Information-disturbance tradeoff in estimating a maximally entangled state

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We derive the amount of information retrieved by a quantum measurement in estimating an unknown maximally entangled state, along with the pertaining disturbance on the state itself. The optimal tradeoff between information and disturbance is obtained, and a corresponding optimal measurement is provided.

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The tradeoff between information retrieved from a quantum measurement and the disturbance caused on the state of a quantum system is a fundamental concept of quantum mechanics and has received a lot of attention in the literature. Such an issue is studied for both foundations and its enormous relevance in practice, in the realm of quantum key distribution and quantum cryptography.

A part from many heuristic statements of the information-disturbance tradeoff, just a few quantitative derivations have been obtained in the scenario of quantum state estimation. The optimal tradeoff has been derived in the following cases: in estimating a single copy of an unknown pure state, many copies of identically prepared pure qubits, a single copy of a pure state generated by independent phase-shifts, and an unknown coherent state. Recently, experiment realization of minimal disturbance measurements has been also reported.

The problem is typically the following. One performs a measurement on a quantum state picked (randomly, or according to an assigned a priori distribution) from a known set, and evaluates the retrieved information along with the disturbance caused on the state. The physical transformation will be described by a quantum operation (in an old-fashioned terminology, a measurement of the first kind, where it is possible to describe the state transformation will be described by a quantum operation with the disturbance caused on the state. The physical known set, and evaluates the retrieved information along, in the realm of quantum key distribution and quantum cryptography.

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In this Letter, we study and provide the optimal tradeoff between estimation and operation fidelities when the state is a completely unknown maximally entangled state of finite-dimensional quantum systems. We also provide a measurement that achieves such an optimal tradeoff.

The interest in maximally entangled states lies in the fact that they represent a major resource in quantum information technology, e.g. in quantum teleportation and quantum cryptography. The study of the information-disturbance tradeoff for maximally entangled states can become of practical relevance for posing general limits in information eavesdropping and for analyzing security of quantum cryptographic communications.

Our results will be obtained by exploiting the group symmetry of the problem, which allows us to restrict our analysis on covariant measurement instruments. In fact, the property of covariance generally leads to a striking simplification of problems that may look intractable, and has been thoroughly used in the context of state and parameter estimation.

A measurement process on a quantum state $\rho$ with outcomes $\{r\}$ is described by an instrument, namely a set of trace-decreasing completely positive (CP) maps $\{\mathcal{E}_r\}$. Each map can then be written in the Kraus form

$$\mathcal{E}_r(\rho) = \sum_{\mu} A_{r\mu} \rho A_{r\mu}^\dagger,$$

and provides the state after the measurement

$$\rho_r = \frac{\mathcal{E}_r(\rho)}{\text{Tr}\left[\mathcal{E}_r(\rho)\right]} ,$$

along with the probability of outcome

$$p_r = \text{Tr}\left[\mathcal{E}_r(\rho)\right] = \text{Tr}\left[\sum_{\mu} A_{r\mu}^\dagger A_{r\mu} \rho\right].$$

The set of positive operators $\{\Pi_r = \sum_{\mu} A_{r\mu}^\dagger A_{r\mu}\}$ is known as positive operator-valued measure (POVM), and normalization requires the completeness relation $\sum_r \Pi_r = I$. This is equivalent to require that the map $\sum_r \mathcal{E}_r$ is trace-preserving.

When considering bipartite systems it is convenient to exploit the natural isomorphism between operators $A$ on the Hilbert space $\mathcal{H}$ and vectors $|A\rangle$ in $\mathcal{H}^{\otimes 2}$, defined through the equation

$$|A\rangle \equiv \sum_{m,n} \langle m|A|n\rangle |m\rangle |n\rangle .$$
We will make repeated use of the following identities \[ A \otimes B | C \rangle = | A C B^\tau \rangle, \]
\[ \text{Tr}[ A|B \rangle \langle B| C ] = A^\tau B, \]
\[ \text{Tr}_{x} [ A|B \rangle \langle B| C ] = A B^\dagger, \]
\[ \langle A|B \rangle = \text{Tr}[A^\dagger B], \]
where \( \tau \) and \( * \) denote transposition and complex conjugation with respect to the fixed basis in Eq. (4), and \( \text{Tr} \) represents the partial trace over the \( i \)th Hilbert space. A maximally entangled state in \( \mathcal{H} \otimes \mathcal{H} \), with \( \text{dim}(\mathcal{H}) = d \) will then be written as \( \sum_{g} |U_{g}\rangle \rangle \), where \( U_{g} \) is a unitary \( d \times d \) matrix, i.e. \( g \) denotes an element of the group \( SU(d) \). When performing averages on group parameters, for convenience we will take the normalized invariance \( SU(d) \) and we use \( \int d g = 1 \), and we will also omit \( SU(d) \) from the symbol of integral.

