Some entanglement features of a three-atom Tavis–Cummings model: a cooperative case

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Abstract

In this paper we consider a system of three identical two-level atoms interacting at resonance with a single mode of the quantized field in a lossless cavity. The initial cavity field is prepared in the coherent state while the atoms are taken initially to be in either the uppermost excited state |eee⟩, the GHZ-state or the W-state. For this system we investigate different kinds of atomic inversion and entanglement, which arise between the different parts of the system due to the interaction. Also, we discuss the relationship between the entanglement and some other nonclassical effects in the statistical properties, such as collapses and revivals in the atomic inversion where superharmonic effects appear. The Q-functions for different cases are discussed. Most remarkably it is found that the GHZ-state is more robust against energy losses, showing almost coherent trapping, and Schrödinger cat states cannot be produced from such a state. Also the entanglement of the GHZ-state is more robust than that of the W-state.

1. Introduction

The quantum entanglement phenomenon is not only one of the most interesting features of quantum theory [1], which distinguishes it from classical theory, but it also lies at the heart of the new rapidly developing area known as quantum information processing [2]. It is one of the crucial resources required in the applications in this new area of science, which include quantum computation [3], quantum teleportation [4], quantum dense coding [5] and quantum cryptography [6]. In the quantum optics domain, the interaction of an atom with a quantized electromagnetic field mode described by the Jaynes–Cummings model [7, 8] leads to an entanglement of these two systems such that the total state vector cannot be written as a product of the time-dependent atomic and field component vectors [9–11]. To quantify the entangled states, one should know whether they are pure or mixed states. Thus, if the entangled state is a pure state, then it is sufficient to use the von Neumann entropy as a measure of entanglement. Much effort has been devoted to quantifying entanglement, particularly for mixed states of a bipartite system, and a number of measures have been proposed, such as entanglement of formation, relative entropy of entanglement and negativity. The Peres–Horodecki criterion for separability [12, 13] leads to a natural computable measure of entanglement, called negativity [14–16]. It has been proved that the negativity N(ρ) is an entanglement monotone and therefore can be used as a good measure of entanglement [16].

The cooperative nature of the interaction of a quantized radiation field with a system of two-level atoms was first
treated by Dicke [17]. A particular case of the Dicke model, where the atoms interact with a single-mode radiation field inside a cavity, was investigated by Tavis and Cummings (TCM) [18, 19] and recently by [20, 21]. The interaction between a field and an ensemble of atoms develops correlations between the field and the atomic systems and between the atomic system parties as well in the course of their dynamics. Quantifying these quantum correlations (entanglement) is an open problem. Multi-qubit systems are of interest both theoretically and experimentally. In recent years, great achievements have been made in the application of three-qubit states, because understanding of entanglement and dynamics of a three-qubit system is important, for example, in the many applications in quantum cryptography [22], quantum computation [23] and quantum gates [24, 25].

In this paper, we consider a system of three identical two-level atoms interacting at resonance with a single mode of the quantized field in a lossless cavity. The initial cavity field is prepared in the coherent state while the atoms may assume different initial states. For this system, we investigate some kinds of entanglement which arise between the different parts of the system due to the interaction. The emphasis in our investigation is mainly on two things: first we want to test the dynamics of a pairwise, tripartite entanglement in the atomic ensemble and the cooperative entanglement between different system parties. We point out that as the atomic system starts either from a maximally pairwise entangled state or a maximally tripartite entangled state or a fully separable state, a competition starts between the cooperative nature of the Dicke model interaction and the pairwise entanglement. In fact we will find how this nature will support one sort of entanglement over the other.

The other issue is the relationship between the previously mentioned entanglement and the energy exchange between the field and the atomic ensemble monitored by the atomic ensemble inversions. In order to enrich our investigation, we will also discuss the field–atom entanglement through the Q-function.

This paper is organized as follows. In section 2 we introduce the system and its solution. Section 3 is devoted to the atomic inversions. In section 4 we study the cooperative entanglement between different parties versus the atoms’ pairwise entanglement, using the generalized I-concurrency and the concurrence. Section 5 is devoted to the residual three-particle entanglement, where we will use the negativity to quantify this type of entanglement. The field dynamics in phase space, the possibility of having a cat state and the atom–field entanglement are addressed in section 6. Finally in section 7, we conclude the paper with a discussion of the results.

2. The model and its time evolution

We consider a system of three identical two-level atoms interacting with a quantized single-mode electromagnetic field. Under the rotating-wave approximation and resonant condition, the Hamiltonian of this system reads

$$\hat{H} = \hat{H}_0 + \hat{H}_I = \omega_f \left( \hat{a} \hat{a}^\dagger + \frac{1}{2} \sum_{i=a,b,c} \sigma_z^{(i)} \right) + g \left( \hat{a} \sum_{i=a,b,c} \sigma_x^{(i)} + \hat{a}^\dagger \sum_{i=a,b,c} \sigma_x^{(i)} \right)$$  (1)

where $\hbar = 1$, the terms $\hat{H}_0$ and $\hat{H}_I$ represent the free and interaction Hamiltonians respectively, $\omega_f$ is the field frequency, and equals the atomic transition frequency on resonance, $\sigma_x^{(i)}$ and $\sigma_z^{(i)}$ are the usual raising, lowering and inversion operators for the $i$th atom, satisfying $[\sigma_x^{(i)}, \sigma_x^{(j)}] = [\sigma_z^{(i)}, \sigma_z^{(j)}] = 0$ while $\hat{a} \hat{a}^\dagger (+\hat{a}^\dagger \hat{a})$ is the Bose creation (annihilation) operator for the quantized field mode satisfying the commutation relations $[\hat{a}, \hat{a}^\dagger] = 1$, and $g$ is the coupling constant. Since $[\hat{H}_0, \hat{H}_I] = 0$, it follows that Hamiltonian (1) conserves the total number of excitations $N$, i.e. the total excitation operator

