Uncertainty relations in quantum optics. Is the photon intelligent?∗

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Abstract

The Robertson – Schrödinger, Heisenberg – Robertson and Trifonov uncertainty relations for arbitrary two functions $f_1$ and $f_2$ depending on the quantum phase and the number of photons respectively, are given. Intelligent states and states which minimize locally the product of uncertainties $(\Delta f_1)^2 \cdot (\Delta f_2)^2$ or the sum $(\Delta f_1)^2 + (\Delta f_2)^2$ are investigated for the cases $f_1 = \phi, \exp (i\phi), \exp (-i\phi), \cos \phi, \sin \phi$ and $f_2 = n$.

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∗ In memory of Jerzy F. Plebański
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I. INTRODUCTION

The principle of uncertainty for the position and the momentum formulated by Werner Heisenberg in 1927 [1] is one of the most fundamental results not only in quantum mechanics but also in all the physics. It is not strange that this principle has attracted a great deal of interest. First, it was proved with the mathematical rigour by E.H. Kennard [2] and H. Weyl [3]. Then the original uncertainty relations for canonical variables were generalized to the case of any two observables by H.P. Robertson [4, 5] and E. Schrödinger [6] and also to the case of any number of observables by H.P. Robertson [7]. The Robertson approach was thoroughly analysed and developed by D.A. Trifonov in the series of papers [8–15] especially in the context of intelligent, coherent and squeezed states in order to extend the notion of these states on the cases when the generalized uncertainty relations are considered (see also [16]).

As we remember the (usual) coherent state is a state which minimizes the Heisenberg uncertainty relation for the canonical variables $\hat{q} = \frac{1}{\sqrt{2}}(\hat{a} + \hat{a}^\dagger)$ and $\hat{p} = \frac{i}{\sqrt{2}}(\hat{a}^\dagger - \hat{a})$ with $\Delta q = \Delta p$, where $\Delta q$ and $\Delta p$ stand for the uncertainties of $q$ and $p$, respectively [17–20]. The (usual) squeezed states minimize the Heisenberg uncertainty relation for $\hat{q}$ and $\hat{p}$ but $\Delta q \neq \Delta p$ [19–24]. The coherent and squeezed states are called the intelligent states and they can be defined as the states which give the strict equality in the Heisenberg uncertainty relation for $\hat{q}$ and $\hat{p}$ or, equivalently, as the states which minimize globally the product $\Delta q \cdot \Delta p$. It is obvious that in the case of a generalized uncertainty relation for arbitrary observables when the right hand side of the respective inequality depends on the state, these two definitions of an intelligent state given above are in general not equivalent. According to the convention accepted in the previous works [9–11, 25–30] we refer to a generalized intelligent state as the one which satisfies the equality in the respective generalized uncertainty relation.

It is worthwhile mentioning that the Robertson approach to the uncertainty relations and also the concept of the generalized intelligent states can be carried over to the deformation quantization formalism [27, 31].

In the present paper we investigate various generalized uncertainty relations in quantum optics for quantities depending on a phase and a number of photons. Then we study the corresponding intelligent states and the states which minimize (locally) the product or sum...
of uncertainties. The interest in this issue starts with the works of P. Carruthers and M.M. Nieto [32, 33] and R. Jackiw [34]. Then the problem has been explored by many authors [26, 28, 29, 35–45]. Since the generalized uncertainty relations for functions of the phase and the number of photons depend on the quantum state, the usual procedure in searching for the states minimizing these relations consists in modifying the respective inequality to obtain a constant parameter on the right hand side of this inequality. Then one minimizes such an inequality [32–34, 42].

In our paper we are going to use another technique. We simply leave the generalized uncertainty relations as they stand and we ask for the states which give the local minimum of these relations. To the best of our knowledge such a question has not been yet considered in quantum optics but it has been widely investigated for the ‘angular momentum - angle position’ uncertainty relation in [30, 46]. In fact our work is strongly motivated by Ref. [30].

As we will see, the results obtained in our paper are drastically different from the corresponding results in [30], since in contrast to the case of angular momentum operator $\hat{L}_z$, which has the eigenvalues $n\hbar$, $n = 0, \pm 1, \pm 2, \ldots$, the photon number operator $\hat{n}$ admits only the eigenvalues $n = 0, 1, 2, \ldots$. Moreover, if one is going to study the ‘photon number – phase function’ uncertainty relations, then he/she must first decide on the formalism in which he/she considers the quantum phase. He/she may deal within the Susskind – Glogower formalism [19, 33, 34, 47], the Garrison – Wong formalism [35] or he/she can apply the Pegg – Barnett approach [48–50], which is equivalent to the POV – measure approach [45, 51] and to the formalism based on extending the Fock space to the Hilbert space $L^2(S^1)$ [52, 53] (see also [54] and references therein). In the present paper we employ the results of [52, 53] and our analysis is consistent with the celebrated Pegg – Barnett approach to the quantum phase.

The paper is organized as follows. In Sec. 2 we first recall some results of [52, 53] and then we quote the Robertson, Hadamard – Robertson and Trifonov theorems. In the next step, using these theorems we derive the Robertson – Schrödinger, Heisenberg – Robertson and Trifonov uncertainty relations for any two functions $f_1 = f_1(\phi)$ and $f_2 = f_2(n)$ depending on the phase $\phi$ and the number of photons $n$. Finally, we employ these general results to the case of the phase and the number of photons (Example 2.1).

The number – phase Robertson – Schrödinger, Heisenberg – Robertson and Trifonov intelligent states are found (Theorem 2.1). Intelligent states for arbitrary $f_1$ and $f_2$ are
investigated in Sec 3. In particular the cases: \( f_1 = \exp(i\phi) \) and \( f_2 = n \) (Example 3.1), 
\( f_1 = \exp(-i\phi) \) and \( f_2 = n \) (Example 3.2), and finally \( f_1 = \cos \phi \) (\( f_1 = \sin \phi \)) and \( f_2 = n \) (Example 3.3) are considered in all details.

Section 4 is devoted to searching for the states which minimize locally the product of uncertainties \((\Delta f_1)^2 \cdot (\Delta f_2)^2\). The general equation (see Eq. (4.5)) defining these states has been found and the cases of \( f_1 = \exp(-i\phi) \) (Example 4.1) and \( f_1 = \phi \) (Example 4.2) with \( f_2 = n \) (in both cases), have been studied in detail. The same is done in the next section, where the states minimizing locally the sum of uncertainties \((\Delta f_1)^2 + (\Delta f_2)^2\) are investigated. Concluding remarks end our paper.

It is worth reminding that our results are valid within the Pegg – Barnett approach to quantum phase as well as within the POV – measure approach or within the formalism based on extending the Fock space to the Hilbert space \( L^2(S^1) \) [52, 53]. However, the Susskind – Glogower or Garrison – Wong formalisms lead, in general, to different results than the ones presented here.

We devote this modest work to the memory of our Teacher and Friend Jerzy F. Plebański into the tenth anniversary of his death. Professor Plebański was not only a great relativist but he was also the first, with Leopold Infeld, who considered already in the years 1954 – 1955 the squeezed states and the squeeze operator for a harmonic oscillator [21, 22]. Hardly anyone knows this fact. We have found comments about the mentioned publications in [55, 56].

II. UNCERTAINTY RELATIONS

In the recent work [53] it has been argued that given a number – phase function \( f = f(\phi, n), -\pi \leq \phi < \pi, n = 0, 1, \ldots \), the average value of this function in a state defined by a density operator \( \hat{\rho} \) is given by

\[
\langle f(\phi, n) \rangle = \sum_{n=0}^{\infty} \int_{-\pi}^{\pi} f(\phi, n) \rho_W(\phi, n) d\phi = \text{Tr}\{\hat{f}(\phi, n) \hat{\rho}\} = \langle \hat{f}(\phi, n) \rangle,
\]

where \( \rho_W(\phi, n) \) is the number – phase Wigner function for the state \( \hat{\rho} \)

\[
\rho_W(\phi, n) = \text{Re}\{\langle \phi | \hat{\rho} | n \rangle \langle n | \phi \rangle\}.
\]
The operator $\hat{f}(\phi, n)$ reads

$$\hat{f}(\phi, n) = \frac{1}{2} \sum_{n=0}^{\infty} \int_{-\pi}^{\pi} f(\phi, n) \cdot \left( |n\rangle \langle \phi| + |\phi\rangle \langle n| \right) d\phi. \quad (2.3)$$

The vectors $|\phi\rangle$ and $|n\rangle$ stand for the phase state vector [19, 29, 33, 40, 53, 54, 57–59]

$$|\phi\rangle = \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \exp(in\phi)|n\rangle \quad (2.4)$$

and the normalized eigenvector of the number operator $\hat{n}$

$$\hat{n}|n\rangle = n|n\rangle, \quad n = 0, 1, \ldots$$

$$\langle n'|n\rangle = \delta_{n'n} \quad (2.5)$$

respectively.

**Remark** In the present work we do not use the ‘underbar notation’ $\rho_{\psi}, |n\rangle, |\phi\rangle, \ldots$ etc. which has been employed in [52, 53]; we also omit the kernel symbol $K_S$.

In particular if the function $f$ is independent of the photon number $n$ i.e. $f = f(\phi)$ then the formula (2.1) reduces to

$$\langle f(\phi) \rangle = \int_{-\pi}^{\pi} f(\phi)\langle \phi|\hat{\rho}|\phi\rangle d\phi = \text{Tr}\{\hat{f}(\phi)\hat{\rho}\} = \langle \hat{f}(\phi) \rangle \quad (2.6)$$

where $\hat{f}(\phi)$, according to (2.3), is given by

$$\hat{f}(\phi) = \int_{-\pi}^{\pi} f(\phi)|\phi\rangle \langle \phi| d\phi. \quad (2.7)$$

Recall that the set of vectors $\{|\phi\rangle\}_{-\pi}^{\pi}$ is not orthogonal

$$\langle \phi|\phi'\rangle \sim \delta(\phi - \phi') \quad (2.8)$$

but still it gives a resolution of the identity operator

$$\int_{-\pi}^{\pi} |\phi\rangle \langle \phi| d\phi = \hat{1}. \quad (2.9)$$

If the function $f$ is independent of the phase $\phi$ i.e. $f = f(n)$ then (2.1) takes the form

$$\langle f(n) \rangle = \sum_{n=0}^{\infty} f(n)\langle n|\hat{\rho}|n\rangle = \text{Tr}\{\hat{f}(n)\hat{\rho}\} = \text{Tr}\{f(\hat{n})\hat{\rho}\} = \langle f(\hat{n}) \rangle. \quad (2.10)$$

Note that because of (2.8) if $f = f(\phi)$ and $g = g(\phi)$ then

$$\hat{f}g(\phi) \neq \hat{f}(\phi) \cdot \hat{g}(\phi) \quad (2.11)$$
in general, and consequently it may happen that
\[ \langle f(\phi)g(\phi) \rangle = \langle \hat{f}(\phi) \hat{g}(\phi) \rangle \neq \langle \hat{f}(\phi) \cdot \hat{g}(\phi) \rangle. \] (2.12)

In particular
\[ \langle \phi^k \rangle = \langle \hat{\phi}^k \rangle \neq \langle (\hat{\phi})^k \rangle, \] (2.13)
where
\[ \hat{\phi} = -i \sum_{j,l=0}^{\infty} \frac{(-1)^{j-l}}{j-l} |j\rangle \langle l| \quad \text{with} \quad j \neq l \] (2.14)
is the self–adjoint Garrison–Wong phase operator [19, 35, 51, 52].

