Gröbner bases and syzygies on bimodules

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Abstract

A new more efficient method for the computation of two-sided Gröbner bases of ideals and bimodules shifting the problem to the enveloping algebra is proposed. Arising from the ideas this method involves, we introduce the notion of two-sided syzygy, which reveals to be useful in the computation of the intersection of bimodules. Further applications are left for a sequel.

1 Introduction

Though first developed in the ring of polynomials, the methods based on Gröbner bases also work in some noncommutative rings, e.g. the Weyl algebras or, more generally, the so-called Poincaré-Birkhoff-Witt rings (PBW, for short), including some classical quantum groups. After first results in the Weyl algebra (Galligo, 1982) and in tensor algebras of finite-dimensional Lie algebras (Apel and Lassner, 1985), Kandri-Rodi and Weispfenning were the first to introduce Gröbner bases in the more general class of algebras where the degree of a skew-commutator $p_{ij} = x_j x_i - c_{ij} x_i x_j$ is bounded by the degree of the product of generators $x_i x_j$, for $1 \leq i < j \leq n$ (see Kandri-Rody and Weispfenning, 1990). The theory has recently been surveyed in Bueso et al. (1998). Algorithms to compute the Gelfand-Kirillov dimension, to check whether a two-sided ideal is prime or not and to compute the projective dimension of a module have also been developed (see Bueso et al., 1996; Lobillo, 1998; Bueso et al., 1999; Gago-Vargas, 2003).

On these generalizations, authors were mainly interested in one-sided ideals and modules, whereas methods for the two-sided counterparts are merely patches in order to cope with the two-sided input data (cf. Pesch, 1998; Bueso et al., 2003).

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In this note we show that those mends are not necessary, due to the very well known fact that two-sided ideals and bimodules may be seen as left modules on the enveloping algebra. First, we show that the enveloping algebra of a PBW algebra is another PBW algebra. Second, we find a method to shift the data back and forth through the morphism

\[ m^s : (R_{\text{env}})^s \to R^s ; \ (f_i \otimes g_i)^s_{i=1} \mapsto (f_i g_i)^s_{i=1} \]

in order to carry out the computations on the enveloping algebra using one-sided techniques.

This philosophy allows, for example, to compute Gröbner bases for bimodules with only one call to the left Buchberger algorithm, instead of the a priori unknown number of calls typical of the aforementioned methods.

The techniques we use have led us to study the syzygy bimodule, which is the two-sided counterpart of the left syzygy module. Amongst its applications, we show that it can be used in the computation of intersections of bimodules when one starts, as usual, from two-sided input data.

Throughout this paper we will use the following notation. We will denote by \( \epsilon_i \) the element \((0, ..., \hat{1}, ..., 0) \in \mathbb{N}^n \). The symbol \( x^\alpha \) will denote the monomial \( x_1^{\alpha_1} \cdots x_n^{\alpha_n} \) in the free algebra \( k\langle x_1, \ldots, x_n \rangle \) or in any of its epimorphic images, \( k \) being a field. \( R \) will be a PBW \( k \)-algebra, \( R^{\text{op}} \) its opposite algebra and \( R_{\text{env}} \) its enveloping algebra \( R \otimes_k R^{\text{op}} \). Finally, for any subset \( F \) of the free left \( R \)-module \( R^s \), we will denote by \( R\langle F \rangle \), resp. \( R\langle F \rangle_R \), the left \( R \)-module, resp. the \( R \)-bimodule, generated by \( F \).

The computations of the examples shown in this paper were done using a library of procedures built by the authors using the package of symbolic computation Maple 6. The computation times correspond to a Pentium III 700 Mhz personal computer with 256 Mb RAM.

2 The enveloping algebra of a PBW algebra

A subset \( Q = \{ x_j x_i - q_{ji} x_i x_j - p_{ji} ; \ 1 \leq i < j \leq n \} \) of the free algebra \( k\langle x_1, \ldots, x_n \rangle \) is a set of quantum relations bounded by the admissible order “\( \preceq \)” on \( \mathbb{N}^n \) if \( q_{ji} \in k^* \) and \( p_{ji} \) is a finite \( k \)-linear combination of standard monomials \( x^\alpha (\alpha \in \mathbb{N}^n) \) such that \( \exp (p_{ji}) < \epsilon_i + \epsilon_j \), for all \( i < j \), where \( \exp (f) \) denotes the exponent of the leading term of the element \( f \). As may be found in the literature, (see e.g. Kandri-Rody and Weispfenning, 1990; Bueso et.al., 1998, 2003), a Poincaré-Birkhoff-Witt algebra (PBW algebra, for short) is a \( k \)-algebra \( R \) where the set of standard monomials \( \{ x^\alpha ; \alpha \in \mathbb{N}^n \} \) is a \( k \)-basis and
such that there exists a set of quantum relations $Q$ bounded by an admissible order "$\preceq$" satisfying

$$R = \frac{k\langle x_1, \ldots, x_n \rangle}{k\langle x_1, \ldots, x_n \rangle \langle Q \rangle k\langle x_1, \ldots, x_n \rangle}.$$

This algebra is usually denoted by $k\{x_1, \ldots, x_n; Q, \preceq\}$.

