Grid coupling mechanism in the semi-implicit adaptive Multi-Level Multi-Domain method

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Abstract. The Multi-Level Multi-Domain (MLMD) method is a semi-implicit adaptive method for Particle-In-Cell plasma simulations. It has been demonstrated in the past in simulations of Maxwellian plasmas, electrostatic and electromagnetic instabilities, plasma expansion in vacuum, magnetic reconnection [1, 2, 3]. In multiple occasions, it has been commented on the coupling between the coarse and the refined grid solutions. The coupling mechanism itself, however, has never been explored in depth. Here, we investigate the theoretical bases of grid coupling in the MLMD system. We obtain an evolution law for the electric field solution in the overlap area of the MLMD system which highlights a dependance on the densities and currents from both the coarse and the refined grid, rather than from the coarse grid alone: grid coupling is obtained via densities and currents.

1. Introduction

Fully kinetic simulations of space plasmas have greatly benefitted, over the last decades, of the almost predictable technological improvements commonly known as Moore’s law: semiconductor technology doubles the number of transistors per unit area every 18 months [4]. It is however becoming clear that the technological trend that allowed computational scientists to perform larger and more resolved simulations every year is not going to last. [5] blames the prohibitive power consumption of High Performance Computing infrastructures, the need to improve the resiliency of the booming number of components and the increasing gap between the increment of computing performance and that of memory and storage. Even innovative architectures, such as GPUs and Xeon Phi’s, may not be an easy solution to smoothly achieve exascale computing ($> 10^{18}$ floating point operations per second): as remarked in [6], very few applications can fully exploit concurrency. Hence, algorithmic improvement, as opposed to computational improvement alone, becomes a key factor for larger, more resolved fully kinetic plasma simulations in the coming years.

The development of more efficient algorithms for Particle In Cell (PIC) simulations of plasmas [7, 8] has moved, over the decades, in two directions: (semi-) implicit and adaptive algorithms.

Explicit PIC algorithms are subject to rather strict stability constraints which result from the explicit temporal discretisation of the equations for field and particle advancement. Fully implicit and semi-implicit methods can afford more relaxed stability constraints. They can focus
on processes of interest at scales larger than the ones (Debye length, inverse electron plasma frequency) which explicit algorithms need to resolve for stability. The stability constraint of (semi)-implicit algorithms is usually of the form

$$\sigma < \frac{v_{th,e} dt}{dx} < 1,$$

(1)

with $dt$ and $dx$ temporal and spatial resolution, $v_{th,e}$ the average thermal electron velocity and $\sigma \sim 0.01$. This is achieved through rather advanced ways of dealing with the coupling between the field and particle system of equations. The coupling arises from the implicit time discretisation. In fully implicit methods [9, 10] this coupling is not removed and the equation for the field and particle systems are solved concurrently. In semi-implicit methods, the coupling is removed by approximating the particle sources in Maxwell's equations around the particle position at the previous (Implicit Moment Methods – IMM [11, 12, 13]) or next (direct implicit methods [14, 15, 16, 17]) time step.

Adaptive methods rely on locally increased resolution as a way to contain computational costs. The alternative, in fact, would be to use high resolution everywhere. Adaptive techniques can be grouped into two main families. Moving Mesh Adaptation algorithms [18, 19, 20, 21] keep the number of grid points fixed; points are attracted in the “interesting” part of the domain, identified monitoring a relevant quantity. Adaptive Mesh Refinement (AMR) algorithms [22, 23] change the number of grid points on the fly, again according to the variation of a monitor quantity. As the number of cells changes, particles, whose shape function is usually tailored on the local grid size, are split or coalesced.