On the contrary, if \( f = f(n) \) and \( g = g(n) \) then
\[ \hat{f}g(n) = \hat{f}(n) \cdot \hat{g}(n) = f(n) \cdot g(n). \] (2.15)

It can be easily proved that the average value (2.6) is equal exactly to the one calculated within the celebrated Pegg–Barnett formalism [45, 48–53, 58]. Then the formulae (2.11), (2.12) and (2.13) show that the Pegg–Barnett formalism predicts different experimental results than the formalism which assumes that the quantum phase is defined by the Garrison–Wong operator (2.14) and also different than the results predicted by the famous Susskind–Glogower formalism, where the quantum phase is given by two hermitian (self–adjoint) operators \( \hat{\cos} \phi \) and \( \hat{\sin} \phi \) [60–64].

Consequently, one expects that all those three approaches lead also to different uncertainty relations. In this work we study the uncertainty relations which follow from the Pegg–Barnett approach.

Let
\[ \hat{\rho} = |\psi\rangle \langle \psi|, \quad \langle \psi | \psi \rangle = 1 \] (2.16)
be the density operator for a pure state \( |\psi\rangle \). Given any two complex functions \( f_1 = f_1(\phi) \) and \( f_2 = f_2(n) \) one quickly gets from (2.6) and (2.10) their average values
\[ \langle f_1(\phi) \rangle = \int_{-\pi}^{\pi} \psi^*(\phi)f_1(\phi)\psi(\phi)d\phi, \] (2.17a)
\[ \langle f_2(n) \rangle = \int_{-\pi}^{\pi} \psi^*(\phi)f_2 \left( i \frac{\partial}{\partial \phi} \right) \psi(\phi)d\phi, \] (2.17b)
where, with \( |\phi\rangle \) defined by (2.4), the function
\[ \psi(\phi) = \langle \phi | \psi \rangle \] (2.18)
is the wave function in the phase representation.

Introduce two operators acting in the space of such wave functions

\[ \hat{F}_1 \psi(\phi) := f_1(\phi)\psi(\phi), \]  
\[ \hat{F}_2 \psi(\phi) := f_2 \left( i \frac{\partial}{\partial \phi} \right) \psi(\phi) \]

and then the following two operators

\[ \delta \hat{F}_1 := \hat{F}_1 - \langle \hat{F}_1 \rangle = \hat{F}_1 - \langle f_1(\phi) \rangle, \]  
\[ \delta \hat{F}_2 := \hat{F}_2 - \langle \hat{F}_2 \rangle = \hat{F}_2 - \langle f_2(n) \rangle. \]

Let us define now the $2 \times 2$ Hermitian matrix

\[ F_{\mu\nu} := \int_{-\pi}^{\pi} (\delta \hat{F}_\mu \psi(\phi))^* \delta \hat{F}_\nu \psi(\phi) d\phi \quad \text{with} \quad \mu, \nu = 1, 2. \]

It is obvious that the Hermitian form $\mathcal{F}$

\[ \mathcal{F} : \mathbb{C}^2 \times \mathbb{C}^2 \ni (x, y) \mapsto \mathcal{F}(x, y) = \sum_{\mu, \nu=1}^{2} F_{\mu\nu} x_\mu^* y_\nu \in \mathbb{C} \]

is positive semi-definite i.e. $\mathcal{F}(x, x) \geq 0 \ \forall \ x \in \mathbb{C}^2$. We split the matrix $(F_{\mu\nu})$ into two matrices

\[ (F_{\mu\nu}) = (a_{\mu\nu}) + i(b_{\mu\nu}), \]
\[ a_{\mu\nu} := \frac{1}{2}(F_{\mu\nu} + F_{\nu\mu}) = a_{\nu\mu} = a_{\mu\nu}^*, \]
\[ b_{\mu\nu} := \frac{1}{2i}(F_{\mu\nu} - F_{\nu\mu}) = -b_{\nu\mu} = b_{\mu\nu}^*. \]

The matrix $(a_{\mu\nu})$ is real symmetric and $(b_{\mu\nu})$ is real anti-symmetric therefore Hermitian and anti-Hermitian respectively.

Uncertainty relations for $f_1(\phi)$ and $f_2(n)$ follow directly from the general theorems known as: the Robertson theorem [7, 14, 15, 27], the Hadamard–Robertson theorem [7, 9, 10, 15, 27] and the Trifonov theorem [15, 27].

In our present case the Robertson theorem states that

\[ \det(a_{\mu\nu}) \geq \det(b_{\mu\nu}). \]
Then the Hadamard – Robertson theorem shows that

$$a_{11}a_{22} \geq \det(b_{\mu \nu})$$  \hspace{1cm} (2.25)

and the Trifonov theorem reduces now to

$$a_{11} + a_{22} \geq 2|b_{12}|.$$  \hspace{1cm} (2.26)

Notice that (2.24) leads to a stronger estimation of $\det(b_{\mu \nu})$ than (2.25).

Employing (2.21) and (2.23) with (2.19a), (2.19b), (2.20a) and (2.20b) after some elementary algebraic manipulations one gets from (2.24) the following *Robertson – Schrödinger uncertainty relation*

$$\int_{-\pi}^{\pi} \left| \left( \delta \hat{F}_1 \psi(\phi) \right)^* \delta \hat{F}_2 \psi(\phi) d\phi \right|^2$$

$$= \int_{-\pi}^{\pi} \left| \psi^*(\phi) f_1^*(\phi) f_2 \left( \frac{i}{\partial \phi} \right) \psi(\phi) d\phi - \langle f_1(\phi) \rangle \langle f_2(n) \rangle \right|^2,$$  \hspace{1cm} (2.27)

where

$$(\Delta f_\mu)^2 := \int_{-\pi}^{\pi} \left( \delta \hat{F}_\mu \psi(\phi) \right)^* \delta \hat{F}_\mu \psi(\phi) d\phi = F_{\mu \mu} = a_{\mu \mu}, \hspace{0.5cm} \mu = 1, 2$$  \hspace{1cm} (2.28)

is defined as the *variance of* $f_\mu$ and, as usually, $\Delta f_\mu = \sqrt{(\Delta f_\mu)^2}$ is the *uncertainty in* $f_\mu$.

Note that the inequality (2.27) can be understood as the *Schwarz inequality*. Analogously (2.25) leads to the *Heisenberg – Robertson uncertainty relation*

$$\left( \Delta f_1 \right)^2 \cdot \left( \Delta f_2 \right)^2 \geq \left( \int_{-\pi}^{\pi} \left( \delta \hat{F}_1 \psi(\phi) \right)^* \delta \hat{F}_2 \psi(\phi) d\phi \right)^2$$

$$\hspace{1cm} = \left( \int_{-\pi}^{\pi} \psi^*(\phi) f_1^*(\phi) f_2 \left( \frac{i}{\partial \phi} \right) \psi(\phi) d\phi - \langle f_1(\phi) \rangle \langle f_2(n) \rangle \right)^2,$$  \hspace{1cm} (2.29)

Finally, the inequality (2.26) gives the following *Trifonov uncertainty relation*

$$\left( \Delta f_1 \right)^2 + \left( \Delta f_2 \right)^2 \geq 2 \left| \int_{-\pi}^{\pi} \psi^*(\phi) f_1^*(\phi) f_2 \left( \frac{i}{\partial \phi} \right) \psi(\phi) d\phi - \langle f_1(\phi) \rangle \langle f_2(n) \rangle \right|.$$  \hspace{1cm} (2.30)

**Example 2.1 Uncertainty relations for the phase and number of photons**

Here we assume that $f_1 = \phi$ and $f_2 = n$. First, observe that we should modify slightly the definition (2.28) for $\Delta \phi$ to get a physically acceptable concept of the uncertainty in phase. To this end we follow the results of D. Judge in his pioneering work [65] and of H.S. Sharatchandra [44].
Consider then that $-\pi \leq \gamma < \pi$ and define $\delta \phi_\gamma := \phi - \gamma$. Since $|\delta \phi_\gamma|$ can be greater than $\pi$, we propose the following object

$$
\widetilde{\delta \phi}_\gamma = \begin{cases} 
\phi - \gamma + 2\pi & \text{for } -\pi \leq \phi \leq -\pi + \gamma \\
\phi - \gamma & \text{for } -\pi + \gamma \leq \phi < \pi 
\end{cases}
$$

(2.31a)

if $\gamma \geq 0$ and

$$
\widetilde{\delta \phi}_\gamma = \begin{cases} 
\phi - \gamma & \text{for } -\pi \leq \phi \leq -\pi + \gamma \\
\phi - \gamma - 2\pi & \text{for } \pi + \gamma \leq \phi < \pi 
\end{cases}
$$

(2.31b)

if $\gamma < 0$. Then the variance of $\phi$ is defined by

$$
(\widetilde{\Delta} \phi)^2 := \min_{-\pi \leq \gamma < \pi} \int_{-\pi}^\pi (\widetilde{\delta \phi}_\gamma \psi(\phi))^* \widetilde{\delta \phi}_\gamma \psi(\phi) d\phi. 
$$

(2.32)

Performing simple calculations and employing the periodicity of $\psi(\phi)$ i.e. $\psi(\phi \pm 2\pi) = \psi(\phi)$ one obtains

$$
(\widetilde{\Delta} \phi)^2 = \min_{-\pi \leq \gamma < \pi} \int_{-\pi}^\pi \phi^2 |\psi(\phi + \gamma)|^2 d\phi.
$$

(2.33)

This result suggests that it is convenient to introduce a new wave function

$$
\widetilde{\psi}(\phi) := \psi(\phi + \gamma_0),
$$

(2.34)

where $\gamma_0$ minimizes (2.33).