$$\hat{N} = \hat{a}^\dagger \hat{a} + \frac{1}{2} \sum_{i=a,b,c} \sigma_z^{(i)}$$  (2)

is a constant of motion. This provides a decomposition for the system Hilbert space as $\mathcal{H} = \bigoplus_{n=0}^{\infty} \mathcal{H}_n$, such that $\mathcal{H}_0 = \{|g, g, g; 0\}, \mathcal{H}_1 = \{|g, g, g; 1\}, |g, e, g; 0\}, |g, g, e; 0\}, |e, g, g; 0\}$ and $\mathcal{H}_2 = \{|g, g, g; 2\}, |g, e, g; 1\}, |g, g, e; 1\}, |e, g, g; 0\}, |e, g, e; 0\}, |g, e, g; 0\}, |g, e, e; 0\}$ are the one-dimensional, four-dimensional and seven-dimensional eigensubspaces of $\hat{N}$ respectively and $\mathcal{H}_{n+3} \bigoplus_{n=0}^{\infty} \{|eeg; n+1\}, |eeg; n+1\}, |gee; n+1\}, |geg; n+1\}, |geg; n+2\}, |geg; n+2\}, |ggg; n+3\}$ are the eight-dimensional eigensubspaces of $\hat{N}$. In the Hilbert space constituted by the above basis, the interaction $\hat{H}_I$ is a diagonal block matrix made up of $8 \times 8$ sub-matrices; every sub-matrix represents a subspace corresponding to a definite excitation number $N$. However, we found that it is more appropriate for this system to use the Dicke states [17] as a basis, because of the degeneracy produced by the symmetry. For an $N$-spin 1/2 particle system, the Dicke states are defined as the states $|S, m_S\rangle$ that are common eigenstates of both the square of the total spin operator $\hat{S}^2$ and its component along the $z$-axis $S_z$ with the corresponding eigenvalues $S(S+1)$ and $m_S \hbar$. For our Dicke system the states of the above-mentioned product states are given by

- $|D_1\rangle = |eeg; n\rangle$
- $|D_2\rangle = \frac{1}{\sqrt{2}} (|eeg; n+1\rangle + |egg; n+1\rangle + |gee; n+1\rangle)$
- $|D_3\rangle = \frac{1}{\sqrt{2}} (|eeg; n+2\rangle + |egg; n+2\rangle + |gee; n+2\rangle)$
- $|D_4\rangle = |ggg; n+3\rangle$
- $|D_5\rangle = \frac{1}{\sqrt{2}} (2|gee; n+1\rangle - |eeg; n+1\rangle - |egg; n+1\rangle)$
- $|D_6\rangle = \frac{1}{\sqrt{2}} (|eeg; n+1\rangle - |egg; n+1\rangle - |gee; n+1\rangle)$
- $|D_7\rangle = \frac{1}{\sqrt{2}} (|gee; n+2\rangle + |ggg; n+2\rangle - 2|egg; n+2\rangle)$
- $|D_8\rangle = \frac{1}{\sqrt{2}} (|ggg; n+2\rangle - |egg; n+2\rangle).$  (3)

The first four states are fully symmetric with respect to the permutation of particles, while the other four states are in
fact two states of mixed symmetry, which correspond to two degenerate representations, each of dimension 2 [26]. By substituting into the Schrödinger equation $-i \hbar \frac{d}{dt} \psi(t) = \hat{H}_f \psi(t)$, where $r = gt$, the wavefunction generally can be written as $\psi(t) = \sum_{\pm} X^{(\pm)}_i(t) D_{\pm i}$. We found, as expected, that the equations of the probability amplitudes which correspond to the two degenerate states are decoupled from each other and from the four amplitudes which correspond to the fully symmetric Dicke states. The initial state of the system can be described as follows:

$$|\psi(0)\rangle = \left|\psi_g(0)\right\rangle \otimes \left|\psi_f(0)\right\rangle$$

$$|\psi_g(0)\rangle = C_1|e, e, e\rangle + \frac{1}{\sqrt{3}} C_{w1}|e, e, g\rangle + |e, g, e\rangle + |g, e, e\rangle$$

$$+ \frac{1}{\sqrt{3}} C_{w2}|g, e, g\rangle + |g, e, g\rangle + |e, g, g\rangle + C_g|g, g, g\rangle$$

$$|\psi_f(0)\rangle = \sum_{n=0}\alpha_n|n\rangle; \quad \alpha_n = e^{(-n|n|^2/2)} \frac{a_n^2}{\sqrt{n!}}$$

where $|C_1|^2 + |C_{w1}|^2 + |C_{w2}|^2 + |C_g|^2 = 1$. For the above initial atomic states the wavefunction evolve only inside the subspace spanned by the fully symmetric Dicke states and the probability amplitudes of the other state vanishes; hence our system is equivalent to a four-level Dicke atom. This is a generalization of the case of the two identical atoms [27], where the antisymmetric states do not participate in the dynamics. Solving the four coupled equations of the probability amplitudes we get the wavefunction of the system at any time

$$|\psi(t)\rangle = \sum_{n=0}^{\infty} \sum_{i=1}^4 X_i^{(n)}(t)|D_i\rangle$$

$$X_i(t) = \begin{bmatrix} X_i^{(1)}(t) & X_i^{(2)}(t) & X_i^{(3)}(t) & X_i^{(4)}(t) \end{bmatrix}^T$$

$$= U X(0)$$

where the evolution matrix $U$ is given by

$$U = \begin{bmatrix} U_{11} & U_{12} & U_{13} & U_{14} \\ U_{21} & U_{22} & U_{23} & U_{24} \\ U_{31} & U_{32} & U_{33} & U_{34} \\ U_{41} & U_{42} & U_{43} & U_{44} \end{bmatrix}$$

