A GRADIENT DESCENT PERSPECTIVE ON SINKHORN

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Abstract. We present a new perspective on the popular Sinkhorn algorithm, showing that it can be seen as a Bregman gradient descent (mirror descent) of a relative entropy (Kullback-Leibler divergence). This viewpoint implies a new sublinear convergence rate with a robust constant.

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1. Introduction

The Sinkhorn algorithm has been used to solve matrix scaling problems [Yul12, Kru37, DS40, Bac65] and in particular regularized optimal transport problems [Wil69, Erl80, ES90, GS10, Cut13]. Its convergence was studied in [Sin64, Rü95] and rates of convergence were first established in [FL89].

The Sinkhorn algorithm can be seen as a solver for the minimum entropy problem

$$H^*(\mu, \nu, R) = \inf_{\pi \in \Pi(\mu, \nu)} H(\pi|R),$$

where \((X, \mu)\) and \((Y, \nu)\) are two probability spaces, \(R\) is a measure on \(X \times Y\) and \(\Pi(\mu, \nu)\) denotes the space of probability measures on \(X \times Y\) (sometimes called couplings or plans) having \(X\)-marginal \(\mu\) and \(Y\)-marginal \(\nu\). Moreover,
$H$ is the relative entropy (also known as Kullback–Leibler divergence) defined by $H(\pi|R) = \int \ln(\pi/R) \pi$. The Sinkhorn method constructs a sequence of couplings $\pi_0, \pi_1, \pi_2, \ldots$ by alternative projections onto couplings with $Y$-marginal $\nu$ (these are $\pi_1, \pi_2$, etc) and couplings with $X$-marginal $\mu$ ($\pi_1, \pi_2$, etc).

Our contribution is a new perspective on the Sinkhorn algorithm. We show that it can be seen as a Bregman gradient descent (mirror descent) of the relative entropy $\rho \mapsto H(\rho|\nu) = \int \ln(\rho/\nu) \rho$. This allows us to derive a new sublinear convergence rate

$$H(\rho_n|\nu) \leq \frac{H^*(\mu, \nu, R)}{n},$$

where $\rho_n$ is the $Y$-marginal of the iterate $\pi_n$. Contrary to all previously known global rates, our result features a robust constant $H^*$ which is always finite. In particular this new rate can be used for general reference measures $R$, without needing lower or upper bounds on the entries of $R$. We also obtain a new bound

$$H(\rho_n|\nu) \leq \frac{M_2(\mu) + M_2(\nu)}{n \varepsilon}$$

in the regularization of quadratic optimal transport

$$\inf_{\pi \in H(\mu, \nu)} \int \int \frac{1}{2} |x - y|^2 \pi(dx, dy) + \varepsilon H(\pi|\mu \otimes \nu).$$

Here $\varepsilon > 0$ and $M_2$ denotes second moments. This is of particular interest in the limit $\varepsilon \to 0$.

In regard to convergence rates of the Sinkhorn algorithm, linear rates were obtained in [FL89] by using the so-called Hilbert projective metric. This elegant approach yields for instance bounds of the form $H(\rho_n|\nu) \lesssim \lambda^n$ for a constant $\lambda \in (0, 1)$. In a large number of situations $\lambda$ is away from 1 and this rate is much stronger than our new sublinear rate. However all the known linear rates deteriorate when the reference measure $R$ (i.e. the “matrix” we wish to scale) contains large or small (nonnegative) entries. As an example $\lambda = 1$ if $R$ contains zero entries, in which case the linear rate is unusable. Therefore there is a dichotomy where either $R$ has good lower and upper bounds, in which case fast linear rates exist, or $R$ contains small or large values, in which case Hilbert metric theory might not even imply convergence of the iterates. Note that in many cases one might be interested in scaling matrices $R$ which contain many zeros, or even are sparse. Our new results remedy this situation by providing a convergence rate which is applicable to any problem.

