Metrics with vanishing quantum corrections

A A Coley¹, G W Gibbons², S Hervik¹,³ and C N Pope²,⁴

¹ Department of Mathematics & Statistics, Dalhousie University, Halifax N.S. B3H 3J5, Canada
² DAMTP, Cambridge University, Wilberforce Road, Cambridge CB3 0WA, UK
³ Department of Mathematics and Natural Sciences, University of Stavanger, N-4036 Stavanger, Norway
⁴ George P & Cynthia W Mitchell Institute for Fundamental Physics, Texas A&M University, College Station, TX 77843-4242, USA

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Abstract

We investigate solutions of the classical Einstein or supergravity equations that solve any set of quantum corrected Einstein equations in which the Einstein tensor plus a multiple of the metric is equated to a symmetric conserved tensor \( T_{\mu \nu}(g_{\alpha \beta}, \partial_\tau g_{\alpha \beta}, \partial_\tau \partial_\sigma g_{\alpha \beta}, \ldots) \) constructed from sums of terms, involving contractions of the metric and powers of arbitrary covariant derivatives of the curvature tensor. A classical solution, such as an Einstein metric, is called universal if, when evaluated on that Einstein metric, \( T_{\mu \nu} \) is a multiple of the metric. A Ricci flat classical solution is called strongly universal if, when evaluated on that Ricci flat metric, \( T_{\mu \nu} \) vanishes. It is well known that pp-waves in four spacetime dimensions are strongly universal. We focus attention on a natural generalization; Einstein metrics with holonomy \( \text{Sim}(n - 2) \) in which all scalar invariants are zero or constant. In four dimensions we demonstrate that the generalized Ghanam–Thompson metric is weakly universal and that the Goldberg–Kerr metric is strongly universal; indeed, we show that universality extends to all four-dimensional \( \text{Sim}(2) \) Einstein metrics. We also discuss generalizations to higher dimensions.

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1. Introduction

It has been realized for some time that certain solutions of the classical Einstein equations, or some variant of them such as the classical supergravity equations, remain valid solutions in the quantum theory, despite our ignorance of precisely what that quantum theory might be. Indeed, some solutions of the classical Einstein equations have such a restricted curvature structure that they remain valid solutions of almost any set of covariant equations involving the metric and its derivatives. We shall call such metrics universal, with a further subdivision into strongly universal and weakly universal, which we shall explain below.
We assume that the field equations for an $n$-dimensional spacetime metric take the form

$$G_{\mu\nu} = -\Lambda g_{\mu\nu} + 2T_{\mu\nu},$$  \hspace{1cm} (1)

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$ is the Einstein tensor, $\Lambda$ is a classical cosmological constant term, and $T_{\mu\nu}$ represents possible quantum corrections to the classical Einstein equations in the case of classical vacuum. Necessarily the symmetric second rank tensor $T_{\mu\nu} = T_{\mu\nu}(g_{\alpha\beta}, \partial_{\tau}g_{\alpha\beta}, \partial_{\eta}g_{\alpha\beta}, \ldots)$, which we assume to be made up of sums of terms each constructed from the metric, the curvature tensor $R_{\mu\nu\sigma\tau}$ and its covariant derivatives, is conserved; i.e.,

$$T_{\mu\nu;\nu} = 0.$$  \hspace{1cm} (2)

Now suppose we have a solution $g_{\mu\nu}$ of the classical equations, obtained by omitting the tensor $T_{\mu\nu}$ from the right-hand side of (1); i.e., an Einstein metric for which

$$R_{\mu\nu} = \frac{2}{n-2}\Lambda g_{\mu\nu},$$  \hspace{1cm} (3)

We ask whether the classical metric $g_{\mu\nu}$ (possibly rescaled by a constant factor $h$) solves the full quantum corrected equations (1). This requires that

$$T_{\mu\nu}(hg_{\rho\sigma}) = F(h)g_{\mu\nu}$$  \hspace{1cm} (4)

for some function $F(h)$ and that $h$ may be chosen to satisfy

$$\Lambda(h - 1) = 2F(h).$$  \hspace{1cm} (5)

A sufficient condition that (4) hold is that it hold for any symmetric conserved tensor constructed from the classical metric $g_{\mu\nu}$ and its derivatives. We call this condition weak universality because, subject to there being real solutions of (5), we can simply rescale the metric to get a solution of any set of corrected field equations.

An example of a weakly universal metric is a maximally symmetric space, such as de Sitter spacetime, for which

$$R_{\mu\nu\sigma\tau} = c(g_{\mu\sigma}g_{\nu\tau} - g_{\nu\sigma}g_{\mu\tau}),$$  \hspace{1cm} (6)

where $c$ is necessarily a constant, and hence

$$R_{\mu\nu\sigma\tau;\lambda_1\ldots\lambda_k} = 0,$$  \hspace{1cm} (7)

for arbitrary integers $k$. It follows that $c$ must satisfy

$$T_{\mu\nu} = f(c)g_{\mu\nu},$$  \hspace{1cm} (8)

for some function $f(c)$, and if a value of $c$ can be found satisfying

$$f(c) + \frac{(n-1)(n-2)}{4}c = 0,$$  \hspace{1cm} (9)

we have a solution. This argument can obviously be generalized to cover the case of any symmetric space $M = G/H$, where $G$ is the isometry group with stabilizer $H$, for which by definition

$$R_{\mu\nu\sigma\tau;\lambda} = 0,$$  \hspace{1cm} (10)

if, in addition, we assume that $H$ acts irreducibly on the tangent space. In fact, as long as $H$ acts irreducibly (10) is not necessary; it suffices that the space be locally homogeneous. Bleecker [1] has called the restricted class of Riemannian metrics of the sort we are considering critical metrics, as long as the field equation may be derived from a diffeomorphism invariant action functional, which in our case means

$$\frac{1}{4} \int \sqrt{|g|}(R - 2\Lambda) + I(g)$$  \hspace{1cm} (11)
such that
\[ T_{\mu\nu} = -\frac{2}{\sqrt{|g|}} \frac{\delta I(g)}{\delta g^{\mu\nu}}. \] (12)

In what follows we shall use Bleecker’s term critical metric for a metric of any signature.

A necessary and sufficient condition for a metric to be weakly universal is that any conserved symmetric tensor constructed from the metric and its derivatives should be a multiple of the metric. A necessary and sufficient condition for a metric to be critical is that any conserved symmetric tensor constructed from the metric and its derivatives that is a variational derivative of an invariant functional should be a multiple of the metric. Clearly a sufficient condition for a metric to be critical is that it be universal. It is not completely obvious whether or not the converse is true.

