Optimal design of lottery with cumulative prospect theory

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Abstract

A lottery is a popular form of gambling between a seller and multiple buyers, and its profitable design is of primary interest to the seller. Designing a lottery requires modeling the buyer decision-making process for uncertain outcomes. One of the most promising descriptive models of such decision-making is the cumulative prospect theory (CPT), which represents people’s different attitudes towards gain and loss, and their overestimation of extreme events. In this study, we design a lottery that maximizes the seller’s profit when the buyers follow CPT. The derived problem is nonconvex and constrained, and hence, it is challenging to directly characterize its optimal solution. We overcome this difficulty by reformulating the problem as a three-level optimization problem. The reformulation enables us to characterize the optimal solution. Based on this characterization, we propose an algorithm that computes the optimal lottery in linear time with respect to the number of lottery tickets. In addition, we provide an efficient algorithm for a more general setting in which the ticket price is constrained. To the best of the authors’ knowledge, this is the first study that employs the CPT framework for designing an optimal lottery.

1 Introduction

A lottery is a common form of gambling between a single seller and multiple buyers. The seller, typically a government in modern lotteries, announces the price per ticket and the prizes in advance; each buyer purchases a ticket if he/she feels that the ticket is worth the price. The buyers then randomly receive prizes according to predefined rules. In the lottery event, the seller profits from the difference between the sales amount and the total prizes. Thus, a profitable lottery design is of primary interest to the seller. From this viewpoint, the optimal design of lotteries has been investigated [6, 13]. For this purpose, it is necessary to model when a buyer purchases a lottery ticket that yields an uncertain outcome.

Several models have been proposed to explain people’s decisions under uncertainty. The expected utility hypothesis (EU) is a seminal framework [8, 9, 23]. This framework introduces a utility function to describe people’s satisfaction with outcomes. Agents choose actions based on the expected utility, that is, the sum of the products of the utilities of outcomes and their corresponding probabilities. Although EU is a standard model, it does not satisfactorily explain the behavior of people who buy lottery tickets [16].

Cumulative prospect theory (CPT) employed in this study is a celebrated model that originates from EU [22]. CPT incorporates the cumulative manner to calculate decision weights and introduces sign-dependent preferences. These improvements incorporate CPT with preferable properties, such as first-order stochastic dominance [7]. This theory is shown to capture decisions in the context of the lottery; for example, consider Powerball (https://www.powerball.com/), a popular lottery sold in the United States. When the grand prize is $40 million, the default jackpot, it can be explained through the CPT framework that risk-seeking Americans feel that the lottery is worth buying but standard Americans do not. Note that, if we assume that people make decisions according to the expected value, nobody purchases a ticket since the expected value is negative, which is inconsistent with reality.

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1.1 Our contributions

In this study, we consider the optimal design of the lottery to maximize the seller’s profit when buyers follow CPT. Designing optimal (stochastic) mechanisms is a fundamental problem in the field of multi-agent systems [1, 2, 12]. The contributions of this work are summarized as follows.

We model the problem of finding optimal lottery that maximizes the seller’s profit when the buyers follow CPT. To the best of the authors’ knowledge, this is the first study that employs the CPT framework to design an optimal lottery. The derived problem is nonconvex and constrained, and hence, it is challenging to characterize its optimal solution. To overcome this difficulty, we reformulate the problem as a three-level optimization problem. By using this reformulation, we show that the prizes of the optimal lottery are divided into three parts: a polynomially increasing part, a fixed amount higher than the ticket price, and no prizes. We also propose an algorithm that computes the parameters of the characterization in quadratic time with respect to the number of lottery tickets. Moreover, we improve the computational time to linear by further investigating a structure of the optimal lottery.

We also analyze lottery optimization with an extra constraint on the ticket price, which would be more useful in practice. In this case, it is difficult to obtain the optimal lottery analytically, but we characterize it as the optimal solution of a 1-dimensional optimization problem, which can be solved in a simple way, such as the Newton method. Then we give an algorithm that outputs the optimal lottery by solving the 1-dimensional optimization problem quadratic times. Moreover, we give a linear time algorithm that finds the optimal lottery in a specific case.

1.2 Related studies

There are several other frameworks that originate from EU. Rank-dependent expected utility (RDEU) is an improvement of the EU [15]. It successfully captures the overestimation of an unlikely extreme event such as a jackpot in lottery. Mathematically, the expected utility in this model is calculated by using decision weights, which are probabilities distorted by an inverse S shape probability weighting function in a cumulative manner. Another series of studies is prospect theory (PT) in behavioral economics [3, 11, 20]. PT considers the sign-dependent preferences, where risk attitudes depend on whether the individual is evaluating a gain or a loss. Mathematically, the expected utility here is defined by an S-shaped value function instead of the utility functions above, together with distorted probability. CPT stands as an improved version of RDEU and PT. CPT incorporates the cumulative manner to calculate the decision weights from RDEU and introduces sign-dependent preferences from PT.

This study investigates the optimal lottery problem based on the CPT framework, for the first time, whereas previous studies are based on RDEU or other frameworks. There are several other differences between this study and previous studies: Quiggin [16] deals with the maximization of the buyer’s expected utility under a fixed profit for the seller with RDEU. The profit maximization is studied in [6, 13]. Maeda [13] analyzes an optimal design away from the expected utility under restricted settings. While their analysis requires the lottery structure (such as the number of the winning tickets) to be determined in advance, we also optimize it based on CPT. Dennery and Dierer [6] optimize the probability of the prizes in addition to its value and the number of prizes. They concluded that their model is not compatible with a discrete number of prizes. We summarize the comparison between our setting and existing ones in Table 1.

Analysis with CPT becomes more challenging than analysis with RDEU. For example, consider the optimal design of insurance, which is similar to that of lotteries since both problems deal with a fixed payment and random rewards. Although several optimal designs of insurance have been investigated using RDEU [4, 10, 24], the design using CPT is still limited to a simplified CPT setting [21]. One of the reasons is the (inverse) S shape of the probability weighting function and the value function. As stated by Bernard et al. [4], the inverse S shape often causes analytical difficulty owing to the nonconvexity. The lottery problem in this study also encounters the same difficulty: the S shape also results in the nonconvexity. Nevertheless, we resolve it by decomposing the lottery design problem into three-level optimization.
Table 1: Comparison with existing results

|                      | Model | Charact. | Comput. |
|----------------------|-------|----------|---------|
| Quiggin [16]         | RDEU  | ✓        |         |
| Dennery and Dier [6] | RDEU  | ✓        |         |
| Maeda [13]           | original | ✓ ✓   |         |
| This work            | CPT   | ✓ ✓      |         |

2 Preliminaries

2.1 Notations

Let $\mathbb{R}$ be the set of real numbers. We will denote by $\mathbb{R}_{\geq 0}$ (resp., $\mathbb{R}_{\leq 0}$) the set of nonnegative (resp., nonpositive) real numbers. Let $\mathbb{N} := \{1, 2, \ldots\}$ be the set of natural numbers. For $n \in \mathbb{N}$, we denote the set $\{1, \ldots, n\}$ by $[n].$

2.2 Cumulative prospect theory

Let us consider prospects $(w_1, p_1), \ldots, (w_N, p_N)$, where $w_\ell \in \mathbb{R}$ is a potential outcome and $p_\ell (\geq 0)$ is the probability of yielding it. We assume that the outcomes are arranged in the nondecreasing order (i.e., $w_1 \leq w_2 \leq \cdots \leq w_N$) and $\sum_{\ell=1}^N p_\ell = 1$ holds. We call a nonnegative outcome gain and nonpositive one loss. We interpret zero as either gain or loss.

In the CPT framework [22], an agent makes a decision according to the expected utility of the prospects. To define the expected utility, we first introduce two key ingredients in CPT, that is, a value function and decision weights.

Value function A value function represents an agent’s preference for outcomes. It is concave and convex for gain and loss, respectively (see Figure 1). The concavity represents risk aversion of an agent when facing risk related to gain and the convexity does risk-seeking behavior when facing one to loss.

The most widely used value function for CPT is $V(w) := \begin{cases} w^\alpha & (w \geq 0), \\ -\lambda(-w)^\beta & (w < 0), \end{cases}$ (1)

where $w \in \mathbb{R}$ is an outcome and $\alpha, \beta, \lambda$ are hyperparameters satisfying $\alpha, \beta \in (0, 1)$ and $\lambda > 0$. The parameters $(\alpha, \beta, \lambda)$ are usually set for the function to be steeper for loss than gain, which is known as loss aversion. Throughout this paper, we assume that the value function is represented as (1). We will discuss possible extensions to more general value functions in Section 7.

Decision weights Decision weights represent an agent’s subjective probabilities of the corresponding outcomes. We say that $f : [0, 1] \to \mathbb{R}$ is an inverse S-shaped function if it is strictly increasing, continuously differentiable and there exists $x_0 \in [0, 1]$ such that $x \mapsto f'(x)$ is strictly decreasing on $[0, x_0]$ and strictly increasing on $[x_0, 1]$. We call such $x_0$ the inflection point (see Figure 2). Let $W, \overline{W} : [0, 1] \to [0, 1]$ be inverse S-shaped functions satisfying $W(0) = \overline{W}(0) = 0$ and $W(1) = \overline{W}(1) = 1$. We call such $W$ and $\overline{W}$ probability weighting functions for gain and loss, respectively. Typical probability weighting functions for gain and loss are

$W(p) := \frac{p^\gamma}{(p^\gamma + (1-p)^\gamma)^{1/\gamma}}$ and $\overline{W}(p) := \frac{p^\bar{\gamma}}{(p^\bar{\gamma} + (1-p)^\bar{\gamma})^{1/\bar{\gamma}}}$

with $\gamma, \bar{\gamma} \in (0, 1]$, which are originally introduced by Tversky and Kahneman [22].

Let $n$ and $\pi$ be the numbers of nonnegative and nonpositive outcomes, respectively; that is, it follows that $w_{\pi+1} \leq 0$, $w_{\pi+1} \geq 0$, and $\pi + n = N$.  

