AN EXAMPLE OF BIRATIONALLY INEQUIVALENT PROJECTIVE SYMPLECTIC VARIETIES WHICH ARE D-EQUIVALENT AND L-EQUIVALENT

SHINNOSUKE OKAWA

ABSTRACT. We give an example of a pair of projective symplectic varieties in arbitrarily large dimensions which are D-equivalent, L-equivalent, and birationally inequivalent.

1. INTRODUCTION

It is widely believed that the bounded derived category of coherent sheaves $D(X) = D^b\text{coh} X$ is a fundamental invariant of a smooth projective variety $X$. It is hence natural to ask which kind of information of the variety $X$ can be regained from the triangulated category $D(X)$.

The Grothendieck ring of varieties, which will be denoted by $K_0(\text{Var})$ in this paper, is the quotient of the free abelian group generated by the set of isomorphism classes of schemes of finite type over the fixed base field $k$ modulo the relations

$$[X] = [X \setminus Z] + [Z] \quad (1.1)$$

for closed embeddings $Z \subset X$. Multiplication in $K_0(\text{Var})$ is defined by the Cartesian product, which is easily seen to be associative, commutative, and unital with $1 = [\text{Spec } k]$. The localized Grothendieck ring of varieties $K_0(\text{Var})[L^{-1}]$ is the localization of $K_0(\text{Var})$ by the class $L = [A^1]$ of the affine line.

A pair $(X, Y)$ of smooth projective varieties are said to be D-equivalent if they have equivalent derived categories. Similarly, they are said to be L-equivalent if they satisfy the following equivalent conditions.

$$[X] = [Y] \in K_0(\text{Var})[L^{-1}] \iff L^m \cdot ([X] - [Y]) = 0 \in K_0(\text{Var}) \quad \exists m \in \mathbb{N} \quad (1.2)$$

It is asked independently in the first preprint versions of [KS17] and [IMOU16] if D-equivalence should imply L-equivalence. This is motivated by the first such example found in [Bot], [Mar16] and the other examples discovered in [IMOU] (the D-equivalence is shown later in [Kuz16]), [KS17], and [HL16], [MOU16]. In addition, more supporting evidences have been discovered in the works [BCP17], [KR17], [Man17], and [KKM17]. In fact, all known examples are pairs of simply connected Calabi-Yau varieties with $h^{2,0} = 0$ or K3 surfaces. On the other hand, it is shown in [MOU16] and [Efi17] that the pair of an abelian variety of dimension at least two and its dual is a counter-example to the question as soon as the endomorphism ring of the abelian variety is isomorphic to $\mathbb{Z}$. Taking these counter-examples into account, the simply-connectedness is assumed in [KS17, Conjecture 1.6], [IMOU16, Section 7] instead proposes to modify the Grothendieck ring of varieties suitably.

The aim of this paper is to give a first example of a pair of projective symplectic varieties which are both D-equivalent, L-equivalent, and birationally inequivalent in arbitrarily large dimension. Let $X$ be a projective K3 surface. It is well known by [Fog73] that the
Hilbert scheme $X^{[n]}$ of $n$ points on $X$, which very roughly is described as

$$X^{[n]} = \{ I \subset \mathcal{O}_X | \dim \mathcal{O}_X / I = n \},$$

is a smooth projective symplectic variety of dimension $2n$. It is simply connected and satisfies $h^{2,0}(X) = 1$. Below is the main result of this article.

**Theorem 1.1.** Let $(X, Y)$ be a non-isomorphic pair of K3 surfaces of Picard number $1$ and of degree $2d_X, 2d_Y$ respectively which are both D-equivalent and L-equivalent. Then $X^{[n]}$ and $Y^{[n]}$ are D-equivalent, L-equivalent, and birationally inequivalent if either

1. $d_X \neq d_Y$ or
2. $d_X = d_Y$, $n > 2$, and there exists an integer solution to the following Pell’s equation.

$$(n - 1) X^2 - d_X Y^2 = 1$$

**Remark 1.2.** It follows either from [HL16, Theorem 4.1] or [IMOU16, Theorem 1.3] that if $X$ is a very general K3 surface of degree 12 (i.e., $d_X = 6$) and $Y$ is the Fourier-Mukai partner of $X$ ($d_Y = 6$), then $X$ and $Y$ are L-equivalent. Hence, e.g., if $n = 6y^2 + 2$ for some positive integer $y$, the assumption (2) is satisfied by the obvious solution $(X, Y) = (1, y)$. Hence we do have an example as in Theorem 1.1 for $n = 6y^2 + 2$ ($y = 1, 2, \ldots$) = 8, 26, 56, $\ldots$.

