Some remarks on orthogonality of bounded linear operators-II

Debmalya Sain, Anubhab Ray, Subhrajit Dey, and Kallol Paul

Abstract. Let $X, Y$ be normed linear spaces. We continue exploring the validity of the Bhatia–Šemrl (BŠ) Property: “An operator $T \in \mathcal{L}(X, Y)$ satisfies Bhatia–Šemrl Property if for any $A \in \mathcal{L}(X, Y)$, $T \perp_B A$ implies that there exists a unit vector $x \in X$ such that $\|Tx\| = \|T\|$ and $Tx \perp_B Ax$.”

A pair of normed linear spaces $(X, Y)$ is said to be a BŠ pair if for every $T \in \mathcal{L}(X, Y)$, $T$ satisfies the BŠ Property if and only if $M_T = D \cup (-D)$, where $D$ is a non-empty connected subset of $S_X$. We show that $(\ell_1^n, Y)$ is a BŠ pair for any normed linear space $Y$, and also obtain some other results in this context. In particular, using the notion of norm attainment set, we characterize the space $\ell_3^\infty$ among all 3-dimensional polyhedral Banach spaces whose unit ball have exactly eight extreme points.

1. Introduction

Birkhoff–James orthogonality plays a central role in determining the geometry of normed linear spaces in general, and spaces of operators, in particular. One of the most interesting aspects of Birkhoff–James orthogonality is the relation between orthogonality of operators and that of norming elements in the ground space. The purpose of this paper is to continue the investigation of a certain property from [7], as mentioned in the abstract. Before proceeding further, let us fix the notations and the terminologies.

Letters $X$ and $Y$ denote normed linear spaces. Throughout the present article, we will assume the underlying scalar field to be $\mathbb{R}$. Let $B_X = \{x \in X : \|x\| \leq 1\}$ and $S_X = \{x \in X : \|x\| = 1\}$ denote the unit ball and the unit sphere of $X$, respectively. Let $B[x,r] = \{z \in X : \|x - z\| \leq r\}$ and $B(x,r) = \{z \in X : \|x - z\| < r\}$ denote the closed ball and the open ball centered at $x$ and radius $r > 0$, respectively. For a subset $A$ of $X$, let $|A|$ denote the cardinality of $A$. Let $\mathcal{L}(X, Y)$ be the normed space of all bounded
linear operators from $X$ to $Y$, endowed with the usual operator norm. We write $\mathbb{L}(X,Y) = \mathbb{L}(X)$, if $X = Y$. An element $x(\neq 0)$ is said to be smooth point of $X$ if there is unique $f \in S_X^*$ such that $f(x) = \|x\|$. A normed linear space $X$ is said to be smooth if every non-zero element of $X$ is a smooth point. Let $E_X$ denote the collection of all extreme points of the unit ball $B_X$. For a bounded linear operator $T \in \mathbb{L}(X,Y)$, let $M_T$ denote the norm attainment set of $T$, i.e., $M_T = \{x \in S_X : \|Tx\| = \|T\|\}$. For any two points $x_1, x_2 \in X$, $L[x_1, x_2]$ denotes the closed line segment joining $x_1$ and $x_2$, i.e. $L[x_1, x_2] = \{(1-t)x_1 + tx_2 : t \in [0,1]\}$. For $x,y \in X$, $x$ is said to be orthogonal to $y$ in the sense of Birkhoff-James [2], written as $x \perp y$, if $\|x + \lambda y\| \geq \|x\|$ for all $\lambda \in \mathbb{R}$. Similarly, for $T, A \in \mathbb{L}(X,Y)$, $T$ is said to be Birkhoff-James orthogonal to $A$, written as $T \perp_B A$, if $\|T + \lambda A\| \geq \|T\|$ for all $\lambda \in \mathbb{R}$. For an element $x \in X$, by $x^\perp$ we mean the collection of all elements $y \in X$ such that $x \perp y$, i.e., $x^\perp = \{y \in X : x \perp y\}$. Bhatia and Šemrl [1] proved that if $T, A$ are linear operators on a finite-dimensional Hilbert space $\mathbb{H}$ then $T \perp_B A$ if and only if there exists $x \in M_T$ such that $Tx \perp_B Ax$. We refer the readers to [4,6] for alternative approaches in this context. One part of the result is true even if $T, A$ are defined between arbitrary Banach spaces $X$ and $Y$, i.e., if there exists $x \in M_T$ such that $Tx \perp_B Ax$ then $T \perp_B A$. Bhatia and Šemrl conjectured the converse part to be true in the setting of normed spaces, which later turned out to be incorrect, see [3,5]. For studying orthogonality of operators between Banach spaces, the following definition from [8] is very helpful. Given $x, y \in X$, we say that $y \in x^+$ if $\|x + \lambda y\| \geq \|x\|$ for all $\lambda \geq 0$. Similarly, we say that $y \in x^-$ if $\|x + \lambda y\| \geq \|x\|$ for all $\lambda \leq 0$. For an immediate application of these notions towards studying bounded linear operators, which is also relevant to the present work, we refer the readers to [9]. In connection to the conjecture proposed by Bhatia and Šemrl, the term “Bhatia-Šemrl (BŠ) Property” was first coined in [12] and then extended in [7]. We mention the same, for the convenience of the readers.

**Definition 1.1.** Let $X, Y$ be normed linear spaces and let $T \in \mathbb{L}(X,Y)$. We say that $T$ satisfies the Bhatia-Šemrl (BŠ) Property if for any $A \in \mathbb{L}(X,Y)$, $T \perp_B A$ implies that there exists $x \in M_T$ such that $Tx \perp_B Ax$.

Sain and Paul [10] proved that if $T$ is a linear operator on a finite-dimensional Banach space $X$ with $M_T = D \cup (-D)$, where $D$ is a connected subset of $S_X$ then $T$ satisfies the BŠ Property. Moreover, the following characterization was obtained exclusively for two-dimensional Banach spaces:

**Theorem 1.2.** [12, Th. 2.4] A linear operator $T$ on a 2-dimensional real normed linear space $X$ satisfies the Bhatia-Šemrl Property if and only if $T$ attains its norm only on $D \cup (-D)$, where $D$ is a non-empty connected subset of $S_X$.

