Strongly Consistent of Kullback-Leibler Divergence Estimator and Tests for Model Selection Based on a Bias Reduced Kernel Density Estimator

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Abstract

In this paper, we study the strong consistency of a bias reduced kernel density estimator and derive a strongly consistent Kullback-Leibler divergence (KLD) estimator. As application, we formulate a goodness-of-fit test and an asymptotically standard normal test for model selection. The Monte Carlo simulation show the effectiveness of the proposed estimation methods and statistical tests.

Keywords: Bias Reduced Kernel Density Estimator; Kullback-Leibler Divergence; Strong Consistency of Estimators; Hypothesis Testing; Model Selection.

Mathematical Subject Classification: 62F10, 62F12, 62G07, 62G10, 62G20.

1. Introduction

Let $X_1, \ldots, X_n$ be iid random variables and assume that the common distribution function of these variables has an unknown density $f$. One can estimate $f$ using the parametric approach assuming that the data are drawn from a known parametric family of distributions. The density $f$ can then be estimated by finding estimates of the parameters from the data and substituting these estimates into the formula of the density. One can also use non-parametric approach for the density estimation. A well known non-parametric estimator of the pdf (probability density function) is the histogram [1]. It has the advantage of simplicity but it also has some disadvantages, such as: lack of continuity and the choice of the location of intervals and the bandwidth have an effect on the histogram result. To circumvent such difficulties, Rosenblatt and Parzen [1,2] proposed a more general non-parametric estimator which is the widely used kernel density estimator. The asymptotic properties of this estimator has been intensively investigated and many kernel-type estimators have been proposed. Dony and Einmahl [3] showed the uniform consistency of kernel density estimator with general bandwidth sequences. Salim and Issam [4] established the uniform in bandwidth consistency of kernel-type estimators of Shannon Entropy. Einmah and Mason [5] proved the uniform in bandwidth consistency of kernel-type function estimators. Ngom et al. [6] proposed a strong uniformly consistent kernel-type estimator of divergence measures. Xie and Wu [7] focused on improving convergence rate of kernel density estimator by introducing a bias reduced kernel density estimator. The first main purpose of this paper is to prove the strong consistency of this bias reduced kernel density estimator. To choose a practical optimal bandwidth of classical kernel density estimator, one way of determining a simple and attractive smoothing parameter is the cross-validation method introduced by Rudemo and Bowman [8]. Accordingly we shall propose a cross-validation bandwidth selection for the bias reduced kernel density estimator. Next, we adress the model selection problem. Considering a candidate model for some given data generated by an unknown probability distribution, the dissimilarity between those two probability distributions can be measured by the Kullback-Leibler divergence (KLD) introduced by Kullback and Leibler [9]. Since the true density is unknown, various criteria and hypothesis testing were used for model selection purpose (10, 11, 12, 13, 14, 15, 16, 17, 18). In this paper, we shall derive a strongly consistent estimator of KLD between

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two distributions based on the bias reduced kernel density estimator. The proposed KLD estimator is then used to
correct statistics for hypothesis testing in model selection.

The rest of the paper is organized as follows. We give a brief review of the bias reduced kernel density estimator in
Section 2. Cross-validation bandwidth selection for the bias reduced kernel density estimator is obtained in Section 3.
In Section 4, the strong consistency of the bias reduced kernel density estimator is proved and we establish a strongly
consistent Kullback-Leibler divergence estimator in Section 5. Applications for hypothesis testing in models selection
are proposed in Section 6. The simulation study is presented in Section 7 and finally the conclusion appears in Section
8.

2. A review of the bias reduced kernel density estimator

Let \( X_1, \ldots, X_n \) be iid random variables and assume that the unknown distribution function of these variables has a
Lebesgue density, which we shall denote by \( f \). Consider a probability density function \( K \) defined on \( \mathbb{R} \) (the kernel)
and a positive parameter \( h \), the bandwidth. Assuming that the random variable of density \( K \) is centered with finite variance \( \mu_2 \), the kernel density estimator (Rosenblatt [1] and Parzen [2]) of \( f \) is given by

\[
\hat{f}_{n,h}(x) = \frac{1}{nh} \sum_{i=1}^{n} K \left( \frac{x - X_i}{h} \right).
\]

Devroye [19] showed that the optimal bandwidth is \( h \sim O \left( n^{-\frac{1}{5}} \right) \) and then the optimal MSE is of the order \( n^{-\frac{2}{5}} \), under
the conditions \( h \to 0 \) and \( nh \to \infty \) as \( n \to \infty \).

The optimal performance of kernel density estimator has been widely investigated. Farrell [20] obtained the best
asymptotic convergence rate of MSE for orthogonal kernel estimators. Abramson [21] successfully employed larger
smoothing parameters in low density regions to reduce the bias. Samiuddin [22] proposed a very intuitive and feasible kernel density estimator which reduces the bias and variance at the same time which in turn reduces the MSE in using the skewing method. Xie and
Wu [7] proposed a very intuitive and feasible kernel density estimator which reduces the bias and MSE significantly
compared to the ordinary kernel density estimator. It is defined by

\[
\hat{f}^b_{n,h}(x) = \hat{f}_{n,h}(x) - \frac{h^2}{2} \mu_2 \hat{f}''_{n,h}(x).
\]

Assuming that \( f \) is differentiable of order four in a neighbourhood of \( x \), Xie and Wu [7] came up with a convergence rate of order \( n^{-\frac{2}{5}} \) for this estimator. We prove this results under the appropriate regularity conditions on the kernel \( K \).

**Proposition 1.** Suppose that \( f \) is differentiable of order four in a neighbourhood of \( x \). Let \( K \) be the density of a
centered random variable with finite second and third order moment denoted by \( \mu_2 \) and \( \mu_3 \) respectively, satisfying the
following assumptions:

- \( A_1 \) : \( K(x) = K_1(x)1_A(x), \ A \subseteq \mathbb{R} \) such that \( \lim_{x \to \inf A} K^{(0)}(x) = \lim_{x \to \sup A} K^{(1)}(x) = 0, \forall i = 0, 1; \)

- \( A_2 \) : \( \int K^2(u)du < \infty, \int (K''(u))^2 du < \infty, \int u(K''(u))^2 du = 0. \)

Then, we have

\[
Bias(\hat{f}^b_{n,h}(x)) = -\frac{h^3}{6} \mu_3 f''(x) + O(h^4)
\]
and

\[ \text{var}(\hat{f}_{n,h}(x)) \leq \frac{1}{2nh^2} \mu_3^2 f(x) \int (K''(u))^2 du + O(n^{-1}). \]  

(4)

Consequently the optimal MSE (Mean Squared Error) is of the order \( n^{-4} \).

If in addition, \( K \) is a symmetric kernel, \( \mu_3 = 0 \); hence the optimal MSE for the bias reduced estimator is of order \( n^{-8/9} \).

**Proof.** \( \text{Bias}(\hat{f}_{n,h}(x)) = E\hat{f}_{n,h}(x) - f(x), \forall x \in \mathbb{R} \) where \( \hat{f}_{n,h}(x) = \hat{f}_{n,h}(x) - \frac{h^2}{2} \mu_2 \hat{f}_{n,h}(x) \) and \( \hat{f}_{n,h} \) is a kernel density estimator, then

\[
\text{Bias}(\hat{f}_{n,h}(x)) = E\hat{f}_{n,h}(x) - \frac{h^2}{2} \mu_2 E\hat{f}_{n,h}(x) - f(x).
\]

(5)

We have

\[
E\hat{f}_{n,h}(x) = E \left[ \frac{1}{nh} \sum_{i=1}^{n} K \left( \frac{x - X_i}{h} \right) \right]
\]

\[
= \frac{1}{h} \int K(\frac{x - y}{h}) f(y) dy
\]

\[
= \int K(u) f(x - uh) du.
\]

A Taylor expansion of \( f(x - uh) \) yields

\[
f(x - uh) = f(x) - uhf'(x) + \frac{1}{2} (uh)^2 f''(x) - \frac{(uh)^3}{6} f'''(x) + o(h^3).
\]

Thus

\[
E\hat{f}_{n,h}(x) = f(x) \int K(u) du - uhf'(x) \int uK(u) du + \frac{h^2}{2} \mu_2 f''(x) - \frac{h^3}{6} \mu_3 f'''(x) + o(h^3)
\]

\[
= f(x) + \frac{h^2}{2} \mu_2 f''(x) - \frac{h^3}{6} \mu_3 f'''(x) + o(h^3).
\]

On the other hand,

\[
E\hat{f}_{n,h}''(x) = E \left[ \frac{1}{nh^3} \sum_{i=1}^{n} K'' \left( \frac{x - X_i}{h} \right) \right]
\]

\[
= \frac{1}{h^3} \int K''(\frac{x - y}{h}) f(y) dy
\]

\[
= \frac{1}{h^2} \int K''(u) f(x - uh) du.
\]

By integrating by part twice, and using assumption \( A_1 \), we get

\[
E\hat{f}_{n,h}''(x) = \int K(u) f''(x - uh) du.
\]

Using a 2nd order Taylor expansion of \( f''(x - uh) \) about \( x \) we have

\[
E\hat{f}_{n,h}''(x) = \int K(u) \left[ f''(x) - uhf''(x) + \frac{(uh)^2}{2} f'''(x) + o(h^2) \right] du
\]

\[
= f''(x) \int K(u) du - hf''(x) \int uK(u) du + \frac{h^2}{2} \mu_2 f'''(x) + o(h^2)
\]

\[
= f''(x) + \frac{h^2}{2} \mu_2 f'''(x) + o(h^2).
\]
Thus

\[
\text{Bias}(\hat{f}_{n,h}^b(x)) = E\hat{f}_{n,h}^b(x) - f(x) = E(\hat{f}_{n,h}(x)) - \frac{h^2}{2} \mu_2 E\left(\hat{f}_n''(x)\right) - f(x) = \frac{h^2}{2} \mu_2 f''(x) - \frac{h^3}{6} \mu_3 f'''(x) + o(h^3) - \frac{h^2}{2} \mu_2 \left[ f''(x) + \frac{h^2}{2} \mu_2 f'''(x) + o(h^3) \right] = -\frac{h^3}{6} \mu_3 f'''(x) - \frac{h^4}{4} \mu_2^2 f''''(x) + o(h^3). \]

Hence

\[
\text{Bias}(\hat{f}_{n,h}^b(x)) = -\frac{h^3}{6} \mu_3 f'''(x) + O(h^4).
\]

Consider now

\[
\text{var}(\hat{f}_{n,h}^b(x)) = \text{var}\left(\hat{f}_{n,h}(x) - \frac{h^2}{2} \mu_2 \hat{f}_n'(x)\right).
\]

We have

\[
\text{var}(\hat{f}_{n,h}^b(x)) \leq 2\text{var}(\hat{f}_{n,h}(x)) + 2\text{var}\left(\frac{h^2}{2} \mu_2 \hat{f}_n'(x)\right) \leq 2\text{var}(\hat{f}_{n,h}(x)) + \frac{h^4}{2} \mu_2 \text{var}(\hat{f}_n'(x)).
\]

Notice that the variance of \(\hat{f}_{n,h}(x)\) is given by

\[
\text{var}(\hat{f}_{n,h}(x)) = \frac{1}{nh^6} f(x) \int K(u)^2 du + o(\frac{(nh)^{-1}}{}).
\]

Set \(I = \text{var}(\hat{f}_{n,h}(x))\), we have

\[
I = \text{var}\left(\frac{1}{nh^3} \sum_{i=1}^{n} K'(\frac{x-X_i}{h})\right) = \frac{1}{nh^6} \left[ \text{var}\left(\left(K'\left(\frac{x-X_1}{h}\right)\right)^2\right) - \left(\frac{1}{n} \int \left(K'\left(\frac{x-X_1}{h}\right)\right)^2 f(x) dy \right)^2 \right] = \frac{1}{nh^6} \int (K'(\frac{x-y}{h}))^2 f(y) dy - \frac{1}{n} \left[ \frac{1}{h^4} \int K'\left(\frac{x-y}{h}\right) f(y) dy \right]^2 = \frac{1}{nh^3} \int (K''(u))^2 f(x-uh) du - \frac{1}{n} \left[ f'(x) + O(h^2) \right]^2 = \frac{1}{nh^3} \int (K''(u))^2 \left(f(x) - uh f'(x) + O(h^3)\right) du - \frac{1}{n} \left( f'(x) \right)^2 + O\left(\frac{h^2}{n}\right) = \frac{1}{nh^3} f(x) \int (K''(u))^2 du + O\left(\frac{(nh)^{-1}}{}\right).
\]

Therefore

\[
I = \frac{1}{nh^3} f'(x) \int (K''(u))^2 du + O\left(\frac{(nh)^{-1}}{}\right).
\]

From (8), (9) and (7), we get (4).
Now we consider

\[ \text{MISE}(\hat{f}^b_{n,h}(x)) = \text{Bias}^2(\hat{f}^b_{n,h}(x)) + \text{var}(\hat{f}^b_{n,h}(x)). \]

In our case, we have

\[ \text{MISE}(\hat{f}^b_{n,h}(x)) \leq \frac{h^6}{36} \mu_3^2(f)(x) + \frac{1}{2nh^2} \mu_3^2(f(x)) \int (K''(u))^2 du. \quad (10) \]

Minimizing the term on the right-hand side of this inequality yields \( h_{opt} = O(n^{-1}) \).

