DECOMPOSITION OF INFINITE-TO-ONE FACTOR CODES AND
UNIQUENESS OF RELATIVE EQUILIBRIUM STATES

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ABSTRACT. We show that an arbitrary factor map $\pi : X \to Y$ on an irreducible subshift of finite type is a composition of a finite-to-one factor code and a class degree one factor code. Using this structure theorem on infinite-to-one factor codes, we then prove that any equilibrium state $\nu$ on $Y$ for a potential function of sufficient regularity lifts to a unique measure of maximal relative entropy on $X$. This answers a question raised by Boyle and Petersen (for lifts of Markov measures) and generalizes the earlier known special case of finite-to-one factor codes.

1. INTRODUCTION

The usual setting for relative thermodynamic formalism starts with a fixed factor map between topological dynamical systems. Since the non-relative case indicates that symbolic dynamical systems have the easiest thermodynamic properties, it makes sense to work things out for the relative case in symbolic systems first. Indeed, there has been some progress in this direction in recent years, with or without thermodynamic application in mind.

In this paper, we restrict our attention to infinite-to-one factor codes $\pi : X \to Y$ from irreducible SFTs (shifts of finite type) to sofic shifts, and finite-to-one factor codes between irreducible sofic shifts. The structure of factor codes of the latter type is well understood now ([8, Chapter 8]). For example, each finite-to-one factor code $\pi$ can be associated with a number called degree $d$ which is the common number of points in the fiber $\pi^{-1}(y)$ for almost all $y \in Y$, and almost all fibers $\pi^{-1}(y)$ have a certain permutation structure in it. The class of finite-to-one factor codes is important because classification of irreducible sofic shifts up to finite equivalence, where the complete invariant turns out to be the topological entropy, is via finite-to-one factor codes (Finite Equivalence Theorem). This resembles part of the Ornstein theory which classifies Bernoulli systems up to isomorphism, where the complete invariant is the measure theoretical entropy. Finite-to-one factor codes are important also because each surjective cellular
automaton is a finite-to-one factor code, and because the class of finite-to-one factor codes gives the simplest nontrivial examples of principal extensions.

An early relative result on finite-to-one factor codes is a result on uniqueness of preimage of Markov measures by Tuncel [10]. He showed that for any \( \pi : X \to Y \) finite-to-one factor code between mixing SFTs, each Markov measure \( \nu \) on \( Y \) lifts uniquely to an invariant measure on \( X \) and the unique lift is a Markov measure. This is a relative thermodynamic result because Markov measures are just equilibrium states for locally constant functions (equivalently, (invariant) Gibbs measures for such functions or g-measures for such \( g \)).

For infinite-to-one factor codes, there are usually infinitely many invariant measures \( \mu \) on \( X \) that project to the same Markov measure \( \nu \) on \( Y \) and in some cases, even uncountably many Markov measures \( \mu \) [7].

In [6, Problem 3.16], Boyle and Petersen raised the following question. Given a (possibly infinite-to-one) factor map \( \pi : X \to Y \) between irreducible SFTs and a Markov measure \( \nu \) on \( Y \), is there a unique measure \( \mu \) of maximal relative entropy over \( \nu \) (meaning any lift of \( \nu \) with maximal entropy among lifts of \( \nu \)) and does \( \mu \) have full support? The question of full support was answered positively in [11] by the author, and its generalization to relative equilibrium states was also answered positively in [4] by Antonioli. The full support result is a consequence of the following more general phenomenon. For each ergodic measure \( \nu \) of full support (not necessarily an equilibrium state of a sufficiently regular potential function, let alone a Markov measure), all MMREs (i.e., measures of maximal relative entropy) over \( \nu \) have full support [11].

With a possibly non-Markov \( \nu \), in which case there may be more than one MMREs over it, Petersen, Quas, and Shin [9] showed that the number of ergodic MMREs is nonetheless finite. They also found some easy-to-check sufficient condition on \( \pi \) that guarantees uniqueness of MMREs over any fully supported ergodic \( \nu \) rather than just over any Markov \( \nu \).

Allahbakhshi and Quas defined an invariant (for factor codes) called class degree in [2], generalizing the notion of degree for finite-to-one factor codes, and showed that the number of ergodic MMREs is bounded by class degree. This gives a broader sufficient condition on \( \pi \), namely, having class degree one, for uniqueness of MMREs over any fully supported ergodic \( \nu \).

In order to answer the question of uniqueness over Markov measures under arbitrary factor codes, we decompose a factor code \( \pi : X \to Y \) into two factor codes \( \pi_1 : X \to \tilde{Y} \) and \( \pi_2 : \tilde{Y} \to Y \) where we can apply earlier results for class degree one code and finite-to-one code to \( \pi_1 \) and \( \pi_2 \) respectively. This is the first main result of this paper and is in some sense a structure theorem of infinite-to-one factor codes. It reduces the study of arbitrary factor codes into that of finite-to-one factor codes and that of class degree one codes. The second main result is the application of this decomposition theorem: proof of uniqueness of MMREs over Markov measures or over other equilibrium states of sufficiently regular functions.
In the following table, we list known and new conditions for uniqueness of MMREs.

| condition on $\pi$          | condition on $\nu$          |
|-----------------------------|-----------------------------|
| classical (Tuncel)          | finite-to-one               |
| recent result               | class degree one            |
| new result                  | none                        |
|                             | regular equilibrium state   |
|                             | full support                |
|                             | regular equilibrium state   |

In the next section, we introduce necessary definitions and facts. In Section 3, we prove the decomposition result. In Section 4, we prove uniqueness of relative equilibrium states (more general than MMREs) over sufficiently regular equilibrium states.