The validity of the above result remains unknown, when the dimension
of $X$ is strictly greater than two. The following conjecture, first stated in [12], remains open to the best of our knowledge.

**Conjecture 1.3.** [12, Conj. 2.5] A linear operator $T$ on a finite-dimensional Banach space $X$ satisfies the Bhatia–Šemrl Property if and only if $MT = D \cup (-D)$, where $D$ is a connected subset of $S_X$.

Note that the sufficient part of the above conjecture follows from [10, Th. 2.1], but the validity of the necessary part remains unknown. As far as this article is concerned, we begin our investigation with this particular remark. Our main objective is the continuation of the study in [7]. Indeed, we focus on the following problem: If $X$ is a finite-dimensional Banach space with $\dim X > 2$ and if $T \in L(X, Y)$ is such that $MT \neq D \cup (-D)$, where $D$ is a connected subset of $S_X$, then whether $T$ satisfies the BŠ Property or not, for any normed linear space $Y$. In this connection, Property $P_n$ was introduced in [7].

**Definition 1.4.** [7, Defn. 1.6] Let $X$ be a Banach space. Given $n \in \mathbb{N}$, we say that $X$ has Property $P_n$ if for every choice of $n$ vectors $x_1, x_2, \ldots, x_n \in S_X$, $\bigcup_{i=1}^n x_i^\perp \not\subseteq X$.

Trivially, if $X$ has Property $P_n$ then $X$ has Property $P_m$ for all $m \in \mathbb{N}$, with $m \leq n$. Let us now introduce the definition of BŠ pair which plays a crucial role in the whole scheme of things.

**Definition 1.5.** Let $X, Y$ be normed linear spaces. We say that the pair $(X, Y)$ is a BŠ pair if for every $T \in L(X, Y)$, $T$ satisfies the BŠ Property if and only if $MT = D \cup (-D)$, where $D$ is a non-empty connected subset of $S_X$.

Observe that the existence of BŠ pairs substantiates the Conjecture 1.3 to be true. In this article, we investigate operators $T$ which satisfy the BŠ Property. We also exhibit BŠ pairs of spaces $(X, Y)$. Indeed, we show that $(\ell_1^n, Y)$ is a BŠ pair for any normed linear space $Y$. This proves the validity of Conjecture 1.3, whenever the domain space is $\ell_1^n$. Further, we study the BŠ Property of operators on polyhedral Banach spaces. We also characterize the space $\ell_3^3$ among all 3-dimensional polyhedral Banach spaces having exactly eight extreme points in the unit ball. Recall that a finite-dimensional Banach space $X$ is said to be polyhedral if $B_X$ has only finitely many extreme points. We refer the readers to [11] for some standard geometric definitions in this context.

**2. Main results**

We begin with the following theorem which gives a nice connection between Property $P_n$ and the BŠ Property.
Theorem 2.1. Let $T \in \mathcal{L}(X, Y)$, where $\dim X = n \geq 2$ and $Y$ has Property $P_m$, for some $m \geq 2$. If $4 \leq |M_T| \leq 2m$, then $T$ does not satisfy the BŠ Property.

Proof. Let $M_T = \{\pm x_1, \pm x_2, \ldots, \pm x_k\}$, where $2 \leq k \leq m$. Clearly, as any two elements $x_p, x_q$, $p \neq q$, $p, q \in \{1, 2, \ldots, k\}$, are linearly independent, we can extend $\{x_1, x_2\}$ to a basis $\{x_1, x_2, y_3, \ldots, y_n\}$ of $X$. Then for each $x_i \in M_T$, $i = 1, 2, \ldots, k$, we can write $x_i = c_1^i x_1 + c_2^i x_2 + c_3^i y_3 + \cdots + c_n^i y_n$, where $c_j^i$'s are real scalars. Claim that we can find $n$ scalars $\alpha_j > 0$, $j = 1, 2, \ldots, n$, such that $c_1^i \alpha_1 + c_2^i \alpha_2 + \cdots + c_n^i \alpha_n \neq 0$ and $c_1^i \alpha_1 - c_2^i \alpha_2 + \cdots + c_n^i \alpha_n \neq 0$, for all $i = 1, 2, \ldots, k$. Otherwise, if $c_1^i \alpha_1 + c_2^i \alpha_2 + \cdots + c_n^i \alpha_n = 0$, then $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ belongs to the hyperspace $H_1 = \{(z_1, z_2, \ldots, z_n) : c_1^i z_1 + c_2^i z_2 + \cdots + c_n^i z_n = 0\}$ and if $c_1^i \alpha_1 - c_2^i \alpha_2 + \cdots + c_n^i \alpha_n = 0$, then $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ belongs to the hyperspace $H_2 = \{(z_1, z_2, \ldots, z_n) : c_1^i z_1 - c_2^i z_2 + \cdots + c_n^i z_n = 0\}$. These collections of hyperspaces are finite and so $X \neq \bigcup_{i=1}^k (H_1 \cup H_2)$. Therefore, our claim is established.

Now, $\bigcup_{x \in M_T}(Tx)^\perp = \bigcup_{i=1}^k (Tx_i)^\perp \subsetneq Y$, as $Y$ has property $P_m$ and $k \leq m$. Let us take $z \in Y \setminus \bigcup_{i=1}^k (Tx_i)^\perp$. From [8, Prop. 2.1], it follows that $z \in (Tx_1)^+$ or $z \in (Tx_1)^-$ and $z \in (Tx_2)^+$ or $z \in (Tx_2)^-$. Accordingly we consider the following cases.

