Orthogonal Preserving Quadratic Stochastic Operators: Infinite Dimensional Case

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Abstract. In the present paper, we consider orthogonal preserving quadratic stochastic operators defined on infinite dimensional simplex. We provide a full description of such kind of operators. Moreover, certain examples are given.

1. Introduction
The history of quadratic stochastic operators (QSOs) can be traced back to Bernstein’s work [2] where such kind of operators appeared from the problems of population genetics (see also [8]). Such kind of operators describe time evolution of variety species in biology are represented by so-called Lotka-Volterra (LV) systems [16]. Nowadays, scientists are interested in these operators, since they have a lot of applications especially in modeling in many different fields such as biology [6, 13] (population and disease dynamics), physics [14, 15] (non-equilibrium statistical mechanics), economics and mathematics [8, 13, 15] (replicator dynamics and games).

A quadratic stochastic operator is usually used to present the time evolution of species in biology, which arises as follows. Consider evolution of species in biology as given in the following situation. Let \( I = \{1, 2, \ldots, n\} \) be the \( n \) type of species (or traits) in a population and we denote \( x^{(0)} = (x_1^{(0)}, \ldots, x_n^{(0)}) \) to be the probability distribution of the species in an early state of that population. By \( P_{ij,k} \) we mean the probability of an individual in the \( i \)th species and \( j \)th species to cross-fertilize and produce an individual from \( k \)th species (trait). Given \( x^{(0)} = (x_1^{(0)}, \ldots, x_n^{(0)}) \), we can find the probability distribution of the first generation, \( x^{(1)} = (x_1^{(1)}, \ldots, x_n^{(1)}) \) by using a total probability, i.e.,

\[
x_k^{(1)} = \sum_{i,j=1}^{n} P_{ij,k} x_i^{(0)} x_j^{(0)}, \quad k \in \{1, \ldots, n\}.
\]

This relation defines an operator which is denoted by \( V \) and it is called quadratic stochastic operator (QSO). Each QSO can be interpreted as an evolutionary operator that describes the sequence of generations in terms of probability distributions if the values of \( P_{ij,k} \) and the distribution of the current generation are given. The most well-known class in the theory QSO...
is Volterra one, namely

\[ P_{ij,k} = 0 \text{ if } k \notin \{i, j\}. \tag{1.1} \]

The condition (1.1) biologically means that each individual can inherit only the species of the parents. The dynamics of Volterra QSO was studied in [4, 3]. However, not all QSOs are of Volterra-type, therefore dynamics of non-Volterra QSO remains open. In [5, 11], it has given along self-contained exposition of the recent achievements and open problems in the theory of the QSO.

One of the main problems in the nonlinear operator theory is to study limiting behavior of nonlinear operators. Presently, there are only a small number of studies on dynamical phenomena on higher dimensional systems, even though they are very important.

Note that, most of the studies stated before were done on finite set of all probabilistic distributions. However, there are models where the probability distribution is given on countable set, which means that the corresponding QSO is defined on infinite-dimensional space. The simplest case, the infinite-dimensional space should be the Banach space \( \ell_1 \) of absolutely summable sequences. It is worth mentioning that some infinite dimensional QSOs were studied in [7, 9, 10].

In [1, 12] it was showed that any Volterra QSO preserves the orthogonality of the vectors. Moreover, in finite dimensional setting, any orthogonal preserving QSO is a permutation of Volterra QSO. We look the same problem in the infinite dimensional setting, the last statement become wrong. Therefore, in this paper we study the orthogonality preserving infinite dimensional quadratic stochastic operators. Here, we provide full descriptions of OP QSOs. We provide also some examples showing that the set of orthogonal system of \( S \) that has infinite cardinality supports each of \( \mathbb{F}_k \) is not empty.

