Analysis and Comparison of Coverage Probability in the Presence of Correlated Nakagami-m Interferers and Non-identical Independent Nakagami-m Interferers

Suman Kumar Sheetal Kalyani
Dept. of Electrical Engineering
IIT Madras
Chennai 600036, India
{ee10d040,skalyani}@ee.iitm.ac.in

Abstract

In this work, coverage probability expressions are derived in terms of special functions for the following cases: (i) Both the user channel and the $N$ interferers are independent and non identical Nakagami-m distributed random variables (RVs). (ii) The $N$ interferers are correlated Nakagami-m RVs. Using the properties of special functions, the coverage probability expressions are further simplified. The coverage probability expressions for case (i) and (ii) are compared, and it is analytically shown that the coverage probability in the presence of correlated interferers is greater than or equal to the coverage probability in the presence of non-identical independent interferers when the shape parameter of the channel between the user and its base station is not greater than one. We also provide simulation results and these match with the derived theoretical results.

I. INTRODUCTION

Performance degradation of wireless communication is typically caused by multipath fading and co-channel interference. Various fading models have been studied in literature for modeling the interferer and user channels. Among them, the Nakagami-m distribution is a very popular
fading model for system performance prediction due to its analytical tractability, and wide flexibility. It can be used to model severe as well as weak fading. Rayleigh and Rician fading are both special cases of Nakagami-m fading [1].

Coverage probability is an important metric for performance evaluation of cellular systems. The coverage probability in a Nakagami-m fading environment has been studied extensively [2]–[10]. Coverage probability expressions are available in the literature when the fading parameter for the user channel (channel between user and its base station (BS)) is an integer and interferer channel (channel between user and interfering BS) fading parameters are arbitrary [4]. In contrast, the coverage probability expression is given in [7] when the interferer fading parameters are restricted to integer values and user channel fading parameter is arbitrary. In the case where the fading parameters are arbitrary and possibly non-identical for the \( N \) Nakagami-m interferers, coverage probability expression have been derived in terms of integrals in [9], [10] and infinite series in [5], [6], [8].

In a practical scenario, the interferers can be correlated [11] and the extent of correlation depends on various factors such as angle of arrival difference at the mobile, antenna pattern etc. The outage probability expression in presence of correlation among the interferers is derived in [6] and [10], in terms of an infinite series, and in terms of integrals, respectively. In both the independent and non-identically distributed (i.n.i.d.) case and the correlated case the coverage probability expressions given in terms of integral or series are fairly difficult to simplify. Hence these cannot be easily used for further analysis or comparison of the coverage probability as a function of the different parameters in the system.

In this paper, we derive the coverage probability expression in terms of Lauricella’s function\(^1\) of the fourth kind [18] for the following cases:

(a) Interferer and user channels having arbitrary Nakagami-m fading parameters.
(b) Interferers being correlated where the correlation is specified by a correlation matrix.

Further, using the properties of the special functions, the coverage probability expression is significantly simplified for the following special cases in the case of both i.n.i.d. interferers and correlated interferers:

\(^1\) Lauricella hypergeometric function has been extensively used to obtain the bit error probability expression in the multiple antenna system in the presence of generalized fading model [12]–[17].
(a) When the user channel undergoes Rayleigh fading, and the interfering channel undergoes Nakagami-m fading with arbitrary parameters.

(b) When the user channel undergoes Nakagami fading with integer shape parameter and the interfering channel undergoes Nakagami-m fading with arbitrary parameters.

(c) The $N$ interferers are equidistant from the user.

(d) In the presence of only one or two dominant interferers.

Though the coverage probability expression for the general case is in terms of special functions, the expression for special cases are fairly simple. We believe that there could be many other special cases other than those explored here that can be derived by exploiting the various properties of the special functions and hence our representation of coverage probability in terms of specials functions is very useful.

Furthermore, we compare the coverage probability when the interferers are i.n.i.d. with the coverage probability when the interferers are correlated using majorization theory. It is analytically shown that the coverage probability in presence of correlated interferers is higher than the coverage probability when the interferers are i.n.i.d., when the user channel’s shape parameter is lesser than or equal to one, and the interferers have Nakagami-m fading with arbitrary parameters. This analytical result indicates when the presence of correlation among the interferers is better than having independent interferers. We also show that when the user channel’s shape parameter is greater than one, one cannot say whether coverage probability is higher or lower for the correlated case when compared to the independent case. We have also carried out extensive simulations for both the i.n.i.d. interferers case and the correlated interferers case and some of these results are reported in Section VI. In all the cases, the simulation results match with our theoretical results.

II. SYSTEM MODEL

We consider a homogeneous macrocell network with hexagonal structure having inter cell site distance $2R$ as shown in Fig. 1. The Signal-to-Interference-Ratio (SIR) $\eta$ of a user located at $r$ meters from the base station (BS) is given by

$$\eta(r) = \frac{Pr r^{-\beta}}{\sum_{i \in \phi} P h_i d_i^{-\beta}},$$

(1)
where $\phi$ denotes the set of interfering BSs and $N = |\phi|$ denotes the cardinality of the set $\phi$. The transmit power of a BS is denoted by $P$. A standard path loss model $r^{-\beta}$ is considered, where $\beta \geq 2$ is the path loss exponent. An interference limited network is assumed, and hence noise power is neglected. The distance between user to tagged BS (own BS) and the $i$th interfering BS is denoted by $r$ and $d_i$, respectively. The user and the interferers undergo Nakagami-m fading and the probability density function (pdf) of the Nakagami-m distribution is given by

$$f(x; m, \Omega) = \frac{2m^m x^{2m-1} \exp\left(-\frac{m}{\Omega} x^2\right)}{\Gamma(m) \Omega^m},$$  \hspace{1cm} (2)$$

where $m \geq 0.5$ and $\Omega > 0$ denote the shape parameter and spread parameter, respectively, and $\Gamma(.)$ denotes the gamma function. Here, $g$ is the user channel’s power while $h_i$ is the channel power between $i^{th}$ interfering BS and user with $g \sim G(\alpha, \lambda)$ and $h_i \sim G(\alpha_i, \lambda_i)$. The pdf $G(\alpha, \lambda)$ of the gamma RV is given by

$$f_Y(y) = \frac{y^{\alpha-1} e^{-\frac{y}{\lambda}}}{\lambda^\alpha \Gamma(\alpha)},$$  \hspace{1cm} (3)$$

where, $\alpha \geq 0.5$ is the shape parameter, and $\lambda > 0$ denotes the scale parameter. Note that when $X \sim f(x; m, \Omega)$ then $X^2 \sim G(\alpha, \lambda)$ with $\alpha = m$, $\lambda = \frac{m}{\Omega}$.

The coverage probability of a user located at distance $r$ meters from the BS is defined as

$$P_c(T, r) = P(\eta(r) > T) = P\left(\frac{gr^{-\beta}}{I} > T\right) = P\left(I < \frac{gr^{-\beta}}{T}\right)$$  \hspace{1cm} (4)$$

where $T$ denotes the target SIR, and $I = \sum_{i \in \phi} h_i d_i^{-\beta}$. Since $h_i \sim G(\alpha_i, \lambda_i)$, hence $I$ is the sum of weighted i.n.i.d. gamma variates with weights $d_i^{-\beta}$. We will use the fact that weighted gamma variates $h'_i = w_i h_i$ can be written as gamma variates with weighted scale parameter i.e., $h'_i \sim G(\alpha_i, \lambda_i d_i^{-\beta})$ \cite{19}. Thus, $I = \sum_{i \in \phi} h'_i$ is the sum of i.n.i.d gamma variates. The pdf of sum of i.n.i.d. gamma RVs has been extensively studied in \cite{6, 12, 20–24}. In this context of paper, we use the confluent Lauricella function representation of the pdf of the sum of gamma variates.

