THE COHOMOLOGY OF UNRAMIFIED RAPOPORT-ZINK SPACES OF EL-TYPE AND HARRIS’S CONJECTURE

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Abstract. We study the $l$-adic cohomology of unramified Rapoport-Zink spaces of EL-type. These spaces were used in Harris and Taylor’s proof of the local Langlands correspondence for $GL_n$ and to show local-global compatibilities of the Langlands correspondence. In this paper we consider certain morphisms, $\text{Mant}_{b,\mu}$, of Grothendieck groups of representations constructed from the cohomology of the above spaces, as studied by Harris and Taylor, Mantovan, Fargues, Shin, and others. Due to earlier work of Fargues and Shin we have a description of $\text{Mant}_{b,\mu}(\rho)$ for $\rho$ a supercuspidal representation. In this paper, we give a conjectural formula for $\text{Mant}_{b,\mu}(\rho)$ for $\rho$ an admissible representation and prove it when $\rho$ is essentially square integrable. Our proof works for general $\rho$ conditionally on a conjecture appearing in Shin’s work. We show that our description agrees with a conjecture of Harris in the case of parabolic inductions of supercuspidal representations of a Levi subgroup.

1. Introduction

Our goal in this paper is to give a description of the $l$-adic cohomology of unramified Rapoport-Zink spaces of EL-type. These spaces are moduli spaces of $p$-divisible groups associated to unramified Weil-restrictions of general linear groups and can be thought of as generalizations of Lubin-Tate spaces.

This work generalizes, for these particular spaces, the Kottwitz conjecture stated in [RV14 Conj 7.3]. The Kottwitz conjecture describes the supercuspidal part of the $l$-adic cohomology of Rapoport-Zink spaces, and is known in the cases we consider by work of Shin [Shi12 Cor 1.3]. We prove our description of this cohomology is compatible with a conjecture of Harris [Har01 Conj 5.4] generalizing the Kottwitz conjecture to parabolic inductions of supercuspidal representations.

Our result describes the cohomology of these Rapoport-Zink spaces as a formal alternating sum (indexed by certain root theoretic data) of representation-theoretic constructions including the local Langlands correspondence, parabolic inductions, and Jacquet modules.

We prove our result inductively using two formulas from the literature. The first of these is Shin’s averaging formula [Shi12 Thm 7.5] which is proven using Mantovan’s formula [Man05 Thm 22] connecting the cohomology of Rapoport-Zink spaces, Igusa varieties and Shimura Varieties. The second formula is the Harris-Viehmann conjecture of [RV14 Conj 8.4] which relates the cohomology of so-called non-basic Rapoport-Zink spaces to a product of Rapoport-Zink spaces of lower dimension. A proof of this conjecture is expected to appear in a forthcoming paper of Scholze.

To carry out our induction, we prove combinatorial analogues of the above formulas phrased purely in terms of root-theoretic data. Interestingly, we are able to
prove these analogues for general quasisplit reductive groups, though at present we can only connect them to the cohomology of Rapoport-Zink spaces of unramified EL-type.

We now describe our main results more precisely. We study Rapoport-Zink spaces of unramified EL-type which we denote \( M_{b,\mu} \). These are moduli spaces of \( p \)-divisible groups coming from an unramified EL-datum consisting of

1. a finite unramified extension \( F \) of \( \mathbb{Q}_p \),
2. a finite dimensional \( F \)-vector space \( V \) which defines the group \( G = \text{Res}_{F/\mathbb{Q}_p} \text{GL}(V) \),
3. a conjugacy class of cocharacters \( [\mu] \) with \( \mu : \mathbb{G}_m \to \overline{\mathbb{G}_m} \),
4. an element \( b \) of a finite set \( \mathcal{B}(G, \mu) \) which defines a group \( J_b \) which is an inner twist of a Levi subgroup \( M_b \) of \( G \).

Roughly one can think of \( b, \mu \) as specifying the Newton and Hodge polygons of a \( p \)-divisible group and \( J_b \) as the automorphism group of the isocrystal \( b \).

The spaces \( M_{b,\mu} \) are formal schemes over \( \mathbb{Q}^{ur}_p \). One constructs a tower of rigid spaces \( M_{rig,U,b,\mu} \) over the generic fiber \( M_{rig,b,\mu} \) of \( M_{b,\mu} \), where the index \( U \) runs over compact open subgroups of \( G_{\mathbb{Q}_p} \). Associated to such a tower we have a cohomology space \( H^*_{rig}(G,b,\mu) \) which is an element of the Grothendieck group \( \text{Groth}^{rig}_G \mathbb{Q}_p \Simeq \mathbb{Q}_p \mathbb{W}_E \) of admissible representations of \( G_{\mathbb{Q}_p} \) and \( J_b(\mathbb{Q}_p) \) and \( \mathbb{W}_E \), where the latter group is the Weil group of the reflex field, \( E \), of \( [\mu] \). This construction can be thought of as an alternating sum of a direct limit over \( U \subset G \) of \( l \)-adic cohomology groups with the actions of \( G_{\mathbb{Q}_p} \) and \( J_b(\mathbb{Q}_p) \) arising from Hecke correspondences and isogenies of \( p \)-divisible groups, respectively.

The cohomology object \( H^*_{rig}(G,b,\mu) \) gives rise to a map of Grothendieck groups

\[
\text{Mant}_{G,b,\mu} : \text{Groth}(J_b(\mathbb{Q}_p)) \to \text{Groth}(G(\mathbb{Q}_p) \times W_E)
\]

which maps a representation \( \rho \) to the alternating sum of the \( J_b(\mathbb{Q}_p) \)-linear \( \text{Ext} \) groups of \( H^*_{rig}(G,b,\mu) \) and \( \rho \). We refer to section 3.1 for a precise definition.

The map \( \text{Mant}_{G,b,\mu} \) has been studied by many authors. Harris and Taylor [HT01] used this construction to prove the local Langlands correspondence for general linear groups. It also appears naturally in Mantovan’s work relating the cohomology of Shimura varieties, Igusa varieties, and Rapoport-Zink spaces [Man05]. Fargues studied \( \text{Mant}_{G,b,\mu} \) for basic \( b \) in some EL and PEL-cases in [Far04]. Shin combined Mantovan’s formula with his trace formula description of the cohomology of Igusa varieties to prove instances of local-global Langlands compatibilities [Shi11].

In [Shi12], Shin proved an averaging formula for \( \text{Mant}_{G,b,\mu} \) which is key to our work. He defined a map

\[
\text{Red}_b : \text{Groth}(G(\mathbb{Q}_p)) \to \text{Groth}(J_b(\mathbb{Q}_p))
\]

which up to a character twist is given by composing the un-normalized Jacquet module

\[
\text{Jac}^G_{\mu} : \text{Groth}(G(\mathbb{Q}_p)) \to \text{Groth}(M_b(\mathbb{Q}_p))
\]

with the Jacquet-Langlands map of Badulescu [Bad1]

\[
\text{LJ} : \text{Groth}(M_b(\mathbb{Q}_p)) \to \text{Groth}(J_b(\mathbb{Q}_p)).
\]

Shin uses global methods and so necessarily works with a large but inexplicit class of representations which he denotes \textit{accessible}. This set loosely consists of those representations appearing as the \( p \)-component of an automorphic representation of a
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certain global group. In particular, the essentially square integrable representations in $\text{Groth}(\mathcal{G}(\mathbb{Q}_p))$ are accessible.

In what follows $r_{-\mu}$ is a finite dimensional representation of $\hat{G} \times W_E$ which restricts to the representation of highest weight $-\mu$ on $\hat{G}$, and $LL$ is the semisimplified local Langlands correspondence. Shin shows the following result.

\textbf{Theorem 1.0.1 (Shin).} Assume $\pi$ is an accessible representation of $G(\mathbb{Q}_p)$. Then

$$\sum_{b \in \mathcal{B}(G,\mu)} \text{Mant}_{G,b,\mu}(\text{Red}_b(\pi)) = [\pi][r_{-\mu} \circ LL(\pi)|_{W_E}],$$

where the above formula is correct up to a Tate twist which we omit for clarity.

Additionally we have the conjecture of Harris and Viehmann which allows us to write $\text{Mant}_{G,b,\mu}$ for non-basic $b$ ($b$ is basic when it corresponds to an isocrystal with a single slope) in terms of $\text{Mant}_{G',b',\mu'}$ such that $G'$ is a general linear group of smaller rank than $G$. This conjecture was formulated in work of [Har01] and [RV14] and is expected to be proven in forthcoming work of Scholze. In what follows, $\text{Ind}$ is the un-normalized parabolic induction functor.

\textbf{Conjecture 1.0.2 (Harris-Viehmann).}

$$\text{Mant}_{G,b,\mu} = \sum_{(M_{b',\mu'}) \in \mathcal{I}_{M_{b,\mu}}} \text{Ind}_{\mathcal{I}_{M_{b,\mu}}}^G(\otimes_{i=1}^k \text{Mant}_{M_{b',\mu'}}),$$

where we omit a Tate twist which we discuss at length in section 3.2. The finite set $\mathcal{I}_{M_{b,\mu}}$ is described in proposition 2.5.6.

Shin’s averaging formula and the Harris Viehmann conjecture allow one to compute $\text{Mant}_{G,b,\mu} \circ \text{Red}_b$ recursively. The latter lets us compute $\text{Mant}_{G,b,\mu}$ for non-basic $b$ given that we know $\text{Mant}_{G',b',\mu'}$ for $G'$ of lower rank and the former lets us compute $\text{Mant}_{G,b,\mu}$ for the unique basic $b \in \mathcal{B}(G,\mu)$ if we know it for all non-basic $b \in \mathcal{B}(G,\mu)$. One of our main results is to give a non-recursive description of $\text{Mant}_{G,b,\mu} \circ \text{Red}_b$, which we now describe.

Let $G = \text{Res}_{F/\mathbb{Q}_p} \text{GL}(V)$ as before and choose a rational Borel subgroup $B$ of $G$, and a rational maximal torus $T \subset B \subset G$. Then we consider pairs $(M_S,\mu_S)$ where $M_S \supset T$ is a Levi subgroup of a parabolic subgroup $P_S$ containing $B$, and $\mu_S \in X_*(T)$ is dominant as a cocharacter of $M_S$. We call a pair of the above form a cocharacter pair for $G$.

We associate to a cocharacter pair $(M_S,\mu_S)$ the map of representations $[M_S,\mu_S] : \text{Groth}(G(\mathbb{Q}_p)) \to \text{Groth}(G(\mathbb{Q}_p) \times W_{E_{\mu_S}})$, which up to a Tate twist is given by

$$\pi \mapsto [(\text{Ind}^{G}_{P_S} \circ [\mu_S] \circ \text{Jac}^G_{P_S})(\pi)]$$

and

$$[\mu_S] : \text{Groth}(M_S(\mathbb{Q}_p)) \to \text{Groth}(M_S(\mathbb{Q}_p) \times W_{E_{\mu_S}})$$

given by

$$\pi \mapsto [\pi][r_{-\mu_S} \circ LL(\pi)].$$

Then our main result, which follows from theorem 3.3.7 in this paper is
Theorem 1.0.3. Suppose $\text{Mant}_{G,b,\mu}$ corresponds to a tower of unramified Rapoport-Zink spaces of EL-type. We assume that the Harris-Viehmann conjecture is true. Then if $\rho \in \text{Groth}(G(\mathbb{Q}_p))$ is essentially square-integrable, we have

$$\text{Mant}_{G,b,\mu}(\text{Red}_b(\rho)) = \sum_{(M_S, \mu_S) \in \mathcal{R}_{G,b,\mu}} (-1)^{L_{M_S,M_b}} [M_S, \mu_S](\rho),$$

where $\mathcal{R}_{G,b,\mu}$ is a collection of cocharacter pairs with a combinatorial definition and $(-1)^{L_{M_S,M_b}}$ is an easily determined sign.

Based on a conjecture of Taylor, Shin conjectures ([Shi12, Conj 8.1]) that the averaging formula holds for all admissible representations of $G(\mathbb{Q}_p)$. If this were indeed the case, then our result would also immediately hold for all admissible representations of $G(\mathbb{Q}_p)$.

A crucial part of the proof of the above theorem is the following unconditional result, which is perhaps interesting in its own right.

Theorem 1.0.4 (Imprecise version of theorem 2.5.4 and corollary 2.5.8 of our paper). For general quasisplit $G$ and a cocharacter $\mu$ (not necessarily minuscule), combinatorial analogues of Shin’s formula and the Harris-Viehmann conjecture hold true.

This result suggests that perhaps the combinatorics of cocharacter pairs is related to $\text{Mant}_{G,b,\mu}$ in cases more general than Rapoport-Zink spaces of unramified EL-type. However, we caution the reader that the existence of nontrivial $L$-packets and nontrivial endoscopy in more general groups will likely complicate the situation.

In section 4 of the paper, we use our combinatorial formula to prove the EL-type cases of a conjecture of Harris ([Har01, Conj 5.4]) describing $\text{Mant}_{G,b,\mu}(\text{I}_G(\rho))$ for $\rho$ a supercuspidal representation of $M(\mathbb{Q}_p)$ for $M$ a Levi subgroup of $G$. In this case, $\text{I}_G^G$ denotes normalized parabolic induction. In particular, we show the following result, which is conjecture 4.0.4 in our paper.

Theorem 1.0.5 (Harris conjecture). We assume that Shin’s averaging formula holds for all admissible representations of $G(\mathbb{Q}_p)$ and that the Harris-Viehmann conjecture is true. Let $\rho$ be a supercuspidal representation of $M(\mathbb{Q}_p)$. Then up to a precise Tate twist,

$$\text{Mant}_{G,b,\mu}(\text{JL}^{-1} \delta_{G,P_b}^{1/2} \text{I}_M^h(\rho)) = [\text{I}_M^G(\rho)] \Bigg[ \bigoplus_{(M,\mu') \in \text{Rel}_{G,M,b}^{G,\mu}} r_{-\mu'} \circ LL(\rho) \Bigg]$$

for an explicit set of cocharacter pairs $\text{Rel}_{G,M,b}^{G,\mu}$.

We prove our result for $\text{I}_M^G(\rho)$ not necessarily irreducible and $b$ not necessarily basic, which is a generalization of what Harris conjectured for the $G$ we consider.

Finally, in Appendix A we give an example to show that for general representations, one cannot hope for an expression as simple as that in Harris’s conjecture.

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the Harris-Viehmann conjecture arises naturally in his work. This work is partially supported by NSF grant DMS-1646385 (RTG grant).

2. Cocharacter Formalism

In this section we define and study the notion of a cocharacter pair. This notation will be used in the third and fourth sections of this paper, where we describe the cohomology of certain Rapoport-Zink spaces in terms of cocharacter pairs. We endeavor to use a similar notation to [Kot97].

This section is divided into five subsections. These are structured so that the first contains the basic definitions and the fourth and fifth subsections contain the most important results. The second and third subsections prove a number of technical lemmas that the reader may want to skip at first and refer to as necessary.

2.1. Notation and Preliminary Definitions. For the remainder of this section, we fix $G$ a connected quasisplit reductive group defined over $\mathbb{Q}_p$. This is a significantly more general setting than we will need for applications in this paper. However, we choose to work in this generality because doing so is both conceptually clearer and potentially useful for future applications. The ideas in §5 of [Kot97] might allow one to remove the quasisplit assumption, but we do not attempt this here as it is unnecessary for the applications. Moreover, Kottwitz’s study of the set $\mathcal{B}(G)$ in that section relies on understanding the quasisplit case first.

Remark 2.1.1. The reader will notice that most of this section makes sense over an arbitrary field. The assumption that we work over $\mathbb{Q}_p$ is used in section 2.4 when we connect cocharacter pairs to the set $\mathcal{B}(G)$ defined by Kottwitz. However, in §5.1 of [Kot97], Kottwitz shows that over $\mathbb{Q}_p$, the set $\mathcal{B}(G)$ is parametrized by a disjoint union of sets of the form $X^*(Z(M_S))^+$ for $M_S$ a standard Levi subgroup of $G$. These latter sets make sense over general fields and one could make sense generally of all the results of this section by replacing $\mathcal{B}(G)$ with the sets parametrizing it.

Since $G$ is quasisplit, we can pick a Borel subgroup $B \subset G$ defined over $\mathbb{Q}_p$ and a maximal split torus $A \subset B$ of $G$. We choose $T$ to be a maximal torus defined over $\mathbb{Q}_p$ satisfying $A \subset T \subset B$. We define $X^*(A)$ and $X_*(A)$ respectively to be the character and cocharacter groups of $A_{\overline{\mathbb{Q}_p}}$.

The group $G$ has a relative root datum $(X^*(A), \Phi^*(G, A), X_*(A), \Phi_*(G, A))$, where $\Phi^*(G, A)$ and $\Phi_*(G, A)$ respectively denote the set of relative roots and relative coroots of $G$ and the torus $A$. Our choice of Borel subgroup $B$ determines a decomposition $\Phi^*(G, A) = \Phi^*(G, A)^+ \amalg \Phi^*(G, A)^-$ of positive and negative roots and a subset $\Delta \subset \Phi^*(G, A)^+$ of simple roots. Analogous statements are also true for the coroots. The set of parabolic subgroups $P \supset B$ defined over $\mathbb{Q}_p$ are called standard parabolic subgroups. We define $P_S$ to be the unique standard parabolic subgroup such that $\Phi^*(P_S, A) = \Phi^*(G, A)^+ \amalg (\Phi_*(G, A)^- \cap \text{Span}_\mathbb{Q}(S))$. There is an inclusion preserving bijection between the set of standard parabolic subgroups and subsets of $\Delta$ given by $S \mapsto P_S$.

We let $N_S$ be the unipotent radical of the standard parabolic subgroup $P_S$. It is a standard result that there exists a connected reductive subgroup $M \subset P_S$ so that the natural map $M \to P_S/N_S$ is an isomorphism. In particular, this gives us a Levi decomposition $P_S = MN_S$ and the subgroup $M$ is called a Levi subgroup of $P_S$. The subgroup $M$ is not unique but any two Levi subgroups of $P_S$ are conjugate by an element of $N_S$. However, we have fixed a maximal torus $T$ and there is a unique
Levi subgroup $M_S$ containing $T$. The subgroup $M_S$ is constructed explicitly as the centralizer $C_G(Z)$, where $Z \subseteq T$ is the connected component of the intersection of the kernels of the roots in $S$. We refer to the Levi subgroups $M_S$ that we produce in this way as *standard Levi subgroups*.

Define

$$\mathfrak{A} := X_*(A).$$

We have the closed rational Weyl chamber

$$\overline{C}_Q = \{ x \in \mathfrak{A}_Q : \langle x, \alpha \rangle \geq 0, \alpha \in \Delta \}. $$

We define for each standard Levi subgroup,

$$\mathfrak{A}_{M_S,Q} := \{ x \in \mathfrak{A}_Q : \langle x, \alpha \rangle = 0, \alpha \in S, \langle x, \alpha \rangle \neq 0, \alpha \in \Delta \setminus S \},$$

and denote the *dominant* elements of $\mathfrak{A}_{M_S,Q}$ by

$$\mathfrak{A}^+_{M_S,Q} = \mathfrak{A}_{M_S,Q} \cap \overline{C}_Q.$$ Equivalently,

$$\mathfrak{A}^+_{M_S,Q} = \{ x \in \mathfrak{A}_Q : \langle x, \alpha \rangle = 0, \alpha \in S, \langle x, \alpha \rangle > 0, \alpha \in \Delta \setminus S \},$$

and we have

$$\bigsqcup_{M_S} \mathfrak{A}^+_{M_S,Q} = \overline{C}_Q.$$ There is a partial ordering of $\mathfrak{A}_Q$ given by $\mu \leq \mu'$ if $\mu' - \mu$ is a non-negative rational combination of simple roots.

**Definition 2.1.2.** We define a *cocharacter pair* for a group $G$ (relative to some fixed choice of $T$ and $B$ defined over $\mathbb{Q}_p$) to be a pair $(M_S, \mu_S)$ such that $M_S \subseteq G$ is a standard Levi subgroup and $\mu_S \in X_*(T)$ satisfies $\langle \mu_S, \alpha \rangle \geq 0$ for each positive absolute root $\alpha$ of $T$ in the Lie algebra of $M_S$. Positivity for absolute roots is determined by the Borel subgroup $B$ which we have fixed.

We denote the set of cocharacter pairs for $G$ by $C_G$.

**Remark 2.1.3.** We caution the reader that the cocharacter $\mu_S$ need not be an element of $X_*(A)$, even though $M_S$ is defined over $\mathbb{Q}_p$.

We could define cocharacter pairs more canonically as conjugacy classes of cocharacters of Levi subgroups. We choose not to as in practice we will often need to work with the unique dominant cocharacter in a conjugacy class relative to a fixed based root datum.

Let $\Gamma = \text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$. Since we have assumed $T$ and $B$ are defined over $\mathbb{Q}_p$, $\Gamma$ acts on $T_{\mathbb{Q}_p}$ and $B_{\mathbb{Q}_p}$. This gives us a natural left action of $\Gamma$ on $X_*(T)$ given explicitly by $(\gamma \cdot \mu)(g) = \gamma(\mu(\gamma^{-1}(g)))$ for $\mu \in X_*(T)$ and $\gamma \in \Gamma$. We get an analogous left action on $X^*(T)$ and one can easily check that the pairing $X^*(T) \times X_*(T) \to \mathbb{Z}$ is $\Gamma$ invariant under these actions.

We have

$$X_*(T)^\Gamma = \mathfrak{A}.$$ Indeed, a $\Gamma$-invariant cocharacter $\mu$ factors through the identity component of $T^\Gamma$, where $T^\Gamma$ is the subscheme defined by $T^\Gamma(\overline{\mathbb{Q}}_p) = T(\overline{\mathbb{Q}}_p)^\Gamma$. But the identity component of $T^\Gamma$ is the torus $A$. Conversely any cocharacter of $A$ induces a $\Gamma$-invariant cocharacter via the natural inclusion $A \hookrightarrow T$. 
Given $\mu \in X_*(T)$, we construct an element $\mu^\Gamma$ of $\mathfrak{A}_Q$ as follows:

$$\mu^\Gamma = \frac{1}{|\Gamma : \Gamma_{\mu}|} \sum_{\gamma \in \Gamma_{\mu}} \gamma(\mu)$$

where $\Gamma_{\mu}$ is the stabilizer of $\mu$ in $\Gamma$. Then $\mu^\Gamma \in X_*(T)_{Q} = \mathfrak{A}_Q$.

Given a standard Levi subgroup $M_S$, we let $W^\text{rel}_{M_S}$ be the subgroup of the relative Weyl group, $W^\text{rel}$, which is generated by the transpositions corresponding to simple roots in $S$.

**Definition 2.1.4.** We define a map

$$\theta_{M_S} : X_*(T) \to \mathfrak{A}_Q,$$

given by

$$\theta_{M_S}(\mu) = \frac{1}{|W^\text{rel}_{M_S}|} \sum_{\sigma \in W^\text{rel}_{M_S}} \sigma(\mu^\Gamma).$$

We are now ready to describe a formalism that will prove useful in studying the cohomology of certain Rapoport-Zink spaces. Crucial to everything that follows is a partial ordering on the set $C_G$ of cocharacter pairs for $G$.

**Definition 2.1.5.** We define a partial ordering on $C_G$ which we denote by the symbol $\leq$. Unfortunately, our definition is somewhat indirect: we first define when $(M_{S_2}, \mu_{S_2}) \leq (M_{S_1}, \mu_{S_1})$ for $M_{S_2} \subset M_{S_1}$ and $S_1 \setminus S_2$ contains a single element (in other words, $M_{S_2}$ is a maximal proper Levi subgroup of $M_{S_1}$). We then extend the relation to all cocharacter pairs by taking the transitive closure.

Let $M_{S_2}, M_{S_1}$ be standard Levi subgroups of $G$ such that $M_{S_2} \subset M_{S_1}$ and $S_1 \setminus S_2$ is a singleton. For cocharacter pairs $(M_{S_2}, \mu_{S_2}), (M_{S_1}, \mu_{S_1}) \in C_G$, we write $(M_{S_2}, \mu_{S_2}) \leq (M_{S_1}, \mu_{S_1})$ if $\mu_{S_2}$ is conjugate to $\mu_{S_1}$ in $M_{S_1} \overline{\chi}$ and $\theta_{M_{S_2}}(\mu_{S_2}) > \theta_{M_{S_1}}(\mu_{S_1})$. We then take the transitive closure to extend to a partial ordering on $C_G$.

**Example 2.1.6.** Consider $G = \text{GL}_4$ with $T$ the diagonal torus and $B$ the upper triangular matrices. We can pick a basis for $X_*(T)$ of cocharacters $\hat{e}_i$ defined so that $\hat{e}_i(g)$ is the diagonal matrix with 1 in every position except for the $i$th, which equals $g$. Then we can identify an element of $X_*(T)$ with its coordinate vector in this basis. Finally, we use additional parenthesis to indicate the product structure of the standard Levi subgroup $M_S$. Using this notation, the set of cocharacter pairs that are less than or equal to $\text{GL}_4(1^2, 0^2)$ is given in the diagram at the start of appendix A.

