A general approach for constructing robust virtual element methods for fourth order problems
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Abstract

We present a class of nonconforming virtual element methods for general fourth order partial differential equations in two dimensions. We develop a generic approach for constructing the necessary projection operators and virtual element spaces. Optimal error estimates in the energy norm are provided for general linear fourth order problems with varying coefficients. We also discuss fourth order perturbation problems and present a novel nonconforming scheme which is uniformly convergent with respect to the perturbation parameter without requiring an enlargement of the space. Numerical tests are carried out to verify the theoretical results. We conclude with a brief discussion on how our approach can easily be applied to nonlinear fourth order problems.

Keywords: virtual element method, fourth order problems, nonconforming, perturbation problem, DUNE.

1 Introduction

In recent years the discretization of partial differential equations via the virtual element method (VEM) has seen a rapid increase. Introduced in [8] VEM began as an extension and generalization of both finite element and mimetic finite difference methods as discussed in [13]. In [8] the appropriate local and global VEM spaces are constructed and the approximation properties analysed for the Laplace equation. Another discretization of the Laplace problem was suggested in [1] while a nonconforming approach can be found in [9]. An extension to general, nonlinear second order elliptic PDEs for both conforming and nonconforming spaces is discussed in [10]. Similarly, another approach is taken in [11] for diffusion-convection-reaction problems.

The versatility of VEM has been showcased through the wide variety of problems it has been applied to over recent years. This has led to the construction of $H(\text{div})$ and $H(\text{curl})$-conforming virtual element spaces in [11], conforming virtual elements for polyharmonic problems in [5], and the construction of methods for Stokes flow in [14] [15] [21], to name but a few. Especially the case in which VEM spaces can be constructed to enforce desirable properties of the discrete functions even on general polygonal meshes makes the approach very interesting for a wide range of problems. An example of this is the construction of divergence free vector spaces in [14]. A further example is the construction of discrete spaces with higher order continuity conditions. The construction of even a lowest order $C^1$ conforming space is not straightforward within the standard finite element setting and higher order nonconforming spaces suitable for fourth order problems are also not readily available. Consequently, many software packages provide a large number of spaces for second order problems but often only provide the lowest order Morley element [28] for discretizing fourth order problems without requiring the use of splitting methods.

To construct conforming elements for fourth order problems, $C^1$ continuity is required which makes the methods highly complex. It is known that using traditional finite element methods, polynomials of at least degree five are needed to construct $C^1$ approximations which are piecewise polynomials. In contrast, it is shown in [17] that the virtual element construction of $C^1$ approximations to fourth order plate bending problems is much simpler and arguably more elegant. Additionally, the conforming virtual element method for polyharmonic problems, $\Delta^p u = f$ for $p \geq 1$, has been addressed in [5] where the global VEM space consists of $C^{p-1}$ functions. As well as this, the study of linear elliptic fourth order problems in three dimensions is considered in the conforming case in [14]. Another example of $C^1$ conforming elements can be seen in the application of the lowest order VEM space to the Cahn-Hilliard equation, investigated in [8]. Further studies of the application of virtual elements to the Cahn-Hilliard equation can be found in [26] [27].

In this work we focus on studying nonconforming virtual element methods but for a wide range of problems including nonlinear models. Although we focus on nonconforming VEM for fourth order problems, we highlight that due to the general framework we present, only minor modifications are needed to also include the study of $C^1$ conforming elements for these problems. Existing works which study nonconforming fourth order problems include the nonconforming approximation of the biharmonic plate bending problem, which is considered in [4] [35] [59].
Hence it follows from the Lax-Milgram Lemma that (2.2) has a unique solution. That the bilinear form is coercive and continuous with respect to the energy norm, allows us to prove convergence in the discrete broken energy norm in Section 5. We refer throughout this paper to two well known VEM spaces, the nonconforming space discussed in both [4, 36] and the mixture of spaces have been suggested, some fully nonconforming [3] and others which include some level of continuity [35] though not the full $C^1$ continuity you would see in a fully conforming space. More recently, we see a $C^0$ conforming approach to fourth order perturbation problems being considered in [34]. To our knowledge, the application of higher order VEM also to more general nonconstant coefficients and nonlinear fourth order problems is not available at the time of writing.

Arguably the most important ingredient of VEM is the construction of projection operators. In the available literature on fourth order problems, projection operators are constructed based on the underlying variational problem. The main idea of this approach is to construct only one projection which depends on the local contribution to the bilinear form. In [19] a different approach was taken for discretizing second order problems, which makes it straightforward to apply the method to nonlinear models. In this paper we generalize this approach and demonstrate how it can be applied to a wide range of fourth order problems. A major advantage of this approach is that it can be included more easily into existing software frameworks. A central building block for implementing Galerkin type schemes is the evaluation of nodal basis functions and their derivatives at given quadrature points. To extend this to our VEM setting, these methods have to be replaced with the evaluation of projection operators defined on each element. We implemented this approach within the DUNE [24] software framework, requiring little change to the existing code base. From the user perspective switching between a finite element to a virtual element discretization is seamless especially within the available Python frontend [25, 23].

We begin this paper by detailing the continuous problem (Section 2) - giving the PDE problem and defining the continuous bilinear form. We then move onto the discrete problem in Section 3 and again give details of an abstract set up including a Strang-type Lemma. In the subsequent section, Section 4, we set up the virtual element spaces, projections, and discrete bilinear form. Although the same approach has already been considered the problems so far only include second order PDEs. We therefore extend the VEM enhancement approach to fourth order problems here and show that this gives us certain $L^2$ projection properties for our projection operators. This then allows us to prove convergence in the discrete broken energy norm in Section 5. We refer throughout this paper to two well known VEM spaces, the nonconforming space discussed in both [4, 36] and the $C^0$ conforming space discussed in [35, 33] by demonstrating how they fit into our generalized framework. In Section 6 we discuss the fourth order perturbation problem [20, 23, 33, 34] and present a new robust nonconforming scheme which remains convergent as the perturbation parameter $\epsilon \to 0$. The modified space we present is obtained by an adjustment to the gradient projection. Numerical tests are carried out in Section 7 to confirm the a priori error analysis.

### 2 The continuous problem

Throughout this paper, we adopt the standard notation for Sobolev spaces $H^s(D)$ for non negative integers $s$, and for a domain $D$. We denote the norm and seminorm by $\| \cdot \|_s$ and $\| \cdot \|_{s,D}$ respectively. If $D = \Omega$ then the subscript shall be omitted. The notation $(\cdot, \cdot)_\Omega$ will be used to denote the $L^2(\Omega)$ inner product. For a non negative integer $l$, let $P_l(\Omega)$ denote the set of all polynomials up to degree $l$ over $D$. We use the convention that $P_{-1}(D) = \{ 0 \}$. We denote the standard $L^2(\Omega)$ orthogonal projection onto the polynomial space $P_l(\Omega)$ by $\pi_l^\Omega$. The tensor of all derivatives of a given order $|\mu|$ is denoted with $D^{|\mu|}\varphi$. Let $\partial_s\varphi = \nabla \varphi \cdot n$ denote the normal derivative of a function $\varphi$ over $\partial D$ and let $\partial_t\varphi = \nabla \varphi \cdot \tau$ denote its tangential derivative where we use $\tau$ to denote a tangential vector.

Consider a general linear fourth order problem defined on a polygonal domain $\Omega \subset \mathbb{R}^2$ described by a bilinear form

$$a(u, v) := \int_\Omega \kappa(x) D^2 u : D^2 v \, dx + \int_\Omega \beta(x) Du \cdot Dv \, dx + \int_\Omega \gamma(x) uv \, dx$$

for $u, v \in H_0^2(\Omega)$ with $H_0^2(\Omega) = \{ v \in H^2(\Omega) : v = \partial_n v = 0 \text{ on } \partial \Omega \}$.

We make the minimal assumptions that $\kappa, \beta, \gamma \in L^\infty(\Omega)$. In later sections we impose further conditions on the coefficients. For now assume that the coefficients satisfy $\kappa \geq \kappa_0 > 0$, for a constant $\kappa_0$ and $\beta, \gamma \geq 0$. Note that we could also consider an even more general setting, e.g., take $\beta \in L^\infty(\Omega)^{2\times 2}$ as in [19]. The results in this paper can be easily extended to cover this case but to keep the presentation simple we only consider scalar coefficients.

The variational problem for a given $f \in L^2(\Omega)$ reads as follows: find $u \in H_0^2(\Omega)$ such that

$$a(u, v) = (f, v) \quad \forall v \in H_0^2(\Omega).$$

Since the bilinear form is symmetric, we can define an energy norm $\| \cdot \|$ by $\| v \|^2 = a(v, v)$. It follows easily that the bilinear form is coercive and continuous with respect to the energy norm,

$$a(u, v) \leq \| u \| \| v \|, \quad \text{for all } u, v \in H_0^2(\Omega),$$

$$a(v, v) \geq \| v \|^2, \quad \text{for all } v \in H_0^2(\Omega).$$

Hence it follows from the Lax-Milgram Lemma that (2.2) has a unique solution.
Remark 2.1. Assuming that the solution \( u \) to (2.2) is smooth enough, we can derive the corresponding strong form of the PDE

\[
\sum_{i,j=1}^{2} \partial_{ij}(\kappa \partial_{ij} u) - \sum_{i=1}^{2} \partial_{i}(\beta \partial_{i} u) + \gamma u = f \quad \text{in } \Omega,
\]

\[
u = \partial_{n} u = 0 \quad \text{on } \partial \Omega.
\]

Note that if we were considering constant coefficients, taking \( \kappa, \beta, \gamma \in \mathbb{R} \), as in [12], then the strong form would reduce to the PDE studied there

\[
\kappa \Delta^{2} u - \beta \Delta u + \gamma u = f \quad \text{in } \Omega,
\]

\[
u = \partial_{n} u = 0 \quad \text{on } \partial \Omega.
\]

3 The discrete problem and an abstract convergence result

In this section we provide some general ingredients needed for the discretization of our problem and present a Strang-type abstract error estimate. Let \( \mathcal{T}_{h} \) denote a tessellation of the computational domain \( \Omega \) and denote the set of all edges in \( \mathcal{T}_{h} \) by \( \mathcal{E}_{h} \). We split this set into boundary edges, \( \mathcal{E}_{h}^{\text{bdry}} := \{ e \in \mathcal{E}_{h} : e \subset \partial \Omega \} \) and internal edges \( \mathcal{E}_{h}^{\text{int}} := \mathcal{E}_{h} \setminus \mathcal{E}_{h}^{\text{bdry}} \). Similarly, denote the set of vertices in \( \mathcal{T}_{h} \) by \( \mathcal{V}_{h} = \mathcal{V}_{h}^{\text{int}} \cup \mathcal{V}_{h}^{\text{bdry}} \), which again is made up of interior and boundary vertices.