Amongst the examples of PBW algebras, we find the commutative polynomial ring $k[x_1, \ldots, x_n]$, some iterated Ore extensions or a pretty large class of quantum groups just as the multiparameter $n$-dimensional quantum space $O_q(\mathbb{A}^n)$, the bialgebra of quantum matrices $M_q(2)$, the Weyl algebra $A_n(k)$, the enveloping algebra of traceless matrices $U(\mathfrak{sl}(2))$, etc.

The tensor product of PBW algebras is a new PBW algebra:

**Proposition 1** If $R = k\{x_1, \ldots, x_m; Q_R, \preceq_R\}$ and $S = k\{y_1, \ldots, y_n; Q_S, \preceq_S\}$ are PBW algebras with quantum relations

$$Q_R = \{x_j^i x_i - q_j^i x_i x_j - p_{ji}; 1 \leq i < j \leq m\},$$

$$Q_S = \{y_j^i y_i - q_j^i y_i y_j - p_{ji}'; 1 \leq i < j \leq n\},$$

then $R \otimes_k S$ is the PBW algebra denoted by

$$k\{x_1 \otimes 1, \ldots, x_m \otimes 1, 1 \otimes y_1, \ldots, 1 \otimes y_n; Q, \preceq\},$$

where

$$Q = \begin{cases} 
(x_j \otimes 1)(x_i \otimes 1) - q_{ji}(x_i \otimes 1)(x_j \otimes 1) - p_{ji} \otimes 1; & 1 \leq i < j \leq m \\
(1 \otimes y_j)(x_i \otimes 1) - (x_i \otimes 1)(1 \otimes y_j); & 1 \leq i \leq m, 1 \leq j \leq n \\
(1 \otimes y_j)(1 \otimes y_i) - q_{ji}'(1 \otimes y_i)(1 \otimes y_j) - 1 \otimes p_{ji}'; & 1 \leq i < j \leq n. 
\end{cases}$$

and "$\preceq$" is one amongst the elimination orders (as defined, e.g., in [Cox et al., 1994]) arising from "$\preceq_R$" and "$\preceq_S$".

As a first example, note that $A_{n+m}(k)$ is the PBW algebra $A_n(k) \otimes A_m(k)$ constructed in the proposition. Another example of this construction is the enveloping algebra $R^\text{env}$ of $R = k\{x_1, \ldots, x_n; Q, \preceq\}$. Before we describe it, let us define the composition orders. For any $\alpha = (\alpha_1, \ldots, \alpha_1) \in \mathbb{N}^n$, denote by $\alpha^\text{op}$ the $n$-tuple $(\alpha_n, \ldots, \alpha_1)$.

**Definition 2** Let "$\preceq$" be an order on $\mathbb{N}^n$. The up-component composition
order in \( \mathbb{N}^{2n} \), denoted “\( \preceq_c \)”, is defined by

\[
(\alpha, \beta) \preceq_c (\gamma, \delta) \Leftrightarrow \begin{cases} 
\alpha + \beta^{\text{op}} < \gamma + \delta^{\text{op}}, \text{ or } \\
\alpha + \beta^{\text{op}} = \gamma + \delta^{\text{op}} \text{ and } \beta^{\text{op}} < \delta^{\text{op}}
\end{cases}
\]

The down-component composition order “\( \preceq_c \)” is defined by

\[
(\alpha, \beta) \preceq_c (\gamma, \delta) \Leftrightarrow \begin{cases} 
\alpha + \beta^{\text{op}} < \gamma + \delta^{\text{op}}, \text{ or } \\
\alpha + \beta^{\text{op}} = \gamma + \delta^{\text{op}} \text{ and } \alpha < \gamma
\end{cases}
\]

If “\( \preceq \)” is an admissible order on \( \mathbb{N}^n \), then both composition orders “\( \preceq_c \)” and “\( \preceq_c \)” are admissible orders on \( \mathbb{N}^{2n} \).

Note that the opposite algebra \( R^{\text{op}} \) is the PBW algebra \( k\{x_n, \ldots, x_1; Q^{\text{op}}, \preceq^{\text{op}} \} \), where the elements of \( Q^{\text{op}} \) are those of \( Q \) written oppositely and “\( \preceq^{\text{op}} \)” is the order in \( \mathbb{N}^n \) given by \( \alpha \preceq^{\text{op}} \beta \iff \alpha^{\text{op}} \preceq \beta^{\text{op}} \).

**Proposition 3** If \( R = k\{x_1, \ldots, x_n; Q, \preceq \} \) is a PBW algebra with quantum relations \( Q = \{x_jx_i - q_{ji}x_i x_j - p_{ji}; 1 \leq i < j \leq n\} \), then \( R^{\text{env}} \) is the PBW algebra \( k\{x_1 \otimes 1, \ldots, x_n \otimes 1, 1 \otimes x_n, \ldots, 1 \otimes x_1; Q^*, \preceq\} \), where

\[
Q^* = \left\{ \begin{array}{ll}
(x_j \otimes 1)(x_i \otimes 1) - q_{ji}(x_i \otimes 1)(x_j \otimes 1) - p_{ji} \otimes 1; & 1 \leq i < j \leq n \\
(1 \otimes x_j)(x_i \otimes 1) - (x_i \otimes 1)(1 \otimes x_j); & 1 \leq i, j \leq n \\
(1 \otimes x_i)(1 \otimes x_j) - q_{ji}(1 \otimes x_j)(1 \otimes x_i) - 1 \otimes p_{ji}; & 1 \leq i < j \leq n.
\end{array} \right\
\]

and “\( \preceq \)” is, either any of the elimination orders “\( \preceq^* \)” or “\( \preceq_s \)” in \( \mathbb{N}^{2n} \) corresponding to “\( \preceq \)” and “\( \preceq^{\text{op}} \)” or any of the composition orders “\( \preceq_c \)” or “\( \preceq_c \)” on \( \mathbb{N}^{2n} \) corresponding to “\( \preceq \)”.

