Pricing and hedging game options in currency models with proportional transaction costs

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Abstract

The pricing, hedging, optimal exercise and optimal cancellation of game or Israeli options are considered in a multi-currency model with proportional transaction costs. Efficient constructions for optimal hedging, cancellation and exercise strategies are presented, together with numerical examples, as well as probabilistic dual representations for the bid and ask price of a game option.

Keywords: game options, Israeli options, proportional transaction costs, currency model, superhedging, optimal exercise.

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1 Introduction

Game options, also known as Israeli options, were first introduced by Kifer (2000) and have been well studied; the recent survey paper by Kifer (2013a) provides a complete chronology and literature review. In addition to being of interest as derivative securities in their own right, game options have also played an important role in the study of other derivatives, for example callable options (e.g. Kühn & Kyprianou 2007) and convertible bonds (e.g. Kallsen & Kühn 2003, Bielecki, Crépey, Jeanblanc & Rutkowski 2008, Wang & Jin 2009).

A game option is a contract between a writer (the seller) and the holder (the buyer) whereby a pre-specified payoff is delivered by the seller to the buyer at the earliest of the exercise time (chosen by the buyer) and the cancellation time (chosen by the seller). If the game option is cancelled before or at the same time as being exercised, then the seller also pays a cancellation penalty to the buyer. A game option is thus essentially an American option with the additional provision that the seller can cancel the option at any time before expiry, thus forcing early exercise at a price (the penalty). In practice, this feature tends to reduce the cost of hedging for both the seller and the buyer, which makes game options an attractive alternative to American options.

It has been well observed that arbitrage pricing of European and American options in incomplete friction-free models and models with proportional transaction costs result in a range of arbitrage-free prices, bounded from below by

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the bid price and from above by the ask price (see e.g. Föllmer & Schied 2002, Bensaid, Lesne, Pagès & Scheinkman 1992, Chalasani & Jha 2001). The same holds true for game options (Kallsen & Kühn 2005, Kifer 2013b).

The pricing and hedging of game options in the presence of proportional transaction costs also share a number of other important properties with their European and American counterparts. (The properties for European and American options mentioned below were all established by Roux & Zastawniak (2015) in a similar technical setting to the present paper.) Firstly, similar to European options, the hedging of game options is symmetric in the sense that the hedging problem for the buyer is exactly the same as the hedging problem for the seller (of a different game option with related payoff). Kifer (2013b) observed this property in a two-asset model.

Kifer (2013b) also showed that the probabilistic dual representations of the bid and ask prices of game options contain so-called randomised stopping times, a feature shared with the ask price of an American option (for which it was first observed by Chalasani & Jha 2001). Randomised (or mixed) stopping times have been studied by Baxter & Chacon 1977 and many others, primarily as an aid to show the existence and properties of optimal ordinary stopping times. Randomised stopping times can be thought of as convex combinations of ordinary stopping times in a well-defined sense. The reason for the appearance of randomised stopping times in the probabilistic dual representations of the bid and ask prices is that, in the presence of transaction costs, the most expensive exercise (cancellation) strategy for the seller (buyer) of a game option to hedge against is not necessarily the same as the exercise (cancellation) strategy that is most attractive to the buyer (seller). As a result, it generally costs the seller (buyer) more to hedge against all exercise (cancellation) strategies than against the best exercise (cancellation) strategy for the buyer. It turns out that the seller (buyer) must in effect be protected against a certain randomised exercise (cancellation) strategy.

Furthermore, similar to a long American option (i.e. the buyer’s case), the pricing and hedging problems for both the buyer and seller of game option are inherently non-convex. Thus ideas beyond convex duality are needed to study these problems. Nevertheless, the link between game options and short American options (i.e. the seller’s case, a convex problem) means that convex duality methods still have an important role to play in establishing the probabilistic dual representations.

In this paper we consider the pricing and hedging of game options in the numéraire-free discrete-time model of foreign exchange markets introduced by Kabanov 1999, where proportional transaction costs are modelled as bid-ask spreads between currencies. This model has been well studied by Kabanov & Stricker 2001, Kabanov, Rásonyi & Stricker 2002, Schachermayer 2004 and others (see also Kabanov & Safarian 2009). The main aim of our work is to present constructive algorithms for computing optimal exercise and cancellation strategies together with optimal hedging strategies for both the buyer and seller of a game option in this model. The algorithmic constructions in this paper are closely related to previously developed algorithms for the pricing and hedging of European and American options under proportional transaction costs (see e.g. Löhne & Rudloff 2014, Roux & Zastawniak 2009, 2013). These existing constructions yield efficient numerical algorithms; in particular they are known to price path-independent op-
tions in polynomial time in recombinant models (which typically have exponentially-sized state spaces). Numerical examples that illustrate the constructions are provided. A secondary aim is to establish probabilistic dual representations for the bid and ask prices of game options. In both these contributions we extend the recent results of Kifer (2013) for game options from two-asset to multi-asset models. Our proofs are rigorous, thus closing two gaps in the arguments of Kifer (2013); see Remark 3.8 and the comments below Proposition 3.12 for further details.

The methods used in this paper come from convex analysis and dynamic programming, and in particular we will use recent results from Roux & Zastawniak (2015) for an American option with random expiration date. The restriction to finite state space is motivated by the desire to produce computationally efficient algorithms for pricing and hedging. The restriction to discrete time is justified by a recent negative result by Dolinsky (2013) that the super-replication price of a game option in continuous time under proportional transaction costs is the initial value of a trivial buy-and-hold superhedging strategy.

The structure of this paper is as follows. Section 2 specifies the currency model with proportional transaction costs, and reviews various notions concerning randomised stopping times and approximate martingales. The main algorithms for pricing and hedging together with theoretical results for the seller’s and buyer’s position are presented in Section 3, with the proofs of all results deferred to Section 4. Section 5 concludes the paper with two numerical examples.

2 Preliminaries

2.1 Proportional transaction costs

The numéraire free currency model of Kabakov (1999) has discrete trading dates \( t = 0, \ldots, T \) and is based on a finite probability space \((\Omega, F, P)\) with filtration \((F_t)_{t=0}^T\). The model contains \(d\) currencies (or assets), and at any time \(t\), one unit of currency \(j = 1, \ldots, d\) may be obtained by exchanging \(\pi_{ij}^t > 0\) units of currency \(i = 1, \ldots, d\). We assume that \(\pi_{ii}^t = 1\) for \(i = 1, \ldots, d\), i.e. every currency may be freely exchanged for itself.

Assume that the filtration \((F_t)_{t=0}^T\) is generated by \((\pi_{ij}^t)_{t=0}^T\) for \(i, j = 1, \ldots, d\), and assume for simplicity that \(F_0 = \{\emptyset, \Omega\}\), that \(F_T = F = 2^\Omega\) and that \(P(\omega) > 0\) for all \(\omega \in \Omega\). Write \(\mathcal{L}_\tau\) for the family of \(F_\tau\)-measurable \(\mathbb{R}^d\)-valued random variables for every stopping time \(\tau\), and write \(\mathcal{L}_\tau^+\) for the family of non-negative random variables in \(\mathcal{L}_\tau\).

Let \(\Omega_t\) be the set of atoms of \(F_t\) for \(t = 0, \ldots, T\). The elements of \(\Omega_t\) are called the nodes of the model at time \(t\). A node \(\nu \in \Omega_{t+1}\) is called a successor to a node \(\mu \in \Omega_t\) if \(\nu \subset \mu\). The collection of successors of \(\mu\) is denoted \(\text{succ} \mu\). We shall implicitly and uniquely identify random variables \(f\) in \(\mathcal{L}_t\) with functions \(\mu \mapsto f^\mu\) on \(\Omega_t\), and likewise every set \(A \in \mathcal{L}_t\) that we will consider will be uniquely identified by the collection of projections

\[ A^\mu := \{f^\mu : f \in A\} \subseteq \mathbb{R}^d \text{ for all } \mu \in \Omega_t. \]

A portfolio \(x = (x^1, \ldots, x^d) \in \mathcal{L}_\tau\) is called solvent at a stopping time \(\tau\) if it can be exchanged into a portfolio in \(\mathcal{L}_\tau^+\) without any additional investment, i.e. if
there exist non-negative $\mathcal{F}_\tau$-measurable random variables $\beta_{ij}$ for $i, j = 1, \ldots, d$ such that
\[
x^i + \sum_{j=1}^d \beta_{ji} - \sum_{j=1}^d \beta_{ij} \pi_{ij} \geq 0 \text{ for all } i = 1, \ldots, d.
\]
Write $\mathcal{K}_\tau$ for the family of solvent portfolios at time $\tau$; then the solvency cone $\mathcal{K}_\tau$ is the convex cone generated by the canonical basis $e^1, \ldots, e^d$ of $\mathbb{R}^d$ and the vectors $\pi_{ij} e^i - e^j$ for $i, j = 1, \ldots, d$. Observe that $\mathcal{K}_\tau$ is a polyhedral cone, hence closed.

