Time-limited pseudo-optimal $\mathcal{H}_2$-model order reduction

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Abstract—A model order reduction algorithm is presented that generates a reduced-order model of the original high-order model, which ensures high-fidelity within the desired time interval. The reduced model satisfies a subset of the first-order optimality conditions for time-limited $\mathcal{H}_2$-model reduction problem. The algorithm uses a computationally efficient Krylov subspace-based framework to generate the reduced model, and it is applicable to large-scale systems. The reduced-order model is parameterized to enforce a subset of the first-order optimality conditions in an iteration-free way. The efficacy of the algorithm is validated on benchmark model reduction problems.

Index Terms—$\mathcal{H}_2$-norm, model reduction, optimal, time-limited.

I. INTRODUCTION

The intricacies and complexity of the physical systems are increasing each year. The modern-day physical systems are mathematically described by several hundred or thousands of differential equations resulting in a large-scale state-space model. The computing power of the modern-day computers is also increasing at an increasing rate; however, the complexity of the physical systems still poses a computational challenge to the efficient simulation, analysis, and design. Model order reduction (MOR) techniques are used to obtain a reduced order approximation of the original high-order model, which retains most of its input-output properties. The reduced-order model (ROM) can then be used as a surrogate for the original high-order system in the design and analysis [1]-[6]. Balanced truncation (BT) [7] is among the most popular MOR techniques. The preservation of stability, the existence of an a priori error bound, and high accuracy are among the most significant features of BT. In BT, the states which have significant Hankel singular values are retained in the ROM, and the remaining states are truncated. BT requires the solution of two large-scale Lyapunov equations which is a computationally intensive task. The high computational cost of BT hinders its applicability to large-scale systems. Several extensions of BT exist in the literature to reduce its computational cost like [8]-[12] which suggest replacing the exact solution of the Lyapunov equations with their low-rank approximations. BT is generalized to preserve several other system characteristics like passivity, second-order structure, contractivity, etc. Reference [1] provides an in-depth survey of BT and its extensions.

The modal configuration is an important mathematical property of the system model, which is related to several physical phenomena. For instance, the interconnected power systems exhibit low-frequency oscillations like local, interplant, and interarea oscillations. These are associated with the modes in the frequency interval of $0 - 2$ Hz and are poorly damped [13]. The small-signal stability analysis and the damping controller design rely heavily on these modes. Various modes can also be associated with the power system components in the network, like generators and power system stabilizers. Therefore, their preservation in the ROM is important from a physical perspective. Moreover, their preservation in the ROM is also beneficial for the accuracy of ROM both in the time and frequency domains [14]-[17]. Recently, several eigensolvers are developed which exploit the sparse structure of the large-scale models and efficiently compute the dominant modes [18]-[20] which are required to be preserved in the ROM for achieving good accuracy. Modal truncation based on these eigensolvers can efficiently generate a ROM for large-scale systems which preserves the dominant modes of the original model. In terms of accuracy, modal truncation is way inferior to BT. However, in many applications, the preservation of important modes of the original system is more important than the overall accuracy in terms of error.