In what follows, we will work on the free \( R \)-bimodule \( R^s \), being \( s \) a positive integer and \( R \) a PBW algebra, and we will use the \( R \)-module basis \( \{e_i\}_{i=1}^s \) consisting of \( e_i = (0, \ldots, 0, \underbrace{1}_{i \text{ position}}, \ldots, 0) \in R^s \) for all \( 1 \leq i \leq s \).

The notion and some applications of left Gröbner bases in PBW algebras and left modules may be found, e.g., in [Bueso et al. (2003)]. For convenience, just recall that if \( M \subset R^s \) is an \( R \)-bimodule, then \( G = \{g_1, \ldots, g_r\} \subset M \setminus \{0\} \) is a two-sided Gröbner basis if one of the following equivalent statements holds:

1. \( M = R\langle G \rangle_R \) and \( \text{Exp}(M) = \bigcup_{k=1}^r (\mathbb{N}^n + \text{exp}(g_k)) \);
2. \( G \) is a left Gröbner basis and \( M = R\langle G \rangle_R = R\langle G \rangle \).
(3) $G$ is a left Gröbner basis, $M = \langle G \rangle_R$ and $g_{k} x_{i} \in \langle G \rangle$, for all $k \in \{1, \ldots, r\}$ and $i \in \{1, \ldots, n\}$.

A set $G \subset R^s$ is said to be a two-sided Gröbner basis if so it is for the $R$-bimodule $\langle G \rangle_R$.

3 Computing two-sided Gröbner Bases

This section is devoted to the methods for computation of two-sided Gröbner bases for $R$-subbimodules of $R^s$. We denote by $f \otimes g$ the element $(f_1 \otimes g_1, \ldots, f_s \otimes g_s) \in (R_{env}^s)$, where $f = (f_1, \ldots, f_s), g = (g_1, \ldots, g_s) \in R^s$.

As a consequence of the third characterization of two-sided Gröbner bases above, some authors have proposed an algorithm to compute them (see Bueso et.al., 2003). Alternatively, we propose a new algorithm which improves that one, since it calls only once the left Buchberger Algorithm, although it uses more variables and input elements. The philosophy is to transform the problem into computing a left Gröbner basis in the free module $(R_{env}^s)$. This may be done since, just as we saw in the previous section, $R_{env}$ has a PBW structure.

It is known that $R$-bimodules are exactly left $R_{env}$-modules. Notice that, in particular, the free module $R^s$ is a left $R_{env}$-module with the action $(r \otimes r') f = (rf_1 r', \ldots, rf_s r')$, and $(R_{env}^s)$ possesses an $R$-bimodule structure whose multiplications are given by $r (f \otimes g) r' = (rf_1 \otimes g_1 r', \ldots, rf_s \otimes g_s r')$, where $r, r' \in R$ and $f = (f_1, \ldots, f_s), g = (g_1, \ldots, g_s) \in R^s$.

Likewise, the map $m^s = m \times \cdots \times m : (R_{env}^s) \to R^s$, where $m(r \otimes r') = rr'$, for $r, r' \in R$, is an epimorphism of left $R_{env}$-modules. Thus there exists a bijection

$$\{ N \subseteq (R_{env}^s); \text{Ker}(m^s) \subseteq N \in R_{env}^s \text{-Mod} \} \to \{ M \subseteq R^s; M \in R\text{-Bimod} \}$$

$$N \to M_N := m^s(N),$$

$$N_M := (m^s)^{-1}(M) \leftrightarrow M$$

Using this bijection, for each $R$-bimodule $M \subset R^s$, we have a left $R_{env}$-module $N_M \subset (R_{env}^s)$. Moreover, from a finite generator system for $M$ it is possible to obtain one for $N_M$, just as the following results show.

**Lemma 4** Let $R$ be a $k$-algebra.

(1) If $M = R(f_1, \ldots, f_t)_R \subset R^s$, then $N_M = R_{env}(f_1 \otimes 1, \ldots, f_t \otimes 1) + \text{Ker}(m^s)$;

(2) $\text{Ker}(m^s) = R_{env}(f \otimes 1 - 1 \otimes f; f \in R^s)$;
Corollary 5

If \( R = k\{x_1, ..., x_n; Q; \preceq\} \) is a PBW algebra, then \( R_{\text{rev}}(f \otimes 1 - 1 \otimes f; f \in R^s) = R_{\text{rev}}(x^{(e_j,k)} \otimes 1 - 1 \otimes x^{(e_j,k)}; 1 \leq j \leq n, 1 \leq k \leq s) \).