Consequently, (2.33) reads now

$$
(\widetilde{\Delta} \phi)^2 = \int_{-\pi}^\pi \phi^2 |\widetilde{\psi}(\phi)|^2 d\phi.
$$

(2.35)

Then

$$
\langle \tilde{n} \rangle := \int_{-\pi}^\pi \psi^*(\phi) i \frac{\partial}{\partial \phi} \widetilde{\psi}(\phi) d\phi = \int_{-\pi}^\pi \psi^*(\phi) i \frac{\partial}{\partial \phi} \psi(\phi) d\phi = \langle n \rangle.
$$

(2.36)

Since

$$
\frac{\partial}{\partial \gamma} \int_{-\pi}^\pi \phi^2 |\psi(\phi + \gamma)|^2 d\phi \bigg|_{\gamma = \gamma_0} = 0,
$$

one quickly gets

$$
0 = \int_{-\pi}^\pi \phi^2 \frac{\partial}{\partial \gamma} |\psi(\phi + \gamma)|^2 d\phi \bigg|_{\gamma = \gamma_0} = \int_{-\pi}^\pi \phi^2 \frac{\partial}{\partial \phi} |\psi(\phi + \gamma_0)|^2 d\phi = -2 \int_{-\pi}^\pi \phi |\psi(\phi + \gamma_0)|^2 d\phi
$$

$$
= -2 \int_{-\pi}^\pi |\widetilde{\psi}(\phi)|^2 d\phi = -2 \int_{-\pi}^\pi |\widetilde{\psi}(\phi)|^2 d\phi.
$$

Finally, one has

$$
\langle \tilde{\phi} \rangle := \int_{-\pi}^\pi \phi |\widetilde{\psi}(\phi)|^2 d\phi = 0.
$$

(2.37)
For the present case we redefine the $2 \times 2$ Hermitian matrix (2.21) as
\[
\tilde{F}_{11} = \int_{-\pi}^{\pi} (\phi \tilde{\psi}(\phi))^* \phi \tilde{\psi}(\phi) d\phi = (\tilde{\Delta}\phi)^2,
\]
\[
\tilde{F}_{12} = \int_{-\pi}^{\pi} (\phi \tilde{\psi}(\phi))^* \left( i \frac{\partial}{\partial \phi} - \langle n \rangle \right) \tilde{\psi}(\phi) d\phi = \tilde{F}_{21},
\]
\[
\tilde{F}_{22} = \int_{-\pi}^{\pi} \left[ \left( i \frac{\partial}{\partial \phi} - \langle n \rangle \right) \tilde{\psi}(\phi) \right]^* \left[ \left( i \frac{\partial}{\partial \phi} - \langle n \rangle \right) \tilde{\psi}(\phi) \right] d\phi = (\tilde{\Delta}n)^2 = (\Delta n)^2. \tag{2.38}
\]
Using (2.38) one can rewrite the Robertson – Schrödinger uncertainty relation (2.27) in the form
\[
(\tilde{\Delta}\phi)^2 \cdot (\Delta n)^2 \geq \left| \int_{-\pi}^{\pi} (\phi \tilde{\psi}(\phi))^* \left( i \frac{\partial}{\partial \phi} - \langle n \rangle \right) \tilde{\psi}(\phi) d\phi \right|^2 = \left| \int_{-\pi}^{\pi} \tilde{\psi}^*(\phi) \phi \frac{\partial \tilde{\psi}(\phi)}{\partial \phi} d\phi - \langle \tilde{\psi} \rangle \langle n \rangle \right|^2. \tag{2.39}
\]
Then, integrating by parts
\[
i \int_{-\pi}^{\pi} \tilde{\psi}^*(\phi) \phi \frac{\partial \tilde{\psi}(\phi)}{\partial \phi} d\phi = \frac{i}{2} \int_{-\pi}^{\pi} \left( \tilde{\psi}^*(\phi) \phi \frac{\partial \tilde{\psi}(\phi)}{\partial \phi} + \tilde{\psi}(\phi) \frac{\partial \tilde{\psi}^*(\phi)}{\partial \phi} \right) d\phi
\]
\[
+ \frac{i}{2} \int_{-\pi}^{\pi} \left( \tilde{\psi}^*(\phi) \phi \frac{\partial \tilde{\psi}(\phi)}{\partial \phi} - \tilde{\psi}(\phi) \phi \frac{\partial \tilde{\psi}^*(\phi)}{\partial \phi} \right) d\phi
\]
\[
= -\frac{i}{2} \left( 1 - 2\pi |\tilde{\psi}(\pi)|^2 \right) + \frac{i}{2} \int_{-\pi}^{\pi} \left( \tilde{\psi}^*(\phi) \phi \frac{\partial \tilde{\psi}(\phi)}{\partial \phi} - \tilde{\psi}(\phi) \phi \frac{\partial \tilde{\psi}^*(\phi)}{\partial \phi} \right) d\phi. \tag{2.40}
\]
Substituting (2.40) into (2.39) and using (2.37) we obtain the Robertson – Schrödinger uncertainty relation for $\phi$ and $n$ in the final form
\[
(\tilde{\Delta}\phi)^2 \cdot (\Delta n)^2 \geq \frac{1}{4} \left( 1 - 2\pi |\tilde{\psi}(\pi)|^2 \right)^2 + \left[ \frac{i}{2} \int_{-\pi}^{\pi} \left( \tilde{\psi}^*(\phi) \phi \frac{\partial \tilde{\psi}(\phi)}{\partial \phi} - \tilde{\psi}(\phi) \phi \frac{\partial \tilde{\psi}^*(\phi)}{\partial \phi} \right) d\phi \right]^2. \tag{2.41}
\]
One immediately concludes that the Heisenberg – Robertson uncertainty relation (2.29) gives now
\[
(\tilde{\Delta}\phi)^2 \cdot (\Delta n)^2 \geq \frac{1}{4} \left( 1 - 2\pi |\tilde{\psi}(\pi)|^2 \right)^2 \tag{2.42}
\]
and the Trifonov relation (2.30) reads
\[
(\tilde{\Delta}\phi)^2 + (\Delta n)^2 \geq \left| 1 - 2\pi |\tilde{\psi}(\pi)|^2 \right|. \tag{2.43}
\]

**Remark** The Heisenberg – Robertson uncertainty relation for the phase $\phi$ and the number of photons $n$ or for the angle $\theta$ and the angular momentum $L_z$ were considered by many...
authors and formulae analogous to (2.42) have been found [30, 40, 43, 65–67]. The problem of physically acceptable definition of the uncertainty in angle θ or in phase φ was considered by D. Judge [65], H.S. Sharatchandra [44] or B-S. K. Skagerstam and B.A. Bergsjordet [43]. Our choice of the uncertainty $\tilde{\Delta}\phi$ (2.33) is in accordance with those works.

Now we are at the point, where the intelligent states for the phase and the number of photons should be investigated. According to the commonly used definition the *Robertson–Schrödinger intelligent state for φ and n* is a state represented by a function $\psi(\phi)$ such that the inequality (2.41) reduces to the strict equality. From the general results found in [10, 27] or from a careful analysis of the origin of the Schwarz inequality one concludes that $\psi(\phi)$ is an intelligent state if and only if there exists $\lambda \in \mathbb{C}$ such that the following equation

$$\left( i \frac{\partial}{\partial \phi} - \langle n \rangle + i\lambda \phi \right) \tilde{\psi}(\phi) = 0$$

is satisfied by a function $\tilde{\psi}(\phi)$ related to the function $\psi(\phi)$ according to the rule (2.34).

The general solution of Eq. (2.44) reads

$$\tilde{\psi}(\phi) = \tilde{a}e^{-i\langle n \rangle \phi}e^{-\frac{1}{2}\phi^2}, \quad -\pi \leq \phi < \pi,$$

where $\tilde{a} \in \mathbb{C}$.

Now one should remember that some restrictions must be imposed on the function $\psi(\phi)$ if this function is going to represent a photon state. First, the function $\psi(\phi)$ must be a periodic function with period $2\pi$. Hence, $\tilde{\psi}(\phi)$ is also periodic with the same period $2\pi$ and, consequently

$$\tilde{\psi}(\pi) = \tilde{\psi}(-\pi).$$

From (2.45) and (2.46) one quickly gets the condition

$$\langle n \rangle = N, \quad N = 0, 1, 2, ...$$

The second restriction follows immediately from (2.18) with (2.4). Namely, writing the state $|\psi\rangle$ in the form

$$|\psi\rangle = \sum_{n=0}^{\infty} c_n |n\rangle, \quad c_n \in \mathbb{C}$$

and using (2.18) and (2.4) one obtains

$$\psi(\phi) = \langle \phi | \psi \rangle = \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} c_n e^{-in\phi}. $$
This means that the Fourier expansion of the photon wave function \( \psi(\phi) \) must be of the form (2.49) i.e. it does not involve the exponents of the form \( e^{in\phi}, \, n = 1, 2, \ldots \)

Analogously

\[
\tilde{\psi}(\phi) = \psi(\phi + \gamma_0) = \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} c_n e^{-in(\phi + \gamma_0)} = \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \tilde{c}_n e^{-in\phi}, \quad \tilde{c}_n = c_n e^{-in\gamma_0}. \tag{2.50}
\]

From (2.45), (2.47) and (2.50) one infers that

\[
\int_{-\pi}^{\pi} e^{-\frac{\lambda}{2} \phi^2} e^{-i(N+k)\phi} d\phi = 0 \quad \text{for} \quad k = 1, 2, \ldots \tag{2.51}
\]

and, consequently, also

\[
\int_{-\pi}^{\pi} e^{-\frac{\lambda}{2} \phi^2} e^{i(N+k)\phi} d\phi = 0 \quad \text{for} \quad k = 1, 2, \ldots \tag{2.52}
\]

So the Fourier expansion of the function \( e^{-\frac{\lambda}{2} \phi^2} \) satisfying (2.51) and (2.52) takes the form

\[
e^{-\frac{\lambda}{2} \phi^2} = \sum_{n=0}^{N} b_n \cos n\phi, \quad b_n \in \mathbb{C}. \tag{2.53}
\]

Differentiating both sides of (2.53) with respect to \( \phi \) and putting \( \phi = -\pi \) we get

\[
\lambda \pi e^{-\frac{\lambda}{2} \pi^2} = \sum_{n=1}^{N} nb_n \sin n\pi = 0. \tag{2.54}
\]

Eq. (2.54) holds true if and only if \( \lambda = 0 \). Hence, by (2.45) and (2.34) one finds \( \psi(\phi) \) normalized to 1 as

\[
\psi(\phi) = \frac{1}{\sqrt{2\pi}} e^{-iN\phi}, \quad N = 0, 1, \ldots \tag{2.55}
\]

(Note that now we can put \( \gamma_0 = 0 \)). So the respective ket \( |\psi\rangle \) reads

\[
|\psi\rangle = |N\rangle, \quad N = 0, 1, \ldots \tag{2.56}
\]

One quickly shows that for any of the states (2.56) the relations are satisfied

\[
\Delta n = 0, \tag{2.57a}
\]

\[
\tilde{\Delta}\phi = \Delta\phi = \frac{\pi}{\sqrt{3}} \tag{2.57b}
\]
and also that the states (2.56) are the *Heisenberg–Robertson intelligent states* for \( \phi \) and \( n \) i.e. they fulfill the strict equality in (2.42) but by (2.57a) and (2.57b) none of them gives the strict equality for (2.43), so none of them can be a *Trifonov intelligent state* for \( \phi \) and \( n \). Then, since the following implication Eq. (2.26) \( \Rightarrow \) Eq. (2.25) \( \Rightarrow \) Eq. (2.24) holds true (see [27]), one easily concludes that any number–phase Trifonov intelligent state is a number–phase Heisenberg–Robertson intelligent state and also a number–phase Robertson–Schrödinger intelligent state. Finally, gathering all that we arrive at the result:

**Theorem 2.1** The only number–phase Robertson–Schrödinger intelligent states are the eigenstates of the number operator \( \hat{n} \), \( |n\rangle, n = 0, 1, ... \). These states are also the only number–phase Heisenberg–Robertson intelligent states. There are no number–phase Trifonov intelligent states. ■

### III. INTELLIGENT STATES FOR ARBITRARY \( f_1 \) AND \( f_2 \)

From the results of [8–12, 16, 27] we conclude that the *Robertson–Schrödinger intelligent state for \( f_1 = f_1(\phi) \) and \( f_2 = f_2(n) \) can be found as a solution of the following equation

\[
\left( \delta \hat{F}_2 + i\lambda \delta \hat{F}_1 \right) \psi(\phi) = 0, \tag{3.1}
\]

where \( \lambda \in \mathbb{C} \), \( \delta \hat{F}_1 \) and \( \delta \hat{F}_2 \) are defined by (2.20a) and (2.20b) with (2.19a) and (2.19b). Eq. (3.1) can be rewritten in the form

\[
\left[ f_2 \left( i \frac{\partial}{\partial \phi} \right) + i\lambda f_1(\phi) - \mu \right] \psi(\phi) = 0, \quad \lambda \in \mathbb{C} \tag{3.2}
\]

with

\[
\mu = \langle f_2(n) \rangle + i\lambda \langle f_1(\phi) \rangle. \tag{3.3}
\]

Moreover, as in the previous case when \( f_1 = \phi \) and \( f_2 = n \), on the state \( \psi(\phi) \) the conditions

\[
\psi(\pi) = \psi(-\pi) \tag{3.4}
\]

and (2.49) are imposed. One quickly gets that Eq. (3.2) restricted to \( \lambda \in \mathbb{R} \) defines the Heisenberg–Robertson intelligent states.

