MONOMIAL CONVERGENCE FOR HOLOMORPHIC FUNCTIONS ON $\ell_r$

By

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Abstract. Let $\mathcal{F}$ be either the set of all bounded holomorphic functions or the set of all $m$-homogeneous polynomials on the unit ball of $\ell_r$. We give a systematic study of the sets of all $u \in \ell_r$ for which the monomial expansion $\sum_{\alpha} \frac{\partial^{|\alpha|} f(0)}{|\alpha|!} u^\alpha$ of every $f \in \mathcal{F}$ converges. Inspired by recent results from the general theory of Dirichlet series, we establish as our main tool independently interesting, upper estimates for the unconditional basis constants of spaces of polynomials on $\ell_r$ spanned by finite sets of monomials.

1 Introduction

Let $X$ be a Banach sequence space (i.e., $\ell_1 \subset X \subset c_0$ such that the canonical sequences $(e_k)$ form a 1-unconditional basis) and $R \subset X$ a Reinhardt domain (i.e., a nonempty open set such that any complex sequence $u$ belongs to $R$ whenever there exists $z \in R$ with $|u| \leq |z|$; for instance, the open unit ball $B_X$ of $X$). Then each holomorphic (i.e., Fréchet differentiable) function $f : R \to \mathbb{C}$ has a power series expansion $\sum_{\alpha \in \mathbb{N}_0^n} c^{(n)}_\alpha z^\alpha$ on every finite-dimensional section $R_n$ of $R$, and for example from the Cauchy formula we can see that $c^{(n)}_\alpha = c^{(n+1)}_\alpha$ for $\alpha \in \mathbb{N}_0^n \subset \mathbb{N}_0^{n+1}$. Thus there is a unique family $(c_\alpha(f))_{\alpha \in \mathbb{N}_0^n}$ such that, for all $n \in \mathbb{N}$ and all $z \in R_n$,

$$f(z) = \sum_{\alpha \in \mathbb{N}_0^n} c_\alpha z^\alpha,$$

where the series converges absolutely. The power series $\sum_{\alpha} c_\alpha z^\alpha$ is called the monomial expansion of $f$, and $c_\alpha = c_\alpha(f)$ are its monomial coefficients.

Contrary to what happens on finite-dimensional domains, the monomial expansion of $f$ does not necessarily converge at every point of $R$. This, in [15], motivated the introduction of the following definition: Given a subset $\mathcal{F}(R)$ of $H(R)$, the set
of all holomorphic functions on $R$, we call
\[ \text{mon } F(R) = \left\{ z \in R : \sum_{a \in \mathbb{N}_0^{(N)}} |c_a(f)z^a| < \infty \text{ for all } f \in F(R) \right\} \]
the domain of monomial convergence with respect to $F(R)$. For $z \in \text{mon } F(R)$ we have now
\[ f(z) = \sum_{a \in \mathbb{N}_0^{(N)}} c_a(f)z^a. \]
This can be seen as follows: Let $z_n$ denote the truncation of $z$ to the first $n$ variables. Then, by continuity of $f$ and the fact that the desired equality holds on $R_n$,
\[ f(z) \leftarrow f(z_n) = \sum_{a \in \mathbb{N}_0^{m}} c_a(f)z_n^a = \sum_{a \in \mathbb{N}_0^{(N)}} c_a(f)z^a \rightarrow \sum_{a \in \mathbb{N}_0^{(N)}} c_a(f)z^a. \]

We are mostly interested in determining $\text{mon } F(R)$ when $F(R) = P(mX)$ or $H_\infty(B_X)$ for $X = \ell_r$ with $1 \leq r < \infty$ or $X = c_0$; we denote by $H_\infty(B_X)$ the Banach space of all bounded holomorphic functions $f : B_X \to \mathbb{C}$, and by $P(mX)$ its closed subspace of all $m$-homogeneous polynomials (i.e., all restrictions of bounded $m$-linear forms on $X^m$ to their diagonals).

The case $X = \ell_1$ was solved completely by Lempert in [19], and the case $X = c_0$ seems fairly well-understood through the results of [5] (for more on these results see the introductions of the Sections 5.1 and 5.2). However, for $X = \ell_r$ with $1 < r < \infty$, despite the results of [15], the description of $\text{mon } P(m\ell_r)$ and $\text{mon } H_\infty(\ell_r)$ remains mysterious. In this paper, we improve the knowledge on these cases.

Of course, for $X = \ell_1$ the fact that each element of $\ell_1$ is, as a sequence, by definition absolutely summable is a big advantage, and for $X = c_0$ the crucial tool is the Bohnenblust–Hille inequality (an inequality for $m$-linear forms on $\ell_\infty$) together with all its recent improvements. But for $X = \ell_r$ with $1 < r < \infty$ we need alternative techniques.

The problem is to find for each $u \in B_{\ell_r}$ an additional summability condition which guarantees full control of all sums $\sum_{a} |c_a(f)u^a|$, $f \in H_\infty(B_{\ell_r})$. The general idea is simple. Split the set $\mathbb{N}_0^{(N)}$ of all multi-indices $\alpha$ into a union of finite sets $\Lambda_n$, and then each $\Lambda_n$ into the disjoint union of all its $m$-homogeneous parts $\Lambda_{n,m}$ (i.e., all $\alpha \in \Lambda_n$ with order $|\alpha| = m$). The challenge now is as follows: Find a clever decomposition
\[
\mathbb{N}_0^{(N)} = \bigcup_{m,n} \Lambda_{m,n},
\]
which guarantees full control of all sums $\sum_{a} |c_a(f)u^a|$, $f \in H_\infty(B_{\ell_r})$. The general idea is simple. Split the set $\mathbb{N}_0^{(N)}$ of all multi-indices $\alpha$ into a union of finite sets $\Lambda_n$, and then each $\Lambda_n$ into the disjoint union of all its $m$-homogeneous parts $\Lambda_{n,m}$ (i.e., all $\alpha \in \Lambda_n$ with order $|\alpha| = m$). The challenge now is as follows: Find a clever decomposition
\[
\mathbb{N}_0^{(N)} = \bigcup_{m,n} \Lambda_{m,n},
\]
which allows, in a sense, uniform control over all possible partial sums

$$\sum_{\alpha \in \Lambda_{n,m}} |c_{\alpha}(f)u^\alpha|, \quad f \in H_\infty(B_\ell_r),$$

such that under the additional summability property of $u \in B_\ell_r$, for all functions $f$ we can finally conclude that

$$\sum_{\alpha \in \Lambda_{n,m}} |c_{\alpha}(f)u^\alpha| \leq \sum_n \sum_m \sum_{\alpha \in \Lambda_{n,m}} |c_{\alpha}(f)u^\alpha| < \infty.$$

In order to study domains mon $\mathcal{F}(\mathbb{R})$ of monomial convergence, the decomposition in (1), which for our purposes is crucial, is inspired by the work of Konyagin and Queffélec from [18] on Dirichlet series (see (16)). In fact, convergence of Dirichlet series and sets of monomial convergence of holomorphic functions in infinitely many variables are intimately connected by what is now sometimes called ‘Bohr’s vision’ (see, e.g., [5], [8], or [20]). This somewhat explains why some of our arguments use analytic number theory.

In order to handle (2), we study for arbitrary finite index sets $\Lambda$ of multi indices upper bounds of the unconditional basis constant of the subspace in $\mathcal{P}(m \ell_r)$ spanned by all monomials $z^\alpha, \alpha \in \Lambda$. Two tools of seemingly independent interest are established. The first one is a fairly general upper estimate whenever all $\alpha \in \Lambda$ are $m$-homogeneous (i.e., $|\alpha| = m$) (Theorem 3.2). The second one leads to such estimates for certain sets $\Lambda$ of nonhomogeneous $\alpha$’s, needed to apply the above technique of Konyagin and Queffélec (Theorem 3.7 and Theorem 4.1).

Finally, we present our new results on sets of monomial convergence for homogeneous polynomials and bounded holomorphic functions on $\ell_r$ (for polynomials see part (3) and (4) of Theorem 5.1 and Theorem 5.3, and for holomorphic functions Theorem 5.5 with its corollaries 5.6 and 5.7).

2 Preliminaries

We use standard notation from Banach space theory. As usual, we denote the conjugate exponent of $1 \leq r \leq \infty$ by $r'$, i.e., $\frac{1}{r} + \frac{1}{r'} = 1$. Given $m, n \in \mathbb{N}$ we consider the following sets of indices:

$$\mathcal{M}(m, n) = \{j = (j_1, \ldots, j_m); 1 \leq j_1, \ldots, j_m \leq n \} = \{1, \ldots, n\}^m,$$

$$\mathcal{M}(m) = \mathbb{N}^m,$$

$$\mathcal{M} = \mathbb{N}^\mathbb{N}.$$
and

\[ J(m,n) = \{ j \in M(m,n); 1 \leq j_1 \leq \cdots \leq j_m \leq n \}, \]
\[ J(m) = \bigcup_n J(m,n), \]
\[ J = \bigcup_m J(m). \]

For indices \( i, j \in M \) we denote by \((i, j) = (i_1, i_2, \ldots, j_1, j_2, \ldots)\) the concatenation of \( i \) and \( j \). An equivalence relation is defined in \( M(m) \) as follows: \( i \sim j \) if there is a permutation \( \sigma \) such that \( i_{\sigma(k)} = j_k \) for all \( k \). We write \(|i|\) for the cardinality of the equivalence class \([i]\). Moreover, we note that for each \( i \in M(m) \) there is a unique \( j \in J(m) \) such that \( i \sim j \).

