Phase space picture of neutrino mixing and oscillations

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Abstract. We consider a simple classical model in phase-space resembling the quantum one used for the description of neutrino oscillations and investigate the possibility of defining an analogue to the mixing transformation and to the oscillation formula in terms of generalized coordinates.

1. Introduction
The phenomenon of neutrino oscillations [1] is one of the most intriguing and fascinating ones in the context of particle physics. It is indeed a beautiful manifestation of quantum mechanics at work on macroscopic scales and it is a continuous source for novel physical insights and deeper analysis. In fact, in recent years many interesting features emerged in the study of neutrino mixing and oscillations in the context of Quantum Field Theory [2]: it was found indeed that the mixing transformations at level of fields contain Bogoliubov transformations [3, 4], which produce dramatic changes in the structure of the vacuum inducing a condensate of particle-antiparticle pairs, reflecting into additional terms in the oscillation formulas [5, 6]. Another interesting feature is represented by the geometric phase(s) associated to neutrino oscillations, which are present already at QM level and could provide important information if detectable [7]. Finally, it has been shown that the phenomenon of neutrino mixing and oscillations can be equivalently understood and described in quantum information language: time dependent (single–particle) entanglement among neutrino flavor states is generated by the time evolution of the system and could be in principle used for quantum information tasks [8, 9].

In view of these developments, it is definitely interesting any attempt to reformulate such a phenomenon in a new language and from alternative points of view. In this report, we consider the correspondence of neutrino mixing transformations with transformations in (classical) phase space. This can be interesting since it can help to better understand a phenomenon which has a specific quantum nature by means of some classical system analogue for it. Also, the (classical) phase space picture could be at some point implemented with non-commutativity in order to fully reproduce the quantum mechanical framework. Here we make a first step in this direction, defining a consistent classical phase space picture of neutrino mixing and oscillations leading to the same oscillation formulas obtained in the quantum formulation.

Our analysis is limited to the case of oscillations between two flavors only. This report is structured as follows. In Section 2 we summarize the essential features of neutrino mixing and oscillations which then we rephrase for a system of two coupled oscillators. In Section 3 and 4 we...
present the mixing transformation and the oscillation formula in terms of generalized coordinates. Finally, in Section 5 we draw our conclusions.

2. Flavor mixing and oscillations
In this section we briefly review the simplest description for neutrino oscillations and we rephrase this in terms of a system of coupled oscillators.

2.1. Neutrino oscillations
Pontecorvo mixing transformations [1] are written as a rotation of the states with definite masses $m_1, m_2$, $|\nu_1\rangle, |\nu_2\rangle$, into those with definite flavors $|\nu_e\rangle$ and $|\nu_\mu\rangle$ as:

$$|\nu_e\rangle = \cos \theta |\nu_1\rangle + \sin \theta |\nu_2\rangle,$$

$$|\nu_\mu\rangle = \cos \theta |\nu_2\rangle - \sin \theta |\nu_1\rangle.$$  

(1)

(2)

Neutrino oscillations arise from time evolution of the state $|\nu_e\rangle$ in Eq.(1) which gives:

$$|\nu_e(t)\rangle = \cos \theta e^{-i\omega_1 t} |\nu_1\rangle + \sin \theta e^{-i\omega_2 t} |\nu_2\rangle.$$  

(3)

When computing the probability amplitude of finding a given neutrino, e.g. $\nu_e$, at a time $t$ from its generation, one obtains flavor oscillations:

$$P_{\nu_e \rightarrow \nu_e}(t) = |\langle \nu_e |\nu_e(t)\rangle|^2 = 1 - \sin^2 2\theta \sin^2 \left(\frac{\Delta \omega}{2} t\right) = 1 - P_{\nu_e \rightarrow \nu_\mu}(t).$$  

(4)

One can also verify that flavor conservation holds:

$$|\langle \nu_e |\nu_e(t)\rangle|^2 + |\langle \nu_\mu |\nu_e(t)\rangle|^2 = 1.$$  

(5)

