A Simple Grand Unified Relation between Neutrino Mixing and Quark Mixing

S.M. Barr and Heng-Yu Chen
Department of Physics and Astronomy and
The Bartol Research Institute
University of Delaware
Newark, Delaware 19716

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Abstract

It is proposed that all flavor mixing is caused by the mixing of the three quark and lepton families with vectorlike fermions in $5 + \overline{5}$ multiplets of $SU(5)$. This simple assumption implies that both $V_{CKM}$ and $U_{MNS}$ are generated by a single matrix. The entire $3 \times 3$ complex mass matrix of the neutrinos $M_\nu$ is then found to have a simple expression in terms of two complex parameters and an overall scale. Thus, all the presently unknown neutrino parameters are predicted. The best fits are for $\theta_{atm} \sim 40^\circ$. The leptonic Dirac CP phase is found to be somewhat greater than $\pi$.

It is a striking fact that the leptonic mixing angles of the MNS matrix \cite{1} are much larger than the corresponding quark mixing angles of the CKM matrix \cite{2}. Grand unification suggests a simple explanation for this. In $SU(5)$, a family of quarks and leptons is contained in the multiplets $10 + \overline{5}$, with the left-handed leptons contained in the $\overline{5}$ and the left-handed quarks contained in the $10$. Thus, if there is more mixing among the $\overline{5}$ multiplets of different families than among the $10$ multiplets, the disparity between leptonic and quark mixing angles would be explained. This idea can be implemented in models based on any grand unified group, since all such groups contain $SU(5)$ as a subgroup. Several ways of implementing this basic idea have been proposed in the literature \cite{3,4}.

Here we propose a model in which the three $10 + \overline{5}$ families of fermions are supplemented by three $5 + \overline{5}$ pairs. (The possible existence of such additional “vectorlike” fermions has been much discussed in the literature in a variety of contexts \cite{3,5,6,7,8}.) The central idea of the model proposed here is that all inter-family mixing is caused by the mixing between the $\overline{5}$ multiplets of the ordinary families and the $\overline{5}$ multiplets of the additional vectorlike pairs. As a consequence of having this common source, both quark mixing and lepton mixing are controlled in this model by a single matrix, which we call $A$. This matrix can be determined from the masses and mixing angles of the quarks alone, and this allows the entire $3 \times 3$ complex mass matrix $M_\nu$ of the known neutrinos (which contains 9 real physical observables) to be predicted in terms of just two complex parameters and an overall mass scale. The resulting formula turns out to be quite simple. In the “flavor basis” of the neutrinos, i.e. the basis $(\nu_e, \nu_\mu, \nu_\tau)$, the neutrino mass matrix is given by

$$M_\nu \approx \begin{pmatrix}
\frac{m_e}{m_d}|V_{us}| & 0 & 0 \\
\frac{m_e}{m_d}|V_{ub}|e^{i\delta} & \frac{m_d}{m_s}|V_{cb}| & 1
\end{pmatrix}
\begin{pmatrix}
qe^{i\beta} & 0 & 0 \\
0 & pe^{i\alpha} & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & \frac{m_s}{m_d}|V_{us}| & \frac{m_d}{m_s}|V_{ub}|e^{i\delta} \\
0 & 1 & \frac{m_s}{m_d}|V_{cb}|
\end{pmatrix}
\mu_\nu, \quad (1)$$
where $p e^{i\alpha}$, $q e^{i\beta}$, and the overall scale $\mu_{\nu}$ are free parameters of the model, $\delta$ is the Kobayashi-Maskawa CP phase, and the $V_{ij}$ are the CKM matrix elements. The expression $\sqrt{|V_{us}|}$ stands for $\sin\theta_{us} \cos\theta_{us} = V_{us}/\sqrt{1 - |V_{us}|^2}$. (Since the other quark mixing angles are small, their cosines can be set to one, and we can use simply $V_{ub}$ and $V_{ub}$.) Note that the CKM elements $V_{ij}$ describing quark mixing enter this formula in such a way that if they vanish the neutrino mass matrix becomes diagonal and the neutrino mixing vanishes also. But while these CKM mixing parameters are small, they are here multiplied by large ratios of quark masses, explaining naturally why the neutrino mixing is of order one. ($\frac{m_{\nu}}{m_q} |V_{us}| \sim 4$, $\frac{m_{\nu}}{m_q} |V_{cb}| \sim 2$, and $\frac{m_{\nu}}{m_q} |V_{ub}| \sim 3$.) We shall see later how this arises.

