Training Over-parameterized Models with Non-decomposable Objectives

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Abstract

Many modern machine learning applications come with complex and nuanced design goals such as minimizing the worst-case error, satisfying a given precision or recall target, or enforcing group-fairness constraints. Popular techniques for optimizing such non-decomposable objectives reduce the problem into a sequence of cost-sensitive learning tasks, each of which is then solved by re-weighting the training loss with example-specific costs. We point out that the standard approach of re-weighting the loss to incorporate label costs can produce unsatisfactory results when used to train over-parameterized models. As a remedy, we propose new cost-sensitive losses that extend the classical idea of logit adjustment to handle more general cost matrices. Our losses are calibrated, and can be further improved with distilled labels from a teacher model. Through experiments on benchmark image datasets, we showcase the effectiveness of our approach in training ResNet models with common robust and constrained optimization objectives.

1 Introduction

The misclassification error is the canonical performance measure in most treatments of classification problems [88]. While appealing in its simplicity, practical machine learning applications often come with more complex and nuanced design goals. For example, these may include minimizing the worst-case error across all classes [60], satisfying a given precision or recall target [102], enforcing minimal coverage for minority classes [29], or imposing group-fairness constraints [101]. Unlike the misclassification error, such objectives are non-decomposable, i.e., they cannot be expressed as a sum of losses over individual samples. This poses a non-trivial optimization challenge, which is typically addressed by reducing the problem into a sequence of cost-sensitive learning tasks [11, 24, 4, 15, 67]. Such reduction-based approaches have been successfully employed in many open-source libraries [3, 2, 1] to provide drop-in replacements for standard loss functions.

In this paper, we point out that the standard approach of re-weighting the training loss to incorporate label costs can produce unsatisfactory results when used with high capacity models, which are often trained to memorize the training data. Such over-parameterized models are frequently encountered in the use of modern neural networks, and have been the subject of considerable recent study [103, 69, 62]. As a remedy, we provide new calibrated losses for cost-sensitive learning that are better equipped at training over-parameterized models to optimize non-decomposable metrics, and demonstrate their effectiveness on benchmark image classification tasks. Our main contributions are as follows:

(i) we illustrate the pitfalls of using loss re-weighting in over-parameterized settings, particularly with diagonal class weights (Section 3).

(ii) we propose new logit-adjusted losses for cost-sensitive learning for both diagonal and non-diagonal gain matrices, and show that they are calibrated (Section 4).

(iii) we demonstrate that our losses provide significant gains over loss re-weighting in training ResNet models to optimize worst-case recall and to enforce coverage constraints (Section 5).

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(iv) we show that the proposed approach compares favorably to post-hoc correction strategies, and can be further improved by distilling with a novel loss that we provide (Section 6).

1.1 Related Work
We cover prior work on class-imbalanced learning, cost-sensitive learning, and complex metrics.

Learning under class imbalance. A common setting where metrics beyond the misclassification error have been studied is the problem of class imbalance [46, 10, 31]. Here, the label distribution $P(y)$ is skewed, and one seeks to ensure that performance on rare classes do not overly suffer. To this end, common metrics include the average of per-class recalls (also known as the balanced accuracy) [6, 58], quadratic mean of per-class accuracies [48], and the F-score [51].

A panoply of different techniques have been explored for this problem, spurred in part by recent interest in the specific context of neural networks (where the problem is termed long-tail learning) [87, 8], with the focus largely on optimizing balanced accuracy. An exhaustive survey is beyond the scope of this paper (see, e.g., [31, 37]), but we may roughly identify three main strands of work: data modification [45, 10, 89, 99, 104], loss modification [103, 17, 9, 84, 35, 75, 93, 59, 20, 42, 90], and prediction modification [26, 40, 72, 56, 12, 38, 106]. A related recent thread seeks to improve performance on rare subgroups [78, 79, 82], as a means to ensure model fairness [23].

Amongst loss modification techniques, two strategies are of particular relevance. The first is re-weighting techniques [96, 61, 17]. As has been noted [52, 92, 57] (and as we shall subsequently verify), these may have limited effect on training samples that are perfectly separable under a given model class (as is the case with overparameterized models). The second is margin techniques, which enforce a class-dependent margin in the loss [9, 84, 75, 59, 90]. This seeks to ensure that rare classes are separated with greater confidence, to account for the higher uncertainty in their decision boundary.

Cost-sensitive learning. Cost-sensitive learning is a classical extension of standard multiclass classification, wherein errors on different classes incur different costs [25, 22, 81]. Strategies for this problem largely mirror those for class imbalance, which can be seen as imposing a particular set of costs dependent on the label frequencies [55]. In particular, loss modification techniques based on asymmetric weights [54, 94, 100, 5, 18, 21, 81, 108] and margins [57, 41, 34, 42] have been explored, with the latter proving useful on separable problems. Amongst these, Lin et al. [54] handle a general cost matrix for multiclass problems, but require the class scores to sum to 0, a constraint that might be difficult to impose with neural networks. Khan et al. [41] propose a multiclass loss similar to the logit adjustment idea used in this paper, but only handle a specific type of cost matrix (e.g., in their setup, the default cost matrix is a matrix of all 1s). The standard multiclass loss of Crammer and Singer [16] can also be extended to handle a general cost matrix [86], but unfortunately is not calibrated [74].

Complex metrics. There has been much work on extending the class imbalance literature to handle more complex metrics and to impose data-dependent constraints. These methods can again be divided into loss modifications [36, 70, 39, 65, 29, 24, 4, 15, 67, 47] and post-hoc prediction modifications [98, 64, 43, 66, 68, 30, 19, 63, 97, 85], and differ in how they decompose the problem into simpler cost-sensitive formulations (see [67] for a unified treatment of common reduction strategies). Most of these papers experiment with simple models, with the exception of Sanyal et al. [80], who use re-weighting strategies to train shallow DNNs, Kumar et al. [47], who train CNNs on binary-labeled problems, and Eban et al. [24], who train InceptionNets to optimize an AUC-based ranking metric. There have also been other attempts at training neural networks with ranking metrics [83, 33, 7, 73], in contrast, our focus is on handling a general family of metrics popular in multiclass problems.

2 Non-decomposable Objectives

Notations. Let $[m] = \{1, \ldots, m\}$. Let $\Delta_m$ be the $(m - 1)$-dimensional simplex with $m$ coordinates. Let $1_m \in \mathbb{R}^m$ denote an all 1s vector of size $m$, and $\text{diag}(u_1, \ldots, u_m)$ denote a $m \times m$ diagonal matrix with diagonal elements $u_1, \ldots, u_m$. For vectors $a, b \in \mathbb{R}^m$, $a/b$ denotes element-wise division. Let $\text{softmax}_i(s) = \frac{\exp(s_i)}{\sum_{j=1}^m \exp(s_j)}$ denote a softmax transformation of scores $s \in \mathbb{R}^m$.

Consider a multiclass problem with an instance space $X \subseteq \mathbb{R}^d$ and a label space $Y = [m]$. Let $D$ denote the underlying data distribution over $X \times [m]$, $D_X$ denote the marginal distribution over the instances $X$, and $\pi_i = P(y = i)$ denote the class priors. We will use $p_i(x) = P(y = i | x)$ to denote the conditional-class probability for instance $x$. Our goal is to learn a classifier $h : X \rightarrow \{m\}$, and will
measure its performance in terms of its confusion matrix $C[h]$, where
\[ C_{ij}[h] = E_{(x,y) \sim D}[1(y = i, h(x) = j)]. \]

**Complex learning problems.** In the standard setup, one is often interested in maximizing the overall classification accuracy $\text{acc}[h] = \sum_i C_{ii}[h]$. However, in many practical settings, one may care about other metrics such as the recall for class $i$, $\text{rec}_i[h] = \frac{C_{ii}[h]}{\pi_i}$, the precision on class $i$, $\text{prec}_i[h] = \frac{C_{ii}[h]}{\sum_j C_{ij}}$, or the proportion of predictions made on class $i$, i.e. the coverage, $\text{cov}_i[h] = \sum_j C_{ji}$. Below are common examples of real-world design goals based on these metrics.

**Example 1** (Maximizing worst-case recall). In class-imbalanced settings, where the classifier tends to generalize poorly on the rare classes, we may wish to change the training objective to directly maximize the minimum recall across all classes [11]:
\[ \max_h \min_{i \in [m]} \frac{C_{ii}[h]}{\pi_i}. \]  

**Example 2** (Constraining per-class coverage). Another consequence of label imbalance could be that the proportion of predictions that the classifier makes for the tail classes is lower than the actual prevalence of that class. The following learning problem seeks to maximize the average recall while explicitly constraining the classifier’s coverage for class $j$ to be at least 95% of its prior $\pi_j$ [15, 67]:
\[ \max_h \frac{1}{m} \sum_{i=1}^m \frac{C_{ii}[h]}{\pi_i} \quad \text{s.t.} \quad \sum_{i=1}^m C_{ij}[h] \geq 0.95 \times \pi_j, \forall j \in [m]. \]  

Other examples include constraints on recall and precision (see Table 1), as well as fairness constraints like equal opportunity, which are computed on group-specific confusion matrices [14, 30].

