Bayesian equilibria of axisymmetric plasmas

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Abstract. Bayesian models of axisymmetric plasmas using Gaussian processes and force balance equations have been developed. These models give the full joint posterior probability distributions over plasma current distributions and pressure profiles given the magnetic field and pressure measurements simultaneously. The toroidal currents such as plasma and magnetic field coil currents are modelled as a grid of toroidal current carrying solid beams. The plasma pressure and poloidal current flux profiles are given as a function of the poloidal magnetic flux surface, determined by the toroidal currents. Inference of all these physics parameters is a tomographic problem, thus, in order to exclude unreasonable solutions, two different prior distributions have been exploited: a Gaussian process prior and an equilibrium prior. The Gaussian process prior constrains the plasma current distributions by their covariance (smoothness) function whose hyperparameters have been optimally selected by Bayesian Occam’s razor. On the other hand, the equilibrium prior imposes the magnetohydrodynamic force balance by introducing observations that the differences between the magnetic force and the plasma pressure gradient are almost zero at every plasma current beam. These virtual observations emphasise equilibrium solutions \textit{a priori} as a part of the prior knowledge. These models with the two different priors employ predictive models of magnetic sensors and other plasma diagnostics in order to find all possible solutions consistent with all the measurements. The complex, high dimensional posterior distributions are explored by a new method based on the Gibbs sampling scheme.

Keywords: Plasma equilibria, Plasma diagnostics, JET, Bayesian inference, Physics priors, Virtual observations, Gaussian processes, Forward modelling, Occam’s razor

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1. Introduction

In magnetic confinement fusion research, inference of plasma current distributions is critical to control and to understand the underlying physics and the plasma [1, 2, 3]. The plasma current distributions determine the magnetic field geometry of the plasma that plays an important role in plasma control [4] and provide the canonical coordinate system [1], in which to express physics parameters and models for further research, for example energy transport. This magnetic field geometry can be represented as a set of poloidal magnetic flux surfaces often normalised to zero at the plasma centre, known as the magnetic axis, and to one at the plasma boundary, known as the last closed flux surface (LCFS).

The conventional approach to infer the plasma current distributions is to find a single solution to magnetohydrodynamic (MHD) force balance equations such as the Grad-Shafranov equation [5, 6] consistent with magnetic field measurements [7]. This approach has been providing a plasma equilibrium solution, nevertheless, it has the following limitations: typically it makes use of 1D parameterisations of the plasma pressure and poloidal current flux with a handful of parameters, which are often incapable of representing the shape of spatial profiles of them, and it usually takes into account only the magnetic measurements, thus this equilibrium solution might be inconsistent with plasma pressure and poloidal current measurements. Moreover, this approach finds only a single solution, not all possible solutions given the force balance equations and measurements. In other words, the conventional approach does not provide uncertainties of either the magnetic field geometry or the physics parameters such as the plasma current distributions or pressure profiles.

In this work, we will demonstrate Bayesian inference of axisymmetric plasma current distributions and pressure profiles consistent with a number of measurements from magnetic sensors and other plasma diagnostics in a large-scale fusion experiment. This Bayesian approach makes use of an axisymmetric current beam model [8], which represents all kind of toroidal currents, for example plasma and poloidal field coil currents, as a grid of toroidal current carrying solid beams. These toroidal current distributions determine the poloidal magnetic flux surface on which other physics parameters such as the plasma pressure and poloidal current flux are often assumed to be constant [3], thus they are given as a function of the magnetic flux surface. The plasma pressure and poloidal current flux are modelled by Gaussian processes, which are capable of representing a great variety of 1D spatial profiles of these physics parameters [9, 10]. Since inference of all these physics parameters is a tomographic problem, in order to exclude unreasonable solutions, we have introduced two different prior distributions: a Gaussian process prior and an equilibrium prior. The Gaussian process prior constrains the plasma current distributions by the covariance (smoothness) function which determines the covariance (smoothness) between any two plasma current beams. The parameters of the covariance function, also known as the hyperparameters of Gaussian processes, can be optimised to maximise the posterior probability of the model, which takes into account
the principle of Occam’s razor [11, 12, 13]. In other words, we can infer the plasma current distributions with the optimal smoothness from the measurements. On the other hand, the equilibrium prior imposes the force balance between the magnetic force and the plasma pressure gradient, given by the Grad-Shafranov equation, by introducing virtual observations that the differences between the two forces should be almost zero at every plasma current beam. These virtual observations emphasise equilibrium solutions as a part of the prior knowledge [14]. Here, the axisymmetric plasma model with the Gaussian process prior and the one with the equilibrium prior are called the current tomography model and the equilibrium model, respectively. In addition to these prior distributions, both models employ predictive models of magnetic sensors (pickup coils, saddle coils and flux loops) [8], polarimeters [14, 15], interferometers [14], lithium beam emission spectroscopy [16, 17] and high-resolution Thomson scattering (HRTS) systems [10] in order to find all possible plasma current distributions as well as pressure and poloidal current flux profiles consistent with all their measurements simultaneously at one of the large-scale fusion experiments, the Joint European Torus (JET) [18]. Since these models involve a large number of unknown parameters and observations as well as multiple predictive models of scientific instruments, it is, therefore, inevitable to use a framework that is capable of handling and keeping track of all these parameters, assumptions, predictive models and observations in such a complex model. For this reason, these models have been implemented in the Minerva framework.

The Minerva framework [19, 20] has been developed to achieve consistent scientific inference in a complex system by providing a standardised format for model components, for example probability and forward functions, and a standardised interface for component dependencies, input parameters, which can be connected from output of other model components. Minerva automatically manages all the model components and connections and represents them by a Bayesian graphical model [21], as shown in Figure 1. The modular structure, graphical representation and automatic model administration allow us to handle a complex model and to keep track of a large number of parameters, assumptions, predictive models and observations in a systematic way. Furthermore, we can easily build and compare models with different model specifications, for example various prior distributions. In this work, we present the two axisymmetric plasma models with different priors. The Minerva framework has been used for a number of scientific applications to magnetic sensors [8], interferometers [10, 13, 14], Thomson scattering systems [10, 22], soft X-ray spectroscopy [23], beam emission spectroscopy [16, 17], X-ray imaging crystal spectroscopy [24], electron cyclotron emission diagnostics [25] and effective ion charge diagnostics [26] in nuclear fusion research. These Bayesian models implemented in Minerva can be accelerated by a field-programmable gate array (FPGA) [27] and an artificial neural network [28, 29].

These Bayesian models of axisymmetric plasmas provide the full joint posterior distributions over the plasma current distributions, poloidal current flux and pressure profiles consistent with the magnetic field and pressure measurements simultaneously. However, exploration of such complex, high dimensional posterior distributions is
Figure 1. A simplified version of Minerva graph representing the Bayesian equilibrium model of axisymmetric plasmas at Joint European Torus (JET). The unknown parameters and observations are shown as the red and blue circles, respectively. The toroidal currents of the plasma $J_\phi$, iron core $J_{\text{iron}}$ and magnetic field coils $J_{\text{coils}}$ are modelled as a 2D grid of toroidal current carrying solid beams in $R,Z$ coordinates which determines the magnetic field $B$ and the normalised poloidal magnetic flux surface $\psi_N$. The poloidal current flux $F$ and electron density $n_e$ and temperature $T_e$ are modelled by Gaussian processes, whose hyperparameters are denoted as $\sigma_f$ and $\sigma_x$. All these physics parameters are provided as 3D fields in $x,y,z$ Cartesian coordinates, and given these 3D fields, the predictive models of the magnetic sensors, polarimeters, interferometers, lithium beam emission spectroscopy and high-resolution Thomson scattering (HRTS) systems make their predictions, which are directly compared to the corresponding observations. In order to emphasise equilibrium solutions, the model imposes the force balance between the magnetic force and the plasma pressure gradient, given by the Grad-Shafranov equation, at every plasma current beam by introducing the virtual observations that the differences between the two forces should be almost zero. In the same way, the model introduces empirical constraints that all the physics parameters should be almost zero at the first wall. The nodes in this graph represent larger, collapsed, subgraphs that model the internals of the different functions.
computationally challenging \[14, 30, 31\]. To overcome these problems, we have developed a new method to explore these posterior distributions based on the Gibbs sampling scheme \[32\]. In short, the method splits the full joint posterior distribution into several low dimensional conditional posterior distributions and sample them consecutively. These conditional posterior distributions are in general much simpler to sample, and some of them can be expressed in analytic functions obtained by the linear Gaussian inversion \[8, 14\]. The difficulty of sampling the full joint posterior distributions, therefore, can be substantially reduced by this method.

