Towards an understanding of ramified extensions of structured ring spectra

BY BJØRN IAN DUNDAS

Department of Mathematics, University of Bergen,
Postboks 7800, 5020 Bergen, Norway.
e-mail: dundas@math.uib.no

AYELET LINDENSTRAUSS

Mathematics Department, Indiana University,
831 East Third Street Bloomington, IN 47405, U.S.A.,
e-mail: alindens@indiana.edu

AND BIRGIT RICHTER

Fachbereich Mathematik der Universität Hamburg,
Bundesstraße 55, 20146 Hamburg, Germany.
e-mail: birgit.richter@uni-hamburg.de

(Received 22 February 2017; revised 21 February 2018)

Abstract

We propose topological Hochschild homology as a tool for measuring ramification of maps of structured ring spectra. We determine second order topological Hochschild homology of the $p$-local integers. For the tamely ramified extension of the map from the connective Adams summand to $p$-local complex topological K-theory we determine the relative topological Hochschild homology and show that it detects the tame ramification of this extension. We show that the complexification map from connective topological real to complex K-theory shows features of a wildly ramified extension. We also determine relative topological Hochschild homology for some quotient maps with commutative quotients.

1. Introduction

For a $G$-Galois extension of number fields $K \subset L$ the associated extension of rings of integers $\mathcal{O}_K \to \mathcal{O}_L$ will not be unramified in general. Greither shows in [14, chapter 0, theorem 4.1] that the ramification of such an extension can be detected with the help of the map

$$ h : \mathcal{O}_L \otimes_{\mathcal{O}_K} \mathcal{O}_L \to \prod_G \mathcal{O}_L. \tag{1.1} $$

Here, $h$ is defined as $h(b_1 \otimes b_2) = (b_1 g(b_2))_{g \in G}$ for $b_1, b_2 \in \mathcal{O}_L$. The extension is unramified if $h$ is an isomorphism. For more general extensions of commutative rings this still gives an adequate notion of ramification. The Hochschild homology of $\mathcal{O}_L$ over $\mathcal{O}_K$ is an invariant that behaves differently depending on whether the extension is tamely or wildly ramified. For
instance $\mathbb{H}^\mathbb{Z}(\mathbb{Z}[i])$ is a square-zero extension of $\mathbb{Z}[i]$ with additive two-torsion in positive odd degrees (see Remark 5.3 for details).

In the following we consider cohomology theories with a multiplicative structure that can be represented by a commutative monoid object in one of the symmetric monoidal categories of spectra, for instance the one presented in [10]. The representing objects are called commutative ring spectra. Examples of such cohomology theories are singular cohomology with coefficients in a commutative ring, topological (real or complex) $K$-theory, and several cobordism theories. There is an analogue of Hochschild homology in the context of ring spectra, topological Hochschild homology. It was defined by Bökstedt [6] and a published account can for instance be found in [10, chapter IX].

Let $A$ be a commutative ring spectrum and let $B$ be a commutative (cofibrant) $A$-algebra with an action of a finite group $G$ via maps of commutative $A$-algebras. Then the extension $A \to B$ is called unramified [23, (4.1.2)], if the map

$$h: B \wedge_A B \to \prod_G B$$

is an equivalence. Here, $h$ is the analogue of (1.1) in the context of spectra.

Rognes shows [23, 9.2-6, proof of 9.1.2] that the condition for $B$ to be unramified over $A$ ensures that the map from $B$ to relative topological Hochschild homology of $B$ over $A$, $\text{THH}^A(B)$, is a weak equivalence. Thus the failure of the map

$$B \to \text{THH}^A(B)$$

to be a weak equivalence is a measure of the ramification of the extension $A \to B$. It also makes sense to study $\text{THH}^A(B)$ in more general situations, for instance in the absence of a group action. At the moment we do not have any conceptual notion of tame or wild ramification of maps between commutative ring spectra. The calculations in this paper provide important examples of ramified extensions and we hope that they will be helpful for a general theory of ramification.

Algebraic $K$-theory of an ordinary commutative ring $R$, $K(R)$, contains a lot of arithmetic information about $R$, such as the Picard group of $R$, its Brauer group and its units. Trace methods have been useful for studying $K(R)$: There are trace maps

$$K(R) \xrightarrow{\tr} TC(R) \xrightarrow{\tr} \text{THH}(R)$$

that allow us to approximate $K(R)$ by invariants that are easier to compute, by topological Hochschild homology, $\text{THH}(R)$, and by topological cyclic homology, $TC(R)$. Trace methods also work well for connective commutative ring spectra, i.e., commutative ring spectra whose homotopy groups are concentrated in non-negative degrees.

Galois extensions of commutative $S$-algebras in the sense of Rognes [23, 4.1.3] are unramified. A prominent example is given by the complexification map from real to complex periodic $K$-theory, $c: KO \to KU$. Here, complex conjugation on complex vector bundles induces a $C_2$-action on $KU$ by maps of commutative $KO$-algebra spectra. But a result of Akhil Mathew [19, theorem 6.17] tells us that finite Galois extensions of a connective
spectrum are purely algebraic. So taking the connective cover of the complexification map

\[
\begin{array}{ccc}
k_0 & \xrightarrow{c} & ku \\
\downarrow j & & \downarrow j \\
KO & \xrightarrow{c} & KU
\end{array}
\]

does not yield a $C_2$-Galois extension $k_0 \to ku$ because algebraically

\[
k_0^* = \mathbb{Z}[\eta, y, w]/2\eta, \eta^3, \eta y, y^2 - 4w \longrightarrow \mathbb{Z}[u] \cong ku^*
\]
is certainly not étale. (Here, the degrees are $|\eta| = 1, |y| = 4, |w| = 8$ and $|u| = 2$.) In fact, we will see that $k_0 \to ku$ behaves like a wildly ramified extension of number rings with ramification at 2.

For a commutative $A$-algebra $B$ we denote by $\text{THH}^{[n], A}(B)$ the higher order topological Hochschild homology of $B$ as a commutative $A$-algebra, i.e.,

\[
\text{THH}^{[n], A}(B) = B \otimes S^n,
\]

where $(-) \otimes S^n$ denotes the tensor with the (unbased) $n$-sphere in the category of commutative $A$-algebras. This can be viewed as the realisation of the simplicial commutative $A$-algebra whose $q$-simplices are given by

\[
\bigsqcup_{x \in S^n_q} B,
\]

where the coproduct is the smash product over $A$.

The content of the paper is as follows. Higher $\text{THH}$ of the Eilenberg–MacLane spectra of local number rings also detects ramification \[9\], but after we take coefficients in the residue field we cannot see the difference anymore between tame and wild ramification in higher $\text{THH}$. We offer some partial results towards calculations of higher $\text{THH}$ with unreduced coefficients. We calculate second order $\text{THH}$ of the $p$-local integers:

\[
\text{THH}^{[2]}_s(\mathbb{Z}(p)) \cong \mathbb{Z}(p)[x_1, x_2, \ldots]/p^\infty x_n, x_n^p = px_{n+1}.
\]

See Theorem 2:1. It is possible to determine $\text{THH}^{[2]}_s(\mathbb{Z}(p))$ additively using that $H\mathbb{Z}(p)$ can be constructed as a Thom spectrum of a double-loop map and applying the methods of \[3, 25\]. Blumberg, Cohen and Schlichtkrull identify $\text{THH}(\mathbb{Z}(p))$ with $H\mathbb{Z}(p) \wedge \Omega S^3(3)_+$ \[3, theorem 3:8\], and Klang uses their work to calculate $\text{THH}^{[2]}_s(\mathbb{Z}(p))$ in \[16, proposition 4:4\]. However, this views $H\mathbb{Z}(p)$ as an $E_2$-spectrum and not as a commutative $S$-algebra, so with this method the multiplicative structure of $\text{THH}^{[2]}_s(\mathbb{Z}(p))$ cannot be determined. The multiplicative structure is essential if one aims at a calculation of $\text{THH}^{[n]}_s(\mathbb{Z}(p))$ for larger $n$.

We study the examples of the connective covers of the Galois extensions \[23\] $KO \to KU$ and $L_p \to KU_p$. In the latter case, the connective cover behaves like an extension of the corresponding rings of integers. We test ramification with relative (higher) topological Hochschild homology and for $\ell \to ku(p)$ we see that it looks like tame ramification (see Theorem 4:1): $\text{THH}_s(\ell ku(p))$ is a square zero extension of $\pi_* ku(p)$ of bounded $u$-exponent. The complexification map $c: k_0 \to ku$, however, shows features in its relative $\text{THH}$ that are similar to the behavior of a wildly ramified extension of number rings, e.g., the extension $\mathbb{Z} \hookrightarrow \mathbb{Z}[i]$ (see Theorem 5:2).
Working with structured ring spectra means working in a derived setting, so quotient maps can be thought of as extensions. We offer some calculations of relative $\text{THH}$ in situations where we kill generators of homotopy groups. We consider a version of $\text{ku}/(p, v_1)$ and quotients of the form $R/x$ where $x$ is a regular element in $\pi_s(R)$ where $R$ is a commutative ring spectrum such that $R/x$ is still commutative.

2. Second order $\text{THH}$ of the $p$-local integers

This section consists of a proof of the following somewhat surprising result. In the context of this paper, this calculation is a starting point for comparing with future calculations for other rings of integers. See Remark 2.4 for a discussion of the fact that the answer agrees with topological Hochschild cohomology.

**Theorem 2.1.** For all primes $p$:

$$\text{THH}_s^2(\mathbb{Z}(p)) \cong \mathbb{Z}(p)[x_1, x_2, \ldots]/p^nx_n, x_n^p - px_{n+1}$$

with $|x_n| = 2p^n$. Globally this yields

$$\text{THH}_s^2(\mathbb{Z}) \cong \Gamma_\mathbb{Z}(x)/(x),$$

where $\Gamma_\mathbb{Z}(x)$ denotes the divided power algebra on a generator $x$ of degree 2 and $(x)$ denotes the ideal generated by $x$.

