On the $U(1)$ duality anomaly and the S-matrix of $\mathcal{N} = 4$ supergravity

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ABSTRACT: $\mathcal{N} = 4$ Poincaré supergravity has a global SU(1,1) duality symmetry that acts manifestly only on shell as it involves duality rotations of vector fields. A U(1) subgroup of this symmetry is known to be anomalous at the quantum level in the presence of a non-trivial gravitational background. We first derive this anomaly from a novel perspective, by relating it to a similar anomaly in conformal supergravity where SU(1,1) acts off shell, using the fact that $\mathcal{N} = 4$ Poincaré supergravity has a superconformal formulation. We explicitly construct the corresponding local and nonlocal anomalous terms in the one-loop effective action. We then study how this anomaly is reflected in the supergravity S-matrix. Calculating one-loop $\mathcal{N} = 4$ supergravity scattering amplitudes (with and without additional matter multiplets) using color/kinematics duality and the double-copy construction we find that a particular U(1) symmetry which was present in the tree-level amplitudes is broken at the quantum level. This breaking manifests itself in the appearance of new one-loop $\mathcal{N} = 4$ supergravity amplitudes that have non-vanishing soft-scalar limits (these amplitudes are absent in $\mathcal{N} > 4$ supergravities). We discuss the relation between these symmetry-violating amplitudes and the corresponding U(1) anomalous term in the one-loop supergravity effective action.

KEYWORDS: Supersymmetry and Duality, Extended Supersymmetry, Anomalies in Field and String Theories

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1 Introduction

Extended Poincaré supergravity theories and their ultraviolet behavior have been a subject of renewed interest in recent years. Although, in general, supersymmetry softens ultraviolet divergences, it cannot overcome the effects of the two-derivative stress tensor coupling
characteristic to gravitational theories. It has been suggested that, apart from supersymmetry, non-compact global symmetries \([1–3]\) of the equations of motion of \(\mathcal{N} \geq 4\) classical supergravity theories also play an important role in constraining the ultraviolet behavior of theories exhibiting them. These duality symmetries, involving electric/magnetic duality transformations of abelian vector fields, are specific to four-dimensional extended supergravity theories; their origin is related to dimensional reduction from higher-dimensional supergravities.

The largest duality group is that of \(\mathcal{N} = 8\) supergravity, \(E_{7(7)}\). Other supergravities can be obtained as consistent truncations from \(\mathcal{N} = 8\) supergravity and their duality groups may be thought of as subgroups of \(E_{7(7)}\). While the duality groups of \(\mathcal{N} > 4\) supergravities are expected to be preserved at the quantum level due to the absence of anomalies, an abelian subgroup of the classical \(SU(1,1)\) duality group of \(\mathcal{N} = 4\) supergravity was argued in \([4]\) to be anomalous.

Quite generally, one may say that an anomaly is present whenever a classical (global or local) symmetry is broken by quantum corrections. It may either be a symmetry of the Lagrangian or of the equations of motion or a symmetry of scattering amplitudes (i.e. a symmetry realized on asymptotic states). It would undoubtedly be interesting to identify the consequences of an anomaly such as that of \(U(1) \subset SU(1,1)\) at the level of the scattering matrix of the theory and to address an open question of whether its presence has any consequences for the ultraviolet properties of higher-order S-matrix elements (for a recent discussion see \([5]\) and references therein).

Anticipating our results, we will find a close relation between the \(U(1)\) anomaly and special classes of amplitudes of \(\mathcal{N} = 4\) supergravity which vanish at tree-level but not at one-loop; through the double-copy construction of supergravity amplitudes they are related to non-supersymmetric gauge theory amplitudes that have similar properties.

As it is well-known, \(\mathcal{N} = 4\) supergravity can be formulated in different classically-equivalent ways. In the “covariant” formulation, the global symmetry \(SU(1,1)\) acts linearly on two complex scalar fields \(\Phi_\alpha\) (satisfying \(\Phi_1^* \Phi_1 - \Phi_2^* \Phi_2 = 1\)) while an auxiliary local \(U(1)\) symmetry (with field-dependent composite gauge field) acts also on other fields. The physical complex scalar field of the \(\mathcal{N} = 4\) supergravity multiplet parameterizes the coset \(SU(1,1)/U(1)\). An alternative “unitary-gauge” formulation corresponds to a particular gauge choice for the auxiliary \(U(1)\) gauge symmetry. The action of the global \(U(1)\) symmetry on vector fields has the interpretation of an electric/magnetic duality rotation. The \(U(1)\) anomaly has different — but related — interpretations in these two formulations \([6, 7]\): in the former it is the chiral anomaly of the auxiliary \(U(1)\) gauge symmetry, while in the latter it is the chiral anomaly of a particular global \(U(1)\) duality subgroup whose precise embedding into \(SU(1,1)\) depends on the gauge choice for the auxiliary gauge symmetry.

The crucial point in the computation of the duality symmetry anomaly in the “unitary-gauge” formulation in \([4]\) was the inclusion of the vector field contribution to the anomaly. As the vectors transform under the duality in a nonlocal way, via a \(\delta F = \epsilon F^*\) rotation of their on-shell field strength, this was done indirectly, via a topological count of anomaly for self-dual tensors. The corresponding anomaly may indeed be interpreted as the vector field contribution to the chiral gravitational anomaly, i.e. as the non-vanishing of the
expectation value of the divergence of the corresponding current in a non-trivial gravitational background as was understood in unrelated work in [8–12]. This anomaly was also rederived using a Lorentz non-covariant doubled-vector formulation in [13].

Here we shall present a novel way of understanding this anomaly, by relating it to a similar anomaly [14] in the conformal supergravity (CSG) [15], which admits a formulation where SU(1,1) is a global off-shell symmetry. The key observation is that the classical $\mathcal{N} = 4$ Poincaré supergravity (PSG) has a superconformal formulation [16] as the $\mathcal{N} = 4$ conformal supergravity coupled to six $\mathcal{N} = 4$ vector multiplets (with the higher-derivative action of pure $\mathcal{N} = 4$ CSG not added). In this superconformal framework the six vector fields, which (upon gauge-fixing of the conformal symmetry and S-supersymmetry and elimination of the auxiliary fields) become the six vector fields of the PSG multiplet, couple via their field strengths $F$ to the six (anti)self-dual rank-2 tensors $T$ appearing in the CSG multiplet and that provides a possible link between the $F$ and $T$ contributions to the corresponding anomalies.

For $\mathcal{N} = 8$ Poincaré supergravity, it was argued in [17–19] that, at the tree-level, the $E_{7(7)}$ duality symmetry group implies that all scattering amplitudes vanish in the single soft scalar limit (i.e. the limit in which the momentum of a scalar field goes to zero). This may be viewed as a consequence of the on-shell SU(8) R-symmetry group of the theory. It has also been argued in [18] and further detailed in [20] that the double soft scalar limit (i.e. the limit in which the momenta of two scalar fields vanish simultaneously) probes the commutation relations of the duality group.

A similar analysis has not yet been carried out for $\mathcal{N} = 4$ Poincaré supergravity. As was shown in [21], the $\mathcal{N} = 4$ supergravity can be interpreted as an orbifold truncation of $\mathcal{N} = 8$ supergravity. From this perspective, the one physical complex scalar field of the former theory is a linear combination of the scalar fields of the latter. As such, the tree-level scattering amplitudes of the $\mathcal{N} = 4$ supergravity are a subset of those of the $\mathcal{N} = 8$ supergravity and thus they should vanish in the soft scalar limit. Probing the commutation relations of the $\mathcal{N} = 4$ duality group SU(1,1) through a double-soft limit would require a careful construction and analysis of non-MHV one-loop amplitudes with at least six external particles.

The next step would be to study directly the soft scalar limits of the one-loop amplitudes in $\mathcal{N} = 4$ supergravity. As we shall see below, the U(1) $\subset$ SU(1,1) symmetry that requires the vanishing of the tree-level soft scalar limits is broken at one-loop level and there exist amplitudes with nontrivial U(1) charge which are non-vanishing; we shall refer to them as “anomalous amplitudes”. The mechanism through which these amplitudes are non-vanishing is quite similar to the one leading to the chiral anomaly — a divergence in a loop integral is compensated by a zero in its coefficient (in the amplitude case from the momenta of states running in the loop).\(^1\) We will construct examples of such amplitudes and

\(^1\)One may, in a sense, interpret these anomalous amplitudes as representing an anomaly in an on-shell bosonic symmetry (which is a remnant of the extended supersymmetry of the supersymmetric Yang-Mills theory) of tree-level scattering amplitudes of pure Yang-Mills theory. Indeed, at tree level the S-matrices of the pure Yang-Mills and the super Yang-Mills theories are the same (in the bosonic vector sector), but this is no longer so at the one-loop level.
find their soft scalar limit. We will then use the resulting soft scalar functions to construct the one-loop all-multiplicity U(1)-violating amplitudes with all external legs belonging to a single on-shell chiral multiplet.

We shall also consider the scattering amplitudes in \( \mathcal{N} = 4 \) Poincaré supergravity coupled to an arbitrary number \( n_v \) of abelian \( \mathcal{N} = 4 \) matter (vector) multiplets. While in the presence of matter multiplets \( \mathcal{N} = 4 \) Poincaré supergravity has ultraviolet divergences already at the one-loop supergravity level (the relevant counterterm is proportional to the square of the matter-field stress tensor \([22, 23]\)) we may restrict consideration to a finite sector with all external states belonging to the supergravity multiplet and track down the contribution of the matter multiplets to the U(1) anomalous amplitudes. The U(1) charges of matter fields are related to those of the fields of the supergravity multiplet by the SO(6, \( n_v \)) symmetry of matter-coupled \( \mathcal{N} = 4 \) supergravity.

The construction of one-loop amplitudes in \( \mathcal{N} = 4 \) supergravity with and without matter multiplets is made straightforward by the use of the duality between color and kinematics of super Yang-Mills theory amplitudes, and the corresponding double-copy construction of (super)amplitudes in related supergravity theories uncovered in \([24, 25]\). According to the color/kinematics duality conjecture, the integrands of super Yang-Mills theory amplitudes can be organized in terms of graphs with only cubic vertices such that there is a one-to-one correspondence between the Jacobi identities obeyed by the graphs’ color factors and their kinematic numerator factors. Whenever such a representation is available, the (super)amplitudes of a related supergravity theory are obtained by simply replacing the color factors with the kinematic numerator factors of a second-factor gauge theory. The validity of the construction can be easily confirmed through the evaluation of the \( D \)-dimensional unitarity cuts.

This paper is organized as follows. In section 2 we will discuss the U(1) anomaly of the Poincaré supergravity from the perspective of the superconformal formulation of the theory. We will then construct in detail the anomaly-induced terms in the one-loop effective action in the “unitary-gauge” formulation of the theory with manifest SU(4) \( R \)-symmetry. We shall assume a general reparametrization-invariant regularization scheme and will comment on the structure of other (parity-even, non-anomalous, finite) terms that should also appear in the effective action to maintain supersymmetry. We shall discuss a consistent assignment of the U(1) charges to extra matter multiplets coupled to \( \mathcal{N} = 4 \) Poincaré supergravity and identify a certain U(1) symmetry of asymptotic states such that their corresponding charges are the same as their charges with respect to U(1) \( \subset \) SU(1, 1).

In section 3, after a general discussion on the structure and double-copy construction of the scattering amplitudes in matter-coupled \( \mathcal{N} = 4 \) supergravity, we will proceed to compute explicitly, through the double-copy construction and the generalized unitarity method, the three-, four- and five-point amplitudes that break the asymptotic-state U(1) symmetry previously identified and which are forbidden in the \( \mathcal{N} \geq 5 \) supergravities. These amplitudes are UV finite and have rational dependence on external momenta. As usual in scattering superamplitude calculations, the result preserves manifestly the supersymmetry of the asymptotic states. We also discuss in some detail the same amplitudes in matter-coupled \( \mathcal{N} = 4 \) supergravity.
In section 4 we will analyse the soft scalar limits of these amplitudes and use them to present a well-motivated conjecture for the all-multiplicity one-loop superamplitudes with all fields in one of the two $\mathcal{N} = 4$ on shell supergravity multiplets. These amplitudes correspond to a particular term in the one-loop effective action containing two gravitons and having holomorphic scalar-field dependence. We compare this effective action term with the general form of the anomaly-induced effective action found in section 2.

We close in section 5 with remarks on nontrivial contributions of the one-loop anomalous amplitudes to higher-loop amplitudes. We also discuss the existence of an analogous U(1) (electric/magnetic) duality symmetry in certain matter-coupled supergravity theories, with fewer supercharges, which can be obtained by truncation from $\mathcal{N} = 8$ supergravity. This symmetry appears to be broken (via a mechanism similar to the one described above) by finite graviton-matter amplitudes.

In appendix A we shall comment on the vector field contribution to the chiral anomaly on a gravitational background. In appendix B we shall discuss the local parity-even scalar-curvature-curvature term which is a natural superpartner of the local part of the parity-odd anomalous term in the effective action. Appendices C, D and E contain details on the construction and evaluation of $n = 3, 4, 5$-point anomalous amplitudes.

2 U(1) anomalies in conformal and Poincaré $\mathcal{N} = 4$ supergravities

Our aim here will be to study the structure of the U(1) anomaly in $\mathcal{N} = 4$ Poincaré supergravity (PSG) [4, 6] (with or without additional $\mathcal{N} = 4$ matter multiplets) and its consequences for the corresponding scattering amplitudes. By U(1) anomaly here we mean the anomaly of an auxiliary gauge symmetry in covariant formulation in which a global SU(1, 1) duality symmetry is realized linearly or, equivalently, the anomaly of a particular global U(1) $\subset$ SU(1, 1) symmetry in a “physical gauge”, where the local U(1) symmetry is gauge-fixed.

As was shown in [16], pure $\mathcal{N} = 4$ PSG theory [26, 27] may be interpreted as a spontaneously broken (and gauge-fixed) version of $\mathcal{N} = 4$ conformal supergravity (CSG) [15] coupled to six $\mathcal{N} = 4$ vector multiplets (assuming the higher-derivative “kinetic” term of the CSG theory is not included). A similar formulation also holds for $\mathcal{N} = 4$ PSG coupled to $n$ matter $\mathcal{N} = 4$ vector multiplets: one is to start with $n + 6$ vector multiplets (with six vector multiplets having “wrong” sign of kinetic term) coupled to conformal supergravity.

Given that (i) the anomalies of spontaneously broken and unbroken phases of CSG may be expected to be the same (as suggested by the classical equivalence of the two phases), and also that (ii) anomalies are usually controlled by lowest derivative terms (i.e. are they are unchanged by addition of higher-derivative terms with same symmetry), one may conjecture that the U(1) anomaly of $\mathcal{N} = 4$ PSG [4] can be understood in terms of the corresponding anomaly [14] of the system of higher-derivative CSG coupled to $\mathcal{N} = 4$ vector multiplets.

We shall demonstrate that this is so in sections 2.1 and 2.2 below. Then, we shall discuss the detailed structure of the corresponding anomalous effective action of $\mathcal{N} = 4$ PSG in section 2.4.
2.1 U(1) anomaly in $N = 4$ conformal supergravity

Let us begin by recalling the field content of $N = 4$ CSG [15, 28]. In a “unitary gauge” formulation the scalar sector is described by a complex scalar $C$ parametrizing the coset $SU(1,1)/U(1)$. The fields with non-zero chiral U(1) weights are then:

- 1 complex scalar $C$ (-2)
- 4 left spinors $\Lambda_i$ ($-\frac{7}{2}$)
- 10 complex scalars $E_{(ij)}$ (-1)
- 6 (anti)self-dual tensors $T^{-ij}_{\mu\nu}$ (-1)
- 20 spinors $\chi^{ij}_k$ ($-\frac{1}{2}$)
- 4 left-handed gravitini $\psi_i^{\mu}$ ($-\frac{1}{2}$)

In the “covariant” formulation with manifest (linearly-acting) SU(1,1) symmetry [15] the scalar $C$ is replaced by a doublet of complex scalars $\Phi_\alpha$ with

\[
\eta^{\alpha\beta} \Phi_\alpha^* \Phi_\beta \equiv \Phi_1^* \Phi_1 - \Phi_2^* \Phi_2 = \Phi^\alpha \Phi_\alpha = 1, \quad \alpha = 1, 2, \tag{2.1}
\]

by adding an extra U(1) gauge symmetry. Then $\Phi_\alpha$ transforms under the global SU(1,1) (with matrix $U_\alpha^\beta$) as well as under the local U(1) (with parameter $\gamma(x)$) as follows: $\Phi'_\alpha = e^{-i\gamma(x)} U_\beta^\alpha \Phi_\beta$. The field $\Phi_\alpha$ is assigned the U(1) chiral weight $-1$ while other fields have the same weights as above. This assignment is consistent with supersymmetry transformations at full non-linear level as given in [15] (with the parameter $\epsilon_i$ of Poincaré supersymmetry having weight 1/2). Only the scalar doublet $\Phi_\alpha$ transforms under SU(1,1), but all other fields transform under the local U(1) (with the corresponding weights). That means that all the fields with derivative couplings and non-zero chiral weights couple to the composite U(1) gauge field $a_\mu$ through covariant derivatives (we ignore the fermionic term $\frac{i}{2} A^\mu \gamma_\mu A_i$ in $a_\mu$)

\[
D_\mu = \partial_\mu + ia_\mu, \quad a_\mu = \frac{i}{2}(\Phi^\alpha \partial_\mu \Phi_\alpha - \Phi_\alpha \partial_\mu \Phi^\alpha) = i\Phi^\alpha \partial_\mu \Phi_\alpha. \tag{2.2}
\]

The composite real field $a_\mu$ transforms under U(1) (by a gradient) and is invariant under SU(1,1).

For example, in the particular U(1) gauge

\[
\Phi_1 = \Phi_1^*,
\]

one may parametrize $\Phi_\alpha$ in terms of the above complex scalar $C = \Phi_2/\Phi_1$, i.e.

\[
\Phi_1 = (1 - |C|^2)^{-1/2}, \quad \Phi_2 = C(1 - |C|^2)^{-1/2}. \tag{2.4}
\]
An SU(1, 1) transformation requires a compensating local U(1) transformation to preserve the gauge (2.3). Then

$$a_\mu = \frac{i}{2} (1 - |C|^2)^{-1} (C \partial_\mu C^* - C^* \partial_\mu C) ,$$

with field strength $da \sim dC \wedge dC^* + \ldots$, $da \wedge da = 0$. This $a_\mu$ is no longer a singlet of redefined SU(1, 1); indeed, under SU(1, 1) acting (non-linearly) on $C$ the field $a_\mu$ is shifted by the gradient of field-dependent function multiplied by a rigid SU(1, 1) parameter, i.e. by an induced U(1) transformation with field-dependent parameter. An anomaly in the latter U(1) symmetry implies that the finite part of the effective action breaks the rigid SU(1, 1) symmetry.\(^4\)

The anomalies on CSG were discussed in [14] using this “unitary gauge” formulation. The anomaly of the rigid U(1) corresponding to the above U(1) weights comes from the (chiral) spinors and the self-dual tensors only. They are coupled to $a_\mu$ and the gravitational connection but since $da \wedge da$ here is zero, only the gravitational anomaly is present. We can normalize it to the anomaly of a single left-handed fermion having the chiral weight $c = +1$

$$A_{1/2} = \partial_\mu j^\mu = -\frac{k}{24(4\pi)^2} RR^* , \quad k = c = 1 .$$

where $(R^*)_{\mu\nu\rho\sigma} = \frac{1}{2} \varepsilon_{\rho\sigma\kappa\lambda} R_{\mu\nu}^{\kappa\lambda}$ is the dual curvature tensor.

For a massless spin (helicity) $s$ field with standard kinetic term and chiral weight $c_s$, the anomaly of the corresponding axial current obeys (2.6) with ($k_s$ is related [30] to the difference of the number of left- and right- handed zero modes of the corresponding Laplace operators, with ghost contributions properly accounted for)

$$k = c_s k_s , \quad k_s = (-1)^{2s} 4 (2s^3 - s) ,$$

so that the spinor:vector:gravitino anomalies are related as 1:4:(-21). In general, for a collection of fields with chiral weights $c_s$ and multiplicities $m_s$ we get

$$k = \sum_s c_s m_s k_s .$$

Note that since $j^\mu = \frac{\delta F}{\delta a_\mu}$, this anomaly means that the corresponding term in the finite part of the one-loop effective action is\(^5\)

$$\Gamma_{\text{an}} = \kappa \int RR^* \nabla^2 \nabla_\mu a^\mu , \quad \kappa = \frac{k}{24(4\pi)^2} .$$

We shall discuss in more detail below the structure of this and related terms in the effective action.

\(^4\)A similar anomaly of the continuous SU(1, 1) appears in 10d type IIB supergravity [29].