The explicit form of the matrix elements is given in the appendix, where we have

$$\mu_{1,2} = \frac{1}{2}(\delta \pm \sqrt{\delta^2 - 36\eta^2\gamma^2})$$

$$\delta = (4\beta^2 + 3\gamma^2 + 3\eta^2)$$

$$\gamma = \sqrt{n} + 1, \quad \beta = \sqrt{n} + 2, \quad \eta = \sqrt{n} + 3$$

Note that $X_i^{(n)}$ has $q_{n+i-1}$ as a factor, where $i = 1, 2, 3, 4$. By using this wavefunction, we discuss different aspects of the system in what follows.

3. Atomic inversions

Energy is the primary quantity determining the properties of physical systems. The atomic inversion and level occupation probabilities are the simplest nontrivial physical quantities in the atom–field interaction that display the exchange of energy between the field and the atoms. More important for our investigation is that the atomic inversions display the primary nonclassical effect, namely the collapse and revival structure [28–31] from which we can have information about the atom–field entanglement and disentanglement through the dynamics [9–11, 32–34]. Finally the collapses and revivals will be clearly connected to the evolution of the Q-function [35, 36].

In this section we shall discuss these quantities. We shall deal with three quantities, namely

$$W_T(t) = \sum_{n=0}^{\infty}\left(\left|X_i^{(n)}(t)\right|^2 - \left|X_i^{(n)}(0)\right|^2\right)$$

$$W_a(t) = \left\langle\psi(t)|\hat{\sigma}_z^{(n)}|\psi(t)\right\rangle$$

$$= \sum_{n=0}^{\infty}\left(\left|X_i^{(n)}(t)\right|^2 + \frac{1}{3}\left|X_i^{(n)}(t)\right|^2 - \frac{1}{3}\left|X_i^{(n)}(t)\right|^2\right)$$

where $W_T(t)$ is the total atomic inversion, $W_a(t)$ is the inversion of the atom $a$ and $P_m(t)$ is the probability of occupation of the initial state. The two-atom inversion is double the one-atom inversion. In figure 1 we show these quantities for the atoms being initially (a) in the excited state $|\psi_g(0)\rangle = |eee\rangle$, or (b) the atoms initially in a GHZ-state [37], $|\psi_g(0)\rangle = |\text{GHZ}\rangle = \frac{1}{\sqrt{2}}(|ggee\rangle + |eeee\rangle)$, which has the property that tracing over any one qubit results in a maximally mixed state containing no entanglement between the remaining two qubits, or (c) in a genuine entangled W-state ‘Werner state’ [38], $|\psi_g(0)\rangle = |W\rangle = \frac{1}{\sqrt{2}}(|eegg\rangle + |gee\rangle + |eee\rangle)$; in this state when tracing over any one qubit the average remaining bipartite entanglement is maximal. Before we dwell on the discussion of the figures we present a rough analysis of the time-dependent quantities in the expressions (8)–(10). Since we are dealing with a coherent state for which $\bar{n} = 100$, it is well known that the effective excitation will be due to the photons within the range $|n - \bar{n}| \leq \Delta n = \sqrt{\bar{n}}$. For these photons and those within a reasonable range of them, the radicands appearing in the expressions for $\beta, \gamma$ and $\eta$ in equation (7) can be approximated to $\sqrt{\bar{n}}$ and the expressions for $\mu_1, \mu_2$ are then given by ‘$9n$’ and ‘$n$’ respectively. Thus, the Rabi frequencies appearing in the expressions for $U_{ij}$ are to be approximated as $3\sqrt{\bar{n}}$ and $\sqrt{\bar{n}}$ respectively. Keeping these points in mind, we find the following for the case of the atomic initial state $|\psi_g(0)\rangle = |eee\rangle$, i.e. the three atoms are in their excited states: regarding the quantity $W_T(t)$, the time-dependent summand is proportional to $\frac{1}{\bar{n}}(\cos 6\sqrt{\bar{n}}t + 15\cos 2\sqrt{\bar{n}}t)$, containing super harmonics which results in three revival times $\frac{1}{\bar{n}}(3\sqrt{\bar{n}} + 2\sqrt{\bar{n}}$ and $2\pi\sqrt{\bar{n}}$ related to the trigonometric functions with two equal heights for the 1st and 2nd revival and 16 times larger at the 3rd revival. This appears clearly in figure 1(a1). On the other hand for the single atom population inversion $W_a(t)$, the time-dependent summand is proportional to $\cos 2\sqrt{\bar{n}}t$, which results in a single revival time at $2\pi\sqrt{\bar{n}}$ which is depicted in figure 1(a2). The $P_m(t)$ has the term $\frac{1}{\bar{n}}(10 + \cos 6\sqrt{\bar{n}}t + 6\cos 4\sqrt{\bar{n}}t + 15\cos 2\sqrt{\bar{n}}t)$
in its summand, composed of two and three superharmonics giving rise to revival times at \( \frac{2\pi}{3}\sqrt{n}\), \( \pi\sqrt{n}\), \( \frac{4\pi}{3}\sqrt{n}\) and \( 2\pi\sqrt{n}\). The second revival is larger than the 1st and 3rd because of the coefficients of \( \cos 6n\tau\) and \( \cos 4\sqrt{n}\pi\) while the 4th, at \( 2\pi\sqrt{n}\), is the highest as the four revivals coincide. This is shown clearly in figure 1(a3) where the quantity \( W_{\text{ini}}(t)\) fluctuates around \( \frac{\pi}{5}\).

