The Dynamics and Analysis of Stage-Structured Predator-Prey Model Involving Disease and Refuge in Prey Population

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Abstract. Start your abstract here the objective of this paper is to study the dynamical behaviour of an eco-epidemiological system. A prey-predator model involving infectious disease with refuge for prey population only, the (SI) infectious disease is transmitted directly, within the prey species from external sources of the environment as well as, through direct contact between susceptible and infected individuals. Linear type of incidence rate is used to describe the transmission of infectious disease. While Holling type II of functional responses are adopted to describe the predation process of the susceptible and infected predator respectively. This model is represented mathematically by the set of nonlinear differential equations. The existence, uniqueness and boundedness of the solution of this model are investigated. The local and global stability conditions of all possible equilibrium points are established. Finally, numerical simulation is used to study the global dynamics of the model.

Keywords: prey-predator, functional response, stage-structure, stability analysis, Lyapunov function.

1. Introduction
The first pattern formation of prey-predator model was introduced in 1925 by the will-known Lotka [15] and Volterra [26] which starting from a simple and classical supposition. More complexity, but factual predator–prey models have been constructed by mathematicians and ecologists. In 1992, Berryman [1] considered that the dynamic interactions between predators and their preys have long taken and go on to be one of the most important and central subjects which plays a great role in each of the mathematical ecology, natural, social, and technological sciences, especially in the research on biology and ecology [4, 5, 13, 14, 17, 24, 28].

The relation between the densities of a predator and its prey is an essential topic in ecology. Mixture of disease into this predator-prey relationship is an fundamental factor to be investigated in the relatively new field of eco-epidemiology, which comprises aspects of both epidemiology and ecology, to study of how diseases communicate [21]. Majeed and Shawka [2] studied the stability analysis of eco -
epidemiological system with disease. Sun and Jin [10, 11] studied chaos induced by breakup of waves in a spatial epidemic model with non-linear incidence rate. Naji and Mustafa [22] studied the dynamics of an eco-epidemiological model with non-linear incidence rate.

Many researchers working in these fields paying a great amount of attention to improve and generalize the pattern formation of prey-predator models to more involvement and realistic system, and merging this advantage into many applications that give the nature of existence and global asymptotic stability of predator-prey models a great interest.

On the other hand, the dynamical behaviour of prey-predator models has been investigated by a lot of articles and papers which results a significant expansion in the structural of the prey–predator models to involve multi-species; different functional responses, several types of diseases, refuge and stage-structure, etc. [19, 20].

Lately, Samanta [8] studied the existence and global asymptotic stability of a delay predator-prey model with disease in the prey. In fact, the influence of refuge, stage-structure and the multiplicity of functional responses in the prey–predator ecosystem are the most important topics of interest. In recent 3 years, stage structure models have been studies widely by a lot of researchers, Aiello and Freedman (1990) studied the single species model with stage structure [11]. The prey–predator models with prey refuge have been investigated by Kar [25]. Many authors discussed a prey–predator model with Holling type II functional response incorporating a constant prey refuge [4, 5, 16, 23, 27, and 29].

Moreover, some of the experimental and theoretical work have studied the effects of prey refuges and drawn a conclusion that the prey refuges have a positively effect on the stability of the considered interactions, and prey extinction can be conserved by the addition of refuges [3, 6, 7, 12, 18, 24]. Kadhim, Majeed and Naji [30] studied the stability analysis of food web stage structured prey-predator model with refuge involving Holling type II of function response.

In spite of the effect of prey refuges on the dynamical behaviour of the system is very difficult in reality, but it has been considered and analysis in our proposed model.

In this paper an eco-epidemiological mathematical system consisting of prey-predator model involving SI disease in prey [susceptible species \( S(T) \) and infected species \( I(T) \)] with refuge in prey population and stage-structured in predator [immature \( Z(T) \) and mature \( W(T) \)] population is proposed.

Actually, due to the complexity in the proposed model especially between the disease of prey and the stages of predator it was not easy to find an example identical of this model in the nature, but in any case is not impossible, so after more efforts and with the help of the biologist, we could find an example across two different environments represented by Bee with a disease is prey and stage structured of predator is a red Hornet (Vespa orientials), where bees are considered organisms that are exposed to different diseases. Most of these disease that affect the adult stages and difficult to cure and end to death. Nosema disease is most common bee disease in the world and increase the incidence of this disease in the winter. Red Hornet (Vespa orientials) is the most dangerous predators of the bees and lives in social communities in the cells and feed on him and feed his young).

2. The mathematical model

In this section, an eco-epidemiological model is proposed for study. The model consists of a prey, whose total population density at time \( T \) is denoted by \( N(T) \), interacting with stag-structured predator. It is assumed that the prey population is infected by infectious disease with the prey refuge. Now, the following assumptions are adopted in formulating the basic eco-epidemiology model:

1. there is an SI epidemic disease in the prey population divides the prey population into 3 classes namely \( S(T) \) that represents the density of susceptible prey at time \( T \) and \( I(T) \) which represents the density of infected prey at time \( T \). Therefore at any time \( T \), we have \( N(T) = S(T) + I(T) \). The predator population is divided into two classes namely \( Z(T) \) that represents the density of immature predator at time \( T \) and \( W(T) \) which represents the density of mature predator at time \( T \).
2. it is assumed that only susceptible prey \( S \) is capable of reproducing in logistic growth with carrying capacity \( K > 0 \) and intrinsic growth rate constant \( r > 0 \), the infected prey \( I \) is removed before
having the possibility of reproducing. However, the infected prey population I still contribute with S to population growth toward the carrying capacity.

3. The disease is transmitted within the same species by contact with an infected individual at infection rates $\alpha > 0$, for the prey.

4. The mature predator $W(T)$ consumes the susceptible prey $S(T)$ and infected prey $I(T)$ according to Holling type-II of functional responses with maximum attack rate $a > 0$ and half saturation rate $b > 0$ for susceptible prey, and maximum attack rate $c > 0$ and half saturation rate $d > 0$ for infected prey.

5. The disease may causes mortality with a constant mortality rate $\frac{g_1}{g_2} > 0$ for the infected prey.

6. The immature predator depends completely in its feeding on his parents, so that it feeds on the portion of up taken food by mature predator from the susceptible and infected prey with portion rates $0 < n_1 < 1$ and $0 < n_2 < 1$ with uptake rates $0 < e_1 < 1$ and $0 < e_2 < 1$ respectively. The immature predator individuals grown up and become mature predator individuals with grown up rate $u > 0$.

7. There is type of protection of the prey species from facing predation by the mature predator with refuge rates constants $\frac{g_1}{g_2} > 0$ for susceptible and infected prey respectively.

8. Finally, in the absence of prey the predator facing death with natural death rate $\frac{g_1}{g_2} > 0$ and $\frac{g_2}{g_2} > 0$ for immature and mature predator respectively.

Therefore, the dynamics of the above proposed model can be represented by the following set of first order nonlinear differential equations

$$\begin{align*}
\frac{dS}{dT} &= rS \left( 1 - \frac{S + I}{k} \right) - \alpha SI - \frac{a(1-m_1)SW}{b + (1-m_1)S} \\
\frac{dI}{dT} &= \alpha SI - \frac{c(1-m_2)IW}{d + (1-m_2)I} - d_1 I \\
\frac{dZ}{dT} &= \frac{e_1a(1-n_1)(1-m_1)}{b + (1-m_1)S} SW + \frac{e_2c(1-n_2)(1-m_2)}{d + (1-m_2)I} IW - uZ - d_2Z \\
\frac{dW}{dT} &= \frac{e_1a n_1(1-m_1)}{b + (1-m_1)S} SW + \frac{e_2c n_2(1-m_2)}{d + (1-m_2)I} IW - uZ - d_3 W
\end{align*}$$

(1)

For the simplicity in the above model it is assumed that $m_1 = m_2 = m$.

Note that, above proposed model has sixteen parameters in all which make the analysis difficult. So in order to simplify the system, the number of parameters is reduced by using the following dimensionless variables and parameters:

- $t = rT$, $a_1 = \frac{ak}{r}$, $a_2 = \frac{a(1-m)}{r}$, $a_3 = \frac{b}{k}$, $a_4 = \frac{c(1-m)}{r}$, $a_5 = \frac{d}{k}$
- $a_6 = \frac{d_1}{r}$, $a_7 = \frac{e_1a(1-n_1)(1-m)}{r}$, $a_8 = \frac{e_2c(1-n_2)(1-m)}{r}$
- $a_9 = \frac{u}{r}$, $a_{10} = \frac{d_2}{r}$, $a_{11} = \frac{e_1a n_1(1-m)}{r}$, $a_{12} = \frac{e_2c n_2(1-m)}{r}$
- $a_{13} = \frac{d_3}{r}$

then dimensional form of system (1) can be written as:
\[
\frac{ds}{dt} = s\left(1 - s - (1 + a_4) s - \frac{a_2 w}{a_3} \right) = f_1(s, i, z, w)
\]
\[
\frac{di}{dt} = i\left[a_1 s - \frac{a_4 w}{a_5} + (1 - m)i - a_6 \right] = f_2(s, i, z, w)
\]
\[
\frac{dz}{dt} = \frac{a_3 i w}{a_5} + \frac{a_6 i w}{a_7 + (1 - m)i} - a_9 z - a_{10} z = f_3(s, i, z, w)
\]
\[
\frac{dw}{dt} = \frac{a_{11} s w}{a_3} + \frac{a_{12} i w}{a_3 + (1 - m)s} - a_0 z - a_{13} w = f_4(s, i, z, w)
\]

With \( s(0) \geq 0, i(0) \geq 0, z(0) \geq 30 \text{ and } w(0) \geq 0.33 \). It is observed that the number of parameters have been reduced from sixteen in system (1) to fourteen in system (2). Obviously, the interaction functions of system (2) are continuous and have continuous partial derivatives on the following positive four dimensional space.

\[3R_4^+ = \{ (s, i, z, w) \in R^4 : s(0) \geq 0, i(0) \geq 0, z(0) \geq 0, w(0) \geq 0 \} .\]

Therefore, these functions are lipschitzian on \( R^4_+ \), and hence the solution of system (2) exists and unique. Furthermore, all the solutions of system (2) with non-negative initial conditions are uniformly bounded as demonstrated in the following theorem.