3
Expected utility

Let \( \bar{\omega} := (w_1, \ldots, w_{\bar{n}})^\top \in \mathbb{R}_{\geq 0}^{\bar{n}} \) be the first \( \bar{n} \) components of \( \omega \) that correspond to the losses for the agent. We also define the gains as \( \omega := (w_{\bar{n}+1}, \ldots, w_N)^\top \in \mathbb{R}_{\geq 0}^n \). Then the expected utility of the agent is defined as

\[
\sum_{i=1}^{\bar{n}} \bar{h}_i V(\bar{\omega}_i) + \sum_{j=1}^{n} h_j V(\omega_j).
\]

Agents make decisions according to their expected utilities based on their subjective value and probability. We emphasize that the expected utility is different from the expected value.

### 3 Formulation

We formulate the optimal lottery design as an optimization problem of maximizing the seller’s profit under the constraint that the lottery attracts buyers from the CPT perspective. To guarantee that the tickets are sold, we suppose that the expected utility of a ticket must be nonnegative. Note that this condition is called individual rationality in the auction theory, and our problem can be viewed as a revenue maximizing auction \([14, 19]\) with no items. We will discuss the case where the expected utility is at least a certain positive value in Section 7.

Let \( N \in \mathbb{N} \) be the number of the lottery tickets. We consider the setting in which all the tickets are sold; that is, there are sufficiently many potential buyers. More general cases, such as the expected profit maximization (in the usual sense of expected value) for the seller when partial tickets are sold, can be treated in the same way. Moreover, we assume that all buyers (or at least \( N \) potential buyers) have the same expected utility according to CPT, that is, all buyers have the same utility function \( V \) and the same probability weighting functions \( W \) and \( \overline{W} \). This homogeneity of agents is generally assumed in previous studies that model agents’ behavior by the CPT framework \([17, 18, 22]\).

Let \( \omega := (w_1, \ldots, w_N)^\top \in \mathbb{R}^N \) be the outcomes of tickets, i.e., the differences between lottery prizes and the ticket price. We consider the optimal design of a lottery that maximizes the seller’s profit as follows:

\[
\max_{\pi, \ n \in \mathbb{N} \cup \{0\}} P(\pi, n) \quad \text{s.t.} \quad \pi + n = N.
\]
Here, \( P(\pi, n) \) is defined by the optimal value of the following problem with fixed \( \pi \) and \( n \):

\[
\begin{align*}
\max_{\vec{\omega} \in \mathbb{R}^n, \omega \in \mathbb{R}} & \quad \sum_{i=1}^{n} (\vec{\omega}_i) - \sum_{j=1}^{n} \omega_j \\
\text{s.t.} & \quad \sum_{i=1}^{n} \bar{h}_i V(\vec{\omega}_i) + \sum_{j=1}^{n} h_j V(\omega_j) \geq 0, \\
& \quad 0 \leq \omega_1 \leq \omega_2 \leq \cdots \leq \omega_n, \\
& \quad \bar{\omega}_1 \leq \bar{\omega}_2 \leq \cdots \leq \bar{\omega}_n \leq 0.
\end{align*}
\] (4a)

The objective function (4a) is the seller’s profit, that is, the sum of the seller’s sales and payment. (4b) ensures that for the buyers, the lottery is not worse than 0, the utility when not purchasing the ticket (individual rationality). Constraints (4c) and (4d) impose the signs and the orders on the outcomes.

It is not difficult to see that \( P(0, N) = P(N, 0) = 0 \). Hence, in what follows, we only consider the case where \( n \) and \( \pi \) are positive.

We introduce a new variable \( v \) that satisfies \( v = -\sum_{i=1}^{n} \bar{h}_i V(\vec{\omega}_i) \). Since the objective function (4a) is monotone decreasing with respect to \( \omega_1, \ldots, \omega_n \), the optimal solution satisfies the following equality:

\[ v = \sum_{j=1}^{n} h_j V(\omega_j). \]

Let \( \bar{y}_i = -V(\vec{\omega}_i) \) for \( i \in [n] \) and \( y_j = V(\omega_j) \) for \( j \in [n] \). Then, we can reformulate (4) as the following bilevel optimization problem:

\[
\begin{align*}
\min_{v \in \mathbb{R}, \bar{y} \in [\pi]} & \quad \bar{f}_\pi(v) + f_n(v) \\
\text{s.t.} & \quad \bar{y}_1 \geq \bar{y}_2 \geq \cdots \geq \bar{y}_n \geq \bar{\pi} \geq 0, \\
& \quad \sum_{i=1}^{n} \bar{h}_i \bar{y}_i = v.
\end{align*}
\] (5)

where \( \bar{f}_\pi(v) \) is the optimal value of

\[
\begin{align*}
\min_{\bar{y} \in [\pi]} & \quad \sum_{i=1}^{n} V^{-1}(\bar{y}_i) \\
\text{s.t.} & \quad \bar{y}_1 \geq \bar{y}_2 \geq \cdots \geq \bar{y}_n \geq \bar{\pi} \geq 0, \\
& \quad \sum_{i=1}^{n} \bar{h}_i \bar{y}_i = v
\end{align*}
\] (6a)

and \( f_n(v) \) is the optimal value of

\[
\begin{align*}
\min_{y \in \mathbb{R}^n} & \quad \sum_{j=1}^{n} V^{-1}(y_j) \\
\text{s.t.} & \quad 0 \leq y_1 \leq y_2 \leq \cdots \leq y_n, \\
& \quad \sum_{j=1}^{n} h_j y_j = v.
\end{align*}
\] (7a)

Note that the optimal value of (5) equals that of (4) and it follows from the structure of (5) that we can independently solve (6) and (7).

In summary, we formulate the design of the optimal lottery as a three-level optimization problem; the high-level problem is (3), where we optimize the number of gains and losses to maximize the seller’s profit. The middle-level problem is (5), which is equivalent to (4). The problem (5) is devoted to calculate the optimal profit of the seller under the fixed number of gains and losses. To solve (5), we introduce the two low-level problems (6) and (7), where we essentially compute the optimal design of lotteries under the fixed number of gains and losses, respectively. In the next section, we consider the solutions of the problems above one by one.

### 4 Analysis and algorithms

In this section, we propose a linear-time algorithm to solve the three-level optimization problem consisting of the high-level one (3), the middle-level one (5), and the two low-level ones (6) and (7). To this end, in Sections 4.1 and 4.2, we provide the algorithmic solutions of (6) and (7), respectively. In Section 4.3, we derive the analytic solution of (5). In Section 4.4, we propose a linear-time algorithm to solve (3) by combing the solutions above. In Section 5, we introduce an improvement of the proposed algorithm, in which we can determine the price of lottery tickets beforehand.
4.1 Analysis of low-level problem (7)

In this subsection, we provide the optimal solution of (7). If \( v = 0 \), the optimal value is clearly zero. Thus, in the following, we only consider the case \( v > 0 \). If \( n = 1 \), the feasible region is a singleton \( \{ v/h_1 \} \). Otherwise (i.e., \( n > 1 \)), the feasible region has an interior point, for example, \( y_j = 2jv \cdot [n(n + 1) \sum_{j=1}^{n} h_j]^{-1} \) for \( j \in [n] \). Since the objective function is convex and the constraints are linear, (7) is a convex optimization problem. Therefore, the Karush-Kuhn-Tucker (KKT) conditions are necessary and sufficient one for the optimality; see, e.g., [5].

To start with, we define a key index for our analysis.

\textbf{Definition 4.1.} The transitional index \( J \in [n] \) is defined as

\[
J := \min \{ j \in [n] \mid j \cdot h_{j+1} \geq \sum_{j'=1}^{j} h_{j'} \},
\]

(8)

where we treat \( h_{n+1} \) as \( \infty \).

In words, \( J \) is the index in which \( h_{J+1} \) is equal to or higher than the previous average. Note that, since \( h_j \) only depends on \( n \) for every \( j \in [n] \), the transitional index \( J \) can be parameterized by \( n \) if we regard \( n \) as a variable. In Sections 4.3 and 4.4, we will use \( J_n \) for the transitional index instead of \( J \) to denote this parameterization. Now, we are ready to give the optimal solution of (7).

\textbf{Theorem 4.2.} The optimal solution of (7) is \( y^* \in \mathbb{R}^n \) such that

\[
y^*_j = \begin{cases} 
Y & (1 \leq j \leq J), \\
\left( \frac{J h_j}{\sum_{j'=1}^{J} h_{j'}} \right)^{\frac{1}{1-\alpha}} Y & (J + 1 \leq j \leq n) 
\end{cases}
\]

(9)

for all \( j \in [n] \), where the constant \( Y \geq 0 \) is defined as

\[
Y := \frac{v \left( \sum_{j'=1}^{J} h_{j'} \right)^{\frac{1}{1-\alpha}}}{\left( \sum_{j'=1}^{J} h_{j'} \right)^{\frac{1}{1-\alpha}} + J^{\frac{1}{1-\alpha}} \sum_{j'=J+1}^{n} h_{j'}^{\frac{1}{1-\alpha}}}
\]

Theorem 4.2 shows that, under the fixed number of gain tickets \( n \), the optimal solution consists of uniform prizes and increasing ones. In the proof, we explicitly identify the corresponding Lagrange multipliers and confirm that they satisfy the KKT conditions. We analyze the behaviors of \( y^*_j \) and the multipliers by dividing them into several cases with one auxiliary index called the flexional one. From Theorem 4.2, the optimal value of (7) is given as follows.

\textbf{Corollary 4.3.} The optimal value of the problem (7) is

\[
v^{\frac{1}{\alpha}} \cdot \frac{J}{\left( \sum_{j=1}^{J} h_j \right)^{\frac{1}{1-\alpha}} + \sum_{j=J+1}^{n} (\int h_j)^{\frac{1}{1-\alpha}}}^{\frac{1}{1-\alpha}}.
\]

(10)

In the rest of this subsection, we provide the proof of Theorem 4.2. Firstly we introduce an additional definition, which is convenient to the latter proofs.

\textbf{Definition 4.4 (flexional index).} We say that \( k_0 \in [n] \) is a flexional index if it satisfies that \( h_n > h_{n-1} > \cdots > h_{k_0} \) and \( h_{k_0} \leq h_{k_0-1} < h_{k_0-2} < \cdots < h_1 \).

The following lemma ensures that the flexional index exists uniquely.

