Admissible solutions to augmented nonsymmetric $k$-Hessian type equations II. A priori estimates and the Dirichlet problem

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Abstract

Using the established $d$-concavity of the $k$-Hessian type functions $F_k(R) = \log(S_k(R))$, whose variables are nonsymmetric matrices, we prove $C^{2,\alpha}(\Omega)$ estimates for strictly $(\delta, \tilde{\gamma}_k)$-admissible solutions to the Dirichlet problem without the well-known regularity condition. A necessary condition for the existence of strictly $\delta$-admissible solutions to the equations is given. By the method of continuity, we provide some sufficient conditions for the unique solvability in the class of strictly $(\delta, \tilde{\gamma}_k)$-admissible solutions to the Dirichlet problem, provided that those skew-symmetric matrices in the equations are sufficiently small in some sense.

Keywords: nonsymmetric $k$-Hessian type equation, strictly $\delta$-admissible solution, strictly $\tilde{\gamma}_k$-admissible solution, strictly $(\delta, \tilde{\gamma}_k)$-admissible solution.

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1. Introduction

This paper is a continuation of our previous one [2]. We consider the Dirichlet problem for the following nonsymmetric augmented $k$-Hessian type equations

$$S_k \left[ D^2 u - A(x, u, Du) - B(x, u, Du) \right] = f(x, u, Du) \quad \text{in } \Omega \subset \mathbb{R}^n$$

(1.1)

$$u(x) = \varphi(x) \quad \text{on } \partial\Omega,$$

(1.2)

where $2 \leq k \leq n$, $\Omega$ is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, $Du$ and $D^2 u$ are respectively gradient vector and the Hessian matrix of the unknown function $u : \Omega \to \mathbb{R}$, $A(x, z, p) = [A_{ij}(x, z, p)]_{n \times n}$, $B(x, z, p) = [B_{ij}(x, z, p)]_{n \times n}$ and $f(x, z, p)$ are respectively smooth symmetric, skew-symmetric matrices and scalar valued functions, defined on $\mathcal{D} = \Omega \times \mathbb{R} \times \mathbb{R}^n$, $\varphi(x)$ is given smooth scalar valued defined on smooth $\partial\Omega$. We use $x, z, p, R$ to denote points in $\Omega, \mathbb{R}, \mathbb{R}^n$ and $\mathbb{R}^{n \times n}$ respectively. Here,

$$S_k(R) = \sigma_k(\lambda(R)),$$

where $\lambda(R) = (\lambda_1, \lambda_2, \cdots, \lambda_n) \in \mathbb{C}^n$ is the vector of eigenvalues of the matrix $R = [R_{ij}]_{n \times n} \in \mathbb{R}^{n \times n}$,

$$\sigma_k(\lambda) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k}$$
is the elementary symmetric polynomial of degree \(k\). Noting that, since \(R \in \mathbb{R}^{n \times n}\), \(S_k(R)\) is real-valued. When \(B(x, z, p) \equiv 0\) the equations (1.1) become

\[S_k(D^2u - A(x, u, Du)) = f(x, u, Du) \text{ in } \Omega,\]  

which are symmetric augmented \(k\)-Hessian type equations. When \(k = n\) the equations (1.3) are the Monge-Ampère type equations:

\[\det (D^2u - A(x, u, Du)) = f(x, u, Du) \text{ in } \Omega,\]

the Dirichlet problem for which had been studied in [3]-[6], [8], [13]-[15], [17].

For \(u(x) \in C^2(\Omega)\) and \(x \in \Omega\) we set

\[\omega(x, u) = D^2u(x) - A(x, u(x), Du(x)) = [\omega_{ij}(x, u)]_{n \times n}.\]  

(1.4)

For \(1 \leq k \leq n\) we denote by \(\Gamma_k\) the following cone in \(\mathbb{R}^n\):

\[\Gamma_k = \{ \lambda \in \mathbb{R}^n : \sigma_j(\lambda) > 0, j = 1, \ldots, k \}.\]

When \(k = n\) we have

\[\Gamma_n = \{ \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n : \lambda_j > 0, j = 1, \ldots, n \}.\]

The equations (1.3) have been considered in [7], [9], [16], [18]. A function \(u(x) \in C^2(\Omega)\) is said to be an admissible solution (9) to the equation (1.3) if \(\lambda(\omega(x, u)) \in \Gamma_k\) for any \(x \in \Omega\). Under the assumption of regularity condition (see (7.16)) for the matrix \(A(x, z, p)\) and that of existence of an admissible subsolution, by using the concavity of the function \(\sqrt[k]{\sigma_k(\omega)}\), the authors of [9] has proved the unique existence of admissible solution to the Dirichlet problem for (1.3).

The nonsymmetric Monge-Ampère type equations

\[\det (D^2u - A(x, u, Du) - B(x, u, Du)) = f(x, u, Du)\]  

(1.5)

has been considered in [11], [12]. The main difficulty in this case is that the both functions \(\sqrt[k]{\det R}\) and \(\log(\det R)\) are not concave. To overcome this difficulty, the following class of elliptic solutions to (1.5) are introduced as follows.

**Definition 1.1** ([11], [12]). Suppose \(u(x) \in C^2(\Omega)\). Then

(i) The function \(u(x)\) is said to be an elliptic solution to (1.5) if the following condition holds

\[\lambda_u := \inf_{x \in \Omega} \lambda_{\min}(\omega(x, u)) > 0,\]  

(1.6)

where \(\lambda_{\min}(\omega)\) is the least eigenvalue of \(\omega\);

(ii) Suppose \(0 < \delta < 1\). The function \(u(x)\) is said to be \(\delta\)-elliptic solution to (1.5) if it is elliptic one and it holds

\[\mu(B) \leq \delta \lambda_u,\]  

(1.7)

where the matrix \(B(x, z, p)\) is assumed to belong to \(BC(D)\) and

\[\mu(B) := \sup_{D} \|B(x, z, p)\|,\]

here \(\|B\|\) stands for the operator norm of the matrix \(B\).
For \( u(x) \in C^2(\Omega) \) and \( x \in \Omega \) we set
\[
R(x, u) = \omega(x, u) - B(x, u, Du) = [R_{ij}(x, u)]_{n \times n},
\]
where \( \omega(x, u) \) is defined by (14).

In connection with the \( \delta \)-elliptic solutions, the following convex and unbounded set of nonsymmetric matrices \( R \) had been introduced for \( 0 < \delta < 1, \mu > 0 \) (11)
\[
D_{\delta, \mu} = \{ R \in \mathbb{R}^{n \times n} : R = \omega + \beta, \omega^T = \omega, \beta^T = -\beta, \omega > 0, \| \beta \| \leq \mu, \mu \leq \delta \lambda_{\min}(\omega) \}
\]
as a domain for \( F(R) = \log(\det R) \). We note that if \( u(x) \) is a \( \delta \)-elliptic solution then \( R(x, u) \in D_{\delta, \mu(R)} \) for any \( x \in \Omega \). The notion of \( d \)-concavity for the function \( F(R) = \log(\det R) \) for \( d \geq 0 \) had been introduced in (11) as follows:

**Definition 1.2.** The function \( F(R) \) is said to be \( d \)-concave on \( D_{\delta, \mu} \) if for any \( R^{(0)} = [R_{ij}^{(0)}] = \omega^{(0)} + \beta^{(0)}, R^{(1)} = [R_{ij}^{(1)}] = \omega^{(1)} + \beta^{(1)} \in D_{\delta, \mu} \) the following inequality holds:
\[
F(R^{(1)}) - F(R^{(0)}) \leq \sum_{i,j=1}^{n} \frac{\partial F(R^{(0)})}{\partial R_{ij}} (R^{(1)}_{ij} - R^{(0)}_{ij}) + C \frac{|\beta^{(1)} - \beta^{(0)}|^2}{\lambda_{\min}^2(\omega^{(\tau)})},
\]
where \( \omega^{(\tau)} = (1 - \tau)\omega^{(0)} + \tau \omega^{(1)}, 0 < \tau < 1 \).

The \( d \)-concavity of the function \( F(R) = \log(\det R) \) had been established (11, Theorems 2 and 3), where \( C \) depends only on \( \delta, n \) and does not depend on \( \mu \). Then the \( d \)-concavity, the regularity condition for the matrix \( A(x, z, p) \) and the assumption on existence of an elliptic subsolution \( \underline{u}(x) \) to the problem (1.5)-(1.2) with \( B(x, z, p) = 0 \), enable to get \( C^{2,\alpha}(\Omega) \)-estimates for \( \delta \)-elliptic solutions to the Dirichlet problem (1.5)-(1.2) with some \( 0 < \alpha < 1 \) and then to get the solvability of the problem (12, Theorems 3 and 4).

In this paper, for the cases \( 2 \leq k \leq n \) we prefer to replace the notions of elliptic and \( \delta \)-elliptic solutions respectively by the notions of strictly admissible and strictly \( \delta \)-admissible solutions to the equations (11) that are defined respectively as the same as elliptic and \( \delta \)-elliptic solutions for the Monge-Ampère type equations (1.5). But to get the \( d \)-concavity of the functions \( F_k(R) = \log(S_k(R)) \) we have to restrict more on these classes of strictly admissible solutions. To do this we define a subcone \( \Sigma_{(\bar{\gamma}_k)} \) in \( \Gamma_n, 0 < \bar{\gamma}_k < 1 \), as follows:

**Definition 1.3 (2).** Suppose \( 1 \leq k \leq n \). The subcone \( \Gamma_{(\bar{\gamma}_k)} \) consists of all \( \lambda = (\lambda_1, \cdots, \lambda_n) \in \Gamma_n \), such that
\[
\lambda_{\min} \geq \bar{\gamma}_k \lambda_{\max},
\]
where \( \lambda_{\min} = \min_{1 \leq j \leq n} \lambda_j \), \( \lambda_{\max} = \max_{1 \leq j \leq n} \lambda_j \), and \( \bar{\gamma}_k \) is chosen appropriately in each concrete problem and satisfies the following conditions:

(i) If \( k \in \{2, 3, n - 1, n\} \), then \( 0 < \gamma_k < \bar{\gamma}_k < 1 \), where \( \gamma_k \) is a some positive number that is less than 1 and must be also determined in each case;

(ii) If \( \left[ \frac{n}{2} \right] + 1 \leq k \leq n - 2 \), then
\[
\gamma_k = \frac{n - k}{k} < \bar{\gamma}_k < 1; \quad (1.9)
\]
(iii) If \( 4 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor \), then
\[
\gamma_k = \gamma_{n-k+2} = \frac{k-2}{n-(k-2)} < \tilde{\gamma}_k < 1,
\]  
(1.10)
where \( \gamma_k, 2 \leq k \leq n-1 \), have been already defined in [2] as above.

Now the domain of the function \( F_k(R) = \log(S_k(R)) \) is introduced as follows:

**Definition 1.4** [2] Suppose \( 0 < \delta < 1, \mu > 0 \) and \( 0 < \gamma_k < \tilde{\gamma}_k < 1 \) that have been defined as above. We set
\[
D_{\delta, \mu, \tilde{\gamma}_k} = \{ R = \omega + \beta \in D_{\delta, \mu}; \lambda(\omega) \in \Sigma(\tilde{\gamma}_k) \}.
\]
Noting that all the sets \( \Sigma(\tilde{\gamma}_k) \), \( D_{\delta, \mu} \), and \( D_{\delta, \mu, \tilde{\gamma}_k} \) are convex and unbounded.

We recall now some following results from [2] (Theorem 1) and [11] (Proposition 5.1 and Theorem 1.6) for the functions \( F_k(R) = \log(S_k(R)) \).

**Theorem 1.5** [2, 11]. Suppose \( 2 \leq k \leq n \) and \( 0 < \gamma_k < \tilde{\gamma}_k < 1 \) are defined as in Definition 1.3. Then there exist \( \delta_k, 0 < \delta_k < 1, \delta_k = \delta_k(k, n, \tilde{\gamma}_k) \) if \( 2 \leq k \leq (n-1) \) and \( \delta_k \) may be any positive number that is less than 1 when \( k = n \) and \( C_j > 0, C_j = C_j(k, n, \tilde{\gamma}_k, \delta_k) \) such that for all \( \delta, 0 < \delta < \delta_k \) and

(i) for all \( R = \omega + \beta \in D_{\delta, \mu, \tilde{\gamma}_k}, M = P + Q \in \mathbb{R}^{n \times n}, P^T = P, Q^T = -Q \) the following estimates hold
\[
d^2 F_k(R, P) \leq -C_1 \frac{|P|^2}{\lambda_\text{max}^2(\omega)},
\]
(1.11)
\[
d^2 F_k(R, M) \leq C_2 \frac{|Q|^2}{\lambda_\text{min}^2(\omega)},
\]
(1.12)
where for \( M = [M_{ij}] \in \mathbb{R}^{n \times n}, |M|^2 = \sum_{i,j=1}^n |M_{ij}|^2; \)

(ii) for all \( R^{(0)}, R^{(1)} \in D_{\delta, \mu, \tilde{\gamma}_k}, R^{(0)} = \omega^{(0)} + \beta^{(0)}, R^{(1)} = \omega^{(1)} + \beta^{(1)} \), the following \( d \)-concavity of the function \( F_k(R) \) that is a consequence of (1.12), holds
\[
F_k(R^{(1)}) - F_k(R^{(0)}) \leq \sum_{i,j=1}^n \frac{\partial F_k(R^{(0)})}{\partial R_{ij}} \left( R^{(1)}_{ij} - R^{(0)}_{ij} \right) + C_2 \frac{|\beta^{(1)} - \beta^{(0)}|^2}{\lambda_\text{min}^2(\omega(\tau))},
\]
(1.13)
where \( \omega^{(\tau)} = (1-\tau)\omega^{(0)} + \tau \omega^{(1)}, 0 < \tau < 1. \)

From here and throughout the paper we always assume that the parameters \( \delta, \tilde{\gamma}_k \) are defined as follows:
\[
0 < \gamma_k < \tilde{\gamma}_k < 1, \quad 0 < \delta < \delta_k < 1,
\]
(1.14)
where \( 0 < \gamma_k < 1 \) have been defined in Definition 1.3 and \( 0 < \delta_k < 1 \) has been determined in Theorem 1.5.

Now other types of strictly admissible solutions to the equations (1.1) are introduced.

**Definition 1.6**. Suppose \( u(x) \in C^2(\overline{\Omega}) \) is a strictly admissible solution to (1.1), \( 0 < \delta < 1, 0 < \tilde{\gamma}_k < 1 \) as above, then
(i) It is said to be strictly $\tilde{\gamma}_k$-admissible solution to (1.1) if

$$\gamma_u := \inf_{x \in \Omega} \left[ \frac{\lambda_{\min}(\omega(x, u))}{\lambda_{\max}(\omega(x, u))} \right] \geq \tilde{\gamma}_k,$$  \hspace{1cm} (1.15)

where $\omega(x, u)$ is defined by (1.4), i.e. $\lambda(\omega(x, u)) \in \sigma(\tilde{\gamma}_k), \forall x \in \Omega$;

(ii) It is said to be strictly $(\delta, \tilde{\gamma}_k)$-admissible solution to (1.1) if it is both strictly $\delta$-admissible and strictly $\tilde{\gamma}_k$-admissible solution to (1.1), i.e. (1.7), (1.15) hold and $R(x, u) \in D_{\delta, \mu(B), \tilde{\gamma}_k}, \forall x \in \Omega$.

