Multiple barrier-crossings of an Ornstein-Uhlenbeck diffusion in consecutive periods

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Abstract

We investigate the joint distribution and the multivariate survival functions for the maxima of an Ornstein-Uhlenbeck (OU) process in consecutive time-intervals. A PDE method, alongside an eigenfunction expansion is adopted, with which we are able to calculate the distribution and the survival functions for the maxima of a homogeneous OU-process in a single interval. By a deterministic time-change and a parameter translation, this result can be extended to an inhomogeneous OU-process. Moreover, we derive a general formula for the joint distribution and the survival functions for the maxima of a continuous Markov process in consecutive periods. With these results, one can obtain semi-analytical expressions for the joint distribution and the multivariate survival functions for the consecutive maxima of an OU-process with piecewise constant parameter functions. The joint distribution and the survival functions can be evaluated numerically by an iterated quadrature scheme, which can be implemented efficiently by matrix multiplications. Moreover, we show that the computation can be further simplified to the product of single quadratures if the filtration is enlarged with additional conditions. Such results may be used for the modelling of heat waves and related risk management challenges.

Keywords: Ornstein-Uhlenbeck process; first-passage-time; multiple barrier-crossings and joint survival function; time-dependent barriers, Markov process, infinite series approximation and tail convergence, quadrature and Monte Carlo schemes, numerical efficiency.

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1 Introduction

The Ornstein–Uhlenbeck (OU) process is a well-known diffusion process, widely used in physics, finance, biology and other fields. Due to its extensive use, the study of its first-passage-time (FPT) arises, naturally. The FPT density of a homogeneous OU-process to particular cases of barrier functions can be found in closed-form. For example, if the barrier is equal to the OU long-term mean, its closed-form probability density function (PDF) can be found in [Ricciardi and Sato (1988), Göing-Jaeschke and Yor (2003)] and in [Yi (2010)]. However, it is more involved to obtain the PDF of the FPT of the homogeneous OU-process to an arbitrary constant barrier. Leblanc et al. (2000) claim that the closed-form solution is found, but Göing-Jaeschke and Yor (2003) publish another paper in the

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same journal to point out the results in [Leblanc et al. (2000)] were wrong due to the errors encountered when using the property of 3D Bessel bridges, see [Pitman and Yor (1981)] and [Pitman and Yor (1982)]. [Göing-Jaeschke and Yor (2003)] provide a Bessel bridge representation for the FPT-PDF of a homogeneous OU-process to an arbitrary constant barrier. Since the closed-form expression of the moment generating function for the homogeneous OU-FPT has been well-studied, see for instance [Ricciardi and Sato (1988)], [Alili et al. (2005)] and [Patie (2004)], one may obtain an infinite series representation for the PDF of the FPT of a homogeneous OU-process crossing an arbitrary constant barrier by the inverse Laplace transform, see [Alili et al. (2005)]. The same infinite series representation is also obtained in [Linetsky (2004b)] based on the spectral theory for options pricing in [Linetsky (2004a)]. However, to our knowledge, the properties of the infinite series solution, especially the tail behaviour, have not yet been studied. The tail behaviour of the infinite series representation is practically important since it determines whether one can use the truncated series as a robust approximation.

The case of a time-inhomogeneous OU-process passing a time-dependent barrier tends to be more complicated. [Tuckwell and Wan (1984)] study the FPT of a time-homogeneous Ito process where the barrier function must satisfy a first-order linear ODE. Under such conditions, [Tuckwell and Wan (1984)] provide a solution by numerical PDE methods. [Buonocore et al. (1987)] show that the FPT-PDF of a diffusion process passing a time-dependent boundary satisfies a Volterra integral equation of the second kind involving two arbitrary continuous functions. By this method, the FPT-PDF for a homogeneous OU-process passing some special barrier specifications, e.g., the barrier function is hyperbolic with respect to time, can be found analytically. [Gutiérrez et al. (1997)] generalise this integral equation approach to time-inhomogeneous diffusion processes. The FPT-PDF of a time-inhomogeneous diffusion process passing a constant barrier can be obtained numerically by solving a PDE, see e.g. [Karlin and Taylor (1981)] and [Wenocur (1987)]. [Lo and Hui (2006)] study the Fokker–Planck equation associated with an inhomogeneous OU-process passing a time-dependent barrier and introduce the method of images to derive the solution. However, the generalisation to an unconstraint time-dependent barrier cannot be produced due to the strict conditions imposed by the method of images. [Hernandez-del-Valle (2012)] deals with the FPT of Ito processes whose local drift can be modelled in terms of a solution of the Burgers’ equation. However, the OU-process family does not belong to such a process family.

Since all continuous functions can be approximated to arbitrary precision by piecewise constant functions, it is worthwhile to study the FPT of a homogeneous or inhomogeneous OU-process passing a piecewise constant barrier function. In such cases, the probability that the FPT is within a certain threshold is equal to the probability that the maxima in all intervals determined by the piecewise constant function is less than the barrier level in the corresponding intervals. In this paper, one of the main focuses is put on the joint probability that the running maximum is above arbitrary fixed thresholds in pre-specified consecutive time intervals. Such a probabilistic problem arises for example in applications to environmental and climate risk, to which the insurance industry, but more importantly general global welfare, is exposed. Heat waves, or repeated prolonged periods of droughts, can have substantial impact on economies, be these regional or (supra-)national. A heat wave is an event that often is defined by the temperature passing a pre-specified threshold on a number of consecutive days. This is a unequivocal case where the joint probability of the running maximum is above arbitrary fixed thresholds in pre-specified consecutive time intervals. However, to our knowledge, this mathematical problem has not yet been solved or tackled.

In this paper, we study the multivariate survival function associated with an OU-process crossing arbitrary barriers in multiple time intervals. In Section 2, we adopt a PDE approach to deduce the infinite series representation of the survival function for the FPT of a homogeneous OU-process with lower reflection barrier passing a constant upper barrier. By considering the lower reflection barrier set at $-\infty$, we produce the same infinite series representation as in [Linetsky (2004b)] and
This can be viewed as a generalisation and an alternative derivation of the infinite series representation. Moreover, we analyse the distributional properties of the deduced survival function, especially its tail behaviour and the truncation error. In Section 3, we provide a theorem that transforms the FPT of an inhomogeneous OU-process passing a time-dependent barrier to the FPT of a homogeneous OU-process with a different time-dependent barrier. This transfers the time-inhomogeneity from the process to the time-dependent barrier only, which simplifies the original problem. In Section 4 we deduce an integral representation of the joint distribution and joint survival function for the maxima of a continuous Markov process in consecutive intervals. With the knowledge of the integral representation, the FPT density function and the numerical integration method, the joint distribution and joint survival function for the maxima of an OU-process with piecewise constant parameters in consecutive intervals can be efficiently obtained. We also show that under certain assumptions, the nested integration can be further simplified to become a product of single integrals, which leads to improved computational efficiency. Finally, we present the quadrature scheme and the Monte Carlo integration method for the numerical integration in Section 5. Comparing with the direct Monte Carlo approach, the results obtained by either the quadrature scheme or the Monte Carlo integration method show higher accuracy and robustness. This is especially true in the rare-event cases, where the direct Monte Carlo approach fails to reduce the approximation error efficiently.

2 Survival function for the FPT of a homogeneous OU-process passing a constant barrier

We begin by considering the first-passage-time (FPT) of a homogeneous Ornstein-Uhlenbeck (OU) process crossing a constant barrier. The definitions for the homogeneous and standardised OU-processes are given in Definition 2.1. The FPT of a general continuous stochastic process is specified in Definition 2.2.

**Definition 2.1.** An $\mathbb{R}$-valued stochastic process $(X_t)_{t \geq 0}$ is called a homogeneous OU-process if it satisfies the stochastic differential equation

$$dX_t = (\mu - \lambda X_t) dt + \sigma dW_t, \quad (2.1)$$

where $X_0 = x \in \mathbb{R}$, for $\mu \in \mathbb{R}$, $\lambda > 0$ and $\sigma > 0$, where $(W_t)_{t \geq 0}$ is a Brownian motion on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. When $\mu = 0$, $\lambda = \sigma = 1$, we call the process $(e^{X_t})_{t \geq 0}$ that satisfies

$$de^{X_t} = -e^{X_t} dt + dW_t, \quad (2.2)$$

a standardised OU-process.

**Definition 2.2.** The first-passage-time (FPT) of a continuous process $(X_t)_{t \geq 0}$ to an upper constant barrier $b > X_0 = x$ is defined by $\tau_{X,b} := \inf \{ t \geq 0 : X_t \geq b \}$. $\tilde{F}_{\tau_{X,b}}(t;x) = \mathbb{P}(\tau_{X,b} \geq t \mid X_0 = x)$.

As shown in Alili et al. (2005), Patie (2004) and Linetsky (2004b), if $(X_t)_{t \geq 0}$ is a homogeneous OU-process, the random variable $\tau_{X,b}$ is “properly” defined in the sense that $\mathbb{P}(\tau_{X,b} < \infty) = 1$.

Next we present a relation between the survival functions of the FPTs for two different homogeneous OU-processes. With Lemma 2.1, if one knows the FPT distribution of a homogeneous OU-process to a given barrier, the FPT distribution of another homogeneous OU-process to a shifted barrier can also be obtained.

**Lemma 2.1.** The random variable $\tau_{X,b}$ is equal to $\tau_{\tilde{X},\tilde{b}}$ in distribution for

$$\tilde{t} = \lambda t, \quad \tilde{x} = \sqrt{\frac{\lambda}{\sigma^2}} (x - \frac{\mu}{\lambda}), \quad \tilde{b} = \sqrt{\frac{\lambda}{\sigma^2}} (b - \frac{\mu}{\lambda}).$$
that is, \( \bar{F}_{\tau_x,b} (t, x) = \bar{F}_{\tau_x,b} (\tilde{t}, \tilde{x}) \).

Proof. We have

\[
\bar{F}_{\tau_x,b} (t, x) = P \left( \sup_{s \in [0,t]} X_s < b \, \big| \, X_0 = x \right).
\]

Then by a change of time, it follows

\[
P \left( \sup_{s \in [0,t]} X_s < b \, \big| \, X_0 = x \right) = P \left( \sup_{s \in [0,\lambda t]} X_{s/\lambda} < b \, \big| \, X_{0/\lambda} = \frac{x}{\lambda} \right).
\]

Then,

\[
P \left( \sup_{s \in [0,\lambda t]} X_{s/\lambda} < b \, \big| \, X_{0/\lambda} = \frac{x}{\lambda} \right) = \frac{1}{\lambda} \sqrt{\frac{\lambda}{\sigma^2}} \left( \frac{\lambda x}{\sigma^2} \right) \left( \frac{\lambda x}{\sigma^2} \right).
\]

The dynamics of the process \( (\sqrt{\frac{\lambda}{\sigma^2}} X_{s/\lambda} - \mu / (\sigma \sqrt{\lambda}))_{s \geq 0} \) are given by

\[
d \left( \sqrt{\frac{\lambda}{\sigma^2}} X_{s/\lambda} - \frac{\mu}{\sigma \sqrt{\lambda}} \right) = \sqrt{\frac{\lambda}{\sigma^2}} dX_{s/\lambda} = - \left( \sqrt{\frac{\lambda}{\sigma^2}} X_{s/\lambda} - \frac{\mu}{\sigma \sqrt{\lambda}} \right) dt + dW_t.
\]

This means that, in law, the process \( (\sqrt{\frac{\lambda}{\sigma^2}} X_{s/\lambda} - \mu / (\sigma \sqrt{\lambda}))_{s \geq 0} \) is a standardised OU-process. Therefore, \( \bar{F}_{\tau_x,b} (t, x) = \bar{F}_{\tau_x,b} (\tilde{t}, \tilde{x}) \).

Lemma 2.1 provides a relationship between the survival functions – that is between the distributions of the FPTs – of the homogeneous, respectively, the standardised OU-process. In order to calculate the FPT survival function for a homogeneous OU-process, one can first calculate the FPT survival function for a standardised OU-process. Therefore, from now on in this section, we consider the case of a standardised OU-process.

2.1 The FPT survival function of the standardised OU-process with a constant barrier

On the Hilbert space \( \mathbb{H}^{1,2} \left( [0, \infty], [-\infty, \tilde{b}] \right) \), the function \( \bar{F}_{\tau_{\tilde{x},\tilde{b}}} (\tilde{t}, \tilde{x}) \) satisfies the backward PDE

\[
\frac{\partial \bar{F}_{\tau_{\tilde{x},\tilde{b}}}}{\partial \tilde{t}} = \mathcal{A} \bar{F}_{\tau_{\tilde{x},\tilde{b}}} \tag{2.3}
\]

subject to the initial and boundary conditions

\[
\bar{F}_{\tau_{\tilde{x},\tilde{b}}} (0, \tilde{x}) = 1, \quad \bar{F}_{\tau_{\tilde{x},\tilde{b}}} (\tilde{t}, \tilde{b}) = 0. \tag{2.4, 2.5}
\]
Here $\mathcal{A}$ is the infinitesimal operator of a standardised OU-process $(\tilde{X}_t)_{t \geq 0}$ given by

$$\mathcal{A} = -\tilde{x} \frac{\partial}{\partial \tilde{x}} + \frac{1}{2} \frac{\partial^2}{\partial \tilde{x}^2}.$$ 

In order to solve this PDE, we add the lower boundary condition

$$\frac{\partial \tilde{F}_{\tau \tilde{x}, \tilde{b}}}{\partial \tilde{x}} (\tilde{t}, \tilde{a}) = 0,$$

which is the condition for a reflecting lower boundary at location $\tilde{a} < \tilde{b}$.

**Proposition 2.1.** The analytic solution to the PDE (2.3), subject to the initial condition and boundary conditions (2.4), (2.5) and (2.6) is given by

$$\tilde{F}_{\tau \tilde{x}, \tilde{b}} (\tilde{t}, \tilde{x}) = \sum_{k=1}^{\infty} c_k e^{-\alpha_k \tilde{t}} H(\alpha_k, \tilde{x}, \tilde{a}),$$

for $k \in \mathbb{N}$, where

$$H(\alpha_k, \tilde{x}, \tilde{a}) = \frac{2\sqrt{\pi}}{(\frac{1}{2} - \alpha_k)} \left( _1F_1 \left( -\alpha_k; \frac{1}{2}; \tilde{x}^2 \right) + y(\alpha_k, \tilde{a}) \tilde{x} \ _1F_1 \left( 1 - \alpha_k; \frac{3}{2}; \tilde{x}^2 \right) \right),$$

$$y(\alpha_k, \tilde{a}) = \left. \frac{2\alpha_k \tilde{a} \ _1F_1 \left( 1 - \alpha_k; \frac{3}{2}; \tilde{a}^2 \right) - \ _1F_1 \left( \frac{1}{2} - \alpha_k; \frac{3}{2}; \tilde{a}^2 \right) \tilde{x} \ _1F_1 \left( 1 - \alpha_k; \frac{3}{2}; \tilde{x}^2 \right) }{\ _1F_1 \left( 1 - \alpha_k; \frac{3}{2}; \tilde{a}^2 \right) + \frac{2}{3}(1 - \alpha_k) \tilde{a}^2 \ _1F_1 \left( \frac{3}{2} - \alpha_k; \frac{5}{2}; \tilde{a}^2 \right) } \right|_{\tilde{x} = \tilde{a}}.$$ 

Here, $_1F_1$ is the confluent hypergeometric function of the first kind and $\alpha_k$ are the ordered solutions to the equation

$$\ _1F_1 \left( -\alpha_k; \frac{1}{2}; \tilde{b}^2 \right) + y(\alpha_k, \tilde{a}) \tilde{b} \ _1F_1 \left( 1 - \alpha_k; \frac{3}{2}; \tilde{b}^2 \right) = 0$$

with respect to $\alpha$. Furthermore, the coefficient $c_k$ is given by $c_k = -1 / [\alpha_k \, \partial \alpha_k H(\alpha_k, \tilde{x}, \tilde{a})]$.

**Remark 2.1.** This proposition provides a generalization to the infinite series representation in Alili et al. (2005) and Linetsky (2004b). It recovers the previous result when $\tilde{a} \to -\infty$, which will be shown in Theorem 2.1.

