepsilon} \in L^2_{\text{loc}} \left( V_p ; |\log |z_1|^2|^{-1} \right) \). By Lemma 2.2.3, the equation (eq 2.2.3) holds also on \( V_p \). It follows that equation (eq 2.2.3) holds true on \( X \setminus R \), where \( R := \text{Sing}(P_\psi) \), after considering all \( p \in P_\psi \setminus \text{Sing}(P_\psi) \).

To show that the equation holds true also across \( R \), solve the equation \( v_\varepsilon \equiv w_\varepsilon^{(0,1)} = \partial U \) on a sufficiently small local neighbourhood \( V \) intersecting \( R \). Note that \( R \) is of codimension 2 in \( X \). The function \( U - v_\varepsilon \) is holomorphic on \( V \setminus R \) and thus can be continued to a holomorphic function on \( V \) by the Riemann continuation theorem for holomorphic functions. This, in turn, implies that the equation (eq 2.2.3) holds on all of \( X \). \( \square \)

The theorem of holomorphic extension from the codimension-1 lc centres of \( (X, S) \) is summarised in the following theorem.

**Theorem 2.2.5.** Assume the assumptions (1) and (2) in Theorem 2.2.1. Let \( f \) be any holomorphic section in \( H^0 \left( S, K_X \otimes L \otimes \frac{\mathcal{I}(\varphi_L)}{\mathcal{I}(\varphi_L + \psi)} \right) \). If one has

\[
\int_S |f|^2 d \text{lcv}_{\omega, \varphi_L} [\psi] < \infty ,
\]

then there exists a holomorphic section \( F \in H^0(X, K_X \otimes L \otimes \mathcal{I}(\varphi_L)) \) such that

\[
F \equiv f \quad \text{mod} \ \mathcal{I}(\varphi_L + \psi)
\]

with the estimate

\[
\int_X \frac{|F|^2 e^{-\varphi_L - \psi}}{|\psi|(\log |\ell\psi|)^2 + 1} \leq \int_S |f|^2 d \text{lcv}_{\omega, \varphi_L} [\psi] .
\]

**Proof.** Given any local holomorphic liftings \( \left\{ \widetilde{f}_\gamma \right\}_\gamma \) of \( f \) (such that each \( \widetilde{f}_\gamma \in \mathcal{I}(\varphi_L) \) on some open set \( V_\gamma \subset X \) and \( \left( \widetilde{f}_\gamma \text{ mod } \mathcal{I}(\varphi_L + \psi) \right) = f \) if \( V_\gamma \cap S \neq \emptyset \)) and a partition of unity \( \left\{ \chi_\gamma \right\}_\gamma \) subordinated to an open cover \( \left\{ V_\gamma \right\}_\gamma \) of \( X \), the smooth section \( \widetilde{f} := \sum \gamma \chi_\gamma \widetilde{f}_\gamma \) of the coherent sheaf \( K_X \otimes L \otimes \mathcal{I}(\varphi_L) \) satisfies the properties

\[
f \equiv \widetilde{f} \quad \text{mod} \ \mathcal{E}_X \otimes \mathcal{I}(\varphi_L + \psi) \quad \text{and} \quad \overline{\partial} \widetilde{f} \equiv 0 \quad \text{mod} \ \mathcal{E}_X \otimes \mathcal{I}(\varphi_L + \psi)
\]

as shown in [11] Proof of Thm. 2.8]. Notice that one has the inequality \( \log \frac{1}{|\psi|} \leq \frac{\delta'}{4} e^{-\delta' \psi} \) using (eq 2.2.1) for any \( \delta' > 0 \), and the assumption (1) in Theorem 2.2.1 infers that \( \varphi_L + (1 + \delta') \psi \) is psh for all \( \delta' \in [0, \delta] \). Upper-boundedness of \( \psi \) also implies that \( \varphi_L + (1 + \delta') \psi \leq \varphi_L + \psi \). Therefore, by the strong effective openness property of multiplier ideal sheaves of psh functions (see [20] Main Thm.], also [18]), it follows that

\[
\overline{\partial} \widetilde{f} \in \mathcal{E}_X \otimes \mathcal{I}(\varphi_L + (1 + \delta') \psi) \quad \text{for } 0 < \delta' \ll 1 ,
\]
which in turn implies that
\[ |\bar{\partial}f|_{\omega}^2 e^{-\varphi_L - \psi} \log \left| \frac{f}{e} \right| \]
is integrable over \( X \).

Theorem 2.2.1 can then be invoked to provide the sections \( u_{\varepsilon'} \) and \( w_{\varepsilon'}^{(0,1)} \) with the estimate as stated in the theorem. Proposition 2.2.4 shows that the equation in (eq 2.2.3), namely \( \bar{\partial}u_{\varepsilon'} + w_{\varepsilon'}^{(0,1)} = v_{\varepsilon} \), holds true on the whole of \( X \). Both \( u_{\varepsilon'} \) and \( w_{\varepsilon'}^{(0,1)} \) are smooth on \( X \) by the regularity of the \( \bar{\partial} \) operator and the smoothness of \( u_{\varepsilon} \). Notice that \( \log (\varphi_L - \psi) \) is not integrable at every point of \( S \) for any \( \varepsilon > 0 \), the finiteness of the integral of \( u_{\varepsilon',\varepsilon} \) implies that \( u_{\varepsilon',\varepsilon} \in C^\infty_X \otimes \mathcal{F}(\varphi_L + \psi) \) (see Prop. 3.0.1 and the remarks afterwards for the more explicit calculation).

Recall that \( |\theta'_\omega| \leq \frac{A - B}{AB} + \varepsilon_0 \) on \( X \) by the choice of \( \theta_\varepsilon \). Setting \( F_{\varepsilon',\varepsilon} := \theta_\varepsilon \tilde{f} - u_{\varepsilon',\varepsilon} \) (which is an extension of \( f \)) and using the inequality
\[ |F_{\varepsilon',\varepsilon}|^2 \leq (1 + \alpha^{-1}) \left| \theta_\varepsilon \tilde{f} \right|^2 + (1 + \alpha) |u_{\varepsilon',\varepsilon}|^2 \]
for any positive real number \( \alpha \), one obtains the estimate
\[ \int_X \frac{|F_{\varepsilon',\varepsilon}|^2 e^{-\varphi_L - \psi}}{|\psi|((\log |\ell\psi|)^2 + 1)} + \frac{1}{\varepsilon'} \int_X |w_{\varepsilon'}^{(0,1)}|^2 e^{-\varphi_L - \psi} \log \left| \frac{\ell\psi}{e} \right| \]
\[ \leq (1 + \alpha^{-1}) \int_X \frac{|\theta_\varepsilon \tilde{f}|^2 e^{-\varphi_L - \psi}}{|\psi|((\log |\ell\psi|)^2 + 1)} + (1 + \alpha) \int_X \frac{|u_{\varepsilon',\varepsilon}|^2 e^{-\varphi_L - \psi}}{|\psi|1 - \varepsilon((\log |\ell\psi|)^2 + 1)} \]
\[ + \frac{1}{\varepsilon'} \int_X |w_{\varepsilon'}^{(0,1)}|^2 e^{-\varphi_L - \psi} \log \left| \frac{\ell\psi}{e} \right| \]
\[ \leq (1 + \alpha^{-1}) \int_X \frac{|\theta_\varepsilon \tilde{f}|^2 e^{-\varphi_L - \psi}}{|\psi|((\log |\ell\psi|)^2 + 1)} + \frac{1 + \alpha}{\varepsilon'} \int_X |\theta_\varepsilon \bar{\partial} \tilde{f}|^2 e^{-\varphi_L - \psi} \log \left| \frac{\ell\psi}{e} \right| \]
\[ + (1 + \alpha) \left( \frac{A - B}{AB} + \varepsilon_0 \right)^2 \varepsilon \int_X \left| \tilde{f} \right|^2 e^{-\varphi_L - \psi} \frac{1 - \varepsilon}{|\psi|1 + \varepsilon}. \]

The assumption that \( \int_S |f|^2 d\lambda_{\varepsilon'}^{1/2} \) being well-defined and finite infers that the integral \( \int_X |\tilde{f}|^2 e^{-\varphi_L - \psi} \frac{1}{|\psi|1 + \varepsilon} \) converges for all \( \varepsilon > 0 \), and thus so is \( \int_X |\tilde{f}|^2 e^{-\varphi_L - \psi} \frac{1}{|\psi|((\log |\ell\psi|)^2 + 1)} \). As a result, the first two terms on the right-hand-side both converge to 0 as \( \varepsilon \to 0^+ \) by the dominated convergence theorem, and the last term converges to const. \( \times \int_S |f|^2 d\lambda_{\varepsilon'}^{1/2} \) which is finite by assumption.

Set \( \varepsilon' := \left( \int_X |\theta_\varepsilon \bar{\partial} \tilde{f}|^2 e^{-\varphi_L - \psi} \log \left| \frac{\ell\psi}{e} \right| \right)^{1/2} \), which converges to 0 as \( \varepsilon \to 0^+ \). All the subscripts “\( \varepsilon' \)” are omitted in what follows. Then, it follows from the above estimate that \( w_{\varepsilon}^{(0,1)} \to 0 \) in \( L^2(X; e^{-\varphi_L - \psi}) \) as \( \varepsilon \to 0^+ \). One can also extract a weakly convergent subsequence from \( \{ F_{\varepsilon} \}_\varepsilon \) such that \( F := \lim_{\varepsilon \to 0^+} F_{\varepsilon} \) exists as a weak limit in \( L^2 \left( X; \frac{e^{-\varphi_L - \psi}}{|\psi|((\log |\ell\psi|)^2 + 1)} \right) \), which turns out to be the desired holomorphic extension of \( f \), as is justified below.
That $F$ is truly a holomorphic extension of $f$ can be seen using the argument similar to that in [11, (5.24)]. On any open set $V$ (which can be assumed to be a polydisc on which $L$ is trivialised without loss of generality) in the given open cover $\{V_i\}_i$ of $X$, one can solve $\overline{\partial}s = w^{(0,1)}$ for $s_\varepsilon$ with the $L^2$ Hörmander estimate $\|s_\varepsilon\|_{L^2(V; \overline{\partial} + \psi)} \leq C \|w\|_{L^2(V; \overline{\partial} + \psi)}$ (which implies $s_\varepsilon \in C^\infty_X \otimes \mathcal{S}(\varphi L + \psi)$ on $V$, where $\|\cdot\|_{L^2(V; \overline{\partial} + \psi)}$, resp. $\|\cdot\|_{L^2(V; \overline{\partial} + \psi)}$, denotes the $L^2$ norm on $V$, resp. on $X$, with the weight $e^{-\varphi L - \psi}$). Therefore, $s_\varepsilon \to 0$ in $L^2(V; e^{-\varphi L - \psi})$ as $\varepsilon \to 0^+$, and, passing to suitable subsequences of $\{F_{\varepsilon}^s\}_{\varepsilon}$ and $\{s_\varepsilon\}_{\varepsilon}$, one has $s_{\varepsilon \mu_k} \to 0$ pointwisely almost everywhere (a.e.) on $V$ while $F_{\varepsilon \mu_k} \to F$ weakly in the weighted $L^2$ space on $X$ as $\varepsilon \mu_k \to 0^+$. Moreover, $F_{\varepsilon \mu_k} - s_{\varepsilon \mu_k}$ is a holomorphic extension of $f$ on $V$ with both norm-squares
\[
\int_V \frac{|F_{\varepsilon \mu_k} - s_{\varepsilon \mu_k}|^2}{|\psi|((\log |\psi|)^2 + 1)} \leq \int_V \frac{|F_{\varepsilon \mu_k} - s_{\varepsilon \mu_k}|^2 e^{-\varphi L - \psi}}{|\psi|((\log |\psi|)^2 + 1)}
\]
being bounded above uniformly in $\varepsilon \mu_k$. As $|\psi|((\log |\psi|)^2 + 1)$ belongs to $L^1(V)$ (or $L^1(\psi X)$), the Cauchy–Schwarz inequality applied to the norm-square on the left-hand-side above assures that $F_{\varepsilon \mu_k} - s_{\varepsilon \mu_k}$ is also bounded above in $L^1(V)$ uniformly in $\varepsilon \mu_k$. Being holomorphic, Cauchy’s estimate and the above boundedness guarantee that the sequence $\{F_{\varepsilon \mu_k} - s_{\varepsilon \mu_k}\}_{\varepsilon}$ is locally bounded above in $V$. Montel’s theorem then assures that there is a subsequence which converges locally uniformly in $V$ to a holomorphic function $F_V$ on $V$. Notice that, if $V \cap S \neq \emptyset$, then $F_V \equiv f \mod \mathcal{S}(\varphi L + \psi)$ on $V$, as assured by Fatou’s lemma applied to the norm-square on the right-hand-side above. As a result, there is a subsequence of $\{F_{\varepsilon s}\}_{\varepsilon}$ which converges pointwisely a.e. on $V$ to the holomorphic extension $F_V$ of $f$. It turns out that $F = F_V$ a.e. on $V$. By considering all open sets $V$ in a cover of $X$, it follows that $F$ is indeed a holomorphic extension of $f$ on $X$, after possibly altering its values on a measure 0 set.