Let us compare our index notation with the multi index notation usually used in the context of polynomials. There is a one-to-one correspondence between \( J(m) \) and \( \Lambda_1(m) = \{ \alpha \in \mathbb{N}_0^n; |\alpha| = m \} \); indeed, given \( j \), one can define \( \alpha \) by letting \( \alpha_r = \{|q| j_q = r\} \); conversely, for each \( \alpha \), we consider \( j_\alpha = (1, \alpha_1, 1, 2, \alpha_2, 2, \ldots, n, \alpha_n, n, \ldots) \). In the same way we may identify \( \Lambda(m, n) = \{ \alpha \in \mathbb{N}_0^n; |\alpha| = m \} \) with \( J(m, n) \). Note that \(|j_\alpha| = \frac{m!}{\alpha!} \) for every \( \alpha \in \Lambda(m) \). Taking this correspondence into account, for every Banach sequence space \( X \) the monomial series expansion of a \( m \)-homogeneous polynomial \( P \in \mathcal{P}(mX) \) can be expressed in different ways (we write \( c_\alpha = c_\alpha(P) \)),

\[
\sum_{\alpha \in \Lambda(m)} c_\alpha z^\alpha = \sum_{j \in J(m)} c_j z_j = \sum_{1 \leq j_1 \leq \cdots \leq j_m} c_{j_1 \cdots j_m} z_{j_1} \cdots z_{j_m}.
\]

Given a Banach sequence space \( X \) and some index subset \( J \subset J \), we write \( \mathcal{P}(J X) \) for the closed subspace of all holomorphic functions \( f \in H_\infty(B_X) \) for which \( c_j(f) = 0 \) for all \( j \in J \setminus J \). Clearly, \( \mathcal{P}(mX) = \mathcal{P}(J(m)X) \). If \( J \subset J \) is finite, then

\[
\mathcal{P}(JX) = \text{span}\{ z_j : j \in J \},
\]

where \( z_j \) for \( j = (j_1, \ldots, j_\ell) \) stands for the monomial \( z_j : u \mapsto u_{j_1}u_{j_2} \cdots u_{j_\ell} \). For \( J \subset J(m) \), we call

\[
J^* = \{ j \in J(m-1); \exists k \geq 1, (j,k) \in J \}
\]

the reduced set of \( J \).
3 Unconditionality

Given a compact group $G$, the Sidon constant of a finite set $\mathcal{C}$ of characters $\gamma$ (in the dual group) is the best constant $c \geq 0$, denoted by $S(\mathcal{C})$, such that for every choice of scalars $c_\gamma, \gamma \in \mathcal{C}$, we have that

$$\sum_{\gamma \in \mathcal{C}} |c_\gamma| \leq c \left\| \sum_{\gamma \in \mathcal{C}} c_\gamma \gamma \right\|_\infty.$$  

An immediate consequence of the Cauchy-Schwarz inequality is that

$$(4) \ 1 \leq S(\mathcal{C}) \leq |\mathcal{C}|^{1/2}.$$  

For the circle groups $G = \mathbb{T}, \mathbb{T}^n$ and $\mathbb{T}^\infty$ different values are possible:

- A well-known result of Rudin shows that for the set $\mathcal{C} = \{1, z, \ldots, z^{n-1}\}$ of characters on $G = \mathbb{T}$ we have, up to constants independent of $n$,

  $$(5) \ S(\mathcal{C}) \asymp \sqrt{n}.$$  

- In [12] it was proved that for every $m, n$ the Sidon constant of the monomials $\mathcal{C} = \{z^\alpha : \alpha \in \Lambda(m, n)\}$ on $G = \mathbb{T}^n$, up to the $m$-th power $C^m$ of some absolute constant $C$, satisfies

  $$(6) \ S(\mathcal{C}) \asymp |\Lambda(m - 1, n)|^{1/2}.$$  

- In contrast, a reformulation of a result of Aron and Globevnik [1, Thm 1.3] shows that for every $m$ the Sidon constant of the sparse set $\mathcal{C} = \{z^m_j : j \in \mathbb{N}\}$ fulfills

  $$(7) \ S(\mathcal{C}) = 1.$$  

Let us transfer some of these results into terms of unconditional basis constants of spaces of polynomials on sequence spaces. Recall that a Schauder basis $(x_n)$ of a Banach space $X$ is said to be unconditional whenever there is a constant $c \geq 0$ such that $\| \sum_k \varepsilon_k a_k x_k \| \leq c \| \sum_k a_k x_k \|$ for every $x = \sum_k a_k x_k \in X$ and all choices of $(\varepsilon_k) \subset \mathbb{C}$ with $|\varepsilon_k| = 1$. In this case, the best constant $c$ is denoted by $\chi((x_n))$ and called the unconditional basis constant of $(x_n)$. If such a constant doesn’t exist, i.e., if the basis is not unconditional, we set $\chi((x_n)) = \infty$.

Given a Banach sequence space $X$ and index set $J \subset \mathcal{J}$, such that the set $\mathcal{C} = \{z^j : j \in J\}$ of all monomials associated with $J$ (ordered in a suitable way) forms a basis of $\mathcal{P}(JX)$, we write

$$\chi_{\text{mon}}(\mathcal{P}(JX)) = \chi(\mathcal{C}).$$
If we interpret each of these monomials \( z_j \) as a character on the group \( \mathbb{T}^\infty \), then a straightforward calculation (using the distinguished maximum modulus principle) proves that

\[ S(\mathcal{E}) = \chi_{\text{mon}}(\mathcal{P}(Jc_0)). \]

A simple but useful lemma shows that \( \chi_{\text{mon}}(\mathcal{P}(Jc_0)) \) is an upper bound of all \( \chi_{\text{mon}}(\mathcal{P}(JX)) \).

**Lemma 3.1.** Let \( X \) be a Banach sequence space and \( J \subset \mathcal{J} \) such that the monomials form a basis of \( \mathcal{P}(JX) \). Let \( P \in \mathcal{P}(JX) \) such that for some constant \( c \geq 1 \) and every \( u \in Bc_0^\infty \)

\[
\sum_{j \in J} |c_j(P)| |u_j| \leq c \sup_{w \in Bc_0} |P(w)|.
\]

Then for any \( u \in B_X \)

\[
\sum_{j \in J} |c_j(P)| |u_j| \leq c \sup_{w \in B_X} |P(w)|.
\]

In particular,

\[
\chi_{\text{mon}}(\mathcal{P}(JX)) \leq \chi_{\text{mon}}(\mathcal{P}(Jc_0)).
\]

**Proof.** For each \( v \in B_X \) define \( Q_0 \in \mathcal{P}(Jc_0) \), \( Q_0(w) = P(vw) \); since \( B_X \) is a Reinhardt domain, these polynomials are well-defined and satisfy \( \|Q_0\|_\infty \leq \|P\|_\infty \). Now take some finite set \( J' \subset J \). Since \( B_X \) is a Reinhardt domain in \( \mathbb{C}^n \), we easily see that \( B_X = B_{c_0}^\infty \cdot B_X \) (pointwise multiplication). Then for each \( u \in B_X \) by assumption

\[
\sum_{j \in J} |c_j(P)u_j| \leq \sup_{v \in B_{c_0}^\infty} \sup_{w \in B_X} \sum_{j \in J} |c_j(P)v_j||w_j| = \sup_{v \in B_{c_0}^\infty} \sum_{j \in J} |c_j(Q_0)||w_j| \leq c\|Q_0\|_\infty \leq c\|P\|_\infty,
\]

which obviously gives the conclusion. \( \square \)

Note that by (4), (8) and this lemma for every finite index set \( J \subset \mathcal{J} \) we have

\[ 1 \leq \chi_{\text{mon}}(\mathcal{P}(JX)) \leq \sqrt{|J|}, \]

and one of our aims in the following will be to improve these trivial bounds in more particular situations.

Let us again present some examples: Given \( X \), an immediate consequence of (5) is that for \( \mathcal{P}(JX) = \text{span}\{z_j^i : 0 \leq j \leq n - 1\} \) we have, up to a universal constant,

\[
\chi_{\text{mon}}(\mathcal{P}(JX)) \asymp \sqrt{n},
\]
and from (7) we may deduce that for \( J = \{ (k, \ldots, k); k \in \mathbb{N} \} \subset \mathcal{J}(m) \)
\[
\chi_{\text{mon}}(\mathcal{P}(^t X)) = 1.
\]

Generalizing (6) is much more complicated. In the scale of all \( \ell_r \)-spaces the results from [4] (lower estimates) and [11, 12] (upper estimates) show that for \( 1 \leq r \leq \infty \)
\[
(9) \quad \chi_{\text{mon}}(\mathcal{P}(^t JX)) \asymp |\mathcal{J}(m - 1, n)|^{\frac{1}{1 - \min(r, 2)}},
\]
where \( \asymp \) means that the left and the right side are equal up to the \( m \)-th power \( C^m \) of a constant only depending on \( r \) (and neither on \( m \) nor on \( n \)).

Replacing the index set \( \mathcal{J}(m, n) \) by an arbitrary finite subset \( J \subset \mathcal{J}(m, n) \) the following result generalizes the upper bound of (9); it will be our main tool within our later study of sets of monomial convergence.

3.1 The main tool. The following result improves (9) considerably (see the remarks below), and it is crucial for everything following.

**Theorem 3.2.** Given \( 1 \leq r \leq \infty \) and \( m \geq 1 \), there is a constant \( C(m, r) \geq 1 \) such that for every \( n \geq 1 \), every \( P \in \mathcal{P}(^t JX_n) \), every \( J \subset J(m, n) \), and every \( u \in \ell_r^n \) we have
\[
(10) \quad \sum_{j \in J} |c_j(P)||u_j| \leq C(m, r)|J^*|^{1 - \frac{1}{\min(r, 2)}}\|u\|_r\|P\|_\infty,
\]
where
\[
C(m, r) \leq \begin{cases} 
  \text{e}m^{(m-1)/r} & \text{if } 1 \leq r \leq 2, \\
  \text{e}m^{2(m-1)/2} & \text{if } 2 \leq r \leq \infty.
\end{cases}
\]

In particular, for every \( J \subset \mathcal{J}(m, n) \)
\[
(11) \quad \chi_{\text{mon}}(\mathcal{P}(^t JX)) \leq C(m, r)|J^*|^{1 - \frac{1}{\min(r, 2)}}.
\]

The proof is given in the following two subsections; it is different for \( r \leq 2 \) and for \( r \geq 2 \). For \( r = \infty \) and \( J = \mathcal{J}(m, n) \) the asymptotic from (9) is given in [12], and it uses the hypercontractive Bohnenblust–Hille inequality. The general case \( 1 \leq r \leq \infty \) and \( J = \mathcal{J}(m, n) \) from [11] needs sophisticated tools from local Banach space theory (as Gordon–Lewis and projection constants). These arguments in fact only work for the whole index set \( \mathcal{J}(m, n) \), and they seem to fail in full generality for subsets \( J \) of \( \mathcal{J}(m, n) \). We here provide a tricky, but quite elementary, argument which works for arbitrary \( J \); moreover, we point out that even for the special case \( J = \mathcal{J}(m, n) \) we obtain better constants \( C(m, r) \) for (11) than in [11].
From [14] we know that for each infinite-dimensional Banach sequence space $X$, the Banach space $\mathcal{P}(mX)$ never has an unconditional basis. In particular, the unconditional basis constant $\chi_{\text{mon}}(\mathcal{P}(mX))$ of all monomials $(z_j)_{j \in \mathcal{J}(m)}$ is not finite. But let us note that in contrast to this there are $X$ such that for each $m$

$$\sup_n \chi_{\text{mon}}(\mathcal{P}(\mathcal{J}(m,n)X)) < \infty$$

(this can be easily shown for $X = \ell_1$, but following [14] there are even examples of this type different from $\ell_1$).