Moment matching is another important class of MOR techniques. In moment matching methods, a ROM is constructed which interpolates the original transfer function and a few of its moments using computationally efficient rational Krylov subspace based-framework. These techniques can easily handle large-scale systems and can be applied even if the model is unknown, and only the input-output data is known [21]. Moment matching methods have been significantly advanced over the last two decades, and several generalizations and extensions exist which can preserve various system properties while ensuring good accuracy as well. Reference [22] provides a detailed survey of moment matching methods. The moment matching methods are normally not as accurate as balancing based methods; however, the iterative rational Krylov algorithm (IRKA) is considered among the gold standards of moment matching. IRKA was first proposed by Gugercin et al. for single-input single-output (SISO) systems in [23], and it is as accurate as BT even if the algorithm is initialized with fairly random interpolation points. The ROM generated by IRKA satisfies the first-order optimality conditions for the $\mathcal{H}_2$-MOR problem. IRKA was later generalized for multi-input multi-output (MIMO) systems in [24]. In general, the convergence
is not guaranteed in IRKA, and it significantly slows down as the number of inputs and outputs increases. Moreover, the stability of the ROM is not guaranteed. Günercin presented a modification to IRKA, i.e., iterative SVD rational Krylov algorithm (ISRKA) which satisfies a subset of the first-order optimality conditions for the $H_2$-MOR problem and the stability is also guaranteed [25]. However, the algorithm presented is still an iterative algorithm with no guarantee on convergence. Also, it requires the computation of one large-scale Lyapunov equation which is computationally not feasible in a large-scale setting. In [26], an iteration-free pseudo-optimal rational Krylov (PORK) algorithm is presented, which generates a ROM that satisfies a subset of the first-order optimality conditions like ISRKA, and the ROM is also guaranteed to be stable. Unlike ISKRA, it does not require the solution of a large-scale Lyapunov equation, and thus it is computationally efficient. Practically, no system or its simulation is run over an infinite time interval. It is, therefore, reasonable to ensure a superior accuracy within the actual range of operation. To ensure high-fidelity in a desired limited time interval, BT is generalized to time-limited BT (TLBT) in [27]. TLBT is computationally expensive as it requires the solution of two large-scale dense Lyapunov equations. In [28], the applicability of TLBT is extended to large-scale systems using Krylov subspace-based methods and low-rank approximation of large-scale Lyapunov equations. The ROM is not guaranteed to be stable in TLBT. In [29], IRKA is generalized to time-limited scenario to approximately achieve the first-order optimality conditions for the time-limited $H_2$-MOR problem. The first-order optimality conditions for the time-limited $H_2$-MOR problem are expressed as bi-tangential Hermite interpolation conditions in [30], a descent-based iterative algorithm is presented for SISO systems, which generates a ROM which satisfy these conditions. The algorithm is computationally not feasible in a large-scale setting as it requires the computation of pole-residue form of the original transfer function, which is an expensive task. Moreover, the stability of the original system is not guaranteed to be preserved in the ROM generated by both the algorithms, and there is no guarantee on the convergence of the algorithms. In this paper, we present a time-limited MOR algorithm which satisfies a subset of the first-order optimality conditions for the time-limited $H_2$-MOR problem (as derived in [29] and [30]). The algorithm is iteration-free, and it guarantees the stability of ROM. The proposed algorithm does not involve any large-scale Lyapunov equation, and it uses a computationally efficient rational Krylov algorithm to construct the ROM. Therefore, it is applicable to large-scale systems. The proposed algorithm uses the parametrization of the ROM approach [31], [32] to enforce a subset of the optimality conditions and to place the poles and their associated input or output residues to the specified locations. Thus it can also preserve the modes and their associated input or output residues of the original systems like modal truncation. Therefore, the proposed algorithm can also be used for the applications wherein modal preservation is an important property to be preserved in the ROM. We have tested our algorithm on benchmark MOR problems, and the simulation results confirm the efficacy of the proposed algorithm.

II. Preliminaries

Let $H(s)$ be the $n^{th}$ order model of the original high-order system. The MOR problem is to find a $r^{th}$ ($r << n$) order ROM $\hat{H}_r(s)$ of $H(s)$ such that the error $\|H(s) - \hat{H}_r(s)\|$ is small in some defined sense. Let $x \in \mathbb{R}^{n \times 1}$, $u \in \mathbb{R}^{1 \times m}$, and $y \in \mathbb{R}^{p \times 1}$ be the state, input, and output vectors of the state-space representation of $H(s)$, i.e.,

$$\dot{x} = Ax + Bu,$$

$$y = Cx + Du,$$

$$H(s) = C(sI - A)^{-1}B + D.$$

Let $x_r \in \mathbb{R}^{r \times 1}$ and $y_r \in \mathbb{R}^{p \times 1}$ be the state and output vectors of the state-space representation of $H_r(s)$, i.e.,

$$\dot{x}_r = \hat{A}_r x_r + \hat{B}_r u,$$

$$\hat{y}_r = \hat{C}_r x_r + \hat{D}_r u.$$

$\hat{W}_r$ and $\hat{V}_r$ project $H(s)$ onto a reduced subspace such that the dominant dynamics of $H(s)$ are retained in $H_r(s)$, i.e.,

$$\hat{H}_r(s) = C\hat{V}_r(sI - \hat{W}_r^* A\hat{V}_r)^{-1}\hat{W}_r^* B + \hat{D}_r.$$

The time-limited MOR problem is to compute a ROM $(\hat{A}_r, \hat{B}_r, \hat{C}_r, \hat{D}_r)$ such that $\gamma \approx \hat{y}_r$ during the desired time interval $[0, t]$ sec.

