Strongly Correlated States of Ultracold Rotating Dipolar Fermi Gases

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We study strongly correlated ground and excited states of rotating quasi-2D Fermi gases constituted of a small number of dipole-dipole interacting particles with dipole moments polarized perpendicular to the plane of motion. As the number of atoms grows, the system enters an intermediate regime, where ground states are subject to a competition between distinct bulk-edge configurations. This effect obscures their description in terms of composite fermions and leads to the appearance of novel composite fermion quasi-hole states. In the presence of dipolar interactions, the principal Laughlin state at filling \( \nu = 1/3 \) exhibits a substantial energy gap for neutral (total angular momentum conserving) excitations, and is well-described as an incompressible Fermi liquid. Instead, at lower fillings, the ground state structure favors crystalline order.

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Some of the most fascinating challenges of modern atomic and molecular physics arguably concern ultracold dipolar quantum gases [1]. The recent experimental realization of a quantum degenerate dipolar Bose gas of Chromium [2], and the progress in trapping and cooling of dipolar molecules [3] have opened the path towards ultracold quantum gases with dominant dipole interactions. Particularly interesting in this context are rotating dipolar gases (RDG). Bose-Einstein condensates of RDGs exhibit novel forms of vortex lattices, e.g., square, stripe- and bubble-"crystal" lattices [4]. The stability of these phases in the lowest Landau level was recently investigated [5]. We have demonstrated that the quasi-hole gap survives the large \( N \) limit for fermionic RDGs [6]. This property makes them perfect candidates to approach the strongly correlated regime, and to realize Laughlin liquids (cf. [7]) at filling \( \nu = 1/3 \), and quantum Wigner crystals at \( \nu \leq 1/7 \) [8] for a mesoscopic number of atoms \( N \approx 50 - 100 \). Lately, Rezayi et al. [9] have shown that the presence of a small number of dipole-dipole interacting particles with dipole moments polarized perpendicular to the plane of motion. As the number of atoms grows, the system enters an intermediate regime with respect to the axis of rotation. Along this \( z \)-axis, the dipole moments, as well as spins are assumed to be aligned. Various ways of experimental realizations of ultracold dipolar gases are discussed in [1]. In case of low temperature \( T \) and weak chemical potential \( \mu \) with respect to the axial confinement \( \omega_z \), the gas is effectively two-dimensional, and the Hamiltonian of the system in the rotating reference frame reads

\[
\mathcal{H} = \sum_{j=1}^{N} \frac{1}{2M} (\vec{p}_j - M \Omega \hat{\epsilon}_z \times \vec{r}_j)^2 + \frac{M}{2} \left( \omega_0^2 - \Omega^2 \right) r_j^2 + V_d(1)
\]

Here, \( \omega_0 \ll \omega_z \) is the radial trap frequency, \( \Omega \) is the frequency of rotation, \( M \) is the mass of the particles, \( V_d = \sum_{j<k} \frac{\mathbf{d}^2}{\left| \vec{r}_j - \vec{r}_k \right|^2} \) is the dipolar interaction potential (rotationally invariant with respect to the \( z \)-axis), \( d \) is the dipole moment, and \( \vec{r}_j = x_j \hat{e}_x + y_j \hat{e}_y \) is the position vector of the \( j \)-th particle. The first term of \( \mathcal{H} \) is formally equivalent to the Landau Hamiltonian of particles with mass \( M \) and charge \( e \) moving in a constant magnetic field of strength \( B = 2 M \Omega c/e \) perpendicular to their plane of motion. The spectrum of \( \mathcal{H}_{\text{Landau}} \) consists of equidistantly spaced and highly degenerate Landau levels (LL) with energies \( \varepsilon_n = \hbar \omega_c (n + 1/2) \) where \( \omega_c = 2 \Omega \). We denote by \( N_{\text{LL}} = 1/2 \pi l^2 \) the number of states per unit area in each LL, where \( l = \sqrt{\hbar/m \omega_c} \) is the magnetic length. Given a homogeneous fermionic surface density \( n_f \), the filling factor \( \nu = 2 \pi l^2 n_f \) can be defined and denotes the fraction of occupied Landau levels. Even though the above definition applies to infinite homogeneous systems, it may be used for finite systems as a suitable truncation of the Hilbert space at specific angular momenta. The second term in \( \mathcal{H} \) accounts for a rotationally induced effective reduction of the trap strength. For critical rotation \( \Omega \rightarrow \omega_0 \), the confining potential vanishes.
In the following, it is assumed that particles solely occupy the LLL. Restricted to the LLL and rewritten in second quantized form, the Hamiltonian reads