The entire section is devoted to proving this result. For all primes $p$ the exact sequence

$$\text{THH}_s^2(\mathbb{Z}(p)) \xrightarrow{\partial} \text{THH}_s^2(\mathbb{Z}(p)) \xrightarrow{r} \text{THH}_s^2(\mathbb{Z}(p), \mathbb{F}_p) \xrightarrow{\delta} \Sigma \text{THH}_s^2(\mathbb{Z}(p))$$

(2.1)

is a sequence of $\text{THH}_s^2(\mathbb{Z}(p))$-modules; in particular, $\delta$ is a module map. Furthermore, from [9] we have that

$$\text{THH}_s^2(\mathbb{Z}(p), \mathbb{F}_p) \cong \Gamma_{\mathbb{F}_p}(y) \otimes \Lambda_{\mathbb{F}_p}(z),$$

(2.2)

where $|y| = 2p$ and $|z| = 2p + 1$. We denote the generator $\gamma_p(y)$ in the divided power algebra $\Gamma_{\mathbb{F}_p}(y)$ in degree $2p^{i+1}$ by $y_p^i$ and if $t = t_0 + t_1p + \cdots + t_sp^n$ is the $p$-adic expansion of $t$, then we set $y_i = y_{0i}^1y_{1i}^2\cdots y_{ni}^n$. Since $y_p^0 = 0$ for all $i$, we also get $y_i^p = 0$.

By the $\text{Tor}$ spectral sequence,

$$\text{Tor}_{s,*}^\text{THH}_s(\mathbb{Z}(p), \mathbb{Z}(p)) \Longrightarrow \text{THH}_s^2(\mathbb{Z}(p))$$

we know that $\text{THH}_s^2(\mathbb{Z}(p))$ is finite $p$-torsion for positive $s$ because

$$\text{THH}_s(\mathbb{Z}(p)) = \begin{cases} 
\mathbb{Z}(p), & \ast = 0, \\
\mathbb{Z}(p)/i, & \ast = 2i - 1, \\
0, & \text{otherwise}.
\end{cases}$$

By Equation (2.2) and using the notation introduced below it, this implies that there are integers $a_1, a_2, \ldots$ such that

$$\text{THH}_s^2(\mathbb{Z}(p)) = \begin{cases} 
0, & 2p \not| s, \\
\mathbb{Z}/p^{a_s}\{\tilde{y}_i\}, & s = 2pt,
\end{cases}$$

where the $\tilde{y}_i$ are generators of the given cyclic groups which are sent to the corresponding generators $y_i$ in $\text{THH}_s^2(\mathbb{Z}(p), \mathbb{F}_p)$.
The exponents \( a_i \) that we get below can now be independently verified by comparison with the result of Klang [16, proposition 4.4]. However, even knowing the order of the torsion in each dimension and knowing the multiplicative structure of the reduction from our previous calculation of \( \text{THH}_*^2(\mathbb{Z}(\mathfrak{p}), \mathbb{F}_p) \cong \pi_*(\text{THH}_*^2(\mathbb{Z}(\mathfrak{p})); \mathbb{F}_p) \) does not determine the multiplicative structure of \( \text{THH}_*^2(\mathbb{Z}(\mathfrak{p})) \): if we have \( y_i y_j = y_{i+j} \) in the reduced calculation, then we could by a careful choice of liftings get that \( \tilde{y}_i \tilde{y}_j = \tilde{y}_{i+j} \). However, if we have \( y_i y_j = 0 \), then the product of the liftings might be zero as well, or it might be \( p^e \) times a lifting of the generator for some \( e \geq 1 \). We prove formulas for the order of the torsion and the multiplicative structure in tandem, inductively:

**Lemma 2.2.** The function \( a : \mathbb{N} \to \mathbb{N} \) factors over the \( p \)-adic valuation \( v : \mathbb{N} \to \mathbb{N} \), \( a_t = b_0(t) \), with \( b : \mathbb{N} \to \mathbb{N} \) a strictly increasing function with positive values and \( b_0 = 1 \).

**Proof.** To this end we use induction on the following statement \( P(n) \) for positive integers \( n \). The generators \( \tilde{y}_i \) are chosen inductively.

\( P(n) \): For positive integers \( s, t \) such that \( v(s), v(t) \) are less than \( n \) the following properties hold:

1. if \( v(s) = v(t) \), then \( a_s = a_t \);
2. if \( v(s) > v(t) \), then \( a_s > a_t \);
3. if \( s = s_0 + s_1 p + \cdots + s_{n-1} p^{n-1} \) is the \( p \)-adic expansion of \( s \) (so that \( 0 \leq s_0, \ldots, s_n < p \)), then \( \tilde{y}_s = \tilde{y}_1^{s_0} \cdots \tilde{y}_p^{s_{n-1}} \);
4. if \( n > v \), then \( \tilde{y}_p^{t_{n-1}} = p^t \).

We will repeatedly be considering the exact sequence of Equation (2.1). In homotopy, the maps are trivial except in degrees of the form \( 2pt \) (for varying \( t \)) in which case they are

\[
\begin{array}{cccccc}
0 & \to & \mathbb{F}_p \{ z y_i \} & \to & \text{THH}_*^2(\mathbb{Z}(\mathfrak{p})) & \to & \mathbb{F}_p \{ y_i \} & \to & 0
\end{array}
\]

forcing all the \( a_i \) to be positive. For any generator \( w \), \( \mathbb{F}_p \{ w \} \) denotes the graded vector space generated by \( w \). Here \( r \) is multiplicative and \( \delta \) is a \( \text{THH}_*^2(\mathbb{Z}(\mathfrak{p})) \)-module map. By the surjectivity of \( r \) we have that the \( y_i \)’s can be lifted to integral classes.

**Establishing \( P(1) \)**

Let \( t = 1 \). The sequence

\[
\begin{array}{cccccc}
0 & \to & \mathbb{F}_p \{ z \} & \to & \text{THH}_*^2(\mathbb{Z}(\mathfrak{p})) & \to & \mathbb{F}_p \{ y_1 \} & \to & 0
\end{array}
\]

does not determine the \( \tilde{y}_1 \) and we have that \( \tilde{y}_1 = \tilde{y}_1 \) and \( r(\tilde{y}_1) = y_1 \). In the \( \text{Tor} \)-spectral sequence we only get a \( \mathbb{Z}/p \mathbb{Z} \) in bidegree \( (1, 2p - 1) \) which survives and shows that \( a_1 = 1 \), and so \( \delta(z) = \tilde{y}_1 \).

If \( 1 < t < p \) the sequence

\[
\begin{array}{cccccc}
0 & \to & \mathbb{F}_p \{ z y_1^{t-1} \} & \to & \text{THH}_*^2(\mathbb{Z}(\mathfrak{p})) & \to & \mathbb{F}_p \{ y_1^{t-1} \} & \to & 0
\end{array}
\]

gives that \( \delta(z y_1^{t-1}) = \delta(z) \cdot y_1^{t-1} = \tilde{y}_1 \neq 0 \). \( p \tilde{y}_1^{t-1} = 0 \) and \( r(\tilde{y}_1^{t-1}) = y_1^{t-1} \neq 0 \). The last point shows that \( \tilde{y}_1 \) is not divisible by \( p \) and hence we can choose it as our generator: \( \tilde{y}_1 = \tilde{y}_1 \), and furthermore, this generator is killed by \( p \), so \( a_t = 1 \).

If \( t = t_0 + t_1 p \) with \( 0 < t_0 < p \), then the sequence

\[
\begin{array}{cccccc}
0 & \to & \mathbb{F}_p \{ z y_1^{t_0-1} y_{t_1} \} & \to & \text{THH}_*^2(\mathbb{Z}(\mathfrak{p})) & \to & \mathbb{F}_p \{ y_1^{t_0} y_{t_1} \} & \to & 0
\end{array}
\]
gives that \( \delta(zy_{1}^{n-1}y_{i_{1}}p) = \delta(z) \cdot y_{1}^{n-1}y_{i_{1}}p = y_{1}^{n}y_{i_{1}}p \neq 0 \), \( p\tilde{y}_{i_{1}}y_{i_{1}}p = 0 \) and \( r(\tilde{y}_{1}^{n}y_{i_{1}}p) = y_{1}^{n}y_{i_{1}}p \neq 0 \) for any choice of \( \tilde{y}_{i_{1}}p \). The last point shows that \( \tilde{y}_{1}^{n}y_{i_{1}}p \) is not divisible by \( p \) and hence we can choose it as our generator: \( \tilde{y}_{i} = \tilde{y}_{1}^{n}y_{i_{1}}p \), and furthermore, this generator is killed by \( p \), so \( a_{i} = 1 \).

Note that we may reconsider our choice of \( \tilde{y}_{i_{1}}p \) later, and so the exact choice of \( \tilde{y}_{i} \) may still change within these bounds, but the choices of \( \tilde{y}_{1}, \ldots, \tilde{y}_{p-1} \) remain fixed from now on. Hence \( P(1)(1) - P(1)(3) \) are established and as \( P(1)(4) \) is vacuous we have shown \( P(1) \).

**Establishing \( P(n + 1) \)**

Now, assume \( P(n) \). First, consider the case \( t = p^{n} \). For \( P(n + 1)(4) \) we only have to show that

\[ p^{a_{p^{n}} - a_{p^{n-1}}} \tilde{y}_{p^{n}} = \tilde{y}_{p^{n-1}}, \]

and that \( a_{p^{n}} > a_{p^{n-1}} \). Consider the sequence

\[ 0 \longrightarrow \mathbb{F}_{p} \{ y_{1}^{p-1}, \ldots, y_{p^{n-1}} \} \longrightarrow \text{THH}_{2p^{n}}(\mathbb{Z}_{(p)}) \longrightarrow \text{THH}_{2p^{n}}(\mathbb{Z}_{(p)}) \longrightarrow \mathbb{F}_{p} \{ y_{p^{n}} \} \longrightarrow 0. \]

Firstly, by induction we have that

\[ \delta(zy_{1}^{p-1} \cdots y_{p^{n-1}}p) = p^{a_{p^{n}} - a_{p^{n-1}}}y_{p^{n}} = p^{a_{p^{n}} - a_{p^{n-1}}} \tilde{y}_{p^{n}} = \tilde{y}_{p^{n-1}} = 0. \]

Together this shows that (up to a unit) \( \delta(zy_{1}^{p-1} \cdots y_{p^{n-1}}p) = p^{a_{p^{n}} - a_{p^{n-1}}} \tilde{y}_{p^{n}} \), and that \( \tilde{y}_{p^{n-1}} = p^{a_{p^{n}} - a_{p^{n-1}}} \tilde{y}_{p^{n}} \), and since \( y_{p^{n-1}} = 0 \) that \( a_{p^{n}} > a_{p^{n-1}} \).