2. Definitions

$(X, T)$ is a topological dynamical system (or TDS, for short) if $T$ is a homeomorphism of a compact metric space $X$. A map $\pi: X \to Y$ between two TDS $(X, T)$ and $(Y, S)$ is a factor map if $\pi$ is continuous, surjective and equivariant (i.e., $\pi \circ T = S \circ \pi$). A factor code is a factor map between shift spaces.

Given a factor map $\pi: X \to Y$ and a probability measure $\mu$ on $X$, we denote by $\pi_* \mu$ the pushforward measure (image measure) from $\mu$ under $\pi$. So $\pi_* \mu$ is a probability measure on $Y$ defined by $\pi_* \mu(B) = \mu(\pi^{-1}(B))$ for each Borel subset $B \subset Y$. Thus, the map $\pi: X \to Y$ induces the pushforward map

$$\pi_* : M(X, T) \to M(Y, S),$$

where $M(X, T)$ is the set of all invariant probability measures on $X$.

Shift spaces in this paper always means one-dimensional two-sided shift spaces, i.e., subsystems of the full shift $A^\mathbb{Z}$ with some finite alphabet $A$. The two-sided assumption is because the degree theory and class degree theory rely on it. Not much is lost by the two-sided assumption, since two-sided irreducible sofic shifts and their one-sided versions are essentially interchangeable in terms of invariant measures, entropy and other thermodynamical notions. A shift of finite type (SFT, for short) means a shift space defined by local rules of uniformly bounded range, or equivalently, a shift space defined by a finite set of forbidden blocks. A sofic shift means a shift space that is an image of a SFT under a factor code.

A shift space $X$ is irreducible if each pair of $X$-words $u, v$ can be connected by some third $X$-word $w$ so that $uwv$ is also an $X$-word. A shift space $X$ is mixing if it is topologically mixing. The shift map on a shift space $X$ is denoted by $\sigma_X$ or just $\sigma$.

Definitions of infinite-to-one factor codes, finite-to-one factor codes, and degrees of finite-to-one factor codes are as follows. For the general theory of these notions, we refer to [8].

**Definition 2.1** ([8]). A factor code $\pi: X \to Y$ is finite-to-one if the fiber $\pi^{-1}(y)$ over each point $y \in Y$ is a finite set. Otherwise, it is called infinite-to-one.

To define the degree of a finite-to-one factor code, we need transitive points.
**Definition 2.2.** Given an irreducible sofic shift $Y$, a point $y \in Y$ is right transitive if its forward orbit is dense (in $Y$). A point $y \in Y$ is doubly transitive if both its forward orbit and its backward orbit are dense.

It is well known that the doubly transitive points of $Y$ form a residual subset of $Y$.

**Definition 2.3** ([8]). Given a finite-to-one factor code $\pi$ from an irreducible sofic $X$ onto a sofic shift $Y$, the degree of $\pi$ is defined to be the unique number $d \in \mathbb{N}$ such that $|\pi^{-1}(y)| = d$ for all doubly transitive points $y \in Y$. If $X$ is an SFT, then this number is also the minimum of $|\pi^{-1}(y)|$ over all $y \in Y$.

Given an irreducible sofic shift $Y$, there is always an irreducible SFT $X$ and a factor code $\pi : X \to Y$ which has degree one. The minimal right resolving presentation of $Y$ is such an example.

Now we move on to definitions for infinite-to-one factor codes.

**Definition 2.4** ([2]). Given a (possibly infinite-to-one) factor code $\pi$ from an irreducible SFT $X$ onto a sofic shift $Y$, we may define an equivalence relation on $X$ as follows. For $x, x' \in X$, we say $x \sim x'$ if $\pi(x) = \pi(x') = y$ for some $y \in Y$ and for each $n \in \mathbb{N}$ there is $x'' \in \pi^{-1}(y)$ such that $x''_{(-\infty,n]} = x_{(-\infty,n]}$ and $x''_{[i,\infty)} = x_{[i,\infty)}$ for some $i > n$. We say $x \sim x'$ if $x \sim x'$ and $x' \sim x$. The equivalence classes from the equivalence relation $\sim$ on $X$ are called transition classes. For each $y \in Y$, the transition classes in $\pi^{-1}(y)$ will be called transition classes over $y$.

For each $y \in Y$, the number of transition classes over $y$ is finite. The minimum of this number over all $y \in Y$ is called the class degree of $\pi : X \to Y$. This is also the number of transition classes over each right transitive point of $Y$ (Corollary 4.23 in [2]).

**Definition 2.5** ([2]). Let $\pi : X \to Y$ be a 1-block factor code from a 1-step SFT $X$ onto a sofic $Y$. Let $w = w_0w_1 \cdots w_p$ be a $Y$-block of length $1 + p$. Let $n$ be an integer with $0 < n < p$. Let $M$ be a subset of $\pi^{-1}(w_n)$. We say an $X$-word $u \in \pi^{-1}(w)$ is routable through $a \in M$ at time $n$ if there is a block $u' \in \pi^{-1}(w)$ such that $u'_0 = u_0$ and $u'_n = a$ and $u'_p = u_p$. A triple $(w, n, M)$ is called a transition block of $\pi$ if every $u \in \pi^{-1}(w)$ is routable through a symbol of $M$ at time $n$. The cardinality of $M$ is called the depth of the transition block $(w, n, M)$. A minimal transition block is a transition block of minimal depth. The minimal depth is the same as the class degree if $X$ is irreducible.

We will rely on the following unique routing property for minimal transition blocks.

