EQUATIONS IN VIRTUALLY ABELIAN GROUPS: LANGUAGES AND GROWTH

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Abstract. This paper explores the nature of the solution sets of systems of equations in virtually abelian groups. We view this question from two angles. From a formal language perspective, we prove that the set of solutions to a system of equations forms an EDT0L language, with respect to a natural normal form. Looking at growth, we show that the growth series of the language of solutions is rational. Furthermore, considering the set of solutions as a set of tuples of group elements, we show that it has rational relative growth series with respect to any finite generating set.

1. Introduction

An equation with set of variables $V$ in a group $G$ is an identity $w = 1$, for any $w \in G \ast F_V$, where $F_V$ denotes the free group on $V$. Equations in groups have been a significant area of research, particularly since Makanin proved that it is decidable whether a system of equations in a free group admits a solution, in a series of papers including [21], [22] and [23]. Following on from this, Razborov described the set of solutions to systems of equations in free groups ([29], [30]). More recently Ciobanu, Diekert and Elder [5] proved that the solution set to a system of equations in a free group is an EDT0L language (described as reduced words over a basis). This was generalised to virtually free groups in [15], and to all hyperbolic groups in [7]. Diekert, Jeż and Kufleitner [16] showed that this also holds for right-angled Artin groups.

The class of EDT0L languages was introduced by Rozenberg [31] in 1973. They are an example of L-systems, which where themselves introduced in order to model growth of organisms. A key fact about this growth is that it occurs in parallel across the organism, and this is reflected in the definition of EDT0L systems. Every regular language is EDT0L, and every EDT0L language is an indexed language, and thus context-sensitive. There exist context-free languages which are not EDT0L, and EDT0L languages which are not context-free.

The use of EDT0L languages in describing solutions to equations, rather than the more studied class of context-free languages, arises from the fact that equations can have arbitrarily many variables. When expressing the solution $(x_1, \ldots, x_n)$ as a word, we write it in the form $x_1 \# \cdots \# x_n$, for some new letter $. It follows that even in $\mathbb{Z} = \langle a \mid \rangle$, equations with 3 or more variables will not necessarily be context-free, as the system $X = Y = Z$ will have the solution language $\{a^m \# a^m \# a^m \mid m \in \mathbb{Z}\}$, which is not a context-free language over $\{a, a^{-1}, \#\}$.

It has long been regarded as ‘folklore’ that it is decidable whether systems of equations in virtually abelian groups admit solutions, however it is unclear when this was first proved. In [18] the stronger result that virtually abelian groups have decidable first order theory is shown. A more direct proof

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of the solubility of equations in virtually abelian groups can be found in Lemma 5.4 of [11]. In this paper we study the properties of solution sets of systems of equations in finitely generated virtually abelian groups. Such sets are also known as \textit{algebraic sets}.

Given a choice of finite generating set, and a corresponding normal form, we study the language of representatives for algebraic sets. These will be called solution languages (see Definition 2.19). In Section 3 we show that the solution languages (with respect to a suitable generating set and normal form) are EDT0L. This will be a consequence of the stronger result that they are accepted by \textit{multivariable finite-state automata} (see Definition 2.12):

\textbf{Theorem 3.12.} The solution language to any system of equations in virtually abelian group is accepted by a multivariable finite-state automaton.

\textbf{Corollary 3.13.} The solution language to any system of equations in a virtually abelian group is EDT0L.

Many of the papers that show that solution languages to systems of equations are EDT0L also consider the space complexity of the algorithms which constructs the EDT0L systems. We also show that both the multivariable finite-state automata and the EDT0L systems that accept these solution languages can be constructed in non-deterministic quadratic space (see Proposition 3.15).

It is a standard fact that every regular language has rational growth series. That is, the generating function which counts the number of words in the language with increasing length lies in the ring of rational functions \(\mathbb{Q}(z)\). A result of Chomsky and Schützenberger [4] asserts that the growth series of every unambiguous context-free language is algebraic over \(\mathbb{Q}(z)\). In contrast, there is no reason to expect that those EDT0L languages which do not fall under these two cases have well-behaved growth series. Indeed, Corollary 8 of [8] implies that there are EDT0L languages with transcendental (i.e. non-algebraic) growth series. A priori, the language obtained in Corollary 3.13 is neither regular nor context-free. Nevertheless, we prove that its growth series is rational.

\textbf{Proposition 3.17.} The solution language to any system of equations in a virtually abelian group has rational growth series.

Algebraic sets in groups can be seen as an analogue of the fundamental notion of algebraic varieties – the zero-loci of systems of equations. Meuser [26], and later Denef [13], proved the rationality of the Poincaré series of varieties over the \(p\)-adic integers, which can be thought of as a form of growth series. In Section 4 we prove an analogous result for algebraic sets of virtually abelian groups, using a notion of growth appropriate to the setting of finitely generated groups, namely word growth. We will use the notion of a \textit{polyhedral set}, which has its roots in the model theory of Presburger (see Section 2 for definitions).

Word growth in finitely generated groups is a much-studied topic. The growth function counts the number of group elements of length \(n\), with respect to the metric arising from a choice of finite generating set. The asymptotics of this function are well understood, but many questions remain about the properties of the corresponding formal power series. For an introduction to the topic, the reader is directed to Mann’s book [24].

Any subset of a group has a growth function, inherited from the group itself. This \textit{relative growth} has been studied in many papers, including [12]. The relative growth series of any subgroup of a virtually abelian group was shown to be rational in [19]. In Section 4 we consider the relative
growth of the algebraic sets of a virtually abelian group, as sets of tuples of group elements (with an appropriate metric). We show that the growth series of an algebraic set is always a rational function, regardless of the choice of finite weighted generating set.

**Theorem 4.3.** Let $G$ be a virtually abelian group. Then every algebraic set of $G$ has rational weighted growth series with respect to any finite generating set.

Moreover, we consider the natural *multivariate* growth series of the algebraic set, and demonstrate how recent results of Bishop imply that this series is holonomic (a class which includes algebraic functions and some transcendental functions).

**Corollary 4.21.** Every algebraic set of a virtually abelian group has holonomic weighted multivariate growth series.

We note that it may be useful for other purposes to have an explicit description of the algebraic sets of the groups in question, since this does not appear cleanly in the proofs. For such a statement, the interested reader is directed to Corollary 4.16 where the general structure of algebraic sets is noted, using the terminology of polyhedral sets.

## 2. Preliminaries

In this section we lay out the key definitions and basic results that will be required for the rest of the paper.

**Notation 2.1.** We will write functions to the right of their arguments, with the exception of growth functions and the generating functions of growth series.

We will use $\mathbb{Z}_{>0}$ and $\mathbb{Z}_{\geq 0}$ to denote the positive and non-negative integers, respectively.

If $w \in S^*$ is a word in the generators of some group $G$, we write $\overline{w} \in G$ for the group element that the word $w$ represents.

### 2.1. Polyhedral sets. Our fundamental tool for proving that languages of representatives have rational growth series in Proposition 3.17 and Section 4 will be the theory of polyhedral sets. These ideas appear in model theory as early as Presburger [28]. Results regarding rationality can be found in [13], and the ideas also appear in the theory of Igusa local zeta functions (see [9]). The following definitions and results follow Benson’s work [2], where it is proved that virtually abelian groups have rational (standard) growth series. More recently, polyhedral sets have again been used to prove rationality of various growth series of groups ([17], [19]).

**Definition 2.2.** Let $r \in \mathbb{Z}_{>0}$, and let $\cdot$ denote the Euclidean scalar product. Then we define the following.

1. Any subset of $\mathbb{Z}^r$ of the form $\{z \in \mathbb{Z}^r \mid u \cdot z = a\}$, $\{z \in \mathbb{Z}^r \mid u \cdot z > a\}$, or $\{z \in \mathbb{Z}^r \mid u \cdot z \equiv a \mod b\}$, for any $u \in \mathbb{Z}^r$, $a \in \mathbb{Z}$, $b \in \mathbb{Z}_{>0}$, will be called an **elementary set**;
2. any finite intersection of elementary sets will be called a **basic polyhedral set**;
3. any finite union of basic polyhedral sets will be called a **polyhedral set**.

If $\mathcal{P} \subset \mathbb{Z}^r$ is polyhedral and additionally no element contains negative coordinate entries, we call $\mathcal{P}$ a **positive polyhedral set**.
It is not hard to prove the following closure properties.

**Proposition 2.3** (Proposition 13.1 and Remark 13.2 of [2]). Let \( P, Q \subseteq \mathbb{Z}^r \) and \( R \subseteq \mathbb{Z}^s \) be polyhedral sets for some positive integers \( r \) and \( s \). Then the following are also polyhedral: \( P \cup Q \subseteq \mathbb{Z}^r \), \( P \cap Q \subseteq \mathbb{Z}^r \), \( \mathbb{Z}^r \setminus P \), \( P \times R \subseteq \mathbb{Z}^{r+s} \).

Benson also shows that polyhedral sets behave well under affine transformations, as follows.

**Definition 2.4.** We call a map \( A : \mathbb{Z}^r \to \mathbb{Z}^s \) an integral affine transformation if there exists an \( r \times s \) matrix \( M \) with integer entries and some \( q \in \mathbb{Z}^s \) such that \( pA = pM + q \) for \( p \in \mathbb{Z}^r \).

**Proposition 2.5** (Propositions 13.7 and 13.8 of [2]). Let \( A \) be an integral affine transformation. If \( P \subseteq \mathbb{Z}^r \) is a polyhedral set then \( PA \subseteq \mathbb{Z}^s \) is a polyhedral set. If \( Q \subseteq \mathbb{Z}^s \) is a polyhedral set then the preimage \( QA^{-1} \subseteq \mathbb{Z}^r \) is a polyhedral set.

We note that projection onto any subset of the coordinates of \( \mathbb{Z}^r \) is an integral affine transformation.

**Notation 2.6.** We will now introduce weight functions. When talking about weighted lengths of elements of free abelian or virtually abelian groups, we will use \( \| \cdot \| \) instead of \( | \cdot | \), which will be used for ‘standard’ length of elements.

Let \( P \subseteq \mathbb{Z}^r \) be a polyhedral set. Given some choice of weight function \( \| e_i \| \in \mathbb{Z}_{>0} \) for the standard basis vectors \( \{ e_i \}_{i=1}^r \) of \( \mathbb{Z}^r \), we assign the weight \( \sum_{i=1}^r a_i \| e_i \| \) to the element \( (a_1, \ldots, a_r) \in P \). Define the spherical growth function

\[
\sigma_P(n) = \# \{ p \in P \mid \| p \| = n \},
\]

and the resulting weighted growth series

\[
S_P(z) = \sum_{n=0}^\infty \sigma_P(n) z^n.
\]

Our argument will rely on the following crucial proposition.

**Proposition 2.7** (Proposition 14.1 of [2], and Lemma 7.5 of [13]). If \( P \) is a positive polyhedral set, then the weighted growth series \( S_P(z) \) is a rational function of \( z \).