2.2. Coupled oscillators
Since we are interested in the mixing transformation properties and in the above treatment no reference appears to the fermionic or bosonic nature of the particles, we treat such particles as bosons and consider the simplest case of two harmonic oscillators with different frequencies. The (normal ordered) Hamiltonian for such system is

$$H(\hat{a}_{1,2}:\hat{a}^\dagger_{1,2}) := \omega_1 \hat{a}^\dagger_1 \hat{a}_1 + \omega_2 \hat{a}^\dagger_2 \hat{a}_2,$$  

with $|a_1\rangle = \hat{a}^\dagger_1 |0\rangle_1$, $|a_2\rangle = \hat{a}^\dagger_2 |0\rangle_2$, $[\hat{a}_i, \hat{a}^\dagger_j] = \delta_{ij}$. We use $\hbar = 1$. Eqs.(1),(2) can be seen as arising by the application to the vacuum state $|0\rangle_{1,2} = |0\rangle_1 \otimes |0\rangle_2$ of the following flavor operators (adopting the conventional neutrino terminology):

$$\hat{a}_1^\dagger = \cos \theta \hat{a}_1^\dagger + \sin \theta \hat{a}_2^\dagger, \quad \hat{a}_2^\dagger = \cos \theta \hat{a}_2^\dagger - \sin \theta \hat{a}_1^\dagger.$$  

(7)

The transformation in Eq.(7) is a canonical one. The Hamiltonian in Eq.(6) is written in terms of the flavor ladder operators as:

$$H(\hat{a}_{e,\mu}:\hat{a}^\dagger_{e,\mu}) := \omega_e \hat{a}^\dagger_e \hat{a}_e + \omega_\mu \hat{a}^\dagger_\mu \hat{a}_\mu + \omega_{e\mu} (\hat{a}^\dagger_e \hat{a}_\mu + \hat{a}^\dagger_\mu \hat{a}_e),$$  

with

$$\omega_e = \omega_1 \cos^2 \theta + \omega_2 \sin^2 \theta,$$

$$\omega_\mu = \omega_2 \cos^2 \theta + \omega_1 \sin^2 \theta,$$

$$\omega_{e\mu} = (\omega_2 - \omega_1) \sin \theta \cos \theta.$$  

(9)

(8)
Thus we also have $\omega_1 + \omega_2 = \omega_\epsilon + \omega_\mu$. We remark that the hermiticity of $H(\hat{a}_{e,\mu}; \hat{a}_{e,\mu}^\dagger)$ is preserved, it is not spoiled by the non–diagonal mixing terms.

In order to discuss flavor oscillations within the classical phase–space picture of Section 3, we consider here the Heisenberg picture. Thus we introduce the flavor number operator $\hat{N}_e(t) = \hat{a}_e^\dagger(t)\hat{a}_e(t)$ on the one–particle state $|a_e\rangle$ using - cf. Eq.(7):

$$\hat{a}_e(t) = \cos \theta \hat{a}_1 e^{-i\omega_1 t} + \sin \theta \hat{a}_2 e^{-i\omega_2 t}, \quad \hat{a}_\mu(t) = \cos \theta \hat{a}_2 e^{-i\omega_2 t} - \sin \theta \hat{a}_1 e^{-i\omega_1 t}. \quad (10)$$

and h.c. We obtain the expectation value:

$$\langle a_e|\hat{N}_e(t)|a_e\rangle = \langle a_e| (\cos \theta \hat{a}_1^\dagger e^{i\omega_1 t} + \sin \theta \hat{a}_2^\dagger e^{i\omega_2 t})(\cos \theta \hat{a}_1 e^{-i\omega_1 t} + \sin \theta \hat{a}_2 e^{-i\omega_2 t}) |a_e\rangle$$

$$= 1 - 2 \sin^2 2\theta \sin^2 \left(\frac{\omega_2 - \omega_1}{2} t\right). \quad (11)$$

In the same way we obtain

$$\langle a_e|\hat{N}_\mu(t)|a_e\rangle = 2 \sin^2 2\theta \sin^2 \left(\frac{\omega_2 - \omega_1}{2} t\right)$$

where $\hat{N}_\mu(t) = \hat{a}_\mu^\dagger(t)\hat{a}_\mu(t)$.

