Solutions of the Quantum Yang-Baxter Equation with Extra Non-Additive Parameters

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Abstract:

We present a systematic technique to construct solutions to the Yang-Baxter equation which depend not only on a spectral parameter but in addition on further continuous parameters. These extra parameters enter the Yang-Baxter equation in a similar way to the spectral parameter but in a non-additive form.

We exploit the fact that quantum non-compact algebras such as $U_q(su(1,1))$ and type-I quantum superalgebras such as $U_q(gl(1|1))$ and $U_q(gl(2|1))$ are known to admit non-trivial one-parameter families of infinite-dimensional and finite dimensional irreps, respectively, even for generic $q$.

We develop a technique for constructing the corresponding spectral-dependent R-matrices. As examples we work out the the $R$-matrices for the three quantum algebras mentioned above in certain representations.

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1 Introduction

Quantized universal enveloping algebras (quantum algebras) provide a powerful tool for finding solutions to the spectral-dependent quantum Yang-Baxter equation (QYBE). There exists one such solution, with trigonometric dependence on the spectral parameter, for every pair of representations of any quantum affine Lie algebra. Through the work of many authors a large number of such solutions has now been constructed, see references in . These solutions depend on a parameter \( q \), the deformation parameter of the quantum algebras. This parameter \( q \) is very different in nature from the spectral parameter, because \( q \) does not enter into the Yang-Baxter equation.

It is clearly desirable to have families of solutions of the Yang-Baxter equation depending continuously on extra parameters, entering in a similar way to the spectral parameter. In this paper we develop a method for the construction of such families of solutions. The extra parameters enter the Yang-Baxter equation in a non-additive form .

Solutions of the Yang-Baxter equation have various different applications, for example as Boltzmann weights of integrable lattice models or as scattering matrices in integrable quantum field theories. In all these applications the freedom of having extra continuous parameters opens up new and exciting possibilities. We will come back to this in the discussion section.

The origin of the extra parameters in our solutions are the parameters which are carried by the irreps of the associated quantum algebra. Murakami describes such parameters as colors carried by irreps. For the type-I quantum superalgebras, there are nontrivial one-parameter families of finite-dimensional irreps for generic \( q \). For quantum simple bosonic Lie algebras families of unitary finite-dimensional representations are possible only when the deformation parameter \( q \) is a root of unity. However, they do admit parametrized families of unitary infinite-dimensional irreps even for generic \( q \).

The main aim of this paper is to find solutions to the QYBE with extra non-additive parameters associated with both the infinite-dimensional irreps of a quantum simple Lie algebra and the finite-dimensional irreps of quantum superalgebras. In section we develop a systematic and useful technique which is very much in the spirit of the techniques in designed to find solutions of the QYBE acting on the tensor product module of three different irreps of a quantum algebra. As concrete examples, we work out the solutions (\( R \)-matrices) associated with a one-parameter family of infinite dimensional irreps for the quantum non-compact algebra \( U_q(su(1,1)) \) in section , and the \( R \)-matrices associated with a one-parameter family of finite-dimensional irreps for the quantum superalgebras \( U_q(gl(1|1)) \) and \( U_q(gl(2|1)) \) in section .

\footnote{The chiral Potts model has a spectral parameter which enters the Yang-Baxter equation in a non-additive form. This solution arises from quantum groups at \( q \) a root of unity only . These are not related to our solutions however, in which the spectral parameter stays additive and the extra parameters are non-additive. Our solutions exist for generic \( q \).}
2 General Formalism

Let $G$ denote a simple Lie (super)algebra of rank $r$ with generators $\{e_i, f_i, h_i\}$ and let $\alpha_i$ be its simple roots. Then the quantum (super)algebra $U_q(G)$ can be defined with the structure of a ($\mathbb{Z}_2$-graded) quasi-triangular Hopf algebra \[1\] \[4\]. We will not give the full defining relations of $U_q(G)$ here but mention that $U_q(G)$ has a coproduct structure given by

$$
\Delta(q^{h_i/2}) = q^{h_i/2} \otimes q^{h_i/2}, \quad \Delta(a) = a \otimes q^{-h_i/2} + q^{h_i/2} \otimes a, \quad a = e_i, f_i.
$$

(2.1)

The multiplication rule for the tensor product is defined for elements $a, b, c, d \in U_q(G)$ by

$$
(a \otimes b)(c \otimes d) = (-1)^{|b||c|}(ac \otimes bd)
$$

(2.2)

where $|a| \in \mathbb{Z}_2$ denotes the degree of the element $a$.

Let $\pi_{\Phi}$ be a one-parameter family of irreps of $U_q(G)$ afforded by the irreducible module $V(\Phi)$ in such a way that the highest weight of the irrep depends on the parameter $\Phi$. Assume for any parameter $\Phi$ that the irrep $\pi_{\Phi}$ is affinizable, i.e. it can be extended to an irrep of the corresponding quantum affine (super)algebra $U_q(\hat{G})$. Consider an operator $R^{\Phi_1 \Phi_2}(x) \in \text{End}(V(\Phi_1) \otimes V(\Phi_2))$, where $x \in \mathbb{C}$ is the usual spectral parameter and $\pi_{\Phi_1}, \pi_{\Phi_2}$ are two irreps from the one-parameter family. It has been shown by Jimbo \[3\] that a solution to the linear equations

$$
R^{\Phi_1 \Phi_2}(x) \Delta^{\Phi_1 \Phi_2}(a) = \Delta^{\Phi_1 \Phi_2}(a) R^{\Phi_1 \Phi_2}(x), \quad \forall a \in U_q(G),
$$

$$
R^{\Phi_1 \Phi_2}(x) \left( x \pi_{\Phi_1}(e_0) \otimes \pi_{\Phi_2}(q^{-h_0/2}) + \pi_{\Phi_1}(q^{h_0/2}) \otimes \pi_{\Phi_2}(e_0) \right)
$$

$$
= \left( x \pi_{\Phi_1}(e_0) \otimes \pi_{\Phi_2}(q^{h_0/2}) + \pi_{\Phi_1}(q^{-h_0/2}) \otimes \pi_{\Phi_2}(e_0) \right) R^{\Phi_1 \Phi_2}(x)
$$

