AN APPLICATION OF THE PARTIAL MALLIAVIN CALCULUS TO BAOUENDEI–GRUSHIN OPERATORS

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Abstract. The existence and the continuity of the transition density function of the diffusion process generated by the Baouendi–Grushin operator is shown with the help of the partial Malliavin calculus. For this purpose, the partial Malliavin calculus is reformulated in terms of Watanabe’s distribution theory on Wiener spaces.

1. Introduction

Let \(d_1, d_2 \in \mathbb{N}, d = d_1 + d_2\) and \(\gamma > 0\). The Baouendi–Grushin operator with parameter \(\gamma\) is the following operator on \(\mathbb{R}^d = \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}\):

\[
\Delta_{(\gamma)} = \sum_{i=1}^{d_1} V_i^2 + \sum_{j=1}^{d_2} W_j^2,
\]

where the vector fields are given by

\[
V_i = \frac{\partial}{\partial x^i}, \quad i = 1, \ldots, d_1, \quad W_j = |x|^{\gamma} \frac{\partial}{\partial y^j}, \quad j = 1, \ldots, d_2,
\]

with the standard coordinate systems \(x = (x^1, \ldots, x^{d_1})\) of \(\mathbb{R}^{d_1}\) and \(y = (y^1, \ldots, y^{d_2})\) of \(\mathbb{R}^{d_2}\). The operator is also represented as

\[
\Delta_{(\gamma)} = \Delta_x + |x|^{2\gamma} \Delta_y,
\]

where \(\Delta_x\) and \(\Delta_y\) are the Laplacians in the variables \(x \in \mathbb{R}^{d_1}\) and \(y \in \mathbb{R}^{d_2}\), respectively. While the ellipticity of the operator degenerates on \(\{0\} \times \mathbb{R}^{d_1}\), if \(\gamma\) is an even natural number, then \(\Delta_{(\gamma)}\) satisfies Hörmander’s finite rank condition:

\[
\dim(\text{Lie}[V_1, \ldots, V_{d_1}, W_1, \ldots, W_{d_2}])_z = d, \quad z \in \mathbb{R}^d,
\]

where \(\text{Lie}[V_1, \ldots, V_{d_1}, W_1, \ldots, W_{d_2}]\) stands for the Lie algebra generated by the vector fields \(V_1, \ldots, V_{d_1}, W_1, \ldots, W_{d_2}\), and \((\text{Lie}[V_1, \ldots, V_{d_1}, W_1, \ldots, W_{d_2}])_z\) is the subspace of the tangent space \(T_z \mathbb{R}^N\) given by

\[(\text{Lie}[V_1, \ldots, V_{d_1}, W_1, \ldots, W_{d_2}])_z = \{U_z \in T_z \mathbb{R}^N \mid U \in \text{Lie}[V_1, \ldots, V_{d_1}, W_1, \ldots, W_{d_2}]\}.
\]

The studies of the Baouendi–Grushin operator go back to those by Baouendi in 1967 [1] and Grushin in the beginning of the 1970s [7, 8]. After them, many researches corresponding

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to the operator have been made. Among them, when \( d_1 = d_2 = 1 \) and \( \gamma = 1 \), a geometric analysis was made by Chang et al \([6]\), and when \( \gamma \) is an even natural number, a concrete form of the fundamental solution of the heat equation associated with the operator was given by Bauer et al \([2]\). The fundamental solution is precisely the transition density function of the diffusion process generated by the operator. The main aim of this paper is to show, in the case when \( \gamma > 0 \) is a general real number, that the diffusion process generated by \( \Delta(\gamma)/2 \) possesses a continuous transition density function.

The diffusion process \( \{Z_t = (X_t, Y_t) : t \geq 0\} \) generated by \( \Delta(\gamma)/2 \) can be constructed via the stochastic differential equation of Itô type:

\[
dZ_t = \sum_{i=1}^{d_1} V_i(Z_t) \, d\theta^i_t + \sum_{j=1}^{d_2} W_j(Z_t) \, d\theta^{d_1+j}_t, \quad Z_0 = z = (x, y) \in \mathbb{R}^d = \mathbb{R}^{d_1} \times \mathbb{R}^{d_2},
\]

where \( \{\theta_t = (\theta^1_t, \ldots, \theta^{d_1}_t) : t \geq 0\} \) is the coordinate process of the classical Wiener space \( \mathcal{W}^d = \{w : [0, \infty) \rightarrow \mathbb{R}^d : w \text{ is continuous and } w(0) = 0\} : \theta_t(w) = w(t) \text{ for } w \in \mathcal{W}^d \text{ and } t \geq 0, \) and \( \mathcal{W}^d \) is equipped with the Wiener measure. If \( \gamma \) is not an even natural number, then the \( W_j \) are no longer smooth, and hence we cannot apply the Malliavin calculus directly. It is easy to see that the solution to (1) is explicitly expressed as

\[
X^i_t = x^i + \theta^i_t, \quad i = 1, \ldots, d_1,
\]

\[
Y^j_t = y^j + \int_0^t |X_s|^\gamma \, d\theta^{d_1+j}_s, \quad j = 1, \ldots, d_2.
\]

This expression indicates that if we freeze \( \{(\theta^1_t, \ldots, \theta^{d_1}_t) : t \geq 0\} \) and hence \( \{X_t : t \geq 0\} \), then the integrand in the second identity, which causes the non-smoothness in the sense of the Malliavin calculus, may be thought of as a deterministic function, and does no harm from the point of view of the Malliavin calculus. This is exactly the idea of the partial Malliavin calculus.

The use of the partial Malliavin calculus goes back to the 1980s. Bismut and Michel \([3, 4]\) introduced it by using the integration by parts formula corresponding only to solutions of stochastic differential equations, and Kusuoka and Stroock \([11]\) gave another systematic formulation of it by using the derivation in the direction of the Cameron–Martin subspace and the Ornstein–Uhlenbeck operator on the Wiener space. They both applied it to the filtering theory. Another application of the partial Malliavin calculus was one to the long time asymptotics of the heat kernel by Ikeda et al \([9]\). These works are all before the establishment of Watanabe’s distribution theory on the Wiener space \([17]\), which is the most systematic framework to develop the Malliavin calculus. However, no reformulation of the partial Malliavin calculus in terms of Watanabe’s distribution theory has been made. Hence, revisiting the partial Malliavin calculus with the help of Watanabe’s distribution theory and reconstructing it in terms of the theory have their own interests and importance. Such a reconstruction is another aim of this paper.

In Section 2, we shall discuss the reformulation of the partial Malliavin calculus with the help of Watanabe’s distribution theory. Its application to the diffusion process generated by \( \Delta(\gamma)/2 \) will be shown in Section 3.
2. Partial Malliavin calculus

The aim of this section is to revisit the partial Malliavin calculus with the help of the distribution theory on the Wiener space developed by S. Watanabe [10, 12, 14].

