Whitney’s theorem for local anisotropic polynomial
$L_p$-approximation, $0 < p < 1$

Dinh Dũng$^a$, Nguyen Van Dũng$^b$ and Nguyen Dinh Hoa$^c$

$^a,c$ Vietnam National University, Hanoi, Information Technology Institute
144, Xuan Thuy, Hanoi, Vietnam

$b$ University of Transport and Communications
Lang Thuong, Dong Da, Hanoi, Vietnam

May 27, 2013 -- Version 0.2

Abstract

Dinh Dũng and T. Ullrich have proven a multivariate Whitney’s theorem for the local anisotropic polynomial approximation in $L_p(Q)$ for $1 \leq p \leq \infty$, where $Q$ is a $d$-parallelepiped in $\mathbb{R}^d$ with sides parallel to the coordinate axes. They considered the error of best approximation of a function $f$ by algebraic polynomials of fixed degree at most $r_i - 1$ in variable $x_i$, $i = 1, \ldots, d$. The convergence rate of the approximation error when the size of $Q$ going to 0 is characterized by a so-called total mixed modulus of smoothness. The method of proof used by these authors is not suitable to the case $0 < p < 1$. In the present paper, by a different method we proved this theorem for $0 < p \leq \infty$.

Keywords Whitney’s theorem; Anisotropic approximation by polynomials; Total mixed modulus of smoothness; Marchaud’s inequality.

Mathematics Subject Classifications (2010) 41A10; 41A50; 41A63.

1 Introduction

Let $\omega_r(f, \cdot)_{p,I}$ be the $r$th modulus of smoothness of a function $f \in L_p(I)$, and $E_r(f)_{p,I}$ is the error of best $L_p$-approximation $E_r(f)_{p,I}$ of $f$ by algebraic polynomials of degree at most $r - 1$, where $I := [a, b]$ is an interval in $\mathbb{R}$. Whitney’s theorem establishes a convergence characterization for a local polynomial approximation when the degree $r - 1$ of polynomials is fixed and the length $\delta := b - a$ of the interval $I$ is small. Namely, if $0 < p \leq \infty$, we have for every $f \in L_p(I)$,

$$C' \omega_r(f, \delta)_{p,I} \leq E_r(f)_{p,I} \leq C \omega_r(f, \delta)_{p,I}$$

*Corresponding author. Email: dinhzung@gmail.com
with constant $C, C'$ depending only on $r$ and $p$. This result was first proved by Whitney [14] for $p = \infty$ and then extended by Brudnyi [1], to $1 \leq p < \infty$ and by Storozhenko [11] to $0 < p < 1$. Whitney’s theorem was generalized for multivariate isotropic approximations in [2], [3], [13] and other. We refer the reader to [8], [6] for surveys on univariate and multivariate Whitney’s theorem and recent achievements on this topic.

The present paper is a continuation of the paper [6]. In the latter one, Dinh Dũng and T. Ullrich have proven a multivariate Whitney’s theorem for the local anisotropic polynomial approximation in $L_p(Q)$ for $1 \leq p \leq \infty$, where $Q$ is a $d$-parallelepiped in $\mathbb{R}^d$ with sides parallel to the coordinate axes. They considered the error of best approximation of a function $f$ by algebraic polynomials of fixed degree at most $r_i - 1$ in variable $x_i$, $i = 1, \ldots, d$. The convergence rate of the approximation error when the size of $Q$ going to 0 is characterized by a so-called total mixed modulus of smoothness. The method of proof in [6] based on application of a technique in [7], is not suitable to the case $0 < p < 1$. In this paper, by a different method we prove this theorem for $0 < p \leq \infty$.

To formulate the main result of the present paper we preliminarily introduce some necessary notations. As usual, $\mathbb{N}$ is reserved for the natural numbers, by $\mathbb{Z}$ we denote the set of all integers, and by $\mathbb{R}$ the real numbers. Furthermore, $\mathbb{Z}_+$ and $\mathbb{R}_+$ denote the set of non-negative integers and real numbers, respectively. Elements $x$ of $\mathbb{R}^d$ are denoted by $x = (x_1, \ldots, x_d)$. For a domain $D \subset \mathbb{R}^d$, let $L_p(D)$, $0 < p \leq \infty$, be the quasi-normed space of functions on $D$ with the usual $p$-th integral quasi-norm $\|f\|_{p,D}$ to be finite if $0 < p < \infty$, whereas we use the ess sup norm if $p = \infty$.

For $r \in \mathbb{N}^d$, denote by $\mathcal{P}_r$ the set of algebraic polynomials of degree at most $r_i - 1$ at variable $x_i$, $i \in [d]$, where $[d]$ stands for the natural numbers from 1 to $d$. We are interested in the $L_p$-approximation of a function $f \in L_p(Q)$ defined on a $d$-parallelepiped

$$Q := [a_1, b_1] \times \cdots \times [a_d, b_d]$$

by polynomials from $\mathcal{P}_r$. The error of the best approximation of $f \in L_p(Q)$ by polynomials from $\mathcal{P}_r$ is measured by

$$E_r(f)_{p,Q} := \inf_{\varphi \in \mathcal{P}_r} \|f - \varphi\|_{p,Q}.$$

For $r \in \mathbb{Z}_+$, $h \in \mathbb{R}$, and a univariate functions $f$, the $r$th difference operator $\Delta^{(r)}_h(f)$ is defined by

$$\Delta^{(r)}_h(f, x) := \sum_{j=0}^{r} (-1)^{r-j} \begin{pmatrix} r \\ j \end{pmatrix} f(x + jh), \quad \Delta^0_h(f, x) := f(x).$$

