Mathematical structure of the temporal gauge in quantum electrodynamics

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Abstract

The conflict between Gauss’ law constraint and the existence of the propagator of the gauge fields, at the basis of contradictory proposals in the literature, is shown to lead to only two alternatives, both with peculiar features with respect to standard quantum field theory. In the positive (interacting) case, the Gauss’ law holds in operator form, but only the correlations of exponentials of gauge fields exist (non-regularity) and the space translations are not strongly continuous, so that their generators do not exist. Alternatively, a Källen-Lehmann representation of the two point function of $\mathcal{A}_i$ satisfying locality and invariance under space time translations, rotations and parity is derived in terms of the two point function of $F_{\mu\nu}$; positivity is violated, the Gauss’ law does not hold, the energy spectrum is positive, but the relativistic spectral condition does not hold. In the free case, $\theta$-vacua exist on the observable fields, but they do not have time translationally invariant extensions to the gauge fields; the vacuum is faithful on the longitudinal field algebra and defines a modular structure (even if the energy is positive). Functional integral representations are derived in both cases, with the alternative between ergodic measures on real random fields or complex Gaussian random fields.
1 Introduction

In the treatment of gauge quantum field theories, even if the choice of the gauge, a basic ingredient for the control of the dynamical problem, is irrelevant for the physical conclusions, it crucially affects the mathematical structure of the formulation as well as the way the various mechanisms (mass generation, gauge symmetry breaking, $\theta$-vacua, chiral symmetry breaking etc.) are effectively realized. In the discussion of the non-perturbative aspects of Quantum Chromo-Dynamics (QCD) [1] and of the Higgs mechanism [2], the temporal gauge has been widely used and it is therefore worthwhile to investigate its mathematical structure.

From a technical point of view, (the only relevant for the gauge choice), such a gauge has been preferred to others because it is believed to satisfy locality, positivity, the Gauss’ law constraint in operator form, at the only expense of manifest Lorentz covariance. As such, it appears as intermediate between the Coulomb gauge, where locality is lost (besides manifest Lorentz covariance), and the Feynman-Gupta-Bleuler (FGB) gauge [3, 4] where locality holds but positivity and the operator Gauss’ law constraint are lost.

The aim of this paper is to critically examine the mathematical structure of the temporal gauge and the status of general properties like positivity, operator Gauss’ law, positivity of the energy, relativistic spectrum condition.

The usual formulation of the temporal gauge relies either on canonical quantization as a basis of the perturbative expansion or on a functional integral approach to the interacting theory, with a space lattice regularization, which also gives a (lattice regularized) canonical structure. Thus, in both cases one has a CCR algebra at equal times; actually, also in the presence
of the interaction, the subalgebra generated by $\text{div}A$ and $\text{div}E - j_0$ remains canonical, at all times, with an interaction independent commutator.

Contrary to the standard case, the CCR structure of the temporal gauge does not uniquely identify its vacuum representation; as a matter of fact, the form of the propagator of the gauge potential has been debated in the literature, but a classification of the possibilities is lacking especially in connection with basic structural properties, so that also recent textbook presentations of the temporal gauge \[5\] leave such basic points unsettled.

The analysis of such a problem is the main content of this note: we shall classify all time translation invariant states on the CCR algebra in the free case and indicate the implications on the interacting abelian theory.

The results are the following. Both in the free and the interacting case positivity and time translation invariance exclude the existence of the correlation functions of the field $\text{div}A$, only its exponentials being defined. Positivity also implies that the vacuum satisfies the Gauss’ law constraint in operator form and that the space translations are not strongly continuous, so that one cannot define their generators (the momentum) and the relativistic spectrum condition cannot even be defined. In the free case the condition of positivity of the energy spectrum is shown to uniquely select the (non regular) state considered in Ref. \[6\] (see also \[7, 8\]); other time translational invariant pure states exist, which satisfy the spectral condition only on the observables.

In view of the perturbative expansion and the standard functional integral approach, a problem widely discussed in the literature is the form of the propagator of the gauge potential, with no general sharp conclusion and with proposals often in conflict with basic principles of standard quantum
field theory, even in the free case (see Ref.[5]). To clarify the problem, we shall derive a Källen-Lehmann representation of the two point function of the gauge potential in the interacting case under the general conditions of locality and invariance under space-time translations, rotations and parity. The resulting two point function violates positivity and the relativistic spectral condition (but not the positivity of the energy spectrum) and the vacuum cannot be annihilated by the Gauss operator \( \text{div} E - j_0 \) (such features are shared by the FGB gauge, where, however, there is no violation of the relativistic spectral condition). In the free field case, the quasi free state defined by the two point function gives rise to an indefinite inner product structure which can be discussed as in the FGB gauge in terms of a Hilbert-Krein structure.

The euclidean functional integral representation is discussed in the positive and in the indefinite case, also with the aim of clarifying the unsatisfactory proposals in the literature (which ignore the violation of Nelson positivity, involve infinite normalizations, formal Faddev-Popov ghosts, improper realization of the Gauss constraint etc.). In the indefinite case, the representation of the euclidean fields requires, besides real gaussian fields \( A^r_i(x, \tau), \partial_l \varphi(x), \xi(x, \tau), \) with \( \xi(f, \tau) \) the Wiener process, a complex Gaussian field \( z(x) \). In the positive case, the complex Gaussian field \( z(x) \) is replaced by a real random field \( \Xi(x) \) with functional measure defined by ergodic means. The correlation functions of the exponentials of the (smeared) fields are therefore represented by integration with the product of the above Gaussian measures and a measure over the spectrum of the Bohr algebra generated by the exponentials of \( \Xi(g) \).
2 Algebraic structure

At a formal level the temporal gauge is defined by the gauge condition $A_0 = 0$, by the canonical commutation relations (CCR)

$$[A_i(x), \partial_t A_j(y)] = i \delta_{ij} \delta(x - y),$$

(2.1)

and by the CAR relations of the charged fermion fields $\psi(x), \bar{\psi}(y)$. The gauge fields satisfy the following equations of motion

$$\partial_t^2 A_i - \Delta A_i + \partial_i \text{div} A = j_i,$$

(2.2)

where $j_{\mu}$ is the conserved gauge invariant electromagnetic current constructed in terms of the charged fermion fields. We shall denote by $\mathcal{F}_c$ the polynomial algebra generated by the zero time fields $A_i, \dot{A}_i, \psi, \bar{\psi}, j_{\mu}$, smeared with test functions in $\mathcal{S}(\mathbb{R}^3)$, hereafter called canonical field algebra.