2.1. Theorem (1): all the solutions of system (2) are uniformly bounded.

Proof: Let \( (s(t), i(t), z(t), w(t)) \) be any solution of system (2) with non-negative initial condition \( (s(0), i(0), z(0), w(0)) \in R^4_+ \). According to the first equation of system (2) we have:

\[
\frac{ds}{dt} \leq s(1 - s).
\]

Now, by using the comparison theorem of differential inequality, we get that:

\[
\lim_{t \to \infty} \sup s(t) \leq 1.
\]

Now, define the function: \( H(t) = s(t) + i(t) + z(t) + w(t) \).

\[
\frac{dH}{dt} < 2s - [a_2 - (a_7 + a_{11})] \frac{sw}{a_3} - [a_4 - (a_8 + a_{12})] \frac{iw}{a_5} i - a_6 i
\]

\[
< a_0 z - a_{13} w.
\]

Now, since the conversion rate constant from prey population to predator population can not be exceeding the maximum predation rate constant of predator population to prey population, hence from the biological point of view, always \( a_7 + a_{11} < a_2 \) and \( a_8 + a_{12} < a_4 \), hence it is obtained that:

\[
\frac{dH}{dt} < 2 - LH , \quad \text{where } L = \min \{ 1, a_6, a_{10}, a_{13} \}.
\]

Now, by solving this differential inequality for the initial value \( H(0) = H_0 \), we get that:

\[
H(t) \leq \frac{2}{L} + \left( H(0) - \frac{2}{L} \right) e^{-L}.
\]

Thus \( 0 \leq H(t) \leq \frac{2}{L} \) as \( t \to \infty \).

hence all the solutions of system (2) are uniformly bounded and the proof is complete.

3. Existence of equilibrium points

In this section, the existence of all possible equilibrium points of system (2) is discussed. It is observed that, system (2) has at most five equilibrium points, which are mentioned in the following:
The equilibrium point $E_0 = (0, 0, 0, 0)$, which is known as the vanishing point is always exists.

The axial equilibrium point $E_1 = (1, 0, 0, 0)$, exists unconditionally.

The free predators' equilibrium point $E_2 = (\tilde{s}, \tilde{i}, \tilde{z}, \tilde{w})$ exists uniquely in the interior of $R_+^2$ (Interior of $R_+^2$) of si-plane if there is a positive solution to the following set of equations:

$$s(1 - s - (1 + a_1)i) = 0$$  \hspace{1cm} (3.1a)

$$i( a_4 s - a_6 ) = 0$$ \hspace{1cm} (3.1b)

From equation (3.1b), we have:

$$s = \frac{a_6}{a_1}$$ \hspace{1cm} (3.1c)

now by substituting equation (3.1c) in equation (3.1a), we obtain that:

$$\tilde{i} = \frac{a_1 - a_6}{a_1(1 + a_1)}$$ \hspace{1cm} (3.1d)

note that equation (3.1d) is positive, provided that:

$$a_1 > a_6$$ \hspace{1cm} (3.1e)

The disease-free equilibrium point $E_3 = (\bar{s}, 0, \bar{z}, \bar{w})$ exists uniquely in the interior of $R_+^2$ of szw-space if there is a positive solution to the following set of equations:

$$1 - 3s - \frac{a_2 w}{a_3 + (1 - 3m)s} = 0$$ \hspace{1cm} (3.2a)

$$\frac{a_7 sw}{a_3 + (1 - m)s} - (a_9 + a_{10}) z = 0$$ \hspace{1cm} (3.2b)

$$\frac{a_{11} sw}{a_3 + (1 - m)s} + a_9 z - a_{13} w = 0$$ \hspace{1cm} (3.2c)

From equation (3.2a), we have:

$$w = \frac{(1 - s3)[a_3 + (1 - m3)s]}{a_{32}}.$$ \hspace{1cm} (3.2d)

Also, from equation (3.2b), we have:

$$z = \frac{a_7 sw}{(a_9 + a_{10}) [a_3 + (1 - m)s]}.$$ \hspace{1cm} (3.2e)

Now, by substituting equations (3.2d) and (3.2e) in equation (3.2c), we obtain:

$$\bar{s} = \frac{a_3 a_{13}(a_9 + a_{10})}{(a_9 + a_{10})[a_{11} - a_{13}(1 - m)] + a_7 a_9},$$
Therefore, disease free equilibrium point $E_3 = (\bar{s}, 0, \bar{z}, \bar{w})$ exists uniquely in the interior $R^4_+$ of $szw$-space where $\bar{s} = s(\bar{z})$ and $\bar{w} = w(\bar{s})$ if the following conditions:

$$\bar{s} < 1, \quad (3.2f)$$

$$a_{11} > a_{13} (1-m). \quad (3.2h)$$

Finally, the positive (coexistence) equilibrium point $E_4 = (\bar{s}, \bar{i}, \bar{z}, \bar{w})$ exists if there is a positive solution to the following set of equations:

$$\left[1 - s - (1 + a_1)i - \frac{a_2w}{a_3 + (1-m)s}\right] = 0 \quad (3.3a)$$

$$\left[a_1s + \frac{a_4w}{a_5 + (1-m)i} - a_6\right] = 0 \quad (3.3b)$$

$$\left[\frac{a_7sw}{a_3 + (1-m)s} + \frac{a_8iw}{a_5 + (1-m)i} - (a_9 + a_{10})z\right] = 0 \quad (3.3c)$$

$$\left[\frac{a_{11}sw}{a_3 + (1-3m)s} + \frac{a_{12}iw}{a_5 + (1-3m)3i} + a_9z - a_{13}w\right] = 0 \quad (3.3d)$$

From equation $(3.3b)$, we have:

$$w = \frac{1}{a_4}(a_1s - a_6)[a_5 + (1-m)i] \quad (3.e)$$

From equation $(3.3c)$, we have:

$$z = \frac{1}{a_9 + a_{10}} \left[\frac{a_7sw}{a_3 + (1-3m)s} + \frac{a_8iw}{a_5 + (1-3m)3i}\right] \quad (3.3f)$$

Now, by substituting equations $(3.e)$ in equation $(3.3a)$, we obtain:

$$Q_1(s, i) = a_4[1 - (1 + a_1)i][a_3 + (1-m)s] - a_2(a_1s - a_6)[a_5 + (1-m)i] = 0. \quad (3.g)$$

Also, by substituting equations $(3.e)$ and $(3.f)$ in equation $(3.3d)$, we obtain:

$$Q_2(s, i) = \left[\frac{a_{11}sw}{a_3 + (1-3m)s} + \frac{a_{12}iw}{a_5 + (1-3m)3i} + (a_9 + a_{10})a_3 + (1-3m)3s\right]$$

$$\quad \quad + (a_9 + a_{10})[a_5 + (1-m)i] - a_{13} = 0 \quad (3.3h)$$

Now from equation $(3.g)$ we notice that, when $s \to 0$, then $i \to i_1$, where:

$$i_1 = \frac{-(a_3a_4 + a_2a_5a_6)}{a_2a_6(1-m) - a_3a_4(1 + a_1)}$$

Which is positive under the following condition:

$$a_2a_6(1-m) < a_3a_4(1 + a_1). \quad (3.i)$$
Also, from equation (3.1h) we notice that, when \( s \to 0 \), then \( i \to i_2 \). Where:

\[
i_2 = \frac{a_5 a_{13} (a_9 + a_{10})}{(a_9 + a_{10}) [a_{12} - a_{13} (1 - m)] + a_9 a_9},
\]

Which is positive under the following condition:

\[
a_{12} > a_{13} (1 - m) \quad (3.3j)
\]

Now, from equation (3.3g) we have:

\[
\frac{ds}{di} = - \frac{\partial Q_1}{\partial i} \left( \frac{\partial Q_1}{\partial s} \right). So, \quad \frac{ds}{di} > 0 
\]

if one set of the following sets of conditions hold:

\[
\left( \frac{\partial Q_1}{\partial i} \right) > 0, \left( \frac{\partial Q_1}{\partial s} \right) < 0 OR \left( \frac{\partial Q_1}{\partial s} \right) < 0, \left( \frac{\partial Q_1}{\partial s} \right) > 0 \quad (3.3k)
\]