**Remark** Our equation (3.2) is different from the respective equation which one could find by using the considerations analogous to those given by C. Brif and Y. Ben–Aryeh [29].
The reason lies in the fact that, in general

\[
\langle \phi | \hat{f}_1(\phi) | \psi \rangle \neq f_1(\phi) \langle \phi | \psi \rangle = f_1(\phi) \cdot \psi(\phi),
\]

where \( \hat{f}_1(\phi) \) is defined by (2.7). Note that contrary to (3.5) one gets

\[
\langle \phi | \hat{f}_2(n) | \psi \rangle = f_2 \left( i \frac{\partial}{\partial \phi} \right) \langle \phi | \psi \rangle = f_2 \left( i \frac{\partial}{\partial \phi} \right) \psi(\phi).
\]

As the first example of Eq. (3.2) let us consider the following case

**Example 3.1**

\[
f_1(\phi) = e^{i\phi}, \quad f_2(n) = n.
\]

Eq. (3.2) reads now

\[
\left( i \frac{\partial}{\partial \phi} + i\lambda e^{i\phi} - \mu \right) \psi(\phi) = 0, \quad \lambda \in \mathbb{C}.
\]

The general solution of (3.8) is

\[
\psi(\phi) = ae^{-i\mu \phi} \cdot e^{i\lambda e^{i\phi}}, \quad a \in \mathbb{C}.
\]

The condition (3.4) yields \( \mu \in \mathbb{Z} \). Expanding the term \( e^{i\lambda e^{i\phi}} \) in (3.9) one has

\[
\psi(\phi) = ae^{-i\mu \phi} \sum_{k=0}^{\infty} \frac{(i\lambda)^k}{k!} e^{ik\phi}.
\]

Then \( \psi(\phi) \) given by (3.10) fulfills the condition (2.49) iff

\[
\lambda = 0, \quad \mu = n, \quad n = 0, 1, ...
\]

So the only normalized states \( \psi(\phi) \) satisfying Eq. (3.8) are again the eigenstates of \( \hat{n} \), \( \psi(\phi) = \frac{1}{\sqrt{2\pi}} e^{-in\phi} \) with \( n = 0, 1, ... \) Gathering, analogously as in Example 2.1 the only Robertson – Schrödinger and Heisenberg – Robertson intelligent states for \( e^{i\phi} \) and \( n \) are the eigenstates of \( \hat{n} \), \( |n\rangle \), \( n = 0, 1, ... \). There are no Trifonov intelligent states for \( e^{i\phi} \) and \( n \).

**Example 3.2**

Here we assume

\[
f_1(\phi) = e^{-i\phi}, \quad f_2(n) = n.
\]
Substituting (3.12) into Eq. (3.2) we have
\[ \left( i \frac{\partial}{\partial \phi} + i \lambda e^{-i \phi} - \mu \right) \psi(\phi) = 0, \quad \lambda \in \mathbb{C}. \] (3.13)

The general normalized solution of (3.13) satisfying the conditions (3.4) and (2.49) reads
\[ \psi(\phi) = (2\pi I_0(2|\lambda|))^{-\frac{1}{2}} e^{-i\phi} e^{-i\lambda e^{-i\phi}} = (2\pi I_0(2|\lambda|))^{-\frac{1}{2}} \sum_{k=0}^{\infty} \frac{(-i\lambda)^k}{k!} e^{-i(n+k)\phi}, \] (3.14)
\[ \mu = n = 0, 1, ..., \quad \lambda \in \mathbb{C}, \]
where \( I_0 \) is the zeroth modified Bessel function of the first kind. In the Dirac notation the state \( |\psi\rangle \) is of the form
\[ |\psi\rangle = (I_0(2|\lambda|))^{-\frac{1}{2}} \sum_{k=0}^{\infty} \frac{(-i\lambda)^k}{k!} |n + k\rangle, \quad \lambda \in \mathbb{C}, \quad n = 0, 1, ... \] (3.15)

**Remark** The solution (3.14) was found by C. Brif and Y. Ben – Aryeh [29, 40] and then, with the use of Hardy space formalism, by S. Luo [42]. Note that in the present case one has that
\[ \hat{e}^{-i\phi} = \int_{-\pi}^{\pi} e^{-i\phi} |\phi\rangle \langle \phi| \, d\phi = \sum_{n=0}^{\infty} |n + 1\rangle \langle n| \] (3.16)
and this is an exceptional case when, contrary to (3.5), we get the equality
\[ \langle \phi| \hat{e}^{-i\phi} |\psi\rangle = e^{-i\phi} \langle \phi| \psi\rangle = e^{-i\phi} \psi(\phi). \] (3.17)

So our equation (3.13) is equivalent to Eq. (4.29) of Ref. [29] and, consequently, our solution (3.14) is the same as the respective solution (4.31) of [29].

Straightforward calculations give
\[ \langle n \rangle = n + |\lambda| \frac{I_1(2|\lambda|)}{I_0(2|\lambda|)}, \] (3.18a)
\[ \langle e^{-i\phi} \rangle = i \frac{\lambda^* I_1(2|\lambda|)}{|\lambda| I_0(2|\lambda|)}, \] (3.18b)
\[ \langle n^2 \rangle = n^2 + |\lambda|^2 + 2n|\lambda| \frac{I_1(2|\lambda|)}{I_0(2|\lambda|)}, \] (3.18c)
\[ (\Delta n)^2 = |\lambda|^2 \left[ 1 - \left( \frac{I_1(2|\lambda|)}{I_0(2|\lambda|)} \right)^2 \right], \] (3.18d)
\[(\Delta e^{-i\phi})^2 = 1 - \left( \frac{I_1(2|\lambda|)}{I_0(2|\lambda|)} \right)^2, \quad (3.18e)\]

\[
\left( \text{Im} \left\{ \int_{-\pi}^{\pi} \psi^*(\phi)(e^{-i\phi})^*i\frac{\partial}{\partial \phi}\psi(\phi)d\phi - \langle e^{-i\phi} \rangle \langle n \rangle \right\} \right)^2 = (\text{Re}\lambda)^2 \left[ 1 - \left( \frac{I_1(2|\lambda|)}{I_0(2|\lambda|)} \right)^2 - \frac{n I_1(2|\lambda|)}{|\lambda| I_0(2|\lambda|)} \right]^2 \quad (3.18f)\]

(compare with respective formulae of [29, 40]).

Using the above relations one quickly arrives at the following result.

**Theorem 3.1** The Robertson – Schrödinger intelligent states for \(e^{-i\phi}\) and \(n\) are given by the wave functions (3.14) (or equivalently, by the vectors (3.15)). The Heisenberg – Robertson intelligent states for \(e^{-i\phi}\) and \(n\) are given by the functions (3.14) \(\equiv\) the vectors (3.15) with \(\lambda \in \mathbb{R}\). Finally, the Trifonov intelligent states for \(e^{-i\phi}\) and \(n\) are given by the functions (3.14) \(\equiv\) the vectors (3.15)) with \(\lambda = \pm 1\).

From (3.18d) and (3.18e) it follows that

\[\Delta n = \Delta e^{-i\phi} \iff |\lambda| = 1. \quad (3.19)\]

For completeness, we give now the uncertainties \(\Delta \cos \phi\) and \(\Delta \sin \phi\) in the state (3.14).

Simple calculations show that (see [29])

\[(\Delta \cos \phi)^2 = \langle \cos^2 \phi \rangle - \langle \cos \phi \rangle^2 = \frac{1}{2} + \frac{(\text{Im}\lambda)^2 - (\text{Re}\lambda)^2}{2|\lambda|^2} \frac{I_2(2|\lambda|)}{I_0(2|\lambda|)} - \frac{(\text{Im}\lambda)^2}{|\lambda|^2} \left( \frac{I_1(2|\lambda|)}{I_0(2|\lambda|)} \right)^2 \quad (3.20)\]

and

\[(\Delta \sin \phi)^2 = \langle \sin^2 \phi \rangle - \langle \sin \phi \rangle^2 = \frac{1}{2} + \frac{(\text{Re}\lambda)^2 - (\text{Im}\lambda)^2}{2|\lambda|^2} \frac{I_2(2|\lambda|)}{I_0(2|\lambda|)} - \frac{(\text{Re}\lambda)^2}{|\lambda|^2} \left( \frac{I_1(2|\lambda|)}{I_0(2|\lambda|)} \right)^2, \quad (3.21)\]

where \(I_1\) and \(I_2\) are the first and the second modified Bessel functions of the first kind, respectively.

Adding (3.20) and (3.21), and comparing with (3.18e) one gets

\[(\Delta \cos \phi)^2 + (\Delta \sin \phi)^2 = (\Delta e^{-i\phi})^2 = 1 - \left( \frac{I_1(2|\lambda|)}{I_0(2|\lambda|)} \right)^2. \quad (3.22)\]

In particular, from (3.18d), (3.20) and (3.21) for \(\lambda = \pm 1\) we have

\[(\Delta \cos \phi)^2 = \frac{1}{2} - \frac{I_2(2)}{2I_0(2)} \approx 0.349, \quad (3.23a)\]
\[(\Delta \sin \phi)^2 = \frac{1}{2} + \frac{1}{2} \frac{I_2(2)}{I_0(2)} - \left( \frac{I_1(2)}{I_0(2)} \right)^2 = (\Delta n)^2 - \frac{1}{2} + \frac{1}{2} \frac{I_2(2)}{I_0(2)} \approx 0.164, \quad (3.23b)\]

\[(\Delta \cos \phi)^2 + (\Delta \sin \phi)^2 = 1 - \left( \frac{I_1(2)}{I_0(2)} \right)^2 \approx 0.513. \quad (3.23c)\]

**Example 3.3**

We consider here the following case

\[f_1(\phi) = \cos \phi, \quad f_2(n) = n. \quad (3.24)\]

Then Eq. (3.2) reads now

\[\left( i \partial_{\phi} + i \lambda \cos \phi - \mu \right) \psi(\phi) = 0, \quad \lambda \in \mathbb{C}. \quad (3.25)\]

The normalized to 1 and satisfying the periodicity condition (3.4) solution of (3.25) is

\[\psi(\phi) = \frac{1}{\sqrt{2\pi I_0(2\text{Re}\lambda)}} e^{-i\phi} e^{-\lambda \sin \phi}, \quad \mu = n = 0, 1, ... \quad (3.26)\]

Note that \(|\psi(\phi)|^2 = \frac{1}{2\pi I_0(2\text{Re}\lambda)} e^{-2\text{Re}\lambda \sin \phi}\) is the von Mises circular distribution.