(2.3)

satisfies the QYBE in the tensor product module $V(\Phi_1) \otimes V(\Phi_2) \otimes V(\Phi_3)$ of three irreps from the one-parameter family:

$$
R^{\Phi_1 \Phi_2}(x)R^{\Phi_3}(xy)R^{\Phi_2 \Phi_3}(y) = R^{\Phi_2 \Phi_3}(y)R^{\Phi_3}(xy)R^{\Phi_1 \Phi_2}(x).
$$

(2.4)

In the above, $\Delta = T \cdot \Delta$, with $T$ the twist map defined by $T(a \otimes b) = (-1)^{|a||b|}b \otimes a$, $\forall a, b \in U_q(G)$ and $\Delta^{\Phi_1 \Phi_2}(a) = (\pi_{\Phi_1} \otimes \pi_{\Phi_2})\Delta(a)$; also, if $R^{\Phi_1 \Phi_2}(x) = \sum_i \pi_{\Phi_1}(a_i) \otimes \pi_{\Phi_2}(b_i)$, then $R^{\Phi_1 \Phi_2}(x) = \sum_i \pi_{\Phi_1}(a_i) \otimes \pi_{\Phi_2}(b_i) \otimes I$ etc. Jimbo also showed that the solution to (2.3) is unique, up to scalar functions. The multiplicative spectral parameter $x$ can be transformed into an additive spectral parameter $u$ by $x = \exp(u)$.

In all our equations we implicitly use the "graded" multiplication rule of eq. (2.2). Thus the $R$-matrix of a quantum superalgebra satisfies a "graded" Yang-Baxter equation which, when written as an ordinary matrix equation, contains extra signs:

$$
\left( R^{\Phi_1 \Phi_2}(x) \right)_{\alpha \beta}^{\alpha' \beta'} \left( R^{\Phi_1 \Phi_3}(xy) \right)_{\alpha' \gamma'}^{\alpha'' \gamma''} \left( R^{\Phi_2 \Phi_3}(y) \right)_{\beta' \gamma'}^{\beta'' \gamma''} ( -1 )^{[\alpha][\beta]+[\gamma][\alpha']+[\gamma'][\beta']}
$$

$$
= \left( R^{\Phi_2 \Phi_3}(y) \right)_{\beta \gamma}^{\beta' \gamma'} \left( R^{\Phi_1 \Phi_3}(xy) \right)_{\alpha' \gamma'}^{\alpha'' \gamma''} \left( R^{\Phi_1 \Phi_2}(x) \right)_{\alpha'' \beta'}^{\alpha' \beta'} ( -1 )^{[\beta][\gamma]+[\gamma'][\alpha]+[\beta'][\alpha']},
$$

(2.5)
both sides of which act from $V = 0$ which is usually called the unitarity condition in the literature. Consider three special cases:

The equation, if written in matrix form, does not have extra signs in the superalgebra case. This means that the signs disappear from the equation. Thus any solution of the “graded” Yang-Baxter equation $\alpha$ arising from the R-matrix of a quantum superalgebra provides also a solution of the standard Yang-Baxter equation after the redefinition in eq. (2.6).

Now introduce the (graded) permutation operator $P^{\Phi_1 \Phi_2}$ on the tensor product module $V(\Phi_1) \otimes V(\Phi_2)$ such that

$$P^{\Phi_1 \Phi_2}(v_\alpha \otimes v_\beta) = (-1)^{[\alpha][\beta]} v_\beta \otimes v_\alpha, \ \forall v_\alpha \in V(\Phi_1), \ v_\beta \in V(\Phi_2)$$

and set

$$\tilde{R}^{\Phi_1 \Phi_2}(x) = P^{\Phi_1 \Phi_2}R^{\Phi_1 \Phi_2}(x).$$

Then (2.3) can be rewritten as

$$\tilde{R}^{\Phi_1 \Phi_2}(x)\Delta^{\Phi_1 \Phi_2}(a) = \Delta^{\Phi_2 \Phi_1}(a)\tilde{R}^{\Phi_1 \Phi_2}(x), \ \forall a \in U_q(G),$$

$$\tilde{R}^{\Phi_1 \Phi_2}(x) \left( x\pi_{\Phi_1}(e_0) \otimes \pi_{\Phi_2}(q^{-h_0/2}) + \pi_{\Phi_1}(q^{h_0/2}) \otimes \pi_{\Phi_2}(e_0) \right) = \left( \pi_{\Phi_2}(e_0) \otimes \pi_{\Phi_1}(q^{-h_0/2}) + x\pi_{\Phi_2}(q^{h_0/2}) \otimes \pi_{\Phi_1}(e_0) \right) \tilde{R}^{\Phi_1 \Phi_2}(x)$$

and in terms of $\tilde{R}^{\Phi_1 \Phi_2}(x)$ the QYBE becomes

$$(I \otimes \tilde{R}^{\Phi_1 \Phi_2}(x))(\tilde{R}^{\Phi_1 \Phi_3}(x y) \otimes I)(I \otimes \tilde{R}^{\Phi_2 \Phi_3}(y)) = (\tilde{R}^{\Phi_2 \Phi_3}(y) \otimes I)(I \otimes \tilde{R}^{\Phi_1 \Phi_3}(x y))(\tilde{R}^{\Phi_1 \Phi_2}(x) \otimes I)$$

both sides of which act from $V(\Phi_1) \otimes V(\Phi_2) \otimes V(\Phi_3)$ to $V(\Phi_3) \otimes V(\Phi_2) \otimes V(\Phi_1)$. Note that this equation, if written in matrix form, does not have extra signs in the superalgebra case. This is because the definition of the graded permutation operator in eq. (2.7) includes the signs of eq. (2.6).