We will need the following more general result.

**Corollary 2.8.** Let \( P \subseteq \mathbb{Z}^r \) be any polyhedral set (not necessarily positive). Then the weighted growth series \( S_P(z) \) is a rational function of \( z \).

**Proof.** We show that \( P \) may be expressed as a disjoint union of polyhedral sets, each in weight-preserving bijection with a positive polyhedral set. Let \( Q_1 = \{ z \in \mathbb{Z}^r \mid z \cdot e_i \geq 0, \ 1 \leq i \leq r \} = \bigcap_{i=1}^r \{ z \in \mathbb{Z}^r \mid z \cdot e_i \geq 0 \} \) denote the non-negative orthant of \( \mathbb{Z}^r \), and note that it is polyhedral. Let \( Q_2, \ldots, Q_{2^r} \) denote the remaining orthants (in any order) obtained from \( Q_1 \) by (compositions of) reflections along hyperplanes perpendicular to the axes and passing through the origin. By Proposition 2.5 these are also polyhedral sets. Let \( P_1 = P \cap Q_1 \) and for each \( 2 \leq j \leq 2^r \), inductively define

\[
P_j = \left( P \setminus \bigcup_{k<j} P_k \right) \cap Q_j.
\]

Each \( P_j \) is a polyhedral set by Proposition 2.3 and we have a disjoint union \( P = \bigcup_{j=1}^{2^r} P_j \). Each \( P_j \) is in weight-preserving bijection with a positive polyhedral set (by compositions of reflections along hyperplanes) and so \( S_{P_j}(z) \) is rational. The result follows since \( S_P(z) = \sum_{j=1}^{2^r} S_{P_j}(z) \). □
2.2. Equations.

**Definition 2.9.** Let $G$ be a group. A finite system of equations in $G$ is a finite subset $\mathcal{E}$ of $G * F_V$, where $F_V$ is the free group on a finite set $V$. If $\mathcal{E} = \{w_1, \ldots, w_n\}$, we denote the system $\mathcal{E}$ by $w_1 = w_2 = \cdots = w_n = 1$. Elements of $V$ are called variables. A solution to a system $w_1 = \cdots = w_n = 1$ is a homomorphism $\phi: F_V \to G$, such that $w_1 \phi = \cdots = w_n \phi = 1_G$, where $\phi$ is the extension of $\phi$ to a homomorphism from $G * F_V \to G$, defined by $g \phi = g$ for all $g \in G$.

A finite system of twisted equations in $G$ is a finite subset $\mathcal{E}$ of $G * (F_V \times \text{Aut}(G))$, and is again denoted $w_1 = \cdots = w_n = 1$. Elements of $V$ are called variables. Define the function

$$p: G \times \text{Aut}(G) \to G$$

$$(g, \psi) \mapsto g\psi.$$

If $\phi: F_V \to G$ is a homomorphism, let $\tilde{\phi}$ denote the homomorphism from $G * (F_V \times \text{Aut}(G))$ to $G \times \text{Aut}(G)$, defined by $(h, \psi)\tilde{\phi} = (h\phi, \psi)$ for $(h, \psi) \in F_V \times \text{Aut}(G)$ and $g\phi = g$ for all $g \in G$. A solution is a homomorphism $\phi: F_V \to G$, such that $w_1 \phi p = \cdots = w_n \phi p = 1_G$.

For the purposes of decidability, in finitely generated groups, the elements of $G$ will be represented as words over a finite generating set, and in twisted equations, automorphisms will be represented by their action on the generators.

**Remark 2.10.** We will often display a solution to an equation with variables $X_1, \ldots, X_n$ as a tuple $(x_1, \ldots, x_n)$ of group elements, rather than a homomorphism. We can obtain the homomorphism from the tuple by defining $X_i \mapsto x_i$ for each $i$.

**Definition 2.11.** The set of solutions to a finite system of equations in $n$ variables, expressed as a subset of $G^n$, is called an algebraic set of $G$.

2.3. Multivariable finite-state automata. Since solutions to equations can be thought of as tuples, one method that can be used to study the language complexity of sets of solutions is using multivariable languages, which are sets of tuples of words over an alphabet. We start with the formal definition.

**Definition 2.12.** Let $\Sigma$ be an alphabet, and $n \in \mathbb{Z}_{>0}$. An $n$-variable word over $\Sigma$ is an element of the Cartesian product $(\Sigma^*)^n$, and an $n$-variable language over $\Sigma$ is any subset of $(\Sigma^*)^n$.

We continue with a generalisation of a finite-state automaton to accept $n$-variable languages, for some positive integer $n$: the (asynchronous, non-deterministic) $n$-variable finite-state automaton.

**Definition 2.13.** Let $n \in \mathbb{Z}_{>0}$. An $n$-variable finite-state automaton is a tuple $A = (\Sigma, \Gamma, q_0, F)$, where

1. $\Sigma$ is an alphabet;
2. $\Gamma$ is a finite edge-labelled graph, where labels are $n$-variable words in $(\Sigma^*)^n$, with at most one non-empty word entry. The vertices of $\Gamma$ are called states;
3. $q_0 \in V(\Gamma)$ is called the start state;
4. $F \subseteq V(\Gamma)$ is called the set of accept states.

When tracing a path in $\Gamma$, we trace an $n$-variable word to be the concatenation of the labels of each edge traversed. Since each edge has at most one non-empty entry, the word will only get longer in one coordinate at a time. An $n$-variable word $w \in (\Sigma^*)^n$ is accepted by $A$ if there is a path $\gamma$ in $\Gamma$ from $q_0$ to a state in $F$, such that the $n$-variable word obtained by reading the labels in $\gamma$ is $w$. The language accepted by $A$ is the set of all $n$-variable words accepted by $A$. 
Figure 1. The start state is $q_{(0,0)}$, and $q_{(1,0)}$ is the unique accept state.

We give an example of a language accepted by a 3-variable finite-state automaton. In this case, the language represents the set of solutions to a system of equations in $\mathbb{Z}$.

**Example 2.14.** Let $\mathcal{E}$ be the following system of equations in $\mathbb{Z}$ (using additive notation):

$$
X - Y + Z = 1 \\
-Y + Z = 0.
$$

Note that by subtracting the second equation from the first, it is not difficult to show that the set of solutions to this system is $\{(1, y, y) \mid y \in \mathbb{Z}\}$. To demonstrate a more general method we will use later on, we will construct the set of solutions, and show that $L = \{(a^x, a^y, a^z) \mid (x, y, z) \text{ is a solution to } \mathcal{E}\}$ is accepted by a 3-variable finite-state automaton over the alphabet $\{a, a^{-1}\}$, using a different method. We will show

1. The language $\{(a^x, a^y, a^z) \mid (x, y, z) \text{ is a solution to } \mathcal{E} \text{ and } x, y, z \geq 0\}$ is accepted by a 3-variable finite-state automaton;
2. To show $L$ is accepted by a 3-variable finite-state automaton, we take the finite union across the possible configurations of signs of $X$, $Y$ and $Z$ and use the fact that finite unions of languages accepted by 3-variable finite-state automata are also accepted by 3-variable finite-state automata.

To show (1), consider the 3-variable finite-state automaton in Figure 1. This finite-state automaton works as follows:

1. Traversing an edge labelled by $(a, \varepsilon, \varepsilon)$, $(\varepsilon, a, \varepsilon)$ or $(\varepsilon, \varepsilon, a)$ corresponds to increasing $x$, $y$ or $z$ by 1, respectively. The states $q_{(i, j)}$ correspond to the value of $(x - y + z, -y + z)$, with the current values of $x$, $y$ and $z$.
2. Once we have increased $x$, $y$ and $z$ to the desired values, if this is a solution to $\mathcal{E}$, then we must be in the accept state $q_{(1,0)}$. 
(3) Note that we cannot increase the $x$s, $y$s and $z$s in any order, otherwise we would need an unbounded size of FSA. For example, the element $(a, a', a')$ of $L$, where $l \in \mathbb{Z}$ and $l > 1$ cannot be reached in the above system by traversing one $(a, \epsilon, \epsilon)$ edge, then $l$ $(\epsilon, a, \epsilon)$ edges, and then $l$ $(\epsilon, \epsilon, a)$ edges, as after the $l$ $(\epsilon, a, \epsilon)$ edges we would need a state $q_{(-l+1, -l)}$, which does not lie in the finite-state automaton. Moreover, we cannot add them to the finite-state automaton, as there are infinitely many such states. We prove the existence of an ordering of the edges (up to considering two edges with the same label equivalent) that works in Lemma 3.2. In this specific case, it is not hard to show that the ordering that starts with $(a, \epsilon, \epsilon)$, followed by $l$ traversals of a path comprising one $(\epsilon, a, \epsilon)$ edge and one $(\epsilon, \epsilon, a)$ edge, for all $l > 0$, and a similar ‘reversed’ order would work if $l < 0$.

(4) Note that not all states may be necessary, but it is simpler to construct them all.

2.4. EDT0L languages. The ultimate aim of using the $n$-variable finite-state automata is in order to show that the set of solutions to a system of equations in a virtually abelian group can be described as an EDT0L language, thus adding an additional class of groups to many classes of groups already known to have this property. We start by defining an EDT0L system; a grammar that generates an EDT0L language.

Definition 2.15. An EDT0L system is a tuple $\mathcal{H} = (\Sigma, C, w, R)$, where

1. $\Sigma$ is an alphabet, called the (terminal) alphabet;
2. $C$ is a finite superset of $\Sigma$, called the extended alphabet of $\mathcal{H}$;
3. $w \in C^*$ is called the start word;
4. $R$ is a regular (as a language) subset of $\text{End}(C^*)$, called the rational control of $\mathcal{H}$.

The language accepted by $\mathcal{H}$ is

$$L(\mathcal{H}) = \{w\phi \mid \phi \in R\}.$$ 

A language accepted by an EDT0L system is called an EDT0L language.

There are a number of different definitions of an EDT0L system, that all generate the same class of languages. In [5] and [8], the definition is the same as given here, except for the insistence that the start word is a single letter. To show that these are equivalent, adding a single homomorphism preconcatenated to the rational control of an EDT0L system (as defined here) that maps a new letter $\perp$ to the start word, and then defining the new start symbol to be $\perp$ gives that any EDT0L language is accepted by an EDT0L system with a single letter as the start word. In [32], and many earlier publications, the definition is what is given above, except they only allow rational controls of the form $\Delta^*$, for some finite set of endomorphisms $\Delta$. This definition is again equivalent to the definition we have given [1], but proves to be cumbersome when proving languages are EDT0L.

To streamline the definition of specific EDT0L systems, we introduce the following notation convention for specifying endomorphisms of a given free monoid.

Notation 2.16. When defining endomorphisms of $C^*$ for some extended alphabet $C$, within the definition of an EDT0L system, we will usually define each endomorphism by where it maps each letter in $C$. If any letter is not assigned an image within the definition of an endomorphism, we will say that it is fixed by that endomorphism.

