CONCAVITY OF MINIMAL $L^2$ INTEGRALS RELATED TO MULTIPLIER IDEAL SHEAVES

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Abstract. In this note, we present the concavity of the minimal $L^2$ integrals related to multiplier ideals sheaves on Stein manifolds. As applications, we obtain a necessary condition for the concavity degenerating to linearity, a characterization for 1-dimensional case, and a characterization for the equality in 1-dimensional optimal $L^2$ extension problem to hold.

1. Introduction

The multiplier ideal sheaf associated to plurisubharmonic functions plays an important role in complex geometry and algebraic geometry (see e.g. [34], [29], [31], [9], [10], [6], [11], [27], [32], [33], [5], [23]). We recall the definition of the multiplier ideal sheaves as follows.

Let $\varphi$ be a plurisubharmonic function (see [4]) on a complex manifold. It is known that the multiplier ideal sheaf $\mathcal{I}(\varphi)$ was defined as the sheaf of germs of holomorphic functions $f$ such that $|f|^2 e^{-\varphi}$ is locally integrable (see [5]).

In [6], Demailly posed the so-called strong openness conjecture on multiplier ideal sheaves (SOC for short) i.e. $\mathcal{I}(\varphi) = \mathcal{I}_+(\varphi) := \bigcup_{\epsilon > 0} \mathcal{I}((1 + \epsilon)\varphi)$. When $\mathcal{I}(\varphi) = \mathcal{O}$, SOC degenerates to the openness conjecture posed by Demailly-Kollár [10].

The dimension two case of OC was proved by Favre-Jonsson [13], and the dimension two case of SOC was proved by Jonsson-Mustaţă [25]. OC was proved by Berndtsson [2]. SOC was proved by Guan-Zhou [19], see also [28] and [24].

In [11], Berndtsson established an effectiveness result of OC. Simulated by Berndtsson’s effectiveness result of OC, continuing the solution of SOC [19], Guan-Zhou [20] establish an effectiveness result of SOC.

Recently, Guan [17] established a sharp version of the effectiveness result of SOC by considering the minimal $L^2$ integrals defined on the sub-level set of plurisubharmonic function, and established the concavity of the minimal $L^2$ integrals on pseudoconvex domain in $\mathbb{C}^n$.

In the present note, we generalize the above concavity property.

1.1. A general concavity property. Let $X$ be an $n$-dimensional Stein manifold, and let $K_X$ be the canonical line bundle on $X$. Let $dV_X$ be a continuous volume form with no zero point on $X$. We define $|g|^2 = r^n \frac{\partial^2 \varphi}{\partial r^2}$ for any holomorphic $(n,0)$ form $g$. Let $\psi < -T$ be a plurisubharmonic function on $X$, and let $\varphi$ be a Lebesgue

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measurable function on $X$, such that $\varphi + \psi$ is a plurisubharmonic function on $X$, where $T \in (-\infty, +\infty)$.

We call a positive smooth function $c$ on $(T, +\infty)$ in class $G_T$ if the following three statements hold:

1. $\int_T^{+\infty} c(t)e^{-t}dt < +\infty$;
2. $c(t)e^{-t}$ is decreasing with respect to $t$;
3. for any compact subset $K \subset X$, $e^{-\varphi}c(-\psi)$ has a positive lower bound on $K$.

Especially, if $\varphi \equiv 0$, then (3) is equivalent to $\liminf_{t \to +\infty} c(t) > 0$.

Let $Z_0$ be a subset of $\{\psi = -\infty\}$ such that $Z_0 \cap \text{Supp} (\mathcal{O}/\mathcal{I}(\varphi + \psi)) \neq \emptyset$. Let $U \supset Z_0$ be an open subset of $X$ and let $f$ be a holomorphic $(n, 0)$ form on $U$. Let $\mathcal{F} \supset \mathcal{I}(\varphi + \psi)|_U$ be a coherent subsheaf of $\mathcal{O}$ on $U$.

Denote

$$
\inf \{ \int_{\{\psi < -t\}} |\tilde{f}|^2 e^{-\tilde{\varphi}}c(-\psi)dV_X : \tilde{f} \in H^0(\{\psi < -t\}, \mathcal{O}(K_X)) \\
\text{and } (\tilde{f} - f) \in H^0(\{\psi < -t\} \cap U', \mathcal{O}(K_X) \otimes \mathcal{F}) \}
$$

by $H(t; c)$ ($H(t)$ for short without misunderstanding), where $c \in G_T$.

If there is no holomorphic $(n, 0)$ form $\tilde{f}$ on $\{\psi < -t\}$ satisfying $(\tilde{f} - f) \in H^0(\{\psi < -t\} \cap U', \mathcal{O}(K_X) \otimes \mathcal{F})$ for some open subset $U'$ which satisfies $Z_0 \subset U' \subset U$, then we set $H(t) = -\infty$.

In the present note, we obtain the following concavity of $H(t)$.

Theorem 1.1. $H(h^{-1}(r))$ is concave with respect to $r \in (0, \int_T^{+\infty} c(t)e^{-t}dt]$, where $h(t) = \int_t^{+\infty} c(t_1)e^{-t_1}dt_1, t \in [T, +\infty)$.

Especially, when $c(t) \equiv 1$ and $T = 0$, Theorem 1.1 degenerates to the concavity of the minimal $L^2$ integrals related to multiplier ideals in [17] (Proposition 4.1 in [17]).

Theorem 1.1 implies the following.

Corollary 1.2. For any $c \in G_T$, the following three statements are equivalent

1. $H(h^{-1}(r))$ is linear with respect to $r \in (0, \int_T^{+\infty} c(t)e^{-t}dt]$, i.e.,

$$
H(t) = \frac{H(T)}{\int_T^{+\infty} c(t)e^{-t}dt} \int_t^{+\infty} c(t_1)e^{-t_1}dt_1
$$

holds for any $t \in [T, +\infty)$;

2. $\frac{H(h^{-1}(r_0))}{r_0} \leq \frac{H(T)}{\int_T^{+\infty} c(t)e^{-t}dt}$ holds for some $r_0 \in (0, \int_T^{+\infty} c(t)e^{-t}dt)$, i.e.,

$$
\frac{H(t_0)}{\int_t^{+\infty} c(t_1)e^{-t_1}dt_1} \leq \frac{H(T)}{\int_T^{+\infty} c(t)e^{-t}dt}
$$

holds for some $t_0 \in (T, +\infty)$;

3. $\lim_{r \to 0^+} \frac{H(h^{-1}(r))}{r} \leq \frac{H(T)}{\int_T^{+\infty} c(t)e^{-t}dt}$ holds, i.e.,

$$
\lim_{t \to +\infty} \frac{H(t)}{\int_t^{+\infty} c(t_1)e^{-t_1}dt_1} \leq \frac{H(T)}{\int_T^{+\infty} c(t)e^{-t}dt}
$$

holds.
1.2. Applications. Following the notations and assumptions in Section 1.1, we present some applications of Theorem 1.1.

1.2.1. linear case: necessary condition.

Theorem 1.3. Assume that $H(T; c) < +\infty$. If $H(h^{-1}(r); c)$ is linear with respect to $r \in \{0, \int_T^{+\infty} c(t)e^{-t}dt\}$, then there exists a holomorphic $(n, 0)$ form $F$ on $X$ such that $(F - f) \in H^0(U', K_M \otimes F)$, where $U'$ is an open subset of $X$.

Following the notations and assumptions in Section 1.1, we have

\[ \int_{\{\psi < -t\}} c(-\psi)|F|^2e^{-\psi}dV_X = H(t) = H(T) \frac{\int_t^{+\infty} c(t_1)e^{-t_1}dt_1}{\int_T^{+\infty} c(t_1)e^{-t_1}dt_1} \] (1.5)

holds for any $t \in [T, +\infty)$.

When $c(t) \equiv 1$, $\varphi$ is a smooth plurisubharmonic function on $X$, and $\{\psi = -\infty\}$ is a closed subset of $X$, Xu [33] also get the Theorem 1.3 independently.

We now consider the linearity of $H(h^{-1}(r); c)$ for various $c \in G_T$ and $c \in C^\infty[T, +\infty)$, where $h_c(t) = \int_t^{+\infty} c(t_1)e^{-t_1}dt_1$. We have the following result.

Corollary 1.4. Let $c \in G_T$ and $c \in C^\infty[T, +\infty)$. If $H(T; c) < +\infty$ and $H(h^{-1}(r); c)$ is linear with respect to $r \in \{0, \int_T^{+\infty} c(t)e^{-t}dt\}$. Let $F$ be the holomorphic $(n, 0)$ form on $X$ such that $\int_{\{\psi < -t\}} c(-\psi)|F|^2e^{-\psi}dV_X = H(t; c)$ for any $t \geq T$. Then for any other $\tilde{c} \in G_T$ and $\tilde{c} \in C^\infty[T, +\infty)$, which satisfies $H(T; \tilde{c}) < +\infty$ and $(\log \tilde{c}(t))' \geq (\log c(t))'$, we have

\[ \int_{\{\psi < -t\}} \tilde{c}(-\psi)|F|^2e^{-\psi}dV_X = H(t; \tilde{c}) = \frac{H(T; \tilde{c})}{\int_T^{+\infty} \tilde{c}(t_1)e^{-t_1}dt_1} \int_t^{+\infty} \tilde{c}(t_1)e^{-t_1}dt_1 \] (1.6)

holds for any $t \in [T, +\infty)$, where $k = \frac{H(T; c)}{\int_T^{+\infty} c(t)e^{-t}dt_1}$.

We now consider the relation between the linearity of $H(t)$ and the weight function $\varphi$. Let $c(t) \in G_T$. Denote

\[ \inf\{\int_{\{\psi < -t\}} c(-\psi)|\tilde{f}|^2e^{-\psi}dV_X : \tilde{f} \in H^0(\{\psi < -t\}, O(K_X)) \land \exists \text{ open set } U' \text{ s.t. } Z_0 \subset U' \subset U \text{ and } (\tilde{f} - f) \in H^0(\{\psi < -t\} \cap U', O(K_X) \otimes F) \} \]

by $H(t; \varphi)$. We have the following result.

Corollary 1.5. If there exists a Lebesgue measurable function $\tilde{\varphi}$ such that $\psi + \tilde{\varphi}$ is a plurisubharmonic function on $X$ and satisfies

1. There exists constant $C_1, C_2 > T$ such that
\[ \tilde{\varphi}|_{\{\psi < -C_1\} \cup \{\psi \geq -C_2\}} = \varphi|_{\{\psi < -C_1\} \cup \{\psi \geq -C_2\}}. \]

2. $\tilde{\varphi} \geq \varphi$ on $X$ and $\tilde{\varphi} > \varphi$ on a open set $U$ of $X$. 

3. $\tilde{\varphi} - \varphi$ is bounded on $X$.

Then $H(h^{-1}(r); \varphi)$ can not be linear with respect to $r \in (0, \int_T^{+\infty} c(t)e^{-t}dt]$. 

If $\varphi + \psi$ is a plurisubharmonic function on $X$ and $\varphi + \psi$ is strictly plurisubharmonic at $z_0 \in X$. Denote
\[
\inf\left\{ \int_{\{\psi < -t\}} |\tilde{f}|^2 e^{-\varphi} c(-\psi) dV_X : \tilde{f} \in H^0(\{\psi < -t\}, \mathcal{O}(K_X)), \right.
\left. & \exists \text{ open set } U' \text{ s.t } Z_0 \subset U' \subset U \text{ and} \right.
\left. (\tilde{f} - f) \in H^0(\{\psi < -t\} \cap U', \mathcal{O}(K_X) \otimes \mathcal{F}) \right\}
\]
by $H(\ell; \varphi)$. It follows from Corollary 1.5 that we have

**Corollary 1.6.** $H(h^{-1}(r); \varphi)$ can not be linear with respect to $r \in (0, f_T^{+\infty} c(t)e^{-t} dt]$.

1.2.2. **Equality in optimal $L^2$ extension problem: necessary condition.** Following Guan-Zhou [22], for a suitable pair $(X, Y)$, where $Y$ is a closed complex subvariety of a complex manifold $X$, given a holomorphic function $f$ (or a holomorphic section of some vector bundle) on $Y$ satisfying suitable $L^2$ conditions, we can find an $L^2$ holomorphic extension $F$ on $X$ together with an optimal $L^2$ estimate for $F$ on $X$.

For example, let $X$ be a Stein manifold, and let $Y$ be a $n-k$ dimensional complex submanifold of $X$. Let $\psi < 0$ be a plurisubharmonic function on $X$, such that for any point $Y$ of $X$, $\psi - 2k \log |\omega''|$ is bounded near $x$, where $\omega = (\omega', \omega'')$ is the local coordinate near $x$ such that $\{\omega'' = 0\} = Y$ near $x$.

Following [30] (see also [22]), one can define the measure $dV_X[\psi]$ on $Y$
\[
\int_Y f dV_X[\psi] = \lim_{t \to \infty} \frac{2(n-k)}{\sigma_{2n-2k-1}} \int_X |f|^2 e^{-\varphi} \mathbb{I}_{\{1-t < \psi < -t\}} dV_X
\]  
for any nonnegative continuous function $f$ with $supp f \subset X$, where $\mathbb{I}_{\{1-t < \psi < -t\}}$ is the characteristic function of the set $\{1-t < \psi < -t\}$. Here denote by $\sigma_{m}$ the volume of the unit sphere in $\mathbb{R}^{m+1}$. Let $\varphi$ be a locally upperbounded Lebesgue measurable function on $X$, such that $\varphi + \psi$ is plurisubharmonic on $X$.

Let $c(t) \in \mathcal{G}_T$. It was established in [24] (see also [21]) that for any holomorphic $(n, 0)$ form $f$ on $Y$, such that
\[
\int_Y |f|^2 e^{-\varphi} dV_X[\psi] < +\infty
\]  
there exists a holomorphic $(n, 0)$ form $F$ on $X$ such that $F|_Y = f$ and
\[
\int_X c(-\psi)|F|^2 e^{-\varphi} dV_X \leq \left( \int_0^{+\infty} c(t)e^{-t} dt \right) \frac{\pi^k}{k!} \int_Y |f|^2 e^{-\varphi} dV_X[\psi] < +\infty
\]  

To simplify our notation, denote that $\|f\|_{L^2} := \left( \int_0^{+\infty} c(t)e^{-t} dt \right) \frac{\pi^k}{k!} \int_Y |f|^2 e^{-\varphi} dV_X[\psi]$ and $\|F\|_{L^2} := \int_X c(-\psi)|F|^2 e^{-\varphi} dV_X$. We will consider the following question

**Question 1.7.** (Equality in optimal $L^2$ extension problem) Under which (necessary or sufficient) condition, equality $\|f\|_{L^2} = \inf \{\|F\|_{L^2} : F$ is a holomorphic extension of $f$ from $Y$ to $X\}$ holds? Moreover, can one obtain the characterization (necessary and sufficient condition)?

Theorem 1.3 shows that the following necessary condition for the equality $\|f\|_{L^2} = \inf \{\|F\|_{L^2} \}$ to hold.
Theorem 1.8. Let \( f \) be holomorphic \((n,0)\) form on \( Y \), such that
\[
\int_Y |f|^2 e^{-\varphi} dV_X[\psi] < +\infty
\] (1.10)
If for any holomorphic \((n,0)\) form \( \tilde{F} \) on \( X \), which is a holomorphic extension of \( f \) from \( Y \) to \( X \) i.e. \( \tilde{F}|_Y = f \), then \( \tilde{F} \) satisfies
\[
\int_X c(-\psi)|\tilde{F}|^2 e^{-\varphi} dV_X \geq (\int_0^{+\infty} c(t)e^{-t} dt)\frac{\pi^k}{k!} \int_Y |f|^2 e^{-\varphi} dV_X[\psi]
\] (1.11)
and there exists a holomorphic \((n,0)\) form \( F \) on \( X \) such that
\[
\int_X c(-\psi)|F|^2 e^{-\varphi} dV_X = (\int_0^{+\infty} c(t)e^{-t} dt)\frac{\pi^k}{k!} \int_Y |f|^2 e^{-\varphi} dV_X[\psi]
\] (1.12)
Then for any \( t \geq 0 \), there exists a unique holomorphic \((n,0)\) form \( F_t \) on \( \{ \psi < -t \} \) such that \( F_t|_Y = f \) and
\[
\int_{\{\psi < -t\}} c(-\psi)|F_t|^2 e^{-\varphi} dV_X = (\int_t^{+\infty} c(t_1)e^{-t_1} dt_1)\frac{\pi^k}{k!} \int_X |f|^2 e^{-\varphi} dV_M[\psi]
\] (1.13)
In fact, \( F_t = F|_{\{\psi < -t\}} \).