Let us now mention some related works. In [ANWR17, CK18, DGK18] the authors derive sublinear estimates for the relative entropy $H(\nu|\rho_n)$. Our result improves on these estimates by obtaining an inequality $H(\rho_n|\nu) \leq H^*/n$, and more importantly by identifying the robust constant $H^*$. Indeed the constants appearing in these papers all contain a $-\ln(\min_{ij} R_{ij})$ term which
blows up as \( \min_{ij} R_{ij} \to 0 \) (their setting is finite-dimensional). Finally let us mention that a different mirror descent approach to the Sinkhorn scheme was recently presented in [Mis19]. Mishchenko’s interesting viewpoint is that of an alternating mirror descent; this seemingly doesn’t allow to derive convergence results.

2. Background

2.1. Bregman divergences.

**Definition 1.** Consider a differentiable function \( F : \mathbb{R}^N \to \mathbb{R} \). Its Bregman divergence is defined by

\[
F(\phi_2|\phi_1) = F(\phi_2) - F(\phi_1) - \langle F'(\phi_1), \phi_2 - \phi_1 \rangle,
\]

for any \( \phi_1, \phi_2 \in \mathbb{R}^N \). Here \( F' \) denotes the derivative (or gradient) of \( F \), i.e. the vector \( F'(\phi) = (\partial_i F(\phi))_i \), and \( \langle \cdot, \cdot \rangle \) is the usual dot product.

Let us gather below some well-known results in the theory of Bregman divergences.

**Proposition 1.** Let \( F : \mathbb{R}^N \to \mathbb{R} \) be a convex and differentiable function, and denote by \( F^* \) its convex conjugate

\[
F^*(\rho_1|\rho_2) = \sup_{\phi} \langle \phi, \rho_1 \rangle - F(\phi).
\]

Then

(i) \( F(\phi_2|\phi_1) = F^*(\rho_1|\rho_2) \) for all \( \phi_1, \phi_2 \in \mathbb{R}^N \), where we set \( \rho_i = F'(\phi_i) \).

(ii) Fix \( a \in \mathbb{R}^N \) and define \( F_a(\phi) = F(\phi|a) \). Then \( F_a(\phi_2|\phi_1) = F(\phi_2|\phi_1) \).

2.2. Bregman gradient descent. Consider a differentiable function \( H : \mathbb{R}^N \to \mathbb{R} \) that we wish to minimize without constraints. Let \( G : \mathbb{R}^N \to \mathbb{R} \) be a differentiable strictly convex function which will be used as a movement limiter. The gradient descent iteration with a Bregman divergence based on \( G \), also called mirror descent [NY83, BT03], takes the form

\[
\rho_{n+1} = \arg\min_{\rho} H(\rho_n) + \langle H'(\rho_n), \rho - \rho_n \rangle + G(\rho|\rho_n),
\]

for all \( n \geq 0 \), where \( H' \) denotes the derivative of \( H \) (see previous section). The optimality conditions are given by

\[
G'(\rho_{n+1}) - G'(\rho_n) = -H'(\rho_n).
\]

The following result gathers well-known facts in first-order optimization theory.

**Theorem** (Unconstrained gradient descent). Consider the gradient descent method (1) under the previous hypotheses on \( H \) and \( G \).

i) If the objective function is dominated by the movement limiter, i.e. \( H(\tilde{\rho}|\rho) \leq G(\tilde{\rho}|\rho) \) for all \( \rho, \tilde{\rho} \), then we have the descent property

\[
H(\rho_{n+1}) \leq H(\rho_n) - G(\rho_n|\rho_{n+1}),
\]

for all \( n \geq 0 \).
ii) If in addition $H$ is convex then we have the convergence rate $H(\rho_n) \leq \inf_\rho H(\rho) + \frac{G(\rho_0)}{n}$, for all $n \geq 1$. Therefore if $H$ admits a minimizer $\nu \in \mathbb{R}^N$ then

$$H(\rho_n) - H(\nu) \leq \frac{G(\nu|\rho_0)}{n}.$$ 

2.3. Entropic regularization of optimal transport. Let $(X, \mu)$ and $(Y, \nu)$ be two probability spaces and consider a cost function $c: X \times Y \to \mathbb{R}$. We are interested in the regularized optimal transport problem \cite{GS10, Cut13, PC19}

\begin{equation}
\inf_{\pi \in \Pi(\mu, \nu)} \iint_{X \times Y} c(x, y) \pi(dx, dy) + \varepsilon \iint_{X \times Y} \ln \left( \frac{\pi(dx, dy)}{\mu(dx)\nu(dy)} \right) \pi(dx, dy),
\end{equation}

