Damage spreading transition in an opinion dynamics model

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We study the damage spreading phenomena in two different ways in a opinion dynamics model introduced recently. This kinetic exchange type model is characterized by a fraction $q$ of negative interactions and shows the presence of an order-disorder transition at $q_c$. In the traditional method, the initial opinions are identical for all agents but the two replicas are evolved independently. Two replicas of the population are considered in which the opinion of all the agents are identical initially except for a single agent. The systems are then allowed to evolve identically. In the other method, the initial opinions are identical for all agents but the two replicas are evolved independently. In both cases, a damage spreading transition occurs at $q_d$ where $q_d \approx 0.18$ in the traditional method and $q_d = 0$ for the other; the damage increases with $q$ above $q_d$ and attains a constant value for $q \geq q_c$. However, the correlation between the evolved states above $q_c$ is clearly different in the two methods.

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I. INTRODUCTION

In systems with randomness and disorder, damage spreading is an important dynamical study that was first introduced in the context of biologically motivated systems $[1]$. Later, in physics, a number of studies were conducted in the Ising model and cellular automata $[2, 22]$ in which two copies of the system were made different by a small amount and were evolved using the same random numbers. The time development of the “damage” $D(t)$, which is a measure of the difference between the two replicas, is one of the important features that is studied. One expects the damage will reach a constant value at long times, i.e. $D(t \to \infty) = D_{sat}$ where $D_{sat}$ can be zero or finite. In case it is zero, the damage does not survive. Thus the question whether there could be a damage spreading transition, i.e. whether damage is nonzero only above a certain value of the driving parameter (which may be temperature in physical systems), becomes an important study also. Such a transition point may not necessarily coincide with the order-disorder transition point, if any.

Traditionally, in spin systems, damage spreading is studied in two ways. One can either let the system equilibrate and then make a slightly damaged replica and study the evolution of both. Or, two slightly different replicas may be allowed to evolve right from the beginning $[3]$. Important feature in either case is that the replicas are evolved identically in this so called traditional method (TM).

In some recent works, the fate of two identical copies of the Ising model which were allowed to evolve independently was studied $[23, 24]$. This is done to study the effect of environment which can introduce differences in two systems born with identical features. This has been termed as a “nature versus nurture” (NVN) phenomena $[24]$. In one and two dimensions, this leads to a power law decay of the overlap between the two systems in time signifying that the copies become more and more ‘damaged’ in time. However, such studies have been limited to low temperatures. We use this as an alternative method of damage spreading study calling it the NVN method.

We intend to study whether the two methods lead to any qualitative and quantitative difference as far as damage spreading is concerned in a particular system. We choose an opinion dynamics model to study the damage spreading phenomenon employing both the methods. In contrast to spin models, where different dynamical algorithms (e.g., heat bath, metropolis etc.) may lead to qualitatively different results $[25, 26]$ as far as damage spreading is concerned, opinion dynamics models have well-defined dynamical rules. It is also important to study damage spreading in opinion dynamics models to explore whether an initial small difference in opinion can induce drastic changes in the opinions of all the agents $[27, 28]$. Various models of opinion dynamics exist in the literature, although damage spreading has been studied in comparatively less extent $[29, 31]$. The model studied in this work is relatively new $[32]$. It contains a single parameter and shows an order-disorder transition. Thus the model has the property that individual opinions may go on changing even when the global average opinion reaches a steady state, hence we expect that the time dependence of the damage itself may show interesting features. The model has the additional advantage that it can be studied using both discrete and continuous opinions and the nature of randomness here can also be modified. It therefore gives us the opportunity to study different cases within a common framework.

In the next section, the model and the method are described. Results are presented in section III followed by a summary and discussion in the last section.

II. MODEL AND METHOD

In this work, we study the opinion dynamics model proposed in $[32]$, where opinions can be modeled as discrete or continuous variables. Here two individuals mod-
ify their opinions by the so called “kinetic exchange” scheme 24, 33, 34. The opinions are subject to change due to the mutual binary interactions which can be both positive as well as negative. Let \( O_i(t) \) be the opinion of the \( i \)th agent at time \( t \), then after an interaction of the \( i \)th and \( j \)th agents, their opinions at time \( t + 1 \) are changed according to

\[
O_i(t + 1) = O_i(t) + \mu_{ij} O_j(t) \\
O_j(t + 1) = O_j(t) + \mu_{ij} O_i(t),
\]

(1)

where \( \mu_{ij} \) is random, either +1 or −1. We consider both discrete \( (O_i = 0, \pm 1) \) and continuous opinion \((-1 \leq O_i(t) \leq 1)\). The opinions of the two agents are modified simultaneously. After \( N \) such interactions, one time step is said to be completed. The interacting agents are chosen randomly from the \( N \) agents and thus one may consider the topology of the system to be like a fully connected graph.

