OPTIMAL STOP-LOSS REINSURANCE WITH JOINT
UTILITY CONSTRAINTS

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ABSTRACT. We investigate the optimal reinsurance problems in this paper, specifically, the stop-loss strategies that can bring mutual benefit to both the insurance company and the reinsurance company. The utility improvement constraints are adopted by both contracting parties to guarantee that a reinsurance contract will bring higher expected utilities of wealth to the two participants. We also introduce five risk criteria that reflect the interests of both parties. Under each optimality criterion, we obtain explicit expressions of optimal stop-loss retentions and the corresponding optimised value of objective functions. The upper and lower bounds of expected utility increments under the optimal stop-loss retentions are provided. In the numerical example, we analyse the expected utility improvements under the criterion of minimising total Value-at-Risk. Notable increases in the lower bound of total utility increments are observed after adopting the joint utility improvement constraints.

1. Introduction. Reinsurance is the insurance among two insurance companies of contractual liabilities incurred under contracts of reinsurance. As an effective risk management tool, reinsurance protects insurance firms from underwriting losses and stabilise their underwriting results. It also increases insurance firm’s underwriting capacity and further spreads their risk of loss. Forms of reinsurance treaty include

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quota-share, excess-of-loss, stop-loss, change-loss, etc. Specifically, stop-loss reinsurance deals with the problem of accumulations and caps the aggregate amount of losses that a ceding company is responsible for, and would only apply when the value of claims occurrences reaches the retention level.

An optimal reinsurance is a specified form of the ceded loss which satisfies a certain optimisation criterion. Optimisation criteria commonly used in the literature include minimising the ruin probability, minimising the variance of insurance firm’s risk, maximising the expected utility function of a company’s wealth, and minimising the regulatory capital determined by risk measures such as Value-at-Risk (VaR) and Tail Value-at-Risk (TVaR). However, existing studies on optimal reinsurance mainly focus on the insurance company’s interest. See [1], [7], [29], [24], [14], [30], [3], [15], [13], [36], [20], [23], [33], [26], [27], [28], [32], [35].

In reality, insurance firms and reinsurance firms may have conflicts of interest when optimising their objectives. [9] pointed out that an optimal reinsurance treaty for an insurance company might not be optimal for a reinsurance company and it might not be acceptable for a reinsurance firm. [5] studied two stochastic investment and proportional reinsurance optimisation problems under the assumption that the insurance company has inside information. One of the problems was to maximise the insurance company’s expected utility function, the other was to maximise the expected utility of the reinsurance company’s terminal wealth. The optimal strategies of these two problems are not equivalent, which indicate that the optimal strategy relies on the point of view that we consider.

Therefore, it is more interesting and practical to design an optimal reinsurance contract that is beneficial to both the insurance firm and the reinsurance firm. In recent years there are increasing interests in studying the optimal reinsurance contracts from the point of view of both an insurance firm and a reinsurance firm. [10] studied the optimal reinsurance that is in the interest of both parties by maximising the joint survival and profitable probabilities of insurance firms and reinsurance firms. If the cedent and reinsurance firm evaluate a reinsurance contract by the exponential utility function and adopt the criterion based on the comparison between expected utilities, [16] showed that the indifference prices defined the preference thresholds of the exchange. Thus the range of variability of the reinsurance price was restricted and the margins for achieving Pareto-optimal solutions in trading insurance risks were identified. [34] investigated the optimal reciprocal quota-share reinsurance strategies that would bring mutual benefit to both an insurance firm and a reinsurance firm. [21] studied the reciprocal reinsurance optimisation problem when the risk is measured by the GlueVaR distortion risk measures. Further investigation regarding reciprocal reinsurances can be found in [8], [18], [4], [6], etc.

In this paper, we consider the optimal stop-loss reinsurance strategy that is acceptable for both insurance firm and reinsurance firm to balance the interest of the two conflicting parties. An acceptable reinsurance strategy should satisfy that expected utilities of the wealth after insurance/reinsurance settlement, for both insurance firm and reinsurance firm, are no less than the utilities of their initial wealth. Since the wealth after insurance/reinsurance payment could be negative, we choose the exponential utility function that was used in [25]. As to the optimisation criteria, we also consider the interests of both parties. We firstly study the optimal stop-loss retentions that minimising the summation of the variance of insurance firm’s and reinsurance firm’s potential losses. Thereafter we analyse the stop-loss retentions that minimising various risk measurers, namely VaR, TVaR and
generalised Dutch type I risk measure, of loss functions of both an insurance firm and a reinsurance firm. Finally, we investigate the optimal retentions that maximise the joint survival probability of both insurance firm and reinsurance firm.

An insurance company is solvent if there is an acceptable probability that it will meet its random liabilities. Therefore, solvency requires determining the necessary capital that an insurance company must hold to absorb risks, or at the very least, to meet the legal minimum standards. One of the main aspects of Solvency II, which is a fundamental review of the capital adequacy regime for the European insurance industry, is the calculation of the Solvency Capital Requirements (SCR) and the Minimum Capital Requirements (MCR). Thus, risk measures, considered as important functions of determining the required regulatory capital, have been used extensively in insurance and finance as a tool of risk management. Recently, decisions based on minimising the Value-at-Risk (VaR) and Conditional Tail Expectation (CTE) of the total risk of the insurance firm were studied in detail by many literatures. Particularly, [12] developed such optimisation criteria and explicitly obtained the optimal retentions in a stop-loss reinsurance.

In the directive of the [17], which is the binding framework for Solvency II, the SCR is required to be corresponding to the VaR of the basic own funds of an insurance or reinsurance company subject to a confidence level of 99.5% over a one-year period. However, [2] showed that VaR was not a coherent risk measure since it failed to satisfy the sub-additivity property. Moreover, the VaR-regulated capital might not be sufficient, particularly when an extreme loss event occurs, the company with the VaR-regulated capital could incur huge losses and may even get bankruptcy. Accordingly, Tail Value-at-Risk (TVaR), the smallest of coherent risk measures which only depend on the distribution of underlying random variables and are no less than VaR at the same confidence level (see e.g., [22]), is intuitively appealing in that it captures the expected magnitude of loss given that risk exceeds or equal to its VaR.

In addition to the VaR and TVaR, regulators or practitioners may implement different risk measures in determining the regulatory capitals because one of the objectives of the European industry for Solvency II is to provide incentives to insurance firms to adopt more advanced risk monitoring and risk management tools. This would include developing full and partial internal capital models that have better risk-based capital assessment and a closer link between capital and risk. In this paper, we introduce a possible internal model for optimal reinsurance, where the optimal criterion is based on a generalised Dutch type I risk measure-proposed by [11]. Specifically, we take the expectation of the wealth of insurance/reinsurance firm as the benchmark risk level, and choose TVaR as the risk measure for the loss above benchmark.

When investigating the expected utility increments under optimal stop-loss reinsurance that minimises the total VaR, we find that the insurance/reinsurance firm can easily suffer from utility decrements if we do not set up any constraints. After adopting the utility improvement constraints, even though the upper bound of potential utility increments might decrease for either an insurance firm or a reinsurance firm, the lower bound of increments in expected utility for both parties are at least zero. This guarantees that the stop-loss reinsurance can bring mutual benefit to both parties’ interest in utility. Furthermore, when analysing the total utility increments of insurance firm and reinsurance firm, we find that in the worst case
scenario, the optimal reinsurance strategies following utility constraints will always lead to more utility improvements than those without constraints.

Comparing with the existing relative literature, the main innovations of our paper are summarised as follows. First, we introduce utility constraints for both the insurance firm and the reinsurance firm in our optimisation problems. To be specific, we require that underwriting a reinsurance contract shall not lead to decreases in the expected utilities for either the insurance company or the reinsurance company. The idea of comparing expected utilities before and after reinsurance action is similar with [16], but their work focused on identifying the margins for achieving Pareto-optimal solutions in trading insurance risks. While we use such idea to set up constraints and further study reinsurance optimisation under various criteria. Second, there are quite a few literature investigating optimal reinsurance problems under risk measures such as VaR, CTE and TVaR, the readers may refer to [12], [14], and references therein. However, those studies are mainly from the point of view of the direct insurance company. When using risk measures to set up objective functions in this paper, we take the interest of both the insurance firm and the reinsurance firm into consideration by summing up both parties’ measures of risks. As to the joint survival probability maximisation we examined, the criterion is similar to [10], where optimal reinsurance treaties that maximise the joint survival and profitable probabilities are investigated in the unconstrained model. By introducing utility constraints, our work is different from [10], and more complicated to deal with. Thirdly, our work is different from [34], where the optimal reciprocal quota-share reinsurance strategies with mutual benefit is investigated, because we study the stop-loss reinsurance contract.

The rest of this paper is organised as follows. In Section 2, we provide a general formulation of optimal stop-loss reinsurance problem under the insurance firm’s and reinsurance firm’s utility constraints. Then we determine the range of the stop-loss retention under both parties’ utility improvement constraints in Section 3. The optimal stop-loss reinsurance treaties under five different criteria are investigated in Section 4. Some numerical examples are provided in Section 5 to demonstrate the expected utility improvement under the utility constraints. Finally, some additional remarks are provided in Section 6.

2. Formulation. Let $X$ be a nonnegative random variable representing the (aggregate) claim for an insurance firm in a fixed time period, with distribution function $F(x) = \Pr(X \leq x)$ ($x \geq 0$), mean $\mathbb{E}X = \mu$ and variance $\text{Var}X = \sigma^2$, where $\mu > 0$ and $\sigma^2 > 0$. We assume that random variable $X$ has a continuous and strictly increasing distribution function on $(0, \infty)$ with a possible mass at 0 in order to avoid tedious discussions. Under a reinsurance treaty, a reinsurance firm will cover part of the loss, say $f(X)$, $0 \leq f(X) \leq X$, and the insurance firm will retain the rest of the loss, which is denoted by $I_f(X) = X - f(X)$. The reinsurance premium $P_R^f$ is determined by the expected value principle, i.e. $P_R^f = (1 + \theta_R)\mathbb{E}f(X)$, where $\theta_R > 0$ is the loading factor of the reinsurance company. The net insurance premium for insurance firm is $P_I^f = P_0 - P_R^f$, where $P_0$ is the insurance premium received by the insurance firm from the policyholder (policyholders). Since the main task for an insurance company is to make profit for shareholders, we require $P_0 \geq P_R^f$. In this paper, the set of admissible reinsurance contracts is given by

$$\Pi := \left\{ f : 0 \leq f(x) \leq x, \ 0 \leq f(x_2) - f(x_1) \leq x_2 - x_1 (\forall \ 0 \leq x_1 \leq x_2), \ P_0 \geq P_R^f \right\}.$$
In practice, an insurance company would like to mitigate risks and seek protection from the reinsurance company by purchasing reinsurance contracts, while the intention for a reinsurance company to underwrite such business is more about making profit. To a certain extent, interests of the two parties involved in an reinsurance contract are conflict. Contracts exist in the market should bring mutual benefit to both the insurance company and the reinsurance company. In this paper, we consider the mutual benefit of an reinsurance contract in two aspects, namely the objective function and utility constraints.

To make the reinsurance treaty beneficial to both an insurance company and a reinsurance firm, our objective function will take both parties’ interests into consideration. Let $J^I$ and $J^R$ denote variance, Value-at-Risk (VaR), Tail Value-at-Risk (TVaR) or a generalised Dutch type I risk measure, of the insurance company and the reinsurance company, respectively. In our optimization problems, we focuses on finding a reinsurance contract $f$ that makes $J^I$, which is defined as the summation of $J^I$ for the insurance firm’s potential loss and $J^R$ for potential loss of the reinsurance firm, obtains its minimum value, i.e.,

$$\min_{f \in \Pi} \left\{ J^I \left( I_f(X) - P^I_f \right) + J^R \left( f(X) - P^R_f \right) \right\},$$

(1)

or to find a reinsurance contract $f$ that maximises the joint survival probability, i.e.,

$$\max_{f \in \Pi} \Pr \left\{ I_f(X) \leq u^I_0 + P^I_f, f(X) \leq u^R_0 + P^R_f \right\},$$

(2)

under both parties’ utility constraints

$$\mathbb{E}U_I(u^I_0 + P_0 - X) \leq \mathbb{E}U_I(u^I_0 + P^I_f - I_f(X)),$$

(3)

$$\mathbb{E}U_R(u^R_0) \leq \mathbb{E}U_R(u^R_0 + P^R_f - f(X)),$$

(4)

where $u^I_0 > 0$ and $u^R_0 > 0$ are the initial wealth of the insurance firm and reinsurance firm respectively; $U_I$ and $U_R$ are the utility functions of insurance firm and reinsurance firm, respectively. Constraints (3) and (4) are introduced to guarantee that the reinsurance underwriting is beneficial to both the insurance firm and the reinsurance firm. Specifically, we require expected utilities of both party after entering a reinsurance treaty would be no less than that without reinsurance. For mathematical tractability, we suppose that the claim size follows an exponential distribution with a positive parameter $\beta$, i.e., $X \sim \text{exp}(\beta)$, $\beta > 0$. Then we have $F(x) = 1 - e^{-\beta x}$, $\mu = 1/\beta$ and $\sigma^2 = 1/\beta^2$. As with [25], we choose the exponential utility function

$$U_I(x) = U_R(x) = \lambda - \frac{\alpha}{v} e^{-vx},$$

(5)

where the parameters $\lambda$, $\alpha$, $v$ are positive constants.

