Curvature positivity of invariant direct images of Hermitian vector bundles

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Abstract
We prove that the invariant part, with respect to a compact group action satisfying certain condition, of the direct image of a Nakano positive Hermitian holomorphic vector bundle over a bounded pseudoconvex domain is Nakano positive. We also consider the action of the noncompact group $\mathbb{R}^m$ and get the same result for a family of tube domains, which leads to a new method to the matrix-valued Prekopa’s theorem originally proved by Raufi.

Keywords Group action · Hermitian vector bundle · Nakano positivity · Direct image · Minimum principle of plurisubharmonic functions

Mathematics Subject Classification 32U05 · 32T99 · 32M05

1 Introduction

Let $p : \mathbb{C}^n \times \mathbb{C}^m \to \mathbb{C}^n$ be the natural projection. For a domain $\Omega \subset \mathbb{C}^n \times \mathbb{C}^m$, we denote the fiber $p^{-1}(t) \cap \Omega$ of $\Omega$ over $t$ by $\Omega_t$ for $t \in p(\Omega) \subset \mathbb{C}^n$. In 1998, Berndtsson proved the following remarkable result which stimulates a series of important works on positivity of direct image sheaves of positively curved Hermitian holomorphic vector bundles.

Theorem 1.1 ([1]) Let $\varphi(t, z)$ be a plurisubharmonic (p.s.h for short) function on a pseudoconvex domain $\Omega \subset \mathbb{C}^n \times \mathbb{C}^m$.
(1) If all fibers $\Omega_t (t \in p(\Omega))$ are (connected) Reinhardt domains, and $\varphi(t, z)$ is independent of $\text{Arg}(z_j)$, $j = 1, \cdots, m$, then the function $\tilde{\varphi}$ defined by

$$e^{-\tilde{\varphi}(t)} = \int_{\Omega_t} e^{-\varphi(t, z)} dV_z$$

is a p.s.h function on $p(\Omega)$, where $dV_z$ is the Lebesgue measure on $\mathbb{C}^m$. Moreover, if all $\Omega_t$ contain the origin, we only assume $\Omega_t$ and $\varphi$ are invariant under the transform $z \mapsto e^{i\theta} z$, $\forall \theta \in \mathbb{R}$, then the same result still holds.

(2) If all fibers $\Omega_t$ are tube domains,

$$\Omega_t = X_t + i\mathbb{R}^m$$

and $\varphi$ is independent of $\text{Im}(z_j)$, $j = 1, \cdots, m$, then the function $\tilde{\varphi}$ defined by

$$e^{-\tilde{\varphi}(t)} = \int_{X_t} e^{-\varphi(t, \text{Re}(z))} dV_{\text{Re}(z)}$$

is a p.s.h function on $p(\Omega)$.

The above result is motivated by and generalizes Kiselman’s minimum principle for plurisubharmonic functions [12] and Prekopa’s theorem for convex functions [15]. Kiselman’s minimum principle states that, under the condition of (2) in Theorem 1.1, the function

$$\varphi^*(t) = \inf_{z \in \Omega_t} \varphi(t, z)$$

is a plurisubharmonic function on $p(\Omega)$, and Prekopa’s theorem states that for a convex function $\phi(x, y)$ on $\mathbb{R}^n_x \times \mathbb{R}^m_y$, the function $\tilde{\phi}(x)$ defined by

$$e^{-\tilde{\phi}(x)} = \int_{\mathbb{R}^n} e^{-\phi(x, y)} dV_y$$

is a convex function on $\mathbb{R}^n_x$.

In [2, 3, 9, 10], Theorem 1.1 was generalized along different directions.

Motivated by Raufi’s work on matrix-valued Prekopa’s theorem [16] and the recent work of the first author and collaborators on the characterization of Nakano positivity of Hermitian holomorphic vector bundles [7, 8], we generalize (1) in Theorem 1.1 to the following:

**Theorem 1.2** Let $\Omega \subseteq \mathbb{C}^n_t \times \mathbb{C}^m_z$ be a pseudoconvex domain, such that $\Omega_t$ are connected for all $t$ in $D := p(\Omega)$. Let $E = \Omega \times \mathbb{C}^r$ be the trivial holomorphic vector bundle of rank $r$ over $\Omega$, and let $\bar{h}(t, z)$ be an Hermitian metric on $\bar{E}$ which is viewed as a smooth map from $\Omega$ to the space of positive Hermitian matrices. Let $E = D \times \mathbb{C}^r$ be the trivial bundle over $D$ with the Hermitian metric given by

$$h(t) = \int_{p^{-1}(t)} \bar{h}(t, z) dV_z.$$
Assume that there is a compact Lie group $K$ acting holomorphically on $\Omega$ by acting on the second variable $z$, such that

(i) $\tilde{h}(t, z)dV_z$ is $K$-invariant for $t \in D$, and
(ii) all $K$-invariant holomorphic functions on $\Omega$, are constant, $t \in D$.