Eq.(2.2) implies that $\text{div} E - j_0$ is time independent so that $\forall g \in \mathcal{S}(\mathbb{R}^3)$

$$(\text{div} E - j_0)(g, t) \equiv (\text{div} E - j_0)(g, h), \ h \in \mathcal{S}(\mathbb{R}), \int ds \ h(s) = 1,$$

is a well defined time independent operator and therefore its equal time commutators with the fields are well defined operator valued distributions. Such commutators are fixed by the condition that $G \equiv \text{div} E - j_0$ generates time independent gauge transformations; such a property follows from canonical quantization if the gauge invariant point splitting regularization of the current amounts to the addition of terms linear in $A_i, \partial_0 A_i$ to the canonical fermion current. Under such a condition one has

$$[A_i(x, t), G(y, t)] = -i \partial_t \delta(x - y).$$

(2.3)
As we shall see below, a positive realization of the temporal gauge can only be done in terms of Weyl algebras. We then introduce the algebras:

1) \( \mathcal{A} \equiv \) the polynomial algebra generated by \( A(f) \equiv A_i(f_i) \), \( \partial_t A(g) = E(g) \), \( f_i, g_i \in \mathcal{S}(\mathbb{R}^3) \) (gauge field algebra), and \( \mathcal{W} \equiv \) the corresponding gauge Weyl algebra generated by \( \exp i[A(f) + E(g)] \equiv W(f,g) \);

2) \( \mathcal{A}_l \equiv \) the polynomial algebra generated by \( A_i(\partial_i h) \), \( (\text{div} E - j_0)(g) \), \( h, g \in \mathcal{S}(\mathbb{R}^3) \), called the longitudinal field algebra, and \( \mathcal{W}_l \) the corresponding longitudinal Weyl algebra, generated by \( \exp i[A(\partial h) + (\text{div} E - j_0)(g)] \equiv W_l(h,g) \).

By decomposing test functions into longitudinal and transverse (non-local) components and by an analysis in momentum space, it is not difficult to see that in the free case the time evolution of \( \mathcal{W} \) is relativistically local, i.e.

\[
\alpha_t(A(f) + E(g)) = A(f_t) + E(g_f),
\]

with \( \text{supp} \, f_t \cup \text{supp} \, g_t \) contained in the causal shadow of \( \text{supp} \, f \cup \text{supp} \, g \). The same analysis shows that eq.(2.2) has a relativistically causal Green function, so that, in the interacting case, the relativistic locality of the gauge fields of the temporal gauge follows from the relativistic locality of the observable field \( j_i \); this implies local commutativity for the Wightman field algebra \( \mathcal{F} \), since the fermion coupling is local. In contrast, in the Coulomb gauge, the fermion coupling is non local and local commutativity is lost.

The free time evolution of the longitudinal Weyl algebra is

\[
\alpha_t(W_l(h, k)) = W_l(h, k + t h)
\]

so that the longitudinal fields describe an infinite set of free non-relativistic particles. In fact, given a complete set \( \{f_n\} \) in \( L^2(\mathbb{R}^3) \), with \( (f_n, -\Delta f_m) = \)
the variables
\[ q_n \equiv \text{div}A(f_n), \quad p_n \equiv \text{div}E(f_n), \quad (2.5) \]
are canonical and the time evolution is that of free particles
\[ \dot{q}_n = p_n, \quad \dot{p}_n = 0. \]

The above algebraic structure follows from canonical quantization at equal times. Its validity is independent of the presence of the interaction, provided an ultraviolet regularization (e.g. by a space lattice cutoff) is introduced, so that the time evolution of the above algebras is well defined. Actually, by eq.(2.3), 
\[ [e^{iA(\partial,\mu)}, (\text{div}E - j_0)(g, s)] \]
is independent of \( s \) and therefore such a canonical commutator extends to unequal times and is independent of the interaction
\[ [e^{iA(\partial,\mu)}], (\text{div}E - j_0)(g, s)] = -\int d^4x \Delta h(x) g(x) e^{iA(\partial,\mu)}, \quad h \in \mathcal{S}(\mathbb{R}^4). \quad (2.6) \]

The field algebra \( \mathcal{F} \) of the temporal gauge has the following infinite dimensional group of automorphisms (time independent (small) gauge transformations): \( \gamma^A, \Lambda(x) \in \mathcal{S}(\mathbb{R}^3) \)
\[ \gamma^A(A(f)) = A(f) - \int d^3x \Lambda \text{div}f, \quad \gamma^A(E(g)) = E(g), \quad \gamma^A\psi(f) = \psi(e^{iA}f). \quad (2.7) \]
The \( \gamma^A \) commute with the time translations, as a consequence of the gauge invariance of the Lagrangean; they are generated by \( G(\Lambda) \) and are unitarily implemented by elements of the longitudinal Weyl algebra.