In general, there are many methods for selecting the practical bandwidth for the ordinary kernel density estimator:

i. One can experiment by using different bandwidths and simply select one that "looks right" for the type of data under investigation (subjective selection) \([27]\).

ii. One can refer to some given distribution, i.e. one selects the bandwidth that would be optimal for a particular pdf.

iii. One can use the cross-validation method introduced by Rudemo and Bowman \([8]\) which provides an optimal bandwidth defined by

\[ h_{CV} = \arg \min_{h>0} CV(h) \quad (11) \]

where \( CV(h) \) is cross-validation given by

\[ CV(h) = \int \hat{f}^2 dx - \frac{1}{2} \sum_{i=1}^n \hat{f}_{-i} (X_i) \quad \text{and} \quad \hat{f}_{-i} (x) = \frac{1}{(n-1)h} \sum_{j \neq i}^n K \left( \frac{x-X_j}{h} \right). \]


Following this idea, we propose a cross-validation bandwidth selection for the bias reduced kernel density estimator

### 3. Cross-validation bandwidth selection for the bias reduced kernel density estimator

The expression of Mean Integrated Squared Error (MISE) is defined by

\[ \text{MISE}(\hat{f}^b_{n,h}(x)) := \int \left( \text{Bias}^2(\hat{f}^b_{n,h}(x)) + \text{Var}(\hat{f}^b_{n,h}(x)) \right) dx. \]

Write \( \text{MISE}^b = \text{MISE}^b(h) \) to indicate that the mean integrated squared error is a function of bandwidth. First, note that

\[
\text{MISE}^b(h) = \mathbb{E}_f \left[ (\hat{f}^b_{n,h}(x) - f(x))^2 dx \right] \\
= \mathbb{E}_f \left[ \int (\hat{f}^b_{n,h}(x))^2 dx - \int \hat{f}^b_{n,h}(x) f(x) dx \right] + \int f^2(x) dx.
\]

Since the integral \( \int f^2(x) dx \) does not depend on \( h \), the minimizer of \( \text{MISE}^b(h) \) also minimizes the function

\[ J(h) = \text{MISE}^b(h) - \int f^2(x) dx = \mathbb{E}_f \left[ \int (\hat{f}^b_{n,h}(x))^2 dx - 2 \int \hat{f}^b_{n,h} f(x) dx \right]. \]

Since \( J(h) \) depends on the unknown density \( f \), we rather use a Modified Cross-Validation estimator (MCV) \( MCV(h) \) of \( J(h) \). For this purpose, it is sufficient to consider the following estimators of each of the quantities \( \mathbb{E}_f \left[ \int (\hat{f}^b_{n,h}(x))^2 dx \right] \) and \( \mathbb{E}_f \left[ \int \hat{f}^b_{n,h} f(x) dx \right] :\]

- An unbiased estimator of \( \mathbb{E}_f \left[ \int (\hat{f}^b_{n,h}(x))^2 dx \right] \) is given by \( \int (\hat{f}^b_{n,h}(x))^2 dx \).
- An unbiased estimator of \( \mathbb{E}_f \left[ \int \hat{f}^b_{n,h} f(x) dx \right] \) is given by \( \frac{1}{n} \sum_{i=1}^n \int \hat{f}^b_{n,h} f(x) dx \).

\[ 5 \]
where
\[ f_{n,h}^{b}(x) = \frac{1}{(n-1)h} \sum_{j=1}^{n} K \left( \frac{x-X_j}{h} \right) - \frac{1}{2(n-1)h^2} \sum_{j<i}^{n} K'' \left( \frac{x-X_j}{h} \right). \]

Consequently the estimator \( MCV(h) \) is given by
\[ MCV(h) = \int \left( \hat{f}_{n,h}^{b}(x) \right)^2 dx - \frac{2}{n} \sum_{i=1}^{n} \hat{f}_{n,h}^{b}(X_i). \] (12)
We deduce from (12) the expression of optimal bandwidth \( h_{MCV}^{b} \) as follows
\[ h_{MCV}^{b} = \arg \min_{h>0} MCV(h), \] (13)
and the corresponding bias reduced kernel density estimator \( \hat{f}_{n,h}^{b}(x) \) of \( f \) is written as:
\[ \hat{f}_{n,h}^{b}(x) = \frac{1}{n h_{MCV}^{b}} \sum_{i=1}^{n} K \left( \frac{x-X_i}{h_{MCV}^{b}} \right) - \frac{1}{2n h_{MCV}^{b}} \sum_{i=1}^{n} K'' \left( \frac{x-X_i}{h_{MCV}^{b}} \right). \]

In this following section, we establish the strong consistency of the bias reduced kernel density estimator.

4. Strong consistency of a bias reduced kernel density estimator

Let \( X_1, X_2, \ldots, X_n \) be iid random variables of unknown density \( f \). Under some conditions on \( f \) and \( K \), one obtains a strongly consistent estimator \( \hat{f}_{n,h}^{b} \) of \( f \). For proving such consistency results, we shall consider the following regularity conditions.

(H.1) \( K \) is a density of a centered random variable with finite variance \( \mu_2 \).
(H.2) Set \( \varphi = K - \frac{\mu_2}{2} K'' \); \( \| \varphi \|_\infty = \sup_{x \in \mathbb{R}} |\varphi(x)| < +\infty \) and \( \| \varphi \|_2 := (\int \varphi^2(u) du)^{1/2} < +\infty \).

Consider the class of functions \( \Phi = \{ f \mapsto \varphi ((x-t)/h) : h > 0, \ x \in \mathbb{R} \} \). For \( \varepsilon > 0 \), let \( N(\varepsilon, \Phi) = sup_Q N(\varepsilon, \Phi, d_{2Q}) \)
where the supremum is taken over all probability measures \( Q \) on \((\mathbb{R}, \mathcal{B}), d_{2Q} \) is the \( L_2(Q)\)-metric and \( N(\varepsilon, \Phi, d_{2Q}) \) is the minimal number of balls of radius \( \varepsilon \) needed to cover \( \Phi \).
(H.3) For some \( C > 0 \) and \( \varepsilon > 0 \), \( N(\varepsilon, \Phi) \leq Ce^{-\gamma \varepsilon}, 0 < \varepsilon < 1 \).
This condition discussed in [28, 29] holds whenever \( \varphi : \mathbb{R} \rightarrow \mathbb{R} \) is a function of bounded variation.
(H.4) \( \Phi \) is a pointwise measurable class, that is, there exists a countable subclass \( \Phi_0 \) of \( \Phi \) such that we can find for any function \( \phi \in \Phi \) a sequence of functions \( \phi_m \in \Phi_0 \) for which \( \phi_m(y) \rightarrow \phi(y), \ y \in \mathbb{R} \).
This condition is satisfied whenever \( \varphi \) is right continuous.
(H.5) \( f \) is four differentiable in neighbourhood of \( x \).

**Theorem 1.** Assuming (H.1-H.5) are satisfied. For each pair of sequences \((a_n)_{n \geq 1}\) and \((b_n)_{n \geq 1}\) such that \( \forall n \geq 1, 0 < a_n < b_n \leq 1 \) and \( a_n \leq h \leq b_n \) we have with probability 1,
\[ \limsup_{n \to \infty} \sup_{a_n \leq h \leq b_n} \frac{\sqrt{n} h \| \hat{f}_{n,h}^{b} - E \hat{f}_{n,h}^{b} \|_\infty}{\sqrt{\log (1/h) \log \log n}} =: \omega < \infty. \] (14)

The proof of this theorem follows along the lines of the proof of theorem 1 [3] and requires the following two lemmas that provide some results on pointwise measurable class of bounded functions.

Let \( X_i, 1 \leq i \leq n \) be iid random variables defined from a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) to a measurable space \((S, \mathcal{S})\). Let \( \mathcal{G} \) be a pointwise measurable class of bounded functions and \( G \) be a finite-valued measurable function satisfying
for all $x \in S$, $G(x) \geq \sup_{g \in \mathcal{G}} |g(x)|$. Define $\alpha_n$ to be the empirical process based on the sample $X_1, ..., X_n$, that is, if $g : S \rightarrow \mathbb{R}$, we have

$$\alpha_n(g) = \sum_{i=1}^{n} \left( g(X_i) - \mathbb{E}g(X) \right) / \sqrt{n},$$

and set for any class $\mathcal{G}$ of such functions

$$\| \sqrt{n} \alpha_n \|_{\mathcal{G}} = \sup_{g \in \mathcal{G}} \| \sqrt{n} \alpha_n(g) \|.$$

**Lemma 1.** (Corollary 4 [5]) Let $\mathcal{G}$ be a pointwise measurable class of bounded functions. If $C, \nu \geq 1$ and $0 < \sigma \leq \beta$,

the following conditions hold:

1. $\mathbb{E} \left[ G(X)^2 \right] \leq \beta^2$
2. $N(\epsilon; \mathcal{G}) \leq C \epsilon^{-\nu}, \ 0 < \epsilon < 1$
3. $\sigma^2_0 := \sup_{g \in \mathcal{G}} \mathbb{E} \left[ g(X)^2 \right] \leq \sigma^2$
4. $\sup_{g \in \mathcal{G}} \|g\|_{\infty} \leq U$, where $\sigma_0 \leq U \leq C_2 \sqrt{n} \beta$, and $C_2 = \frac{1}{4} \sqrt{\nu \log C_1}$, $C_1 = C^{1/\nu}$.

Then we have for some absolute constant $A$,

$$\mathbb{E} \left\| \sum_{i=1}^{n} e_i g(X_i) \right\|_{\mathcal{G}} \leq A \left\{ \frac{\sqrt{n} \sigma_0^2 \log(C_1 \beta/\sigma_0) + 2 A n U \log \left( C_3 n (\beta/U)^2 \right)}{\sigma^2} \right\} + 2 \exp \left( \frac{-A t^2}{n \sigma^2} \right) + \exp \left( \frac{-A t}{M} \right),$$

where $e_i, \ 1 \leq i \leq n$, is a sequence of independent Rademacher random variables $X_1, ..., X_n$.

**Lemma 2.** (Inequality of Talagrand [30]) Let $\mathcal{G}$ be a pointwise measurable class of functions satisfying for some $0 < M < \infty$,

$$\|g\|_{\infty} \leq M, \ g \in \mathcal{G}.$$

Then we have for all $t > 0$,

$$\mathbb{P} \left\{ \max_{1 \leq m \leq n} \sum_{i=1}^{m} e_i g(X_i) \geq A_1 \left( \mathbb{E} \left\| \sum_{i=1}^{m} e_i g(X_i) \right\|_{\mathcal{G}} + t \right) \right\} \leq 2 \left\{ \exp \left( \frac{-A_2 t^2}{n \sigma^2} \right) + \exp \left( \frac{-A_2 t}{M} \right) \right\},$$

where $\sigma^2_{\mathcal{G}} = \sup_{g \in \mathcal{G}} \text{Var}(g(X))$ and $A_1, A_2$ are universal constants.