Remark 1.9. It follows from Corollary [1.4] and Theorem [1.5] that for any \( \tilde{c} \in \mathcal{G}_T \) which satisfies \(( \log \tilde{c}(t) )' \geq ( \log c(t) )'\), the holomorphic \((n,0)\) form \( F \) satisfies
\[
H(t; \tilde{c}) = \int_{\{\psi < -t\}} \tilde{c}(-\psi)|F|^2 e^{-\varphi} dV_X = (\int_t^{+\infty} \tilde{c}(t_1)e^{-t_1} dt_1)\frac{\pi^k}{k!} \int_Y |f|^2 e^{-\varphi} dV_X[\psi].
\]
Recall that the pluricomplex Green function \( G(z, \omega) \) on a pseudoconvex domain \( D \subset \mathbb{C}^n \) satisfies \( G_D(z, \omega) < 0 \) and \( G_D(z, \omega_0) = \log |z - \omega_0| + O(1) \) near \( \omega_0 \in D \) (see [3]). Let \( \psi(z) = 2nG_D(z,0) \), \( f \equiv 1 \) and \( F = (z_1, \ldots, z_n) \), and let \( \varphi \equiv 0 \) and \( c(t) \equiv 1 \). Let \( D_t = \{ \psi(z) < t \} \). Note that \( K_{D_t}(0,0) = \frac{1}{\pi(\varphi)} \), then the combination of Corollary 1.2 and Theorem 1.3 implies the following restriction property of Bergman kernels.

Corollary 1.10. The following two statements are equivalent
\[
(1) \quad \frac{K_{D_t}(0,0)}{K_{D_t}(z_0,0)} = e^{t_0} \text{ holds for some } t_0 \in (0, +\infty);
\]
\[
(2) \quad \frac{K_{D_t}(z,0)}{K_{D_t}(z_0,0)} = e^t \text{ holds for any } t \in (0, +\infty) \text{ and any } z \in D_t.
\]

1.2.3. Characterizations for 1-dimensional case. In this section, we present a characterization for the concavity degenerating to linearity for 1-dimensional case, and a characterization for the equality in 1-dimensional optimal \( L^2 \) extension problem to hold.

Let \( X \) be an open Riemann Surface which admits a nontrivial Green function \( G_X(z, \omega) \).

Let \( \psi = kG_X(z, z_0) \), where \( k \geq 2 \) is a real number and \( z_0 \) is a point of \( X \).

Let \( U \) be a open neighborhood of \( z_0 \) in \( X \) and \( f \) be a holomorphic \((1,0)\) form on \( U \). Let \( \varphi \) be a subharmonic function on \( X \). Let \( c(t) \in C^\infty[0, +\infty) \) and \( c(t) \in \mathcal{G}_0 \).
Denote
\[ H(t; c, 2\varphi) := \inf \left\{ \int_{\{\psi < -t\}} c(-\psi)|\tilde{F}|^2 e^{-2\varphi} dV_X : \tilde{F} \in H^0(\{\psi < -t\}, \mathcal{O}(K_X)) \right\} \]
and \( \exists \) open set \( U' \) s.t. \( Z_0 \subset U' \subset U \) and
\[ (\tilde{F} - f) \in H^0(\{\psi < -t\} \cap U', \mathcal{O}(K_X) \otimes I(\psi + 2\varphi)_{|U'}) \],
(1.14)

We have the following necessary conditions for the minimal \( L^2 \) integrals \( H(h^{-1}(r); c, 2\varphi) \) to be linear with respect to \( r \in (0, \int_0^{+\infty} c(t_1)e^{-t_1} dt_1] \).

**Theorem 1.11.** Assume that \( 0 < H(h^{-1}(r); c, 2\varphi) \) is linear with respect to \( r \in (0, \int_0^{+\infty} c(t_1)e^{-t_1} dt_1] \), then \( \varphi = \log |f_\varphi| + v \), where \( f_\varphi \) is a holomorphic function on \( X \) and \( u \) is a harmonic function on \( X \).

Now, in the definition of \( H(t; c, 2\varphi) \), we take \( \psi = 2G_X(z, z_0) \), where \( z_0 \in X \) is a point.

Let \( (V_{z_0}, w) \) be a local coordinate neighborhood of \( z_0 \) satisfying \( w(z_0) = o \) and \( G_X(z, z_0) = \log |w| + u(w) \) on \( V_{z_0} \), where \( u(w) \) is a harmonic function on \( V_{z_0} \). Let \( U = V_{z_0} \). Let \( f \) be a holomorphic \((n,0)\) form on \( X \). Let \( \varphi \) be a subharmonic function on \( X \).

Let \( c_\beta(z) \) be the logarithmic capacity which is locally defined by
\[ c_\beta(z_0) := \exp \left( \lim_{z \to z_0} G_X(z, z_0) - \log |w(z)| \right) \]

To state our result, we introduce the following notations (see [12]).

Let \( p : \Delta \to X \) be the universal covering from unit disc \( \Delta \) to \( X \). We call the holomorphic function \( f \) (resp. holomorphic \((1,0)\) form \( F \)) on \( \Delta \) a multiplicative function (resp. multiplicative differential (Prym differential)) if there is a character \( \chi \), where \( \chi \in Hom(\pi_1(X), C^*) \) and \( |\chi| = 1 \), such that \( g^*f = \chi(g)f \) (resp. \( g^*F = \chi(g)F \)) for every \( g \in \pi_1(X) \) which naturally acts on the universal covering of \( X \). Denote the set of such kinds of \( f \) (resp. \( F \)) by \( \mathcal{O}_X(X) \) (resp. \( \Gamma_X(X) \)).

As \( p \) is a universal covering, then for any harmonic function \( h \) on \( X \), there exists a \( \chi_h \) and a multiplicative function \( f_h \in \mathcal{O}^{\chi_h}(X) \), such that \( |f_h| = p^*e^h \). And if \( g \in \mathcal{O}(X) \) and \( g \) has no zero points on \( X \). Then \( \log |g| \) is harmonic function on \( X \) and we know \( \chi_h = \chi_{h+\log |g|} \) (for the proof, see Appendix [13]).

For Green function \( G_X(\cdot, z_0) \), one can find a \( \chi_{z_0} \) and a multiplicative function \( f_{z_0} \in \mathcal{O}^{\chi_{z_0}}(X) \), such that \( |f_{z_0}| = p^*e^{G_X(\cdot, z_0)} \).

Using Theorem [11] and the solution of extend Suita conjecture in [22] (see Theorem [3.8]), we have the following characterization for \( H(h^{-1}(r); c, 2\varphi) \) to be linear.

**Theorem 1.12.** Assume that \( 0 < H(h^{-1}(r); c, 2\varphi) < +\infty \). The minimal \( L^2 \) integral function \( H(h^{-1}(r); c, 2\varphi) \) is linear with respect to \( r \) if and only if the following statements hold:

1. \( \varphi = \log |f_\varphi| + v \), where \( f_\varphi \) is a holomorphic function on \( X \) and \( v \) is a harmonic function on \( X \).
2. \( \chi_{-v} = \chi_{z_0} \).

The representation \( \varphi = \log |f_\varphi| + v \) is not unique. If \( f_1 \in \mathcal{O}(X) \) and \( f_1 \) has no zero points on \( X \). Then \( \varphi = \log |\tilde{F}^1| + (\log |f_1| + v) \) is another representation of \( \varphi \).

Since \( \chi_{-v} = \chi_{-v-\log |f_1|} \) (see Lemma [3.8]), we know the condition (2) in Theorem [1.12] is free for the choice of the specific representation of \( \varphi \).
Let $f \equiv dw$ on $V_{z_0}$ under the local coordinate $w$ on $V_{z_0}$. We also assume that $\varphi(z_0) > -\infty$.

Now we illustrate the relation between $H(h^{-1}(r); c, 2\varphi)$ is linear with respect to $r$ and the equality $\|f\|_{L^2} = \inf\{\|F\|_{L^2} \}$ holds, where

$$\|f\|_{L^2} = \left( \int_0^{+\infty} c(t)e^{-t}dt \right)^{\frac{1}{2}} \int_Y |f|^2 e^{-2\varphi} dV_X[\psi]$$

and $\|F\|_{L^2} = \int_X c(-\psi)|F|^2 e^{-2\varphi} dV_X$. Direct calculation shows that when $\psi = 2G_X(z, z_0)$, $f = dw$, the $L^2$ norm $\|f\|_{L^2}$ of $f$ defined by (1.7) is

$$\|f\|_{L^2} = \left( \int_0^{+\infty} c(t_1)e^{-t_1}dt_1 \right) \pi \frac{e^{-2\varphi(z_0)}}{c^2_0(z_0)}.$$

We will show that (see Proposition 3.5) that $\lim_{t \to +\infty} \frac{H(t; c, 2\varphi)}{\int_1^{+\infty} e^{-t}dt} = \pi \frac{e^{-2\varphi(z_0)}}{c^2_0(z_0)}$.

We also want to point out that, when $H(-\log r; c, 2\varphi)$ is linear with respect to $r$, there exists (see Lemma 2.5) a holomorphic extension $F$ of $f$ on $X$ such that the $L^2$ norm of $F$ is equal to $\left( \int_0^{+\infty} c(t_1)e^{-t_1}dt_1 \right) \pi \frac{e^{-2\varphi(z_0)}}{c^2_0(z_0)}$ and the $L^2$ norm of $F$ is minimal among all the holomorphic extension of $f$ from $z_0$ to $X$. This shows that $H(-\log r; c, 2\varphi)$ is linear with respect to $r$ implies $\|f\|_{L^2} = \inf\{\|F\|_{L^2} \}$.

When we have $\|f\|_{L^2} = \inf\{\|F\|_{L^2} \}$, it follows from $\lim_{t \to +\infty} \frac{H(t; c, 2\varphi)}{\int_1^{+\infty} e^{-t}dt} = \pi \frac{e^{-2\varphi(z_0)}}{c^2_0(z_0)}$, $\|f\|_{L^2} = \left( \int_0^{+\infty} c(t_1)e^{-t_1}dt_1 \right) \pi \frac{e^{-2\varphi(z_0)}}{c^2_0(z_0)}$ and the concavity of $H(-\log r; 2\varphi)$ that $H(-\log r; 2\varphi)$ is linear with respect to $r$.

Theorem 1.12 shows the following characterization for the equality in optimal $L^2$ extension problem to hold.

**Theorem 1.13.** The equality $\|f\|_{L^2} = \inf\{\|F\|_{L^2} : F$ is a holomorphic extension of $f$ from $Y$ to $X\}$ holds if and only if the following statements hold

1. $\varphi = \log|f_\varphi| + v$, where $f_\varphi$ is a holomorphic function on $X$ and $v$ is a harmonic function on $X$.
2. $\chi - v = \chi_{z_0}$.

When $\varphi \equiv 0$, Theorem 1.13 is the solution of equality part of Suita conjecture [22]. When $\varphi$ is harmonic, Theorem 1.13 is the solution of extended Suita conjecture [22].

2. **Proof of Theorem 1.1**

In this section, we modify some techniques in [17] and prove the Theorem 1.1

2.1. **$L^2$ methods related to $L^2$ extension theorem.** Let $c(t)$ be a positive function in $C^\infty((T, +\infty))$ satisfying $\int_T^\infty c(t)e^{-t}dt < \infty$ and

$$\left( \int_T^t c(t_1)e^{-t_1}dt_1 \right)^2 > c(t)e^{-t} \int_T^t \left( \int_T^{t_2} c(t_1)e^{-t_1}dt_1 \right)dt_2 \quad (2.1)$$

for any $t \in (T, +\infty)$, where $T \in (0, +\infty)$. This class of functions is denoted by $C_T$. Especially, if $c(t)e^{-t}$ is decreasing with respect to $t$ and $\int_T^\infty c(t)e^{-t}dt < \infty$, then inequality (2.1) holds.

In this section, we present the following Lemma, whose various forms already appear in [17] [22] [18] etc.
Lemma 2.1. Let $B \in (0, +\infty)$ and $t_0 \geq T$ be arbitrarily given. Let $X$ be an $n$-dimensional Stein manifold. Let $d\lambda_n$ be a continuous volume form on $X$ with no zero point. Let $\psi < -T$ be a plurisubharmonic function on $X$. Let $\varphi$ be a plurisubharmonic function on $X$. Let $F$ be a holomorphic $(n,0)$ form on $\{ \psi < -t_0 \}$, such that
\[
\int_{K \cap \{ \psi < -t_0 \}} |F|^2 d\lambda_n < +\infty
\] (2.2)
for any compact subset $K$ of $X$, and
\[
\int_X \frac{1}{B} |(1 - b(\psi))|F|^2 e^{-\varphi + v(\psi)} c(-v(\psi)) d\lambda_n \leq C \int_{t_0 - B}^{t_0 + B} c(t)e^{-t} dt
\] (2.3)
Then there exists a holomorphic $(n,0)$ form $\tilde{F}$ on $X$, such that
\[
\int_X |\tilde{F} - (1 - b(\psi))|F|^2 e^{-\varphi + v(\psi)} c(-v(\psi)) d\lambda_n \leq C \int_{t_0 - B}^{t_0 + B} c(t)e^{-t} dt
\] (2.4)
where $b(t) = \int_{-\infty}^t \frac{1}{B} \int_{-t_0 - s - t_0}^{t_0 + B} ds$, $v(t) = \int_0^t b(s)ds$ and $c(t) \in C_T$.

It is clear that $I_{(-t_0, +\infty)} \leq b(t) \leq I_{(-t_0 - B, +\infty)}$ and max$\{q, -t_0 - B\} \leq v(t) \leq$ max$\{t, t_0\}$.

2.2. Some properties of $H(t)$. Following the notations and assumption in Section 1.1, we present some properties related to $H(t)$ in the present section.

Let $Z_0$ be a subset of $\{ \psi = -\infty \}$ such that $Z_0 \cap \text{Supp}(O/\mathcal{I}(\varphi + \psi)) \neq \emptyset$. Let $U \supset Z_0$ be an open subset of $X$. Let $\mathcal{F} \supset \mathcal{I}(\varphi + \psi)|_U$ be a coherent subsheaf of $O$ on $U$.

We firstly introduce a property of coherent analytic sheaves which will be used frequently in our discussion of $H(t)$.

Lemma 2.2. (Closedness of Submodules, see [10]) Let $N$ be a submodule of $\mathcal{O}^q_{K|U}$, $1 \leq q < +\infty$, let $f_j \in \mathcal{O}^q_{K|U}$ be a sequence of $q$-tuples holomorphic in an open neighborhood $U$ of the origin. Assume that the $f_j$ converge uniformly in $U$ towards a $q$-tuple $f \in \mathcal{O}^q_{K|U}$, assume furthermore that all germs $f_{j0}$ belong to $N$. Then $f_0 \in N$.

Lemma 2.3. For any $t_0 \in [T, +\infty)$, assume that $\{f_n\}_{n \in \mathbb{N}^+}$ is a family of holomorphic $(n,0)$ form on $\{ \psi < -t_0 \}$, which compactly convergent to $\hat{f}$ on $\{ \psi < -t_0 \}$.

Assume that for any $n$, there exists open set $U_n$ such that $Z_0 \subset U_n \subset U$ and $f_n \in H^0(\{ \psi < -t_0 \} \cap U_n, \mathcal{O}(K_X) \otimes \mathcal{F})$. Then there exists an open set $U'$ which satisfies $Z_0 \subset U' \subset U$ such that $f \in H^0(\{ \psi < -t_0 \} \cap U', \mathcal{O}(K_X) \otimes \mathcal{F})$.

Proof. As $K_X$ is a holomorphic line bundle on $X$, then $\mathcal{O}(K_X) \otimes \mathcal{F}$ is a coherent analytic sheaf.

For any $z \in Z_0$, we know the germ $(\hat{f}_n, z) \in (\mathcal{O}(K_X) \otimes \mathcal{F})_z$. It follows from Lemma 2.2. and $\hat{f}_n$ compactly convergent to $\hat{f}$ (when $n \rightarrow +\infty$) on $\{ \psi < -t_0 \}$ that $(\hat{f}, z) \in (\mathcal{O}(K_X) \otimes \mathcal{F})_z$.

As $\mathcal{O}(K_X) \otimes \mathcal{F}$ is coherent analytic sheaf, there exists a small open neighborhood $U_z$ of $z$ such that $(\mathcal{O}(K_X) \otimes \mathcal{F})|_{U_z}$ is finite generated i.e $\exists f^1_z, \cdots , f^k_z \in H^0(U_z, \mathcal{O}(K_X) \otimes \mathcal{F})$ such that $\forall y \in U_z$, $(\mathcal{O}(K_X) \otimes \mathcal{F})_y$ is generated by $f^1_y, \cdots , f^k_y$.

Hence for $\hat{f}$, there exists $g^i \in \Gamma(U_z, \mathcal{O})$ such that $\hat{f}_z = \sum g^i_z f^i_z$, i.e $\exists$ small open neighborhood $\hat{U}'_z$ of $z$ satisfies $\hat{U}'_z \subset U_z$ and $\hat{f}|_{\hat{U}'_z} = (\sum g^i_z f^i_z)|_{\hat{U}'_z}$, which implies
\[ \hat{f} \in H^0(\hat{U}', O(K_X) \otimes \mathcal{F}). \]  
Take \( \hat{U}' = (\bigcup_{z \in Z_0} \hat{U}_z') \cap U \). We now find a open set \( \hat{U}' \) 
satisfies \( Z_0 \subset \hat{U}' \subset U \) such that \( f \in H^0(\{\psi < -t_0\} \cap \hat{U}', O(K_X) \otimes \mathcal{F}). \)

The following lemma is a characterization of \( H(T) \neq 0 \).

