WICK-TYPE STOCHASTIC PARABOLIC EQUATIONS WITH RANDOM POTENTIALS

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ABSTRACT. The stochastic parabolic equations with random potentials, driving forces and initial conditions are considered. The Wick product is used to give sense to the product of two generalized stochastic processes, and the existence and uniqueness of solutions are proved via the chaos expansion method from white noise analysis. The estimates on coefficients in the chaos expansion form of the solutions are provided.

1. Introduction and preliminaries

For given generalized stochastic processes of Kondratiev-type \(F\) and \(G\), and a bounded in space generalized stochastic process of Kondratiev-type \(Q\), we consider Cauchy problems for stochastic parabolic evolution equations

\[(\partial_t - \mathcal{L})U + Q \Diamond U = F, \quad U|_{t=0} = G,\]

where \(\mathcal{L}\) is an elliptic operator acting on the space variable. Since the unknown generalized stochastic process \(U\) is involved in product with another generalized stochastic process, potential \(Q\), one must give sense to such product. Here, the Wick product denoted by \(\Diamond\) is used, see [5].

The special case, when \(\mathcal{L}\) is the Laplacian, is the stochastic heat equation with random potential which, due to its various applications in biology, financial mathematics, aerodynamics, structural acoustics, has been widely studied, e.g. [1, 2, 3, 6]. Stochastic evolution equations with multiplicative noise are studied in [7] and stochastic evolution problems with polynomial nonlinearities are studied in [8].

In this work we study problem (1). Using the chaos expansion method from the white noise analysis developed in [5] we prove the theorem on existence of unique generalized stochastic process and provide estimates of coefficients in chaos expansion form of the solution.

To start with, we recall some basic notions from the white noise analysis, and for more details and proofs we refer to [5]. Denote by \(\mathcal{I} := \mathbb{N}_0^m\) the set of multi-indices having finite number of nonzero components, the zero vector by \(\mathbf{0}\), and the length of a multi-index \(\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m, 0, 0, \ldots)\), \(\alpha_i \in \mathbb{N}_0, i = 1, 2, \ldots, m, m \in \mathbb{N}\), by \(|\alpha| = \sum_{i=1}^m \alpha_i\). If \(\alpha = (\alpha_1, \alpha_2, \ldots) \in \mathcal{I}\) and \(\beta = (\beta_1, \beta_2, \ldots) \in \mathcal{I}\), then \(\alpha \leq \beta\) if and only if \(\alpha_k \leq \beta_k\) for all \(k \in \mathbb{N}\). The following lemma collects results needed later. For proofs and details see [5, 8, 9].
Lemma 1. Let \( \alpha \in \mathbb{I} \) and \( k \in \mathbb{N}_0 \).
(a) Let \( k \leq |\alpha| \), and denote by \( N(\alpha, k) \) the number of possibilities in which a multi-index \( \alpha \) can be written as a sum of \( k \) strictly smaller and nonzero multi-indices. Then \( N(\alpha, k) \leq 2^{|\alpha|} \).
(b) Define \( (2\mathbb{N})^\alpha := \prod_{i=1}^\infty (2i)^{\alpha_i} \). Then \( |\alpha| \leq (2\mathbb{N})^\alpha \) and

\[
\sum_{\alpha \in \mathbb{I}} (2\mathbb{N})^{-p|\alpha|} < \infty \iff p > 1.
\]

Moreover, for every \( c > 0 \) there exists \( s \geq 0 \) such that \( c^{|\alpha|} \leq (2\mathbb{N})^{s\alpha} \) and \( c^\alpha \leq (2\mathbb{N})^{s\alpha} \).