In particular, we see that $(\text{GL}_4^4, (1)(1)(0)(0)) \leq (\text{GL}_4^4, (1^2, 0^2))$ since we have a chain of cocharacter pairs where each Levi subgroup is maximal in the next: $(\text{GL}_4^4, (1)(1)(0)(0)) \leq (\text{GL}_1 \times \text{GL}_2 \times \text{GL}_1, (1)(1)(0)(0)) \leq (\text{GL}_3 \times \text{GL}_1, (1^2, 0)(0)) \leq (\text{GL}_4, (1^2, 0^2))$. However, it is not the case that $(\text{GL}_4^4, (1)(0)(1)(0)) \leq (\text{GL}_4^4, (1^2, 0^2))$ even though $\theta_{\text{GL}_4^4}(1, 0, 1, 0) > \theta_{\text{GL}_4^4}(1, 1, 0, 0)$ and the cocharacters are conjugate in $G$.

Finally, we remark that the fact that all the related cocharacter pairs in the above example have equal (as opposed to just conjugate) cocharacters is very much a result of us choosing a fairly small group $G$. Even for $G = \text{GL}_5$, this is not the case.
Definition 2.1.7. We define a cocharacter pair \((M_S, \mu_S)\) for \(G\) to be strictly decreasing if \(\theta_{M_S}(\mu_S) \in \mathfrak{A}_{M_S, Q}^\Lambda\). We denote by \(SD \subset C_G\) the strictly decreasing elements of \(C_G\) and by \(SD_{\mu}\) (for dominant \(\mu \in X_*(T)\)) the strictly decreasing elements \((M_S, \mu_S) \in C_G\) such that \((M_S, \mu_S) \leq (G, \mu)\).

Remark 2.1.8. The \(\theta_{M_S}\) map can be thought of as associating a tuple of slopes to a cocharacter pair. Then the strictly decreasing cocharacter pairs with Levi subgroup \(M_S\) are the ones whose slope tuple lies in the image of the Newton map \(\nu : B(G)_{M_S} \to \mathfrak{A}_{M_S, Q}\). The above statement is made precise by proposition 2.4.3.

2.2. An Alternate Characterization of the Averaging Map. The following two subsections consist of a collection of lemmas developing the theory of the map \(\theta_{M_S}\) and the set of strictly decreasing elements \(SD\) of \(C_G\).

In this section, we give an alternate description of the map \(\theta_{M_S}\). To do so, we will need several properties of cocharacters and root data which we record in the following lemma. For this lemma only, we consider \(T\) and \(G\) defined over a more general class of fields so that these results also apply to the complex dual groups \(\hat{T}\) and \(\hat{G}\).

Lemma 2.2.1. Let \(F \ni \mathbb{Q}\) be a field and \(G\) a connected quasisplit reductive group defined over \(F\). Suppose that \(T \subset G\) is a maximal torus defined over \(F\) and that the group scheme \(T_\mathbb{F}\) admits an action defined over \(F\) by a finite group \(\Lambda\). Let \(X*(T^\Lambda)\) denote the characters of the subgroup scheme of \(\Lambda\)-fixed points of \(T_\mathbb{F}\). The anti-equivalence of categories between tori and finitely generated free Abelian groups given by \(T_\mathbb{F} \mapsto X*(T^\Lambda)\) induces an action of \(\Lambda\) on \(X*(T^\Lambda)\). We then have the following.

1. There is a unique isomorphism \(X*(T^\Lambda) \cong X*(T^\Lambda)_\Lambda\) such that the following diagram commutes.

\[
\begin{array}{ccc}
X*(T) & \xrightarrow{res} & X*(T^\Lambda) \\
\downarrow{proj} & & \downarrow{res} \\
X*(T^\Lambda)_\Lambda & & \\
\end{array}
\]

2. Let \(M_S \subset G\) be a standard Levi subgroup. Let \(W_{M_S}^{\mathrm{abs}}, W_{M_S}^{\mathrm{rel}}\) denote the absolute and relative Weyl groups of \(M_S\) and let \(\Gamma = \mathrm{Gal}(\mathbb{F}/F)\). Then \((X_*(T)^{W_{M_S, \mathrm{abs}}} \Gamma) \cong (X_*(T)^{W_{M_S, \mathrm{rel}}} \Gamma)\).

3. The natural map \(X_*(T^\Lambda)_\mathbb{Q} \hookrightarrow X_*(T)_\mathbb{Q} \twoheadrightarrow X_*(T)_\mathbb{Q, \Lambda}\) induces an isomorphism \(X_*(T)^\Lambda_\mathbb{Q} \cong X_*(T)^\Lambda_\Lambda_\mathbb{Q}\).

Proof. There is an anti-equivalence of categories between diagonalizable groups over \(\mathbb{F}\) and finitely generated Abelian groups. The diagram for the universal property for \(\Lambda\)-invariants is that of \(\Lambda\)-coinvariants but with all the arrows reversed. Thus, there must exist a unique isomorphism between \(X*(T^\Lambda)\) and \(X*(T^\Lambda)_\Lambda\) that makes the diagram commute. This proves (1).
In [Kot84] Lem 1.1.3, Kottwitz proves that
\[ (X_\bullet(T)^\Gamma)/W_{M_S}^{rel} \cong (X_\bullet(T)/W_{M_S}^{abs})^\Gamma. \]

Thus, to prove (2), we need only show that this isomorphism gives a bijection of the singleton orbits. Kottwitz’s isomorphism maps the \( W_{M_S}^{rel} \)-orbit of \( \mu \in X_\bullet(T)^\Gamma \) to its \( W_{M_S}^{abs} \) orbit in \( X_\bullet(T) \). Thus, it suffices to show that if \( \mu \in X_\bullet(T)^\Gamma \) is invariant by \( W_{M_S}^{rel} \) then it is also invariant by \( W_{M_S}^{abs} \). If \( \mu \) is invariant by \( W_{M_S}^{rel} \), then the pairing of \( \mu \) with each relative root of \( M_S \) is 0. Thus the image of \( \mu \) lies in the intersection of the kernels of the relative roots of \( M_S \) which is \( Z(M_S) \cap A \). Therefore, \( \mu \) is invariant under the action of \( W_{M_S}^{abs} \).

For (3), we need to construct an inverse to the map
\[ X_\bullet(T)^\Delta \hookrightarrow X_\bullet(T)_Q \rightarrow X_\bullet(T)_{Q,\Lambda}. \]
Take \([\mu] \in X_\bullet(T)_{Q,\Lambda}\) for \( \mu \in X_\bullet(T)_Q \). Then
\[ \frac{1}{\Lambda} \sum_{\lambda \in \Lambda} \lambda(\mu) \in X_\bullet(T)^\Delta_Q \]
is independent of the choice of lift of \([\mu]\) to \( X_\bullet(T)_Q \) and gives an inverse to the map above. □

Let \( A_{M_S} \) be the maximal split torus in the center of \( M_S \). Then
\[ X_\bullet(A_{M_S})_Q \cong \mathfrak{A}_{M_S,Q}. \]
We now prove a lemma that we will need to use to describe the alternate characterization of \( \theta_{M_S} \).

**Lemma 2.2.2.** There is a natural isomorphism \( X^\bullet(Z(M_S)^\Gamma)_Q \cong \mathfrak{A}_{M_S,Q}. \)

**Proof.** By [2.2.1], we have the following isomorphisms.
\[
X^\bullet(T_{Q,\mathbb{A}_{M_S,Q}}) \cong X^\bullet(T_{Q,W_{M_S}^{abs},\Gamma}) = X_\bullet(T_{Q,W_{M_S}^{rel},\Gamma}) \\
\cong X_\bullet(T_{Q,W_{M_S}^{rel}})^\Gamma = X_\bullet(T_{Q,W_{M_S}^{rel}}) \\
\cong X_\bullet(A_{M_S})_Q \cong \mathfrak{A}_{M_S,Q}.
\]
We explicate the isomorphism \( X_\bullet(T_{Q,W_{M_S}^{rel}}) \cong X_\bullet(A_{M_S})_Q \). This follows from the isomorphism \( X_\bullet(A)^{W_{M_S}^{rel}} \cong X_\bullet(A_{M_S}) \) which we now describe. Suppose we have \( \mu \in X_\bullet(A)^{W_{M_S}^{rel}} \). Equivalently, for each relative root \( \alpha \) of \( M_S \), we have \( \sigma_\alpha(\mu) = \mu \) (where \( \sigma_\alpha \) is the reflection in the Weyl group corresponding to \( \alpha \)). Since \( \sigma_\alpha(\mu) = \mu - \langle \mu, \alpha \rangle \alpha \), this is equivalent to \( \langle \mu, \alpha \rangle = 0 \) for all relative roots \( \alpha \) of \( M_S \), which in turn is equivalent to the statement that \( \text{im}(\mu) \subseteq \bigcap_\alpha \ker_\alpha \). Finally, this is equivalent to \( \text{im}(\mu) \subseteq Z(M_S) \cap A \). Since the image of a cocharacter is connected, we in fact have that \( \mu \in X_\bullet(A_{M_S}) \).

To finish the argument, we need to construct an isomorphism
\[ X^\bullet(Z(M_S)^\Gamma)_Q \cong X^\bullet(T_{Q,W_{M_S}^{rel}}^\Gamma)_Q. \]
Note that it is necessary to take the tensor product with \( \mathbb{Q} \) here as \( Z(M_S) \) and \( T_{Q,W_{M_S}^{rel}} \) need not be isomorphic.
It suffices to show that
\[ X^*(Z(M_S))_Q \cong X^*(\hat{T}^{W_{abS}})_Q. \]

The group \(Z(M_S)\) is equal to the intersection of the kernels of the roots of \(\hat{M}_S\) and so \(X^*(Z(M_S))\) is identified with \(X^*(\hat{T})/R\) where \(R\) is the \(\mathbb{Z}\)-module spanned by the roots of \(M_S\). By 2.2.1, \(X^*(\hat{T}^{W_{abS}}) \cong X^*(\hat{T})_{W_{abS}} = X^*(\hat{T})/D\) where \(D\) is the \(\mathbb{Z}\) module spanned by \(w(\mu) - \mu\) for \(w \in W_{abS}\) and \(\mu \in X^*(\hat{T})\). Since \(Z(M_S) \subset \hat{T}^{W_{abS}}\), we have a natural surjection
\[ X^*(\hat{T}^{W_{abS}}) \twoheadrightarrow X^*(Z(M_S)). \]

By our previous discussion, the kernel of this map is \(R\). Thus, to prove our claim, it suffices to show that \(R/D\) is finite. But if \(\alpha\) is a root of \(M_S\), then \(\sigma_{\alpha}(\alpha) - \alpha = -2\alpha\). Thus \(2R \subset D\) and so we have the desired result. \(\Box\)

**Remark 2.2.3.** We show that this is the same isomorphism as the one in §4.4.3 of [Ko97]. There the author gives the map
\[ \mathfrak{A}_{M_S,Q} \twoheadrightarrow X_*(T)_Q = X_*(\hat{T})_Q \xrightarrow{\text{res}} X^*(Z(M_S)^\Gamma)_Q, \]

where the final map is restriction of characters. By 2.2.1 (1), this last map is the same as the composition
\[ X^*(\hat{T})_Q \twoheadrightarrow X^*(\hat{T})_QW_{rel}^\Gamma \cong X^*(\hat{T}_{W_{abS}}^\Gamma)_Q \cong X^*(Z(M_S)^\Gamma)_Q, \]

Thus, by applying 2.2.1 and the proof of 2.2.2, we get that the entire map is given by
\[ \mathfrak{A}_{M_S,Q} \cong X_*(T)_Q^{W_{rel}^\Gamma} \cong X_*(T)^{W_{rel}^\Gamma}_Q \cong X_*(T)^{W_{rel}^\Gamma}_{Q, W_{abS}^\Gamma}, \]

\[ \cong X^*(\hat{T}_{W_{abS}}^\Gamma)_Q \cong X^*(Z(M_S)^\Gamma)_Q. \]

We observe that this is the inverse of what we wrote down above.

We are now ready to give our alternate characterization of the map \(\theta_{M_S}\).

**Proposition 2.2.4.** [Alternate Characterization of \(\theta_{M_S}\)] The map \(\theta_{M_S}\) that was introduced in 2.1.4 is equal to the composition
\[ X_*(T) = X_*(\hat{T}) \xrightarrow{\text{res}} X^*(Z(M_S)^\Gamma) \twoheadrightarrow X^*(Z(M_S)^\Gamma)_Q \cong \mathfrak{A}_{M_S,Q} \subset \mathfrak{A}_Q, \]

where the final isomorphism is the one described in 2.2.2.

**Proof.** We recall 2.1.3 where \(\theta_{M_S}\) is defined to be the composition
\[ X_*(T) \twoheadrightarrow X_*(T)^{W_{rel}^\Gamma}_Q \twoheadrightarrow X_*(T)^{W_{rel}^\Gamma}_{Q, W_{abS}^\Gamma} \subset \mathfrak{A}_Q, \]

where both maps are averages over the relevant group. As we now show, this is the same as the composition
\[ X_*(T) \twoheadrightarrow X_*(T)^{W_{rel}^\Gamma}_{Q} \twoheadrightarrow X_*(T)^{W_{rel}^\Gamma}_{Q, W_{abS}^\Gamma} \cong X_*(T)^{W_{rel}^\Gamma}_{Q, W_{abS}^\Gamma} \subset \mathfrak{A}_Q, \]

where the first two maps are averages and the third is as in 2.2.1 (2). Indeed,
\[ \frac{1}{|W_{rel}^\Gamma|} \sum_{w \in W_{rel}^\Gamma} \sum_{\gamma \in \Gamma} w(\gamma(\mu)), \]
The benefit of this is that now we can write $\theta_{M_S}$ by \[2.2.1\] (2) and so equals (keeping in mind that $W^\text{rel}_{M_S} \subset W^\text{abs}_{M_S}$ by \[3.0.2\])

$$\frac{1}{|W^\text{abs}_{M_S}|} \sum_{w \in W^\text{abs}_{M_S}} \sum_{\gamma \in \Gamma} w(\gamma(\mu)) = \frac{1}{|W^\text{abs}_{M_S}|} \sum_{w \in W^\text{abs}_{M_S}} \sum_{\gamma \in \Gamma} \gamma(w(\gamma(\mu)))$$

$$= \frac{1}{|W^\text{abs}_{M_S}|} \sum_{w \in W^\text{abs}_{M_S}} \sum_{\gamma \in \Gamma} \gamma(w(\mu)) = \frac{1}{|W^\text{abs}_{M_S}|} \sum_{\gamma \in \Gamma} \sum_{w \in W^\text{abs}_{M_S}} \gamma(w(\mu)).$$

Now, we consider the following commutative diagram.

\[ X_*(T)_Q \xrightarrow{\text{avg}} X_*(T)^{W^\text{abs}_{M_S}}_Q \xrightarrow{\text{avg}} X_*(T)^{W^\text{abs}_{M_S}}_{Q, \Gamma} \]

\[ X_*(T)_{Q, W^\text{abs}_{M_S}} \xrightarrow{\text{avg}} X_*(T)^{W^\text{abs}_{M_S}}_{Q, \Gamma} \]

\[ X_*(T)^{W^\text{abs}_{M_S}}_{Q, \Gamma} \xrightarrow{\text{avg}} X_*(T)^{W^\text{abs}_{M_S}}_{Q, \Gamma} \]

The commutativity essentially follows from the definition of the averaging maps.

The benefit of this is that now we can write $\theta_{M_S}$ as the composition of

$$X_*(T) \to X_*(T)^{W^\text{abs}_{M_S}} \to X_*(T)^{W^\text{abs}_{M_S} \Gamma} \to X_*(T)^{W^\text{abs}_{M_S} \Gamma} \to X_*(T)^{W^\text{abs}_{M_S} \Gamma} \equiv X_*(T)^{W^\text{rel}_{M_S}} \subset \mathfrak{A}_Q$$

where we no longer need to base change the first three spaces to $Q$ because denominators are not introduced in the maps until later.

Using the equality between cocharacters of $T$ and characters of $\hat{T}$, we rewrite this as

$$X_*(T) = X_*(\hat{T}) \to X_*(\hat{T})^{W^\text{abs}_{M_S}} \to X_*(\hat{T})^{W^\text{abs}_{M_S} \Gamma} \to X_*(\hat{T})^{W^\text{abs}_{M_S} \Gamma} \equiv X_*(T)^{W^\text{rel}_{M_S}} \subset \mathfrak{A}_Q.$$
Thus, the previous expression equals

\[ \mathcal{X}_*(T) = \mathcal{X}^*(\hat{T}) \xrightarrow{\text{res}} \mathcal{X}^*(\hat{T}^{W_{\text{abs}}}) \xrightarrow{\text{res}} \mathcal{X}^*(Z(\bar{M})^\Gamma) \rightarrow \mathcal{X}^*(Z(\bar{M})^\Gamma)_\mathbb{Q} \]

\[ \cong \mathcal{X}^*(\hat{T}^{W_{\text{abs}}})_{\mathbb{Q}} \cong \mathcal{X}^*(\hat{T})_{\mathbb{Q}, W_{\text{abs}}^{\text{rel}}} \rightarrow \mathcal{X}^*(\hat{T})_{\mathbb{Q}, W_{\text{abs}}^{\text{rel}}}^{W_{\text{abs}}} = \mathcal{X}_*(T)_{\mathbb{Q}, W_{\text{abs}}^{\text{rel}}} \cong \mathcal{X}_*(T)^{\Gamma, W_{\text{abs}}^{\text{rel}}} \subset \mathfrak{A}_\mathbb{Q}. \]

comparing with \( \text{2.2.2} \) we can rewrite \( \theta_{M_S} \) as

\[ \mathcal{X}_*(T) = \mathcal{X}^*(\hat{T}) \xrightarrow{\text{res}} \mathcal{X}^*(Z(\bar{M})^\Gamma) \rightarrow \mathcal{X}^*(Z(\bar{M})^\Gamma)_\mathbb{Q} \cong \mathfrak{A}_{M_S, \mathbb{Q}} \subset \mathfrak{A}_\mathbb{Q} \]

as desired.

We record the following useful corollary of the ideas discussed in the above argument.

**Corollary 2.2.5.** Suppose that \( \mu, \mu' \in \mathcal{X}_*(T) \) are conjugate in \( M_{S_{\text{rep}}} \). Then \( \theta_{M_S}(\mu) = \theta_{M_S}(\mu') \).

**Proof.** By the observation at the start of 2.2.4 \( \theta_{M_S} \) is equivalently defined as the composition

\[ \mathcal{X}_*(T) \rightarrow \mathcal{X}_*(T)_{\mathbb{Q}, W_{\text{abs}}^{\text{rel}}} \rightarrow \mathcal{X}_*(T)_{\mathbb{Q}, W_{\text{abs}}^{\text{rel}}}^{W_{\text{abs}}} \cong \mathcal{X}_*(T)_{\mathbb{Q}, W_{\text{abs}}^{\text{rel}}} \subset \mathfrak{A}_\mathbb{Q}. \]

In particular, \( \mu \) and \( \mu' \) are mapped to the same element under the first map in the above composition. \( \square \)

### 2.3. Strictly Decreasing Cocharacter Pairs

In this section, we prove a number of properties of strictly decreasing cocharacter pairs and their relation to the partial order we defined in \( \text{2.1.5} \). As always, we let \( \sigma_\alpha \) denote the reflection in the relative Weyl group corresponding to the relative root \( \alpha \).

**Lemma 2.3.1.** If \( x \in \mathfrak{A}_\mathbb{Q} \) is dominant, then

\[ y = \frac{1}{|W_{\text{abs}}^{\text{rel}}|} \sum_{\sigma \in W_{\text{abs}}^{\text{rel}}} \sigma(x) \]

is also dominant. If in addition, \( \langle x, \alpha \rangle > 0 \) for some \( \alpha \in \Delta \setminus S \), then we also have \( \langle y, \alpha \rangle > 0 \).

**Proof.** For the first part of the lemma, we claim that if we can show that \( \langle \sigma(x), \alpha \rangle \geq 0 \) for each \( \sigma \in W_{\text{rep}}^{\text{rel}} \) and \( \alpha \in \Delta \setminus S \), then we are done. This follows because if a collection of cocharacters pair non-negatively with \( \alpha \), then so will their average. Thus for \( \alpha \in \Delta \setminus S \), we get \( \langle y, \alpha \rangle \geq 0 \). For \( \alpha \in S \), we automatically have \( \langle y, \alpha \rangle = 0 \) since \( 0 = y - \sigma_\alpha(y) = \langle y, \alpha \rangle \alpha \).

Pick \( \alpha \in \Delta \setminus S \). Then the root group of \( \alpha \) is contained in the unipotent radical \( N_S \) of \( P_S \). The group \( N_S \) is normalized by \( M_S \) and therefore by \( W_{\text{rep}}^{\text{rel}} \). In particular, for any \( \sigma \in W_{\text{rep}}^{\text{rel}} \), the root group of \( \sigma^{-1}(\alpha) \) is contained in \( N_S \) and hence is also positive. Thus \( \langle \sigma(x), \alpha \rangle = \langle x, \sigma^{-1}(\alpha) \rangle \geq 0 \) as desired.

To prove the second part, we notice since \( \langle x, \alpha \rangle > 0 \), the term in \( y \) corresponding to \( \sigma = 1 \) has positive pairing with \( \alpha \). Since all the other terms have non-negative pairing with \( \alpha \), we must have that \( \langle y, \alpha \rangle > 0 \). \( \square \)
Lemma 2.3.2. If x as in the previous lemma is dominant, then
\[ \frac{1}{|W_{M_S}|} \sum_{\sigma \in W_{M_S}} \sigma(x) \leq x \]

Proof. It suffices to show that for any \( \sigma \in W_{M_S} \), we have \( \sigma(x) \leq x \). This is a standard fact \([\text{Bou68}, \text{Ch} 6.1.18, \text{p. 158}]\).

Corollary 2.3.3. Let \( (M_S, \mu_S) \in SD \) be a strictly decreasing cocharacter pair and let \( (M_{S'}, \mu_{S'}) \in C_G \) and suppose that \( (M_S, \mu_S) \leq (M_{S'}, \mu_{S'}) \). Then \( (M_S, \mu_S) \in SD \).

Proof. We need to show that for each \( \beta \in \Delta \setminus S' \), that \( \langle \theta_{M_{S'}}(\mu_{S'}), \beta \rangle > 0 \). By 2.2.5 \( \theta_{M_{S'}}(\mu_{S'}) = \theta_{M_{S'}}(\mu_S) \). Further, we observe that

(1) \[ \theta_{M_{S'}}(\mu_S) = \frac{1}{|W_{M_{S'}}|} \sum_{\sigma \in W_{M_{S'}}} \sigma(\theta_{M_S}(\mu_S)). \]

Since \( \theta_{M_{S'}}(\mu_S) \) is dominant by assumption and satisfies \( \langle \theta_{M_S}(\mu_S), \beta \rangle > 0 \), we can apply 2.3.1 to get the desired result.

The following easy uniqueness result is quite useful.

Lemma 2.3.4. Let \( (M_{S_1}, \mu_{S_1}), (M_{S_2}, \mu_{S_2}), (M_{S_2'}, \mu_{S_2'}) \in C_G \). Suppose further that \( (M_{S_1}, \mu_{S_1}) \leq (M_{S_2}, \mu_{S_2}) \), that \( (M_{S_2}, \mu_{S_2}) \leq (M_{S_2'}, \mu_{S_2'}) \). If \( M_{S_2} = M_{S_2'} \), then \( (M_{S_2}, \mu_{S_2}) = (M_{S_2'}, \mu_{S_2'}) \).

Proof. By definition, \( \mu_{S_1}, \mu_{S_2}, \mu_{S_2'} \) are all conjugate in \( M_{S_2} \). But also, \( \mu_{S_2} \) and \( \mu_{S_2'} \) are dominant in the absolute root system. Thus they are equal.

We now define the notion of a cocharacter pair being strictly decreasing relative to a Levi subgroup.

Definition 2.3.5. Let \( M_S \subseteq M_{S'} \) be standard Levi subgroups of \( G \). We say \( (M_S, \mu_S) \) is strictly decreasing relative to \( M_{S'} \) if \( \langle \theta_{M_S}(\mu_S), \alpha \rangle > 0 \) for \( \alpha \in S' \setminus S \).

Remark 2.3.6. Recall that by construction, \( \langle \theta_{M_S}(\mu_S), \alpha \rangle = 0 \) for \( \alpha \in S \). Thus, \( (M_S, \mu_S) \in SD \) exactly when it is strictly decreasing relative to \( G \).

Lemma 2.3.7. Let \( (M_{S_1}, \mu_{S_1}), (M_{S_1'}, \mu_{S_1'}) \in C_G \) be cocharacter pairs such that \( (M_{S_1}, \mu_{S_1}) \leq (M_{S_1'}, \mu_{S_1'}) \). Let \( M_{S_2} \supset M_{S_1} \) be a standard Levi subgroup of \( G \) and suppose \( (M_{S_1}, \mu_{S_1}) \) is strictly decreasing relative to \( M_{S_2} \). Then \( (M_{S_1}, \mu_{S_1}) \) is strictly decreasing relative to \( M_{S_1'} \setminus S_2 \).