The sum, $X$ of $N$ i.n.i.d. gamma RVs, $h'_i \sim G(\alpha_i, \lambda'_i)$ where $\lambda'_i = \lambda_i d_i^{-\beta}$ has a pdf given by \cite{12, 18, 25},

$$f_X(x) = \frac{\sum_{N=1}^{N} \alpha_i^{-1}}{\prod_{i=1}^{N} (\lambda'_i)^{\alpha_i} \Gamma\left(\sum_{i=1}^{N} \alpha_i\right)} \phi_2^{(N)}\left(\alpha_1, \cdots, \alpha_N; \sum_{i=1}^{N} \alpha_i; \frac{-x}{\lambda'_1}, \cdots, \frac{-x}{\lambda'_N}\right),$$  \hspace{1cm} (5)$$
where $\phi_2^{(N)}(\cdot)$ is the confluent Lauricella function [18], [25], [26]. The cumulative distribution function (cdf) of $X$ is given by

$$F_X(x) = \frac{\sum_{i=1}^{N} \alpha_i}{\prod_{i=1}^{N} (\lambda_i')^{\alpha_i} \Gamma \left( \sum_{i=1}^{N} \alpha_i + 1 \right)} \phi_2^{(N)} \left( \alpha_1, \cdots, \alpha_N; \sum_{i=1}^{N} \alpha_i + 1; -\frac{x}{\lambda_1'}, \cdots, -\frac{x}{\lambda_N'} \right).$$  \hspace{1cm} (6)$$

Fig. 1: Macrocell network with hexagonal structure having inter cell site distance $2R$

### III. Coverage Probability with I.N.I.D Fading

In this section, the coverage probability expression is derived in terms of special functions. Then using the properties of these functions, various special cases are studied.

The coverage probability expression can be written as $P \left( I < \frac{gr^{-\beta}}{T} \right)$. Using the fact that $I$ is the sum of i.n.i.d. gamma variates, one obtains,

$$P_c(T, r) = E_g \left[ \left( \frac{gr^{-\beta}}{T} \right)^{\sum_{i=1}^{N} \alpha_i} \prod_{i=1}^{N} (\lambda_i')^{\alpha_i} \Gamma \left( \sum_{i=1}^{N} \alpha_i + 1 \right) \phi_2^{(N)} \left( \alpha_1, \cdots, \alpha_N; \sum_{i=1}^{N} \alpha_i + 1; -\frac{gr^{-\beta}}{T\lambda_1'}, \cdots, -\frac{gr^{-\beta}}{T\lambda_N'} \right) \right],$$  \hspace{1cm} (7)$$

where $E_g$ denotes expectation with respect to RV $g$. Since $g$ is gamma distributed, (7) can be further simplified as

$$P_c(T, r) = \int_{0}^{\infty} \left( \frac{gr^{-\beta}}{T} \right)^{\sum_{i=1}^{N} \alpha_i} \prod_{i=1}^{N} (\lambda_i')^{\alpha_i} \Gamma \left( \sum_{i=1}^{N} \alpha_i + 1 \right) \phi_2^{(N)} \left( \alpha_1, \cdots, \alpha_N; \sum_{i=1}^{N} \alpha_i + 1; -\frac{gr^{-\beta}}{T\lambda_1'}, \cdots, -\frac{gr^{-\beta}}{T\lambda_N'} \right) g^{\alpha-1} e^{-\frac{g}{\lambda}} \frac{\lambda^\alpha}{\Gamma(\alpha)} dg$$  \hspace{1cm} (8)
\[ K' \int_0^\infty \frac{g^{\alpha_1+\alpha-1}}{\Gamma\left(\sum_{i=1}^N \alpha_i + \alpha\right)} e^{-\frac{g}{\lambda}} \phi_2^{(N)} \left( \alpha_1, \ldots, \alpha_N; \sum_{i=1}^N \alpha_i + 1; -\frac{gr^{-\beta}}{T\lambda_1'}, \ldots, -\frac{gr^{-\beta}}{T\lambda_N'} \right) dg \quad (9) \]

where, \( K' = \frac{1}{\lambda^\alpha} \frac{\Gamma\left(\sum_{i=1}^N \alpha_i + \alpha\right)}{\Gamma\left(\sum_{i=1}^N \alpha_i + 1\right)} \prod_{i=1}^N \left(\frac{1}{T\lambda_i'}\right)^{\alpha_i} \). Using transformation of variables with \( \frac{g}{\lambda} = t \) and rewriting (9), one obtains

\[ P_c(T, r) = K' \lambda^{\sum_{i=1}^N \alpha_i + \alpha} \int_0^\infty \frac{e^{-t}}{\Gamma\left(\sum_{i=1}^N \alpha_i + \alpha\right)} \phi_2^{(N)} \left( \alpha_1, \ldots, \alpha_N; \sum_{i=1}^N \alpha_i + 1; -\frac{\lambda tr^{-\beta}}{T\lambda_1'}, \ldots, -\frac{\lambda tr^{-\beta}}{T\lambda_N'} \right) dt. \quad (10) \]

In order to simplify (10), we use the following relationship between confluent Lauricella function and Lauricella’s function of the fourth kind [25, P. 286, Eq 43]

\[ F_d^{(N)}[\alpha, \beta_1, \ldots, \beta_N; \gamma; x_1, \ldots, x_N] = \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-t^{\alpha-1}} \phi_2^{(N)}[\beta_1, \ldots, \beta_2, \gamma; x_1 t, \ldots, x_N t] dt, \quad (11) \]

max\{Re(x_1), \ldots, Re(x_N)\} < 1, Re(\alpha) > 0;

Note that Re(\alpha) > 0 since \( \alpha_i > 0 \) and max\{Re(x_1), \ldots, Re(x_N)\} < 1 since max\{\(-\frac{\lambda tr^{-\beta}}{T\lambda_1'}, \ldots, -\frac{\lambda tr^{-\beta}}{T\lambda_N'}\}\) < 1. Hence applying (11) to evaluate (10), we obtain

\[ P_c(T, r) = \frac{\Gamma\left(\sum_{i=1}^N \alpha_i + \alpha\right)}{\Gamma\left(\sum_{i=1}^N \alpha_i + 1\right)} \prod_{i=1}^N \left(\frac{1}{T\lambda_i'}\right)^{\alpha_i} F_d^{(N)} \left[ \sum_{i=1}^N \alpha_i + \alpha_1, \ldots, \alpha_N; \sum_{i=1}^N \alpha_i + 1; -\frac{\lambda tr^{-\beta}}{T\lambda_1'}, \ldots, -\frac{\lambda tr^{-\beta}}{T\lambda_N'} \right], \quad (12) \]

Here \( F_d^{(N)}[a, b_1, \ldots, b_N; c; x_1, \ldots, x_N] \) is the Lauricella’s function of the fourth kind [27]. It can be evaluated by following single integral expression, i.e.,

\[ \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1} \prod_{i=1}^N (1-x_i t)^{-b_i} dt, \quad \text{where } Re(c) > Re(a) > 0. \quad (13) \]

and also by following multiple integral expression,

\[ \int \cdots (N) \cdots \int u_1^{b_1-1} \cdots u_N^{b_N-1}(1-u_1-\cdots-u_N)^{c-b_1-\cdots-b_N-1}(1-u_1 x_1-\cdots-u_N x_N)^{-a} du_1 \cdots du_N. \quad (14) \]
Here, the range of integration limits is \( u_1 \geq 0, \cdots u_N \geq 0, 1 - u_1 - \cdots - u_N \geq 0 \) and \( \text{Re}(b_1), \cdots, \text{Re}(b_N) \), and \( \text{Re}(c - b_1 - \cdots - b_N) \) are positive. A series expression for \( F^{(N)}_D \) involving \( N \)-fold infinite sums is given by

\[
F^{(N)}_D[a, b_1, \cdots, b_N; c; x_1, \cdots, x_N] = \sum_{i_1, \cdots, i_N=0}^{\infty} \frac{(a)_{i_1+\cdots+i_N}}{(c)_{i_1+\cdots+i_N}} \frac{(b_1)_{i_1} \cdots (b_N)_{i_N}}{i_1! \cdots i_N!} \frac{x_1^{i_1}}{i_1!} \cdots \frac{x_N^{i_N}}{i_N!},
\]

\[
\max\{|x_1|, \cdots |x_N|\} < 1,
\]

where, \((a)_n\) denotes the Pochhammer symbol which is defined as \( (a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} \). The series expression for Lauricella’s function of the fourth kind converges if \( \max\{|x_1|, \cdots |x_N|\} < 1 \). However from (12) it is apparent that convergence condition, i.e., \( \max\{|x_1|, \cdots |x_N|\} < 1 \) is not always satisfied, since \( r < d_\alpha \). Hence in order to obtain a series expression for \( F^{(N)}_D \) which converges, we use the following property of the Lauricella’s function of the fourth kind [18, p.286].