Now we look at the initial state \( |\Psi_e(0)\rangle = \frac{1}{\sqrt{2}}(|egg\rangle + |eee\rangle)\) in figures 1(b1)–(3b), and applying the same analysis as in the previous case, we find the following: the quantity \( W_{\tau}(t)\) as well as \( W_{\text{ini}}(t)\) is almost zero as can be seen from figures 1(b1) and (b2). In contrast, the quantity \( P_{\text{ini}}(t)\) depends on the summand of the form \( \frac{1}{3}(5 + \cos 4\sqrt{n}\pi)\) which results in revival times \( \pi\sqrt{n}\) and \( 2\pi\sqrt{n}\) shown clearly in figure 1(b3). Also it exhibits the fluctuations around \( \frac{\pi}{3}\). These results show that starting from the GHZ-state would result in coherent trapping and the atoms almost do not interact with the field. This is shown in the value of \( \frac{\pi}{3}\) for the probability for the atoms staying in the initial state.

The case of the atomic initial Werner state, namely \( |\Psi_e(0)\rangle = \frac{1}{\sqrt{2}}(|eeeg\rangle + |ege\rangle + |gee\rangle)\), is shown in figures 1(c1)–(c3). Analysis of the summand of \( W_{\tau}(t)\) reveals that the time dependence is of the form \( \cos 6\sqrt{n}\pi + \cos 2\sqrt{n}\pi\). This amounts to the revival times \( \frac{2\pi}{3}\sqrt{n}, \frac{2\pi}{3}\sqrt{n}, \pi\sqrt{n}\) and \( 2\pi\sqrt{n}\) with the amplitude at the 3rd revival twice as much as the 1st revival. However, the single atom population inversion has a single revival at \( 2\pi\sqrt{n}\) because the time-dependent term in the summand is proportional to \( \cos 2\sqrt{n}\pi\).

Besides superharmonics a notable feature appearing in figure 1 is that the fluctuations in the population inversion are the highest for the \( |eee\rangle\) initial state and the lowest for the GHZ initial state showing an almost coherent trapping for the latter in such a way as to resist the exchange of energy with the field, whereas the W-state is in the middle, showing a modest degree of exchange of the energy with the field. The results of the dynamics of attractor states [21] support these findings. The previous feature of the GHZ noted for the other two states of the tendency of the energy exchange with the field will reflect itself in the next sections, when we will investigate the system entanglement.

4. Cooperative versus atoms’ pairwise entanglement

The system under investigation is a multipartite system initially in an overall pure state, occupying the following Hilbert space \( \mathcal{H} = \mathcal{H}_a \otimes \mathcal{H}_{bc} \otimes \mathcal{H}_d \otimes \mathcal{H}_f \) with dimension \( 2 \otimes 2 \otimes 2 \otimes \infty \). Since we consider here only the cooperative case, whereas the Hamiltonian is symmetric under atom exchange, there are only eight nonequivalent partitions of bipartite subsystems, namely (i) \( f \otimes (abc) \), (ii) \( f \otimes ab \), (iii) \( f \otimes a \), (iv) \( f \otimes b \), (v) \( v \otimes bc \) and (vi) \( v \otimes b \), corresponding to the field times the whole atomic system, two atoms times the field with the remaining atom, one atom times the field with the other two atoms, the field times two atoms, the field times one atom, one atom times the field with one atom, one atom times the other two atoms and one atom times one atom respectively.

These partitions except (i) are obtained by tracing over one or more Hilbert subspaces of the total Hilbert space and hence are generally mixed states. For convenience and simplicity we will only study the entanglement evolution of the partitions
(i) and the entanglement of the last partition, i.e. atom–atom entanglement. In the following section the entanglement of the bipartite partition (v) will be discussed together with the phenomenon of entanglement sharing [3, 40] and the residual the-particle entanglement using the negativity.

For the bipartite partitions in (i), each of them starts from an overall pure state and a number of widely accepted measures of entanglement are available. An easier way to quantify the entanglement in this bipartite partitions is to use the square of the pure-state I-concurrence introduced in [41],

$$C^2(\Psi) = 2v_{d_1}v_{d_2}[1 - \text{Tr}\{\hat{\rho}^2_{J}\}] = 2v_{d_1}v_{d_2}[1 - \text{Tr}\{\hat{\rho}^2_{abc}\}].$$

(11)