**Remark 1.** When exposed to uncertainty, risk aversion is a common risk attitude for a rational decision-maker. Practitioners tend to choose suitable utility functions to measure the risk attitudes. There are three types of commonly used utility functions, namely exponential, power and generalised logarithmic. Since both parties’ wealth after insurance/reinsurance claim settlements might be negative, only the exponential utility function is suitable in our model. Furthermore, we choose the same utility function for both insurance firm and reinsurance firm for the mathematical tractability.

The exponential utility (5) exhibits constant absolute risk aversion (CARA) and the absolute risk aversion coefficient is $v$. The coefficient $v$ represents an entity’s risk
preference. The larger the value of $v$, the more risk-averse an entity is. Because of the nature of insurance products, insurance and reinsurance industry bear risks to sell insurance policies. Thus we exclude the unrealistic case of extreme risk aversion and assume $v < \beta$.

**Remark 2.** The survival function involved in equation (2) corresponds to the ultimate joint survival probability of the insurance firm and the reinsurance firm. It is defined by the probability that surpluses of both the insurance company and the reinsurance company are positive. Under a reinsurance treaty $f$, the expression of joint survival probability is $Pr\{I_f(X) \leq u_0^I + P_f^I, f(X) \leq u_0^R + P_R^f\}$. In the context of stop-loss reinsurance contracts, we will give a detailed discussion on the joint survival probability in subsection 4.5.

In a stop-loss reinsurance, $f(X) = (X-d)_+$ and $I_f(X) = X \wedge d$, where $d \geq 0$ is the stop-loss retention. Here and throughout this paper, we denote $(x)_+ = \max\{x, 0\}$ and $x \wedge y = \min\{x, y\}$. We denote the reinsurance premium and net insurance premium by $P_R(d) = P_{R|f(X) = (X-d)_+}^f = (1 + \theta_R) \int_d^\infty (1 - F(x))dx$

$$= (1 + \theta_R) \int_d^\infty e^{-\beta x}dx = (1 + \theta_R)e^{-\beta d}/\beta$$

and

$$P_f(d) = P_{f|f(X) = (X-d)_+}^f = P_0 - (1 + \theta_R)e^{-\beta d}/\beta,$$

respectively.

3. **The range of stop-loss retention.** The range of stop-loss retention $d (d \geq 0)$ is narrowed by insurance firm’s constraint (3) and reinsurance firm’s constraint (4). Let $D_I$ be the unique root in $(0, 1)$ of $f_I(x) = e^{-(1+\theta_R)x}/\beta + \frac{v}{\beta} x \frac{\beta x}{\beta} - 1$ and $D_R$ be the unique positive root of $f_R(x) = e^{(1+\theta_R)x}/\beta - \frac{v}{\beta-\beta} x - 1$. Then we have the following Lemma.

**Lemma 3.1.** The range of the stop-loss retention $d$, determined by both parties’ utility constraints, is as follows:

(i) if $0 < \theta_R < \frac{\beta}{\alpha} \ln \frac{\beta}{\beta-v} - 1$, any stop-loss retention is not acceptable for a reinsurance company (i.e., no stop-loss reinsurance);

(ii) if $\frac{\beta}{\alpha} \ln \frac{\beta}{\beta-v} - 1 \leq \theta_R < \frac{\beta}{\alpha} \ln \frac{\beta}{\beta-v}$, we have $-\ln D_I/\beta \leq d \leq -\ln D_R/\beta$;

(iii) if $\theta_R \geq \frac{\beta}{\alpha} \ln \frac{\beta}{\beta-v}$, we have $d \geq -\ln D_I/\beta$.

**Proof.** In our stop-loss reinsurance treaty, when the utility function is defined by (5), we have

$$g_I(d) \equiv U_I(u_0^I + P_0 - X) - U_I(u_0^I + P_I(d) - X \wedge d)$$

$$= \mathbb{E} \left[ e^{-v(u_0^I + P_0 - X)} - \mathbb{E} e^{-v(u_0^I + P_0 - (1+\theta_R)e^{-\beta d}/\beta - X \wedge d)} \right]$$

$$= \mathbb{E} e^{-v(u_0^I + P_0)} \mathbb{E} e^{(1+\theta_R)ve^{-\beta d}/\beta e^v(X \wedge d) - M_X(v)}$$
and
\[
g_R(d) \triangleq \mathbb{E}U_R(u_R^d) - \mathbb{E}U_R(u_R^0) + P_R(d) - (X - d)_+
\]
\[
= \mathbb{E} \left[ \lambda - \frac{\alpha}{v} e^{-v u_R^0} - \lambda + \frac{\alpha}{v} e^{-v (u_R^0 + (1 + \theta_R)e^{-\beta d} - \theta)(X - d)_+} \right]
\]
\[
= \frac{\alpha}{v} e^{-v u_R^0} \left[ e^{-(1 + \theta_R) v e^{-\beta d}/\beta} \mathbb{E} e^{v (X - d)_+} - 1 \right].
\]

When \( \beta > v > 0 \), we have \( M_X(v) = \frac{\beta}{\beta - v} \), \( \mathbb{E} e^{v (X - d)_+} = \int_0^d e^{v x} e^{-\beta x} dx + e^v e^{-\beta d} = \frac{\beta - v}{\beta - v} e^{-(\beta - v) d} \) and \( \mathbb{E} e^{v (X - d)_+} = 1 - e^{-\beta d} + \int_d^{\infty} e^{v (x - d)} \beta e^{-\beta x} dx = 1 + \frac{v}{\beta - v} e^{-\beta d} \).

Then the insurance firm’s utility constraint (3), which is equivalent to \( g_f(d) \leq 0 \), becomes
\[
e^{(1 + \theta_R) v e^{-\beta d}/\beta} \left[ \frac{\beta}{\beta - v} - \frac{v}{\beta - v} e^{-(\beta - v) d} \right] - \frac{\beta}{\beta - v} \leq 0,
\]
and the reinsurance firm’s utility constraint (4), i.e. \( g_R(d) \leq 0 \), becomes
\[
e^{-(1 + \theta_R) v e^{-\beta d}/\beta} \left[ 1 + \frac{v}{\beta - v} e^{-\beta d} \right] - 1 \leq 0.
\]

Let \( D \triangleq e^{-\beta d} \), then \( 0 < D \leq 1 \) and the constraints can be simplified as
\[
1 - \frac{v}{\beta} D^{\frac{\beta}{\beta - v}} \leq e^{-(1 + \theta_R) v D/\beta} \leq \frac{1}{1 + v D/(\beta - v)},
\]
i.e., \( f_f(D) \geq 0 \) and \( f_R(D) \geq 0 \), where \( f_f(D) \triangleq e^{-(1 + \theta_R) v D/\beta} + \frac{v}{\beta} D^{\frac{\beta}{\beta - v}} - 1 \) and \( f_R(D) \triangleq e^{(1 + \theta_R) v D/\beta} - \frac{v}{\beta - v} D - 1 \).

First we consider reinsurance firm’s constraint. We have \( f_R(0) = 0 \), \( f_R(1) = e^{(1 + \theta_R) v/\beta} - \beta/(\beta - v) \) and \( f_R(D) = \frac{(1 + \theta_R) v}{\beta} e^{(1 + \theta_R) v D/\beta} - v/(\beta - v) \). For \( \theta_R \geq \frac{\beta}{\beta - v} \), we have \( f_R(D) \geq 0 \). Therefore, for any stop-loss retention \( d \geq 0 \), reinsurance firm’s constraints could be satisfied because the reinsurance is expensive enough for reinsurance firm. For \( 0 < \theta_R < \frac{\beta}{\beta - v} \), \( f_R(D) \) is decreasing with \( D \) when \( 0 < D \leq \frac{\beta}{(1 + \theta_R) v} \ln \left( \frac{\beta}{\beta - v} \right) \) and is increasing with \( D \) when \( D \geq \frac{\beta}{(1 + \theta_R) v} \ln \left( \frac{\beta}{(1 + \theta_R) v (\beta - v)} \right) \). Since \( f_R(0) = 0 \) and \( f_R(+\infty) = +\infty \), \( f_R(D) \) admits a unique positive root \( D_R \) and \( f_R(D) < 0 \) when \( 0 < D < D_R \). Furthermore, when \( \frac{\beta}{v} \ln \left( \frac{\beta}{\beta - v} \right) - 1 \leq \theta_R < \frac{\beta}{\beta - v} \), \( f_R(1) \geq 0 \) thus \( 0 < D_R < 1 \). Then, to satisfy the reinsurance firm’s constraint \( f_R(D) \geq 0 \), we need \( D_R \leq D \leq 1 \). When \( 0 < \theta_R < \frac{\beta}{v} \ln \left( \frac{\beta}{\beta - v} \right) - 1 \), we have \( f_R(1) < 0 \) and \( D_R > 1 \). Hence, for \( \forall D \in (0,1] \), \( f_R(D) < 0 \), which indicates that reinsurance firm’s utility cannot be improved by the stop-loss reinsurance. This is because the reinsurance premium is too cheap for reinsurance company.

As to the insurance firm’s utility constraints, we only need to consider the case that \( \theta_R \geq \frac{\beta}{v} \ln \left( \frac{\beta}{\beta - v} \right) - 1 \). From Lemma A.1, we know that for \( \theta_R \geq \frac{\beta}{v} \ln \left( \frac{\beta}{\beta - v} \right) - 1 \), \( f_f(D) \geq 0 \) if and only if \( 0 \leq D \leq D_f \), where \( D_f \in (0,1) \) is the root of \( f_f(D) \). Therefore, to satisfy both parties’ constraints, for \( \beta/v \ln \left( \frac{\beta}{\beta - v} \right) - 1 \leq \theta_R < \frac{\beta}{\beta - v} \), we need \( D_R \leq D \leq D_f \), i.e., \(-\ln D_f/\beta \leq d \leq -\ln D_R/\beta \); for \( \theta_R \geq \frac{\beta}{\beta - v} \), we need \( 0 \leq D \leq D_f \), i.e., \( d \geq -\ln D_f/\beta \). □
Remark 3. The reinsurance premium is reflected by $\theta_R$. When $\theta_R < \frac{\beta}{\nu} \ln \left( \frac{\beta}{\beta - \nu} \right) - 1$, the premium is too cheap to stimulate a reinsurance company entering the business. When $\theta_R \geq \frac{\nu}{\beta - \nu}$, insurance firm will always retain part of the risks because the stop-loss reinsurance premium is expensive. When $\theta_R \geq \frac{\nu}{\beta - \nu}$, an insurance company is happy to enter the stop-loss treaty with any retention because the premium is high enough; with such high premium rate, an insurance firm can still get benefits from purchasing reinsurance policy because the stop-loss reinsurance will cap the aggregate amount of losses that an insurance company is responsible for.

4. Optimal strategies under different criteria. In this section, we investigate the optimal stop-loss retentions and the corresponding objective functions under different optimisation criteria.