2. The model

In Bayesian inference \[21, 33, 34\], the model can be defined by a joint probability of unknown parameters and observations \(P(H, D)\). The joint probability consists of the predictive probability \(P(D|H)\) and the prior probability \(P(H)\), which can be written as:

\[
P(H, D) = P(D|H) P(H).
\]

The prior probability \(P(H)\) encodes the prior knowledge of the unknown parameters such as physical/empirical assumptions. For example, density or temperature must be positive, thus the probability of any negative density or temperature must be zero. Given a hypothetical value of the unknown parameters, a prediction can be made as a predictive distributions \(P(D|H)\) over the observations. Typically, the mean of predictive distributions can be given as a function, which encapsulates the physical processes happening during an experiment by taking into account physics phenomena as well as experimental setup, known as a forward model \(f(H)\). The prior probability of unknown parameters can be updated to the posterior probability \(P(H|D)\) through Bayes formula:

\[
P(H|D) = \frac{P(D, H)}{P(D)} = \frac{P(D|H) P(H)}{P(D)},
\]

where \(P(D)\) is a marginal probability of the observation, also known as the model evidence, which is a normalisation constant in this context.

If the model contains a large number of unknown parameters and heterogeneous data sets, the model can be written as a product of individual prior and predictive distributions, conditional on their parent variables, also known as factorisation:

\[
P(\{D_i\}, \{H_j\}) = \left(\prod_i P(D_i|H)\right) \left(\prod_j P(H_j)\right).
\]

Each of the predictive distributions contains a forward model of the scientific instruments which includes model parameters, for example calibration factors. The prior distributions encode prior knowledge such as physics/empirical assumptions. The prior and predictive distributions together constitute the model, the joint distributions \(P(\{D_i\}, \{H_j\})\). In other words, the model specification consists of not only the predictive distributions which make predictions over the observations but also the prior distributions in which
we reflect the prior knowledge and assumptions. All these model components and their conditional dependencies can be represented by a Bayesian graphical model, as shown in Figure 1, which is a transparent way of unfolding the complexity of the model.

In this work, we have developed Bayesian models of axisymmetric plasmas, the current tomography and equilibrium models, which involve a large number of unknown parameters, assumptions, predictive models and observations, as shown in Figure 1. The unknown parameters (the red circles) and observations (the blue circles) are connected via the forward models and functions (the white boxes), and the arrows visualise their dependencies. These two models share the following components: the axisymmetric current beam model \( J_\phi, J_{\text{iron}}, J_{\text{coils}} \) and Magnetic model, the Gaussian process priors of the poloidal current flux and pressure profiles \( F, n_e, T_e \), the wall constraints \( D_{\text{wall}} \) and the predictive distribution of all the plasma diagnostics (e.g. Thomson model). Given all these model components, the current tomography and equilibrium models employ the Gaussian process prior of the plasma current distributions and the equilibrium prior, respectively. All these model components are briefly described in the following subsections.

2.1. The axisymmetric current beam model

In an axisymmetric magnetic confinement fusion experiment, all kind of toroidal currents such as plasma and magnetic field coil currents can be modelled as a grid of toroidal solid beams with finite rectangular cross sections, each beam carrying a uniform current. This current beam model, previously developed in [8], has been implemented in the current tomography and equilibrium models. This current beam model takes into account the toroidal currents of the plasma \( J_\phi \), iron core \( J_{\text{iron}} \) and magnetic field coils \( J_{\text{coils}} \) of the JET fusion experiment, as shown in Figure 2.

Given these toroidal current distributions, the magnetic vector potential \( \mathbf{A} \) at a spatial location \( \mathbf{r} = [x, y, z] \) generated by the toroidal current density \( J \) is given by the Biot-Savart law:

\[
\mathbf{A} (\mathbf{r}) = \frac{\mu_0}{4\pi} \iiint \frac{J(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \, d^3r,
\]

where \( \mu_0 \) is the vacuum permeability. The magnetic field \( \mathbf{B} \) can be derived from the vector potential, which is:

\[
\mathbf{B} = \nabla \times \mathbf{A}.
\]

The fast calculation of the magnetic vector potential and the magnetic field can be carried out by multiplying current density by a unit current response factor for an arbitrary fixed spatial location given the beam grids. These response factors are pre-calculated for all spatial locations where the model needs to determine the magnetic vector potential and the magnetic field, for instance the spatial locations of plasma current beams and magnetic sensors. From the magnetic vector potential, the poloidal magnetic flux \( \psi \) is given by:

\[
\psi (x, y, z) = \oint A \cdot d\ell.
\]
Figure 2. The JET toroidal current model. The beam grids of unknown toroidal currents of the plasma $J_\phi$ and iron core $J_{\text{iron}}$ are shown in thin black lines. The beam grids of the poloidal magnetic field coil currents $J_{\text{coils}}$, which are retrieved from the JET database, are shown in thin purple lines. The equilibrium virtual observations $D_{\text{equi}}$, which impose the force balance constraints, are introduced at every plasma current beam. The wall virtual observations $D_{\text{wall}}$ force the currents of the outermost beams (with grey shade) whose cross sections are intersected by the first wall of the JET machine (thick black line) to zero.

The poloidal magnetic flux can be represented as a set of flux surfaces, normalised to zero at the centre of plasma or the magnetic axis and to one at the boundary or the last closed flux surface (LCFS). These normalised poloidal magnetic flux surfaces $\psi_N$ are often considered as the canonical coordinates, which plays an important role in diagnostic data analysis, physics studies and plasma control in nuclear fusion research. The current tomography and equilibrium models make use of the normalised flux $\psi_N$ coordinates to express the other physics parameters, the poloidal current flux $F$ and electron density $n_e$ and temperature $T_e$ by using Gaussian processes.

2.2. The Gaussian process prior

A Gaussian process [35, 36, 37] is a non-parametric function which associates any set of points of the domain of the function, for example space and time, with a random
vector following a multivariate Gaussian distribution. The properties of Gaussian process are specified by the mean and covariance function of the Gaussian distribution. The covariance function determines the covariance of output values at any two points which can be seen as smoothness of the Gaussian process. Unlike a parametric model, which might severely constrain the posterior distribution, the Gaussian process, not depending on any specific parameterisation, puts less constraints on the posterior distribution. In nuclear fusion research, Gaussian processes were introduced by non-parametric tomography of the electron density and current distribution [13], followed by a number of applications [9, 10, 16, 23, 24, 38]. Gaussian processes are also the standard way to model profiles in Minerva.

The zero mean and squared exponential covariance function are one of the most common specifications of a Gaussian process $f(x)$, which can be given by:

$$f(x) \sim \mathcal{N}(\mu(x), \Sigma(x,x)),$$

$$\mu(x) = 0,$$

$$\Sigma(x_i, x_j) = \sigma_f^2 \exp \left( -\frac{(x_i - x_j)^2}{2\sigma_x^2} \right) + \sigma_y^2 \delta_{ij},$$

where $\mu$ is the mean function and $x$ is an arbitrary (scalar or vector) point in the domain. The covariance function $\Sigma(x_i, x_j)$ is defined between any two arbitrary points $x_i$ and $x_j$. The parameters of the covariance function $\sigma_f$, $\sigma_x$ and $\sigma_y$ are the hyperparameters of the Gaussian process. The overall scale $\sigma_f$ and the length scale $\sigma_x$ determine smoothness of the function, and $\sigma_y$ is chosen to be relatively small number with respect to $\sigma_f$, for example $\sigma_y/\sigma_f = 10^{-3}$ to avoid numerical instabilities. The overall and length scale (the smoothness) of this Gaussian process is constant over the domain. The prior distribution of the plasma current distributions $J_\phi$ is modelled by this Gaussian process, which is:

$$P(J_\phi|\sigma_f, J_\phi, \sigma_x, J_\phi) \sim \mathcal{N}(\mu_{J_\phi}(x), \Sigma_{J_\phi}(x,x)),$$

where the mean $\mu_{J_\phi}$ and covariance function $\Sigma_{J_\phi}$ are given by Equation (8) and Equation (9), respectively. The domain of these functions is given in $R, Z$ coordinates $x = [R, Z]$, and the length scale contains two components for $R$ and $Z$ direction $\sigma_{x,J_\phi} = [\sigma_{R,J_\phi}, \sigma_{Z,J_\phi}]$. The prior distributions of the hyperparameters $\sigma_{f,J_\phi}$ and $\sigma_{x,J_\phi}$ are given by uniform distributions. In a similar way, the prior distribution of the poloidal current flux $F$ can be modelled by this Gaussian process as well, which is:

$$P(F|\sigma_{f,F}, \sigma_{x,F}) \sim \mathcal{N}(\mu_{F}(\psi_N), \Sigma_{F}(\psi_N,\psi_N)),$$

and the prior distributions of the hyperparameters $\sigma_{f,F}$ and $\sigma_{x,F}$ are given by uniform distributions.