If $(\tilde{E}, \tilde{h})$ is Nakano semi-positive and $h$ is $C^2$, then $(E, h)$ is Nakano semi-positive.

**Remark 1.1** In Theorem 1.2, if $K$ is not compact as assumption, then $h$ would be identically equal to $+\infty$.

Our method to Theorem 1.2 is different from those in [1] and [3]. The idea of the proof is as follows. Since $(\tilde{E}, \tilde{h})$ is Nakano semi-positive, it satisfies the optimal $L^2$-estimate condition (see § 2 for definition) by Hörmander’s $L^2$-estimate of $\bar{\partial}$, which can imply that $(E, h)$ also satisfies the optimal $L^2$-estimate condition. By [8, Theorem 1.1], we see that $(E, h)$ is Nakano semi-positive.

With the same method, Theorem 1.2 can be generalized to general Kähler fibrations and Nakano semi-positive Hermitian vector bundles. But we just restrict ourselves to the basic context as above. If the fibers $\Omega_t$ in Theorem 1.2 are not assumed to be connected, we can also get a similar result by considering the GIT quotient $\Omega//K$, as in [9]. Under the assumption as in Theorem 1.2, we can canonically identify $\Omega//K$ with $D$.

Theorem 1.2 considers actions of compact groups. However, following the idea in [9, 10], it can be generalized to certain noncompact group actions. In the present context, we consider tube domains as in Theorem 1.1 which corresponds to the action of the group $\mathbb{R}^m$.

**Theorem 1.3** Let $\Omega \subseteq \mathbb{C}^n_t \times \mathbb{C}^m_z$ be a pseudoconvex domain, such that $\Omega_t = U_t \times i\mathbb{R}^m$ are (connected) tube domains for all $t \in D := p(\Omega)$. Let $\tilde{E} = \Omega \times \mathbb{C}^r \rightarrow \Omega$ be the trivial holomorphic vector bundle of rank $r$ on $\Omega$. Let $\tilde{h}(t, z)$ be an Hermitian metric on $\tilde{E}$, which is independent of the imaginary part $\text{Im} z$ of $z$. Let $h(t)$ be the Hermitian metric on the trivial vector bundle $E = D \times \mathbb{C}^r \rightarrow D$ over $D$, given by

$$h(t) := \int_{U_t} \tilde{h}(t, \text{Re} z)dV_{\text{Re} z}.$$ 

If $(\tilde{E}, \tilde{h})$ is Nakano semi-positive and $h$ is $C^2$, then $(E, h)$ is Nakano semi-positive.

A Corollary of Theorem 1.3 is the following

**Theorem 1.4** Let $\Omega_0 \subseteq \mathbb{R}^n_t \times \mathbb{R}^m_x$ be a convex domain, let $p_0 : \Omega_0 \rightarrow \mathbb{R}^n_t$ be the natural projection, and let $\Omega_{0,t} := p_0^{-1}(t)$ for $t \in D_0 := p_0(\Omega_0)$. Let $\tilde{g}(t, x) : \Omega_0 \rightarrow GL(r, \mathbb{C})$ be an Hermitian metric on the trivial vector bundle $\tilde{E}_0 = \Omega_0 \times \mathbb{C}^r \rightarrow \Omega_0$. Let $g(t) : D_0 \rightarrow GL(r, \mathbb{C})$ be the Hermitian metric on the trivial vector bundle $E_0 = D_0 \times \mathbb{C}^r \rightarrow D_0$ over $D_0$ given by

$$g(t) := \int_{\Omega_{0,t}} \tilde{g}(t, x)dV_x.$$ 

If $(\tilde{E}_0, \tilde{g})$ is Nakano semi-positive and $g$ is $C^2$, then $(E_0, g)$ is Nakano semi-positive.
The concept of Nakano positivity of \((\tilde{E}_0, \tilde{g}_0)\) was introduced in [16](phrased as Nakano log concave there) and will be recalled in § 2.