2.5. Space complexity. We give a brief definition of space complexity. We refer the reader to [27] for a comprehensive introduction to space complexity, or to [5] for the consideration of space complexity when constructing EDT0L systems.
Definition 2.17. Let \( f : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{Z}_{\geq 0} \) be a function. We say that an algorithm runs in \( \text{NSPACE}(f) \) if it can be performed by a non-deterministic Turing machine with the following:

1. A read-only input tape;
2. A write-only output tape;
3. A read-write work tape that has a length of at most \( O(nf) \) for an input of length \( n \).

An algorithm is said to run in non-deterministic quadratic space if it runs in \( \text{NSPACE}(f) \), for some quadratic function \( f : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{Z}_{\geq 0} \).

We will use this definition to show that we can construct the multivariable finite-state automaton from Theorem 3.12, and hence the EDT0L system from Corollary 3.13, in non-deterministic quadratic space. Recall that a multivariable finite-state automaton has a set of vertices, edges, an assignment of labels to edges, a specified start state, and a set of accept states that all must be constructed.

We will later need the following lemma that allows us to take finite unions of languages that are accepted by multivariable finite-state automata without changing the space complexity.

Lemma 2.18. Let \( f : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{Z}_{\geq 0} \) be a function. A finite union of languages accepted by multivariable finite-state automata that are all constructible in \( \text{NSPACE}(f) \) is also accepted by a multivariable finite-state automaton that is constructible in \( \text{NSPACE}(f) \).

Proof. The automaton \( M \) we use is the automaton obtained by taking the union of all of the automata of the languages in the union, and collapsing the start states to a single state, which will be the start state. All accept states will remain accept states. We can construct \( M \) by constructing each of the automata in the union one at a time, which can be done in \( \text{NSPACE}(f) \). \qed

2.6. Languages of solutions to equations. We now define the languages that we will be studying, which are derived from the set of solutions. We need to choose a finite generating set and normal form in order to do this, although any can work. We consider two methods: one can either look at the language of \( n \)-variable words representing solutions, or one can look at the language of words that comprise the solutions concatenated with one another, delimited by an additional letter \( \# \).

Definition 2.19. Let \( G \) be a finitely generated group, with a finite monoid generating set \( \Sigma \), and a normal form \( \eta : G \rightarrow \Sigma^* \). Let \( E \) be a system of (twisted) equations in \( G \), and let \( n \) be the number of variables in \( E \). Let \( V = \{ X_1, \ldots, X_n \} \) be the set of variables, and let \( S \) be the set of solutions, which are homomorphisms from \( F_V \ast G \) to \( G \).

The multivariable solution language to \( E \) with respect to \( \Sigma \) and \( \eta \), is defined to be
\[
\{(X_1\psi\eta, X_2\psi\eta, \ldots, X_n\psi\eta) \mid \psi \in S\} \subset \Sigma^* \times \Sigma^* \times \cdots \times \Sigma^*.
\]
The \#-joined solution language to \( E \) with respect to \( \Sigma \) and \( \eta \), is defined to be
\[
\{X_1\psi\eta\#X_2\psi\eta\#\cdots X_n\psi\eta \mid \psi \in S\} \subset (\Sigma \cup \{\#\})^*.
\]

We now show that a multivariable solution language being accepted by an \( n \)-variable finite-state automaton is sufficient for the corresponding \#-joined solution language to be EDT0L.
Lemma 2.20. Let $L$ be an $n$-variable language over an alphabet $\Sigma$ (where $n \in \mathbb{Z}_{>0}$), that is accepted by an $n$-variable finite-state automaton, constructible in $\text{NSPACE}(f)$, for some $f: \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$. Then

1. The language $M = \{w_1\# \cdots \# w_n \mid (w_1, \ldots, w_n) \in L\}$ is an EDT0L language over $\Sigma \cup \{\#\}$;
2. An EDT0L system for $M$ can be constructed in $\text{NSPACE}(f)$.

Proof. We will construct an EDT0L system $H$ for $M$ as follows. The terminal alphabet will be $\Sigma \cup \{\#\}$, the extended alphabet will be $C = \Sigma \cup \{\#\}$, and the start word will be $\perp_1 \# \cdots \# \perp_n$.

Let $A = (\Sigma, \Gamma, q_0, F)$ be an $n$-variable finite-state automaton that accepts $L$. We will use $A$ to define the rational control of $H$. Let $W$ be the set of all $n$-variable words that appear as edge labels within $\Gamma$. For each $w = (w_1, \ldots, w_n) \in W$, define $\varphi_w \in \text{End}(C^*)$ by

$$\perp_1 \varphi_w = w_1 \perp_1$$

$$\vdots$$

$$\perp_n \varphi_w = w_n \perp_n.$$ 

Also define $\psi \in \text{End}(C^*)$

$$\perp_i \psi = \varepsilon,$$ 

for all $i \in \{1, \ldots, n\}$. Our rational control $R$ will be a regular language over the set $\{\varphi_w \mid w \in W\}$. Let $\Gamma'$ be the edge-labelled graph obtained from $\Gamma$, by replacing the label $w$ on each edge with $\varphi_w$.

Consider the (1-variable) finite-state automaton $B = (\Sigma, \Gamma', q_0, F)$. Let $K$ be the language accepted by $B$. We have that $K$ is precisely the set of all endomorphisms $\theta$ of $C^*$ that can be written as products of endomorphisms $\varphi_w$, for $w \in W$, such that $(\perp_1 \# \cdots \perp_n)\theta = u_1 \perp_1 \# \cdots \# u_n \perp_n$, for some $(u_1, \ldots, u_n) \in L$. Therefore, the regular language $K\psi$ is the set of all endomorphisms that map $\perp_1 \# \cdots \# \perp_n$ to an element of $M$, and so taking $R = K\psi$ gives the desired EDT0L system.

For (2), since a multivariable finite state automaton contains an alphabet $\Sigma$, this can be obtained and output in $\text{NSPACE}(f)$, and thus the alphabet for the EDT0L language, $\Sigma \cup \{\#\}$, and the extended alphabet $C = \Sigma \cup \{\#\}$ can also be constructed and written to the output tape in $\text{NSPACE}(f)$. The start word will always be $\perp_1 \# \cdots \# \perp_n$, regardless of the input, and we can just output this.

It remains to construct the rational control. As in the construction of $H$, we use the same set of vertices and edges, but whenever the rational control in $H$ labels an edge using $\varphi_w$, we instead label it using $w$, and note that $\varphi_w$ can be effectively computed from $w$. To record $\varphi_w$, we only need to know where each $\perp_i$ maps (as they always fix everything else), and that is precisely the information that $w$ contains. \[\square\]

3. Solution languages in virtually abelian groups

The purpose of this section is to prove that the multivariable solution languages to systems of equations in virtually abelian groups are accepted by multivariable finite-state automata, and so #-joined solution languages are EDT0L, all with respect to a natural generating set and normal form. We do this by first showing that the multivariable solution languages for systems of twisted
equations in free abelian groups are recognised by finite-state automata, and then prove that equations in virtually abelian groups reduce to twisted equations in free abelian groups. Throughout this section, when referring to free abelian groups, we will use additive notation. This means that equations in free abelian groups will be expressed as sums rather than ‘products’. When representing solution languages, we will express them using multiplicative notation, as this is more natural with languages, using $a_1, \ldots, a_k$ to be the standard generators of $\mathbb{Z}^k$.

The next lemmas are used to prove that systems of equations, and therefore twisted equations, in free abelian groups have multivariable solution languages accepted by $\text{ EDT0L } \#$-joined languages, and Diekert [16] has a more direct method for systems of equations in virtually abelian groups reduce to twisted equations in free abelian groups. Throughout equations in free abelian groups are recognised by finite-state automata, and Diekert [14] shows that right-angled Artin groups have EDT0L $\#$-joined languages, which can easily be generalised to all free abelian groups. For the sake of completeness, we give our own argument here.

We begin with the following technical definition.

**Definition 3.1.** Let $B = [b_{ij}]$ be an $n \times m$ integer matrix. Then define a function $| \cdot |_B : \mathbb{R}^n \to \mathbb{R}$ via

$$|(y_1, \ldots, y_n)|_B = \max \left( \sum_{i=1}^{n} y_i b_{i1}, \sum_{i=1}^{n} y_i b_{i2}, \ldots, \sum_{i=1}^{n} y_i b_{im} \right).$$

In other words, $|y|_B$ is the maximal absolute value of the coordinates of the vector $yB$.

Note that if $y \in \mathbb{Z}^n$ then $|y|_B \in \mathbb{Z}$, and that $| \cdot |_B$ satisfies the triangle inequality.

We now show that we can construct any solution to a system of equations while controlling the value of $| \cdot |_B$ at each intermediate point.

**Lemma 3.2.** Let $B$ be an $n \times m$ integer matrix, $X$ be a vector of $n$ variables, $c \in \mathbb{Z}^m$, and consider the system of $n$ equations over $\mathbb{Z}$ given by $XB = c$. Write $b_{\text{max}} = \max_{i,j} |b_{ij}|$ and let $K = \max(|c_1|, \ldots, |c_m|) + n^{3/2} \cdot b_{\text{max}}$.

Then, for each $x \in \mathbb{Z}^n$ such that $xB = c$, there is a sequence

$$\{0 = x^{(0)}, x^{(1)}, \ldots, x^{(k)} = x\} \subset \mathbb{Z}^n$$

with each $x^{(j)} = x^{(j-1)} + e_j$ for some positive or negative standard basis vector $e_j$, such that $|x^{(j)}|_B \leq K$ for each $j \in \{1, \ldots, k\}$.

**Proof** First, consider the straight line segment $L \subset \mathbb{R}^n$ from 0 to $x$. Since $B$ defines a linear transformation $\mathbb{R}^n \to \mathbb{R}^m$, the function $| \cdot |_B : \mathbb{R}^n \to \mathbb{R}$ is monotone non-decreasing as we move along $L$ from 0 to $x$. Therefore, for each $y \in L$, we have $|y|_B \leq |x|_B = \max(|c_1|, \ldots, |c_m|)$. To obtain the required sequence, approximate $L$ with a piecewise linear path comprised of (positive and negative) standard basis vectors.