This generalizes the original notion of concurrence introduced by Hill and Wooters [42] $C(\Psi) = \sqrt{\langle\Psi|S_2 \otimes S_2(|\Psi\rangle\langle\Psi|)|\Psi\rangle} = \sqrt{|\langle\Psi|\sigma_y \otimes \sigma_y|\Psi\rangle|}$ for pairs of qubits in a joint pure state $|\Psi\rangle$, to be applied for pairs of quantum systems of arbitrary dimension $d_1 \otimes d_2$ in a joint pure state. The concurrence is defined with the help of a superoperator $S_\sigma$, whose action on a qubit density operator $\rho = |\Psi\rangle\langle\Psi|$ is to flip the spin of the qubit density operator $S_\sigma(\rho) = \rho^{*}; \rho^{*} = |\Psi\rangle\langle\Psi|$ where $|\Psi\rangle = \sigma_y \otimes \sigma_y|\Psi\rangle$, the asterisk denotes the complex conjugate and $\sigma_y$ is the Pauli matrix. Rungta et al [41] use the formalism for superoperators [43] to generalize the spin-flip superoperator $S_\sigma$ for a qubit to a superoperator $S_\sigma$ that acts on qudit states ($d$-dimensional states). For defining an I-concurrence, one should choose the scaling factor $v_d$ to be independent of $d$, otherwise, the pure state I-concurrence could be changed simply by adding extra, unused dimensions to one of the subsystems. To be consistent with the qubit concurrence, one should choose $v_d = 1$. With this choice the pure-state I-concurrence runs from zero for product states to $I_{max} = \sqrt{2m-1}$, where $m = \min(d_1, d_2)$, for a maximally entangled state. Henceforth, we will use only the term I-concurrence when referring to it.

On the other hand, Wooters extended the concurrence notation to the case of a two qubits in an arbitrary joint mixed state; he showed that the entanglement of formation [44, 45] of an arbitrary two-qubit mixed state $\rho$ can be written in terms of the minimum average pure-state concurrence of ensemble decompositions of $\rho$, and derived an explicit expression for this minimum in terms of the eigenvalues of $\rho\hat{\rho}$ [46].

$$C(\rho) = \max(0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4)$$

(12)

where $\lambda_i$ are the eigenvalues, in decreasing order, of the Hermitian matrix $R = \sqrt{\rho\hat{\rho}}\sqrt{\rho\hat{\rho}}$. Alternatively, one can say that the $\lambda_i$ are the square roots of the eigenvalues of the matrix $\rho\hat{\rho}$ and each $\lambda_i$ is a non-negative real number. The spin-flipped state $\hat{\rho}$ is obtained by spin flipping, namely $\hat{\rho} = (\sigma_y \otimes \sigma_y)\rho^{*}(\sigma_y \otimes \sigma_y)$, where again the asterisk denotes the complex conjugate. For a pure state $|\Psi\rangle$, $R$ has only one eigenvalue that may be nonzero, namely $C(\Psi)$ (equation (11)). Wooters called this minimum average, the concurrence of the mixed state. The entanglement of the last partition, i.e. atom–atom entanglement, can be investigated using this formula.

It is worth noting that the remaining partitions are all in a mixed state with dimensions $2 \otimes \infty, 4 \otimes \infty$ and $2 \otimes 4$, which cannot be quantified by any of the above-mentioned entanglement measures. Following Tessier et al [3], we can quantify the entanglement for all the above partitions using Osborne’s formula [47] for the I-concurrence $\tau(\rho_{AB})$ for mixed states $\rho_{AB}$ of a pair AB of qudits, of dimensions $d_A$ and $d_B$, having no more than two nonzero eigenvalues, i.e. $\rho_{AB}$ with rank not greater than 2. The I-concurrence $\tau$ between A and B is given by the expression

$$\tau(\rho_{AB}) = \text{Tr}\{\rho_{AB}\hat{\rho}_{AB}\} + 2\lambda_{min}(1 - \text{Tr}\{\rho_{AB}^2\})$$

(13)

where $\lambda_{min}$ is the smallest eigenvalue of the real symmetric $3 \times 3$ matrix $M$ as defined in [47].

In figure 2 we show the I-concurrence of the reduced density matrices $\rho_{abc}, \rho_{ab}$ and $\rho_a, (a)$ for the $|\text{eee}\rangle$ case, (b) the $|\text{GHZ}\rangle$ case and (c) the $|\text{W}\rangle$ case. This type of entanglement may be called a cooperative entanglement. Figures 2(a1)–(a3), (b1) and (c1) begin from zero which corresponds to the initial product state while figures 2(b2), (b3), (c2) and (c3) start from the three-particle maximum entanglement state. They suddenly increase in the collapse region to reach the maximum values, $I_{max} = 1.34, 1.23$ and $f$ for $f \otimes (abc)$, $(f c) \otimes ab$ and $(f c b) \otimes a$ respectively. The exception is the entanglement between the field and the total atomic system in the case of the GHZ-state in figure 2(b1), which does not reach this maximum. This reflects the robustness of the initial entanglement of this state, i.e. the initial entanglement between the atoms resists the interaction and the energy exchange between them and the field as shown in the discussion of the population inversion of the last section. When this entanglement becomes rather weak the field–atom entanglement increases rapidly to the maximum value, $I_{max}$. On the other hand, the W-state case did not show such robustness. There are many local minima and maxima asymptotically increasing to reach the maximum values, $I_{max}$. Finally, it is noted that the GHZ-state shows more periodicity than the other two cases.

Figure 3 shows the concurrence of the reduced density matrix $C(\rho_{ab})$ for the different initial states. In contrast to the above cooperative entanglement, the pairwise atoms’ entanglement asymptotically decreases and vanishes. The initially excited atoms and the GHZ case start from zero while the $|\text{W}\rangle$ case starts from maximum entanglement $C(\rho_{ab}) = \frac{3}{2}$ and then there is a sudden death [48–50] and sudden rebirth ‘anabiosis’ [51]. This phenomenon appears in this case as a bipartite subsystem where the third atom and the field are traced.

