Infrared Zero of $\beta$ and Value of $\gamma_m$ for an SU(3) Gauge Theory at the Five-Loop Level

Thomas A. Ryttov$^a$ and Robert Shrock$^b$

(a) CP$^*$-Origins and Danish Institute for Advanced Study
Southern Denmark University, Campusvej 55, Odense, Denmark
(b) C. N. Yang Institute for Theoretical Physics
Stony Brook University, Stony Brook, NY 11794, USA

We calculate the value of the coupling at the infrared zero of the beta function of an asymptotically free SU(3) gauge theory at the five-loop level as a function of the number of fermions. Both a direct analysis of the beta function and analyses of Padé approximants are used for this purpose. We then calculate the value of the five-loop anomalous dimension, $\gamma_m$, of the fermion bilinear at this IR zero of the beta function.

The evolution of an asymptotically free gauge theory from the ultraviolet (UV) to the infrared (IR) is of fundamental importance. The evolution of the running gauge coupling $g = g(\mu)$, as a function of the Euclidean momentum scale, $\mu$, is described by the renormalization-group (RG) beta function \[ \frac{d\beta}{dt} = \frac{g^2}{16\pi^2} \left( \frac{1}{\alpha} - 1 \right) \]

where $\alpha = g^2 / (16\pi^2)$ is the running coupling, $\beta$ is the RG beta function, and $dt = d\ln \mu$ (the argument $\mu$ will often be suppressed in the notation). Here we consider a vectorial gauge theory, $G = \text{SU}(3)$ and $N_f$ flavors of fermions $\psi_i$, $i = 1, ..., N_f$, transforming in the fundamental (triplet) representation. We impose the condition of asymptotic freedom (AF) for the self-consistency of the perturbative calculation of $\beta$. For simplicity, we take the fermions to be massless. This theory is quantum chromodynamics (QCD) with $N_f$ massless quarks.

The beta function of this theory has the series expansion \[ \beta = -2\alpha \sum_{\ell=1}^{\infty} b_\ell \alpha^\ell = -2\alpha \sum_{\ell=1}^{\infty} b_\ell \alpha^\ell, \] (1)

where $a = g^2 / (16\pi^2)$ is the $\ell$-loop coefficient, $b_\ell$ is the $\ell$-loop coefficient, and we extract a minus sign for convenience. The $n$-loop ($n\ell$) beta function, denoted $\beta_{n\ell}$, is obtained from Eq. (1) by changing the upper limit on the $\ell$-loop summation from $\infty$ to $n$. The (scheme-independent) one-loop and two-loop coefficients are $b_1 = 11 - (2/3)N_f$ \[ \text{and } b_2 = 102 - (38/3)N_f. \]

The AF condition implies the upper bound $N_f < N_f, 0 \leq 33/2$ \[ \text{i.e., the integer upper bound } N_f \leq 16 \text{, which we impose.} \]

We denote the interval $0 \leq N_f \leq 16$ as $\text{AF}_\ell$. The $b_\ell$ with $\ell \geq 3$ are scheme-dependent \[ K \text{ and } B \text{ (and checked in } \ell \text{), in the } \text{MS scheme } \ell \text{, e.g., } b_3 = (2857/2) - (5033/18)N_f + (325/54)N_f^2. \]

As $N_f \in \text{AF}_\ell$, increases from 0, $b_\ell$ decreases, vanishing at $N_f, 0 = 153/19 = 8.05$, and is negative in the real interval $153/19 < N_f < 33/2$, i.e., the integer interval $1 IRZ : 9 \leq N_f \leq 16$. If $N_f \in \text{IRZ}$, then the two-loop beta function $\beta_{2\ell}$ has an IR zero (IRZ), at $\alpha = \alpha_{2\ell}$. Here we denote the IR zero (if it exists) of the $n\ell$-loop beta function $\beta_{n\ell}$ as $\alpha_{n\ell}$. For $N_f$ near the upper end of $\text{IRZ}_\ell$, $\alpha_{2\ell}$ is small and can be studied perturbatively \[ 4 \text{.} \]