Consider the set of unit $n$-cubes with integer-valued corners, which intersect $L$. From among the corners of these cubes, we can find a sequence $\{x^{(j)}\} \subset \mathbb{Z}^n$ of integer-valued points, where subsequent terms share a cube edge (and so each $x^{(j)} = x^{(j-1)} + e_j$ for some $e_j$), such that $x^{(0)} = 0$ and $x^{(k)} = x$, for some $k$. We will show that each point in this sequence satisfies the required bound.
Since the diameter of a unit \( n \)-cube is \( \sqrt{n} \), each point \( x(i) \) is a Euclidean distance of at most \( \sqrt{n} \) from the line \( L \). In other words, for each \( j \) we have \( x(i) = y + d \) for some \( y \in L \) and \( d = (d_1, \ldots, d_n) \in \mathbb{R}^n \) with \( |d_i| \leq \sqrt{n} \). Then note that for any such \( d \) we have

\[
|d|_B = \max \left( \sum_{i=1}^{n} d_i b_{1i}, \ldots, \sum_{i=1}^{n} d_i b_{im} \right)
\]

\[
\leq \max \left( \sum_{i=1}^{n} |d_i||b_{1i}|, \ldots, \sum_{i=1}^{n} |d_i||b_{im}| \right)
\]

\[
\leq \max \left( \sum_{i=1}^{n} \sqrt{n} \cdot b_{\text{max}}, \ldots, \sum_{i=1}^{n} \sqrt{n} \cdot b_{\text{max}} \right) = (n \sqrt{n}) b_{\text{max}}.
\]

We can then bound each element of the sequence as follows:

\[
|x(i)|_B = |y + d|_B \leq |y|_B + |d|_B \leq (|c_1|, \ldots, |c_m|) + (n \sqrt{n}) b_{\text{max}} = K.
\]

Thus the sequence \( \{x(i)\} \) satisfies the requirements of the Lemma. \( \square \)

We now show that a system of twisted equations in \( \mathbb{Z}^k \) can be reduced to a system of (non-twisted) equations in \( \mathbb{Z} \).

**Lemma 3.3.** Let \( S_E \) be the solution set of a finite system \( E \) of twisted equations in \( \mathbb{Z}^k \) in \( n \) variables. Then there is a finite system of equations \( F \) in \( \mathbb{Z}^n \) with \( k n \) variables and solution set \( S_F \) such that

\[
S_E = \{ ((x_1, \ldots, x_k), (x_{k+1}, \ldots, x_{2k}), \ldots, (x_{(k-1)n}, \ldots, x_{kn})) | (x_1, \ldots, x_{kn}) \in S_F \}.
\]

**Proof** Consider a twisted equation in \( \mathbb{Z}^k \)

\[
(1) \quad c_0 + Y_{i_1} \Phi_1 + c_1 \cdots + Y_{i_n} \Phi_n + c_n = 0,
\]

where \( Y_1, \ldots, Y_n \) are variables, \( c_0, \ldots, c_n \in \mathbb{Z}^n \) are constants, and \( \Phi_1, \ldots, \Phi_n \in \text{GL}_k(\mathbb{Z}) \). Set \( c = c_0 + \cdots + c_n \). By grouping the occurrences of each \( Y_i \), we have that (1) is equivalent to the following identity

\[
(2) \quad Y_1 B_1 + \cdots + Y_n B_n + c = 0,
\]

where \( B_1 = [b_{i_1j}], \ldots, B_n = [b_{ni_1j}] \) are \( k \times k \) integer-valued matrices, although not necessarily in \( \text{GL}_k(\mathbb{Z}) \). We will first show that the solution set of (2) is equal to the solution set of a system of \( k \) equations in \( \mathbb{Z} \). Write \( Y_i = (Y_{i1}, \ldots, Y_{ik}) \) and \( c = (c_1, \ldots, c_n) \) for variables \( Y_{ij} \) over \( \mathbb{Z} \) and constants \( c_i \in \mathbb{Z} \), for each \( i \). Then \( Y_i B_i = \left( \sum_{j=1}^{k} b_{ij1} Y_{ij}, \ldots, \sum_{j=1}^{k} b_{ijk} Y_{ij} \right) \), for each \( i \). It follows that the solution set of (2) is equal to the solution set of the following system of equations in \( \mathbb{Z} \):

\[
\sum_{i=1}^{n} \sum_{j=1}^{k} b_{ij1} Y_{ij} + c_i = 0
\]

\[
\vdots
\]

\[
\sum_{i=1}^{n} \sum_{j=1}^{k} b_{ijk} Y_{ij} + c_i = 0.
\]

We can conclude that the lemma holds for single twisted equations in \( \mathbb{Z}^k \). It follows that the solution set to a system of \( m \) twisted equations in \( \mathbb{Z}^k \) will be constructible as stated in the lemma, from the solution set to a system of \( m \) of the above systems; that is a system of \( km \) equations in \( \mathbb{Z} \). \( \square \)
Before we can prove Lemma 3.5 we need a slightly altered version of modular arithmetic, where we replace 0 with the quotient.

**Notation 3.4.** For each \( n, r \in \mathbb{Z}_{\geq 0} \) with \( r > 0 \), define

\[
    n \mod^+ r = \begin{cases} 
    n \mod r & \text{if } n \mod r \neq 0 \\
    r & \text{if } n \mod r = 0 
    \end{cases}
\]

We are now in a position to prove that multivariable solution languages to twisted equations in free abelian groups are accepted by multivariable finite state automata. We do this by expressing our equation as an identity of matrices, where the coefficients of the matrix determine the equation. This allows us to use the bound from Lemma 3.2 to construct our automaton.

**Lemma 3.5.** The multivariable solution language to a system of twisted equations in a free abelian group, with respect to the standard generating set and normal form, is accepted by a multivariable finite-state automaton.

**Proof** Let \( \mathcal{E} \) be a system of \( m \) twisted equations in \( \mathbb{Z}^k \) in \( n \) variables. Let \( \{a_1, \ldots, a_k\} \) denote the standard generating set for \( \mathbb{Z}^k \). By Lemma 3.3, there is a system of \( km \) equations \( \mathcal{F} \) in \( \mathbb{Z} \) with \( kn \) variables, such that the solution language to \( \mathcal{E} \) is equal to

\[
    \mathcal{S}_\mathcal{E} = \left\{ \left( a_1^{t_1} \cdots a_k^{t_k}, a_1^{t_1+1} \cdots a_k^{t_k}, \ldots, a_1^{t_{k(n-1)+1}} \cdots a_k^{t_{kn}} \right) \mid (t_1, \ldots, t_{kn}) \text{ is a solution to } \mathcal{F} \right\}.
\]

We represent this new system \( \mathcal{F} \) via the identity \( X \mathbf{B} = \mathbf{c} \) where

- \( X = (X_1, \ldots, X_{kn}) \) is a row vector of \( kn \) variables,
- \( \mathbf{B} = [b_{ij}] \) is a \( kn \times km \) matrix of coefficients, and
- \( \mathbf{c} \in \mathbb{Z}^{km} \) is a row vector of constants.

The constant of Lemma 3.2 is then \( K = \max(|c_1|, \ldots, |c_{km}|) + (kn)^{3/2}b_{\text{max}} \).

We can now show that the multivariable solution language is accepted by a \( kn \)-variable finite-state automaton, using the method described in Example 2.14. We define our automaton \( \mathcal{A} \) to have the set of states

\[
    \{ q_\mathcal{X} \mid \mathcal{X} = (x_1, \ldots, x_{kn}) \in \mathbb{Z}^{kn}, |x_i| \leq K \},
\]

Our start state will be \( q_\mathcal{X} \) and \( q_\mathcal{E} \) will be our only accept state. Let \( \mathbf{w}_i \) be the \( kn \)-variable word with \( a_i \mod^+ k \) in the \( i \)th position, and \( \varepsilon \) elsewhere. We have an edge from \( q_\mathcal{X} \) to \( q_\mathcal{Y} \) labelled with \( \mathbf{w}_j \) for all \( j \) such that \( \mathcal{X} + (b_{j1}, \ldots, b_{j(kn)}) = \mathcal{Y} \). By construction, the language accepted by \( \mathcal{A} \) is contained within \( \mathcal{S}_\mathcal{E} \). On the other hand, any word in \( \mathcal{S}_\mathcal{E} \) is accepted by \( \mathcal{A} \), by following an appropriate sequence as given by Lemma 3.2. \( \square \)

We now consider the space complexity that is needed to construct the multivariable finite-state automaton defined in the proof of Lemma 3.5.

**Remark 3.6.** Before we can attempt to show anything about space complexity, we need to define the length of our input. Often, when talking about equations \( w \in F_V \ast G \), for a group \( G \) and set of variables \( V \), we take the length of \( w \) to be \( |w| \), with respect to the word metric on \( F_V \ast G \), which is inherited from \( F_V \) with respect to \( V \), and \( G \) with respect to a specified generating set.

With free abelian, and also virtually abelian groups, we can write our equations more efficiently. Recall that we use \( a_1, \ldots, a_k \) to be the standard generating set of \( \mathbb{Z}^k \), when using multiplicative
notation. By reordering an equation in \( n \) variables in \( \mathbb{Z}^k \), and we can assume it is in the form
\[
X_1^{b_1} \cdots X_n^{b_n} a_1^{c_1} \cdots a_k^{c_k} = 1,
\]
where \( X_1, \ldots, X_n \in V \), and \( b_1, \ldots, b_n \in \mathbb{Z} \). The length of this equation with respect to the word metric is
\[
\sum_{i=1}^{n} |b_i| + \sum_{j=1}^{k} |c_j|.
\]
However, since an integer \( r \in \mathbb{Z} \) can be stored using \( \log |r| + C \) bits, for some constant \( C \), and we only need to store \( b_1, \ldots, b_n, c_1, \ldots, c_k \), we can write this equation using
\[
\sum_{i=1}^{n} \log |b_i| + \sum_{j=1}^{k} \log |c_j| + C kn
\]
bits. To write a twisted equation, we can first rearrange it to the form
\[
(X_1 B_1) \cdots (X_n B_n) a_1^{c_1} \cdots a_k^{c_k} = 1,
\]
for the matrices \( B_r = [b_{rij}] \) as described in the proof of Lemma 3.3. The matrices are all \( k \times k \) matrices, and therefore \( B_r \) can be stored using
\[
\sum_{i,j=1}^{k} \log |b_{rij}| + C'k^2
\]
bits, for some constant \( C' \).

**Lemma 3.7.** The multivariable finite-state automaton defined in Lemma 3.5 can be constructed in non-deterministic quadratic space.

**Proof** Let \( k \) be the rank of the free abelian group, \( E \) be the system of equations, \( n \) be the number of variables, and \( m \) be the number of equations. We start by converting \( E \) into the form \( X B = c \) (all we need to store is \( B \) and \( c \)). Let \( I \in \mathbb{Z}_{\geq 0} \) be the length of the input.

Index the equations \( w_1, \ldots, w_m \). We copy each equation in \( E \) into the work tape, so our work tape will now have the same size as our input. We have assumed our equations are already in the form stated in Remark 3.6 and converting them to additive notation means they will be in the form
\[
Y_1 B_1 + \cdots Y_n B_n = d,
\]
where each \( B_i \) is a \( k \times k \) matrix, each \( Y_i \) is a variable, and \( d \in \mathbb{Z}^k \). We will now construct the matrix \( B \) and the vector \( c \). We write \( Y_i = (Y_{i1}, \ldots, Y_{ik}) \), and \( B_1 = [b_{1ij}], \ldots, B_n = [b_{nij}] \). For each equation
\[
Y_1 B_1 + \cdots Y_n B_n = d,
\]
add the following vectors as columns to \( B \), and store them in the work tape:
\[
(b_{111}, \ldots, b_{nk1}), (b_{11k}, \ldots, b_{nnk}).
\]
The matrix \( B \) will at this point be a \( kn \times km \) matrix. For each equation, we also append the entries of \( d \) to the vector \( c \).