In fact, as we mention in the next remark, one can determine if $X^{[n]}$ and $Y^{[n]}$ are birationally equivalent or not by checking the existence of solutions to certain set of Pell’s equation. Hence for those $n$ where the birationality holds, we do not have an example in dimension $2n$ yet. Note, however, if there is a Fourier-Mukai pair $(X, Y)$ of K3 surfaces of Picard number one which are L-equivalent and $d_X \neq d_Y$, then by Theorem 1.1 we can construct examples for every $n$.

**Remark 1.3.** In the recent preprint [MMY18], the authors gave a criterion for the birationality of the Hilbert schemes of points $X^{[n]}$ and $Y^{[n]}$ on K3 surfaces $X, Y$ of Picard number one. In [MMY18, Proposition 1.2] they apply the criterion to the pair $(X, Y)$ which appeared in the previous remark, to construct examples of pairs of Hilbert schemes of points which are D-equivalent and birationally inequivalent. They in particular show that $X^{[n]}$ and $Y^{[n]}$ are birationally equivalent to each other for some $n$, starting with $n = 2, 3, 4, 10, 12, 14, 15, 18, 20, \ldots$. In fact, they are even isomorphic to each other for $n = 2$ and 3 as shown in [Yos01, Example 7.2].

The proof of Theorem 1.1 is given in the next section. In fact the D-equivalence is nothing but [Plo07, Proposition 8], which in turn is an application of [BKR01]. The L-equivalence immediately follows from the description of the generating series of the Hilbert scheme of points on a smooth quasi-projective variety $Z$ as the $(L - \dim Z[Z])$-th (!) power of the generating series for $A^{\dim Z}$ due to [GZLMH06]. In order to show the birational inequivalence of $X^{[n]}$ and $Y^{[n]}$, we use the very detailed description of the movable cone of $X^{[n]}$ due to [BM14]. This is the only place where we use the assumption on the Picard number and the degrees, and will be discussed in Proposition 2.2.

Throughout this paper the base field $k$ will be the field of complex numbers $\mathbb{C}$. Varieties are always assumed to be connected.

**Acknowledgements.** The author is indebted to Kota Yoshioka, Michal Kapustka, and Grzegorz Kapustka for pointing out a crucial error in Proposition 2.2 of the first draft and informing him of references. He is also grateful to Kota Yoshioka for sending the preprint [MMY18] to the author. The author was partially supported by Grants-in-Aid for Scientific Research (16H05994, 16K13746, 16H02141, 16K13743, 16K13755, 16H06337) and the Inamori Foundation.
2. Proof of Theorem \[\text{[1.1]}\]

Let \((X, Y)\) be a pair of smooth projective surfaces which are both D-equivalent and L-equivalent. Then \((X[n], Y[n])\) is a pair of smooth projective 2n-folds by [Fog68], which are also D-equivalent by [Plo07, Proposition 8]. On the other hand, one can show the L-equivalence as follows. This is essentially due to [GZLMH06], and it applies to smooth quasi-projective varieties of arbitrary dimension. In the proof we consider the generating series of the Hilbert scheme, which is defined for an arbitrary variety \(Z\) as follows.

\[
\mathbb{H}_Z(T) := \sum_{n=0}^{\infty} [Z^n] T^n = 1 + [Z]T + \cdots \in 1 + T \cdot K_0(\text{Var}) [T].
\]  

(2.1)

Lemma 2.1. Let \((X, Y)\) be a pair of L-equivalent smooth quasi-projective varieties. Then \((X[n], Y[n])\) also is a pair of L-equivalent varieties.