\textbf{Lemma 4.5.} Suppose that the probability weighting function \( W : [0, 1] \rightarrow [0, 1] \) is inverse S-shaped. Then, there exists a unique flexional index \( k_0 \in [n] \).
Proof. Since $W(0) = 0$ holds, we have
\[ h_j = W\left(\frac{n - j + 1}{N}\right) - W\left(\frac{n - j}{N}\right) \]
for every $j \in [n]$. Then, for all $j \in [n]$, the mean value theorem to the left hand side in (11) implies that there exists $\theta_j \in \left(\frac{n-j}{N}, \frac{n-j+1}{N}\right)$ such that
\[ h_j = \frac{dW(\theta_j)}{N}. \]

Let $x_0$ be the inflection point of $W$ and $\tilde{j}$ be the minimum index satisfying $\theta_{\tilde{j}} \leq x_0$ ($\tilde{j} = 1$ if $\theta_{\tilde{j}} > x_0$). Since $W$ is an inverse S-shaped function, we have
\[ h_n > h_{n-1} > \cdots > h_{\tilde{j}} \quad \text{and} \quad h_{\tilde{j}+1} < h_{\tilde{j}+2} < \cdots < h_1. \]
Thus,
\[ k_0 = \begin{cases} \tilde{j} & \text{if } \tilde{j} = 1 \text{ or } h_{\tilde{j}} \leq h_{\tilde{j}+1}, \\ \tilde{j} - 1 & \text{otherwise} \end{cases} \]
is the unique flexional index.

We note that by the inverse S shape of the weighting function, it holds that $J \geq k_0$ (see Definition 4.4). Especially, we have $J = 1$ if and only if $k_0 = 1$.

The optimality condition of (7) Here we prepare the proof of Theorem 4.2: we give the optimality condition of (7). For notation simplicity, we define $\hat{\alpha} := 1/\alpha$. It follows from the definition that $\hat{\alpha} > 1$. As we see in the subsection 4.1, it is sufficient to consider the KKT conditions. The Lagrangian of (7) is given by
\[ L(y, \mu, \Lambda) = \sum_{j=1}^{n} (y_j)^{\hat{\alpha}} - \mu_1 y_1 + \sum_{j=1}^{n-1} \mu_{j+1} (y_j - y_{j+1}) + \Lambda \left( \sum_{j=1}^{n} h_j y_j - v \right), \]
where $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{R}^n$ and $\Lambda \in \mathbb{R}$ are Lagrange multipliers. Then the KKT conditions are
\begin{align*}
\hat{\alpha}(y_j)^{\hat{\alpha} - 1} + \mu_{j+1} - \mu_j + \Lambda h_j &= 0 \quad (\forall j \in [n-1]), \tag{12} \\
\hat{\alpha}(y_n)^{\hat{\alpha} - 1} - \mu_n + \Lambda h_n &= 0, \tag{13} \\
\mu_{j+1}(y_j - y_{j+1}) &= 0, \quad \mu_1 y_1 = 0, \tag{14} \\
\sum_{j=1}^{n} h_j y_j &= v, \tag{15} \\
\mu_j &\geq 0, \quad 0 \leq y_1 \leq \cdots \leq y_n. \tag{16}
\end{align*}
The first and second equalities are obtained by $\nabla_y L = 0$. The third equality is the complementary condition. The fourth and fifth are obtained by the condition for the Lagrange multipliers w.r.t. inequality constraints and the constraints of the original problem.

Then we give Lagrange multipliers satisfying (12)–(16) with $y_j^*$ defined in Theorem 4.2.

Lemma 4.6. Define $y^* \in \mathbb{R}^n$ as in Theorem 4.2. Then $y^*$ and Lagrange multipliers $(\mu_j)_{j=1}^n$ and $\Lambda$ defined by
\begin{align*}
\mu_j &= -\frac{\hat{\alpha}}{v} \left[ \sum_{j'=1}^{n} (y_j^*)^{\hat{\alpha}} \right] \left( \sum_{j'=j}^{n} h_{j'} \right) + \hat{\alpha} \left[ \sum_{j'=j}^{n} (y_j^*)^{\hat{\alpha} - 1} \right], \tag{17} \\
\Lambda &= -\frac{\hat{\alpha}}{v} \sum_{j=1}^{n} (y_j^*)^{\hat{\alpha}} \tag{18}
\end{align*}
satisfy the KKT conditions (12)–(16).
Therefore, we have that $y^*$ introduced in Theorem 4.2 is the KKT point of (7), which induces the optimality of $y^*_j$ as we prove in the last of this subsection. To this end, we give the auxiliary lemma as follows:

**Lemma 4.7.** Define $(\mu_j)_{j=1}^n$ as in Lemma 4.6 and $(t_j)_{j=1}^n$ by $t_j = (y_j^*)^{\alpha - 1}/h_j$ for all $j \in [n]$. Then, we have

\[
-\frac{v}{\alpha} \mu_n = \sum_{j'=1}^n h_{j'} h_n y^*_j (t_{j'} - t_n),
\]

(19)

\[
\frac{v}{\alpha} (\mu_{j+1} - \mu_j) = \sum_{j'=1}^n h_{j'} h_j y^*_j (t_{j'} - t_j) \quad (\forall j \in [n-1]).
\]

(20)

**Proof.** By the construction of $y^*$, a straightforward calculation shows

\[
\sum_{j=1}^n h_j y^*_j = v.
\]

By (17), it holds that

\[
-\frac{v}{\alpha} \mu_n = h_n \sum_{j'=1}^n (y_{j'}^*)^{\alpha} - v (y_n^*)^{\alpha - 1}
\]

\[
= h_n \sum_{j'=1}^n (y_{j'}^*)^{\alpha} - \left( \sum_{j'=1}^n h_{j'} y_{j'}^* \right) (y_n^*)^{\alpha - 1}
\]

\[
= \sum_{j'=1}^n y_{j'}^* \left[ h_n (y_{j'}^*)^{\alpha - 1} - h_{j'} (y_n^*)^{\alpha - 1} \right]
\]

\[
= \sum_{j'=1}^n h_{j'} h_n y^*_j (t_{j'} - t_n).
\]

Moreover, (17) means that for each $j \in [n-1],$

\[
\frac{v}{\alpha} (\mu_{j+1} - \mu_j) = h_{j} \sum_{j'=1}^n (y_{j'}^*)^{\alpha} - v (y_j^*)^{\alpha - 1}
\]

\[
= h_{j} \sum_{j'=1}^n (y_{j'}^*)^{\alpha} - \left( \sum_{j'=1}^n h_{j'} y_{j'}^* \right) (y_j^*)^{\alpha - 1}
\]

\[
= \sum_{j'=1}^n y_{j'}^* \left[ h_{j} (y_{j'}^*)^{\alpha - 1} - h_{j'} (y_j^*)^{\alpha - 1} \right]
\]

\[
= \sum_{j'=1}^n h_{j'} h_{j} y^*_j (t_{j'} - t_j).
\]

Thus, we get the conclusion. \qed

Then we move to the proof of Lemma 4.6.

**Proof of Lemma 4.6.** By straightforward calculations, we can check that (12), (13) and (15) hold. Then, it remains to confirm (14) and (16).

Since $J \geq k_0$, we can observe that $(y_j^*)_{j=1}^n$ in Theorem 4.2 satisfies

\[
0 \leq y_1^* \leq \cdots \leq y_n^*.
\]

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in other words, the inequality for \( y \) in (16). Moreover, it holds that \( y_j^* - y_{j+1}^* = 0 \) for \( j \in [J - 1] \). Therefore, it suffices to check the \( (\mu_j)_{j=1}^n \) determined by (17) satisfies complementary condition (14) and the inequality in (16): \( \mu_j \geq 0 \) for all \( j \in [n] \) and \( \mu_j = 0 \) for \( j = 1 \) and \( J + 1 \leq j \leq n \).

We consider the case \( J < n \). It holds that by the definition of \( y^* \),

\[
t_{J+1} = \cdots = t_n = \frac{JY^\alpha - 1}{\sum_{j=1}^n h_j} =: T.
\]

Then it follows from (19) that

\[
-\frac{v}{\alpha} \mu_n = \sum_{j=1}^n h_j y_j^* (t_j - t_n)
= h_n \sum_{j' = 1}^J h_j' Y (t_j' - t_n)
= h_n \sum_{j' = 1}^J \left( Y^\alpha - \frac{Jh_j'}{\sum_{j'=1}^n h_j'} Y^\alpha \right)
= h_n (JY^\alpha - JY^\alpha) = 0.
\]

The same argument gives \( \mu_{j+1} - \mu_j = 0 \) for \( j = J + 1, \ldots, n - 1 \) (Replace \( t_n \) by \( t_j \) in the above transformation). Hence we get \( \mu_j = 0 \) for \( J + 1 \leq j \leq n \).

Moreover, (17) implies

\[
-\frac{v}{\alpha} \mu_1 = -\frac{v}{\alpha} \mu_n + \sum_{j=1}^{n-1} (\mu_{j+1} - \mu_j)
= \sum_{j_1, j_2 = 1}^J h_{j_1} h_{j_2} y_{j_1}^* (t_{j_1} - t_{j_2})
= \sum_{j_1 < j_2} h_{j_1} h_{j_2} (y_{j_1}^* - y_{j_2}^*) (t_{j_1} - t_{j_2})
= \sum_{j_1 = 1}^J \sum_{j_2 = J+1}^n h_{j_1} h_{j_2} (y_{j_2}^* - y_{j_1}^*) (t_{j_2} - t_{j_1})
= \sum_{j_1 = 1}^J \sum_{j_2 = J+1}^n h_{j_1} h_{j_2} (y_{j_2}^* - Y) (T - t_{j_1})
= \sum_{j_1 = 1}^J h_{j_1} (T - t_{j_1}) \sum_{j_2 = J+1}^n h_{j_2} (y_{j_2}^* - Y)
= 0,
\]

where the second equality follows from (19) and (20), the fourth equality from \( y_{j_1}^* = y_{j_2}^* \) when \( j_2 \leq J \) and \( t_{j_1} = t_{j_2} \) when \( j_1 \geq J + 1 \), and the last equality from

\[
\sum_{j_1 = 1}^J h_{j_1} Y (T - t_{j_1}) = JY^\alpha - JY^\alpha = 0.
\]

Now it remains to prove that \( \mu_j \geq 0 \) for \( 1 < j < J \). For \( 1 \leq j \leq J \), we have that

\[
\mu_{j+1} - \mu_j = \bar{\alpha} \left[ \frac{h_j}{v} \sum_{j' = 1}^n (y_{j'}^*)^\alpha - (y_j^*)^\alpha - 1 \right] = \bar{\alpha} \left[ \frac{h_j}{v} \sum_{j' = 1}^n (y_{j'}^*)^\alpha - Y^\alpha - 1 \right].
\]

