Symmetry of Quadratic Homogeneous Differential Systems

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Abstract

In this paper, the symmetry group of a differential system of \( n \) quadratic homogeneous first order ODEs of \( n \) variables is studied. For this purpose, we consider the action of both point and contact transformations to signify the corresponding Lie algebras. We also find the independent differential invariants of these actions.

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1 Introduction

A quadratic homogeneous differential system (QHDS) is a generalization of second degree polynomial system of differential equations that is rooted in natural science, and also in mathematics by virtue of Hilbert’s sixteen problem of his famous list of problems (See, e.g., [1] and references therein). They are studied in some different view point of their applications. For example, in physical sciences, in van der Pol oscillator as an important example of qualitative theory of ordinary differential equations (ODEs), in mathematical ecology and in particular in Volterra-Lotka equations, and many other applications in astrophysics and fluid mechanics [1].

There are many aspects for studying a QHDS. For instance, in [3] geometric classification of trajectories of a given two dimensional QHDS has studied for determining the invariant lines through the origin of the QHDS and also its location in the plane of parameters for classification of the geometry of its trajectories. In [8], the necessary conditions for the existence of polynomial first integrals and a polynomial symmetry field of a n-dimensional QHDS have given.

At the present work, the symmetry group of a QHDS of every finite dimension is studied. We specify symmetry groups of both point and contact

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transformation groups.

Assume that we have a differential system $\mathcal{E}$ of $n$ quadratic homogeneous first order ODEs of $n$ variables, in which each of them is a function of $t$:

$$
\mathcal{E} : \dot{x}_i = F_i(x_1, \ldots, x_n), \quad F_i = \sum_{j,k} a_{jk}^i x_j x_k, \quad (1.1)
$$

where the coefficients, $a_{jk}^i$ for $1 \leq i, j, k \leq n$, are arbitrary real constants. The solution space of differential system (1.1) is an immersed submanifold of the first order jet space, $J^1(\mathbb{R}, \mathbb{R}^n)$ of dimension $2n + 1$ and local coordinates $(t, x, p) := (t, x_1, \ldots, x_n, p_1, \ldots, p_n)$, when each $p_j$ is just $\dot{x}_j$.

According to [6], the action of contact transformation group is introduced on the jet space $J^1(\mathbb{R}, \mathbb{R}^n)$, and any element of this group is as the following transformation:

$$
T = \chi(t, x, p), \quad X_i = \psi_i(t, x, p), \quad P_j = \pi_j(t, x, p), \quad (1.2)
$$

where $1 \leq i, j \leq n$. An important special case of contact transformation group is point transformation group, where the action reduces to the following change of coordinates

$$
T = \chi(t, x), \quad X_i = \psi_i(x, p),
$$

when the variables come from the jet space $J^1(\mathbb{R}, \mathbb{R}^n)$.

**Remark 1.1** In general, the point and contact symmetry groups of a geometric object are not necessarily equal. A jet space $J^k(\mathbb{R}^m, \mathbb{R}^n)$ of order $k$ can be imbedded in other jet spaces $J^l(\mathbb{R}^m, \mathbb{R}^n)$ of order $l > k$ as a submanifold. Also a contact transformation of $J^l(\mathbb{R}^m, \mathbb{R}^n)$ can be restricted to $J^k(\mathbb{R}^m, \mathbb{R}^n)$ and then acts as a point transformation. Thus, the point transformation group of an geometric object is a Lie group of the contact transformation group and in the special case of symmetries, we conclude that the point symmetry group is a Lie subgroup of the contact symmetry group.

In the next section, we will see that point symmetry group of the differential system $\mathcal{E}$ is of dimension 1, where its contact symmetry group has infinite dimension.

## 2 Point Symmetry of a QHDS

As is indicated above, an important question of a quadratic homogeneous system is about its symmetry group. There are various methods for signifying symmetry group based on Lie’s method of infinitesimal generators, that some of them are exist in [2], [3], [5] and [6]. In this work, we will follow the method of [6] for finding point and contact symmetries.
In jet space $J^1(\mathbb{R}, \mathbb{R}^n)$ the system of equations (1.1) will be as follows

$$p_i = \sum_{j,k} a^i_{jk} x_j x_k, \quad 1 \leq i, j, k \leq n. \quad (2.3)$$

For finding symmetry group of system $\mathcal{E}$, we consider the general form of the infinitesimal generators provided from a point transformation as follows:

$$v = T \frac{\partial}{\partial t} + \sum_{i=1}^{n} K_i \frac{\partial}{\partial x_i}, \quad (2.4)$$

in which, $T$ and $K_i$s are smooth functions of variables $t, x_1, \ldots, x_n$.