Proof. Let \(m\) be a natural number such that

\[
\mathbb{L}^m \cdot ([X] - [Y]) = 0 \iff \mathbb{L}^m \cdot [X] = \mathbb{L}^m \cdot [Y] \in K_0(\text{Var}).
\]  

(2.2)

Since both \(\mathbb{L}^m \cdot [X]\) and \(\mathbb{L}^m \cdot [Y]\) are primitive elements in the sense of [Yas03, Definition 1.1], their dimensions are well-defined and the same. Hence we see \(\dim X = \dim Y\).

By [GZLMH06 COROLLARY], for any smooth quasi-projective variety \(Z\) there is an equality

\[
\mathbb{H}_Z(T) = (\mathbb{H}_{\text{dim } Z}(T))^{1 - \text{dim } Z} \in K_0(\text{Var}) [\mathbb{L}^{-1}] [T]
\]  

(2.3)

(see [GZLMH06] and references therein for the notion of the power structure of \(K_0(\text{Var})\)). Combining it with the assumption

\[
[X] = [Y] \in K_0(\text{Var}) [\mathbb{L}^{-1}],
\]  

(2.4)

we obtain the following sequence of equalities.

\[
\mathbb{H}_X(T) = (\mathbb{H}_{\text{dim } X}(T))^{1 - \text{dim } X} \mathbb{H}_Y(T) = \mathbb{H}_Y(T) \in K_0(\text{Var}) [\mathbb{L}^{-1}] [T].
\]  

(2.5)

Comparing the coefficients of \(T^n\), we obtain the L-equivalence of \(X[n]\) and \(Y[n]\). \(\square\)

Let us now specialize to the pair \((X, Y)\) as in Theorem \[\text{[1.1]}\]. In the rest we assume \(n \geq 2\) (for the case \(n = 1\) being trivial). Since we assumed that \(X\) is of Picard number 1, we can and will use the results in [BM14, Section 13] to understand the movable cone of \(X[n]\).

Proposition 2.2. Let \(X\) and \(Y\) be a pair of non-isomorphic K3 surfaces of Picard number 1 and of degree \(2d_X, 2d_Y\) respectively. Then \(X[n]\) and \(Y[n]\) are not birationally equivalent if either

(1) \(d_X \neq d_Y\) or

(2) \(d_X = d_Y, n > 2\), and there exists an integer solution to the following Pell’s equation.

\[
(n - 1) X^2 - d_X Y^2 = 1
\]  

(2.6)

Proof. Let us briefly recall the results in [BM14, Section 13]. The Picard group \(\text{Pic}(X[n])\) is freely generated by the two divisors \(\tilde{H}\) and \(B\), where \(\tilde{H}\) is the pull-back of the ample generator of \(\text{Pic}(\text{Sym}^n X)\) by the Hilbert-Chow morphism, and \(B\) is the half of the exceptional divisor. Moreover there exists a primitive embedding

\[
\text{Pic}(X[n]) \hookrightarrow H^*(X, \mathbb{Z}),
\]  

(2.7)
where $H^* (X, \mathbb{Z}) = H^0 (X, \mathbb{Z}) \oplus \text{Pic} (X) \oplus H^4 (X, \mathbb{Z})$ is the Mukai lattice of $X$ equipped with the Mukai pairing
\[
(r, L, s) \cdot (r', L', s') = LL' - rs' - sr' \in \mathbb{Z}.
\] (2.8)

The embedding is an isometry with respect to this pairing and the Beauville-Bogomolov-Fujiki form $q$ on $\text{Pic} (X^{[n]})$, and the embedding sends $\tilde{H}$ to $(0, -H, 0)$ and $B$ to $(-1, 0, 1 - n)$.

As explained in [BHL14 Proposition 13.1], there are three possibilities (a), (b), and (c) for the two extremal rays of the movable cone $\text{Mov} (X^{[n]})$. In any case, one of the rays is spanned by the primitive vector $\tilde{H}$ corresponding to the Hilbert-Chow morphism.