**Lemma 2.4.** The following two statements are equivalent:

1. For any open set \( U' \) satisfying \( Z_0 \subset U' \subset U \), \( f \notin H^0(U', O(K_X) \otimes \mathcal{F}). \)
2. \( H(T) \neq 0 \).

**Proof.** (2) \( \Rightarrow \) (1) If there exists open set \( U' \) satisfies \( Z_0 \subset U' \subset U \) and \( f \in H^0(U', O(K_X) \otimes \mathcal{F}) \), then \( H(T) = 0 \) (just take \( \hat{f} = 0 \)).

Now we prove (1) \( \Rightarrow \) (2) by contradiction.

Assume \( H(T) = 0 \), then there exist holomorphic \((n,0)\) forms \( \{\hat{f}_n\}_{n \in \mathbb{N}^+} \) on \( X \) such that \( \lim_{n \to +\infty} \int_X |\hat{f}_n|^2 e^{-\varphi} c(-\psi) dV_X = 0 \) and for each \( n \), \( \exists U'_n \) satisfies \( Z_0 \subset U'_n \subset U \) and \( \hat{f}_n - f \in H^0(U'_n, O(K_X) \otimes \mathcal{F}). \) As \( e^{-\varphi} c(-\psi) \) has positive lower bound on any compact subset of \( X \), then (by diagonal method) there exists a subsequence of \( \{\hat{f}_n\}_{n \in \mathbb{N}^+} \) denoted by \( \{\hat{f}_k\}_{k \in \mathbb{N}^+} \) compactly convergent to 0 on \( X \) when \( k \to +\infty \). Hence \( \hat{f}_k - f \) is compactly convergent to 0 on \( X \) which contradicts the condition.

The following lemma shows the uniqueness of the holomorphic \((n,0)\) form related to \( H(t) \).

**Lemma 2.5.** Assume that \( H(t) < +\infty \) for some \( t \in [T, +\infty) \). Then there exists a unique holomorphic \((n,0)\) form \( F_t \) on \( \{\psi < -t\} \) satisfying

\[ (F_t - f) \in H^0(\{\psi < -t\} \cap \hat{U}', O(K_X) \otimes \mathcal{F}), \]

for some open set \( U' \) such that \( Z_0 \subset U' \subset U \) and

\[ \int_{\{\psi < -t\}} |F_t|^2 e^{-\varphi} c(-\psi) dV_X = H(t). \]

Furthermore, for any holomorphic \((n,0)\) form \( \hat{F} \) on \( \{\psi < -t\} \) satisfying

\[ \int_{\{\psi < -t\}} |\hat{F}|^2 e^{-\varphi} c(-\psi) dV_X < +\infty \]

and

\[ (\hat{F} - f) \in H^0(\{\psi < -t\} \cap \hat{U}', O(K_X) \otimes \mathcal{F}) \]

for some open set \( \hat{U}' \) such that \( Z_0 \subset \hat{U}' \subset U \), the following equality holds

\[ \int_{\{\psi < -t\}} |F_t|^2 e^{-\varphi} c(-\psi) dV_X + \int_{\{\psi < -t\}} |\hat{F} - F_t|^2 e^{-\varphi} c(-\psi) dV_X \]

\[ = \int_{\{\psi < -t\}} |\hat{F}|^2 e^{-\varphi} c(-\psi) dV_X. \]

**Proof.** As \( H(t) < +\infty \), then there exist holomorphic \((n,0)\)-forms \( \{f_n\}_{n \in \mathbb{N}^+} \) on \( \{\psi < -t\} \) such that \( \lim_{n \to +\infty} \int_{\{\psi < -t\}} |f_n|^2 e^{-\varphi} c(-\psi) dV_X = H(t) \) and for each \( n \), there exists \( U'_n \) such that \( Z_0 \subset U'_n \subset U \) and \( (\hat{f}_n - f) \in H^0(\{\psi < -t\} \cap U'_n, O(K_X) \otimes \mathcal{F}). \)
As $e^{-\varphi}c(-\psi)$ has positive lower bound on any compact subset of $\{\psi < -t\}$, then (by diagonal method) there exist a subsequence of $\{f_j\}$ also denoted by $\{f_j\}$ compact convergence to a holomorphic $(n,0)$ form $F$ (when $j \to +\infty$) on $\{\psi < -t\}$ satisfying

$$
\int_K |F|^2 e^{-\varphi}c(-\psi) dV_X \leq \liminf_{j \to +\infty} \int_K |f_j|^2 e^{-\varphi}c(-\psi) dV_X \\
\leq \liminf_{j \to +\infty} \int_{\{\psi < -t\}} |f_j|^2 e^{-\varphi}c(-\psi) dV_X \quad (2.6)
$$

Lemma 2.3 shows that there exists an open subset $U'$ such that $Z_0 \subset U' \subset U$ and $(F-f) \in H^0(\{\psi < -t\} \cap U'; \mathcal{O}(K_X) \otimes \mathcal{F})$ which implies $H(t) \leq \int_{\{\psi < -t\}} |F|^2 e^{-\varphi}c(-\psi) dV_X$.

Hence we obtain the existence of $F_t(= F)$.

We prove the uniqueness of $F_t$ by contradiction: if not, there exists two different holomorphic $(n,0)$ forms $f_1$ and $f_2$ on $\{\psi < -t\}$ satisfying $\int_{\{\psi < -t\}} |f_1|^2 e^{-\varphi}c(-\psi) dV_X = \int_{\{\psi < -t\}} |f_2|^2 e^{-\varphi}c(-\psi) dV_X = H(t)$, $(f_1 - f) \in H^0(\{\psi < -t\} \cap U_1', \mathcal{O}(K_X) \otimes \mathcal{F})$ and $(f_2 - f) \in H^0(\{\psi < -t\} \cap U_2', \mathcal{O}(K_X) \otimes \mathcal{F})$ where both $U_1', U_2'$ are open set satisfy $Z_0 \subset U_1' \subset U$ and $Z_0 \subset U_2' \subset U$. Note that

$$
\int_{\{\psi < -t\}} \left| \frac{f_1 + f_2}{2} \right|^2 e^{-\varphi}c(-\psi) dV_X + \int_{\{\psi < -t\}} \left| \frac{f_1 - f_2}{2} \right|^2 e^{-\varphi}c(-\psi) dV_X \\
= \frac{1}{2} \int_{\{\psi < -t\}} |f_1|^2 e^{-\varphi}c(-\psi) dV_X + \int_{\{\psi < -t\}} |f_2|^2 e^{-\varphi}c(-\psi) dV_X = H(t) 
$$

then we obtain that

$$
\int_{\{\psi < -t\}} |f_1 + f_2|^2 e^{-\varphi}c(-\psi) dV_X \leq H(t) 
$$

and $(f_1 + f_2 - f) \in H^0(\{\psi < -t\} \cap (U_1' \cap U_2'), \mathcal{O}(K_X) \otimes \mathcal{F})$, which contradicts to the definition of $H(t)$.

Now we prove the equality (2.8). For any holomorphic $(n,0)$ form $h$ on $\{\psi < -t\}$ satisfying $\int_{\{\psi < -t\}} |h|^2 e^{-\varphi}c(-\psi) dV_X < +\infty$ and $h \in H^0(\{\psi < -t\} \cap U_1', \mathcal{O}(K_X) \otimes \mathcal{F})$ for some open subset $U_h$ which $Z_0 \subset U_h' \subset U$. It is clear that for any complex number $\alpha$, $F_t + \alpha h$ satisfying $((F_t + \alpha h) - f) \in H^0(\{\psi < -t\} \cap (U_1' \cap U_2'), \mathcal{O}(K_X) \otimes \mathcal{F})$ and $\int_{\{\psi < -t\}} |F_t|^2 e^{-\varphi}c(-\psi) dV_X \leq \int_{\{\psi < -t\}} |F_t + \alpha h|^2 e^{-\varphi}c(-\psi) dV_X$. Note that

$$
\int_{\{\psi < -t\}} |F_t + \alpha h|^2 e^{-\varphi}c(-\psi) dV_X - \int_{\{\psi < -t\}} |F_t|^2 e^{-\varphi}c(-\psi) dV_X \geq 0 
$$

(By considering $\alpha \to 0$) implies

$$
\Re \int_{\{\psi < -t\}} F_t \bar{h} e^{-\varphi}c(-\psi) dV_X = 0 
$$

then we have

$$
\int_{\{\psi < -t\}} |F_t + h|^2 e^{-\varphi}c(-\psi) dV_X = \int_{\{\psi < -t\}} (|F_t|^2 + |h|^2) e^{-\varphi}c(-\psi) dV_X 
$$

(2.11)

Letting $h = \hat{F} - F_t$ (and $U_1' = \hat{U}' \cap U_2'$), we obtain equality (2.8). 

Now we show the lower semi-continuity property of $H(h^{-1}(r))$. 

Lemma 2.6. Assume that $H(T) < +\infty$. Then $H(t)$ is decreasing with respect to $t \in [T, +\infty)$, such that $\lim_{t \to t_0^+} H(t) = H(t_0)$ ($t_0 \in [T, +\infty)$), $\lim_{t \to t_0^-} H(t) \geq H(t_0)$ ($t_0 \in (T, +\infty)$), and $\lim_{t \to +\infty} H(t) = 0$. Especially, $H(h^{-1}(r))$ is lower semicontinuous with respect to $r$.

Proof. By the definition of $H(t)$, it is clear that $H(t)$ is decreasing on $[T, +\infty)$ and $\lim_{t \to t_0^-} H(t) = H(t_0)$. It suffices to prove $\lim_{t \to t_0^+} H(t) = H(t_0)$ . We prove it by contradiction: if not, then $\lim_{t \to t_0^+} H(t) < H(t_0) < +\infty$.

By Lemma 2.5, for any $t > t_0$, there exists a unique holomorphic $(n, 0)$ form $F_t$ on $\{\psi < -t\}$ satisfying $\int_{\{\psi < -t\}} |F_t|^2 e^{-\varphi} c(-\psi) dV_X = H(t)$ and $(F_t - f) \in H^0(\{\psi < -t\} \cap U_t', \mathcal{O}(K_X) \otimes \mathcal{F})$ where open set $U_t'$ satisfies $Z_0 \subset U_t' \subset U$. Note that $H(t)$ is decreasing implies that $\int_{\{\psi < -t\}} |F_t|^2 e^{-\varphi} c(-\psi) dV_X \leq \lim_{t \to t_0^+} H(t) < +\infty$ for any $t > t_0$.

For any compact subset $K$ of $\{\psi < -t_0\}$, as $e^{-\varphi} c(-\psi)$ has positive lower bound on $K$, there exists $F_{j_t}$ ($t_j \to t_0 + 0$, as $j \to +\infty$) uniformly convergent on $K$, then (by diagonal method) there exists a subsequence of $F_{j_t}$ (also denoted by $F_{j_t}$) convergent on any compact subset of $\{\psi < -t_0\}$.

Let $F_{t_0} := \lim_{j \to +\infty} F_{j_t}$, which is a holomorphic $(n, 0)$ form on $\{\psi < -t_0\}$. By Lemma 2.5, we conclude that there exists an open set $\hat{U}'$ such that $Z_0 \subset \hat{U}' \subset U$ and $(F_{t_0} - f) \in H^0(\{\psi < -t_0\} \cap \hat{U}'', \mathcal{O}(K_X) \otimes \mathcal{F})$. Then it follows from the decreasing property of $H(t)$ that

$$\int_K |\hat{F}_{t_0}|^2 e^{-\varphi} c(-\psi) dV_X \leq \liminf_{j \to +\infty} \int_K |F_{j_t}|^2 e^{-\varphi} c(-\psi) dV_X \leq \lim_{j \to +\infty} H(t_j) \leq \lim_{t \to t_0^+} H(t)$$

(2.12)

for any compact set $K \subset \{\psi < -t_0\}$. It follows from Levi’s theorem that

$$\int_{\{\psi < -t_0\}} |\hat{F}_{t_0}|^2 e^{-\varphi} c(-\psi) dV_X \leq \lim_{t \to t_0^+} H(t)$$

(2.13)

Then we obtain that $H(t_0) \leq \int_{\{\psi < -t_0\}} |\hat{F}_{t_0}|^2 e^{-\varphi} c(-\psi) dV_X \leq \lim_{t \to t_0^+} H(t)$ which contradicts $\lim_{t \to t_0^+} H(t) < H(t_0)$.

We consider the derivatives of $H(t)$ in the following lemma.

Lemma 2.7. Assume that $H(T) < +\infty$. Then for any $t_0 \in (T, +\infty)$, we have

$$\frac{H(T) - H(t_0)}{\int_T^{t_0} c(t) e^{-c(t)} dt - \int_{t_0}^{+\infty} c(t) e^{-c(t)} dt} \leq \liminf_{B \to 0^+} \frac{H(t_0) - H(t_0 + B)}{B}$$

(2.14)

Proof. By Lemma 2.5, there exists a holomorphic $(n, 0)$ form $F_{t_0}$ on $\{\psi < -t_0\}$, such that $\int_{\{\psi < -t_0\}} |F_{t_0}|^2 e^{-\varphi} c(-\psi) dV_X = H(t_0)$ and $(F_{t_0} - f) \in H^0(\{\psi < -t_0\} \cap U_{t_0}', \mathcal{O}(K_X) \otimes \mathcal{F})$ where open set $U_{t_0}'$ satisfies $Z_0 \subset U_{t_0}' \subset U$. 

Note that \( \liminf_{B \to 0+} \frac{H(t_0) - H(t_0 + B)}{B} \in [0, +\infty) \) because of the decreasing property of \( H(t) \). Then there exist \( 1 \geq B_j \to 0+ \) (as \( j \to +\infty \)) such that

\[
\lim_{j \to +\infty} \frac{H(t_0) - H(t_0 + B_j)}{B_j} = \liminf_{B \to 0+} \frac{H(t_0) - H(t_0 + B)}{B} \tag{2.15}
\]

and \( \{ \frac{H(t_0) - H(t_0 + B_j)}{B_j} \}_{j \in \mathbb{N}^+} \) is bounded.

As \( \int_{\{\psi < -t_0\}} \left| F_{t_0} \right|^2 e^{-\varphi} c(-\psi) dV \) = \( H(t_0) < +\infty \) and \( e^{-\varphi} c(-\psi) \) has positive lower bound on any compact set \( K \) of \( X \). Then \( \int_{K \cap \{\psi < -t_0\}} \left| F_{t_0} \right|^2 dV_X < +\infty \) for any compact set \( K \). Note that \( c(t) \) is smooth on \((T, +\infty)\), hence bounded on \([t_0, t_0 + 1] \), so \( \int_X \frac{1}{B_j} \left| \tilde{F}_j \right|^2 e^{-\varphi} c(-\psi) dV_X < +\infty \).

By Lemma 2.1 (\( \varphi \prec \varphi + \psi \)), for any \( B_j \), there exists holomorphic \((n, 0)\) form \( \tilde{F}_j \) on \( X \) such that

\[
\int_X \left| \tilde{F}_j - (1 - b_{t_0, B_j} (\psi)) F_{t_0} \right|^2 e^{-(\varphi + \psi) + v_j (\psi)} c(-v_j (\psi)) dV_X < +\infty \tag{2.16}
\]

where \( b_{t_0, B_j} (t) = \int_{-\infty}^{t} \frac{1}{B_j} \left| -t_0 - B_j < s < -t_0 \right| ds \) and \( v_j (t) = \int_{0}^{t} b_{t_0, B_j} (s) ds \).

It follows from (2.10) that \( \int_{\{\psi < -t_0 - B_j\}} \left| \tilde{F}_j - (1 - b_{t_0, B_j} (\psi)) F_{t_0} \right|^2 e^{-(\varphi + \psi) + v_j (\psi)} c(-v_j (\psi)) dV_X < +\infty \), and note that \( e^{-t} c(t) \) is decreasing with respect to \( t \) and \( v_j (\psi) \geq \max \{\psi, t_0 - B_j\} \geq t_0 - 1 \). Hence \( e^{v_j (\psi)} c(-v_j (\psi)) \) has positive lower bound, which implies

\[
\int_{\{\psi < -t_0 - B_j\}} \left| \tilde{F}_j - (1 - b_{t_0, B_j} (\psi)) F_{t_0} \right|^2 e^{-(\varphi + \psi)} dV_X < +\infty \tag{2.17}
\]

As \( \{\psi < -t_0 - B_j\} \) is open, there exists an open subset \( U_j' \subset (\{\psi < -t_0 - B_j\} \cap U) \) such that \( \tilde{F}_j - F_{t_0} \in H^{0} (\{\psi < -t_0\} \cap U_j' \cap \mathcal{O}(K_X) \otimes \mathcal{F}) \subset H^{0} (\{\psi < -t_0\} \cap U_j' \cap \mathcal{O}(K_X) \otimes \mathcal{F}) \), which implies \( (\tilde{F}_j - f) \in H^{0} (\{\psi < -t_0\} \cap (U_j' \cap U_{t_0}), \mathcal{O}(K_X) \otimes \mathcal{F}) \).

