A non self-referential expression of Tsallis’ probability distribution function

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Abstract. The canonical probability distribution function (pdf) obtained by optimizing the Tsallis entropy under the linear mean energy constraint (first formalism) or the escort mean energy constraint (third formalism) suffer self-referentiality. In a recent paper [Phys. Lett. A 335 (2005) 351-362] the authors have shown that the pdfs obtained in the two formalisms are equivalent to the pdf in non self-referential form. Based on this result we derive an alternative expression, which is non self-referential, for the Tsallis distributions in both first and third formalisms.

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1 Introduction

Since the pioneering work of Tsallis [1], a large number of papers have been published based on the framework of the so-called nonextensive thermostatistics [2,3,4]. Tsallis’ entropy is a generalization of the Boltzmann-Gibbs (BG) entropy defined by

\[ S_q \equiv \sum_i \left( p^q_i - p_i \right) \left( 1 - q \right), \]

where \( p_i \) is the probability of \( i \)-th state of the system and \( 0 \leq q \leq 2 \) is a real deformed parameter. Here, for the sake of simplicity, the Boltzmann constant is set to unity. In the \( q \to 1 \) limit, \( S_q \) reduces to the standard BG-entropy \( S = -\sum_i p_i \ln(p_i) \).

The current formulation of Tsallis thermostatistics has been established in Ref. [2]. In the first formalism the probability distribution function (pdf) is obtained by maximizing \( S_q \) with the conditions posed by the normalization

\[ \sum_i p_i = 1 , \]

and by the linear average energy

\[ \sum_i p_i E_i = U , \]

as follows

\[ \frac{\delta}{\delta p_i} \left( S_q - \tilde{\beta} \sum_j p_j E_j - \tilde{\gamma} \sum_j p_j \right) = 0 , \]

where \( \tilde{\gamma} \) and \( \tilde{\beta} \) are the Lagrange multipliers associated to the constraints (2) and (3), respectively.

The solution of this MaxEnt problem is

\[ p_i = \frac{1}{\bar{Z}_q^{(1)}} \cdot \exp_{2-q} \left( -\tilde{\beta} (E_i - U) \right) , \]

where \( \bar{Z}_q^{(1)} \) is the generalized partition function in the first formalism, and the following relation is satisfied.

\[ \bar{Z}_q^{(1)} = \left( \sum_i p_i^q \right)^{\frac{1}{1-q}} . \]

\( \bar{Z}_q^{(1)} \) is also related to the Lagrange multiplier \( \tilde{\gamma} \) through the relation

\[ 1 + \tilde{\gamma} = q \ln_q \bar{Z}_q^{(1)} - \tilde{\beta} U . \]

In Eqs. (5) and (7) we have introduced the \( q \)-deformed exponential

\[ \exp_{q}(x) = [1 + (1-q) x]^{\frac{1}{1-q}} , \]

and its inverse function the \( q \)-deformed logarithm

\[ \ln_{q}(x) = \frac{x^{1-q} - 1}{1-q} . \]

Note that Eq. (5) is the pdf which Di Sisto et al. [12] and Bashkirov [13] have independently obtained by modifying the treatment of the original version of Tsallis [1].
On the other hand, in the third formalism the Tsallis pdf is obtained by maximizing $S_q$ with the conditions posed by the normalization (2) and by the normalized $q$-average energy
\[ \sum_i p_i^q E_i = U_q , \] as follows
\[ \frac{\delta}{\delta p_i} \left( S_q - \beta \sum_j p_j^q E_j - \gamma \sum_j p_j \right) = 0 , \] where $\gamma$ and $\beta$ are the Lagrange multipliers associated to the constraints (2) and (3), respectively. From Eq. (11) we obtain
\[ p_i = \frac{1}{Z^{(3)}_q} \cdot \exp_q \left( -\beta (E_i - U_q) \right) , \] where $Z^{(3)}_q$ is the generalized partition function in the third formalism, and
\[ Z^{(3)}_q = \left( \sum_i p_i^q \right)^{\frac{1}{q-1}} . \]
Both expressions (5) and (12) are explicitly self-referential, which cause some difficulties. One of them is concerning with numerical convergence when one calculates the $p_i$.

In order to overcome this problem, Tsallis et al. [2] proposed the two different calculation methods: “the iterative procedure”; and “$\beta \to \beta'$ transformation”. Both methods were further studied by Lima and Penna [4].

An even more serious difficulty is that a maximum is not necessarily guaranteed due to the nondiagonal Hessian matrix associated to the above Lagrange procedure. In order to overcome this difficulty Martínez et al. [6] have introduced an alternative Lagrange route, which is so called optimal Lagrange multiplier (OLM) formalism [7, 8, 9, 10].

In our previous work [11], the equivalences of some different expressions of the Tsallis pdfs have been studied. Among them, it was explicitly shown that the non self-referential pdf
\[ p_i = \alpha \exp_q \left( -\beta E_i - \gamma \right) , \] with $\alpha = (2 - q)^{1/(q-1)}$, maximizes the entropy
\[ S_{2-q} = -\sum_i p_i \ln p_i = \sum_i \left( \frac{p_i^{2-q} - p_i}{q-1} \right) , \] under the constraints (2) and (3). Furthermore it was shown that Eq. (14) is equivalent to the self-referential forms (3) and (12) arising in the first and third formalism, respectively.