As $N_f \in \text{IRZ}$ decreases, $\alpha_{2\ell}$ increases toward strong coupling. Hence, to study the IR zero for $N_f$ toward the middle and lower part of $\text{IRZ}_\ell$ with reasonable accuracy, one requires higher-loop calculations. These were carried out to four-loop order in \[ \text{IRZ}_\ell \text{.} \]

Clearly, such a perturbative calculation of the IR zero of $\beta_{n\ell}$ is only reliable if the resultant $\alpha_{n\ell}$ is not excessively large. Since the $b_\ell$ with $\ell \geq 3$ are scheme-dependent, it is necessary to assess the sensitivity of the value obtained for $\alpha_{n\ell}$ for $n \geq 3$ to the scheme used for the calculation. This was done in \([18-21]\) (see also \([22-23]\)). In \([18-19]\), a set of conditions that an acceptable scheme transformation must satisfy were presented, and it was shown that although these are automatically satisfied in the local vicinity of the origin, $\alpha = 0$ (as in optimized schemes for perturbative QCD calculations \([23-24]\)), they are not automatically satisfied, and indeed, are quite restrictive conditions, when one applies the scheme transformation at an IR zero away from the origin.

Here we report the first calculation of the five-loop IR zero of $\beta$ and resultant five-loop evaluation of the anomalous dimension of the fermion bilinear at this IR zero, for $N_f \in \text{IRZ}_\ell$, making use of the recent calculation of $b_5$ in the \text{MS} scheme from \([26]\). The results are of fundamental importance for understanding the RG evolution of SU(3) gauge theory with variable fermion content.

The anomalous dimension $\gamma_m$ of the fermion bilinear operator $\psi_i \psi_i$ (no sum on $i$) is defined as $D(\psi_i \psi_i) = 3 - \gamma_m$, where $D$ is the full scaling dimension. Knowing $\alpha_{1\ell, 1\ell}$, one can then evaluate $\gamma_m$ (calculated to the same $n\ell$-loop order) at $\alpha = \alpha_{1\ell, 1\ell}$; we denote this as $\gamma_{1\ell, 1\ell}$. This anomalous dimension is of particular interest, since (if calculated to all orders) it is a scheme-independent physical quantity. (Unless indicated otherwise hereafter, the scheme taken for the $b_n$ and resultant $\beta_{n\ell}$, $\alpha_{n\ell}$, and $\gamma_{n\ell}$ with $n \geq 3$ is the \text{MS} scheme.)

Our previous work showed the usefulness of higher-loop calculations of $\gamma_{1\ell, 1\ell}$. For example, for a (vectorial) SU(3) gauge theory with $N_f = 12$ massless Dirac fermions, the values of $\gamma_{1\ell, 1\ell}$ at the two-loop, three-loop, and four-loop level were found to be 0.773, 0.312, and 0.253, respectively \([14, 15]\). Our four-loop result, $\gamma_{4\ell, 4\ell}$, is in good agreement with the fully nonperturbative lattice calculations $\gamma_R = 0.27 \pm 0.03$ \([27]\), $\gamma_R = 0.25$ \([28]\), and $\gamma_R = 0.235 \pm 0.046$ \([29, 30]\). These measure-
ments are part of an intensive lattice program to elucidate the properties of asymptotically free gauge theories with various fermion contents, in particular, those exhibiting quasiconformal behavior; besides their intrinsic field-theoretic interest, such theories might play a role in physics beyond the Standard Model [50]. Similar agreement was found for four-loop calculations in other schemes [18–21]. An iterative method to calculate $\gamma_{IR}$ in a scheme-independent manner has been presented in [22]. It allows for a direct comparison of perturbative methods with exact results in $N = 1$ supersymmetric QCD, for which it was shown that $\gamma_{IR}$ is very well described already at a few loops level throughout the entire conformal interval.