The behavior of the model was shown to be independent of the distribution from which \( \mu_{ij} \) are drawn; in the present work, we consider \( \mu_{ij} = \pm 1 \). If the opinion of an agent becomes higher (lower) than +1(−1) following an interaction, then it is made equal to +1(−1) for both continuous and discrete opinions. If the discrete opinions are taken as 0 and \( \pm 1 \) initially, \( \mu_{ij} = \pm 1 \) ensures that at subsequent times the opinion values will take up one of these values only.

In models which are defined through dynamical rules, the question of equilibration may not be relevant and hence one introduces a damage in the beginning only in the traditional method (TM). We simulate two systems of \( N \) individuals using the same initial random discrete/continuous opinions except for one randomly chosen individual. Then the two systems are allowed to evolve using the same random numbers. In the other (NVN) method, the initial systems are identical but different random numbers are used in the time evolution. This implies that the agents who interact in the two replicas are in general different in the NVN method.

The damage at time \( t \) is defined as

\[
\langle D(t) \rangle = \frac{1}{N} \sum_i |O_i^{(1)}(t) - O_i^{(2)}(t)|,
\]

(2)

where \( O_i^{(1)}(i) \) and \( O_i^{(2)}(i) \) are the \( i \)th agent’s opinion in the two replicas. We also calculate \( P_D(t) \), the fraction of agents for which \( O_i^{(1)} \neq O_i^{(2)} \).

In this model a parameter \( q \) is used which denotes the fraction of negative interaction \( (\mu_{ij} = -1) \). It was found in [32] that below a particular value \( q = q_c = 0.25 \), the system becomes ordered, while a disordered phase exists for higher values of \( q \). For \( q \geq q_c \), the distributions of opinions are symmetric in both discrete and continuous cases. For discrete opinions, above \( q_c \), all three fractions of opinions \((0, \pm 1)\) are equally probable with probability \( 1/3 \). For the continuous case also, all opinions \(-1 \leq O_i(t) \leq 1\) are equally probable in the disordered phase, barring \( O_i = 0, \pm 1 \). \( O(i) = \pm 1 \) has a higher probability compared to other values due to the imposed boundary condition. When the interaction \( \mu_{ij} \) between two agents particular agents \( i \) and \( j \) is kept unchanged throughout the time evolution, we call it quenched randomness. On the other hand, when a new value of \( \mu_{ij} \) is chosen every time the two agents interact, it is a case of annealed randomness. It was shown in [32] that the order-disorder phase transition which occurs with mean field critical behavior is independent of this choice.

In the simulation, we have taken systems with size \( 2^8 \leq N \leq 2^{12} \). For the same system, 400 different choices of initial random opinion of agents have been taken and quantities are averaged over all the configurations.

### III. RESULTS AND ANALYSIS

#### A. Results for the Traditional Method

1. **Discrete opinion**

   ![FIG. 1. (Color online) (Traditional method (TM); quenched randomness) Plot of the average damage (left panel) and fraction of damaged agents (right panel) as a function of time for different values of \( q \) for discrete opinion. All data are for \( N = 2048 \).](image)

   ![FIG. 2. (Color online) (TM; quenched randomness) Variation of saturation value of damage (left panel) and fraction of damaged agents (right panel) with \( q \) for discrete opinion.](image)

We first discuss the case when \( \mu \) is chosen randomly in...
a quenched manner. For the discrete opinion case the initial damage is introduced in one randomly chosen agent say, X, in the following way: if the opinion $O_X$ of this agent is 1 in replica A, then the opinion in replica B becomes 0, otherwise it is $(O_X + 1)$. We calculate average damage $D(t)$ and fraction of average damaged agents $P_D(t)$ as functions of time (Fig. 1). For the first few time steps both $D(t)$ and $P_D(t)$ increase sharply and then decrease for small values of $q$. Finally $D(t)$ ($P_D(t)$) reaches a saturation value $D_{sat}$ ($P_{sat}$) which depends on $q$ up to $q \approx 0.25$. The saturation value $D_{sat}$ is nonzero for all $q$. We plot the saturation values for different system size $N$ as functions of $q$ (Fig. 2). Up to $q_d \approx 0.18$, $D_{sat}$ and $P_{sat}$ decrease with $N$, above $q_d$ they show system size independent behavior. For $q \geq q_c = 0.25$, $D_{sat}$ and $P_{sat}$ are independent of both $q$ and $N$.