A self-financing trading strategy $y = (y_t)_{t=0}^T$ is an $\mathbb{R}^d$-valued predictable process with initial endowment $y_0 \in \mathcal{L}_0$ satisfying $y_t - y_{t+1} \in \mathcal{K}_t$ for all $t = 0, \ldots, T-1$. Denote the family of self-financing trading strategies by $\Phi$.

A self-financing trading strategy $y = (y_t) \in \Phi$ is called an arbitrage opportunity if $y_0 = 0$ and there exists some $x \in \mathcal{L}_T^+ \setminus \{0\}$ such that $y_T - x \in \mathcal{K}_T$. This definition of arbitrage is consistent with (though formally different to) that of Schachermayer (2004) and Kabanov & Stricker (2001) (who called it weak arbitrage).

For any non-empty convex cone $A \subseteq \mathbb{R}^d$, write $A^*$ for the positive polar of $A$, i.e.
\[
A^* := \{ x \in \mathbb{R}^d : x \cdot y \geq 0 \text{ for all } y \in A \}.
\]

**Theorem 2.1** (Kabanov & Stricker (2001)). The model is free of arbitrage if and only if there exists a probability measure $\mathbb{P}$ equivalent to $\mathbb{P}$ and an $\mathbb{R}^d$-valued $\mathbb{P}$-martingale $S = (S_t)_{t=0}^T$ such that $S_t \in \mathcal{K}_t^* \setminus \{0\}$ for $t = 0, \ldots, T$.

Any pair $(\mathbb{P}, S)$ satisfying the conditions of Theorem 2.1 is called an equivalent martingale pair. Denote the family of equivalent martingale pairs by $\mathcal{P}$; then $\mathcal{P} \neq \emptyset$ in the absence of arbitrage.

**Remark 2.2.** Theorem 2.1 and the other results in this paper can equivalently be formulated in terms of consistent pricing processes $(Z_t)_{t=0}^T$ where
\[
Z_t = S_t \mathbb{E}_\mathbb{P} \left( \frac{d\mathbb{P}}{d\mathbb{P}} \bigg| \mathcal{F}_t \right) \text{ for all } t = 0, \ldots, T.
\]

Schachermayer (2004, pp. 24–25) provides further details on this equivalence.

Assume for the remainder of this paper that the model contains no arbitrage.

### 2.2 Randomised stopping times

**Definition 2.3** (Randomised stopping time). A randomised (or mixed) stopping time $\chi = (\chi_t)_{t=0}^T$ is an adapted nonnegative process satisfying
\[
\sum_{t=0}^T \chi_t = 1.
\]

Denote the set of randomised stopping times by $\mathcal{X}$, and the set of (ordinary) stopping times by $\mathcal{T}$. Every stopping time $\tau \in \mathcal{T}$ corresponds to a randomised stopping time $\chi^\tau = (\chi_t^{\tau})_{t=0}^T$ defined as
\[
\chi_t^{\tau} := 1_{\{\tau = t\}} \text{ for } t = 0, \ldots, T,
\]
where \(1\) is the indicator function on \(\Omega\). The set \(X\) is the convex hull of \(\{ \chi^\tau : \tau \in T \}\) and so \(X\) can be thought of as the linear relaxation of the set of ordinary stopping times in this sense.

Fix any process \(A = (A_t)_{t=0}^T\) and randomised stopping time \(\chi \in \mathcal{X}\). The value of \(A\) at \(\chi\) is defined as

\[
A_\chi := \sum_{t=0}^T \chi_t A_t.
\]

Define the processes \(\chi^* = (\chi^*_t)_{t=0}^T\) and \(A^{\chi^*} = (A^{\chi^*_t})_{t=0}^T\) by

\[
\chi^*_t := \sum_{s=t}^T \chi_s, \quad A^{\chi^*_t} := \sum_{s=t}^T \chi_s A_s
\]

for \(t = 0, \ldots, T\). For convenience also define \(\chi^*_{T+1} := 0\) and \(A^{\chi^*}_{T+1} := 0\). Observe that \(A_\chi = A^{\chi^*_0}\) and that \(\chi^*\) is a predictable process since

\[
\chi^*_t = 1 - \sum_{s=0}^{t-1} \chi_s \text{ for } t = 1, \ldots, T.
\]

Note moreover that if \(\chi = \chi^\tau\) for some \(\tau \in T\), then

\[
\chi^*_t = 1_{\{\tau \geq t\}}, \quad A^{\chi^*}_t = \sum_{s=t}^T 1_{\{\tau = s\}} A_s = A_\tau 1_{\{\tau \geq t\}}
\]

for \(t = 0, \ldots, T\), and in particular \(A^{\chi^\tau}_\tau = A_\tau\).

**Definition 2.4** (Approximate martingale pair). Fix any \(\chi \in \mathcal{X}\). A pair \((\mathbb{P}, S)\) consisting of a probability measure \(\mathbb{P}\) and an adapted \(\mathbb{R}^d\)-valued process \(S\) is called a \(\chi\)-approximate martingale pair if

\[
S_t \in K^*_t \setminus \{0\}, \quad \mathbb{E}_\mathbb{P}(S^{\chi^*_t+1}_t | F_t) \in K^*_t
\]

for all \(t = 0, \ldots, T\). If \(\mathbb{P}\) is in addition equivalent to \(P\), then \((\mathbb{P}, S)\) is called a \(\chi\)-approximate equivalent martingale pair.

Denote the family of \(\chi\)-approximate equivalent pairs \((\mathbb{P}, S)\) by \(\mathcal{P}(\chi)\) and the set of \(\chi\)-approximate pairs by \(\bar{\mathcal{P}}(\chi)\). For any \(\chi \in \mathcal{X}\) and \(i = 1, \ldots, d\) define

\[
\mathcal{P}^i(\chi) := \{ (\mathbb{P}, S) \in \mathcal{P}(\chi) : S^i_t = 1 \text{ for } t = 0, \ldots, T \},
\]

\[
\bar{\mathcal{P}}^i(\chi) := \{ (\mathbb{P}, S) \in \bar{\mathcal{P}}(\chi) : S^i_t = 1 \text{ for } t = 0, \ldots, T \}.
\]

Since \(\mathcal{P} \subseteq \mathcal{P}(\chi) \subseteq \bar{\mathcal{P}}(\chi)\) and \(K^*_t\) is a cone for all \(t = 0, \ldots, T\), the no-arbitrage assumption implies that \(\mathcal{P}^i(\chi)\) and \(\bar{\mathcal{P}}^i(\chi)\) are non-empty.

**Definition 2.5** (Truncated stopping time). Fix any \(\chi \in \mathcal{X}\), \(\sigma \in T\). The truncated randomised stopping time \(\chi \wedge \sigma = ((\chi \wedge \sigma)_t)_{t=0}^T\) is defined as

\[
(\chi \wedge \sigma)_t := \chi_t 1_{\{t < \sigma\}} + \chi^*_t 1_{\{t = \sigma\}} \text{ for } t = 0, \ldots, T.
\]
The process $\chi \land \sigma$ is clearly adapted and nonnegative, and moreover
\[ \sum_{t=0}^{T}(\chi \land \sigma)_t = \sum_{t=0}^{\sigma - 1} \chi_t + \sigma^* = 1, \]
so it is indeed a randomised stopping time. If $\chi = \chi^\tau$ for some $\tau \in T$, then clearly $(\chi \land \sigma)_t = 1_{(\sigma \land \tau = t)}$ for all $t = 0, \ldots, T$, and so $\chi^\tau \land \sigma = \chi^{\tau \land \sigma}$. Denote the set of randomised stopping times truncated at $\sigma \in T$ by $\mathcal{X} \land \sigma := \{ \chi \land \sigma : \chi \in \mathcal{X} \}$.

