EFFECTIVE INVARIANTS OF BRAID MONODROMY AND TOPOLOGY OF PLANE CURVES

ENRIQUE ARTAL BARTOLO, JORGE CARMONA RUBER, AND JOSÉ IGNACIO COGOLLUDO AGUSTÍN

Abstract. In this paper we construct effective invariants for braid monodromy of affine curves. We also prove that, for some curves, braid monodromy determines their topology. We apply this result to find a pair of curves with conjugate equations in a number field but which do not admit any orientation-preserving homeomorphism.

Let $C \subset \mathbb{C}^2$ be an algebraic affine curve. We say a property $P(C)$ is an invariant of $C$ if it is a topological invariant of the pair $(\mathbb{C}^2, C)$, in other words, if $P(C) = P(C')$ whenever $(\mathbb{C}^2, C)$ and $(\mathbb{C}^2, C')$ are homeomorphic as pairs. Analogously, we define the concept of invariants of projective algebraic curves.

Our purpose in this paper is the construction of new and effective invariants for algebraic curves that reveal that the position of singularities is not enough to determine the topological type of the pair $(\mathbb{P}^2, C)$.

These invariants will be derived from a refinement of a well-known invariant of curves such as braid monodromy. Roughly speaking, the braid monodromy of a curve $\tilde{C} \subset \mathbb{P}^2$ with respect to a pencil of lines $\mathcal{H}$, is a representation of a free group $\mathbb{F}$ on the braid group on $d$ strings – where $d$ is the degree of $\tilde{C}$ restricted to the generic fiber of $\mathcal{H}$ after resolution of its base point. The free group $\mathbb{F}$ corresponds to the fundamental group of an $r$-punctured complex line, where the punctures come from the non-generic elements of $\mathcal{H}$.

Braid monodromy is a strong invariant for projective (or affine) plane curves. It is fair to say that the main ideas leading to this invariant have already appeared in the classic works of Zariski [23] and Van Kampen [12] as a tool to find the fundamental group of the complement of a curve. Nevertheless, its consideration as an invariant itself is due to B. Moishezon – see [19] for definitions and other related references. This author defines the invariant and obtains from it a number of applications.

Date: October 31, 2018.

1991 Mathematics Subject Classification. 14D05,14H30,14H50,68W30.

Key words and phrases. Braid monodromy, plane curve, group representations.

First and second authors are partially supported by DGES PB97-0284-C02-02; third author is partially supported by DGES PB97-0284-C02-01.
of beautiful results in several papers, both as single author and with M. Teicher. Later on, A. Libgober \cite{Libgober} proves that the homotopy type of the complement of an affine curve is an invariant of its braid monodromy. Several invariants of the fundamental group have been found to be effective as invariants of affine curves, such as the Alexander polynomial, the Alexander module or the sequence of characteristic varieties \cite{Libgober}. A second group of effective invariants was described by A. Libgober in \cite{Libgober2}. They depend on polynomial representations of the braid group and are invariants of the conjugation class of the image of the representation defining the braid monodromy.

By fixing a particular class of bases of the aforementioned free group $\mathbb{F} - a$ geometric basis -- braid monodromy may be represented by an $r$-tuple of braids. This sequence is not unique; there is a natural action of $B_d \times B_r$ on $B_d^r$ such that the admissible $r$-tuples form an orbit of this action. In general, it is not easy to decide whether two elements of $B_d^r$ are in the same orbit or not, since these orbits are infinite.

Our purpose is to find finer invariants that are sensitive to changes within conjugation classes. Using finite representations of braid groups we obtain new effective invariants for braid monodromies. Effectiveness is obtained by means of the free software GAP4 \cite{GAP4}.

For curves having only nodes and cusps as singularities, Teicher and Kulikov \cite{TeicherKulikov} have recently proved that the diffeomorphism type of their embedding is determined by their braid monodromy. The general case has been resolved independently by the second named author in \cite{Cogolludo}. In this work, we prove a kind of converse of this result; we will prove that braid monodromy of an affine curve $C$ determines the oriented topological type of the pair $(\mathbb{P}^2, \overline{C} \cup L_{\infty})$, where $\overline{C}$ is the union of $C$ and all non-transversal vertical lines and $L_{\infty}$ is the line at infinity.

Braid monodromy may also be useful to study the moduli of curves with prescribed degree and topological types of singularities. Although braid monodromy is defined for affine curves, one can define it in the projective case by choosing generic lines at infinity. Let $d$ be a positive integer and let $T_1, \ldots, T_r$ be topological types of singularities of curves. Let us denote by $\Sigma(T_1, \ldots, T_r; d)$ the space of all projective plane curves of degree $d$ with $r$ singular points of type $T_1, \ldots, T_r$. Let $\mathcal{M}(T_1, \ldots, T_r; d)$ be the quotient of $\Sigma(T_1, \ldots, T_r; d)$ by the action of the projective group. Sometimes we will restrict ourselves to $\Sigma^{\text{irr}}$ and $\mathcal{M}^{\text{irr}}$ by considering only irreducible curves. Note that braid monodromy is an invariant of each connected component of these moduli spaces.
In this work we also address the problem of the topology type of conjugate varieties. Let \((V, W)\) be a pair of projective varieties, \(W \subset V\) (\(W\) may be empty), with defining equations in a number field \(K\). For each embedding \(j\) of \(K \subset \mathbb{C}\) (if we consider \(K\) already embedded in \(\mathbb{C}\), for each Galois action of the normal closure of \(K/\mathbb{Q}\)), one obtains a pair \((V_j, W_j)\) of complex projective varieties, which essentially shares all the algebraic properties of \((V, W)\).

It is well known that a great number of topological properties of \((V_j, W_j)\) are of algebraic nature and they depend only on the pair \((V, W)\). Nevertheless, examples by Serre [20] and Abelson [1] show that it is possible to find examples of non-homeomorphic conjugate varieties, i.e., different embeddings of \(K\) provide different topological pairs.

Serre distinguishes the complex manifolds by means of the fundamental group (although the associated algebraic fundamental groups must be isomorphic). Abelson’s examples have the same fundamental groups but they differ on the cup product in cohomology. Serre’s examples are surfaces whereas Abelson’s are in higher dimensions. Applying generic projection and Chisini’s conjecture, Serre’s result implies that there are conjugate projective plane curves which do not have the same embedded topological type in the projective plane, probably with a large degree. Recently, A. Kharlamov and V. Kulikov [13] have proven the existence of pairs of (complex) conjugate algebraic curves which are of course diffeomorphic but not isotopic. Note that the homeomorphism does not respect curve orientations. In addition, they prove that their examples have non-equivalent braid monodromies; the smallest degree of their examples is 825.

We want to construct such examples with smaller degrees. It is well known, from Degtyarev’s work [14], that no example of non-homeomorphically embedded conjugate curves can exist up to degree 5. In [2], several examples were found as candidates to produce such kind of curves. For example, it was proved that \(\mathcal{M}(A_{19}; 6)\) has two elements and representatives were found having conjugate equations in \(\mathbb{Q}(\sqrt{5})\). In other cases, such as \(\mathcal{M}(A_{18}, A_1; 6)\), there are three elements conjugated in a degree 3 extension of \(\mathbb{Q}\). In [3], we have also studied the family \(\mathcal{M}(A_9, A_9, A_1; 6)\). It is easily seen that if \(C \in \mathcal{M}(A_9, A_9, A_1; 6)\) then \(C\) is the union of a quintic curve with two singular points (one \(A_9\) and one \(A_1\)) and a line which intersects the quintic at a smooth point with intersection multiplicity 5 (providing the second \(A_9\) point). Following the ideas used in [3] we can prove that \(\mathcal{M}(A_9, A_9, A_1; 6)\) has three points; the first one contains curves such that the tangent line to \(C\) at the first \(A_9\) point passes through the second \(A_9\) point. This is not the case for the last two cases, where we can find representatives having
conjugate equations with coefficients in \( \mathbb{Q}(\sqrt{5}) \). H. Tokunaga has proven that curves in the first case are not topologically equivalent to those in the other ones, by means of finding in an algebraic way all possible coverings corresponding to the dihedral group of 10 elements.

In this work we will deal with curves in \( \mathcal{M} := \mathcal{M}(E_6, A_7, A_3, A_2, A_1; 6) \). It is easily seen that if \( C \in \mathcal{M} \) then \( C \) is the union of a quintic curve with three singular points (of types \( E_6, A_3 \), and \( A_2 \)) and a line which intersects the quintic at two smooth points with intersection multiplicities 4 and 1.

**Theorem 1.** The space \( \mathcal{M} \) has exactly two points \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \). There exist representatives \( C_i \in \mathcal{M}_i \), \( i = 1, 2 \), having conjugate equations with coefficients in \( \mathbb{Q}(\sqrt{2}) \).

We will not give a complete proof of this theorem since it is obtained by applying a standard Cremona transformation and the ideas used in [3] to the family \( \mathcal{M}(A_{15}, A_3, A_1; 6) \). In this paper we also compute the necessary tools to prove that the fundamental groups of the complements are both isomorphic to \( \mathbb{Z} \times SL(2, \mathbb{Z}/7\mathbb{Z}) \).

Let \( C \in \mathcal{M} \); we construct a curve \( \tilde{C} \) of degree 12 as follows: let us consider the pencil of curves through the \( E_6 \) point. There are exactly six non-generic lines in this pencil which are: the lines joining this point with the four other singular points of \( C \), an ordinary tangent line and the tangent line to \( C \) at the base point. Then, \( \tilde{C} \) is the union of the quintic and seven lines. Note that if \( C_i \in \mathcal{M}_i \), \( i = 1, 2 \), are conjugate curves, it is also the case for \( \tilde{C}_i \). We will prove the following:

**Theorem 2.** Let us consider two conjugate curves \( C_i \in \mathcal{M}_i \), \( i = 1, 2 \). Then, \( \tilde{C}_1 \) and \( \tilde{C}_2 \) are two conjugate curves such that the pairs \( (\mathbb{P}^2, \tilde{C}_1) \) and \( (\mathbb{P}^2, \tilde{C}_2) \) are not homeomorphic by an orientation-preserving homeomorphism.