Proof. We first reduce to the case where \( M_{S_1} \) is a maximal Levi subgroup of \( M_{S_1'} \) (i.e. \( S_1' = S_1 \cup \{ \alpha \} \) for some \( \alpha \in \Delta \setminus S_1 \)). To do so, we recognize that the relation \( (M_{S_1}, \mu_{S_1}) \leq (M_{S_1'}, \mu_{S_1'}) \) definitionally implies that there is a finite sequence of cocharacter pairs

\[ (M_{S_1}, \mu_{S_1}) = (M_{S_0}, \mu_{S_0}) \leq \ldots \leq (M_{S_k}, \mu_{S_k}) = (M_{S_1'}, \mu_{S_1'}) \]

where each \( M_{S_i} \) is a maximal Levi subgroup of \( M_{S_{i+1}} \). Thus, if we prove the lemma in the maximal Levi subgroup case, we can inductively prove it in the general case.

We now assume that \( M_{S_1} \subseteq M_{S_1'} \) is a maximal Levi subgroup so that \( S_1' = S_1 \cup \{ \alpha \} \) for some \( \alpha \in \Delta \setminus S_1 \). We need to show that \( \langle \theta_{M_{S_1'}}(\mu_{S_1'}), \beta \rangle > 0 \) for each \( \beta \in S_1' \setminus S_2 \setminus S_1' \). First note that any such \( \beta \) is an element of \( S_2 \setminus S_1 \). By 2.2.5, since
\(\mu_S\) and \(\mu_S'\) are conjugate in \(M_{S_1}\), we have \(\theta_{M_{S_1}}(\mu_{S_1}) = \theta_{M_{S_1}}(\mu_{S_1})\). Thus we are 
reduced to showing \(\langle \theta_{M_{S_1}}(\mu_{S_1}), \beta \rangle > 0\) for \(\beta \in S_2\backslash S_1\).

Note that since \((M_{S_1}, \mu_{S_1})\) is strictly decreasing relative to \(M_{S_2}\), we have 
\(\theta_{M_{S_1}}(\mu_{S_1})\) is dominant relative to the root datum of \(M_{S_2}\) and 
\(\langle \theta_{M_{S_1}}(\mu_{S_1}), \beta \rangle > 0\). Therefore, by equation \(\ref{equation1}\) and lemma \(\ref{lemma2.3.1}\) \(\langle \theta_{M_{S_1}}(\mu_{S_1}), \beta \rangle > 0\) as desired. \(\square\)

**Proposition 2.3.8.** Let \((M_S, \mu_S) \in C_G\) and suppose it is strictly decreasing relative 
to some standard Levi subgroup \(M_{S'} \supseteq M_S\). Then there is a unique \((M_{S'}, \mu_{S'}) \in C_G\) 
such that \((M_S, \mu_S) \leq (M_{S'}, \mu_{S'})\). We call \((M_{S'}, \mu_{S'})\) the extension of \((M_S, \mu_S)\) to 
\(M_{S'}\).

In the case where \(S' = S \cup \{\alpha\}\) for \(\alpha \in \Delta \backslash S\), the converse is true. Specifically, 
if \((M_S, \mu_S) \in C_G\) and there exists \((M_{S'}, \mu_{S'}) \in C_G\) satisfying \((M_{S'}, \mu_{S'}) \succeq (M_S, \mu_S)\) 
with \(S' = S \cup \{\alpha\}\), then \((M_S, \mu_S)\) is strictly decreasing relative to \(M_{S'}\).

**Proof.** We begin by proving the first statement. Uniqueness follows from \(\ref{equation2.3.4}\) For existence, we first reduce to the case where \(M_S\) is a maximal Levi subgroup of \(M_{S'}\). Suppose we have proven the proposition in this reduced case. We might then try to 
prove the general case by iteratively applying the reduced case of the proposition 
to a chain of standard Levi subgroups \(M_S = M_{S_0} \subset \cdots \subset M_{S_n} = M_{S'}\) such that 
each is maximal in the next. Such a chain clearly exists, but to apply the reduced case of the proposition we need to show that if we have constructed a cocharacter 
pair \((M_{S_i}, \mu_{S_i}) \succeq (M_S, \mu_S)\) then \((M_{S_i}, \mu_{S_i})\) is strictly decreasing relative to \(M_{S'}\). This follows from \(\ref{equation2.3.4}\).

Now, we let \(\mu_{S'}\) be the unique conjugate of \(\mu_S\) which is dominant in \(M_{S'}\). If we 
can show that \(\theta_{M_{S'}}(\mu_{S'}) < \theta_{M_S}(\mu_S)\), then \((M_{S'}, \mu_{S'})\) will satisfy the conditions of the 
proposition. By \(\ref{equation2.2.5}\) and equation \(\ref{equation1}\)

\[
\theta_{M_{S'}}(\mu_{S'}) = \theta_{M_{S'}}(\mu_S) = \frac{1}{|W_{M_{S'}}|} \sum_{\sigma \in W_{M_{S'}}} \sigma(\theta_{M_{S'}}(\mu_S)),
\]

so we can reduce to showing that

\[
\frac{1}{|W_{M_{S'}}|} \sum_{\sigma \in W_{M_{S'}}} \sigma(y) < y,
\]

for any \(y\) satisfying \(\langle y, \alpha \rangle > 0\) for \(\alpha \in S' \backslash S\) and \(\langle y, \alpha \rangle = 0\) for \(\alpha \in S\). Any such \(y\) is 
dominant in the root datum of \(M_{S'}\) and so by lemma \(\ref{lemma2.3.2}\)

\[
\frac{1}{|W_{M_{S'}}|} \sum_{\sigma \in W_{M_{S'}}} \sigma(y) \leq y.
\]

Further, the above equation cannot be an equality because \(y\) has positive pairing 
with each root of \(S' \backslash S\) while \(\frac{1}{|W_{M_{S'}}|} \sum_{\sigma \in W_{M_{S'}}} \sigma(y)\) has 0 pairing with these roots.

To prove the converse, suppose that \((M_S, \mu_S) \leq (M_{S'}, \mu_{S'})\) and \(S' = S \cup \{\alpha\}\) for 
some \(\alpha \in \Delta \backslash S\). Then by \(\ref{equation2.2.5}\)

\[
\theta_{M_{S'}}(\mu_{S'}) = \theta_{M_S}(\mu_S) = \frac{\theta_{M_{S}}(\mu_S) + \sigma_\alpha(\theta_{M_{S}}(\mu_S))}{2},
\]

and so

\[
\theta_{M_{S}}(\mu_S) - \theta_{M_{S'}}(\mu_{S'}) = \frac{\theta_{M_{S}}(\mu_S) - \sigma_\alpha(\theta_{M_{S}}(\mu_S))}{2} = \frac{1}{2} \langle \theta_{M_S}(\mu_S), \alpha \rangle \alpha.
\]
Since by assumption \( \theta_{M_{\alpha'}}(\mu_{\alpha'}) < \theta_{M_{\alpha'}}(\mu_{\alpha'}) \), it follows that \( \langle \theta_{M_{\alpha'}}(\mu_{\alpha'}), \alpha \rangle > 0 \). \( \square \)

**Remark 2.3.9.** Note that the converse of the above proposition is not true in the general case.

**Corollary 2.3.10.** Fix a standard Levi subgroup \( M_S \) and roots \( \alpha_1, \alpha_2 \in \Delta \setminus S \). Suppose we have cocharacter pairs \( (M_S, \mu_S), (M_{S \cup \{ \alpha_1 \}}, \mu_{S \cup \{ \alpha_1 \}}), (M_{S \cup \{ \alpha_1, \alpha_2 \}}, \mu_{S \cup \{ \alpha_1, \alpha_2 \}}) \in C_G \) satisfying

\[
(M_S, \mu_S) \leq (M_{S \cup \{ \alpha_1 \}}, \mu_{S \cup \{ \alpha_1 \}}) \leq (M_{S \cup \{ \alpha_1, \alpha_2 \}}, \mu_{S \cup \{ \alpha_1, \alpha_2 \}})
\]

and that \( (M_S, \mu_S) \) is strictly decreasing relative to \( M_{S \cup \{ \alpha_2 \}} \).

Then the extension of \( (M_S, \mu_S) \) to \( M_{S \cup \{ \alpha_2 \}} \), which we denote \( (M_{S \cup \{ \alpha_2 \}}, \mu_{S \cup \{ \alpha_2 \}}) \), satisfies

\[
(M_S, \mu_S) \leq (M_{S \cup \{ \alpha_2 \}}, \mu_{S \cup \{ \alpha_2 \}}) \leq (M_{S \cup \{ \alpha_1, \alpha_2 \}}, \mu_{S \cup \{ \alpha_1, \alpha_2 \}})
\]

**Proof.** By the second statement of 2.3.8, we have that \( (M_S, \mu_S) \) is strictly decreasing relative to \( M_{S \cup \{ \alpha_1 \}} \). Then by 2.3.7 \( (M_{S \cup \{ \alpha_2 \}}, \mu_{S \cup \{ \alpha_2 \}}) \) is strictly decreasing relative to \( M_{S \cup \{ \alpha_1, \alpha_2 \}} \). Thus by 2.3.8 we have \( (M_{S \cup \{ \alpha_2 \}}, \mu_{S \cup \{ \alpha_2 \}}) \leq (M_{S \cup \{ \alpha_1, \alpha_2 \}}, \mu_{S \cup \{ \alpha_1, \alpha_2 \}}) \) as desired. \( \square \)

**Proposition 2.3.11.** Let \( S \subset S_1 \subset S_2 \) be subsets of \( \Delta \) and suppose \( (M_S, \mu_S), (M_{S_2}, \mu_{S_2}) \in C_G \) with

\[
(M_S, \mu_S) \leq (M_{S_2}, \mu_{S_2})
\]

and \( (M_S, \mu_S) \) is strictly decreasing relative to \( M_{S_1} \). Then the unique extension \( (M_{S_1}, \mu_{S_1}) \) of \((M_S, \mu_S)\) to \( M_{S_1} \) satisfies

\[
(M_{S_1}, \mu_{S_1}) \leq (M_{S_2}, \mu_{S_2}).
\]

**Proof.** Since \( (M_S, \mu_S) \leq (M_{S_2}, \mu_{S_2}) \), there is an increasing chain of cocharacter pairs

\[
(M_S, \mu_S) = (M_{S_0}, \mu_{S_0}) \leq \ldots \leq (M_{S_k}, \mu_{S_k}) = (M_{S_2}, \mu_{S_2})
\]

such that each standard Levi subgroup is maximal in the next. The content of this proposition is that we can pick a chain such that \( (M_{S_1}, \mu_{S_1}) \) appears. By 2.3.7 we can assume that \( M_S \) is maximal in \( M_{S_1} \). Let \( \alpha \) be the unique element of \( S_1 \setminus S \).

Pick a chain of cocharacter pairs \( (M_S, \mu_S) = (M_{S_0}, \mu_{S_0}) \leq \ldots \leq (M_{S_k}, \mu_{S_k}) = (M_{S_2}, \mu_{S_2}) \) as above. Chains of cocharacter pairs are determined by an ordering on the roots in \( S \setminus S = \{ \alpha_1, \ldots, \alpha_k \} \), such that the \( S^i = S \cup \{ \alpha_1, \ldots, \alpha_i \} \). The root \( \alpha \) appears in this chain so \( \alpha = \alpha_i \) for some \( i \). If \( i = 1 \) we are done. Otherwise, we consider \( (M_{S^i-1}, \mu_{S^i-1}) \leq (M_{S^i-2}, \mu_{S^i-2}) \leq (M_S, \mu_S) \). By 2.3.7 \( (M_{S^i-2}, \mu_{S^i-2}) \) is strictly decreasing relative to \( M_{S^i-2} \) and so by 2.3.10 (applied so that \( (M_{S^i-2}, \mu_{S^i-2}) \) takes the place of \( (M_S, \mu_S) \) in 2.3.10), we get a new chain of cocharacter pairs between \( (M_S, \mu_S) \) and \( (M_{S_2}, \mu_{S_2}) \) where we switch the positions of \( \alpha, \alpha_{i-1} \) in the corresponding ordering of \( S_2 \setminus S \). By repeating this argument, we can construct a chain where \( \alpha = \alpha_1 \), which is what we need. \( \square \)

The preceding propositions give us the following picture. Given a cocharacter pair \( (M_S, \mu_S) \) we check for which simple roots \( \alpha \) satisfy \( \langle \theta_{M_S}(\mu_S), \alpha \rangle > 0 \). Suppose there are \( n \) such simple roots. Then we get \( n \) standard Levi subgroups containing \( M_S \) corresponding to adding different subsets of these simple roots. The cocharacter pair \( (M_2, \mu_S) \) has a unique extension to each of the Levi subgroups and the poset lattice of these co-character pairs can be thought of as a directed \( n \) dimensional cube (such that the vertices are cocharacter pairs and the directed edges correspond to the < relation).
2.4. Connection With Isocrystals. We now investigate the relation between strictly decreasing cocharacter pairs and Kottwitz’s theory of isocrystals with additional structure. See [Kot97] for omitted details on the theory of isocrystals.

An isocrystal is a pair \((V, \Phi)\) where \(V\) is a finite dimensional \(\mathbb{Q}_p^\text{ur}\) vector space and \(\Phi : V \to V\) is an additive transformation satisfying \(\Phi(av) = \sigma(a)\Phi(v)\) for \(a \in \mathbb{Q}_p^\text{ur}, v \in V\) and \(\sigma\) a lift of Frobenius. As before, let \(G\) be a connected reductive group defined over \(\mathbb{Q}_p\) and consider the set of isomorphism classes of exact \((\otimes)\)-functors from \(\text{Rep}(G)\) to \(\text{Isoc}\), the category of isocrystals. Such isomorphism classes are classified by \(H^1(W_{\mathbb{Q}_p}, G(\mathbb{Q}_p^{\text{ur}}))\) which we denote \(\text{B}(G)\) (where \(W_{\mathbb{Q}_p}\) is the Weil group of \(\mathbb{Q}_p\)).

In §4.2 of [Kot97], Kottwitz constructs the Newton map \(\nu : \text{B}(G) \to \overline{\mathbb{C}}_p\) and the Kottwitz map \(\kappa : \text{B}(G) \to X^*(\hat{Z}(\hat{G})^\Gamma)\). An element of \(\text{B}(G)\) is uniquely determined by its image under these maps.

We say that the standard Levi subgroup \(M_S\) is associated to \(b \in \text{B}(G)\) if \(\nu(b) \in \mathfrak{A}_{M_S, \mathbb{Q}}\). Henceforth, we will often denote the standard Levi subgroup associated to \(b\) by \(M_b\). Notice that many elements of \(\text{B}(G)\) could be associated to the same Levi subgroup. We call \(b\) basic if \(M_b = G\). We write

\[
\text{B}(G) = \bigsqcup_{S \subset \Delta} \text{B}(G)_{M_S}
\]

such that \(\text{B}(G)_{M_S}\) consists of those \(b \in \text{B}(G)\) associated to \(M_S\). We denote by \(\text{B}(M_S)^+\) the maximal subset of \(\text{B}(M_S)\) such that \(\nu(\text{B}(M_S)^+) \subset \overline{\mathbb{C}}_p\). In §5.1 of [Kot97], Kottwitz uses the Kottwitz map for \(M_S\) to construct canonical bijections

\[
\text{B}(G)_{M_S} \cong \text{B}(M_S)^+_{M_S} \cong X^*(Z(\hat{M}_S)^\Gamma)^+
\]

where Kottwitz constructs a canonical isomorphism

\[
X^*(Z(\hat{M}_S)^\Gamma)_Q \cong \mathfrak{A}_{M_S, \mathbb{Q}}
\]

and \(X^*(Z(\hat{M}_S)^\Gamma)^+\) denotes the subset of \(X^*(Z(\hat{M}_S)^\Gamma)\) mapping to \(\mathfrak{A}_{M_S, \mathbb{Q}}^+\). In fact, Kottwitz shows that the composition of the above isomorphisms gives the Newton map

\[
\text{B}(G)_{M_S} \to \mathfrak{A}_{M_S, \mathbb{Q}}^+ \leftarrow \overline{\mathbb{C}}_p.
\]

For a further discussion of equation 3, we refer the reader to 2.2.2 and the subsequent remark.

We now prove an important lemma that will be used to relate the set \(\text{B}(G)\) to the strictly decreasing elements of \(C_G\).

\textbf{Lemma 2.4.1.} Fix a standard Levi subgroup \(M_S\) of \(G\) and let \((M_S, \mu_S) \in \mathcal{SD}\). Then \(\theta_{M_S}(\mu_S) \in \nu(\text{B}(G)_{M_S})\).

\textbf{Proof.} We first describe the set \(\nu(\text{B}(G)_{M_S})\). By equations 2 and 3, the set \(\nu(\text{B}(G)_{M_S})\) is equal to the image of \(X^*(Z(\hat{M}_S)^\Gamma)^+\) in \(\mathfrak{A}_{M_S, \mathbb{Q}}\). Thus, to prove this proposition, it suffices to show that \(\theta_{M_S}\) factors through the map \(X^*(Z(\hat{M}_S)^\Gamma) \hookrightarrow X^*(Z(\hat{M}_S)^\Gamma)_Q \cong \mathfrak{A}_{M_S, \mathbb{Q}}\) where the isomorphism is as in equation 3 or lemma 2.2.2. Then, since \((M_S, \mu_S)\) is strictly decreasing, the factoring of \(\theta_{M_S}\) will map \(\mu_S\) to an element of \(X^*(Z(\hat{M}_S)^\Gamma)^+\) as desired. That \(\theta_{M_S}\) factors in this way follows from the alternate characterization of \(\theta_{M_S}\) given in 2.2.4. □
Definition 2.4.2. Fix $\mu \in X_*(T)$. Then we recall the following definition of Kottwitz [Kot97 §6.2]:

$$B(G, \mu) := \{ b \in B(G) : \nu(b) \leq \theta_T(\mu), \kappa(b) = \mu|_{Z(G)^r} \}. $$

Now we prove the key result of this section, which permits us to associate an element of $B(G)$ to each strictly decreasing cocharacter pair.

Proposition 2.4.3. We have a natural map

$$T : SD \to B(G)$$

defined as follows. Let $(M_S, \mu_S) \in SD$. Then there exists a $b \in B(G)$ so that $\kappa(b) = \mu_S|_{Z(G)^r}$ and $\nu(b) = \theta_{M_S}(\mu_S)$. We note that by construction, $b$ is unique. Then we define $T((M_S, \mu_S)) = b$. In particular we show that

$$T(SD_{\mu}) \subset B(G, \mu).$$

Proof. We first define $b$. Note that since $(M_S, \mu_S)$ is strictly decreasing, $\theta_{M_S}(\mu_S) \in \mathfrak{A}_{M_S, \mathbb{Q}}^+$. By 2.2.1 it follows that $\mu_S|_{Z(M)^r} \in X^*(Z(M)^r)^+$ so we can define $b$ to be the element of $B(G)$ corresponding to $\mu_S|_{Z(M)^r}$ under the isomorphism $B(G)_{M_S} \cong X^*(Z(M)^r)^+$ of equation 2. Recall that the composition of this isomorphism with equation 3 induces the Newton map restricted to $B(G)_{M_S}$. Thus, we have $\theta_{M_S}(\mu_S) = \nu(b)$. Equation (4.9.2) of [Kot97] implies that $\kappa(b) = \mu_S|_{Z(G)^r}$.

It remains to show that if $(M_S, \mu_S) \in SD_{\mu}$ then the element $b \in B(G)$ that we have constructed lies in the set $B(G, \mu)$. For this, we need to show that $\nu(b) = \theta_{M_S}(\mu_S) \leq \theta_T(\mu)$.

We claim that $\theta_T(\mu) \geq \theta_T(\mu_S)$. After all, by (Bou68 Ch 1.6.18, p. 158), we have $\mu \geq \mu_S$. Then the claim follows from 3.0.4.

Now we claim that $\theta_T(\mu_S)$ is dominant in the relative root system of $M_S$. To prove the claim, we first observe that $\mu_S$ is dominant relative to the absolute root system of $M_S$. As above, the Galois group $\Gamma$ preserves the Weyl chamber corresponding to the positive absolute roots given by $B$. Thus, $\gamma(\mu_S)$ is dominant for each $\gamma \in \Gamma$, and so $\theta_T(\mu_S)$ is dominant relative to the absolute roots of $M_S$. The intersection of the closed positive Weyl chamber for the absolute root datum of $M_S$ with $\mathfrak{A}_\mathbb{Q}$ is the Weyl chamber for relative root datum of $M_S$. Thus, $\theta_T(\mu_S)$ is dominant with respect to the relative roots as desired.

Finally, we apply lemma 2.3.2 and equation 1 to get

$$\theta_T(\mu_S) \geq \theta_{M_S}(\mu_S),$$

which finishes the proof. \qed

Question 2.4.4. Is it true that

$$T(SD_{\mu}) = B(G, \mu)?$$

This would be an intrinsically interesting result. However, the author does not know a proof of this statement.

2.5. The Induction and Sum Formulas. We are now ready to prove our main theorems on cocharacter pairs. We begin by defining some key subsets of $C_G$, the set of cocharacter pairs for $G$. 

Definition 2.5.1. Fix a dominant $\mu \in X_\bullet(T)$ and $b \in \mathcal{B}(G, \mu)$. We define the sets $T_{G,b,\mu}$ and $R_{G,b,\mu}$ as follows:

$$T_{G,b,\mu} := T^{-1}(b) \cap SD_\mu$$

and

$$R_{G,b,\mu} = \{(M_{S_1}, \mu_{S_1}) \in \mathcal{C}_G : (M_{S_1}, \mu_{S_1}) \leq (M_{S_2}, \mu_{S_2}) \text{ for some } (M_{S_2}, \mu_{S_2}) \in T_{G,b,\mu}\}.$$ 

Definition 2.5.2. Let $\mathbb{Z}\langle \mathcal{C}_G \rangle$ denote the free Abelian group generated by the set of cocharacter pairs for $G$.

We define $M_{G,b,\mu} \in \mathbb{Z}\langle \mathcal{C}_G \rangle$ by

$$M_{G,b,\mu} = \sum_{(M_{S_1}, \mu_{S_1}) \in R_{G,b,\mu}} (-1)^{L_{M_{S_1},M_b}}(M_{S_1}, \mu_{S_1})$$

such that for $M_{S_1} \subset M_{S_2}$, $L_{M_{S_1},M_{S_2}}$ is defined to be $|S_2 \setminus S_1|$.

Remark 2.5.3. We observe that for $(M_{S_1}, \mu_{S_1}) \in SD_\mu$, if $T((M_{S_1}, \mu_{S_1})) = b$, then $M_S = M_b$.

We will show in theorem 3.3.7 that at least in the case where $G$ is a restriction of scalars of a general linear group, $M_{G,b,\mu}$ is related to the cohomology of Rapoport-Zink spaces for $G$. Thus one expects there to be an analogue of the Harris-Viehmann conjecture (which in this setting we call the induction formula). Perhaps the more surprising result is that there is also an analogue of Shin’s averaging formula (which we call the sum formula) [Shi12, Thm 7.5]. We first prove the sum formula.

Theorem 2.5.4 (Sum Formula). The following holds in $\mathbb{Z}\langle \mathcal{C}_G \rangle$:

$$\sum_{b \in \mathcal{B}(G, \mu)} M_{G,b,\mu} = (G, \mu).$$

Proof. We need to show that

$$\sum_{b \in \mathcal{B}(G, \mu)} M_{G,b,\mu} = (G, \mu),$$

or equivalently

$$\sum_{b \in \mathcal{B}(G, \mu)} \sum_{(M_{S_1}, \mu_{S_1}) \in R_{G,b,\mu}} (-1)^{L_{M_{S_1},M_b}}(M_{S_1}, \mu_{S_1}) = (G, \mu).$$

We prove this equality by counting how many times a given cocharacter pair shows up on the left-hand side. The pair $(G, \mu)$ shows up exactly once in the left-hand sum as an element of $R_{G,b,\mu}$ for $b$ the unique basic element of $\mathcal{B}(G, \mu)$. Suppose $(M_{S_1}, \mu_{S_1}) \in \mathcal{C}_G$ is some other cocharacter pair. Then define

$$Y_{(M_{S_1}, \mu_{S_1})} = \{ b \in \mathcal{B}(G, \mu) : (M_{S_1}, \mu_{S_1}) \in R_{G,b,\mu}\}.$$

We are reduced to showing

$$(4) \quad \sum_{b \in Y_{(M_{S_1}, \mu_{S_1})}} (-1)^{L_{M_{S_1},M_b}} = 0.$$ 

Our general strategy will be to show that the left-hand side of equation (4) vanishes for each $(M_{S_1}, \mu_{S_1}) < (G, \mu)$ by inducting on the size of $\Delta \setminus S$. However, in the case that $(M_{S_1}, \mu_{S_1}) \in SD_\mu$, we can prove the vanishing without an inductive argument. We show this first before discussing the induction.
Suppose now that \((M_S, \mu_S) \in SD_{\mu}\). By 2.3.3 every pair \((M_{S'}, \mu_{S'}) \in CG\) satisfying \((M_S, \mu_S) \leq (M_{S'}, \mu_{S'}) \leq (G, \mu)\) is strictly decreasing and thus by 2.4.3 we have \(T((M_{S'}, \mu_{S'})) \in B(G, \mu)\). These are precisely the elements \(b \in B(G, \mu)\) so that \((M_S, \mu_S) \in R_{G,b,\mu}\). By the discussion after 2.3.11 this set forms a cube and it is a standard fact that if we associate one of \(\{1, -1\}\) to each vertex of a cube so that adjacent vertices have opposite signs, then the sum of all the signs is 0. This implies that the left-hand side of (3) vanishes in the strictly decreasing case.