\[
F^{(N)}_D[a, b_1, \cdots, b_N; c; x_1, \cdots, x_N] = \left[ \prod_{i=1}^{N} (1 - x_i)^{-b_i} \right] F^{(N)}_D \left( c - a, b_1, \cdots, b_N; c; \frac{x_1}{x_1 - 1}, \cdots, \frac{x_N}{x_N - 1} \right)
\]

and rewrite (12) as

\[
P_c(T, r) = \frac{\Gamma(\sum \alpha_i)}{\Gamma(\sum (\alpha_i+1))} \prod_{i=1}^{N} \frac{\lambda}{\lambda + \lambda_i r^\beta T} \alpha_i \left[ 1 - \alpha, \alpha_1, \cdots, \alpha_N; \sum_{i=1}^{N} \alpha_i + 1; \frac{\lambda}{\lambda + r^\beta T}, \cdots, \frac{\lambda}{\lambda + r^\beta T} \right] F^{(N)}_D
\]

The coverage probability can now be evaluated using the integral expression given by (13) and by the series expression in (15). However the main utility of our expression in terms of special function is that it can be significantly simplified for various special cases and one would not require to evaluate either the integral or series expression. Hence, we will now show how the expression can be simplified using various properties of Lauricella’s function for a variety of cases.

A. Coverage probability when the user channel undergoes Rayleigh fading while the interfering channels undergo Nakagami-m fading with arbitrary parameters

When for the user channel \( \alpha = 1 \), then the coverage probability given in (17) reduces to

\[
P_c(T, r) = \prod_{i=1}^{N} \left( \frac{\lambda}{\lambda + \lambda_i r^\beta T} \right)^{\alpha_i} F^{(N)}_D \left[ 0, \alpha_1, \cdots, \alpha_N; \sum_{i=1}^{N} \alpha_i + 1; \frac{\lambda}{\lambda + r^\beta T}, \cdots, \frac{\lambda}{\lambda + r^\beta T} \right].
\]
The equivalent series expression is
\[
P_c(T, r) = \prod_{i=1}^{N} \left( \frac{\lambda}{\lambda + \lambda d_i^{-\beta} r^\beta T} \right)^{\alpha_i} \sum_{i_1 + \cdots + i_N = 0}^{\infty} \left( \sum_{i=1}^{N} \alpha_i + 1 \right) \left( \lambda \lambda_i \lambda_{i_1} \cdots \lambda_{i_N} \right)^{i_1} \cdots \left( \lambda \lambda_i \lambda_{i_1} \cdots \lambda_{i_N} \right)^{i_N} \left( \lambda + \lambda d_i^{-\beta} r^\beta T \right)^{i_1 + \cdots + i_N}.
\]

This can now be simplified using the fact that \( \left( 0 \right)_0 = 1 \) and \( \left( 0 \right)_k = 0 \) \( \forall k \geq 1 \). Hence, coverage probability is given by
\[
P_c(T, r) = \prod_{i=1}^{N} \left( \frac{\lambda}{\lambda + \lambda d_i^{-\beta} r^\beta T} \right)^{\alpha_i}.
\]

The above expression is equivalent to [4] eq. 30. Based on the simple expression in (20) the following can be inferred:

(a) Since \( \frac{\lambda}{\lambda + \lambda d_i^{-\beta} r^\beta T} < 1 \), as \( \alpha_i \) increases, coverage probability decreases. In other words, as interferer shape parameter \( \alpha_i \) increases the signal power in the line of sight (LOS) component coming from the \( i \)-th interferer increases, and this results in a lower coverage probability.

(b) The coverage probability is dominated by the smallest term in \( \prod_{i=1}^{N} \left( \frac{\lambda}{\lambda + \lambda d_i^{-\beta} r^\beta T} \right)^{\alpha_i} \), i.e., if \( \lambda_i \) or \( \alpha_i \to \infty \) then \( P_c(T, r) \to 0 \).

(c) When interferer channels also undergoes i.n.i.d. Rayleigh fading, coverage probability further simplifies to \( \prod_{i=1}^{N} \left( \frac{\lambda}{\lambda + \lambda d_i^{-\beta} r^\beta T} \right) \).

**B. Coverage probability when the user channel undergoes Nakagami-m fading with shape parameter \( \alpha = 2 \) and interfering channel can have Nakagami-m fading with arbitrary parameter**

When \( g \sim \mathcal{G}(2, \lambda) \) while \( h_i \sim \mathcal{G}(\alpha_i, \lambda_i^\prime) \) then the coverage probability given in (17) reduces to
\[
P_c(T, r) = \prod_{i=1}^{N} \left( \frac{\lambda}{\lambda + \lambda d_i^{-\beta} r^\beta T} \right)^{\alpha_i} \left( \sum_{i=1}^{N} \alpha_i + 1 \right) F_D^{(N)} \left[ -1, -1, \cdots, \alpha_N \sum_{i=1}^{N} \alpha_i + 1; \lambda, \frac{\lambda}{\lambda + \lambda d_i^{-\beta} r^\beta T} \right].
\]

The equivalent series expression is
\[
P_c(T, r) = \left( \sum_{i=1}^{N} \alpha_i + 1 \right) \prod_{i=1}^{N} \left( \frac{\lambda}{\lambda + \lambda d_i^{-\beta} r^\beta T} \right)^{\alpha_i} \sum_{i_1 + \cdots + i_N = 0}^{\infty} \left( \sum_{i=1}^{N} \alpha_i + 1 \right)^{\alpha_i} \left( \lambda \lambda_i \lambda_{i_1} \cdots \lambda_{i_N} \right)^{i_1} \cdots \left( \lambda \lambda_i \lambda_{i_1} \cdots \lambda_{i_N} \right)^{i_N} \left( \lambda + \lambda d_i^{-\beta} r^\beta T \right)^{i_1 + \cdots + i_N}.
\]

\( F_D^{(N)} (.) \) can now be simplified using the following property of Pochhammer symbol [25, P. 14]
\[
(-n)_k = \begin{cases} 
\frac{(-1)^k n!}{(n-k)!}, & 0 \leq k \leq n, \\
0, & k > n,
\end{cases}
\]

(23)
and hence $F_D^{(N)}[-1, b_1, \cdots, b_N; c; x_1, \cdots, x_N] = 1 - \frac{1}{c} \sum_{i=1}^{N} b_i x_i$. Thus, the coverage probability given in (21) can be rewritten as

$$P_c(T, r) = \left( \sum_{i=1}^{N} \alpha_i + 1 - \sum_{i=1}^{N} \frac{\lambda \alpha_i}{\lambda + \lambda d_i^{-\beta} r^\beta T} \right) \prod_{i=1}^{N} \left( \frac{\lambda}{\lambda + \lambda d_i^{-\beta} r^\beta T} \right)^{\alpha_i}. \quad (24)$$

Rearranging the above expression one obtains

$$P_c(T, r) = \left( 1 + \sum_{i=1}^{N} \frac{\lambda_i d_i^{-\beta} r^\beta T \alpha_i}{\lambda + \lambda d_i^{-\beta} r^\beta T} \right) \prod_{i=1}^{N} \left( \frac{\lambda}{\lambda + \lambda d_i^{-\beta} r^\beta T} \right)^{\alpha_i}. \quad (25)$$

The above expression is equivalent to [4, eq. 31]. Now, the expressions given in (20) and (25) will be compared to see the impact of user channel’s shape parameter on coverage probability.

We see that the ratio of coverage probability when user channel undergoes Nakagami-m fading with shape parameter 2 to the coverage probability when user channel undergoes Nakagami-m fading with shape parameter 1 is $\left( 1 + \sum_{i=1}^{N} \frac{\lambda_i d_i^{-\beta} r^\beta T \alpha_i}{\lambda + \lambda_i d_i^{-\beta} r^\beta T} \right)$, and it is greater than 1. In other words, as the user channel’s shape parameter increases, coverage probability increases.