\(^5\)The effective action $\Gamma$ is given by the logarithm of the determinant of a chiral operator constructed in terms of $a_\mu$ and gravitational Lorentz connection $\omega_\mu$. The finite anomalous part of $\Gamma$ depends on both the longitudinal part of $a_\mu$ and the curvature of $\omega_\mu$. The Minkowski one-loop effective action is defined as $\epsilon^{\alpha \beta} = (\epsilon^{\beta})$ and thus the anomalous term in is real.
The chiral U(1) anomaly of the conformal gravitino was found in [31, 32] to be $-20A_{1/2}$ in terms of the anomaly of a single chiral fermion. This anomaly is related to the anomaly of the standard (Poincaré) gravitino [33–35] which is $-21A_{1/2}$ (in agreement with (2.7) setting $s = 3/2$).

The chiral anomaly of the selfdual tensor $T_{\mu \nu}$ can be found, e.g., by replacing it with a complex transverse vector $\zeta = \xi + i\eta$ as $T = d\xi + *d\eta$ (see section 3.2 in [36]) and using the result [35] for the anomaly of the chiral vector in four dimensions. An alternative way [14] is to replace $T$ with a symmetric product of two two-component left-handed spinors, finding that the anomaly of a single $T^-$ field is $4A_{1/2}$.

Taking into account the chiral weights $c_s$ and the number $m_s$ of components of each field, the total U(1) anomaly count $\sum c_s m_s k_s A_{1/2}$ of $\mathcal{N} = 4$ CSG is then [14]:

$$A_{\text{CSG}} = A_{A_i} + A_T + A_\chi + A_{\psi^\mu_n}(c)$$

$$= \left[ \left(-\frac{3}{2}\right)(4) \times 1 + (-1)(6) \times 4 + \left(-\frac{1}{2}\right)(20) \times 1 + \left(-\frac{1}{2}\right)(4) \times (-20) \right] A_{1/2} = 0,$$

i.e. the pure $\mathcal{N} = 4$ CSG theory has no “external” U(1) anomaly. This would formally imply that the global SU(1, 1) symmetry is preserved in the corresponding quantum theory. However, pure $\mathcal{N} = 4$ CSG has conformal [36] and SU(4) [14] gauge anomalies and thus is inconsistent (e.g. non-unitarity) at the quantum level.

\footnote{The two anomalies are simply related [31, 32] by adding gauge-fixing terms, taking into account the relevant ghosts and their chiralities, etc. More explicitly, the axial anomaly depends only of $\gamma^\mu D_\mu$ factor in $D^3$-type conformal gravitino operator: taking into account chiralities (or keeping track of $\gamma_5 \partial_\mu$ coupling in covariant derivatives) one has $D^+ D^- D^+$ so that only one spinor helicity contributes. That means gauge (or SO(4) Lorentz connection) gravitational anomalies of the gauge-fixed operators of the conformal ($D^3$) gravitino and of the standard ($D$) gravitino actually coincide, and are the same as the anomaly of 4 chiral spinors (as the gravitino has an extra 4-vector index). In the case of the gravitational anomaly there is additional “non-minimal” curvature contribution to anomaly that happens to be -24 times the chiral spinor anomaly. Thus the total gravitational anomaly from the gauge-fixed conformal or standard gravitino is the same $4-24 = -20$. The difference between the two gravitini comes from the ghost sector. In the conformal gravitino case the total contribution of $\gamma^\mu D_\mu$’s from all (FP and NK) ghosts, with chiralities properly taken into account, happens to vanish [31, 32]. Thus the gauge-field anomaly of the conformal gravitino is the same as of 4 chiral spinors [14], while its gravitational axial anomaly is $4-24 = -20$ of the single spinor value [31, 32]. In the standard gravitino case the FP ghosts have the same chirality as the gravitino but the opposite statistics, while the NK ghosts have the opposite chirality (and their contribution is 1/2 of the FP ghost one), so that the final count of the gauge anomaly is $4-2+1 = 4-1 = 3$ times $A_{1/2}$. This agrees with the expectation that the gauge anomaly should be proportional to the helicity, implying that there is a factor of 3 between the standard gravitino and the single spinor gauge anomaly [4]. In the gravitational anomaly case one is still to add the non-minimal term contribution leading to $(4-24)-1 = -20-1 = -21$ coefficient in (2.7). It should be noted that a count of anomalies is of course not correlated with a count of physical degrees of freedom (or overall power of box operator in the partition function which is 8 in the case of conformal gravitino and 2 = 2(4-2-1) in the case of the standard gravitino) as there one counts all derivative operators without taking into account the corresponding chiralities.}

\footnote{In the CSG action one has a $DT^- DT^+$ kinetic term but extra $D^5$ that appear in the kinetic terms for chiral vectors will not influence the chiral anomaly.}

\footnote{We use labels (c) and (s) to distinguish the conformal and the standard gravitino anomalies.}

\footnote{This U(1) symmetry has no associated dynamical gauge field in $\mathcal{N} = 4$ CSG [15].}
Let us consider now the system of $N = 4$ CSG coupled to $n N = 4$ vector multiplets (VM) with fields $(A_\mu, \psi^i, \phi^{ij})$. Following the conventions of [16] we shall assume that left fermions $\psi^i$ of VM have U(1) chiral weight $+\frac{1}{2}$, i.e. opposite to the one of $\chi_k^{ij}$ or the conformal gravitino $\psi^i_\mu$.\footnote{The chiral weight of a right-handed fermion $\psi_i$ is the same as that of the gravitino $\psi^i_\mu$ but we are counting the anomaly relative to a left-handed spinor.} This gives the additional VM contribution to the anomaly as

$$A_{VM} = n \left( + \frac{1}{2} \right) (4) A_{1/2} = 2n A_{1/2}, \quad A_{CSG+VM} = A_{VM}. \quad (2.11)$$

The total anomaly is thus

$$A_{CSG+VM} = A_{VM} = 2n A_{1/2}, \quad (2.12)$$

and it never vanishes for any non-zero number of VMs. In particular, the system of $N = 4$ CSG plus $n = 4$ VM that has no conformal [36] and SU(4) [14] gauge anomalies. Yet, it still has a U(1) gravitational anomaly, implying [6] the breaking of the rigid SU(1, 1) symmetry in the combined system at the quantum level. Since SU(1, 1) here acts on the vectors from VM as an on-shell duality rotation, it is present only on equations of motion (unless one gives up manifest Lorentz symmetry and uses doubled formulation) and in any case it does not survive generalization to non-abelian VM case.

2.2 $U(1)$ anomaly in $N = 4$ Poincaré supergravity

Let us now consider the $N = 4$ system of $n = 6$ VM coupled to CSG multiplet but without adding the higher-derivative (Weyl tensor squared + . . . ) action of the CSG itself. As was argued in [16], in the spontaneously-broken phase in which (at least some of) the VM scalars have constant vacuum values, this theory is classically equivalent (upon Weyl and S-supersymmetry gauge fixing and solving for the auxiliary fields $E, T, V$ and $\chi, D$ which are not propagating here) to the standard $N = 4$ Poincaré supergravity (PSG).

Anomalies of $N = 4$ PSG without any reference to this conformal supergravity construction were discussed in [4]. Our aim below will be to explain how they can be understood using the above results [14] about the anomalies of the CSG + VM system.

In general, one may expect that there should be anomaly matching between spontaneously broken and unbroken phases. In the unbroken phase, anomalies of CSG and VM can be counted separately as all massless modes are obvious — only fermions from the VMs then contribute to the anomaly (gravitino, vectors, and $\Lambda_i$ do not have free kinetic terms) because the reality of the VM precludes assigning a nonzero U(1) charge to the vectors [16]. In the broken phase we get kinetic terms for the metric, vectors, gravitino and $\Lambda$ from the VM part, meaning they explicitly contribute. In the broken phase there is some mode rearrangement but at the end anomalies should match.

In $N = 4$ PSG the contributions to the gravitational $U(1)$ anomaly (the non-gravitational anomaly is zero because $da \wedge da = 0$) come from the gravitini, spinors and vectors. The chiral weights of the gravitino, $T_{\mu\nu}$ and $\Lambda_i$ in the CSG setting were $-\frac{1}{2}, -1, -\frac{3}{2}$ ($\Lambda_i$ that had $D^3$ kinetic term in the CSG becomes the physical spinor of $N = 4$ PSG). These are the same weights used in [4], up to overall rescaling by -2. Since the gravitino is now
standard (not conformal) its anomaly is \((-21)A_{1/2}\) (rather than \((-20)A_{1/2}\)). Finally, the contribution of vectors (or the self-dual tensor contribution in \([4]\) ) is the same as that of the self-dual tensor in CSG.\(^{11}\)

Thus the total U(1) anomaly count is (cf. (2.10))

\[
A_{\text{PSG}} = A_A + A_A + A^{(s)}_{\psi^T} \quad \text{(2.13)}
\]

This is the same (up to an overall \(-2\) factor due to different normalization of anomaly) as in \([4]\). This is also exactly the same as the anomaly of \(n = 6\) \(\mathcal{N} = 4\) VM coupled to CSG in (2.11) which should be the anomaly in the unbroken phase.

We can understand this in detail by restoring the CSG multiplet contribution (which is zero) and then tracking how the anomaly rearrangement takes place in the \(\mathcal{N} = 4\) PSG derived following \([16]\), i.e. by assuming the broken phase and gauge fixing and solving for the non-dynamical fields. In the broken phase we have: (i) \(A_i\) and \(T_{ij}^{\mu\nu}\) contribute as before; (ii) gravitino contribution is reduced by \(-1\) from \(-20\) to \(-21\) due to breaking of S-susy (and solving for non-dynamical fields)\(^{12}\) (iii) 20 fermions \(\chi_{ij}^{\mu}\) of CSG and the rest of \(\psi_I^i\) fermions of 6 VMs should no longer contribute (they are integrated out or set to zero in the spontaneously broken phase).\(^{13}\)

The count of anomaly in the broken phase then proceeds as follows:

\[
(A_{\text{CSG}} + A_{\text{VM}})_{\text{br.ph.}} = A_A + A_T + 0 \times A_\chi + [A_{\psi^\mu} + A_{\psi^I}] + 0 \times (6 - 1)A_\psi
\]

\[
= \left( -\frac{3}{2} \right) (4)A_{1/2} + (-1)(6)4A_{1/2} + \left[ \left( -\frac{1}{2} \right) (4)(-20) + \left( +\frac{1}{2} \right) (4) \right]A_{1/2}
\]

\[
= 12A_{1/2} \quad \text{(2.14)}
\]

where the terms in the square brackets represent the standard gravitino contribution, in perfect agreement with direct \(\mathcal{N} = 4\) PSG anomaly count (2.13).\(^{14}\)

The conclusion is therefore that the U(1) anomaly of \(\mathcal{N} = 4\) PSG can be understood as the anomaly in the superconformal phase and that gives an alternative justification of the claim of \([4]\).

\(^{11}\)There we had two transverse vectors equivalent to \(T\). Explicitly (cf. eq. (3.49) in \([36]\)), we have \([16, 37]\) a vector-tensor coupling of the type \(FT + TT\). If we add the \(\partial T\bar{T}\) terms in the CSG action, we will get a \(\zeta^2(\partial^2 + m^2)\zeta\) type action for the complex transverse vector \(\zeta = \xi + i\eta\). It is important also to note that while in CSG case the \(T\)-tensor couples to the scalar connection \(a_\mu\) directly via covariant derivatives and thus the corresponding determinant contains scalar contributions, there is no such coupling in the PSG context: here scalar coupling enters originally as prefactors in the kinetic terms of the vectors. However, the consistent phase space formulation requires specific measure factors that are cancelled if we first redefine the vector fields to absorb these scalar couplings (cf. two-dimensional sigma models). Then we get instead derivative couplings of (doubled) vectors to scalar connection (a related remark appeared in \([13]\)).

\(^{12}\)In the spontaneously broken phase the gravitino should be absorbing the massless Goldstino of S-supersymmetry to become the standard gravitino.

\(^{13}\)Note that since \(\psi^I\) has opposite chirality to the gravitino one, adding its contribution is equivalent to subtracting the contribution of the same-chirality fermion, as was the case in the gauge anomaly count for the standard gravitino (4-2+1-3 due to FP and NK ghosts).

\(^{14}\)Let us add a few remarks. One could try to argue that adding the CSG action to the VM action should not change anomalies as this is like adding a higher-derivative regulator (e.g., for the fermions \(D \rightarrow D(1 + M^{-2}D^2)\)). Indeed, one may think that as anomalies may appear only at one-loop they thus
Let conclude this section with some comments on the SU(4) anomalies. While the local SU(4) symmetry is gauge-fixed in the unitary gauge by the condition completely fixing \[ \text{SU}(4) \] the VM scalars to be constant or zero in the broken phase, one may (as in \cite{4}) still formally consider the “external” axial anomaly of the SU(4) current.\(^{15}\) The conclusion of \cite{4} is that this anomaly is non-vanishing (but has no real consequences).\(^{16}\) Let us see how to relate the SU(4) anomaly count in \( \mathcal{N} = 4 \) Poincaré supergravity \cite{4} to the discussion of the SU(4) anomaly in the \( \mathcal{N} = 4 \) conformal supergravity \cite{14}, viewing \cite{16} the \( \mathcal{N} = 4 \) PSG from the superconformal point of view.

In the pure \( \mathcal{N} = 4 \) CSG the count of the non-abelian SU(4) anomaly goes as follows \cite{14}: normalizing the anomaly to the \( d_{abc} \) symbols in the fundamental representation, \( d_{abc}^4 \) a left-handed spinor \( \psi^i \) (or Majorana spinor whose left-handed part transforms in \( 4 \) of SU(4)) contributes +1; then, the CSG spinor \( \chi_i \) contributes \( -1 \); the left-handed gravitino \( \Psi_{\mu}^i \) contributes +4; \( T_{\mu\nu}^{(} \) does not contribute as the two-index antisymmetric representation of SU(4) is real; \( \chi_k^{[ij]} \) gives\(^ {17} \) -7, and thus the total axial gauge anomaly is

\[
 A_{\text{CSG}}^{(\text{gauge})} = A_{\Lambda} + A_{\chi} + A_{\psi_{\mu}} = (1)A_{1/2} + (7)A_{1/2} + 4A_{1/2} = -4A_{1/2}, \quad \text{(2.15)}
\]

\[
 A_{1/2} = \frac{1}{24(4\pi)^2} \text{Tr}(F F^*). \quad \text{(2.16)}
\]

If we add coupling to \( n \) of \( \mathcal{N} = 4 \) VMs with the left-handed spinor \( \psi^i \) being in \( 4 \) representation of SU(4) (like the gravitini), then

\[
 A_{\text{CSG}}^{(\text{gauge})} = nA_{1/2}, \quad A_{\text{VM}}^{(\text{gauge})} + A_{\text{CSG}}^{(\text{gauge})} = (n - 4)A_{1/2}. \quad \text{(2.17)}
\]

Thus for \( n = 4 \) we have the cancellation of the SU(4) anomaly while for \( n = 6 \) we have the anomaly equal to \( 2A_{1/2} \).

---

\(^{15}\)To preserve the gauge-equivalence, one would need to add a local counterterm depending on compensating scalars if one does not fix the “unitary” gauge \cite{7}.

\(^{16}\)If we solve for \( V_{\mu}^{ij} \) at the classical level, as it enters the action only algebraically (see \cite{16}), we find: \( V_{\mu}^{ij} \sim \phi^{ij} \delta_{\mu} \phi_{1i} + \psi^{i1} \gamma_{\mu} \psi_{1i} + \Lambda^{i} \gamma_{\mu} \Lambda_{i} \). After gauge-fixing SU(4) by imposing a condition on scalars, we are left only with global SU(4) and as long as there are no scalars left transforming non-trivially under this SU(4) there are no immediate consequences of the SU(4) anomaly. One reason for looking at the SU(4) anomaly is that it should be in same multiplet as the Weyl anomaly and thus, understanding why it is not relevant or how it is cancelled by a local counterterm depending on the compensator multiplet fields in the case of the \( n = 6 \) VM system may be relevant in a more general context.

\(^{17}\)One may use the relation between the \( d_{abc} \) symbols in a mixed-symmetry representation and in the fundamental representation, \( d_{abc}^{[ij]} = \frac{1}{2}(N^2 - 7N - 2)_{abc}N_{N=4}^N = -7d_{abc}^4 \)
Let us now compare this with the count of the SU(4) anomaly in N = 4 PSG \[16\] interpreted as a superconformal system in the broken phase \[16\] with all extra gauge symmetries fixed. Here we have the same four spinors Λ, in the fundamental representation of SU(4) and the same gravitino ψ_μ, but no spinors χ. The standard gravitino contribution to the gauge anomaly is proportional to the helicity so it should have a relative factor of 3 compared to the chiral spin 1/2 fermion. This implies
\[
A^{(\text{gauge})}_{\text{PSG}} = A_\Lambda + A^{(s)}_{\psi_\mu} = (-1)A_{1/2} + 3A_{1/2} = 2A_{1/2}. \tag{2.18}
\]
This reproduces the count of the SU(4) anomaly in conformal supergravity in (2.17) for n = 6.

2.3 U(1) anomaly in Poincaré supergravity coupled to n_v vector multiplets

In the case when CSG is coupled to n = 6 + n_v rather than just six vector multiplets, i.e. to extra n_v “matter” multiplets, the anomaly relation (2.12) does not directly apply. Indeed, the matter VMs surviving as dynamical fields in the broken phase may acquire different U(1) charges than the ones assumed in the unbroken phase.\[18\] As the chiral weights should be consistent also with the supersymmetry of the PSG theory, a way to fix them is to use that for n_v = 6 the resulting \(\mathcal{N} = 4\) PSG + matter theory can be interpreted as a truncation of \(\mathcal{N} = 8\) supergravity.

Namely, let us decompose the \(\mathcal{N} = 8\) graviton multiplet into the \(\mathcal{N} = 4\) components following the embedding
\[
\text{SU}(8) \supset \text{SU}(4) \times \text{SU}(4) \times \text{U}(1). \tag{2.19}
\]
This decomposition encodes the double-copy structure of supergravity theories, first realized in the string-theory KLT relations \[38, 39\], which imply that the spectrum of the theory and its on-shell interactions may be represented in terms of two copies of the \(\mathcal{N} = 4\) sYM theory, each of which has an SU(4) symmetry. The lone U(1) in (2.19) will be identified with the U(1) symmetry of \(\mathcal{N} = 4\) supergravity (at least up to conjugation by SU(1, 1) elements) and the first SU(4) will be identified with the R-symmetry of \(\mathcal{N} = 8\) supergravity. This is possible because the charges of the PSG and matter multiplet asymptotic states are the same under the two symmetries.

The decomposition of the SU(8) representations appearing in \(\mathcal{N} = 8\) theory in representations of SU(4) × SU(4) × U(1), denoted by (SU(4), SU(4)′) U(1), is:
\[
\begin{align*}
1 &= (1, 1)^0 \\
8 &= (4, 1)^q \oplus (1, 4)^{-q} \\
28 &= (6, 1)^{2q} \oplus (1, 6)^{-2q} \oplus (4, 4)^0 \\
56 &= (4, 1)^{3q} \oplus (1, 4)^{-3q} \oplus (6, 4)^q \oplus (4, 6)^{-q} \\
70 &= (1, 1)^{4q} \oplus (1, 1)^{-4q} \oplus (4, 4)^{2q} \oplus (4, 4)^{-2q} \oplus (6, 6)^0.
\end{align*}
\]
\[18\]In the presence of matter multiplets, the U(1) \(\subset\) SU(1, 1) symmetry is a combination of the U(1) symmetry in the matter-free theory and the U(1) duality symmetries of the matter VMs. It is also possible that, in the process of fixing the conformal (super)symmetry, the matter vector fields absorb some power of a CSG field that was charged under the U(1) symmetry and thus acquire different charges than in eq. (2.11).
Here \( q \) is the normalization of the U(1) charge which will be fixed below. All of the components invariant under the second SU(4) group form the \( \mathcal{N} = 4 \) supergravity multiplet:

\[
(1, 1)^0, (4, 1)^q, (6, 1)^{2q}, (4, 1)^{3q}, (1, 1)^{4q}, (1, 1)^{-4q}.
\] (2.21)

We can also identify four \( \mathcal{N} = 4 \) gravitino multiplets, transforming in the 4 of the second SU(4) group,

\[
(1, 4)^{-q}, (4, 4)^0, (6, 4)^q, (4, 4)^{2q}, (4, 4)^{-2q};
\] (2.22)

as well as six \( \mathcal{N} = 4 \) vector multiplets transforming in the 6 of the second SU(4) group,

\[
(1, 6)^{-2q}, (4, 6)^{-q}, (6, 6)^0.
\] (2.23)

The conjugate representations (i.e. asymptotic states with opposite helicity) have opposite U(1) charges. We note that, as expected from the discussion in the beginning of this section, the charges of the fields of the matter vector multiplets under the U(1) are different from the U(1) charges in the unbroken phase of CSG coupled to \( n = 6 + n_v \) vector multiplets.