On the base of these equivalences, we can obtain a non self-referential expression of the Tsallis pdf not only for the first formalism but also for the third formalism. This is the purpose of the present note.

## 2 The non self-referential expressions

We begin with the first formalism. Let us briefly review the equivalence between the pdf in the form (14) and the distribution (5). From the MaxEnt problem
\[ \frac{\delta}{\delta p_i} \left( S_{2-q} - \bar{\beta} \sum_j p_j E_j - \bar{\gamma} \sum_j p_j \right) = 0 , \] we obtain
\[ (2 - q)p_i^{1-q} - 1 \bar{\beta} E_i - \bar{\gamma} = 0 . \]

Multiplying the both sides by $p_i$ and summing over $i$, taking into account of the constraints (2) and (3), it follows the solution
\[ p_i = \frac{1}{Z^{(1)}_{2-q}} \cdot \exp_q \left( -\bar{\beta}(E_i - U) \right) , \] where
\[ \frac{1}{Z^{(1)}_{2-q}} = \alpha \exp_q \left( -\bar{\gamma} - \bar{\beta} U \right) , \]
which is equivalent to Eq. (9) with $q$ replaced by $2 - q$. Note that by replacing $q$ with $2 - q$ in Eq. (13), we recover the form of Eq. (4).

On the other hand, Eq. (17) can be immediately solved w.r.t. $p_i$ as
\[ p_i = \exp_q \left( \frac{-1 + \bar{\gamma} + \bar{\beta} E_i}{2 - q} \right) \] which is equivalent to the non self-referential expression (14) in the first formalism.

We consider now the expression of the pdf in the third formalism. The derivation of the non self-referential expression equivalent to the pdf (12) is very simple, and the point is to express the quantity $\sum_j p_j^q$ in terms of $\gamma$. From Eq. (11) we have
\[ \frac{q p_i^{q-1}}{1 - q} - \bar{\beta} \frac{q^{q-1}}{\sum_j p_j^q} (E_i - U_q) - \bar{\gamma} = 0 . \]

Multiplying the both sides of this equation by $p_i$ and taking summation over $i$, we obtain
\[ \sum_i p_i^q = 1 + (1 - q) \frac{\gamma}{q} = 1 + (1 - q) \left( \frac{\gamma + 1}{q} \right) . \]

From Eq. (13) this relation is also expressed as
\[ Z^{(3)}_q = \exp_q \left( \frac{\gamma + 1}{q} \right) . \]

By utilizing Eq. (13), Eq. (12) can be written as
\[ p_i = \frac{1}{Z^{(3)}_q} \exp_q \left( -\frac{\beta}{Z^{(3)}_q} (E_i - U_q) \right) = \frac{1}{\left( Z^{(3)}_q \right)^{1-q}} \exp_q \left( \ln \left( Z^{(3)}_q - \beta (E_i - U_q) \right) \right) , \]
Substituting Eq. (23) into this we finally obtain

\[ p_i = \left[ \exp_q \left( \frac{\gamma + 1}{q} \right) \right]^{-2} \exp_q \left( \frac{\gamma + 1}{q} - \beta(E_i - U_q) \right). \] (25)

We remark that \( \gamma \) depends on only \( U_q \) and \( \beta \). Eq. (25) is thus the non self-referential expression of the pdf in the framework of the third formulation.

3 Conclusions

We have obtained the non self-referential expressions of the Tsallis pdf for both first and third formalisms based on the equivalences among the expression given in Eq. (14), which is non self-referential, the ones given in Eqs. (5) and (12), which are self-referential. These non self-referential expressions permit us to overcome some computational problems as well as to simplify the treatment of other related difficulties which arise from employing the self-referenced expressions (5) and (12). Notwithstanding, we remark that although implicitly defined, both pdfs (5) and (12) remain still useful in the analytical computations of many properties.

Note added

We would like to thank Dr. Suyari for informing his work concerning on this issue. During proceeding this work, T.W. had a chance to know that Dr. Suyari has independently performed the similar work [14]. After almost completing this work, Dr. Suyari and we exchanged the results each other, and noticed that both he and we obtained the same results by different methods.

Appendix

Let us show here the equivalence between Eq. (25) and Eq. (30) in Ref. [14], which can be written in our notation as follows.

\[ p_i = \frac{1}{Z^{(3)}_q} \exp_q \left( -\beta_q(E_i - U_q) \right), \] (A.1)

where

\[ \beta_q \equiv \frac{\beta}{1 + (1 - q) \left( \frac{\gamma + 1}{q} \right) \gamma q}. \] (A.2)

By utilizing the identity

\[ \exp_q(x + y) = \exp_q(x) \exp_q \left( \frac{y}{1 + (1 - q)x} \right), \] (A.3)

it follows

\[ \exp_q \left( \frac{\gamma + 1}{q} - \beta(E_i - U_q) \right) \]

\[ = \exp_q \left( \frac{\gamma + 1}{q} \right) \cdot \exp_q \left( -\beta_q(E_i - U_q) \right). \] (A.4)

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