Optimal Fuel Saving in 4D Waypoint Networks
Kawser Ahmed, Milca de Freitas Coelho, and Kouamana Bousson
University of Beira Interior

Abstract
The purpose of this work is to develop a trajectory optimization method that generates a fuel optimal trajectory from a predefined 4D waypoint networks, where the arrival time is specified for each waypoint in the network. A single source shortest path algorithm is presented to generate the optimal flight trajectory that minimizes fuel burn. Generating such trajectories enables the airlines to cope with increasing fuel costs and to reduce aviation induced climate change, as emission is directly related to the amount of fuel burn. Two case studies were considered and the simulation results showed that flying a fuel optimal trajectory based on the proposed algorithm leads to a reduction of average fuel consumption on international flights by 2-4% compared with the conventional trip fuel.

Keywords: Fuel saving, Cost index, 4D trajectory optimization, Waypoint network, Dijkstra’s algorithm

1. Introduction
Improving aircraft operational efficiency has become a dominant topic in air transportation, as the airlines around the world have seen the price of fuel has risen sharply during the last decades. Currently, air transportation accounts for about 2% of total global CO₂ emissions and about 12% of the CO₂ from all transportation source [1]. The increased fuel prices and environmental concerns have pushed airlines to reduce fuel consumption and to find margins for performance improvements. Efforts to modernize the aircraft fleet are limited by an extremely slow and expensive process of new aircraft adoption, which can take decades, therefore it is important to find different alternatives to reduce the fuel consumption in current aircraft, which will likely to share the sky with most modern aircraft in near future. One of these alternatives is to optimize flight trajectories and traffic control procedure. The existing flight planning techniques are suboptimal. Hence, a fuel optimal flight path can significantly save fuel.

A practical solution that reduces the cost associated with time and fuel consumption during flight is the Cost Index (CI). The value of the CI reflects the relative effects of fuel cost on overall trip cost as compared to time-related direct operating cost. For all aircraft models, the minimum value of cost index equal to zero results in maximum range...
airspeed and minimum trip fuel, but this configuration ignores the time cost. If the cost index is maximum, the flight time is minimum, the velocity and the Mach number are maximum, but ignores the fuel cost [2]. In this study, the Cost Index assumes to be zero as only fuel cost is taken into consideration.

\[ CI = \frac{\text{TimeCost} \sim (€/hr)}{\text{FuelCost} \sim (€/kg)} \]  

Recent studies propose that, during the take-off and climb phase of the flight, accelerating and flap retraction at a lower altitude than the typical 3000 ft decrease the fuel consumption, lower flap setting cause low drag, resulting less fuel burn during climb, it also suggest that descending at a higher slope angle than 30 enable the aircraft to save fuel [3], [4]. By improving the cruise speed and altitude profiles is possible to reduce fuel burn in cruise phase, Hagelauer and Mora-Camino [3] conducted a study based on a constant value of Cost Index for a given arrival time, in order to find the optimum cruise speed and altitude profile. An alternative way to conserve fuel in current aircraft is by flying optimal trajectories. The trajectory optimization problem can be solved by various kinds of methods, however, these methods can be classified into two basic approaches: the indirect approach and the direct approach [5], [6].

The trajectory optimization problem is solved by the pontryagin maximum principle [7] in the indirect approach, where the original optimal control problem is converted into Eular- Lagrange system (boundary value problem) by formulating the first-order necessary condition which derived from pontryagin maximum principle. Generally, the indirect approach leads to more accurate results than the direct approach. However, in general, a rather good initial approximation of the co-state equation is required in order to convergence, which is quite difficult to guess as the physical meaning of co-estate equations are not well established [8]. Besides for many practical optimal control problems, these boundary values problems are quite difficult to solve, because of complex dynamics and constraints structure, which results in two-point boundary value problem (TPBVPs), it demands computationally intensive iterative numerical procedures.

On the other hand, the direct approach is based on the transformation of optimal control problem into a parameter optimization problem [9]. Which is done by discretizing the infinite-dimensional problem into a finite-dimensional problem and later on solving it by the nonlinear programming. Direct methods tend to have better convergence properties over indirect methods. Another great advantage of direct methods is that they do not have to deal with the co-state equation. The parameterization techniques have an important role in the convergence and accuracy of the solution. The most known direct approaches are based on Runge-Kutta scheme [10] and collocation methods [11]. Recently, some works have been presented for higher nonlinear dynamic system called
a Chebyshev pseudo-spectral method [12], [13], [14]. That procedure is based on the approximation of both controls and state by interpolating polynomials at the Chebyshev nodes. However experimental results show that the approximation of controls by higher-order polynomials give rise to excessive wavy curves for the states.