Suppose for a contradiction that there exists a birational map $\varphi : X^{[n]} \dasharrow Y^{[n]}$. Since both $X^{[n]}$ and $Y^{[n]}$ are smooth and have trivial canonical bundles, $\varphi$ is an isomorphism in codimension one. Hence by [Huy99, Lemma 2.6] it induces an isometry
\[
\varphi_* : \left( \text{Pic} (X^{[n]}), q_X \right) \sim \left( \text{Pic} (Y^{[n]}), q_Y \right),
\] (2.9)
which by its construction also respects the movable cones. In particular $\varphi^{-1}$ sends the base point free divisor $\tilde{H}_Y$, the primitive ample divisor on $Y^{[n]}$ corresponding to the Hilbert-Chow morphism of $Y$, to either
\begin{enumerate}
  \item $\tilde{H}$ or
  \item the primitive generator of the other extremal ray $\rho$ of $\text{Mov} (X^{[n]})$.
\end{enumerate}

In the case (i), the birational map $\varphi$ respects the exceptional divisors of the Hilbert-Chow morphisms of $X$ and $Y$. Hence by [Deb84, Theorem 2.1], $\varphi$ should be induced from an isomorphism from $X$ to $Y$ (note that any birational map between $X$ and $Y$ is an isomorphism). Since $X$ and $Y$ are not isomorphic to each other by the assumption, this is a contradiction.

In the rest of the proof we assume (ii) and show that we end up with a contradiction, to conclude that $Y^{[n]}$ is birationally inequivalent to $X^{[n]}$. Let us now assume
\[
d_X \geq d_Y
\] (2.10)
without loss of generality.

In the case (a), $\rho$ corresponds to the (rational) Lagrangian fibration of $X^{[n]}$. Hence this case can not occur under our assumptions.

In the case (b), $\rho$ is spanned by the integral divisor
\[
x_1 (n - 1) \tilde{H} - d_X y_1 B, \tag{2.11}
\]
where $x_1, y_1 > 0$ is the integer solution of the Pell’s equation (2.6) with the smallest $x_1$.

It is easy to see that $\gcd (x_1 (n - 1), d_X y_1) = 1$, since otherwise $(x_1, y_1)$ can not be a solution of (1.4). Hence (2.11) is the primitive generator of $\rho$. Now since (2.9) is an isometry, we obtain the following equality.
\[
2d_Y = q_Y \left( \tilde{H}_Y, \tilde{H}_Y \right) = q_X \left( x_1 (n - 1) \tilde{H} - d_X y_1 B, x_1 (n - 1) \tilde{H} - d_X y_1 B \right) = (x_1)^2 (n - 1)^2 (2d_X) - (d_X)^2 (y_1)^2 (2(n - 1)) = 2d_X (n - 1) (x_1 (n - 1) - d_X (y_1)^2) = 2d_X (n - 1). \tag{2.12}
\]

For the last equality, we use that $(x_1, y_1)$ is a solution of the Pell’s equation (2.6). Since it follows from (2.10) that $2d_X (n - 1) \geq 2d_Y$, we should have $n = 2$ and $d_X = d_Y$. This contradicts both of the assumptions (1) and (2). Since the existence of a solution to the Pell’s equation is assumed in the case (2), here we conclude the proof in that case because of the trichotomy in [BHL14 Proposition 13.1].
Finally, suppose that we are in the case (c) under the assumption (1). Then $\rho$ is spanned by the integral divisor

$$x'_1 \tilde{H} - y'_1 d_X B,$$

(2.13)

where $x'_1, y'_1$ is the integer solution of the Pell’s equation

$$X^2 - d_X (n - 1) Y^2 = 1$$

(2.14)

with the smallest $\frac{y'_1}{x'_1} > 0$. One can easily check as in the case (b) that (2.13) is the primitive generator of $\rho$. Thus we obtain the following contradiction.

$$2d_Y = q_Y \left( \tilde{H}_Y, \tilde{H}_Y \right) = q_X \left( x'_1 \tilde{H} - y'_1 d_X B, x'_1 \tilde{H} - y'_1 d_X B \right) = 2d_X.$$

(2.15)

□

**Remark 2.3.** If $d_X = d_Y$ and there is no solution to the Pell’s equation (2.6), then we are in the case (c) but get no contradiction. In fact this does occur, e.g., when $n = 3$ and $(X, Y)$ is a Fourier-Mukai pair of K3 surfaces of degree 12 as mentioned in Remark 1.3.

**REFERENCES**

[BBCP17] Lev A. Borisov, Andrei Caldararu, and Alexander Perry, *Intersections of two Grassmannians in $P^9$*, arXiv:1707.00534, 2017.