The electron density and temperature can have substantially different smoothness (gradient) in the core and edge regions [39]. A non-stationary covariance function [40] is able to represent such spatially varying smoothness, which is given by:

$$\Sigma(x_i, x_j) = \sigma_f^2 \left( \frac{2\sigma_x(x_i)\sigma_x(x_j)}{\sigma_x(x_i)^2 + \sigma_x(x_j)^2} \right) \cdot \exp \left( -\frac{(x_i - x_j)^2}{\sigma_x(x_i)^2 + \sigma_x(x_j)^2} \right) + \sigma_y^2 \delta_{ij},$$

where $\sigma_x$ is the standard deviation of $x$. The overall scale $\sigma_f$ and the length scale (the smoothness) of this Gaussian process is constant over the domain.
where the length scale $\sigma_x(x)$ is given as an arbitrary function. We need a function with different smoothness (gradient) in the core and edge regions and a smooth transition between the two for the length scale, and for this reason, we choose a hyperbolic tangent function \[9, 10\], which is:

$$
\sigma_x(x) = \frac{\sigma_{x,\text{core}} + \sigma_{x,\text{edge}}}{2} - \frac{\sigma_{x,\text{core}} - \sigma_{x,\text{edge}}}{2} \tanh \left( \frac{x - x_0}{x_w} \right),
$$

where $\sigma_{x,\text{core}}$ and $\sigma_{x,\text{edge}}$ are the length scale value in the core and edge regions. The position and width of the transition are denoted as $x_0$ and $x_w$. The prior distributions of the electron density $n_e$ and temperature $T_e$ can be modelled by this Gaussian process, which is:

$$
P(n_e|\sigma_{f,n_e}, \sigma_{x,n_e}) \sim \mathcal{N}(\mu_{n_e}(\psi_N), \Sigma_{n_e}(\psi_N, \psi_N)),
$$

$$
P(T_e|\sigma_{f,T_e}, \sigma_{x,T_e}) \sim \mathcal{N}(\mu_{T_e}(\psi_N), \Sigma_{T_e}(\psi_N, \psi_N)),
$$

where their mean and covariance functions given by Equation (8) and Equation (12), respectively. The length scale functions $\sigma_{x,n_e}$ and $\sigma_{x,T_e}$ contain the four parameters of $\sigma_{x,\text{core}}$, $\sigma_{x,\text{edge}}$, $x_0$ and $x_w$. Again, the prior distributions of all these hyperparameters are given by uniform distributions.

All the physics parameters are modelled by either 2D beam grids in $R, Z$ coordinates ($J_\phi$ and $J_{\text{iron}}$) or 1D Gaussian processes in the $\psi_N$ coordinates ($F$, $n_e$ and $T_e$). These physics parameters can be given by 3D fields in $x, y, z$ coordinates through the coordinate transformations. Since we have such 3D fields of all these physics parameters, the predictions of observed quantities can be made at any point in space and time by the predictive models. Furthermore, physics/empirical laws can be predicted, for example the force balance equation. These predictions of the physics/empirical laws can be formally introduced as a part of the prior distributions by making virtual observations [14, 41].

2.3. The equilibrium prior

A magnetic confinement fusion device confines the plasma particles with high kinetic energy by the magnetic force of the plasma currents and the magnetic field. This magnetic force counteracts the kinetic force of the plasma particles due to the plasma pressure gradient and keeps the plasma in a macroscopic equilibrium state, which can be described by the MHD force balance equation:

$$
J \times B - \nabla p \simeq 0,
$$

where $J$ is the plasma current density, $B$ is the magnetic field and $p$ is the isotropic pressure of the plasma particles. For axisymmetric plasmas, this force balance can be

\[\text{\S}\] These observations are introduced to prescribe physics/empirical assumptions, not observed during the experiments.
given in term of the toroidal current density $J_\phi$, poloidal current flux $F$ and pressure $p$, by the Grad-Shafranov equation [5, 6], which can be written as:

$$J_\phi - Rp' - \frac{\mu_0}{R} FF' \simeq 0,$$

(17)

where $p' = \frac{\partial p}{\partial \psi}$ and $F' = \frac{\partial F}{\partial \psi}$. In this work, the isotropic pressure of plasma particles are assumed to be twice of the electron pressure $p = 2n_eT_e$.

In order to impose the MHD force balance, the model evaluates the integral of the Grad-Shafranov equation over cross section of every plasma current beam. These force balance constraints are introduced by virtual observations [14], which are given by:

$$P(D_{\text{equi}}|J_\phi, F, n_e, T_e) = \prod_i \mathcal{N} \left( \int_{Z_{\text{min},i}}^{Z_{\text{max},i}} \int_{R_{\text{min},i}}^{R_{\text{max},i}} J_\phi - Rp' - \frac{\mu_0}{R} FF' \, dR \, dZ, \sigma_{\text{equi}} \right),$$

(18)

where $R_{\text{min},i}, R_{\text{max},i}, Z_{\text{min},i}$ and $Z_{\text{max},i}$ are the minimum and maximum values of $R$ and $Z$ positions of the cross section of the $i^{th}$ plasma current beam. All virtual measurements of the force balance constraints $D_{\text{equi}}$ are set to be zero, which mean the force balance should be fulfilled. The standard deviation of these virtual observations $\sigma_{\text{equi}}$, to which we reflect the epistemic uncertainties of the force balance, is set to be 50 kA m$^{-2}$, a few per cent of the typical average plasma current density. These virtual observations together with the prior distributions of $J_\phi, F, n_e$ and $T_e$ constitute the equilibrium prior [14], which can be written as:

$$P(J_\phi, F, n_e, T_e|D_{\text{equi}}) = \frac{P(D_{\text{equi}}|J_\phi, F, n_e, T_e) P(J_\phi) P(F) P(n_e) P(T_e)}{P(D_{\text{equi}})}.$$

(19)

Here, the prior distribution of the toroidal current density $P(J_\phi)$ is not given by Equation (10) but Gaussian distributions with large standard deviations, for instance $300 \times 10^6$ kA m$^{-2}$, which is much higher than typical average plasma current density of the order of $10^3$ kA m$^{-2}$ at JET, thus these Gaussian priors are effectively uniform.

2.4. The wall constraints

The plasma current density $J_\phi$ and the electron density $n_e$ and temperature $T_e$ are not expected to be significantly high on the material surface facing the plasma inside the machine, also known as the first wall. These empirical expectations are taken into account by another set of virtual observations $D_{\text{wall}}$ on the first wall. These virtual observations $D_{\text{wall}} = [D_{\text{wall},J_\phi}, D_{\text{wall},n_e}, D_{\text{wall},T_e}]$ are introduced at either the outermost plasma current beams whose cross section are intersected by the first wall (the beams with grey shade in Figure 2) or some spatial locations on the first wall except the divertor
regions (the black dots in Figure 3), which are given by:

\[
P(\text{D}_{\text{wall}}|J_\phi, n_e, T_e) = P(\text{D}_{\text{wall}}, J_\phi|J_\phi) P(\text{D}_{\text{wall}}, n_e|n_e) P(\text{D}_{\text{wall}}, T_e|T_e),
\]

\[
P(\text{D}_{\text{wall}}, J_\phi|J_\phi) = \prod_i N(J_\phi(R_i, Z_i), \sigma_{\text{wall}}, J_\phi),
\]

\[
P(\text{D}_{\text{wall}}, n_e|n_e) = \prod_i N(n_e(x_i, y_i, z_i), \sigma_{\text{wall}}, n_e),
\]

\[
P(\text{D}_{\text{wall}}, T_e|T_e) = \prod_i N(T_e(x_i, y_i, z_i), \sigma_{\text{wall}}, T_e),
\]

where \(J_\phi(R_i, Z_i)\) is the plasma current density of the \(i^{th}\) outermost current beam and \(n_e(x_i, y_i, z_i)\) and \(T_e(x_i, y_i, z_i)\) are the electron density and temperature at the \(i^{th}\) spatial location on the first wall. All the virtual observations and uncertainties at the first wall are set to be reasonably low: \(D_{\text{wall}}, J_\phi = 0.0 \text{ kA m}^{-2} \), \(\sigma_{\text{wall}}, J_\phi = 1.0 \text{ kA m}^{-2} \), \(D_{\text{wall}}, n_e = 10^{15} \text{ m}^{-3} \), \(\sigma_{\text{wall}}, n_e = 10^{15} \text{ m}^{-3} \), \(D_{\text{wall}}, T_e = 0.1 \text{ eV} \) and \(\sigma_{\text{wall}}, T_e = 0.1 \text{ eV} \).