Finally, to obtain the acclaimed estimate for $F$, noting that $F$ comes with the estimate
\[
\int_X \frac{|F|^2 e^{-\varphi L - \psi}}{|\psi|((\log |\psi|)^2 + 1)} \leq (1 + \alpha) \left( A - B \frac{A - 1}{AB} + \varepsilon_0 \right)^2 \int_S |f|^2 d\mathcal{L}^1_{\omega,\varphi L} \psi
\]
and letting $\alpha \to 0^+$, $A \to +\infty$, $B \to 1^+$ and $\varepsilon_0 \to 0^+$ (and choosing the limit of $F$ suitably such that it converges locally uniformly) yield the desired result.

**Remark 2.2.6.** In some applications, it is necessary to control how fast the estimate grows when the constant $\delta$ in the normalisation of $\psi$ shrinks. The constant $\ell$ in the estimate is there to give a more precise control. Choose $\ell := \delta$ and write
\[
\psi = \psi_0 - \frac{a}{\delta}
\]
where $a > 0$ is a constant and $\sup_X \psi_0 = 0$. Then $a$ can be chosen independent of $\delta$ such that the assumption (2) in Theorem 2.2.1 is satisfied. Indeed, choosing $a$ such that
\[
a > e \quad \text{and} \quad \frac{2}{a \log \frac{a}{e}} + \frac{1}{a} = 1
\]
suffices (thus $a \approx 4.6805$). In this case, the estimate obtained is
\[
\int_X \frac{|F|^2 e^{-\varphi L - \psi_0}}{|\delta \psi_0 - a|((\log |\delta \psi_0 - a|)^2 + 1)} \leq \frac{1}{\delta} \int_S |f|^2 d\mathcal{L}^1_{\omega,\varphi L} \psi_0.
\]
Note that \( e^{-\nu_\delta} \) is bounded below by a positive constant independent of \( \delta \) (which can be seen easily by considering the cases \( \delta < a \) and \( \delta \geq a \) separately and applying (eq 2.2.7) suitably in both cases).

**Remark 2.2.7.** Concerning the weight in the norm of the extension \( F \), McNeal and Varolin prove in [28] some estimates with better weights. More precisely, for the case \( \psi := \psi_s = \phi_s - \varphi_s \) (which is suitably normalised for each of the weights below), they obtain holomorphic extension with an estimate in the norm with any of the following weights:

\[
\frac{\delta' e^{-\psi_s}}{|\psi_s|^{1+\delta'}}, \frac{\delta' e^{-\psi_s}}{|\psi_s|(\log |\psi_s|)^{1+\delta'}}, \ldots, \frac{\delta' e^{-\psi_s}}{|\psi_s| \cdot \log |\psi_s| \cdot \log \delta \cdot \log \delta \cdot \ldots \cdot \log \delta^{(N-1)} \cdot |\psi_s| \cdot (\log \delta^{(N)} |\psi_s|)^{1+\delta'}},
\]

where \( \delta' \in (0, 1) \) is a fixed number in each case, and \( \log^j \) denotes the composition of \( j \) copies of log functions here. It would be interesting to see if it is possible to obtain these weights in the setting of this paper.

**Remark 2.2.8.** It is not clear to the author whether Theorem 2.2.5, if allowing \( X \) to be non-compact, does include the results in [5] and [17] on the optimal constant for the estimate, although the constant in the current estimate looks “optimal”.

### 2.3. Extension from lc centres of higher codimension

**Definition 2.3.1.** Define \( \text{lc}_X^\sigma(S) \) to be the union of all lc centres of \((X, S)\) with codimension \( \sigma \) in \( X \) for any natural number \( \sigma \). Set also \( \text{lc}_X^0(S) := X \) for convenience. Moreover, let \( \mathcal{I}_{\text{lc}_X^\sigma(S)} \) be the ideal sheaf defining the reduced structure of \( \text{lc}_X^\sigma(S) \).

**Remark 2.3.2.** If \( \pi: \tilde{X} \to X \) is a log-resolution of \((X, \varphi_L, \psi)\) and if \( \tilde{S} \) is the reduced divisor with snc defined as in Section 2.1, then the subvariety \( \text{lc}_X^\sigma(S) \) given in Section 1.3 is indeed given by

\[
\text{lc}_X^\sigma(S) := \pi \left( \text{lc}_X^\sigma \left( \tilde{S} \right) \right),
\]

where the right-hand-side is given by Definition 2.3.1. Notice that the two definitions coincide when \( S \) itself is a reduced divisor with snc.

When mlc of \((X, S)\) have codimension > 1 (or when \( \sigma_f > 1 \), see Definition 1.3.1) and if one insists in using the Bochner–Kodaira inequality directly as above, the author *cannot* find any choices of the functions \( \nu, \chi_\epsilon, \ell_\epsilon \) and even \( \theta_\epsilon \) such that inequalities similar to (eq 2.2.5a) and (eq 2.2.5b) would hold while the integral of \( f \) on the right-hand-side of the estimate is replaced by

\[
\int_{\text{lc}_X^\sigma(S)} f^2 d\text{lcv}_{\varphi_L}^\sigma[\psi] := \lim_{\epsilon \to 0^+} \frac{\epsilon}{1-\epsilon} \int_X \frac{|f|^2}{|\psi|^{\sigma+\epsilon}} e^{-\varphi_L-\psi},
\]

where \( \sigma \) is some integer \( \geq \sigma_f > 1 \).

Indeed, a natural attempt to prove this higher codimensional case is to replace every \( \psi \) in the proof of Theorem 2.2.1 by \( -|\psi|^\sigma \). However, this would give rise to a negative term in the curvature form \( \Theta \) (see (eq 2.2.4) and compare it with (eq 2.3.2)) and the form can no longer be semi-positive. Fortunately, the magnitude of the negative term is small on the support of \( \theta_\epsilon \) (notation as in and before Theorem 2.2.1) and it goes to 0 as \( \epsilon \to 0^+ \). One just has to make use of the function \( \theta_\epsilon \) in the expression of \( \psi_\epsilon^{(1)} \) to restrict the support of the relevant integrand. This is the contents in the proof of Theorem 2.3.3.

For the purpose of stating the next theorem and for notational convenience,
\begin{itemize}
  \item set \(\omega_b := \omega + b \left( \sqrt{-1} \partial \overline{\partial}(\varphi_L + \psi) - \sqrt{-1} \partial \overline{\partial} \log |\psi| \right)\) for every \(b \geq 0\);
  \item redefine \(\theta_x\) and \(\theta'_x\) as \(\theta_x := \theta \circ |\psi|^{-\sigma_x}\) and \(\theta'_x := \theta' \circ |\psi|^{-\sigma_x}\), and set also \(\theta''_x := \theta'' \circ |\psi|^{-\sigma_x}\);
  \item set \(M_{\theta'} := \left( \frac{A - B}{AB} + \varepsilon_0 \right)^2\) and take a constant \(C_{\theta'} > 0\) such that \(|\theta' (t)|^2 \leq M_{\theta'}\) and \(|\theta'' (t)|^2 \leq C_{\theta'}\) for all \(t \leq 0\);
  \item let \(e_\sigma := \sqrt{\text{e}}\).
\end{itemize}

**Theorem 2.3.3.** Let \(\sigma \geq 2\) be a positive integer and assume the assumption (1) in Theorem 2.2.1 as well as the following variance of the assumption (2):

\((2)_a\) for any given constant \(\ell > 0\) and using the notation \(e_\sigma := \sqrt{\text{e}}\), the function \(\psi\) is normalised (by adding to it a suitable constant) such that

\[
\psi < -\frac{e_\sigma}{\ell} \quad \text{and} \quad \frac{5}{|\psi| \log |\psi|} + \frac{\sigma}{|\psi|} \leq \delta .
\]

Suppose again that \(\tilde{f}\) is the \(K_X \otimes \mathbb{L}\)-valued smooth section on \(X\) such that \(|\overline{\partial} \tilde{f}|^2_{\omega} e^{-\varphi_L - \psi} \log |\ell|_{e_\sigma}|\) is integrable over \(X\). Let \(v_\varepsilon\) be the section again given by

\[
v_\varepsilon := \overline{\partial} \left( \theta \left( \frac{1}{|\psi|^{\sigma}} \right) \tilde{f} \right) = -\varepsilon \theta'_x \overline{\partial} \left| \psi \right|^{\sigma} \wedge \tilde{f} + \theta_x \overline{\partial} \tilde{f} .
\]

Then, for any (sufficiently small\(^7\) numbers \(\varepsilon, \varepsilon', \alpha, b > 0\), there exist a smooth \(K_X \otimes \mathbb{L}\)-valued sections \(u^{(0,1)}_{x,\varepsilon}\) and a smooth \((0,1)\)-form \(w^{(0,1)}_{x,\varepsilon}\) on \(X^0\) such that

\[
\text{eq 2.3.1} \quad \overline{\partial} u^{(0,1)}_{x,\varepsilon} + w^{(0,1)}_{x,\varepsilon} = v_\varepsilon \quad \text{on } X^0
\]

in the sense of currents, with the estimates

\[
\int_{X^0} \left| u^{(0,1)}_{x,\varepsilon} \right|^2 e^{-\varphi_L - \psi} \left( \frac{c_\alpha}{\varepsilon^{\beta} + \varepsilon'} \right) \int_{X^0} \left| w^{(0,1)}_{x,\varepsilon} \right|^2 e^{-\varphi_L - \psi} \left( \frac{1}{\varepsilon^{\beta} + \varepsilon'} \right) \leq \frac{\sigma}{\ell} \int_{X^0} \left| \theta_x \overline{\partial} \tilde{f} \right|^2 e^{-\varphi_L - \psi} \log |\ell|_{e_\sigma} + \frac{(1 + \alpha) M_{\theta'} \varepsilon}{(1 - \varepsilon)} \int_{\text{supp} \theta'} \left| \psi \right|^{\sigma + \sigma_x} ,
\]

where \(c_\alpha > 0\) is a constant independent of \(\varepsilon, \varepsilon', \alpha, b\) (but could possibly go to 0 when \(\alpha \to 0^+\)).