Case I: Let $z \in (Tx_1)^+ \cap (Tx_2)^+$ or $z \in (Tx_1)^- \cap (Tx_2)^-$. Let us define a linear operator $A : X \to Y$ by

$$Ax_1 = \alpha_1 z, \quad Ax_2 = -\alpha_2 z \quad \text{and} \quad Ay_j = \alpha_j z \quad \text{for} \quad j = 3, 4, \ldots, n.$$ 

If $z \in (Tx_1)^+ \cap (Tx_2)^+$, then $Ax_1 \in (Tx_1)^+$ and $Ax_2 \in (Tx_2)^-$. Also, if $z \in (Tx_1)^- \cap (Tx_2)^-$, then $Ax_1 \in (Tx_1)^-$ and $Ax_2 \in (Tx_2)^+$. In both the cases, it follows from [8, Th. 2.2] that $T \perp_B A$. Clearly $Ax_i = (c_1^i \alpha_1 - c_2^i \alpha_2 + \cdots + c_n^i \alpha_n) z$, for all $i = 1, 2, \ldots, k$. As $c_1^i \alpha_1 - c_2^i \alpha_2 + \cdots + c_n^i \alpha_n \neq 0$ and $Tx_i \not\perp_B z$, for all $i = 1, 2, \ldots, k$, we conclude that $Tx_i \not\perp_B Ax_i$, for all $i = 1, 2, \ldots, k$. Thus $T$ does not satisfy the BŠ Property.

Case II: Let $z \in (Tx_1)^+ \cap (Tx_2)^-$ or $z \in (Tx_1)^- \cap (Tx_2)^+$. Let us define a linear operator $A : X \to Y$ by $Ax_1 = \alpha_1 z, \quad Ax_2 = \alpha_2 z$ and $Ay_j = \alpha_j z$ for $j = 3, 4, \ldots, n$. Proceeding similarly as in Case I, we can conclude that $T \perp_B A$ but there exists no $x \in M_T$ such that $Tx \perp_B Ax$. Therefore, $T$ does not satisfy the BŠ Property.

\[\square\]

Remark 2.2. We note that Theorem 2.1 improves [7, Cor. 2.2.1].

Our next example illustrates the applicability of Theorem 2.1 in studying the BŠ Property of bounded linear operators.

Example 2.3. Let $X = \ell_\infty^n$ and let $Y = \ell_2^n$. Consider a bounded linear operator $T : X \to Y$, defined by

$$Tx = \frac{x}{\sqrt{n}}, \quad x \in X.$$
Then it is easy to check that \(|T| = 1\) and \(M_T = E_X\). Clearly \(Y\) has Property \(P_m\) for any \(m \in \mathbb{N}\). Therefore, by using Theorem 2.1, we conclude that \(T\) does not satisfy the BS Property.

We next present a generalized version of [12, Lemma 2.1], which will be essential for our purpose of studying orthogonality of bounded linear operators.

**Lemma 2.4.** Let \(M\) be a countable subset of a Banach space \(X\) of dimension \(n \geq 2\). Then for any given \(m \in \{1, 2, \ldots, n\}\), there exist \((n-m)\) linearly independent vectors \(y_{m+1}, y_{m+2}, \ldots, y_n\) such that \(\{x_1, x_2, \ldots, x_m, y_{m+1}, y_{m+2}, \ldots, y_n\}\) is a basis of \(X\), whenever \(\{x_1, \ldots, x_m\}\) is any linearly independent set in \(M\).

**Proof.** For \(m = 2\), the proof of the lemma directly follows from the proof of Lemma 2.1 of [12]. All the other cases can be proved similarly.

We next obtain another class of operators not satisfying the BS Property.

**Theorem 2.5.** Let \(X\) be an \(n\)-dimensional Banach space and let \(Y\) be any smooth Banach space. Let \(T \in \mathbb{L}(X, Y)\) be such that \(M_T\) satisfies the following conditions:

1. \(M_T\) has more than two and countably many components.
2. \(M_T\) has at most two non-singleton components \(\pm D_i\), for some \(i \in \mathbb{N}\). If \(D_i = -D_i\), then \(M_T\) has exactly one non-singleton component.
3. \(M_T\) contains at least one pair of singleton components \(\pm D_j\), \(j(\neq i) \in \mathbb{N}\) such that \(\pm D_j \cap \text{span}\{D_i\} = \phi\).

Then \(T\) does not satisfy the BS Property.

**Proof.** If \(M_T\) does not contain any non-singleton component, then the desired result follows from [7, Th. 2.3]. Without loss of generality, we assume that \(M_T = \bigcup_{j \neq i}^n (\pm D_j) \cup (\pm D_i)\), where \(D_i\) is the non-singleton component and \(D_j\), \(j \neq i\), are the singleton components of \(M_T\). Without loss of generality we assume that \(\pm D_1\) are the non-singleton components and \(\pm D_2\) are the singleton components such that \(\pm D_2 \cap \text{span}\{D_i\} = \phi\). Let us assume that \(\dim (\text{span} \{D_i\}) = l\), where \(l < n\). Let \(x_1, x_2, \ldots, x_l \in D_1\) be linearly independent and let \(\pm D_2 = \{\pm x_{l+1}\} \subseteq S_X\). We would like to apply Lemma 2.4 in our present setting. Let \(M = \{x_1, x_2, \ldots, x_l\} \cup \{z \in M_T : z \notin \text{span}\{x_1, x_2, \ldots, x_l\}\} = P \cup Q\), where \(P = \{x_1, x_2, \ldots, x_l\}\) and \(Q = \{z \in M_T : z \notin \text{span}\{x_1, x_2, \ldots, x_l\}\}\). Clearly \(M\) is a countable set, as \(\pm D_1 \subseteq \text{span}\{x_1, x_2, \ldots, x_l\}\) and \(M_T \setminus \{\pm D_1\}\) is a countable set. Let \(m = l + 1\). Therefore, by Lemma 2.4, we can fix \(n - (l + 1)\) elements \(z_{l+2}, z_{l+3}, \ldots, z_n \in X\) such that \(\{x_1, x_2, \ldots, x_l, z, z_{l+2}, z_{l+3}, \ldots, z_n\}\) is a basis of \(X\) for all \(z \in Q\). Then \(\{x_1, x_2, \ldots, x_l, x_{l+1}, z_{l+2}, z_{l+3}, \ldots, z_n\}\) is a basis of \(X\) and so we can fix scalars \(c_{x,k}(x \in X, 1 \leq k \leq n)\) such that for each
We will show that there is a non-zero $z$, also it follows that if $z \in Q$, then $c_{z,l+1} \neq 0$. For each $v \in Y$, let $A_v$ be the linear operator defined by

\[ A_v x_k = Tx_k, \quad k = 1, 2, \ldots, l, \]
\[ A_v x_{l+1} = v, \]
and \[ A_v z_k = Tz_k, \quad k = l + 2, l + 3, \ldots, n. \]