**Proof.** By the method of eigenfunction expansion, $\tilde{F}_{\tau \tilde{x}, \tilde{b}} (\tilde{t}, \tilde{x})$ admits the following representation

$$\tilde{F}_{\tau \tilde{x}, \tilde{b}} (\tilde{t}, \tilde{x}) = \sum_{k=1}^{\infty} c_k e^{-\alpha_k \tilde{t}} \phi_k(\tilde{x}),$$

where $c_k$ are the constant coefficients, and $\alpha_k$ and $\phi_k(\tilde{x})$ are the eigenvalues and eigenfunctions that satisfy the general eigenfunction equation

$$\mathcal{A} \phi_k(\tilde{x}) = -\alpha_k \phi_k(\tilde{x})$$

subject to $\phi_k'(\tilde{a}) = \phi_k(\tilde{b}) = 0$. The pair $(\phi(\cdot), \alpha)$ satisfies

$$\frac{d^2 \phi}{d \tilde{x}^2} - 2\tilde{x} \frac{d \phi}{d \tilde{x}} + 2\alpha \phi = 0$$

subject to

$$\phi'(\tilde{a}) = \phi(\tilde{b}) = 0.$$
As shown in [Zaitsev and Polyanin 2002], the ODE (2.9) is known as the Hermite differential equation, whose general solution is given by

\[
\phi(\tilde{x}) = A \, _1F_1\left(-\frac{\alpha}{2}; \frac{1}{2}; \tilde{x}^2\right) + B \tilde{x} \, _1F_1\left(-1; \frac{3}{2}; \tilde{x}^2\right) \tag{2.11}
\]

where \(A\) and \(B\) are independent of \(\tilde{x}\). After substituting Equation (2.11) into condition (2.10), we obtain the system

\[
\begin{align*}
A \, _1F_1\left(-\frac{a}{2}; \frac{1}{2}; b^2\right) + B b \, _1F_1\left(-\frac{a}{2}; \frac{3}{2}; b^2\right) &= 0, \\
B \left[ _1F_1\left(-\frac{a}{2}; \frac{3}{2}; \tilde{a}\right) + \frac{2}{3} (1 - \alpha) \tilde{a} \, _1F_1\left(-\frac{3-a}{2}; \frac{5}{2}; \tilde{a}\right) - 2a \tilde{a} \, _1F_1\left(-\frac{2-a}{2}; \frac{3}{2}; \tilde{a}\right) \right] &= 0.
\end{align*}
\]

Therefore, the eigenvalues \(\alpha_k\) must be the zeros of the equation

\[
_1F_1\left(-\frac{\alpha}{2}; \frac{1}{2}; b^2\right) + y(\alpha, \tilde{a}) b \, _1F_1\left(-1; \frac{3}{2}; b^2\right) = 0
\]

with respect to \(\alpha\). So we let

\[
\phi_k(\tilde{x}) = H(\alpha_k, \tilde{x}; \tilde{a}) = \frac{2^\alpha \sqrt{\pi}}{\Gamma\left(\frac{1-a}{2}\right)} \left( _1F_1\left(-\frac{\alpha_k}{2}; \frac{1}{2}; \tilde{x}^2\right) + y(\alpha_k, \tilde{a}) \tilde{x} \, _1F_1\left(-\frac{1-a_k}{2}; \frac{3}{2}; \tilde{x}^2\right) \right),
\]

which is convenient for later use. Similar to [Linetsky 2004a], the coefficient of each term can be calculated by tedious but simple steps and are given by \(c_k = -1/[\alpha_k \partial_{\alpha} H(\alpha_k, \tilde{x}; \tilde{a})]\). \(\square\)

Proposition 2.1 gives the survival function of the FPT for a homogeneous OU-process passing a given upper barrier subject to a lower reflection boundary. We re-derive the formulae after removing the lower reflection boundary by taking a limit in the following theorem. This can be treated as a different derivation of the infinite series representation in [Alili et al. 2005] and [Linetsky 2004b] based on relaxing specific conditions. Here, the definition of Hermite function \(\mathcal{H}_\alpha(x)\) is given in [Abramowitz and Stegun 1964].

**Theorem 2.1.** The analytic solution to the PDE (2.3) subject to the initial and boundary conditions (2.4) and, respectively, (2.5) is given by

\[
\tilde{F}_{\tilde{t}, \tilde{x}}(\tilde{t}, \tilde{x}) = \sum_{k=1}^{\infty} c_k e^{-\alpha_k \tilde{t}} \mathcal{H}_{\alpha_k}(\tilde{x})
\]

for \(k \in \mathbb{N}\), where \(\mathcal{H}_\alpha(\cdot)\) is the Hermite function with parameter \(\alpha\), and \(\alpha_k\) are the solutions to the equation \(\mathcal{H}_\alpha(-\tilde{b}) = 0\) with respect to \(\alpha\), and \(c_k = -1/[\alpha_k \partial_{\alpha} \mathcal{H}_{\alpha_k}(-\tilde{b})]\).

**Proof.** By [Abramowitz and Stegun 1964], we have the asymptotic

\[
_1F_1\left(x; y; z\right) \sim \frac{\Gamma(y) e^x z^{y-x}}{\Gamma(x)}
\]

for \(z \to \infty\), that is,

\[
\lim_{z \to \infty} \frac{_1F_1\left(x; y; z\right) \Gamma(x)}{\Gamma(y) e^x z^{y-x}} = 1.
\]

Therefore,

\[
\lim_{\tilde{a} \to -\infty} y(\alpha, \tilde{a}) = \lim_{\tilde{a} \to -\infty} \frac{2a \tilde{a}}{\Gamma\left(\frac{1-a}{2}\right)} \left( _1F_1\left(-\frac{a}{2}; \frac{3}{2}; \tilde{a}\right) + \frac{2}{3} (1 - \alpha) \tilde{a} \, _1F_1\left(-\frac{3-a}{2}; \frac{5}{2}; \tilde{a}\right) - 2a \tilde{a} \, _1F_1\left(-\frac{2-a}{2}; \frac{3}{2}; \tilde{a}\right) \right)
\]

\[
= \lim_{\tilde{a} \to -\infty} \frac{\mathcal{H}_{\alpha_k}(\tilde{x})}{\mathcal{H}_{\alpha_k}(\tilde{b})} + \frac{2}{3} (1 - \alpha) \lim_{\tilde{a} \to -\infty} \frac{\mathcal{H}_{\alpha_k}(\tilde{x})}{\mathcal{H}_{\alpha_k}(\tilde{b})}
\]

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\[ \frac{2\Gamma\left(\frac{1-a}{2}\right)}{\Gamma\left(-\frac{a}{2}\right)}, \]

in particular, when \( a = a_k \). For \( \tilde{a} \to -\infty \) and \( \alpha = a_k \), the eigenvalues \( \alpha_k \) are required to satisfy

\[ 2^a \sqrt{\pi} \left( \frac{1 F_1 \left( -\frac{a}{2}; \frac{1}{2}; \tilde{b}^2 \right)}{\Gamma \left( \frac{1-a}{2} \right)} + 2\tilde{b} \frac{1 F_1 \left( \frac{1-a}{2}; \frac{3}{2}; \tilde{b}^2 \right)}{\Gamma \left( -\frac{a}{2} \right)} \right) = 0, \]

which turn out to be the zeros of the Hermite function \( \mathcal{H}_a(-\tilde{b}) \) with respect to \( \alpha \):

\[ \mathcal{H}_a(-\tilde{b}) := 2^a \sqrt{\pi} \left[ \frac{1 F_1 \left( -\frac{a}{2}; \frac{1}{2}; \tilde{b}^2 \right)}{\Gamma \left( \frac{1-a}{2} \right)} + 2\tilde{b} \frac{1 F_1 \left( \frac{1-a}{2}; \frac{3}{2}; \tilde{b}^2 \right)}{\Gamma \left( -\frac{a}{2} \right)} \right] = 0. \]

Thus, the eigenfunctions are represented by \( \phi_k(\tilde{x}) = \mathcal{H}_{a_k}(-\tilde{x}) \). The coefficients can be then obtained by Proposition 2.1.

**Remark 2.2.** The Hermite function \( \mathcal{H}_a(x) \) is equal to the limit

\[ \lim_{\tilde{a} \to -\infty} H(\alpha; x; \tilde{a}) \]

in Proposition 2.1.

**Example 2.1.** Here we consider the PDF for the FPT of a standardised OU-process hitting the upper barrier \( \tilde{b} \) with different lower reflection barriers \( \tilde{a} \). Figure 1 shows the distances of the ordered eigenvalues tend to increase, regardless of the value \( \tilde{a} \) takes. We can observe from Figure 2 that when \( \tilde{a} \) becomes smaller, the PDF with lower reflection barrier approaches the PDF without lower reflection barrier.

![Figure 1: Distance between the first eleven ordered eigenvalues for the PDE (2.3) with upper barrier \( \tilde{b} = 1.5 \) and different lower reflection barriers \( \tilde{a} \).](image-url)
Figure 2: The probability density function for the first-passage-time of a standardised OU-process crossing the upper barrier $\tilde{b} = 1.5$ with lower reflection barrier $\tilde{a}$.

**Corollary 2.1.** The analytic form of the FPT survival function of the OU-process (2.1) is given by

$$F_{\tau, \tilde{a}, \tilde{b}}(t, x) = \sum_{k=1}^{\infty} c_k e^{-\lambda \alpha_k t} \mathcal{H}_{\alpha_k}(x - \frac{\mu}{\lambda})$$

where $\mathcal{H}_{\alpha_k}(\cdot)$ is the Hermite function, and the $\alpha_k$'s are the ordered solutions to the equation

$$\mathcal{H}_{\alpha_k}\left(-\sqrt{\frac{\lambda}{\sigma^2}}\left(b - \frac{\mu}{\lambda}\right)\right) = 0.$$

Furthermore, the coefficient $c_k$ is given by

$$c_k = -\frac{1}{\alpha_k \cdot \partial_{\alpha_k} \mathcal{H}_{\alpha_k}\left(-\sqrt{\frac{\lambda}{\sigma^2}}\left(b - \frac{\mu}{\lambda}\right)\right)}.$$

**Proof.** Based on the relationship between the survival functions of the homogeneous, respectively, the standardised OU-process in Lemma 2.1, one can obtain this result by substituting the parameters in Lemma 2.1 into Theorem 2.1.

In the following theorem, we show the absolute convergence of the infinite series (2.12) and the bound of the truncation error utilising Corollary 2.1.

**Theorem 2.2.** The infinite series in formula (2.12) is absolutely convergent. The truncated series

$$\sum_{k=1}^{K} c_k e^{-\lambda \alpha_k t} \mathcal{H}_{\alpha_k}(x - \frac{\mu}{\lambda})$$

has truncation error $O(e^{-2\lambda t})$, as $K \to \infty$. Moreover, the absolute value of the truncation error is bounded by

$$e(a_K) = \frac{\exp\left(\frac{x^2 - b^2}{2}\right)}{\sqrt{2}|b|} \left[\frac{\exp(-\lambda t a_K)}{a_K} + (1 - \lambda t)\Gamma(0, \lambda t a_K)\right],$$

where $x' = \sqrt{\frac{\lambda}{\sigma^2}}(x - \frac{\mu}{\lambda})$, $b' = \sqrt{\frac{\lambda}{\sigma^2}}(b - \frac{\mu}{\lambda})$, and $\Gamma(a, x)$ is the upper incomplete Gamma function with parameter $a$. 

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Proof. By Lebedev and Silverman (1972), as $\alpha_k \to \infty$, which means $k \to \infty$, we have

$$\mathcal{H}_{\alpha_k}(-x') = 2^{\alpha_k + 1/2} e^{x'^2 - \alpha_k/2 - 1/4} \left( \frac{\alpha_k}{2} + \frac{1}{4} \right)^{\alpha_k/2} \cos \left( 2x' \sqrt{\frac{\alpha_k}{2} + \frac{1}{4} - \frac{\alpha_k \pi}{2}} \right) \left[ 1 + O \left( \frac{1}{\sqrt{\alpha_k/2 + 1/4}} \right) \right].$$

Hence, for a large enough $k \in \mathbb{N}$, we have

$$c_k = -\left[ 1 + O \left( \frac{1}{\sqrt{\alpha_k/2 + 1/4}} \right) \right] \left( \alpha_k 2^{\alpha_k + 1/2} \exp \left( \frac{b^2}{2} - \frac{\alpha_k}{2} - \frac{1}{4} \right) \left( \frac{\alpha_k}{2} + \frac{1}{4} \right)^{\alpha_k/2} \times \sin \left( 2b' \sqrt{\alpha_k/2 + 1/4} \right) \left( \frac{\alpha_k}{2} + \frac{1}{4} - \frac{\alpha_k \pi}{2} \right) \right]^{-1}.$$

We obtain the asymptotic behaviour of $\alpha_k$

$$\alpha_k = 2k + 1 + \frac{4b^2}{\pi^2} + \frac{2b'}{\pi} \sqrt{4k + 3 + \frac{4b^2}{\pi^2}} \quad (2.13)$$

when $k \to \infty$. Therefore, for large enough $K \in \mathbb{N}$, the exact truncation error of Equation (2.12) is

$$\sum_{k=K}^{\infty} c_k \mathcal{H}_{\alpha_k}(-x') e^{-\lambda \alpha_k t} = \sum_{k=K}^{\infty} -\exp \left( \frac{x'^2 - b^2}{2} \right) \frac{\cos \left( 2x' \sqrt{\alpha_k/2 + 1/4} - \frac{\alpha_k \pi}{2} \right) e^{-\lambda \alpha_k t}}{\alpha_k \sin \left( 2b' \sqrt{\alpha_k/2 + 1/4} - \frac{\alpha_k \pi}{2} \right) \left( \frac{\alpha_k}{2} + \frac{1}{4} + \frac{\alpha_k \pi}{2} \right)}.$$

Since large $\alpha_k$'s satisfy Equation (2.13), we have

$$\left| \sin \left( 2b' \sqrt{\frac{\alpha_k}{2} + \frac{1}{4} + \frac{\alpha_k \pi}{2}} \right) \right| = 1.$$

Therefore, we have the asymptotic inequality

$$\sum_{k=K}^{\infty} \left| c_k \mathcal{H}_{\alpha_k}(-x') e^{-\lambda \alpha_k t} \right| \leq \sum_{k=K}^{\infty} \frac{\exp \left( \frac{x'^2 - b^2}{2} - \lambda \alpha_k t \right)}{\alpha_k \left( \frac{\alpha_k}{2} + \frac{b' \alpha_k}{\sqrt{\alpha_k/2 + 1/4}} \right)} \leq \exp \left( \frac{x'^2 - b^2}{2} \right) C_1 \sum_{k=K}^{\infty} \frac{\exp (-\lambda \alpha_k t)}{\alpha_k^{3/2}} \leq \exp \left( \frac{x'^2 - b^2}{2} \right) C_1 \exp (-\lambda \alpha_K t) \sum_{k=K}^{\infty} \frac{1}{\alpha_k^{3/2}} \leq \exp \left( \frac{x'^2 - b^2}{2} \right) C_1 C_2 \exp (-\lambda \alpha_K t) \sum_{k=K}^{\infty} \frac{1}{k^{3/2}} \leq \exp \left( \frac{x'^2 - b^2}{2} \right) C_1 C_2 C_3 \exp (-\lambda \alpha_K t) \leq O \left( \exp (-\lambda \alpha_K t) \right) = O \left( e^{-2Kt} \right)$$

Meanwhile, since $\alpha_K > 0$, we have

$$\sum_{k=K}^{\infty} \left| c_k \mathcal{H}_{\alpha_k}(-x') e^{-\lambda \alpha_k t} \right| \leq \sum_{k=K}^{\infty} \frac{\exp \left( \frac{x'^2 - b^2}{2} - \lambda \alpha_k t \right)}{\alpha_k \left( \frac{\alpha_k}{2} + \frac{b' \alpha_k}{\sqrt{\alpha_k/2 + 1/4}} \right)}.$$
\[
\begin{align*}
&\leq \exp \left( \frac{x'^2 - b'^2}{2} \right) \sum_{k=K}^{\infty} \frac{\exp(-\lambda \alpha k t) \sqrt{\alpha_k + 1}}{\sqrt{2} \alpha_k \alpha_k^2} \\
&\leq \exp \left( \frac{x'^2 - b'^2}{2} \right) \int_{\alpha_K}^{\infty} \frac{\exp(-\lambda \alpha t) (\alpha + 1)}{\sqrt{2} \alpha t \alpha^2} d\alpha \\
&= \frac{\exp \left( \frac{x'^2 - b'^2}{2} \right)}{\sqrt{2} \alpha K} \left[ \frac{\exp(-\lambda t \alpha K)}{\alpha K} + (1 - \lambda t) \Gamma(0, \lambda t \alpha K) \right].
\end{align*}
\]

Theorem 2.2 gives an upper bound for the truncation error. One can determine the terms need to be truncated with an explicit function. An example of the error and its upper bound can be found in Figure 3.

Figure 3: The log-scale plot between the truncation terms and the error when \( x' = 0 \) and \( b' = 1 \).