For \( \gamma \in \mathbb{I} \), the \( \gamma \)th Fourier-Hermite polynomial is defined by \( H_\gamma(\omega) := \prod_{k=1}^\infty h_\gamma_k(\langle \omega, \xi_k \rangle) \), where \( \xi_k, k \in \mathbb{N} \), is the Hermite function of order \( k \), and \( h_k, k \in \mathbb{N}_0 \), is the Hermite polynomial. For a normed space \( X \), the tensor product \( X \otimes (S)_{-p} \) is the space of \( X \)-valued generalized stochastic processes of Kondratiev-type, and the space \( X \otimes (S)_{-1} \) is the inductive limit of spaces \( X \otimes (S)_{-p}, p \geq 0 \). Every \( X \)-valued generalized stochastic process of Kondratiev-type, \( F \in X \otimes (S)_{-1} \), can be represented in the chaos expansion form \( F(x, \omega) = \sum_{\gamma \in \mathbb{I}} f_\gamma(x) H_\gamma(\omega), f_\gamma \in X \), with \( \|F\|_{X \otimes (S)_{-p}}^2 := \sum_{\gamma \in \mathbb{I}} \|f_\gamma\|^2_X (2\mathbb{N})^{-p|\gamma|} \) finite for some \( p \geq 0 \). If it is finite for \( p_0 \), then it is finite for all \( p \geq p_0 \). The minimal such \( p \) we call the critical exponent. For \( X \) a Banach space, \( X \otimes (S)_{-p} \) is a Banach space for every \( p \geq p_0 \), and \( X \otimes (S)_{-1} \) is a Fréchet space.

The Wick product is introduced to overcome the multiplication problem for random variables in \([3]\) and it is generalized to the set of generalized stochastic processes in \([7]\). Recall, if \( F,G \in X \otimes (S)_{-p} \), \( p \geq 0 \), are generalized stochastic processes given in chaos expansion forms \( F = \sum_{\alpha \in \mathbb{I}} f_\alpha H_\gamma \) and \( G = \sum_{\beta \in \mathbb{I}} g_\beta H_\beta \), then the Wick product \( F \triangleleft G \), is defined by \( F \triangleleft G = \sum_{\gamma \in \mathbb{I}} \left( \sum_{\alpha+\beta=\gamma} f_\alpha g_\beta \right) H_\gamma \).

We conclude the introductory section stating a theorem on the deterministic parabolic evolution problems, and a technical Lemma, both necessary for the later analysis. The proof of the theorem is similar to the proof of Theorem 3 in \([4]\), while proof of the lemma is straightforward, thus both are omitted.

Theorem 2. Let the unbounded and closed operator \( \mathcal{L} \) with a dense domain \( D \subseteq L^2(\mathbb{R}^d) \), be an infinitesimal generator of a \( C_0 \)-semigroup \( (T_t)_{t \geq 0} \) on \( L^2(\mathbb{R}^d) \). Let the force term \( f \in AC([0,T]; L^2(\mathbb{R}^d)) \), i.e., being differentiable a.e. on \([0,T]\) with \( f' \in L^1([0,T], L^2(\mathbb{R}^d)) \). Let the initial condition \( g \in D \), and the potential \( q \in L^\infty(\mathbb{R}^d) \). Then, the deterministic parabolic initial value problem

\[
\left( \frac{\partial}{\partial t} - \mathcal{L} \right) u(t, x) + q(x) \cdot u(t, x) = f(t, x), \quad u(0, x) = g(x),
\]

has a unique bounded nonnegative solution \( u \in AC([0,T]; L^2(\mathbb{R}^d)) \) which satisfies

\[
\|u(t, \cdot)\|_{L^2} \leq M(t) \left( \|g(\cdot)\|_{L^2} + \int_0^t \|f(s, \cdot)\|_{L^2} \, ds \right), \quad t \in [0, T),
\]

where \( M(t) := M \exp \left( \|w + M\|_{L^\infty} t \right) \), and \( w \in \mathbb{R} \) and \( M > 0 \) are the stability constants from the semigroup estimate \( \|T_t\|_{L(L^2(\mathbb{R}^d))} \leq Me^{Mt}, t \geq 0 \).