We will show that there is a non-zero $v \in Y$ such that $T \perp_B A_v$ but $Tx \not\perp_B A_v x$ for each $x \in M_T$. For any $\lambda \geq 0$ and $v \in B(-Tx_{l+1}, \|T\|)$, we have

\[ \|T + \lambda A_v\| \geq \|(T + \lambda A_v)x_1\| = \|(1 + \lambda)Tx_1\| = (1 + \lambda)\|T\| \geq \|T\| \]
and

\[ \|T - \lambda A_v\| \geq \|(T - \lambda A_v)x_{l+1}\| = \|Tx_{l+1} - \lambda v\| \geq \|(1 + \lambda)T(x_{l+1} - \lambda (T_{x_{l+1}} + v))\| \]
\[ \geq (1 + \lambda)\|T_{x_{l+1}} + \|T\| - \lambda \|T\| = \|T\|. \]

Therefore, $T \perp_B A_v$ for each $v \in B(-Tx_{l+1}, \|T\|)$. We next show that there is at least one $v \in B(-Tx_{l+1}, \|T\|)$ such that $Tx \not\perp_B A_v x$ for each $x \in M_T$. Firstly, $Tx \not\perp_B A_v x$ for each $x \in M_T \cap \text{span}\{x_1, x_2, \ldots, x_l\}$, since $A_v x = Tx \neq 0$ for all $x \in M_T \cap \text{span}\{x_1, x_2, \ldots, x_l\}$. Let $x \in M_T \setminus \text{span}\{x_1, x_2, \ldots, x_l\}$, i.e., $x \in Q$. Define $H_x = \{y \in Y : Ty \perp_B y\}$. By smoothness of $Y$, $H_x$ is an unique closed hyperspace of $Y$. Hence $H_x$ is nowhere dense set of $Y$. Put $P_x = \{v \in Y : A_v x \in H_x\}$. Since $A_v x = c_{x,1}Tx_1 + c_{x,2}Tx_2 + \cdots + c_{x,l+1}v + c_{x,l+2}Tz_{l+2} + \cdots + c_{x,n}Tz_n$ and $c_{x,l+1} \neq 0$, we have

\[ P_x = \frac{1}{c_{x,l+1}} \left( H_x - (c_{x,1}Tx_1 + c_{x,2}Tx_2 + \cdots + c_{x,l+2}Tz_{l+2} + \cdots + c_{x,n}Tz_n) \right), \]

for each $x \in Q$. The set $P_x = \{v \in Y : Tx \perp_B A_v x\}$, being homeomorphic to $H_x$, is also nowhere dense. As $Q$ is countable, it follows from Baire category theorem that the non-empty open set $B(-Tx_{l+1}, \|T\|)$ contains an element $v$ such that $v \not\in P_x$ for each $x \in Q$. Thus we found $v \in B(-Tx_{l+1}, \|T\|)$ such that $T \perp_B A_v$ but $Tx \not\perp_B A_v x$ for all $x \in M_T$. Hence $T$ does not satisfy the BŠ Property.

\[ \square \]

**Remark 2.6.** We note that Theorem 2.5 improves on [7, Th. 2.3].

For linear operators between a polyhedral Banach space and a smooth Banach space, we have the following corollary.
Corollary 2.7. Let \( X \) be an n-dimensional polyhedral Banach space and let \( Y \) be any smooth Banach space. Let \( T \in \mathbb{L}(X, Y) \) be such that \( M_T \) satisfies the following conditions:

1. \( M_T \) has more than two components.
2. \( M_T \) has at most two non-singleton components \( \pm D_i \), for some \( i \in \mathbb{N} \). If \( D_i = -D_i \), then \( M_T \) has exactly one non-singleton component. All the other components of \( M_T \) are singleton.
3. \( M_T \) contains at least one pair of singleton components \( \pm D_j \), \( j(\neq i) \in \mathbb{N} \) such that \( \pm D_j \cap \text{span}\{D_i\} = \emptyset \).

Then \( T \) does not satisfy the BS Property.

Proof. We note that in a finite-dimensional polyhedral Banach space \( X \), \( B_X \) contains finitely many extreme points and each component of \( M_T \) contains extreme points of \( B_X \). Therefore, \( M_T \) has finitely many components. Thus \( M_T \) satisfies all the conditions of Theorem 2.5 and so, \( T \) does not satisfy the BS Property.

We next present one of the main results of the article that shows that \( X = \ell_1^n \) acts as universal domain space for the pair \((X, Y)\) to be a BS pair.

Theorem 2.8. Given any normed linear space \( Y \), the pair \((\ell_1^n, Y)\) is a BS pair.

Proof. If \( \dim Y = 1\) then given any \( T \in \mathbb{L}(\ell_1^n, Y) \), it is easy to check that \( M_T = D \cup (-D) \), where \( D \) is a connected subset of \( S_{\ell_1^n} \). From this it follows that the pair \((\ell_1^n, Y)\) is a BS pair. Let us assume that \( \dim Y = 1 \). Let \( T \in \mathbb{L}(\ell_1^n, Y) \) be such that \( M_T \neq D \cup (-D) \), where \( D \) is a connected subset of \( S_{\ell_1^n} \). We prove that \( T \) does not satisfy the BS Property. Since \( M_T \) is not of the form \( D \cup (-D) \), it must be of the form \( \bigcup_{i=1}^k D_i' \cup \bigcup_{j=1}^l E_j \), where \( D_i \) and \( E_j \) are components of \( M_T \), \( D_i' = D_i \cup (-D_i) \), \( E_j = -E_j \) and \( D_i \neq -D_i \). Let us now complete the proof of the theorem by considering the following three exhaustive cases.