4.1. Variance. Variance is always used by a financial institution to assess the risks that it faces. Therefore, we adopt variance to evaluate future losses of both an insurance firm and a reinsurance firm. In this subsection, we consider the stop-loss reinsurance strategies under the optimality criterion of minimising the total variance of an insurance firm’s loss and a reinsurance firm’s loss. To be specific, we investigate the optimisation problem $\min_{d \geq 0} J_1(d)$, with objective (1) being

$$J_1(d) \triangleq \text{Var}(X \land d - P_I(d)) + \text{Var}((X - d)_+ - P_R(d))$$

$$= \text{Var} \left( X \land d - P_0 + \frac{(1 + \theta_R)e^{-\beta d}}{\beta} \right) + \text{Var} \left( (X - d)_+ - \frac{(1 + \theta_R)e^{-\beta d}}{\beta} \right)$$

$$= \text{Var}(X \land d) + \text{Var}((X - d)_+) = \left[ 1 + 2(1 - \beta d) e^{-\beta d} - 2e^{-2\beta d} \right]/\beta^2,$$

where the last equation is derived from

$$\text{Var}(X \land d) = \mathbb{E}((X \land d)^2) - (\mathbb{E}(X \land d))^2 = \left[ 1 - 2\beta de^{-\beta d} - e^{-2\beta d} / \beta^2 \right]$$

and

$$\text{Var}((X - d)_+) = \mathbb{E}((X - d)_+^2) - (\mathbb{E}((X - d)_+))^2 = (2 - e^{-\beta d}) e^{-\beta d} / \beta^2.$$ 

These two variances are obtained by the fact that

$$\mathbb{E}(X \land d) = \int_0^d x \beta e^{-\beta x} dx + de^{-\beta d} = (1 - e^{-\beta d})/\beta,$$

$$\mathbb{E}((X \land d)^2) = \int_0^d x^2 \beta e^{-\beta x} dx + d^2 e^{-\beta d} = 2(1 - e^{-\beta d} - \beta de^{-\beta d}) / \beta^2,$$

$$\mathbb{E}((X - d)_+) = \int_d^{+\infty} (x - d) \beta e^{-\beta x} dx = e^{-\beta d} / \beta,$$

$$\mathbb{E}((X - d)_+^2) = \int_d^{+\infty} (x - d)^2 \beta e^{-\beta x} dx = 2e^{-\beta d} / \beta^2.$$ 

Let $r$ be the unique positive root of the function $f(x) = 2e^{-x} + x - 2$. We derive the optimal stop-loss strategies in the following theorem.

Theorem 4.1. When minimising the total variance of both parties’ potential risks, the optimal stop-loss retention level and corresponding minimum total variance are as follows.
Case 1. If $0 < \theta_R < \frac{\beta}{\nu} \ln \frac{\beta - \nu}{\beta - \nu - 1}$, the optimal stop-loss retention does not exist and the total variance without reinsurance treaty is $1/\beta^2$.

Case 2. If $\frac{\beta}{\nu} \ln \frac{\beta}{\beta - \nu} - 1 \leq \theta_R < \frac{\nu}{\beta - \nu}$, the optimal stop-loss retention and minimum total variance are

(i) when $f_1(\nu^{-}) < 0$, $d^* = -\ln D_I/\beta$ and $\min J_1(d) = \left[1 + 2(1 + \ln D_I)D_I - 2D_I^2\right]/\beta^2$;

(ii) when $f_1(\nu^{-}) \geq 0$ and $f_R(\nu^{-}) \geq 0$, $d^* = r/\beta$ and $\min J_1(d) = (1 - re^{-r})/\beta^2$;

(iii) when $f_R(\nu^{-}) < 0$, $d^* = -\ln D_R/\beta$ and $\min J_1(d) = \left[1 + 2(1 + \ln D_R)D_R - 2D_R^2\right]/\beta^2$.

Case 3. If $\theta_R \geq \frac{\nu}{\beta - \nu}$, the optimal stop-loss retention and minimum total variance are

(i) when $f_1(\nu^{-}) < 0$, $d^* = -\ln D_I/\beta$ and $\min J_1(d) = \left[1 + 2(1 + \ln D_I)D_I - 2D_I^2\right]/\beta^2$;

(ii) when $f_1(\nu^{-}) \geq 0$, $d^* = r/\beta$ and $\min J_1(d) = (1 - re^{-r})/\beta^2$.

Proof. Let $r$ be the unique positive root of $f(x) = 2e^{-x} + x - 2$. Then $J'_1(d)$ has a unique positive root $\bar{d} = r/\beta$. Besides, $J_1(d)$ is decreasing for $d \in [0, \bar{d}]$ and increasing for $d \geq \bar{d}$.

Case 1. If $0 < \theta_R < \frac{\beta}{\nu} \ln \frac{\beta}{\beta - \nu} - 1$, from Lemma 3.1 we know that the reinsurance premium is too low for a reinsurance company to benefit from entering into the stop-loss contract. Thus the optimal stop-loss retention does not exist. Without reinsurance, the total variance of both parties is $\min J_1(d) = \lim_{d \rightarrow +\infty} J_1(d) = 1/\beta^2$, i.e., the variance of insurance firm’s wealth.

Case 2. If $\frac{\beta}{\nu} \ln \frac{\beta}{\beta - \nu} - 1 \leq \theta_R < \frac{\nu}{\beta - \nu}$, from the utility constraints of both insurance firm and reinsurance firm, we need $-\ln D_I/\beta \leq d \leq -\ln D_R/\beta$. When $f_1(\nu^{-}) < 0$, we have $D_I < e^{-r}$, i.e., $-\ln D_I/\beta > \bar{d}$, then $J_1(d)$ is increasing for $d \in [-\ln D_I/\beta, -\ln D_R/\beta]$ and the optimal retention $d^* = -\ln D_I/\beta$. The corresponding minimum total variance is $\min J_1(d) = \left[1 + 2(1 + \ln D_I)D_I - 2D_I^2\right]/\beta^2$; when $f_1(\nu^{-}) \geq 0$ and $f_R(\nu^{-}) \geq 0$, we have $D_R \leq e^{-r} \leq D_I$, i.e., $-\ln D_I/\beta \leq \bar{d} \leq -\ln D_R/\beta$. Then $d^* = \bar{d} = r/\beta$ and $\min J_1(d) = (1 - re^{-r})/\beta^2$; when $f_R(\nu^{-}) < 0$, we have $e^{-r} < D_R$, i.e., $-\ln D_R/\beta < \bar{d}$, then $J_1(d)$ is decreasing for $d \in [-\ln D_I/\beta, -\ln D_R/\beta]$. Thus the optimal retention is $d^* = -\ln D_R/\beta$ and $\min J_1(d) = \left[1 + 2(1 + \ln D_R)D_R - 2D_R^2\right]/\beta^2$.

Case 3. If $\theta_R \geq \frac{\nu}{\beta - \nu}$, expected utility constraints require that stop-loss retention $d$ should satisfy $d \geq -\ln D_I/\beta$. (i) When $f_1(\nu^{-}) < 0$, from Lemma 3.1, we have $e^{-r} > D_I$, rearrange this inequation gives $-\ln D_I/\beta > \bar{d}$. Therefore, $J_1(d)$ is an increasing function for $d \geq -\ln D_I/\beta$. Thus, the minimum of objective function is obtained at optimal retention level $d^* = -\ln D_I/\beta$, and $\min J_1(d) = \left[1 + 2(1 + \ln D_I)D_I - 2D_I^2\right]/\beta^2$. (ii) When $f_1(\nu^{-}) \geq 0$, we have $-\ln D_I/\beta \leq \bar{d}$. Then, $J_1(d)$ is decreasing for $-\ln D_I/\beta \leq d \leq \bar{d}$, and increasing for $d \geq \bar{d}$. Therefore, the optimal retention is $d^* = r/\beta$, and $\min J_1(d) = (1 - re^{-r})/\beta^2$. \(\Box\)

4.2. Value-at-risk. The Value-at-Risk (VaR) is the maximum expected loss over a given horizon period at a given confidence level. Because of its simplicity and intuitive appeal, VaR is adopted by many financial regulators to set the requirement on capital reserves. The directive of the [17] adopts VaR for setting up provisions and capital requirements of a financial institution to ensure solvency. More specifically, the SCR is required to be corresponding to the VaR of the basic own funds of an insurance or reinsurance company subject to a confidence level of 99.5% over a
one-year period. The VaR of a loss random variable \( Y \) at a confidence level \( 1 - \gamma \), \( 0 < \gamma < 1 \) (e.g. \( \gamma = 0.5\% \)), is defined as
\[
\text{VaR}_\gamma(Y) = \inf\{L : \Pr(Y > L) \leq \gamma\},
\]
When we consider the total VaR of the decrement in wealth of insurance firm and reinsurance firm, at the confidence level \( 1 - \alpha_I \) and \( 1 - \alpha_R \) respectively, as the optimality criterion, we are aiming to find the optimal retention level \( d^* \) over admissible control set \([0, \infty)\) that minimising the objective function
\[
J_2(d) \triangleq \text{VaR}_{\alpha_I}(X \wedge d - P_I(d)) + \text{VaR}_{\alpha_R}((X - d)_+ - P_R(d))
\]
\[
= \text{VaR}_{\alpha_I}((X \wedge d) - P_I + (1 + \theta_R)e^{-\beta d}/\beta + \text{VaR}_{\alpha_R}((X - d)_+ - (1 + \theta_R)e^{-\beta d}/\beta
\]
\[
= \text{VaR}_{\alpha_I}((X \wedge d) + \text{VaR}_{\alpha_R}((X - d)_+ - P_R).
\]
Let \( Y_1 \triangleq (X \wedge d) \) and \( Y_2 \triangleq (X - d)_+ \). Since \( X \sim \exp(\beta) \), we have
\[
\Pr(Y_1 = d) = e^{-\beta d}, \quad F_1(y) = \Pr(Y_1 \leq y) = 1 - e^{-\beta y}, \quad 0 \leq y < d,
\]
\[
\Pr(Y_2 = 0) = 1 - e^{-\beta d}, \quad F_2(y) = \Pr(Y_2 \leq y) = 1 - e^{-\beta(y+d)}, \quad y > 0.
\]
Then,
\[
\text{VaR}_{\alpha_I}(Y_1) = \inf\{L : \Pr(Y_1 > L) \leq \alpha_I\} = \left\{ \begin{array}{ll}
\{d, & \text{if } d \leq -\ln \alpha_I/\beta, \\
-\ln \alpha_I/\beta, & \text{if } d > -\ln \alpha_I/\beta.
\end{array} \right.
\]
\[
\text{VaR}_{\alpha_R}(Y_2) = \inf\{L : \Pr(Y_2 > L) \leq \alpha_R\} = \left\{ \begin{array}{ll}
-d - \ln \alpha_R/\beta, & \text{if } d < -\ln \alpha_R/\beta, \\
0, & \text{if } d \geq -\ln \alpha_R/\beta.
\end{array} \right.
\]
Therefore,
\[
J_2(d) = \min \{d, -\ln \alpha_I/\beta\} + \max \{-d - \ln \alpha_R/\beta, 0\} - P_0.
\]
We derive the optimal stop-loss retention and the corresponding total VaR in the following theorem.

**Theorem 4.2.** The optimal stop-loss retention level \( d^* \) that minimises both parties’ VaR in total and the corresponding minimum total VaR are as follows.

**Case 1.** For \( \alpha_I > \alpha_R \), when \( \beta/v \ln \beta/\beta - 1 < \theta_R < \beta/v \ln \beta/\beta - 1 \),

(i) if \( \alpha_I \leq D_R, \forall d^* \in [-\ln D_I/\beta, -\ln D_R/\beta] \) is the optimal stop-loss retention;

(ii) if \( \alpha_R < D_I < \alpha_I \), the optimal stop-loss retention \( d^* = -\ln D_I/\beta \);

(iii) if \( \alpha_R \geq D_R, \forall \max\{-\ln D_I/\beta, -\ln \alpha_R/\beta\} \leq d^* \leq -\ln D_R/\beta \) is the optimal retention;

and the minimum total VaR is
\[
\min J_2(d) = \left\{ \begin{array}{ll}
-\ln \alpha_R/\beta - P_0, & \text{if } \alpha_I \leq D_R, \\
-\ln \alpha_I/\beta - \ln \alpha_R/\beta + \ln D_R/\beta - P_0, & \text{if } \alpha_R < D_R < \alpha_I, \\
-\ln \alpha_I/\beta - P_0, & \text{if } \alpha_R \geq D_R.
\end{array} \right.
\]

Otherwise, \( \min J_2(d) = -\ln \alpha_I/\beta - P_0 \). Specifically, when \( 0 < \theta_R < \beta/v \ln \beta/\beta - 1 \), the stop-loss retention does not exist; when \( \theta_R \geq \beta/v \ln \beta/\beta - 1 \), any retention \( d \) that is greater than \( \max\{-\ln D_I/\beta, -\ln \alpha_R/\beta\} \) is the optimal retention.