In this model, there are three families of fermions denoted by $10_i + \overline{5}_i$ ($i = 1, 2, 3$), and three vectorlike pairs of fermion multiplets denoted by $5_A' + \overline{5}_A$ ($A = 1, 2, 3$). In the absence of the vectorlike pairs, the Yukawa couplings and mass matrices of the three families would be flavor diagonal, due to discrete symmetries that distinguish the three families from each other. All flavor mixing is indirectly caused by mass terms that mix the $\overline{5}_i$ with the $\overline{5}_A'$. The model is defined by the following quark and lepton Yukawa terms:

$$L_{Yuk} = Y_i(10, 10_i)\langle 5_H \rangle + y_i(10, \overline{5}_i)\langle 5_H^\dagger \rangle$$
$$+ \overline{Y}_i(10, 10_i)\langle 5_H \rangle + \overline{y}_i(10, \overline{5}_i)\langle 5_H^\dagger \rangle$$
$$+ \frac{\lambda}{M_{R}}\langle 5_i, \overline{5}_i \rangle\langle 5_H \rangle\langle 5_H \rangle^\dagger$$
$$+ Y_{AB}(5_A, \overline{5}_B)\langle 1_H \rangle + y_{AB}(5_A, \overline{5}_B)\langle 1_H \rangle^\dagger,$$  (2)

where the subscript $H$ denotes Higgs multiplets. The renormalizable Yukawa terms in Eq. (2) are the most general allowed by the symmetry $K_1 \times K_2 \times K_3 \times K'$, where (for a given $i$ equal to 1, 2, or 3) $K_i$ is a $Z_2$ symmetry under which $10_i$, $\overline{5}_i$, and $1_i'$ are odd and all other fields even. $K'$ is a $Z_N$ symmetry ($N > 2$) under which $5_A' \rightarrow e^{2\pi i/N}5_A'$, $\overline{5}_A \rightarrow e^{2\pi i/N}\overline{5}_A$, $1_H \rightarrow e^{-4\pi i/N}1_H$, and $1_i' \rightarrow e^{-2\pi i/N}1_i'$. (This is just one example. Many other simple discrete symmetries would give the form in Eq. (2).)

The first four terms in Eq. (2) are the standard Yukawa terms of $SU(5)$ grand unification, and are the minimal terms needed to give mass to the known quarks and leptons. (As already noted in the original Georgi-Glashow paper on $SU(5)$ unification, the presence of a $45$ of Higgs fields avoids the unrealistic relations between down quark and charged lepton masses that would arise if only a $5$ of Higgs fields existed.) The fifth term is the standard dimension-5 Weinberg operator that gives the left-handed neutrinos Majorana masses. (The symmetry $K'$ prevents other dimension-5 operators that would give neutrino masses, such as $5_A'\overline{5}_B5_H^\dagger5_H^\dagger$.) Note that all these standard terms are forced to be flavor diagonal by the $K_i$ symmetries.

The last two terms in Eq. (2) are the only ones peculiar to this model. The first of these simply gives masses to the vectorlike fermions, and the second gives masses that mix these vectorlike fermions with the three families. The Higgs fields in these two terms are gauge singlets, so that their VEVs would naturally be superlarge. All that matters for the purposes of this paper is that the masses coming from these two terms be roughly of the same scale, which we shall call $M_4$, and that this scale be large compared to the masses of the down quarks and charged leptons. Note that the Yukawa matrices in these two terms are in general not diagonal.

This is the model; all that remains is to extract its predictions. First, let us examine the mass matrix of the down quarks that emerges from Eq. (2). There are left-handed anti-down quarks in both $\overline{5}_i$ and $\overline{5}_A$, which will be denoted $d_i^c$ and $D_A^c$ respectively. There are left-handed down quarks in both $10_i$ and $5_A$, which will be denoted $d_i$ and $D_A$ respectively. Altogether, then, there is a $6 \times 6$
mass matrix for the down quarks, given by

\[
\mathcal{L}_{(d \ mass)} = (d_i \ D_A') \begin{pmatrix}
(m_D)_i \delta_{ij} & 0 \\
\Delta A_j & M_{AB}
\end{pmatrix}
\begin{pmatrix}
d_i \\
D_B'
\end{pmatrix},
\] (3)

where \((m_D)_i = y_i \langle 5_H^\dagger \rangle + \tilde{y}_i \langle 45_H^\dagger \rangle\), \(M_{AB} = Y'_{AB} \langle 1_H \rangle\), and \(\Delta A_j = y'_{A_j} \langle 1'_{H_j} \rangle\). Here and throughout the paper, Dirac mass matrices are multiplied from the left by the left-handed fermions and from the right by the right-handed fermions (or, equivalently, the left-handed anti-fermions).