In the UV to IR evolution, as $\mu$ decreases, $\alpha(\mu)$ approaches the IR zero in $\beta$. If this zero occurs at relatively weak coupling, it can be an exact IR fixed point (IRFP) of the RG, and the corresponding IR phase is a chirally symmetric, deconfined non-Abelian Coulomb phase (NACP, conformal interval). If the IR zero in $\beta$ occurs at a sufficiently large value of $\alpha$, then the IR phase has confinement and spontaneous chiral symmetry breaking (S\&SB) associated with a nonzero bilinear fermion condensate formed at a scale $\Lambda$. In this case, the fermions gain dynamical masses and are integrated out of the low-energy effective theory applicable for $\mu < \Lambda$. The IR zero in $\beta$ is then only an approximate IRFP and similarly, $\gamma_{IR}$ is only an effective quantity describing the RG flow near this approximate IRFP.

We next describe the behavior of $\beta_5$ as a function of $N_f$ (for the behavior of $b_3$ and $b_4$, see [14,15].) As $N_f \in I_{AF}$ increases from 0, $b_5$ initially decreases through positive values, reaches a minimum at $N_f = 6.074$ [5], where $b_5 = 0.640 \times 10^{-3}$, and then increases. For all $N_f \in I_{AF}, b_5$ is negative-definite, while $b_4$ and $b_5$ are positive-definite. We list values of the $b_5$ for $1 \leq \ell \leq 5$ in Table I.

For our analysis of the IR zero of $\beta$, it is convenient to extract a prefactor and define a reduced $n$-loop beta function as

$$\beta_{r,n\ell} \equiv \frac{\beta_{n\ell}}{-2\alpha^2 b_1} = 1 + \sum_{\ell=2}^{n} \tilde{\rho}_\ell \alpha^{\ell-1}$$

(2)

where $\tilde{\rho}_\ell = b_\ell/b_1$. The equation $\beta_{r,n\ell} = 0$ determines the IR zero and is a polynomial equation of degree $n - 1$ in $\alpha$. Among the $n - 1$ roots, the smallest positive (real) root, if there is such a root, is $\alpha_{IR,n\ell}$. The nature of the roots at the $n = 3$ and $n = 4$ loop level has been discussed in [14,15].

We present our results for $\alpha_{IR,5\ell}$ in Table II. We begin the discussion at the upper end of the interval $I_{IRZ}$. For $14 \leq N_f \leq 16$, we find that $\alpha_{IR,5\ell}$ is close to, and slightly larger than, $\alpha_{IR,4\ell}$. For $N_f = 13$, $\alpha_{IR,5\ell}$ is about 20 % larger than $\alpha_{IR,4\ell}$. If $9 \leq N_f \leq 12$, we find that the five-loop beta function (in the MS scheme, with $b_5$ from [22]) has no physical IR zero; instead, the roots of the quartic polynomial $\beta_{r,5\ell}$ consist of two complex-conjugate (c.c.) pairs. This is a surprising result, since at all of the lower-loop orders, namely $n = 2, n = 3,$ and $n = 4$, for $N_f \in I_{IRZ}$, the $n$-loop beta functions (in the MS scheme and also other schemes [18–21]) have physical IR zeros $\alpha_{IR,n\ell}$, and one would naturally expect that as one extends the calculation of $\beta_{n\ell}$ to higher-loop order, this behavior would continue. Specifically, we find the following: $N_f = 9 \Rightarrow \alpha_{IR,5\ell} \approx 0.863 \pm 0.515 i$; $N_f = 10 \Rightarrow \alpha_{IR,5\ell} \approx 0.715 \pm 0.382 i$; $N_f = 11 \Rightarrow \alpha_{IR,5\ell} \approx 0.609 \pm 0.277 i$; and $N_f = 12 \Rightarrow \alpha_{IR,5\ell} \approx 0.528 \pm 0.176 i$. Although these roots are unphysical if $9 \leq N_f \leq 12$, the respective real parts are similar to lower-loop values; for example, $\text{Re}(\alpha_{IR,5\ell}) = 0.609$ for $N_f = 11$, which is close to $\alpha_{IR,4\ell} = 0.626$, etc. As $N_f$ increases in this interval $9 \leq N_f \leq 12$, the real part and the magnitude of the imaginary part decrease, consistent with the approach to the real value $\alpha_{IR,5\ell} = 0.406$ at $N_f = 13$. Formally extending $N_f$ to real numbers, we find that as $N_f$ approaches the value $N_f \approx 12.8944$ from below, the two complex-conjugate roots approach the real axis, with the real part approaching 0.47, and for larger $N_f \in I_{IRZ}$, the two c.c. roots are replaced by two real roots, which respectively decrease and increase from $\alpha_{IR,5\ell} \approx 0.47$ as $N_f$ increases beyond 12.8944. At the next physical integer value, $N_f = 13$, the lower root in this pair occurs at $\alpha_{IR,5\ell} = 0.406$, as listed in Table III while the upper one occurs at 0.5195.