where $\varepsilon > 0$. Here $\Pi(\mu, \nu)$ denotes the set of couplings $\pi$ having $X$-marginal $\mu$ and $Y$-marginal $\nu$, i.e. $\int_Y \pi(dx, dy) = \mu(dx)$ and $\int_X \pi(dx, dy) = \nu(dy)$. The above problem can be written as

$$\inf_{\pi \in \Pi(\mu, \nu)} \varepsilon H(\pi|R),$$

by defining $R(dx, dy) = e^{-c(x,y)/\varepsilon} \mu(dx)\nu(dy)$. The relative entropy is defined by $H(\pi|R) = \iint_{X \times Y} \ln(\pi/R) d\pi$ when $\pi$ is absolutely continuous with respect to $R$, and $\infty$ otherwise. We will focus primarily on the dual formulation of (2), which takes the form

$$\varepsilon \sup_{\phi, \psi} \int_Y \phi(y) \nu(dy) + \int_X \psi(x) \mu(dx) - \ln \left( \iint_{X \times Y} e^{\phi(y) + \psi(x)} R(dx, dy) \right).$$

The supremum is here taken over functions $\phi: Y \to \mathbb{R}$ and $\psi: X \to \mathbb{R}$.

3. Sinkhorn as a gradient descent method

3.1. Definitions and notations. Let $(X, \mu)$ and $(Y, \nu)$ be two probability spaces, and $R$ a reference measure on $X \times Y$ (note that we do not assume that $R$ has necessarily mass 1). Define the dual functional

$$D(\phi, \psi) = \langle \phi, \nu \rangle - \langle \psi, \mu \rangle - \ln \left( \iint_{X \times Y} e^{\phi(y) - \psi(x)} R(dx, dy) \right),$$

over functions $\phi: Y \to \mathbb{R}$ and $\psi: X \to \mathbb{R}$. Here $\langle \phi, \nu \rangle = \int_Y \phi \nu$ and $\langle \psi, \mu \rangle = \int_X \psi \mu$. The Sinkhorn method can be seen as an iterative solver for the optimization problem $\sup_{\phi, \psi} D(\phi, \psi)$ (note that this is a concave maximization problem). We can write the Sinkhorn iterations concisely by making use of two transforms $\phi \to \phi^+$ and $\psi \to \psi^-$, defined by

$$\phi^+(x) = \ln \left( \int_Y e^{\phi(y)} R(dx, dy)/\mu(dx) \right),$$

$$\psi^-(y) = -\ln \left( \int_X e^{-\psi(x)} R(dx, dy)/\nu(dy) \right).$$
The fractions in the expression above should be interpreted in the Radon–Nikodym sense and are assumed to be well-defined. Then, the Sinkhorn iteration takes the form

\[ \phi_{n+1} = (\phi_n)^{+-}. \]

Note that the “+”-transform maps a potential \( \phi \) defined on \( Y \) to a potential \( \phi^+ \) defined on \( X \) (and vice versa for the “−”-transform). These two transformations play similar roles as the \( c \)-transforms from optimal transport [Vil09].

To each pair of potential \( (\phi, \psi) \) is associated a primal quantity: the coupling or “plan”

\[ \pi(\phi, \psi)(dx, dy) = Z^{-1}e^{\phi(y) - \psi(x)} R(dx, dy), \]

where the scalar \( Z = \iint e^{\phi(y) - \psi(x)} R(dx, dy) \) ensures that the measure \( \pi(\phi, \psi) \) has mass 1.

In order to relate potentials and densities let us first define

\[ J(\phi) = \sup_{\psi} D(\psi, \phi), \]

where the supremum is taken over all function \( \psi: X \to \mathbb{R} \). Since \( D \) is concave, \( J \) is easily seen to be concave as well. Second, we define a functional \( F \) by \( J(\phi) = \langle \phi, \nu \rangle - F(\phi) \). Written more explicitly, we have

\[ F(\phi) = \langle \phi^+, \mu \rangle. \]

The important role played by the convex functional \( F \) lies in its derivative \( F' \) which is a bridge between potentials and densities, as shown by the following result.