The \(6 \times 6\) matrix in Eq. (3) can be block-diagonalized by multiplying it from the right by a unitary matrix \(U\) whose elements are of order one (since the elements of the matrices \(\Delta\) and \(M\) are of the same order) and from the left by a unitary matrix whose angles are of order \(m_D/M \ll 1\) and which therefore can be neglected. Specifically, the unitary matrix \(U\) is such that

\[
\begin{pmatrix}
m_D & 0 \\
\Delta & M
\end{pmatrix} U \equiv \begin{pmatrix}
m_D & 0 \\
\Delta & M
\end{pmatrix}
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix} = \begin{pmatrix}
m_D A & m_D B \\
0 & M
\end{pmatrix},
\] (4)

where \(A = (I + \Delta \Delta^{-1} M^{-1} \Delta^{-1/2})\). The off-diagonal block \(m_D B\) in the last matrix in Eq. (4) can be removed by a rotation from the left that is of order \(m_D/M\), which is negligible, as already noted. Thus, after block-diagonalization, the upper-left \(3 \times 3\) block that describes the masses of the three observed down quarks becomes simply

\[M_D = m_D A.\] (5)

In other words, the net effect of the mixing of the three families with the heavy vectorlike fermions is to multiply the diagonal mass matrix \(m_D\) from the right by a non-diagonal matrix \(A\). This can be understood diagramatically from Fig. 1.

**Fig. 1** Diagrams showing how the mass terms \((\Delta)\) that mix the \(\bar{5}_i\) with the \(\bar{5}'_A\) lead to insertions of the matrix \(A\) on external \(\bar{5}\) fermion lines.

From these diagrams, it is easy to see that such factors of \(A\) accompany external fermion lines that are in the \(\bar{5}\) representation of \(SU(5)\). Thus the mass matrices for the up quarks, down quarks,
charged leptons and neutrinos (which come, respectively, from \((10 \ 10)5_H\), \((10 \ 5)5_H^\dagger\), \((\overline{5} \ 10)5_H\), and \((\overline{5} \overline{5})5_H5_H\) terms) have the form

\[
M_U = m_U,
M_D = m_D A,
M_\ell = A^T m_\ell,
M_\nu = A^T m_\nu A
\]

where the matrices \(m_U\), \(m_D\), \(m_\ell\), and \(m_\nu\) are all diagonal. This can also easily be shown by block-diagonalizing the full mass matrices of the charged leptons and neutrinos in the same way that we did for the down quarks.

It might seem that the matrix we have called \(A\) should be different for the different types of fermions due to renormalization effects. At the unification scale, the same \(3 \times 3\) matrices \(\Delta\) and \(M\) appear in the \(6 \times 6\) mass matrices of the charged leptons and the down quarks. But, due to gluon loops, the \(\Delta\) and \(M\) of the down quarks should run more strongly between the unification scale and the scale \(\mathcal{M}\) than the corresponding matrices of the leptons. The crucial point, however, is that \(A\) depends on the ratio \(M^{-1}\Delta\); and since gauge boson loops cause \(\Delta\) and \(M\) to run in the same way, these effects cancel out in \(A\). Moreover, the renormalization effects due to Yukawa couplings (which are small for the \(\overline{5}\) fermions) can be neglected. Thus, it really is the same matrix \(A\) that appears in \(M_D\), \(M_\ell\), and \(M_\nu\). Ultimately, this is due to \(SU(5)\) symmetry. It is this fact that makes this model so predictive.