In the case that \(\Omega = \mathbb{R}^n \times \mathbb{R}^m\), Theorem 1.4 is proved and called the matrix-valued Prekopa’s theorem by Raufi in [16]. The proof in [16] contains two main ingredients—Berndtsson’s method to the positivity of direct image bundles [3] and a Fourier transform technique. To avoid the Fourier transform technique and complex analysis in the proof, Cordero-Erausquin recently produced a new proof of the matrix-valued Prekopa’s theorem based on \(L^2\)-methods in real analysis [5].

Our method to Theorem 1.4 is different from those in [16] and [5]. One of the main novelties of our method is to reduce the noncompact group \(\mathbb{R}^n\) action to the action of the compact group of torus of dimension \(n\), by considering the covering map \(\pi : \mathbb{C}^m \to (\mathbb{C}^*)^n\) given by the exponential map. (This idea is motivated by the work in [9].) In this way, we can remarkably simplify the argument.

In Theorem 1.1, the original p.s.h function \(\varphi\) and the derived function \(\tilde{\varphi}\) are not required to be smooth. But by a suitable approximation process, the general case can be reduced to the case that \(\Omega\) is bounded and has smooth boundary, and \(\varphi\) is smooth up to the boundary of \(\Omega\). It follows that Theorem 1.1 is essentially a special case of Theorem 1.2.

In the setting of vector bundles, we can also consider singular Hermitian metrics with Nakano positive curvature in certain sense. But the problem is that there is no standard definition of Nakano positivity for singular metrics on vector bundles (see, e.g., [17]). Here, we introduce two different definitions as follows.

In [14], Lempert gives a definition as follows (see Definition 2.2 for details). A singular Hermitian metric \(h\) on a holomorphic vector bundle \(E\) is called Nakano semi-positive if \(h\) is the limit of a sequence of increasing smooth Hermitian metrics on \(E\) whose curvatures are semi-positive in the sense of Nakano. In the case that \(h\) itself is smooth, Lempert rises the question that whether the curvature of \(h\) is Nakano semi-positive in the ordinary sense, if it is Nakano semi-positive in the above sense. Applying the characterization of Nakano positivity for smooth Hermitian metrics in [8], it is shown in [13] that the answer of the above question is positive. It follows that Lempert’s definition of Nakano positivity coincides with the ordinary definition for smooth Hermitian metrics. Under Lempert’s definition, it is clear that a metric with local form \(e^{-\varphi}\) on a line bundle is Nakano semi-positive if and only if \(\varphi\) is p.s.h, as desired naturally.

As mentioned as above, it is shown in [8] that a smooth Hermitian metric on a holomorphic vector bundle has semi-positive curvature in the sense of Nakano if and only if it satisfies the optimal \(L^2\)-estimate condition, which is an integration condition introduced in [7] and [8]. Note that the optimal \(L^2\)-estimate condition also makes sense for singular Hermitian metrics on holomorphic vector bundles, so it is nature to consider this condition as the definition of Nakano semi-positivity for singular Hermitian metrics on holomorphic vector bundles. However, the difficult point in this expectation lies in the fact that, for the basic and testing case of singular metrics on a line bundle, we do not know whether the optimal \(L^2\)-estimate condition is equivalent to the plurisubharmonicity of the weight of the metric. In [11], Inayama bypasses this difficulty by requiring Griffiths positivity as a part in his definition. In other words, a singular Hermitian metric \(h\) on a holomorphic vector bundle \(E\) is defined to be Nakano semi-positive by Inayama if it is Griffiths semi-positive and satisfies the optimal \(L^2\)-estimate condition. Here, according to Berndtsson-Paun [4], \(h\) is called Griffiths semi-positive if the logarithmic of the length function on the dual bundle \(E^*\) of \(E\) with respect to the dual metric \(h^*\) is plurisubharmonic.

In the present note, we generalize Theorem 1.2 to singular Hermitian metrics which are Nakano semi-positive in the sense of Lempert’s definition presented as above.
Theorem 1.5 Let $\Omega \subseteq \mathbb{C}^n \times \mathbb{C}^m$ be a pseudoconvex domain, such that $\Omega_t$ are connected for all $t \in D := p(\Omega)$. Let $\tilde{E} = \Omega \times \mathbb{C}^r$ be the trivial holomorphic vector bundle of rank $r$ over $\Omega$, and let $\tilde{h}(t, z)$ be a singular Hermitian metric on $\tilde{E}$. We assume that $\tilde{h}(t, z)$ is the limit of an increasing sequence of smooth Hermitian metrics $\tilde{h}_j$ on $E$ with Nakano semi-positive curvature. Let $E = D \times \mathbb{C}^r$ be the trivial bundle over $D$ with the singular Hermitian metric given by

$$h(t) = \int_{p^{-1}(t)} \tilde{h}(t, z) dV_z.$$ 

Assume that there is a compact Lie group $K$ acting holomorphically on $\Omega$ by acting on the second variable $z$, such that

(i) $\tilde{h}_j(t, z) dV_z$ is $K$-invariant for $t \in D$, and

(ii) all $K$-invariant holomorphic functions on $\Omega_t$ are constant, $t \in D$.