To utilise the infinite series approximation for a probability density function, one needs to analyse how many terms one should truncate in order to reach a certain precision level for the approximated distribution. Since one can transform the homogeneous OU-process barrier-crossing problem to the standardised OU-process barrier-crossing problem, see Lemma 2.1, we here study the truncation precision of the standardised OU-process. Here, we proceed with Algorithmic 1 in the appendix to study the relationship among the process initial values, barrier levels and the number of truncations. We plot the number of truncations required for various initial values and barrier levels in Figure 4, where the \( \alpha \)-zeros are taken in \([0, 70]\). We observe that when the barrier level is far away from the initial value, the number of truncations required becomes smaller. Figure 4 can be treated as a benchmark to determine how many terms one should truncate for a given quantile precision requirement.
Remark 2.3. The $\alpha$-zeros can be obtained by the bisection method. However, these $\alpha$-zeros do not need to be obtained with high precision. Through the numerical test, we notice that if the $\alpha$-zeros are accurate up to $10^{-4}$, the approximation can be stable and reliable.

Remark 2.4. For a barrier level $b$ which is larger than 5, numerically solving the higher orders of $\alpha$-zeros (for $\alpha \geq 70$) becomes unstable. This is due to the value of $H_{\alpha}(b)$ becomes too large to be stored in the computer, leading to the overflow of the mantissa under double precision issue. For these cases, if the initial value is not next to the barrier level, one can truncate with fewer terms. This cannot cause larger errors due to the empirical results shown in Figure 4, in which one only needs to truncate with a few terms to reach a 5% quantile precision.

2.2 Tail behaviour of the FPT distribution for a homogeneous OU-process passing a constant barrier

With the given infinite series representation in Equation (2.12), we can analyse the property of the FPT distribution for a homogeneous process passing a constant barrier. With the method given in page 114 of Peters and Shevchenko (2015), its tail behaviour can be characterised by the “hazard rate function”.

Lemma 2.2. The distribution of the FPT of OU-process (2.1) to a constant barrier $b$ is light-tailed, that is, the exponential moments exist up to the $\lambda \alpha_1$ order, where $\alpha_1$ is given in Corollary 2.1 i.e.

$$\mathbb{E}[e^{\theta \tau_{x,b}}] \leq \infty, \quad \forall \theta < \lambda \alpha_1.$$

Proof. We consider the hazard rate function given in Peters and Shevchenko (2015), p. 114. The hazard rate function $r(t)$ for the FPT of OU-process (2.1) is given by

$$r(t) := \frac{-\partial_t \tilde{F}_X(x,t;b)}{\tilde{F}_X(x,t;b)} = \sum_{k=1}^{\infty} B_k e^{-\lambda \alpha_k t}$$

where $B_k = \lambda \alpha_k c_k H_{\alpha_k}(-\sqrt{\frac{\alpha_k}{\sigma^2}(x-\mu)}(\cdot))$ and $C_k = c_k \alpha_k H_{\alpha_k}(-\sqrt{\frac{\alpha_k}{\sigma^2}(x-\mu)}(\cdot))$. As shown in Remark 3.6 of Peters and Shevchenko (2015), if $\lim_{t \to \infty} r(t) > 0$ exists, then the distribution is light-tailed and the exponential moments exist up to $\lim_{t \to \infty} \inf r(t)$. In our case, since $\lambda > 0$ is given by the
definition of an OU-process and \( \{\alpha_k\}_{k \in \mathbb{N}} \) are ordered positive solutions to the equation

\[
\mathcal{H}_\alpha \left( -\sqrt{\frac{\lambda}{\sigma^2}} \left( b - \frac{\mu}{\lambda} \right) \right) = 0
\]

with respect to \( \alpha \), we have

\[
\lim_{t \to \infty} r(t) = \lim_{t \to \infty} \sum_{k=1}^{\infty} \frac{B_k e^{-\lambda \alpha_t t}}{\sum_{k=1}^{\infty} C_k e^{-\lambda \alpha_t t}} = \lim_{t \to \infty} \frac{B_1 + \sum_{k=2}^{\infty} B_k e^{-\lambda (\alpha_k - \alpha_t) t}}{C_1 + \sum_{k=2}^{\infty} C_k e^{-\lambda (\alpha_k - \alpha_t) t}} = \lambda \alpha_1.
\]

3  FPT transformation between an inhomogeneous and homogeneous OU-process crossing time-dependant barriers

In Section 2 we derive the eigenvalue expansion formulae to compute the survival function of the FPT for a homogeneous OU-process hitting a constant barrier. However, the homogeneous condition is usually too strong to model real events. For example, if we want to model the process with periodic structure by a homogeneous OU-process, the periodic structure in the process cannot be modelled. On the other hand, the barriers in reality will also vary with time. It is therefore natural and practically important to study the FPT for an inhomogeneous OU-process hitting a time-dependent barrier. The inhomogeneous OU-process is defined in Definition 3.2. We also present the definition of the FPT that a stochastic process hits a time-dependent barrier in Definition 3.1.

**Definition 3.1.** Let \((Z_t)_{t \geq 0}\) be a continuous Markov process. The first-passage-time of \((Z_t)_{t \geq 0}\) with initial value \(Z_0 = z\) to an upper time-dependent barrier \(b(t)\), where \(b(0) > z\), is defined by

\[
\mathcal{F}_{Z,b}(t) := \inf\{t \geq 0 : Z_t \geq b(t)\}.
\]

The survival function of \(\mathcal{F}_{Z,b}(t)\) is denoted by \(\hat{F}_{\mathcal{F}_{Z,b}}(t; z)\) and is given by

\[
\hat{F}_{\mathcal{F}_{Z,b}}(t; z) = P(\mathcal{F}_{Z,b}(t) > t).
\]

We focus on in particular inhomogeneous OU-process, which is defined as follows.

**Definition 3.2.** Let \((W_t)_{t \geq 0}\) denote Brownian motion on the probability space \((\Omega, \mathcal{F}, \mathbb{P})\). A solution \((Y_t)_{t \geq 0}\) to the stochastic differential equation

\[
dY_t = (\mu(t) - \lambda(t) Y_t) \, dt + \sigma(t) \, dW_t,
\]

where \(Y_0 = y \in \mathbb{R}\), is called inhomogeneous Ornstein-Uhlenbeck process. For \(\mu(t) : \mathbb{R}^+ \to \mathbb{R}\), \(\lambda(t) : \mathbb{R}^+ \to \mathbb{R}^+\) and \(\sigma(t) : \mathbb{R}^+ \to \mathbb{R}^+\) satisfying (a) \(|\mu(t) - \lambda(t) x| + |\sigma(t)| \leq C(1 + |x|)\) for \(C \in \mathbb{R}\), and (b) \(\lambda(t)\) is bounded, \(\forall t \geq 0\), the solution \((Y_t)_{t \geq 0}\) exists and is unique.

By Theorem 5.3.2 in Øksendal (2003), the properties (a) and (b) in Definition 3.2 ensure that the SDE (3.3) has a unique \(t\)-continuous solution. A sufficient condition for \(t\)-continuity is for \(\mu(t), \lambda(t)\) and \(\sigma(t)\) to be bounded. Next we show that the FPT distribution of an inhomogeneous OU-process crossing a time-dependent barrier is equivalent to the FPT distribution of a standardised OU-process crossing another time-dependent barrier.

**Definition 3.3.** The mean-reverting scaling function \(\alpha(t) : \mathbb{R}^+ \to \mathbb{R}^+\), the shift function \(\beta(t) : \mathbb{R}^+ \to \mathbb{R}\), and the time-compensation function \(\gamma(t) : \mathbb{R}^+ \to \mathbb{R}^+\) are to satisfy.
a) \(a(t), \beta(t)\) and \(\gamma(t)\) \(\in C^1(\mathbb{R}^+)\) for \(t > 0\);

b) \(\gamma(t)\) is non-decreasing for \(t > 0\);

c) \(a(t), \beta(t)\) and \(\gamma(t)\) satisfy the ODE system

\[
\begin{align*}
\sigma(\lambda(t))a(t)\sqrt{\gamma'(t)} & = 1 \\
\lambda(\gamma(t))\gamma'(t) - \frac{a'(t)}{a(t)} & = 1 \\
a(t)\mu(\gamma(t))\gamma'(t) + \frac{\alpha'(t)\beta(t)}{a(t)} - \beta'(t) - \beta(t)\lambda(\gamma(t))\gamma'(t) & = 0
\end{align*}
\]

subject to the initial condition

\[
a(0) = a_0 \in \mathbb{R}^+, \quad \beta(0) = \beta_0 \in \mathbb{R}, \quad \gamma(0) = 0
\]

where the constants \(a_0\) and \(\beta_0\) are pre-determined.

The time-dependent parameters \(\mu(t), \lambda(t)\) and \(\sigma(t)\) are specified in Definition 3.2

**Lemma 3.1.** Sufficient condition for the uniqueness and existence of \(a(t), \beta(t)\) and \(\gamma(t)\).

For \(\mu(t), \lambda(t)\) and \(\sigma(t)\) given in Definition 3.2 if \(\lambda(t) \in C^1(\mathbb{R}^+)\) and \(\sigma(t) \in C^2(\mathbb{R}^+)\), then the ODE system (3.4) has a unique local solution with initial conditions \(a(0) = a \in \mathbb{R}^+, \beta(0) = b \in \mathbb{R}, \gamma(0) = 0\).

**Proof.** The first two equations can be rearranged such that

\[
a(t) = \frac{1}{\sigma(t)\sqrt{\gamma'(t)}}, \quad \lambda(\gamma(t))\gamma'(t) = 1 + \frac{a'(t)}{a(t)}.
\]

Substituting the first equation into the second, we obtain

\[
\gamma'(t) = \frac{1}{\lambda(\gamma(t))}\left(1 - \frac{\sigma'(t)}{\sigma(t)} - \frac{1}{2\gamma(t)}\right).
\]

Next we consider

\[
f(t, \gamma) := \frac{1}{\lambda(\gamma)}\left(1 - \frac{\sigma'(t)}{\sigma(t)} - \frac{1}{2\gamma}\right).
\]

Since \(\lambda(t) \in C^1(\mathbb{R}^+)\) and \(\sigma(t) \in C^2(\mathbb{R}^+)\), \(f(t, \gamma)\) is continuous in \(t\) and \(\partial f / \partial \gamma\) is continuous in \(\gamma\).

By the Picard-Lindelöf theorem, see [Lindelöf [1894]], the unique local solution to \(\gamma(t)\) is guaranteed, which implies the solution to \(a(t)\) also exists and is unique. Since

\[
a(t)\mu(\gamma(t))\gamma'(t) + \frac{\alpha'(t)\beta(t)}{a(t)} - \beta'(t) - \beta(t)\lambda(\gamma(t))\gamma'(t) = 0
\]

is a first-order linear ODE with respect to \(\beta(t)\), whose solution is guaranteed to be unique, the ODE system (3.4) admits a unique solution. \(\square\)

**Remark 3.1.** The necessary and sufficient conditions for the uniqueness and existence of \(a(t), \beta(t)\) and \(\gamma(t)\) are non-trivial.

For \(a(t), \beta(t)\) and \(\gamma(t)\) specified as in Definition 3.3, an inhomogeneous OU-process is transformed into a standardised one as follows.

**Proposition 3.1.** Consider the inhomogeneous OU-process \((Y_t)_{t \geq 0}\) given in Definition 3.2. Assume \(a(t), \beta(t)\) and \(\gamma(t)\), given in Definition 3.3, satisfy the sufficient conditions in Lemma 3.1. Then the transformed process \((a(t)Y_t - \beta(t))_{t \geq 0}\) is a standardised OU-process \((\tilde{X}_t)_{t \geq 0}\) with initial value \(\tilde{X}_0 = ay - b\).
Proof. The solution to the SDE \((3.3)\) is given by
\[
Y_t = e^{-\int_0^t \lambda(u)du} \left[ x + \int_0^t \mu(s) \exp \left( \int_0^s \lambda(u)du \right) ds + \int_0^t \sigma(s) \exp \left( \int_0^s \lambda(u)du \right) dW_s \right]
\]
Since \(\gamma(\cdot)\) satisfies the sufficient condition in Lemma 3.1, \(\gamma(\cdot)\) exists. Therefore,
\[
Y_{\gamma(t)} = e^{-\int_0^{\gamma(t)} \lambda(u)du} \left[ x + \int_0^{\gamma(t)} \mu(s) \exp \left( \int_0^s \lambda(u)du \right) ds + \int_0^{\gamma(t)} \sigma(s) \exp \left( \int_0^s \lambda(u)du \right) dW_s \right],
\]
and
\[
Y_{\gamma(t)} = \exp \left( -\int_0^t \lambda(\gamma(u))\gamma'(u)du \right) \left[ x + \int_0^t \mu(\gamma(s))\gamma'(s) \exp \left( \int_0^s \lambda(\gamma(u))\gamma'(u)du \right) ds + \int_0^t \sigma(\gamma(s))\sqrt{\gamma'(s)} \exp \left( \int_0^s \lambda(\gamma(u))\gamma'(u)du \right) dW_s \right].
\]
It follows that
\[
dY_{\gamma(t)} = [\mu(\gamma(t))\gamma'(t) - \lambda(\gamma(t))\gamma'(t)] Y_{\gamma(t)} dt + \sigma(\gamma(t))\sqrt{\gamma'(t)} dW_t,
\]
and hence
\[
d\tilde{X}_t = \left[ \alpha'(t) Y_{\gamma(t)} - \beta'(t) \right] dt + \alpha(t)dY_{\gamma(t)}
\]
\[
= \left[ \alpha'(t) \frac{\tilde{X}_t + \beta(t)}{\alpha(t)} - \beta'(t) + \alpha(t)\mu(\gamma(t))\gamma'(t) - \alpha(t)\lambda(\gamma(t))\gamma'(t) \right] dt
\]
\[
+ \alpha(t)\sigma(t)\sqrt{\gamma'(t)} dW_t
\]
\[
= -\tilde{X}_t dt + dW_t.
\]
In the last step Definition 3.3 is used. We thus have that \((\tilde{X}_t)_{t\geq 0}\) is a standardised OU-process. \(\Box\)

Now we are in the position to present the main theorem that links the FPT distribution functions of the inhomogeneous and the standardised OU-processes.

**Theorem 3.1.** Let \((Y_t)_{t\geq 0}\) be the inhomogeneous OU-process in Definition 3.2 and assume that the ODE \((3.4)\) has a unique solution. Then,
\[
F_{X_{\gamma(t)},a(t)}(t; \gamma) = F_{\mathcal{F}_{\tilde{X},b(t)}}(\gamma^{-1}(t); \tilde{x})
\] (3.6)
where \((\tilde{X}_t)_{t\geq 0}\) is a standardised OU-process with initial value \(\tilde{X}_0 = \tilde{x} = \alpha_0Y - \beta_0\), and
\[
g(t) = \alpha(t)b(\gamma(t)) - \beta(t).
\]
An equivalent statement is that \(\mathcal{F}_{Y,b(t)}\) and \(\gamma(\mathcal{F}_{\tilde{X},b(t)})\) are equal in distribution.