Given an admissible order in \( \mathbb{N}^n \), from here on we will call TOP (term over position), resp. POT (position over term) the orders in \( \mathbb{N}^{n(s)} \) given by

\[
(\alpha, i) \prec (\beta, j) \iff \begin{cases} \alpha \prec \beta, & \text{or} \quad i > j, & \text{resp.} \quad \alpha = \beta \text{ and } i > j, \text{ or } \alpha = \beta \text{ and } i < j \end{cases}
\]

Lemma 6

Let \( R = k\{x_1, ..., x_n; Q; \preceq\} \) be a PBW algebra and consider the order TOP (or POT) on both \( R^s \) and \((R_{\text{rev}})^s\).

- Taking \( \preceq^* \) or \( \preceq^c \) on \( R_{\text{rev}}^s \), if \( h \in (R_{\text{rev}})^s \) is such that \( \exp_{(R_{\text{rev}})^s}(h) = ((\alpha, 0), i) \in \mathbb{N}^{2n(s)} \), then \( h \not\in \ker(m^s) \) and \( \exp_{R^s}(m^s(h)) = (\alpha, i) \);
- Taking \( \preceq_* \) or \( \preceq_* \) on \( R_{\text{rev}}^s \), if \( h \in (R_{\text{rev}})^s \) is such that \( \exp_{(R_{\text{rev}})^s}(h) = ((0, \alpha), i) \in \mathbb{N}^{2n(s)} \), then \( h \not\in \ker(m^s) \) and \( \exp_{R^s}(m^s(h)) = (\alpha, i) \).

Using these results we have:

Theorem 7

Let \( R = k\{x_1, ..., x_n; Q; \preceq\} \) be a PBW algebra, \( M \subset R^s \) be an \( R \)-bimodule and consider in \( R_{\text{rev}}^s \) the PBW structure given in the proposition 3 (where the order is one of \( \preceq^*, \preceq^c, \preceq_* \) or \( \preceq_* \)).

If \( G \) is a left Gröbner basis for \( N_M = (m^s)^{-1}(M) \) with TOP (resp. POT), then the set \( m^s(G) \setminus \{0\} \) is a two-sided Gröbner basis for \( M \) with TOP (resp. POT).

The theorem provides a method of construction of two-sided Gröbner bases for bimodules. For convenience we write explicitly under the name of Algorithm 1.

Algorithm 1

Two-sided Gröbner bases

Require: \( F = \{f_1, ..., f_t\} \subseteq R^s \setminus \{0\} \);
Ensure: \( G = \{g_1, ..., g_{\nu'}\} \), a two-sided Gröbner basis for \( R\langle F \rangle_R \) such that \( F \subseteq G \);

INITIALIZATION: \( B := \{f_i \otimes 1\}_{i=1}^t \cup \{x^{(e_j,k)} \otimes 1 - 1 \otimes x^{(e_j,k)}\}_{1 \leq j \leq n, 1 \leq k \leq s} \);

Using the Left Buchberger Algorithm, compute a left Gröbner basis \( G' \) in the PBW algebra \((R_{\text{rev}})^s\) for the input data \( B \);

If \( G' = \{g'_1, ..., g'_{\nu'}\} \) with \( g'_i = (\sum_{j \in \mathbb{N}} p_{ij}^1 \otimes q_{i1}^1, ..., \sum_{j \in \mathbb{N}} p_{ij}^s \otimes q_{ij}^s) \), take
\[ g_i := (\sum_{j\in\mathbb{N}} p^{1}_{ij} q^{1}_{ij}, \ldots, \sum_{j\in\mathbb{N}} p^{s}_{ij} q^{s}_{ij}); \]

\[ G := \emptyset; \]

\textbf{for all} \( i = 1 \) to \( r \) \textbf{do}

\textbf{if} \( g_i \neq 0 \) \textbf{then}

\[ G := G \cup \{ g_i \}; \]

\textbf{end if}

\textbf{end for}

The advantage offered by this algorithm is that only one call to the left Buchberger Algorithm is done, whereas the one shown in \( \text{[Bueso et. al., 2003]} \) makes an, a priori, unknown number of calls.

\textbf{Example 8} Let \( R \) be the quantum plane, i.e., \( R = \mathbb{C}\{x, y; \{yx - qxy\}, \leq_{(1,3)} \} \)

(take \( q = i\sqrt{2} \)) where \( \leq_{(1,3)} \) is the \((1,3)\)-weighted lexicographical order. Let \( F = \{(2x, x^2y, xy^2 + y^2), (xy, 0, -x^2y^2), (x^2, 2, 0)\} \subset R^3 \) and consider the order \( \text{TOP} \) in \( R^3 \).