Then, $(E, h)$ is Nakano semi-positive in the sense of Lempert’s definition.

We do not know whether the tensor product of two vector bundles with Nakano semi-positive singular Hermitian metrics is also Nakano semi-positive, in the sense of Inayama’s definition. This is the obstruction to prove a result similar to Theorem 1.5 to such singular metrics.

Following the same way of deriving Theorem 1.3 from Theorem 1.2, we can deduce from Theorem 1.5 the following

Corollary 1.6 Let $\Omega \subseteq \mathbb{C}^n \times \mathbb{C}^m$ be a pseudoconvex domain, such that $\Omega_t = U_t \times i\mathbb{R}^m$ are (connected) tube domains for all $t \in D := p(\Omega)$. Let $\tilde{E} = \Omega \times \mathbb{C}^r \to \Omega$ be the trivial holomorphic vector bundle of rank $r$ on $\Omega$. Let $\tilde{h}(t, z)$ be a singular Hermitian metric on $\tilde{E}$, which is independent of the imaginary part $\text{Im} z$ of $z$. Let $h(t)$ be the singular Hermitian metric on the trivial vector bundle $E = D \times \mathbb{C}^r \to D$ over $D$, given by

$$h(t) := \int_{U_t} \tilde{h}(t, \text{Re } z) dV_{\text{Re } z}.$$ 

If $(\tilde{E}, \tilde{h})$ is Nakano semi-positive in the sense of Lempert’s definition, then $(E, h)$ is Nakano semi-positive in the sense of Lempert’s definition.

2 Preliminaries

We recall some notions and known results that will be used later.

Definition 2.1 ([8]) Let $\Omega \subseteq \mathbb{C}^n$ be a bounded domain. Let $E = \Omega \times \mathbb{C}^r$ be the trivial holomorphic vector bundle over $\Omega$ whose canonical frame is denoted by $\{e_1, \ldots, e_r\}$. Let $h$ be a Hermitian metric on $E$ such that $h(e_{\lambda}, e_{\mu}) = h_{\lambda\mu}$. We say that $(E, h)$ satisfies the optimal $L^2$-estimate condition, if for any smooth strictly plurisubharmonic function $\psi$ on $\Omega$, for any $f \in C^0_c(\Omega, \Lambda^{0,1} T^* \Omega \otimes E)$ with $\bar{\partial} f = 0$, there is a locally integrable section $u$ of $E$, satisfying $\bar{\partial} u = f$ in the sense of distribution, and
\[
\int_{\Omega} |u|_h^2 e^{-\psi} \, dV \leq \int_{\Omega} \sum_{i,j,\mu} \psi^{ij} f_{ij} \bar{f}_{i\mu} \psi_{j\mu} \, dV,
\]
where \( dV \) is the Lebesgue measure, \( (\psi^{ij}) = (\psi_{ij})^{-1} = (\frac{\partial \psi}{\partial \bar{z}_i \partial z_j})^{-1} \), and \( f = \sum_{\lambda=1}^r (\sum_{j=1}^n f_{ij} \, d\bar{z}_j) \otimes e_{\lambda}. \)

By the \( L^2 \)-estimate of \( \bar{\partial} \) by Hörmander and Demailly, we have

**Lemma 2.1** (c.f. [6, Theorem 4.5]) Let \( \Omega, E, h \) be as in Definition 2.1. If \( \Omega \) is pseudoconvex and \( (E, h) \) is Nakano semi-positive, then \( (E, h) \) satisfies the optimal \( L^2 \)-estimate condition.

For the definition of curvature and Nakano positivity for Hermitian holomorphic vector bundles, see [6]. Recently, the converse of Lemma 2.1 was established.

**Lemma 2.2** ([8, Theorem 1.1]) If \( (E, h) \) satisfies the optimal \( L^2 \)-estimate condition, then \( (E, h) \) is Nakano semi-positive.