Lemma 3. Let \( M(t) \) be as in Theorem 2 with the potential \( q_0 \in L^\infty(\mathbb{R}^d) \). Then, the following estimates hold

(a) \( \bar{M}(t) := \int_0^t M(s) \, ds = \frac{M(t) - M}{w + M\|q_0\|_{L^\infty}}, \quad \int_0^t sM(s) \, ds \leq t\bar{M}(t), \)

(b) \( \int_0^t M(s)\bar{M}(s)^n \, ds \leq \bar{M}(t)^{n+1}, \quad \int_0^t sM(s)\bar{M}(s)^n \, ds \leq t\bar{M}(t)^{n+1} \) for all \( n \in \mathbb{N} \).
2. Stochastic parabolic equations with random and space depending bounded potential

Now we turn our attention to stochastic initial value problem (1), more precisely we consider

$$ \left( \frac{\partial}{\partial t} - \mathcal{L} \right) U(t, x, \omega) + Q(x, \omega) \triangledown U(t, x, \omega) = F(t, x, \omega), \quad t \in (0, T], \quad x \in \mathbb{R}^d, \quad \omega \in \Omega $$

$$ U(0, x, \omega) = G(x, \omega), \quad x \in \mathbb{R}^d, \quad \omega \in \Omega. $$

The main result follows.

**Theorem 4.** Let $\mathcal{L}$ be an unbounded closed operator with dense domain $D \subseteq L^2(\mathbb{R}^d)$, acting on the space component and generating a $C_0$-semigroup on $L^2(\mathbb{R}^d)$ satisfying $\|T_t\|_{L(L^2(\mathbb{R}^d))} \leq M e^{\omega t}$, and let $\hat{M}(T)$ be as in Lemma 3. Let the potential $Q \in L^\infty(\mathbb{R}^d) \otimes (S)_{-1}$ be a generalized stochastic process of Kondratiev-type with the critical exponent $p_1$, and the chaos expansion $F \in AC([0, T]; L^2(\mathbb{R}^d)) \otimes (S)_{-1}$ and the initial condition $G \in D \otimes (S)_{-1}$ are generalized stochastic processes of Kondratiev-type, with critical exponents $p_2$ and $p_3$, respectively. Then, there exists a unique generalized stochastic process $U \in AC((0, T]; D) \otimes (S)_{-1} \subseteq AC((0, T]; L^2(\mathbb{R}^d)) \otimes (S)_{-1}$ satisfying the stochastic evolution initial value problem (3). Moreover, for all $t \in [0, T]$ the coefficients $u_\gamma, \gamma \in \mathcal{I}$ satisfy

$$ \|u_\gamma(t, \cdot)\|_{L^2} \leq M(t) \left\{ a_\gamma(t) + \sum_{k=1}^{[\gamma]} \hat{M}(t)^k \left( \sum_{0 \leq |\beta| \leq |\gamma| - k} a_{\beta}(t) \left( \sum_{\substack{\theta_1, \ldots, \theta_k \in \mathbb{N} \cup \{0\}, i=1, \ldots, k}} \prod_{i=1}^{k} q_{\theta_i} \|q_{\theta_i} \|_{L^\infty} \right) \right) \right\}, $$

where $a_\gamma(t) := \|g_\gamma\|_{L^2} + t\|f_\gamma\|, \gamma \in \mathcal{I}.$

**Proof.** Representing stochastic processes $Q, F$ and $G$ appearing in the problem (3) in their chaos expansion forms, assuming the solution $U$ in the form $U(t, x, \omega) = \sum_{\gamma \in \mathcal{I}} u_\gamma(t, x) H_\gamma(\omega)$, and using the definition of the Wick product, we formally obtain

$$ \sum_{\gamma \in \mathcal{I}} \left( \frac{\partial}{\partial t} - \mathcal{L} \right) u_\gamma(t, x) H_\gamma(\omega) + \sum_{\gamma \in \mathcal{I}} \sum_{\alpha + \beta = \gamma} q_\alpha(x) u_\beta(t, x) H_\gamma(\omega) = \sum_{\gamma \in \mathcal{I}} f_\gamma(t, x) H_\gamma(\omega), $$

$$ \sum_{\gamma \in \mathcal{I}} u_\gamma(0, x) H_\gamma(\omega) = \sum_{\gamma \in \mathcal{I}} g_\gamma(x) H_\gamma(\omega). $$