(22)
Moreover, since \((J - 1)h_J < \sum_{j'=1}^{J-1} h_{j'}\) by definition of \(J\), we have that
\[
t_J = \frac{Y_{\hat{a} - 1}}{h_J} > \frac{JY_{\hat{a} - 1}}{\sum_{j'=1}^{J-1} h_{j'}} = T,
\]
which implies
\[
\frac{v}{\hat{\alpha}} (\mu_{j+1} - \mu_j) = \sum_{j'=1}^{n} h_{j'} h_J y_{j'}^*(t_{j'} - t_J) < \sum_{j'=1}^{n} h_{j'} h_J y_{j'}^*(t_{j'} - T) = -\frac{h_J}{\hat{\alpha}} \mu_n = 0.
\]
Combining this with \(\mu_{j+1} = 0\), we get \(\mu_J > 0\). In addition, since the increase/decrease of the right hand side in (22) is determined by that of \(h_J\), the definition of \(k_0\) gives
\[
\mu_2 - \mu_1 > \cdots > \mu_{k_0} - \mu_{k_0-1} \geq \mu_{k_0+1} - \mu_{k_0}
\]
and
\[
\mu_{k_0+1} - \mu_{k_0} < \cdots < \mu_{J+1} - \mu_J < 0.
\]
Then we can show that there exists an integer \(J_0 \in [J - 1]\) such that
\[
\begin{cases}
\mu_{j+1} - \mu_j \geq 0 & 1 \leq j \leq J_0, \\
\mu_{j+1} - \mu_j < 0 & J_0 \leq j \leq J.
\end{cases}
\]
Indeed, if there are no such indices, it holds that \(\mu_{j+1} - \mu_j < 0\) for all \(j \in [J]\) by (23) and (24). This implies that \((\mu_j)_{j=1}^{J+1}\) is monotonically increasing, which contradicts to \(\mu_1 = \mu_{J+1} = 0\). Then (25) induces
\[
0 = \mu_1 \leq \mu_2 \leq \cdots \leq \mu_{J_0+1}
\]
and
\[
\mu_{J_0+1} > \mu_{J_0+2} > \cdots > \mu_{J+1} = 0.
\]
Thus we get \(\mu_j \geq 0\) for \(1 < j \leq J\). Hence (14) and the inequality in (16) hold, which gives the conclusion for the case \(J < n\).

If \(J = n\), (9) in Theorem 4.2 means \(y_j^* = Y\) for all \(j \in [\bar{m}]\). Then it suffices to show that \(\mu_j \geq 0\) for all \(j \in [n]\). We can get \(\mu_1 = 0\) by
\[
-\frac{v}{\hat{\alpha}} \mu_J = \sum_{j_1=1}^{J} \sum_{j_2=J+1}^{n} h_{j_1} h_{j_2} (y_{j_2}^* - y_{j_1}^*) (t_{j_2} - t_{j_1}) = 0,
\]
and
\[
\mu_J = 0.
\]
where the first equality is the same as (21). For $j \geq 2$, (17) gives

$$
\frac{v}{\hat{\alpha}} \mu_j = -\left[ \sum_{j=1}^{n} (y_j^*)^{\hat{\alpha}} \right] \left( \sum_{j'=j}^{n} h_j' \right) + \hat{\alpha} \left[ \sum_{j'=j}^{n} (y_j^*)^{\hat{\alpha}-1} \right]
$$

$$
= -nY^{\hat{\alpha}} \left( \sum_{j'=1}^{j} h_j' \right) + v(n - j + 1)Y^{\hat{\alpha}-1}
$$

$$
= Y^{\hat{\alpha}-1} \left[ v(n - j + 1) - n \left( \sum_{j'=j}^{n} h_j'Y \right) \right]
$$

$$
= Y^{\hat{\alpha}-1} \left[ v(n - j + 1) - n \left( v - \sum_{j'=1}^{j-1} h_j'Y \right) \right]
$$

$$
= (j - 1)Y^{\hat{\alpha}-1} \left( \sum_{j'=1}^{j-1} h_j'Yn - v \right)
$$

$$
\geq (j - 1)Y^{\hat{\alpha}-1} \left( \frac{\sum_{j'=1}^{j-1} h_j'}{n} Yn - v \right)
$$

$$
= (j - 1)Y^{\hat{\alpha}-1}(v - v) = 0.
$$

Here, the inequality follows from the fact that the average value $\sum_{j'=1}^{j-1} h_j'Yn/j$ is monotonically non-increasing with respect to $j$ (see the definition of $J$). Thus, we get $\mu_j \geq 0$ for all $j \in [n]$, which gives the conclusion for the case $J = n$.

**proof of Theorem 4.2.** Lemma 4.6 shows that $y^*$ defined in Theorem 4.2 is the KKT point of (7). Since (7) is the convex optimization problem and its objective function is strongly convex, $y^*$ is a unique optimum. □

### 4.2 Analysis of low-level problem (6)

In this subsection, we provide the optimal solution of (6). Note that (6) is a nonconvex optimization problem. However, it has a global optimum because the objective function is continuous and the feasible region is a compact subset of $\mathbb{R}^n$. The following theorem provides the optimum of (6).

**Theorem 4.8.** There exists an index $\bar{J} \in [\bar{n}]$ such that

$$
\bar{y}_i = \begin{cases} 
Y & (1 \leq i \leq \bar{J}), \\
0 & (\bar{J} + 1 \leq i \leq \bar{n})
\end{cases}
$$

is a global optimum of (6), where $Y := v/\sum_{i=1}^{\bar{n}} \bar{h}_i \geq 0$.

**Proof.** For notation simplicity, we define $\hat{\beta} := 1/\beta$. It follows from the definition that $\hat{\beta} > 1$.

We show that a global optimum is represented by (26) for some $Y \in \mathbb{R}$. Let $\bar{y} = (\bar{y}_1, \ldots, \bar{y}_{\bar{n}})$ and $F(\bar{y})$ be the objective function. Suppose that

$$
\bar{y}_1 \geq \cdots \geq \bar{y}_{L_1-1},
\bar{y}_{L_1-1} > \bar{y}_{L_1} = \cdots = \bar{y}_{L_2} > 0,
\bar{y}_{L_2+1} = \cdots = \bar{y}_{\bar{n}} = 0
$$

(27)
for integers $L_1$ and $L_2$ such that $1 \leq L_1 \leq L_2 \leq n$. It is clear that this $\bar{y}$ satisfies the inequality constraints. Then let us consider to parameterize $\bar{y}$ as

$$
\bar{y}(t) = \begin{cases} 
\bar{y}_i - at & 1 \leq i \leq L_1 - 1, \\
\bar{y}_i + t & L_1 \leq i \leq L_2, \\
0 & L_2 + 1 \leq i \leq n,
\end{cases}
$$

where $t$ is a variable that moves across an interval in which the inequality constraints hold and $a > 0$ is a constant determined by the equality constraint. We note that $t = 0$ is the interior point of the interval. Then, we have

$$F(\bar{y}(t)) = -\sum_{i=1}^{L_1-1} (\bar{y}_i - at)^{\hat{\beta}} - \sum_{i=L_1}^{L_2} (\bar{y}_i + t)^{\hat{\beta}}.
$$

This function is concave with respect to $t$, because

$$
d^2 d^{-2} F(\bar{y}(t)) = -a^{2\hat{\beta}}(\hat{\beta} - 1) \sum_{i=1}^{L_1-1} (\bar{y}_i - at)^{\hat{\beta} - 2} - \hat{\beta}(\hat{\beta} - 1) \sum_{i=L_1}^{L_2} (\bar{y}_i + t)^{\hat{\beta} - 2} < 0,
$$

where we use $\hat{\beta} > 1$ for the last inequality. This implies that $F(\bar{y}(t))$ takes its minimum value at one endpoint of the interval. Using this argument, we can prove that a global optimum is represented by (26) by contradiction. If the optimal value is obtained by $\bar{y}$ that does not satisfy (26), there exist $L_1$ and $L_2$ for which $\bar{y}$ satisfies (27). Then we can continuously transform $\bar{y}$ while decreasing the function value. This contradicts that $\bar{y}$ is a global optimum.

It remains to specify $Y$. For $\bar{J} \in [n]$, the equality constraint implies

$$
\left( \sum_{i=1}^{\bar{J}} \bar{h}_i \right) Y = v.
$$

Hence, it holds that

$$Y = \frac{v}{\sum_{i=1}^{\bar{J}} \bar{h}_i},
$$

which gives the conclusion.

Theorem 4.8 shows that, under the fixed number of loss tickets $\bar{n}$, the optimal solution is located at a vertex of the polyhedron defined by the constraints. The following corollary specifies the optimal value of (6).

**Corollary 4.9.** *The optimal value of the problem (6) is*

$$
-(\frac{v}{\chi})^{\frac{1}{\hat{\beta}}} \cdot \max_{\bar{J} \in [\bar{n}]} \bar{J} \cdot \left( \sum_{i=1}^{\bar{J}} \bar{h}_i \right)^{-\frac{1}{\hat{\beta}}}. \tag{28}
$$

Note that, since $\bar{h}_i$ only depends on $\bar{n}$ for every $i \in [\bar{n}]$, the index that realizes (28) can be parameterized by $\bar{n}$ if we regard $\bar{n}$ as a variable. In Sections 4.3 and 4.4, we will use the symbol $\bar{J}_{\bar{n}}$ instead of $\bar{J}$ to denote the parameterization; that is, we define

$$\bar{J}_{\bar{n}} := \arg \max_{\bar{J} \in [\bar{n}]} \bar{J} \cdot \left( \sum_{i=1}^{\bar{J}} \bar{h}_i \right)^{-\frac{1}{\hat{\beta}}}. \tag{29}
$$

If there are multiple indexes that attain the maximum value, we assume that $\arg \max$ returns the smallest one of them.
4.3 Analysis of middle-level problem (5)

Define \( \Gamma_{n} := \Gamma_{n,J} \) and \( \Gamma_{n,\bar{J}} \) where

\[
\Gamma_{n,J} := \left[ \left( \sum_{j=1}^{J} h_j \right)^{1/\alpha} + \sum_{j=J+1}^{n} (J^\alpha h_j)^{1/\alpha} \right]^{\frac{\beta}{\beta - \alpha}}, \tag{30}
\]

\[
\Gamma_{n,\bar{J}} := \bar{J} \cdot \left( \lambda \sum_{i=1}^{\bar{J}} \bar{h}_i \right)^{1/\beta} \tag{31}
\]

We provide the optimal solution of (5). It follows from Corollaries 4.3 and 4.9 that (5) reduces to

\[
\min_{v \in \mathbb{R} \geq 0} \Gamma_{n} \cdot v^\frac{1}{\alpha} - \Gamma_{\bar{n}} \cdot v^\frac{1}{\beta}. \tag{32}
\]

This is a 1-dimensional and possibly nonconvex optimization problem. The following theorem provides the optimal solution of (32).