**Proof of the theorem**

We first write that

$$\mathbb{E} \left[ \varphi^2 \left( \frac{x-X}{h} \right) \right] = h \int \frac{1}{h} \varphi^2 \left( \frac{x-y}{h} \right) f(y) dy \leq h \|f\|_{\infty} \int \varphi^2(u) du \leq h \|f\|_{\infty} \|\varphi\|_2^2.$$
For \(j, k \geq 0\) and \(c > 0\), set \(n_k = 2^k\), \(h_{jk} = \left(\frac{2/c \log n_k}{n_k}\right)\) and
\[
\Phi_{jk} = \left\{ \varphi \left( \frac{x - \cdot}{h} \right) \right\}, \quad h_{jk} \leq h \leq h_{j+1,k}, \quad x \in \mathbb{R}\).
\]

Therefore for \(h_{jk} \leq h \leq h_{j+1,k}\), one has
\[
\mathbb{E} \left[ \varphi^2 \left( \frac{x - X}{h} \right) \right] \leq 2h_{jk} \|f\|_\infty \|\varphi\|_2^2.
\] (15)

On the other hand, using (H.2),
\[
\mathbb{E} \left[ \varphi^2 \left( \frac{x - X}{h} \right) \right] \leq \gamma^2.
\] (16)

Combining (15) and (16), we have
\[
\mathbb{E} \left[ \varphi^2 \left( \frac{x - X}{h} \right) \right] \leq \gamma^2 \wedge 2h_{jk} \|f\|_\infty \|\varphi\|_2^2 \wedge \mathbb{E} \left[ \varphi^2 \left( \frac{x - X}{h} \right) \right] \leq \gamma^2 \wedge B_0h_{jk} := \sigma^2_{jk}\text{ where } B_0 = 2 \|f\|_\infty \|\varphi\|_2^2 \text{ and } a \wedge b := \min(a, b).
\]

We now use the lemma to bound
\[
\mathbb{E} \left[ \left\| \sum_{i=1}^{n_k} e_i g(X_i) \right\|_{\Phi_{jk}} \right].
\]

We first note that each \(\Phi_{jk}\) satisfies the condition 1 with \(G = \beta = \gamma\). Further, since \(\Phi_{jk} \subseteq \Phi\), we see by (H.3) that each \(\Phi_{jk}\) also fulfills the condition 2. Without loss of generality we assume that \(v, C \geq 1\) in (H.3). Noting that
\[
C_1 \sqrt{\beta} \|\sigma_0\| \leq \left(\beta^2 / \|\sigma_0^2 \wedge C_1^2\right) \text{ with } a \vee b := \max(a, b).
\]

Applying lemma with \(U = \beta = \gamma\) and using the bound \(\sigma_0 \leq \sigma_{jk} \leq \sqrt{B_0h_{jk}}\), we have for \(j \geq 0\),
\[
\mathbb{E} \left[ \left\| \sum_{i=1}^{n_k} e_i g(X_i) \right\|_{\Phi_{jk}} \right] \leq A \sqrt{\gamma n_k B_0 h_{jk} \log \left( \frac{\beta^2 / \|\sigma_0^2 \wedge C_1^2\} \right)} + 2Avy \log (C_3 n_k)
\]

which is written for \(B_1 = A \sqrt{\gamma B_0}\) and \(B_2 = B_0/\beta^2\) as
\[
\mathbb{E} \left[ \left\| \sum_{i=1}^{n_k} e_i g(X_i) \right\|_{\Phi_{jk}} \right] \leq B_1 \sqrt{n_k h_{jk} \log \left( \frac{n_k}{B_2 h_{jk} \vee C_1} \right)} + 2Avy \log (C_3 n_k).
\]

Using the fact that \(h_{jk} \geq (c \log n_k) / n_k\), for large \(k\),
\[
B_1 \sqrt{c \log n_k} \sqrt{\log \left( \frac{n_k}{cB_2 \log n_k} \right)} \leq B_1 \sqrt{n_k h_{jk} \log \left( \frac{1}{B_2 h_{jk} \vee C_1} \right)}
\]

And for large \(k\),
\[
\frac{2Avy \log (C_3 n_k)}{B_1 \sqrt{c \log n_k} \sqrt{\log \left( \frac{n_k}{cB_2 \log n_k} \right)}} \leq B'
\]

where \(B' = \frac{2Avy}{n \sqrt{c}}\). This implies that
\[
2Avy \log (C_3 n_k) \leq B' B_1 \sqrt{c \log n_k} \sqrt{\log \left( \frac{n_k}{cB_2 \log n_k} \right)} \leq B' B_1 \sqrt{n_k h_{jk} \log \left( \frac{1}{B_2 h_{jk} \vee C_1} \right)}.
\]
Therefore
\[
\mathbb{E} \left\| \sum_{i=1}^{n_k} \varepsilon_i g(X_i) \right\|_{\Phi_{j,k}} \leq B_1 \sqrt{n_k h_{j,k} \log \left( \frac{1}{B_2 h_{j,k}} \lor C_1^2 \right)} + B B_1 \sqrt{n_k h_{j,k} \log \left( \frac{1}{B_2 h_{j,k}} \lor \log \log n_k \right)} \\
\leq B_1 \sqrt{n_k h_{j,k} \log \left( \frac{1}{B_2 h_{j,k}} \lor \log \log n_k \right)} \\
:= B_1 a_{j,k}
\]
where \(B_1 = (B_1 + B B_1)\).

Using the lemma with \(M = \gamma\) and \(\sigma_{h,j,k}^2 = \sigma_{h,j,k}^2 \leq B_0 h_{j,k}\), we get for any \(t > 0\),
\[
P \left\{ \max_{n_k \leq n \leq 2n_k} \| \sqrt{n} a_n \|_{\Phi_{j,k}} \geq A_1 (B_3 \sigma_{h,j,k} + t) \right\} \leq 2 \left\{ \exp \left( -\frac{A_2 t^2}{n_k B_0 h_{j,k}} \right) + \exp \left( -\frac{A_2 t}{K} \right) \right\}.
\]
Setting for any \(\delta > 1\), with \(t = \delta a_{j,k}\), \(j \geq 0\) and \(k \geq 1\),
\[
p_{j,k}(\delta) = P \left\{ \max_{n_k \leq n \leq 2n_k} \| \sqrt{n} a_n \|_{\Phi_{j,k}} \geq A_1 (B_3 \delta a_{j,k}) \right\}
\]
and using the fact that \(\sigma_{h,j,k}^2 \geq \log \log n_k\), we can infer that for large \(k\),
\[
p_{j,k}(\delta) \leq 2 \left\{ \exp \left( -\frac{A_2 \delta^2 a_{j,k}^2}{n_k B_0 h_{j,k}} \right) + \exp \left( -\frac{A_2 \delta a_{j,k}}{K} \right) \right\} \\
\leq 2 \left\{ \exp \left( -\frac{A_2 \delta^2}{B_0} \log \log n_k \right) + \exp \left( -\frac{A_2 \delta}{K} \sqrt{n_k h_{j,k} \log \log n_k} \right) \right\} \\
\leq 2 (\log n_k)^{-\delta} \text{ where } \delta = A_2^- \delta^2.
\]
Set \(l_k = \max \{ j : h_{j,k} \leq 2 \}\). For large \(k\)
\[
l_k \leq 2 \log n_k.
\]
Hence (17) and (18) give for large \(k\) and for \(\delta \geq 1\)
\[
P_k(\delta) := \sum_{j=0}^{l_k-1} p_{j,k}(\delta) \leq 4 (\log n_k)^{1-\delta}
\]
which implies that if we choose \(\delta \geq 2(B_0/A_2)^{1/2}\), we have
\[
P_k(\delta) \leq \frac{4}{k^{(\log 2)^2}} \text{ and } \sum_{k=1}^{\infty} P_k(\delta) < \infty.
\]
Notice from (5) that by definition of \(l_k\) for large \(k\), \(h_{j,k} \leq 2 \Rightarrow 2 h_{j,k} \geq 2 \text{ and } h_{j,k} \geq 1\). Consequently, we then have for \(n_{k-1} \leq n \leq n_k\),
\[
\left[ \frac{c \log n}{n}, 1 \right] \subset \left[ \frac{c \log n_k}{n_k}, h_{j,k} \right].
\]
Thus for all large enough \(k\) and \(n_{k-1} \leq n \leq n_k\),
\[
A_k(\delta) := \max \sup_{n_k \leq n \leq 2n_k} \frac{\sqrt{n} a_n}{\| f_n/h \|_{H_n} \sqrt{\log (1/h) \lor \log \log n}} > 2 A_1 (B_3 + \delta) \}
\]
\[
\subset \bigcup_{j=0}^{l_k-1} \left\{ \max_{n_k \leq n \leq 2n_k} \| \sqrt{n} a_n \|_{\Phi_{j,k}} \geq A_1 (B_3 + \delta) a_{j,k} \right\}.
\]
Therefore

$$P(A_\delta) := P\left\{ \max_{h \leq k \leq 1} \sup_{n \to \infty} \frac{\sqrt{nh} \|f_{n,h}^b - E\hat{f}_{n,h}^b\|_\infty}{\sqrt{\log (1/h) \vee \log \log n}} > 2A_1(B_3 + \delta) \right\} \leq \sum_{j=0}^{\infty} P\left\{ \max_{n \to \infty} \|na_{n,j}\|_{\Phi^\infty} \geq A_1(B_3 + \delta) a_{n,j} \right\} = P_\delta(\delta).$$

By (19), \( \sum_{k=1}^{\infty} P(A_k(\delta)) < \infty \). Via the Borel-Cantelli lemma

$$P\left\{ \lim \sup_{n \to \infty} \max_{n \to \infty} \sqrt{nh} \|f_{n,h}^b - E\hat{f}_{n,h}^b\|_\infty > 2A_1(B_3 + \delta) \right\} = 0.$$ 

This implies that

$$P\left\{ \lim \sup_{n \to \infty} \max_{n \to \infty} \frac{\sqrt{nh} \|f_{n,h}^b - E\hat{f}_{n,h}^b\|_\infty}{\sqrt{\log (1/h) \vee \log \log n}} \leq 2A_1(B_3 + \delta) \right\} = 1$$

and

$$\lim \sup_{n \to \infty} \frac{\sqrt{nh} \|f_{n,h}^b - E\hat{f}_{n,h}^b\|_\infty}{\sqrt{\log (1/h) \vee \log \log n}} \leq 2A_1(B_3 + \delta). \quad (20)$$

As \( n \to \infty \), \( (c \log n)/n \to 0 \). From (20), we can write for \( 0 < h < 1 \) such that \( a_n \leq h \leq b_n \)

$$\lim \sup_{n \to \infty} \sup_{a_n \leq h \leq b_n} \frac{\sqrt{nh} \|f_{n,h}^b - E\hat{f}_{n,h}^b\|_\infty}{\sqrt{\log (1/h) \vee \log \log n}} \leq 2A_1(B_3 + \delta).$$

**Remark 1.** We further note that Theorem [1] implies for any sequences \( 0 < a_n < b_n \leq 1 \), satisfying \( b_n \to 0 \) and \( na_n/\log(n) \to \infty \), with probability 1,

$$\sup_{a_n \leq h \leq b_n} \|f_{n,h}^b - E\hat{f}_{n,h}^b\|_\infty = 0 \left( \frac{\log(1/a_n) \vee \log \log n}{na_n} \right); \quad (21)$$

which in turn implies

$$\lim_{n \to \infty} \sup_{a_n \leq h \leq b_n} \|f_{n,h}^b - E\hat{f}_{n,h}^b\|_\infty = 0 \ a.s.$$ 

**Theorem 2.** Let \( f \) be Lipschitz function on \( \mathbb{R} \). Assume that (H.1) and (H.5) are satisfied and the derivative of order \( j \) of \( f \) are bounded, \( \forall j = 2,3,4 \). For any sequences \( 0 < a_n < b_n \leq 1 \) satisfying \( a_n \leq h \leq b_n \) together with \( b_n \to 0 \), we have

$$\sup_{a_n \leq h \leq b_n} \|f_{n,h}^b - E\hat{f}_{n,h}^b\|_\infty = 0 (b_n).$$