We conclude this section by the following: first, the cooperative nature of the considered system supports the cooperative entanglement on the expense of the pairwise atoms’ entanglement. While the first develops to its maxima the other develops to its minima. Second, and more important, we find that a resistance of energy exchange with the cavity field mode ‘and hence a resistance to the interaction’ which the GHZ-state shows in section 3 is accompanied here with a clear entanglement robustness. This behaviour nominates this state for the quantum information tasks which require a resistance to the environment dissipation. In fact we conjecture that this state will resist the interaction with the modes of the environment. It should be noted that the previous behaviour...
Figure 2. The $I$-concurrence for the three bipartite partitions in (i), (a) $|eee\rangle$-initial state, (b) $|\text{GHZ}\rangle$-initial state and (c) $|W\rangle$-initial state.

Figure 3. Evolution of the atoms’ pairwise entanglement, by the concurrence of $\hat{\rho}_{ab}$. For (a) $|eee\rangle$-initial state, (b) $|\text{GHZ}\rangle$-initial state and (c) $|W\rangle$-initial state.

is closely related to the phenomenon of entanglement sharing \cite{52, 53}. This behaviour clearly shows a connection between the energy exchange and entanglement.

5. Three-particle residual entanglement

In the previous section we studied the entanglement shared in the whole system, i.e. the atomic ensemble plus the field. Here we will look at the same behaviour but inside the atomic ensemble, after tracing out the field.

By using the notion of negativity \cite{14, 16}, we shall study the phenomena of entanglement sharing and the residual three-particle entanglement ‘three-way entanglement’ \cite{3, 40}. Coffman et al \cite{40} introduced the tangle notation to quantify the entanglement of three qubits A, B and C in a joint pure state and they discussed the entanglement sharing between three particles. They found that unlike classical correlation, quantum entanglement cannot be freely shared among the particles. More precisely they found that the following inequality

\begin{equation}
\tau_{A(BC)} \geq \tau_{AB} + \tau_{AC} \tag{14}
\end{equation}

holds for any three qubits in a joint pure state. Here the tangle $\tau_{A(BC)}$ is the $I$-concurrence of equation (11), between A and the other two qubits as one entity, while the other two tangles $\tau_{AB}$ and $\tau_{AC}$ are the squares of concurrence of equation (12), between A and B, and A and C respectively. Consequently they define the 3-tangle (or residual tangle) $\tau_{ABC}$ as follows:

\begin{equation}
\tau_{ABC} = \tau_{AB} + \tau_{AC} \tag{15}
\end{equation}

which states that the entanglement of A with the rest of the system is equal to the entanglement of A with B alone, plus the entanglement of A with C alone plus the 3-tangle of the whole system. The 3-tangle quantifies the residual three-particle entanglement, which cannot be accounted for by
Figure 4. Evolution of the negativity $N_{(ab)}(\rho_{ab}), N_{a-b}(\rho_{ab})$ and the residual three-particle negativity $N_{abc}$ for different initial atomic states: (a) $|eee\rangle$-initial state, (b) $|\text{GHZ}\rangle$-initial state and (c) $|W\rangle$-initial state.

Figure 5. The $Q$-function for the initial atomic state $|\Psi(0)\rangle = |eee\rangle$. 
the pairwise entanglement. For the GHZ-state we have \(1 = 0 + 0 + \tau_{ABC}\). Also they showed that the inequality (14) is true in case of three particles in a joint mixed state.

On the other hand the Peres–Horodecki criterion for separability [12, 13] leads to a natural computable measure of entanglement, called negativity [14–16]. The negativity is based on the trace norm of the partial transpose \(\rho^{T_x}\) of the bipartite mixed state \(\rho_{AB}\), and measures the degree to which \(\rho^{T_x}\) fails to be positive. The density matrices which represent physical systems are non-negative matrices with unit trace, the partial transpose also satisfies \(\text{Tr}\{\rho^{T_x}\} = 1\), but since it may have negative eigenvalues \(\mu_i\) ‘for entangled states’ its trace norm reads in general

\[
\|\rho^{T_x}\|_1 = 1 + 2\sum_i |\mu_i| = 1 + \mathcal{N}(\rho) \tag{16}
\]

where \(\mathcal{N}(\rho)\) is the negativity, i.e. the absolute value of twice the sum of the negative eigenvalues. Vidal and Werner [16] proved that the negativity \(\mathcal{N}(\rho)\) is an entanglement monotone and therefore is a good measure of entanglement. Following [16], the residual three-particle entanglement \(\mathcal{N}_{abc}\) in our case can be quantified using the relation

\[
\mathcal{N}_{abc} = \mathcal{N}_{a-bc}(\rho_{abc}) = \mathcal{N}_{a-b}(\rho_{ab}) = \mathcal{N}_{a-c}(\rho_{ac}) \tag{17}
\]

The term \(\mathcal{N}_{a-bc}(\rho_{abc})\) quantifies the strength of quantum correlations between the atom ‘a’ and the other two atoms. The remaining two terms quantify the pairwise entanglement between the atom ‘a’ and b or c. Note that the partial trace operation belongs to the set of local operations and classical communication (LOCC) under which the entanglement cannot increase. Therefore, the left-hand side of equation (17) is a residual three-particle entanglement. In our case \(\mathcal{N}_{a-b}(\rho_{ab}) = \mathcal{N}_{a-c}(\rho_{ac})\) because of the symmetry. In figure 4 we show the results for the different initial states. We see that the residual three-particle entanglement changes between local maxima and minima during the evolution. The local maxima occur at half-revival times, when the atoms became most disentangled from the field. Asymptotically this residual entanglement vanishes and the atomic system becomes maximally entangled with the field. From figures 3 and 4 it is shown that the negativity measure of the two-atom entanglement is less than or at most equal to the concurrence as is conjectured in [15].