As \( t \leq v(t) \), the decreasing property of \( c(t)e^{-t} \) shows that

\[
c(t)e^{-t} \leq c(-v(-t))e^{v(-t)} \tag{2.18}
\]

for any \( t \geq 0 \), which implies

\[
e^{-\psi + v(\psi)} c(-v(\psi)) \geq c(-\psi) \tag{2.19}
\]
So we have

\[
\int_X |\tilde{F}_j - (1 - b_{t_0, B_j} (\psi)) F_{t_0}|^2 e^{-\varphi} c(-\psi) dV_X \\
\leq \int_X |\tilde{F}_j - (1 - b_{t_0, B_j} (\psi)) F_{t_0}|^2 e^{-\varphi} e^{-\psi + \psi(\psi)} c(-\psi) dV_X \\
\leq \int_T c(t) e^{-t} dt \int_X \frac{1}{B_j} \{t_0 - B_j < \psi < -t_0\} |F_{t_0}|^2 e^{-\varphi - \psi} dV_X \\
\leq e^{t_0 + B_j} \inf_{t \in (t_0, t_0 + B_j)} c(t) e^{-t} dt \int_X \frac{1}{B_j} \{t_0 - B_j < \psi < -t_0\} |F_{t_0}|^2 e^{-\varphi} c(-\psi) dV_X \\
= \frac{e^{t_0 + B_j} \int_t^{t_0 + B_j} c(t) e^{-t} dt}{\inf_{t \in (t_0, t_0 + B_j)} c(t)} \times \left( \int_X \frac{1}{B_j} \{t_0 - B_j < \psi < -t_0\} |F_{t_0}|^2 e^{-\varphi} c(-\psi) dV_X \right) \\
- \frac{1}{B_j} \{\psi < -t_0 - B_j\} |F_{t_0}|^2 e^{-\varphi} c(-\psi) dV_X \\
\leq \frac{e^{t_0 + B_j} \int_t^{t_0 + B_j} c(t) e^{-t} dt}{\inf_{t \in (t_0, t_0 + B_j)} c(t)} \times \frac{H(t_0) - H(t_0 + B_j)}{B_j} 
\tag{2.20}
\]

After the estimate for \(\int_X |\tilde{F}_j - (1 - b_{t_0, B_j} (\psi)) F_{t_0}|^2 e^{-\varphi} c(-\psi) dV_X\), we can prove the main result.

Firstly, we will prove that \(\int_X |\tilde{F}_j|^2 e^{-\varphi} c(\psi) dV_X\) is uniformly bounded with respect to \(j\).

Note that

\[
(\int_X |\tilde{F}_j - (1 - b_{t_0, B_j} (\psi)) F_{t_0}|^2 e^{-\varphi} c(-\psi) dV_X)^{1/2} \\
\geq (\int_X |\tilde{F}_j|^2 e^{-\varphi} c(-\psi) dV_X)^{1/2} - (\int_X |(1 - b_{t_0, B_j} (\psi)) F_{t_0}|^2 e^{-\varphi} c(-\psi) dV_X)^{1/2} 
\tag{2.21}
\]

then it follows from inequality \(2.20\) that

\[
(\int_X |\tilde{F}_j|^2 e^{-\varphi} c(-\psi) dV_X)^{1/2} \\
\leq \frac{e^{t_0 + B_j} \int_t^{t_0 + B_j} c(t) e^{-t} dt}{\inf_{t \in (t_0, t_0 + B_j)} c(t)}^{1/2} \times \left( \frac{H(t_0) - H(t_0 + B_j)}{B_j} \right)^{1/2} \\
+ (\int_X |(1 - b_{t_0, B_j} (\psi)) F_{t_0}|^2 e^{-\varphi} c(-\psi) dV_X)^{1/2} 
\tag{2.22}
\]

Since \(\left\{\frac{H(t_0) - H(t_0 + B_j)}{B_j}\right\}_{j \in \mathbb{N}^+}\) is bounded and \(0 \leq b_{t_0, B_j} (\psi) \leq 1\), then \(\int_X |\tilde{F}_j|^2 e^{-\varphi} c(-\psi) dV_X\) is uniformly bounded with respect to \(j\).

Now we will prove the main result.
It follows from $b_{t_0,B_j}(\psi) = 1$ on $\{\psi \geq -t_0\}$ that
\[
\int_X |\hat{F}_j - (1 - b_{t_0,B_j}(\psi))F_{t_0}|^2 e^{-\varphi}c(-\psi) dV_X \\
= \int_{\{\psi \geq -t_0\}} |\hat{F}_j|^2 e^{-\varphi}c(-\psi) dV_X \\
+ \int_{\{\psi < -t_0\}} |\hat{F}_j - (1 - b_{t_0,B_j}(\psi))F_{t_0}|^2 e^{-\varphi}c(-\psi) dV_X
\]
(2.23)

It is clear that
\[
\int_{\{\psi < -t_0\}} |\hat{F}_j - (1 - b_{t_0,B_j}(\psi))F_{t_0}|^2 e^{-\varphi}c(-\psi) dV_X \\
\geq (\int_{\{\psi < -t_0\}} |\hat{F}_j - F_{t_0}|^2 e^{-\varphi}c(-\psi) dV_X)^{1/2} - \left(\int_{\{\psi < -t_0\}} |b_{t_0,B_j}(\psi)F_{t_0}|^2 e^{-\varphi}c(-\psi) dV_X\right)^{1/2})^2 \\
\geq \int_{\{\psi < -t_0\}} |\hat{F}_j - F_{t_0}|^2 e^{-\varphi}c(-\psi) dV_X \\
- 2\left(\int_{\{\psi < -t_0\}} |\hat{F}_j - F_{t_0}|^2 e^{-\varphi}c(-\psi) dV_X\right)^{1/2}\left(\int_{\{\psi < -t_0\}} |b_{t_0,B_j}(\psi)F_{t_0}|^2 e^{-\varphi}c(-\psi) dV_X\right)^{1/2} \\
\geq \int_{\{\psi < -t_0\}} |\hat{F}_j - F_{t_0}|^2 e^{-\varphi}c(-\psi) dV_X \\
- 2\left(\int_{\{\psi < -t_0\}} |\hat{F}_j - F_{t_0}|^2 e^{-\varphi}c(-\psi) dV_X\right)^{1/2}\left(\int_{\{-t_0 - B_j < \psi < -t_0\}} |F_{t_0}|^2 e^{-\varphi}c(-\psi) dV_X\right)^{1/2} \\
(2.24)
\]

where the last inequality follow from $0 \leq b_{t_0,B_j}(\psi) \leq 1$ and $b_{t_0,B_j}(\psi) = 0$ on $\{\psi \leq -t_0 - B_j\}$. Combining equality (2.23), inequality (2.24) and equality (2.5), we obtain that
\[
\int_X |\hat{F}_j - (1 - b_{t_0,B_j}(\psi))F_{t_0}|^2 e^{-\varphi}c(-\psi) dV_X \\
= \int_{\{\psi \geq -t_0\}} |\hat{F}_j|^2 e^{-\varphi}c(-\psi) dV_X + \int_{\{\psi < -t_0\}} |\hat{F}_j - (1 - b_{t_0,B_j}(\psi))F_{t_0}|^2 e^{-\varphi}c(-\psi) dV_X \\
\geq \int_{\{\psi \geq -t_0\}} |\hat{F}_j|^2 e^{-\varphi}c(-\psi) dV_X + \int_{\{\psi < -t_0\}} |\hat{F}_j - F_{t_0}|^2 e^{-\varphi}c(-\psi) dV_X \\
- 2\left(\int_{\{\psi < -t_0\}} |\hat{F}_j - F_{t_0}|^2 e^{-\varphi}c(-\psi) dV_X\right)^{1/2}\left(\int_{\{-t_0 - B_j < \psi < -t_0\}} |F_{t_0}|^2 e^{-\varphi}c(-\psi) dV_X\right)^{1/2} \\
= \int_{\{\psi \geq -t_0\}} |\hat{F}_j|^2 e^{-\varphi}c(-\psi) dV_X + \int_{\{\psi < -t_0\}} |\hat{F}_j - F_{t_0}|^2 e^{-\varphi}c(-\psi) dV_X - \int_{\{\psi < -t_0\}} |F_{t_0}|^2 e^{-\varphi}c(-\psi) dV_X \\
- 2\left(\int_{\{\psi < -t_0\}} |\hat{F}_j - F_{t_0}|^2 e^{-\varphi}c(-\psi) dV_X\right)^{1/2}\left(\int_{\{-t_0 - B_j < \psi < -t_0\}} |F_{t_0}|^2 e^{-\varphi}c(-\psi) dV_X\right)^{1/2} \\
= \int_X |\hat{F}_j|^2 e^{-\varphi}c(-\psi) dV_X - \int_{\{\psi < -t_0\}} |F_{t_0}|^2 e^{-\varphi}c(-\psi) dV_X \\
- 2\left(\int_{\{\psi < -t_0\}} |\hat{F}_j - F_{t_0}|^2 e^{-\varphi}c(-\psi) dV_X\right)^{1/2}\left(\int_{\{-t_0 - B_j < \psi < -t_0\}} |F_{t_0}|^2 e^{-\varphi}c(-\psi) dV_X\right)^{1/2} \\
(2.25)
It follows from equality (2.25) that

\[
\left( \int_{\{\psi < -t_0\}} |\tilde{F}_j - F_{t_0}|^2 e^{-\varphi} c(-\psi) dV_X \right)^{1/2} \nonumber
\]

\[
= \left( \int_{\{\psi < -t_0\}} (|\tilde{F}_j|^2 - |F_{t_0}|^2) e^{-\varphi} c(-\psi) dV_X \right)^{1/2} \nonumber
\]

\[
\leq \left( \int_{\{\psi < -t_0\}} |\tilde{F}_j|^2 e^{-\varphi} c(-\psi) dV_X \right)^{1/2} \nonumber
\]

\[
\leq \left( \int_X |\tilde{F}_j|^2 e^{-\varphi} c(-\psi) dV_X \right)^{1/2} \nonumber
\]

(2.26)

Since \( \int_X |\tilde{F}_j|^2 e^{-\varphi} c(-\psi) dV_X \) is uniformly bounded with respect to \( j \), inequality (2.25) implies that \( (\int_{\{\psi < -t_0\}} |\tilde{F}_j - F_{t_0}|^2 e^{-\varphi} c(-\psi) dV_X)^{1/2} \) is uniformly bounded with respect to \( j \).

It follows from \( \int_{\{\psi < -t_0\}} |F_{t_0}|^2 e^{-\varphi} c(-\psi) dV_X = H(t_0) \leq H(T) < +\infty \) and the dominated convergence theorem that \( \lim_{j \to +\infty} \int_{\{-t_0 - B_j < \psi < -t_0\}} |F_{t_0}|^2 e^{-\varphi} c(-\psi) dV_X \)

\[ dV_X = 0. \]

Hence

\[
\lim_{j \to +\infty} \left( \int_{\{\psi < -t_0\}} |\tilde{F}_j - F_{t_0}|^2 e^{-\varphi} c(-\psi) dV_X \right)^{1/2} \left( \int_{\{-t_0 - B_j < \psi < -t_0\}} |F_{t_0}|^2 e^{-\varphi} c(-\psi) dV_X \right)^{1/2} = 0
\]

(2.27)

Combining with inequality (2.23), we obtain

\[
\lim_{j \to +\infty} \int_X |\tilde{F}_j - (1 - b_{t_0, B_j}(\psi))F_{t_0}|^2 e^{-\varphi} c(-\psi) dV_X \nonumber
\]

\[
\geq \lim_{j \to +\infty} \int_X |\tilde{F}_j|^2 e^{-\varphi} c(-\psi) dV_X - \int_{\{\psi < -t_0\}} |F_{t_0}|^2 e^{-\varphi} c(-\psi) dV_X \nonumber
\]

(2.28)

Using inequality (2.24) and inequality (2.28), we obtain that

\[
\frac{\int_{t_0}^{t_0 + B_j} c(t) e^{-t} dt}{c(t_0) e^{-t_0}} \lim_{j \to +\infty} \frac{H(t_0) - H(t_0 + B_j)}{B_j} \nonumber
\]

\[
= \lim_{j \to +\infty} \inf_{t \in (t_0, t_0 + B_j)} \frac{\int_{t_0 + B_j}^{t_0} c(t) e^{-t} dt}{c(t_0) e^{-t_0}} \times \frac{H(t_0) - H(t_0 + B_j)}{B_j} \nonumber
\]

\[
\geq \lim_{j \to +\infty} \inf_{t \in (t_0, t_0 + B_j)} e^{t_0 + B_j} \frac{\int_{t_0 + B_j}^{t_0 + B_j} c(t) e^{-t} dt}{c(t_0) e^{-t_0}} \times \int_X \frac{1}{B_j} \|(-t_0 - B_j < \psi < -t_0)\| \int_X |\tilde{F}_j - (1 - b_{t_0, B_j}(\psi))F_{t_0}|^2 e^{-\varphi} c(-\psi) dV_X \nonumber
\]

\[
\geq \lim_{j \to +\infty} \int_X |\tilde{F}_j|^2 e^{-\varphi} c(-\psi) dV_X - \int_{\{\psi < -t_0\}} |F_{t_0}|^2 e^{-\varphi} c(-\psi) dV_X \nonumber
\]

\[
\geq H(T) - H(t_0)
\]

(2.29)

This proves Lemma 2.7

\[ \square \]

Lemma 2.7 implies the following lemma.
Lemma 2.8. Assume that $H(T) < +\infty$. Then for any $t_0, t_1 \in [T, +\infty)$, where $t_0 < t_1$, we have
\[
\frac{H(t_0) - H(t_1)}{\int_{t_0}^{t_1} c(t)e^{-t}dt} \leq \lim_{B \to +0} \frac{\inf_{B \to +0} \left( \frac{H(t_0) - H(t_1 + B)}{H(t_1 + B)} \right)}{\int_{t_1}^{t_1} c(t)e^{-t}dt} \quad (2.30)
\]
i.e.
\[
\frac{H(t_0) - H(t_1)}{\int_{t_0}^{t_1} c(t)e^{-t}dt} \leq \lim_{B \to +0} \frac{H(t_1) - H(t_1 + B)}{\int_{t_1}^{t_1 + B} c(t)e^{-t}dt - \int_{t_1}^{t_1} c(t)e^{-t}dt} \quad (2.31)
\]

2.3. Proof of Theorem 1.1. As $H(h^{-1}(r); c(t))$ is lower semicontinuous (Lemma 2.6), then it follows from the following well-known property of concave functions that Lemma 2.8 implies Theorem 1.1.

Lemma 2.9. (see [17]) Let $H(r)$ be a lower semicontinuous function on $(0, R]$. Then $H(r)$ is concave if and only if
\[
\frac{H(r_1) - H(r_2)}{r_1 - r_2} \leq \inf_{r_3 \to r_2 - 0} \frac{H(r_3) - H(r_2)}{r_1 - r_2} \quad (2.32)
\]
holds for any $0 < r_2 < r_1 \leq R$.

3. Proof of Theorem 1.3

In this section, we will prove Theorem 1.3.

Proof of Theorem 1.3. We firstly recall some basic construction in the proof of Lemma 2.7.

Given $t_0 \in (T, +\infty)$. By Lemma 2.5, there exists a holomorphic $(n, 0)$ form $F_{t_0}$ on $\{\psi < -t_0\}$, such that $\int_{\{\psi < -t_0\}} |F_{t_0}|^2 e^{-\varphi c(-\psi)}dV_X = H(t_0)$ and $(F_{t_0} - f) \in H^0(\{\psi < -t_0\} \cap U'_{t_0} \cap O(K_X) \cap F)$, where open subset $U'_{t_0}$ satisfies $Z_0 \subset U'_{t_0} \subset U$.

Note that $\lim_{B \to +0} \frac{H(t_0) - H(t_0 + B)}{B} \in [0, +\infty)$ because of the decreasing property of $H(t)$. Then there exist $1 \geq B_j \to 0 + 0$ (as $j \to +\infty$) such that
\[
\lim_{j \to +\infty} \frac{H(t_0) - H(t_0 + B_j)}{B_j} = \lim_{B \to +0} \frac{H(t_0) - H(t_0 + B)}{B} \quad (3.1)
\]
and $(\frac{H(t_0) - H(t_0 + B_j)}{B_j})_{j \in \mathbb{N}^+}$ is bounded.

As $\int_{\{\psi < -t_0\}} |F_{t_0}|^2 e^{-\varphi c(-\psi)}dV_X = H(t_0) < +\infty$ and $e^{-\varphi c(-\psi)}$ has positive lower bounded on any compact set $K$ of $X$. Then $\int_{K \cap \{\psi < -t_0\}} |F_{t_0}|^2 dV_X < +\infty$ for any compact set $K$. Note that $c(t)$ is smooth on $(T, +\infty)$, hence bounded on $[t_0, t_0 + 1]$, so $\int_X \frac{1}{B_j} \int_{-t_0 - B_j < \psi < -t_0} |F_{t_0}|^2 e^{-\varphi c(-\psi)}dV_X < +\infty$.