Recently Some research activities have been done for 4D optimal trajectory generation. Bousson and Gameiro [15] presented a quintic spline approach for 4D trajectory generation for UAVs. Boukraa, Bestaoui and Azouz [16] proposed a 3D optimal trim trajectories planner algorithm to generate trajectories for a set of predefined waypoints in space. Ahmed and Bousson [17] generated a time-optimal trajectory from 4D predefined networks.

In this present paper, applying shortest path algorithms in graph theory, an optimal trajectory has been approximated by the path that minimizes the total link cost connecting the origin and destination in a pre-defined network. The graph methods often require large computation time and memory space but guarantee global optimal solutions. In this paper, the single source shortest path algorithm was used to generate the fuel optimal trajectory.

This study is restricted to the climb, cruise and descent phases of the flight and ignores the take-off and landing approach, and assuming the initial and final waypoints are at an altitude of 3000 feet, where, in the initial waypoint the aircraft begins the climb phase and in the final waypoint the aircraft begins the landing approach. This work primarily attempts to quantify benefits of fuel optimal trajectory which was found by implying the Dijkstra’s shortest path algorithm. In this work, a benefit is meant to imply a reduction in fuel burn due to using the Dijkstra’s shortest path algorithm to the actual unimproved flight.

2. Problem Formulation

The main goal of this paper is to find a fuel optimal path from a predefined 4D waypoint networks. A representation of waypoint networks is shown in Figure 1, where \( P_1 \) is the initial waypoint and \( P_N \) is the final waypoint of the networks.

Most of the approaches consider the waypoints defined by tri-dimensional coordinate positions. \( P_k = (\lambda_k, \varphi_k, h_k)^T \) where, \( k = 1, 2, ..., i, j, ..., N \) and do not take into account the time. By adding the arrival time restriction to the tri-dimensional waypoint it is possible to define the 4D waypoints as \( P_k = (\lambda_k, \varphi_k, h_k, \tau_k)^T \). Where, \( \lambda_k, \varphi_k, h_k, \tau_k \) are respectively longitude, latitude, altitude and arrival time at waypoint \( P_k \).
As trajectory generation requires a geocentric coordinates system, the 4D waypoints need to be transformed from the accustomed geodetic coordinate system to geocentric coordinates. Now to transform the geodetic coordinates the following equations need to be applied [18].

\[x_j = (N_j + h_j) \cos \phi_j \cos \lambda_j\]  
\[y_j = (N_j + h_j) \cos \phi_j \sin \lambda_j\]  
\[z_j = [N_j(1 - e^2) + h_j] \sin \phi_j\]

Being \(a\) the Earth semi-major axis and \(e\) its eccentricity, \(N_j\) can be calculated as follows:

\[N_j = \frac{a}{\sqrt{1 - e^2 \sin^2 \phi_j}}\]

Now the 4D waypoints can be demonstrated in geocentric coordinates as follows:

\[P_j = (x_j, y_j, z_j, \tau_j)^T\]

The problem to be solved is to navigate the aircraft along with 4D waypoints as in Eq. (6) starts from the initial waypoint \(P_1\) to the final waypoint \(P_N\) such that it minimizes the total fuel consumption by the aircraft. The performance index to be minimized in this problem can be written in the integral form as:

\[J = \int_{\tau_0}^{\tau_f} (f + CI^* \tau) dt\]

Where, \(f\) and \(\tau\) represent the fuel burn and flight time of the full trajectory from waypoint \(P_1\) to waypoint \(P_N\). \(CI^*\) is the cost index as in Eq. (1), it is an adjustable parameter which is chosen by the airlines to balance the fuel and time costs. In this problem, the Cost Index assumes to be zero as only fuel cost is taken into consideration.
The following section proposes a method that will determine the fuel optimal path along with specified waypoints from a 4D waypoint network by implying the Dijkstra's shortest path algorithm.