2.5. The plasma diagnostics

In order to find all possible plasma current distributions as well as the poloidal current flux and pressure profiles consistent with magnetic field and pressure measurements simultaneously, the current tomography and equilibrium models employ a number of predictive models of the following plasma diagnostics: magnetic sensors (pickup coils, saddle coils and flux loops) [8], polarimeters [15], interferometers [10, 13, 14], lithium beam emission spectroscopy [16, 17] and high-resolution Thomson scattering (HRTS) systems [10]. All the lines of sight and observation positions of all these plasma diagnostics are shown in Figure 3. Each of these predictive models includes forward models of measurement techniques which encapsulate the relevant physical phenomena and experimental setup. These models make the predictions of these measurements given the 3D fields of these physics parameters in \(x, y, z\) coordinates. These 3D fields of the poloidal current flux \(F\) and electron density \(n_e\) and temperature \(T_e\) are transformed from the \(\psi_N\) coordinates, thus all the measurements related to those parameters from the polarimeters, interferometers, lithium beam emission spectroscopy and HRTS systems depend on the toroidal current distributions \(J_\phi\) and \(J_{\text{iron}}\) as well as the coil currents \(J_{\text{coils}}\), which are known and taken from the JET database.

2.5.1. The magnetic sensors

The JET magnetic sensors consist of the pickup coils, saddle coils and flux loops (the red dots, lines and diamonds in Figure 3). These magnetic sensors measure the magnetic flux through the surrounding coils. The pickup coils, saddle coils and flux loops provide, respectively, local magnetic field measurements at the coil positions (the red dots), flux differences between the two \(R, Z\) positions (the two endpoints of the red lines) and poloidal magnetic flux through the toroidal loops.
Figure 3. The lines of sight and observation positions, projected into the poloidal plane, of all the JET diagnostics in the model. The magnetic sensors (in red) consist of pickup coils, saddle coils and flux loops, which measure the magnetic field strength. The interferometers and polarimeters share the lines of sight (in yellow) and provide $\int n_e \, d\ell$ and $\int n_e B_\parallel \, d\ell$, respectively. The HRTS system measures the electron density and temperature at 63 spatial locations (in orange) along the horizontal injected laser path. The lithium beam emission spectroscopy system provides edge electron density at 26 spatial locations (in pink) along the vertically injected beam from the top of the machine. The first wall and the wall constraints are shown in the black line and dots.

(the red diamonds). The predictive model of all these magnetic sensors \cite{8} is given by:

$$P (D_{\text{mag}} | J_\phi, J_{\text{iron}}) = P (D_{\text{pickup}} | J_\phi, J_{\text{iron}}) P (D_{\text{saddle}} | J_\phi, J_{\text{iron}}) P (D_{\text{fluxloop}} | J_\phi, J_{\text{iron}}),$$  \hspace{1cm} (24)  

$$P (D_{\text{pickup}} | J_\phi, J_{\text{iron}}) = \prod_i \mathcal{N} \left( B_R (R_i, Z_i) \cos \theta_i + B_Z (R_i, Z_i) \sin \theta_i, \sigma_{\text{pickup}}, i \right),$$  \hspace{1cm} (25)  

$$P (D_{\text{saddle}} | J_\phi, J_{\text{iron}}) = \prod_i \mathcal{N} \left( G_{\text{saddle}}, i \left( \psi (R_{2,i}, Z_{2,i}) - \psi (R_{1,i}, Z_{1,i}) \right), \sigma_{\text{saddle}}, i \right),$$  \hspace{1cm} (26)  

$$P (D_{\text{fluxloop}} | J_\phi, J_{\text{iron}}) = \prod_i \mathcal{N} \left( \psi (R_i, Z_i), \sigma_{\text{fluxloop}}, i \right),$$  \hspace{1cm} (27)
where $B_R(R_i, Z_i)$ and $B_Z(R_i, Z_i)$ are the $R$ and $Z$ direction of the magnetic field at the position of the $i^{th}$ pickup coil with the angle of the normal vector $\theta_i$, $\psi(R_{2,i}, Z_{2,i})$ and $\psi(R_i, Z_i)$ are the poloidal magnetic flux at the two $R, Z$ positions of the $i^{th}$ saddle coil with the correction factor for the actual 3D coil geometry $G_{saddle,i}$ and $\psi(R_i, Z_i)$ is the poloidal magnetic flux through the $i^{th}$ flux loop. All the observations and uncertainties of all these magnetic sensors $D_{mag} = [D_{pickup}, D_{saddle}, D_{fluxloop}]$ and $\sigma_{mag} = [\sigma_{pickup}, \sigma_{saddle}, \sigma_{fluxloop}]$ are retrieved from the JET database.

2.5.2. The interferometers and polarimeters  The JET far-infrared (FIR) interferometer-polarimeter system [42, 43, 44] launches electromagnetic waves into the plasma and measures the phase differences and polarisation angles between these injected waves and the reference wave. These phase differences and polarisation angles are proportional to the line integrated quantities $\int n_e d\ell$ and $\int n_e B_\parallel d\ell$, respectively, along four lateral and four vertical lines of sight (the yellow lines in Figure 3). These quantities are pre-calculated and stored in the JET database with their uncertainties. The predictive models of the JET interferometer-polarimeter system [14, 15] are given by:

$$P(D_{int}|n_e(\psi_N)) = \prod_i \mathcal{N}\left(\int n_e d\ell_i, \sigma_{int,i}\right),$$

$$P(D_{pol}|J_\phi, n_e(\psi_N)) = \prod_i \mathcal{N}\left(\int n_e B_\parallel d\ell_i, \sigma_{pol,i}\right),$$

where $\int d\ell_i$ is the line integral along the $i^{th}$ line of sight and $B_\parallel$ is the magnetic field strength parallel to the line of sight. All the observations and uncertainties of all these line integrated quantities $D_{int}$, $D_{pol}$, $\sigma_{int}$, $\sigma_{pol}$ are retrieved from the JET database.

2.5.3. The high-resolution Thomson scattering (HRTS) system  The JET high-resolution Thomson scattering (HRTS) system [45] launches laser pulses into the plasma and collects Thomson scattered photons [46] with polychromators with four spectral channels from 63 spatial locations (the orange dots in Figure 3) with a spatial resolution of 0.8 cm to 1.6 cm and a temporal resolution of 20 Hz. The intensity and width of Thomson scattering spectra provide the electron density $n_e$ and temperature $T_e$ measurements. Since the electron density calibration factor $C_{TS}$ and the position shift of all spatial channels along the laser path $S_{TS}$ of the HRTS system are cross-calibrated with other plasma diagnostics, they are regards as additional unknown parameters. The predictive model of the JET HRTS system [10] is given by:

$$P(D_{TS}|n_e(\psi_N), T_e(\psi_N), C_{TS}, S_{TS}) = \prod_i \prod_j \mathcal{N}\left(A_{TS,i,j}(n_e(R_i, Z_i, S_{TS}), T_e(R_i, Z_i, S_{TS}), C_{TS}), \sigma_{TS,i,j}\right),$$

where $A_{TS,i,j}(n_e(R_i, Z_i, S_{TS}), T_e(R_i, Z_i, S_{TS}), C_{TS})$ is the amplitude of the Thomson scattering spectrum of the $j^{th}$ spectral channel of the $i^{th}$ spatial position and $\sigma_{TS,i,j}$ is
the corresponding uncertainty. The electron density and temperature at the shifted position of the $i$\textsuperscript{th} spatial channel are given by:

\begin{align}
n_e (R_i, Z_i, S_{TS}) &= n_e (R_i + S_{TS} \cos \theta_{TS}, Z_i + S_{TS} \sin \theta_{TS}), \quad (31) \\
T_e (R_i, Z_i, S_{TS}) &= T_e (R_i + S_{TS} \cos \theta_{TS}, Z_i + S_{TS} \sin \theta_{TS}), \quad (32)
\end{align}

where $R_i$ and $Z_i$ the $R$ and $Z$ position of the $i$\textsuperscript{th} spatial channel and $\theta_{TS}$ is the angle of the laser path. If the position shift $S_{TS}$ is positive, the shifted positions will be closer to the first wall than the original positions. The range of $S_{TS}$ is set not to allow any shifted position to be beyond the first wall. The amplitude of the Thomson scattering spectrum can be written as:

$$A_{TS,i,j} (n_e, T_e, C_{TS}) = C_{TS} n_e E_{\text{laser}} \int \phi_{i,j} (\lambda) \frac{\lambda}{h c} r_e^2 \frac{S(\lambda, \theta, T_e)}{\lambda_{\text{laser}}} d\lambda,$$ (33)

where $E_{\text{laser}}$ is the laser energy, $\phi_{i,j} (\lambda)$ spectral response function of the $j$\textsuperscript{th} spectral channel of the $i$\textsuperscript{th} spatial position, $\lambda$ the scattered wavelength, $h$ the Planck constant, $c$ the speed of light, $r_e$ the classical electron radius, $S (\lambda, \theta, T_e)$ the spectral density function \cite{47}, $\theta$ the scattering angle and $\lambda_{\text{laser}}$ the laser wavelength. The prior distributions of $C_{TS}$ and $S_{TS}$ are given by uniform distributions.