Then, the solution (3.26) fulfills the condition (2.49) iff

\[\int_{-\pi}^{\pi} e^{-\lambda \sin \phi} e^{-i(n+k)\phi} d\phi = 0, \quad \text{for } k = 1, 2, ... \quad (3.27)\]

As is well known

\[\int_{-\pi}^{\pi} e^{-\lambda \sin \phi} e^{-im\phi} d\phi = 2\pi J_m(i\lambda), \quad m = 0, 1, ... \quad (3.28)\]

where \(J_m\) denotes the \(m\)-th Bessel function of the first kind. From (3.27) and (3.28) one infers that \(\psi(\phi)\) given by (3.26) satisfies the condition (2.49) iff

\[J_{n+k}(i\lambda) = 0, \quad \text{for all } k = 1, 2, ... \quad (3.29)\]

But the system of constraints (3.29) holds true iff \(\lambda = 0\).

So the only Robertson – Schrödinger and Heisenberg – Robertson intelligent states for \(\cos \phi\) and \(n\) are given by the normalized wave functions \(\psi(\phi) = \frac{1}{\sqrt{2\pi}} e^{-i\phi}, n = 0, 1, ..., \) i.e. the eigenfunctions of the number operator \(\hat{n}\).
Then one can easily find that the uncertainty $\Delta \cos \phi$ in the state $\psi(\phi) = \frac{1}{\sqrt{2\pi}} e^{-in\phi}$, $n = 0, 1, ...$, is given by

$$\Delta \cos \phi = \frac{1}{\sqrt{2}}$$

and, consequently

$$(\Delta n)^2 + (\Delta \cos \phi)^2 = \frac{1}{2}.$$ (3.31)

As can be quickly shown, the right hand side of the Trifonov uncertainty relation (2.30) in our present case vanishes. Thus the formula (3.31) shows immediately that the states $\psi(\phi) = \frac{1}{\sqrt{2\pi}} e^{-in\phi}$, $n = 0, 1, ...$ are not the Trifonov intelligent states and, therefore, there are no Trifonov intelligent states for $\cos \phi$ and the photon number $n$.

Finally, we observe that some conclusions on intelligent states hold true when

$$f_1 = \sin \phi, \quad f_2 = n.$$ (3.32)

IV. THE MINIMUM UNCERTAINTY PRODUCT STATES

In this section we deal with the problem of searching for the states which minimize locally the product of uncertainties $(\Delta f_1)^2 \cdot (\Delta f_2)^2$. To this end we consider the functional

$$\mathcal{H}[\psi(\phi), \psi^*(\phi)] := (\Delta f_1)^2 \cdot (\Delta f_2)^2.$$ (4.1)

Thus our problem reduces to the following isoperimetric variational problem

$$\delta \mathcal{H}[\psi(\phi), \psi^*(\phi)] = 0, \quad \int_{-\pi}^{\pi} \psi^*(\phi)\psi(\phi)d\phi = 1.$$ (4.2)

Inserting Eq. (4.1) into Eq. (4.2) and employing Def. (2.28) one easily proves that the variational problem (4.2) leads to the following equation

$$\left[ (\Delta f_1)^2 \cdot (\delta \hat{F}_2)^\dagger \cdot \delta \hat{F}_2 + (\Delta f_2)^2 \cdot (\delta \hat{F}_1)^\dagger \cdot \delta \hat{F}_1 - \sigma \right] \psi(\phi) = 0,$$ (4.3)

where $\sigma \in \mathbb{R}$ is a Lagrange multiplier. The operators $\delta \hat{F}_1$ and $\delta \hat{F}_2$ are defined by Eqs. (2.20a) and (2.20b), respectively.

Multiplying (4.3) by $\psi^*(\phi)$ and integrating we find that

$$\sigma = 2(\Delta f_1)^2 \cdot (\Delta f_2)^2.$$ (4.4)
Substituting (4.4) into (4.3) and dividing by \((\Delta f_1)^2 \neq 0\) one finally gets
\[
\left[ (\delta \hat{F}_2) \cdot \delta \hat{F}_2 + \frac{(\Delta f_2)^2}{(\Delta f_1)^2} (\delta \hat{F}_1) \cdot \delta \hat{F}_1 - 2(\Delta f_2)^2 \right] \psi(\phi) = 0. \tag{4.5}
\]

Eq. (4.5) for Hermitian operators \(\hat{F}_1^\dagger = \hat{F}_1\) and \(\hat{F}_2^\dagger = \hat{F}_2\) was previously found by R. Jackiw [34] and then also analyzed extensively by P. Carruthers and M.M. Nieto [33]. Our Eq. (4.5) is an obvious generalization of their results on the case of non-hermitian operators.

Now we consider the first important example of the application of Eq. (4.5).

**Example 4.1**

Here we assume that \(f_1\) and \(f_2\) are given by (3.12) i.e. \(f_1(\phi) = e^{-i\phi}\) and \(f_2(n) = n\). Then Eq. (4.5) reads now
\[
\left\{ \left( i \frac{\partial}{\partial \phi} - \langle n \rangle \right)^2 + \frac{(\Delta n)^2}{(\Delta e^{-i\phi})^2} \left( e^{i\phi} - \langle e^{i\phi} \rangle \right) \left( e^{-i\phi} - \langle e^{-i\phi} \rangle \right) - 2(\Delta n)^2 \right\} \psi(\phi) = 0. \tag{4.6}
\]

One should remember that the wave function \(\psi(\phi)\) must be of the form given by (2.49).

Substituting (2.49) into (4.6) we quickly note that the three following cases should be analyzed:

(i). \(\Delta n \neq 0, \quad \langle e^{-i\phi} \rangle \neq 0. \tag{4.7}\)

Here one immediately finds that all coefficients \(c_n\) in (2.49) vanish, so \(\psi(\phi) = 0\).

(ii). \(\Delta n \neq 0, \quad \langle e^{-i\phi} \rangle = 0. \tag{4.8}\)

Then
\[
(\Delta e^{-i\phi})^2 = \int_{-\pi}^{\pi} \psi^*(\phi) \left( e^{i\phi} - \langle e^{i\phi} \rangle \right) \left( e^{-i\phi} - \langle e^{-i\phi} \rangle \right) \psi(\phi) d\phi = 1 \tag{4.9}
\]

and Eq. (4.6) reduces to
\[
\left\{ \left( i \frac{\partial}{\partial \phi} - \langle n \rangle \right)^2 - (\Delta n)^2 \right\} \psi(\phi) = 0. \tag{4.10}
\]

The general normalized to 1 solution of (4.10) satisfying (4.8) and (2.49) reads
\[
\psi(\phi) = \frac{1}{\sqrt{4\pi}} e^{i\alpha} \left( e^{-ik\phi} + e^{i\beta} e^{-il\phi} \right), \tag{4.11}
\]
where $\alpha, \beta \in \mathbb{R}, \mathbb{N} \ni k, \ell \geq 0, |\ell - k| \geq 2$. Straightforward calculations give
\[
\langle n \rangle = \frac{k + \ell}{2}, \quad \Delta n = \frac{|\ell - k|}{2}.
\] (4.12)

We will show now that although the wave function (4.11) fulfills Eq. (4.6), the respective quantum state is not a minimum uncertainty product state. To this end assume that in (4.11) the natural numbers $k$ and $\ell$ are chosen so that $k < \ell$. Define the normalized function
\[
\psi'(\phi) := \sqrt{1 - \varepsilon} \psi(\phi) + \sqrt{\varepsilon} \frac{1}{\sqrt{2\pi}} e^{-im\phi},
\] (4.13)
where $m \in \mathbb{N}, k < m < \ell, \varepsilon \in \mathbb{R}, 0 < \varepsilon \leq 1,$ and $\psi(\phi)$ is given by (4.11).

Denoting the uncertainties of the photon number $n$ and the phase function $e^{-i\phi}$ for the state (4.13) by $(\Delta n)'$ and $(\Delta e^{-i\phi})'$, respectively one easily gets
\[
(\Delta n)'^2 = \left(\frac{k - \ell}{2}\right)^2 + \varepsilon(k - m)(\ell - m) - \varepsilon^2 \left(\frac{k + \ell}{2} - m\right)^2
\]
\[
= (\Delta n)^2 + \varepsilon(k - m)(\ell - m) - \varepsilon^2(\langle n \rangle - m)^2 < (\Delta n)^2;
\] (4.14a)
\[
(\Delta e^{-i\phi})'^2 = 1 - |\langle e^{-i\phi} \rangle'|^2 \leq 1 = (\Delta e^{-i\phi})^2,
\] (4.14b)

where $(\Delta e^{-i\phi})^2$, $\Delta n$ and $\langle n \rangle$ are given by (4.9) and (4.12), and $\langle e^{-i\phi} \rangle' := \int_{-\pi}^{\pi} e^{-i\phi} |\psi'(\phi)|^2 d\phi$.

The norm of the difference $\psi' - \psi$ is given by
\[
\|\psi' - \psi\|_0 := \sup_{-\pi \leq \phi < \pi} |\psi'(\phi) - \psi(\phi)| \leq |\sqrt{1 - \varepsilon} - 1| \|\psi\|_0 + \sqrt{\varepsilon} \frac{1}{\sqrt{2\pi}}
\]
\[
\leq 1 - \sqrt{1 - \varepsilon} + \sqrt{\frac{\varepsilon}{2\pi}} = \sqrt{\frac{\varepsilon}{2\pi}} \left(\frac{\sqrt{2\varepsilon}}{1 + \sqrt{1 - \varepsilon}} + 1\right), \quad 0 < \varepsilon \leq 1
\] (4.15)
and it can be done arbitrarily small by taking $\varepsilon$ sufficiently small. Since by (4.14a) and (4.14b)
\[
(\Delta n)' \cdot (\Delta e^{-i\phi})' < \Delta n \cdot \Delta e^{-i\phi}, \quad \forall \quad 0 < \varepsilon \leq 1,
\] (4.16)
the state (4.11) is not a minimum uncertainty product state for $n$ and $e^{-i\phi}$. Note that taking in (4.13)
\[
m > \ell + 1 > k + 1
\] (4.17)
and putting $\varepsilon$ sufficiently small one easily infers from Eqs. (4.14a) and (4.14b) that, with (4.17) assumed, the relations

$$(\Delta n)' > \Delta n, \quad (\Delta e^{-i\phi})' = 1$$

imply

$$(\Delta n)' \cdot (\Delta e^{-i\phi})' > \Delta n \cdot \Delta e^{-i\phi}.$$  \hspace{1cm} (4.18)

Therefore, the state (4.11) is also not a maximum uncertainty product state. Remember that we deal with a local minimum and a local maximum uncertainty product states.