For \( i = 1, 2 \), let \( (\mathcal{W}^{(i)}, \mathcal{H}^{(i)}, \mu^{(i)}) \) be an abstract Wiener space; \( \mathcal{W}^{(i)} \) is a real separable Banach space, \( \mathcal{H}^{(i)} \) is a real separable Hilbert space, which is embedded in \( \mathcal{W}^{(i)} \) continuously and densely, and \( \mu^{(i)} \) is a Gaussian measure on \( \mathcal{W}^{(i)} \) such that each \( \ell \in (\mathcal{W}^{(i)})^* \) (\( \equiv \) the dual space of \( \mathcal{W}^{(i)} \)), being thought of as a random variable on \( \mathcal{W}^{(i)} \), is a Gaussian random variable with mean zero and variance \( \|\ell\|^2_{\mathcal{H}(i)} \), where \( \| \cdot \|_{\mathcal{H}(i)} \) denotes the norm of \( \mathcal{H}^{(i)} \), and we have used the identification between \( \mathcal{H}^{(i)} \) and \( (\mathcal{H}^{(i)})^* \) and the inclusion

\[
(\mathcal{W}^{(i)})^* \subset (\mathcal{H}^{(i)})^* = \mathcal{H}^{(i)} \subset \mathcal{W}^{(i)}.
\]

The product space

\[
(\mathcal{W}, \mathcal{H}, \mu) = (\mathcal{W}^{(1)} \times \mathcal{W}^{(2)}, \mathcal{H}^{(1)} \times \mathcal{H}^{(2)}, \mu^{(1)} \times \mu^{(2)})
\]

is also an abstract Wiener space.

Let \( E \) be a separable Hilbert space and

\[
\mathbb{D}^{k,p}_{(i)}(E) = \left\{ F : \mathcal{W}^{(i)} \rightarrow E : \text{ \( k \)-times differentiable in the sense of the Malliavin calculus, and its derivatives of all orders up to \( k \) are \( p \)-th integrable with respect to \( \mu^{(i)} \)} \right\}.
\]

For details of the definition, see [10, 12, 14]. Set

\[
\mathbb{D}^{\infty,-}_{(i)}(E) = \bigcap_{k \in \mathbb{N}, 1 < p < \infty} \mathbb{D}^{k,p}_{(i)}(E).
\]

Denote by \( \nabla_{(i)} : \mathbb{D}^{\infty,-}_{(i)}(E) \rightarrow \mathbb{D}^{\infty,-}_{(i)}(E \otimes \mathcal{H}^{(i)}) \) the Malliavin derivative on \( \mathcal{W}^{(i)} \), where \( E \otimes \mathcal{H}^{(i)} \) denotes the real Hilbert space of Hilbert–Schmidt operators from \( E \) to \( \mathcal{H}^{(i)} \), and by \( \nabla^*_i \) its dual. Such spaces and operators on \( \mathcal{W} \) are denoted by \( \mathbb{D}^{k,p}(E), \mathbb{D}^{\infty,-}(E), \nabla, \) and \( \nabla^* \) without scripts \( (i) \), respectively.

Remember that those dual operators are all continuous [12, Theorems 5.1.9 and 5.2.1], i.e., for each \( k \in \mathbb{N} \) and \( 1 < p < \infty \), there exist constants \( C_{(i),k,p} \) such that

\[
\|\nabla^* G\|_{\mathbb{D}^{k+1,p}(E \otimes \mathcal{H})} \leq C_{k,p} \|G\|_{\mathbb{D}^{k+1,p}(E \otimes \mathcal{H})},
\]

\[
\|\nabla^*_i G_i\|_{\mathbb{D}^{k,p}_{(i)}(E)} \leq C_{(i),k,p} \|G_i\|_{\mathbb{D}^{k+1,p}_{(i)}(E \otimes \mathcal{H}^{(i)})},
\]

for \( G \in \mathbb{D}^{k+1,p}(E \otimes \mathcal{H}) \) and \( G_i \in \mathbb{D}^{k+1,p}_{(i)}(E \otimes \mathcal{H}^{(i)}) \), \( i = 1, 2 \), where

\[
\|K\|_{\mathbb{D}^{k,p}(E)} = \sum_{j=0}^{k} \|\nabla^j K\|_{L^p(\mu; E \otimes \mathcal{H}^{(i)})}, \quad \|K_i\|_{\mathbb{D}^{k,p}_{(i)}(E)} = \sum_{j=0}^{k} \|\nabla^j K_i\|_{L^p(\mu; (E \otimes \mathcal{H}^{(i)})^{\otimes j})},
\]

for \( K \in \mathbb{D}^{k,p}(E), K_i \in \mathbb{D}^{k,p}_{(i)}(E), \ i = 1, 2, \)

and \( L^p(\mu; E \otimes \mathcal{H}^{(i)}) \) and \( \| \cdot \|_{L^p(\mu; E \otimes \mathcal{H}^{(i)})} \) are the \( L^p \)-space of \( E \otimes \mathcal{H}^{(i)} \)-valued functionals with respect to \( \mu \) and its norm, respectively.
We now proceed to define partial Malliavin derivatives. Denote by $\mathcal{P}^{(i)}(E)$, $i = 1, 2,$ and $\mathcal{P}(E)$ the spaces of $E$-valued polynomial Wiener functionals on $\mathcal{W}^{(i)}$, $i = 1, 2,$ and $\mathcal{W}$, respectively. For example, $\mathcal{P}^{(1)}(E)$ consists of all $F : \mathcal{W}^{(1)} \to \mathbb{R}$ of the form

$$F(w^{(1)}) = \sum_{i=1}^{n} p_i(\ell_1(w^{(1)}), \ldots, \ell_m(w^{(1)}))e_i, \quad w^{(1)} \in \mathcal{W}^{(1)},$$

where each $p_i : \mathbb{R}^m \to \mathbb{R}$ is a polynomial, $\ell_1, \ldots, \ell_m \in (\mathcal{W}^{(1)})^*$, and $e_i \in E$, $i = 1, \ldots, n$.

Define $\partial_{(2)} : \mathcal{P}(E) \to \mathcal{P}(E \otimes \mathcal{H})$ by

$$\partial_{(2)} = \pi_{(2)} \circ \nabla,$$

where $\pi_{(2)}$ is the projection from $\mathcal{H}$ onto $\mathcal{H}^{(2)}$. Denote by $\mathcal{D}^{k,p}_{(2)}(E)$ the completion of $\mathcal{P}(E)$ with respect to

$$\|F\|_{(2),k,p} = \sum_{j=0}^{k} \|\partial_{(2)}^j F\|_{L^p(\mu;E \otimes \mathcal{H}^{(s)})}.$$

Set

$$\mathcal{D}^{\infty, \infty}_{(2)}(E) = \bigcap_{k \in \mathbb{N}, 1 < p < \infty} \mathcal{D}^{k,p}_{(2)}(E).$$

For any $k \in \mathbb{N}$ and $1 < p < \infty$, by the very definitions of $\partial_{(2)}$ and $\mathcal{D}^{k,p}_{(2)}(E)$, $\partial_{(2)} : \mathcal{P}(E) \to \mathcal{P}(E \otimes \mathcal{H})$ is extended to a continuous linear operator from $\mathcal{D}^{k,p}_{(2)}(E)$ to $\mathcal{D}^{k-1,p}_{(2)}(E \otimes \mathcal{H})$, which will be denoted by $\partial_{(2)}$ again. In particular, $\partial_{(2)}$ is a continuous linear operator from $\mathcal{D}^{\infty, \infty}_{(2)}(E)$ to $\mathcal{D}^{\infty, \infty}_{(2)}(E \otimes \mathcal{H})$, where both spaces are equipped with the Fréchet norms inherited from the $\mathcal{D}^{k,p}_{(2)}$.

**Theorem 2.1.**

(i) For any $k \in \mathbb{N}$ and $1 < p < \infty$, $\partial_{(2)}^*$, the dual of $\partial_{(2)}$, is extended to a continuous linear operator from $\mathcal{D}^{k,p}_{(2)}(E \otimes \mathcal{H})$ to $\mathcal{D}^{k-1,p}_{(2)}(E)$, which will be denoted by $\partial_{(2)}^*$ again. In particular, $\partial_{(2)}^*$ is a continuous linear operator from $\mathcal{D}^{\infty, \infty}_{(2)}(E \otimes \mathcal{H})$ to $\mathcal{D}^{\infty, \infty}_{(2)}(E)$.