### 3.1.1 The case $r \leq 2$

We need several lemmas. The first one is a Cauchy estimate and can be found in [7, p. 323]. For the sake of completeness we include a streamlined argument.

**Lemma 3.3.** Let $1 \leq r \leq \infty$ and $\alpha \in \mathbb{N}_0^n$ with $|\alpha| = m$. Then for each $P \in \mathcal{P}(m\ell_r^n)$ we have

$$|c_\alpha(P)| \leq \left( \frac{m^m}{\alpha^\alpha} \right)^{1/r} \|P\|_{\infty}.$$

In particular, for each $j \in \mathcal{J}(m,n)$ we have that

$$|c_j(P)| \leq e^{m/r} |j|^{1/r} \|P\|_{\infty}.$$

**Proof.** Define $u = m^{-1/r}(\alpha_1^{1/r}, \ldots, \alpha_n^{1/r}) \in B_{\ell_r^n}$. Then by the Cauchy integral formula for each $P \in \mathcal{P}(m\ell_r^n)$

$$c_\alpha(P) = \frac{1}{(2\pi i)^n} \int_{|z| = u} \cdots \int_{|z| = u} \frac{P(z)}{z_1^{\alpha_1} \cdots z_n^{\alpha_n}} dz.$$

Hence we obtain

$$|c_\alpha(P)| \leq \frac{1}{|u^\alpha|} \|P\|_{\infty} = \left( \frac{m^m}{\alpha^\alpha} \right)^{1/r} \|P\|_{\infty},$$

the conclusion. For the second inequality note first that

$$\left( \frac{m^m}{\alpha^\alpha} \right)^{1/r} \leq e^{m/r} \left( \frac{m!}{\alpha!} \right)^{1/r},$$

and recall that if we associate to $j$ the multi-index $\alpha$, then $\frac{m!}{\alpha!} = |j|$. \qed

**Corollary 3.4.** Consider the linear operator $Q \in \mathcal{L}(\ell_r^n, \mathcal{P}(m^{-1}\ell_r^n))$ defined by

$$Qw(z) = \sum_{j \in \mathcal{J}(m-1,n)} \left( \sum_{k=1}^n b_{j,k} w_k \right) z_j.$$

Then for any $j \in \mathcal{J}(m-1,n)$,

$$\left( \sum_{k=1}^n |b_{j,k}|^{r'} \right)^{1/r'} \leq e^{m-1/r} |j|^{1/r} \|Q\|.$$
Proof. Let us fix \( w \in B_{\ell^r} \). Then \( Qw \in \mathcal{P}(m-1,\ell^r) \). Thus, by the preceding lemma for any \( j \in \mathcal{J}(m-1,n) \),

\[
\left| \sum_{k=1}^{n} b_{(j,k)} w_k \right| \leq e^{\frac{m-1}{r}} \| j \|^{1/r} \sup_{z \in B_{\ell^r}} |Qw(z)| \leq e^{\frac{m-1}{r}} |j|^{1/r} \| Q \|.
\]

We now take the supremum over all possible \( w \in B_{\ell^r} \). \( \square \)

Lemma 3.5. Let \( P \in \mathcal{P}(m,\ell^r) \). Then for any \( j \in \mathcal{J}(m-1,n) \),

\[
\left( \sum_{k=j_{m-1}}^{n} |c_{(j,k)}(P)|^{r'} \right)^{1/r'} \leq me^{1+\frac{m-1}{r}} |j|^{1/r} \| P \|_\infty.
\]

Proof. Let \( A : \ell^n \times \cdots \times \ell^n \to \mathbb{C} \) be the symmetric \( m \)-linear form associated to \( P \),

\[
A(z^{(1)}, \ldots, z^{(m)}) = \sum_{i \in \mathcal{N}(m,n)} a_i(A) z_{i_1}^{(1)} \cdots z_{i_m}^{(m)},
\]

in particular, for each \( j \in \mathcal{J}(m, n) \) we have \( a_j(A) = \frac{c_{(P)}(j)}{|j|} \). For \( z, w \in \ell^n \) define the linear operator

\[
Q(z, w) = A(z, \ldots, z, w) \in L(\ell^n, \mathcal{P}(m-1,\ell^r));
\]

then a simple calculation proves

\[
Q(z, w) = \sum_{i \in \mathcal{N}(m,n)} a_i(A) z_{i_1} \cdots z_{i_{m-1}} w_{i_m}
= \sum_{i \in \mathcal{N}(m-1,n)} \sum_{k=1}^{n} a_{(i,k)}(A) z_{i_1} \cdots z_{i_{m-1}} w_k
= \sum_{j \in \mathcal{J}(m-1,n)} \sum_{i \in [j]} \sum_{k=1}^{n} a_{(i,k)}(A) z_{i_1} \cdots z_{i_{m-1}} w_k
= \sum_{j \in \mathcal{J}(m-1,n)} \sum_{k=1}^{n} \left( \sum_{i \in [j]} a_{(i,k)}(A) z_{i_1} \cdots z_{i_{m-1}} \right) w_k
= \sum_{j \in \mathcal{J}(m-1,n)} \sum_{k=1}^{n} a_{(j,k)}(A) z_{j_1} \cdots z_{j_{m-1}} w_k.
\]

Now note that for every \( j \in \mathcal{J}(m-1,n) \) we have \(|(j, k)| \leq m|j|\), and hence by the
preceding corollary for such \( j \)

\[
\left( \sum_{k: j_{m-1} \leq k} |c_{j, k}(P)|^{r} \right)^{1/r} = \left( \sum_{k: j_{m-1} \leq k} |a_{j, k}(A)|^{r} |j|^{r} \right)^{1/r} \\
\leq m \left( \sum_{k=1}^{n} |a_{j, k}(A)|^{r} \right)^{1/r} \\
\leq m e^{\frac{m-1}{r}} |j|^{1/r} \| Q \|.
\]

Finally, by Harris’ polarization formula we know that \( \| Q \| \leq e \| P \|_{\infty} \), and hence we obtain the desired conclusion.

Now we are ready to give the

**Proof of Theorem 3.2 for \( 1 \leq r \leq 2 \).** Take \( P \in \mathcal{P}(\mathcal{H}(m, n)) \), \( J \subset \mathcal{H}(m, n) \) and \( u \in \ell_{r}^{n} \). Then, by Lemma 3.5, for any \( j \in J^{*} \),

\[
\left( \sum_{k: j_{k} \in J} |c_{j, k}(P)|^{r} \right)^{1/r} \leq \left( \sum_{k=1}^{n} |c_{j, k}(P)|^{r} \right)^{1/r} \leq me^{1+\frac{m-1}{r}} |j|^{1/r} \| P \|_{\infty}.
\]

Now by Hölder’s inequality (two times) and the multinomial formula we have

\[
\sum_{j \in J} |c_{j}(P)||u_{j}| = \sum_{j \in J^{*}} \sum_{k: j_{k} \in J} |c_{j, k}||u_{j}||u_{k}| \\
\leq \sum_{j \in J^{*}} |u_{j}| \left( \sum_{k: j_{k} \in J} |c_{j, k}|^{r} \right)^{1/r} \left( \sum_{k} |u_{k}|^{r} \right)^{1/r} \\
\leq me^{1+\frac{m-1}{r}} \sum_{j \in J^{*}} |j|^{1/r} |u_{j}| \| u \|_{r} \| P \|_{\infty} \\
\leq me^{1+\frac{m-1}{r}} \left( \sum_{j \in J^{*}} |j| |u_{j}|^{r} \right)^{1/r} \left( \sum_{j \in J^{*}} 1 \right)^{1/r} \| u \|_{r} \| P \|_{\infty} \\
\leq me^{1+\frac{m-1}{r}} \left( \sum_{j \in J^{*(m-1,n)}} |j| |u_{j}|^{r} \right)^{1/r} \left( \sum_{j \in J^{*}} 1 \right)^{1/r} \| u \|_{r} \| P \|_{\infty} \\
= me^{1+\frac{m-1}{r}} |J^{*}|^{1-\frac{1}{r}} \| u \|_{r}^{m} \| P \|_{\infty}.
\]

In order to deduce (11), note that for every finite \( J \subset \mathcal{H}(m) \) there is \( n \) such that \( J \subset \mathcal{H}(m, n) \). Then every \( P \in \mathcal{P}(J, \ell_{r}^{n}) \) can be considered as a polynomial in \( \mathcal{P}(\mathcal{H}(m, n)) \) with equal norm, which implies the conclusion. \( \square \)

### 3.1.2 The case \( r \geq 2 \)

Note first that Lemma 3.1 shows that we only have to deal with the case \( r = \infty \). For \( r = \infty \) we need another lemma which substitutes the argument (by Cauchy’s estimates) from Lemma 3.3. It is an improvement of Parseval’s identity, and its proof can be found in [5, Lemma 2.6].
Lemma 3.6. Let $P \in \mathcal{P}^J(m,n \ell_\infty^n)$. Then

$$
\sum_{k=1}^{n} \left( \sum_{j \in J(m-1,n) \atop j_m \leq k} |c_{j,k}(P)|^2 \right)^{1/2} \leq e m^{2^{m-1}} \|P\|_\infty.
$$

We are now ready for the Proof of Theorem 3.2 for $r = \infty$. Let $P \in \mathcal{P}^J(m,n \ell_\infty^n)$. Then, for any $u \in B_{\ell_\infty^n}$, by the Cauchy-Schwarz inequality and the preceding Lemma 3.6 we have

$$
\sum_{j \in J} |c_j(P)||u_j| \leq \sum_{k=1}^{n} \left( \sum_{j \in J^* \atop (j,k) \in J} |c_{j,k}| \right)^{1/2} \leq \sum_{k=1}^{n} \left( \sum_{j \in J^* \atop (j,k) \in J} |c_{j,k}|^2 \right)^{1/2} |\{j \in J^* : (j,k) \in J\}|^{1/2} \leq \sum_{k=1}^{n} \left( \sum_{j \in J(m-1,n) \atop j_m \leq k} |c_{j,k}|^2 \right)^{1/2} |J^*|^{1/2} \leq e m^{2^{m-1}} |J^*|^{1/2} \|P\|_\infty.
$$

For the second statement, see again the argument from the proof in the case $1 \leq r \leq 2$. This finally completes the proof of Theorem 3.2.