3. Classical phase–space picture

We now consider a classical analogue of the above system, i.e. two harmonic oscillators, with the same mass $m$ but different frequencies $\omega_1 \neq \omega_2$. The total Hamiltonian is:

$$H(Q_{1,2}; P_{1,2}) = \frac{P_1^2}{2m} + \frac{\omega_1^2 m}{2} Q_1^2 + \frac{P_2^2}{2m} + \frac{\omega_2^2 m}{2} Q_2^2. \quad (13)$$

Defining

$$\alpha_i = \sqrt{\frac{m\omega_i}{2}} \left( Q_i + i \frac{P_i}{m\omega_i} \right), \quad \alpha_i^* = \sqrt{\frac{m\omega_i}{2}} \left( Q_i - i \frac{P_i}{m\omega_i} \right),$$

one can rewrite the Hamiltonian in Eq.(13) as

$$H(\alpha_{1,2}; \alpha_{1,2}^*) = \omega_1 |\alpha_1|^2 + \omega_2 |\alpha_2|^2. \quad (15)$$

where $\alpha_i$ are c–number counterparts of the above ladder operators and may be thought as eigenvalues of a coherent state corrisponding to the annihilation operator $\hat{a}_i$. In analogy with Eq.(7) we write the following transformations:

$$\alpha_e = \cos \theta \alpha_1 + \sin \theta \alpha_2, \quad \alpha_\mu = \cos \theta \alpha_2 - \sin \theta \alpha_1. \quad (16)$$

and c.c., and substitute them in Eq.(15), obtaining the Hamiltonian in the “flavor” variables:

$$H(\alpha_{e,\mu}; \alpha_{e,\mu}^*) = \omega_\epsilon |\alpha_e|^2 + \omega_\mu |\alpha_\mu|^2 + \omega_{e\mu}(\alpha_e^* \alpha_\mu + \alpha_\mu^* \alpha_e). \quad (17)$$

where $\omega_\epsilon, \omega_\mu$ and $\omega_{e\mu}$ are given in Eq.(8). Eq.(15) and Eq.(17) represent alternative forms of the Hamiltonian Eq.(13) in terms of $(\alpha_1, \alpha_2, \alpha_1^*, \alpha_2^*)$, $(\alpha_e, \alpha_\mu, \alpha_e^*, \alpha_\mu^*)$, respectively. It is now interesting to ask whether the same Hamiltonian can be written in terms of other canonical variables $(Q_e, Q_\mu, P_e, P_\mu)$ in such a way to close the chain depicted in the following graph:
Inspired by Eq. (7) and Eq. (16), we introduce the following canonical transformations:

\[ H(Q_{1,2}; P_{1,2}) \leftrightarrow H(\alpha_{1,2}; \alpha^*_{1,2}) \]

\[ \downarrow 1 \quad \uparrow 3 \]

\[ H(Q_{e,\mu}; P_{e,\mu}) \leftrightarrow H(\alpha_{e,\mu}; \alpha^*_{e,\mu}) \]

In analogy with Eq. (14) we define (\( \sigma = e, \mu \)):

\[ \alpha_\sigma = \sqrt{\frac{m \omega_\sigma}{2}} \left( Q_\sigma + i \frac{P_\sigma}{m \omega_\sigma} \right), \quad \alpha^*_\sigma = \sqrt{\frac{m \omega_\sigma}{2}} \left( Q_\sigma - i \frac{P_\sigma}{m \omega_\sigma} \right). \]  

(19)

Inspired by Eq. (7) and Eq. (16), we introduce the following canonical transformations:

\[ Q_e = \cos \theta \sqrt{\frac{\omega_1}{\omega_e}} Q_1 + \sin \theta \sqrt{\frac{\omega_2}{\omega_e}} Q_2, \quad P_e = \cos \theta \sqrt{\frac{\omega_1}{\omega_2}} P_1 + \sin \theta \sqrt{\frac{\omega_1}{\omega_2}} P_2, \]