The automorphisms of eq.(2.7), with \( \Lambda(x) = \alpha \cdot x \), are called large gauge transformations and are still denoted by \( \gamma^A \). They commute with the time
and space translations and are locally generated by the local charges

\[ G^\Lambda_R \equiv G(\Lambda f_R), \quad f_R(x) = f(|x|/R), \quad f \in \mathcal{D}(\mathbb{R}), \]

in the sense that the variations of the fields \( A \) are given by

\[
\delta^\Lambda A = \lim_{R \to \infty} -i [Q^\Lambda_R, A].
\] (2.8)

The observable subalgebras \( \mathcal{F}_{\text{obs}}, \mathcal{A}_{\text{obs}}, \mathcal{W}_{\text{obs}} \) are characterized by pointwise invariance under all \( \gamma^\Lambda \). \( \mathcal{A}_{\text{obs}} \) is the algebra generated by \( A(f), \text{div} f = 0 \) and by \( E(g); \mathcal{F}_{\text{obs}} \) has a non trivial center which contains the algebra generated by \( G(f), f \in \mathcal{S}(\mathbb{R}^3) \). The invariance of the vacuum under large gauge transformations is incompatible with the existence of the correlation functions of the field algebra \( \mathcal{F} \) and, as we shall see explicitly in the free case, only holds in the non regular positive formulation.

The gauge field algebra \( \mathcal{A} \), as well as the gauge Weyl algebra \( \mathcal{W} \), have the following three parameter group of automorphisms \( \beta^\theta, \theta \in \mathbb{R}^3 \):

\[
\beta^\theta (A(f)) = A(f), \quad \beta^\theta (E(g)) = E(g) + \theta_i \int d^3 x g_i,
\] (2.9)

which generate a background constant (classical) electric field.

The automorphisms \( \beta^\theta \), for simplicity called \( \theta \) automorphisms, commute with the space translations and have the following commutation relations with the gauge transformations and with the free time evolution

\[
\beta^\theta \gamma^\Lambda = \gamma^\Lambda \beta^\theta, \quad \beta^\theta \alpha_t = \alpha_t \beta^\theta \gamma^t x,
\] (2.10)

so that that they commute with the free time evolution on the observable algebra. The \( \theta \) automorphisms are generated on \( \mathcal{W} \) (and on \( \mathcal{A} \)) by the local charges

\[ Q_R \equiv A_i(\theta_i f_R). \] (2.11)
Even if the automorphisms $\beta^\theta$ commute with the gauge transformations, the corresponding generators do not. In conclusion, at least in the free case, such transformations have similar properties to those of the chiral transformations in quantum chromodynamics.

3 States and representations

In the following we shall adopt the physicist terminology by which a linear normalized functional on a $*$-algebra $\mathcal{A}$ is briefly called a state, even if it is not positive with respect to the intrinsic $*$-operation defined on the algebra. A state in the above sense defines a representation $\pi_\omega$ of $\mathcal{A}$ with a cyclic vector $\Psi_\omega$ and an inner product $(\pi_\omega(A) \Psi_\omega, \pi_\omega(B) \Psi_\omega) = \omega(A^* B)$. A representation of a $*$-algebra $\mathcal{A}$ is called irreducible if any bounded operator which commutes with $\mathcal{A}$ is a multiple of the identity.

We shall show that the gauge condition $A_0 = 0$ does not uniquely fix the vacuum representation of the longitudinal algebra, i.e. its actual realization in terms of operators. We shall discuss the general obstructions which arise if, given a positive state on the observable field algebra $\mathcal{F}_{\text{obs}}$, one looks for extensions to the field algebra $\mathcal{F}$. Actually, independently of the interaction, we shall show that existence of the correlation functions of the fields requires a non positive vacuum state, which cannot be annihilated by the Gauss operator $\text{div}E - j_0$, whereas a positive representation requires a non regular state on the longitudinal algebra $\mathcal{L}$ generated by $\exp i(A_i(\partial_i h))$, $h \in \mathcal{S}(\mathbb{R}^4)$. The two alternatives are shown to have very different mathematical features and can in fact be distinguished on the basis of the structural properties that one wants to preserve.
Proposition 3.1  Let $\omega$ be a positive vacuum state on $F_{obs}$, satisfying the cluster property, then

i) $\omega$ does not have a positive extension to $F$

ii) any positive extension $\Omega$ to $L$ is non regular and satisfies

\[
\Omega(W_l(0, g)) = \omega(W_l(0, g)), \quad \Omega(W_l(f, g)) = 0, \quad if \quad f \neq 0 \quad (3.1)
\]

iii) all positive extensions of $\omega$ to an algebra containing $L$ define a GNS representation $\pi_\Omega$ in which the space translations are not implemented by strongly continuous unitary operators $U(x), \ x \in \mathbb{R}^3$, and therefore the generator, the momentum, cannot be defined

iv) all positive extensions of $\omega$ are invariant under the large (and small) gauge transformations:

\[
(\gamma^A)^* \Omega = \Omega, \quad (\text{div}E(x) - j_0(x)) \Psi_{\Omega} = 0. \quad (3.2)
\]

Proof. i). Since $[W_l(0, h), F_{obs}] = 0$, by Theor. 4.4 of \[9\]

\[
W_l(0, h) \Psi_\omega = c_h \Psi_\omega, \quad c_h \in \mathbb{C}.
\]

By Schwarz’ inequality a positive extension $\Omega$ satisfies

\[
\Omega(W_l(0, h) BW_l(0, g)) = c_g \overline{c}_{-h} \Omega(B), \quad \forall \ B \in F. \quad (3.4)
\]

This implies

\[
\Omega([\text{div}A(f), (W_l(0, h) - c_h)]) = 0, \quad \forall f, h \in \mathcal{S}(\mathbb{R}^4),
\]

whereas the algebraic structure, eq.(2.6) gives

\[
\Omega([\text{div}A(f), W_l(0, h)]) = i \int d^4 x f(x) \Delta h(x) \omega(W_l(0, h)) = c_h \int d^4 x f \Delta h,
\]
which cannot vanish since $|c_h| = 1$.