(ii) Let $F \in \mathcal{D}^{\infty, \infty}_{(2)}(E_1)$, $G \in \mathcal{D}^{\infty, \infty}_{(2)}(E_2)$, and $K \in \mathcal{D}^{\infty, \infty}_{(2)}(E_2 \otimes \mathcal{H})$. Then $F \otimes G \in \mathcal{D}^{\infty, \infty}_{(2)}(E_1 \otimes E_2)$ and $F \otimes K \in \mathcal{D}^{\infty, \infty}_{(2)}(E_1 \otimes E_2 \otimes \mathcal{H})$. Moreover,

$$\partial_{(2)}(F \otimes G) = (\partial_{(2)} F) \otimes G + F \otimes (\partial_{(2)} G),$$

$$\partial_{(2)}^*(F \otimes K) = F \otimes \partial_{(2)}^* K - (\partial_{(2)} F, K)_{\mathcal{H}},$$

where in the first identity we have identified $(E_1 \otimes \mathcal{H}) \otimes E_2$ and $E_1 \otimes (E_2 \otimes \mathcal{H})$ with $(E_1 \otimes E_2) \otimes \mathcal{H}$.

(iii) Let $1 < p < \infty$ and $\mathcal{F}^{(1)}$ be the $\sigma$-field on $\mathcal{W}$ generated by all $\ell \in (\mathcal{W}^{(1)})^*$, where such an $\ell$ is thought to be an element in $\mathcal{W}^*$ by $\ell((w^{(1)}, w^{(2)})) = \ell(w^{(1)})$. If $F \in L^p(\mu; E)$ is $\mathcal{F}^{(1)}$-measurable, then $F \in \mathcal{D}^{k,p}_{(2)}(E)$ for any $k \in \mathbb{N}$, and $\partial_{(2)} F = 0$.

**Proof.** (i) Without loss of generality, we may and will assume $E = \mathbb{R}$ [12, Lemma 5.2.7]. Moreover, it suffices to show that there exists a constant $A_{k,p}$ such that

$$\|\partial_{(2)}^j(\partial_{(2)}^* G)\|_{L^p(\mu;\mathcal{H}^{(s)})} \leq A_{k,p} \|G\|_{(2),j+1,p}, \quad j = 0, \ldots, k - 1, \quad (4)$$
if } G \in \mathcal{P}(\mathcal{H}) \text{ is of the form } G = \sum_{i=1}^{n} G_i \ell_i \text{ for some } n \in \mathbb{N}, \; G_1, \ldots, G_n \in \mathcal{P}(\mathbb{R}), \text{ and } \\
\ell_1, \ldots, \ell_n \in \mathcal{W}^*.

Let } G \in \mathcal{P}(\mathcal{H}) \text{ be as above. Since } \nabla^* (\pi(2) \ell_i) = \pi(2) \ell_i \text{ [12], by [12, Theorem 5.2.8] we have }

\begin{align*}
(\nabla^* (\pi(2) G))(w^{(1)}, w^{(2)}) &= \sum_{i=1}^{n} \left\{ G_i(w^{(1)}, w^{(2)})(\pi(2) \ell_i)(w^{(1)}, w^{(2)}) - \langle \nabla G_i(w^{(1)}, w^{(2)}), \pi(2) \ell_i \rangle_{\mathcal{H}} \right\} \\
&= \sum_{i=1}^{n} \left\{ G_i(w^{(1)}, w^{(2)})(\pi(2) \ell_i)(w^{(1)}, w^{(2)}) - \langle \nabla (G_i(w^{(1)}, w^{(2)})), \pi(2) \ell_i \rangle_{\mathcal{H}} \right\} \\
&\quad \text{for } (w^{(1)}, w^{(2)}) \in \mathcal{W}^{(1)} \times \mathcal{W}^{(2)} = \mathcal{W}, \text{ where } \langle \cdot, \cdot \rangle_{\mathcal{H}} \text{ is the inner product of } \mathcal{H}. \text{ Since } \\
\pi(1) \ell_i(w^{(1)}, w^{(2)}) = \pi(2) \ell_i(0, w^{(2)}), \text{ we can think of } \pi(2) \ell_i \text{ as an element of } (\mathcal{W}^{(2)})^* \subset \mathcal{H}^{(2)}.
\end{align*}

Moreover, it is easily checked that

\[ \pi(2)(\nabla G(w^{(1)}, w^{(2)})) = \nabla (G(w^{(1)}, \cdot))(w^{(2)}), \]

where the right-hand side means ‘applying \( \nabla \) to \( G(w^{(1)}, \cdot) \) with \( w^{(1)} \) being frozen, and then substitute \( w^{(2)} \) into the resulting functional’. Hence the right-hand side of (5) is written as

\[ \sum_{i=1}^{n} \left\{ G_i(w^{(1)}, w^{(2)})(\pi(2) \ell_i)(w^{(1)}, w^{(2)}) - \langle \nabla (G_i(w^{(1)}, w^{(2)})), \pi(2) \ell_i \rangle_{\mathcal{H}} \right\}. \]

Applying [12, Theorem 5.2.8] to \( \mathcal{W}^*_2 \), we obtain from this expression that

\[ (\nabla^* (\pi(2) G))(w^{(1)}, \cdot) = \nabla^* (\pi(2) G)(w^{(1)}, \cdot), \quad w^{(1)} \in \mathcal{W}^*(1). \]

Notice that \( \partial^*_2 = \nabla^* \circ \pi(2) \). Thus we obtain that

\[ (\partial^*_2 G)(w^{(1)}, \cdot) = \nabla^* (\pi(2) G)(w^{(1)}, \cdot). \]

Moreover, the above observation implies that \( \partial^*_2 G \in \mathcal{P}(\mathbb{R}) \).

Let } j \leq k - 1. \text{ By the Fubini theorem, (3), (6), and the fact that }

\[ (\partial^j_i F)(w^{(1)}, \cdot) = \nabla^j_i (F(w^{(1)}, \cdot)) \quad \text{for any } F \in \mathcal{P}(\mathbb{R}), \; w^{(1)} \in \mathcal{W}^{(1)}, \; i = 1, 2, \ldots, \]

we have

\[ \int_{\mathcal{W}} \| \partial^j_i (\partial^* G) \|_{\mathcal{H}^{(2)} \otimes j}^p d\mu \]

\[ = \int_{\mathcal{W}^{(1)}} \left( \int_{\mathcal{W}^{(2)}} \| \nabla^j_i (\pi(2) G)(w^{(1)}, \cdot) \|_{\mathcal{H}^{(2)} \otimes j}^p d\mu_{(\mathcal{H}^{(2)})^{(2)}} \right) \mu^{(1)}(dw^{(1)}) \]

\[ \leq \int_{\mathcal{W}^{(1)}} \left( C_{(2), j+1, p}(j+2)^p \sum_{i=0}^{j+1} \int_{\mathcal{W}^{(2)}} \| \nabla^j_i (\pi(2) G)(w^{(1)}, \cdot) \|_{\mathcal{H}^{(2)} \otimes i}^p d\mu_{(\mathcal{H}^{(2)})^{(2)}} \right) \mu^{(1)}(dw^{(1)}) \]

\[ = \int_{\mathcal{W}^{(1)}} \left( C_{(2), j+1, p}(j+2)^p \sum_{i=0}^{j+1} \| (\partial^j_i (\pi(2) G))(w^{(1)}, \cdot) \|_{\mathcal{H}^{(2)} \otimes i}^p d\mu_{(\mathcal{H}^{(2)})^{(2)}} \right) \mu^{(1)}(dw^{(1)}) \]

\[ \leq C_{(2), j+1, p}(j+2)^p \sum_{i=0}^{j+1} \int_{\mathcal{W}} \| \partial^j_i G \|_{\mathcal{H}^{(2)}}^p d\mu. \]

Thus (4) holds.
(ii) The assertion holds for $F \in \mathcal{P}(E_1)$, $G \in \mathcal{P}(E_2)$, and $K \in \mathcal{P}(E_2 \otimes \mathcal{H})$. By the continuity of $\partial_{(2)}$ and $\partial^*_{(2)}$, the assertion for general $F$, $G$, and $K$ follows immediately.