The old algorithm (see \text{[Bueso et. al., 2003]} \) takes 66.371 seconds to compute the two-sided Gröbner basis, consisting of 17 elements:

\[ G = \{(2x, x^2y, xy^2 + y^2), (xy, 0, -x^2y^2), (x^2, 2, 0), (xy - 2x, x^3y - x^2y - 4, -y^2), (xy, x^4y + y - 4x, 0), (0, (i\sqrt{2} + 2)x^3y - 12, 0), (0, (i\sqrt{2} + 2)y, 0), ((-i\sqrt{2} - 2)xy, (-i\sqrt{2} - 2)x^3y + (a + 4i\sqrt{2})x + 12, 0), (0, (-4 + 4i\sqrt{2})x^2, 0), ((i\sqrt{2} - 1)xy^2 + (2 - 2i\sqrt{2})xy, (2 - 2i\sqrt{2})x^3y^2 + 3x^2y^2 + (a - 2i\sqrt{2})y, 0), ((1 - i\sqrt{2})xy^2 + (-2 + 2i\sqrt{2})xy, (1 - 2i\sqrt{2})x^3y^2 - 3x^2y^2, 0), ((1 + i\sqrt{2})x^2y^2, 0, 0), (3y, 0), ((1 - i\sqrt{2}/4)xy^2, 3/4y^2 + (-4 + i\sqrt{2})xy, 0), (xy, -2i\sqrt{2}x, 0), (0, (-4 + 4i\sqrt{2})x, 0), (0, -4 + 2i\sqrt{2}, 0)\}. \]

The algorithm 1 takes 46.972 seconds to compute a two-sided Gröbner basis with 12 elements (considering the order \( \preceq^c \) (see remark 2) in \( R^\env \)):

\[ G = \{(2x, x^2y, xy^2 + y^2), (xy, 0, -x^2y^2), (x^2, 2), (xy - 2x, x^3y - x^2y - 4, -y^2), (0, (i\sqrt{2} + 2)x^3y - 12, 0), (i\sqrt{2} - 1)xy^2 + (2 - 2i\sqrt{2})xy, 3x^2y^2 + (2 - 2i\sqrt{2})y, 0), (2xy, -i\sqrt{2}y^2 - 4i\sqrt{2}x, 0), (0, (-i\sqrt{2}/4 - 1/2)y^2 + (2 - 2i\sqrt{2})xy, 0), (0, 3y, 0), (0, -2i\sqrt{2}x^2, 0), (0, -6i\sqrt{2}x, 0), (0, -16i\sqrt{2} - 16, 0)\}. \]

\textbf{Example 9} Let \( R \) be the PBW algebra \( \mathcal{M}_q(2) = \mathbb{C}\{x, y, z, t; Q, \preceq_{\text{glex}} \} \) of quantum matrices (with \( q = i\sqrt{2} \)), where \( Q = \{yx - qxy, ty - qyt, zy - zt - qzt, zy - yz, tx - xt - (q^{-1} - q)yz\} \) and consider the order “POT” in \( R^2 \). Let \( F = \{(-xzt + 1, 2y^3), (x^2t, y^2)\} \subset R^2 \).
The old algorithm takes 216.433 seconds to compute the two-sided Gröbner basis, consisting of 26 elements:

\[
G = \{ (-xzt + 1, 2y^3), (x^2t, y^2), (-x, -2xy^3 + 1/2y^2z), (-3/2yz^2 - 1, 4i\sqrt{2}xy^3zt - 3y^4z^2 - y^2z^2t - 2y^3), (0, 4i\sqrt{2}xy^3t - i\sqrt{2}xy^2zt + 3/8i\sqrt{2}y^3z^2 + y^2), (0, 3i\sqrt{2}xy^4z^2 + 3/8y^3z^3 - (4i\sqrt{2} + 2)xy^3 + (1/2 - 1/2i\sqrt{2})y^2z), (0, -3i\sqrt{2}xy^4z - 3/8i\sqrt{2}y^3z^2 - 3xy^2), (0, (4i\sqrt{2} + 2)x^2y^3
\]

\[-(i\sqrt{2} + 1/2)xy^2z), (0, (12 - 6i\sqrt{2}xy^4z^2 - 3/4(i\sqrt{2} + 1)y^3z^3 + (2 + 4i\sqrt{2})xy^3 + 3/2y^2z), (0, -12x^2y^4z - 3/2xy^3z^2 + (-2 + 2i\sqrt{2})xy^2), (0, (1 + 1/2i\sqrt{2})y^3), (0, (1/2 + 1/4i\sqrt{2})y^3z), (0, (1 - i\sqrt{2})y^3z^2t), (0, (i\sqrt{2} + 2)xy^3zt + (3/8i\sqrt{2} - 3/8)y^4z^2 + (1 + 1/2i\sqrt{2})y^3), (0, (1 + 1/2i\sqrt{2})y^2z), (0, (1/2 + 1/4i\sqrt{2})y^2z^2), (0, (1 - i\sqrt{2})y^2z^3t), (0, (i\sqrt{2} + 2)xy^2z^2 t + (3/8i\sqrt{2} - 3/8)y^3z^3 + (1 + 1/2i\sqrt{2})y^2z), ((1 + 1/4i\sqrt{2})t, -8xy^3zt^2 + 3i\sqrt{2}y^4z^2 - i\sqrt{2}y^2z^2t^2 + 4y^2t), (3/2y, 3y^4 + 3yt), (3/2i\sqrt{2}yz, (-4i\sqrt{2} - 2)xy^3t + 3i\sqrt{2}y^4z + (1/2 + i\sqrt{2})y^2zt), ((-1 + 1/4i\sqrt{2})t, (4i\sqrt{2} + 8)xy^3zt^2 + (-3 - 3i\sqrt{2})y^4z^2t + (i\sqrt{2} - 1)y^2z^2t), (0, -3i\sqrt{2}xy^4zt + 3/8i\sqrt{2}y^3z^2t + (1 + 1/2i\sqrt{2})y^2t) \}.