ii). In fact,

$$|c_h|^2 \Omega(e^{iA_i(\partial_i f)}) = \Omega(W_i(0, -h) e^{iA_i(\partial_i f)} W_i(0, h)) =$$

$$= e^{i \int d^4 x \Delta h} \Omega(e^{iA_i(\partial_i f)}), \quad \forall f, h \in S(\mathbb{R}^4),$$

and therefore, for $f \neq 0$,

$$\Omega(e^{iA_i(\partial_i f)}) = 0.$$  \hfill (3.5)

iii). In fact, one has

$$\Omega(e^{iA_i(\partial_i h)} U(-x) e^{-iA_i(\partial_i h)} U(x)) = \Omega(e^{iA_i(\partial_i h - \partial_i h x)})$$

and the right hand side vanishes for all $x \neq 0$ and it is $= 1$ for $x = 0$.

iv). In fact, by eq.(2.7) one has

$$\frac{d}{d\lambda} \Omega(\gamma^\Lambda (B)) = \lim_{R \to \infty} i \Omega([G_R^\Lambda, \gamma^\Lambda(B)]) = 0$$

since eq.(3.4) implies $G(\Lambda f_R) \Psi_\Omega = C_{\Lambda f_R} \Psi_\Omega$. Furthermore, Lorentz invariance implies $C_h = \omega(G(h)) = 0$.

The above Proposition clarifies the roots of the problems which arise in the quantization of the temporal gauge, by reducing them to very basic structures. The solutions proposed in the literature, relying on an analysis of the free case, involve a non normalizable vacuum, or the violation of time translation invariance, etc., (for an extensive review see Ref.[5]), so that new problems are somewhat arbitrarily added, hiding the basic issues. The following analysis of the free case makes clear that general properties, like time translation invariance and either positivity or existence of the correlation functions of the fields select exactly two alternatives, yielding solutions...
in terms of a time translation invariant vacuum over well-defined operator algebras.

In particular, the existence of the ground state correlation functions of the fields requires an indefinite inner product space as in the Gupta-Bleuler gauge; alternatively, positivity can be achieved at the price of regularity of the representation of the longitudinal Weyl algebra. A close analog of such a situation appears for free non-relativistic particles when one asks for the existence of a ground state (see [12]).

In general, given an algebra $\mathcal{A}$, a time translation automorphism $\alpha_t$ and a time translationally invariant hermitean linear functional $\Omega$ on $\mathcal{A}$, we shall say that the energy spectral condition holds if the expectations $G_{AB}(t) \equiv \Omega(A\alpha_t(B))$ are continuous in $t$ and their Fourier transform

$$\tilde{G}_{AB}(\omega) = (2\pi)^{-1/2} \int dt G_{AB}(t) e^{-i\omega t}$$

are supported in $\mathbb{R}^+$. 

**Proposition 3.2**

1. Let $\Omega$ be a state on the gauge field algebra $\mathcal{A}$ invariant under the free time evolution, then

   i) $\Omega$ cannot be positive

   ii) if the restriction $\omega$ of $\Omega$ to the observable gauge algebra $\mathcal{A}_{obs}$ is semidefinite and satisfies the energy spectral condition, then the GNS representation $\pi_\omega$ of $\mathcal{A}_{obs}$ is irreducible and coincides with the standard vacuum representation of the electromagnetic field algebra.

2. Let $\Omega$ be a state on the Weyl gauge algebra $\mathcal{W}$ invariant under the free time evolution, satisfying the energy spectral condition and its restriction $\omega$ to $\mathcal{W}_{obs}$ be semidefinite, then $\pi_\Omega(divE) = 0$, the GNS representation $\pi_\omega$ of
$\mathcal{W}_{\text{obs}}$ is irreducible and coincides with the standard vacuum representation of the free electromagnetic field (Weyl) algebra.

**Proof.** 1.i). In fact, by time translation invariance, $\Omega(\partial_t \mathcal{O}) = 0$, $\forall \mathcal{O} \in \mathcal{A}$, and, since by the equations of motion $\partial_t \text{div} E = 0$, one has

$$\Omega((\text{div} E(f))^2) = \Omega(\partial_t (\text{div} A(f) \text{div} E(f))) = 0,$$

so that positivity implies that the Hilbert space vector $\Psi_{\Omega}$, which represents $\Omega$ (in the GNS representation space), satisfies

$$\text{div} E(f) \Psi_{\Omega} = 0, \quad \forall f \in \mathcal{S}(\mathbb{R}^3).$$

This is incompatible with the CCR since

$$\Omega([A_i(x), \text{div} E(y)]) = -i \partial_i \delta(x - y).$$

1.ii). By Schwarz’ inequality, eq.(3.7) gives

$$\Omega(\mathcal{O} \text{div} E(h)) = 0, \quad \forall \mathcal{O} \in \mathcal{A}_{\text{obs}}.$$

Thus, the restriction $\omega$ of $\Omega$ to $\mathcal{A}_{\text{obs}}$ yields a representation $\pi_{\omega}$ such that $\pi_{\omega}(F_{\mu\nu})$ is a free electromagnetic field with energy spectral condition and the usual argument gives the standard Fock representation; the one-point function $\omega(E) = \omega(\hat{A})$ vanishes by the time translation invariance of $\Omega$.