From the uniqueness of the chaos expansion representations, the problem is reduced to a triangular system of deterministic equations which can be solved recursively with respect to the length of $\gamma \in \mathcal{I}$. For $|\gamma| = 0$:

$$ \left( \frac{\partial}{\partial t} - \mathcal{L} \right) u_0(t, x) + q_0(x) u_0(t, x) = f_0(t, x), \quad u_0(0, x) = g_0(x). $$

By assumptions, $q_0 \in L^\infty(\mathbb{R}^d), f_0 \in AC([0, T]; L^2(\mathbb{R}^d))$, and $g_0 \in D$. Theorem 3 implies a unique solution $u_0 \in AC([0, T], D)$ to (5) given by $u_0(t, x) = S_t g_0(x) + \int_0^t S_{t-s} f_0(s, x) ds, t \in [0, T], x \in \mathbb{R}^d$, where $(S_t)_{t \geq 0}$ is a $C_0$-semigroup on $L^2(\mathbb{R}^d)$ generated by $\mathcal{L} - q_0 \text{Id}$, satisfying

$$ \|u_0(t, \cdot)\|_{L^2} \leq M(t) \left( \|g_0\|_{L^2} + \int_0^t \|f_0(s, \cdot)\|_{L^2} ds \right) \leq M(t) (\|g_0\|_{L^2} + t\|f_0\|) = M(t)a_0(t). $$
For $|\gamma| > 1$ we have
\[
\left( \frac{\partial}{\partial t} - L \right) u_\gamma(t, x) + q_0(x) u_\gamma(t, x) = \tilde{f}_\gamma(t, x), \quad u_\gamma(0, x) = g_\gamma(x),
\]
where $\tilde{f}_\gamma(t, x) = f_\gamma(t, x) - \sum_{\alpha+\beta=\gamma} q_\alpha(x) u_\beta(t, x) = f_\gamma(t, x) - \sum_{\alpha<\gamma} q_{\beta-\gamma}(x) u_\beta(t, x)$, with $u_0$ and $u_\beta$, $\beta < \gamma$ being the solutions obtained in the previous steps. Thus, we obtain a deterministic problem of the same form as for $|\gamma| = 0$ satisfying, by assumptions, the conditions of Theorem 2 yielding a unique solution $u_\gamma \in AC([0, T], D)$ given by $u_\gamma(t, x) = S_t g_\gamma(x) + \int_0^t S_{t-s} \tilde{f}_\gamma(t, x) ds$ and satisfying
\[
\|u_\gamma(t, \cdot)\|_{L^2} \leq M(t) \left( a_\gamma(t) + \sum_{0 \leq \beta < \gamma} \|q_{\gamma-\beta}\|_{L^\infty} \int_0^t \|u_\beta(s, \cdot)\|_{L^2} ds \right).
\]
Next, by induction, we will prove that $u_\gamma$, $\gamma \in I$, satisfy the estimate (1). The estimate for $|\gamma| = 0$ boils down to (6). We assume that the estimate (1) holds for every $\beta \in I$ with $|\beta| \leq n$, i.e.,
\[
\|u_\beta(t, \cdot)\|_{L^2} \leq M(t) \left\{ a_\beta(t) + \sum_{l=1}^{[\beta]} M_l(t)^{l+1} \left( \sum_{0 \leq |\alpha| \leq |\beta| - l} \sum_{\alpha < \beta} a_\alpha(t) \prod_{\theta_i \neq 0 \atop \theta_i \in I} |\theta_i| \right) \right\},
\]
and want to show that (1) holds for $\gamma \in I$ with $|\gamma| = n + 1$. Integrating (8) and using Lemma 3 we obtain