For $r \in \mathbb{Z}_+$, $h \in \mathbb{R}^d$ and a $d$-variate function $f : \mathbb{R}^d \to \mathbb{R}$, the mixed $r$th difference operator $\Delta^r_h$ is defined by

$$\Delta^r_h := \prod_{i=1}^{d} \Delta^{r_i}_{h_i},$$

where the univariate operator $\Delta^{r_i}_{h_i}$ is applied to the univariate function $f$ by considering $f$ as a function of variable $x_i$ with the other variables held fixed. Let

$$\omega_r(f, t)_{p,Q} := \sup_{|h_i| \leq t, i \in [d]} \|\Delta^{r_i}_{h_i}(f)\|_{p,Q,h_i}, \quad t \in \mathbb{R}_+^d,$$
be the mixed $r$th modulus of smoothness of $f$, where for $y, h \in \mathbb{R}^d$, we write $yh := (y_1h_1, ..., y_dh_d)$ and $Q_y := \{x \in Q : x_i, x_i + y_i \in [a_i, b_i], i \in [d]\}$. For $r \in \mathbb{Z}_+^d$ and $e \subset [d]$, denote by $r(e) \in \mathbb{Z}_+^d$ the vector with $r(e)_i = r_i, i \in e$ and $r(e)_i = 0, i \notin e$ ($r(\emptyset) = 0$). We define the total mixed modulus of smoothness of order $r$ \cite{1, 6} by

$$
\Omega_r(f, t)_{p, Q} := \sum_{e \subset [d] \setminus \emptyset} \omega_{r(e)}(f, t)_{p, Q}, \ t \in \mathbb{R}_+^d.
$$

Let the size $\delta(Q)$ of $Q$ be defined as $\delta(Q) := (b_1 - a_1, ..., b_d - a_d) \in \mathbb{R}_+^d$.

The main result of the present paper is read as follows.

**Theorem 1.1** \textit{Let} $0 < p \leq \infty$, $r \in \mathbb{N}^d$. \textit{Then there are constants} $C, C'$ \textit{depending only on} $r, d, p$ \textit{such that for every} $f \in L_p(Q)$,

$$
C'\Omega_r(f, \delta)_{p, Q} \leq E_r(f)_{p, Q} \leq C\Omega_r(f, \delta)_{p, Q}, \quad (1.1)
$$

\textit{where} $\delta = \delta(Q)$ \textit{is the size of} $Q$.

Theorem 1.1 shows that for $0 < p \leq \infty$, $r \in \mathbb{N}^d$, the total mixed modulus of smoothness $\Omega_r(f, t)_{p, Q}$ completely characterizes the convergence rate of the best anisotropic polynomial $L_p$-approximation when the degree $r$ of polynomials is fixed and the size $\delta(Q)$ of $Q$ is going to 0. It may have applications in multivariate approximations of functions with bounded mixed smoothness or differences by piecewise polynomials or splines.

Theorem 1.1 extends a result of the paper \cite{6} proving it for $1 \leq p \leq \infty$, and is a multivariate generalization of a result of the paper \cite{11} proving it for $d = 1$ and $0 < p < 1$. In the latter paper, to prove her result the author used inductive arguments and Marchaud’s inequality \cite{9} to reduce the problem to the approximation of functions by constants. We will develop this method to prove Theorem 1.1. It turns out that in its proof we should overcome certain difficulties by employing some auxiliary results in particular, a version of Marchaud’s inequality for mixed modulus of smoothness, an upper bound for the error of the anisotropic approximation by Taylor polynomials for functions from Sobolev spaces of mixed smoothness, a basic relationship between the $r$th mixed difference operators $\Delta^r_h$ and the polynomials from $P_r$ (see Lemma 2.5), etc.

It is worth to notice that there was another proof of Whitney’s theorem for $d = 1$ and $0 < p < 1$ given in \cite{12} based on a technique used in the original proof of Whitney \cite{14}: he estimated the deviation of the function from an interpolating polynomial with equally spaced nodes by means of finite differences. It is interesting to develop it to prove Theorem 1.1 However, this would go beyond the scope of the present paper.

The paper is organized as follows. In Section 2, we prove Marchaud’s inequality for mixed modulus of smoothness and other auxiliary facts. In Section 3 we prove Theorem 1.1.
2 Marchaud’s inequality and other auxiliary results

The total mixed modulus of smoothness $\Omega_r(f,t)_{p,Q}$ is not suitable when we want to estimate the error of anisotropic approximations by by polynomials from $\mathcal{P}_r$. We therefore introduce a modification as follows. For $r \in \mathbb{Z}_+^d$, $h \in \mathbb{R}^d$ and a $d$-variate function $f : \mathbb{R}^d \to \mathbb{R}$, the mixed $p$-mean modulus of smoothness of order $r(e)$ is given by

$$w_r(f,t)_{p,Q} := \left( \left( \prod_{i=1}^d t_i^{i-1} \int_{U(t)} \int_{Q_i(h)} |\Delta_{h}^r(f,x)|^p \, dx \, dh \right)^{1/p} , \ t \in \mathbb{R}^d, \right.$$  

where $U(t) := \{ h \in \mathbb{R}^d : |h_i| \leq t_i, \ i \in [d] \}$, with the usual change of the outer mean integral to sup if $p = \infty$. This leads to the definition of the total mixed $p$-mean modulus of smoothness of order $r$ by

$$W_r(f,t)_{p,Q} := \sum_{e \subset [d], e \neq \emptyset} w_{r(e)}(f,t)_{p,Q}, \ t \in \mathbb{R}^d.$$  

We use letters $C, C', C_1, C_2, ...$ to denote a positive constant independent of the parameters and/or functions which are relevant in the context.