**Reduction to cost-sensitive learning.** All the problems described above are non-decomposable, in the sense that, they cannot be written as minimization of a sum of errors on individual examples, and hence cannot be tackled using common off-the-shelf learning algorithms. The dominant approach for solving such problems is to formulate a sequence of cost-sensitive objectives, which do take the form of an average of training errors [66, 11, 4, 15]. For example, the robust learning problem in (1) can be equivalently re-written as the following saddle-point optimization problem:
\[ \max_h \min_{\lambda \in \Delta_m^m} \sum_{i=1}^m \lambda_i \frac{C_{ii}[h]}{\pi_i}, \]  

where $\lambda_i$ is a multiplier for class $i$. One can then find a saddle-point for (3) by jointly minimizing the weighted objective over $\lambda \in \Delta_m$ (using e.g. exponentiated-gradient updates) and maximizing the objective over $h$. Notice that for a fixed $\lambda$, the optimization over $h$ is a cost-sensitive (or a gain-weighted) learning problem:
\[ \max_h \sum_{i,j} G_{ij} C_{ij}[h], \]  

where $G_{ii} = \frac{\lambda_i}{\pi_i}$ and $G_{ij} = 0, \forall i \neq j$ are the rewards associated with predicting class $j$ when the true class is $i$. We will refer to $G \in \mathbb{R}^{m \times m}$ as the “gain” matrix, which in this case is diagonal. In practice, the cost-sensitive learning problem in (4) is usually replaced with multiple steps of gradient descent, interleaved with the updates on $\lambda$.

Similar to (1), the constrained learning problem in (2) can be written as an equivalent (Lagrangian) min-max problem with Lagrange multipliers $\lambda \in \mathbb{R}^m_+$:
\[ \max_h \min_{\lambda \in \mathbb{R}^m_+} \frac{1}{m} \sum_{i=1}^m \frac{1}{\pi_i} C_{ii}[h] + \sum_{j=1}^m \lambda_j \left( \sum_{i=1}^m C_{ij}[h] - 0.95 \pi_j \right). \]

One can find a saddle-point for this problem by jointly minimizing the Lagrangian over $\lambda \in \mathbb{R}^m_+$ (using e.g. gradient updates) and maximizing it over $h$, with the maximization over $h$ taking the form of a cost-sensitive learning problem. In this case, the gain matrix $G$ is non-diagonal and is given by $G_{ii} = \frac{1}{m \pi_i} + \lambda_i, \forall i$ and $G_{ij} = \lambda_j, \forall i \neq j$. See Table 1 for the form of the gain matrix for other common constraints. In Appendix A, we provided detailed descriptions of the algorithms discussed.
While this weighted loss is calibrated for
We now point out the problems in applying the algorithms described in the previous section to
Table 1: Examples of complex learning problems, the associated gain matrices
to denote the minimum allowed recall/precision,

| Problem | Gain Matrix | Losses |
|---------|-------------|--------|
| 1 \(\max h, \min y \, \text{rec}_y[h]\) | \(\text{diag}(\lambda/\pi)\) | \(\ell^G\) |
| 2 \(\max h, \text{acc}[h]\) s.t. \(\text{rec}_y[h] \geq \tau, \forall y\) | \(\text{diag}(1_m + \lambda/\pi)\) | \(\ell^G\) |
| 3 \(\max h, \text{acc}_y[h]\) s.t. \(\text{rec}_y[h] \geq \tau, \forall y\) | \(\text{diag}(1_m + \lambda) - \tau 1_m\lambda^T\) | \(\ell^\text{bwb}, \ell^\text{bms}\) |
| 4 \(\max h, \sum_y \text{rec}_y[h]\) s.t. \(\text{cov}_y[h] \geq 0.95\pi_y, \forall y\) | \(\text{diag}(1_m/\pi) + 1_m\lambda^T\) | \(\ell^\text{bwb}, \ell^\text{bms}\) |
| 5 \(\max h, \sum_y \text{rec}_y[h]\) s.t. \(\text{beov}_y[h] \geq 0.95\pi_y, \forall y\) | \(\text{diag}(1_m/\pi) + (1_m/\pi)\lambda^T\) | \(\ell^\text{bwb}\) |

Table 1: Examples of complex learning problems, the associated gain matrices \(G\), and the proposed losses that are applicable to the setting. We use \(\tau\) to denote the minimum allowed recall/precision, and \(\text{beov}_y[h] = \sum_{i=1}^m \frac{1}{\pi_i} C_{iy}[h]\) is the balanced coverage metric we use in our experiments.

Cost-sensitive losses. A standard approach for solving the cost-sensitive learning problem in (4) is to use a surrogate loss function \(\ell : [m] \times \mathbb{R}^m \rightarrow \mathbb{R}_+\) that takes a label \(y\) and a \(m\)-dimensional score \(u \in \mathbb{R}^m\), and outputs a real value \(\ell(y, u)\). One would then minimize the expected loss

\[
\mathcal{L}(s) = \mathbb{E}_{y, u} [\ell(y, s(x))] \quad \text{over a class of scoring function} \ s : \mathcal{X} \rightarrow \mathbb{R}^m \text{ that map each instance to an} \ m\text{-dimensional score. The final classifier} h^* \text{ can then be obtained from the learned scoring function} \ s^* \text{ by taking an argmax of its predicted scores, i.e. by constructing} \ h^*(x) \in \argmax_{i \in [m]} s^*_i(x).

In practice, we are provided a finite sample \(S = \{(x_1, y_1), \ldots, (x_n, y_n)\}\) drawn from \(D\), and will seek to minimize the average loss on \(S\), given by

\[
\hat{\mathcal{L}}(s) = \frac{1}{|S|} \sum_{(x, y) \in S} \ell(y, s(x)).
\]

We will also assume access to a held-out validation sample \(S_{\text{val}} = \{(x_1, y_1), \ldots, (x_{n_{\text{val}}}, y_{n_{\text{val}}})\}\).

One common sanity check for a “good” loss function is to confirm that it is classification calibrated for the learning problem of interest, i.e. to check if in the large sample limit, optimizing the loss over all scoring functions \(s : \mathcal{X} \rightarrow \mathbb{R}^m\) would result in the Bayes-optimal classifier for the problem. For the cost-sensitive objective in (4), the Bayes-optimal classifier is given below [50].

**Proposition 1.** The optimal classifier for (4) for a general gain matrix \(G \in \mathbb{R}^{m \times m}\) is of the form:

\[
h^*(x) \in \argmax_{y \in [m]} \sum_{i=1}^m G_{iy} p_i(x) = \argmax_{y \in [m]} (G^T \mathbf{p}(x))_y.
\]

3 The Perils of Over-parameterization

We now point out the problems in applying the algorithms described in the previous section to
over-parameterized models, focusing particularly on the loss used for cost-sensitive learning.

Limitations of loss re-weighting. One of the most common loss function for a diagonal gain matrix \(G\) is a simple re-scaling of the standard cross-entropy loss with the diagonal weights:

\[
\ell^{\text{wt}}(y, s) = -G_{y, y} \log \left( \frac{\exp(s_y)}{\sum_j \exp(s_j)} \right).
\]

The following is a natural extension of this re-weighted loss to a general gain matrix \(G\), used, for example, in constrained optimization libraries [3], and also for mitigating label noise [71]:

\[
\ell^{\text{wt}}(y, s) = - \sum_{i=1}^m G_{y, i} \log \left( \frac{\exp(s_i)}{\sum_j \exp(s_j)} \right).
\]

While this weighted loss is calibrated for \(G\) (see, e.g., [71]), it is often inadequate when training an over-parameterized model on a finite sample. This is evident when \(G\) is a diagonal matrix: such a model will usually memorize the training labels and achieve zero training loss for every class, irrespective of what we choose for the outer weighting. In such separable settings, it is unclear how the outer weighting will impact the model’s out-of-sample performance on different classes.

For clarity, we provide a simple illustration in Figure 1, where we use a re-weighted loss to train ResNet-56 models on 10000 images from CIFAR-10 with different diagonal gain matrices \(G\). We assign a weight of \(\omega\) to the first class “airplane”, and a weight of 1 on all other classes (i.e. set \(G_{1,1} = \omega\) and \(G_{y,y} = 1, \forall y \neq 1\)), and plot the normalized weighted accuracy \(\sum_y G_{y,y} C_{yy}[h]/\sum_y G_{y,y}\) on the test set as we vary \(\omega\). Compared to an unweighted cross-entropy loss, the re-weighted loss does
not produce a significant change to the test metric, whereas the logit-adjusted loss (which we will discuss in Section 4) yields substantially better values. Probing closer into the test accuracy for each class, we see that the re-weighted loss yields slightly better accuracies for class 1, but is significantly worse-off on the other classes, suggesting that re-weighting has the effect of excessively focusing on the class with the higher weight at the cost of the other classes. Sagawa et al. [79] also make a similar observation when using importance weights to improve worst-group generalization, while Cui et al. [17], Cao et al. [9] observe that up-weighting the minority classes can lead to unstable optimization.