The first order prolongation of $v$ will be then as the following expression:

$$v^{(1)} = v + \sum_{i=1}^{n} K'_i \frac{\partial}{\partial p_i}, \quad (2.5)$$

where for each $1 \leq i \leq n$ we have

$$K'_i = K'_{i,t} + \sum_{j} K_{i,x_j} - p_i(T_t + \sum_{j} T_{i,x_j}). \quad (2.6)$$

According to [6], $v^{(1)}$ as a prolongation of $v$ is a symmetry of system $\mathcal{E}$, if and only if it satisfies in relation $v^{(1)}[\mathcal{E}] = 0$. By effecting $v^{(1)}$ on the equations of the system we conclude the following system of equations for $i = 1, \ldots, n$:

$$K_1 \left( \sum_k (a^i_{1k} + a^i_{k1}) x_k \right) + \cdots + K_n \left( \sum_k (a^i_{nk} + a^i_{kn}) x_k \right) - K_{i,t} - p_1 K_{i,x_1} - p_2 K_{i,x_2} - \cdots - p_n K_{i,x_n} + p_i(T_t + p_1 T_{x_1} + \cdots + p_n T_{x_n}) = 0. \quad (2.7)$$

In the last equation, since $t, x_1, p_j$ ($1 \leq i, j \leq n$) are arbitrary and functions $K_i$ and $N_j$ only depend on $t, x_i$, so equations (2.7) will be satisfied if and only if we have the following system of equations

$$K_1 \left( \sum_k (a^i_{1k} + a^i_{k1}) x_k \right) + \cdots + K_n \left( \sum_k (a^i_{nk} + a^i_{kn}) x_k \right) - K_{i,t} = 0, \quad (2.8)$$

$$K_{i,x_1} = 0, \ K_{i,x_2} = 0, \ldots, K_{i,x_{i-1}} = 0, \ K_{i,x_{i+1}} = 0, \ldots, K_{i,x_n} = 0, \quad (2.9)$$

$$T_t - K_{i,x_i} = 0, \ T_{x_1} = 0, \ldots, T_{x_n} = 0. \quad (2.10)$$

for every $1 \leq i \leq n$. At first, from Eqs. (2.9)–(2.10) we understand that $T$ depends only to $t$ and for each $i$, $K_i$ is a function of $t$ and $x_i$. For a fixed $1 \leq i \leq n$, by replacing the Eq. $T_t = K_{i,x_i}$ in (2.8) and then differentiating with respect to $x_i$, we will find the following expression

$$T_t \left( \sum_k (a^i_{1k} + a^i_{k1}) x_k \right) = T_{tt}, \quad (2.11)$$
By solving the last equation in respect to \(t\), we have the following solution

\[
T(t) = c_1 \exp\left(t \sum_k (a_{ik}^i + a_{ki}^j)x_k\right) + c_2,
\]

for constants \(c_1\) and \(c_2\). Since \(T\) is just a function of \(t\), so \(c_1\) must be zero, and therefore

\[
T(t, x_1, \cdots, x_n) = c,
\]

for arbitrary constant \(c\). Then for each \(i\), we see that \(K_{i,x_i} = 0\) and so is dependent to \(t\) only. But by solving Eqs. (2.8) with respect to \(K_i(t)\) we obtain the below solutions when \(i\) varies between 1 and \(n\):

\[
K_i(t) = \exp\left(t \sum_k (a_{ik}^i + a_{ki}^j)x_k\right)
\]

\[
\left(\int \exp \left(-t \sum_k (a_{ik}^i + a_{ki}^j)x_k\right) F \, dt + c_3\right), \tag{2.11}
\]

where we assumed that \(F = \sum_{j \neq i} K_j(t)\left(\sum_k (a_{jk}^i + a_{kj}^i)x_k\right)\) and \(c_3\) be arbitrary constant. Eqs. (2.11) and the fact that all of \(K_i\)s are just depend to \(t\) implies that

\[
K_i(t, x_1, \cdots, x_n) = 0 \quad \text{for} \quad 1 \leq i \leq n.
\]

when \(c\) is an arbitrary constant. The symmetry group of the general infinitesimal generators, that provided as \(v^\frac{\partial}{\partial t}\) will be just the translation of \(t\) by a constant coefficient of the parameter \(s\) of the 1-parameter subgroup:

\[
(t, x_1, \cdots, x_n) \mapsto (cs + t, x_1, \cdots, x_n)
\]

**Theorem 2.1** Set of all infinitesimal generators of the point symmetry group of the system (1.1) is a 1-dimensional Lie algebra with the base \(\{\frac{\partial}{\partial t}\}\). Therefore, the symmetry group of the system (1.1) is a 1-dimensional Lie group of time translations.