The algorithm 1 takes 59.909 seconds to compute the two-sided Gröbner basis (considering the order ≤* in R^{rev}), consisting of 11 elements:

\[
G = \{ (-xzt + 1, 2y^3), (x^2t, y^2), (-x, -2xy^3 + 1/2y^2z), (0, -3i\sqrt{2}xy^4z^2 - 3/8y^3z^3 + (4i\sqrt{2} + 2)xy^3 + (1/2i\sqrt{2} - 1/2)y^2z), (-3/2yz^2t + (-1/2i\sqrt{2} - 1)t, -6y^3t), (0, -3i\sqrt{2}x^2y^4z - 3/8i\sqrt{2}y^3z^2 - 3xy^2), (0, (1 + 1/2i\sqrt{2})y^3), (0, (1 + 1/2i\sqrt{2})y^2z), (3/2y, 3yt), (-1, 0), (0, -i\sqrt{2})y^2t), (0, (7/9 - 4/9i\sqrt{2})y^2) \}.
\]

4 Syzygy Bimodules

In this section we study the notion of *syzygy bimodule* of a subset of $R^s$, $R$ being a PBW algebra and $s \in \mathbb{N}^s$. This notion can be viewed as the analogous one of *left syzygy module* for left modules, since it presents some similar properties.

There exists an algorithm (see again [Bueso et al. 2003](#)) which computes a generator system of the left syzygy module $Syz^l(F)$, provided a finite set of input data $F \subset R^s$ is given. This algorithm is shown below and will be used within Algorithm 3.
Algorithm 2 Left Syzygy Module

Require: $F = \{f_1, \ldots, f_t\} \subseteq R^s \setminus \{0\}$
Ensure: $H$, a finite left generator system of $\text{Syz}^l(F)$

**INITIALIZATION:** Run the Left Buchberger Algorithm for the input data $F$ in order to compute:
- a left Gröbner basis $G = \{g_1, \ldots, g_r\} \subseteq R^s$ for $R(F)$,
- the elements $h_{ij}^k \in R$ such that $SP(g_i, g_j) = \sum_{k=1}^r h_{ij}^k g_k$ for all $1 \leq i < j \leq r$, and
- the matrix $Q \in M_{r \times t}(R)$ such that $(g_1, \ldots, g_r) = (f_1, \ldots, f_t)Q^t$;

for all $1 \leq i < j \leq r$ do
  if level (exp ($g_i$)) = level (exp ($g_j$)) then
    Compute $r_{ij}, r_{ji}$ such that $SP(g_i, g_j) = r_{ij}g_i - r_{ji}g_j$;
    Let $p_{ij} := (0, \ldots, r_{ij}^i, \ldots, 0) - (0, \ldots, r_{ji}^j, \ldots, 0) - (h_{ij}^1, \ldots, h_{ij}^r)$;
  end if
end for
Let $H := \{p_{ij}Q \text{ such that } 1 \leq i < j \leq r\}$, and level (exp ($g_i$)) = level (exp ($g_j$))$\}$.

**Definition 10** Let $f_1, \ldots, f_t \in R^s$. The syzygy bimodule of the matrix

$$F = \begin{bmatrix} f_1 \\ \vdots \\ f_t \end{bmatrix} \in M_{t \times s}(R),$$

denoted by $\text{Syz}(F)$ or $\text{Syz}(f_1, \ldots, f_t)$, is the kernel of the homomorphism of left $R^\text{env}$-modules $(R^\text{env})^t \longrightarrow R^s; (h_1, \ldots, h_t) \longmapsto \sum_{i=1}^t h_i f_i$.

We can compute the syzygy bimodule of a matrix $F$ using again the techniques showed in section 3, that is, we will move the problem to the context of the enveloping algebra in order to use the methods on the left side.

**Proposition 11** Let $M \subseteq R^s$ be a $R$-bimodule and $N = (m^s)^{-1}(M)$. Let

$\{h_1, \ldots, h_r\} \subset (R^\text{env})^{t+s n}$ be a generator system of $\text{Syz}^l(\{f_i \otimes 1\}_{i=1}^t, \{x^{(e_j, k)} \otimes 1 - 1 \otimes x^{(e_j, k)}\}_{1 \leq j \leq s, 1 \leq k \leq s})$ as a left $R^\text{env}$-module.

Then $\text{Syz}(f_1, \ldots, f_t) = R(\pi(h_1), \ldots, \pi(h_r))_R$, where $\pi$ is the projection homomorphism $\pi : (R^\text{env})^t \times (R^\text{env})^{sn} \longrightarrow (R^\text{env})^t$.

**Proof.** Notice that $\sum_{i=1}^t g_i(f_i \otimes 1) \in \text{Ker}(m)$ for any $g = (g_1, \ldots, g_t) \in$
On the other hand, it has been proved, first in the commutative case and then order is unavoidably used. In Universal Theory, they are computationally inefficient, mainly because elimination techniques are useful at solving several problems in Module Theory.

Example 12 Let \( R \) be the quantum plane with the PBW algebra structure \( \mathbb{C}\{x, y; (yx - qxy), \prec_{(2, 1)}\} \), where \( q = i\sqrt{2} \), and consider the order POT in \( R^2 \). Let \( F = \{(x + 1, y), (xy, 0)\} \subset R^2 \).