When $G$ is not diagonal, the minimum training loss may not be zero. Nonetheless, the following proposition sheds some light on the scores learned by a model that achieves minimum training loss.

**Proposition 2.** Let $\hat{L}^w_t(s) = \frac{1}{|S|} \sum_{(x,y) \in S} \ell^w(y, s(x))$ denote the average loss on training sample $S$ with $\ell^w$ defined as in (6) for a gain matrix $G$. Then a scoring function $\hat{s}$ that achieves the minimum value of $\hat{L}^w_t(s)$ over all $s: X \rightarrow \mathbb{R}^m$ is of the form: $\text{softmax}_i(\hat{s}(x)) = \frac{G_{y,i}}{\sum_j G_{y,j}}, \forall (x, y) \in S$.

Notice that the model output is invariant to re-scaling of the rows in $G$, i.e., one can multiply each row of the gain matrix $G$ by a different scalar, and the values memorized by the model for each training example will remain unchanged. On the other hand, re-scaling a column of $G$ does substantially change the score learned for each example. While this does not tell us about how the model would behave on unseen new examples, our experience has been that loss re-weighting is usually effective when used to control the amount of out-of-sample predictions that a model makes for a certain class (by suitably scaling the column for that class in $G$), but does not work well when used to emphasize greater accuracy on a particular class. Consequently, we find that loss re-weighting usually fares better with metrics like coverage that depend only on the model predictions and not on the true labels, than with metrics like recall which depend on both the predictions and labels.

**Existing remedies for better generalization.** The literature offers some remedy to improve model generalization with non-standard objectives. Sagawa et al. [79] improve performance on rare subgroups by regularizing the losses based on the size of each group, but their solution applies to a specific learning problem. Other remedies such as smarter re-weighting strategies [53, 17] or deferred re-weighting schedules [9] have proven to work well in specialized imbalanced settings.

The work that most closely relates to our setting is Cotter et al. [13], who propose a simple modification to the algorithms discussed in Section 2, with the use of two datasets: a training sample $S$, and a held-out validation sample $S^{val}$. They suggest using the validation sample to perform updates on the multipliers and in turn the gain matrix $G$, and the training sample to solve the resulting cost-sensitive learning problem in (4). The latter would typically involve optimizing the empirical risk on the training set $\frac{1}{|S|} \sum_{(x,y) \in S} \ell_G(y, s(x))$, for some cost-sensitive loss $\ell_G$. The intuition here is that, even when a model achieves very low training error, the estimate of $G$ will accurately reflect the model’s performance on held-out examples. While this modification improves our estimate of $G$, it only provides a partial solution for over-parameterized settings, where the model would still struggle to generalize well if $\ell_G$ happens to be the simple re-weighted loss in (6).

# 4 Cost-sensitive Losses Based on Logit Adjustment

We present new cost-sensitive losses that seek to avoid the problems mentioned above. We build on recent work on logit adjustment, shown previously to be effective in long tail settings [75, 59, 90].

**Diagonal gain matrix.** When the gain matrix $G$ is diagonal, Proposition 1 tells us that the Bayes-optimal classifier for the resulting weighted accuracy metric is of the form $h^*(x) \in \text{argmax}_y \hat{s}(x)$. This tells us that the Bayes-optimal classifier can be determined by simply selecting the class with the highest logit-adjusted score for each example. However, when $G$ is not diagonal, the situation becomes more complex.

**Proposition 2.** Let $\hat{L}^w_t(s) = \frac{1}{|S|} \sum_{(x,y) \in S} \ell^w(y, s(x))$ denote the average loss on training sample $S$ with $\ell^w$ defined as in (6) for a gain matrix $G$. Then a scoring function $\hat{s}$ that achieves the minimum value of $\hat{L}^w_t(s)$ over all $s: X \rightarrow \mathbb{R}^m$ is of the form: $\text{softmax}_i(\hat{s}(x)) = \frac{G_{y,i}}{\sum_j G_{y,j}}, \forall (x, y) \in S$.

Notice that the model output is invariant to re-scaling of the rows in $G$, i.e., one can multiply each row of the gain matrix $G$ by a different scalar, and the values memorized by the model for each training example will remain unchanged. On the other hand, re-scaling a column of $G$ does substantially change the score learned for each example. While this does not tell us about how the model would behave on unseen new examples, our experience has been that loss re-weighting is usually effective when used to control the amount of out-of-sample predictions that a model makes for a certain class (by suitably scaling the column for that class in $G$), but does not work well when used to emphasize greater accuracy on a particular class. Consequently, we find that loss re-weighting usually fares better with metrics like coverage that depend only on the model predictions and not on the true labels, than with metrics like recall which depend on both the predictions and labels.

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argmax_{y \in [m]} G_{yy} p_y(x), \text{ where } p_y(x) = P(y|x). \text{ Intuitively, we would like the learned scoring function } s(x) \text{ to mimic the Bayes-optimal scores } G_{yy} p_y(x) \text{ for each } x. \text{ In particular, we would like the maximizer of } s_y(x) \text{ over labels } y \text{ to match the Bayes-optimal label for } x. \text{ One way to facilitate this is to adjust the scores } s_y(x) \text{ based on the diagonal weights } G_{yy}, \text{ and to compute a loss on the adjusted scores. Specifically, we shift } s_y(x) \text{ to } \hat{s}_y(x) = s_y(x) - \log(G_{yy}), \text{ and optimize the shifted scores so that their softmax transformation matches the class probabilities } p(x). \text{ This would then encourage the original scorer } s \text{ to be monotonic in the Bayes optimal scores:}

\[
\frac{\exp(\hat{s}_y(x))}{\sum_j \exp(\hat{s}_j(x))} = p_y(x) \iff \hat{s}_y(x) \propto \ln(p_y(x)) \iff s_y(x) \propto \ln(G_{yy} p_y(x)).
\]

In practice, } p(x) \text{ is not directly available to us, and so we employ the following logit-adjusted cross-entropy loss that implements this idea with labels } y \text{ drawn according to } p(x):

\[
\ell^{LA}(y, s) = -\log \left( \frac{\exp(s_y - \log(G_{yy}))}{\sum_j \exp(s_j - \log(G_{jj}))} \right).
\]

This loss is a simple generalization of the one analyzed in Ren et al. [75], Menon et al. [59], Wang et al. [90], in which the logits are adjusted based on the class priors to optimize the balanced error rate. Such approaches have historical precedent in the class imbalance literature [72, 107, 12].

**Proposition 3.** The logit-adjusted loss \( \ell^{LA} \) is calibrated for a diagonal gain matrix \( G \).

Unlike the weighted loss in (5), changing the diagonal entries of \( G \) in (7) changes the operating point of the learned scoring function. In separable settings, this would mean that the diagonal weights will determine the operating point at which the model achieves zero training error, and tend to push the separator towards classes that have higher weights, thus helping improve model generalization. An alternate explanation posed by Menon et al. [59] re-writes the loss in a pair-wise form:

\[
\ell^{LA}(y, s) = \log \left( 1 + \sum_{j \neq y} \exp(\delta_{yj} - (s_y - s_j)) \right),
\]

where \( \delta_{yj} = \log(G_{yy}/G_{jj}) \) can be seen as the relative margin between class \( y \) and class \( j \). This tells us that the loss encourages a larger separation between classes that have different weights. In Appendix C, we elaborate on this margin interpretation and point out that this loss can in fact be seen as a soft-approximation to the more traditional margin-based loss of Crammer and Singer [16].

**General gain matrix.** When the gain matrix \( G \) is non-diagonal, a simple logit adjustment no longer works. In this case, we propose a hybrid approach that combines logit adjustment with an outer weighting. To this end, we prescribe factorizing the gain matrix into a product \( G = MD \) for some diagonal matrix \( D \in \mathbb{R}^{m \times m} \), and \( M = GD^{-1} \). The proposed loss then adjusts the logits to account for the diagonal entries of \( D \) and applies an outer weighting to account for \( M \):

\[
\ell^{hyb}(y, s) = -\sum_{i=1}^{m} M_{yi} \log \left( \frac{\exp(s_i - \log(D_{ii}))}{\sum_j \exp(s_j - \log(D_{jj}))} \right).
\]

In practice, \( D \) can be chosen to reflect the relative importance of the classes and include information such as the class priors, which cannot be effectively incorporated as part of the outer weighting. One simple choice could be \( D = \text{diag}(1/\pi_1, \ldots, 1/\pi_m) \). Another intuitive choice could be to \( D = \text{diag}(G_{11}, \ldots, G_{mm}) \), so that the residual matrix \( M = GD^{-1} \) that serves as the outer weighting has 1s in all its diagonal entries, and thus equal weights on the diagonal loss terms.