In order to prove this theorem, we construct braid monodromies for the affine curves resulting from \( C \in \mathcal{M}_1 \cup \mathcal{M}_2 \) and considering the tangent line at the \( E_6 \) point as the line at infinity. The pencil used for this purpose is the one given by the lines through the \( E_6 \) point. Braid monodromies of curves in different connected components turn out not to be equivalent – note that this is also the case in the examples by Serre and Kharlamov-Kulikov.

In §1, we set notations and definitions. In §2, we define the concepts of geometric and lexicographic bases and lexicographic braids. In §3, we translate the definition of braid monodromy in terms of \( r \)-tuples of braids and prove the theorem about the determination of the braid monodromy by the topology in the fibered case. In §4 and §5, we introduce a way to produce effective invariants for
braid monodromies and finite representations of braid groups. In §6, we find braid monodromies for representatives in \( M_1 \) and \( M_2 \). In the Appendix, we give the GAP4 programs which help to distinguish braid monodromies. The results of §3, §6 and the Appendix prove Theorem 2.

1. Settings and definitions

Let \( G \) be a group and \( a, b \in G \). For the sake of simplicity we use the following notation \( a^b := b^{-1}ab \), \([a, b] := a^{-1}b^{-1}ab\) and \( b \ast a := bab^{-1} \). Given any set \( A \) we denote by \( \Sigma A \) the group of all bijective maps from \( A \) onto itself. As usual we denote \( \Sigma \{1, \ldots, n\} \) by \( \Sigma_n \).

We also denote by \( F_n \) the free group on \( n \) generators, say \( x_1, \ldots, x_n \), and by \( B_n \) the braid group on \( n \) strings given by the following presentation:

\[
\left\langle \sigma_1, \ldots, \sigma_{n-1} : [\sigma_i, \sigma_j] = 1, \left| i - j \right| \geq 2, \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \ i = 1, \ldots, n-2 \right\rangle.
\]

There is a natural right action \( \Phi : F_n \times B_n \to F_n \) such that

\[
x_i^{\sigma_i} = \begin{cases} 
x_{i+1} & \text{if } j = i \\
x_{i+1} * x_i & \text{if } j = i + 1 \\
x_j & \text{if } j \neq i, i + 1. \end{cases}
\]

The induced antihomomorphism \( B_n \to \text{Aut} F_n \) is injective. Identifying \( B_n \) with its image one has the following characterization, which may be found in any classic text on braids, e.g. \[4\] or \[6\]:

**Proposition 1.1.** Let \( \tau : F_n \to F_n \) be an automorphism. Then \( \tau \in B_n \) if and only if \( \tau(x_n \cdot \ldots \cdot x_1) = x_n \cdot \ldots \cdot x_1 \) and there exists \( \sigma \in \Sigma_n \) such that \( \tau(x_i) \) is conjugate to \( x_{\sigma(i)} \).

Next we describe a geometrical interpretation of these definitions. All the results in this section appear in J. Birman’s book \[3\]. Let us fix \( y := \{t_1, \ldots, t_n\} \) a subset of \( \mathbb{C} \) of exactly \( n \) elements. Let us consider also a big enough geometric closed disk \( \Delta \) such that \( y \) is contained in the interior of \( \Delta \) and let \(* \) be a point on \( \partial \Delta \).

**Notation 1.2.** We will denote by \( \gamma \in \pi_1(\mathbb{C} \setminus y; *) \) the homotopy class of the loop based at \( * \) which surrounds the circle \( \partial \Delta \) counterclockwise.

We recall:

**Fact 1.3.** The group \( \pi_1(\mathbb{C} \setminus y; *) \) is isomorphic to \( F_n \).
Definition 1.4. A \( y \)-special homeomorphism is an orientation-preserving homeomorphism \( \mathbb{C} \to \mathbb{C} \) which globally fixes \( y \) and is the identity on \( \mathbb{C} \setminus \Delta \). A \( y \)-special isotopy is an isotopy \( H : \mathbb{C} \times [0, 1] \to \mathbb{C} \times [0, 1] \) with the above properties.

The quotient of the set of \( y \)-special homeomorphisms module \( y \)-special isotopy has a natural structure of group and is denoted by \( B_y \). There is a natural left action of \( B_y \) on \( \pi_1(\mathbb{C} \setminus y; \ast) \) as a result of the definitions.

Fact 1.5. The group \( B_y \) is isomorphic to \( B_n \).

Let \( V \) be the space of monic polynomials in \( \mathbb{C}[t] \) of degree \( n \) (\( V \) and \( \mathbb{C}^n \) are naturally isomorphic as affine spaces). Let \( D \) be the discriminant space of \( V \), i.e., the set of elements in \( V \) with multiple roots. The space \( D \) is an algebraic hypersurface of \( V \). We will use the following notation: \( X := V \setminus D \), \( p(t) := \prod_{j=1}^n(t - t_j) \), \( B_y := \pi_1(X; p(t)) \). By means of the theorem of continuity of roots, \( X \) can be naturally identified with the subsets of \( \mathbb{C} \) having exactly \( n \) elements.

Remark 1.6. From now on, \( X \) will denote, depending on the context, either \( V \setminus D \) or the set \( \{ y \subset \mathbb{C} \mid \#y = n \} \) with quotient topology resulting from the \( n \)-th-symmetric product of \( \mathbb{C} \).

An element \( \tau \in B_y \) is identified with the homotopy class, relative to \( y \), of sets of paths \( \{ \gamma_1, \ldots, \gamma_n \} \), \( \gamma_j : [0, 1] \to \mathbb{C} \), starting and ending at \( y \) and such that \( \forall t \in [0, 1], \{ \gamma_1(t), \ldots, \gamma_n(t) \} \) is a set of \( n \) distinct points. The elements of \( B_y \) are called braids based at \( y \) and are represented as a set of non-intersecting paths in \( \mathbb{C} \times [0, 1] \), as usual. Products are also defined in the standard way.

Since any orientation-preserving homeomorphism of \( \mathbb{C} \) is isotopic to the identity, we can associate to any element of \( B_y \) the braid representing the motion of \( y \) along the isotopy.

Fact 1.7. The groups \( B_y \) and \( B_y \) are naturally anti-isomorphic.

Thus one can construct a right action

\[ \Phi_y : \pi_1(\mathbb{C} \setminus y; \ast) \times B_y \to \pi_1(\mathbb{C} \setminus y; \ast). \]

As above, exponential notation will be used to describe this action.

Example 1.8. Let us consider:

- \( y^0 = \{ -1, \ldots, -n \} \),
- \( \Delta = \{ t \in \mathbb{C} \mid |t| \leq R \} \), \( R \gg 0 \),
- \( \ast = R \).
Let $\delta_j$ be the path obtained by surrounding counterclockwise the circle of radius $\frac{1}{4}$ centered at $-j$ and starting at $-j + \frac{1}{4}$, $j = 1, \ldots, n$. The path $\delta_j^+$ will denote the first half-circle and $\delta_j^-$ the second one.

For any given $j = 1, \ldots, n$, one can define a path $\eta_j$ from $*$ to $-j + \frac{1}{4}$ constructed from a straight segment along the $x$-axis after replacing the segments joining $-k + \frac{1}{4}$ and $-k - \frac{1}{4}$ by $\delta_k^+$, $k = 1, \ldots, j - 1$.

The following set of paths based on $*$ will be useful: $x_j := \eta_j \cdot \delta_j \cdot (\eta_j)^{-1}$, $j = 1, \ldots, n$. It is well known that $x_1, \ldots, x_n$ is a basis of the free group $\pi_1(\mathbb{C} \setminus \mathbf{y}^0, *)$. Moreover, $\gamma = x_n \cdot \ldots \cdot x_1$.

Let us consider the group $B_{y^0}$. For $j = 1, \ldots, n - 1$, the braid $\sigma_j$ will be defined as a set $\{\gamma_j^{(1)}, \ldots, \gamma_j^{(n)}\}$ of paths such that:

- $\gamma_j^{(1)}$ is constant with value $-k$ if $k \neq j, j + 1$;
- $\gamma_j^{(j)}$ is $\delta_j^+$ and
- $\gamma_j^{(j)}$ is $\delta_j^-$.

It is well known that $\sigma_1, \ldots, \sigma_{n-1}$ generate $B_{y^0}$, providing a way to identify $B_{y^0}$ and $B_n$. These identifications are equivariant with respect to the actions $\Phi$ and $\Phi_{y^0}$, which also become identified.

Based on this example we will describe isomorphisms from $\pi_1(\mathbb{C} \setminus \mathbf{y}; *)$ to $\mathbb{F}_n$ and from $B_{y^0}$ to $B_n$ for any $\mathbf{y} \in X$, commuting with respect to $\Phi$ and $\Phi_{y^0}$. This can be achieved by defining braids with different ends.

**Definition 1.9.** Let $\mathbf{y}^1$ and $\mathbf{y}^2$ be subsets of $\mathbb{C}$ with exactly $n$ elements – that is, $\# \mathbf{y}^1 = \# \mathbf{y}^2 = n$. A braid from $\mathbf{y}^1$ to $\mathbf{y}^2$ is a homotopy class, relative to $\mathbf{y}^1 \cup \mathbf{y}^2$, of sets $\{\gamma_1, \ldots, \gamma_n\}$ of paths $\gamma_j : [0, 1] \to \mathbb{C}$ starting at $\mathbf{y}^1$, ending at $\mathbf{y}^2$ and such that $\forall t \in [0, 1]$, $\# \{\gamma_1(t), \ldots, \gamma_n(t)\} = n$. The sets $\mathbf{y}^1$ and $\mathbf{y}^2$ are usually referred to as the ends of the braid.