**Lemma 2.6.** ([3, Lemma 4.1]) Let $\pi$ be a factor code from an irreducible SFT $X$ onto a sofic $Y$. Suppose $\pi$ is 1-block and $X$ is 1-step. Let $(w, n, M)$ be a minimal transition block. Then each preimage of $w$ is routable through a unique symbol of $M$ at $n$.

We will also rely on the following properties of transition classes over right transitive points.
Lemma 2.7. ([3, Theorem 4.4]) Let \( \pi \) be a factor code from an irreducible SFT \( X \) onto a sofic \( Y \). Suppose \( \pi \) is 1-block and \( X \) is 1-step. Let \( y \in Y \) be right transitive. Then any two points from two distinct transition classes over \( y \) are mutually separated.

The unique routing property for transition classes over right transitive points is as follows.

Lemma 2.8. ([3, Lemma 5.1]) Let \( \pi \) be a factor code from an irreducible SFT \( X \) onto a sofic \( Y \). Suppose \( \pi \) is 1-block and \( X \) is 1-step. Let \( y \in Y \) be right transitive. Suppose \( y[i, i + |w|) = w \) for some \( i \) and some minimal transition block \((w, n, M)\).

Let \( C \) be a transition class over \( y \). Then there is a unique symbol \( b \in M \) such that for each \( x \in C \), \( x[i, i + |w|) \) is routable through \( b \) at \( n \).

3. Decomposition of infinite-to-one factor codes

To decompose an infinite-to-one factor code \( \pi \) into \( \pi_1 \) and \( \pi_2 \) and verify that the two resulting factor codes have the desired properties, we need to establish convenient characterizations of those properties.

Definition 3.1. A point \( x \in X \) in a shift space is left-asymptotic to \( x' \in X \) if \( d(\sigma^{-i}x, \sigma^{-i}x') \to 0 \) as \( i \to \infty \) and right-asymptotic to \( x' \in X \) if \( d(\sigma^ix, \sigma^ix') \to 0 \) as \( i \to \infty \) or, equivalently, if there is some \( m \) such that \( x[m, \infty) = x'[m, \infty) \).

In the following lemma, we establish characterizations of class degree one factor codes that do not rely on any recoding assumption on the factor code.

Lemma 3.2. Let \( \pi \) be a factor code from an irreducible SFT \( X \) onto a sofic shift \( Y \). The following are equivalent.

1. The class degree of \( \pi \) is 1.
2. For each doubly transitive point \( y \in Y \) and for each ordered pair \( x, x' \) in the fiber \( \pi^{-1}(y) \) there is \( x'' \in \pi^{-1}(y) \) that is left asymptotic to \( x \) and right asymptotic to \( x' \).
3. For each right transitive point \( y \in Y \) and for each ordered pair \( x, x' \) in the fiber \( \pi^{-1}(y) \) there is \( x'' \in \pi^{-1}(y) \) that is left asymptotic to \( x \) and right asymptotic to \( x' \).
4. There is a doubly transitive point \( y \in Y \) such that for each ordered pair \( x, x' \) in the fiber \( \pi^{-1}(y) \) there is \( x'' \in \pi^{-1}(y) \) that is left asymptotic to \( x \) and right asymptotic to \( x' \).
5. There is a right transitive point \( y \in Y \) such that for each ordered pair \( x, x' \) in the fiber \( \pi^{-1}(y) \) there is \( x'' \in \pi^{-1}(y) \) that is left asymptotic to \( x \) and right asymptotic to \( x' \).

Proof. Each of the five conditions is invariant under conjugacy, and so we may assume that \( \pi \) is 1-block and \( X \) is 1-step. Among the last four conditions, the seemingly strongest condition is (3) and the seemingly weakest is (5), so it suffices to show that (5) implies (1) and that (1) implies (3).
(5) → (1)): Suppose \( y \) is a right transitive point satisfying the condition (5). Since \( y \) is right transitive, distinct transition classes over it are mutually separated, but then the condition forces all transition classes over it to be the same, i.e., there is only one transition class over \( y \). Since the class degree of \( \pi \) is the minimum number of transition classes over points in \( Y \), condition (1) follows.

((1) → (3)): Suppose class degree of \( \pi \) is 1 and let \( y \in Y \) a right transitive point and \( x, x' \) an ordered pair in the fiber \( \pi^{-1}(y) \). Since \( y \) is right transitive, there is only one transition class over it. Therefore \( x, x' \) are in the same transition class over \( y \) and so the conclusion of condition (3) follows from the definition of transition classes.

In the following lemma, we establish similar characterizations of degree \( d \) factor codes.

**Lemma 3.3.** Let \( \pi \) be a factor code from an irreducible sofic shift \( X \) onto another \( Y \) and let \( d \in \mathbb{N} \). The following are equivalent.

1. The factor code \( \pi \) is finite-to-one and its degree is \( d \).
2. For each doubly transitive point \( y \in Y \), the fiber \( \pi^{-1}(y) \) contains exactly \( d \) points.
3. There is a doubly transitive point \( y \in Y \) such that the fiber \( \pi^{-1}(y) \) contains exactly \( d \) points.

**Proof.** It is known that under the assumption of our lemma, \( \pi^{-1}(y) \) being finite for all \( y \in Y \) implies the seemingly stronger property that there is a uniform upper bound on the size of \( \pi^{-1}(y) \) over all \( y \in Y \) ([8, Theorem 8.1.19]) and this property in turn implies the property that the finite size of \( \pi^{-1}(y) \) over any doubly transitive \( y \in Y \) is the same and this size is defined to be the degree of \( \pi \) ([8, Corollary 9.1.14]). Therefore condition (1) implies (2). Condition (2) trivially implies (3).