**Proposition 4.** For any diagonal matrix \( D \in \mathbb{R}^{m \times m} \) with \( D_{yy} > 0, \forall y, \text{ and } M = GD^{-1} \), the hybrid loss \( \ell^{hyb} \) is calibrated for \( G \).

In some special cases, we may be able to avoid the outer-weighting: e.g., when the gain matrix is the sum of a diagonal matrix and a matrix with equal rows, i.e. \( G = \text{diag}(\alpha) + \beta I \) (cf. Table 1, rows 3 & 4), the Bayes-optimal classifier from Proposition 1 is of the form \( h^*(x) = \argmax_{y \in [m]} \alpha_y p_y(x) + \beta_y \). The additive terms \( \beta_y \) in the optimal classifier can then be encoded in the loss as a shift to the softmax output, giving us the following softmax-shifted loss: for constant \( C > 0 \),

\[
\ell^{SMS}(y, s) = -\log \left( C \frac{\exp(s_y - \log(\alpha_y))}{\sum_j \exp(s_j - \log(\alpha_j))} - \beta_y/\alpha_y \right).
\]
### Table 2: Results of maximizing worst-case recall on CIFAR-10 and the minimum of the head and tail recalls on CIFAR-100 and Tiny-ImageNet. The highest entry in each column is shaded. The proposed logit-adjusted loss yields significantly better worst-case recall than all the baselines.

| Method                  | CIFAR-10-LT | CIFAR-100-LT | TinyImgNet-LT |
|-------------------------|-------------|--------------|---------------|
|                         | Avg Rec     | Min Rec      | Avg Rec       | Min HT Rec | Avg Rec     | Min HT Rec |
| ERM                     | 0.754       | 0.589        | 0.444         | 0.076      | 0.319       | 0.010       |
| LA with class priors    | 0.790       | 0.658        | 0.489         | 0.083      | 0.350       | 0.095       |
| CSL [Re-weighted]       | 0.731       | 0.592        | 0.415         | 0.083      | 0.306       | 0.098       |
| CSL [Logit-adjusted]    | 0.773       | 0.721        | 0.466         | 0.048      | 0.326       | 0.310       |

### Table 3: Results of maximizing average recall, while constraining the coverage for each class to be at least 95% of \( \frac{1}{m} \) on CIFAR-10, and constraining the head and tail coverages to be at least 95% of \( \frac{1}{m} \) on CIFAR-100 and TinyImageNet. The models are evaluated on a balanced test set and hence we set the coverage target to \( 0.95 \frac{1}{m} \). All metrics and targets are rounded off to 3 decimal places. The highest entry and those comparable to it (within 0.002) are shaded. The maximum recall among rows where the coverage is closest to the target is underlined. The proposed hybrid losses yield coverage values \( \geq \) target, while achieving a higher average recall than the re-weighted loss.

| Method                  | CIFAR-10-LT | CIFAR-100-LT | TinyImgNet-LT |
|-------------------------|-------------|--------------|---------------|
|                         | Avg Rec     | Min Cov      | Avg Rec       | Min HT Cov | Avg Rec     | Min HT Cov |
| ERM                     | 0.754       | 0.058        | 0.444         | 0.001      | 0.319       | 0.000       |
| LA with class priors    | 0.790       | 0.076        | 0.489         | 0.005      | 0.350       | 0.002       |
| CSL [Re-weighted]       | 0.739       | 0.092        | 0.408         | 0.010      | 0.289       | 0.004       |
| CSL [Hybrid A]          | 0.759       | 0.095        | 0.474         | 0.010      | 0.350       | 0.005       |
| CSL [Hybrid B]          | 0.760       | 0.097        | 0.475         | 0.010      | 0.349       | 0.005       |

**Proposition 5.** \( \ell_{\text{SMS}} \) is calibrated for the gain matrix \( G = \text{diag}(\alpha) + 1\beta^T \) when \( C = 1 + \sum_y \beta_y / \alpha_y \).

See Appendix D for a practical variant this loss that avoids a negative value within the log.

## 5 Experimental Comparison with Loss Re-weighting

We now showcase that the proposed losses provide substantial gains over loss re-weighting through experiments on three benchmark datasets: CIFAR-10, CIFAR-100 [44], and TinyImageNet [49, 76] (a subset of the ImageNet dataset with 200 classes). Similar to [17, 9, 59], we use long-tail versions of these datasets by downsampling examples with an exponential decay in the per-class sizes. We set the imbalance ratio \( \frac{\max_i P(y=i)}{\min_i P(y=i)} \) to 100 for CIFAR-10 and CIFAR-100, and to 83 for TinyImageNet (the slightly smaller ratio is to ensure that the smallest class is of a reasonable size). In each case, we use a balanced validation sample of 5000 held-out images, and a balanced test set of the same size. We trained ResNet-56 models on CIFAR-10 and CIFAR-100, and ResNet-18 models on TinyImageNet, using SGD with momentum. See Appendix E for details of the hyper-parameters used.

**Cost-sensitive learning and baselines.** We consider two learning objectives: maximizing worst-case recall, and maximizing average recall subject to coverage constraints. We employ the “two dataset” approach of Cotter et al. [13] discussed in Section 3 to solve these problems, with the validation set used for updates on the gain matrix \( G \), and the training set used for the cost-sensitive learning (CSL) step. Since our goal here is to demonstrate that the the proposed losses fair better at solving the CSL step than loss re-weighting, we use a large validation sample to get good estimates of the gain matrix \( G \). For the CIFAR datasets, we perform 32 SGD steps on the cost-sensitive loss for every update on \( G \), and for TinyImageNet, we perform 100 SGD steps for every update on \( G \). In each case, we compare the results with empirical risk minimization (ERM) with the standard cross-entropy loss, and as a representative method that seeks to maximize the average per-class recall (i.e. balanced accuracy), we include the logit-adjusted loss of Menon et al. [59] in which the adjustments are based on the class priors alone (LA with class priors).

**Maximizing worst-case recall.** In our first task, we maximize the worst-case recall over all 10 classes on CIFAR-10 (cf. (1)). Given the larger number of classes, this can be a very restrictive goal for CIFAR-100 and TinyImageNet, leading to poor overall performance. So for these datasets, we will consider a simpler goal, where we measure the average recall on the bottom 10% of the
Table 4: Improvements with post-shifting: Results of maximizing the minimum recall over all classes (col 1–2) and over just the head and tail classes (col 3–4). Proposed approach compares favorably to ERM + PS, and is the best at maximizing worst-case recall over all 100 classes on CIFAR-100.

| Method                      | CIFAR-10-LT | CIFAR-100-LT | CIFAR-100-LT | TIN-LT |
|-----------------------------|-------------|--------------|--------------|--------|
|                             | Per-class Recall | Head-Tail Recall | Head-Tail Recall | Head-Tail Recall |
|                             | Avg | Min | Avg | Min | Avg | Min | Avg | Min |
| ERM + PS                    | 0.771 | 0.722 | 0.286 | 0.090 | 0.450 | 0.447 | 0.311 | 0.310 |
| CSL [Logit-adjusted]        | 0.773 | 0.721 | 0.342 | 0.120 | 0.450 | 0.446 | 0.325 | 0.323 |
| CSL [Logit-adjusted] + PS   | 0.755 | 0.708 | 0.315 | 0.120 | 0.457 | 0.451 | 0.331 | 0.319 |

Table 5: Improvements with self-distillation: Performance of student model on maximizing worst-case recall on CIFAR-10 and the minimum of the head and tail recalls on CIFAR-100. Both the teacher and student use a ResNet-56 architecture. The highest entry in each column is shaded.

| Method                          | CIFAR-10-LT | CIFAR-100-LT |
|---------------------------------|-------------|--------------|
|                                | Avg Rec | Min Rec | Avg Rec | Min Rec | Avg Rec | Min HT Rec |
| Distilled ERM                  | 0.757   | 0.585   | 0.455   | 0.058   |
| Distilled LA with teacher priors | 0.817   | 0.708   | 0.509   | 0.248   |
| Distilled CSL [Re-weighted]    | 0.736   | 0.571   | 0.451   | 0.082   |
| Distilled CSL [Logit-adjusted] | 0.781   | 0.734   | 0.462   | 0.458   |
| Distilled CSL [Hybrid-distilled]| 0.775   | 0.744   | 0.473   | 0.467   |