To avoid confusion when the number of Hilbert spaces proliferates, we will also use the notation \( |A\rangle_{ij} \) when it is necessary to identify the vector in the Hilbert space \( \mathcal{H}_{i} \otimes \mathcal{H}_{j} \). Similarly, \( A\langle^i \rangle \) will denote a linear operator acting on \( \mathcal{H}_{i} \otimes \mathcal{H}_{j} \).

The operation fidelity \( F \) evaluates on average how much the state after the measurement resembles the original one, in terms of the squared modulus of the scalar product. Hence, for a measurement of an unknown maximally entangled state, one has
\[ F = \frac{1}{d^2} \int d g \sum_{r \mu} | \langle U_{g} | A_{r \mu} | U_{g} \rangle |^2, \]
where \( \{ A_{r \mu} \} \) are the Kraus operators of the measurement instrument \( \{ U_{g} \} \). For each measurement outcome \( r \), one guesses a maximally entangled state \( \frac{1}{\sqrt{d}} | U_{r} \rangle \rangle \) and the corresponding average estimation fidelity is given by
\[ G = \frac{1}{d^3} \int d g \sum_{r \mu} | \langle U_{g} | A_{r \mu}^\dagger A_{r \mu} | U_{g} \rangle | \langle U_{r} | U_{g} \rangle |^2. \]
Without loss of generality, we can restrict out attention to covariant instruments, that satisfy
\[ \mathcal{E}_{h}(U_{g} \otimes I \rho U_{g}^\dagger \otimes I) = (U_{g} \otimes I)\mathcal{E}_{-1h}(\rho)(U_{g}^\dagger \otimes I). \]
In fact, for any instrument \( \{ U_{g} \} \) and guess \( \frac{1}{\sqrt{d}} | U_{r} \rangle \rangle \), one can easily show that the covariant instrument
\[ \mathcal{E}_{h}(\rho) = \sum_{r \mu} (U_{h} U_{r}^\dagger \otimes I) A_{r \mu}(U_{r} U_{h}^\dagger \otimes I) \rho \times (U_{h} U_{r}^\dagger \otimes I) A_{r \mu}^\dagger(U_{r} U_{h}^\dagger \otimes I), \]
with continuous outcome \( h \in SU(d) \), along with the guess \( \frac{1}{\sqrt{d}} | U_{h} \rangle \rangle \), provides the same values of \( F \) and \( G \) as the original instrument \( \{ U_{g} \} \).

It is useful now to consider the Jamiołkowski representation \( \{ U_{g} \} \), that gives a one-to-one correspondence between a CP map \( \mathcal{E} \) from \( \mathcal{H}_{in} \) to \( \mathcal{H}_{out} \) and a positive operator \( R \) on \( \mathcal{H}_{out} \otimes \mathcal{H}_{in} \) through the equations
\[ \mathcal{E}(\rho) = \text{Tr}_{in}[(I_{out} \otimes \rho^{*}) R], \]
\[ R = (\mathcal{E} \otimes I_{in} ) | I \rangle \langle I|. \]
When \( \mathcal{E} \) is trace preserving, one has also \( \text{Tr}_{out}[R] = I_{in} \).

For covariant instruments \( \mathcal{E}_{\tau} \) acting on \( \mathcal{H}_{1} \otimes \mathcal{H}_{2} \) as in Eq. (12), the operator \( R_{g} \) acts on \( \mathcal{H}_{\tau} \otimes 1 \), and has the form
\[ R_{g} = U^{(1)}_{g} \otimes U^{*}_{g}(3) R_{0} U^{(1)}_{g} \otimes U^{*}_{g}(3), \]
with \( R_{0} \geq 0 \), and the trace-preserving condition
\[ \int d g \text{Tr}_{34}[R_{g}] = I^{(12)} \]
From Eq. (14) and the identity (Schur’s lemma for irreducible group representations \( \{ U_{g} \} \))
\[ \int d g U_{g} X U_{g}^{\dagger} = \frac{1}{d} \text{Tr}[X] I, \]
it follows that condition (15) is equivalent to
\[ \text{Tr}_{134}[R_{0}] = d I^{(2)}, \]
which implies \( \text{Tr}[R_{0}] = d^2 \).