It remains the last case to be considered

(iii).

$$\Delta n = 0.$$ \hspace{1cm} (4.19)

Here of course, $\psi(\phi) = \frac{1}{\sqrt{2\pi}} e^{-in\phi}, \; n = 0, 1, 2, \ldots, \langle n \rangle = n, \langle e^{-i\phi} \rangle = 0, \Delta e^{-i\phi} = 1$ and $\Delta n \cdot \Delta e^{-i\phi} = 0.$

Thus we get now the (global) minimum uncertainty product states. Summarizing, one arrives at

**Theorem 4.1** The only minimum uncertainty product states for the photon number $n$ and the phase function $e^{-i\phi}$ are the eigenstates $|n\rangle, \; n = 0, 1, 2, \ldots$ of the photon number operator $\hat{n}$.

The states $|n\rangle$ are the global minimum uncertainty product states for $n$ and $e^{-i\phi}$. $\blacksquare$

From this theorem one immediately gets

**Corollary 4.1** If $|\psi\rangle, \langle \psi|\psi\rangle = 1$, is a state different from the eigenstate of $\hat{n}$, and $\Delta n$ and $\Delta e^{-i\phi}$ are uncertainties for the number of photons and for the phase function $e^{-i\phi}$ respectively, then in any neighborhood of $|\psi\rangle$ (in the sense of the norm $\| \cdot \|_0$) there exists a state $|\psi'\rangle, \langle \psi'|\psi'\rangle = 1$, such that the product of uncertainties $(\Delta n)'$ and $(\Delta e^{-i\phi})'$ is less than the product of $\Delta n$ and $\Delta e^{-i\phi}$ i.e. $(\Delta n)' \cdot (\Delta e^{-i\phi})' < \Delta n \cdot \Delta e^{-i\phi}$. $\blacksquare$

One can quickly observe that Theorem 4.1 and Corollary 4.1 are also true, *mutatis mutandi*, for the cases $f_1 = e^{i\phi}, \; f_1 = \cos \phi$ or $f_1 = \sin \phi$ and $f_2 = n$.

The second important example we deal with is
Example 4.2

Here we look for the minimum uncertainty product states for the quantum phase and the number of photons.

A careful analysis of Example 2.1 shows that one should consider now the following isoperimetric variational problem

$$\delta \left\{ \int_{-\pi}^{\pi} \phi^2 |\psi(\phi)|^2 d\phi \cdot \int_{-\pi}^{\pi} \psi^*(\phi) \left( i \frac{\partial}{\partial \phi} - \langle n \rangle \right)^2 \psi(\phi) d\phi \right\} = 0, \quad (4.20)$$

with the constraint $\int_{-\pi}^{\pi} \psi^*(\phi) \psi(\phi) d\phi = 1$.

Then in the present case the counterpart of Eq. (4.5) reads

$$\left\{ \left( i \frac{\partial}{\partial \phi} - \langle n \rangle \right)^2 + (\Delta n)^2 \right\} \psi(\phi) = 0. \quad (4.21)$$

Substituting

$$\chi(\phi) = e^{i\langle n \rangle \phi} \psi(\phi) \quad (4.22)$$

we turn Eq. (4.21) to the form

$$\frac{d^2 \chi(\phi)}{d\phi^2} - \left[ \frac{(\Delta n)^2}{\langle \phi^2 \rangle} \phi^2 - 2(\Delta n)^2 \right] \chi(\phi) = 0. \quad (4.23)$$

Assume first that $\Delta n \neq 0$ and define a new variable

$$z = \left( \frac{2\Delta n}{\sqrt{\langle \phi^2 \rangle}} \right)^{1/2} \phi. \quad (4.24)$$

Then Eq. (4.23) reads

$$\frac{d^2 \chi(z)}{dz^2} - \left( \frac{z^2}{4} - \Delta n \cdot \sqrt{\langle \phi^2 \rangle} \right) \chi(z) = 0. \quad (4.25)$$

This is the parabolic cylinder equation [68, 69] and the same equation appears in the work by D.T. Pegg, S.N. Barnett et al on the minimum uncertainty states of the angular momentum and the angular position (see [30] Eq. (37)). The general solution of (4.25) is well known [68] and using formulas (4.22) and (4.23) one gets the general solution of (4.21) with $\Delta n \neq 0$ as

$$\psi(\phi) = e^{-i\langle n \rangle \phi} e^{-\frac{1}{2} \frac{\Delta n}{\sqrt{\langle \phi^2 \rangle}} \phi^2} \left\{ a_1 \cdot \ _1F_1 \left( \frac{1}{2}, \left[ \frac{1}{2} - \Delta n \cdot \sqrt{\langle \phi^2 \rangle} \right], \frac{1}{2} \cdot \frac{\Delta n}{\sqrt{\langle \phi^2 \rangle}} \phi^2 \right) \right. + a_2 \sqrt{\frac{2\Delta n}{\sqrt{\langle \phi^2 \rangle}}} \phi \cdot \ _1F_1 \left( \frac{1}{2}, \left[ \frac{3}{2} - \Delta n \cdot \sqrt{\langle \phi^2 \rangle} \right], \frac{3}{2} \cdot \frac{\Delta n}{\sqrt{\langle \phi^2 \rangle}} \phi^2 \right) \left. \right\}, \quad a_1, a_2 \in \mathbb{C}, \quad (4.26)$$
where \( \text{I}_1 \) stands for the confluent hypergeometric function.

In order to find a physically acceptable solution \( \psi(\phi) \) we must extract from the general formula (4.26) a function which fulfills the periodicity conditions

\[
\psi(\pi) = \psi(-\pi), \quad \frac{d\psi(\pi)}{d\phi} = \frac{d\psi(-\pi)}{d\phi} \tag{4.27}
\]

and has the Fourier expansion of the form (2.49). Therefore

\[
\int_{-\pi}^{\pi} \psi(\phi)e^{-ik\phi}d\phi = 0, \quad \text{for} \quad k = 1, 2, \ldots \tag{4.28}
\]

This last condition makes our problem drastically different from the problem stated in [30], where the minimum uncertainty product states of the angular momentum and the angular position are studied. Substituting Eq. (4.26) into (4.27) and using the abbreviations

\[
y_1(\phi) = e^{-\frac{1}{2}\Delta n \sqrt{\langle \phi^2 \rangle}} \cdot \text{I}_1 \left( \frac{1}{2} \cdot \left[ 1 - \Delta n \cdot \sqrt{\langle \phi^2 \rangle} \right], \frac{1}{2}, \frac{\Delta n \sqrt{\langle \phi^2 \rangle}}{2} \right), \tag{4.29a}
\]

\[
y_2(\phi) = e^{-\frac{1}{2}\Delta n \sqrt{\langle \phi^2 \rangle}^2} \sqrt{\frac{2\Delta n}{\sqrt{\langle \phi^2 \rangle}}} \cdot \text{I}_1 \left( \frac{1}{2} \cdot \left[ 3 - \Delta n \cdot \sqrt{\langle \phi^2 \rangle} \right], 3, \frac{2\Delta n \sqrt{\langle \phi^2 \rangle}}{2} \right) \tag{4.29b}
\]

one gets the following system of equations (remember that \( y_1(-\phi) = y_1(\phi) \) and \( y_2(-\phi) = -y_2(\phi) \))

\[
a_2 \cos \left( \langle n \rangle \pi \right) \cdot y_2(\pi) = ia_1 \sin \left( \langle n \rangle \pi \right) \cdot y_1(\pi), \tag{4.30a}
\]

\[
a_1 \cos \left( \langle n \rangle \pi \right) \cdot \frac{dy_1(\pi)}{d\phi} = ia_2 \sin \left( \langle n \rangle \pi \right) \cdot \frac{dy_2(\pi)}{d\phi}. \tag{4.30b}
\]

The Wronski determinant \( W(y_1(\phi), y_2(\phi)) \) is given by

\[
W(y_1(\phi), y_2(\phi)) = y_1(\phi) \frac{dy_2(\phi)}{d\phi} - y_2(\phi) \frac{dy_1(\phi)}{d\phi} = \text{const.}
\]

\[
= y_1(0) \frac{dy_2(0)}{d\phi} = \left( \frac{2\Delta n}{\sqrt{\langle \phi^2 \rangle}} \right)^{1/2}. \tag{4.31}
\]

Without any loss of generality we can put \( a_1 \) real

\[
a_1^* = a_1. \tag{4.32}
\]

Assuming that the wave function \( \psi(\phi) \) given by (4.26) is normalized to 1, and taking into account Eqs. (4.29a), (4.29b) and (4.32) one quickly finds

\[
\langle n \rangle = \int_{-\pi}^{\pi} \psi^*(\phi) i \frac{\partial}{\partial \phi} \psi(\phi) d\phi = \langle n \rangle + i \int_{-\pi}^{\pi} a_1 \cdot \left( a_2 y_1(\phi) \frac{dy_2(\phi)}{d\phi} + a_2 y_2(\phi) \frac{dy_1(\phi)}{d\phi} \right) d\phi. \tag{4.33}
\]
Hence
\[ a_1 \cdot \int_{-\pi}^{\pi} \left( a_2 y_1(\phi) \frac{dy_2(\phi)}{d\phi} + a_2^* y_2(\phi) \frac{dy_1(\phi)}{d\phi} \right) d\phi = 0. \] (4.34)

Multiplying the Wronskian (4.31) by \(a_1 a_2\), integrating out over \(d\phi\) and comparing with Eq. (4.34) we get
\[ a_1 \cdot \int_{-\pi}^{\pi} (a_2 + a_2^*) y_2(\phi) \frac{dy_1(\phi)}{d\phi} d\phi = -2\pi a_1 a_2 \left( \frac{2\Delta n}{\sqrt{\langle \phi^2 \rangle}} \right)^{1/2}. \] (4.35)

Hence, the product \(a_1 a_2\) is a real number. Concluding, without any loss of generality one can put the coefficients \(a_1\) and \(a_2\) real i.e.
\[ \chi^\ast(\phi) = \chi(\phi) \] (4.36)

with \(\chi(\phi)\) defined by (4.22).

Then returning to the periodicity conditions (4.30a), (4.30b) we easily realize that the analysis of these conditions splits into three cases:

(i). \(a_1 y_1(\pi) = 0 = a_2 y_2(\pi)\) and \(a_1 \frac{dy_1(\pi)}{d\phi} = 0 = a_2 \frac{dy_2(\pi)}{d\phi}\),

(ii). \(\sin(\langle n \rangle \pi) = 0, a_2 y_2(\pi) = 0\) and \(a_1 \frac{dy_1(\pi)}{d\phi} = 0\),

(iii). \(\cos(\langle n \rangle \pi) = 0, a_1 y_1(\pi) = 0\) and \(a_2 \frac{dy_2(\pi)}{d\phi} = 0\).

Consider the case (i). Here one immediately infers that from the theorem on existence and uniqueness of a solution of the initial value problem for Eq. (4.23) it follows that the unique solution of this equation fulfilling the conditions posed in (i) is \(\chi(\phi) = 0\). So \(\psi(\phi)\) is also equal to zero and such a wave function does not represent any quantum state.