**Proof.** The first half of the proof goes as in the proof of Theorem 2.2.1. All notations used in this proof carry the same meanings as in the proof there unless stated otherwise.

Set, as before,

\[
\tilde{\nu}(t) := -\log |\ell|_{e} , \quad \tilde{\eta}_x(t) := |t| e^{-\tilde{\nu}} = |t| e^{-\tilde{\nu}} , \quad \nu := \tilde{\nu} (-|\psi|^{\sigma}) , \quad \eta_x := \tilde{\eta}_x (-|\psi|^{\sigma}) \quad \text{and} \quad \lambda_x := \tilde{\lambda}_x (-|\psi|^{\sigma}) .
\]

\(^7\)The estimate is still valid when the numbers \(\varepsilon, \varepsilon', \alpha, b\) are not small, but the constants appearing in the estimate may look more tedious.
Let \( L \) be endowed with a metric with potential \( \varphi := \varphi_L + \psi + \nu \). The curvature form as in \( \text{eq 2.3.2} \) is then

\[
\Theta := \eta_e i \partial \overline{\partial} (\varphi_L + \psi + \nu) - i \partial \eta_e - \frac{1}{\lambda_e} i \partial \eta_e \wedge \overline{\partial} \eta_e
\]

\[
= \eta_e i \partial \overline{\partial} (\varphi_L + \psi) + \eta_e (\overline{\nu}' - (\log \overline{\eta}_e)') \circ (\frac{1}{|\psi|^{\sigma}}) \ i \partial \overline{\partial} (-|\psi|^{\sigma}) + \frac{2}{\lambda_e} \ i \partial \overline{\partial} \psi
\]

\[
+ \left( \frac{1}{|\psi|^{\sigma}} + \frac{2}{\lambda_e} \right) \ i \partial \overline{\partial} \psi
\]

As in the proof of Theorem 2.2.1 (with every formula with variable \( t \) being unchanged except that \( \ell \) is replaced by \( \ell^\sigma \) ), \( \lambda_e \) is chosen such that

\[
\Gamma = \frac{\varepsilon}{e^\nu |t|^{1+\varepsilon}} \quad \text{for all } t < -\frac{e}{\ell^\sigma}.
\]

One then has

\[
0 < \overline{\lambda}_e = \overline{\eta}_e \left( 1 - \frac{1}{\log |\ell^\sigma|} \right)^2 \leq |t|^{1-\varepsilon} (\log |\ell^\sigma t|)^2 e^{-\overline{\nu}},
\]

and thus \( (\eta_e + \lambda_e) e^{\nu} \leq |\psi|^{\sigma - \varepsilon} (|\sigma \log |\ell^\sigma|) + 1 \) . By the positivity assumption \( \text{[1]} \) in Theorem 2.2.1 and the normalisation assumption \( \text{[2]} \), it also follows that

\[
\Theta \geq -\Lambda(\psi) \ i \partial \psi \wedge \overline{\partial} \psi + \Gamma(-|\psi|^{\sigma}) \ i \partial |\psi|^{\sigma} \wedge \overline{\partial} |\psi|^{\sigma} \quad \text{on } X.
\]

Although \( \omega \) is not assumed to be complete in the statement, the standard argument (see, for example, \[10\] Ch. VIII, \( \S 6 \)) reduces the problem to the case where \( \omega \) is complete on \( X^\circ \), which is assumed to be the case in what follows. Moreover, it follows from the positivity assumption \( \text{[1]} \) and the normalisation assumption \( \text{[2]} \) that the \( d \)-closed \((1,1)\)-form

\[
i \partial \overline{\partial} (\varphi_L + \psi) - i \partial \overline{\partial} |\psi| = i \partial \overline{\partial} (\varphi_L + \psi) + \frac{1}{|\psi|} i \partial \overline{\partial} \psi + \frac{1}{|\psi|^2} i \partial \psi \wedge \overline{\partial} \psi
\]
is non-negative. Therefore, \( \omega_b \) is a Kähler metric on \( X^o \) (which is assumed to be complete in the rest of the proof). Notice that one has

\[
\text{eq } 2.3.4 \quad |\partial \psi|_{\omega_b}^2 \leq \frac{|\psi|}{b}.
\]

Write \( \langle \cdot, \cdot \rangle_b := \langle \cdot, \cdot \rangle_{X^o, \omega_b, \psi} \), the global inner product on \( X^o \) induced by the potential \( \varphi = \varphi_L + \psi + \nu \) and the metric \( \omega_b \). Moreover, let

\[
e^{-\bar{\mu}(t)} := \bar{\ell}_t (-|t|^\sigma) \cdot (\log |\ell t|)^3 \quad \text{and} \quad \mu := \bar{\mu}(\psi)
\]

and write

- \( \cdot \) ker \( \sigma \) and \( \partial \) as, respectively, the orthogonal projection onto the kernel of \( \mathcal{D} \) and the formal adjoint of \( \mathcal{D} \) with respect to the inner product induced by the metrics \( e^{-\varphi} \) and \( \omega_b \);
- \( \cdot \) ker \( \sigma, \mu \) and \( \partial \mu \) as the same operators but with respect to the inner product induced by the metrics \( e^{-\varphi-\mu} \) and \( \omega_b \); note that \( \partial \mu(\cdot) = \partial(\cdot) + \bar{\mu}(\psi) (\partial \psi)^{\omega_b} \cdot \cdot \) and the image of \( \cdot \) ker \( \sigma, \mu \) is \( L^2 \) with respect to \( \langle \cdot, \cdot \rangle_b \) as \( e^{-\mu} \) is strictly positive on \( X^o \).

It remains to obtain a suitable bound for the inner product \( \langle \zeta, v_\epsilon \rangle_b \) for any compactly supported smooth \( K_X \otimes L \)-valued \((0,1)\)-form \( \zeta \) on \( X^o \) in order to invoke the Riesz Representation Theorem. First note that, for such a given \( \zeta \), there exists a sequence \( \{ \xi_k^{(\mu)} \}_k \) of \( \mathcal{D} \)-closed \( K_X \otimes L \)-valued \( L^2 \) \((0,2)\)-forms (with respect to \( \langle \cdot, \cdot \rangle_b \)) such that

\[
\zeta - \partial \mu \xi_k^{(\mu)} \rightarrow (\zeta) \text{ ker } \sigma, \mu \quad \text{with respect to } \langle \cdot, \cdot \rangle_b \text{ as } k \rightarrow \infty
\]

and

\[
\langle \zeta, v_\epsilon \rangle_b \xrightarrow{k \rightarrow \infty} \langle (\zeta) \text{ ker } \sigma, \mu + \partial \mu \xi_k^{(\mu)}, \mathcal{D}(\theta_\epsilon \bar{f}) \rangle_b
\]

\[
= \langle (\zeta) \text{ ker } \sigma, \mu + \bar{\mu}(\psi) (\partial \psi)^{\omega_b} \cdot \xi_k^{(\mu)}, \mathcal{D}(\theta_\epsilon \bar{f}) \rangle_b
\]

\[
= \langle (\zeta) \text{ ker } \sigma, \mu, \xi_k^{(1)} \rangle_b + \langle (\zeta) \text{ ker } \sigma, \mu + \bar{\mu}(\psi) (\partial \psi)^{\omega_b} \cdot \xi_k^{(2)} \rangle_b.
\]

The last two terms are estimated separately below.

To estimate \( \langle (\zeta) \text{ ker } \sigma, \mu, \xi_k^{(1)} \rangle_b \), notice that

\[
\partial(\zeta) \text{ ker } \sigma, \mu = \partial \mu \zeta - \bar{\mu}(\psi) (\partial \psi)^{\omega_b} \cdot (\zeta) \text{ ker } \sigma, \mu
\]

\[
= \partial \zeta + \bar{\mu}(\psi) (\partial \psi)^{\omega_b} \cdot (\zeta - \bar{\mu}(\psi) (\partial \psi)^{\omega_b} \cdot (\zeta) \text{ ker } \sigma, \mu
\]

\[
\text{eq } 2.3.5 \quad = \partial \zeta + \left( \sigma - \sigma \epsilon + \frac{1}{\log |\psi|_\sigma^2} + \frac{3}{\log |\ell \psi|} \right) \frac{(\partial \psi)^{\omega_b}}{|\psi|^2} \cdot (\zeta - (\zeta) \text{ ker } \sigma, \mu).
\]

Moreover, on \( \text{ supp } \theta' \epsilon \) and when \( \epsilon \ll 1 \), one has the estimates

\[
\frac{1}{|\psi|^2} \leq \frac{1}{B} \iff \log (\ell ^2 \sigma) \leq \sigma \epsilon \log |\ell \psi|,
\]
\[
\tilde{\mu}'(\psi) \leq \left( \frac{\sigma - \sigma \varepsilon}{\log(\ell \sigma B^2)} + \frac{3\sigma}{\log(\ell \sigma B^2)} \right) \frac{1}{|\psi|} \leq \frac{\sigma + 1}{|\psi|},
\]
\[
\Lambda(\psi) \leq \left( 1 - \varepsilon + \frac{2}{\log(\ell \sigma B^2)} \right) \sigma(\sigma - 1) \frac{\eta_{\varepsilon}}{|\psi|^2} \leq 2\sigma(\sigma - 1) \frac{\eta_{\varepsilon}}{|\psi|^2},
\]
\[
\eta_{\varepsilon} + \lambda_{\varepsilon} \leq C\eta_{\varepsilon}(\log |\ell \psi|)^2 \quad \text{for some positive constant } C.
\]