C. Coverage probability when the user channel undergoes Nakagami-m fading with integer shape parameter and interfering channel undergoes Nakagami-m fading with arbitrary parameters

When $g \sim G(M, \lambda)$ where $M \geq 1$ is an integer, while interfering channel can have arbitrary fading parameter then the coverage probability given in (17) can be rewritten as

$$P_c(T, r) = \frac{\Gamma \left( \sum_{i=1}^{N} \alpha_i + M \right)}{\Gamma \left( \sum_{i=1}^{N} \alpha_i + 1 \right) \Gamma(M)} \prod_{i=1}^{N} \left( \frac{\lambda}{\lambda + \lambda d_i^{-\beta} r^\beta T} \right)^{\alpha_i} F_D^{(N)} \left[ 1 - M, \alpha_1, \cdots, \alpha_N; \sum_{i=1}^{N} \alpha_i + 1; \frac{\lambda}{\lambda + \lambda d_i^{-\beta} r^\beta T}, \cdots, \frac{\lambda}{\lambda + \lambda d_i^{-\beta} r^\beta T} \right]. \quad (26)$$

Similar to the above case, using (23), Equation (26) can now be simplified as

$$P_c(T, r) = \frac{\Gamma \left( \sum_{i=1}^{N} \alpha_i + M \right)}{\Gamma \left( \sum_{i=1}^{N} \alpha_i + 1 \right) \Gamma(M)} \prod_{i=1}^{N} \left( \frac{\lambda}{\lambda + \lambda d_i^{-\beta} r^\beta T} \right)^{\alpha_i} \left[ 1 + \sum_{i_1, \cdots, i_N = 0}^{\frac{1}{M-1}} \sum_{i_1 = 1}^{M-1} \cdots \frac{(-1)^1 (\alpha_1)_{i_1} \cdots (\alpha_N)_{i_N} \left( \frac{\lambda}{\lambda + \lambda d_i^{-\beta} r^\beta T} \right)^{i_1}}{i_1!} \cdots \left( \frac{\lambda}{\lambda + \lambda d_i^{-\beta} r^\beta T} \right)^{i_N} \frac{i_N!}{i_N!} \right]. \quad (27)$$
Note that when user channel’s shape parameter is an integer, the coverage probability is given in terms of $M - 1$ sums each sum having a finite number of terms and hence it can be easily evaluated.

D. Coverage probability with equidistant interferers

In this subsection, we study the coverage probability in a scenario where the set of dominant interferers are equidistant from the user. This type of scenario is of practical interest and has been studied in [28] where they look at an interfering cluster of co-located single or multiple antenna terminals. We consider two different cases: case (a). All the interferers are equidistant, and (b). There are two layers of equidistant interferers.

Case (a): There are $N$ interferers equidistant from user and the distribution corresponding to $i$th interference is $G(\alpha_i, \lambda_c)$. Since interferers are equidistant $d_i = d \ \forall i$. Hence, the coverage probability is given by

$$P_c(T, r) = \prod_{i=1}^{N} \left( \frac{\lambda}{\lambda + \lambda_c d - \beta r T} \right)^{\alpha_i} \Gamma \left( \frac{\sum_{i=1}^{N} \alpha_i + \alpha}{\sum_{i=1}^{N} \alpha_i + 1} \right) \Gamma(\alpha) \frac{1}{\lambda^{\alpha_1} \cdots \cdots \lambda^{\alpha_N}} F_D^{(N)} \left[ 1 - \alpha, \alpha_1, \cdots, \alpha_N; \sum_{i=1}^{N} \alpha_i + 1; \sum_{i=1}^{N} \lambda_c \lambda d - \beta r T \right].$$

(28)

The coverage probability can now be simplified using the following property [29]:

$$F_D^{(N)}[a, b_1, \cdots, b_M, b_{M+1}, \cdots, b_N; c; x, \cdots, x] = 2F_1 \left( a, \sum_{i=1}^{N} b_i; c; x \right)$$

(29)

where $2F_1(a; b; c; z)$ denotes the Gauss hypergeometric function and hence (28) can be simplified as

$$P_c(T, r) = \prod_{i=1}^{N} \left( \frac{\lambda}{\lambda + \lambda_c d - \beta r T} \right)^{\alpha_i} \Gamma \left( \frac{\sum_{i=1}^{N} \alpha_i + \alpha}{\sum_{i=1}^{N} \alpha_i + 1} \right) \Gamma(\alpha) \frac{1}{\lambda^{\alpha_1} \cdots \cdots \lambda^{\alpha_N}} 2F_1 \left( 1 - \alpha, \sum_{i=1}^{N} \alpha_i; \sum_{i=1}^{N} \alpha_i + \lambda_c \lambda d - \beta r T \right).$$

(30)

The coverage probability with equidistant interferers is in terms of the well known Gauss hypergeometric function which can be easily evaluated using software such as Mathematica.

Case (b): Let us consider that $M$ interferers out of the $N$ interferers are in one layer and remaining $N - M$ interferers are in the second layer. Distance between the user and the first layer and the second layer are denoted by $d_1$ and $d_2$, respectively.

Using the property from [29] of Lauricella’s function

$$F_D^{(N)}[a, b_1, \cdots, b_M, b_{M+1}, \cdots, b_N; c; x_1, \cdots x_1, x_2 \cdots x_2] = F_1 \left( a, \sum_{i=1}^{M} b_i; \sum_{i=M+1}^{N} b_i; c; x_1, x_2 \right)$$

(31)
where $F_1(a_1, b_1, b_2; c; x_1, x_2)$ denotes the Appell function \[18\], the coverage probability given in \[17\] reduces to

$$P_c(T, r) = \prod_{i=1}^{N} \left( \frac{\lambda}{\lambda + \lambda a_i d_i^{-\beta r^\beta T}} \right)^{\alpha_i \Gamma \left( \sum_{i=1}^{N} \alpha_i + 1 \right) \Gamma \left( \sum_{i=1}^{N} \alpha_i \right)} F_1 \left( 1 - \alpha, \sum_{i=1}^{N} \alpha_i, \sum_{i=M+1}^{N} \alpha_i; \sum_{i=1}^{N} \alpha_i + 1; \frac{\lambda}{\lambda + \lambda a_i d_i^{-\beta r^\beta T}}, \frac{\lambda}{\lambda + \lambda a_i d_i^{-\beta r^\beta T}} \right).$$ \(32\)

Here also we see that coverage probability with two layers of equidistant interferers can be expressed in terms of the Appell function which again can be evaluated using Mathematica.

In many practical scenarios, where one has a few dominating interferers, the interferers can be modeled as two sets of equidistant interferers and in such case coverage probability can be calculated using simpler function than the $F_D(.)$. Similarly when $N$ interferers can be modeled by $M$ layers of equidistant interferers ($M < N$), the coverage probability expression can be simplified since it now involves $F_D^M(.)$ instead of $F_D^N(.)$.

E. Coverage probability in presence of single strong interferer

In certain scenarios, a user may only see a single strong interferer. For example, in heterogeneous networks, small cell typically experience very strong interference from the nearest macrocell. In this subsection, we derive the coverage probability when the user see a very strong interferer and the other interferers can be ignored. Assume the strong interferer channel power is distributed as $G(\alpha_a, \lambda_a)$, coverage probability in \[17\] reduces to following expression,

$$P_c(T, r) = \left( \frac{\lambda}{\lambda + \lambda a d^{-\beta r^\beta T}} \right)^{\alpha \alpha \Gamma \left( \alpha + 1 \right) \Gamma \left( \alpha \right)} F_1 \left( 1 - \alpha, \alpha; \alpha + 1; \frac{\lambda}{\lambda + \lambda a d^{-\beta r^\beta T}} \right),$$ \(33\)

This expression is similar to expression obtained for coverage probability with an equidistant interferers. Similarly, one can find out the coverage probability in the case of two strong interferers and it will be similar to the coverage probability expression given in \(32\).