Now, for \( P(n + 1)(1) \) and \( P(n + 1)(2) \), consider a general \( t \) with \( o(t) = n \) and write \( t = t_{n}p^{n} + sp^{n+1} \) with \( 0 < t_{n} < p \). The exact sequence

\[ \mathbb{F}_{p} \{ y_{1}^{p-1}, \ldots, y_{p^{n-1}}p, y_{p^{n}}p^{n+1} \} \longrightarrow \text{THH}_{2p^{n}}(\mathbb{Z}_{(p)}) \longrightarrow \text{THH}_{2p^{n}}(\mathbb{Z}_{(p)}) \longrightarrow \mathbb{F}_{p} \{ y_{p^{n}}p^{n+1} \} \]

gives that

\[ \delta(zy_{1}^{p-1} \cdots y_{p^{n-1}}p, y_{p^{n}}p^{n+1}) = y_{1} \tilde{y}_{1}^{p-1} \cdots y_{p^{n-1}}p, y_{p^{n}}p^{n+1} = 0 \]

but \( p\delta(zy_{1}^{p-1} \cdots y_{p^{n-1}}p, y_{p^{n}}p^{n+1}) = p^{a_{p^{n}} - a_{p^{n-1}}} \tilde{y}_{p^{n}}p^{n+1} = 0 \) and \( r(\tilde{y}_{p^{n}}p^{n+1}) = y_{p^{n}}p^{n+1} \neq 0 \).

Again, the last point shows that \( \tilde{y}_{p^{n}}p^{n+1} \) is not divisible by \( p \), and so we may choose \( \tilde{y}_{i} = \tilde{y}_{p^{n}}p^{n+1} \), and furthermore that this generator is annihilated by \( p^{a_{p^{n}} - a_{p^{n-1}}} \), so that \( a_{i} = a_{p^{n}} \).

Lastly, by \( P(n)(3) \), we have that if \( s = s_{0} + s_{1}p + \cdots + s_{n-1}p^{n-1} \) is the \( p \)-adic expansion of \( s \), then \( \tilde{y}_{s} = \tilde{y}_{s_{0}} \cdots \tilde{y}_{s_{n-1}}p^{n-1} \). If \( t = s + s_{n}p^{n} \), then \( r(\tilde{y}_{t}p^{n}) = y_{t} \), so we can choose \( \tilde{y}_{t} = \tilde{y}_{s}p^{n} \), as desired in \( P(n + 1)(3) \).
Background on Böksteds

Let \((C_s, \partial)\) be a complex of free abelian groups and assume \(\alpha \in C_n\) has the property that \(\alpha \otimes 1\) is a cycle in \(C_s \otimes \mathbb{F}_p\). That the Bökstein \(\beta_{i-1}([\alpha \otimes 1])\) is defined and equal to zero for some \(i \geq 2\), means that there exist \(\gamma \in C_n\) and cycle \(\delta \in C_{n-1}\) so that \(\partial(\alpha + p\gamma) = p^i\delta\), and in that case, \(\beta_i([\alpha \otimes 1]) = [\delta \otimes 1]\).

Assume we have a short exact sequence of complexes of free abelian groups \(0 \rightarrow B_s \rightarrow C_s \rightarrow A_s \rightarrow 0\). Choosing a section in each degree, we may assume \(C_n = A_n \oplus B_n\) for all \(n\). Suppose we have \(a \in A_n\) and \(b \in B_n\) so that \([a \otimes 1] + [b \otimes 1]\) represents a cycle in \(C_s \otimes \mathbb{F}_p\) with \(\beta_{i-1}([((a+b) \otimes 1)]) = [0] \in H_{n-1}(C_s \otimes \mathbb{F}_p)\). As above, there exist \(c \in A_n, d \in B_n, e \in A_{n-1}, f \in B_{n-1}\) with \(c + d = A_{n-1}\) a cycle with \(\partial(c + d + p(c + d)) = p^i(e + f)\), and in that case \(\beta_i([(a+b) \otimes 1]) = [(e + f) \otimes 1]\). Then if \([e \otimes 1] \neq [0] \in H_{n-1}(A_s \otimes \mathbb{F}_p)\), we get that \(\beta_i([(a+b) \otimes 1]) \neq [0]\), since \([(e + f) \otimes 1] \mapsto [e \otimes 1] \neq [0]\) by the homomorphism induced by the projection \(C_s \rightarrow A_s\).

More generally, consider a filtered complex \(C_s\) of free abelian groups. Assume we have a chain \(a \in E(C_s)_{i,j}\) in the associated spectral sequence such that \([a \otimes 1]\) survives to \(E(C_s \otimes \mathbb{F}_p)_{i,j}^\infty\) in the mod \(p\) spectral sequence. If we know that the class \([a+b] \otimes 1\) represents \([a \otimes 1]\) in \(E(C_s \otimes \mathbb{F}_p)_{i,j}\) satisfies \(\beta_{i-1}([(a+b) \otimes 1]) = [0] \in H_{n-1}(C_s \otimes \mathbb{F}_p)\), but that \(d^0(a \otimes 1) = p^i(e \otimes 1)\) and \([e \otimes 1] \neq [0] \in E(C_s \otimes \mathbb{F}_p)_{i,j-1}\), then \(\beta_i([(a+b) \otimes 1]) \neq [0]\).

The \(p\)-order of the multiplicative generators

We will calculate \(\text{THH}^{[2]}_s(\mathbb{Z}((p))\) by studying its Hurewicz image in \(H_s(\text{THH}^{[2]}(\mathbb{Z}((p)); \mathbb{F}_p)\), using the model

\[
\text{THH}^{[2]}(\mathbb{Z}((p)) \simeq B(H\mathbb{Z}((p), \text{THH}(\mathbb{Z}((p)), H\mathbb{Z}((p))).
\]

We use the filtration by simplicial skeleta. We denote \(H_s(\text{THH}(\mathbb{Z}((p)); \mathbb{F}_p)\) by \(\tilde{A}\), and by Bökstede,

\[
H_s(\text{THH}(\mathbb{Z}((p)); \mathbb{F}_p) \cong \tilde{A} \otimes \mathbb{F}_p[x_{2^p-1}] \otimes \Lambda[x_{2^p-1}]_1,
\]

where the augmentation \(\text{THH}(\mathbb{Z}((p)) \rightarrow H\mathbb{Z}((p))\) induces the projection \(\tilde{A} \otimes \mathbb{F}_p[x_{2^p}] \otimes \Lambda[x_{2^p-1}] \rightarrow \tilde{A}\) sending \(x_{2^p-1}\) and \(x_{2^p-1}\) to zero. We get that

\[
E_{s,s}^1 \cong B(\tilde{A}, \tilde{A} \otimes \mathbb{F}_p[x_{2^p}] \otimes \Lambda[x_{2^p-1}], \tilde{A})
\]

is isomorphic to

\[
B(\tilde{A}, \tilde{A} \otimes \mathbb{F}_p[x_{2^p}], \mathbb{F}_p) \otimes B(\mathbb{F}_p, \mathbb{F}_p[x_{2^p}], \mathbb{F}_p) \otimes B(\mathbb{F}_p, \Lambda[x_{2^p-1}], \mathbb{F}_p),
\]

and so its homology is

\[
E_{s,s}^2 \cong \tilde{A} \otimes \Lambda[y_{2^p+1}] \otimes \Gamma[y_{2^p}]
\]

with \(y_{2^p+1} = 1 \otimes x_{2^p} \otimes 1\) and \(y_{2^p} = 1 \otimes x_{2^p-1}\) and \(y_{2^p}^{(a)} = 1 \otimes x_{2^p-1}^{(a)} \otimes 1\).

The dimensions in each total degree in the \(E^2\)-term account for \(p\)-torsion of rank 1 in each positive dimension divisible by \(2^p\), and from knowing \(\text{THH}^{[2]}_s(\mathbb{Z}((p)); \mathbb{F}_p)\) \([9, \text{theorem 3-1}]\) we get that this agrees with the abutment of the spectral sequence, so it has to collapse at \(E^2\).

We use this to prove Theorem 2-1. By Lemma 2-2, the only remaining problem is to determine the order of the \(p\)-torsion in each dimension divisible by \(2^p\).

**Lemma 2-3.** The \(p\)-torsion in \(\text{THH}^{[2]}_{2^p}(\mathbb{Z}((p))\) is precisely \(\mathbb{Z}((p))/\mathbb{Z}((p))/p^{3(p-1)}+1\).
On the Eilenberg MacLane space $K\Omega$, the order of the torsion is divisible by $p^{a+1}$. We will use the general observation about Bocksteins above for

$$C_s = C_s(\Omega^\infty(\text{THH}^2(\mathbb{Z}(p)); \mathbb{Z})); C_s(\Omega^\infty B(H\mathbb{Z}(p), \text{THH}(\mathbb{Z}(p), H\mathbb{Z}(p)); \mathbb{Z}),$$

filtered by simplicial skeleta of the bar construction, to get that the torsion is exactly $p^{a+1}$.

Fixing a $\tau$, we have two quasi-isomorphisms (letting $s$ vary)

$$C_s(\Omega^\infty B_1(H\mathbb{Z}(p), \text{THH}(\mathbb{Z}(p), H\mathbb{Z}(p)); \mathbb{Z}) \to \cdots \to C_{s+\tau}((\Delta^1)^s \times \Omega^\infty B_1(H\mathbb{Z}(p), \text{THH}(\mathbb{Z}(p), H\mathbb{Z}(p)); \mathbb{Z}) \to C_{s+\tau}((\Delta^1)^s \times \Omega^\infty B_1(H\mathbb{Z}(p), \text{THH}(\mathbb{Z}(p), H\mathbb{Z}(p)); \mathbb{Z}),$$

and we call their composition $\phi$.