3. Proposed Method

To generate a fuel optimal trajectory from a set of waypoints in 4D waypoint network requires finding the associated fuel consumed the other, defined as: $df_k$ by the aircraft to go from one waypoint to

$$df_k = f_{nom} \times d\tau_k$$  \hspace{1cm} (8)

Where, $f_{nom}$ [kg/min] is the nominal fuel flow rate, $df_k$ [kg] is the amount of fuel consumed and $d\tau_k$ is the amount of time needed by the aircraft to go from waypoints $P_{k-1}$ to $P_k$ and, which can be described in the following equations:

$$df_k = f_k - f_{k-1}$$  \hspace{1cm} (9)

$$d\tau_k = \tau_k - \tau_{k-1}$$  \hspace{1cm} (10)

Where, $f_k$ [kg] and $\tau_k$ [min] are respectively the fuel burn and flight time required to get to waypoint $P_k$ from initial waypoint. The nominal fuel flow rate $f_{nom}$, can be estimated by the thrust and thrust specific fuel consumption as follows:

$$f_{nom} = \eta \times T/hr$$  \hspace{1cm} (11)

However, the $f_{nom}$ varies with specific aircraft and with different flight phases, as the thrust in Eq.(11) is different in different phases of flight. The Base of Aircraft Data (BADA) model provides coefficients that allow to calculate the thrust specific fuel consumption $\eta$ and different thrust level $T/hr$, which can be used to calculate the $f_{nom}$ in different phases of the flight [19], [20].

4. Dijkstra's Algorithm

Dijkstra's algorithm, was first proposed by the Dutch computer scientist Edsger Dijkstra in 1956 and published in 1959, is the most well-known shortest path algorithm. This is a graph search algorithm that solves the single-source shortest path problem for a graph with non-negative edge path costs, producing a shortest-path tree. The most common variant of the algorithm fixes one vertex as the source and another as the destination vertex and find the shortest path between them.
Dijkstra’s algorithm solves the single-source shortest-paths problem on a weighted, directed graph \( G(V, E) \) where \( V \) is a set of vertices and \( E \) is a set of edges on the graph. This algorithm requires 3 variables as input in order to find the path with the lowest cost between the source and destination vertices, they are respectively the graph, the source vertex, and the destination vertex, and at the end, it returns a reduced graph as output.

This algorithm will determine the global optimal (best route to take), given a number of vertices and edges as long as it has the graph as an input, no matter how large the graph is. In addition to the basic formulation of Dijkstra's algorithm, the following aspects must be defined specifically for the flight trajectory optimization problem. The number of vertices \( V \), the edges \( E \) between the vertices and the source and destination vertices. In this paper, the waypoints of the 4D waypoint networks are the vertices \( V \), the initial waypoint is the source vertex \( s \), the final waypoint is the destination vertex and the associated travel time \( d_{r_k} \) by the aircraft between the pairs of waypoints are the edges \( E \) between these vertices (waypoints).

In Figure 2 a full execution of the Dijkstra’s shortest path algorithm operation is shown. The circles represent the vertices or nodes and the lines with arrows are the edges. Each edge has a non-negative cost associated with it. The problem is to find the most cost-efficient route from the source vertex to any other vertex.

![Figure 2: The execution of Dijkstra's algorithm](image)

In this example, the source vertex \( s \) is the leftmost vertex. The value with low-cost estimates appear within the vertices, and shaded edges indicate predecessor values. Black vertices are already examined thus they have the value of the lowest cost associated with them to go from the source vertex, and the white vertices are going to be examined. The first step (a) shows the situation just before the first iteration of the
while loop. Form step (b) to step (f) shows the situation after each successive iteration of the while loop. The value of lowest cost and predecessors shown in last step (f), and these are the final values of the lowest cost to go to that vertex from the source vertex [21–23].

5. Modeling of 4D Waypoints Network

The following differential equations are the dynamic model used to model the problem:

\[
\dot{x} = V \cos \gamma \cos \psi \\
\dot{y} = V \cos \gamma \sin \psi \\
\dot{z} = V \sin \gamma \\
\dot{V} = u_1 \\
\dot{\gamma} = u_2 \\
\dot{\psi} = u_3
\]

where, (x, y, z) are the geocentric coordinate system, the V, \( \gamma \), and \( \psi \) are the velocity, flight path angle, and heading respectively, the variables \( u_1 \), \( u_2 \), and \( u_3 \) are respectively the acceleration, the flight path angle rate, and the heading rate. The state and control vectors are composed by \( X = [x, y, z, V, \gamma, \psi] \) and \( U = [u_1, u_2, u_3] \) respectively. Considering the following constraints: Due to aerodynamic, structural and propulsive limitations, bound constraints are imposed on the state and control vectors as follow:

\[
V_{\text{min}} \leq V \leq V_{\text{max}}
\]

\[
\gamma_{\text{min}} \leq \gamma \leq \gamma_{\text{max}}
\]

\[
\psi_{\text{min}} \leq \psi \leq \psi_{\text{max}}
\]

\[
u_{i}^{\text{min}} \leq u_i \leq u_{i}^{\text{max}}, \ i = 1, 2, 3
\]
6. Simulation and Result