**Lemma 1.** Consider a potential \( \phi: Y \to \mathbb{R} \). Then \( F'(\phi) = p_Y \pi(\phi, \phi^+) \), where \( p_Y \) denotes the \( Y \)-marginal projection. In other words,

\[ F'(\phi)(dy) = \int_X e^{\phi(y) - \phi^+(x)} R(dx, dy). \]

The \( X \)-marginal (sum along the rows) of \( \pi(\phi, \phi^+) \) is always \( \mu \).

In the above lemma and in the rest of this note we use \( F' \) to denote the derivative (or first variation) of \( F \). It is defined for instance by \( F'(\phi)h = \lim_{\epsilon \to 0} (F(\phi + \epsilon h) - F(\phi))/\epsilon \).

3.2. **Main results.** We recall that \((X, \mu)\) and \((Y, \nu)\) are two probability spaces and that \( R \) is a measure on \( X \times Y \). In the previous section, we defined two transformations “+” and “−” by (3), a coupling function \( \pi(\phi, \psi) \) by (5) and a functional \( F(\phi) = \langle \phi^+, \mu \rangle \).

Our starting point is the observation, already present in [Ber17], that the Sinkhorn iteration \( \phi_{n+1} = (\phi_n)^{+-} \) can be written as

\[ \phi_{n+1} - \phi_n = -\ln(\rho_n/\nu), \]

where \( \rho_n \) denotes the probability measure associated with \( \phi_n \), i.e. \( \rho_n = F'(\phi_n) \) (see Lemma 1). Then the main result of this paper says that the
Sinkhorn iteration (6) can be seen as a gradient descent method of a relative entropy (Kullback–Leibler divergence):

**Theorem 1.** Let \( \pi_n = \pi(\phi_n, \phi_n^+) \) be the coupling produced by the Sinkhorn iteration \( \phi_n \to \phi_n + 1 \) (see (4) and (6)) and denote by \( \rho_n \) its \( Y \)-marginal (we recall that the \( X \)-marginal of \( \pi_n \) is always \( \mu \)). Then the Sinkhorn scheme can be seen as the gradient descent

\[
(F^*)'(\rho_{n+1}) - (F^*)'(\rho_n) = -H'_\nu(\rho_n).
\]

Here \( H_\nu(\rho) = H(\rho|\nu) = \int \ln(\rho/\nu) \rho \) denotes the relative entropy of \( \rho \) with respect to \( \nu \), and \( F^* \) is the convex conjugate of \( F \). Moreover \( ' \) denotes derivative, see (1).

**Corollary 1** (Sublinear rate). Let \( H^*(\mu, \nu, R) = \inf_{\pi \in \Pi(\mu, \nu)} H(\pi|R) \) be the value of the minimum entropy problem. Assume that \( R \) has total mass 1. Then the gradient descent formulation (7) implies decrease of the relative entropies

\[
H(\rho_{n+1}|\nu) \leq H(\rho_n|\nu),
\]

and a sublinear convergence rate with a strong constant:

\[
H(\rho_n|\nu) \leq \frac{H^*(\mu, \nu, R)}{n},
\]

for all \( n \geq 1 \). In particular this bound is always finite (if there exists a solution to the minimum entropy problem).

Let us point out that general measures \( R \) which don’t necessarily sum up to 1 can be considered in Corollary 1. In that case (8) should be replaced by

\[
H(\rho_n|\nu) \leq \frac{H^*(\mu, \nu, R) + \ln(\iint R)}{n}.
\]

In fact a slightly stronger bound valid for any measure \( R \) is implied by the gradient descent viewpoint, namely

\[
H(\rho_n|\nu) \leq \frac{H^*(\mu, \nu, R) - H(\mu|\bar{\mu})}{n},
\]

where \( \bar{\mu} \) is the \( X \)-marginal of \( R \). Note that when \( \bar{\mu} \) is not a probability measure the entropy \( H(\mu|\bar{\mu}) \) can be positive or negative.

Let us now mention briefly two applications which are developed in more details in the discussion section (Examples 1 and 2).