(20)

\[ Q_\mu = \cos \theta \sqrt{\frac{\omega_2}{\omega_\mu}} Q_2 - \sin \theta \sqrt{\frac{\omega_1}{\omega_\mu}} Q_1, \quad P_\mu = \cos \theta \sqrt{\frac{\omega_1}{\omega_2}} P_2 - \sin \theta \sqrt{\frac{\omega_1}{\omega_2}} P_1. \]

(21)

They indeed guarantee the conservation of the Poisson brackets. For example:

\[ \{Q_e, P_\mu\} = \sum_{i=1}^{2} \left( \frac{\partial Q_e}{\partial Q_i} \frac{\partial P_\mu}{\partial \dot{Q}_i} - \frac{\partial Q_e}{\partial \dot{Q}_i} \frac{\partial P_\mu}{\partial Q_i} \right) = \cos^2 \theta \sqrt{\frac{\omega_1}{\omega_e}} \sqrt{\frac{\omega_1}{\omega_2}} + \sin^2 \theta \sqrt{\frac{\omega_2}{\omega_e}} \sqrt{\frac{\omega_1}{\omega_2}} = 1. \]  

(22)

The generator of the canonical (point) transformation in Eqs. (20), (21) is of the \( F_2 \)-type [10]:

\[ F_2(Q_{1,2}, P_{e,\mu}) = \left( \cos \theta \sqrt{\frac{\omega_1}{\omega_e}} Q_1 + \sin \theta \sqrt{\frac{\omega_2}{\omega_e}} Q_2 \right) P_e + \left( \cos \theta \sqrt{\frac{\omega_2}{\omega_\mu}} Q_2 - \sin \theta \sqrt{\frac{\omega_1}{\omega_\mu}} Q_1 \right) P_\mu; \]

\[ \dot{P}_{1,2} = \frac{\partial F_2}{\partial Q_{1,2}}, \quad \dot{Q}_{e,\mu} = \frac{\partial F_2}{\partial P_{e,\mu}}. \]

In order for the chain in Eq. (18) to close, we invert the transformations in Eqs. (20), (21) and substitute them in Eq. (13). We obtain:

\[ H(Q_{e,\mu}; P_{e,\mu}) = \frac{P^2}{2m} + \frac{P^2}{2m} + \frac{m \omega^2}{2} Q^2 + \frac{m \omega^2}{2} Q^2 + \omega_{e,\mu} \left( \frac{P_e}{\sqrt{m \omega_e \sqrt{m \omega_\mu}}} + \frac{P_\mu}{\sqrt{m \omega_\mu \sqrt{m \omega_e}}} \right), \]  

(23)

We also verify that, in turn, substituting Eq. (19) in Eq. (17), we obtain again Eq. (23). Consistently with the canonicity of the used transformations, \( H(Q_{e,\mu}; P_{e,\mu}) \) turns out to be a real quantity; the imaginary terms proportional to \( Q_e P_\mu \) and \( Q_\mu P_e \) cancel. From Eq. (23) one can derive the equations of motion for these variables:

\[ \dot{Q}_e = \frac{P_e}{m} + \omega_{e,\mu} \frac{P_\mu}{m \sqrt{\omega_e \omega_\mu}}, \quad \dot{P}_e = -m \omega^2 e - \omega_{e,\mu} m \sqrt{\omega_e \omega_\mu} Q_\mu, \]

(24)

\[ \dot{Q}_\mu = \frac{P_\mu}{m} + \omega_{e,\mu} \frac{P_e}{m \sqrt{\omega_e \omega_\mu}}, \quad \dot{P}_\mu = -m \omega^2 \mu - \omega_{e,\mu} m \sqrt{\omega_e \omega_\mu} Q_e, \]

(25)

which can be rewritten in the following form:

\[ \left\{ \begin{array}{l}
\dot{Q}_e = - (\omega^2 e + \omega_{e,\mu}^2) Q_e - \omega_{e,\mu} Q_\mu \left( \frac{\omega_{e,\mu}}{\sqrt{\omega_e \omega_\mu}} \right) \\
\dot{Q}_\mu = - (\omega^2 \mu + \omega_{e,\mu}^2) Q_\mu - \omega_{e,\mu} Q_e \left( \frac{\omega_{e,\mu}}{\sqrt{\omega_e \omega_\mu}} \right)
\end{array} \right. \]