**Definition 2.2** ([16, Definition 2]) Let \( g : \Omega \to GL(r, \mathbb{C}) \) be an Hermitian metric on the trivial complex vector bundle \( E = \Omega \times \mathbb{C}^r \) over an open set \( \Omega \subset \mathbb{R}^n. \) Let

\[
\Theta^g_{jk} = -\frac{\partial}{\partial x_k} \left( g^{-1} \frac{\partial g}{\partial x_j} \right), \quad 1 \leq j, k \leq n,
\]

where differentiation should be interpreted elementwise. We say that \( (E, g) \) is Nakano semi-positive if for any \( n \)-tuple of vectors \( \{u_j\}_{j=1}^n \subset \mathbb{C}^r \)

\[
\sum_{j,k=1}^n (\Theta^g_{jk} u_j, u_k)_g \geq 0.
\]

**Remark 2.1** Let \( U \subset \mathbb{R}^n \) be a connected open set and let \( \Omega = U + i\mathbb{R}^n \subset \mathbb{C}^n \) be the tube domain with base \( U. \) Let \( h(z) : \Omega \to GL(r, \mathbb{C}) \) be an Hermitian metric on the trivial holomorphic vector bundle \( E = \Omega \times \mathbb{C}^r. \) Assume that \( h(z) \) is independent of the imaginary part of \( z. \) By definition, one can see that \( (E, h) \) is Nakano semi-positive as an Hermitian holomorphic vector bundle if and only if \( (E|_U = U \times \mathbb{C}^r, h|_U) \) is Nakano semi-positive in the sense of Definition 2.2.

**Definition 2.3** ([14, Definition 4.2]) Let \( g : \Omega \to \mathcal{M}_{r \times r}, \) where \( \mathcal{M} \) is a matrix, and each component of \( \mathcal{M} \) is in \([0, \infty) \), \( g \) is called a singular Hermitian metric on the trivial complex vector bundle \( E = \Omega \times \mathbb{C}^r \) over an open set \( \Omega \subset \mathbb{C}^n(\mathbb{R}^n) \) if there is a sequence \( g_1 \leq g_2 \leq \cdots \) of Hermitian metrics of class \( C^2 \) on \( E \) such that \( g(e) = \lim_{j \to \infty} g_j(e) \) for all \( e \in E. \) We say that \( (E, g) \) is Nakano semi-positive in the sense of Lempert if \( (E, g_j) \) is Nakano semi-positive.

### 3 Positivity of invariant direct images with compact group actions

The aim of this section is to prove Theorem 1.2. For convenience, we restate it here.
Theorem 3.1 (=Theorem 1.2) Let $\Omega \subseteq \mathbb{C}^n \times \mathbb{C}^m$ be a pseudoconvex domain, such that $\Omega_t$ are connected for all $t$ in $D := p(\Omega)$. Let $\tilde{E} = \Omega \times \mathbb{C}^r$ be the trivial holomorphic vector bundle of rank $r$ over $\Omega$, and let $\tilde{h}(t, z)$ be an Hermitian metric on $\tilde{E}$ which is viewed as a smooth map from $\Omega$ to the space of positive Hermitian matrices. Let $E = D \times \mathbb{C}^r$ be the trivial bundle over $D$ with the Hermitian metric given by

$$h(t) = \int_{p^{-1}(t)} \tilde{h}(t, z)dV_z.$$ 

Assume that there is a compact Lie group $K$ acting holomorphically on $\Omega$ by acting on the second variable $z$, such that

(i) $\tilde{h}(t, z)dV_z$ is $K$-invariant for $t \in D$, and

(ii) all $K$-invariant holomorphic functions on $\Omega_t$ are constant, $t \in D$.

If $(\tilde{E}, \tilde{h})$ is Nakano semi-positive and $h$ is $C^2$, then $(E, h)$ is Nakano semi-positive.

Proof By Lemma 2.2, it suffices to prove that $(E, h)$ satisfies the optimal $L^2$ estimate condition.

Assume that $(e_1, ..., e_r)$ is the canonical holomorphic frame of $E$, and $(\tilde{e}_1, ..., \tilde{e}_r)$ is the canonical holomorphic frame of $\tilde{E}$. Let $\psi$ be a smooth strictly plurisubharmonic function on $D$. Setting $\tilde{\psi}(t, z) = \psi(t)$, $(t, z) \in \Omega$, we get a smooth plurisubharmonic function $\tilde{\psi}$ on $\Omega$.