To extract the predictions of this model, let us consider first the quarks. The up quarks are already in the “mass basis”, since \(M_U = m_U\) is diagonal. The down quark mass matrix can be written

\[
M_D = m_D A = \mu_d \begin{pmatrix} \delta_d & 0 & 0 \\ 0 & \epsilon_d & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}
\]

\[
\longrightarrow \bar{\nu}_d \begin{pmatrix} \bar{\delta}_d & 0 & 0 \\ 0 & \bar{\tau}_d & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & b & ce^{i\theta} \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix} = \bar{\nu}_d \begin{pmatrix} \bar{\delta}_d \bar{\tau}_d & \bar{\delta}_d b & \bar{\delta}_d ce^{i\theta} \\ 0 & \bar{\tau}_d & \bar{\tau}_d a \\ 0 & 0 & 1 \end{pmatrix},
\]

where \(\mu_d\) is the 33 element of the diagonal matrix \(m_D\). As we have indicated in the second line of Eq. (7), the matrix \(A\) can be brought to triangular form by rotations of the right-handed quarks. Then, by rescaling the parameters \(\mu_d\), \(\epsilon_d\), and \(\delta_d\), one can make the diagonal elements of \(A\) equal to 1. And finally, by rephasings of the left-handed and right-handed quarks, one can remove all phases except one, which we can place in the 13 element, as shown.

Since the up quark mass matrix is already diagonal, the CKM mixing matrix of the quarks comes entirely from diagonalizing the mass matrix in Eq. (7). It is easy to show from that equation that

\[
|V_{cb}| = \sin \theta_{cb} \approx \tan \theta_{cb} \approx \tau_d a, \quad |V_{ub}| = \sin \theta_{ub} \approx \tan \theta_{ub} \approx \delta_{dc} c, \quad |V_{us}|\sqrt{1 - |V_{us}|^2} = \tan \theta_{us} \approx \delta_{db}/\tau_d,
\]

and \(\theta = \delta\), the CP phase of the quarks. (Notice that the basic structure predicted by the model, given in Eq.(6), explains why \(V_{ub} \approx V_{us}/V_{cb}\), since \(|V_{us}| = \Omega(\delta_d / \tau_d), \quad |V_{cb}| = \Omega(\tau_d), \quad \text{and} \quad |V_{ub}| = \Omega(\delta_d)\).

It is also clear from Eq. (7) that \(\tau_d \approx m_s / m_b\) and \(\delta_d \approx m_d / m_b\). Notice that this implies that \(\delta_d \ll \tau_d \ll 1\), so that the matrix \(m_D\) is not only diagonal (because of family symmetry) but “hierarchical”.

\[
A
\]
(Though the model as presented in Eq. (2) does not explain the mass hierarchy among the families, we shall see that a simple extension of the model can do this.) Combining the above relations gives

\[
\begin{align*}
    a & \approx \frac{m_a}{m_s} |V_{cb}| \sim 2, \\
    b & \approx \frac{m_b}{m_d} |V_{us}| \sqrt{1 - |V_{us}|^2} \sim 4, \\
    c & \approx \frac{m_c}{m_d} |V_{ub}| \sim 3, \\
    \theta & \approx \delta.
\end{align*}
\] (8)

Turning to the leptons, one sees that the mass matrix of the charged leptons can be written

\[
M_\ell = A m_\ell = \begin{pmatrix}
    A_{11} & A_{12} & A_{13} \\
    A_{21} & A_{22} & A_{23} \\
    A_{31} & A_{32} & A_{33}
\end{pmatrix}
\left(\begin{array}{ccc}
    \delta_\ell & 0 & 0 \\
    0 & \epsilon_\ell & 0 \\
    0 & 0 & 1
\end{array}\right)
\mu_\ell
\] (9)

We have transformed \(A\) to have the same form as in the second line of Eq. (7), by doing the same combination of rotations to the left-handed leptons as we did to the right-handed down quarks, followed by analogous rescalings and rephasings. If we do the same rotations to the left-handed charged leptons and left-handed neutrinos, no MNS mixing is induced at this stage; but by doing so, it is clear that we also make the matrix \(A\) appearing in \(M_\nu = A^T m_\nu A\) have the same form as in the second line of Eq. (7) and Eq. (9). What results is precisely the form shown in Eq. (1) for the neutrino mass matrix. Since we only have the freedom to rephase the left-handed neutrinos, there are three physical phases in Eq. (1), rather than one as in the other mass matrices. The extra two phases are the ones called \(\alpha\) and \(\beta\) in Eq. (1).

