Representation spaces for the membrane matrix model

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Abstract. The $SU(N)$-invariant matrix model potential is written as a sum of squares with only four frequencies (whose multiplicities and simple $N$-dependence are calculated).

Difficult problems, unless one is willing to give up on them, should be viewed from different perspectives. For the membrane matrix model\footnote{[4]} a Lax-pair was recently given in [1], a set of 'dual' variables, in which the Hamiltonian is of the form $\frac{1}{2}(\vec{p}^2 + \vec{q}^2)$, introduced in [2], and an $r$-matrix derived in [3]. Here I would like to point out that if one introduces real symmetric matrices $Y = (Y_{ab})_{a,b=1...N^2-1=n} = (\vec{x}_a \cdot \vec{x}_b)$ as variables, the potential

$$- \sum_{i,j=1}^{d} \text{Tr}[X_i, X_j]^2 = f_{abc}f_{ade}x_{ib}x_{jc}x_{id}x_{je}$$

(1)

$$= \text{Tr}(YF(Y))$$

$$= W(Y)$$

becomes a diagonalizable quadratic form in $Y$, as the map $Y \rightarrow F(Y) = -\sum_a F_a Y F_a$

$$(F(Y))_{ab} = \text{Tr}(F_a Y F_b)$$

(2)

$$(F_a)_{bc} = -f_{abc}$$

is symmetric. When trying to calculate the eigenvalues of $F$, I noticed that for each pair $(ab)$ the 4-dimensional subspace spanned by the symmetric $n \times n$ matrices (for the definition of the invariant $d$-tensor and many useful identities, see [6])

$$E_{ab} := \delta_{ab}1_{n \times n}$$

$$\Delta_{ab}^{cd} := 2\delta_{ab}\delta_{cd} - \delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}$$

$$H_{ab} := d_{abc}D_c, \ (D_c)_{eg} := d_{ceg}$$

$$Z_{ab} := D_a D_b + D_b D_a$$

\footnote{[3]}(see also [5])
is left invariant by \( F \), and diagonalization of the simple \( 4 \times 4 \) matrix

\[
\begin{pmatrix}
\frac{2}{N} & \frac{4}{N^2} & 1 & 0 \\
1 & \frac{4}{N} & \frac{2}{N} & 0 \\
\frac{N^2-8}{2N} & -2 & \frac{N}{2} & 0 \\
0 & 2N & 0 & N
\end{pmatrix}
\]

(4)

gives, apart from the immediate

\[
F(E_{ab}) = NE_{ab}, \quad F(H_{ab}) = \frac{N}{2}H_{ab}
\]

(5)

the eigenmatrices (with eigenvalue \( \mp 1 \))

\[
K_{ab} := Z_{ab} - \frac{N+2}{N}\Delta_{ab} - \frac{N+4}{N+2}H_{ab} + 2\frac{N+2}{N+1}E_{ab}
\]

\[
M_{ab}^{(N \geq 3)} := Z_{ab} + \frac{N-2}{N}\Delta_{ab} - \frac{N-4}{N-2}H_{ab} - 2\frac{N-2}{N-1}E_{ab},
\]

(6)

with the understanding that \( Z_{ab} \) and \( H_{ab} \) are (put to) zero when \( N = 2 \).

It is also not difficult to see that

\[
M_{ab}^{(N=3)} = 0, \quad \sum_a M_{aa} = 0 = \sum_a K_{aa}, \quad d_{abc}M_{ab} = 0 = d_{abc}K_{ab},
\]

(7)

and to determine for small \( N \) the number of independent matrices of \( (E, H, M, K) \)-type, namely \( (1,0,5,0) \) for \( N = 2 \), \( (1,8,27,0) \) for \( N = 3 \), and \( (1,15,84,20) \) for \( N = 4 \) (in each of these cases together spanning the \( \frac{n(n+1)}{2} \)-dimensional space of symmetric \( n \times n \) matrices).