\[
\mathcal{N}_{a-bc}(\rho_{abc}) = \|\rho_{abc}^{T_x}\|_1 - 1 \tag{18}
\]

\[
\mathcal{N}_{a-b}(\rho_{ab}) = \|\rho_{ab}^{T_x}\|_1 - 1 = \|\rho_{ab}^{T_x}\|_1 - 1 \tag{19}
\]

\[
\mathcal{N}_{a-c}(\rho_{ac}) = \|\rho_{ac}^{T_x}\|_1 - 1 = \|\rho_{ac}^{T_x}\|_1 - 1 \tag{20}
\]
With respect to the entanglement sharing, a remarkable feature pointed out here is that, as in the previous section, the cooperative nature of our system supports the three-particle versus the pairwise atoms’ entanglement inside the atomic ensemble. This has also been noted in the study of entanglement dynamics with relation to attractor states for the case of three-qubit states [21]. In the next section we will try to connect the entanglement dynamics with field dynamics demonstrated by the field $Q$-function.

6. $Q$-function

In this section we return to the entanglement between the field and the atomic system but through the field dynamic; namely we will discuss the field quasi-probability distribution function, the $Q$-function [54], defined as $Q(\beta, t) = \frac{1}{\pi} \langle \beta | \rho_f (t) | \beta \rangle$. Using equation (5) we get the following form of the $Q$-function in terms of probability amplitudes:

$$ Q(\alpha, t) = e^{-|\beta|^2} \sum_{i=1}^{4} \sum_{n=0}^{\infty} \frac{(-1)^{n+i-1}}{\sqrt{(n+i-1)!}} X_i^{(n)} . \quad (21) $$

Figures 5–7 show the $Q$-function for different initial atomic states at different characteristic times $t_0 = 0$, $t_1 = \frac{\pi}{2} \sqrt{n}$, $t_2 = \pi \sqrt{n}$, $t_3 = \frac{2\pi}{3} \sqrt{n}$, $t_4 = 2\pi \sqrt{n}$, $t_5 = 2\pi \sqrt{n}$. The field is initially in a coherent state with the average photon number $|\alpha_0|^2 = \bar{n} = 100$. At $\tau = 0$ the shape of the $Q$-function is a Gaussian centred at $(\sqrt{n}, 0)$ as expected for the initial coherent state. But as time develops we find different types of behaviour associated with each initial atomic state. For the state $|\text{eee}\rangle$ depicted in figure 5, we note that the single peak is split into two pairs, a pair with large amplitude moving slowly in opposite directions and the other pair of the smaller amplitudes moving fastly in opposite directions. To shed some light on this behaviour, rough analysis of formula (21), similar to that used in the atomic inversion, shows that time-dependent factors in the summand behave like $\{ \cos 3(\sqrt{n} - \sqrt{m}) \tau + 3 \cos(\sqrt{n} - \sqrt{m}) \tau \}$. This form exhibits clearly the appearance of the pairs and the first term represents the faster pair while the second term represents the slower pair with amplitude three times larger than the faster ones. Note the appearance of the Schrödinger cat-state [34], at half-revival ($\tau = t_3$). The case of the state $\sqrt{2}(|\text{ggg}\rangle + |\text{eee}\rangle)$ is quite interesting. Here the peak splits into only two peaks; a small faster peak and a large slower peak. The analysis shows that the time-dependent factor in the summand behaves as $e^{i(\sqrt{m} - \sqrt{n}) \tau} + 3e^{i(\sqrt{n} - \sqrt{m}) \tau}$ which exhibits the two single peaks; they move in opposite directions and the
larger one is the slower peak. The third case of the Werner state as depicted in figure 7 shows an opposite behaviour to that of the $|eee\rangle$ state. The pair with the larger amplitude moves faster than the pair with smaller amplitude. The time-dependent factor in the summand in this case behaves like $\{3 \cos(\sqrt{m_1} - \sqrt{m_2}) + \cos(\sqrt{m_1} - \sqrt{m_2})\}$ which demonstrates the fast movement of the larger components and the slow movement of the smaller components. It is to be remarked that at half revival time starting from either the uppermost excited state ($|eee\rangle$) or the Werner state the Schrödinger cat-state is produced. On the other hand starting from the GHZ-state we find that at half revival time the field state returns to its original state. Because as the larger component moves $\frac{1}{2}$ in the phase space the smaller and faster component would move $\frac{1}{2}$ on the other direction and meets the larger one. The previous behaviour is also demonstrated in figure 1(b) regarding the atomic inversions.

Now looking at the various entanglement evolutions in figures 2 and 4 we find that there is a clear connection between the $Q$-function dynamics and various entanglement evolutions. For the $|eee\rangle$ and $|W\rangle$ initial states, at half the first-revival time $t_1$ which corresponds to a local minima of $I$-Concurrence, $I_{I(ab)}$ in the $|eee\rangle$ case and to the minimum value in the $|W\rangle$ case, the faster peaks became most apart from each other. Also at that time the residual three-particle entanglement has a local maximum. At the first-revival time $t_2$ these two peaks collide and recombine, and this corresponds to a local minima of $I_{I(ab)}$ in the two cases, a local minima of the residual 3-particle in $|eee\rangle$ case and a maximum value in the $|W\rangle$ case. Also we note that the amplitudes of these two peaks increase significantly in the $|W\rangle$ case. This behaviour can be connected to figures 3(c1)–(c3), where it is clear that at time $t_3$ the entanglement of all the atomic ensembles with the field is minimum and this is not the case in the $|eee\rangle$ state. The last interesting feature we want to mention here for the $|eee\rangle$ and Wener cases is that, at $t_5$, the $Q$-function shows phase squeezing.