2. In fact, time translation invariance implies that for $h \neq 0$, $\Omega(W_t(h, k))$ is independent of $k$, say $F(h)$. On the other hand, one has

$$e^{i[(\partial h, \partial k')-(\partial k, \partial h')]/2} e^{i t (\partial h, \partial h')/2} \Omega(W_t(h, k) \alpha_t(W_t(h', k'))))$$

$$= F(h + h'), \quad \text{if} \quad h + h' \neq 0.$$
Thus, the energy spectral condition requires $F(h+h') = 0$, whenever $h+h' \neq 0$, since otherwise, by taking $h = h'$ one would get a negative point of the energy spectrum. It also requires that the distributional Fourier transform $\tilde{G}^t(\omega)$ of $G(t h)$, with respect to the variable $t$, has support in $\omega = 0$. In fact, putting $h' = -h$, $k = k' = 0$, $d \equiv (\partial h, \partial h)/2$, in the above formula we have

$$G_h(\alpha t) \equiv G(\alpha h) = \Omega(W_l(\alpha h, 0) W_l(-\alpha h, \alpha h)) e^{it \alpha^2 d} \equiv H(t) e^{it \alpha^2 d}.$$ 

Taking the Fourier transform with respect to $t$, and using the positive support of the Fourier transform of $H(t)$, we get

$$\text{supp} \tilde{G}_h(\omega) = \text{supp} \tilde{H}((\omega - \alpha^2 d)/\alpha) \subseteq \mathbb{R}^+,$$

so that

$$\text{supp} \tilde{G}((\omega + \alpha^2 d)/\alpha) \subseteq \mathbb{R}_+, \quad \text{supp} \tilde{G}(\omega) \subseteq \alpha^{-1} \mathbb{R}_+ - \alpha d, \quad \forall \alpha \in \mathbb{R}.$$ 

Then $\text{supp} \tilde{G} \subseteq [-\alpha d, \infty]$, for $\alpha > 0$ and $\text{supp} \tilde{G} \subseteq [\infty, -\alpha d]$ for $\alpha < 0$, which implies $\text{supp} \tilde{G} = \{0\}$. Since, by positivity of $\Omega$, $G(t)$ is bounded, one has $G(\omega) = \delta(\omega)$ and

$$\Omega(W_l(0, t k)) = 1,$$ 

so that $\text{div} E$ is a regular variable and all its correlation functions vanish.

The above Propositions imply that the representations of the temporal gauge in the free case with positive energy are the following.

### 3.1 Positive gauge invariant representation

**Proposition 3.3** Invariance under free time evolution and positivity of the energy uniquely determine the positive states $\Omega$ on the Weyl field algebra to
be of the following form

\[ \Omega(W(f,g)) = 0, \quad \text{if} \quad \text{div} f \neq 0, \quad (3.8) \]

\[ \Omega(W(f, g + \partial k)) = \Omega(W(f, g)), \quad (3.9) \]

\[ \Omega(W(f,g)) = e^{-w(f,g)}, \quad \text{if} \quad \text{div} f = 0, \quad (3.10) \]

where \((\partial k)_i = \partial_i k, k \in \mathcal{S}_{\text{real}}(\mathbb{R}^3)\) and \(w(f,g)\) is the standard transverse two-point function \(<(A(f) + E(g))(A(f) + E(g))>\), \(\text{div} f = 0\).

**Proof.** In fact, by eqs.(3.7), (2.1), one has

\[ 0 = \Omega([W(f,g), \text{div} E(k)]) = - \int d^3 x \, k \, \text{div} f \, \Omega(W(f,g)), \]

which implies eq.(3.10). Equation (2.4) and the invariance under time translation implies eq.(3.9). The last equation follows from Proposition 3.2 which fixes the representation of the observable algebra to be the standard Fock one.

It is not difficult to see that \(\Omega\) is pure \(\mathfrak{S}\) and coincides with the state considered in Ref. [6]. Thus, as anticipated, we have a non regular representation and the ground state correlation functions of the vector potential do not exist. Non regularity also follows from the requirement of Gauss’ law constraint by the results of Refs. [7, 8]. However, the selection of the above representation of \(\mathcal{W}\), eqs.(3.8 - 3.10), crucially depends on the condition of positive energy; in fact, one may find other (non regular) time translationally invariant pure states which define disjoint representations in which the energy spectral condition is violated.

**Proposition 3.4** In the free case, the \(\theta\) automorphisms are not unitarily implementable in the GNS representation \(\pi_\Omega\) given by the state \(\Omega\) defined above,
The states $\beta^\theta \ast \Omega$ are space time translationally invariant and define disjoint non regular representations of the Weyl field algebra, in which the energy spectral condition is violated.

**Proof.** In fact, by using eqs. (3.8)-(3.10) one has

$$\Omega(e^{iE_j(f_R)/R^3}) = e^{-w(0,f_R/R^3)} \xrightarrow{R \to \infty} 1,$$

which implies

$$s - \lim_{R \to \infty} e^{iE_j(f_R)/R^3} \Psi_\Omega = \Psi_\Omega.$$  

By the CCR, the same equation holds for any $\Psi$ of the form $A \Psi_\Omega, A \in \mathcal{W}$, i.e. on a dense set, and therefore on any vector of the representation (since $W(0,R^{-3}f_R)$ is a unitary operator). On the other hand, by eq.(2.10)

$$\lim_{R \to \infty} \beta^\theta \ast \Omega(e^{iE_j(f_R)/R^3}) = e^{i\theta_j}.$$  

Thus, the states $\beta^\theta \ast \Omega$ define disjoint representations.

Space translation invariance follows from $\beta^\theta \alpha_x = \alpha_x \beta^\theta$ and time translation invariance follows from eq.(2.9)

$$(\beta^\theta \ast \Omega)(\alpha_t(W(f,g))) = \Omega(\beta^\theta \alpha_t(W(f,g))) = \Omega(\alpha_t \gamma^{t \theta x} \beta^\theta(W(f,g))) =$$

$$= \Omega(\beta^\theta(W(f,g)) = (\beta^\theta \ast \Omega)((W(f,g))).$$

The energy spectral condition is violated as a consequence of Proposition 3.2.