Case I: \( l = 0 \). If \( k = 1 \), then \( M_T = D_1 \cup (-D_1) \), which contradicts our hypothesis. So we assume \( k \geq 2 \). Then \( M_T = \bigcup_{i=1}^k D_i' \). Since \( D_i \) is a connected subset of \( S_X \), where \( X = \ell_1^n \), an easy application of the Krein-Milman theorem shows that each \( D_i \) must contain at least one extreme point of \( B_X \). Let \( D_i \cap E_X = \{e_{i1}, e_{i2}, \ldots, e_{im_i}\} \), \( 1 \leq i \leq k \), i.e., \( |D_i \cap E_X| = m_i \). Clearly, \( \sum_{i=1}^k m_i \leq n \). Let us write \( X_1 = \text{span}\{e_{11}, e_{12}, \ldots, e_{im_1}\} \) and \( X_2 = \text{span}\{e_{21}, e_{22}, \ldots, e_{2m_2}, \ldots, e_{k1}, e_{k2}, \ldots, e_{km_k}\} \). It is easy to see that \( D_1 \cup (-D_1) = D_1' \subseteq X_1 \) and \( \bigcup_{i=2}^k D_i' \subseteq X_2 \). Thus we have \( M_T \subseteq X_1 \cup X_2 \) and moreover it is immediate that \( X_1 \cap X_2 = \{\theta\} \), as \( X = \ell_1^n \). Therefore, \( M_T \) is partitioned into two non-empty subsets \( Y_1 = M_T \cap X_1 \) and \( Y_2 = M_T \cap X_2 \) of \( X \), which are contained in complementary subspaces of \( X \). Then by [12, Prop. 2.1], we conclude that \( T \) does not satisfy the BS Property.
Case II: $k = 0$. If $l = 1$, then $M_T = E_1$, a contradiction to our hypothesis. So we assume $l \geq 2$. In this case $M_T = \bigcup_{i=1}^{l} E_j$. Proceeding similarly as in Case I, we can show that that $T$ does not satisfy BŠ property.

Case III: $l \geq 1, k \geq 1$. In this case $M_T = (\bigcup_{i=1}^{k} D'_i) \cup (\bigcup_{j=1}^{l} E_j)$ and once again proceeding as above, we conclude that $T$ does not satisfy the BŠ property.

Thus, in all the possible cases, we can conclude that $T \in \mathbb{L}(\ell_1^n, \mathbb{Y})$ satisfies the BŠ Property if and only if $M_T = D \cup (-D)$, where $D$ is a connected subset of $S_{\ell_1^n}$.

Remark 2.9. Observe that $\ell_1^n$ is the unique (upto isometric isomorphisms) $n$-dimensional Banach space having the minimum possible number of extreme points of its unit ball. This is the fundamental reason that the above theorem works exclusively for $\ell_1^n$ spaces.

In the next theorem, we obtain another class of BŠ pairs of Banach spaces, when the domain space is $\ell_3^{\infty}$. For the sake of convenience of the reader, we give the definition of adjacent edges of the unit sphere of a finite-dimensional polyhedral space.

Definition 2.10. Let $X$ be finite-dimensional polyhedral Banach space. Two edges $E_1, E_2$ of $S_X$ are said to be adjacent if $E_1 \cap E_2 = \{v\}$, where $v$ is an extreme point of $B_X$. Similarly, the edges $E_1, E_2, \ldots E_n$ are said to be adjacent if $E_1 \cap E_2 \cdots \cap E_n = \{v\}$. An extreme point of $B_X$ is also called a vertex of $S_X$. If $v$ is a vertex and $v \in E_1 \cap E_2 \cdots \cap E_n$, then we also say that the edges $E_1, E_2, \ldots E_n$ are adjacent to the vertex $v$.

Theorem 2.11. Given any strictly convex and smooth Banach space $Y$, the pair $(\ell_3^{\infty}, Y)$ is a BŠ pair.
Proof. Observe that $B_{\ell_\infty^2}$ has eight vertices $\pm v_1 = \pm (1, 1, 1), \pm v_2 = \pm (-1, 1, 1), \pm v_3 = (-1, -1, 1)$ and $\pm v_4 = \pm (1, -1, 1)$ and twelve edges $\pm E_{12} = \pm L[v_1, v_2], \pm E_{23} = \pm L[v_2, v_3], \pm E_{34} = \pm L[v_3, v_4], \pm E_{41} = \pm L[v_4, v_1], \pm E_{1(-3)} = \pm L[v_1, -v_3], \pm E_{2(-4)} = \pm L[v_2, -v_4]$. We prove that if $M_T$ is not of the form $D \cup (-D)$, where $D$ is a connected subset of $S_{\ell_\infty^2}$, then $T$ does not satisfy the BŠ Property. Given any $T \in L(\ell_\infty^2, \mathbb{Y})$, if $M_T$ is not of the form $D \cup (-D)$, then it is easy to see that $M_T$ must be one of the following forms:

(i) $M_T$ contains exactly two pairs of vertices of $B_{\ell_\infty^2}$ and no other points of $B_{\ell_\infty^2}$.

(ii) $M_T$ contains exactly three pairs of vertices of $B_{\ell_\infty^2}$ and no other points of $B_{\ell_\infty^2}$.

(iii) $M_T$ contains exactly four pairs of vertices of $B_{\ell_\infty^2}$ and no other points of $B_{\ell_\infty^2}$.

(iv) $M_T$ contains exactly one pair of vertices and exactly one pair of edges of $B_{\ell_\infty^2}$ such that the vertices do not belong to the concerned edges. $M_T$ contains no other points of $B_{\ell_\infty^2}$.

(v) $M_T$ contains exactly two pairs of vertices and exactly one pair of edges of $B_{\ell_\infty^2}$ such that the vertices do not belong to any of the concerned edges. $M_T$ contains no other points of $B_{\ell_\infty^2}$.

(vi) $M_T$ contains exactly two pairs of non-adjacent edges of $B_{\ell_\infty^2}$ and no other points of $B_{\ell_\infty^2}$.