**Examples.** Take \( X = Y = \mathbb{R}^d \).

1. Let \( \mu \) and \( \nu \) have finite second moments \( M_2(\mu) = \int |x|^2 \mu(dx) \) and \( M_2(\nu) = \int |y|^2 \nu(dy) \). Fix \( \varepsilon > 0 \) and consider the regularized optimal transport problem

\[
\inf_{\pi \in \Pi(\mu, \nu)} \iint \frac{1}{2}|x - y|^2 \pi(dx, dy) + \varepsilon \iint \ln \left( \frac{\pi(dx, dy)}{\mu(dx)\nu(dy)} \right) \pi(dx, dy).
\]
Let \( \rho_n \) denote the sequence of \( Y \) densities corresponding to the Sinkhorn iterates, as in Theorem 1. Then

\[
H(\rho_n | \nu) \leq \frac{M_2(\mu) + M_2(\nu)}{n \varepsilon},
\]

for all \( n \geq 1 \).

2. Let \( U \) be a strongly convex potential energy \( D^2U(x) \geq \lambda I \), normalized so that \( m(dx) = e^{-U(x)} \, dx \) is a probability measure, and let \( \mu \) and \( \nu \) be two absolutely continuous probability measures with respect to \( m \). Let \( \mathcal{R} \) be the joint measure at times 0 and \( T \) associated with the SDE

\[
dX_t = -\nabla U(X_t) \, dt + dW_t, \quad X_0 \sim m.
\]

Then we have the Sinkhorn convergence rate

\[
H(\rho_n | \nu) \leq \frac{H(\mu|m) + H(\nu|m)}{n (1 - e^{-\lambda T})}.
\]

We develop below discussions and examples related to these results.

Discussion on the gradient descent formulation. We refer to Section 2 for a short introduction on Bregman divergences and gradient descent methods.

Our gradient descent perspective in Theorem 1 shifts the focus of the Sinkhorn method from potentials to probability measures. It is based on the “semi-dual” formulation (6) which eliminates one of the two potentials (here \( \psi \)) and provides a description of the Sinkhorn algorithm based only on \( Y \)-variables (\( \phi \) and \( \rho \)). By symmetry it is possible of course to state an analogue of Theorem 1 using instead variables defined on \( X \).

A rather nonstandard aspect of the theorem’s gradient scheme is the movement limiter based on \( F^* \). First recall from Section 2.2 that the gradient descent update (7) admits the variational formulation

\[
\rho_{n+1} = \arg\min_{\rho} H_\nu(\rho_n) + \langle H'_\nu(\rho_n), \rho - \rho_n \rangle + F^*(\rho | \rho_n),
\]

which highlights the form of movement limiter \( F^*(\rho | \rho_n) \): a Bregman divergence based on the function \( F^* \). Here \( F^* \) is specific to the the problem at hand; from the optimization point of view it is natural to have movement limiters well-adapted to the objective function. The next result might shed some light on this Bregman divergence by expressing it as a relative entropy (Kullback–Leibler divergence) of the corresponding couplings.

Lemma 2. Let \( \phi \) and \( \tilde{\phi} \) be two potentials defined over \( Y \). Denote \( \pi = \pi(\phi, \phi^+) \), \( \tilde{\pi} = \pi(\tilde{\phi}, \tilde{\phi}^+) \), and set the \( Y \)-marginals \( \rho = p_Y \pi \) and \( \tilde{\rho} = p_Y \tilde{\pi} \). Then

\[
F(\phi | \tilde{\phi}) = F^*(\tilde{\rho} | \rho) = H(\tilde{\pi} | \pi),
\]

with \( H(\tilde{\pi} | \pi) = \int \int \ln(\tilde{\pi}/\pi) \, \tilde{\pi} \).
A benefit of a gradient descent framework is that obtaining a convergence rate becomes a clearly defined problem: the movement limiter should dominate (in the convex sense) the objective function. Here it means roughly speaking obtaining the inequality over Hessians

\[ H''_{\psi} \leq (F^*)''. \]

(Note that we don’t actually need these functions to be twice-differentiable). This is proven in the next section (see Lemma 3), but let us point out here that it relies on the following simple fact: the relative entropy decreases when taking marginals, thus

\[ H(\widetilde{\pi} | \pi) \geq H(p_Y \widetilde{\pi} | p_Y \pi). \]