A further argument for the identification of the supergravity U(1) symmetry with the U(1) symmetry appearing in (2.19) is that they have similar consequences on scattering amplitudes. Indeed, the supergravity U(1) symmetry acts on vector fields as electric/magnetic duality rotation; as such it implies (see e.g. [40]) that scattering amplitudes of vector fields of the same flavor vanish identically unless they have an equal number of positive and negative helicity fields. If scattering amplitudes preserve the U(1) symmetry of eq. (2.19), then they must carry vanishing charge. Restricting, as above, to the scattering of a single type of vector field it immediately follows that the amplitude vanishes unless one scatters an equal number of positive helicity and negative helicity fields. Thus, the two symmetries have the same consequences on scattering amplitudes.

With the U(1) charges in eq. (2.21) we can compute the anomaly contribution of the \( \mathcal{N} = 4 \) graviton multiplet following (2.7), (2.8) as

\[
k_{\text{PSG}} = 4k_{3/2}(q) + 6k_1(2q) + 4k_{1/2}(3q) = -24q = 12,
\] (2.24)

where we have chosen the chiral weight of the gravitino to be \( q = -1/2 \), i.e. the same as in (2.10) and (2.13), to reproduce the anomaly coefficient 12 in eqs. (2.13) and (2.14).

With this choice the anomaly contribution of one \( \mathcal{N} = 4 \) vector multiplet is:

\[
k_v = k_1(-2q) + 4k_{1/2}(-q) = -12q = 6,
\] (2.25)

which is different from the anomaly (2.11) of a VM in the conformal phase.\(^{19}\) The total anomaly coefficient for the \( \mathcal{N} = 4 \) supergravity coupled with \( n_v \) vector multiplets is then

\[
k_{\text{PSG}+\text{VM}} = k_{\text{PSG}} + n_v k_v = -12q(2 + n_v) = 6(2 + n_v).
\] (2.26)

\(^{19}\)One may understand this and the fact that the vector fields in the supergravity and matter multiplets have the same U(1) charges as a consequence of the \( O(6,n_v) \) symmetry of matter-coupled \( \mathcal{N} = 4 \) supergravity.
This, along with eq. (2.9), implies that the corresponding anomalous part of the effective action is
\[
\Gamma_{\text{an}}^{N=4,n_v} = \frac{1}{2} (2 + n_v) \Gamma_{\text{an}}^{N=4,n_v=0} = \frac{2 + n_v}{4(4\pi)^2} \int RR^* \nabla^{-2} \nabla \mu a^\mu. \tag{2.27}
\]

In sections 3 and 4 we will reproduce this dependence on the number of vector multiplets from scattering amplitude calculations.

2.4 Structure of the anomalous part of effective action

Let us now comment further on the meaning of the above U(1) anomaly and the the related breaking of the global SU(1,1) symmetry of \( N = 4 \) PSG theory in the context of the one-loop supergravity effective action in an external scalar and gravitational background.

2.4.1 General comments

Let us first not fix a U(1) gauge, so that the composite U(1) gauge field \( a_\mu \) in (2.2) transforms by a gradient under a chiral rotation of \( \Phi_\alpha \). Suppose we consider the one-loop effective action for a Majorana fermion coupled to gravity (though the Lorentz connection) and chirally (i.e. with \( \gamma_5 \)) to \( a_\mu \). If we split \( a_\mu \) into the longitudinal and transverse parts, \( a_\mu = a_\mu^\parallel + a_\mu^\perp \), \( \nabla_\mu a_\mu^\perp = 0 \) then, integrating the U(1) anomaly to obtain the corresponding effective action (2.9), we find that the latter may be written as
\[
\Gamma[a;g] = \Gamma_{\text{an}}[a^\parallel;g] + \Gamma_{\text{inv}}[a^\perp;g], \quad \Gamma_{\text{an}}[a^\parallel;g] = \kappa \int RR^* \nabla^{-2} \nabla^{\mu} a_\mu, \tag{2.28}
\]
where \( \kappa \) stands for the overall coefficient in (2.9). As both \( a_\mu^\parallel \) and \( a_\mu^\perp \) are separately SU(1,1) invariant, the same applies to \( \Gamma_{\text{an}} \) and \( \Gamma_{\text{inv}} \).

Let us parametrize the scalar doublet \( \Phi_\alpha \) as
\[
\Phi_1 = \sqrt{1 + r^2} \ e^{i(a-b)}, \quad \Phi_2 = r \ e^{i(a+b)}, \tag{2.29}
\]
where \( r, a, b \) are three real fields. Then only \( a \) is transforming under the local U(1) symmetry (by a shift), while all the three fields transform under the SU(1,1). The connection (2.2) and the PSG scalar Lagrangian take the form:
\[
a_\mu = -\partial_\mu a + (1 + 2r^2) \partial_\mu b, \quad L = D_\mu \Phi_\alpha D^\mu \Phi_\alpha = |D_\mu \Phi_1|^2 - |D_\mu \Phi_2|^2 = -\frac{(\partial_\mu r)^2}{1 + r^2} - 4(1 + r^2) r^2 (\partial_\mu b)^2. \tag{2.30}
\]
Both \( a_\mu \) and \( L \) are SU(1,1) invariant, while \( L \) is also invariant under local U(1) transformations. Then (2.28) implies
\[
\Gamma = \Gamma_{\text{an}}[a, b, r; g] + \Gamma_{\text{inv}}[b, r; g], \quad \Gamma_{\text{an}} = -\kappa \int RR^* a + \kappa \int RR^* \nabla^{-2} \nabla^{\mu} [(1 + 2r^2) \partial_\mu b], \tag{2.32}
\]

20 For a real fermion there is no gravitational (i.e. local Lorentz) anomaly, so the local Lorentz symmetry is unbroken and the effective action depends on the metric \( g \) rather than on the vierbein.

21 An alternative parametrization is \( (r = \sinh \rho) \): \( \Phi_1 = \cosh \rho \ e^{i(a-b)}, \quad \Phi_2 = \sinh \rho \ e^{i(a+b)} \), with \( a_\mu = -\partial_\mu a + \cosh 2\rho \partial_\mu b \) and \( L = -\partial_\mu \rho)^2 - \sinh^2 2\rho (\partial_\mu b)^2. \)
so that while $\Gamma$ is SU(1, 1) invariant, it is not invariant under the local U(1) transformations because it depends on the scalar field $a$. This anomalous term, however, is local, and thus can be cancelled by a local counterterm. The important difference compared to a standard gauge theory (where the gauge field is a fundamental field and the anomalous term is nonlocal) is that here the basic variables in the path integral are the scalar fields $a, b, r$ rather than $a_\mu$. Thus, defining the effective action as

$$\Gamma' = \Gamma'_{an}[b, r; g] + \Gamma_{inv}[b, r; g],$$

$$\Gamma'_{an} = \Gamma_{an} + S_{c.t.}, \quad S_{c.t.} = \kappa \int RR^* \left[ a + f(b, r) \right],$$

we may restore the local U(1) invariance, i.e. the absence of dependence on the gauge degree of freedom $a$.\footnote{The “minimal” counterterm, $\int RR^* a$, may be written in terms of the original $\Phi_\alpha$ as $\frac{1}{4} \int RR^* \ln \frac{\Phi_1 \Phi_2^*}{\Phi_1^2 \Phi_2}$, illustrating its non-invariance under SU(1, 1).}

For example, we may choose to also cancel the local $b$-dependent term in $\Gamma_{an}$ in (2.33) ending up with $\Gamma'_{an} = \kappa \int RR^* \nabla^{-2} \nabla^\mu (2r^2 \partial_\mu b)$. However, being nonlocal, $\Gamma'_{an}$ cannot be completely eliminated by a local counterterm. Furthermore, since $S_{c.t.}$ is not SU(1, 1) invariant, the resulting effective action $\Gamma'$ is not invariant under SU(1, 1) transformations. This is thus an illustration of a general “compensator” mechanism discussed, e.g., in [7].

Equivalently, the U(1) anomaly implies that starting with a classical theory in two different physical U(1) gauges one finds two different quantum effective actions, but the possibility to cancel the anomaly by a local counterterm means that these two effective actions differ only by local terms. We may then choose a particular U(1) gauge from the start and interpret $\Gamma_{an}$ as the part of the effective action that breaks SU(1, 1) invariance (this is the framework used in [4]).

For example, the gauge where $a = b$ corresponds to (2.4), i.e. the gauge in which $\Phi_1$ is real and $C = \frac{r}{\sqrt{1 + r^2}} e^{2ib}$. Another gauge is, e.g., $a = 0$ where $\Phi_1 \sim \Phi_2^*$, i.e. $\Phi_1 = \sqrt{1 + r^2} e^{-ib}$, $\Phi_2 = r e^{ib}$. In these two cases we find

$$\langle \Gamma_{an} \rangle_{a=b} = \kappa \int RR^* \nabla^{-2} \nabla^\mu (2r^2 \partial_\mu b),$$

$$\langle \Gamma_{an} \rangle_{a=0} = \kappa \int RR^* b + \kappa \int RR^* \nabla^{-2} \nabla^\mu (2r^2 \partial_\mu b).$$

The two effective actions differ by a local term which is linear in the scalar field $b$ (implying, e.g., that in the first gauge the “anomalous” S-matrix describes the scattering of a smaller

\footnote{In general, a local counterterm may also contain many other terms, including derivative-dependent and curvature-independent ones that should be constrained by the requirement of preservation of desired symmetries of the theory (i.e. satisfaction of the respective Ward identities). Note, however, that (i) we cannot introduce local derivative terms without extra powers of a dimensionful parameter, and (ii) we cannot construct an SU(1, 1) invariant using just algebraic functions of $a, b, r$, i.e. adding $f(b, r)$ cannot restore the SU(1, 1) symmetry.}
number of external states). One may of course rule out this difference by requiring that we add a local counterterm as part of the definition of the theory.\textsuperscript{24}

Such a finite local counterterm also appears in the relation between the SO(4) \textsuperscript{26, 27} and SU(4) \textsuperscript{41} formulations of \( \mathcal{N} = 4 \) supergravity: the map between the two theories requires a local chiral redefinition of spinors and a duality rotation of vectors (resulting in a local contribution of the type \( \int RR^* f(b, r) \) to the anomaly) as well as a local reparametrization of the scalar fields in the \( a = 0 \) gauge.

\subsection{2.4.2 The case of SU(4) invariant version of \( \mathcal{N} = 4 \) Poincaré supergravity}

Let us recall that there are two “unitary-gauge” formulations of \( \mathcal{N} = 4 \) PSG: one with SO(4) symmetry \textsuperscript{26, 27} and one with SU(4) symmetry \textsuperscript{41}. They are related by a field redefinition of the scalar fields, a chiral rotation of fermions and a duality rotation of vectors. They may also be understood as corresponding to two different U(1) gauges in the superconformal formulation \textsuperscript{16}. Because of the duality anomaly the corresponding quantum effective action should be different, but only by local terms (explaining the puzzle of “quantum-inequivalence” found in \textsuperscript{42}).

Let us discuss in detail the SU(4) invariant version of \( \mathcal{N} = 4 \) supergravity \textsuperscript{41}. It corresponds to the following U(1) gauge \textsuperscript{43–45}:

\[ \text{Im} \Phi_1 = \text{Im} \Phi_2. \]  

(2.38)

Introducing the two independent scalar fields as\textsuperscript{25}

\[ \tau = B + ie^{-\varphi} = \frac{\Phi_1 + \Phi_2}{\Phi_1 - \Phi_2}, \]  

(2.39)

we find that the composite connection and the scalar kinetic term are

\[ a_\mu = -\frac{\partial_\mu (\tau + \bar{\tau})}{4 \text{Im} \tau} = -\frac{1}{2} e^{\varphi} \partial_\mu B, \]  

(2.40)

\[ -4D^\mu \Phi^\alpha D_\mu \Phi_\alpha = \frac{\partial^\mu \tau \partial_\mu \bar{\tau}}{(\text{Im} \tau)^2} = (\partial_\mu \varphi)^2 + e^{2\varphi} (\partial_\mu B)^2, \]  

(2.41)

while the vector-scalar interaction terms are \( (F^\pm = \frac{1}{2} (F \pm i F^*)) \)

\[ L = \frac{1}{4} i \tau F^\pm_{\mu \nu} F^\mp_{\mu \nu} - \frac{1}{4} i \bar{\tau} F^\pm_{\mu \nu} F^\mp_{\mu \nu} = -\frac{1}{4} e^{-\varphi} F_{\mu \nu} F_{\mu \nu} - \frac{1}{4} BF_{\mu \nu} F^\mu_{\nu}. \]  

(2.42)

To recall, we started with a formulation with manifest linearly realised SU(1, 1) and the U(1) gauge symmetry (with \( a_\mu \) being SU(1, 1) invariant); once we fixed a U(1) gauge, \( a_\mu \) starts transforming by a gradient under a subgroup of SU(1, 1) which is broken in this gauge.

Then, at the quantum level, we find the anomaly of that subgroup, whose precise embedding into SU(1, 1) depends on a the particular gauge choice. In the gauge (2.38)

\textsuperscript{24}As already mentioned above, extra counterterms may be required to make the anomaly consistent with supersymmetry (see \textsuperscript{5, 7}).

\textsuperscript{25}The field redefinition relating \( \tau \) to the complex scalar \( C \) of the SO(4) invariant formulation (corresponding to the gauge (2.3)) is \( \tau = i \frac{1 + e^{-C}}{1 - e^{-C}} \), with \( \frac{e^{\varphi} \partial_\mu \tau}{(\text{Im} \tau)^2} = i \frac{e^{\varphi} \partial_\mu C}{(1 - e^{-C})^2 \tau}. \)
the SU(1, 1) symmetry becomes SL(2, \mathbb{R}) acting on \( \tau \) in the usual way, through Möbius transformations,
\[
\tau' = \frac{a\tau + b}{c\tau + d}, \quad ad - bc = 1.
\] (2.43)
In SL(2, \mathbb{R}) language we have 3 subgroups of the duality group acting on the scalars (and vectors and fermions):

1. shift of \( \tau \) by a real constant (shift of \( B \) only): \( \tau' = \tau + b \) (\( c = 0, a = d = 1 \))

2. rescaling of \( \tau \) (and vectors, in the opposite way): \( \tau' = a^2 \tau \) (\( c = 0, b = 0, d = a^{-1} \))

3. non-linear transformation: \( \tau' = \frac{\tau}{1 + \epsilon \tau} \) (\( c = \epsilon, b = 0, a = d = 1 \)) or, in an infinitesimal form, \( \delta \tau \sim \epsilon \tau^2 \), with vectors transforming as \( \delta F = \epsilon F^* \) and fermions rotating chirally, \( \delta \psi \sim \epsilon \gamma_5 \psi \), see \[41\].

The U(1) connection (2.40) is invariant under first two subgroups and transforms under the third one by a gradient, \( \delta \alpha_\mu = -\frac{1}{2} \epsilon \partial_\mu (\tau - \bar{\tau}) \). Since \( \tau - \bar{\tau} = 2 i e^{-\varphi} \) the anomalous term in the effective action (2.9) thus transforms as \( \delta \Gamma \sim \epsilon \int RR^* e^{-\varphi} \).

The above relations imply that in this gauge we find the following anomalous term (2.9) in the effective action (cf. (2.33), (2.35))
\[
\Gamma_{\text{an}} = \kappa \int RR^* \nabla^2 \varphi \partial_\mu B
\]
\[
= \kappa \int RR^* B + \kappa \int RR^* \nabla^2 \varphi \left[ \left( \varphi + \frac{1}{2} \varphi^2 + \ldots \right) \partial_\mu B \right],
\] (2.44)
\[
\kappa = -\frac{1}{2} \times 12 \times \frac{1}{24(4\pi)^2} = -\frac{1}{4(4\pi)^2}.
\] (2.45)

Here the extra factor \(-\frac{1}{2}\) in \( \kappa \) comes from (2.40) and 12 is the total value of coefficient \( k \) in (2.9) corresponding to PSG in (2.13) or (2.24).

Recall that the total coefficient 12 in (2.13) includes the contribution \(-24 = (-1)6 \times 4\) of six vectors with chiral weight \(-1\). We can then check the normalization of \( \Gamma \) in (2.45) by relating the above discussion to the known anomalous correlator \[8, 9\] (see also \[10–12\]) of the quantum Maxwell theory in a gravitational background
\[
\langle F^{\mu\nu} F_{\mu\nu}^* \rangle = \frac{1}{3(4\pi)^2} RR^* ,
\] (2.46)
which also implies the presence in the effective action of a vector contribution which is anomalous under the duality rotation \( \delta F_{\mu\nu} = \epsilon F_{\mu\nu}^* \) (see appendix A).\textsuperscript{26} Expanding the effective action corresponding to the classical action (2.42) in powers of the scalar fields, we then conclude that the term linear in \( B \) in the corresponding effective action is given by\textsuperscript{27}
\[
\Gamma_{\text{an}}^{(\text{vec})} = -\frac{1}{4} \int \langle (F^{\mu\nu} F_{\mu\nu}^*) (x) \rangle B(x) + \ldots = -\frac{1}{12(4\pi)^2} \int RR^* B(x) + \ldots .
\] (2.47)
\textsuperscript{26}This conclusion may be reached by relating the discussion above to the quantum anomaly of the chirality current \( K_\mu = e^{\mu\lambda} A_\lambda \partial_\mu A_\rho, \partial^\rho K_\mu = \frac{1}{2} F^{\mu\rho} F_{\mu\rho} \).
\textsuperscript{27}We note in passing that the result for the vector loop contribution to the corresponding term in the effective action found by a diagrammatic method in \[42\] disagrees with this by an extra 3/2 factor (which should be due to the use of a regularization which is not reparametrization-invariant).
Recalling again that according to (2.13) the anomalous contribution of the full $\mathcal{N} = 4$ PSG multiplet should be 3 times a single vector contribution (i.e. 4, if vector has weight +1) we conclude that (2.47) is indeed in agreement with (2.44), (2.45).

While the quantum vector field contribution to $\int RR^* B$ term in (2.44) follows directly from (2.46), one can find similar spinor contributions from the form of the corresponding $\gamma_5 \partial_\mu B$ covariant derivative couplings in [41] (producing $\int RR^* B$ terms in $\Gamma$ via fermion triangle loop diagram). It is worth emphasizing that here one need not go through the duality group anomaly discussion of [4] — one is simply computing certain leading terms in the one-loop effective action.28

With an expression for the anomalous part of the effective action in hand we may consider extracting the corresponding contribution to various one-loop scattering amplitudes by expanding in powers of the scalar fields,

$$\Gamma_{\text{an}} = \Gamma_1 + \Gamma_2 + \ldots, \quad \Gamma_1 = \kappa \int RR^* B, \quad (2.48)$$

$$\Gamma_2 = \kappa \int RR^* \nabla^{-2}(\partial^\mu \varphi \partial_\mu B + \varphi \nabla^2 B), \quad (2.49)$$

and constructing tree-level amplitudes with the resulting effective vertices. These 1-PI vertices give particular contributions to the $S$-matrix with at least 2 gravitons on external lines. For example, we get $h h B$ 3-point amplitudes and $h h \varphi B$ 4-point amplitudes.

As follows from the $\mathcal{N} = 4$ supergravity Lagrangian (here we set $K = \frac{1}{2}$ in [41]; $r = 1, \ldots, 6$)

$$\mathcal{L} = -R - \frac{1}{2} (\partial^\mu \varphi \partial_\mu \varphi + e^2 \varphi \partial^\mu B \partial_\mu B) - \frac{1}{4} e^{-\varphi} F^r_{\mu\nu} F^{r*}_{\mu\nu} - \frac{1}{4} B F^r_{\mu\nu} F^{r*}_{\mu\nu} + \ldots, \quad (2.50)$$

we get the following linearized equation of motion for $B$: $\nabla^2 B = \frac{1}{4} F^r_{\mu\nu} F^{r*}_{\mu\nu} + \ldots$. Thus connecting the one-loop $RR^* B$ vertex and the tree-level $BF^r_{\mu\nu} F^{r*}_{\mu\nu}$ vertex by the $B$-field propagator we get also $h h A A$ 4-point amplitude represented by the $\int RR^* \nabla^{-2} F^r_{\mu\nu} F^{r*}_{\mu\nu}$ term in the generating functional.

It should be noted that these are not the complete $S$-matrix elements with a given choice of external states: there are obviously other nonlocal terms in the effective action not directly related to the duality anomaly term (2.44) discussed above. In particular, the 4-point matrix element $h^{++} h^{++} A^+ A^+$, while receiving contributions from the $B$ intermediate state, can be shown to vanish identically when all the contributions are included, in agreement with the consequences of supersymmetry Ward identities.