As above, we will usually not distinguish between a braid and a representative of its class. The set of all braids from $\mathbf{y}^1$ to $\mathbf{y}^2$ is denoted by $B(\mathbf{y}^1, \mathbf{y}^2)$. Note that $B_{\mathbf{y}} = B(\mathbf{y}, \mathbf{y})$ for any $\mathbf{y} \in X$. Given $\mathbf{y}^1, \mathbf{y}^2, \mathbf{y}^3 \in X$, there are natural definitions

$$B(\mathbf{y}^1, \mathbf{y}^2) \times B(\mathbf{y}^2, \mathbf{y}^3) \to B(\mathbf{y}^1, \mathbf{y}^3), \quad B(\mathbf{y}^1, \mathbf{y}^2) \to B(\mathbf{y}^2, \mathbf{y}^1),$$

for the product and inverse of braids, which have the natural properties which identify them with the fundamental groupoid of $X$ and will be called the braid groupoid on $n$ strings.

Moreover, let us consider $\mathbf{y}^1, \mathbf{y}^2 \in X$ and let $\Delta$ be a geometric disk containing $\mathbf{y}^1 \cup \mathbf{y}^2$ in its interior.
Definition 1.10. A \((y^2, y^1)\)-special homeomorphism is an orientation-preserving homeomorphism \(f : \mathbb{C} \to \mathbb{C}\) such that \(f(y^1) = y^2\) and \(f\) is the identity on \(\mathbb{C} \setminus \Delta\). Analogously, \((y^2, y^1)\)-special isotopies can be defined.

The quotient set of special isotopy classes of \((y^2, y^1)\)-special homeomorphisms of \(\mathbb{C}\) will be denoted by \(\mathbb{B}(y^2, y^1)\). With the natural operations, one obtains a groupoid on \(X\) which is naturally anti-isomorphic to the braid groupoid. By choosing \(\ast \in \partial \Delta\) as a base point, one has a natural mapping

\[
\mathbb{B}(y^2, y^1) \times \pi_1(\mathbb{C} \setminus y^1; \ast) \to \pi_1(\mathbb{C} \setminus y^2; \ast)
\]

which may be viewed as a groupoid action. It induces a right groupoid action

\[
\pi_1(\mathbb{C} \setminus y^1; \ast) \times B(y^1, y^2) \to \pi_1(\mathbb{C} \setminus y^2; \ast),
\]

which becomes \(\Phi_y\) when \(y^1 = y^2 = y\). This action will be denoted using exponential notation.

Let us fix ends \(y^1, y^2 \in X\) and a braid \(\tau \in B(y^1, y^2)\). There are two natural isomorphisms

\[
R_\tau : B_{y^1} \to B_{y^2} \\
\sigma \mapsto \tau^{-1} \cdot \sigma \cdot \tau
\]

and

\[
\Psi_\tau : \pi_1(\mathbb{C} \setminus y^1; \ast) \to \pi_1(\mathbb{C} \setminus y^2; \ast) \\
x \mapsto x^\tau.
\]

If \(y^0 = y^1 = y\), then \(\tau \in B_y\), \(R_\tau\) is the inner automorphism \(\sigma \mapsto \tau^{-1} \cdot \sigma \cdot \tau\) and \(\Psi_\tau\) is the automorphism \(\Phi_y(\bullet, \tau)\) induced by the right action \(\Phi_y\) with respect to \(\tau\). We summarize these results.

Proposition 1.11. Let \(y^0 = \{-1, \ldots, -n\}\) as in example 1.8, and let \(y \in X\). We identify \(\pi_1(\mathbb{C} \setminus y^0; \ast)\) with \(\mathbb{F}_n\) and \(B_{y^0}\) with \(B_n\).

Then, for any \(\tau \in B(y^0, y)\) there is a canonical identification of \(\pi_1(\mathbb{C} \setminus y; \ast)\) with \(\mathbb{F}_n\) and of \(B_y\) with \(B_n\) by means of \(\Psi_\tau\) and \(R_\tau\) respectively. These identifications are equivariant with respect to \(\Phi\) and \(\Phi_y\).

Moreover, two such identifications differ by an inner automorphism of \(B_n\), with respect to a given \(\hat{\tau} \in B_n\), and the corresponding automorphism of \(\mathbb{F}_n\) is \(\Phi(\bullet, \hat{\tau})\).

2. Geometric bases and lexicographic braids

The notations introduced along the previous section will also be used in the present one. We recall an important definition from algebraic geometry.
**Definition 2.1.** Let $X$ be a connected projective manifold and let $H$ be a hypersurface of $X$. Let $* \in X \setminus H$ and let $K$ be an irreducible component of $H$. A homotopy class $\gamma \in \pi_1(X \setminus H; *)$ is called a **meridian about** $K$ with respect to $H$ if $\gamma$ has a representative $\delta$ satisfying the following properties:

(a) there is a smooth disk $\Delta \subset X$ transverse to $H$ such that $\Delta \cap H = \{*\}' \subset K$ is a smooth point of $H$
(b) there is a path $\alpha$ in $X \setminus H$ from $*$ to $*'' \in \partial \Delta$
(c) $\delta = \alpha \cdot \beta \cdot \alpha^{-1}$, where $\beta$ is the closed path obtained by traveling from $*''$ along $\partial \Delta$ in the positive direction.

It is well known that any two meridians about $K$ with respect to $H$ are conjugate in $\pi_1(X \setminus H; *)$. Moreover, the converse also holds.

In example (1.8), $x_j$ is a meridian loop about $-j$ in $\pi_1(\mathbb{C} \setminus y^0; n + 1)$ where $y^0 = \{-1, \ldots, -n\}$, for $j = 1, \ldots, n$. Let us note that if $y^1 \in X$ and $\tau \in B(y^0, y^1)$, then $x_1^\tau, \ldots, x_n^\tau$ are meridian loops about the points in $y^1$, and of course they also form a basis for the free group $\pi_1(\mathbb{C} \setminus y^1; *)$.

**Definition 2.2.** Let us consider $y$ a point of $X$, $\Delta \subset \mathbb{C}$ a geometric closed disk containing $y$ in its interior, and $*$ a point in $\partial \Delta$. Also, let $\gamma \in \pi_1(\mathbb{C} \setminus y, *)$ be as in notation (1.2). A **geometric basis** of the free group $\pi_1(\mathbb{C} \setminus y, *)$ (with respect to $\Delta$) is an ordered basis $\mu_1, \ldots, \mu_n$ such that:

1. $\mu_1, \ldots, \mu_n$ are meridians of the points in $y$;
2. $\mu_n \cdot \ldots \cdot \mu_1 = \gamma$; in particular, it is the inverse of a meridian about the point at infinity of $\mathbb{C}$.

**Proposition 2.3.** Under the conditions and notations of example (1.8) the following holds:

(i) The base $x_1, \ldots, x_n$ is geometric with respect to $\Delta$.
(ii) For any $\tau \in B_n$, the basis $x_1^\tau, \ldots, x_n^\tau$ is geometric with respect to $\Delta$.
(iii) Any geometric basis with respect to $\Delta$ is obtained as in (ii).

Let us denote by $\mathcal{S}^n(\Delta)$ the set of all geometric bases with respect to $\Delta$. The map $\Phi$ induces an action of $B_n$ on $\mathbb{F}_n^n$ where $\mathcal{S}^n(\Delta)$ becomes the orbit of $(x_1, \ldots, x_n)$. The group $B_n$ acts freely on $\mathcal{S}^n(\Delta)$. The next result is a direct consequence of propositions 1.11 and 2.3.

**Corollary 2.4.** Let $y \in X$, $\Delta$ and $*$ be as in definition (2.2). Let us also assume that $\Delta$ contains $y^0$ in its interior. Then the set $\mathcal{S}^n(\Delta)$ is an orbit under the action of $B_y$ induced by $\Phi_y$ on $\pi_1(\mathbb{C} \setminus y; *)^n$. 
Note that, for any braid \( \tau \in B(y^0, y) \), the set \( x_1^\tau, \ldots, x_n^\tau \) is a geometric basis of \( \pi_1(C \setminus y; *) \). Moreover, any element of the orbit can be obtained in this manner.

Our next purpose is to construct canonical isomorphisms between \( B_y \) and \( B_n \) for any \( y \in X \). This can be done by fixing a special braid \( \tau \in B(y^0, y) \). We must emphasize that even if the process to produce such a braid is canonical, this method is not invariant by orientation-preserving homeomorphisms of \( C \) (not even by rotations).

Construction 2.5. We will start by constructing lexicographic bases and braids. Let us fix \( y \in X \) and let us consider \( y^0 \) as above. Let \( \Delta \) be a geometric disk (centered at 0) containing \( y \cup y^0 \) and choose \( * \) as the only positive real number in \( \partial \Delta \).

Let us consider the basis \( x_1, \ldots, x_n \) of \( \pi_1(C \setminus y^0, *) \) given in example 1.8.

One can order the points \( t_1, \ldots, t_n \in y \) such that \( \Re t_1 \geq \cdots \geq \Re t_n \) and if \( \Re t_j = \Re t_{j+1} \), then \( \Im t_j > \Im t_{j+1} \). This ordering is called the lexicographic ordering of \( y \).