The important mathematical notations which are used throughout the text are tabulated in Table I.

| Notation | Meaning |
|----------|---------|
| $\lambda_i(\cdot)$ | Eigenvalues of the matrix |
| $\text{Ran}(\cdot)$ | Range of the matrix |
| $\text{orth}(\cdot)$ | Orthogonal basis for the range of the matrix |
| $\text{span} \{ \cdot \}$ | Span of the set of $r$ vectors |

A. PORK [26]

$\hat{H}_r(s)$ interpolates $H(s)$ at the interpolation points $\sigma_i$ in the respective (right) tangential directions $\hat{c}_i \in \mathbb{C}^{m \times 1}$ for any output rational Krylov subspace $\hat{W}_r$ such that $\hat{W}_r^* \hat{V}_r = I$ if the input rational Krylov subspace $\hat{V}_r$ satisfies the following property

$$\text{Ran}(\hat{V}_r) = \text{span} \{ (\sigma_i I - A)^{-1} B \hat{c}_i \}. \quad (1)$$

Choose any $\hat{W}_r$, for instance, $\hat{W}_r = \hat{V}_r$, and compute the following matrices

$$\hat{E} = \hat{W}_r^T \hat{V}_r, \quad \hat{A} = \hat{W}_r^T \hat{A} \hat{V}_r, \quad \hat{B} = \hat{W}_r^T B, \quad (2)$$

$$B_{\perp} = B - \hat{V}_r \hat{E}^{-1} \hat{B}, \quad (3)$$

$$\hat{L}_r = (B_{\perp}^T B_{\perp})^{-1} B_{\perp}^T (A\hat{V}_r - \hat{V}_r \hat{E}^{-1} \hat{A}), \quad (4)$$

$$\hat{S}_r = \hat{E}^{-1} (\hat{A} - \hat{B} \hat{L}_r). \quad (5)$$
Then $\hat{V}_r$ satisfies the following Sylvester equation:

$$A\hat{V}_r + \hat{V}_r(-\hat{S}_r) + B(-\hat{L}_r) = 0$$

where $\{\sigma_1, \cdots, \sigma_r\}$ are the eigenvalues of \( \hat{S}_r \). A family of ROMs which satisfy the interpolation condition $\hat{H}_r(\sigma_i) \hat{c}_i = H(\sigma_i) \hat{c}_i$ can be obtained by parametrizing the ROM in $\xi$ if all the interpolation points $\sigma_i$ have positive real parts, and $(\hat{S}_r, \hat{L}_r)$ is observable, i.e.,

$$\hat{A}_r = \hat{S}_r + \xi \hat{L}_r, \quad \hat{B}_r = \xi, \quad \hat{C}_r = C\hat{V}_r, \quad \hat{D}_r = D.$$

$\hat{H}_r(s)$ ensures that

$$||H(s) - \hat{H}_r(s)||^2_{\mathcal{H}_2} = ||H(s)||^2_{\mathcal{H}_2} - ||\hat{H}_r(s)||^2_{\mathcal{H}_2}$$

if $\xi$ is set to $\xi = -\hat{Q}_s^{-1}\hat{L}_r^T$ where $\hat{Q}_s$ solves

$$-\hat{S}_r^T\hat{Q}_s - \hat{Q}_s\hat{S}_r + \hat{L}_r^T\hat{L}_r = 0.$$

Equation (6) is a subset of the first-order optimality conditions for the local optimality problem $||H(s) - \hat{H}_r(s)||^2_{\mathcal{H}_2}$ and thus $\hat{H}_r(s)$ is a pseudo-optimal ROM of $H(s)$.

B. TLBT [27]

TLBT [27] is a generalization of BT [17] wherein the standard controllability and observability Gramians, which are defined over the infinite time horizon, are replaced by the ones defined over the time interval of interest. Let $P_T$ and $Q_T$ be the time-limited controllability Gramian and time-limited observability Gramian respectively defined over the time interval $[0, t]$ wherein a superior accuracy is desired, i.e.,

$$P_T = \int_0^t e^{A^Tt}BB^Te^{A^Tt}dt \text{ and } Q_T = \int_0^t e^{A^Tt}CTCe^{A^Tt}dt$$

which solve the following Lyapunov equations

$$AP_T + P_AA^T + BB^T - e^{A^Tt}BB^Te^{A^Tt} = 0$$

$$A^TQ_T + Q_TAC^T + C^TC - e^{A^Tt}CTCe^{A^Tt} = 0.$$

$\hat{V}_r$ and $\hat{W}_r$ in TLBT are computed as $\hat{V}_r = \hat{T}_r\hat{R}_T$ and $\hat{W}_r = \hat{T}_r^{-1}\hat{R}_T^T$, respectively, where $\hat{R}_T = [R_{r \times r} \ldots R_{r \times (n-r)}]$, $\hat{T}_r^{-1}P_T\hat{T}_r = \hat{T}_r^TQ_T\hat{T}_r = diag(\sigma_1, \sigma_2, \ldots, \sigma_n)$, and $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$. The ROM is then obtained as

$$\hat{A}_r = \hat{W}_r^TAV_r, \quad \hat{B}_r = \hat{W}_r^TB, \quad \hat{C}_r = C\hat{V}_r, \quad \hat{D}_r = D.$$