Let $f \in C^\infty(D, \Lambda^{0,1} T^* D \otimes E)$ with compact support and $\bar{\partial}f = 0$. We can write $f$ as

$$f = \sum_{\lambda=1}^r w_\lambda \otimes e_\lambda = \sum_{\lambda=1}^r \left( \sum_{j=1}^n f_{j\lambda} \tilde{d}_j \right) \otimes e_\lambda,$$

where $f_{j\lambda}$ are smooth functions on $D$. Let $\tilde{w}_\lambda = p^*(w_\lambda)$, $\tilde{f}_{j\lambda} = p^*(f_{j\lambda}) := f_{j\lambda} \circ p$, and let

$$\tilde{f} = \sum_{\lambda=1}^r \tilde{w}_\lambda \otimes \tilde{e}_\lambda = \sum_{\lambda=1}^r \left( \sum_{j=1}^n \tilde{f}_{j\lambda} \tilde{d}_j \right) \otimes \tilde{e}_\lambda \in C^\infty(\Omega, \Lambda^{0,1} T^* \Omega \otimes \tilde{E}),$$

then $\bar{\partial}\tilde{f} = 0$.

Let $\Theta_{E, \tilde{h}}$ be the curvature operator of $(\tilde{E}, \tilde{h})$. Then, $\Theta + \bar{\partial}\tilde{\psi}$ is the curvature operator of $\tilde{E}$ with the Hermitian metric $e^{-\tilde{\psi}} \cdot \tilde{h}$. Since $(\tilde{E}, \tilde{h})$ is Nakano semi-positive, by Lemma 2.1, there exists $\tilde{u} \in L^2(\Omega, \tilde{E})$ such that

$$\bar{\partial}\tilde{u} = \tilde{f},$$

and

$$\int_{\Omega} |\tilde{u}|^2 e^{-\tilde{\psi}}dV_{(t,z)} \leq \int_{\Omega} \langle i[\Theta_{E, \tilde{h}} + i\bar{\partial}\tilde{\psi}, \Lambda]^{-1} \tilde{f}, \tilde{f} \rangle e^{-\tilde{\psi}}dV_{(t,z)} \leq \int_{\Omega} \langle i\bar{\partial}\tilde{\psi}, \Lambda \rangle^{-1} \tilde{f}, \tilde{f} \rangle e^{-\tilde{\psi}}dV_{(t,z)} \leq \int_{\Omega} \sum_{i,j=1}^n \sum_{\lambda,\mu=1}^r \tilde{\psi}_{ij} \tilde{f}_{j\lambda} \tilde{f}_{i\mu} \tilde{h}_{\lambda\mu} e^{-\tilde{\psi}}dV_{(t,z)},$$

(2)
where $\Lambda$ is the adjoint of the operator given by the wedge product of the flat Kähler form on $\Omega$, and $(\bar{\partial}^j)^{n \times n}$ is the inverse of the matrix $(\bar{\partial}^{ij})^{n \times n}$.

We assume that $\bar{u}$ is minimal, in the sense that the left hand side in the top line in (2) is minimal. We write $\bar{u}$ as

$$\bar{u} = \sum_{\lambda=1}^r \bar{u}_\lambda \bar{e}_\lambda,$$

where $\bar{u}_\lambda$ are functions on $\Omega$. From equation (1), we have $\bar{\partial} \bar{u}_\lambda = \bar{w}_\lambda$, which means that $\bar{\partial} \bar{u}_\lambda = 0$ and hence $\bar{u}_\lambda$ are holomorphic with $z_1, \cdots, z_m$.

For any $g \in K$, let

$$\bar{u}_g = \sum_{\lambda=1}^r \bar{u}_\lambda(t, gz) \bar{e}_\lambda,$$

then it is clear that $\bar{u}_g$ also satisfies the equation $\bar{\partial} \bar{u}_g = \bar{f}$. By assumption, we also have

$$\int_{\Omega} |\bar{u}_g|^2 e^{-\bar{\psi}} dV(t, z) = \int_{\Omega} |\bar{u}|^2 e^{-\psi} dV(t, z).$$

By the uniqueness of the minimal solution, we have $\bar{u}_g = \bar{u}$ and hence $\bar{u}_\lambda(t, gz) = \bar{u}_\lambda(t, z)$ for all $g \in K$. By assumption, $\bar{u}_\lambda(t, z)$ must be independent of $z$. So we can view $\bar{u}_\lambda(t, z)$ as a function on $D$, denoted by $u_\lambda(t)$.