Constraining coverage. The next task we consider for CIFAR-10 seeks to ensure that when evaluated on a balanced dataset, the model makes the same proportion of predictions for each class. This leads us to the optimization problem shown in row 5 of Table 1, where we wish to maximize the average recall, constraining the model’s “balanced coverage” on each class: \( \text{bcov}_y[h] = \sum_{i=1}^{m} \frac{1}{\pi_i} C_{iy}[h] \) to be at least 95% of \( \frac{1}{m} \), where \( m \) is the number of classes. Because the validation and test samples are already balanced, the model’s coverage on these datasets is the same as its balanced coverage. For CIFAR-100 and TinyImageNet, we consider the simpler goal of maximizing the average recall over all classes, with constraints on the model’s average coverage over the head labels, and its average coverage over the tail labels to be both at least 95% of \( \frac{1}{m} \):

\[
\max_h \frac{1}{m} \sum_{y \in [m]} \text{rec}_y[h] \quad \text{s.t.} \quad \frac{1}{|\mathcal{H}|} \sum_{y \in \mathcal{H}} \text{bcov}_y[h] \geq 0.95 \frac{1}{m}, \quad \frac{1}{|\mathcal{T}|} \sum_{y \in \mathcal{T}} \text{bcov}_y[h] \geq 0.95 \frac{1}{m}.
\]

The gain matrix \( G \) for these problems (as shown in Table 1) is non-diagonal, and does not have a special structure that we can exploit. We will therefore use the hybrid loss function that we provide in (9) for a general \( G \), and try out both variants suggested: with the diagonal matrix \( D = \text{diag}(1/\pi_1, \ldots, 1/\pi_m) \) (variant “A”), and with \( D = \text{diag}(G_{11}, \ldots, G_{mm}) \) (variant “B”). In this case, loss re-weighting is able to match the coverage targets for CIFAR-100 and TinyImageNet and comes in a close-second on CIFAR-10, but fares poorly on the average recall. The proposed losses perform significantly better on this metric, while yielding coverage values that are closest to the target. Between the two hybrid variants, there isn’t a clear winner.

### 6 Improvements with Post-shifting and Distillation

Having demonstrated that the proposed losses can provide significant gains over loss re-weighting, we explore ways to further improve their performance.
Does post-shifting provide further gains? Post-hoc correction strategies have generally shown to be very effective in optimizing evaluation metrics [64, 43, 97, 85, 59], often matching the performance of more direct methods that modify the training loss. They are implemented in two steps: (i) train a base scoring model \( s: \mathcal{X} \rightarrow \mathbb{R}^m \) using ERM, (ii) construct a classifier that estimates the Bayes-optimal label for a given \( x \) by applying a gain matrix \( G \in \mathbb{R}^{m \times m} \) to the predicted probabilities:

\[
h(x) \in \arg\max_{y \in [m]} \sum_{i=1}^{m} G_{iy} \eta_i(x), \quad \text{where} \quad \eta(x) = \text{softmax}(s(x)).
\]

The coefficients \( G \) are usually chosen to maximize the given evaluation metric on a held-out validation set, using either a simple grid search (when the number of classes is small) or more sophisticated optimization tools [66, 85]. Table 4 shows the results of post-shifting the ERM-trained model for the tasking of maximizing worst-case recall (see Appendix E for details of how we fit the post-shift coefficients). While post-shifting does considerably improve the performance of ERM, on the more difficult problem of maximizing the worst-case recall over all 100 classes in CIFAR-100, our proposed approach of modifying the training loss fairs significantly better on both the average and minimum recalls. In some cases, our model over-fits to the validation sample as a result of post-shifting.

New cost-sensitive loss for distillation. Another technique that has proven effective in boosting the performance of neural networks is knowledge distillation, wherein soft predictions from a “teacher” model \( p^t: \mathcal{X} \rightarrow \Delta_m \) are used as labels to train a “student” model [32, 77, 27, 95, 105]. All the losses discussed so far can be easily applied to a distillation setup, where the training labels can be replaced with an expectation over the teacher’s label distribution: \( E_{x \sim D_X} \left[ \sum_{y=1}^{m} p^t_y(x) \ell(y, s(x)) \right] \).

In applying these losses for cost-sensitive learning, it is important that the class priors \( \pi y \) used to construct the gain matrices (see Table 1) be replaced with the teacher’s prior distribution \( \pi^t_y = \frac{1}{\vert S \vert} \sum_{(x,y) \in S} p^t_y(x) \). This is particularly important when the teacher is trained on a dataset with a different prior distribution, or its outputs are re-calibrated to yield soft predictions.

We also provide an additional loss for distillation that seeks to better exploit the teacher predictions by encoding them as a part of the logit adjustment. This is an extension of the hybrid loss proposed in (9) for a general gain matrix \( G \), where we pick a diagonal matrix \( D \in \mathbb{R}^{m \times m} \), and compute the residual matrix \( M = GD^{-1} \). For given teacher labels \( z \in \Delta_m \) and student scores \( s \in \mathbb{R}^d \), the proposed loss takes the following form with a hyper-parameter \( \gamma \in [0, 1] \):

\[
\ell_{\text{dist}}(z, s) = -\sum_{y=1}^{m} \tilde{z}_y^{1-\gamma} \log \left( \frac{\exp(s_y - \log(D_{yy}) - \gamma \log(\tilde{z}_y))}{\sum_j \exp(s_j - \log(D_{jj}) - \gamma \log(\tilde{z}_j))} \right), \quad \text{where} \quad \tilde{z} = M^\top z. \tag{11}
\]

The loss applies the matrix \( M \) to the teacher labels, and uses a portion of the transformed teacher scores (determined by the parameter \( \gamma \)) as an outer weighting, and uses the remainder to perform an additional adjustment to the logits. The calibration properties of this surrogate rely on the teacher labels mimicking the conditional-class probabilities \( p(y|x) \), and are discussed in Appendix F.

We re-run the worst-case recall experiments on CIFAR-10 and CIFAR-100 from Section 5 with distillation. We employ self-distillation and use the same ResNet-56 architecture for both the teacher and the student. We first train the teacher using ERM, and use its labels to train a student on the same dataset. For the hybrid-distilled loss in (11), we set \( D \) to a diagonal matrix of inverse teacher priors \( 1/\pi_1^t, \ldots, 1/\pi_m^t \) and pick the parameter \( \gamma \) from \( \{0.1, 0.2, 0.3, 0.4, 0.5\} \) using the validation set. As shown in Table 5, distillation uniformly improves the performance for all methods, with the hybrid-distilled loss yielding the best worst-case recall.

In conclusion, we have proposed new cost-sensitive losses for training over-parameterized models with non-decomposable metrics, and have shown significant gains over standard loss re-weighting. Our approach compares favorably to post-shifting, and can be further improved with distillation. A limitation of our losses is that while they are calibrated, these guarantees only hold in the large sample limit. In the future, we wish to probe further into why our losses improve generalization even in finite sample settings, and to develop a more formal understanding of their margin properties. More broadly, when applied to objectives with fairness constraints, the techniques presented here could be used to improve performance of complex neural networks on under-represented samples. Carefully studying their empirical behaviour in such settings, to ensure they do not introduce unforeseen additional biases, is an important direction for future study.
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Checklist

1. For all authors...
   (a) Do the main claims made in the abstract and introduction accurately reflect the paper's contributions and scope? [Yes]
   (b) Did you describe the limitations of your work? [Yes] See last para of Section 6
   (c) Did you discuss any potential negative societal impacts of your work? [Yes] See last para of Section 6
   (d) Have you read the ethics review guidelines and ensured that your paper conforms to them? [Yes]

2. If you are including theoretical results...
   (a) Did you state the full set of assumptions of all theoretical results? [Yes] See Section 4
   (b) Did you include complete proofs of all theoretical results? [Yes] See Appendix B

3. If you ran experiments...
   (a) Did you include the code, data, and instructions needed to reproduce the main experimental results (either in the supplemental material or as a URL)? [No] Unfortunately, due to internal library dependencies, we are unable to provide code at this stage. We'll provide a self-contained codebase with the final version of the paper.
   (b) Did you specify all the training details (e.g., data splits, hyperparameters, how they were chosen)? [Yes] See Appendix E
   (c) Did you report error bars (e.g., with respect to the random seed after running experiments multiple times)? [No] Unfortunately, owing to the non-trivial computational requirements of training on large-scale image datasets (particularly in distillation setups), it was prohibitive to run all methods for multiple trials. Thus, for consistency, our reported results in Sections 5–6 are limited to one trial. However, as a sanity check, we did perform multiple runs of the baseline method, and find that the gains shown by our method are significantly greater than the variance across trials. For example, in the illustrative experiments in Section 3, where we do average our results over 5 trials and report error bars, one can see that the gains our proposed approach offers are substantially greater than the variance.
   (d) Did you include the total amount of compute and the type of resources used (e.g., type of GPUs, internal cluster, or cloud provider)? [Yes] See Appendix E

4. If you are using existing assets (e.g., code, data, models) or curating/releasing new assets...
   (a) If your work uses existing assets, did you cite the creators? [Yes] See Section 5
   (b) Did you mention the license of the assets? [N/A] The datasets we experiment with are widely used in the literature and have their licences publicly available.
   (c) Did you include any new assets either in the supplemental material or as a URL? [No]
   (d) Did you discuss whether and how consent was obtained from people whose data you’re using/curating? [N/A]
   (e) Did you discuss whether the data you are using/curating contains personally identifiable information or offensive content? [No] The data we use has no personally identifiable information or offensive content