**Theorem 4.10.** The optimal value of (32) is (i) \( H \) if \( \alpha < \beta \), (ii) 0 if \( \alpha = \beta \) and \( \Gamma_{n} \geq \Gamma_{\bar{n}} \), and (iii) \( -\infty \) otherwise; where

\[
H = -\left( \frac{\Gamma_{\bar{n}}}{\Gamma_{n}} \right)^{\frac{\alpha}{\beta}} \left( \frac{\alpha}{\beta} \right)^{\frac{\beta}{\beta - \alpha}} \left( 1 - \frac{\alpha}{\beta} \right) \tag{33}
\]

**Proof.** As for the case \( \alpha < \beta \), we can easily check that the global optimum of (32) is \( \left( \frac{\Gamma_{\bar{n}}}{\beta} \right)^{\frac{\alpha}{\beta}} \) by the first derivative test. For the other cases, the proofs are straightforward. \( \square \)

Theorem 4.10 shows that the optimal value of (32) varies according to the order of \( \alpha \) and \( \beta \): if \( \alpha < \beta \) holds, that is, the buyers are more sensitive to minor losses than minor gains, the optimal value takes nonzero finite one. Otherwise, the seller can make infinite profit or cannot make any gain, which seems to be inconsistent with reality. Although \( \alpha = \beta \) holds in the original CPT setting [22], the subsequent surveys [17, 18] reveal that the relationship \( \alpha < \beta \) holds in many countries. In Section 5, we will propose another modeling that includes an extra constraint on the ticket price, where the optimal solution of the middle-level problem does not diverge no matter how the relationship of \( \alpha \) and \( \beta \) is.

4.4 Analysis of high-level problem (3) and linear-time algorithm

In this subsection, we propose a linear-time algorithm to solve the entire problem of the optimal lottery by analyzing an algorithmic framework for the high-level problem (3). We first give a naive quadratic time algorithm and then improve it to run in linear time.

**Naive algorithm** This naive algorithm computes the optimal solution of (3) by the brute-force search for \((\bar{n}, n) \in \mathbb{N}^2 \) under \( \bar{n} + n = N \). Under each fixed \((\bar{n}, n)\), it computes the optimal value of (5) as follows: it first computes \( J_{n} \) and \( \bar{J}_{n} \) by (8) and (29), respectively. Then, it computes \( \Gamma_{n} \) and \( \bar{\Gamma}_{n} \) by (30) and (31) and the optimal value under the fixed \((\bar{n}, n)\) as Theorem 4.10. Since the computation of \( \Gamma_{n} \) and \( \bar{\Gamma}_{n} \) takes \( O(N) \) time and the algorithm computes them at every \((\bar{n}, n)\), the overall time complexity is \( O(N^2) \).

**Improved algorithm** The bottleneck of the above algorithm is the calculation of \( J_{n}, \bar{J}_{n}, \Gamma_{n}, \) and \( \bar{\Gamma}_{n} \). We can efficiently calculate \( (J_1, \Gamma_1), \ldots, (J_N, \Gamma_N) \) all at once in \( O(N) \) time by Algorithm 2. As we state soon later, the time complexity attributes to the following lemma:

**Lemma 4.11.** \((J_k)_{k \in [N]}\) is monotonically nondecreasing.
For the proof, we first prepare another auxiliary lemma. Hereafter, we use the symbol \( h_{j,n} \) to denote \( h_j \) under \( n \) gain tickets, i.e., \( h_{j,n} = W \left( \frac{(n - j + 1)}{N} \right) - W \left( \frac{(n - j)}{N} \right) \). Note that, by definition, it holds that
\[
h_{j,n} = h_{j-1,n-1}. \tag{33}
\]

**Lemma 4.12.** Suppose that \( k \geq 2 \) and \( J_k \leq k - 1 \) hold. Then, for all \( J_k \leq j \leq k - 1 \), it follows that
\[
j \cdot h_{j+1,k} \geq \sum_{j'=1}^{j} h_{j',k}. \tag{34}
\]

**Proof.** By Definitions 4.1 and 4.4, we have \( J_k \geq k_0 \) and
\[
\frac{\sum_{j'=1}^{k_0} h_{j',k}}{k_0} \leq h_{k_0,k} \leq h_{k_0+1,k} \leq \cdots \leq h_{k,k}.
\]
Since \( J_k \leq j \), we obtain
\[
\sum_{j'=1}^{j} h_{j',k} = k_0 \cdot \frac{\sum_{j'=1}^{k_0} h_{j',k}}{k_0} + h_{k_0,k} + \cdots + h_{j,k} \leq j \cdot h_{j+1,k}.
\]

**proof of Lemma 4.11.** We only need to show \( J_{k+1} \geq J_k \) when \( J_k \geq 2 \) holds. Note that it follows from the definition (8) that
\[
(J_k - 1) \cdot h_{J_k,k} < \sum_{j=1}^{J_k-1} h_{j,k}. \tag{35}
\]

We first prove that
\[
h_{J_k,k} < h_{1,k} \tag{36}
\]
by contradiction; suppose that \( h_{J_k,k} \geq h_{1,k} \) holds. Then, we have \( h_{J_k,k} \geq h_{j,k} \) for all \( j \in [J_k - 1] \) by Definition 4.4 and \( J_k \geq k_0 \), which contradicts to (35).

Then, it follows that
\[
J_k \cdot h_{J_k+1,k+1} = J_k \cdot h_{J_k,k} < h_{J_k+1,k+1} + \sum_{j=1}^{J_k-1} h_{j,k} < h_{1,k+1} + \sum_{j=1}^{J_k-1} h_{j+1,k+1} = \sum_{j=1}^{J_k} h_{j,k+1}, \tag{37}
\]
where the first equality follows from (33), the first inequality from (35), and the second one from (33) and (35). (37) implies that \( J_{k+1} \geq J_k \). Indeed, if \( J_{k+1} < J_k \) holds, since we have
\[
J_{k+1} < J_k \leq k = (k + 1) - 1 \tag{38}
\]
it follows from Lemma 4.12 that
\[
J_k \cdot h_{J_k+1,k+1} \geq \sum_{j=1}^{J_k} h_{j,k+1}, \tag{39}
\]
which contradicts to (37). \qed

Now, we explain the detail of Algorithm 2. Let \( h_{j,n} \) be \( h_j \) under \( n \) gain tickets, i.e., \( h_{j,n} := W\left(\frac{n-j+1}{N}\right) - W\left(\frac{n-j}{N}\right) \).

We first compute prefix sums \( \eta_0, \ldots, \eta_N \) such that

\[
\eta_t := \sum_{\ell=1}^t \left( W\left(\frac{\ell}{N}\right) - W\left(\frac{\ell-1}{N}\right) \right),
\]

in \( O(N) \) time. Note that, for each \( k \in [N] \) and \( J \in [k] \), it follows from the definition of \( h_j \) that

\[
\sum_{j=1}^J h_{j,k} = W\left(\frac{k}{N}\right) - W\left(\frac{k-J}{N}\right).
\]

Next, we compute \( (J_k)_{k \in [N]} \): for every \( k \)th iteration, we set \( J = J_{k-1} \) and increase \( J \) until it reaches \( k \)

\[
W\left(\frac{k}{N}\right) - W\left(\frac{k-J}{N}\right) \leq J \cdot (W\left(\frac{k-J-1}{N}\right) - W\left(\frac{k-J}{N}\right))
\]

From Lemma 4.11, the computation takes \( O(N) \) time. We compute each \( \Gamma_k \) in constant by using (40) and (41). Algorithm 2 formally states this procedure.

Let us go back to the explanation of the algorithm. We can prove that there are no zero components in \( (\omega^*_1, \ldots, \omega^*_n, \omega^*_1, \ldots, \omega^*_n) \) unless the optimal value of (3) is zero as follows:

**Proposition 4.13.** Let \( (\pi^*, n^*) \) be a global optimum solution of (3) and suppose that the optimal value \( P^* \) is positive. Then, for every \( (\omega^*_1, \ldots, \omega^*_n, \omega^*_1, \ldots, \omega^*_n) \) that is a global optimum of (4) with \( (\pi^*, n^*) \), we have \( \omega^*_i \neq 0 \) for any \( i \in [\pi^*] \) and \( \omega^*_j \neq 0 \) for any \( j \in [n^*] \).

**Proof.** Let \( \bar{y}^*_i = V(\omega^*_i) \) for \( i \in [\pi^*] \) and \( y^*_j = V(\omega^*_j) \) for \( j \in [n^*] \). Then, \( (\bar{y}^*_i)_{i \in [\pi^*]} \) and \( (y^*_j)_{j \in [n^*]} \) are respectively global optimum of (6) and (7) with \( v = v^* \) such that

\[
v^* := -\sum_{i=1}^{\pi^*} \bar{h}_i V(\omega^*_i) = \sum_{j=1}^{n^*} h_j V(\omega^*_j).
\]

We prove the proposition by contradiction. Suppose that there exists a zero component in \( (\omega^*_1, \ldots, \omega^*_n, \omega^*_1, \ldots, \omega^*_n) \) (i.e., also in \( (\bar{y}^*_1, \ldots, \bar{y}^*_{\pi^*}, y^*_1, \ldots, y^*_{n^*}) \)). Since \( P^* > 0 \), we have \( \pi^* > 0 \), \( n^* > 0 \), and \( v^* > 0 \). By Theorem 4.2, we have \( y^*_j > 0 \) for any \( j \in [n] \), and hence \( \omega^*_j \neq 0 \) for any \( j \in [n] \). Therefore, we may assume that \( \omega^*_i = 0 \) for some \( i \in [\pi^*] \), which implies \( \bar{y}^*_i = 0 \) by (6b). In what follows, we show

\[
P(\pi^* - 1, n^* + 1) > P(\pi^*, n^*),
\]

which contradicts the optimality of \( (\pi^*, n^*) \). To see this inequality, let us consider \( (\bar{y}^*_1, \ldots, \bar{y}^*_{\pi^* - 1}) \) and \( (\bar{y}^*_1, y^*_1, \ldots, y^*_{n^*}) \), which are respectively feasible solutions of (6) and (7) with \( \bar{\pi} = \pi^* - 1, n = n^* + 1 \), and \( v = v^* \) (recall that \( \bar{y}^*_{\pi^* - 1} = 0 \)). By the feasibility of \( (\bar{y}^*_1, \ldots, \bar{y}^*_{\pi^* - 1}) \), we have

\[
\bar{f}_{\pi^*}(v^*) \leq \bar{f}_{\pi^* - 1}(v^*).
\]

As \( f_{n^*, v^*} \leq f_{n^* - 1, v^*} \) is feasible but is not optimum of (6) by Theorem 4.2, we have

\[
f_{n^*, v^*} < f_{n^* - 1, v^*}.
\]

Combining these inequalities, as for the optimization problem (5) with \( \pi = \pi^* - 1, n = n^* + 1 \), it holds that \( v = v^* \) achieves the objective value smaller than \(-P(\pi^*, n^*)\), which is the optimal value of (5) with \( \pi = \pi^* \) and \( n = n^* \). This concludes (43).