2.5.4. The lithium beam emission spectroscopy system The JET lithium beam emission spectroscopy system \cite{48, 49} injects lithium beam atoms into the plasma and collect line emission at 26 spatial locations (the pink dots in Figure 3) with a spatial resolution of approximately 1.0 cm and a temporal resolution of 10 ms to 20 ms. The lithium beam atoms interact with the plasma electrons and ions via collisions and produce spontaneous emission from the first excited state. The intensity of the line emission can provide the electron density and temperature measurements, but the JET lithium beam emission spectroscopy system is designed to provide only the electron density measurements in the edge region at the top of the machine. The predictive model of the JET lithium beam emission spectroscopy system \cite{16, 17} is given by:

$$P (D_{Li} | n_e (\psi_N), T_e (\psi_N)) = \prod_i \mathcal{N} (A_{Li,i} (n_e (R_i, Z_i), T_e (R_i, Z_i)), \sigma_{Li,i}), \quad (34)$$

where $A_{Li,i} (n_e (R_i, Z_i), T_e (R_i, Z_i))$ is the amplitude of the lithium line emission of the $i$\textsuperscript{th} spatial position and $\sigma_{Li,i}$ is the corresponding uncertainty. The amplitude of the line emission is predicted by the collisional-radiative model which takes into account excitation and de-excitation, ionisation and spontaneous emission.

2.6. The joint distribution

All these prior and predictive distributions represent the prior knowledge of the unknown parameters and the predictions of the observations and together constitute the joint distribution which embodies the full relationship between the unknown parameters and the observations. Therefore, the model is defined as the joint distribution.
The axisymmetric plasma model with the Gaussian process prior of the plasma current distributions, the current tomography model, is given by:

\[
P(J_\phi, \sigma_{J_\phi}, J_{iron}, n_e, \sigma_{n_e}, T_e, \sigma_{T_e}, C_{TS}, S_{TS}, D_{mag}, D_{int}, D_{pol}, D_{Li}, D_{TS}, D_{wall})
= P(D_{mag}|J_\phi, J_{iron}) P(D_{int}|n_e (\psi_N)) P(D_{pol}|J_\phi, n_e (\psi_N)) P(D_{Li}|n_e (\psi_N), T_e (\psi_N))
\times P(D_{TS}|n_e (\psi_N), T_e (\psi_N), C_{TS}, S_{TS}) P(C_{TS}) P(S_{TS}) P(D_{wall}|J_\phi, n_e (\psi_N), T_e (\psi_N))
\times P(J_\phi|\sigma_{J_\phi}, \sigma_{x,J_\phi}) P(\sigma_{J_\phi}) P(J_{iron})
\times P(n_e|\sigma_{f,n_e}, \sigma_{x,n_e}) P(\sigma_{f,n_e}) P(\sigma_{x,n_e}) P(T_e|\sigma_{f,T_e}, \sigma_{x,T_e}) P(\sigma_{f,T_e}) P(\sigma_{x,T_e})
, \tag{35}
\]

where \( \sigma_{J_\phi} = [\sigma_{J_\phi}, \sigma_{x,J_\phi}] \), \( \sigma_F = [\sigma_{f,F}, \sigma_{x,F}] \), \( \sigma_{n_e} = [\sigma_{f,n_e}, \sigma_{x,n_e}] \) and \( \sigma_{T_e} = [\sigma_{f,T_e}, \sigma_{x,T_e}] \).

The other model with the equilibrium prior shown in Figure 11 the equilibrium model, is given by:

\[
P(J_\phi, J_{iron}, F, \sigma_F, n_e, \sigma_{n_e}, T_e, \sigma_{T_e}, C_{TS}, S_{TS}, D_{mag}, D_{int}, D_{pol}, D_{Li}, D_{TS}, D_{equi}, D_{wall})
= P(D_{mag}|J_\phi, J_{iron}) P(D_{int}|n_e (\psi_N)) P(D_{pol}|J_\phi, n_e (\psi_N)) P(D_{Li}|n_e (\psi_N), T_e (\psi_N))
\times P(D_{TS}|n_e (\psi_N), T_e (\psi_N), C_{TS}, S_{TS}) P(C_{TS}) P(S_{TS}) P(D_{wall}|J_\phi, n_e (\psi_N), T_e (\psi_N))
\times P(D_{equi}|J_\phi, n_e (\psi_N), T_e (\psi_N), F (\psi_N)) P(J_\phi) P(J_{iron}) P(F|\sigma_{f,F}, \sigma_{x,F}) P(\sigma_{f,F}) P(\sigma_{x,F})
\times P(n_e|\sigma_{f,n_e}, \sigma_{x,n_e}) P(\sigma_{f,n_e}) P(\sigma_{x,n_e}) P(T_e|\sigma_{f,T_e}, \sigma_{x,T_e}) P(\sigma_{f,T_e}) P(\sigma_{x,T_e})
. \tag{36}
\]

We emphasise that the main difference between these two models is the choice of the prior distribution to emphasise solutions based on either optimal hyperparameters based on Bayesian Occam’s razor or the MHD force balance. The current tomography model makes use of the Gaussian process prior of the plasma current distributions with the optimal hyperparameters, and on the other hand, the equilibrium model imposes the MHD force balance by the equilibrium prior.

3. The inference

Given the joint distributions, when observations are made by an experiment, the posterior distributions can be calculated through Bayes formula, given by Equation [2]. The full joint posterior distribution of the current tomography model is:

\[
P(J_\phi, \sigma_{J_\phi}, J_{iron}, n_e, \sigma_{n_e}, T_e, \sigma_{T_e}, C_{TS}, S_{TS}, D_{mag}, D_{int}, D_{pol}, D_{Li}, D_{TS}, D_{wall})
= \frac{P(J_\phi, \sigma_{J_\phi}, J_{iron}, n_e, \sigma_{n_e}, T_e, \sigma_{T_e}, C_{TS}, S_{TS}|D_{mag}, D_{int}, D_{pol}, D_{Li}, D_{TS}, D_{wall})}{P(D_{mag}, D_{int}, D_{pol}, D_{Li}, D_{TS}, D_{wall})}, \tag{37}
\]

and the one of the equilibrium model is:

\[
P(J_\phi, J_{iron}, F, \sigma_F, n_e, \sigma_{n_e}, T_e, \sigma_{T_e}, C_{TS}, S_{TS}|D_{mag}, D_{int}, D_{pol}, D_{Li}, D_{TS}, D_{equi}, D_{wall})
= \frac{P(J_\phi, J_{iron}, F, \sigma_F, n_e, \sigma_{n_e}, T_e, \sigma_{T_e}, C_{TS}, S_{TS}|D_{mag}, D_{int}, D_{pol}, D_{Li}, D_{TS}, D_{equi}, D_{wall})}{P(D_{mag}, D_{int}, D_{pol}, D_{Li}, D_{TS}, D_{equi}, D_{wall})}, \tag{38}
\]
where the denominators are a normalisation constant in this context. These posterior distributions can be explored by optimisation or sampling algorithms, for example pattern search \[50\] and Markov chain Monte Carlo (MCMC) algorithms \[51, 52, 53\], in order to find the optimal solution, which maximises the posterior probability, also known as the maximum *a posteriori* (MAP) solution or all possible solutions, which are drawn from the posterior distribution, also known as posterior samples. However, these posterior distributions are high dimensional (more than 1000 dimensions) and highly correlated through all these observations. In addition, these distributions involve all these forward models, which often require a significant amount of computation time. For this reason, exploration of such complex, high dimensional posterior distributions is computationally challenging \[14, 30, 31\]. A number of numerical recipes and algorithms has been applied to the equilibrium problem \[14, 30, 31, 41, 54, 55\], but, none of these algorithms manage to sample from these complex joint posterior distributions for cases such as H-mode plasmas.

In this work, we have developed a new method to explore these posterior distributions. The main idea of the new method is to separate a high dimensional target distribution \(P(X_1, X_2, \cdots, X_n)\) into a number of low dimensional conditional distributions \(P(X_i|X_1, \cdots, X_{i-1}, X_{i+1}, \cdots, X_n)\) and to sample them consecutively based on the Gibbs sampling scheme \[32\] as follow:

(i) Begin with initial values \(X_1^{(k)}, X_2^{(k)}, \cdots, X_n^{(k)}\).

(ii) Sample the first parameter \(X_1^{(k+1)}\) from \(P(X_1^{(k+1)}|X_2^{(k)}, X_3^{(k)}, \cdots, X_n^{(k)})\). Update the first parameter to \(X_1^{(k+1)}\) and sample the second parameter \(X_2^{(k+1)}\) from \(P(X_2^{(k+1)}|X_1^{(k+1)}, X_3^{(k)}, \cdots, X_n^{(k)})\). Update the second parameter to \(X_2^{(k+1)}\) and sample the third parameter \(X_3^{(k+1)}\) from \(P(X_3^{(k+1)}|X_1^{(k+1)}, X_2^{(k+1)}, X_4^{(k)}, \cdots, X_n^{(k)})\). Likewise, sample all the other parameters consecutively until to update all the parameters to \(X_1^{(k+1)}, X_2^{(k+1)}, \cdots, X_n^{(k+1)}\), which will be the \((k+1)^{th}\) sample.