Choose \( \varepsilon > 0 \) such that the closed disks centered at \( t_j, j = 1, \ldots, n \) of radius \( \varepsilon \) are pairwise disjoint and contained in the interior of \( \Delta \). Let us consider the polygonal path \( \Gamma \) joining \( *, t_1, \ldots, t_n \) and the circles of radius \( \varepsilon \) centered at \( t_1, \ldots, t_n \). For each \( j = 1, \ldots, n \), we construct \( y_j \), a meridian about \( t_j \) with respect to \( C \) based at \( * \). For this purpose, we will follow the process shown in example (1.8), but replacing the segment on the real line by \( \Gamma \) and the disks of radius \( \frac{1}{4} \) centered at \( -j \) by the disks of radius \( \varepsilon \) centered at \( t_j, j = 1, \ldots, n \).

In this way a geometric basis \( y_1, \ldots, y_n \) of \( \pi_1(C \setminus y; *) \) is produced. Such a basis will be called the lexicographic basis for \( y \). Note that, in particular, \( x_1, \ldots, x_n \) is the lexicographic basis for \( y^0 \).

Note that there is a unique braid \( \tau_y \in B(y^0, y) \) such that \( y_j = x_j^{\tau_y}, \forall j = 1, \ldots, n \). This braid \( \tau_y \) is called the lexicographic braid associated to \( y \). The corresponding isomorphism \( R_y := R_{\tau_y} : B_n \to B_y \) will be called lexicographic isomorphism from \( B_n \) to \( B_y \).

Remark 2.6. Let us consider \( y_1, y_2 \in X \). Lexicographic braids allow us to consider canonical bijections of \( B(y_1, y_2) \) where \( B_n = B_{y^0} \):

\[
B(y^1, y^2) \to B_n, \quad \sigma \mapsto \tau_{y^1} \cdot \sigma \cdot \tau_{y^2}^{-1}.
\]

Let us say we have a braid in \( \mathbb{C} \times [0, 1] \), then a generic projection onto \( \mathbb{R} \times [0, 1] \) is obtained as follows:
• Take the projection $\mathbb{R} : \mathbb{C} \to \mathbb{R}$ given by the real part.
• For any isolated double point of the projection, the branch with the smallest imaginary part will be drawn using a continuous line.
• If a line of double points occurs (e.g., a pair of conjugate arcs), we slightly perturb the projection in order to have greater imaginary parts to the right and smaller imaginary parts to the left.

3. Braid monodromy and fibered curves

In order to define braid monodromy and fibered curves, we will consider the affine plane $\mathbb{C}^2$ and the projection $\pi : \mathbb{C}^2 \to \mathbb{C}$ given by $\pi(x, y) = x$. Instead of fixing the second coordinate of $\mathbb{C}^2$ we will allow affine transformations, preserving $\pi$, to act on $\mathbb{C}^2$.

Definition 3.1. A reduced affine curve $C \subset \mathbb{C}^2$ is said to be horizontal with respect to $\pi$ if $\pi|_C : C \to \mathbb{C}$ is a proper map. The horizontal degree $\deg_\pi(C)$ of $C$ is the degree of $\pi|_C$, i.e., the number of preimages of a regular value. The set of regular values of $\pi|_C$ will be denoted by $\mathbb{C}_C$.

Let $C$ be a horizontal curve of horizontal degree $d$. Note that, after fixing the second coordinate of $\mathbb{C}^2$ we can assume that an equation $f(x, y) = 0$ for $C$ is given by

$$f(x, y) = y^d + f_1(x)y^{d-1} + \cdots + f_1(x)y + f_0(x),$$

where $f_j(x) \in \mathbb{C}[x]$, $j = 0, 1, \ldots, d - 1$. The condition of being horizontal is equivalent to the non-existence of vertical asymptotes (vertical lines included). Note that the set of critical values is the set of zeros of the discriminant of $f$ with respect to $y$, which is a polynomial $\mathcal{D}(x) \in \mathbb{C}[x]$. Denoting this set of zeros by $x_1, \ldots, x_r$, one has $\mathbb{C}_C = \mathbb{C} \setminus \{x_1, \ldots, x_r\}$. Therefore the special fibers of $\pi$ are exactly the vertical lines $L_i := \pi^{-1}(x_i)$ of equation $x = x_i$, $i = 1, \ldots, r$.

Definition 3.2. Let $C \subset \mathbb{C}^2$ be a horizontal curve with respect to $\pi$. With the above notation, the fibered curve associated to $C$ is the curve $C^\pi := C \cup L_1 \cup \cdots \cup L_r$.

The motivation behind this definition is the following. Let $\bar{\pi} := \pi|_C^{-1}(\mathbb{C}_C)$. Since $\pi|_C$ is proper, $\bar{\pi} : \bar{\mathbb{C}} \to \mathbb{C}_C$ is a (possibly non-connected) covering map. The mapping $\bar{\pi}^\varphi : \mathbb{C}^2 \setminus \mathcal{C} \to \mathbb{C}_C$ is a locally trivial fibration whose fiber is diffeomorphic to $\mathbb{C} \setminus \{1, \ldots, d\}$. The polynomial $f$ induces an algebraic mapping $\bar{f} : \bar{\mathbb{C}} \to X$ defined as $\bar{f}(x_0) := \{x = x_0\} \cap \mathcal{C}$—or equivalently as the polynomial $f(x_0, t) \in \mathbb{C}[t]$.

Let us fix $* \in \mathbb{C}_C$, a complex regular value on the boundary of a geometric disk $\Delta$ containing $\{x_1, \ldots, x_r\}$ in its interior. Let us denote by $y^*$ the set of roots of the polynomial $f(*, t)$. 


Definition 3.3. The homomorphism $\nabla_* : \pi_1(\mathbb{C}; \ast) \to B_{\mathfrak{y}^*}$ induced by $\tilde{f}$ is called a braid monodromy of $\mathcal{C}$ with respect to $\pi$.

Remark 3.4. Note that $\nabla_*$ classifies the locally trivial fiber bundle $\pi^\varphi$.

Let us briefly explain how to construct this braid monodromy. Note that there is a particular class of basis for $\pi_1(\mathbb{C}; \ast)$, namely the geometric bases with respect to $\Delta$. Let us fix one of these bases, say $\mu_1, \ldots, \mu_r$ (e.g., one can choose the lexicographic basis). Let us consider the lexicographic isomorphism $R_{\mathfrak{y}^*} : B_n \to B_{\mathfrak{y}^*}$. These facts allow us to represent $\nabla_*$ by an element $(\tau_1, \ldots, \tau_r) \in B_r^d$ such that:

$$\tau_j := R_{\mathfrak{y}^{-1}}(\nabla_*(\mu_j)), \ \forall j = 1, \ldots, r.$$

Definition 3.5. We say that $(\tau_1, \ldots, \tau_r) \in B_r^d$ represents the braid monodromy of $\mathcal{C}$ if there exist:

- a geometric disk $\Delta$ containing $\{x_1, \ldots, x_r\}$ in its interior;
- an element $\ast \in \partial \Delta$ such that $\mathfrak{y}^*$ is the set of roots of the polynomial $\tilde{f}(\ast) \in \mathbb{C}[t]$;
- a geometric basis $\mu_1, \ldots, \mu_r$ of $\pi_1(\mathbb{C}, \ast)$;
- a braid $\beta \in B(\mathfrak{y}^*, \mathfrak{y}^0)$

for which $\tau_j = R_\beta(\nabla_*(\mu_j)), \ \forall j = 1, \ldots, r$.

Remark 3.6. Two natural right actions on $B_r^d$ are related to the concept of $r$-tuples of braids representing the monodromy.

The first one is an action produced by $B_r$ as follows:

$$(\tau_1, \ldots, \tau_r)^{\sigma_i} := (\tau_1, \ldots, \tau_{i-1}, \tau_{i+1}, \tau_{i+1}\tau_i\tau_{i+1}^{-1}, \tau_{i+2}, \ldots, \tau_r),$$

where $\sigma_i \in B_r, i = 1, \ldots, r - 1$ is a canonical generator and $(\tau_1, \ldots, \tau_r) \in B_r^d$. The second action is given by $B_n$ and it is defined by conjugation on each coordinate as follows:

$$(\tau_1, \ldots, \tau_r)^{\beta} := (\tau_1^\beta, \ldots, \tau_r^\beta),$$

where $\beta \in B_d$ and $(\tau_1, \ldots, \tau_r) \in B_r^d$. These actions commute with each other and hence they define a new action of $B_r \times B_d$. The first action represents the change of geometric basis. The second action represents both the change of the chosen braid in $B(\mathfrak{y}^*, \mathfrak{y}^0)$ and also the change of the base point.

This can be summarized as follows:
Proposition 3.7. Let \((\tau_1, \ldots, \tau_r)\) be an \(r\)-tuple of braids representing the braid monodromy of \(C\) w.r.t. \(\pi\). Then \((\tilde{\tau}_1, \ldots, \tilde{\tau}_r)\) \(\in B_r^d\) represents the braid monodromy of \(C\) if and only if \((\tilde{\tau}_1, \ldots, \tilde{\tau}_r)\) is in the orbit of \((\tau_1, \ldots, \tau_r)\) by the action of \(B_r \times B_d\) on \(B_r^d\).

Remark 3.8. Moreover, note that cyclic permutations of \(r\)-tuples representing a braid monodromy have essentially the same information as their original representatives. This remark leads to the concept of pseudogeometric bases, that is, a basis \(\mu_1, \ldots, \mu_r\) such that \(\mu_j\) is a meridian about \(x_{\sigma(j)}\) for some \(\sigma \in \Sigma_r\) and \((\mu_r \cdots \mu_1)^{-1}\) is a meridian about the point at infinity. Sometimes, braid monodromies are easier to compute for pseudogeometric bases.