According to theorem 2.74 of [6], the invariants \(u = I(t, x_1, x_2, \cdots, x_n)\) of one–parameter group with infinitesimal generators of the form (2.4) satisfy the linear, homogeneous partial differential equations of first order:

\[
v[I] = 0.
\]

The solutions of the last equation, are found by the method of characteristics (See [6] and [4] for details). So we can replace the last equation by the following characteristic system of ordinary differential equations

\[
\frac{dt}{T} = \frac{dx_1}{K_1} = \frac{dx_2}{K_2} = \cdots = \frac{dx_n}{K_n}. \tag{2.12}
\]
By solving the equations (2.12) of the differential generator \( v = c \frac{\partial}{\partial t} \), we (locally) find the following general solutions

\[ I_1(t, x_1, x_2, \cdots, x_n) = x_1 = d_1, \quad \cdots \quad I_n(t, x_1, x_2, \cdots, x_n) = x_n = d_n, \quad (2.13) \]

for constants \( d_i \) when \( i = 1, \cdots, n \).

**Theorem 2.2** The independent first integrals of the characteristic system of the infinitesimal generator \( v = c \frac{\partial}{\partial t} \) are the derived invariants (2.13).

### 3 Contact Symmetry of a QHDS

On the other hand, we consider that the infinitesimal generators comes from contact transformations, that is, the contact transformation group is acting on the jet space \( J^1 \). Hence, we may suppose general form of a infinitesimal generator as

\[ v = T \frac{\partial}{\partial t} + \sum_{i=1}^{n} K_i \frac{\partial}{\partial x_i} + \sum_{j=1}^{n} P_j \frac{\partial}{\partial p_j}, \quad (3.14) \]

for arbitrary smooth functions \( T, K_i, P_j \) of variables \( t, x, p \).

Since this expression depends to variables of 1-jet space and so is a infinitesimal generator of the jet space, so it dose not need to be prolonged. Therefore, according to [6] \( v \) is a symmetry of \( E \), if and only if, it satisfies in

\[ v[E] = 0. \]

After acting \( v \) on the system we obtain the relation

\[ K_1 \sum_k (a_{1k}^i + a_{k1}^i) x_k + \cdots + K_n \sum_k (a_{nk}^i + a_{kn}^i) x_k - P_i = 0 \]

where \( i \) varies between 1 and \( n \). Thus the general form of each symmetry of the system will be as follows

\[ v = T \frac{\partial}{\partial t} + \sum_{i=1}^{n} \left\{ K_i \frac{\partial}{\partial x_i} + \left( \sum_{j,k} K_j (a_{jk}^i + a_{kj}^i) x_k \frac{\partial}{\partial p_j} \right) \right\}, \quad (3.15) \]

in which \( T, K_1, K_2, \cdots, K_n \) are arbitrary functions of \( t, x_1, x_2, \cdots, x_n \). One may divides \( v \) to the following vector fields

\[
\begin{align*}
v_1 &= T \frac{\partial}{\partial t} \\
v_2 &= K_1 \left( \frac{\partial}{\partial x_1} + \sum_{j,k} (a_{1k}^j + a_{k1}^j) x_k \frac{\partial}{\partial p_j} \right) \\
&\quad \vdots \\
v_{n+1} &= K_n \left( \frac{\partial}{\partial x_n} + \sum_{j,k} (a_{nk}^j + a_{kn}^j) x_k \frac{\partial}{\partial p_j} \right),
\end{align*}
\]