$$\hat{H} = \hbar \omega_0 \hat{N} + \hbar (\omega_0 - \Omega) \hat{L}^z + \sum_{m_1, \ldots, m_4} V_{1234} a_{m_1}^\dagger a_{m_2}^\dagger a_{m_3} a_{m_4} (2)$$

where $\hat{N}$ and $\hat{L}^z$ are the total number and $z$-component angular momentum operators, $a_{m_i}^\dagger$ creates a particle with angular momentum $m_i$, and $V_{1234} = \frac{1}{2} \langle m_1 m_2 | V_4 | m_3 m_4 \rangle$ is the matrix element of interaction expressed in the Fock–Darwin single particle angular momentum basis. Due to circular symmetry, (2) can be diagonalized blockwise for a given $L^z$. Calculations have been performed for $N = 3$ to 12 particles with complete and Davidson block diagonalization techniques, respectively. In the LLL Hamiltonian, besides the first constant term, there are two competing contributions to the energy: the kinetic term linear in $\hat{L}$ and the strength of interaction, which scales as $d^2/|l|^3 = 2\hbar \omega_0 (a_d/l)$ with $a_d = M d^2/\hbar^2$. The natural unit of energy is $\hbar \omega_0$, whereas distances are measured in $l$; from now on, we set $a_d/l \equiv 1$.

We analyze the ground state interaction energy for a given $L^z$. It reveals plateaus, clearly visible for small $N$ (see Fig. 1).

![FIG. 1: Interaction contribution to the GS energy as a function of $L^z$](image)

Ground state candidates are the first states of these plateaus where a downward cusp occurs in the spectrum. By tuning the rotational frequency, some of these states are selected as true ground states at specific “magic” angular momenta as depicted in Fig. 2. For relatively low values of $\Omega$, the ground state is the filled LL state at $\nu = 1$, which is completely insensitive to type and strength of interaction as long as the LLL approximation holds.

When $\Omega$ is continuously increased, the system evolves from the weakly interacting regime (where the second term of Eq. 2 is larger than the third) to strongly correlated states (where the interaction is dominant). In case of short-range interactions, this process terminates at $L_0^z = \nu_0^{-1} N (N - 1)/2$, where the fermionic (bosonic) Laughlin states at filling $\nu_0 = 1/3(1/2)$ become the true ground states.

The existence of a final $L^z$ is due to the fact that the ground state contact interaction energy vanishes for $L^z \geq L_0^z$.

![FIG. 2: Ground state angular momentum series over $\alpha \equiv \omega_0 - \Omega$ (the divergence at $\alpha = 0$ is not shown).](image)

The long range nature of dipolar interactions lifts this degeneracy. Thus, the whole principal series of fillings, i.e., $\nu = 1/(2m + 1)$ for fermionic gases, is accessible. To reveal the internal structures of relevant states, we consider the density-density correlation function $\hat{\rho}(\vec{r}, \vec{r}_0)$, which represents the conditional probability to find one atom at $\vec{r}$ when another is simultaneously located at $\vec{r}_0$:

$$\hat{\rho}(\vec{r}, \vec{r}_0) = \sum_{j<k}^{N} \delta(\vec{r}_j - \vec{r}) \delta(\vec{r}_k - \vec{r}_0).$$ (3)

It is crucial to analyze second order correlations, since the GS density only reveals radially symmetric contributions. Fig. 3 depicts $\hat{\rho}(\vec{r}, \vec{r}_0)$ for a selection of ground states with $N = 10$ particles. Generally, for small $N$, starting from the unstruc-
tered maximum-density-droplet at $L^z = N(N - 1)/2$ (top left) with $\nu = 1$, a fraction of atoms starts to arrange itself in a “crystal” on the edge leaving a residual “Fermi sea” at the center (top right), until the correlations are homogeneously established at $L_{\theta}^z = 3N(N - 1)/2$. For larger $N$, however, the energetically favorable number of edge atoms in the crystal changes “irregularly” with $L^z$ (right column); the system seems to be “frustrated” in this respect. In this regime, the bulk has to reorganize itself accordingly, and novel fascinating states with a hole at the center of the bulk appear (centered and bottom left).