Now we discuss the inductive argument. The base case will be for pairs \((M_S, \mu_S) < (G, \mu)\) satisfying \(|\Delta \setminus S| = 1\). The second statement of proposition 2.3.8 implies that in this case \((M_S, \mu_S)\) is strictly decreasing relative to \(G\), which means that \((M_S, \mu_S) \in SD_{\mu}\). Thus, the base case is proven by the previous paragraph.

We now discuss the inductive step. Suppose \((M_S, \mu_S) < (G, \mu)\). If \((M_S, \mu_S)\) is strictly decreasing, then we are done by the above. Suppose now that \((M_S, \mu_S)\) is not strictly decreasing. We claim that \((M_S, \mu_S)\) must be strictly decreasing with respect to at least some standard Levi subgroup of \(G\) that properly contains \(M_S\). After all, since \((M_S, \mu_S) < (G, \mu)\), there must exist at least some \(\alpha \in \Delta \setminus S\) and \((M_{S \cup \{\alpha\}}, \mu_{S \cup \{\alpha\}}) \in CG\) so that \((M_S, \mu_S) \leq (M_{S \cup \{\alpha\}}, \mu_{S \cup \{\alpha\}})\). Then by 2.3.3 this implies that \((M_S, \mu_S)\) is strictly decreasing relative to \(M_{S \cup \{\alpha\}}\).

Thus, let \(M_{S'}\) be the maximal standard Levi subgroup of \(G\) such that \((M_S, \mu_S)\) is strictly decreasing relative to \(M_{S'}\). We can write \(S' = S \cup \{\alpha_1, ..., \alpha_n\}\) where \(\alpha_i \neq \alpha_j\) for \(i \neq j\) and each \(\alpha_i \in \Delta \setminus S\). We denote by \(X\) the \(n\)-cube of cocharacter pairs above \((M_S, \mu_S)\) as in the discussion after 2.3.11.

We claim that

\[
\sum_{b \in Y(M_S, \mu_S)} \sum_{(M_{S'}, \mu_{S'}) \in X \setminus \{(M_S, \mu_S)\}} (-1)^{L_{M_S, M_{S'}}} = -\sum_{(M_{S'}, \mu_{S'}) \in X \setminus \{(M_S, \mu_S)\}} \sum_{b \in Y(M_{S'}, \mu_{S'})} (-1)^{L_{M_{S'}, M_{S'}}}.
\]

Given this claim, we see that to finish the proof, it suffices to show that the right-hand side is identically 0. However, the right-hand side consists of a sum of a number of terms similar to the left-hand side but for pairs \((M_{S'}, \mu_{S'})\) in place of \((M_S, \mu_S)\). Note that each \(S'\) is strictly larger than \(S\) and thus we are done by induction.

We now prove the claim. Moving all the terms to one side, we need only show that

\[
\sum_{(M_{S'}, \mu_{S'}) \in X} \sum_{b \in Y(M_{S'}, \mu_{S'})} (-1)^{L_{M_{S'}, M_{S'}}} = 0.
\]

Fix \(b \in B(G, \mu)\). Then it suffices to show the contribution from \(b\) in the above formula vanishes. Thus, we must show

\[
\sum_{(M_{S'}, \mu_{S'}) \in X \cap R_{G,b,\mu}} (-1)^{L_{M_{S'}, M_{S'}}} = 0. \tag{5}
\]

We examine the structure of \(X \cap R_{G,b,\mu}\) when it is nonempty. If we can show that the cocharacter pairs in this set form a sub-cube of \(X\) of positive dimension, then we will be done by the standard fact that if we place alternating signs on the vertices of a cube and add up all the signs we get 0.

Clearly, any \((M_{S'}, \mu_{S'}) \in X \cap R_{G,b,\mu}\) must satisfy \(M_S \subset M_{S'} \subset M_b\). The subset of \(X\) satisfying this latter property forms a sub-cube of \(X\) since its elements are indexed by subsets of \(S_b \setminus S\), where \(S_b\) is the subset of \(\Delta\) corresponding to \(M_b\) in the
standard way (note that by \[2.5.4\] there is at most one element of \(X \cap R_{G,b,\mu}\) for each standard Levi \(M_{S'}\)). Moreover, this latter set cannot form a cube of dimension 0 for then we would have \(M_S = M_b\) and so \(X \cap R_{G,b,\mu} = \{(M_S,\mu_S)\}\) which would imply that \((M_S,\mu_S)\) is strictly decreasing contrary to assumption.

Thus to finish the proof, we need only show that every \((M_{S'},\mu_{S'})\) such that
\[
(1) \quad M_S \subset M_{S'} \subset M_b, \\
(2) \quad (M_S,\mu_S) \leq (M_{S'},\mu_{S'}), \\
(3) \quad (M_S,\mu_S)\text{ is strictly decreasing relative to } M_{S'},
\]
satisfies \((M_{S'},\mu_{S'}) \leq (M_b,\mu_b)\) for some \((M_b,\mu_b) \in T_{G,b,\mu}\). Since we assumed that \(X \cap R_{G,b,\mu} \neq \emptyset\), then in fact there is an \((M_b,\mu_b) \in T_{G,b,\mu}\) with \((M_S,\mu_S) \leq (M_b,\mu_b)\). Then the desired result follows from \[2.3.11\]

We now turn to the induction formula. Fix a standard Levi subgroup \(M_S\) of \(G\). Then our choice of maximal torus \(T\) and Borel subgroup \(B\) of \(G\) provides us with natural choices \(B \cap M_S\) and \(T\) of a Borel subgroup and maximal torus of \(M_S\). This allows us to define the set \(C_{M_S}\) of cocharacter pairs for \(M_S\). There is a natural inclusion
\[
(6) \quad i_{M_S}^G : C_{M_S} \rightarrow C_G.
\]
The image of this inclusion is precisely the set of cocharacter pairs \((M_{S'},\mu_{S'})\) where \(S' \subset S\). This inclusion preserves the partial ordering of cocharacter pairs. The strictly decreasing elements of \(C_{M_S}\) map to the elements of \(C_G\) which are strictly decreasing relative to \(M_S\).

Now choose a \(b \in B(G,\mu)\) and rational Levi \(M_S\) such that \(M_b \subset M_S \subset G\). We have a unique \(b' \in B(M_b)^\times\) corresponding to \(b\) under the isomorphism given by equation \[2\] The inclusion \(M_b \subset M_S\) induces a map
\[
B(M_b) \rightarrow B(M_S).
\]
Let \(b_S\) be the image of \(b'\) under this map.

The following definition will be important in relating cocharacter pairs of a group \(G\) to those of a standard Levi. Compare with [RV14 eqn 8.1].

**Definition 2.5.5.** Let \(M_S\) be a standard Levi subgroup of \(G\), let \(\mu \in X_w(T)\) be a dominant cocharacter and choose \(b \in B(G,\mu)\). We take \(b_S \in B(M_S)\) as constructed in the previous paragraph and define the set
\[
\mathcal{I}_{M_S,b_S} = \{(M_S,\mu_S) \in C_{M_S} : b_S \in B(M_S,\mu_S), \mu_S\text{ is conjugate to }\mu\text{ in }G\}.
\]

We first check the following transitivity property of \(\mathcal{I}_{M_S,b_S}\).

**Proposition 2.5.6.** Fix \((G,\mu) \in C_G\) and \(b \in B(G,\mu)\). Suppose \(M_{S_2}\) and \(M_{S_1}\) are standard Levi subgroups of \(G\) such that \(M_b \subset M_{S_2} \subset M_{S_1}\). Then
\[
\mathcal{I}_{M_{S_2},b_{S_2}}^{G,\mu} = \bigsqcup_{(M_{S_1},\mu_{S_1})} \mathcal{I}_{M_{S_2},b_{S_2}}^{M_{S_1},\mu_{S_1}}.
\]

**Proof.** We claim that for distinct \((M_{S_1},\mu_{S_1}), (M_{S_1},\mu'_{S_1}) \in \mathcal{I}_{M_{S_2},b_{S_2}}^{G,\mu}\), the sets \(\mathcal{I}_{M_{S_2},b_{S_2}}^{M_{S_1},\mu_{S_1}}, \mathcal{I}_{M_{S_2},b_{S_2}}^{M_{S_1},\mu'_{S_1}}\) are disjoint. Indeed suppose \((M_{S_2},\mu_{S_2})\) is an element
of both sets. Then \( \mu_{S_2} \) is conjugate to both \( \mu_{S_1} \) and \( \mu'_{S_1} \) in \( M_{S_1} \). Since \( \mu_{S_1}, \mu'_{S_1} \) are dominant in \( M_{S_1} \), this implies they are equal.

Thus to prove the proposition, it suffices to show each set is a subset of the other. Take \((M_{S_2}, \mu_{S_2}) \in T_{M_{S_2}, b_{S_2}}^{G, \mu} \). Let \( \mu_{S_1} \) be the unique dominant cocharacter conjugate to \( \mu_{S_2} \) in \( M_{S_1} \). Then we consider \((M_{S_1}, \mu_{S_1}) \) as an element of \( C_{M_{S_1}} \) and just need to show that \( b_{S_1} \in B(M_{S_1}, \mu_{S_1}) \) since we already know that \( b_{S_2} \in B(M_{S_2}, \mu_{S_2}) \) by assumption. Thus, we need only show that \( \nu(b_{S_1}) \leq \theta_T(\mu_{S_1}) \) and \( \kappa(b_{S_1}) = \mu_{S_1} \mid Z(M_{S_1}) \).

We prove the inequality first. By assumption, \( \nu(b_{S_2}) \leq \theta_T(\mu_{S_2}) \) and by equations \([2]\) and \([3]\) \( \nu(b_{S_1}) = \nu(b_{S_2}) \). Since \( \mu_{S_1} \) and \( \mu_{S_2} \) are conjugate in \( M_{S_1} \) and \( M_{S_1} \) is dominant, it follows from \([\text{Bou68}]\) Ch6 1.6.18, p. 158 that \( \mu_{S_2} \leq \mu_{S_1} \). Then, by \([\text{B.0.4}]\) it follows that \( \theta_T(\mu_{S_2}) \leq \theta_T(\mu_{S_1}) \) in the relative root system. Combining all this data, we get

\[
\nu(b_{S_1}) = \nu(b_{S_2}) \leq \theta_T(\mu_{S_2}) \leq \theta_T(\mu_{S_1}),
\]

as desired.

To prove \( \kappa(b_{S_1}) = \mu_{S_1} \mid Z(M_{S_1}) \), we note that by equation (4.9.2) of \([\text{Kot97}]\) and the fact that \( b_{S_2} \in B(M_{S_2}, \mu_{S_2}) \), we have

\[
\kappa(b_{S_1}) = \mu_{S_1} \mid Z(M_{S_1}) \,
\]

Then \( \mu_{S_1} \) and \( \mu_{S_2} \) are conjugate in \( M_{S_1} \) so there exists a \( w \in W_{M_{S_1}}^{\text{abs}} \) so that \( w(\mu_1) = \mu_2 \). This implies that \( \mu_1 \) and \( \mu_2 \) are conjugate in \( M_{S_1} \) and in particular equal when restricted to \( Z(M_{S_1}) \). This implies the desired equality.

To show the converse inclusion, we start with \((M_{S_2}, \mu_{S_2}) \in T_{M_{S_2}, b_{S_2}}^{G, \mu} \) for some \((M_{S_1}, \mu_{S_1}) \in T_{M_{S_1}, b_{S_1}}^{G, \mu} \) and need to show that \( b_{S_2} \in B(M_{S_2}, \mu_{S_2}) \) and that \( \mu_{S_2} \) is conjugate to \( \mu \) in \( G \). But \((M_{S_2}, \mu_{S_2}) \in T_{M_{S_2}, b_{S_2}}^{G, \mu} \) implies that \( b_{S_2} \in B(M_{S_2}, \mu_{S_2}) \) and also that \( \mu_{S_2} \) is conjugate to \( \mu_{S_1} \) in \( M_{S_1} \). Further, \((M_{S_1}, \mu_{S_1}) \in T_{M_{S_1}, b_{S_1}}^{G, \mu} \) implies that \( \mu_{S_1} \) is conjugate to \( \mu \) in \( G \). Thus, \( \mu_{S_2} \) is conjugate to \( \mu \) in \( G \) as desired. \( \square \)

The set \( T_{M_{S}, b_{S}}^{G, \mu} \) will primarily be useful because it allows us to relate the set \( T_{G, b, \mu} \) to analogous constructions in \( M_{S} \). This is encapsulated in the following proposition.

**Proposition 2.5.7.** Fix \( M_{S}, \mu \) and \( b \) as in \([2.5.3]\). The natural inclusion \( i_{M_S}^G : C_{M_S} \to C_G \) of equation \([2]\) induces a bijection

\[
\bigcup_{(M_{S}, \mu_{S}) \in T_{M_{S}, b_{S}, \mu_{S}}^{G, \mu}} T_{M_{S}, b_{S}, \mu_{S}} \cong T_{G, b, \mu}.
\]

**Proof.** We first show that

\[
i_{M_S}^G \left( \bigcup_{(M_{S}, \mu_{S}) \in T_{M_{S}, b_{S}, \mu_{S}}^{G, \mu}} T_{M_{S}, b_{S}, \mu_{S}} \right) \supseteq T_{G, b, \mu}.
\]

Since \( M_b \subset M_{S} \), it follows from the discussion after equation \([6]\) that

\[
T_{G, b, \mu} \subset i_{M_S}^G (C_{M_S}).
\]

Thus, pick an arbitrary element of \( T_{G, b, \mu} \) of the form \( i_{M_S}^G (M_b, \mu_b) \) for \((M_b, \mu_b) \in C_{M_S} \). The cocharacter pair \( i_{M_S}^G (M_b, \mu_b) \) is strictly decreasing, and therefore so
is $(M_b, \mu_b) \in \mathcal{C}_{M_S}$. By \ref{2} we can find $(M_S, \mu_S) \in \mathcal{C}_{M_S}$ such that $(M_b, \mu_b) \leq (M_S, \mu_S)$. Observe that since $i_{M_S}^G(M_b, \mu_b) \leq (G, \mu)$, the cocharacter $\mu_b$ is conjugate to $\mu$ in $G$ and therefore $\mu_b$ must be as well by construction. If we can show that $\mathcal{T}((M_b, \mu_b)) = b_S$, then we will be done because by \ref{2} this implies that $b_S \in B(M_S, \mu_S)$ and so therefore that $(M_S, \mu_S) \in \mathcal{T}^G_{M_S, b_S}$ and $(M_b, \mu_b) \in \mathcal{T}_{M_S, b_S, \mu_S}$.

By assumption, $\mathcal{T}(i_{M_S}^G(M_b, \mu_b)) = b \in B(G, \mu)$. Recall that the map $\mathcal{T}$ is defined so that a strictly decreasing $(M_b, \mu_b) \in \mathcal{C}_G$ which satisfies $(M_b, \mu_b) \leq (G, \mu)$ is mapped first to the element $\mu_b|_{Z(\widehat{M}_b)^\Gamma} \in X^*(Z(\widehat{M}_b)^\Gamma)^+$. Then, this element is identified with an element of $B(G)$ via the isomorphisms of equation \ref{2}. $$X^*(Z(\widehat{M}_b)^\Gamma)^+ \cong B(M_b)_{M_b} \cong B(G).$$ where the second isomorphism above is induced by the inclusion $M_b \hookrightarrow G$. We have the commutative diagram

$$\begin{array}{ccc}
B(M_b) & \longrightarrow & B(M_S) \\
\downarrow & & \downarrow \\
B(G) & & 
\end{array}$$

where each map is induced from the inclusion of groups. By definition, the element $b' \in B(M_b)^+$ maps to $b \in B(G)$ and $b_S \in B(M_S)$ respectively. Thus, we see that by construction, $\mathcal{T}((M_b, \mu_b)) = b_S$.

Conversely, suppose $(M_b, \mu_b) \in \mathcal{T}_{M_S, b_S, \mu_S}$ for some $(M_S, \mu_S) \in \mathcal{T}^G_{M_S, b_S}$. Since $b' \in B(M_b)^+$, it follows from the definition of $b_S$ and $\mathcal{T}_{M_S, b_S, \mu_S}$ that $\mu_b|_{Z(\widehat{M}_b)^\Gamma}$ is an element of $X^*(Z(\widehat{M}_b)^\Gamma)^+$. This implies that $i_{M_S}^G(M_b, \mu_b) \in \mathcal{S}D$. By \ref{2} we have an extension of $i_{M_S}^G(M_b, \mu_b)$ to $G$, and since $\mu_b$ and $\mu$ are conjugate in $G$ by assumption, it follows that this extension is $(G, \mu)$ so that $i_{M_S}^G(M_b, \mu_b) \leq (G, \mu)$. It follows from these facts that $i_{M_S}^G(M_b, \mu_b) \in \mathcal{T}_{G, b, \mu}$.

Finally, we remark that for distinct $(M_S, \mu_S), (M_S, \mu'_S) \in \mathcal{T}^G_{M_S, b_S}$ the sets $\mathcal{T}_{M_S, b_S, \mu_S}$ and $\mathcal{T}_{M_S, b_S, \mu'_S}$ are indeed disjoint by \ref{2}.

As a corollary of this result, we have the induction formula.

**Corollary 2.5.8 (Induction Formula).** We continue using the notation of the previous proposition. The natural map

$$i_{M_S}^G : \mathcal{C}_{M_S} \to \mathcal{C}_G$$

induces a map

$$i_{M_S}^G : \mathbb{Z}(\mathcal{C}_{M_S}) \to \mathbb{Z}(\mathcal{C}_G)$$

which gives an equality

$$\sum_{(M_S, \mu_S) \in \mathcal{T}^{G, \mu}_{M_S, b_S}} i_{M_S}^G(M_S, b_S, \mu_S) = \mathcal{M}_{G, b, \mu}$$

**Proof.** It follows from \ref{2.5.7} that the map $i_{M_S}^G$ induces a bijection

$$\bigsqcup_{(M_S, \mu_S) \in \mathcal{T}^{G, \mu}_{M_S, b_S}} \mathcal{R}_{M_S, b_S, \mu_S} \cong \mathcal{R}_{G, b, \mu}.$$ 

We remark that for distinct $(M_S, \mu_S), (M_S, \mu'_S) \in \mathcal{T}^{G, \mu}_{M_S, b_S}$ we have $\mathcal{R}_{M_S, b_S, \mu_S} \cap \mathcal{R}_{M_S, b_S, \mu'_S} = \emptyset$ by \ref{2}.
The corollary then follows from the definition of $\mathcal{M}_{G,b,\mu}$. □

This result can be thought of as an analogue of the Harris-Viehmann conjecture which we discuss in the next section.

In the cases we are interested in, we will also need a description of how cocharacter pairs behave with respect to products.

Suppose $G = G_1 \times ... \times G_k$ and $T = T_1 \times ... \times T_k$ such that $T_i$ is a maximal torus for $G_i$. Then

$$X_*(T) \cong X_*(T_1) \oplus ... \oplus X_*(T_k),$$

and any standard Levi subgroup admits a product decomposition

$$M_S \cong M_{S_1} \times ... \times M_{S_k},$$

such that $T_i \subset M_{S_i} \subset G_i$. Then any cocharacter pair $(M_S,\mu_S)$ of $G$ corresponds to a tuple of cocharacter pairs

$$((M_{S_1},\mu_{S_1}),..., (M_{S_k},\mu_{S_k})) \in \mathcal{C}_{G_1} \times ... \times \mathcal{C}_{G_k},$$

in the obvious way. The pair $(M_S,\mu_S) \in \mathcal{C}_G$ is strictly decreasing if and only if each pair $(M_{S_i},\mu_{S_i}) \in \mathcal{C}_{G_i}$ is, and if $T((M_{S_i},\mu_{S_i})) = b_i \in B(G_i,\mu_i)$, then we also have $T_i((M_{S_i},\mu_{S_i})) = b_i \in B(G_i,\mu_i)$ where $T_i$ is the map $T$ defined for the group $G_i$.

Thus, $b \mapsto (b_1,...,b_k)$ under the natural bijection

$$B(G) \cong B(G_1) \times ... \times B(G_k).$$

We record the following proposition

**Proposition 2.5.9.** We use the notation of the previous two paragraphs.

The natural bijection

$$\mathcal{C}_G \cong \mathcal{C}_{G_1} \times ... \times \mathcal{C}_{G_k},$$

induces bijections

$$\mathcal{T}_{G,b,\mu} \cong \mathcal{T}_{G_1,b_1,\mu_1} \times ... \times \mathcal{T}_{G_k,b_k,\mu_k},$$

and

$$\mathcal{R}_{G,b,\mu} \cong \mathcal{R}_{G_1,b_1,\mu_1} \times ... \times \mathcal{R}_{G_k,b_k,\mu_k}.$$

Further, under the natural isomorphism $\mathbb{Z}\langle \mathcal{C}_G \rangle \cong \mathbb{Z}\langle \mathcal{C}_{G_1} \rangle \otimes ... \otimes \mathbb{Z}\langle \mathcal{C}_{G_k} \rangle$ we have

$$\mathcal{M}_{G,b,\mu} \cong \mathcal{M}_{G_1,b_1,\mu_1} \otimes ... \otimes \mathcal{M}_{G_k,b_k,\mu_k}.$$

3. **Cohomology of Rapoport-Zink spaces and the Harris-Viehmann Conjecture**

In this section, we define the Rapoport-Zink spaces we will work with and show how we can describe their cohomology using the language developed in the previous section. We also give a statement of the Harris-Viehmann conjecture, and explain the necessity of a small correction to the conjecture. We follow [Far04], [Shi12], and [RV14].
3.1. Rapoport-Zink Spaces of EL-Type. We fix the following notation. Suppose $G$ is a reductive group defined over a field $k$ and $\mu \in X_*(G)$. Then if $H$ is a subgroup of $G$, we denote by $\{\mu\}_H$ the $H(\overline{k})$ conjugacy class of $\mu$ and by $E(\mu)_H$ the field of definition of $\{\mu\}_H$ (i.e. the smallest extension of $k$ so that each element of $\text{Gal}(\overline{k}/E(\mu)_H)$ stabilizes $\{\mu\}_H$).

Now we define the Rapoport-Zink data we consider.

**Definition 3.1.1.** An unramified Rapoport-Zink datum of EL type is a tuple $(F, V, \{\mu\}_G, b)$ where

1. $F$ is a finite unramified extension of $\mathbb{Q}_p$,
2. $V$ is a finite dimensional $F$ vector space,
3. $G := \text{Res}_{F/\mathbb{Q}_p}(\text{GL}_F(V))$,
4. $\mu : \mathbb{G}_{m, \mathbb{Q}_p} \to G_{\mathbb{Q}_p}$ is a cocharacter inducing a weight decomposition $V \otimes \overline{\mathbb{Q}_p} \cong V_0 \oplus V_1$ where $\mu(z)$ acts by $z^i$ on $V_i$,
5. $b \in \mathcal{B}(G, \mu)$.

Let $X$ be a $p$-divisible group defined over $\overline{\mathbb{F}_p}$ with an action of $\mathcal{O}_F$ and covariant isocrystal isomorphic to $(V_F, b)$. We consider the moduli functors $\mathcal{M}_{\mathbb{Q}_p}$ such that for $S$ a scheme over $\mathcal{O}_{\overline{\mathbb{Q}_p}}$ with $p$ locally nilpotent, $\mathcal{M}_{\mathbb{Q}_p}(S) = \{(X, i, \rho) \}/\sim$. Where $X$ is a $p$-divisible group defined over $S$, $i : \mathcal{O}_F \to \text{End}_F(X)$, and $\rho : \mathfrak{X} \times_{\mathbb{F}_p} S \to \mathfrak{X}$ is a quasi-isogeny ($\mathfrak{S}, \mathfrak{X}$ are the reductions modulo $p$).

By work of Rapoport and Zink [RZ96 Thm 3.25], the above moduli problem is represented by a formal scheme over $\mathcal{O}_{\overline{\mathbb{Q}_p}}$ which we also denote by $\mathcal{M}_{\mathbb{Q}_p}$. We have the generic fiber $\mathcal{M}_{\mathbb{Q}_p}^{rig}$ which is a rigid analytic space over $\overline{\mathbb{Q}_p}$. Further, we get a tower of coverings $\mathcal{M}_{\mathbb{Q}_p}^{rig}$ for each compact open subgroup $U \subset G(\mathbb{Q}_p)$.

For a fixed prime $l \neq p$, we denote by $H^i_l(\mathcal{M}_{\mathbb{Q}_p}^{rig}, \overline{\mathbb{Q}_l} \otimes \mathcal{T})$ the etale cohomology with compact supports. This is a $\mathbb{Q}_l$ vector space which is a smooth representation of $J^l_b(\mathbb{Q}_p) \times W_{E(\mu)_G}$, where $J^l_b$ is the inner form of $M_b$ associated to $b$ (as constructed in §3.3 of [Kot97]) and $W_{E(\mu)_G}$ is the Weil group of $E(\mu)_G$ (for example see [RV14 Prop 6.1]).

We use the notation Groth$(\cdot)$ for the Grothendieck group of admissible representations of topological groups. See §I.2 of [HT01] for the precise definition of these Grothendieck groups.