In this section, the simulation and result of the fuel optimal trajectories are presented for two different case studies. In the first example, a short-haul flight Lisbon to Geneva and in the second example a medium-haul flight Lisbon to Stockholm were considered. In both examples, the fuel optimal trajectories were generated by using Dijkstra’s algorithm. All the analysis of the simulation has been done using Matlab 2016.

Example 1

This subsection presents the simulation and results of example 1 where a short-haul flight, Lisbon to Geneva was considered. The 4D waypoint network of this short-haul flight consists of two trajectories, and has total of 22 waypoints including the initial and final waypoints, and each trajectory has 12 waypoints including the initial and final waypoints.

| Waypoint | \(x\) [m] | \(y\) [m] | \(z\) [m] | \(d_\tau\) [min] | \(df\) [kg] |
|----------|-----------|-----------|-----------|----------------|-------------|
| Initial (P1) | 2647.235288 | -421.992558 | 2155.785264 | 0 | 0 |
| P2 | 2644.86006 | -414.5681541 | 2161.780769 | 2.371751 | 291.0612 |
| P3 | 2639.688421 | -400.346416 | 2173.301325 | 3.437191 | 348.0156 |
| P4 | 2626.183424 | -359.68012 | 2199.831101 | 7.295251 | 535.3803 |
| P5 | 2617.384267 | -321.3434642 | 2217.611029 | 5.748997 | 294.6361 |
| P6 | 2513.665622 | -148.0478476 | 2351.006175 | 32.48854 | 1176.085 |
| P7 | 2455.327051 | 5.585254511 | 2415.971579 | 23.71969 | 858.6528 |
| P8 | 2440.964861 | 153.4156637 | 2425.576695 | 19.97808 | 723.2065 |
| P9 | 2432.450512 | 164.6407432 | 2431.936166 | 2.058735 | 782.3191 |
| P10 | 2415.587867 | 198.5338065 | 2443.073738 | 5.71202 | 32.05871 |
| P11 | 2404.139607 | 224.5489423 | 2449.734117 | 5.275779 | 42.20624 |
| Final (P22) | 2389.738702 | 240.8587387 | 2460.555523 | 5.618537 | 54.83692 |
| Total | 113.7046 | 4363.963 | |

Boeing 737-700 (B737) aircraft was used to analyze the flight trajectories. (Table 1 and 2) Show the waypoints lists for both of the trajectories. Each waypoint is defined in geocentric coordinates \((x, y, z)\), the travel time \(d_\tau\) and consumed fuel \(df\) between the waypoints are also shown. To find the fuel optimal trajectory from the 4D waypoint network possible connection between waypoints in both trajectories were established,
and their travel time $\tau_k$ and consumed fuel $df_k$ between these possible waypoints connections were calculated.

### Table 2: List of waypoints in 2nd trajectory for short-haul flight

| Waypoint   | $x$ [m]          | $y$ [m]          | $z$ [m]          | $\tau_k$ [min] | $df_k$ [kg] |
|------------|------------------|------------------|------------------|----------------|-------------|
| Initial (P1) | 2647.235288       | -421.1992558     | 2155.785264      | 0              | 0           |
| P12        | 2646.148436       | -411.3007568     | 2160.834393      | 2.862798       | 351.3225    |
| P13        | 2641.765637       | -394.8066635     | 2171.800352      | 3.66612        | 371.1947    |
| P14        | 2631.351084       | -350.7310454     | 2195.127395      | 7.374258       | 541.1783    |
| P15        | 2624.264915       | -308.9768094     | 2211.270337      | 6.036433       | 309.3672    |
| P16        | 2581.499623       | -79.54841825     | 2280.219078      | 32.6647        | 1182.462    |
| P17        | 2485.884636       | 24.1455534       | 2384.614747      | 23.55432       | 852.6664    |
| P18        | 2445.165301       | 155.0221788      | 2421.268939      | 19.04446       | 689.4095    |
| P19        | 2437.263718       | 170.4015841      | 2426.749665      | 2.415762       | 9.179894    |
| P20        | 2421.774537       | 205.9872305      | 2436.367639      | 5.787862       | 32.48437    |
| P21        | 2407.152816       | 231.6764483      | 2446.133024      | 5.625951       | 45.00761    |
| Final (P22)| 2389.738702       | 240.8587387      | 2460.555523      | 2.415762       | 852.6664    |
| Total      |                  |                  |                  | 114.6753       | 4439.345    |