**Lemma 2.1** Let $0 < p \leq \infty$, $r \in \mathbb{N}^d$. Then there are constants $C, C'$ depending only on $r, d$ such that for every $f \in L_p(Q)$,

$$CW_r(f,t)_{p,Q} \leq \Omega_r(f,t)_{p,Q} \leq C'W_r(f,t)_{p,Q}, \ t \in \mathbb{R}^d.$$  

**Proof.** It is enough to show that for $r \in \mathbb{Z}_+^d$,

$$C w_r(f,t)_{p,Q} \leq \omega_r(f,t)_{p,Q} \leq C'w_r(f,t)_{p,Q}, \ t \in \mathbb{R}^d. \tag{2.1}$$

The first inequality in (2.1) follows directly from the definitions of $\omega_r(f,t)_{p,Q}$ and $w_r(f,t)_{p,Q}$. For simplicity let us prove the second one for $d = 2$. This inequality was proven in [10] for the univariate case ($d = 1$). Therefore, we have for $|h_i| \leq t_i$ and for almost all $x_i' \in I_i'$,

$$\|\Delta_{h_i}^r(f)\|_{p,x_i} \leq C_1 t_i^{-1} \int_{U(t_i)} \int_{I_i(r_i h_i)} |\Delta_{h_i}^r(f,x)|^p \, dx_i \, dh_i, \ i = 1, 2,$$

where $i' = 2$ if $i = 1$, and $i' = 1$ if $i = 2$, and the quasi-norm $\|\Delta_{h_i}^r(f)\|_{p,x_i}$ is applied to the function $f$ by considering $f$ as a univariate function in variable $x_i$ with the other variable held fixed. Hence, by using of the identity $\|\Delta_{h_i}^r(f)\|_p = \|\Delta_{h_2}^{r_2}(\Delta_{h_1}^{r_1}(f))\|_{p,x_1,p,x_2}$ and Fubini's theorem we prove the second inequality in (2.1). 

**Lemma 2.2** If $I = \bigcup_{j=1}^n I_j$, where $I_j$ are cubes with disjoint interiors, $j = 1, ..., n$, then there is a constant $C'$ depending only on $r, d, p$ such that

$$\sum_{j=1}^n W_r(f,t)_{p,I_j} \leq CW_r(f,t)_{p,\mathbb{R}^d}.$$  

4
Proof. Indeed, we have
\[ \sum_{j=1}^{n} W_r(f, t)_{p,1,j}^p \leq C \sum_{j=1}^{n} \sum_{e \subset [d], e \neq \emptyset} w_r(e)(f, t)_{p,1,j}^p = C \sum_{e \subset [d], e \neq \emptyset} \sum_{j=1}^{n} w_r(e)(f, t)_{p,1,j}^p, \]
and
\[ \sum_{j=1}^{n} w_r(e)(f, t)_{p,1,j}^p = \prod_{i=1}^{d} t_i^{-1} \int_{U(t)} \sum_{j=1}^{n} \int_{(I_j)_r(e)} |\Delta_h^{r(e)}(f, x)|^p dx dh \]
\[ \leq \prod_{i=1}^{d} t_i^{-1} \int_{U(t)} \int_{(I_r)_h} |\Delta_h^{r(e)}(f, x)|^p dx dh \]
\[ \leq w_r(e)(f, t)_{p,1}^p. \]
Hence,
\[ \sum_{j=1}^{n} W_r(f, t)_{p,1,j}^p \leq C \sum_{e \subset [d], e \neq \emptyset} w_r(e)(f, t)_{p,1}^p \leq C' W_r(f, t)_{p,1}^p. \]
\[ \square \]

The following Marchaud’s inequality for mixed modulus of smoothness gives upper bounds of a mixed modulus of smoothness of a function on \( d \)-parallelepiped \( Q \) by its higher order’s mixed modulus of smoothness and \( L_p \)-quasi-norm (for \( d = 1 \) see \cite{[2]} and also \cite{[4]} Theorems II.8.1 & II.8.2)). It allows us to reduce the general case of Theorem \( 1.1 \) to the simplest case where \( r = (1, 1, ..., 1) \) and therefore we deal with the \( L_p \)-approximation by constant functions.

Lemma 2.3 Let \( 0 < p \leq \infty, i \in [d], \delta(Q) := (\delta_1, ..., \delta_d) \) be the size of \( Q \) and \( k, r \in \mathbb{Z}_+^d \) such that \( 1 \leq k_i < r_i \) and \( k_j = r_j \) \( j \neq i \). Then there is a constant \( C \) depending only on \( r, d \) and \( p \) such that for each \( f \in L_p(Q), \ t = (t_1, ..., t_d) > 0, \)
\[ \omega_k(f, t)_{p,Q} \leq C \int_{t_i}^{\delta_i} \omega_r(f, (t_1, ..., t_i-1, u, t_{i+1}, ..., t_d))_{p,Q} du \]
\[ + \|f\|_{p,Q} \left[ \frac{\|f\|_{p,Q}}{\delta_i^{k_i}} \right], \quad 1 \leq p \leq \infty, \quad (2.2) \]
and
\[ \omega_k(f, t)_{p,Q}^p \leq C' \int_{t_i}^{\delta_i} \omega_r(f, (t_1, ..., t_i-1, u, t_{i+1}, ..., t_d))^p_{p,Q} du \]
\[ + \|f\|_{p,Q}^p \left[ \frac{\|f\|_{p,Q}^p}{\delta_i^{pk_i}} \right], \quad 0 < p < 1. \quad (2.3) \]

Proof. We will prove the inequality \( (2.3) \). The inequality \( (2.2) \) can be proven in a similar way with a slight modification. For simplicity let us prove for \( d = 2 \) when \( Q = [a_1, b_1] \times [a_2, b_2] \). The case \( d > 2 \) can be proven analogously by induction on \( d \). Set \( Q_i := [a_i, b_i], \ i = 1, 2. \)

We first prove \( (2.3) \) for the special case \( r = (k_1 + 1, k_2) \). Let the shift operator \( T_h \) be defined by \( T_h(f, x) := f(x + h) \). From the identity
\[ (x - 1)^{k_1} = 2^{-k_1}(x^2 - 1)^{k_1} + P(x)(x - 1)^{k_1 + 1} \]
for the polynomial $P(x) := [1 - 2^{-k_1} (x + 1)^{k_1}] / (x - 1)$ of degree $k_1 - 1$, we have

$$(T_{h_1} - I)^{k_1} = 2^{-k_1} (T_{2h_1} - I)^{k_1} + P(T_{h_1})(T_{h_1} - I)^{k_1+1}. \tag{2.4}$$