For an integer \( s > 0 \), define the broken Sobolev space \( H^{s}(\mathcal{T}_{h}) \) by

\[
H^{s}(\mathcal{T}_{h}) := \{ v \in L^{2}(\Omega) : v|_{K} \in H^{s}(K), \forall K \in \mathcal{T}_{h} \},
\]

and on this space define the broken \( H^{s} \) seminorm

\[
|v|^{2}_{s,h} = \sum_{K \in \mathcal{T}_{h}} |v|^{2}_{s,K}.
\]

For a function \( v \in H^{2}(\mathcal{T}_{h}) \) we define the jump operator \( [\cdot] \) across an edge \( e \in \mathcal{E}_{h} \) as follows. For an internal edge, \( e \in \mathcal{E}_{h}^{\text{int}} \), define \( [v] := v^{+} - v^{-} \) where \( v^{\pm} \) denotes the trace of \( v|_{K_{\pm}} \) where \( e \subset \partial K^{+} \cap \partial K^{-} \). For boundary edges, \( e \in \mathcal{E}_{h}^{\text{bdry}} \), let \( [v] := v|_{e} \). We denote with \( \mathbb{P}_{k}(K) \) the space of polynomials over a grid element \( K \) and define the piecewise polynomial space \( \mathbb{P}_{k}(\mathcal{T}_{h}) \) for any \( k \in \mathbb{N} \) with

\[
\mathbb{P}_{k}(\mathcal{T}_{h}) := \{ p \in L^{2}(\Omega) : p|_{K} \in \mathbb{P}_{k}(K), \forall K \in \mathcal{T}_{h} \}.
\]

We now make the following basic assumptions. In particular, we stress that throughout the paper the polynomial order \( l \) is fixed.

Assumption 3.1. Assume the following holds for any fixed \( h > 0 \) and for a fixed \( l \geq 2 \).

(A1) The mesh \( \mathcal{T}_{h} \) consists only of simple polygons.

(A2) The finite dimensional function space \( V_{h,l} \) satisfies \( \mathbb{P}_{l}(\mathcal{T}_{h}) \subset V_{h,l} \) for some fixed \( l \geq 2 \) and \( V_{h,l} \subset H_{l}^{2,nc}(\mathcal{T}_{h}) \).

We define the nonconforming space \( H_{l}^{2,nc}(\mathcal{T}_{h}) \subset H^{2}(\mathcal{T}_{h}) \) as

\[
H_{l}^{2,nc}(\mathcal{T}_{h}) := \left\{ v \in H^{2}(\mathcal{T}_{h}) : v \text{ continuous at internal vertices, } v(v') = 0 \quad \forall v' \in \mathbb{P}_{l}^{\text{bdry}} \right\},
\]

\[
\int_{e} [\partial_{n} v] p \, ds = 0 \quad \forall p \in \mathbb{P}_{l-2}(e), \quad \int_{e} [v] p \, ds = 0 \quad \forall p \in \mathbb{P}_{l-3}(e), \quad \forall e \in \mathcal{E}_{h}.
\]

(A3) There exists \( f_{h} \in \mathcal{V}^{0}_{h,l} \), which approximates the right hand side of our variational problem (2.2).

(A4) There exists a discrete bilinear form \( a_{h} : V_{h,l} \times V_{h,l} \to \mathbb{R} \), such that for any \( u_{h}, v_{h} \in V_{h,l} \),

\[
a_{h}(u_{h}, v_{h}) = \sum_{K \in \mathcal{T}_{h}} a_{K}^{h}(u_{h}, v_{h}).
\]

The bilinear form \( a_{K}^{h} : V_{h,l}|_{K} \times V_{h,l}|_{K} \to \mathbb{R} \) is the restriction of \( a_{h} \) to an element \( K \). We denote the restriction of the VEM space \( V_{h,l} \) to an element \( K \) by \( V_{h,l}|_{K} := V_{h,l}|_{K} \).

(A5) Stability property: assume that there exists two constants \( \alpha_{+}, \alpha_{-} \) such that

\[
\alpha_{+} a_{K}^{h}(v_{h}, v_{h}) \leq a_{K}^{h}(u_{h}, v_{h}) \leq \alpha_{-} a_{K}^{h}(v_{h}, v_{h})
\]

for all \( v_{h} \in V_{h,l}^{K} \).

The criteria in the stability property (A5) is required to show that the discrete bilinear form is coercive and continuous.
Lemma 3.2. The broken Sobolev norm $|·|_{2,h}$ is a norm on the spaces $H^2_0(\Omega)$ and $H^2_{2,nc}(T_h)$. Define the element wise discrete energy norm $\|u_h\|_K^2 = \sum_{K \in T_h} \|u_h\|_{K}^2$ for functions $u_h \in H^2_{2,nc}(T_h)$, where the element wise contributions are given by

$$\|u_h\|_{K}^2 = (\kappa D^2 w_h, D^2 w_h)_{K} + (\beta D w_h, D w_h)_{K} + (\gamma w_h, w_h)_{K}.$$

Then, we have that $\|·\|_h$ is a norm on $H^2_{2,nc}(T_h)$. Therefore, under Assumption [A3], it follows that both $|·|_{2,h}$ and $\|·\|_h$ are a norm on $V_{h,l}$.

Proof. From [6, 8, 10] it follows that $|·|_{2,h}$ is a norm on both $H^2_0(\Omega)$ and $H^2_{2,nc}(T_h)$. Consequently, $\|·\|_h$ is a norm on $H^2_{2,nc}(T_h)$ under the given conditions on the coefficients $\kappa, \beta, \gamma$ stated in Section 2.

The following is now a direct consequence of the stability assumption [A5]

Theorem 3.3 (Existence and uniqueness of solutions to the discrete problem). Under Assumption [3, 7], the discrete problem: find $u_h \in V_{h,l}$ such that

$$a_h(u_h, v_h) = \langle f_h, v_h \rangle \quad \forall v_h \in V_{h,l} \tag{3.1}$$

admits a unique solution.

We now have the following Strang-type error bound, the proof of which is standard and identical to the method from e.g. [9].

Theorem 3.4 (A priori error bound). Under Assumption [3, 4] it holds that

$$\alpha_{\Delta} \|u - u_h\|_h \leq \inf_{v_h \in V_{h,l}} \alpha^* \|u - v_h\|_h + \sup_{w_h \in V_{h,l}} \frac{|(f_h, w_h) - (f, w_h)|}{\|w_h\|_h} + \sup_{w_h \in V_{h,l}, w_h \neq 0} \frac{|N(u, w_h)|}{\|w_h\|_h}$$

$$+ \inf_{p \in P_l(T_h)} (\alpha^* + 1) \|u - p\|_h + \sum_{K \in T_h} \sup_{w_h \in V_{h,l}^K} \frac{|a^K(p, w_h) - a^K_h(p, w_h)|}{\|w_h\|_K}, \tag{3.2}$$

where $\alpha_{\Delta}$ and $\alpha^*$ are from the stability property [A5] The nonconformity error is given by

$$N(u, w_h) = a(u, w_h) - (f, w_h). \tag{3.3}$$

We finish this section by collecting the remaining technicalities needed for the rest of the paper. In particular, we make the following regularity conditions on the mesh $T_h$ which are standard in the virtual element framework, see e.g., [3].

Assumption 3.5 (Mesh assumptions). Assume there exists some $\rho > 0$ such that the following hold.

(A6) For every element $K \in T_h$ and every edge $e \subset \partial K$, $h_e \geq \rho h_K$ where $h_e = |e|$ and $h_K$ is the diameter of $K$.

(A7) Assume that each element is star shaped with respect to a ball of radius $\rho h_K$.

Finally, we recall some standard results for the $L^2$ projection operator.

Definition 3.6. For any $K \in T_h$ define the $L^2(K)$ orthogonal projection onto the polynomial space $P_l(K)$, that is $P^l_K : L^2(K) \to P_l(K)$ by,

$$(P^l_K v, p)_K = (v, p)_K \quad \text{for all } p \in P_l(K),$$

and for any edge $e \subset \partial K$ define the $L^2(e)$ orthogonal projection onto $P_l(e)$, $P^l_e : L^2(e) \to P_l(e)$ by,

$$(P^l_e v, p)_e = (v, p)_e \quad \text{for all } p \in P_l(e).$$

A proof of the following error estimates can be obtained using for example the theory in either [13, 20].

Theorem 3.7. Under Assumption [3, 3] for $l \geq 0$ and for any $w \in H^m(K)$ with $1 \leq m \leq l + 1$, it follows that

$$|w - P^l_K w|_{s,K} \lesssim h^{m-s}_K |w|_{m,K} \tag{3.4}$$

for $s = 0, 1, 2$. Further, for any edge shared by $K^+, K^- \in T_h$ and for any $w \in H^m(K^+ \cup K^-)$, with $1 \leq m \leq l + 1$, it follows that

$$|w - P^l_e w|_{s,e} \lesssim h^{-s-\frac{1}{2}}_e |w|_{m,K^+ \cup K^-} \tag{3.5}$$

for $s = 0, 1, 2$. 


4 The virtual element spaces

We dedicate the next section to the virtual element discretization. We specify the chosen degrees of freedom by a dof tuple which allows us to easily encode a number of different local VEM spaces. A major part of the VEM method is the construction of projection operators. We detail a new construction of projection operators suitable for general VEM discretization of a wide range of fourth order problems with nonconstant coefficients.

Throughout this section we provide examples of nonconforming VEM spaces for fourth order problems ($C^1$ nonconforming spaces). In particular, we use as examples the original nonconforming space detailed in [4, 36] with $V_{h,l} \not\subset H^1_0(\Omega)$. As well as this space, we look at the $C^0$ conforming space detailed in [33, 34], such that $V_{h,l} \subset H^1_0(\Omega)$. We conclude this section with defining the global VEM spaces and the bilinear forms.

4.1 Degrees of freedom tuple

We begin this section by introducing the concept of a degrees of freedom (dof) tuple, used to generically describe the degrees of freedom relating to a VEM space on each element of the grid. So let $K \in \mathcal{T}_h$ be fixed in the following.

Definition 4.1. For the $C^1$ virtual element spaces, let the degrees of freedom tuple, $M \in \mathbb{Z}^5$, be defined as

$$M = (d_0^v, d_1^v, d_0^e, d_1^e, d_0^p).$$

The entries correspond to the number of moments used in the definition of our degrees of freedom, with $d_j^v$, for $j = 0, 1$, encoding the information for the vertex moments, $d_j^e$ for $j = 0, 1$, for the edge moments, and $d_0^p$ for the inner moments. The subscript $j = 0$ corresponds to moments of the function values and $j = 1$ to derivative moments on the vertices and edges.

From the dof tuple, we are able to define the corresponding degrees of freedom.

Definition 4.2. For a function $v_h \in H^2(K)$, the local degrees of freedom corresponding to the degrees of freedom tuple in (4.1) are given by the following.

1. The values $h |_{\partial e} D^j_1 v_h$ for each vertex $v$ of $K$ for $j = 0, 1$. Here $h_v$ is some local length scale associated to the vertex $v$, e.g., an average of the diameters of all surrounding elements.

2. The moments of $\partial^j_v v_h$ up to order $d_j^v$ on each $e \subset \partial K$, for $j = 0, 1$,

$$|e|^{-1+j} \int_{e} \partial^j_v v_h p\ ds \quad \forall p \in \mathbb{P}_e(e).$$

3. The moments of $v_h$ up to order $d_0^p$ inside $K$,

$$\frac{1}{|K|} \int_{K} v_h p\ dx \quad \forall p \in \mathbb{P}_d^e(K).$$

Remark 4.3. We use the convention that $D^0 v_h = v_h$ and $\partial^j_v v_h = v_h$. If any of the entries in the dof tuple $M$ are zero, this implies that we take constant moments, if an entry is less than or equal to $-1$, then this corresponds to no moments. So for example $d_0^v = -1$ implies that no vertex values are used. The case $d_0^e = -1$ is for example relevant for the $C^0$ nonconforming space presented in [19] but will not be considered in the discussion here.