(iii) Repeat the above.

The samples approximate the target distribution \(P(X_1, X_2, \cdots, X_n)\). These conditional distributions are in general much simpler to sample than the target distribution, and for some of them, analytic expressions can be found. In this way, the difficulty of sampling the high dimensional target distribution can be substantially reduced.

For this reason, the current tomography model is divided by the following parts:

\[
P(J_\phi, J_{\text{iron}}|n_e, \sigma_{n_e}, T_e, \sigma_{T_e}, C_{TS}, S_{TS}, D_{\text{mag}}, D_{\text{int}}, D_{\text{pol}}, D_{\text{Li}}, D_{TS}, D_{\text{wall}}),
\]

\[
P(n_e, \sigma_{n_e}, T_e, \sigma_{T_e}, C_{TS}, S_{TS}|J_\phi, J_{\text{iron}}, D_{\text{mag}}, D_{\text{int}}, D_{\text{pol}}, D_{\text{Li}}, D_{TS}, D_{\text{wall}}),
\]
and the equilibrium model is divided by the following parts:

\[
P(J_\phi, J_{\text{iron}}|F, \sigma_F, n_e, \sigma_{n_e}, T_e, \sigma_{T_e}, C_{TS}, S_{TS}, D_{\text{mag}}, D_{\text{int}}, D_{\text{pol}}, D_{\text{Li}}, D_{\text{TS}}, D_{\text{equi}}, D_{\text{wall}}),
\]

\( (41) \)

\[
P(n_e, \sigma_{n_e}, T_e, \sigma_{T_e}, C_{TS}, S_{TS}|J_\phi, J_{\text{iron}}, F, \sigma_F, D_{\text{mag}}, D_{\text{int}}, D_{\text{pol}}, D_{\text{Li}}, D_{\text{TS}}, D_{\text{equi}}, D_{\text{wall}}),
\]

\( (42) \)

\[
P(F, \sigma_F|J_\phi, J_{\text{iron}}, n_e, \sigma_{n_e}, T_e, \sigma_{T_e}, C_{TS}, S_{TS}, D_{\text{mag}}, D_{\text{int}}, D_{\text{pol}}, D_{\text{Li}}, D_{\text{TS}}, D_{\text{equi}}, D_{\text{wall}}).
\]

\( (43) \)

In order to make analytic expressions available for the conditional posterior distributions of electron density and temperature, given by Equation (40) and Equation (42), through the linear Gaussian inversion [8, 14], the observations of the HRTS and lithium beam spectroscopy systems are given as electron density and temperature, which are pre-calculated from individual inference applications. In addition, the hyperparameters \( \sigma_F, \sigma_{n_e} \) and \( \sigma_{T_e} \) and the HRTS model parameters \( C_{TS} \) and \( S_{TS} \) are set to be the MAP solutions which are found through the inversion procedures, described in the next paragraph.

These conditional posterior distributions of the current tomography model can be now written as:

\[
P(J_\phi, J_{\text{iron}}|n_e, T_e, C_{TS}, S_{TS}, D_{\text{mag}}, D_{\text{int}}, D_{\text{pol}}, D_{\text{Li}}, D_{\text{TS}}, D_{\text{wall}}),
\]

\( (44) \)

\[
P(n_e, T_e|J_\phi, J_{\text{iron}}, \sigma_{n_e}, \sigma_{T_e}, C_{TS}, S_{TS}, D_{\text{mag}}, D_{\text{int}}, D_{\text{pol}}, D_{\text{Li}}, D_{\text{TS}}, D_{\text{wall}}),
\]

\( (45) \)

and those of the equilibrium model:

\[
P(J_\phi, J_{\text{iron}}|F, n_e, T_e, C_{TS}, S_{TS}, D_{\text{mag}}, D_{\text{int}}, D_{\text{pol}}, D_{\text{Li}}, D_{\text{TS}}, D_{\text{equi}}, D_{\text{wall}}),
\]

\( (46) \)

\[
P(n_e, T_e|J_\phi, J_{\text{iron}}, F, \sigma_{n_e}, \sigma_{T_e}, C_{TS}, S_{TS}, D_{\text{int}}, D_{\text{pol}}, D_{\text{Li}}, D_{\text{TS}}, D_{\text{equi}}, D_{\text{wall}}),
\]

\( (47) \)

\[
P(F|J_\phi, J_{\text{iron}}, \sigma_F, n_e, T_e, D_{\text{equi}}).
\]

\( (48) \)

Most of these conditional posterior distributions can be sampled from their analytic expressions through the linear Gaussian inversion. The conditional posterior distribution of the toroidal current distributions whose analytic forms are not available through the linear Gaussian inversion can be explored by MCMC algorithm, specifically the adaptive Metropolis–Hastings algorithm [51, 52, 53]. Still approximated analytic distribution functions are available for these distributions, the MCMC algorithm can use these approximated distribution functions as initial proposal distributions. The posterior samples of the full joint posterior distributions will be drawn by sampling these conditional posterior distributions consecutively.

The MAP solutions can be found by a set of optimisation steps to explore the conditional posterior distributions based on the same scheme. Here, the procedure makes use of the pattern search algorithm [50] and the linear Gaussian inversion as follow:

(i) Infer the toroidal current density \( J_\phi \) and \( J_{\text{iron}} \) by the current tomography [8] with Gaussian processes given the magnetic field measurements \( D_{\text{mag}} \) and the wall constraints \( D_{\text{wall}} \) through the linear Gaussian inversion. The optimal
hyperparameters are found through the pattern search algorithm that explores
the model evidence, which can be analytically calculated through the linear Gaussian
inversion.

(ii) Calculate the magnetic flux surface $\psi_N$ given the toroidal current density.

(iii) Infer the electron density $n_e$ and temperature $T_e$ given the $\psi_N$ coordinates as well
as the electron density and temperature measurements $D_{\text{int}}, D_{\text{Li}}, D_{\text{TS}}$ and the wall
constraints $D_{\text{wall}}$ through the linear Gaussian inversion. The hyperparameters $\sigma_{n_e}$
and $\sigma_{T_e}$ and the HRTS model parameters $C_{TS}$ and $S_{TS}$ are optimised through the
pattern search algorithm with the linear Gaussian inversion to explore the model
evidence.

(iv) Infer the poloidal current flux $F$ and the hyperparameters $\sigma_F$ given all the other
unknown parameters from the previous steps and equilibrium prior $D_{\text{equi}}$ if the
equilibrium model. Otherwise, skip this step.

(v) Infer the toroidal current density $J_\phi$ and $J_{\text{iron}}$ given all the other unknown parameters
given the previous steps and the magnetic field measurements $D_{\text{mag}}$ in addition to
the polarimeter measurements $D_{\text{pol}}$ as well as the wall constraints $D_{\text{wall}}$.

(vi) Explore the full joint posterior distribution by the pattern search algorithm from
these initial guesses.

(vii) Repeat the above from (iii) until finding the (local) maximum.

The MAP solutions are relatively easier to be found than the posterior samples and
can be used as inference solutions or initial guesses for further exploration. The new
sampling method uses these MAP solutions as initial guesses in order to sample the full
joint posterior distribution. In the following subsections, the MAP solutions and the
posterior samples of the current tomography and equilibrium models found by these
inversion procedures will be presented.