We know by Bökstedt that additively $\text{THH}(\mathbb{Z}(p)) \simeq H\mathbb{Z}(p) \vee \Sigma^{2p-1} H\mathbb{F}_p \vee \cdots$, so we can map

$$S^0 \wedge K(\mathbb{F}_p, 2p-1)^{\wedge \tau} \wedge S^0 = S^0 \wedge (\Omega^\infty(\Sigma^{2p-1} H\mathbb{F}_p))^{\wedge \tau} \wedge S^0 \to \Omega^\infty H\mathbb{Z}(p) \wedge (\Omega^\infty(\text{THH}(\mathbb{Z}(p)))^{\wedge \tau} \wedge \Omega^\infty H\mathbb{Z}(p)$$

We call this composition $\psi$. It induces

$$\psi_s : C_s(\Omega^\infty B_1(H\mathbb{Z}(p), \text{THH}(\mathbb{Z}(p), H\mathbb{Z}(p)); \mathbb{Z}) \to \cdots \to C_s(\Omega^\infty(\text{THH}(\mathbb{Z}(p)))^{\wedge \tau} \wedge H\mathbb{Z}(p)); \mathbb{Z}),$$

so composing we get a map of complexes

$$\phi \circ \psi_s : C_s(\Omega^\infty B_1(H\mathbb{Z}(p), \text{THH}(\mathbb{Z}(p), H\mathbb{Z}(p)); \mathbb{Z}) \to \cdots \to E^0_{t,s}.$$

On the Eilenberg MacLane space $K(\mathbb{F}_p, 2p-1)$, we have a 2p-chain with integer coefficients $\tilde{x}_{2p}$ so that $[\tilde{x}_{2p}]$ (mod $p$) generates $H_{2p}(K(\mathbb{F}_p, 2p-1); \mathbb{F}_p) \simeq \mathbb{F}_p$ and $\partial \tilde{x}_{2p} = p \tilde{x}_{2p-1}$ for a chain $\tilde{x}_{2p-1}$ so that $[\tilde{x}_{2p-1}]$ (mod $p$) generates $H_{2p-1}(K(\mathbb{F}_p, 2p-1); \mathbb{F}_p) \simeq \mathbb{F}_p$. For these elements, $\beta_l([\tilde{x}_{2p}]) = [\tilde{x}_{2p-1}]$. Note that these elements map to generators of the stable homology in the correct dimensions. Thus, $\phi \circ \psi_s(\tilde{x}_{2p}) \otimes 1$ can be taken as a representative of $x_{2p}$, and $\phi \circ \psi_s(\tilde{x}_{2p-1}) \otimes 1$ can be taken as a representative of $x_{2p-1}$, and we still have $d^0(\phi \circ \psi_s(\tilde{x}_{2p})) = p(\phi \circ \psi_s(\tilde{x}_{2p-1}))$ in $E^0_{1,s}$. And more generally, for any $a, b \geq 0$, in $E^0_{a+b+1, s}$ we also have

$$d^0(\phi \circ \psi_s(\tilde{x}_{2p-1}^a \otimes \tilde{x}_{2p} \otimes \tilde{x}_{2p-1}^b)) = p(\phi \circ \psi_s(\tilde{x}_{2p-1}^{a+b})).$$

We know that the class $(\phi \circ \psi_s(\tilde{x}_{2p-1}^a \otimes \tilde{x}_{2p} \otimes \tilde{x}_{2p-1}^b)) \otimes 1$ represents the class $1 \otimes x_{2p-1}^{\otimes a} \otimes x_{2p} \otimes x_{2p-1}^{\otimes b}$ which survives to $E^\infty_{a+b+1, s}$ and therefore to $E^\infty_{a+b+1, s}$, and similarly for $(\phi \circ \psi_s(\tilde{x}_{2p-1}^{a+b})) \otimes 1$ and $1 \otimes x_{2p-1}^{\otimes a+b+1} \otimes 1$.

And so, if $t = p^m m$ with $(p, m) = 1$,

$$d^t \sum_{i=0}^{t-1} (-1)^i (\phi \circ \psi_s(\tilde{x}_{2p-1}^i \otimes \tilde{x}_{2p} \otimes \tilde{x}_{2p-1}^{t-1-i})) \otimes 1 = pt \cdot (\phi \circ \psi_s(\tilde{x}_{2p-1}^t)) \otimes 1 = p^{a+1} m \cdot (\phi \circ \psi_s(\tilde{x}_{2p-1}^t)) \otimes 1.$$
The mod $p$ homology class which is the image under the Hurewicz map of $z\gamma_{1-1}(y)$ can be expressed as

$$(1 \otimes x_{2p} \otimes 1)(1 \otimes x_{2p-1}^{\otimes n-1} \otimes 1)$$

via the bar construction and it is represented by $\sum_{i=0}^{l-1}(-1)^i \varphi \circ \psi_{a}(\tilde{x}_{p-1} \wedge \tilde{x}_{2p} \wedge \tilde{x}_{2p-1}^{i-1}) \otimes 1$. From Lemma 2.2 we have a lower bound on the order of the torsion and hence $\beta_a(z\gamma_{1-1}(y)) = 0$ and by the $d^0$ calculation above $\beta_{a+1}(z\gamma_{1-1}(y)) = \gamma_1(y)$ up to a unit.

This result is a result on stable mod $p$ homology rather than on stable mod $p$ homotopy, but since we are applying it to the images under the Hurewicz map of the two stable mod $p$ homotopy classes of an Eilenberg–MacLane space of rank 1 $p$-torsion, the Bockstein operators have to do the same on the mod $p$ homotopy.

**Proof of Theorem 2.1** Set $x_n = \tilde{y}_{p^n}$. Then we get the $p$-order of these elements from Lemma 2.3 and we worked out the multiplicative relations in Lemma 2.2.

**Remark 2.4.** Mike Hill noticed that $\text{THH}_s^2(\mathbb{Z}_{(p)})$ is abstractly isomorphic to $\text{THH}^* (\mathbb{Z}_{(p)})$; the calculation of $\text{THH}^* (\mathbb{Z}_{(p)})$ is due to Franjou and Pirashvili [12]. We are not sure whether this is a coincidence or whether (for some commutative $S$-algebras) there is a duality between $\text{THH}_s^2$ and topological Hochschild cohomology. Note, however, that $\text{THH}_s^2(\mathbb{F}_p)$ is an exterior algebra over $\mathbb{F}_p$ on a class in degree three whereas $\text{THH}^* (\mathbb{F}_p)$ is much larger:

$$\text{THH}^* (\mathbb{F}_p) \cong \mathbb{F}_p[e_0, e_1, \ldots]/(e_0^p, e_1^p, \ldots), \quad |e_i| = 2p^i,$$

[11, 7.3], [7], so there is no isomorphism of these groups in general.

3. **Greenlees’ approach to THH**

There is a relative version of the cofiber sequence from [13, lemma 7.1] already mentioned in [9]. We make it explicit for later use. Here and elsewhere $S$ denotes the sphere spectrum.

**Lemma 3.1.** Let $R$ be a commutative $S$-algebra and let $C \to B \to k$ be a sequence of maps of commutative $R$-algebras. Then there is a cofiber sequence of commutative $k$-algebras

$$B \wedge_{C}^R k \to \text{THH}^R(C, k) \to \text{THH}^R(B, k).$$

The proof is obtained from the one of [13, lemma 7.1] by replacing the sphere spectrum by $R$.

**Remark 3.2.** Note that there are two cofiber sequences for any such sequence $C \to B \to k$, because we can forget the commutative $R$-algebra structures on $C$ and $B$ and consider them as commutative $S$-algebras. This gives a commutative diagram of cofiber sequences

$$
\begin{array}{ccc}
B \wedge_{C}^R k & \to \text{THH}(C, k) & \to \text{THH}(B, k) \\
\downarrow & & \downarrow \\
B \wedge_{C}^R k & \to \text{THH}^R(C, k) & \to \text{THH}^R(B, k),
\end{array}
$$

so $B \wedge_{C}^R k$ measures the difference of the absolute and also of the relative THH-terms of $C$ and $B$.

Let us abbreviate $B \wedge_{C}^R k$ by $A$. Lemma 3.1 provides an equivalence

$$\text{THH}^R(B, k) \simeq \text{THH}^R(C, k) \wedge_{A}^L k.$$
and thus we get a spectral sequence whose $E^2$-term is

$$\text{Tor}^A_{a_1}(\text{THH}_s(C, k), k_s)$$

which converges to $\text{THH}_s(B, k)$.

We will consider the following examples.

(i) Let $\ell$ denote the Adams summand of $p$-local connective topological complex K-theory, $ku_{(p)}$, for some odd prime $p$. For

$$R = \ell \longrightarrow C = \ell \longrightarrow B = ku_{(p)} \longrightarrow k$$

with $k = H\mathbb{Z}_{(p)}$ or $k = H\mathbb{F}_p$ we obtain calculations for $\text{THH}_s(ku_{(p)}, k)$. We determine $\text{THH}_s(ku_{(p)})$ by different means.

(ii) The complexification map from real to complex topological K-theory $c : ko \rightarrow ku$ is a map of commutative $S$-algebras. Wood’s theorem displays the $ko$-module $ku$ as the cofiber of the Hopf map $\eta : \Sigma ko \rightarrow ko$. Consequently, the $ku$-module $ku \wedge_{ko}^L ku$ is the cofiber of $\eta : \Sigma ku \rightarrow ku$, and the resulting short exact sequences

$$0 \longrightarrow \pi_{2m} ku \longrightarrow \pi_{2m}(ku \wedge_{ko}^L ku) \longrightarrow \pi_{2m-1}(\Sigma ku) \longrightarrow 0$$

are split via the multiplication map on $ku$, because the map $ku \rightarrow ku \wedge_{ko}^L ku$ above is induced by the unit map of $ku$ as a commutative $ko$-algebra so we get

$$\pi_{2m}(ku \wedge_{ko}^L ku) \cong \pi_{2m} ku \oplus \pi_{2m-2}(ku).$$

We will determine the $ku_s$-algebra structure of $\pi_s(ku \wedge_{ko}^L ku)$ in Lemma 5.1. This is the input for the Tor-spectral sequence computing $\text{THH}_s(ko)$ and we will identify $\text{THH}_s(ko)$ in Theorem 5.2.

We will also use the cofiber sequences of commutative $k$-algebras

$$ku \wedge_{ko}^L k \longrightarrow ku \longrightarrow \text{THH}_s(ko, k)$$

for $k = H\mathbb{Z}_{(2)}$ and $k = H\mathbb{F}_2$ and we will calculate THH of $ku$ over $ko$ with coefficients in $H\mathbb{Z}_{(2)}$ and $H\mathbb{F}_2$ (see Proposition 5.4).

(iii) We propose $ku_{(p)} \wedge_{\ell}^L H\mathbb{F}_p$ as a model for $ku/\langle p, v_1 \rangle$ and use the sequence

$$S \longrightarrow H\mathbb{F}_p \longrightarrow ku_{(p)} \wedge_{\ell}^L H\mathbb{F}_p \longrightarrow H\mathbb{F}_p$$

for calculating its THH with coefficients in $H\mathbb{F}_p$ (Proposition 6.2).

(iv) In Section 7 we determine relative topological Hochschild homology of quotient maps $R \rightarrow R/x$.