Discussion on the convergence rate. The strength of our convergence rate

\[ H(\rho_n | \nu) \leq \frac{H^*}{n} \]

lies in the robust constant \( H^* \) rather than its sublinear nature, since linear rates are well-known to exist (see the next paragraph). Indeed, the constant \( H^* = H^*(\mu, \nu, R) \) is finite as soon as the feasibility set of the entropic problem (2) is non-empty. In other words, when there is a solution then the convergence rate \( H^*/n \) holds. For instance, this allows to deal with reference measures \( R \) with zero entries. To the best of our knowledge this improves on all the known global rates for the Sinkhorn algorithm which are sensitive to zero entries of \( R \).

A classical approach to obtain rates on the convergence of the Sinkhorn method is to use the Hilbert projective metric [FL89]. Then one can derive linear convergence rates of the form \( H(\rho_n | \nu) \lesssim \lambda^{-2n} \) for some \( \lambda \in (0, 1] \); however the constant \( \lambda \) can be weak in practice. We refer to [PC19] for precise formulas but let us point out that \( \lambda \to 1 \) as \( \min_{ij} R_{ij} \to 0 \).

More recently, a series of work [ANWR17, CK18, DGK18] have derived sublinear estimates for the relative entropy \( H(\nu | \rho_n) \) in the same spirit as our convergence rate. In these works is proven, roughly speaking, that \( O(1/\delta) \) iterations are needed to obtain an accuracy of \( \delta > 0 \), measured in a KL divergence. The convergence rate obtained from our gradient descent viewpoint improves on these estimates on two fronts. First we obtain that the quantities \( H(\rho_n | \nu) \) decrease as \( n \) grows, as well as a true inequality \( H(\rho_n | \nu) \leq \frac{H^*}{n} \). Second the constants appearing in the literature slightly differ from one another but all have in common a \( -\ln(\min_{ij} R_{ij}) \) term which blows up as \( \min_{ij} R_{ij} \to 0 \) (their setting is finite-dimensional so that \( R \) is a matrix with entries \( R_{ij} \)).

We now present some examples which allow a more explicit bound on the rate constant \( H^* \).

Example 1 (Regularization of quadratic optimal transport). Take \( X = Y = \mathbb{R}^d \) and let \( \mu \) and \( \nu \) be two probability measures on \( \mathbb{R}^d \) with finite second
Fix $\varepsilon > 0$ and consider the problem
\[
E^* = \inf_\pi \int \frac{1}{2}[|x-y|^2 \pi(dx,dy)] + \varepsilon \int \int \ln \left( \frac{\pi(dx,dy)}{\mu(dx)\nu(dy)} \right) \pi(dx,dy).
\]
As usual the infimum runs over couplings $\pi \in \Pi(\mu,\nu)$. To fit into the framework of this paper define $R(dx,dy) = e^{-\frac{|x-y|^2}{2\varepsilon}} \mu(dx)\nu(dy)$ with $Z = \int e^{-\frac{|x-y|^2}{2\varepsilon}} \mu(dx)\nu(dy)$. Then $E^* = \varepsilon \inf_R H(\pi|R)$. The simple upper bound $H^*(\mu,\nu,R) = \inf_\pi H(\pi|R) \leq M_2(\mu) + M_2(\nu)$ can be obtained with $\pi(dx,dy) = \mu(dx)\nu(dy)$. Also note that the total mass of $R$ satisfies $\int \int R \leq 0$. Using the general form (9) of our main result we obtain the Sinkhorn convergence rate
\[
H(\rho_n|\nu) \leq \frac{M_2(\mu) + M_2(\nu)}{n\varepsilon},
\]
for all $n \geq 1$.

Often one is interested in the limit $\varepsilon \to 0$. Then the above inequality provides a $O\left(\frac{1}{n\varepsilon}\right)$ bound which can be compared to the $O((1 - e^{-1/\varepsilon})^{2n})$ bound from the Hilbert metric theory [PC19]. For instance, with $\varepsilon = 10^{-4}$, assuming all the other constants are $O(1)$, one can guarantee an accuracy $H(\rho_n|\nu) < 10^{-3}$ in

- $n \sim 10^7$ iterations with our $O\left(\frac{1}{n\varepsilon}\right)$ bound; and
- $n \sim e(10^4)$ iterations with a $O((1 - e^{-1/\varepsilon})^{2n})$ bound.