**Proof.** First, we show \(\mathcal{F}_{Y,b(t)}\) and \(\gamma(\mathcal{F}_{\tilde{X},a(t)b(\gamma(t)) - \beta(t)})\) are equal in distribution. We have that
\[
\mathcal{F}_{Y,b(t)} = \inf \{ t > 0 : Y_t \geq b(t) \} = \inf \{ \gamma(t) > 0 : Y_{\gamma(t)} \geq b(\gamma(t)) \}
\]
\[
= \inf \{ \gamma(t) > 0 : \alpha(t)Y_{\gamma(t)} - \beta(t) \geq \alpha(t)b(\gamma(t)) - \beta(t) \}.
\]
Since \(\gamma(\cdot)\) is monotone, non-decreasing and positive, we deduce
\[
\mathcal{F}_{Y,b(t)} = \gamma \left( \inf \{ t > 0 : \alpha(t)Y_{\gamma(t)} - \beta(t) \geq \alpha(t)b(\gamma(t)) - \beta(t) \} \right).
\]
By Proposition \[3.1\] we know that the process \((a(t)Y_{\gamma(t)} - \beta(t))_{t \geq 0}\) has the law of a standardised OU-process. Therefore,

\[
\mathcal{F}_{Y, b(t)} = \mathcal{G} \left( \inf \left\{ t > 0 : a(t)Y_{\gamma(t)} - \beta(t) \geq a(t)b(\gamma(t)) - \beta(t) \right\} \right) = \mathcal{G} \left( \mathcal{F}_{X, a(t)b(\gamma(t)) - \beta(t)} \right).
\]

Then, it follows that

\[
\tilde{F}_{\mathcal{F}_{Y, b(t)}}(t; x) = \mathbb{P} \left( \mathcal{F}_{Y, b(t)} > t \mid Y_0 = x \right) = \mathbb{P} \left( \gamma \left( \mathcal{F}_{X, a(t)b(\gamma(t)) - \beta(t)} > t \mid X_0 = \bar{x} \right) \right) = \tilde{F}_{\mathcal{F}_{X, b(t)}}(\gamma^{-1}(t); \bar{x}).
\]

\[\square\]

**Example 3.1 (Seasonal trend).** One example in practice is to apply a seasonality function to the mean-reverting level function \(\mu(t)\). In this example, we show how we can utilise Theorem \[3.1\] to transform the problem for an inhomogeneous OU-process hitting a constant barrier to the problem for a standardised OU-process hitting a periodic barrier. We consider the inhomogeneous OU-process \((Y_t)_{t \geq 0}\), parametrised by \(\mu(t) = A \sin(\theta t + \varphi)\), \(\lambda(t) = \lambda\) and \(\sigma(t) = \sigma\), with initial value \(Y_0 = x\), where \(A, \theta, \varphi \in \mathbb{R}\) and \(\lambda, \sigma > 0\). The constant barrier is denoted by \(b\). The mean-reverting scaling function \(a(t)\) and time-compensation function \(\gamma(t)\) are given by \(a(t) = \sqrt{\lambda}/\sigma\) and \(\gamma(t) = t/\lambda\). Then \(\beta(t)\) satisfies

\[
\beta'(t) + \beta(t) = \mu \left( \frac{t}{\lambda} \right) \frac{1}{\sigma \sqrt{\lambda}}.
\]

The associated ODE \[(3.7)\], in this particular case, has unique solution

\[
\beta(t) = Be^{-t} + \frac{A\sqrt{\lambda}}{\sigma \sqrt{\lambda^2 + \theta^2}} \sin \left( \frac{\theta}{\lambda} t + \varphi - \arctan \left( \frac{\theta}{\lambda} \right) \right),
\]

where \(B\) is a constant so to match the initial condition. For convenience, we let \(B = 0\) by imposing the initial condition

\[
\beta(0) = \frac{A\sqrt{\lambda}}{\sigma \sqrt{\lambda^2 + \theta^2}} \sin \left( \varphi - \arctan \left( \frac{\theta}{\lambda} \right) \right),
\]

which, by Theorem \[3.1\] means that the standardised OU-process \((\tilde{X}_t)_{t \geq 0}\) starts from

\[
\tilde{X}_0 = \frac{\sqrt{\lambda}}{\sigma} x_0 - \frac{A\sqrt{\lambda}}{\sigma \sqrt{\lambda^2 + \theta^2}} \sin \left( \varphi - \arctan \left( \frac{\theta}{\lambda} \right) \right).
\]

Then we have a particular solution for \(\beta(t)\):

\[
\beta(t) = \frac{A\sqrt{\lambda}}{\sigma \sqrt{\lambda^2 + \theta^2}} \sin \left( \frac{\theta}{\lambda} t + \varphi - \arctan \left( \frac{\theta}{\lambda} \right) \right).
\]

By Theorem \[3.1\] we only need to study the probability of a standardised OU-process with initial value

\[
\tilde{X}_0 = \bar{x} = \frac{x\sqrt{\lambda}}{\sigma} - \frac{A\sqrt{\lambda}}{\sigma \sqrt{\lambda^2 + \theta^2}} \sin \left( \varphi - \arctan \left( \frac{\theta}{\lambda} \right) \right)
\]

crossing a periodic barrier

\[
g(t) = \frac{b\sqrt{\lambda}}{\sigma} - \frac{A\sqrt{\lambda}}{\sigma \sqrt{\lambda^2 + \theta^2}} \sin \left( \frac{\theta}{\lambda} t + \varphi - \arctan \left( \frac{\theta}{\lambda} \right) \right).
\]

The new problem for a standardised OU-process hitting a periodic barrier can be solved by consid-
\[
\frac{\partial \tilde{F}_{x,g(t)}}{\partial t} = -\bar{x} \frac{\partial \tilde{F}_{x,g(t)}}{\partial \bar{x}} + \frac{1}{2} \frac{\partial^2 \tilde{F}_{x,g(t)}}{\partial \bar{x}^2}
\]

subject to the initial condition \( \tilde{F}_{x,g(t)}(0, \bar{x}) = 1 \) and the boundary condition \( \tilde{F}_{x,g(t)}(t, g(t)) = 0 \).

### 4 Multiple crossings of an inhomogeneous OU-process

For an inhomogeneous OU-process with parameter functions in \( C^1(\mathbb{R}^+) \), one can transform its barrier-crossing problem to one associated with the standardised OU-process and a time-dependent barrier function. The time-dependent barrier can be approximated by a piece-wise constant function. This is due to the fact that any continuous function can be approximated with a piecewise constant function to arbitrary accuracy given a sufficiently large number of partitions. The problem thus reduces to a multiple-crossing problem for a standardized OU-process.

Alternatively, one may directly use the piecewise constant approximation for the parameter functions of the inhomogeneous OU-process. This alternative method leads to a multiple-crossing problem for a locally-homogeneous OU-process.

The first method transforms the inhomogeneous OU-process to a global standardized OU-process by solving an ODE system given in Definition 3.3. It uses piecewise constant functions to approximate the time-varying barriers. This scheme requires further conditions to be satisfied, such as continuity for parameter functions, for the transformation to be well-defined. In addition, solving the ODE system can be difficult. The second method does not rely on such a transformation, however, it results in a locally homogeneous OU-process with piecewise constant barriers, where the time steps for the barriers and OU-parameters may not necessarily match.

In applications, the method one should select is decided on a case by case basis. In Figure 5, we simulate a time-inhomogeneous OU-process with a time-dependent barrier. The application of the first method can offset most of the time-dependences from the parameters and the barrier, however, the direct approximation method will lead to higher approximation error. In order to reach the same level of accuracy, one may have to approximate using more time segments, which complicates the problem.

Figure 5: Inhomogeneous OU-process with \( b(t) = 1 + 0.65 \sin(10t + \arctan(10)) \), \( \mu(t) = \sin(10t) \) and \( \lambda = \sigma = 1 \).
However, this does not mean that the first method is always better than the direct approximation approach. For example in Figure 6, with the same number of discretizations, the transformation method leads to a higher approximation error.

In general, if any of the three OU parameter functions are not in $C^1(\mathbb{R}^+)$, one should use the direct approximation method. Otherwise, when applying the transformation method, the transformed barrier function $g(t) = \alpha(t)b(\gamma(t)) - \beta(t)$ can be written as

$$g(t) = e^{-t} \left[ \alpha_0 e^{\int_0^t \lambda(s) ds} b(\gamma(t)) - \beta_0 - \alpha_0 \beta_0 \int_0^{\gamma(t)} e^{\int_0^s \lambda(u) du} \mu(s) ds \right],$$

where $\gamma(t)$ is obtained from the equation

$$\alpha_0 e^{\int_0^t \lambda(s) ds - t} \sigma(\gamma(t)) \sqrt{\gamma'(t)} = 1.$$

Although it is difficult to come up with a general principle on which method one should adopt, we can conclude a rule for some special cases, such as in Proposition 4.1.

**Proposition 4.1.** Assume that the inhomogeneous OU-process in Definition 3.2 has constant coefficients $\sigma(t) = \sigma$, $\lambda(t) = \lambda$, $\mu(t) \in C^2([t_1, t_2])$, and constant barrier $b(t) = b$. The transformed barrier function is denoted by $g(t) = \alpha(t)b(\gamma(t)) - \beta(t)$, where $\alpha(t), \beta(t)$ and $\gamma(t)$ are defined in Definition 3.3. Then in the bounded time interval $[t_1, t_2]$, if $\forall t \in [t_1, t_2]$, $g'(\lambda t)$ and $\mu'(t)$ are both positive or negative, $g''(\lambda t)$ and $\mu''(t)$ are both positive or negative, and $\mu(t)$ satisfies

$$\max_{t \in [t_1, t_2]} \frac{e^{-\lambda t} \beta_0 (\sigma \sqrt{\lambda} + \lambda \int_0^t e^{2s} \mu(s) ds) + \beta_0 \mu'(t) - \beta_0 \mu(t)}{\max_{t \in [t_1, t_2]} \mu''(t)} < \sigma \sqrt{\lambda},$$

g(t) is less convex than $\mu(t)$, i.e. it is better to use the transformation method.

**Proof.** Here we prove the case for positive $g'(\lambda t), \mu'(t), g''(\lambda t)$ and $\mu''(t)$, since the other case can be shown in the same way. For $\sigma(t) = \sigma$, $\lambda(t) = \lambda$ and $b(t) = b$, we can solve the ODE system (3.4), and we obtain

$$\alpha(t) = \frac{\sqrt{\lambda}}{\sigma}, \quad \beta(t) = \beta_0 e^{-t} \left[ 1 + \frac{1}{\sigma \sqrt{\lambda}} \int_0^t e^{\mu(\frac{s}{\lambda})} ds \right], \quad \gamma(t) = \frac{t}{\lambda},$$

Figure 6: Inhomogeneous OU-process with $b(t) = 0.5$, $\mu(t) = \frac{1}{1 + e^{-t}}$ and $\lambda = \sigma = 1$. 

\[\text{Graphs showing barrier functions.}\]
which are defined for \( t \in [\lambda t_1, \lambda t_2] \). Since \( \mu(t) \in C^2([t_1, t_2]) \), we have

\[
g(t) = \frac{\sqrt{\lambda}}{\sigma} b - \beta(t) \in C^2([t_1, t_2]).
\]

Therefore,

\[
\inf_{t \in [t_1, t_2]} g''(t) \leq \max_{t \in [t_1, t_2]} \beta_0 e^{-t} \left[ \frac{\beta_0 e^{-t} + \beta_0}{\sigma} \int_0^t e^s \mu' \left( \frac{e^s}{\lambda} \right) ds \right] + \beta_0 \mu' - \beta_0 \mu(t)
\]

\[
\inf_{t \in [t_1, t_2]} \mu''(t) \leq \max_{t \in [t_1, t_2]} \mu''(t)
\]

\[
= \frac{1}{\sigma \sqrt{\lambda}} \left( \sigma \sqrt{\lambda} + \lambda \int_0^t e^{\lambda s} ds \right) + \beta_0 \mu' - \beta_0 \mu(t)
\]

Since \( g(t) \) and \( \mu(t) \) are monotone increasing and convex, if

\[
\max_{t \in [t_1, t_2]} e^{-t} \beta_0 \left( \sigma \sqrt{\lambda} + \lambda \int_0^t e^{\lambda s} ds \right) + \beta_0 \mu' - \beta_0 \mu(t) < \sigma \sqrt{\lambda},
\]

we have

\[
\inf_{t \in [t_1, t_2]} \frac{g''(t)}{\nu''(t)} < 1.
\]

Hence \( g(t) \) is less convex than \( \mu(t) \). It means that for a fixed level of accuracy, the transformed barrier \( g(t) \) can be approximated a piecewise constant function with with fewer segments than \( \mu(t) \), i.e. \( g(t) \) can be approximated better with the transformation method. \( \square \)

**Example 4.1.** Let \( \lambda = \sigma = \beta_0 = 1 \) and \( \mu(t) = t^3 \in C([t_1, t_2]) \). Then \( g(t) = b - \beta(t) \in C([t_1, t_2]) \).

Thus,

\[
\max_{t \in [t_1, t_2]} \frac{|g''(t)|}{|\mu''(t)|} = \max_{t \in [t_1, t_2]} \frac{|e^{-t} + \int_0^t e^s \mu' \mu(s) ds + \mu'(t) - \mu(t)|}{\max_{t \in [t_1, t_2]} |\mu''(t)|} = \max_{t \in [0, 1]} \frac{|-6 + 7e^{-t} + 6t|}{\max_{t \in [0, 1]} |6t|}.
\]

If \( t_1 = 0 \) and \( t_2 = 1 \), then

\[
\max_{t \in [t_1, t_2]} \frac{|g''(t)|}{|\mu''(t)|} = \frac{7e^{-1}}{6} < 1
\]

In this case, it is more accurate to use the transformation method. This is because \( \mu(t) \) is convex, monotone increasing, and more convex than \( g(t) \). Hence \( g(t) \) tends to be smoother than \( \mu(t) \), where \( g(t) \) is better approximated by piecewise constant functions than \( \mu(t) \). However, if \( t_1 = 0 \) and \( t_2 = 0.1 \), then

\[
\max_{t \in [0, 1]} \frac{|-6 + 7e^{-t} + 6t|}{\max_{t \in [0, 1]} |6t|} = \frac{35e^{-0.1}}{3} - 9 > 1
\]

In this case it is less accurate to use the transformation method.

Next we investigate the case where \( \mu(t), \lambda(t) \) and \( \sigma(t) \), the parameter functions of the inhomogeneous OU-process \((Y_t)_{t \geq 0}\), and the barrier function \( b(t) \) are càdlàg piecewise constant functions.
Let the parameter functions \( b \) specified by
\[
\mu(t) = \sum_{i=1}^{N^{(\mu)}} \mu_i \mathbb{1}(t \in [t^{(\mu)}_{i-1}, t^{(\mu)}_i)), \quad \lambda(t) = \sum_{i=1}^{N^{(\lambda)}} \lambda_i \mathbb{1}(t \in [t^{(\lambda)}_{i-1}, t^{(\lambda)}_i)), \quad \sigma(t) = \sum_{i=1}^{N^{(\sigma)}} \sigma_i \mathbb{1}(t \in [t^{(\sigma)}_{i-1}, t^{(\sigma)}_i))
\]
and
\[
b(t) = \sum_{i=1}^{N} b_i \mathbb{1}(t \in [t_{i-1}, t_i))
\]
for all \( i = 1, 2, \ldots, N \), where \( \lambda_i, \sigma_i \in \mathbb{R}^+, \mu_i, b_i \in \mathbb{R} \) and \( b_0 > Y_0 \). Here, we consider a finite-time horizon where \( t^{(\mu)}_{N^{(\mu)}} = t^{(\lambda)}_{N^{(\lambda)}} = t^{(\sigma)}_{N^{(\sigma)}} = T \), and \( \mathbb{1}(\cdot) \) denotes the indicator function. We study the following probabilities:

\[
\mathbb{P}\left(M^{Y}_{t_0} < b_1, M^{Y}_{t_1,t_2} < b_2, \ldots, M^{Y}_{t_{N-1},t_N} < b_N\right), \quad \text{(4.1)}
\]
\[
\mathbb{P}\left(M^{Y}_{t_0,t_1} \geq b_1, M^{Y}_{t_1,t_2} \geq b_2, \ldots, M^{Y}_{t_{N-1},t_N} \geq b_N\right), \quad \text{(4.2)}
\]
where \( M^{Y}_{t_{i-1},t_i} = \sup_{t \in (t_{i-1},t_i]} Y \). In particular, probability (4.1) is equal to the probability that the FPT of this inhomogeneous OU-process is larger than \( t_N \); probability (4.2) is the probability that the inhomogeneous OU-process crosses the barrier in each interval, i.e. the multiple crossing probability. We may consider the following discretisation schemes:

1) Matching time-discretization for \( \mu(t), \lambda(t), \sigma(t) \) and \( b(t) \), i.e. \( t^{(\mu)}_i = t^{(\lambda)}_i = t^{(\sigma)}_i = t_i \) for all \( i = 0, 1, \ldots, N \).

2) Matching time-discretization for \( \mu(t), \lambda(t) \) and \( \sigma(t) \) only. i.e. \( t^{(\mu)}_i = t^{(\lambda)}_i = t^{(\sigma)}_i \) for all \( i = 0, 1, \ldots, N^{(\mu)} \).

3) Non-matching time-discretizations for any of \( \mu(t), \lambda(t), \sigma(t) \) and \( b(t) \).

One can show that the probabilities (4.1) and (4.2) in the last two cases can be further reduced to the first case by utilising Theorem 4.1. Therefore, in what follows, we will focus on the first case unless specified otherwise, and we shall show the reduction methods from the case 2) and 3) to case 1) in Section 4.3.

### 4.1 The joint distribution and multivariate survival function for multiple maxima of a continuous Markov process in consecutive intervals

Now we start from a general theorem for continuous Markov process. We recall the definition of a Markov process, see for instance Bingham and Kiesel (2013). In this section, we consider the probability space \((\Omega, \mathcal{F}, \mathbb{P})\) that is equipped with a filtration \(\mathcal{F}_t\).

**Definition 4.1.** Let \((\mathcal{F}_t)\) be the natural filtration generated by \((Z_u)_{u \in [0, t]}\). A Markov process \((Z_t)_{t \geq 0}\) satisfies \(\mathbb{P}(A | \mathcal{F}_t) = \mathbb{P}(A | Z_t)\), for all \(A \in \sigma(Z_u : u \geq t)\).

**Lemma 4.1.** If \((Z_t)_{t \geq 0}\) is a Markov process, and \(\forall A \in \sigma(Z_u : u \geq t), B \in \sigma(Z_u : u \leq t), \mathbb{P}(A \cap B | Z_t) = \mathbb{P}(A | Z_t) \mathbb{P}(B | Z_t)\). Hence it follows that \(\mathbb{P}(A | B, Z_t) = \mathbb{P}(A | Z_t)\).