The results for the case when $\mu$ is considered as a annealed random variable are similar (Figs 3 and 4). However, the saturation values below $q_d$ appear to be almost independent of $q$ which is in contrast to the quenched case. Both $D$ and $P_D$ also show more pronounced non-monotonic behavior against time close to $q_d$.

It is interesting to note that even for $q = 0$, there is a small non-zero value of damage in finite systems. We will discuss this issue later in this section.

![FIG. 3. (Color online)(TM; annealed randomness) Plot of the average damage (left panel) and fraction of damaged agents (right panel) as a function of time for different values of $q$ for discrete opinion. All data are for $N = 2048$.](image)

2. Continuous opinion

In the continuous opinion case the results do not depend qualitatively and quantitatively on the manner in which $\mu$ are chosen (annealed or quenched). Here the initial difference between two opinions of a single agent in the two replicas is taken as 0.01. In this case, we take two opinions to be equal if their difference is less than $10^{-6}$. Also, the results do not depend qualitatively on the initial damage. It is observed that below a certain value of $q$, the value of $D(t)$ and $P_D(t)$ go sharply to zero after an initial increase and beyond this value of $q$, both reach a finite saturation value (Fig. 5). We conclude that this value of $q$ corresponds to the damage spreading transition point but it has a finite size dependence and therefore is denoted as $q_d(N)$. Plotting $q_d(N)$ versus $1/N$ (Fig. 6), $q_d(N \rightarrow \infty)$ is obtained as $\sim 0.17$ in the thermodynamic limit. For $q > q_d$, $D_{sat}$ and $P_{sat}$ show nominal system size dependence (which is not systematic) and for $q \geq q_c$ these are also independent of $q$ (Fig. 7).

![FIG. 4. (Color online) (TM; annealed randomness) Variation of saturation value of damage (left panel) and fraction of damaged agents (right panel) with $q$ for discrete opinion.](image)

![FIG. 5. (Color online) (TM; quenched randomness) Plot of the average damage (left panel) and fraction of damaged agents (right panel) as a function of time for different values of $q$ for continuous opinion. All data are for $N = 2048$.](image)

B. Nature versus nurture method

Having observed in the TM that the nature of randomness does not affect the results significantly, we have considered only quenched randomness in $\mu_{ij}$ when using the NVN method (this involves less time to get the results computationally).

1. Discrete opinions

Initially a random configuration with opinion $\pm 1, 0$ is chosen. Starting with two identical configurations and allowing them to evolve independently, we observe that
the time dependence of $D(t)$ and $P_D(t)$ is qualitatively similar to that in the TM (Fig. 8). However, when saturation values are considered, we find that both $D_{\text{sat}}$ and $P_{\text{Sat}}$ show no appreciable systematic dependence on $N$; even for $q \to 0$, both remain finite (Fig. 9). Thus we conclude that the damage spreading transition occurs at $q_d = 0$ in this case. There is an increase with $q$ before $D_{\text{sat}}$ or $P_{\text{Sat}}$ reaches a constant value close to $q \approx 0.25$.

2. Continuous opinions

For the continuous opinion, a similar procedure is followed. In this case, like the TM, we take two opinions to be equal if their difference is less than $10^{-6}$. Time evolution of $D(t)$ and $P_D(t)$ is shown in Fig. 10. Both saturation values are finite for all $q > 0$ and show system size independent behavior. For $q \gtrsim 0.25$ saturation value of $D(t)$ and $P_D(t)$ show a $q$ independent behavior (Fig. 11).

For small $q$, both $D_{\text{sat}}$ and $P_{\text{Sat}}$ are order of magnitude smaller than that obtained in the discrete case, being $O(10^{-2})$. This is similar to the TM result. However, in TM, the saturation values are zero up to a finite value of $q$. Here in contrast, $D_{\text{sat}}$ and $P_{\text{Sat}}$ grow from zero at $q = 0$ itself with no system size dependence. So here too, the damage spreading transition occurs at $q_d = 0$ as in the discrete case.
C. Theoretical estimates and comparison

One can easily estimate $D_{sat}$ and $P_{sat}$ for $q > 0.25$ theoretically when the system becomes completely disordered. We denote the average damage and the fraction of disagreeing agents obtained theoretically by $D_{est}$ and $P_{est}$ respectively.