A characteristic property of the $\theta$-vacua is that they yield a non vanishing expectation of the electric field, which is the time derivative of the vector potential. This is not incompatible with time translation invariance, because
\[ \beta^a \Omega(E(f)) = 0, \text{ if } \text{div} f = 0 \text{ and, if } \text{div} f \neq 0, A(f) \text{ is non regularly represented, namely its expectations do not exist, only those of its exponentials do.} \]

Since \( \beta^a \) commutes with \( \alpha_t \) on the observable fields, the energy spectral condition holds for the correlation functions of observables and in fact each observable sector \( \mathcal{H}_\theta \) has a unique translationally invariant state, which is the lowest energy state.
3.2 Indefinite regular representations

The perturbative expansion as well as the standard functional integral computations rely on the use of the field variables and therefore implicitly make use of a representation of the field algebra (otherwise the propagator of the vector potential would not exist). However, even in the free case there is a rich literature on the possible form of the propagator of the gauge field $A_i$ and no general agreement on the conclusion (for a review of the contributions and a detailed bibliography see Ref.[5]).

At the roots of the problem debated in the literature is the identification of gauge invariance with the vacuum being annihilated by the Gauss operator $G = \text{div}E - j_0$ and the conflict of this condition with canonical quantization. The solutions proposed, often in conflict with basic features of standard quantum field theory, do not seem to realize that the vanishing of the Gauss operator on the vacuum is only compatible with a non regular representation, precluding the existence of the propagator of $A_i$. As a consequence, see Proposition 3.1, a representation of the field algebra requires to abandon positivity, to admit that not all vectors obtained by applying the fields to the vacuum have a physical interpretation and to require the Gauss operator constraint only in expectations on the physical states (a feature common to other non positive gauges like the Feynman-Gupta-Bleuler gauge).

Motivated by the lack in the literature of a satisfactory characterization of the two point function of the gauge potential (even in the free case), we shall analyze it under the general condition of space time translational invariance. In our opinion it is difficult to live without such a condition, as required by a momentum space analysis of the correlation functions or of
the Feynman diagrams, according to the general wisdom of quantum field theory (e.g. the positive energy spectral condition needed for the analytic continuation to imaginary times and the functional integral representation of the so obtained Schwinger functions).

In the following, we shall characterize the two point function in the temporal gauge with interaction, under the assumption of locality discussed in Sect.2, in terms of a Källen-Lehmann representation under the additional condition of rotational and parity invariance. The result shows that i) positivity of the energy spectrum is satisfied by the two point function, but not the relativistic spectral condition, ii) the vacuum is a non positive functional on the field algebra, iii) the Gauss’ law constraint does not hold as an operator equation on the physical states and can only be required to hold in expectations of such states.

**Proposition 3.5** Let $\Omega$ be a state on the local field algebra $\mathcal{F}$ invariant under space time translations, rotations and parity, whose restriction to the observable field algebra satisfies the standard Wightman axioms for vacuum expectation values, then the two point function of the gauge potential has the following representation, ($y \equiv x' - x$),

$$< A_i(x) A_j(x') > \equiv \Omega(A_i(x) A_j(x')) =$$

$$= \int d^4k e^{iky} \int d\rho(m^2) \left( \delta_{ij} - \frac{k_i k_j}{k^2 + m^2} \right) \delta(k^2 + m^2) \theta(k_0) +$$

$$+ \frac{1}{2} i y_0 \left[ \partial_i \partial_j \mathcal{P}(\Delta) \delta(y) + \int d^3k e^{-iky} \int d\rho(m^2) k_i k_j (k^2 + m^2)^{-1} \right] + \partial_i \partial_j a(x^2),$$

(3.11)

**where** $\mathcal{P}$ **is a polynomial and** $d\rho$ **is the spectral measure of the two point function.**
function of the electromagnetic field

\[ < F_{\mu\nu} F^{\rho\sigma} > (y) = \left( g_{\rho\sigma} \partial_\mu \partial_\nu + g_{\mu\nu} \partial_\rho \partial_\sigma - g_{\rho\nu} \partial_\mu \partial_\sigma - g_{\mu\sigma} \partial_\nu \partial_\rho \right) F(y); \]

\[ \tilde{F}(k) = \int d\rho (m^2) \delta(k^2 + m^2) \theta(k_0). \quad (3.12) \]

The condition of a canonical structure at equal times, apart from renormalization constants, requires

\[ \mathcal{P} = \text{constant} \equiv Z. \]

The arbitrary function \( a(x) \) can be removed by a time independent operator gauge transformation.

Such a two point function satisfies the positive energy but not the relativistic spectral condition.

In particular, in the free field case, we have

\[ < A_i A_j > (y) = (\delta_{ij} - \partial_i \partial_j (\Delta)^{-1}) D^+(y) + \frac{i}{2} y_0 \partial_i \partial_j (\Delta)^{-1} \delta(y). \quad (3.13) \]

**Proof.** Invariance under space time translation, rotations and parity implies that the two point function can be written in the form

\[ < A_i A_j > (x) = \delta_{ij} H(x) + \partial_i \partial_j L(x), \quad (3.14) \]

with \( H, L \) rotationally invariant distributions; such a decomposition is unique up to a redefinition \( H \to H + h(x_0), \ L \to L - \frac{i}{2} h(x_0) x^2, \) \( L \) being defined up to constants. A comparison between the two point function of the electric field given by eq.(3.12) and that derived from eq.(3.14) (using \( E_i = \partial_0 A_i \)) yields

\[ \delta_{ij} \partial_0^2 (H - F) + \partial_i \partial_j (F + \partial_0^2 L) = 0. \]
Such an equation implies

\[ H = F + h(t), \quad \partial_i \partial_j \partial_0^2 L = -\partial_i \partial_j F - \delta_{ij} h(t) \]

and one can use the arbitrariness in the definition of \( H, L \) to remove \( h(t) \). Hence one can write

\[ \partial_i \partial_j L = -\left( \partial_i \partial_j / \partial_0^2 \right) F + a_{ij}(x) + i t b_{ij}(x), \]

since the operator \( \partial_i \partial_j / \partial_0^2 \) is well defined in momentum space, where it corresponds to multiplication of the spectral measure by the bounded function \( k_i k_j (k^2 + m^2)^{-1} \); furthermore, by taking the curl one gets \( a_{ij}(x) = \partial_i \partial_j a(x^2), \ b_{ij}(x) = \partial_i \partial_j b(x^2) \).