C. TLIRKA

The time-limited squared $\mathcal{H}_2$-norm ($\mathcal{H}_{2,t}$) of the error system $H(s) - \hat{H}_r(s)$ is given by

$$||H(s) - \hat{H}_r(s)||^2_{\mathcal{H}_{2,t}} = \text{trace}(CP_TCT) + \text{trace}(\hat{C}_r\hat{P}_T\hat{C}_r^T) - 2\text{trace}(C\hat{P}_T\hat{C}_r^T)$$

$$= \text{trace}(B^TQ_TB) + \text{trace}(\hat{B}_r^T\hat{Q}_T\hat{B}_r) - 2\text{trace}(\hat{B}_r^T\hat{Q}_T\hat{B}_r)$$

where

$$\hat{A}_r\hat{P}_T + \hat{P}_TA^T_r + \hat{B}_r\hat{B}_r^T - e^{A_T\hat{t}}\hat{B}_r\hat{B}_r^Te^{A^T\hat{t}} = 0$$

$$\hat{A}_r^T\hat{Q}_T + \hat{Q}_T\hat{A}_r + \hat{C}_r^T\hat{C}_r - e^{A^T\hat{t}}\hat{C}_r^T\hat{C}_r e^{A\hat{t}} = 0$$

$$\hat{A}_r^T\hat{Q}_T + \hat{Q}_T\hat{A}_r + \hat{C}_r^T\hat{C}_r - e^{A^T\hat{t}}\hat{C}_r^T\hat{C}_r e^{A\hat{t}} = 0.$$
A. TLPORT

Let us define $\hat{S}_r, \hat{L}_r, B_T, H_T(s), \hat{H}_T(s)$, and $L_T$ as

$$\hat{S}_r = diag(\sigma_1, \cdots, \sigma_r), \quad \hat{L}_r = [\hat{c}_1 \cdots \hat{c}_r]$$

$$B_T = [B - e^{At}B], \quad H_T(s) = C(sI - A)^{-1}B_T + D,$$

$$\hat{H}_T(s) = C\hat{V}_r(sI - \hat{W}^*_rA\hat{V}_r)^{-1}\hat{W}^*_rB_T + D$$

$$L_T = \begin{bmatrix} \hat{L}_r \\ \hat{L}_re^{-\hat{S}_t} \end{bmatrix} = [c_1 \cdots c_r]. \quad (16)$$

$H_T(s)$ interpolates $H_T(s)$ at the interpolation points $\sigma_i$ in the respective (right) tangential directions $c_i \in \mathbb{C}^{2m \times 1}$ for any output rational Krylov subspace $\hat{W}_r$ such that $\hat{W}_r \hat{V}_r = I$ if the input rational Krylov subspace $\hat{V}_r$ is defined as

$$\hat{V}_r = [(A - \sigma_1I)^{-1}B_T c_1 \cdots (A - \sigma_rI)^{-1}B_T c_r].$$

Owing to the relation with the Sylvester equation [33], $\hat{V}_r$ solves the following Sylvester equation

$$A\hat{V}_r + \hat{V}_r(-\hat{S}_r) + B_T(-L_T) = 0. \quad (17)$$

A family of ROMs which satisfy the interpolation condition $\hat{H}_T(\sigma_i)c_i = H_T(\sigma_i)c_i$ can be obtained by parametrizing $\hat{H}_T(\sigma_i)$ in $\xi$ if all the interpolation points $\sigma_i$ have positive real parts, and $(\hat{S}_r, L_T)$ is observable, i.e.,

$$\hat{A}_r = \hat{S}_r + \xi L_T \quad \hat{B}_r = \xi \quad \hat{C}_r = C\hat{V}_r.$$

This can be verified by multiplying (17) with $\hat{W}_r^*$ from the left; see [33] for more details. Now set $\xi$ to

$$\xi = \begin{bmatrix} -\hat{Q}_s^{-1}\hat{L}_r^* & \hat{Q}_s^{-1}e^{-\hat{S}_t}\hat{L}_r^* \end{bmatrix} \quad (18)$$

where

$$-\hat{S}_r^*\hat{Q}_s - \hat{Q}_s\hat{S}_r + \hat{L}_r^*\hat{L}_r - e^{-\hat{S}_t}\hat{L}_r^*\hat{L}_re^{-\hat{S}_t} = 0. \quad (19)$$

Then, the ROM $\hat{H}_r(s)$ in TLPORT can be extracted from $\hat{H}_T(s)$ by removing $\hat{Q}_s^{-1}e^{-\hat{S}_t}\hat{L}_r^*$ from $\hat{B}_T$, i.e.,

$$\hat{A}_r = \hat{S}_r + \xi L_T, \quad \hat{B}_r = -\hat{Q}_s^{-1}\hat{L}_r^*, \quad \hat{C}_r = C\hat{V}_r, \quad \hat{D}_r = D. \quad (20)$$

Theorem 1: If $(\hat{A}_r, B_r, \hat{C}_r, \hat{D}_r)$ is defined as in equation (20), $\hat{H}_r(s)$ has the following properties:

(i) $\hat{H}_r(s)$ has poles at the mirror images of the interpolation points.