Indeed, in the above discussion we concentrated just on a particular set of terms in the effective action which are directly related to the anomaly. The effective action should contain, of course, many other local and nonlocal terms. Supersymmetry may require that some of them be natural partners, being parts of the same superinvariants. The

28Note also that while one might argue that the local $\int RR^* B$ term is ambiguous as we can eliminate it by adding a local counterterm, doing this would effectively drive us away from the SU(4) version of supergravity — once we have fixed the U(1) gauge (2.38), and thus decided about which subgroup of the SU(1, 1) duality group is anomalous (with shifts of $B$ and rescalings being non-anomalous) we should not add further local parity-odd counterterms.
contributions to scattering amplitudes coming just from the anomalous terms in (2.9) may therefore appear to break supersymmetry (separating the field configurations mentioned above in helicity components one may notice that some of them are forbidden by the analysis of [46]). Since we do not expect supersymmetry to be anomalous, the effective action should thus contain also other terms related by supersymmetry transformations to the anomalous ones in (2.48), (2.49). For example, we may expect to find also the $\int RR\varphi$ term [42, 47, 48] as a partner of $\int RR^*B$ as well as an $\int R^{\mu\nu\lambda\rho}F_{\mu\nu}F_{\lambda\rho}$ term corresponding to the graviton-vector-vector amplitudes. We will return to this discussion in sections 3 and 4 below where we will evaluate the relevant scattering amplitudes in a scheme that manifestly preserves supersymmetry at the linearized level.

3 U(1) symmetry violating scattering amplitudes

Let us now address the question of how the U(1) anomaly of the $N=4$ Poincaré supergravity is reflected in the S-matrix of the theory. As we have seen above, the anomaly implies that the effective action for gravitons and scalars should contain anomalous terms which, in turn, should correspond to particular U(1) symmetry violating one-loop scattering amplitudes.

To construct the S-matrix we will not start directly from the PSG action but instead use the (generalized) unitarity method [49–51], color/kinematics duality and the double-copy construction to express it in terms of the S-matrix of the $N=4$ supersymmetric YM theory and pure YM theory coupled to scalar fields [24, 25].

3.1 Generalities on gauge theory and supergravity scattering amplitudes

The ($D$-dimensional) generalized unitarity method together with the KLT [38, 39] relations, determining the tree-level amplitudes of a (super)gravity theory in terms of the scattering amplitudes of two (supersymmetric) gauge theories, provide a sure way for constructing one- and higher-loop amplitudes in $N=4$ supergravity (pure or coupled to vector multiplets). In this approach the spectrum of $N=4$ supergravity is realized as a tensor product of the fields of $N=4$ and $N=0$ supersymmetric YM (sYM) theories, i.e. as a product of an $N=4$ vector multiplet and a single vector field (we denote YM or sYM gluons by $g$ and supergravity abelian vector fields by $A$):

$$(g^+, \lambda^+_{ABC}, s_{AB}, \lambda^-_A, g^-) \otimes (g^+, g^-) = (h^+, \psi^+_{ABC}, A^+_{AB}, \chi^+_A, \bar{t}) \oplus (t, \chi^-_{ABC}, A^-_{AB}, \psi^-_A, h^-).$$

Here and below $(t, \bar{t})$ is the complex field that labels the external scalar states in the supergravity scattering amplitudes; in terms of the two vector fields it is

$$t = g^+_{N=4} \otimes g^-_{N=0}, \quad \bar{t} = g^+_{N=4} \otimes g^-_{N=0}.$$  \hspace{1cm} (3.2)

Additional $n_v$ vector multiplets may be described as

$$(g^+, \lambda^+_{ABC}, s_{AB}, \lambda^-_A, g^-) \otimes \phi_p = (A^+_p, \lambda^+_{p,ABC}, s_{p,AB}, \lambda^-_{p,A}, A^-_p)$$

where $\phi_p$ with $p = 1, \ldots, n_v$ are real scalar fields.
While the construction of the scattering amplitudes of the $\mathcal{N} = 4$ sYM factor is clear (and will be reviewed shortly), this is less clear for the bosonic factor since one may consider several different self-couplings of the scalar fields. The correct choice follows from the observation that, on the one hand, half-maximal supersymmetry implies unique consistent coupling of vector multiplets to supergravity and, on the other, that the models of $\mathcal{N} = 4$ supergravity coupled to $n_v = 2, 4, 6$ vector multiplets can be realized as orbifolds of $\mathcal{N} = 8$ supergravity. This construction implies that the tree-level scattering amplitudes of the bosonic factor must be chosen to be the same as those of the $\mathcal{N} = 4$ sYM theory, i.e. that the $n_v$ scalars should have quartic self-couplings.

The scattering amplitudes of $\mathcal{N} = 4$ sYM theory manifestly preserve linearized $\mathcal{N} = 4$ supersymmetry. The on-shell fields of this theory can be combined into a chiral superfield

$$
\Phi(\eta) = g^{-} + \eta^{A} \lambda_{A}^{-} + \frac{1}{2!} \eta^{A} \eta^{B} \phi_{AB} + \frac{1}{3!} \eta^{A} \eta^{B} \eta^{C} \epsilon_{ABCD} \lambda^{+D} + \frac{1}{4!} \eta^{A} \eta^{B} \eta^{C} \eta^{D} \epsilon_{ABCD} g^{+},
$$

where $\lambda^{A}_{ABC} = \epsilon_{ABCD} \lambda^{+D}$ and $\eta^{A}$ are four Grassmann variables. The scattering amplitudes of component fields are assembled into a superamplitude

$$
A_{n} = i \frac{\delta^{(4)}}{(\sum_{i=1}^{n} \lambda_{i} \dot{\lambda}_{i}) \delta^{(8)}}(\sum_{i=1}^{n} \lambda_{i} \eta^{A}_{i}) \sum_{r=0}^{n-4} P_{n}^{r},
$$

where $\lambda_{i}$ are the spinors corresponding to $i$-th momentum, $k_{i \mu}(\sigma^{\mu})_{a \dot{a}} = \lambda_{ai} \bar{\lambda}_{\dot{a}i}$ and $P_{n}^{r}$ are degree-$4r$ polynomials in the Grassmann variables $\eta^{A}_{i}$. The invariance under the $R$-symmetry implies that $P_{n}^{r}$ are invariant under $SU(4)$ rotations of the Grassmann variables $\eta^{A}_{i}$. The lowest-order term in the $\eta$ expansion has Grassmann weight 8, while the highest-order term has Grassmann weight $4n - 8$. CPT conjugation exchanges weight $4r + 8$ with weight $4n - 4r - 8$. The $r = 0$ term in eq. (3.5) has $P_{n}^{0} = 1$ and contains all the $n$-point maximally helicity-violating amplitudes.

Component amplitudes may be extracted by multiplying the superamplitude with the appropriate product of superfields and integrating over all Grassmann parameters:

$$
A_{n}(k_{1}, h_{1} ; \ldots , k_{n}, h_{n}) = \int \prod_{i=1}^{n} d^{4} \eta_{i} \prod_{i=1}^{n} \Phi_{h_{i}}(\eta_{i}) A_{n}(k_{1}, \eta_{1}, \ldots , k_{n}, \eta_{n}).
$$

The superfields $\Phi_{h_{i}}(\eta_{i})$ have a single non-vanishing term corresponding to the field with helicity $h_{i}$. For example, NMHV $n$-point amplitudes appear inside the superamplitude $A_{n}$ as

$$
A_{n}(k_{1}, \eta_{1}, \ldots , k_{n}, \eta_{n}) = \cdots + (\eta_{1})^{4}(\eta_{2})^{4}(\eta_{3})^{4} A_{n}(-, -, -, +, \ldots , +) + (\eta_{1})^{4}(\eta_{2})^{4}(\eta_{4})^{4} A_{n}(-, -, -, +, \ldots , +) + \cdots ,
$$

where $(\eta)^{4}$ stands for the $SU(4)$-invariant expression $\frac{1}{24} \epsilon_{ABCD} \eta^{A} \eta^{B} \eta^{C} \eta^{D}$.

The (super)amplitudes of $\mathcal{N} = 4$ sYM theory can also be formulated in anti-chiral superspace; they are obtained from the chiral superspace expressions by conjugating all spinors (i.e. interchanging $\lambda$ and $\bar{\lambda}$ and their corresponding spinor products) and Fourier-transforming all Grassmann variables $\eta^{A}_{i}$. In the corresponding superfield the positive
helicity gluon wave function comes multiplied by $(\tilde{\eta})^4$ and the positive helicity gluon wave function has no $\tilde{\eta}$ factors.

The organization of non-supersymmetric amplitudes is less compact at a generic loop order. Tree-level gluon amplitudes may however be obtained from tree-level amplitudes of $\mathcal{N} = 4$ sYM by requiring that only gluons appear on the external lines. Similarly, for a scalar-coupled YM theory with scalars having the same quartic self-interaction as the $\mathcal{N} = 4$ scalars, tree-level scattering amplitudes can be obtained by truncating, e.g., eq. (3.5) to gluon and scalar external states. Since at tree level and for such external states no fermions can appear on the internal lines, the resulting bosonic amplitudes formally obey the same supersymmetry Ward identities as the corresponding $\mathcal{N} = 4$ sYM amplitudes with only bosons on external lines. In this restricted sense one may say that there is a linearized (extended) supersymmetry algebra (relating bosons to bosons) acting on the asymptotic space of states of the scalar-coupled YM theory.

The double-copy structure of the supergravity amplitudes implied at the tree-level by the KLT relations was clarified recently in [24], where it was realized that the gauge theory amplitudes can be arranged in a graph-organized representation so as to make manifest a duality between their color and kinematic factors. In the same organization, the $L$-loop amplitude is

$$A_{m}^{L-\text{loop}} = i^L g^{m-2+2L} \sum_{i \in \Gamma} \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D} \frac{1}{S_i} \prod_{\alpha} p_{\alpha}^2 C_i.$$  (3.8)

Here the sum runs over the complete set $\Gamma$ of $m$-point $L$-loop graphs with only cubic (trivalent) vertices, including all permutations of external legs, the integration is over the $L$ independent loop momenta $p_l$ and the denominator is given by the product of all propagators of the corresponding graph. The coefficients $C_i$ are the color factors obtained by assigning to every three-vertex in a graph a factor of the structure constant $\tilde{f}^{abc} = i\sqrt{2} f^{abc} = \text{Tr}([T^a, T^b]T^c)$ while respecting the cyclic ordering of edges at the vertex. The hermitian generators $T^a$ of the gauge group are normalized so that $\text{Tr}(T^a T^b) = \delta^{ab}$.

The coefficients $n_i$ are kinematic numerator factors depending on momenta, polarization vectors and spinors. For supersymmetric amplitudes in an on-shell superspace, they will also contain Grassmann parameters. The symmetry factors $S_i$ of each graph remove any overcount introduced by summing over all permutations of external legs (included by definition in the set $\Gamma$), as well as any internal automorphisms of the graph, i.e. symmetries of the graph with fixed external legs.

The color/kinematics duality [24] is manifest when the kinematic numerators of a graph representation of the amplitude satisfy antisymmetry and (generalized) Jacobi relations around each propagator — in one-to-one correspondence with the color-factors. That is, schematically for cubic-graph representations, it requires that

$$C_i + C_j + C_k = 0 \quad \Rightarrow \quad n_i + n_j + n_k = 0.$$  (3.9)

Related supergravity amplitudes are then trivially given in the same graph organization but with the color factors replaced by another copy of (a potentially different) gauge theory.
kinematic factors (which are not required to satisfy the duality):

\[ \mathcal{M}^{L-\text{loop}}_m = i^{L+1} \left( \frac{\kappa}{2} \right)^{m-2+2L} \sum_{i \in \Gamma} \prod_{l=1}^L \frac{d^D p_l}{(2\pi)^D} \frac{1}{S_l} \prod_{\alpha \in i} p_{\alpha}^{-i}. \]  

(3.10)

Here \( \kappa \) is the gravitational coupling. The Grassmann parameters that may appear in the two gauge theory factors are therefore inherited by the corresponding supergravity amplitudes.

The organization of \( \mathcal{N} = 4 \) supergravity states in on shell multiplets follows directly from eqs. (3.1) and (3.4). The states in each parenthesis on the right-hand side can be organized in an on shell chiral \( \mathcal{N} = 4 \) multiplet:

\[ \Phi^+ (\eta) = \bar{t} + \eta A^A \chi_A + \frac{1}{2!} \eta A^B \eta C \epsilon_{ABCD} \psi^{+,D} + \frac{1}{4!} \eta A^B \eta C \eta D \epsilon_{ABCD} h^{++}, \]

(3.11)

\[ \Phi^- (\eta) = h^{--} + \eta A^A \psi^- A - \frac{1}{2!} \eta A^B \eta C \epsilon_{ABCD} \chi^{-,D} + \frac{1}{4!} \eta A^B \eta C \eta D \epsilon_{ABCD} \bar{t}. \]

(3.12)

They are CPT conjugates of each other.\(^{29}\)

The double-copy construction of \( \mathcal{N} = 4 \) supergravity amplitudes thus manifestly preserves the linearized \( \mathcal{N} = 4 \) supersymmetry and the resulting amplitudes have manifest SU(4) R-symmetry. The relation to \( \mathcal{N} = 8 \) supergravity discussed in section 2.3 implies the existence, at least at tree level, of an additional U(1) symmetry — the lone U(1) in (2.19). We can identify the corresponding charges from the double-copy perspective as a combination of the little-group transformations on the two gauge theory factors:

\[ q_{U(1)} (\Phi \otimes \tilde{\Phi}) = 2q (h(\tilde{\Phi}) - h(\Phi)). \]

(3.13)

Here \( \Phi \) denotes a field in the \( \mathcal{N} = 4 \) factor, \( \tilde{\Phi} \) denotes a field in the non-supersymmetric factor, \( h(\Phi) \) is the helicity of \( \Phi \). \( \Phi \otimes \tilde{\Phi} \) denotes the supergravity field with helicity \( h(\tilde{\Phi}) + h(\Phi) \) and other quantum numbers given by the quantum numbers of \( \Phi \) and \( \tilde{\Phi} \).

Half-maximal supersymmetry is substantially less restrictive than maximal supersymmetry. Since, in particular, it acts only on one gauge theory factor, supersymmetry cannot relate amplitudes that differ in the field content or helicity assignment of the non-supersymmetric factor. For example, the four-graviton amplitudes \( \mathcal{M}(1^{++}2^{--}3^{++}4^{--}) \) and \( \mathcal{M}(1^{++}2^{--}3^{--}4^{++}) \), while related by supersymmetry in \( \mathcal{N} = 8 \) supergravity, belong to independent superamplitudes in a half-maximal \( \mathcal{N} = 4 \) supergravity theory. Because of this, the standard organization of amplitudes in \( \mathcal{N}^k \)MHV, following the number of Grassmann variables is no longer sufficiently descriptive. We will instead refer to amplitudes as \( \mathcal{N}^k \)MHV\(^{(p,q)} \) where \( p \) and \( q \) are the number of \( \Phi^+ \) and \( \Phi^- \) external states, respectively, and the total number of external legs is \( p + q \).

\(^{29}\)Since both multiplets are written in chiral superspace, a CPT transformation also assumes a Fourier transform of the Grassmann variables \( \eta \).
3.2 Special classes of anomalous amplitudes

The fact that one of the gauge theory factors is non-supersymmetric implies that certain amplitudes that vanish identically at tree level, are non-vanishing at one-loop level. Restricting ourselves to external gluons only, these amplitudes are the so-called all-plus and single-minus amplitudes: assuming all participating gluons are either incoming or outgoing, these amplitudes have all and all-but-one gluon of identical helicity, respectively. It is the supergravity amplitudes built out of such special non-supersymmetric amplitudes that we shall be focusing on.

As we shall see, through the double-copy construction, these gauge theory amplitudes lead to non-vanishing supergravity amplitudes with nonzero U(1) charge; they thus break this U(1) symmetry, i.e. it is anomalous. We stress that this anomaly cannot be interpreted as a failure of the double-copy construction because the results are checked against a direct evaluation of \( D \)-dimensional unitarity cuts.

It is interesting to note that, even from a gauge theory perspective, non-vanishing all-plus and single-minus amplitudes may be interpreted as a consequence of an anomaly of a symmetry of on-shell asymptotic states. Indeed, the tree-level scattering amplitudes of pure YM theory as well as of pure YM theory coupled to scalars with the same self-interactions as in \( \mathcal{N} = 4 \) sYM theory obey the same supersymmetry Ward identities as the tree-level amplitudes of \( \mathcal{N} = 4 \) sYM theory with only external bosons. One of the consequences of these Ward identities is the vanishing of the all-plus and single-minus amplitudes. Absence of the fermionic superpartners in the bosonic theory leads to one-loop amplitudes that no longer obey supersymmetry Ward identities — e.g. non-vanishing all-plus and single-minus amplitudes — and thus to a formal breaking of the above tree-level supersymmetry understood as acting only on bosonic asymptotic states. In this sense, the bosonic tree-level symmetry of the YM theory “inherited” from supersymmetry of the sYM theory is anomalous.

Moreover, non-vanishing one-loop amplitudes arise in dimensional regularization through an \( \epsilon/\epsilon \) mechanism similar to the one leading to chiral anomalies. For the all-plus amplitudes this property is manifest in their dimension-shifting construction \[52\] in terms of the MHV amplitudes of \( \mathcal{N} = 4 \) sYM theory\[30\]  
\[
\mathcal{A}^{(1); \text{YM}}_{n}(1^{+}, \ldots, n^{\mp}) = -2\epsilon(1 - \epsilon)(4\pi)^2 \frac{1}{(ij)^4} \mathcal{A}^{(1); \text{sYM MHV}}_{n}(1^{+}, \ldots, i^{-}, \ldots, j^{-}, \ldots, n^{+})
\]
\( D \rightarrow D + 4 \).

(3.14)

Here \( \mathcal{A}^{(1)} \) denotes the color-dressed one-loop \( n \)-point amplitude. The dimension shift \( D \rightarrow D + 4 \) yields an \( \epsilon^{-1} \) UV divergence which is then cancelled by the overall \( \epsilon \) factor above.\[31\]

To see that the all-plus and single-minus YM amplitudes potentially can lead to amplitudes breaking the U(1) symmetry of (2.19) it suffices to discuss an example. The external legs of an \( (n + 2) \)-point MHV amplitude are two positive and \( n \) negative helicity gluons. Tensoring each external state with the positive helicity gluon of an all-plus amplitude and

\[30\]While extensively tested, the origin of this rather curious relation remains to be understood. The amplitude representations obtained this way can be easily checked against the results of explicit calculations.

\[31\]One starts with the amplitude in \( D = 4 - 2\epsilon \) and then performs the shift \( D \rightarrow D + 4 \) so that all integrals over loop momenta are evaluated in \( 8 - 2\epsilon \) dimensions while the external momenta remain in \( D = 4 \).
identifying the supergravity states using (3.1) and (3.2) leads to two positive-helicity gravitons and \( n \) \( \bar{n} \) scalars (see (3.2)). This amplitude carries the charge \( 4qn = -2n \) under the U(1) symmetry in (2.19) and thus, if non-vanishing, is an anomalous amplitude, i.e. it breaks this symmetry.

Anticipating the result of the next section that such superamplitudes are non-vanishing, we can classify the supergravity superamplitudes that can be constructed on the basis of the corresponding gravity superamplitudes as well. Denoting the double-copy operation by the tensor product symbol \( \otimes \), the independent one-loop anomalous supergravity superamplitudes (with rational momentum dependence) can be organized in the two classes,

\[
\begin{align*}
A^{(1)}_{n}; \text{sYM } N^{4}_{\text{MHV}} & \otimes A^{(1)}_{n}; \text{YM } (1^+, \ldots, i^+, \ldots n^+) , \\
A^{(1)}_{n}; \text{sYM } N^{4}_{\text{MHV}} & \otimes A^{(1)}_{n}; \text{YM } (1^+, \ldots, i^-, \ldots n^+) ,
\end{align*}
\]

with \( k = 0, \ldots, n-4 \). Since there is no symmetry relating a positive-helicity gluon to a negative-helicity one in pure YM theory these superamplitudes are unrelated to each other. Two more classes of amplitudes can be constructed by conjugation; in the language of the double-copy construction their pure YM components are the all-minus and single-plus amplitudes.

It is interesting to note that, starting with five external legs, it is, in principle, possible that other supergravity amplitudes are also anomalous. A potential example is provided by the supergravity amplitude obtained from the one-loop five-point MHV \( \mathcal{N} = 4 \) sYM superamplitude and the one-loop five-point MHV pure YM amplitude; for any choice of external helicities this superamplitude carries nonzero U(1) charge. Similarly to the amplitudes in eqs. (3.15) and (3.16), this amplitude also vanishes at the tree level. If non-vanishing at loop level, it can have non-rational dependence on momentum invariants. While it would be interesting to construct and analyze it, we will not do it here and focus instead on superamplitudes in the two classes (3.15) and (3.16).