### 2. Preliminaries and Basic Definitions

Recall that, finite dimensional simplex is defined by:

\[ S^{n-1} = \left\{ x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid x_i \geq 0, \sum_{i=1}^{n} x_i = 1 \right\}. \]

Analogously, the infinite simplex is defined as follows

\[ S = \left\{ x = (x_1, x_2, \ldots, x_n, \ldots) \in \mathbb{R}^\infty \mid x_i \geq 0, \sum_{i=1}^{\infty} x_i = 1 \right\}. \tag{2.1} \]

Next, let \( V \) be a mapping defined by

\[ V(x)_k = \sum_{i,j=1}^{\infty} P_{ij,k} x_i x_j, \quad k = 1, 2, \ldots . \tag{2.2} \]

Here, \( \{P_{ij,k}\} \) are hereditary coefficients which satisfy

\[ P_{ij,k} \geq 0, \quad P_{ij,k} = P_{ji,k}, \quad \sum_{k=1}^{\infty} P_{ij,k} = 1, \quad i, j, k = 1, 2, \ldots, n . \tag{2.3} \]

One can see that \( V \) maps \( S \) into itself, such kind of mapping is called Quadratic Stochastic Operator (QSO) [9].

By \( \text{support of } x = (x_1, \ldots, x_n, \ldots) \) we mean a set \( \text{Supp}(x) = \{ i \in \mathbb{N} \mid x_i \neq 0 \} \). Now, let \( A_1, \ldots, A_n, \ldots \) and \( B \) be sets such that \( A_k \subset B \) for every \( k \in \mathbb{N} \). The sequence of \( \{A_k\} \) is called \( \text{cover of } B \) if \( \bigcup_{k=1}^{\infty} A_k = B \) and \( A_i \cap A_j = \emptyset \) for \( i, j \in \mathbb{N} \).
Definition 2.1. Two vectors $x, y \in S$ are called orthogonal (denoted by $x \perp y$) if $\text{Supp}(x) \cap \text{Supp}(y) = \emptyset$.

Let $x = (x_1, \ldots, x_n, \ldots)$ and $y = (y_1, \ldots, y_n, \ldots)$. It is clear that $x \perp y$ (i.e., $x, y \in S$) if and only if $x \circ y = 0$ (or $x_k \cdot y_k = 0$ for all $k \in \mathbb{N}$). Here, $\circ$ is referring to usual dot product.

Definition 2.2. A QSO given by (2.2) is called orthogonal preserving QSO (OP QSO) if for any $x, y \in S$ with $x \perp y$, it implies $V(x) \perp V(y)$.

Remark 2.3. Let $\{F_k\}$ and $\{F'_k\}$ be two orthogonal systems. Observes that, by the definition of OP QSOs one gets OP QSOs map from one orthogonal system to another orthogonal system i.e.,

$$F_1 \perp F_2 \perp \cdots \perp F_n \perp \cdots \Rightarrow V(F_1) \perp V(F_2) \perp \cdots \perp V(F_n) \perp \cdots$$

Moreover, the permutation $\pi : \mathbb{N} \rightarrow \mathbb{N}$ of an orthogonal system is still an orthogonal system, therefore we always consider OP QSOs map from one orthogonal system to another orthogonal system by some permutation $\pi$ i.e.,

$$\{V(F_k)\} = \{F'_{\pi(k)}\}$$

Let us recall linear stochastic operators (LSOs) defined by

$$T(x)_k = \sum_{i=1}^{\infty} t_{i,k} x_i \quad \text{for} \quad k \in \mathbb{N}$$

where $x \in S$, $t_{i,k} \geq 0$ and $\sum_{k=1}^{\infty} t_{i,k} = 1$ for any $i \in \mathbb{N}$.

Remark 2.4. Note that, to study OP of a LSO on infinite dimensional simplex it is enough for us to consider a mapping $T$ on the standard basis $\{e_i\}$ of $S$ to an arbitrary orthogonal system in $S$. This fact is due to for any $x \in S$, $T$ can be written

$$T(x) = (x_1 T(e_1),\ldots,x_n T(e_n),\ldots).$$

Hence, we want to know this kind of property for the quadratic case. The answer is given in the next remark.