It remains to show that (3) implies (1). Suppose \( y \) is a doubly transitive point satisfying condition (3). It is enough to show that \( \pi \) is finite-to-one. Let \( \pi_R : X_R \to X \) be the minimal right resolving presentation of \( X \). Since \( \pi_R \) is finite-to-one and \( \pi^{-1}(y) \) is finite, the fiber \( (\pi \circ \pi_R)^{-1}(y) \) is finite. So \( \pi \circ \pi_R \) is a factor code from an irreducible SFT \( X_R \) to \( Y \) with a finite fiber over some doubly transitive point. Since \( \pi \circ \pi_R \) is a factor code on an irreducible SFT with a finite fiber over at least one doubly transitive point, \( \pi \circ \pi_R \) must be finite-to-one. Therefore \( \pi \) is also finite-to-one.

Now we are ready to prove the first main theorem.

**Theorem 3.4.** Let \( \pi \) be a factor code from an irreducible SFT \( X \) onto a sofic shift \( Y \). Let \( c_\pi \) be its class degree. Then there is an irreducible sofic shift \( \tilde{Y} \) and factor codes \( \pi_1 : X \to \tilde{Y} \) and \( \pi_2 : \tilde{Y} \to Y \) such that \( \pi = \pi_2 \circ \pi_1 \) and \( \pi_1 \) has class degree 1 and \( \pi_2 \) is finite-to-one and has degree \( c_\pi \).

\(^1\) Exercise 9.1.2 in [8].
where the sliding block code (which are preimages of \(w\) of all \(\pi\)) that for each right transitive point \(\tilde{y}\) denoted by \(\pi\). We define a sliding block code \(\pi: X \to Y \times \tilde{M}^Z\) (whose image will be denoted by \(Y\)) by letting

\[
\pi_1(x) = (\pi(x), \alpha(x))
\]

where the sliding block code \(\alpha: X \to \tilde{M}^Z\) is defined in the following way. For each \(x \in X\) and \(i \in \mathbb{Z}\), let \((\alpha(x))_i = 0\) if \(\pi(x)_{i-n,i+|w|-n-1} \neq w\), otherwise let \((\alpha(x))_i\) be the unique symbol in \(M\) that the word \(x_{i-n,i+|w|-n-1}\) (which is a preimage of \(w\) via \(\pi\)) is routable through (at \(n\)).

It is easy to check that \(\pi_1\) just defined is a sliding block code. Let \(\tilde{Y}\) be its image. This image is an irreducible sofic shift in \(Y \times \tilde{M}^Z\) because \(X\) is an irreducible SFT. Define \(\pi_2: \tilde{Y} \to Y\) to be the restriction of the projection \(p_1\). It is easy to check that \(\pi = \pi_2 \circ \pi_1\). Since the composition \(\pi = \pi_2 \circ \pi_1\) is surjective the map \(\pi_2\) is surjective as well and hence \(\pi_2\) is a factor code onto \(Y\). We have obtained a decomposition \(\pi = \pi_2 \circ \pi_1\) into factor codes. It remains to show that the factor codes \(\pi_1, \pi_2\) have the desired properties.

\[\square\]

Stage 1. We construct \(\pi_1, \pi_2, \tilde{Y}\) first.

\[\square\]

Proof. We define a sliding block code \(\pi_1: X \to Y \times \tilde{M}^Z\) (whose image will be denoted by \(\tilde{Y}\)) by letting

\[
\pi_1(x) = (\pi(x), \alpha(x))
\]

where the sliding block code \(\alpha: X \to \tilde{M}^Z\) is defined in the following way. For each \(x \in X\) and \(i \in \mathbb{Z}\), let \((\alpha(x))_i = 0\) if \(\pi(x)_{i-n,i+|w|-n-1} \neq w\), otherwise let \((\alpha(x))_i\) be the unique symbol in \(M\) that the word \(x_{i-n,i+|w|-n-1}\) (which is a preimage of \(w\) via \(\pi\)) is routable through (at \(n\)).

It is easy to check that \(\pi_1\) just defined is a sliding block code. Let \(\tilde{Y}\) be its image. This image is an irreducible sofic shift in \(Y \times \tilde{M}^Z\) because \(X\) is an irreducible SFT. Define \(\pi_2: \tilde{Y} \to Y\) to be the restriction of the projection \(p_1\). It is easy to check that \(\pi = \pi_2 \circ \pi_1\). Since the composition \(\pi = \pi_2 \circ \pi_1\) is surjective the map \(\pi_2\) is surjective as well and hence \(\pi_2\) is a factor code onto \(Y\). We have obtained a decomposition \(\pi = \pi_2 \circ \pi_1\) into factor codes. It remains to show that the factor codes \(\pi_1, \pi_2\) have the desired properties.

\[\square\]

Stage 2. We claim \(\pi_1\) has class degree one.

Proof. Note \(\pi_1\) may not be a 1-block code. By Lemma 3.2, it is enough to show that for each right transitive point \(\tilde{y} \in \tilde{Y}\) and for each ordered pair \(x, x' \in \pi_1^{-1}(\tilde{y})\) there is \(x'' \in \pi_1^{-1}(\tilde{y})\) that is left asymptotic to \(x\) and right asymptotic to \(x'\).

Let \(\tilde{y} = (y, s) \in \tilde{Y}\) be right transitive and let \(x, x' \in \pi_1^{-1}(\tilde{y})\). The point \(y \in Y\) is right transitive because it is the image of right transitive \(\tilde{y}\) under the factor code \(\pi_2\). From the definition of \(\pi_1\) we have \(x, x' \in \pi^{-1}(y)\). Let \(J \subset \mathbb{Z}\) be the set of all \(i\) for which \(y_{i-n,i+|w|-n-1} = w\), or equivalently, the set of all \(i\) for which \(s_i \neq 0\). The set \(J\) marks the occurrences of the block \(w\) along \(y\). The set \(J\) is non-empty (in fact, infinite to the right) because \(y\) is right transitive. Fix one \(i_s \in J\).