\[
\int_0^t \|u_\beta(s, \cdot)\|_{L^2} ds \leq a_\beta(t) \tilde{M}(t) + \sum_{l=1}^{[\beta]} \tilde{M}(t)^{l+1} \left( \sum_{0 \leq |\alpha| \leq |\beta| - l} \sum_{\alpha < \beta} a_\alpha(t) \prod_{\theta_i \neq 0 \atop \theta_i \in I} |\theta_i| \right) \|q_\beta\|_{L^\infty}.
\]
Starting from (7) we obtain that $\|u_\gamma(t, \cdot)\|_{L^2}$ is bounded from above by $M(t)$ multiplied with
\[
a_\gamma(t) + \sum_{0 \leq \beta < \gamma} \|q_{\gamma-\beta}\|_{L^\infty} a_\beta(t) \tilde{M}(t) + \sum_{0 < \beta < \gamma} \|q_{\gamma-\beta}\|_{L^\infty} \sum_{l=1}^{[\beta]} \tilde{M}(t)^{l+1} \left( \sum_{0 \leq |\alpha| \leq |\beta| - l} \sum_{\alpha < \beta} a_\alpha(t) \prod_{\theta_i \neq 0 \atop \theta_i \in I} |\theta_i| \right) \|q_\beta\|_{L^\infty}.
\]
We want to sum terms with respect to powers of $\tilde{M}(t)$. The first step is to change the order of the first two sums in the third term, where, while $0 < \beta < \gamma$, the length $|\beta|$ varies from 1 to $n$, thus the third term becomes
\[
\sum_{l=1}^n \tilde{M}(t)^{l+1} \sum_{0 \leq \beta < \gamma} \left( \sum_{0 \leq |\alpha| \leq |\beta| - l} a_\alpha(t) \prod_{\theta_i \neq 0 \atop \theta_i \in I} |\theta_i| \right) \left( \sum_{l=1}^{\gamma - |\beta| - l} \prod_{\theta_i \neq 0 \atop \theta_i \in I} |\theta_i| \right) \|q_\beta\|_{L^\infty}.
\]
Next we merge the sums over $0 < \beta < \gamma$ (which implies $|\beta| \leq |\gamma| - 1$) and $0 \leq \alpha < \beta$ with $|\alpha| \leq |\beta| - l$ into the sum over $0 \leq \alpha < \gamma$ with $|\alpha| \leq |\gamma| - (l + 1)$. Denoting $q_{\theta_{l+1}} := q_{\gamma-\beta}$ we obtain
\[
\sum_{l=1}^n \tilde{M}(t)^{l+1} \sum_{0 \leq |\alpha| \leq |\gamma| - (l+1)} a_\alpha(t) \prod_{\theta_i \neq 0 \atop \theta_i \in I} |\theta_i| \|q_\beta\|_{L^\infty}.
\]
\[
= \sum_{k=2}^{\gamma} \tilde{M}(t)^k \sum_{0 \leq |\alpha| \leq |\gamma| - k} a_\alpha(t) \prod_{\theta_i \neq 0 \atop \theta_i \in I} |\theta_i| \|q_\beta\|_{L^\infty}.
\]
Therefore, $\|u_\gamma(t, \cdot)\|_{L^2}$ is bounded by

$$M(t) \left\{ a_\gamma(t) + \sum_{0 \leq \beta < \gamma} \|q_{\gamma - \beta}\|_{L^\infty} a_\beta(t) \tilde{M}(t) + \sum_{k=2}^{|\gamma|} \tilde{M}(t)^k \sum_{0 \leq |\alpha| \leq |\gamma| - k} \sum_{0 \leq |\beta| \leq |\gamma| - k} \sum_{\theta_1 + \ldots + \theta_k = \gamma - \alpha} \prod_{i=1}^{k} \|q_{\theta_i}\|_{L^\infty} \right\},$$