Further, from (3.3h) we notice that

\[
\frac{ds}{di} = - \frac{\partial Q_2}{\partial i} \left( \frac{\partial Q_2}{\partial s} \right). So, \quad \frac{ds}{di} < 0 
\]

if one set of the following sets of conditions hold:

\[
\left( \frac{\partial Q_2}{\partial i} \right) > 0, \left( \frac{\partial Q_2}{\partial s} \right) > 0 OR \left( \frac{\partial Q_2}{\partial s} \right) < 0, \left( \frac{\partial Q_2}{\partial s} \right) < 0 \quad (3.3l)
\]

then the two isoclines (3.3g) and (3.3h) intersect at a unique positive point \( \left( s', i' \right) \) if in addition to conditions (3.i) and (3.3j) the following condition hold:

\[
i_1 < i_2. \quad (3.m)
\]

If we substituting the value of \( s \) and \( i \) in (3.3f) and (3.3e) respectively, yield that \( \bar{z} = z \left( \bar{s}, \bar{i} \right) \) is positive and \( \bar{w} = w \left( \bar{i} \right) \) is positive by condition:

\[
s > \frac{a_6}{a_1} \quad (3.3n)
\]

These are presents the conditions of existence of \( E_4 = \left( \bar{s}, \bar{s', \bar{i}', \bar{z}, \bar{w} \right) \).

4. Local stability analysis

In this section, we shall analyzed the local stability of the system (2) around each3 equilibrium point and discussed through computing the 3Jacobian matrix \( J(s, i, z, w) \) and determined the eigenvalues of system (2) at each of them. The Jacobian matrix \( J(s, i, z, w) \) of the system (2) can be written:
\[ J = \begin{bmatrix}
\frac{\partial f_1}{\partial s} & \frac{\partial f_1}{\partial i} & \frac{\partial f_1}{\partial z} & \frac{\partial f_1}{\partial w} \\
\frac{\partial f_2}{\partial s} & \frac{\partial f_2}{\partial i} & \frac{\partial f_2}{\partial z} & \frac{\partial f_2}{\partial w} \\
\frac{\partial f_3}{\partial s} & \frac{\partial f_3}{\partial i} & \frac{\partial f_3}{\partial z} & \frac{\partial f_3}{\partial w} \\
\frac{\partial f_4}{\partial s} & \frac{\partial f_4}{\partial i} & \frac{\partial f_4}{\partial z} & \frac{\partial f_4}{\partial w}
\end{bmatrix}, \quad (4.1) \]

Where \( f_i \), 1,2,3,4 are given in right hand side of system (2) and
\[
\frac{\partial f_1}{\partial s} = 1 - 2s - (1 + a_1) i - \frac{a_2 a_3 w}{[a_3 + (1-m)s]^2}, \quad \frac{\partial f_1}{\partial i} = -(1 + a_1)s,
\]
\[
\frac{\partial f_2}{\partial z} = a_1 s - \frac{a_5 a_4 w}{a_3 + (1 - 3 m)s^2} - a_6, \quad \frac{\partial f_2}{\partial w} = a_5 (1 - 3 m) i',
\]
\[
\frac{\partial f_3}{\partial s} = \frac{[a_3 + (1 - 3 m)s^2]}{a_3 a_1 w} \frac{\partial f_3}{\partial i} = \frac{a_7 s}{a_5 (1 - 3 m) i^2} + \frac{a_6 i}{a_5 (1 - 3 m) i^2},
\]
\[
\frac{\partial f_4}{\partial s} = \frac{a_1 s}{a_3 + (1 - 3 m)s^2} \frac{\partial f_4}{\partial w} = \frac{a_7 s}{a_5 (1 - 3 m) i^2} + \frac{a_6 i}{a_5 (1 - 3 m) i^2}, \quad \frac{\partial f_4}{\partial z} = a_9,
\]
\[
\frac{\partial f_4}{\partial w} = \frac{a_3 + (1 - 3 m)s^2}{a_3 + (1 - 3 m)s^2} + \frac{a_5 + (1 - 3 m) i^2}{a_5 + (1 - 3 m) i^2} - a_13.
\]

4.1. The local stability analysis at \( E_0(0, 0, 0, 0) \):
the Jacobian matrix of system (2.2) at \( E_0 \) can be written as,
\[
J_0 = J(E_0) = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & -a_6 & 0 & 0 \\
0 & 0 & -(a_9 + a_{10}) & 0 \\
0 & 0 & a_9 & -a_13
\end{bmatrix}, \quad (4.2)
\]
Then the eigenvalues of \( J(E_0) \) are \( \lambda_{0x} = 1 > 0, \lambda_{0i} = -a_6 < 0, \lambda_{0y} = -(a_9 + a_{10}) < 0 \) and \( \lambda_{0w} = -a_{13} < 0 \). Thus, the equilibrium point \( E_0 \) is unstable.

4.2. The local stability analysis at \( E_1(1, 0, 0, 0) \):
The Jacobian matrix of system (2) at \( E_1 \) can be written as:
\[
J_1 = J(E_1) = \begin{bmatrix}
-1 & -(1 + a_1) & 0 & \frac{-a_2}{a_3 + (1 - m)} \\
0 & a_1 - a_6 & 0 & 0 \\
0 & 0 & -(a_9 + a_{10}) & \frac{a_2}{a_3 + (1 - m)} \\
0 & 0 & a_9 & \frac{a_2}{a_3 + (1 - m)} - a_13
\end{bmatrix} \quad (4.3)
\]
Then the characteristic equation of \( J(E_1) \) is given by:
\[
(-1 - \lambda) \left( a_1 - a_6 - \lambda \right) \left[ \lambda^2 - tr(A) \lambda + \det(A) \right] = 0,
\]
Where \( A = \begin{bmatrix} -(a_9 + a_{10}) & a_\gamma \\ a_\gamma & a_9 + (1 - m) \end{bmatrix} \)

So, either 

\[
(\lambda^2 - \text{tr}(A)\lambda + \det(A)) = 0,
\]

Which gives the first two eigenvalues of \( f_1 \) by:

\[
\lambda_{1x} = -1 < 0 \quad \text{and} \quad \lambda_{1y} = 1 - a_6 < 0
\]

Or

\[
[\lambda^2 - \text{tr}(A)\lambda + \det(A)] = 0,
\]

Where:

\[
\text{tr}(A) = a_{11} \frac{a_{12}}{a_3 + (1 - m)} - (a_9 + a_{10} + a_{13}),
\]

and

\[
\det(A) = a_{11} a_{13} (a_9 + a_{10} a_{13} - \frac{(a_{11} (a_9 + a_{10}) + a_9 a_\gamma)}{a_3 + (1 - m)}).
\]

Thus, the equilibrium point \( E_1 \) becomes stable, provided that:

\[
\frac{a_{11}}{a_3 + (1 - m)} < (a_{13} + a_9 + a_{10}),
\]

\[
a_{13} (a_9 + a_{10}) > \frac{(a_{11} (a_9 + a_{10}) + a_9 a_\gamma)}{a_3 + (1 - m)}.
\]

Otherwise, \( E_1 \) is unstable. However, it is a saddle point other.

4.3. The local stability analysis at \( E_2 (s, i, 0, 0) \):

The Jacobian matrix of system (2) at \( E_2 \) can be written as:

\[
f_2 = f(E_2) = \begin{bmatrix} u_{ij} \end{bmatrix}_{4 \times 4},
\]

where:

\[
u_{11} = 1 - 2s - (1 + a_i)i, \quad u_{12} = -(1 + a_i)s < 0, \quad u_{13} = 0,
\]

\[
u_{14} = -\frac{a_2 s}{a_5 + (1 - m)i} < 0, \quad u_{21} = a_i i > 0, \quad u_{22} = a_i s - a_6 = 0,
\]

\[
u_{23} = 0, \quad u_{24} = -\frac{a_4 i}{a_5 + (1 - m)i} < 0, \quad u_{31} = 0, \quad u_{32} = 0,
\]

\[
u_{33} = -(a_9 + a_{10}) < 0, \quad u_{34} = \frac{a_2 s}{a_5 + (1 - m)i} + \frac{a_4 i}{a_5 + (1 - m)i} > 0,
\]

\[
u_{41} = 0, \quad u_{42} = 0, \quad u_{43} = a_9, \quad u_{44} = \frac{a_{11} s}{a_5 + (1 - m)i} + \frac{a_{12} i}{a_5 + (1 - m)i} - a_{13}.
\]

Then the characteristic equation of \( f(E_2) \) is given by:

\[
[\lambda^2 + \text{tr}(\hat{A})\lambda + \det(\hat{A})][\lambda^2 + \text{tr}(\hat{B})\lambda + \det(\hat{B})] = 0
\]

where:

\[
\hat{A} = \begin{bmatrix} 1 - 23s - (31 + a_i)i & -(1 + a_i)s \\ a_i i & 0 \end{bmatrix}
\]

\[
\hat{B} = \begin{bmatrix} -(a_9 + a_{10}) & a_\gamma s + a_9 i \\ a_\gamma s + a_9 i & a_{33} + (1 - 3s) + a_{31} i \end{bmatrix}
\]