(iii) By the very definition of $\partial_{(2)}$, the assertion holds for $F \in \mathcal{P}(1)(E) \subset \mathcal{P}(E)$, where the last inclusion is realized by extending elements in $(\mathcal{W}(1))^* \otimes \mathcal{W}^*$ as stated in the assertion. For an $\mathcal{F}(1)$-measurable $F \in L^p(\mu; E)$, we can take a sequence $\{F_n\}_{n=1}^\infty \subset \mathcal{P}(1)(E)$ such that $\|F_n - F\|_{L^p(\mu; E)} \to 0 (n \to \infty)$. Since $\partial_{(2)}F_n = 0$, this implies the desired assertion. \hfill \Box

Let $N \in \mathbb{N}$. We say $F = (F_1, \ldots, F_N) \in D_{(2)}^{\infty, \infty}((\mathbb{R}^N)$ is non-degenerate if

$$
\det((\partial_{(2)}F_{i}, \partial_{(2)}F_{j}H))_{1 \leq i, j \leq N})^{-1} \in L^{\infty}(\mu) = \bigcap_{1 < p < \infty} L^p(\mu).
$$

**Lemma 2.1.** If $F = (F_1, \ldots, F_N) \in D_{(2)}^{\infty, \infty}((\mathbb{R}^N)$ is non-degenerate, then

$$
((\partial_{(2)}F_{i}, \partial_{(2)}F_{j}H))_{1 \leq i, j \leq N})^{-1} \in D_{(2)}^{\infty, \infty}((\mathbb{R}^N) \otimes \mathbb{R}^N).
$$

**Proof.** In exactly the same manner as in the proof of [12, Lemma 5.4.4], we have

$$
(\det((\partial_{(2)}F_{i}, \partial_{(2)}F_{j}H))_{1 \leq i, j \leq N})^{-1} \in D_{(2)}^{\infty, \infty}(\mathbb{R}),
$$

which implies the desired assertion. \hfill \Box

**Theorem 2.2.** Suppose $F = (F_1, \ldots, F_N) \in D_{(2)}^{\infty, \infty}((\mathbb{R}^N)$ is non-degenerate. For $1 \leq i \leq N$ and $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{Z}_0^N$, where $\mathbb{Z}_0 = \{k \in \mathbb{Z} : k \geq 0\}$, define $I_i, I_\alpha : D_{(2)}^{\infty, \infty}(\mathbb{R}) \to D_{(2)}^{\infty, \infty}(\mathbb{R})$ by

$$
I_i(G) = \partial^*_{(2)} \left( G \sum_{j=1}^N y_{F}^{ij} \partial_{(2)}F_{j} \right), \quad i = 1, \ldots, N, \quad \text{and} \quad I_\alpha = I_{\alpha_1} \circ \cdots \circ I_{\alpha_N},
$$

where

$$
y_{F}^{ij} = ((\partial_{(2)}F_{i}, \partial_{(2)}F_{j}H))_{1 \leq i, j \leq N})^{-1}
$$

and

$$
I_{\alpha_i} = \underbrace{I_{i} \circ \cdots \circ I_{i}}_{a_i \text{ times}}.
$$

Then it holds that

$$
\int_{\mathcal{W}} (\partial_\alpha f)(F)G \, d\mu = \int_{\mathcal{W}} f(F)I_\alpha(G) \, d\mu
$$

for any $f \in C_b^\infty(\mathbb{R}^N)$ and $G \in D_{(2)}^{\infty, \infty}(\mathbb{R})$, where

$$
\partial^\alpha = \left( \frac{\partial}{\partial y^1} \right)^{\alpha_1} \cdots \left( \frac{\partial}{\partial y^N} \right)^{\alpha_N},
$$

$(y^1, \ldots, y^N)$ being the standard coordinate system of $\mathbb{R}^N$.

**Proof.** Approximating $F$ by elements in $\mathcal{P}(\mathbb{R}^N)$, we can easily show that

$$
\partial_{(2)}(f(F)) = \sum_{i=1}^N \frac{\partial f}{\partial y^i}(F) \partial_{(2)}F_i.
$$
Then we have
\[
\frac{\partial f}{\partial y^i}(F) = \sum_{j=1}^{N} \gamma_{ij}^F \langle \partial_j(f(F)), \partial_j F_j \rangle_{\mathcal{H}}, \quad i = 1, \ldots, N.
\]
By the duality of $\hat{\alpha}_{(2)}$, this identity implies (7) for $\alpha = (\delta_{ij})_{1 \leq j \leq N}, i = 1, \ldots, N$. The general case follows by induction.

As an application, we have the partial Malliavin calculus version of ‘S. Watanabe’s pull-back’ of tempered distributions. To state the result, we introduce some notation. Let $\mathcal{S}(\mathbb{R}^N)$ be the space of rapidly decreasing functions on $\mathbb{R}^N$ and $\mathcal{S}'(\mathbb{R}^N)$ be the space of tempered distributions on $\mathbb{R}^N$. For $k \in \mathbb{N}$, let $\mathcal{S}_{-2k}(\mathbb{R}^N)$ be the completion of $\mathcal{S}(\mathbb{R}^N)$ by the norm
\[
\| f \|_{-2k} = \sup_{x \in \mathbb{R}^N} \left| (1 + |x|^2) - \frac{1}{2} \Delta \right|^{-k} f(x),
\]
where $\Delta$ is the Laplacian on $\mathbb{R}^N$. Then $\mathcal{S}'(\mathbb{R}^N) = \bigcup_{k \in \mathbb{N}} \mathcal{S}_{-2k}(\mathbb{R}^N)$.

**Theorem 2.3.** Suppose $F = (F_1, \ldots, F_N) \in D_{(2)}^{\infty, \infty^\prime}(\mathbb{R}^N)$ is non-degenerate. Let $k \in \mathbb{N}$ and $1 < p < \infty$. There exists a constant $A_{k,p}$ such that
\[
\| f(F) \|_{(2), ((k,p))} \leq A_{k,p} \| f \|_{-2k} \text{ for any } f \in \mathcal{S}(\mathbb{R}^N),
\]
where $\| \cdot \|_{(2), ((k,p))}$ is the norm of the dual space $(D_{(2)}^{k,p}(\mathbb{R}))^*$ of $D_{(2)}^{k,p}(\mathbb{R})$:
\[
\| f(F) \|_{(2), ((k,p))} = \sup \left\{ \int_{\mathcal{W}} f(F)G \, d\mu : G \in \mathcal{P}(\mathbb{R}), \| G \|_{(2), k,p} \leq 1 \right\}.
\]
In particular, the mapping $\mathcal{S}(\mathbb{R}^N) \ni f \mapsto f(F) \in D_{(2)}^{k,p}(\mathbb{R})$ is extended to a continuous linear mapping $\mathcal{C}_F : \mathcal{S}_{-2k}(\mathbb{R}^N) \to (D_{(2)}^{k,p}(\mathbb{R}))^*$. Moreover, the function $\mathbb{R}^N \ni y \mapsto (\mathcal{C}_F(\delta_y))(1)$, where $\delta_y$ denotes Dirac’s delta function concentrated at $y \in \mathbb{R}^N$ and 1 denotes the constant function with value 1, realizes the smooth density function of the distribution of $F$.