We let $P_b$ be the standard parabolic subgroup with Levi factor $M_b$ and denote the opposite parabolic by $P_b^{op}$. We define $J^G_{M_b}, \text{Jac}^G_{P_b}$ to be the normalized and unnormalized Jacquet module functors, and we define $I^G_{P_b}, \text{Ind}^G_{P_b}$ to be the normalized and un-normalized parabolic induction functors. Often, if $M \subset P$ is the standard Levi subgroup of $P$ and we are taking $I^G_{M_b}, J^G_{P_b}$ to be maps of Grothendieck groups, we will write $I^G_{M_b}, J^G_{P_b}$ to remind the reader that these maps do not depend on choice of $P, P^{op}$ when considered as maps of Grothendieck groups.

In [Man05], Mantovan considers the following construction (see also [Shi12]). We define a map

$$\text{Mant}_{G, b, \mu} : \text{Groth}(J_b(\mathbb{Q}_p)) \to \text{Groth}(G(\mathbb{Q}_p) \times W_{E(\mu)_G}),$$

by

$$\text{Mant}_{G, b, \mu}(\rho) = \sum_{i, j \geq 0} (-1)^{i+j} \lim_{U \subset G(\mathbb{Q}_p)} \text{Ext}^j_{J_b_U}(H^i_l(\mathcal{M}_{\mathbb{Q}_p}^{rig}, \overline{\mathbb{Q}_l} \otimes \mathcal{T}, \rho), (-\dim \mathcal{M}_{\mathbb{Q}_p}^{rig}).$$
In §6.2 of [Shi12] and §2.4 of [Shi11], Shin considers a map
\[ \text{Red}_b : \text{Groth}(G(\mathbb{Q}_p)) \to \text{Groth}(J_b(\mathbb{Q}_p)). \]
We follow the construction given in [Shi11]. We define \( \text{Red}_b \) by
\[ \pi \mapsto e(J_b)(LJ \circ J^G_{\mathbb{Q}_p}(\pi) \otimes \delta^\frac{1}{2}_p), \]
where
\[ LJ : \text{Groth}(M_b(\mathbb{Q}_p)) \to \text{Groth}(J_b(\mathbb{Q}_p)), \]
is the map defined by Badulescu extending the inverse Jacquet-Langlands correspondence (see [Bad07 Prop 3.2]) and \( e(J_b) \) is the Kottwitz sign as defined in [Kot83]. Since the maps \( J^G_{\mathbb{Q}_p}, J^G_{\mathbb{Q}_p} \) induce the same maps on Grothendieck groups, we can rewrite the above as
\[ \text{Red}_b : \pi \mapsto e(b)(LJ \circ \text{Jac}^G_{\mathbb{Q}_p}(\pi)). \]

We now describe the main result of [Shi12]. The cocharacter \( \mu \) of \( G \) is a map \( \mu : \mathfrak{g}^{\text{op},\mathbb{Q}_p} \to \prod_{r \in \text{Hom}(F,\mathbb{Q}_p)} \text{GL}_{n,\mathbb{Q}_p} \) such that the weights in each \( \text{GL}_n \) factor are 0s or 1s. Thus we let \( p_r, q_r \) denote the number of 1 and 0 weights respectively in the factor corresponding to \( r \).

The following formula is the main theorem in [Shi12 Thm 7.5].

**Theorem 3.1.2** (Shin). *We have the following equality for accessible representations in \( \text{Groth}(G(\mathbb{Q}_p) \times \mathcal{W}_{E(\nu)G}) \).

\[
\sum_{b \in B(G,\mu)} \text{Mant}_{b,\mu}(\text{Red}_b(\pi)) = [\pi][r_{-\mu} \circ \text{LL}(\pi)]_{\mathcal{W}_{E(\nu)G}} \otimes | \cdot |_{\sum_p p \cdot q_r / 2}.
\]

Loosely speaking, accessible representations in Shin’s paper are character twists of the local components of global representations that can be found within the cohomology of Shimura varieties. Shin shows that all essentially square-integrable representations are accessible.

In this case, LL is the semisimplified local Langlands correspondence (known by the work of [HT10] for instance). The map \( r_{-\mu} \) is the algebraic representation of \( \hat{G} \times \mathcal{W}_{E(\nu)G} \subset LG \) defined by Kottwitz ([Kot84 Lem 2.1.2]). It is characterized by the fact that \( r_{-\mu}|_{\hat{G}} \) is the irreducible representation of extreme weight \( -\mu \) and if we take a \( \Gamma \)-invariant splitting of \( \hat{G} \), then the subgroup \( \mathcal{W}_{E(\nu)G} \) of \( LG \) acts trivially on the highest weight vector of \( r_{-\mu} \) associated with this splitting.

**Remark 3.1.3.** The Tate twist appearing on the right-hand side of the above formula comes from the dimension formula for Shimura varieties and is equal to \( -\langle \rho_G, \mu \rangle \) where \( \rho_G \) is the half sum of the positive roots in \( G \).

The above theorem is analogous to the sum formula for cocharacter pairs. The induction formula is related to the Harris-Viehmann conjecture. A proof of this conjecture is expected to appear in forthcoming work of Scholze.

---

1We believe the construction given before lemma 6.2 of [Shi12] has a slight typo. There, \( \text{Red}_b \) is defined by \( \pi \mapsto e(J_b)(LJ \circ \text{Jac}^G_{\mathbb{Q}_p}(\pi)) \). However as maps of Grothendieck groups, \( \text{Jac}^G_{\mathbb{Q}_p} = J^G_{\mathbb{Q}_p} \otimes \delta^\frac{1}{2}_p = J^G_{\mathbb{Q}_p} \otimes \delta^\frac{1}{2}_p \). But this is not equal to \( J^G_{\mathbb{Q}_p}(\pi) \otimes \delta^\frac{1}{2}_p \), which is the construction given in [Shi11].
3.2. Harris-Viehmann Conjecture. We now state the Harris-Viehmann conjecture following Rapoport and Viehmann in [RV14]. We return to the notation of section 2 so that in particular, $G$ is a connected, quasisplit reductive group defined over $\mathbb{Q}_p$. Choose a dominant $\mu \in X_a(T)$ with weights as in 3.1.1 (where we can consider $\mu$ as a cocharacter of $G$ since $T \subset G$) and a $b \in B(G, \mu)$. Associated to $b$, we have the standard Levi subgroup $M_b$. Suppose we have a standard rational Levi subgroup $M_S$ so that $M_b \subset M_S \subset G$. We define $b', b_S$ as we did before 2.5.5.

In [RV14, (6.2)], the authors associate a cohomological construction to the triple $(G, b, \mu)$ which they denote $H^*((G, [b], \{\mu\}))$. This construction agrees with $\text{Mant}_{G,b,\mu}$ in the case above. We will denote this construction $H^*(G, b, \mu)$ since we deal with dominant cocharacters instead of conjugacy classes. Then they have the following conjecture.

**Conjecture 3.2.1 (Harris-Viehmann).** We have the equality

$$H^*(G, b, \mu) = \sum_{(M_S, \mu_S) \in \mathcal{E}_{G,M,b}} \langle \text{Ind}_{M_S}^{G}(M_S, b_S, \mu_S) \rangle \otimes [1] \cdot \langle \rho, \mu_S \rangle - \langle \rho, \mu \rangle,$$

in $\text{Groth}(G(\mathbb{Q}_p) \times W_{E_\rho(G)})$. The parabolic induction only modifies the $\text{Groth}(G(\mathbb{Q}_p))$ parts of these representations.

**Remark 3.2.2.** We need to explain several things in the above conjecture. First we explain why the right-hand side is a representation of $W_{E_\rho(G)}$, second we check that the conjecture satisfies a transitivity property, and third we give an example justifying the extra character twist appearing in our formulation. This twist is not present in the original formulation of the conjecture.

We first explain why the right-hand side is a representation of $W_{E_\rho(G)}$. We start with a general lemma.

**Lemma 3.2.3.** Suppose a group $\Lambda$ acts on a finite set $S$. Suppose further that for each $s \in V$, we attach a vector space $V_s$ and for each $\lambda \in \Lambda$ and $s \in S$ we have an isomorphism

$$i(s, \lambda) : V_s \to V_{\lambda(s)}.$$

We suppose further that $i(s, 1)$ is the identity map and that $i(\lambda_1(s), \lambda_2) \circ i(s, \lambda_1) = i(s, \lambda_2 \lambda_1)$. Then $\bigoplus_{s \in S} V_s$ is naturally a representation of $\Lambda$. Let $\{s_1, ..., s_k\} \subset \Lambda$ be a set containing one representative from each $\Lambda$-orbit in $S$. Then

$$\bigoplus_{s \in S} V_s \cong \bigoplus_{i=1}^k \text{Ind}_{\text{stab}(s_i)}^\Lambda V_{s_i},$$

where $\text{Ind}$ refers to the induced representation (not parabolic induction).

**Proof.** The proof is more or less clear from the definition of induced representation. \qed

Moreover, we record the following transitivity property for later use.

**Lemma 3.2.4.** Suppose that $\Lambda$ acts on $S$ as before. Let $S_1 \bigcup ... \bigcup S_k = S$ be a partition of $S$ so that $\Lambda$ acts on $\{S_1, ..., S_k\}$. Suppose we have for each $s \in S$ a vector space $V_s$ and isomorphisms $i(s, \lambda)$ as above. Then by 3.2.3 we can consider the $\text{stab}(S_i) \subset \Lambda$ representation $V_{S_i} = \bigoplus_{s \in S_i} V_s$. For each $\lambda \in \Lambda$, we get isomorphisms
we observe that the twist by $r$ and therefore on $\mathbb{V}_S$ or orbits of $W$ is not an obstacle to defining the associated to $p$ these actions induce morphisms of the corresponding towers of rigid spaces and therefore the spaces Rapoport-Zink data. We remark that the character twist by $M$ elements relative to $\mu$ mean that if we have standard Levi subgroups $W$ under this action is $\mu S$ that stabilizes $\sigma_S$ and $S$. We observe that we have now shown that $W$ Now we discuss the $W_{E_{(\nu)}G}$-action in the Harris-Viehmann conjecture. Observe that for $\mu \in X_\ast(G)$, if $\gamma \in W_{E_{(\nu)}G}$ stabilizes $\{\mu\}_G$ then it also stabilizes $\{\mu\}_G$ so that $W_{E_{(\nu)}G} < W_{E_{(\nu)}G}$. We remark that the character twist by $M$ that stabilizes $\{\mu_S\}_MS$. Conversely, if $\gamma$ stabilizes $\{\mu_S\}_MS$ then since it maps dominant elements relative to $M_S$ to dominant elements, we must have $\gamma(\mu_S) = \mu_S$. We observe that we have now shown that $W_{E_{(\nu)}G}$ acts on the collection of Rapoport-Zink data $(M_S, b_S, \mu_S)$ for $(M_S, \mu_S) \in \mathbf{I}_{M_S, b_S}^{G,\mu}$. By [RV14 prop 5.3.iv], these actions induce morphisms of the corresponding towers of rigid spaces and therefore the spaces $H^\ast(M_S, b_S, \mu_S)$. Thus by 3.2.3 we get an action of $W_{E_{(\nu)}G}$ on the sum of vector spaces

$$
\sum_{(M_S, \mu_S) \in \mathbf{I}_{M_S, b_S}^{G,\mu}} H^\ast(M_S, b_S, \mu_S),
$$

and therefore on

$$
\sum_{(M_S, \mu_S) \in \mathbf{I}_{M_S, b_S}^{G,\mu}} \text{Ind}_G^{M_S}(H^\ast(M_S, b_S, \mu_S)).
$$

We remark that the character twist by $-\dim M_{b, \mu, U}$ in the definition of $H^\ast(M_S, b_S, \mu_S)$ is not an obstacle to defining the $W_{E_{(\nu)}G}$-action as the dimensions of the spaces associated to $(M_S, b_S, \mu_S)$ and $(M_S, b_S, \gamma(\mu_S))$ are the same (for $\gamma \in W_{E_{(\nu)}G}$). Also we observe that the twist by $[1][\langle \rho_G, \mu_S \rangle - \langle \rho_G, \mu \rangle]$ is harmless as it is constant over orbits of $W_{E_{(\nu)}G}$. This concludes our discussion of the $W_{E_{(\nu)}G}$ action.

We now check that the Harris-Viehmann conjecture is transitive. By this, we mean that if we have standard Levi subgroups $M_{S_1}$ and $M_{S_2}$ of $G$ such that $M_S < M_{S_2} < M_{S_1} < G$, then first applying the conjecture to $(G, b, \mu)$ and the inclusion $M_{S_1} < G$ and then applying the conjecture to each resulting $(M_{S_1}, b_S, \mu_S)$ for $(M_S, b_S, \gamma(\mu_S))$ in $\mathbf{I}_{M_S, b_S}^{G,\mu}$, and the inclusion $M_{S_2} < M_{S_1}$ should be the same as applying the conjecture to $(G, b, \mu)$ and the inclusion $M_{S_2} < G$.

We need to check that the character twists match, that

$$
\mathbf{I}_{M_{S_2}, b_{S_2}}^{G,\mu} = \bigcup_{(M_{S_1}, \mu_{S_1}) \in \mathbf{I}_{M_{S_1}, b_{S_1}}^{G,\mu}} \mathbf{I}_{M_{S_2}, b_{S_2}}^{M_{S_1}, \mu_{S_1}}.
$$
and that the $W_{E(\mu,G)}$ actions are the same.

To check the characters match, it suffices to check that for $(M_{S_1}, \mu_{S_1}), (M_{S_2}, \mu_{S_2}), (G, \mu) \in C_G$ such that $\mu_{S_2} \sim M_{S_1} \mu_{S_1}$ and $\mu_{S_1} \sim G \mu$, we have

$$\langle \rho_G, \mu_{S_2} \rangle - \langle \rho_G, \mu \rangle = \langle \rho_G, \mu_{S_1} \rangle - \langle \rho_G, \mu \rangle + \langle \rho_{M_{S_1}}, \mu_{S_2} \rangle - \langle \rho_{M_{S_1}}, \mu_{S_1} \rangle.$$  

This reduces to showing the equality

$$\langle \rho_{G,M_{S_1}}, \mu_{S_1} \rangle = \langle \rho_{G,M_{S_1}}, \mu_{S_2} \rangle,$$

where $\rho_{G,M_{S_1}}$ is the half-sum of the absolute roots of $G$ that are not roots of $M_{S_1}$. Since $\mu_{S_2}$ and $\mu_{S_1}$ are conjugate in $M_{S_1}$, there exists a $w \in W_{M_{S_1}}$ so that $w(\mu_1) = \mu_2$. Then the desired equality follows from the fact that the pairing $\langle \cdot, \cdot \rangle$ is $W_{M_{S_1}}$-invariant and that $W_{M_{S_1}}$ stabilizes the set of positive absolute roots in $G$ but not $M_{S_1}$. To prove this second fact, note that $M_{S_1}$ normalizes the unipotent radical $U_{S_1}$ of $P_{S_1}$ and that the roots of $\text{Lie}(U_{S_1})$ are precisely the positive absolute roots of $G$ that are not contained in $M_{S_1}$.

The second check is precisely 2.5.6 and the third check follows from 2.5.6 and 3.2.4.

Now we compute an example to illustrate the necessity of the extra Tate twist in our statement of 3.2.1.

**Example 3.2.5.** Let $n_1 < n_2$ be coprime positive integers and let $G = \text{GL}_{n_1+n_2}$. Fix $T$ the standard maximal torus of diagonal matrices and $B$ the Borel subgroup of upper triangular matrices. Let $\mu$ be the minuscule cocharacter with weight vector $(1^2, 0^{n_1+n_2-2})$ and $b \in B(G, \mu)$ satisfying $\nu_b = ((4/n_1)^{n_1}, 1/(n_2)^{n_2})$. Let $\rho_1, \rho_2$ be supercuspidal representations of $\text{GL}_{n_1}(\mathbb{Q}_p), \text{GL}_{n_2}(\mathbb{Q}_p)$ respectively. Define the standard Levi subgroup $M_b = \text{GL}_{n_1} \times \text{GL}_{n_2}$, and consider the representation $\pi = I_{M_b}^G(\rho_1 \boxtimes \rho_2)$. We will be interested in computing $\text{Mant}_{G,b,\mu}(\text{Red}_b(\pi))$.

The key point is that we can use Shin’s formula (Theorem 3.1.2 of this paper) and known cases of the Harris-Viehmann conjecture due to Mantovan (Man08) to do this computation, even though the Harris-Viehmann conjecture is not known to be true in the case of $M_b$ since $b$ is not of Hodge-Newton type.

We observe that there are only 3 elements $b' \in B(G, \mu)$ that satisfy

$$\text{Mant}_{G,b',\mu}(\text{Red}_{b'}(\pi)) \neq 0.$$  

After all, the fact that $\rho_1, \rho_2$ are supercuspidal and the geometric lemma of Bernstein-Zelevinski (§2.11 of [BZ77]) forces $M_{b'}$ to be one of $G, \text{GL}_{n_1} \times \text{GL}_{n_1}, \text{GL}_{n_2} \times \text{GL}_{n_1}$. In the case where $M_{b'} = G$, we also get $0$ since $LJ(\pi) = 0$. Thus, if we write out Shin’s formula for our $\pi$, the only elements of $B(G, \mu)$ whose terms contribute to the left-hand side are $b, b_1, b_2$ where $b$ is as before and $b_1, b_2$ are defined by

$$\nu_{b_1} = ((2/n_1)^{n_1}, 0^{n_2}), \nu_{b_2} = ((2/n_2)^{n_2}, 0^{n_1}).$$

Thus, we have $M_{b_1} = M_b = \text{GL}_{n_1} \times \text{GL}_{n_2}$ and $M_{b_2} = \text{GL}_{n_2} \times \text{GL}_{n_1}$. Note that $b_1, b_2$ are both of Hodge-Newton type so that we can apply the results of Mantovan.

We have

$$\text{Mant}_{G,b_1,\mu}(\text{Red}_{b_1}(\pi)) = \text{Mant}_{G,b_1,\mu}(LJ(\delta_{\rho_1}^{1/2} I_{M_b}^G(\rho_1 \boxtimes \rho_2))).$$

By the geometric lemma of Bernstein-Zelevinski (§2.11 of [BZ77]) we have that the above equals

$$\text{Mant}_{G,b_1,\mu_1}(LJ((\rho_1 \boxtimes \rho_2) \otimes \delta_{\rho_1}^{1/2})).$$
We recall that $\delta_P = (| \cdot |^{n_2} \circ \det) \boxtimes (| \cdot |^{-n_1} \circ \det)$ and henceforth use the notation $\rho(n)$ to mean $(| \cdot |^{n} \circ \det) \otimes \rho$. Thus, we can rewrite the above formula as

$$\operatorname{Mant}_{G, b, \mu} (LJ(\rho_1(n_2/2)) \boxtimes LJ(\rho_2(-n_1/2))).$$

Then applying the Harris-Viehmann formula we get that the above equals (8)

$$\operatorname{Ind}^G_{M_b} (\operatorname{Mant}_{\operatorname{GL}_{n_1}(12, 0^{n_1-2})}(LJ(\rho_1(n_2/2))) \boxtimes \operatorname{Mant}_{\operatorname{GL}_{n_2}(0^{n_2})}(LJ(\rho_2(-n_1/2))))).$$

Since $\rho_1$ and $\rho_2$ are supercuspidal, we can compute (by an easy application of Shin’s formula for instance) that

$$\operatorname{Mant}_{\operatorname{GL}_{n_1}(12, 0^{n_1-2})}(LJ(\rho_1(n_2/2))) = [\rho_1(n_2/2)][r_{(-12, 0^{n_1-2})} \circ LL(\rho_1(n_2/2)) \boxtimes |^{2-n_1}],$$

and hence equation (8) becomes equal to

$$[\pi][r_{(-12, 0^{n_1-2})} \circ LL(\rho_1(n_2/2)) \boxtimes |^{2-n_1} \circ r_{(0^{n_2})} \circ LL(\rho_2(-n_1/2))].$$

Pulling the twists through the $r_{-\mu}$ maps, we get

$$[\pi][r_{(-12, 0^{n_1-2})} \boxtimes r_{(0^{n_2})} \circ (LL(\rho_1) \oplus LL(\rho_2)) \boxtimes |^{2-n_1-n_2}].$$

Repeating this computation for the $b_2$ term, we get

$$\operatorname{Mant}_{G, b_2, \mu} (\operatorname{Red}_{b_2}(\pi)) = [\pi][r_{(-12, 0^{n_1-2})} \boxtimes r_{(0^{n_1})} \circ (LL(\rho_2) \oplus LL(\rho_1)) \boxtimes |^{2-n_1-n_2}].$$

We now compare these terms to the righthand side of Shin’s formula. There the term is

$$[\pi][r_{-\mu} \circ LL(\pi) \boxtimes |^{2-n_1-n_2}].$$

Now $LL(\pi) = LL(\rho_1) \oplus LL(\rho_2)$. Thus, we need to restrict $r_{-\mu}$ to $\hat{M}_b \subset \hat{G}$ (we have been ignoring the Galois part of $^LG$ in this example since $G$ is a split group). Using the theory of weights, we get

$$r_{-\mu}|_{\hat{M}_b} = [r_{(-12, 0^{n_1-2})} \boxtimes r_{(0^{n_2})}] \boxplus [r_{(-1, 0^{n_1-1})} \boxtimes r_{(-1, 0^{n_2-1})}] \boxplus [r_{(0^{n_2})} \boxtimes r_{(-12, 0^{n_2-2})}],$$

and so we see that the contributions for $b_1, b_2$ which we computed above will cancel terms on the righthand side of Shin’s formula leaving us with

$$\operatorname{Mant}_{G, b, \mu} (\operatorname{Red}_{b}(\pi)) = [\pi][r_{(-1, 0^{n_1-1})} \boxtimes r_{(-1, 0^{n_2-1})} \circ (LL(\rho_1) + LL(\rho_2)) \boxtimes |^{2-n_1-n_2}].$$

However, if the Harris-Viehmann conjecture without the extra Tate twist were to hold for $b$, we would get

$$\operatorname{Mant}_{G, b, \mu} (\operatorname{Red}_{b}(\pi)) = \operatorname{Mant}_{G, b, \mu} (LJ(\rho_1(2/2)) \boxtimes LJ(\rho_2(-1/2)))$$

$$= [\pi][r_{(-1, 0^{n_1-1})} \boxtimes r_{(-1, 0^{n_2-1})} \circ (LL(\rho_1) + LL(\rho_2)) \circ |^{1-n_2}].$$

Thus, we see the Tate twists do not agree.

In general, the righthand side of Shin’s formula has a twist of $-\langle \rho_G, \mu \rangle$ where $\rho_G$ is the half sum of the positive roots of $G$. If we have a term on the lefthand side of Shin’s formula corresponding to some $b \in \mathcal{B}(G, \mu)$, then we would expect the Galois part of $\operatorname{Mant}_{M_b, b, \mu}$ terms to come with a twist of $-\langle \rho_{M_b}, \mu_b \rangle$ and that we would get an extra twist of $-\langle \frac{\det(\operatorname{Ad}_{M_b})}{2}, \mu_b \rangle$ corresponding to the twist on $r_{-\mu_b} \circ LL(\rho)$ that we get by twisting $\rho$ by $\delta_{\mu_b}$. We note that

$$\langle \rho_{M_b}, \mu_b \rangle + \langle \frac{\det(\operatorname{Ad}_{M_b})}{2}, \mu_b \rangle = \langle \rho_G, \mu \rangle,.$$
Thus, we see that the difference between these Tate twists is
\[ \langle \rho_G, \mu_b \rangle - \langle \rho_G, \mu \rangle. \]
which is the twist in conjecture \ref{conjecture:3.2.7}.

**Remark 3.2.6.** We note that in the Hodge-Newton case studied by Mantovan, $\mu = \mu_b$ so that this extra twist vanishes, agreeing with Mantovan’s results.

We now give an alternate version of the Harris-Viehmann conjecture that we will use in numerous arguments in this paper. Suppose that $G, b, \mu$ are as in theorem \ref{thm:1.0.3}. The standard Levi subgroup $M_b$ has a natural product decomposition
\[ M_b = M_1 \times \ldots \times M_k \]
so that under the natural isomorphism
\[ \text{B}(M_b) \cong \text{B}(M_1) \times \text{B}(M_k), b' \mapsto (b'_1, \ldots, b'_k), \]
each $\nu(b_i)$ has a single slope. Now pick $(M_b, \mu_b) \in \mathcal{X}_{M_b, b}^{G, \mu}$. Then the local Shimura variety datum $(M_b, b', \mu_b)$ decomposes into a collection $(M_1, b'_1, \mu_{1b_1}), \ldots, (M_k, b'_k, \mu_{kb_k})$. In §5.2.(ii) of \cite{MV14}, the authors show that the local Shimura variety associated to $(M_b, b', \mu_b)$ is the product of those associated to $(M_i, b'_i, \mu_{ib_i})$. Furthermore using the Kunneth formula (as in \cite{Man08} 15), we get that
\[ \text{Mant}_{M_b, b', \mu_b} = \bigotimes_{i=1}^k \text{Mant}_{M_i, b'_i, \mu_{ib_i}}. \]
Thus, we have the following alternate form of the Harris-Viehmann conjecture for the Rapoport-Zink spaces we consider.