Therefore, to obtain the overall optimal solution, we can skip the computation of \( \bar{J}_{\pi} \) by replacing \( \bar{J}_{\pi} \) with \( \bar{\pi} \) (i.e., replacing \( \bar{\pi} \) with \( \bar{\pi}_{\pi, \pi} \)). With this replacement, we can compute \( \bar{\Gamma}_{\pi, \pi} \) in \( O(1) \) time for each \( \bar{\pi} \in [\pi] \) by

\[
\bar{\Gamma}_{\pi, \pi} = \bar{\pi} \cdot \left( \lambda \sum_{i=1}^{\pi} \bar{h}_i \right)^{-\frac{1}{\beta}} = \pi \cdot \left( \lambda W\left(\frac{\bar{\pi}}{\pi^*}\right) \right)^{-\frac{1}{\beta}}.
\]
Hence, where we use (31) and set the procedure above. Let us formally state the time complexity of the algorithm.

\[ \sum_{i=1}^{\pi} \bar{b}_i = \sum_{i=1}^{\pi} \left( W \left( \frac{N - i + 1}{N} \right) - W \left( \frac{N - i}{N} \right) \right) = W \left( \frac{\pi}{N} \right). \]

Hence, \( \Gamma_{1,1}, \ldots, \Gamma_{N,N} \) can be computed in \( O(N) \) time. Note that we can also compute \( (\bar{y}_i^*)^{\pi_i} \) in \( O(N) \) time according to Theorem 4.8. Then, it conducts the linear search with respect to \( n \) in \( O(N) \) time. Algorithm 1 formally states the procedure above. Let us formally state the time complexity of the algorithm.

**Proposition 4.14.** Algorithm 1 runs in \( O(N) \) time.

**Proof.** Recall that we can compute \( \Gamma_{1,1}, \ldots, \Gamma_{N,N} \) in \( O(N) \) time by (44). Since Algorithm 2 runs in \( O(N) \) time, we conclude that Algorithm 1 also does in \( O(N) \) time.

Based on Algorithm 1, we conduct numerical experiments with specific parameters \( (N, \alpha, \beta, \lambda, W \) and \( \bar{W} )\) and give optimal lotteries; see Section 6 for the results.

### 5 Fixed ticket price

In the previous sections, the ticket price is one of the decision variables and can take an arbitrarily large value. In this section, we analyze the setting where the ticket price is determined in advance. We first provide its formulation and
then propose an algorithm for solving this problem. Moreover, we propose a linear-time algorithm that outputs the optimal solution in a specific case.

5.1 Problem formulation and characterization of the optimal solution

Let $-\tilde{\omega}_{\text{min}}$ denote the fixed ticket price. Then we add $\tilde{\omega}_{\text{min}} \geq \tilde{\omega}_{\text{min}}$ to the middle-level problem (4), which results in the following low-level problem instead of (6):

\[
\min_{\bar{y} \in \mathbb{R}^n} \sum_{i=1}^{n} V^{-1}(\bar{y}_i)
\]
\[
\text{s.t. } y_{\min} \leq \bar{y}_1 \leq \bar{y}_2 \geq \cdots \geq \bar{y}_n \geq 0,
\]
\[
\sum_{i=1}^{n} \bar{h}_i \bar{y}_i = v
\]

with $y_{\min} := -V(\tilde{\omega}_{\text{min}})$. We can also characterize the optimal value of (45) as follows:

**Theorem 5.1.** The problem (45) is feasible if and only if $0 \leq v \leq y_{\min} \sum_{i=1}^{n} \bar{h}_i$ holds. Under the condition, the optimal value of (45) is given by

\[
\min_{\ell_1, \ell_2 \in \mathbb{N} \cup \{0\}} -\ell_1 y_{\min} + (\ell_2 - \ell_1) \left( \frac{v - y_{\min} \sum_{i=1}^{\ell_1} \bar{h}_i}{\sum_{i=\ell_1+1}^{\ell_2} \bar{h}_i} \right)^{\frac{1}{2}}
\]
\[
\text{s.t. } y_{\min} \sum_{i=1}^{\ell_1} \bar{h}_i \leq v \leq y_{\min} \sum_{i=1}^{\ell_2} \bar{h}_i,
\]
\[
\ell_1 \leq \ell_2 \leq n.
\]

For the proof of Theorem 5.1, we introduce a result, which is a modification of Theorem 4.8.

**Proposition 5.2.** Suppose that $v \leq y_{\min} \sum_{i=1}^{n} \bar{h}_i$. There exist a positive integer $\ell_1, \ell_2 \in [n]$ and $0 \leq \gamma \leq y_{\min}$ such that the optimal solution of (45) is obtained by

\[
\bar{y}_i = \begin{cases} 
y_{\min} & 1 \leq i \leq \ell_1, 
\gamma & \ell_1 + 1 \leq i \leq \ell_2, 
0 & \ell_2 + 1 \leq i \leq n,
\end{cases}
\]

where

\[
\gamma = \frac{v - y_{\min} \sum_{i=1}^{\ell_1} \bar{h}_i}{\sum_{i=\ell_1+1}^{\ell_2} \bar{h}_i}.
\]

While Theorem 4.8 states that the optimal solution consists of two values $\gamma$ and 0, this proposition implies that there are three values: $\gamma$, 0, and $y_{\min}$. Under the inequality constraint of (45), we have $\sum_{i=1}^{\ell_1} \bar{h}_i \bar{y}_i \leq y_{\min} \sum_{i=1}^{\ell_2} \bar{h}_i$, therefore we need $v \leq y_{\min} \sum_{i=1}^{n} \bar{h}_i$ for the feasibility. We abbreviate the rest of the proof since it is essentially equal to that of Theorem 4.8.

We move to explain how Theorem 5.1 is derived. Under fixed $\ell_1$ and $\ell_2$, with specifying $\gamma$ by the equality constraint, (45) comes down to the 1-dimensional optimization problem w.r.t. $v$. Then the brute-force search over the optimal values under all-possible $\ell_1$ and $\ell_2$ gives the global optimum, which is represented by (46) in Theorem 5.1. The range of $v$ is derived by the following argument: under fixed $\ell_1$ and $\ell_2$, it holds that $y_{\min} \sum_{i=1}^{\ell_1} \bar{h}_i \leq \gamma \sum_{i=\ell_1+1}^{\ell_2} \bar{h}_i \leq y_{\min} \sum_{i=1}^{n} \bar{h}_i$, therefore the equality constraint gives $y_{\min} \sum_{i=1}^{\ell_1} \bar{h}_i \leq v \leq y_{\min} \sum_{i=1}^{\ell_2} \bar{h}_i$.

Theorem 5.1 enables us to reduce (45) to an optimization problem with three variables regardless of the dimension of $\bar{y}$. Using the result, we transform the middle-level problem into the following form:

\[
\min_{\ell_1, \ell_2 \in \mathbb{N}, \gamma \in \mathbb{R}_{\geq 0}} \Gamma_n \gamma^\frac{1}{2} - \Gamma_\pi(\ell_1, \ell_2, v)
\]
\[
\text{s.t. } (46b) \text{ and } (46c),
\]

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Proof. Recall that we can compute Proposition 5.5. Algorithm 3 runs in \( O(n) \) time. We summarize the result on the computational complexity of Algorithm 3.

\[ \Gamma_n^*(\ell_1, \ell_2, v) := \ell_1 y_{\min} + (\ell_2 - \ell_1) \left( \frac{v - y_{\min} \sum_{i=1}^{\ell_1} h_i}{\sum_{i=1}^{\ell_2 - \ell_1} h_i} \right)^{\frac{1}{3}}. \]

5.2 Algorithms

According to the above result, we provide the algorithm for optimal lottery design with a fixed ticket price. We first give a naive algorithm and then improve it to run faster (in which we solve a simple 1-dimensional optimization problem for quadratic time). Moreover, for the case \( \alpha \geq \beta \), we give a linear time algorithm that ensures the optimal solution. The results of numerical experiments can be seen in Section 6.

Naive algorithm To obtain the optimal solution of (47), we solve (47) with respect to \( v \) for every fixed \( \ell_1, \ell_2 \in \mathbb{N}^2 \) such that \( \ell_1 < \ell_2 \leq \pi \). By simple transformations, this problem is reduced to a 1-dimensional problem

\[ \min_{\kappa \in [0,1]} (A + \kappa)^{\frac{1}{3}} - B\kappa^{\frac{1}{3}}, \tag{48} \]

where \( A \) and \( B \) are positive constants depending on \( \ell_1 \) and \( \ell_2 \); see Section 5.3 for their explicit forms and derivation of them. We can prove that this problem has only three candidates as a global optimum: the two endpoints of the constraints (i.e., \( \kappa = 0, 1 \)) and the local minimum obtained by the first derivative test. Hence, we can compute the optimal solution of (48) by using gradient methods, such as the Newton method. Since there are at most \( O(\pi^2) \) values of \( (\ell_1, \ell_2) \in \mathbb{N}^2 \) to consider, we numerically obtain a global optimum of (47) by solving \( O(\pi^2) \) instances of (48). In addition, by considering every possible \( n \in [N] \), we can find the optimal lottery with a fixed price by solving \( O(N^3) \) instances of (48) in total. Note that we can compute the optimal \( \bar{y} \) from the optimal solution of (47); see Proposition 5.2 for the detail.