**Proof.** The assertions follow from the integration by parts formula in Theorem 2.2 in exactly the same manner as Watanabe’s pull-back of tempered distributions through a non-degenerate smooth Wiener functional was established. For example, see [12, Section 5.4]. We omit the details.

We shall close this section by applying the above theorem to stochastic integrals. In what follows, let $T > 0$, $d_1, d_2 \in \mathbb{N}$ and $d = d_1 + d_2$. For $m \in \{d, d_1, d_2\}$, let
\[
\mathcal{W}_T^m = \{ w : [0, T] \to \mathbb{R}^m : \text{continuous and } w(0) = 0 \},
\]
let $\mu_T^m$ be the associated Wiener measure, and denote by $\mathcal{H}_T^m$ the Cameron–Martin subspace of $\mathcal{W}_T^m$, which consists of all absolutely continuous $h \in \mathcal{W}_T^m$ with square integrable derivative $\dot{h}$ with respect to the Lebesgue measure on $[0, T]$. The $\mathcal{H}_T^m$ is a Hilbert space with inner product
\[
\langle h, g \rangle_{\mathcal{H}_T^m} = \int_0^T \langle \dot{h}(t), \dot{g}(t) \rangle_{\mathbb{R}^m} \, dt.
\]
Obviously \( \mathcal{W}_T^d = \mathcal{W}_{T_1}^d \times \mathcal{W}_{T_2}^d, \mathcal{H}_T^d = \mathcal{H}_{T_1}^d \times \mathcal{H}_{T_2}^d \), and \( \mu_T^d = \mu_{T_1}^d \times \mu_{T_2}^d \), and hence under the above notation

\[
(\mathcal{W}^{(i)}, \mathcal{H}^{(i)}, \mu^{(i)}) = (\mathcal{W}_T^d, \mathcal{H}_T^d, \mu_T^d), \quad i = 1, 2, \quad \text{and} \quad (\mathcal{W}, \mathcal{H}, \mu) = (\mathcal{W}_T^d, \mathcal{H}_T^d, \mu_T^d).
\]

The coordinate process on \( \mathcal{W}_T^d \) is denoted by \( \{\theta_t = (\theta_t^1, \ldots, \theta_t^d)\}_{t \leq T}; \theta_t(w) = w(t) \), the position of \( w \in \mathcal{W}_T^d \) at time \( t \). Then the stochastic processes \( \theta^{(1)}(t) = (\theta_t^1, \ldots, \theta_t^d) \) and \( \theta^{(2)}(t) = (\theta_t^{d+1}, \ldots, \theta_t^d) \) are the coordinate processes of \( \mathcal{W}_T^{d_1} \) and \( \mathcal{W}_T^{d_2} \), respectively. Denote by \( \mathcal{B}_t \) the \( \sigma \)-field generated by \( \theta_s, 0 \leq s \leq t \), and denote by \( \mathcal{B}_{t}^{(i)} \) that generated by \( \theta_t^{(i)}(s), 0 \leq s \leq t \) for \( i = 1, 2 \).

Let \( \phi = \{\phi_t = (\phi_t^{ij})_{1 \leq i \leq N, d+1 \leq j \leq d} \}_{t \leq T} \) be an \( \mathbb{R}^{N \times d_2} \)-valued \( (\mathcal{B}_t^{(1)}) \)-progressively measurable process satisfying that

\[
\int_0^T \|\phi_t\|^2 \, dt \in L^{\infty-}(\mu_T^d),
\]

where \( \mathbb{R}^{N \times d_2} \) stands for the space of real \( N \times d_2 \) matrices and

\[
\|\phi_t\|^2 = \sum_{1 \leq i \leq N, d+1 \leq j \leq d} |\phi_t^{ij}|^2.
\]

Since it is also \( (\mathcal{B}_t) \)-progressively measurable, we can define the \( \mathbb{R}^N \)-valued stochastic integral

\[
F = \int_0^T \phi_t \, d\theta_t^{(2)},
\]

i.e.,

\[
F_i = \sum_{j=d+1}^d \int_0^T \phi_t^{ij} \, d\theta_t^j, \quad 1 \leq i \leq N.
\]

**Theorem 2.4.**

(i) \( F \in \mathcal{D}_{(2)}^{\infty,-}(\mathbb{R}^N) \) and \( \partial_2^2(\mathcal{B}_t^{(2)}) F = 0 \).

(ii) If

\[
\left( \det \left( \int_0^T \phi_t^i \phi_t^i \, dt \right) \right)^{-1} \in L^{\infty-}(\mu_T^{d_1}),
\]

where \( A^\top \) stands for the transposed matrix of \( A \), then \( F \) admits a smooth density function \( q_F \). Moreover, there exists a polynomial \( \Psi: \mathbb{R}^N \times \mathbb{R}^{N \times N} \rightarrow \mathbb{R} \), independent of \( F \), such that

\[
q_F(y) = \int_{\mathcal{W}^d} \Psi(\Gamma_F, \ldots, \Gamma_F, y_F) \prod_{j=1}^N |F_j^i - y_j^i| \, d\mu_T^d \quad \text{for } y = (y^1, \ldots, y^N) \in \mathbb{R}^N,
\]

where

\[
\gamma_F^i = (y_F^i)_{1 \leq i, j \leq N} = \left( \int_0^T \phi_t \phi_t^i \, dt \right)^{-1}, \quad \Gamma_F^i = \sum_{j=1}^N \gamma_F^i F_j, \quad i = 1, \ldots, N.
\]
Proof. (i) Fix $2 \leq p < \infty$. Approximate $\{\phi_t\}_{t \leq T}$ by $(\mathcal{B}^p_1)$-progressively measurable processes $\psi^n = \{\psi^n_i = (\psi^n_{ij})_{1 \leq i \leq N, 1 \leq j \leq d} \}_{t \leq T}$ of the form $\psi^n = \sum_{m=0}^{M_n} \psi^m_{0n} 1_{[t_m^n, t_{m+1}^n)}(t)$ with $\psi^m_{0n} \in \mathcal{P}(\mathbb{R}^N \otimes \mathbb{R}^d)$, $m = 0, \ldots, M_n$, and $0 = t_0^n < \ldots < t_{M_n+1}^n = T$ such that

$$\lim_{n \to \infty} \int_{\mathcal{Y} \times \mathcal{Y}^d} \left( \int_0^T \|\phi_t - \psi^n_t\|^{dp} \, dt \right) d\mu_T^n = 0.$$ 

Set

$$F^{(n)} = \int_0^T \psi^n_t \, d\theta^{(2)}_t = \sum_{m=0}^{M_n} \psi^m_{t_{m+1}}(\theta^{(2)}_{t_{m+1}} - \theta^{(2)}_{t_m}).$$

Then, by Theorem 2.1, we have $F^{(n)}(F^{(1)}_1, \ldots, F^{(N)}_N) \in D^{(2),\infty}_\mathcal{H}$ and

$$\partial_j F^{(n)}_i[h] = \sum_{j=1}^d \sum_{i=1}^n \psi^n_{ij} h^j(t) \, dt \quad \text{for} \quad h = (h^1, \ldots, h^d) \in \mathcal{H}^d_T, \quad \text{and} \quad \partial^2 F^{(n)}_i = 0.$$

Letting $n \to \infty$, we have $F \in D^{(2),\infty}_\mathcal{H}$ for any $k \in \mathbb{N}$ and

$$\partial_j F_i[h] = \sum_{j=1}^d \sum_{i=1}^n \phi^n_{ij} h^j(t) \, dt \quad \text{for} \quad h = (h^1, \ldots, h^d) \in \mathcal{H}^d_T, \quad \text{and} \quad \partial^2 F_i = 0. \quad (12)$$

Since $1 < p < \infty$ is arbitrary, $F \in D^{(2),\infty}_\mathcal{H}$. Thus the assertion has been shown.