In fact, Bleecker showed that critical compact orientable Riemannian manifolds must be homogeneous spaces \( G/H \), where \( H \) acts irreducibly. However, the metrics we usually encounter in physics are Lorentzian, and there are many more possibilities. For example, one might consider what in this paper we shall call a vacuum Brinkmann wave [3–5]. This is a Ricci flat Lorentzian metric admitting a covariantly constant null vector field (CCNV) \( n^\nu \);
\[ \nabla_\mu n^\nu = 0. \] (13)
The most general metric admitting a covariantly constant null vector can be written as [6, 7]
\[ ds^2 = 2 du [dv + H(u, x^k) du + A_i(u, x^k) dx^i] + g_{ij}(u, x^k) dx^i dx^j, \] (14)
and is a subclass of the Kundt metrics [8]. Requiring the vanishing of the Ricci tensor implies that the transverse metric \( g_{ij}(u, x^k) \) is Ricci flat.

A commonly studied subclass of this class of metrics is the class for which \( g_{ij}(u, x^k) \) is independent of \( u \) and flat, and \( A_i = 0 \). Horowitz and Steif [9] have shown that all such metrics are not only weakly universal but they possess an even stronger property, which we shall call strong universality.

Clearly for Brinkmann waves, since the Ricci tensor \( R_{\mu\nu} \) vanishes, the classical value of \( \Lambda \) also vanishes. Equally clearly, any constant multiple \( h \) of the classical metric is also a solution. Horowitz and Steif showed, when evaluated on a Brinkmann wave background, all other conserved tensors \( T_{\mu\nu}(g_{\rho\sigma}) \) (except of course the metric itself) will vanish. Thus, in distinction to the case of maximally symmetric spaces such as de Sitter spacetime, no rescaling of the metric is required when passing from the classical to the quantum-corrected metric. To make the notion of strong universality precise we need to consider, for a general spacetime, the vector space (over real constants) of all symmetric conserved second rank tensors constructed from the metric and its derivatives, modulo constant multiples of the metric itself. If, when restricted to a classical metric \( g_{\mu\nu} \) such as a Brinkmann wave, all such tensors vanish, then we say that the classical metric is strongly universal.

A result related to that of Horowitz and Steif has been obtained by Torre [10], who showed that any metric with the same isometry group as plane (or Rosen) waves, but which does not necessarily solve the vacuum Einstein equations, is strongly universal. Some physical implications of these facts for quantum field theory and string theory may be found in [9, 11, 12].

One of the aims of the present paper is to see whether these results for Brinkmann waves can be extended to the newly constructed class of solutions of the Einstein equations [13]

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5 Interestingly, Bleecker [2] also discussed the idea of critical maps between Riemannian manifolds, which have obvious relevance to nonlinear sigma models and string theory.

6 We shall define a Brinkmann wave as a spacetime admitting a covariantly constant null vector. Vacuum pp-waves are examples of vacuum Brinkmann spacetimes.
which admit a null vector field $n^\nu$ that is covariantly constant merely in direction, sometimes called \textit{recurrent}; i.e.,

$$\nabla_\mu n^\nu = B_\mu n^\nu,$$

(15)

for some recurrence 1-form $B_\mu$. An equivalent way of expressing this condition is to say that the corresponding metrics have holonomy contained in $\text{Sim}(n-2)$, the maximal proper sub-group of the Lorentz group $SO(n-1,1)$. In the Brinkmann case, for which $B_\mu = 0$, the holonomy is contained in the subgroup of Euclidean motions $E(n-2)$. In some cases, the holonomy reduces even further. For example, for some simple Brinkmann metrics the holonomy group could be contained in the Abelian subgroup $\mathbb{R}^{n-2}$ of $\text{Sim}(n-2)$.

A principal result of our work is the discovery of two new classes of four-dimensional metrics, one of which is strongly universal and the other of which is weakly universal.

Before embarking on our discussion we wish to make a brief remark on the issue of vanishing local counterterms. In our context the integrand of $I(g)$ in equation (11) would be called a diffeomorphism invariant local counterterm, and its vanishing subject to the Ricci flat condition $R_{\mu\nu} = 0$ (i.e., ‘shell’) is often taken as an indication that quantum corrections are finite. However, this does not necessarily mean that quantum corrections vanish, since one can conceive of cases for which $I(g)$ vanishes while its variational derivative $\frac{1}{2}T_{\mu\nu}$ does not.

In fact, this is precisely what happens in the case of Calabi–Yau compactifications in string theory. The specific scalar invariants $I(g)$ that arise as higher-order string corrections to the effective action all vanish for the product of four-dimensional Minkoswki spacetime with a manifold of $SU(3)$ holonomy, but their variations $\delta I \delta g_{\mu\nu}$ do not. The classical Ricci flatness condition is consequently modified by quantum corrections, although the supersymmetry is preserved (see, for example, [15] for an extensive discussion). A similar phenomenon arises in compactifications involving manifolds with $G_2$ and $\text{Spin}(7)$ holonomy [16, 17].

Another example of this phenomenon is the Ricci type N, Weyl type N Brinkmann waves for which all scalar invariants vanish [6]. However, for an action containing the invariant $R_{\mu\nu} R^{\mu\nu}$, its variation will contain a term $\nabla^2 R_{\mu\nu}$ which may or may not vanish. For example, for the metric (14) with $u$-independent and flat $g_{ij}(u, x^k)$ with $A_i = 0$, this term gives a contribution $(\nabla^2)^2 H$ which will not vanish in general.

It is nevertheless of interest to ask whether all scalar invariants formed from the metric and its derivatives vanish on shell; the so-called \textit{vanishing scalar invariants}, or VSI, condition. An Einstein metric with vanishing scalar invariants must, of course, be Ricci flat. Examples of VSI spaces are all Ricci-flat Brinkmann waves in four and five dimensions. A weaker, but nevertheless still interesting and potentially important condition, is when all scalar invariants are constant; the so-called \textit{constant scalar invariants}, or CSI, condition. For example, we can imagine having a symmetric tensor of the form

$$T_{\mu\nu} = \mathcal{I} g_{\mu\nu},$$

where $\mathcal{I}$ is a curvature invariant. Requiring this tensor to be conserved, $T^{\mu\nu}_{\ ;\nu} = 0$, immediately implies $\mathcal{I}_{\mu\nu} = 0$, and hence that $\mathcal{I}$ is a constant. It is therefore natural to search among the CSI spacetimes for weakly universal spacetimes. It is not obvious whether or not all weakly universal spacetimes are CSI spacetimes.

