Effective action in a quantized metric

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Abstract

We calculate the effective action in Yang-Mills and scalar $\phi^4$ quantum field theory with quantized scale invariant metric treated nonperturbatively in $d = 4$ dimensions. There is no charge renormalization in the one-loop order for matter fields. We show that the electromagnetic energy of point charges can be finite. The temperature dependence of the effective action in inflationary models is changed substantially as a result of an interaction with quantum gravity.

1 Introduction

The effective action is derived as a result of an integration over quantum fields with some classical sources. At the same time it can be given a meaning of the classical action whose tree approximation takes into account the quantum effects. It gives a useful illustration of the quantum corrections to classical phenomena. In this paper we show that the form of the effective action for fields interacting with gravity follows from the scaling properties of quantum gravity. We assume that quantum gravity is scale invariant at short distances with the scale dimension $\gamma$. When expressed in terms of the metric $g$ this means that $\lambda^{2\gamma} g^{\mu\nu}(\lambda x)$ and $g^{\mu\nu}(x)$ have the same correlation functions. We calculate the 1-loop generating functional for matter fields in an external gravitational field. Then, the (quantum) correlation functions of the gravitational field are assumed to be scale invariant. An approximate evaluation of an average of the generating functional over the gravitational field leads (after the Legendre transformation) to an effective action which is substantially different from the perturbative one. The inverse propagator grows faster in momenta than the free one. Moreover, the 1-loop corrections to the effective action do not receive the logarithmic corrections characteristic to the renormalizable quantum field theories.
2 The scalar heat kernel

We repeat some steps of our earlier paper [1] where an interaction with a scale invariant gravity has been discussed. We express an average of the heat kernel of the Laplace-Beltrami operator \(2A\) over the metric \(g\) by means of the path integral

\[
\langle K_\tau(x,y) \rangle = \langle \exp(\tau A)(x,y) \rangle = \int Dg \exp \left( -\int dx L_g(g(x)) \right)
\]

where \(L_g\) is the gravitational Lagrangian. In order to obtain the behavior of \(\langle K_\tau(x,y) \rangle\) for a small \(\tau\) it is sufficient to assume the scale invariance of the metric \(g\) at short distances. We can also derive this short time asymptotics directly from the functional integral (1) assuming a scale invariance of the action

\[
\int dx L_g(\lambda^2 g(\lambda x)) = \int dx L_g(g(x)) \tag{2}
\]

In fact, introducing in eq.(1) a new functional integration variable \(\tilde{g} = \lambda^{2\gamma} g(\lambda x)\) and a new path \(\tilde{q}\) defined on an interval \([0, 1]\) by

\[
q(\tau s) - x = \tau^\sigma (\tilde{q}(s) - x) \tag{3}
\]

where

\[
\sigma = \frac{1}{2}(1 + \gamma)^{-1}
\]

we can explicitly extract the \(\tau\)-dependence of \(\langle K_\tau(x,y) \rangle\)

\[
\langle K_\tau(x,y) \rangle = \langle E[\delta(y - x - \tau^\frac{1}{2\gamma}(\tilde{q}(1)))] \rangle \tag{4}
\]

where \(E[.]\) denotes an expectation value over the paths \(\tilde{q}\). Then, an average over the propagator (in the proper time representation) behaves as

\[
\langle A^{-1}(x,y) \rangle = \int_0^\infty d\tau \langle K_\tau(x,y) \rangle \simeq |x - y|^{-2 + 2\gamma} \tag{5}
\]

For the effective action we need only the diagonal of the heat kernel. From eq.(4)

\[
\langle K_\tau(x,x) \rangle = \tau^{-\frac{1}{2\gamma}} \langle E[\delta(\tilde{q}(1))]) \rangle \equiv \tau^{-\frac{1}{2\gamma}} v(x) \tag{6}
\]

where the mean value \(v\) depends only on \(x\).