Locality of the commutator \( < [A_i(x), \partial_0 A_j(y)] > \) requires

\[ 2\tilde{b}(k) = \int d\rho (m^2)(k^2 + m^2)^{-1} + \mathcal{P}(k^2) \]

and a canonical structure at equal times requires \( \mathcal{P}(k^2) = Z \). The residual gauge invariance of the equations of motion and of the CCR’s under time independent operator gauge transformations

\[ A_i(x) \rightarrow A_i(x) + \partial_i \varphi(x), \ \psi(x) \rightarrow e^{i\varphi(x)} \psi(x) \]

allows to eliminate the function \( a(x^2) \).

The Fourier transform of the term linear in time has support on the plane \( \omega = 0, \ k \) arbitrary, so that the positivity of the energy spectrum is satisfied, but not the relativistic spectral condition.

In the free field case both \( \text{div} A \Psi_0 \) and \( \text{div} E \Psi_0 \) are vectors of zero indefinite product with themselves, briefly of zero norm or null vectors, which however cannot vanish.
As one should a priori expect, whenever a state yields a non-trivial representation of a gauge dependent field algebra \([11]\), the above indefinite states on the field algebra are not gauge invariant. In fact, one has \(\Omega(\gamma^A(A_i)) \neq \Omega(A_i) = 0\).

**Proposition 3.6** In the free case, the states \(\beta^{\theta*}\Omega\), with \(\Omega\) any quasi free (indefinite) state defined by eq.(3.13) are space translationally invariant on the field algebra, but not time translationally invariant. Only their restrictions to the gauge invariant field algebra are time translationally invariant.

**Proof.** In fact, \(\forall f \in \mathcal{S}(\mathbb{R}^3)\), one has
\[
\beta^{\theta*}\Omega(\alpha_t(A(f))) = \Omega(A(f)) + t \int d^3x \theta_i f_i(x)
\]
and, if \(\text{div} f = 0\), \(\int dx \theta_i f_i(x) = -\int dx \theta \cdot x \text{div} f(x) = 0\).

In conclusion, the space and time translationally invariant \(\theta\)-states on the observable field algebra do not have regular time translationally invariant extensions to the field algebra (the time invariant extension are non regular); in this sense they display a mechanism which is crucial for solving the problem arising in the Ward identities of chiral symmetry breaking in quantum chromodynamics \([10, 11]\).

Since the new structures emerging with respect to the standard case are connected with the longitudinal algebra, it is worthwhile to have a better mathematical control on the properties of its GNS representation given by the state \(\Omega\) of Proposition 3.5, at least in the free case. As mentioned in Section 2, eqs.(2.5), the longitudinal algebra can be discussed in terms of the field variables \(\text{div} A_l(f_n), \text{div} \dot{A}_l(f_n), f_n \in \mathcal{S}(\mathbb{R}^3)\). The problem is then reduced to the unique ground state (indefinite) representation of the Heisenberg algebra.
associated to a countable number of free particles. Such GNS representation has been analyzed in \cite{[12]} and the result is

**Proposition 3.7** In the free case the quasi free (indefinite) state $\Omega$ defined by eq. (3.13) is faithful on the longitudinal algebra $\mathcal{A}_l$ generated by $\text{div}A, \text{div}E$ and the commutant of $\mathcal{A}_l$ in the corresponding GNS representation is isomorphic to $\mathcal{A}_l$.

The GNS representation is given as an infinite tensor product of Fock and anti-Fock representations \cite{[13], [14]} of the canonical variables

$$Q_{n,\pm} \equiv (q_n \pm p'_n)/\sqrt{2}, \quad P_{n,\pm} \equiv (\pm p_n + q'_n)/\sqrt{2},$$

with

$$q'_n \equiv iS q_n S, \quad p'_n \equiv -iS p_n S, \quad \forall n,$$

and $S$ the antiunitary KMS operator defined by

$$S A \Psi_0 = A^* \Psi_0, \quad \forall A \in \mathcal{A}_l.$$

**Proof.** The proof is the same as for a single free particle. \cite{[12]}
We start by discussing the functional integral representation of the temporal gauge in the indefinite case with free time evolution.

By analytic continuation to imaginary time the two point correlation function, eq.(3.13), gives rise to the following Schwinger function

\[
S_{ij}(x - y) = (\delta_{ij} - \Delta^{-1} \partial_i \partial_j) S(x - y) - \partial_i \partial_j \Delta^{-1} \delta(x - y) |x_0 - y_0|/2,
\]

where \( S \) is the standard Schwinger function of a scalar field. The Schwinger function eq.(4.1) defines an inner product in \( S^3_{\text{real}}(\mathbb{R}^4) \)

\[
<f, f> = <f, f>_{tr} + <f, f>_{l},
\]

\[
<f, f>_{tr} = \int d^4x d^4y f_i(x) (\delta_{ij} - \Delta^{-1} \partial_i \partial_j) f_j(y) S(x - y), \quad (4.2)
\]

\[
<f, f>_{l} = \int d^4x d^4y \partial f(x) \partial f(y) \Delta^{-1} \delta(x - y) |x_0 - y_0|/2. \quad (4.3)
\]

The transverse inner product \(<., .>_{tr}\) is semidefinite and therefore it defines a Gaussian integral with measure \( d\mu(A_{tr}(x, \tau)) \) and an euclidean Gaussian field \( A_{tr}(x, \tau) \).