**Remark 1.7.** The condition (1.15) seems to be rather strict one, because the equation (1.1) becomes indeed uniformly elliptic at solutions of this kind and the $C^2(\Omega)$-estimates for solutions are easily obtained. But the condition (1.15) is actually needed, because it allows the function $F_k(R(x, u)) = \log (S_k(R(x, u)))$ to be $d$-concave with respect to $R(x, u)$ for $x \in \Omega$, with the aid of which one can prove the Holder continuity of $D^2u(x)$ in $\Omega$. The condition (1.15) is only a structural one for solutions of the problem (1.1)-(1.2), but it is not structural one for the equations (1.1). The most important structural conditions for the data $A(x, z, p), f(x, z, p)$ and $\Omega$, as it will be clear later in an example at the last section of the paper, must be those ones, under which there exists a strictly $\tilde{\gamma}_k$-admissible subsolution $\underline{u}(x)$ of the problem (1.1)-(1.2).

**Remark 1.8.** When $k = n$, in [8] the authors did not assume the uniform ellipticity condition for elliptic solutions to the Monge-Ampère type equations. But the elliptic solution $u(x)$, that exists and is unique in [8], is actually a strictly $\tilde{\gamma}_n$-admissible one, where $\tilde{\gamma}_n$ is some positive number, that is less than 1. Indeed, thanks to assumptions on regularity condition (7.16) and some additional structural conditions on $A(x, z, p), f(x, z, p)$ and the assumption on existence of elliptic subsolution $\underline{u}(x)$ to the problem (1.1)-(1.2), the authors had proved that there exist $M_0 > 0, M_1 > 0, M_2 > 0, 0 < M_3 < M_4$ such that

$$\sup_{x \in \Omega}|u(x)| \leq M_0, \sup_{x \in \Omega}|Du(x)| \leq M_1, \sup_{x \in \Omega}|D^2u(x)| \leq M_2,$$

from which one obtains

$$\inf_{x \in \Omega}\lambda_{\min}(\omega(x, u)) \geq M_3, \sup_{x \in \Omega}\lambda_{\max}(\omega(x, u)) \leq M_4,$$

and therefore (1.15) follows with $\tilde{\gamma}_n = \frac{M_3}{M_4}$.

The purpose of the paper is to study the solvability of the problem (1.1)-(1.2) in the class of strictly $(\delta, \tilde{\gamma}_k)$-admissible solutions without regularity condition for the matrix $A(x, z, p)$. The paper is organized as follows. In Section 2 we establish the comparison principle (Theorem 2.2) for strictly $\delta$-admissible solutions. This principle is analogous to that for Monge-Ampère type (1.3) equations (1.1). In Section 3, for strictly $(\delta, \tilde{\gamma}_k)$-admissible solutions to the Dirichlet problem (1.1)-(1.2), we estimate eigenvalues of the matrices $\omega(x, u)$ at any $x \in \Omega$. It is interesting that for this kind of solutions, to do this, we do not need neither $d$-concavity of the function $F_k(R)$, nor regularity condition for the matrix $A(x, z, p)$.
Proposition 1.9. Suppose $A(x, z, p), f(x, z, p) \in C(\Omega), B(x, z, p) \in BC(\Omega), f(x, z, p) > 0$. Suppose $u(x) \in C^2(\Omega)$ is a strictly admissible solution to the equation (1.1) and there exist $M_0 > 0, M_1 > 0$ such that

$$\sup_{\Omega} |u(x)| \leq M_0, \ \sup_{\Omega} |Du(x)| \leq M_1.$$ 

We set

$$f_0 = \inf_{|z| \leq M_0, |p| \leq M_1} f(x, z, p), \quad f_1 = \sup_{|z| \leq M_0, |p| \leq M_1} f(x, z, p).$$

Then the following assertions hold for any $x \in \Omega$:

(i) $0 < \lambda_{\min}(\omega(x, u)) \leq \left[ \frac{f_1}{\binom{n}{k}} \right]^\frac{1}{k}$; 

(ii) If $u(x)$ is a strictly $\delta$-admissible solution, then besides (1.14), the following inequality is true

$$\left[ \frac{(1 + \delta^2) - \left[ \frac{f_1}{n} \right]}{\binom{n}{k}} \right]^\frac{1}{k} \leq \lambda_{\max}(\omega(x, u));$$

(iii) If $u(x)$ is a strictly $\tilde{\gamma}_k$-admissible solution, then besides (1.17), the following inequality is true

$$\lambda_{\max}(\omega(x, u)) \leq \frac{1}{\tilde{\gamma}_k} \left[ \frac{f_1}{\binom{n}{k}} \right]^\frac{1}{k};$$

(iv) If $u(x)$ is a strictly $(\delta, \tilde{\gamma}_k)$-admissible solution, then

$$\tilde{\gamma}_k \left[ \frac{(1 + \delta^2) - \left[ \frac{f_1}{n} \right]}{\binom{n}{k}} \right]^\frac{1}{k} \leq \lambda_{\min}(\omega(x, u)) \leq \left[ \frac{f_1}{\binom{n}{k}} \right]^\frac{1}{k};$$

$$\left[ \frac{(1 + \delta^2) - \left[ \frac{f_1}{n} \right]}{\binom{n}{k}} \right]^\frac{1}{k} \leq \lambda_{\max}(\omega(x, u)) \leq \frac{1}{\tilde{\gamma}_k} \left[ \frac{f_1}{\binom{n}{k}} \right]^\frac{1}{k}.$$

Under some structure conditions on the matrix $A(x, z, p)$, proposed by N.S Trudinger and his colleagues in [8], by using the comparison principle (Theorem 2.2) we obtain $C^2(\Omega)$-estimates for strictly $(\delta, \tilde{\gamma}_k)$-admissible solutions $u(x)$ in the following theorem.

Theorem 1.10. Assume that $0 < \delta < 1, 0 < \tilde{\gamma}_k < 1$ are defined as in (1.14) and the following conditions fulfill:

(i) $A(x, z, p) \in C^3(\Omega)$ and satisfies structure conditions:

$$A(x, z, p) \geq -\gamma_0 (1 + |p|^2) E_n, \quad \gamma_0 > 0,$$

$$\lambda_{\max}(A(x, z, 0)) \geq 0, \quad D_z A(x, z, p) \geq 0;$$
(ii) \( f(x, z, p) \in C^3(D) \) and
\[
f(x, z, p) > 0 \text{ in } D,
\]
\[
\inf_D \left[ \frac{D_z f(x, z, p)}{f(x, z, p)} \right] \geq \frac{k\delta}{(1 + \delta^2)^{\beta_1}}, \beta_1 > 0,
\]

(iii) There exists a strictly \( \tilde{\gamma}_k \)-admissible subsolution \( \underline{u}(x) \) to the problem
\[
S_k \left( D^2 u - A(x, u, Du) \right) = f(x, u, Du) \quad \text{in } \Omega,
\]
\[u = \varphi \quad \text{on } \partial \Omega; \quad (1.22)
\]

(iv) Suppose \( u(x) \in C^2(\overline{\Omega}) \) is a strictly \( (\delta, \tilde{\gamma}_k) \)-admissible solution to the problem (1.1)-(1.2);

(v) \( B(x, z, p) \in BC^3(D) \) and
\[
\mu(B) \leq \delta \min (\lambda_u, \lambda_u),
\]
\[
\mu(D_z B) \leq \beta_1 \min (\lambda_u, \lambda_u).
\]
Then there exist \( M_0 > 0, M_1 > 0, C_3 > 0 \), that depend only on \( \delta, k, n, \tilde{\gamma}_k, \beta_1, A, \underline{u}, f, \varphi \) such that
\[
\sup_{\overline{\Omega}} |u(x)| \leq M_0, \quad \sup_{\overline{\Omega}} |Du(x)| \leq M_1
\]
and
\[
\|u\|_{C^3(\overline{\Omega})} \leq C_3. \quad (1.24)
\]

Using (1.24), the ellipticity estimate (3.5) and the \( d \)-concavity in the sense of (1.13) of the function \( F_k(R) = \log (S_k(R)) \), at the end of Section 3 we show the Hölder continuity of second-order derivatives \( D^2 u \) with some \( 0 < \alpha < 1 \) inside \( \Omega \), if \( u(x) \in C^4(\Omega) \). Here, besides the quantities, on which \( C_3 \) depends, \( \alpha \) depends also on \( \mu_2(B) \), where
\[
\mu_2(B) = \|B(x, z, p)\|_{BC^2(D)}. \quad (1.25)
\]

In Section 4 we consider \( x^0 \in \partial \Omega \). By translation and rotation, we can assume that \( x^0 \) is the origin of coordinates and the unit inner normal at \( x^0 \) is on the axis \( Ox_n \). Suppose that in a neighborhood \( N \) of \( x^0 \), the boundary \( \partial \Omega \) is the graph of the function
\[
x_n = h(x'), \quad x' = (x_1, \ldots, x_{n-1}),
\]
where \( h(x') \in C^4 \) and
\[
h(0') = 0, \quad Dh(0') = 0.
\]
We change \( x = (x_1, \ldots, x_n) \) into \( y = (y_1, \ldots, y_n) \) by the mapping
\[
y = \psi(x) = (\psi_1(x), \ldots, \psi_n(x)) = (x', x_n - h(x')),
\]
where \( y \in \tilde{N} := \psi(N) \). From (1.26) we have
\[
x = \tilde{\psi}(y) = (\tilde{\psi}_1(y), \ldots, \tilde{\psi}_n(y)) = (y', y_n + h(y')). \quad (1.27)
\]
We set
\[
v(y) = u(x) - \varphi(x), \quad (1.28)
\]
\[
J(x) = \frac{D\psi(x)}{Dx} = \left[ \frac{\partial \psi_i(x)}{\partial x_j} \right]_{n \times n} = \left[ \begin{array}{ccc}
\frac{\partial \psi_1(x)}{\partial x_1} & \cdots & \frac{\partial \psi_1(x)}{\partial x_n} \\
\vdots & \ddots & \vdots \\
\frac{\partial \psi_n(x)}{\partial x_1} & \cdots & \frac{\partial \psi_n(x)}{\partial x_n}
\end{array} \right],
\]

where \(x\) and \(y\) are related by (1.26) and (1.27). In (1.28) we assume that the function \(\varphi(x)\) has been extended smoothly from \(\partial \Omega\) into some neighborhood of \(\partial \Omega\).

Then we have

\[
J(x) = \begin{bmatrix}
E_{(n-1)} & 0 \\
-Dh(x') & 1
\end{bmatrix} = J(y), \quad J^T(x) = \begin{bmatrix}
E_{(n-1)} & -(Dh)^T(x') \\
0 & 1
\end{bmatrix} = J^T(y), \tag{1.29}
\]

\[
J^{-1}(x) = \begin{bmatrix}
E_{(n-1)} & 0 \\
Dh(x') & 1
\end{bmatrix} = J^{-1}(y), \quad (J^{-1})^T(x) = \begin{bmatrix}
E_{(n-1)} & (Dh)^T(x') \\
0 & 1
\end{bmatrix} = (J^{-1})^T(y), \tag{1.30}
\]

where \(E_{(n-1)}\) is the unit matrix of size \((n-1)\) and \(Dh\) stands for the row vector

\[
Dh(x') = (D_{x_1}h(x'), \ldots, D_{x_n}h(x')) = (D_{y_1}h(y'), \ldots, D_{y_n}h(y')) = Dh(y').
\]

We have

\[
D_xu = (D_yv)J + D\varphi, \tag{1.31}
\]

\[
D_x^2u = J^TD_y^2vJ + \sum_{m=1}^n D_{ym}vD^2\psi_m + D_x^2\varphi, \tag{1.32}
\]

where \(Du = (D_{x_1}u, \ldots, D_{x_n}u), Dv = (D_{y_1}v, \ldots, D_{y_n}v)\). We set further on the base of (1.31), (1.32):

\[
\begin{align*}
\bar{A}(y, z, p) &= (J^{-1})^T \left[ A(\tilde{\psi}(y), z + \varphi(\tilde{\psi}(y)), pJ + D_x\varphi(\tilde{\psi}(y))) 
\right. \\
&\quad \left. - \sum_{m=1}^n p_mD_x^2\psi_m(\tilde{\psi}(y)) - D_x^2\varphi(\tilde{\psi}(y)) \right] J^{-1}, \\
\bar{B}(y, z, p) &= (J^{-1})^T B(\tilde{\psi}(y), z + \varphi(\tilde{\psi}(y)), pJ + D_x\varphi(\tilde{\psi}(y))) (J^{-1})^T, \\
\bar{f}(y, z, p) &= f(\tilde{\psi}(y), z + \varphi(\tilde{\psi}(y)), pJ + D_x\varphi(\tilde{\psi}(y))).
\end{align*}
\]

It follows from (1.32)–(1.33) that

\[
D^2u - A(x, u, Du) - B(x, u, Du) = J^T \left[ D^2v - \bar{A}(y, v, Dv) - \bar{B}(y, v, Dv) \right] J.
\]

The equation (1.1) becomes

\[
S_k \left[ J^T \left( D^2v - \bar{A}(y, v, Dv) - \bar{B}(y, v, Dv) \right) J \right] = \bar{f}(y, v, Dv) \text{ in } \bar{\Omega}_\rho, \tag{1.34}
\]

where \(\bar{\Omega}_\rho = \{(y', y_n) : |y| < \rho, y_n > 0\}\), \(v(y)\) satisfies condition:

\[
v(y', y_n) = 0 \text{ when } y_n = 0, |y'| < \rho, \rho > 0. \tag{1.35}
\]

We set for \(v(y) \in C^2(\bar{\Omega}_\rho)\)

\[
\tilde{\omega}(y, v) = D^2v - \bar{A}(y, v, Dv), \\
\tilde{R}(y, v) = \tilde{\omega}(y, v) - \bar{B}(y, v, Dv) = \left[ \tilde{R}_{ij} \right]_{n \times n}.
\]

8
Suppose $i_1i_2\cdots i_k$ and $j_1j_2\cdots j_k$ are indices such that

\[1 \leq i_1 < i_2 < \cdots < i_k \leq n, \quad 1 \leq j_1 < j_2 < \cdots < j_k \leq n.\]

We denote

\[\tilde{R}^{(k)}_{i_1\cdots i_k, j_1\cdots j_k} = \left[\tilde{R}_{p,q}\right]_{p,q=1}^k.\]