3 The electromagnetic heat kernel

For a calculation of the effective action we need the heat kernel of an operator arising from the electromagnetic Lagrangian

\[
W = \frac{1}{4} \int d^4x \sqrt{g} g^{\mu\nu} g^{\sigma\rho} F_{\mu\sigma} F_{\nu\rho} \tag{7}
\]
We choose the Feynman gauge which results from an addition to the action of the term

\[ W_0 = \frac{1}{2} \int d^4x (g^{\mu\nu} \partial_\nu (\sqrt{g} A_\mu))^2 \]  

(8)

We write \( \tilde{W} = W + W_0 \) as

\[ \tilde{W} = \frac{1}{2} \int d^4x \tilde{A}_a (g^{\mu\nu} (-\partial_\mu \delta_{ac} + \omega^c_{\mu a}) (\partial_\nu \delta_{cb} + \omega^b_{\mu c}) + R_{ab}) \tilde{A}_b \equiv \frac{1}{2} \int \tilde{A} \Delta_{EM} \tilde{A} \]  

(9)

where \( g^{\mu\nu} = e^{\mu a} e^{\nu a} \), \( \tilde{A}_a = e^{\mu a} A_\mu \), \( R_{ab} \) is the Ricci tensor and the spin connection \( \omega \) is just a transformation of the Christoffel symbol to the fixed frame

\[ \omega^a_{\mu c} = e^\nu c \Gamma^\sigma_{\mu \nu l} a + l^o \nu \partial_\mu e^\nu c \]

where \( l \) is the inverse matrix to \( e \). We are interested in the heat equation

\[ \partial_{\tau} \tilde{A} = \frac{1}{2} \Delta_{EM} \tilde{A} \]  

(10)

The solution can be expressed in the form [3]

\[ \hat{A}(\tau, x) \equiv (T_\tau \hat{A})(x) = E[\mathcal{T}(\tau) \hat{A}(q_\tau(x))] \]  

(11)

where \( \mathcal{T}(\tau) \) is a solution of the equation

\[ dT_{ac} = \omega^b_{\mu a} T_{bc} dq^\mu + \frac{1}{2} R_{ab} T_{bc} d\tau \]  

(12)

From eq.(11) (cp. with eq.(6))

\[ \langle K_\tau(x, x) \rangle = \langle E[\mathcal{T}(\tau) \delta (q_\tau)] \rangle \]  

(13)

We can apply now the scaling of sec.2 in order to conclude that

\[ \langle K_\tau(x, x) \rangle = \tau^{-\frac{2}{\nu}} v(\tau, x) \]  

(14)

where \( v(\tau, x) \) is equal to (with the notation as in eq.(6))

\[ v(\tau, x) = \langle E[\mathcal{T}(\tau) \delta (\tilde{q}(1))] \rangle \]

It is a regular function of \( \tau \).

4 The effective action for the \( \Phi^4 \) scalar field

We wish to calculate the generating functional (where \( A \) may denote either the gauge field or the scalar field)

\[ Z[J, \Theta] = \int DA Dg \exp \left( -\frac{1}{\hbar} (L + JA + g\Theta) \right) \]  

(15)
In the conventional approach we make a shift $A \rightarrow A + B$ and $g \rightarrow g + h$. We choose $B$ and $h$ as solutions of classical equations. Then, the linear terms in $A$ and $g$ vanish. The integration over the quadratic term in $A$ gives a determinant. We restrict ourselves to the effective action for the matter fields in the 1-loop approximation. Then, $\Theta = 0$ and $h = 0$. The standard loop expansion is an expansion in the Planck constant $\hbar$. We assume that $\hbar$ is small but the gravitational coupling $\kappa$ is large and of the order $1/\hbar$. This assumption is a mathematical trick which allows us to treat gravity beyond the one loop in order to see its non-perturbative effects. It can however have a physical interpretation. The parameters $\kappa$ and $\hbar$ have a dimension. Hence, large and small depends on the physical context. A small length $\sim \sqrt{\hbar}$ (with small $\hbar$ and large $\kappa$) means that it is small in comparison to the Planck length $\sim \sqrt{\kappa \hbar}$.