Let $Q'_1 := [a_1, c_1], \ c_1 := (a_1 + b_1) / 2$. Notice that $\|P(T_{h_1})(g)\|_p \leq M\|g\|_p$ for $g \in L_p(Q'_1)$. Hence, by (2.4) we obtain

$$\|\Delta_{h_1}^{k_1} \Delta_{h_2}^{k_2}(f)\|^p_{p,Q'_1} \leq 2^{-k_1} \|\Delta_{2h_1}^{k_1} \Delta_{h_2}^{k_2}(f)\|^p_{p,Q'_1} + M^p \|\Delta_{h_1}^{k_1+1} \Delta_{h_2}^{k_2}(f)\|^p_{p,Q'_1} \tag{2.5}$$

$$\leq M^p \sum_{j=0}^{m} 2^{-k_1j} \|\Delta_{2^{j+1}h_1}^{k_1} \Delta_{h_2}^{k_2}(f)\|^p_{p,Q_1} + 2^{-k_1(m+1)} \|\Delta_{2^{m+1}h_1}^{k_1} \Delta_{h_2}^{k_2}(f)\|^p_{p,Q_1} \tag{2.6}$$

provided that $2^{m+2}k_1h_1 \leq \delta_1$. Here we consider $f$ as a function of variable $x_1$ with $x_2$ held fixed. We see that (2.6) holds also if $Q'_1$ is replaced by $Q''_1 := [c_1, b_1]$. This can be obtained by applying (2.6) to the function $g(x) := f(b_1 - x)$ which has the same moduli of smoothness as $f$. Therefore, it follows that (2.6) also holds with $Q_1$ in place of $Q'_1$ and with additional multiplier 2 in the right-hand side:

$$\|\Delta_{h_1}^{k_1} \Delta_{h_2}^{k_2}(f)\|^p_{p,Q_1} \leq 2M^p \sum_{j=0}^{m} 2^{-k_1j} \|\Delta_{2^{j+1}h_1}^{k_1} \Delta_{h_2}^{k_2}(f)\|^p_{p,Q_1} + 2.2^{-k_1(m+1)} \|\Delta_{2^{m+1}h_1}^{k_1} \Delta_{h_2}^{k_2}(f)\|^p_{p,Q_1}.$$  

For $h = (h_1, h_2) \in \mathbb{R}^2, \ t = (t_1, t_2) \in \mathbb{R}^2, \ |h_1| \leq t_1, \ |h_2| \leq t_2$, from the last inequality we derive

$$\int_{Q_2} \int_{Q_1} |\Delta_{h_1}^{k_1} \Delta_{h_2}^{k_2}(f)|^p \ dx_1 \ dx_2 \leq 2M^p \sum_{j=0}^{m} 2^{-k_1j} \int_{Q_2} \int_{Q_1} |\Delta_{2^{j+1}h_1}^{k_1} \Delta_{h_2}^{k_2}(f)|^p \ dx_2 \ dx_1$$

$$+ 2.2^{-k_1(m+1)} \int_{Q_2} \int_{Q_1} |\Delta_{h_2}^{k_1} \Delta_{2^{m+1}h_1}^{k_1}(f)|^p \ dx_2 \ dx_1$$

$$\leq C^{'} \left[ 2^{k_1p} \sum_{j=0}^{m} (2^{pj}t_1)^{-k_1} \int_{Q_2} \int_{Q_1} |\Delta_{h_2}^{k_1} \Delta_{2^{j}h_1}^{k_1}(f)|^p \ dx_2 \ dx_1 \right.$$

$$+ 2^{-k_1mp} \int_{Q_2} \int_{Q_1} |f|^p \ dx_2 \ dx_1 \left. \right]$$

Since

$$\int_{Q_2} \int_{Q_1} |\Delta_{h_2}^{k_1} \Delta_{2^{j}h_1}^{k_1}(f)|^p \ dx_2 \ dx_1 \leq 2^{k_1p+1} \int_{2^{j}t_1}^{2^{j+1}t_1} \omega_{k_1,1,k_2}(f, (u, t_2))^p |u|^{-k_1p-1} \ du,$$
if we take \( m \) to be the last integer for which \( 2^{m+2}k_1t_1 \leq \delta_1 \), then exists a constant \( C \) such that

\[
\omega_k(f, t)^p_{x, y} \leq C^{p, k_1} \left[ \int_{t_1}^{\delta_1} \frac{\omega_{k_1+1, k_2}(f, (u, t_2))^p}{u^{k_1p+1}} du + \frac{\| f \|_{p, Q}^p}{\delta_1^{k_1p}} \right].
\]

Thus, the inequality (2.3) has been proven for the case \( r = (k_1 + 1, k_2) \).

We now prove it for arbitrary \( k, r \) by induction on \( r_1 \). We assume that (2.3) holds true for \( r = (r_1, k_2) \) and prove it for \( r = (r_1 + 1, k_2) \). Hence, by the inequality (2.3) for \( r = (k + 1, k_2) \) we get

\[
\omega_k(f, t)^p_{x, y} \leq C^{p, k_1} \left[ \int_{t_1}^{\delta_1} \frac{\omega_{r_1+1, k_2}(f, (u, t_2))^p}{u^{r_1p+1}} du + \frac{\| f \|_{p, Q}^p}{\delta_1^{r_1p}} \right] + \int_{t_1}^{\delta_1} \frac{\omega_{r_1+1, k_2}(f, (u, t_2))^p}{u^{r_1p+1}} du + \frac{\| f \|_{p, Q}^p}{\delta_1^{r_1p}}
\]