In order to solve eqs. (2.9), we use a method similar to the one developed in [6] for a quite different problem. We will normalize the $R$-matrix $\tilde{R}^{\Phi_1 \Phi_2}(x)$ in such a way that

$$\tilde{R}^{\Phi_1 \Phi_2}(x)\tilde{R}^{\Phi_2 \Phi_1}(x^{-1}) = I$$

which is usually called the unitarity condition in the literature. Consider three special cases: $x = 0, \ x = \infty$ and $x = 1$. For these special values of $x$, $\tilde{R}^{\Phi_1 \Phi_2}(x)$ satisfies the spectral-free, but extra non-additive parameter-dependent QYBE,

$$(I \otimes \tilde{R}^{\Phi_1 \Phi_2})(\tilde{R}^{\Phi_1 \Phi_3} \otimes I)(I \otimes \tilde{R}^{\Phi_2 \Phi_3}) = (\tilde{R}^{\Phi_2 \Phi_3} \otimes I)(I \otimes \tilde{R}^{\Phi_1 \Phi_3})(\tilde{R}^{\Phi_1 \Phi_2} \otimes I).$$

Moreover, from (2.9), we have respectively, for $x = 0$,

$$\tilde{R}^{\Phi_1 \Phi_2}(0)\Delta^{\Phi_1 \Phi_2}(a) = \Delta^{\Phi_2 \Phi_1}(a)\tilde{R}^{\Phi_1 \Phi_2}(0), \ \forall a \in U_q(G),$$

$$\tilde{R}^{\Phi_1 \Phi_2}(0) \left( \pi_{\Phi_1}(q^{h_0/2}) \otimes \pi_{\Phi_2}(e_0) \right) = \left( \pi_{\Phi_2}(e_0) \otimes \pi_{\Phi_1}(q^{-h_0/2}) \right) \tilde{R}^{\Phi_1 \Phi_2}(0)$$

(2.13)
for $x = \infty$,
\[ \hat{R}^{\Phi_1 \Phi_2}(\infty) \Delta^{\Phi_1 \Phi_2}(a) = \Delta^{\Phi_2 \Phi_1}(a) \hat{R}^{\Phi_1 \Phi_2}(\infty), \quad \forall a \in U_q(G), \]
\[ \hat{R}^{\Phi_1 \Phi_2}(\infty) \left( \pi_{\Phi_1}(e_0) \otimes \pi_{\Phi_2}(q^{-h_0/2}) \right) = \left( \pi_{\Phi_2}(q^{h_0/2}) \otimes \pi_{\Phi_1}(e_0) \right) \hat{R}^{\Phi_1 \Phi_2}(\infty) \quad (2.14) \]

and for $x = 1$,
\[ \hat{R}^{\Phi_1 \Phi_2}(1) \Delta^{\Phi_1 \Phi_2}(a) = \Delta^{\Phi_2 \Phi_1}(a) \hat{R}^{\Phi_1 \Phi_2}(1), \quad \forall a \in U_q(G), \]
\[ \hat{R}^{\Phi_1 \Phi_2}(1) \left( \pi_{\Phi_1}(e_0) \otimes \pi_{\Phi_2}(q^{-h_0/2}) + \pi_{\Phi_1}(q^{h_0/2}) \otimes \pi_{\Phi_2}(e_0) \right) = \left( \pi_{\Phi_2}(e_0) \otimes \pi_{\Phi_1}(q^{-h_0/2}) + \pi_{\Phi_2}(q^{h_0/2}) \otimes \pi_{\Phi_1}(e_0) \right) \hat{R}^{\Phi_1 \Phi_2}(1). \quad (2.15) \]

Eqs. (2.13), (2.14) and (2.15) respectively admit a unique solution for any given one-parameter family of irreps of $U_q(G)$, provided such representations are consistently affinizable. In the case of a multiplicity-free tensor product decomposition we may write
\[ V(\Phi_1) \otimes V(\Phi_2) = \bigoplus_{\mu} V(\mu), \quad (2.16) \]
where $\mu$ denotes a highest weight depending on the parameters $\Phi_1$ and $\Phi_2$. Let $\{ |e_\mu^\alpha \rangle_{\Phi_1 \otimes \Phi_2} \}$ be an orthonormal basis for $V(\mu)$ in $V(\Phi_1) \otimes V(\Phi_2)$. $V(\mu)$ is also embedded in $V(\Phi_2) \otimes V(\Phi_1)$ through the opposite coproduct $\hat{\Delta}$. Let $\{ |e_\alpha^\mu \rangle_{\Phi_2 \otimes \Phi_1} \}$ be the corresponding orthonormal basis $\footnote{For the precise definition of this basis see Appendix C of [3].}$.

Using these bases we define operators $P^{\Phi_1 \Phi_2}$ and $P^{\Phi_1 \Phi_2}$
\[ P^{\Phi_1 \Phi_2} = \sum_\alpha |e_\alpha^\mu \rangle_{\Phi_2} \otimes \Phi_1 \otimes \Phi_2 \langle e_\alpha^\mu |, \]
\[ P^{\Phi_1 \Phi_2} = \sum_\alpha |e_\alpha^\mu \rangle_{\Phi_2} \otimes \Phi_1 \otimes \Phi_2 \langle e_\alpha^\mu |. \quad (2.17) \]

Clearly the $P^{\Phi_1 \Phi_2} : V(\Phi_1) \otimes V(\Phi_2) \rightarrow V(\mu) \subset V(\Phi_1) \otimes V(\Phi_2)$ are projection operators. The $P^{\Phi_1 \Phi_2} : V(\Phi_1) \otimes V(\Phi_2) \rightarrow V(\mu) \subset V(\Phi_2) \otimes V(\Phi_1)$ are the elementary intertwiners, i.e.,
\[ P^{\Phi_1 \Phi_2} \Delta^{\Phi_1 \Phi_2}(a) = \Delta^{\Phi_2 \Phi_1}(a) P^{\Phi_1 \Phi_2}, \quad \forall a \in U_q(G) \quad (2.18) \]

and $P^{\Phi_1 \Phi_2}$ and $P^{\Phi_1 \Phi_2}$ satisfy the relations
\[ P^{\Phi_1 \Phi_2} = \delta_{\mu \mu'} P^{\Phi_1 \Phi_2} = \delta_{\mu \mu'} P^{\Phi_1 \Phi_2}, \]
\[ P^{\Phi_1 \Phi_2} = \delta_{\mu \mu'} P^{\Phi_1 \Phi_2}, \quad \sum_\mu P^{\Phi_1 \Phi_2} = I. \quad (2.19) \]