The hierarchy among the charged lepton masses tells us that \(\delta_\ell \ll \tau_\ell \ll 1\). So, the diagonal matrix \(m_\ell\) is hierarchical, just as \(m_D\) and \(m_U\) are. By comparing Eqs. (7) and (9), we see how this model explains the disparity between the neutrino mixing angles and quark mixing angles. Because \(M_D = m_D A\), whereas \(M_\ell = A^T m_\ell A\), the mass matrix of the down quarks has a hierarchy among the rows, whereas the charged lepton mass matrix has a hierarchy among the columns. Since rotations of the left-handed fermions (which are the ones relevant to the CKM and MNS mixing angles) are rotations among the rows, we see that small quark mixing angles and large lepton mixing angles arise. (This is a realization of the basic idea of “lopsided” models [3, 4].)

The charged lepton mass matrix in the second line of Eq. (9) is not yet diagonal, but to a very good approximation it can be diagonalized by rotations done only to the right-handed charged leptons. Rotations of the left-handed charged leptons are also required, but they are rotations by angles that are proportional to \(\tau_\ell^2\), \((\delta_\ell/\tau_\ell)^2\), and \(\delta_\ell^2\). The only one of these that is numerically significant is a rotation in the \(\mu_\ell, \tau_\ell\) plane by angle \(\theta_{\nu\tau} \approx a \tau_\ell^2 \sim 2(m_\mu/m_\tau)^2 \sim 0.4^\circ\). This contributes at the 1% level to the atmospheric neutrino mixing angle, and this has only a minor effect on the predictions of the model, as we shall see.

Thus, in effect, the mass matrix in the second line of Eq. (9) is in the mass basis of the left-handed charged leptons. Consequently, the mass matrix \(M_\nu\) shown in Eq. (1) contains all the information about the masses, mixings and CP-violating phases of the neutrinos. There are nine physical observables involved: the three neutrino masses, the three MNS angles, the Dirac CP phase, and the two majorana CP phases of the neutrinos. These are all determined by the five model parameters in Eq. (1): \(\mu_\nu, pe^{i\alpha}\), and \(qe^{i\beta}\).
Since five neutrino observables have already been measured ($\theta_{\text{sol}}$, $\theta_{\text{atm}}$, $\theta_{13}$, $\delta m_{23}^2$ and $\delta m_{12}^2$), we can use them to determine the five model parameters, and then predict the four as-yet-unmeasured neutrino observables. Since the equations are non-linear (they involve trigonometric functions), there is no guarantee that they can fit the five measured neutrino properties with five adjustable parameters. (To put it another way, the adjustable parameters $\alpha$ and $\beta$ are bound within the range $[0, 2\pi]$.) Nevertheless, good fits are obtained. This is only true, however, if some of the measured neutrino properties have values that lie within a smaller range than that presently allowed by experiment. For example, although the current experimental range of the atmospheric neutrino mixing angle is $\theta_{\text{atm}} = 45 \pm 6.5^\circ$ [11], the model only obtains good fits for $\theta_{\text{atm}} \lesssim 43^\circ$, with values near $40^\circ$ preferred, as we shall see. The fits also prefer a value of $\theta_{\text{sol}}$ greater than or equal to $34^\circ$, i.e. greater than the present experimental central value. The quark properties are also constrained: the best fits are obtained with $m_s/m_d \lesssim 20$, and $\delta$ greater than or equal to its present experimental central value. Thus, in addition to predicting the four as-yet-unmeasured neutrino observables, the model places non-trivial and testable constraints on the values of quantities that have been measured.

In Table I, we show a representative fit in which all the input quark parameters and the neutrino observables obtained as output are in their experimentally allowed ranges (and in most cases at their central values). The experimental values are taken from the 2012 Review of Particle Properties [11], except for $\delta_{\text{lep}}$ (the neutrino Dirac CP phase) where we use the result of a recent global analysis of neutrino data [12]. For $m_d/m_s$ we have used the renormalization group results of [13] to obtain $m_s(2\text{GeV})$, which is given in [11].

**Table I.** A fit to the quark and neutrino data. $\mu_\nu$, $p e^{i\alpha}$, and $g e^{i\beta}$ are model parameters. $\delta_{\text{lep}}$ is the neutrino Dirac CP phase, and $(M_\nu)_{ee}$ the mass that comes into neutrinoless double beta decay.