By defining the projector on the unnormalized maximally entangled vector of \( \mathcal{H}_{1} \otimes \mathcal{H}_{2} \) as
\[ \mathcal{T}^{(ij)} = | I \rangle \langle I |, \]
the fidelities \( F \) and \( G \) can be written as \( F = \text{Tr}[R_{F} R_{0}] \) and \( G = \text{Tr}[R_{G} R_{0}] \), where \( R_{F} \) and \( R_{G} \) are the following positive operators
\[ R_{F} = \frac{1}{d^2} \int d g U^{(1)}_{g} \otimes U^{*}_{g}(3) T^{(12)} \otimes T^{(34)} U^{(1)}_{g} \otimes U^{*}_{g}(3), \]
\[ R_{G} = \frac{1}{d^3} \int d g | I \langle U_{g} |)^{2} U^{*}_{g}(3) T^{(12)} \otimes T^{(34)} U_{g}(3) \]
\[ = \frac{1}{d} \{ I^{(12)} \otimes \text{Tr}_{12}[T^{(12)} \otimes T^{(34)} R_{F}] \}. \]
Using the identity (Schur’s lemma for reducible group representations \( \{ U_{g} \} \))
\[ \int d g U_{g} \otimes U_{g}^* Y U_{g}^\dagger \otimes U_{g}^* = \text{Tr}[Y T_{d}/d] I - [I d]/d^2 - 1, \]
one obtains
\[ R_{F} = \frac{1}{d^2(d^2 - 1)} \left[ I + T^{(13)} \otimes T^{(24)} \right. \]
\[ - \frac{1}{d^2} \{ I^{(13)} \otimes T^{(24)} + T^{(13)} \otimes I^{(24)} \}, \]
\[ R_{G} = \frac{1}{d^2(d^2 - 1)} \left[ \left( 1 - \frac{2}{d^2} \right) I + \frac{1}{d^2} I^{(12)} \otimes T^{(34)} \right]. \]
The optimal tradeoff between $F$ and $G$ can be found by looking for a positive operator $R_0$ that satisfies the trace-preserving condition \[ \text{Tr}[R] = 1 \] and maximizes a convex combination
\[ pG + (1 - p)F = \text{Tr}\{(pR_G + (1 - p)R_F)R_0\}, \tag{20} \]
where $p \in [0, 1]$ controls the tradeoff between the quality of the state estimation and the quality of the output replica of the state. Then, $R_0$ will provide a covariant instrument that achieves the optimal tradeoff. It turns out that for any $p$ the eigenvector corresponding to the maximum eigenvalue of $C(p) = pR_G + (1 - p)R_F$ is of the form \[ |\chi\rangle = x|\lambda\rangle|I\rangle_{12}|I\rangle_{34} + y|\lambda\rangle|I\rangle_{13}|I\rangle_{24}, \tag{21} \]
with suitable positive $x$ and $y$. Upon taking $R_0$ proportional to $|\chi\rangle|\chi\rangle$, the covariant instrument will then be optimal. In fact, condition \[ \text{Tr}[R] = 1 \] can be easily verified, and the normalization can be derived from the condition \[ \text{Tr}[R_0] = d^2. \]

From Eqs. (13) and (14), it follows that the optimal tradeoff can be reached by an instrument with Kraus operators
\[ A_y = a|U_y\rangle\langle U_y| + bI, \tag{22} \]
where $0 \leq a \leq 1$, and $b = \frac{d^2}{2} - \frac{1}{d^2}(\sqrt{2d^2(1 - a^2)(a^2 - 1)}).$ In fact, condition \[ \text{Tr}[R_0] = d^2 \] is equivalent to \[ (a^2 + b^2)d^2 + 2abd = d^2. \] The corresponding fidelities are given by
\[ F = \frac{1}{d^2(d^2 - 1)}[d^2 + (d^2 - 2)(a + bd)] = 1 - \frac{d^2 - 2 + a^2}{d^2}, \]
\[ G = \frac{1}{d^2(d^2 - 1)}[d^2 - 2 + (ad + b)^2] = 1 - \frac{d^2 - 2 + a^2}{d^2}. \]

Notice that the instrument given by operators (22) is pure, in the sense that it leaves pure states as pure. When no measurement is performed ($a = 0$), one has $F = 1$ and $G = \frac{d^2}{2}$, which is equivalent to randomly guessing the unknown state. The optimal estimation can be obtained by a Bell measurement ($b = 0$), namely by projectors on maximally entangled states, and gives $F = G = \frac{d^2}{2}$. Upon eliminating $a$ and $b$, we obtain the optimal tradeoff between $F$ and $G$
\[ \sqrt{(d^2 - 2)(2 - d^2)G} = \sqrt{(d^2 - 1)F - 1 - \sqrt{1 - F}}, \tag{23} \]
or, equivalently,
\[ \sqrt{\frac{d^2}{d^2 - 2}} \left( F - \frac{1}{d^2 - 1} \right) = \sqrt{G - \frac{d^2 - 2}{d^2(d^2 - 1)}} + \sqrt{(d^2 - 1) \left( \frac{2}{d^2} - G \right)}. \tag{24} \]