We recently introduced the Multi-Level Multi-Domain (MLMD) method [1, 2, 3], a method for fully kinetic PIC simulations which combines the benefits of implicit and adaptive algorithms. Different grid levels are simulated with different spatial and temporal resolution: comparing the spatial and temporal resolution used at the different grid levels in previously published work, e.g. [2, 3], one can see that rather large “jumps” in resolution are obtained between the levels. Large resolution jumps are used because the processes one wants to resolve at the different levels operate at different scales, for example ion vs. electron scales. Large and sudden changes in resolution could prove a fatal challenge for an explicit Particle In Cell algorithm, where stability constraints require to resolve fractions of the Debye length and of the inverse electron plasma frequency. The problem is circumvented with the IMM, which is used in each grid. The stability region of the IMM is larger than that of explicit methods, see Equation 1. Hence, large jumps in temporal and spatial resolution between the grid can be achieved, while all grids stay in the stability region of the method. The different grid levels are coupled through information exchange operations: exchange of field and boundary conditions from the less resolved to the more resolved grid and projection of the more resolved field solution to the less resolved grid level. This last operation in particular has profound implication for moment conservation in the MLMD system, as explored in [24]. It also constitutes one of the key ingredients of the grid coupling mechanism. In this paper, we will explore how the projection operation results into grid coupling in the MLMD system.

The paper is structured as follows. Section 2 recaps the basics of the MLMD method and recalls the types of scenarios to which it has been applied so far. Section 3 is the core of the paper. It analyses the mechanism that couples the grids simulated at different resolution levels. This analysis has never been shown before. It is demonstrated that the projection (restriction) operation implemented in the MLMD method (Equation 32-33 in [1]) results into grid coupling because it introduces a dependency of the mixed grid electric field solution on the refined grid density and currents. In Section 4, conclusions are drawn.

2. The Multi-Level Multi-Domain method
Figure 1 shows a sketch of a two-level MLMD system.
The Coarse Grid (CG) covers the entire domain and is simulated with “lower” temporal and spatial resolution. The Refined Grid (RG) covers a fraction of the entire domain and is resolved with “higher” resolution. Small scale processes (for example, electron scale processes as opposed to ion scale processes) need higher resolution to be properly reproduced in the simulation. Consequently, the Refined Grid is placed in correspondence of the areas where one knows, from previous experience, that smaller scale processes are going to develop. Small scale processes are therefore simulated twice, by the Coarse and by the Refined Grid. They are reproduced without approximation (i.e., at the appropriate resolution) by the RG. They are simulated in an approximated way by the Coarse Grid where, as typical of semi-implicit methods, selective damping and spatial compression affect the under-resolved scales [25]. Larger scale processes are simulated without approximation by both grids.

The jumps in spatial and temporal resolution between the two grids are indicated as Refined Factor (RF) and Time Ratio (TR). All grids are fully simulated with fields and particles, whose shape function is tailored on the local grid size for momentum conservation purposes [8]. Information exchange operations ensure proper coupling between the different grid levels. They consist of field and particle boundary condition exchange from the coarse to the refined grid (red and yellow dotted arrows in Figure 1) and electric field projection from the refined to the coarse grid (magenta arrow) [1]. The projection operation affects the area in the coarse grid simulated by both the coarse and the refined grid, the so called “overlap area” shaded in pink in Figure 1. It produces a “mixed grid” electric field obtained by merging the CG and RG electric field solution as recapped in Section 3.2. In this paper, we will study in particular the effects of this projection operation in terms of grid coupling.

The Implicit Moment Method (IMM) is used to advance fields and particles at all resolution levels. Equation 1 is the stability condition for the IMM: as long as the $dt$ and $dx$ are changed together, quite high jumps in resolution between the grids may be achieved. In [3], $RF = 12$ and $TR = 1 \rightarrow 6$ are used in simulations of magnetic reconnection where a realistic ion to electron mass ratio $m_r = 1836$ is used. The aim is to resolve ion scale processes on the CG and electron scale processes on the RG. The symbols $\Delta$ and $\delta$ stand for the coarse and refined grid resolution, $d_i$ and $d_e$ are the ion and electron skin depths, related to the ion to electron mass ratio $m_r$ as $d_i/d_e = \sqrt{m_r}$, and $\omega_{pi}$ is the ion plasma frequency. On the CG, the resolution used is $\Delta x/d_i = 0.078$, $\omega_{pi}\Delta t = 0.05 \rightarrow 0.3$, on the RG is $\delta x/d_e = 0.28$, $\omega_{pi}\delta t = 0.05$. A reduced ion to electron mass ratio compresses ion and electron scales and allows to perform shorter and smaller simulations [28]. This is obviously an advantage, which may however be paid for in terms of inaccurate results: since the growth rate of different instabilities scales differently with the mass ratio, the risk exists of altering the hierarchy of the growth rates of competing instabilities [29].
For this reason, realistic mass ratios are our target. Figure 2 illustrates a measure of performance for the MLMD method. The execution times of MLMD simulations (solid line) and single level simulations (dotted line) are compared as a function of the RF. At a fixed RF, the size of the domain is the same for the two simulations. The MLMD simulations are initialised with the RF indicated and $TR = 1$. A fraction $1/RF^2$ of the entire domain is simulated with higher resolution $\delta x = \Delta x/RF$, $\delta y = \Delta y/RF$. In the corresponding single level simulation, the RG resolution is used in the entire domain. The aim is to compare how much time is gained with a MLMD simulation if the higher resolution $\Delta x/RF$ is needed only in a fraction $1/RF^2$ of the domain. The benchmark is the single level alternative, i.e. using high resolution everywhere in the domain. Since, at increasing RF, the RG resolution increases, so does the execution time of the single level simulations. The execution time of the MLMD simulations increases only slightly, since the number of particles in the system is independent on the RF (the computational cost is proportional to the number of particles in the simulation). The increase is due to the higher impact of grid exchange information on the total execution time with increasing RFs. The savings in computational time granted by the MLMD method are remarkable at the RF values normally used in our simulations. At $RF = 14$, the MLMD execution time is about 70 times shorter that its single level counterpart. Additional information on performance evaluation can be found in [3].