3 Main results and discussion

In this section we formally define what we mean by a game option, and present the constructions and main results.

**Definition 3.1 (Game option).** A game option is a derivative security that is exercised at a stopping time $\tau \in T$ chosen by the buyer and cancelled at a stopping time $\sigma \in T$ chosen by the seller. At time $\sigma \land \tau$ the buyer receives the payoff $Q_{\sigma \tau}$ from the seller, where
\[ Q_{st} \equiv Q_{st}^{Y,X,X'} := Y_t 1_{\{s > t\}} + X_s 1_{\{s < t\}} + X'_s 1_{\{s = t\}} \]
for all $s, t = 0, \ldots, T$ (3.1) and $Y = (Y_t)_{t=0}^T$, $X = (X_t)_{t=0}^T$ and $X' = (X'_t)_{t=0}^T$ are adapted $\mathbb{R}^d$-valued processes such that
\[ X_t - X'_t \in \mathcal{K}_t, \quad X'_t - Y_t \in \mathcal{K}_t \quad (3.2) \]
for all $t = 0, \ldots, T$.

In the event that the buyer exercises before the option is cancelled, i.e. on $\{ \tau < \sigma \}$, the buyer receives the payoff $Y_\tau$ from the seller at his exercise time $\tau$. If the seller cancels the option before it is exercised, i.e. on $\{ \sigma < \tau \}$, the seller is required to deliver the payoff $X_\sigma$ to the buyer at the cancellation time $\sigma$, which consists of $Y_\sigma$ and a penalty
\[ X_\sigma - Y_\sigma = (X_\sigma - X'_\sigma) + (X'_\sigma - Y_\sigma) \in \mathcal{K}_{\sigma}. \]

In the event that the option is exercised and cancelled simultaneously, i.e. on $\{ \sigma = \tau \}$, the seller pays $X'_\sigma$ to the buyer, consisting of $Y_\sigma$ and a penalty $X'_\sigma - Y_\sigma \in \mathcal{K}_{\sigma}$. The assumptions (3.2) mean that, at any time $t$, the portfolio $X'_t$ payable on cancellation is at least as attractive to the buyer as the portfolio $X'_t$ payable in the event of simultaneous cancellation and exercise, which in turn is at least as attractive as the portfolio $Y_t$ payable on exercise. It is therefore clear from Definition 3.1 that a game option is essentially an American option with payoff process $Y$ with the additional feature that it may be cancelled by the seller at any time (upon payment of a cancellation penalty).

**Remark 3.2.** Definition 3.1 is slightly more general than the usual approach followed in the literature (see e.g. Kifer 2000, 2013), where the standard assumption is that no penalty is paid if cancellation and exercise takes place simultaneously (i.e. $X'_t = Y_t$ for all $t$) and that no penalty is paid on maturity (i.e. $X_T = X'_T = Y_T$). The main motivation for the generalization in the present paper is that it enables exploitation of the symmetry between the seller’s and buyer’s hedging problems; see Proposition 3.12 below.
### 3.1 Pricing and hedging for the seller

A hedging strategy for the seller of a game option \((Y, X, X')\) comprises a cancellation time \(\sigma\) and a self-financing trading strategy \(y\) that allows delivery of the payoff without risk, for every exercise time chosen by the buyer.

**Definition 3.3 (Hedging strategy for the seller).** A hedging strategy for the seller is a pair \((\sigma, y) \in T \times \Phi\) satisfying

\[
y_{\sigma \wedge \tau} - Q_{\sigma \wedge \tau} \in K_{\sigma \wedge \tau} \quad \text{for all} \quad \tau \in T.
\]  

There exists at least one hedging strategy for the seller. Indeed, fixing \(i = 1, \ldots, d\) and defining

\[
m := \max \left\{ \sum_{j=1}^{d} \pi_{ij}^j(\omega) \max \{ |Y^j_t(\omega)|, |X^j_t(\omega)|, |X'^j_t(\omega)| \} : t = 0, \ldots, T, \omega \in \Omega \right\},
\]

the (possibly expensive) buy-and-hold strategy \(y = (y_t)_{t=0}^T\) with \(y_t = me^t\) for \(t = 0, \ldots, T\) hedges the game option for the seller with any choice of the cancellation time \(\sigma \in T\).

Consider now the following construction.

**Construction 3.4.** Construct adapted set-valued mappings \((Y^a_t)_{t=0}^T, (X^a_t)_{t=0}^T, (U^a_t)_{t=0}^T, (V^a_t)_{t=0}^T, (W^a_t)_{t=0}^T, (Z^a_t)_{t=0}^T\) as follows. For all \(t = 0, \ldots, T\) let

\[
y^a_t := Y_t + K_t, \quad X^a_t := \begin{cases} X'_t + K_T & \text{if } t = T, \\ X_t + K_t & \text{if } t < T. \end{cases}
\]

Define

\[
W^a_T := V^a_T := L_T, \quad Z^a_T := X^a_T.
\]

For \(t = T - 1, \ldots, 0\) define by backward iteration

\[
W^a_t := Z^a_{t+1} \cap L_t, \quad Y^a_t := W^a_t + K_t, \quad Z^a_t := (Y^a_t \cap Z^a_t) \cup X^a_t.
\]

For each \(t = 0, \ldots, T\), the set \(Y^a_t\) is the collection of portfolios in \(L_t\) that allows the seller to settle the option in the event that the buyer exercises at time \(t\) and the seller does not cancel the option at time \(t\). The set \(X^a_t\) is the collection of portfolios that allows the seller to settle the option upon cancellation at time \(t\), irrespective of whether the buyer exercises at time \(t\) or not.

The following result shows that \(Z^a_0\) is the set of initial endowments that allow the seller to hedge the game option.

**Proposition 3.5.** We have

\[
Z^a_0 = \{ y_0 : (\sigma, y) \text{ hedges } (Y, X, X') \text{ for the seller} \}.
\]
It is demonstrated in the proof of Proposition 3.5, which is deferred to Section 4, that for each \( t < T \) the sets \( V_{at} \), \( W_{at} \) and \( Z_{at} \) have natural interpretations that are important to the seller of the option. The set \( W_{at} \) consists of those portfolios at time \( t \) that allow the seller to hedge the option in the future (at time \( t + 1 \) or later), and \( V_{at} \) consists of those portfolios that may be rebalanced at time \( t \) into a portfolio in \( W_{at} \). The set \( Z_{at} \) consists of all portfolios that allow the seller to settle the option at time \( t \) or any time in the future without risk.

Construction 3.4 is essentially an iteration (backwards in time) over the nodes of the price tree generated by the exchange rates; note in particular that (3.5) could equivalently be written

\[
W_{at}^{\mu} := \bigcap_{\nu \in \text{succ} \mu} Z_{t+1}^{\nu} \quad \text{for} \quad \mu \in \Omega_t.
\]

This property makes the construction particularly efficient for recombinant models, for which the number of nodes grow only polynomially with the number of steps in the model, despite the state space growing exponentially.

The sets \( X_{at} \) and \( Y_{at} \) in Construction 3.4 are clearly polyhedral and non-empty for all \( t = 0, \ldots, T \), as are \( V_T \), \( W_T \) and \( Z_T \). The operations in Construction 3.4 are direct addition of polyhedral cones in (3.4) and (3.6), intersection in (3.5) and (3.7), and union in (3.7). The appearance of the union in (3.7) means that the sets \( V_{at} \), \( W_{at} \) and \( Z_{at} \) are generally not convex for all \( t \). However, it is clear that these sets can be written as the finite union of non-empty (closed) polyhedra, and are therefore closed. In particular the closedness of \( Z_0 \) is essential to Theorem 3.9 below.

The ask price of the game option in terms of any currency is defined as the infimal initial endowment in that currency that would allow the seller to hedge the game option without risk.

**Definition 3.6** (Ask price). The ask price or seller’s price or upper hedging price of a game option \((Y, X, X')\) at time 0 in terms of currency \(i = 1, \ldots, d\) is

\[
\pi^a_i(Y, X, X') := \inf\{z \in \mathbb{R} : (\sigma, y) \in T \times \Phi \text{ with } y_0 = ze^i \text{ hedges } (Y, X, X') \text{ for the seller}\}.
\]

The existence of the buy-and-hold strategy for the seller means that the ask price is well defined. We now present a dual representation for the ask price in terms of randomised stopping times and approximate martingale pairs.