**Example 2** (Entropic Talagrand inequality). Take $X = Y = \mathbb{R}^d$, let $\mu$ and $\nu$ be two probability measures absolutely continuous with respect to the Lebesgue measure and let $R$ be the joint measure at times 0 and $T$ associated with the SDE
\[
dX_t = -\nabla U(X_t) dt + dW_t, \quad X_0 \sim m(dx) := e^{-U(x)} dx.
\]
We assume that the potential energy $U$ is normalized and satisfies the strong convexity bound $D^2U(x) \geq \lambda I$ for some $\lambda > 0$. Here $I$ denotes the $d \times d$ identity matrix. One can have in mind for instance the Ornstein–Uhlenbeck process corresponding to $U(x) = \frac{1}{2}|x|^2$.

In this setting we can obtain more precise bounds for our Sinkhorn convergence rate (8) by using recent results in [Con19, CT19]. These works provide an entropic version of the Talagrand inequality from optimal transport. Specifically, they obtain the following bound on the entropic cost:
\[
H^*(\mu,\nu,R) \leq \frac{H(\mu|m) + H(\nu|m)}{1 - e^{-\lambda T}},
\]
where $m(dx) = e^{U(x)} dx$. For our purposes, this implies the Sinkhorn convergence rate
\[
H(\rho_n|\nu) \leq \frac{H(\mu|m) + H(\nu|m)}{n(1 - e^{-\lambda T})}.
\]
3.3. Proofs.

Proof of Lemma 1. The result of this lemma can be obtained from an elementary computation of derivative, using the expression

\[ F(\phi) = \langle \phi^+, \mu \rangle = \int_X \ln \left( \int_Y e^{\phi(y)} R(dx, dy)/\mu(dx) \right) \mu(dx). \]

\[ \square \]

Proof of Lemma 2. Let \( \phi \) and \( \tilde{\phi} \) be two potentials on \( Y \), and denote \( \pi \) and \( \tilde{\pi} \) the corresponding couplings, as well as \( \rho \) and \( \tilde{\rho} \) the corresponding probability measures on \( Y \).

Firstly, the identity \( F(\phi|\tilde{\phi}) = F^*(\tilde{\rho}|\rho) \) is a general property of Bregman divergences, see Prop. 1(i). Here it follows from Lemma 1 which says that \( \rho = F'(\phi) \) and \( \tilde{\rho} = F'(\tilde{\phi}) \).

Secondly, we prove that \( F(\phi|\tilde{\phi}) = H(\tilde{\pi}|\pi) \). We write

\[ H(\tilde{\pi}|\pi) = \iint \ln \left( \frac{\tilde{\pi}}{\pi} \right) \tilde{\pi}. \]

Using the expression \( \pi(dx, dy) = e^{\phi(y) - \phi^+(x)} R(dx, dy) \) and the corresponding one for \( \tilde{\pi} \) we obtain

\[ H(\tilde{\pi}|\pi) = \iint \left[ (\tilde{\phi}(y) - \phi(y)) - (\tilde{\phi}^+(x) - \phi^+(x)) \right] \tilde{\pi}(dx, dy) \]
\[ = \langle \tilde{\phi} - \phi, \tilde{\rho} \rangle - \langle \tilde{\phi}^+ - \phi^+, \mu \rangle, \]

since the \( X \)-marginal of the couplings \( \pi(\phi, \phi^+) \) we construct is always \( \mu \). Continuing,

\[ H(\tilde{\pi}|\pi) = (\tilde{\phi} - \phi, F'(\tilde{\phi})) - F(\tilde{\phi}) + F(\phi) \]
\[ = F(\phi|\tilde{\phi}). \]

\[ \square \]

Proof of Theorem 1. Let \( n \geq 0 \) and consider a Sinkhorn iterate \( \phi_n \). We start by proving the observation (6). Set \( \psi_n = (\phi_n)^+ \), so that \( \phi_{n+1} = (\psi_n)^- \). Let \( \rho_n \) denote the \( Y \)-marginal of \( \pi(\phi_n, \psi_n) \), i.e.

\[ \rho_n(dy) = \int_X e^{\phi_n(y) - \psi_n(x)} R(dx, dy). \]

This can be written, recognizing the \( "-" \)-transform, as

\[ \rho_n(dy) = e^{\phi_n(y) - \phi_{n+1}(y)} \nu(dy). \]

After some algebra we obtain the desired equality. We now start the proof of the theorem.