Such an optimal tradeoff overcomes the corresponding one for a completely unknown state \[ \text{in a Hilbert space with dimension } d^2, \] i.e. for a fixed value of the estimation fidelity $G$ one can achieve here a better value of the operation fidelity $F$. In other words, when a partial knowledge of the set of states is available (here, the fact that the states are maximally entangled), one can obtain the same estimation fidelity with a smaller disturbance of the state.

We can introduce two normalized quantities—a sort of visibilities—that can be interpreted as the average information $I$ retrieved from the quantum measurement and the average disturbance $D$ affecting the original quantum state as follows:
\[ I = \frac{G - G_0}{G_{\text{max}} - G_0} = d^2G - 1 = 1 - b^2, \tag{25} \]
where $G_0 = \frac{d^2}{2}$ is the value of $G$ for random guess and $G_{\text{max}} = \frac{d^2}{4}$ is the maximum value attainable by $G$;
\[ D = \frac{1 - F}{1 - F_{\text{min}}} = \frac{d^2(1 - F)}{d^2 - 2} = a^2, \tag{26} \]
where $F_{\text{min}} = \frac{d^2}{4}$ represents the average fidelity with the maximally chaotic state $\frac{I}{d^2}$. Clearly, one has $0 \leq I \leq 1$, and $0 \leq D \leq 1$. In this way, after some algebra one obtains the quadratic expression
\[ d^2(D - 1)^2 - 4D(1 - I) = 0 \tag{27} \]
that gives the optimal information-disturbance tradeoff. We plot in Fig. 1 the behavior of the tradeoff for dimension $d = 2, 4,$ and $8$. For a given value of the information $I$, the curves $D(I)$ represent a lower bound for the disturbance of any measurement instrument.

![FIG. 1: Optimal information-disturbance tradeoff in estimating an unknown maximally entangled state for dimension $d = 2$ (solid line), $d = 4$ (dashed line), and $d = 8$ (dotted), where $I$ and $D$ are defined through Eqs. (25) and (26) in terms of the estimation and operation fidelities $G$ and $F$, respectively. For given value of the retrieved information $I$, the curves $D(I)$ are a lower bound for the disturbance of any measurement instrument.](image)
and the optimal map for estimating an unknown maximally entangled state. It can be easily shown that the discrete version \( \{E_r\} \) of such an instrument with Kraus operators
\[
A_r = \frac{1}{d} (a |U_r\rangle \langle U_r| + b I), \quad r = 1, 2, \ldots, d^2, \quad (28)
\]
and orthogonal \( \{ |U_r\rangle\} \), namely \( \langle U_r | U_s\rangle = d \delta_{rs} \), achieves the same values of \( F \) and \( G \), and hence the optimal tradeoff as well. Notice that the POVM \( A_r A_r^\dagger \) corresponding to this instrument is made of projectors on so-called Werner states \(29\), i.e. convex mixtures of maximally entangled and maximally chaotic states.

The experimental realization of such a kind of measurement could be investigated for hyperentangled two-photon states, for which Bell measurements have been already demonstrated \(30\).

In conclusion, a tight bound between the quality of estimation of an unknown maximally entangled state and the degree the initial state has to be changed by this operation has been derived. Such a bound can be achieved by noisy Bell measurements, where the noise continuously controls the tradeoff between the information retrieved by the measurement and the disturbance on the original state.

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\† Also at CNISM - Consorzio Nazionale Interuniversitario per le Scienze Fisiche della Materia

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25. It can be shown analytically that the operator norm of \( C(p) \) (i.e. its maximum eigenvalue) satisfies
\[
\|C(p)\| = \frac{1}{2d^2(d^2 - 1)} \left\{ d^4 - 4d + 3d^2p - d^2p \right\}
\]
\[
+ d\sqrt{d^2[2 - d^2(1 - p)^2 + 4(d^2 - 2)p(1 - p)]}
\]
for \( d = 2, 3 \), and the corresponding eigenvector is of the form as in Eq. 21, with \( x = 1 \) and
\[
y = \frac{1}{2p} \left\{ d^2(1 - p) - d(2 - p) \right\}
\]
\[
\sqrt{d^2[2 - d^2(1 - p)^2 + 4(d^2 - 2)p(1 - p)]}
\]
For higher values of \( d \), such a result can be checked, for example, by means of the power method.
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