And \( \text{tr}(\hat{A}) = \lambda_{2s} + \lambda_{2i} = 1 - 2s - (1 + a_i)i \),
\[
\det(\hat{A}) = \lambda_{2s} \lambda_{2i} = a_1 (1 + a_4)s > 0,
\]
and
\[
\text{tr}(\hat{B}) = \lambda_{2s} + \lambda_{2i} = \frac{a_{11} s}{a_3 + (1 - m)s} + \frac{a_{12} i}{a_5 + (1 - m)i} - (a_9 + a_{10} + a_{13}),
\]
\[
\det(\hat{B}) = \lambda_{2s} \lambda_{2i} = (a_9 + a_{10}) \left[ a_{13} - \left( \frac{a_{11} s}{a_3 + (1 - m)s} + \frac{a_{12} i}{a_5 + (1 - m)i} \right) \right] - a_9 \left[ \frac{a_{11} s}{a_3 + (1 - m)s} + \frac{a_{12} i}{a_5 + (1 - m)i} \right].
\]
So, either
\[
\lambda^2 - \text{tr}(\hat{A}) \lambda + \det(\hat{A}) = 0, \quad (4.4a)
\]
which gives the first two eigenvalues of \( J \) with negative real parts due to the following condition:
\[
\tilde{s} + (1 + a_4) i > 1, \quad (4.4b)
\]
Or,
\[
\lambda^2 - \text{tr}(\hat{B}) \lambda + \det(\hat{B}) = 0, \quad (4.4c)
\]
which gives the second two3 eigenvalues of \( J \) with negative real parts due to the following conditions:
\[
a_{13} > \frac{a_{11} s}{a_3 + (1 - m)s} + \frac{a_{12} i}{a_5 + (1 - m)i}, \quad (4.4e)
\]
\[
\varphi_1 > \varphi_2, \quad (4.4 f)
\]
Where:
\[
\varphi_1 = (a_9 + a_{10}) \left[ a_{13} - \left( \frac{a_{11} s}{a_3 + (1 - m)s} + \frac{a_{12} i}{a_5 + (1 - m)i} \right) \right],
\]
\[
\varphi_2 = a_9 \left[ \frac{a_{11} s}{a_3 + (1 - m)s} + \frac{a_8 i}{a_5 + (1 - m)i} \right].
\]
Therefore, \( E_3 \) is stable equilibrium point if conditions (4.4b), (4.4c), (4.4e) and (4.4f) are hold. However it is unstable otherwise.

4.4. The local stability analysis at \( E_3(\bar{s}, 0, \bar{z}, \bar{w}) \):

The 3Jacobian matrix of system (2) at the free disease equilibrium point
\( E_3 = (\bar{s}, 0, \bar{z}, \bar{w}) \), Can be written as:
\[
f_3 = f(E_3) = \begin{bmatrix} n_{ij} \end{bmatrix}_{3 \times 4}, \quad (4.5)
\]
Where:
\[
n_{11} = 1 - 2\bar{s} - \frac{a_2 a_3 \bar{w}}{a_3 (1 - m)\bar{s}}^2, n_{12} = -(1 + a_4) \bar{s} < 0, n_{13} = 0,
\]
\[
n_{14} = \frac{-a_2 \bar{s}}{a_3 + (1 - m)\bar{s}} < 0, n_{21} = 0, n_{22} = a_1 \bar{s} - \frac{a_4 \bar{w}}{a_5} - a_6, n_{23} = 0,
\]
\[
n_{24} = 0, n_{31} = \frac{a_2 \bar{s}}{a_3 + (1 - m)\bar{s}}^2 > 0, n_{32} = \frac{a_2 \bar{w}}{a_5} > 0, n_{33} = -(a_9 + a_{10}) < 0,
\]
\[
n_{34} = \frac{a_2 \bar{s}}{a_3 + (1 - m)\bar{s}} > 0, n_{41} = \frac{a_5 a_1 \bar{w}}{a_5} > 0, n_{42} = \frac{a_5 \bar{w}}{a_5} > 0,
\]
\[
n_{43} = a_9 > 0, n_{44} = \frac{a_{11} s}{a_3 + (1 - m)s} - a_{13}.
\]
then the characteristic equation of \( f(E_3) \) is given by:
\[(n_{22} - \lambda)[\lambda^3 + R_1 \lambda^2 + R_2 \lambda + R_3] = 0, \quad (4.5a)\]

where:
\[
R_1 = -(n_{11} + n_{33} + n_{44}),
R_2 = n_{11}(n_{33} + n_{44}) + n_{33}n_{44} - n_{14}n_{41},
R_3 = -n_{11}(n_{33}n_{44} - n_{43}n_{34}) - n_{14}(n_{31}n_{43} - n_{41}n_{33}).
\]

So, either \[(n_{22} - \lambda) = 0, \text{ that is, } \lambda_{3\text{i}} = n_{22}, \text{ which is negative}\]

Provided that:
\[
\tilde{s} < \frac{a_4\bar{a} + a_6a_6}{a_1a_1}, \quad (4.5b)
\]

or \[\lambda^3 + R_1 \lambda^2 + R_2 \lambda + R_3 = 0\] however by using Routh Hurwitz criterion all the other eigenvalues, which represent the roots of second part of eq. \[(4.5a),\] have negative real parts if and only if \[R_1 > 0, \quad R_3 > 0\]
and \[R_1R_2 - R_3 > 0.\]

Straightforward computation shows that:
\[
R_1R_2 - R_3 = -n_{11}^2(n_{33} + n_{44}) - n_{33}n_{44}(n_{33} + n_{44}) - n_{11}(n_{33} + n_{44})^2 - n_{14}[n_{41}(n_{11} + n_{44}) + n_{31}n_{43}] \quad (4.5d)
\]

Now, \[R_1 > 0\] and \[R_3 > 0\] provided that:
\[
2\tilde{s} + \frac{|a_2a_3|}{|a_3 + (1 - m)\tilde{s}|^2} > 1, \quad (4.5e)
\]

Moreover, eq. \[(4.5d)\] is positive if in addition to conditions \[(4.5e)\] and \[(4.5f)\] the following condition hold:
\[
n_{41}(n_{11} + n_{44}) > n_{31}n_{43}. \quad (4.5h)
\]

So, all the eigenvalues of \(f(E_3)\) have negative real part if in addition to the given conditions and hence \(E_3\) is locally asymptotically stable. However, it is unstable otherwise.

4.5. The local stability analysis at \(3E_4(\tilde{s}, \bar{i}, \bar{z}, \bar{w})\):

At \(E_4\) the Jacobian matrix becomes: \(f_4 = f(E_4) = [m_{ij}]_{4\times4}\), \[(4.6)\]

Where:
\[
m_{11} = 1 - 2\tilde{s} - (1 + a_1)i - \frac{a_2a_3\bar{w}}{(a_3 + (1 - m)\tilde{s})^2}, \quad m_{12} = -(1 + a_1)i < 0, \quad m_{13} = 0, \quad m_{14} = \frac{-a_2\tilde{s}}{a_3 + (1 - m)\tilde{s}} < 0, \quad m_{21} = a_1i > 0, \quad m_{22} = a_1 \tilde{s} - \frac{a_4\bar{a} + a_6a_6}{(a_3 + (1 - m)\tilde{s})^2} - a_6, \quad m_{23} = 0, m_{24} = -\frac{a_4a_1}{a_5 + (1 - m)i} < 0, \quad m_{31} = \frac{a_3a_7\bar{w}}{(a_3 + (1 - m)\tilde{s})^2} > 0, m_{32} = \frac{a_5a_6\bar{w}}{(a_5 + (1 - m)i)^2} > 0, \quad m_{33} = (a_3 + (1 - m)\tilde{s}) > 0, \quad m_{34} = \frac{a_7\tilde{s}}{a_3 + (1 - m)\tilde{s}} + \frac{a_8\bar{i}}{a_5 + (1 - m)i} > 0, \quad m_{41} = a_1 \tilde{s} - \frac{a_4}{a_3 + (1 - m)\tilde{s}} - a_6, \quad m_{42} = 0, m_{43} = -\frac{a_8\bar{i}}{a_5 + (1 - m)i} < 0, \quad m_{44} = a_1i > 0, \quad m_{12} = -(1 + a_1)i < 0, \quad m_{14} = \frac{-a_2\tilde{s}}{a_3 + (1 - m)\tilde{s}} < 0, \quad m_{21} = a_1i > 0, \quad m_{22} = a_1 \tilde{s} - \frac{a_4\bar{a} + a_6a_6}{(a_3 + (1 - m)\tilde{s})^2} - a_6, \quad m_{23} = 0, m_{24} = -\frac{a_4a_1}{a_5 + (1 - m)i} < 0, \quad m_{31} = \frac{a_3a_7\bar{w}}{(a_3 + (1 - m)\tilde{s})^2} > 0, m_{32} = \frac{a_5a_6\bar{w}}{(a_5 + (1 - m)i)^2} > 0, \quad m_{33} = (a_3 + (1 - m)\tilde{s}) > 0, \quad m_{34} = \frac{a_7\tilde{s}}{a_3 + (1 - m)\tilde{s}} + \frac{a_8\bar{i}}{a_5 + (1 - m)i} > 0, \quad m_{41} = a_1 \tilde{s} - \frac{a_4}{a_3 + (1 - m)\tilde{s}} - a_6, \quad m_{42} = 0, m_{43} = -\frac{a_8\bar{i}}{a_5 + (1 - m)i} < 0, \quad m_{44} = a_1i > 0,
\]
\[ m_{41} = \frac{a_3a_{11}w}{(a_3 + (1 - m)s)^2} > 0, \quad m_{42} = \frac{a_5a_{12}i\bar{w}}{(a_5 + (1 - m)i)^2} > 0, \]
\[ m_{43} = a_0 > 0, \quad m_{44} = \frac{a_{11}s}{a_3 + (1 - 3m)s} + \frac{a_{12}i}{a_3 + (1 - 3m)i} = -a_{13}. \]