**Proposition 1.11.** In a neighborhood of the origin $y^0 = 0$ the equation (1.36) can be rewritten in the form:

\[S_k(\tilde{R}(y, v)) + H_k \left(y', \tilde{R}(y, v)\right) = \tilde{f}(y, v, Dv), \quad y \in \tilde{\Omega}_\rho,\]

where $H_n(y', \tilde{R}) = 0$ and if $2 \leq k \leq n - 1$ then

\[
H_k(y', \tilde{R}) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \sum_{m=1}^{k} \left( (D_m h(y'))^2 \right) \det \tilde{R}^{(k)}_{i_1\cdots i_k, j_1\cdots j_k} + \prod_{1 \leq \ell \leq \ell' \leq k} \left( (D_{m_\ell} h(y'))^2 \right) \det \tilde{R}^{(k)}_{i_1\cdots i_k, j_1\cdots j_k} \times \left( \sum_{m=1}^{k} (D_m h(y')) \right)
\]

\[\left( \sum_{\ell=1}^{m-1} (-1)\ell (D_{j_\ell} h(y')) \delta_i_{j_1j_2} \cdots \delta_i_{j_{\ell-1}j_\ell-1} \delta_i_{j_\ell j_{\ell+1}} \cdots \delta_i_{j_{m-1}j_m} \delta_i_{j_m j_{m+1}} \cdots \delta_i_{j_k j_{k-1}} \right) \]

In Section 5 we prove that if $u(x)$ is a strictly $(\delta, \tilde{\gamma}_k)$-admissible solution to the equation (1.1); then $v(y)$, defined by (1.28), is a strictly $(\tilde{\delta}, \tilde{\gamma}_k)$-admissible solution to the equation (1.36) in $\tilde{\Omega}_\rho$ with $\tilde{\delta} = (1 + \varepsilon)^2 \delta, \tilde{\gamma}_k = \frac{1}{(1 + \varepsilon)^2} \gamma_k$, where $\varepsilon > 0$ is sufficiently small if $\rho$ is chosen sufficiently small. We denote by $\tilde{F}_k \left(y', \tilde{R}\right)$ the corresponding new $k$-Hessian type function of the equation (1.36), which is

\[\tilde{F}_k(y', \tilde{R}) = \log \left[ S_k(\tilde{R}) + H_k \left(y', \tilde{R}\right) \right],\]

where $H_k \left(y', \tilde{R}\right)$ is defined by (1.37). The $\tilde{d}$-concavity of the function $\tilde{F}_k(y', \tilde{R})$ will be proved in the following.
PROPOSITION 1.12. Suppose $\rho$ is chosen sufficiently small so that

$$0 < \gamma_k < \tilde{\gamma}_k = \frac{1}{(1 + \varepsilon)^2} \tilde{\gamma}_k < 1, \quad 0 < \tilde{\delta} = (1 + \varepsilon)^2 \delta < \delta_k < 1,$$

where $0 < \gamma_k < \tilde{\gamma}_k < 1$ are defined in Definition 1.3, $0 < \delta < \delta_k < 1$ are determined in Theorem 1.5. Then for any $y'$, $|y'| \leq \rho$, the function $\tilde{F}_k(y', \tilde{R})$ is $\tilde{d}$-concave on the set $D_{\tilde{\sigma}, \mu(\tilde{B})} \tilde{\gamma}_k$ in the sense of (1.13), where

$$\mu(\tilde{B}) = \sup_{y \in \tilde{\Omega}_\rho, z \in \mathbb{R}, p \in \mathbb{R}^n} \| \tilde{B}(y, z, p) \|$$

and $C'_2 > 0$ in (1.13) does not depend on $y', |y'| < \rho$.

Using (1.35) and the $\tilde{d}$-concavity of $\tilde{F}_k(y', \tilde{R})$, we show the Hölder continuity of $D^2 v(y)$ in $\tilde{\Omega}$ with some $0 < \alpha < 1$, if $v(y) \in C^4(\tilde{\Omega}) \cap C^2(\tilde{\Omega}_\rho)$ and $\tilde{A}(y, z, p), \tilde{B}(y, z, p), \tilde{f}(y, z, p) \in C^3(\tilde{D}_\rho)$, $\tilde{D}_\rho = \tilde{\Omega}_\rho \times \mathbb{R} \times \mathbb{R}^n$. So we will obtain the following theorem at the end of Section 5.

THEOREM 1.13. Under the assumptions of Theorem 1.10 there exist $C_4 > 0$, $0 < \alpha < 1$, that depend on $n, k, \delta, \tilde{\gamma}_k, \beta_1, \Omega, A(x, z, p), f(x, z, p), u(x), \varphi, \mu_2(B)$, such that if $u(x)$ is any strictly $(\delta, \tilde{\gamma}_k)$-admissible solution to the problem (1.1)-(1.2), the following estimate holds

$$\|u\|_{C^2, \alpha(\Omega)} \leq C_4,$$

(1.39)

where $\mu_2(B)$ is defined by (1.25).

In Section 6 we study the solvability of the Dirichlet problem (1.1)-(1.2) in the classes of strictly admissible solutions. A necessary condition and some sufficient conditions on $B(x, z, p)$ have been found as follows.

THEOREM 1.14 (A necessary condition). Suppose $0 < \delta < 1$ and there exists a strictly $\delta$-admissible solution $u(x)$ to the equation (1.1), which satisfies the following conditions:

(i) $\lambda_u = \inf_{x \in \Omega} \lambda_{\min}(\omega(x, u)) > 0$,

(ii) $\mu(B) \leq \delta \lambda_u$,

(iii) $\sup_{\Omega} |u(x)| \leq M_0$, $\sup_{\Omega} |Du(x)| \leq M_1$.

Then it is necessary that

$$\mu(B) \leq \delta \left[ \frac{1}{(k)} f_1 \right]^{1/2},$$

(1.40)

where $f_1$ is defined by (1.16).

The following theorem is the main result of the paper.

THEOREM 1.15 (Sufficient conditions). Suppose $2 \leq k \leq n$, $0 < \delta < 1$, $0 < \tilde{\gamma}_k < 1$ are defined as in (1.14), $A(x, z, p), f(x, z, p) \in C^3(D)$. Assume that the following conditions hold:
(i) \( A(x, z, p) \geq -\gamma_0 (1 + |p|^2) E_n, \gamma_0 > 0, \lambda_{\text{max}}(A(x, z, 0)) \geq 0, D_z A(x, z, p) \geq 0; \)

(ii) \( f(x, z, p) > 0 \) in \( D \) and

\[
\inf_D \left[ \frac{D_z f(x, z, p)}{f(x, z, p)} \right] \geq \frac{k \delta}{(1 + \delta^2)^{\beta_1}} \beta_1 > 0;
\]

(iii) There exists a strictly \( \tilde{\gamma}_k \)-admissible subsolution \( u(x) \in C^4(\Omega) \) to the problem

\[
S_k \left( D^2 u - A(x, u, Du) \right) = f(x, u, Du) \text{ in } \Omega,
\]

\( u = \varphi \) on \( \partial \Omega, \)

that satisfies the following conditions:

\[
\lambda_u > 0
\]

and

\[
\gamma_u > \tilde{\gamma}_k + \varepsilon_0, \varepsilon_0 > 0, \tag{1.41}
\]

where \( \lambda_u \) and \( \gamma_u \) are defined by (1.6), (1.15) respectively. Here we assume that \( \partial \Omega \in C^4, \varphi \in C^4; \)

(iv) Suppose \( B(x, z, p) \in BC^3(D) \) is a skew-symmetric and satisfies the following conditions:

\[
\mu(B) < \delta \min (\lambda_u, \lambda_*), \tag{1.42}
\]

\[
\mu(D_z B) < \beta_1 \min (\lambda_u, \lambda_*), \tag{1.43}
\]

where

\[
\lambda_* = \tilde{\gamma}_k \left[ \frac{(1 + \delta^2)^{-\frac{k}{2}}}{\beta_1} \right]^{\frac{1}{k}}, \tag{1.44}
\]

\( f_0 \) is defined by (1.16) with \( M_0, M_1 \) as in Theorem 1.10.

Then there exists unique strictly \( (\delta, \tilde{\gamma}_k) \)-admissible solution \( u(x) \) to the problem (1.11) - (1.12) that belongs to \( C^{2, \alpha}(\Omega) \) with some \( 0 < \alpha < 1, \) where \( \alpha \) depends on \( n, k, \delta, \tilde{\gamma}_k, \beta_1, \Omega, A(x, z, p), f(x, z, p), u(x), \varphi, \mu_2(B). \)

In the last Section 7, we consider an example of the Dirichlet problem for a nonsymmetric \( k \)-Hessian type equation in the cases \( 2 \leq k \leq n \) and in the separated case \( k = 2. \)

2. The comparison principle for the strictly \( \delta \)-admissible solutions

First, we prove the following lemma on ellipticity of the equation \( \log F_k(R(x, u)) = \log f(x, u, Du) \) at a strictly \( \delta \)-admissible solution.

**Lemma 2.1.** Suppose \( 0 < \delta < 1, \mu > 0 \) and \( R = \omega + \beta \in D_{\delta, \mu}. \) Then for \( F_k(R) = \log (S_k(R)) \) we have

\[
\frac{k}{n} (1 + \delta^2)^{-2 \left[ \frac{k}{2} \right]} \lambda_{\min}^k(\omega) \gamma^{k+1}(\omega) |\xi|^2 \leq \frac{1}{2} \sum_{i,j=1}^n \left( \frac{\partial F_k(R)}{\partial R_{ij}} + \frac{\partial F_k(R)}{\partial R_{ji}} \right) \xi_i \xi_j \leq \frac{(1 + \delta^2)^{\frac{k}{2}}}{\lambda_{\min}(\omega)} |\xi|^2 \tag{2.1}
\]

for any \( \xi = (\xi_1, \cdots, \xi_n)^T \in \mathbb{R}^n. \)
Proof. Suppose

\[ R = \omega + \beta = C^{-1}(D + C\beta C^{-1})C = C^{-1}(D + \widetilde{\beta})C = C^{-1}\widetilde{R}C = [R_{ij}]_{n\times n}, \]

where \( C \) is an orthogonal matrix, \( D = \text{diag}(\lambda_1, \ldots, \lambda_n) \), \( \lambda_j > 0 \). Since \( S_k(R) = S_k(\widetilde{R}) \), then \( F_k(R) = F_k(\widetilde{R}) \). We denote

\[ \eta = (\eta_1, \ldots, \eta_n)^T = C\xi. \]

Then we have

\[ \frac{1}{2} \sum_{i,j=1}^{n} \left( \frac{\partial F_k(R)}{\partial R_{ij}} + \frac{\partial F_k(R)}{\partial R_{ji}} \right) \xi_i \xi_j = \frac{1}{2} \sum_{i,j=1}^{n} \left( \frac{\partial F_k(\widetilde{R})}{\partial R_{ij}} + \frac{\partial F_k(\widetilde{R})}{\partial R_{ji}} \right) \eta_i \eta_j. \]

So, we can assume that \( R = D + \beta \in D_{\delta, \mu} \). We note that if \( \sigma = D^{-\frac{1}{2}}\beta D^{-\frac{1}{2}} \), then \( \|\sigma\| \leq \delta \).

To prove (2.1) we recall some facts from [2]. If for indices \( i_1i_2\ldots i_k \) with \( 1 \leq i_1 < \cdots < i_k \leq n \) we set

\[ R_{i_1\ldots i_k} = [R_{ipiq}]_{p,q=1}^k, \quad G_{i_1\ldots i_k}(R) = \det (R_{i_1\ldots i_k}), \quad (R_{i_1\ldots i_k})^{-1} = \left[(R_{i_1\ldots i_k})^{-1}ight]^k \]

then we have

\[ \frac{\partial F_k(R)}{\partial R_{ij}} = \frac{1}{S_k(R)} \sum_{1 \leq i_1 < \cdots < i_k \leq n} G_{i_1\ldots i_k}(R) \sum_{p,q=1}^k (R_{i_1\ldots i_k})^{-1}_{iqp} \delta_{iip} \delta_{jiq}. \quad (2.2) \]

There are some following relations:

\[ \frac{(R_{i_1\ldots i_k})^{-1} + [(R_{i_1\ldots i_k})^{-1}]^T}{2} = D_{i_1\ldots i_k}^{-\frac{1}{2}} \left( E_{i_1\ldots i_k} - \sigma_{i_1\ldots i_k}^2 \right) D_{i_1\ldots i_k}^{-\frac{1}{2}}, \]

\[ (1 + \delta^2)\left[ \frac{k}{2} \right] E_{i_1\ldots i_k} \leq (E_{i_1\ldots i_k} - \sigma_{i_1\ldots i_k}^{-1}) \leq E_{i_1\ldots i_k}, \quad (2.3) \]

where

\[ E_{i_1\ldots i_k} = [\delta_{ipiq}]_{p,q=1}^k, \quad D_{i_1\ldots i_k}^{-\frac{1}{2}} = \text{diag} \left( \lambda_{i_1}^{-\frac{1}{2}}, \ldots, \lambda_{i_k}^{-\frac{1}{2}} \right), \]

\[ \sigma_{i_1\ldots i_k} = D_{i_1\ldots i_k}^{-\frac{1}{2}} \beta_{i_1\ldots i_k} D_{i_1\ldots i_k}^{-\frac{1}{2}}, \quad \|\sigma_{i_1\ldots i_k}\| \leq \delta, \]

\[ G_{i_1\ldots i_k}(D) \leq G_{i_1\ldots i_k}(R) \leq (1 + \delta^2)\left[ \frac{k}{2} \right] G_{i_1\ldots i_k}(D), \quad (2.4) \]

\[ S_k(D) \leq S_k(R) \leq (1 + \delta^2)\left[ \frac{k}{2} \right] S_k(D), \quad (2.5) \]

\[ \frac{(1 + \delta^2)\left[ \frac{k}{2} \right] G_{i_1\ldots i_k}(D)}{S_k(D)} \leq \frac{G_{i_1\ldots i_k}(R)}{S_k(R)} \leq (1 + \delta^2)\left[ \frac{k}{2} \right] \frac{G_{i_1\ldots i_k}(D)}{S_k(D)}. \quad (2.6) \]

From (2.2)-(2.6) it follows that

\[ \frac{1}{2} \sum_{i,j=1}^{n} \left( \frac{\partial F_k(R)}{\partial R_{ij}} + \frac{\partial F_k(R)}{\partial R_{ji}} \right) \xi_i \xi_j \leq \frac{1 + \delta^2}{{\lambda_{\text{min}}}^2} \sum_{1 \leq i_1 < \cdots < i_k \leq n} \frac{\lambda_{i_1} \cdots \lambda_{i_k}}{\sigma_k(\lambda)} \left( \sum_{p=1}^k \xi_{ip}^2 \right) \]

\[ \leq \frac{(1 + \delta^2)\left[ \frac{k}{2} \right]}{{\lambda_{\text{min}}}^2} |\xi|^2. \]
Here we have used the facts that \( \sum_{p=1}^{k} \xi_{ip}^2 \leq |\xi|^2 \) and \( \frac{1}{\sigma_k(\lambda)} \sum_{1 \leq i_1 < \cdots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k} = 1. \)

On the other side, we also have from (2.2)-(2.6) that

\[
\frac{1}{2} \sum_{i,j=1}^{n} \left( \frac{\partial F_k(R)}{\partial R_{ij}} + \frac{\partial F_k(R)}{\partial R_{ji}} \right) \xi_i \xi_j \geq \frac{(1 + \delta^2)^{-2[\frac{k}{2}]}}{\lambda_{\max}} \sum_{1 \leq i_1 < \cdots < i_k \leq n} \frac{\lambda_{i_1} \cdots \lambda_{i_k}}{\sigma_k(\lambda)} \sum_{p=1}^{n} \xi_{ip}^2
\]

\[
= \frac{(1 + \delta^2)^{-2[\frac{k}{2}]}}{\lambda_{\max}} \sum_{\ell=1}^{n} \left[ \sum_{1 \leq i_1 < \cdots < i_k \leq n} \frac{\lambda_{i_1} \cdots \lambda_{i_k}}{\sigma_k(\lambda)} \right] \xi_{i_{\ell}}^2
\]

\[
= \frac{(1 + \delta^2)^{-2[\frac{k}{2}]}}{\lambda_{\max}} \sum_{\ell=1}^{n} \frac{\lambda_{\ell} \sigma_{k-1}(\lambda)}{\sigma_k(\lambda)} \lambda_{i_{\ell}}^k \xi_{i_{\ell}}^2
\]

\[
\geq \frac{(1 + \delta^2)^{-2[\frac{k}{2}]}}{\lambda_{\max}} \left( \frac{n-1}{k} \right) \lambda_{\min}^k |\xi|^2
\]

\[
= \frac{k}{n} \frac{(1 + \delta^2)^{-2[\frac{k}{2}]}}{\lambda_{\max}^k} \lambda_{\min}^k |\xi|^2.
\]

Then (2.1) follows from (2.7) and (2.8).