Let $u = \sum_{\lambda=1}^r u_\lambda e_\lambda \in L^2(D, E)$, then it is clear that $\partial u = f$. By Fubini’s theorem, we get

$$\int_{\Omega} |\bar{u}|^2 e^{-\psi} dV(t, z) = \int_{\Omega} \bar{u}_\lambda(t, z) \bar{w}_\lambda(t, z) e^{-\psi} dV(t, z) = \int_D u_\lambda h_{\lambda} e^{-\psi} dV_t$$

$$= \int_D |u|^2 e^{-\psi} dV_t, \int_D \sum_{\lambda=1}^r \sum_{j=1}^n \bar{\psi}^{ij} f_{i,j} \bar{w}_\lambda e^{-\psi} dV(t, z)$$

$$= \int_D \sum_{i,j=1}^n \sum_{\lambda=1}^r \psi^{ij} f_{i,j} h_{\lambda} e^{-\psi} dV_t,$$

Combing through the above identities with estimate (2), we get

$$\int_D |u|^2 e^{-\psi} dV_t \leq \int_D \sum_{i,j=1}^n \sum_{\lambda=1}^r \psi^{ij} f_{i,j} h_{\lambda} e^{-\psi} dV_t,$$

which implies that $(E, h)$ satisfies the optimal $L^2$ condition. By Lemma 2.2, $(E, h)$ is Nakano semi-positive.

**Theorem 3.2 (Theorem 1.5)** Let $\Omega \subseteq \mathbb{C}^n \times \mathbb{C}^m$ be a pseudoconvex domain, such that $\Omega_t$ are connected for all $t$ in $D := p(\Omega)$. Let $\tilde{E} = \Omega \times \mathbb{C}$ be the trivial holomorphic vector bundle of rank $r$ over $\Omega$, and let $\tilde{h}(t, z)$ be a singular Hermitian metric on $\tilde{E}$. Let $E = D \times \mathbb{C}$ be the trivial bundle over $D$ with the singular Hermitian metric given by

$$h(t) = \int_{p^{-1}(t)} \tilde{h}(t, z) dV_c.$$
Assume that there is a compact Lie group $K$ acting holomorphically on $\Omega$ by acting on the second variable $z$, such that

(i) $\tilde{h}_j(t, z) dV_z$ is $K$-invariant for $t \in D$, and

(ii) all $K$-invariant holomorphic functions on $\Omega_t$ are constant, $t \in D$.

If $(\tilde{E}, \tilde{h})$ is Nakano semi-positive in the sense of Lempert’s definition, then $(E, h)$ is Nakano semi-positive in the sense of Lempert’s definition.

**Proof** From the hypothesis, there exists a sequence $\tilde{h}_1 \leq \tilde{h}_2 \leq \cdots$ of Hermitian metrics of class $C^2$ on $\tilde{E}$, such that

(i) $\tilde{h}_j(t, z) dV_z$ is $K$-invariant for $t \in D$,

(ii) $\tilde{h}(\tilde{e}) = \lim_{j} \tilde{h}_j(\tilde{e})$, for all $\tilde{e} \in \tilde{E}$, and

(iii) $(\tilde{E}, \tilde{h}_j)$ is Nakano semi-positive, $j = 1, 2$...

By approximation, we may assume $\Omega$ has smooth boundary and $\tilde{h}_j$ are smooth up to the boundary of $\Omega$. Let

$$h_j(t) = \int_{p^{-1}(t)} \tilde{h}_j(t, z) dV_z, \quad j \geq 1,$$

then $h_j$ are smooth. It is clear that $h_1 \leq h_2 \leq \cdots$ and $h(e) = \lim_{j} h_j(e)$, for all $e \in E$. By Theorem 3.1, $h$ is Nakano semi-positive in the sense of Lempert’s definition. $\Box$

### 4 The case of tube domains

We give the proof of Theorem 1.3 and Theorem 1.4.

**Theorem 4.1** (=Theorem 1.3) Let $\Omega \subseteq \mathbb{C}^n \times \mathbb{C}^m$ be a pseudoconvex domain, such that $\Omega_t = U_t \times i \mathbb{R}^m$ are (connected) tube domains for all $t \in D := p(\Omega)$. Let $\tilde{E} = \Omega \times \mathbb{C}^r \to \Omega$ be the trivial holomorphic vector bundle of rank $r$ on $\Omega$. Let $\tilde{h}(t, z)$ be an Hermitian metric on $\tilde{E}$, which is independent of the imaginary part $\text{Im} z$ of $z$. Let $h(t)$ be the Hermitian metric on the trivial vector bundle $E = D \times \mathbb{C}^r \to D$ over $D$, given by

$$h(t) := \int_{U_t} \tilde{h}(t, \text{Re} z) dV_{\text{Re} z}.$$  

If $(\tilde{E}, \tilde{h})$ is Nakano semi-positive and $h$ is smooth, then $(E, h)$ is Nakano semi-positive.