Therefore, one obtains
\[
\left| \langle (\zeta)_{\ker \varpi_{\mu}}, \nu_{\varepsilon}^{(1)} \rangle \right|_b = \left| \left( -\varepsilon \frac{\tilde{f}}{|\psi|^{\sigma + \sigma \varepsilon}} \middle| \frac{\partial \psi}{|\psi|} \right) \right| \leq \left( \int_{|\psi|^2 \geq 2} \left( \int_{\sup \theta_{\varepsilon}^*} \frac{|\tilde{f}|^2}{|\psi|^2} e^{-\frac{\varepsilon}{\psi - \varepsilon} - \varepsilon} \right)^{\frac{1}{2}} \right).
\]

by \eqref{eq.2.3.3}
\[
\left( \int_{X_0} \left| \varrho_{\theta_{\varepsilon}}^\mu \right|^2 \lambda(\psi) \right)^{\frac{1}{2}} \leq \left( \int_{X_0} \left( \partial \varrho_{\theta_{\varepsilon}}^\mu \right) \lambda(\psi) \right)^{\frac{1}{2}} \leq \left( \int_{X_0} \left| \varrho_{\theta_{\varepsilon}}^\mu \right|^2 \lambda(\psi) \right)^{\frac{1}{2}} \leq \left( \int_{X_0} \left( \partial \varrho_{\theta_{\varepsilon}}^\mu \right) \lambda(\psi) \right)^{\frac{1}{2}}.
\]

by \eqref{eq.2.3.2}
\[
\left( \int_{X_0} \left| \varrho_{\theta_{\varepsilon}}^\mu \right|^2 \lambda(\psi) \right)^{\frac{1}{2}} \leq \left( \int_{X_0} \left| \varrho_{\theta_{\varepsilon}}^\mu \right|^2 \lambda(\psi) \right)^{\frac{1}{2}} \leq \left( \int_{X_0} \left| \varrho_{\theta_{\varepsilon}}^\mu \right|^2 \lambda(\psi) \right)^{\frac{1}{2}} \leq \left( \int_{X_0} \left| \varrho_{\theta_{\varepsilon}}^\mu \right|^2 \lambda(\psi) \right)^{\frac{1}{2}}.
\]

by \eqref{eq.2.3.3}
\[
\left( \int_{X_0} \left| \varrho_{\theta_{\varepsilon}}^\mu \right|^2 \lambda(\psi) \right)^{\frac{1}{2}} \leq \left( \int_{X_0} \left| \varrho_{\theta_{\varepsilon}}^\mu \right|^2 \lambda(\psi) \right)^{\frac{1}{2}} \leq \left( \int_{X_0} \left| \varrho_{\theta_{\varepsilon}}^\mu \right|^2 \lambda(\psi) \right)^{\frac{1}{2}} \leq \left( \int_{X_0} \left| \varrho_{\theta_{\varepsilon}}^\mu \right|^2 \lambda(\psi) \right)^{\frac{1}{2}}.
\]

and
\[
\left( (1 + \alpha) M_{\theta_{\varepsilon}} \int_{X_0} |\partial \psi|_{\varphi, \omega_{\varphi, \omega}}^2 \right)^{\frac{1}{2}} \leq \left( \int_{X_0} \left| \varrho_{\theta_{\varepsilon}}^\mu \right|^2 \lambda(\psi) \right)^{\frac{1}{2}} \leq \left( \int_{X_0} \left| \varrho_{\theta_{\varepsilon}}^\mu \right|^2 \lambda(\psi) \right)^{\frac{1}{2}}.
\]
Moreover, the normalisation assumption also infers that the untwisted version of the Bochner–Kodaira inequality with respect to the norm with assumption (2) via

$$\Theta := \left( \int_{X^o} \left| \partial \zeta \right|^2 (\eta + \lambda \varepsilon) + \frac{C_{\alpha}}{b} \varepsilon \int_{X^o} |\zeta|^2 \bar{\omega}_b \eta \varepsilon (\log |\ell \psi|)^3 \right)^{\frac{1}{2}} \cdot \left( \frac{1 + \alpha}{1 - \varepsilon} \int_{\text{supp} \phi} |\ell \psi|^{\sigma + \sigma \varepsilon} \right)^{\frac{1}{2}},$$

where \( \alpha \) and \( C_{\alpha} \) are some positive constants which does not depend on \( \varepsilon \).

Next is to estimate \( \left( \zeta_{\text{ker} \phi, \mu} + \bar{\mu}'(\psi) (\partial \psi)^{\omega_b} \right) \xi_{\mu} \), \( v_{(2)} \). It is easy to get the bound for

$$\left| \left( \zeta_{\text{ker} \phi, \mu} + \bar{\mu}'(\psi) \right) (\partial \psi)^{\omega_b} \right|^2_b \leq \int_{X^o} \left| \zeta_{\text{ker} \phi, \mu} \right|^2 \omega_b \left( e^{-\varphi} \int \theta \omega_b \right)^2 e^{-\varphi} \leq \int_{X^o} \left| \zeta \right|^2 \omega_b e^{-\varphi - \mu} \cdot \int_{X^o} \left| \theta \omega_b \right|^2 e^{-\varphi}.$$

Notice that the fact \( \left| \theta \omega_b \right|^2 \leq \left| \theta \omega_b \right|^2 \) is invoked here. To estimate the remaining term, the untwisted version of the Bochner–Kodaira inequality with respect to the norm with weight \( e^{-\varphi - \mu} \) obtained from (eq 2.2.2) simply by replacing \( \xi \) by 1, \( \lambda \) by 0 and \( \varphi \) by \( \varphi + \mu \) is considered. The curvature form in this case, denoted by \( \Theta_{\mu} \), is given by

$$\Theta_{\mu} := i\bar{\partial} \left( \varphi_L + \psi + \nu + \mu \right) = i\bar{\partial} \left( \varphi_L + \psi \right) + \left( \sigma |\psi|^{\sigma-1} \bar{\vartheta}(-|\psi|^{\sigma}) + \bar{\mu}'(\psi) \right) i\bar{\partial} \psi + \left( \sigma |\psi|^{2\sigma - 2} \bar{\vartheta}'(-|\psi|^{\sigma}) - \sigma(\sigma - 1) |\psi|^{\sigma - 2} \bar{\vartheta}(-|\psi|^{\sigma}) + \bar{\mu}''(\psi) \right) i\bar{\partial} \psi \wedge \bar{\partial} \psi$$

$$= i\bar{\partial} \left( \varphi_L + \psi \right) + \left( \frac{\sigma - \sigma \varepsilon}{|\psi|} + \frac{2}{|\psi| \log \left( \frac{\psi}{|\psi|} \right)} + \frac{3}{|\psi| \log |\ell \psi|} \right) i\bar{\partial} \psi + \left( \frac{\sigma - \sigma \varepsilon}{\log \left( \frac{\psi}{|\psi|} \right)} + \frac{2}{\log \left( \frac{\psi}{|\psi|} \right)^2} + \frac{3}{\log |\ell \psi|} + \frac{3}{\log |\ell \psi|} \right) i\bar{\partial} \psi \wedge \bar{\partial} \psi \right) \left( \frac{\sigma - \sigma \varepsilon}{\log \left( \frac{\psi}{|\psi|} \right)} + \frac{2}{\log \left( \frac{\psi}{|\psi|} \right)^2} + \frac{3}{\log |\ell \psi|} + \frac{3}{\log |\ell \psi|} \right) i\bar{\partial} \psi \wedge \bar{\partial} \psi.$$

It follows from the positivity assumption \( (1) \) in Theorem 2.2.2 and the normalisation assumption \( (2)_{\sigma} \) (which infers that the coefficient of \( i\bar{\partial} \psi \wedge \bar{\partial} \psi \) is \( \leq \delta \)) that

$$\Theta_{\mu} \geq \left( \sigma - \sigma \varepsilon + \frac{1}{\log \left( \frac{\psi}{|\psi|} \right)} + \frac{3}{\log |\ell \psi|} \right) i\bar{\partial} \psi \wedge \bar{\partial} \psi \leq \left( \sigma - \sigma \varepsilon + \frac{1}{\log \left( \frac{\psi}{|\psi|} \right)} + \frac{3}{\log |\ell \psi|} \right)$$

on \( X^o \).

Moreover, the normalisation assumption also infers that

$$|\bar{\mu}'(\psi)|^2 e^{\mu} = \frac{1}{|\psi|^2} \left( \frac{\sigma - \sigma \varepsilon + \frac{1}{\log \left( \frac{\psi}{|\psi|} \right)} + \frac{3}{\log |\ell \psi|}}{\sigma |\psi|^{\sigma -\sigma \varepsilon} \log \left( \frac{\psi}{|\psi|} \right) \log |\ell \psi|^{\sigma -\sigma \varepsilon} \log \left( \frac{\psi}{|\psi|} \right) \log |\ell \psi|} \right)^2$$

$$\leq \frac{1}{|\psi|^2} \frac{\delta \sigma^2}{|\psi|^{\sigma -\sigma \varepsilon} \log \left( \frac{\psi}{|\psi|} \right) \log |\ell \psi|^{\sigma -\sigma \varepsilon} \log \left( \frac{\psi}{|\psi|} \right) \log |\ell \psi|} \left( \sigma - \sigma \varepsilon + \frac{1}{\log \left( \frac{\psi}{|\psi|} \right)} + \frac{3}{\log |\ell \psi|} \right)$$

\( ^8 \)This is the place which explains why the coefficient “5” instead of “2” occurs in the normalisation condition \( (2)_{\sigma} \).
\[
\frac{\sigma^{-1} \geq 1}{|\psi|^2} \left( \sigma - \sigma \varepsilon + \frac{1}{\log \left| \frac{\psi}{|\psi|^2} \right|} + \frac{3}{\log |\ell\psi|} \right)\]
on $X^\circ$ for some constant $C > 0$ which depends only on $\psi, \ell$ and $\sigma$. These, together with the Bochner–Kodaira inequality and the fact that all $\xi_k^{(\mu)}$s are $\partial$-closed, yield
\[
\left| \left\langle \vec{\mu}'(\psi)(\partial \psi) \right\rangle \right|_{b}^2 \leq \int_{X^\circ} \left| (\partial \psi) \right|_{\omega, b}^2 \xi_{k}^{(\mu)}(\psi)^2 |\vec{\mu}'(\psi)|^2 e^{C} \cdot \int_{X^\circ} \left| \psi \partial f \right|_{\omega}^2 e^{-\psi}
\leq \delta \sigma^2 C \int_{X^\circ} \Theta_{\mu}^{(\psi)}(\xi_{k}^{(\mu)}, \xi_{k}^{(\mu)}) \cdot \int_{X^\circ} \left| \psi \partial f \right|_{\omega}^2 e^{-\psi}
\leq \delta \sigma^2 C \int_{X^\circ} \left| \partial \mu \xi_{k}^{(\mu)} \right|_{b_{\omega, b}}^2 \cdot \int_{X^\circ} \left| \psi \partial f \right|_{\omega}^2 e^{-\psi}.
\]
Notice that the integral $\int_{X^\circ} \left| \partial \mu \xi_{k}^{(\mu)} \right|_{b_{\omega, b}}^2 e^{-\mu}$ converges to $\int_{X^\circ} \left| \xi - (\xi)_{\theta} \right|^2_{b_{\omega, b}} e^{-\mu}$ as $k \to \infty$, which is dominated by $\int_{X^\circ} |\xi|^2_{\varphi, \omega b} e^{-\mu}$. Collecting all the estimates above with the treatment as in part of the proof of Theorem 2.2.1 where the constant $\varepsilon' > 0$ is introduced, one has, after letting $k \to \infty$,
\[
|\langle \zeta, v \rangle_b| \leq \left( \int_{X^\circ} |\partial \zeta|^2 \left( \eta_c + \lambda_c \right) + \left( \frac{C_{\zeta}}{b} \varepsilon + \left( 1 + \delta \sigma^2 C \right) 2\varepsilon' \right) \int_{X^\circ} |\xi|^2_{b_{\omega, b}} \eta_c (\log |\ell\psi|)^2 \right)^{1/2}
\cdot \left( \frac{2\sigma}{2\varepsilon' \int_{X^\circ} \left| \psi \partial f \right|_{\omega}^2 e^{-\psi}} \log \left| \frac{\psi}{|\psi|^2} \right| e^{\sigma} + \left( 1 + \alpha \right) M_{\varphi, \omega} \int_{\sup \theta_c^b} \left| \psi \right|^{\sigma, \omega} \right)^{1/2}.
\]
As a result, the Riesz Representation Theorem then provides the sections and estimates as asserted. \qed

Next is to show that (eq.2.3.1) still holds true on the whole of $X$. However, Lemma 2.2.3 is not sufficient for the purpose when $\sigma > 1$. The following strengthened lemma is needed.