For \( u(x) \in C^2(\overline{\Omega}) \) we set

\[
G_k[u](x) = \log(S_k(R(u))) - \log f(x, u, Du),
\]

where \( R(x, u) \) is defined by (1.8).

**Theorem 2.2.** Suppose \( A(x, z, p), f(x, z, p) \in C^1(\mathcal{D}), B(x, z, p) \in BC^1(\mathcal{D}), \) \( 0 < \delta < 1 \) and suppose \( u(x), v(x) \in C^2(\overline{\Omega}) \) and satisfy the following conditions

(i) \( G_k[u](x) \leq G_k[v](x), \ x \in \Omega; \)

(ii) \( \lambda_u > 0, \ \lambda_v > 0; \)

(iii) \( D_z A(x, z, p) \geq 0, \ (x, z, p) \in \mathcal{D}; \)

(iv) \( \mu(B) \leq \delta \min(\lambda_u, \lambda_v); \)

(v) \( \mu(D_z B) \leq \beta_1 \min(\lambda_u, \lambda_v), \ \beta_1 > 0; \)

(vi) \( f(x, z, p) > 0, \ (x, z, p) \in \mathcal{D}; \)

(vii) \( \inf_{\mathcal{D}} \left[ \frac{D_z f(x, z, p)}{f(x, z, p)} \right] \geq \frac{k\delta}{(1+\delta^2)} \beta_1. \)

Then the following assertions are true:

(a) If \( u(x) \geq v(x) \) on \( \partial \Omega, \) then \( u(x) \geq v(x) \) in \( \Omega, \)
(b) If \( u(x) = v(x) \) on \( \partial \Omega \), then
\[
\frac{\partial u(x)}{\partial \nu} \geq \frac{\partial v(x)}{\partial \nu} \text{ on } \partial \Omega,
\]
where \( \nu \) is the unit inward normal at \( x \in \partial \Omega \).

**Proof.** From the assumptions (ii) and (iv) it follows that for any \( x \in \Omega \)
\[
R(x, u), R(x, v) \in D_{\delta, \mu(B)}
\]
and
\[
\lambda_{\min}(\omega(x, u)) \geq \lambda_u > 0, \quad \lambda_{\min}(\omega(x, v)) \geq \lambda_v > 0.
\]
Then, by using (2.1)-(2.6) and the following relation ([11])
\[
\left( R_{i_1 \cdots i_k}^{-1} - \left[ (R_{i_1 \cdots i_k}^{-1})^{-1} \right]^T \right)_{i_1 \cdots i_k} = D_{i_1 \cdots i_k}^{-\frac{1}{2}}(-\sigma_{i_1 \cdots i_k})(E_{i_1 \cdots i_k} - \sigma_{i_1 \cdots i_k}^2)D_{i_1 \cdots i_k}^{-\frac{1}{2}}, \quad (2.9)
\]
the proof of the theorem will go analogously as in the proof of the comparison principle for nonsymmetric Monge-Ampère type equations (1.5) in [11] (Theorem 4).

3. The \( C^2(\overline{\Omega}) \)-estimates for strictly \((\delta, \tilde{\gamma}_k)\)-admissible solutions and the Hölder continuity of their second-order derivatives inside the domain

We recall that for \( u(x) \in C^2(\overline{\Omega}), x \in \overline{\Omega} \) the matrices \( \omega(x, u) \) and \( R(x, u) \) are defined respectively by (1.4) and (1.8).

The equation (1.1) can be written as
\[
S_k(R(x, u)) = f(x, u, Du), \quad x \in \overline{\Omega}. \quad (3.1)
\]

3.1. Proof of Proposition 1.9

(i) Suppose \( u(x) \) is a strictly admissible solution to (3.1), i.e. \( \lambda_{\min}(\omega(x, u)) \geq \lambda_u > 0 \). We have from (2.5) and (3.1) that
\[
\left( \begin{array}{c}
\binom{n}{k} \lambda_k^k(\omega(x, u)) \\
\binom{n}{k} \lambda_k^k(\omega(x, u)) \\
\binom{n}{k} \lambda_k^k(\omega(x, u)) \\
\end{array} \right) \leq S_k(\omega(x, u)) \leq S_k(R(x, u)) = f(x, u, Du)
\]
\[
\leq \sup_{x \in \overline{\Omega}, |z| \leq M_0, |p| \leq M_1} [f(x, z, p)] = f_1,
\]
if \( |u(x)| \leq M_0, |Du(x)| \leq M_1 \). So, (1.17) is proved.

(ii) Suppose \( u(x) \) is a strictly \( \delta \)-admissible solution to (3.1). Then
\[
\mu(B) \leq \delta \lambda_u
\]
and \( R(x, u) \in D_{\delta, \mu(B)} \) for any \( x \in \overline{\Omega} \). From (3.1) and (2.5) we obtain
\[
\left( \begin{array}{c}
\binom{n}{k} (1 + \delta^2)^{\frac{k}{2}} \lambda_k^k(\omega(x, u)) \\
\binom{n}{k} (1 + \delta^2)^{\frac{k}{2}} \lambda_k^k(\omega(x, u)) \\
\binom{n}{k} (1 + \delta^2)^{\frac{k}{2}} \lambda_k^k(\omega(x, u)) \\
\end{array} \right) \geq (1 + \delta^2)^{\frac{k}{2}} S_k(\omega(x, u))
\]
\[
\geq S_k(R(x, u)) = f(x, u, Du) \geq f_0,
\]
from which it follows (1.18).
(iii) If \( u(x) \) is a strictly \( \tilde{\gamma}_k \)-admissible solution to (3.1), then from (1.15) implies that
\[
\lambda_{\min}(\omega(x,u)) \geq \tilde{\gamma}_k \lambda_{\max}(\omega(x,u)), \ x \in \overline{\Omega}
\]
and (1.19) follows therefore from (1.17) and the last inequality.

(iv) If \( u(x) \) is a strictly \((\delta, \tilde{\gamma}_k)\)-admissible solution to (3.1), then (1.20) and (1.21) follow from (1.17), (1.18), (1.19) and the last inequality.

\[\square\]

3.2. Proof of Theorem 1.10

Suppose \( u(x) \) is a strictly \((\delta, \tilde{\gamma}_k)\)-admissible solution to the problem (1.1)-(1.2), it is also a strictly \( \delta \)-admissible one. Since there exists a strictly \( \tilde{\gamma}_k \)-admissible subsolution \( \underline{u}(x) \) to (1.22)-(1.23), this function due to (2.5) is also a strictly \( \tilde{\gamma}_k \)-admissible subsolution to the problem (1.1)-(1.2). From the condition (v) it follows that the function \( u(x) \) is also a strictly \( \delta \)-admissible solution to (1.1)-(1.2). Therefore we can apply the comparison principle (Theorem 2.2) for \( u(x) \) and \( \underline{u}(x) \) to conclude that \( u(x) \geq \underline{u}(x) \) in \( \Omega \), \( \frac{\partial u}{\partial \nu} \geq \frac{\partial \underline{u}}{\partial \nu} \) on \( \partial \Omega \), where \( \nu \) is the unit inner normal to \( \partial \Omega \).

By using this fact and by following the same arguments as in [8], from the structure conditions for \( A(x,z,p) \), we can obtain the following estimates
\[
\sup_{\Omega} |u| \leq M_0, \quad \sup_{\Omega} |Du| \leq M_1,
\]
where \( M_0 \) depends on \( |u|_{0,\Omega}, |\varphi|_{0,\Omega} \) and \( M_1 \) depends on \( n, \gamma_0, |\underline{u}|_{1,\Omega}, |\varphi|_{2,\Omega} \) and \( \Omega \). We prove that there exists \( M_2 > 0 \) such that
\[
\sup_{\Omega} |D^2 u| \leq M_2. \quad (3.2)
\]

Indeed, since \( u(x) \) is a strictly \((\delta, \tilde{\gamma}_k)\)-admissible solution, then it follows from (1.21) that for any \( x \in \overline{\Omega} \) we have
\[
|\omega(x,u)| \leq \sqrt{n} \|\omega(x,u)\| = \sqrt{n} \lambda_{\max}(\omega(x,u)) \leq \frac{\sqrt{n}}{\tilde{\gamma}_k} \left[ \frac{1}{k} \right]^{\frac{1}{k}}. \quad (3.3)
\]
From the equality
\[
D^2 u = \omega(x,u) + A(x,u(x),Du(x)), \ x \in \overline{\Omega}
\]
and from (3.3), we obtain (3.2), where \( M_2 \) depends on \( n, k, \tilde{\gamma}_k, M_0, M_1, A(x,z,p) \) and \( f(x,z,p) \).

\[\square\]

3.3. Hölder continuity of the second-order derivatives inside the domain

PROPOSITION 3.1. Suppose \( A(x,z,p), f(x,z,p) \in C^3(\mathcal{D}), B(x,z,p) \in BC^3(\mathcal{D}), u(x) \in C^4(\Omega) \) is a strictly \((\delta, \tilde{\gamma}_k)\)-admissible solution to the problem (1.1)-(1.2). Then for any \( \Omega' \subset \subset \Omega \) there exist \( C'_4 > 0, 0 < \alpha < 1 \) such that
\[
\|D^2 u\|_{C^{2,\alpha}(\Omega')} \leq C'_4, \quad (3.4)
\]
where \( C'_4 \) and \( \alpha \) depend on \( n, k, \delta, \tilde{\gamma}_k, \beta_1, \Omega', A, f, \underline{u}, \mu_2(B) \).
First we prove the following lemma on uniform ellipticity of the equation \( \log \left(F_k(R(x,u))\right) = \log f(x,u,Du)\) at a \((\delta,\tilde{\gamma}_k)\)-admissible solution \(u(x)\) by improving (2.1).

**Lemma 3.2.** Suppose \(u(x)\) is a strictly \((\delta,\tilde{\gamma}_k)\)-admissible solution to the problem (1.1)-(1.2). Then for \(R = R(x,u) = [R_{ij}(x,u)]\) the following estimates are true for \(x \in \Omega\):

\[
k\left(\frac{n}{k} + \delta \frac{(1 + \delta^2)^{-2}}{\tilde{\gamma}_k}\right) \|\xi\|^2 \leq \frac{n}{k} \sum_{i,j=1}^{n} \left(\frac{\partial F_k(R)}{\partial R_{ij}} + \frac{\partial F_k(R)}{\partial R_{ji}}\right) \xi_i \xi_j \leq \frac{(n \delta^2 k)}{\tilde{\gamma}_k^2 f_0^{\frac{1}{2}}} \|\xi\|^2,
\]

where \(f_0, f_1\) are defined by (1.16).

**Proof.** Since \(R \in D_{\delta,\mu(B)}\), the inequalities (2.1) are true. Then (3.5) follows from (2.1), the relation \(\lambda_{\min}(\omega(x,u)) \geq \tilde{\gamma}_k \lambda_{\max}(\omega(x,u))\), \(x \in \Omega\) and from (1.20), (1.21).

**Proof of Proposition 3.1.** To prove the Hölder continuity inside \(\Omega\) for second-order derivatives of the solution \(u(x)\) we consider the equation \(F_k(R(x,u)) = \log f(x,u,Du)\) in \(\Omega\) and we can use the following already established facts:

(i) The \(C^2(\Omega)\)-estimates (1.24)

\[
\|u\|_{C^2(\Omega)} \leq C_3;
\]

(ii) The uniform ellipticity (3.5) of the equation (1.1) at \(R = R(x,u)\) for any \(x \in \Omega\);

(iii) The strict concavity (1.11) of \(F_k(\omega + \beta) = \log (S_k(\omega + \beta))\) as a function of \(\omega > 0\) when \(\beta^T = -\beta\) is fixed, i.e

\[
d^2F_k(R, P) \leq -\frac{C_1}{\lambda_{\max}(\omega)} |P|^2, \quad P^T = P, \quad C_1 > 0,
\]

where \(\lambda_{\max}(\omega(x,u))\) satisfies the estimates (1.21);

(iv) The following version (1.13) of the \(d\)-concavity of the function \(F_k(R)\) on the set \(D_{\delta,\mu(B),\tilde{\gamma}_k}\):

\[
F_k\left(R^{(1)}\right) - F_k\left(R^{(0)}\right) \leq \frac{\sum_{i,j=1}^{n} \frac{\partial F_k\left(R^{(0)}\right)}{\partial R_{ij}} \left(R_{ij}^{(1)} - R_{ij}^{(0)}\right) + C_2 \left|\beta^{(1)} - \beta^{(0)}\right|^2}{\lambda_{\min}(\omega(\tau))}\,
\]

where \(R^{(0)} = \omega^{(0)} + \beta^{(0)}, R^{(1)} = \omega^{(1)} + \beta^{(1)} \in D_{\delta,\mu(B),\tilde{\gamma}_k}, \omega(\tau) = (1 - \tau)\omega^{(0)} + \tau \omega^{(1)}, 0 < \tau < 1, \lambda_{\min}(\omega(x))\) satisfies the estimates (1.20).