(ii) By (12), it holds that

$$(\partial_j F_i, \partial_j F_j)_{\mathcal{H}^d_T} = \int_0^T \phi^n_{ij} \, dt.$$

Applying Theorem 2.3, we obtain the first assertion.

To see the second assertion, recall that

$$\delta_{\gamma}(d\eta) = \left( \frac{\partial}{\partial \gamma_i^1} \right)^2 \cdots \left( \frac{\partial}{\partial \gamma_i^N} \right)^2 \left( \prod_{j=1}^N |y^j - \eta^j|/2 \right) d\eta.$$

By Theorems 2.2 and 2.3, the second assertion is proven once we show that there exists a polynomial $\Psi$, independent of $F$, such that

$$2^{-N}(\mathcal{I}^2_{\gamma_f} \circ \cdots \circ \mathcal{I}^2_{\gamma_f})(1) = \Psi(\Gamma^1_{\gamma_F}, \ldots, \Gamma^N_{\gamma_F}),$$

where the $\mathcal{I}_i$ are the operators defined in Theorem 2.2. To do this, we show by induction the more general fact that for each $i_1, \ldots, i_m \in \{1, \ldots, n\}$, there is a polynomial $\Psi_{i_1, \ldots, i_m} : \mathbb{R}^m \times \mathbb{R}^N \to \mathbb{R}$, independent of $F$, such that

$$(\mathcal{I}_i \circ \cdots \circ \mathcal{I}_i)(1) = \Psi_{i_1, \ldots, i_m}(\Gamma^i_{\gamma_F}, \ldots, \Gamma^i_{\gamma_F}) \quad (13)$$

From the first identity in (12), repeating the argument in the proof of [12, Theorem 5.3.3(1)] we see that

$$\partial^2 F_i = F_i.$$
Moreover, the second identity in (12) implies that $\partial_{(2)} \langle \partial_{(2)} F_i, \partial_{(2)} F_j \rangle_{\mathcal{H}^d_T} = 0$ and hence

$$\partial_{(2)} \gamma_F = 0.$$  \hfill (14)

In conjunction with Theorem 2.1, these yield that

$$\mathcal{I}_t(G) = G T^F_\ell - \sum_{j=1}^{N} \gamma^T_j \langle \partial_{(2)} G, \partial_{(2)} F_j \rangle_{\mathcal{H}^d_T},$$  \hfill (15)

By this, $\mathcal{I}_t(1) = \Gamma_i^i, i = 1, \ldots, N$, and (13) holds if $m = 1$.

Suppose that (13) holds for $m$. For $i_{m+1} \in \{1, \ldots, n\}$, by (14) and (15),

$$(\mathcal{I}_{i_{m+1}} \circ \cdots \circ \mathcal{I}_i)(1)$$

$$= \mathcal{I}_{i_{m+1}}(\Psi_{i_1, \ldots, i_m}(\Gamma^{i_1}_F, \ldots, \Gamma^{i_m}_F, \gamma_F))$$

$$= \Psi_{i_1, \ldots, i_m}(\Gamma^{i_1}_F, \ldots, \Gamma^{i_m}_F, \gamma_F)\Gamma^{i_{m+1}}_F - \sum_{j=1}^{N} \gamma^T_j \sum_{\ell=1}^{m} \frac{\partial \Psi_{i_1, \ldots, i_m}(\Gamma^{i_1}_F, \ldots, \Gamma^{i_m}_F, \gamma_F)}{\partial \xi^\ell} \delta^i_{i_j}$$

$$= \Psi_{i_1, \ldots, i_m}(\Gamma^{i_1}_F, \ldots, \Gamma^{i_m}_F, \gamma_F)\Gamma^{i_{m+1}}_F - \sum_{\ell=1}^{m} \gamma^T_{i_{m+1}} \frac{\partial \Psi_{i_1, \ldots, i_m}(\Gamma^{i_1}_F, \ldots, \Gamma^{i_m}_F, \gamma_F)}{\partial \xi^\ell},$$

where $(\xi^1, \ldots, \xi^m, (\alpha^{ij})_{1 \leq i, j \leq N})$ is the standard coordinate of $\mathbb{R}^m \times \mathbb{R}^{N \times N}$. This means that (13) holds for $m + 1$, which completes the proof. \hfill \Box

3. Smoothness

In this section, as an application of the partial Malliavin calculus, we shall investigate the density functions of the distributions of $Y_t$ and $Z_t$ described in (2). To specify the initial condition $z = (x, y)$, we shall write $Y^{(x,y)}_t$ and $Z^{(x,y)}_t$ for $Y_t$ and $Z_t$, respectively. Our goal of this section is the following assertion.

**Theorem 3.1.** Let $z = (x, y) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ and $T > 0$. Then:

(i) the distribution of $Y^{(x,y)}_T$ possesses a smooth density function;

(ii) there exists a continuous function $p_T : \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \to [0, \infty)$ such that

$$\mu_T^d(Z^{(x,y)}_T) \in d\xi \ d\eta = p_T(\xi, \eta) \ d\xi \ d\eta.$$

For the proof, we continue to work on the classical Wiener spaces $(\mathcal{W}^{d_1}_T, \mathcal{H}^{d_1}_T, \mu^{d_1}_T), i = 1, 2$, and their product space $(\mathcal{W}^{d}_T, \mathcal{H}^{d}_T, \mu^{d}_T)$ as at the end of the last section. We need several lemmas for the proof, and start with a lemma on Hölder continuity. The proof of Theorem 3.1 is then given at the end of this section.

**Lemma 3.1.** Let $r, \alpha > 0$. For a continuous $\phi : [0, T] \to \mathbb{R}^d$ with

$$B^{r,\alpha}_{\phi,T} := \int_0^T \int_0^T \frac{|\phi(t) - \phi(s)|^r}{|t-s|^\alpha} \ d\xi \ d\eta < \infty,$$

it holds that

$$|\phi(t) - \phi(s)| \leq 2^{3+2/r}(B^{r,\alpha}_{\phi,T})^{1/r} \frac{r}{\alpha - 2} |t-s|^{(\alpha-2)/r}.$$
In particular, if $B_{\phi,T}^{6,3} < \infty$, then

$$|\phi(t) - \phi(s)| \leq 96(B_{\phi,T}^{6,3})^{1/6}|t - s|^{1/6}.$$ 

**Proof.** This is a direct conclusion of [16, Theorem 2.1.3], which asserts the following. Let $p, q : [0, \infty) \to [0, \infty)$ be strictly increasing continuous functions with $p(0) = q(0) = 0$. For any continuous $\phi : [0, T] \to \mathbb{R}^d$ with

$$B_{\phi,T} := \int_0^T \int_0^T q\left(\frac{\phi(t) - \phi(s)}{p(|t - s|)}\right) ds \, dt < \infty,$$

it holds that

$$|\phi(t) - \phi(s)| \leq 8 \int_0^{[t-s]} q^{-1}\left(\frac{4B_{\phi,T}}{u^2}\right) p(du).$$

Hence

$$\int_0^{[t-s]} q^{-1}\left(\frac{4B_{\phi,T}}{u^2}\right) p(du) = (4B_{\phi,T})^{1/r} \frac{r}{\alpha - 2} |t - s|^{(\alpha - 2)/r}. \quad \Box$$

Let $\beta^{(1)} = \{(\beta_t^1, \ldots, \beta_t^{d_1})\}_{t \leq T}$ be the Brownian bridge with $\beta_0 = \beta_T = 0$ realized by the stochastic differential equation

$$d\beta_t = d\theta_t^{(1)} - \frac{\beta_t}{T-t} dt, \quad \beta_0 = 0.$$ 

See [10, p. 243].