**Conjecture 3.2.7 (Alternate Form of Harris-Viehmann Conjecture).** We use the notation of the previous paragraphs so that in particular, $(G, b, \mu)$ comes from an unramified Rapoport-Zink space of EL-type as in \ref{thm:3.1.1}. Then we have the following equality in Groth($G(\mathbb{Q}_p) \times W_{E(\mu_G)})$:
\[ \text{Mant}_{G,b,\mu} = \sum_{(M_b, \mu_b) \in \mathcal{X}_{M_b, b}^{G, \mu}} \text{Ind}_G^{G_b} (\bigotimes_{i=1}^k \text{Mant}_{M_i, b'_i, \mu_{ib_i}}) \otimes [1] \cdot [\langle \rho_G, \mu_b \rangle - \langle \rho_G, \mu \rangle]. \]

3.3. **Proof of Theorem 1.0.3.** The combination of the Harris-Viehmann conjecture and sum formula allows us to relate the cohomology of Rapoport-Zink spaces to the cocharacter pairs studied in section 2. To do so, we attach a map of Grothendieck groups to each cocharacter pair.

Fix a cocharacter pair $(G, \mu) \in C_G$. Suppose $(M_S, \mu_S) \in C_G$ and satisfies $\mu_S \sim_G \mu$. We associate $(M_S, \mu_S)$ to a map of representations
\[ [M_S, \mu_S] : \text{Groth}(G(\mathbb{Q}_p)) \to \text{Groth}(G(\mathbb{Q}_p) \times W_{E(\mu_S)}), \]
given by
\[ \pi \mapsto (\text{Ind}_S^{G_S} \circ [\mu_S] \circ \text{Jac}_S^{G_S})(\pi) \otimes [1] \cdot [\langle \rho_G, \mu_S \rangle - \langle \rho_G, \mu \rangle], \]
with
\[ [\mu_S] : \text{Groth}(G(\mathbb{Q}_p)) \to \text{Groth}(G(\mathbb{Q}_p) \times W_{E(\mu_S)}), \]
given by
\[ \pi \mapsto [\pi][r_{-\mu_S} \circ LL(\pi)]_{W_{E(\mu_S)}} \otimes [\cdot [\langle \rho_{MS}, \mu_S \rangle]. \]
Remark 3.3.1. We note that the map \([M_S, \mu_S]\) is only defined relative to a cocharacter pair \((G, \mu)\). Also, in the case where \(G = G_1 \times \ldots \times G_k\) where each \(G_i\) is a general linear group or Weil restriction of a general linear group, \(LL(\pi_1 \boxtimes \ldots \boxtimes \pi_k)\) is defined to be \(LL(\pi_1) \oplus \ldots \oplus LL(\pi_k)\) for \(\pi_1 \boxtimes \ldots \boxtimes \pi_k\) a smooth irreducible representation of \(G(\mathbb{Q}_p)\).

Remark 3.3.2. We observe an interesting property of the maps \([M_S, \mu_S]\). Fix \((G, \mu)\) and consider \((M_S, \mu_S)\) such that \(\mu_S \sim_G \mu\). Since the normalized Jacquet module and parabolic induction functors behave better with respect to the local Langlands correspondence, it makes sense to rewrite \([M_S, \mu_S]\) in terms of these maps. We get

\[
[M_S, \mu_S] = (I_{M_S}^G \otimes \delta_{P_S}^{-\frac{1}{2}} \circ [\mu_S] \circ \delta_{P_S}^\frac{1}{2} \otimes J_{P_S}^G) \otimes [1][\cdot \langle \rho_G, \mu_S - \mu \rangle].
\]

Note that the twists by the modular character cancel in the admissible part but do not cancel in the Galois part. Thus, the total Tate twist of the Galois part is

\[
\langle \rho_G, \mu_S - \mu \rangle - \langle \rho_{M_S}, \mu_S \rangle - \langle \frac{\det(Ad_{N_S}(M_S))}{2}, \mu_S \rangle = -\langle \rho_G, \mu \rangle.
\]

This twist does not depend on \((M_S, \mu_S)\) but rather only on \((G, \mu)\). Thus, as we will see in the computations of the next section, it is possible for large cancellations to occur in computations of \(\text{Mant}_{G,h,\mu}(\rho)\) for various \(\rho\).

We now prove some lemmas relating to these maps before tackling the main theorem.

Lemma 3.3.3. Let \(M_{S_1}, M_{S_2}\) be standard Levi subgroups of \(G\) satisfying \(M_{S_2} \subset M_{S_1}\). Consider the natural map

\[
i_{M_{S_1}}^G : C_{M_{S_1}} \rightarrow C_G,
\]

as defined in equation\(\square\). Let \((M_{S_2}, \mu_{S_2}) \in C_{M_{S_1}}\). Suppose further that we have fixed pairs \((M_{S_1}, \mu_{S_1}) \in C_{M_{S_1}}\) and \((G, \mu) \in C_G\) so that \(\mu_{S_2} \sim_{M_{S_1}} \mu_{S_1}\) and \(\mu_{S_2} \sim_G \mu\).

Then for \(\pi \in \text{Groth}(G(\mathbb{Q}_p))\),

\[
i_{M_{S_1}}^G([M_{S_2}, \mu_{S_2}])(\pi) = (\text{Ind}_{P_{S_1}}^G \circ [M_{S_2}, \mu_{S_2}] \circ \text{Jac}_{P_{S_1}}^G)(\pi) \otimes [1][\cdot \langle \rho_G, \mu_{S_1} \rangle - \langle \rho_G, \mu \rangle],
\]

where we write

\[
i_{M_{S_1}}^G([M_{S_2}, \mu_{S_2}] : \text{Groth}(G(\mathbb{Q}_p)) \rightarrow \text{Groth}(G(\mathbb{Q}_p) \times W_{E_{(p, S_2), M_{S_2}}}),
\]

to denote the map associated to \(i_{M_{S_1}}^G([M_{S_2}, \mu_{S_2}]\) in the manner above.

Proof. We just need to check that the twists on both sides match. On the left we get a twist of

\[
\langle \rho_G, \mu_{S_2} \rangle - \langle \rho_G, \mu \rangle - \langle \rho_{M_{S_2}}, \mu_{S_2} \rangle,
\]

and on the right we have a twist of

\[
\langle \rho_G, \mu_{S_1} \rangle - \langle \rho_G, \mu \rangle + \langle \rho_{M_{S_1}}, \mu_{S_2} \rangle - \langle \rho_{M_{S_1}}, \mu_{S_1} \rangle - \langle \rho_{M_{S_2}}, \mu_{S_2} \rangle.
\]

Thus, it suffices to check that

\[
\langle \rho_G, \mu_{S_1}, \mu_{S_2} \rangle = \langle \rho_G, \mu_{S_1}, \mu_{S_2} \rangle.
\]

This is the same as equation\(\square\).
Lemma 3.3.4. Suppose we are in the situation of 2.5.3 so that $G = G_1 \times \cdots \times G_k$ is a connected reductive group with standard Levi subgroup $M_S = M_{S_1} \times \cdots \times M_{S_k}$. Fix cocharacter pairs $(M_{S_i}, \mu_S), (G, \mu) \in C_G$ with $\mu_S \sim_G \mu$. The bijection $C_G \cong C_{G_1} \times \cdots C_{G_k}$ takes $(M_{S_i}, \mu_S)$ to $((M_{S_1}, \mu_{S_1}), \ldots, (M_{S_k}, \mu_{S_k}))$ and $(G, \mu)$ to $((G_1, \mu_1), \ldots, (G_k, \mu_k))$ and we have $\mu_{S_i} \sim_G \mu_i$. For each $i$, we have $W_{E_{(\mu_S)M_{S_i}}} \subset W_{E_{(\mu_S)M_{S_i}}}$. Then we have the following equality of homomorphisms of Grothendieck groups:

$$\boxtimes_{i=1}^k [M_{S_i}, \mu_{S_i}] : \text{Groth}(G(Q_p)) \to \text{Groth}(G(Q_p) \times W_{E_{(\mu_S)M_{S_i}}})$$

by

$$\pi_1 \boxtimes \cdots \boxtimes \pi_k \mapsto [M_{S_1}, \mu_{S_1}]((\pi_1)) \boxtimes \cdots \boxtimes [M_{S_k}, \mu_{S_k}]((\pi_k)).$$

Then we have the following equality of homomorphisms of Grothendieck groups:

$$\boxtimes_{i=1}^k [M_{S_i}, \mu_{S_i}] = [M_S, \mu_S].$$

Proof. We claim that $\boxtimes_{i=1}^k [\mu_{S_i}] = [\mu_S]$ as maps

$$\text{Groth}(M_S(Q_p)) \to \text{Groth}(M_S(Q_p) \times W_{E_{(\mu_S)M_S}}).$$

Indeed, for $\pi = \pi_1 \boxtimes \cdots \boxtimes \pi_k$ a smooth irreducible representation of $G(Q_p)$, we have

$$\boxtimes_{i=1}^k [\mu_{S_i}]((\pi)) = (\boxtimes_{i=1}^k \pi_i) \otimes (LL(\pi_1) \otimes \cdots \otimes LL(\pi_k)) \otimes |\cdot|^{\langle \rho_{M_{S_i}}, \mu_{S_i} \rangle}$$

$$= [\pi][\boxtimes_{i=1}^k \mu_{S_i}] \otimes (LL(\pi_1) \otimes \cdots \otimes LL(\pi_k)) \otimes |\cdot|^{\langle \rho_{M_S}, \mu_S \rangle}$$

$$= [\mu_S](\pi).$$

Now, we have

$$\boxtimes_{i=1}^k [M_{S_i}, \mu_{S_i}] = \boxtimes_{i=1}^k \text{Ind}_{M_{S_i}}^{G_{M_{S_i}}} \circ [\mu_{S_i}] \circ \text{Jac}_{M_{S_i}}^{G_{M_{S_i}}} \otimes [1][|\cdot|^{\langle \rho_{G_{S_i}}, \mu_{S_i} - \mu \rangle}]$$

$$= \text{Ind}_{M_S}^G \circ [\mu] \circ \text{Jac}_{M_S}^G \otimes [1][|\cdot|^{\langle \rho_{G}, \mu_S - \mu \rangle}]$$

$$= [M_S, \mu_S].$$

For some finite subset $C \subset C_G$, such that each $(M_S, \mu_S) \in C$ satisfies $\mu_S \sim_G \mu$, we would like to make sense of a sum

$$\sum_{(M_S, \mu_S) \in C} [M_S, \mu_S].$$

This makes sense as a map $\text{Groth}(G(Q_p)) \to \text{Groth}(G(Q_p) \times W_E)$ where $W_E = \bigcap_{(M_S, \mu_S) \in C} W_{E_{(\mu_S)M_S}}$. However, for our purposes, we would like to understand when we can extend the image of this map to a representation in $\text{Groth}(G(Q_p) \times W_{E_{(\mu)G}})$.

Lemma 3.3.5. Fix a pair $(G, \mu) \in C_G$. Consider a finite subset $C \subset C_G$ such that if $(M_S, \mu_S) \in C$ then $\mu_S \sim_G \mu$. Furthermore, suppose that for each $\gamma \in W_{E_{(\mu)G}}$ and element $(M_S, \mu_S) \in C$, we have $(M_S, \gamma(\mu_S)) \in C$. Then

$$\sum_{(M_S, \mu_S) \in C} [M_S, \mu_S],$$

is a map

$$\text{Groth}(G(Q_p)) \to \text{Groth}(G(Q_p) \times W_{E_{(\mu)G}})$$

in a natural way.
Proof: Our construction is analogous to that of \[3.2.3\] We fix \( \rho \in \text{Groth}(G(\mathbb{Q}_p)) \) and give

\[ V_C = \bigoplus_{(M_S, \mu_S) \in C} [M_S, \mu_S](\rho), \]

the structure of a \( G(\mathbb{Q}_p) \times W_{E_{(\mu)}} \)-representation. Suppose that \( C = C_1 \cdots C_n \) where each \( C_i \) is a single \( W_{E_{(\mu)}} \)-orbit. Then for each \( i \), we give

\[ V_{C_i} = \bigoplus_{(M_S, \mu_S) \in C_i} [M_S, \mu_S](\rho), \]

the structure of a \( G(\mathbb{Q}_p) \times W_{E_{(\mu)}} \)-representation and then define the \( G(\mathbb{Q}_p) \times W_{E_{(\mu)}} \)-structure on \( V_C \) to be the direct sum of the \( V_{C_i} \).

Suppose now that \( C \) contains a single \( W_{E_{(\mu)}} \) orbit. In this case, we will show that

\[ \bigoplus_{(M_S, \mu_S) \in C} [M_S, \mu_S](\rho), \]

can be given the structure of a \( \text{Groth}(G(\mathbb{Q}_p) \times W_{E_{(\mu)}}) \) representation equal to

\[ [\text{Ind}_{\tilde{E}}(\text{Jac}_{\tilde{E}}(\rho))] \cdot [r \circ L\text{L}(\text{Jac}_{\tilde{E}}(\rho))] \cdot [\text{Ind}_{\tilde{E}}(\tilde{M}_S \times W_{E_{(\mu)}}) \cdot \text{Ind}_{\tilde{E}}(\tilde{M}_S \times W_{E_{(\mu)}})], \]

where \( r \) is the induced representation (not parabolic induction) given by

\[ \text{Ind}_{\tilde{E}}(\tilde{M}_S \times W_{E_{(\mu)}}) \cdot \text{Ind}_{\tilde{E}}(r_{-\mu_S}), \]

for a fixed choice of \( (M_S, \mu_S) \in C \). The isomorphism class of \( r \) will not depend on this choice.

We study the representation \( r \). Fix representatives \( \gamma_1, ..., \gamma_k \in W_{E_{(\mu)}} / W_{E_{(\mu)}} \) so that \( \gamma_1 = 1 \). Then \( r \) is defined to be the sum of \( k \) copies of \( r_{-\mu_S} \) indexed by the \( \gamma_i \) and acted on by \( W_{E_{(\mu)}} \) in the standard way. We check that the i\( i \)th copy of \( r_{-\mu_S} \) is a representation of \( \tilde{M}_S \times W_{E_{(\gamma_i(\mu_S))}} \) and isomorphic to \( r_{-\gamma_i(\mu_S)} \). Let \( V_i \) be the underlying vector space of the \( i \)th copy of \( r_{-\mu_S} \). Then \( V_i \) is naturally a representation of \( \tilde{M}_S \times \gamma_i W_{E_{(\mu_S)}} \). Let \( \tilde{V}_i = \tilde{M}_S \times W_{E_{(\gamma_i(\mu_S))}} \).

Now suppose \( v \in V_i \) is a weight vector of \( \tilde{T} \subset \tilde{M}_S \) of weight \( \mu' \). Then we show that \( (1, \gamma_i)v \in V_i \) has weight \( \gamma_i(\mu') \). After all, for \( t \in \tilde{T} \), we have

\[ r((t, 1))(1, \gamma_i)v = (t, \gamma_i)v \]

\[ = (1, \gamma_i)(\gamma_i^{-1}(t), 1)v \]

\[ = (1, \gamma_i)r_{-\mu_S}((\gamma_i^{-1}(t), 1))(v) \]

\[ = (1, \gamma_i)(\mu' \gamma_i^{-1}(t)v) \]

\[ = \gamma_i(\mu')(t, 1)v. \]

In particular, we have shown that \( V_i \) is irreducible of extreme weight \( -\gamma_i(\mu_S) \) as an \( \tilde{M}_S \)-representation (since \( r_{-\mu_S} \) is irreducible of extreme weight \( -\mu_S \) as an \( \tilde{M}_S \)-representation). It is a simple check similar to the above that \( W_{E_{(\gamma_i(\mu_S))}} \) acts trivially on the highest weight space of \( V_i \). This proves that \( V_i \) is isomorphic to \( r_{-\gamma_i(\mu_S)} \).
In particular, this shows that we can give
\[ \bigoplus_{\gamma \in W_{E(\nu)|G}/W_{E(\nu)|M_S}} \rho_{-\gamma_1(\mu_S)} \circ LL(Jac^G_{P_S}(\rho))|_{W_{E(\nu)|M_S}}, \]
the structure of a $W_{E(\nu)|G}$ representation isomorphic to
\[ \rho \circ LL(Jac^G_{P_S}(\rho))|_{W_{E(\nu)|G}}. \]
To conclude the proof, we just need to check that the $| \cdot |$ twists on each $[M_S, \gamma_i(\mu_S)]$-term are the same. This follows because $\rho_G$ and $\rho_{M_S}$ are both invariant by $W_{E(\nu)|G}$.

We write $[M_{G,b,\mu}]$ for the map of representations given by replacing each term in the formal sum for $M_{G,b,\mu}$ by the corresponding map of representations. We would like to check the following:

**Lemma 3.3.6.** The sum $M_{G,b,\mu}$ gives a map
\[ [M_{G,b,\mu}] : \text{Groth}(G(\mathbb{Q}_p)) \to \text{Groth}(G(\mathbb{Q}_p) \times W_{E(\nu)|G}). \]

**Proof.** By Theorem 2.5.5, it suffices to show that $M_{G,b,\mu}$ is invariant under the natural action of $W_{E(\nu)|G}$ on $\mathbb{Z}(\mathcal{C}_G)$. Pick $\gamma \in W_{E(\nu)|G}$. Since the action of $\gamma$ on a cocharacter pair fixes the standard Levi subgroup in the first factor, signs will not be an issue and we will be done if we can check that $R_{G,b,\mu}$ is $\gamma$-invariant. But if $(M_b, \mu_b) \in \mathcal{T}_{G,b,\mu}$ then it is a consequence of the definition of $\mathcal{T}$ that so is $(M_b, \gamma(\mu_b))$. Furthermore, if $(M_S, \mu_S) \leq (M_b, \mu_b)$ then $(M_S, \gamma(\mu_S)) \leq (M_b, \gamma(\mu_b))$ by definition of the partial order relation (remarking that $\theta_{M_S}(\mu_S) = \theta_{M_S}(\gamma(\mu_S))$). This shows that $R_{G,b,\mu}$ is $\gamma$-invariant as desired.

If we combine the previous lemma with 2.5.9 and 3.3.4 we get
\[ \bigoplus_{b=1}^{\infty} [M_{G,b,\mu}] = [M_{G,b,\mu}]. \]

We now prove the key result of this section which provides the connection between Mant and cocharacter pairs.

**Theorem 3.3.7.** Assume that Theorem 3.1.2 holds for all admissible representations of $\text{Groth}(G(\mathbb{Q}_p))$ and that the Harris-Viehmann conjecture is true for the general linear groups we consider. Then we have the following equality of morphisms $\text{Groth}(G(\mathbb{Q}_p)) \to \text{Groth}(G(\mathbb{Q}_p) \times W_{E(\nu)|G})$:
\[ \text{Mant}_{G,b,\mu} \circ \text{Red}_b = [M_{G,b,\mu}]. \]

See the remark after the proof for a discussion of which of the conditions we can remove in the essentially square integrable case.

**Proof.** We prove this result by induction on the rank of $X_{\phi}(T)$.

If the rank of $X_{\phi}(T)$ is 1, then $B(G, \mu)$ is a singleton and so the result follows from Theorem 3.1.2.

Suppose the result holds for all non-basic $b \in B(G, \mu)$ with $\text{Rk}(X_{\phi}(T)) \leq r$. Then by Theorem 3.1.2 and 2.5.4, the result holds for all $b \in B(G, \mu)$ with $\text{Rk}(X_{\phi}(T)) \leq r$.

Finally, suppose the result holds for all $b \in B(G, \mu)$ with $\text{Rk}(X_{\phi}(T)) \leq r$. Then suppose $X_{\phi}(T)$ has rank $r + 1$ and choose $b \in B(G, \mu)$ such that $b$ is not basic. By the Harris-Viehmann formula,
\[ \text{Mant}_{G,b,\mu} \circ \text{Red}_b. \]
By inductive assumption we get
\[ \sum_{(M_b, \mu_b) \in \mathcal{M}_{G, b}} \left( \text{Ind}^G_{\rho_b} \circ [\mathcal{M}_{M_b, b', \mu_b}] \right) \left( \text{Red}_{\rho_b} \circ \text{Jac}^G_{\rho_b} \right) \otimes [1][\cdot \langle \rho_G, \mu_b \rangle, \langle \rho_G, \mu_b' \rangle]. \]
and now by equation 9
\[ \sum_{(M_b, \mu_b) \in \mathcal{M}_{G, b}} \left( \text{Ind}^G_{\rho_b} \circ [\mathcal{M}_{M_b, b', \mu_b}] \right) \left( \text{Jac}^G_{\rho_b} \right) \otimes [1][\cdot \langle \rho_G, \mu_b \rangle, \langle \rho_G, \mu_b' \rangle], \]
and so by 3.3.8 and 3.3.8
\[ = \left[ \mathcal{M}_{G, b, \mu} \right]. \]
We must check that the $W_{E(\nu)G}$ structure coming from 3.3.8 is compatible with that of 3.3.5. Pick $\rho \in \text{Groth}(G(\mathbb{Q}_p))$. By inductive assumption and 3.3.3 for each $(M_b, \mu_b) \in \mathcal{I}_{G, b}$, the $W_{E(\nu)G}$ structures on
\[ (\text{Ind}^G_{\rho_b} \circ \text{Mant}_{M_b, b', \mu_b} \circ \text{Red}_{\rho_b} \circ \text{Jac}^G_{\rho_b})(\rho) \otimes [1][\cdot \langle \rho_G, \mu_b \rangle, \langle \rho_G, \mu_b' \rangle], \]
and
\[ \iota^G_{M_b, \mu_b}([\mathcal{M}_{M_b, b', \mu_b}]) \rho \]
are the same. Thus by 3.2.3 the $W_{E(\nu)G}$ structure on $\text{Mant}_{G, b, \mu}(\text{Red}_b(\rho))$ is a direct sum over the $W_{E(\nu)G}$-orbits of $\mathcal{I}_{G, b}$ of induced representations of the form
\[ \text{Ind}_{W_{E(\nu)G}} \iota_{M_b}^G([\mathcal{M}_{M_b, b', \mu_b}]) \rho. \]
This $W_{E(\nu)G}$ structure matches the one on $\mathcal{M}_{G, b, \mu}$ (coming from 3.3.5) by the transitivity of the induced representation construction (see 3.2.3 for instance).

Remark 3.3.8. To conclude the proof of theorem 1.0.3 we need to show that if we restrict ourselves to the essentially square integrable representations $\text{Irr}^2(G(\mathbb{Q}_p)) \subseteq \text{Groth}(G(\mathbb{Q}_p))$, then we can remove the first assumption. In particular, these representations are accessible, so we have 3.1.2 unconditionally. In the above proof we need only observe that the Jacquet module $J^G_M(\rho)$ is a sum of essentially square integrable representations for $\rho \in \text{Irr}^2(G(\mathbb{Q}_p))$. Thus, to get the result for $\text{Irr}^2(G(\mathbb{Q}_p))$ by induction, our inductive assumption need only hold for all $\text{Irr}^2(G(\mathbb{Q}_p))$ for $rkG' < rkG$. This shows that under the condition that the Harris-Viehmann conjecture is true in the cases we consider, the theorem is true for essentially square integrable representations without any other assumptions.
4. Harris’s Generalization of the Kottwitz Conjecture (proof of Theorem 1.5)

In this section, we discuss an explicit computation using the above results. In particular, we prove that Shin’s formula for all admissible representations combined with the Harris-Viehmann conjecture proves Harris’s conjecture for the general linear groups considered in section 3. This conjecture is distinct from the Harris-Viehmann conjecture and is \[\text{[Har01, Conj 5.4]}\].

We begin by discussing the Kottwitz conjecture, which appears as \[\text{[Shi12, Cor 7.7]}\] in the cases we consider, and more generally as \[\text{[RV14, Conj 7.3]}\].

Fix \(G\) as in section 3 of this paper and a cocharacter pair \((G, \mu)\) such that \(\mu\) is minuscule. Let \(b \in B(G, \mu)\) be the unique basic element.

Now, consider \(\rho\) a representation of \(J_b(\mathbb{Q}_p)\) such that \(JL(\rho)\) is a supercuspidal representation of \(G(\mathbb{Q}_p)\). Then

\[
\text{Mant}_{G,b,\mu}(\text{Red}_b(JL(\rho))) = \text{Mant}_{G,b,\mu}(\rho),
\]

but by theorem 3.3.7 the lefthand side equals \([M_{G,b,\mu}](JL(\rho))\).

Now we see that since \(JL(\rho)\) is supercuspidal, each term of the form \([M_S, \mu_S](JL(\rho))\) is 0 when \(M_S\) is a proper Levi subgroup of \(G\). Thus,

\[
\text{Mant}_{G,b,\mu}(\rho) = [M_{G,b,\mu}](JL(\rho)) = [JL(\rho)](r_\mu \circ LL(\rho)) \cdot [\rho_{G,b,\mu}].
\]

This result is the Kottwitz conjecture for \(G\). Alternatively, if \(b \in B(G, \mu)\) is not basic, then no cocharacter pairs with \(G\) as the Levi subgroup will appear in \(M_{G,b,\mu}\) and so

\[
\text{Mant}_{G,b,\mu}(\rho) = 0.
\]

Of course, these results are already known by \[\text{[Shi12]}\], but we review them as motivation for Harris’s conjecture.

We begin with the following useful definition.

**Definition 4.0.1.** Fix \((G, \mu) \in C_G\) and \(b \in B(G, \mu)\). Let \(M_S\) be a standard Levi subgroup such that \(M_S \subset M_b\). We define the subset \(\text{Rel}_{M_S, b}^{G, \mu} \subset C_G\) as the set

\[
\{(M_S, \mu_S) \in C_G : \exists (M_b, \mu_b) \in T_{G,b,\mu} \text{ with } \theta_{M_b}(\mu_b) = \theta_{M_S}(\mu_S), \mu_b \sim_{M_b} M_S\}.
\]

The notation \(\mu_S \sim_{M_b} \mu_b\) is defined to mean that \(\mu_S\) and \(\mu_b\) are conjugate in \(M_b\).