All external states of the amplitudes of the first type, eq. (3.15), belong to \( \Phi^{+} \) on shell chiral multiplets, eq. (3.11), whereas those of the CPT-conjugate amplitudes belong to on shell chiral multiplets of the type \( \Phi^{-}(\eta) \), eq. (3.12). It is perhaps natural to use an anti-chiral superspace for one of them. We will later use the anti-chiral superspace for the former. For \( k = 0 \) it is easy to write a superspace expression for the contribution of such amplitudes to the effective Lagrangian; it consists of a sum of a chiral and anti-chiral superspace integrals

\[
\mathcal{L}_{\text{eff}} = \frac{1}{(4\pi)^2} \left( \int d^8\theta \sum_n d_n W^n + \int d^8\bar{\theta} \sum_n d_n \bar{W}^n \right),
\]

so that the action is hermitian for real coefficients \( d_n \). They are to be determined by explicit calculations. The component expansion of the chiral superfield \( W(x, \theta) \) is

\[
W(x, \theta) = \tau + \theta^A \chi^A_{\alpha} + \theta^A \phi^B F_{\alpha\beta} C D {\epsilon}^{ABCD} + \theta^A \theta^B \theta^C \psi_{\alpha\beta\gamma\delta} D {\epsilon}^{ABCD} + \theta^A \theta^B \theta^C \phi^D C_{\alpha\beta\gamma\delta} {\epsilon}^{ABCD},
\]

(3.18)
and it contains the same states as the momentum space on shell superfield \((3.12)\). Here \(C_{\alpha\beta\gamma\delta}\) is totally symmetric spinor component of the self-dual Weyl tensor which on shell is the same as the self-dual curvature tensor \(R^{-}\), describing the negative helicity graviton. The Grassmann variables \(\theta\) and \(\eta\) are formally related by Fourier transform. Note that the dimension of \(W\) is zero and therefore any power of this superfield has dimension zero so that the one-loop multi-point amplitude has a correct dimension.

The effective Lagrangian for amplitudes of the second type, eq. \((3.16)\), is somewhat more involved. It depends on the anti-chiral superfield \(W\) and its space-time derivatives as well as on the anti-chiral superfield \(C_{\alpha\beta\gamma\delta}(x, \theta, \bar{\theta})\) whose first component is the self-dual Weyl tensor \(C_{\alpha\beta\gamma\delta}(x)\):\(^{32}\)

\[
D_{\eta}^{\dagger}W(x, \theta, \bar{\theta}) = 0, \quad D_{\eta}^{\dagger}C_{\alpha\beta\gamma\delta}(x, \theta, \bar{\theta}) = 0. \tag{3.19}
\]

Since the dimension of \(C_{\alpha\beta\gamma\delta}(x)\) is 2 and its indices must be contracted with some derivative factors, a certain amount of non-locality is necessary to write down the effective Lagrangian; because of this it is naturally written in momentum space. For \(k = 0\) in \((3.16)\) we can write a superspace expression for the contribution of such amplitudes to the momentum space effective Lagrangian:

\[
L(p) = f_{4}\left(\sum_{i=1}^{4} p_{i}\right) \left(\int d^{8}\theta C_{\alpha\beta\gamma\delta}(p_{1}, \bar{\theta}) D_{\alpha}^{\dagger} D_{\beta}^{\dagger} W(p_{2}, \bar{\theta}) D_{\gamma}^{\dagger} D_{\delta}^{\dagger} W(p_{3}, \bar{\theta}) W(p_{4}, \bar{\theta}) + \int d^{8}\theta \bar{C}_{\bar{\alpha}\bar{\beta}\bar{\gamma}\bar{\delta}}(p_{1}, \theta) D_{\bar{\alpha}}^{\dagger} D_{\bar{\beta}}^{\dagger} W(p_{2}, \theta) D_{\bar{\gamma}}^{\dagger} D_{\bar{\delta}}^{\dagger} W(p_{3}, \theta) W(p_{4}, \theta)\right). \tag{3.20}
\]

The coefficient \(f_{4}\) can be determined by an explicit scattering amplitude calculation (see section 3.4). The effective Lagrangian is again a sum of a chiral and anti-chiral superspace integrals and therefore, if \(f_{4} \neq 0\), it breaks the \(U(1)\) symmetry. Eq. \((3.20)\) can be easily extended to capture higher-point amplitudes of the type \((3.16)\) with \(k = 0\): one simply multiplies the chiral and anti-chiral integrands above by further factors of \(W\) and \(\bar{W}\), respectively. Other generalizations may also be possible, but we will not discuss them here.

It is worth mentioning that, while in the discussion above we assumed that one of the two gauge theory factors is the \(\mathcal{N} = 4\) SYM theory, the same analysis can be carried out if \(\mathcal{N} = 4\) theory replaced by the \(\mathcal{N} = 1\) or the \(\mathcal{N} = 2\) SYM theory. As argued in the previous section, the resulting supergravity theories should have a \(U(1)\) symmetry analogous to that in eq. \((2.19)\) and the scattering amplitudes constructed as above should provide candidates for anomalous amplitudes breaking it. The complete classification of the supergravity amplitudes in this case is slightly more involved than just \((3.15)\), \((3.16)\) due to the existence of several types of \(\mathcal{N}^{k}\) MHV amplitudes in \(\mathcal{N} = 1, 2\) SYM theories.

The discussion above can also be extended trivially to a non-supersymmetric SYM theory coupled with scalars; we will do this in section 3.5.

Let us now turn to examples of anomalous amplitudes in pure \(\mathcal{N} = 4\) supergravity.

\(^{32}\)The existence of a linearized anti-chiral superfield \(C_{\alpha\beta\gamma\delta}\) is a consequence of the fact that the spectrum of \(\mathcal{N} = 4\) supergravity does not contain spin-5/2 states and that the graviton is on shell. Indeed, assuming that the right-hand side of the second equation in \((3.19)\) is non-zero we can decompose it into a completely symmetric spin-5/2 term and a term proportional to \(D^{\alpha\dagger}C_{\alpha\beta\gamma\delta}\) which is set to zero by on-shell conditions. Absence of spin-5/2 fields sets to zero the first term as well, leading to the second equation in \((3.19)\).
3.3 Graviton-scalar amplitudes in $\mathcal{N} = 4$ supergravity

There are many graviton-scalar amplitudes in $\mathcal{N} = 4$ supergravity and most of them have counterparts in $\mathcal{N} = 8$ supergravity. Here we shall focus on the one-loop amplitudes described in the previous section, which carry non-zero U(1) charge. All-plus and single-minus amplitudes have been constructed in [53]; the all-plus amplitudes can also be obtained by dimension-shifting (3.14) the color/kinematics-satisfying representations of the one-loop four-point [54], five-point [55] and six- and seven-point [56] $\mathcal{N} = 4$ MHV superamplitudes. The six- and seven-point calculations use the general framework [57] which, in principle, can be used at higher multiplicities.

We will collect here the results for the three-, four- and five-point supergravity superamplitudes which contain component amplitudes with two positive-helicity gravitons and one, two and three scalar fields $\bar{t}$. Relegating some of the details of the calculations to appendices C, D and E, we find:

\[
\mathcal{M}^{(1);\mathcal{N}=4 PSG}_{3}(1, 2, 3) = \frac{i}{(4\pi)^2} \left(\frac{\kappa}{2}\right)^3 \delta(8) \left(\sum_{i=1}^{3} \tilde{\eta}_{i,A} \tilde{\lambda}_{i}\right), \tag{3.21}
\]

\[
\mathcal{M}^{(1);\mathcal{N}=4 PSG}_{4}(1, 2, 3, 4) = \frac{i}{(4\pi)^2} \left(\frac{\kappa}{2}\right)^4 \delta(8) \left(\sum_{i=1}^{4} \tilde{\eta}_{i,A} \tilde{\lambda}_{i}\right), \tag{3.22}
\]

\[
\mathcal{M}^{(1);\mathcal{N}=4 PSG}_{5}(1, 2, 3, 4, 5) = \frac{2i}{(4\pi)^2} \left(\frac{\kappa}{2}\right)^5 \delta(8) \left(\sum_{i=1}^{5} \tilde{\eta}_{i,A} \tilde{\lambda}_{i}\right). \tag{3.23}
\]

Since they are proportional to eight powers of $\tilde{\eta}$ we may refer to them as MHV$^{(n,0)}$ superamplitudes, with $n = 3, 4, 5$, because all their external legs are in the $\Phi^+$ multiplet. The MHV$^{(0,n)}$ amplitudes, with negative-helicity gravitons and scalars $t$ are obtained by conjugation (i.e. through $\tilde{\lambda} \rightarrow \lambda$, $\tilde{\eta} \rightarrow \eta$).

It should be mentioned that, as discussed in section C, the three-point superamplitude is not constructed directly via the double-copy relation [25]. This is because, at least naively, the three-point (one-)loop amplitudes vanish identically in $\mathcal{N} = 4$ sYM theory while the three-point all-plus amplitude is formally singular [58]. As a means of regularizing this 0/0 situation we construct this superamplitude as the soft-graviton limit of the $k = 0$ four-point one-loop amplitude of the type (3.16).

For completeness, we mention that the four- and five-point superamplitudes (3.22) and (3.23) are of type (3.15) with $k = 0$ and $k = 1$, respectively.

These amplitudes are forbidden in $\mathcal{N} = 8$ supergravity by the SU(8) on-shell R-symmetry. In particular, since they are by construction invariant under the SU(4) R-symmetry of one $\mathcal{N} = 4$ sYM theory factor and the external states are inert under the SU(4) R-symmetry of the other $\mathcal{N} = 4$ sYM theory factor, it is the U(1) that appears in the decomposition (2.19), SU(8) $\supset$ SU(4) $\times$ SU(4) $\times$ U(1), that forbids the amplitudes (3.21), (3.22), (3.23) in the $\mathcal{N} = 8$ theory. We therefore see explicitly that this symmetry, while present at tree-level in $\mathcal{N} = 4$ supergravity, is broken at the one-loop level.

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33We thank Z. Bern and L. Dixon for sharing their unpublished notes on this calculation.
To extract component amplitudes we multiply the superamplitude by the appropriate on-shell superfields and integrate over all \( \eta \) variables. The terms in the superamplitude that are proportional to \((\hat{\eta}_1)^4(\hat{\eta}_2)^4\) are the component amplitudes \(\mathcal{M}(1^{++}2^{++}3^+)\), \(\mathcal{M}(1^{++}2^{++}3^+4^1)\) and \(\mathcal{M}(1^{++}2^{++}3^+4^15^1)\), respectively. Presence of external scalar fields is, however, not a requirement as among the superpartners of, e.g., \(\mathcal{M}(1^{++}2^{++}3^+)\) we can find the graviton-vector-vector amplitude \(\mathcal{M}(1^{++}2^{++}3^+)\). This term can also be found in the superspace expression (3.17).

### 3.4 Nonlocal amplitudes in \(\mathcal{N} = 4\) supergravity

The characteristic property of the amplitudes discussed in the previous section is that they are local. As such, while still anomalous, they may be adjusted or even eliminated completely by simply defining the theory to contain finite local counterterms that simply set them to zero. The same cannot be immediately said about nonlocal amplitudes; it is therefore of interest to see whether there exist such amplitudes which are also anomalous.

Perhaps the simplest such amplitude is \(\mathcal{M}(1^{--}2^{++}3^+4^1)\), whose soft graviton limit leads to (3.21). The calculation in appendix C implies that, in anti-chiral superspace, this superamplitude is

\[
\mathcal{M}^{(1):\mathcal{N}=4\text{PSG}}_{4}(1, 2, 3, 4) = -\frac{i}{(4\pi)^2} \frac{1}{[31][14]} \frac{32[24]}{[21]} \delta^{(8)} \left( \sum_{i=1}^{4} \hat{\eta}_{i,A} \hat{\lambda}_{i} \right). \tag{3.24}
\]

Extracting component amplitudes (see appendix C for details) does not remove the apparent nonlocality of this expression. Note that the spinor product ratio prefactor can also be written as \(<1|\hat{k}_2\hat{k}_3|1>/(stu)\) which reproduces the momentum dependence of the four-point superamplitude following from the superfield expression (3.20). This implies that, as anticipated, \(f_4 \neq 0\).

It is also possible to construct the five-point supergravity superamplitudes of type (3.16); the result appears to be nonlocal and rather unwieldy and we will not present it here.

The five-point nonlocal superamplitude of type (3.15) can also be easily constructed: it corresponds to choosing \(k = 0\) in (3.15). The result of its calculation is (see appendix E)

\[
\mathcal{M}^{(1):\mathcal{N}=4\text{PSG}}_{5}(1, 2, 3, 4, 5) = i \left(\frac{\kappa}{2}\right)^5 \frac{\delta^{(8)}(Q_5)}{(4\pi)^2} \sum_{s_5} \frac{1}{s_{12}} \left(\hat{\gamma}_{12}\right)^2
\]

\[
= i \left(\frac{\kappa}{2}\right)^5 \frac{\delta^{(8)}(Q_5)}{(4\pi)^2} \left[\frac{\tilde{\gamma}_{12}^2}{s_{12}} + \frac{\tilde{\gamma}_{13}^2}{s_{13}} + \frac{\tilde{\gamma}_{14}^2}{s_{14}} + \frac{\tilde{\gamma}_{15}^2}{s_{15}} + \frac{\tilde{\gamma}_{23}^2}{s_{23}} + \frac{\tilde{\gamma}_{24}^2}{s_{24}} + \frac{\tilde{\gamma}_{25}^2}{s_{25}} + \frac{\tilde{\gamma}_{34}^2}{s_{34}} + \frac{\tilde{\gamma}_{35}^2}{s_{35}} + \frac{\tilde{\gamma}_{45}^2}{s_{45}}\right] \tag{3.25}
\]

with

\[
\tilde{\gamma}_{12} = \frac{[12]^2[34][45][35]}{(12)[23][35][15] - [12][23][35]} \tag{3.26}
\]

and the other \(\tilde{\gamma}_{ij}\) obtained by relabeling. \(Q_5\) is defined in eq. (E.4). A component S-matrix element contained in this superamplitude is \(\mathcal{M}^{(1)}_{5}(h^{++}, h^{++}, t, A^+, A^+)\).

It is interesting to note that, up to a trivial numerical coefficient, the square bracket in eq. (3.25) is also the divergence of the five-point amplitude of \(\mathcal{N} = 8\) supergravity in
$D = 8 - 2\epsilon$ [55]. The collinear limits have the expected behavior dictated by the fact that the amplitudes evaluated here and their four-point counterparts vanish exactly at tree level, implying that only the tree-level splitting amplitudes are necessary. These features suggest that, as in the case of $\mathcal{N} = 8$ supergravity, the amplitude (3.25) is related to the covariantization of a four-point term in the effective action.

### 3.5 Amplitudes in $\mathcal{N} = 4$ supergravity coupled to $n_v$ vector multiplets

We may explore further the relation between the matrix element\footnote{We use the definition $R^\pm = \frac{1}{2}(R \pm iR^\ast)$ for the “self-dual” curvature components.} $\langle(R^+)^2\bar{t}\rangle$ and the anomaly of the U(1) symmetry in (2.19) by coupling $\mathcal{N} = 4$ supergravity with $n_v$ vector multiplets and comparing the S-matrix elements with the corresponding anomalous term in the effective action.

As discussed above, in the KLT-based generalized unitarity approach to supergravity scattering amplitudes as well as in the double-copy construction, the scattering amplitudes in $\mathcal{N} = 4$ supergravity coupled to $n_v$ vector multiplets may be found by supplementing the non-supersymmetric gauge theory factor by $n_v$ real scalar fields,

\[
(\mathcal{N} = 4 \text{ sYM}) \otimes [(\mathcal{N} = 0 \text{ sYM}) \oplus n_v \text{ real scalars}].
\] (3.27)

The $n_v$ scalars couple to the $\mathcal{N} = 0$ gluons though the standard minimal coupling and also have quartic self-couplings similar to the scalar fields of $\mathcal{N} = 4$ sYM theory.\footnote{Since we are interested only in scattering amplitudes of $\mathcal{N} = 0$ gluons the scalar self-couplings need not be specified.}

At one-loop level all fields of a gauge theory make independent contributions to gluon scattering amplitudes. Moreover, for the finite helicity amplitudes, supersymmetric Ward identities imply that the contributions of particles of different spin circulating around the loop are related, $A_{n_{1}}^{(1);s=1} = -A_{n_{1}}^{(1);s=1/2} = A_{n_{1}}^{(1);s=0}$, where the spin zero particle is a complex scalar. Thus, we may express the all-plus and single-minus pure YM amplitudes in the previous section in terms of the contribution of the scalar fields circulating in the loop. We can do this directly by using $A_{n_{1}}^{(1);s=1} = A_{n_{1}}^{(1);s=0}$ or by writing the spectrum of free YM theory in terms of the spectra of supersymmetric YM theories:

\[
(\mathcal{N} = 0 \text{ sYM}) = (\mathcal{N} = 4 \text{ sYM}) - 4(\mathcal{N} = 1 \text{ chiral multiplet}) + 2 \text{ real scalars}.
\] (3.28)

Here the second term on the right-hand side cancels the contribution of the $\mathcal{N} = 4$ fermions and the third term cancels the contribution of the $\mathcal{N} = 4$ and $\mathcal{N} = 1$ scalars. Due to supersymmetry, the contribution of the first two multiplets to the all-plus and single-minus amplitudes vanishes identically and one concludes that these amplitudes are given by two real scalars (or one complex scalar) running in the loop.

Thus, in the theory $(\mathcal{N} = 0 \text{ sYM}) \oplus (n_v \text{ real scalars})$ the all-plus and single-minus one-loop gluon amplitudes are given by $n_v + 2$ real scalars running in the loop and can therefore be obtained by multiplying the amplitudes in pure YM theory by $\frac{1}{2}(n_v + 2)$. Proceeding to the $\mathcal{N} = 4$ supergravity coupled to $n_v$ vector multiplets, since in the double-copy approach the integrands of the anomalous S-matrix elements are proportional to the
all-plus or single-minus YM amplitudes, they can be obtained from those of pure $\mathcal{N} = 4$ supergravity by multiplication by $\frac{1}{2}(n_v + 2)$.

The $n_v$-dependence of the overall coefficient matches the one in eq. (2.27); this suggests that the U(1) anomaly captured by the scattering amplitude calculation is that of the U(1) subgroup appearing in (2.19).

4 Effective action from scattering amplitudes

As reviewed in section 1, it was shown in [18] that the tree-level amplitudes of $\mathcal{N} = 8$ supergravity vanish identically in the single soft scalar limits (i.e. the limit in which the momentum of a single scalar field goes to zero). Moreover, it was argued that this is a manifestation of the $E_{7(7)}$ duality symmetry of the theory and that, in the absence of $E_{7(7)}$ anomalies, this should extend to all-loop orders. One may also understand the vanishing of the soft scalar limit in $\mathcal{N} = 8$ supergravity as a consequence of the SU(8) symmetry which, if unbroken, requires that all amplitudes corresponding to field configurations that are not SU(8) invariant vanish identically. To see this, let us consider a general $n$-point scattering amplitude that is SU(8) invariant and has at least an external scalar. Then, since the scalar fields transform under SU(8), the field configuration obtained by dropping this scalar field cannot be SU(8) invariant and thus the corresponding $(n - 1)$-point amplitude should vanish identically. Thus the soft scalar limit of the $n$-point amplitude should be zero if SU(8) is not anomalous.

Let us study the single soft-scalar limits of the anomalous amplitudes constructed in the previous section. We will find that, while vanishing at the tree level, the soft scalar limits are non-vanishing at the one-loop level. We will then use them to find which “anomalous” terms in the effective action correspond to amplitudes breaking the U(1) symmetry.

4.1 Soft scalar limit and the single-multiplet soft scalar function

Since $\mathcal{N} = 4$ supergravity is a consistent orbifold projection of $\mathcal{N} = 8$ supergravity [21], its tree-level amplitudes are a subset of those of $\mathcal{N} = 8$ supergravity and therefore should also have vanishing soft scalar limits at tree level. This is a consequence of the U(1) symmetry (2.19) because the scalar field labeling the $\mathcal{N} = 4$ supergravity amplitudes is not charged under the SU(4) R-symmetry. The fact that there exist sequences of loop-level amplitudes that break this U(1) and differ only in the number of external scalars implies that their soft scalar limits are non-vanishing. Below we will construct the resulting soft scalar function and use it to find all one-loop terms in the effective action with holomorphic dependence on $t$.