By \( f^{(k)} \), \( k \in \mathbb{Z}^d_+ \), we denote the \( k \)-th order generalized mixed derivative of a locally integrable function \( f \), i.e.,

\[
\int_Q f^{(k)}(x) \varphi(x) \, dx = (-1)^{k_1 + \ldots + k_d} \int_Q f(x) \frac{\partial^{k_1 + \ldots + k_d}}{\partial x_1^{k_1} \ldots \partial x_d^{k_d}} \varphi(x) \, dx
\]

for all test functions \( \varphi \in C^\infty_0(Q) \), where \( C^\infty_0(Q) \) is the space of infinitely differentiable functions on \( Q \) with compact support which is interior to \( Q \). For \( r \in \mathbb{Z}_+^d \) and \( 0 < p \leq \infty \), the Sobolev space \( W^r_p(Q) \) of mixed smoothness \( r \) is defined as the set of locally integrable functions \( f \in L_p(Q) \), for which the generalized derivative \( f^{(r(e))} \) exists as a locally integrable function and the following quasi-norm is finite

\[
\| f \|_{W^r_p(Q)} := \sum_{e \subseteq [d]} \| f^{(r(e))} \|_{p, Q}.
\]

For \( x, y \in \mathbb{R}^d \), the inequality \( x \leq y \) \( (x < y) \) means that \( x_j \leq y_j \) \( (x_j < y_j) \), \( j \in [d] \). Let \( f \in W^r_p(Q) \) and \( x^0 \in Q \). Then \( f \) has continuous derivatives of order \( k \) for each \( k < r \), therefore,
we can define the Taylor polynomial \( P_k(f) \) of order \( k \) by

\[
P_k(f, x) := P_k(f, x_0, x) = \sum_{0 \leq s < k} f^{(s)}(x_0)e_s(x - x_0),
\]

where \( e_s(x) := \prod_{i=1}^d e_{s_i}(x_i) \) and \( e_m(t) := t^m/m! \).

We will need an estimate of the error of the approximation of a function \( f \in W_r^p(Q) \) by Taylor polynomials via the size of \( Q \) and the \( L_p \)-quasi-norm of its derivatives. The following lemma on multivariate anisotropic Taylor polynomial approximation is a nontrivial generalization of the well-known univariate result. For a proof of this lemma we refer the reader to [6].

**Lemma 2.4** Let \( 1 \leq p \leq \infty, r \in \mathbb{N}^d \). Then there is a constant \( C \) depending only on \( r, d \) such that for every \( f \in W_r^p(Q) \),

\[
E_r(f)_{p,Q} \leq \|f - P_r(f)\|_{p,Q} \leq C \sum_{e \subset [d], e \neq \emptyset} \prod_{i \in e} \delta_{r_i} f^{(r(e))}\|_{p,Q},
\]

where \( \delta = \delta(Q) \) is the size of \( Q \).

We denote by \( L_p^{\text{loc}}(\mathbb{R}^d) \) the set of all functions \( f \) on \( \mathbb{R}^d \) such that for every \( x \in \mathbb{R}^d \), there exists a neighborhood \( V_x \) of \( x \) such that \( f \in L_p(V_x) \). A basic property of the univariate \( r \)-th difference operator \( \Delta_h^r \) is that it turns a function \( f \in L_p^{\text{loc}}(\mathbb{R}) \) to an almost everywhere zero function, i.e., \( \Delta_h^r(f, x) = 0 \) for almost every \( x \), if and only if \( f \) is almost everywhere equal to a polynomial from \( P_r \) (see, e.g., [5, Proposition II.7.1]). The following lemma generalizes this property to the multivariate mixed difference operator \( \Delta_h^r \).

**Lemma 2.5** Let \( 0 < p \leq \infty, r \in \mathbb{N}^d \) and \( f \in L_p^{\text{loc}}(\mathbb{R}^d) \). Then \( f \) is almost everywhere equal to a polynomial in \( P_r \) if and only if

\[
\Delta_h^r(f, x) = 0
\]

for all non-empty \( e \subset [d] \) and for almost all \( x, h \in \mathbb{R}^d \).

**Proof.** The statement "only if" can be verified directly. Let us prove the other one. We need the following auxiliary fact. There are coefficients \( a_k, 0 < k \leq r \) and \( b_e, e \subset [d], e \neq \emptyset \), such that

\[
1 = \sum_{0 < k \leq r} a_k x^k + \sum_{e \subset [d], e \neq \emptyset} b_e P_e(x)
\]

where \( x^k := \prod_{j=1}^d x_j^{k_j} \) and

\[
P_e(x) := \prod_{i \in e} (x_i - 1)^{r_i}.
\]

Indeed, putting

\[
A_e(x) := \prod_{i \in e} [(x_i - 1)^{r_i} - (-1)^{r_i}],
\]

we get

\[
1 = \sum_{0 < k \leq r} a_k x^k + \sum_{e \subset [d], e \neq \emptyset} b_e P_e(x).
\]
we have
\[ P_{[d]}(x) = \prod_{i=1}^{d} (x_i - 1)^{r_i} = \prod_{i=1}^{d} [(x_i - 1)^{r_i} - (-1)^{r_i} + (-1)^{r_i}] \]
\[ = \sum_{e \subseteq [d]} \prod_{i \in [d] \setminus e} (-1)^{r_i} \prod_{i \in e} [(x_i - 1)^{r_i} - (-1)^{r_i}] \]
\[ = A_{[d]}(x) + \sum_{i=1}^{d} (-1)^{r_i} + \sum_{e \subseteq [d], e \neq [d]} \prod_{i \in e \setminus u} (-1)^{r_i} A_e(x), \]
and
\[ A_e(x) = \sum_{u \subseteq e, u \neq \emptyset} \prod_{i \in e \setminus u} (-1)^{r_i} P_u(x) + (-1)^{|e|} \prod_{i \in e} (-1)^{r_i}. \]

Hence, it is easy to verify that
\[ P_{[d]}(x) = A_{[d]}(x) + \sum_{e \subseteq [d], e \neq [d], \emptyset} (-1)^{r_i} \sum_{u \subseteq e, u \neq \emptyset} \prod_{i \in e \setminus u} (-1)^{r_i} P_u(x) - (-1)^d \prod_{i=1}^{d} (-1)^{r_i}, \]
or equivalently,
\[ 1 = (-1)^d \prod_{i=1}^{d} (-1)^{r_i} \left\{ A_{[d]}(x) + \sum_{e \subseteq [d], e \neq [d], \emptyset} \sum_{u \subseteq e, u \neq \emptyset} \prod_{i \in e \setminus u} (-1)^{r_i} P_u(x) - P_{[d]}(x) \right\}. \]

Notice that the polynomial in the right-hand side of the last equality is of the form (2.8), what is desired.