Definition 3.9. Two affine horizontal curves are said to have equivalent braid monodromies if they have the same representatives for their braid monodromies.

Note that this equivalence relation is finer than the one arising from curves with the same monodromy representation.

In general, it is difficult to find effective invariants to compare braid monodromies. For example, the invariants suggested by Libgober depend essentially on the conjugation class of the image of the braid monodromy. We recall how braid monodromy is related to the fundamental group.

Theorem of Zariski-Van Kampen 3.10. Let \(\mathbb{F}_d\) be the free group generated by a geometric basis \(g_1, \ldots, g_d\) and let \((\tau_1, \ldots, \tau_r)\) \(\in B_r^d\) be an \(r\)-tuple of braids representing the braid monodromy of \(C\). Then the fundamental group of \(\mathbb{C}^2 \setminus C\) is isomorphic to

\[ \langle g_1, \ldots, g_d | g_j^{\tau_i} = g_j, \quad i = 1, \ldots, r, \quad j = 1, \ldots, d \rangle, \]

The main tools required to prove this theorem are the classical Van Kampen theorem and this fibered version of Zariski-Van Kampen’s theorem.

Proposition 3.11. The fundamental group of \(\mathbb{C}^2 \setminus C^\varphi\) is isomorphic to

\[ \langle g_1, \ldots, g_d, \alpha_1, \ldots, \alpha_r | g_j^{\tau_i} = \alpha_i^{-1} g_j \alpha_i, \quad i = 1, \ldots, r, \quad j = 1, \ldots, d \rangle. \]

The key point in the proof of this fibered version is the long exact sequence of homotopy associated with a fibration.

More information about the pair \((\mathbb{C}^2, C^\varphi)\) can be obtained by choosing the paths representing \(g_1, \ldots, g_d, \alpha_1, \ldots, \alpha_r\) in an appropriate manner.
Definition 3.12. Let \( \mu_1, \ldots, \mu_r \) be a geometric basis of \( \pi_1(\mathbb{C}C, *) \). We assume that these paths have their support in a simply connected compact set \( K \subset \mathbb{C} \) satisfying \( * \in \partial K \). Let \( L \subset \mathbb{C}^2 \) be a simply connected compact subset of \( \mathbb{C}^2 \) such that \( \pi^{-1}(\mathbb{C} \cap K) \subset L \) and let \( * \in \mathbb{C} \) such that \( (\ast, \hat{*}) \notin L \). We say that \( \alpha_1, \ldots, \alpha_r \in \pi_1(L^2 \cup \mathbb{C}^2; (\ast, \hat{*})) \) is a suitable lifting of \( \mu_1, \ldots, \mu_r \) if

1. \( \pi \phi^* (\alpha_j) = \mu_j, \ j = 1, \ldots, r \);
2. the support of \( \alpha_1, \ldots, \alpha_r \) is in \( \pi^{-1}(K) \setminus L \);
3. for any \( j = 1, \ldots, r \), the closed path \( \alpha_j \) is a meridian about the line \( L_{\sigma(j)} \) with respect to \( \mathbb{C}^2 \), where \( \sigma \) is the permutation associated to the original geometric basis.

Lemma 3.13. Let \( \mu_1, \ldots, \mu_r \) be a geometric basis of \( \pi_1(\mathbb{C}C, *) \). Then, a suitable lifting of this basis is unique up to homotopy.

Proposition 3.14. Let \( L \phi^* := (\pi \phi^*)^{-1}(*) \) be a generic fiber of \( \pi \phi^* \) and let \( g_1, \ldots, g_d \) be a geometric basis of \( \pi_1(L \phi^* \cup \mathbb{C}^2; (\ast, \hat{*})) \). Then both the elements \( g_1, \ldots, g_d \) and a suitable lifting \( \alpha_1, \ldots, \alpha_r \) of \( \mu_1, \ldots, \mu_r \) may be chosen for the presentation of \( \pi_1(L^2 \cup \mathbb{C}^2; (\ast, \hat{*})) \) in theorem 3.11.

Corollary 3.15. Any presentation of \( \pi_1(L^2 \setminus \mathbb{C}^2; (\ast, \hat{*})) \) in terms of a geometric basis of \( \pi_1(L \phi^* \cup \mathbb{C}^2; (\ast, \hat{*})) \) and a suitable lifting \( \alpha_1, \ldots, \alpha_r \) of \( \mu_1, \ldots, \mu_r \), determines the braid monodromy \( \Phi \) of \( \mathcal{C} \).

The following definition and lemma will help to give a more geometrical construction for the suitable lifting described in proposition 3.14.

Definition 3.16. Let \( A, B \gg 0 \). A polydisk \( \tilde{\Delta} := \Delta_A \times \Delta_B \subset \mathbb{C}^2 \) of multiradius \( (A, B) \) is well adapted to \( \mathcal{C} \) if \( \{x_1, \ldots, x_r\} \) is contained in the interior of \( \Delta_A \) and \( \{(x, y) \in \mathcal{C} \mid x \in \Delta_A\} \subset \Delta_A \times \Delta_B \).

Lemma 3.17. There exists a real number \( A_0 > 0 \) such that for any \( A \geq A_0 \) there exists \( B_0(A) > 0 \) satisfying that for any \( B \geq B_0(A) \) the polydisk \( \tilde{\Delta} := \Delta_A \times \Delta_B \) is well adapted to \( \mathcal{C} \).

Remark 3.18. If \( \tilde{\Delta} \) is a well-adapted polydisk as above, we can set \( (\ast, \hat{*}) = (A, B) \) and choose the paths \( \alpha_i \) in \( \Delta_A \times \{B\} \).

Now we can state the main theorem relating equivalence of pairs of fibered curves and equivalence of braid monodromies. This is a partial converse to [15] and [14].
Theorem 3.19. Let \( C_1, C_2 \subset \mathbb{C}^2 \) be two horizontal affine curves and let us consider the standard embedding \( \mathbb{C}^2 \subset \mathbb{P}^2 \). If \( F : (\mathbb{C}^2, C_1^\varphi) \to (\mathbb{C}^2, C_2^\varphi) \), is an orientation preserving homeomorphism that extends to a homeomorphism of \( \mathbb{P}^2 \), then \( C_1 \) and \( C_2 \) have equivalent braid monodromies.

Proof. For the proof we will use the notation introduced in statements [3.10, 3.11] and [3.14].

Let us consider a horizontal curve \( C \). The elements \( g_1, \ldots, g_d \) freely generate a subgroup \( H \), which is normal since \( H = \ker \pi_1^\varphi \). Note that \( g_1, \ldots, g_d \) are meridians about (all) the (irreducible) components of \( C \). Since \( H \) is normal, it is generated by all the meridians about all irreducible components of \( C \). The group \( H \) is, hence, identified with the fundamental group of the generic fiber. Therefore, the basis \( g_1, \ldots, g_d \) is geometric. Note that \( g_d \cdot \ldots \cdot g_1 \) is the boundary of a sufficiently big disk on the fiber. One obtains a natural exact sequence:

\[
1 \to H \to \pi_1(\mathbb{C}^2 \setminus C^\varphi; (*, \hat{*})) \to \pi_1(\mathbb{C} \setminus C; *) \to 1. \tag{3.1}
\]

Given any two curves \( C_1, C_2 \) as in the statement, let us take the elements \( g_1^{(1)}, \ldots, g_d^{(1)} \) and \( \alpha_1^{(1)}, \ldots, \alpha_r^{(1)} \) described in proposition [3.14]. We recall that the elements \( g_1^{(1)}, \ldots, g_d^{(1)} \) form a geometric basis of \( H^{(1)} \) and \( \pi_1^\varphi(\alpha_1^{(1)}), \ldots, \pi_1^\varphi(\alpha_r^{(1)}) \) is a geometric basis of \( \pi_1(\mathbb{C} \setminus C_1; *) \). Let us choose a big disk \( \Delta_{A_1} \) of radius \( A_1 \gg 0 \), such that the values of the non-transversal vertical lines to \( C_1 \) are in the interior of \( \Delta_{A_1} \). Note that we can construct a polydisk \( \tilde{\Delta}_1 \) well adapted to \( C_1 \) and two polydisks \( \tilde{\Delta}_2 \) and \( \hat{\Delta}_2 \) well adapted to \( C_2 \) such that

\[
\tilde{\Delta}_2 \subset F(\tilde{\Delta}_1) \subset \hat{\Delta}_2.
\]

Let us define

\[
g_j^{(2)} := F_*(g_j^{(1)}), \quad j = 1, \ldots, d.
\]

From the discussion above, \( g_1^{(2)}, \ldots, g_d^{(2)} \) is a basis for the free group \( H^{(2)} \). Moreover, let us suppose that the generic fiber \( L_i^{(1)} \) (of \( \pi \)) is very close to a non-transversal vertical line \( L_i^{(1)} \) (if no such line exists there is nothing to prove). Since the boundary of a big disk in \( L_i^{(1)} \) is sent to the boundary of a big disk in \( L_i^{(2)} \), we easily deduce that \( g_d^{(2)} \cdot \ldots \cdot g_1^{(2)} \) is homotopic to the boundary of a big disk in a generic fiber for \( \pi^{\varphi(2)} \). This argument proves that \( g_1^{(2)}, \ldots, g_d^{(2)} \) is a geometric basis of \( H^{(2)} \).