Note that for the $C^1$ nonconforming spaces, we always have $d_0^v = 0$ and $d_0^e = -1$. We only begin to prescribe the vertex derivative values when we consider $C^1$ conforming spaces or even higher order conforming spaces see for example [3] [12].

We now give some examples of degrees of freedom tuples relating to some common VEM spaces to illustrate the idea.

Example 4.4. For $l \geq 2$, consider the original $C^1$ nonconforming space introduced in [4]. The dof tuple describing this space is

$$M_{\text{nc}}^{C^1} = (0, -1, l - 3, l - 2, l - 4).$$

This dof tuple also corresponds to the space considered in [30]. A visualization of these dofs on triangles can be seen in Figure 1.

For $l \geq 2$, consider the $C^1$ space introduced in [33, 34] which is $C^0$ conforming. This space is described by the dof tuple

$$M_{\text{conf}}^{C^1} = (0, -1, l - 2, l - 2, l - 4).$$

The dofs for this space are shown in Figure 2.

For $l \geq 3$, consider the $C^1$ conforming space given in [5]. The dof tuple describing the local dofs for this space is given by

$$M_{\text{con}}^{C^1} = (0, 0, l - 4, l - 3, l - 4).$$
Definition 4.6. The enlarged virtual element space to ensure that our projection operators satisfy certain \( L^2 \) projection properties which are stated at the end of this section.

### 4.2 Local spaces and projection operators

We now focus on the crucial aspect of defining the VEM spaces and projection operators. Following the approach in \[ 1 \] \[ 19 \] we introduce an enlarged local virtual element space to ensure that our projection operators satisfy certain \( L^2 \) projection properties which are stated at the end of this section.

**Remark 4.5.** As mentioned already, the concept of defining a dof tuple extends to describing the dofs relating to \( C^0 \) VEM spaces. Consider the simplest of the \( C^0 \) conforming serendipity spaces discussed in \[ 9 \]. The dofs for this space can be described by the dof tuple

\[
M_{C^0}^{IC} = (0, -1, l - 2, -1, l - 3).
\]

### 4.2 Local spaces and projection operators

We now focus on the crucial aspect of defining the VEM spaces and projection operators. Following the approach in \[ 1 \] \[ 19 \] we introduce an enlarged local virtual element space to ensure that our projection operators satisfy certain \( L^2 \) projection properties which are stated at the end of this section.

**Definition 4.6.** The enlarged virtual element space \( \mathring{V}_h^{K} \) on \( K \in T_h \) is defined as follows.

\[
\mathring{V}_h^{K} := \{ v_h \in H^2(K) : \Delta^2 v_h \in P_l(K), v_h|_c \in P_l(e), \Delta v_h|_c \in P_{l-2}(e), \forall c \subset \partial K \}
\]

with dimension \( \dim \mathring{V}_h^{K} := \tilde{N}_V^K = n^K_e (2l - 1) + \frac{1}{2}(l + 1)(l + 2) \), where \( n^K_e \) denotes the number of edges in the polygon \( K \).

The enlarged space \( \mathring{V}_h^{K} \) is characterized by the following extended degrees of freedom tuple, denoted by \( \mathring{M}^{K} \), where

\[
\mathring{M}^{K} = (0, -1, l - 2, -1, l, l).
\]

Note that the number of extended dofs is equal to \( \tilde{N}_{dofs} = n^K_e (2l - 1) + \frac{1}{2}(l + 1)(l + 2) \) so that we have \( \tilde{N}_V^K = \tilde{N}_{dofs} \). We denote with \( \tilde{L}^{K} \) the set of extended degrees of freedom described by \( \mathring{M}^{K} \) as given by Definition 4.2.

We show next that this set of degrees of freedom is unisolvent in \( \mathring{V}_h^{K} \).

**Lemma 4.7.** The set of extended degrees of freedom \( \tilde{L}^{K} \) is unisolvent over the space \( \mathring{V}_h^{K} \).

**Proof.** We show that if all the degrees of freedom vanish for \( v_h \in \mathring{V}_h^{K} \) then \( v_h \equiv 0 \). Using Green’s formula \[ 20 \], for a function \( v_h \in \mathring{V}_h^{K} \) it follows that

\[
|v_h|^2_{2,K} = \int_K D^2 v_h : D^2 v_h = \int_K \Delta v_h \Delta v_h + \int_K 2 \partial_{12} v_h \partial_{12} v_h - \partial_{11} v_h \partial_{22} v_h - \partial_{22} v_h \partial_{11} v_h
\]

\[
= \int_K \Delta^2 v_h v_h - \int_{\partial K} v_h \partial_n (\Delta v_h) + \int_{\partial K} \partial_n v_h (\Delta v_h - \partial_s v_h) + \int_{\partial K} \partial_n v_h \partial_s v_h
\]

\[
= 0
\]

\[
\Rightarrow v_h \equiv 0.
\]
using integration by parts on the last term we see that
\[ \int_{\partial K} \partial_{mx} v_h \partial_x v_h = \sum_{c \subset \partial K} \int_{c} \partial_{mx} v_h \partial_x v_h = - \sum_{c \subset \partial K} \int_{c} \partial_{mx}(v_h) v_h + [v_h \partial_{mx} v_h](e^+) + [v_h \partial_{mx} v_h](e^-) \]
where \( e^\pm \) denote the vertices of an edge \( e \). Now assuming that all the degrees of freedom \( \tilde{\Lambda}^K \) vanish, implies that the vertex values vanish - see (D1). Therefore,
\[ |v_h|^2_{2,K} = \int_K \Delta^2 v_h v_h - \int_{\partial K} v_h (\partial_n (\Delta v_h + \partial_s v_h)) + \int_{\partial K} \partial_s v_h (\Delta v_h - \partial_s v_h). \]
Using integration by parts on the second term, it holds that
\[ \int_{e} v_h (\partial_n (\Delta v_h + \partial_s v_h)) = - \int_{e} \partial_s v_h (\Delta v_h + \partial_s v_h) + [(\Delta v_h + \partial_s v_h)v_h](e^+) + [(\Delta v_h + \partial_s v_h)v_h](e^-) \]
where the last two terms are zero as well. Finally, it follows that
\[ |v_h|^2_{2,K} = \int_K \Delta^2 v_h v_h + \int_{\partial K} 2\partial_s v_h \Delta v_h. \]
Since \( v_h \in \tilde{V}^K_{h,l} \), it follows from (4.2) that \( \Delta^2 v_h \in \mathcal{P}_1(K) \) and \( \Delta v_h \in \mathcal{P}_{l-2}(e) \). We therefore see that \( |v_h|^2_{2,K} = 0 \).
As in [4], this implies that \( v_h = 0 \) due to the boundary conditions.

We now turn our attention to the local VEM space \( V^K_{h,l} \), which we define as a subspace of the enlarged virtual element space \( \tilde{V}^K_{h,l} \). First, we assume that we have a degrees of freedom tuple \( M^K = (0, \ldots, d_0, d_1, d_2, d_3) \) such that \( d_0 \leq l \). This degrees of freedom tuple gives us a set of degrees of freedom, \( \Lambda^K \), such that \( \Lambda^K \subset \tilde{\Lambda}^K \).

**Remark 4.8.** In this paper we focus on the construction for \( C^1 \) nonconforming spaces. However, note that the following discussion on defining projection operators also covers some conforming spaces suggested in the literature for solving second order problems, e.g., the spaces from [9]. As well as this, as discussed above a conforming \( C^1 \) space can also be described in this framework (using the dof tuple \( (0, 0, l - 4, l - 4, l - 4, l - 4) \)). The nonconforming \( C^0 \) space from [19] also fits the framework using the dof tuple \( (-1, -1, -1, -1, -1, -1, 2) \). Note that both of these spaces require a different enlarged space. Since we are not going to analyse these two spaces, we do not discuss these choices further but we would like to note that our definition of projection operators cover these cases as well with only minimum modifications.

In order to define the local VEM space, we first construct the following projections: an interior value projection, \( \Pi^K_0 : \tilde{V}^K_{h,l} \rightarrow \mathcal{P}_l(K) \), an edge value projection \( \Pi^K_0 : \tilde{V}^K_{h,l} \rightarrow \mathcal{P}_l(e) \), and an edge normal projection \( \Pi^K_1 : \tilde{V}^K_{h,l} \rightarrow \mathcal{P}_{l-1}(e) \). These projections have to be computable from the degrees of freedom \( \Lambda^K \) of a given \( v_h \in \tilde{V}^K_{h,l} \). Using \( \Pi^K_0, \Pi^K_0, \Pi^K_1 \) we can then define the VEM space \( V^K_{h,l} \), the gradient projection, \( \Pi^K_0 : \tilde{V}^K_{h,l} \rightarrow (\mathcal{P}_{l-1}(K))^2 \), and finally the hessian projection \( \Pi^K_1 : \tilde{V}^K_{h,l} \rightarrow (\mathcal{P}_{l-2}(K))^2 \), which satisfy certain \( L^2 \) projection properties discussed in the following. These projections are then used to define the discrete bilinear form.

**Definition 4.9.** We say that a value projection \( \Pi^K_0 \), an edge projection \( \Pi^K_0 \), and an edge normal projection \( \Pi^K_1 \) are *dof compatible* if for \( v_h \in \tilde{V}^K_{h,l} \) they are a linear combination of the original dofs \( \Lambda^K(v_h) \), and satisfy the following additional properties.

- The value projection \( \Pi^K_0 v_h \in \mathcal{P}_l(K) \) satisfies
  \[ \int_K \Pi^K_0 v_h p = \int_K v_h p \quad \forall p \in \mathcal{P}_{d_0}(K), \]
  and \( \Pi^K_0 q = q \) for all \( q \in \mathcal{P}_l(K) \).

- For each edge \( e \subset \partial K \), the edge projection \( \Pi^K_0 v_h \in \mathcal{P}_l(e) \) satisfies
  \[ \int_e \Pi^K_0 v_h p = \int_e v_h p \quad \forall p \in \mathcal{P}_{d_0}(e) \quad \Pi^K_0 v_h(e^+) = v_h(e^+), \]
  and \( \Pi^K_0 q = q|_e \) for all \( q \in \mathcal{P}_l(K) \).

- For each \( e \subset \partial K \), the edge normal projection \( \Pi^K_1 v_h \in \mathcal{P}_{l-1}(e) \) satisfies
  \[ \int_e \Pi^K_1 v_h p = \int_e \partial_n v_h p \quad \forall p \in \mathcal{P}_{d_1}(e), \]
  and \( \Pi^K_1 q = \partial_n q|_e \) for all \( q \in \mathcal{P}_l(K) \).
Note that there are multiple choices for defining the value, edge, and edge normal projections such that they are dof compatible. We provide an example for defining these projections based on constraint least squares problems after defining the gradient and hessian projections.

**Definition 4.10.** The gradient projection $\Pi_h^K : \tilde{V}_{h,l}^K \to (P_{l-1}(K))^2$ is now taken to be

$$
\int_K \Pi_h^K v_h p = - \int_K \Pi_h^K v_h \nabla p + \sum_{e \subseteq \partial K} \int_e \Pi_h^K v_h \nabla n, \quad \forall p \in P_{l-1}(K)^2
$$

and the hessian projection $\Pi_h^2 : \tilde{V}_{h,l}^K \to (P_{l-2}(K))^{2 \times 2}$ to be

$$
\int_K \Pi_h^2 v_h p = - \int_K \Pi_h^2 v_h \nabla \nabla p + \sum_{e \subseteq \partial K} \int_e \Pi_h^2 v_h n_n + \sum_{e \subseteq \partial K} \int_e \Pi_h^2 v_h n_n \cdot \nabla \nabla n + \partial_n(\Pi_h^2 v_h) n_n) \cdot \nabla \nabla n, \quad \forall p \in (P_{l-2}(K))^{2 \times 2}.
$$

Here $n, \tau$ denote the unit normal and tangent vectors of $e$, respectively.