**Proof.** Set \( \Psi_n(x) = E\hat{f}_{n,h}^b(x) - f(x), \forall x \in \mathbb{R} \). Using the formula (1) and (2), we have

$$\Psi_n(x) = E\hat{f}_{n,h}^b(x) - \frac{h^2}{2} \mu_2 E\hat{f}_{n,h}^b(x) - f(x).$$
\[
E \left[ \frac{1}{nh} \sum_{i=1}^{n} K \left( \frac{x - X_i}{h} \right) \right] - \frac{h^2}{2 \mu_2} E \left[ \frac{1}{nh^3} \sum_{i=1}^{n} K' \left( \frac{x - X_i}{h} \right) \right] = f(x)
\]
\[
= \frac{1}{h} \int K \left( \frac{x - y}{h} \right) f(y) dy - \frac{h^2}{2 \mu_2} \frac{1}{h^3} \int K' \left( \frac{x - y}{h} \right) f(y) dy - f(x)
\]
\[
= \frac{1}{h} \int K \left( \frac{x - y}{h} \right) f(y) dy - \frac{h^2}{2 \mu_2} \frac{1}{h^3} \int K \left( \frac{x - y}{h} \right) f''(y) dy - f(x)
\]
\[
= \int K(u) f(x - uh) du - \frac{h^2}{2 \mu_2} \int K(u) f''(x - uh) du - f(x).
\]

Applying a Taylor approximation on the second term, we have
\[
\Psi_{n,h}(x) = \int K(u) \left[ f(x - uh) - f(x) + f(x) \right] du + \frac{h^2}{2 \mu_2} \int K(u) \left( f'(x) - uh f''(x) + \frac{1}{2}(uh)^2 f'''(x) + o(h^3) \right) du - f(x)
\]
\[
= \int K(u) \left[ f(x - uh) - f(x) \right] du - \frac{h^2}{2 \mu_2} f''(x) + \frac{h^3}{4 \mu_2} f'''(x) + o(h^4).
\]

Since \( f'' \) and \( f''' \) are bounded on \( R \), i.e. \( \forall x \in R \), there exist two constants \( M \) and \( N \) such that \( |f''(x)| \leq M \) and \( |f'''(x)| \leq N \). We then have
\[
|\Psi_{n,h}(x)| \leq \int K(u) |f(x - uh) - f(x)| du + \frac{h^2}{2 \mu_2} |f''(x)| + \frac{h^3}{4 \mu_2} |f'''(x)| + |o(h^4)|. \quad (22)
\]

For small enough \( h \), (22) gives
\[
|\Psi_{n,h}(x)| \leq \int K(u) |f(x - uh) - f(x)| du.
\]

Note that \( f \) is Lipschitz function on \( R \), i.e. for \( \alpha > 0 \) and for \( x, y \in R \), \( |f(x) - f(y)| \leq \alpha |x - y| \). Consequently
\[
|\Psi_{n,h}(x)| \leq \int K(u) \alpha |uh| du
\]
\[
\leq \alpha |h| \int |u| K(u) du = \alpha c |h| \text{ where } c = \int |u| K(u) du < \infty.
\]

For any sequences \( 0 < a_n < b_n \leq 1 \) satisfying \( a_n \leq h \leq b_n \) together with \( b_n \to 0 \), we have
\[
|\Psi_{n,h}(x)| \leq A |b_n| \text{ with } A = \alpha c.
\]

Which means that
\[
\Psi_{n,h}(x) = O(b_n).
\]

This finally implies that
\[
\sup_{a_n \leq h \leq b_n} \left\| \Psi_{n,h} \right\|_\infty = O(b_n).
\]

It concludes the proof of the theorem.
5. **Strongly Consistent Kullback-Leibler divergence Estimator**

Let $X_1, \ldots, X_n$ be a random sample of unknown density function $f$ defined on $\mathbb{R}$ and let $f_\theta$ be a parametric candidate model. Denote by $D_{KL}(f, f_\theta)$, the Kullback-Leibler divergence between $f$ and $f_\theta$ defined by

$$
D_{KL}(f, f_\theta) = \int f(x) \log \left( \frac{f(x)}{f_\theta(x)} \right) dx.
$$

(23)

Notice that the Kullback-Leibler divergence does not obey the triangle inequality and in general $D_{KL}(f, f_\theta)$ does not equal to $D_{KL}(f_\theta, f)$. The unknown density function $f$ can be estimated by the bias reduced kernel density estimator. Using this estimator, we then define the Kullback-Leibler divergence estimator of $D_{KL}(f, f_\theta)$ as follows

$$
\hat{D}_{KL}(f_n^h, f_\theta) := \int f_n^h(x) \ln \left( \frac{f_n^h(x)}{f_\theta(x)} \right) dx,
$$

where $A_n = \{x \in \mathbb{R}; f_n^h(x) \geq \varepsilon_n\}$ with $(\varepsilon_n)$ a sequence of positive constants such that $\varepsilon_n \to 0$ as $n \to \infty$. Since $f_n^h$ is strongly consistent as shown in preceding section, we shall prove the strong consistency of Kullback-Leibler divergence estimator defined by (24). Throughout the remainder of this paper $\hat{\hat{D}}_{KL}(f_n^h, f_\theta)$ is given by

$$
\hat{\hat{D}}_{KL}(f_n^h, f_\theta) := \int \hat{f}_n^h(x) \ln \left( \frac{\hat{f}_n^h(x)}{f_\theta(x)} \right) dx,
$$

where $A_n$ is defined in (24). Hence the following theorem.

**Theorem 3.** Let the conditions (H.1-H.5) be satisfied and let $f$ be bounded and Lipschitz density function on $\mathbb{R}$. For each pair of sequence $(a_n)_{n \geq 1}$ and $(b_n)_{n \geq 1}$ and for $0 < h < 1$ such that $0 < a_n < h \leq b_n \leq 1$ together with $b_n \to 0$ and $n a_n / \log(n) \to \infty$ as $n \to \infty$, we have with probability 1;

$$
\sup_{a_n \leq h \leq b_n} \left| \hat{\hat{D}}_{KL}(f_n^h, f_\theta) - D_{KL}(f, f_\theta) \right| = O \left( \sqrt{\frac{\log(1/a_n) \vee \log \log n}{n a_n}} \vee b_n \right).
$$

The proof of this theorem is based on two following lemmas and the methods developed in [4] will be helpful.

**Lemma 3.** Suppose that the conditions (H.1-H.5) hold and let $f$ be continuous and bounded density on $\mathbb{R}$. We have with probability 1, for each pair of sequence $0 < a_n < b_n \leq 1$ and for $0 < h < 1$ such that $a_n \leq h \leq b_n$ together with $n a_n / \log(n) \to \infty$ as $n \to \infty$

$$
\sup_{a_n \leq h \leq b_n} \left| \hat{\hat{D}}_{KL}(f_n^h, f_\theta) - \hat{D}_{KL}(f_n^h, f_\theta) \right| = O \left( \sqrt{\frac{\log(1/a_n) \vee \log \log n}{n a_n}} \right).
$$

We need the following proposition in order to prove this lemma.

**Proposition 2. (Theorem 9.1, [37].)** Let $K$ be an arbitrary integrable function on $\mathbb{R}^d$ (i.e., $\int |K| < \infty$), and let $f$ be a density on $\mathbb{R}^d$. Denoting $K_h(x) = (1/|h|^d) K(x/h)$, $x \in \mathbb{R}^d$, $h > 0$, we have

$$
\lim_{h \to 0} \int f * K_h - f \int K = 0.
$$

Proof of the lemma [3]

Define

$$
\Gamma_{n 1} := \hat{\hat{D}}_{KL}(f_n^h, f_\theta) - \hat{\hat{D}}_{KL}(f_n^h, f_\theta).
$$
One has
\[
\begin{align*}
\Gamma_{n1} &= \int_{A_n} \left[ \frac{\hat{p}_n^b(x)}{\hat{f}_{n,h}^b(x)} \ln \left( \frac{\hat{f}_{n,h}^b(x)}{\hat{f}_{n,h}^b(x)} \right) - \frac{\hat{p}_n^b(x)}{\hat{f}_{n,h}^b(x)} \ln \left( f_0(x) \right) \right] \, dx - \int_{A_n} \left[ \frac{\hat{f}_{n,h}^b(x)}{\hat{f}_{n,h}^b(x)} \ln \left( \frac{\hat{f}_{n,h}^b(x)}{\hat{f}_{n,h}^b(x)} \right) - \frac{\hat{f}_{n,h}^b(x)}{\hat{f}_{n,h}^b(x)} \ln \left( f_0(x) \right) \right] \, dx \\
&= \int_{A_n} \left[ \ln \left( \frac{\hat{f}_{n,h}^b(x)}{\hat{f}_{n,h}^b(x)} \right) - \ln \left( \frac{\hat{f}_{n,h}^b(x)}{\hat{f}_{n,h}^b(x)} \right) \right] \, dx + \int_{A_n} \left[ \frac{\hat{p}_n^b(x)}{\hat{f}_{n,h}^b(x)} - \frac{\hat{f}_{n,h}^b(x)}{\hat{f}_{n,h}^b(x)} \right] \ln \left( f_0(x) \right) \, dx + \\
&\quad - \int_{A_n} \left[ \frac{\hat{p}_n^b(x)}{\hat{f}_{n,h}^b(x)} - \frac{\hat{f}_{n,h}^b(x)}{\hat{f}_{n,h}^b(x)} \right] \ln \left( f_0(x) \right) \, dx \\
&:= \Gamma_{n11} + \Gamma_{n12} - \Gamma_{n13}.
\end{align*}
\]

We first prove that \( \sup_{\epsilon_n, \epsilon_n \geq \epsilon_n} |\Gamma_{n1}| = O \left( \sqrt{\frac{\log \log n}{n \log n}} \right) \). Observing that for all \( y > 0, |\ln(y)| \leq \left| \frac{1}{y} - 1 \right| + |y - 1| \), we have
\[
\left| \ln \left( \frac{\hat{f}_{n,h}^b(x)}{\hat{f}_{n,h}^b(x)} \right) - \ln \left( \frac{\hat{f}_{n,h}^b(x)}{\hat{f}_{n,h}^b(x)} \right) \right| \leq \frac{2}{\epsilon_n} \left( \frac{\hat{f}_{n,h}^b(x)}{\hat{f}_{n,h}^b(x)} - \frac{\hat{f}_{n,h}^b(x)}{\hat{f}_{n,h}^b(x)} \right).
\]

Recalling that \( A_n = \{ x \in \mathbb{R}, f_{n,h}^b(x) \geq \epsilon_n \} \), we readily obtain from these relations that, for any \( x \in A_n \),
\[
\left| \ln \left( \frac{\hat{f}_{n,h}^b(x)}{\hat{f}_{n,h}^b(x)} \right) - \ln \left( \frac{\hat{f}_{n,h}^b(x)}{\hat{f}_{n,h}^b(x)} \right) \right| \leq \frac{2}{\epsilon_n} \left( \frac{\hat{f}_{n,h}^b(x)}{\hat{f}_{n,h}^b(x)} - \frac{\hat{f}_{n,h}^b(x)}{\hat{f}_{n,h}^b(x)} \right).
\]