Let us define

\[
\alpha_i^{(2)} := F_*(\alpha_i^{(1)}), \quad i = 1, \ldots, r.
\]
By the naturality of the exact sequence (3.1) we deduce that $\pi^*(2)\alpha^*(2), \ldots$, $\pi^*(2)(\alpha^*(2)_r)$ is a basis of the free group $\pi_1(\mathbb{C} \setminus \mathbb{C}_{C^2}; \ast)$. Since the extension of $F$ to $\mathbb{P}^2$ preserves the line at infinity and since the inverse of $\pi^*(1)\alpha^*(1) \cdot \ldots \cdot \alpha^*(1)_1$ is a meridian about the point at infinity in $\mathbb{C}$, we deduce that it is also the case for $\pi^*(2)(\alpha^*(2)_r \cdot \ldots \cdot \alpha^*(2)_1)^{-1}$. Then, $\pi^*(2)(\alpha^*(2)_1), \ldots, \pi^*(2)(\alpha^*(2)_r)$ is a pseudogeometric basis of $\pi_1(\mathbb{C} \setminus \mathbb{C}_{C^2}; \ast)$.

From the exact sequence (3.1) and the system of generators $g_i^{(2)}, \alpha_j^{(2)}$, we obtain a representation of $F_r$ in $B_d$ which provides an element of $M_2 \in (B_d)^r$. We have proven that this element represents the braid monodromy of $C_2$.

Because of the way the generators are chosen, one can apply the same argument to $C_1$, obtaining an element $M_1 \in (B_d)^r$ representing the braid monodromy of $C_1$. Since the two families of generators are related by $F_r$, we deduce that $M_1 = M_2$.

This theorem will be used to compare curves in $\mathcal{M}_1$ and $\mathcal{M}_2$. We must compute their braid monodromies and find suitable invariants in order to be able to compare them.

4. Effective invariants of braid monodromy

Let $G$ be a group and let $r$ be a positive integer. Let us consider the sets $G^r$ and $G \setminus G^r$ which is the quotient of $G^r$ by the diagonal action of $G$ by conjugation. The braid group $B_r$ acts on $G^r$ by the so-called Hurwitz action:

$$\sigma_j \cdot (g_1, \ldots, g_r) := (g_1, \ldots, g_{j-1}, g_{j+1}, g_{j+1}g_j^{-1}g_{j+1}^{-1}, g_{j+2}, \ldots, g_r), \ 1 \leq j < r,$$

where $g_1, \ldots, g_r \in G$. Since Hurwitz and conjugation actions commute, $B_r$ also acts on $G \setminus G^r$.

Let us denote by $\mathcal{M}_r(G)$ (resp. $\mathcal{M}_r(G)$) the quotient of $G^r$ (resp. $G \setminus G^r$) by the Hurwitz action. The index $r$ will be dropped if no ambiguity seems likely to arise. The elements of $\mathcal{M}_r(G)$ will be called $G$-monodromies (of order $r$). Before we use the main result to produce effective invariants, let us set the main definitions of this last section.

**Definition 4.1.** Let $\mathcal{C}$ be a horizontal curve with deg$_\pi(\mathcal{C}) = d$ and possessing $r$ non-transversal vertical lines. The braid monodromy of $\mathcal{C}$ is defined as the element in $\mathcal{M}_r(B_d)$ determined by any $r$-tuple representing the braid monodromy of $\mathcal{C}$.

**Remark 4.2.** This point of view is inspired by the work of Brieskorn on automorphic sets [8]. We restrict our attention to automorphic sets defined by conjugation on groups. We have also modified the conjugation action defined in [8].
Definition 4.3. Let \( g := (g_1, \ldots, g_r) \in G^r \). The pseudo Coxeter element associated with \( g \) is defined as \( c(g) := g_r \cdot \ldots \cdot g_1 \).

Note that pseudo Coxeter element is also well defined in \( \mathcal{N}(G) \) and its conjugation class is well defined in \( M_r(G) \).

Let \( \phi : G_1 \to G_2 \) be a group homomorphism. It induces in a functorial way mappings \( \phi_{\mathcal{N}} : \mathcal{N}(G_1) \to \mathcal{N}(G_2) \) and \( \phi_{\mathcal{M}} : \mathcal{M}(G_1) \to \mathcal{M}(G_2) \).

Definition 4.4. Let \( \mathcal{C} \) be as in definition 4.1 and let \( \Phi : B_d \to G \) be a representation of \( B_d \) onto a group \( G \). Then the \((\Phi, G)\)-monodromy of \( \mathcal{C} \) is the image in \( M_r(G) \) by \( \Phi_{\mathcal{M}} \) of the braid monodromy of \( \mathcal{C} \).

Proposition 4.5. Let \( \mathcal{C}_1, \mathcal{C}_2 \) be horizontal curves with \( \deg_\pi(\mathcal{C}_1) = \deg_\pi(\mathcal{C}_2) = d \) having \( r \) non-transversal vertical lines. Let \( \Phi : B_d \to G \) be a representation. If the \((\Phi, G)\)-monodromies of \( \mathcal{C}_1 \) and \( \mathcal{C}_2 \) are not equal then there is no orientation-preserving homeomorphism of \( (\mathbb{C}^2, \mathcal{C}_1) \) and \( (\mathbb{C}^2, \mathcal{C}_2) \) that extends to a homeomorphism of \( \mathbb{P}^2 \).

Proof. This is a straightforward consequence of theorem 3.19. \qed

If \( G \) is a finite group then \( \mathcal{M}(G) \) is a finite set and hence, knowing the braid monodromies of two given curves would allow us to compare their \((\Phi, G)\)-monodromies, up to computational capacity. In the Appendix we give an algorithm implemented on GAP4 [11]. Its input consists of braid monodromies of two curves and a finite representation of the braid group. Its output affirms or negates the equality of their \((\Phi, G)\)-monodromies. We sketch the general lines of the algorithm:

(i) Compute the \((\Phi, G)\)-monodromies of the curves \( \mathcal{C}_1, \mathcal{C}_2 \).

(ii) Compute their pseudo Coxeter elements \( c_G(\mathcal{C}_1), c_G(\mathcal{C}_2) \in G \). If they are not conjugate, braid monodromies are not equal. If they are conjugate to each other, choose an element \( g \in G \) such that \( c_G(\mathcal{C}_2)^g = c_G(\mathcal{C}_1) =: h \). Let \( H \) be the centralizer of \( h \) in \( G \) and consider the set \( H_{\mathcal{C}_2} \) of conjugates of the \((\Phi, G)\)-monodromy of \( \mathcal{C}_2 \) by \( gx, x \in H \).

(iii) Compute the orbit of the \((\Phi, G)\)-monodromy of \( \mathcal{C}_1 \) by the action of \( B_r \). Note that since Hurwitz action preserves pseudo Coxeter elements it is enough to consider conjugation by \( H \). Since \( B_r \) admits a generator system with two elements, an algorithm can be easily programmed to construct the finite orbit. For each new element of the orbit, verify if it is in \( H_{\mathcal{C}_2} \) and in that case stop the program.

(iv) If no element of the orbit is in \( H_{\mathcal{C}_2} \) then the two \((\Phi, G)\)-monodromies are not equivalent.
Strategy 4.6. In general, in order to distinguish braid monodromies for two horizontal curves $C_1$ and $C_2$ with $\deg_\pi(C_1) = \deg_\pi(C_2) = d$, we proceed as follows:

- Verify if the curves (with the projections) have the same combinatorics (also at infinity).
- If this is the case, then compute the fundamental group of the curves.
- If either the groups are isomorphic or we cannot determine that they are not, then we compute the image of the braid monodromy.
- If we cannot determine whether or not the images are conjugate, we look for Libgober invariants which provide easy-to-compare polynomials. Also the sequence of characteristic varieties might help to distinguish the groups.
- If the previous steps do not work, we try the methods described in this section.

5. Finite representations of the braid group

Finding finite representations of braid groups is an interesting problem already studied by several authors: Assion [5], Kliutmann [14], Birman-Wajnryb [1], Wajnryb [21, 22]. Infinite families of presentations have been obtained, for instance the isomorphism $B_3 \to SL(2; \mathbb{Z})$ produces finite representations on $SL(2; \mathbb{Z}/n\mathbb{Z})$.

These presentations can be carried over $B_4$ via the epimorphism $B_4 \to B_3$. Analogously, homomorphisms onto symplectic groups provide finite representations.

As it is well known, for a group $G$, Hurwitz actions on $G^r$ and $G \setminus G^r$ provide finite representations of $B_r$. Let $g \in G^r$ and let us denote by $\Omega_g$ its orbit. Then the Hurwitz action defines a homomorphism $\tilde{\theta}_g : B_r \to \Sigma_{\Omega_g}$. Let us denote by $G_g$ the image $\tilde{\theta}_g(B_d)$, which is a transitive subgroup of $\Sigma_{\Omega_g}$. The induced mapping $\theta_g : B_d \to G_g$ is a surjective representation of $g$.

Analogously, considering the class $[g]$ of $g$ in $G \setminus G^r$, one can construct another finite representation $\theta_{[g]} : B_d \to G_{[g]}$ which factors through the canonical mapping $G_g \to G_{[g]}$.

Example 5.1. Let us suppose that $g_1, \ldots, g_d$ commute pairwise. Then, Hurwitz action factors through the canonical map $B_d \to \Sigma_d := \Sigma_{\{1,\ldots,d\}}$ onto the permutation action of $\Sigma_d$ on the ordered coordinates of $g$. If $g_1, \ldots, g_d$ are pairwise distinct, then $G_g$ is naturally isomorphic to $\Sigma_d$.

In the general case, one can understand this braid action as a lifting of the permutation action on the abelianized group of $G$. In order to see an easy, but not trivial, example we consider the case $r = 3$ and $G = \Sigma_3$. 
Let us take $g := [(1, 2), (1, 3), ()]$. Using GAP, we find that $\#\Omega_g = 9$ and $G_g$ is a group of order 162 given by the following presentation:

$$\langle a, b : aba = bab, a^6 = 1, [a^2, b^2] = 1, (ab^{-1})^3 = 1 \rangle.$$ 

Note that only three relations (including the first two) are needed.