3.2 Lower bound. It is natural to ask for lower bounds of $\chi_{\text{mon}}(\mathcal{P}^J(\ell_r^n))$ using $|J|$ or $|J^*|$. For the whole set of $m$-homogeneous polynomials, this has been done in [13] for $r \geq 2$ and in [4] for $1 \leq r \leq 2$. Using the Kahane–Salem–Zygmund inequality, we can give such a lower bound at least for the case $r = \infty$: For $J \subset J(m,n)$, by Theorem 3, Chapter 6 of [17], there exists some absolute constant $C > 0$ and signs $(\varepsilon_j)_{j \in J}$ such that

$$
\sup_{u \in B_{\ell_\infty^n}} \left| \sum_{j \in J} \varepsilon_j u_j \right| \leq C n^{1/2} |J|^{1/2} (\log m)^{1/2}.
$$

Now, the inequality

$$
|J| = \sup_{u \in B_{\ell_\infty^n}} \sum_{j \in J} |\varepsilon_j||u_j| \leq \chi_{\text{mon}}(\mathcal{P}^J(\ell_\infty^n)) \sup_{u \in B_{\ell_\infty^n}} \left| \sum_{j \in J} \varepsilon_j u_j \right|
$$

yields

$$
\chi_{\text{mon}}(\mathcal{P}^J(\ell_\infty^n)) \geq \frac{|J|^{1/2}}{C n^{1/2} (\log m)^{1/2}}.
$$

However, the inequality given by Theorem 3.2 is very bad if $J$ involves many independent variables; see in particular (7).
3.3 Examples. We shall now apply Theorem 3.2 to several examples of sets \( J \) beyond \( J(m, n) \). A natural example appears in the work of Balasubramanian, Calado and Queffélec [2] about Dirichlet series. It is linked to our subject by the Bohr point of view. Define, for each \( x > 0 \), the index set \( J(x) = \{ j \in \mathbb{J} : p_j \leq x \} \). Then to each Dirichlet polynomial

\[
D(s) = \sum_{n=1}^{x} a_n n^{-s} = \sum_{j \in J(x)} a_{p_j^{-s}}
\]

we can associate a polynomial

\[
P(z) = \sum_{j \in J(x)} a_{p_j^{-s}} z^j \in \mathbb{P}(J(x)c_0).
\]

Kronecker’s theorem ensures that \( \|P\|_{\infty} = \sup_{t \in \mathbb{R}} |D(it)| \). Balasubramanian, Calado and Queffélec were particularly interested in \( m \)-homogeneous Dirichlet polynomials, which means that \( P \) belongs to \( \mathbb{P}(mc_0) \). Define for \( m \in \mathbb{N} \) and \( x > 0 \) the index set \( J(x, m) = \{ j \in \mathbb{J}(m) : p_j \leq x \} \). Theorem 1.4 of [2] translates into

\[
\chi_{\text{mon}}(\mathbb{P}(J(x, m)c_0)) \asymp \frac{x^{m-1}}{(\log x)^{m-1}},
\]

with constants only depending on \( m \). Our aim in this subsection is to extend (12) to the scale of \( \ell_r \)-spaces and also to replace in the definition of \( J(x, m) \) the sequence of prime numbers by a more general sequence \( q \); we are going to see (e.g. in Theorem 5.1) that a particular choice of \( q \) depending on the ambient space \( X \) may help to study the domains of convergence of polynomials and holomorphic functions.

Hereinafter let \( q = (q_k)_k \) denote a strictly increasing sequence with \( q_1 > 1 \) and \( q_k \to \infty \) for \( k \to \infty \). For \( x > 0 \), let

\[
\rho(x) = \max \{ k \in \mathbb{N}_0 : q_k \leq x \}
\]

\[
L(x) = \sum_{k: q_k \leq x} \frac{1}{q_k}.
\]

We assume that there exists \( c > 0 \) such that, for all \( 1 \leq s \leq t \),

\[
s \rho(t) \leq c \rho(st)
\]

which means that \( q \) does not tend too quickly to \( \infty \). As in (12) we define for \( m \in \mathbb{N} \) and \( x > 0 \) the index set

\[
J(x, m) = \{ j = (j_1, \ldots, j_m) \in \mathbb{J}(m) : q_j \leq x \}.
\]
Theorem 3.7. Let $X = \ell_r$ with $1 \leq r < \infty$ and set $\sigma = 1 - \frac{1}{\min(r, 2)}$, or let $X = c_0$ and set $\sigma = 1/2$. Let also $m \in \mathbb{N}$. Then there exists $C(m, X) > 0$ such that, for all $P \in \mathcal{P}^m X$, for all $x > 0$ and all $u \in X$,

$$\sum_{j: q_j \leq x} |c_j(P)u_j| \leq C(m, X)\left(\rho\left(\frac{m-1}{m}\right)L^{m-2}\left(x^{\frac{m}{2m}}\right)\right)^\sigma \|u\|^m.$$  

In particular,

$$\chi_\text{mon}(\mathcal{P}(J(x, m)X)) \leq C(m, X)\left(\rho\left(\frac{m-1}{m}\right)L^{m-2}\left(x^{\frac{m}{2m}}\right)\right)^\sigma.$$  

We need the following lemma on this set.

Lemma 3.8. For the reduced index set $J(x, m)^*$ (see (3)) we have

$$J(x, m)^* \subset J\left(x^{\frac{m-1}{m}}, m-1\right).$$  

Proof. Let $j = (j_1, \ldots, j_{m-1}) \in J(x, m)^*$. Then there exists $k \geq j_{m-1}$ such that $(j, k) \in J(x, m)$. Hence $q_j \cdot q_k = q_{j_k} \leq x$. Since $q_k \geq q_{j_m}$, this implies either $q_k > x^{\frac{1}{m}}$ or $q_{j_1} \leq \cdots \leq q_{j_{m-1}} \leq q_k \leq x^{\frac{1}{m}}$. This finally yields $q_{j_1} \cdots q_{j_{m-1}} \leq x^{\frac{m-1}{m}}$. □

Lemma 3.9. Let $x > 0$ and $m \in \mathbb{N}$. Then $|J(x, m)| \leq c^{m-1} \rho(x)L^{m-1}(\sqrt{x})$, where $c$ denotes the constant of (13).

Proof. The proof is done by induction on $m$, the inequality being clear for $m = 1$. Assume that it is true for some $m \geq 1$. Any $j = (j_1, \ldots, j_{m+1}) \in J(x, m+1)$ satisfies $q_{j_1} \leq \sqrt{x}$. Hence

$$|J(x, m+1)| \leq \sum_{i: q_i \leq x} |J\left(\frac{x}{q_i}, m\right)| \leq \sum_{i: q_i \leq x} c^{m-1} \rho\left(\frac{x}{q_i}\right)L^{m-1}\left(\sqrt{\frac{x}{q_i}}\right) \leq c^{m-1}L^{m-1}\left(\sqrt{x}\right) \sum_{i: q_i \leq x} \rho\left(\frac{x}{q_i}\right)q_i \frac{1}{q_i} \leq c^m L^{m-1}\left(\sqrt{x}\right) \rho(x) \sum_{i: q_i \leq x} \frac{1}{q_i} \leq c^m L^m\left(\sqrt{x}\right) \rho(x),$$

which completes the argument. □

Theorem 3.7 follows now immediately from Theorem 3.2, Lemma 3.8 and Lemma 3.9.

Remark 3.10. Many sequences $q$ satisfy the assumptions given above. This is the case for instance if $q_k = k^\alpha (\log(k + 1))^\beta$ for $\alpha, \beta \in (0, \infty)$, and it is easy to estimate in that case $\rho(x)$ and $L(x)$. This is also the case if $(q_k)$ is the sequence...
of prime numbers. Then $\rho(x)$ is asymptotic to $\frac{x}{\log x}$ and $L(x)$ is asymptotic to $\log \log x$. But observe that, for $X = c_0$, our more general method does not give the optimal logarithmic term if we compare it with (12) (on the other hand, it seems unknown whether the constant involved in the asymptotic of (12) is exponential in $m$, whereas the constant we obtain has that property).

Note that, taking into account Lemma 3.1, we in fact have, for any $2 \leq r < \infty$, for any $m \in \mathbb{N}$, any $P \in \mathcal{P}(m \ell_r)$ and any $u \in \ell_r$,

$$\sum_{j: p_j \leq x} |c_j(P)u_j| \leq C(m, r \frac{x^{\sigma - 1}}{\log x}) \|u\|^m \|P\|_\infty.$$

### 3.4 Bohr radii.

Here we briefly sketch how Theorem 3.2 is connected with multi-dimensional Bohr radii.

Recall that, given an index set $J \subset \mathcal{J}$, the Bohr radius of a Reinhardt $R$ in $\mathbb{C}^n$ with respect to $J$ is defined by

$$K(R; J) = \sup \left\{ 0 \leq r \leq 1 \mid \forall f \in H_\infty(R) : \sup_{u \in R} \sum_{j \in J} |c_j(f)u_j| \leq \|f\|_\infty \right\}.$$

The standard multi-variable Bohr radius is then denoted by $K(R) = K(R; \mathcal{J})$. Let us recall the two most important results on Bohr radii: For the open unit disc $R = \mathbb{D}$, Bohr’s power series theorem states that $K(\mathbb{D}) = 1/3$, and in [6] (following the main idea of [12]) it was recently proved that

$$\lim_{n \to \infty} \frac{K(B_{\ell_p^n})}{\sqrt{\frac{\log n}{n}}} = 1.$$

For every $1 \leq r \leq \infty$ and every $n$ (with constants depending on $r$ only) we have

$$K(B_{\ell_r^n}) \asymp \left( \frac{\log n}{n} \right)^{1 - \frac{1}{\min(r, 2)}}.$$

The probabilistic argument for the upper estimate is due to [7] (see also [13]), and the proof of the lower estimate from [11] uses symmetric tensor products and local Banach space theory. We sketch here a simplified argument based on Theorem 3.2.