A necessary condition for the perturbative calculation of the IR zero to be reliable is that the magnitude of the fractional difference

$$\Delta_{IR,n-1,n} = \frac{\alpha_{IR,(n-1)\ell} - \alpha_{IR,n\ell}}{2|\alpha_{IR,(n-1)\ell} + \alpha_{IR,n\ell}|}$$

(3)
TABLE II: Values of $\alpha_{IR,n}$ as a function of $N_f$ for $N_f \in I_{IRZ}$ and loop order $2 \leq n \leq 5$. See text for discussion of $\alpha_{IR,5}$ for $9 \leq N_f \leq 12$.

| $N_f$ | $\alpha_{IR,2}$ | $\alpha_{IR,3}$ | $\alpha_{IR,4}$ | $\alpha_{IR,5}$ |
|-------|-----------------|-----------------|-----------------|-----------------|
| 9     | 5.24            | 1.028           | 1.072           | —               |
| 10    | 2.21            | 0.764           | 0.815           | —               |
| 11    | 1.23            | 0.578           | 0.626           | —               |
| 12    | 0.754           | 0.435           | 0.470           | —               |
| 13    | 0.468           | 0.317           | 0.337           | 0.406           |
| 14    | 0.278           | 0.215           | 0.224           | 0.233           |
| 15    | 0.143           | 0.123           | 0.126           | 0.127           |
| 16    | 0.0416          | 0.0397          | 0.0398          | 0.0398          |

TABLE III: Values of $\Delta_{IR,n-1,n}$ as a function of $N_f$ for $N_f \in I_{IRZ}$. See text for discussion of $\Delta_{IR,4,5}$ for $9 \leq N_f \leq 12$.

| $N_f$ | $\Delta_{IR,2,3}$ | $\Delta_{IR,3,4}$ | $\Delta_{IR,4,5}$ |
|-------|--------------------|--------------------|--------------------|
| 9     | 1.344              | −0.04175           | −                  |
| 10    | 0.971              | −0.0642            | −                  |
| 11    | 0.723              | −0.0791            | −                  |
| 12    | 0.537              | −0.0785            | −                  |
| 13    | 0.386              | −0.0639            | −0.185             |
| 14    | 0.258              | −0.0415            | −0.0404            |
| 15    | 0.146              | −0.0185            | −0.00770           |
| 16    | 0.0461             | −0.00255           | −0.000288          |

should be reasonably small and should tend to decrease with increasing loop order, $n$ \cite{31}. We have calculated the various $\Delta_{IR,n-1,n}$ and list the values in Table III. As is evident, this necessary condition is satisfied if $14 \leq N_f \leq 16$. If $N_f = 13$, then the requisite behavior is observed for $\Delta_{IR,23}$ and $|\Delta_{IR,34}|$, but $|\Delta_{IR,45}|$ is actually about three times larger than $|\Delta_{IR,34}|$. For lower values of $N_f \in I_{IRZ}$, the $|\Delta_{IR,n-1,n}|$ criterion is not applicable, since $\beta_{5f}$ is complex.

These results are a consequence of the properties of the relevant coefficients $b_n$ in $\beta$. In general, if, as a function of $N_f \in I_{IRZ}$, $|b_n|$ becomes very small in magnitude, then the $n$-loop contribution to $\beta$ will tend to be a commensurately small correction to the $(n−1)$-loop beta function, so $\Delta_{IR,n-1,n}$ will also be small. As $N_f$ decreases from 16 to 9, $|b_3|$ decreases by a factor of 2.5 and $b_4$ decreases sharply, by a factor of 45. This strong decrease in $b_4$ means that although the overall size of $\alpha_{IR,4}$ increases as $N_f$ decreases in this interval $I_{IRZ}$, the fractional difference $\Delta_{IR,3,4}$ remains small, as is evident in Table III. In contrast, although $b_5$ also decreases as $N_f$ decreases in $I_{IRZ}$, it is still considerably larger than $b_4$, leading to the larger value of $|\Delta_{IR,4,5}|$ observed for $N_f = 13$.