Algorithm 3 takes 23.622 seconds to compute the R-bimodule generator system \( H \) of Syz\((F)\) consisting of 8 elements:

\[
H = \{ (1 \otimes y - y \otimes 1, (-1 + i\sqrt{2})1 \otimes 1), \\
\quad ((i\sqrt{2}/6 - 1/3)y \otimes x + (-2/3 - i\sqrt{2}/3)xy \otimes 1, 1 \otimes x + 1 \otimes 1), (0, 1 \otimes y + i\sqrt{2}/2 y \otimes 1), \\
\quad ((i\sqrt{2}/6 - 1/3)y \otimes x + (-2/3 + i\sqrt{2}/3)xy \otimes 1, i\sqrt{2} x \otimes 1 + 1 \otimes 1), \\
\quad ((-1/3 - i\sqrt{2}/3)y \otimes x + (-2i\sqrt{2}/3 - 2/3)xy \otimes 1, i\sqrt{2} x \otimes x + i\sqrt{2} \otimes 1), \\
\quad (-y \otimes y + y^2 \otimes 1 + i\sqrt{2} y \otimes y, -i\sqrt{2}y \otimes 1), \\
\quad ((1/3 + i\sqrt{2}/3)1 \otimes x^2 + (1/3 - 2i\sqrt{2}/3)x \otimes x + (-2/3 + i\sqrt{2}/3)x^2 \otimes 1, 0), \\
\quad ((1/3 + i\sqrt{2}/3)1 \otimes xy + (i\sqrt{2}/6 - 1/3)y \otimes x + (-1/3 - i\sqrt{2}/3)x \otimes y + (1/3 + i\sqrt{2}/3)xy \otimes 1, 0) \}.
\]

Although elimination techniques are useful at solving several problems in Module Theory, they are computationally inefficient, mainly because elimination orders are unavoidably used.

On the other hand, it has been proved, first in the commutative case and then using left syzygy R-modules being R a non-commutative ring (such as a PBW
algebra), that syzygies provide a much more efficient treatment, for example, in the computation of the intersection of left $R$-submodules of $R^s$, ideal quotients, kernels of homomorphisms of left $R$-submodules, etc. (see Bueso et.al., 2003).

In what follows, we will see that some applications of left syzygies can be generalized using the new definition of syzygy bimodules, so that, for example, it is possible to give an algorithm to compute a finite intersection of $R$-subbimodules of $R^s$ when, as natural, two-sided input data are given. Further applications will be studied in the sequels.

The following result states a general property which the above-mentioned algorithm will be based on.

**Lemma 13** Let $M$ be an $R$-subbimodule of $R^s$ such that there exist $p, q \geq 1$ and $H = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} \in M_{(s+p)\times q}(R)$, where $H_1 \in M_{s\times q}(R)$ and $H_2 \in M_{p\times q}(R)$, satisfying the following two conditions:

i) $(m^s(h) \otimes 1)H_1 = hH_1$, $\forall h \in (R^{\text{env}})^s$;

ii) $M = \{ h \in R^s ; \exists h'' \in (R^{\text{env}})^p$ such that $(h \otimes 1, h'') \in \text{Syz}(H) \}$.

Let us split up each element $h \in (R^{\text{env}})^{s+p}$ into $h = (h', h'')$ with $h' \in (R^{\text{env}})^s$ and $h'' \in (R^{\text{env}})^p$.

1) If $\{ h_1, ..., h_t \} \subseteq (R^{\text{env}})^{s+p}$ is an $R$-bimodule generator system of $\text{Syz}(H)$ then $M = \langle m^s(h_1'), ..., m^s(h_t') \rangle_R$.

2) Furthermore, if $\{ h_1, ..., h_t \}$ is a left Gröbner basis of $\text{Syz}(H)$ (as a left $R^{\text{env}}$-module) for the order POT in $(R^{\text{env}})^{s+p}$ and any of $\preceq^s, \preceq^c, \succeq_s, \succeq_c$ in $R^{\text{env}}$, then $\{ m^s(h_1'), ..., m^s(h_t') \} \setminus \{ 0 \}$ is a two-sided Gröbner basis of $M$ for POT.

**Theorem 14** Let $\{ M_i \}_{i=1}^r$ be a family of $R$-subbimodules of $R^s$ and suppose that $M_i = \langle f_1^i, ..., f_{t_i}^i \rangle_R \subseteq R^s$. Then

$$\bigcap_{i=1}^r M_i = \{ h \in R^s ; \exists h'' \in (R^{\text{env}})^{\sum_{j=1}^r t_j}$ such that $(h \otimes 1, h'') \in \text{Syz}(H) \},$$
where

\[
H = \begin{pmatrix}
I_s & \cdots & I_s \\
f_1^1 & \cdots & 0 \\
\vdots & & \vdots \\
f_{t_1}^1 & \cdots & 0 \\
\vdots & & \vdots \\
0 & \cdots & f_1^r \\
\vdots & & \vdots \\
0 & \cdots & f_{t_r}^r
\end{pmatrix} \in M_{(s + \sum_{i=1}^r t_i) \times rs}(R)
\]

From 13 and 14, an algorithm to compute finite intersections of \(R\)-subbimodules of \(R^s\) may be formulated (see Algorithm 4).