For any \( n \geq 1 \), we can therefore write the inequalities
\[
\begin{align*}
|\Gamma_{n11}| &= \left| \int_{A_n} \left[ \ln \left( \frac{\hat{f}_{n,h}^b(x)}{\hat{f}_{n,h}^b(x)} \right) - \ln \left( \frac{\hat{f}_{n,h}^b(x)}{\hat{f}_{n,h}^b(x)} \right) \right] \, dx \right| \\
&\leq \int_{A_n} \frac{2}{\epsilon_n} \left( \frac{\hat{f}_{n,h}^b(x)}{\hat{f}_{n,h}^b(x)} - \frac{\hat{f}_{n,h}^b(x)}{\hat{f}_{n,h}^b(x)} \right) \, dx \\
&\leq \frac{2}{\epsilon_n} \sup_{x \in A_n} \left( \frac{\hat{f}_{n,h}^b(x)}{\hat{f}_{n,h}^b(x)} - \frac{\hat{f}_{n,h}^b(x)}{\hat{f}_{n,h}^b(x)} \right) \int_{A_n} \frac{\hat{f}_{n,h}^b(x)}{\hat{f}_{n,h}^b(x)} \, dx \\
&\leq \frac{2}{\epsilon_n} \sup_{x \in \mathbb{R}} \left( \frac{\hat{f}_{n,h}^b(x)}{\hat{f}_{n,h}^b(x)} - \frac{\hat{f}_{n,h}^b(x)}{\hat{f}_{n,h}^b(x)} \right) \int_{\mathbb{R}} \frac{\hat{f}_{n,h}^b(x)}{\hat{f}_{n,h}^b(x)} \, dx.
\end{align*}
\]

An application of proposition \( 2 \) gives
\[
\lim_{n \to 0} \int_{\mathbb{R}} \left| \frac{\hat{f}_{n,h}^b(x)}{\hat{f}_{n,h}^b(x)} - f_0(x) \right| \int \left( K - \frac{\mu^2}{2} K' \right)(x) \, dx = 0.
\]

This implies that
\[
\lim_{n \to 0} \int_{\mathbb{R}} \frac{\hat{f}_{n,h}^b(x)}{\hat{f}_{n,h}^b(x)} \, dx = \int f_0(x) \int \left( K - \frac{\mu^2}{2} K' \right)(x) \, dx.
\]

Assuming that \( \xi := \int \left( K - \frac{\mu^2}{2} K' \right)(x) \, dx < \infty \), one has
\[
\lim_{n \to 0} \int_{\mathbb{R}} \frac{\hat{f}_{n,h}^b(x)}{\hat{f}_{n,h}^b(x)} \, dx = \xi \int f_0(x) \, dx.
\]

Thus
\[
\begin{align*}
|\Gamma_{n11}| &\leq \frac{2\xi}{\epsilon_n} \sup_{x \in \mathbb{R}} \left( \frac{\hat{f}_{n,h}^b(x)}{\hat{f}_{n,h}^b(x)} - \frac{\hat{f}_{n,h}^b(x)}{\hat{f}_{n,h}^b(x)} \right) \\
&\leq \frac{2\xi}{\epsilon_n} \left\| \frac{\hat{f}_{n,h}^b(x)}{\hat{f}_{n,h}^b(x)} - \frac{\hat{f}_{n,h}^b(x)}{\hat{f}_{n,h}^b(x)} \right\|_{\infty}.
\end{align*}
\]
Therefore
\[
\sup_{x_n \leq h \leq b_n} |\Gamma_{n1}| = \frac{2\zeta}{n} \sup_{x_n \leq h \leq b_n} \left\| \tilde{p}_{n,h}(x) - E\tilde{p}_{n,h}(x) \right\|_{\infty}.
\] (25)

Substituting (21) in (25) the result follows.

We next prove that \( \sup_{a_n \leq h \leq b_n} |\Gamma_{n2}| = O\left(\sqrt{\frac{\log(1/a_n) \vee \log \log n}{n\alpha_n}}\right) \). Since \( |h(y)| \leq \frac{1}{y} \), for all \( y > 0 \), one has
\[
|\Gamma_{n2}| = \left| \int_{A_n} \left[ \tilde{p}_{n,h}(x) - E\tilde{p}_{n,h}(x) \right] \ln \left( \frac{\tilde{p}_{n,h}(x)}{f_{n,h}(x)} \right) dx \right|
\leq \int_{A_n} \left[ \tilde{p}_{n,h}(x) - E\tilde{p}_{n,h}(x) \right] \left[ \frac{1}{f_{n,h}(x)} + \tilde{p}_{n,h}(x) \right] dx.
\]

Similarly as above, we get for any \( x \in A_n \),
\[
\frac{1}{f_{n,h}(x)} + \tilde{p}_{n,h}(x) = \left( \frac{1}{f_{n,h}(x)} \tilde{p}_{n,h}(x) + 1 \right) \tilde{p}_{n,h}(x)
\leq \left( \frac{1}{n^2} + 1 \right) \tilde{p}_{n,h}(x).
\]

Therefore, we have
\[
|\Gamma_{n2}| \leq \left( \frac{1}{n^2} + 1 \right) \int_{A_n} |\tilde{p}_{n,h}(x) - E\tilde{p}_{n,h}(x)| \frac{1}{f_{n,h}(x)} dx
\leq \left( \frac{1}{n^2} + 1 \right) \sup_{x \in A_n} |\tilde{p}_{n,h}(x) - E\tilde{p}_{n,h}(x)| \int_{A_n} f_{n,h}(x) dx
\leq \left( \frac{1}{n^2} + 1 \right) \zeta \left\| \tilde{p}_{n,h}(x) - E\tilde{p}_{n,h}(x) \right\|_{\infty}.
\]

Since \( a_n \leq h \leq b_n \) and \( b_n \to 0 \), as \( n \to \infty \) we have
\[
\sup_{a_n \leq h \leq b_n} |\Gamma_{n2}| \leq \left( \frac{1}{n^2} + 1 \right) \zeta \left| \log(1/a_n) \vee \log \log n \right| \sup_{x \leq A_n} |\tilde{p}_{n,h}(x) - E\tilde{p}_{n,h}(x)|.
\] (26)

Substituting (21) in (26) we have
\[
\sup_{a_n \leq h \leq b_n} |\Gamma_{n2}| = \left( \frac{1}{n^2} + 1 \right) \zeta \left| \log(1/a_n) \vee \log \log n \right| \sup_{x \leq A_n} |\tilde{p}_{n,h}(x) - E\tilde{p}_{n,h}(x)|.
\]

Hence
\[
\sup_{a_n \leq h \leq b_n} |\Gamma_{n2}| = 0 \left( \sqrt{\frac{\log(1/a_n) \vee \log \log n}{n\alpha_n}} \right).
\] (27)

We evaluate now the last term \( \sup_{a_n \leq h \leq b_n} |\Gamma_{n1}| = 0 \left( \sqrt{\frac{\log(1/a_n) \vee \log \log n}{n\alpha_n}} \right) \).

Consider
\[
|\Gamma_{n1}| = \left| \int_{A_n} \left[ \tilde{p}_{n,h}(x) - E\tilde{p}_{n,h}(x) \right] \ln \left( f_{n,h}(x) \right) \left( f_{n,h}(x) \right) dx \right|
\leq \sup_{x \in A_n} |\tilde{p}_{n,h}(x) - E\tilde{p}_{n,h}(x)| \int_{A_n} \left( \frac{1}{f_{n,h}(x)} + f_{n,h}(x) \right) dx.
\]
Therefore

$$|\Gamma_{n1}| \leq \sup_{x \in \mathbb{R}} \left| \int_{\mathbb{R}} \left( \frac{1}{f_{\theta}(x)} + f_{\theta}(x) \right) dx \right|$$

For $a_n \leq h \leq b_n$ and $b_n \to 0$ as $n \to \infty$ there exists a constant $C_1 = \int_{\mathbb{R}} \left( \frac{1}{f_{\theta}(x)} + f_{\theta}(x) \right) dx < \infty$ such that

$$\sup_{a_n \leq h \leq b_n} |\Gamma_{n1}| \leq C_1 \sup_{a_n \leq h \leq b_n} \| f_{\theta}(x) - \hat{f}_{\theta}(x) \|_{\infty}.$$ 

Thus in view of (21), we get

$$\sup_{a_n \leq h \leq b_n} |\Gamma_{n1}| = 0 \left( \sqrt{\frac{\log(1/a_n) \lor \log \log n}{n a_n}} \right). \quad (28)$$

Finally, the combination of (25), (27) and (28) gives

$$\sup_{a_n \leq h \leq b_n} |\Gamma_{n1}| = 0 \left( \sqrt{\frac{\log(1/a_n) \lor \log \log n}{n a_n}} \right).$$

It concludes the proof of the lemma.

**Lemma 4.** Assuming (A.1), (H.1) and (H.5) and let $f$ be Lipschitz density function on $\mathbb{R}$. For each pair of sequence $0 < a_n < b_n \leq 1$ and for $0 < h < 1$ such that $a_n \leq h \leq b_n$, together with $b_n \to 0$ as $n \to \infty$, we have with probability 1:

$$\sup_{a_n \leq h \leq b_n} \left| \hat{\mathbb{D}}_{KL}(f_{\theta,h}^{b}, f_{\theta}) - \mathbb{D}(f, f_{\theta}) \right| = O(b_n).$$

**Proof.** We set $\Gamma_{n2} = \hat{\mathbb{D}}_{KL}(f_{\theta,h}^{b}, f_{\theta}) - \mathbb{D}(f, f_{\theta})$, therefore

$$\Gamma_{n2} = \int_{A_n} \left[ \mathbb{E} f_{\theta,h}^{b}(x) \ln \left( \frac{f_{\theta,h}^{b}(x)}{f_{\theta,h}^{b}(x)} \right) - \mathbb{E} f_{\theta,h}^{b}(x) \ln(f_{\theta}(x)) - f_{\theta}(x) \ln(f_{\theta}(x)) + f_{\theta}(x) \ln(f_{\theta}(x)) \right] dx \right.$$ 

$$= \int_{A_n} \mathbb{E} f_{\theta,h}^{b}(x) \ln \left( \frac{\mathbb{E} f_{\theta,h}^{b}(x)}{f_{\theta,h}^{b}(x)} \right) - \ln(f_{\theta}(x)) \right] dx + \int_{A_n} \ln(f_{\theta}(x)) \left[ \mathbb{E} f_{\theta,h}^{b}(x) - f_{\theta}(x) \right] dx +$$

$$- \int_{A_n} \left[ \mathbb{E} f_{\theta,h}^{b}(x) - f_{\theta}(x) \right] \ln(f_{\theta}(x)) dx$$

$$:= \Gamma_{n21} + \Gamma_{n22} - \Gamma_{n23}.$$ 

Our purpose is to show that $\sup_{a_n \leq h \leq b_n} |\Gamma_{n21}| = O(b_n)$, $\sup_{a_n \leq h \leq b_n} |\Gamma_{n22}| = O(b_n)$ and $\sup_{a_n \leq h \leq b_n} |\Gamma_{n23}| = O(b_n)$. Begin by the first term. We can write

$$|\Gamma_{n21}| = \left| \int_{A_n} \left[ \ln \left( \frac{\mathbb{E} f_{\theta,h}^{b}(x)}{f_{\theta,h}^{b}(x)} \right) - \ln(f_{\theta}(x)) \right] \mathbb{E} f_{\theta,h}^{b}(x) dx \right|$$

$$\leq \int_{A_n} \left| \ln \left( \frac{\mathbb{E} f_{\theta,h}^{b}(x)}{f_{\theta,h}^{b}(x)} \right) - \ln(f_{\theta}(x)) \right| dx \mathbb{E} f_{\theta,h}^{b}(x)$$

Repeating the arguments above in the terms $|\Gamma_{n11}|$ with the formal change of $f_{\theta,h}^{b}$ by $f$, one has

$$|\Gamma_{n11}| \leq \sup_{x \in A_n} |\mathbb{E} f_{\theta,h}^{b}(x) - f(x)| \int_{A_n} \left( \frac{1}{x_n} + \frac{1}{f(x)} \right) dx.$$ 

There exists a constant $C_2 = \int_{A_n} \left( \frac{1}{x_n} + \frac{1}{f(x)} \right) dx < \infty$ such that

$$|\Gamma_{n11}| \leq C_2 \sup_{x \in A_n} |\mathbb{E} f_{\theta,h}^{b}(x) - f(x)|$$

$$\leq C_2 \| \mathbb{E} f_{\theta,h}^{b}(x) - f(x) \|_{\infty}. \quad (29)$$
Applying the theorem \(2\) we have for each \(a_n \leq h \leq b_n\), as \(n \to \infty\),
\[
\sup_{a_n \leq h \leq b_n} \left\| \mathbb{E} f_{n,h}^{b_n} - f \right\|_{\infty} = 0(b_n).
\] (30)

(29) combined with (30) gives
\[
\sup_{a_n \leq h \leq b_n} |\Gamma_{n11}| \leq C_2 O(b_n).
\]

Finally
\[
\sup_{a_n \leq h \leq b_n} |\Gamma_{n11}| = O(b_n).
\] (31)

Now we can prove the second term \(\sup_{a_n \leq h \leq b_n} |\Gamma_{n22}| = O(b_n)\).