If $M_T$ is of the form described in either of the Cases (i), (ii), (iii), then $T$ does not satisfy the BŠ Property and the proof of it follows directly from [7, Th. 2.3]. Also, if $M_T$ is of the form described in either of the Cases (iv), (v), then $T$ does not satisfy the BŠ Property and the proof of it follows directly from Corollary 2.7. Here we only consider the Case (vi). Without loss of generality, we may assume that $M_T = \pm E_{12} \cup \pm E_{34}$, for some $T \in L(\ell_\infty^2, \mathbb{Y})$. As $\mathbb{Y}$ is strictly convex, we must have $T(\pm E_{12}) = \pm u_1, T(\pm E_{34}) = \pm u_2$, where $u_1, u_2 \in \mathbb{Y}$ are linearly independent and $\|u_1\| = \|u_2\|$. Now we define a linear operator $A : \ell_\infty^2 \to \mathbb{Y}$ by $A(\pm E_{12}) = T(\pm E_{12}) = \pm u_1, A(\pm E_{34}) = -T(\pm v_3) = \mp u_2$. Then $A(\pm E_{34}) = -T(\pm E_{34}) = \mp u_2$, since $v_4 = v_1 - v_2 + v_3$. Here $Av_1 \in (Tv_1)^+$ and $Av_3 \in (Tv_3)^-$. Therefore, using [8, Th. 2.2] we get $T \perp_B A$. From the construction of the operator $A$, it is clear that $Tx \not\in_B Ax$, for all $x \in M_T$. This completes the proof of the theorem.

In order to obtain further examples of BŠ pairs of polyhedral Banach spaces $(X, \mathbb{Y})$, we require the following lemma.

Lemma 2.12. Let $X$ be any finite-dimensional Banach space and let $\mathbb{Y}$ be any polyhedral Banach space such that $B_{\mathbb{Y}}$ has exactly $2m$ facets. Let $T \in L(X, \mathbb{Y})$ be such that $M_T$ is not of the form $D \cup (-D)$, where $D$ is a connected subset of $S_X$. Then $M_T$ can have at most $2m$ components.
Proof. Suppose on the contrary that $M_T$ has $(2m + 1)$ components, say $D_1, D_2, \ldots, D_{2m+1}$. Let us consider a subset $\{x_1, x_2, \ldots, x_{2m+1}\}$ of $M_T$, where $x_i \in D_i$ for $i = 1, 2, \ldots, 2m+1$. Let $Tx_i \in F$, where $F$ is a facet of $B_Y$. Then we must have $Tx_i \notin F$ for $i \in \{2, 3, \ldots, 2m+1\}$. If not then, $(1-\lambda)x_1 + \lambda x_i \in M_T$, for all $\lambda \in [0,1]$, contradicting the fact that $D_1$ and $D_i (i > 1)$ are distinct components of $M_T$. Thus $Tx_i$ and $Tx_j \ (i \neq j)$ can not belong to the same facet of $B_Y$. Let us denote the facets of $B_Y$ as $F_1, F_2, \ldots, F_{2m}$ such that $Tx_i \in F_i$ for $i = 1, 2, \ldots, 2m$. Thus $Tx_{2m+1}$ can not belong to any facet of $B_Y$, which is a contradiction to the fact that $x_{2m+1} \in M_T$. This completes the proof of the lemma.

Using the above lemma, we obtain the following theorem:

**Theorem 2.13.** $(\ell_\infty^3, \ell_\infty^2)$ is a BS̃ pair.

**Proof.** $B_{\ell_\infty^3}$ has eight vertices $\pm v_1 = \pm(1,1,1)$, $\pm v_2 = \pm(-1,1,1)$, $\pm v_3 = (-1,-1,1)$ and $\pm v_4 = \pm(1,-1,1)$ and twelve edges $\pm E_{12} = \pm L[v_1,v_2], \pm E_{23} = \pm L[v_2,v_3], \pm E_{34} = \pm L[v_3,v_4], \pm E_{41} = \pm L[v_4,v_1], \pm E_{1(-3)} = \pm L[v_1,-v_3], \pm E_{2(-4)} = \pm L[v_2,-v_4]$. Let $T \in \mathbb{L}(\ell_\infty^3, \ell_\infty^2)$ be such that $M_T$ is not of the form $D \cup (-D)$, where $D$ is a connected subset of $S_\infty$. Then by Lemma 2.12, $M_T$ must be one of the following forms:

(i) $M_T$ contains exactly two pairs of vertices of $B_{\ell_\infty^3}$ and no other points of $B_{\ell_\infty^3}$.

(ii) $M_T$ contains exactly one pair of edges and exactly one pair of vertices of $B_{\ell_\infty^3}$ such that the concerned vertices do not belong to the concerned edges. $M_T$ contains no other points of $B_{\ell_\infty^3}$.

(iii) $M_T$ contains exactly two pairs of edges of $B_{\ell_\infty^3}$ and no other points of $B_{\ell_\infty^3}$. If $M_T$ is of the form described in either of the Cases (i) and (ii), then $T$ does not satisfy the BS̃ Property and the proof of it follows directly from [12, Prop. 2.1]. We only consider the case (iii) in which $M_T$ contains exactly two pairs of edges of $B_{\ell_\infty^3}$. Without loss of generality, we may and do assume that $M_T = \pm E_{12} \cup \pm E_{34}$. Clearly, we have $T(M_T) \cap E_{\ell_\infty^2} = \phi$ and hence $Tx \in Sm(S_{\ell_\infty^3})$ for any $x \in M_T$, where $Sm(S_{\ell_\infty^3})$ denotes the collection of all smooth points of $S_{\ell_\infty^3}$. As $E_{12} \subseteq M_T$ and $E_{34} \subseteq M_T$, we have $Tv_1$ and $Tv_2$ belong to the same edge of $B_{\ell_\infty^3}$ and also $Tv_3$ and $Tv_4$ belong to the same edge of $B_{\ell_\infty^3}$. Let us define an operator $A \in \mathbb{L}(\ell_\infty^3, \ell_\infty^2)$ as follows:

$$Av_1 = Tv_1, Av_2 = Tv_1, Av_3 = -Tv_3.$$ 

Clearly, $Av_1 \in (Tv_1)^+$ and $Av_3 \in (Tv_3)^-$. Therefore, using [8, Th. 2.2], we get that $T \perp_B A$. Now, we have $A(E_{12}) = Tv_1$ and $A(E_{34}) = -Tv_3$, as $v_4 = v_1 - v_2 + v_3$. Then it is easy to check that $Tu \not\perp_B Au$ for any $u \in M_T$. Hence $T$ does not satisfy the BS̃ Property.