The (longitudinal) inner product \(<., .>_{l}\) is indefinite but not degenerate on \( S_{l}(\mathbb{R}^4) \equiv \{ h = \partial_i g_i, g_i \in S_{\text{real}}(\mathbb{R}^4) \} \). Therefore, the longitudinal inner product defines the two point Schwinger function of a Gaussian vector field \( \partial_i \phi(x, \tau) \) with

\[
<f, f>(\phi(x, \tau), \phi(y, \sigma)) = \Delta^{-1} \delta(x - y) |\tau - \sigma|/2.
\]

Thus, \( \forall f \in S(\mathbb{R}^3), \phi(f, \tau) \) is the analog of the variable \( q(\tau) \) describing the position of a free particle and eq.(4.4) corresponds to the ground state euclidean representation of the Heisenberg algebra with free evolution \[12\].
Following the results of Ref. [12], a functional integral representation is obtained by representing $A_i(x, \tau)$ by the random field

$$\tilde{A}_i(x, \tau) = A_{tr}^i(x, \tau) + \partial_i[\xi(x, \tau) + z(x) - \bar{z}(x)|\tau|],$$

(4.5)

where $z(x)$ is a complex Gaussian field with the following expectations

$$<z(x)z(y)> = 0, \quad <z(x)\bar{z}(y)> = -\frac{i}{2}\Delta^{-1}\delta(x - y),$$

corresponding to $z = z_1 + iz_2$, $z_1, z_2$ independent real Gaussian fields with

$$<z_1^2> = <z_2^2> = -\Delta^{-1}\delta(x - y)/4,$$

and $\xi(x, \tau)$ is a real Gaussian field with

$$<\xi(x, \tau)\xi(y, \sigma)> = -\frac{i}{2}\Delta^{-1}\delta(x - y) (-|\tau - \sigma| + |\tau| + |\sigma|).$$

Clearly, the covariance of $\xi$ is a positive kernel, being the product of the positive kernel $-\Delta^{-1}\delta$ and of the Wiener kernel. Hence, one has

$$<A_{i_1}(x_1, \tau_1) \ldots A_{i_n}(x_n, \tau_n)> = \int d\mu(A_{tr}(x, \tau)) dw(\xi(x, \tau)) d\nu(z(x))$$

$$\prod_{k=1}^{n} (A_{tr}^i(x_k, \tau_k) + \partial_i(\xi(x_k, \tau_k) + z(x_k) - \bar{z}(x_k)|\tau_k|)),$$

(4.6)

where $d\mu, d\nu, dw$ are the functional measures defined by the processes introduced above.

In the positive (non regular) formulation of Section 3.1 the construction of a functional integral representation for the euclidean correlation functions essentially reduces to the case of the euclidean correlation functions given by the (non regular) positive ground state of a non relativistic particle, discussed
in Ref. [12]. In fact, the euclidean correlation functions of exponentials of fields

\[ \Omega(e^{iA(f_1, \tau_1)} \cdots e^{iA(f_n, \tau_n)}) \]

obtained from eqs (3.8) - (3.10) have the same form as in Ref. [12], eq.(C.2), with \( \alpha_k \) replaced by \( \partial_i \phi(f_k) \) and vanish unless

\[ \partial_i f^i_1(x) + \cdots + \partial_i f^i_n(x) = 0. \]  

(4.7)

If this condition is satisfied, by Proposition 3.6 and eqs.(4.2-3) they coincide with the correlation functions of the indefinite case. Moreover, eq.(4.7) implies that in the exponential the variable \( z \) is smeared with a vanishing test function and the two point function of \( \bar{z} \) vanishes. Therefore, as in Ref. [12], the above correlation functions coincide with those of the exponentials of gaussian fields

\[ A^{tr}_i(f^i, \tau) + \xi(-\partial_i f^i, \tau) \]  

(4.8)

with the measures \( d\mu, dw \) introduced above in eq.(4.6).

As for a free particle, the above correlation functions are therefore given by the ergodic mean over the real variables \( \Xi(g), \Xi \in \mathcal{S}_{real}(\mathbb{R}^3) \) of the correlation functions of exponentials

\[ \exp i A^{tr}_i(f^i, \tau) \exp -i (\xi(\partial_i f^i, \tau) + \Xi(\partial_i f^i)) \]  

(4.9)

and therefore, by the Riesz-Markov theorem, they can be represented as integrals over the spectrum \( \Sigma \) of the \( C^* \)-algebra generated by \( \exp i\Xi(g), g \in \mathcal{S}_{real}(\mathbb{R}^3) \). \( \Sigma \) is the generalization of the spectrum of the Bohr algebra [15], generated by \( \exp i\alpha x, x \in \mathbb{R} \), with \( \Xi \) corresponding to \( x \) and \( g \) to \( \alpha \).
In conclusion,

\[ \Omega(e^{iA(f_1,\tau_1)} \ldots e^{iA(f_n,\tau_n)}) = \int d\mu(A^{tr}(x,\tau)) \, dw(\xi(x,\tau)) \]

\[ \int d\nu_\Sigma(\Xi(g)) \prod_{s=1}^n e^{iA^{tr}(f_s^i,\tau)} e^{-i\xi(\partial_k f_s^i,\tau)} e^{-i\Xi(\partial_k f_s^i)} \]

with \( d\nu_\Sigma \) the measure on \( \Sigma \) representing the ergodic mean in all the variables \( \Xi(g) \) \[ \[12 \]; the integral vanishes if eq.(4.7) does not hold and otherwise coincides with the expectation of a product of exponentials of fields of the form (4.8).
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