Construction of Lie Algebras and Invariant Tensors through Abelian Semigroups

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Abstract. The Abelian Semigroup Expansion Method for Lie Algebras is briefly explained. Given a Lie Algebra and a discrete abelian semigroup, the method allows us to directly build new Lie Algebras with their corresponding non-trivial invariant tensors. The Method is especially interesting in the context of M-Theory, because it allows us to construct M-Algebra Invariant Chern–Simons/Transgression Lagrangians in \( d = 11 \).

1. Introduction
In the context of M-Theory, there is an interesting interplay between different supersymmetries. As a matter of fact, the M-Algebra and the \( \mathfrak{osp}(32|1) \) algebra are related through the Maurer–Cartan forms power-series expansion procedure (See Refs. [1, 2]). This procedure is formulated in terms of the Maurer–Cartan forms of the Lie Algebra. Due to the fact that the invariant tensor is written using the generators of the algebra, finding an explicit expression for the M-Algebra invariant tensor (and therefore a Chern–Simons Lagrangian) becomes non-trivial.

This problem has been treated through several approaches in the past; for example, in Refs. [1, 2], the relationship between expansion and Chern–Simons forms has been treated through free-differential algebras. In Refs. [5, 6] the problem has been treated applying the Noether Method on Chern–Simons forms; other approaches can be found in Refs. [4, 3].

All these approaches have focused directly in the construction of the Chern–Simons form for the expanded algebra. Here we will face the problem in a different way: we show a new general method for Lie Algebra manipulation in terms of the generators: the \( S \)-Expansion, which provides us automatically with expressions for a non-trivial invariant tensor. From it, the construction of a Chern–Simons/Transgression form for the algebra becomes straightforward. The \( S \)-Expansion method requires as input an abelian semigroup \( S \) and a Lie algebra \( \mathfrak{g} \), and gives as output a new, and in general larger, symmetry \( \mathfrak{G} \). More details on the \( S \)-Expansion method can be found in Refs. [7, 9, 10]; for the construction of a gauge theory for the M-Algebra in \( d = 11 \) using this method, see Refs. [8, 9, 10].

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2. Making Algebras Smaller: Reduction
Let \( g \) be a Lie algebra of the form \( g = V_0 \oplus V_1 \), with \( \{ T_{a_0} \} \) being the generators of \( V_0 \) and \( \{ T_{a_1} \} \) the ones of \( V_1 \). When \( [V_0, V_1] \subset V_1 \), i.e., when

\[
[T_{a_0}, T_{b_0}] = C_{a_0b_0}^{c_0} T_{c_0} + C_{a_0b_1}^{c_1} T_{c_1},
\]

\[
[T_{a_0}, T_{b_1}] = C_{a_0b_1}^{c_1} T_{c_1},
\]

\[
[T_{a_1}, T_{b_1}] = C_{a_1b_1}^{c_1} T_{c_1},
\]

then it is straightforward to show that the structure constants \( C_{a_0b_0}^{c_0} \) satisfy automatically the Jacobi identity by themselves. Therefore,

\[
[T_{a_0}, T_{b_0}] = C_{a_0b_0}^{c_0} T_{c_0}
\]

also correspond to a Lie algebra. This algebra, with structure constants \( C_{a_0b_0}^{c_0} \), will be called reduced algebra of \( g \), and denoted by \( |V_0| \).

It is important to notice that in general \( |V_0| \) does not correspond to a subalgebra. In some way it could be regarded as an “ideal division” or “inverse extension” but we have to notice that \( V_1 \) in general does not need to be an ideal.

3. Making Algebras Bigger: S-Expansion
In order to construct larger Lie Algebras, the key ingredients in the present approach are a semigroup \( S \) and a Lie algebra \( g \). A semigroup \( S \) is a set provided with a closed, associative product. From now on, the conditions of abelianity and finiteness will be also imposed. Provided with an arbitrary Lie Algebra \( g \),

\[
[T_A, T_B] = C_{AB}^{C} T_{C},
\]

and an abelian Semigroup \( S = \{ \lambda_{\alpha} \} \), with 2-Selectors

\[
K_{\alpha\beta}^{\gamma} = \begin{cases} 
1, \text{ when } \lambda_{\alpha}\lambda_{\beta} = \lambda_{\gamma} \\
0, \text{ otherwise},
\end{cases}
\]

it is possible to prove (See Refs. [7, 9, 10]) that the product \( \Theta = S \times g \) corresponds to the Lie Algebra given by

\[
[T_{(A,\alpha)}, T_{(B,\beta)}] = K_{\alpha\beta}^{\gamma} C_{AB}^{C} T_{(C,\gamma)}.
\]