Hence, the facts mentioned above and the methods of L.C. Evans and N.V. Krylov allow ones with the aid of (2.2)-(2.9) to get the desired Hölder continuity (3.4) of \(D^2u\) inside \(\Omega\) (see [10], Section 17.4).

\[\square\]
4. A new kind of the $k$-Hessian type equation in a neighborhood of the boundary

4.1. The $k$-compound of a square matrix

Let $M = [M_{ij}]$ be an $n \times n$ matrix with entries in $\mathbb{R}$ or $\mathbb{C}$. Suppose that $i_1 i_2 \cdots i_k$ and $j_1 j_2 \cdots j_k$ are indices such that

$$1 \leq i_1 < \cdots < i_k \leq n, \quad 1 \leq j_1 < \cdots < j_k \leq n.$$ 

We denote

$$M^{(k)}_{i_1 \cdots i_k, j_1 \cdots j_k} = \left[ M_{i_p j_q} \right]_{p,q=1}^k.$$

Then $\det \left( M^{(k)}_{i_1 \cdots i_k, j_1 \cdots j_k} \right)$ is a minor at the intersection of the rows $i_1, i_2, \cdots, i_k$ and the columns $j_1, j_2, \cdots, j_k$. When the indices $i_1 i_2 \cdots i_k$ are arranged in the lexical order, the resulting $\binom{n}{k} \times \binom{n}{k}$ square matrix, that consists of corresponding minors, is called the $k$-compound of the matrix $M$ and written as $M^{(k)}$. That means

$$M^{(k)} = \left[ \det \left( M^{(k)}_{i_1 \cdots i_k, j_1 \cdots j_k} \right) \right]_{\binom{n}{k} \times \binom{n}{k}}.$$

We list here some properties of the $k$-compounds.

**Proposition 4.1 ([1])**. Let $M$ and $N$ be matrices in $\mathbb{C}^{n \times n}$. Then the following assertions are true:

(i) Binet-Cauchy Theorem

$$(MN)^{(k)} = M^{(k)}N^{(k)};$$

(ii) $$(M^{(k)})^T = (M^T)^{(k)};$$

(iii) $$(M^{(k)})^* = (M^*)^{(k)};$$

(iv) $$(M^{(k)})^{-*} = (M^*)^{(k)};$$

(v) $M$ is non-singular if and only if $M^{(k)}$ is non-singular, and

$$[M^{(k)}]^{-1} = (M^{-1})^{(k)};$$

(vi) $$(hM)^{(k)} = h^kM^{(k)},$$ for any $h \in \mathbb{C};$

(vii) $M^{(k)}$ is symmetric if $M$ is symmetric;

(viii) If $M = \text{diag} (\lambda_1, \lambda_2, \cdots, \lambda_n) \in \mathbb{C}^{n \times n}$, then

$$M^{(k)} = \text{diag} (\lambda_{i_1}, \lambda_{i_2}, \cdots, \lambda_{i_k}; 1 \leq i_1 < \cdots < i_k \leq n).$$

(ix) If $M \in \mathbb{C}^{n \times n}$, then

$$S_k(M) = \sigma_k(\lambda(M)) = \text{Tr}(M^{(k)}) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \det \left( M^{(k)}_{i_1 \cdots i_k, i_1 \cdots i_k} \right).$$
4.2. Proof of Proposition 1.11

By using Proposition 1.11, we rewrite the left-hand side of (1.34) as follows

\[
S_k \left[ J^T (D^2 v - \bar{A}(y, v, Dv) - \bar{B}(y, v, Dv)) J \right] = S_k \left( J^T \bar{R} J \right) = \text{Tr} \left( (J^T \bar{R} J)^{(k)} \right)
\]

\[
= \text{Tr} \left( (J^T)^{(k)} (\bar{R})^{(k)} J^{(k)} \right) = \text{Tr} \left( J^{(k)} (J^T)^{(k)} (\bar{R})^{(k)} \right)
\]

\[
= \text{Tr} \left( (JJ)^{(k)} (\bar{R})^{(k)} \right)
\]

\[
= \sum_{1 \leq i_1 < \cdots < i_k \leq n} \det \left( (JJ)^{(k)}_{i_k \cdots i_1 j_1 \cdots j_k} \right) \det \left( (\bar{R})^{(k)}_{i_k \cdots i_1 j_1 \cdots j_k} \right)
\]

From (1.29) it follows

\[
JJ = \begin{bmatrix}
E_{n-1} & -(Dh)^T \\
-Dh & 1 + |Dh|^2
\end{bmatrix}.
\]  \tag{4.2}

Then (1.36) and the proposition 1.11 follow from (4.1) and the following lemma. \qed

**Lemma 4.2.** The entries of \((JJ)^{(k)}\) are of the following values:

(i) If \(1 \leq i_1 < \cdots < i_k \leq n - 1, 1 \leq j_1 < \cdots < j_k \leq n - 1\) then

\[
\det(JJ)^{(k)}_{i_1 \cdots i_k j_1 \cdots j_k} = \delta_{i_1 j_1} \delta_{i_2 j_2} \cdots \delta_{i_k j_k};
\]

(ii) If \(1 \leq i_1 < \cdots < i_{k-1} \leq n - 1, \) then

\[
\det(JJ)^{(k)}_{i_1 \cdots i_{k-1} n j_1 \cdots j_k} = \left( 1 + \sum_{m \in \{1, \cdots, n-1\} \setminus \{i_1, \cdots, i_{k-1}\} } (D_m h)^2 \right).
\]

(iii) If \(1 \leq i_1 < \cdots < i_k \leq n - 1, 1 \leq j_1 < \cdots < j_{k-1} < n = j_k\) then

\[
\det(JJ)^{(k)}_{i_1 \cdots i_k j_1 \cdots j_{k-1} n} = \det(JJ)^{(k)}_{i_1 \cdots i_k j_1 \cdots j_{k-1} n} = (-1)^{k-1} \sum_{m=1}^{k} (-1)^m (D_{i_m} h) \delta_{i_1 j_1} \cdots \delta_{i_{m-1} j_{m-1}} \delta_{i_m+1 j_m} \cdots \delta_{i_{k-1} j_{k-1}};
\]

(iv) If \(1 \leq i_1 < \cdots < i_{k-1} \leq n - 1, 1 \leq j_1 < \cdots < j_{k-1} \leq n - 1 \) with \((i_1, \cdots, i_{k-1}) \neq (j_1, \cdots, j_{k-1})\), then

\[
\det(JJ)^{(k)}_{i_1 \cdots i_{k-1} n j_1 \cdots j_{k-1} n} = \det(JJ)^{(k)}_{j_1 \cdots j_{k-1} n i_1 \cdots i_{k-1} n} = (-1)^{k-1} \sum_{m=1}^{k-1} (D_{i_m} h) (-1)^m \left[ \sum_{\ell=1}^{m-1} (-1)^\ell (D_{j_\ell} h) \delta_{i_1 j_1} \cdots \delta_{i_{\ell-1} j_{\ell-1}} \delta_{i_{\ell+1} j_{\ell+1}} \cdots \delta_{i_{m-1} j_{m-1}} \delta_{i_m+1 j_m} \cdots \delta_{i_{k-1} j_{k-1}} \right]
\]

\[+ (-1)^m (D_{j_m} h) \delta_{i_1 j_1} \cdots \delta_{i_{m-1} j_{m-1}} \delta_{i_m+1 j_m} \cdots \delta_{i_{k-1} j_{k-1}} \]

\[+ \sum_{\ell=m+1}^{k-1} (-1)^\ell (D_{j_\ell} h) \delta_{i_1 j_1} \cdots \delta_{i_{m-1} j_{m-1}} \delta_{i_m+1 j_m} \cdots \delta_{i_{k-1} j_{k-1}} \cdots \delta_{i_{k-1} j_{k-1}} \right].\]
Proof. First, we prove that if $1 \leq \ell \leq n$, $1 \leq i'_1 < i'_2 < \cdots < i'_\ell \leq n$, $1 \leq j'_1 < j'_2 < \cdots < j'_\ell \leq n$, then

$$
\det \begin{bmatrix}
\delta_{i'_1 j'_1} & \delta_{i'_1 j'_2} & \cdots & \delta_{i'_1 j'_\ell} \\
\delta_{i'_2 j'_1} & \delta_{i'_2 j'_2} & \cdots & \delta_{i'_2 j'_\ell} \\
\vdots & \vdots & \ddots & \vdots \\
\delta_{i'_\ell j'_1} & \delta_{i'_\ell j'_2} & \cdots & \delta_{i'_\ell j'_\ell}
\end{bmatrix} = \delta_{i'_1 j'_1} \cdots \delta_{i'_\ell j'_\ell}.
$$

(4.3)

Indeed, the determinant is not zero if and only if all the following conditions hold: there exists $j'_{m_1}$, $1 \leq m_1 \leq \ell$, such that $j'_{m_1} = i'_1$, there exists $j'_{m_2}$, $1 \leq m_2 \leq \ell$, $m_2 > m_1$ such that $j'_{m_2} = i'_2$, $\cdots$, there exists $j'_{m_\ell}$, $1 \leq m_\ell \leq \ell$, $m_\ell > m_p$, $1 \leq p \leq \ell - 1$ such that $j'_{m_\ell} = i'_\ell$. But these conditions hold if and only if

$$\{i'_1, i'_2, \cdots, i'_\ell\} = \{j'_1, j'_2, \cdots, j'_\ell\}.$$ By using (4.3), the entries of the matrix $(JJ^T)^{(k)}$ can be calculated directly from (4.2).

5. The $\tilde{d}$-concavity of the new kind of the $k$-Hessian type function and the $C^{2,\alpha}$ estimates

5.1. Proof of Proposition 1.12

Suppose $J$ and $J^T$ are defined by (1.29). We set

$$S = JJ^T = \begin{bmatrix}
E_{n-1} & -Dh \\
-(Dh)^T & 1 + |Dh|^2
\end{bmatrix}.$$ We denote the eigenvalues of $S$ as $s_1, \cdots, s_n$ with $s_1 \geq s_2 \geq \cdots \geq s_n$. One can verify that $s_2 = s_3 = \cdots = s_{n-1} = 1$ and

$$s_1 = \frac{(2 + |Dh|^2) + \sqrt{(2 + |Dh|^2)^2 - 4}}{2} \geq 1,$$

$$s_n = \frac{(2 + |Dh|^2) - \sqrt{(2 + |Dh|^2)^2 - 4}}{2} = \frac{1}{s_1}.$$ We have

$$\frac{1}{\sqrt{s_1}} \leq \|J\| = \|J^T\| = \sqrt{\|J^T J\|} = \sqrt{s_1}.$$ But $s_1, \cdots, s_n$ are also the eigenvalues of the matrix $S^{-1} = (J^T)^{-1} J^{-1}$. So we have

$$\frac{1}{\sqrt{s_1}} \leq \|J^{-1}\| = \|(J^T)^{-1}\| = \sqrt{s_1}.$$ Therefore, we can assume that the neighborhood $\tilde{\Omega}_p$ is chosen sufficiently small so that

$$\frac{1}{\sqrt{1 + \varepsilon}} \leq \|J\| = \|J^T\| \leq \sqrt{1 + \varepsilon},$$

(5.1)

$$\frac{1}{\sqrt{1 + \varepsilon}} \leq \|J^{-1}\| = \|(J^T)^{-1}\| \leq \sqrt{1 + \varepsilon},$$

(5.2)
where $\varepsilon > 0$ is sufficiently small.

Since
\[ \tilde{\omega}(y, v) = D^2 v - \tilde{A}(y, v, Du) = J^{-1}\omega(x, u)(J^{-1})^T, \]
\[ \tilde{B}(y, v, Du) = J^{-1}B(x, u, Du)(J^{-1})^T, \]
from (5.2) we have
\[ \frac{1}{(1 + \varepsilon)} \lambda_u \leq \lambda_v \leq (1 + \varepsilon) \lambda_u, \]
\[ \frac{1}{(1 + \varepsilon)} \mu(B) \leq \mu(\tilde{B}). \leq (1 + \varepsilon) \mu(B). \]

Suppose $u(x)$ is a strictly $(\delta, \tilde{\gamma}_k)$-admissible solution, i.e.
\[ \mu(B) \leq \delta \lambda_u, \]
\[ \lambda_{\min}(\omega(x, u)) \geq \tilde{\gamma}_k \lambda_{\max}(\omega(x, u)), \ x \in \overline{\Omega}. \] (5.3)

From (5.1)-(5.3) we obtain
\[ \mu(\tilde{B}) \leq (1 + \varepsilon)^2 \delta \lambda_v, \]
\[ \lambda_{\min}(\tilde{\omega}(y, v)) \geq \frac{\tilde{\gamma}_k}{(1 + \varepsilon)^2} \lambda_{\max}(\tilde{\omega}(y, v)). \]

So, $v(y)$ is a strictly $(\overline{\delta}, \overline{\tilde{\gamma}}_k)$-admissible solutions to (1.34), where
\[ \overline{\delta} = (1 + \varepsilon)^2 \delta, \overline{\tilde{\gamma}}_k = \frac{\tilde{\gamma}_k}{(1 + \varepsilon)^2}. \]

Since $0 < \gamma_k < \tilde{\gamma}_k < 1$, where $\gamma_k$ is defined in Definition 1.3 and $0 < \delta < \delta_k < 1$, $\delta_k$ is determined in Theorem 1.5, we can assume that $\varepsilon$ is chosen sufficiently small so that
\[ 0 < \gamma_k < \tilde{\gamma}_k < 1, \quad 0 < \overline{\delta} < \delta_k < 1. \]

We prove now that the function $\tilde{F}_k(\tilde{R})$, defined by (1.38), is $\overline{\delta}$-concave in the sense of (1.13). We rewrite (1.38) as follows
\[ \tilde{F}_k \left( y', \tilde{R} \right) = \log \left[ S_k(\tilde{R}) + H_k \left( y', \tilde{R} \right) \right], \]
where $H_k(y', \tilde{R})$ is homogeneous of degree $k$ with respect to $\tilde{R} = \left[ \tilde{R}_{ij} \right]_{n \times n}$. \hfill (5.4)

Suppose $\tilde{R} = \tilde{\omega} + \tilde{\beta} \in D_{\sigma, \mu(B), \tilde{\gamma}_k}$. Then we have
\[ \tilde{R} = C^{-1} \tilde{D} C + \tilde{\beta} = C^{-1} \left( \tilde{D} + C \tilde{\beta} C^{-1} \right) C = C^{-1} (\tilde{D} + \tilde{\sigma}) C, \]
where $\tilde{D} + \tilde{\sigma} \in D_{\delta, \mu(B)}$. So we can assume that $\tilde{R} = \tilde{D} + \tilde{\sigma}$, $\tilde{D} = \text{diag}(\tilde{\lambda}_1, \tilde{\lambda}_2, \ldots, \tilde{\lambda}_n) > 0$,
\[ \tilde{\lambda}_{\min} \geq \tilde{\gamma}_k \tilde{\lambda}_{\max}, \ ||\tilde{\sigma}|| \leq \mu(\tilde{B}) \leq \overline{\delta} \tilde{\lambda}_{\min}. \]

From (5.4) we have
\[ \frac{\partial \tilde{F}_k(y', \tilde{R})}{\partial R_{ij}} = \frac{1}{(S_k(\tilde{R}) + H_k(y', \tilde{R}))} \frac{\partial (S_k(\tilde{R}) + H_k(y', \tilde{R}))}{\partial \tilde{R}_{ij}}, \]