Part a): By Lemma 1 we know that \( F'(\phi_n) = \rho_n \). Convex conjugation inverts derivatives, therefore \( \phi_n = (F^*)'(\rho_n) \).
Part b): The crucial element needed to derive a $O(1/n)$ convergence rate for a gradient descent scheme is showing that the movement limiter dominates (in a convex sense) the objective function. We refer to Section 2.2 for a more detailed explanation. For the problem at hand, this means proving the following result.

**Lemma 3.** For all probability measures $\rho, \tilde{\rho}$ on $Y$,

$$H_\nu(\tilde{\rho}|\rho) \leq F^*(\tilde{\rho}|\rho),$$

where $H_\nu(\rho) = H(\rho|\nu)$.

To show this, first use Lemma 2 to write

$$F^*(\tilde{\rho}|\rho) = H(\pi^*|\pi),$$

where $\pi$ and $\pi^*$ are defined in accordance with Lemma 2. Then we use a property of the relative entropy (true more generally for $f$-divergences) that relative entropy decreases when taking marginals, thus

$$H(\pi^*|\pi) \geq H(p_Y \pi^*|p_Y \pi).$$

This property is a simple consequence of Jensen’s inequality and is left as an exercise to the reader. We have obtained

$$F^*(\tilde{\rho}|\rho) \geq H^{*}(\tilde{\rho}|\rho).$$

Next we can use Prop. 1(ii) to say that

$$H(\pi^*|\pi) = H(\nu|\rho_0),$$

so that the above lemma is proven. From general gradient descent theory (see for instance the theorem in Section 2.2) we deduce that

$$H(\rho_\pi|\nu) \leq \frac{F^*(\nu|\rho_0)}{n},$$

for all $n \geq 1$. Thus we have derive the $O(1/n)$ convergence bound, but we would like to obtain a more tractable inequality. To this end, assume that the initial iterate $\phi_0$ is identically zero. Let $\pi_0$ be the coupling associated to $\rho_0$ and let $\pi^*$ be the coupling associated to $\nu$, i.e. $\pi^*$ is the minimizer to the entropic problem $\inf_{\pi} H(\pi|R)$ (we assume in this paper that the minimizer exists). By Lemma 2 we know that $F^*(\nu|\rho_0) = H(\pi^*|\pi_0)$. Denote $\psi_0 = (\phi_0)^+; \quad \pi_0(dx,dy) = e^{\phi_0(y) - \psi_0(x)} R(dx,dy)$ and we have

$$H(\pi^*|\pi_0) = \iint \ln \left( \frac{\pi^*}{\pi_0} \right) \pi^* = \iint \ln \left( \frac{\pi^*}{R} \right) \pi^* - \iint \phi_0 \pi^* + \iint \psi_0 \pi^* = H(\pi^*|R) - \langle \phi_0, \nu \rangle + \langle \psi_0, \mu \rangle.$$

Since we assume that $\phi_0 = 0$ the second term cancels, and the third term is

$$\langle \psi_0, \mu \rangle = \int \ln \left( \int e^{\phi_0(x) - \psi_0(x)} R(dx) \right) \mu = -H(\mu|\tilde{\mu}) \leq 0$$

where $\tilde{\mu}$ is the $X$-marginal of $R$. Therefore $H(\pi^*|\pi_0) = H(\pi^*|R) - H(\mu|\tilde{\mu})$. If $R$ has total mass 1 then so does its marginal $\tilde{\mu}$, which implies that the
relative entropy $H(\mu|\bar{\mu})$ is nonnegative. Thus $H(\pi^*|\pi_0) \leq H(\pi^*|R)$ which concludes the proof.

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