By Lemma 2.1 ($\varphi \sim \varphi + \psi$), for any $B_j$, there exists holomorphic $(n, 0)$ form $\tilde{F}_{t_0}$ on $X$ such that
\[
\int_X |\tilde{F}_{t_0} - (1 - b_{t_0, B_j}(|\psi|))F_{t_0}|^2 e^{-(\varphi + \psi + v_j(|\psi|))c(-v_j(|\psi|))}dV_X \leq \int_{Q}^{t_0 + B_j} c(t)e^{-t}dt \int_X \frac{1}{B_j} \int_{-t_0 - B_j < \psi < -t_0} |F_{t_0}|^2 e^{-\varphi c(-\psi)}dV_X < +\infty \quad (3.2)
\]
where $b_{t_0, B_j}(|\psi|) = \int_{-\infty}^{t_0 + B_j} \frac{1}{B_j} \int_{-t_0 - B_j < \psi < -t_0} ds$ and $v_j(t) = \int_{0}^{t} b_{t_0, B_j}(s)ds$. 
It follows from (3.2) that \( \int_{\{\psi < -t_0 - B_j\}} |\tilde{F}_j - (1 - b_{t_0,B_j}(\psi))F_{t_0}|^2 e^{-(\varphi + \psi) + v_j(\psi)c(-v_j(\psi))} dV_X < +\infty \), and note that \( e^{-t}c(t) \) is decreasing with respect to \( t \) and \( v_j(\psi) \geq \max\{\psi, -t_0 - B_j\} \geq -t_0 - 1 \). Hence \( e^{-\psi}c(-\psi) \) has positive lower bound, which implies
\[
\int_{\{\psi < -t_0 - B_j\}} |\tilde{F}_j - (1 - b_{t_0,B_j}(\psi))F_{t_0}|^2 e^{-(\varphi + \psi)} dV_X < +\infty \tag{3.3}
\]

As \( \{\psi < -t_0 - B_j\} \) is open, there exists an open subset \( U_j' \subset (\{\psi < -t_0 - B_j\} \cap U) \) such that \( (\tilde{F}_j - F_{t_0}) \in H^0(\{\psi < -t_0\} \cap U_j') \), \( \mathcal{O}(K_X) \otimes \mathcal{I}(\varphi + \psi) \subset H^0(\{\psi < -t_0\} \cap U_j', \mathcal{O}(K_X) \otimes \mathcal{F}) \), which implies \( (\tilde{F}_j - f) \in H^0(\{\psi < -t_0\} \cap (U_j' \cap U'_0), \mathcal{O}(K_X) \otimes \mathcal{F}) \).

We have already proved in Lemma 2.7 that \( \int_X |\tilde{F}_j|^2 e^{\psi} c(-\psi) dV_X \) is uniformly bounded with respect to \( j \).

As \( e^{-\psi}c(-\psi) \) has positive lower bound on any compact subset \( K \) of \( X \), (by diagonal method) there exist a subsequence of \( \{\tilde{F}_j\} \) (also denoted by \( \{\tilde{F}_j\} \)) compact convergence to a holomorphic \((n, 0)\) form \( \tilde{F}_0 \) (when \( j \to +\infty \)) on \( X \). Since \( (\tilde{F}_j - f) \in H^0(\{\psi < -t_0\} \cap (U_j' \cap U'_0), \mathcal{O}(K_X) \otimes \mathcal{F}) \), it follows from Lemma 2.3 that there exists an open set \( U' \) which satisfies \( Z_0 \subset U' \subset U \) such that \( (\tilde{F}_0 - f) \in H^0(\{\psi < -t_0\} \cap U', \mathcal{O}(K_X) \otimes \mathcal{F}) \).

It follows from (3.2) that
\[
\int_X |\tilde{F}_0 - (1 - b_{t_0}(\psi))F_{t_0}|^2 e^{-\varphi} e^{-\psi + v_{t_0}(\psi)c(-v_{t_0}(\psi))} dV_X \tag{3.4}
\]
\[
\leq \liminf_{j \to +\infty} \int_X |\tilde{F}_j - (1 - b_{t_0,B_j}(\psi))F_{t_0}|^2 e^{-(\varphi + \psi) + v_j(\psi)c(-v_j(\psi))} dV_X
\]
\[
\leq \liminf_{j \to +\infty} \int_{t_0+\beta j} c(t) e^{-t} dt \int_X \frac{1}{B_j} \mathbb{I}_{\{-t_0 - B_j < \psi < -t_0\}} |F_{t_0}|^2 e^{-\varphi} dV_X
\]
\[
\leq \liminf_{j \to +\infty} \frac{e^{t_0+\beta j} \int_{t_0}^{t_0+\beta j} c(t) e^{-t} dt}{\inf_{t \in (t_0, t_0+\beta j)} c(t)} \int_X \frac{1}{B_j} \mathbb{I}_{\{-t_0 - B_j < \psi < -t_0\}} |F_{t_0}|^2 e^{-\psi} dV_X
\]
\[
\leq \liminf_{j \to +\infty} \frac{e^{t_0+\beta j} \int_{t_0}^{t_0+\beta j} c(t) e^{-t} dt}{\inf_{t \in (t_0, t_0+\beta j)} c(t)} \frac{H(t_0) - H(t_0 + B_j)}{B_j}
\]
\[
= \left( \int_{t_0}^{t_0+\beta j} c(t) e^{-t} dt \right) \frac{\liminf_{B \to 0 + 0} \frac{H(t_0) - H(t_0 + B)}{B}}{c(t_0) e^{-t_0}}
\]
the first "\( \leq \)" holds because of Fatou Lemma, where \( b_{t_0}(t) = \mathbb{I}_{\{t \geq -t_0\}} \) and \( v_{t_0}(t) = \int_{t_0}^{t} b_{t_0}(s) ds \). Note that \( 1 - b_{t_0}(\psi) = \mathbb{I}_{\{\psi < -t_0\}} \).

Note that \( v_{t_0}(t) \geq t \) and \( c(t) e^{-t} \) is decreasing with respect to \( t \), then \( e^{-\psi + v_{t_0}(\psi)c(-v_{t_0}(\psi))} \geq c(-\psi) \) holds on \( X \). Hence we have
\[
\int_X |\tilde{F}_0 - (1 - b_{t_0}(\psi))F_{t_0}|^2 e^{-\varphi} c(-\psi) dV_X
\]
\[
\leq \int_X |\tilde{F}_0 - (1 - b_{t_0}(\psi))F_{t_0}|^2 e^{-\varphi} e^{-\psi + v_{t_0}(\psi)c(-v_{t_0}(\psi))} dV_X \tag{3.5}
\]
\[
\leq \int_{t_0+\beta j} c(t) e^{-t} dt \liminf_{B \to 0 + 0} \frac{H(t_0) - H(t_0 + B)}{B} \]
However,

\[
\int_X |\tilde{F}_0 - \mathbb{1}_{\{\psi < -t_0\}} F_{t_0}|^2 e^{-\varphi} c(-\psi)dV_X
\]

\[
= \int_{\{\psi < -t_0\}} |\tilde{F}_0|^2 e^{-\varphi} c(-\psi)dV_X + \int_{\{\psi < -t_0\}} |\tilde{F}_0 - F_{t_0}|^2 e^{-\varphi} c(-\psi)dV_X
\]

\[
= \int_{\{\psi < -t_0\}} |\tilde{F}_0|^2 e^{-\varphi} c(-\psi)dV_X + \int_{\{\psi < -t_0\}} |\tilde{F}_0|^2 e^{-\varphi} c(-\psi)dV_X - \int_{\{\psi < -t_0\}} |F_{t_0}|^2 e^{-\varphi} c(-\psi)dV_X
\]

\[
\geq H(T) - H(t_0)
\]  

(3.6)

Combining with (3.5) and (3.6), we have

\[
H(T) - H(t_0)
\]

\[
\leq \int_X |\tilde{F}_0 - \mathbb{1}_{\{\psi < -t_0\}} F_{t_0}|^2 e^{-\varphi} c(-\psi)dV_X - \int_{\{\psi < -t_0\}} |F_{t_0}|^2 e^{-\varphi} c(-\psi)dV_X
\]

\[
= \int_X |\tilde{F}_0 - \mathbb{1}_{\{\psi < -t_0\}} F_{t_0}|^2 e^{-\varphi} c(-\psi)dV_X
\]

\[
\leq \int_X |\tilde{F}_0 - \mathbb{1}_{\{\psi < -t_0\}} F_{t_0}|^2 e^{-\varphi} c(-\psi)dV_X + \int_{\{\psi < -t_0\}} |F_{t_0}|^2 e^{-\varphi} c(-\psi)dV_X
\]

\[
\leq \frac{\int_{t_0 + B}^{t_0 + B} c(t)e^{-t}dt}{c(t_0)e^{-t_0}} \liminf_{B \rightarrow 0+0} \frac{H(t_0) - H(t_0 + B)}{B}
\]

(3.7)

As \(H(h^{-1}(r))\) is linear with respect to \(r\), hence

\[
\frac{H(T) - H(t_0)}{\int_{t_0 + B}^{t_0 + B} c(t)e^{-t}dt} = \liminf_{B \rightarrow 0+0} \frac{H(t_0) - H(t_0 + B)}{B}
\]

, then all “\(\leq\)” in (3.7) should be “\(=\)”, i.e.

\[
H(T) - H(t_0)
\]

\[
= \int_X |\tilde{F}_0|^2 e^{-\varphi} c(-\psi)dV_X - \int_{\{\psi < -t_0\}} |F_{t_0}|^2 e^{-\varphi} c(-\psi)dV_X
\]

\[
= \int_X |\tilde{F}_0 - \mathbb{1}_{\{\psi < -t_0\}} F_{t_0}|^2 e^{-\varphi} c(-\psi)dV_X
\]

\[
= \int_X |\tilde{F}_0 - \mathbb{1}_{\{\psi < -t_0\}} F_{t_0}|^2 e^{-\varphi} c(-\psi)dV_X + \int_{\{\psi < -t_0\}} |F_{t_0}|^2 e^{-\varphi} c(-\psi)dV_X
\]

\[
= \frac{\int_{t_0 + B}^{t_0 + B} c(t)e^{-t}dt}{c(t_0)e^{-t_0}} \liminf_{B \rightarrow 0+0} \frac{H(t_0) - H(t_0 + B)}{B}
\]

(3.8)

It follows from the first “\(=\)” in (3.8) and \(H(t_0) = \int_{\{\psi < t_0\}} |F_{t_0}|^2 e^{-\varphi} c(-\psi)dV_X\) that

\[
H(T) = \int_X |\tilde{F}_0|^2 e^{-\varphi} c(-\psi)dV_X
\]
It follows from \( c(\psi) = e^{-\psi + v_0(\psi)}c(-v_0(\psi)) \) on \( \{\psi \geq t_0\} \) and
\[
\int_X |\tilde{F}_0 - 1_{\{\psi < -t_0\}}F_{t_0}|^2 e^{-\varphi}c(-\psi)dV_X \\
= \int_X |\tilde{F}_0 - 1_{\{\psi < -t_0\}}F_{t_0}|^2 e^{-\varphi} e^{-\psi + v_0(\psi)}c(-v_0(\psi))dV_X
\]
that
\[
\int_{\{\psi < -t_0\}} |\tilde{F}_0 - F_{t_0}|^2 e^{-\varphi}c(-\psi)dV_X \\
= \int_{\{\psi < -t_0\}} |\tilde{F}_0 - F_{t_0}|^2 e^{-\varphi} e^{-\psi + v_0(\psi)}c(-v_0(\psi))dV_X \tag{3.9}
\]
Note that, on \( \{\psi < t_0\} \),
\[
c(-\psi) < e^{-\psi + v_0(\psi)}c(-v_0(\psi))
\]
and the integrand in (3.9) is nonnegative, we must have \( \tilde{F}_0|_{\{\psi < -t_0\}} = F_{t_0} \).

Theorem 1.4. is proved.

3.1. Proof of Corollary 1.4. To prove Corollary 1.4, we need the following Propositions.

**Proposition 3.1.** If \( H(h^{-1}(r); c) \) is linear with respect to \( r \in (0, \int_{-\infty}^{+\infty} c(t)e^{-t}dt] \).

Let \( t_0 \geq T \) be given. Let \( \tilde{F} \) be a holomorphic \((n,0)\) form on \( \{\psi < -t_0\} \) which satisfies \( \tilde{F} \not\equiv F|_{\{\psi < -t_0\}}, (\tilde{F} - f) \in H^0(U' \cap \{\psi < -t_0\}, K_M \otimes F) \), where \( U' \) is an open subset of \( X \) satisfies \( Z_0 \subset \tilde{U}' \subset U \) and \( \int_{\{\psi < -t_0\}} c(-\psi)|\tilde{F}|^2 e^{-\varphi}dV_X < +\infty \).

Then for any \( t_0 \leq t_1 \leq t_2 \leq +\infty \), we have
\[
\int_{\{-t_2 \leq \psi < -t_1\}} c(-\psi)|\tilde{F}|^2 e^{-\varphi}dV_X > \int_{\{-t_2 \leq \psi < -t_1\}} c(-\psi)|F|^2 e^{-\varphi}dV_X
\]

**Proof.** when \( t_2 = +\infty \), it follows form Lemma 2.5 that
\[
\int_{\{\psi < -t_1\}} c(-\psi)|\tilde{F}|^2 e^{-\varphi}dV_X - \int_{\{\psi < -t_1\}} c(-\psi)|F|^2 e^{-\varphi}dV_X \\
= \int_{\{\psi < -t_1\}} c(-\psi)|\tilde{F} - F|^2 e^{-\varphi}dV_X
\]
As \( \tilde{F} - F \neq 0 \) on \( \{\psi < -t_1\} \), the zero set of \( \tilde{F} - F \) (denoted by \( Z(\tilde{F} - F) \)) is an analytic set of \( \{\psi < -t_1\} \) and the measure of \( Z(\tilde{F} - F) \) is zero. Then
\[
\int_{\{\psi < -t_1\}} c(-\psi)|\tilde{F} - F|^2 e^{-\varphi}dV_X > 0,
\]
hence
\[
\int_{\{\psi < -t_1\}} c(-\psi)|\tilde{F}|^2 e^{-\varphi}dV_X > \int_{\{\psi < -t_1\}} c(-\psi)|F|^2 e^{-\varphi}dV_X
When \( t_0 \leq t_1 < t_2 < +\infty \), we have
\[
\int_{\{t_2 \leq \psi < t_1\}} c(-\psi)|\tilde{F}|^2 e^{-\varphi} dV_X - \int_{\{t_2 \leq \psi < t_1\}} c(-\psi)|F|^2 e^{-\varphi} dV_X
= \int_{\{\psi < t_1\}} c(-\psi)|\tilde{F}|^2 e^{-\varphi} dV_X - \int_{\{\psi < t_2\}} c(-\psi)|\tilde{F}|^2 e^{-\varphi} dV_X
- \left( \int_{\{\psi < t_1\}} c(-\psi)|F|^2 e^{-\varphi} dV_X - \int_{\{\psi < t_2\}} c(-\psi)|F|^2 e^{-\varphi} dV_X \right)
= \int_{\{\psi < t_1\}} c(-\psi)|\tilde{F} - F|^2 e^{-\varphi} dV_X - \int_{\{\psi < t_2\}} c(-\psi)|\tilde{F} - F|^2 e^{-\varphi} dV_X
= \int_{\{-t_2 \leq \psi < -t_1\}} c(-\psi)|\tilde{F} - F|^2 e^{-\varphi} dV_X.
\]
As \( \tilde{F} - F \neq 0 \) on \( \{\psi < -t_1\} \), the zero set of \( \tilde{F} - F \) is an analytic set of \( \{\psi < -t_1\} \).

Note that the measure of the set \( \{-t_2 \leq t < t_1\} \) is positive and the measure of \( Z(\tilde{F} - F) \) is zero, we know
\[
\int_{\{-t_2 \leq \psi < -t_1\}} c(-\psi)|\tilde{F} - F|^2 e^{-\varphi} dV_X > 0,
\]
hence
\[
\int_{\{t_2 \leq \psi < t_1\}} c(-\psi)|\tilde{F}|^2 e^{-\varphi} dV_X > \int_{\{t_2 \leq \psi < -t_1\}} c(-\psi)|F|^2 e^{-\varphi} dV_X.
\]

Now we begin to prove Corollary 1.4.