F. Coverage probability in presence of single strong interferer when the user channel’s shape parameter is an integer

We have already derived the coverage probability expression in presence of one strong interferer in terms of Gauss hypergeometric function(given by \(33\)). The series expression of Gauss hypergeometric function is given by

$$2 F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} z^n, \text{ where } c \neq 0, -1, -2, \cdots.$$ \(34\)
Using the properties of Pochhammer symbol given in (23), and the fact that \( \frac{(a)_n}{(a+1)_n} = \frac{a}{a+n} \), the coverage probability given in (33) can be reduced to

\[
P_c(T, r) = \left( \frac{\lambda}{\lambda + \lambda_a d^{-\beta} r^{-\beta} T} \right)^{\alpha_a} \left( \frac{\Gamma(\alpha_a + \alpha)}{\Gamma(\alpha)} \right) \frac{1}{\Gamma(\alpha + n + 1)} \left( \frac{(1 - \alpha)_n}{\alpha + n + 1} \right) \left( \frac{\lambda}{\lambda + \lambda_a d^{-\beta} r^{-\beta} T} \right)^n.
\]

The above expression matches with [4, eq. 33]. Similarly one can also obtain a simplified expression for the case of only two dominating interferers.

G. Impact of scale parameter

It is evident from the coverage probability expression given in (17) that as scale parameter \( \lambda \) of the user channel increases coverage probability increases whereas, the coverage probability decreases with the increase in scale parameter of the interferer channels. It is also apparent that if the scale parameter of the interferer channels are equal to the scale parameter of the user channel then in (17) both will cancel each other, i.e., coverage probability will be independent of the scale parameter.

H. Coverage probability in presence of log normal shadowing

So far we have derived coverage probability in presence of small scale fading and path loss. In this subsection, we derive the approximate coverage probability when large scale fading, i.e., log-normal shadowing is also taken into account. In general, the large scale fading in the received signal power is modeled by zero-mean log-normal distribution and is given by,

\[
f_X(x) = \frac{1}{x \sqrt{2\pi(\sigma_{dB})^2}} \exp \left( -\frac{(\ln(x))^2}{2(\sigma_{dB})^2} \right), \quad x > 0,
\]

where \( \sigma \) is the shadow standard deviation represented in dB. Typically the value of \( \sigma_{dB} \) varies from 3 dB to 10 dB [30], [31]. It has been shown in [32], [33] that the channel power due to shadowing can be reasonably approximated by Gamma distribution with shape parameter \( m_s \) and scale parameter \( \frac{\Omega_s}{m_s} \), i.e., \( g_s \sim G(m_s, \frac{\Omega_s}{m_s}) \). Using moment matching between the moments of lognormal and gamma pdf, [33] shows that

\[
m_s = \frac{1}{(\exp(\frac{\sigma}{8.686})^2 - 1)} \quad \text{(36)}
\]

\[
\Omega_s = \sqrt{(m_s + 1)/m_s} \quad \text{(37)}
\]
Now, using the result in [34] which approximate the product of two gamma RV by another gamma RV, the product of two gamma RV \( g \sim G(\alpha, \lambda) \) and \( g_s \sim G(\beta_s, \Omega_s) \) can be approximated by gamma RV \( g_l \sim G(\alpha_l, \lambda_l) \), where \( \alpha_l \) and \( \lambda_l \) are given by

\[
\alpha_l = \frac{1}{(\alpha_l + 1) \exp((\sigma_{.686})^2) - 1} \quad \text{and} \quad \lambda_l = (1 + \alpha)\lambda \exp \left( \frac{3(\sigma_{.686})^2}{2} \right) - \alpha \lambda \exp \left( \frac{(\sigma_{.686})^2}{2} \right) \quad (38)
\]

Hence, in the presence of \( N \) interferers each undergoing small scale fading (Nakagami-m fading) and large scale fading (log normal shadowing), the sum interference \( I = \sum_{i=1}^{N} k_i \) where \( k_i \sim G(\tilde{\alpha}_i, \tilde{\lambda}_i) \) with

\[
\tilde{\alpha}_i = \frac{1}{(\tilde{\alpha}_i + 1) \exp((\sigma_{.686})^2) - 1} \quad \text{and} \quad \tilde{\lambda}_i = (1 + \tilde{\alpha}_i)\lambda_i \exp \left( \frac{3(\sigma_{.686})^2}{2} \right) - \alpha_i \lambda_i \exp \left( \frac{(\sigma_{.686})^2}{2} \right) \quad (39)
\]

The user channel also undergoes both small scale fading and large scale fading hence \( g_u \sim G(\alpha_u, \lambda_u) \) where \( \alpha_u \) and \( \lambda_u \) are given by (38). Consequently, the process of finding coverage probability in the presence of shadowing is similar to finding the coverage probability in an i.n.i.d. case. Therefore, one can derive the coverage probability in presence of the shadowing by applying the methods used for finding the coverage for the i.n.i.d. case.

**IV. COVERAGE PROBABILITY IN PRESENCE OF CORRELATED INTERFERERS**

In this section, we obtain the coverage probability expression in presence of correlated interferers, when the shape parameter of all the interferers are identical and analyze various special cases. The sum, \( Z \) of \( N \) correlated not necessarily identically distributed gamma RVs \( Y_i \sim G(\alpha_c, \lambda'_i) \) has a cumulative distribution function given by [23], [24],

\[
F_Z(z) = \frac{z^{N\alpha_c}}{\det(A)^{\alpha_c} \Gamma(N\alpha_c + 1)} \phi_2^{(N)} \left( \frac{\alpha_i, \cdots, \alpha_i; N\alpha_c + 1; \frac{-z}{\lambda_i}, \cdots, \frac{-z}{\lambda_N}}{} \right),
\]

here, \( A = DC \), where \( D \) is the diagonal matrix with entries \( \lambda'_i \) and \( C \) is the symmetric positive definite (s.p.d.) \( N \times N \) matrix defined by

\[
C = \begin{bmatrix}
1 & \sqrt{\rho_{12}} & \cdots & \sqrt{\rho_{1N}} \\
\sqrt{\rho_{21}} & 1 & \cdots & \sqrt{\rho_{2N}} \\
\cdots & \cdots & \cdots & \cdots \\
\sqrt{\rho_{N1}} & \cdots & \cdots & 1
\end{bmatrix},
\]

(40)
where $\rho_{ij}$ denotes the correlation coefficient between $Y_i$ and $Y_j$, and is given by,
\[
\rho_{ij} = \frac{\text{cov}(Y_i, Y_j)}{\sqrt{\text{var}(Y_i)\text{var}(Y_j)}}, \quad 0 \leq \rho_{ij} \leq 1, \quad i, j = 1, 2, \ldots, N.
\]

$cov(Y_i, Y_j)$ and $\text{var}(Y_i)$ denote the covariance between $Y_i$ and $Y_j$ and variance of $Y_i$, respectively.

$\text{det}(A) = \prod_{i=1}^{N} \hat{\lambda}_i$ is the determinant of the matrix $A$, and $\hat{\lambda}_i$'s are the eigenvalues of $A$. Note that $\hat{\lambda}_i > 0 \ \forall i$, since $C$ is s.p.d. and the diagonal elements of $A$ are equal to $\lambda_i'$. The functional form of cdf of sum of correlated gamma RVs is similar to the cdf of sum of i.n.i.d. gamma RVs. Hence the coverage probability in the presence of correlated interferers can be derived by applying the methods given in Section III, and one obtains
\[
P_c(T, r) = \frac{\Gamma(N\alpha_c + \alpha)}{\Gamma(N\alpha_c + 1)} \prod_{i=1}^{N} \left( \frac{\lambda}{\lambda + \hat{\lambda}_i r^{\beta} T} \right)^{\alpha_c} F_D^{(N)} \left[ 1 - \alpha, \alpha_c, \ldots, \alpha_c; N\alpha_c + 1; \frac{\lambda}{\lambda + \hat{\lambda}_i r^{\beta} T}, \ldots, \frac{\lambda}{\lambda + \hat{\lambda}_N r^{\beta} T} \right],
\]

However, note that here the coverage probability is a function of the eigenvalues of $A$ and the shape parameter of the user and interferer channels while in the i.n.i.d. case it was only a function of the shape parameters and scale parameters. Now, similar to independent case we will obtain the coverage probability expression for some special cases.

A. Coverage probability when the user channel undergoes Rayleigh fading and interfering channels undergo Nakagami-m fading.

Similar to Section IIIA, using the fact that $F_D^{(N)}[0, b_1, \ldots, b_N; c; x_1, \ldots, x_N] = 1$, one obtains
\[
P_c(T, r) = \prod_{i=1}^{N} \left( \frac{\lambda}{\lambda + \hat{\lambda}_i r^{\beta} T} \right)^{\alpha_c}.
\]

When user channel undergoes only Rayleigh fading, based on (43) the following can be inferred.