(ii) $\hat{Q}_s^{-1}$ is the time-limited controllability Gramian of the pair $(\hat{A}_r, B_r)$.

(iii) $\hat{H}_r(s)$ is a time-limited pseudo-optional ROM of $H(s)$.

(iv) $\hat{c}_r$ is the input-residual of $-\sigma_i^*.$

Proof: (i) By multiplying $\hat{Q}_s^{-1}$ from the left side of equation (19) yields

$$-\hat{Q}_s^{-1}\hat{S}_r^*\hat{Q}_s - \hat{S}_r + \hat{Q}_s^{-1}\hat{L}_r^*\hat{L}_r - \hat{Q}_s^{-1}e^{-\hat{S}_t}\hat{L}_r^*\hat{L}_re^{-\hat{S}_t} = 0$$

$$-\hat{Q}_s^{-1}\hat{S}_r^*\hat{Q}_s - \hat{S}_r + \hat{Q}_s^{-1}\hat{L}_r^*\hat{L}_r - e^{-\hat{S}_t}\hat{L}_r^*\hat{L}_re^{-\hat{S}_t} = 0.$$

Thus, $\hat{A}_r = -\hat{Q}_s^{-1}\hat{S}_r^*\hat{Q}_s,$ and hence, $\lambda_i(\hat{A}_r) = -\lambda_i(\hat{S}_r^*).$

(ii) The time-limited controllability Gramian of $(\hat{A}_r, B_r)$ solves the following Lyapunov equation (as in equation (7))

$$\hat{A}_r \hat{P}_T + \hat{P}_T \hat{A}_r^* + \hat{B}_r \hat{B}_r^* - e^{A_T} \hat{B}_r \hat{B}_r^* e^{A_T^T} = 0.$$

By pre- and post-multiplying equation (7) with $\hat{Q}_s$, by putting $\hat{A}_r = -\hat{Q}_s^{-1}\hat{S}_r^*\hat{Q}_s$ and $\hat{B}_r = -\hat{Q}_s^{-1}\hat{L}_r^*,$ and also by noting that $\hat{Q}_s e^{A_T} \hat{Q}_s^{-1} = e^{\hat{S}_t},$ equation (7) becomes

$$-\hat{S}_r^*\hat{Q}_s \hat{P}_T \hat{Q}_s - \hat{Q}_s \hat{P}_T \hat{Q}_s \hat{S}_r + \hat{L}_r^* \hat{L}_r - e^{-\hat{S}_t} \hat{L}_r^* \hat{L}_re^{-\hat{S}_t} = 0.$$

Due to uniqueness, $\hat{Q}_s \hat{P}_T \hat{Q}_s = \hat{Q}_s$, $\hat{Q}_s \hat{P}_T = I,$ and $\hat{P}_T = \hat{Q}_s^{-1}.$

(iii) Consider the following equation

$$AV_r \hat{P}_T + \hat{V}_r \hat{P}_T \hat{A}_r^* + \hat{B}_r \hat{B}_r^* - e^{At} \hat{B}_r \hat{B}_r^* e^{A_T^T} = 0.$$

Due to uniqueness, $\hat{V}_r \hat{P}_T = \hat{P}_r,$ and therefore, $\hat{C}_r \hat{P}_T = C\hat{P}_r.$ Hence, $\hat{H}_r(s)$ is a time-limited pseudo-optimal model ROM of $H(s)$.

(iv) $(-\hat{Q}_s^{-1})(-\hat{S}_r^*)(-\hat{Q}_s)$ is actually the spectral factorization of $\hat{A}_r.$ Also, $\hat{B}_r = -\hat{Q}_s^{-1}[\hat{c}_1 \cdots \hat{c}_r]^T.$ Thus, $\hat{c}_i$ is the input-residual of $-\sigma_i^*.$