We now construct the states. Since our set of states is the set of all \( q_x \) such that \( x \in \mathbb{Z}^{kn} \) with each coordinate having absolute value at most \( K \), where \( K \) is from Lemma 3.2 we can construct the set of states by remembering the last state constructed, together with the bound \( K \), and proceeding in any ‘sensible’ systematic manner, such as starting in one ‘corner’, and running down each line in the ‘cube’. To do this, we need a memory that can store a vector of length \( kn \) at any time, with entries within \([-K, K]\).
As in the proof of Lemma 3.5, \( K = \max(|c_1|, \ldots, |c_{kn}|) + (kn)^{3/2} \cdot b_{\max} \), where \( b_{\max} = \max_{i,j} |b_{ij}| \).

Recall \( I = \sum_i \log |c_i| + \sum_{i,j} \log |b_{ij}| + Ckn \), where \( C \) is a constant, as mentioned in Remark 3.6.

Then
\[
\log K \leq \log |c_1| + \cdots + \log |c_{kn}| + \frac{3}{2} \log(kn) + \log |b_{\max}| \leq \frac{3}{2} I
\]

So storing an integer within \([-K, K]\) requires \( \frac{3}{2} I \) bits, ignoring constants. Since \( kn \leq I \), storing a vector of length \( kn \) with entries in \([-K, K]\) requires at most \( \frac{3}{2} I^2 \) bits.

We can simply assign \( 0 \) and \( c \) as the start and accept states.

We now need to compute the edges. Recall that we have an edge from \( q_x \) to \( q_y \) labelled with \( w_j \) for all \( j \) such that \( x + (b_{j1}, \ldots, b_{j(kn)}) = y \), where \( w_j \) is the \( kn \)-variable word with \( a_{i \text{mod} + k} \) in the \( i \)th position and \( \varepsilon \) elsewhere. Therefore, we can go through the states systematically and add all of the outgoing edges, and we only need to remember the state we are on in order to compute and output its outgoing edges and their labels. To do this, we only need to record a vector of length \( kn \), the entries of which will lie in \([-K, K]\). As discussed before, this requires at most \( \frac{3}{2} I^2 \) bits to store.

In the next lemma, we show that the solution set to a system of equations in an arbitrary group can be expressed in terms of the solution set to a system of twisted equations in a finite-index subgroup.

**Lemma 3.8.** Let \( G \) be a group, and \( T \) be a finite transversal of a normal subgroup \( H \) of finite index. Let \( S \) be the solution set to a finite system of equations with \( n \) variables in \( G \). Then there is a finite set \( B \subseteq T^n \), and for each \( t \in B \), there is a solution set \( S_t \) to a system of twisted equations in \( H \), such that
\[
S = \bigcup_{(t_1, \ldots, t_n) \in B} \{(h_1t_1, \ldots, h_nt_n) \mid (h_1, \ldots, h_n) \in S_t(t_1, \ldots, t_n)\}.
\]

**Proof** Let
\[
X_{x_{ij}}^{e_{ij}} g_{ij} \cdots X_{y_{pj}}^{e_{pj}} g_{pj} = 1
\]

be a system of equations in \( G \), with variables \( X_1, \ldots, X_n \), where \( j \in \{1, \ldots, k\} \) for some constant \( k \), and \( e_{ij} \in \{-1, 1\} \) for all \( i \) and \( j \). Note that we can fold the constants that can occur before the first occurrence of a variable \( X_{ij} \) in each equation into \( g_{pj} \) by conjugating. For each \( X_i \), define new variables \( Y_i \) and \( Z_i \) over \( H \) and \( T \) respectively such that \( X_i = Y_iZ_i \). We have that \( g_i = h_it_i \) for some \( h_i \in H \) and \( t_i \in T \). Replacing these in (3) gives
\[
(Y_{i_{ij}} Z_{i_{ij}})^{e_{ij}} h_{ij} t_{ij} \cdots (Y_{y_{pj}} Z_{y_{pj}})^{e_{pj}} h_{pj} t_{pj} = 1.
\]

For each \( g \in G \), define \( \psi_g : G \to G \) by \( h\psi_g = ghg^{-1} \). We will abuse notation, and extend this notation to define \( \psi Z_1, \ldots, \psi Z_n \). For each \( i \) and \( j \), let
\[
\delta_{ij} = \begin{cases} 
0 & \epsilon_{ij} = 1 \\
1 & \epsilon_{ij} = -1.
\end{cases}
\]

It follows that (4) is equivalent to
\[
(Y_{i_{ij}}^{e_{ij}} \psi_{Z_{i_{ij}}}^{\delta_{ij}} Z_{i_{ij}}^{e_{ij}}) h_{ij} t_{ij} \cdots (Y_{y_{pj}}^{e_{pj}} \psi_{Z_{y_{pj}}}^{\delta_{pj}} Z_{y_{pj}}^{e_{pj}}) h_{pj} t_{pj} = 1.
\]
By pushing all \( Y \)'s and \( h \)'s to the left, we obtain
\[
(5) \quad (Y_{i_1j}^{ε_{i_1j}} ψ_{1i_1j}^{δ_{1i_1j}})(h_{1j}^{ε_{h_1j}} ψ_{1i_1j}^{δ_{1i_1j}}) \cdots (Y_{p_j}^{ε_{p_j}} ψ_{1p_j}^{δ_{1p_j}})(h_{p_j}^{ε_{h_{p_j}}})(ψ_{t(p-1)_j}^{ε_{ψ_{t(p-1)_j}}} \cdots ψ_{t_1j}^{ε_{ψ_{t_1j}}}) Z_{i_1j}^{ε_{Z_{i_1j}}} = 1.
\]
Note that if \((u_1, \ldots, u_n) \in T^n\) is a solution to the \( Z \)'s within a solution to (5), then \( u_1^{ε_{i_1j}} t_1 \cdots u_n^{ε_{i_nj}} t_p \in H\). Let \( A \) be the set of all such tuples. Note that as \( T \) is finite, so is \( A \). Plugging a fixed \((u_1, \ldots, u_n) \in A\) into (5) gives a twisted equation in \( H\):
\[
(6) \quad (Y_{i_1j}^{ε_{i_1j}} ψ_{1i_1j}^{δ_{1i_1j}})(h_{1j}^{ε_{h_1j}} ψ_{1i_1j}^{δ_{1i_1j}}) \cdots (Y_{p_j}^{ε_{p_j}} ψ_{1p_j}^{δ_{1p_j}})(h_{p_j}^{ε_{h_{p_j}}})(ψ_{1}^{ε_{ψ_{1}}} \cdots ψ_{t_1j}^{ε_{ψ_{t_1j}}}) Z_{i_1j}^{ε_{Z_{i_1j}}} = 1.
\]
Let \( B(u_1, \ldots, u_n) \) be the solution set to the above system. It follows that the solution set to (6) equals
\[
\bigcup_{(u_1, \ldots, u_n) \in A} \{(f_1 u_1, \ldots, f_n u_n) \mid (f_1, \ldots, f_n) \in B(u_1, \ldots, u_n)\}.
\]
\[\Box\]

The following proposition reflects a well-known fact about decidability of systems of equations in groups: if a group \( G \) has a finite index normal subgroup \( H \), such that there is an algorithm that determines if any system of twisted equations in \( H \) admits a solution, then there is an algorithm that determines if any system of (untwisted) equations \( G \) admits a solution. This fact turns out to be true regarding EDT0L solutions, and a variant of it is used in [15].

**Proposition 3.9.** Let \( G \) be a group with a finite index normal subgroup \( H \), such that the multivariable solution language to any system of twisted equations in \( H \) is accepted by an \( n \)-variable finite-state automaton, for some \( n \in \mathbb{Z}_{>0} \), with respect to a generating set \( π \), and normal form \( η \).

Then the multivariable solution language to any system of equations in \( G \) is accepted by an \( n \)-variable finite-state automaton, for some \( n \in \mathbb{Z}_{>0} \), with respect to the generating set \( Σ \cup T \), for any right transversal \( T \) of \( H \), and the normal form \( ζ \), where \( gζ = (hη)t \), where \( h \in H \) and \( t \in T \) are (unique) such that \( g = ht \).

**Proof** We have from Lemma 3.8 that the solution language is a finite union across valid choices of transversal vectors \((t_1, \ldots, t_n)\) of
\[
(6) \quad \{((h_1 η)t_1, \ldots, (h_n η)t_n) \mid (h_1, \ldots, h_n) \in A(t_1, \ldots, t_n)\},
\]
where \( A(t_1, \ldots, t_n) \) is the solution set to a system of twisted equations in \( H \). Since the class of languages accepted by \( n \)-variable finite-state automata is closed under finite unions, it suffices to show that (6) is accepted by an \( n \)-variable finite-state automaton.

By our assumptions on \( H \), the language
\[
\{(h_1 η, \ldots, h_n η) \mid (h_1, \ldots, h_n) \in A(t_1, \ldots, t_n)\}
\]
is accepted by an \( n \)-variable finite-state automaton \( M \), for any valid choice of transversal vector \((t_1, \ldots, t_n)\). We can therefore modify this automaton to accept
\[
\{((h_1 t_1, \ldots, h_n t_n) \mid (h_1, \ldots, h_n) \in A(t_1, \ldots, t_n)\}
\]
We do this by adding a new state \( q \), and with an edge labelled \((t_1, \ldots, t_n)\) from every accept state of \( M \), and making \( q \) the only accept state. By construction, this accepts the stated language. \[\Box\]
Remark 3.10. Before we can talk about the space complexity of equations in virtually abelian groups, we need to define the size of our input. We do this by using the finite-index normal free abelian subgroup, as these have an efficient way of storing equations.

Let be a group with a finite-index subgroup , and suppose that there is a convention for input sizes of twisted equations in (see Remark 3.6 for free abelian groups). We define the length of a system of equations in to be the length of the system of twisted equations in derived in Lemma 3.8.

Lemma 3.11. Let , and be defined as in Proposition 3.9, and suppose the multivariable finite-state automaton that accepts a system of twisted equations in , in the statement of Proposition 3.9, is constructible in \( \text{NSPACE}(f) \), where \( f : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{Z}_{\geq 0} \) is a function. Then the automaton that accepts a system of equations in is also constructible in \( \text{NSPACE}(f) \).