Therefore, the pair $(\ell_\infty^3, \ell_\infty^2)$ is a BS̃ pair.
Using similar arguments, we can also prove the next result, the proof of which is omitted as it follows in similar spirit to the above theorem.

**Theorem 2.14.** $(\ell^3_\infty, \ell^3_\infty)$ is a BŠ pair.

**Remark 2.15.** Given $m, n \in \mathbb{N}$ such that $m, n > 3$, it is still unknown whether the pair $(\ell^m_\infty, \ell^n_\infty)$ is a BŠ pair.

We would like to end the article with a related result that characterizes the unit cube among all 3-dimensional convex polyhedrons having eight vertices. This is of independent interest and illustrates the connection between operator norm attainment and geometries of the domain space and the co-domain space.

**Theorem 2.16.** Let $\mathcal{X}$ be a three-dimensional polyhedral Banach space such that $B_{\mathcal{X}}$ has exactly eight vertices. Then $\mathcal{X}$ is isometrically isomorphic with $\ell^3_\infty$ if and only if given any strictly convex Banach space $\mathcal{Y}$ and given any two pairs of non-adjacent edges $\pm E_1$ and $\pm E_2$ of $S_{\mathcal{X}}$, there is a rank two linear operator $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ such that $M_T = \pm E_1 \cup \pm E_2$.

**Proof.** Let us first prove the necessary part of the theorem. We use the notations for the vertices and the edges of $B_{\ell^3_\infty}$ as used in Theorem 2.11 (see Figure 2.1). Now for any two pairs of non-adjacent edges of $B_{\ell^3_\infty}$, the following three cases may arise:

(i) $\pm E_{12} = \pm L[v_1, v_2]$ and $\pm E_{34} = \pm L[v_3, v_4].$

(ii) $\pm E_{14} = \pm L[v_1, v_4]$ and $\pm E_{23} = \pm L[v_2, v_3].$

(iii) $\pm E_{1(-3)} = \pm L[v_1, -v_3]$ and $\pm E_{2(-4)} = \pm L[v_2, -v_4].$

We only consider the Case (i), as the other two cases will follow similarly.

Define a linear operator $T: \ell^3_\infty \to \mathcal{Y}$ by $T(v_1) = T(v_2) = u_1, T(v_3) = u_2$, where $u_1, u_2 \in S_{\mathcal{Y}}$ are linearly independent. So, we have $T(\pm E_{12}) = \pm u_1$ and $T(\pm E_{34}) = \pm u_2$. Hence it can be easily shown that $\|T\| = 1, M_T = \pm E_{12} \cup \pm E_{34}$ and $T$ is a rank two linear operator.

Next we prove the sufficient part of the theorem. Let $\pm v_1, \pm v_2, \pm v_3, \pm v_4$ be the eight vertices of $B_{\mathcal{X}}$. Using the Krein-Milman theorem, we conclude that the set $\{v_1, v_2, v_3, v_4\}$ contains a basis of $\mathcal{X}$. Therefore, the following two cases arise in this context:

**Case (i)** Any three elements of the set $\{v_1, v_2, v_3, v_4\}$ are linearly independent.

**Case (ii)** Case (i) is not satisfied.

**Case (i):** Let $\{v_1, v_2, v_3\}$ be a basis of $\mathcal{X}$ and let $v_4 = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$, where $\alpha_i \in \mathbb{R}, i = 1, 2, 3$. Then each $\alpha_i$ is non-zero, as any three elements of the set $\{v_1, v_2, v_3, v_4\}$ are linearly independent. In a three-dimensional polyhedral Banach space every vertex has at least two adjacent edges. Let $\pm E_1 = \pm L[v_1, v_2]$ and $\pm E_2 = \pm L[v_3, v_4].$ Let $T_1$ be a rank two linear operator such that $M_{T_1} = \pm E_1 \cup \pm E_2.$ As $\mathcal{Y}$ is a strictly convex Banach space and $\pm E_1 \subseteq M_{T_1}, T_1(\pm E_1) = \pm u_1$ for some non-zero $u_1 \in \mathcal{Y}$. Also $T_1(\pm E_2) = \pm u_2,$
for some non-zero \( u_2 \in \mathbb{Y} \). As \( T_1 \) is of rank two, \( u_1, u_2 \) are linearly independent. As \( v_4 = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 \), we have \( u_2 = T_1(v_4) = (\alpha_1 + \alpha_2)u_1 + \alpha_3 u_2 \). Therefore, \( (1 - \alpha_3)u_2 = (\alpha_1 + \alpha_2)u_1 \). So we must have \( \alpha_1 = -\alpha_2 \) and \( \alpha_3 = 1 \), as \( u_1, u_2 \) are linearly independent. Hence \( v_4 = \alpha_1(v_1 - v_2) + v_3 \). Without loss of generality, we assume that \( L[v_1, v_3], L[v_2, v_4] \) are two edges of \( B_X \). Now for this two pairs of non-adjacent edges \( \pm E_1' = \pm L[v_1, v_3] \) and \( \pm E_2' = \pm L[v_2, v_4] \), there is a rank two linear operator \( T_2 \in \mathbb{L}(X, Y) \) such that \( M_{T_2} = \pm E_1' \cup \pm E_2' \).

In a similar argument like above, we have \( T_2(\pm E_1') = u_1' \) for some non-zero \( u_1' \in \mathbb{Y} \) and \( T_2(\pm E_2') = u_2' \) for some non-zero \( u_2' \in \mathbb{Y} \). Therefore, we have \( u_2' = T_2(v_4) = \alpha_1 u_1' - \alpha_1 u_2' + u_1' \). So \( \alpha_1 = -1 \), as \( u_1', u_2' \) are linearly independent. Hence \( v_4 = -v_1 + v_2 + v_3 \). Now we define a linear map \( S: X \to \ell_\infty^2 \) by

\[
S(v_1) = (1, 1, 1), \quad S(v_2) = (-1, 1, 1), \quad S(v_3) = (1, 1, -1).
\]

Then \( S(v_4) = (-1, 1, -1) \). Clearly, \( S \) is an isomorphism, as it maps a basis to a basis and \( \|S\| = 1 \), as \( S(E_X) = E_{\ell_\infty^2} \). Also \( \|S^{-1}\| = 1 \), as \( S^{-1}(E_{\ell_\infty^2}) = E_X \). Thus we have

\[
\|x\| = \|S^{-1}S(x)\| \leq \|S(x)\| \leq \|x\|.
\]

Therefore, \( S \) is an isometric isomorphism.