**Proof** Let us consider the following map:

$$f : \Omega \to \Omega^{*}_{(t, w)}$$

$$(t_1, \ldots, t_n, z_1, \ldots, z_m) \mapsto (t_1, \ldots, t_n, e^{z_1}, \ldots, e^{z_m}),$$  

where $\Omega^* = f(\Omega)$. Since $\tilde{h}(t, z)$ is independent of the imaginary part of $z$, it induces a metric $\tilde{h}'(t, w) : \Omega^* \to GL(r, \mathbb{C})$ on the trivial bundle...
which is given by
\[
\tilde{h}'(t, w) = \tilde{h}(t, \ln |w|).
\]

Then we have
\[
h(t) = \int_{U_t} \tilde{h}(t, x)dV_x
= \frac{1}{(2\pi)^m} \int_{\Omega^*_t} \frac{1}{|w_1|} \cdots \frac{1}{|w_m|} \tilde{h}'(t, w)dV_w
= \int_{\Omega^*_t} \tilde{h}''(t, w)dV_w,
\]
where \(\tilde{h}''(t, w) := \frac{1}{(2\pi)^m} \frac{1}{|w_1|} \cdots \frac{1}{|w_m|} \tilde{h}'(t, w)\) can be viewed as an Hermitian metric on \(\tilde{E}'\).

The curvature of \((\tilde{E}', \tilde{h}'')\) is given by
\[
\Theta_{\tilde{E}', \tilde{h}''} = \Theta_{\tilde{E}', \tilde{h}'} - \sum_{j=1}^m \partial \bar{\partial}(\ln |w_j|) - \partial \bar{\partial}\ln(2\pi)^m = \Theta_{\tilde{E}', \tilde{h}'}.
\]

So \((\tilde{E}', \tilde{h}'')\) is also Nakano semi-positive.

Considering the compact Lie group \(K := (S^1)^m\) which acts on \(p^{-1}(t)\) as
\[
(\alpha_1, \cdots, \alpha_m)(t, w_1, \cdots, w_m) = (t, \alpha_1w_1, \cdots, \alpha_mw_m),
\]
and applying Theorem 4.1, we see that \((E, h)\) is Nakano semi-positive. \(\square\)

**Theorem 4.2** (Theorem 1.4) Let \(\Omega_0 \subseteq \mathbb{R}_+^n \times \mathbb{R}_x^m\) be a convex domain, let \(p_0 : \Omega_0 \to \mathbb{R}_+^n\) be the natural projection, and let \(\Omega_{0,t} = p_0^{-1}(t)\) for \(t \in D_0 := p_0(\Omega_0)\). Let \(\tilde{g}(t, x) : \Omega_0 \to GL(r, \mathbb{C})\) be an Hermitian metric on the trivial vector bundle \(\tilde{E}_0 = \Omega_0 \times \mathbb{C}^r \to \Omega_0\). Let \(g(t) : D_0 \to GL(r, \mathbb{C})\) be the Hermitian metric on the trivial vector bundle \(E_0 = D_0 \times \mathbb{C}^r \to D_0\) over \(D_0\) given by
\[
g(t) := \int_{\Omega_{0,t}} \tilde{g}(t, x)dV_x.
\]

If \((\tilde{E}_0, \tilde{g})\) is Nakano semi-positive and \(g\) is \(C^2\), then \((E_0, g)\) is Nakano semi-positive.

**Proof** Let \(\Omega = \Omega_0 + i\mathbb{R}^{n+m} \subseteq \mathbb{C}^n \times \mathbb{C}^m\). We extend \(\tilde{g}\) to an Hermitian metric \(\tilde{h}\) on the trivial bundle \(\tilde{E} := \Omega \times \mathbb{C}^r\) such that \(\tilde{h}(t, z)\) is independent of the imaginary part of \(t, z\) and \(\tilde{h}|_{\Omega_0} = \tilde{g}\). Let \(p : \mathbb{C}^n \times \mathbb{C}^m \to \mathbb{C}^n\) be the natural projection and let \(D = p(\Omega) \subseteq \mathbb{C}^n\) and \(\Omega_x = p^{-1}(t) = U_t \times i\mathbb{R}^m\). We define an Hermitian metric on the trivial bundle \(E = D \times \mathbb{C}^r\) by setting
\[
h(t) = \int_{U_t} \tilde{h}(t, Rez)dV_{Rez}.
\]
By Theorem 4.1, \((E, h)\) is Nakano semi-positive. It is clear that \(h(t)\) is independent of the imaginary part of \(t\) and \(h|_{D_0} = g\), and thus, \((E_0, g)\) is Nakano semi-positive (see Remark 2.1).

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