**Lemma 3.2.**

(i) $B_{\phi^{(1)},T}^{6,3} \in L_{\infty}^\infty(\mu_T^{d_1})$, where $B_{\phi^{(1)},T}^{6,3}$ stands for $B_{\phi,T}^{6,1}$ in Lemma 3.1 with $\phi(t) = \theta_t^{(1)}$.

(ii) For $\xi \in \mathbb{R}^{d_1}$, set $\ell_t^{T,\xi} = (t/T)\xi$, $t \in [0, T]$, and $\ell^{T,\xi} = \{\ell_t^{T,\xi}\}_{t \leq T}$. Then $B_{\beta^{(1)} + \ell^{T,\xi}}^{6,3} \in L_{\infty}^\infty(\mu_T^{d_1}).$

**Proof.** For $k \in \mathbb{N}$, set

$$A_k = \int_{\mathbb{R}^{d_1}} \frac{|x|^{2k}}{\sqrt{2\pi}^{d_1}} e^{-|x|^2/2} \, dx.$$

(i) Since

$$\int_{\mathbb{W}_T^{d_1}} |\theta_t^{(1)} - \theta_s^{(1)}|^{2k} d\mu_T^{d_1} = A_k |t - s|^k,$$

we have

$$\int_{\mathbb{W}_T^{d_1}} \left(\int_0^T \int_0^T \frac{|\theta_t^{(1)} - \theta_s^{(1)}|}{|t - s|^3} dt ds\right)^k d\mu_T^{d_1} \leq T^{2(k-1)} \int_0^T \int_0^T \left(\int_{\mathbb{W}_T^{d_1}} |\theta_t^{(1)} - \theta_s^{(1)}|^{6k} d\mu_T^{d_1} \right) dt ds = A_{3k} T^{2k}.$$ 

Thus $B_{\phi^{(1)},T}^{6,3} \in L_{\infty}^\infty(\mu_T^{d_1}).$
Thus we obtain
\[
\int_{\mathcal{W}_T^d} \left( \int_0^T \left( \int_0^T \frac{|(\beta^{(1)}_t + \ell_t^{T,\xi} - (\beta^{(1)}_s + \ell_s^{T,\xi})|^6}{|t - s|^3} \, dt \right) ds \right) \, d\mu_T^d
\]
\[
\leq 2^{6k} T^{2(k-1)} \int_0^T \left( \int_{\mathcal{W}_T^d} \frac{|(\beta^{(1)}_t - \beta^{(1)}_d|\ell_t^{T,\xi} - \ell_s^{T,\xi}|6k}{|t - s|^3} \, d\mu_T^d \right) dt \, ds
\]
\[
\leq 2^{6k} A_3k T^{2k} + 2^{6k} T^{-k} |\xi|^{6k}. \tag{16}
\]
Thus $B_{\beta^{(1)} + \ell T^{\xi}, T}^6, \ell T^{\xi}, \in L^\infty(-\mu_T^d)$. \hfill \Box

For $\gamma > 0$ and continuous $\phi : [0, T] \ni t \mapsto \phi_t \in \mathbb{R}^d$, put
\[
h(\gamma, \phi) = \int_0^T |\phi_t|^{2\gamma} \, dt.
\]

**Lemma 3.3.**

(i) $h(\gamma, \theta^{(1)})^{-1} \in L^\infty(-\mu_T^d)$.

(ii) Let $\xi \in \mathbb{R}^d$ and $\ell T^{\xi}$ be as described in Lemma 3.2. Then

\[
\sup_{|\xi| \leq R} \int_{\mathcal{W}_T^d} h(\gamma, \beta^{(1)} + \ell T^{\xi})^{-p} \, d\mu_T^d < \infty \quad \text{for any } 1 < p < \infty \text{ and } R > 0. \tag{17}
\]

In particular, $h(\gamma, \beta^{(1)} + \ell T^{\xi})^{-1} \in L^\infty(-\mu_T^d)$.

**Proof.** For $\theta^{(1)}_t = (\theta^{(1)}_t, \ldots, \theta^{(d)}_t)$, $\beta^{(1)}_t + \ell_t^{T,\xi} = (\beta^{(1)}_t + \ell_t^{T,\xi,1}, \ldots, \beta^{(1)}_t + \ell_t^{T,\xi,d})$, and $\xi = (\xi^1, \ldots, \xi^d) \in \mathbb{R}^d$, $|\theta^{(1)}_t| \leq |\theta^{(1)}_t|$, $|\beta^{(1)}_t + \ell_t^{T,\xi,1}| \leq |\beta^{(1)}_t + \ell_t^{T,\xi}|$, and $|\xi^1| \leq |\xi|$. Hence, without loss of generality, we may and will assume $d = 1$.

(i) Let $M = \max_{t \leq T} |\theta^{(1)}_t|$, and take $\sigma \in [0, T]$ such that $|\theta^{(1)}_\sigma| = M$. By Lemma 3.1, we have

\[
|\theta^{(1)}_t - \theta^{(1)}_s| \leq 96(B_{\theta^{(1)}, T}^6)^{1/6}|t - s|^{1/6} \quad \text{for any } s, t \in [0, T].
\]

Hence $|\theta^{(1)}_t| \geq M/2$ if $|t - \sigma| \leq M^6/(192B_{\theta^{(1)}, T}^6)$. Thus we have

\[
h(\gamma, \theta^{(1)}) \geq \left( \frac{M}{2} \right)^{2\gamma} \frac{M^6}{192B_{\theta^{(1)}, T}^6},
\]

which implies that

\[
h(\gamma, \theta^{(1)})^{-1} \leq 2^{2\gamma} 192B_{\theta^{(1)}, T}^6 M^{-6+2\gamma}.
\]

Now, by Lemma 3.2, it suffices to show that $M^{-1} \in L^\infty(-\mu_T^d)$. To see this, recall [15, p. 34] that

\[
\mu_T^1(M \leq u) = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k + 1} \exp \left( -\frac{(2k + 1)^2 \pi^2 T}{8u^2} \right).
\]

This immediately implies $M^{-1} \in L^\infty(-\mu_T^d)$. 