We can show $\footnote{For the precise definition of this basis see Appendix C of [3].}$ that the solutions of eqs. (2.13), (2.14) and (2.15) take the particularly simple forms,
\[ \hat{R}^{\Phi_1 \Phi_2}(0) = \sum_\mu \epsilon(\mu) q^{C(\mu) - C(\Phi_1) - C(\Phi_2)} P^{\Phi_1 \Phi_2} \]
\[ \hat{R}^{\Phi_1 \Phi_2}(\infty) = \sum_\mu \epsilon(\mu) q^{C(\mu) - C(\Phi_1) - C(\Phi_2)} P^{\Phi_1 \Phi_2} \]
\[ \hat{R}^{\Phi_1 \Phi_2}(1) = \sum_\mu P^{\Phi_1 \Phi_2} \quad (2.20) \]
where $C(\Lambda) = (\Lambda, \Lambda + 2\rho)$ is the eigenvalue of the quadratic Casimir invariant of $G$ in the irrep with highest weight $\Lambda$, $\rho$ is the half-sum of positive roots of $U_q(G)$, and $\epsilon(\mu)$ is the parity of $V(\mu)$ in $V(\Phi_1) \otimes V(\Phi_2)$.

Here we illustrate the proof of the last relation in (2.20). From the unitarity condition (2.11) it follows that for $x = 1$

$$\tilde{R}^{\Phi_1_\Phi_2(1)}(1)\tilde{R}^{\Phi_2\Phi_1}(1) = I.$$  \hspace{1cm} \text{(2.21)}

We write $\tilde{R}^{\Phi_1_\Phi_2(1)}$ in the general form

$$\tilde{R}^{\Phi_1_\Phi_2(1)} = \sum_{\mu} \rho(1) P_{\mu}^{\Phi_1_\Phi_2}.$$  \hspace{1cm} \text{(2.22)}

Observing (2.21) we at once see that $\rho(1)$ satisfies $(\rho(1))^2 = 1$, so that $\rho(1) = \pm 1$. By examining the limit $\Phi_1 \to \Phi_2$, which we can do because $\Phi_1$ and $\Phi_2$ are continuous parameters (such arguments are not valid for the case considered in [6] and thus the derivation of $\tilde{R}(1)$ there is much more subtle), and using that when $\Phi_1 = \Phi_2$, $P_{\mu}^{\Phi_1_\Phi_1}$ are the usual projection operators and $\tilde{R}^{\Phi_1_\Phi_1}(1)$ is the identity, one can conclude that the $\rho(1)$ appearing in (2.22) must equal 1 identically, thus completing the proof.

We remark that in the present case $\Phi_1$ etc. are continuous parameters and so the parities $\epsilon(\mu)$ in (2.20) can easily be worked out by examining the limit $\Phi_1 \to \Phi_2$, in contrast to the case considered in [6].

Multiplying the second equation in (2.13) by $P_{\mu}^{\Phi_1_\Phi_2}$ from the left and by $P_{\nu}^{\Phi_1_\Phi_2}$ from the right, and using (2.20) and (2.13) we obtain

$$P_{\mu}^{\Phi_1_\Phi_2} \left( \pi_{\Phi_1}(e_0) \otimes \pi_{\Phi_2}(q^{h_0/2}) \right) P_{\mu}^{\Phi_1_\Phi_2} = P_{\mu}^{\Phi_2_\Phi_1} \left( \pi_{\Phi_2}(e_0) \otimes \pi_{\Phi_1}(q^{-h_0/2}) \right) P_{\mu}^{\Phi_1_\Phi_2},$$

$$\epsilon(\mu) q^{C(\mu)/2} P_{\mu}^{\Phi_1_\Phi_2} \left( \pi_{\Phi_1}(q^{h_0/2}) \otimes \pi_{\Phi_2}(e_0) \right) P_{\nu}^{\Phi_1_\Phi_2}$$

$$= \epsilon(\nu) q^{-C(\nu)/2} P_{\mu}^{\Phi_2_\Phi_1} \left( \pi_{\Phi_2}(e_0) \otimes \pi_{\Phi_1}(q^{-h_0/2}) \right) P_{\mu}^{\Phi_1_\Phi_2}, \quad \forall \mu \neq \nu.$$ \hspace{1cm} \text{(2.23)}

Similarly, from (2.14) and (2.13), we obtain

$$P_{\mu}^{\Phi_1_\Phi_2} \left( \pi_{\Phi_1}(e_0) \otimes \pi_{\Phi_2}(q^{h_0/2}) \right) P_{\mu}^{\Phi_1_\Phi_2} = P_{\mu}^{\Phi_2_\Phi_1} \left( \pi_{\Phi_2}(q^{h_0/2}) \otimes \pi_{\Phi_1}(e_0) \right) P_{\mu}^{\Phi_1_\Phi_2},$$

$$\epsilon(\mu) q^{-C(\mu)/2} P_{\mu}^{\Phi_1_\Phi_2} \left( \pi_{\Phi_1}(e_0) \otimes \pi_{\Phi_2}(q^{-h_0/2}) \right) P_{\nu}^{\Phi_1_\Phi_2}$$

$$= \epsilon(\nu) q^{-C(\nu)/2} P_{\mu}^{\Phi_2_\Phi_1} \left( \pi_{\Phi_2}(q^{h_0/2}) \otimes \pi_{\Phi_1}(e_0) \right) P_{\mu}^{\Phi_1_\Phi_2}, \quad \forall \mu \neq \nu.$$ \hspace{1cm} \text{(2.24)}

and

$$P_{\mu}^{\Phi_1_\Phi_2} \left( \pi_{\Phi_1}(e_0) \otimes \pi_{\Phi_2}(q^{-h_0/2}) \right) P_{\mu}^{\Phi_1_\Phi_2} = P_{\mu}^{\Phi_2_\Phi_1} \left( \pi_{\Phi_2}(e_0) \otimes \pi_{\Phi_1}(q^{-h_0/2}) \right) P_{\mu}^{\Phi_1_\Phi_2},$$

$$P_{\mu}^{\Phi_1_\Phi_2} \left( \pi_{\Phi_1}(e_0) \otimes \pi_{\Phi_2}(q^{-h_0/2}) \right) P_{\nu}^{\Phi_1_\Phi_2}$$

$$= P_{\mu}^{\Phi_2_\Phi_1} \left( \pi_{\Phi_2}(e_0) \otimes \pi_{\Phi_1}(q^{-h_0/2}) \right) P_{\nu}^{\Phi_1_\Phi_2}, \quad \forall \mu \neq \nu.$$ \hspace{1cm} \text{(2.25)}