As the eigenspaces with different eigenvalues can not mix (as the map commutes with the group/algebra–action) it is immediate that they correspond to representation spaces under the action of \( SU(N) \), and as there are (for \( N \geq 3 \)) precisely 4 (for \( N = 3 \) only 3) such irreducible spaces occurring in the symmetric part of the tensor product of 2 adjoint–representations \((1,0\ldots0,1)\) of \( A_l \cong SU(l+1) \), for \( l > 3 \): \((10\ldots01) \times (10\ldots01)\), \((0\ldots0) \oplus (10\ldots01) \oplus (2,0,\ldots,0,2) \oplus (0,1,0\ldots0,1,0)\), these must precisely be the spaces spanned by the symmetric matrices of \( (E, H, K, M) \)-type to which we will, apart from using the above–mentioned Dynkin–labels, refer to as the \( E, H, K, \) resp. \( M \)–representations. While the dimension of the \( H \) (= adjoint)–representation is of course \( n = N^2 - 1 \) and that of \( E \) trivially = 1, the dimension of the \( K \) (= \( (2,0,\ldots,0,2) \)) and \( M \) (= \( (0,1,0\ldots0,1,0) \)) representations is slightly less trivial (though of course known; elementary
derivations are given in the appendix):

$$\dim(\mathbb{K}) = \frac{N^2(N - 1)(N + 3)}{4}$$

$$\dim(\mathbb{M}) = \frac{N^2(N + 1)(N - 3)}{4}$$

(giving indeed $\frac{N^2(N^2 - 1)}{2}$ for the total dimension of the space of symmetric $n \times n$ matrices).

$$Y = y_0 W_0 + \bar{y}_H \bar{W}_H + \bar{y}_K \bar{W}_K + \bar{y}_M \bar{W}_M,$$

with the $W$’s orthonormal bases for the respective irreducible representation–spaces, then gives

$$W(Y) = Ny_0^2 + \frac{N}{2} \bar{y}_H^2 - \bar{y}_K^2 + \bar{y}_M^2.$$  

The at first surprising $-$ sign ($W(Y) = -\text{Tr}[X_i, X_j]^2 \geq 0$ for traceless hermitean $N \times N$ matrices $X_i$) brings one to the important issue that the $Y$’s in (11) are not arbitrary (symmetric) matrices; they are (as $= QQ^T$) positive–semidefinite, and in fact, if $n > d$, necessarily of smaller than general rank.

As an example, consider the $d = 2$, $N = 3$ matrix model; then the singular value decomposition gives

$$Y = \lambda_1 \bar{u}_1 \bar{u}_1^T + \lambda_2 \bar{u}_2 \bar{u}_2^T$$

where $\bar{u}_1$ and $\bar{u}_2$ are orthonormal vectors in $\mathbb{R}^8$, and $\lambda_1 \geq \lambda_2 \geq 0$ the 2 eigenvalues of $Y$; so containing only $7 + 6 + 2 = 15$ parameters.

Nevertheless (9), which is of a tantalizing simple form, should be useful.

What about $N \to \infty$? Despite of the, simple $N$ dependence of the frequencies (and multiplicities; naively one should think that it is easy to see which modes are the most important ones as $N \to \infty$; note that when summing their products the leading power of $N$ cancels), and (6) converging to well–defined expressions, the $N \to \infty$ limit seems difficult, for a variety of reasons. As indicated already by (7), and clear from general considerations, the 4 invariant subspaces $\mathbb{E}, \mathbb{H}, \mathbb{K}$ and $\mathbb{M}$ should most conveniently be discussed by corresponding projectors $P_{\alpha=1,2,3,4}$ (resp. $\alpha = E, H, K, M$), forming a partition of the identity, with e.g.

$$P_H = \frac{N}{N^2 - 4} d_{abc} d_{a'b'c'} = P_H^2, \quad F(P_H Y) = \frac{N}{2}(P_H Y).$$