To summarize, we have isolated five conditions on a metric: strongly universal; weakly universal; critical; vanishing scalar invariants (VSI) and constant scalar invariants (CSI). In what follows, we shall study the extent to which these conditions hold for the Einstein metrics with $\text{Sim}(n-2)$ holonomy, and particularly the new solutions of [13].
Table 1. The different conditions on the metric considered in this paper. Here, $T_{\mu\nu}$ is a conserved symmetric 2-tensor made from the metric and its derivatives; $I$ is a scalar curvature invariant; and $c$ is a constant.

| Definition       | Conditions                                                                 |
|------------------|-----------------------------------------------------------------------------|
| Strongly universal | $T_{\mu\nu} = 0$.                                                         |
| Weakly universal  | $T_{\mu\nu} = c g_{\mu\nu}$.                                               |
| Critical         | $T_{\mu\nu} = c g_{\mu\nu}$, where $T_{\mu\nu} = -\frac{2}{\sqrt{|g|}} \frac{\delta I(g)}{\delta g_{\mu\nu}}$. |
| VSI              | $I = 0$.                                                                    |
| CSI              | $I = \text{constant}$.                                                      |

2. Previous work and the situation in four dimensions

As shown in [13], by rescaling the null vector field $n^\mu$ of a metric with Sim$(n - 2)$ holonomy we may always arrange things so that

$$\nabla_\mu n^\nu = \kappa n_\mu n^\nu. \quad (16)$$

It follows that the null congruence with tangent vector $n^\mu$ is geodesic, hypersurface orthogonal (i.e., twist free), expansion free and shear free and hence, in all spacetime dimensions, belongs to the class of Kundt metrics [8]. However not every Kundt metric has holonomy in Sim$(n - 2)$. Metrics with Sim$(n - 2)$ holonomy may be cast in the Walker form [18]

$$ds^2 = 2 du [dv + H(v, u, x^i) du + A_i(u, x^k) dx^i] + g_{ij}(u, x^k) dx^i dx^j, \quad (17)$$

which is a special case of a Kundt metric.\(^7\)

For four-dimensional vacuum Kundt spacetimes, the null vector $n^\mu$ is, in addition, a thrice repeated principal null direction of the Weyl tensor. It follows from a result of Pravda [19] (see also Pravda et al. [20]) that all invariants formed from the Weyl tensor (and since it is Ricci flat, the Riemann tensor) necessarily vanish.

It was pointed out in [11] that the vanishing of all invariants of type N vacuum spacetimes with geodesic, non-twisting, non-expanding, null congruences implies that all counterterms must vanish, no matter what theory of gravity one is considering. The question was raised in [11] as to whether this was true for type III metrics. This was answered in the affirmative in [19] (see also [21]). The question of whether such ‘all loop finite’ metrics have reduced holonomy was also raised in [11]. In the present Sim(2) Petrov type III case we see that, as in the more familiar E(2) Petrov type N case, the answer is yes. However, VSI spaces that do not possess a recurrent null vector (the case $\epsilon \neq 0$ in [6]) are ‘all loop finite’ examples having general holonomy.

3. Boost-weight decomposition

In what follows, it is useful to consider the boost-weight decomposition of tensors [22]. Consider an arbitrary covariant tensor $T$ and a null frame $e^a_\mu = \{\ell^\mu, n^\mu, m^\mu\}$; i.e.,

$$\ell_\mu n^\mu = 1, \quad m^\mu m^{\mu\nu} = \delta^{\mu}_{\nu}, \quad \ell_\mu t^\mu = n_\mu n^\mu = \ell_\mu m^{\mu\nu} = n_\mu m^{\mu\nu} = 0. \quad (18)$$

Consider now a boost in the plane spanned by $\ell^\mu$ and $n^\mu$:

$$e^a_\mu = \{\tilde{\ell}^\mu, n^\mu, m^\mu\} = \{e^\mu \ell^\mu, e^\mu n^\mu, m^\mu\}. \quad (18)$$

Note that there exists, in general, a coordinate transformation [7] that allows us to set $A_i = 0$. However, this requires that the transverse metric be $u$-dependent. In our case, and especially for the VSI and CSI metrics, it is more useful to keep $A_i \neq 0$ so that we can make the transverse metric $u$-independent [8].
We can consider the vector-space decomposition of the tensor $T$ in terms of the boost weight with respect to the transformation (18) \cite{22}:

$$T = \sum_b (T)_b,$$

(19)

where $(T)_b$ denotes the projection of the tensor $T$ onto the vector space of boost-weight $b$. The components of the tensor $(T)_b$ with respect to the null frame will transform according to

$$(T)_{b_1 b_2 \ldots} = e^{-b_1} (T)_{b_0 b_2 \ldots}.$$  

(20)

Furthermore, we note that

$$(T \otimes S)_b = \sum_{b_0 + b' = b} (T)_{b'} \otimes (S)_{b'}.$$  

(21)

First, note that the metric has boost-weight decomposition $g = (g)_0$, so that raising and lowering indices does not change the boost weight. Second, note that any invariant (complete contraction) must be boost invariant, so that

$$\text{contr}[T] = \text{contr}[(T)_0],$$

where contr means complete contraction (i.e., only boost-weight 0 components will contribute).

Consider the Riemann tensor $R$ (analogously for the Ricci tensor or the Weyl tensor $C$). Recall that the Riemann tensor is algebraically special (otherwise it is of type G) of a certain type if there exists a frame in which the following hold (in terms of the boost-weight decomposition) \cite{23, 24}:

- type I: $R = (R)_1 + (R)_0 + (R)_{-1} + (R)_{-2},$
- type II: $R = (R)_0 + (R)_{-1} + (R)_{-2},$
- type D: $R = (R)_0,$
- type III: $R = (R)_{-1} + (R)_{-2},$
- type N: $R = (R)_{-2},$
- type O: $R = 0.$

4. Sim$(n - 2)$ VSI and CSI metrics

We shall consider the $n$-dimensional metric (17), which is a special Kundt metric. For the Sim$(n - 2)$ metrics there are no further restrictions on the metric functions. However, if we make the requirement that all the scalar polynomial curvature invariants constructed from the Riemann tensor and its covariant derivatives be constants, then the metric function $H(v, u, x^k)$ only contains terms polynomial in $v$ to second order \cite{6, 8}; hence,

$$H(v, u, x^k) = v^2 \sigma + v H^{(1)}(u, x^k) + H^{(0)}(u, x^k),$$

where $\sigma$ is a constant.

Let us choose the null frame

$$\hat{e}^+ = dv + H \, du + A_j \, dx^j,$$
$$\hat{e}^- = du,$$
$$\hat{e}^i = e^i,$$

(22)

where $e^i$ is a vielbein basis for the transverse metric, $g_{ij}(u, x^k) \, dx^i \, dx^j = \delta_{ij} e^i e^j.$ In general, with respect to the above null frame, the Riemann tensor of this metric has the boost-weight decomposition

$$R = (R)_0 + (R)_{-1} + (R)_{-2},$$

which means it is of Riemann type II (and hence, of Ricci type II and of Weyl type II).
4.1. Vanishing curvature invariants (VSI)

In this case all curvature invariants to all orders vanish, so we require that boost-weight 0 components vanish \[25\]. This, in turn, implies that we can set \[6\]

\[
\sigma = 0, \quad g_{ij} k \; dx^i \; dx^j = \delta_{ij} \; dx^i \; dx^j.
\]

(This is the $W_i^{(1)} = 0$ case of the general higher-dimensional VSI metrics.) The Riemann tensor is of type III, N or O and has, along with all higher-derivative curvature tensors, only negative boost-weight components.

Considering the vacuum case, it must be of Weyl type III. These metrics would therefore be the Weyl type III, $\epsilon = 0$, vacuum solutions in \[6\] (see table 1 therein).