One possible dof compatible choice for the value and two edge projections is shown in the following example.

**Example 4.11.** We consider projection operators obtained from constraint least squares problems. Consider the dof tuple $(0, -1, d_0, d_1, d_2)$.

- We define the value projection $\Pi_h^0 v_h \in P_l(K)$ as the solution to the problem

  $$
  \begin{align*}
  \text{Minimise: } & \sum_{i=1}^{N_{dof}} (\text{dof}_i(\Pi_h^0 v_h) - \text{dof}_i(v_h))^2, \\
  \text{subject to: } & \int_K \Pi_h^0 v_h p = \int_K v_h p, \quad \forall p \in P_{d_0}(K).
  \end{align*}
  $$

  From this definition it is clear that (4.3) holds.

- It is clear that if we choose the edge projection to be the unique solution in $P_l(e)$ of

  $$
  \int_e \Pi_e^0 v_h p = \int_e v_h p, \quad \forall p \in P_{d_0}(e), \quad \Pi_e^0 v_h(e^+) = v_h(e^+),
  $$

  then $\Pi_e^0 v_h \in P_l(e)$ and (4.4) is satisfied.

- Finally, if we define the normal edge projection $\Pi_e^1 v_h \in P_{l-1}(e)$ to be the unique solution of

  $$
  \int_e \Pi_e^1 v_h p = \int_e \partial_n v_h p, \quad \forall p \in P_{d_1}(e), \quad \text{and} \quad \int_e \Pi_e^2 v_h p = \int_e \partial_n(\Pi_h^0 v_h) p, \quad \forall p \in P_{l-1}(e) \setminus P_{d_1}(e)
  $$

  then we satisfy (4.5). Note that we could replace the final constraint with

  $$
  \int_e \Pi_e^1 v_h p = \int_e \Pi_h^0 v_h \cdot n_p, \quad \forall p \in P_{l-1}(e) \setminus P_{d_1}(e)
  $$

  since $\Pi_h^K v_h$ does not depend on $\Pi_e^1 v_h$. This is what we use in our implementation.

Note that these definitions also cover the case of the $C^0$ nonconforming space with $d_1 = -1$. The gradient projection in this case is identical to the one given in (4.9) but we get a projection for the hessian as well.

We can finally use given *dof compatible* projections $\Pi_h^0, \Pi_h^1$, and $\Pi_h^2$ to define the local virtual element space on $K$.

**Definition 4.12.** The local virtual element space $V_{h,l}^K$ is given as the following subset of the enlarged space.

$$
V_{h,l}^K := \{ v_h \in \tilde{V}_{h,l}^K : (v_h - \Pi_h^K v_h, p)_K = 0 \quad \forall p \in P_l(K) \setminus P_{d_0}(K),

(\partial_{n_j} v_h - \Pi_h^j v_h, p)_e = 0 \quad \forall p \in P_{l-2}(e) \setminus P_{d_1}(e) \text{ for } j = 0, 1 \}.
$$

We now show that the subset of dofs $\Lambda^K$ are unisolvent for our local VEM space $V_{h,l}^K$.

**Lemma 4.13.** The original set of degrees of freedom $\Lambda^K$ is unisolvent for $V_{h,l}^K$. 

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Proof. Let \( v_h \in V_{h,l}^{K} \), and set the original dofs \( \Lambda^K \) to zero. Then, noting that both the value and edge projection are computed using the dofs, it follows that \( \Pi^K_0 v_h = 0, \Pi^0 v_h = 0, \) and \( \Pi^1 v_h = 0 \). From (4.8), we see that

\[
(\nu^h - \Pi^K_0 v_h, p)_K = (\nu^h, p)_K = 0 \quad \forall p \in \mathbb{P}_l(K) \backslash \overline{\mathbb{P}}_{d_0}^{\nu}(K),
\]

and for \( j = 0, 1 \),

\[
(\partial^T \nu^h - \Pi_j^0 v_h, p)_e = (\partial^T \nu^h, p)_e = 0 \quad \forall p \in \mathbb{P}_{l-2}(e) \backslash \overline{\mathbb{P}}_{d_0}^{0}(e),
\]

i.e. the inner moments of order \( l, \ldots, l - (d_0 + 1) \) and the edge moments of order \( l - 2, \ldots, l - (d_0 + 1) \) of \( v_h \) are also zero. Similarly, the higher order edge normal moments are also zero. So viewing \( v_h \) as a function in the enlarged space, \( v_h \in V_{h,l}^{K} \subset V_{h,l}^{K} \) it follows that \( v_h \equiv 0 \) as a function in \( V_{h,l}^{K} \) with the extended degrees of freedom set to zero, using Lemma 4.7. Therefore the original dofs are unisolvent for \( V_{h,l}^{K} \).

This leads us to a final crucial result of this subsection. We see in the next Lemma that by construction, our value, gradient, and hessian projections possess important \( L^2 \) projection properties. In particular, due to construction of the VEM space, the value projection is identical to the \( L^2 \) projection into the space of polynomials of degree \( l \) for all VEM functions.

Lemma 4.14. If the value, edge, and edge normal projections are dof compatible, then for any \( v_h \in V_{h,l}^{K} \) it follows that, for \( s = 0, 1, 2 \),

\[
\Pi^K_0 v_h \in \mathbb{P}^{l-s}_K(D^s v_h).
\]

Due to the dof compatibility, and the fact that \( \mathbb{P}_l(K) \subset \tilde{V}_{h,l}^{K} \), it holds that \( \mathbb{P}_l(K) \subset V_{h,l}^{K} \). This implies that \( \Pi^K_0 p = D^s p \) for \( p \in \mathbb{P}_l(K) \) for each \( s = 0, 1, 2 \).

Proof. Using the dof compatibility for the value projection \( \Pi^K_0 \), Definition 4.9, it is clear that

\[
\int_K \Pi^K_0 v_h p = \int_K v_h p \quad \forall p \in \mathbb{P}_{d_0}^s(K)
\]

while the definition of the space \( V_{h,l}^{K} \) (4.8) leads to

\[
\int_K \Pi^K_0 v_h p = \int_K v_h p \quad \forall p \in \mathbb{P}_l(K) \backslash \overline{\mathbb{P}}_{d_0}^0(K).
\]

Combining the two we obtain the stated \( L^2 \) projection property of \( \Pi^K_0 \). Similar arguments are used for the other projections: due to the polynomial exactness of \( \Pi^K_0 \) and the definition of the extended space \( \tilde{V}_{h,l} \) we have \( \Pi_0 v_h = v_h |_e \) and therefore for a polynomial \( p \in \mathbb{P}_{l-1}(K) \),

\[
\int_K \Pi^K_2 v_h p = -\int_K \Pi^K_2 v_h \nabla p + \sum_{e \subseteq \partial K} \int_e \Pi^K_0 v_h np = -\int_K v_h \nabla p + \sum_{e \subseteq \partial K} \int_e v_h np
\]

where we used the \( L^2 \) projection property of \( \Pi^K_0 \). Applying integration by parts to the RHS completes the proof. Finally we have for the hessian projection using the results already proven and the dof compatibility of the edge normal projection, \( \Pi^2 \),

\[
\int_K \Pi^K_2 v_h p = -\int_K \Pi^K_2 v_h \nabla p + \sum_{e \subseteq \partial K} (\int_e \Pi^K_0 v_h \nabla \nabla p + \partial_e(\Pi^K_0 v_h, \nabla \nabla p))
\]

so that the result follows using integration by parts.

4.3 Global spaces and the discrete bilinear form

We conclude with the definition of the global virtual element space and the discrete bilinear form, ensuring that it satisfies the abstract assumptions given in (4.41)-(4.45). We keep the presentation brief, since it follows the general construction of VEM found in the literature.

We define the global space \( V_{h,l} \) by

\[
V_{h,l} := \{ v_h \in H^2_{l,nc}(T_h) : v_h |_K \in V_{h,l}^{K} \quad \forall K \in \mathcal{T}_h \}.
\]

Recall that \( H^2_{l,nc}(T_h) \) is the \( H^2 \) nonconforming space detailed in (A2).

Remark 4.15. In the definition of the global VEM space in (4.9) taking \( v_h \in H^2_{l,nc}(T_h) \) restricts us to values in our dof tuple which satisfy the following: \( d_0^l = 0, d_0^l = -1, d_0 \geq l - 3, d_0^l \geq l - 2, \) and \( d_0 \geq l - 4 \). We assume that our dof tuple satisfies these criteria later on in Section 5.
We extend Definition 4.16 to arrive at the global degrees of freedom.

Definition 4.16. Global degrees of freedom are given by the following.

1. The values \( h_j^jD^jv_h \) for each internal vertex \( v \) of \( T_h \) for \( j = 0, 1 \).
2. The moments of \( \partial_1^j v_h \) up to order \( d_j^j \) for \( j = 0, 1 \), on each internal edge \( e \in E_h \)

\[
|e|^{-1+j} \int_e \partial_1^j v_h p \, ds \quad \forall p \in P_{d_j^j}(e).
\]

3. The moments of \( v_h \) up to order \( d_0^j \) inside each \( K \in T_h \)

\[
\frac{1}{|K|} \int_K v_h p \, dx \quad \forall p \in P_{d_0^j}(K).
\]

We set the local degrees of freedom which correspond to boundary vertices and boundary edges, \( e \in E_h^{bdry} \), to zero. Note that the global degrees of freedom are unisolvent - this follows from the unisolvency of the local stabilization term.

Remark 4.18. Furthermore, notice that due to Lemma 4.14, it holds that \( \Pi_u \) of (Approximation error estimate) is presented in Theorem 5.7. Proving convergence involves bounding all of the terms in the Strang-type error. In this section we collect all the building blocks needed to prove a general convergence result in the energy norm, which is presented in Theorem 5.1. Proving convergence involves bounding all of the terms in the Strang-type estimate presented in Theorem 5.3. In the following we consider each term from (3.2) in the order in which they appear, starting with the approximation error.

5 Error analysis

5.1 Interpolation error estimate

The following estimate for the interpolation error, the first term in equation (3.2), is simply a consequence of standard scaling arguments.

Theorem 5.1 (Approximation error estimate). Let \((A1)\) - \((A7)\) hold, defined in Assumptions 3.1 and 3.5. Let \( u \in H^{l+1}(\Omega) \) be the solution to the continuous problem (2.2). Then, it follows that

\[
\inf_{v_h \in V_h^l} \| u - v_h \|_h \leq C(h^{l-1}\| u \|_L^p + h^l\| \beta \|_{L^\infty} + h^{l+1}\| \gamma \|_{L^\infty})|u|_{l+1}.
\]

for a constant \( C \) independent of \( h \).
5.2 Load term

The next term to appear in the a priori bound (3.2) is an error estimate for the load term. To treat this term, following the methods in [11 19], we define $f_h$ to be the piecewise $L^2$ projection of $f$ on $T_h$, $f_h := P^l_K f$. Using Lemma 4.14 we observe that for $w_h \in V_{h,l}$,

$$\langle f_h, w_h \rangle = \sum_{K \in T_h} (P^l_K f, w_h)_K = \sum_{K \in T_h} (f, P^l_K w_h)_K = \sum_{K \in T_h} (f, \Pi^K_0 w_h)_K.$$ 

Consequently, the right hand side of the discrete variational problem (5.1) is computable. The following estimate now follows easily using for example, the method in [4].