Using various $SU(N)$–identities, in particular (cp. [7])

$$f_{abc} f_{cde} = \frac{2}{N} (\delta_ac \delta_bd - \delta_ad \delta_bc) + (d_{ace} d_{bde} - d_{ade} d_{bce})$$

\[2\text{many thanks to R. Suter for a related discussion}\]
the corresponding projectors $P_K$ and $P_M$ are not difficult to work out, and in fact (I noticed that after finding (6)) have been worked out, in the context of QCD [8]. The projectors however do not converge as $N \to \infty$. Moreover, the following general problem exists: while there do exist bases of $SU(N)$ in which the structure constants $f_{abc}$ converge (to those, $g_{abc}$, of the Lie–algebra of area preserving diffeomorphisms; (the fuzzy sphere [5] was invented in precisely this context), and $d_{abc}^{(N)} \to d_{abc}^{\infty}$ too, as well as the central object $f_{abc} f_{ade}$ (sum over $a$, cp. (11)) converging to $g_{abc} g_{ade}$, similarly (cp. (11)) $d_{abc} d_{a'b'e}$ (the sum over $e$ is finite for fixed $ab, a'b'$) their action on the 4 subspaces, resp. projectors, involves multiple sums where the range of the indices is not finite. Another aspect of the arising subtleties, and difficulties, can be demonstrated by looking at (12). As explained e.g. in [9], the normalisation for the $f$’s and $d$’s suitable to take the limit is such that (12) becomes

$$\frac{1}{N^2} f_{abc} f_{cde} = 2(\delta_{ac} \delta_{bd} - \delta_{ad} \delta_{bc}) + (d_{ace} d_{bde} - d_{ade} d_{bce}).$$

Indeed, with $\tilde{d}_{ace} = \int Y_a(\varphi) Y_c(\varphi) Y_e(\varphi) \rho \, d^2 \varphi$, the $Y_a(\varphi)$ being orthonormal eigenfunctions of the Laplacian on the parameter–surface, the rhs. is zero for $N = \infty$. Vice versa this however shows that if decomposing the relevant operator, $\tilde{f}_{abc} \tilde{f}_{cde}$ in the normalisation where $\tilde{f}_{abc}$ is finite (and the sum over $e$ as well) decomposing it with respect to $f$’s and $d$’s, which effectively is done in [8] (for finite $N$), involves (for infinite $N$) a finite part of $\infty \cdot 0$. Let me at this point go back to how I came to consider the matrices given in (3). The adjoint action (of the $a$–th generator of $SU(N)$) on the symmetric $n \times n$ matrix $Y$ (the symmetric part of the tensor–product of two copies of the Lie–algebra) is, possibly up to an overall sign, commutation with the matrix $F_a$ (cp. (2)), i.e. $[F_a, Y]$ (and the map $F$ commutes with the $SU(N)$ action: $- [F_a, F(Y)] = [F_a, F_c Y F_e] = [F_a, F_c] Y F_e + F_c Y [F_a, F_e] + F_e [F_a, Y] F_c = \pm f_{abc} (F_c Y F_e + F_e Y F_c) + F_e [F_a, Y] F_c = - F([F_a, Y])$). Due to $[F_a, D_b]$ being (again, not worrying about the signs in this qualitative argument) $f_{abc} D_c$, the $n$ dimensional subspace consisting of linear combinations of the $D$’s is clearly invariant (giving the $(1 0 \ldots 0 1)$, resp. $H$–space). This being so easy, the obvious next step was to consider $D_a D_b$ ($+ D_b D_a$, to get symmetric matrices), i.e. $Z_{ab}$. Calculating $F(Z_{ab})$, which involves $\Delta_{ab}$ then led to (3), resp. (4–7). If on the other hand one wants (‘only’) to understand the representation theory, it is natural to look for identities involving the $Z_{ab}$ (which can not be linearly independent when taken together with the first order polynomials in
the $D$’s, as too many), and there one finds that

\begin{equation}
\label{eq:14}
d_{abc}Z_{bc} = 2d_{abc}D_bD_c = D_a \frac{N^2 - 12}{N}
\end{equation}

(which then is already most of the final answer). Unfortunately the scaling, $\tilde{d}_{abc}^{(N)} = \sqrt{N}d_{abc}$, (cp.e.g.\cite{17}), that is known to converge to the totally symmetric tensor,

\begin{equation}
\label{eq:15}
\tilde{d}_{abc} = \tilde{d}_{abc} = \int Y_a(\varphi)Y_b(\varphi)Y_c(\varphi)\rho \, d^2\varphi := h_{abc}
\end{equation}

for functions on $\sum$ (compact orientable, surface of genus $g$) does not cancel (actually: enhances) the diverging factor on the rhs. of (14), and while the naive analogue of the $Z_{ab}$,