Note that we do not require \((M_S, \mu_S) \leq (G, \mu)\) or \((M_S, \mu_S) \leq (M_b, \mu_b)\).

We record the following useful properties of \(\text{Rel}_{M_S, b}^{G, \mu}\).

**Lemma 4.0.2.** We use the same notation as in the previous definition. Then

\[
\text{Rel}_{M_S, b}^{G, \mu} = \prod_{(M_b, \mu_b) \in I_{M_b, b}^{G, \mu}} \text{Rel}_{M_S, b}^{M_b, \mu_b}.
\]

**Proof.** If \((M_S, \mu_S) \in \text{Rel}_{M_S, b}^{G, \mu}\), then there is an \((M_b, \mu_b) \in T_{G,b,\mu}\) such that \(\theta_{M_b}(\mu_b) = \theta_M(M_S)\) and \(\mu_S \sim_{M_b} \mu_b\). Then by 2.5.7 there is a unique \((M_b, \mu') \in I_{M_b, b}^{G, \mu}\) such that \((M_b, \mu_b) \in T_{M_b, b'}\) and so \((M_S, \mu_S) \in \text{Rel}_{M_S, b'}^{M_b, \mu_b}\). The reverse inclusion is analogous.

**Lemma 4.0.3.** The set \(\text{Rel}_{M_S, b}^{G, \mu}\) is invariant under the action of \(W_{E(\theta)G}\).
Assumption. Of 3.3.5. Mant conjecture (3.2.1). We will first assume that $G$ acquires the structure of a supercuspidal representation, the following representations are equal in Groth($G(\mathbb{Q}_p) \times W_{E(\nu)|G}$):

$$\text{Mant}_{G,b,\mu}(LJ(\delta^{^{1/2}}_{G,P_b}I_{M_S}^b(\rho)))$$

and

$$[I_{M_S}^G(\rho)] \left[ \bigoplus_{(M_S,\mu_S) \in \text{Rel}^G_{M_S,b}} r_{-\mu_S} \circ LL(\rho)|_{W_{E(\nu)|M_S}} \cdot |^{-\langle \rho_G, \mu \rangle} \right].$$

where $r_{-\mu_S}$ is a representation of $\tilde{M}_S \times W_{E(\nu)|M_S}$ but the righthand side naturally acquires the structure of a $G(\mathbb{Q}_p) \times W_{E(\nu)|G}$ representation from 4.0.3 and the proof of 3.3.6.

In particular, for $b$ basic, this says that

$$\text{Mant}_{G,b,\mu}(LJ(I_{M_S}^G(\rho))) = [I_{M_S}^G(\rho)] \left[ \bigoplus_{(M_S,\mu_S) \in \text{Rel}^G_{M_S,b}} r_{-\mu_S} \circ LL(\rho)|_{W_{E(\nu)|M_S}} \cdot |^{-\langle \rho_G, \mu \rangle} \right].$$

We will prove this conjecture assuming that Shin’s formula (theorem 3.1.2 of this paper) holds for all admissible representations.

We proceed by induction on the rank of $T$. The key observation will be that Harris’s conjecture is compatible with the Harris-Viehmann conjecture and Shin’s formula. We will first assume that $I_{M_S}^G(\rho)$ is irreducible and later remove this assumption.

The following proposition shows that 4.0.4 is compatible with the Harris-Viehmann conjecture 3.2.1.

**Proposition 4.0.5.** Fix $b \in \mathcal{B}(G, \mu)$ non-basic and fix a standard Levi subgroup $M_S \subset M_b$. Pick $\rho \in \text{Groth}(M_S(\mathbb{Q}_p))$ and suppose that $I_{M_S}^G(\rho)$ is irreducible. Suppose that conjecture 4.0.4 for $\rho$ holds for Mant$_{M_b,\nu,\mu_b}$ for each $(M_b,\mu_b) \in I_{M_b,b'}^G$. Then 4.0.4 holds for Mant$_{G,b,\mu}$.

**Proof.** We compute

$$\text{Mant}_{G,b,\mu}(LJ(\delta^{^{1/2}}_{G,P_b}I_{M_S}^b(\rho))) = \sum_{(M_b,\mu_b) \in I_{M_b,b'}^G} \text{Ind}_{P_b}^G(\text{Mant}_{M_b,\nu,\mu_b}(LJ(\delta^{^{1/2}}_{G,P_b}I_{M_S}^b(\rho)))) \otimes [1]|\cdot|^{-\langle \rho_G, \mu_b \rangle},$$

so by assumption

$$= \sum_{(M_b,\mu_b) \in I_{M_b,b'}^G} \left[ \text{Ind}_{P_b}^G \delta^{^{1/2}}_{G,P_b}I_{M_S}^b(\rho) \right] \left[ \bigoplus_{(M_S,\mu_S) \in \text{Rel}^G_{M_S,b'}} r_{-\mu_S} \circ LL(I_{M_S}^G(\rho))|_{W_{E(\nu)|M_S}} \cdot |^{-\langle \rho_G, \mu \rangle} \right].$$

Equipped with the above definition, we can now make the following restatement and slight generalization of [Har01, Conj 5.4] for the $G$ that we consider.

**Conjecture 4.0.4 (Harris).** Fix $a,b \in \mathcal{B}(G, \mu)$ and a standard Levi subgroup $M_S \subset M_b$. Then for $\rho \in \text{Groth}(M_S(\mathbb{Q}_p))$ a supercuspidal representation, the following representations are equal in Groth($G(\mathbb{Q}_p) \times W_{E(\nu)|G}$):

$$\text{Mant}_{G,a,b,\mu}(LJ(\delta^{^{1/2}}_{G,P_a}I_{M_S}^b(\rho)))$$

and

$$[I_{M_S}^G(\rho)] \left[ \bigoplus_{(M_S,\mu_S) \in \text{Rel}^G_{M_S,a,b}} r_{-\mu_S} \circ LL(\rho)|_{W_{E(\nu)|M_S}} \cdot |^{-\langle \rho_G, \mu \rangle} \right].$$

This finishes the proof.

We compute

$$\text{Mant}_{G,b,\mu}(LJ(I_{M_S}^G(\rho))) = [I_{M_S}^G(\rho)] \left[ \bigoplus_{(M_S,\mu_S) \in \text{Rel}^G_{M_S,b}} r_{-\mu_S} \circ LL(\rho)|_{W_{E(\nu)|M_S}} \cdot |^{-\langle \rho_G, \mu \rangle} \right].$$

We will prove this conjecture assuming that Shin’s formula (theorem 3.1.2 of this paper) holds for all admissible representations.

We proceed by induction on the rank of $T$. The key observation will be that Harris’s conjecture is compatible with the Harris-Viehmann conjecture and Shin’s formula. We will first assume that $I_{M_S}^G(\rho)$ is irreducible and later remove this assumption.

The following proposition shows that 4.0.4 is compatible with the Harris-Viehmann conjecture 3.2.1.
where $S = -\langle \rho_{M_b}, \mu_b \rangle + \langle \rho_G, \mu_b - \rho \rangle - \langle \text{det}(\text{Ad}_{M_b}(\mu_b)), \mu_b \rangle = -\langle \rho_G, \mu \rangle$ (following the discussion in §3.3.2). Now simplifying the above expression, we get

$$= \sum_{(M_b, \mu_b) \in T^{\mu_b}_{M_b, b'}} [T^G_{M_b}(\rho)] \left[ \bigoplus_{(M_S, \mu_S) \in \text{Re}^{M_b}_{M_b, b'}} r_{-\mu_S} \circ LL(I^G_M(\rho))|_{W_{E(\rho)^G}} \right].$$

Thus, we are reduced to showing that

$$\text{Rel}^G_{M_b, b} = \bigcap_{(M_b, \mu_b) \in T^{\mu_b}_{M_b, b'}} \text{Re}^{M_b}_{M_b, b'}.$$

This is just 4.0.2.

With 4.0.5 in hand, it remains to show that if 4.0.4 holds for all non-basic $b \in B(G, \mu)$ then it holds for the basic $b$. The key to proving this is theorem 3.1.2.

We begin by making some observations about $r_{-\mu}$. Since we assumed $I^G_{M_b}(\rho)$ is irreducible, we have $LL(I^G_{M_b}(\rho)) = LL(\rho)$ and the image of this representation lies inside $E_M \subset E_G$. Thus, the term $[r_{-\mu} \circ LL(I^G_{M_b}(\rho))|_{W_{E(\rho)^G}}]$ depends only on the restriction $r_{-\mu}|_{\tilde{M}_S \times W_{E(\rho)^G}}$. Since $\mu$ is assumed to be minuscule, we have the following equality of $\tilde{M}_S$ representations.

$$r_{-\mu}|_{\tilde{M}_S} = \bigoplus_{(M_S, \mu_S) \in C_G : \mu_S \sim_G \mu} r_{-\mu_S}|_{\tilde{M}_S}.$$

We further note that each $r_{-\mu_S}$ is a representation of $\tilde{M}_S \times W_{E(\rho)^G}$. Since $\{(M_S, \mu_S) \in C_G : \mu_S \sim_G \mu\}$ is invariant under the natural action of $W_{E(\rho)^G}$, it follows from the proof of 3.3.3 that the right-hand side of the above equation can be promoted to a representation of $\tilde{M}_S \times W_{E(\rho)^G}$ so that 10 is an equality of $W_{E(\rho)^G}$ representations.

Now we recall the following subsets of $W_{\text{rel}}$ defined in §2.11 of [BZ77].

**Definition 4.0.6.** Let $M_S, N_S$ be standard Levi subgroups of $G$. We define

$$W^{M_S} = \{w \in W_{\text{rel}} : w(M_S \cap B) \subset B\},$$

$$W^{M_S, N_S} = \{w \in W_{\text{rel}} : w(M_S \cap B) \subset B, w^{-1}(N_S \cap B) \subset B\}.$$

We record the following lemma:

**Lemma 4.0.7.** [BZ77] Lem 2.11 Suppose $M_S, N_S$ are standard Levi subgroups of $G$ and $w \in W^{M_S, N_S}$. Then $w(M_S) \cap N_S$ and $w^{-1}(N_S) \cap M_S$ are standard Levi subgroups.

**Lemma 4.0.8.** Suppose $M_S$ is a standard Levi subgroup of $G$. Then $W^{M_S}$ contains a unique representative of each left coset of $W_{\text{rel}}^{M_S}$. Equivalently, $(W^{M_S})^{-1}$ contains a unique representative of each right coset of $W_{\text{rel}}^{M_S}$.

**Proof.** Suppose $w \in W_{\text{rel}}$. Then $B' = w^{-1}(B)$ is a Borel subgroup of $G$ containing the maximal torus $T$. Since $B'$ contains exactly one of each root and its negative, $B' \cap M_S$ is a Borel subgroup of $M_S$. In particular, since $B' \cap M_S, B \cap M_S$ are both Borel subgroups of $M_S$ containing $T$, there exists a $w_m \in W^{M_S}_{\text{rel}}$ so that

$$w_m(B \cap M_S) = B' \cap M_S.$$
Then \( w w_m(B \cap M_S) = B \cap M_S \subset B \), so that \( w w_m \in W^{M_S} \). Thus the coset \( w W^{rel}_{M_S} \)
contains at least one element of \( W^{M_S} \).

Suppose \( w w_m, w w'_m \in W^{rel}_{M_S} \cap W^{M_S} \). In particular, \( w w'_m = (w w_m)(w_m^{-1} w'_m) \). But \( w w_m \) takes all positive roots of \( M_S \) to positive roots of \( G \), and equivalently, negative roots of \( M_S \) to negative roots of \( G \). Thus, if \( w_m^{-1} w'_m \) takes any positive root of \( M_S \) to a negative root of \( M_S \), then \( w w'_m \) cannot be an element of \( W^{M_S} \). In particular, this implies that \( w_m^{-1} w'_m = 1 \) which shows uniqueness.

**Lemma 4.0.9.** Suppose \( M_S \) is a standard Levi subgroup of \( G \) and \( x \in \mathfrak{A}^+_\mathbb{Q}, M_S \) and \( w \in W^{rel} \). Then \( w(x) = x \) if and only if \( w \in W^{rel}_{M_S} \).

**Proof.** Recall that by assumption, \( G \) is quasi-split over \( \mathbb{Q}_p \) and \( A \) is a split torus of \( G \) of maximal rank. Pick \( g \in N_G(A)(\mathbb{Q}_p) \) so that \( g \) projects to \( w \in W^{rel} = N_G(A)(\mathbb{Q}_p)/Z_G(A)(\mathbb{Q}_p) \). Then the equation \( w(x) = x \) implies that \( g \in Z_G(x)(\mathbb{Q}_p) \).

This centralizer is a Levi subgroup, and since \( x \in \mathfrak{A}^+_\mathbb{Q}, M_S \), we have \( Z_G(x) = M_S \). In particular, \( g \in N_{M_S}(A)(\mathbb{Q}_p) \) and so \( w \in W^{rel}_{M_S} \).

We remark that strictly speaking, \( x \) is not a cocharacter, but that \( Z_G(x) \) still makes sense as there is an induced action of \( G \) on \( X_s(A) \mathbb{Q} \).

We can now prove the following key proposition.

**Proposition 4.0.10.** Fix \((G, \mu) \in \mathcal{C}_G \) and suppose \((M_S, \mu_S) \in \mathcal{C}_G \) satisfies \( \mu_S \sim_G \mu \). Then there exists a unique \( b \in \mathcal{B}(G, \mu) \) and a unique \( w \in W^{M_S,M_b} \) so that \((w(M_S), w(\mu_S)) \in \text{Re}^{G,\mu}_{w(M_S),b} \).

**Proof.** We note that \( W^{M_S,M_b} \subset W^{rel} \) acts on \( X_s(T) \) by \( \mathcal{B} \).

We prove uniqueness first. By assumption, \( w \) must map \( M_S \) to a standard Levi subgroup \( w(M_S) \). This induces an equality \( W^{rel}_{M_S} w^{-1} = W^{rel}_{w(M_S)} \). In particular, it follows that \( w(\theta_{M_S}(\mu_S)) = \theta_{w(M_S)}(w(\mu_S)) \).

Since \((w(M_S), w(\mu_S)) \in \text{Re}^{G,\mu}_{w(M_S),b} \), it follows that \( \theta_{w(M_S)}(w(\mu_S)) \) is dominant in the relative roots system. In particular, \( \theta_{w(M_S)}(w(\mu_S)) \) must be equal to the unique element \( x \) in the \( W^{rel} \) orbit of \( \theta_{M_S}(\mu_S) \) which is dominant in \( \mathfrak{A}. \) Now \( x \in \mathfrak{A}^+_b, \mathbb{Q} \) for a unique \( M_{S'} \). Since any \((M_b, \mu) \in T_G(b, \mu) \) is definitionally strictly decreasing, it follows that even though we can’t yet conclude the uniqueness of \( b \), we have shown that any other \( b_1 \) must satisfy \( M_{b_1} = M_b = M_{S'} \).

Now, suppose we had \( w, w' \in W^{M_S,M_b} \) such that \( w(\theta_{M_S}(\mu_S)) = x = w'(\theta_{M_S}(\mu_S)) \).

Then in particular, \( w' w^{-1} \) stabilizes \( x \) and so by 4.0.9 \( w' w^{-1} \in W^{rel}_{M_b} \). So \( w \) and \( w' \) are in the same right coset \( W^{rel}_{M_b} \). However, \( W^{M_S,M_b} \subset (W^{M_b})^{-1} \). By lemma 4.0.8 \((W^{M_b})^{-1} \) contains a unique representative of each right coset of \( (W^{M_b})^{-1} \) and so there is a unique \( w \in (W^{M_b})^{-1} \) satisfying \( w(\theta_{M_S}(\mu_S)) = x \). In particular, this implies that \( w = w' \). Thus, we have shown that \( w \) is unique, if it exists. There exists exactly one \( \mu' \in X_s(T) \) such that \( \mu' \sim_{M_b} w(\mu) \) and \( \mu' \) is dominant in \( M_b \). Then \((M_b, \mu') \in T_G(b, \mu) \) for at most one \( b \in \mathcal{B}(G, \mu) \). This shows uniqueness.

To prove existence, we again define \( x \) to be the unique dominant element in the \( W^{rel} \)-orbit of \( \theta_{M_S}(\mu_S) \). Define \( M_{S'} = Z_G(x) \) and take the unique \( w \in (W^{M_{S'}})^{-1} \) such that \( w(\theta_{M_S}(\mu_S)) = x \). We would like to show that \( w \in W^{M_S,M_{S'}} \).
By definition,
\[ w(M_S) \subset w(Z_G(\theta_{M_S}(\mu_S))) = Z_G(x) = M_{S'} . \]
Suppose it is not the case that \( w(M_S \cap B) \subset B \). In particular, \( w \) maps a positive root \( r \) of \( M_S \) to a root \( w(r) \) of \( M_{S'} \) which is not positive. In particular, \( -w(r) \) is positive and so \( w^{-1}(-w(r)) = -r \) is positive (since \( w \in (W^{M_{S'}})^{-1} \)). But this is clearly a contradiction. Thus, in fact \( w \in W^{M_S,M_{S'}} \).

By \ref{thm:1.2} \( w(M_S) \cap M_{S'} = w(M_S) \) is a standard Levi. It remains to show that the previous expression is equal to \( \mathcal{C}_G \). For \( b \in B(G,\mu) \), define \( W_b \) by \( \{ w \in W^{M_{S'},M_b} : w(M_S) \subset M_b \} \). Then the previous lemma gives a bijection
\[ \{ (M_S,\mu_S) \in \mathcal{C}_G : \mu_S \sim_G \mu \} \cong \bigoplus_{b \in B(G,\mu)} \bigoplus_{w \in W_b} \text{Rel}^{G,\mu}_{w(M_S),b} . \]

**Proof.** By the construction in the previous proposition, it is clear that given an \( (M_S,\mu_S) \in \mathcal{C}_G \), we get an element of the right-hand side of the above equation. Conversely, an element \( (w(M_S),\mu_S) \) of the right-hand side comes with a fixed \( w \) in \( W_b \) and so we can recover \( (M_S, w^{-1}(\mu_S)) \) on the left-hand side. \qed

We are now ready to finish the proof of conjecture \ref{conj:4.0.4}. By inductive assumption we assume we’ve shown \ref{thm:4.0.4} for \( G \) with maximal torus of rank less than \( n \). Then proposition \ref{thm:4.0.8} implies that \ref{thm:4.0.4} holds for \( G \) with maximal torus of rank \( n \) in the case where \( b \) is not basic. It remains to prove the basic case, for which it suffices to show that theorem \ref{thm:3.1.2} is compatible with \ref{thm:4.0.4}. We have
\[ \sum_{b \in B(G,\mu)} \text{Mant}_{G,b,\mu}(\text{Red}_b(I^G_{M_S}(\rho))) \]
\[ = \sum_{b \in B(G,\mu)} \text{Mant}_{G,b,\mu}(LJ(\delta^{\frac{1}{2}}_{P_b} \otimes J^G_{M_S}(\rho))). \]
By the geometric lemma of \cite{BZ77}, we have
\[ J^G_{M_S}I^G_{M_S}(\rho) = \sum_{w \in W^{M_S,M_b}} I^M_{M_b}(w(J^M_{M_S}(\rho))), \]
where \( M'_S = M_S \cap w^{-1}(M_b), M'_b = w(M_S) \cap M_b \). By the assumption that \( \rho \) is supercuspidal we must have \( M'_S = M_S \) and \( M'_b = w(M_S) \). In this case, we have from the geometric lemma that \( w(M_S) \) is a standard Levi subgroup. Thus we get that the previous expression is equal to
\[ \sum_{b \in B(G,\mu)} \text{Mant}_{G,b,\mu}(\sum_{w \in W_b} LJ(\delta^{\frac{1}{2}}_{P_b} \otimes I^M_{w(M_S)}(w(\rho))), \]

**Corollary 4.0.11.** Fix a cocharacter pair \( (G,\mu) \in \mathcal{C}_G \) and a standard Levi subgroup \( M_S \) of \( G \). For \( b \in B(G,\mu) \), define \( W_b \) by \( \{ w \in W^{M_{S'},M_b} : w(M_S) \subset M_b \} \). Then the previous lemma gives a bijection
\[ \{ (M_S,\mu_S) \in \mathcal{C}_G : \mu_S \sim_G \mu \} \cong \bigoplus_{b \in B(G,\mu)} \bigoplus_{w \in W_b} \text{Rel}^{G,\mu}_{w(M_S),b} . \]
where \( W_b \subset W^{M_{S'}, M_b} \) is the subset of \( w \) such that \( w(M_S) \subset M_b \). We now apply 4.0.4 to get

\[
\sum_{b \in \text{B}i(G, \mu)} \sum_{w \in W_b} [I^G_{w(M_S)}(w(\rho))] \left[ \bigoplus_{(w(M_S), \mu') \in \text{Red}_{w(M_S), b}^{G, \mu}} r_{-\mu'} \circ LL(I^G_{w(M_S)}(w(\rho))) \, |w_{E(w(\mu'))}^{M_{S'}}(M_S) \cdot |-^{\langle \rho_G, \mu \rangle} \right].
\]

By \( \text{BZ77}, \text{Thm 2.9} \), we have that

\[
[I^G_{w(M_S)}(w(\rho))] = [I^G_{M_S}(\rho)],
\]

and since \( I^G_{M_S}(\rho) \) is assumed to be irreducible, we have

\[
LL(I^G_{M_S}(\rho)) = LL(\rho).
\]

Finally, we note that \( W_{E((w^{-1}(\mu')) M_S)} = W_{E(w(\mu'))} \) and we have an equality

\[
[r_{-\mu'} \circ LL(w(\rho))]_{w(M_S)} = [r_{-w^{-1}(\mu')} \circ LL(\rho)]_{w^{-1}(\mu')} M_S.
\]

Thus the above expression becomes

\[
\sum_{b \in \text{B}i(G, \mu)} \sum_{w \in W_b} [I^G_{M_S}(\rho)] \left[ \bigoplus_{(w(M_S), \mu') \in \text{Red}_{w(M_S), b}^{G, \mu}} r_{-w^{-1}(\mu')} \circ LL(\rho) \, |w_{E(w(\mu'))}^{M_{S'}}(M_S) \cdot |-^{\langle \rho_G, \mu \rangle} \right].
\]

By corollary 4.0.11 this equals

\[
[I^G_{M_S}(\rho)] \left[ \bigoplus_{(M_S, \mu_S) : \mu_S \sim \mu} r_{-\mu_S} \circ LL(\rho) \, |w_{E(w(\mu_S))}^{M_{S'}}(M_S) \cdot |-^{\langle \rho_G, \mu \rangle} \right].
\]

Finally, we apply the decomposition given by equation 10 to get

\[
[I^G_{M_S}(\rho)] [r_{-\mu_S} W_{E(w(\mu_S))}^{M_{S'}}(M_S) \circ LL(\rho) \, |w_{E(w(\mu_S))}^{M_{S'}}(M_S) \cdot |-^{\langle \rho_G, \mu \rangle}]
\]

which is the desired result.

Finally, we show that conjecture 4.0.4 holds even if \( I^G_{M_S}(\rho) \) is not irreducible. Our verification that conjecture 4.0.4 is compatible with the Harris-Viehmann conjecture did not rely on the irreducibility of \( I^G_{M_S}(\rho) \). Thus in the general case, it would suffice to show that 4.0.4 is true in the basic case. If \( b \) is basic, then \( M_b = G \) so we have

\[
\text{Mant}_{G, b, \mu}(LL(\delta_{G, P_b}^G, I^G_{M_S}(\rho))) = \text{Mant}_{G, b, \mu}(\text{Red}_{b}(I^G_{M_S}(\rho))).
\]

This can now be computed by cocharacter pairs using the results of section 3. If \( I^G_{M_S}(\rho) \) is assumed to be irreducible, then for each cocharacter pair \( (M_{S'}, \mu_{S'}) \) of \( G \), we have

\[
[M_{S'}, \mu_{S'}] I^G_{M_S}(\rho) = (\text{Ind}_{P_{S'}}^G \circ [\mu_{S'}])(\delta_{P_{S'}}^\chi \otimes J_{M_{S'}}^G(\rho)) \otimes [1][\cdot |-^{\langle \rho_{G, \mu_{S'}} - \mu \rangle}]
\]

\[
= (\text{Ind}_{P_{S'}}^G \circ [\mu_{S'}])(\bigoplus_{w \in W_{P_{S'}}} \delta_{P_{S'}}^\chi \otimes J_{M_{S'}}^{w(M_S)}(w(\rho))) \otimes [1][\cdot |-^{\langle \rho_{G, \mu_{S'}} - \mu \rangle}]
\]

where \( W_\rho \) is the subset of \( w \in W^{M_{S'}, M_{S'}} \) such that \( w(M_S) \subset M_{S'} \). Then the above equals

\[
[I^G_{M_S}(\rho)] \left[ \bigoplus_{w \in W_\rho} r_{-\mu_{S'}} \circ LL(w(\rho)) \cdot |-^{\langle \rho_G, \mu \rangle} \right].
\]

Thus we see that applying various \([M_{S'}, \mu_{S'}] \) to \( I^G_{M_S}(\rho) \) in the irreducible case will always yield the same term of Groth(G(Q_p)) (namely \([I^G_{M_S}(\rho)] \)) and so when
evaluating $\text{Mant}_{G,b,\mu}(\text{Red}_b(I_{G_M^G}(\rho)))$ as a sum of cocharacter pairs, the different Galois terms must cancel to give conjecture \[4.0.4\] Thus, if we can show that in the reducible case, the Groth$(G(Q_p))$ part of each $[M_{S'}, \mu_{S'}](I_{G_M^G}(\rho))$ is fixed and the Galois part is identical to the irreducible case, then conjecture \[4.0.4\] must hold for this case as well.