Improved algorithm Similarly to the analysis of the improved algorithm in Section 4.3, we can conclude that there are no zero components in the optimal solution unless the optimal value is zero. Therefore, it suffices to check the case \( \ell_2 = \pi \), which reduces the computational cost. Formally, we obtain the following proposition.

Proposition 5.3. The optimal solution of the lottery optimization with a fixed ticket price can be obtained by solving \( O(N^2) \) instances of (48).

Improved algorithm \( (\alpha \geq \beta) \) The bottleneck of the above algorithm is the linear search for \( \ell_1 \) and solving (47). In the case \( \alpha \geq \beta \), we only need to consider the case \( \ell_i = y_{\min} \) for any \( i \in [\pi] \) by following the above proposition.

Proposition 5.4. Let \( (\pi^*, n^*) \) be a global optimum solution of (3) under the fixed price constraint and suppose that the optimal value \( P^* \) is positive. Then, if \( \alpha \geq \beta \) holds, for every \( (\omega_1^*, \ldots, \omega_\pi^*, \omega_1^*, \ldots, \omega_n^*) \) that is a global optimum of (4) with the additional constraint \( \omega_1 \geq \omega_{\min} \) under \( (\pi, n) = (\pi^*, n^*) \), we have \( \omega_1^* = \omega_{\min} \) for any \( i \in [\pi^*] \).

The proof can be seen in Section 5.3. We note that \( \omega_1^* = \omega_{\min} \) implies \( \ell_i = y_{\min} \). This fact provides us a linear time algorithm to obtain the optimal solution. We write the algorithm in Algorithm 3. The notable points are line 7 and line 8, in which we compute \( v \) and \( P(\pi, n) \) by letting \( \ell_i = y_{\min} \) for any \( i \in [\pi] \). In this algorithm, we no longer need to compute the optimal value of (45). In line 7, we use the equality

\[ v = \sum_{i=1}^{\pi} h_i y_{\min} = \sum_{i=1}^{\pi} \left[ W\left( \frac{i}{N} \right) - W\left( \frac{i-1}{N} \right) \right] y_{\min} = W\left( \frac{\pi}{N} \right) y_{\min}. \]

We summarize the result on the computational complexity of the Algorithm 3.

Proposition 5.5. Algorithm 3 runs in \( O(N) \) time.

Proof. Recall that we can compute \( (J_1, \Gamma_1), \ldots, (J_N, \Gamma_N) \) and \( \Gamma_{1,1}, \ldots, \Gamma_{N,N} \) in \( O(N) \) time. Since for each \( \pi \in [N] \), the computational time of \( v \) and \( P(\pi, n) \) are \( O(1) \), Algorithm 3 runs in \( O(N) \) time.
5.3 Proof of Proposition 5.4

We remind the optimization problem related to \( v \) and \( \ell_1, \ell_2 \) introduced in Section 5:

\[
\begin{align*}
\min_{\ell_1, \ell_2 \in \mathbb{N} \cup \{0\}} & \quad -\ell_1 y_{\min}^\beta - (\ell_2 - \ell_1) \left( \frac{v - y_{\min} \sum_{i=1}^{\ell_1} \bar{h}_i}{\sum_{i=1}^{\ell_2} \bar{h}_i} \right)^{\frac{1}{\beta}} & (46a) \\
\text{s.t.} & \quad y_{\min} \sum_{i=1}^{\ell_1} \bar{h}_i \leq v \leq y_{\min} \sum_{i=1}^{\ell_2} \bar{h}_i, & (46b) \\
& \quad \ell_1 < \ell_2 \leq \pi. & (46c)
\end{align*}
\]

and

\[
\begin{align*}
\min_{\ell_1, \ell_2 \in \mathbb{N}, v \in \mathbb{R}_{\geq 0}} & \quad \Gamma_n v^\frac{1}{\beta} - \Gamma_n^\alpha (\ell_1, \ell_2, v) & (47) \\
\text{s.t.} & \quad (46b) \text{ and } (46c).
\end{align*}
\]

We consider the optimal solution of \( v \) under fixed \( \ell_1 \) and \( \ell_2 \) satisfying the constraint (46c). Let

\[
g_{\ell_1, \ell_2}(v) := \Gamma_n v^\frac{1}{\beta} - \Gamma_n^\alpha (\ell_1, \ell_2, v) = \Gamma_n v^\frac{1}{\beta} - (\ell_2 - \ell_1) \left( \frac{v - y_{\min} \sum_{i=1}^{\ell_1} \bar{h}_i}{\sum_{i=1}^{\ell_2} \bar{h}_i} \right)^{\frac{1}{\beta}} - \ell_1 y_{\min}^\beta,
\]

and consider the optimization problem

\[
\begin{align*}
\min_{v \in \mathbb{R}_{\geq 0}} & \quad g_{\ell_1, \ell_2}(v) \\
\text{s.t.} & \quad (46b).
\end{align*}
\]

Lemma 5.6. Suppose that \( \alpha \geq \beta \). Then, for any \( \ell_1, \ell_2 \) satisfying (46c), the optimal solution of (50) is at a boundary of the interval (46b), i.e., the optimal solution \( v^\star \) satisfies

\[
v^\star \in \left\{ y_{\min} \sum_{i=1}^{\ell_1} \bar{h}_i, y_{\min} \sum_{i=1}^{\ell_2} \bar{h}_i \right\},
\]

(51)

Proof. By letting \( \kappa := \frac{v - y_{\min} \sum_{i=1}^{\ell_1} \bar{h}_i}{y_{\min} \sum_{i=1}^{\ell_2} \bar{h}_i} \), we have

\[
g_{\ell_1, \ell_2}(v) = \Gamma_n y_{\min}^{\frac{1}{\beta}} \left( \sum_{i=1}^{\ell_1} \bar{h}_i + \kappa \sum_{i=1}^{\ell_2} \bar{h}_i \right)^{\frac{\alpha}{\beta}} - (\ell_2 - \ell_1) y_{\min}^{\frac{\alpha}{\beta}} \kappa^{\beta} - \ell_1 y_{\min}^\beta,
\]

where we use the notation \( \tilde{\alpha} = 1/\alpha \) and \( \tilde{\beta} = 1/\beta \). By ignoring the constant factor \(-\ell_1 y_{\min}^\beta\) and dividing \( g_{\ell_1, \ell_2}(v) \) by \( \Gamma_n y_{\min}^{\frac{1}{\beta}} \left( \sum_{i=1}^{\ell_2} \bar{h}_i \right)^{\tilde{\alpha}} \), we obtain the equivalent problem to (50) as follows:

\[
\begin{align*}
\min_{\kappa \in \mathbb{R}} & \quad \tilde{g}_{\ell_1, \ell_2}(\kappa) := \left( \frac{\sum_{i=1}^{\ell_1} \bar{h}_i}{\sum_{i=1}^{\ell_2} \bar{h}_i} + \kappa \right)^{\tilde{\alpha}} - \frac{(\ell_2 - \ell_1) y_{\min}^{\tilde{\alpha} - \beta}}{\Gamma_n \left( \sum_{i=1}^{\ell_2} \bar{h}_i \right)^{\tilde{\alpha}}} \kappa^{\tilde{\beta}} & (52) \\
\text{s.t.} & \quad 0 \leq \kappa \leq 1.
\end{align*}
\]

Then, we first show that \( \tilde{g}_{\ell_1, \ell_2}(\kappa) = 0 \) for \( 0 < \kappa < 1 \) implies \( \tilde{g}_{\ell_1, \ell_2}(\kappa) < 0 \). To this end, we simplify \( \tilde{g}_{\ell_1, \ell_2}(\kappa) \) by denoting

\[
\begin{align*}
\tilde{g}_{\ell_1, \ell_2}(\kappa) := (A + \kappa)^{\tilde{\alpha}} - B \kappa^{\tilde{\beta}},
\end{align*}
\]

where

\[
A := \frac{\sum_{i=1}^{\ell_1} \bar{h}_i}{\sum_{i=1}^{\ell_2} \bar{h}_i} \quad \text{and} \quad B := \frac{(\ell_2 - \ell_1) y_{\min}^{\tilde{\alpha} - \beta}}{\Gamma_n \left( \sum_{i=1}^{\ell_2} \bar{h}_i \right)^{\tilde{\alpha}}}.
\]
Algorithm 3: Optimal design of the lottery with the fixed price constraint

1. **Input:** Number of lottery tickets $N$, CPT parameters $\alpha, \beta, \lambda, W$ and $W$, fixed price $-\bar{\omega}_{\min}$.
2. **Output:** $P^*, n^*, \pi^*, \omega^*_1, \ldots, \omega^*_n, \bar{\omega}^*_0, \ldots, \bar{\omega}^*_{\pi^*}$.
3. Compute $J_j, \Gamma_j$ for all $j \in [N]$ by Algorithm 2.
4. Set $P^* \leftarrow 0$ and $(n^*, \pi^*) \leftarrow (N, 0)$.
5. for $n \leftarrow 1$ to $N - 1$ do
   6. Set $\pi \leftarrow N - n$;
   7. $v \leftarrow W(\pi/N)y_{\min}$;
   8. $P(\pi, n) \leftarrow \Gamma_n v^{\frac{1}{2}} - \pi(y_{\min}/\lambda)^{\frac{1}{2}}$;
   9. if $P(\pi, n) < P^*$ then
      10. $P^* \leftarrow P(\pi, n)$ and $(\pi^*, n^*) \leftarrow (\pi, n)$;
   11. Compute $y^* \in \mathbb{R}^n$ by (9) and $\bar{y}^* \in \mathbb{R}^{p^*}$ by (26);
   12. $\omega_j \leftarrow (y^*_j)^{\frac{1}{2}} (\forall j \in [n^*])$;
   13. $\bar{\omega}_i^* \leftarrow -(y_{\min}/\lambda)^{\frac{1}{2}} (\forall i \in [\pi^*])$;

We note that $A, B > 0$ by their definition. A straightforward calculation shows that

\[ \bar{g}'_{\ell_1, \ell_2}(\kappa) = \hat{\alpha}(A + \kappa)^{\frac{a}{1}} - \hat{\beta}B\kappa^{\beta - 1}, \]
\[ \bar{g}''_{\ell_1, \ell_2}(\kappa) = \hat{\alpha}(\hat{\alpha} - 1)(A + \kappa)^{\frac{a}{2}} - \hat{\beta}(\hat{\beta} - 1)B\kappa^{\beta - 2}. \]