**Proof.** This is straightforward and shown in most standard textbooks on probability, see, e.g., Bingham and Kiesel (2013).

We prove Theorem 4.1 by use of the conditional independence property. We consider the time steps \(0 = t_0 < t_1 < \cdots < t_N = T\), and denote the barrier level in the interval \([t_{i-1}, t_i]\) by \(b_i\). In
Theorem 4.1: we calculate the joint distribution function, or the survival function, of the maxima of a continuous Markov process in each interval. Here, we denote \( M_{t_{i-1}, t_i} := \sup_{t \in [t_{i-1}, t_i]} Z_t. \)

**Theorem 4.1.** Let \( b_1, \ldots, b_N \in \mathcal{D} := \text{Dom}(Z_t). \) The joint distribution and survival functions of the maxima of a continuous Markov process \((Z_t)_{t \geq 0}\) in consecutive intervals are given, respectively, by

\[
P(M_{t_0, t_1} < b_1, \ldots, M_{t_{N-1}, t_N} < b_N | Z_0 = z_0) = \int_{z_0} \psi_1(t_0, t_1, z_0, z_1, b_1) \cdots \int_{z_1} \psi_N(t_{N-2}, t_{N-1}, z_{N-2}, z_{N-1}, b_{N-1}) \times q(t_{N-1}, t_N, Z_{N-1}) dz_{N-1} \cdots dz_1,
\]

\[
P(M_{t_0, t_1} \geq b_1, \ldots, M_{t_{N-1}, t_N} \geq b_N | Z_0 = z_0) = \int_{z_0} \kappa_1(t_0, t_1, z_0, z_1, b_1) \cdots \int_{z_1} \kappa_N(t_{N-2}, t_{N-1}, z_{N-2}, z_{N-1}, b_{N-1}) \times \bar{q}(t_{N-1}, t_N, Z_{N-1}) dz_{N-1} \cdots dz_1,
\]

where

\[
\psi_i(t_{i-1}, t_i, z_{i-1}, z_i, b_i) = P(M_{t_{i-1}, t_i} < b_i | Z_{t_{i-1}} = z_{i-1}, Z_{t_i} = z_i) p(t_{i-1}, t_i, z_{i-1}, z_i),
\]

\[
\kappa_i(t_{i-1}, t_i, z_{i-1}, z_i, b_i) = P(M_{t_{i-1}, t_i} \geq b_i | Z_{t_{i-1}} = z_{i-1}, Z_{t_i} = z_i) p(t_{i-1}, t_i, z_{i-1}, z_i),
\]

\[
q(t_{N-1}, t_N, Z_{N-1}) = P(M_{t_{N-1}, t_N} < b_N | Z_{t_{N-1}} = z_{N-1}),
\]

\[
\bar{q}(t_{N-1}, t_N, Z_{N-1}) = 1 - q(t_{N-1}, t_N, Z_{N-1}).
\]

Here \( p(t_{i-1}, t_i, z_{i-1}, z_i) \), where \( i = 1, \ldots, N-1 \), is the transition density function of the process \((Z_t)_{t \geq 0}\) from state \( z_{i-1} \) at time \( t_{i-1} \) to state \( z_i \) at time \( t_i \).

**Proof.** Here we show the proof for the joint distribution function. The proof for the joint survival function is similar, since we also utilise the Markov conditional independence property. We proceed with a proof by induction.

1. Case \( N = 2 \): We know that

\[
P(M_{t_0, t_1} < b_1, M_{t_1, t_2} < b_2 | Z_0 = z_0) = \int_{z_0} P(M_{t_0, t_1} < b_1, M_{t_1, t_2} < b_2 | Z_{t_1} = z_1, Z_0 = z_0) \mathbb{P}(Z_{t_1} \in dz_1 | Z_0 = z_0).
\]

Since \( \{M_{t_0, t_1} < b_1\} \in \sigma(Z_s : s \leq t_1) \) and \( \{M_{t_1, t_2} < b_2\} \in \sigma(Z_s : s \geq t_1) \), we have

\[
P(M_{t_0, t_1} < b_1, M_{t_1, t_2} < b_2 | Z_0 = z_0) = \int_{z_0} P(M_{t_0, t_1} < b_1 | Z_{t_1} = z_1, Z_0 = z_0) P(M_{t_1, t_2} < b_2 | Z_{t_1} = z_1, Z_0 = z_0) \mathbb{P}(Z_{t_1} \in dz_1 | Z_0 = z_0),
\]

by Lemma 4.1. Then by the Markov property, we obtain

\[
P(M_{t_0, t_1} < b_1, M_{t_1, t_2} < b_2 | Z_0 = z_0) = \int_{z_0} \psi_1(t_0, t_1, z_0, z_1, b_1) q(t_1, t_2, Z_1) dz_1,
\]

which is the case \( N = 2 \) in Theorem 4.1.

2. Consider Theorem 4.1 for \( N = K \) such that

\[
P(M_{t_0, t_1} < b_1, M_{t_1, t_2} < b_2, \ldots, M_{t_{K-1}, t_K} < b_K | Z_0 = z_0) = \int_{z_0} \psi_1(t_0, t_1, z_0, z_1, b_1) \cdots \int_{z_1} \psi_{K-1}(t_{K-2}, t_{K-1}, z_{K-2}, z_{K-1}, b_{K-1}) q(t_{K-1}, t_K, Z_{K-1}) dz_{K-1} \cdots dz_1.
\]

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Now consider the case when \( N = K + 1 \). We have
\[
\mathbb{P}(M_{t_0,t_1} < b_1, M_{t_1,t_2} < b_2, \cdots, M_{t_{K-1},t_K} < b_K, M_{t_K,t_{K+1}} < b_{K+1} \mid Z_0 = z_0) = \int_{\mathbb{R}^N} \mathbb{P}(M_{t_0,t_1} < b_1, M_{t_1,t_2} < b_2, \cdots, M_{t_{K-1},t_K} < b_K, M_{t_K,t_{K+1}} < b_{K+1} \mid Z_{t_1} = z_1, Z_0 = z_0) \mathbb{P}_0(z_1 \mid Z_0 = z_0) dz_1.
\]
(4.3)

Then, by Lemma 4.1, since \( \{M_{t_0,t_1} < b_1\} \in \sigma(Z_s : s \leq t_1) \) and \( \{M_{t_1,t_2} < b_2, \cdots, M_{t_{K-1},t_K} < b_K, M_{t_K,t_{K+1}} < b_{K+1} \} \in \sigma(Z_s : s \geq t_1) \), we have
\[
\mathbb{P}(M_{t_0,t_1} < b_1, M_{t_1,t_2} < b_2, \cdots, M_{t_{K-1},t_K} < b_K, M_{t_K,t_{K+1}} < b_{K+1} \mid Z_{t_1} = z_1, Z_0 = z_0) = \mathbb{P}(M_{t_0,t_1} < b_1 \mid Z_{t_1} = z_1, Z_0 = z_0) \mathbb{P}(M_{t_1,t_2} < b_2, \cdots, M_{t_{K-1},t_K} < b_K, M_{t_K,t_{K+1}} < b_{K+1} \mid Z_{t_1} = z_1, Z_0 = z_0).
\]

For \( N = K \), by the Markov property, we have
\[
\mathbb{P}(M_{t_1,t_2} < b_2, M_{t_2,t_3} < b_3, \cdots, M_{t_{K-1},t_K} < b_K, M_{t_K,t_{K+1}} < b_{K+1} \mid Z_{t_1} = z_1) = \int_{\mathbb{R}} \psi_1(t_1, t_2, z_1, z_2, b_2) \cdots \int_{\mathbb{R}} \psi_K(t_{K-1}, t_K, z_{K-1}, z_K, b_K) q(t_K, t_{K+1}, z_K) dz_K \cdots dz_2.
\]

By iterated substitutions, we obtain the integration formulae (4.3), and the proof is complete for the case \( N = K + 1 \).

\[\square\]

Now we decompose the joint distribution and survival functions of the maxima of a continuous Markov process in consecutive intervals into three components:

a) The distribution or survival function of the maximum of the continuous Markov process in a given interval conditional on its starting value and terminal value;

b) The transition density function in a given interval;

c) The distribution function of the maximum of the continuous Markov process in a given interval conditional on its starting value only.

The third item is equivalent to the FPT distribution for a Markov process to cross a constant barrier in a given interval. Essentially, the first item involves the calculation of the maximum of a continuous Markov bridge, which involves Proposition 4.2.

**Proposition 4.2.** Let \((Z_t)_{t \geq 0}\) be a continuous Markov process where \( Z_0 = z \), and \( \tau_{Z,b} := \inf\{t \geq 0 : Z_t \geq b\} \). Then,
\[
\mathbb{P}(M_{0,T} \geq b \mid Z_T = z', Z_0 = z) = \left\{ \begin{array}{ll}
\int_0^T \frac{p(t,T,b,z')}{p(0,T,z,z')} f_{\tau_{Z,b}}(t;z) dt, & \text{if } z, z' < b, \\
1, & \text{otherwise},
\end{array} \right.
\]
where \( p(t, T, b, z') \) denotes the transition density function of \((Z_t)_{t \geq 0}\) from state \( b \) at time \( t \) to state \( z' \) at time \( T \), and \( f_{\tau_{Z,b}}(t;z) \) is the probability density function of first-passage-time \( \tau_{Z,b} \) with \( Z_0 = z \).

**Proof.** We first consider the case that \( z, z' < b \). Since the two events \( \{M_{0,T} \geq b\} \) and \( \{\tau_{Z,b} \leq T\} \) are equivalent,
\[
\mathbb{P}(M_{0,T} \geq b \mid Z_T = z', Z_0 = z) = \mathbb{P}(\tau_{Z,b} \leq T \mid Z_T = z', Z_0 = z) = \int_0^T f_{\tau_{Z,b}}(t \mid Z_T = z', Z_0 = z) dt,
\]
where \( f_{Z,b}(t \mid Z_T = z', Z_0 = z) \) denotes the conditional density function of first-passage-time \( \tau_{Z,b} \). By Bayes' theorem, we have

\[
P(M_{0,T} \geq b \mid Z_T = z', Z_0 = z) = \int_0^T p\left(0, T, z, z' \mid \tau_{Z,b} = t\right) \frac{f_{Z,b}(t; z)}{p(0, T, z, z')} dt.
\]

Here, \( p\left(0, T, z, z' \mid \tau_{Z,b} = t\right) \) denotes the conditional transition density of \((Z_t)_{t \geq 0}\) from state \( b \) at time \( t \) to state \( z' \) at time \( T \). Since \((Z_t)\) is a continuous process, we have \( \{\tau_{Z,b} \geq t\} \cap \{Z_t \geq b\} = \{\tau_{Z,b} \geq t\} \).

Hence,

\[
p\left(0, T, z, z' \mid \tau_{Z,b} = t\right) = p\left(0, T, z, z' \mid \tau_{Z,b} = t, Z_t = b\right).
\]

Because \( \{\tau_{Z,b} \geq t\} \in \sigma(Z_s : 0 \leq s \leq t) \) and \( \{Z_T \geq z'\} \in \sigma(Z_s : t < s \leq T) \), by Lemma 4.1, we have

\[
p\left(0, T, z, z' \mid \tau_{Z,b} = t, Z_t = b\right) = p\left(0, T, z, z' \mid Z_T = b\right) = p\left(t, T, b, z'\right).
\]

Therefore,

\[
P(M_{0,T} \geq b \mid Z_T = z', Z_0 = z) = \int_0^T p(t, T, b, z') \frac{f_{Z,b}(t; z)}{p(0, T, z, z')} dt.
\]

In the case either \( z \geq b \) or \( z' \geq b \), because of the continuous property of process \((Z_t)_{t \geq 0}\), the probability turns to be 1 obviously.

With Proposition 4.2 and Theorem 4.1, we are able to at least approximate the joint distribution and survival functions of the maxima of a continuous Markov process in consecutive intervals, provided that we know its transition density function and its FPT density for a constant barrier.

### 4.2 Simplified computation of survival functions via restrictions

Now we present a theorem which simplifies the calculation of the survival function in Theorem 4.1 if the restriction \( P(Z_{t_0} < b_1, Z_{t_1} < b_2, \ldots, Z_{t_N} < b_N) = 1 \) is introduced. We can prove that if, at the end of each interval, the terminal value of the process is lower than the barrier level in the subsequent time interval, the nested integral simplifies to a product of single integrals.

**Theorem 4.2.** Given that \( P(Z_{t_0} < b_1, Z_{t_1} < b_2, \ldots, Z_{t_N} < b_N) = 1 \), the joint survival function of the maxima of a continuous Markov process \((Z_t)_{t \geq 0} \in \mathbb{R}\) in consecutive left-open-and-right-closed time intervals is given by

\[
P(M_{t_0,t_1} \geq b_1, \ldots, M_{t_{N-1},t_N} \geq b_N \mid Z_0 = z_0) = \prod_{i=2}^N \int_{-\infty}^b P(M_{t_{i-2},t_{i-1}} \geq b_{i-1}|Z_{t_{i-1}} = x_i) P(M_{t_{i-1},t_i} \geq b_i|Z_{t_i} = x_i, Z_0 = z_0) P(Z_{t_{i-1}} \in dx_i|Z_0 = z_0)
\]

\[
\times P(M_{t_{0},t_1} \geq b_1 \mid Z_0 = z_0)
\]
Furthermore, since \( \{M_{t_0,t_1} \geq b_1, \ldots, M_{t_{i-3},t_{i-2}} \geq b_{i-2}\} \subset \mathcal{F}_{t_{i-1}} \) and \( \{M_{t_{i-1},t_i} \geq b_1\} \subset \mathcal{F}_{t_i} \setminus \mathcal{F}_{t_{i-1}} \), by Lemma 4.1, we obtain
\[
\mathbb{P}(M_{t_{0},t_1} \geq b_1, \ldots, M_{t_{i-1},t_i} \geq b_{i-1} | Z_0 = z_0, C, M_{t_{0},t_1} \geq b_1, \ldots, M_{t_{i-2},t_{i-1}} \geq b_{i-2})
= \mathbb{P}(M_{t_{0},t_1} \geq b_1 | Z_0 = z_0, C, \tau^{(i-1)} \leq t_{i-1}, Z_{t_{i-1}} = b_{i-1})
= \mathbb{P}(M_{t_{i-1},t_i} \geq b_i | Z_0 = z_0, C, \tau^{(i-1)} \leq t_{i-1}) = \mathbb{P}(M_{t_{i-1},t_i} \geq b_i | Z_0 = z_0, C, M_{t_{i-2},t_{i-1}} \geq b_{i-1}).
\]
It means that conditional on event \( C \), the discrete process \( (L_i)_{i \in \mathbb{N}} \) defined by \( L_i = 1_{M_{t_{i-1},t_i} \geq b_i} \) is a discrete Markov process. Hence,
\[
\mathbb{P}(M_{t_{0},t_1} \geq b_1, \ldots, M_{t_{i-1},t_i} \geq b_{i} | Z_0 = z_0, C) = \prod_{i=2}^{N} \mathbb{P}(M_{t_{i-1},t_i} \geq b_i | Z_0 = z_0, C, M_{t_{i-2},t_{i-1}} \geq b_{i-1})
\]
Therefore, if \( \mathbb{P}(Z_{t_1} < b_2, \ldots, Z_{t_N} < b_N) = 1 \),
\[
\mathbb{P}(M_{t_{0},t_1} \geq b_1, \ldots, M_{t_{N-1},t_N} \geq b_N | Z_0 = z_0)
= \mathbb{P}(M_{t_{0},t_1} \geq b_1 | Z_0 = z_0) \prod_{i=2}^{N} \mathbb{P}(M_{t_{i-1},t_i} \geq b_i | Z_0 = z_0, M_{t_{i-2},t_{i-1}} \geq b_{i-1})
= \mathbb{P}(M_{t_{0},t_1} \geq b_1 | Z_0 = z_0)
\times \prod_{i=2}^{N} \int_{-\infty}^{b_i} \mathbb{P}(M_{t_{i-1},t_i} \geq b_i | Z_{t_{i-1}} = x) \mathbb{P}(Z_{t_{i-1}} \in dx | Z_0 = z_0, M_{t_{i-2},t_{i-1}} \geq b_{i-1})
= \mathbb{P}(M_{t_{0},t_1} \geq b_1 | Z_0 = z_0)
\times \prod_{i=2}^{N} \int_{-\infty}^{b_{i-1}} \mathbb{P}(M_{t_{i-1},t_i} \geq b_i | Z_{t_{i-1}} = x, Z_0 = z_0) \mathbb{P}(Z_{t_{i-1}} \in dx | Z_0 = z_0).
\]

Here, with the additional condition \( \mathbb{P}(Z_{t_0} < b_1, Z_{t_1} < b_2, \ldots, Z_{t_N} < b_N) = 1 \), the discrete process \( (L_i)_{i \in \mathbb{N}} \) defined by \( L_i = 1_{M_{t_{i-1},t_i} \geq b_i} \) is a discrete Markov process. We can simplify the previous nested integral to the product of multiple single integrals under restrictions. This is much more efficient from a computational perspective. However, the joint distribution function
\[
\mathbb{P}(M_{t_{0},t_1} < b_1, \ldots, M_{t_{N-1},t_N} < b_N | Z_0 = z_0)
\]
does not admit such a simplification. One has to utilise the nested integral formula in Theorem 4.1 to compute it, although this can also be computed efficiently with the two schemes in Section 5.