**Lemma 5.2.** For $l \geq 2$, let $s$ be an integer with $f \in H^s(\Omega)$ and define $r := \min(s - 1, l)$. Then, the following estimate holds

$$|(f_h, w_h) - (f, w_h)| \leq Ch^{r+2}|f|_{r+1}|w_h|_{1,h} \tag{5.1}$$

for a constant $C$ independent of $h$.

**Proof.** We can show the following using the bounds of Theorem 3.7

$$|(f_h, w_h) - (f, w_h)| = \sum_{K \in T_h} \int_K (P^l_K f - f) w_h = \sum_{K \in T_h} \int_K (P^l_K f - f)(w_h - P^l_K w_h)|K|.$$ 

$$\leq \sum_{K \in T_h} \|P^l_K f - f\|_{0,K} \|w_h - P^l_K w_h\|_{0,K} \leq Ch^{r+1}|f|_{r+1}h^l|w_h|_{1,h},$$

hence (5.1) holds, as required. \qed

5.3 Nonconformity error

We now turn our attention to the next term in the a priori bound (3.2), the nonconformity error $N(u, w_h)$. From now on we assume that $l \geq 2$ and as per Remark 4.14 the values in our degrees of freedom tuple satisfy the following $d_0^e = 0$, $d_0^e = -1$, $d_0^e \geq l - 3$, $d_0^e \geq l - 2$, and $d_0^e \geq l - 4$. Recall that as a consequence of this for $w_h \in V_{h,l}$, and for any edge $e$ in the grid

$$\int_e [w_h] p ds = 0 \quad \forall p \in P_{l-3}(e), \tag{5.2}$$

$$\int_e [\partial_n w_h] q ds = 0 \quad \forall q \in P_{l-2}(e). \tag{5.3}$$

Applying integration by parts gives us, for any $q \in P_{l-2}(e)$,

$$\int_e [\partial_n w_h] q ds = 0. \tag{5.4}$$

This follows since the jump is zero at the vertices on the edges and $\partial_s q \in P_{l-3}(e) = P_{l-5}(e)$ which means that we can apply (5.2).

**Lemma 5.3** (Nonconformity error). We assume that the coefficients satisfy $\kappa \in W^{2,\infty}(\Omega)$ and $\beta \in W^{1,\infty}(\Omega)$. Assume that the solution $u$ to (2.2) satisfies $u \in H^4(\Omega)$. Then, for $w_h \in V_{h,l}$ the nonconformity error $N(u, w_h)$ satisfies

$$N(u, w_h) = \sum_{e \in E_h} \left\{ \int_e \kappa (\Delta u - \partial_s u)[\partial_n w_h] + \int_e \kappa \partial_s u [\partial_n w_h] + \int_e (\partial_n w_h) \kappa \partial_s u - \partial_n (\kappa \Delta u)[w_h] \right\}. \tag{5.5}$$

**Proof.** Using integration by parts, we can express the hessian and gradient terms as follows

$$\int_K \kappa D^2 u : D^2 w_h = -\int_K \sum_{i,j=1}^2 \partial_{ij} (\kappa \partial_{ij} u) \partial_i w_h + \int_{\partial K} \sum_{i,j=1}^2 \kappa \partial_{ij} u \partial_i w_n_j \tag{5.6}$$

$$\int_K \beta D u \cdot D w_h = -\int_K D \cdot (\beta D u) w_h + \int_{\partial K} \beta \partial_n w_h. \tag{5.7}$$

Since we assume $u \in H^4(\Omega)$ we use (2.3) and an application of integration by parts to see that

$$(f, w_h) = \sum_{K \in T_h} -\int_K \sum_{i,j=1}^2 \partial_{ij} (\kappa \partial_{ij} u) \partial_i w_h + \int_{\partial K} \sum_{i,j=1}^2 \partial_{ij} (\kappa \partial_{ij} u) w_h n_j - \int_K D \cdot (\beta D u) w_h + \int_K \gamma u w_h.$$
Therefore the nonconformity error is equal to

\[ N(u, w_h) = a(u, w_h) - (f, w_h) = \sum_{K \in \mathcal{T}_h} \left\{ \int_{\partial K} \sum_{i,j=1}^2 \kappa \partial_{ij} u \partial_{ij} w_h n_j - \int_{\partial K} \sum_{i,j=1}^2 \partial_j (\kappa \partial_{ij} u) w_h n_i + \int_{\partial \Omega} \beta \partial_u w_h \right\}. \]  
(5.8)

We also use the following identities for rewriting the boundary terms in a way that is useful later on in Theorem 5.5.

\[ \int_{\partial K} \sum_{i,j=1}^2 \kappa \partial_{ij} u \partial_{ij} w_h n_j = \int_{\partial K} \kappa (\Delta u - \partial_u u) \partial_u w_h + \partial_u u \partial_u w_h \]  
(5.9)

\[ \int_{\partial K} \sum_{i,j=1}^2 \partial_j (\kappa \partial_{ij} u) w_h n_i = \int_{\partial K} (\partial_u u \partial_u \kappa - \partial_u u \partial_u \kappa + \partial_u (\kappa \Delta u)) w_h \]  
(5.10)

It is straightforward albeit tedious to show that (5.9) and (5.10) hold. It is now clear that the result (5.5) follows from these expressions and equation (5.8).

The next corollary looks at how (5.5) simplifies when our VEM space is $C^0$ conforming.

**Corollary 5.4.** Under the assumptions of Lemma 5.3 and assuming that $V_{h,l} \subset H^1_0(\Omega)$, it follows that

\[ N(u, w_h) = \sum_{e \in \mathcal{E}_h} \int_{\partial e} \kappa (\Delta u - \partial_u u) | \partial_u w_h | \ ds. \]  
(5.11)

We now bound each term in (5.5) to achieve an error estimate for the nonconformity error. This essentially involves the jump properties of the VEM space (5.2) - (5.4) as well as standard interpolation estimates detailed in Theorem 5.7.

**Theorem 5.5 (Nonconformity error bound).** Let (AJ1 - AJ7) hold. Assume that the solution $u$ to (2.2) satisfies $u \in H^{l+1}(\Omega)$ and assume that the coefficients satisfy $\kappa \in W^{l+1, \infty}(\Omega)$ and $\beta \in W^{l-2, \infty}(\Omega)$. Then, for $l \geq 3$ the nonconformity error satisfies the following estimate

\[ |N(u, w_h)| \leq \begin{cases} C h^l (|\kappa|_{W^{l+1, \infty}(\Omega)} |u|_{l+1} + |\beta|_{W^{l-2, \infty}(\Omega)} |w_h|_{2l+1}) & \text{if } V_{h,l} \nsubseteq H^1_0(\Omega), \\ C h^{l-1} (|\kappa|_{W^{l-1, \infty}(\Omega)} |u|_{l+1} + |\beta|_{W^{l-2, \infty}(\Omega)} |w_h|_{2l+1}) & \text{if } V_{h,l} \subseteq H^1_0(\Omega), \end{cases} \]  
(5.12)

for any $w_h \in V_{h,l}$, for a constant $C$ independent of $h$. For $l = 2$, the nonconformity error satisfies the following estimate

\[ |N(u, w_h)| \leq \begin{cases} C(\|\kappa\|_{W^{l+1, \infty}(\Omega)} |u|_{l+1} + |\beta|_{L^\infty} |w_h|_{2l+1} + L_\infty |f|) |w_h|_{l+1} & \text{if } V_{h,l} \nsubseteq H^1_0(\Omega), \\ C |\kappa|_{W^{l+1, \infty}(\Omega)} |u|_{l+1} |w_h|_{2l+1} & \text{if } V_{h,l} \subseteq H^1_0(\Omega), \end{cases} \]  
(5.13)

for a constant $C$ independent of $h$.

**Proof.** Firstly, consider the case $l \geq 3$. Then since $u \in H^4(\Omega)$, using Lemma 5.3, it holds that

\[ N(u, w_h) = \sum_{e \in \mathcal{E}_h} \int_{\partial e} \kappa (\Delta u - \partial_u u) | \partial_u w_h | \ + \sum_{e \in \mathcal{E}_h} \int_{\partial e} \kappa \partial_{ij} u | \partial_{ij} w_h | \]

\[ + \sum_{e \in \mathcal{E}_h} \int_{\partial e} (\partial_u u \partial_u \kappa - \partial_u u \partial_u \kappa + \partial_u (\kappa \Delta u)) w_h =: I_1 + I_2 + I_3. \]

For $I_1$, we apply Lemma 5.3, to see that,

\[ I_1 \leq \sum_{e \in \mathcal{E}_h} \int_{\partial e} \left( \kappa (\Delta u - \partial_u u) - \mathcal{P}_e^{l-2} (\kappa (\Delta u - \partial_u u)) \right) \left( | \partial_u w_h | - \mathcal{P}_e^l | \partial_u w_h | \right) \]

\[ \leq \sum_{e \in \mathcal{E}_h} \| \kappa (\Delta u - \partial_u u) - \mathcal{P}_e^{l-2} (\kappa (\Delta u - \partial_u u)) \|_{0,e} \| \partial_u w_h | - \mathcal{P}_e^l | \partial_u w_h | \|_{0,e}. \]

Where we have used the properties of the $L^2$ projection, as well as Cauchy-Schwarz in the last step. Using the estimates

\[ \| \kappa (\Delta u - \partial_u u) - \mathcal{P}_e^{l-2} (\kappa (\Delta u - \partial_u u)) \|_{0,e} \leq C h^{l-2+1-1/2} | \kappa (\Delta u - \partial_u u) |_{l-1}, \]

\[ | | \partial_u w_h | - \mathcal{P}_e^l | \partial_u w_h | \|_{0,e} \leq C h^{l-1/2} | \partial_u w_h |_{l-1}. \]
It therefore follows that,
\[ I_1 \leq \left| \sum_{e} \int_{e} \kappa (\Delta u - \partial_{ss} u) \partial_{ss} w_h \, ds \right| \leq C h^{-1} \| \kappa \|_{W^{1,\infty}} |u|_{1+1} |w_h|_{2,h}. \]  

(5.14)

For the term \( I_2 \), we apply (5.4), introduce the polynomial \( P^2_v(\kappa \partial_{ss} u) \) and use standard interpolation estimates to get,
\[ I_2 \leq \left| \sum_{e \in \mathcal{E}_h} \int_{e} (\kappa \partial_{ss} u - P^2_v(\kappa \partial_{ss} u)) \left( \partial_{ss} w_h - P^2_v(\partial_{ss} w_h) \right) \, ds \right| \leq C h^{-1} \| \kappa \|_{W^{1,\infty}} |u|_{1+1} |w_h|_{2,h}. \]  

(5.15)

Finally, consider the term \( I_3 \). For ease of notation, let us set \( u^* := \partial_{ss} u \partial_{ss} \kappa - \partial_{ss} u \partial_{ss} \kappa - \partial_{ss} (\kappa \Delta u) + \beta \partial_{ss} u \) and note that
\[ |u^*|_{-2} \leq \| \kappa \|_{W^{1,\infty}} (|u|_0 + |u|_{1+1}) + \| \beta \|_{W^{1,\infty}} |u|_{-1}. \]

For this term, \( I_3 \), we follow the approach taken in [36] and introduce the interpolation of \( w_h \) into the lowest order conforming VEM space \( \Pi^1 w_h \in H^1_0(\Omega) \).
\[ I_3 \leq \left| \sum_{e \in \mathcal{E}_h} \int_{e} (\partial_{ss} u \partial_{ss} \kappa - \partial_{ss} u \partial_{ss} \kappa - \partial_{ss} (\kappa \Delta u) - \beta \partial_{ss} u)(w_h - \Pi^1 w_h) \, ds \right| \leq C h^{-1} \| u^* \|_{-2} |w_h|_{2,h}. \]  

(5.16)

Hence, when \( l \geq 3 \), combining (5.14), (5.15), and (5.16), the result (5.12) follows.