We discuss first the scalar field Lagrangian (it is technically simpler and may be relevant for inflationary models). Let

$$L = \frac{1}{2} g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi - \frac{\alpha}{6} \Phi^4$$

So, taking only diagrams with one scalar loop we obtain

$$Z[J] = \langle \exp \left( -\frac{1}{\hbar} W(\phi_c) \right) \det \mathcal{M}^{-\frac{1}{2}} \rangle$$

where

$$\mathcal{M} = -A + \alpha \phi_c^2$$

and $\phi_c$ is the solution of the classical equation

$$-A\phi_c + \frac{1}{3} \alpha \phi_c^3 = \frac{1}{2} J$$

$2A$ is the Laplace-Beltrami operator. $W(\phi_c)$ denotes the classical action. Clearly, it is not simple to calculate the average in eq.(16) exactly. We can apply the cummulant expansion

$$\langle \exp U \rangle = \exp \left( \langle U \rangle + \frac{1}{2} \langle (U - \langle U \rangle)(U - \langle U \rangle) \rangle + ... \right)$$

The first order approximation in eq.(16) is

$$Z[J] = \exp \left( -\frac{1}{\hbar} (W(\phi_c)) - \frac{1}{2} \langle \text{Tr} \ln \mathcal{M} \rangle \right)$$

We write

$$W_q = \frac{1}{2} \langle \text{Tr} \ln \mathcal{M} \rangle = -\frac{1}{2} \int dx \int_0^\infty d\tau \frac{d\tau}{\tau} \langle \exp (\tau \mathcal{M})(x, x) \rangle$$

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Applying eq.(3) and (20) we obtain

\[ W_q = -\frac{1}{2} \int dx \int_0^\infty \frac{d\tau}{\tau} \tau^{-\frac{d-2}{d-4}} \langle E[\delta(\hat{q}(1))] \exp \left( -\tau \int_0^1 ds \alpha \phi_c^2(x + \tau^\sigma (\hat{q}_s - x)) \right) \rangle \]  

(21)

\( W_q \) as written is infinite. In order to define a finite expression we make the \( \zeta \)-function regularization replacing \( \frac{d\tau}{\tau} \) by \( \frac{d\tau}{\tau^{1-z}} \). Then, \( \frac{d\tau}{\tau^{1-z}} M_{|z=0} \) which can be considered as a definition of \( \text{Tr} \ln M \) is well-defined. Up to a finite renormalization the \( \zeta \)-function definition is equivalent to a definition by a counterterm \[ W_q = -\frac{1}{2} \int dx \int_0^\infty \frac{d\tau}{\tau} \tau^{-\frac{d-2}{d-4}} \langle E[\delta(\hat{q}(1))] \exp \left( -\tau \int_0^1 ds \alpha \phi_c^2(x + \tau^\sigma (\hat{q}_s - x)) \right) \rangle \]  

(22)

The behavior for a large \( \phi \) depends on a small \( \tau \). Then, we can use the approximation \( \hat{q}_s - x \approx 0 \) in eq.(22). In such a case

\[ W_q \simeq -\int dx \langle E[\delta(\hat{q}(1))] \int_0^\infty \frac{d\tau}{\tau} \tau^{-\frac{d-2}{d-4}} \exp(\tau^\sigma (\hat{q}_s - x)) \rangle \]  

(23)

Note that with \( \gamma > 0 \) the subtraction of the term \( \phi_c^4(x) \) in eq.(22) is unnecessary (there is no coupling constant renormalization). The generating functional (23) has the same form as the one resulting from a \( \Phi^4 \) theory in \( d = 4/(1 + \gamma) \) dimensions. In principle, we can solve eq.(18) as a power series in \( J \) and calculate the expectation value over the gravitational field again in a power series in \( J \). In this way we obtain the generating functional \( Z[J] \) for the correlation functions \( \langle \Phi(x_1)\ldots\Phi(x_n) \rangle \).

It is interesting to do it in the zeroth order in \( \alpha \). Then,

\[ Z[J] = \langle \exp \left( \frac{1}{4} \int J A^{-1} J \right) \rangle \simeq \exp \left( \frac{1}{4} \int J \langle A^{-1} \rangle J \right) \]  

(24)

The r.h.s. of eq.(24) will give the correct two-point function for the scalar field in a quantum gravitational field but only an approximate formula for \( n \)-point correlation functions (although the short distance behavior will be the same).