In deriving (2.25), eqs (2.23) and eqs. (2.24) have been considered.
Now the most general $\hat{R}^{\Phi_1, \Phi_2}(x)$ satisfying the first equation in (2.26) may be written in the form

$$\hat{R}^{\Phi_1, \Phi_2}(x) = \sum_{\nu} \rho_\mu(x) \mathbf{D}^{\Phi_1, \Phi_2}_\mu$$

(2.26)

where $\rho_\mu(x)$, are unknown functions depending on $x$, $q$ and the extra non-additive parameters. Inserting the above equation into the second equation of (2.29) and multiplying the resultant equation by $\mathcal{P}^{\Phi_1, \Phi_2}_\mu$ from the left and by $\mathcal{P}^{\Phi_1, \Phi_2}_\nu$ from the right, and then using (2.23), (2.24) and (2.25) to simplify the resulting equation, one finally finds

$$\left\{ \rho_\mu(x) \left( xq^{C(\mu)/2} + \epsilon(\mu)e(\nu)q^{C(\nu)/2} \right) - \rho_\nu(x) \left( q^{C(\mu)/2} + \epsilon(\mu)e(\nu)xq^{C(\nu)/2} \right) \right\} \times \mathcal{P}^{\Phi_1, \Phi_2}_\mu \left( \pi_{\Phi_1}(e_0) \otimes \pi_{\Phi_2}(q^{-h_0/2}) \right) \mathcal{P}^{\Phi_1, \Phi_2}_\nu = 0, \quad \forall \mu \neq \nu$$

(2.27)

In many cases it is possible to determine when $\mathcal{P}^{\Phi_1, \Phi_2}_\mu \left( \pi_{\Phi_1}(e_0) \otimes \pi_{\Phi_2}(q^{-h_0/2}) \right) \mathcal{P}^{\Phi_1, \Phi_2}_\nu \neq 0$ and thus to obtain a solution $\rho_\mu(x)$ to the system of equations (2.27), recursively given by

$$\rho_\mu(x) = \rho_\nu(x) \frac{q^{C(\mu)/2} + \epsilon(\mu)e(\nu)xq^{C(\nu)/2}}{xq^{C(\mu)/2} + \epsilon(\mu)e(\nu)q^{C(\nu)/2}}, \quad \forall \mu \neq \nu.$$ 

(2.28)

In the following sections we will work out three examples.

3 $U_q(su(1,1))$, Its One-Parameter Family of Irreps and $R$-Matrix with Non-Additive Parameter

The quantum algebra $U_q(su(1,1))$ is defined by generators $\{e, f, q^h\}$ and relations

$$q^h e q^{-h} = q e, \quad q^h f q^{-h} = q^{-1} f, \quad [e, f] = -\frac{q^{2h} - q^{-2h}}{q - q^{-1}}$$

(3.1)

with the following coproduct

$$\Delta(q^h) = q^h \otimes q^h, \quad \Delta(e) = e \otimes q^{-h} + q^h \otimes e, \quad \Delta(f) = f \otimes q^{-h} + q^h \otimes f$$

(3.2)

Note here that we have used a slightly different normalization for the generator $h$ compared with the one in section 2.

Infinite-dimensional unitary irreps of $U_q(su(1,1))$ have been described by several authors [12] [13] [14]. The classification is similar to that in the classical case, but there are some new features, notably the appearance of a ‘strange’ series of unitary irreps [14] with no classical analogue. For simplicity, we shall assume here that $q \neq 1$ is real and positive, and consider the infinite-dimensional unitary irreps $D^\pm(\Phi)$ from the so-called discrete series. The irrep $D^-(\Phi)$ has a highest weight $\Phi$ but no lowest weight, and $D^+(\Phi)$ has a lowest weight $-\Phi$ but no highest weight. In each case, the real parameter $\Phi$ can take any negative value.
The structure of $D^{-}(\Phi)$ is very simple: Let $V_{\Phi}$ be a complex Hilbert space with orthonormal basis $\{v_{\Phi+\mu} : \mu = 0, -1, -2, \cdots \}$, and set
\[
\begin{align*}
h v_{\Phi+\mu} &= (\Phi + \mu) v_{\Phi+\mu}, \\
e v_{\Phi+\mu} &= ([\mu]_{q} [\mu + 2\Phi + 1]_{q})^{1/2} v_{\Phi+\mu+1}, \\
f v_{\Phi+\mu} &= ([\mu - 1]_{q} [\mu + 2\Phi]_{q})^{1/2} v_{\Phi+\mu-1}.
\end{align*}
\]
where and throughout this paper,
\[
[q]_{q}^{-n} = \frac{q^{n} - q^{-n}}{q - q^{-1}}.
\]
The tensor product $D^{-}(\Phi_{1}) \otimes D^{-}(\Phi_{2})$ is completely reducible, and is easily seen to be
\[
D^{-}(\Phi_{1}) \otimes D^{-}(\Phi_{2}) = \bigoplus_{\mu=0}^{-\infty} D^{-}(\Phi_{1} + \Phi_{2} + \mu).
\]
The vector $\Omega_{\Phi_{1} + \Phi_{2} + \mu}$, corresponding to the highest weight $\Phi_{1} + \Phi_{2} + \mu$, $\mu = 0, -1, -2, \cdots$ in the component $D^{-}(\Phi_{1} + \Phi_{2} + \mu)$, has the form
\[
\Omega_{\Phi_{1} + \Phi_{2} + \mu} = c \sum_{i=0}^{\mu} (-1)^{i} q^{i(\Phi_{1} + \Phi_{2} + \mu + 1)} \Gamma_{q}^{1/2}(2\Phi_{1} + \mu - i + 1) \Gamma_{q}^{1/2}(\mu - i)
\]
\[
\cdot \Gamma_{q}^{1/2}(2\Phi_{2} + i + 1) \Gamma_{q}^{1/2}(i) v_{\Phi_{1}+\mu-i} \otimes v_{\Phi_{2}+i},
\]
where $c$ is a normalization constant, and $\Gamma_{q}(z)$ is a $q$-gamma function satisfying $\Gamma_{q}(z+1) = [z]_{q} \Gamma_{q}(z)$. To verify (3.4), it is enough to check that $\Omega_{\Phi_{1} + \Phi_{2} + \mu}$ has the correct weight and is annihilated by $\Delta(e)$.