(26)

Summarizing, we have found the analogue, at classical level, of Pontecorvo mixing transformations. These are given by the the canonical (classical) transformations Eqs. (20), (21).
4. Oscillations in phase space

We wonder whether it is possible to obtain the oscillation formulas Eqs.(11),(12) within the classical model in terms of the $Q_\sigma$, $P_\sigma$ variables. We know that the (real) solutions of Hamilton’s equations for the Hamiltonian Eq.(13) are:

$$Q_1(t) = A_1 e^{i\omega_1 t} + A_1^* e^{-i\omega_1 t}, \quad Q_2(t) = A_2 e^{i\omega_2 t} + A_2^* e^{-i\omega_2 t},$$

and

$$P_1(t) = i\omega_1 A_1 e^{i\omega_1 t} - i\omega_1 A_1^* e^{-i\omega_1 t}, \quad P_2(t) = i\omega_2 A_2 e^{i\omega_2 t} - i\omega_2 A_2^* e^{-i\omega_2 t}. \quad (27)$$

We now set $m = 1$ for simplicity and consider the following quantities:

$$\alpha_1(t) = \frac{1}{2} \left( \sqrt{\omega_1} Q_1(t) + i \frac{1}{\sqrt{\omega_1}} P_1(t) \right) = A_1^* \sqrt{\omega_1} e^{-i\omega_1 t} \quad (29)$$

$$\alpha_2(t) = \frac{1}{2} \left( \sqrt{\omega_2} Q_2(t) + i \frac{1}{\sqrt{\omega_2}} P_2(t) \right) = A_2^* \sqrt{\omega_2} e^{-i\omega_2 t} \quad (30)$$

and c.c.. Using Eqs.(20),(21) the quantity $\alpha_\epsilon(t)$ and $\alpha_\mu(t)$ take the following form:

$$\alpha_\epsilon(t) = \frac{1}{2} \left( \sqrt{\omega_\epsilon} Q_\epsilon(t) + i \frac{1}{\sqrt{\omega_\epsilon}} P_\epsilon(t) \right) = \cos \theta A_1^* \sqrt{\omega_1} e^{-i\omega_1 t} + \sin \theta A_2^* \sqrt{\omega_2} e^{-i\omega_2 t}, \quad (31)$$

$$\alpha_\mu(t) = \frac{1}{2} \left( \sqrt{\omega_\mu} Q_\mu(t) + i \frac{1}{\sqrt{\omega_\mu}} P_\mu(t) \right) = \cos \theta A_2^* \sqrt{\omega_2} e^{-i\omega_2 t} - \sin \theta A_1^* \sqrt{\omega_1} e^{-i\omega_1 t}. \quad (32)$$

Setting as initial conditions

$$\epsilon: \quad A_1 = e^{i\gamma \cos \theta}, \quad A_2 = e^{i\gamma \sin \theta} \quad (33)$$

with $\gamma$ arbitrary real, we obtain

$$|\alpha_\epsilon(t)|^2_\epsilon = 1 - 2 \sin^2 2\theta \sin^2 \left( \frac{\omega_2 - \omega_1}{2} t \right), \quad (34)$$

which shows that already at a classical level we can reproduce flavor oscillation formulas. By the same reasoning, we have:

$$|\alpha_\mu(t)|^2_\mu = 2 \sin^2 2\theta \sin^2 \left( \frac{\omega_2 - \omega_1}{2} t \right). \quad (35)$$

Choosing as initial conditions:

$$\mu: \quad A_1 = e^{i\gamma \sin \theta} \sqrt{\omega_1}, \quad A_2 = e^{i\gamma \cos \theta} \sqrt{\omega_2}, \quad (36)$$

we obtain similar results as above, cf. Eqs.(34),(35), by exchanging $\epsilon \leftrightarrow \mu$.

$$|\alpha_\mu(t)|^2_\mu = 1 - 2 \sin^2 2\theta \sin^2 \left( \frac{\omega_2 - \omega_1}{2} t \right), \quad (37)$$

and

$$|\alpha_\epsilon(t)|^2_\mu = 2 \sin^2 2\theta \sin^2 \left( \frac{\omega_2 - \omega_1}{2} t \right). \quad (38)$$
We note that, in the above derivation, the quantities $|\alpha_\sigma(t)|^2_\rho$ behave exactly as the expectation values of the operators $\hat{N}_\sigma(t)$ on the state $|a_\rho\rangle$.