For $g := [(1, 2), (1, 3), (1, 2)]$, we find that $\#\Omega_g = 8$ and $G_g$ is a group of order 24 given by the following presentation:

$$\langle a, b : aba = bab, a^3 = 1 \rangle.$$ 

For $g := [(1, 2), (1, 2), (1, 2)]$, we find that $\#\Omega_g = 12$ and $G_g$ is a group of order 48 given by the following presentation:

$$\langle a, b : aba = bab, a^4 = 1, (a^2b^{-2})^2 = 1, (ab^{-1})^3 = 1 \rangle.$$ 

As in the first case, only three relations including the first two are needed.

For $g := [(1, 2), (1, 2), (1, 2, 3)]$, we find that $\#\Omega_g = 18$ and $G_g$ has order 648.

In this way, we have obtained all the $G_g$, $g \in G^3$, up to conjugation.

6. Construction of curves in $\mathcal{M}$

We follow the ideas in [3] to construct curves in $\mathcal{M}$. Any curve in $\mathcal{M}$ is projectively equivalent to exactly one of the following two projective curves $\tilde{C}_\beta := \{f_\beta g_\beta = 0\}$ with:

$$f_\beta(x, y, z) := x^2y^3 + (303 - 216 \beta)xy^2z^2 + (-636 + 450 \beta)xyz^3 +$$

$$+ (-234 \beta + 331)xz^4 + (-18 \beta + 27)yz^4 + (18 \beta - 26)z^5$$

$$g_\beta(x, y, z) := x + \left(\frac{10449}{196} - \frac{3645}{98} \beta\right)y + \left(-\frac{432}{7} + \frac{297}{7} \beta\right)z$$

where $\beta^2 = 2$. The line $y = 0$ is the tangent line to both curves $\tilde{C}_\beta$ at the $E_6$ point which is $[1 : 0 : 0]$. We take the affine plane of coordinates $(x, z)$ and consider the projection $\pi(x, z) = z$. Since $\text{mult}_{x_0}(C_\beta) = 3$, one has $\text{deg}_\pi(C_\beta) = 3$. There are 4 non-transversal vertical lines corresponding to the singular points of types $A_7$, $A_3$, $A_2$, $A_1$. There is also an ordinary tangent vertical line intersecting $\tilde{C}_\beta$ at a point of tangency denoted by $A_0$. The values for $z$ at these lines are shown in table 1.

| $A_7$ | $A_3$ | $A_2$ | $A_1$ | $A_0$ |
|---|---|---|---|---|
| $904+9\Delta$ | 0 | 1 | $18+27\Delta$ | $-45+36\Delta$ |

Table 1.
Let us denote by $C_\beta$ the affine curve $f_\beta(x, 1, z)g_\beta(x, 1, z) = 0$. In both cases we obtain $C_{C_\beta}$ by eliminating five real points of $\mathbb{C}$. We choose $* \in \mathbb{R}$, $* \gg 0$ and we choose geometric bases for $\pi_1(C_{C_\beta}; *)$ using the lexicographic construction \[ and remark \] with respect to a segment in the real axis. We will denote by $\alpha_j^\beta$ the meridian about the non-generic line passing through the point $A_j$. By computing numerical values we obtain that the geometric bases are

$$
\alpha_1^{\sqrt{2}}, \alpha_1^{-\sqrt{2}}, \alpha_2^{\sqrt{2}}, \alpha_2^{-\sqrt{2}}, \alpha_3^{\sqrt{2}}, \alpha_3^{-\sqrt{2}}
$$

and

$$
\alpha_2^{\sqrt{2}}, \alpha_3^{-\sqrt{2}}, \alpha_2^{-\sqrt{2}}, \alpha_3^{\sqrt{2}}, \alpha_1^{-\sqrt{2}}, \alpha_1^{\sqrt{2}}.
$$

Figures 1 and 2 show the real parts of $C_{\sqrt{2}}$ and $C_{-\sqrt{2}}$; we have drawn their topological behavior. The dotted curves represent the real parts of the imaginary solutions and the thick point is the tacnode. The branch at infinity corresponding to the $E_6$ point is represented in both cases by the branches of the curves going to $-\infty$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{Real part of $C_{\sqrt{2}}$}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.png}
\caption{Real part of $C_{-\sqrt{2}}$}
\end{figure}
Braid monodromy of $C_{\sqrt{2}}$ is computed from figure 1:

$\alpha_{7\sqrt{2}} \mapsto \sigma_2^8$

$\alpha_{1\sqrt{2}} \mapsto \sigma_2^4 \ast \sigma_1^2$

$\alpha_{2\sqrt{2}} \mapsto \sigma_2^4 \sigma_1 \ast \sigma_2^3 = \sigma_2^3 \ast \sigma_1^3$

$\alpha_{3\sqrt{2}} \mapsto \sigma_2^4 \sigma_1 \sigma_2 \sigma_1^{-1} \sigma_2 \ast \sigma_2^4 = \sigma_2 \ast \sigma_1^4$

$\alpha_{0\sqrt{2}} \mapsto \sigma_2^4 \sigma_1 \sigma_2 \sigma_1^{-1} \sigma_2 \sigma_2^{-1} \sigma_1 \ast \sigma_2 = \sigma_1^{-3} \ast \sigma_2$.

Braid monodromy of $C_{-\sqrt{2}}$ is computed from figure 2:

$\alpha_{2-\sqrt{2}} \mapsto \sigma_2^3$

$\alpha_{0-\sqrt{2}} \mapsto \sigma_2 \sigma_1^{-1} \sigma_2 \ast \sigma_1$

$\alpha_{7-\sqrt{2}} \mapsto \sigma_2 \sigma_1^{-1} \sigma_2 \sigma_1 \ast \sigma_2^8 = \sigma_2 \ast \sigma_1^8$

$\alpha_{3-\sqrt{2}} \mapsto \sigma_2 \sigma_1^{-1} \sigma_2 \sigma_1 \sigma_2 \sigma_1 \sigma_2 \sigma_2^{-1} \sigma_1 \ast \sigma_2 \sigma_1^{-1} \sigma_2 \sigma_2 \sigma_1 \sigma_2 \sigma_2 \sigma_2 = \sigma_1^{-3} \ast \sigma_2^2$.

Applying Zariski-Van Kampen Theorem 3.10, we obtain the fundamental group of the complement to the projective curve $C_\beta \cup T_\beta$, where $T_\beta$ is the tangent line at $E_6$. In order to obtain $\pi_1(\mathbb{P}^2 \setminus C_\beta)$ we must kill a meridian about the line at infinity. Let $g_1, g_2, g_3$ denote the generators of the lexicographic geometric basis in a generic line $z = K$, $K \gg 0$. Let $e$ denote the meridian about the point at infinity on this line, obtained by taking $e := (g_3 \cdot g_2 \cdot g_1)^{-1}$. On a neighbourhood of the center of the projection one can check that

$((e^{-1}g_3e) \cdot g_3 \cdot (eg_3e^{-1}) \cdot g_3 \cdot g_2 \cdot g_1)^{-1}$

is a meridian of the line at infinity. Using GAP4 we obtain that both fundamental groups are isomorphic to $\mathbb{Z} \times SL(2;\mathbb{Z}/7\mathbb{Z})$. Moreover, there is an isomorphism preserving meridians – the image of a meridian in $C_{\sqrt{2}}$ is a meridian in $C_{-\sqrt{2}}$.

**Theorem 6.1.** The braid monodromies of $C_{\sqrt{2}}$ and $C_{-\sqrt{2}}$ with respect to $\pi$ are not equivalent.

**Proof.** It is enough to apply the final method described in strategy 4.6 using the representation

$B_3 \to SL(2;\mathbb{Z}) \to SL(2;\mathbb{Z}/32\mathbb{Z})$.

The orbits of both braid monodromies have size 15360 and are disjoint as shown by the GAP4 program in the Appendix. □
Finally, we are in the position to prove theorem 2.

**Proof of Theorem 2.** Let $C_1$ and $C_2$ in the statement of theorem 2 correspond to $C_{\sqrt{2}}$ and $C_{-\sqrt{2}}$ respectively. By theorems 6.1 and 3.19 there is no homeomorphism $(\mathbb{P}^2, C_{\phi_1}) \rightarrow (\mathbb{P}^2, C_{\phi_2})$ preserving orientations on $\mathbb{P}^2$, $C_1$ and $C_2$.

Since both curves have real equations, they are invariant by complex conjugation which preserves orientations on $\mathbb{P}^2$ but exchanges orientations on the curves. Hence there is no homeomorphism $(\mathbb{P}^2, C_{\phi_1}) \rightarrow (\mathbb{P}^2, C_{\phi_2})$ preserving the orientation of $\mathbb{P}^2$ but reversing it on $C_1$ and $C_2$.

Since algebraic braids always turn in the same direction – that is, they are positive – it is not possible to have a homeomorphism $(\mathbb{P}^2, C_{\phi_1}) \rightarrow (\mathbb{P}^2, C_{\phi_2})$ preserving the orientation of $\mathbb{P}^2$ and only some, but not all, of the components of the curves.

**APPENDIX**

In this appendix we provide the program source in GAP4 which produces the result stated in theorem 6.1. The authors include it for completeness and the text file can be distributed upon request. The execution of the program took about ten hours on a Pentium III 866Mhz running with GNU/Linux and sharing CPU time with other computations.