Scattering amplitudes have universal factorization properties in the limit in which the momentum of an external particle (say, the $n$-th one) is soft. In general, the $L$-loop scattering amplitudes behave as [59–62]

$$M_n^{(L)}(1, 2, \ldots n - 1, n) \xrightarrow{k_n \to 0} \frac{\kappa}{2} \sum_{l=0}^{L} S_n^{(l)} M_{n-1}^{(L-l)}(1, 2, \ldots n - 1). \quad (4.1)$$
In this expression the coupling constant $\kappa/2$ is assumed to be placed in vertices. It has been argued in [63] that, in pure supergravity theories, the soft scalar function does not receive loop corrections; therefore, the expression above collapses to a single term

$$\mathcal{M}^{(L)}(1, 2, \ldots n - 1, n) \xrightarrow{L=0} \frac{\kappa}{2} S^{(0)}_n \mathcal{M}^{(L)}_{n-1}(1, 2, \ldots n - 1). \quad (4.2)$$

For color-ordered amplitudes in a gauge theory the soft scalar factor depends on the label of the soft leg (here $n$) and the legs adjacent to it (i.e. 1 and $(n-1)$). In supergravity (as in any unordered theory) the soft factor $S_n$ depends on all the external legs. In tree-level (super)gravity it is typically given by

$$S_n = \sum_{i=2}^{n-2} s(1, i, n - 1, n). \quad (4.3)$$

Despite appearances, $S_n$ is symmetric under the interchange of legs 1 and $(n-1)$ with the others. This expression captures the fact that, in an unordered amplitude, some fixed external leg can formally be “adjacent” to any two other legs.

The examples of amplitudes in eqs. (3.21), (3.22) and (3.23) are sufficient to determine the soft scalar functions for the case when all particles belong to the same $\mathcal{N} = 4$ multiplet as the soft scalar. We should stress that the functions $s(1, i, n - 1, n)$ determined from them may not be correct when any of the external legs 1, $i$, $(n-1)$ belongs to a different multiplet than the soft scalar leg, $n$.

Regardless of the arguments of [63] on the non-renormalization of soft functions, the fact that the anomalous amplitudes vanish identically at tree level implies that only the leading order soft scalar function is important at one-loop level. Extracting the component amplitudes and fitting the soft scalar limit onto eq. (4.2) and with $S^{(0)}_n$ of the form (4.3) implies that

$$S^{(0)}_n = \sum_{s=2}^{n-2} 0 + \sum_{s=3/2}^{1} \frac{1}{4} + \sum_{s=1}^{1/2} \frac{1}{2} + \sum_{s=1/2}^{3/4} \frac{3}{4} + \sum_{s=0}^{1/2} 1 = \sum_{i=2}^{n-2} \frac{1}{2} (2 - h_{\text{external}, i}), \quad (4.4)$$

where the sums run over all the external legs of the amplitude except for the one on which the soft scalar limit is taken and $h_{\text{external}, i}$ is the helicity of the $i$-th external leg. We stress that, due to its derivation, this expression holds only for amplitudes in which all external legs are in the same on-shell $\mathcal{N} = 4$ multiplet. By construction, this soft scalar function allows one to obtain the superamplitudes (3.21)–(3.23) from each other.

The soft scalar function (4.4) may also be written in superspace. To this end one selects the $n$-th external leg of the superamplitude and chooses its Grassmann parameter

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36 It is interesting to also consider soft scalar limits of one-loop amplitudes that do not vanish at tree-level, i.e. of amplitudes with vanishing total $U(1)$ charge. The fact that the anomalous $U(1)$ is a symmetry of tree-level amplitudes implies that the structure of the amplitude in the soft scalar limit is that in eq. (4.2) with $L = 1$ and with $\mathcal{M}^{(1)}_{n-1}$ being a one-loop anomalous amplitude. It seems therefore that we can expect that the one-loop amplitudes that have a non-zero tree-level counterpart (and thus are allowed in $\mathcal{N} = 8$ supergravity) also have non-zero soft scalar limits.

37 We have also successfully tested this expression for the spin-2 fields in the conjugate $\mathcal{N} = 4$ on-shell multiplet.
dependence such that it corresponds to a scalar field (i.e. one multiplies the anti-chiral superspace superamplitude by \((\bar{\eta}_n)^4\) and integrates over \(\bar{\eta}_n\)) and then sets \(k_n = 0\). The result is:

\[ S_n^{(0)} = n - 3. \]  

(4.5)

We can use this soft scalar function to construct higher-point superamplitudes with all fields belonging to the same multiplet by starting from the lower-point ones. This is analogous to the inverse soft limit used at tree level to construct all tree amplitudes. In that case a detailed analysis was necessary to identify the origin of the soft limit and the relevant BCFW shifts. A similar analysis is not needed here because there is no momentum dependence associated with the additional scalar leg.

4.2 Inverse soft scalar limit and the \(\text{MHV}^{(0,n+2)}\) and \(\text{MHV}^{(n+2,0)}\) superamplitudes

The general form of an \((n + 2)\)-point \(\text{MHV}^{(n+2,0)}\) amplitude (i.e. the MHV with all fields belonging to the \(\Phi^+\) multiplet, (3.11)) is

\[ \mathcal{M}_{n+2}^{(1); N=4} (1, 2, \ldots, n + 2) = c_{n+2} \delta^{(n)} \left( \sum_{i=1}^{n+2} \bar{\eta}_i A \bar{\lambda}_i \right), \]  

(4.6)

where \(c_n\) can be a function of momentum invariants which has no multi-particle poles. The analogous MHV\((0,n+2)\) amplitude, containing the matrix element \(\langle h^{--} h^{--} (t)^n \rangle\), is equally simple and is naturally written in chiral superspace: it is simply obtained by the transformation \(\lambda_{\alpha i} \leftrightarrow \bar{\lambda}_{\dot{\alpha}, i}\) and \(\eta \leftrightarrow \bar{\eta}\). These superamplitudes are related by soft scalar limits. Using the superspace form of the soft scalar function (4.5) and accounting for the absence of multi-particle poles, it is not difficult to see that all the coefficient functions are constant

\[ c_3 = \frac{i}{(4\pi)^2} \left( \frac{\kappa}{2} \right)^3, \quad c_{n+2} = \frac{i}{(4\pi)^2} (n - 1)! \left( \frac{\kappa}{2} \right)^{n+2} \text{ for } n \geq 2, \]  

(4.7)

where for \(n = 1\) we used eq. (3.21). Thus, the \((n + 2)\)-point \(\text{MHV}^{(n+2,0)}\) superamplitude is

\[ \mathcal{M}_{n+2}^{(1); N=4} (1, 2, \ldots, n + 2) = \frac{i}{(4\pi)^2} (n - 1)! \left( \frac{\kappa}{2} \right)^{n+2} \delta^{(n)} \left( \sum_{i=1}^{n+2} \bar{\eta}_i A \bar{\lambda}_i \right), \]  

(4.8)

and its graviton-scalar component is\(^3\)

\[ \mathcal{M}_{n+2}^{(1); N=4} (h_1^{++}, h_2^{++}, \tilde{t}_3, \ldots, \tilde{t}_{n+2}) = \frac{i}{(4\pi)^2} (n - 1)! \left( \frac{\kappa}{2} \right)^{n+2} [12]^4. \]  

(4.9)

Let us now construct the graviton-scalar effective action that reproduces the amplitudes in eq. (4.9). We assume that the classical supergravity action is normalized as \[^6\]

\[ S = \frac{1}{2} \left( \frac{\kappa}{2} \right)^{-2} \int d^4x \sqrt{g} R + \ldots, \]  

(4.10)

\(^3\)It may be useful to compare our approach with other inverse soft-limit loop-level constructions, that in contrast to what we have done, use soft-functions visible at tree-level. Using the tree-level graviton soft function, e.g., the rational terms in the one-loop \(n\)-graviton amplitudes in \(N = 4\) supergravity were recently constructed in [64].
i.e. $\kappa$ is the gravitational coupling. The term in the effective action that contributes to the matrix element $\langle h^{++}h^{++} (\bar{t})^n \rangle$ in eqs. (4.9) and has the highest number of fields is $(R^\pm = \frac{1}{2}(R \pm iR^*))$

$$\Gamma_{n+2}^{(1)} = \int d^4x \, s_n \, (R^+)^2 \bar{t}^n + c.c. \quad (4.11)$$

It is not difficult to construct the contribution to the two-graviton–$n$-scalar S-matrix element of such an effective action term:

$$\mathcal{M}_{\Gamma_{n+2}^{(1)}} (h_1^{++}, h_2^{++}, \bar{t}_3 \ldots \bar{t}_{n+2}) = 2i \, s_n \, n! \left( \frac{\kappa}{2} \right)^{n+2} \quad [12]^4 \quad (4.12)$$

where $k_1$ and $k_2$ are the momenta of the two gravitons and the overall $n!$ factor accounts for the nonstandard normalization of an (effective) action term containing the $n$-th power of a scalar field (the overall factor of 2 has a similar origin).

To determine the numerical coefficient $s_n$ we compare (4.9) with the same S-matrix element computed from the effective action; the latter is given by the sum of (4.12) and the contribution of Feynman graphs with one vertex from the lower-point effective action and the other vertices from the tree-level Lagrangian. It is not difficult to see that the tree-level part of such Feynman graphs is forbidden by supersymmetry if all its external lines are on shell. Since in Feynman graphs the internal lines are off shell, these tree-level Green’s functions give a contribution proportional to the square of the momentum of the internal line, i.e. they are proportional to the tree level equation of motion for the off-shell leg.\(^{39}\) Such Green’s functions can be set to zero by a local field redefinition in the corresponding effective action.

Thus (4.12) represents the complete contribution of $\Gamma^{(1)}$ to the one-loop two-graviton–$n$-scalar S-matrix element. Comparing it with (4.9) we find that the coefficient $s_n$ is

$$s_n = \frac{1}{2(4\pi)^2} \frac{1}{n} \quad (4.13)$$

Therefore, the one-loop effective action with two gravitons and any number of scalar fields is

$$\Gamma_{hh}^{(1)} = \sum_{n=1}^{\infty} \Gamma_{n+2}^{(1)} = \frac{1}{2(4\pi)^2} \int d^4x \, (R^+)^2 \left( \bar{t} + \sum_{n \geq 2} \frac{1}{n} \bar{t}^n \right) + c.c. \quad (4.14)$$

This expression may be summed up as

$$\Gamma_{hh}^{(1)} = -\frac{1}{2(4\pi)^2} \int d^4x \, (R^+)^2 \ln(1 - \bar{t}) + c.c. \quad (4.15)$$

While bosonic, it was constructed from scattering amplitudes that manifestly preserve linearized (asymptotic-state) supersymmetry and thus may be promoted to a superspace expression in terms of analogs of the superfields (3.18) whose fields are identified with the

\(^{39}\)For example, using the supergravity action, one can see that an amplitude with a single $\tau$ (the full supergravity field) and any number of $\bar{\tau}$ fields is indeed proportional to $\square \tau$. Also, it is impossible to draw tree-level graphs with the field configuration $h^{++}h^{--}\bar{\tau}^m$ because all vertices contain both $\tau$ and $\bar{\tau}$. The matrix elements of such operators are set to zero by the supersymmetry Ward identities.
double-copy fields (3.1). Extracting the two-graviton component of (3.17) it is easy to see that the coefficients \(d_n\) in that equation are given by

\[
d_{n+2} = \frac{1}{(n+2)(n+1)} s_{n+2} = \frac{1}{2(4\pi)^2} \frac{1}{n(n+1)(n+2)}.
\]

The resulting resumed one-loop effective action in linearized superspace has a relatively simple but unilluminating expression and we will not present it here.

The derivation above can be repeated with minimal changes in the presence of additional \(n_v\) \(\mathcal{N} = 4\) vector multiplets. The discussion in section 3.5 implies that the analogs of eqs. (4.8) and (4.9) pick up an overall factor of \((2 + n_v)/2\). Consequently, the only change to \(\Gamma^{(1)}_{hh}\) and the \(d_{n+2}\) coefficients is a multiplicative factor of \((2 + n_v)/2\).

Interpreting eq. (4.15) as the local off-shell extension of the local on-shell effective action one may proceed to construct the nonlocal part of the on-shell effective action from non-local amplitudes, some of which were described in section 3.4. Given a particular field configuration, one should first compute the one-loop amplitude with that external field configuration and then subtract from it the corresponding tree-level amplitude with exactly one vertex from the local one-loop effective action and all the others from the tree-level action. Whenever nonzero, this difference represents a new contribution to the on-shell effective action. While potentially non-local, such terms are dependent on the local effective action being chosen not to contain terms proportional to the classical equations of motion; relaxing this requirement will modify the non-local effective action by further local terms which are not necessarily proportional to the classical equations of motion.

### 4.3 Comparison with the anomaly-induced term in the effective action

In general, the local part of the effective action depends on the regularization scheme used to construct it. Assuming that the scheme selected by the double-copy construction preserves the shift symmetry of the supergravity axion field \(B (2.39)\), we can compare (4.15) with the anomaly-induced effective action (2.44). Locality of \(\Gamma^{(1)}_{hh}\) implies that it can be compared only with the first local term \(\int RR^* B\) in eqs. (2.44), (2.48). Furthermore, while \(\Gamma^{(1)}_{hh}\) was derived in a chiral superspace framework which is manifestly supersymmetric, the parity-odd anomalous term in (2.44) needs to be supplemented by other parity-even terms to make the resulting effective action consistent with supersymmetry. One obvious candidate for such an extra term is \(\int RR\varphi\), which should indeed be present in the one-loop effective action \(\Gamma\) as discussed in appendix B.\(^{40}\)

Since the parity-odd anomalous term (2.48) is linear in \(B\), it is natural to expect that its extension which is consistent with supersymmetry should be linear in \(\tau = B + ie^{-\varphi}\), namely

\[
\Gamma = \frac{i}{4(4\pi)^2} \int d^4x \left[ (R^+)^2 \tau - (R^-)^2 \tau \right] = -\frac{1}{4(4\pi)^2} \int d^4x \left( RR^* B - RR e^{-\varphi} \right).
\]

\(^{40}\)Note that up to terms proportional to equations of motion (i.e. Ricci tensor terms) \(RR\) is equal to \(R^* R^*\) which is a total derivative in four dimensions and thus does not contribute to the S-matrix.
Indeed, if one makes a general ansatz like

$$\Gamma = \int d^4x (R^+)^2 \sum_{n,m} a_{nm} \tau^n \bar{\tau}^m + \text{c.c.}, \quad (4.18)$$

imposes the linearized supersymmetry restrictions on the resulting scattering amplitudes and requires that for $\tau = B + ie^{-\varphi}$ as in eq. (2.39) this effective action $\Gamma$ is linear in $B$, one is led to (4.17).

Terms containing other fields, such as $R^+ F^+ F^+$ and others which are required by linearized supersymmetry and identified at linearized level in the amplitude calculation are to be determined separately (or found by expanding in components the supersymmetric anomaly-related action constructed in [5]).

Comparing (4.17) and $\Gamma_{hh}^{(1)}$ in eq. (4.15) and adjusting the normalization of the Einstein term (4.10) to match the one in eq. (2.50) (i.e. absorbing a factor of $\sqrt{2}$ in each graviton wave function and thus producing an extra factor of $1/2$ in $\Gamma_{hh}^{(1)}$), we see that they match provided one makes the identification

$$\tau = i - i \ln(1 - t). \quad (4.19)$$

Here the constant term accounts for the fact that we assume that $B = \varphi = 0$ should correspond to $t = 0$. Thus, the assumption that the regularization scheme employed in amplitude calculations preserves the shift symmetry of the axion implies this symmetry is realized as a shift and a rescaling of $t$.

Assuming (4.19), the $U(1)$ transformation of $t$ does not translate into a simple transformation of the supergravity field $\tau$. At the linearized level, however, when $\tau - i = i t$, an infinitesimal $U(1)$ transformation of $t$ is the same as an infinitesimal transformation of $\tau$ under the anomalous $U(1) \subset SU(1,1)$.

In the presence of additional $n_v$ vector multiplets $\Gamma$ in (4.17) must be multiplied by a factor of $(2 + n_v)/2$ as in (2.27).

5 Summary and concluding remarks

In this paper we first discussed in detail the $U(1)$ anomaly of the $\mathcal{N} = 4$ Poincaré supergravity realized as conformal supergravity coupled to vector multiplets. In the SU(4)-invariant formulation of the theory we identified the anomalous $U(1)$ subgroup of the duality group $SU(1,1)$. We have also identified a $U(1)$ symmetry acting on the on-shell asymptotic states under which all the fields carry the same charges as under the anomalous $U(1) \subset SU(1,1)$.

\footnote{At the linearized level, setting $\tau = i + c t + O(t)$ (with some constant $c$), supersymmetry requires that the only non-vanishing matrix elements are

$(R^+)^2 \bar{t}^{1+n}$, \quad $(R^-)^2 \bar{t}^{1+n}$ with $n \geq 0$,

$(R^+)^2 \bar{t}^{n+2} \bar{t}^m$, \quad $(R^-)^2 \bar{t}^{n+2} \bar{t}^m$ with $n \geq 0$, $m \geq 1$.}
Then, using the double-copy construction, we computed particular one-loop supergravity scattering amplitudes whose tree-level counterparts vanish identically. These amplitudes break the asymptotic-state U(1) symmetry. Interestingly, this breaking is related to an anomaly in an asymptotic-state tree-level bosonic symmetry of pure YM theory which, in the context of the $\mathcal{N} = 4$ super YM theory, follows from supersymmetry.

The dependence of the symmetry-breaking amplitudes (and of the symmetry-breaking effective action) on the number of additional vector multiplets matches the dependence of the anomaly of the U(1) subgroup of the duality group.

We have shown that the soft-scalar limits of the anomalous amplitudes are non-vanishing, which is in line with the expectation that a non-anomalous duality symmetry requires that this type of limit vanishes. The soft scalar function extracted this way is momentum-independent and can be used to construct a class of higher-point one-loop anomalous amplitudes of any multiplicity.

Symmetries analogous to the anomalous asymptotic state U(1) symmetry discussed here are present in other supergravity theories which, for certain matter content, can be obtained through factorized orbifolding (consistent truncation) from $\mathcal{N} = 8$ supergravity. Examples include $\mathcal{N} = 2$ supergravity with $(1 + n_v)$ vector multiplets (for $n_v = 2, 4, 6$) and $\mathcal{N} = 1$ supergravity with one chiral and $n_v$ vector multiplets (for $n_v = 2, 4, 6$). From a double-copy perspective these theories are realized as $(\mathcal{N} = 2 \text{ sYM}) \otimes (\text{pure YM} \oplus \phi_p)$ and $(\mathcal{N} = 1 \text{ sYM}) \otimes (\text{pure YM} \oplus \phi_p)$ (with the U(1) charges of fields determined as in (3.13)). The four-point anomalous amplitudes can be found using the color/kinematics-satisfying representations of the one-loop four-gluon amplitudes in $\mathcal{N} = 1$ and $\mathcal{N} = 2$ sYM theory together with the color/kinematics-satisfying representations of the one-loop all-plus amplitudes obtained, e.g., by dimension shifting (see appendix D). They are:

$$M_{4}^{(1):\mathcal{N}}(h_1^{++}, h_2^{++}, t_3, t_4) = \frac{i}{(4\pi)^2} \left( \frac{\kappa}{2} \right)^4 [12]^4 \left[ 1 + \left( 4 - \mathcal{N} \right) \left( 1 - \frac{tu}{3s^2} \right) \right],$$  

(5.1)

where $\mathcal{N} = 1, 2$, the first term in parenthesis is the contribution of the $\mathcal{N} = 4$ supergravity multiplet and the second term represents the subtracted contribution of two $\mathcal{N} = 2$ gravitino multiplets (for $\mathcal{N} = 2$) or three $\mathcal{N} = 1$ gravitino and three $\mathcal{N} = 1$ vector multiplets (for $\mathcal{N} = 1$); $s, t$ and $u$ are the usual Mandelstam variables. The scalar fields $t$ belong to a vector multiplet for an $\mathcal{N} = 2$ theory or a chiral multiplet for an $\mathcal{N} = 1$. As in the case considered in section 3.5, the dependence on the number of vector multiplets enters through an $(n_v + 2)/2$ multiplicative factor.

It would be interesting to understand whether the quantities defined by eq. (3.13) correspond to the charges of a physical symmetry even if both gauge theory factors appearing in the double-copy construction are supersymmetric. For a symmetric (i.e. with identical gauge theory factors) construction with $\mathcal{N} = 1 \text{ sYM}$ factors, the charges $q(\Phi \otimes \tilde{\Phi})$ of the fields in the supergravity multiplet appear to be given by a combination of the U(1) R-symmetry groups of the two gauge theory factors which can be identified with the Cartan generator of the SU(2) R-symmetry group of $\mathcal{N} = 2$ supergravity. For a symmetric construction with $\mathcal{N} = 2 \text{ sYM}$ factors, which realizes $\mathcal{N} = 4$ supergravity with two vector multiplets [21, 66], the $q(\Phi \otimes \tilde{\Phi})$ of fields in the supergravity multiplet are such that they
combine into representations of $SU(4) \supset SU(2) \otimes SU(2) \otimes U(1)$ eq. (3.13). The scalar field in the supergravity multiplet is uncharged under this $U(1)$ and therefore the anomalous amplitudes discussed in this paper, while non-vanishing, do not break it. The charges of the matter vector multiplets are different from the expected ones, suggesting that for them eq. (3.13) assigns charges corresponding to a linear combination of the $U(1) \subset SU(4)$ R-symmetry and the duality symmetry of each vector multiplet. It would be useful to clarify this structure in more detail as well as understand the meaning of the quantities defined by eq. (3.13) for asymmetric double-copy constructions.