Now consider the case \( l = 2 \). Assume that \( u \) solves problem (2.2) in \( H^{-1}(\Omega) \). For a test function \( v \in H^1_0(\Omega) \) (and for \( u \in H^1(\Omega) \)) it holds that
\[ (f, v) = \sum_{K \in \mathcal{T}_h} \left\{ - \int_{K} \sum_{i,j=1}^{2} \partial_j (\kappa \partial_{s} u) \partial_{i} v + \int_{K} \beta D u \cdot D v + \int_{K} \gamma u w_h \right\}. \]  

(5.17)

Using (5.6) to express the hessian term in the bilinear form, it follows that
\[ \mathcal{N}(u, w_h) = a(u, w_h) - (f, w_h) \]

\[ = \left( \sum_{K \in \mathcal{T}_h} \left\{ - \int_{K} \sum_{i,j=1}^{2} \partial_j (\kappa \partial_{s} u) \partial_{i} w_h + \int_{K} \beta D u \cdot D w_h + \int_{K} \gamma u w_h \right\} - (f, w_h) \right) \]

\[ + \sum_{K \in \mathcal{T}_h} \int_{\partial K} \sum_{i,j=1}^{2} \kappa \partial_{s j} u \partial_{s i} w_h n_j =: E_1 + E_2. \]

Using arguments as in the proof of Lemma (5.3) it follows that \( E_2 = I_1 + I_2 \). These terms can be bounded as in (5.14) and (5.15) respectively. Therefore, it follows that
\[ E_2 \leq C h \| \kappa \|_{W^{1,\infty}} |u|_3 |w_h|_{2,h}. \]

Note that when the VEM space is conforming, \( V_{h,l} \subset H^1_0(\Omega) \), it follows that \( w_h \) is a viable test function in \( H^1_0(\Omega) \) and therefore we can take \( v = w_h \) in (5.17). Hence, the nonconformity error reduces to the following
\[ \mathcal{N}(u, w_h) = E_2 \leq C h \| \kappa \|_{W^{1,\infty}} |u|_3 |w_h|_{2,h}. \]

To treat \( E_1 \) in the case that \( V_{h,l} \not\subset H^1_0(\Omega) \) we now introduce the interpolation of \( w_h \) into the lowest order conforming VEM space, \( \Pi^1 w_h \in H^1_0(\Omega) \). Then, it holds that
\[ E_1 = \sum_{K \in \mathcal{T}_h} \left\{ \int_{K} \text{div}(\kappa D^2 u) \cdot D(\Pi^1 w_h - w_h) + \int_{K} \beta D u \cdot D(w_h - \Pi^1 w_h) + \int_{K} \gamma u (w_h - \Pi^1 w_h) \right\} \]

\[ + (f, \Pi^1 w_h - w_h). \]

Using standard estimates, we see that
\[ \sum_{K \in \mathcal{T}_h} \int_{K} \text{div}(\kappa D^2 u) \cdot D(\Pi^1 w_h - w_h) \leq C h \| \kappa \|_{W^{1,\infty}} |u|_3 |w_h|_{2,h}, \]

\[ \sum_{K \in \mathcal{T}_h} \int_{K} \beta D u \cdot D(w_h - \Pi^1 w_h) \leq C h \| \beta \|_{L^\infty} |u|_1 |w_h|_{2,h}, \]
\[
\sum_{K \in T_h} \int_K \gamma a(w_h - \Pi^1 w_h) \leq C h \|\gamma\|_{L^\infty} \|u\|_{W^{1,1}} \|w_h\|_{1,h},
\]

and finally,
\[
(f, \Pi^1 w_h - w_h) \leq C h \|f\|_{0} \|w_h\|_{1,h}.
\]

This concludes the proof. \(\square\)

### 5.4 Energy norm estimate

Now that we have successfully bounded the nonconformity error, we look at the final term in Theorem 4.3 and prove convergence in the energy norm.

Before we state the results, observe that the following equivalence between the discrete norms and the energy norm holds as a consequence of a discrete Poincaré inequality, see [4]. To ease the presentation of the next Theorems and Corollary we define \(\beta_{0} = \min(\beta(x), \gamma(x))\) and note that
\[
|w_h|_{s,h} \leq C \eta_h \|w_h\|_h
\]
for each \(s = 0, 1, 2\), where \(\eta_0 = \frac{1}{\sqrt{2\pi}}, \eta_1 = \frac{1}{\max(\frac{1}{\sqrt{2\pi}}, \sqrt{\nu})}, \) and \(\eta_2 = \frac{1}{\sqrt{2\pi}}\).

**Theorem 5.6.** Assume that (A1) - (A7) hold, defined in Assumptions 3.1 and 3.5. Let \(u \in H^{1+1}(\Omega)\) be the solution to continuous problem (2.2). Assume that the coefficients satisfy \(\kappa \in W^{1-\infty}(\Omega), \beta \in W^{1-\infty}(\Omega), \) and \(\gamma \in W^{1+1,\infty}(\Omega)\). Then it holds that
\[
\inf_{p \in P_1(T_h)} \left[ \|u - p\|_h + \sum_{K \in T_h} \sup_{w_h \in V^h_K} \frac{|a^K(p, w_h) - a^K(p, w_h)|}{\|w_h\|_K} \right] \leq C (h^{-1} c_2 + h c_1 + h^{1+1} c_0) \|u\|_{1+1} \tag{5.18}
\]
for a constant \(C\) independent of \(h\). We define the remaining constants \(c_0, c_1,\) and \(c_2\) as follows; let \(c_2 = \|\kappa\|_{L^\infty}^\frac{2}{\nu} + \eta_2 \|\kappa\|_{W^{1-\infty}} , c_1 = \|\beta\|_{L^\infty} + \eta_2 \|\beta\|_{W^{1-\infty}},\) and \(c_0 = \|\gamma\|_{L^\infty} + \eta_2 \|\gamma\|_{W^{1+1-\infty}}.\)

**Proof.** To show that (5.18) holds, we have that
\[
\inf_{p \in P_1(T_h)} \left[ \|u - p\|_h + \sum_{K \in T_h} \sup_{w_h \in V^h_K} \frac{|a^K(p, w_h) - a^K(p, w_h)|}{\|w_h\|_K} \right] \leq \left\| u - p^h \right\|_h + \sum_{K \in T_h} \sup_{w_h \in V^h_K} \frac{|a^K(p^h_K u, w_h) - a^K(p^h_K u, w_h)|}{\|w_h\|_K}.
\]
Note that the first term is bounded easily by standard interpolation estimates. For the other term, we use the property that \(\Pi^0 p = p\) which implies that the stabilization part of the discrete bilinear form vanishes, see Remark 4.18. Therefore,
\[
|a^K(p^h_K u, w_h) - a^K(p^h_K u, w_h)| \leq \left| \int_K \kappa D(D(P^h_K u) - D^2 w_h) - \int_K \kappa \Pi^K (P^h_K u) : \Pi^K w_h \right| + \left| \int_K \beta D(P^h_K u) \cdot D w_h - \int_K \beta \Pi^K (P^h_K u) : \Pi^K w_h \right| + \left| \int_K \gamma (P^h_K u) w_h - \int_K \gamma \Pi^K (P^h_K u) \Pi^K w_h \right| =: T_2 + T_1 + T_0.
\]

Then, denoting the coefficients as \(\alpha_2 := \kappa, \alpha_1 := \beta\) and \(\alpha_0 := \gamma\), we can show that
\[
T_0 \leq C h^{1+1} \|a\|_{W^{1+1,\infty}} \|u\|_{1+1} \|w_h\|_{1,h} \tag{5.19}
\]
To see that (5.19) holds, we use the results of Lemma 4.14 to express our projections as \(L^2\) projections. Hence, it follows that
\[
T_0 = \left| \int_K \alpha_2 (D^2(P^h_K u) - D^2 w_h - P^{1-s}_K D^2(P^h_K u) : P^{1-s}_K D^2 w_h) \right| = \left| \int_K \alpha_2 D^2(P^h_K u) : (I - P^{1-s}_K) D^2 w_h \right| \leq \|I - P^{1-s}_K\| \alpha_2 \|D^2(P^h_K u)\|_{0,K} \|D^2 w_h\|_{0,K} \leq C h^{1+1} \|a\|_{W^{1+1,\infty}} \|u\|_{1+1} \|w_h\|_{1,h}.
\]
The result now follows. \(\square\)

We now have the following convergence theorem which is a result of Theorems 5.1, 5.5, 5.6 and Lemma 5.2.
Theorem 5.7 (Convergence in the energy norm). Assume that (4.1) and (4.7) hold, defined in Assumptions 3.3 and 2.2. Let \( u \in H^{l+1}(\Omega) \) be the solution to the continuous problem and suppose that \( u_h \in V_{h,l} \) is the solution to the discrete problem (3.1). Assume that the coefficients satisfy \( \kappa \in W^{l-1,\infty}(\Omega), \beta \in W^{l,\infty}(\Omega), \) and \( \gamma \in W^{l,\infty}(\Omega) \). Let \( f \in H^l(\Omega) \) and define \( r := \min(s-1, l) \). Then, under these assumptions there exists a constant \( C \) independent of \( h \) such that

\[
\| u - u_h \|^2 \leq C \{ h^{-1} (c_2 |u|_{l+1} + c_3 |u_l|_l + c_4 |u_{l-1}|_{l-1}) + h^l c_1 |u|_{l+1} + h^{l+1} c_0 |u_{l+1}| + h^{l+2} \eta_l |f|_{l+1} \}. \tag{5.20}
\]

Recall that the constants \( c_0, c_1, \) and \( c_2 \) are defined in Theorem 5.6 and define \( c_3 = \eta_2 \| \kappa \|_{W^{l-1,\infty}} \) and \( c_4 = \eta_2 \| \beta \|_{W^{l,\infty}} \).