Dually, a time-limited pseudo-optimal ROM can also be achieved by using a fixed $\hat{W}$ and then parameterizing the ROM. We refer to it as “Output-TLPORT (O-TLPORT)” to differentiate with TLPORT. Let $\sigma_i$ be the interpolation points in the (left) tangential directions $\hat{b}_i \in \mathbb{C}^{1 \times p}.$ Let us define $\hat{L}_T, \hat{B}_T,$ and $\hat{D}_T$ as

$$\hat{L}_T = \begin{bmatrix} \begin{bmatrix} C \\ \xi \end{bmatrix}e^{At} \end{bmatrix}, \quad \hat{B}_T = \begin{bmatrix} \hat{b}_1^* \\ \cdots \\ \hat{b}_r^* \end{bmatrix}, \quad \hat{D}_T = \begin{bmatrix} \hat{b}_1^* \\ \cdots \\ \hat{b}_r^* \end{bmatrix}. \quad (21)$$

$\hat{W}_r^*$ solves the following Sylvester equation

$$\hat{W}_r^*A + (-\hat{S}_r)\hat{W}_r^* + (-\hat{B}_T)(\hat{L}_T) = 0. \quad (22)$$

If all the interpolation points $\sigma_i$ have positive real parts, and the pair $(\hat{S}_r, \hat{B}_T)$ is controllable, a time-limited pseudo-optimal ROM $\hat{H}_r(s)$ of $H(s)$ can be obtained as

$$\hat{A}_r = \hat{S}_r + \hat{B}_T \xi, \quad \hat{B}_r = \hat{W}_r^*\hat{B}_r, \quad \hat{C}_r = -\hat{B}_T \hat{P}_S^{-1}, \quad \hat{D}_r = D \quad (23)$$

where

$$\xi = \begin{bmatrix} -\hat{B}_T \hat{P}_S^{-1} \\ \hat{B}_T e^{-\hat{S}_t} \hat{P}_S^{-1} \end{bmatrix}, \quad (24)$$

and $\hat{P}_S$ solves

$$-\hat{S}_r \hat{P}_S - \hat{P}_S \hat{S}_r^* + \hat{B}_r \hat{B}_r^* \hat{D}_T e^{-\hat{S}_T} - e^{-\hat{S}_T} \hat{B}_r \hat{B}_r^* e^{-\hat{S}_T} = 0. \quad (25)$$
Theorem 2: If \((A_r, B_r, C_r, D_r)\) is defined as in equation (23), \(H_r(s)\) has the following properties:

(i) \(H_r(s)\) has poles at the mirror images of the interpolation points.

(ii) \(P_S^{-1}\) is the time-limited observability Gramian of the pair \((A_r, C_r)\).

(iii) \(H_r(s)\) is a time-limited pseudo-optimal ROM of \(H(s)\).

(iv) \(b_i\) is the output residual of the pole \(-\sigma_i^r\).

Proof: (i) By multiplying \(P_S^{-1}\) from the right, equation (25) becomes

\[
-\dot{S}_r - \dot{P}_S^T \dot{S}_r P_S^{-1} + B_T B_T^T P_S^{-1} - e^{-S_r^r} B_T \dot{B}_T e^{-S_r^r} P_S^{-1} = 0
\]

\[
-\dot{P}_S S_r^r P_S^{-1} - \dot{A}_r = 0.
\]

Thus, \(\dot{A}_r = -\dot{P}_S S_r^r P_S^{-1}\), and hence, \(\lambda_i(\dot{A}_r) = -\lambda_i(\dot{S}_r)\).

(ii) The time-limited observability Gramian of \((A_r, C_r)\) solves the following Lyapunov equation (as in equation (8))

\[
\dot{A}_r^T \dot{Q}_T + \dot{Q}_T^T \dot{A}_r + \dot{C}_r^T C_r - e^{A_T^r t} \dot{C}_r^T C_r e^{A_T^r t} = 0.
\]

By pre- and post-multiplying equation (8) with \(P_S\), by putting \(\dot{A}_r = -\dot{P}_S S_r^r P_S^{-1}\) and \(\dot{C}_r = -B_T^T P_S^{-1}\), and also by noting that \(\dot{P}_S^{-1} e^{-A_T^r t} \dot{P}_S = e^{-S_r^r t}\), equation (8) becomes

\[
-\dot{S}_r \dot{P}_S \dot{Q}_r P_S - \dot{P}_S \dot{Q}_r P_S S_r^r + B_T B_T^T
\]

\[
= -e^{-S_r^r} B_T \dot{B}_T e^{-S_r^r} t = 0.
\]

Due to uniqueness, \(\dot{P}_S \dot{Q}_r P_S = \dot{P}_S\), \(\dot{P}_S \dot{Q}_r P_S = I\), and \(\dot{Q}_T = P_S^{-1}\).