The MLMD method is used in [2, 3] to simulate magnetic reconnection. Increased resolution is used in a small area surrounding the Electron Diffusion Region [26]. It is demonstrated that electron scale features, such as electron jet speed [27], can be recovered in the higher resolution area at a moderate cost even for realistic mass ratios.

Recently, the MLMD method has been used to simulate turbulence produced by the Lower Hybrid Drift Instability, LHDI [30], again at a realistic mass ratio [31]. The LHDI is notorious for being unstable over a large wavenumber and frequency range and can therefore be regarded as a “turbulence generator” since it breaks large scale fields into smaller and smaller scales. The MLMD method has been used to build “mixed grid spectra”, obtained joining the power spectra calculated on the grids simulated with higher and lower resolution. In “mixed grid spectra”, the CG provides the low wavenumber solution. Large domains, low wavenumbers are needed in LHDI simulations to allow the development of the ion-ion kink instability, responsible for the well known current sheet kinking at late times [32]. The RG provides the high wavenumber part, when the electromagnetic and electrostatic LHDI branches develop. Notice that at realistic mass ratio the wavenumber of the ion-ion kink instability and that of the electromagnetic LHDI, usually compressed at reduced mass ratio, are well separated. Both LHDI and the ion-ion kink instability have wavenumber in the perpendicular direction with respect to the ambient magnetic field. Let’s indicate with $k_\perp$ the perpendicular wavenumber and with $\rho_i$ and $\rho_e$ the ion and electron gyroradius. The ion-ion kink instability develops at $k_\perp L_H \sim 0.5$, the electromagnetic LHDI at $k_\perp \sqrt{\rho_i \rho_e} \sim 1$. With our choice of initial parameters, they correspond to $k_\perp d_i \sim 1$ and $k_\perp d_i \sim 20$. The coarse and refined grid spectra are shown to join seamlessly in the wavenumber range reliably simulated at both resolution levels. Using a RG in conjunction with a CG extends of a factor $RF$ the wavenumber range covered, at a minimum computational cost. In fact, the cost increases only of about two times with respect to the execution time of the CG alone. A 2D3V (two spatial components, three velocity components) single level simulation reaching the same maximum wavenumber would cost $RF^2$ times more.

3. Grid coupling mechanism in the MLMD method
In this Section, the evolution law for the mixed grid electric field is derived. We define the mixed grid electric field as the electric field in the overlap area (pink in Figure 1) after projection.

First, in Section 3.1 we will show how the equation solved at each level of the MLMD system, Equation 2, is derived from Maxwell’s equations. This constitutes an excellent exercise for the
calculations shown later. Then, the details of the projection operation in the MLMD method are recalled in Section 3.2. Finally, the evolution law for the mixed grid electric field is derived and discussed in Section 3.3.