If \( V_{h,l} \subset H^l_0(\Omega) \) then it follows that

\[
\| u - u_h \|_k \leq C \{ (h^{-1} c_2 + h^l c_1 + h^{l+1} c_0) |u|_{l+1} + h^{l+2} \eta_l |f|_{l+1} \}. \tag{5.21}
\]

6 Perturbation problem

We turn our attention to the following fourth order perturbation problem. For a polygonal domain \( \Omega \subset \mathbb{R}^2 \) the perturbation problem reads as follows

\[
e^2 \Delta^2 u - \Delta u = f, \quad \text{in } \Omega,
\]

\[
u = \partial_n u = 0, \quad \text{on } \partial \Omega. \tag{6.1}
\]

We make the minimal assumptions that \( f \in L^2(\Omega) \) and \( \epsilon \in \mathbb{R} \) such that \( 0 < \epsilon \leq 1 \). Taking \( \kappa(x) = \epsilon^2, \beta(x) = 1, \) and \( \gamma(x) = 0 \) we can examine the error analysis from the previous section, with the energy norm now becoming

\[
\| u \|^2 = \epsilon^2 |u|_2^2 + |u|_2^2.
\]

It is well known that for example the lowest order \( C^1 \) nonconforming space on triangles (the Morley element, \[25\]) does not lead to a scheme that is robust with respect to \( \epsilon \to 0 \) (see for example \[13\] \[20\]). There have been a range of modifications suggested to the original Morley element, for example in \[29\] \[32\] \[33\] \[34\]. In \[29\] a modification is suggested which on triangles corresponds to our \( C^1-C^0 \) conforming in the lowest order setting and convergence of the method is proven in this case. We give error estimates for the higher order version of those two spaces in the following. In addition we study a new modified \( C^1 \) nonconforming discretization which has the same degrees of freedom as the original \( C^1 \) nonconforming space but is stable with respect to \( \epsilon \to 0 \). This is achieved by a modification to the gradient projection, \( \Pi^K_1 \), given next.

Definition 6.1. We define the modified gradient projection to be the following

\[
\int_K \Pi^K_1 v_h p = - \int_K \Pi^K_1 v_h \nabla p + \sum_{e \subset \partial K} \int_e \Pi^{l-1} v_h pm, \quad \forall p \in P_{l-1}(K)^2 \tag{6.2}
\]

for any \( v_h \in V^K_{h,l} \). We denote with \( \Pi^{l-1} \) the interpolation into a \( H^l \) conforming VEM space of order \( l-1 \). We use the \textquotedblleft lazy\textquotedblright\ version of the serendipity spaces discussed in \[9\]. The dofs for this space were mentioned in Remark 4.5 and for order \( l-1 \) are described by the dof tuple \((0,-1,l-3,-1,l-4)\). Therefore the dofs are a subset of the dofs defining the nonconforming \( C^1 \) space. The vertex values and the \( l-3 \) moments on the edges uniquely define \( \Pi^{l-1} v_h \in P_{l-1}(\epsilon) \) so that the gradient projection given above is computable using the dofs for the \( C^4 \) nonconforming space.

Remark 6.2. This gradient change could also be achieved by replacing the edge projection \( \Pi^K_0 v_h \) with \( (\Pi^{l-1} v_h)|e \) which is dof compatible (see Definition 4.9 with the exception that \( \Pi^K_0 q_h = q_h \) only holds for \( q \in P_{l-1}(K) \). The resulting virtual element space satisfies \( V_{h,l} \subset H^l_0(\Omega) \) but only \( P_{l-1}(T_h) \subset V_{h,l} \). The \( L^2 \) projection properties of the value and the hessian projection are still satisfied (in fact they are both not changed by the use of \( \Pi^{l-1} \) for the edge projection). But the property given in Lemma 4.14 does not hold anymore for the gradient projection. While, due to the continuity of the discrete function space, we could use many of the bounds from the previous section avoiding the scaling with \( \epsilon^{-1} \), we cannot use all of the results due to the missing polynomial exactness. Therefore, we briefly describe a convergence proof based on using the original \( C^1 \) nonconforming VEM space with the modified gradient projection given above.

The following property is now obtained for the modified gradient projection.

Lemma 6.3. For the modified gradient projection detailed in Definition 6.1 it holds that

\[
\Pi^K_1 v_h = P_r^{l-2}(D\Pi^{l-1} v_h) \quad \text{for any } v_h \in V^K_{h,l}. \tag{6.3}
\]

To prove convergence in the energy norm for this modified scheme we use the ideas seen in \[33\] \[32\]. We consider a modified bilinear form by changing the lower order contribution.

\[
b(u,v) := \epsilon^2 \int_{\Omega} D^2 u : D^2 v + \int_{\Omega} D\Pi^{l-1} u \cdot D\Pi^{l-1} v
\]

for all \( u, v \in H^l_0(\Omega) \). Using \( b(\cdot,\cdot) \) we can prove a a Strang-type lemma similar to Theorem 3.4.
Theorem 6.4 (Abstract a priori error bound). Let \( l \geq 1 \) be an integer. Under Assumption 2.4 it holds that
\[
\|u - u_h\|_h \leq \inf_{v_h \in V_{h,l}} \|u - v_h\|_h + \sup_{w_h \in V_{h,l}} \frac{|f_h, w_h| + |f, \Pi^{l-1} w_h|}{\|w_h\|_h} + \|b(u, w_h) - (f, \Pi^{l-1} w_h)\| \tag{6.3}
\]
\[
\int_\Omega \Delta^2 u \Pi^{l-1} w_h - \int_\Omega \Delta u \Pi^{l-1} w_h = \int_\Omega (-\Delta^2 u + D(u) + D(\Pi^{l-1} u) + D(\Pi^{l-1} w_h))
\]
and similarly
\[
b(u, w_h) = -\sum_K (\epsilon^2 \int_K D(\Delta u) \cdot D w_h - \epsilon^2 \int_K (\Delta u - \partial_s u) \partial_s w_h + \partial_s u \partial_s w_h - \int_K D(\Pi^{l-1} u) \cdot D(\Pi^{l-1} w_h))
\]

Therefore, it holds that
\[
b(u, w_h) - (f, \Pi^{l-1} w_h) = \epsilon^2 \sum_{K \in \mathcal{T}_h} \|D(\Delta u) - D(\Pi^{l-1} u) - D(\Pi^{l-1} w_h)\| \leq C \epsilon \|D(\Delta u) - D(\Pi^{l-1} w_h)\|_h.
\]

Notice that \( J_2 \) can be bounded as before in the nonconformity error proof (Theorem 5.3). Then for \( J_1 \) using Cauchy-Schwarz and standard estimates,
\[
J_1 \leq \epsilon \|D(\Delta u) - D(\Pi^{l-1} u) - D(\Pi^{l-1} w_h)\|_{0, K} \leq C \epsilon \|D(\Delta u) - D(\Pi^{l-1} w_h)\|_{1, K}
\]

Now, for \( J_3 \) we bound this term using the optimal interpolation properties of the lower order VEM space as well as stability of the interpolation operator.
\[
J_3 \leq \sum_{K \in \mathcal{T}_h} \|u - \Pi^{l-1} u\|_{1, K} \leq C \|u\|_{1, K}
\]

Finally, for the last term in (6.3) we can show that
\[
\inf_{p \in V_{h,l}} \left(\|u - p\|_h + \sup_{K \in \mathcal{T}_h} \frac{|b_K(p, w_h) - a_h^K(p, w_h)|}{\|w_h\|_K}\right) \leq C \epsilon \|u\|_{1, K} + \|u\|_h.
\]
This follows from
\[ b^K(P_K^i u, w_h) - a^K_h(P_K^i u, w_h) = \epsilon^2 \int_K D^2 P_K^i u : D^2 w_h - \epsilon^2 \int_K \Pi^K_h P_K^i u : \Pi^K_h w_h \\
+ \int_K D \Pi^{-1} P_K^i u : D \Pi^{-1} w_h - \int_K \Pi^K_h P_K^i u : \Pi^K_h w_h. \]

Due to the \( L^2 \) properties in Lemma 4.14 the hessian terms cancel. Secondly, using the property in Lemma 6.3 it holds that \( \Pi^K_h v_h = P_K^{h^{-2}}(D \Pi^{-1} v_h) \). Therefore for any \( v_h \in V_h^K \) it holds that
\[ \int_K D \Pi^{-1} P_K^i u : D \Pi^{-1} w_h - \int_K \Pi^K_h P_K^i u : \Pi^K_h w_h \leq C h^{-1} |\epsilon| \|w_h\|_K. \]

Therefore, the result follows from combining all of the intermediary results.

Finally, we finish this section by presenting the results for all the considered spaces applied to the perturbation problem (6.1). Due to keeping track of the coefficients in the previous error analysis section, the following corollary is simply a consequence of Theorem 5.7 and Theorem 6.3.

**Corollary 6.6.** Under the same assumptions as Theorem 5.7 with \( f \in H^{-1}(\Omega) \) the approximate solution in the \( C^1 \) nonconforming space \( C^1,nc \) satisfies
\[ |||u - u_h|||_h \leq C \{(h^{-1} \epsilon + h^4)|u|_{i+1} + \frac{h^{-1}}{\epsilon}|u|_{i-1} + h^{-1}|f|_{i-2}\}. \]

The solution in the \( C^1,C^0 \) conforming space, assuming that \( f \in H^{-1}(\Omega) \), satisfies the following error estimate
\[ |||u - u_h|||_h \leq C \{(h^{-1} \epsilon + h^4)|u|_{i+1} + h^4|f|_{i-1}\}. \]

Finally, for the modified \( C^1 \) nonconforming scheme defined by the modified gradient projection in Definition 6.4 and assuming that \( f \in H^{-1}(\Omega) \), it holds that
\[ |||u - u_h|||_h \leq C \{h^{-1} \epsilon|u|_{i+1} + h^{-1}|u|_{i} + h^{-1}(|f|_{i-1} + |f|_{i-2})\}. \]

In all three cases, the constant \( C \) remains independent of \( h \) and \( \epsilon \).

## 7 Numerical results

In this section we show some numerical results to verify the a priori bounds from the previous sections for three virtual element spaces.

- **C1-nc**: the nonconforming space from [4] [36] defined by the dof tuple \( (0, -1, l - 3, l - 2, l - 4) \) (see Example 4.4) and using the default projection operators (see Example 4.11).
- **C1-mod**: the nonconforming space defined by the dof tuple \( (0, -1, l - 3, l - 2, l - 4) \) (see Example 4.4) and using the modified gradient projection (see Section 6).
- **C1-C0**: the continuous space from [35] [34] defined by the dof tuple \( (0, -1, l - 2, l - 2, l - 4) \) (see Example 4.4) and using the default projection operators (see Example 4.11).

We focus on results for \( l = 2, 3, 4 \). Most of the simulations are performed on structured triangular grids but to demonstrate the flexibility of the virtual element method we include some simulations using grids consisting of remapped hexagonal elements also used in [19] [4].

We first study the simple linear problem with constant coefficients (6.1), showing results with varying \( h, \epsilon \). As a second example we study convergence of the methods for a linear problem of the general form (2.1) with varying coefficients.

Finally we highlight how our approach based on general projection operators allows us to handle nonlinear problems by showing results for the Willmore flow of graphs and the Cahn-Hilliard equation. Note that no special linearization is required, in both cases we use a standard Newton solver to handle the fourth order nonlinear problems arising from an implicit time discretization. This final section is only a first investigation into applying our approach to more complicated settings and it is beyond the scope of this paper to do a detailed analysis. We therefore restrict the presentation to the C1-nc space with \( l = 3 \) on triangular grids.