4. Relative THH of $ku_{(p)}$ as a commutative $\ell$-algebra

Let $p$ be an odd prime. On the level of coefficients, the map from the connective Adams summand to $p$-local connective topological complex K-theory is $\ell_s = \mathbb{Z}_{(p)}[v_1] \rightarrow \mathbb{Z}_{(p)}[u] = (ku_{(p)})_s, v_1 \mapsto u^{p-1}$. The corresponding $p$-complete periodic extension is a $C_{p-1}$-Galois extension [23]. However, the connective extension is not unramified, but it is a topological analogue of a tamely ramified extension. Rognes defined a notion of THH-étale extensions in [23, 9.2-1]: A map of commutative $S$-algebras $A \rightarrow B$ is formally THH-étale, if the canonical map from $B$ to $\text{THH}_s^A(B)$ is an equivalence. For instance, Galois extensions are formally THH-étale [23, 9.2-6]. We will show that the map $\ell \rightarrow ku_{(p)}$ is not formally THH-étale by determining $\text{THH}_s^\ell(ku_{(p)})$. Rognes mentions in [23, p. 59] that $ku_{(p)} \rightarrow \text{THH}_s^\ell(ku_{(p)})$
is a $K(1)$-local equivalence and Sagave showed in [24] that the map $\ell \to ku(p)$ is log-étale.
Ausoni proved that the $p$-completed extension even satisfies Galois descent for $\text{THH}$ and
algebraic K-theory [2, theorem 1.5]:

$$\text{THH}(ku(p))^{hC_{p^{-1}}} \simeq \text{THH}(\ell_p), \quad K(ku(p))^{hC_{p^{-1}}} \simeq K(\ell_p).$$

The tame ramification is visible in $\text{THH}$:

**Theorem 4.1.**

$$\text{THH}_s^s(ku(p)) \simeq (ku(p))_s \rtimes ((ku(p))_s/u^{p-2})_{\{y_0, y_1, \ldots\}},$$

where $(ku(p))_s \rtimes M$ denotes a square-zero extension of $(ku(p))_s$ by a $(ku(p))_s$-module $M$. The degree of $y_i$ is $2pi + 3$.

**Proof.** We can apply the Bökstedt spectral sequence with $\pi_s$ as the homology theory
because $(ku(p))_s$ is projective over $\ell_s$. The $E^2$-page consists of

$$E^2_{s,t} = \text{HH}_{s,t}^{\ell_s}((ku(p))_s, (ku(p))_s).$$

As an $\ell_s$-algebra $(ku(p))_s$ is isomorphic to $\ell_s[u]/(u^{p-1} - v_1)$. From [17] we know that we
can use the following complex in order to calculate Hochschild homology:

$$\ldots \xrightarrow{\Delta(u)} \Sigma^2p(ku(p))_s \xrightarrow{0} \Sigma^2p-2(ku(p))_s \xrightarrow{\Delta(u)} \Sigma^2(ku(p))_s \xrightarrow{0} (ku(p))_s,$$

where $\Delta(u) = (p - 1)u^{p-2}$. As $(p - 1)$ is a unit in $\ell_s$, this yields:

$$\text{HH}_s^s((ku(p))_s, (ku(p))_s) = \begin{cases}
(ku(p))_s, & \text{if } i = 0, \\
\Sigma^{2mp-2m+2}(ku(p))_s/u^{p-2}, & \text{if } i = 2m + 1, m \geq 0, \\
0, & \text{otherwise}.
\end{cases}$$

As $\text{THH}_s^s(ku(p))$ is an augmented commutative $ku(p)$-algebra, we know that $ku(p)$ splits off
$\text{THH}_s^s(ku(p))$. Therefore the copy of the homotopy groups of $ku(p)$ in the zero column of the
spectral sequence has to survive and cannot be hit by any differentials. For degree reasons,
there are no other possible non-trivial differentials and the spectral sequence collapses at the
$E^2$-page.

In every fixed total degree there is only one term in the $E^2$-page contributing to this
degree: If we consider an element $u^{k_1}$ in homological degree $2m_1 + 1$ and another element
$u^{k_2}$ in homological degree $2m_2 + 1$ for $m_1 \neq m_2$, then their total degrees are $2m_1 p + 2k_1 + 3$ and $2m_2 p + 2k_2 + 3$. These degrees can only be equal if $2p(m_1 - m_2) = 2(k_2 - k_1)$. Thus
$p$ has to divide $k_2 - k_1 \neq 0$. But $0 \leq k_1, k_2 \leq p - 3$, so this cannot happen.

Thus there are no additive extensions and therefore additively we get the desired result.

As $\text{THH}_s^s(ku(p))$ is an augmented graded commutative $(ku(p))_s$-algebra and as everything
in the augmentation ideal is concentrated in odd degrees there cannot be any non-trivial
multiplication of any two elements in the augmentation ideal.

The spectral sequence is a spectral sequence of $(ku(p))_s$-modules and elements of the form
$u^b \cdot \Sigma^{2mp-2m+2}u^m$ are cycles, thus the copy of $(ku(p))_s$ in homological degree zero acts on
$ku(p)_s/u^{p-2}y_m$ in the standard way.

**Remark 4.2.** For Galois extensions of non-connective commutative ring spectra we would like to have a good notion of rings of integers. In the above case $ku(p)$ behaves like the ring of integers of $KU(p)$, and similarly for the connective Adams summand. The result for relative
THH corresponds to the one of ordinary rings of integers \([18]\). In other cases, taking the connective cover does not seem to give good results.

For coefficients in \(H\mathbb{Z}_{(p)}\) and \(H\mathbb{F}_p\) we obtain a rather different result.

**Proposition 4.3.**

\[
\operatorname{THH}^\ell_s(ku_{(p)}, H\mathbb{Z}_{(p)}) \cong \Lambda_{\mathbb{Z}_{(p)}}(\varepsilon u) \otimes \Gamma_{\mathbb{Z}_{(p)}}(\varphi^0 u)
\]

and also

\[
\operatorname{THH}^\ell_s(ku_{(p)}, H\mathbb{F}_p) \cong \Lambda_{\mathbb{F}_p}(\varepsilon u) \otimes \Gamma_{\mathbb{F}_p}(\varphi^0 u).
\]

**Proof.** We consider the sequence of commutative \(\ell\)-algebras

\[
R = \ell \longrightarrow C = \ell \longrightarrow B = ku_{(p)} \longrightarrow k
\]

with \(k = H\mathbb{Z}_{(p)}\) and \(k = H\mathbb{F}_p\). In both cases, by Lemma 3.1 we get that

\[
\operatorname{THH}^\ell(ku_{(p)}, k) \cong \operatorname{THH}^\ell(\ell, k) \wedge_{ku_{(p)}, \ell}^L k \cong k \wedge_{ku_{(p)}, \ell}^L k
\]

which gives a Tor-spectral sequence

\[
\operatorname{Tor}^{\pi_*(ku_{(p)}, \ell)}(\pi_* k, \pi_* k) \Longrightarrow \operatorname{THH}^\ell_s(ku_{(p)}, k).
\]

For \(k = H\mathbb{Z}_{(p)}\) homological algebra tells us that

\[
\operatorname{Tor}^{\pi_*(u)/u=1}_{(p), \ell}(\mathbb{Z}_{(p)}, \mathbb{Z}_{(p)}) \cong \Lambda_{\mathbb{Z}_{(p)}}(\varepsilon u) \otimes \Gamma_{\mathbb{Z}_{(p)}}(\varphi^0 u).
\]

Here, \(|\varepsilon u| = 3\) and \(|\varphi^0 u| = 2p\). There are no differentials in this spectral sequence for degree reasons and there are no multiplicative extensions, hence we get the claim.

For \(k = H\mathbb{F}_p\) the same method gives

\[
\operatorname{THH}^\ell_s(ku_{(p)}, H\mathbb{F}_p) \cong \Lambda_{\mathbb{F}_p}(\varepsilon u) \otimes \Gamma_{\mathbb{F}_p}(\varphi^0 u).
\]

**Remark 4.4.** At the prime 3 we get

\[
\operatorname{THH}^{[0], \ell}_{s}(ku_{(3)}, H\mathbb{F}_3) \cong \mathbb{F}_3[u]/u^2
\]

hence with the methods of [9] we can deduce that

\[
\operatorname{THH}^{[0], \ell}_{s}(ku_{(3)}, H\mathbb{F}_3) \cong H\mathbb{F}_3 \vee \Sigma^2 H\mathbb{F}_3
\]

as an augmented commutative \(H\mathbb{F}_3\)-algebra and that we can calculate higher \(\operatorname{THH}\) as iterated Tor-algebras. Hence we get

\[
\operatorname{THH}^{[n+1], \ell}_{s}(ku_{(3)}, H\mathbb{F}_3) \cong \operatorname{Tor}^{\pi_*(ku_{(3)}, H\mathbb{F}_3)}_{s}(\mathbb{F}_3, \mathbb{F}_3)
\]

for all \(n \geq 0\).

Using Greenlees’ spectral sequence [13, lemma 3.1] one can actually deduce that this is true at all odd primes.

5. Relative \(\operatorname{THH}\) of the complexification map

The graded commutative ring \(ko_s\) is \(\mathbb{Z}[\eta, y, w]/(2\eta, \eta y, \eta^3, y^2 - 4w)\) with \(|\eta| = 1\), \(|y| = 4\) and \(w\) is the Bott class in degree 8. The complexification map \(c : ko \to ku\) induces a map \(c_s : ko_s \to ku_s = \mathbb{Z}[u]\) and it sends \(\eta\) to zero, \(y\) to \(2u^2\) and the Bott class \(w\) to \(u^4\).

Note that the homotopy fixed points of \(ku\) with respect to complex conjugation are not equivalent to \(ko\). The homotopy fixed points spectral sequence yields generators in negative degrees in the homotopy groups of \(ku^hC_2\) [23, 5.3].
Lemma 5.1. As a graded commutative augmented ku*-algebra
\[(ku \wedge^{L}_{ko} ku)_* \cong ku_*[\tilde{u}]/\tilde{u}^2 - u^2\]
with |\tilde{u}| = 2.

Proof. As we saw in the introduction, Wood’s theorem gives that \((ku \wedge^{L}_{ko} ku)_*\) is additively isomorphic to \(ku_* \oplus \pi_*(\Sigma^2 ku)\). The Tor spectral sequence
\[\text{Tor}_{*,*}^{ku}(ku_*, ku_*) \Rightarrow (ku \wedge^{L}_{ko} ku)_*\]
allows us to determine the multiplicative structure.

The tensor product \(ku_* \otimes_{ko_*} ku_* \cong ku_*[\tilde{u}]/(2\tilde{u}^2 - 2u^2, \tilde{u}^4 - u^4)\) has three generators in degree four: \(u^2, u\tilde{u}, \tilde{u}^2\). The element \(\tilde{u}^2 - u^2\) is 2-torsion, but there is no 2-torsion in the abutment \((ku \wedge^{L}_{ko} ku)_*\). Hence this class has to die via a differential in the spectral sequence.