**Theorem 3.7.** The ask price of a game option \((Y, X, X')\) in terms of currency \(i = 1, \ldots, d\) is

\[
\pi^a_i(Y, X, X') = \min_{\sigma \in T} \max_{\chi \in \mathcal{A}} \sup_{(P,S) \in \mathcal{P}^i(\chi, T)} \mathbb{E}_P((Q_{\sigma} \cdot S)_\chi)
\]

\[
= \min_{\sigma \in T} \max_{\chi \in \mathcal{A}} \sup_{(P,S) \in \mathcal{P}^i(\chi, T)} \mathbb{E}_P((Q_{\sigma} \cdot S)_\chi),
\]

where \(Q_{\sigma} \cdot S\) denotes the process \((Q_{\sigma t} \cdot S_t)_{t=0}^T\).

The proof of Theorem 3.7 appears in Section 4.

**Remark 3.8.** The proof of Theorem 3.7 hinges on the fact that a pair \((\sigma, y)\) hedges the game option for the seller if and only if \(y\) hedges an American option
with payoff process $H_x$ (defined in (4.1)) and random expiration date $\sigma$ for the seller. By contrast, in proving a similar result in a two asset model, Kifer (2013) claims that $(\sigma, y)$ hedges the game option for the seller if and only if $y$ hedges an American option with expiration date $T$, rather than $\sigma$, and with the same payoff process up to time $\sigma$, and payoff 0 after $\sigma$. This claim does not hold true in general, because the additional hedging requirement after time $\sigma$ places additional solvency constraints on $y$, for example in cases where the values of the payoff processes $Y$, $X$ and $X'$ are not all solvent (a common occurrence when using the machinery developed in this section to price game options for the buyer). This is also why the formulation of Theorem 3.7 above contains the truncated stopping time $\chi \wedge \sigma$ rather than $\chi$ obtained by Kifer (2013, Theorem 3.1).

Let us return to the problem of computing the ask price of a game option. The following result is a direct consequence of Proposition 3.5 and the closedness of $Z^a_0$.

**Theorem 3.9.** We have

$$\pi^a_i(Y, X, X') = \min\{x \in \mathbb{R} : xe^i \in Z^a_0\}. \quad (3.9)$$

Moreover, there exists a hedging strategy $(\hat{\sigma}, \hat{y})$ for the seller such that $\hat{y}_0 = \pi^a_i(Y, X, X') e^i$.

A hedging strategy $(\sigma, y)$ for the seller is called optimal if it satisfies the properties in Theorem 3.9. A procedure for constructing such a strategy can be extracted from the proof of Proposition 3.5.

**Construction 3.10.** Construct an optimal strategy $(\hat{\sigma}, \hat{y})$ for the seller as follows. Let

$$\hat{y}_0 := \pi^a_i(Y, X, X') e^i.$$  

For each $t = 0, \ldots, T - 1$ and $\mu \in \Omega$, if $\hat{y}_t^\mu \in Z^a_0 \setminus X^a_t$, then choose any

$$\hat{y}_{t+1}^\mu \in W^a_t \cap [\hat{y}_t^\mu - K^a_t], \quad (3.10)$$

otherwise put $\hat{y}_{t+1}^\mu := \hat{y}_t^\mu$. Also define

$$\hat{\sigma} := \min \{t : \hat{y}_t \in X^a_t\}.$$  

The optimal strategy for the seller is not unique in general; this is reflected in the choice (3.10). In practice the seller might use secondary considerations, such as appetite for holding certain currencies over others, or optimality of a secondary hedging criterion, to guide the construction of a suitable optimal hedging strategy.

A toy example illustrating Constructions 3.4 and 3.10 as well as a second example with a more realistic flavour can be found in Section 5.

### 3.2 Pricing and hedging for the buyer

Consider now the hedging, pricing and optimal exercise problem for the buyer of a game option $(Y, X, X')$. 


Definition 3.11 (Hedging strategy for the buyer). A hedging strategy for the buyer is a pair \((\tau, y) \in T \times \Phi\) satisfying
\[
y_{\sigma \land \tau} + Q_{\sigma \tau} \in \mathcal{K}_{\sigma \land \tau} \quad \text{for all } \sigma \in T,
\]
where the payoff process \(Q\) is defined in (3.1).

Observe from (3.1) that
\[
Q_{\sigma \tau} = Q_{\sigma \tau}^{Y, X, X'} = -Q_{\tau \sigma}^{-X, -Y, -X'} \quad \text{for all } \sigma, \tau \in T,
\]
and moreover from (3.2) that for all \(t = 0, \ldots, T\)
\[
-Y_t - (-X'_t) = X'_t - Y_t \in \mathcal{K}_t,
- X'_t - (-X_t) = X_t - X'_t \in \mathcal{K}_t.
\]
Thus if \((Y, X, X')\) is the payoff of a game option, then so is \((-X, -Y, -X')\), and (3.11) is equivalent to
\[
y_{\tau \land \sigma} - Q_{\tau \sigma}^{-X, -Y, -Z} \in \mathcal{K}_{\tau \land \sigma} \quad \text{for all } \sigma \in T.
\]
Thus we arrive at the following result.

Proposition 3.12. A pair \((\tau, y) \in T \times \Phi\) hedges the game option \((Y, X, X')\) for the buyer if and only if \((\tau, y)\) hedges the game option \((-X, -Y, -X')\) for the seller.

This symmetry means that the results and constructions developed in the previous subsection for the seller’s case can also be applied to the hedging and pricing problem for the buyer, thus substantiating the claim by Kifer (2013b, pp. 679–80). In particular, Construction 3.4 can be applied directly, provided that \(Y^b_t\) and \(X^b_t\) in (3.4) is redefined to take into account the fact that the option is now \((-X, -Y, -X')\) rather than \((Y, X, X')\). The resulting construction reads as follows.

Construction 3.13. For all \(t\) let
\[
\mathcal{X}^b_t := -X_t + \mathcal{K}_t,
\]
\[
\mathcal{Y}^b_t := \begin{cases} -X'_t + \mathcal{K}_T & \text{if } t = T, \\
-Y_t + \mathcal{K}_t & \text{if } t < T. \end{cases}
\]
Define
\[
\mathcal{W}^b_t := \mathcal{Y}^b_t := \mathcal{L}_T, \quad \mathcal{Z}^b_t := \mathcal{X}^b_t.
\]
For \(t = T-1, \ldots, 0\) let
\[
\mathcal{W}^b_t := \mathcal{Z}^b_{t+1} \cap \mathcal{L}_t, \quad \mathcal{Y}^b_t := \mathcal{W}^b_t + \mathcal{K}^b_t, \quad \mathcal{Z}^b_t := (\mathcal{Y}^b_t \cap \mathcal{Y}^b_t) \cup \mathcal{X}^b_t.
\]
It follows directly from Theorem 3.5 that
\[
\mathcal{Z}^b_0 = \{ z : (\tau, y) \in T \times \Phi \text{ with } y_0 = ze^t \text{ hedges } (Y, X, X') \text{ for the buyer}\}.
\]

The bid price of a game option in any currency is the largest amount that the buyer can raise in that currency at time 0 by using the payoff of the option as a guarantee.
Definition 3.14 (Bid price). The bid price or lower hedging price or buyer’s price of a game option \((Y, X, X')\) in currency \(i = 1, \ldots, d\) is defined as

\[
\pi^b_i(Y, X, X') := \sup\{-z : (\tau, y) \in T \times \Phi \text{ with } y_0 = ze^i \text{ superhedges } (Y, X, X') \text{ for the buyer}\}.
\]

It also follows from Proposition 3.12 and Construction 3.13 that

\[
\pi^b_i(Y, X, X') = -\pi^a_i(-X, -Y, -X') = -\inf\{z : ze^i \in Z^b_0\}.
\]

If we call a hedging strategy \((\tau, y)\) for the buyer optimal if \(y_0 = -\pi^b_i(Y, X, X')e^i\), then optimal hedging strategies can be generated by rewriting Construction 3.10 as follows.