**Proof.**

**Step 1:**
Given \( t_2 \geq T \). It follows from Lemma 2.4 that there exists a holomorphic \((n,0)\) form \( \tilde{F} \) on \( \{\psi < -t_2\} \) such that \( (\tilde{F} - f) \in H^0(U' \cap \{\psi < -t_2\}, K_M \otimes I(\psi + \varphi)|\nu) \), where \( U' \) is an open subset of \( X \) satisfies \( Z_0 \subset U' \subset U \) and
\[
H(t_2; c) = \int_{\{\psi < -t_2\}} \tilde{c}(-\psi)|\tilde{F}|^2 e^{-\varphi} dV_X < +\infty
\]
As \( (\log \tilde{c}(t))' \geq (\log c(t))' \), we have \( \tilde{c}(t) \geq Mc(t) \) for some constant \( M > 0 \). It follows from \( \int_{\{\psi < -t_2\}} \tilde{c}(-\psi)|\tilde{F}|^2 e^{-\varphi} dV_X < +\infty \) that we have
\[
\int_{\{\psi < -t_2\}} c(-\psi)|\tilde{F}|^2 e^{-\varphi} dV_X < +\infty.
\]

**Step 2:**
Denote \( I(t) = \int_{\{\psi < -t\}} c(-\psi)|\tilde{F}|^2 e^{-\varphi} dV_X \), where \( t \geq t_2 \). For any \( t_0 > t_1 \geq t_2 \), Proposition 3.1 shows that
\[
\int_{\{-t_0 \leq \psi < -t_1\}} c(-\psi)|\tilde{F}|^2 e^{-\varphi} dV_X \geq \int_{\{-t_0 \leq \psi < -t_1\}} c(-\psi)|F|^2 e^{-\varphi} dV_X,
\]
the equality holds if and only if \( \tilde{F} = F|_{\{\psi < -t_2\}} \). Hence we know
\[
\frac{I(t_1) - I(t_0)}{\int_{t_0}^{t_1} c(t)e^{-t} dt} \geq \frac{H(t_1; c) - H(t_0; c)}{\int_{t_1}^{t_0} c(t)e^{-t} dt} = k,
\]
(3.10)
the equality holds if and only if $\tilde{F} = F|_{\{\psi < -t_2\}}$.

Note that we also have

$$H(t_2; \tilde{c}) - H(t_1; \tilde{c}) \geq \int_{\{t_1 \leq \psi < -t_2\}} \tilde{c}(\psi) |\tilde{F}|^2 e^{-\varphi} dV_X$$

$$= \sum_{i=1}^{n} \int_{\{t_1 + (i-1) \frac{t_1-t_2}{n} \leq \psi < t_1 + \frac{t_1-t_2}{n}\}} \frac{\tilde{c}(\psi)}{c(\psi)} |\tilde{F}|^2 e^{-\varphi} dV_X$$

(3.11)

As $c(t) \in \mathcal{G}_T$, it follows from condition (2) and (3) of $\mathcal{G}_T$ that $c(t) \neq 0$ for any $t \geq T$. Then $\frac{\tilde{c}(\psi)}{c(\psi)}$ is uniformly continuous on $[t_2, t_1]$. When $n$ big enough, we have

$$H(t_2; \tilde{c}) - H(t_1; \tilde{c}) \geq \sum_{i=1}^{n} \left( \int_{\{t_1 + (i-1) \frac{t_1-t_2}{n} \leq \psi < t_1 + \frac{t_1-t_2}{n}\}} c(\psi) |\tilde{F}|^2 e^{-\varphi} dV_X \times \frac{\tilde{c}(t_1 - i \frac{t_1-t_2}{n})}{c(t_1 - i \frac{t_1-t_2}{n})} - c \right)$$

$$= S_{1,n} + S_{2,n}$$

where

$$S_{1,n} = \sum_{i=1}^{n} \left( \int_{\{t_1 + (i-1) \frac{t_1-t_2}{n} \leq \psi < t_1 + \frac{t_1-t_2}{n}\}} c(\psi) |\tilde{F}|^2 e^{-\varphi} dV_X \right) \frac{\tilde{c}(t_1 - i \frac{t_1-t_2}{n})}{c(t_1 - i \frac{t_1-t_2}{n})}$$

and

$$S_{2,n} = -c \sum_{i=1}^{n} \int_{\{t_1 + (i-1) \frac{t_1-t_2}{n} \leq \psi < t_1 + \frac{t_1-t_2}{n}\}} c(\psi) |\tilde{F}|^2 e^{-\varphi} dV_X.$$ 

It is easy to see that $\lim_{n \to +\infty} S_{2,n} = 0$. For $S_{1,n}$, we have

$$S_{1,n} = \sum_{i=1}^{n} \int \left[ \frac{c(t)}{c(t_1 - i \frac{t_1-t_2}{n})} e^{-t_1 + \frac{t_1-t_2}{n} t_1 - t_2} \tilde{c}(t_1 - i \frac{t_1-t_2}{n}) e^{-t_1 + \frac{t_1-t_2}{n} t_1 - t_2} \right]$$

$$\geq \sum_{i=1}^{n} k \int \left[ \frac{c(t)}{c(t_1 - i \frac{t_1-t_2}{n})} e^{-t_1 + \frac{t_1-t_2}{n} t_1 - t_2} \tilde{c}(t_1 - i \frac{t_1-t_2}{n}) e^{-t_1 + \frac{t_1-t_2}{n} t_1 - t_2} \right]$$

(3.12)

The “$\geq$” holds because of (3.10). Let $n \to +\infty$ in (3.12) we have $\lim_{n \to +\infty} S_{1,n} \geq k \int_{t_2}^{t_1} \tilde{c}(t)e^{-t} dt$. Hence we have

$$H(t_2; \tilde{c}) - H(t_1; \tilde{c}) \geq k \int_{t_2}^{t_1} \tilde{c}(t)e^{-t} dt$$

i.e.

$$\frac{H(t_2; \tilde{c}) - H(t_1; \tilde{c})}{\int_{t_2}^{t_1} \tilde{c}(t)e^{-t} dt} \geq k.$$
Let $t_1 \to +\infty$, then

$$
\frac{H(t_2; \tilde{c})}{\int_{t_2}^{+\infty} \tilde{c}(t)e^{-t}dt} \geq k
$$

(3.13)

Recall that $F$ is the holomorphic $(n, 0)$ form on $X$ such that $H(t; c) = \int_{\{\psi < -t\}} c(\psi)|F|^2e^{-\varphi}dV_X$, for any $t \geq T$. Let $T \leq t_2 < t_1 < +\infty$, we have

$$
\frac{\int_{\{\psi \leq t_1 \leq t_2\}} c(-\psi)|F|^2e^{-\varphi}dV_X}{\int_{t_2}^{t_1} c(t)e^{-t}dt} = k
$$

Note that

$$
\int_{\{\psi \leq t_1 \leq t_2\}} \tilde{c}(\psi)|F|^2e^{-\varphi}dV_X
$$

(3.14)

Let $n$ be big enough, the right hand side of (3.14) is bounded by

$$
\sum_{i=1}^{n} \left( \int_{\{-t_1+(i-1)\frac{t_2-t_1}{n} \leq \psi \leq -t_1+i\frac{t_2-t_1}{n}\}} c(-\psi)|F|^2e^{-\varphi}dV_X \right) \frac{\tilde{c}(t_1-i\frac{t_2-t_1}{n})}{c(t_1-i\frac{t_2-t_1}{n})} \pm \epsilon
$$

$$
\sum_{i=1}^{n} \left| k \int_{t_1-i\frac{t_2-t_1}{n}}^{t_1-(i-1)\frac{t_2-t_1}{n}} c(t)e^{-t}dt \frac{\tilde{c}(t_1-i\frac{t_2-t_1}{n})}{c(t_1-i\frac{t_2-t_1}{n})} \pm \epsilon \right|
$$

$$
\sum_{i=1}^{n} \left[ k \int_{t_1-i\frac{t_2-t_1}{n}}^{t_1-(i-1)\frac{t_2-t_1}{n}} c(t)e^{-t}dt \right] (\tilde{c}(t_1-i\frac{t_2-t_1}{n})e^{-t_1+i\frac{t_2-t_1}{n}t_1-t_2/n} \pm \epsilon)
$$

$$
k \int_{t_1-i\frac{t_2-t_1}{n}}^{t_1-(i-1)\frac{t_2-t_1}{n}} c(t)e^{-t}dt
$$

(3.15)

When $n \to +\infty$, combining (3.14) and (3.15), we have

$$
\int_{\{-t_1 \leq \psi \leq -t_2\}} \tilde{c}(\psi)|F|^2e^{-\varphi}dV_X = k \int_{t_2}^{t_1} \tilde{c}(t)e^{-t}dt
$$

(3.16)

Let $t_1$ goes to $+\infty$ in (3.16), we know

$$
\frac{\int_{\{\psi < -t_2\}} \tilde{c}(\psi)|F|^2e^{-\varphi}dV_X}{\int_{t_2}^{+\infty} \tilde{c}(t)e^{-t}dt} = k
$$

Hence

$$
\frac{H(t_2; \tilde{c})}{\int_{t_2}^{+\infty} \tilde{c}(t)e^{-t}dt} \leq \frac{\int_{\{\psi < -t_2\}} \tilde{c}(\psi)|F|^2e^{-\varphi}dV_X}{\int_{t_2}^{+\infty} \tilde{c}(t)e^{-t}dt} = k
$$

(3.17)

It follows from (3.13) and (3.17) that for any $t_2 \geq T$,

$$
H(t_2; \tilde{c}) \leq k \int_{t_2}^{+\infty} \tilde{c}(t)e^{-t}dt
$$
holds, i.e. \( H(h^{-1}(r); \tilde{c}) \) is linear with respect to \( r \). Hence there exists a holomorphic \((n,0)\) form \( F_\tilde{c} \) on \( X \) such that

\[
H(t_2; \tilde{c}) = \int_{\{\psi < -t_2\}} \tilde{c}(\psi)|F_\tilde{c}|^2 e^{-\psi} dV_X
\]

and we also have

\[
\frac{H(t_2; \tilde{c}) - H(t_1; \tilde{c})}{\int_{t_2}^{t_1} \tilde{c}(t)e^{-t} dt} = \frac{\int_{\{\psi < -t_2\}} \tilde{c}(\psi)|F_\tilde{c}|^2 e^{-\psi} dV_X}{\int_{t_2}^{t_1} \tilde{c}(t)e^{-t} dt}
\]

(3.18)

If \( F_\tilde{c} \neq F \) on \( X \), it follows from Proposition 3.8 and 3.9 and (3.18) that

\[
k = \frac{\int_{\{\psi < -t_2\}} \tilde{c}(\psi)|F_\tilde{c}|^2 e^{-\psi} dV_X}{\int_{t_2}^{t_1} \tilde{c}(t)e^{-t} dt} > \frac{\int_{\{\psi < -t_2\}} \tilde{c}(\psi)|F_\tilde{c}|^2 e^{-\psi} dV_X}{\int_{t_2}^{t_1} \tilde{c}(t)e^{-t} dt} = k,
\]

which is a contradiction. Hence we must have \( F_\tilde{c} = F \) on \( X \).

Corollary 1.4 is proved.

\[ \square \]

3.2. **Proof of Corollary 1.5 and Corollary 1.6**

In this section, we prove Corollary 1.5 and Corollary 1.6.

**Proof.** We prove the Corollary 1.5 by contradiction.

Assume \( H(h^{-1}(r); \varphi) \) is linear with respect to \( r \in (0, \int_{-\infty}^{+\infty} c(t)e^{-t} dt) \). Then it follows from Theorem 1.3 that there exists a holomorphic \((n,0)\) form \( F \) on \( X \) such that

\[
H(t; \varphi) = \int_{\{\psi < -t\}} c(\psi)|F|^2 e^{-\psi} dV_X
\]

holds for any \( t \geq T \).

Denote

\[
\inf\{\int_{\{\psi < -t\}} c(\psi)|\tilde{f}|^2 e^{-\psi} dV_X : \tilde{f} \in H^0(\{\psi < -t\}, \mathcal{O}(K_X)),
\]

& \( \exists \) open set \( U' \) s.t \( \emptyset \subset U' \subset U \) and

\[
(\tilde{f} - f) \in H^0(\{\psi < -t\} \cap U', \mathcal{O}(K_X) \otimes \mathcal{F})
\]

by \( H(t; \tilde{\varphi}) \). As \( e^{-\tilde{\varphi} - \varphi} \leq e^{-\varphi} \), we know \( H(T; \tilde{\varphi}) < +\infty \).

Let \( C_2 > t_1 > t_2 \geq T \). It follows from Lemma 2.5 that there exists a holomorphic \((n,0)\) form \( \tilde{F}_{t_2} \) on \( \{\psi < -t_2\} \) such that

\[
H(t_2; \tilde{\varphi}) = \int_{\{\psi < -t_2\}} c(\psi)|\tilde{F}_{t_2}|^2 e^{-\tilde{\varphi}} dV_X < +\infty.
\]

As \( \tilde{\varphi} - \varphi \) is bounded on \( X \), we have

\[
H(t_2; \tilde{\varphi}) = \int_{\{\psi < -t_2\}} c(\psi)|\tilde{F}_{t_2}|^2 e^{-\tilde{\varphi}} dV_X < +\infty.
\]

Note that on \( \{-t_1 \leq \psi < -t_2\} \subset \{\psi \geq -C_2\} \), we have \( \tilde{\varphi} = \varphi \), hence

\[
H(t_2; \tilde{\varphi}) - H(t_1; \tilde{\varphi}) \geq \int_{\{-t_1 \leq \psi < -t_2\}} c(\psi)|\tilde{F}_{t_2}|^2 e^{-\tilde{\varphi}} dV_X
\]

\[
\geq \int_{\{-t_1 \leq \psi < -t_2\}} c(\psi)|F|^2 e^{-\varphi} dV_X
\]

(3.19)
The second inequality holds because of Proposition \([3.1]\). It follows from \([3.19]\) that
\[
\frac{H(t_2; \tilde{\varphi}) - H(t_1; \tilde{\varphi})}{\int_{t_2}^{t_1} c(t)e^{-t}dt} = k
\] (3.20)

Let \(t_2 = T\), it follows from Theorem 1.1 and note that \(\tilde{\varphi} \geq \varphi\) on \(X\), we have
\[
\frac{H(T; \tilde{\varphi}) - H(t_1; \tilde{\varphi})}{\int_{t_2}^{t_1} c(t)e^{-t}dt} \leq \frac{H(T; \varphi) - H(t_1; \varphi)}{\int_{t_2}^{t_1} c(t)e^{-t}dt} = k
\] (3.21)

It follows from \([3.20]\) and \([3.21]\) that
\[
\frac{H(T; \tilde{\varphi})}{\int_{t_2}^{t_1} c(t)e^{-t}dt} = k.
\]

Let \(t_3\) be big enough such that \(\{\psi < -t_3\} \subset \{\psi < C_1\}\). Then, on \(\{\psi < -t_3\}\), we have \(\tilde{\varphi} = \varphi\). When \(t \geq t_3\), we have \(H(t; \tilde{\varphi}) = H(t; \varphi)\) and
\[
\frac{H(t; \tilde{\varphi})}{\int_{t_2}^{t_1} c(t)e^{-t}dt} = k.
\]

Recall that \(\frac{H(T; \tilde{\varphi})}{\int_{t_2}^{t_1} c(t)e^{-t}dt} = k\), we know \(\lim_{t \to +\infty} \frac{H(t; \tilde{\varphi})}{\int_{t_2}^{t_1} c(t)e^{-t}dt} = \frac{H(T; \tilde{\varphi})}{\int_{t_2}^{t_1} c(t)e^{-t}dt} = k\), then we know \(H(-\log r; \tilde{\varphi})\) is linear with respect to \(r\). Then there exist a holomorphic \((n, 0)\) form \(\tilde{F}\) on \(X\) such that for any \(t \geq T\), we have
\[
H(t; \tilde{\varphi}) = \int_{\{\psi < -t\}} c(\tilde{\varphi})|\tilde{F}|^2e^{-\tilde{\varphi}}dV_X.
\]

When \(t_0\) big enough such that \(\tilde{\varphi} = \varphi\) on \(\{\psi < -t_0\}\), then \(H(t_0; \tilde{\varphi}) = H(t_0; \varphi)\), hence we have (note that \(\tilde{\varphi} = \varphi\))
\[
\int_{\{\psi < -t_0\}} c(\tilde{\varphi})|\tilde{F}|^2e^{-\tilde{\varphi}}dV_X = H(t_0; \tilde{\varphi}) = H(t_0; \varphi) = \int_{\{\psi < -t_0\}} c(\tilde{\varphi})|F|^2e^{-\varphi}dV_X
\]

which (by Proposition \([3.1]\) implies \(\tilde{F} = F\) on \(\{\psi < -t_0\}\). Note that \(\{\psi < -t_0\}\) is an open subset of \(X\), \(F\) and \(\tilde{F}\) are holomorphic \((n, 0)\) form on \(X\), it follows from \(\tilde{F} = F\) on \(\{\psi < -t_0\}\) that \(\tilde{F} = F\) on \(X\).

However \(e^{-\varphi} > e^{-\tilde{\varphi}}\) on \(U \subset X\), we must have
\[
k = \frac{H(T; \tilde{\varphi})}{\int_{t_2}^{t_1} c(t)e^{-t}dt} = \frac{\int_X |\tilde{F}|^2e^{-\tilde{\varphi}}dV_X}{\int_{t_2}^{t_1} c(t)e^{-t}dt} = \frac{\int_X |F|^2e^{-\varphi}dV_X}{\int_{t_2}^{t_1} c(t)e^{-t}dt} = \frac{H(T; \varphi)}{\int_{t_2}^{t_1} c(t)e^{-t}dt} = k
\]

This is a contradiction. Hence \(H(h^{-1}(r); \varphi)\) can not be linear with respect to \(r\). Corollary \([1.5]\) is proved.

To prove Corollary \([1.6]\), we only need to construction a function \(\tilde{\varphi}\) on \(X\) which satisfies the condition of Corollary \([1.5]\).