Then the characteristic equation of \( J(E_4) \) is given by:
\[ \lambda^4 + H_1 \lambda^3 + H_2 \lambda^2 + H_3 \lambda + H_4 = 0, \quad (4.6a) \]

Where:
\[ H_1 = - (\gamma_0 + \gamma_1) \]
\[ H_2 = \gamma_0\gamma_1 + \gamma_2 + \gamma_3 - (\gamma_4 + \gamma_5 + \gamma_6 + \gamma_7) \]
\[ H_3 = -[(\gamma_0(\gamma_2 - \gamma_4) + \gamma_1(\gamma_3 - \gamma_6) - \gamma_5)(\gamma_8 - \gamma_9 + \gamma_{10}) + \gamma_6 \gamma_{11} + m_{14}(\gamma_{12} - \gamma_{13} + \gamma_{14})] \]
\[ H_4 = (\gamma_2 - \gamma_4)(\gamma_3 - \gamma_6) + (\gamma_9 - \gamma_{10})(\gamma_{15} - \gamma_{16}) + \gamma_{17}(\gamma_4 - \gamma_{18}) + \gamma_{19}(\gamma_{13} - \gamma_{14}). \]

With
\[ \gamma_0 = m_{11} + m_{22}, \gamma_1 = m_{33} + m_{44}, \gamma_2 = m_{33}m_{44}, \gamma_3 = m_{11}m_{22}, \]
\[ \gamma_4 = m_{34}m_{43} > 0, \gamma_5 = m_{24}m_{42} < 0, \gamma_6 = m_{12}m_{21} < 0, \]
\[ \gamma_7 = m_{14}m_{41} < 0, \gamma_8 = m_{11}m_{42}, \gamma_9 = m_{32}m_{43} > 0, \gamma_{10} = m_{42}m_{33} < 0, \]
\[ \gamma_{11} = m_{14}m_{41}, \gamma_{12} = m_{22}m_{41} < 0, \gamma_{13} = m_{31}m_{42} > 0, \]
\[ \gamma_{14} = m_{41}m_{33} < 0, \gamma_{15} = m_{11}m_{24} < 0, \gamma_{16} = m_{14}m_{21} < 0, \gamma_{17} = m_{12}m_{24} > 0, \gamma_{18} = m_{31}m_{43} > 0, \gamma_{19} = m_{14}m_{22} > 0. \]

Now, \( H_i > 0, i = 1,3 \) and 4, provided that
\[ \min \left\{ W_1, W_2, W_3, W_4 \right\} > \frac{s^2}{W_3}, \quad (4.6b) \]
\[ a_{13} > \frac{a_{11}s\frac{a_5 + (1 - m)i}{a_3 + (1 - m)i} + a_{12}\frac{a_3 + (1 - m)i}{a_3 + (1 - m)i}}{a_3 + (1 - 3m)i}, \quad (4.6c) \]
\[ W_4 > a_9 \left[ \frac{a_7s}{a_3 + (1 - 3m)i} \right] \left[ \frac{a_8i}{a_3 + (1 - 3m)i} \right] \] 
\[ a_9 \frac{a_7s}{a_3 + (1 - 3m)i} > a_9 \frac{a_3w - s\frac{a_3 + (1 - 3m)i}{a_3 + (1 - 3m)i}}{a_3 + (1 - m)s^2}, \quad (4.6d) \]

Where:
\[ W_1 = \frac{a_3a_{11}w}{a_2\left[ a_3 + (1 - m)s \right]}, \]
\[ W_2 = \frac{a_4\left[ (1 - 2s - (1 + a_1)i)[a_3 + (1 - m)s] - a_2a_3w \right]}{a_1a_2\left[ a_3 + (1 - 3m)s \right] \left[ a_5 + (1 - 3m)i \right]}, \]
\[ W_3 = \frac{1 - (1 + a_1)i}{2\left[ a_3 + (1 - m)s \right]^2 - a_2a_3w}, \]
\[ \frac{\left[ (1 - 2s - (1 + a_1)i)[a_3 + (1 - m)s] - a_2a_3w \right]}{a_1a_2\left[ a_3 + (1 - 3m)s \right] \left[ a_5 + (1 - 3m)i \right]}, \]
\[ W_4 > a_9 \left[ \frac{a_7s}{a_3 + (1 - 3m)i} \right] \left[ \frac{a_8i}{a_3 + (1 - 3m)i} \right] \] 
\[ a_9 \frac{a_7s}{a_3 + (1 - 3m)i} > a_9 \frac{a_3w - s\frac{a_3 + (1 - 3m)i}{a_3 + (1 - 3m)i}}{a_3 + (1 - m)s^2}, \quad (4.6d) \]

Where:
\[ W_1 = \frac{a_3a_{11}w}{a_2\left[ a_3 + (1 - m)s \right]}, \]
\[ W_2 = \frac{a_4\left[ (1 - 2s - (1 + a_1)i)[a_3 + (1 - m)s] - a_2a_3w \right]}{a_1a_2\left[ a_3 + (1 - 3m)s \right] \left[ a_5 + (1 - 3m)i \right]}, \]
\[ W_3 = \frac{1 - (1 + a_1)i}{2\left[ a_3 + (1 - m)s \right]^2 - a_2a_3w}, \]
\[ \frac{\left[ (1 - 2s - (1 + a_1)i)[a_3 + (1 - m)s] - a_2a_3w \right]}{a_1a_2\left[ a_3 + (1 - 3m)s \right] \left[ a_5 + (1 - 3m)i \right]}, \]
Straightforward computation shows that:

\[
\Delta = \frac{a_{11}}{a_3 + (1 - m)s} \cdot \frac{a_{12}}{a_5 + (1 - m)i}.
\]

Hence, \( \Delta \) will be positive if in addition to conditions (4.6a) – (4.6e), the following condition hold

\[
K_1 > K_2,
\]

Therefore, all the eigenvalues of \( J \) have negative real part under the given conditions and hence \( E_4 \) is locally asymptotically stable.

However, it is unstable otherwise.

5. Global stability analysis:

In this section the global stability analysis for the equilibrium points, which are locally asymptotically stable, of system (2) is studied analytically with the help of lyapunov method, as it is shown in the following theorems:

Theorem (2):

Assume that the disease and predator free equilibrium point \( E_1 = (1,0,0,0) \) of system (2) is locally asymptotically stable in \( R^4_+ \). Then \( E_1 \) is globally asymptotically stable on the sub region \( \omega_1 \subset R^4_+ \) provided that the following condition hold:

\[
\frac{a_0}{a_1} > (1 + a_1).
\]

Proof: consider the following function:

\[
Q_1(s, i, z, w) = (s - 1 - \ln s) + i + z + w.
\]

It is clearly to see that \( Q_1(s, i, z, w) \in C^1(R^4_+, R), Q_1(E_1) = 0 \) and \( Q_1(s, i, z, w) > 0; \forall (s, i, z, w) \neq E_1 \).

Now by differentiating \( Q_1 \) with respect to time \( t \) and going some algebraic handing, given that

\[
\frac{dQ_1}{dt} = -(s - i)^2 - si - [a_6 - (1 + a_1)]\frac{s^2}{a_2 + (1 - m)s} - [a_4 - (a_0 + a_{12})] \frac{iw}{a_5 + (1 - m)i}.
\]

Now, according to the conditions in theory (1), and (5.1a) we obtain \( \frac{dQ_1}{dt} < 0 \). Hence \( E_1 \) is a globally asymptotically stable and then the proof is complete.