Construction 3.15. Construct an optimal strategy \((\tilde{\tau}, \tilde{y})\) for the buyer as follows. Let

\[
\tilde{y}_0 := -\pi^b_i(Y, X, X')e^i.
\]

For each \(t = 0, \ldots, T - 1\) and \(\mu \in \Omega_t\), if \(\tilde{y}^\mu_t \in Z^{b\mu}_t \setminus X^{b\mu}_t\), then choose any

\[
\tilde{y}^\mu_{t+1} \in W^{b\mu}_t \cap [\tilde{y}^\mu_t - K^{\mu}_t],
\]

(3.14)

otherwise put \(\tilde{y}^\mu_{t+1} := \tilde{y}^\mu_t\). Also define

\[
\tilde{\tau} := \min\{t : \tilde{y}_t \in X^b_t\}.
\]

A toy example illustrating Constructions 3.13 and 3.15, which also demonstrates that the optimal cancellation and exercise times for the seller and buyer do not necessarily coincide, can be found in Section 5.

Finally, combining Theorem 3.7 with (3.12) and (3.13) immediately gives the following dual representation for the bid price.

Theorem 3.16. We have

\[
\pi^b_i(Y, X, X') = \max_{\tau \in T} \min_{X \in \mathcal{X}(P, S) \in \mathcal{P}(\chi \wedge \tau)} \inf \mathbb{E}^\mathbb{P}_\tau((Q \cdot \tau \cdot S)\chi),
\]

\[
= \max_{\tau \in T} \min_{X \in \mathcal{X}(P, S) \in \mathcal{P}(\chi \wedge \tau)} \min_{\chi \in \mathcal{X}(\tau \wedge \tau)} \mathbb{E}^\mathbb{P}_\tau((Q \cdot \tau \cdot S)\chi),
\]

where \(Q \cdot \tau \cdot S\) denotes the process \((Q_{Y,X,X'}^T \cdot S_t)_{t=0}^T\).

4 Proofs of main results

Proof of Proposition 3.5. Fix any \(z \in Z^a_0\). We claim that there exists a hedging strategy \((y, \sigma)\) for the seller with \(y_0 = z\). To this end, we construct \(y = (y_t)_{t=0}^T\) together with a non-decreasing sequence \((\sigma_t)_{t=0}^T\) of stopping times. Define

\[
y_0 := z, \quad \sigma_0 := \begin{cases} 0 & \text{if } z \in X^a_0, \\ 1 & \text{if } z \in Z^a_0 \setminus X^a_0. \end{cases}
\]

If \(\sigma_0 = 0\), let \(y_1 := y_0\). If \(\sigma_0 = 1\), then

\[
y_0 \in Z^a_0 \setminus X^a_0 \subseteq Y^a_0 \cap J^a_0 \subseteq W^a_0 + K_0,
\]
so that there exists some \( y_1 \in W_0^a \) such that \( y_0 - y_1 \in K_0 \) and \( y_1 \in Z_1^a \).

Suppose by induction that for some \( t > 0 \) we have constructed \( y_0, \ldots, y_t \) and non-decreasing \( \sigma_0, \ldots, \sigma_{t-1} \) such that for \( s = 0, \ldots, t-1 \) we have \( y_{s+1} \) being \( \mathcal{F}_s \)-measurable, \( \sigma_s \leq s + 1 \), \( y_s - y_{s+1} \in K_s \), \( y_{s+1} \in X_{\sigma_s}^a \) on \( \{ \sigma_s \leq s \} \) and \( y_u \in Z_u^a \setminus X_u^a \) on \( \{ \sigma_u > u \} \) for all \( u = 0, \ldots, s \). Define

\[
\sigma_t := \sigma_{t-1} 1_{\{ \sigma_{t-1} < t \}} + t 1_{\{ \sigma_{t-1} = t \}} \cap \{ y_t \in Z_t^a \setminus X_t^a \} + (t + 1) 1_{\{ \sigma_{t-1} = t \}} \cap \{ y_t \in Z_t^a \setminus X_t^a \}.
\]

On the set

\[
\{ \sigma_t > t \} = \{ \sigma_t = t + 1 \} = \{ \sigma_{t-1} = t \} \cap \{ y_t \in Z_t^a \setminus X_t^a \}
\]

we have

\[
y_t \in Z_t^a \setminus X_t^a \subseteq V_t^a \cap Y_t^a \subseteq V_t^a = W_t^a + K_t,
\]

so there exists a \( \mathcal{F}_t \)-measurable random variable \( x \in Z_{t+1}^a \) such that \( y_t - x \in K_t \) on this set. Now define the \( \mathcal{F}_t \)-measurable random variable

\[
y_{t+1} := x 1_{\{ \sigma_t = t+1 \}} + y_t 1_{\{ \sigma_t \leq t \}}.
\]

Since \( 0 \in K_t \) we have \( y_t - y_{t+1} \in K_t \). Moreover,

\[
y_{t+1} = y_{t+1} 1_{\{ \sigma_t < t \}} + y_t 1_{\{ \sigma_t = t \}} + y_{t+1} 1_{\{ \sigma_t = t+1 \}} = y_{t+1} 1_{\{ \sigma_t < t \}} + y_{t+1} 1_{\{ \sigma_t = t \}} \cap \{ y_t \in X_t^a \} + y_{t+1} 1_{\{ \sigma_t = t+1 \}}
\]

so \( y_{t+1} \in X_{\sigma_t}^a \) on the set

\[
\{ \sigma_t \leq t \} = \{ \sigma_t < t \} \cup \{ \sigma_t = t \} \cap \{ y_t \in X_t^a \}.
\]

This concludes the inductive step.

Let \( \sigma := \sigma_T \); then the pair \( (\sigma, y) \in \mathcal{T} \times \Phi \) satisfies

\[
y_\sigma \in X_\sigma^a, \quad y_t \in Z_t^a \setminus X_t^a \text{ on } \{ t < \sigma \} \text{ for } t = 0, \ldots, T - 1.
\]

Fix any stopping time \( \tau \). On \( \{ \tau < \sigma \} \) we have

\[
y_\tau \in Z_\tau^a \setminus X_\tau^a \subseteq V_\tau^a \cap Y_\tau^a \subseteq Y_\tau + K_\tau.
\]

On the set \( \{ \tau = \sigma \} \) we have \( y_\sigma \in X_\sigma^a \subseteq X_\sigma + K_\sigma \). On the set \( \{ \tau > \sigma \} \) we have \( \sigma < \tau \leq T \) and \( y_\sigma \in X_\sigma^a = X_\sigma + K_\sigma \). Thus \( y_{\sigma \land \tau} - \bar{Q}_{\sigma \land \tau} \in K_{\sigma \land \tau} \), from which it follows that \( (\sigma, y) \) hedges the game option for the seller.

Conversely, suppose that \( (\sigma, y) \) hedges the game option for the seller. We show by backward induction that \( y_t \in Z_t^a \) on \( \{ t \leq \sigma \} \) and \( y_t \in W_t^a \cap Y_t^a \) on \( \{ t < \sigma \} \) for all \( t = 0, \ldots, T \), from which it can be deduced that \( z = y_0 \in Z_0^a \), which completes the proof. At time \( T \) we have \( \{ T = \sigma \} = \{ T \leq \sigma \} \), so clearly \( y_T \in X_T^a + K_T = \bigcup_{\tau \leq \sigma} Z_\tau^a \) on \( \{ T \leq \sigma \} \).

For any \( t < T \), suppose that \( y_{t+1} \in Z_{t+1}^a \) on \( \{ t + 1 \leq \sigma \} = \{ t < \sigma \} \). This means that \( y_{t+1} \in W_{t+1}^a \) on \( \{ t < \sigma \} \) as \( y_{t+1} \in L_t \). Moreover \( y_t - y_{t+1} \in K_t \) implies that \( y_t \in V_t^a \) on \( \{ t < \sigma \} \), and therefore

\[
y_t \in V_t^a \cap [Y_t + K_t] = V_t^a \cap Y_t^a \subseteq Z_t^a \text{ on } \{ t < \sigma \}.
\]

On the set \( \{ \tau = \sigma \} = \{ t \leq \sigma \} \setminus \{ t < \sigma \} \) we have \( y_t \in X_t + K_t = X_t^a \subseteq Z_t^a \). This concludes the induction.
The following observation will play an important role in the proof of Theorem 3.7 below. Define the auxiliary process

\[ H_{st} \equiv H_{st}^{(Y,X,X')} := Y_s 1_{\{s > t\}} + X_s 1_{\{s = t < T\}} + X'_s 1_{\{s = T\}} \]  

for all \( s, t = 0, \ldots, T \). Observe that the process \( H_{\sigma} = (H_{\sigma t})_{t=0}^T \) is adapted for any \( \sigma \in \mathcal{T} \), and that \( H_{\sigma \tau} = H_{\sigma, \sigma \land \tau} \) for all \( \tau \in \mathcal{T} \).