**Lemma 2.3.4.** On an open set $\Omega \subset \mathbb{C}^n$, let $z_1$ be a complex coordinate on $\Omega$ such that $\sup \Omega |z_1|^2 < 1$. If there are
- a $(n, 1)$-form $v$ with coefficients in $L^2_{\text{loc}}(\Omega)$ (notice that it requires $v$ to be in $L^2_{\text{loc}}$ instead of just $L^1_{\text{loc}}$, and it is restricted to an $(n, 1)$-form instead of a $(p, q)$-form; cf. Lemma 2.2.3 and
- a $(n, 0)$-form $u$ with coefficients in $L^2_{\text{loc}}(\Omega; |\log |z_1|^2|^{-s})$, where $s > 1$,
satisfying
\[
\overline{\partial} u = v \quad \text{on} \quad \Omega \setminus \{ z_1 = 0 \}
\]
in the sense of currents, then the same equation holds in the sense of currents on the whole of $\Omega$.

**Proof.** The objective is to prove that, under the given assumptions, coefficients of $u$ are in fact in $L^2_{\text{loc}}(\Omega; |\log |z_1|^2|^{-s-1+\delta})$ for some given $\delta \in (0, 1]$, and thus in $L^2_{\text{loc}}(\Omega; |\log |z_1|^2|^{-1-\delta'})$ for some $\delta' \in [0, 1]$ by induction. The lemma can then be concluded using Lemma 2.2.3.
Using Lemma 2.2.3, it can be seen that the equations
\[
\frac{v}{|\log z_1|^\frac{1}{2+\delta}} = \frac{\bar{\partial} u}{|\log z_1|^\frac{1}{2+\delta}} = \partial \left( \frac{u}{|\log z_1|^\frac{1}{2+\delta}} \right) - \frac{s - 1 + \delta}{2} \frac{dz_1 \wedge u}{z_1 |\log z_1|^\frac{1}{2+\delta+1}}
\]
hold on the whole of \(\Omega\), as coefficients of \(\frac{u}{|\log z_1|^\frac{1}{2+\delta}}\) are apparently in \(L^2_{\text{loc}}(\Omega; |\log z_1|^-(1-\delta))\), and, on any compact \(K \subset \Omega\), the Cauchy–Schwarz inequality yields
\[
\int_K \frac{|dz_1 \wedge u|}{|\log z_1|^\frac{1}{2+\delta+1}} \leq \left( \int_K \frac{|dz_1|^2}{|\log z_1|^2} |\log z_1|^2 |\log z_1|^2 \right)^{\frac{1}{2}} \left( \int_K |u|^2 \right)^{\frac{1}{2}},
\]
which shows that the coefficients of \(\frac{dz_1 \wedge u}{|\log z_1|^\frac{1}{2+\delta+1}}\) are in \(L^1_{\text{loc}}(\Omega)\) (the first integral on the right-hand-side is convergent when \(\delta > 0\)). Those of \(\frac{v}{|\log z_1|^\frac{1}{2+\delta}}\) are also in \(L^1_{\text{loc}}(\Omega)\) since so are those of \(v\).

To show that coefficients of \(u\) are actually in \(L^2_{\text{loc}}(\Omega; |\log z_1|^-(s-1+\delta))\), let \(\Omega' \subset \Omega\) be a smaller polydisc. Choose a complete Kähler metric on \(\Omega'\) which is modified from a smooth complete Kähler metric \(\bar{\omega}\), given by
\[
\omega := \bar{\omega} + \sqrt{-1} \frac{dz_1 \wedge d\bar{z}_1}{|z_1|^2},
\]
such that
\[
|dz_1|_\omega^2 \leq |z_1|^2.
\]
Choose also a smooth strictly psh function \(\varphi\) on \(\Omega\) (so it is continuous on \(\bar{\Omega}\)). Let \(\|\cdot\|_{\omega, -\varphi}\) be the global \(L^2\) norm on \(\Omega'\) induced from \(\omega\) with the weight \(e^\varphi\) (instead of \(e^{-\varphi}\)). Then, for every compactly supported smooth \((n, 0)\)-form \(\zeta\) on \(\Omega'\), one has
\[
\partial \bar{\partial} \zeta = -\nabla^\omega \nabla_{\bar{\omega}} \zeta = -\nabla^\omega \nabla_{\bar{\zeta}} + \left( \nabla^\omega \nabla_{\bar{\zeta}} - \nabla^\bar{\omega} \nabla_{\zeta} \right) \zeta = -\nabla^\omega \nabla_{\bar{\zeta}} + \text{Tr}^\omega (\partial \bar{\partial} \varphi) \zeta,
\]
where \(\nabla_j\) and \(\nabla_{\bar{j}}\) are the covariant differential operators along respectively the tangent fields \(\frac{\partial}{\partial z_j}\) and \(\frac{\partial}{\partial \bar{z}_j}\) (\(z_j\) being a holomorphic coordinate) induced from the Chern connection with respect to \(\omega\) and \(e^\varphi\), and \(\nabla^\omega\) and \(\nabla^\bar{\omega}\) are the operators obtained from raising the indices of \(\nabla_j\) and \(\nabla_{\bar{j}}\) via the inverse of \(\omega\). \(\text{Tr}^\omega\) is the trace operator with respect to \(\omega\) on \((1, 1)\)-forms. Note that the Einstein summation convention is used in the above formula. Moreover, the formula is well-defined everywhere on \(\Omega'\) even though \(\omega\) is singular along \(z_1 = 0\), since only the inverse of \(\omega\) is involved when differentiating differential forms covariantly. Contracting the above formula with \(\zeta e^\varphi\) and applying integration by parts, one obtains the \(\nabla^{(1,0)}\)-Bochner–Kodaira formula (see [34], (1.3.4) and (1.3.5))
\[
\text{(eq 2.3.6)} \quad \| \bar{\partial} \zeta \|_{\omega, -\varphi}^2 = \| \nabla^{(1,0)} \zeta \|_{\omega, -\varphi}^2 + \int_{\Omega'} \text{Tr}^\omega (\partial \bar{\partial} \varphi) |\zeta|^2 e^\varphi \geq C_{K, \varphi} \| \zeta \|_{K}^2,
\]
where \(K \subset \Omega'\) is some compact subset, and \(C_{K, \varphi} := \inf_K e^\varphi \text{Tr}^\omega (\partial \bar{\partial} \varphi) > 0\). Note that \(C_{K, \varphi}\) is positive since \(i \partial \bar{\partial} \varphi\) is (strictly) positive in every directions but \(\omega\) is singular only
in the $z_1$-direction on $K$ (i.e. the inverse of $\omega$ only vanishes in the $z_1$-direction). By the completeness of $\omega$, the inequality is valid for all $L^2$ forms $\zeta \in \text{Dom}(\overline{\partial})$ on $\Omega'$. 

Now notice that the coefficients of $\frac{u}{|\log |z_1|^2|^{s+1+\delta}}$ are in $L^2(\Omega')$. One also has

$$\int_{\Omega'} \left( \frac{d\zeta \wedge u}{|z_1| \log |z_1|^2} \right)^2 \leq \int_{\Omega'} \frac{|d\zeta|^2}{|z_1|^2} |u|^2 \leq \int_{\Omega'} \frac{|u|^2}{|\log |z_1|^2|^{s+1+\delta}} < \infty,$$

which implies that coefficients of $\overline{\partial}\left(\frac{u}{|\log |z_1|^2|^{s+1+\delta}}\right)$ are in $L^2((\Omega', \omega; \epsilon^\varphi))$. Therefore, substituting $\zeta = \frac{u}{|\log |z_1|^2|^{s+1+\delta}}$ in the $\nabla^{(1,0)}$-Bochner–Kodaira inequality (eq 2.3.6) yields that coefficients of $u$ are in $L^2_{\text{loc}}(\Omega'; |\log |z_1|^2|^{-s+1+\delta})$ and hence in $L^2_{\text{loc}}(\Omega; |\log |z_1|^2|^{-s+1+\delta})$ as $\Omega' \subset \Omega$ is arbitrary. This completes the proof.

**Proposition 2.3.5.** Under the assumption (1) in Theorem 2.2.1 and assumption (2) in Theorem 2.3.3, the equation [eq 2.3.1], namely $\overline{\partial} u_{\epsilon, \varphi} + w_{\epsilon, \varphi}^{(0,1)} = v_{\epsilon}^{(0,1)}$, which comes with the estimate given in Theorem 2.3.3 holds true on the whole of $X$.

**Proof.** A similar argument as in the proof of Proposition 2.2.3 shows that the equation [eq 2.3.1] holds on $X \setminus P_\psi$. It remains to invoke Lemma 2.3.4 to have the equation continued to the complement of a codimension-2 subset in $X$, and thus consequently the whole of $X$. Notice that, using [eq 2.2.7] and the fact that $\varphi_L + \psi$ is locally bounded above, coefficients of $u$ and $w_{\epsilon, \varphi}^{(0,1)}$ are in $P_{\text{loc}}^2(X; |\psi|^\sigma)$ and $P_{\text{loc}}^2((X, \omega_\psi); |\psi|^{-\sigma+\sigma_\epsilon}(\log |\zeta\psi|)^3)$ respectively. As coefficients of $w_{\epsilon, \varphi}^{(0,1)}$ are not in $L^2_{\text{loc}}(X)$ (unweighted), one has to solve

$$\overline{\partial} s_{\epsilon, \varphi} = w_{\epsilon, \varphi}^{(0,1)}$$

locally with suitable estimate for $s_{\epsilon, \varphi}$ (which is proved below), and then invoke Lemma 2.3.4 with $u_{\epsilon, \varphi} + s_{\epsilon, \varphi}$ in place of $u$ in the Lemma.

First notice that $\overline{\partial} w_{\epsilon, \varphi}^{(0,1)} = 0$ on $X^\circ$. Since there is an analytic subset $R_2 \subset P_\psi$ of codimension 2 in $X$ such that $\psi \equiv_{\text{loc}} c \log |z|^2$ mod $\mathcal{C}^\infty$ for some constant $c > 0$ and holomorphic coordinate $z$ around any points in $P_\psi \setminus R_2$, Lemma 2.3.4 then assures that $w_{\epsilon, \varphi}^{(0,1)}$ is $\overline{\partial}$-closed also on $X \setminus R_2$.

Next is to solve $\overline{\partial} s_{\epsilon, \varphi} = w_{\epsilon, \varphi}^{(0,1)}$ using Hörmander’s $L^2$ method. For later use, a more general claim is proved here.