Similar results hold for the structure of $D^{+}(\Phi)$ and for the tensor product $D^{+}(\Phi_{1}) \otimes D^{+}(\Phi_{2})$, but will not be used here.

All irreps of $U_{q}(su(1,1))$ are affinizable. In particular, we have the identification of $e_{0}$, $h_{0}$ in section 2 with $f$, $h$ in this section: $e_{0} \leftrightarrow f$, $h_{0}/2 \leftrightarrow -h$. In the case of (3.5), eqs. (2.20) take the form
\[
\begin{align*}
\tilde{R}_{\Phi_{1}\Phi_{2}}(0) &= \sum_{\mu=0}^{-\infty} (-1)^{\mu} q^{I(\Phi_{1}+\Phi_{2}+\mu)-I(\Phi_{1})-I(\Phi_{2})} P_{\mu}^{\Phi_{1}\Phi_{2}} \\
\tilde{R}_{\Phi_{1}\Phi_{2}}(\infty) &= \sum_{\mu=0}^{-\infty} (-1)^{\mu} q^{-I(\Phi_{1}+\Phi_{2}+\mu)+I(\Phi_{1})+I(\Phi_{2})} P_{\mu}^{\Phi_{1}\Phi_{2}} \\
\tilde{R}_{\Phi_{1}\Phi_{2}}(1) &= \sum_{\mu=0}^{-\infty} P_{\mu}^{\Phi_{1}\Phi_{2}},
\end{align*}
\]
where $I(\Lambda) = \Lambda(\Lambda + 1)$ is the eigenvalue of the $su(1,1)$ quadratic Casimir invariant, $I = h(h + 1) - fe = h(h-1) - ef$, in the irrep with highest weight $\Lambda$, and now $P_{\mu}^{\Phi_{1}\Phi_{2}} : D^{-}(\Phi_{1}) \otimes D^{-}(\Phi_{2}) \rightarrow D^{-}(\Phi_{1} + \Phi_{2} + \mu)$. Eqs. (3.5) imply the identification of $\epsilon(\mu)$, $C(\mu)$ in (2.27): $\epsilon(\mu) \leftrightarrow (-1)^{\mu}$ and $C(\mu)/2 \leftrightarrow I(\Phi_{1} + \Phi_{2} + \mu)$.

We will now determine $\rho_{\mu}(x)$ in (2.27) for this case. Observe 3 that the tensor operator $\Delta(q^{-h})(f \otimes q^{h})$ behaves like a component of the adjoint tensor operator of $U_{q}(su(1,1))$. Therefore, $P_{\mu}^{\Phi_{1}\Phi_{2}}(\pi_{\Phi_{1}}(f) \otimes \pi_{\Phi_{2}}(q^{h})) P_{\nu}^{\Phi_{1}\Phi_{2}}$, $\mu \neq \nu$, vanishes unless the two highest weights $\Phi_{1} + \Phi_{2} + \mu$
and $\Phi_1 + \Phi_2 + \nu$, associated with $P_{\mu}^{\Phi_1\Phi_2}$ and $P_{\nu}^{\Phi_1\Phi_2}$, respectively, differ by a non-zero weight of the adjoint representation of $U_q(su(1,1))$, that is by a root of $su(1,1)$. This implies that $P_{\mu}^{\Phi_1\Phi_2}(\pi_{\Phi_1}(f) \otimes \pi_{\Phi_2}(q^h))P_{\nu}^{\Phi_1\Phi_2} \equiv 0$ for $\mu \neq \nu$ unless $\mu = \nu \pm 1$. Therefore, from (2.27),

$$\frac{\rho_\mu(x)}{\rho_\nu(x)} = \frac{q^{I(\Phi_1+\Phi_2+\mu)} x q^{I(\Phi_1+\Phi_2+\nu)} - (-1)^{\mu+\nu} x q^{I(\Phi_1+\Phi_2+\nu)}}{x q^{I(\Phi_1+\Phi_2+\nu)}}, \quad \mu = \nu \pm 1. \quad (3.8)$$

It follows immediately that

$$\rho_\mu(x) = \rho_0(x) \prod_{\nu=-1}^{\mu} \frac{1 - x q^{2(\Phi_1+\Phi_2+\nu)}}{x q^{2(\Phi_1+\Phi_2+\nu)}}, \quad \mu = -1, -2, \ldots \quad (3.9)$$

which in turn leads to the quantum $R$-matrix

$$\hat{R}_{\Phi_1\Phi_2}(x) = \sum_{\mu=0}^{\infty} \prod_{\nu=-1}^{\mu} \frac{1 - x q^{2(\Phi_1+\Phi_2+\nu)}}{x q^{2(\Phi_1+\Phi_2+\nu)}} P_{\mu}^{\Phi_1\Phi_2} \quad (3.10)$$

(it should be understood that $\prod_{\nu=-1}^{0} (\cdots) = 1$), where the scalar factor $\rho_0(x)$ has been absorbed.