The code used to perform the simulations is based on the DUNE software framework [7]. We implemented our VEM approach within the module DUNE-FEM [24]. This is an extension module for DUNE that provides interfaces for the implementation of general grid based numerical schemes on general unstructured grids. It is open source software implemented in C++. Recently a Python based frontend was added to DUNE [23] [24]. The domain specific language UFL [2] can be used to describe mathematical models. A detailed tutorial including some VEM examples, e.g., for linear elasticity and also the Willmore flow example described here, showcases the flexibility of the approach [23].
7.1 Perturbation problem

Consider problem (6.1) with right hand side given by \( f = \epsilon^2 \Delta^2 u - \Delta u \). As exact solution we use \( u(x,y) = \sin(2\pi x)^2 \sin(2\pi y)^2 \) which satisfies the Dirichlet boundary conditions on the domain \( \Omega = [0,1]^2 \). Note that we are only considering problems without boundary layers, a discussion of problem (6.1) with boundary layers can be found in [34]. We start by taking \( \epsilon = 1 \) to study the convergence of our three VEM spaces. The results on the structured simplex grid and on the remapped hexagon grid are show in Figures 3 and 4, respectively. The methods show the expected experimental order of convergence of \( l^{-1} \). The C1-nc and the C1-mod methods are mostly indistinguishable, which is to be expected since in the case \( \epsilon = 1 \) the main cause of error is due to the hessian projection which is the same for both methods. This also holds for the C1-C0 method which consequently on a fixed grid produces a very similar error compared to the other two methods but requires more degrees of freedom due to using additional edge moments.

Results for \( \epsilon = 10^{-2} \) and \( \epsilon = 10^{-8} \) on the structured triangular grid are summarized in Figures 5 and 6, respectively. For the larger \( \epsilon \) the order of convergence on the final grids seem to be again approaching \( l^{-1} \) for all three methods in accordance with Theorem 6.6. The errors in the energy norm on a fixed grid differ considerably demonstrating the differences in the gradient projection. It is clear that when \( \epsilon \) is reduced further, the observed convergence rates of the three schemes differ, while the convergence rate for the C1-nc scheme drops to \( l^{-2} \), it remains at \( l^{-1} \) for the C1-mod method, while increasing to \( l \) for the C1-C0 method.

7.2 Linear varying coefficient problem

We now study the more general linear fourth order problem with varying coefficients using

\[
\kappa(x,y) = \frac{1}{1 + x^2 + y^2}, \quad \beta(x,y) = e^{-xy}, \quad \gamma(x,y) = (\sin(x^2 + y^2))^2.
\]

We again choose the forcing so that \( u(x,y) = \sin(2\pi x)^2 \sin(2\pi y)^2 \) is the exact solution on the domain \([0,1]^2\). Overall the results show the same picture as we already saw for the simple constant coefficient setting with \( \epsilon = 1 \) so we only show results here for the simplex grid in Tables 1, 2, and 3. As expected, all methods converge with an order of \( l^{-1} \) and produce very similar errors on a given grid with C1-C0 requiring more degrees of freedom.

| Table 1: Problem varying-coefficients of order 2 |
|-----------------------------------------------|
|     | C1-nc      |     | C1-mod     |     | C1-C0     |     |
|     | dofs | error | eoc | dofs | error | eoc | dofs | error | eoc |
| 49  | 6.173e+01 | —   | 49  | 5.990e+01 | —   | 82  | 5.326e+01 | —   |
| 169 | 4.273e+01 | 0.53| 169 | 4.247e+01 | 0.50| 289 | 3.736e+01 | 0.51|
| 625 | 2.486e+01 | 0.78| 625 | 2.486e+01 | 0.77| 1081| 2.181e+01 | 0.78|
| 2401| 1.310e+01 | 0.93| 2401| 1.311e+01 | 0.92| 4177| 1.161e+01 | 0.91|
| 9409| 6.644e+00 | 0.98| 9409| 6.645e+00 | 0.98| 16417| 5.916e+00 | 0.97|
| 37249| 3.334e+00 | 0.99| 37249| 3.336e+00 | 0.99| 45089| 2.960e+00 | 1.00|

| Table 2: Problem varying-coefficients of order 3 |
|-----------------------------------------------|
|     | C1-nc      |     | C1-mod     |     | C1-C0     |     |
|     | dofs | error | eoc | dofs | error | eoc | dofs | error | eoc |
| 115 | 3.892e+01 | —   | 115 | 3.864e+01 | —   | 148 | 4.242e+01 | —   |
| 409 | 1.182e+01 | 1.72| 409 | 1.182e+01 | 1.71| 529 | 1.207e+01 | 1.81|
| 1537| 3.386e+00 | 1.80| 1537| 3.386e+00 | 1.80| 1993| 3.391e+00 | 1.83|
| 5953| 9.103e-01 | 1.90| 5953| 9.107e-01 | 1.90| 7729| 9.116e-01 | 1.90|
| 23425| 2.338e-01 | 1.96| 23425| 2.339e-01 | 1.96| 30433| 2.342e-01 | 1.96|
| 92929| 5.888e-02 | 1.99| 92929| 5.896e-02 | 1.99| 120769| 5.897e-02 | 1.99|

| Table 3: Problem varying-coefficients of order 4 |
|-----------------------------------------------|
|     | C1-nc      |     | C1-mod     |     | C1-C0     |     |
|     | dofs | error | eoc | dofs | error | eoc | dofs | error | eoc |
| 199 | 2.234e+01 | —   | 199 | 2.223e+01 | —   | 232 | 3.416e+01 | —   |
| 721 | 5.275e+00 | 2.08| 721 | 5.251e+00 | 2.08| 841 | 5.527e+00 | 2.63|
| 2737| 9.090e-01 | 2.53| 2737| 9.090e-01 | 2.53| 3183| 8.790e-01 | 2.65|
| 10657| 1.236e-01 | 2.88| 10657| 1.233e-01 | 2.88| 12433| 1.192e-01 | 2.88|
| 42049| 1.582e-02 | 2.97| 42049| 1.579e-02 | 2.97| 49057| 1.590e-02 | 2.96|
Figure 3: Perturbation problem on a structured simplex grid with $\epsilon = 1$ and $l = 2, 3, 4$ (top to bottom). Left column shows error in the energy norm with respect to number of degrees of freedom for the three spaces. Right column shows experimental order of convergence again for the energy norm versus the grid spacing $h$. 
Figure 4: Perturbation problem on the remapped hexagonal grid with $\epsilon = 1$ and $l = 2, 3, 4$ (top to bottom). Left column shows error in the energy norm with respect to number of degrees of freedom for the three spaces. Right column shows experimental order of convergence again for the energy norm versus the grid spacing $h$. 
Figure 5: Perturbation problem with $\epsilon = 10^{-2}$ and $l = 2, 3, 4$ (top to bottom). Left column shows error in the energy norm with respect to number of degrees of freedom for the three spaces. Right column shows experimental order of convergence again for the energy norm versus the grid spacing $h$. 
Figure 6: Perturbation problem with $\epsilon = 10^{-8}$ and $l = 2, 3, 4$ (top to bottom). Left column shows error in the energy norm with respect to number of degrees of freedom for the three spaces. Right column shows experimental order of convergence again for the energy norm versus the grid spacing $h$. 
7.3 Nonlinear problems

We conclude this section with some preliminary results, which demonstrate that the method discussed in this paper is well suited to solve complex nonlinear fourth order problems. We choose two problems which have been studied in the literature, both energy minimization problems solved by a gradient descent algorithm. In both cases the mathematical models are time dependent fourth order problems. We use a Rothe approach in which first the problem is discretized in time. The resulting spatial problems are stationary fourth order problems with linearizations of the form studied here. In both cases we only show results for the standard C1-nc space with \( l = 3 \) using a structured simplex grid on \( \Omega = [0, 1]^2 \).

7.3.1 Cahn-Hilliard problem

For our first problem we solve the Cahn-Hilliard equation using a fully implicit backward Euler method to discretize the problem in time. A virtual element method for this problem was studied in [3] where a different approach for defining the projection operators, requiring a special linearization, restricted the method to \( l = 2 \).

Let \( \psi : \mathbb{R} \to \mathbb{R} \) be defined as \( \psi(x) = \frac{(1-x^2)^2}{4} \) and define \( \phi(x) = \psi(x)' \). Then, the Cahn-Hilliard problem reads as follows: find \( u : \Omega \times [0,T] \to \mathbb{R} \) such that

\[
\partial_t u - \Delta (\phi(u) - \gamma^2 \Delta u) = 0 \quad \text{in} \quad \Omega \times [0,T],
\]

\[
u(\cdot, 0) = u_0(\cdot) \quad \text{in} \quad \Omega,
\]

\[
\partial_n u = \partial_n (\phi(u) - \gamma^2 \Delta u) = 0 \quad \text{on} \quad \partial \Omega \times [0,T].
\]

Note that this problem requires slightly different boundary conditions compared to the problems studied so far. Some snapshots from a simulation with \( \gamma = 0.02 \) on a 60 x 60 grid and time step \( \tau = 10^{-3} \) are displayed in Figure 7. The initial conditions were a small perturbation in the centre of the domain. The first snapshot is taken at a point in time where the phase separation is already well developed. The following figures then show the usual coarsening ending with the red phase concentrated in approximately a circle in the centre of the domain.

![Figure 7: Some snapshots from the simulation of the Cahn-Hilliard problem.](image)

7.3.2 Willmore flow of graphs

In our final example we study the minimization of the Willmore energy of a surface, in the case where the surface is given by a graph over a flat domain \( \Omega \). The corresponding Euler-Lagrange equations can be rewritten as a fourth order problem for a function \( u \) defined over \( \Omega \). We use a second order two stage implicit Runge-Kutta method as suggested in [22]. The resulting problem is a system of two nonlinear fourth order partial differential equations. The corresponding linearized problem is a system of equations for the two Runge-Kutta stages which is of the form (2.1).

As detailed for example in [22], the Willmore functional for the graph of a function \( u \in W^{2,\infty}(\Omega) \) is given by

\[
W(u) = \frac{1}{2} \int_{\Omega} \| E(Du) : D^2 u \|^2 \, dx.
\]

The function \( E : \mathbb{R}^2 \to Sym(2) \), where \( Sym(2) \) denotes the space of symmetric \( 2 \times 2 \) matrices, is given by

\[
E_{ij}(w) := \frac{1}{(1+|w|^2)^2} \left( \delta_{ij} - \frac{w_i w_j}{1+|w|^2} \right) \quad \text{for} \ i,j = 1,2, \text{and} \ w \in \mathbb{R}^2.
\]

As initial condition we use \( \sin(2\pi x)^2 \sin(2\pi y)^2 \).

Figure 8 shows the evolution of the surface and a graph showing the decay of the Willmore energy over time. Overall the results indicate the method is well suited for solving this problem.
8 Conclusion

We have presented a general approach for constructing nonconforming VEMs for solving a general fourth order PDE problem with nonconstant coefficients. We have analysed these nonconforming VEMs and shown that the virtual element solution in each of the considered spaces converges to the true solution in the energy norm with optimal convergence rates. We also introduced a novel modified nonconforming scheme for solving the fourth order perturbation problem which remained convergent as $\epsilon \to 0$. Unlike modifications seen in the literature, our change did not require an enlargement of the space and was obtained via an adjustment to the gradient projection. These results were verified with numerical experiments on a variety of polygonal meshes.

We introduced a new concept of describing the degrees of freedom via a dof tuple which allowed us to easily encode a variety of different VEM spaces. We followed the approach taken in [1, 19] and introduced dof compatible projections which were constructed in a way that made them computable entirely from the available degrees of freedom. In particular, we gave examples of how dof compatible projections could be constructed based on constraint least squares problems. We were then able to construct the VEM spaces in a way that ensured that the value, gradient, and hessian projections were all identical to $L^2$ projections.

The ease with which our approach can extend to the application of nonlinear fourth order problems was also demonstrated with some additional numerical experiments. However, note that the theory behind the numerical results displayed is out of the scope of this paper and is future work.

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