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Then, for $\tilde{M} = [\tilde{M}_{ij}]_{n \times n} \in \mathbb{R}^{n \times n}$:

$$d^2 \tilde{F}_k (y', \tilde{R}, \tilde{M}) = -\frac{1}{(S_k + H_k)^2} \left[ dS_k (\tilde{R}, \tilde{M}) + dH_k (y', \tilde{R}, \tilde{M}) \right]^2$$

$$+ \frac{1}{(S_k + H_k)} \left[ d^2 S_k (\tilde{R}, \tilde{M}) + d^2 H_k (y', \tilde{R}, \tilde{M}) \right].$$

(5.5)

We have the following relations:

$$\frac{1}{S_k} = \frac{1}{S_k + H_k} - \frac{H_k}{S_k (S_k + H_k)},$$

$$\frac{1}{(S_k + H_k)^2} = \frac{1}{S_k^2} - \frac{2}{S_k (S_k + H_k)} + \frac{H_k^2}{S_k (S_k + H_k)^2},$$

(5.6)

$$S_k (\tilde{R}) \geq S_k (\tilde{D}) \geq \left( \frac{n}{k} \delta_{\min} \right)^k \geq \left( \frac{n}{k} \delta_{\tilde{y}} \right)^k (\tilde{\lambda}_{\max})^k.$$  

(5.7)

Since the function $H_k (y, \tilde{R})$ is a linear combination of $\det \left( \tilde{R}^{(k)}_{ij} \right)$ with coefficients, that are polynomials with respect to $Dh(y')$ of degree at the least 1 and at the most 2, $Dh(y')$ is small, and

$$|\tilde{R}_{ij}| \leq \delta_{ij} \tilde{\lambda}_{\max} + |\tilde{\sigma}_{ij}| \leq \delta_{ij} \tilde{\lambda}_{\max} + \sqrt{n} \delta \tilde{\lambda}_{\min} \leq (1 + \sqrt{n} \delta) \tilde{\lambda}_{\max},$$

we have

$$\left| H_k (y', \tilde{R}) \right| \leq C_7 |Dh(y')| (\tilde{\lambda}_{\max})^k,$$

(5.8)

$C_7 > 0$ and does not depend on $y'$.

From (5.7) and (5.8) we can assume that the neighborhood $\tilde{\Omega}_{\rho}$ is chosen small so that for any $|y'| \leq \rho$

$$S_k (\tilde{R}) + H_k (y', \tilde{R}) \geq C_8 (\tilde{\lambda}_{\max})^k,$$

(5.9)

$C_8 > 0$ and does not depend on $y'$.

It follows from (5.5)–(5.6) that

$$d^2 \tilde{F}_k (y', \tilde{R}, \tilde{M}) = d^2 ( \log S_k ) (\tilde{R}, \tilde{M})$$

$$+ \sum_{i,j,\ell,m=1}^n \left[ \sqrt{\tilde{\lambda}_i \tilde{\lambda}_j \tilde{\lambda}_\ell \tilde{\lambda}_m} g_{ij,\ell,m}(y', \tilde{R}) \right] \left( \frac{\tilde{M}_{ij}}{\sqrt{\tilde{\lambda}_i \tilde{\lambda}_j}} \right) \left( \frac{\tilde{M}_{\ell m}}{\sqrt{\tilde{\lambda}_\ell \tilde{\lambda}_m}} \right),$$

(5.10)
where \( g_{ij,\ell m}(y', \bar{R}) \) are homogeneous of degree \((-2)\) with respect to \( \bar{R} \). From (5.5)-(5.9) we can assume that for any \( i, j, \ell, m \)

\[
\sup_{\tilde{\lambda}_{\min} \leq \tilde{\lambda} \leq \tilde{\lambda}_{\max}} \left| \sqrt{\tilde{\lambda}_i \tilde{\lambda}_j \tilde{\lambda}_\ell \tilde{\lambda}_m} g_{ij,\ell m}(y', \bar{D} + \bar{\sigma}) \right| \leq C_9 |Dh(y')|,
\]

\( C_9 > 0 \) and does not depend on \( y', i, j, \ell, m \).

We know from Theorem 1.5 that for the function \( F_k(\bar{R}) = \log(S_k(\bar{R})) \) when \( \bar{R} \in D_{\delta, \mu(\bar{B}), \tilde{\gamma}_k} \), where \( 0 < \gamma_k < \tilde{\gamma}_k < 1 \), \( 0 < \bar{\delta} < \delta_k < 1 \), the estimates (1.11), (1.12) hold, i.e.

\[
d^2 F_k(\bar{R}, \bar{P}) \leq -C_1 \frac{\bar{P}^2}{\tilde{\lambda}_{\max}}, \bar{P}^T = \bar{P},
\]

\[
d^2 F_k(\bar{R}, \bar{P} + \bar{Q}) \leq C_2 \frac{\bar{Q}^2}{\tilde{\lambda}_{\min}}, \bar{Q}^T = -\bar{Q}.
\]

(5.11)

From (5.10)-(5.11) it follows that if we choose \( \bar{\Omega}_\rho \) sufficiently small, then we have the following estimates for any \( |y'| \leq \rho \)

\[
d^2 \bar{F}_k \left( y', \tilde{\bar{R}}, \tilde{\bar{P}} \right) \leq -C_{10} \frac{\tilde{\bar{P}}^2}{\tilde{\lambda}_{\max}(\tilde{\bar{\omega}})}, \tilde{\bar{P}}^T = \tilde{\bar{P}},
\]

(5.12)

\[
d^2 \bar{F}_k \left( y', \tilde{\bar{R}}, \tilde{\bar{P}} + \tilde{\bar{Q}} \right) \leq C_{11} \frac{\tilde{\bar{Q}}^2}{\tilde{\lambda}_{\min}(\tilde{\bar{\omega}})}, \tilde{\bar{Q}}^T = -\tilde{\bar{Q}}^T,
\]

(5.13)

where \( C_{10} > 0, C_{11} > 0 \) depend on \( C_1, C_2, C_9, \rho, Dh \) and do not depend on \( y' \) and \( \mu(\bar{B}) \). From (5.13) it is easy to obtain the following version of \( \tilde{\bar{d}} \)-concavity for the function \( \bar{F}_k(y', \bar{R}) \) on the set \( D_{\tilde{\bar{\delta}}, \mu(\bar{B}), \tilde{\gamma}_k} \) :

\[
\bar{F}_k \left( y', \tilde{\bar{R}}^{(1)} \right) - \bar{F}_k \left( y', \tilde{\bar{R}}^{(0)} \right) \leq \sum_{i,j=1}^n \frac{\partial \bar{F}_k \left( y', \tilde{\bar{R}}^{(0)} \right)}{\partial \tilde{\bar{R}}_{ij}} (\tilde{\bar{R}}^{(1)}_{ij} - \tilde{\bar{R}}^{(0)}_{ij})
\]

\[
+ C_{11} \frac{|\tilde{\bar{\beta}}^{(1)} - \tilde{\bar{\beta}}^{(0)}|^2}{\tilde{\lambda}_{\min}(\tilde{\bar{\omega}})}, \quad 0 < \tau < 1,
\]

\( \text{where} \ |y'| \leq \rho, \bar{R}^{(0)} = \tilde{\bar{\omega}}^{(0)} + \tilde{\bar{\beta}}^{(0)}, \tilde{\bar{R}}^{(1)} = \tilde{\bar{\omega}}^{(1)} + \tilde{\bar{\beta}}^{(1)} \in D_{\tilde{\bar{\delta}}, \mu(\bar{B}), \tilde{\gamma}_k}, \tilde{\bar{\omega}}^{(\tau)} = (1-\tau)\tilde{\bar{\omega}}^{(0)} + \tau \tilde{\bar{\omega}}^{(1)} \).

5.2. The Hölder continuity of \( D^2 v(y) \)

Since \( \|u(x)\|_{C^2(\bar{\Omega})} \leq C_3 \) and \( v(y) = u(x) - \varphi(x) \), where \( y \) and \( x \) are related by (1.27), \( h(y') \in C^4 \), then we have

\[
\|v(y)\|_{C^2(\bar{\Omega}_\rho)} \leq C_{12}.
\]

(5.15)

From (1.30) it follows that

\[
\|J^{-1}(y')\|_{C^2(|y'| \leq \rho)} \leq C'_{12}.
\]

(5.16)
From (5.15), (5.16) and (1.33) we obtain
\[ \|A(y, v(y), Dv(y))\|_{C^2(\Omega)} + \|B(y, v(y), Dv(y))\|_{C^2(\Omega)} \leq C_{12}, \]
where \( C_{12}, C_2 \) are uniformly bounded when \( 0 < \rho \leq \rho_0 \).

The matrix \( \frac{1}{2} \left( \frac{\partial F_k(R)}{\partial R_{ij}} + \frac{\partial F_k(R)}{\partial R_{ji}} \right) \) satisfies the ellipticity conditions (3.5). But, by definition
\[ \bar{F}_k(l', \bar{R}) = F_k(R) = F_k(J^{-1}\bar{R}(J^{-1})^T), \]
where \( \frac{1}{\sqrt{1+\varepsilon}} \leq \|J^{-1}\| = \|(J^{-1})^T\| \leq \sqrt{1+\varepsilon} \), it follows from (3.5) that for any \( |l'| \leq \rho \)
\[ C_{13}|\xi|^2 \leq \frac{1}{2} \sum_{i,j=1}^n \left( \frac{\partial \bar{F}_k(l', \bar{R})}{\partial R_{ij}} + \frac{\partial \bar{F}_k(l', \bar{R})}{\partial R_{ji}} \right) \xi_i \xi_j \leq C_{14}|\xi|^2, \]
where \( C_{13}, C_{14} \) depend on \( n, k, \delta, \bar{\gamma}_k, f_0, f_1, \varepsilon \) and do not depend on \( l' \). We have just proved above the strict concavity (5.12) of the function \( \bar{F}_k \) \( (l', \bar{R} + \bar{\beta}) \) when \( \bar{\beta} \) fixed and the \( \bar{d} \)-concavity (5.13) of \( \bar{F}_k \) \( (l', \bar{R}) \) on the set \( \bar{D}_{\delta, \mu(\bar{R}), \bar{\gamma}_k}, \rho \). From the facts listed above with the aid of (5.15)-(5.17) and (2.2)-(2.9), applied for \( \bar{R}(y, v) \) and \( \bar{F}_k(\bar{R}) \), one can prove (10), Section 17.8) that from the equation \( \bar{F}_k(l', \bar{R}(y, v)) = \log \bar{f}(y, v, Dv) \) in \( \Omega_\rho, v(y)|_{y_\rho} = 0 \) it follows
\[ \|D^2v(y)\|_{C^\alpha(\Omega_\rho)} \leq C_{15}, \]
where \( C_{15} > 0, 0 < \alpha < 1 \) do not depend on \( \mu(\bar{B}) \). From the last inequality and (1.27), (1.28), we have:
\[ \|D^2u\|_{C^\alpha(\Omega_\rho)} \leq C_{15}. \]

5.3. Proof of Theorem 1.13

In Sections 3 and 5 we have obtained the following estimates for a strictly \( (\delta, \bar{\gamma}_k) \)-admissible solution \( u(x) \) to the problem (1.1)-(1.2):
\[ \|u\|_{C^2(\Omega)} \leq C_3, \]
\[ \|D^2u\|_{C^\alpha(\Omega')} \leq C_5, \quad \Omega' \subset \subset \Omega, \]
\[ \|D^2u\|_{C^\alpha(\Omega_\rho)} \leq C_{15}', \quad \Omega_\rho = B_\rho(x) \cap \Omega, \ x \in \partial \Omega \]
From these estimates it follows the desired inequality (1.39):
\[ \|u\|_{C^{2, \alpha}(\Omega_\rho)} \leq C_4, \]
where \( 0 < \alpha < 1, C_4 \) depend on \( n, k, \delta, \bar{\gamma}_k, \beta_1, \Omega, A(x, z, p), f(x, z, p), w(x), \varphi, \mu_2(B) \).
6. The solvability of the Dirichlet problem

6.1. A necessary condition for the existence of a strictly $\delta$-admissible solution

We give here proof for Theorem 1.14. Suppose there exists a strictly $\delta$-admissible solution $u(x)$ to the equation (1.1) and it satisfies the conditions (i),(ii) and (iii) of the theorem.

Since $S_k(\omega(x, u) - B(x, u, Du)) \geq S_k(\omega(x, u))$, from (1.1) it follows that $S_k(\omega(x, u)) \leq f(x, u, Du)$. But $\omega(x, u) > 0$, $S_k(\omega(x, u)) \geq \left(\frac{n}{k}\right) \lambda_{\text{min}}(\omega(x, u))$ and $\mu(B) \leq \delta \lambda_u \leq \delta \lambda_{\text{min}}(\omega(x, u))$, then we have

$$\mu(B) \leq \delta \left[\frac{f(x, u, Du)}{\left(\frac{n}{k}\right)}\right]^\frac{1}{k}$$

and consequently

$$\mu(B) \leq \delta \left[f_1\left(\frac{n}{k}\right)\right]^\frac{1}{k},$$

where $f_1$ is defined by (1.16). The inequality (1.40) is proved. \(\square\)

6.2. Some sufficient conditions for unique existence of the strictly $(\delta, \tilde{\gamma}_k)$-admissible solution

We prove here Theorem 1.15 on the unique solvability of the problem (1.1)-(1.2) in the class of strictly $(\delta, \tilde{\gamma}_k)$-admissible solutions that belong to $C^{2,\alpha}(\Omega)$ for some $0 < \alpha < 1$. The uniqueness follows from the comparison principle, Theorem 2.2.

Suppose $B(x, z, p)$ satisfies (1.42), (1.43). By using the method of continuity ([10], Section 17.2) we will prove the existence of strictly $(\delta, \tilde{\gamma}_k)$-admissible solution $u(x)$ to the problem (1.1)-(1.2).

Since $S_k(R(x, u)) \geq S_k(\omega(x, u))$, it follows from the conditions (iii) and (iv) that the function $u(x)$ is also strictly $(\delta, \tilde{\gamma}_k)$-admissible subsolution to the problem (1.1)-(1.2). Now for each $t \in [0, 1]$ we consider the following Dirichlet problem:

$$S_k \left[D^2 u^{(t)} - A(x, u^{(t)}, Du^{(t)}) - B(x, u^{(t)}, Du^{(t)})\right] = f^{(t)}(x, u^{(t)}, Du^{(t)}) \text{ in } \Omega, \quad (6.1)$$

$$u^{(t)} = \varphi \text{ on } \partial \Omega, \quad (6.2)$$

where

$$f^{(t)}(x, z, p) = f(x, z, p)e^{(1-t)G[w](x)}, \quad (6.3)$$

$$G[w](x) = \log (S_k(R(x, w))) - \log f(x, w, Dw). \quad (6.4)$$

From (6.1)-(6.4) it follows that the function $u^{(0)} = \underline{u}(x)$ is the solution to the problem (6.1)-(6.2) with $t = 0$ and if the function $u^{(1)}(x)$ is solution to the problem (6.1)-(6.2) when $t = 1$, then $u(x) = u^{(1)}(x)$ is a solution to the problem (1.1)-(1.2).