Theorem 5.2. The Tor spectral sequence
\[E^2_{*,*} = \text{Tor}^{ku}_{*,*}((ku \wedge^{L}_{ko} ku)_*, ku_*) \Rightarrow \text{THH}^{ko}_*(ku)\]
collapses at the \(E^2\)-page and \(\text{THH}^{ko}_*(ku)\) is a square zero extension of \(ku_*\):
\[\text{THH}^{ko}_*(ku) \cong ku_* \times (ku_* \langle y_0, y_1, \ldots \rangle)\]
with |y_j| = (1 + |u|)(2j + 1) = 3(2j + 1).

Proof. Lemma 5.1 implies that the \(E^2\)-term of the Tor spectral sequence is
\[E^2_{*,*} = \text{Tor}^{ku}_{*,*}((ku \wedge^{L}_{ko} ku)_*, ku_*) = \text{Tor}^{ku_*[\tilde{u}]}/\tilde{u}^2 - u^2 (ku_*, ku_*),\]
where \(\varepsilon: ku_*[\tilde{u}]/\tilde{u}^2 - u^2 \rightarrow ku_*, \varepsilon(\tilde{u}) = u\) gives the \((ku \wedge^{L}_{ko} ku)_*\)-module structure of \(ku_*\). We have a periodic free resolution of \(ku_*\) as a module over \(ku_*[\tilde{u}]/(\tilde{u}^2 - u^2)\)
\[\ldots \rightarrow \Sigma^4 ku_*[\tilde{u}]/(\tilde{u}^2 - u^2) \rightarrow \Sigma^2 ku_*[\tilde{u}]/(\tilde{u}^2 - u^2) \rightarrow ku_*[\tilde{u}]/(\tilde{u}^2 - u^2).\]
Tensoring this down to \(ku_*\) yields
\[\ldots \rightarrow \Sigma ku_* \rightarrow ku_* \rightarrow ku_* .\]
As \(ku_*\) splits off \(\text{THH}^{ko}_*(ku)\) the zero column has to survive and cannot be hit by differentials and hence all differentials are trivial.

For the \(E^\infty\)-term we therefore get \(E^0_{0,*} \cong ku_*\), \(E^\infty_{j,*} = 0\) for \(j > 0\), and \(E^\infty_{2j+1,*} \cong (ku_* \langle y_j \rangle)\) for \(y_j\) in bidegree \((2j + 1, 4j + 2)\) if \(j > 0\). Thus we have multiple contributions when the odd total degree is greater than or equal to 9; we claim that the additive extensions are all trivial.

The spectral sequence is one of \(ku_*\)-algebras, so in particular, one of \(ku_*\)-modules. In total degree 9 we only have the generators \(y_1 \in E^\infty_{3,6}\) generating a copy of \(\mathbb{Z}\) and \(u^3 y_0 \in E^\infty_{1,8}\) generating a copy of \(\mathbb{Z}/2\mathbb{Z}\). Since the only extension of \(\mathbb{Z}/2\mathbb{Z}\) by \(\mathbb{Z}\) is the trivial one, we conclude that
\[\text{THH}^{ko}_0(ku) \cong \mathbb{Z}/2\mathbb{Z}\langle u^3 y_0 \rangle \oplus \mathbb{Z}\langle y_1 \rangle\]
and moreover, since the image of \(\text{THH}^{ko}_0(ku)\) under the multiplication by powers of \(u\) gives \(F_3(\text{THH}^{ko}_{9+2i}(ku))\) for all \(i \geq 1\), that
\[F_3(\text{THH}^{ko}_{9+2i}(ku)) \cong \mathbb{Z}/2\mathbb{Z}\langle u^3 y_0 \rangle \oplus \mathbb{Z}/2\mathbb{Z}\langle u^i y_1 \rangle\]
for all such $i$, concluding the calculation of $\text{THH}^{ko}_{i1}(ku)$ and $\text{THH}^{ko}_{i3}(ku)$. In total degree 15, we also get $y_2 \in E_{3,10}^\infty$, but since the only extension of $\mathbb{Z}/2\mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z}$ by $\mathbb{Z}$ is the trivial one, we conclude that

$$\text{THH}^{ko}_{15}(ku) \cong \mathbb{Z}/2\mathbb{Z}\langle u^6 y_0 \rangle \oplus \mathbb{Z}/2\mathbb{Z}\langle u^3 y_1 \rangle \oplus \mathbb{Z}(y_2)$$

and similarly that $F_5(\text{THH}^{ko}_{i+2}(ku))$ splits as a direct sum of all the $E^\infty$ contributions in filtration degree less than or equal to 5 for all $i \geq 1$, and we continue inductively.

Since the generators $y_i$ over $ku$ are all in odd degree, and their products cannot hit the direct summand $ku_s$ in filtration degree zero, their products are all zero.

**Remark 5.3.** The relative $\text{THH}$-groups above are similar to the Hochschild homology groups of the Gaussian integers:

$$\text{HH}^Z_*(\mathbb{Z}[i]) \cong \text{THH}^H_{*+i}(H\mathbb{Z}[i]) = \begin{cases} \mathbb{Z}[i], & \text{for } * = 0, \\ \mathbb{Z}[i]/2i, & \text{for odd } *, \\ 0, & \text{otherwise.} \end{cases}$$

(5.1)

Hence

$$\text{HH}^Z_*(\mathbb{Z}[i]) \cong \mathbb{Z}[i] \times (\mathbb{Z}[i]/2i)(y_j, j \geq 0)$$

with $|y_j| = 2 j + 1$. Hence we might view $ko \to ku$ as being wildly ramified.

We consider the sequence of commutative $ko$-algebras $R = ko \to C = ko \to B = ku$ with $k = H\mathbb{F}_2$ or $k = H\mathbb{Z}_{(2)}$ and, (since $\text{THH}^{ko}(ko, k) \cong k$), we get cofiber sequences of commutative $k$-algebras

$$ku \wedge^{L}_{ko} k \to k \to \text{THH}^{ko}(ku, k).$$

This yields a Tor-spectral sequence

$$E^2_{s,t} = \text{Tor}_{s+t}^{\pi_*(ku \wedge^{L}_{ko} k)}(k, k) \Longrightarrow \text{THH}^{ko}_{s+t}(ku, k).$$

(5.2)

Wood’s cofiber sequence identifies $ku$ as the cone on $\eta: \Sigma ko \to ko$. Thus we get a cofiber sequence

$$\Sigma k \to k \to ku \wedge^{L}_{ko} k$$

and $\pi_*(ku \wedge^{L}_{ko} k) \cong \pi_*(k \vee \Sigma^2 k) \cong \Lambda_{x_2}(x_2)$ where $x_2$ is a generator of degree two.

For $k = H\mathbb{F}_2$ and $H\mathbb{Z}_{(2)}$ we can deduce with [9, 2.1] that as a commutative augmented $k$-algebra $ku \wedge^{L}_{ko} k$ is weakly equivalent to the square-zero extension $k \vee \Sigma^2 k$. Thus

$$\text{THH}^{ko}(ku, k) \simeq k \wedge^{L}_{k \vee \Sigma^2 k} k$$

and the spectral sequence (5.2) reduces to

$$E^2_{s,t} = \text{Tor}_{s+t}^{\pi_*(k \vee x_2 \wedge^{2}_{\Sigma^2})}(\pi_* k, \pi_* k) \Rightarrow \text{THH}^{ko}_{s+t}(ku, k).$$

But $\text{Tor}_{s+t}^{\pi_*(k \vee x_2 \wedge^{2}_{\Sigma^2})}(\pi_* k, \pi_* k) \cong \Lambda_{x_2}(x_2) \otimes \Gamma_{x_2}(\phi^0 x_2)$ with $|x_2| = 3$, $|\phi^0 x_2| = 6$, and we know from [4] combined with the methods from [9, section 3] that there cannot be any differentials in this spectral sequence. Hence we obtain

**Proposition 5.4.**

$$\text{THH}^{ko}_{*}(ku, H\mathbb{Z}_{(2)}) \cong \Lambda_{x_2}(x_2) \otimes \Gamma_{x_2}(\phi^0 x_2)$$
and

\[ \text{THH}_k^{ko}(ku, H\mathbb{F}_2) \cong \Lambda_{\mathbb{F}_2}(\varepsilon x_2) \otimes \Gamma_{\mathbb{F}_2}(\varphi^0 x_2). \]

Over \( \mathbb{F}_2 \) we can also iterate the calculation and obtain

\[ \text{THH}_k^{[n+1],ko}(ku, H\mathbb{F}_2) \cong \text{Tor}^{\text{THH}_k^{[0],ko}}(ku, H\mathbb{F}_2)(\mathbb{F}_2, \mathbb{F}_2). \]

**Remark 5.5.** To the eyes of \( \text{THH} \) with coefficients in \( H\mathbb{F}_p \) coefficients (for \( p = 2 \) resp. \( p = 3 \)) the extensions \( ko \to ku \) and \( \ell \to ku_{(3)} \) show a similar behaviour. This is analogous to the algebraic case: Hochschild homology homology of the 2-local Gaussian integers with coefficients in \( \mathbb{F}_2 \) is isomorphic to \( \Lambda_{\mathbb{F}_2}(x_2) \otimes \Gamma_{\mathbb{F}_2}(x_2) \) and \( \text{HH}^{[3]}_*(\mathbb{Z}[\sqrt{3}], \mathbb{F}_3) \cong \Lambda_{\mathbb{F}_3}(x_2) \otimes \Gamma_{\mathbb{F}_3}(x_2) \). Thus Hochschild homology (and also higher Hochschild homology) with reduced coefficients doesn’t distinguish tame from wild ramification either.

6. \( ku(p) \wedge_\ell^L H\mathbb{F}_p \) as a model for \( ku/(p, v_1) \)

John Greenlees asks in [13, example 8.4] for a commutative \( S \)-algebra model of \( ku/(p, v_1) \). We suggest \( ku/(p, v_1) = ku(p) \wedge_\ell^L H\mathbb{F}_p \) which is a commutative \( S \)-algebra (even an augmented commutative \( H\mathbb{F}_p \)-algebra, which might not be what Greenlees had in mind) and satisfies \( \pi_* (ku(p) \wedge_\ell^L H\mathbb{F}_p) \cong \mathbb{F}_p[u]/u^{p-1} \).