The one-loop $U(1) \subset SU(1, 1)$ anomaly discussed above is reflected also in higher-loop scattering amplitudes in $\mathcal{N} = 4$ supergravity (as well as in all other theories with fewer supercharges that exhibit this $U(1)$ symmetry at the tree level). Indeed, an inspection of the two-particle cuts shows that, beginning at three loops, amplitudes that do not a priori break the $U(1)$ symmetry (such as the four-graviton amplitude) receive nontrivial contributions from anomalous matrix elements like (3.22), (3.23), cf. figure 1(a). Higher-loop amplitudes receive contributions from higher-point one-loop anomalous matrix elements as well as from higher-loop four-point ones, cf. figure 1(b). It would undoubtedly be interesting to consider higher-loop anomalous amplitudes. Based on available gauge theory amplitudes [67] it should not be too difficult to find the two-loop counterpart of (3.22). Unlike the one-loop all-plus YM amplitude, the two-loop all-plus YM amplitude is UV-divergent. This divergence will cancel in the supergravity amplitude, presumably through the mechanism discussed in [68].

A full understanding of the implications of the $U(1)$ anomaly for the ultraviolet properties of the theory remains an open question, cf. [13]. It is nevertheless interesting to note that, due to the expression for the anomalous amplitudes in (3.22), the super-cut in figure 1(a) is independent of the cut momenta.\footnote{Up to irrelevant numerical factors this cut is just $\langle 34 \rangle^4$. Up to a further numerical factor, the cut in figure 1(b) has the same expression.} This implies that the part of the three-loop amplitude detected by this cut is effectively (i.e. after the integrals of each of the one-loop amplitude factors are evaluated leading to (3.22)) a one-loop bubble integral and thus is divergent in the UV. The result of [69] then implies that, in the complete amplitude, this apparent divergence is cancelled by the contribution of other intermediate states not related by supersymmetry to a two-scalar state as well as by the contribution of other cuts, neither one of them being obviously related to the $U(1)$ anomaly. This might hint at the existence of a larger symmetry in $\mathcal{N} = 4$ supergravity.
It would also be interesting to construct higher-point $\mathcal{N} = 4$ supergravity amplitudes which carry a non-zero U(1) charge (and therefore also break U(1) invariance) and which are obtained from gauge theory amplitudes with non-rational momentum dependence. As already mentioned in section 3.2, if such amplitudes are non-vanishing, they potentially have nonlocal dependence on external momenta. The first candidate has five external legs. We note that such amplitudes first appear in unitarity cuts of four-point U(1)-preserving amplitudes at four-loop order (cf. figure 1(b) for a different field assignment to the cut legs). Whether such anomalous amplitudes affect the UV behavior of the theory (which, at four loops, will be unambiguously determined by an explicit calculation currently in progress [70]) remains an open question.

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A On vector contribution to the gravitational axial anomaly

The contribution of vectors to the U(1) duality anomaly was originally not included in [6] and once accounted for in [4] led to the conclusion that the duality anomaly does not cancel in $\mathcal{N} = 4$ PSG but cancels in $\mathcal{N} = 8$ PSG. Since the duality symmetry acts only on-shell (unless one gives up Lorentz symmetry and considers a doubled formulation as, e.g., in [13]) the vector field contribution to the anomaly may look unfamiliar and it was not derived explicitly in [4]. Below we shall make few clarifying comments and mention some relevant references.

Let us start with quantum Maxwell theory in a curved background and consider a (nonlocal) transformation $\delta A_\mu = \epsilon K_\mu A^\nu$ where $K_\mu$ is a differential operator such that the corresponding field strength transforms as $\delta F_{\mu\nu} = \epsilon F_{\mu\nu}^*$, i.e. as in the duality transformation. Then the corresponding Noether current is given by $j^\mu = \frac{\partial L}{\partial A_\mu} \delta A_\nu$ and its divergence is equal to $\partial_\mu j^\mu = F_{\mu\nu}^* F^{\mu\nu} + O(\partial_\mu F^{\mu\nu})$. Thus its quantum expectation value is (after omitting the equations of motion term under path integral)

$$\langle \partial_\mu j^\mu \rangle = \langle F_{\mu\nu}^* F_{\mu\nu} \rangle.$$  \hspace{1cm} (A.1)

Naively, the expectation value $\langle F_{\mu\nu}^* F_{\mu\nu} \rangle$ of a parity-odd operator in the parity-even Maxwell theory should vanish. However, as in the case of the chiral spinor current anomaly in a theory of a real spinor, a reparametrization-invariant regularization leads to the conclusion that this correlator is proportional to the parity-odd curvature tensor contraction $RR^s \equiv \frac{1}{7} e^{\mu\nu\rho\lambda} R^s_{\mu\nu} R^{\rho\sigma\lambda}$. 

\[ -37 - \]
To find the proportionality coefficient one may use, e.g., the standard perturbation theory by expanding near flat space to second order in $\hbar_{\mu\nu} = g_{\mu\nu} - \delta_{\mu\nu}$. The problem reduces then to the computation of the correlator $\langle (FF^*)(x_1)T_{\mu\nu}(x_2)T_{\kappa\lambda}(x_3) \rangle$ (where $T_{\mu\nu}$ is the stress tensor of the Maxwell field) in the flat-space free abelian vector theory. This correlator vanishes at separated points but receives a contact term contribution: in momentum space it corresponds to a one-loop triangular diagram that gives a non-vanishing finite contribution if computed in a way consistent with reparametrization invariance (e.g. in dimensional regularization).

Alternatively, one may determine the integrated value of $\langle FF^* \rangle$ in a curved background using topological anomaly considerations, i.e. by relating it to the difference between the number of self-dual and anti-self-dual zero modes of the differential operator acting on (anti) self-dual rank-2 tensors. It can then be related to the spectral index (using $\zeta$-function regularization) as in [30, 34] and [14]^{43,44}

$$n(1,0) - n(0,1) = \tau = \frac{1}{3}P = \frac{1}{3(4\pi)^2} \int d^4x \ RR^*, \quad (A.2)$$

which is equivalent to the integrated form of the local relation in eq. (2.46). To compare, for the chiral spinors one gets $n(\frac{1}{2},0) - n(0,\frac{1}{2}) = -\frac{1}{2\pi} P$ which is consistent with eq. (2.6).

The anomaly of a spin 1 field can also be found as follows [14, 71, 72]: since a self-dual tensor can be counted as a direct product of two chiral spinors, its anomaly is $(2 + 2)$ times the anomaly of a single spinor, i.e. there is a factor of 4 difference between its animal and the anomaly of a single chiral spinor (an extra factor of 2 comes in the case of a full vector field). To apply this count to $\mathcal{N} = 4$ supergravity as in (2.13) one needs also to take into account the chiral weights of fields, which need not be same.

The same (“topological”) count of the vector field contribution to the U(1) chiral anomaly (4 times that of a spinor) was used in [4].

The vector contribution to the chiral gravitational anomaly was made explicit in [8, 9], where it was computed directly by expanding near a flat background and using a diagrammatic unitarity-based method. The equivalent result was found using standard covariant methods in [10, 11] and also by computing the correlator $\langle FF^*T_{\mu\nu}T_{\kappa\lambda} \rangle$ in coordinate space in [12]. A computation of the vector contribution to the U(1) duality anomaly using the “first-order” doubled formalism was given in [13].

Let us mention again that viewing the U(1) anomaly from the effective action point of view where it corresponds, in particular, to the presence of a local term $\int RR^*B$ in (2.44)\footnote{Using that $\langle FF^* \rangle = \langle F_2^2 - F_2^2 \rangle$ this correlator can be expressed as $\text{tr}(P_+ G - P_- G)$ where $G$ is the field strength Green’s function (i.e. $F_{\mu\nu}(x_1)F_{\nu\lambda}(x_2)$) at coincident points and $P_{\pm}$ are projectors onto (anti)self-dual components, i.e. it is related to the index of the second-order order operator acting on $F$. This operator is found by starting with $D_{\nu}F^{\mu\nu} = 0$ equation (which in the standard Lorentz gauge leads to the $-D^2A_\mu + R_{\mu\nu}A^\nu$ operator) and acting with another derivative $D_\sigma$. The resulting equation $D_\sigma D^\sigma F_{\mu\nu} = D_\sigma D^\sigma F_{\mu\nu}$ can be simplified using $\epsilon^{\mu\nu\sigma\lambda}D_\sigma F_{\lambda} = 0$ leading to the operator $(-D^2 + X R)F = 0$ where $R$ is the full curvature and $X$ is proportional to the generator of SO(4) in the vector representation (see, e.g., [30]). The final result for the correlator is then given by $\text{tr}(P_+ X R X R)$ where the trace is over the corresponding representation (the familiar analog in spinor case is $\text{tr}(\gamma_\mu \gamma_\nu \gamma_\rho)R^{\mu\nu\rho}R^{\rho\mu\nu}$).}

\footnote{Here $\tau$ and $P$ are the Hirzebruch signature and the Pontryagin number, respectively and the first equal sign is the statement of the Hirzebruch signature theorem.}
(which for the vector field contribution is proportional $\int \langle FF^* \rangle$ $B$) removes the mystery related to the vector field anomaly count in [4].

B On the $\int R^* R^* \varphi$ contribution to the effective action

Starting with a supersymmetric theory with a classical action containing the bosonic terms in (2.50) one should expect that the effective action should also be supersymmetric so that the anomalous terms in (2.44) should be also accompanied by other terms that are parts of the same superinvariant. This supersymmetric aspect of the duality anomaly was discussed, e.g., in [7] and explained in detail in the present context in [5].

The anomalous terms in (2.44) should thus have their “supersymmetry partners” in the full effective action. In particular, the presence of the local anomalous parity-odd term $\int R R^* B$ in (2.44) implies that there should be also another local parity-even term $\int R R \varphi$. Below we shall discuss how one can directly find such term in the one-loop effective action.

Consider the Lagrangian $L = k(x)(\partial \phi)^2 = -k(x)\phi \Delta \phi + \ldots$ where $\phi$ is a quantum field on a curved background and $k$ is a background field. If we integrate over $\phi$ we get a complicated dependence on the derivatives of $k$, but one may wonder if the effective action contains also a contribution that survives if $k$ is constant, such as $\int \sqrt{g} \ln k R R$. This question depends of course on the choice of regularization scheme and path integral measure: since the term we are interested in is local it can be changed by adding a local counterterm.

Indeed, if we make a local field redefinition $\phi' = k^{-1/2} \phi$ then the remaining dependence on $k$ will be only through its derivatives. The resulting Jacobian contribution to the effective action can be regularized as follows:

$$\Gamma_1 = \frac{1}{2} \text{Tr} \ln \left( k^{-1/2} e^{-\Lambda^{-2} \Delta} \right), \quad \Lambda \to \infty. \quad (B.1)$$

Using the asymptotic expansion of the heat kernel in four dimensions $e^{-\Lambda^{-2} \Delta} = \Lambda^4 b_0 + \Lambda^2 b_0 + b_4 + O(\Lambda^{-2})$ and assuming that all cut-off dependent terms cancel between different quantum fields (or are removed by adding divergent counterterms or by some further regularization prescription like $\zeta$-function regularization) we are then left with $\Gamma'_1 = -\frac{1}{4} \int d^4 x \sqrt{g} \ln k b_4(x,x)$ where $b_4$ is the familiar conformal anomaly coefficient.

In the case of a vector field in four dimensions with $k = e^{-\varphi}$, i.e. with a Lagrangian

$$L = -\frac{1}{4} e^{-\varphi} F_{\mu\nu} F^{\mu\nu} \quad (B.2)$$

as in (2.50), one can show (by similar arguments as in the two-dimensional case [73]) that the effective action found by integrating out $A_\mu$ satisfies [47, 48]

$$\Gamma[\varphi, g] - \Gamma[-\varphi, g] = -\frac{1}{64\pi^2} \int d^4 x \sqrt{g} R^* R^* \varphi, \quad (B.3)$$

where $R^* R^* = R_{\mu\nu\rho\lambda}^2 - 4 R^2_{\mu\nu} + R^2$ is the Euler number density. This implies that

$$\Gamma[\varphi, g] = -\frac{1}{128\pi^2} \int d^4 x \sqrt{g} R^* R^* \varphi + O((\partial \varphi)^2), \quad (B.4)$$

An alternative argument is based on counting only 0-mode contributions, assuming $\delta(0)$ term with sum over all modes is set to zero.
where all other terms should be even in $\varphi$ (as follows from the fact that a vector-vector duality transformation reverses the sign of $\varphi$) and should depend only its derivatives.

Note that the coefficient of the local term in (B.4) disagrees with the one found in [42] where a different regularization was used. This illustrates again that this coefficient is, in general, scheme-dependent. In a supersymmetric theory one should use a regularization preserving supersymmetry (which was not a priori the case in [42]).

To compute the coefficient of this leading $\int RR\varphi$ term such that it is consistent with the anomalous term discussed in appendix A one may expand in powers of $\varphi$ and near-flat metric $h_{\mu\nu} = g_{\mu\nu} - \delta_{\mu\nu}$ so that the leading contribution will be determined by the correlator $\langle F_{\mu\nu}(x_1)T_{\kappa\lambda}(x_2)T_{\rho\sigma}(x_3) \rangle$ in free Maxwell theory. This correlator vanishes at separated points but will contain a local $\delta$-function term consistent with (B.4) under a particular regularization.

Apart from the vector-scalar kinetic coupling of the type discussed above, the $\mathcal{N} = 4$ supergravity Lagrangian (2.50) also contains the scalar-scalar term $e^{2\varphi} \partial^\mu B \partial_\mu B$ in (2.41). Its contribution was also computed in [42] but again depends on a particular regularization used. For example, given that (2.41) is a sigma model (based on the $\text{SL}(2,\mathbb{R})/\text{U}(1)$ coset) one may assume that the corresponding path integral measure should contain the usual $\sqrt{G}$ factor cancelling the effect of redefinition of $B$ by $e^\varphi$ and thus suggesting that there should be no quantum scalar $B$ contribution to (B.4).47

One may wonder if the coefficient of the local term in (B.4) in $\mathcal{N} = 4$ supergravity is controlled by the total “conformal anomaly” coefficient $b_4$. Indeed, the sum of $b_4$ coefficients in the supergravity multiplet can be written as [30, 74]48

$$b_{4\text{tot}} = \frac{1}{32\pi^2} a_{\text{tot}} R^* R^* , \quad a_{\text{tot}} = \frac{1}{24} (58n_2 - 17n_{3/2} - 2n_1 - n_{1/2} + n_0) , \quad (B.5)$$

where $N_s$ are the numbers of spin $s$ fields. In $\mathcal{N} = 4$ PSG

$$n_2 = 1 , \quad n_{3/2} = n_{1/2} = 4 , \quad n_1 = 6 , \quad n_0 = 2 . \quad a = -1 , \quad (B.6)$$

Thus, if all fields were coupling to $\varphi$ with the same weight as the vector fields, then the resulting coefficient $a_{\text{tot}}$ is twice the one in (B.4), i.e. $-\frac{1}{64\pi^2}$, which is, incidentally, the same as the coefficient of $\int RR^* B$ term in (2.44).

C The one-loop superamplitude containing $\mathcal{M}_4^{(1)}(h^{--}, h^{++}, h^{++}, \bar{u})$

In this appendix we shall use the double-copy construction [25] reviewed in section 3.1 to compute the superamplitude containing the three-graviton-scalar amplitude. Because

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46 Here we do not distinguish between $R^* R^*$ and $RR$ as they differ by Ricci-tensor terms that do not contribute to on-shell scattering amplitudes or the on-shell effective action.

47 At the same time, having no contribution from fermions (assuming that they have canonically normalized kinetic terms) appears to be in conflict with supersymmetry — spin 3/2 and 1/2 fermions did contribute to similar $\int RR^* B$ term in (2.44).

48 Here $\frac{1}{32\pi^2} RR$, etc., terms in $b_4$ cancel out between different spin contributions.
the $\mathcal{N} = 4$ sYM amplitude has loop-momentum-independent numerator factors of its one-loop four-point amplitude, we can use the formulation [68] of the double-copy construction which expresses the four-point supergravity amplitude as

\[ M_{\mathcal{N}=4 \text{PSG}}^{(1)}(1,2,3,4) = i \left( \frac{\kappa}{2} \right)^4 s_{12}s_{23} A_{\mathcal{N}=4 \text{tree}}^{(1)}(1,2,3,4) \left( A_{\mathcal{N}=0}^{(1)}(1,2,3,4) + A_{\mathcal{N}=0}^{(1)}(1,3,4,2) + A_{\mathcal{N}=0}^{(1)}(1,4,2,3) \right) \]

where $A_{\mathcal{N}=0}^{(1)}$ is a pure or matter-coupled YM one-loop color-ordered amplitude. The helicities of supergravity fields are determined in the usual way, by adding the helicities of the individual gauge theory fields, cf. section 3.1.

The only $\mathcal{N} = 4$ sYM amplitude is the MHV one, which we choose to write in antichiral superspace

\[ A_{\mathcal{N}=4 \text{tree}}^{(1)}(1,2,3,4) = \frac{1}{i[23][34][41]} \delta^{(8)} \left( \sum_{i=1}^{4} \tilde{\eta}_i A \tilde{\lambda}_i \right) . \]  

(C.2)

For the field configuration we are interested in, $M_{\mathcal{N}=4 \text{PSG}}^{(1)}(h^-,h^+,\bar{t})$, the relevant $\mathcal{N} = 0$ gauge theory amplitude is the single-minus amplitude [53]

\[ A_{\mathcal{N}=0}^{(1)}(1^-,2^+,3^+,4^+) = (1 + \mathcal{O}(\epsilon)) \frac{i}{3(4\pi)^2 \left[ \begin{array}{c} 2 \end{array} \right] \left[ \begin{array}{c} 3 \end{array} \right] \left[ \begin{array}{c} 4 \end{array} \right] \left[ \begin{array}{c} 4 \end{array} \right]} \left[ \begin{array}{c} 12 \end{array} \right] \left[ \begin{array}{c} 23 \end{array} \right] \left[ \begin{array}{c} 34 \end{array} \right] \left[ \begin{array}{c} 41 \end{array} \right] \right] . \]

(C.3)

It is not difficult to see that it is symmetric under permutations of 2,3,4, as Bose symmetry requires. In the presence of $n_v$ scalar fields this amplitude acquires an extra factor of $(2 + n_v)/2$.

Therefore, using (C.1), the $\mathcal{N} = 4$ supergravity amplitude with the $\mathcal{N} = 0$ amplitude factor being (C.3) is

\[ M_{\mathcal{N}=4 \text{PSG}}^{(1)}(1,2,3,4) = -i \left( \frac{4\pi}{2} \right)^2 \frac{1}{\left[ \begin{array}{c} 12 \end{array} \right] \left[ \begin{array}{c} 23 \end{array} \right] \left[ \begin{array}{c} 34 \end{array} \right] \left[ \begin{array}{c} 41 \end{array} \right]} \delta^{(8)} \left( \sum_{i=1}^{4} \tilde{\eta}_i A \tilde{\lambda}_i \right) \]

\[ = -i \left( \frac{4\pi}{2} \right)^2 \frac{1}{\left[ \begin{array}{c} 32 \end{array} \right] \left[ \begin{array}{c} 24 \end{array} \right] \left[ \begin{array}{c} 21 \end{array} \right]} \delta^{(8)} \left( \sum_{i=1}^{4} \tilde{\eta}_i A \tilde{\lambda}_i \right) . \]

(C.4)

This is the superamplitude quoted in eq. (3.24).

We may extract several instances of $M_{\mathcal{N}=4 \text{PSG}}^{(1)}(h^{++},h^{++},h^{--},\bar{t})$, which superficially differ only by their momentum assignment. Two examples are

\[ M_{\mathcal{N}=4 \text{PSG}}^{(1)}(1^{--},2^{++},3^{++},4^{+}) = -i \left( \frac{4\pi}{2} \right)^2 \frac{1}{\left[ \begin{array}{c} 32 \end{array} \right] \left[ \begin{array}{c} 24 \end{array} \right] \left[ \begin{array}{c} 21 \end{array} \right]} \left[ \begin{array}{c} 23 \end{array} \right] \delta^{(8)} , \]

(C.5)

\[ M_{\mathcal{N}=4 \text{PSG}}^{(1)}(1^{--},2^t,3^{++},4^{++}) = -i \left( \frac{4\pi}{2} \right)^2 \frac{1}{\left[ \begin{array}{c} 32 \end{array} \right] \left[ \begin{array}{c} 24 \end{array} \right] \left[ \begin{array}{c} 21 \end{array} \right]} \left[ \begin{array}{c} 23 \end{array} \right] \delta^{(8)} . \]

(C.6)

In the limit in which the momentum of the scalar field is soft the second expression (C.6) vanishes identically since it scales like some positive power of scalar momentum $k_2$. The

\[ \text{This result was obtained independently by Z. Bern and L. Dixon [75].} \]
first expression appears to give a finite expression. However, since the 3-point amplitude with two positive-helicity gravitons is MHV, momenta should be continued such that the products \( ab \neq 0 \) while \( \langle a \ b \rangle = 0 \). The right-hand side of (C.5) is proportional to \( \langle 1 \ 2 \rangle \) and therefore vanishes in the soft limit as well, as required by consistency with supersymmetry and (C.6).