4.2. Constant curvature invariants (CSI)

In this case all curvature invariants to all orders are constants. Unfortunately, we do not have a crisp theorem determining all of these spacetimes (except in three dimensions \[26\]), but we have three useful conjectures \[8\]:

(1) A Lorentzian spacetime with all constant scalar invariants (CSI) is either locally homogeneous or a subclass of the Kundt spacetimes.

(2) For a Kundt spacetime with all constant scalar invariants (CSI), there exists a null frame such that for curvature tensors of all orders, all the positive boost-weight components are zero, while all the boost-weight 0 components are constants.

(3) For a spacetime, $(M, g)$, with all constant scalar invariants, there exists a related locally homogeneous spacetime, $(\tilde{M}_{\text{Hom}}, \tilde{g})$, having identical curvature invariants to those of $(M, g)$.

We recall that the class of CSI$_F$ spacetimes are those spacetimes for which there exists a null frame such that for curvature tensors to all orders, all the positive boost-weight components are zero, while all the boost-weight 0 components are constants. The second conjecture therefore suggests that all Kundt CSI spacetimes are actually CSI$_F$ spacetimes (and consequently, this is usually referred to as the CSI$_F$ conjecture).

Note also that for a Kundt CSI spacetime, this would imply that it is of Riemann type II (or simpler). It is easy to see that this is sufficient for a Kundt spacetime to be CSI. However, the problematic part of the proof is to establish the necessity. (In three dimensions, all of these conjectures have been proven \[26\]). In what follows, we shall assume these conjectures are true.

An important theorem regarding the CSI$_F$ spacetimes states \[8\]: For a Kundt CSI$_F$ there exists a (u-dependent) diffeomorphism $\tilde{x}^i = f^i(u, x^k)$ such that the transverse metric can be made $u$-independent. Furthermore, the transverse metric is locally homogeneous. Hence, there is no loss of generality in assuming that $g_{ij}(x^k) \; dx^i \; dx^j$ is a locally homogeneous space, $M_{\text{Hom}}$.

Regarding the invariants, there is generally no further simplification; they can all be non-zero. However, there is a related locally homogeneous spacetime, $(\tilde{M}_{\text{Hom}}, \tilde{g})$ which has invariants that are identical to those of the Kundt CSI metric. For the Sim($n-2$) metrics, this is given by setting the functions $A_i$, $H^{(1)}$ and $H^{(0)}$ to zero:

\[
\tilde{ds}^2 = 2du[dv + \sigma v^2 \; du] + g_{ij}(x^k) \; dx^i \; dx^j.
\]

For the Kundt CSI metrics with holonomy Sim($n-2$), $\sigma$ must be constant. Interestingly, this is the space $M_2 \times M_{\text{Hom}}$, where $M_2$ is two-dimensional de Sitter, Minkowski, or anti-de Sitter space, depending on the sign of $\sigma$. 


This is an important observation because to study invariants of Kundt CSI spacetimes of the form (17), it is sufficient (provided the conjectures are true) to study the invariants of the associated locally homogeneous space (23). In particular, the locally homogeneous part, defined by (23), is the only part that contributes to the boost-weight 0 components of the curvatures of the metric (17). In this way, the invariants will only depend on the locally homogeneous metric (23). Because of the Kundt form, this will also be valid for the higher-derivative curvature tensors.

Special case: $M_{\text{Hom}}$ is a symmetric Riemannian space. Let us consider the case where the transverse space $g_{ij}(x^k)\,dx^i\,dx^j$ is a symmetric space. This means that the product space (23) is also symmetric and hence,

$$\tilde{\nabla}_\lambda \tilde{R}_{\mu\nu\alpha\beta} = 0.$$  

Consequently, for the metric (17), all higher-order curvature invariants will vanish. The only possible non-zero invariants are those constructed from the Riemann tensor itself (i.e., the Ricci, Weyl and mixed tensor invariants).

Also note that assuming $g_{ij}$ is a (Riemannian) symmetric space implies that $g_{ij}$ is a locally homogeneous space.

Other examples. There are many examples of CSI metrics with Sim$(n-2)$ holonomy (although many are not written down explicitly). Assuming the metric is Einstein, we find that the transverse metric must also be Einstein. To simplify matters, we can set $A_i = 0 = H^{(1)}$ and still have Sim$(n-2)$ holonomy, provided that $H^{(0)}$ is sufficiently general and the transverse metric $g_{ij}(x^k)$ has general holonomy. These constitute the case $\beta_i = 0$ in [27] described in section 3.4.

Assuming $g_{ij}$ is the metric of a negatively curved space, we can take the transverse space to be an Einstein solvmanifold [28–30]. There are many examples of these, and most of them have general holonomy.

5. Conserved symmetric 2-tensors

For any 2-tensor we have, in general, the boost-weight decomposition

$$T = \sum_{b=-2}^{2} (T)_b.$$  

If this tensor is constructed from contractions of curvature tensors from the previous metrics, then there are no positive boost-weight components. Note that the divergence has the decomposition

$$\text{div} \, T = (\text{div} \, T)_1 + (\text{div} \, T)_0 + (\text{div} \, T)_{-1},$$

all of which components we require to be zero independently (i.e., they must vanish to each boost weight order separately).

For the Riemann tensor we also have the Bianchi identity, which upon contraction, gives the useful identity

$$R^\nu_{\mu\nu\alpha\beta} = R_{\mu\nu\beta\alpha} - R_{\mu\nu\alpha\beta}.$$  

Furthermore, for a tensor $T_{a_1...a_k}$, we have

$$[\nabla_\mu, \nabla_\nu]T_{a_1...a_k} = T_{b_1...a_k} R^b_{\alpha_1\mu} + \cdots + T_{a_1...b_k} R^b_{\alpha_k\mu}.$$  

(24)

The latter identity implies one may permute the order of covariant derivatives of curvature tensors, at the cost of possibly getting products of lower-order curvature tensors.
The various Ricci types and Weyl types restrict the possible non-zero tensors constructed from the metric and the curvature tensors. This is perhaps best illustrated with examples.

5.1. Four dimensions

5.1.1. Generalized Ghanam–Thompson. This is a generalization of a solution found in [31] and is defined as a four-dimensional Sim(2) Einstein metric, \( R_{\mu\nu} = \lambda g_{\mu\nu} \), where \( A_i = 0 \). The Einstein equations imply that \( H^{(0)}(u, x^k) \) is harmonic in the two-dimensional space [13]. Regarding the Weyl tensor, it is of Weyl (or Petrov) type II:

\[
C = (C)_0 + (C)_{-2}.
\]

However, the transverse space is the hyperbolic plane [13], and hence symmetric, and so \((\nabla R)_0 = 0\). As can be checked,

\[
\nabla^{(k)} R = (\nabla^{(k)} R)_{-2} + (\nabla^{(k)} R)_{-3} + \cdots.
\]

Any boost-weight components of order lower than \(-2\) cannot contribute to a symmetric 2-tensor, so the only one that can contribute from the higher derivatives is \((\nabla^{(k)} R)_{-2}\). (In fact, \(\nabla^{(k)} R = \nabla^{(k)} C\) since this metric is Einstein.) It follows that the non-zero invariants are the Ricci scalar, zeroth-order Weyl invariants, and arbitrary functions of these, \(f(I_i)\). These are all constants.