\begin{equation}
\label{eq:16}
(\tilde{Z}_{ab})_{cd} = (\tilde{D}_a\tilde{D}_b + \tilde{D}_b\tilde{D}_a)_{cd} = (\tilde{d}_{ace}\tilde{d}_{bde} + a \leftrightarrow b)
\end{equation}

is well–defined,

\begin{equation}
\label{eq:17}
(\tilde{d}_{abc}\tilde{Z}_{bc})_{fg} = \tilde{d}_{abc}(\tilde{d}_{fde}\tilde{d}_{cge} + b \leftrightarrow c)
\end{equation}

is not (seen by inserting (15) resp. indicated by the triple sum over $bce$ in (17), involving truly infinite sums). Similarly, concerning the decomposition of adjoint$\otimes$adjoint for $\text{sdiff} \Sigma$: defining infinite matrices $G_\alpha$ and $H_\beta$ ($\alpha, \beta = 1 \ldots \infty$) by

\begin{equation}
\label{eq:18}
(G_\alpha)_{\beta\gamma} := -g_{\alpha\beta\gamma} \quad (H_\alpha)_{\beta\gamma} := h_{\alpha\beta\gamma}
\end{equation}

satisfying (note: no convergence–problems, as each row and column of the matrices $G_\alpha$ and $H_\beta$ has only a finite number of non–zero entries)

\begin{equation}
\label{eq:19}
[G_\alpha, G_\beta] = g_{\alpha\beta\gamma}G_\gamma, \quad [G_\alpha, H_\beta] = -g_{\alpha\beta\gamma}H_\gamma,
\end{equation}

$G(X) := -G_\alpha XG_\alpha$, resp. $(G(X))_{\alpha\beta} := \text{Tr}G_\alpha G_\beta X$, is formally $\text{sdiff}$–invariant, $-[G_\alpha, G(x)] = [G_\alpha, G_\varepsilon XG_\varepsilon] = -G([G_\alpha, X])$ and, due to (19), the subspace consisting of linear combinations of the $H_\gamma$ certainly corresponds to an adjoint representation, $G(H_\varepsilon) = \ldots = +\frac{1}{2}g_{\alpha\beta\varepsilon}g_{\alpha\beta\gamma}H_\gamma$ gives a diverging eigenvalue on that $H$–space (‘consistent’ with having gotten $\frac{N}{2}$ for finite $N$). My reason for, still, being fairly optimistic about understanding the $N \rightarrow \infty$ limit this way are twofold: firstly, pure mathematics (understanding $\text{sdiff}$, whose structure strongly depends on the genus, hence must be reflected by the representation theory); secondly: as for classical motions of given energy the potential is by default finite/for a (regular) minimal surface (without singularities) all local quantities are finite/the apparent divergencies one gets above may actually tell one how to proceed, i.e. which collective degrees of freedom the system chooses.
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Appendix

The $K$– and $M$– representations

The easiest way to calculate the dimensions, and see that for $SU(N)$ the symmetric part of the tensor product of 2 adjoints contains only 4 irreducible representations is

\[(A^i_j A^k_l)_s = (A^i_j A^k_l)_u = \frac{1}{2}(A^i_j + A^k_l)\]

where both $\hat{A}$ and $\tilde{A}$ are symmetric under $(\ell^j_i) \leftrightarrow (\ell^k_l)$, $\hat{A}$ is traceless with respect to any upper and lower index (while $A^i_j$ is traceless only with respect to $(\ell^i_j)$ and $(\ell^k_l)$), $\hat{A}^{ik}_{jl}$ is a linear combination of the $N^2$ quantities $A^{pq}_{nm} (= A^{qp}_{mn}$; $p, q = 1 \ldots N$), $\hat{A}^{(ik)}_{(jl)}$ is symmetric in the upper, and lower indices (i.e. taking into account the traceless–condition, giving rise to a \(\frac{N(N+1)}{2} - N^2 = \frac{N^2}{4}(N+3)(N-1)\) dimensional space, the \((2 \ldots 0 \ldots 2)\) representation $K$) while $\hat{A}^{[ik]}_{[jl]}$ is antisymmetric in the upper and lower indices (and traceless) giving rise to a \(\frac{N^2}{4} (N+1)(N-3)\) dimensional space, the \((0 \ldots 0 \ldots 10)\) representation $M$, which is part of the tensor product \((0 \ldots 0 \ldots 0) \times (0 \ldots 0 \ldots 0)\) of the 2 fundamental representations $\omega_2 = (0 \ldots 0 \ldots 0)$ (known to be realized on the exterior product of two defining representations, corresponding to $A^{[ij]}_s$) and $\omega_{N-2} = (0 \ldots 0 \ldots 10)$ (which by duality of the Dynkin–diagram corresponds to the $A^{[kl]}_s$ space), and the traceless–ness conditions making it irreducible, i.e. \((0 \ldots 0 \ldots 10)\); while the $N^2$ dimensional space of $\hat{A}^{ik}_{jl}$’s (traceless with respect to $(\ell^i_j)$ and $(\ell^k_l)$, but not $(\ell^i_j)$ and $(\ell^k_l)$) gives an $N^2 - 1$ dimensional adjoint, \((1 \ldots 0 \ldots 1)\), and a singlet. $N = 3$ (and $N = 4$) are slightly special, as for $N = 3$ (cp. [10])