| Quantity | Values in fit | Experiment |
|----------|---------------|------------|
| $\mu_\nu$ | 0.1428 eV | — |
| $p e^{i\alpha}$ | $0.1525 e^{-2.134\pi}$ | — |
| $g e^{i\beta}$ | $0.01405 e^{-0.352\pi}$ | — |
| $m_d/m_s$ | 52.9 | $52.9 \pm 2.6$ |
| $m_s/m_d$ | 19 | 17 to 22 |
| $|V_{As}|$ | 0.2252 | $0.2252 \pm 0.0009$ |
| $|V_{Cb}|$ | 0.0409 | $0.0409 \pm 0.0011$ |
| $|V_{ub}|$ | 0.00415 | $0.00415 \pm 0.00049$ |
| $\delta$ | $1.30 \text{ rad}$ | $1.187^{+0.110}_{-0.109} \text{ rad}$ |
| $\theta_{\text{sol}}$ | $34.1^\circ$ | $33.89^\circ +0.59^\circ -0.971^\circ$ |
| $\theta_{\text{atm}}$ | $40^\circ$ | $45^\circ +6.5^\circ$ |
| $\theta_{13}$ | $9.12^\circ$ | $9.122^\circ +0.68^\circ -0.647^\circ$ |
| $\delta m_{23}^2$ | $2.32 \times 10^{-3} \text{ eV}^2$ | $2.32^{+0.12}_{-0.08} \times 10^{-3} \text{ eV}^2$ |
| $\delta m_{12}^2$ | $7.603 \times 10^{-5} \text{ eV}^2$ | $(7.5^{+0.2}) \times 10^{-5} \text{ eV}^2$ |
| $\delta_{\text{lep}}$ | $1.15\pi \text{ rad}$ | $1.1\pi^{+0.3\pi}_{-0.4\pi} \text{ rad}$ |
| $(M_\nu)_{ee}$ | 0.0020 eV | |

Note that the model’s prediction for $\delta_{\text{lep}}$ is $1.15\pi$ radians, which accords remarkably well with the one-sigma range found in [12] of $1.1\pi^{+0.3\pi}_{-0.4\pi} \text{ rad}$. The value of $(M_\nu)_{ee}$ (to which the amplitude of neutrinoless double beta decay is proportional) is much smaller than the experimental limits, which tend to be in the range of a few tenths of an eV to several eVs for different experiments [11]. This prediction of the model is not very sensitive to variation of the model’s input parameters.

Figs. 2-4 show the degree of sensitivity of the $\delta_{\text{lep}}$ prediction to the values of $\theta_{\text{atm}}$, $\theta_{\text{sol}}$, and $\delta$.
(the quark CP phase). In Fig. 2, we have fixed the values of all the quark mass ratios and CKM parameters, and of \( \theta_{\text{sol}} \) and \( \theta_{13} \), but have allowed \( \theta_{\text{atm}} \) and the ratio \( \delta m_{12}^2/\delta m_{23}^2 \) (which we henceforth call \( r \)) to take different values. The curves are the relation of \( r \) to the predicted \( \delta_{\text{lep}} \) for different values of \( \theta_{\text{atm}} \). The horizontal lines are the one-sigma limits for \( r \). One sees that \( \delta_{\text{lep}} \) is predicted to be roughly \( 1.15\pi \) radians and that values of \( \theta_{\text{atm}} \approx 41^\circ \) are preferred.

In Fig. 3, we have done a similar thing, but this time fixing \( \theta_{\text{atm}} \) to be \( 40^\circ \) and allowing \( \theta_{\text{sol}} \) and \( r \) to vary. One can see a preference for values of \( \theta_{\text{sol}} \) equal or above the present experimental central value. In Fig. 4, we have allowed the quark CP phase \( \delta \) and \( r \) to vary. One sees that the best-fit value of \( \delta_{\text{lep}} \) is rather insensitive to the assumed values of the measured quark and neutrino properties, but the width of the range of \( \delta_{\text{lep}} \) values that give good fits is quite sensitive. A more detailed analysis of the predictions of the model will be given in another paper.

**Fig. 2** The result of fits with the values of quark parameters given in Table I, \( \theta_{\text{sol}} = 34.2^\circ \), and \( \theta_{13} = 9.12^\circ \). The curves are the relation of \( r \) (\( = \delta m_{12}^2/\delta m_{23}^2 \)) to the predicted \( \delta_{\text{lep}} \) for different values of \( \theta_{\text{atm}} \). The horizontal lines are the one-sigma limits for \( r \).