3.1. Resonant Subalgebras
In order to systematically extract subalgebras from \( \Theta = S \times g \), it is necessary to codify the subspace structure of the original algebra \( g \). In order to do this, let us consider the Lie algebra \( g = \bigoplus_{p \in I} V_p \), where \( I \) is a set of indexes. Let \( i \) be the mapping \( i : I \times I \rightarrow 2^I \), such that the subspace structure of \( g \) can be written as \(^1\)

\[
[V_p, V_q] \subset \bigoplus_{r \in i(p,q)} V_r.
\]

In this way, the mapping \( i \) codifies all the information on the subspace structure of \( g \).

\(^1\) Here \( 2^I \) denotes the set of all subsets of \( I \).
It is possible to prove (See Refs. [7, 9, 10]) that when a subset decomposition of \( S = \bigcup_{p \in I} S_p \), such that the condition

\[ S_p \cdot S_q \subset \bigcap_{r \in (p,q)} S_r, \]

is fulfilled can be found, then \( \mathfrak{g}_R = \bigoplus_{p \in I} S_p \times V_p \) is a subalgebra of \( \mathfrak{g} = S \times \mathfrak{g} \), called resonant subalgebra of \( \mathfrak{g} \).  

3.2. Resonant Reduction

The systematic codification of the subspace structure of \( \mathfrak{g} \) through the mapping \( i : I \times I \rightarrow 2^I \) allows us to go further, defining also reduced algebras from the resonant subalgebra \( \mathfrak{g}_R \). Let \( \mathfrak{g}_R = \bigoplus_{p \in I} S_p \times V_p \) be a resonant subalgebra, and let \( S_p = \hat{S}_p \cup \check{S}_p \) be a partition of the subsets \( S_p, \hat{S}_p \cap \check{S}_p = \emptyset \), such that the condition

\[ S_p \cdot \check{S}_q \subset \bigcap_{r \in (p,q)} \hat{S}_r \]

is fulfilled. Then, it is possible to show (See Refs. [7, 9, 10]) that \( [\hat{\mathfrak{g}}_R, \check{\mathfrak{g}}_R] \subset \hat{\mathfrak{g}}_R \), where

\[ \hat{\mathfrak{g}}_R = \bigoplus_{p \in I} \hat{S}_p \times V_p, \quad \check{\mathfrak{g}}_R = \bigoplus_{p \in I} \check{S}_p \times V_p. \]

Therefore, \( [\hat{\mathfrak{g}}_R] \) corresponds to a reduction of the resonant subalgebra \( \mathfrak{g}_R \).

Let us consider a semigroup \( S \) provided with an element \( 0_S \) such that for every \( \lambda \alpha \in S \), \( 0_S \lambda \alpha = 0_S \), and let \( S = \bigcup_{p \in I} S_p \) be a subset decomposition satisfying eq. (5) and such that each \( S_p \) includes the element \( 0_S \). Then, \( \hat{S}_p = \{ 0_S \} \) and \( \check{S}_p = S_p - \{ 0_S \} \) satisfies eq. (6), and the associated reduced algebra corresponds to imposing the condition \( 0_S T_A = 0 \) on \( \mathfrak{g}_R \); we will call this particular case \( 0_S \)-reduced algebra.

The present approach provides us with non-trivial invariant tensors for \( S \)-expanded algebras and in particular, for \( 0_S \)-reduced ones. It is possible to prove (See Refs. [7, 9, 10]) that for a \( 0_S \)-reduced algebra the invariant tensor reads

\[ \langle T(a_{p_1},i_{p_1}) \cdots T(a_{p_n},i_{p_n}) \rangle = \alpha_j K_{i_{p_1} \cdots i_{p_n}}^{j} \langle T(a_{p_1},i_{p_1}) \cdots T(a_{p_n},i_{p_n}) \rangle, \]

where \( T(a_p) \in V_p, \langle T(a_p) \cdots T(a_p) \rangle \) corresponds to the invariant tensor of \( \mathfrak{g} \), the index \( i_p \) is such that \( \lambda_{i_p} \in \check{S}_p \), the index \( j \) is such that \( \lambda_j \neq 0_S \), and \( K_{i_{p_1} \cdots i_{p_n}}^{j} \) corresponds to the \( n \)-Selector associated to \( S \).