Remark 2.5. Let us denote $\mathcal{V}$ be the set of OP QSO that generated by $V(e_k) = F_k$ for any orthogonal system $F_k$ in $S$. Now, from the beginning we assume an OP QSO $V$ is generated by $\tilde{V}(F_k) = F'_k$ where $\{F_k\}$ and $\{F'_k\}$ are orthogonal system in $S$. On the other hand, if one consider $\{\tilde{V}(e_k)\}$, the sequence also has to form an orthogonal system in $S$ i.e.,

$$\tilde{V}(e_k) = I_k$$

where the sequence of $\{I_k\}$ is an orthogonal system in $S$. Hence $\tilde{V}$ is an element of $\mathcal{V}$. Indeed, it is one specific case of OP QSO generated by $\tilde{V}(e_k) = I_k$ in such a way $\tilde{V}(F_k) = F'_k$. Therefore, to fully describe all OP QSO, it is enough for us to consider OP QSOs $V$ that generated by $V(e_k) = F_k$ for arbitrary orthogonal system $\{F_k\}$ in $S$.

Recall that a QSO $V : S \rightarrow S$ is called Volterra if and only if

$$P_{ij,k} = 0 \quad \text{if} \quad k \notin \{i, j\} \quad \text{(2.4)}$$

In [10] it was given an alternative definition Volterra operator in terms of extremal elements of $S$. 

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Definition 2.6. A QSO $V : S^{n-1} \to S^{n-1}$ is called $\pi$-Volterra if there is a permutation $\pi$ of the set $\mathbb{N}$ such that $V$ has the following form

$$V(x)_k = x_{\pi(k)} \left( 1 + \sum_{i=1}^{n} a_{\pi(k);i} x_i \right)$$

where $a_{\pi(k);i} = 2P_{\pi(k),k} - 1$ and $a_{\pi(k);i} = -a_{\pi(k);i}$ for any $i, k \in \{1, \ldots, n\}$. Set $a_{\pi(k);\pi(k)} = 0$.

Analogously, one can define infinite dimensional $\pi$-Volterra QSO as follow.

Definition 2.7. A QSO $V$ defined on $S$ is called $\pi$-Volterra if there is a permutation $\pi$ of the set $\mathbb{N}$ such that $V$ has the following form

$$V(x)_k = x_{\pi(k)} \left( 1 + \sum_{i=1}^{\infty} a_{\pi(k);i} x_i \right)$$

where $a_{\pi(k);i} = 2P_{i\pi(k),k} - 1$ and $a_{\pi(k);i} = -a_{i\pi(k);i}$ for any $k, i \in \mathbb{N}$. Set $a_{\pi(k);\pi(k)} = 0$.

In [1, 12] we have proved the following result.

Theorem 2.8 ([1, 12]). Let $V$ be a QSO on $S^{n-1}$. Then the following statements are equivalent:

(i) $V$ is orthogonal preserving;
(ii) $V$ is $\pi$-Volterra QSO.

3. Description of infinite dimensional OP QSOs

In this section, we want to describe infinite dimensional OP QSOs. Note that in [1] it was considered a special class of OP QSOs namely the following theorem has been proved.

Theorem 3.1 ([1]). Let $V$ be a QSO on $S$ such that $V(e_i) = e_{\pi(i)}$ for some permutation $\pi : \mathbb{N} \to \mathbb{N}$. Then $V$ is an OP QSO if and only if $V$ is $\pi$-Volterra QSO.

Remark 3.2. Note that, the vertices of the infinite simplex $S$ are described by the elements $e_k = (\delta_{ik})_{i \in \mathbb{N}}$ where $\delta_{ik}$ is the Kronecker’s delta. If $V$ is an OP QSO on $S$, then the system $\{V(e_k)\}_{k \in \mathbb{N}}$ is also form an orthogonal system on $S$. However, in $S$ there are many orthogonal system which differs from the system $\{e_k\}_{k \in \mathbb{N}}$. For example,

$$\mathbb{F}_1^{(1/2)} = \left( \begin{array}{c} 1 \\ 1 \\ 1/2 \\ 0, \ldots \end{array} \right), \mathbb{F}_2^{(1/2)} = \left( \begin{array}{c} 0, 0 \\ 1/2 \\ 0, \ldots \end{array} \right), \ldots, \mathbb{F}_k^{(1/2)} = \left( \begin{array}{c} 0, 0, \ldots, 1 \\ 1/2 \\ 0, \ldots \end{array} \right), \ldots \quad (3.1)$$

This makes the description of OP QSOs is more challenging than the finite case.