In the case (ii) we find that \(\langle n \rangle = N, N = 0, 1, 2, \ldots\) Hence \(\psi(\phi) = e^{-iN\phi} \chi(\phi), N = 0, 1, 2, \ldots\) Inserting this \(\psi(\phi)\) into (4.28), taking then the complex conjugate of both sides and employing (4.36) one obtains
\[ \int_{-\pi}^{\pi} \chi(\phi)e^{i(N+k)\phi} d\phi = \int_{-\pi}^{\pi} \psi(\phi)e^{i(2N+k)\phi} d\phi = 0, \quad k = 1, 2, \ldots \] (4.37)

Finally, by (4.28) and (4.37) we conclude that in the case (ii) the respective wave function must be of the form
\[ \psi(\phi) = \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{2N} c_n e^{-in\phi}. \] (4.38)
Substituting (4.38) into Eq. (4.21), taking the first and then the second derivative of both sides and comparing the values of left hand sides at $\phi = \pi$ and $\phi = -\pi$ one gets that if $\Delta n \neq 0$ then
\begin{equation}
\psi(\pi) = 0 \quad \text{and} \quad \frac{d\psi(\pi)}{d\phi} = 0. \tag{4.39}
\end{equation}
Consequently, as in the preceding case (i), by the uniqueness of the solution of the initial value problem for Eq. (4.21), the only solution for the case (ii) is $\psi(\phi) = 0$. Finally, in the case (iii) one has $\langle n \rangle = N + \frac{1}{2}$, $N = 0, 1, 2, \ldots$ So $\psi(\phi) = e^{-iN\phi}e^{-\frac{i}{2}\phi}\chi(\phi)$. Inserting this wave function into (4.28) and taking the complex conjugate of the integral obtained, using also (4.36) we get
\begin{equation}
\int_{-\pi}^{\pi} \chi(\phi)e^{iN\phi}e^{\frac{i}{2}\phi}e^{ik\phi}d\phi = \int_{-\pi}^{\pi} \psi(\phi)e^{i(2N+1+k)\phi}d\phi = 0, \quad k = 1, 2, \ldots \tag{4.40}
\end{equation}
From (4.28) and (4.40) it follows that in the case (iii) the wave function is of the form
\begin{equation}
\psi(\phi) = \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{2N+1} c_n e^{-in\phi}. \tag{4.41}
\end{equation}
Consequently, the analogous arguments as in the case (ii) lead to the conclusion that the only solution of the case (iii) is $\psi(\phi) = 0$.

Gathering our rather long discussion we find that: \textit{there is no minimum uncertainty product state for the quantum phase and the number of photons if $\Delta n \neq 0$}. Thus one arrives at the following

\textbf{Theorem 4.2} The only minimum uncertainty product states for the quantum phase and the number of photons are the eigenstates $|n\rangle$, $n = 0, 1, \ldots$, of the photon number operator $\hat{n}$.

The eigenstates $|n\rangle$ are the global minimum uncertainty product states for the quantum phase and the number of photons. ■

From this theorem we get an important

\textbf{Corollary 4.2} Let $|\psi\rangle$, $\langle \psi | \psi \rangle = 1$, be any state which is not an eigenstate of the photon number operator $\hat{n}$, i.e. $\Delta n \neq 0$ and let the product of uncertainties $\Delta n$ and $\Delta \phi$ in $|\psi\rangle$ be $\Delta n \cdot \Delta \phi = b$. Then in any neighborhood of $|\psi\rangle$ there exists a state $|\psi'\rangle$, $\langle \psi' | \psi' \rangle = 1$, such that the product of uncertainties $(\Delta n)'$ and $(\Delta \phi)'$ in $|\psi'\rangle$ is less than $b$ i.e. $(\Delta n)' \cdot (\Delta \phi)' < \Delta n \cdot \Delta \phi = b$. ■

(For the definition of $\Delta \phi$ see Example 2.1).
V. THE MINIMUM UNCERTAINTY SUM STATES

In this section we are going to find the states which minimize locally the sum of uncertainties \((\Delta f_1)^2 + (\Delta f_2)^2\). This problem reduces to the isoperimetric variational problem

\[
\delta G[\psi(\phi), \psi^*(\phi)] = 0, \quad \int_{-\pi}^{\pi} \psi^*(\phi)\psi(\phi)d\phi = 1,
\]

where

\[
G[\psi(\phi), \psi^*(\phi)] = (\Delta f_1)^2 + (\Delta f_2)^2 = \int_{-\pi}^{\pi} \left( \delta \hat{F}_1 \psi(\phi) \right)^* \delta \hat{F}_1 \psi(\phi) d\phi + \int_{-\pi}^{\pi} \left( \delta \hat{F}_2 \psi(\phi) \right)^* \delta \hat{F}_2 \psi(\phi) d\phi,
\]

It leads to the following equation

\[
\left[ (\delta \hat{F}_1)^\dagger \delta \hat{F}_1 + (\delta \hat{F}_2)^\dagger \delta \hat{F}_2 - \tau \right] \psi(\phi) = 0,
\]

where \(\tau \in \mathbb{R}\) is a Lagrange multiplier. Multiplying (5.2) by \(\psi^*(\phi)\) and integrating over \(d\phi\) we obtain that \(\tau = (\Delta f_1)^2 + (\Delta f_2)^2\). Finally, Eq. (5.2) reads

\[
\left\{ (\delta \hat{F}_1)^\dagger \delta \hat{F}_1 + (\delta \hat{F}_2)^\dagger \delta \hat{F}_2 - \left[ (\Delta f_1)^2 + (\Delta f_2)^2 \right] \right\} \psi(\phi) = 0.
\]

Note that for \(\Delta f_1 = \Delta f_2\) Eqs. (4.5) and (5.3) are equivalent.

As the first example we consider

**Example 5.1**

Here we take \(f_1 = f_1(\phi) = e^{-i\phi}\) and \(f_2 = f_2(n) = n\).

Then Eq. (5.3) gives now

\[
\left\{ \left( i \frac{\partial}{\partial \phi} - \langle n \rangle \right)^2 - \langle e^{i\phi} \rangle e^{-i\phi} - \langle e^{-i\phi} \rangle e^{i\phi} + 2\langle e^{i\phi} \rangle \langle e^{-i\phi} \rangle \right\} \psi(\phi) = 0.
\]

Assume that \(\Delta n \neq 0\). Then one quickly shows that the results of our considerations done in (i) and (ii) of Example 4.1 hold true if we change the ‘product of uncertainties’ to the ‘sum of uncertainties’. Thus, for example, one should change (4.16) to

\[
(\Delta n)^2 + (\Delta e^{-i\phi})^2 < (\Delta n)^2 + (\Delta e^{-i\phi})^2
\]

and (4.18) to

\[
(\Delta n)^2 + (\Delta e^{-i\phi})^2 > (\Delta n)^2 + (\Delta e^{-i\phi})^2.
\]
It remains only to study the case when $\Delta n = 0$. Now we have of course $\psi(\phi) = \frac{1}{\sqrt{2\pi}} e^{-in\phi}$, $n = 0, 1, \ldots$, $\langle n \rangle = n$, $\langle e^{-i\phi} \rangle = 0$ and $\Delta e^{-i\phi} = 1$. So

$$(\Delta n)^2 + (\Delta e^{-i\phi})^2 = 1.$$  \hspace{1cm} (5.7)$$

The question is, if the states $\psi(\phi) = \frac{1}{\sqrt{2\pi}} e^{-in\phi}$, $n = 0, 1, \ldots$, are really the minimum uncertainty sum states for the phase function $e^{-i\phi}$ and the number of photons.

To answer this question let us consider the following state

$$\psi''(\phi) := \frac{1}{\sqrt{2\pi}} \left( \sqrt{1 - \varepsilon} e^{-in\phi} + \sqrt{\varepsilon} e^{-i(n+1)\phi} \right), \quad 0 < \varepsilon < 1. \hspace{1cm} (5.8)$$

Simple calculations lead to the following results

$$(\Delta n)''^2 = \varepsilon \cdot (1 - \varepsilon), \hspace{2cm} (5.9a)$$

$$\langle e^{-i\phi} \rangle'' = \sqrt{\varepsilon \cdot (1 - \varepsilon)}, \hspace{2cm} (5.9b)$$

$$(\Delta e^{-i\phi})''^2 = 1 - |\langle e^{-i\phi} \rangle''|^2 = 1 - \varepsilon \cdot (1 - \varepsilon). \hspace{2cm} (5.9c)$$

Hence

$$(\Delta n)^2 + (\Delta e^{-i\phi})''^2 = 1,$$ \hspace{1cm} (5.10)

as in (5.7). However, the function (5.8) does not satisfy Eq. (5.4) for any $0 < \varepsilon < 1$. Therefore the state $\psi''(\phi)$ given by (5.8) is not a (local) minimum uncertainty sum state.

On the other hand $\lim_{\varepsilon \to 0} \psi''(\phi) = \frac{1}{\sqrt{2\pi}} e^{-in\phi}$ and the sum (5.10) of uncertainties for $\psi''(\phi)$ is equal to the sum (5.7) of the uncertainties for $\psi(\phi) = \frac{1}{\sqrt{2\pi}} e^{-in\phi}$. Since the state $\psi''(\phi)$ is not a minimum uncertainty sum state, for any $0 < \varepsilon < 1$ in an arbitrary neighborhood of $\psi''(\phi)$ (in the sense of the norm $\| \cdot \|_0$) there exists a state such that the respective sum of uncertainties is less than 1. Consequently, in any neighborhood of the state $\psi(\phi) = \frac{1}{\sqrt{2\pi}} e^{-in\phi}$ lies a state for which this sum is less than that given in (5.7) i.e. 1. This means that the states $\psi(\phi) = \frac{1}{\sqrt{2\pi}} e^{-in\phi}$, $n = 0, 1, \ldots$ are not the minimum uncertainty sum states.

Gathering the results obtained in the present example one arrives at the conclusion.

**Theorem 5.1** There are no minimum uncertainty sum states for the phase function $e^{-i\phi}$ and the number of photons $n$. 

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It is an easy matter to show that the analogous theorems hold true in the cases of phase functions $e^{i\phi}$, $\cos \phi$ or $\sin \phi$ and the number of photons $n$.

Theorem 5.1 leads immediately to the following

**Theorem 5.1** Let $|\psi\rangle$, $\langle \psi | \psi \rangle = 1$, be any state, and let $\Delta n$ and $\Delta e^{-i\phi}$ be the uncertainties of the number of photons $n$ and the phase function $e^{-i\phi}$, respectively in $|\psi\rangle$. Then in any neighborhood of $|\psi\rangle$ (in the sense of the norm $\| \cdot \|_0$) there exists a state $|\psi'\rangle$, $\langle \psi' | \psi' \rangle = 1$, such that $\left( \Delta n \right)'^2 + (\Delta e^{-i\phi})'^2 < \left( \Delta n \right)^2 + (\Delta e^{-i\phi})^2$, where $(\Delta n)'$ and $(\Delta e^{-i\phi})'$ stand for the respective uncertainties in $|\psi'\rangle$.

The same holds true in the cases of phase functions $e^{i\phi}$, $\cos \phi$ or $\sin \phi$ and the number of photons $n$. ■

We end the considerations of this section with an important

**Example 5.2**

We are looking now for the minimum uncertainty sum states for the quantum phase and the number of photons.