**Theorem 3.11.** Let $1 \leq r \leq \infty$ and $\sigma = 1 - \frac{1}{\min(r, 2)}$. Then there is a constant $C = C(r)$ such for every $J \subset \mathcal{J}$ and every $n$

$$\frac{C}{\sup_m \|J(m, n)\|_{\ell^r_2}^{1/\sigma}} \leq K(B_{\ell_r^n}; J),$$

where $J(m, n) = J \cap \mathcal{J}(m, n)$ and $C \geq \frac{1}{3e^{1/2}} > 0$. 
Proof. By a simple analysis of [13, Theorem 2.2] as well as [13, Lemma 2.1]
we have
\[ \frac{1}{3} \inf_m K(B_{\ell_r}; J(m, n)) \leq K(B_{\ell_r}; J), \]
and
\[ K(B_{\ell_r}; J(m, n)) = \frac{1}{\sqrt[\chi_{\text{mon}}(\mathcal{P}(J(m, n)\ell_r))}. \]
Then the conclusion is an immediate consequence of Theorem 3.2 and the simple
fact that for the constant \(C(m, r) \leq e^m e^{(m-1)/\min\{r, 2\}} \leq e^{2m}.\)
\[ \square \]
Now the proof for the lower bound in (14) follows from the special case
\( J = \mathcal{J}. \) Indeed,
\[ J^*(m, n) = \mathcal{J}(m - 1, n) = \left( \frac{(m - 1) + n - 1}{m - 1} \right) \leq e^{m-1} \left(1 + \frac{n}{m-1}\right)^{m-1}, \]
hence inserting this estimate into (15) and minimizing over \(m\) gives what we want.

4 The Konyagin-Queffélec method

In this section, we are interested in estimates like those of Theorem 3.7 but without
the restriction of considering homogeneous polynomials. Our starting point is the
work [18] of Konyagin and Queffélec devoted to the study of the correct asymptotic
order of the Sidon constant of Dirichlet polynomials of length \(x.\) Indeed they
proved that there exists a constant \(\beta > 0,\) such that for every Dirichlet polynomial
\(\sum_{n=1}^{x} a_n n^{-s},\)
\[ \sum_{n=1}^{x} |a_n| \leq \sqrt{x} \exp(-\beta + o(1)) \sqrt{\log x \log \log x} \sup_{t \in \mathbb{R}} \left| \sum_{n=1}^{x} a_n n^{it} \right|. \]
This was improved in [9], [10], and [12] where, in particular, it turned out that
\(\beta = \frac{1}{\sqrt{2}}\) is the best possible value. Translating this in terms of polynomials using
the Bohr point of view as before (recall the notation \(J(x) = \{ j \in \mathcal{J} : p_j \leq x \} \)), this
means that
\[ \chi_{\text{mon}}(\mathcal{P}(J(x)c_0)) = \sqrt{x} \exp \left( \left( -\frac{1}{\sqrt{2}} + o(1) \right) \sqrt{\log x \log \log x} \right); \]
in other terms, the latter expression gives the precise asymptotic order of the Sidon
constant \(S(x)\) for the characters \(z_j, j \in J(x)\) on the group \(\mathbb{T}^\infty.\)

In this section, we extend (17): We replace \(c_0\) by the scale of \(\ell_r\)-spaces, and
the sequence of primes in the index \(J(x) = \{ j \in \mathcal{J} : p_j \leq x \}\) by more general
sequences \((q_k)\). Although the method applies to very general \((q_k)\)'s, the last step
of our method needs precise estimates, and hence we formulate our result just for
two specific sequences.
Theorem 4.1. Let $X = \ell_r$ with $1 \le r < \infty$ and set $\sigma = 1 - \frac{1}{\min(r, 2)}$, or let $X = c_0$ and set $\sigma = \frac{1}{2}$. Then for every $f \in H_\infty(B_X)$, every $u \in B_X$, and every $x > e$ we have

1. for $p$ denoting the sequence of primes,

$$\sum_{j : p_j \le x} |c_j(f)u_j| \le x^\sigma \exp(-\sqrt{2\sigma + o(1)}\log x \log \log x) \|f\|_\infty;$$

2. for $q = (q_k)_k$, defined by $q_k = k \cdot (\log(k + 2))^\theta$ with some $\theta \in (\frac{1}{2}, 1]$,

$$\sum_{j : q_j \le x} |c_j(f)u_j| \le x^\sigma \exp\left(\left( -2\sigma\sqrt{\theta - \frac{1}{2}} + o(1)\right) \sqrt{\log x \log \log x} \right) \|f\|_\infty.$$

In both cases, the $o$-term does not depend on $f$.

The proof will be given in Section 4.2; the next section prepares it.

4.1 Size of some index sets. The method of the proof uses a decomposition of integers inspired by the work of Konyagin and Queffélec in [18]. Let $q = (q_k)_k$ denote again a strictly increasing sequence with $q_1 > 1$ and $q_k \to \infty$ for $k \to \infty$. For technical reasons, we introduce the index of length zero $\vartheta = ()$, for which $q_\vartheta = 1$ and $(i, \vartheta) = (\vartheta, i) = i$ by convention. Following Konyagin and Queffélec [18], we denote, for $x > 0$ and $l \in \mathbb{N}$,

$$J(x) = \{ j : q_j \le x \} \cup \{ \vartheta \},$$

$$J^-(x; l) = \{ j = (j_1, \ldots, j_k) \in \mathbb{J}(k) : k \in \mathbb{N}, q_j \le x, j_k \le l \} \cup \{ \vartheta \},$$

and for $m \in \mathbb{N}$,

$$J^+(x, m; l) = \{ j = (j_1, \ldots, j_m) \in \mathbb{J}(x, m) : l < j_1 \},$$

respectively, for $m = 0$, $J^+(x, 0; l) = \{ \vartheta \}$.

From the general construction of these sets we can already say something about their size—we need a couple of lemmas.

Lemma 4.2. Let $x > 2$, $m \in \mathbb{N}$ and $l \in \mathbb{N}$.

1. $|J^-(x; l)| \le \left( 1 + \frac{\log x}{\log q_1} \right)^l$,

2. $J(x, m) = \emptyset$ whenever $m > \frac{\log x}{\log q_1}$.

Proof. (1) Using the correspondence between $\mathbb{J}(m)$ and $\Lambda(m)$, the set $J^-(x; l)$ has the same cardinal number as

$$\Gamma^-(x; l) = \{ \alpha \in \mathbb{N}_0^l | q_1^{\alpha_1} \cdots q_l^{\alpha_l} \le x \}. $$
Now, for $\alpha \in \Gamma^-(x; l)$ and $1 \leq j \leq l$,

$$q_1^{\alpha_j} \leq q_1^{\alpha_1} \cdots q_l^{\alpha_l} \leq x,$$

so that $\alpha_j \leq \frac{\log x}{\log q_1}$ for all $j$.

(2) Note that for every $j \in J^+(x, m; l)$ we have $q_1^m \leq q_j \leq x$, which immediately gives the conclusion. \qed

The next lemma is a version of Lemma 3.8 for $J^+(x, m; l)$. The proof is identical.

**Lemma 4.3.** For the reduced index sets,

$$J^+(x, m; l)^* \subset J^+\left(x^{m-1}, m-1; l\right).$$

When $q$ is increasing sufficiently fast, we can say the following about the size of $J^+(x, m; l)$:

**Lemma 4.4.** Assume that $(q_k)$ satisfies

(18) \quad $q_{j+k} - q_j \geq q_k$

for any $j, k \geq 1$. Then for every $x > 2$ and every $l, m \in \mathbb{N}$,

$$|J^+(x, m; l)| \leq xq_l^{-m}\exp(q_lL(x)).$$

**Proof.** We introduce a completely multiplicative function,

$$|J^+(x, m; l)| = \sum_{j \in J^+(x, m; l)} 1 \leq \frac{x}{q_l^m} \sum_{j \in J^+(x, m; l)} \frac{q_l}{q_j} \cdots \frac{q_l}{q_{j+m}} \leq \frac{x}{q_l^m} \prod_{l < k < x} \left(\sum_{v=1}^{\infty} \left(\frac{q_l}{q_k}\right)^v\right) \leq \frac{x}{q_l^m} \exp\left(-\sum_{l < k < x} \log\left(1 - \frac{q_l}{q_k}\right)\right).$$

Using the series expansion of the logarithm around 1, we obtain for the exponent

$$-\sum_{l < k < x} \log\left(1 - \frac{q_l}{q_k}\right) = \sum_{l < k < x} \sum_{v=1}^{\infty} \frac{1}{v} \left(\frac{q_l}{q_k}\right)^v \leq \sum_{l < k < x} \frac{q_l}{q_k} \frac{1}{1 - \frac{q_l}{q_k}}.$$

With (18), this leads to

$$-\sum_{l < k < x} \log\left(1 - \frac{q_l}{q_k}\right) \leq q_l \sum_{l < k < x} \frac{1}{q_k - q_l} \leq q_l \sum_{l < k < x} \frac{1}{q_k - l} \leq q_l \sum_{k < x} \frac{1}{q_k} = q_l L(x),$$

which completes the proof. \qed

Finally, we mention that by [3, Corollaire 1] the previous result remains true for the sequence of prime numbers. Note that in this case $L(x) \sim \log \log x$.

**Lemma 4.5.** Let $q$ denote the sequence of primes. Then there exists a constant $c > 0$ such that for every $x > 2$ and every $l, m \in \mathbb{N}$

$$|J^+(x, m; l)| \leq xq_l^{-m}\exp(q_l \cdot \left(\log \log x + c\right)).$$
4.2 Proof of Theorem 4.1. To present the Konyagin–Queffélec technique in general we need one more additional lemma.