5. If you used crowdsourcing or conducted research with human subjects...
   (a) Did you include the full text of instructions given to participants and screenshots, if applicable? [N/A]
   (b) Did you describe any potential participant risks, with links to Institutional Review Board (IRB) approvals, if applicable? [N/A]
   (c) Did you include the estimated hourly wage paid to participants and the total amount spent on participant compensation? [N/A]
Supplementary Material

Algorithm 1 Reductions-based Algorithm for Maximizing Worst-case Recall (1)

Inputs: Training set $S$, Validation set $S^{\text{val}}$, Step-size $\omega \in \mathbb{R}_+$, Class priors $\pi$, CS-loss $\ell_G$

Initialize: Classifier $h^0$, Multipliers $\lambda^0 \in \Delta_m$

for $t = 0$ to $T - 1$ do

Update $\lambda$:

$\lambda_i^{t+1} = \lambda_i^t \exp \left(-\omega \frac{\hat{C}_{ij}[h^t]}{\pi_i}\right), \forall i$,

where $\hat{C}_{ij}[h] = \frac{1}{|S^{\text{val}}|} \sum_{(x,y) \in S^{\text{val}}} 1(y = i, h(x) = j)\lambda_i^{t+1}$

$G = \text{diag}(\lambda_1^{t+1}/\pi_1, \ldots, \lambda_m^{t+1}/\pi_m)$

Cost-sensitive Learning (CSL):

$s_i^{t+1} \in \arg\min_s \frac{1}{|S|} \sum_{(x,y) \in S} \ell_G(y,s(x))$ // Replaced by few steps of SGD

$h_i^{t+1}(x) \in \arg\max_{s \in [m]} s_i^{t+1}(x), \forall x$

end for

return $h^T$

Algorithm 2 Reductions-based Algorithm for Constraining Coverage (2)

1: Inputs: Training set $S$, Validation set $S^{\text{val}}$, Step-size $\omega \in \mathbb{R}_+$, Class priors $\pi$, CS-loss $\ell_G$

2: Initialize: Classifier $h^0$, Multipliers $\lambda^0 \in \mathbb{R}_m^+$

3: for $t = 0$ to $T - 1$ do

4: Update $\lambda$:

5: $\lambda_i^{t+1} = \lambda_i^t - \omega \left(\sum_{j=1}^m \hat{C}_{ji}[h^t] - 0.95\pi_i\right), \forall i$

where $\hat{C}_{ij}[h] = \frac{1}{|S^{\text{val}}|} \sum_{(x,y) \in S^{\text{val}}} 1(y = i, h(x) = j)$

6: $G_{ij} = \frac{1}{m\pi_j} 1(i = j) + \lambda_i^{t+1}, \forall i, j$

7: $G_{ij} = \frac{1}{m\pi_j} 1(i = j) + \lambda_i^{t+1}$, Projection to $\mathbb{R}_+$

8: Cost-sensitive Learning (CSL):

9: $s_i^{t+1} \in \arg\min_s \frac{1}{|S|} \sum_{(x,y) \in S} \ell_G(y,s(x))$ // Replaced by few steps of SGD

10: $h_i^{t+1}(x) \in \arg\max_{s \in [m]} s_i^{t+1}(x), \forall x$

end for

11: return $h^T$

A Algorithms for Robust and Constrained Learning

Recall that the algorithms discussed in Section 2 have two intuitive steps: (i) update the multipliers $\lambda$ based on the current classifier’s performance and construct a gain matrix $G$; (ii) train a new classifier by optimizing a cost-sensitive loss $\ell_G$ for $G$. Algorithm 1 outlines this procedure for the problem of maximizing the worst-case recall in (1), and Algorithm 2 outlines this procedure for the problem of maximizing the average recall subject to coverage constraints in (2). These algorithms additionally incorporate the “two dataset” trick suggested by Cotter et al. [13] for better generalization, wherein the updates on $\lambda$ are performed using a held-out validation set $S^{\text{val}}$, and the minimization of the resulting cost-sensitive loss is performed using the training set $S$.

In Algorithm 1, we seek to find a saddle-point for the max-min problem for (1). For this, we jointly minimize the weighted objective over $\lambda \in \Delta_m$ using exponentiated-gradient descent and maximize the objective over $h$. The latter can be equivalently formulated as the minimization of a cost-sensitive loss $\ell_G$ with $G = \text{diag}(\lambda_1/\pi_1, \ldots, \lambda_m/\pi_m)$. In Algorithm 2, we seek to find a saddle-point for the Lagrangian max-min problem for (2). In this case, we jointly minimize the Lagrangian over $\lambda \in \Delta_m$ using projected gradient descent and maximize the Lagrangian over $h$. The latter is equivalent to minimizing a cost-sensitive loss $\ell_G$ with $G = \text{diag}(\lambda_m/\pi_m) + 1_m \lambda^T$. In our experiments, the class priors $\pi$ were estimated from the training sample.
The cost-sensitive learning steps optimizes a scoring function \( s: \mathcal{X} \rightarrow \mathbb{R}^m \) over a class of scoring models, and constructs a classifier \( h(x) \in \operatorname{arg\max}_{i \in [m]} s_i(x) \) from the learned scoring function. In practice, we do not perform a full optimization for this step, and instead perform a few steps of stochastic gradient descent (SGD) on the loss \( \ell_{C_j} \), warm-starting each time from the scoring function from the previous iteration.

See Chen et al. [11], Cotter et al. [15] for theoretical guarantees for the learned classifier, which usually require the algorithms to output a stochastic classifier that averages over the individual iterates \( h^1, \ldots, h^T \). Since a stochastic classifier can be difficult to deploy [14], in practice, for the problems we consider, we find it sufficient to simply output the last iterate \( h^T \).

**B  Proofs**

We will provide the standard result to be useful in our proofs. Since the negative log is a strictly proper, in the sense of Gneiting and Raftery [28], Williamson et al. [91], we have that:

**Lemma 6** (Gneiting and Raftery [28], Williamson et al. [91]). For any distribution \( u \in \Delta_m \), the minimizer of the expected risk

\[
E_{y \sim u} [− \log(v_y)] = − \sum_{i=1}^{m} u_i \log(v_i)
\]

over all distributions \( v \in \Delta_m \) is unique and achieved at \( v = u \).

**B.1 Proof of Proposition 1**

**Proof.** We reproduce the proof from Narasimhan et al. [66]. Expanding the weighted accuracy in (4),

\[
\sum_{i,j} G_{ij} C_{ij}[h] = E_{x,y} \left[ \sum_{i,j} G_{ij} \mathbf{1}(y = i, h(x) = j) \right] = E_{x,y} \left[ \sum_j G_{yj} \mathbf{1}(h(x) = j) \right] = E_x \left[ E_{y|x} \left[ \sum_j G_{yj} \mathbf{1}(h(x) = j) \right] \right] = E_x \left[ \sum_{i,j} p_i(x) G_{ij} \mathbf{1}(h(x) = j) \right].
\]

To compute the Bayes-optimal classifier for (4), it suffices to maximize the above objective point-wise, and to predict for each \( x \), the label which maximizes the term within the expectation:

\[
h^*(x) \in \operatorname{arg\max}_{y \in [m]} \sum_i p_i(x) G_{ij} = \operatorname{arg\max}_{i \in [m]} (G^\top p(x))_j,
\]

as desired. \( \square \)

**B.2 Proof of Proposition 2**

**Proof.** The average training loss \( \hat{\ell}^{\text{avg}}(s) = \frac{1}{|S|} \sum_{(x,y) \in S} \ell^{\text{avg}}(y, s(x)) \) is minimized by a scoring function \( s \) that yields the minimum loss \( \ell^{\text{avg}}(y, s(x)) \) for each \( (x, y) \in S \). For a fixed \( (x, y) \in S \), the loss can be expanded as:

\[
\ell^{\text{avg}}(y, s(x)) = -\sum_{i=1}^{m} G_{y,i} \log \left( \frac{\exp(s_i(x))}{\sum_j \exp(s_j(x))} \right) = -C_y \sum_{i=1}^{m} \frac{G_{y,i}}{\sum_j G_{y,j}} \log \left( \frac{\exp(s_i(x))}{\sum_j \exp(s_j(x))} \right),
\]

where \( C_y = \sum_j G_{y,j} \) can be treated as a constant for a fixed \( y \). We then have from Lemma 6, that any scoring function \( \hat{s} \) that minimizes \( \hat{\ell}^{\text{avg}}(s) \) evaluates to

\[
\frac{\exp(\hat{s}_i(x))}{\sum_j \exp(\hat{s}_j(x))} = \frac{G_{y,i}}{\sum_j G_{y,j}}, \quad \forall i \in [m] \text{ on the examples } (x, y) \text{ in } S.
\]

\( \square \)