From the definition of \(\pi_1\), the two blocks \(x_{i_s-n,i_s+|w|-n-1}\) and \(x'_{i_s-n,i_s+|w|-n-1}\) (which are preimages of \(w\) via \(\pi\)) are routable through the common symbol \(s_{i_s} \in M\). Using this routing, we can obtain a point \(x'' \in \pi^{-1}(y)\) that is left asymptotic to \(x\) and right asymptotic to \(x'\) and \(x''_{i_s} = s_{i_s}\). Since different transition classes over \(y\) (via \(\pi\)) must be mutually separated, \(x, x', x''\) are in the same transition.
class over \(y\). Therefore, since \(y\) is right transitive and \(x, x', x''\) are in the same transition class, for each \(i \in J\), the block \(x''_{[i-n,i+n-1]}\) is routable through \(s_i\), by Lemma 2.8. (Without using that lemma, by definition of \(x''\), it is obvious that the block \(x''_{[i-n,i+n-1]}\) is routable through \(s_i\) for \(i = i^*\) and for those \(i \in J\) with \(|i - i^*| \geq |w|\). The lemma takes care of the remaining case \(0 < |i - i^*| < |w|\) where the two occurrences of \(w\) may overlap.) Therefore, \(\pi_1(x'') = (y, s)\) and the proof of \(\pi_1\) having class degree 1 is complete.

**Stage 3.** It remains to show that \(\pi_2\) is finite-to-one and has degree \(c\).

**Proof.** Let \(y \in Y\) be doubly transitive. By Lemma 3.3, we only need to show that \(\pi_2^{-1}(y)\) contains exactly \(c\) points.

Since \(y \in Y\) is doubly transitive, there are exactly \(c\) transition classes in \(X\) over \(y\). Fix \(x^{(1)}, \ldots, x^{(c)} \in \pi_2^{-1}(y)\) to be representatives of the distinct transition classes. We have \(\pi_1(x^{(k)}) = (y, s^{(k)})\) for some \(s^{(k)} \in \hat{M}^2\) for each \(x^{(k)}\). We will show that \((y, s^{(1)}), \ldots, (y, s^{(c)})\) are distinct \(c\) points in \(\pi_2^{-1}(y)\) and that there are no other points in \(\pi_2^{-1}(y)\).

Let \(J \subseteq \mathbb{Z}\) be the set of all \(i\) for which \(y_{[i-n,i+n-1]} = w\). \(J\) is bi-infinite because \(y\) is doubly transitive. For each \(i \in J\), \((s_i^{(k)})_{1 \leq k \leq c}\) are distinct \(c\) symbols in \(M\) because transition classes over \(y\) are mutually separated. Therefore, \((y, s^{(k)})\) are distinct \(c\) points in \(\pi_2^{-1}(y)\).

It remains to show that there are no other points in \(\pi_2^{-1}(y)\). Suppose \((y, s^*)\) is in \(\pi_2^{-1}(y)\). Since \(\pi_1\) is onto, there is some \(x^{*} \in X\) such that \(\pi_1(x^{*}) = (y, s^*)\). The point \(x^{*}\) must belong to one of the \(c\) transition classes in \(\pi^{-1}(y)\). We may assume that \(x^{*}\) is in the same transition class as \(x^{(1)}\) (with respect to \(\pi\)). Therefore, since \(y\) is right transitive, for each \(i \in J\), \(s_i^{*}\) and \(s_i^{(1)}\) must be the same symbol in \(M\), by Lemma 2.8. Therefore \(s^{*} = s^{(1)}\) and we have \((y, s^{*}) = (y, s^{(1)})\). Since \((y, s^*)\) was arbitrarily chosen, we have shown that there are no points in \(\pi_2^{-1}(y)\) other than \((y, s^{(1)}), \ldots, (y, s^{(c)})\).

We have shown that the two factor codes have the desired properties, and this completes the proof of Theorem 3.4.

**Definition 3.5.** Let \(\pi : X \to Y\) be a factor code from an irreducible SFT onto a sofic shift. Any irreducible sofic shift \(\hat{Y}\) and factor codes \(\pi_1, \pi_2\) satisfying the conclusion of the theorem above are called a class degree decomposition of \(\pi\). In this case, the sofic shift space \(\hat{Y}\) is called a class degree factor of \(X\) over \(Y\) with respect to \(\pi\).

Class degree decompositions are not unique up to conjugacy in general. Depending on the choice of the minimal transition block \(w\), we may get different decompositions.

Can we strengthen the decomposition theorem by constructing a class degree factor as an SFT rather than merely sofic? Since the occurrences of the block \(w\) along \(y\) may have unbounded gaps, the class degree factor constructed from \(w\) is usually strictly sofic. But that does not rule out the possibility of an alternative construction, so we raise the following question.
**Question 3.6.** Is there a factor code \( \pi \) from an irreducible SFT \( X \) to a sofic \( Y \) such that every class degree factor is strictly sofic?

Such a factor code is necessarily infinite-to-one because otherwise \( X \) is vacuously a class degree factor.

4. **Uniqueness of relative equilibrium states over regular equilibrium states**

Let \( (X, T) \) be a TDS. Then \( M(X, T) \) denotes the set of all invariant (probability) measures on \( X \). This set is a compact metrizable space under the weak star topology (same as the vague topology in our case). If \( T \) is understood (usually when \( X \) is a shift space so that \( T \) is the shift map \( \sigma_X \) on \( X \)), then we denote it by \( M(X) \).