Our calculation of $\alpha_{IR,5}$ thus reveals new complexities with the IR zero in $\beta$ for $N_f \in I_{IRZ}$ that were not observed at lower-loop order and hence were not anticipated at five-loop order, since one expects that (in a non-perturbative scheme) calculations at higher-loop order should exhibit greater stability than those at lower-loop order \cite{31}. In view of our finding, we next make use of the powerful method of Padé approximants (PAs) \cite{32} to study the IR zero in $\beta$ at the five-loop level. The $[p, q]$ PA to $\beta(\alpha, n)$ is the rational function

$$[p, q]_{\beta_n} = \frac{1 + \sum_{j=1}^{p} n_j \alpha^j}{1 + \sum_{k=1}^{q} d_k \alpha^k}$$

(4)

with $p+q = n−1$, where the $n_j$ and $d_j$ are $\alpha$-independent coefficients. For a given $[p, q]_{\beta_n}$, there are thus $n$ PAs, namely the set $\{[n−k, k−1]\beta_{n,k−1}\}_{1 \leq k \leq n}$. For $n = 3$ loops, this is the set $\{[4, 0], [3, 1], [2, 2], [1, 3], [0, 4]\}$. The $[4, 0]$ PA is just $\beta_{5f}$ itself, which we have already analyzed, and the $[0, 4]$ PA has no zero and hence cannot be used for the analysis of the IR zero of $\beta_{5f}$, which leaves us with the remaining three PAs. We have calculated and analyzed these. If a $[p, q]_{\beta_n}$ PA has a physical IR zero at this $n = 5$ level, it is denoted as $\alpha_{IR,5,\{p,q\}}$. Clearly, if a PA has a pole closer to the origin (indicated as $p(\ell)$) than a zero, then this zero is not a reliable guide to the UV to IR evolution of the theory from weak coupling. Furthermore, a PA may contain an essentially coincident pair of a zero and pole (indicated by $zp$); in this case, the zero and pole factors cancel and may be neglected.

We present the results of our Padé analysis in Table IV. Importantly, we find that in several cases the PAs yield results for the IR zero at the five-loop level that are physical and/or more stable than the zeros of $\beta_{5f}$ themselves. For $N_f = 16$ and $N_f = 15$, all of the three $\alpha_{IR,5,\{p,q\}}$ listed in Table IV agree very well with the respective values of $\alpha_{IR,5}$, and this is also true for $\alpha_{IR,5,\{2,2\}}$ and $\alpha_{IR,5,\{1,3\}}$ in the case of $N_f = 14$. For $N_f = 13$, the values of $\alpha_{IR,5,\{2,2\}}$ and $\alpha_{IR,5,\{1,3\}}$ lie roughly midway between $\alpha_{IR,4}$ and $\alpha_{IR,5}$. For $9 \leq N_f \leq 12$, where there is no physical IR zero of $\beta_{5f}$, at least one of the PAs, namely $\{3, 1\}$ yields physical IR zeros, and the respective values of $\alpha_{IR,5,\{3,1\}}$ are reasonably close to, and somewhat smaller than, the corresponding values of $\alpha_{IR,4}$. (PAs that yield negative or complex zeros are marked with $-$. Thus, using the physical results from the Padé approximants helps to circumvent the problem with complex $\alpha_{IR,5}$ in this lower region of $I_{IRZ}$.