3.1. Derivation of the single grid equation for electric field advancement

Each resolution level of the MLMD system solves the following equation to advance in time the electric field \( \mathbf{E} \) [33]:

\[
\frac{1}{c} \frac{\partial}{\partial t} \left( \frac{4\pi}{c} \mathbf{J} + \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} \right) = \nabla^2 \mathbf{E} - \nabla (4\pi \rho),
\]

where \( \mathbf{J} \) is the total current, \( \rho \) the total density and \( c \) the speed of light. It is obtained by applying the gradient operator to Gauss’ law,

\[
\nabla \cdot \mathbf{E} = 4\pi \rho,
\]

and then using the vector identity

\[
\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}
\]

to substitute the \( \nabla(\nabla \cdot \mathbf{E}) \) term. \( \nabla \times \mathbf{E} \) is obtained from Maxwell-Faraday equation

\[
\nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t},
\]

with \( \mathbf{B} \) the magnetic field. The resulting \( \nabla \times \mathbf{B} \) term in

\[
\frac{1}{c} \frac{\partial}{\partial t} \nabla \times \mathbf{B} = \nabla^2 \mathbf{E} - \nabla (4\pi \rho)
\]

is then substituted away using Ampere’s law

\[
\nabla \times \mathbf{B} = \frac{4\pi}{c} \mathbf{J} + \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t}.
\]

Equation 2 describes the time evolution of \( \mathbf{E} \) as a function of \( \mathbf{J} \) and \( \rho \). The magnetic field, which is conveniently eliminated from the evolution law, is then obtained from the electric field through Maxwell-Faraday. The implicit temporal discretization of Equation 2 and the IMM approximation allow to recover the familiar-looking Equation 26 in [1], solved in IMM in general [34] and in the MLMD method in particular. [33] remark that the time discretisation of Equation 2 leads to a system of linear equation which is better conditioned than alternative formulations of the evolution law for \( \mathbf{E} \). Also, it is noticed there that initial perturbations of Gauss’ law are damped in Equation 2. We immediately think here of perturbations related to numerical noise.

3.2. Projection in the MLMD method

To analyse the coupling of the coarse and refined grid solution in the overlap area (pink in Figure 1), it is convenient to preliminarily recall the details of the projection operation (magenta line). The three components of the electric field are projected from the refined (RG) to the coarser (CG) grids as in

\[
\mathbf{E}_{P,CG} = \frac{1}{2} \left( \mathbf{E}_{N,CG} + \mathbb{P}^{RG\rightarrow CG}(\mathbf{E}_{N,RG}) \right),
\]

where \( \mathbf{E}_P \) denotes the projected electric field and \( \mathbf{E}_N \) the electric field calculated natively on the grid. The symbol \( \mathbb{P}^{RG\rightarrow CG} \) is a short notation for the projection operator, implemented as in

\[
\mathbb{P}^{RF\rightarrow CG}(\mathbf{E}_{N,RG}) = \frac{\sum_{RG} \mathbf{E}_{N,RG} W_{CG}(x_{CG} - x_{RG})}{\sum_{RG} W_{CG}(x_{CG} - x_{RG})}.
\]
The same interpolation function \( W(\mathbf{x}_{CG} - \mathbf{x}) \) is used in the coarse grid for grid to particle and particle to grid interpolation. The “mixed grid electric field” mentioned earlier in the paper is indicated as \( \mathbf{E}_{P,CG} \). One can see that, neglecting the operations necessary to interpolate the RG solution to the different gridding used in the CG, the projection operation is essentially an average between the coarse and the refined grid electric field values. A formulation of Equation 8 in the continuum (i.e., neglecting the spatial discretisation) therefore reads:

\[
\mathbf{E}^* = \frac{\mathbf{E} + \mathbf{E}_{RG}}{2},
\]

where \( \mathbf{E}^* \) (which corresponds to \( \mathbf{E}_{P,CG} \) in Equation 9) is the mixed grid solution in the overlap area, \( \mathbf{E} \) is now used to label the “native” coarse grid solution (the \( CG \) notation is dropped for easier reading) and \( \mathbf{E}_{RG} \) is the refined grid solution.