To study the problem (6.1)-(6.2), for $\varepsilon > 0$ we introduce a class $U^{(\varepsilon)} = U(k, n, \delta, \tilde{\gamma}_k, \varepsilon, \beta_1, B)$ that consists of functions $u(x) \in C^2(\Omega)$, that satisfy the following conditions:

$$\lambda_u > 0, \quad (6.5)$$

$$\mu(B) < \delta \lambda_u, \quad (6.6)$$

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\[ \mu(D_zB) < \beta_1 \lambda_u, \quad (6.7) \]

\[ \gamma_u > \gamma_k + \varepsilon, \quad (6.8) \]

where \( \lambda_u \) and \( \gamma_u \) are defined by (1.6) and (1.15) respectively.

We note that if \( \varepsilon_1 > \varepsilon_2 > 0 \) then \( U^{(\varepsilon_1)} \subset U^{(\varepsilon_2)} \). It is obvious that \( U^{(\varepsilon)} \) is open in \( C^2(\Omega) \).

From (1.41)-(1.43) it follows that \( u(x) \in U^{(\varepsilon_0)} \). If \( u^{(t)} \in U^{(\varepsilon)} \) and it is a solution to the problem (6.1)-(6.2), then from (6.5)-(6.8) we see that it is also a strictly \((\delta, \gamma_k)\)-admissible solution.

**Lemma 6.1.** Assume that all conditions of Theorem 1.15 are fulfilled. Then, \( A(x, z, p), B(x, z, p), f^{(t)}(x, z, p) \) satisfy all assumptions of Proposition 1.9 and of Theorems 1.10, 1.13 and the function \( \underline{u}(x) \) is a strictly \((\delta, \gamma_k)\)-admissible subsolution to all problems (6.1)-(6.2).

**Proof.** Since \( f^{(t)}(x, z, p) = f(x, z, p)e^{(1-t)G[\underline{u}](x)} \), then \( f^{(t)}(x, z, p) > 0 \) and

\[ \inf_D \left[ \frac{D_zf^{(t)}(x, z, p)}{f^{(t)}(x, z, p)} \right] = \inf_D \left[ \frac{D zf(x, z, p)}{f(x, z, p)} \right] \geq \frac{k\delta}{(1+\delta^2)^2}\beta_1. \]

We show that the function \( \underline{u}(x) \) is strictly \((\delta, \gamma_k)\)-admissible subsolution to all equations (6.1).

Indeed, since \( G[\underline{u}](x) \geq 0 \), we have for \( 0 \leq t \leq 1 \):

\[ S_k(R(x, \underline{u})) = f(x, \underline{u}, D\underline{u}) \cdot \frac{S_k(R(x, \underline{u}))}{f(x, \underline{u}, D\underline{u})} = f(x, \underline{u}, D\underline{u})e^{G[\underline{u}](x)} \geq f(x, \underline{u}, D\underline{u})e^{(1-t)G[\underline{u}](x)} = f^{(t)}(x, \underline{u}, D\underline{u}). \]

Moreover, from (1.42), (1.43), (6.6), (6.7) it follows that

\[ \mu(B) < \delta \min(\lambda_u, \lambda_\underline{u}), \]

\[ \mu(D_zB) < \beta_1 \min(\lambda_u, \lambda_\underline{u}). \]

□

**Corollary 6.2.** Suppose \( u^{(t)} \) is a strictly \((\delta, \gamma_k)\)-admissible solution to the problem (6.1)-(6.2). Then there exist \( M_0 > 0, M_1 > 0, 0 < \alpha < 1, C_4 > 0 \) that depend on \( n, k, \delta, \gamma_k, \beta_1, \Omega, A(x, z, p), f(x, z, p), \underline{u}(x), \varphi, \mu_2(B) \), and do not depend on \( t \) such that

\[ \sup_{\Omega} |u^{(t)}| \leq M_0, \quad \sup_{\Omega} |Du^{(t)}| \leq M_1, \]

\[ \|u^{(t)}\|_{C^{2,\alpha}(\Omega)} \leq C_4, \quad (6.9) \]

\[ \lambda_{u^{(t)}} \geq \gamma_k \left[ \frac{(1+\delta^2)^{-\frac{1}{2}}}{(n_k)} f_0 \right]^\frac{1}{2}. \quad (6.10) \]
Here, to get (6.10) we have used from (1.15) the fact that
\[ f_0 = \inf_{x \in \Omega, |x| \leq M_0, |p| \leq M_1} f(x, z, p) \leq \inf_{x \in \Omega, |x| \leq M_0, |p| \leq M_1} f(x, z, p)e^{(1-t)G[u](x)}. \]

We rewrite the problem (6.1)-(6.2) as follows
\[ \log (S_k (R (x, u(t)))) - \log f (x, u(t), Du(t)) = (1 - t)G[u](x) \text{ in } \Omega, \quad (6.11) \]
\[ u(t) = \phi \text{ on } \partial \Omega. \quad (6.12) \]

We consider the operator:
\[ G[u](x) : C^{2,\alpha} (\Omega) \to C^{0,\alpha} (\Omega), \]
where \( G[u](x) \) is defined by (6.4), which is connected to the left-hand side of (6.11) and \( 0 < \alpha < 1 \) is the same as in (6.9).

**Lemma 6.3.** Suppose \( u \in C^{2,\alpha} (\Omega) \) is a strictly \( (\delta, \gamma_k) \)-admissible solution to the problem (6.11)-(6.12). Then the operator \( G[u](x) \) is Frechet continuously differentiable at \( u \) and its differential \( G_u \) is defined as follows
\[ G_u : C^{2,\alpha}_0 (\Omega) \to C^{0,\alpha} (\Omega), \]
where \( C^{2,\alpha}_0 (\Omega) = \{ h \in C^{2,\alpha} (\Omega) : h = 0 \text{ on } \partial \Omega \} \).

\[ G_u(h) = \sum_{i,j=1}^n a^{ij}(x) D_{ij} h + \sum_{i=1}^n b^i(x) D_i h + c(x) h, \]

\[ a^{ij}(x) = \frac{1}{2} \left[ F^{ij}[u](x) + F^{ji}[u](x) \right], \quad i, j = 1, \ldots, n, \]

\[ b^i(x) = - \sum_{\ell, m=1}^n F^{\ell m}[u](x) D_{\ell} (A_{\ell m} + B_{\ell m})(x, u, Du) - \left( \frac{D_{\ell f}}{f} \right) (x, u, Du), \quad i = 1, \ldots, n, \]

\[ c(x) = - \sum_{\ell, m=1}^n F^{\ell m}[u](x) D_{\ell} (A_{\ell m} + B_{\ell m})(x, u, Du) - \left( \frac{D_{z f}}{f} \right) (x, u, Du), \]

\[ F^{ij}[u](x) = \frac{\partial F_k(R(x, u))}{\partial R_{ij}}, \quad F_k(R) = \log(S_k(R)). \]

The operator \( G_u(h) \) is uniformly elliptic on \( \Omega \), all the coefficients \( a^{ij}, b^i, c \) are from \( C^{0,\alpha} (\Omega) \) and \( c(x) \leq 0 \). Moreover, it is invertible.

**Proof.** Since \( u(x) \in C^{2,\alpha}(\Omega) \) and it is a strictly \( (\delta, \gamma_k) \)-admissible solution to the problem (6.1)-(6.2), then the uniform ellipticity of the operator \( G_u(h) \) follows from (3.5). Due to \( A(x, z, p), B(x, z, p), f(x, z, p) \in C^3(\mathcal{D}), u(x) \in C^{2,\alpha} (\Omega) \), then the coefficients \( a^{ij}(x), b^i(x), c(x) \) are from \( C^{0,\alpha} (\Omega) \). As in the proof of the comparison principle, from the assumptions on \( D_z A, D_z B \) and \( D_z f \), the assertion \( c(x) \leq 0 \) can be verified, from which it follows that \( G_u \) is invertible. \( \square \)
We rewrite the problem (6.11)-(6.12) in the form
\[ H(u^{(t)}, t) = 0 \text{ in } \Omega, \quad u^{(t)} = \phi \text{ on } \partial \Omega \] (6.13)
where \( H : C^{2,\alpha}(\Omega) \times [0, 1] \to C^{0,\alpha}(\Omega) \),
\[ H(u^{(t)}, t) = G[u^{(t)}](x) - (1 - t)G[u](x). \] (6.14)

We consider a set of solutions to the problem (6.13) as follows
\[ \mathcal{V}(\varepsilon) = \mathcal{U}(\varepsilon) \cap C^{2,\alpha}(\Omega), \]
where \( 0 < \alpha < 1 \) as in (6.9), fixed and is the same for all \( u^{(t)}, 0 \leq t \leq 1 \).

We introduce the following set
\[ I = \{ t \in [0, 1] : \exists u^{(t)} \in \mathcal{V}(\varepsilon), \varepsilon = \varepsilon(u^{(t)}) > 0, H(u^{(t)}, t) = 0, u^{(t)} = \phi \text{ on } \partial \Omega \}. \]
The solvability of the problem (6.11)-(6.12) is equivalent to the fact that \( t \in I \). When \( t = 0 \) the function \( u^{(0)} = u \) is a solution to (6.13), i.e.
\[ H(u^{(0)}, 0) = 0. \]

This means that \( t = 0 \in I \) and \( I \neq \emptyset \). The following lemma shows that \( I \) is "open".

**Lemma 6.4.** Suppose \( t' \in I \),
\[ u^{(t')} \in \mathcal{V}(\varepsilon'), \] (6.15)
and \( \varepsilon' > \varepsilon'' > 0 \). Then there exists \( \tau' > 0 \) such that \([t', t' + \tau'] \subset I\) and
\[ u^{(t)} \in \mathcal{V}(\varepsilon'') \] (6.16)
for any \( t \in [t', t' + \tau'] \). Moreover, all \( u^{(t)} \) are in some \( C^{2,\alpha}(\Omega) \)-neighborhood of \( u^{(t')} \) and \( u^{(t)} \) is continuous mapping from \([t', t' + \tau']\) to \( C^{2,\alpha}(\Omega) \).

**Proof.** From (6.14) and Lemma 6.3 it follows that the derivative \( H_{u^{(t')}} = G_{u^{(t')}} \) is invertible, so we can apply the implicit function Theorem to conclude that there exist \( \tau' > 0 \) and continuous mapping \( u^{(t)} \) from \([t', t' + \tau']\) to \( C^{2,\alpha}(\Omega) \) such that
\[ H(u^{(t)}, t) = 0, \ t \in [t', t' + \tau'], \ u^{(t)} = \phi \text{ on } \partial \Omega. \]
We have
\[ \lambda_{\min}(\omega(x, u)) = \inf_{|\xi| = 1} \sum_{i,j=1}^n D_{x_i x_j} u(x) - A_{ij}(x, u, Du) \xi_i \xi_j, \]
\[ \lambda_{\max}(\omega(x, u)) = \sup_{|\xi| = 1} \sum_{i,j=1}^n D_{x_i x_j} u(x) - A_{ij}(x, u, Du) \xi_i \xi_j \]
and \( A(x, z, p) \in C^3(D) \). So from (6.15), (1.15) and \( \varepsilon' > \varepsilon'' > 0 \) it follows that, if there is a necessity, we may decrease \( \tau \) in that way so that (6.16) is satisfied.

The desired conclusion of the theorem will be derived from the following lemma.
Lemma 6.5. The assertion

\[ I = [0, 1] \]

is true.

Proof. We now apply consecutively Lemma 6.4. Since \( u^{(0)} = u \in V^{(\varepsilon_0)} \), \( \varepsilon'' = \varepsilon_1 = \varepsilon_0 - \frac{\varepsilon_0}{4} < \varepsilon_0 = \varepsilon' \), then for \( t' = 0 \) there exists \( \tau_1 > 0 \) such that if \( t_1 = t' + \tau_1 \), then \( [0, t_1] \subset I \) and \( u^{(t)} \in V^{(\varepsilon_1)} \), for any \( t \in [0, t_1] \). Now we choose \( t' = t_1 \) and \( \varepsilon'' = \varepsilon_2 = \varepsilon_0 - \left( \frac{\varepsilon_0}{4} + \frac{\varepsilon_0}{8} \right) < \varepsilon_1 = \varepsilon' \).

Then there exists \( \tau_2 > 0 \) such that if \( t_2 = t_1 + \tau_2 \) then \( [t_1, t_2] \subset I \) and \( \varepsilon' \). Then from (6.19), (6.20) we obtain

\[ \| u^{(t_m)} \|_{C^2,\alpha(\Omega)} \leq C, \]  

(6.17)

From (6.9), (6.10), (6.11) we have

\[ \| u^{(t_m)} \|_{C^2,\alpha(\Omega)} \leq C, \]  

(6.18)

\[ \lambda_{u^{(t_m)}} \geq \tilde{\gamma}_k \left[ \frac{1 + \delta^2 \cdot \left( \frac{n}{k} \right)}{f_0} \right]^{\frac{1}{4}}, \]  

(6.19)

and

\[ G \left[ u^{(t_m)} \right] (x) - (1 - t_m) G [u] (x) = 0. \]  

(6.20)

From (6.18) it follows that there exist \( \{ t_{m'} \} \subset \{ t_m \} \) and \( u(x) \in C^{2,\alpha}(\Omega) \) such that \( t_{m'} \to t^* \), \( u^{(t_{m'})} \to u(x) \) as \( m' \to \infty \) in \( C^{2,\alpha}(\Omega) \).

Then from (6.19), (6.20) we obtain

\[ \lambda_{u} \geq \tilde{\gamma}_k \left[ \frac{1 + \delta^2 \cdot \left( \frac{n}{k} \right)}{f_0} \right]^{\frac{1}{4}}, \]  

(6.21)

\[ G [u] (x) - (1 - t^*) G [u] (x) = 0. \]  

(6.22)

But from (6.17) we have

\[ \lambda_{u} \geq \tilde{\gamma}_k + \frac{\varepsilon_0}{2} > \tilde{\gamma}_k + \frac{\varepsilon_0}{4}. \]  

(6.23)

We will verify the conditions (6.6), (6.7).
From (1.42), (1.43) and (6.21) it follows that
\[ \mu(B) < \delta \lambda_u, \]  
(6.24)

\[ \mu(D_z B) < \beta_1 \lambda_u. \]  
(6.25)

Therefore, the conditions (6.6), (6.7) are satisfied. Since \( u(x) \in C^{2,\alpha}(\Omega) \), from (6.22)-(6.25) it follows that \( u(t^*) = u \in \mathcal{V}(\frac{\alpha}{4}) \) and \( t^* \in I \).