The anomalous dimension $\gamma_m$ has the series expansion

$$\gamma_m = \sum_{\ell=1}^{\infty} c_\ell \alpha^\ell.$$  

The $n$-loop $\gamma_m$ is $\gamma_m,n,\ell = \sum_{\ell=1}^{\infty} c_\ell \alpha^\ell$. The coefficient $c_1 = 8$ is scheme-independent, while the $c_\ell$ with $\ell \geq 2$ are scheme-dependent \cite{10}. In the MS scheme, the $c_\ell$ have been calculated up to $\ell = 4$ \cite{33} and recently to $\ell = 5$ \cite{54} e.g., $c_2 = (40/3) − (40/9) N_f$, etc.

As noted above, we define $\gamma_{IR,\ell}$ evaluated at $\alpha = \alpha_{IR,\ell}$. We calculate $\gamma_{IR,5}$ here. For $14 \leq N_f \leq 16$, we use our values of $\alpha_{IR,5}$. For $N_f = 13$, we use $\alpha = \alpha_{IR,5,\{1,3\}}$ and for $10 \leq N_f \leq 12$ we use $\alpha = \alpha_{IR,5,\{3,1\}}$. In both the chirally symmetric and chirally broken IR phases, the IR value of $\gamma_m$ has the upper bound \cite{35} $\gamma_{IR,\ell} < 2$. Since $\gamma_{IR,2}$ violates this for $N_f = 10$ \cite{14}, we
TABLE IV: Values of $\alpha_{IR, n\ell, [p, q]}$ from $[p, q]$ Padé approximants to $\beta_{\ell}$, as a function of $N_f \leq 16$, including comparison with $\alpha_{IR, 4\ell}$ and $\alpha_{IR, 5\ell}$. The symbols (i) $zp$ and (ii) $pcl$ mean that the Padé approximant has (i) a coincident zero-pole pair close to the origin in the complex $\beta$ plane. Entries with $-\beta$ are unphysical.

| $N_f$ | $\alpha_{IR, 2\ell}$ | $\alpha_{IR, 5\ell}$ | $\alpha_{IR, 5\ell} [1, 1]$ | $\alpha_{IR, 5\ell} [2, 2]$ | $\alpha_{IR, 5\ell} [3, 3]$ |
|-------|-------------------|-------------------|------------------|------------------|------------------|
| 9     | 1.072             | $-1.02_{zp}$      | $-1$             | $-1$             | $-1$             |
| 10    | 0.815             | $-0.75_{zp}$      | $-pcl$           | $-pcl$           | $-pcl$           |
| 11    | 0.626             | $-0.56_{zp}$      | $-pcl$           | $-pcl$           | $-pcl$           |
| 12    | 0.470             | $-0.407_{zp}$     | 0.634            | 0.614            | $-\beta$         |
| 13    | 0.337             | 0.406             | 0.376            | 0.375            | $-\beta$         |
| 14    | 0.224             | 0.233             | $-0.232$         | 0.232            | $-\beta$         |
| 15    | 0.126             | 0.127             | 0.127            | 0.127            | $-\beta$         |
| 16    | 0.0398            | 0.0398            | 0.0398           | 0.0398           | 0.0398           |

TABLE V: Values of the five-loop anomalous dimension for the fermion bilinear, $\gamma_{IR, 5\ell}$, evaluated at the IR zero of the five-loop beta function, $\beta_{5\ell}$, as a function of $N_f$ for $11 \leq N_f \leq 16$, including comparison with lower-loop values of $\gamma_{IR, n\ell}$.

| $N_f$ | $\gamma_{IR, 2\ell}$ | $\gamma_{IR, 3\ell}$ | $\gamma_{IR, 4\ell}$ | $\gamma_{IR, 5\ell}$ |
|-------|-------------------|-------------------|-------------------|-------------------|
| 11    | 1.61              | 0.439             | 0.250             | 0.294             |
| 12    | 0.773             | 0.312             | 0.253             | 0.255             |
| 13    | 0.404             | 0.220             | 0.210             | 0.239             |
| 14    | 0.212             | 0.146             | 0.147             | 0.154             |
| 15    | 0.0997            | 0.0826            | 0.0836            | 0.0843            |
| 16    | 0.0272            | 0.0258            | 0.0259            | 0.0259            |

In summary, using the recently calculated five-loop term in the SU(3) beta function from [26], we have presented the first calculation of the five-loop IR zero in the beta function for an SU(3) gauge theory and the first five-loop calculation of the anomalous dimension of the fermion bilinear operator at this IR zero.