Put $\mathbb{F}_k = \{f_{k,l}\}$ such that $f_{k,l} = 0$ if $l \notin \text{supp}(\mathbb{F}_k)$. Henceforth, $|A|$ is referred to the cardinality of the set $A$ and the set $\mathcal{C}$ represents the complement of $\mathbb{N}$ with respect to the union of $\text{supp}(\mathbb{F}_k)$ i.e.,

$$\mathcal{C} = \mathbb{N} \setminus \left( \bigcup_{k \in \mathbb{N}} \text{supp}(\mathbb{F}_k) \right)$$

Theorem 3.3. Let $V$ be a QSO such that $V(e_k) = \mathbb{F}_{\pi(k)}$ with $|\text{supp}(\mathbb{F}_k)|$ is finite for any $k \in \mathbb{N}$ and $\pi : \mathbb{N} \to \mathbb{N}$ is some permutation. Then, $V(x)$ is an OP QSO if and only if it has the following form for any $x \in S$:
(a) For any \( m \in \text{supp}(\mathbb{F}_\pi(k)) \)

\[
V(x)_m = x_k \left( f_{\pi(k), m} + \sum_{i=1}^{\infty} a_{ik}^{(m)} x_i \right)
\]

where \( a_{ik}^{(m)} = 2P_{k,m} - f_{\pi(k),m} \) and set \( a_{kk}^{(m)} = 0 \).

(b) For any \( c \in C \), \( V(x)_c \) takes one of the following form

(I) If there is no \( P_{ij,c} > 0 \) for every \( i, j \in \mathbb{N} \), then \( V(x)_c = 0 \) or

(II) If there exist at least one \( P_{ij,c} > 0 \), then \( V(x)_c \) has one of the following form:

\( \text{(i)} \) If there is no \( P_{ij,c} > 0 \) for \( j \in \{i_c, j_c\} \) where \( i \in \mathbb{N} \backslash \{j\} \), then

\[
V(x)_c = 2P_{i,j,c} x_i x_j
\]

(\( \text{ii} \)) If there exist \( P_{ij,c} > 0 \) for either \( j = i_c \) or \( j = j_c \) (here let \( j = i_c \)), then \( V(x)_c \) has one of the following form:

(\( \text{I} \)) If \( P_{i,j,c} > 0 \) then

\[
V(x)_c = 2 \left( P_{i,j,c} x_i x_j + P_{i,j,c} x_{i_0} x_{j_0} + P_{i,j,c} x_{i_0} x_{i_1} \right)
\]

(\( \text{II} \)) If \( P_{i,j,c} = 0 \) then

\[
V(x)_c = 2x_{i_c} \left( P_{i,j,c} x_j + \sum_{i=1}^{\infty} P_{i,j,c} x_i \right)
\]

An immediate consequence of the theorem is the following result.

**Corollary 3.4.** Let \( V \) be a QSO such that \( V(e_k) = \mathbb{F}_{\pi(k)} \) for some permutation \( \pi : \mathbb{N} \rightarrow \mathbb{N} \) with \( |\text{supp}(\mathbb{F}_k)| \) is finite for any \( k \in \mathbb{N} \). Then \( V(x) \) is an OP QSO if and only if heredity coefficients \( P_{ij,k} \) satisfy the following:

(a) \( P_{ii,k} = f_{\pi(i), k} \) for \( k \in \text{supp}(\mathbb{F}_{\pi(i)}) \), and \( P_{ij,k} = 0 \) for \( k \notin \{\text{supp}(\mathbb{F}_{\pi(i)}) \cup \text{supp}(\mathbb{F}_{\pi(j)})\} \)