In order to understand the “magic” $L^z$ numbers, Jain et al. [10] have proposed to model the interacting system in terms of an effective theory of non-interacting composite fermions (CF) for electrons in a quantum dot. In this ansatz, each fermion captures an even number of quantum fluxes, and the wave function reads

$$\Psi_{\text{CF}}(\{z_j\}) = N^\frac{1}{2} P_{\text{LLL}} \left\{ \prod_{j<k} (z_j - z_k)^{2m} \Psi_{\text{Landau}} \right\}. \tag{4}$$

Here, $m$ denotes the number of flux pairs, $\Psi_{\text{Landau}}$ is an $N$-particle eigenfunction of $\mathcal{H}_{\text{Landau}}$, and $P_{\text{LLL}}$ is the LLL projector. Motivated by the success of CFs for fermions, Cooper and Wilkin have nicely adapted this scheme to bosonic contact interacting gases [14]. Following this idea, we compared the series of true GS for non-interacting composites and dipolar fermions. As long as there is no real bulk in the system, i.e., for $N < 7$, the ground state series nearly identically match with the predictions of the effective CF theory. Furthermore, overlap calculations for electrons and short-range interacting bosons [10,14] suggest that similar results will hold for dipolar fermions. For bigger $N$, however, important deviations from CF theory, in particular in the intermediate regime, occur (see, for instance, the CF and true “magic” numbers in the table 5). This deficiency of the CF theory, already commented in Ref. [14], is more clearly related to the “frustration” effect and reorganization of the bulk in the present case. For $N = 10$, ground states with a density defect at the center are found at $L^z = 90, 95$ and 103. They either have no ($L^z = 95$), or are hardly similar to the corresponding “magic” CF states. Instead, the analysis for the accessible range of systems strongly suggests that these states originate from “parent” states which are boosted by $N$ quanta of angular momentum, e.g., $L^z = 80, 85$ and 93 for the above series. None of these parent states has its “magic” analogues in the CF model.

| $L^z$, CF     | 45 | 55 | 63 | 69 | 77 | 83 | 90 | 97 | 103 | 111 | 117 | 125 | 135 |
|---------------|----|----|----|----|----|----|----|----|-----|-----|-----|-----|-----|
| $L^z$, true   | 45 | 52 | 59 | 66 | 73 | 77 | 80 | 85 | 90  | 93  | 95  | 103 | 111 | 125 | 135 |

FIG. 4: Density-density correlation function $\hat{\rho}(\vec{r}, \vec{r}_0)$ of the Laughlin state for $N = 12$ dipolar fermions with $\vec{r}_0$ chosen at the maximum of the density, which occurs at the edge.

To a good approximation, the close connection between parent and boosted state (e.g. $L = 93$ and 103 respectively) can be understood as a quasi-hole excitation, which is analytically represented by

$$\Psi_{\text{qh}}(\{z_j\}) = N^\frac{1}{2} \left\{ \prod_{j=1}^{N} (z_j - z_0)^m \Psi_{L^z} \right\}, \tag{5}$$

where $z_0 = 0$, and $m = 1$ for $\nu = 1/3$. Indeed, the above wave function proves to be a good approximation for the states with $L^z = L^z_{\text{parent}} + N$: even though the exact states do not reveal a true topological defect at the origin due to finite size, the density at the origin scales as $1/N$ and decreases from 0.14 to $\approx 0.10$ as $N$ varies from 7 to 11, while overlaps with the exact ground states grow from 0.6 to 0.7 despite the simultaneous significant increase of the relevant Hilbert space dimension.

We have also studied in detail low energy excitations of the dipolar Laughlin state at $\nu = 1/3$, which for $N \geq 10$ consists of a significant bulk, surrounded by a practically “melted crystal” at the edge (see Fig. 4). In order to relate our results to the ones obtained in the thermodynamical limit in Ref. [3], it is necessary to identify the finite-size quasi-hole excitations in the spectrum. To map out irrelevant candidates, it is reasonable to appropriately truncate the Hilbert space by a finite size