**Case 2.** For \( \alpha_I = \alpha_R \), the total VaR is a constant and \( J_2(d) = -\ln \alpha_I/\beta - P_0 \). The optimal stop-loss retention \( d^* \) does not exist when \( 0 < \theta_R < \beta/v \ln \beta/\beta - 1 \); when \( \beta/v \ln \beta/\beta - 1 < \theta_R < \beta/v \ln \beta/\beta - 1 \), \( \forall d^* \in [-\ln D_I/\beta, -\ln D_R/\beta] \) is the optimal retention; when \( \theta_R \geq \beta/v \ln \beta/\beta - 1 \), any retention \( d^* \in [-\ln D_I/\beta, +\infty) \) is optimal.

**Case 3.** For \( \alpha_I < \alpha_R \), when \( 0 < \theta_R < \beta/v \ln \beta/\beta - 1 \), the stop-loss retention does not exist; when \( \beta/v \ln \beta/\beta - 1 \leq \theta_R < \beta/v \ln \beta/\beta - 1 \),
(i) if \( \alpha_R \leq D_1 \), \( \forall - \ln D_1 / \beta \leq d^* \leq \min\{ - \ln D_R / \beta, - \ln \alpha_R / \beta \} \) is the optimal retention;
(ii) if \( \alpha_I < D_1 < \alpha_R \), the optimal stop-loss retention \( d^* = - \ln D_1 / \beta \);
(iii) if \( \alpha_I \geq D_1 \), \( \forall d^* \in [ - \ln D_1 / \beta, - \ln D_R / \beta ] \) is the optimal stop-loss retention; and the corresponding minimum total VaR is

\[
\min J_2(d) = \begin{cases} 
- \ln \alpha_R / \beta - P_0, & \text{if } \alpha_R \leq D_1, \\
- \ln D_1 / \beta - P_0, & \text{if } \alpha_I < D_1 < \alpha_R, \\
- \ln \alpha_I / \beta - P_0, & \text{if } \alpha_I \geq D_1.
\end{cases}
\]

When \( \theta_R \geq \frac{\beta}{\beta - \frac{v}{\beta}} \), the minimum total VaR is also given by (8). Meanwhile,

(i) if \( \alpha_R \leq D_1 \), \( \forall d^* \in [ - \ln D_1 / \beta, - \ln \alpha_R / \beta ] \) is the optimal stop-loss retention;
(ii) if \( \alpha_I < D_1 < \alpha_R \), the optimal stop-loss retention \( d^* = - \ln D_1 / \beta \);
(iii) if \( \alpha_I \geq D_1 \), \( \forall d^* \geq - \ln D_1 / \beta \) is the optimal stop-loss retention.

**Proof. Case 1.** For \( \alpha_I > \alpha_R \), we have \( - \ln \alpha_I / \beta < - \ln \alpha_R / \beta \). From (7),

\[
J_2(d) = \begin{cases} 
- \ln \alpha_R / \beta - P_0, & \text{if } d < - \ln \alpha_I / \beta, \\
- \ln \alpha_I / \beta - \ln \alpha_R / \beta - d - P_0, & \text{if } - \ln \alpha_I / \beta \leq d \leq - \ln \alpha_R / \beta, \\
- \ln \alpha_R / \beta - P_0, & \text{if } d > - \ln \alpha_R / \beta.
\end{cases}
\]

For \( d \in [ - \ln \alpha_I / \beta, - \ln \alpha_R / \beta ] \), \( J_2(d) \) is decreasing with \( d \) and is continuous at \( d = - \ln \alpha_I / \beta \) and \( d = - \ln \alpha_R / \beta \). When \( 0 < \theta_R < \frac{\beta}{\beta - \frac{v}{\beta}} - 1 \), the stop-loss retention does not exist; when \( \frac{\beta}{\beta} \ln \frac{\beta}{\beta - \frac{v}{\beta}} - 1 \leq \theta_R < \frac{\beta}{\beta - \frac{v}{\beta}} \), the insurance firm’s and reinsurance firm’s utility constraints require that \( d \in [ - \ln D_1 / \beta, - \ln D_R / \beta ] \),

(i) if \( \alpha_I \leq D_R \), we have \( - \ln D_R / \beta \leq - \ln \alpha_I / \beta \), then, for \( d \in [ - \ln D_1 / \beta, - \ln D_R / \beta ] \),

\[ J_2(d) = - \ln \alpha_R / \beta - P_0 \]

and thus \( \forall d^* \in [ - \ln D_1 / \beta, - \ln D_R / \beta ] \) is the optimal stop-loss retention;
(ii) if \( \alpha_R < D_R < \alpha_I \), we have \( - \ln \alpha_I / \beta < - \ln D_R / \beta < - \ln \alpha_R / \beta \). Then, \( J_2(d) \) is a constant when \( d \leq - \ln \alpha_I / \beta \) and is decreasing in \( d \) for \( - \ln \alpha_I / \beta \leq d \leq - \ln D_R / \beta \), therefore, \( J_2(d) \) minimses at \( d^* = - \ln D_R / \beta \) and the minimum total VaR is min \( J_2(d) = J_2(- \ln D_R / \beta) = - \ln \alpha_I / \beta - \ln \alpha_R / \beta + \ln D_R / \beta - P_0 \);
(iii) if \( \alpha_R \geq D_R \), we have \( - \ln D_R / \beta \geq - \ln \alpha_R / \beta \). \( J_2(d) \) is decreasing in \( d \) for \( d \leq - \ln \alpha_R / \beta \), and become a constant for \( - \ln \alpha_R / \beta \leq d \leq - \ln D_R / \beta \). Therefore, \( \forall d \in [ - \ln D_1 / \beta, - \ln \alpha_R / \beta ] \) is the optimal stop-loss retention and min \( J_2(d) = - \ln \alpha_I / \beta - P_0 \).

When \( \theta_R \geq \frac{\beta}{\beta - \frac{v}{\beta}} \), we have \( d \geq - \ln D_1 / \beta \) and hence min \( J_2(d) = - \ln \alpha_I / \beta - P_0 \). Meanwhile, any stop-loss retention \( d \) satisfying that \( d \geq \max\{ - \ln D_1 / \beta, - \ln \alpha_R / \beta \} \) are the optimal strategies.

**Case 2.** For \( \alpha_I = \alpha_R \), the optimality criterion (7) becomes \( J_2(d) = - \ln \alpha_I / \beta - P_0 \), \( \forall d \geq 0 \). When \( 0 < \theta_R < \frac{\beta}{\beta} \ln \frac{\beta}{\beta - \frac{v}{\beta}} - 1 \), reinsurance premium is too low that no reinsurance company will accept the policy. Hence, the stop-loss retention does not exist; when \( \frac{\beta}{\beta} \ln \frac{\beta}{\beta - \frac{v}{\beta}} - 1 \leq \theta_R < \frac{\beta}{\beta - \frac{v}{\beta}} \), we have \( - \ln D_1 / \beta \leq d \leq - \ln D_R / \beta \) thus the optimal stop-loss retention \( d^* \) could be any value in \( [ - \ln D_1 / \beta, - \ln D_R / \beta ] \) when \( \theta_R \geq \frac{\beta}{\beta - \frac{v}{\beta}} \), the utility constraints determine that \( d \geq - \ln D_1 / \beta \). Consequently, \( \forall d^* \in [ - \ln D_1 / \beta, +\infty ) \) is the optimal stop-loss retention.

**Case 3.** For \( \alpha_I < \alpha_R \), results can be proved by similar analysis with the case of \( \alpha_I > \alpha_R \). \( \Box \)
4.3. Tail value-at-risk. In spite of the popularity among regulators and practitioners, VaR is extensively criticised for possessing some undesirable theoretical properties such as not coherent. Moreover, if an extreme loss event occurs, companies with the VaR-regulated capital might not be able to absorb its entire risks. Considering that Tail Value-at-Risk (TVaR) is the smallest of coherent risk measures which only depend on the distribution of underlying random variables and are no less than VaR at the same confidence level, we adopt TVaR as the risk measure in this subsection to reflect the potential catastrophic losses of the tail of the distribution.

The TVaR of a random variable \( Y \) at the confidence level \( 1 - \gamma \) is defined as

\[
TVaR_{\gamma}(Y) = \frac{1}{\gamma} \int_{0}^{\gamma} VaR_{q}(Y)dq.
\]

When we take TVaR as the risk measure, and the confidence level for insurance firm and reinsurance firm are \( 1 - \alpha_I \) and \( 1 - \alpha_R \) respectively, (1) becomes

\[
J_3(d) \triangleq TVaR_{\alpha_I}(X \wedge d - P_I(d)) + TVaR_{\alpha_R}((X - d_+) - P_R(d)) = TVaR_{\alpha_I}(X \wedge d) + TVaR_{\alpha_R}((X - d_+) - P_0),
\]

where

\[
TVaR_{\alpha_I}(X \wedge d) = \frac{1}{\alpha_I} \int_{0}^{\alpha_I} VaR_{q}(X \wedge d) dq = \left\{ \begin{array}{ll}
\frac{1-\ln \alpha_I}{\beta} - \frac{e^{-\beta d}}{\alpha_I \beta}, & \text{if } d \leq -\frac{\ln \alpha_I}{\beta}, \\
\frac{1-\ln \alpha_I}{\beta} - \frac{\ln \alpha_I}{\beta}, & \text{if } d > -\frac{\ln \alpha_I}{\beta},
\end{array} \right.
\]

and

\[
TVaR_{\alpha_R}((X - d_+) = \frac{1}{\alpha_R} \int_{0}^{\alpha_R} VaR_{q}((X - d_+) dq = \left\{ \begin{array}{ll}
\frac{1-\ln \alpha_R}{\beta} - d, & \text{if } d \leq -\frac{\ln \alpha_R}{\beta}, \\
\frac{1-\ln \alpha_R}{\beta} - \frac{\ln \alpha_R}{\beta}, & \text{if } d > -\frac{\ln \alpha_R}{\beta},
\end{array} \right.
\]

We would like to find the retention \( d^* \in [0, \infty) \) that makes the optimality criterion \( J_3(d) \) obtain its minimum over \([0, \infty)\). We derive the optimal stop-loss retention in the following theorem.

**Theorem 4.3.** The optimal stop-loss retention level \( d^* \) that minimise both parties’ TVaR in total and the corresponding minimum total TVaR are as follows.

**Case 1.** For \( \alpha_I > \alpha_R \), when \( \frac{\beta}{\nu} \ln \frac{\beta}{\nu} - 1 \leq \theta_R < \frac{\nu}{\beta - \nu} \), if \( \alpha_R \leq D_I, \forall d^* \in [-\ln D_I/\beta, -\ln D_R/\beta] \) is the optimal stop-loss retention; if \( \alpha_R > D_I \), the optimal retention \( d^* = -\ln D_I/\beta \). The corresponding minimum total TVaR is

\[
\min J_3(d) = \begin{cases} 
\frac{1-\ln \alpha_I}{\beta} - P_0, & \text{if } \alpha_I \leq D_R, \\
\frac{2-\ln(\alpha_I\alpha_R)}{\beta} - \frac{D_R}{\alpha_I \beta} + \frac{D_R}{\alpha_R \beta} - P_0, & \text{if } \alpha_R < D_R < \alpha_I, \\
\frac{1-\ln \alpha_I}{\beta} - \frac{D_R}{\alpha_I \beta} + \frac{D_R}{\alpha_R \beta} - P_0, & \text{if } \alpha_R \geq D_R.
\end{cases}
\]

When \( 0 < \theta_R < \frac{\beta}{\nu} \ln \frac{\beta}{\nu} - 1 \) or \( \theta_R \geq \frac{\nu}{\beta - \nu} \), \( \min J_3(d) = (1 - \ln \alpha_I)/\beta - P_0 \) and the optimal stop-loss retention does not exist.