**Lemma 4.1.** A pair \((\sigma, y) \in \mathcal{T} \times \Phi\) hedges the game option for the seller if and only if

\[ y_{\sigma \land \tau} - H_{\sigma \tau} \in K_{\sigma \land \tau} \text{ for all } \tau \in \mathcal{T}. \]  

**Proof.** Fix any \((\sigma, y) \in \mathcal{T} \times \Phi\) satisfying (4.2). In view of (3.3), this implies that

\[ (y_{\sigma \land \tau} - Y_{\sigma \land \tau}) 1_{\{\sigma > \tau\}} \in K_{\sigma \land \tau} \text{ for all } \tau \in \mathcal{T}, \]  

\[ (y_{\sigma} - X_{\sigma}) 1_{\{\sigma < T\}} \in K_{\sigma}, \]  

\[ (y_{\sigma} - X'_{\sigma}) 1_{\{\sigma = T\}} \in K_{\sigma}. \]  

From (3.2) we have \( X_{\sigma} - X'_{\sigma} \in K_{\sigma} \) and so (4.4)–(4.5) give \( y_{\sigma} - X'_{\sigma} \in K_{\sigma}, \) whence

\[ (y_{\sigma \land \tau} - X'_{\sigma \land \tau}) 1_{\{\sigma = \tau\}} \in K_{\sigma \land \tau} \text{ for all } \tau \in \mathcal{T}. \]  

(4.6)

It also follows from (4.3) that

\[ (y_{\sigma \land \tau} - X_{\sigma \land \tau}) 1_{\{\sigma < \tau\}} \in K_{\sigma \land \tau} \text{ for all } \tau \in \mathcal{T}. \]  

(4.7)

Properties (4.3), (4.6) and (4.7) lead to (3.3), and so \((\sigma, y)\) hedges the game option for the seller.

Suppose conversely that \((\sigma, y) \in \mathcal{T} \times \Phi\) hedges the game option for the seller. Property (3.3) gives (4.3) together with

\[ y_{\sigma} - X_{\sigma} 1_{\{\sigma < T\}} - X'_{\sigma} 1_{\{\sigma = T\}} \in K_{\sigma}, \]  

which leads to

\[ (y_{\sigma \land \tau} - X_{\sigma} 1_{\{\sigma = \tau\}} - X'_{\sigma} 1_{\{\sigma = T\}} 1_{\{\sigma \leq T\}}) \in K_{\sigma \land \tau} \text{ for all } \tau \in \mathcal{T}. \]  

(4.8)

Properties (4.3) and (4.8) then give (4.2), which completes the proof.

**Proof of Theorem 3.7.** Lemma 4.1 shows that, for \( \sigma \in \mathcal{T} \) given, the pair \((\sigma, y) \in \mathcal{T} \times \Phi\) hedges the game option \((Y, X, X')\) for the seller if and only if (4.2) holds, equivalently

\[ y_{\tau} - H_{\sigma \tau} \in K_{\tau} \text{ for all } \tau \in \mathcal{T} \text{ such that } \tau \leq \sigma. \]

**Definition 3.6** gives that

\[ \pi^a_i(Y, X, X') = \min_{\sigma \in \mathcal{T}} \inf \{z \in \mathbb{R} : (\sigma, y) \text{ hedges } (Y, X, X') \text{ for the seller and } y_0 = z \epsilon^i\} \]

\[ = \min_{\sigma \in \mathcal{T}} p^i(\sigma), \]
where
\[ p^0_i(\sigma) := \inf\{ z \in \mathbb{R} : y \in \Phi \text{ such that } \]
\[ y_0 = zv^i, y_\tau - H_{\sigma \tau} \in K_\tau \text{ for all } \tau \in T, \tau \leq \sigma \}. \]

Thus the pair \((\sigma, y)\) hedges \((Y, X, X')\) for the seller if and only if \(y\) super-hedges an American option with payoff process \(H_\sigma = (H_{\sigma t})_{t \in T}\) and \((\text{random})\) expiration date \(\sigma\) for the seller (using the terminology of Roux & Zastawniak (2015)). The quantity \(p^0(\sigma)\) is the ask price of such an American option, and Roux & Zastawniak (2015, Theorem 3) established that
\[
p^0_i(\sigma) = \max_{\chi \in \mathcal{X}} \sup_{(P, S) \in \mathcal{P}(\chi \wedge \sigma)} \mathbb{E}_P((H_\sigma \cdot S)_\chi) = \max_{\chi \in \mathcal{X}} \sup_{(P, S) \in \mathcal{P}(\chi \wedge \sigma)} \mathbb{E}_P((H_\sigma \cdot S)_\chi)
\]
where \(H_\sigma \cdot S = (H_{\sigma t} \cdot S_t)_{t \in T}\), whence
\[
p^0_i(\sigma) = \max_{\chi \in \mathcal{X}} \sup_{(P, S) \in \mathcal{P}(\chi \wedge \sigma)} \mathbb{E}_P((H_\sigma \cdot S)_\chi) = \max_{\chi \in \mathcal{X}} \sup_{(P, S) \in \mathcal{P}(\chi \wedge \sigma)} \mathbb{E}_P((H_\sigma \cdot S)_\chi).
\]

Fix now any \(\chi \in \mathcal{X}\) and \((P, S) \in \mathcal{P}(\chi \wedge \sigma)\) and note that
\[
(H_\sigma \cdot S)_{\chi \wedge \sigma} = (Q_\sigma \cdot S)_\chi + (\chi^*_\sigma 1_{(\sigma < T}) - \chi^*_{\sigma + 1})X_\sigma \cdot S_\sigma + (\chi^*_\sigma 1_{(\sigma = T}) - \chi^*_\sigma)X'_\sigma \cdot S_\sigma
\]
\[
= (Q_\sigma \cdot S)_\chi + (\chi^*_\sigma 1_{(\sigma < T}) - \chi^*_{\sigma + 1} 1_{(\sigma = T}) X_\sigma \cdot S_\sigma - \chi^*_\sigma 1_{(\sigma < T}) X'_\sigma \cdot S_\sigma
\]
\[
= (Q_\sigma \cdot S)_\chi + \chi^*_\sigma 1_{(\sigma < T}) (X_\sigma - X'_\sigma) \cdot S_\sigma.
\]

This follows from the properties of \(\chi^*\): in particular \(\chi^*_{T + 1} = 0\), \(\chi^*_T = \chi_T\) and \(\chi^*_\sigma = \chi^*_{\sigma + 1} + \chi_\sigma\). Since \(X_\sigma - X'_\sigma \in K_\sigma\) by (3.2) and \(S_\sigma \in K^*_\sigma\), we immediately have
\[
(H_\sigma \cdot S)_{\chi \wedge \sigma} \geq (Q_\sigma \cdot S)_\chi.
\]

Define the stopping time \(\chi' = (\chi'_t) \in \mathcal{X}\) by
\[
\chi'_t := \chi_t 1_{(t < \sigma)} + \chi^*_\sigma 1_{(t = T)} \text{ for } t = 0, \ldots, T;
\]
then \(\chi \wedge \sigma = \chi' \wedge \sigma\) and so \((P, S) \in \mathcal{P}(\chi' \wedge \sigma)\). Moreover, since \(\chi'_\sigma 1_{(\sigma < T}) = 0\) it follows from (4.10) that
\[
(H_\sigma \cdot S)_{\chi \wedge \sigma} = (H_\sigma \cdot S)_{\chi' \wedge \sigma} = (Q_\sigma \cdot S)_{\chi'}.
\]

Combining (4.11) and (4.12) then gives
\[
\max_{\chi \in \mathcal{X}} \sup_{(P, S) \in \mathcal{P}(\chi \wedge \sigma)} \mathbb{E}_P((H_\sigma \cdot S)_\chi) = \max_{\chi \in \mathcal{X}} \sup_{(P, S) \in \mathcal{P}(\chi \wedge \sigma)} \mathbb{E}_P((Q_\sigma \cdot S)_\chi),
\]
and the result follows from (4.9).
\[\Box\]

5 Numerical examples

This section contains two examples. The first is a toy example to illustrate the constructions in Section 3; the second has a more realistic flavour.
Example 5.1. A game option $(Y, X, X')$ in a binary two-step two-currency model is presented in Figure 1. The model is recombinant and has transaction costs only at node $u$ at time 1, and the option is path-independent and has no cancellation penalties at time 2.