Given a measure \( \mu \in M(X, T) \) and a function \( \phi \in C(X) \) (where \( C(X) \) is the set of all continuous (real-valued) functions on \( X \), we denote by \( h(\mu, T) \) or \( h(\mu) \) the measure-theoretical entropy of \( \mu \) with respect to \( T \). The expression \( \mu(\phi) \) denotes the integral \( \int \phi \, d\mu \).

Given a continuous function \( \phi \) on a shift space \( X \), we say \( \phi \) is Hölder continuous if \( \text{var}_n \phi \leq Ca^n \) for some constants \( C > 0 \) and \( 0 < \alpha < 1 \), where

\[
\text{var}_n \phi := \max \left\{ |\phi(x) - \phi(x')| : x_{[-n,n]} = x'_{[-n,n]}, \ x, x' \in X \right\}.
\]

For each \( m \in \mathbb{N} \), we denote by \( S_m \phi \) the cocycle sum \( \phi + \phi \circ T + \cdots + \phi \circ T^{m-1} \).

We denote by \( P(X, T, \phi) \) (or \( P(T, \phi) \) if \( X \) is understood, or even \( P(\phi) \)) the topological pressure of \( \phi \) with respect to \( T \).

We denote by \( P(\mu, T, \phi) \) or \( P(\mu, \phi) \) the measure pressure \( h(\mu, T) + \mu(\phi) \), i.e., the free energy of \( \mu \) with respect to \( \phi \).

**Definition 4.1.** Let \( \phi \in C(X) \) where \( (X, T) \) is some TDS. Within the measures \( \mu \) in \( M(X, T) \), those measures maximizing the measure pressure \( P(\mu, \phi) \) are called \emph{equilibrium states} for the potential function \( \phi \). Equilibrium states for the constant function \( \phi = 0 \) are called \emph{measures of maximal entropy} or MMEs for short.

The variational principle for pressure states that the supremum of the measure pressure \( P(\mu, \phi) \) over all \( \mu \in M(X, T) \) is the same as the topological pressure \( P(\phi) \). Therefore, equilibrium states for \( \phi \) are precisely those \( \mu \) satisfying the equality \( P(\mu, \phi) = P(\phi) \).

It is known that if \( (X, T) \) is expansive then the pressure map \( \mu \mapsto P(\mu, \phi) \) is upper semi-continuous on the compact space \( M(X, T) \) and therefore attains a maximum. Therefore, in this case, there is at least one equilibrium state for \( \phi \). For mixing SFTs, there is a unique equilibrium state for \( \phi \) if \( \phi \) is sufficiently regular. A TDS with a unique MME is called \emph{intrinsically ergodic}. Since the unique equilibrium property for sufficiently regular functions is a generalization of this property, it makes sense to introduce the notion of intrinsically ergodicity for a class of functions, as follows.

**Definition 4.2.** Let \( X \) be a shift space and \( \mathcal{F} \subset C(X) \) be a family of continuous functions on \( X \). We say \( X \) is \emph{intrinsically ergodic for the class} \( \mathcal{F} \) if there is a
unique equilibrium state for each $\phi \in \mathcal{F}$. We say $X$ is *intrinsically ergodic for $\mathcal{F}$ with full support* if there is a unique equilibrium state for each $\phi \in \mathcal{F}$ and if that equilibrium state has full support.

Throughout this section, we suppose a class of functions $\mathcal{V}(X) \subset C(X)$ is defined for each irreducible sofic shift $X$ and satisfies the following two conditions:

1. Each irreducible sofic shift $X$ is intrinsically ergodic for $\mathcal{V}(X)$ with full support.
2. It is closed under lifting of functions (for each factor code between irreducible sofic shifts $\pi : X_1 \to X_2$, $\phi \circ \pi \in \mathcal{V}(X_1)$ whenever $\phi \in \mathcal{V}(X_2)$).

The Bowen class (consisting of all functions with the Bowen property) is an example of a fairly large class satisfying the above conditions and contains all Hölder continuous functions.

**Definition 4.3.** A function $\phi \in C(X)$ has the *Bowen property* if there is a constant $C < \infty$ such that

$$|S_n \phi(x) - S_n \phi(x')| < C$$

for all $n$ and all $x, x' \in X$ with $x_{[0,n]} = x'_{[0,n]}$.

It is easy to verify that the Bowen class satisfies the second condition (closed under lifting of functions). It is probably folklore result that it also satisfies the first condition, but for completeness we will include a proof which closely parallels a standard proof of the similar fact for Axiom A diffeomorphisms [5]. The proof works by transferring intrinsic ergodicity of mixing SFTs for the Bowen class to sofic shifts. For that, we need a quick lemma about pressure:

**Lemma 4.4.** Let $\pi : X^* \to X$ be a factor code between shift spaces. Let $\phi : X \to \mathbb{R}$ be continuous and let $\phi^* = \phi \circ \pi$. Then $P(\phi^*) \geq P(\phi)$. Equality holds if $\pi$ is finite-to-one.

**Proof.** For each $\mu^* \in M(X^*)$ and $\mu = \pi \mu^*$, we have

$$P(\mu^*, \phi^*) = P(\mu, \phi) + h(\mu^* | \mu)$$

where $h(\mu^* | \mu) = h(\mu^*) - h(\mu) \geq 0$ is the relative entropy of $\mu^*$ with respect to $\pi$. Since the pushforward map $M(X^*) \to M(X)$ is surjective, we obtain the desired inequality by applying the variational principle on $\phi^*$ and $\phi$ each. Equality in finite-to-one case follows because $h(\mu^* | \mu) = 0$ in that case. \qed

**Lemma 4.5.** Let $X$ be an irreducible sofic shift. Let $\phi : X \to \mathbb{R}$ have the Bowen property. Then $\phi$ has a unique equilibrium state. Furthermore, the unique equilibrium state has full support.