The fuel optimal trajectory was generated from the 4D waypoint network using the Dijkstra’s shortest path algorithm. The fuel optimal trajectory contains 9 waypoints (initial (P1) → P2 → P3 → P4 → P5 → P18 → P19 → P11 → final (P22)). The comparison of fuel consumed in different phases of flight for these two trajectories and fuel optimal trajectory are shown in (Table 3).

### Table 3: Fuel consumed from initial to the final waypoint in different trajectories for short-haul flight.

| Trajectory | Fuel consumed [kg] | Total [kg] |
|------------|--------------------|------------|
|            | Climb              | Cruise     | Descent     |               |
| 1          | 1469.1             | 2757.9     | 136.9       | 4363.9        |
| 2          | 1573.1             | 2724.5     | 141.7       | 4439.3        |
| Fuel optimal | 1469.1             | 2652.8     | 136.1       | 4258          |

As seen in (Table 3), by using the fuel optimal trajectory for the short-haul flight (Lisbon – Geneva) the aircraft consumes 105.9 kg of less fuel than the first trajectory, which is equivalent to 2.4% less fuel than the first trajectory and consumes 181.3 kg of less fuel than the second trajectory, which is equivalent to 4.1% less fuel than the second trajectory. The fuel optimal trajectory in 3D is shown in (Figure 3) where, the fuel optimal trajectory is represented by the blue line and the red circles around the trajectory are the waypoints.
Example 2

In this example a medium-haul flight, Lisbon to Stockholm was considered. There are also two trajectories between the initial and final waypoints in the 4D waypoint network, each trajectory has 13 waypoints including the initial and final waypoints, and total 24 waypoints are there in the 4D waypoint network including the initial and final waypoint. Boeing 777-200 (B772) aircraft was used to analyze the flight trajectories. (table 4 and 5) Show the waypoints lists for both of the trajectories.

Table 4: List of waypoints in 1st trajectory for medium-haul flight

| Waypoint | x[m]      | y[m]      | z[m]      | $\Delta t_k$ [min] | $d_f^k$ [kg] |
|----------|-----------|-----------|-----------|--------------------|--------------|
| Initial  | 2647.235288 | -421.1992558 | 2155.785264 | 0                  | 0            |
| P2       | 2643.010993 | -415.6490463 | 2163.819992 | 2.665515           | 1070.631     |
| P3       | 2628.576005  | -399.552819 | 2187.815353 | 5.529098           | 1724.94      |
| P4       | 2599.967218  | -365.4324126 | 2230.633603 | 8.291856           | 1859.863     |
| P5       | 2586.230618  | -353.564446  | 2249.330174 | 3.243938           | 589.0991     |
| P6       | 2385.564946  | -198.309008  | 2477.014631 | 42.43539           | 5385.051     |
| P7       | 2238.11589   | 42.12084785  | 2617.270998 | 39.21055           | 4975.819     |
| P8       | 2030.506669  | 324.3682322  | 2761.81912  | 47.18143           | 5987.324     |
| P9       | 1782.896372  | 459.677681   | 2908.641706 | 39.59531           | 5024.645     |
| P10      | 1772.18145   | 466.7142505  | 2913.251798 | 1.695762           | 31.54117     |
| P11      | 1731.625729  | 489.6688411  | 2931.15743  | 6.807662           | 161.8522     |
| P12      | 1690.948931  | 517.3407314  | 2947.282952 | 9.043191           | 287.5735     |
| Final    | 1676.867259  | 536.4414256  | 2950.540034 | 5.538229           | 212.0034     |
| Total    |            |            |           | 211.2379           | 27310.34     |
The fuel optimal trajectory contains 9 waypoints [initial waypoint (P1)→ P2→ P3→ P4→ P5→ P20→ P21→ P23→ final waypoint (P24)], which was generated implying Dijkstra’s algorithm. The comparison of consumed fuel in different phases of flight for different trajectories including the fuel optimal trajectory are shown in (table 6).