[BKR01] Tom Bridgeland, Alastair King, and Miles Reid, *The McKay correspondence as an equivalence of derived categories*, J. Amer. Math. Soc. **14** (2001), no. 3, 535–554 (electronic). MR MR1824990 (2002f:14023)

[BM14] Arend Bayer and Emanuele Macrì, *MMP for moduli of sheaves on K3s via wall-crossing: nef and movable cones, Lagrangian fibrations*, Invent. Math. **198** (2014), no. 3, 505–590. MR 3279332

[Bor] Lev Borisov, *Class of the affine line is a zero divisor in the Grothendieck ring*, arXiv:1412.6194 (2014).

[Deb84] Olivier Debarre, *Un contre-exemple au théorème de Torelli pour les variétés symplectiques irréductibles*, C. R. Acad. Sci. Paris Sér. I Math. **299** (1984), no. 14, 681–684. MR 770463

[Efi17] Alexander I. Efimov, *Some remarks on L-equivalence of algebraic varieties*, O7 2017.

[Fog68] John Fogarty, *Algebraic families on an algebraic surface*, Amer. J. Math. **90** (1968), 511–521. MR 0237496

[Fog73] J. Fogarty, *Algebraic Families on an Algebraic Surface, II, the Picard Scheme of the Punctual Hilbert Scheme*, American Journal of Mathematics **95** (1973), no. 3, 660–687.

[GZLMH06] S. M. Gusein-Zade, I. Luengo, and A. Melle-Hernández, *Power structure over the Grothendieck ring of varieties and generating series of Hilbert schemes of points*, Michigan Math. J. **54** (2006), no. 2, 353–359.

[HL16] Brendan Hassett and Kuan-Wen Lai, *Cremona transformations and derived equivalences of K3 surfaces*, 12 2016.

[Huy99] Daniel Huybrechts, *Compact hyper-Kähler manifolds: basic results*, Invent. Math. **135** (1999), no. 1, 63–113. MR 1664696

[IMOU] Atsushi Ito, Makoto Miura, Shinzosuke Okawa, and Kazushi Ueda, *The class of the affine line is a zero divisor in the Grothendieck ring: via $G_2$-Grassmannians*, arXiv:1606.04210 (2016).

[IMOU16] A., *Derived equivalence and Grothendieck ring of varieties: the case of K3 surfaces of degree 12 and abelian varieties*, arXiv:1612.08497, 12 2016.

[KKM17] Grzegorz Kapustka, Michal Kapustka, and Riccardo Moschetti, *Equivalence of K3 surfaces from Veronese threefolds*, 12 2017.

[KR17] Michal Kapustka and Marco Rampazzo, *Torelli problem for Calabi-Yau threefolds with GLSM description*, 11 2017.

[KS17] Alexander Kuznetsov and Evgeny Shinder, *Grothendieck ring of varieties, D- and L-equivalence, and families of quadrics*, Selecta Mathematica (2017).

[Kuz16] Alexander Kuznetsov, *Derived equivalence of Ito-Miura-Okawa-Ueda Calabi-Yau 3-folds*, to appear in JMSJ, 11 2016.
Laurent Manivel, *Double spinor Calabi-Yau varieties*, 09 2017.

Nicolas Martin, *The class of the affine line is a zero divisor in the Grothendieck ring: an improvement*, C. R. Math. Acad. Sci. Paris **354** (2016), no. 9, 936–939. MR 3535349

Ciaran Meachan, Giovanni Mongardi, and Kota Yoshioka, *Derived equivalent Hilbert schemes of points on K3 surfaces which are not birational*, 02 2018.

David Ploog, *Equivariant autoequivalences for finite group actions*, Adv. Math. **216** (2007), no. 1, 62–74. MR 2353249 (2009c:14095)

Takehiko Yasuda, *Dimensions of jet schemes of log singularities*, Amer. J. Math. **125** (2003), no. 5, 1137–1145. MR 2004431

Kōta Yoshioka, *Moduli spaces of stable sheaves on abelian surfaces*, Math. Ann. **321** (2001), no. 4, 817–884. MR 1872531

Department of Mathematics, Graduate School of Science, Osaka University, Machikaneyama 1-1, Toyonaka, Osaka, 560-0043, Japan.

E-mail address: okawa@math.sci.osaka-u.ac.jp