Since \(|ln(y)| \leq \frac{1}{2} + y\), for all \(y > 0\)
\[
|\Gamma_{n22}| = \left| \int_{\mathbb{A}_n} \mathbb{E} \left[ f_{n,h}^{b_n}(x) - f(x) \right] ln(f(x)) dx \right|
\leq \int_{\mathbb{A}_n} \left\| \mathbb{E} f_{n,h}^{b_n}(x) - f(x) \right\| \left( \frac{1}{f(x)} + f(x) \right) dx
\leq \sup_{x \in \mathbb{A}_n} \left\| \mathbb{E} f_{n,h}^{b_n}(x) - f(x) \right\| \int_{\mathbb{A}_n} \left( \frac{1}{f(x)} + f(x) \right) dx
\leq \sup_{x \in \mathbb{R}} \left\| \mathbb{E} f_{n,h}^{b_n}(x) - f(x) \right\| \int_{\mathbb{R}} \left( \frac{1}{f(x)} + f(x) \right) dx.
\]

As assumed above, \(\int_{\mathbb{R}} \left( \frac{1}{f(x)} + f(x) \right) dx < \infty\), this in turn, implies that
\[
\sup_{a_n \leq h \leq b_n} |\Gamma_{n22}| \leq C_3 \sup_{a_n \leq h \leq b_n} \left\| \mathbb{E} f_{n,h}^{b_n}(x) - f(x) \right\|_{\infty}.
\]

where \(C_3 = \int_{\mathbb{R}} \left( \frac{1}{f(x)} + f(x) \right) dx < \infty\). In view of theorem \(2\) one has
\[
\sup_{a_n \leq h \leq b_n} |\Gamma_{n22}| = O(b_n).
\] (32)

The third term is given by
\[
\Gamma_{n23} := \int_{\mathbb{A}_n} \left[ \mathbb{E} f_{n,h}^{b_n}(x) - f(x) \right] ln(f_0(x)) dx.
\]

Repeat the argument in terms of \(\Gamma_{n22}\) with the formal change of \(f\) by \(f_0\) and considering the constant \(C_4 = \int_{\mathbb{R}} \left( \frac{1}{f_0(x)} + f_0(x) \right) dx < \infty\), one has
\[
|\Gamma_{n23}| \leq C_4 \left\| \mathbb{E} f_{n,h}^{b_n}(x) - f(x) \right\|_{\infty}.
\]

Thus
\[
\sup_{a_n \leq h \leq b_n} |\Gamma_{n23}| \leq C_4 \sup_{a_n \leq h \leq b_n} \left\| \mathbb{E} f_{n,h}^{b_n}(x) - f(x) \right\|_{\infty} = 0(b_n).
\] (33)

Finally combining (31), (32), and (33), the proof of lemma (4) is deduced.

Proof of theorem (3)
The combination of the lemma (3) and lemma (4) concludes the proof of the Theorem (3).
6. Applications for Hypothesis Testing in Models Selection

Let \((X, \beta_X, F)\) be the statistical space, \(X = \{x_1, x_2, \ldots, x_M\}, \) \(\forall M_0 \geq 1; \) \(\beta_X\) is the \(\sigma\)-algebra of all the sub-sets of \(X\) and \((X, \beta_X)\), the measurable space.

Let

\[
\Lambda_{M_0} = \left\{ F = (F_1, \ldots, F_{M_0})^T : \forall x \in \mathbb{R} \text{ and } i = 1, \ldots, M_0, F_i(x) > 0 \text{ and } \sum_{i=1}^{M_0} F_i(x) = 1 \right\}
\]

(34)

be the simplex of distributions \(M_0\)-vectors. It is the set of discrete distributions. One can define the parametric family of models as follows

\[
\mathcal{F} = \left\{ F_\theta = (F_1(., \theta), \ldots, F_{M_0}(., \theta))^T : \theta \in \Theta \right\}
\]

(35)

where \(\Theta \subset \mathbb{R}^{M_0}\). To be more explicit, suppose that we are sampling from a distribution \(F_X(x)\). Divide the range of the distribution into \(M_0\) mutually exclusive and exhaustive classes, say \(I_1, \ldots, I_{M_0}\). Each class has a probability of containing the random variable \(X\), \(P(X \in I_i) := F_{i\theta}, i = 1, \ldots, M_0\) and each sample value \(x\) falls into exactly one of the intervals.

6.1. Goodness-of-fit test

The parametric family of models defined by (35) may or may not contain the true model. If \(\mathcal{F}\) contains the true model, then there exists a \(\theta_0 \in \Theta\) such that \(F = F_{\theta_0}\) and the model is said to be correctly specified. We consider now the case when the model is not specified i.e. \(H_1 : F \neq F_{\theta_0}\). Based on Kullback-Leibler divergence, this alternative hypothesis is written as \(H_1 : D_{KL}(F, F_{\theta_0}) \neq 0\) where

\[
D_{KL}(F, F_{\theta_0}) = \sum_{i=1}^{M_0} F_i \log \left( \frac{F_i}{F_{\theta_0,i}} \right) \text{ with } F_{\theta_0,i} = F_i(., \theta_0), i = 1, \ldots, M_0
\]

(36)

We must reject the null hypothesis iff \(D_{KL}(F, F_{\theta_0}) > c\) where \(c\) must be chosen for getting a level \(\alpha\) test. In general it is not possible to get the exact distribution of the statistic \(D_{KL}(F, F_{\theta_0})\) and we must use its asymptotic distribution. Notice that the estimator of \(D_{KL}(F, F_{\theta_0})\) given by (36) is defined as follows

\[
\tilde{D}_{KL}(\hat{F}_n^{\theta}, F_{\theta_0}) = \sum_{i=1}^{M_0} \hat{F}_n^{\theta,i} \log \left( \frac{\hat{F}_n^{\theta,i}}{F_{\theta_0,i}} \right)
\]

where \(\hat{F}_n^{\theta}\) is a bias reduced kernel density estimator of \(F\). In the following theorem we present the asymptotic distribution of \(D_{KL}(F, F_{\theta_0})\).

Let us introduce the two important regularity assumptions:

-(J_1) Under the regularity conditions on the dominated model \(F_{\theta_0}\), the MLE is unique and asymptotically normal under \(F_{\theta_0}, \forall \theta_0\)

1) \(F_{\theta_0} \overset{d}{\rightarrow} F_{\theta_0} \) when \(n \rightarrow \infty\)

2) \(\sqrt{n}(\hat{\theta} - \theta_0) \overset{L}{\rightarrow} N(0, I(\theta_0)^{-1})\)

where \(I(\theta_0)\) is Fisher information and \(n \rightarrow \infty\).

-(J_2) There exists \(\theta_0 \in \Theta\) and \(\Lambda^* = \begin{pmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{pmatrix}\), with \(\Lambda_{12} = \Lambda_{21}\) such that

\[
\sqrt{n} \left( \frac{\hat{F}_n^{\theta} - F}{F_{\theta_0} - F_{\theta_0}} \right) \overset{L}{\rightarrow} N(0, \Lambda^*).
\]

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Theorem 4. Let $\mathcal{D}_{KL}(F,F_0)$ be the Kullback-Leibler divergence between $F$ and $F_0$, and let $\hat{\mathcal{D}}_{KL}(\hat{F}_n,F_0)$ its estimator. Under $H_1: F_0 \neq F$ (we have omitted 0 on $\theta$) and assuming that the conditions $(J_1)$ and $(J_2)$ hold, one has:

$$\sqrt{n} \left[ \mathcal{D}_{KL}(\hat{F}_n,F_0) - \mathcal{D}_{KL}(F,F_0) \right] \xrightarrow{p} N(0, \lambda_2^2)$$

where

$$\lambda_2^2 = A^T \lambda_{11} A + A^T \lambda_{12} W + W^T \lambda_{12} W + W^T \lambda_{22} W,$$

(37)

$A^T = (a_1, \ldots, a_{M_0})$ is the vector of partial derivatives with respect to the components of the first variable with

$$a_i = \left( \frac{\partial}{\partial F_{i0}} \mathcal{D}_{KL}(F,F_0) \right), \quad i = 1, \ldots, M_0$$

and $W^T = (w_1, \ldots, w_{M_0})$ is the vector of partial derivatives with respect to the components of the second variable with

$$w_i = \left( \frac{\partial}{\partial F_{0i}} \mathcal{D}_{KL}(F,F_0) \right), \quad i = 1, \ldots, M_0; \quad F_{0i} = F_{i\theta}. $$

**Proof.** A first order Taylor expansion gives

$$\sqrt{n} \left[ \mathcal{D}_{KL}(\hat{F}_n,F_0) - \mathcal{D}_{KL}(F,F_0) \right] = \sqrt{n} \left[ A^T (\hat{F}_n - F) + W^T (\hat{F}_0 - F_0) \right] + \sqrt{n} \xi_n \left( \| \hat{F}_n - F \| + \| \hat{F}_0 - F_0 \| \right).$$

We observe that

$$\sqrt{n} \left[ \mathcal{D}_{KL}(\hat{F}_n,F_0) - \mathcal{D}_{KL}(F,F_0) \right] = \sqrt{n} \left[ A^T (\hat{F}_n - F) + W^T (\hat{F}_0 - F_0) \right] + \sqrt{n} \xi_n \left( \| \hat{F}_n - F \| + \| \hat{F}_0 - F_0 \| \right).$$

Since $\sqrt{n}(F_0 - F_0) \xrightarrow{p} N(0, \Sigma_{F_0})$, when $n \to \infty$, with $\Sigma_{F_0} = \text{diag}(F_0) - F_0 F_0'$; then $|F_0 - F_0| = O_p(n^{-1/2})$ and $\sqrt{n} \xi_n = o_p(1)$. Therefore $\sqrt{n} \xi_n \left( \| \hat{F}_n - F \| + \| \hat{F}_0 - F_0 \| \right) = o_p(1)$. Hence

$$\sqrt{n} \left[ \mathcal{D}_{KL}(\hat{F}_n,F_0) - \mathcal{D}_{KL}(F,F_0) \right] = \sqrt{n} \left[ A^T (\hat{F}_n - F) + W^T (\hat{F}_0 - F_0) \right] + o_p(1).$$

The random variables $\sqrt{n} \left[ \mathcal{D}_{KL}(\hat{F}_n,F_0) - \mathcal{D}_{KL}(F,F_0) \right]$ and $\sqrt{n} \left[ A^T (\hat{F}_n - F) + W^T (\hat{F}_0 - F_0) \right]$ have the same asymptotic distribution. In view of $J_1$ and $J_2$ we have

$$\sqrt{n} \left[ A^T (\hat{F}_n - F) + W^T (\hat{F}_0 - F_0) \right] \xrightarrow{p} N(0, \lambda_2^2)$$

where $\lambda_2^2$ is given by (37). This completes the proof.

It is possible to choose the model among a collection of candidate models which is close to the true distribution according to the Kullback-Leibler divergence thanks to the goodness-of-fit test.

6.2. Test for model selection based on Kullback-Leibler divergence

We propose now to take two candidate parametric models $F_0^1$ and $F_0^2$, i.e. $F_0^1$ and $F_0^2 \in \mathcal{F}$. For simplicity in the rest of the paper, we will note $\theta$ in place of $\theta_1$ and $\gamma$ in place of $\theta_2$. Based on Kullback-Leibler divergence; we would like to choose the candidate model which is close to the true probability distribution $F$; i.e. the minimized KLD. Our major work is to propose some tests for model selection as follows

$H_0: \mathcal{D}_{KL}(F,F_0^1) = \mathcal{D}_{KL}(F,F_0^2)$ means that the two models are equivalent,

$H_1: \mathcal{D}_{KL}(F,F_0^1) \neq \mathcal{D}_{KL}(F,F_0^2)$ means that $F_0^1$ is not equivalent to $F_0^2$. 