**B.3 Proof of Propositions 3–4**

We provide a proof for Proposition 4. The proof of Proposition 3 follows by setting \( D = G \) in the hybrid loss in (9).
Proof of Proposition 4. We wish to show that for any fixed $x$, the scoring function $s^* : \mathcal{X} \rightarrow \mathbb{R}^m$ that minimizes the expected loss $E_{(x,y) \sim D} [\ell_{\text{hyb}}(y, s(x))]$ recovers the Bayes-optimal classifier for $G$. To this end, we first re-write the expected loss in terms of a conditional risk:

$$\argmax_{y \in [m]} s^*_y(x) \subseteq \argmax_{y \in [m]} (G^T p(x))_y.$$  \hfill (12)

To this end, we first re-write the expected loss in terms of a conditional risk:

$$E_{(x,y) \sim D} [\ell_{\text{hyb}}(y, s(x))] = E_{x \sim D_X} [E_{y \sim p(x)} [\ell_{\text{hyb}}(y, s(x))]].$$

The optimal scoring function $s^*(x)$ therefore minimizes the conditional risk $E_{y \sim p(x)} [\ell_{\text{hyb}}(y, s(x))]$ for each $x$. Expanding the conditional risk for a fixed $x$, we have:

$$E_{y \sim p(x)} [\ell_{\text{hyb}}(y, s(x))] = \sum_{y=1}^m p_y(x) \ell_{\text{hyb}}(y, s(x))$$

$$= - \sum_{i=1}^m \sum_{y=1}^m M_{yi} p_y(x) \log \left( \frac{\exp(s_i(x) - \log(D_{ii}))}{\sum_j \exp(s_j(x) - \log(D_{jj}))} \right)$$

$$= - \sum_{i=1}^m (M^T p(x))_i \log \left( \frac{\exp(s_i(x) - \log(D_{ii}))}{\sum_j \exp(s_j(x) - \log(D_{jj}))} \right)$$

$$= - C \sum_{i=1}^m (M^T p(x))_i \log \left( \frac{\exp(s_i(x) - \log(D_{ii}))}{\sum_j \exp(s_j(x) - \log(D_{jj}))} \right),$$

where $C = \sum_j (M^T p(x))_j$ can be treated as a constant for a fixed $x$. Appealing to Lemma 6, we then have that for any fixed $x$, because $s^*(x)$ minimizes the conditional risk,

$$\frac{\exp(s^*_i(x) - \log(D_{ii}))}{\sum_j \exp(s^*_j(x) - \log(D_{jj}))} = \frac{(M^T p(x))_i}{\sum_j (M^T p(x))_j}, \forall i \in [m].$$

It follows that

$$s^*_i(x) - \log(D_{ii}) = \log ((M^T p(x))_i), \forall i \in [m],$$

or equivalently

$$s^*_i(x) = \log (D_{ii} (M^T p(x))_i), \forall i \in [m].$$

This then gives us:

$$s^*(x) = \log (D^T (M^T p(x))) = \log ((MD)^T p(x)) = \log (G^T p(x)),$$

where $\log$ is applied element-wise, and we use $M = GD^{-1}$ in the last equality. Because $\log$ is a strictly monotonic function, $s^*$ satisfies the required condition in (12) for each $x$. \hfill \square

Proof of Proposition 3. The proof follows by setting $D = G$ and applying Proposition 4. \hfill \square

B.4 Proof of Proposition 5

We will assume that $\alpha_y, \beta_y > 0, \forall y$.

Proof. As shown in the proof of Proposition 4, to compute the optimal scoring function $s^*$ for the expected loss $E_{(x,y) \sim D} [\ell_{\text{SMS}}(y, s(x))]$, it suffices to minimize the conditional risk point-wise for each $x$. For a fixed $x$, the conditional risk for $\ell_{\text{SMS}}$ is given by:

$$E_{y \sim p(x)} [\ell_{\text{SMS}}(y, s(x))] = - \sum_{y \in [m]} p_y(x) \log \left( C \frac{\exp(s_y(x) - \log(\alpha_y))}{\sum_{y'} \exp(s_{y'}(x) - \log(\alpha_{y'}))} - \frac{\beta_y}{\alpha_y} \right).$$

To prove that the loss is calibrated, we need to show that the minimizer $s^*(x)$ for each $x$ recovers the Bayes-optimal prediction for $x$, i.e., satisfies:

$$\argmax_{y \in [m]} s^*_y(x) \subseteq \argmax_{y \in [m]} \alpha_y p_y(x) + \beta_y, \forall x.$$  \hfill (13)
We first consider the case where \( p_y(x) > 0, \forall y \). Ignoring the dependence on \( x \), and letting \( u = \exp(s_y - \log(\alpha_y)) \), consider the problem of maximizing \(- \sum_{y \in [m]} p_y \log(C u - \beta_y/\alpha_y)\) over all \( u \in \Delta_m \), i.e.:

\[
\min_{u \in \mathbb{R}^2} - \sum_{y} p_y \log(C u - \beta_y/\alpha_y) \quad \text{s.t.} \quad u_y \geq 0, \forall y, \quad \sum_{y} u_y = 1. \tag{14}
\]

Introducing Lagrangian multipliers \( \gamma \in \mathbb{R} \) for the equality constraint and \( \mu_y \geq 0 \) for the inequality constraints, the Lagrangian for this problem is given by:

\[
- \sum_{y} p_y \log(C u - \beta_y/\alpha_y) - \sum_{y} \mu_y u_y + \gamma (\sum_{y} u_y - 1). \tag{15}
\]

For this problem, the first-order KKT conditions are sufficient for optimality. We therefore have that any \( u^* \) that satisfies the following conditions for some multipliers \( \mu, \gamma \) is a solution to (15):

\[
\begin{align*}
\frac{C p_y}{C u^*_y - \beta_y/\alpha_y} &= \gamma - \mu_y, \forall y \quad \text{(16)} \\
\mu_y u^*_y &= 0, \forall y \quad \text{(17)} \\
\sum_{y} u^*_y &= 1 \quad \text{(18)}
\end{align*}
\]

We now show that \( u^*_y = \frac{p_y + \beta_y/\alpha_y}{C} \), \( \gamma = C \) and \( \mu_y = 0, \forall y \) satisfies (16)–(18). Plugging \( u^*_y, \gamma \) and \( \mu_y \) into the LHS of (16), we get:

\[
\frac{C p_y}{p_y + \beta_y/\alpha_y - \beta_y/\alpha_y} = C = \gamma - \mu_y,
\]

which is the same as the RHS. It is also easy to see that (17) is satisfied. To see that (18) holds, observe that:

\[
\sum_{y} u^*_y = \frac{\sum_{y} p_y + \sum_{y} \beta_y/\alpha_y}{C} = \frac{1 + \sum_{y} \beta_y/\alpha_y}{C} = \frac{C}{C} = 1.
\]

We can now derive the optimal scoring function \( s^* \) from \( u^* \):

\[
\frac{\exp(s^*_y(x) - \log(\alpha_y))}{\sum_{y'} \exp(s^*_{y'}(x) - \log(\alpha_{y'}))} = \frac{p_y(x) + \beta_y/\alpha_y}{C},
\]

Equivalently,

\[
s^*_y(x) - \log(\alpha_y) = \log \left( \frac{p_y(x) + \beta_y/\alpha_y}{C} \right)
\]

or in other words,

\[
s^*_y(x) \propto \log(\alpha_y p_y(x) + \beta_y),
\]

which clearly satisfies the required condition in (13).

For simplicity, we do not explicitly include in (14) constraints \( C u_y - \beta_y/\alpha_y \geq 0, \forall y \) that would require the terms within the log to be non-negative. The form of the optimal scoring function \( s^* \) does not change when these constraints are included. We were able to avoid including these constraints because we assumed that \( p_y(x) > 0, \forall y \). When this is not the case, the additional constraints will be needed for the proof. \( \square \)

C Margin Interpretation for \( \ell^A \)

A limiting form of the logit-adjusted loss in (8) is given below:

\[
\lim_{\gamma \to \infty} \frac{1}{\gamma} \cdot \log \left[ \sum_{j=1}^{m} \exp \left\{ \gamma \cdot (s_{yj} - (s_y - s_j)) \right\} \right] = \max_{j \in [m]} \delta_{yj} - (s_y - s_j),
\]

which has the same form as the loss function proposed by Crammer and Singer [16], Tsochantaridis et al. [86], where \( \delta_{yj} \) is the penalty associated with predicting class \( j \) when the true class is \( y \). The term \( \delta_{yj} \) can be seen as a margin for class \( y \) relative to class \( j \). The only difference between the limiting form given above and the original loss of Crammer and Singer [16] is that the margin term there is typically non-negative, whereas it is set to \( \delta_{yj} = \log(G_{yj}) - \log(G_{jj}) \) in our formulation and can take negative values.
D Practical Variant of $\ell^{\text{SMS}}$