**Fig. 3** The result of fits with the values of quark parameters given in Table I, \( \theta_{\text{atm}} = 40^\circ \), and \( \theta_{13} = 9.12^\circ \). The curves are the relation of \( r \) (\( = \delta m_{12}^2/\delta m_{23}^2 \)) to the predicted \( \delta_{\text{lep}} \) for different values of \( \theta_{\text{sol}} \).
Besides its great simplicity, one feature of the model proposed here that increases its plausibility is that it allows the simple solution to the Strong CP Problem proposed in [6]. All that is required is that CP be assumed to be a symmetry of the Lagrangian that is spontaneously broken by the VEVs of the singlet scalars $1'_{Hi}$ that produce the off-diagonal mass matrix $\Delta$ in Eq. (3). In fact, the model proposed here has the same structure as the model originally proposed by A.E. Nelson in [6], except that here the $3 \times 3$ mass matrices of the ordinary three families are required to be flavor diagonal by family symmetries.

The model proposed here gives an account of how the CKM and MNS flavor mixings arise, but does not explain the mass hierarchy among the families, since the hierarchies in the diagonal matrices $m_U$, $m_D$, $m_\ell$, and $m_\nu$ are simply assumed. There are, however, several simple ways in which the present model could be extended to give an explanation of the mass hierarchy. One way is to combine the structure in this model with the structure assumed in [3]. In that paper, the mass hierarchies were explained by the three ordinary families mixing with vectorlike $10 + \overline{10}$ fermion pairs in a way analogous to the mixing with $5 + \overline{5}$ assumed here. Combining the structures of the two models would be appealing since it would mean that the vectorlike fermions would comprise entire family-antifamily pairs. (It has been pointed out that this can lead in a simple way to unification of gauge couplings in non-SUSY models [8].)

Another possibility would be a Froggatt-Nielsen scheme [14]. For example, instead of the family symmetry assumed above, suppose that the $K_i$ ($i = 1$ or 2) were $Z_4$ symmetries, under which $\overline{5_i} \rightarrow -\overline{5_i}$, $1'_{Hi} \rightarrow -1'_{Hi}$, $10_i \rightarrow i10_i$, and $S_i \rightarrow iS_i$, where the $S_i$ are Froggatt-Nielsen fields that are $SU(5)$ singlets. Effective Yukawa terms containing factors of $10_i$ would then have to contain equal numbers of factors of $\langle S_i \rangle / M_F \equiv \epsilon_i$, where $M_F$ is some flavor-physics scale. If $\epsilon_1 \ll \epsilon_2 \ll 1$, a mass hierarchy among families would result. Moreover, the hierarchy would be strongest for the up quarks (for which it is quadratic in the $\epsilon_i$), intermediate for the down quarks and charged leptons (for which it is linear in the $\epsilon_i$), and weakest for the neutrinos (which involve no factors of $\epsilon_i$). This is just the pattern that is observed. One should note that the same relationship among the mass hierarchies is obtained in the approach of [3].

Finally it should be noted that the present model could be embedded in many grand unified schemes. For example, in an $SO(10)$ model, the ordinary families could be in three $16$ multiplets, while the vectorlike fermions could be in three $10$ multiplets. In $E_6$, one gets the extra vectorlike fermions “for free”, since the $27$ contains $16 + 10 + 1$ of $SO(10)$. Different patterns of breaking of the grand unified group could be assumed without affecting the predictions for fermion masses and
mixings. For example, in many unified models, an adjoint Higgs field does some of the breaking of the unification group. If that adjoint Higgs multiplet does not transform under the $K'$ symmetry mentioned after Eq. (2), it would not couple renormalizably to $(5', \overline{5}')_B$ or $(5', \overline{5})_I$ and hence not contribute to the matrices $\Delta$ and $M$ in Eq. (4) and the matrix $A$. Consequently, except for negligible higher-order corrections, the matrix $A$ would not “know” that the unification group is broken, and the same $A$ would appear in both the quark and lepton sectors, as is necessary for the model to be predictive.

In conclusion, if all flavor changing in both the quark and lepton sectors arises as a consequence of the mixing of the ordinary families with vectorlike fermions that are in $5 + \overline{5}$ of $SU(5)$, a testable relationship arises between the quark and lepton mixing. This relationship allows the prediction of the four as-yet-unmeasured neutrino observables as well as testable constraints on several quantities that have been measured. Measurement of the Dirac CP phase of the neutrinos $\delta_{\nu\nu}$, as well as more precise determinations of such quantities as $\theta_{\text{atm}}$, $\theta_{\text{sol}}$, $|V_{ub}|$, $m_s/m_d$, and $\delta$ (the quark CP phase) would provide stringent tests of the model.

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