Let us use Construction 3.4 to find the set of hedging endowments for the seller. Clearly

$$Z_{a}^{uu} = \{ (x^1, x^2) \in \mathbb{R}^2 : 16x^1 + x^2 \geq 9 \},$$

$$Z_{a}^{ud} = \{ (x^1, x^2) \in \mathbb{R}^2 : 10x^1 + x^2 \geq 4 \},$$

$$Z_{a}^{dd} = \{ (x^1, x^2) \in \mathbb{R}^2 : 4x^1 + x^2 \geq 0 \}$$

at time $t = 2$.

For time $t = 1$, consider the node $u$. We obtain $W_{a}^{uu}$ from (3.5) and $V_{a}^{uu}$ from (3.6); both procedures are shown graphically in Figure 2. Observe that the magnitude of the transaction costs at this node means that $W_{a}^{uu} + K_{a}^{u} = W_{a}^{uu}$, whence $V_{a}^{uu} = W_{a}^{uu}$. The next step is to compute $Z_{a}^{uu}$ from (3.7). As shown in Figure 2, this can be done by finding the intersection of $V_{a}^{uu}$ and $Y_{a}^{uu}$ first and then taking the union with $X_{a}^{uu}$. The non-convexity of $Z_{a}^{uu}$ is due to the magnitude of the transaction costs and the shape of the payoff at this node. Similar considerations at the node $d$ give that

$$Z_{a}^{ud} = X_{a}^{ud} = \{ (x^1, x^2) : 6x^1 + x^2 \geq 1 \}.$$

Finally, following the same steps for time $t = 0$ results in the sets of portfolios $W_{0}^{uu}$, $V_{0}^{uu}$ and $Z_{0}^{uu}$ depicted in Figure 3. The ask prices of the game option in the two currencies are the intersections of the lower boundary of $Z_{0}^{uu}$ with the axes, and so can be read directly from the final graph in Figure 3, namely

$$\pi_{1}^{a}(Y, X, X') = \frac{2}{5}, \quad \pi_{2}^{a}(Y, X, X') = 4.$$
Figure 2: $W_{1}^{au}$, $Y_{1}^{au}$, $Y_{1}^{au} \cap Y_{1}^{au}$, $Z_{1}^{au}$ and $\hat{y}_{1}$, Example 5.1

Figure 3: $W_{0}^{au}$, $V_{0}^{au}$, $V_{0}^{au} \cap Y_{0}^{au}$, $Z_{0}^{au}$ and $\hat{y}_{0}$, Example 5.1
\[ y \] which means that every choice of \( y \) in other words that \( y \) so there is some choice in the selection of \( y \) that together with the bid prices to exercise the option at time 2 at the node \( uu \) and remain in (or return to) a solvent position. It is also clear that \( y \) and so \( y \) that

For the buyer, Construction 3.13 and similar calculations as above yields

\[
\mathcal{Z}_{2}^{\text{ban}} = \{ (x^1, x^2) \in \mathbb{R}^2 : 16x^1 + x^2 \geq -9 \}, \\
\mathcal{Z}_{2}^{\text{bid}} = \mathcal{Z}_{2}^{\text{bud}} = \{ (x^1, x^2) \in \mathbb{R}^2 : 10x^1 + x^2 \geq -4 \}, \\
\mathcal{Z}_{2}^{\text{bid}} = \{ (x^1, x^2) \in \mathbb{R}^2 : 4x^1 + x^2 \geq 0 \}, \\
\mathcal{Z}_{1}^{\text{bid}} = \{ (x^1, x^2) \in \mathbb{R}^2 : 16x^1 + x^2 \geq -4, 8x^1 + x^2 \geq -4 \}, \\
\mathcal{Z}_{0}^{\text{bid}} = \{ (x^1, x^2) \in \mathbb{R}^2 : 6x^1 + x^2 \geq -1 \}, \\
\mathcal{Z}_{0}^{\text{bid}} = \{ (x^1, x^2) \in \mathbb{R}^2 : 6x^1 + x^2 \geq -\frac{11}{2} \}
\]

together with the bid prices

\[
\pi_1^b(Y, X, X') = \frac{1}{4}, \quad \pi_2^b(Y, X') = \frac{1}{4}.
\]

Let us use Construction 3.13 to find an optimal hedging strategy \((\hat{\tau}, \hat{y})\) for the buyer in the scenario \( uu \) from the initial value \( y_0 := (0, -\frac{1}{3}) \). Clearly \( y_0 \notin X_0^b = \mathcal{K}_0 \) and so \( \hat{\tau} > 0 \). As can be seen from Figure 5.1

\[
\mathcal{W}_0^{b} \cap [y_0 - \mathcal{K}_0] = \{ \left( -\frac{29}{16}, \frac{5}{4} \right) \}
\]

and so \( \hat{y}_1 = \left( -\frac{29}{16}, \frac{5}{4} \right) \) by (3.13). Observing that

\[
y_1 \notin X_1^{bu} = \{ (x^1, x^2) \in \mathbb{R}^2 : 16x^1 + x^2 \geq -3, 8x^1 + x^2 \geq -3 \},
\]

in other words that \( \hat{\tau}(uu) > 1 \), we proceed to select \( \hat{y}_2^b \). Figure 5.1 also shows that

\[
\mathcal{W}_1^{bu} \cap [\hat{y}_1 - \mathcal{K}_1^b] = \text{conv} \{ \hat{y}_1, \left( -\frac{47}{30}, \frac{29}{10} \right), \left( -\frac{13}{5}, \frac{13}{2} \right), (0, -4) \},
\]

so there is some choice in the selection of \( \hat{y}_2^b \). In fact

\[
\mathcal{W}_1^{bu} \cap [\hat{y}_1 - \mathcal{K}_1^b] \subseteq X_2^{ban} = \{ (x^1, x^2) \in \mathbb{R}^2 : 16x^1 + x^2 \geq -9 \},
\]

which means that every choice of \( \hat{y}_2^b \in \mathcal{W}_1^{bu} \cap [\hat{y}_1 - \mathcal{K}_1^b] \) would enable the buyer to exercise the option at time 2 at the node \( uu \) and remain in (or return to) a solvent position. It is also clear that \( \hat{\tau}(uu) = 2 \).

Returning to the seller, it is straightforward to use Construction 3.10 to create an optimal hedging strategy \((\hat{\tau}, \hat{y})\) for the seller from the initial endowment

Figure 4: \( \hat{y}_0 \) and \( \hat{y}_1 \), Example 5.1
\[ \hat{y}_0 := (0, 4). \] From Figure 3 it is clear that \( \hat{y}_0 \notin X^m_0 \) (and so \( \hat{\sigma} > 0 \)), and also that

\[ \mathcal{N}_0^m \cap [\hat{y}_0 - K_0] = \{(0, 4)\} = \{\hat{y}_0\}, \]

and so by (3.10) we must choose \( \hat{y} = \hat{y}_0 \). Figure 2 and (5.1) shows that \( \hat{y}_1 \notin X^m_1 \) and so \( \hat{\sigma} = 1 \). This means that if the seller follows \( (\hat{\sigma}, \hat{y}) \) and the buyer \( (\hat{\tau}, \hat{y}) \), then in scenario uu the option will be cancelled (but not exercised) at the node u, at which time the seller delivers \((0, 4)\) to the buyer.