and thus (4) holds. Next we show that the solution $U$ is a generalized stochastic process of Kondratiev-type, i.e., that the sum $\|U\|^2 := \sum_{\gamma \in I} \|u_\gamma\|^2_{AC([0,T]; L^2(R^d))} (2N)^{-p\gamma}$ is finite for some critical exponent $p$ to be determined below. Using (4) and $(a_1 + a_2 + a_3)^2 \leq 3a_1^2 + 3a_2^2 + 3a_3^2$ we find that $\|U\|^2$ is bounded by

$$3M(T)^2 \sum_{\gamma \in I} \|g_\gamma\|^2_{L^2}(2N)^{-p\gamma} + 3T^2 M(T)^2 \sum_{\gamma \in I} \|f_\gamma\|^2_{L^2}(2N)^{-p\gamma}$$

$$+ 3M(T)^2 \sum_{\gamma \in I} \sum_{k=1}^{\gamma} \tilde{M}(T)^k \left( \sum_{0 \leq |\beta| \leq |\gamma| - k} a_\beta(T) \left( \sum_{\theta_1 + \ldots + \theta_k = \gamma - \beta} \prod_{i=1}^{k} \|q_{\theta_i}\|_{L^\infty} \right) \right)^2 (2N)^{-p\gamma}$$

$$=: 3M(T)^2 \left( S_1 + T^2 S_2 + S_3 \right).$$

Chosing $p \geq \max\{p_2, p_3\}$, according to assumptions on $G$ and $F$, we have

$$S_1 + T^2 S_2 \leq \sum_{\gamma \in I} \|g_\gamma\|^2_{L^2}(2N)^{-p\gamma} + T^2 \sum_{\gamma \in I} \|f_\gamma\|^2_{L^2}(2N)^{-p\gamma} := A < \infty.$$  

For $S_3$, using $(\sum_{k=1}^{\gamma} x_k)^2 \leq |\gamma| \sum_{k=1}^{\gamma} x_k^2$ we obtain

$$S_3 \leq \sum_{\gamma \in I} |\gamma| \sum_{k=1}^{\gamma} \tilde{M}(T)^{2k} \left( \sum_{0 \leq |\beta| \leq |\gamma| - k} a_\beta(T) \left( \sum_{\theta_1 + \ldots + \theta_k = \gamma - \beta} \prod_{i=1}^{k} \|q_{\theta_i}\|_{L^\infty} \right) \right)^2 (2N)^{-p\gamma}.$$

Further, using $(\sum_{\alpha \in I} |x_\alpha y_\alpha|^2) \leq \left( \sum_{\alpha \in I} |x_\alpha|^2 \right) \left( \sum_{\alpha \in I} |y_\alpha|^2 \right)$ and rearranging the powers of $(2N)^{-p\gamma}$ we obtain

$$S_3 \leq \sum_{\gamma \in I} |\gamma| \sum_{k=1}^{\gamma} \tilde{M}(T)^{2k} \sum_{0 \leq |\beta| \leq |\gamma| - k} \sum_{\theta_1 + \ldots + \theta_k = \gamma - \beta} \prod_{i=1}^{k} \|q_{\theta_i}\|_{L^\infty} (2N)^{-p\beta} \left( \sum_{0 \leq |\beta| \leq |\gamma| - k} a_\beta(T)^2 (2N)^{-p\beta} \right)^2 (2N)^{-p\beta}.$$