(b) Coefficients \( P_{ij,c} \) where \( c \in C \), they must satisfy one of the following:

(I) \( P_{ij,c} = 0 \) for all \( i, j \in \mathbb{N} \) or

(II) If there exist \( P_{ij,c} > 0 \), then \( P_{ij,c} = 0 \) for any \( i, j \in \mathbb{N} \backslash \{i_c, j_c\} \). Further, the other coefficients must satisfy one of the following,

(i) \( P_{ij,c} = 0 \) for \( j \in \{i_c, j_c\} \) for all \( i \in \mathbb{N} \backslash \{j\} \) or

(ii) If there exist \( P_{ij,c} > 0 \) for either \( j = i_c \) or \( j = j_c \) (here we let \( j = i_c \)), then \( P_{ij,c} = 0 \) for any \( i \in \mathbb{N} \backslash \{i_c, j_c, i_\alpha\} \). Moreover one of the following must be satisfied:

(1) \( P_{ij,c} > 0 \), then \( P_{i_\alpha,c} = 0 \) for any \( i \in \mathbb{N} \backslash \{i_c, j_c, i_\alpha\} \) or

(2) \( P_{ij,c} = 0 \)

**Remark 3.5.** We point out that if \( \mathbb{F}_k = e_k \) then from Theorem 3.3 we immediately get Theorem 3.1 as a consequence.

**Remark 3.6.** One of the important class of infinite dimensional OP QSO is when the union of the supports \( \{\mathbb{F}_k\} \) forms cover of \( N \). So, let \( V \) be a QSO such that \( V(e_k) = \mathbb{F}_{\pi(k)} \) and the supports of \( \{\mathbb{F}_k\} \) forms cover of \( N \). Then orthogonal preserveness of \( V \) can be described as follow

(i) \( V \) has the form given by

\[
V(x)_m = x_k \left( f_{\pi(k), m} + \sum_{i=1}^{\infty} a_{ik}^{(m)} x_i \right)
\]

for any \( m \in \text{supp}(\mathbb{F}_{\pi(k)}) \).
(ii) The heredity coefficients $P_{ij,k}$ satisfy

$$P_{ni,k} = f_{i,k} \quad \forall k \in \text{supp}(\pi(i)) \quad \text{and} \quad P_{ij,k} = 0 \quad \forall k \notin \{\text{supp}(\pi(i)) \cup \text{supp}(\pi(j))\}$$

**Corollary 3.7.** If $T$ is a linear stochastic operator such that $T(e_i) = \pi(k)$ for any $k \in \mathbb{N}$ and $\pi : \mathbb{N} \rightarrow \mathbb{N}$, then $T$ is an OP linear stochastic operator if and only if $T$ takes the following form:

(i) For any $m \in \text{supp}(\pi(k))$

$$T(x)_m = f_{k,m} x_k$$

(ii) For any $c \in C$

$$V(x)_c = 0$$

Next, we provide two examples of OP QSO.

**Example 3.8.** Let us choose $V$ is an OP QSO associated with $V(e_i) = e_{i+1}$ for any $i \in \mathbb{N}$. From Corollary 3.4 one gets $P_{i,i+1} = 1$ for any $i \in \mathbb{N}$. Choose $P_{i+1,2} = 0$ for any $i \geq 2$. Next, we take for any $k \geq 2$

$$P_{ik,k+1} = 1 \quad \text{for} \quad i \in \{1, 2, \ldots, k - 1\} \quad \text{and} \quad P_{ik,k+1} = 0 \quad \text{for} \quad i \geq k + 1$$