Proof. By Lemma 2.18 it suffices to show that each automaton that accepts a language 
\[
\{(h_1t_1, \ldots, h_nt_n) \mid (h_1, \ldots, h_n) \in A(t_1, \ldots, t_n)\},
\]
where \( A(t_1, \ldots, t_n) \) is as defined in the proof of Proposition 3.9. Recall that this is constructed from the automaton \( M \) that accepts a system of twisted equations in by adding one additional state \( q \), and edges from each accept state to \( q \), all labelled \( (t_1, \ldots, t_n) \), and then by making \( q \) the only accept state. We do this by modifying the algorithm that constructs \( M \) to add the state \( q \) at the beginning, then perform the algorithm that constructs \( M \), except whenever we would label a state \( p \) as an accept state, we instead add an edge from \( p \) to \( q \), labelled by \( (t_1, \ldots, t_n) \). This does not use a longer work tape than the algorithm that constructs \( M \). \( \square \)

We now have enough to prove our first main result. Our generating set is the union of the standard generating set \( \Sigma \) of the finite index free abelian subgroup together with a right transversal \( T \). We use the normal form 
\[
\{at \mid a \in \Sigma, t \in T\}.
\]

Theorem 3.12. Multivariable solution languages to systems of equations in virtually abelian groups with \( n \) variables are accepted by \( n \)-variable finite-state automata.

Proof. This follows from Proposition 3.9 together with the fact that multivariable solution languages to systems of twisted equations in free abelian groups are accepted by \( n \)-variable finite-state automata (Lemma 3.5). \( \square \)

Lemma 2.20 now gives us the following result.

Corollary 3.13. The \#-joined solution languages to systems of equations in virtually abelian groups are EDT0L.

Remark 3.14. Corollary 3.13 uses the normal form defined by writing an element of a virtually abelian group as a product of a word in the finite-index free abelian normal subgroup, written in standard normal form, with an element of the (finite) transversal for that subgroup.

We can change our generating set to any other generating set, and there will exist a normal form such that solution languages are still EDT0L. Adding a new generator does not change the language at all, as we can keep the normal form the same, and so our new generator will not appear in any normal form word. To remove a redundant generator \( c \), we can fix a word \( w_c \) over the remaining generators and their inverses that represents the same element as \( c \), and apply the free monoid homomorphism that maps \( c \) to \( w_c \). This corresponds to changing the normal form used by replacing every occurrence of \( c \) with \( w_c \).
Changing the normal form is more difficult. In [7], Section 5, Ciobanu and Elder show that changing between quasigeodesic normal forms will not affect whether or not the solution language to a given system is EDT0L. This relies on the fact that in a hyperbolic group \( G \), the set of all pairs \((u, v)\) of \((\lambda, \mu)\)-quasigeodesics such that \( u =_G v \) is accepted by a 2-variable finite state automaton. Unfortunately, this doesn’t work in \( \mathbb{Z}^2 \), so a different approach would be required to preserve the EDT0L status of the language when changing between normal forms in virtually abelian groups.

We now combine the various lemmas on the space complexity of the algorithms we have used to construct multivariable finite-state automata and EDT0L systems to show the following.

**Proposition 3.15.** The multivariable finite-state automaton from Theorem 3.12 and the EDT0L system from Corollary 3.13 can be constructed in non-deterministic quadratic space.

**Proof** The fact that the multivariable finite-state automaton can be constructed in non-deterministic quadratic space follows from Lemma 3.11 and Lemma 3.7. Using this fact with Lemma 2.20 gives the second statement.

To understand the the growth of the \#-joined solution language, we need the following Lemma.

**Lemma 3.16.** Let \( A \subseteq (\mathbb{Z}^k)^n \) be the solution set to a system of twisted equations in \( \mathbb{Z}^k \) (with \( n \) variables). Then \( A \) is a polyhedral subset of \( \mathbb{Z}^{kn} \).

**Proof** By Lemma 3.3, \( A \) may be viewed as the set of solutions to a system of (non-twisted) equations in \( \mathbb{Z} \), with \( kn \) variables, with each element of \( A \) given as a vector in \( \mathbb{Z}^{kn} \), with respect to the standard basis of \( \mathbb{Z}^{kn} \). Now a single such equation in \( \mathbb{Z} \) may be expressed as

\[
\sum_{i=1}^{kn} a_i x_i = b
\]

for variables \( x_i \) and constants \( a_i, b \in \mathbb{Z} \). Therefore the solution set to such an equation has the form

\[
\left\{(x_1, \ldots, x_{kn}) \in \mathbb{Z}^{kn} \left| \sum_{i=1}^{kn} a_i x_i = b \right. \right\} = \left\{x \in \mathbb{Z}^{kn} \mid a \cdot x = b \right\}
\]

and is thus an elementary set (see Definition 2.2). The solution set to a system of equations is then the intersection of finitely many elementary sets, and is therefore a polyhedral set by the definition.

We can now use the polyhedral structure of solution sets in \( \mathbb{Z}^k \) to prove the following Proposition about the growth of solution languages in virtually abelian groups.

**Proposition 3.17.** The \#-joined solution language of any system of equations in a virtually abelian group has rational growth series.

**Proof** As before, let \( G \) be a virtually abelian group and let \( \mathbb{Z}^k \) denote a free abelian normal subgroup of finite index, and \( T \) a choice of transversal. The normal form on \( \mathbb{Z}^k \) given by the standard basis vectors is denoted \( \eta \). By Lemma 3.8, the solution language is given by a finite union of sets of the form

\[
\{(h_1 \eta) t_1 \# (h_2 \eta) t_2 \# \cdots \# (h_n \eta) t_n \mid (h_1, \ldots, h_n) \in A_t\}
\]
where \( n \) is the number of variables, \( t = (t_1, \ldots, t_n) \) is some subset of \( T^n \), and each \( A_t \) is the solution set to some system of twisted equations in \( \mathbb{Z}^k \).

Now, the word \((h_1\eta)t_1\#\cdots\#(h_n\eta)t_n \in (T \cup \{\#\} \cup \{\pm e_i \mid 1 \leq i \leq kn\})^* \) has length \( 2n - 1 + |(h_1, \ldots, h_n)| \). So the growth series of the set (7) is equal to the growth series of \( A_t \) multiplied by \( z^{2n-1} \). That is,

\[
 z^{2n-1} \sum_{m=0}^{\infty} \# \{(h_1, \ldots, h_n) \in A_t \mid |(h_1, \ldots, h_t)| = m \} z^m.
\]

Since each \( A_t \) is polyhedral by Lemma 3.16, Corollary 2.8 implies that their growth series (with the weight of each generator equal to 1 in this case) is rational, and hence the growth series of (7) is also rational. So the growth series of the solution language is a finite sum of rational functions, and therefore rational itself.

**Remark 3.18.** We note that the language above will not be context-free in general. For example, suppose the underlying group is \( \mathbb{Z} = \langle x \rangle \), and consider the equation \( X = Y = Z \) (more formally the system of equations \( XY^{-1} = YZ^{-1} = 1 \)). In the notation of this paper, the set of solutions is \( \{a^m \# a^m \# a^m \mid m \in \mathbb{Z} \} \), which is not context-free over the alphabet \( \{a, a^{-1}, \#\} \) by standard techniques.

Thus we have a large class of EDT0L languages, with rational growth series, which are not, in general, context-free.

### 4. Relative Growth of Algebraic Sets

We now study the nature of algebraic sets from a different point of view. Expanding on the theme of Proposition 3.17, we consider the growth of algebraic sets, this time as sets of tuples of group elements, with respect to a natural metric inherited from the word metric on the group.

The usual notion of the growth function of a group can be altered by restricting to a subset. This is known as relative growth. The study of relative growth of subgroups in particular has attracted significant interest, for example Davis-Olshanskii [12], and recently Cordes-Russell-Spriano-Zalloum [10]. Here, we define and study the relative growth of algebraic sets. Since such a set is a subset of \( G^n \), rather than \( G \) itself, we must decide how to assign lengths to tuples. We do this in perhaps the most obvious way, by taking the sum of the lengths of the components (see Definition 4.2).

Since the growth of virtually abelian groups is always polynomial (that is, the number of elements of length \( n \) is at most polynomial in \( n \)), it is clear that the same will be true of algebraic sets. Instead, we study the growth series, the formal power series associated to the relative growth function of an algebraic set, and show that this is always a rational function (see Theorem 4.3). This means that there exists a set of unique geodesic representatives for each algebraic set, which has rational growth series as a language.

An alternative approach which avoids the need to define the length of \( n \)-tuples of group elements is to study the multivariate growth series, the formal power series in \( n \) variables, which correspond to the \( n \) variables of the system of equations in question (see Definition 4.2). In this case, we have the weaker result that the series is always holonomic (Corollary 4.21).

From now on we will assume that \( G \) is virtually abelian with a normal, finite index subgroup isomorphic to \( \mathbb{Z}^k \) for some positive integer \( k \).
Definition 4.1. Let $G$ be generated by a finite set $S$ and suppose $S$ is equipped with a weight function $\| \cdot \| : S \to \mathbb{Z}_{>0}$. This naturally extends to $S^*$ so that $\| s_1 s_2 \cdots s_k \| = \sum_{i=1}^{k} \| s_i \|$.

1. Define the weight of a group element as
   \[
   \| g \| = \min \{ \| w \| : w \in S^*, \; w =_{G} g \}.
   \]
   Any word representing $g$ whose weight is equal to $\| g \|$ will be called geodesic. This coincides with the usual notion of word length when the weight of each non-trivial generator is equal to 1.

2. Let $V \subseteq G$ be any subset. Then the relative weighted growth function of $V$ relative to $G$, with respect to $S$, is defined as
   \[
   \sigma_{V \subseteq G,S}(m) = \# \{ g \in V : \| g \| = m \}.
   \]
   For simplicity of notation, we will write $\sigma_V(m)$ when the other information is clear from context.

3. The corresponding weighted growth series is the formal power series
   \[
   S_{V \subseteq G,S}(z) = \sum_{m=0}^{\infty} \sigma_{V \subseteq G,S}(m)z^m.
   \]

4. Benson proved in [2] that the series $S_{G \subseteq G}(z)$ is always rational (that is, the standard growth series of $G$), and the first named author proved in [19] that for any subgroup $H$ of $G$, the series $S_{H \subseteq G}(z)$ is always rational. Both of these results hold regardless of the choice of finite weighted generating set. As discussed, we wish to apply these ideas to algebraic sets, which are subsets of $G^n$ in general, for some positive integer $n$. Therefore, we extend Definition 4.1 as follows.

Definition 4.2. Let $G$ be generated by a finite set $S$, equipped with a weight function $\| \cdot \|$.