Upon use of the graviton soft function [76] the soft graviton limit \( k_1 \to 0 \) together with the fact that in \( \mathcal{N} = 4 \) antichiral superspace the negative helicity graviton wave function contains no factors of \( \tilde{\eta} \), we find the answer [75] quoted in eq. (3.21):

\[
\mathcal{M}^{(1); \mathcal{N}=4 \text{ PSG}}_{3}(1, 2, 3) = \frac{i}{(4\pi)^2} \delta^{(8)} \left( \sum_{i=1}^{3} \tilde{\eta}_{i; A} \tilde{\lambda}_{i} \right) \tag{C.7}
\]

In the presence of \( n_v \) scalar fields coupled to the YM theory or, equivalently, in the presence of additional \( n_v \) vector multiplets coupled to \( \mathcal{N} = 4 \) supergravity, both this superamplitude and the one in eq. (C.4) acquire an extra factor of \((2 + n_v)/2\).

**D The one-loop superamplitude containing \( \mathcal{M}^{(1)}_{4}(h^{++}, h^{++}, \bar{t}, \bar{t}) \)**

To compute the superamplitude containing the two-graviton-two-scalar amplitude we can use again the double-copy construction in the form [68]:

\[
\mathcal{M}^{(1); \mathcal{N}=4 \text{ PSG}}_{4}(1, 2, 3, 4) = i \left( \frac{k}{2} \right)^4 s_{12}s_{23} A^{\text{tree}}_{\mathcal{N}=4}(1, 2, 3, 4) \left( A^{(1)}_{\mathcal{N}=0}(1, 2, 3, 4) + A^{(1)}_{\mathcal{N}=0}(1, 3, 4, 2) + A^{(1)}_{\mathcal{N}=0}(1, 4, 2, 3) \right). \tag{D.1}
\]

where \( A^{(1)}_{\mathcal{N}=0} \) are pure or matter-coupled YM theory color ordered amplitudes. For the desired field configuration the relevant \( A^{(1)}_{\mathcal{N}=0} \) is the all-plus four-point amplitude given by [53]

\[
A^{(1)}_{\mathcal{N}=0} = \frac{2i}{(4\pi)^2} \frac{[12][34]}{[12][23][34][41]} \delta^{(8)} \left( \sum_{i=1}^{4} \tilde{\eta}_{i; A} \tilde{\lambda}_{i} \right). \tag{D.2}
\]

As in appendix C we shall use the anti-chiral superspace expression of the \( A^{\text{tree}}_{\mathcal{N}=4}(1, 2, 3, 4) \) factor

\[
A^{\text{tree}}_{\mathcal{N}=4}(1, 2, 3, 4) = \frac{i}{[12][23][34][41]} \delta^{(8)} \left( \sum_{i=1}^{4} \tilde{\eta}_{i; A} \tilde{\lambda}_{i} \right). \tag{D.3}
\]

Alternatively, we can use the color/kinematics-satisfying representation of the one-loop \( \mathcal{N} = 0 \) amplitude obtained by dimension shifting [52] from the \( \mathcal{N} = 4 \) four-point MHV gluon amplitude (i.e. the coefficient of e.g. \( \tilde{\eta}_1^2 \tilde{\eta}_2^4 \) in the first equation below):

\[
\mathcal{A}^{(1); \mathcal{N}=4}(1, 2, 3, 4) = \frac{1}{[12][34]} \delta^{(8)} \left( \sum_{i=1}^{4} \tilde{\eta}_{i; A} \tilde{\lambda}_{i} \right) (I_{1234}[\mu^4]C_{1234} + I_{1342}[\mu^4]C_{1342} + I_{1423}[\mu^4]C_{1423}),
\]

\[
\mathcal{A}^{(1); \mathcal{N}=0}(1^+, 2^+, 3^+, 4^+) = 2i \frac{[12][34]}{[12][34]} (I_{1234}[\mu^4]C_{1234} + I_{1342}[\mu^4]C_{1342} + I_{1423}[\mu^4]C_{1423}) \tag{D.4}
\]
Here $C_{abcd}$ are the color factors of a box integral with external legs ordered as $(a, b, c, d)$ and $I_{1234}^{[\mu^4]}$ is

$$I_{abcd}[\mu^4] = -\epsilon(1 - \epsilon)I_{abcd}^{8-2\epsilon} = -\frac{1}{(4\pi)^2} \frac{1}{6}$$  \hspace{1cm} (D.5)$$

with $I_{abcd}^{8-2\epsilon}$ the eight-dimensional box interval with external legs ordered as $(a, b, c, d)$. The argument $\mu^4$ of the integral on the left-hand side represents the insertion in the numerator of a four-dimensional box integral of the fourth power of the $(-2\epsilon)$-dimensional of the loop momentum. All external momenta are taken to be four dimensional.

Putting together the amplitudes (D.4) following the double-copy construction, we find

$$M^{(1;N=4PSG)}_{4,1,2,3,4} = -i \left(\frac{k}{2}\right)^4 \left(-2 \times 3 \times \frac{1}{6}\right) \frac{[12][34]}{[12][34]} \frac{\langle 12 \rangle}{\langle 34 \rangle} \frac{\langle 34 \rangle}{\langle 12 \rangle} \delta^{(8)} \left(\sum_{\lambda_i} \tilde{\eta}_{\lambda_i} \tilde{\lambda_i}\right)$$

$$= i \left(\frac{k}{2}\right)^4 \delta^{(8)} \left(\sum_{\lambda_i} \tilde{\eta}_{\lambda_i} \tilde{\lambda_i}\right),$$  \hspace{1cm} (D.6)$$

which is the expression quoted in eq. (3.22). In the presence of $n_v$ real scalar fields coupled to the YM theory or, equivalently, in the presence of additional $n_v$ vector multiplets in $\mathcal{N} = 4$ supergravity, the amplitudes in eq. (D.2), the second eq. (D.4) and eq. (D.6) each acquire a factor of $(2 + n_v)/2$.

### E Five-point superamplitudes

In this appendix we include some of the details of the calculations leading to eqs. (3.23) and (3.25). The main ingredients are the five-point superamplitude in a form obeying color/kinematic duality \[55\] and the corresponding all-plus amplitude in a similar color/kinematic-satisfying representation obtained through dimension-shifting.

#### Ingredients

The one-loop five-point $\mathcal{N} = 4$ sYM MHV amplitude in a form obeying color/kinematic duality \[55\] is

$$A_{5}^{(1;N=4)} = ig^5 \sum_{S_5} \left(\frac{1}{10} \beta_{12345} C^{(P)} I^{(P)} + \frac{1}{4} \gamma_{12} C^{(B)} I^{(B)}_{12}\right),$$  \hspace{1cm} (E.1)$$

where $g$ is the coupling constant, and the sum is over all 120 permutations, $S_5$, of the external leg labels; the symmetry factors $1/10$ and $1/4$ compensate for the overcount in this sum, $I^{(P)}$ and $I^{(B)}_{12}$ are the integrals shown in figure 2 and given by

$$I^{(P)} = \int \frac{d^D p}{(2\pi)^D} \frac{1}{p^2(p + k_1)^2(p + k_1 + k_2)^2(p - k_4 - k_5)^2(p - k_5)^2},$$

$$I^{(B)} = \frac{1}{s_{12}} \int \frac{d^D p}{(2\pi)^D} \frac{1}{p^2(p + k_1 + k_2)^2(p - k_4 - k_5)^2(p - k_5)^2}.$$

(E.2)

The coefficients $\beta$ and $\gamma$ coefficients are:

$$\beta_{12345} = i \delta^{(8)}(Q_5) \left[\begin{array}{c} [12] \\ [23] \\ [34] \\ [45] \\ [51] \end{array}\right] \frac{\langle 12 \rangle}{\langle 23 \rangle} \frac{\langle 23 \rangle}{\langle 34 \rangle} \frac{\langle 34 \rangle}{\langle 45 \rangle} \frac{\langle 45 \rangle}{\langle 51 \rangle} = \delta^{(8)}(Q_5) \left[\begin{array}{c} [12] \\ [23] \\ [34] \\ [45] \\ [51] \end{array}\right] \frac{4 \varepsilon(1, 2, 3, 4)}{4 \varepsilon(1, 2, 3, 4)},$$

$$\gamma_{12} = \beta_{12345} - \beta_{21345} = \delta^{(8)}(Q_5) \left[\begin{array}{c} [12] \\ [23] \\ [34] \\ [45] \\ [51] \end{array}\right] \frac{4 \varepsilon(1, 2, 3, 4)}{4 \varepsilon(1, 2, 3, 4)}.$$

(E.3)
The argument of the $\delta$-function is the usual supermomentum
\[ Q_5^{\alpha A} = \sum_{i=1}^{5} \lambda_i^\alpha \eta_i^A. \] (E.4)

The color factors can be read directly from the graphs in figure 2:
\[ C^{(P)} = \tilde{f} g_{a_1 b} \tilde{f} a_2 c \tilde{f} a_3 d \tilde{f} a_4 e \tilde{f} a_5 g, \]
\[ C^{(B)} = \tilde{f} a_1 a_2 b \tilde{f} b c g \tilde{f} a_3 d \tilde{f} a_4 e \tilde{f} a_5 g, \] (E.5)
where $a_i$ are the external color labels.

We also need the all-plus five-point color-dressed gluon amplitude in pure Yang-Mills theory; to construct it we may use the following color-ordered all-plus five-point amplitude [52], valid to all orders in $\epsilon$:
\[ A_5^{(1)}(1^+, 2^+, 3^+, 4^+, 5^+ ) = i \frac{\epsilon(1-\epsilon)}{(12)\langle 23 \langle 34 \langle 45 \langle 51 \rangle (4\pi)^2 \epsilon} \times \left[ s_{23}s_{34}I_{4;51}^{D=8-2\epsilon} + s_{34}s_{45}I_{4;12}^{D=8-2\epsilon} + s_{45}s_{51}I_{4;23}^{D=8-2\epsilon} \right. \\
+ s_{51}s_{12}I_{4;34}^{D=8-2\epsilon} + s_{12}s_{23}I_{4;45}^{D=8-2\epsilon} + (4-2\epsilon)\text{Tr}[\gamma_5 k_1 k_2 k_3 k_4]I^{(P),D=10-2\epsilon} \right] \] (E.6)

To find the color-dressing we use [77]; the color factors are given by a color-space pentagon graph with one structure constant at each vertex:
\[ A_5^{(1)} = g^2 \sum_{\sigma \in S_4/R} \text{Tr}[F^{a_1} F^{a_2} F^{a_3} F^{a_4} F^{a_5}] A_5^{(1)}(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5). \] (E.7)

Here the sum is over all non-cyclic permutations and $R$ is the reflection, $R(1, 2, 3, 4, 5) = (5, 4, 3, 2, 1)$, etc. This color dressing applies to all one-loop amplitudes.

The integrals that appear in the all-plus amplitude (E.6) are [52]:
\[ \epsilon(1-\epsilon)I_{4;i,i+1}^{D=8-2\epsilon} = \frac{1}{6}, \quad \epsilon(1-\epsilon)I^{(P),D=10-2\epsilon} = \frac{1}{24}. \] (E.8)
Using them (E.6) becomes

\[
A_5^{(1)}(1^+, 2^+, 3^+, 4^+, 5^+) = \frac{i}{(12)(23)(34)(45)(51)} \frac{1}{6(4\pi)^2} \\
\times \left[ s_{23}s_{34} + s_{34}s_{45} + s_{45}s_{51} + s_{51}s_{12} + s_{12}s_{23} + \text{Tr}[\gamma_5 k_1 k_2 k_3 k_4] \right] \\
= -\frac{i}{48\pi^2} \sum_{1 \leq i_1 < i_2 < i_3 < i_4 \leq 5} \text{Tr}[k_{i_1} k_{i_2} k_{i_3} k_{i_4}] .
\]

(E.9)

We can also construct the color-dressed all-plus one-loop five-point amplitude in a color/kinematics-satisfying form by using its relation to the dimensionally-shifted MHV amplitude in \(N = 4\) sYM theory [52] and starting with (E.1). The result is

\[
A_5^{(1): N = 0} = 2ig^5 \sum_{S_5} \left( \frac{1}{10} \tilde{\beta}_{12345} C_1^{(P)} I^{(P)}[\mu^4] + \frac{1}{4} \tilde{\gamma}_{12} C_1^{(B)} I^{(B)}[\mu^4] \right),
\]

(E.10)

with

\[
\tilde{\beta}_{12345} = \frac{[12][23][34][45][51]}{(12)[23][35][51] - [12][23][35][51]} = \frac{[12][23][34][45][51]}{4\varepsilon(1, 2, 3, 4)},
\]

\[
\tilde{\gamma}_{12} = \tilde{\beta}_{12345} - \tilde{\beta}_{21345} = \frac{[12][23][34][45][35]}{4\varepsilon(1, 2, 3, 4)}.
\]

(E.11)

These quantities are the same as \(\beta\) and \(\gamma\) but with the \(\delta^{(8)}(Q)\) stripped off. This expression as well as (E.10) also match (after suitable manipulations) the one found in [53].

Evaluating the integrals using (E.8)

\[
I^{(P)}[\mu^4] = 0 + \mathcal{O}(\epsilon), \quad I^{(B)}_{i,i+1}[\mu^4] = -\epsilon(1 - \epsilon) \frac{I_{D=8-2\epsilon}^{1\rightarrow[i,i+1]} s_{i,i+1}}{6s_{i,i+1}} = -\frac{1}{(4\pi)^2} \frac{1}{6s_{i,i+1}} + \mathcal{O}(\epsilon),
\]

(E.13)

implies that

\[
A_5^{(1): N = 0}(1^+, 2^+, 3^+, 4^+, 5^+) = -2g^5 \frac{1}{6(4\pi)^2} \sum_{S_5} \frac{1}{4} \tilde{\gamma}_{12} C_1^{(B)} .
\]

(E.14)

From here it is not difficult to extract the color ordered amplitude

\[
A_5^{(1): N = 0}(1^+, 2^+, 3^+, 4^+, 5^+)
\]

and check that reproduces (E.9) after suitable manipulations. In the presence of \(n_\nu\) scalar fields both equations pick up a factor of \((2 + n_\nu)/2\).

**The 5-point superamplitude containing \(M_5^{(1)}(h^{++}, h^{++}, \tilde{t}, \tilde{t}, \tilde{t})\).** Using the ingredients above we can construct the superamplitude containing the two-graviton-three-scalar component amplitude \(M_5^{(1)}(h^{++}, h^{++}, \tilde{t}, \tilde{t}, \tilde{t})\). It is not difficult to see that, to have the
desired field content, we in fact need the MHV \( \mathcal{N} = 4 \) superamplitude. As mentioned previously, it may be obtained from (E.1) by simply \( \lambda_{ai} \leftrightarrow \lambda_{ai} \) and \( \eta \leftrightarrow \tilde{\eta} \). Using [25] or just eq. (3.10) we find

\[
\mathcal{M}_5^{(1); \mathcal{N} = 4 \text{ PSG}}(1, 2, 3, 4, 5) = i \left( \frac{\kappa}{2} \right)^5 \frac{\delta^{(8)}(\mathcal{Q}_5)}{3(4\pi)^2} \sum_{S_5} \frac{1}{4} \frac{s_{12} \gamma_{12}}{s_{12}} \frac{\gamma_{12} \gamma_{13}}{s_{13}} \frac{\gamma_{14} \gamma_{14}}{s_{14}} \frac{\gamma_{15} \gamma_{15}}{s_{15}} \frac{\gamma_{23} \gamma_{23}}{s_{23}} \\
+ \frac{s_{24} \gamma_{24}}{s_{24}} \frac{s_{25} \gamma_{25}}{s_{25}} \frac{s_{34} \gamma_{34}}{s_{34}} \frac{s_{35} \gamma_{35}}{s_{35}} \frac{s_{45} \gamma_{45}}{s_{45}}, \quad (E.16)
\]

where \( \gamma_{ij} \) and \( \tilde{\gamma}_{ij} \) are defined in eqs. (E.11) and (E.12) and

\[
\mathcal{Q}_5 = \sum_{i=1}^{5} \tilde{\eta}_{i, A} \lambda_i, \quad (E.17)
\]

\[
\hat{\gamma}_{12} = \hat{\gamma}_{12345} - \hat{\gamma}_{2345} = \frac{\langle 12 \rangle^2 \langle 34 \rangle \langle 45 \rangle \langle 35 \rangle}{4 \varepsilon(1, 2, 3, 4)} \quad (E.18)
\]

\[
\hat{\gamma}_{12345} = \frac{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 51 \rangle}{\langle 12 \rangle \langle 23 \rangle \langle 35 \rangle \langle 45 \rangle \langle 51 \rangle} = \frac{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 51 \rangle}{4 \varepsilon(1, 2, 3, 4)} \quad (E.19)
\]

The sum in eq. (E.16) evaluates to

\[
\frac{\gamma_{12} \gamma_{13}}{s_{12}} \frac{\gamma_{14} \gamma_{14}}{s_{14}} \frac{\gamma_{15} \gamma_{15}}{s_{15}} \frac{s_{23}}{s_{23}} \frac{s_{24} \gamma_{24}}{s_{24}} \frac{s_{25} \gamma_{25}}{s_{25}} \frac{s_{34} \gamma_{34}}{s_{34}} \frac{s_{35} \gamma_{35}}{s_{35}} \frac{s_{45} \gamma_{45}}{s_{45}} = 2 \quad (E.20)
\]

and thus the superamplitude containing the anomalous amplitude \( \mathcal{M}_5^{(1)}(h^{++}, h^{++}, t, t, t) \) is

\[
\mathcal{M}_5^{(1); \mathcal{N} = 4}(1, 2, 3, 4, 5) = 2i \left( \frac{\kappa}{2} \right)^5 \frac{\delta^{(8)}(\mathcal{Q}_5)}{3(4\pi)^2} \quad (E.21)
\]

This superamplitude is local and is the result quoted in (3.23).

**Five-point superamplitude containing \( \mathcal{M}_5^{(1)}(h^{++}, h^{++}, t, A^+, A^+) \).** Using the ingredients described above we can also construct the superamplitude containing \( \mathcal{M}(h^{++}, h^{++}, t, A^+, A^+) \). This field content implies that it is natural to present this superamplitude in chiral superspace. Using [25] or just eq. (3.10) we find that it is given by:

\[
\mathcal{M}_5^{(1); \mathcal{N} = 4 \text{ PSG}}(1, 2, 3, 4, 5) = i \left( \frac{\kappa}{2} \right)^5 \frac{\delta^{(8)}(\mathcal{Q}_5)}{3(4\pi)^2} \sum_{S_5} \frac{1}{4} \frac{(\gamma_{12})^2}{s_{12}} \frac{s_{12}}{s_{12}} \frac{s_{13}}{s_{13}} \frac{s_{14}}{s_{14}} \frac{s_{15}}{s_{15}} \frac{s_{23}}{s_{23}} \frac{s_{24}}{s_{24}} \frac{s_{25}}{s_{25}} \frac{s_{34}}{s_{34}} \frac{s_{35}}{s_{35}} \frac{s_{45}}{s_{45}} \frac{s_{45}}{s_{45}} \frac{s_{23}}{s_{23}} \frac{s_{24}}{s_{24}} \frac{s_{25}}{s_{25}} \frac{s_{34}}{s_{34}} \frac{s_{35}}{s_{35}} \frac{s_{45}}{s_{45}} \frac{s_{45}}{s_{45}} = 2 \quad (E.22)
\]

This is the expression quoted in eq. (3.25). An additional factor of \((2 + n_v)/2\) appears in the presence of \(n_v\) vector multiplets.
It is not difficult to check (numerically) that this expression agrees with the result of the double-copy construction that uses the standard form of the all-plus five-point YM amplitude $A^{(0)}_{\mathcal{N}=0}(1^+,\ldots,5^+)$ given in [52, 53] and in eq. (E.6).

To extract the amplitude $\mathcal{M}(h^{++},h^{++},\bar{t},A^+,A^+)$ from (3.25) one isolates the appropriate combination of $\eta$-variables, i.e. $\eta_3^A\eta_4^B\eta_5^C\epsilon_{ABCD}$; this leads to the simple replacement $\delta^{(8)}(Q) \mapsto \langle 35 \rangle_2^2 \langle 34 \rangle_2^2$.

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