3.1. The current tomography inference

The full joint posterior distribution of the current tomography model are explored to
find all possible plasma current distributions and electron pressure profiles, which are
modelled by the Gaussian process priors, given all the measurements simultaneously
through the new inversion procedures. The marginal posterior distributions of the
normalised poloidal magnetic flux surfaces, the electron density and temperature profiles
for typical JET L- and H-mode plasmas are shown in Figure 4 and Figure 5. The
blue and light blue lines are the marginal posterior mean and samples, respectively.
The blue dashed lines are the lower and upper boundaries of one standard deviation
($\pm 1\sigma$) marginal posterior uncertainty bands. The magnetic axis and the last closed
flux surface (LCFS) are depicted as big dots and thick lines, and three different sets of
thinner lines are the normalised flux surfaces which are corresponding to $\psi_N = 0.25, 0.50,$
0.75 in Figure 4(a) and Figure 5(a). For comparison, the normalised flux surfaces of
the conventional equilibrium fitting (EFIT) code [7, 56] (in green), the electron density
Figure 4. The inference results of the current tomography model for JET discharge #89709 at 8.0 s (an L-mode plasma): (a) the normalised poloidal magnetic flux surfaces, (b) the electron density and (c) temperature profiles. For comparison, the results of the conventional analysis of the equilibrium fitting (EFIT) code (in green), the HRTS system (in orange) and the lithium beam spectroscopy (in pink) systems are shown here. The orange and pink dots in (a) are the measurement positions of the HRTS and lithium beam spectroscopy systems, respectively. The HRTS system is automatically calibrated with the inferred HRTS model parameters $C_{TS}$ and $S_{TS}$ given all the other measurements, and the posterior mean of $C_{TS}$ and $S_{TS}$ are presented. We note that these inferred HRTS model parameters $C_{TS}$ and $S_{TS}$ are also applied to the electron density profiles of the conventional analysis of the HRTS system (the orange dots), which means that the HRTS electron density profiles (the orange dots) are slightly lower (in this case 0.967 times lower) than the original analysis results.

and temperature profiles of the conventional analysis of the HRTS (in orange) and the lithium beam spectroscopy (in pink) systems are presented. The orange and pink dots in Figure 4(a) and Figure 5(a) are the measurement positions of the HRTS and lithium beam spectroscopy systems, respectively. The HRTS system is automatically calibrated with the inferred electron density calibration factor $C_{TS}$ and the measurement position shift $S_{TS}$ given all the other measurements during the inversion procedures. The posterior mean of $C_{TS}$ and $S_{TS}$ are presented in Figure 4(b) and Figure 5(b). We note that these inferred HRTS model parameters $C_{TS}$ and $S_{TS}$ are also applied to calibrate the electron density profiles of the conventional analysis of the HRTS system shown in Figure 4(b) and Figure 5(b) in order to make reasonable comparison and avoid confusion. This means that the HRTS electron density profiles (the orange dots) are 0.967 times lower than the original analysis results.

The hyperparameters of Gaussian processes of the plasma current distributions and electron density and temperature profiles are optimised by maximising the posterior distributions of the hyperparameters during the inversion procedures, and the marginal
Figure 5. Same as Figure 4 for an H-mode plasma (JET discharge #89709 at 13.5 s).

Figure 6. The marginal posterior distributions of the hyperparameters explored during the MAP inversion procedures: (a) the overall and length scale of the plasma current distributions and (b) the position and width of the transition of the smoothness of the electron density profiles.

posterior distributions of these hyperparameters are shown in Figure 6. These posterior distributions are proportional to the model evidence which embodies Bayesian Occam’s razor [11, 12], and over-complex Gaussian processes are quantitative and automatically rejected. As shown in Figure 5, the Gaussian processes with the optimal hyperparameters are able to represent the low and steep gradient of the electron density and temperature profiles in the core and edge regions without problems of under- or over-fitting.

The comparison between the predictions and observations of the magnetic sensors, polarimeters and interferometers are shown in Figure 7. The blue and light blue lines
are the predictions given the posterior mean and samples, respectively. The valid and invalid observations are shown as the red and orange dots. Some of the magnetic sensors frequently suffer complete failure, for example systematic signal drift over time, the invalid magnetic sensors have been automatically detected by the magnetic diagnostic rejection procedure [14]. The normalised differences between the predictions and observations are presented. As shown in Figure 7, the predictions given the posterior samples well agree with the observations within their predictive uncertainties.

We emphasise that the inference results of the plasma current distributions and pressure profiles are consistent with all the measurements. Typically, the conventional analysis of the plasma diagnostics makes use of the normalised flux surfaces from the EFIT code with the magnetic field measurements in order to map the physics parameters, for example electron density profiles in the EFIT $\psi_N$ coordinates. The results from the conventional analysis might very well be inconsistent with each other, because of not only the individual analysis but also the $\psi_N$ coordinates. The inference solutions from the HRTS and lithium beam spectroscopy analysis codes are not consistent with each other in the EFIT $\psi_N$ coordinates, as shown in Figure 8(c). In such cases, it would be very difficult to figure out which electron density (temperature) profiles is correct and should be used for further physics analysis. In contrast, the current tomography model takes into account all the magnetic field and pressure measurements simultaneously in order to find all possible consistent solutions, as shown in Figure 8(b). Furthermore, since the electron density and temperature are assumed to be constant on the normalised flux surfaces, thus they can give additional information on the normalised flux surfaces or the plasma current distributions. As shown in Figure 8(a), the normalised flux surfaces inferred with the current tomography model and the EFIT code are notably different, especially the LCFS near the top of the machine where the lithium beam spectroscopy system provides edge electron density profiles and the magnetic axis. The comparison of the current tomography $n_e$ and conventional analysis $n_e$ in the current tomography and EFIT $\psi_N$ coordinates during a JET discharge are shown in Figure 9. The electron density profiles are consistent with those of the conventional analysis of the HRTS and lithium spectroscopy systems in the current tomography $\psi_N$ coordinates, whereas not in the EFIT $\psi_N$ coordinates.

We remark that, unlike the conventional EFIT code, all these inference results provide not only a single solution but also all possible solutions with their associated uncertainties. The full uncertainties of all the unknown parameters are calculated from the posterior samples, which can be used for quantification of uncertainties for further physics analysis such as transport analysis with TRANSP [57].

3.2. The equilibrium inference

In the previous subsection, we have explored the full joint posterior distributions of the current tomography model which does not take into account any further prior knowledge such as the MHD force balance. On the other hand, the equilibrium model introduces the
Figure 7. The predictions and observations of (a) the pickup coils, (c) saddle coils, (e) polarimeters and (g) interferometers. The valid and invalid data points are shown as the red and orange dots, respectively. The blue and light blue lines are the predictions given the posterior mean and samples. The normalised differences between the predictions and observations \((P - D)/\sigma\) are presented for (b) the pickup coils, (d) saddle coils, (f) polarimeters and (h) interferometers. We note that the line integrated electron density measurements from the second channel of the interferometers do not exist for this case, nevertheless, the model still makes the corresponding predictions.
Figure 8. The comparison of the electron density profiles in the current tomography and EFIT $\psi_N$ coordinates for JET discharge #92398 at 5.0 s (an L-mode plasma): (a) the normalised flux surfaces inferred with the current tomography model (in blue) and the conventional EFIT code (in green), (b) the electron density profiles in the current tomography $\psi_N$ coordinates and (c) in the EFIT code $\psi_N$ coordinates. The electron density profiles inferred with the current tomography model are consistent with all the measurements.

Figure 9. The comparison of the current tomography $n_e$ and conventional analysis $n_e$ in the current tomography and EFIT $\psi_N$ coordinates during JET discharge #92398: (a) the comparison of the current tomography $n_e$ and conventional analysis $n_e$ of the HRTS (cross) and the lithium beam spectroscopy (dots) systems in the current tomography (in blue) and EFIT (in green) $\psi_N$ coordinates and (b) the time evolution of the electron density profiles in the current tomography $\psi_N$ coordinates and (c) in the EFIT $\psi_N$ coordinates. The black line is $y = x$ in (a).
virtual observations which impose the MHD force balance constraints at every plasma current beam as a part of the prior. The full joint posterior distribution of the equilibrium model are explored through the new inversion procedure. The plasma and equilibrium current distributions of the MAP solution for typical JET L- and H-mode plasmas are shown in Figure 10 and Figure 11. The equilibrium current distributions are calculated given the poloidal current flux and pressure profiles \( p = 2n_e T_e \), and the difference between the two distributions are presented. The difference between these two current distributions are in general less than a few per cent of the typical plasma current values, therefore these solutions fulfil the MHD force balance.

The inference results of the equilibrium model propose a strong current in the edge region for an H-mode plasma, as shown in Figure 11. The electron density and temperature profiles of a typical H-mode plasma have a very steep gradient in the edge region \( (\psi_N \sim 0.95) \), as shown in Figure 12. The strong force due to the steep plasma pressure gradient in the edge region has to be balanced out by strong magnetic force due to high plasma currents. In other words, if the plasma in a stable, macroscopic MHD equilibrium state has steep pressure gradient in the edge region, then it might very well
be that the plasma currents are strong in the edge region. Typically, this inferred edge current density is around $5 \times 10^2 \text{kA}$, which is approximately one-third of the inferred core current density. The edge current density in the low field side is approximately two times higher than the one in the high field side. Furthermore, the normalised flux surfaces, especially near the X-point, are shrunk due to this strong edge current density, as shown in Figure 12.