**Lemma 2.3.6.** Suppose that $w^{(0,1)}$ is a $K \otimes L$-valued $(0, 1)$-form such that

$$\|w^{(0,1)}\|^2_{\sigma,-(3)} := \int_{X^\circ} |w^{(0,1)}|^2 e^{-\varphi_L - \psi}$$

is finite.

Then, for any polydisc $V \subset X$ on which $\overline{\partial} w^{(0,1)} = 0$ and

$$\psi \equiv_{\text{loc}} \sum_{j=1}^n c_j \log |z_j|^2 \mod \mathcal{C}^\infty\]"
for some constants \( c_j \geq 0 \), where \( z_j \) are local holomorphic coordinates, one can solve the \( \overline{\partial} \)-equation \( \overline{\partial}s = w^{(0,1)} \) on \( V \cap X^o \) for \( s \) with an estimate

\[
\|s\|_{V, \sigma, -,(3)}^2 := \int_{V \cap X^o} |s|^2 e^{-\varphi_L - \psi} \leq C \|w^{(0,1)}\|_{\sigma, -,(3)}^2,
\]

where \( C > 0 \) is a constant independent of \( \varepsilon \).

Proof. Notice that

\[
\Theta_{(3)} := i\overline{\partial}(\varphi_L + \psi + \log(|\psi|^{\sigma - \varepsilon}(\log |\ell\psi|)^3))
\]

\[
= i\overline{\partial}(\varphi_L + \psi) - \left(\frac{\sigma - \varepsilon}{|\psi|} + \frac{3}{|\psi| \log |\ell\psi|}\right) i\overline{\partial}\psi
\]

\[
- \left(\frac{\sigma - \varepsilon}{|\psi|^2} + \frac{3}{|\psi|^2 \log |\ell\psi|} + \frac{3}{|\psi|^2 (\log |\ell\psi|)^2}\right) i\overline{\partial}\psi \wedge \overline{\partial}\psi.
\]

Since \( \frac{1}{|\psi|} i\overline{\partial} \log |z_j|^2 = 0 \) as a current for each \( j \), it can be seen that \( \Theta_{(3)} \geq -c_{\omega_b} \) for some constant \( c' > 0 \) on \( V \). Therefore, by choosing a suitable smooth strictly psh function on \( V \), one can solve the \( \overline{\partial} \)-equation \( \overline{\partial}s = w^{(0,1)} \) for a smooth section \( s \) on \( V \cap X^o \) using Hörmander’s \( L^2 \) method which comes with an estimate \( \|s\|_{V, \sigma, -,(3)}^2 \leq C \|w^{(0,1)}\|_{\sigma, -,(3)}^2 \) for some constant \( C > 0 \) independent of \( \varepsilon \). \( \square \)

As a result, on any polydisc \( V \) in \( X \setminus R_2 \), there is a \( K_X \otimes L \)-valued section \( s_{e', \varepsilon} \) on \( V \) such that \( \overline{\partial}s_{e', \varepsilon} = w^{(0,1)}_{e', \varepsilon} \) on \( V \cap X^o \) and \( s_{e', \varepsilon} \in L^2(V; |\psi|^{-\sigma}) \) (and thus \( u_{e', \varepsilon} + s_{e', \varepsilon} \in L^2_{\text{loc}}(V_\varepsilon; |\psi|^{-\sigma}) \)). With all \( u_{e', \varepsilon}, s_{e', \varepsilon} \) and \( v_\varepsilon \) belonging to \( L^2_{\text{loc}}(X \setminus P_\varepsilon) \), it follows that

\[
\overline{\partial}(u_{e', \varepsilon} + s_{e', \varepsilon}) = \overline{\partial}u_{e', \varepsilon} + w^{(0,1)}_{e', \varepsilon} = v_\varepsilon
\]
on \( V \setminus P_\varepsilon \). Lemma 2.3.3 then implies that the above equation, or equivalently (eq 2.3.1), can be continued across \( P_\varepsilon \setminus R_2 \), and therefore continued to \( X \setminus R_2 \). Since \( R_2 \) is of codimension 2, the equation can also be continued to the whole of \( X \) using the argument as in the proof of Proposition 2.2.4 (namely, the Riemann continuation theorem for holomorphic functions). \( \square \)

**Theorem 2.3.7.** Let \( \sigma \geq 2 \) be a positive integer and assume the positivity assumption \( \{1\} \) in Theorem 2.2.1 and the normalisation assumption \( \{2\}_{\varepsilon} \) in Theorem 2.3.3.

Let \( f \) be any holomorphic section in \( H^0(S, K_X \otimes L \otimes \mathcal{J}(\varphi_L)) \). If \( \sigma \) is a positive integer such that

\[
\int_{K_X(S)} |f|^2 d\text{lcv}^\sigma_{\omega_{\varphi_L}}[\psi] < \infty,
\]

then there exists a holomorphic section \( F \in H^0(X, K_X \otimes L \otimes \mathcal{J}(\varphi_L)) \) such that

\[
F \equiv f \mod \mathcal{J}(\varphi_L) \cdot \mathcal{I}_{K_X(S)}
\]

(see Definition 2.3.1 for the notation \( \mathcal{I}_{K_X(S)} \)) with the estimate

\[
\int_X \frac{|F|^2 e^{-\varphi_L - \psi}}{|\psi|^\sigma ((\sigma \log |\ell\psi|)^2 + 1)} \leq \frac{1}{\sigma} \int_{K_X(S)} |f|^2 d\text{lcv}^\sigma_{\omega_{\varphi_L}}[\psi].
\]
Proof. Following the argument in the proof of Theorem 2.2.5, there exists a smooth section \( \tilde{f} \) such that \( f \equiv \tilde{f} \mod \mathcal{E}_X^\infty \otimes \mathcal{A}(\varphi_L + \psi) \) and

\[
\left| \partial \tilde{f} \right|^2_\omega e^{-\varphi_L - \psi} \log \left| \ell_\varphi \right|_\sigma e \leq \left| \tilde{f} \right|^2_\omega e^{-\varphi_L - \psi} \log \left| \ell_\varphi \right|_\sigma e
\]

is integrable over \( X \). Theorem 2.3.3 can then be invoked to obtain the equation (eq 2.3.1) with the estimates. Proposition 2.3.5 also assures that the equation holds on the whole of \( X \).

Set \( \varepsilon' := \left( \int_X \left| \theta_s \partial \tilde{f} \right|^2_\omega e^{-\varphi_L - \psi} \log \left| \ell_\varphi \right|_\sigma e \right)^{\frac{1}{2}} \), which converges to 0 as \( \varepsilon \to 0^+ \), and suppress all the subscripts “\( \varepsilon' \)” in what follows as in the proof of Theorem 2.2.5. Let \( F_\varepsilon := \theta_s \tilde{f} - u_\varepsilon \), such that \( \overline{\partial}F_\varepsilon = w_\varepsilon^{(0,1)} \) on \( X \) in the sense of currents. One then obtains the estimate

\[
\left( 1 + \frac{1}{\alpha} \right) \int_X \left| \theta_s \tilde{f} \right|^2_\omega e^{-\varphi_L - \psi} \leq \left( 1 + \frac{1}{\alpha} \right) \int_X \left| \theta_s \tilde{f} \right|^2_\omega e^{-\varphi_L - \psi} + \left( \frac{c_\alpha}{\varepsilon + \varepsilon'} \right) \int_X \left| \theta_s \tilde{f} \right|^2_\omega e^{-\varphi_L - \psi} \log \left| \ell_\varphi \right|_\sigma e \left( \frac{1}{\varepsilon} \right) ^{\frac{1}{2}} 
\]

The right-hand-side converges to \( \frac{(1 + \alpha)^2 M_\sigma}{\varepsilon} \int_{lcv(S)} |f|^2 d \ell_\varphi \sigma e \), which is independent of \( b \), as \( \tilde{f} \) is treated as an \( (n, 0) \)-form and thus the integral before taking the limit is independent of \( \omega \) or \( \omega_b \).

A weakly convergent subsequence \( \{ F_{\varepsilon, \mu} \}_\mu \) can be extracted from the family \( \{ F_\varepsilon \}_\varepsilon \). Set \( F := \lim_{\mu \to 0^+} F_{\varepsilon, \mu} \). It remains to justify that \( F \) is the desired holomorphic extension of \( f \).

It can be seen that \( \overline{\partial}F_\varepsilon = w_\varepsilon^{(0,1)} \) and that \( w_\varepsilon^{(0,1)} \) converges to 0 in \( L^2 \left( (X, \omega_b) ; \frac{e^{-\varphi_L - \psi}}{|\psi|^{2 - \sigma \varepsilon (\log |\ell_\varphi|)^{\frac{2}{2}}} \right) \) as \( \varepsilon \to 0^+ \). By Lemma 2.3.6 and the fact that the \( \psi \) is bounded above and the polar ideal sheaf of \( \psi \) is given by a snc divisor, on any open polydisc \( V \subset X \), one can solve the \( \overline{\partial} \)-equation \( \overline{\partial}s_\varepsilon = w_\varepsilon^{(0,1)} \) on \( V \cap X^0 \) for a section \( s_\varepsilon \) on \( V \) using Hörmander’s \( L^2 \) method which comes with an estimate \( ||s_\varepsilon||_{V, (\sigma, \cdot)} \leq C' \left| w_\varepsilon^{(0,1)} \right|_{\sigma, \cdot} \) for some constant \( C' > 0 \) independent of \( \varepsilon \).

Following the argument as in the proof of Proposition 2.3.5, since \( \overline{\partial}(F_\varepsilon - s_\varepsilon) = 0 \) on \( V \cap X^0 \) with both \( F_\varepsilon \) and \( s_\varepsilon \) belonging to some suitable weighted-\( L^2 \) space, it follows that the equality holds on \( V \setminus F_\varepsilon \), and then on the complement \( V \setminus R_2 \) of some codimension-2 subset \( R_2 \) by Lemma 2.3.4 and thus on the whole of \( V \) by the Riemann continuation theorem.