For discrete opinion, above $q = 0.25$, all three types of opinion ($0, \pm 1$) have equal probability ($= 1/3$) of occurrence. So one can estimate $P_{sat}$ and $D_{sat}$ for two configurations which are completely uncorrelated (which will happen ideally if the damage spreads through the entire system). Assuming there is no correlation, the opinion of an agent in replica A and replica B can have equal probability of having the following nine possible set of values: $(-1, -1), (-1, 0), (-1, +1), (0, -1), (0, 0), (0, +1), (+1, -1), (+1, 0)$ and $(+1, +1)$. Thus the average damage

$$D_{est} = (2 \times 2/9 + 4 \times 1/9) = 8/9.$$  

Also the probability $P$ that the opinions are different is equal to $P_{est} = 2/3$.

For continuous opinions, the probability that the opinions are exactly equal in the two replicas would be zero for completely uncorrelated replicas and $P_{est} = 1$. Denoting the opinion distribution by $P(x)$, the square of average damage can be calculated as

$$D_{est}^2 = \frac{\int_{-1}^{1} \int_{-1}^{1} (x_1 - x_2)^2 P(x_1)P(x_2) \, dx_1 \, dx_2}{\int_{-1}^{1} \int_{-1}^{1} P(x_1)P(x_2) \, dx_1 \, dx_2},$$

$$= \frac{2}{3}$$

assuming $P(x)$ to be uniform above $q_c$. Therefore the expected value of average $D_{est} = \sqrt{2/3} \approx 0.82$

1. Comparison with TM

The observed values of $P_{sat}$ and $D_{sat}$ for large $q$ ($D_{sat} \sim 0.47, P_{sat} \sim 0.39$ and $D_{sat} \sim 0.50, P_{sat} \sim 0.50$ for the discrete and continuous opinions respectively) for TM however do not match with the theoretical estimates. For both the discrete and continuous cases, the values of $P_{sat}$ and $D_{sat}$ are far less than $P_{est}$ and $D_{est}$. This shows that the damage must have spread only partially (or not at all) in a finite fraction of cases. We study this in detail in the next subsection by calculating the distribution of $P_{sat}$. In case $P_{sat} < P_{est}$, one can say there is positive correlation between the two replicas, otherwise there is negative correlation. The latter is possible only for the discrete opinion case as for the continuous opinions, $P_{est} = 1$.

2. Comparison with NVN method

In comparison, we note that in the NVN method, the saturation values of $P_D$ and $D$ for $q > 0.25$ coincide with the estimated values quite well. Thus the disordered states are simply uncorrelated here, all initial correlation is completely destroyed. This is in stark contrast to the result of the traditional method.

D. Distribution of fraction of damaged site in TM

In the TM, we noted that the estimated saturation values of $D$ and $P$ are clearly different from the observed ones for $q > q_c$. This may happen due to two reasons. Either the systems actually evolve to correlated configurations (conjecture 1), or there may be some configurations for which they reach identical or nearly identical states and some for which they evolve to uncorrelated states (conjecture 2). In order to check this, we conduct a systematic study of the probability distribution $R(P_{sat})$ for all $q$ values.

For the discrete case, $R(P_{sat})$ shows an interesting behavior even when $q$ is quite small. It has non-zero values at $P_{sat} = 0$ and at a fairly large value of $P_{sat}$ close to unity. This signifies that at small values of $q_c$ when the system is almost fully ordered (almost all opinions equal to $+1$ or $-1$), the initial damage leads in a few cases to configurations with strong negative correlations. For the extreme case $q = 0$, these two configurations correspond to the all $+1$ all $-1$ states. However, as previously noted, this effect vanishes as the system size becomes larger. In continuous opinions, no such phenomenon is observed.