4. M-Algebra and \( S \)-Expansions

In order to get the M-Algebra from \( \mathfrak{osp}(32|1) \) using the \( S \)-Expansion procedure, it is necessary to divide \( \mathfrak{osp}(32|1) \) into subspaces. Let us define \( V_0 \) as the Lorentz subalgebra, \( V_1 \) as the fermionic sector and \( V_2 \) as the AdS boosts and M5-brane piece. Let us pick as semigroup \( S_E^{(2)} = \{ \lambda_0, \lambda_1, \lambda_2, \lambda_3 \} \), provided with the product rule

\[ \lambda_\alpha \lambda_\beta = \begin{cases} \lambda_{\alpha + \beta} & \text{when } \alpha + \beta \leq 2, \\ \lambda_3 & \text{when } \alpha + \beta \geq 3. \end{cases} \]

2 Here the name ‘resonant’ is due the formal similarity between eqs. (4) and (5). The product \( S_p \cdot S_q \) denotes the set whose elements correspond to the product of each element of \( S_p \) with each element of \( S_q \).
Figure 1. $S_E^{(2)}$-expansion of $\mathfrak{osp}(32|1)$. (a) The gray region corresponds to the full algebra $S_E^{(2)} \times \mathfrak{osp}(32|1)$. (b) Resonant subalgebra. (c) M-Algebra, obtained after $0_S$-reduction of the resonant subalgebra.

The subset decomposition $S_E^{(2)} = S_0 \oplus S_1 \oplus S_2$ with

$S_0 = \{\lambda_0, \lambda_2, \lambda_3\}$,

$S_1 = \{\lambda_1, \lambda_3\}$,

$S_2 = \{\lambda_2, \lambda_3\}$,

satisfies the resonant condition eq. (5), and therefore, $\mathfrak{g}_R = \bigoplus_{p=0}^2 S_p \times V_p$ corresponds to a resonant subalgebra of $S_E^{(2)} \times \mathfrak{osp}(32|1)$ [see Fig. 1 (a) and (b)]. On the other hand, let us notice that in our case, $\lambda_3 = 0_S$. Therefore, it is possible to apply the reduction procedure, choosing $\check{S}_p = \{0_S\}$ and $\check{S}_p = S_p - \{0_S\}$, or equivalently, applying the condition $0_S T_A = 0$ on $\mathfrak{g}_R$. As a result, we obtain the M-Algebra [see Fig. 1 (c)]. In the present approach, an invariant tensor for the M-Algebra is given by the expression from eq. (8),

$$\langle T_{(a_{p_1},i_{p_1})} \cdots T_{(a_{p_n},i_{p_n})} \rangle = \alpha_j \delta_{i_{p_1} + \cdots + i_{p_n}}^{i_1} \langle T_{a_{p_1}} \cdots T_{a_{p_n}} \rangle,$$

where $\delta$ is the Kronecker delta and the range of the indices is given by $i_0 = 0, 2$, $i_1 = 1$, $i_2 = 2$ and $j = 0, 1, 2$. The construction of a Chern–Simons/Transgression theory for the M-Algebra using this approach has been considered in Refs. [8, 9, 10].

5. Conclusions

The procedure sketched here is completely general and becomes a very practical ‘tool’ in order to construct algebras with some special behaviour. Given an algebra, it is possible to construct bigger symmetries for different choices of semigroup and applying the resonant subalgebra and reduction theorems. The procedure of Maurer–Cartan forms power-series expansion and the İnönü–Wigner contraction can be reobtained from a particular choice of semigroup (See Refs. [7, 9, 10]). In the case of the M-Algebra, it is possible to observe that it belongs to a family of symmetries with similar behaviour which arise from $\mathfrak{osp}(32|1)$ expansions; examples and a deeper analysis of this are provided in Refs. [7, 8, 9, 10]. On the other hand, the procedure not only gives us a new symmetry, but also a non-trivial invariant tensor, in general different from the supertrace. For the case of the M-Algebra, this is a very important feature, since the supertrace provides us only with a trivial (Lorentz-valued only) invariant tensor, which is
completely useless in order to construct a supersymmetric Chern–Simons theory. In the present context, the construction of the theory is straightforward using the invariant tensor from eqs. (8) and (9); see Refs. [8, 9, 10].

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