The research of T.A.R. and R.S. was supported in part by the Danish National Research Foundation grant DNRF90 to CP³-Origins at SDU and by the U.S. NSF Grant NSF-PHY-13-16617, respectively.

[1] Some early RG include E. C. G. Stueckelberg and A. Peterman, Helv. Phys. Acta 26, 499 (1953); M. Gell-Mann and F. Low, Phys. Rev. 95, 1300 (1954); N. N. Bogolubov and D. V. Shirkov, Dokl. Akad. Nauk SSSR 103, 391 (1955); C. G. Callan, Phys. Rev. D 2, 1541 (1970); K. Symanzik, Commun. Math. Phys. 18, 227 (1970); K. Wilson, Phys. Rev. D 3, 1818 (1971).

[2] It is straightforward to include fermion mass terms, but if a given fermion has a mass $m$, it is integrated out of the effective field theory applicable at scales $\mu < m$ and does not affect the further evolution to the IR.

[3] D. J. Gross and F. Wilczek, Phys. Rev. Lett. 30, 1343 (1973); H. D. Politzer, Phys. Rev. Lett. 30, 1346 (1973); ‘t Hooft, unpublished.

[4] W. E. Caswell, Phys. Rev. Lett. 33, 244 (1974); D. R. T. Jones, Nucl. Phys. B 75, 531 (1974).

[5] Here and elsewhere, when an expression is given for $N_f$ that formally evaluates to a non-integral real value, it is understood implicitly that one infers an appropriate integral value from it.

[6] D. J. Gross, in R. Balian and J. Zinn-Justin, eds. Methods in Field Theory, Les Houches 1975 (North Holland, Amsterdam, 1976), p. 141.

[7] O. V. Tarasov, A. A. Vladimirov, and A. Yu. Zharkov, Phys. Lett. B 93, 429 (1980); S. A. Larin and J. A. M. Vermaseren, Phys. Lett. B 303, 334 (1993).

[8] T. van Ritbergen, J. A. M. Vermaseren, and S. A. Larin, Phys. Lett. B 400, 379 (1997).

[9] M. Czakon, Nucl. Phys. B 710, 485 (2005).

[10] W. A. Bardeen, A. J. Buras, D. W. Duke, and T. Muta, Phys. Rev. D 18, 3998 (1978).

[11] T. Banks and A. Zaks, Nucl. Phys. B 196, 189 (1982).

[12] E. Gardi and M. Karliner, Nucl. Phys. B 529, 383 (1998); E. Gardi and G. Grunberg, JHEP 03, 024 (1999).

[13] F. A. Chishie, V. Elias, V. A. Miransky, and T. G. Steele, Prog. Theor. Phys. 104, 603 (2000).

[14] T. A. Ryttov, R. Shrock, Phys. Rev. D 83, 056011 (2011) [arXiv:1011.4522].

[15] C. Pica, F. Sannino, Phys. Rev. D 83, 035013 (2011) [arXiv:1111.5917].

[16] R. Shrock, Phys. Rev. D 87, 105005 (2013); R. Shrock, Phys. Rev. D 87, 116007 (2013).

[17] Refs. [14, 15] studied the IR zero in $\beta$ for general $G$ and fermion representation $R$. Here we restrict to $G = SU(3)$ and $R =$ fundamental.