The effective action \( \Gamma \) is defined as

\[ \Gamma(\phi) = \ln Z[J] - J \phi \]

where

\[ \phi(x) = \frac{\delta \ln Z[J]}{\delta J(x)} \]  

(25)
Hence, in order to obtain the effective action from the generating functional \( Z[J] \) we need to express \( J \) by \( \phi \) from eq.(25). In the zeroth order in \( \alpha \) with the approximation (24) we obtain

\[
\Gamma(\phi) = \frac{1}{4} \int \phi (\langle A^{-1} \rangle)^{-1} \phi
\]

where \((\langle A^{-1} \rangle)^{-1}\) denotes the kernel of an operator inverse to the operator determined by the kernel \( \langle A^{-1}(x,y) \rangle \) (in momentum space \( \langle A^{-1}(k) \rangle \sim k^{2+2\gamma} \)).

When, \( \alpha > 0 \) the inversion is more complicated even with the approximation (19). We have to express \( \langle W(\phi_c) \rangle \) by \( J \). In perturbation theory this is a sum of terms of the form

\[
\langle W(\phi_c) \rangle = \sum \int dx_1 \ldots dx_{2n} \langle A^{-1}(x_1, x_2) \ldots A^{-1}(x_{2n-1}, x_{2n}) \rangle \prod_{k,l} \delta(x_k - x_l) J(x_1) \ldots J(x_r)
\]

where the number of \( \delta \)-functions and \( r \) are adjusted so that we obtain 2\( n \) coordinates altogether. Then, we have to take the Legendre transform (25) of \( \langle W(\phi_c) \rangle \) in order to obtain \( \Gamma_{\text{tree}}(\phi) \). The difference between \( W(\phi) \) and \( \Gamma_{\text{tree}}(\phi) \) is in the replacement of \( A^{-1}(x_1, x_2) \ldots A^{-1}(x_{2n-1}, x_{2n}) \) by \( \langle A^{-1}(x_1, x_2) \ldots A^{-1}(x_{2n-1}, x_{2n}) \rangle \) (and an inversion of the averaged kernels). For the 1-loop term we make the following approximations

\[
\langle W_q(\phi_c) \rangle \simeq W_q(\langle \phi_c \rangle) \simeq W_q(\frac{1}{2} \langle A^{-1} \rangle J) \simeq W_q(\phi)
\]

where the last step follows from eq.(25). Hence, with these approximations

\[
\Gamma(\phi) = \Gamma_{\text{tree}}(\phi) + \hbar W_q(\phi)
\]

In the inflationary models \( \| \) we need the effective potential at finite temperature. We consider a time-independent (three-dimensional) perturbation of the classical expanding metric. In the comoving frame (moving with the speed of the expansion) we may treat the resulting metric as a static metric on a three-dimensional manifold. A calculation of the effective potential for a static metric is a straightforward generalization of the one in eq.(20). We have just to replace \( (\exp \tau M)(x, x) \) by a sum over integer \( n \) of \( (\exp \tau M)(x_0 + n\beta, x; x_0, x) \) (periodic boundary conditions in time, here \( \beta^{-1} = KT \), where \( T \) is the temperature and \( K \) is the Boltzman constant). The part involving \( x_0 \) separates and gives just a factor \( (2\pi \tau)^{-\frac{3}{2}} \exp(-\frac{n^2 \beta^2}{2\tau}) \). The sum over \( n \) leads to a formula resembling the standard one (the summation method and the result are similar to the standard flat case \( \| \) )

\[
W_{q,\beta}(\phi) = W_{q,\infty} + W_{q,\beta}^{(1)} = W_{q,\infty} + \beta^{-1} Tr \ln(1 - \exp(-\beta M))
\]

where \( W_{q,\infty} \equiv W_q \) is defined in eq.(22) and

\[
M^2 = \frac{1}{2} \Delta_3 + \alpha \phi^2 = -\frac{1}{2} g^{kl} \partial_k \partial_l - \frac{1}{2} \Gamma^l \partial_l + \alpha \phi^2
\]
\( \Delta_3 \) denotes the threedimensional Laplace-Beltrami operator and \( \Gamma \) is the Christoffel symbol. We must take an average over the quantum gravitational field in eq.(26). We need some approximations in order to perform this difficult task. We set

\[
\langle Tr \ln (1 - \exp(-\beta M)) \rangle \simeq Tr \ln (1 - \langle \exp(-\beta M) \rangle) \tag{27}
\]