(a) Since $\hat{\lambda}_i$'s are the eigenvalues of a s.p.d matrix, $\hat{\lambda}_i > 0 \ \forall i$ and hence $\frac{\lambda}{\lambda + \hat{\lambda}_i r^{\beta} T} < 1$. Therefore, similar to independent interferers case as interferer shape parameter $\alpha_c$ increases, coverage probability decreases.

(b) The coverage probability is dominated by the smallest term in $\prod_{i=1}^{N} \left( \frac{\lambda}{\lambda + \hat{\lambda}_i r^{\beta} T} \right)^{\alpha_c}$, i.e., if $\hat{\lambda}_i$ or $\alpha_c \rightarrow \infty$ then $P_c(T, r) \rightarrow 0$.

(c) When interferers also undergoes Rayleigh fading, coverage probability further simplifies to $\prod_{i=1}^{N} \left( \frac{\lambda}{\lambda + \hat{\lambda}_i r^{\beta} T} \right)$.
B. Coverage probability when the user channel undergoes Nakagami-m fading with shape parameter $\alpha = 2$ and interfering signal can have Nakagami-m fading with arbitrary parameter

Similar to Section IIIB, using the fact that $F_{D}^{(N)}[-1, b_{1}, \ldots, b_{N}; c; x_{1}, \ldots, x_{N}] = 1 - \frac{1}{c} \sum_{i=1}^{N} b_{i} x_{i}$, one obtains

$$P_{c}^{c}(T, r) = \left( N\alpha_{c} + 1 - \sum_{i=1}^{N} \frac{\lambda\alpha_{c}}{\lambda + \lambda_{i} r^{\beta_{T}}} \right) \prod_{i=1}^{N} \left( \frac{\lambda}{\lambda + \lambda_{i} r^{\beta_{T}}} \right)^{\alpha_{c}}. \quad (44)$$

Rearranging the above expression one obtains

$$P_{c}(T, r) = \left( 1 + \sum_{i=1}^{N} \frac{\hat{\lambda}_{i} r^{\beta_{T}} \alpha_{c}}{\lambda + \hat{\lambda}_{i} r^{\beta_{T}}} \right) \prod_{i=1}^{N} \left( \frac{\lambda}{\lambda + \hat{\lambda}_{i} r^{\beta_{T}}} \right)^{\alpha_{c}}. \quad (45)$$

Coverage probability for various other cases can be obtained in a similar fashion by using the methods in Section III.

V. COMPARISON OF COVERAGE PROBABILITY IN THE PRESENCE OF CORRELATED INTERFERERS WITH COVERAGE PROBABILITY IN THE PRESENCE OF INDEPENDENT INTERFERERS

In this section, we compare the coverage probability in the i.n.i.d. case and correlated case using the derived expressions to better understand the impact of correlation. Note that the coverage probability expression for the correlated case is derived when the interferers shape parameter are all equal and hence for a fair comparison we consider equal shape parameter for the i.n.i.d. case also. We first start with the special case when user channel’s fading is Rayleigh and interferers have Nakagami-m fading with arbitrary parameters. The coverage probability for the i.n.i.d. case and the correlated case are given in (20) and (43), respectively, and for $\alpha_{i} = \alpha_{c} \ \forall i$ they are given by

$$P_{c}(T, r) = \prod_{i=1}^{N} \left( \frac{1}{1 + \lambda_{i}^{\beta_{T}}} \right)^{\alpha_{c}} \quad \text{and} \quad P_{c}^{c}(T, r) = \prod_{i=1}^{N} \left( \frac{1}{1 + \hat{\lambda}_{i}^{\beta_{T}}} \right)^{\alpha_{c}}. \quad (46)$$

We now state and prove the following theorem for the case where the user channel undergoes Rayleigh fading and then generalize it:

**Theorem 1.** The coverage probability in correlated case is higher than that of the i.n.i.d. case, when user’s channel undergoes Rayleigh fading, i.e.,

$$\prod_{i=1}^{N} \left( \frac{1}{1 + k\lambda_{i}} \right)^{\alpha_{c}} \geq \prod_{i=1}^{N} \left( \frac{1}{1 + k\lambda_{i}^{\prime}} \right)^{\alpha_{c}} \quad (47)$$
where \( \hat{\lambda}_i \)s are the eigenvalues of matrix \( A \) and \( \lambda'_i \)s are the scale parameter for the i.n.i.d. case and \( k = \frac{\omega r}{\lambda} \) is a non negative constant.

**Proof:** Note that since \( A = DC \), the diagonal elements of \( A \) are \( \lambda'_i \)s. Recall the following well known theorem from majorization theory [35, P. 300, B.1.].

**Theorem 2.** If \( H \) is an \( n \times n \) Hermitian matrix with diagonal elements \( b_1, \ldots b_n \) and eigenvalues \( a_1, \ldots a_n \) then

\[
\lambda_1 > b
\]  

\[(49)\]

**Proof:** The details of the proof can be found in [35].

In our case, since \( \hat{\lambda}_i \)s are the eigenvalues and \( \lambda'_i \) are the diagonal elements of a symmetric matrix \( A \) so, \( \hat{\lambda} > \lambda' \) where \( \hat{\lambda} = [\hat{\lambda}_1, \ldots, \hat{\lambda}_n] \) and \( \lambda' = [\lambda'_1, \ldots, \lambda'_n] \). We will now use following Proposition from [35, P. 97, C.2.].

**Proposition 1.** If function \( \phi \) is symmetric and convex, then \( \phi \) is Schur-convex function. Consequently, \( x > y \) implies \( \phi(x) \geq \phi(y) \).

**Proof:** For the details of this proof please refer to [35].

Now if it can be shown that \( \prod_{i=1}^{N} \left( \frac{1}{1+k\lambda_i} \right)^{\alpha} \) is a Schur-convex function then by a simple application of Proposition 1 it is evident that \( \prod_{i=1}^{N} \left( \frac{1}{1+k\lambda_i} \right)^{\alpha} \geq \prod_{i=1}^{N} \left( \frac{1}{1+k\lambda'_i} \right)^{\alpha} \). Hence to prove that \( \prod_{i=1}^{N} \left( \frac{1}{1+kx_i} \right)^{\alpha} \) is a Schur convex function we will show that it is a symmetric and convex function.

It is apparent that function \( \prod_{i=1}^{N} \left( \frac{1}{1+kx_i} \right)^{\alpha} \) is a symmetric function due to the fact that any two of its arguments can be interchanged without changing the value of the function. So we need to show that the function \( f(x_1, \ldots, x_n) = \prod_{i=1}^{N} \left( \frac{1}{1+kx_i} \right)^{a_i} \) is a convex function where \( x_i \geq 0, a_i > 0 \). The function \( f(x_1, \ldots, x_n) \) is convex if and only if its Hessian \( \nabla^2 f \) is positive semi-definite [36].

\[\text{The notation } a > b \text{ indicate that vector } b \text{ is majorized by vector } a. \text{ Let } a = [a_1, \ldots, a_n] \text{ and } b = [b_1, \ldots, b_n] \text{ with } a_1 \leq \ldots \leq a_n \text{ and } b_1 \leq \ldots \leq b_n \text{ then } a \succ b \text{ if and only if }\]

\[
\sum_{i=1}^{k} b_i \geq \sum_{i=1}^{k} a_i, \quad k = 1, \ldots, n-1, \text{ and } \sum_{i=1}^{n} b_i = \sum_{i=1}^{n} a_i.
\]  

\[(48)\]
Deriving the Hessian matrix of function $\prod_{i=1}^{N} \left(\frac{1}{1+kx_i}\right)^{a_i}$, one obtains

$$\nabla^2 f = k^2 \prod_{i=1}^{N} \left(\frac{1}{1+kx_i}\right)^{a_i} \begin{bmatrix}
\frac{a_1(a_1+1)}{(1+kx_1)^2} & \frac{a_1a_2}{(1+kx_1)(1+kx_2)} & \cdots & \frac{a_1a_n}{(1+kx_1)(1+kx_n)} \\
\frac{a_1a_2}{(1+kx_1)(1+kx_2)} & \frac{a_2(a_2+1)}{(1+kx_2)^2} & \cdots & \frac{a_2a_n}{(1+kx_2)(1+kx_n)} \\
\cdots & \cdots & \ddots & \cdots \\
\frac{a_1a_n}{(1+kx_1)(1+kx_n)} & \frac{a_2a_n}{(1+kx_2)(1+kx_n)} & \cdots & \frac{a_n(a_n+1)}{(1+kx_n)^2}
\end{bmatrix} (50)$$