Lemma 4.6. Let $m_1, m_2, l \in \mathbb{N}$, $X$ be a Banach sequence space and $P \in \mathcal{P}^{(m_1+m_2)X}$ such that $c_k(P) \neq 0$ for only finitely many $k \in \mathbb{N}(m_1 + m_2)$. Then for every $i \in \mathbb{N}(m_1, l)$ the polynomial

$$P_i = \sum_{j \in \mathbb{N}(m_2) \atop j > l} c_{i,j}(P) z_{i,j} \in \mathcal{P}^{(m_1+m_2)X}$$

satisfies

$$\|P_i\|_\infty \leq \|P\|_\infty.$$

Proof. Given $u \in X$, a straightforward calculation shows that

$$P_i(u) = \int_{T^l} P(\zeta_1 u_1, \ldots, \zeta_l u_l, u_{l+1} \ldots) \tilde{\zeta}_i \cdots \tilde{\zeta}_l \, d(\zeta_1, \ldots, \zeta_l),$$

where $d(\zeta_1, \ldots, \zeta_l)$ stands for the normalized Lebesgue measure on $T^l$. This immediately implies the desired inequality. □

Proof of Theorem 4.1. Recall the setting of our theorem. Let $x > e$ and $l \in \mathbb{N}$. Given $u \in B_X$, at first write $u = u^- + u^+$ where $u_k^- = 0$ for $k > l$ and $u_k^+ = 0$ for $k \leq l$. Any $k \in J(x)$ may be written as $k = (i, j)$ with $i \in J^-(x; l)$ and $j \in J^+(x, m; l)$. Moreover, $|u_i| = |u_i^-|$ and $|u_j| = |u_j^+|$. Hence, with $c_k = c_k(f)$ we have

$$\sum_{q_k \leq x} |c_k u_k| = \sum_{i \in J^-(x; l)} \sum_{m \in \mathbb{N}_0} \sum_{j \in J^+(x, m; l) \atop q_{i,j} \leq x} |c_{i,j} u_{i,j}| = \sum_{i \in J^-(x; l)} \sum_{m \in \mathbb{N}_0} \sum_{j \in J^+(x, m; l) \atop q_{i,j} \leq x} |u_i^-| \sum_{q_{i,j} \leq x} |c_{i,j} u_j^+|.$$

Using Theorem 3.2, we can now estimate the latter sum for every $i \in J^-(x; l)$,

$$|u_i^-| \sum_{j \in J^+(x, m; l) \atop q_{i,j} \leq x} |c_{i,j} u_j^+| \leq |u_i^-| |C^m| J^+(x, m; l)^* |^\sigma \sup_{\|\zeta\|_l \leq \|u^+\|_l \atop \forall k \leq l} \|c_{i,j} \zeta_j\|_{j > l} \sum_{j \in J^+ \atop j > l} \|c_{i,j} u_j^+\|_{j > l} \leq C^m |J^+(x, m; l)^* |^\sigma \sup_{\|\zeta\|_l \leq \|u^+\|_l \atop \forall k \leq l} \|c_{i,j} \zeta_j\|_{j > l} \sum_{j \in J^+(x, m; l) \atop j > l} \sum_{j \in J^+(x, m; l) \atop j > l} \|c_{i,j} u_j^+\|_{j > l} \|c_{i,j} \zeta_j\|_{j > l} \leq C^m |J^+(x, m; l)^* |^\sigma \sum_{j \in J^+(x, m; l) \atop j > l} \|c_{i,j} \zeta_j\|_{j > l},$$

where the last inequality is a consequence of $(u^- + \zeta)_{i,j} = u_i^- \zeta_j$ and

$$\|u^- + \zeta\|_\varphi = \|u^-\|_\varphi + \|\zeta\|_\varphi \leq \|u^-\|_\varphi + \|u^+\|_\varphi \leq 1,$$
in the case $X = \ell_r$ with $1 \leq r < \infty$, respectively $\|u^+ + \zeta\|_\infty = \max\{\|u^-\|_\infty, \|\zeta\|_\infty\} \leq 1$ in the case $X = c_0$. Choose for each $i \in J^{-}(x, l)$ some $m_i \in \mathbb{N}$ such that $i \in \mathcal{B}(m_i)$. By Lemma 4.6 we then obtain

\[
\sum_{q_k \leq x} |c_k u_k| \leq \sum_{i \in J^{-}(x;l)} \sum_{m} C^m |J^+(x, m; l)^*|^\sigma \|\sum_{k \in \mathcal{B}(m+m_i)} c_k u_k\|_\infty.
\]

Moreover, if we decompose $f$ into its sum of homogeneous Taylor polynomials, we deduce by Cauchy estimates that

\[
\sum_{q_k \leq x} |c_k u_k| \leq \left( |J^{-}(x;l)| \sum_{m} C^m |J^+(x, m; l)^*|^\sigma \right) \|f\|_\infty.
\]

Now $J^+(x, m; l)^* \subset J^+(x^{\frac{m-1}{m}}, m - 1; l)$ and $J^+(x, m; l) = \emptyset$ for $m > \frac{\log x}{\log q_1}$ by Lemmas 3.8 and 4.2. Hence

\[
|J^{-}(x;l)| \cdot \sum_{m} C^m |J^+(x, m; l)^*|^\sigma \leq \left( 1 + \frac{\log x}{\log q_1} \right)^{l+1} \sup_{m} C^m |J^+(x^{\frac{m-1}{m}}, m - 1; l)|^\sigma.
\]

We now assume that either $q$ satisfies (18) or that $q$ is the sequence of prime numbers. Using Lemma 4.4 or Lemma 4.5, respectively

\[
\left( 1 + \frac{\log x}{\log q_1} \right)^{l+1} \sup_{m} C^m |J^+(x^{\frac{m-1}{m}}, m - 1; l)|^\sigma
\]

\[
\leq \left( 1 + \frac{\log x}{\log q_1} \right)^{l+1} \sup_{m} \left( C^m x^{\frac{m-1}{m}} q_i^{-m+1} \exp \left( q_i L \left( x^{\frac{m-1}{m}} \right) \right) \right)^\sigma
\]

\[
\leq \left( 1 + \frac{\log x}{\log q_1} \right)^{l+1} x^\sigma q_i^\sigma \exp \left( \sigma q_i L(x) \right) \sup_{m} \left( C^m x^{-\frac{m}{m}} q_i^{-m} \right)^\sigma.
\]

Differentiating

\[
h_{x,i}(m) = m \log C + \frac{1}{m} \log x - m \log q_i,
\]

we see that it attains its maximum at

\[
M = \sqrt{\frac{\log x}{\log q_i} - C} \geq \sqrt{\frac{\log x}{\log q_i}},
\]

and therefore

\[
h_{x,i}(m) \leq h_{x,i}(M) = \log(C) \sqrt{\frac{\log x}{\log q_i}} - 2 \sqrt{\log x \log q_i} = \left( -2 + o(1) \right) \sqrt{\log x \log q_i}
\]

provided $l$ tends to $+\infty$ as $x$ tends to $+\infty$. Therefore, we have shown that

\[
\sum_{q_k \leq x} |c_k u_k| \leq \left( 1 + \frac{\log x}{\log q_1} \right)^{l+1} \exp(\sigma q_i L(x)) \exp((-2\sigma + o(1)) \sqrt{\log x \log q_i}) \|f\|_\infty.
\]
Up to this point, our arguments are independent of the specific choice of \( q \). But now we have to choose \( l \) and we restrict ourselves to the sequences appearing in the statement of Theorem 4.1.

For the sequence \( q = (q_k)_k \) defined by \( q_k = k \cdot (\log(k+2))^{\theta} \) with \( \theta \in (\frac{1}{2}, 1] \) we easily check that
\[
\frac{1}{2^{1+\theta}} \frac{x}{(\log x)^\theta} \leq \rho(x) \leq 2^\theta \frac{x}{(\log x)^\theta}
\]
and that \( L(x) \leq (\log x)^{1-\theta} \) for \( \theta < 1 \) and \( L(x) \leq \log \log x \) for \( \theta = 1 \).

We now treat both cases at once. In the case of \( q \) denoting the sequence of primes, set \( \theta = 1 \). Note that we then have similar estimates for \( \rho(x) \) and \( L(x) \).

Set \( l \) as the largest integer such that \( q_l \leq (\log x)^{\theta-1/2} \log \log x \). Observe that this implies that
\[
l = O(1) \frac{q_l}{(\log q_l)^\theta} = o(1) \frac{\sqrt{\log x}}{\log \log x};
\]
this easily implies the result.

\[\square\]

5 Monomial convergence

In this section we apply the new estimates on the unconditional basis constant of polynomials on \( X = \ell_r \) or \( X = c_0 \) from the preceding two sections, to the analysis of sets \( \text{mon} P(mX) \) and \( \text{mon} H_\infty(B_X) \) of monomial convergence of \( m \)-homogeneous polynomials on \( X \) and bounded holomorphic functions on \( B_X \).

5.1 Polynomials. The next statement gives the state of art for homogeneous polynomials. As usual, we denote \( \ell_{p,\infty} = \{ x \in c_0; \| x \|_{p,\infty} = \sup_k |k^\frac{1}{p} x_k^*| < \infty \} \), where \( x^* \) denotes the decreasing rearrangement of a sequence \( x \), and we endow \( \ell_{p,\infty} \) with \( \| \cdot \|_{p,\infty} \).

**Theorem 5.1.** Let \( m \geq 2 \). Then:

1. \( \text{mon} P(m \ell_{c_0}) = \ell_{\frac{2m}{m-1},\infty} \).
2. \( \text{mon} P(m \ell_1) = \ell_1 \).
3. If \( 2 \leq r < \infty \), then \( \ell_{\frac{2m}{m-1},\infty} \cdot \ell_r \subset \text{mon} P(m \ell_r) \subset \ell_{(\frac{m-1}{\theta}+\frac{1}{2})^{-1},\infty} \).
4. If \( 1 < r < 2 \), then for any \( \epsilon > 0 \), \( \ell_{(mr^\gamma-\epsilon)} \subset \text{mon} P(m \ell_r) \subset \ell_{(mr^\gamma+\epsilon),\infty} \).

Several cases of this theorem are already known: the first one can be found in [5] and the second one in [21] or [15]. The upper estimate in the third and the fourth case can also be found in [15]. The proof of the lower estimate in the third case follows from a general technique inspired by Lemma 3.1. We need to
introduce another notation. For $X$ a Banach sequence space, $R$ a Reinhard domain in $X$ and $\mathcal{F}(R)$ a set of holomorphic functions on $R$, we set

$$\{[\mathcal{F}(R)]_\infty = \{ f_w : u \in B_{c_0} \to f(uw); \ w \in R, \ f \in \mathcal{F}(R) \}.$$  

$[\mathcal{F}(R)]_\infty$ is a set of holomorphic functions on $B_{c_0}$, and the following general result holds true.