The only non-zero symmetric 2-tensors are of the form (non-zero boost-weight components in brackets):

- \(f(I_i)g_{\mu\nu}\) (boost-weight 0).
- Products of Weyl tensors with appropriate contractions (boost-weight 0, \(-2\))
- \(f(I_i)R_{\lambda\beta\gamma\delta}...\) with appropriate contractions (boost-weight \(-2\)). Using the contracted Bianchi identity, we can show that these are zero too, up to possible products of Weyl tensors, which have already been considered.

At this stage, we have only used boost-weight arguments and the Bianchi identities to get possible forms for the conserved 2-tensors. However, there are further simplifications. Consider a 2-tensor, \(T_{\mu\nu}\), made out of an arbitrary product of Riemann tensors. This tensor can therefore only consist of boost-weight 0 components and a boost-weight \(-2\) component. Consider only zeroth-order tensors first. The boost-weight \(-2\) component is the component \(T_{--}\), which can consist of terms

\[
S^\mu_{\nu} C_{--}^\nu - \mu, \quad S^\mu_{\nu \alpha} C_{--}^{\nu \alpha - \mu}, \quad S_{\alpha \beta \mu \nu} C_{--}^{\alpha \beta \nu \mu},
\]

where \(S\) is some tensor consisting of a product of the boost-weight 0 components of the Riemann tensor \((R)_{0}\). Consider the first term (with respect to the null-frame (22))

\[
T_{--} = S^\mu_{\nu} C_{--}^\nu - \mu = S^I_{\ nu} C^i_{--}.
\]

The tensor \(S^\mu_{\nu}\) consists entirely of contractions of curvature tensors of the homogeneous space \(AdS_2 \times H^2\), where \(H^2\) denotes the hyperbolic plane. Hence, by symmetry arguments (in the orthonormal frame)

\[
(S^\mu_{\nu}) = \text{diag}(c_1, c_1, c_2, c_2).
\]

Therefore, since \(C^i_{--} = 0\), we get \(T_{--} = c_2 C^i_{--} = 0\). For the other terms in (25), we note that since \(S = (S)_{0}\), any index ‘\(-\)’ must be accompanied by an index ‘\(+\)’ (downstairs). Therefore, using symmetry arguments we can show that these vanish too. Hence, the boost-weight \(-2\) component vanishes. Consider, therefore, the boost-weight 0 components of \(T_{\mu\nu}\); these must also consist of a product of \((R)_{0}\), and hence, must be of the same form as \(S^\mu_{\nu}\). By a
Wick rotation, we can consider the Euclidean-signature metric $\mathbb{H}^2 \times \mathbb{H}^2$, where the curvatures of each $\mathbb{H}^2$ are identical. It is therefore clear that $c_1 = c_2$, and hence, $S_{\mu \nu} = c_1 g_{\mu \nu}$.

Regarding higher-order invariants, we note that we must consider contractions of $(R^{(2)})_0 \otimes (V^{(1)} C)_{-2}$. We also note that we can write

$$ R_{\mu \nu \tau \sigma} = c \left( g^{(1)}_{\mu \sigma} g^{(1)}_{\nu \tau} - g^{(1)}_{\nu \sigma} g^{(1)}_{\mu \tau} + g^{(2)}_{\mu \sigma} g^{(2)}_{\nu \tau} - g^{(2)}_{\nu \sigma} g^{(2)}_{\mu \tau} \right), \quad (26) $$

where $g^{(1)}$ and $g^{(2)}$ are two-dimensional projection operators such that $g = g^{(1)} \oplus g^{(2)}$. It is therefore also clear that contractions of $(R^{(2)})_0 \otimes (V^{(k)} C)_{-2}$ will vanish. Combining all of this, we get $T_{\mu \nu} = c_1 g_{\mu \nu}$, where $c_1$ is some constant.

We have consequently shown that the four-dimensional generalized Ghanam–Thompson solution is weakly universal.

5.1.2. Goldberg–Kerr. This is a four-dimensional vacuum (Ricci type O) and Weyl (or Petrov) type III metric [32, 33]:

$$ C = (C)_{-1} + (C)_{-2}. $$

Here, all the scalar invariants vanish; i.e., it is a VSI. In general, the higher-order curvature tensors are

$$ \nabla^{(k)} R = (\nabla^{(k)} R)_{-1} + (\nabla^{(k)} R)_{-2} + \cdots $$

(again, it is vacuum so only the Weyl tensor will contribute). Without any further restrictions, we have the following possible 2-tensors:

- $g_{\mu \nu}$ (boost-weight 0).
- $R_{\mu \nu}$ (=0), and appropriate contractions of $R_{\mu \nu \alpha \beta \gamma \delta \ldots}$ (boost-weight $-1, -2$). Using the Bianchi identity these can, without loss of generality, also be assumed to be zero up to quadratic terms of (higher-order) curvature tensors (these are considered below).
- Quadratic terms of curvature tensors (boost-weight $-2$).

Let us study these quadratic terms more carefully. The only non-vanishing component of $T_{\mu \nu}$ is the boost-weight $-2$ component $T_{\mu \nu}$... The space is Ricci flat, so only the Weyl tensor will contribute. Consider therefore the quadratic term $C_{\alpha \beta \gamma \delta \ldots} C^{\alpha \beta \gamma \delta \ldots}$, where $C$ and $\tilde{C}$ can be either the Weyl tensor or its dual. Both of these tensors have the same symmetries and, in particular, the tracelessness implies $C_{i = i'} = C_{ij = \ldots 1}$. In four dimensions (and in four dimensions only) we have as many independent components of $C_{i = i'}$ as of $C_{ij = \ldots}$, and by direct calculation we find

$$ C_{\alpha \beta \gamma \delta \ldots} \tilde{C}^{\alpha \beta \gamma \delta \ldots} = -2 C_{i = i'} \tilde{C}^{i \ldots} \hat{\delta} + C_{ij = \ldots} \tilde{C}^{i \ldots} = 0. $$

This result can also be derived from the dimensionally dependent identities of Lovelock [34], see the appendix. There could also be a term $C_{\alpha \beta \gamma \delta \ldots} \tilde{C}^{\alpha \beta \gamma \delta \ldots}$, but this vanishes too for the same reason. Therefore, because of the symmetries of the Weyl tensor, all zeroth-order quadratic terms will vanish. Each higher-order quadratic term can be written as

$$ A_{a \beta \gamma \delta \ldots} B_{b \gamma \delta \ldots} a^{\beta \gamma \delta \ldots}, \quad (27) $$

where $A$ and $B$ can be assumed to be linear in the curvature tensors. Because of the symmetries of the curvature tensors, the Bianchi identities and equation (24), we can assume the indices are of a certain order. First, this implies that any contraction of $C_{\mu \nu \alpha \beta \gamma \delta \ldots}$ is zero. This further implies that we can, without loss of generality, assume that (or let $\alpha \leftrightarrow \beta$ in $B$)

$$ A_{a \beta \gamma \delta \ldots} = C_{\alpha \beta \gamma \delta \ldots}, \quad B_{b \gamma \delta \ldots} = \tilde{C}_{a \beta \gamma \delta \ldots}. \quad (28) $$