**Proof.** The lemma is known to be true in the case when $X$ is a mixing SFT.

First we assume $X$ is an irreducible SFT. Then we have the spectral decomposition $X = \bigcup_{i=1}^m X_i$, where $X_i$ are disjoint from each other and $\sigma(X_i) = X_{i+1 \mod m}$ and $(X_1, \sigma^m)$ is conjugate to a mixing SFT. A measure $\mu \in M(X, \sigma)$ induces $\mu' \in M(X_1, \sigma^m)$ by restriction to $X_1$ and conversely, any $\mu' \in M(X_1, \sigma^m)$ induces a measure $\mu \in M(X, \sigma)$ as the convex combination of copies of $\mu'$ on
Therefore \( \mu \rightarrow \mu' \) is a bijection between \( M(X, \sigma) \) and \( M(X_1, \sigma^{m_1}) \). We have 
\[
h(\mu', \sigma^{m_1}) = mh(\mu, \sigma) \quad \text{and} \quad \mu'(S_m \phi) = m \mu(\phi).
\]
Therefore, finding \( \mu \) maximizing \( P(\mu, \phi) \) is equivalent to finding \( \mu' \) maximizing \( P(\mu', S_m \phi) \). Also, \( \mu \) has full support iff \( \mu' \) has full support. Since \( S_m \phi \) restricted to \( X_1 \) satisfies the Bowen property and \( (X_1, \sigma^{m_1}) \) is a mixing SFT, we are done. We have shown that any irreducible SFT is intrinsically ergodic for the Bowen class with full support. 

The strictly sofic case remains. Let \( \pi : X^* \rightarrow X \) be the minimal right resolving presentation of \( X \). In particular, \( X^* \) is an irreducible SFT and \( \pi \) has degree one. Let \( \phi^* = \phi \circ \pi \). The function \( \phi^* : X^* \rightarrow \mathbb{R} \) has the Bowen property.

As the first part of this proof showed, we have a unique equilibrium state \( \mu_{\phi^*} \) for \( \phi^* \). Since \( \mu_{\phi^*} \) is a fully supported ergodic measure on \( X^* \), the set of doubly transitive points is a full measure set with respect to \( \mu_{\phi^*} \). Let \( \mu_{\phi} = \pi_* (\mu_{\phi^*}) \). Then \( \mu_{\phi} \) is an invariant measure on \( X \). It has full support because \( \pi \) is onto and \( \mu_{\phi^*} \) has full support. The measure preserving systems arising from \( \mu_{\phi} \) and \( \mu_{\phi^*} \) are conjugate because \( \pi \) is one-to-one except on a \( \mu_{\phi^*} \)-null set, namely, the complement of the set of the doubly transitive points in \( X^* \). (\( \pi \) is one-to-one on doubly transitive points because \( \pi \) has degree one.) In particular, \( h(\mu_{\phi}) = h(\mu_{\phi^*}) \) and \( \mu_{\phi}(\phi) = \mu_{\phi^*}(\phi^*) \). Therefore, we have
\[
P(\mu_{\phi}, \phi) = P(\mu_{\phi^*}, \phi^*) = P(\phi^*) \geq P(\phi)
\]
Hence \( \mu_{\phi} \) is an equilibrium state for \( \phi \).

Suppose \( \mu \) is any other equilibrium state of \( \phi \). Pick an invariant measure \( \mu^* \) on \( X^* \) with \( \pi_* \mu^* = \mu \). Then \( h(\mu^*) \geq h(\mu) \) and we have
\[
P(\mu^*, \phi^*) \geq P(\mu, \phi) = P(\phi) = P(\phi^*)
\]
Therefore \( \mu^* \) is an equilibrium state for \( \phi^* \) and by uniqueness we have \( \mu^* = \mu_{\phi^*} \). Then \( \mu = \pi_* (\mu_{\phi^*}) = \mu_{\phi} \).

Next, we establish the unique lift property of regular equilibriums via finite-to-one factor codes. Tuncel [10] proved this for Markov measures on SFTs, but the same proof works for equilibrium states of regular potential functions on irreducible sofic shifts. We reproduce the proof shortly in our notation in the following lemma.

**Lemma 4.6.** Let \( \pi : X \rightarrow Y \) be a finite-to-one factor code between irreducible sofic shift spaces. Let \( \psi \in \mathcal{V}(Y) \) and let \( \mu_{\psi} \) be its unique equilibrium state. Then there is a unique invariant measure in \( M(X) \) that projects to \( \mu_{\psi} \). The unique measure is the unique equilibrium state for \( \psi \circ \pi \).

**Proof:** Since \( \pi \) is finite-to-one, we have \( P(\psi) = P(\psi \circ \pi) \). Let \( \mu \) be the unique equilibrium state for \( \psi \circ \pi \). This is unique because \( \psi \circ \pi \in \mathcal{V}(X) \). Its image \( \pi_* \mu \) is an equilibrium state for \( \psi \) and so the image must be \( \mu_{\psi} \).

Let \( \mu' \) be another measure in \( M(X) \) whose image is \( \mu_{\psi} \). Then
\[
P(\mu', \psi \circ \pi) = P(\mu_{\psi}, \psi) = P(\psi) = P(\psi \circ \pi).
\]
Therefore \( \mu' \) is an equilibrium state for \( \psi \circ \pi \), but the equilibrium state is unique.
**Definition 4.7.** Let \( \pi : X \to Y \) be a factor map between two TDSs \((X, T)\) and 
\((Y, S)\). Let \( \nu \in M(Y, S) \). A measure \( \mu \in M(X, T) \) is called a measure of maximal relative entropy (MMRE) over \( \nu \) if it maximizes the entropy \( h(\mu) \) subject to the constraint \( \pi_* \mu = \nu \).