**Case 2.** For \( \alpha_I = \alpha_R \), the total TVaR is a constant and \( J_3(d) \equiv (1 - \ln \alpha_I)/\beta - P_0 \). When \( 0 < \theta_R < \frac{\beta}{\nu} \ln \frac{\beta}{\nu} - 1 \), the stop-loss retention does not exist; when \( \frac{\beta}{\nu} \ln \frac{\beta}{\nu} - 1 \leq \theta_R < \frac{\nu}{\beta - \nu} \), the optimal retention could be any value in \([-\ln D_I/\beta, -\ln D_R/\beta]\); when \( \theta_R \geq \frac{\nu}{\beta - \nu} \), \( \forall d^* \geq -\ln D_I/\beta \) is the optimal stop-loss retention.
Case 3. For $\alpha_I < \alpha_R$, when $0 < \theta_R < \frac{\beta}{v} \ln \frac{\beta}{\beta - v} - 1$, the stop-loss retention does not exist; when $\theta_R \geq \frac{\beta}{v} \ln \frac{\beta}{\beta - v} - 1$, the minimum total TVaR is
\[
\min J_3(d) = \begin{cases} 
\frac{1-\ln \alpha_R}{\beta} - P_0, & \text{if } \alpha_R \leq D_I, \\
\frac{D_I}{\alpha_R} - \ln D_I/\beta - P_0, & \text{if } \alpha_I < D_I < \alpha_R, \\
\frac{1-\ln \alpha_I}{\beta} - \frac{D_I}{\alpha_R^2} + \frac{D_I}{\alpha_R} - P_0, & \text{if } \alpha_I \geq D_I;
\end{cases}
\]
regarding the optimal stop-loss retention, if $\alpha_R > D_I$, $d^* = -\ln D_I/\beta$; if $\alpha_R \leq D_I$, when $\frac{\beta}{v} \ln \frac{\beta}{\beta - v} - 1 \leq \theta_R < \frac{\beta}{v}$, the optimal stop-loss retention $d^*$ could be any value satisfying that $-\ln D_I/\beta \leq d^* \leq \min\{-\ln D_R/\beta, -\ln \alpha_R/\beta\}$; when $\theta_R \geq \frac{\beta}{v}$, \(\forall d^* \in \{-\ln D_I/\beta, -\ln \alpha_R/\beta\}\) is the optimal stop-loss retention.

Proof. Case 1. For $\alpha_I > \alpha_R$, we have $-\ln \alpha_I/\beta < -\ln \alpha_R/\beta$. From (10),
\[
J_3(d) = \begin{cases} 
\frac{1-\ln \alpha_R}{\beta} - P_0, & \text{if } d < -\ln \alpha_I/\beta, \\
\frac{2-\ln (\alpha_I/\alpha_R)}{\beta} - \frac{e^{-3d/\alpha_I^2}}{\alpha_I} - d - P_0, & \text{if } -\ln \alpha_I/\beta \leq d \leq -\ln \alpha_R/\beta, \\
\frac{1-\ln \alpha_I}{\beta} - \frac{e^{-3d/\alpha_R^2}}{\alpha_R} - d - P_0, & \text{if } d > -\ln \alpha_R/\beta.
\end{cases}
\]
For $-\ln \alpha_I/\beta \leq d \leq -\ln \alpha_R/\beta$, $J_3(d) = e^{-3d/\alpha_I^2} - 1 \leq 0$ and for $d > -\ln \alpha_R/\beta$, $J_3(d) = (1/\alpha_I - 1/\alpha_R)e^{-3d/\alpha_R^2} < 0$. Therefore, $J_3(d)$ is a constant for $d < -\ln \alpha_I/\beta$, and is decreasing with $d$ for $d \geq -\ln \alpha_I/\beta$. Besides, $J_3(d)$ is continuous at $d = -\ln \alpha_I/\beta$ and $d = -\ln \alpha_R/\beta$. When $0 < \theta_R < \frac{\beta}{v} \ln \frac{\beta}{\beta - v} - 1$, the stop-loss retention does not exist because reinsurance premium is too low for a reinsurance company. When $\frac{\beta}{v} \ln \frac{\beta}{\beta - v} - 1 \leq \theta_R < \frac{\beta}{v}$, from Lemma 3.1 we have the retention $d \in [-\ln D_I/\beta, -\ln D_R/\beta]$. If $\alpha_I \leq D_R$, we have $-\ln D_R/\beta \leq -\ln \alpha_I/\beta$, then $J_3(d) \equiv (1 - \ln \alpha_R)/\beta - P_0$ and $\forall d^* \in [-\ln D_I/\beta, -\ln D_R/\beta]$ is the optimal stop-loss retention; if $\alpha_I > D_R$, then $-\ln D_R/\beta > -\ln \alpha_I/\beta$ and the total TVaR minimises at $d^* = -\ln D_R/\beta$. The corresponding minimum total TVaR is
\[
\min J_3(d) = J_3(-\ln D_R/\beta) = \begin{cases} 
\frac{1-\ln \alpha_R}{\beta} - P_0, & \text{if } \alpha_I \leq D_R, \\
\frac{2-\ln (\alpha_I/\alpha_R)}{\beta} - \frac{\ln D_R}{\alpha_I^2} + \frac{\ln D_R}{\alpha_R} - P_0, & \text{if } \alpha_R < D_R < \alpha_I, \\
\frac{1-\ln \alpha_I}{\beta} - \frac{\ln D_R}{\alpha_R^2} + \frac{\ln D_R}{\alpha_R} - P_0, & \text{if } \alpha_R \geq D_R.
\end{cases}
\]
When $\theta_R \geq \frac{\beta}{v}$, we have $d \geq -\ln D_I/\beta$ and $\min J_3(d) = \lim_{d \to -\infty} J_3(d) = \left(1 - \ln \alpha_I/\beta\right) - P_0$. The total TVaR takes the minimum value when there is no reinsurance and the optimal stop-loss retention does not exist.

Case 2. For $\alpha_I = \alpha_R$, the optimality criterion (10) becomes $J_3(d) \equiv (1 - \ln \alpha_I/\beta - P_0, \forall d \geq 0$. Then the results in this case can be proved by following the reasoning in Theorem 4.2.

Case 3. For $\alpha_I < \alpha_R$, we can prove results by following same procedures with the situation of $\alpha_I > \alpha_R$. \(\square\)

4.4. Generalised Dutch type I risk measure. To obtain better risk-based capital assessments, the Solvency II regime provides incentives for companies and regulators to adopt more sophisticated risk management tools by developing internal capital models. In this subsection, we adopt a generalised Dutch risk measure as an advanced risk monitoring tool. The Dutch risk measure is introduced by [19] under the name of Dutch premium principles in an insurance pricing environment. It is generalised by [31] by replacing expectation with precise previsions and the generalised expression is
\[
\rho_D(X) = P_0(X) + cP_1((X - P_0(X))_+), \quad c \in [0, 1],
\]
where \( P_0 \) and \( P_1 \) are dF-coherent previsions so that \( \rho_D(X) \) is coherent. We may interpret \( P_0 \) as a sort of first approach risk measure for insurance firm’s claim losses, and \( P_1 \) corrects \( P_0 \) by taking account of its shortfall \( (X - P_0(X))^+ \). For mathematical tractability, we construct the optimality criterion based on a generalised Dutch type I risk measure, which is introduced by [11]. We take the expectation of the wealth of insurance/reinsurance firm as the benchmark risk level, and choose TVaR as the risk measure for the loss above benchmark, then \( \rho(X) = \mathbb{E}[X] + t\text{TVaR}_q((X - \mathbb{E}[X])^+) \), \( 0 < t \leq 1 \). Additionally, we assume the confidence level \( \alpha \) of TVaR for insurance firm and reinsurance firm are the same.

Then, our optimisation problem becomes \( \min_{d \geq 0} J_4(d) \), with optimality criterion being:

\[
J_4(d) \triangleq \rho(X \wedge d - P_1(d)) + \rho((X - d)^+ - P_R(d))
\]

\[
= \rho \left( X \wedge d - P_0 + \frac{(1 + \theta_R)e^{-\beta d}}{\beta} \right) + \rho \left( (X - d)^+ - \frac{(1 + \theta_R)e^{-\beta d}}{\beta} \right)
\]

\[
=t\text{TVaR}_\alpha \left( X \wedge d - \frac{1 - e^{-\beta d}}{\beta} \right) + t\text{TVaR}_\alpha \left( (X - d)^+ - \frac{e^{-\beta d}}{\beta} \right) + \frac{1}{\beta} - P_0.
\]

Denote \( Y_1 \triangleq (X \wedge d - (1 - e^{-\beta d})/\beta)^+ \). Since \( X \sim \exp(\beta) \), we have \( \Pr(Y_1 = 0) = \Pr(X \leq 1 - (1 - e^{-\beta d})/\beta) = 1 - e^{-\beta d} - 1 \), \( \Pr(Y_1 = d - (1 - e^{-\beta d})/\beta) = \Pr(X \geq d) = e^{-\beta d} \), for \( 0 < y < d - (1 - e^{-\beta d})/\beta \), the cumulative distribution function of \( Y_1 \) is \( F_1(y) = \Pr(Y_1 \leq y) = \Pr(X \leq y + (1 - e^{-\beta d})/\beta) = 1 - e^{-\beta y\cdot e^{-\beta d}} \). Then we obtain:

\[
\text{Var}_{\alpha}(Y_1) = \begin{cases} 
  d - \frac{1 - e^{-\beta d}}{\beta}, & \text{if } \alpha < e^{-\beta d}, \\
  -\ln \alpha \frac{1 + e^{-\beta d}}{\beta}, & \text{if } e^{-\beta d} \leq \alpha < e^{-\beta d - 1}, \\
  0, & \text{if } e^{-\beta d - 1} \leq \alpha \leq 1,
\end{cases}
\]

and

\[
\text{TVaR}_{\alpha}(Y_1) = \frac{1}{\alpha} \int_0^{\alpha} \text{Var}_{\alpha}(Y_1) dq = \begin{cases} 
  d - \frac{1 - e^{-\beta d}}{\beta}, & \text{if } \alpha < e^{-\beta d}, \\
  e^{-\beta d - 1} - \frac{\ln \alpha}{\beta} - \frac{e^{-\beta d}}{\beta}, & \text{if } e^{-\beta d} \leq \alpha < e^{-\beta d - 1}, \\
  e^{-\beta d - 1}, & \text{if } e^{-\beta d - 1} \leq \alpha < 1.
\end{cases}
\]

Let \( Y_2 \triangleq ((X - d)^+ - e^{-\beta d}/\beta)^+ \), then \( \Pr(Y_2 = 0) = \Pr(X \leq d + e^{-\beta d}/\beta) = 1 - e^{-\beta d} - e^{-\beta d} \). For \( y > 0 \), the cumulative distribution probability is \( F_2(y) = \Pr(Y_2 \leq y) = \Pr(X \leq y + d + e^{-\beta d}/\beta) = 1 - e^{-\beta y - \beta d - e^{-\beta d}} \). Then we have:

\[
\text{Var}_{\alpha}(Y_2) = \begin{cases} 
  -\ln \alpha + \frac{\beta d + e^{-\beta d}}{\beta}, & \text{if } \alpha < e^{-\beta d - e^{-\beta d}}, \\
  0, & \text{if } e^{-\beta d - e^{-\beta d}} \leq \alpha < 1,
\end{cases}
\]

and

\[
\text{TVaR}_{\alpha}(Y_2) = \frac{1}{\alpha} \int_0^{\alpha} \text{Var}_{\alpha}(Y_2) dq = \begin{cases} 
  -\ln \alpha - 1 + \frac{\beta d + e^{-\beta d}}{\beta}, & \text{if } \alpha < e^{-\beta d - e^{-\beta d}}, \\
  e^{-\beta d - e^{-\beta d}} - \frac{\ln \alpha}{\beta}, & \text{if } e^{-\beta d - e^{-\beta d}} \leq \alpha < 1.
\end{cases}
\]

Let \( d_1 \in (0, -\ln \alpha/\beta) \) be the unique root of \( \beta d + e^{-\beta d} + \ln \alpha = 0 \) when \( \alpha < 1/e \) (see Lemma A.2). The optimal stop-loss retention and the corresponding total risk measure are derived in the following theorem.
Theorem 4.4. For $0 < \alpha < 1/e$, the optimal stop-loss retention level that minimises total Dutch type I risk measure $\rho$ and the corresponding minimum total risk are as follows.

Case 1. When $0 < \theta_R < \frac{\alpha}{\alpha - 1} \ln \frac{\alpha - e^{-\alpha}}{1 - e^{-\alpha}} - 1$, the stop-loss retention does not exist.