The first part of our previous computation did not depend on the irreducibility of $I_{M_S}(\rho)$ so we still have

$$[M_{S'}, \mu_{S'}](I_{M_S}(\rho)) = (\text{Ind}_{S'}^{G_P} \circ \mu_{S'})(\bigoplus_{w \in W_P} \delta_{\rho_{S'}(M_P)}(\rho_{S'}(w(\rho))) \otimes [1] \cdot [\rho_{G-M_S}(\mu_{S'})])$$

Suppose now that $I_{M_S}(\rho) = \pi_1 \oplus \ldots \oplus \pi_k$. Then

$$[\mu_{S'}](I_{w(M_S)}(\rho)) = \bigotimes_{i=1}^{\delta} \rho_{S'}(\rho_{S'}(w(\rho))) \otimes [1] \cdot \rho_{G-M_S}(\mu_{S'})$$

Thus, the expression for $[M_{S'}, \mu_{S'}](I_{M_S}(\rho))$ becomes

$$[I_{M_S}(\rho)] \bigoplus_{w \in W_{M_S,M_{S'}}} r_{M_{S'}}(\pi_1) \otimes LL(w(\rho)) \otimes [1] \cdot \rho_{G-M_S}(\mu_{S'})$$

as desired.

### Appendix A. Examples

In this section, we give an example to show that even in the unramified EL-type case, we do not get an expression as simple as Harris’s conjecture for $\text{Mant}_{G,b,\mu}(\rho)$ for general $\rho$. We generally use the same notation as in the computation in example \[3.2.3\]

Let $G = GL_4$, suppose $\mu$ has weights $(1^2, 0^2)$, and take $b$ basic. Let $T$ be the diagonal maximal torus and $B$ be the Borel subgroup of upper triangular matrices. Then the set of cocharacter pairs less than or equal to $(G, \mu)$ is as follows.

$$(GL_4, (1^2, 0^2))$$

$$GL_3 \times GL_3, (1^2, 0^2))$$

$$GL_2 \times GL_2, (1^2, 0^2))$$

Let $\rho \in \text{Groth}(GL_4(Q_p))$ and consider $\pi$ the unique essentially square integrable quotient of $I_{GL_4}(\rho \boxtimes \rho(1) \boxtimes \rho(2) \boxtimes \rho(3))$. We want to compute $\text{Mant}_{G,b,\mu}(\text{Red}_b(\pi))$.

We introduce some notation which will allow us to describe the answer to this question. The results of \[2\] of \[ZKNS\] show that $J_{GL_4}(\rho \boxtimes \rho(1) \boxtimes \rho(2) \boxtimes \rho(3))$ has exactly 8 irreducible subquotients. If $\pi'$ is one such subquotient, then $J_{GL_4}(\rho \boxtimes \rho(1) \boxtimes \rho(2) \boxtimes \rho(3))$ will be a finite sum of representations of the form $\rho(\lambda(0)) \boxtimes \rho(\lambda(1)) \boxtimes \rho(\lambda(2)) \boxtimes \rho(\lambda(3))$ where $\lambda$ is a permutation of $\{0, 1, 2, 3\}$. In particular, if $\Omega$ denotes the set of all
such permutations of $\rho \boxtimes \rho(1) \boxtimes \rho(2) \boxtimes \rho(3)$, then each permutation lies in the Jacquet module of exactly one irreducible subquotient of $I_{GL_3}^G(\rho \boxtimes \rho(1) \boxtimes \rho(2) \boxtimes \rho(3))$ so that the irreducible subquotients correspond to a partition of $\Omega$. We use the following shorthand: we define the notation $(0123)$ to refer to the representation $\rho(0) \boxtimes \rho(1) \boxtimes \rho(2) \boxtimes \rho(3)$. Following Zelevinsky, our 8 irreducible subquotients naturally correspond to vertices of a 3-dimensional cube, and so we denote them by binary strings of length 3. Then if we denote the subset of $\Omega$ corresponding to some subquotient $\rho'$ by $\Omega(\rho')$, we have

$$
\Omega[[111]] = \{(3210)\}
$$

$$
\Omega[[011]] = \{(2310), (2130), (2103)\}
$$

$$
\Omega[[101]] = \{(3120), (1320), (1302), (3102), (1032)\}
$$

$$
\Omega[[110]] = \{(3201), (3021), (0321)\}
$$

$$
\Omega[[001]] = \{(1203), (1023), (1230)\}
$$

$$
\Omega[[010]] = \{(2013), (2031), (0213), (0231), (2301)\}
$$

$$
\Omega[[100]] = \{(3012), (0312), (0132)\}
$$

$$
\Omega[[000]] = \{(0123)\}
$$

In particular, our representation $\pi$ corresponds to $[111]$ under the above notation. A tedious computation using theorem 3.3.7 yields the following

**Proposition A.0.1.**

$$
\text{Mant}_{G,b,\mu}(\text{Red}_b(\pi)) = [111][LL(\rho)^2(-4) + LL(\rho)^2(-3)]
$$

$$
- ([110][LL(\rho)^2(-5)] + [011][LL(\rho)^2(-5)])
$$

$$
+ [010][LL(\rho)^2(-6)]
$$

$$
- [000][LL(\rho)^2(-7)]
$$

We finish by remarking that the set of cocharacter pairs less than or equal to $(G, \mu)$ has some special properties in the above case that make the general case more complicated.

For instance, each $T_{G,b,\mu}$ has at most a single element. However, if $G$ has a nontrivial action by $\Gamma$, this need not be the case.

In the case we consider we have a single cocharacter pair for each Levi subgroup. In general, this need not be the case. For instance, if $G = GL_5, \mu = (1^3, 0^2)$, then $(GL_3 \times GL_2, (1^3)(0^2)), (GL_3 \times GL_2, (1^2, 0)(1, 0))$ are both less than $(G, \mu)$.

Further, in the above example, each cocharacter pair $(M_S, \mu_S)$ had the property that $\mu_S$ was dominant as a cocharacter of $G$ relative to $B$. In general this need not be the case. In fact, $(GL_3^1, (1)(1)(0)(1)) \not\leq (GL_5, (1^3, 0^2))$.

**Appendix B. Relative Root Systems and Weyl Chambers**

In this section we prove a fact about root systems that is needed in the text (for instance in the proof of 2.4.3). We assume that $G$ is a quasisplit group over a field $k$ of characteristic 0 and pick a separable closure $k^{sep}$. We fix a split $k$-torus $A$ of maximal rank in $G$ and choose a maximal torus $T$ and Borel subgroup $B$ both
defined over \( k \) and such that \( A \subset T \subset B \). Associated to this data, we have an absolute root datum

\[
(X^* (T), \Phi^* (G, T), X_u (T), \Phi_u (G, T)),
\]

and a relative root datum

\[
(X^* (A), \Phi^* (G, A), X_u (A), \Phi_u (G, A)).
\]

Our choice of \( B \) also gives sets \( \Delta \) of absolute simple roots and \( k \Delta \) of relative simple roots. Note that we also have a natural restriction map

\[
\text{res} : X^* (T) \to X^* (A),
\]

and that by definition an absolute root in \( \Phi^* (G, T) \) restricts to an element of \( \Phi^* (G, A) \cup \{0\} \).

We record two standard consequences of our assumption that \( G \) is quasisplit.

**Proposition B.0.1.** Let \( G \) be quasisplit and use the notations as above. Then,

1. The centralizer \( Z_G (A) = T \).
2. We have \( \text{res} (\Delta) = k \Delta \). The key point being that no absolute simple root restricts to the trivial character.

We have the following easy consequence on the structure of the Weyl group of the relative root system. Recall that the absolute Weyl group \( W \) equals

\[
N_G (T)(k^\text{sep}) / Z_G (T)(k^\text{sep}),
\]

and the relative Weyl group \( W^{\text{rel}} \) is \( N_G (A)(k) / Z_G (A)(k). \)

**Corollary B.0.2.** We have the following equality: \( W^{\text{rel}} = W^\Gamma \), where \( \Gamma = \text{Gal}(k^\text{sep} / k) \).

*Proof.* It suffices to show that \( Z_G (A) = Z_G (T) \) and that \( N_G (A)(k) = N_G (T)(k) \).

For the first equality, we note that by the quasisplit assumption, \( Z_G (A) = T = Z_G (T) \).

For the second equality, we note that any \( g \in N_G (A)(k) \) must also normalize the centralizer of \( A \) which is \( T \). Conversely, if \( g \in N_G (T)(k) \) then \( g \) normalizes the unique maximal \( k \)-split sub-torus of \( T \) which is \( A \).

\( \square \)

Define the absolute Weyl chamber \( \overline{C}_Q^* \subset X^* (T)_Q \) by \( \{ x \in X^* (T)_Q : \langle \alpha, x \rangle \geq 0, \alpha \in \Delta \} \) and define the relative Weyl chamber \( k \overline{C}_Q^* \subset X^* (A)_Q \) analogously. The key result of this section is that

\[
\text{res} (\overline{C}_Q^*) = k \overline{C}_Q^*.
\]

Despite its simple statement, the author has been unable to locate a convenient reference of this fact. For \( x \in X^* (T)_Q \) and \( \alpha \in \Delta \), we need to relate \( \langle \alpha, x \rangle \) and \( \langle \text{res} (\alpha), \text{res} (x) \rangle \). If we let \( \sigma_\alpha \in W \) be the transposition corresponding to the root \( \alpha \), then we have

\[
(11) \quad x - \sigma_\alpha (x) = \langle \alpha, x \rangle \alpha.
\]

and analogously for \( \text{res} (\alpha) \). Thus it will suffice to relate \( \sigma_\alpha \) and \( \sigma_{\text{res} (\alpha)} \).

Note that since \( B \) is defined over \( k \), we have \( \gamma (\Delta) = \Delta \) for every \( \gamma \in \Gamma \). Moreover, for each \( \alpha \in \Delta \), we have \( \text{res} (\gamma (\alpha)) = \text{res} (\alpha) \). After all, \( \Gamma \) acts trivially on \( X^* (A)_Q \) and the restriction map is \( \Gamma \)-equivariant.

Now fix \( \alpha \in \Delta \) and let \( W_\alpha \) be the subgroup of \( W \) generated by the elements \( \sigma_{\gamma (\alpha)} \) for each \( \gamma \in \Gamma \). We claim that if we can find a nontrivial \( \Gamma \)-invariant element of \( W_\alpha \), then it must equal \( \sigma_{\text{res} (\alpha)} \). To prove this, we first recall the construction of \( \sigma_\alpha \) and
σ_{\text{res}(\alpha)} (see [Bor91 pg 230]) for instance). Given a root \( \alpha \in \Phi^*(G, T) \) we can define a group \( G_{\alpha} = Z_G(T_\alpha) \) where \( T_\alpha = \ker(\alpha)^0 \subset T \). Then \( N_{G_\alpha}(T)(k^{sep})/Z_{G_\alpha}(T)(k^{sep}) \) embeds into \( W \) and has a unique nontrivial element which is \( \sigma_\alpha \). Analogously, we define \( A_{\text{res}(\alpha)} \) and \( G_{\text{res}(\alpha)} = Z_G(A_{\text{res}(\alpha)}) \). Then \( N_{G_{\text{res}(\alpha)}}(A)(k)/Z_{G_{\text{res}(\alpha)}}(A)(k) \) embeds into \( W^{\text{res}} \) and has a unique nontrivial element that is identified with \( \sigma_{\text{res}(\alpha)} \).

Now, by [B0.2] we have

\[
N_{G_{\text{res}(\alpha)}}(A)(k)/Z_{G_{\text{res}(\alpha)}}(A)(k) = N_{G_{\text{res}(\alpha)}}(T)(k)/Z_{G_{\text{res}(\alpha)}}(T)(k).
\]

Thus to complete the proof of the claim, we need to show that

\[
N_{G_\alpha}(T)(k^{sep})/Z_{G_\alpha}(T)(k^{sep}) \hookrightarrow N_{G_{\text{res}(\alpha)}}(T)(k^{sep})/Z_{G_{\text{res}(\alpha)}}(T)(k^{sep}).
\]

After all, the unique nontrivial \( \Gamma \)-invariant element of the group on the right is \( \sigma_{\text{res}(\alpha)} \) and the group on the left contains \( \sigma_\alpha \). Since we get the same equation if we replace \( \alpha \) everywhere with \( \gamma(\alpha) \), this will imply that

\[
W_\alpha \subset N_{G_{\text{res}(\alpha)}}(T)(k^{sep})/Z_{G_{\text{res}(\alpha)}}(T)(k^{sep}).
\]

Now, equation (12) follows from the fact that

\[
Z_{G_\alpha}(T) = Z_{G_{\text{res}(\alpha)}}(T) = T,
\]

and

\[
N_{G_\alpha}(T) \subset N_{G_{\text{res}(\alpha)}}(T).
\]

We are now interested in finding a nontrivial \( \Gamma \)-invariant element of the group \( W_\alpha \) defined above. In fact, \( W_\alpha \) will be a finite Coxeter group and the element we seek is the unique element of longest length. We need to compute this element explicitly, which we now do. We treat two cases. Suppose first that the elements of the \( \Gamma \)-orbit of \( \sigma_\alpha \) commute pairwise. Then clearly the product

\[
\prod_{\gamma \in \Gamma/\ker(\sigma_\alpha)} \sigma_{\gamma(\alpha)}
\]

is \( \Gamma \)-invariant.

In the second case, suppose that the \( \Gamma \)-orbit of \( \sigma_\alpha \) has precisely two elements which we denote \( X \) and \( Y \). Then we have \((XY)^k = 1 \) for some \( k \geq 2 \) which we assume to be minimal. If \( k \) is even, then \((XY)^{k/2} \) is invariant and nontrivial and if \( k \) is odd, then \((XY)^{(k-1)/2} \) is invariant and nontrivial.

We now prove that any \( \Gamma \) action on the simple roots \( \Delta \) of \( G \) is a combination of these cases. The action of \( \Gamma \) on \( \Delta \) induces an action on the associated (not necessarily connected) Dynkin diagram \( D \). Each \( \gamma \in \Gamma \) maps connected components of \( D \) to connected components and so there is an induced action of \( \Gamma \) on the set of connected components \( \pi_0(D) \).

Now fix an \( \alpha \in \Delta \) and consider the \( \Gamma \)-orbit \( \Gamma \alpha \) of \( \alpha \). Suppose \( D^i \) is a connected component of \( D \) such that \( D^i \cap \Gamma \alpha \neq \emptyset \). Then via the classification of connected Dynkin diagrams, we see that \( \Gamma \alpha \cap D^i \) contains either a single node, 2 non-adjacent nodes, 2 adjacent nodes, or 3 nodes where no two are adjacent. In particular, these are all covered by the cases we considered above, so we can find an element \( w_i \) of \( W_\alpha \) that is invariant by the action of \( \text{stab}(D^i) \subset \Gamma \). Then \( \Gamma \alpha \) consists of finitely many disjoint copies of one of the above possibilities and so we see that \( \prod_i w_i \) is \( \Gamma \)-invariant and an element of \( W_\alpha \) and therefore equal to \( \sigma_{\text{res}(\alpha)} \). Equipped with this description, we now give a proof of the main result of this section.
Proposition B.0.3. We continue to observe the assumptions made above. In particular, $G$ is a quasisplit group over $k$. Then the map $\text{res} : X^*(T) \to X^*(A)$ induces an equality

$$\text{res}(\overline{C}_Q^\alpha) = k \overline{C}_Q^\alpha.$$ 

Proof. We first show that $\text{res}(\overline{C}_Q^\alpha) \subset k \overline{C}_Q^\alpha$. Pick $x \in \overline{C}_Q^\alpha$ and $\alpha \in \Delta$. Then we need to show that

$$\langle \text{res}(\alpha), \text{res}(x) \rangle \geq 0$$

or equivalently, that

$$\text{res}(x) - \sigma_{\text{res}(\alpha)}(\text{res}(x))$$

is a non-negative multiple of $\text{res}(\alpha)$. Note that $\text{res}$ is $W\Gamma$-equivariant (where $W\Gamma$ acts as $W^\text{res}$ on $X^*(A)$). Thus, it suffices to show that

$$\text{res}(x - \sigma_{\text{res}(\alpha)}(x))$$

is a non-negative multiple of $\text{res}(\alpha)$. Thus, we need to compute $x - \sigma_{\text{res}(\alpha)}(x)$. We do so using our description of $\sigma_{\text{res}(\alpha)}$.

We first consider the case where the $\Gamma$-orbit of $\sigma_\alpha$ consists of pairwise commuting elements. Equivalently, the elements of $\Gamma \alpha$ are pairwise orthogonal. Then

$$\sigma_{\text{res}(\alpha)} = \sigma_{\alpha_n} \circ \cdots \circ \sigma_{\alpha_1}$$

for $\{\alpha_1, \ldots, \alpha_n\} = \Gamma \alpha$. Since $x$ is dominant in the absolute root system, we have

$$x - \sigma_{\alpha_i}(x) = a_i \alpha_i$$

for some $a_i \geq 0$. Then since $\alpha_i$ is orthogonal to $\alpha_j$ for $i \neq j$, we have $\sigma_{\alpha_i}(\alpha_j) = \alpha_j$. Thus,

$$x - \sigma_{\text{res}(\alpha)}(x) = \sum_{i=1}^{n} (\sigma_{\alpha_1} \circ \cdots \circ \sigma_{\alpha_{i-1}})(x) - (\sigma_{\alpha_1} \circ \cdots \circ \sigma_{\alpha_i})(x)$$

$$= \sum_{i=1}^{n} (\sigma_{\alpha_1} \circ \cdots \circ \sigma_{\alpha_{i-1}})(x - \sigma_{\alpha_i}(x))$$

$$= \sum_{i=1}^{n} (\sigma_{\alpha_1} \circ \cdots \circ \sigma_{\alpha_{i-1}})(a_i \alpha_i)$$

$$= \sum_{i=1}^{n} a_i \alpha_i.$$ 

Thus in this case,

$$\text{res}(x - \sigma_{\text{res}(\alpha)}(x)) = (a_1 + \cdots + a_n) \text{res}(\alpha)$$

and $a_1 + \cdots + a_n \geq 0$ as desired.

Now we consider the case where $\Gamma \alpha = \{\alpha, \beta\}$ and $\alpha$ and $\beta$ are adjacent in $D$ and connected by a single edge. Then $\sigma_\alpha(\beta) = \alpha + \beta = \sigma_{\beta}(\alpha)$. In this case, $\sigma_{\text{res}(\alpha)} = \sigma_{\beta} \circ \sigma_\alpha \circ \sigma_{\beta}$. By assumption, we have that $x - \sigma_\alpha(x) = a \alpha$ and $x - \sigma_{\beta}(x) = b \beta$ for $a$ and $b$ non-negative. Thus,

$$x - \sigma_{\text{res}(\alpha)}(x) = (x - \sigma_{\beta}(x)) + \sigma_{\beta}(x - \sigma_\alpha(x)) + (\sigma_{\beta} \circ \sigma_\alpha)(x - \sigma_{\beta}(x))$$

$$= b \beta + a(\alpha + \beta) + b \alpha$$

$$= (a + b)(\alpha + \beta),$$

which projects to $2(a + b)\text{res}(\alpha)$ and $2(a + b) \geq 0$ as desired.
Finally, we must consider the case where $\Gamma$ equals $\{\alpha_1, \beta_1, ..., \alpha_n, \beta_n\}$ such that $\alpha_i$ and $\beta_i$ are connected by a single edge in $D$ but for $i \neq j$, neither $\alpha_i$ nor $\beta_i$ are connected to either $\alpha_j$ or $\beta_j$. We compute $x - (\sigma_{\beta_i} \circ \sigma_{\alpha_i} \circ \sigma_{\beta_i})(x)$ as in the previous paragraph. Then if we let $w_i = \sigma_{\beta_i} \circ \sigma_{\alpha_i} \circ \sigma_{\beta_i}$, we have

$$\sigma_{\text{res}(\alpha)} = w_1 \circ ... \circ w_n.$$ 

Now we can compute $x - \sigma_{\text{res}(\alpha)}(x)$ as in the commuting case, substituting $w_i$ for $\sigma_{\alpha_i}$. We see in this case that

$$\text{res}(x - \sigma_{\text{res}(\alpha)}(x)) = 2(a_1 + b_1 + ... + a_n + b_n)\text{res}(\alpha).$$

This concludes the proof that $\text{res}(\overline{C}_Q^G) \subset k\overline{C}_Q^G$.

It remains to show that we actually have equality. We claim it suffices to show that $\delta_{\text{res}(\alpha)}$ is an element of $\text{res}(\overline{C}_Q^G)$. Recall that $\delta_{\text{res}(\alpha)}$ is the element in the $\mathbb{Q}$-span of the relative roots defined so that the pairing with $\text{res}(\alpha)$ is 1 and the pairing is 0 with all the other relative simple roots. To show the claim we note there is a natural isomorphism $X^*(A)_Q \iso X^*(A_0)_Q \times X^*(A')_Q$ where $A_0$ is the maximal $k$-split central torus and $A'$ is the identity component of the intersection of $A$ with the derived subgroup of $G$. Then $k\overline{C}_Q^G$ corresponds under this identification to the product of $X^*(A_0)_Q$ with the projection of $k\overline{C}_Q^G$ to $X^*(A')_Q$. Then we have a natural map $X^*(Z(G)^0)_Q \rightarrow X^*(A_0)_Q$ where $Z(G)^0$ is the identity component of the center of $G$ and $X^*(Z(G)^0)_Q \subset \overline{C}_Q^G$. Thus it suffices to show that $\text{res}(\overline{C}_Q^G)$ surjects onto the projection of $k\overline{C}_Q^G$ to $X^*(A')_Q$. This latter space is identified with the set of non-negative linear combinations of the fundamental relative weights, thus proving the claim.

To prove that $\delta_{\text{res}(\alpha)}$ is an element of $\text{res}(\overline{C}_Q^G)$, we make use of an equivalent description of $\delta_{\text{res}(\alpha)}$. It is the unique element in the $\mathbb{Q}$-span of the relative roots so that $\sigma_{\text{res}(\alpha)}(\delta_{\text{res}(\alpha)}) = \delta_{\text{res}(\alpha)}$ for $\text{res}(\alpha)$ and $\sigma(\beta)$ distinct simple roots and $\sigma_{\text{res}(\alpha)}(\delta_{\text{res}(\alpha)}) - \delta_{\text{res}(\alpha)} - \delta_{\text{res}(\beta)}$ when $\text{res}(\alpha) = \text{res}(\beta)$.

In the case where the elements of $\Gamma\alpha$ are mutually orthogonal, we have by the above characterization of fundamental weights that the absolute fundamental weight $\delta_{\alpha}$ restricts to $\delta_{\text{res}(\alpha)}$. In the case where $\Gamma\alpha$ has two elements that are connected in $D$, then $\delta_{\alpha}$ restricts to $2\delta_{\text{res}(\alpha)}$. In the final case, $\delta_{\alpha}$ restricts to $2\delta_{\text{res}(\alpha)}$. Thus, in all cases, we can find an element of $X^*(T)_Q$ that restricts to $\delta_{\text{res}(\alpha)}$. This completes the proof.

We record an important corollary of this proposition.

**Corollary B.0.4.** Suppose $\mu, \mu' \in X_*(T)_Q$ and $\mu \geq \mu'$. Let $\mu^\Gamma$ be the average of $\mu$ over its $\Gamma$ orbit. Then $\mu^\Gamma \geq \mu'^\Gamma$ in $X_*(A)_Q$. We caution that the first inequality means that $\mu - \mu'$ is a non-negative combination of absolute simple roots, while the second means that $\mu^\Gamma - \mu'^\Gamma$ is a non-negative combination of relative simple roots.

**Proof.** Recall that the action of $\Gamma$ stabilizes $\Delta$. Thus for each $\gamma \in \Gamma$, we have $\gamma(\mu) \geq \gamma(\mu')$ and so also $\mu^\Gamma \geq \mu'^\Gamma$ in the absolute root system. Thus, we are reduced to showing that if $x \in X_*(T)_Q$ is a non-negative combination of simple absolute roots, then it is also a non-negative combination of simple relative roots (under the identification $X_*(A)_Q = X_*(T)_Q$).


Equivalently, we need to show that if $x$ has non-negative pairing with every element of $\mathcal{C}_Q^\bullet$, then $x$ has non-negative pairing with every element of $k\mathcal{C}_Q^\bullet$. This is indeed equivalent because $x$ has non-negative pairing with each element of $\mathcal{C}_Q^\bullet$ if and only if it has non-negative pairing with each fundamental weight $\delta_\alpha$ and this is the case if and only if $x$ is a non-negative combination of simple roots.

Finally, this equivalent statement is an immediate consequence of the proposition. □

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