\[\hat{A}^{[ik]}_{[jl]} = e^{ikp} \epsilon_{jql} \hat{A}^p_q,\]

while the traceless–ness condition then says that $\hat{A}^p_q$ must be $= 0$; for $N = 4$, the antisymmetric part of $\hat{A}$ gives the \((0 \ldots 20)\) representation, lying in \((0 \ldots 10) \times (0 \ldots 10)\), the first \((0 \ldots 10)\) viewed as $A^{[ik]}_s$, the exterior square of \((1 \ldots 0)\), the second \((0 \ldots 10)\) as $A^{[ij]}_s$, the exterior square of the \((0 \ldots 01)\) representation–space.

Apart from these simple considerations, one may also calculate the dimensions of the 2 non–trivial representations ($K$ and $M$) as follows: Weyl’s dimension formula (see e.g. [11]) says that if all the roots of a
(semi–)simple Lie–algebra have the same length (which is the case for $SU(N) \cong A_{N-1}$),

$$(A2) \quad \dim V_{\vec{n}} = \prod_{\alpha=\sum_{j=1}^{l} k_j \alpha_j \in \varphi_+} \frac{\sum_{i=1}^{l} k_i (n_i + 1)}{\sum_{i=1}^{l} k_i} = \prod_{\alpha} d_\alpha,$$

where $\vec{n} = (n_1, n_2, \ldots, n_l) \in \mathbb{N}_0^l$ classifies the finite dimensional irreducible representations, $\alpha_1, \ldots, \alpha_l$ are the simple roots ($\alpha_i = \varepsilon_i - \varepsilon_{i+1} = (0 \ldots 1 - 1 0 \ldots 0)$), and $\vec{k} = (k_1, \ldots, k_l) \in \mathbb{N}_0^l$ characterizes the different positive roots – which for $A_l$ are all of the form $\varepsilon_p - \varepsilon_q = (0 1 0 \ldots -1 0)$ where $1 \leq p < q \leq l + 1$.

For $\vec{n} = (2, 0, \ldots, 0, 2)$ the numerator of $d_\alpha$ will be equal to the denominator, $k_\alpha = \sum k_i$, resp. $k_\alpha + 2$ or $k_\alpha + 4$, depending on whether $\alpha$ contains neither $\alpha_1$ nor $\alpha_l$, contains $\alpha_1$ (but not $\alpha_l$), or $\alpha_l$ (but not $\alpha_1$), resp. containing both $\alpha_1$ and $\alpha_l$. As all positive roots are of the form

$$(A3) \quad \varepsilon_p - \varepsilon_q = \alpha_p + \alpha_{p+1} + \ldots + \alpha_{q-1},$$

the 4 factors (corresponding to the just mentioned 4 cases) are

- case 1: 1
- case 2 ($\varepsilon_1 - \varepsilon_q$): $\prod_{q=2}^{l} \frac{q+1}{q-1} = \frac{3 \cdot 4 \cdot \ldots \cdot (l+1)}{2 \cdot 1} = \frac{l(l+1)}{2} = \frac{N(N-1)}{2}$
- case 3 ($\varepsilon_p - \varepsilon_{l+1}$): $\prod_{p=2}^{l} \frac{p+1}{p-1} = \frac{N(N-1)}{2}$
- case 4 ($\varepsilon_1 - \varepsilon_{l+1}$): $\frac{\frac{l+4}{l}}{\frac{N+3}{N-1}}$, hence

$$(A4) \quad \dim V_{(2,0,\ldots,0,2)} = \frac{N^2(N-1)^2}{4} \frac{N + 3}{N - 1} = \frac{N^2}{4} (N - 1)(N + 3).$$