(ii) Let \( M^\xi = \max_{t \leq T} |\beta_t^{(1)} + \ell T, \xi | \). Repeating the above argument, we have

\[
\Theta(Y, \beta^{(1)} + \ell T, \xi)^{-1} \leq 2^{2\nu} 192^6 B^{6,3} \mu_{\beta^{(1)} + \ell T, \xi, T} (M^\xi)^{-(6+2\nu)}. \tag{18}
\]

Since \( \beta^{(1)} + \ell T, \xi \) realizes the pinned Brownian motion starting at zero at time zero and ending at \( \xi \) at time \( T \) [10], by equation (4.12) in [5], we have

\[
\mu_T^1 (M^\xi \leq u) = \sum_{k=-\infty}^\infty (-1)^k \exp \left(- \frac{2ku(ku - \xi)}{T} \right) \quad \text{for } u > 0.
\]

Plugging the relation for the theta function \( \Theta_2 [13, p. 422] \) into this, we obtain

\[
\mu_T^1 (M^\xi \leq u) = e^{\xi^2/(2T)} \frac{\sqrt{2\pi T}}{u} \sum_{k=0}^\infty \exp \left(- \frac{(2k + 1)^2 \pi^2 T}{8u^2} \right) \cos \left( \frac{(2k + 1)\pi \xi}{2u} \right) \quad \text{for } u > 0.
\]

Since \( (2k + 1)^2 \geq k + 1 \) for \( k \geq 0 \), it holds that

\[
\mu_T^1 (M^\xi \leq u) \leq e^{\xi^2/(2T)} \frac{\sqrt{2\pi T}}{u} \exp \left(- \frac{\pi^2 T}{8u^2} \right) \quad \text{for } u > 0.
\]

In conjunction with (16) and (18), this implies (17). \( \square \)

**Proof of Theorem 3.1.** (i) By (2), \( Y_{T}^{(x, y)} \) is of the form

\[
Y_{T}^{(x, y)} - y = \int_0^T \phi_t \, d\theta_t^{(2)}.
\]

where

\[
\phi_t = |x + \theta_t^{(1)}|^\nu I_{d_2},
\]

\( I_{d_2} \) being the \( d_2 \times d_2 \) identity matrix. Then

\[
\det \left( \int_0^T \phi_t \phi_t^\top \, dt \right) = \left( \int_0^T |x + \theta_t^{(1)}|^{2\nu} \, dt \right)^{d_2}.
\]

If \( x = 0 \), then by Lemma 3.3, we have

\[
\left( \det \left( \int_0^T \phi_t \phi_t^\top \, dt \right) \right)^{-1} \in L^{\infty-}(\mu_T^{d_1}). \tag{19}
\]

If \( x \neq 0 \), then setting \( \sigma = \inf \{t : |\theta_t^{(1)}| > |x|/2 \} \), we have

\[
\det \left( \int_0^T \phi_t \phi_t^\top \, dt \right) \geq \left( \frac{|x|^{\nu}}{2^\nu \sigma \wedge T} \right)^{d_2}.
\]

Since \( (\sigma \wedge T)^{-1} \in L^{\infty-}(\mu_T^{d_1}) \) [12, Lemma 5.5.3], (19) holds for \( x \neq 0 \). Now apply Theorem 2.4 to see that \( Y_{T}^{(x, y)} \) possesses a smooth density function.

(ii) By (2), \( \{X_{t}^{(x, y)}\}_{t \leq T} \) is a Brownian motion starting at \( x \). Hence

\[
\int_{\mathbb{W}_T^{d_2}} f(Z_{T}^{(x, y)}) \, d\mu_T^{d_2} = \int_{\mathbb{E}^{d_1}} \mathbb{E}[f(\xi, Y_{T}^{(x, y)}) \mid \theta_T^{(1)} = \xi - x] \frac{1}{\sqrt{2\pi d_1}} e^{-|\xi - x|^2/2} \, d\xi
\]
for any bounded \( f \in C(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}) \), where \( \mathbb{E} \left[ \cdot \mid \theta^{(1)}_T = \xi - x \right] \) denotes the conditional expectation given the condition \( \theta^{(1)}_T = \xi - x \). Thus the proof is completed once we show that there exists a continuous function \( q_T : \mathbb{R}^{d_2} \times \mathbb{R}^{d_1} \rightarrow \mathbb{R} \) such that

\[
\mu^d_T (Y^{(x,y)}_T) \in d\eta \mid \theta^{(1)}_T = \xi - x = q_T(\xi, \eta) \, d\eta,
\]

where \( \mu^d_T (\cdot \mid \theta^{(1)}_T = \xi - x) \) is the conditional probability of \( \mu^d_T \) given the condition \( \theta^{(1)}_T = \xi - x \).

We first fix \( \xi \in \mathbb{R}^{d_1} \) and show the existence of \( q_T(\xi, \eta) \). Notice that the process \( \{X_t^{(x,y)}\}_{t \leq T} \) under the conditional probability measure \( \mu^d_T (\cdot \mid \theta^{(1)}_T = \xi - x) \) is realized by the process \( \beta^{(1)} + \ell^{T,\xi - x} \) under \( \mu^d_T \), where \( \ell^{T,\xi - x} \) is the process defined in Lemma 3.2. Thus, setting

\[
\phi^{T,\xi - x}_s = [\beta^{(1)} + \ell^{T,\xi - x}] Y_{d_2}, \quad \widehat{Y}^{(x,y)}_t,\xi = y + \int_0^t \phi^{T,\xi - x}_s \, d\theta^{(2)}_s, \quad t \in [0, T],
\]
due to (2), we have

\[
\mu^d_T (Y^{(x,y)}_T) \in d\eta \mid \theta^{(1)}_T = \xi - x = \mu^d_T (\widehat{Y}^{(x,y)}_T,\xi) \in d\eta.
\]

By Lemma 3.3, it holds that

\[
\left\{ \det \left( \int_0^T \phi^{T,\xi - x}_s (\phi^{T,\xi - x}_s)^T \, dt \right) \right\}^{-1} = h(\gamma, \beta^{(1)} + \ell^{T,\xi - x})^{-d_2} \in L^{\infty} (\mu^d_T).
\]  (20)

Applying Theorem 2.4, we see the existence of a smooth function \( q_T (\xi, \cdot) \) such that

\[
\mu^d_T (\widehat{Y}^{(x,y)}_T,\xi) \in d\eta = q_T (\xi, \eta) \, d\eta.
\]

We next show the continuity of \( q_T (\xi, \eta) \) in \( (\xi, \eta) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \). By Theorem 2.4, there is a polynomial \( \Psi : \mathbb{R}^{d_2} \times \mathbb{R}^{d_2} \times \mathbb{R} \rightarrow \mathbb{R} \), independent of \( x, y, \) and \( \xi \), such that

\[
q_T (\xi, \eta) = \int_{\mathcal{Y}^{d_2}} \Psi (\Gamma_{x,\xi}) \Gamma_{x,\xi} \prod_{j=1}^{d_2} (Y^{(x,y)}_T)_{\xi}^{-1} \eta_j \, d\mu^d_T,
\]  (21)

where \( \gamma_{x,\xi} = h(\gamma, \beta^{(1)} + \ell^{T,\xi - x})^{-1} I_{d_2} \) and

\[
\Gamma_{x,\xi} = (\Gamma_{x,\xi}^1, \ldots, \Gamma_{x,\xi}^{d_2}) = h(\gamma, \beta^{(1)} + \ell^{T,\xi - x})^{-1} (\widehat{Y}^{(x,y)}_T - y).
\]

It is easy to see the continuity of the mappings

\[
\mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \ni (\xi, \eta) \mapsto \prod_{j=1}^{d_2} (Y^{(x,y)}_T)_{\xi}^{-1} \eta_j, \quad Y^{(x,y)}_T,\xi, \ldots, (Y^{(x,y)}_T)_{\xi}^{d_2} \in L^p (\mu^d_T)
\]

for any \( 1 < p < \infty \). Moreover, by Lemma 3.3(ii), \( \{ h(\gamma, \ell^{T,\xi - x})^{-1} : |\xi| \leq R \} \) is bounded in \( L^p (\mu^d_T) \) for any \( R > 0 \) and \( 1 < p < \infty \). In particular, it is uniformly \( L^p \)-integrable for any \( R > 0 \) and \( 1 < p < \infty \). Hence, the mapping

\[
\mathbb{R}^{d_1} \ni \xi \mapsto h(\gamma, \ell^{T,\xi - x})^{-1} \in L^p (\mu^d_T)
\]

is also continuous for any \( 1 < p < \infty \). Owing to these continuities, the expression (21) implies the continuity of \( q_T \) in \( (\xi, \eta) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \). □
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