Here is the program:

```gap
# Some function definitions. Right conjugation.
cnj:=function(u,v)
    return u*v/u;
end;

# Simultaneous conjugation of a list.
conjorbita:=function(lista,u)
    return List(lista,x->[x[1],x[2]^u]);
end;

# Reverse product of a list.
prdct:=function(lista)
    local j,producto,n;
    n:=Length(lista);
    producto:=();
    for j in [1..n] do
        producto:=producto*lista[n-j+1][2];
    od;
    return producto;
end;

# Action of $\sigma_1$.
q1:=function(lista)
    local resultado,i,n;
    n:=Length(lista);
```
resultado:=[];
resultado[1]:=lista[2];
resultado[2]:=[lista[1][1],cnj(lista[2][2],lista[1][2])];
for i in [3..n] do
    resultado[i]:=lista[i];
od;
return resultado;
end;

# Action of $\sigma_1 \cdot \ldots \cdot \sigma_{n-1}$.

q2:=function(lista)
    local resultado,i,n;
    n:=Length(lista);
    resultado:=[];
    resultado[1]:=lista[n];
    for i in [2..n] do
        resultado[i]:=[lista[i-1][1],cnj(lista[n][2],lista[i-1][2])];
    od;
    return resultado;
end;

# This part produces the images of the standard generators of the braid group by the given representation. This part should be replaced for different representations.

m:=32;
R:=ZmodnZ(m);
fam:=ElementsFamily(FamilyObj(R));;
u:= ZmodnZObj(fam,1);
um:= ZmodnZObj(fam,m-1);
u0:= ZmodnZObj(fam,0);
A:=[[u,u0], [um,u]];
B:=[[u,u], [u0,u]];
g1:=Group(A,B);
iso:=IsomorphismPermGroup(g1);
a:=Image(iso,A);
b:=Image(iso,B);
g:=Group(a,b);

# This part describes both braid monodromies. This part should be replaced for different braid monodromies. Note that the elements in the list have two entries. The first one is a label to control the local topological singularity types.

pr:=[[[1,b^8],[2,cnj(b^4,a^2)],[3,cnj(b^3,a^3)],[4,cnj(b,a^4)],
    [5,b^(a-3)]];
otro:=[[[3,b^3],[5,cnj(b/a*b,a)],[1,cnj(b,a^8)],
    [4,(b^6)^a],[2,(b^2)^a^3]];

# The reversed product is applied to obtain pseudo Coxeter elements.

totpr:=prdct(pr);
tototro:=prdct(otro);
# Verification to check if both pseudo Coxeter elements are conjugate. Then, the list of all conjugates to the second braid monodromy having the same pseudo Coxeter elements as the first one is produced.

Print(IsConjugate(g,tototro,totpr),"n");

vale:=RepresentativeAction(g,tototro,totpr);
conjugar:=List(Elements(Centralizer(g,tototro)),x->x*vale);
segundo:=Unique(List(conjugar,u->conjorbita(otro,u)));
Sort(segundo);

# This part inductively constructs a subset, say $A$, containing the first braid monodromy up to conjugation and stable by the action of the function $q_1$. Note that it is enough to produce conjugations which preserve the pseudo Coxeter element of the first braid monodromy. Then it considers a subset $B$ of $A$ (up to the parameter $s$) such that its image by the function $q_2$ is contained in $A$. It chooses an element $x \in A \setminus B$ and applies $q_2$ to it. If its image is already in $A$ then $x$ is added to $B$. If $q_2(x) \notin A$ then its orbit by $q_1$ is added to $A$. The program stops when $A = B$. In fact, at each step it checks for common elements with the second braid monodromy and stops if there is any common element.

cnm:=Centralizer(g,totpr);
lcnm:=Elements(cnm);
micnj:=function(el)
  return Unique(List(lcnm,x->conjorbita(el,x)));
end;
orbita:=[pr];
quedan:=ShallowCopy(orbita);
Sort(quedan);
s:=0;
t:=Length(orbita);
r:=Length(quedan);

while s<Length(orbita) do
  elemento:=ShallowCopy(orbita[t]);
elmcnj:=micnj(elemento);
elt:=ShallowCopy(orbita[t]);
control:=not (elt in segundo);
if not control then
  s:=Length(orbita);
  Print("Orbits are equal\n");
f1;
control0:=control;
while control do
  elt1:=q1(elt);
  control:=not (elt1 in elmcnj);
  if control then
    control0:=not (elt1 in segundo);
    if control0 then
      Add(orbita,elt1);
BRAID MONODROMY AND PLANE CURVES

Add(quetan,elt1);
elt:=ShallowCopy(elt1);
else
s:=Length(orbita);
Print("Orbits are equal
");
control:=false;
fi;
fi;
od;
if control0 then
t:=Length(orbita)+1;
s:=s+1;
Print(s, " a ",t, "\n");
elt:=ShallowCopy(orbita[s]);
control:=true;
Sort(quetan);
while control do
elt1:=q2(elt);
control1:=not (elt1 in segundo);
control0a:=control1;
if not control1 then
s:=Length(orbita);
Print("Orbits are equal
");
control:=false;
else
cntrlcnj:=true;
j:=1;
orbcnj:=micnj(elt1);
while cntrlcnj do
elt1j:=orbcnj[j];
control1:=not (elt1j in quedan);
j:=j+1;
cntrlcnj:=control1 and (not j>Length(orbcnj));
od;
fi;
if control0a then
if control1 then
control:=false;
Add(orbita,elt1);
Add(quetan,elt1);
Sort(quetan);
elif s<Length(orbita) then
s:=s+1;
Print(s,"b,\n");
elt:=ShallowCopy(orbita[s]);
RemoveSet(quetan,elt1);
else
control:=false;
s:=s+1;
Print("Orbits are different\n");
fi;
fi;
od;
REFERENCES

[1] H. Abelson, Topologically distinct conjugate varieties with finite fundamental group, Topology 13 (1974), 161–176.
[2] E. Artal Bartolo, J. Carmona, and J.I. Cogolludo, On sextic curves with big Milnor number, Preprint, November 2000.
[3] E. Artal Bartolo, J. Carmona, J.I. Cogolludo, and H. Tokunaga, On curves with singular points in special position, available at arXiv:math.AG/0007152, to appear in J. Knot Theory Ramifications.
[4] E. Artin, Theory of braids, Ann. of Math. (2) 48 (1947), 101–126.
[5] J. Assion, Einige endliche Faktorgruppen der Zopfgruppen, Math. Z. 163 (1978), no. 3, 291–302.
[6] J. S. Birman, Braids, links, and mapping class groups, Princeton University Press, Princeton, N.J., 1974, Annals of Mathematics Studies, No. 82.
[7] J. S. Birman and B. Wajnryb, Markov classes in certain finite quotients of Artin’s braid group, Israel J. Math. 56 (1986), no. 2, 160–178.
[8] E. Brieskorn, Automorphic sets and braids and singularities, Braids (Santa Cruz, CA, 1986), Amer. Math. Soc., Providence, RI, 1988, pp. 45–115.
[9] J. Carmona, thesis, preprint.
[10] A. I. Degtyarëv, Isotopic classification of complex plane projective curves of degree 5, Leningrad Math. J. 1 (1990), no. 4, 881–904.
[11] The GAP Group, Aachen, St Andrews, GAP – Groups, Algorithms, and Programming, Version 4.2, 2000, \(\text{\url{http://www-gap.dcs.st-and.ac.uk/~gap}}\).
[12] E.R. van Kampen, On the fundamental group of an algebraic curve, Amer. J. Math. 55 (1933), 255–260.
[13] V. Kharlamov and Vik. S. Kulikov, Diffeomorphisms, isotopies, and braid monodromy factorizations of plane cuspidal curves, available at arXiv:math.AG/0104021.
[14] P. Kluitmann, Hurwitz action and finite quotients of braid groups, Braids (Santa Cruz, CA, 1986), Amer. Math. Soc., Providence, RI, 1988, pp. 299–325.
[15] Vik. S. Kulikov and M. Teicher, Braid monodromy factorizations and diffeomorphism types, Izv. Ross. Akad. Nauk Ser. Mat. 64 (2000), no. 2, 89–120.
[16] A. Libgober, On the homotopy type of the complement to plane algebraic curves, J. Reine Angew. Math. 367 (1986), 103–114.
[17] ———, Invariants of plane algebraic curves via representations of the braid groups, Invent. Math. 95 (1989), no. 1, 25–30.
[18] ———, Characteristic varieties of algebraic curves, Preprint available at arXiv:math.AG/9801070, 1998.
[19] B. G. Moishezon, Stable branch curves and braid monodromies, L.N.M. 862, Algebraic geometry (Chicago, Ill., 1980), Springer, Berlin, 1981, pp. 107–192.
[20] J. P. Serre, Exemples de variétés projectives conjuguées non homéomorphes, C. R. Acad. Sci. Paris 258 (1964), 4194–4196.
[21] B. Wajnryb, *Markov classes in certain finite symplectic representations of braid groups*, Braids (Santa Cruz, CA, 1986), Amer. Math. Soc., Providence, RI, 1988, pp. 687–695.

[22] , *A braidlike presentation of sp(n,p)*, Israel J. Math. 76 (1991), no. 3, 265–288.

[23] O. Zariski, *On the problem of existence of algebraic functions of two variables possessing a given branch curve*, Amer. J. Math. 51 (1929), 305–328.

Departamento de Matemáticas, Campus Plaza de San Francisco s/n, E-5009 Zaragoza SPAIN

E-mail address: artal@posta.unizar.es

Departamento de Sistemas informáticos y programación, Universidad Complutense, Ciudad Universitaria s/n, E-28040 Madrid SPAIN

E-mail address: jcarmona@eucmos.sim.ucm.es

Departamento de Álgebra, Universidad Complutense, Ciudad Universitaria s/n, E-28040 Madrid SPAIN

E-mail address: jicogo@eucmos.sim.ucm.es