4 $U_q(gl(1|1)), \ U_q(gl(2|1))$, One-Parameter Families of Irreps and $R$-Matrix with Non-Additive Parameters

It is well known [10] that type-I superalgebras admit nontrivial one-parameter families of finite-dimensional irreps which deform to provide one-parameter families of finite-dimensional irreps of the corresponding type-I quantum superalgebras [10]. Here we are only concerned with $U_q(gl(m|n))$, all irreps of which are known to be affinizable.

Choose $\{\varepsilon_i\}_{i=1}^{m} \cup \{\bar{\varepsilon}_j\}_{j=1}^{n}$ as a basis for the dual of the Cartan subalgebra of $gl(m|n)$ satisfying

$$(\varepsilon_i, \varepsilon_j) = \delta_{ij}, \quad (\varepsilon_i, \bar{\varepsilon}_j) = -\delta_{ij}, \quad (\varepsilon_i, \bar{\varepsilon}_j) = 0. \quad (4.1)$$

Using this basis, any weight $\Lambda$ may written as

$$\Lambda \equiv (\Lambda_1, \ldots, \Lambda_m | \bar{\Lambda}_1, \ldots, \bar{\Lambda}_n) \equiv \sum_{i=1}^{m} \Lambda_i \varepsilon_i + \sum_{j=1}^{n} \bar{\Lambda}_j \bar{\varepsilon}_j \quad (4.2)$$

and the graded half sum $\rho$ of the positive roots of $gl(m|n)$ is

$$2\rho = \sum_{i=1}^{m} (m - n - 2i + 1) \varepsilon_i + \sum_{j=1}^{n} (m + n - 2j + 1) \bar{\varepsilon}_j. \quad (4.3)$$

In what follows we will consider the one-parameter family of finite-dimensional irreducible $U_q(gl(m|n))$-modules $V(\alpha)$ with highest weights of the form $\Lambda(\alpha) = (0, \ldots, 0|\alpha, \ldots, \alpha)$.

We first consider the one-parameter family of two-dimensional irreps $V(\alpha)$ of $U_q(gl(1|1))$ with the highest weight $\Lambda(\alpha) = (0|\alpha)$. Assuming that $\alpha \neq 0 \neq \beta$, $\alpha + \beta \neq 0$, we have the following decomposition

$$V(\alpha) \otimes V(\beta) = V(\Lambda_1) \bigoplus V(\Lambda_2) \quad (4.4)$$
where \( \Lambda_1 = (0|\alpha + \beta) \) and \( \Lambda_2 = (-1|\alpha + \beta + 1) \). The Casimir operator takes the values

\[
C(\alpha) = -\alpha(\alpha + 1), \quad C(\beta) = -\beta(\beta + 1)
\]

\[
C(\Lambda_1) = -(\alpha + \beta)(\alpha + \beta + 1)
\]

\[
C(\Lambda_2) = -\alpha(\alpha + \beta)^2 - 3(\alpha + \beta) .
\]

(4.5)

By considering the limit \( q \to 1, \alpha \to \beta \), it follows that \( \epsilon(\Lambda_1) = -\epsilon(\Lambda_2) = 1 \). Also it is easy to conclude that \( \mathcal{P}_{\Lambda_1}^{\alpha\beta}(\pi_{\alpha}(e_0) \otimes \pi_{\beta}(q^{-\alpha_0})/2)) \mathcal{P}_{\Lambda_2}^{\alpha\beta} \neq 0 \) and \( \mathcal{P}_{\Lambda_1}^{\alpha\beta}(\pi_{\alpha}(e_0) \otimes \pi_{\beta}(q^{-\alpha_0}/2)) \mathcal{P}_{\Lambda_2}^{\alpha\beta} \neq 0 \). Thus from \( [2,27] \) we have

\[
\rho_{\Lambda_2}(x) = \frac{1 - x\alpha^{\alpha+\beta}}{x - \alpha^{\alpha+\beta}} \rho_{\Lambda_1}(x)
\]

(4.6)

which gives rise to the properly normalized quantum R-matrix

\[
\tilde{R}^{\alpha\beta}(x) = \mathcal{P}_{\Lambda_1}^{\alpha\beta} + \frac{1 - x\alpha^{\alpha+\beta}}{x - \alpha^{\alpha+\beta}} \mathcal{P}_{\Lambda_2}^{\alpha\beta}
\]

(4.7)

where again the scalar factor \( \rho_{\Lambda_1}(x) \) has been absorbed.

It can be shown that the elementary intertwiners in the above equation take the form

\[
\mathcal{P}_{\Lambda_1}^{\alpha\beta} = [\alpha + \beta]_q^{-1} \begin{pmatrix}
[\alpha + \beta]_q & 0 & 0 & 0 \\
0 & ([\alpha]_q[\beta]_q)^{1/2} q^{(\alpha+\beta)/2} & [\alpha]_q & 0 \\
0 & [\beta]_q & ([\alpha]_q[\beta]_q)^{1/2} q^{-(\alpha+\beta)/2} & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
\]

(4.8)

\[
\mathcal{P}_{\Lambda_2}^{\alpha\beta} = [\alpha + \beta]_q^{-1} \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & ([\alpha]_q[\beta]_q)^{1/2} q^{-(\alpha+\beta)/2} & -[\beta]_q & 0 \\
0 & -[\alpha]_q & ([\alpha]_q[\beta]_q)^{1/2} q^{(\alpha+\beta)/2} & 0 \\
0 & 0 & 0 & [\alpha + \beta]_q
\end{pmatrix}.
\]

The details of derivations and other interesting material will be published in a separate paper \( [17] \). With the help of \( (4.8) \) the R-matrix \( (4.7) \) reads

\[
\tilde{R}^{\alpha\beta}(x) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -([\alpha]_q[\beta]_q)^{1/2} q^{(\alpha+\beta)/2} \frac{q^{-1}}{x - \alpha^{\alpha+\beta}} & [\alpha]_q[\beta]_q & 0 \\
0 & [\beta]_q - [\alpha]_q & [\alpha + \beta]_q & \frac{[\alpha]_q[\beta]_q}{[\alpha + \beta]_q} \frac{q^{-1}}{x - \alpha^{\alpha+\beta}} \\
0 & 0 & 0 & \langle \alpha + \beta \rangle
\end{pmatrix}
\]