**Proof.** As \(\varphi + \psi\) is strictly plurisubharmonic at \(z_0\), we can find a small open neighborhood \((U, z)\) of \(z_0\) and \(z = (z_1, \ldots, z_n)\) is the local coordinate on \(U\) such that \(i\partial\bar{\partial}(\varphi + \psi) > \epsilon\omega\) for some \(\epsilon > 0\), where \(\omega = i \sum_{i=1}^{n} dz_i \wedge \bar{dz}_i\) under the local coordinate on \(U\). By shrinking \(U\), we also assume that \(U \subset X\). Take \(z_1 \in U, z_1 \notin \{\psi = -\infty\}\), then we can choose an open subset \(V\) such that \(z_1 \in V\) and \(V\) satisfies

1. \(V \subset U\),

(2) $V \cap \{ \psi = -\infty \} = \emptyset$.

Let $\rho$ be a smooth nonnegative function on $X$ which satisfies $\rho \equiv 1$ on $W \subset V$ and $\text{supp}\rho \subset V$. Let $\delta$ be a small positive constant such that

$$i\partial\bar{\partial}(\varphi + \psi) + i\partial\bar{\partial}(\delta \rho) > \frac{\epsilon}{2} \omega$$

on $V$. Let $\tilde{\varphi} = \varphi + \delta \rho$, note that $0 \leq \delta \rho \leq \delta$ is a smooth function, then $\tilde{\varphi}$ satisfies

1. $\tilde{\varphi} + \psi$ is plurisubharmonic function on $X$.
2. $\tilde{\varphi} > \varphi$ on $W$ and $\tilde{\varphi} = \varphi$ on $X \setminus U$.
3. $\tilde{\varphi} - \varphi$ is bounded on $X$.

It is easy to see that the function $\tilde{\varphi}$ satisfies the conditions (1), (2), (3) in Corollary 1.5. Then it follows Corollary 1.5 that $H(h^{-1}(r); \varphi)$ can not be linear with respect to $r$.

Corollary 1.6 is proved. $\square$

3.3. Proof of Theorem 1.8. Let $c(t) \in \mathcal{G}_0$. Let $Z_0 = Y$. Let $\tilde{f}$ be a holomorphic extension of $f$ from $Y$ to $U$, where $U \supset Y$ is an open subset of $X$. Let $F = \mathcal{I}(\psi)|_U$ on $U$.

Define

$$H(t) := \inf \left\{ \int_{\{ \psi < -t \}} |\tilde{f}|^2 e^{-\varphi} c(-\psi) dV_X : \tilde{f} \in H^0(\{ \psi < -t \}, \mathcal{O}(K_X)) \right\}$$

where $\exists$ open set $U'$ s.t $Z_0 \subset U' \subset U$ and $(\tilde{f} - f) \in H^0(\{ \psi < -t \} \cap U', \mathcal{O}(K_X) \otimes \mathcal{I}(\psi))$.

(3.22)

It follows from condition (1.11) and (1.12) that

$$\int_X c(-\psi) |F|^2 e^{-\varphi} dV_X = H(0).$$

The optimal $L^2$ extension theorem in [21] shows that

$$\int_{\{ \psi < -t \}} c(-\psi) |F|_t^2 e^{-\varphi} dV_X \leq \left( \int_t^{+\infty} c(t) e^{-t} dt \right) \int_Y |f|^2 e^{-\varphi} dV_X [\psi]$$

holds for any $t \in [0, +\infty)$, where $F_t$ is a holomorphic extension of $f$ from $Y$ to $\{ \psi < -t \}$. Note that by the definition of $H(t)$, we have

$$H(t) \leq \int_{\{ \psi < -t \}} c(-\psi) |F_t|^2 e^{-\varphi} dV_X$$

Theorem 1.11 implies that

$$\int_X |F|^2 e^{-\varphi} dV_X = H(0) \leq \int_0^{+\infty} c(t) e^{-t} dt \int_t^{+\infty} c(t) e^{-t} dt H(t)$$
Now we have

\[ H(0) = \int_X c(-\psi)|F|^2 e^{-\varphi} dV_X \]
\[ \leq \frac{\int_0^{+\infty} c(t)e^{-t} dt}{\int_t^{+\infty} c(t)e^{-t} dt} H(t) \]
\[ \leq \frac{\int_0^{+\infty} c(t)e^{-t} dt}{\int_t^{+\infty} c(t)e^{-t} dt} \int_{\{\psi < -t\}} |F|^2 e^{-\varphi} dV_X \]
\[ \leq \left( \int_0^{+\infty} c(t)e^{-t} dt \right) \frac{\pi k}{k!} \int_Y |f|^2 e^{-\varphi} dV_X[\psi] \]

holds for any \( t \in [0, +\infty) \). Recall that \( F \) satisfies

\[ \int_X c(-\psi)|F|^2 e^{-\varphi} dV_X = \left( \int_0^{+\infty} c(t)e^{-t} dt \right) \frac{\pi k}{k!} \int_Y |f|^2 e^{-\varphi} dV_X[\psi] \]

Hence all “\( \leq \)” in (3.23) should be “\( = \)”, i.e.

\[ H(0) = \int_X c(-\psi)|F|^2 e^{-\varphi} dV_X \]
\[ \frac{\int_0^{+\infty} c(t)e^{-t} dt}{\int_t^{+\infty} c(t)e^{-t} dt} H(t) \]
\[ \frac{\int_0^{+\infty} c(t)e^{-t} dt}{\int_t^{+\infty} c(t)e^{-t} dt} \int_{\{\psi < -t\}} |F|^2 e^{-\varphi} dV_X \]
\[ = \left( \int_0^{+\infty} c(t)e^{-t} dt \right) \frac{\pi k}{k!} \int_Y |f|^2 e^{-\varphi} dV_X[\psi] \]

holds for any \( t \in [0, +\infty) \). Especially,

\[ \frac{\int_{\{\psi < -t\}} |F|^2 e^{-\varphi} dV_X}{\int_0^{+\infty} c(t)e^{-t} dt} = \frac{\pi k}{k!} \int_Y |f|^2 e^{-\varphi} dV_X[\psi] \]

holds. It follows from Theorem 1.3 that \( F|_{\{\psi < -t\}} = F_t \).

Theorem 1.8 is proved.

3.4. Proof of Corollary 1.10

It is easy to see that (2) implies (1).

Now we assume that the statement (1) holds. It follows from Corollary 1.2 that \( H(-\log r) \) is linear with respect to \( r \), i.e. \( L_{D_t}(0,0) = e^t \) holds for any \( t \in [0, +\infty) \). Now we only need to show that the linearity of \( H(-\log r) \) implies (2).

It is known that \( L_{D_t}(z,0) \) satisfies \( \int_{D_t} L_{D_t}(z,0) d\lambda_n = H(t) \), where \( d\lambda_n \) is the Lebesgue measure on \( \mathbb{C}^n \). It follows from Theorem 1.3 that the linearity of \( H(-\log r) \) implies \( L_{D_t}(z,0) \) is \( e^t \) holds for any \( t \in [0, +\infty) \), we have \( L_{D_t}(z,0) = e^t \) holds for any \( t \in [0, +\infty) \) and any \( z \in D_t \).

Corollary 1.10 is proved.
3.5. **Proof of Theorem 1.11 and Theorem 1.12** We firstly discuss some property of \( H(t,c,\varphi) \).

Recall that \( X \) is an open Riemann Surface which admits a nontrivial Green function \( G_X(z,w) \).

Let \( \psi = kG_X(z,z_0) \), where \( z_0 \) is a point of \( X \) and \( k \geq 2 \) is a real number.

Let \( U \) be a open neighborhood of \( z_0 \). Let \( f \) be a holomorphic \((n,0)\) form on \( V_{z_0} \).

Let \( \varphi \) be a subharmonic function on \( X \). Let \( c(t) \in \mathcal{G}_0 \). Denote

\[
H(t,c,\varphi) := \inf \{ \int_{\{\psi < -t\}} c(-\psi)|\hat{F}|^2e^{-\varphi}dV_X : \hat{F} \in H^0(\{\psi < -t\}, \mathcal{O}(K_X)) \}
\]

& \exists \text{open set } U' \text{s.t } Z_0 \subset U' \subset U \text{ and } (\hat{F} - f) \in H^0(\{\psi < -t\} \cap U', \mathcal{O}(K_X) \otimes \mathcal{I}(\psi + \varphi)|_U).
\]

(3.25)

We now consider the linearity of \( H(h^{-1}(r);c,\varphi) \) with respect to \( r \) for various \( c \in \mathcal{G}_T \) and \( c \in C^\infty[T, +\infty) \), where \( H(t) = \int_t^{+\infty} c(t_1)e^{-t_1}dt_1 \). We have the following result.

**Proposition 3.2.** Let \( c \in C^\infty[T, +\infty) \) and \( c \in \mathcal{G}_T \). If \( H(T;c,\varphi) < +\infty \) and \( H(h^{-1}(r);c,\varphi) \) is linear with respect to \( r \in (0, \int_T^{+\infty} c(t)e^{-t}dt] \). Let \( F \) be the holomorphic \((n,0)\) form on \( X \) such that \( \int_{\{\psi < -t\}} c(-\psi)|F|^2e^{-\varphi}dV_X = H(t;c,\varphi) \) for any \( t \geq T \). Then for any other \( \hat{c} \in C^\infty[T, +\infty) \) and \( \hat{c} \in \mathcal{G}_T \), which satisfies \( H(T;\hat{c},\varphi) < +\infty \) we have

\[
\int_{\{\psi < -t\}} \hat{c}(-\psi)|\hat{F}|^2e^{-\varphi}dV_X = H(t;\hat{c},\varphi) = \frac{H(T;\hat{c},\varphi)}{\int_t^{+\infty} \hat{c}(t_1)e^{-t_1}dt_1} \int_t^{+\infty} \hat{c}(t_1)e^{-t_1}dt_1
\]

\[
=k \int_t^{+\infty} \hat{c}(t_1)e^{-t_1}dt_1
\]

holds for any \( t \in [T, +\infty) \), where \( k = \frac{H(T;\hat{c},\varphi)}{\int_t^{+\infty} \hat{c}(t_1)e^{-t_1}dt_1} \).

**Proof.**

**Step 1:**

Fix any \( t_2 \geq 0 \), we firstly show that for any holomorphic \((n,0)\) form \( F \) defined on \( \{\psi < -t_2\} \) which satisfied \((\hat{F} - f) \in H^0(\{\psi < -t_2\} \cap U', \mathcal{O}(K_X) \otimes \mathcal{I}(\psi + \varphi)|_U) \) for some open set \( z_0 \subset U' \subset U \) and

\[
\int_{\{\psi < -t_2\}} c(-\psi)|F|^2e^{-\varphi}dV_X < +\infty.
\]

(3.27)

The follows inequality holds,

\[
\int_{\{\psi < -t_2\}} \hat{c}(-\psi)|\hat{F}|^2e^{-\varphi}dV_X < +\infty.
\]

As \( H(T;\hat{c},\varphi) < +\infty \), it follows from Lemma 2.25 that there exists a holomorphic \((n,0)\) form \( \hat{F} \) on \( \{\psi < -t_2\} \) which satisfies

\[
(\hat{F} - f) \in H^0(\{\psi < -t_2\} \cap U', \mathcal{O}(K_X) \otimes \mathcal{I}(\psi + \varphi)|_U)
\]

for some open set \( U \) such that \( z_0 \subset U \subset U \) and

\[
H(t_2;\hat{c},\varphi) = \int_{\{\psi < -t_2\}} \hat{c}(-\psi)|\hat{F}|^2e^{-\varphi}dV_X < +\infty.
\]
Let \( t_1 \) be big enough such that \( \{ \psi = kG_X(z, z_0) < -t_1 \} \subset U' \cap \bar{U} \) and \( \{ \psi < -t_1 \} \) is an relative compact open subset in \( X \) containing \( z_0 \). Then
\[
\int_{\{ \psi < -t_2 \}} \tilde{c}(\psi)|F|^2 e^{-\psi} dV_X = \int_{\{-t_1 \leq \psi < -t_2 \}} \tilde{c}(\psi)|F|^2 e^{-\psi} dV_X + \int_{\{ \psi < -t_1 \}} \tilde{c}(\psi)|F|^2 e^{-\psi} dV_X \tag{3.28}
\]
where \( I_1 = \int_{\{ \psi < -t_1 \}} \tilde{c}(\psi)|F|^2 e^{-\psi} dV_X \) and \( I_2 = \int_{\{ \psi < -t_1 \}} \tilde{c}(\psi)|F|^2 e^{-\psi} dV_X \).

Formula (3.27) implies that
\[
\int_{\{-t_1 \leq \psi < -t_2 \}} c(\psi)|F|^2 e^{-\psi} dV_X < +\infty. \tag{3.29}
\]

As \( c(t) \in G_0 \), it follows from condition (2) and (3) of \( G_0 \) that \( c(t) \neq 0 \) for any \( t > 0 \). \( c(t) \) is also smooth on \( [t_2, t_1] \), hence \( \inf_{t \in [t_2, t_1]} c(t) > 0 \). Then by inequality (3.29), we have
\[
\int_{\{-t_1 \leq \psi < -t_2 \}} |F|^2 e^{-\psi} dV_X < +\infty.
\]

Since \( \tilde{c}(t) \) is smooth on \( [t_2, t_1] \), we know
\[
I_1 \leq \left( \sup_{t \in [t_1, t_2]} \tilde{c}(t) \right) \int_{\{-t_1 \leq \psi < -t_2 \}} |F|^2 e^{-\psi} dV_X < +\infty. \tag{3.30}
\]

For \( I_2 \), we have
\[
I_2 \leq \int_{\{ \psi < -t_1 \}} \tilde{c}(\psi)|F - f|^2 e^{-\psi} dV_X + \int_{\{ \psi < -t_1 \}} \tilde{c}(\psi)|f|^2 e^{-\psi} dV_X \tag{3.31}
\]

where \( S_1 = \int_{\{ \psi < -t_1 \}} \tilde{c}(\psi)|F - f|^2 e^{-\psi} dV_X \) and \( S_2 = \int_{\{ \psi < -t_1 \}} \tilde{c}(\psi)|f|^2 e^{-\psi} dV_X \).

Note that \( c(t) \in G_0 \), we know \( \tilde{c}(t) < C e^t \) for some constant \( C > 0 \). It follows from \( (\tilde{F} - f) \in H^0(\{ \psi < -t_2 \} \cap U', \mathcal{O}(K_X) \otimes \mathcal{I}(\psi + \varphi)|_U) \) and \( \{ \psi < -t_1 \} \) is relatively compact in \( X \) that
\[
S_1 = \int_{\{ \psi < -t_1 \}} \tilde{c}(\psi)|F - f|^2 e^{-\psi} dV_X \leq C \int_{\{ \psi < -t_1 \}} e^{-\psi}|F - f|^2 e^{-\varphi} dV_X < +\infty.
\]

For \( S_2 \), we have
\[
S_2 \leq \int_{\{ \psi < -t_1 \}} \tilde{c}(\psi)|f - \tilde{F}|^2 e^{-\psi} dV_X + \int_{\{ \psi < -t_1 \}} \tilde{c}(\psi)|\tilde{F}|^2 e^{-\psi} dV_X \tag{3.32}
\]

It follows from the set \( \{ \psi < -t_1 \} \) is relatively compact in \( X \) and \((\tilde{F} - f) \in H^0(\{ \psi < -t_2 \} \cap U', \mathcal{O}(K_X) \otimes \mathcal{I}(\psi + \varphi)|_U) \)
for some open set \( \tilde{U} \) such that \( z_0 \subset \tilde{U} \subset U \) and
\[
H(t_2; \tilde{c}, \varphi) = \int_{\{ \psi < -t_2 \}} \tilde{c}(\psi)|\tilde{F}|^2 e^{-\varphi} dV_X < +\infty
\]
that we know $S_2 < +\infty$. Hence we have
\[
\int_{\{\psi < -t_2\}} \tilde{c}(-\psi)|\tilde{F}|^2e^{-\varphi}dV_X < +\infty.
\]

**Step 2:** The following proof is almost the same as the Step 2 in the proof of Corollary [1.4]

Given $t_2 \geq 0$. It follows from Lemma [2.3] that there exists a holomorphic $(n,0)$
form $\tilde{F}$ on $\{\psi < -t_2\}$ which satisfies
\[
(\tilde{F} - f) \in H^0(\{\psi < -t_2\} \cap \bar{U}, \mathcal{O}(K_X) \otimes \mathcal{I}(\psi + \varphi)|_U)
\]
for some open set $\bar{U}$ such that $\omega_0 \subset \bar{U} \subset U$ and
\[
H(t_2; \tilde{c}, \varphi) = \int_{\{\psi < -t_2\}} \tilde{c}(-\psi)|\tilde{F}|^2e^{-\varphi}dV_X < +\infty.
\]
It follows the result in Step 1 that $I(t) = \int_{\{\psi < -t\}} c(-\psi)|\tilde{F}|^2e^{-\varphi}dV_X < +\infty$, for any $t \geq t_2$. Fix $t_0 > t_1 \geq t_2$, Proposition [3.1] shows that
\[
\int_{\{-t_0 \leq \psi < -t_1\}} c(-\psi)|\tilde{F}|^2e^{-\varphi}dV_X \geq \int_{\{-t_0 \leq \psi < -t_1\}} c(-\psi)|\tilde{F}|^2e^{-\varphi}dV_X,
\]
the equality holds if and only if $\tilde{F} = F|_{\{\psi < -t_2\}}$. Hence we know
\[
\frac{I(t_1) - I(t_0)}{\int_{t_1}^{t_0} c(t)e^{-t}dt} \geq \frac{H(t_1; \tilde{c}) - H(t_0; \tilde{c})}{\int_{t_1}^{t_0} c(t)e^{-t}dt} = k,
\]
the equality holds if and only if $\tilde{F} = F|_{\{\psi < -t_2\}}$.