We need to show that $\nabla^2 f$ is a positive semi-definite matrix. In this work, we use the following condition for a matrix being positive definite (p.d.) matrix $[37, \text{P. 566}]$: For a real symmetric matrix $M$, if $x^TMx > 0$ for every $N \times 1$ nonzero real vector $x$, then the matrix $M$ is p.d. matrix. We rewrite the Hessian matrix as sum of two matrices and it is then given by.

$$\nabla^2 f = k^2 \prod_{i=1}^{N} \left(\frac{1}{1+kx_i}\right)^{a_i} [P + Q]. \quad (51)$$

Here,

$$P = \begin{bmatrix}
\frac{a_1^2}{(1+kx_1)^2} & \frac{a_1a_2}{(1+kx_1)(1+kx_2)} & \cdots & \frac{a_1a_n}{(1+kx_1)(1+kx_n)} \\
\frac{a_1a_2}{(1+kx_1)(1+kx_2)} & \frac{a_2^2}{(1+kx_2)^2} & \cdots & \frac{a_2a_n}{(1+kx_2)(1+kx_n)} \\
\cdots & \cdots & \ddots & \cdots \\
\frac{a_1a_n}{(1+kx_1)(1+kx_n)} & \frac{a_2a_n}{(1+kx_2)(1+kx_n)} & \cdots & \frac{a_n^2}{(1+kx_n)^2}
\end{bmatrix} \quad (52)$$

and

$$Q = \begin{bmatrix}
\frac{a_1}{(1+kx_1)^2} & 0 & \cdots & 0 \\
0 & \frac{a_2}{(1+kx_2)^2} & \cdots & 0 \\
\cdots & \cdots & \ddots & \cdots \\
0 & 0 & \cdots & \frac{a_n}{(1+kx_n)^2}
\end{bmatrix}. \quad (53)$$

Here by definition $k^2 \prod_{i=1}^{N} \left(\frac{1}{1+kx_i}\right)^{a_i} > 0$. If now both $P$ and $Q$ are p.d. then the Hessian is p.d. Since $P$ can be written as $U^TU$ where $U = \begin{bmatrix}
\frac{a_1}{(1+kx_1)}, \frac{a_2}{(1+kx_2)}, \cdots, \frac{a_n}{(1+kx_n)}
\end{bmatrix}$ is a $N \times 1$ matrix and hence $x^TMx = x^T(U^TU)x = ||Ux||^2 > 0$ for every $N \times 1$ nonzero real vector $x$. Thus, $P$ is a p.d. matrix. Since $Q$ is a diagonal matrix with positive entries, $Q$ is p.d. matrix. Therefore, $P + Q$ is a p.d. matrix due to the fact that sum of the p.d. matrix is p.d. matrix. Hence Hessian matrix $\nabla^2 f$ is a p.d. matrix. Thus, $f(x_1, \cdots, x_n) = \prod_{i=1}^{N} \left(\frac{1}{1+kx_i}\right)^{a_i}$ is a convex function.
Since \( \prod_{i=1}^{N} \left( \frac{1}{1+kx_i} \right)^{\alpha_c} \) is a convex and symmetric function, so, it is a Schur-convex function. Since \( \hat{\lambda} \succ \lambda' \) and \( \prod_{i=1}^{N} \left( \frac{1}{1+kx_i} \right)^{\alpha_c} \) is a Schur-convex function, thus \( \prod_{i=1}^{N} \left( \frac{1}{1+k\lambda_i} \right)^{\alpha_c} \geq \prod_{i=1}^{N} \left( \frac{1}{1+k\lambda'_i} \right)^{\alpha_c} \).

Thus, the coverage probability in the presence of correlation among the interferers is greater than or equal to the coverage probability in the i.n.i.d. case, when user channel undergoes Rayleigh fading and the interferer shape parameter \( \alpha_i = \alpha_c \forall i \). Now, we compare the coverage probability for general case, i.e., when user’s \( \alpha \) is arbitrary.

**Theorem 3.** The coverage probability in the presence of the correlated interferers is greater than or equal to the coverage probability in presence of i.n.i.d. interferers, when user channel’s shape parameter is less than or equal to 1, i.e., \( \alpha \leq 1 \).

**Proof:** The coverage probability expressions for the scenario when interferers are i.n.i.d. and the scenario when interferers are correlated are given in (17) and (42), respectively and rewriting them for the case when \( \alpha = \alpha_c \forall i \), one obtains

\[
P_c(T, r) = K' F^{(N)}_D \left[ 1 - \alpha, \alpha_c, \ldots, \alpha_c; N\alpha_c + 1; \frac{\lambda}{\lambda + r^\beta \lambda_1 T}, \ldots, \frac{\lambda}{\lambda + r^\beta \lambda_N T} \right]
\]

\[
P_c'(T, r) = \hat{K} F^{(N)}_D \left[ 1 - \alpha, \alpha_c, \ldots, \alpha_c; N\alpha_c + 1; \frac{\hat{\lambda}}{\hat{\lambda} + r^\beta \hat{\lambda}_1 T}, \ldots, \frac{\hat{\lambda}}{\hat{\lambda} + r^\beta \hat{\lambda}_N T} \right],
\]

where \( K' = \frac{\Gamma(N\alpha_c + \alpha)}{\Gamma(N\alpha_c + 1)} \prod_{i=1}^{N} \left( \frac{1}{1+\lambda_i r^\beta} \right)^{\alpha_c} \) and \( \hat{K} = \frac{\Gamma(N\alpha_c + \alpha)}{\Gamma(N\alpha_c + 1)} \prod_{i=1}^{N} \left( \frac{1}{1+\hat{\lambda}_i r^\beta} \right)^{\alpha_c} \). From Theorem 1 it is clear that \( \hat{K} > K' \). Now, we need to compare the Lauricella’s function of the fourth kind of (54) and (55). Here, for comparison we use the series expression for \( F_D(\cdot) \).

We expand the series expression for the Lauricella’s function of the fourth kind in the following form:

\[
F^{(N)}_D[a, b, \ldots, b; c; x_1, \ldots, x_N] = 1 + \sum_{i=1}^{N} x_i + \sum_{i=1}^{N} x_i^2 + \sum_{1 \leq i < j \leq N} x_i x_j + \sum_{i=1}^{N} x_i^3 + \cdots
\]

\[
K_{1,1} = \frac{(a)_1 (b)_1}{(c)_1}, \quad K_{2,1} = \frac{(a)_2 (b)_2}{(c)_2}, \quad K_{2,2} = \frac{(a)_2 (b)_1 (b)_1}{(c)_2}, \quad K_{3,1} = \frac{(a)_3 (b)_1}{(c)_3}, \quad K_{3,2} = \frac{(a)_3 (b)_2}{(c)_3}, \quad K_{3,3} = \frac{(a)_3 (b)_1 (b)_1}{(c)_3}
\]

and so on.
Hence the coverage probability for independent case given in (54) can be written as

\[ P_c(T, r) = K' \left[ 1 + K_{1,1} \sum_{i=1}^{N} \left( \frac{1}{1 + \lambda_i r^\beta T} \right) + K_{2,1} \sum_{i=1}^{N} \left( \frac{1}{1 + \lambda_i r^\beta T} \right)^2 + K_{3,1} \sum_{i=1}^{N} \left( \frac{1}{1 + \lambda_i r^\beta T} \right)^3 \right] + \]

\[ K_{2,2} \sum_{1 \leq i < j \leq N} \left( \frac{1}{1 + \lambda_i r^\beta T} \right) \left( \frac{1}{1 + \lambda_j r^\beta T} \right) + K_{3,2} \sum_{i,j=1}^{N} \left( \frac{1}{1 + \lambda_i r^\beta T} \right)^2 \left( \frac{1}{1 + \lambda_j r^\beta T} \right) + \]

\[ K_{3,3} \sum_{1 \leq i < j < k \leq N} \left( \frac{1}{1 + \lambda_i r^\beta T} \right) \left( \frac{1}{1 + \lambda_j r^\beta T} \right) \left( \frac{1}{1 + \lambda_k r^\beta T} \right) + \ldots \] (57)