To avoid a negative value in the softmax-shifted loss in (10), we provide a practical variant of the loss. Notice that the Bayes-optimal predictions $h^*(x) \in \text{argmax}_{y \in [m]} \alpha_y P_y(x)$ are unchanged when we subtract a constant from each $\beta_y$, and compute $h^*(x) \in \text{argmax}_{y \in [m]} \alpha_y P_y(x) + \beta_y - \max_y \beta_y$. This gives us the following variant of the loss in which the log is always evaluated on a non-negative value:

$$
\ell_{\text{SMS}*}(y, s) = -\log \left( C \exp(s_y - \log(\alpha_y)) \sum_j \exp(s_j - \log(\alpha_j)) \right) + \max_{y'} \beta_{y'}/\alpha_{y'} - \beta_y/\alpha_y,
$$

One practical difficulty with this formulation is that when the shift term $\max_{y'} \beta_{y'}/\alpha_{y'} - \beta_y/\alpha_y$ for class $y$ is large, and the softmax prediction for that class may have minimal effect on the loss. As a remedy, we prescribe a hybrid variant in which we use a combination of an outer weighting and an inner shift to the softmax:

$$
\ell_{\text{SMS}^\dagger}(y, s) = -\sum_{i=1}^{m} (1(y = i) + \kappa_i) \log \left( C \exp(s_i - \log(\alpha_i)) \sum_j \exp(s_j - \log(\alpha_j)) \right) + \max \kappa' - \kappa',
$$

where the $\kappa_i$'s and $\kappa'_i$'s are chosen so that $\kappa_i + \kappa'_i = \beta_i/\alpha_i$. As with Proposition 5, the calibration properties of this loss depend on our choice of the constant $C$, which in practice, we propose be treated as a hyper-parameter.

E Additional Experimental Details

We provide further details for the experiments run in Sections 5–6. The training sample sizes for the long-tail versions of CIFAR-10, CIFAR-100 and TinyImageNet were as follows: 12406, 10847 and 21748. The test and validation samples had 5000 images each for all three datasets. The CIFAR datasets had images of size 32 $\times$ 32, while TinyImageNet had images of size 224 $\times$ 224.

All models were trained using SGD with a momentum of 0.9 and with a batch size of 128. For the CIFAR datasets, we ran the optimizer for a total of 256 epochs, with an initial learning rate of 0.4, and with a weight decay of 0.1 applied at the 96th epoch, at the 192th epoch and at the 224th epoch. We employed the same data augmentation strategy used by Menon et al. [59], with four pixels padded to each side of an image, a random 32 $\times$ 32 patch of the image cropped, and the image flipped horizontally with probability 0.5. For the TinyImageNet dataset, we ran the optimizer for a total of 256 epochs, with an initial learning rate of 0.1, with a weight decay of 0.1 applied at the 75th epoch and at the 135th epoch.

The step size $\omega$ for the reductions-based algorithms (Algorithms 1–2 in Appendix A) that we use to optimize worst-case recall and constrain coverage was set to 0.1 for CIFAR-10, 0.5 for CIFAR-100, and 1.0 for TinyImageNet. For the CIFAR datasets, we perform 32 SGD steps on the cost-sensitive loss $\ell_C$ for every update on the multipliers, and for TinyImageNet, we perform 100 SGD steps for every update on the multipliers.

For the distillation experiments in Section 6, the logit scores from the teacher ResNet models were temperature scaled to produce soft probabilities $\exp(s_y/\tau)/\sum_{y'} \exp(s_{y'}/\tau)$, with the temperature scale parameter $\tau$ was set to 3.

All experiments were run on 8 chips of TPU v3.

E.1 Implementation of Post-shifting

As noted in Section 6, post-shifting is implemented in two steps: (i) train a base scoring model $s : \mathcal{X} \rightarrow \mathbb{R}^m$ using ERM, (ii) construct a classifier that estimates the Bayes-optimal label for a given $x$ by applying a gain matrix $G \in \mathbb{R}^{m \times m}$ to the predicted probabilities:

$$
h(x) \in \text{argmax}_{y \in [m]} \sum_{i=1}^{m} G_{iy} \eta_i(x), \text{ where } \eta(x) = \text{softmax}(s(x)).
$$

To choose coefficients $G$ to maximize worst-case recall on the validation sample $S_{\text{val}}$, we adopt the optimization-based framework of Narasimhan et al. [66]. The idea is to employ a variant of
We show below that the distillation loss in (11) is calibrated when the teacher model mimics the underlying conditional-class probabilities \( p \).

**Proposition 7.** Suppose the teacher probabilities \( p'(x) = p(x), \forall x \). Let \( D \in \mathbb{R}^{m \times m} \) be a diagonal matrix with \( D_{yy} > 0, \forall y \) and \( M = GD^{-1} \). Then for any \( \gamma \in [0, 1] \), the distilled loss \( \ell_{\text{dis}} \) in (11) is calibrated for \( G \).

**Proof.** As with the proof of Proposition 4, we will show that the minimizer \( s^* \) of the expected distilled loss: \( \mathbb{E}_{x \sim D_x} [\ell_{\text{dis}}(p'(x), s(x))] \) recovers the Bayes-optimal classifier for \( G \). This requires us to show that for each \( x \):

\[
\argmax_{y \in [m]} s^*_y(x) \subseteq \argmax_{y \in [m]} (G^T p(x))_y. \tag{20}
\]

It suffices to consider each \( x \) separately as \( s^* \) minimizes \( \ell_{\text{dis}}(p^t(x), s(x)) \) point-wise for each \( x \).

For simplicity, we ignore the dependence on \( x \), and denote the teacher score \( p'(x) \) by \( z \), the student score \( s(x) \) by \( s \), and the transformed teacher score by \( \bar{z} = M^T z \). We then have:

\[
\ell_{\text{dis}}(z, s^*) = - \sum_{y=1}^{m} \bar{z}_y^{1-\gamma} \log \left( \frac{\exp(s^*_y - \log(D_{yy}) - \gamma \log(\bar{z}_y))}{\sum_j \exp(s^*_j - \log(D_{jj}) - \gamma \log(\bar{z}_j))} \right) = -C \sum_{y=1}^{m} \bar{z}_y^{1-\gamma} \log \left( \frac{\exp(s^*_y - \log(D_{yy}) - \gamma \log(\bar{z}_y))}{\sum_j \exp(s^*_j - \log(D_{jj}) - \gamma \log(\bar{z}_j))} \right),
\]

where \( C = \sum_j \bar{z}_j^{1-\gamma} \) can be treated as a constant for a fixed \( x \). We have from Lemma 6 that the minimizer \( s^* \) of the above loss satisfies:

\[
\frac{\exp(s^*_y - \log(D_{yy}) - \gamma \log(\bar{z}_y))}{\sum_j \exp(s^*_j - \log(D_{jj}) - \gamma \log(\bar{z}_j))} = \frac{\bar{z}_y^{1-\gamma}}{\sum_j \bar{z}_j^{1-\gamma}}.
\]

It follows that

\[
s^*_y - \log(D_{yy}) - \gamma \log(\bar{z}_y) = \log(\bar{z}_y^{1-\gamma})
\]

which gives us:

\[
s^*_y = \log(D_{yy} \bar{z}_y^{1-\gamma}) = \log(D_{yy} \bar{z}_y),
\]

and we further have:

\[
s^* = \log(D \bar{z}) = \log(D^T \bar{z}) = \log((MD)^T \bar{z}) = \log(G^T \bar{z}),
\]

where we use \( M = GD^{-1} \). Because of our assumption that \( p'(x) = p(x) \), the above gives us that \( s^*(x) = \log(G^T p(x)) = \log(G^T p(x)) \), which clearly satisfies (20). \( \Box \)

---

**Algorithm 3 Post-shifting to Maximize Worst-case Recall (1)**

Inputs: Validation set \( S_{\text{val}} \), Step-size \( \omega \in \mathbb{R}_+ \), Class priors \( \pi \), Base model \( \eta : \mathcal{X} \to \Delta_m \)

Initialize: Classifier \( h^0 \), Multipliers \( \lambda^0 \in \Delta_m \\
for t = 0 to T - 1 do
\hspace{1cm} \lambda_{t+1}^i = \lambda_t^i \exp \left( -\omega \frac{\hat{C}_{i}[h^t]}{\pi_i} \right), \forall i, \text{ where } \hat{C}_{i[i]}[h] = \frac{1}{|S_{\text{val}}|} \sum_{(x,y) \in S_{\text{val}}} 1(y = i, h(x) = j) \\
\hspace{1cm} \lambda_{t+1}^m = \frac{\sum_{i=1}^{m} \lambda_{t+1}^i}{\sum_{i=1}^{m} \lambda_{t+1}^i, \forall i} \\
\hspace{1cm} G = \text{diag}(\lambda_{t+1}^1/\pi_1, \ldots, \lambda_{t+1}^m/\pi_m) \\
\hspace{1cm} h^{t+1}(x) \in \text{argmax}_{i \in [m]} \sum_{j=1}^{m} G_{ji} \eta_j(x), \forall x \\
end for

return \( h^* \), where \( t^* = \text{argmax}_{t \in [T]} \min \left\{ \frac{\hat{C}_{i}[h^t]}{\pi_i} \right\} \)