### 4.3 Non-matching time-discretization

As discussed before, we may have non-matching time-discretization schemes for the piecewise constant functions \( \mu(t), \lambda(t), \sigma(t) \) and \( b(t) \). In such a situation, the process is still continuous and Markov. By Theorem 4.1, if we can obtain
\[
\mathbb{P}(M_{t_{i-1},t_i} \geq b_i | Z_{t_{i-1}} = z_{i-1}, Z_{t_i} = z_i) \text{ and } \mathbb{P}(M_{t_{N-1},t_N} \geq b_N | Z_{t_{N-1}} = z_{N-1})
\]

∀i = 1, . . . , N − 1, the joint distribution and survival function for the maxima of the inhomogeneous OU-process in consecutive intervals can still be calculated. We have the following two sub-cases for non-matching time-discretizations in the interval [t_{i−1}, t_i].

**Case 1: Matching time-discretization for μ(t), λ(t) and σ(t), but non-matching for b(t).** An example of this case is shown in Figure 7. Here, the time-discretizations for μ, λ and σ are the same.

![Figure 7: Matching time-discretization for μ(t), λ(t) and σ(t), but different for b(t).](image)

In this case, \( P(M_{t_{i−1}, t_i} \geq b_i | Z_{t_{i−1}} = z_{i−1}) \) can still be calculated by Theorem 4.1. For example in Figure 7 we have

\[
P(M_{t_{i−1}, t_i} \geq b_i | Z_{t_{i−1}} = z_{i−1}, Z_{t_i} = z_i) = 1 - P(M_{t_{i−1}, t_j} < b_i, M_{t_j, t_{i−1}} < b_i, M_{t_{j+1}, t_i} < b_i | Z_{t_{i−1}} = z_{i−1}, Z_{t_i} = z_i)
\]

which can be solved by a nested integration formula, see Theorem 4.1, with the local homogeneous property for each sub-interval. In terms of \( P(M_{t_{i−1}, t_i} \geq b_i | Z_{t_{i−1}} = z_{i−1}, Z_{t_i} = z_i) \), we can also obtain follows by Theorem 4.1

\[
P(M_{t_{i−1}, t_i} \geq b_i | Z_{t_{i−1}} = z_{i−1}, Z_{t_i} = z_i) = 1 - \int_{R} P(M_{t_{i−1}, t_j} < b_i, M_{t_j, t_{i−1}} < b_i, M_{t_{j+1}, t_i} < b_i | Z_{t_{i−1}} = z_{i−1}, Z_{t_i} = z_i)
\]

\[
\times P(Z_{t_{j+1}} \in dx | Z_{t_{i−1}} = z_{i−1}, Z_{t_i} = z_i)
\]

\[
= 1 - \int_{R} P(M_{t_{i−1}, t_j} < b_i | Z_{t_{i−1}} = z_{i−1}, Z_{t_{j+1}} = x) P(M_{t_{j+1}, t_i} < b_i | Z_{t_{j+1}} = x, Z_{t_i} = z_i)
\]

\[
\times p(t_{j}, t_{j}, x, z_i)p(t_{i−1}, t_{j+1}, z_{i−1}, x)dx
\]

\[
= 1 - \int_{R} P(M_{t_{i−1}, t_j} < b_i | Z_{t_{i−1}} = z_{i−1}, Z_{t_{j+1}} = x) \int_{R} P(M_{t_{j+1}, t_i} < b_i | Z_{t_{j+1}} = x, Z_{t_i} = z_i)
\]

\[
\times p(t_{j}, t_{j}, x, z_i)p(t_{i−1}, t_{j+1}, z_{i−1}, x)dydx.
\]

Theorem 4.2 simplifies the nested integral in Theorem 4.1 to a product of single integrals, provided that some additional restrictions are satisfied. For non-matching time-discretization, Theorem 4.2 can still be applied. However, the terms \( P(M_{t_{i−2}, t_{i−1}} \geq b_{i−1} | Z_{t_{i−1}} = x_{i−1}, Z_0 = z_0) \) and \( P(M_{t_{i−1}, t_i} \geq b_i | Z_{t_{i−1}} = x_i) \) can only be evaluated by the nested integral in Theorem 4.1.
• Case 2: Non-matching time-discretizations for any of the functions $\mu(t), \lambda(t), \sigma(t)$ and $b(t)$.

An example of this case is shown in Figure 8. This case can be reduced back to Case 1 by taking the union of all the time-discretizations steps as the an overall discretizations scheme. For example in Figure 8, we can consider it as a special case of Case 1 for the time steps $t_{i-1}, t_{j_1}, t_{j_2}, t_{j_3}, t_{j_4}$ and $t_i$.

![Figure 8: Non-matching time-discretizations for any of $\mu(t), \lambda(t), \sigma(t)$ and $b(t)$.

Remark 4.1. When the discretisation is non-matching, we have two layers of nested integration:

a) The nested integration due to non-matching time-discretization;

b) The nested integration arising from the application of Theorem 4.1.

By Theorem 4.2 one can simplify the nested integral in b) to a product of single integrations under some restrictions. This can reduce the computational complexity. Although the nested integral in a) cannot be further reduced, in practice, if the variations of the piecewise constants within a single segment are much smaller than the variations of the piecewise constants among all the segments, one can use matching time-discretization as an efficient approximation.

5 Computational methods and numerical results

For now, we have obtained decomposition formulae for both, the joint distribution and the survival function for the maxima of a continuous Markov process in consecutive intervals. For convenience, in this section we take the survival function for a standardised OU-process in consecutive intervals as an example to illustrate the computational methods. For simplicity, in this section, we consider the case that the lengths of all the time intervals are constant $\Delta t$, i.e. $t_i = i \Delta t$ for $i = 0, 1, 2, ..., N$.

**Corollary 5.1.** Let $(Z_t)_{t \geq 0}$ be a standardised OU-process and $M_{t_{i-1}, t_i} = \sup_{t \in [t_{i-1}, t_i]} Z_t$, where $t_i = i \Delta t$, for $i = 0, 1, 2, ..., N$. Then

$$
P\left(M_{t_0, t_1} \geq b_1, \ldots, M_{t_{N-1}, t_N} \geq b_N \mid Z_{t_0} = z_0\right) = \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \kappa(z_0, z_1, b_1) \cdots \kappa(z_{N-3}, z_{N-2}, b_{N-2}) \kappa(z_{N-2}, z_{N-1}, b_{N-1}) \tilde{q}(z_{N-1}, b_N) dz_{N-2} dz_{N-3} \cdots dz_1,
$$

(5.1)
where
\[
q(z_{N-1}, b_N) = \left[ 1 - \sum_{k=1}^{\infty} c_k^{(N)} e^{-a_k^{(N)} \Delta t} \mathcal{H}_{a_k^{(N)}}^{-1}(-z_{N-1}) \right] \mathbb{1}(z_{N-1} < b_N) + \mathbb{1}(z_{N-1} \geq b_N),
\]
\[
k(z_{i-1}, z_i, b_i) = \sum_{k=1}^{\infty} c_k^{(i)} \alpha_k^{(i)} \mathcal{H}_{a_k^{(i)}}^{-1}(-z_{N-1})
\times \int_{e^{-\Delta t}}^{1} \frac{x^{-\alpha_k^{(i)} \Delta t} \mathbb{1}(z_i < b_i) \mathbb{1}(z_i < b_i)}{\sqrt{\pi(1-x^2)}} \exp \left\{ \frac{-(z_i - b_i x)^2}{1-x^2} - \alpha_k^{(i)} \Delta t \right\} \, dx
+ p(0, \Delta t, z_{i-1}, z_i) (1 - \mathbb{1}(z_{i-1} < b_i) \mathbb{1}(z_i < b_i)).
\]
Here, \(\mathcal{H}_{a}(\cdot)\) is the Hermite function with parameter \(a\), \(\{\alpha_k^{(i)}\}\) are the solutions to the equation \(\mathcal{H}_{a}(-b_i) = 0\) and \(c_k^{(i)} = -1/(\alpha_k^{(i)} \partial_{a_k^{(i)}} \mathcal{H}_{a_k^{(i)}}(-b_i))\).

**Proof.** By Theorem 4.1 we have
\[
P(M_{t_0, t_1} \geq b_1, \ldots, M_{t_{N-1}, t_N} \geq b_N | Z_{t_0} = z_0)
= \int_{\mathbb{R}} P(M_{t_0, t_1} \geq b_1 | Z_{t_0} = z_0, Z_{t_1} = z_1) p(t_0, t_1, z_0, z_1) \ldots
\times \int_{\mathbb{R}} P(M_{t_{N-2}, t_{N-1}} \geq b_{N-1} | Z_{t_{N-2}} = z_{N-2}, Z_{t_{N-1}} = z_{N-1}) p(t_{N-2}, t_{N-1}, z_{N-2}, z_{N-1})
\times (1 - P(M_{t_{N-1}, t_N} < b_N | Z_{t_{N-1}} = z_{N-1})) \, dz_{N-1} \cdots dz_1,
\]
where \(p(t_{i-1}, t_i, z_{i-1}, z_i)\) is the transition density function of the process \((Z_t)_{t \geq 0}\). By Theorem 2.1 we have
\[
1 - P(M_{t_{N-1}, t_N} < b_N | Z_{t_{N-1}} = z_{N-1}) = \left[ 1 - \sum_{k=1}^{\infty} c_k^{(N)} e^{-a_k^{(N)} \Delta t} \mathcal{H}_{a_k^{(N)}}^{-1}(-z_{N-1}) \right] \mathbb{1}(z_{N-1} < b_N) + \mathbb{1}(z_{N-1} \geq b_N).
\]
By the homogeneous property of the standardized OU-process and Proposition 4.2 we have
\[
P(M_{t_{i-1}, t_i} \geq b_i | Z_{t_{i-1}} = z_{i-1}, Z_{t_i} = z_i) = P(M_{0, \Delta t} \geq b_i | Z_{0} = z_{i-1}, Z_{\Delta t} = z_i) p(0, \Delta t, z_{i-1}, z_i)
= \int_{0}^{\Delta t} p(t, \Delta t, b_i) p(\tau_{Z, b_i} \in dt | Z_0 = z_{i-1}) \mathbb{1}(z_{i-1} < b_i) \mathbb{1}(z_i < b_i)
+ P(0, \Delta t, z_{i-1}, z_i) (1 - \mathbb{1}(z_{i-1} < b_i) \mathbb{1}(z_i < b_i))
= \sum_{k=1}^{\infty} c_k^{(i)} \alpha_k^{(i)} \mathcal{H}_{a_k^{(i)}}^{-1}(-z_{N-1}) \int_{e^{-\Delta t}}^{1} \frac{x^{-\alpha_k^{(i)} \Delta t} \mathbb{1}(z_i < b_i) \mathbb{1}(z_i < b_i)}{\sqrt{\pi(1-x^2)}} \exp \left\{ \frac{-(z_i - b_i x)^2}{1-x^2} - \alpha_k^{(i)} \Delta t \right\} \, dx
+ P(0, \Delta t, z_{i-1}, z_i) [1 - \mathbb{1}(z_{i-1} < b_i) \mathbb{1}(z_i < b_i)].
\]
Therefore,
\[
P(M_{t_0, t_1} \geq b_1, \ldots, M_{t_{N-1}, t_N} \geq b_N | Z_{t_0} = z_0)
= \int_{\mathbb{R}} \kappa(z_0, z_1, b_1) \cdots \int_{\mathbb{R}} \kappa(z_{N-3}, z_{N-2}, b_{N-2}) \int_{\mathbb{R}} \kappa(z_{N-2}, z_{N-1}, b_{N-1}) q(z_{N-1}, b_N) \, dz_{N-1} \, dz_{N-2} \cdots dz_1.
\]
The nested integral can be approximated efficiently by quadrature schemes or Monte Carlo integration methods. We describe the two methods in what follows.

5.1 Quadracture schemes

We first present a quadrature scheme to evaluate

\[ I := \mathbb{P}(M_{t_0}, t_1 \geq b_1, \ldots, M_{t_{N-1}}, t_N \geq b_N \mid Z_{t_0} = z_0) \]

\[ = \int_{\mathbb{R}} \kappa(z_{x_0}, z_1, b_1) \cdots \int_{\mathbb{R}} \kappa(z_{N-3}, z_{N-2}, z_{N-2}) \int_{\mathbb{R}} \kappa(z_{N-2}, z_{N-1}, b_{N-1}) \bar{q}(z_{N-1}, b_N) \, dz_{N-1} \cdots \, dz_1, \]

(5.2)

Since the OU-process is defined on \( \mathbb{R} \), we choose a sufficiently large number \( Z_{\max} \) and sufficiently small number \( Z_{\min} \). We partition the domain \( [Z_{\min}, Z_{\max}] \) into \( L \) pieces of equal length \( \delta z \), where the grid points are denoted \( Z_{\min} = z^{(1)} < z^{(2)} < \cdots < z^{(L)} = Z_{\max} \). We can then approximate the integration as follows:

**Proposition 5.1.** The nested integral in Equation (5.1) can be approximated by the product of matrices

\[ I \approx \prod_{i=1}^{N-1} K_i \bar{Q}(\delta z)^{N-1}, \]

where for \( i = 1 \)

\[ K_1 = \left[ \kappa(z_{x_0}, z^{(1)}_1), \kappa(z_{x_1}, z^{(2)}_1), \ldots, \kappa(z_{x_0}, z^{(L)}_1) \right], \]

for \( i = 2, 3, \ldots, N-1 \),

\[ K_i = \left[ \begin{array}{cccc}
\kappa(z^{(1)}_{i-1}, z^{(1)}_i, b_i) & \kappa(z^{(1)}_{i-1}, z^{(2)}_i, b_i) & \cdots & \kappa(z^{(1)}_{i-1}, z^{(L)}_i, b_i) \\
\kappa(z^{(2)}_{i-1}, z^{(1)}_i, b_i) & \kappa(z^{(2)}_{i-1}, z^{(2)}_i, b_i) & \cdots & \kappa(z^{(2)}_{i-1}, z^{(L)}_i, b_i) \\
\cdots & \cdots & \cdots & \cdots \\
\kappa(z^{(L)}_{i-1}, z^{(1)}_i, b_i) & \kappa(z^{(L)}_{i-1}, z^{(2)}_i, b_i) & \cdots & \kappa(z^{(L)}_{i-1}, z^{(L)}_i, b_i) 
\end{array} \right], \]

and

\[ \bar{Q} = \left[ \bar{q}(z^{(1)}_{N-1}, b_N), \bar{q}(z^{(2)}_{N-1}, b_N), \ldots, \bar{q}(z^{(L)}_{N-1}, b_N) \right]^\top, \]

**Proof.** We can approximate the nested integral by

\[ I \approx \sum_{k_1=1}^L \kappa(z_{x_0}, z^{(k_1)}_1, b_1) \delta z \sum_{k_2=1}^L \kappa(z^{(k_1)}_1, z^{(k_2)}_2, b_2) \delta z \cdots \sum_{k_{N-1}=1}^L \kappa(z^{(k_{N-2})}_{N-2}, z^{(k_{N-1})}_{N-1}, b_{N-1}) \bar{q}(z^{(k_{N-1})}_{N-1}, b_N) \delta z. \]

We write

\[ f_i(z^{(k_i)}_i) = \sum_{k_{i+1}=1}^L \kappa(z^{(k_i)}_i, z^{(k_{i+1})}_{i+1}, b_{i+1}) \sum_{k_{i+2}=1}^L \kappa(z^{(k_{i+1})}_{i+1}, z^{(k_{i+2})}_{i+2}, b_{i+2}) \cdots \sum_{k_{N-1}=1}^L \kappa(z^{(k_{N-2})}_{N-2}, z^{(k_{N-1})}_{N-1}, b_{N-1}) \bar{q}(z^{(k_{N-1})}_{N-1}, b_N), \]

and

\[ F_i = \left[ f_i(z^{(1)}_i), f_i(z^{(2)}_i), \ldots, f_i(z^{(L)}_i) \right]^\top. \]

Then we proceed as follows:

1) The integral \( I \) can be approximated to obtain \( I \approx \sum_{k_1=1}^L \kappa(z_{x_0}, z^{(k_1)}_1, b_1) f_1(z^{(k_1)}_1) = K_1 \cdot F_1 \).