Suppose that $\bar{g}'_{\ell_1, \ell_2}(\kappa) = 0$. Then, we have $(A + \kappa)^{\frac{a}{1}} = \hat{\beta}B\kappa^{\beta - 1}$, and hence

\[ (A + \kappa)\bar{g}''_{\ell_1, \ell_2}(\kappa) = \hat{\alpha}(\hat{\alpha} - 1)(A + \kappa)^{\frac{a}{2}} - \hat{\beta}(\hat{\beta} - 1)B\kappa^{\beta - 2}(A + \kappa) \]
\[ = \hat{\alpha}(\hat{\alpha} - 1)\frac{\hat{\beta}}{\alpha}B\kappa^{\beta - 1} - \hat{\beta}(\hat{\beta} - 1)B\kappa^{\beta - 2}(A + \kappa) \]
\[ = \hat{\beta}B\kappa^{\beta - 2}
\[ (\alpha - \hat{\beta})\kappa - (\hat{\beta} - 1)A < 0, \]

where the last inequality follows from $\hat{\alpha} \leq \hat{\beta}$ (since $\alpha \geq \beta$ and $\hat{\beta} > 1$. This inequality and $A + \kappa > A + 0$ give $\bar{g}''_{\ell_1, \ell_2}(\kappa) < 0$ when $\bar{g}'_{\ell_1, \ell_2}(\kappa) = 0$.

We prove the lemma by contradiction. Suppose that the optimal value $v^*$ is not $y_{\min} \sum_{i=1}^{\ell_1} h_i$ nor $y_{\min} \sum_{i=1}^{\ell_2} h_i$. Then, the optimal solution of (52), denoted by $\kappa^*$, satisfies $0 < \kappa^* < 1$. Since $\bar{g}'_{\ell_1, \ell_2}$ is continuously differential in $(0, 1)$, it must hold that $\bar{g}'_{\ell_1, \ell_2}(\kappa^*) = 0$. Nevertheless, this implies $\bar{g}''_{\ell_1, \ell_2}(\kappa^*) < 0$, which means $\bar{g}_{\ell_1, \ell_2}(\kappa^*)$ is a local maximum. This is the contradiction.

**Proof of Proposition 5.4.** By Lemma 5.6, for the global optimum of (4), the corresponding $v, \ell_1$ and $\ell_2$ satisfy the relationship (51). This fact and the definition of $v, \ell_1$ and $\ell_2$ (see Proposition 5.2) imply that $\bar{y}^*_i \in \{y_{\min}, 0\}$ for any $i \in [\pi^*]$. Moreover, by using the same argument as the proof of Proposition 4.13, we have $\bar{\omega}^*_i \neq 0$ ($\iff \bar{y}^*_i \neq 0$) for any $i \in [\pi^*]$. Thus we obtain that $\bar{y}^*_i = y_{\min}$ for any $i \in [\pi^*]$, which gives the conclusion.

6 Experiments

We construct concrete optimal lotteries with the parameters of Canada and the United States estimated in [18], where the value function is (1) and the probability weighting functions for gain and loss are (2), which we use in the theoretical analysis. We confirm that our algorithms output the optimal solution within practical time. The experiments related to the unconstrained cases and the price-constrained case ($\alpha \geq \beta$) are implemented in C++, and that of the
other price-constrained case is implemented in Matlab_R2022a. All the experiments are executed in MacBook Pro 2020 with Apple M1 tip, 8-core CPU, and 16.0 GB memory.

**Canada setting** We conduct a numerical experiment with \( N = 10^9 \) and the CPT parameters \((\alpha, \beta, \lambda, \gamma, \bar{\gamma}) = (0.42, 0.83, 1.62, 0.44, 0.60)\) estimated in [18]. Table 2 describes the distribution of the optimal prize. The computation time is 1648s. The optimal ticket price is $2.30 and the maximum prize is $1.36\times10^8$. The ratio of the gain to the total number of tickets \((n/N)\) is 31.83% and the seller’s profit is $7.76\times10^6$. It shows that in the optimal design, the lottery consists of many small winnings and a few large ones. We can say that this result is in line with real lotteries, which implies the effectiveness of our algorithm.

| Prize Number | Odds          |
|--------------|---------------|
| \(10^8 - 10^9\) | 1 in 1,000,000,000,000 |
| \(10^7 - 10^8\) | 2 in 500,000,000,000 |
| \(10^6 - 10^7\) | 34 in 29,411,764.71 |
| \(10^5 - 10^6\) | 366 in 2,732,240.44 |
| \(10^4 - 10^5\) | 3,872 in 258,264.46 |
| \(10^3 - 10^4\) | 39,432 in 25,360.11 |
| \(10^2 - 10^3\) | 368,846 in 2711.16 |
| \(10 - 10^2\) | 3,398,560 in 294.24 |
| \(>0 - 10\) | 314,492,608 in 3.18 |
| \(0\) | 681,696,279 in 1.47 |

**The United States setting: unconstrained case** We conduct a numerical experiment with \( N = 10^8 \) and the CPT parameters \((\alpha, \beta, \lambda, \gamma, \bar{\gamma}) = (0.42, 0.49, 1.36, 0.44, 0.71)\) estimated in [18]. Table 3 describes the distribution of the optimal prize. The computation time is 79 seconds. The ticket price is $3.53\times10^{39}$ and the maximum prize is $3.95\times10^{39}$. The seller’s profit is $5.05\times10^{38}$. We can see that optimal design consists of almost all the tickets that return gains and the only one enormous loss, which seems unrealistic. One possible cause is insensitivity of large loss in the United States setting; since \(\beta\) of the United States is 0.49 and smaller than that of Canada \((\beta = 0.83)\), we can design the large loss without decreasing the expected utility much. Such immense prices can be seen in other settings where \(\beta\) is relatively small. This phenomenon represents a limitation of the original formulation, and thus we consider an alternative one with an additional constraint on a ticket price.

| Prize Number | Odds          |                   |
|--------------|---------------|-------------------|
| \(10^{39} - 10^{40}\) | 999,999,999 | 1 in 10000000000 |
| \(0\) | 1 | 1 in 1000000000 |

**The United States setting: price-constrained case** To overcome the problems seen in the previous experiment, we add the constraint on a ticket price to the optimal design problem, as Section 5. We conduct a numerical experiment with \( N = 1000, \bar{\omega}_{\text{min}} = 2 \), and the same CPT parameters as the unconstrained case. Table 4 shows the optimal design of the lotteries. The computation time is 819 seconds. The ticket price is $2, which coincides with the fixed-price constraint we imposed. The number of prizes is 576, the maximum prize is $312.41, and the seller’s profit is $339.80. Hence, the additional constraint results in the effective lottery design; the output structure is similar to the Canada setting.
Table 4: The optimal lottery of the United States under price constraints: The ticket price is $2.

| Prize     | Number | Odds          |
|-----------|--------|---------------|
| $10^{2}$ – $10^{3}$ | 1      | 1 in 1000.00  |
| $10^{1}$ – $10^{2}$  | 5      | 1 in 200.00   |
| >0 – 10    | 570    | 1 in 1.75     |
| 0          | 424    | 1 in 2.36     |

The Greece setting: price-constrained case ($\alpha \geq \beta$) Finally, we conduct a numerical experiment with $N = 10^9$ and the CPT parameters $(\alpha, \beta, \lambda, \gamma, \bar{\gamma}) = (0.50, 0.30, 1.29, 0.44, 0.82)$ estimated in [18] under the price constraint $\bar{\omega}_{\text{min}} = 2$. In this case, since $\alpha \geq \beta$, the linear time algorithm (Algorithm 3) can be applied here. Table 5 describes the distribution of the optimal prize. The computation time is 584 seconds. The maximum prize is $4.11 \times 10^7$. The ratio of the gain to the total number of tickets $(n/N)$ is 0.078% and the sellers profit is $1.91 \times 10^9$.

Table 5: The optimal lottery of Greece: the overall odds of gain are 1 in 1.01. The ticket price is $2.00.

| Prize     | Number | Odds          |
|-----------|--------|---------------|
| $10^7$ – $10^8$   | 1      | 1 in 1.000,000,000.00 |
| $10^6$ – $10^7$   | 5      | 1 in 200,000,000.00 |
| $10^5$ – $10^6$   | 44     | 1 in 22,727,272.73 |
| $10^4$ – $10^5$   | 334    | 1 in 2,994,011.98 |
| $10^3$ – $10^4$   | 2577   | 1 in 388,048.12 |
| $10^2$ – $10^3$   | 20,056 | 1 in 49,860.39 |
| 0 – 10           | 239,920| 1 in 4168.06 |
| >0 – 10          | 518,488| 1 in 1928.68 |
| 0                | 999,218,575| 1 in 1.01   |

7 Conclusion and discussion

In this study, we provided a linear-time algorithm to compute the optimal lottery in the CPT framework. For the algorithm construction, we formulated the lottery optimization as a three-level optimization problem and characterized its optimal solution by analyzing the subproblems. This study is the first to employ CPT for lottery design problems without model simplification. We also examined an additional constraint on the ticket price. Under this constraint, we provided a quadratic time (solving $1$-dimensional optimization problem) algorithm for the general case and a linear time algorithm for a specific case.

Finally, we discuss possible extensions and future work. If we require that the expected utility of each buyer is at least $\varepsilon$ ($>0$), that is, the right-hand side of (4b) is $\varepsilon$, the optimal value of (4) is given by $\min_{v \in \mathbb{R}_{\geq 0}} \Gamma_n \cdot (v + \varepsilon)^{1/\alpha} - \Gamma_{\pi} \cdot v^{1/\beta}$. Therefore, we can easily compute the optimal lottery in this case as well. Moreover, we can solve the maximization of the buyers’ expected utilities while ensuring that the seller’s profit is more than a certain amount by conducting a binary search with $\varepsilon$. For the case when the value function is not represented as (1), we can also explicitly solve low-level problems. However, solving middle-level problem is not straightforward because it is not possible to separate $v$ as in (32). We leave this case for future work. In addition, future work includes examining cases where individual buyers have different expected utilities and proposing faster algorithms to the model with a fixed ticket price.

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