**Remark 6-1.** Alternatively one could consider \( ku/(p, v_1) \) defined by an iterated cofiber sequence. This is an \( A_\infty \)-ring spectrum [1, 3.7], hence an associative \( S \)-algebra, but we cannot expect any decent level of commutativity without the price of getting something of the homotopy type of a generalized Eilenberg–MacLane spectrum: if \( ku/(p, v_1) \) were a pseudo-\( H_2 \) spectrum, then it automatically splits as a wedge of suspensions of \( H\mathbb{F}_p \)’s [8, III.4.1]. In particular, an \( E_\infty \)-structure (\textit{i.e.}, a commutative \( S \)-algebra structure) would lead to such a splitting.

We determine \( \text{THH}(ku(p) \wedge_\ell^L H\mathbb{F}_p, H\mathbb{F}_p) \).

**Proposition 6-2.** Topological Hochschild homology of \( ku(p) \wedge_\ell^L H\mathbb{F}_p \) with coefficients in \( H\mathbb{F}_p \) is

\[ \text{THH}_*(ku(p) \wedge_\ell^L H\mathbb{F}_p, H\mathbb{F}_p) \cong \mathbb{F}_p[\mu] \otimes \Lambda_{\mathbb{F}_p}(\varepsilon u) \otimes \Gamma_{\mathbb{F}_p}(\varphi^0 u), \]

where \( \mathbb{F}_p[\mu] = \text{THH}_*(H\mathbb{F}_p). \)

**Proof.** Greenlees’ cofiber sequence [13, 7.1] yields an equivalence

\[ \text{THH}(ku(p) \wedge_\ell^L H\mathbb{F}_p, H\mathbb{F}_p) \cong H\mathbb{F}_p \wedge_{ku(p) \wedge_\ell^L H\mathbb{F}_p} \text{THH}(H\mathbb{F}_p). \]

Therefore, the \( \text{Tor} \)-spectral sequence has \( E^2 \)-term

\[ \text{Tor}^{\mathbb{F}_p[u]/u^{p-1}}_{*,*}((\mathbb{F}_p, \text{THH}_*(H\mathbb{F}_p))). \]

We use the standard periodic resolution of \( \mathbb{F}_p \) over \( \mathbb{F}_p[u]/u^{p-1} \). As \( \text{THH}(H\mathbb{F}_p) \) has the same chromatic type as \( H\mathbb{F}_p \), \( u \) acts by zero on \( \text{THH}_*(H\mathbb{F}_p) = \mathbb{F}_p[\mu] \) and hence the \( E^2 \)-term is isomorphic to

\[ \mathbb{F}_p[\mu] \otimes \Lambda_{\mathbb{F}_p}(\varepsilon u) \otimes \Gamma_{\mathbb{F}_p}(\varphi^0 u). \]

As \( \text{THH}(ku(p) \wedge_\ell^L H\mathbb{F}_p) \) is an augmented commutative \( \text{THH}(H\mathbb{F}_p) \)-algebra, the \( \mathbb{F}_p[\mu] \)-factor splits off and hence there cannot be any differentials and multiplicative extensions.
7. Killing regular generators in $\pi_* R$

Killing regular elements in the homotopy groups of a commutative $S$-algebra rarely gives rise to commutative quotients. However, there are some important examples for which we do get commutative quotients whose relative THH can be calculated.

**Proposition 7.1.** Let $R$ be a connective commutative $S$-algebra whose coefficients $\pi_* R$ are concentrated in even degrees, with a nonzero divisor $x$ of positive degree. If the canonical map $R \to R/x$ is a morphism of commutative $S$-algebras, then the Tor spectral sequence

$$\text{Tor}_{\pi_* R}^*(R/x, (R/x)_*) \Longrightarrow \text{THH}_*(R/x)$$

collapses at the $E^2$-term. Its $E_\infty$-term is isomorphic to $\Gamma_{\pi_+(R/x)}(\rho^0 \varepsilon x)$ with $|\rho^0 \varepsilon x| = |x| + 2$ and there are no additive extensions.

**Proof.** The defining cofiber sequence

$$\Sigma |x| R \xrightarrow{x} R \xrightarrow{x} R/x$$

gives a long exact sequence of stable homotopy groups, and since $x$ is not a zero divisor it splits into short exact sequences yielding $\pi_+(R/x) \cong \pi_+(R)/x$. That also gives us the form of the $E^2$-spectral sequence of $\pi_+(R)/x$-modules $\text{Tor}_{\pi_+(R)}^*(R/x, \pi_+(R)/x) \Rightarrow \pi_+(R/x) \wedge_R^L (R/x)$, which contains only a zeroth and first column and therefore collapses to give

$$\pi_+(R/x) \wedge_R^L (R/x) \cong \Lambda_{\pi_+(R)/x} (\varepsilon x)$$

with $|\varepsilon x| = |x| + 1$. In the spectral sequence calculating the homotopy groups of $\text{THH}_*(R/x) \cong (R/x) \wedge_R (R/x)$, we then have as an $E^2$-term

$$\text{Tor}_{\pi_+(R)/x}^* (\varepsilon x) (\pi_+(R)/x, \pi_+(R)/x).$$

We consider the periodic resolution of $\pi_+(R)/x$

$$\ldots \xrightarrow{\varepsilon x} \Sigma 2|x|+2 \Lambda_{\pi_+(R)/x} (\varepsilon x) \xrightarrow{\varepsilon x} \Sigma |x|+1 \Lambda_{\pi_+(R)/x} (\varepsilon x) \xrightarrow{\varepsilon x} \Lambda_{\pi_+(R)/x} (\varepsilon x)$$

and tensor it down to $\pi_+(R)/x$. As $\pi_+(R)/x$ is concentrated in even degrees, the multiplication by $\varepsilon x$ induces the trivial map and hence our Tor-terms are the homology of the complex

$$\ldots \xrightarrow{\varepsilon x=0} \Sigma 2|x|+2 \pi_+(R)/x \xrightarrow{\varepsilon x=0} \Sigma |x|+1 \pi_+(R)/x \xrightarrow{\varepsilon x=0} \pi_+(R)/x$$

and this gives a divided power algebra $\Gamma_{\pi_+(R)/x} (\rho^0 \varepsilon x)$ with a generator $\rho^0 \varepsilon x$ in degree $|x|+2$. We have to show that there are no non-trivial differentials and no extension problems.

Since the spectral sequence is spanned over $\pi_+(R)/x$, which has only even-dimensional classes, by divided powers of an even-dimensional generator, all the non-zero elements have even dimension and so the spectral sequence has no non-trivial differentials.

The spectral sequence is a spectral sequence of $\pi_+(R)/x = \pi_+(R)/x$-algebras because $R/x$ is assumed to be a commutative $R$-algebra, hence $\text{THH}_*(R/x)$ is a commutative $R/x$-algebra. We do not have additive extensions because the $E_\infty$-term is free over $\pi_+(R)/x$. Thus as an $\pi_+(R)/x$-module we get that $\text{THH}_*^R (R/x) \cong \Gamma_{\pi_+(R)/x} (\rho^0 \varepsilon x)$.

**Corollary 7.2.** If in addition to the assumptions in Proposition 7.1 we have that $R/x$ is an Eilenberg–MacLane spectrum of a commutative ring $k$, then

$$\text{THH}_*^R (Hk, Hk) \cong Hk \wedge_{Hk \wedge \Sigma |x|+1 Hk} Hk$$
as augmented commutative $Hk$-algebras. In particular,
\[ \text{THH}_{\omega}^R(Hk) \cong \Gamma_k(\rho^0 \varepsilon x) \]
with $|\rho^0 \varepsilon x| = |x| + 2$.

**Proof.** Greenlees’ cofiber sequence identifies $\text{THH}_R(Hk)$ as
\[ Hk \wedge_{Hk \wedge Hk} L R Hk \]
using the sequence of commutative ring spectra $R = R \rightarrow Hk = Hk$. The homotopy groups of $Hk \wedge_{R}^L Hk$ are isomorphic to $\Lambda_k(\varepsilon x)$ with $|\varepsilon x| = |x| + 1$. Hence we know from [9, proposition 2.1] that
\[ Hk \wedge_{R}^L Hk \cong Hk \vee \Sigma |\varepsilon x| Hk \]
with the square zero multiplication as augmented commutative $Hk$-algebras. Therefore we get the first claim. This also shows that $\text{THH}_R(Hk)$ can be modeled as the two-sided bar construction
\[ B^{\text{H}k}(Hk, Hk \vee \Sigma |\varepsilon x| Hk, Hk) \]
and by [9] we know that its homotopy groups are the homology groups of the algebraic bar construction $B^k(k, \Lambda(\varepsilon x), k)$. We know from [4, proposition 2.3] that there is a quasiisomorphism between $\Gamma_k(\rho^0 \varepsilon x)$ (with zero differential) and the differential graded commutative algebra associated to $B^k(k, \Lambda(\varepsilon x), k)$.

**COROLLARY 7-3.** If in addition to the assumptions of Corollary 7-2 the ring $k$ is the field $\mathbb{F}_p$ we get
\[ \text{THH}_{\omega}^{[n+1], R}(H\mathbb{F}_p, H\mathbb{F}_p) \cong \text{Tor}_{\omega, \ast}^{\text{THH}_{op}^R(H\mathbb{F}_p, H\mathbb{F}_p)}(\mathbb{F}_p, \mathbb{F}_p) \]
for all $n \geq 0$.

**Remark 7-4.** In the above statement one can consider a slightly more general case of any field of characteristic $p$.

**PROPOSITION 7-5.** Assume in addition to the requirements of Proposition 7-1 that there is a regular sequence $(x, y_1, \ldots, y_n)$ in $\pi_\ast R$ such that $R/(x, y_1, \ldots, y_n)$ is $Hk$ for some field $k$. Then
\[ \text{THH}_{\omega}^{[n+1], R}(R/x, Hk) \cong \Gamma_k(\rho^0 \varepsilon x) \]
with $|\rho^0 \varepsilon x| = |x| + 2$. If $k = \mathbb{F}_p$, then
\[ \text{THH}_{\omega}^{[n+1], R}(R/x, H\mathbb{F}_p) \cong \text{Tor}_{\omega, \ast}^{\text{THH}_{op}^R(R/x, H\mathbb{F}_p)}(\mathbb{F}_p, \mathbb{F}_p) \]
for all $n \geq 0$.

**Proof.** We consider the sequence of commutative $S$-algebras
\[ R \rightarrow R \rightarrow R/x \rightarrow Hk. \]
Then $\pi_\ast(Hk \wedge_R^L R/x) \cong \Lambda_k(\varepsilon x)$ and as before we can conclude with [9, 2.1] that $Hk \wedge_R^L R/x$ is equivalent to the square zero extension $Hk \vee \Sigma |\varepsilon x| Hk$ in the homotopy category of commutative augmented $Hk$-algebras.