8 This can most easily be seen if we note that the isotropy group is $U(1)^2 \times \mathbb{Z}_2$, where $\mathbb{Z}_2$ interchanges the tangent spaces of the $\mathbb{H}^2$. This isotropy group acts irreducibly on the tangent space.
Note that $A_{-i_4\cdots} = A^i_{ij\cdots}$. Therefore, consider the ‘diagonal’ elements (where $i_k$-indices need not be summed)

$$A_{-a\beta\gamma\delta_1\delta_2\cdots} B_{-}{}^{a\beta\gamma\delta_1\delta_2\cdots} = C_{-a\beta\gamma\delta_1\delta_2\cdots} \tilde{C}_{-}{}^{a\beta\gamma\delta_1\delta_2\cdots} = 0.$$

Again, this follows from the dimensionally dependent identities of Lovelock [34] (see the appendix). Thus, all quadratic terms will vanish.

Consequently, the four-dimensional Goldberg–Kerr solution is strongly universal.

5.1.3. General four-dimensional Sim(2) metrics. Indeed, the universality of the Ghanam–Thompson and Goldberg–Kerr solutions extends to all four-dimensional Sim(2) metrics.

**Theorem 5.1.** Consider a four-dimensional Sim(2) metric and assume that the metric is Einstein, $R_{\mu\nu} = \lambda g_{\mu\nu}$. Then:

1. If $\lambda \neq 0$, the metric is weakly universal.
2. If $\lambda = 0$, the metric is strongly universal.

This theorem is proven in the appendix and illustrates the remarkable nature of these metrics.

5.2. General results in higher dimensions

We can also summarize some general results for Kundt metrics in higher dimensions in terms of the algebraic properties of the Weyl and Ricci tensor.

Weyl type $N$, Ricci type $O$ (vacuum). All scalar invariants vanish, all symmetric 2-tensors, except $g_{\mu\nu}$, vanish. Therefore, these are strongly universal.

Weyl type $N$, Ricci type $N$. All scalar invariants vanish. The non-trivial symmetric 2-tensors are $g_{\mu\nu}, R_{\mu\nu}$ and linear in higher-orders of $R_{\mu\nu}$ (such as, for example, $\Box R_{\mu\nu}$).

Weyl type $N$, Einstein space. The only non-vanishing scalar invariant is the Ricci scalar and arbitrary functions thereof. The symmetric 2-tensors are thus of the form $f(I_i)g_{\mu\nu}$. These are therefore weakly universal.

Examples of metrics in this class are the $n$-dimensional ‘anti-de Sitter waves’ with metric

$$ds^2 = \frac{1}{b^2 z^2} [2du(dv + H(u, x^i) du) + dz^2 + \delta_{AB} dx^A dx^B].$$

These metrics are CSI but do not have holonomy in Sim($n - 2$).

Counterexamples. Here we provide counterexamples in higher dimensions which show that not all of the remarkable properties outlined above of the four-dimensional Ghanam–Thompson and Goldberg–Kerr solutions extend to higher dimensions.

Consider first the $n$-dimensional analogue of the Ghanam–Thompson solution, obtained in [13]. This Sim($n - 2$) holonomy metric can be taken to be

$$ds^2 = 2du dv + (H(u, x^k) - 2\sigma v^2) du^2 + ds^2,$$

and it is Einstein, with $\hat{R}_{\mu\nu} = -2\sigma \tilde{g}_{\mu\nu}$, provided that $ds^2$ is also Einstein (with $R_{\mu\nu} = -2\sigma g_{\mu\nu}$), and that $H$ is harmonic over $ds^2$. In the vielbein basis $\hat{e}^+ = dv + \frac{1}{2}(H - 2\sigma v^2) du$, $\hat{e}^- = du$, $\hat{e}^{i} = e^{i}$, the non-zero components of the torsion-free connection, and
the Riemann tensor are given by
\[ \hat{\omega}_{-+} = 2\sigma v^-, \quad \hat{\omega}_{-i} = \frac{1}{2}(\nabla_i H)\hat{v}^-, \quad \hat{\omega}_{ij} = \omega_{ij}, \]
\[ \hat{R}_{-+-} = 2\sigma, \quad \hat{R}_{-ij} = -\frac{1}{2}\nabla_i \nabla_j H, \quad \hat{R}_{ij\ell} = R_{ij\ell}. \] (30)

Furthermore, the Weyl tensor decomposes as \( C = (C)_0 + (C)_{-2} \) in the same manner as the four-dimensional Ghannam–Thompson solution. Let us consider the symmetric tensor \( \hat{S}_{ab} = \hat{C}_{aede} \hat{C}_{dbef} \). If we take the base metric \( d\hat{s}^2 \) to be maximally symmetric, so that \( R_{ij\ell} = -2\sigma/(n-3)(\delta_i \delta_j - \delta_i \delta_{jk}) \), then a simple calculation shows that the non-zero components of \( \hat{S}_{ab} \) are given by
\[ \hat{S}_{++} = \frac{8(n-2)\sigma^2}{n-1}, \quad \hat{S}_{ij} = \frac{16\sigma^2}{(n-1)(n-3)} \delta_{ij}. \] (32)

It is easily verified that this is divergence-free. However, it is clearly only in \( n = 4 \) dimensions that it is proportional to the metric tensor, and so this Sim\((n-2)\) Einstein spacetime is not weakly universal in more than four dimensions.

One can also see, by means of counterexamples, that the generalizations to higher dimensions of the Ricci-flat Goldberg–Kerr solutions do not share the strong universality property of the four-dimensional case. The generalized Goldberg–Kerr metric in \( n \) dimensions can be written as
\[ d\hat{s}^2 = 2du \, dv + (H_0(u, x^4) + vH_1(u, x^4)) \, du^2 + 2A_i(u, x^4) \, du \, dx^i + ds^2, \] (33)
where the base metric \( ds^2 \) is Ricci flat. The conditions for Ricci-flatness of \( d\hat{s}^2 \) reduce to [13]
\[ \nabla^2 H_0 - \frac{4}{3} F_{ij} F_{ij} - 2A_i \nabla_i H_1 - H_1 \nabla^i A_i - 2\nabla^i \dot{A}_i = 0, \] (34)
\[ \nabla^i F_{ij} + \nabla_i H_1 = 0, \] (35)
where \( F_{ij} = \nabla_i A_j - \nabla_j A_i \), and it is assumed that \( ds^2 \) is independent of \( u \). One may choose \( H_1 = -\nabla^i A_i \) as a gauge choice [13], in which case Ricci-flatness is achieved by first solving the linear equation \( (35) \) for \( A_i \), and then solving the Poisson equation \( (34) \) for \( H_0 \).