Theorem (3): assume that the predators free equilibrium point \( E_2 = (\tilde{s}, i, \tilde{z}, \tilde{w}) \) of system (2) is a locally asymptotically stable in \( R^4_+ \). Then \( E_2 \) is a globally asymptotically stable on the sub region \( \omega_2 \subset R^4_+ \) that satisfies the following conditions:

\[
\frac{a_{13}}{a_3 + (1 - 3m)3s} + \frac{a_{12}}{a_5 + (1 - 3m)i}, \quad \frac{a_{41}}{a_3 + (1 - 3m)3s} + \frac{a_{42}}{a_5 + (1 - 3m)i}.
\]

(5.2a)

Therefore, all the eigenvalues of \( J \) have negative real part under the given conditions and hence \( E_4 \) is locally asymptotically stable.
\[ i > i \quad . \] (5.2c)

**Proof:** consider the following function
\[ Q_2(s, i, z, w) = \left( s - \frac{s - \bar{s} \ln \frac{s}{\bar{s}}}{s} \right) + \left( i - \frac{i - \bar{i} \ln \frac{i}{\bar{i}}}{i} \right) + z + w. \]

It is easy to see that \[ Q_2(s, i, z, w) \in C^1(R^4_+, R), \] and \[ Q_3(s, i, z, w) > 0, \forall (s, i, z, w) \neq E_2. \] Now by differentiating \[ Q_2 \] with respect to time \( t \) going some algebraic handling, given that:
\[ a_{13} \left( - \frac{a_3}{a_3 + (1 - 3m)s} + \frac{a_4}{a_5 + (1 - 3m)i} \right), \]
\[ a_{11} \left( a_3 + (1 - m)s \right) \left( a_3 + (1 - 3m)s \right). \]

Now, by using the condition (5.2b) and (5.2c) with the biological facts those are mentioned in theorem (1), we obtain that:
\[ \frac{dQ_2}{dt} < - (s - \bar{s})^2 - a_{10} z - \left[ a_{13} - \left( \frac{a_2}{a_3 + (1 - m)s} + \frac{a_4}{a_5 + (1 - 3m)i} \right) w^3 - a_{10} z - \frac{a_2 - (a_4 + a_{11})}{a_5 + (1 - 3m)i} w \right]. \]

Then \( \frac{dQ_2}{dt} < 0 \) under condition (5.2a).

Hence, \( E_2 \) is a globally asymptotically stable and then the proof is complete.

**Theorem (5.3):** Assume that the free disease equilibrium point \( E_3 = (s, 0, 2, \bar{w}) \) of system (2) is locally asymptotically stable in \( R^4_+ \). Then \( E_3 \) is a globally asymptotically stable on any region \( \mathcal{O}_3 \subset R^4_+ \) that satisfies the following conditions:
\[ C_1 \leq \sqrt{AB}, \quad (5.3a) \]
\[ C_2 \leq \sqrt{BB}, \quad (5.3b) \]
\[ C_3 \leq \sqrt{BB}, \quad (5.3c) \]
\[ a_6 > (1 + a_1) \bar{s}, \quad (5.3d) \]
\[ 1 > \frac{a_2}{a_{13}[a_3 + (1 - m)s][a_3 + (1 - 3m)s]}, \quad (5.3e) \]
\[ a_{13}[a_3 + (1 - m)s] > a_{11}s, \quad (5.5) \]
\[ \dot{A} = 1 - \frac{a_2(1 - m)3\bar{w}}{[a_3 + (1 - 3m)s][a_3 + (1 - 3m)s]}, \]
\[ \dot{B} = \frac{Z}{a_9 + a_{10}}, \]
\[ \dot{C}_1 = \frac{a_{12}a_3 + (1 - m)s - a_{11}s}{w[a_3 + (1 - m)s]}, \]
\[ \dot{C}_2 = \frac{a_7\bar{w} + a_9\bar{w}[a_3 + (1 - m)s]}{a_3a_7\bar{w}}, \]
\[ \dot{C}_3 = \frac{a_7\bar{w} + a_9\bar{w}[a_3 + (1 - m)s]}{a_3a_7\bar{w}}. \]

**Proof:** Consider the following function:
\[ Q_3(s, i, z, w) = \left( 3s - \bar{s} - 3\bar{s} \ln \frac{s}{\bar{s}} \right) + i + \left( z - \bar{z} \ln \frac{z}{\bar{z}} \right) + \left( w - \bar{w} \right) \left( w - \bar{w} \right). \]
It is easy to see that $Q_3(s, i, z, w) \in C^1(R^4, R)$, $Q_3(E_3) = 0$, and $Q_3(s, i, z, w) > 0$, $\forall (s, i, z, w) \neq E_3$ now by differentiating $Q_3$ with respect to time $t$ going some algebraic handling, given that:

$$
\frac{dQ_3}{dt} = - \frac{1}{32} \left( 13 - \frac{a_2(1 - m)\bar{w}}{w[a_3 + (1 - m)s][a_3 + (1 - m)s]} \right) (3s - \bar{s})^2 + \\
\frac{a_3a_1\bar{w} - a_2w[a_3 + (1 - m)s]}{w[a_3 + (1 - m)s][a_3 + (1 - m)s]} (s - \bar{s})(w - \bar{w}) - \\
\left[ \frac{a_3[a_3 + (1 - m)s] - a_{11}\bar{s}}{2w[a_3 + (1 - m)s]} \right] (w - \bar{w})^2 - \\
\frac{1}{2} \left[ \frac{1}{[a_3 + (1 - m)s][a_3 + (1 - m)s]} \right] (s - \bar{s})^2 + \\
\frac{a_3a_7\bar{w}(s - 3\bar{s})(z - 2)}{z[a_3 + (1 - m)s]} - \frac{a_9 + a_10}{2z} (z - 2)(w - \bar{w}) - \\
\frac{1}{2} \left[ \frac{a_3[a_3 - (1 + 3m)s] - a_{11}\bar{s}}{2w[a_3 + (1 - 3m)s]} \right] (w^3 - \bar{w})^2 - \frac{[a_9 - (1 + a_{12})]\bar{w}}{z[a_3 + (1 - m)s]} - \frac{[a_9\bar{w} + a_{12z}\bar{w}]i}{z(a_3 + (1 - m)s)}.
$$

Now, by using the conditions (5.3a) – (5.f) with the biological facts those are mentioned in theorem (1), we obtain that:

$$
\frac{dQ_3}{dt} < - \left[ \sqrt{\frac{A}{2}} (s - \bar{s}) - \sqrt{\frac{B}{2}} (w - \bar{w}) \right]^2 - \left[ \sqrt{\frac{A}{2}} (s - \bar{s}) - \sqrt{\frac{B}{2}} (z - 2) \right]^2 - \\
\left[ \sqrt{\frac{B}{2}} (z - \bar{z}) - \sqrt{\frac{B}{2}} (w - \bar{w}) \right]^2 - \frac{(a_9\bar{w} + a_{12z}\bar{w})i}{z(a_3 + (1 - m)s)}.
$$

Hence $\frac{dQ_3}{dt}$ is negative definite and hence $Q_3$ is strictly lyapunov function. Thus we obtain that $E_3$ is a globally asymptotically stable and then the proof is complete.

Theorem (5): Assume that the positive equilibrium point $E_4(s, \bar{s}, \bar{w}, \bar{w})$ of system (2) is locally asymptotically stable in the $R^4_+$. Then $E_4$ is a globally asymptotically stable on any region $\omega_4 \subset R^4_+$ if that satisfies the following conditions hold:

$$
C_1 \leq \sqrt{x_1x_2}, \quad C_2 \leq \sqrt{x_1x_2}, \\
(5.4a) \\
a_4(a_5 + (1 + m)i) > a_5a_{12}, \quad (5.4d) \\
i > \bar{i}, \quad (5.4e) \\
w > \bar{w}, \quad (5.4f) \\
z > \bar{z}, \quad (5.4g) \\
q_1 > q_2, \quad (5.4h) \\
1 > \frac{a_2(1 - m)\bar{w}}{[a_3 + (1 - m)s][a_3 + (1 - m)s]}, \quad (5.4i)
$$
\[ a_{13} > \frac{a_{11} \bar{s}}{[a_3 + (1 - m)\bar{s}]^2} + \frac{a_{12} i}{[a_5 + (1 - m)\bar{i}]}, \]  

where:

\[ A_i^* = 1 - \frac{a_2(1 - m)\bar{w}}{[a_3 + (1 - m)s][a_3 + (1 - m)\bar{s}]}, \quad B^* = \frac{a_9 + a_{10}}{z} \]

\[ D_i^* = \frac{1}{w} \left[ a_{13} - \left( \frac{a_{11} \bar{s}}{[a_3 + (1 - m)s][a_3 + (1 - m)\bar{s}]} + \frac{a_{12} i}{[a_5 + (1 - m)\bar{i}]} \right) \right], \]

\[ C_i^* = \frac{a_2}{z} \frac{a_5}{[a_3 + (1 - m)s][a_3 + (1 - m)\bar{s}]} - a_3 a_{11} \]

\[ C_3^* = \frac{1}{z} \left( \frac{a_5}{[a_3 + (1 - m)s][a_3 + (1 - m)\bar{s}]} + \frac{a_5 i}{a_5 + (1 - m)i} + \frac{a_5 z}{w} \right) \]

\[ q_1 = \left[ \frac{\sqrt{A^*}}{2} (s - \bar{s}) - \frac{\sqrt{B^*}}{2} (z - \bar{z}) \right]^2 + \left[ \frac{\sqrt{A^*}}{2} (s - \bar{s}) - \frac{\sqrt{B^*}}{2} (w - \bar{w}) \right]^2 + \]

\[ \left[ \frac{\sqrt{B^*}}{2} (z - \bar{z}) - \frac{\sqrt{D^*}}{2} (w - \bar{w}) \right]^2 + \left[ \frac{a_5 a_{10}}{z[a_5 + (1 - m)i][a_5 + (1 - m)\bar{i}]} \right] \left( \begin{array}{c} 3 i - \bar{z} \\ 3w - \bar{w} \end{array} \right). \]

\[ q_2 = \frac{a_4}{[a_5 + (1 - m3)i][a_5 + (1 - m3)i]} \]