(3.16)
Lie bracket of each two of \(v_1, \cdots, v_{n+1}\) is defined to be the commutator 
\([v_i, v_j] := v_i v_j - v_j v_i\) \((1 \leq i, j \leq n + 1)\). It is not hard to see that the commutator of any two of \(v_1, \cdots, v_{n+1}\) is a linear combination of them, and hence \(v_1, \cdots, v_{n+1}\) generate a Lie algebra. For instance for \(1 \leq \alpha, \beta \leq n\) if we assume that 
\[
v_{\alpha+1} = K_{\alpha} \left( \frac{\partial}{\partial x^\alpha} + \sum_{j,k} (a_{\alpha k j} + a_{\alpha j k})x_k \frac{\partial}{\partial p_j} \right),
\]
\[
v_{\beta+1} = K_{\beta} \left( \frac{\partial}{\partial x^\beta} + \sum_{l,m} (a_{\beta m l} + a_{\beta l m})x_m \frac{\partial}{\partial p_l} \right),
\]
then we have 
\[
[v_{\alpha+1}, v_{\beta+1}] = \{K_{\alpha} K_{\beta, x^\alpha} + K_{\alpha} \sum_{j,k} K_{\beta, p_j} (a_{\alpha k j} + a_{\alpha j k})x_k \} \frac{\partial}{\partial x^\beta} 
\]
\[-\{K_{\beta} K_{\alpha, x^\beta} + K_{\beta} \sum_{j,k} K_{\alpha, p_j} (a_{\beta k j} + a_{\beta j k})x_k \} \frac{\partial}{\partial x^\alpha} 
\]
\[+ \sum_j \{K_{\alpha} K_{\beta, x^\alpha} \sum_k (a_{\beta j k} + a_{\beta k j})x_k \} \frac{\partial}{\partial x^\alpha} 
\]
\[-K_{\beta} K_{\alpha, x^\beta} \sum_k (a_{\alpha j k} + a_{\alpha k j})x_k \}
\]
\[+ \left( K_{\alpha} \sum_{l,m} K_{\beta, p_l} (a_{l m \alpha} + a_{m \alpha l})x_m \right) \left( \sum_k (a_{\beta j k} + a_{\beta k j})x_k \right) \}
\]
\[-K_{\beta} \left( \sum_{l,m} K_{\alpha, p_l} (a_{l m \beta} + a_{m \beta l})x_m \right) \left( \sum_k (a_{\alpha j k} + a_{\alpha k j})x_k \right) \} \frac{\partial}{\partial p_j}.
\]

One can compute the table of Lie symmetry algebra of commutators arising from infinitesimal operators (3.16) that is given in table 1.

\[
\begin{array}{cccccc}
  & v_1 & v_2 & \cdots & v_n & v_{n+1} \\
 v_1 & 0 & v_2 - v_1 & \cdots & v_n - v_1 & v_{n+1} - v_1 \\
v_2 & v_1 - v_2 & 0 & \cdots & v_n - v_2 & v_{n+1} - v_2 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
v_n & v_n - v_1 & v_2 - v_n & \cdots & 0 & v_{n+1} - v_n \\
v_{n+1} & v_1 - v_{n+1} & v_2 - v_{n+1} & \cdots & v_n - v_{n+1} & 0 \\
\end{array}
\]

Table 1. The commutators table of Lie algebra of equation (3.14)

Therefore we have the following theorem:

**Theorem 3.1** The symmetry group of the system \(\mathcal{E}\) of quadratic homogeneous differential equations, (3.17), under the action of contact transformations is an
infinite dimensional Lie algebra generated by infinitesimal operators (3.16).

By referring again to theorem 2.74 of [6], and considering the infinitesimal operators of the form (3.15), the invariants \( I(t, \mathbf{x}, \mathbf{p}) \) of their one-parameter group, are signified by the characteristic system of ordinary differential equations as follows \((i, j, k = 1, \cdots, n)\)

\[
\frac{dt}{T} = \frac{dx_i}{K_i} = \frac{dp_j}{K_1 \sum_k (a^1_{ik} + a^1_{ki})x_k + \cdots + K_n \sum_k (a^n_{nk} + a^n_{kn})x_k}
\]

in which determines \(2n\) invariant functions. By solving these equations we find that

\[
I_1(t, \mathbf{x}, \mathbf{p}) = \int (K_1 dt - T dk_1) = d_1,
\]

\[
I_n(t, \mathbf{x}, \mathbf{p}) = \int (K_n dt - T dk_n) = d_n,
\]

\[
I_{n+1}(t, \mathbf{x}, \mathbf{p}) = \int \left( \sum_{j,k} K_j (a^1_{jk} + a^1_{kj})x_k \right) dt - \int T dp_1 = d_{n+1}
\]

\[
I_{2n}(t, \mathbf{x}, \mathbf{p}) = \int \left( \sum_{j,k} K_j (a^n_{jk} + a^n_{kj})x_k \right) dt - \int T dp_n = d_{2n}
\]