**Definition 4.8.** Let \( \pi, \nu \) be as in the previous definition. Let \( \phi \in C(X) \). A measure \( \mu \in M(X, T) \) is called a relative equilibrium state of \( \phi \) over \( \nu \) if it maximizes the measure pressure \( P(\mu, \phi) \) subject to the same constraint \( \pi_* \mu = \nu \).

Now we fix another class of functions. Throughout this section, suppose a class of functions \( \mathcal{U}(X) \subset C(X) \) is defined for each irreducible SFT \( X \) such that the following property hold. For each class degree one factor code \( \pi_1 : X \to Y \) onto a sofic shift and each fully supported ergodic measure \( \tilde{\nu} \) on \( Y \) and each \( \phi \in \mathcal{U}(X) \), there is a unique relative equilibrium state of \( \phi \) over \( \tilde{\nu} \).

This property holds for \( \mathcal{U}(X) = \{0\} \) (see [2]) and this old special case alone already generates a new result if combined with our first main theorem. Recently it was shown that this property also holds for \( \mathcal{U}(X) = \) the Bowen class (see [1]).

We are ready to apply the lemmas so far to prove the second main theorem, which is a consequence of the first main theorem.

**Theorem 4.9.** Let \( \pi : X \to Y \) be a factor code from an irreducible SFT onto a sofic shift. Let \( \phi \in \mathcal{U}(X), \psi \in \mathcal{V}(Y) \). Let \( \nu \in M(Y) \) be the unique equilibrium state for \( \psi \). Then there is a unique relative equilibrium state \( \mu \) of \( \phi \) over \( \nu \).

*Proof.* By expansivity of the shift map for \( X \), the pressure map \( \mu \mapsto P(\mu, \phi) \) is upper semi-continuous. Since the measure fiber \( \pi_*^{-1}(\nu) \) is a compact subset of \( M(X) \), there is at least one \( \mu \) in it maximizing the measure pressure. In other words, there is at least one relative equilibrium state of \( \phi \) over \( \nu \).

Let \( \mu \) be any measure in \( M(X) \) that projects to \( \nu \). Let \( \tilde{Y}, \pi_1, \pi_2 \) be a fixed class degree decomposition for \( \pi \). Let \( \tilde{\nu} \) be the image of \( \mu \) on \( \tilde{Y} \). Since \( \mu \) projects to \( \nu \) on \( Y \), the measure \( \tilde{\nu} \) must project to the same measure \( \nu \). By uniqueness in Lemma 4.6, \( \tilde{\nu} \) is the unique lift of \( \nu \) to \( \tilde{Y} \). In particular, \( \tilde{\nu} \) does not depend on \( \mu \).

The measure \( \tilde{\nu} \) is fully supported and ergodic because it is the unique equilibrium state of \( \psi \circ \pi_2 \in \mathcal{V}(\tilde{Y}) \).

The measure fiber on \( X \) over \( \nu \) and the measure fiber over \( \tilde{\nu} \) are the same subsets of \( M(X) \). Therefore, \( \mu \) is a relative equilibrium state of \( \phi \) over \( \nu \) if and only if it is a relative equilibrium state of \( \phi \) over \( \tilde{\nu} \). But the latter relative equilibrium state is unique, since \( \phi \in \mathcal{U}(X) \) and \( \pi_1 \) has class degree one and \( \tilde{\nu} \) is fully supported.
The proof is complete, but for a later remark, we let \( \pi_R : R \to \tilde{Y} \) be the minimal right resolving presentation of \( \tilde{Y} \). By applying Lemma 4.6 to \( \pi_R \) and \( \tilde{\nu} \), we obtain the unique measure \( \rho \in M(R) \) that projects to \( \tilde{\nu} \). The measure \( \rho \) is the unique equilibrium state of \( \psi \circ \pi_2 \circ \pi_R \in V(R) \).

We state the special case \( \phi = 0 \) (i.e., the case \( \mu(X) = \{0\} \)) of the above main theorem as follows.

**Corollary 4.10.** Let \( \pi : X \to Y \) be a factor code from an irreducible SFT onto a sofic shift. Let \( \nu \) be the unique equilibrium state of some function in \( V(Y) \). Then there is a unique measure of maximal relative entropy on \( X \) over \( \nu \).

This answers [6, Problem 3.16] positively because Markov measures on SFTs are equilibrium states of locally constant functions and such functions have the Bowen property.

We remark that the problem of obtaining a concrete description of the unique MMRE is still open. It would be nice to have a description concrete enough to prove that \( \mu \) is Bernoulli for instance. But the question of whether \( \mu \) is always Bernoulli is also open.

If \( \nu \) is Markov, then \( \rho \) (in the proof) is also Markov. So \( \tilde{\nu} \) is a hidden Markov measure. Since \( \tilde{Y} \) is usually strictly sofic, we cannot say that \( \tilde{\nu} \) is Markov. Nonetheless, the MMRE \( \mu \) over the Markov \( \nu \) is the MMRE over the hidden Markov \( \tilde{\nu} \) with respect to a class degree one map. In order to describe \( \mu \), we only need to describe MMREs over hidden Markov measures under class degree one maps. For certain class degree one maps (maps with singleton clumps [9] for instance), this is doable. For arbitrary class degree one factor codes, this seems to require further investigation, but it is hoped that this will turn out to be easier than using the original arbitrary infinite-to-one factor code directly.

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