**Lemma 5.2.** $R \cdot \text{mon}[\mathcal{F}(R)]_\infty \subset \text{mon} \mathcal{F}(R)$.

**Proof.** Let $w \in R$ and $u \in \text{mon}[\mathcal{F}(R)]_\infty$. For any $f \in \mathcal{F}(R)$ then

$$c_a(f_w) = w^a c_a(f)$$

and therefore

$$\sum_a |c_a(f)||wu|^a = \sum_a |c_a(f_w)||u|^a < +\infty,$$

which yields the claim. \hfill \Box

It is now easy to deduce the lower estimate in the third case, knowing the result of part (1). Indeed, $[\mathcal{P}^m X]_\infty$ is contained in the set of bounded $m$-homogeneous polynomials on $B_{c_0}$, thus in $\mathcal{P}^m c_0$ by the natural extension of a bounded polynomial from $B_{c_0}$ to $c_0$.

The lower inclusion in (4) is a partial solution of a conjecture made in [15] (see the remarks after Example 4.6 in [15]). Its proof seems less simple, and requires some preparation. Note that for $2 \leq r < \infty$ we have that

$$\frac{1}{p^{\frac{m-1}{m}}} \cdot \ell_r \subset \text{mon} \mathcal{P}^m \ell_r$$

and

$$\frac{1}{p^{\frac{m}{m}}} \cdot c_0 \subset \text{mon} \mathcal{P}^m c_0,$$

and the exponent in $p$ is optimal; this is an immediate consequence of Theorem 5.1, (3) and the prime number theorem. For $1 < r < 2$ we can prove this up to an $\varepsilon$:

**Theorem 5.3.** For $1 < r < 2$ and $m \geq 1$ put $\sigma_m = \frac{m-1}{m}(1 - 1/r)$. Then for every $\varepsilon > 1/r$

$$\frac{1}{p^{\sigma_m(\log(p))^{\varepsilon}}} \cdot \ell_r \subset \text{mon} \mathcal{P}^m \ell_r.$$

In particular, for all $\varepsilon > 0$,

$$\frac{1}{p^{\sigma_m+\varepsilon}} \cdot \ell_r \subset \text{mon} \mathcal{P}^m \ell_r.$$
Since the upper estimate in Theorem 5.1, (4) is already proved, the prime number theorem again shows that the given exponent in $p$ is optimal.

**Proof.** Let $P = \sum_{j \in \mathcal{B}(m)} c_j(P)z_j \in \mathcal{P}(m \ell_r)$ and let $u \in \ell_r$. We intend to show that

$$S = \sum_{j \in \mathcal{B}(m)} |c_j(P)| \frac{1}{(p_{j_1} \cdots p_{j_m})^\alpha \log(p_{j_1})} |u_j| \leq C \|u\|_r^m \|P\|_\infty$$

for some constant $C > 0$. Let us observe that, for any $j_1, \ldots, j_m \geq 1$,

$$\log(p_{j_1}) \cdots \log(p_{j_m}) \geq \frac{(\log 2)^{m-1}}{m} \log(p_{j_1} \cdots p_{j_m}).$$

We order the sum over $j \in \mathcal{B}(m)$ with respect to the value of the product $p_{j_1} \cdots p_{j_m}$.

Precisely, using (19), we write

$$S \ll \sum_{N = m}^{+\infty} \sum_{\mathcal{B}(m)} \frac{1}{2^{N \sigma} N^\varepsilon} |c_j(P)||u_j| \ll \sum_{N = m}^{+\infty} \frac{1}{2^{N \sigma} N^\varepsilon} \sum_{p_{j_1} \leq 2^{N+1}} |c_j(P)||u_j|.$$  

We apply Theorem 3.7 to find

$$S \ll \sum_{N = m}^{+\infty} \frac{1}{2^{N \sigma} N^\varepsilon} \frac{2^{N \sigma} \log(N)^{1-m-1(1-\frac{1}{r})}}{N^\varepsilon} \|P\|_\infty \|u\|_r^m.$$  

The series is convergent since $\varepsilon > 1/r$. □

Finally, we are ready to provide the

**Proof of the lower inclusion of Theorem 5.1, (4).** Given $u \in \ell_{(m \ell_r)\gamma-\varepsilon}$, we show that the decreasing rearrangement $u^* \in \text{mon } \mathcal{P}(m \ell_r)$. Then for some $\delta > 0$ we have

$$u_n^* \ll \frac{1}{n^{\frac{1}{m \ell_r - \varepsilon}}} = \frac{1}{n^{\frac{1}{m \ell_r + \delta}}}.$$  

By the prime number theorem we know that $p_n \asymp n \log n$, hence

$$\frac{1}{n^{\frac{1}{m \ell_r + \delta}}} = \frac{1}{p_n^{\frac{m-1}{m \ell_r + \delta}}} \ll \frac{1}{n^{\frac{m-1}{m \ell_r + \delta}}} \frac{(n \log n)^{\frac{m-1}{m \ell_r + \delta}}}{n^{\frac{1}{m \ell_r + \delta}}}.  

But obviously

$$\frac{(n \log n)^{\frac{m-1}{m \ell_r + \delta}}}{n^{\frac{1}{m \ell_r + \delta}}} \in \ell_r,$$

hence by Theorem 5.3

$$\frac{1}{n^{\frac{1}{m \ell_r + \delta}}} \in \text{mon } \mathcal{P}(m \ell_r),$$  

we receive the conclusion. □
**Remark 5.4.** A look at [15] shows that in the case $2 < r < \infty$ the proof of the inclusion $\text{mon} \mathcal{P}^m(\ell_r) \subset \ell_{(\frac{m-1}{2m}, \frac{1}{r})^{-1}, \infty}$ keeps working if we replace $\ell_r$ by $\ell_{r, \infty}$. Indeed, it just uses that
\[
\sup_{u \in B_{\ell_r}^n} \sum_{k=1}^{n} |u_k|^2 = n^{1-2/r}
\]
and this remains true, up to a constant factor, if we replace $B_{\ell_r}$ by $B_{\ell_{r, \infty}}$. If we combine this with Lemma 5.2 and Theorem 5.1, (1) then we find that, for $2 < r < \infty$,
\[
\ell_{(\frac{m-1}{2m}, \frac{1}{r})^{-1}, \infty} = \ell_{\frac{2m}{m-1}, \infty} \subset \text{mon} \mathcal{P}^m(c_0) \cdot \ell_{r, \infty} \subset \text{mon} \mathcal{P}^m(\ell_{r, \infty}) \subset \ell_{(\frac{m-1}{2m}, \frac{1}{r})^{-1}, \infty},
\]
hence we receive the equality
\[
\ell_{(\frac{m-1}{2m}, \frac{1}{r})^{-1}, \infty} = \text{mon} \mathcal{P}^m(\ell_{r, \infty}).
\]

**5.2 Holomorphic functions.** We now study $\text{mon} H_\infty(B_X)$ for $X = \ell_r$ with $1 \leq r < \infty$ or $X = c_0$. The extreme cases are already well-known: By a result of Lempert (see, e.g., [19] and [15]) we have
\[
(20) \quad \text{mon} H_\infty(B_{\ell_1}) = B_{\ell_1}.
\]
Moreover by [5] we know that
\[
(21) \quad B \subset \text{mon} H_\infty(B_{c_0}) \subset \overline{B}
\]
where
\[
B = \left\{ u \in B_{c_0}; \limsup_n \frac{1}{\log n} \sum_{k=1}^{n} |u_k|^2 < 1 \right\},
\]
\[
\overline{B} = \left\{ u \in B_{c_0}; \limsup_n \frac{1}{\log n} \sum_{k=1}^{n} |u_k|^2 \leq 1 \right\}.
\]
For $1 < r < \infty$, it was shown in [15] that, setting $\frac{1}{s} = \frac{1}{2} + \frac{1}{\max\{r, 2\}}$, for every $\varepsilon > 0$
\[
(22) \quad B_{\ell_r} \cap \ell_s \subset \text{mon} H_\infty(B_{\ell_r}) \subset B_{\ell_r} \cap \ell_{s+\varepsilon}.
\]
In the following we improve the previous inclusion, and show in particular that here $\varepsilon = 0$ is not possible. More precisely, we give necessary and sufficient conditions on $\alpha, \beta > 0$ such that
\[
\left( \frac{1}{n^\alpha (\log(n+2))^{\beta}} \right)_n \in \text{mon} H_\infty(B_{\ell_r}).
\]
Note that by (21), for every $\beta > 0$

\[
\left(\frac{1}{n^\frac{1}{\beta} \left(\log(n + 2)\right)^\beta}\right)_n \in \text{mon } H_\infty(B_{c_0})
\]

we do not know whether here $\beta = 0$ is possible. Moreover, by (20)

\[
\left(\frac{1}{n \left(\log(n + 2)\right)^\beta}\right)_n \in \text{mon } H_\infty(B_{\ell_1})
\]

if and only if $\beta > 1$. The following result gathers our knowledge for the remaining cases:

**Theorem 5.5.** For $1 \leq r \leq \infty$ put $\sigma = 1 - \frac{1}{\min(r,2)}$. Then

(1a) For any $\theta > \frac{1}{2}$ and $1 \leq r \leq 2$

\[
\left(\frac{1}{n^\theta \left(\log(n + 2)\right)^\theta}\right)_n \cdot B_{\ell_r} \subset \text{mon } H_\infty(B_{\ell_r}).
\]

In particular, $\left(\frac{1}{n^\theta \left(\log(n + 2)\right)^\theta}\right)_n \in \text{mon } H_\infty(B_{\ell_r})$ whenever $\beta > \frac{1}{2r} + \frac{1}{2}$. (1b) For any $\theta > 0$ and $2 \leq r < \infty$

\[
\left(\frac{1}{n^\theta \left(\log(n + 2)\right)^\theta}\right)_n \cdot B_{\ell_r} \subset \text{mon } H_\infty(B_{\ell_r}).
\]

In particular, $\left(\frac{1}{n^\theta \left(\log(n + 2)\right)^\theta}\right)_n \in \text{mon } H_\infty(B_{\ell_r})$ whenever $\beta > \frac{1}{r}$. (1c) For any $\theta > 0$ and $r = \infty$

\[
\left(\frac{1}{n^\theta \left(\log(n + 2)\right)^\theta}\right)_n \cdot B_{c_0} \subset \text{mon } H_\infty(B_{c_0}).
\]

In particular, $\left(\frac{1}{n^\theta \left(\log(n + 2)\right)^\theta}\right)_n \in \text{mon } H_\infty(B_{c_0})$ whenever $\beta > 0$.