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To define the model selection statistic, we consider
\[ \xi^2 = (U_\theta - U_\gamma; S_\theta - S_\gamma) \overset{\ast}{\sim} (U_\theta - U_\gamma; S_\theta - S_\gamma) \] (38)
the variance of
\[ \sqrt{n}(U_\theta - U_\gamma; S_\theta - S_\gamma)^T \left( F_{n,b} - F_{\theta} \right) \]
where \( U_\theta = (u_1, \ldots, u_{M_0}) \); with
\[ u_i = \left( \frac{\partial}{\partial F_i} D_{KL}(F, F_\theta) \right), \quad i = 1, \ldots, M_0, \]
and \( S_\theta^T = (s_1, \ldots, s_{M_0}) \); with
\[ s_i = \left( \frac{\partial}{\partial F_{\theta,i}} D_{KL}(F, F_\theta) \right), \quad i = 1, \ldots, M_0. \]

Since \( U_\theta, U_\gamma, S_\theta, S_\gamma \) and \( \overset{\ast}{\sim} \) are consistently estimated by their sample analogues \( U_\theta, U_\gamma, S_\theta, S_\gamma \) and \( \overset{\ast}{\sim} \). Hence \( \xi^2 \)
is consistently estimated by
\[ \hat{\xi}^2 = (U_\theta - U_\gamma; S_\theta - S_\gamma)^T \overset{\ast}{\sim} (U_\theta - U_\gamma; S_\theta - S_\gamma). \]

Therefore we propose the model selection statistic \( KL_m \) as follows
\[ KL_m = \frac{\sqrt{n}}{\hat{\xi}} \left[ \overline{D}_{KL}(\hat{F}_{n,h}, F_\theta) - \overline{D}_{KL}(\hat{F}_{n,h}, F_\gamma) \right]. \] (39)
It is possible to get the asymptotic distribution of \( KL_m \). Hence the following theorem.

**Theorem 5.** (Asymptotic distribution of the \( KL_m \)-statistic).
Under the regularity assumptions \((J_1)\) and \((J_2)\), suppose that \( \xi \neq 0 \), then under the null hypothesis \( H_0 \), \( KL_m \overset{d}{\rightarrow} N(0,1) \).

**Lemma 5.** Under the regularity assumptions \((J_1)\) and \((J_1)\), we have

1) for the model \( F_\theta \)
\[ \overline{D}_{KL}(\hat{F}_{n,h}, F_\theta) = D_{KL}(F, F_\theta) + U_\theta^T(\hat{F}_{n,h} - F) + S_\theta^T(F_\theta - F_\theta) + o_p(1). \] (40)

2) for model \( F_\gamma \)
\[ \overline{D}_{KL}(\hat{F}_{n,h}, F_\gamma) = D_{KL}(F, F_\gamma) + U_\gamma^T(\hat{F}_{n,h} - F) + S_\gamma^T(F_\gamma - F_\gamma) + o_p(1). \] (41)

**Proof.** The results follows from a first order Taylor expansion.

Proof of theorem\[\textsuperscript{5}\]
From the lemma\[\textsuperscript{5}\] it follows that
\[ \overline{D}_{KL}(\hat{F}_{n,h}, F_\theta) - \overline{D}_{KL}(\hat{F}_{n,h}, F_\gamma) = D_{KL}(F, F_\theta) - D_{KL}(F, F_\gamma) + U_\theta^T(\hat{F}_{n,h} - F) - U_\gamma^T(\hat{F}_{n,h} - F) + S_\theta^T(F_\theta - F_\theta) - S_\gamma^T(F_\gamma - F_\gamma) + o_p(1). \]

Under \( H_0 \), \( D_{KL}(F, F_\theta) = D_{KL}(F, F_\gamma) \), \( F_\theta = F_\gamma \) and \( F_\theta = F_\gamma \) we have
\[ \overline{D}_{KL}(\hat{F}_{n,h}, F_\theta) - \overline{D}_{KL}(\hat{F}_{n,h}, F_\gamma) = U_\theta^T(\hat{F}_{n,h} - F) - U_\gamma^T(\hat{F}_{n,h} - F) + S_\theta^T(F_\theta - F_\theta) - S_\gamma^T(F_\gamma - F_\gamma) + o_p(1) \]
\[ = (U_\theta - U_\gamma, S_\theta - S_\gamma)^T \left( \hat{F}_{n,h} - F \right) + o_p(1). \]

Finally, applying the Central Limit Theorem and assumptions \((J_1) - (J_2)\), one can get immediately \( KL_m \overset{d}{\rightarrow} N(0,1) \). It concludes the proof of the theorem\[\textsuperscript{5}\]
7. Simulation Study

To illustrate the theory discussed in the preceding section, we consider the family of Poisson distribution and the family of Geometric distribution. For more details, let us introduce them. The probability mass function (PMF) of a Poisson distribution with the parameter \( \lambda \) is given by

\[
p(x; \lambda) \equiv f_\lambda(x) = \frac{e^{-\lambda} \lambda^x}{x!}, \quad x \in \mathbb{N}
\]

and the PMF of geometric distribution with the parameter \( \theta \) is given by

\[
g(x; \theta) \equiv f_\theta(x) = \theta(1-\theta)^{x-1}, \quad x \in \mathbb{N}.
\]

We consider various sets of experiments in which data are generated from the mixture of a Poisson and Geometric distributions. These two distributions are calibrated so that their two means are close (9 and 10 respectively). Hence the Data Generating Process (DGP) has density

\[ t(\pi) = \pi \text{Poisson}(9) + (1 - \pi) \text{Geometric}(0.1) \]

where \( \pi \in (0, 1) \) is specific to each set of experiments. In each set, several random samples are drawn from this mixture. The sample size varies from 20 to 250, and for each sample size the number of replications is 1000. We choose different values of \( \pi \) which are 0.00, 0.25, 0.5, 0.75, 1.00. Although our proposed model selection procedure does not require that the data generating process belong to either of the candidate models. We consider the two limiting cases \( \pi = 0.00 \) and \( \pi = 1.00 \) for they correspond to the correctly specified cases. For \( \pi = 0.25 \) and \( \pi = 0.75 \) both candidate models are misspecified but not at equal distance from the DGP. These cases correspond to a DGP which is Poisson or Geometric distributions but slightly contaminated by the other distribution. The value \( \pi = 0.5 \) is the value for which the Poisson and Geometric distributions are approximately at equal distance to the mixture \( t(\pi) \). In order to perfect fit by the proposed method, for the chosen parameters of these two distributions, we note that most of the mass is concentrated between 0 and 10. Therefore, the chosen partition has eight cells defined by \([c_{i-1}, c_i] = [i - 1, i], i = 1, ..., 7]\) and \([c_7, c_8] = [7, \infty]\. The corresponding maximum likelihood estimators (MLEs) of \( \lambda \) and \( \theta \) are \( \hat{\lambda} = \bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i \) and \( \hat{\theta} = \frac{n \sum_{i=1}^{n} X_i}{n \sum_{i=1}^{n} (X_i - 1)} \). Since the properties of kernel density estimators do not depend much on which particular kernel is used, we choose the standard normal as the kernel function \( K \) without loss of generality. Therefore for the Gaussian kernel,

\[
\hat{f}_{nh}(x) = \frac{1}{2 \sqrt{2\pi} nh} \sum_{i=1}^{n} \left[ 3 - \left( \frac{x - X_i}{h} \right)^2 \right] e^{- \left( \frac{x - X_i}{h} \right)^2}.
\]

To get \( h \) optimal of a bias reduced kernel density estimator, the cross-validation bandwidth selection for a bias reduced kernel density estimator is used as proved in section 5.

### 7.1. Comparative results of \( D_1 \equiv \hat{D}_{KL}(\hat{f}_{nh}, f_\theta) \) and \( D_2 \equiv \hat{D}_{KL}(\hat{f}_{nh}^{b}, f_\theta) \)

In this subsection, we carry out a simulation study designed to demonstrate the performance of the Kullback-Leibler divergence estimator based on the bias reduced kernel density estimator \( \hat{f}_{nh}^{b} \) comparatively to the Kullback-Leibler divergence estimator based on the kernel density estimator \( \hat{f}_{nh} \) given in (2) and (1) respectively. To do this well, we use the Minimized Kullback-Leibler Divergence (MKLD) defined by \( MKLD = \frac{\hat{D}_2}{\hat{D}_1} \) as a measure of efficiency of the estimator, where \( f_\theta \) is a given parametric model. If this MKLD is less than 1, then we conclude that \( \hat{f}_{nh}^{b} \) is a more efficient estimator of \( f \) than \( \hat{f}_{nh} \) in this sense that it has a smaller Kullback-Leibler divergence. In our case, we suppose \( f_\theta \) to be a Geometric distribution. The values in parentheses are standard errors. Note that to be too rigorous, we have used the classical cross-validation method in order to get \( h \) optimal when the kernel density estimator is computed and the cross-validation bandwidth selection for a bias reduced kernel density estimator when the bias reduced kernel density estimator is computed. The results are presented in Tables 1-3.
\[ \hat{D}_1 = 0.3425 \quad 0.5857 \quad 0.7687 \quad 1.1369 \quad 1.8199 \\
(0.2968) \quad (0.5109) \quad (0.6302) \quad (0.1570) \quad (0.8641) \]

\[ \hat{D}_2 = 0.0894 \quad 0.3488 \quad 0.5500 \quad 0.9825 \quad 1.7547 \\
(0.0421) \quad (0.1082) \quad (0.1528) \quad (0.2370) \quad (0.3721) \]

\[ MKLD = 0.2610 \quad 0.5955 \quad 0.7154 \quad 0.8641 \quad 0.9641 \\
(0.2848) \quad (0.3956) \quad (0.4187) \quad (0.3934) \quad (0.3601) \]

\[ \hat{D}_1 = 0.4865 \quad 1.0904 \quad 1.5158 \quad 2.4796 \quad 4.2242 \\
(0.3125) \quad (0.5543) \quad (0.6303) \quad (0.8219) \quad (1.1855) \]

\[ \hat{D}_2 = 0.0597 \quad 0.2283 \quad 0.3710 \quad 0.6707 \quad 1.2102 \\
(0.0295) \quad (0.0669) \quad (0.0889) \quad (0.1374) \quad (0.2205) \]

\[ MKLD = 0.1227 \quad 0.2093 \quad 0.2447 \quad 0.2704 \quad 0.2864 \\
(0.1196) \quad (0.1272) \quad (0.0942) \quad (0.0873) \quad \]

\[ \hat{D}_1 = 0.6187 \quad 1.4410 \quad 2.0764 \quad 3.5116 \quad 6.0540 \\
(0.3373) \quad (0.5570) \quad (0.6995) \quad (0.9557) \quad (1.4543) \]

\[ \hat{D}_2 = 0.0511 \quad 0.1428 \quad 0.2270 \quad 0.4213 \quad 0.7760 \\
(0.0329) \quad (0.0394) \quad (0.0523) \quad (0.0912) \quad (0.1405) \]

\[ MKLD = 0.0825 \quad 0.0990 \quad 0.1093 \quad 0.1199 \quad 0.1281 \\
(0.0520) \quad (0.0519) \quad (0.0477) \quad (0.0402) \quad (0.0334) \]

All tables show that \( \hat{D}_2 \) has a small values comparatively to \( \hat{D}_1 \). As consequence the minimized Kullback-Leibler divergence (MKLD) is less than 1. This proves that the Kullback-Leibler divergence estimator based on a bais reduced kernel density estimator perform competitively well and better than the Kullback-Leibler divergence estimator based on the ordinary kernel density estimator.