3.3. Evolution law for the mixed grid electric field

In the MLMD method, Equation 2 is solved to obtain the native electric field \( \mathbf{E}_{N,CG} \). Therefore, substituting Equation 10 into Equation 2 one gets

\[
\frac{1}{c^2} \frac{\partial^2}{\partial t^2} (2\mathbf{E}^* - \mathbf{E}_{RG}) = \nabla^2 (2\mathbf{E}^* - \mathbf{E}_{RG}) - \nabla (4\pi\rho) - \frac{4\pi}{c^2} \frac{\partial \mathbf{J}}{\partial t},
\]

(11)

where Gauss’ law is assumed to hold in the CG. One should keep in mind that the symbol \( \mathbf{E} \) in Section 3.1 labels the single grid electric field, while it is now used for the native coarse grid solution. Using the vector identity Equation 4 for \( \mathbf{E}^* \) in Equation 12, one obtains

\[
\frac{1}{c^2} \frac{\partial^2}{\partial t^2} (2\mathbf{E}^* - \mathbf{E}_{RG}) = 2(\nabla \cdot (\nabla \cdot \mathbf{E}^*) - \nabla \times \nabla \times \mathbf{E}^*) - \nabla^2 \mathbf{E}_{RG} - \nabla (4\pi\rho) - \frac{4\pi}{c^2} \frac{\partial \mathbf{J}}{\partial t}. \tag{12}
\]

Assuming that Gauss’s law and Maxwell-Faraday equation hold for the mixed grid solution, i.e.

\[
\nabla \cdot \mathbf{E}^* = 4\pi\rho \tag{13}
\]

and

\[
\nabla \times \mathbf{E}^* = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}, \tag{14}
\]

Equation 12 becomes

\[
\frac{1}{c^2} \frac{\partial^2}{\partial t^2} (2\mathbf{E}^* - \mathbf{E}_{RG}) = \nabla (4\pi\rho) + 2\nabla \times \left( \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} \right) - \nabla^2 \mathbf{E}_{RG} - \frac{4\pi}{c^2} \frac{\partial \mathbf{J}}{\partial t} \tag{15}
\]

It is now useful to remember that Equation 2 is solved on the refined grid as well, hence \( \nabla^2 \mathbf{E}_{RG} \) can be written as:

\[
\nabla^2 \mathbf{E}_{RG} = \frac{1}{c^2} \frac{\partial^2 \mathbf{E}_{RG}}{\partial t^2} + \nabla (4\pi\rho_{RG}) + \frac{4\pi}{c^2} \frac{\partial \mathbf{J}_{RG}}{\partial t}, \tag{16}
\]

where \( \rho_{RG} \) and \( \mathbf{J}_{RG} \) are the density and the current in the Refined Grid. Substituting Equation 16 into Equation 15 yields

\[
\frac{1}{c^2} \frac{\partial^2}{\partial t^2} (2\mathbf{E}^* - \mathbf{E}_{RG}) = \nabla (4\pi\rho) + 2\nabla \times \left( \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} \right) - \frac{1}{c^2} \frac{\partial^2 \mathbf{E}_{RG}}{\partial t^2} + \nabla (4\pi\rho_{RG}) + \frac{4\pi}{c^2} \frac{\partial \mathbf{J}_{RG}}{\partial t} - \frac{4\pi}{c^2} \frac{\partial \mathbf{J}}{\partial t}. \tag{17}
\]
and then
\[
\frac{1}{c} \frac{\partial^2 \mathbf{E}^\star}{\partial t^2} = \nabla \times \left( \frac{\partial \mathbf{B}}{\partial t} \right) + \frac{4\pi c}{2} \nabla (\rho - \rho_{RG}) - \frac{1}{2} \frac{4\pi}{c} \frac{\partial}{\partial t} (\mathbf{J}_{RG} + \mathbf{J})
\]  
(18)

after simple math. Integrating in time one obtains
\[
\frac{1}{c} \frac{\partial \mathbf{E}^\star}{\partial t} = \nabla \times \mathbf{B} + \frac{4\pi c}{2} \int dt \nabla (\rho - \rho_{RG}) - \frac{1}{2} \frac{4\pi}{c} (\mathbf{J}_{RG} + \mathbf{J}) ,
\]  
(19)