2) Then \( f_1(z^{(k_1)}_1) = \sum_{k_2=1}^L \kappa(z^{(k_1)}_1, z^{(k_2)}_2, b_2) f_2(z^{(k_2)}_2) \), and therefore \( F_1 = K_2 F_2 \).
3) We repeat step 2) for \( N - 2 \) times until
\[
f_{N-3}(z_{N-3}) = \sum_{k_{N-2}=1}^{L} \kappa(z_{N-3}^{(k_{N-2})}, z_{N-2}^{(k_{N-2})}, b_{N-2}) f_{N-2}(z_{3}^{(k_{N-2})}) = K_{N-2} F_{N-2},
\]
and thus, \( F_{N-2} = K_{N-1} \bar{Q} \).

We finally have \( I \approx \prod_{i=1}^{N-1} K_i \bar{Q}(\delta z)^{N-1} \). Here, we may approximate \( q(z_{N-1}, b_N) \) and \( \kappa(z_{i-1}, z_i, b_i) \) as follows:
\[
\bar{q}(z_{N-1}, b_N) \approx \left[ 1 - \sum_{k=1}^{K} c_k^{(N)} e^{-\alpha_k^{(N)} t} \mathcal{H}_k^{(N)} (-z_{N-1}) \right] 1( z_{N-1} < b_N ) + 1( z_{N-1} \geq b_N ),
\]
\[
\kappa(z_{i-1}, z_i, b_i) \approx \sum_{k=1}^{K} c_k^{(i)} \alpha_k^{(i)} \mathcal{H}_k^{(i)} (-z_{N-1})
\times \sum_{j=1}^{L} x_j^{-\alpha_k^{(i)} t} \mathcal{H}_j^{(i)} 1( z_{i-1} < b_i ) 1( z_i < b_i ) \exp \left\{ -\frac{( z_i - b_i x_j )^2}{1 - x_j^2} - \alpha_k^{(i)} \Delta t \right\} \delta x_j
\]
\[+ p(0, \Delta t, z_{i-1}, z_i) (1 - 1( z_{i-1} < b_i ) 1( z_i < b_i )).\]

where \( e^{-\Delta t} = x_0 < x_1 < \cdots < x_L = 1 \) and \( \delta x_j = x_j - x_{j-1} \). \( \square \)

**Corollary 5.2.** With further condition \( \mathbb{P}(Z_{t_0} < b_1, Z_{t_1} < b_2, \cdots, Z_{t_{N-1}} < b_N) \), the nested integral (5.1) can be reduced to the product of single integrals, which can be evaluated efficiently in vector form as follows:

\[
I \approx C \left( \prod_{i=1}^{N} V_i W_i(\delta z) \right)
\]

where
\[
C = \left( \prod_{i=2}^{N-1} \mathbb{P}(M_{t_{i-1}, t_{i}} \geq b_i | Z_0 = z_0) \right)^{-1}
\]
\[
V_i = \left[ \mathbb{P}(M_{t_{i-1}, t_{i}} \geq b_i | Z_{t_{i-1}} = z_{i}^{(1)}), \cdots, \mathbb{P}(M_{t_{i-1}, t_{i}} \geq b_i | Z_{t_{i-1}} = z_{i}^{(L)}) \right]
\]
\[
W_i = \left[ \mathbb{P}(M_{t_{i-2}, t_{i-1}} \geq b_{i-1} | Z_{t_{i-1}} = z_{i}^{(1)}, Z_0 = z_0) p(0, t_{i-1}; z_0, z_{i}^{(1)}) \cdots \mathbb{P}(M_{t_{i-2}, t_{i-1}} \geq b_{i-1} | Z_{t_{i-1}} = z_{i}^{(L)}, Z_0 = z_0) p(0, t_{i-1}; z_0, z_{i}^{(L)}) \right].
\]

**Proof.** The proof can be shown by rearranging Equation (4.6) in Theorem 4.2. \( \square \)

**Remark 5.1.** One can see from the difference between Proposition 5.1 and Corollary 5.2 that the complexity of the matrices operation with the simplification theorem is \( O(NL) \), while the complexity of the original nested integral is \( O(NL^2) \).

### 5.2 Monte Carlo integration method

The integral (5.2) can also be evaluated efficiently by an importance sampling approximation.

**Proposition 5.2.** Assume \( Z_1, Z_2, \ldots, Z_{N-1} \) are independent and identical random variables with density function \( p : \mathbb{R} \to \mathbb{R}^+ \), which first-order stochastically dominate \( \varphi \left( z, z', u \right) := \kappa \left( z, z', u \right) / p(z') \). Let \( Z_{i-1}^{(k)} \) be the \( k_{i-1} \)-th random number in the sample generated from the random variable \( Z_{i-1} \), and let \( L_i \)
be the sample size of the random variable $Z_{i-1}$. Then the nested integral \((5.1)\) in Proposition \(5.1\) can be approximated by the product of matrices $I \approx \left( \prod_{i=1}^{N-1} \Omega_i / L_i \right) \tilde{\mathcal{Q}}$, where for $i = 1$,

$$
\Omega_1 = \left[ \varphi \left( z_0, Z_1^{(1)} \right), \varphi \left( z_0, Z_1^{(2)} \right), \ldots, \varphi \left( z_0, Z_1^{(K_1)} \right) \right],
$$

for $i = 2, 3, \ldots, N - 1$,

$$
\Omega_i = \left[ \begin{array}{ccc}
\varphi \left( Z_{i-1}^{(1)}, Z_i^{(1)}, b_i \right) & \varphi \left( Z_{i-1}^{(1)}, Z_i^{(2)}, b_i \right) & \cdots & \varphi \left( Z_{i-1}^{(1)}, Z_i^{(K_i)}, b_i \right) \\
\varphi \left( Z_{i-1}^{(2)}, Z_i^{(1)}, b_i \right) & \varphi \left( Z_{i-1}^{(2)}, Z_i^{(2)}, b_i \right) & \cdots & \varphi \left( Z_{i-1}^{(2)}, Z_i^{(K_i)}, b_i \right) \\
\vdots & \vdots & \ddots & \vdots \\
\varphi \left( Z_{i-1}^{(K_{i-1})}, Z_i^{(1)}, b_i \right) & \varphi \left( Z_{i-1}^{(K_{i-1})}, Z_i^{(2)}, b_i \right) & \cdots & \varphi \left( Z_{i-1}^{(K_{i-1})}, Z_i^{(K_i)}, b_i \right)
\end{array} \right],
$$

and

$$
\tilde{\mathcal{Q}} = \left[ \tilde{q} \left( Z_{N-1}^{(1)}, b_N \right), \tilde{q} \left( Z_{N-1}^{(2)}, b_N \right), \ldots, \tilde{q} \left( Z_{N-1}^{(K_{N-1})}, b_N \right) \right]^T.
$$

Proof. The integral can be rewritten as

$$
I = \int_{\mathbb{R}} \frac{\kappa (z_0, z_1, b_1)}{p(z_1)} p(z_1) \cdots \int_{\mathbb{R}} \frac{\kappa (z_{N-3}, z_{N-2}, b_{N-2})}{p(z_{N-2})} p(z_{N-2}) \times \int_{\mathbb{R}} \frac{\kappa (z_{N-2}, z_{N-1}, b_{N-1})}{p(z_{N-1})} \tilde{q} (z_{N-1}, b_N) p(z_{N-1}) dz_{N-1} dz_{N-2} \cdots dz_1,
$$

and further as

$$
I = \int_{\mathbb{R}} \varphi (z_0, z_1, b_1) p(z_1) \cdots \int_{\mathbb{R}} \varphi (z_{N-3}, z_{N-2}, b_{N-2}) p(z_{N-2}) \times \int_{\mathbb{R}} \varphi (z_{N-2}, z_{N-1}, b_{N-1}) \tilde{q} (z_{N-1}, b_N) p(z_{N-1}) dz_{N-1} dz_{N-2} \cdots dz_1.
$$

The proof can now be continued analogous to the one for Proposition \(5.1\). \(\blacksquare\)

### 5.3 Numerical tests

The two methods can be compared with the direct Monte Carlo approach, which can be shown in Algorithm 2 (or a small memory version Algorithm 3). Since the direct Monte Carlo method needs to be implemented by a time discretisation, this method underestimates the real passage-time probability, see Lemma B.1 in Appendix B. We observe that when the number of time steps increase, the results obtained by direct Monte Carlo method become closer to the results obtained by quadrature scheme and the Monte Carlo integration. However, direct Monte Carlo results become increasingly noisy when the joint passage event becomes rarer, while the quadrature scheme and Monte Carlo integration methods remain stable. It shows the quadrature and Monte Carlo integration methods can improve the accuracy if the joint passage is a rare event. Another interesting result is that although the direct Monte Carlo result will be more accurate for larger number of time steps, its Monte Carlo error will be bigger as well, provided that the event occurrence is a rare event, see Lemma B.2 in Appendix B. It means the direct Monte Carlo method is not suitable for the passage-time approximation.

In the comparison of the three methods, i.e. the direct Monte Carlo, quadrature and Monte Carlo integration, we test the following two cases:

1) We fix the number of consecutive intervals and change the level of barriers in each interval;

2) We fix the level of barrier and increase the number of intervals.
In both cases, the probability we want to approximate becomes small when the barriers rise or the number of intervals increase. The direct Monte Carlo estimator will become noisy when the joint event becomes rare, due to Lemma \textcolor{red}{B.1} and \textcolor{red}{B.2} in Appendix \textcolor{red}{B}. In this subsection, we want to show that with the quadrature scheme and Monte Carlo integration estimators, the approximation can still be accurate and robust and the efficiency can also be smaller compared with the direct Monte Carlo methods.

Both the quadrature scheme and the Monte Carlo integration scheme contain two types of error source:

1) The truncation error from the approximation of the FPT density infinite series;

2) The deterministic or stochastic error from the numerical integration.

As we have shown in Section \textcolor{red}{2}, the truncation error in 1) can be reduced to a small level by introducing few truncation terms. The numerical error 2) depends on its discretization size in the quadrature scheme while depends on the number of Monte Carlo paths in the Monte Carlo integration scheme. This type of error can be reduced by introducing more time-discretizations or more Monte Carlo samples.

The first numerical example is to compute the probability of the maxima for a standardised OU-process in $[0, 1)$ and $[1, 2]$ above $b_1$ and $b_2$, whose values vary in the left of Table \textcolor{red}{1}. In this example, we choose the number of paths and discretization numbers to fix the error order of magnitude as $10^{-8}$ for the case $b_1 = 1$ and $b_2 = 1$. It gives the following results:

| $b_1$ | $b_2$ | MC (500)     | MC (1000)    | MC (2000)    | Quad.        | MC int.      |
|-------|-------|--------------|--------------|--------------|--------------|--------------|
| 1     | 1     | $1.417 \times 10^{-1}$ | $1.448 \times 10^{-1}$ | $1.469 \times 10^{-1}$ | $1.517 \times 10^{-1}$ | $1.515 \times 10^{-1}$ |
|       |       | ($2 \times 10^{-4}$)     | ($2 \times 10^{-4}$)     | ($2 \times 10^{-4}$)     | ($3 \times 10^{-4}$)     | ($6 \times 10^{-4}$)     |
| 1     | 2     | $1.27 \times 10^{-2}$ | $1.311 \times 10^{-2}$ | $1.352 \times 10^{-2}$ | $1.426 \times 10^{-2}$ | $1.440 \times 10^{-2}$ |
|       |       | ($7 \times 10^{-5}$)     | ($8 \times 10^{-5}$)     | ($8 \times 10^{-5}$)     | ($2 \times 10^{-5}$)     | ($7 \times 10^{-5}$)     |
| 2     | 1     | $5.08 \times 10^{-3}$ | $5.38 \times 10^{-3}$ | $5.54 \times 10^{-3}$ | $5.837 \times 10^{-3}$ | $5.843 \times 10^{-3}$ |
|       |       | ($5 \times 10^{-5}$)     | ($5 \times 10^{-5}$)     | ($6 \times 10^{-5}$)     | ($2 \times 10^{-5}$)     | ($3 \times 10^{-5}$)     |
| 2     | 2     | $2.35 \times 10^{-3}$ | $2.50 \times 10^{-3}$ | $2.62 \times 10^{-3}$ | $2.72 \times 10^{-3}$ | $2.74 \times 10^{-3}$ |
|       |       | ($4 \times 10^{-5}$)     | ($4 \times 10^{-5}$)     | ($4 \times 10^{-5}$)     | ($2 \times 10^{-5}$)     | ($3 \times 10^{-5}$)     |
| 2     | 3     | $4.1 \times 10^{-5}$ | $5.0 \times 10^{-5}$ | $5.4 \times 10^{-5}$ | $5.08 \times 10^{-5}$ | $5.10 \times 10^{-5}$ |
|       |       | ($4 \times 10^{-6}$)     | ($5 \times 10^{-6}$)     | ($4 \times 10^{-6}$)     | ($2 \times 10^{-7}$)     | ($3 \times 10^{-7}$)     |
| 3     | 2     | $1.1 \times 10^{-3}$ | $1.1 \times 10^{-3}$ | $1.2 \times 10^{-3}$ | $1.455 \times 10^{-3}$ | $1.458 \times 10^{-3}$ |
|       |       | ($3 \times 10^{-6}$)     | ($3 \times 10^{-6}$)     | ($4 \times 10^{-6}$)     | ($1 \times 10^{-7}$)     | ($9 \times 10^{-8}$)     |
| 3     | 3     | $7 \times 10^{-6}$ | $6 \times 10^{-6}$ | $6 \times 10^{-6}$ | $5.47 \times 10^{-6}$ | $5.42 \times 10^{-6}$ |
|       |       | ($2 \times 10^{-6}$)     | ($2 \times 10^{-6}$)     | ($2 \times 10^{-6}$)     | ($9 \times 10^{-8}$)     | ($5 \times 10^{-8}$)     |

Table 1: The probability of the maxima for a standardised OU-process in $[0, 1)$ and $[1, 2]$ are above $b_1$ and $b_2$. The number below in the bracket is the absolute Monte Carlo or quadrature error. The three sets of direct Monte Carlo results are implemented with $2,000,000$ sample paths. The number of time steps for the three sets of direct Monte Carlo results are $500$, $1,000$ and $2,000$, respectively. The quadracture scheme is implemented between the state domain $[-5, 5]$ with state increment $0.005$. The Monte Carlo integration method is implemented with $100,000$ sample paths.

We can also observe from Table \textcolor{red}{1} that in the direct Monte Carlo cases, the error cannot be improved by introducing more time-discretization due to Lemma \textcolor{red}{B.2} in Appendix \textcolor{red}{B}. We also compare the time consumption to obtain the results in Table \textcolor{red}{1}. In Figure \textcolor{red}{9}, we can observe that the quadrature and Monte Carlo integration methods are more efficient than the direct Monte Carlo methods. In fact, the quadrature scheme can be implemented in real-time.
Figure 9: With given absolute error at $2 \sim 4 \times 10^{-5}$ for all schemes, the CPU time consumption for the case that $b_1 = b_2 = 2$.

Our second example is to fix the barrier and increase the number of intervals, which will lead to smaller joint passage probability. We can observe in Table 2 that the numerical results obtained by the direct Monte Carlo methods tends to be less accuracy when the number of intervals increases. However, the quadrature and Monte Carlo integration methods remain reliable compared with the noisy direct Monte Carlo results.

| N  | MC (500)          | MC (1000)         | MC (2000)         | Quadrature       | MC integration   |
|----|-------------------|--------------------|--------------------|------------------|------------------|
| 2  | $2.35 \times 10^{-3}$ (5 x 10^{-5}) | $2.50 \times 10^{-3}$ (4 x 10^{-5}) | $2.62 \times 10^{-3}$ (4 x 10^{-5}) | $2.72 \times 10^{-3}$ (2 x 10^{-5}) | $2.74 \times 10^{-3}$ (3 x 10^{-5}) |
| 3  | $3.1 \times 10^{-4}$ (1 x 10^{-5}) | $3.3 \times 10^{-4}$ (1 x 10^{-5}) | $3.3 \times 10^{-4}$ (1 x 10^{-5}) | $3.23 \times 10^{-4}$ (1 x 10^{-6}) | $3.29 \times 10^{-4}$ (5 x 10^{-6}) |
| 4  | $5.5 \times 10^{-5}$ (4 x 10^{-6}) | $5.7 \times 10^{-5}$ (7 x 10^{-6}) | $5.6 \times 10^{-5}$ (8 x 10^{-6}) | $5.23 \times 10^{-5}$ (3 x 10^{-7}) | $5.30 \times 10^{-5}$ (8 x 10^{-7}) |
| 5  | $8 \times 10^{-6}$ (2 x 10^{-6}) | $7 \times 10^{-6}$ (2 x 10^{-6}) | $8 \times 10^{-6}$ (6 x 10^{-8}) | $8.06 \times 10^{-6}$ (6 x 10^{-8}) | $8.07 \times 10^{-6}$ (9 x 10^{-8}) |

Table 2: The probability of the maxima of a standardised OU-process in $N$ consecutive intervals are all above $b = 2$. The number below is the absolute Monte Carlo or quadrature error. The three sets of direct Monte Carlo results are implemented with 2,000,000 sample paths. The number of time steps for the three sets of direct Monte Carlo results are 500, 1,000 and 2,000, respectively. The quadracture scheme is implemented between the state domain $[-5, 5]$ with state increment 0.005. The Monte Carlo integration method is implemented with 100,000 sample paths.