**Example 5.2.** Consider a three-currency model with \( T \) steps and time horizon 1 based on the two-asset recombinant Korn-Muller model [Korn & Müller 2009] with Cholesky decomposition, i.e. friction-free exchange rates in terms of currency 3 are modelled by the process \( (S_t)_{t=0}^T \), where

\[ S_t = (\varepsilon_t, S_{t-1}^1, S_{t-1}^2, 1) \text{ for } t = 1, \ldots, T \]

and \( (\varepsilon_t)_{t=1}^T = ((\varepsilon_1, \varepsilon_2)^T)_{t=1}^T \) is a sequence of independent identically distributed random variables taking the values

\[
\begin{align*}
&\left(e^{-\frac{1}{2}\sigma_1^2 \Delta - \varepsilon_1 \sqrt{\Delta}}, e^{-\frac{1}{2}\sigma_2^2 \Delta - \sqrt{\Delta}}, e^{-\frac{1}{2}\sigma_2^2 \Delta + \sqrt{\Delta}}, e^{-\frac{1}{2}\sigma_1^2 \Delta + \sqrt{\Delta}}, e^{-\frac{1}{2}\sigma_2^2 \Delta - \sqrt{\Delta}}, e^{-\frac{1}{2}\sigma_1^2 \Delta + \sqrt{\Delta}}, e^{-\frac{1}{2}\sigma_1^2 \Delta - \sqrt{\Delta}}, e^{-\frac{1}{2}\sigma_2^2 \Delta + \sqrt{\Delta}}, \right), \\
&\left(e^{-\frac{1}{2}\sigma_1^2 \Delta - \varepsilon_1 \sqrt{\Delta}}, e^{-\frac{1}{2}\sigma_2^2 \Delta + \sqrt{\Delta}}, e^{-\frac{1}{2}\sigma_2^2 \Delta - \sqrt{\Delta}}, e^{-\frac{1}{2}\sigma_1^2 \Delta + \sqrt{\Delta}}, e^{-\frac{1}{2}\sigma_2^2 \Delta - \sqrt{\Delta}}, e^{-\frac{1}{2}\sigma_1^2 \Delta + \sqrt{\Delta}}, e^{-\frac{1}{2}\sigma_1^2 \Delta - \sqrt{\Delta}}, e^{-\frac{1}{2}\sigma_2^2 \Delta + \sqrt{\Delta}}, \right), \\
&\left(e^{-\frac{1}{2}\sigma_1^2 \Delta + \varepsilon_1 \sqrt{\Delta}}, e^{-\frac{1}{2}\sigma_2^2 \Delta + \sqrt{\Delta}}, e^{-\frac{1}{2}\sigma_2^2 \Delta - \sqrt{\Delta}}, e^{-\frac{1}{2}\sigma_1^2 \Delta + \sqrt{\Delta}}, e^{-\frac{1}{2}\sigma_2^2 \Delta - \sqrt{\Delta}}, e^{-\frac{1}{2}\sigma_1^2 \Delta + \sqrt{\Delta}}, e^{-\frac{1}{2}\sigma_1^2 \Delta - \sqrt{\Delta}}, e^{-\frac{1}{2}\sigma_2^2 \Delta + \sqrt{\Delta}}, \right), \\
&\left(e^{-\frac{1}{2}\sigma_1^2 \Delta + \varepsilon_1 \sqrt{\Delta}}, e^{-\frac{1}{2}\sigma_2^2 \Delta + \sqrt{\Delta}}, e^{-\frac{1}{2}\sigma_2^2 \Delta - \sqrt{\Delta}}, e^{-\frac{1}{2}\sigma_1^2 \Delta + \sqrt{\Delta}}, e^{-\frac{1}{2}\sigma_2^2 \Delta - \sqrt{\Delta}}, e^{-\frac{1}{2}\sigma_1^2 \Delta + \sqrt{\Delta}}, e^{-\frac{1}{2}\sigma_1^2 \Delta - \sqrt{\Delta}}, e^{-\frac{1}{2}\sigma_2^2 \Delta + \sqrt{\Delta}}, \right),
\end{align*}
\]

each with positive probability, where \( \Delta := \frac{1}{T} \) is the step size. The exchange rates with transaction costs are modelled as

\[
\pi_{ij}^t := \begin{cases} 
\frac{S_{ij}^t}{S_{ij}^t} (1 + k) & \text{if } i \neq j, \\
1 & \text{if } i = j,
\end{cases}
\]

for \( i, j = 1, \ldots, 3 \) and \( t \leq T \), where \( k \in [0, 1) \). We take

\[ (S_{01}^1, S_{02}^2) = (40, 50), \quad \sigma_1 = 0.15, \quad \sigma_2 = 0.1, \quad \rho = 0.5. \]

Consider a game put option with physical delivery on a basket containing one unit each of currencies 1 and 2 and with strike \( K \) in currency 3, i.e.

\[ Y_t = (-1, -1, K) \text{ for } t = 0, \ldots, T. \]

On cancellation the seller delivers the above payoff to the buyer, together with a cancellation penalty \( p \geq 0 \) in currency 3, so that

\[ X_t = X_t' = (-1, -1, K + p) \text{ for } t = 0, \ldots, T. \]

We allow for the possibility that the seller may choose not to cancel the option, and the buyer may choose not to exercise, by adding an additional time step \( T + 1 \) and taking

\[ Y_{T+1} = X_{T+1} = X_{T+1}' = (0, 0, 0). \]

Except for the union, the operations in Constructions 3.4 and 3.13 namely intersection and direct addition of a polyhedral cone, are standard geometric
Table 1: Bid and ask prices of game basket put option with $N = 10$, $p = 5$ and different strikes $K$, Example 5.2.

| $K$  | $\pi_a^k$ | $\pi_b^k$ | $\pi_a^{0.005}$ | $\pi_b^{0.005}$ |
|------|-----------|-----------|-----------------|-----------------|
| 100  | 10.043290 | 11.033942 | 9.568590        | 11.687749       |
| 95   | 5.266479  | 6.817389  | 4.706543        | 7.480028        |
| 90   | 0.967824  | 2.774598  | 0.367975        | 3.423798        |
| 85   | -2.934360 | -1.091048 | -3.587214       | -0.444708       |
| 80   | -6.910514 | -5.131149 | -7.584034       | -4.614029       |

Table 2: Bid and ask prices of game basket put option with $N = 10$, $K = 100$ and different penalties, Example 5.2.

| $p$  | $\pi_a^k$ | $\pi_b^k$ | $\pi_a^{0.005}$ | $\pi_b^{0.005}$ |
|------|-----------|-----------|-----------------|-----------------|
| 0    | 10.000000 | 10.000000 | 9.550000        | 10.447761       |
| 1    | 10.014709 | 10.278348 | 9.556726        | 10.790910       |
| 2    | 10.027095 | 10.497310 | 9.562075        | 11.051351       |
| 5    | 10.034290 | 11.033942 | 9.568590        | 11.687749       |
| 10   | 10.050958 | 11.571315 | 9.571850        | 12.297913       |
| 20   | 10.052026 | 11.796921 | 9.572414        | 12.575621       |
| American | 10.052027 | 11.812658 | 9.572414        | 12.589930       |

procedures when applied to polyhedra, and can be implemented using existing software libraries. Both these operations are union-preserving, and so the extension to unions of polyhedra is straightforward. The numerical results below were produced using Maple with the Convex package (Franz 2009).

Table 1 contains bid and ask prices in currency 3 of the basket put with penalty $p = 5$ for a range of strike prices. Both bid and ask prices increase with the strike. Note the appearance of negative bid and ask prices for out-of-the-money options (i.e. $K < S_1^0 + S_2^0 = 90$). The reason for this is that the seller can cancel the option at any time, and cancellation tends to be particularly attractive to the seller (and very costly for the buyer) when the option is far out of the money; for example, if $K = 80$ then, from the point of view of the seller, cancelling the option at time 0 is equivalent to receiving the basket from the buyer and paying $K + p = 85$ in currency 3, which is less than the market price $S_1^0 + S_2^0 = 90$ (ignoring transaction costs) of the basket.

Bid and ask prices of the game basket put with strike $K = 100$ and a range of penalty values are reported in Table 2, together with bid and ask prices of the American basket put with the same strike (using the constructions of Roux & Zastawniak (2015)). In practical terms, game options with large penalties resemble American options (because larger penalties make it less attractive for the seller to cancel the option early), and this explains the convergence of the bid and ask prices of the game option to that of the American option as the penalty increases.
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