**Theorem 3.2** The derived invariants (3.18) as independent first integrals of the characteristic system of the infinitesimal generator (3.17), provide the general solution

\[
S(t, \mathbf{x}, \mathbf{p}) := \varphi(I_1(t, \mathbf{x}, \mathbf{p}), I_2(t, \mathbf{x}, \mathbf{p}), \cdots, I_{2n}(t, \mathbf{x}, \mathbf{p}))
\]

with an arbitrary function \(\varphi\), which satisfies in the equation \(v[\varphi] = 0\) and therefore in

\[
T \frac{\partial S}{\partial t} + \sum_{i=1}^{n} \left( K_i \frac{\partial S}{\partial k_i} + \left( \sum_{j,k} K_j (a^1_{jk} + a^1_{kj})x_k \right) \frac{\partial S}{\partial p_i} \right) = 0.
\]

The solution of the last equation is defined implicitly by \(S(t, \mathbf{x}, \mathbf{p}) = 0\). If \(\frac{\partial S}{\partial t} \neq 0\), then the solution can be written explicitly by \(t = \gamma(\mathbf{x}, \mathbf{p})\).

**Example 3.3** Clearly, by assuming \(T = 1\) and \(K_i = 0\) we return back to a point symmetry, in which as we saw in previous section, the symmetry group provides by translating time and fixing other variables.
Example 3.4 If we suppose $T = 1$ and $K_i = 1$ for some $i$ and $K_j = 0$ for $j \neq i$, then we have the following expression of infinitesimal generators
\[ v = \frac{\partial}{\partial t} + \frac{\partial}{\partial x_i} + \left( \sum_{j,k} (a^i_{jk} + a^j_{ki}) x_k \right) \frac{\partial}{\partial p_j}. \] (3.19)

Its one-parameter group then will be as follows
\[ (t, x, p) \mapsto (t + s, x_1, \cdots, x_i + s, \cdots, x_n, p_m + a^m_{ii} s^2 + s \sum_k (a^m_{ik} + a^i_{mk}) x_k) \]

where $s$ is the parameter of the flow and $m$ changes over $1, \cdots, n$. If the point $(t, x, p)$ be fixed, then the flow of $v$, by restricting the jet space coordinates to the $t, x_i$ and $p_j$ axis, for a $1 \leq j \leq n$ and the fixed value $i$, is a space parabolic.

Example 3.5 Let $T = t$, $K_i = 1$ for some index $i$ and $K_j = 0$ for other indices $j \neq i$. The infinitesimal operator $v$ and its flow are given respectively by changing the coefficient of $\frac{\partial}{\partial t}$ to $t$ in (3.20) and the component $t + s$ of right hand side of (3.20) to $t e^s$.

By fixing $(t, x, p)$ and acting the symmetry group on it, then the flow of $v$, under the projection to the coordinates in terms of $t, x_i$ and $p_j$ for a $1 \leq j \leq n$, will be similar to the space curve $(s, e^s, s^2 + s)$ and has two branches, the first of space exponential map and the second of a space parabolic.

Example 3.6 For the case which we assume that $T = c t$ when $c$ is a constant, and $K_i = 1$ when $i = 1, \cdots, n$, the contact infinitesimal operator $v$ is as follows
\[ v = c t \frac{\partial}{\partial t} + \sum_i \left\{ \frac{\partial}{\partial x_i} + \left( \sum_{j,k} (a^i_{jk} + a^j_{ki}) x_k \right) \frac{\partial}{\partial p_j} \right\} \]

The one-parameter of $v$ then is
\[ (t, x, p) \mapsto (t e^{cs}, x + s, p_m + s^2 \sum_i a^m_{ii} + s \sum_{i,k} (a^m_{ik} + a^k_{mi}) x_k) \]

when in it, $m$ varies from 1 to $n$ and we assumed that $1 \leq k, l \leq n$. The action of the symmetry group on a fixed point, will give a flow that its projection to three arbitrary axis $(t, x_i, p_j)$ is similar to the 3-dimensional flow explained in example 3.5. In the particular case of this example, the characteristic equations tend to the following independent differential invariants for $1 \leq \alpha \leq n$ and $n + 1 \leq \beta \leq 2n$:
\[ I_\alpha(t, x, p) = t(1 - c k_\alpha), \quad I_\beta(t, x, p) = t \left( \sum_{j,k} (a^\beta_{jk} + a^j_{k\beta}) x_k \right) - c t p_{\beta - n} \].
Example 3.7  By assuming $T = 0$ and $K_i = x_i$ we find that the infinitesimal operator as