We can observe from Figure 10 that the time consumption by direct Monte Carlo method increases linearly with respect to the number of intervals. On the other hand, there is a small jump in the time consumption for the quadrature scheme and the Monte Carlo integration method. This is because when the number of intervals is two, we do not need matrix $\Phi$ or $\Omega$ in Proposition 5.1 and 5.2. Once matrix $\Phi$ or $\Omega$ are obtained, they will be saved for future computation. It means that the quadrature scheme and the Monte Carlo integration method remain similar after three intervals. The probability for large number of intervals can be evaluated much more efficiently by the
quadrature and Monte Carlo integration methods.

![Figure 10: Time consumption of different schemes for a different number of intervals in Table 2.](image)

6 Conclusions

In this paper, we consider the multiple barrier-crossing problem of an OU-process in consecutive periods. To analyse it, we first present the formulae for the FPT survival function, which coincide with the formulae given in Alili et al. (2005) and Linetsky (2004b). Furthermore, we also show that the FPT distribution is not heavy-tailed. Afterwards, we show the transformation formulae of the FPTs between an inhomogeneous OU-process to a time-dependent barrier and a standardised OU-process to another time-dependent barrier. We investigate the running maxima of an inhomogeneous OU-process with piecewise constant parameters in consecutive intervals in detail. We present a nested integration formulae for the joint distribution and survival functions for the maxima in consecutive intervals of an inhomogeneous OU-process with piecewise constant parameters. By matrix multiplications, the nested integral can be solved efficiently by quadrature or Monte Carlo integration method. We also show that if we put further constraints on the problem, the event that the maxima in each interval are above the barrier can be simplified to the product of single integrals.

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A Algorithms

Algorithm 1 Truncation terms deviation by quantile.
1: For a given initial value $x$ and barrier level $b$, compute all the $\alpha$-zeros in a given interval.
2: Take all $\alpha$-zeros to approximate the $\omega$ quantile of the hitting time, denoted $q_\omega$, by Theorem 2.1.
3: Fix the relative error $\delta$. Obtain the error tolerance interval $[q_\omega(1-\delta), q_\omega(1+\delta)]$.
4: Denote the approximation with $n$ ordered $\alpha$-zeros by $\hat{q}_\omega$.
5: Starting from $n = 1$:
6: while $\hat{q}_\omega \notin [q_\omega(1-\delta), q_\omega(1+\delta)]$ do
7: $n^+ = 1$.
8: Output $n$.

Algorithm 2 Algorithm of Direct Monte Carlo
1: while path $n \leq N$ do
2: while time step $m \leq M$ do
3: 1, simulate realization of standard normal random variable $\phi^{(n)}_m$
4: 2, evaluate $x^{(n)}_{m+1} = x^{(n)}_m e^{-\lambda \delta} + \mu(1 - e^{-\lambda \delta}) + \sigma \sqrt{\frac{1-e^{-2\lambda \delta}}{2\lambda}} \phi^{(n)}_m$
5: if $\max(x^{(n)}_1, \ldots, x^{(n)}_{M/2}) \geq b_1$ and $\max(x^{(n)}_{M/2+1}, \ldots, x^{(n)}_M) \geq b_2$ then
6: $I^{(n)} = 1$
7: else
8: $I^{(n)} = 0$
9: $\text{Final\_Prob} = \text{Mean}(Ind)$
10: $\text{Final\_Err} = \text{StD}(Ind)/\sqrt{N}$

Here, we have:

1: $N$ : Number of paths for a given simulation set
2: $M$ : Number of time steps per path
3: $b_1, b_2$ : barrier level in interval one and two, consecutively
4: $n$ : $n$-th path
5: $m$ : $m$-th time step
6: $\phi^{(n)}_m$ : realization from standard normal random variable for path $n$ at time step $m$
7: $\delta = T/M$ : time step length
8: $x^{(n)}_m$ : the realised OU-process value of path $n$ at time step $m$
9: $I^{(n)}$ : the indicator
We estimate the probability \( P \)

Algorithm 3 Algorithm of Direct Monte Carlo (low memory requirement)

1: while set \( nb \leq NB \) do
2:     while path \( n \leq N \) do
3:         while time step \( m \leq M \) do
4:             1, simulate realization of standard normal random variable \( \phi_m \)
5:             2, evaluate \( x_{m+1}^{(n)} = x_m^{(n)} e^{-\lambda \delta} + \mu (1 - e^{-\lambda \delta}) + \sigma \sqrt{\frac{1-e^{-2\lambda \delta}}{2\lambda}} \phi_m \)
6:             if \( \max(x_1^{(n)}, \cdots, x_{M/2}^{(n)}) \geq b_1 \) and \( \max(x_{M/2+1}^{(n)}, \cdots, x_M^{(n)}) \geq b_2 \) then
7:                 \( I(n) = 1 \)
8:             else
9:                 \( I(n) = 0 \)
10:        \( \text{Prob}_{(nb)} = \text{Mean}(Ind) \)
11: Final_Prob = Mean(Prob)
12: Final_Err = StD(Prob)/\( \sqrt{NB} \)

Here, we have:

- \( NB \): Number of sets
- \( N \): Number of paths for a given simulation set
- \( M \): Number of time steps per path
- \( b_1, b_2 \): barrier level at interval one and two, consecutively
- \( n \): \( n \)-th path
- \( m \): \( m \)-th time step
- \( \phi_m \): realization from standard normal random variable for path \( n \) at time step \( m \)
- \( \delta = T/M \): time step length
- \( x_m^{(n)} \): the realised OU-process value of path \( n \) at time step \( m \)
- \( I(n) \): the indicator
- \( \text{Prob}_{(nb)} \): the probability of joint crossing in two consecutive intervals for set \( nb \)

**B Error analysis of the direct Monte Carlo method in FPT estimation**

We estimate the probability \( P \left( \sup_{t \in [t_0, t_1]} X_t \geq b_1, \sup_{t \in [t_1, t_2]} X_t \geq b_2 \right) \) by the following algorithm:

a) We discretise the time interval \([t_0, t_2]\) into \( M \) pieces:
   \[
   t_0 = t^{(0)} < t^{(1)} < \cdots < t^{(M/2)} < t_1 = t^{(M/2+1)} < \cdots < t^{(M)} = t_2.
   \]

b) We estimate the maximum in each interval by
   \[
   \sup_{t \in [t_0, t_1]} X_t \approx \max \left\{ X_{t^{(0)}}, X_{t^{(1)}}, \cdots, X_{t^{(M/2)}} \right\}, \quad \sup_{t \in [t_1, t_2]} X_t \approx \max \left\{ X_{t^{(M/2+1)}}, X_{t^{(M/2+2)}}, \cdots, X_{t^{(M)}} \right\}.
   \]

c) We approximate:
   \[
   P \left( \sup_{t \in [t_0, t_1]} X_t \geq b_1, \sup_{t \in [t_1, t_2]} X_t \geq b_2 \right) 
   \approx P \left( \max \left\{ X_{t^{(1)}}, X_{t^{(2)}}, \cdots, X_{t^{(M/2)}} \right\} \geq b_1, \max \left\{ X_{t^{(M/2+1)}}, X_{t^{(M/2+2)}}, \cdots, X_{t^{(M)}} \right\} \geq b_2 \right).
   \]
Lemma B.1. The direct Monte Carlo algorithm (a) - (c) underestimates the real probability due to the time discretisation, that is

\[
\mathbb{P} \left( \sup_{t \in [t_0, t_2]} X_t \geq b_1, \sup_{t \in [t_1, t_2]} X_t \geq b_2 \right) \\
\quad \geq \mathbb{P} \left( \max \left\{ X_{t(1)}, X_{t(2)}, \cdots, X_{t(M)} \right\} \geq b_1, \max \left\{ X_{t(\overline{M} + 1)}, X_{t(\overline{M} + 2)}, \cdots, X_{t(M)} \right\} \geq b_2 \right).
\]

Proof. We observe that

\[
\left\{ \max \left\{ X_{t(1)}, X_{t(2)}, \cdots, X_{t(M)} \right\} \geq b_1, \max \left\{ X_{t(\overline{M} + 1)}, X_{t(\overline{M} + 2)}, \cdots, X_{t(M)} \right\} \geq b_2 \right\}
\subseteq \left\{ \sup_{t \in [t_0, t_1]} X_t \geq b_1, \sup_{t \in [t_1, t_2]} X_t \geq b_2 \right\}.
\]

Since the probability of a sub-event is smaller than that of the event itself, we have

\[
\mathbb{P} \left( \sup_{t \in [t_0, t_1]} X_t \geq b_1, \sup_{t \in [t_1, t_2]} X_t \geq b_2 \right) \\
\quad \geq \mathbb{P} \left( \max \left\{ X_{t(1)}, X_{t(2)}, \cdots, X_{t(M)} \right\} \geq b_1, \max \left\{ X_{t(\overline{M} + 1)}, X_{t(\overline{M} + 2)}, \cdots, X_{t(M)} \right\} \geq b_2 \right).
\]

The larger \( M \), the more time steps, and the approximated probability tends to be closer to the actual probability of the event (that cannot be perfectly estimated by direct Monte Carlo).

For example, let us consider \( M = 500, 1000, 2000 \) partitions in the time interval \([0, 2]\). We then have:

\[
\mathbb{P} \left( \sup_{t \in [0, 1]} X_t \geq b_1, \sup_{t \in [1, 2]} X_t \geq b_2 \right) \\
\geq \mathbb{P} \left( \max \left\{ X_{1/1000}, X_{2/1000}, \cdots, X_{1000/1000} \right\} \geq b_1, \max \left\{ X_{1001/1000}, X_{1002/1000}, \cdots, X_{2000/1000} \right\} \geq b_2 \right)
\geq \mathbb{P} \left( \max \left\{ X_{2/1000}, X_{4/1000}, \cdots, X_{1000/1000} \right\} \geq b_1, \max \left\{ X_{1002/1000}, X_{1004/1000}, \cdots, X_{2000/1000} \right\} \geq b_2 \right)
\geq \mathbb{P} \left( \max \left\{ X_{1/500}, X_{2/500}, \cdots, X_{500/500} \right\} \geq b_1, \max \left\{ X_{501/500}, X_{502/500}, \cdots, X_{1000/500} \right\} \geq b_2 \right)
\geq \mathbb{P} \left( \max \left\{ X_{2/500}, X_{4/500}, \cdots, X_{500/500} \right\} \geq b_1, \max \left\{ X_{502/500}, X_{504/500}, \cdots, X_{1000/500} \right\} \geq b_2 \right)
\geq \mathbb{P} \left( \max \left\{ X_{1/250}, X_{2/250}, \cdots, X_{250/250} \right\} \geq b_1, \max \left\{ X_{251/250}, X_{252/250}, \cdots, X_{500/250} \right\} \geq b_2 \right).
\]

Therefore, \( \text{prob}_{MC(500)} \leq \text{prob}_{MC(1000)} \leq \text{prob}_{MC(2000)} \leq \text{prob}_{\text{actual}} \).

We next address the Monte Carlo error of the algorithm (a) - (c).

Lemma B.2. For a fixed number of paths, the errors of the direct Monte Carlo algorithm (a) - (c), based on a discretisation with \( M_1 \) and \( M_2 \) time steps \( M_1 < M_2 \), satisfy the relations

\[
\text{Err}_{MC(M_1)} \geq \text{Err}_{MC(M_2)} \quad \text{if and only if} \quad \text{prob}_{MC(M_1)} + \text{prob}_{MC(M_2)} \geq 1,
\]

\[
\text{Err}_{MC(M_1)} \leq \text{Err}_{MC(M_2)} \quad \text{if and only if} \quad \text{prob}_{MC(M_1)} + \text{prob}_{MC(M_2)} \leq 1.
\]

Proof. For convenience, we write

\[
A = \left\{ \max \left\{ X_{t(1)}, X_{t(2)}, \cdots, X_{t(M_1)} \right\} \geq b_1, \max \left\{ X_{t(M_1 + 1)}, X_{t(M_1 + 2)}, \cdots, X_{t(M)} \right\} \geq b_2 \right\},
\]

\[
B = \left\{ \max \left\{ X_{t(1)}, X_{t(2)}, \cdots, X_{t(M_2)} \right\} \geq b_1, \max \left\{ X_{t(M_2 + 1)}, X_{t(M_2 + 2)}, \cdots, X_{t(M)} \right\} \geq b_2 \right\}.
\]

Then the Monte Carlo algorithm (a) - (c) computes \( \text{prob}_{MC(M_1)} = \mathbb{P}(A) = \mathbb{E}[1_A] \) for \( M_1 \) time steps,
and \( \text{prob}_{MC(M_1)} = P(B) = \mathbb{E}[\mathbb{1}_B] \) for \( M_2 \) time steps. If we implement the Monte Carlo algorithm (a) - (c) to compute \( \mathbb{E}[\mathbb{1}_A] \) and \( \mathbb{E}[\mathbb{1}_B] \), the ratio between the resulting errors is equal to the ratio between the standard deviation of \( \mathbb{1}_A \) and \( \mathbb{1}_B \). That is:

\[
\frac{\text{Err}(\mathbb{1}_A)}{\text{Err}(\mathbb{1}_B)} = \frac{\text{StD}(\mathbb{1}_A)}{\text{StD}(\mathbb{1}_B)} = \sqrt{\frac{\text{Var}(\mathbb{1}_A)}{\text{Var}(\mathbb{1}_B)}} = \sqrt{\frac{\mathbb{E}[\mathbb{1}_A^2] - \mathbb{E}[\mathbb{1}_A]^2}{\mathbb{E}[\mathbb{1}_B^2] - \mathbb{E}[\mathbb{1}_B]^2}}.
\]

We observe that \( \mathbb{1}_A \) and \( \mathbb{1}_B \), we have \( (\mathbb{1}_A)^2 = \mathbb{1}_A \) and \( (\mathbb{1}_B)^2 = \mathbb{1}_B \). Therefore,

\[
\frac{\text{Err}(\mathbb{1}_A)}{\text{Err}(\mathbb{1}_B)} = \frac{\mathbb{E}[\mathbb{1}_A] - \mathbb{E}[\mathbb{1}_A]^2}{\mathbb{E}[\mathbb{1}_B] - \mathbb{E}[\mathbb{1}_B]^2} = \frac{\mathbb{E}[\mathbb{1}_A](1 - \mathbb{E}[\mathbb{1}_A])}{\mathbb{E}[\mathbb{1}_B](1 - \mathbb{E}[\mathbb{1}_B])} = \frac{P(A)P(\bar{A})}{P(B)P(\bar{B})}.
\]

This shows that

\[
\text{Err}_{MC(M_1)} = \sqrt{\frac{\text{prob}_{MC(M_1)} 1 - \text{prob}_{MC(M_1)}}{\text{prob}_{MC(M_2)} 1 - \text{prob}_{MC(M_2)}}} \text{Err}_{MC(M_2)}.
\]

Since \( M_1 < M_2 \), we have \( \text{prob}_{MC(M_1)} \leq \text{prob}_{MC(M_2)} \). We denote \( \text{prob}_{MC(M_2)} \) by \( p \in \mathbb{R}^+ \), then

\[
\text{prob}_{MC(M_1)} = p - a
\]

for some \( a \in [0, p] \). Therefore,

\[
\text{Err}_{MC(M_1)} = \sqrt{\frac{p - a}{p} \cdot \frac{1 - (p - a)}{1 - p} \text{Err}_{MC(M_2)}} = \sqrt{1 + \frac{a}{p(1 - p)} (\text{prob}_{MC(M_1)} + \text{prob}_{MC(M_2)} - 1) \text{Err}_{MC(M_2)}}.
\]

Since \( a \geq 0 \) and \( p \in [0, 1] \),

\[
\text{Err}_{MC(M_1)} \geq \text{Err}_{MC(M_2)} \quad \text{if and only if} \quad \text{prob}_{MC(M_1)} + \text{prob}_{MC(M_2)} \geq 1,
\]

\[
\text{Err}_{MC(M_1)} \leq \text{Err}_{MC(M_2)} \quad \text{if and only if} \quad \text{prob}_{MC(M_1)} + \text{prob}_{MC(M_2)} \leq 1.
\]

\[\square\]

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