**Corollary 15** Let \(M_i\) and \(H\) be as in 14. If \(\text{Syz}(H) = R\langle g_1, \ldots, g_t \rangle_R\) with \(g_k = (g_k', g_k'') \in (R^\text{env})^s \times (R^\text{env})^{\sum_{j=1}^{t_j}}\) for all \(1 \leq k \leq t\), then

\[
\bigcap_{i=1}^r M_i = R\langle m^s(g_1'), \ldots, m^s(g_t') \rangle_R.
\]

If \(G = \{g_1, \ldots, g_t\}\) is also a left Gröbner basis of \(\text{Syz}(H)\) (as left \(R^\text{env}\)-module) with \(\text{POT}\) in \((R^\text{env})^s + \sum_{j=1}^{t_j}\), then \(\{m^s(g_1'), \ldots, m^s(g_t')\} \setminus \{0\}\) is a two-sided Gröbner basis of \(\bigcap_{i=1}^r M_i\) with \(\text{POT}\) in \(R^s\).

**Algorithm 4** Intersection of \(R\)-subbimodules of \(R^s\)

**Require:** \(\{M_i\}_{i=1}^r\), a family of \(R\)-subbimodules of \(R^s\) with \(M_i = R\langle f_1^i, \ldots, f_{t_i}^i \rangle_R\);

**Ensure:** \(M\), a finite generator system of \(\bigcap_{i=1}^r M_i\) as an \(R\)-bimodule;
INITIALIZATION:

\[
    H := \begin{pmatrix}
    I_s \cdots I_s \\
    f_1^s \cdots 0 \\
    \vdots \\
    f_t^s \cdots 0 \\
    0 \cdots f_r^s \\
    0 \cdots f_r^t
\end{pmatrix} \in M_{(s+\sum_{j=1}^r t_j) \times r_s(R)};
\]

Using the Syzygy Bimodule Algorithm compute a generator system \( G = \{g_1, \ldots, g_r\} \) of \( \text{Syz}(H) \) as an \( R \)-bimodule;

If \( g_k = (g_k^l, g_k^m) \) where \( g_k^l \in (R^\text{env})^s \) and \( g_k^m \in (R^\text{env})^{\sum_{j=1}^r t_j} \) for \( 1 \leq k \leq t \), take \( M := \{m^s(g_k^l), \ldots, m^s(g_k^r)\} \).

Example 16 Let \( R \) be the quantum plane (as in example 12) and consider the order \( \text{POT} \) in \( \mathbb{R}^2 \). Let \( M_1 \) and \( M_2 \) be the \( R \)-submodules of \( \mathbb{R}^2 \) generated by \( \{(2x^2 + 2x, -y), (0, -8), (-3xy, 0)\} \) and \( \{(x + 2, 0), (1, -y)\} \), respectively.

Algorithm 4 takes 111.418 seconds to compute the \( R \)-bimodule generator system \( M \) of \( M_1 \cap M_2 \), consisting of 8 elements:

\[
    M = \{ \frac{4i\sqrt{2}}{3}x^2y + \frac{7}{3}xy, \frac{i\sqrt{2}}{3}y^2, \frac{2i\sqrt{2}}{3}x^3 + \frac{2i\sqrt{2}}{3} - \frac{4}{3}x^2 - \frac{4}{3}x, \frac{i\sqrt{2}}{3}xy + \frac{2}{3}y, \\
    \frac{5}{3}x^2y + \frac{4i\sqrt{2}}{3}xy, -\frac{2}{3}y^2, \frac{-19}{3} + \frac{5i\sqrt{2}}{3}x^2 - \frac{16}{3}x, \\
    \frac{4i\sqrt{2}}{3}xy + \frac{8}{3}y, (5x^2y + (-1 - 3i\sqrt{2})xy, 2y^2, \frac{2}{3}x^2 + \frac{2}{3}x, -\frac{4}{3}y, \\
    \frac{5i\sqrt{2}}{3}x^2y + \frac{8}{3}xy, \frac{2i\sqrt{2}}{3}y^2, \frac{8}{3}x^2y^2 - \frac{7}{3}xy^2, \frac{1}{3}y^3, \frac{-2}{3}x^3 - 2x^2 - \frac{4}{3}x, \\
    \frac{1}{3}xy + \frac{2}{3}y, ((\frac{4i\sqrt{2}}{3} + \frac{8}{3})x^2y + (\frac{4}{3} - \frac{4i\sqrt{2}}{3})xy, (\frac{i\sqrt{2}}{3} + \frac{2}{3})y^2, \\
    \frac{-4i\sqrt{2}}{3}x^2y - \frac{4}{3}x^2y + \frac{4i\sqrt{2}}{3}xy, \frac{-2}{3}y^2, \frac{1}{3}y^3, (-\frac{2}{3} + \frac{2i\sqrt{2}}{3})x^3 + \\
    (-\frac{2}{3} + \frac{2i\sqrt{2}}{3})x^2, 0, (-2x^2y + (-\frac{2}{3} + \frac{2i\sqrt{2}}{3})xy, 0), \frac{2}{3}x^2 + \frac{2}{3}x, -\frac{4}{3}y \}. \}
\]
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