Case 2. When $\frac{\alpha}{\alpha - 1} \ln \frac{\alpha - e^{-\alpha}}{1 - e^{-\alpha}} - 1 \leq \theta_R < \frac{\alpha}{\alpha - 1},$ if $D_I + \ln(\alpha/D_I) \leq 0$, the minimum total risk measure is $\min J_4(d) = -t \ln \alpha + 1/\beta - P_0$ and any $d^*$ such that $- \ln D_I/\beta \leq d^* \leq \min\{1, -\ln D_R/\beta\}$ is the optimal stop-loss retention; if $D_I + \ln(\alpha/D_I) > 0$, in the case that $(a) D_R \geq D_0$, $(b) D_I \geq \alpha$ and $D_I e^{-D_I}/\alpha + D_I - D_I - 1 \leq D_R - \ln \alpha - D_R(1 - e^{-D_R})/\alpha$, $(c) D_I < \alpha$ and $D_I - D_I(1 - e^{-D_I})/\alpha \leq D_R - D_R(1 - e^{-D_R})/\alpha$, the optimal retention $d^* = - \ln D_I/\beta$ and

$$
\min J_4(d) = \begin{cases} 
\frac{D_I e^{-D_I}}{\alpha \beta} + \frac{D_I - \ln D_I - 1}{\beta}, & \text{if } D_I \geq \alpha, \\ \frac{D_I - \ln D_I}{\beta} + \frac{D_I(1 - e^{-D_I})}{\alpha \beta}, & \text{if } D_I < \alpha.
\end{cases}
$$

in the case that $(a) D_I \geq \alpha$ and $D_I e^{-D_I}/\alpha + D_I - \ln D_I - 1 \geq D_R - \ln \alpha - D_R(1 - e^{-D_R})/\alpha$, $(b) D_I < \alpha$ and $D_I - D_I(1 - e^{-D_I})/\alpha \geq D_R - \ln \alpha - D_R(1 - e^{-D_R})/\alpha$, $(c) D_I \leq D_0$, the optimal stop-loss retention is $d^* = - \ln D_R/\beta$ and the corresponding total risk measure is $\min J_4(d) = [(D_R - \ln \alpha)/\beta - D_R(1 - e^{-D_R})/(\alpha \beta)] t + 1/\beta - P_0$.

Case 3. When $\theta_R \geq \frac{\alpha}{\alpha - 1},$ if $D_I + \ln(\alpha/D_I) \leq 0$, $\forall d^* \in [-\ln D_I/\beta, d_I]$ is the optimal stop-loss retention and $\min J_4(d) = -t \ln \alpha + 1/\beta - P_0$; if $D_I + \ln(\alpha/D_I) > 0$, the total Generalised Dutch type I risk measure takes the minimum value when there is no reinsurance, i.e., $\min J_4(d) = -t \ln \alpha + 1/\beta - P_0$ and the optimal stop-loss retention does not exist.

Proof. We start with analysing the objective function.

(i) When $0 \leq d < d_1$, we have $0 < \alpha < e^{-\beta d - e^{-\beta d}}$, then $\text{TVaR}_{\alpha}(Y_1) = d - (1 - e^{-\beta d})/\beta$ and $\text{TVaR}_{\alpha}(Y_2) = - \ln \alpha - 1 + \beta d + e^{-\beta d}/\beta$. Thus $J_4(d) \equiv -t \ln \alpha + 1/\beta - P_0$ is a constant function of the stop-loss retention $d$.

(ii) when $d_1 \leq d < - \ln \alpha/\beta$, we have $e^{-\beta d - e^{-\beta d}} \leq \alpha < e^{-\beta d}$, then $\text{TVaR}_{\alpha}(Y_1) = d - (1 - e^{-\beta d})/\beta$ and $\text{TVaR}_{\alpha}(Y_2) = e^{-\beta d - e^{-\beta d}}/(\alpha \beta)$. The optimality criterion becomes $J_4(d) = \left[\frac{e^{-\beta d - e^{-\beta d}}}{\alpha \beta} + d + \frac{e^{-\beta d}}{\beta} - \frac{1}{\beta}\right] t + \frac{1}{\beta} - P_0$. Taking derivatives with respect to $d$ and rearrange, we obtain $J_4(d) \geq 0$. Thus, $J_4(d)$ is increasing in the retention level $d$ for $d \in (d_1, - \ln \alpha/\beta]$.

(iii) when $d \geq - \ln \alpha/\beta$, we have $\alpha \geq e^{-\beta d}$, and $\alpha < e^{-\beta d - 1}$ will always hold for any non-negative stop-loss retention. Then, $\text{TVaR}_{\alpha}(Y_1) = (e^{-\beta d} - \ln \alpha)/\beta - e^{-\beta d}/(\alpha \beta)$ and $\text{TVaR}_{\alpha}(Y_2) = e^{-\beta d - e^{-\beta d}}/(\alpha \beta)$. The objective $J_4(d) = \left[\frac{e^{-\beta d - \ln \alpha}}{\alpha \beta} + \frac{e^{-\beta d - e^{-\beta d}} - e^{-\beta d}}{\alpha \beta}\right] t + 1/\beta - P_0$. From Lemma A.3, we know that $J_4(d)$ is an increasing function in $d$ when $d \in [- \ln \alpha/\beta, - \ln D_0/\beta]$ and is decreasing with $d$ for $d \geq - \ln D_0/\beta$, where $D_0 \in (0, \alpha)$ is the unique root of $f(D) = (1 - D)e^{-D} + \alpha - 1$.

Therefore, for $0 < \alpha < 1/e$, the optimality criterion

$$
J_4(d) = \begin{cases} 
\frac{- \ln \alpha}{\beta} + \frac{1}{\beta} - P_0, & \text{if } 0 \leq d < d_1, \\
\frac{e^{-\beta d - e^{-\beta d}}}{\alpha \beta} + d + \frac{e^{-\beta d}}{\beta} - \frac{1}{\beta}, & \text{if } d_1 \leq d < - \ln \alpha/\beta, \\
\frac{e^{-\beta d - \ln \alpha}}{\beta} + \frac{e^{-\beta d - e^{-\beta d}} - e^{-\beta d}}{\alpha \beta}, & \text{if } d \geq - \ln \alpha/\beta.
\end{cases}
$$
is a constant for $0 \leq d < d_1$, then increases with $d$ when $d_1 \leq d < -\ln D_0/\beta$, and
will become a decreasing function in retention level for $d \geq -\ln D_0/\beta$. Meanwhile,
\( \lim_{d \to -\infty} J_4(d) = -t\ln \frac{\beta}{\beta-v} + 1/\beta - P_0 \), which is equal to the value of objective
function when $0 \leq d < d_1$. Then, we analyse the optimal stop-loss strategy and the
 corresponding minimum total generalised Dutch type I risk measure.

**Case 1.** When $0 < \theta_R < \frac{1}{\beta} \ln \frac{\beta}{\beta-v} - 1$, the stop-loss retention does not exist
because the reinsurance premium is too low for a reinsurance company.

**Case 2.** When $\frac{1}{\beta} \ln \frac{\beta}{\beta-v} - 1 \leq \theta_R < \frac{1}{\beta} v$, from Lemma 3.1 we know that the
retention should satisfy $-\ln D_1/\beta \leq d \leq -\ln D_R/\beta$.

1. If $D_1 + \ln(\beta/D_1) \leq 0$, we have $-\ln D_1/\beta \leq d_1$. Hence, any $d^*$ such that
\(-\ln D_1/\beta \leq d^* \leq \min\{d_1, -\ln D_R/\beta\} \) is the optimal stop-loss retention, and the
corresponding total risk measure is $\min J_4(d) = -t\ln \frac{\beta}{\beta-v} + 1/\beta - P_0$.

2. If $D_1 + \ln(\beta/D_1) > 0$ and $D_R \geq D_0$, we have $d_1 < -\ln D_1/\beta \leq -\ln D_R/\beta \leq
-\ln D_0/\beta$ and then $J_4(d)$ is an increasing function for $d \in [-\ln D_1/\beta, -\ln D_R/\beta]$. Then,
the optimal retention $d^* = -\ln D_1/\beta$ and

\[
\min J_4(d) = J_4\left( -\ln \frac{D_1}{\beta} \right) = \left\{ \begin{array}{ll}
\frac{D_1 e^{-D_1/\beta}}{\beta} + \frac{D_1 - \ln D_1 - 1}{\beta} & \text{if } D_1 \geq \alpha, \\
\frac{D_1 - \ln D_1 - 1}{\beta} & \text{if } D_1 < \alpha.
\end{array} \right.
\]

3. If $D_1 + \ln(\beta/D_1) > 0$ and $D_R \leq D_0 < D_1$, then $d_1 < -\ln D_1/\beta < -\ln D_0/\beta \leq
-\ln D_R/\beta$. The objective function $J_4(d)$ is increasing for $-\ln D_1/\beta \leq d \leq
-\ln D_0/\beta$ and is decreasing for $-\ln D_0/\beta \leq d \leq -\ln D_R/\beta$. $J_4(-\ln D_R/\beta) =
[(D_R - \ln(\beta/v))/\beta - D_R(1 - e^{-D_0/\beta})/(\beta\beta)]t + 1/\beta - P_0$. When $D_1 \geq \alpha$ and
$D_1 e^{-D_1/\beta} + D_1 - \ln D_1 - 1 \leq D_R - \ln(\beta/v) - D_R(1 - e^{-D_0/\beta})/\beta$, we have
$J_4(-\ln D_1/\beta) = [D_1 e^{-D_1/\beta} + (D_1 - \ln D_1 - 1)/\beta]t + 1/\beta - P_0 \leq J_4(-\ln D_R/\beta)$, hence $d^* = -\ln D_1/\beta$. From the similar reasoning, when
$D_1 < \alpha$ and $D_1 e^{-D_1/\beta} + D_1 - \ln D_1 - 1 \geq D_R - \ln(\beta/v) - D_R(1 - e^{-D_0/\beta})/\beta$, or
when $D_1 < \alpha$ and $D_1 - (1 - e^{-D_1/\beta})/\beta \geq D_R - D_R(1 - e^{-D_0/\beta})/\beta$, we have
$J_4(-\ln D_1/\beta) \geq J_4(-\ln D_R/\beta)$. Thus, $d^* = -\ln D_R/\beta$ and min $J_4(d) =
J_4(-\ln D_R/\beta) = [(D_R - \ln(\beta/v))/\beta - D_R(1 - e^{-D_0/\beta})/(\beta\beta)]t + 1/\beta - P_0$.

4. If $D_1 \leq D_0$, we have $-\ln D_1/\beta \geq -\ln D_0/\beta$ and then $J_4(d)$ is decreasing in $d$.
Therefore, the optimal stop-loss retention is $d^* = -\ln D_R/\beta$ and the corresponding
total risk measure is $\min J_4(d) = J_4(-\ln D_R/\beta) = [(D_R - \ln(\beta/v))/\beta - D_R(1 - e^{-D_0/\beta})/(\beta\beta)]t + 1/\beta - P_0$.

**Case 3.** When $\theta_R \geq \frac{1}{\beta} v$, the utility constraints require that $d \geq -\ln D_1/\beta$.

1. If $D_1 + \ln(\beta/D_1) \leq 0$, i.e., $-\ln D_1/\beta \leq d_1$, the objective $J_4(d)$ is a constant
for $d \in [-\ln D_1/\beta, d_1]$, then increases in the retention for $d \in [d_1, -\ln D_0/\beta]$
and start to diminish for $d \geq -\ln D_0/\beta$. Since $\lim_{d \to -\infty} J_4(d)$ is equal to
the initial constant value of the objective function, then for $d > d_1$, we have
$J_4(d)$ will be always greater than $J_4(d_1)$. Therefore, the optimal stop-loss retention is $\forall d^* \in [-\ln D_1/\beta, d_1]$ and the minimum total risk measure is $\min J_4(d) = -t\ln \frac{\beta}{\beta-v} + 1/\beta - P_0$. 

2. If \( D_1 + \ln(\alpha/D_1) > 0 \), i.e., \(-\ln D_1/\beta > d_1\), the minimum total risk measure is \( \min J_4(d) = \lim_{d \to \infty} J_4(d) = -t \ln(\alpha/\beta + 1/\beta) - P_0 \), and the optimal stop-loss retention does not exist.