[18] T. A. Ryttov and R. Shrock, Phys. Rev. D 86, 065032.
(2012); T. A. Ryttov and R. Shrock, Phys. Rev. D 86, 085005 (2012).
[19] R. Shrock, Phys. Rev. D 88, 036003 (2013); R. Shrock, Phys. Rev. D 90, 045011 (2014); R. Shrock, Phys. Rev. D 91, 125039 (2015); G. Choi and R. Shrock, Phys. Rev. D 90, 125029 (2014); G. Choi and R. Shrock, arXiv:1607.03500.
[20] T. A. Ryttov, Phys. Rev. D 89, 016013 (2014); T. A. Ryttov, Phys. Rev. D 89, 056001 (2014); T. A. Ryttov, Phys. Rev. D 90, 056007 (2014).
[21] J. A. Gracey and R. M. Simms, Phys. Rev. D 91, 085037 (2015).
[22] T. A. Ryttov, Phys. Rev. Lett., in press [arXiv:1604.00687].
[23] P. M. Stevenson, [arXiv:1607.01670].
[24] P. M. Stevenson, Phys. Rev. D 23, 2916 (1981); W. Celmaster and R. J. Gonsalves, Phys. Rev. D 20, 1420 (1979); E. Braaten and J. P. Leveille, Phys. Rev. D 24, 1369 (1981); S. J. Brodsky, G. P. Lepage, and P. B. MacKenzie, Phys. Rev. D 28, 228 (1983); S. G. Gorishy, A. L. Kataev, and S. A. Larin, Phys. Lett. B 194, 429 (1987); G. Grunberg and A. L. Kataev, Phys. Lett. B 279, 352 (1992); J. A. Gracey, J. Phys. A 46, 225403 (2013).
[25] For recent reviews, see, X.-G. Wu, S. J. Brodsky, and M. Mojaza, Prog. Part. Nucl. Phys. 72, 44 (2013); A. Deur, S. J. Brodsky, and G. F. de T´eramond. [arXiv:1604.08082].
[26] P. A. Baikov, K. G. Chetyrkin, and J. H. Kühn, [arXiv:1606.08659]. Our $\beta_l = 4^l/\beta_{l-1}$ in this paper.
[27] A. Hasenfratz, A. Cheng, G. Petropoulos, and D. Schaich, PoS (Lattice 2012) 034 [arXiv:1207.7162].
[28] A. Hasenfratz, A. Cheng, G. Petropoulos, and D. Schaich, contrib. to Lattice 2013 [arXiv:1310.1124].
[29] M. P. Lombardo, K. Miura, T. J. Nunes da Silva, and E. Pallante, JHEP 12, 183 (2014).
[30] See, e.g., talks in the CP3 Workshop at [http://cp3-origins.dk/events/meetings/mass2013 Lattice-2014] at [https://www.bnl.gov/lattice2014; SCGT15 at [http://www.kmi.nagoya-u.ac.jp/workshop/SCGT15 and Lattice-2015 at [http://www.aics.riken.jp/sympo/lattice2015].
[31] Since the perturbative series expansion for a function in quantum field theory is not, in general, a Taylor series expansion with finite radius of convergence, but instead, only an asymptotic expansion, one cannot apply the Taylor-Maclaurin remainder theorem to bound the difference between the true function and the n-loop series approximation. Here $\Delta_{IR,n-1,n}$ is only intended to give an estimate of the stability of the n-loop calculation.
[32] A review is G. A. Baker and P. Graves-Morris, Padé Approximants, Encyclopedia of Math. v. 13 (Addison-Wesley, Reading, 1981). Our notation follows I-H. Lee and R. E. Shrock, Phys. Rev. B 36, 3712 (1987); J. Phys. A 21, 3139 (1988); V. Matveev and R. Shrock, J. Phys. A 28, 1557 (1995).
[33] J. A. M. Vermaseren, S. A. Larin, and T. van Ritbergen, Phys. Lett. B 405, 327 (1997).
[34] P. A. Baikov, K. G. Chetyrkin, and J. H. Kühn, JHEP 10, 076 (2014) [arXiv:1402.6611]. Our $c_l = 2^{l+2}\beta_{l-1}$ in this paper.
[35] In the conformal phase, see S. Ferrara, R. Gatto, A. F. Grillo, Phys. Rev. D 9, 3564 (1974); G. Mack, Commun. Math. Phys. 55, 1 (1977); B. Grinstein, K. Intriligator, and I. Rothstein, Phys. Lett. B 662, 367 (2008). In the phase with $S_{\chi_{SB}}$, the dynamically generated momentum-dependent fermion mass is $m(k) \sim \Lambda(\Lambda/k)^{2-\gamma_m}$ up to logs, and the requirement that $\lim_{k \to \infty} m(k) = 0$ yields the same bound.