$$v = \sum_i \left\{ x_i \frac{\partial}{\partial x_i} + \left( \sum_{j,k} (a^i_{jk} + a^k_{ij}) x_j x_k \right) \frac{\partial}{\partial p_i} \right\}$$

and the one-parameter group of $v$ as

$$(t, x, p) \mapsto (t, e^s x, p_i + (e^{2s} - 1) \sum_{j,k} (a^i_{jk} + a^k_{ij}) x_j x_k)$$

where $i$ varies on values $1, \cdots, n$. Its flow also signifies a parabolic if we reduce the jet coordinate $(t, x, p)$ to $(t, x_l, p_k)$ for some specified $l$ and $k$. The concluded invariants then will be as follows $(1 \leq \alpha, \beta \leq n)$

$$I_1 = t,$$
$$I_{\alpha} = \ln x_\alpha - \ln x_{\alpha - 1},$$
$$I_{\beta + n} = \sum_{j,k \neq \alpha} (a^\beta_{jk} + a^\beta_{kj}) x_j x_k + \sum_{k \neq \beta} \left( a^\beta_{\beta k} + a^\beta_{k \beta} \right) x_k - p\beta.$$

Finally, we present an example of the case in which $K_i$s are functions dependent to $p_j$s resp.

Example 3.8  As we indicate, we suppose that $T = 0$ and $K_i = p_i$ for $i = 1, \cdots, n$. So the infinitesimal generator is in the form of

$$v = \sum_i \left\{ p_i \frac{\partial}{\partial x_i} + \left( \sum_{j,k} p_j (a^i_{jk} + a^k_{ij}) x_k \right) \frac{\partial}{\partial p_i} \right\},$$

that has the following trajectory in term of the parameter $s$

$$(t, x, p) \mapsto (t, x + ps, p')$$

where $p'$ consists of $n$ component, which its $i^{th}$ component is

$$p'_i = p_i + \left( \exp \left( (\sum_k (a^i_{ik} + a^i_{ki}) x_k) s \right) - 1 \right) \left( \sum_{j,k} p_j (a^i_{jk} + a^i_{kj}) x_k \right).$$

One can derive the following independent differential invariants

$$I_1 = t,$$
$$I_{\alpha} = p_\alpha x_{\alpha - 1} - p_{\alpha - 1} x_\alpha,$$
$$I_{\beta + n} = \frac{p\beta}{\sum_k (a^\beta_{jk} + a^\beta_{kj}) x_k} - \ln \left( \sum_{j,k} p_j (a^\beta_{jk} + a^\beta_{kj}) x_k \right) \frac{\sum_{j,k} p_j (a^\beta_{jk} + a^\beta_{kj}) x_k}{(\sum_k (a^\beta_{jk} + a^\beta_{kj}) x_k)^2},$$

for $\alpha, \beta = 1, \cdots, n$. Clearly, projection of the jet coordinate to the coordinate $(x_m, p_l)$ for specified amounts $m$ and $l$, is the graph of the exponential map.
According to [7], if $G$ acts smoothly and transitively on a manifold $M$, then $M$ is isomorphic to $G/H$ as a homogeneous space, which obtained by quotienting by a closed Lie subgroup that is, an isotropy subgroup provided by element of $G$. Thus, we can use of the exponential map, $\exp : TM \to M$, defined in a small neighborhood of identity element of $M$ as a Lie group. From Baker–Campbell–Hausdorff formula we know that for vector fields $X,Y$ of the Lie algebra of a Lie group, if we have $[X,Y] = 0$, then the flow of $X+Y$ is $\xi_t = \phi_t \circ \psi_t = \psi_t \circ \phi_t$ where $\phi_t$ and $\psi_t$ are flows of resp. $X$ and $Y$.

**Remark 3.8** With similar method as indicated in above examples, by different selections of coefficients of (3.15) as the linear combinations of 1-jet space variables, their corresponding invariants are in the forms of (3.18), and their proper projections of trajectories to 2 or 3-dimensional space of their variables, providing their commutator be zero, are similar to parabolic, exponential map or compositions of two branches in the similar shape to them.

This result is inferred, since the symmetric group prepared in the above examples act transitively, if we add the condition that the Lie bracket of every two infinitesimal operators be zero, then the flow of every linear combination of any two of infinitesimal generators is a composition of flows made by these two operators.

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