7.2. Simulation results: Model selection procedure

To illustrate the model selection procedure discussed in the subsection [6.2] we consider the problem of choosing between the family of Poisson distribution and the family of Geometric distribution. The KLD between the bias reduced kernel density estimator and each model are defined as follows

\[ \overline{D}_{KL}(\hat{f}_{n,h}^b, f_P) = \sum_{i=1}^{m} \hat{f}_{n,h,i}^b \log \left( \frac{\hat{f}_{n,h,i}^b}{f_P,i} \right) \]

and

\[ \overline{D}_{KL}(\hat{f}_{n,h}^b, f_G) = \sum_{i=1}^{m} \hat{f}_{n,h,i}^b \log \left( \frac{\hat{f}_{n,h,i}^b}{f_G,i} \right) \]

where \( m \) is the number of cells considered. The results of our five sets of experiments are presented in Tables 1-5.
### Table 1. DGP = Poisson (9)

| n   | 20   | 60   | 90   | 150  | 250  |
|-----|------|------|------|------|------|
| $\lambda$ | 8.9727 | 8.9817 | 8.9967 | 8.9946 | 8.9928 |
|       | (0.6482) | (0.3858) | (0.3194) | (0.2331) | (0.1877) |
| $\hat{\theta}$ | 0.1006 | 0.1003 | 0.1001 | 0.1001 | 0.1001 |
|       | (0.0065) | (0.0039) | (0.0031) | (0.0023) | (0.0018) |
| $\hat{D}_{KL_1}$ | 0.1221 | 0.4186 | 0.6527 | 1.1472 | 1.9759 |
|       | (0.0339) | (0.0707) | (0.1000) | (0.1570) | (0.2549) |
| $\hat{D}_{KL_2}$ | 0.1142 | 0.1626 | 0.2104 | 0.3060 | 0.4400 |
|       | (0.0962) | (0.1055) | (0.1231) | (0.1457) | (0.1836) |
| $KL_n$ | 0.0774 | 2.3677 | 3.5413 | 5.0650 | 6.2434 |
|       | (0.4555) | (0.8375) | (1.1850) | (2.0340) | (3.8894) |
| Model selection | Correct 4.6% | 72.4% | 94.8% | 99.7% | 100% |
|       | Indecisive 95.3% | 27.0% | 5.1% | 0.3% | 0.0% |
|       | Incorrect 0.1% | 0.6% | 0.1% | 0.0% | 0.0% |

### Table 2. DGP = Geometric (0.1)

| n   | 20   | 60   | 90   | 150  | 250  |
|-----|------|------|------|------|------|
| $\lambda$ | 8.9831 | 8.9340 | 8.9504 | 8.9955 | 8.9791 |
|       | (2.0556) | (1.2048) | (1.0019) | (0.7643) | (0.5925) |
| $\hat{\theta}$ | 0.10458 | 0.1021 | 0.1015 | 0.1006 | 0.1005 |
|       | (0.02241) | (0.0123) | (0.0102) | (0.0077) | (0.0059) |
| $\hat{D}_{KL_1}$ | 0.1163 | 0.4183 | 0.6827 | 1.2375 | 2.2480 |
|       | (0.0732) | (0.1359) | (0.1852) | (0.2660) | (0.4032) |
| $\hat{D}_{KL_2}$ | 0.5433 | 1.2596 | 1.8269 | 2.9174 | 4.9253 |
|       | (0.5215) | (0.6907) | (0.9123) | (1.0476) | (1.4205) |
| $KL_n$ | -0.8593 | -1.2391 | -1.3335 | -1.7367 | -2.1483 |
|       | (2.2220) | (5.2589) | (8.1391) | (11.8464) | (19.7045) |
| Model selection | Correct 11% | 19.5% | 19.4% | 34.6% | 50.4% |
|       | Indecisive 89% | 80.5% | 80.6% | 65.4% | 49.6% |
|       | Incorrect 0.0% | 0.0% | 0.0% | 0.0% | 0.0% |

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Table 3. DGP = 0.25 Poisson (9) + 0.75 Geometric (0.1)

| n  | 20   | 60   | 90   | 150  | 250  |
|----|------|------|------|------|------|
| $\hat{\lambda}$ | 8.6053 | 8.5710 | 8.5857 | 8.6201 | 8.6073 |
|      | (1.5539) | (0.9112) | (0.7599) | (0.5790) | (0.4481) |
| $\hat{\theta}$ | 0.1068 | 0.1054 | 0.1049 | 0.1043 | 0.1043 |
|      | (0.0173) | (0.0099) | (0.0082) | (0.0062) | (0.0048) |
| $\hat{D}_{KL_1}$ | 0.0944 | 0.3112 | 0.5147 | 0.9447 | 1.7313 |
|      | (0.0696) | (0.1047) | (0.1456) | (0.2234) | (0.3607) |
| $\hat{D}_{KL_2}$ | 0.2619 | 0.6287 | 0.9044 | 1.5169 | 2.5895 |
|      | (0.2344) | (0.2725) | (0.3477) | (0.4544) | (0.6205) |
| $KL_n$ | -0.8308 | -1.1985 | -1.1990 | -1.4620 | -1.7481 |
|      | (0.9019) | (2.0524) | (3.0837) | (4.7931) | (7.7619) |

Model selection based on $KL_n$

- Geometric: 9.7% 17% 16.5% 22.4% 33.8%
- Indecisive: 90.3% 83% 83.5% 77.6% 66.2%
- Poisson: 0.0% 0.0% 0.0% 0.0% 0.0%

Table 4. DGP = 0.5 Poisson (9) + 0.5 Geometric (0.1)

| n  | 20   | 60   | 90   | 150  | 250  |
|----|------|------|------|------|------|
| $\hat{\lambda}$ | 8.7256 | 8.7074 | 8.7229 | 8.7445 | 8.6073 |
|      | (1.0837) | (0.6383) | (0.5351) | (0.4048) | (0.4481) |
| $\hat{\theta}$ | 0.1040 | 0.1034 | 0.1031 | 0.1027 | 0.1043 |
|      | (0.0115) | (0.0067) | (0.0056) | (0.0042) | (0.0048) |
| $\hat{D}_{KL_1}$ | 0.1488 | 0.4729 | 0.7290 | 1.2969 | 2.3195 |
|      | (0.0665) | (0.1321) | (0.1806) | (0.2681) | (0.4092) |
| $\hat{D}_{KL_2}$ | 0.0873 | 0.1865 | 0.2730 | 0.4921 | 0.8988 |
|      | (0.0716) | (0.0885) | (0.0902) | (0.1435) | (0.2424) |
| $KL_n$ | 0.7491 | 2.3460 | 2.8037 | 3.3866 | 3.9257 |
|      | (0.3675) | (0.9455) | (1.5428) | (2.9105) | (5.7219) |

Model selection based on $KL_n$

- Geometric: 1.3% 0.1% 0.0% 0.0% 0.0%
- Indecisive: 92.8% 37.1% 20.6% 5.3% 1.2%
- Poisson: 5.9% 62.8% 79.4% 94.7% 98.8%
Table 5. DGP=0.75 Poisson (9) + 0.25 Geometric (0.1)

| n   | 20   | 60   | 90   | 150  | 250  |
|-----|------|------|------|------|------|
| \( \hat{\lambda} \) | 8.6021 | 8.5956 | 8.9967 | 8.6201 | 8.6141 |
|     | (0.7146) | (0.4257) | (0.3583) | (0.2656) | (0.2081) |
| \( \hat{\theta} \) | 0.1047 | 0.1044 | 0.1042 | 0.1040 | 0.1040 |
|     | (0.0078) | (0.0046) | (0.0038) | (0.0028) | (0.0022) |
| \( \hat{D}_{KL1} \) | 0.1298 | 0.4627 | 0.7190 | 1.2914 | 2.3190 |
|     | (0.0374) | (0.0954) | (0.1345) | (0.2211) | (0.3751) |
| \( \hat{D}_{KL2} \) | 0.1074 | 0.1558 | 0.2060 | 0.3037 | 0.4972 |
|     | (0.0917) | (0.0939) | (0.1117) | (0.1312) | (0.1641) |
| \( KL_n \) | 0.2273 | 2.8437 | 3.7625 | 4.6600 | 5.4338 |
|     | (0.4395) | (0.8358) | (1.2935) | (2.5960) | (5.3012) |

Model selection based on \( KL_n \):
- Geometric: 3.9% 0.2% 0.1% 0.0% 0.0%
- Indecisive: 95.2% 14.2% 1.5% 0.4% 0.0%
- Poisson: 0.9% 85.6% 98.4% 99.6% 100%

The first half of each table gives the average values of the MLE \( \hat{\lambda} \) and \( \hat{\theta} \), the Kullback-Leibler divergence test statistics \( \hat{D}_{KL1} \) and \( \hat{D}_{KL2} \), and the model selection statistic \( KL_n \). The second half of each table gives in percentage the number of times our proposed model selection procedure based on \( KL_n \) favors the Poisson model, the Geometric model and indecisive. The tests are conducted at 5% nominal significance level. In the first two sets of experiments where the model is correctly specified, we use the labels correct, incorrect and indecisive when a choice is made. The first halves of Tables 1-5 confirm our asymptotic results. They all show that the MLE \( \hat{\lambda} \) and \( \hat{\theta} \) converge rapidly to their true values in the correctly specified cases and to their pseudo-true values in the misspecified cases as the sample size increases. The statistics \( \hat{D}_{KL1} \) and \( \hat{D}_{KL2} \) increase at the rate of \( n \), as expected when the models are correctly specified and when the models are misspecified. As expected, our statistic divergence \( KL_n \) converge to \( -\infty \) (Tables 2 and 3) and to \( +\infty \) (Tables 1, 4 and 5) as the sample size increases.

Turning to the second halves of Tables 1 and 2, we note that the percentage of correct choices using model selection statistic steadily increases and ultimately converge to 100%. As a consequence, the probability of correct choice (PCS) based on Monte Carlo simulation is found to be significantly higher in choosing the correct model in this selection procedure based on KLD. This preceding comments can be applied to the second halves of Tables 3, 4 and 5. In all tables, as sample size increases, the percentage of incorrect model still keeping the same, i.e. 0.0%. This is because in KLD, the correct model represents the "true" distribution of observations while the incorrect model represents an approximation of the true model. Except in Table 4 the percentage of incorrect model converges to zero. This is because the Poisson and Geometric distributions are approximately at equal distance to the mixture \( t(\pi) \) according to statistics \( \hat{D}_{KL1} \) and \( \hat{D}_{KL2} \).

For \( n = 90 \), we plot the histogram of datasets and overlay the curves for Poisson and Geometric distributions. As can be observed in Figure 1 and 3, the Geometric distribution is distinguished from Poisson distribution and is closely approximates the data sets. In Figure 2 and 5 the Poisson distributions is much closer to the data sets. When \( \pi = 0.5 \) (Figure 4) the Poisson distribution and Geometric distribution try to be closer to the data set. This follows from the fact that they are equidistant from the DGP.
Figure 1: Histogram of DGP = Geom(0.1).

Figure 2: Histogram of DGP = Pois(9).

Figure 3: Histogram of DGP = 0.25 Pois(9) + 0.75Geom(0.1).

Figure 4: Histogram of DGP = 0.5Pois(9) + 0.5 Geom(0.1).

Figure 5: Histogram of DGP = 0.75 Pois(9) + 0.25 Geom(0.1).
8. Conclusion

This paper shows the strong consistency of the bias reduced kernel density estimator and establishes a strongly consistent Kullback-Leibler divergence estimator based on the bias reduced kernel density estimator. Furthermore, we have considered an application in the problem of selecting estimated models using Kullback-Leibler divergence estimator type statistics proposed. Specifically, we have proposed some asymptotically standard normal and hypothesis testing based on Kullback-Leibler divergence estimator constructed in terms of the bias reduced kernel density estimator. We have also proposed a cross-validation bandwidth selection for the bias reduced kernel density estimator. Our tests are based on testing whether the candidate models are equally close to the true distribution against the alternative hypotheses that one model is closer than the other where closeness of a model is measured according to the discrepancy implicit in the Kullback-Leibler divergence type statistics used. The simulations studies show that the Kullback-Leibler divergence estimator based on the bias reduced kernel density estimator is more efficient estimator of Kullback-Leibler divergence than the Kullback-Leibler divergence estimator based on an ordinary kernel density estimator. The model selection procedure based on the Kullback-Leibler divergence estimator proposed is competitively especially in small samples. Since the Kullback-Leibler divergence is a special case of $f$-divergences as well as the class of Bregman divergences. It would be interesting to propose the methods for discriminations based on others divergence measures.

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