which gives a measure of coupling between coarse and refined grid solution in the overlap area, again neglecting spatial discretisation issues. As a preliminary sanity check, one should notice that Equation 19 reduces to Ampere’s law if \( \mathbf{E}^\star \rightarrow \mathbf{E} \) (the mixed grid solution tends to the native coarse grid solution) and \( \mathbf{J}_{RG} \rightarrow \mathbf{J}, \rho_{RG} \rightarrow \rho \) (the zeroth and first order refined grid moments tend to their coarse grid counterparts). Equation 19 provides an evolution law for the mixed grid electric field solution \( \mathbf{E}^\star \) in terms of the native CG magnetic field solution \( \mathbf{B} \) and, more interestingly, as function of the coarse and refined \( \rho \) and \( \rho_{RG} \) and \( \mathbf{J} \) and \( \mathbf{J}_{RG} \). Grid coupling in the overlap area depends on the coarse and refined grid densities and currents. The RG density and currents participate in determining the mixed grid electric field solution.

Notice that, apart from pathological cases, one expects the term \( \int dt \nabla (\rho - \rho_{RG}) \) not to diverge in time. However, differences in the coarse and refined grid densities (and between the mixed grid and refined grid electric field solution) are expected and have been observed and explained in the overlap area.

As an example, it is useful to revisit here the discussion on the mismatch between the CG and RG electric field and density traces observed in the 1D3V simulation of a Two Stream Instability, Section 6.3 of [1]. Figure 10 in [1] is reproduced here in Figure 3. The \( \mathbf{E}^\star \) and \( \mathbf{E}_{RG} \) fields are shown in panel a and b as a function of space (vertical axis) and time (horizontal axis). The symbols \( d_e \) and \( \omega_{pe} \) indicate the electron skin depth and the electron plasma frequency. The RG overlaps the CG at \( x/d_e < 6.95 \). The contours of the CG density are superimposed to the electric field in both panels. One can notice a mismatch in \( \mathbf{E}^\star \) and \( \mathbf{E}_{RG} \) especially in the area marked with the red rectangle. In [1] the mismatch is related to different electron hole merging patterns in the phase spaces of the CG and RG. At later times the CG, RG differences subside in longitudinal electric fields, densities, phase spaces.

The possibility of accommodating slightly different evolutions in the different grids, possibly in response to processes at different scales simulated differently by the two grids, is considered one of the strengths of the MLMD method and has been amply documented in [2, 3].

Of particular relevance for this discussion is the case of grid coupling documented in Figure 9 of [2], reproduced here as Figure 4. Two reconnection points are simulated in the CG. One of the two reconnection points, Figure 4 panel (a), is simulated with a RG also, while in Figure 4 panel (b) the “control reconnection point”, simulated only with the CG, is shown. The presence of the RG makes the mixed grid solution (panel a) stable against the finite grid instability which develops instead in the control reconnection point (panel b) due to the absence of field smoothing. We now interpret this stabilising effect as a result of the influence of the RG density and currents on the mixed grid solution through the projection operation.

4. Conclusions

The MLMD method, a semi-implicit adaptive method for fully kinetic plasma simulations, relies on two pillars: the capability of the CG to drive RG evolution through CG-to-RG boundary conditions and good RG-CG coupling in the overlap area. The latter property is achieved through electric field projection from the refined to the coarse grid. In this paper, we investigate the mechanism of grid coupling through projection. We neglect spatial discretisation at the different grid levels and work in the continuum, where the CG electric field after projection is
Figure 3. The $E_x^*$ (panel a) and $E_{x,RG}$ (panel b) fields as a function of space (vertical axes) and time (horizontal axis) in a 1D3V simulation of Two Stream Instability. Coarse grid density contours are superimposed to both panels. Reproduced from [1].

Figure 4. $E_y$ component of the mixed grid solution (panel a) vs the coarse grid solution (panel b) in a simulation of magnetic reconnection where the absence of field smoothing results in the development of the finite grid instability at the X point simulated only with the CG. The presence of the Refined Grid stabilises the mixed grid solution. Reproduced from [2].

simply a mixed grid solution $E^*$ obtained by averaging the native CG solution $E$ and the RF solution $E_{RG}$. We write the evolution law for $E^*$ in terms of CG and RG quantities, Equation 19. Equation 19 shows that CG-RG coupling is achieved through densities and currents: both coarse grid and refined grid currents and densities contribute to the time evolution of $E^*$. These results give a theoretical framework to the cases of excellent CG-RG coupling documented in [1, 2, 3].

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