From the initial waypoint to reach the final waypoint using the fuel optimal trajectory the aircraft consumes 579.2 kg of less fuel than the first trajectory and consumes 637.4 kg of less fuel than the second trajectory. In another word by using the fuel optimal trajectory for the medium-haul flight, the aircraft consumes 2.1% less fuel than the first trajectory, and 2.3% less fuel than the second trajectory. The fuel optimal trajectory in 3D is shown in (figure 4).

**TABLE 5: List of waypoints in 2nd trajectory for medium-haul flight**

| Waypoint | x[m]       | y[m]       | z[m]       | d\(\tau\) [min] | df\(_f\) [kg] |
|----------|------------|------------|------------|-----------------|---------------|
| Initial (P1) | 2647.235288 | -421.1992558 | 2155.785264 | 0               | 0             |
| P13      | 2645.498539 | -411.4598298 | 2161.594618 | 2.87            | 4146         |
| P14      | 2632.238338 | -391.2773722 | 2184.925324 | 5.74            | 8028         |
| P15      | 2604.511324 | -357.1441614 | 2226.697876 | 8.14            | 48028        |
| P16      | 2592.536805 | -337.3252479 | 2244.591415 | 3.64            | 2769         |
| P17      | 2454.226514 | -11.8794812  | 2417.526975 | 49.00           | 00666       |
| P18      | 2214.11542  | 178.8695328  | 2631.773393 | 46.56           | 6667         |
| P19      | 1954.552046 | 377.266035   | 2809.170706 | 46.27           | 6667         |
| P20      | 1805.357884 | 485.112201   | 2890.720914 | 25.06           | 6374         |
| P21      | 1792.670589 | 493.4471453  | 2896.374754 | 2.01            | 6462         |
| P22      | 1747.783059 | 517.5842066  | 2916.832487 | 7.48            | 8843         |
| P23      | 1699.492739 | 535.9784473  | 2939.082013 | 9.82            | 27318       |
| Final (P24) | 1676.867259 | 536.4414256  | 2950.540034 | 5.86            | 4875         |
| Total    | 212.5119    | 27368.51     |            |                 |               |

The blue curve in (Figure 4) corresponds to the fuel optimal trajectory for the medium-haul flight and the red circles around the curve are the waypoints of the fuel optimal trajectory.

**TABLE 6: Fuel consumed from initial to the final waypoint in different trajectories for medium-haul flight**

| Trajectory | Fuel consumed [kg] | Total [kg] |
|------------|--------------------|------------|
|            | Climb              | Cruise     | Descent   |               |
| 1          | 5244.5             | 21372.8    | 692.97    | 27310.3       |
| 2          | 5435.3             | 21180.6    | 752.6     | 27368.5       |
| Fuel optimal | 5244.5             | 20744.4    | 742.2     | 26731.1       |

The blue curve in (Figure 4) corresponds to the fuel optimal trajectory for the medium-haul flight and the red circles around the curve are the waypoints of the fuel optimal trajectory.
7. Conclusion

This study is based on finding the fuel optimal trajectories of the climb, cruise and descent phases of the flight, but ignores the takeoff and landing phases of the flight. In this work, several steps were made in order to achieve a complete trajectory from a 4D waypoint network that optimizes the fuel consumption. This study uses Dijkstra's shortest path algorithm that finds a fuel optimal trajectory from a given 4D waypoints network, this technique was used to compare different length (short and medium-haul) flights.

The analysis results show promising potential for reduction of consumed fuel in different flights via using the Dijkstra's shortest path algorithm, across a range of common aircraft and routes. The results suggest that by flying fuel optimal trajectory for a short-haul flight, it is possible to save 2.4-4.1% on fuel burn, which is equivalent to 105.9 – 181.3 kilograms of fuel for B737 aircraft. In medium-haul flight by flying the fuel optimal trajectory can potentially save 2.1-2.3% fuel, reducing fuel burn by 579.2 – 637.4 kilograms for B772 aircraft. In general, the savings of the fuel is proportional to the trip lengths, and depends on the aircraft types.