With this approximation a calculation of the effective potential at finite temperature is already simple. We may use the formula

\[
\langle \exp(-\beta M) \rangle = \frac{1}{\pi} \int_0^\infty dp \exp(ip\beta) \int_0^\infty ds_1 ds_2 s_1^{-\frac{3}{2}} \exp(-s_2p^2) \left( \exp(-s_1M^2) - \exp(-s_2M^2) \right)
\]

As follows from eq.(6) the diagonal part of the heat kernel in quantum gravitational field in \( d \) dimensions is the same as the one without the gravitational field but in \( d(1 + \gamma)^{-1} \) dimensions. Applying the Dolan-Jackiw formula [4] to \( 3(1 + \gamma)^{-1} \) dimensions we obtain the following integral representation of the 1-loop effective action

\[
W^{(1)}_{\Phi,\beta} = 2\beta^{-\frac{2}{1+\gamma}-1} \int_0^\infty duu^{-\frac{1}{1+\gamma}-1} \ln \left( 1 - \exp(-\sqrt{u^2 + \beta^2\phi^2}) \right) \tag{28}
\]

It follows from eq.(28) that there is no logarithmic correction to the effective action characteristic of renormalizable models. As can be seen from eq.(28) the expansion for high temperature (and small values of the field) starts with the term

\[
\beta^{-\frac{2}{1+\gamma}} \phi^2
\]

The temperature dependence of the phase transition in inflationary models can be derived either from this expansion or from the formula for the correlation function at high and intermediate temperature

\[
\langle \Phi(t, x)\Phi(t, y) \rangle \simeq \frac{1}{2} \left( M^{-1} \exp(-\beta M) \right)(x, y) = \frac{1}{2\pi} \int_0^\infty d\tau \int dp \left( \exp(-\frac{\tau}{2}p^2 - \frac{\tau}{2}M^2) \right)(x, y) \exp(ip\beta)
\]

To the r.h.s. we can apply the method of sec.2 which leads to the formula

\[
\langle \Phi(t, x) \rangle^2 \sim \beta^{-\frac{2}{1+\gamma}}
\]

Let us note that also the temperature dependence of the energy-momentum tensor is modified by quantum gravity

\[
\langle T_{\mu\nu} \rangle \sim \beta^{-\frac{2}{1+\gamma}}
\]

The Planck spectrum of particles at temperature \( T \) will be deformed to

\[
\rho(E) \simeq E^\frac{3}{1+\gamma} \left( \exp(\beta E) - 1 \right)^{-1}
\]

We can conclude that the temperature-dependence of the phase transition in inflationary models will be modified by an inclusion of quantum gravity.
5 The effective action for gauge fields

We consider next the non-Abelian gauge theory at zero temperature. With the same approximations as for the scalar field we obtain

\[ Z[J] = \exp \left( -\frac{1}{\hbar} (W(A_{cl})) - \frac{1}{2} (\text{Tr} \ln \mathcal{M}) + \langle \text{Tr} \ln \mathcal{M}_{FP} \rangle \right) \]  

(29)

where \( A_{cl} \) is a solution of the Yang-Mills equations with an external source \( J \) (we shall omit the index \( cl \) furtheron), \( \det \mathcal{M}_{FP} \) is the Fadeev-Popov determinant (Feynman gauge) and

\[ \mathcal{M}^{ab}_{\mu \nu} = g_{\mu \nu} \sigma^{ab} D_{\mu \nu}^{ac} D_{\rho}^{a} + 2 f^{abc} F_{\mu \nu}^{c} + R_{\mu \nu} \delta_{ab} \]  