(2) Suppose that $\left(\frac{1}{n^\theta \left(\log(n + 2)\right)^\theta}\right)_n \in \text{mon } H_\infty(B_{\ell_r})$. Then $\beta \geq \frac{1}{r}$.

Note that we cannot replace $\log(n + 2)$ by $\log(n + 1)$ in the previous statement (indeed, it can be easily seen that $\text{mon } H_\infty(B_{\ell_r}) \subset \text{mon } H_\infty(B_{c_0}) \subset B_{c_0}$).

**Proof.** At first note that (1c) follows directly from (21). We proceed to prove (1b): We have that $\theta > 0$ and $2 \leq r < \infty$. Then an easy argument yields

\[
\left(\frac{1}{n^\frac{1}{\theta} \left(\log(n + 2)\right)^\theta}\right)_n \in B.
\]

From $[H_\infty(B_{\ell_r})]_\infty \subset H_\infty(B_{c_0})$ we get $\text{mon } H_\infty(B_{c_0}) \subset \text{mon } [H_\infty(B_{\ell_r})]_\infty$, and hence by (21) and Lemma 5.2

\[
\left(\frac{1}{n^\frac{1}{\theta} \left(\log(n + 2)\right)^\theta}\right)_n \cdot B_{\ell_r} \subset B \cdot B_{\ell_r} \subset \text{mon } [H_\infty(B_{\ell_r})]_\infty \cdot B_{\ell_r} \subset \text{mon } H_\infty(B_{\ell_r}).
\]
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Let us now prove part (1a): Assume that $\theta > \frac{1}{2}$ and $1 \leq r \leq 2$. We shall apply Theorem 4.1, (2) with the sequence $q$ defined by $q_j = j \cdot (\log(j+2))^\theta$. Then for $f \in H_\infty(\ell_r)$ and $u \in B_{\ell_r}$ the conclusion follows from

$$\sum_j \frac{1}{q_j^\sigma} |c_j(f)u_j|$$

$$= \sum_{N=0}^\infty \sum_{j : e_N N \leq j \leq e_{N+1}} \frac{1}{q_j^\sigma} |c_j(f)u_j|$$

$$\leq \sum_{N=0}^\infty \frac{1}{e^{N\sigma}} \sum_{j \in J(e_{N+1})} |c_j(f)u_j|$$

$$\leq \sum_{N=0}^\infty \frac{1}{e^{(N-1)\sigma}} e^{N\sigma} \exp \left( (-2\sigma \sqrt{\theta - \frac{1}{2}} + o(1)) \sqrt{N+1} \log(N+1) \right) \cdot \|f\|_\infty < \infty.$$ 

Finally, we check part (2): For $r = 1$, (20) already proves the claim. At first we will treat the case $1 < r \leq 2$ with a probabilistic argument. Afterwards we reduce the case $r \geq 2$ to the case $r = 2$. Let $1 < r \leq 2$. We shall apply Corollary 3.2 of [4] (with $p = r$). Then there is an absolute constant $C \geq 1$ such that for any $m, n$ there are $(\epsilon_j)_j \in T_{\beta(m,n)}$ for which

$$\sup_{u \in B_{\ell_r}} \left| \sum_j \epsilon_j |j| u_j \right| \leq C(n \log m)^\sigma m^{\sigma}.$$ 

Let now $x = (k^{-1}(\log(k+2))^{-\beta})_k$ denote the sequence in question and assume $x \in \text{mon } H_\infty(B_{\ell_r})$. Then, by a closed graph argument, there exists a constant $\tilde{C} \geq 1$, such that for every $f \in H_\infty(B_{\ell_r})$,

$$\sum_j |c_j(f) x^\epsilon| \leq \tilde{C} \|f\|.$$ 

For any $n \in \mathbb{N}$ now,

$$\left( \sum_{k=1}^n |x_k| \right)^m = \sum_{j \in T_{\beta(m,n)}} |\epsilon_j |j| |x_j| | \leq \tilde{C} \sup_{u \in B_{\ell_r}} \left| \sum_j \epsilon_j |j| u_j \right| \leq \tilde{C} C(n \log m)^\sigma m^{\sigma},$$

by (26) and (25). Taking the $m$-th root, we obtain

$$\sum_{k=1}^n \frac{1}{k(\log(k+2))^{\beta}} \leq (\tilde{C})^\frac{1}{m} (n \log m)^\sigma m^{\sigma}$$

for every $n, m \in \mathbb{N}$. It now suffices to notice that with $m = \lceil \log n \rceil$, the right-hand side is asymptotically equivalent to $(\log n)^\sigma$ and the left-hand side to $(\log n)^{1-\beta}$ as $n \to \infty$. Hence $\beta > -\sigma + 1 = 1/r$. 

Now suppose \( r \geq 2 \) and set \( \xi = (k^{\frac{1}{r}}(\log(k+2))^{-\frac{1}{r}+\varepsilon})_k \) for \( \frac{1}{r} + \frac{1}{\ell_2} = \frac{1}{2} \) and \( \varepsilon > 0 \). Consider \( f \in H_\infty(B_{\ell_2}) \) and let us set \( g = f \circ D_\xi \), where \( D_\xi \) denotes the diagonal operator \( \ell_r \rightarrow \ell_2 \) induced by \( \xi \), which is bounded by Hölder’s inequality. Thus \( g \in H_\infty(B_{\ell_2}) \). We have

\[
\sum_j |c_j(f)| \frac{1}{j_1(\log(j_1+2))^{\frac{1}{r}+\beta+\varepsilon}} \cdots \frac{1}{j_m(\log(j_m+2))^{\frac{1}{r}+\beta+\varepsilon}} = \sum_j |(c_j(f) \xi_j) x_j| < \infty,
\]

under the assumption that \( x = (k^{\frac{1}{r}+\frac{1}{\ell_2}}(\log(k+2))^{-\beta})_k \in \text{mon} \ H_\infty(B_{\ell_2}) \) (note that \( \frac{1}{\ell_2} \ll \frac{1}{r} + \frac{1}{\ell_2} = 1 \)). Hence \( (k(\log(k+2))^{\frac{1}{r}+\beta+\varepsilon})_k \in \text{mon} \ H_\infty(B_{\ell_2}) \) and by our result in the case \( r = 2, \frac{1}{r} + \beta + \varepsilon \geq \frac{1}{2} \) for every \( \varepsilon > 0 \). □

We are now able to give an answer to our previously stated question: the inclusion (22) holds not true for \( \varepsilon = 0 \).

**Corollary 5.6.** Let \( 1 < r < \infty \) and \( \frac{1}{s} = \frac{1}{r} + \frac{1}{\max\{r,2\}} \). Then

\[
B_{\ell_r} \cap \ell_s \subset \subset \text{mon} \ H_\infty(B_{\ell_r}).
\]

**Proof.** Assume equality. Let \( q = (k(\log(k+2)))_k \). By Theorem 5.5 this implies that the diagonal operator \( \ell_r \rightarrow \ell_s \) induced by the sequence \( q^{-\sigma} \), where \( \sigma = 1 - \frac{1}{\min\{r,2\}} \), is bounded. Hence

\[
\left( \sum_{k=1}^\infty |q_k^{-\sigma}|^t \right)^{\frac{1}{t}} = \sup_{x \in B_{\ell_\rho}} \left( \sum_{k=1}^\infty |x_k q_k^{-\sigma}|^t \right)^{\frac{1}{t}} = \|D_{q^{-\sigma}} : \ell_r \rightarrow \ell_s\| < \infty,
\]

where \( \frac{1}{s} = \frac{1}{r} + \frac{1}{\ell_2} \). Therefore \( q^{-\sigma} \in \ell_r \). But

\[
\sum_{k=1}^\infty q_k^{-\sigma} = \sum_{k=1}^\infty \frac{1}{k \log(k+2)} = \infty,
\]

a contradiction. □

Using the same technique as in the proof of (1a) in Theorem 5.5, we easily obtain the following analog of Theorem 5.3.

**Corollary 5.7.** Let \( 1 < r < \infty \) and let \( \sigma = 1 - \frac{1}{\min\{r,2\}} \). Then

\[
p^{-\sigma} : B_{\ell_r} \subset \subset \text{mon} \ H_\infty(B_{\ell_r}),
\]

and here \( \sigma \) is best possible.
Proof. We proceed analogously to the proof of (1a) in Theorem 5.5 and obtain, for \( f \in H_\infty(B_{\ell_r}) \) and \( u \in B_{\ell_r} \) by Theorem 4.1,
\[
\sum_j \frac{1}{p^j} |c_j(f)u_j| = \sum_{N=0}^{\infty} \sum_{j: e^N < q_j \leq e^{N+1}} \frac{1}{p^j} |c_j(f)u_j| \leq \sum_{N=0}^{\infty} \frac{e^{N\sigma}}{\exp((-\sqrt{2}\sigma + o(1))\sqrt{N \log N})} \|f\|_\infty < \infty.
\]

Remark 5.8. Analogously to the result (21) for \( c_0 \) and in view of Theorem 5.5, a plausible conjecture would be that for all \( 2 \leq r < \infty \)
\[
B_r \subset \text{mon} H_\infty(B_{\ell_r}) \subset \overline{B}_r,
\]
where for \( \frac{1}{s} = \frac{1}{2} + \frac{1}{r} \)
\[
B_r = \left\{ u \in B_{c_0}; \limsup_n \frac{1}{(\log n)^{s+1}} \sum_{k=1}^n |u_k^n|^s < 1 \right\},
\]
\[
\overline{B}_r = \left\{ u \in B_{c_0}; \limsup_n \frac{1}{(\log n)^{s+1}} \sum_{k=1}^n |u_k^n|^s \leq 1 \right\}.
\]

Remark 5.9. In Theorem 5.5, the cases \( 1 \leq r \leq 2 \) and \( 2 \leq r < \infty \) do not really fit for \( r = 2 \). This is due to the fact that when we apply Theorem 4.1 (2) (within the proof of Theorem 5.5, (1a)), then we need that \( \theta > 1/2 \). It would be nice to extend the statement of this last theorem to \( \theta \in (0, 1/2] \).

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