(iii) Consider the following equation

\[
\dot{A}_r^T \dot{Q}_T \dot{W}_r^* + \dot{Q}_T \dot{W}_r^* A + \dot{C}_r^T C - e^{A_T^r t} \dot{C}_r^T C e^{A_T^r t} = 0
\]

\[
-\dot{Q}_T \dot{S}_r \dot{W}_r^* + \dot{Q}_T [\dot{S}_r \dot{W}_r^* + B_T L_T]
\]

\[
= 0.
\]

Due to uniqueness, \(\dot{Q}_T \dot{W}_r^* = \dot{Q}_T\), and therefore, \(\dot{Q}_T \dot{B}_T = \dot{Q}_T B\). Hence, \(H_r(s)\) is a time-limited pseudo-optimal model ROM of \(H(s)\).

(iv) \((-\dot{P}_S)(-\dot{S}_r)(-\dot{P}_S^{-1})\) is actually the spectral factorization of \(A_r\). Moreover, \(\dot{C}_r = [b_1^r \ldots b_r^r] (-\dot{P}_S^{-1})\). Thus, \(\dot{b}_i\) is the output residual of \(-\sigma_i^r\).

Remark 1: If the interpolation points are selected as the mirror images of \(r\)-modes of \(H(s)\), TLPORK and O-TLPORK preserve these modes in \(H_r(s)\) like modal truncation. Additionally, TLPORK and O-TLPORK preserve the input and output residues, respectively of these modes if the tangential directions are selected as such. Moreover, if \(t\) is set to \(\infty\), TLPORK and O-TLPORK reduce to PORK.

Remark 2: The time-limited pseudo-optimality (equation (15)) does not depend on the realization of \(H(s)\) or \(H_r(s)\).

B. Algorithmic Aspects

We allow the state-space matrices to be complex so far in this section; however, one can obtain a real ROM for a real original model. For instance, a real \(V_r\) can be computed by any rational Krylov subspace method instead of actually solving the Sylvester equation (17), i.e.,

\[
\text{Ran}(\dot{V}_r) = \text{span} \{((\sigma_i - A)^{-1} B_T c_i)\}.
\]

The next step then is to compute the matrices of Sylvester equation which this \(\dot{V}_r\) satisfies (as it may not satisfy equation (17)). This can be accomplished in a few simple steps. Choose any \(W_r\), for instance, \(\dot{W}_r = \dot{V}_r\). Then compute the following matrices

\[
\dot{E} = \dot{W}_r^T \dot{V}_r, \quad \dot{A} = \dot{W}_r^T A \dot{W}_r, \quad \dot{C} = \dot{L}_T \dot{V}_r,
\]

\[
\dot{C}_r = \dot{L}_T - \dot{C} \dot{E} \dot{W}_r^T.
\]
Then, \( \tilde{B}_T \) and \( \tilde{S}_r \) for the Sylvester equation of this \( \tilde{W}_r \) can be computed as

\[
\begin{align*}
\tilde{B}_T &= \left( \tilde{W}_r^T A - \tilde{A} \tilde{E}^{-1} \tilde{W}_r^T \right) C_\perp^T (C_\perp C_\perp^T)^{-1} \\
\tilde{S}_r &= \left( \tilde{A} - \tilde{B}_T \tilde{C} \right) \tilde{E}^{-1}.
\end{align*}
\]  

(36)

(37)

Now partition \( \tilde{B}_T \) as \( \tilde{B}_T = \begin{bmatrix} \tilde{B}_{m \times r}^+ & \tilde{B}_{m \times r}^- \end{bmatrix} \), and define \( \tilde{B}_T^- \) as \( \tilde{B}_T^- = \begin{bmatrix} \tilde{B}_{m \times r}^+ & -\tilde{B}_{m \times r}^- \end{bmatrix} \). Then \( \tilde{P}_S \) can be computed from the following Lyapunov equation

\[
\tilde{S}_r \tilde{P}_S + \tilde{P}_S \tilde{S}_r^T + \tilde{B}_T \tilde{B}_T^- = 0.
\]

(38)

The ROM is obtained as

\[
\begin{align*}
\hat{A}_r &= -\tilde{P}_S \tilde{S}_r^T \tilde{P}_S^{-1}, \\
\hat{B}_r &= \tilde{W}_r^T B, \\
\hat{C}_r &= -\left( \tilde{B}_{m \times r}^+ \right)^T \tilde{P}_S^{-1}, \\
\hat{D}_r &= D.
\end{align*}
\]

(39)

**Algorithm 2 O-TLPORK**

**Input:** Original model: \((A, B, C, D)\), interpolation points: \(\{\sigma_1, \cdots, \sigma_r\}\), tangential directions: \(\{b_1, \cdots, b_r\}\).