filling factor. We use the approach of Ref. [15], and fix $\nu$ by imposing the constraint $m_\nu \leq (N - 1)/\nu$ on the maximum single particle angular momentum in the $N$-particle Fock basis. If a quasi-hole (quasi-particle) is nucleated, the filling factor is lowered (raised) by a correction of the order of $1/N$, i.e., $m_\nu \rightarrow m_\nu \pm 1$. This significantly reduces the basis dimension of the system and maps out irrelevant states. If the rotational frequency $\Omega$ is tuned to favor $\Psi_{1/3}^\nu$ as the true ground state, the lower lying excitations for $L_{1/3}^z < L^z < L_{1/3}^L + N$ turn out to be gapped by $\Delta \varepsilon \approx 1/N$. For short-range interacting
bosonic systems, these states have been partially identified as
dge excitations that carry quanta of angular momentum $[6]$. To
tify the excited states with a quasi-hole at the centre, the
ground states at $L_{qh}^L = L_{1/3}^L + N$ are calculated with
adapted $m_{\nu} + 1$. These states closely follow the lowest en-
ergy branch of edge excitations, and should be regarded as
such, rather than quasi-holes. This is strongly supported by
the fact that the substantial density dip for $N = 3$ continu-
ously vanishes for increasing number of particles. Additional
information about these edge excitations can be obtained by
direct ”naked-eye” inspection of Fig. 4. For the ”standard”
Laughlin state, the edge excitations are described by a hydro-
dynamical Luttinger liquid theory$[7]$, and their amplitude de-
cays as $\sin^{-3}(\phi)$, where $\phi$ is the ”angular distance” from $r_0$
along the edge; here the decay is much slower, which is yet
another signature of the finite size effects.

The above considerations imply that the only reliable quan-
tity that remains to estimate the quasi-hole energy in the large
$N$ limit is the neutral gap at fixed $L^z$. This gap remains more
or less constant for smaller $N$s, and gradually increases as
the number of bulk particles in the system and $N$ grow. For
the data of Ref. [6] ($M = 30$ a.m.u., $d = 0.5$ Debye, and a trap
frequency of $2\pi \times 10^3$ Hz), the gap is of the order of percents
of $2\hbar \omega_0$, i. e., it is substantial, but an order of magnitude less
compared to the value estimated in [6] in the large $N$ limit.
Obviously, the lack of true bulk behavior in the investigated
mesoscopic samples is responsible for this effect.

In the semiconductor fractional quantum Hall effect,
Wigner crystals, i. e., specific charge-density waves, were
discussed as competing ground states to Laughlin liquids.
The interplay of quantum fluctuations with the interaction en-
ergy have proven crystalline Wigner order to be favorable for
low enough fillings for electrons. This behavior intuitively
changes in the case of dipolar particles as the interaction en-
ergy scales differently in the density of the particles. A very
recent detailed analysis of this issue has confirmed the sta-
bility estimates for Laughlin and Wigner states of Ref. [6]
for systems constituted of 50-200 particles [8].

Fehrmann et al. have proven that phonon excitations destabilize
the Wigner crystal for fillings $\nu > 1/7$, which melts into a Laugh-
lin liquid. In the microsystems discussed in this letter, this
crossover is strongly supported. For fillings $\nu \leq 1/5$, den-
sity profiles significantly deviate from the Laughlin state and
show a clear pattern of hexagonal order for $N = 6$ particles
as depicted in Fig. 5 where the outer edge ring of the state
is constituted of five particles. Even though speaking of a true
“crystal” is certainly not applicable in such a system, signa-
tures of crystalline order are clearly present.

Summarizing, we have studied in detail ground and excited
states of quasi-2D ultracold rotating dipolar Fermi gases. By
exact diagonalization methods, we studied systems up to 12
particles. We have identified novel kinds of strongly corre-
lated states in the intermediate regime, i. e., pseudo-hole ex-
citations of CF states. Calculation of the substantial gap in the
excitation spectrum of the dipolar Laughlin state at $\nu = 1/3$
prove the accessibility of fractional quantum Hall states in

![FIG. 5: Radial density $\rho(r)$ of Laughlin states (circles) and true dipolar ground states (triangles) for $N = 6$ at different filling factors. Apart from five particles which constitute the edge ring, localization of the sixth particle at the origin is clearly visible for $\nu \leq 1/5$.]

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$\omega_+ = \sqrt{\omega_0^2 + \omega_3^2/4 + \omega_5 (\hbar = 1)}$) has to be much larger
than the interaction energy, thermal interaction, and the single
particle kinetic energy inside the LLL. The latter is given by
\[ \omega_+ = \sqrt{\omega_0^2 + \omega_c^2 / 4} = \frac{1}{2} \omega_c; \] see A. G. Morris and D. L. Feder, cond-mat/0602037.

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