For $\vec{n} = (0, 1, 0 \ldots 0, 1, 0)$ there are, apart from $\alpha$’s containing neither $\alpha_2 = \varepsilon_2 - \varepsilon_3$ nor $\alpha_{l-1} = \varepsilon_{l-1} - \varepsilon_l$ (trivially contributing factors $d_\alpha = 1$), the following cases:
\[
\varepsilon_1 - \varepsilon_{q>2} = \alpha_1 + \alpha_2 + \ldots + \alpha_{q-1} : \prod_{q=3}^{l-1} \frac{q}{q-1} = \frac{3 \cdot 4 \cdots \cdot l - 1}{2 \cdot 3 \cdots \cdot l - 2} = \frac{(l - 1)}{2}
\]

\[
\varepsilon_2 - \varepsilon_{q>2} = \alpha_2 + \ldots + \alpha_{q-1} : \prod_{q=3}^{l-1} \frac{q - 1}{q - 2} = \frac{2 \cdot 3 \cdots \cdot l - 2}{1 \cdot 2 \cdots \cdot l - 3} = (l - 2)
\]

\[
\varepsilon_1 - \varepsilon_l = \alpha_1 + \alpha_2 + \ldots + \alpha_{l-1} : \frac{l + 1}{l - 1}
\]

\[
\varepsilon_1 - \varepsilon_{l+1} = \alpha_1 + \ldots + \alpha_l : \frac{l + 2}{l}
\]

\[
\varepsilon_2 - \varepsilon_l = \alpha_2 + \ldots + \alpha_{l-1} : \frac{l}{l - 2}
\]

\[
\varepsilon_2 - \varepsilon_{l+1} = \alpha_2 + \ldots + \alpha_l : \frac{l + 1}{l - 1}
\]

\[
\varepsilon_{p>2} - \varepsilon_l = \alpha_p + \ldots + \alpha_{l-1} : (l - 2) \left( = \prod_{p=3}^{l-1} \frac{l - p + 1}{l - p} \right)
\]

\[
\varepsilon_{p>2} - \varepsilon_{l+1} = \alpha_p + \ldots + \alpha_l : \frac{(l - 1)}{2} \left( = \prod_{p=3}^{l-1} \frac{l - p + 2}{l - p + 1} \right), \text{ hence}
\]

(A5)

(A6)

\[
\dim V_{(0,1,0,\ldots,0,1,0)} = \left( \frac{1}{2} (l - 1)(l - 2) \right)^2 \left( \frac{l + 1}{l - 1} \right)^2 \frac{l + 2}{l} \frac{l}{l - 2}
\]

\[
= \frac{1}{4} (l - 2)(l + 1)^2(l + 2) = \frac{N^2}{4} (N + 1)(N - 3).
\]

References

[1] J.Hoppe, arXiv:2101.01803
[2] J.Hoppe, arXiv:2101.04495
[3] J.Hoppe, arXiv:2101.11510
[4] J.Hoppe, Ph.D. thesis, MIT 1982 [http://dspace.mit.edu/handle/1721.1/15717]
[5] T.Banks, W.Fischler, D.Shenker, L.Susskind, Phys.Rev.D 55, 1997
[6] H.E.Haber, arXiv:1912.13302
[7] A.J.Macfarlane, A.Sudbery, P.H.Weisz, Com.Math.Phys.11, 1968
[8] P.Arnold, arXiv:1904.04264
[9] J.Hoppe, M.Trzetrzelewski, arXiv:1101.4403
[10] S.Coleman, Fun with SU(3), Seminar held in Trieste, 1965
[11] R.Carter, Lie Algebras of Finite and Affine Type, Cambridge studies in advanced mathematics 96, Cambridge University Press, 2005

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