On the same line of reasoning, we can consider the “expectation value” of the Hamiltonian Eq.(17) on the “flavor state”, corresponding to the initial conditions in Eq.(33):

$$H(\alpha_{e,\mu}; \alpha_{e,\mu}^*)\big|_e = \omega_e + (\omega_\mu - \omega_e)|\alpha_\mu(t)|^2_e + \omega_{e\mu}\left(\alpha_\mu^*(t)\alpha_\mu(t) + \alpha_{\mu}^*(t)\alpha_e(t)\right)\big|_e,$$

where we used $|\alpha_e(t)|^2_e = 1 - |\alpha_\mu(t)|^2_e$.

Following Ref.[11], we can regard the “free” Hamiltonians $\tilde{H}_e = \omega_e |\alpha_e(t)|^2$ and $\tilde{H}_\mu = \omega_\mu |\alpha_\mu(t)|^2$ as free energies $F_\sigma$, and write:

$$H = \sum_{\sigma=e,\mu} \left( F_\sigma(t) + TS_\sigma(t) \right),$$

where we make the identification $T = \tan 2\theta$ and

$$S_\sigma = \frac{1}{4}\delta\omega\left(\alpha_\mu^*(t)\alpha_\mu(t) + \alpha_e^*(t)\alpha_e(t)\right),$$

with $\delta\omega = \frac{2}{\tan 2\theta} \omega_{e\mu}$. We have:

$$S_e(t)\big|_e = -\frac{1}{4}\delta\omega \sin 4\theta \sin^2 \left( \frac{\omega_2 - \omega_1}{2} t \right).$$

The other results are summarized in Table 1, from which we see how the energetic balance is recovered.

| init.cond. | $H$ | $F_e$ | $F_\mu$ | $TS_e = TS_\mu$ |
|------------|-----|------|-------|----------------|
| $e$        | $\omega_e$ | $\omega_e |\alpha_e(t)|^2_e$ | $\omega_\mu |\alpha_\mu(t)|^2_e$ | $\frac{1}{2}\delta\omega |\alpha_\mu(t)|^2_e$ |
| $\mu$      | $\omega_\mu$ | $\omega_e |\alpha_e(t)|^2_\mu$ | $\omega_\mu |\alpha_\mu(t)|^2_\mu$ | $-\frac{1}{2}\delta\omega |\alpha_e(t)|^2_\mu$ |

Note finally, that the integral of the entropy expectation value over an oscillation cycle, is only dependent on the mixing angle:

$$\int_0^\tau S_e(t)\big|_e dt = \pi \cos^2 2\theta \sin 2\theta,$$

where the period is $\tau = \frac{2\pi}{\omega_2 - \omega_1}$. Such a quantity, being independent of dynamical parameters, can be related to other geometric invariants as geometric phase for oscillating neutrinos [7].
5. Conclusions
In this paper we have considered a simple classical model in phase-space resembling the quantum one used for the description of neutrino oscillations. We have found that it is indeed possible to obtain flavor oscillation formulas by considering classical analogues of the (time–dependent) flavor number operators and (initial) neutrino states. By resorting to previous results [11], we have given a thermodynamical interpretation of the phenomenon of flavor oscillations in terms of energy fluxes between two open subsystems.

One interesting output of the present work is a form for the classical hamiltonian describing flavor oscillations. This can be useful for identifying classical (hamiltonian) systems analogue for this phenomenon. Another possible direction of investigation which is suggested by the above hamiltonian form Eq.(23) is the mapping of such system into an equivalent non–commutative one. On such aspects, work is in progress.

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