**Proof:** Consider the following function:

\[ Q_4(s, i, z, w) = \left( s - \bar{s} - s \ln \frac{s}{\bar{s}} \right) + \left( i - \bar{i} - i \ln \bar{i} \right) + \left( z - \bar{z} - z \ln \bar{z} \right) + \]

\[ \left( w - \bar{w} - w \ln \bar{w} \right). \]

Clearly \( Q_4(s, i, z, w) \in C^4(R^4, R), \ Q_4(E_4) = 0, \text{ and } Q_4(s, i, z, w) > 0, \forall (s, i, z, w) \neq E_4. \) Now by differentiating \( Q_4 \) with respect to time \( t \) going some algebraic handling, given that:

\[
\frac{dQ_4}{dt} = - \left( \frac{31}{[a_3 + (1 - m)s][a_3 + (1 - m)\bar{s}]} \right) (s3 - \bar{s})^2 - \]

\[
\frac{a_2(1 - m)\bar{w}}{[a_3 + (1 - m)s][a_3 + (1 - m)\bar{s}] - a_3 a_{11} (s3 - \bar{s})(w3 - \bar{w})} - \]

\[
\frac{a_4}{[a_3 + (1 - m3)i][a_3 + (1 - m3)i]} \left( \begin{array}{c} 3 i - \bar{z} \\ 3w - \bar{w} \end{array} \right). \]

\[
\frac{(a_9 + a_{10})}{z} (x3 - \bar{x})^2 + \frac{a_5 a_{10}}{z[a_5 + (1 - m3)i][a_5 + (1 - m3)i]}. \]
Now, by using the conditions (5.4a) – (5.4d) and (5.4i) – (5.4j) we obtain that:

\[
\frac{dQ_4}{dt} \leq -\left[ \frac{A^*}{2} (s - \bar{s}) + \frac{B^*}{2} (z - \bar{z}) \right]^2 - \left[ \frac{A^*}{2} (s3 - \bar{s}) + \frac{D^*}{2} (3w - \bar{w}) \right]^2 + \frac{a_5 a_9 w (i - \bar{i}) (3z - \bar{z})}{z [a_5 + (1 - m)i] [a_5 + (1 - m)i]} + \frac{a_4 (1 - m) w (i - \bar{i})^2}{[a_5 + (1 - m)i] [a_5 + (1 - m)i]} + \frac{a_4 (1 - m) w (i - \bar{i})}{[a_5 + (1 - m)i] [a_5 + (1 - m)i]} = -q_1 + q_2.
\]

Clearly, \(\frac{dQ_4}{dt}\) is negative definite on the region \(\hat{\omega}_4\) due to the conditions (5.4e) – (5.4h). Hence \(Q_4\) is strictly Lyapunov function. Thus \(E_4\) is a globally asymptotically stable on the region \(\hat{\omega}_4\) and the proof is complete.

6. numerical simulation:
In this section, the dynamical behaviour of system (2) is studied numerically for a set of parameters and different sets of initial points. The objectives of this study are: investigate the effect of varying the value of each parameter on the dynamical behaviour of system (2) and confirm the obtained analytical results.

It is observed that, for the following set of hypothetical parameters that satisfies stability conditions of the positive equilibrium point, system (2) has a globally asymptotically stable positive equilibrium point as shown in figure (36.1) (a – d).

\[
\begin{align*}
\alpha_3 & = 2, \quad \alpha_2 = 0.33, \quad \alpha_3 = 0.5, \quad \alpha_4 = 0.2, \quad \alpha_5 = 0.5, \\
\alpha_6 & = 0.01, \quad \alpha_7 = 0.1, \quad \alpha_8 = 0.1, \quad \alpha_9 = 0.5, \quad \alpha_{10} = 0.01, \\
\alpha_{11} & = 0.1, \quad \alpha_{12} = 0.1, \quad \alpha_{13} = 0.1, \quad m = 0.7.
\end{align*}
\]

(6.1)
Figure (6.1): The time series of the solution of system (2) started from the three different initial points (0.4, 0.5, 0.6, 0.6), (0.4, 0.5, 0.7, 0.39) and (0.8, 0.9, 1.5, 0.5) for the data given by eq. (6.1). (a) the trajectories of $s$ as a function of time, (b) the trajectories of $i$ as a function of time, (c) trajectories of $z$ as a function of time, (d) the trajectories of $w$ as a function of time.

Clearly, figure (6.1) shows that system (2) has a globally asymptotically stable as the solution of system (2) approaches asymptotically to the positive equilibrium point $E_4 = (0.233, 0.050, 0.116, 1.173)$ starting from three different initial points and this is confirming our obtained analytical results.

Now, in order to discuss the effect of the parameters values of system (2) on the dynamical behaviour of the system. The system is solved numerically for the data given in eq. (6.1) with varying one parameter at each time and sometime two parameters the obtained results are given in table (1).

Table (1): Numerical Behaviours of System (2) for the Data Given in (6.1) with Varying One Parameter at each Time

| Range of Parameter | Numerical Behavior of System (2) | Bifurcation Point |
|--------------------|---------------------------------|-------------------|
| $0.1 \leq a_1 < 1.9$ $1.9 \leq a_1 < 2.2$ | Approaches to the infected prey free equilibrium point $E_3$ approaches to the positive equilibrium point $E_4$ | $a_1 = 1.9$ |
| $0.1 \leq a_2 < 0.29$ $0.29 \leq a_2 < 1$ | Approaches to the infected prey free equilibrium point $E_3$ approaches to the positive equilibrium point $E_4$ | $a_2 = 0.29$ |
| $0.25 < a_3 \leq 0.4$ $0.4 < a_3 \leq 1$ | Approaches to the infected prey free equilibrium point $E_3$ approaches to the positive equilibrium point $E_4$ | $a_3 = 0.4$ |
| $0.1 \leq a_4 < 0.22$ $0.22 \leq a_4 < 1$ | Approaches to the positive equilibrium point $E_4$ Approaches to the infected prey free equilibrium point $E_3$ | $a_4 = 0.22$ |
Some text. The effect of varying the predation rate on susceptible prey $a_2$ in the range $0.1 \leq a_2 < 0.29$ keeping other parameters as data given in eq. (6.1), causes extinction in the infected prey and the system will approach to the infected prey free equilibrium point $E_3$, as shown in fig. (6.2) (a), for typical value $a_2 = 0.2$. In the range $0.29 \leq a_2 < 1$, it is observed that the solution of system (2) approaches asymptotically to the positive equilibrium point $E_4$, as shown in fig. (6.2) (b) for typical value $a_2 = 0.35$.

Fig. (6.2) (a): Time series of the solution of system (2) approaches asymptotically to the infected prey free equilibrium point $E_3 = (0.298, 0, 0.205, 2.070)$ in the positive quadrant of $szw$ – space. For the data given in eq. (6.1) with $a_2 = 0.2$. (b): time series of the solution of system (2) approaches asymptotically to the positive equilibrium point $E_4 = (0.130, 0.144, 0.067, 0.677)$ in the int. of $R_+^4$. For the data given in eq. (6.1) with $a_2 = 0.35$.

The varying of the parameter $a_7$ which represents the conversion rate from the susceptible prey to the immature predator in the range $0.0001 \leq a_7 < 0.11$, and keeping the rest of parameters values as data given in eq. (6.1), the solution of system (2) still approaches asymptotically to the positive equilibrium point $E_4$, as shown in fig. (6.3) (a), for typical value $a_7 = 0.08$, however increasing this parameter further $0.11 \leq a_7 \leq 0.15$ it is observed that the system (2) still approach to the infected prey free equilibrium point $E_3$, as shown in fig. (6.3) (b), for typical value $a_7 = 0.12$.