In general, for $q >> q_d$, $R(P_{sat})$ has a bimodal nature having non-zero values for $P_{sat} = 0$ and a few larger values of $P_{sat}$ close to $P_{est}$. We indeed find that even for large values of $q$, $R(P_{sat} = 0)$ is non zero which explains the discrepancy between the observed and theoretical results and indicates that the second conjecture is true. For the continuous case, one gets $R(P_{sat} = 1) = 0$ even when...
$q$ is very large as the high density of agents with opinions equal to $\pm 1$ induce some positive correlation (not considered in the theoretical estimates). The results are plotted in Figs 12 and 13. $R(P_{sat} = 0)$ has an interesting behavior; it is almost equal to 1 up to $q_d$ and shows a sharp fall to another constant non-zero value beyond $q_d$ in both discrete and continuous cases. The constant value of $R(P_{sat} = 0)$ for $q > q_d$ is about 0.5 and 0.4 for the discrete and continuous opinions respectively.

![FIG. 12. (Color online) Histogram showing $R(P_{sat})$, the probability of fraction of damaged agents for different values of $q$ using TM. Data are for $N = 2048$.](image1)

![FIG. 13. (Color online) Variation of probability of zero damage with $q$ in TM. Data are for $N = 2048$.](image2)

**IV. SUMMARY AND DISCUSSIONS**

We have studied the damage spreading phenomena in an opinion dynamics model which has a known order disorder transition and where one can use both discrete and continuous opinions. We employ two different methods, the traditional method and the nature versus nurture method [24] to study the phenomena. In both cases, the dynamics of the damage $D$ shows a non-monotonicity which makes it difficult to comment on the exact nature of $D(t)$ as a function of time. Thus unlike in many other studies, no estimate of exponents associated with $D(t)$ can be made. This is perhaps a feature which makes this model different from conventional dynamical models. At large times, $D(t)$ reaches a saturation value $D_{sat}$. The behavior of $D_{sat}$ as a function of the model parameter $q$ reveals the existence of a damage spreading transition at $q = q_d$. We find that $q_d$ is less than $q_c$, the order disorder phase transition point. This is true for both the methods, in fact for NVN, $q_d = 0$. It is difficult to ensure the nature of transition, apparently $D_{sat}$ varies continuously with $q$ above $q_d$ before attaining a $q$ independent value for $q > q_c$, where the system enters the disordered phase. However, there is no system size dependence of $D(t)$ and $P(t)$ above $q_d$ so that conventional finite size scaling analysis for continuous phase transition is also not possible to do.

As mentioned before, in many physical systems, the question whether the damage spreading transition coincides with the order-disorder transition point has been investigated. Here we clearly get $q_d < q_c$. The significance of this result in TM is that for $q_d < q < q_c$, even though consensus is reached, very small changes in even a single agent may lead to a different consensus state with a finite probability. In NVN, $q_d = 0$ implies that if the same agent goes through a different sequence of interactions, the result will be different for any $q$ with finite probability. The NVN result is consistent with the finding of [24] in the Ising model, where even at zero temperature, the overlap between identical states vanishes, albeit in a much slower manner. Above $q_c$, i.e., in the disordered state, even when systems are slightly different initially, if they follow the same environment, they have a finite probability of ending up in configurations with large overlap while that is never true if the environment is completely different. This shows that if the initial states are damaged and in addition they are evolved independently, they will always lead to uncorrelated states. Although the damage spreading transition occurs before the order-disorder transition ($q_d < q_c$), we find that the damage is sensitive to the static critical point as it attains a saturation value for $q \geq q_c$. This is true in both methods.

We have considered different nature of randomness and opinions. The nature of randomness apparently does not play any role. Rather, the results depend on the nature of the opinions. Not only is the dynamical evolution quite different for the two cases below $q_d$ in finite systems, the estimated damage spreading transition point also appears to be slightly different as noted in TM. For NVN too, we find the fluctuations to be larger in case of discrete opinions which, however, is not surprising. However, $q_d = 0$ for both discrete and continuous in NVN. The distribution of $P_{sat}$ is also different for the two types of opinions as checked in TM, in particular, for $P_{sat} = 0$, the values differ markedly for $q > q_d$.

One of the main issues in the present work is the comparison of the two cases of damage spreading. Both qualitative and quantitative differences are noted in the results. The transition occurs at $q_d = 0$ for NVN in con-
contrast to a finite value of $q_d$ obtained in TM. The overlap of the evolved states above $q_c$ also shows an interesting difference. In the disordered phase ($q > q_c$), the expected value of $D_{sat}$ and $P_{sat}$ considering totally uncorrelated opinions in the two time evolved configurations are much larger compared to the values obtained in TM but for NVN these are very close to the observed values. Further study of the distribution shows that in TM, there are configurations where damage does not spread at all leading to values less than that expected for completely uncorrelated cases.

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