7. Discussion and conclusion

In this paper we considered a system of three identical two-level atoms interacting at resonance with a single-mode of the quantized field in a lossless cavity. The initial cavity field is prepared in the coherent state while the atoms are taken to be in different initial states, namely in the excited state, $|eee\rangle$, the GHZ entangled state and the Werner entangled state. For this system we investigated different kinds of entanglement, atom field cooperative and atoms’ pairwise entanglements. We use the concurrence, the generalized $I$-concurrence and the negativity as the measures of these types of entanglements. The relationship between this entanglement and the collapse and revival in the atomic inversion is investigated. Also the $Q$-functions for different cases are discussed and connected to different entanglements evolutions of the system. Most significantly, we found that the GHZ-state is more robust against energy losses, and shows almost coherent trapping. Also one can say that the entanglement of GHZ-state is more robust than that of the W-state. These two different behaviours have been distinctly shown through the study, clearly depicted in figures 1(b1) and 4(b1) respectively. This suggests that the GHZ-state may show more resistance to the decoherence phenomena and energy dissipation to the environment than any other three-partite states. Consequently it may be a primary candidate for many quantum information tasks, which need a three-partite entangled state. In fact the GHZ-state is a resource for many applications; these include quantum secret sharing [6], open destination teleportation [55] and quantum computation [56]. On the other hand for the other two cases ($|eee\rangle$ and $|W\rangle$ states) the Schrödinger cat-state is produced. Another interesting feature is the clear link between the $Q$-function dynamics and various entanglement evolutions. Finally we found that, while the W-state is the state with the maximal possible bipartite entanglement in the reduced two-qubit states, its initial entanglement vanishes very rapidly. Moreover the production of such a pairwise entanglement through the evolution is very small. This in contrast to the other two cases which have no pairwise entanglement initially but such entanglements increase greatly through the evolution. Sudden death and sudden revival of the atoms’ pairwise entanglement are produced with the W-state.

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Appendix. Evolution matrix for the probabilities amplitudes

In this appendix we give the explicit form of the elements of the evolution matrix $U$ appearing in equation (6):

$$U_{11} = \left[ \frac{\mu_1 - 4\beta^2 - 3\eta^2}{(\mu_1 - \mu_2)(\mu_2 - \mu_1)} \cos(\sqrt{\mu_1} \tau) + \frac{\mu_2 - 4\beta^2 - 3\eta^2}{(\mu_2 - \mu_1)} \cos(\sqrt{\mu_2} \tau) \right]$$

$$U_{12} = -i\sqrt{3}\gamma \left[ \frac{\mu_1 - 3\eta^2}{\sqrt{\mu_1}(\mu_1 - \mu_2)} \sin(\sqrt{\mu_1} \tau) + \frac{\mu_2 - 3\eta^2}{\sqrt{\mu_2}(\mu_2 - \mu_1)} \sin(\sqrt{\mu_2} \tau) \right]$$

$$U_{13} = 2\sqrt{3}\beta\gamma \left[ \frac{1}{(\mu_1 - \mu_2)} \cos(\sqrt{\mu_1} \tau) + \frac{1}{(\mu_2 - \mu_1)} \cos(\sqrt{\mu_2} \tau) \right]$$

$$U_{14} = -6i\beta\gamma \eta \left[ \frac{1}{\sqrt{\mu_1}(\mu_1 - \mu_2)} \sin(\sqrt{\mu_1} \tau) + \frac{1}{\sqrt{\mu_2}(\mu_2 - \mu_1)} \sin(\sqrt{\mu_2} \tau) \right]$$

$$U_{22} = \left[ \frac{\mu_1 - 3\eta^2}{(\mu_1 - \mu_2)} \cos(\sqrt{\mu_1} \tau) + \frac{\mu_2 - 3\eta^2}{(\mu_2 - \mu_1)} \cos(\sqrt{\mu_2} \tau) \right]$$
\[ U_{33} = -2i\beta \left[ \frac{\sqrt{\mu_1}(\mu_1 - \mu_2)}{\mu_1 - \mu_2} \sin(\sqrt{\mu_1} \tau) \right] \\
+ \frac{1}{\sqrt{\mu_2}} \sin(\sqrt{\mu_2} \tau) \right] \\
U_{34} = 2\sqrt{3}\beta n \left[ \frac{1}{\mu_1 - \mu_2} \cos(\sqrt{\mu_1} \tau) \right] \\
+ \frac{1}{\mu_2 - \mu_1} \cos(\sqrt{\mu_2} \tau) \right] \\
U_{33} = \left[ \frac{\mu_1 - 3\gamma^2}{\mu_1 - \mu_2} \cos(\sqrt{\mu_1} \tau) + \frac{\mu_2 - 3\gamma^2}{\mu_2 - \mu_1} \cos(\sqrt{\mu_2} \tau) \right] \\
U_{34} = -i\sqrt{3} \left[ \frac{-\mu_1 - 3\gamma^2}{\mu_1 - \mu_2} \sin(\sqrt{\mu_1} \tau) \right] \\
+ \frac{\mu_2 - 3\gamma^2}{\sqrt{2}(\mu_2 - \mu_1)} \sin(\sqrt{\mu_2} \tau) \right] \\
U_{44} = \left[ \frac{\mu_1 - 4\beta^2 - 3\gamma^2}{\mu_1 - \mu_2} \cos(\sqrt{\mu_1} \tau) \\
+ \frac{\mu_2 - 4\beta^2 - 3\gamma^2}{\mu_2 - \mu_1} \cos(\sqrt{\mu_2} \tau) \right]. \]

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