The section \( s_\varepsilon \) is smooth according to the regularity of the \( \overline{\partial} \) operator and the smoothness of \( w_\varepsilon^{(0,1)} \). The weight in the norm of \( s_\varepsilon \) and the smoothness of \( w_\varepsilon^{(0,1)} \). The weight in the norm of \( s_\varepsilon \) is a smooth section \( s_\varepsilon \in \mathcal{E}_X^\infty \otimes \mathcal{A}(\varphi_L) \cdot lcv[S] \) on \( V \) for each \( \varepsilon \) and the estimates afterwards for the more explicit calculation, and the estimate implies that \( s_\varepsilon \to 0 \) in the \( L^2 \) norm as \( \varepsilon \to 0^+ \). By passing to suitable subsequences, one has \( s_{\varepsilon, \mu_k} \to 0 \) pointwisely a.e. on \( V \) while \( F_{\varepsilon, \mu_k} \to F \) weakly in the weighted \( L^2 \) norm on \( X \) as \( \varepsilon, \mu_k \to 0^+ \).
The rest of the argument is similar to that in the proof of Theorem 2.2.5. The section $F_{\varepsilon_{\mu_k}} - s_{\varepsilon_{\mu_k}}$ is a holomorphic extension on $V$ from $\mathrm{lc}^\varepsilon(V)$ of $f$ (which means, more precisely, $F_{\varepsilon_{\mu_k}} - s_{\varepsilon_{\mu_k}} \equiv f \mod \mathcal{I}(\varphi_L) \cdot \mathcal{I}_{\mathrm{lc}^\varepsilon(S)}$ on $V$) such that both norm-squares

$$
\int_V \frac{|F_{\varepsilon_{\mu_k}} - s_{\varepsilon_{\mu_k}}|^2}{|\psi|^2((\sigma \log |\ell\psi|)^2 + 1)} \leq \int_V \frac{|F_{\varepsilon_{\mu_k}} - s_{\varepsilon_{\mu_k}}|^2 e^{-\varphi_L - \psi}}{|\psi|^2((\sigma \log |\ell\psi|)^2 + 1)}
$$

are bounded above uniformly in $\varepsilon_{\mu_k}$. The denominator in the integrand being $L^1$ implies that $F_{\varepsilon_{\mu_k}} - s_{\varepsilon_{\mu_k}}$ is also bounded above in $L^1(V)$ uniformly in $\varepsilon_{\mu_k}$ via the use of the Cauchy–Schwarz inequality on the norm-square on the left-hand-side above. Cauchy’s estimate together with Montel’s theorem assures that a subsequence of $\{F_{\varepsilon_{\mu_k}} - s_{\varepsilon_{\mu_k}}\}_k$ converges locally uniformly in $V$ to a holomorphic extension $F_V$ such that $F_V \equiv f \mod \mathcal{I}(\varphi_L) \cdot \mathcal{I}_{\mathrm{lc}^\varepsilon(S)}$ on $V$ (via an application of Fatou’s lemma on the norm-square on the right-hand-side above), which then infers that a subsequence of $\{F_{\varepsilon_{\mu_k}}\}_k$ converges pointwisely a.e. on $V$ to $F_V$, and thus $F = F_V$ a.e. on $V$. Considering all open polydiscs $V \subset X$, it follows that $F$, after possibly altering its values on a measure 0 set, is a holomorphic extension in the sense that $F \equiv f \mod \mathcal{I}(\varphi_L) \cdot \mathcal{I}_{\mathrm{lc}^\varepsilon(S)}$ on $X$.

Finally, the required estimate is obtained, after letting $\varepsilon \to 0^+$, by letting $\alpha \to 0^+$, $b \to 0^+$, $A \to +\infty$, $B \to 1^+$ and $\varepsilon_0 \to 0^+$ (so that $M_{\varepsilon_0} \to 1^+$) subsequently.

By an induction on the codimension of the lc centres from which a section is extended from, one can obtain the required holomorphic extension of $f$ from $S$ to $X$ with estimates.

**Theorem 2.3.8.** Suppose that the positivity assumption (1) in Theorem 2.2.5 and the normalisation assumption (2) in Theorem 2.3.3 (with $\sigma$ being at most the codimension of $\mathrm{mlc}$ of $(X, S)$) hold. Then, for any holomorphic section $f \in H^0(S, K_X \otimes L \otimes \frac{\mathcal{I}(\varphi_L)}{\mathcal{I}(\varphi_L + \psi)})$, one has

$$
\int_{\mathrm{lc}^\varepsilon(S)} |f|^2 \, d\mathrm{lcv}_{\omega, \varphi_L} [\psi] < \infty,
$$

and there exists a sequence $\{F_{\sigma}\}_{\sigma = 1}^\infty \subset H^0(X, K_X \otimes L \otimes \mathcal{I}(\varphi_L + \psi))$ such that

$$
\sum_{\sigma = 1}^{\sigma_f} F_{\sigma} \equiv f \mod \mathcal{I}(\varphi_L + \psi)
$$

with estimates

$$
\int_X |F_{\sigma}|^2 e^{-\varphi_L - \psi} \leq \frac{1}{\sigma} \int_{\mathrm{lc}^\varepsilon(S)} |f| \, d\mathrm{lcv}_{\omega, \varphi_L} [\psi] < \infty
$$

for all $\sigma = 1, \ldots, \sigma_f$.

**Proof.** Simply apply Theorem 2.3.7 successively, where $\sigma$ is running from $\sigma_f$ to 1, to extend $f - \sum_{\sigma = 1}^{\sigma_f} F_i \mod \mathcal{I}(\varphi_L + \psi)$ from $\mathrm{lc}^\varepsilon(S)$ to $X$ and obtain $F_{\sigma}$ with the acclaimed estimate (with $f - \sum_{i=\sigma+1}^{\sigma_f} F_i$ being set to $f$ when $\sigma = \sigma_f$). Notice that the integral

$$
\varepsilon \int_X F_{\sigma} \left| f - \sum_{i=\sigma+1}^{\sigma_f} F_i \right| e^{-\varphi_L - \psi} |\psi|^2 \, d\omega < \infty
$$

is finite for all $\varepsilon > 0$ and for each $\sigma = \sigma_f, \ldots, 1$, where $\tilde{f}$ is the smooth extension of $f$ as in Theorem 2.3.7 because
each \( F_\sigma \) belongs to \( \mathcal{J}(\varphi_L) \) (as can be seen from the estimate on it by applying the inequality (eq 2.2.7) to the weight \( e^{-\psi} \left( (\sigma \log |\psi|)^{r+1} \right) \)),

- \( f - \sum_{i=\sigma+1}^{\sigma'} F_i \in \mathcal{J}(\varphi_L) \cdot \mathcal{I}_{k,\alpha^+(S)} \), and
- \( \varphi_L \) and \( \psi \) both have only neat analytic singularities.

The computation is similar to the proof of Proposition 3.0.1 with Remarks 3.0.3 and 3.0.4 taken into account.

The claim can then be concluded by noticing that \( \mathcal{J}(\varphi_L + \psi) = \mathcal{J}(\varphi_L) \cdot \mathcal{I}_{k,\alpha^+(S)} \).

**Remark 2.3.9.** In order to obtain Theorem 1.3.2 which is applicable to general values of \( m_0 \) and \( m_1 \), simply replace \( \varphi_L \) and \( \varphi_L + \psi \) in the proofs of Theorems 2.3.7 and 2.3.8 by, respectively, \( \varphi_L + m_0 \psi \) and \( \varphi_L + m_1 \psi \). All arguments remain the same.

### 3. The measures on lc centres

In this section, \( \varphi_L \) and \( \psi \) are assumed again to have their associated polar ideal sheaves being principal with their corresponding divisors having only snc with each other, according to the discussion in Section 2.2.

The well-definedness of the measure on lc centres of \((X, S)\) of codimension \( \sigma \) (called the “lc-measure” or the “\( \sigma \)-lc-measure” for short in this section) is justified below. Define \( \tilde{\varphi}_L \) by

\[
\tilde{\varphi}_L + \psi_S := \varphi_L + m_1 \psi,
\]

where \( \psi_S := \phi_S - \varphi_S^{sm} < 0 \) (see Notation 0.0.3 for the meaning of \( \phi_S \) and \( \varphi_S^{sm} \)).

A potential \( \varphi \) is said to have Kawamata log-terminal (klt) singularities if \( \mathcal{J}(\varphi) = \mathcal{O}_X \).

**Proposition 3.0.1.** Given the snc assumption on \( \varphi_L \) and \( \psi \), suppose that

((†) \( \tilde{\varphi}_L \) has only klt singularities and \( \tilde{\varphi}_L^{-1}(-\infty) \cup \tilde{\varphi}_L^{-1}(\infty) \) does not contain any lc centres of \((X, S)\).

Suppose also that \( V \) is an open coordinate neighbourhood on which

\[
\psi|_V = \sum_{j=1}^{\sigma_V} \nu_j \log |z_j|^2 + \sum_{k=\sigma_V+1}^{n} c_k \log |z_k|^2 + \alpha \quad \text{and}
\]

\[
\tilde{\varphi}_L|_V = \sum_{k=\sigma_V+1}^{n} \ell_k \log |z_k|^2 + \beta,
\]

where

- each \( z_j \) is a holomorphic coordinate and \((r_j, \theta_j)\) its corresponding polar coordinates on \( V \) for \( j = 1, \ldots, n \),
- \( \alpha \) and \( \beta \) are smooth functions such that \( \sup_V \alpha < 0 \),
- \( S \cap V = \{ z_1 \cdots z_{\sigma_V} = 0 \} \),
- \( \sup_V \frac{\partial \nu_j}{\partial z_j} \alpha > -1 \) (i.e. \( \sup_V r_j \) is sufficiently small) for \( j = 1, \ldots, \sigma_f \),
- \( \sup_V \log |z_k|^2 < 0 \) for \( k = \sigma_f + 1, \ldots, n \),
- \( \nu_j \)’s are constants such that \( \nu_j > 0 \) for \( j = 1, \ldots, \sigma_V \), and
- \( c_k \)’s are constants such that \( c_k \geq 0 \) for \( k = \sigma_V + 1, \ldots, n \),
- \( \ell_k \)’s are constants such that \( \ell_k < 1 \) (due to the klt assumption, possibly negative) for \( k = \sigma_V + 1, \ldots, n \).
Then, for any compactly supported smooth function $f$ on $V$ such that

$$|f| = \prod_{k=\sigma_f+1}^{\sigma_V} |z_k|^{1+a_k} \cdot g \quad \text{with} \quad \inf_{V} g > 0$$

for some non-negative integer $\sigma_f \leq \sigma_V$ and non-negative integers $a_{\sigma_f+1}, \ldots, a_{\sigma_V}$ (and set $\prod_{k=\sigma_f+1}^{\sigma_V} |z_k|^{1+a_k} = 1$ when $\sigma_f = \sigma_V$) and compactly supported smooth real-valued function $g$, one has

$$\int_{c^*_V(S)} |f|^2 d \text{vol}_{\omega, \varphi_L}^{(m)} [\psi] := \lim_{\varepsilon \to 0^+} \varepsilon \int_{X} \frac{|f|^2 e^{-\varepsilon \varphi_L - m \psi} \text{vol}_{X, \omega}}{|\psi|^\alpha + \varepsilon}$$

where $S^{\sigma_f} := \{ z_1 \cdots z_{\sigma_f} = 0 \}$.

Remark 3.0.2. With the snc assumption on $\varphi_L$ and $\psi$, it is easy to see that $X$ can be covered by the kind of open coordinate neighbourhoods described in the proposition.