From the selected heredity coefficients, we have $P_{ij,l} = 0$ for any $i, j \in \mathbb{N}$ and it is clear that they satisfy (2.3) hence $V$ is well-defined. Thus, using Theorem (3.3) one gets

$$V(x) = \begin{cases} 
V(x)_1 = 0 \\
V(x)_2 = x_1^2 \\
V(x)_{k+1} = x_k \left(1 + \sum_{i=1, i \neq k}^{\infty} (2P_{ik,k+1} - 1)x_i \right) \quad k \geq 3 \\
\end{cases}$$

$$= \begin{cases} 
V(x)_1 = 0 \\
V(x)_2 = x_1^2 \\
V(x)_{k+1} = x_k \left(1 + \sum_{i=1, i \neq k}^{\infty} (2P_{ik,k+1})x_i - (1 - x_k) \right) \quad k \geq 3 \\
\end{cases}$$

$$= \begin{cases} 
V(x)_1 = 0 \\
V(x)_2 = x_1^2 \\
V(x)_{k+1} = x_k \left(\sum_{i=1}^{k-1} 2x_i + x_k \quad \forall \right) \quad k \geq 3 \\
\end{cases}$$

Interestingly, this is a concrete example of nonlinear shifting operator.

**Example 3.9.** Let the permutation $\pi : k \rightarrow k + 1$ for every $k \in \mathbb{N}$. Let $V$ be an OP QSO associated by $V(e_i) = \pi^{(1)}$ where

$$\pi^{(1)} = \begin{pmatrix} 
0, \ldots, 0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, \ldots \\
3k - 2, 3k - 1, 3k 
\end{pmatrix}$$

From Theorem 3.3 one sees that for any $m \in \text{supp}(\pi^{(1)})$

$$V(x)_m = x_k \left(\frac{1}{3} + \sum_{i=1}^{\infty} \left(\frac{1}{3}P_{ik,m} - \frac{1}{3}\right)x_i \right) = x_k \left(\frac{1}{3}x_k + \sum_{i=1}^{\infty} \frac{1}{3}P_{ik,m}x_i \right)$$
Now, we choose

\[ P_{21,1} = 1, \quad P_{11,1} = 0, \quad P_{22,1} = 0 \quad \forall \ i \in \mathbb{N}\setminus\{1, 2\}; \quad P_{31,2} = 1, \quad P_{41,2} = 1, \quad P_{43,2} = 1, \quad P_{5,j,2} = 0 \quad \forall \ i, j \notin \{1, 3, 4\}; \]

\[ P_{51,3} = 1, \quad P_{61,3} = 1, \quad P_{65,3} = 0, \quad P_{11,3} = 1, \quad i \geq 7; \quad P_{22,7} = 1, \quad i \geq 3; \quad P_{33,10} = 1, \quad i \geq 5; \]

\[ P_{i,k,3k+1} = 1, \quad i \geq k + 1 \quad \text{for} \ k \geq 4. \]

One can check the operator above is well-defined. Further, for any \( x = (x_1, \ldots, x_n, \ldots) \in S \)

\( V(x) \) can be written in the following form

\[ V(x)_1 = 2x_1x_2; \quad V(x)_2 = 2(x_1x_3 + x_3x_4 + x_1x_4); \quad V(x)_3 = 2x_1 \left( x_5 + \sum_{i=6}^{\infty} x_i \right); \]

\[ V(x)_4 = \frac{1}{3}x_1^2; \quad V(x)_5 = \frac{1}{3}x_1^2; \quad V(x)_6 = \frac{1}{3}x_1^2; \quad V(x)_7 = x_2 \left( \frac{1}{3}x_2 + \sum_{i=3}^{\infty} x_i \right); \quad V(x)_8 = \frac{1}{3}x_2^2; \quad V(x)_9 = \frac{1}{3}x_2^2; \]

\[ V(x)_{10} = x_3 \left( \frac{1}{3}x_3 + \sum_{i=5}^{\infty} x_i \right); \quad V(x)_{11} = \frac{1}{3}x_3^2; \quad V(x)_{12} = \frac{1}{3}x_3^2; \]

\[ V(x)_{3k+1} = x_k \left( \frac{1}{3}x_k + \sum_{i=k+1}^{\infty} x_i \right); \quad V(x)_{3k+2} = \frac{1}{3}x_k^2; \quad V(x)_{3k+3} = \frac{1}{3}x_k^2; \]

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