(30)

We obtain a path integral representation as in secs. 2-3

\[ W_q = \frac{1}{2} \text{Tr} \ln \mathcal{M} = -\frac{1}{4} \int dx \int_{0}^{\infty} d\tau \left( \frac{du}{\sinh(u)} \right) \left( \frac{du}{\sinh(u)} \right) \]  

(31)

where \( T \) is the solution of the equation

\[ dT_{apcr} = A^{m}_{\mu} f_{psm} \mathcal{T}_{ascr} dq^{\mu} + \omega^{b}_{\mu} \mathcal{T}_{bpcr} dq^{\mu} + \frac{1}{2} R_{ab} \mathcal{T}_{bpcr} d\tau + f_{psm} F^{m}_{ab} \mathcal{T}_{bscr} d\tau \]  

(32)

In order to perform the integral over the gravitational field in eq.(29) we need some rough approximations. First, in the stochastic representation of the trace (29) we apply the same method as in eq.(22) writing \( q_s = x + \tau (q_s - x) \). Then, for small \( \tau \) we may assume that \( F(q_s) \) depends only on \( x \). Next, we assume that only one component of \( F \) is different from zero (e.g. \( f_{psm} F^{m}_{ab} = \epsilon_{ps3} F^{3} \) ). There remains the main difficulty to take an average of \( T \) over the gravitational field. In these computations we make the approximation \( \omega \simeq \langle \omega \rangle \simeq 0 \), \( q_{\mu} \simeq b_{\mu} \) and set \( R_{\mu \nu} \simeq \langle R_{\mu \nu} \rangle = Q \delta_{\mu \nu} \). Then, we can calculate \( W_q \) exactly

\[ W_q = -\frac{1}{2} \int dx v(x) \int_{0}^{\infty} d\tau \left[ \frac{du}{\sinh(u)} \right] \left( \left. \left( F_{\mu \nu} F_{\mu \nu} + F_{\mu \nu}^{*} F_{\mu \nu}^{*} \right) \right|_{q_{\mu}}^{q_{\mu}} + \frac{1}{2} R_{\mu \nu} \right] \]  

(33)

where

\[ u_{\pm} = \frac{1}{4\sqrt{2}} \left( (F_{\mu \nu} F_{\mu \nu} + F_{\mu \nu}^{*} F_{\mu \nu}^{*}) \right) \pm \left( F_{\mu \nu} F_{\mu \nu} - F_{\mu \nu}^{*} F_{\mu \nu}^{*} \right) \]  

(34)

It follows from the behavior of the integrand (33) for a small \( \tau \) (assuming that the integral (33) is convergent for a large \( \tau \) that for large \( F \)

\[ W_q \approx F^{2} \frac{\tau^{2}}{\tau^{2}} \]  

(35)
In the tree approximation we have to perform the Legendre transform in order to obtain the effective action from the generating functional (see the discussion at the scalar field). In the Abelian case we would obtain

$$\Gamma_{tree}(A) \simeq \frac{1}{4} \int A(\mathcal{M}^{-1})^{-1} A$$

(36)

Note that the behavior (5) for the $\langle AA \rangle$ correlations means the behavior $\sim |k|^{-2\gamma}$ for the $\langle F(k)F(k) \rangle$ correlations (the spectral function). This can lead to the simplest experimental check of the predictions of quantum gravity if the photon spectral density would be measured in astrophysical observations. Summarizing, for these values of the fields where the $\tau$ integral in eq.(33) is convergent the effective action consists of the tree part which grows quadratically for large fields $F$, and the 1-loop part with slower growth (35).

On the basis of the generating functional (24) we can give an argument for a finiteness of the electromagnetic energy. From eq.(5) in the static approximation it follows that the potential between point charges is $V(r) \simeq r^{-1+2\gamma}$. Hence, the electric energy density $\epsilon \simeq r^{-4+4\gamma}$. The electromagnetic energy of the point charge (which includes the interaction energy between the charge and the gravitons) will be finite if $\gamma > \frac{1}{4}$.

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