Let us consider the symmetric tensor
\[ \hat{T}_{ab} = \hat{R}_{aede} \hat{R}_{dbef} - \frac{1}{2} \hat{G}_{ab} \hat{R}_{def}, \] (36)
which can easily be seen to be conserved, after imposing the condition of Ricci-flatness. Consider first the case where \( ds^2 \) in \( (33) \) is the flat Euclidean metric. We then have
\[ \hat{T}_{---} = -\frac{1}{2}(\nabla_i H_1)(\nabla_i H_1) + \frac{1}{4}(\nabla_i F_{jk})(\nabla^i F^{jk}), \]
\[ = -\frac{1}{2}(\nabla_i F_{ij})(\nabla^i F^{ij}) + \frac{1}{4}(\nabla_i F_{jk})(\nabla^i F^{jk}), \] (37)
where the second line follows after using \( (35) \). For any dimension \( n > 4 \) this will in general be non-zero, thus showing that there exist non-vanishing conserved symmetric 2-tensors for these metrics. Thus, the generalization of the Goldberg–Kerr metrics to dimensions higher than four do not satisfy the strong universality principle. (The four-dimensional case is an exception since then we have \( F_{ij} = f \epsilon_{ij} \), which implies that \( \hat{T}_{---} \) vanishes.)

We will now consider one specific example where this can be seen explicitly. Consider the \( n > 4 \) dimensional metric with flat transverse metric \( ds^2 = \delta_{ij} \, dx^i \, dx^j \). Furthermore, let
\[ A_i \, dx^i = xy \, dx - xyz \, dy, \] (38)
where \( x, y, z, \ldots \) are Cartesian coordinates on \( dx^2 \), and \( a \) is a constant. We note that \( H_1 = -y/(1 - a) \), so one can verify that equation \( (35) \) is indeed satisfied, and equation \( (34) \) then reduces to
\[ \nabla^2 H_0 - x^2 + y^2(1 - a)^2 - a^2 z^2 = 0. \] (39)
This equation has the solution
\[ H_0 = \frac{1}{12} \left[ x^4 - y^4 (1 - a)^2 + a^2 z^4 \right] + F(u, x^4), \] (40)
where \( F(u, x^4) \) is a solution to \( \nabla^2 F = 0 \). Using equation (37) gives \( \hat{T}_{\ldots} = a \) and hence, unless \( a = 0 \), this tensor does not vanish. Therefore, the solution presented above is, for \( a \neq 0 \), not strongly universal.

**Semi-universality.** The (counter)examples of the higher-dimensional Ghanam–Thompson solution naturally leads us to the possibility of another class of metrics with relatively simple quantum corrections.

Let us consider the case where the transverse space is a symmetric space in which the isotropy group acts irreducibly on the tangent space, and define the projection operators,
\[ g^{(1)} = 2 \hat{e}^i \hat{e}^j, \quad g^{(2)} = \delta_{ij} \hat{e}^i \hat{e}^j, \]
so that \( g_{\mu\nu} = g^{(1)}_{\mu\nu} + g^{(2)}_{\mu\nu} \). Let us try to generalize the Ghanam–Thompson case to higher-dimensions. Therefore, we assume the Ricci tensor is of type D and
\[ R_{\mu\nu} = \lambda_1 g^{(1)}_{\mu\nu} + \lambda_2 g^{(2)}_{\mu\nu}. \]
We also assume that the Weyl tensor is of the form \( C = (C)_0 + (C)_{-2} \). Note that most of the arguments in the derivation of the symmetric 2-tensors of the Ghanam–Thompson solution go through, and we find the following general form:
\[ T_{\mu\nu} = f_1(\lambda_1, \lambda_2) g^{(1)}_{\mu\nu} + f_2(\lambda_1, \lambda_2) g^{(2)}_{\mu\nu}. \] (41)
We note than in four dimensions, and for \( \lambda_1 = \lambda_2 \), we can argue that \( f_1 = f_2 \). In higher dimensions the equations of motion, including quantum corrections, reduce to solving
\[ \lambda_1 + f_1(\lambda_1, \lambda_2) = 0, \quad \lambda_2 + f_2(\lambda_1, \lambda_2) = 0. \] (42)
If such solutions exist, we shall call these metrics semi-universal.

**6. Discussion**

In this paper we have investigated solutions of the classical Einstein or supergravity equations that remain valid solutions in the quantum theory. We have been particularly interested in solutions of the classical Einstein equations which are universal (weakly or strongly), and consequently have a restricted curvature structure such that they remain solutions of almost any set of covariant equations involving the metric and its derivatives. An example of such solutions are the Brinkmann wave class of solutions of the Einstein equations which admit a recurrent null vector field [13], so that the metrics have holonomy contained in \( \text{Sim}(n-2) \).

In particular, we have focussed attention on the subsets of Einstein metrics which are VSI or CSI.

In four dimensions we have found two new classes of metrics, one of which is strongly universal and the other of which is weakly universal. In more detail, in the four-dimensional generalized Ghanam–Thompson spacetime, which is a \( \text{Sim}(2) \) Einstein CSI metric (with \( A_i = 0 \) and a harmonic \( H^{(0)}(u, x^4) \) in which the Weyl tensor is of type II), the only non-zero invariants are the Ricci scalar, zeroth-order Weyl invariants, and arbitrary functions of these, which are all constants. By a direct calculation we then found that the only non-zero conserved symmetric 2-tensors are of the form \( cg_{\mu\nu} \), where \( c \) is a constant. Therefore, the four-dimensional generalized Ghanam–Thompson solution is weakly universal. In the four-dimensional vacuum and Weyl type III Goldberg–Kerr VSI metric, all of the scalar invariants vanish. By a direct calculation we then found that the metric (modulo a constant rescaling)
is the only non-zero conserved symmetric 2-tensor. Consequently, the four-dimensional Goldberg–Kerr solution is strongly universal.

We also showed that all four-dimensional Sim(2) Einstein metrics are universal, and in the future it will be of interest to consider four-dimensional Einstein metrics with general holonomy and investigate under what circumstances such metrics are weakly universal ($\lambda \neq 0$) or strongly universal ($\lambda = 0$). Since the Goldberg–Kerr and the Ghanam–Thompson solutions are of different algebraic type there is hope that their properties will extend to an even bigger class of metrics.

We then discussed universality in higher dimensional spacetimes. We first summarized some general results in higher dimensions and discussed some examples. In particular, we noted that the higher dimensional generalizations of the Ghanam–Thompson and Goldberg–Kerr solutions are not (weakly and strongly, respectively) universal. It is therefore clear that theorem 5.1 cannot be generalized to higher-dimensions. It was suggested that the notion of semi-universality may be of relevance in studying the quantum corrections of higher-dimensional classical solutions. In future we hope to further study the higher dimensional case.

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Appendix. Proof of theorem 5.1

Here we intend to prove the following.
Consider a four-dimensional Sim(2) metric and assume that the metric is Einstein, $R_{\mu\nu} = \lambda g_{\mu\nu}$. Then

(i) If $\lambda \neq 0$, the metric is weakly universal.
(ii) If $\lambda = 0$, the metric is strongly universal.