For a non-negative integer \( s \) and a function \( f \) on \( \mathbb{R} \), let the operator \( T_h^s \), \( h \in \mathbb{R} \), be defined by \( T_h^s(f, x) := f(x + sh) \). For a \( k \in \mathbb{Z}_+^d \) and a function \( f \) defined on \( \mathbb{R}^d \), let the mixed operator \( T_h^k \), \( h \in \mathbb{R}^d \), be defined by
\[ T_h^k(f) := \prod_{i=1}^{d} T_{h_i}^{k_i}(f), \]
where the univariate operator \( T_{h_i}^{k_i} \) is applied to \( f \) as a univariate function in variable \( x_i \) with the other variables held fixed. By using the correspondence between the operator \( T_h^k \) and the monomial \( x^k \), and between the operator \( \Delta_h^{r(e)} \) and the polynomial \( P_e(x) \), from (2.8) we get the following equality for a function \( f \) defined on \( \mathbb{R}^d \),
\[ f(x) = \sum_{0 < k \leq r} a_k f(x + kh) + \sum_{e \subseteq [d], e \neq \emptyset} b_e \Delta_h^{r(e)}(f, x). \]

Let \( f \in L_p^{bc}(\mathbb{R}^d) \) satisfying the condition (2.7). Then we have for almost all \( x, h \in \mathbb{R}^d \),
\[ f(x) = \sum_{0 < k \leq r} a_k f(x + kh). \]
We first let $1 \leq p \leq \infty$. Take a function $g \in C_0^\infty(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} g(h) \, dh = 1$. Then by (2.9) we have for almost all $x$,

$$f(x) = f(x) \int_{\mathbb{R}^d} g(h) \, dh = \sum_{0 < k \leq r} a_k \int_{\mathbb{R}^d} g(h) f(x + kh) \, dh$$

$$= \sum_{0 < k \leq r} a_k \prod_{i=1}^d k_i^{-1} \int_{\mathbb{R}^d} g((y - x)/k) f(y) \, dy,$$

where $(y - x)/k = ((y_1 - x_1)/k_1, ..., (y_d - x_d)/k_d)$. Notice that each term in the right-hand side is a function in $C^\infty(\mathbb{R}^d)$. Hence, after redefinition on a set of measure zero $f \in C^\infty(\mathbb{R}^d)$. From (2.9) and the equality

$$f^{(k)}(x) = \lim_{h \to 0} \prod_{i=1}^d h_i^{-k_i} \Delta_h^k(f, x)$$

for any $k \in \mathbb{Z}_+$ we conclude that $f^{(r(e))}(x) = 0$ for all non-empty $e \subset [d]$. Applying Lemma 2.4 gives for all $N > 0$,

$$\|f - P_r(f)\|_{p,Q_N} \leq C \sum_{e \subset [d], e \neq \emptyset} \prod_{i \in e} N_i \|f^{(r(e))}\|_{p,Q_N} = 0,$$

where $Q_N$ the $d$-cube of the size $N$. This implies that $f(x) = P_r(f, x) \in \mathcal{P}_r$ almost everywhere on $\mathbb{R}^d$. We now consider the case $0 < p < 1$. By (2.9) and Jensen’s inequality we have for almost all $x, h \in \mathbb{R}^d$,

$$|f(x)|^p \leq \sum_{0 < k \leq r} |a_k|^p |f(x + kh)|^p.$$

Hence,

$$|f(x)|^p \leq \sum_{0 < k \leq r} |a_k|^p \int_{|h| \leq 1} |f(x + kh)|^p \, dh.$$

Since the right-hand side is a locally bounded function on $\mathbb{R}^d$, so is $f$. From the proven case $p = \infty$ it follows that $f(x) = P_r(f, x) \in \mathcal{P}_r$ almost everywhere on $\mathbb{R}^d$. \qed

3 Proof of Theorem 1.1

In this section we prove Theorem 1.1. To the end we need two auxiliary lemmas more. The first lemma gives an upper bound of the error of the approximation of a function $f \in L_p(Q)$ by constants functions. The second one establishes sufficient conditions of pre-compactness of a subset in $L_p(Q)$ via the total mixed modulus of smoothness of $\Omega_r(f, t)_{p,Q}$.

Lemma 3.1 Let $0 < p \leq \infty$. Then there is a constants $C$ depending only on $r, d, p$ such that for every $f \in L_p(Q)$, exists a constant $\beta$ such that

$$\|f - \beta\|_{p,Q} \leq C \Omega_{1}(f, \delta)_{p,Q},$$

where $\delta = \delta(Q)$ is the size of $Q$ and $1 = (1, ..., 1) \in \mathbb{N}^d$. 10
Proof. Let us prove for the case $0 < p \leq 1$ and $d = 2$. The general case can be proven in a similar way with a slight modification. Recall that $Q = [a_1, b_1] \times [a_2, b_2]$ and $\delta(Q) = (\delta_1, \delta_2) = (b_1 - a_1, b_2 - a_2)$ for $d = 2$. We have

\[
\int_{Q} \int_{Q} |f(x) - f(y)|^p \, dx \, dy \\
\leq \int_{Q} \int_{Q} |f(x_1, x_2) - f(y_1, x_2)|^p \, dx_1 \, dy_1 + \int_{Q} \int_{Q} |f(y_1, x_2) - f(y_1, y_2)|^p \, dx_2 \, dy_2 \\
\leq \delta_2 \int_{a_2}^{b_2} \int_{a_1}^{b_1} |f(x_1, x_2) - f(y_1, x_2)|^p \, dx_1 \, dy_2 + \delta_1 \int_{a_1}^{b_1} \int_{a_1}^{b_1} |f(y_1, x_2) - f(y_1, y_2)|^p \, dx_2 \, dy_1 =: I_1 + I_2.
\]