Employing the results of Examples 2.1 and 4.2 one finds that in the present case Eq. \(5.3\) reads

\[
\left\{ \left( i \frac{\partial}{\partial \phi} - \langle n \rangle \right)^2 + \phi^2 - \left[ \langle \phi^2 \rangle + (\Delta n)^2 \right] \right\} \psi(\phi) = 0.
\] (5.11)

We quickly recognize that Eq. (5.11) can be obtained from (4.21) by substitutions $\frac{(\Delta n)^2}{\langle \phi^2 \rangle} \to 1$ and $2(\Delta n)^2 \to \langle \phi^2 \rangle + (\Delta n)^2$. So we define $\chi(\phi)$ as in (4.22) and, according to (4.24), the variable

\[ x = \sqrt{2}\phi. \] (5.12)

Then Eq. (5.11) reduces to the parabolic cylinder equation analogous to (4.25)

\[
\frac{d^2 \chi(x)}{dx^2} - \left( \frac{x^2}{4} - \frac{\langle \phi^2 \rangle + (\Delta n)^2}{2} \right) \chi(x) = 0.
\] (5.13)

Finally, the general solution of Eq. (5.11) reads (compare with (4.26))

\[
\psi(\phi) = e^{-i\langle n \rangle \phi} e^{-\frac{1}{2} \phi^2} \left\{ a_1 \cdot \text{\large 1}\text{\large F1} \left( \frac{1}{2}, \left[ \frac{1}{2} - \frac{\langle \phi^2 \rangle + (\Delta n)^2}{2} \right], \frac{1}{2}, \phi^2 \right) \right.
\]

\[ + a_2 \sqrt{2}\phi \cdot \text{\large 1}\text{\large F1} \left( \frac{1}{2}, \left[ \frac{3}{2} - \frac{\langle \phi^2 \rangle + (\Delta n)^2}{2} \right], \frac{3}{2}, \phi^2 \right) \}, \quad a_1, a_2 \in \mathbb{C}. \] (5.14)

The further analysis is, *mutatis mutandi* the same as in Example 4.2. Thus one arrives at the following
Theorem 5.2 There is no minimum uncertainty sum state for the quantum phase and the number of photons.

From this theorem we get also an important

Corollary 5.2 For any state $|\psi\rangle$, $\langle \psi | \psi \rangle = 1$, and for any neighbourhood of $|\psi\rangle$ (in the sense of the norm $\| \cdot \|_0$) there exists a state $|\psi'\rangle$, $\langle \psi' | \psi' \rangle = 1$, such that $(\Delta n)'^2 + (\tilde{\Delta} \phi)'^2 < (\Delta n)^2 + (\tilde{\Delta} \phi)^2$.

VI. CONCLUDING REMARKS

Comparing our results with the ones concerning intelligent states and the minimum uncertainty product or sum states for the angular momentum and the angular position [30, 70, 71] one quickly notes that most of the states, which have been found in those papers and which play an important role in quantum mechanics on the circle, are not admitted in quantum optics. Of course, the reason of this lies in the fact that, in contrast to angular momentum $L_z$ of the particle on the circle, which can assume values $L_z = 0, \pm \hbar, \pm 2\hbar, \ldots$, the number of photons $n$ can be only a natural number $n = 0, 1, 2, \ldots$. So in quantum optics the wave function $\psi(\phi)$ must satisfy the condition (2.49) which, as we have seen in the present paper, is highly restrictive. In particular one can see this from Theorems 2.1 and 4.2 which prove that the only number phase Robertson – Schrödinger and Heisenberg – Robertson intelligent states are the eigenstates $|n\rangle$, $n = 0, 1, 2, \ldots$ of the photon number operator $\hat{n}$ and the same states are also the only minimum uncertainty product states for the quantum phase and the number of photons. We can succinctly state that the photon is intelligent.

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[1] W. Heisenberg, Z. Phys. 43 (1927) 172.
[2] E.H. Kennard, Z. Phys. 44 (1973) 326.
[3] H. Weyl, *Gruppentheorie und Quantenmechanik* (Hirzel Verlag, Leipzig, 1928).
[4] H.P. Robertson, Phys. Rev. 34 (1929) 163.
[5] H.P. Robertson, Phys. Rev. 35 (1930) 667A.
[6] E. Schrödinger, Sitz. Preus. Acad. Wiss. (Phys.-Math. Klasse) 19 (1930) 296.
[7] H.P. Robertson, Phys. Rev. 46 (1934) 794.
[8] D.A. Trifonov, J. Math. Phys. 34 (1993) 100.
[9] D.A. Trifonov, J. Math. Phys. 35 (1994) 2297.
[10] D.A. Trifonov, J. Phys. A: Math. Gen. 30 (1997) 5941.
[11] D.A. Trifonov, Phys. Scripta 58 (1998) 246.
[12] D.A. Trifonov, S.G. Donev, J. Phys. A: Math. Gen. 31 (1998) 8041.
[13] D.A. Trifonov, J. Phys. A: Math. Gen. 33 (2000) L299; J. Phys. A: Math. Gen. 34 (2001) L75.
[14] D.A. Trifonov, in: *Geometry, Integrability and Quantization, Varna 1999* (Coral Press Sci. Publ., Sofia 2000) pp. 257-282.
[15] D.A. Trifonov, Eur. Phys. J. B 29 (2002) 349.
[16] V.V. Dodonov, E.V. Kurmyshev, V.I. Man’ko, Phys. Lett. A 79 (1980) 150.
[17] J.R. Klauder, B.S. Skagerstam, *Coherent States*, (World Scientific, Singapore 1985).
[18] W.-M. Zang, D.H. Feng, R. Gilmore, Rev. Mod. Phys. 62 (1990) 867.
[19] R. Tanaś, *Lectures on Quantum Optics* (in polish), http://zonz8.physd.amu.edu.pl/~tanas, 2007.
[20] S. Kryszewski, Quantum Optics, Lecture notes for students, iftia9.uviv.gda.pl/~sjk/QO~SK.pdf, Gdańsk 2009-2010.
[21] L. Infeld, J. Plebański, Acta Phys. Polon. XIV (1955) 41.
[22] J. Plebański, Bull. Acad. Polon. Sci. Cl. III, II (1954) 213; J. Plebański, Acta Phys. Polon. XIV (1955) 275; J. Plebański, Phys. Rev. 101 (1956) 1825.
[23] H.P. Yuen, Phys. Rev. A 13 (1976) 2226.
[24] R. Loudon, P. Knight, J. Mod. Opt. 34 (1987) 709.

[25] C. Aragone, E. Chalbaud, S. Salam, J. Math. Phys. 17 (1976) 1963.

[26] J.A. Vaccaro, D.T. Pegg, J. Mod. Opt. 37 (1990) 17.

[27] M. Przanowski, F.J. Turrubiates, J. Phys. A: Math. Gen. 35 (2002) 10643.

[28] D.T. Smithey, M. Beck, J. Cooper, M.G. Raymer, Phys. Rev A 48 (1993) 3159.

[29] C. Brif, Y. Ben-Aryeh, Phys. Rev. A 50 (1994) 3505.

[30] D.T. Pegg, S.N. Barnett, R. Zambrini, S. Franke-Arnold, M. Padgett, New J. Phys. 7 (2005) 62.

[31] T. Curtright, C. Zachos, Mod. Phys. Lett. A 16 (2001) 2381.

[32] P. Carruthers, M.M. Nieto, Phys. Rev. Lett. 14 (1965) 387.

[33] P. Carruthers, M.M. Nieto, Rev. Mod. Phys. 40 (1968) 411.

[34] R. Jackiw, J. Math. Phys. 9 (1968) 339.

[35] J.C. Garrison, J. Wong, J. Math. Phys. 11 (1970) 2242.

[36] J.M. Levy-Leblond, Ann. Phys. (N.Y.) 101 (1976) 319.

[37] Y. Yamamoto, S. Machida, N. Imoto, M. Kitagawa, G. Biork, J. Opt. Soc. Am. B 4 (1987) 1645.

[38] A. Lukš, V. Peřinová, J. Krépelka, Czech. J. Phys. 42 (1992) 59.

[39] A. Lukš, V. Peřinová, Phys. Rev. A 45 (1992) 6710.

[40] C. Brif, Class. Quantum Grav. 12 (1995) 803.

[41] I. Mendaš, D.B. Popović, Phys. Rev. A 52 (1995) 4356.

[42] S. Luo, Phys. Lett. A 275 (2000) 165.

[43] B.-S. K. Skagerstam, B. Bergsjoerd, Phys. Scripta 70 (2004) 26.

[44] H.S. Sharatchandra, arXiv:quant-ph/9710020v1 (1997).

[45] J.H. Shapiro, S.R. Shepard, Phys. Rev. A 43 (1991) 3795.

[46] T. Pereira, D.H.U. Marchetti, Prog. Theor. Phys. 122 (2009) 1137.

[47] L. Susskind, J. Glogower, Physics 1 (1964) 49.

[48] D.T. Pegg, S.M. Barnett, Europhys. Lett. 6 (1988) 483.

[49] D.T. Pegg, S.M. Barnett, Phys. Rev. A 39 (1989) 1665.

[50] S.M. Barnett, D.T. Pegg, J. Modern Opt. 36 (1989) 7.

[51] P. Busch, M. Grabowski, P. Lahti, Ann. Phys. 237 (1995) 1.

[52] M. Przanowski, P. Brzykcy, Ann. Phys. 337 (2013) 34.
[53] M. Przanowski, P. Brzykcy, J. Tosiek, Ann. Phys. 351 (2014) 919.

[54] V. Peřínová, A. Lukš, J. Peřina, *Phase in Optics*, (World Scientific, Singapore, 1998).

[55] A. Miranowicz, *Lectures on Quantum Optics* (in polish), http://zon8.physd.amu.edu.pl/~miran, 2008.

[56] O. Castaños, R. López – Peña, M. A. Man’ko and V. Man’ko, *Squeezing Operator and Squeeze Tomography*, in *Topics in Mathematical Physics, General Relativity and Cosmology in Honor of Jerzy Plebański*, H. García – Compeán, B. Mielnik, M. Montesinos, M. Przanowski (Eds.), (World Scientific, London, 2006) 109.

[57] F. London, Z. Phys. 37 (1926) 915; Z. Phys. 40 (1927) 193.

[58] A. Luis, L.L. Sánchez – Soto, Phys. Rev. A 48 (1993) 752.

[59] Z. Białynicka – Birula, I. Białynicki – Birula, J. Appl. Phys. B 60 (1995) 275.

[60] H. Gerhardt, H. Welling, D. Frölich, Appl. Phys. 2 (1973) 91.

[61] M.M. Nieto, Phys. Lett. A 60 (1977) 401.

[62] R. Lynch, Phys. Rev. A 41 (1990) 2841.

[63] C.C. Gerry, K.E. Urbański, Phys. Rev. A 42 (1990) 662.

[64] T.S. Gantsog, A. Miranowicz, R. Tanaś, Phys. Rev. A 46 (1992) 2870.

[65] D. Judge, Il Nuovo Cim. XXXI (1964) 332.

[66] A. Lukš, V. Peřínová, Czech. J. Phys. 41 (1991) 1205.

[67] A. Lukš, V. Peřínová, Phys. Scripta T 48 (1993) 94.

[68] A. Abramowitz, I.A. Stegun, (Eds.) *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, (Dover, New York, 1972).

[69] D. Zwillinger, *Handbook of Differential Equations*, (Academic Press, Boston, 1997).

[70] J.A. González, M.A. del Olmo, J. Phys. A: Math. Gen. 31 (1998) 8841.

[71] K. Kowalski, J. Rembieliński, J. Phys. A: Math. Gen. 35 (2002) 1405.