1. Let $x = (x_1, \ldots, x_n) \in G^n$ be any $n$-tuple of elements of $G$. Define the weight of $x$ as follows:
   \[
   \| x \| = \min \left\{ \sum_{i=1}^{n} \| v_i \| : v_i \in S^*, \; v_i =_{G} x_i, \; 1 \leq i \leq n \right\} = \sum_{i=1}^{n} \| x_i \|.
   \]

2. Let $V \subseteq G^n$ be any set of $n$-tuples of elements. Then the relative weighted growth function of $V$ is defined as the function
   \[
   \sigma_{V \subseteq G^n,S}(m) = \# \{ x \in V : \| x \| = m \}.
   \]

3. The corresponding (univariate) weighted growth series is
   \[
   S_{V \subseteq G^n,S}(z) = \sum_{m=0}^{\infty} \sigma_{V \subseteq G^n,S}(m)z^m \in \mathbb{Q}[[z]].
   \]

4. The multivariate growth series is
   \[
   M_{V \subseteq G^n,S}(z_1, \ldots, z_n) = \sum_{x \in V \subseteq G^n} z_1^{\| x_1 \|} \cdots z_n^{\| x_n \|} \in \mathbb{Q}[[z_1, z_2, \ldots, z_n]].
   \]

We will suppress some or all of the subscripts when it is clear what the notation refers to.

With these definitions, we can state the main result of this section.

Theorem 4.3. Let $G$ be a virtually abelian group. Then every algebraic set of $G$ has rational weighted growth series with respect to any finite generating set.
4.1. **Structure of virtually abelian groups.** To prove the Theorem, we will extend the framework used in [2] and [19] to apply to our setting. We give the necessary definitions and results below, and refer the reader to the above mentioned articles for full details.

**Definition 4.4.** As above, fix a finite generating set $S$ for $G$.

1. We define $A = S \cap \mathbb{Z}^k$ and $B = S \setminus A$. Any word in $B^*$ will be called a pattern.
2. Let $A = \{x_1, \ldots, x_r\}$, and $\pi = y_1 y_2 \cdots y_l$ be some pattern (with each $y_i \in B$). Then a word in $S^*$ of the form
   \[
   w = x_1^{i_1} x_2^{i_2} \cdots x_r^{i_r} y_1 x_1^{i_{r+1}} x_2^{i_{r+2}} \cdots x_r^{i_{2r}} y_2 \cdots y_l x_1^{i_{l+r}} x_2^{i_{l+r+1}} \cdots x_r^{i_{l+r+r}}
   \]
   for non-negative integers $i_j$ is called a $\pi$-patterned word. For a fixed $\pi \in B^*$, denote the set of all such words by $W^\pi$.

This definition allows us to identify patterned words with vectors of non-negative integers, by focussing on just the powers of the generators in $A$ as follows.

**Definition 4.5.** Fix a pattern $\pi$ of length $l$, and write $m_\pi = l r + r$. Define a bijection $\phi_\pi : W^\pi \to \mathbb{Z}_{\geq 0}^m$ via
   \[
   \phi_\pi : x_1^{i_1} x_2^{i_2} \cdots x_r^{i_r} y_1 x_1^{i_{r+1}} x_2^{i_{r+2}} \cdots x_r^{i_{2r}} y_2 \cdots y_l x_1^{i_{l+r}} x_2^{i_{l+r+1}} \cdots x_r^{i_{l+r+r}} \mapsto (i_1, i_2, \ldots, i_{l+r+r}).
   \]

This bijection will allow us to count subsets of $\mathbb{Z}^m_\pi$ in place of sets of words. We apply the weight function $\| \cdot \|$ to $\mathbb{Z}^m_\pi$ in the natural way, weighting each coordinate with the weight of the corresponding $x \in A$. More formally, we have
   \[
   \| (i_1, \ldots, i_m) \| \ := \sum_{j=1}^{m_\pi} i_j \| x_j \mod + r \|.
   \]

Then $\phi_\pi$ preserves the weight of words in $W^\pi$, up to a constant:
   \[
   \|w\phi_\pi \| = \|w\| - \|\pi\|.
   \]

Fix a transversal $T$ for the cosets of $\mathbb{Z}^k$ in $G$. Note that, since $\mathbb{Z}^k$ is a normal subgroup, we can move each $y_i$ in the word [8] to the right, modifying only the generators from $A$, and we have $\overline{w} \in \mathbb{Z}^k \pi$. Thus $\overline{W^\pi} \subset \mathbb{Z}^k t_\pi$ for some $t_\pi \in T$ where $\overline{\pi} \in \mathbb{Z}^k t_\pi$.

It turns out that we can pass from a word $w \in W^\pi$ to the normal form (with respect to $T$ and the standard basis for $\mathbb{Z}^k$) of the element $\overline{w}$ using an integral affine transformation.

**Proposition 4.6** (Section 12 of [2]). For each pattern $\pi \in B^*$, there exists an integral affine transformation $A^\pi : \mathbb{Z}_{\geq 0}^m \to \mathbb{Z}^k$ such that $\overline{w} = (w\phi_\pi A^\pi) t_\pi$ for each $w \in W^\pi$.

Observe that the union $\bigcup W^\pi$ of patterned sets taken over all patterns $\pi$ contains a geodesic representative for every group element (since any geodesic can be arranged into a patterned word without changing its image in the group). However, this is an infinite union, since patterns are simply elements of $B^*$.

Consider the extended generating set $\tilde{S}$ defined as follows:
   \[
   \tilde{S} = \{s_1 s_2 \cdots s_c \mid s_i \in S, \ 1 \leq c \leq [G : \mathbb{Z}^k]\}.
   \]

Define a weight function $\| \cdot \|_\infty : \tilde{S} \to \mathbb{Z}_{\geq 0}$ via $\|s_1 s_2 \cdots s_c\|_\infty = \sum_{i=1}^c \|s_i\|$. Notice that although group elements will have different lengths with respect to this new generating set, we have $\|g\|_\infty = \|g\|$.
for any $g \in G$. Thus the weighted growth functions, and hence series, of any subset $V \subseteq G$ with respect to $S$ and $\overline{S}$ are equal. The following fact shows that passing to this extended generating set means we only need consider finitely many patterns.

**Proposition 4.7** (11.3 of [2]). Every element of $G$ has a geodesic representative with a pattern whose length (with respect to $\overline{S}$) does not exceed $[G: \mathbb{Z}^k]$.

**Definition 4.8.** Let $P$ denote the set of patterns of length at most $[G: \mathbb{Z}^k]$ (with respect to $\overline{S}$).

From now on we will implicitly work with the extended generating set, allowing us to restrict ourselves to the finite set of patterns $P$.

We now reduce each $W^\pi$ so that we have only a single geodesic representative for each element of $G$.

**Theorem 4.9** (Section 12 of [2]). For each $\pi \in P$, there exists a set $U^\pi \subset W^\pi$ such that every word in $U^\pi$ is geodesic, every element in $G$ is represented by some word in $\bigcup_{\pi \in P} U^\pi$, and no two words in $\bigcup_{\pi \in P} U^\pi$ represent the same element. Furthermore, each $U^\pi \phi_\pi$ is a polyhedral set in $\mathbb{Z}^{m_\pi}$.

**Corollary 4.10.** The weighted growth series $S_{G \subseteq G}(z)$ of $G$ is rational, with respect to all generating sets.

**Proof** The growth series $S_{G \subseteq G}$ is precisely the growth series of $\bigcup_{\pi \in P} U^\pi$ as a set. From Definition 4.5 we have

$$S_{U^\pi \subseteq G}(z) = z\|\pi\| S_{U^\pi \phi_\pi}(z)$$

and thus

$$S_{G \subseteq G}(z) = \sum_{\pi \in P} z\|\pi\| S_{U^\pi \phi_\pi}(z)$$

is rational, since each $S_{U^\pi \phi_\pi}(z)$ is a positive polyhedral set and hence rational by Proposition 2.7. $\square$

### 4.2. Univariate growth series of algebraic sets

We can now demonstrate our main result. This will be a consequence of a more general rationality criterion. First, we make the following definitions, extending the framework explained above to $n$-tuples of group elements.

**Definition 4.11.** Let $\underline{\pi} = (\pi_1, \ldots, \pi_n) \in P^n$ be a tuple of patterns, with respect to $\overline{S}$.

1. Let $W_{\underline{\pi}} = W^{\pi_1} \times \cdots \times W^{\pi_n}$ and $U_{\underline{\pi}} = U^{\pi_1} \times \cdots \times U^{\pi_n} \subset (S^*)^n$. Note that $U_{\underline{\pi}}$ is a polyhedral set by Proposition 2.3.
2. Let $m_{\underline{\pi}} = \sum_{i=1}^n m_{\pi_i}$, and $\|\underline{\pi}\| = \sum_{i=1}^n \|\pi_i\|$.
3. Define a map $\phi_{\underline{\pi}}: W_{\underline{\pi}} \to \mathbb{Z}_{\geq 0}^{m_{\underline{\pi}}}$ in the natural way via

$$(w_1, \ldots, w_n) \mapsto (w_1 \phi_{\pi_1}, \ldots, w_n \phi_{\pi_n}).$$

As in the above discussion, $\phi_{\underline{\pi}}$ preserves the weight of words, up to a constant, i.e.

$$\|(w_1, \ldots, w_n)\phi_{\underline{\pi}}\| = \sum_{i=1}^n \|w_i\| - \|\underline{\pi}\|.$$

4. Given $A^{\pi_i}$ as in Proposition 4.6, define an integral affine transformation $A_{\underline{\pi}}: \mathbb{Z}_{\geq 0}^{m_{\underline{\pi}}} \to \mathbb{Z}^{kn}$ in the natural way via

$$(x_1, \ldots, x_n) \mapsto (x_1 A^{\pi_1}, \ldots, x_n A^{\pi_n}) \in \mathbb{Z}^k \times \cdots \times \mathbb{Z}^k.$$
Now we define a class of subsets of finitely generated virtually abelian groups which is particularly amenable to study using the tools we have described.

**Definition 4.12.** Let $T$ be a choice of transversal for the finite index normal subgroup $\mathbb{Z}^k$. A subset $V \subseteq G^n$ will be called *coset-wise polyhedral* if, for each $t = (t_1, \ldots, t_n) \in T^n$, the set

$$V_t = \left\{ (g_1 t_1^{-1}, g_2 t_2^{-1}, \ldots, g_n t_n^{-1}) \mid (g_1, \ldots, g_n) \in V, g_i \in \mathbb{Z}^k t_i \right\} \subseteq \mathbb{Z}^{kn}$$

is polyhedral.

**Remark 4.13.** Note that the definition is independent of the choice of $T$. Indeed, suppose that we chose a different transversal $T'$ so that for each $t_j \in T$ we have $t'_j \in T'$ with $\mathbb{Z}^k t_j = \mathbb{Z}^k t'_j$. Then there exists $y_j \in \mathbb{Z}^k$ with $t_j = y_j t'_j$ for each $j$, and so $g t_j^{-1} = g_j t'_j^{-1} y_j$ for any $g \in \mathbb{Z}^k t_j = \mathbb{Z}^k t'_j$. So changing the transversal changes the set $V_t$ by adding a constant vector $(y_1, \ldots, y_n)$, and so it remains polyhedral by Proposition 2.5.

As an example of Definition 4.12, we provide a brief proof that subgroups are coset-wise polyhedral.

**Proposition 4.14.** Let $G$ be a virtually abelian group, with normal free abelian subgroup $\mathbb{Z}^k$, and let $H$ be any subgroup. Then $H$ is coset-wise polyhedral.