The varying of the parameter $a_7$ which represents the conversion rate from the susceptible prey to the immature predator in the range $0.0001 \leq a_7 < 0.11$, and keeping the rest of parameters values as data given in eq. (6.1), the solution of system (2) still approaches asymptotically to the positive equilibrium point $E_4$, as shown in fig. (6.3) (a), for typical value $a_7 = 0.08$, however increasing this parameter further $0.11 \leq a_7 \leq 0.15$ it is observed that the system (2) still approach to the infected prey free equilibrium point $E_3$, as shown in fig. (6.3) (b), for typical value $a_7 = 0.12$. 

| $0.1 \leq a_5 < 0.45$ | Approaches to the infected prey free equilibrium point $E_3$ approaches to the positive equilibrium point $E_4$ $a_5 = 0.45$ |
|-----------------------|------------------------------------------------------------------|
| $0.45 \leq a_5 < 0.6$ | $0 \leq a_6 < 0.3$ approaches to the positive equilibrium point $E_4$ approaches to the infected prey free equilibrium point $E_3$ $a_7 = 0.11$ |
| $0.3 \leq a_6 < 1$   | $0.0001 \leq a_7 < 0.11$ approaches to the positive equilibrium point $E_4$ approaches to the infected prey free equilibrium point $E_3$ $a_7 = 0.11$ |
| $0.11 \leq a_7 < 0.15$ | $0.01 \leq a_{13} < 0.97$ approaches to the infected prey free equilibrium point $E_3$ approaches to the positive equilibrium point $E_4$ $a_{13} = 0.97$ |
| $0.97 \leq a_{13} < 0.13$ |
The effect of varying the infection rate of prey in the range $0.1 \leq a_1 < 1.9$ keeping other parameter as data given in eq.(6.1) is studied, it is observed that system (2) approach asymptotically to the infected prey free equilibrium point $E_2$, as shown in figure (6.2)(a), for typical value $a_1 = 0.8$. In the range $1.9 \leq a_1 < 2.2$, it is observed that the solution of system (2) approaches asymptotically to the positive equilibrium point $E_4$, as shown in figure (6.2) (b) for typical value $a_1 = 1.97$.

The effect of varying the predation rate on susceptible prey $a_2$ in the range $0.1 \leq a_2 < 0.29$ keeping other parameters as data given in eq.(6.1), causes extinction in the infected prey and the system will approach to the infected prey free equilibrium point $E_3$, as shown in fig.(6.3) (a), for typical value $a_2 = 0.2$. In the range $0.29 \leq a_2 < 1$, it is observed that the solution of system (2) approaches asymptotically to the positive equilibrium point $E_4$, as shown in fig. (6.3) (b) for typical value $a_2 = 0.35$.

![Fig. (6.3): (a) Time series of the solution of system (2) for the data given by (6.1) with $a_7 = 0.08$, which approaches to $E_4 = (0.150, 0.139, 0.074, 0.785)$. (b) Time series of the solution of system (2) for the data given by eq.(6.1) with $a_7 = 0.12$, which approaches to $E_3 = (0.267, 0, 0.153, 1.460)$](image-url)
Figure (6.3) (a): Time series of the solution of system (2) approaches asymptotically to the infected prey free equilibrium point \( E_3 = (0.298, 0, 0.205, 2.070) \) in the positive quadrant of.szw–space. For the data given in eq. (6.1) with \( a_2 = 0.2 \). Figure (b): Time series of the solution of system (2) approaches asymptotically to the positive equilibrium point \( E_4 = (0.130, 0.144, 0.067, 0.677) \) in the int. of \( R_+^4 \). For the data given in eq. (6.1) with \( a_2 = 0.35 \).

Moreover varying the number of prey inside the refuge parameter \( m \) and keeping the rest of parameters values as data given in eq. (6.1), while the increasing of this parameter in the range \( 0.4 < m \leq 1 \) the solution of system (2) approaches asymptotically to the positive equilibrium point \( E_4 \), as shown in Fig. (6.15), for typical value \( m = 0.5 \).

Moreover varying the parameters \( a_1, a_2, a_3, a_4, a_6, a_7, a_8, a_{11}, a_{12}, \) and \( m \) into the following values which satisfies conditions (4.4b), (4.4c), (4.4f) and (5.2a) – (5.2c).

| Parameter | Value       |
|-----------|-------------|
| \( a_1 \) | 0.6         |
| \( a_2 \) | 0.05        |
| \( a_3 \) | 0.7         |
| \( a_4 \) | 0.7         |
| \( a_5 \) | 0.5         |
| \( a_6 \) | 0.1         |
| \( a_7 \) | 0.05        |
| \( a_8 \) | 0.05        |
| \( a_9 \) | 0.5         |
| \( a_{10} \) | 0.01   |
| \( a_{11} \) | 0.05       |
| \( a_{12} \) | 0.05       |
| \( a_{13} \) | 0.1        |
| \( m \)   | 0.2         |

(6.2)

It is observed that system (2) approach asymptotically to the free predators equilibrium point \( E_2 = (s, \tilde{i}, 0, 0, 0) \), as shown in Fig. (6.16) and this is confirming our obtained analytical results.

Figure (6.16): Time series of the solution of system (2) approaches asymptotically to the axial equilibrium point \( E_2(\tilde{s}, \tilde{i}, 0, 0, 0) \) for the data given in eq. (6.2)

Finally, varying the parameters \( a_1, a_2, a_3, a_4, a_6, a_7, a_8, a_9, a_{10} \) and \( a_{13} \) into the following values which satisfies conditions (4.a – 4.c) and (5.1a),
It is observed that system (2) approach asymptotically to the axial equilibrium point $E_1 = (1,0,0,0)$, as shown in fig. (6.17) and this is confirming our obtained analytical results.

Fig.(4.17): Time series of the solution of system (2) approaches asymptotically to the axial equilibrium point $E_1(1,0,0,0)$ for the data given in eq.(6.3).

7. Conclusions and Discussion:

In this paper, an eco-epidemiological mathematical model consisting of prey-predator model involving SI infectious disease in prey and stage structured predator species with prey refuge has been proposed and analysed. Further, in this model, Holling type II of functional responses for the predation of susceptible and infected prey which are outside refuge as well as linear incidence rate for describing the transition of disease are used. Our aim is to study the role of infectious disease on the dynamics. Also system (2) has been solved numerically for different sets of initial points and different sets of parameters starting with the hypothetical set of data given by eq. System (6.1) and the following observations are obtained.

- System (2) has only one type of attractor in $\mathbb{R}^4$. It approaches to globally stable point.
- For the set hypothetical parameters value given in eq. (6.1), the system (2) approaches asymptotically to globally stable positive point $E_4 = (0.234,0.049,0.117,1.178)$. 
- As the infection rate of prey $a_1$ increasing to 1.9 keeping the rest of parameters as in eq. (6.1), the solution of system (2) approaches to infected free equilibrium point $E_3$. however if $1.9 \leq a_1 < 2.2$, then the infected prey will grow again and then the trajectory transferred from infected prey free equilibrium point to the positive equilibrium point $E_4$, thus, the parameter $a_1 = 1.9$ is a bifurcation point.
- As the attack rate of mature predator on susceptible prey $a_2$ increasing to 0.29 keeping the rest of parameters as in eq. (6.1), the solution of system (2) approaches to infected free equilibrium point $E_3$. however if $0.29 \leq a_2 < 1$, then the infected prey will grow again and then the trajectory transferred from infected prey free equilibrium point to the positive equilibrium point $E_4$, thus, the parameter $a_2 = 0.29$ is a bifurcation point.
- As the half saturation rate of the predator upon the susceptible prey increasing to 0.4 keeping the rest of parameters as in eq.(6.1), the solution of system (2) approaches to the disease free equilibrium point $E_3 = (\hat{s},0,\hat{z},\bar{w})$, in the range $0.4 < a_3 \leq 1$ the trajectory transferred from the disease free equilibrium point $E_3$ to the positive equilibrium point $E_4$ ,thus $a_3 =0.4$ is a bifurcation point.
- As the attack rate of mature predator on infected prey $a_4$ increasing to 0.22 keeping the rest of parameters as in eq. (6.1), the solution of system (2) approaches to the positive equilibrium point...
in the range \(0.22 \leq a_4 < 1\) the trajectory transferred from the positive equilibrium point \(E_4\) to the disease free equilibrium point \(E_3\), thus \(a_4 = 0.22\), is a bifurcation point.

- As the half saturation rate of the predator upon the infected prey increasing to 0.45 while keeping the rest of parameters as in eq. (6.1), the solution of system (2) approaches to the disease free equilibrium point \(E_3 = (\tilde{s}, 0, \tilde{z}, \tilde{w})\), in the range \(0.45 \leq a_5 < 0.6\) and the trajectory transferred from the disease free equilibrium point \(E_3\) to the positive equilibrium point \(E_4\), thus \(a_5 = 0.45\), is a bifurcation point.

- Moreover increasing the parameter \(a_6\), which represent the death rate of the infected prey due to the disease to 0.3, and keeping the rest of parameter values as data given in eq. (6.1), the solution of system (2) approaches to the positive equilibrium point \(E_4\). Therefore, \(a_6 = 0.3\) is a bifurcation point.

- Now increasing the conversion rate of food from susceptible prey to immature predator to 0.11, keeping the rest of parameters as in eq. (6.1), the solution of system (2) approaches to the infected prey free equilibrium point \(E_3\), in the range \(0.097 \leq a_{13} < 0.15\), the trajectory transferred from the infected prey free equilibrium point \(E_3\) to the positive equilibrium point \(E_4\), thus, \(a_{13} = 0.097\) is a bifurcation point.

As the natural death rate of mature predators \(a_{13}\) increasing to 0.097 while keeping the rest of parameters as in eq. (6.1), the solution of system (2) approaches to the infected prey free equilibrium point \(E_4\), in the range \(0.097 \leq a_{13} < 0.15\), the trajectory transferred from the infected prey free equilibrium point \(E_3\) to the positive equilibrium point \(E_4\), thus, \(a_{13} = 0.097\) is a bifurcation point.

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