Dimensionally dependent identities. In four dimensions, and four dimensions only, we have the identity [34]
\[ \delta^{[\mu}_{[\nu} C^{\rho\sigma]}_{\rho\sigma} = 0. \] (A.1)

This identity is what Lovelock calls a dimensionally dependent identity and is valid for a Weyl tensor, or any Weyl-like tensor with the same symmetries as the Weyl tensor. By considering covariant derivatives of this identity we obtain therefore
\[ \delta^{[\mu}_{[\nu} C^{\rho\sigma]}_{\rho\sigma} \epsilon_1 \cdots \epsilon_k = 0. \]

Consider a tensor $W_{\alpha\beta\gamma\epsilon_1 \cdots \epsilon_k}$, where
\[ W_{\alpha\beta\gamma\epsilon_1 \cdots \epsilon_k} = W_{\alpha\beta\gamma\epsilon_1 \cdots \epsilon_k}, \quad W_{\beta\alpha\gamma\epsilon_1 \cdots \epsilon_k} = 0. \]

By contraction, we get the identity
\[ 0 = g_{\mu\nu} \left( W_{\alpha\beta\gamma\epsilon_1 \cdots \epsilon_k} C_{\alpha\beta\gamma\delta_1 \cdots \delta_k} + 4 W_{\alpha\beta\gamma\epsilon_1 \cdots \epsilon_k} C_{\mu}^{\alpha} \right) + 4 W_{\alpha\beta\gamma\epsilon_1 \cdots \epsilon_k} C_{\mu}^{\alpha} \delta_{\gamma, \eta_1 \cdots \eta_k}. \] (A.2)

Note that if we are considering the diagonal components (i.e., $\epsilon_1 = \eta_1, \ldots, \epsilon_k = \eta_k$), then the boost-weight of the first term in equation (A.2) is zero. Hence, the second term can only
have boost-weight 0 terms also. This identity is very useful in proving universality for Sim(2) metrics.

**Einstein equations.** For a Sim(2) metric the Einstein equations imply that

\[ H(v, u, x^k) = v^2 \lambda + v H^{(1)}(u, x^k) + H^{(0)}(u, x^k), \]

and the transverse metric, \( g_{ij}(u, x^k) \) is a two-dimensional Einstein space. In particular, this means that the transverse space is symmetric and either \( S^2, \mathbb{R}^2, \mathbb{H}^2 \). Moreover, the Sim(2)-metric must be CSI (VSI if \( \lambda = 0 \)) and we can use a coordinate transformation to get rid of the \( u \)-dependence in the transverse metric (i.e., \( g_{ij}(x^k) \)).

**Boost-weight decomposition.** Regarding the zeroth-order curvature tensors, these are of Ricci type D and Weyl (Petrov) type II (for \( \lambda = 0 \) it is of type III):

\[ C = (C)_0 + (C)_{-1} + (C)_{-2}. \]

Since the transverse metric is symmetric, higher-order curvature tensors have the following boost-weight decomposition \((k > 0)\):

\[ \nabla^{(k)} C = (\nabla^{(k)} C)_{-1} + (\nabla^{(k)} C)_{-2} + \cdots. \]

For the symmetric tensor \( T_{\mu\nu} \) we have that

\[ T = (T)_0 + (T)_{-1} + (T)_{-2}, \]

where each boost-weight component will be considered in turn.

**Boost-weight 0.** The boost-weight 0 components of \( T_{\mu\nu} \) are identical to those of the symmetric space \( ds^2 = 2du[du + v^2 du] + g_{ij}(x^k) dx^i dx^j \). Considering the isotropy group, we can see that any curvature 2-tensor must be of the form \( S'_{\mu\nu} = \text{diag}(c_1, c_1, c_2, c_2) \). Since the curvatures match, we must have \( c_1 = c_2 \), and hence

\[ S_{\mu\nu} = c_1 g_{\mu\nu}. \]

**Boost-weight \(-1\).** These components must arise as contractions of terms like

\[ (R^{(0)}_0)_{0} \otimes (C)_{-1}, \quad (R^{(0)}_0)_{0} \otimes (\nabla^{(k)} C)_{-1}. \]

First, consider zeroth-order terms. In this case it is advantageous to switch to the Weyl canonical frame. This will not change the boost-weight 0 terms but might alter the negative boost weight terms. For \( \lambda \neq 0 \), the spacetime is of Weyl type II, and in the Weyl canonical frame we have \( C = (C)_0 + (C)_{-2} \). Therefore, in this frame the boost-weight \(-1\) components are zero. For \( \lambda = 0 \) (which implies Ricci flat) the Weyl canonical frame gives \( C = (C)_{-1} \) and therefore zeroth-order terms give no contributions to the boost-weight \(-1\) components of the symmetric tensor \( T_{\mu\nu} \).

Next, consider the higher order terms \((R^{(0)}_0)_{0} \otimes (\nabla^{(k)} C)_{-1}\). We easily see that the number of covariant derivatives must be even in order for this term to contribute to the quantum corrections. Now, the boost-weight 0 curvature tensor has the form equation (26) which implies that \((R^{(0)}_0)_{0}\) must be a tensor product of \( g^{(1)} \) and \( g^{(2)} \). For the Sim(2) metrics we note that the components \( C_{a\beta\gamma\delta,i_1\cdots i_1} = 0 \) if there is an \( i_1 \)-index being '+' (or else, the holonomy would not be a subgroup of Sim(2)). Therefore, if \( g^{(1)} = 2\hat{x}^2 \hat{e}^- \), we have \( C_{a\beta\gamma\delta,i_1\cdots i_1} g^{1,i_1} = 0 \). Thus, since \( g = g^{(1)} + g^{(2)} \),

\[ C_{a\beta\gamma\delta,i_1\cdots i_1\cdots i_1} g^{2,i_1} = C_{a\beta\gamma\delta,i_1\cdots i_1\cdots i_1} g^{i_1}. \]

By permuting the indices \( i_1 \) and \( i_1 \) we can show, up to lower-order products, that this must vanish too. Consequently, any contraction of \( g^{(2)} \) with \((\nabla^{(k)} C)_{-1}\) is also zero. Since \((R^{(0)}_0)_{0}\)
is a tensor product of $g^{(1)}$ and $g^{(2)}$, $(R^0)_{0} \otimes (\nabla^{(k)} C)_{-1}$ cannot contribute to a symmetric 2-tensor.

Hence, we are led to the conclusion that all boost-weight $-1$ components must vanish.

Boost-weight $-2$. This component contains contractions of terms such as 

$$(R^0)_{0} \otimes (C)_{-2}, \quad (\nabla^{(k)} C)_{-2}, \quad (\nabla^{(k)} C)_{-1} \otimes (\nabla^{(k)} C)_{-1}.$$  

Contributions from the first and second terms must vanish (this follows from an identical argument as for the Ghanam–Thompson solution). The vanishing of the third term follows from permuting the indices using equation (24), using the Bianchi identity, and using equation (A.2).

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