(4.9)

where

\[
\langle \alpha + \beta \rangle \equiv \frac{1 - x\alpha^{\alpha+\beta}}{x - \alpha^{\alpha+\beta}} .
\]

(4.10)

In the limit \( x = 0 \), we obtain the braid group representation \( \sigma^{\alpha\beta} \equiv \tilde{R}^{\alpha\beta}(0) \):

\[
\sigma^{\alpha\beta} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & ([\alpha]_q[\beta]_q)^{1/2} q^{-(\alpha+\beta)/2} (q - q^{-1}) & q^{-\beta} & 0 \\
0 & q^{-\alpha} & 0 & 0 \\
0 & 0 & 0 & -q^{-\alpha - \beta}
\end{pmatrix}
\]

(4.11)
For the special case $\alpha = \beta = 1$ this braid group representation has been known. It was obtained from $U_q(gl(1|1))$ in [18, 19].

We now come to the one-parameter family of four-dimensional irreps $V(\alpha)$ [20] of $U_q(gl(2|1))$ with the highest weight $\Lambda(\alpha) = (0,0|\alpha)$. Assuming that $\alpha \neq 0 \neq \beta$, $\alpha + \beta \neq -1$, we have the decomposition [21]

$$V(\alpha) \otimes V(\beta) = V(\Lambda_1) \bigoplus V(\Lambda_2) \bigoplus V(\Lambda_3)$$ (4.12)

where $\Lambda_1 = (0, 0|\alpha + \beta)$, $\Lambda_2 = (-1, -1|\alpha + \beta + 2)$ and $\Lambda_3 = (0, -1|\alpha + \beta + 1)$. Thus

$$C(\alpha) = -\alpha(\alpha + 2), \quad C(\beta) = -\beta(\beta + 2)$$
$$C(\Lambda_1) = - (\alpha + \beta)(\alpha + \beta + 2)$$
$$C(\Lambda_2) = 4 - (\alpha + \beta + 2)(\alpha + \beta + 4)$$
$$C(\Lambda_3) = 3 - (\alpha + \beta + 1)(\alpha + \beta + 3).$$ (4.13)

By considering the limit $q \to 1$, $\alpha \to \beta$, we see that $\epsilon(\Lambda_1) = \epsilon(\Lambda_2) = -\epsilon(\Lambda_3) = 1$. Also we can conclude in this case that

$$P_{\Lambda_1}^{\alpha\beta}(\pi_\alpha(e_0) \otimes \pi_\beta(q^{-h_0/2}))P_{\Lambda_3}^{\alpha\beta} \neq 0$$
$$P_{\Lambda_2}^{\alpha\beta}(\pi_\alpha(e_0) \otimes \pi_\beta(q^{-h_0/2}))P_{\Lambda_1}^{\alpha\beta} \neq 0$$
$$P_{\Lambda_3}^{\alpha\beta}(\pi_\alpha(e_0) \otimes \pi_\beta(q^{-h_0/2}))P_{\Lambda_2}^{\alpha\beta} \neq 0$$
$$P_{\Lambda_1}^{\alpha\beta}(\pi_\alpha(e_0) \otimes \pi_\beta(q^{-h_0/2}))P_{\Lambda_3}^{\alpha\beta} \neq 0.$$ (4.14)

Thus from (2.27) we get

$$\rho_{\Lambda_3}(x) = \frac{1 - xq^{\alpha+\beta}}{x - q^{\alpha+\beta}} \rho_{\Lambda_1}(x)$$
$$\rho_{\Lambda_2}(x) = \frac{1 - xq^{\alpha+\beta}}{x - q^{\alpha+\beta}} \frac{1 - xq^{\alpha+\beta+2}}{x - q^{\alpha+\beta+2}} \rho_{\Lambda_1}(x)$$ (4.15)

which lead to the quantum $R$-matrix

$$\check{R}^{\alpha\beta}(x) = P_{\Lambda_1}^{\alpha\beta} + \frac{1 - xq^{\alpha+\beta}}{x - q^{\alpha+\beta}} P_{\Lambda_3}^{\alpha\beta} + \frac{1 - xq^{\alpha+\beta}}{x - q^{\alpha+\beta}} \frac{1 - xq^{\alpha+\beta+2}}{x - q^{\alpha+\beta+2}} P_{\Lambda_2}^{\alpha\beta}$$ (4.16)

where again the scalar factor $\rho_{\Lambda_1}(x)$ has been absorbed. We have explicit expressions for the elementary intertwiners appearing in this expression as $16 \times 16$ matrices which we will publish in [17].

5 Conclusion

We have developed a systematic technique for constructing solutions to the QYBE with extra non-additive parameters. The technique is a generalization of that used in [3] for a different problem. We have treated in particular solutions associated to families of representations of the quantum algebras $U_q(su(1,1))$, $U_q(gl(1|1))$ and $U_q(gl(2|1))$. The technique can be applied
to other quantum (super)algebras with families of irreducible highest weight or lowest weight representations. We can also apply the technique to families of representations of Yangians and will then obtain rational \( R \)-matrices with extra parameters. Our general expressions (2.20) for the braid generators can also be used to construct multivariable link invariants from quantum (super)algebras.

We expect that the possibility of extra parameters in solutions of the Yang-Baxter equation will open up many new applications. The physical interpretation of the extra parameters will depend on the particular application. An example:

The scattering matrices for quantum excitations in integrable 2-dimensional quantum field theories are given by solutions of the Yang-Baxter equation. The ratios between the masses of the quantum excitations are determined by the locations of the poles of the S-matrices. Because these poles are fixed in all known non-trivial crossing symmetric \( R \)-matrices, these \( R \)-matrices are able to describe only theories in which the ratios of the quantum masses are fixed to particular values. In our \( R \)-matrices the locations of the poles depend on the extra parameters. So if our \( R \)-matrices are used as S-matrices, these parameters have the interpretation of adjustable quantum masses in integrable theories. We intend to study this possibility in detail.

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