**Proof.** By the Second Isomorphism Theorem, $H$ is itself virtually abelian, with finite-index (free) abelian subgroup $H \cap \mathbb{Z}^k$. Furthermore, $c := [H : H \cap \mathbb{Z}^k] \leq [G : \mathbb{Z}^k] =: d$. Choose a set of representatives $\{t_1, \ldots, t_c\}$ for the cosets of $H \cap \mathbb{Z}^k$ in $H$, and extend this to a set of representatives $\{t_1, \ldots, t_c, t_{c+1}, \ldots, t_d\}$ for the cosets of $\mathbb{Z}^k$ in $G$. For each $t_i$ with $i \leq c$, the set

$$H_{t_i} = \left\{ h t_i^{-1} \mid h \in H, h \in \mathbb{Z}^k t_i \right\} = \left\{ h t_i^{-1} \mid h \in \left( H \cap \mathbb{Z}^k \right) t_i \right\} = H \cap \mathbb{Z}^k.$$

For $i > c$, $H_{t_i}$ is empty. Now since $H \cap \mathbb{Z}^k$ is free abelian, it is a polyhedral set when viewed as a subset of $\mathbb{Z}^k$. The empty set is also polyhedral (as, say, the intersection of a pair of disjoint hyperplanes). Hence $H$ is coset-wise polyhedral. □

In light of Proposition 4.14, the following Theorem is in some sense a generalisation of Theorem 3.3 of [19], namely that every subgroup has rational relative growth series.

**Theorem 4.15.** Let $G$ be virtually abelian, with normal free abelian subgroup $\mathbb{Z}^k$, and let $S$ be any finite weighted generating set. If $V \subseteq G^n$ is coset-wise polyhedral, then the weighted growth series $S_{V \subseteq G^n, S}(z)$ is a rational function.

**Proof.** Fix a transversal $T$. For each $t \in T^n$, let $P_t \subset P^n$ denote the set of $n$-tuples of patterns of the form $\pi = (\pi_1, \ldots, \pi_n)$ where each $\pi_i \in \mathbb{Z}^k t_i$. Let $U^\pi = U^{\pi_1} \times \cdots \times U^{\pi_n} \subset (S^n)^{\pi}$. Then by Theorem 4.9, the disjoint union $\bigcup_{\pi \in P_t} U^{\pi}$ consists of exactly one $n$-tuple of geodesic representatives for each $n$-tuple in $\mathbb{Z}^k t_1 \times \cdots \times \mathbb{Z}^k t_n$. We are only interested in $n$-tuples of elements which lie in the set $V$. Each element of $V$ lies in a unique product of cosets, so we partition $V$ into such products:

$$V = \bigcup_{t \in T^n} \left\{ (g_1, \ldots, g_n) \in V \mid g_i \in \mathbb{Z}^k t_i \right\} = \bigcup_{t \in T^n} \left\{ (g_1, \ldots, g_n) \in G^n \mid (g_1 t_1^{-1}, \ldots, g_n t_n^{-1}) \in V_t \right\}. \quad (9)$$

Now, for a fixed $t$, $(g_1, \ldots, g_n)$ has a unique geodesic representative in the set $U^\pi$, for some $\pi \in P_t$ determined by $t$. So the growth series of each component in the union (9) is equal to the growth series of the set

$$\bigcup_{\pi \in P_t} \left\{ (u_1, \ldots, u_n) \in U^\pi \mid (u_1 \phi_{\pi_1} A^{\pi_1}, \ldots, u_n \phi_{\pi_n} A^{\pi_n}) \in V_t \right\} = \bigcup_{\pi \in P_t} V_t (\phi_\pi A^\pi)^{-1} \cap U^\pi.$$
Applying the map \( \phi_\pi \) to a component of the union yields the set
\[
\left\{ (u_1\phi_1, \ldots, u_n\phi_n) \in U^\pi \phi_\pi \mid (u_1\phi_1^\pi, \ldots, u_n\phi_n^\pi) \in V_t \right\} = V_t (A^\pi)^{-1} \cap U^\pi \phi_\pi.
\]
Now by Propositions 2.3 and 2.5, this last set is polyhedral, and so has rational growth. Since both \( T^n \) and \( P_t \) are finite, the growth series of \( V \) is a finite sum of growth series of sets of the form \( V_t (A^\pi)^{-1} \cap U^\pi \phi_\pi \) (each multiplied by \( z^{|\pi|} \) for the appropriate \( \pi \)) and is therefore rational, finishing the proof. \( \square \)

We can now prove the main result of this section.

**Proof (of Theorem 4.3)** Let \( S \) denote an algebraic set. By Theorem 4.15 it suffices to show that \( S \) is coset-wise polyhedral. By Lemma 3.3 we have
\[
S = \bigcup_{(t_1, \ldots, t_n) \in B} \left\{ (h_1 t_1, \ldots, h_n t_n) \mid (h_1, \ldots, h_n) \in S_{(t_1, \ldots, t_n)} \right\}
\]
\[
= \bigcup_{(t_1, \ldots, t_n) \in T^n} \left\{ (h_1 t_1, \ldots, h_n t_n) \mid (h_1, \ldots, h_n) \in S_{(t_1, \ldots, t_n)} \right\}
\]
where each \( S_{(t_1, \ldots, t_n)} \) is the solution set to some system of twisted equations in \( \mathbb{Z}^k \) (and is empty for \( (t_1, \ldots, t_n) \notin B \). By Lemma 3.16 each \( S_{(t_1, \ldots, t_n)} \) is a polyhedral subset of \( \mathbb{Z}^{kn} \), and thus \( S \) is coset-wise polyhedral as required. \( \square \)

For clarity, we explicitly state the description of algebraic sets in terms of polyhedral sets, which is a consequence of the proof of Theorem 4.3.

**Corollary 4.16.** Let \( G \) be a finitely generated virtually abelian group (with a finite-index free abelian normal subgroup \( \mathbb{Z}^k \) for some \( k \)). Choose a transversal \( T \). Suppose \( S \subset G^n \) is an algebraic set. Then for each \( t = (t_1, \ldots, t_n) \in T^n \), there exists a polyhedral set \( S_t \subset \mathbb{Z}^{kn} \) such that \( S \) decomposes as a finite disjoint union:
\[
S = \bigcup_{t \in T^n} \left\{ (g_1, \ldots, g_n) \in \mathbb{Z}^k t_1 \times \cdots \times \mathbb{Z}^k t_n \mid (g_1 t_1^{-1}, \ldots, g_n t_n^{-1}) \in S_t \right\}.
\]

**4.3. Multivariate Growth Series.** We now turn to the multivariate growth series (see Definition 4.2) and demonstrate that for an algebraic set \( V \), the multivariate growth series \( M_{V \subseteq G^n, S} (z) \) is a holonomic function.

**Definition 4.17.** For clarity, we also define the multivariate growth series of a language. Let \( L \) be a language over some finite weighted alphabet \( A = \{ a_1, \ldots, a_r \} \) (with weights denoted \( ||a_i|| \)) and let \( |w|_i \) denote the number of occurrences of \( a_i \) in a word \( w \in L \). The **weighted multivariate growth series** of \( L \) is the formal power series
\[
\sum_{w \in L} z_1^{||a_1|| |w|_1} z_2^{||a_2|| |w|_2} \cdots z_r^{||a_r|| |w|_r} \in \mathbb{Q}[[z_1, z_2, \ldots, z_r]].
\]

Let \( z = (z_1, \ldots, z_n) \) and \( \partial_{z_i} \) denote the partial derivative with respect to \( z_i \).

**Definition 4.18.** A multivariate function \( f(z) \) is **holonomic** if the span of the set of partial derivatives
\[
\{ \partial_{z_1}^{j_1} \partial_{z_2}^{j_2} \cdots \partial_{z_n}^{j_n} f(z) \mid j_i \in \mathbb{Z}_{\geq 0} \}
\]
over the ring of rational functions \( \mathbb{C}(z) \) is finite-dimensional.
From this definition, we see that a function is holonomic if and only if it satisfies a linear differential equation involving partial derivatives of finite order, and rational coefficients, for each variable $z_i$. Holonomic functions thus generalise the class of algebraic functions. For a more complete introduction to this topic, see [20].

In recent work [3], Bishop extends results of Massazza [25] to show that a certain class of formal languages has holonomic multivariate growth series. The following Lemma follows easily from Proposition 4.3 of [3], and the fact that holonomic functions are closed under algebraic substitution (Theorem B.3 of [20]).

**Lemma 4.19.** The weighted multivariate growth series of a polyhedral set (viewed as a formal language over the alphabet consisting of standard basis vectors) is holonomic.

As in the univariate case, we prove a more general statement about coset-wise polyhedral subsets.

**Theorem 4.20.** Let $V \subset G^n$ be a coset-wise polyhedral set of tuples of elements of a virtually abelian group $G$. Then the weighted multivariate growth series $M_{V \subseteq G^n, S}$ is holonomic, with respect to any generating set $S$.

**Proof** Following the proof of Theorem 4.15, the coset-wise polyhedral set $V$ is represented by a finite disjoint union of polyhedral sets in $\mathbb{Z}^{kn}$, where $k$ is the dimension of the finite-index free abelian normal subgroup of $G$.

Lemma 4.19 implies that the weighted multivariate growth series of each of these polyhedral sets (in the sense of Definition 4.17) is holonomic. These series will involve $kn$ variables, say

$$(z_{11}, \ldots, z_{1k}, z_{21}, \ldots, z_{2k}, \ldots, z_{n1}, \ldots, z_{nk}).$$

To obtain the weighted multivariate growth series of $V$ (in the sense of Definition 4.2), we need only set each $z_{ij} = z_i$ and multiply each of the finitely many growth series by an appropriate constant to account for the contribution from each pattern $\pi$. The closure properties of holonomic functions (Theorem B.3 of [20]) ensure that the resulting growth series is still holonomic (with variables $z_1, \ldots, z_n$ corresponding to the variables in the system of equations). □

**Corollary 4.21.** An algebraic set in a virtually abelian group has holonomic weighted multivariate growth series.

**Proof** The proof of Theorem 4.3 shows that any algebraic set is coset-wise polyhedral. □

5. Further work

It is hoped that Corollary 4.21 can be improved upon. Many power series associated to structures in virtually abelian groups turn out to be rational (see [2], [19], [3]) and therefore we make the following conjecture, noting that an affirmative answer would immediately imply Theorem 4.3.

**Conjecture 5.1.** The weighted multivariate growth series of any algebraic set of a finitely generated virtually abelian group is rational.

A system of equations in a group $G$ is an example of a first order sentence, part of the first order theory of $G$. For groups with decidable first order theory (including virtually abelian groups [15]), a natural generalisation is to study the sets of tuples of elements of $G$ that satisfy more general first order sentences. The present paper is intended to be the first step in an investigation of the formal language properties, and the growth series behaviour, of such more general definable sets.
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