Future work will deal with the extension of the proposed concept with computational intelligence methods such as the A* algorithm, reinforcement learning, and adaptive dynamic programming.
Acknowledgment

This research work was conducted in the Laboratory of Avionics and Control of the University of the Beira interior (Covilhã, Portugal) and supported by the Portuguese Foundation for Sciences and Technology (FCT) and Santander-Totta/UBI doctoral grant and the Aeronautics and Astronautics Research Group (AeroG) of the Associated Laboratory for Energy, Transports and Aeronautics (LAETA).

References

[1] ICAO. ICAO Environmental Report: Aviation and Climate change. ICAO, 2013.
[2] Robenson B. Fuel Conservation Strategies: Cost Index Explained. Boeing, 2007, pp.26-28.
[3] Roberson W. and Johns J. A. Fuel Conservation strategies: Takeoff and Climb. Boeing, 2008, pp. 25-28.
[4] Roberson W. and Johns J. A. Fuel Conservation Strategies: Descent and Approach. Boeing, 2008, pp. 25-28.
[5] Bryson A. E and Ho Y. C. Applied Optimal Control: Optimization, Estimation and, Control. Taylor & Francis, New York, 1975.
[6] Betts, J.T.: “Survey of numerical methods for trajectory optimization” Journal of Guidance, Control, and Dynamics, 1998, pp. 193-207.
[7] Pontryagin, L.S., Boltyanskii V.G., Gamkrelidze R.V., Mishchenko E.F. The Mathematical Theory of Optimal Processes. Wiley-Interscience, New-York, 1962.
[8] Von Stryk, O. and Bulirsch R.: “Direct and indirect methods for trajectory optimization” Annals of Operations Research, 37, 1992, pp. 357-373.
[9] Hull, D.G.: “Conversion of Optimal Control Problems into Parameter Optimization Problems” Journal of Guidance, Control, and Dynamics, 1997, pp. 57-60.
[10] Schwartz, A. and Polak E.: “Consistent approximations for Optimal Control Problems Based on Runge-Kutta Integration” SIAM Journal on Control and Optimization, vol.34, No.4, 1996, pp. 1235-1269.
[11] Hargraves, C.R. and Paris S.W.: “Direct Trajectory Optimization Using Nonlinear Programming and Collocation” Journal of Guidance, Control and Dynamics, vol. 10, No.4, 1987, pp. 338-342.
[12] Bousson, K. Chebyshev pseudospectral trajectory optimization of differential inclusion models. SAE World Aviation Congress, Montreal, Canada, paper no. 2003-01-3044, 2003.
[13] Fahroo, F. and Ross I.M.: “Direct trajectory optimization by a Chebyshev pseudospectral method” Journal of Guidance, Control, and Dynamics, 2002, pp. 160-166.

[14] Bousson K. and Machado P.: "4D Flight Trajectory Generation and Tracking for Waypoint- Based Aerial Navigation" WSEAS transactions on system and control, Vol 8, No 3, July 2013, pp 105-119.

[15] Bousson K, and Gameiro T. A.: " A Quintic Spline Approach to 4D Trajectory Generation for Unmanned Aerial Vehicles" International Review of Aerospace Engineering (IREASE), Vol 8, No 1, 2015.

[16] Boukraa D., Bestaoui Y. and Azouz N.: "Three Dimensional Trajectory Generation for an Autonomous Plane" International Review of Aerospace Engineering (IREASE), Vol 2, No 4, 2014.

[17] Ahmed K, and Bousson K.: " Generating Time Optimal Trajectory from Predefined 4D waypoint Networks" International Review of Aerospace Engineering (IREASE), Vol 10, No 4, 2017.

[18] Seemkooei A. A.: "Comparison of different algorithm to transform geocentric to geodetic coordinates" Survey Review 36, 286 October 2002. pp. 627-632.

[19] Aircraft Performance Summary Tables for the Base of Aircraft Data (BADA). Eurocontrol Experimental Centre, Revision 3.4, June 2002.

[20] User Manual for the Base of Aircraft Data (BADA). Eurocontrol Experimental Centre, Revision 3.9, April 2011.

[21] Cormen T. H, Leiserson C. E, Rivest R. L and Stein C. Introduction to algorithms. London, England: The MIT Press, 2009, pp. 658-659.

[22] Hart C. Graph Theory Topics in Computer Networking. 2013, pp 13-20.

[23] Dasgupta S., Papadimitriou C. H. and Vazirani U. V. Algorithms. McGraw-Hill, New York, July 18, 2006, pp. 112-118.