Similarly, for the correlated case the coverage probability given in (55) can be written as

\[ P_c^c(T, r) = K' \left[ 1 + K_{1,1} \sum_{i=1}^{N} \left( \frac{1}{1 + \lambda_i r^\beta T} \right) + K_{2,1} \sum_{i=1}^{N} \left( \frac{1}{1 + \lambda_i r^\beta T} \right)^2 + K_{3,1} \sum_{i=1}^{N} \left( \frac{1}{1 + \lambda_i r^\beta T} \right)^3 \right] + \]

\[ K_{2,2} \sum_{1 \leq i < j \leq N} \left( \frac{1}{1 + \lambda_i r^\beta T} \right) \left( \frac{1}{1 + \lambda_j r^\beta T} \right) + K_{3,2} \sum_{i,j=1}^{N} \left( \frac{1}{1 + \lambda_i r^\beta T} \right)^2 \left( \frac{1}{1 + \lambda_j r^\beta T} \right) + \]

\[ K_{3,3} \sum_{1 \leq i < j < k \leq N} \left( \frac{1}{1 + \lambda_i r^\beta T} \right) \left( \frac{1}{1 + \lambda_j r^\beta T} \right) \left( \frac{1}{1 + \lambda_k r^\beta T} \right) + \ldots \] (58)

Here \( K_{1,1} = \frac{(1-\alpha_1)(\alpha_3)}{(N_{\alpha_3}+1)!!} \), \( K_{2,1} = \frac{(1-\alpha_2)(\alpha_3)}{(N_{\alpha_3}+1)2!!} \), \( K_{2,2} = \frac{(1-\alpha_2)(\alpha_1)}{(N_{\alpha_1}+1)2!!} \), \( K_{3,1} = \frac{(1-\alpha_3)(\alpha_1)}{(N_{\alpha_1}+1)3!!} \), \( K_{3,2} = \frac{(1-\alpha_3)(\alpha_2)}{(N_{\alpha_2}+1)3!!} \), and so on. Note that here \( K_{i,j} \) are the same for both \( P_c(T, r) \) and \( P_c^c(T, r) \). Now, we want to show that each summation term in the series expression is a Schur-convex function.

Each summation term in the series expression is symmetrical due to the fact that any two of its argument can be interchanged without changing the value of the function. We have already shown that \( \sum_{i=1}^{N} \left( \frac{1}{1 + \lambda_i r^\beta T} \right)^a \) is a convex function \( \forall x_i \geq 0 \) and \( \forall a_i > 0 \). Now, the terms in the summation terms in (57) and (58) are of the form \( \prod_{i=1}^{M} \left( \frac{1}{1 + k x_i} \right)^{a_i} \) where \( M \leq N \). To show that these functions are convex function we need to show that the corresponding Hessian are p.d.

The corresponding Hessians are nothing but principal sub-matrices of the matrix in (50). Hence using the fact that every principal sub-matrix of a s.p.d. matrix is a s.p.d. matrix [37], one can show that each term of each summation term is a convex function. Using the fact that convexity is preserved under summation one can show that each summation term is a convex function. Thus, each summation term in series expression is a Schur-convex function.

Now we consider following two cases.

Case I when \( \alpha < 1 \): Since \( \alpha < 1 \), so \( 1 - \alpha > 0 \) and hence all the constant \( K_{i,j} > 0 \ \forall \ i, j \). Each
summation term in series expression of coverage probability for correlated case is greater than or equal to the corresponding summation term in the series expression of coverage probability for independent case. Thus, if user channel’s shape parameter $\alpha < 1$ then coverage probability of correlated case is greater than or equal to the coverage probability for independent case.

Case II when $\alpha > 1$: Since $\alpha > 1$, then $1 - \alpha < 0$ and hence $K_{i,j} < 0 \forall i \in 2|\mathbb{Z}| + 1$ and $\forall j$ where set $\mathbb{Z}$ denote the integer number, due to the fact that $(a)_N < 0$ if $a < 0$ and $N \in 2|\mathbb{Z}| + 1$. Whereas, $K_{i,j} > 0 \forall i \in 2|\mathbb{Z}|$ and $\forall j$ due of the fact that $(a)_N > 0$ if $a < 0$ and $N \in 2|\mathbb{Z}|$. Thus, if user channel’s shape parameter $> 1$, we can not state whether the coverage probability of one case is greater than or lower than the other case.

Summarizing, the coverage probability in the presence of correlated interferers is greater than or equal to the coverage probability in presence of independent interferers, when user channel’s shape parameter is less than or equal to 1, i.e., $\alpha \leq 1$. When $\alpha > 1$, one can not say whether coverage probability is better in correlated interferer case or independent interferer case.

VI. NUMERICAL ANALYSIS AND APPLICATION

In this section, we give some simulation results for the coverage probability for both cases. We will see that simulation results match the derived analytical results. The behavior of shape parameter and the impact of correlation among interferers on the coverage probability is also discussed.

For the simulations, we consider a 19 cell system with hexagonal structure having inter cell site distance $2R = 1732$ meters as shown in Fig. 1. We calculate the coverage probability of a user which is connected to the 0th cell and at a distance $r$ from the 0th BS. The distance between the user and the 18 interfering BS are given in [38].

Fig. 2 shows the impact of shape parameter on the coverage probability in the i.n.i.d. case. Here, three combinations of shape parameter have been considered. We first note that the simulation results exactly match with the analytical results. Secondly, it can be observed that as user channel’s shape parameter increases while keeping the interferer shape parameters fixed, the coverage probability increases. Whereas, when interferer channel’s shape parameter increases and the user channel’s shape parameter is fixed, the coverage probability decreases as expected.

Fig. 3 and Fig. 4 depict the impact of correlation among the interferers on the coverage probability for different values of shape parameter. The correlation among the interferers is
defined by the correlation matrix in (40) with $\rho_{pq} = \rho^{|p-q|}$ where $p, q = 1, \ldots, N$ [39]. From Fig. 3 we first note that for user channel’s shape parameter 0.5 and 1, coverage probability in presence of correlation is higher than that of independent scenario (which match our analytical result). Secondly, as shape parameter decreases, the relative increase in coverage probability due to correlation increases. Whereas, observing Fig. 4 one cannot say that coverage probability in presence of correlation is higher or lower than that of independent scenario when user channel’s shape parameter is greater than one. However, it can be seen that when the user channel’s shape parameter is significantly higher than the interferers shape parameter, coverage probability in the presence of independent interferers dominates over the coverage probability in the presence of correlated interferers. While if user channel’s shape parameter is comparable to the interferers channel shape parameter, the coverage probability of independent interferers is higher than the coverage probability of correlated interferers when user is close to the BS. However, the coverage probability of independent interferers is significantly lower than the coverage probability of correlated interferers when the user is far from the BS.

We now compare the coverage probability of correlated interferers with the coverage proba-
Fig. 3: Impact of correlation among the interferers on the coverage probability for different values of shape parameter, when $\alpha \leq 1$.

Fig. 5 plots the coverage probability of correlated interferers case and coverage probability of SIMO network. It can be seen that for $\rho = 0.98$ coverage probability of correlated case is higher than the SIMO network. However, for $\rho = 0.81$ it is close to the SIMO network at the edge.

VII. CONCLUSION

In this work, expressions for the coverage probability in terms of Lauricella’s function of the fourth kind have been derived for following two cases: (a) Interferers and user channel having arbitrary Nakagami-$m$ fading parameters. (b) Interferers being correlated where the correlation is specified by a correlation matrix. These expressions have been further simplified for a wide
variety of special cases. The coverage probability of the correlated case and the independent case have been compared analytically. We have shown that the coverage probability in correlated interferer case is higher than that of the independent case, when the user channel’s shape parameter is lesser than or equal to one, and the interferers have Nakagami-m fading with arbitrary parameters. The results of this work will be useful to analyzing the performance of cellular networks in presence of Nakagami-m fading environment and in studying the impact of correlated interferers when compared with independent interferers.

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Fig. 5: Comparison of coverage probability of correlated interferers with the coverage probability of SIMO network. Here $\alpha = \alpha_i = 1$.

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