Since $\beta < \gamma$ and $p \geq \max\{p_2, p_3\}$ (chosen in (9)) we have

$$\sum_{0 \leq |\beta| \leq |\gamma| - k} a_\beta(T)^2 (2N)^{-p\beta} \leq 2A < \infty,$$

and by Lemma (1) we find

$$S_3 \leq 2A \sum_{\gamma \in I} |\gamma| \sum_{k=1}^{\gamma} \tilde{M}(T)^{2k} \left( \sum_{0 \leq |\beta| \leq |\gamma| - k} \|q_{\beta}\|_{L^\infty} 2^{2k|\gamma - \beta|} (2N)^{-p\beta} \right)(2N)^{-p\beta}. $$
By Lemma [11] there exists $s \geq 0$ such that \(2^{2k|\gamma-\beta|} \leq (2N)^{s(\gamma-\beta)/2}\), and choosing \(p\) to satisfy \(p > s + 3\) we have

\[
S_3 \leq 2A \sum_{\gamma \in I} |\gamma| \left( \sum_{k=1}^{2^\beta} (2N)^{\frac{p(\gamma-\beta)}{2}} \left( \sum_{\beta < k} (2N)^{-\frac{p(\gamma-\beta)}{2}} \right) (2N)^{-\frac{p(\gamma-\beta)}{2}} \right) (2N)^{-\frac{p(\gamma-\beta)}{2}} 
\]

\[
\leq 2AC \sum_{\gamma \in I} |\gamma| \sum_{k=1}^{2^\beta} \tilde{M}(T)^{2k} \|q_0\|_{L^\infty}^{2k}(2N)^{-\frac{p(\gamma-\beta)}{2}},
\]

where \(A\) is defined in (9) and \(C := \sum_{\omega \in I} (2N)^{-\frac{p(\gamma-\beta)}{2}}\) is finite by (2). The assumption \(\tilde{M}(T)\|q_0\|_{L^\infty} \neq 1\) allows to sum up the inner sum leading to

\[
S_3 \leq 2AC \frac{\tilde{M}(T)^2 \|q_0\|_{L^\infty}^2}{1 - \tilde{M}(T)^2} \left( \sum_{\gamma \in I} |\gamma| (2N)^{-\frac{p(\gamma-\beta)}{2}} - \sum_{\gamma \in I} |\gamma| \tilde{M}(T)^{2|\gamma|} \|q_0\|_{L^\infty}^{2|\gamma|} (2N)^{-\frac{p(\gamma-\beta)}{2}} \right).
\]

By Lemma [1] for \(\tilde{M}(T)^2 \|q_0\|_{L^\infty}^2 > 0\) there exists \(s_1 > 0\) so that

\[
(10) \quad \sum_{\gamma \in I} |\gamma| \tilde{M}(T)^{2|\gamma|} \|q_0\|_{L^\infty}^{2|\gamma|} (2N)^{-\frac{p(\gamma-\beta)}{2}} \leq \sum_{\gamma \in I} (2N)^{s_1 \gamma} (2N)^{-\frac{p(\gamma-\beta)}{2}} \leq \sum_{\gamma \in I} (2N)^{s_1 \gamma} (2N)^{-\frac{p(\gamma-\beta)}{2}} \leq \sum_{\gamma \in I} (2N)^{s_1 \gamma} (2N)^{-\frac{p(\gamma-\beta)}{2}} \leq \sum_{\gamma \in I} (2N)^{s_1 \gamma} (2N)^{-\frac{p(\gamma-\beta)}{2}} \leq \sum_{\gamma \in I} (2N)^{\frac{(\xi-1)\gamma}{2}} < \infty,
\]

yielding that \(S_3\) is finite. Finally, \(\|U\|_{L^1}^2 < \infty\) for \(p \geq \max\{p_2, p_3, s + 3s_1 + 6\}\), and \(U \in AC([0, T]; D) \otimes (S)_{-1}\).

The uniqueness of the solution \(U\) follows from the uniqueness of its coefficients \(u_\gamma, \gamma \in I\) and the uniqueness of the chaos expansion representation in the Fourier–Hermite basis of orthogonal stochastic polynomials.

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