On the other hand,

\[
\int_{a_1}^{b_1} \int_{a_1}^{b_1} |f(x_1, x_2) - f(y_1, x_2)|^p \, dx_1 \, dy_1 \\
= \int_{a_1}^{b_1} \int_{a_1}^{x_2} |f(x_1, x_2) - f(y_1, x_2)|^p \, dx_1 \, dy_1 + \int_{a_1}^{b_1} \int_{x_1}^{b_1} |f(x_1, x_2) - f(y_1, x_2)|^p \, dx_1 \, dy_1 \\
= \int_{0}^{\delta_1} \int_{a_1+u}^{b_1} |f(x_1, x_2) - f(x_1 - u, x_2)|^p \, dx_1 \, du + \int_{0}^{\delta_1} \int_{a_1}^{b_1-u} |f(x_1, x_2) - f(x_1 + u, x_2)|^p \, dx_1 \, du \\
= 2 \int_{0}^{\delta_1} \int_{a_1}^{b_1-u} |f(x_1 + u, x_2) - f(x_1, x_2)|^p \, dx_1 \, du.
\]

Hence,

\[
I_1 \leq 2\delta_2 \int_{a_2}^{b_2} \int_{a_1}^{b_1} |f(x_1 + u, x_2) - f(x_1, x_2)|^p \, dx_1 \, dx_2 \, du \leq 2\delta_1 \delta_2 \omega_{(1,0)}(f, \delta(Q))_{p, Q}^p.
\]

In a similar way we prove that

\[
I_2 \leq 2\delta_1 \delta_2 \omega_{(0,1)}(f, \delta(Q))_{p, Q}^p.
\]

Thus, we have proven the following inequality

\[
\int_{Q} \int_{Q} |f(x) - f(y)|^p \, dx \, dy \leq 2\delta_1 \delta_2 \left[ \omega_{(1,0)}(f, \delta(Q))_{p, Q}^p + \omega_{(0,1)}(f, \delta(Q))_{p, Q}^p \right]. \quad (3.1)
\]

The function

\[
g(y) := \int_{Q} |f(x) - f(y)|^p \, dx
\]
is continuous on $Q$. Consequently, by (3.1) there is a $y^* \in Q$ such that for $\beta = f(y^*)$,
\[
\int_Q |f(x) - \beta|^p \, dx \leq 2 \left[ \omega_{(1,0)}(f, \delta(Q))_p^p + \omega_{(0,1)}(f, \delta(Q))_p^p \right] \leq C\Omega_1(f, \delta(Q))^p_{p,Q}.
\]
\[\square\]

**Lemma 3.2** Let $0 < p \leq \infty$, $r \in \mathbb{N}^d$ and $F$ is a set of functions in $L^p(Q)$. Then $F$ is pre-compact in $L^p(Q)$ if $F$ is bounded, i.e., $\|f\|_{p,Q} \leq M$ for a constant $M$, and
\[
\lim_{t \to 0} \Omega_r(f, t)_{p,Q} = 0, \text{ for } t > 0, \ t \in \mathbb{R}_+^d, \text{ uniformly for } f \in F. \tag{3.2}
\]

**Proof.** For simplicity we prove the lemma for the case $0 < p \leq 1$ and $Q = [0, 1]^2$. The general case can be proven in a similar way. Put $r^1 = (r_1, 0)$, $r^2 = (0, r_2)$. By Lemma 2.3
\[
\omega_{(1,0)}(f, t)_{p,Q} \leq C_1 t_1^p \left[ \int_{t_1} \omega_{(1,0)}(f, (u, t_2))_{p,Q}^p \, du + M^p \right],
\]
\[
\omega_{(0,1)}(f, t)_{p,Q} \leq C_2 t_2^p \left[ \int_{t_2} \omega_{(0,1)}(f, (t_1, u))_{p,Q}^p \, du + M^p \right].
\]
Hence, by (3.2) and the inequality $\omega_{(1,0)}(f, t)_{p,Q} \leq 2 \omega_{(1,0)}(f, t)_{p,Q}^p$ we have that
\[
\lim_{t \to 0} \Omega_1(f, t)_{p,Q} = 0, \text{ for } t > 0, \ t \in \mathbb{R}_+^d, \text{ uniformly for } f \in F. \tag{3.3}
\]