\( \square \)

**Remark 4.** The confidence level is close to 0, we only consider the case of \( 0 < \alpha < 1/e \). For the case of \( 1/e \leq \alpha < 1 \), TVaR\(\alpha(Y_2) \equiv e^{-\beta d - e^{-\beta d}/(\alpha \beta)} \), then,

\[
J_4(d) = \begin{cases} 
\frac{e^{-\beta d - e^{-\beta d}}}{\alpha \beta} + d + \frac{e^{-\beta d}}{\beta} - \frac{1}{\beta} t + \frac{1}{\beta} - P_0, & \text{if } 0 \leq d < -\frac{\ln \alpha}{\beta}, \\
\frac{e^{-\beta d - \ln \alpha}}{\beta} + \frac{e^{-\beta d - e^{-\beta d} - e^{-\beta d}}}{\beta} + \frac{1}{\beta} - P_0, & \text{if } -\frac{\ln \alpha}{\beta} \leq d < -\frac{\ln(1 + \ln \alpha)}{\beta}, \\
\frac{e^{-\beta d - 1} + e^{-\beta d - e^{-\beta d} - e^{-\beta d}}}{\beta} + \frac{1}{\beta} - P_0, & \text{if } d \geq -\frac{\ln(1 + \ln \alpha)}{\beta}.
\end{cases}
\]

\( J_4(d) \) is initially increasing with \( d \) and becomes decreasing after a turning point. We also have \( \lim_{d \to +\infty} J_4(d) = J_4(0) \). Therefore, when \( \theta_R < \theta_1 \), optimal stop-loss reinsurance does not exist; when \( \theta_1 \leq \theta_R < \theta_2 \), the optimal stop-loss retention is either \(-\ln D_1/\beta \) or \(-\ln D_R/\beta \); when \( \theta_R \geq \theta_2 \), the optimal stop-loss retention does not exist and the total risk measures take the minimum value when there is no reinsurance.

### 4.5. Joint survival probability

[10] investigated optimal reinsurance contract in the interest of both parties by maximising their joint survival probability. Enlightened by their work, we consider the optimal stop-loss reinsurance by studying the joint survival probability of an insurance firm and a reinsurance firm. In our stop-loss reinsurance treaty, the joint survival probability of an insurance firm and a reinsurance firm is

\[
J_S(d) \equiv \Pr\{X \land d \leq u_0^I + P_l(d), (X - d)_+ \leq u_0^R + P_R(d)\}
\]

\[
= \Pr\{X \land d \leq u_0^I + P_l(d), (X - d)_+ \leq u_0^R + P_R(d), X \leq d\}
\]

\[
+ \Pr\{X \land d \leq u_0^I + P_l(d), (X - d)_+ \leq u_0^R + P_R(d), X > d\}
\]

\[
= \begin{cases} 
F(u_0^I + d + P_R(d)), & \text{if } d \leq u_0^I + P_l(d), \\
F(u_0^I + P_l(d)), & \text{if } d > u_0^I + P_l(d).
\end{cases}
\]

**Lemma 4.5.** The equation

\[
d + (1 + \theta_R)e^{-\beta d}/\beta = u_0^I + P_0
\]

has roots in \([0, +\infty)\) if and only if \( \theta_R \leq e^{\beta(u_0^I + P_0) - 1} - 1 \). Specifically, when \( \beta(u_0^I + P_0) - 1 < \theta_R < e^{\beta(u_0^I + P_0) - 1} - 1 \), \( (11) \) has two positive roots; otherwise, \( (11) \) admits a unique positive root.

**Proof.** The first order derivative of the left hand side of equation \( (11) \), i.e., \( d + P_R(d) \), with respect to \( d \) is \([d + P_R(d)]' = 1 - (1 + \theta_R)e^{-\beta d} \), which is positive if and only if \( d > \frac{\ln(1 + \theta_R)}{\beta} \). Thus, \( d + P_R(d) \) is decreasing with \( d \) when \( 0 < d < \frac{\ln(1 + \theta_R)}{\beta} \) and is increasing in \( d \in \left[\frac{\ln(1 + \theta_R)}{\beta}, +\infty\right)\). The function \( d + P_R(d) \) attains its minimum at \( d = \frac{\ln(1 + \theta_R)}{\beta} \) and the minimum value is \( 1 + \frac{\ln(1 + \theta_R)}{\beta} \). Therefore, when \( \frac{1 + \ln(1 + \theta_R)}{\beta} > u_0^I + P_0 \), i.e., \( \theta_R > e^{\beta(u_0^I + P_0) - 1} - 1 \), equation \( (11) \) admits no root in \((0, +\infty)\); when \( \frac{1 + \ln(1 + \theta_R)}{\beta} < u_0^I + P_0 < [d + P_R(d)]_{d=0} = \frac{1 + \theta_R}{\beta} \), i.e., \( \beta(u_0^I + P_0) - 1 < \theta_R < e^{\beta(u_0^I + P_0) - 1} - 1 \), \( (11) \) admits two roots in \((0, +\infty)\); when \( u_0^I + P_0 \geq [d + P_R(d)]_{d=0} \)
or \( u_0^I + P_0 = \frac{1+\ln(1+\theta R)}{\theta} \), i.e., \( \theta R \leq \beta(u_0^I + P_0) - 1 \) or \( \theta R = e^{\beta(u_0^I + P_0)^{-1} - 1} \), equation (11) has a unique positive root.

We denote the unique root, or the bigger root (when \( \beta(u_0^I + P_0) - 1 \leq \theta R < e^{\beta(u_0^I + P_0)^{-1} - 1} \)), of equation (11) as \( \bar{d} \).

**Theorem 4.6.** The optimal stop-loss retention \( d^* \in [0, +\infty) \), which maximise the joint survival probability \( J_S(d) \), exists if and only if \( \frac{\bar{d}}{\beta} \ln \frac{\beta}{\beta - v} - 1 \leq \theta R < \frac{\bar{d}}{\beta - v} \) or \( \frac{v}{\beta - v} \leq \theta R \leq e^{\beta(u_0^I + P_0)^{-1} - 1} \) and \( -\ln D_I \leq \bar{d} \).

**Proof.** For \( 0 < \theta R < \frac{\bar{d}}{\beta} \ln \frac{\beta}{\beta - v} - 1 \), the reinsurance firm’s utility constraint requires no stop-loss reinsurance. Thus, optimal stop-loss treaty does not exist.

For \( \frac{\bar{d}}{\beta} \ln \frac{\beta}{\beta - v} - 1 \leq \theta R < \frac{v}{\beta - v} \), the utility constraints of insurance firm and reinsurance firm require \( -\ln D_I / \beta \leq d \leq -\ln D_R / \beta \). The maximum of joint survival probability \( J_S(d) \) exists on this closed interval and the maximum is achieved by the left or right endpoint if \( d \leq \min{\{\text{left endpoint} \mid \text{left endpoint} < 1\}} \) and \( d \geq \min{\{\text{right endpoint} \mid \text{right endpoint} > 1\}} \). Otherwise, \( J_S(d) = \max{\{J_S(-\ln D_I / \beta), J_S(-\ln D_R / \beta)\}} \).

For \( \theta R \geq \frac{v}{\beta - v} \), we have \( d \geq -\ln D_I / \beta \). When \( \theta R \geq e^{\beta(u_0^I + P_0)^{-1} - 1} \), (11) has no root and \( J_S(d) \) is increasing for \( d \in [-\ln D_I / \beta, +\infty) \). Thus the optimal stop-loss retention does not exist. When \( \theta R < e^{\beta(u_0^I + P_0)^{-1} - 1} \), if \( -\ln D_I > \beta \bar{d} \), \( J_S(d) \) is increasing with \( d \) and optimal retention does not exist; if we have \( -\ln D_I \leq \beta \bar{d} \), the optimal stop-loss retention is \( \bar{d} \) and the maximum joint survival probability is \( F(u_0^I + u_0^R + P_0) \).

**5. Numerical examples.** This section investigates the expected utility improvement of both an insurance company and a reinsurance company after underwriting stop-loss reinsurance under the total VaR minimisation criterion. In both examples we assume the common parameters \( u_0^I = u_0^R = 0 \), \( P_0 = 0.4 \), \( \alpha = 1 \), \( v = 1 \), and \( \beta = 3 \). If the insurance firm’s premium is also determined under expected premium principle, its safety loading is 0.2. Let \( \theta_1 = \frac{\bar{d}}{\beta} \ln \frac{\beta}{\beta - v} - 1 \) and \( \theta_2 = \frac{v}{\beta - v} \).

Under a stop-loss reinsurance with retention \( d \), the insurance firm’s expected utility increment is

\[
\Delta U_I(d) = EU_I(u_0^I + P_1(d) - X \land d) - EU_I(u_0^I + P_0 - X)
\]

\[
= \frac{\alpha}{v} e^{-v(u_0^I + P_0)} \left[ \frac{\beta}{\beta - v} - e^{\frac{1+\theta R}{\beta} e^{-\beta d}} \left( \frac{\beta}{\beta - v} - \frac{v}{\beta - v} e^{-(\beta - v)d} \right) \right];
\]

the reinsurance firm’s expected utility increment is

\[
\Delta U_R(d) = EU_R(u_0^R + P_R(d) - (X - d)_+) - EU_R(u_0^R)
\]

\[
= \frac{\alpha}{v} e^{-vP_0} \left[ 1 - e^{-\frac{1+\theta R}{\beta} e^{-\beta d}} \left( 1 + \frac{v}{\beta - v} e^{-\beta d} \right) \right];
\]

and the total increment of both insurance firm’s and reinsurance firm’s utility is

\[
\Delta U_T(d) \triangleq \Delta U_I(d) + \Delta U_R(d).
\]

**Example 1.** In this example we consider the utility improvement under the criterion of minimising the total VaR in the case that the insurance company operates under a stricter confidence level than that of reinsurance company, i.e., \( \alpha_I < \alpha_R \). We let \( \alpha_I = 0.05 \) and \( \alpha_R = 0.15 \).
If we do not take any utility constraint into account, the optimal stop-loss retention $d^*$ that minimises the total VaR of both parties’ potential loss is any values satisfying that $0 \leq d^* \leq -\ln \alpha_R / \beta$. We draw the graph of insurance firm’s or reinsurance firm’s utility increments with respect to the safety loading of the reinsurance company. We let $\theta_R > 0.2$ because reinsurance company normally charges a higher safety loading in premium than that of direct insurance firm. The grey area in Figure 1(a) describes the insurance company’s possible expected utility increment after purchasing the stop-loss retention with an optimal retention level and Figure 1(b) describes that of reinsurance firm. After underwritten an optimal stop-loss reinsurance, when $\theta_R < \theta_1$, insurance firm’s potential expected utility increments are always positive, while reinsurance firm has to face a decrement in its expected
utility. This is because the reinsurance premium is too low to cover the tail part of claim losses.

The unique positive root of $f_R(D)$, i.e., $D_R$, can be regarded as a function of $\theta_R$. Then $D_R = \alpha_R$ admits an unique root $\theta_r$ and $\theta_1 < \theta_r < \theta_2$. When $\theta_1 \leq \theta_R < \theta_r$, the expected utility fluctuation might be either upwards or downwards for both insurance firm and reinsurance firm. When $\theta_R \geq \theta_r$, the premium is high enough for a reinsurance firm to guarantee its increase in expected utility. In the meantime, considering that stop-loss reinsurance can cap the aggregate losses, insurance company also has the possibility of expecting increase in utility even though purchasing reinsurance is relatively expensive.

![Graph](a) insurance firm's utility increment

![Graph](b) reinsurance firm's utility increment

**Figure 2.** The expected utility increment with utility constraints when $\alpha_I < \alpha_R$

After requiring utility improvement for both insurance and reinsurance companies, the optimal retentions are described in Theorem 4.2. Figures 2(a) and 2(b)
depict the possible utility changes of an insurance firm and a reinsurance firm, respectively, after entering a stop-loss reinsurance treaty with an optimal retention level that can bring mutual benefit to both players. These two graphs indicate that both parties' expected utilities after reinsurance will be no less than that without entering into the reinsurance market. When comparing Figure 1 with Figure 2, we find that the upper bound of insurance firm’s expected utility increment after reinsurance will decrease after adopting the utility constraints, while reinsurance firm’s lower bound of expected increment in utility will increase after doing optimal constrained stop-loss reinsurance.

When \( \theta_R < \theta_1 \), both parties’ expected utility increments are zero since the reinsurance premium is too low to encourage any reinsurance company entering into market. When \( \theta_1 \leq \theta_R < \theta_r \), both parties’ lower bound of expected utility increments are zero and the upper bound of expected utility increments increase with \( \theta_R \). When the reinsurance premium becomes higher, a reinsurance firm can gain more profits from underwriting stop-loss policies and is willing to accept a lower retention level. Therefore, an insurance firm has the opportunity to receive more protection of the tail claim losses and thence end up with higher expected utility.

When \( \theta_R \geq \theta_r \), insurance firm’s upper bound of expected utility increment will shrink with the growth of reinsurance firm’s safety loading because stop-loss reinsurance is too expensive. For the same reason, a reinsurance firm can always expect utility increment by underwriting stop-loss policies with any optimal retentions.