Proof. Choosing the canonical section defining $\phi_S$ suitably, it follows that

$$\varepsilon \int_{X} \frac{|f|^2 e^{-\varepsilon \varphi_L - m \psi} \text{vol}_{X, \omega}}{|\psi|^\alpha + \varepsilon} = \varepsilon \int_{V} \frac{F_0}{|\psi|^\alpha + \varepsilon} \det \omega \bigwedge_{j=1}^{n} \left( \frac{1}{\sqrt{-1}} d z_j \wedge d \overline{z}_j \right) \prod_{k=\sigma_f+1}^{\sigma_V} \frac{|z_k|^{\alpha_k} \cdot g^2 e^{-\varepsilon \varphi_L + \psi_S} \text{vol}_{S^{\sigma_f}, \omega}}{|\psi|^\alpha + \varepsilon}$$

where $\ell_{\sigma_f+1}, \ell_{\sigma_f+2}, \ldots, \ell_{\sigma_V}$ and $a_{\sigma_V+1}, a_{\sigma_V+2}, \ldots, a_n$ are all defined to be 0. In view of Fubini’s Theorem, integrations with respect to the variables $z_{\sigma_f+1}, \ldots, z_n$ are done at the last step. Since all $(\ell_k - a_k)$’s are $< 1$, the integral with respect to all variables is convergent as soon as the integral with respect to variables $z_1, \ldots, z_{\sigma_f}$ is bounded. The differentials corresponding to $z_{\sigma_f+1}, \ldots, z_n$ are made implicit in what follows. Notice that $F_0$ is a smooth function.

Observe that, if $\sigma_f = 0$, the integral above is convergent and bounded above by $O(\varepsilon)$. Therefore, it goes to 0 when $\varepsilon \to 0^+$.

Assume that $\sigma_f \geq 1$ in what follows. Set

$$t_j := \nu_j \log r_j^2 = \nu_j \log |z_j|^2.$$
The integrand is integrated with respect to each \( r_j \) over \([0, 1]\) (thus to each \( t_j \) over \((-\infty, 0]\)) and to each \( \theta_j \) over \([0, 2\pi]\). Write also \( \partial_j \) for \( \frac{\partial}{\partial j} \). The integral in question then becomes

\[
\frac{\varepsilon}{\prod_{j=1}^{\sigma_f} t_{j}} \int \frac{F_{0}}{|\psi|^{\sigma_f}} \prod_{j=1}^{\sigma_f} d\tau_{j} \cdot \prod_{j=1}^{\sigma_f} dt_{j} \cdot d\theta = \frac{\varepsilon}{\nu} \int \frac{F_{0}}{1 + \frac{r_{1}}{2\nu_{1}} \partial_{\tau_{1}}} \frac{d|\psi|^{\sigma_f}}{|\psi|^{\sigma_f - \varepsilon}} \prod_{j=2}^{\sigma_f} dt_{j} \cdot d\theta
\]

(\*)

\[
= \frac{\varepsilon}{\nu(\sigma - 1 + \varepsilon)} \int \frac{F_{0}}{1 + \frac{r_{1}}{2\nu_{1}} \partial_{\tau_{1}}} \frac{d}{d\theta} \left( \frac{1}{|\psi|^{\sigma_f - \varepsilon}} \right) dr_{1} \prod_{j=3}^{\sigma_f} dt_{j} \cdot d\theta
\]

(\**)

\[
= \cdots = \frac{(-1)^{\sigma_f} \varepsilon}{\nu \prod_{j=1}^{\sigma_f} (\sigma - j + \varepsilon)} \int \frac{F_{\sigma_f}}{|\psi|^{\sigma_f - \varepsilon}} \prod_{j=1}^{\sigma_f} dr_{j} \cdot d\theta.
\]

Note that the \( F_{j} \)'s are defined inductively by

\[
F_{j} := \partial_{r_{j}} \left( \frac{F_{j-1}}{1 + \frac{r_{j}}{2\nu_{j}} \partial_{r_{j}}} \right),
\]

and all of them are smooth functions. When \( \sigma > \sigma_{f} \), the last expression in (\***) is bounded above by \( O(\varepsilon) \). Therefore, the integral tends to 0 again as \( \varepsilon \to 0^{+} \). When \( \sigma = \sigma_{f} \), the last expression in (\***) is bounded above, but the multiple constant in front of the integral does not converge to 0 as \( \varepsilon \to 0^{+} \). After letting \( \varepsilon \to 0^{+} \), the dominated convergence theorem and the fundamental theorem of calculus gives

\[
\frac{(-1)^{\sigma_f}}{(\sigma_{f} - 1)!} \nu \int F_{\sigma_{f}} \prod_{j=1}^{\sigma_{f}} dr_{j} \cdot d\theta = \frac{(-1)^{\sigma_{f} - 1}}{(\sigma_{f} - 1)!} \nu \int F_{\sigma_{f} - 1} \bigg|_{r_{\sigma_{f} - 1} = 0} \prod_{j=1}^{\sigma_{f} - 1} dr_{j} \cdot d\theta
\]

\[
= \cdots = \frac{1}{(\sigma_{f} - 1)!} \nu \int F_{0} |S_{\sigma_{f}} d\theta = \frac{2\pi}{(\sigma_{f} - 1)!} \nu \int |S_{\sigma_{f}} \prod_{k=\sigma_{f} + 1}^{\sigma_{f} - 1} |z_{k}|^{a_{k}} \cdot g^{2} e^{-\tilde{\varphi}_{L + \varphi_{m}}^{a_{m}}} d\text{vol}_{S^{\sigma_{f}, \omega}},
\]

which is the desired result.

It remains to check for the case \( \sigma < \sigma_{f} \) but \( \sigma_{f} \geq 1 \). Consider a further change of variables

\[
|\psi| = |\psi|, \quad q_{j} := \frac{t_{j}}{|\psi|} = \frac{|t_{j}|}{|\psi|} \quad \text{for} \quad j = 2, \ldots, \sigma_{f}.
\]
where each $q_j$ varies within $[0, 1]$ on $V$. The expression in (3.0.5) then becomes

$$\mathcal{X} = \frac{(-1)^{\sigma_f} \varepsilon}{V} \int \frac{F_0}{1 + \frac{1}{2r_1} \partial r_1 \alpha} \frac{|\psi|^{\sigma_f-1} d|\psi|}{|\psi|^{\sigma + \varepsilon}} \prod_{j=2}^{\sigma_f} dq_j \cdot d\theta,$$  

$$= \frac{(-1)^{\sigma_f} \varepsilon}{V (\sigma_f - \sigma - \varepsilon)} \int \frac{F_0}{1 + \frac{1}{2r_1} \partial r_1 \alpha} d\left(|\psi|^{\sigma_f-\varepsilon}\right) \prod_{j=2}^{\sigma_f} dq_j \cdot d\theta.$$

Notice that the factor $(-1)^{\sigma_f}$ is there only to account for the difference in orientation between the coordinate systems $(r_1, \ldots, r_{\sigma_f})$ and $(|\psi|, q_2, \ldots, q_{\sigma_f})$. The whole expression is itself non-negative. As $\frac{F_0}{1 + \frac{1}{2r_1} \partial r_1 \alpha} > 0$ on $V$ and $d(|\psi|^{\sigma_f-\varepsilon})$ is non-integrable on $V$ when $\varepsilon$ is sufficiently small, the expression above tends to $\infty$ as $\varepsilon \to 0^+$. □

**Remark 3.0.3.** For a general compactly supported function $f$ on $X$, on every local coordinate neighbourhood $V$ such that $S \cap V = \{z_1 \ldots z_{\sigma_V} = 0\}$, there is an integer $\sigma_f$ such that

$$f|_V = \sum_{p \in \mathcal{S}_{\sigma_V}(\mathcal{E}_r x \times \mathcal{E}_r f)} \prod_{k=\sigma_f+1}^{\sigma_V} |z_{p(k)}|^{1+q_{p(k)}} \cdot g_p \quad \text{with} \quad \max_{p, x \in V} |g_p(x)| > 0,$$

where every $p$ is a choice of $\sigma_V - \sigma_f$ elements from the set $\{1, 2, \ldots, \sigma_V\}$, and each $g_p$ is a bounded function on $V$ which is smooth in the variables other than $z_{p(\sigma_f+1)}, \ldots, z_{p(\sigma_V)}$. It is not difficult to see that sumsmands of the sum over $p \in \mathcal{S}_{\sigma_V}/(\mathcal{S}_{\sigma_V-\sigma_f} \times \mathcal{S}_{\sigma_f})$ are mutually orthogonal with respect to the inner product induced from $dlcv_{\omega_{\varphi_L}} [\psi]$ when $\sigma \geq \sigma_f$ or $\sigma_f - 2 < \sigma < \sigma_f$. Therefore, using a partition of unity, the results in the proposition still hold for each $f|_V$, except that the integral in the case $\sigma = \sigma_f$ is now the sum of integrals over all lc centres in $V$ of codimension $\sigma_f$, namely, $\mathcal{S}^{\sigma_f} := \{z_{p(1)} \ldots z_{p(\sigma_f)} = 0\}$. Note that the largest $\sigma_f$ among all different local neighbourhoods covering $X$ is the codimension of mlc of $(X, S)$ with respect to $f$ (see Definitions 3.0.1 or 3.0.2). Considering all $f|_V$’s, the proposition also holds true for $f$ with $\sigma_f$ being the codimension of mlc of $(X, S)$ with respect to $f$ in all cases of $\sigma$ (after the modification for the case $\sigma = \sigma_f$).

**Remark 3.0.4.** If $f$ vanishes to suitable orders along the polar subspaces of $\tilde{\varphi}_L$ and $\psi$, the assumption (3.0.4) is not necessary in the proposition. For example, if $f \in \mathcal{G}^{\infty} \otimes \mathcal{F}(\varphi_L + m_0 \psi)$, which implies that $f \in \mathcal{G}^{\infty} \otimes \mathcal{F}(\tilde{\varphi}_L)$ as

$$G := |f|^2 e^{-\tilde{\varphi}_L} = |f|^2 e^{-\varphi_L-m_1 \psi-\varphi_3^m+\phi_S} = |f \otimes s_S|^2 e^{-\varphi_L-m_1 \psi-\varphi_3^m}$$

and $S$ is defined by $\text{Ann}_{\mathcal{S}_X}(\mathcal{A}_{\varphi_L+m_0 \psi}/\mathcal{A}_{\varphi_L+m_1 \psi})$ by definition. Moreover, $G^{-1}(\infty)$ does not contain any lc centres of $(X, S)$, as can be seen from the assumptions that $m_1$ is a jumping number and the polar ideal sheaves of $\varphi_L$ and $\psi$ are principal. Let $\sigma_f$ be the maximal codimension in $X$ of all lc centres of $(X, S)$ contained in $G^{-1}(0)$. Then the integral in the proposition still converges even without the assumption (3.0.5).

**Definition 3.0.5.** Given any function or vector-bundle-valued section $f$ on $S$ such that $f \in \mathcal{G}_X^{\infty} \otimes \mathcal{A}_{\varphi_L+m_0 \psi}$ with $\tilde{f} \in \mathcal{G}_X^{\infty} \otimes \mathcal{F}(\varphi_L + m_0 \psi)$ denoting any local lifting of $f$, define the **codimension of mlc of $(X, S)$ (or of $(X, \varphi_L, \psi, m_1)$) with respect to $f$, denoted by $\sigma_f$, to be the maximal codimension in $X$ of all the lc centres of $(X, S)$ which are not contained**
in the zero locus of $G := \left| \tilde{f} \right|^2 e^{-\tilde{\varphi}_L}$. An lc centre of $(X, S)$ with such codimension in $X$ which does not lie in $G^{-1}(0)$ is called an \textit{mlc of $(X, S)$ with respect to $f$}.

It can be seen from Proposition 3.0.1 and the remarks that follow that this definition coincides with Definition 1.3.1 when the snc assumption on $\varphi_L$ and $\psi$ holds.

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