For $n \in \mathbb{N}$, we define the space $S_n$ of all piecewise constant functions $f$ on $Q$ such that $f(x) = c_{i,j}$ for $x \in I_{i,j} := [i/n, (i+1)/n] \times [j/n, (j+1)/n]$, where $c_{i,j}$ are a constant. Due to to Lemmas 3.1 and 2.1 for each cube $I_{i,j}$ there is a constant $\beta_{i,j}$ satisfying
\[
\|f - \beta_{i,j}\|_{p,I_{i,j}} \leq C_2 \Omega_1(f, (1/n, 1/n))_{p,I_{i,j}} \leq C_3 W_1(f, (1/n, 1/n))_{p,I_{i,j}}. \tag{3.4}
\]
Let $P_n(f, x) := \beta_{i,j}$, for $x \in I_{i,j}$. From (3.1) and Lemmas 2.2 and 2.1 we obtain
\[
\|f - P_n(f)\|_{p,Q} = \sum_{i,j} \|f - \beta_{i,j}\|_{p,I_{i,j}} \leq C_3 \sum_{i,j} W_1(f, (1/n, 1/n))_{p,I_{i,j}}^p \leq C_4 W_1(f, (1/n, 1/n))_{p,Q}^p \leq C_5 \Omega_1(f, (1/n, 1/n))_{p,Q}^p.
\]
Hence, by (3.3) for arbitrary $\varepsilon > 0$, we can choose $n$ so that
\[
\|f - P_n(f)\|_{p,Q} \leq \varepsilon, \ f \in F. \tag{3.5}
\]
Moreover, since $F$ is bounded, so is the set $E := \{P_n(f) : f \in F\}$. Consequently, $E$ is pre-compact as a bounded subset in a finite dimensional subspace. Hence, there is a finite $\varepsilon$-net for $E$, which by (3.5) is a finite $2^{1/p} \varepsilon$-net for $F$. This means that $F$ is pre-compact in $L^p(Q)$. \[\square\]
We are now able to prove Theorem 1.1.

**Proof of Theorem 1.1.** The first inequality in (1.1) is trivial. Indeed, if \( f \in L^p(Q) \) then for every non-empty \( e \subset [d] \) and every \( \varphi \in \mathcal{P}_r \) we have

\[
\omega_r(e)(f, \delta)_{p,Q} = \omega_r(e)(f - \varphi, \delta)_{p,Q} \ll \|f - \varphi\|_{p,Q}.
\]

Hence, we obtain the first inequality in (1.1).

Let us prove the second inequality. It is sufficient to prove it for \( Q = I^d := [0,1]^d \). Suppose that it is not true. Then for each \( n \in \mathbb{N} \), there would exist a function \( f_n \in L^p(I^d) \) such that

\[
E_r(f_n)_{p,1^d} = \|f_n\|_p = 1, \quad \Omega_r(f_n, 1)_{p,1^d} \leq 1/n.
\]

From the convergence \( \Omega_r(f_n, t)_{p,1^d} \to 0, t \to 0 \), for each \( n \), and the inequality \( \Omega_r(f_n, t)_{p,1^d} \leq 1/n \) for all \( t \in \mathbb{R}^d_+ \), we can see that this convergence is uniform in \( n \). By Lemma 3.2 the set \( F = \{f_n\}_{n=1}^\infty \) is precompact. Therefore, there is a subsequence \( \{f_{n_k}\}_{k=1}^\infty \) such that

\[
f_{n_k} \to f \in L^p(I^d), \quad k \to \infty.
\]

We have \( \Omega_r(f, t)_{p,1^d} = 0, \quad t \in \mathbb{R}^d_+ \). This implies that \( \Delta^{r(e)}_h(f, x) = 0 \) for all non-empty \( e \subset [d] \) and for almost all \( x, h \in \mathbb{R}^d \). Then by Lemma 2.5 \( f \) is almost everywhere equal to a polynomial in \( \mathcal{P}_r \). But this is a contradiction because

\[
E_r(f)_{p,1^d} = \lim_{n \to \infty} E_r(f_n)_{p,1^d} = 1.
\]

\( \square \)

**Acknowledgements** This research work is funded by Vietnam National Foundation for Science and Technology Development (NAFOSTED) under Grant No. 102.01-2012.15.

**References**

[1] Yu.A. Brudnyi, On a theorem on best local approximations, *Kazanskii Gosudarstvennyi Universitet. Uchenye Zapiski* **124**, No. 6 (1964), 43–49.

[2] Yu.A. Brudnyi, A multidimensional analogue of a certain theorem of Whitney, *Math. USSR-Sb.* **2** (1970), 157–170.

[3] Yu.A. Brudnyi, Approximation of functions of \( n \) variables by quasipolynomials, *Izv. Akad. Nauk SSSR Ser. Mat.* **34**, No. 3 (1970), 564–583.

[4] O.V. Davydov, Sequences of rectangular Fourier sums of continuous functions with given majorants of the mixed moduli of smoothness, *Mat. Sb.* **187**(1996), No7, 35–58.

[5] R.A. DeVore and G.G. Lorentz, Constructive approximation, Springer-Verlag, New York, 1993.
[6] Dinh Dung and T. Ullrich, On Whitney type inequalities for local anisotropic polynomial approximation, Journal of Approximation Theory 163(2011), 1590–1565.

[7] H. Johnen and K. Scherer, On the equivalence of the K-functional and moduli of continuity and some applications, in: Constructive Theory of Functions of Several Variables, Proc. Conf. Math. Res. Inst., Oberwolfach, 1976, 119–140; Lecture Notes 571, Springer Berlin, 1977.

[8] L. I. Hedberg and Yu. Netrusov, An axiomatic approach to function spaces, spectral synthesis, and Luzin approximation, Mem. Amer. Math. Soc. 188, No. 882 (2007).

[9] A. Marchaud, Sur les dérivées et sur les differences des fonctions de variables réelles J. Math. pures et appl. 6(1927), 337–425.

[10] V. A. Popov and P. Petrushev, Rational approximation of real valued functions, Encyclopedia of Math, and Applications, vol. 28, Cambridge Univ. Press, Cambridge, 1987.

[11] È.A. Storozhenko, Approximation by algebraic polynomials of functions in the class $L^p$, $0 < p < 1$, Izv. Akad. Nauk SSSR, Ser. Mat. 41 (1977), 652–662.

[12] È.A. Storozhenko, Yu. V. Kryakin, Whitney’s theorem in the $L^p$-metric, $0 < p < \infty$, Mat. Sb. 186:3(1995), 131–142.

[13] È.A. Storozhenko and P. Oswald, Jackson’s theorem in the spaces $L^p(\mathbb{R}^k), 0 < p < 1$, Siberian Math. J. 19 (1978), 630-640.

[14] H. Whitney, On functions with bounded $n$th difference, J. Math. Pures Appl. 36 (1957), 67–95.