**Output:** ROM: \((\hat{A}_r, \hat{B}_r, \hat{C}_r, \hat{D}_r)\).

1. Define \( \hat{B}_T \) and \( L_T \) as in equation (21).
2. Compute \( \tilde{W}_r \) from equation (33).
3. Update \( \tilde{S}_r \) and \( \tilde{B}_T \) from equations (34)–(37).
4. Partition \( \tilde{B}_T \) as \( \tilde{B}_T = \begin{bmatrix} \tilde{B}_{m \times r}^+ & \tilde{B}_{m \times r}^- \end{bmatrix} \).
5. Define \( \hat{B}_T^- \) as \( \hat{B}_T^- = \begin{bmatrix} \tilde{B}_{m \times r}^+ & -\tilde{B}_{m \times r}^- \end{bmatrix} \).
6. Solve equation (38) to calculate \( \tilde{P}_S \).
7. Obtain the ROM from equation (39).

**C. Computational Cost**

TLBT [27] is computationally not feasible for large-scale systems because of the cubic complexity associated with the solution of large-scale Lyapunov equations. In [28], the applicability of TLBT is extended to large-scale systems by using the low-rank approximation of Lyapunov equations. Various methods to efficiently compute \( e^{A \bar{t}} B \) and \( C e^{A \bar{t}} \) are also discussed in [28]. TLPORK and O-TLPORK do not involve large-scale Lyapunov equations like TLBT [27]. The main computational effort in TLPORK and O-TLPORK is spent on calculating the rational Krylov subspaces \( \hat{V}_r \) and \( \tilde{W}_r \) for which several efficient methods are available [33]. Thus, TLPORK and O-TLPORK are easily applicable to large-scale systems.

**IV. Numerical Results**

In this section, we perform three experiments to test the efficacy of TLPORK, and we compare its performance with the well-known existing techniques. The test models are taken from the benchmark collection of [34]. In all the experiments, we initialize IRKA randomly and use its final interpolation points and tangential directions to initialize TLIRKA. We use the same interpolation points and tangential directions for PORK. We use the final interpolation points and tangential directions of TLIRKA for TLPORK. This helps to investigate the impact of satisfying only a subset of the optimality conditions on the accuracy of the ROM. The results of O-TLPORK are indistinguishable from TLPORK and hence, not shown for brevity. All the experiments are conducted on a computer with Intel(R) Core(TM) i7-8550U 1.80GHz \times 8 processors and 16GB memory using MATLAB 2016.

**Heat equation in a thin rod:** This is a 200\(^{th}\) order SISO model taken from [34]. A 5\(^{th}\) order ROM is obtained using BT, TLBT, IRKA, PORK, TLIRKA, and TLPORK. The desired time interval is specified as \([0, 2]\) sec. Note that unlike TLBT and TLIRKA, the stability of the ROM is guaranteed in TLPORK. The \(H_{2, \tau}\) norm of the error, i.e., \(\|H(s) - \hat{H}_r(s)\|_{H_{2, \tau}}\), is tabulated in Table II. Interestingly, TLPORK ensures less \(H_{2, \tau}\)-norm error than TLIRKA in this example. This is not a surprising result as TLIRKA does not yield an optimal ROM but only tends to yield an optimal ROM and nearly satisfies the optimality conditions.

**Clamped beam:** This is a 348\(^{th}\) order SISO model taken from [34]. A 12\(^{th}\) order ROM is obtained using BT, TLBT, IRKA, PORK, TLIRKA, and TLPORK. The desired time interval is specified as \([0, 4]\) sec. The \(H_{2, \tau}\) norm of the error, i.e., \(\|H(s) - \hat{H}_r(s)\|_{H_{2, \tau}}\), is tabulated in Table III. It can be seen that TLPORK ensures the least \(H_{2, \tau}\)-norm error.

**International space station:** This is a 270\(^{th}\) order MIMO model taken from [34] with three inputs and three outputs. A 10\(^{th}\) order ROM is obtained using BT, TLBT, IRKA, PORK, TLIRKA, and TLPORK. The desired time interval is specified as \([0, 1]\) sec. The \(H_{2, \tau}\) norm of the error, i.e., \(\|H(s) - \hat{H}_r(s)\|_{H_{2, \tau}}\), is tabulated in Table IV. It can be seen that TLPORK ensures the least \(H_{2, \tau}\)-norm error.

**V. Conclusion**

We present an iteration-free tangential interpolation algorithm which places the poles of the ROM at the mirror images of the interpolation points and makes the tangential directions the residues associated with these poles. The algorithm thus enforces a subset of the first-order optimality conditions for
time-limited $\mathcal{H}_2$-MOR problem. The proposed algorithm can also preserve the desired modes and their associated residues in the ROM like modal truncation. The Krylov subspace-based implementation ensures its applicability to large-scale systems. The numerical results confirm the theory presented in the paper.

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