![Figure 3](image.png)

**Figure 3.** The lower bound of total expected utility increments when \( \alpha_I < \alpha_R \)

Adding the insurance firm's and reinsurance firm's expected utility increments together, we obtain the total increments of both parties after entering into an optimal stop-loss reinsurance contract. Now we look at the worst case scenario, i.e., the lower bound of total expected utility increments. When no utility constraint is considered, the lower bound of total expected utility increments are described by the blue curve in Figure 3, while the red curve represents the lower bound of increments under mutual utility constraints. Figure 3 attests that the lower bound
will be negative without considering any constraint. By setting up utility improvement requirements, the lower bound will always be positive and the total utility increments will be guaranteed.

Example 2. In this example we investigate the utility improvements under minimising total VaR criterion in the case that $\alpha_I > \alpha_R$. We let $\alpha_I = 0.12$ and $\alpha_R = 0.03$.

Figure 4. The expected utility increment without utility constraint when $\alpha_I > \alpha_R$.

If there is no utility constraint, $\forall d^* \geq -\ln \alpha_R / \beta$ is the optimal stop-loss retention that minimises the total VaR of two parties. Figure 4(a) and Figure 4(b) depict the utility movements of an insurance firm and a reinsurance firm after entering into a stop-loss reinsurance contract with an optimal retention without constraint. The upper bound of utility increment of an insurance company will shrink with the
increase of reinsurance safety loading. Let $\theta_3$ and $\theta_4$ be the unique root of $D_R = \alpha_I$ and $D_R = \alpha_R$, respectively. We have $\theta_1 < \theta_3 < \theta_4 < \theta_2$. Regarding the expected utility of a reinsurance firm, Figure 4(b) demonstrates that it will decrease after selling stop-loss contract when $\theta_R < \theta_4$, and will increase when $\theta_R \geq \theta_4$.

Figure 5. The expected utility increment with utility constraints when $\alpha_I > \alpha_R$.

Theorem 4.2 gives the optimal stop-loss retention after both insurance and reinsurance companies adopted utility improvement requirements. Figure 5(a) and Figure 5(b) describe the possible utility changes after entering into an optimal constrained stop-loss reinsurance contract. Figure 5 indicates that both parties’ expected utilities after reinsurance will be no less than that without entering reinsurance contracts.

When reinsurance is relatively cheap, i.e., $\theta_R < \theta_1$, both parties’ expected utility increments are zero since no reinsurance company is willing to offer stop-loss
product. When $\theta_1 \leq \theta_R < \theta_3$, $\forall d^* \in [-\ln D_I/\beta, -\ln D_R/\beta]$ is an optimal stop-loss retention level. Both parties upper bound of expected utility increments will increase with premium safety loading $\theta_R$. When $\theta_3 \leq \theta_R < \theta_4$, the optimal stop-loss retention $d^* = -\ln D_R/\beta$. Under the optimal strategy, insurance company will expect a higher utility and reinsurance firm’s expected utility will stay at the same level as that without doing reinsurance. When $\theta_R \geq \theta_4$, the optimal retention should satisfy that $d^* \geq \max\{-\ln D_I/\beta, -\ln \alpha_R/\beta\}$, if $\theta_R < \theta_2$, we also require $d^* \leq -\ln D_R/\beta$. With the increase of reinsurance premium safety loading, insurance firm’s upper bound of expected utility increment will shrink but reinsurance firm will expect even higher utility increment, which is manifested by Figure 5(a) and Figure 5(b).

![Figure 6](image)

**Figure 6.** The lower bound of total expected utility increment when $\alpha_I > \alpha_R$

Now we investigate the lower bound of total utility increment of both parties after entering into an optimal stop-loss reinsurance contract. In Figure 6, the blue curve describes the lower bound of total utility increment without considering utility constraint. For the reinsurance firm’s safety loading we draw in the figure ($0.2 < \theta_R < 0.55$), the total utility stays at the same level with not entering any reinsurance contract, but if the reinsurance premium is extremely high ($\theta_R > 14$), insurance firm will face the risk of utility decrement. The red curve in Figure 6 depicts the total utility increments in the worst scenario under utility improvement constraints. This curve indicates that we can always expect improvement in total utility when $\theta_1 \leq \theta_R \leq \theta_2$.

6. **Concluding remarks.** We investigate the optimal stop-loss reinsurance problem under five different criteria, namely the summation of the variance, VaR, TVaR or a generalised Dutch type I risk measure, and the joint survival probability, of an insurance firm and a reinsurance firm. Some of them are popularly used by regulators and practitioners in insurance industry, while the one based on generalised Dutch type I risk measure is proposed as a new internal model for companies’ capital
assessments, which is highly appreciated under the Solvency II regime. Additionally, the utility improvement constraints are adopted by both contracting parties, which guarantee that both the insurance firm and the reinsurance firm end up with higher utility of wealth after reinsurance purchase.

When calculating the reinsurance premium, the expected value premium principle is adopted in this paper. In future studies, we can consider the optimal reinsurance problems under other suitable premium principles. Further, it will also be of interest to investigate the reciprocal reinsurance under each optimality criterion and to find the most efficient type of reinsurance treaty. With more complex optimality criteria and constraints, analytical reinsurance strategies may not be available. Nevertheless, numerical approximation method can provide a viable alternative and obtain useful economic insights to practitioners and decision-makers.

Another possible extension of this paper is to investigate the more practical insurance markets. For mathematical tractability, this paper considers the simplified insurance market by assuming that there exists only one insurance firm and one reinsurance firm. In reality, the insurance industry is a complex unit involving plenty of insurance institutions and multiple reinsurance companies. Besides the conflict of interests between an insurance entity and its reinsurance firm, insurance/reinsurance companies inevitably compete with others for higher profits and market shares. In the meantime, cooperation is sometimes necessary in order to keep the sustainable and stable development of the insurance industry as a whole. Therefore, incorporating multiple insurance companies and reinsurance firms into our model makes the formulation too sophisticated to derive valuable conclusions. However, it is possible to investigate such problem under the framework of game theory. We leave this as an open problem for future study.

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Appendix A. appendix. To prove Lemma 3.1, we need the following Lemma.

Lemma A.1. When \( \theta_R \geq \frac{\beta}{v} \ln \frac{\beta}{v} - 1 \), the function \( f_1(D) = e^{-\frac{\beta R + D}{\beta}} - 1 \) admits a unique root \( D_1 \) in the range \((0, 1)\). Moreover, when \( \frac{\beta}{v} \ln \frac{\beta}{v} - 1 \leq \theta_R < \frac{\beta}{v} \), we have \( D_1 \geq D_R \), where \( D_R \) is the unique positive root of \( f_R(D) \).

Proof. The first order derivative of function \( f_1(D) \) with respect to \( D \) is

\[
 f'_1(D) = v(\beta - v) e^{-\frac{\beta R + D}{\beta}} g(D),
\]

where \( g(D) \triangleq D^{-\frac{1}{\beta}} e^{-\frac{\beta R + D}{\beta}} - \frac{\beta R + D}{\beta} \). Since \( \lim_{D \to 0^+} g(D) = +\infty \) and \( \min g(D) = g\left( \frac{1}{1+\theta_R} \right) < 0 \), we consider function \( f_1(D) \) in two cases: \( g(1) \leq 0 \) and \( g(1) > 0 \). If \( g(1) \leq 0 \), function \( g(D) \) admits a unique root \( D^* \in (0, 1) \) \( (D^* < \frac{1}{1+\theta_R}) \) and \( f_1(D) \) is increasing in \( D \in (0, D^*) \) and decreasing in \( D \in [D^*, 1] \). If \( g(1) > 0 \), function \( g(D) \) admits two and only two roots \( D^* \) and \( D^* \) in \((0, 1) \) \( (0 < D^* < \frac{1}{1+\theta_R} < D^* < 1) \) and \( g(D) < 0 \) if and only if \( D^* < D < D^* \). Therefore, \( f_1(D) \) is decreasing in \( D \in [D^*, D^*] \) and is increasing with \( D \) for \( 0 < D < D^* \) and \( D^* < D \leq 1 \). In both cases, since \( \lim_{D \to 0^+} f_1(D) = 0 \) and \( f_1(1) \leq 0 \), the function \( f_1(D) \) admits a unique root, denoted as \( D_1 \), in \((0, 1] \).
To prove $D_I \geq D_R$, we only need to prove $f_I(D_R) \geq 0$. For notational convenience, we denote $\zeta \triangleq \frac{(1+e\theta)e}{\beta}$ and $\tau \triangleq \frac{\theta}{\beta} \ (0 < \tau < 1)$. Then $f_I(D) = e^{-\zeta D} + (1 - \tau)D^{\tau} - 1$ and $f_R(D) = e^{\zeta D} - (\frac{1}{\tau} - 1)D - 1$. We have

$$f_I(D_R) = \frac{1 - \tau}{(1 - \tau)D_R + \tau}h(\tau),$$

where $h(\tau) \triangleq (1 - \tau)D_R^{\tau+1} - D_R + \tau D_R^{\tau}$. When $\frac{\theta}{\beta} \ln \frac{\beta}{\beta - e} - 1 \leq \theta_R < \frac{\beta}{\beta - e}$, the root of $f_R(D)$ satisfies $0 < D_R \leq 1$. Then we have $(1 - D_R)I_D R \leq 0$, $1 - D_R + D_R R \in D_R \geq 0$ and $-(1 - D_R) \ln D_R \geq 1 - D_R + D_R R \ln D_R$. Hence, $h'(\tau) < 0$ if and only if $\tau > \tau^*$, where $\tau^* = \frac{1 - D_R + D_R R \ln D_R}{1 - (1 - D_R) \ln D_R} \in (0, 1)$. Consequently, $h(\tau)$ is increasing in $\tau \in (0, \tau^*)$ and is decreasing in $\tau \in [\tau^*, 1]$. Since $h(0) = h(1) = 0$, we obtain that $h(\tau)$ is always positive for $\tau \in (0, 1)$. Therefor $f_I(D_R) > 0$ and $D_I \geq D_R$.

\square

To prove Theorem 4.4, we need the following two Lemmas.

**Lemma A.2.** Let $f(x) = x + e^{-x} + \ln \alpha, \ 0 < \alpha < 1$, then,

(i) for $0 < \alpha < 1/e$, $f(x)$ admits a unique root $x_0 \in (0, \alpha)$;

(ii) for $1/e < \alpha < 1$, $f(x) \geq 0$ always holds when $x \geq 0$.

**Proof.** The derivative of $f(x)$ with respect to $x$, $f'(x) = 1 - e^{-x}$, is positive for any positive $x$, then the function $f(x)$ is an increasing function in $x$.

For $0 < \alpha < 1/e$, we have $f(0) = 1 + \ln \alpha < 0$ and $f(\alpha) = \alpha > 0$. Hence $f(x)$ admits a unique root $x_0$ and $0 < x_0 < \alpha$.

For $1/e < \alpha < 1$, we have $f(0) = 1 + \ln \alpha > 0$, therefore, $f(x)$ is always positive when $x > 0$.

\square

**Lemma A.3.** When $0 < \pi < 1/e$, the function $J(D) = De^{-D} + \pi D - D$ is increasing in $D$ for $0 \leq D \leq D_0$ and is decreasing in $D$ for $D \geq D_0$, where $D_0 \in (0, \pi)$ is the unique root of $f(D) = (1 - D)e^{-D} + \pi - 1$.

**Proof.** Taking the derivative of $J(D)$ with respect to $D$, we obtain $J'(D) = (1 - D)e^{-D} + \pi - 1 = f(D)$. Since $f'(x) = (D - 2)e^{-D} < 0$, $f(D)$ is decreasing in $D$. Together with the boundary conditions that $f(0) = \pi > 0$, $f(\pi) = (1 - \pi)(e^{-\pi} - 1) < 0$, we draw the conclusion that $f(D)$ admits a unique root $D_0 \in (0, \pi)$. Moreover, $f(D)$ is positive when $0 < D < D_0$ and is negative when $D > D_0$. Therefore, $J(D)$ is increasing with $D \in [0, D_0]$ and is decreasing in $D$ for $D \geq D_0$.

\square

**Remark 5.** Let $D \triangleq e^{-\beta d}$, then we have $J_4(d) = |J(D) - \pi \ln \pi|/((\pi \beta) + 1/\beta - P_0$ when $0 < \alpha < 1/e$ and $d \geq -\ln \pi/\beta$ (i.e., $0 \leq D \leq \pi$). From Lemma A.3, $J_4(d)$ is an increasing function in $d$ when $d \in [-\ln \pi/\beta, -\ln D_0/\beta]$ and is decreasing with $d$ for $d \geq -\ln D_0/\beta$.

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