The case \( t^* < 1 \) is impossible, because if \( t^* < 1 \) then we can apply again Lemma 6.4 with \( t' = t^*, \varepsilon' = \frac{\alpha}{4}, \varepsilon'' = \frac{\alpha}{8} < \varepsilon' \) and deduce that there exists \( \tau > 0 \) such that \([t^*, t^* + \tau] \subset I \).

Hence \( t^* = 1 \) and the function \( u(x) = u^{(1)} \in \mathcal{V}(\frac{\alpha}{4}) \)

is a strictly \((\delta, \tilde{\gamma}_k + \frac{\alpha}{4})\)-admissible solution to the Dirichlet problem (1.1)-(1.2). The lemma and Theorem 1.15 are proved.

**Remark 6.6 (On simplified sufficient conditions).** Since \( u^{(0)} = u \), from (6.10) it follows that
\[ \lambda_u \geq \tilde{\gamma}_k \left[ \frac{1 + \delta^2 - \lfloor \frac{1}{k} \rfloor}{\left( \frac{n}{k} \right)} \right] f_0 \]  
(6.26)

where \( \lambda_u \) is defined by (1.44).

Then, from (6.26), (1.42), (1.43) we deduce that for the existence of strictly \((\delta, \tilde{\gamma}_k)\)-admissible solution to the problem (1.1)-(1.2), the matrices \( B(x,z,p) \) must satisfy the following simplified sufficient conditions:
\[ \mu(B) < \delta \lambda_u, \]  
(6.27)

\[ \mu(D_z B) < \beta_1 \lambda_u. \]  
(6.28)

The condition (6.27) is stricter than the necessary condition (1.40).

**Remark 6.7 (On the choice of \( \tilde{\gamma}_k \) and \( \delta \)).** From (6.26), (6.27) it follows that to have a broader class of the matrices \( B(x,z,p) \) we must increase \( \tilde{\gamma}_k \) and \( \delta \) as much as possible. If \( k \in \{2, 3, n - 1, n\} \), then we have to determine \( \gamma_k \) before we do it for \( \tilde{\gamma}_k \). The parameter \( \tilde{\gamma}_k \), \[ 0 < \gamma_k < \tilde{\gamma}_k < 1, \]
depends on the choice of the subsolution \( u(x) \). When \( \tilde{\gamma}_k \) has been chosen, the parameter \( \delta_k \), \[ 0 < \delta_k < 1, \]
is determined as in Theorem 1.3. Then we should choose \( \delta = \delta_k - \varepsilon_1 \), where \( \varepsilon_1 > 0 \) is sufficiently small such that \( 0 < \delta < \delta_k \).

7. An example

7.1. A k-Hessian type equation in an ellipsoid

Consider the following problem with \( 2 \leq k \leq n \)
\[ S_k \left( D^2 u - A(x,u,Du) - B(x,u,Du) \right) = f(x,u,Du) \text{ in } \Omega, \]  
(7.1)

\[ u = 0 \text{ on } \partial \Omega, \]  
(7.2)
where
\[ A(x, z, p) = (\arctan z) \frac{|p|^2}{(1 + |p|^2)^{\frac{3}{2}}} E_n, \] (7.3)
\[ f(x, z, p) = e^z (1 + |p|^2)^m, \quad 0 \leq m < \frac{k}{2}, \]
\[ \Omega = \left\{ x \in \mathbb{R}^n : \sum_{j=1}^n \mu_j^2 x_j^2 < 1, \mu_j > 0 \right\}. \]

We will show how to determine \( \gamma_k \) for \( k \in \{2, 3, n-1, n\} \) and how to construct a strictly \( \tilde{\gamma}_k \)-admissible subsolution \( u(x) \), where \( 0 < \gamma_k < \tilde{\gamma}_k < 1 \), in this concrete case.

We set \( \mu_{\min} = \min_{1 \leq j \leq n} \mu_j, \mu_{\max} = \max_{1 \leq j \leq n} \mu_j, \gamma_{\Omega} = \frac{\mu_{\min}^2}{\mu_{\max}} \) and assume that
\[ \gamma_k < \gamma_{\Omega} \leq 1, \quad \text{if} \quad 4 \leq k \leq n-2, \] (7.4)
where \( \gamma_k, 4 \leq k \leq n-2, \) is defined in (1.9) and (1.10). In the cases \( k \in \{2, 3, n-1, n\} \) we may choose \( \gamma_k = \gamma_{\Omega} - 3\varepsilon_0 \), where \( \varepsilon_0 > 0 \) is sufficiently small such that \( \gamma_k > 0 \).

Then for all \( k, 2 \leq k \leq n, \) from (7.4) we can choose
\[ \tilde{\gamma}_k = \gamma_{\Omega} - 2\varepsilon_0, \] (7.5)
where \( \varepsilon_0 > 0 \) is assumed to be sufficiently small such that \( 0 < \gamma_k < \tilde{\gamma}_k < 1 \). For \( x \in \overline{\Omega} \) we set
\[ v(x) = \sum_{j=1}^n \mu_j^2 x_j^2 - 1, \]
\[ u(x) = \frac{c}{2} v(x), \quad c > 0. \] (7.6)

We show that if \( c > 0 \) is chosen sufficiently large, then \( u(x) \) is a strictly \((\tilde{\gamma}_k + \varepsilon_0)\)-admissible subsolution to the equation
\[ S_k \left[ D^2 u - A(x, u, Du) \right] = f(x, u, Du) \quad \text{in} \ \Omega. \] (7.7)
This means that \( u(x) \) satisfies the following conditions
\[ S_k \left[ D^2 u - A(x, u, Du) \right] \geq f(x, u, Du) \quad \text{in} \ \Omega, \] (7.8)
\[ \lambda_{\underline{u}} > 0, \] (7.9)
\[ \gamma_{\underline{u}} > \tilde{\gamma}_k + \varepsilon_0, \quad \varepsilon_0 > 0. \] (7.10)
Indeed, from (7.6) we have
\[ Du = c \left( \mu_1^2 x_1, \cdots, \mu_n^2 x_n \right), \]
\[ D^2 u = c \operatorname{diag} \left( \mu_1^2, \cdots, \mu_n^2 \right). \]
Since
\[ A(x, u, Du) = \frac{(\arctan u(x)) |Du(x)|^2}{(1 + |Du(x)|^2)^{\frac{3}{2}}} E_n \]
and \(-\frac{c}{2} \leq u(x) \leq 0\) in \(\Omega\), \(|D_u(x)| \leq c\sqrt{n}\mu_{\text{max}}\), then

\[
0 \leq -A(x, u, Du) \leq \frac{\pi |D_u(x)|^2}{2(1 + |D_u(x)|^2)^{\frac{3}{4}}} E_n \leq \frac{\pi}{2} \sqrt{c\sqrt{n}\mu_{\text{max}}} E_n.
\]

Hence, with \(\omega(x, u) = D^2 u - A(x, u, Du)\) we have

\[
\lambda_{\text{min}}(\omega(x, u)) \geq c\mu_{\text{min}}^2, \quad (7.11)
\]

\[
\lambda_{\text{max}}(\omega(x, u)) \leq c\mu_{\text{max}}^2 + \frac{\pi}{2} \sqrt{c\sqrt{n}\mu_{\text{max}}}. \quad (7.12)
\]

From (7.11), (7.12) and (7.5) it follows that if we choose \(c\) so that \(c > c_1\), where

\[
c_1 = \left(\frac{2(\tilde{\gamma}_k + \varepsilon_0)}{2\varepsilon_0}\right)^2 \frac{\sqrt{n}}{\mu_{\text{max}}^3},
\]

then

\[
\gamma_u = \inf_{x \in \Omega} \left(\frac{\lambda_{\text{min}}(\omega(x, u))}{\lambda_{\text{max}}(\omega(x, u))}\right) > \tilde{\gamma}_k + \varepsilon_0
\]

and (7.10) is satisfied. From (7.11) we have \(\lambda_u \geq c\mu_{\text{min}}^2\) and (7.9) holds.

Now we consider the condition (7.8). Since \(u \leq 0, -A \geq 0\) and \(A(x, z, p)\) is a multiple of \(E_n, A(x, u, Du)\) and \(D^2 u\) commute, we have

\[
S_k \left[D^2 u - A(x, u, Du)\right] - f(x, u, Du) \geq S_k(D^2 u) - f(x, u, Du)
\]

\[
= c^k \left[\sigma_k(\mu_1^2, \ldots, \mu_n^2) - \frac{c^k(1 + |Du|^2)^m}{c^k}\right]
\]

\[
\geq c^k \left[\sigma_k(\mu_1^2, \ldots, \mu_n^2) - \frac{(1 + c^2n\mu_{\text{max}}^2)^m}{c^k}\right].
\]

Since \(0 \leq 2m < k\), the equation

\[
\frac{(1 + c^2n\mu_{\text{max}}^2)^m}{c^k} = \sigma_k(\mu_1^2, \ldots, \mu_n^2)
\]

(7.13)

has at least one positive root. We denote by \(c_2\) the largest positive root of the equation (7.13). Then, when \(c > c_2\) we have

\[
S_k \left[D^2 u - A(x, u, Du)\right] \geq f(x, u, Du)
\]

and (7.8) holds. Hence, if \(c > \max(c_1, c_2)\) then the function \(u(x) = cv(x)\) is a strictly \((\tilde{\gamma}_k + \varepsilon_0)\)-admissible subsolution to the equation (7.7). The function \(u(x)\) is also a strictly \((\tilde{\gamma}_k + \varepsilon_0)\)-admissible subsolution to the problem (7.1)-(7.2) for any skew-symmetric matrix \(B(x, z, p) \in BC^3(D)\).

Suppose \(0 < \delta < \delta_k\), where \(0 < \delta_k < 1\) is determined as in Theorem 1.5. It is obvious that the matrix \(A(x, z, p)\) satisfies the condition (i) of Theorem 1.15. The function \(f(x, z, p) > 0\) and \(\frac{D_i f(x, z, p)}{f(x, z, p)} = 1\). So we choose \(\beta_1 = \frac{1 + \delta_2}{k\delta}\).
By \( A(x, z, p), u(x) \) and \( \varphi = 0 \) we determine \( M_0 > 0, M_1 > 0 \) as in Theorem 1.10. Then

\[
f_0 = \inf_{x \in \Omega, |x| \leq M_0, |p| \leq M_1} f(x, z, p) = e^{-M_0}
\]

and

\[
\lambda_* = \tilde{\gamma}_k \left[ \frac{(1 + \delta^2)^{-\frac{n}{2}} e^{-M_0}}{n^k} \right]^\frac{1}{k}.
\]

Theorem 1.15 and Remark 6.6 state that the problem (7.1)-(7.2) has unique strictly \((\delta, \tilde{\gamma}_k)\)-admissible solution \( u(x) \), that belongs also to \( C^{2,\alpha}(\Omega) \) for some \( 0 < \alpha < 1 \), if the skew-symmetric matrices \( B(x, z, p) \), by (6.27), (6.28), satisfy the following conditions:

\[
\mu(B) < \delta \lambda_*, \quad (7.14)
\]

\[
\mu(D_z B) < \frac{(1 + \delta^2)}{k\delta} \lambda_* \quad (7.15)
\]

The parameter \( 0 < \alpha < 1 \) depends on \( n, k, \delta, \tilde{\gamma}_k, \mu_2(B) \).

**Remark 7.1.** Since the matrix \( A(x, z, p) \), defined by (7.3), does not satisfy the regularity condition (8),

\[
\sum_{i,j,\xi,m=1}^n \frac{\partial A_{ij}(x, z, p)}{\partial p_i \partial p_m} \xi_i \xi_j \eta_m \eta_n \geq 0, \quad (x, z, p) \in D, \quad \xi, \eta \in \mathbb{R}^n, \quad \xi \perp \eta,
\]

then the equations (7.1) have not yet been considered in [9] of the case \( B(x, z, p) = 0 \) and in [11] of the case \( k = n, B(x, z, p) \neq 0 \). So, the result of the Theorem 1.15 is new even for symmetric \( k \)-Hessian type equations and nonsymmetric Monge-Ampère type equations. We note that it is the geometric structure condition (7.4), that allows one to drop out the condition (7.16) for the matrix \( A(x, z, p) \).

### 7.2. The case \( k = 2 \)

We consider the same equation (7.1)-(7.2), but in the case \( k = 2 \). It is well-known that if \( \tilde{\omega}, \tilde{\beta} \) are any matrices of size \( 2 \times 2 \) with \( \tilde{\omega}^T = \tilde{\omega}, \tilde{\beta}^T = -\beta \), then

\[
det(\tilde{\omega} + \tilde{\beta}) = det \tilde{\omega} + det \tilde{\beta}.
\]

Since \( D^2 u(x) + A(x, u, Du) \) is symmetric, \( B(x, u, Du) \) is skew-symmetric, from the assertions (vi), (ix) of Proposition 4.1 it follows that the equation (7.1) becomes the following

\[
S_2 \left(D^2 u - A(x, u, Du)\right) = f(x, u, Du) - S_2(B(x, u, Du)), \quad x \in \Omega, \quad (7.17)
\]

where for \( B(x, z, p) = [B_{ij}(x, z, p)]_{n \times n}, \ B^T = -B \) we have

\[
S_2(B(x, z, p)) = \sum_{i<j} B_{ij}^2(x, z, p).
\]
That means, we have reduced a nonsymmetric 2-Hessian type equation to a symmetric one with a new right-hand side. Suppose $A(x, z, p)$ and $\Omega$ are the same as in the problem (7.17), (7.2) and $\tilde{\gamma}_2, \gamma_2$ are chosen as the same as above, i.e.

$$0 < \gamma_2 = \gamma_\Omega - 3\varepsilon_0 < \tilde{\gamma}_2 = \gamma_\Omega - 2\varepsilon_0 < 1, \varepsilon_0 > 0,$$

where $\gamma_\Omega = \frac{\mu_2^2}{\mu_{\text{max}}}.$

We assume that the function

$$g(x, z, p) = f(x, z, p) - S_2(B(x, z, p))$$

satisfies the following conditions:

$$g(x, z, p) > 0 \text{ in } \mathcal{D}, \quad (7.18)$$

$$D_zg(x, z, p) \geq 0 \text{ in } \mathcal{D}, \quad (7.19)$$

$$g(x, z, p) \leq C \left(1 + |p|^2\right)^h, \ 0 \leq h < 1, \ C > 0. \quad (7.20)$$

Then, as for the problem (7.1)-(7.2), we can show that the function $u(x, z) = \frac{c_2}{\tilde{\gamma}_2}v(x),$ where $c$ is sufficiently large positive number, is a strictly $(\tilde{\gamma}_2 + \varepsilon_0)$-admissible subsolution to the problem (7.17), (7.2). Then we can apply the result of Subsection 7.1 in the case $k = 2$ and $B(x, z, p) = 0$ to conclude the unique solvability of the problem (7.17), (7.2) in the class of strictly $(\tilde{\gamma}_2 + \varepsilon_0)$-admissible solutions. In this case the matrices $B(x, z, p)$ need not to be sufficiently small as in (7.14), (7.15), they satisfy only the conditions (7.18)-(7.20) and must not to be bounded on $\mathcal{D}.$

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