**Case (ii):** Without loss of generality, we assume that \( v_1, v_2, v_3 \) are linearly dependent. Consider the subspace spanned by \( v_1, v_2 \). Let \( Z = \text{span}\{v_1, v_2\} \). Then \( \dim Z = 2 \) and \( \pm v_4 \notin Z \), otherwise \( \dim X = 2 \), which is a contradiction. Now, \( \pm v_1, \pm v_2, \pm v_3 \in S_X \cap Z \). Therefore, \( B_X \) is of the form of hexagonal pyramid, where \( \pm v_1, \pm v_2, \pm v_3 \) are the vertices of the hexagonal base. Now for any two edges \( E_1 \) and \( E_2 \) on the hexagonal base, either \( E_1 \) and \( E_2 \) are adjacent or \( -E_1 \) and \( -E_2 \) are adjacent. Therefore, two pairs of non-adjacent edges \( \pm E_1 \) and \( \pm E_2 \) are not possible from hexagonal base of \( B_X \). Also any two edges, which are not in the subspace \( Z \), are adjacent as they have a common vertex, either \( v_4 \) or \(-v_4 \). Hence there is only one possibility for two pairs of non-adjacent edges, one edge is in the two dimensional subspace \( Z \) and one edge is not in the two dimensional subspace \( Z \). Without loss of generality, we assume \( \pm E_1 = \pm L[v_1, v_2] \) and \( \pm E_2 = \pm L[v_3, v_4] \). Now we claim that for any operator \( T \in \mathbb{L}(X, Y) \), with \( \pm E_1 \cup \pm E_2 \subseteq M_T \), \( T \) is a rank one linear operator. As \( Y \) is strictly convex and \( \pm E_1 \subseteq M_T \), we must have \( T(\pm E_1) = \pm u_1 \) for some non-zero \( u_1 \in Y \). Also for similar reason \( T(\pm E_2) = \pm u_2 \) for some non-zero \( u_2 \in Y \). As \( v_1, v_2, v_3 \) are linearly dependent, we have \( v_3 = \alpha_1v_1 + \alpha_2v_2 \), for some non-zero \( \alpha_1, \alpha_2 \in \mathbb{R} \). Then \( T(v_3) = (\alpha_1 + \alpha_2)u_1 \). As \( \|Tv_3\| = \|T\| = \|u_1\| \), we must have \( |\alpha_1 + \alpha_2| = 1 \). Hence \( u_2 = \pm u_1 \). Therefore, \( T \) is a rank one linear operator. So there do not exist any rank two linear operator \( T \in \mathbb{L}(X, Y) \) such that \( M_T = \pm E_1 \cup \pm E_2 \), where \( \pm E_1 \) and \( \pm E_2 \) are any two pairs of non-adjacent edges and hence Case-(ii) is not possible. This establishes the theorem.
Acknowledgments. The research of Dr. Debmalya Sain is sponsored by DST-SERB N-PDF Fellowship under the mentorship of Professor Apporva Khare. Dr. Sain feels elated to acknowledge the tremendous positive impact of his childhood friend Dr. Prateep Phadikar, a renowned physician in Kolkata, in various aspects of his life!

References

[1] R. Bhatia and P. Šemrl, Orthogonality of matrices and some distance problems, *Linear Algebra Appl.*, 287 (1999), no. 1–3, 77–85.
[2] R. C. James, Orthogonality and linear functionals in normed linear spaces, *Trans. Amer. Math. Soc.*, 61 (1947), 265–292.
[3] C. K. Li and H. Schneider, Orthogonality of matrices, *Linear Algebra Appl.*, 347 (2002), 115–122.
[4] K. Paul, Translatable radii of an operator in the direction of another operator, *Sci. Math.*, 2 (1999), no. 1, 119–122.
[5] K. Paul and D. Sain, Orthogonality of operators on \((\mathbb{R}^n, \| \cdot \|_{\infty})\), *Novi Sad J. Math.*, 43 (2013), 121–129.
[6] K. Paul, Sk. M. Hossein and K. C. Das, Orthogonality on \(B(H,H)\) and Minimal-norm Operator, *J. Anal. Appl.*, 6 (2008), no. 3, 169–178.
[7] A. Ray, D. Sain, S. Dey and K. Paul, Some remarks on orthogonality of bounded linear operators, *J. Convex Anal.*, 29 (2022), no. 1, 165–181.
[8] D. Sain, Birkhoff-James orthogonality of linear operators on finite dimensional Banach spaces, *J. Math. Anal. Appl.*, 447 (2017), 860–866.
[9] D. Sain, On the norm attainment set of a bounded linear operator, *J. Math. Anal. Appl.*, 457 (2018), 67–76.
[10] D. Sain and K. Paul, Operator norm attainment and inner product spaces, *Linear Algebra Appl.*, 439 (2013), no. 8, 2448–2452.
[11] D. Sain, K. Paul, P. Bhunia and S. Bag, On the numerical index of polyhedral Banach spaces, *Linear Algebra Appl.*, 577 (2019), 121–133.
[12] D. Sain, K. Paul and S. Hait, Operator norm attainment and Birkhoff-James orthogonality, *Linear Algebra Appl.*, 476 (2015), 85–97.

D. Sain, Department of Mathematics, Indian Institute of Science, Bengaluru 560012, Karnataka, India; e-mail: saindebmalya@gmail.com

A. Ray, Assistant Professor, Department of Basic Science & Humanities (Mathematics), Institute of Engineering & Management, Kolkata 700091, West Bengal, India; e-mail: anubhab.jumath@gmail.com

S. Dey, Department of Mathematics, Muralidhar Girls’ College, Kolkata 700029, West Bengal, India; e-mail: subhrajitdeyjumath@gmail.com

K. Paul, Department of Mathematics, Jadavpur University, Kolkata 700032, West Bengal, India; e-mail: kalloldada@gmail.com
Springer Nature or its licensor holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.