Greenlees’ cofiber sequence identifies $\text{THH}_R(R/x, Hk)$ as
\[ Hk \wedge_{Hk \vee \Sigma |\varepsilon x| Hk}^L Hk \]
and we know from [9, 4] that this gives $\text{THH}_a^R(R/\mathbb{F}_p, Hk) \cong \Gamma_k(\rho^0 e\mathcal{x})$.

Higher $\text{THH}$ can be calculated using the Tor spectral sequence associated to the 2-sided bar construction: A simplicial model for $\text{THH}^{[n+1], K}(R/\mathbb{F}_p, H\mathbb{F}_p)$ is

$$B(\mathbb{F}_p, \text{THH}^{[n], K}(R/\mathbb{F}_p, H\mathbb{F}_p), H\mathbb{F}_p)$$

and we know from the methods established in [9] and [4] that these Tor-spectral sequences all collapse at the $E^2$-term with no non-trivial extensions.

**Examples 7-6.** We end the section with some examples.

(i) Let $R$ be an Eilenberg–MacLane spectrum $HA$ with $A$ a commutative ring and let $x$ be regular in $A$. Then $\text{THH}^H(A/HA/x)$ is isomorphic to Shukla-homology of $A/x$ over $A$, $SH^A_x(A/x)$. In this case we obtain

$$\text{THH}^H(A/HA/x) \cong SH^A_x(A/x) \cong \Gamma_{A/x}(\rho^0 e\mathcal{x})$$

with $|\rho^0 e\mathcal{x}| = 2$. An explicit generator of $SH^A_x(A/x)$ is given by

$$\sum_{i=0}^{m} (-1)^i \tau \otimes i \otimes 1 \otimes \tau^{m-i}.$$ 

Here, we consider the resolution of $A/x$ that is of the form $(A[\tau]/\tau^2, d(\tau) = x)$.

Higher order Shukla homology is crucial for determining higher order $\text{THH}$ of $\mathbb{Z}/p^m\mathbb{Z}$ with coefficients in $\mathbb{Z}/p\mathbb{Z}$, see [5].

(ii) Recall that the connective covers of the Morava $E$-theories, $e_n$, have coefficients

$$\pi_* e_n \cong W^F_p[[u_1, \ldots, u_{n-1}]][u]$$

with $|u| = 2$, where $W^F_p$ denotes the Witt vectors over $\mathbb{F}_p$ and where the $u_i$ are generators in degree zero. Hence $\pi_0 e_n = W^F_p[[u_1, \ldots, u_{n-1}]]$. The quotient $e_n/u = H^F_p W^F_p[[u_1, \ldots, u_{n-1}]]$ is a commutative $S$-algebra and the map $e_n \to e_n/u$ can be realized as a map of commutative $S$-algebras.

The residue field $H^F_p$ is the quotient $e_n/(u, u_1, \ldots, u_{n-1}, p)$ and thus the results of Section 7 allow us to calculate $\text{THH}_e^H(e_n/u, e_n/u)$ and $\text{THH}_e^{[m]}(e_n/u, H^F_p, p)$ for all $m \geq 1$.

(iii) Lawson and Naumann show in [20] that $BP(2)$ at the prime two is a commutative $S$-algebra by identifying it with the 2-localised connective spectrum of topological modular forms together with a level three structure, $\text{tmf}_1(3)_{(2)}$. They prove in [21, section 5] that there is a map of commutative $S$-algebras $\varphi: \text{tmf}_1(3)_{(2)} \to ku(2)$ and there is a complex orientation of $\text{tmf}_1(3)_{(2)}$ such that the effect of $\varphi$ on homotopy groups is as follows [21, section 5]:

$$\pi_* \varphi: \text{tmf}_1(3)_{(2)} \to \mathbb{Z}_{(2)}[a_1, a_3] \to \mathbb{Z}_{(2)}[u], \quad a_i \mapsto u, \quad a_3 \mapsto 0.$$ 

Here the degree of $a_i$ is $2i$.

With Propositions 7.1 and 7.5 we can determine $\text{THH}^\text{tmf}_1(3)_{(2)}(ku(2))$ additively and we get explicit formulae for higher relative $\text{THH}$ of $ku(2) \cong \text{tmf}_1(3)_{(2)}/a_3$ with respect to $\text{tmf}_1(3)_{(2)}$ and with coefficients in $H^F_2 = \text{tmf}_1(3)_{(2)}/(a_3, a_1, 2)$.

(iv) The discretisation map from $ku$ to $H\mathbb{Z} = ku/u$ gives rise to another example of a regular quotient with a commutative $S$-algebra structure with residue field $H^F_p = ku/(u, p)$ for any prime $p$, and so does the map from the connective Adams summand $\ell$ to $H\mathbb{Z}_{(p)} = \ell/\wp_1$ with residue field $H^F_p = \ell/(\wp_1, p)$. Thus in these
cases we can determine $\text{THH}^{[a], ku}(H\mathbb{Z}, H\mathbb{Z}/p\mathbb{Z}), \text{THH}^{[n], ku}(H\mathbb{Z}/p\mathbb{Z}, H\mathbb{Z}/p\mathbb{Z})$ for all primes and $\text{THH}^{[a], f}(H\mathbb{Z}_{(p)}, H\mathbb{Z}/p\mathbb{Z}), \text{THH}^{[n], f}(H\mathbb{Z}/p\mathbb{Z}, H\mathbb{Z}/p\mathbb{Z})$ for all odd primes and all $n$.

Acknowledgements. The second author acknowledges the support of a Simons Foundation Collaboration Grant. The last named author thanks the Hausdorff Research Institute for Mathematics in Bonn for its hospitality during the Trimester Program Homotopy theory, manifolds, and field theories. She also thanks the Department of Mathematics of the Indiana University Bloomington for an invitation in the spring of 2016.

REFERENCES

[1] VIGLEIK ANGELTVÆT. Topological Hochschild homology and cohomology of $A_\infty$ ring spectra. Geom. Topol. 12 (2008), 987–1032.
[2] CHRISTIAN AUSONI. Topological Hochschild homology of connective complex K-theory. Amer. J. Math. 127 (2005), 1261–1313.
[3] ANDREW BLUMBERG, RALPH L. COHEN and CHRISTIAN SCHLICHTKRULL. Topological Hochschild homology of Thom spectra and the free loop space. Geom. Topol. 14 (2010), 1165–1242.
[4] IRINA BOBKova, AYELET LINDENSTRAUSS, KATE POIRIER, BIRGIT RICHTER and INNA ZAKHAREVICH. On the higher topological Hochschild homology of $\mathbb{F}_p$ and commutative $\mathbb{F}_p$-group algebras, Women in Topology: Collaborations in Homotopy Theory. Contemp. Math. 641 AMS, (2015), 97–122.
[5] IRINA BOBKova, EVA HÖNING, AYELET LINDENSTRAUSS, KATE POIRIER, BIRGIT RICHTER and INNA ZAKHAREVICH. Higher $\text{THH}$ and higher Shukla homology of $\mathbb{Z}/p^n\mathbb{Z}$ and of truncated polynomial algebras over $\mathbb{F}_p$, in preparation.
[6] MARCEL BOKSTEDT. Topological Hochschild homology, preprint.
[7] MARCEL BOKSTEDT. The topological Hochschild homology of $\mathbb{Z}$ and of $\mathbb{Z}/p\mathbb{Z}$, preprint.
[8] ROBERT R. BRUNER, J. PETER MAY, JAMES E. McCLURE and MARK STEINBERGER. $H_\infty$ ring spectra and their applications. Lecture Notes in Math., 1176 (Springer-Verlag, Berlin, 1986), viii+388 pp.
[9] BJØRN IAN DUNDAS, AYELET LINDENSTRAUSS and BIRGIT RICHTER. On higher topological Hochschild homology of rings of integers, to appear in Math. Res. Lett. arXiv:1502.02504.
[10] ANTHONY D. ELMENDORF, IGOR KRIZ, MICHAEL A. MANDELL and J. PETER MAY. Rings, modules, and algebras in stable homotopy theory, With an appendix by M. Cole. Math. Surv. Monogr. 47 (American Mathematical Society, Providence, RI, 1997), xii+249 pp.
[11] VINCENT FRANJOU, JEAN LANNES and LIONEL SCHWARTZ. Autour de la cohomologie de Mac Lane des corps finis. Invent. Math. 115 (1994), 513–538.
[12] VINCENT FRANJOU and TEMURAZ PIRASHVILI. On the MacLane cohomology for the ring of integers. Topology 37 (1998), 109–114.
[13] JOHN P. C. GREENLEES. Ausoni-Bökstedt duality for topological Hochschild homology. J. Pure Appl. Alg. 220 (2016), 1382–1402.
[14] CORNELIUS GREITHER. Cyclic Galois extensions of commutative rings. Lecture Notes in Math. 1534 (Springer-Verlag, Berlin, 1992), x+145 pp.
[15] MICHAEL HILL and TYLER LAWSON. Automorphic forms and cohomology theories on Shimura curves of small discriminant. Adv. Math. 225 (2010), 1013–1045.
[16] INBAR KLANG. The factorisation homology of Thom spectra and twisted non-abelian Poincaré duality, preprint, arXiv:1606.03805.
[17] MICHAEL LARSEN and AYELET LINDENSTRAUSS. Cyclic homology of Dedekind domains. K-Theory 6 (1992), 301–334.
[18] AYELET LINDENSTRAUSS and IB MADSEN. Topological Hochschild homology of number rings. Trans. Amer. Math. Soc. 352 (2000), 2179–2204.
[19] AKHIL MATHEW. The Galois group of a stable homotopy theory. Adv. Math. 291 (2016), 403–541.
[20] TYLER LAWSON and NIKO NAUMANN. Commutativity conditions for truncated Brown-Peterson spectra of height 2. J. Topol. 5 (2012), 137–168.
[21] TYLER LAWSON and NIKO NAUMANN. Strictly commutative realisations of diagrams over the Steenrod algebra and topological modular forms at the prime 2. Int. Math. Res. Not. 10 (2014), 2773–2813.
[22] ALAN ROBINSON. Gamma homology, Lie representations and $E_\infty$ multiplications. Invent. Math. 152 (2003), 331–348.
[23] John Rognes. Galois extensions of structured ring spectra. Stably dualizable groups. *Mem. Amer. Math. Soc.* **898** (2008), viii+137 pp.

[24] Steffen Sagave. Logarithmic structures on topological $K$-theory spectra. *Geom. Topol.* **18** (2014), 447–490.

[25] Christian Schlichtkrull. Higher topological Hochschild homology of Thom spectra. *J. Topol.* **4** (2011), 161–189.