We emphasise that the electron density and temperature profiles in the equilibrium $\psi_N$ coordinates are consistent with all the measurements. For comparison, the electron density and temperature profiles of the conventional analysis of the HRTS system (in orange), the lithium beam spectroscopy (in pink) and reflectometer (in grey) systems are shown in Figure 12. The electron density and temperature profiles are inferred given the interferometer, HRTS and lithium beam spectroscopy measurements, nevertheless, these electron density profiles are also consistent with the reflectometer measurements. As explained in the previous subsection, the HRTS system is automatically calibrated with the inferred model parameters $C_{TS}$ and $S_{TS}$ given all the other measurements. In this case, the posterior mean of the electron density calibration factor $C_{TS}$ is 0.897, which means that the conventional analysis of the HRTS system overestimates the electron density by around ten per cent. The electron density profiles of the HRTS measurements
Figure 12. The inference results of the equilibrium model for JET discharge #92398 at 7.0 s (an H-mode plasma): (a) the normalised poloidal magnetic flux surfaces, (b) the electron density, (c) temperature and (d) safety factor $q$ profiles. For comparison, the results of the conventional analysis of the equilibrium fitting (EFIT) code (in green), the HRTS (in orange), the lithium beam spectroscopy (in pink) and the reflectometer (grey) systems are shown here. The orange and pink dots in (a) are the measurement positions of the HRTS and lithium beam spectroscopy systems. We note that the reflectometer measurements are not included in the model, nevertheless, the electron density profiles are consistent with the reflectometer measurements.

(the orange dots) shown in Figure 12(b) are calibrated with the inferred model parameters $C_{TS}$ and $S_{TS}$ in order to make reasonable comparison and avoid confusion, and these HRTS profiles are lower than the original analysis results. Importantly, the electron density profiles inferred with the equilibrium model are consistent with all the other measurements in the model as well as with the reflectometer measurements. In other words, the automatic calibration of the HRTS system has been carried out with all the other measurements in the model and confirmed by the reflectometer measurements. Furthermore, the equilibrium $\psi_N$ coordinates map all these electron density profiles in a consistent way.

The safety factor $q$, which is the number of poloidal winding per a single toroidal winding of the magnetic field line, can be calculated given the poloidal current flux.
The safety factor $q$ can be used to check certain stabilities of the plasma, which can be written as:

$$q = \frac{r B_\phi}{R B_\theta},$$

where $B_\phi$ and $B_\theta$ are the toroidal and poloidal magnetic field, respectively, and $r$ is the minor radius. The $q$ profiles inferred with the equilibrium model are similar to the one of the EFIT code. In the core region, $q$ profiles and their posterior uncertainties may be determined by the Gaussian process priors. Again, all these physics parameters are provided with their associated uncertainties which can be used for further physics analysis.

3.3. The equilibrium predictions given the current tomography solutions

The current tomography model does not impose further assumptions like the equilibrium model, which prescribes the MHD force balance. In this work, we do not have any poloidal current measurement technique such as motional Stark effect (MSE) diagnostics, thus the current tomography model has no information on the poloidal current distributions. Nevertheless, we can explore the joint posterior distribution of the poloidal current flux profiles and the other physics parameters drawn from the current tomography posterior distributions by using the equilibrium virtual observations:

$$P(F, J_\phi, J_{\text{iron}}, n_e, T_e | \sigma_F, D_{\text{equi}}, D_{\text{CT}}) \approx P(D_{\text{equi}} | F, J_\phi, n_e, T_e) P(F | J_\phi, J_{\text{iron}}, n_e, T_e, \sigma_F) P(J_\phi, J_{\text{iron}}, n_e, T_e | D_{\text{CT}}),$$

where $P(J_\phi, J_{\text{iron}}, n_e, T_e | D_{\text{CT}})$ is the conditional posterior distributions of the current tomography model which has been already explored in Section 3.1 and all the measurements, which are taken into account the current tomography model, are denoted as $D_{\text{CT}} = [D_\text{mag}, D_\text{int}, D_\text{pol}, D_\text{Li}, D_\text{TS}, D_\text{wall}]$. Given the pre-calculated current tomography posterior samples, we can easily calculate Equation (50) and explore the joint posterior distribution of the poloidal current flux profiles and the other physics parameters. Here we make use of the same equilibrium virtual observations of the equilibrium model. We note that this is a new way to explore equilibrium solutions given the current tomography posterior samples.

The MHD force balance predictions given the current tomography posterior samples are shown in Figure 13. The difference between the plasma and equilibrium current density is small in general, but in the edge region, this difference can be notable. Interestingly, the equilibrium solutions of the current tomography model propose reverse bumps of the poloidal current flux profiles in the edge region, as shown in Figure 13(d), which attempt to
cancel out the strong equilibrium current contributions due to the steep pressure gradient (the brown reverse bumps) in the edge region as shown in Figure 13(a). These equilibrium current contributions \( J_{p'} = R p' \) can be seen as the toroidal current preconditions to balance out the plasma pressure gradient by the magnetic force of the plasma currents and the magnetic field. On the other hand, the equilibrium current contributions due to the poloidal current flux profiles \( J_{F_F'} = \frac{\mu_0}{R} F F' \) can cancel these toroidal current preconditions. In this case, the inference results of the current tomography model explain plasma equilibria without a strong current density in the edge region. In contrast, the inference results of the equilibrium model propose a strong toroidal current density in the edge region, approximately \( 4 \times 10^2 \) kA, as shown in Figure 14. The plasma and equilibrium current distributions inferred with the equilibrium model are consistent with each other better than those with the current tomography model, therefore equilibrium model offers solutions consistent with the MHD force balance within small equilibrium prior uncertainties. Nevertheless, the predictions of both models consistent with all the measurements, therefore, the poloidal current measurement technique such as the MSE diagnostics would be crucial to understand these equilibrium solutions further. These current tomography and equilibrium models give different solutions for plasma equilibria because of the different model priors.

4. Conclusions

The Bayesian models of axisymmetric plasmas using Gaussian processes and magnetohydrodynamics force balance equations have been developed. These models give the full joint posterior distributions of the plasma current distributions and pressure profiles consistent with the magnetic field and pressure measurements from the following plasma diagnostics: the magnetic sensors, polarimeters, interferometers, high-resolution Thomson scattering and lithium beam emission spectroscopy systems. The plasma current distributions are modelled as a grid of toroidal solid beams carrying a uniform current, and the other physics parameters such as the plasma pressure and poloidal current flux profiles are given as a function of the normalised poloidal magnetic flux surfaces, determined by the toroidal currents. Since inference of all these physics parameters is a tomographic problem, in order to exclude unreasonable solutions, we have introduced two different prior distributions: the Gaussian process prior and the equilibrium prior. The current tomography model makes use of the Gaussian process prior with the optimal hyperparameters obtained by Bayesian Occam’s razor. On the other hand, the equilibrium model imposes the Grad-Shafranov force balance constraints as a part of the equilibrium prior by introducing the virtual observations. These complex, high dimensional full joint posterior distributions have been explored by the new inversion procedures based on the Gibbs sampling scheme.

Unlike the conventional approach such as the EFIT code and the analysis code of each individual diagnostics, this approach provides the consistent solutions of the plasma current distributions as well as the poloidal current flux and electron density and
Figure 13. The MHD force balance predictions given the current tomography posterior samples for JET discharge #89709 at 13.5 s (an H-mode plasma): (a) and (b) the plasma and equilibrium current distributions across the plasma, (c) the pressure and (d) poloidal current flux profiles and (e) the comparison between the plasma and equilibrium current density. The blue and thick red lines in (a) and (b) are the plasma and equilibrium current distributions, respectively. The brown and purple lines in (a) and (b) are the equilibrium current contributions $J_p' = Rp'$ and $J_{FF'} = \frac{\mu_0}{R} FF'$ due to the pressure and poloidal current flux profiles shown in (c) and (d). For comparison, the results of the EFIT code and the HRTS analysis are shown as the green lines and orange dots in (c) and (d). The black line is $y = x$ in (e).
temperature profiles with all the magnetic field and pressure measurements simultaneously. The plasma current distributions all the other physics parameters are optimally inferred from all the measurements. As a result, the plasma current distributions and all the other physics parameters such as the electron density and temperature expressed in the normalised flux coordinates are self-consistent with all the measurements. For this reason, these inference solutions provide extra information of the plasma current distributions from the electron density and temperature measurements and vice versa. Furthermore, the HRTS system is automatically calibrated with the inferred model parameters.

The equilibrium solutions inferred with the equilibrium model propose a strong toroidal current density in the edge region due to the steep pressure gradient of H-mode plasmas. This edge current density is approximately one-third of the core current density.
and provides a strong magnetic force which can balance out the steep pressure gradient in a stable, macroscopic MHD equilibrium state. On the other hand, the equilibrium solutions predicted given the current tomography posterior samples propose the poloidal current flux hole (the reverse bumps) in the edge region which can strengthen the magnetic force to balance out the steep gradient without a strong toroidal current in the edge region. Nevertheless, the predictions of both models agree with all the measurements, therefore the poloidal current measurements would be crucial to understand these equilibrium solutions further.

All these solutions are provided with the optimal hyperparameters which are optimally chosen by Bayesian Occam’s razor. Moreover, all these inferred physics parameters are provided with the full uncertainties, which can be used to explore all possible solutions of high-level physics parameters, for example the energy transport coefficient, in further physics studies.

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