Note that we also have
\[
H(t_2; \tilde{c}) - H(t_1; \tilde{c}) \geq \int_{\{-t_1 \leq \psi < -t_2\}} \tilde{c}(-\psi)|\tilde{F}|^2e^{-\varphi}dV_X
\]
\[
= \sum_{i=1}^{n} \int_{\{-t_1 + (i-1)\frac{t_1 - t_2}{n} \leq \psi < -t_1 + i\frac{t_1 - t_2}{n}\}} \frac{\tilde{c}(-\psi)}{c(-\psi)}c(-\psi)|\tilde{F}|^2e^{-\varphi}dV_X
\]
(3.34)
As $c(t) \in \mathcal{G}_T$, it follows from condition (2) and (3) of $\mathcal{G}_T$ that $c(t) \neq 0$ for any $t \geq T$. Then $\frac{\tilde{c}(-\psi)}{c(-\psi)}$ is uniformly continuous on $[t_2, t_1]$. When $n$ big enough, we have
\[
H(t_2; \tilde{c}) - H(t_1; \tilde{c}) \geq \sum_{i=1}^{n} \left( \int_{\{-t_1 + (i-1)\frac{t_1 - t_2}{n} \leq \psi < -t_1 + i\frac{t_1 - t_2}{n}\}} c(-\psi)|\tilde{F}|^2e^{-\varphi}dV_X \right) \times
\]
\[
\frac{\tilde{c}(t_1 - i\frac{t_1 - t_2}{n})}{c(t_1 - i\frac{t_1 - t_2}{n})} - \frac{\tilde{c}(t_1 - i\frac{t_1 - t_2}{n})}{c(t_1 - i\frac{t_1 - t_2}{n})} - \frac{\tilde{c}(t_1 - i\frac{t_1 - t_2}{n})}{c(t_1 - i\frac{t_1 - t_2}{n})} - c
\]
\[
= S_{1,n} + S_{2,n}
\]
where
\[
S_{1,n} = \sum_{i=1}^{n} \left( \int_{\{-t_1 + (i-1)\frac{t_1 - t_2}{n} \leq \psi < -t_1 + i\frac{t_1 - t_2}{n}\}} c(-\psi)|\tilde{F}|^2e^{-\varphi}dV_X \right) \cdot \frac{\tilde{c}(t_1 - i\frac{t_1 - t_2}{n})}{c(t_1 - i\frac{t_1 - t_2}{n})},
\]
and
\[
S_{2,n} = -c \sum_{i=1}^{n} \left( \int_{\{-t_1 + (i-1)\frac{t_1 - t_2}{n} \leq \psi < -t_1 + i\frac{t_1 - t_2}{n}\}} c(-\psi)|\tilde{F}|^2e^{-\varphi}dV_X \right).
\]
It is easy to see that \( \lim_{n \to +\infty} S_{2,n} = 0 \). For \( S_{1,n} \), we have

\[
S_{1,n} = \sum_{i=1}^{n} I(t_1 - i \frac{t_1 - t_2}{n}) - I(t_1 - (i-1) \frac{t_1 - t_2}{n}) \times \\
\int_{t_1 - (i-1) \frac{t_1 - t_2}{n}}^{t_1 - i \frac{t_1 - t_2}{n}} c(t) e^{-t} dt \\
\left[ \frac{\int_{t_1 - (i-1) \frac{t_1 - t_2}{n}}^{t_1 - i \frac{t_1 - t_2}{n}} c(t) e^{-t} dt}{\int_{t_1 - (i-1) \frac{t_1 - t_2}{n}}^{t_1 - i \frac{t_1 - t_2}{n}} e^{-t_1 + i \frac{t_1 - t_2}{n} t_1 - t_2} e^{-t_1 + i \frac{t_1 - t_2}{n} t_1 - t_2} dt} \right] \\
\geq \sum_{i=1}^{n} k \left[ \frac{\int_{t_1 - (i-1) \frac{t_1 - t_2}{n}}^{t_1 - i \frac{t_1 - t_2}{n}} c(t) e^{-t} dt}{\int_{t_1 - (i-1) \frac{t_1 - t_2}{n}}^{t_1 - i \frac{t_1 - t_2}{n}} e^{-t_1 + i \frac{t_1 - t_2}{n} t_1 - t_2} e^{-t_1 + i \frac{t_1 - t_2}{n} t_1 - t_2} dt} \right] \\
(3.35)
\]

The “\( \geq \)” holds because of (3.33). Let \( n \to +\infty \) in (3.35) we have \( \lim_{n \to +\infty} S_{1,n} \geq k \int_{t_1}^{t_2} \tilde{c}(t) e^{-t} dt \). Hence we have

\[
H(t_2; \tilde{c}) - H(t_1; \tilde{c}) \geq k \int_{t_2}^{t_1} \tilde{c}(t) e^{-t} dt \\
i.e. \\
\frac{H(t_2; \tilde{c}) - H(t_1; \tilde{c})}{\int_{t_2}^{t_1} \tilde{c}(t) e^{-t} dt} \geq k.
\]

Let \( t_1 \to +\infty \), then

\[
\frac{H(t_2; \tilde{c})}{\int_{t_2}^{+\infty} \tilde{c}(t) e^{-t} dt} \geq k \\
(3.36)
\]

Recall that \( F \) is the holomorphic \((n,0)\) form on \( X \) such that \( H(t; c) = \int_{\psi < -t} c(-\psi) |F|^2 e^{-\varphi} dV_X \), for any \( t \geq T \). Let \( T \leq t_2 < t_1 < +\infty \), we have

\[
\frac{\int_{\{-t_2 \leq \psi < -t_2\}} c(-\psi) |F|^2 e^{-\varphi} dV_X}{\int_{t_2}^{t_1} c(t) e^{-t} dt} = k
\]

Note that

\[
\int_{\{-t_1 \leq \psi < -t_2\}} \tilde{c}(-\psi) |F|^2 e^{-\varphi} dV_X \\
= \sum_{i=1}^{n} \int_{\{-t_1 + (i-1) \frac{t_1 - t_2}{n} \leq \psi < -t_1 + i \frac{t_1 - t_2}{n}\}} \tilde{c}(-\psi) |F|^2 e^{-\varphi} dV_X \\
(3.37)
\]
Let \( n \) be big enough, the right hand side of (3.37) is bounded by

\[
\sum_{i=1}^{n} \left( \int_{t_i+(i-1)\frac{t_1-t_2}{n}}^{t_i} c(-\psi)|F|^2 e^{-r} dV_X \right) \left( \frac{\tilde{c}(t_1-i\frac{t_1-t_2}{n})}{c(t_1-i\frac{t_1-t_2}{n})} \pm \epsilon \right)
\]

\[
= \sum_{i=1}^{n} k \int_{t_1-(i-1)\frac{t_1-t_2}{n}}^{t_1-(i-1)\frac{t_1-t_2}{n}} c(t)e^{-t} dt \left( \frac{\tilde{c}(t_1-i\frac{t_1-t_2}{n})}{c(t_1-i\frac{t_1-t_2}{n})} \pm \epsilon \right)
\]

\[
= \sum_{i=1}^{n} \left[ k - \int_{t_1-(i-1)\frac{t_1-t_2}{n}}^{t_1-(i-1)\frac{t_1-t_2}{n}} c(t)e^{-t} dt \right] \left( \frac{\tilde{c}(t_1-i\frac{t_1-t_2}{n})}{c(t_1-i\frac{t_1-t_2}{n})} \pm \epsilon \right)
\]

\[
k \epsilon \int_{t_1-(i-1)\frac{t_1-t_2}{n}}^{t_1-(i-1)\frac{t_1-t_2}{n}} c(t)e^{-t} dt
\]

(3.38)

When \( n \to +\infty \), combining (3.37) and (3.38), we have

\[
\int_{\{\psi \leq -t_2\}} \tilde{c}(-\psi)|F|^2 e^{-r} dV_X = k \int_{t_2}^{t_1} \tilde{c}(t)e^{-t} dt
\]

(3.39)

Let \( t_1 \) goes to \( +\infty \) in (3.39), we know

\[
\frac{\int_{\{\psi < -t_2\}} \tilde{c}(-\psi)|F|^2 e^{-r} dV_X}{\int_{t_2}^{\infty} \tilde{c}(t)e^{-t} dt} = k
\]

Hence

\[
\frac{H(t_2; \tilde{c})}{\int_{t_2}^{\infty} \tilde{c}(t)e^{-t} dt} \leq \frac{\int_{\{\psi < -t_2\}} \tilde{c}(-\psi)|F|^2 e^{-r} dV_X}{\int_{t_2}^{\infty} \tilde{c}(t)e^{-t} dt} = k
\]

(3.40)

It follows from (3.39) and (3.40) that for any \( t_2 \geq T \),

\[
\frac{H(t_2; \tilde{c})}{\int_{t_2}^{\infty} \tilde{c}(t)e^{-t} dt} = k
\]

holds, i.e. \( H(h^{-1}(r); \tilde{c}) \) is linear with respect to \( r \). Hence there exists a holomorphic \((n,0)\) form \( F_{\tilde{c}} \) on \( X \) such that

\[
H(t_2; \tilde{c}) = \int_{\{\psi < -t_2\}} \tilde{c}(-\psi)|F_{\tilde{c}}|^2 e^{-r} dV_X
\]

and we also have

\[
\frac{H(t_2; \tilde{c}) - H(t_1; \tilde{c})}{\int_{t_2}^{t_1} \tilde{c}(t)e^{-t} dt} = \frac{\int_{\{\psi < -t_2\}} \tilde{c}(-\psi)|F_{\tilde{c}}|^2 e^{-r} dV_X}{\int_{t_2}^{t_1} \tilde{c}(t)e^{-t} dt}
\]

(3.41)

If \( F_{\tilde{c}} \neq F \) on \( X \), it follows from Proposition 3.31, 3.39 and 3.41 that

\[
k = \frac{\int_{\{\psi < -t_2\}} \tilde{c}(-\psi)|F|^2 e^{-r} dV_X}{\int_{t_2}^{t_1} \tilde{c}(t)e^{-t} dt} > \frac{\int_{\{\psi < -t_2\}} \tilde{c}(-\psi)|F_{\tilde{c}}|^2 e^{-r} dV_X}{\int_{t_2}^{t_1} \tilde{c}(t)e^{-t} dt} = k,
\]

which is a contradiction. Hence we must have \( F_{\tilde{c}} = F \) on \( X \).
Remark 3.3. Proposition 3.2 shows that if there exists $c_1(t) \in C^\infty(0, +\infty)$ and $c_1(t) \in \mathcal{G}_0$ such that $H(h_1^{-1}(r); c_1, \varphi)$ is linear with respect to $r$, where $h_1(t) = \int_t^{+\infty} c_1(t_1) e^{-t_1} dt_1$. Then for any other $c(t) \in C^\infty(0, +\infty)$ and $c(t) \in \mathcal{G}_0$, we know $H(h_1^{-1}(r); c, \varphi)$ is also linear with respect to $r$, where $h(t) = \int_t^{+\infty} c(t_1) e^{-t_1} dt_1$.

Let $c(t) \equiv 1$, then $h_1^{-1}(r) = \log r$. It follows from Proposition 3.2 that to prove Theorem 1.11 and Theorem 1.12, we only need to consider the necessary and sufficient condition for the function $H(-\log r; 1, \varphi)$ being linear with respect to $r$. We denote $H(t; 1, \varphi)$ by $H(t; \varphi)$ for simplicity.

Now we begin to prove Theorem 1.11.

As $\varphi$ is a plurisubharmonic function on $X$ and $i\partial \bar{\partial} \varphi \neq 0$ on $X$. By Siu’s decomposition theorem, we have

$$i \pi \partial \bar{\partial} \varphi = \sum_{j \geq 1} \lambda_j [x_j] + R, \quad \lambda_j > 0$$

where $x_j \in X$ is a point, $\lambda_j = v(i\partial \bar{\partial} \varphi, x_j)$ is the Lelong number of $i\partial \bar{\partial} \varphi$ at $x_j$, $R$ is a closed positive $(1, 1)$ current with $v(R, x) = 0$ for $x \in X$. Note that $E_1(T) = \{x \in X | v(i\partial \bar{\partial} \varphi, x) \geq 1\} = \{x_j | \lambda_j \geq 1\}$ is an analytic subset of $X$, hence $E_1(T)$ a set of isolated points. Denote $E := \{x \in X | v(T, x) is a positive integer\}, E \subset E_1(T)$ is also a set of isolated points.

We need the following Lemma to prove Theorem 1.11.

Lemma 3.4. If $(i\partial \bar{\partial} \varphi)|_{X \setminus E} \neq 0$. Then there exists a function $\hat{\varphi} \in PSH(X)$ $\hat{\varphi} \geq \varphi$ and $\mathcal{I}(2\hat{\varphi})_x = \mathcal{I}(2\varphi)_x$ for any $x \in X$. Let $z \in \{-t < kG_X(z, z_0) < 0\}$, when $t \to 0$, we have $\hat{\varphi}(z) \to \varphi(z)$. Moreover, there exists a relatively compact open subset $U \subset X$ such that $\varphi - \hat{\varphi}$ has lower bound $-A$ ($A > 0$ is a constant) for any $z \in X \setminus U$.

Lemma 3.4 will be proved in the Appendix (see Section 4.2). Now we prove Theorem 1.11.

Proof of Theorem 1.11. We only need to show that if $H(-\log r; 2\varphi)$ is linear with respect to $r$, then we have $\varphi = \log |f_\varphi| + v$, where $f_\varphi$ is a holomorphic function on $X$ and $v$ is a harmonic function on $X$.

Our proof will be divided into two steps.

Step 1:

In step 1, we will show that if $(i\partial \bar{\partial} \varphi)|_{X \setminus E} \neq 0$, then $H(-\log r; 2\varphi)$ can not be linear with respect to $r \in (0, 1]$.

Assume that $H(-\log r; 2\varphi)$ is linear with respect to $r \in (0, 1]$. As $H(-\log r; 2\varphi)$ is linear with respect to $r$, it follows from Theorem 1.10 that there exists a holomorphic $(1, 0)$ form $F$ on $X$ such that $\forall t \geq 0$,

$$H(t; \varphi) = \int_{\{\psi < -t\}} |F|^2 e^{-2\varphi} dV_X$$

holds. As $e^{-2\hat{\varphi}} < e^{-2\varphi}$, we have

$$k = \frac{H(t; 2\varphi)}{e^{-t}} > \frac{\int_{\{\psi < -t\}} |F|^2 e^{-2\varphi} dV_X}{e^{-t}} \geq \frac{H(t; 2\varphi)}{e^{-t}}$$

(3.42)

When $t = 0$, there exists a holomorphic $(1, 0)$ form $\tilde{F}$ on $X$ such that

$$H(t; 2\varphi) = \int_X |\tilde{F}|^2 e^{-2\varphi} dV_X < +\infty$$
By Lemma [3.3], there exist $U \subset X$ such that $\varphi - \tilde{\varphi}$ has lower bound $-A$ ($A > 0$ is a constant) for any $z \in X \setminus U$.

Denote

$$ I_1 = \int_U |\tilde{F}|^2 e^{-2\varphi} dV_X $$

and

$$ I_2 = \int_{X \setminus U} |\tilde{F}|^2 e^{-2\varphi} dV_X. $$

As $U$ is relatively compact in $X$, $\int_U |\tilde{F}|^2 e^{-2\varphi} dV_X < +\infty$ and $I(2\tilde{\varphi})_x = I(2\varphi)_x$, for any $x \in X$, then we know

$$ I_1 = \int_U |\tilde{F}|^2 e^{-2\varphi} dV_X < +\infty. $$

On $X \setminus U$, we have

$$ I_2 = \int_{X \setminus U} |\tilde{F}|^2 e^{-2\varphi} dV_X \leq e^{2A} \int_{X \setminus U} |\tilde{F}|^2 e^{-2\varphi} dV_X < +\infty. $$

Hence

$$ \int_X |\tilde{F}|^2 e^{-2\varphi} dV_X = \int_U |\tilde{F}|^2 e^{-2\varphi} dV_X + \int_{X \setminus U} |\tilde{F}|^2 e^{-2\varphi} dV_X = I_1 + I_2 < +\infty. $$

Let $t_1 > 0$ be small enough such that $|\tilde{\varphi} - \varphi(z)| < \epsilon$, then we have

$$ H(0; 2\tilde{\varphi}) - H(t_1; 2\tilde{\varphi}) \geq \int_{\{t_1 \leq \psi < 0\}} |\tilde{F}|^2 e^{-2\varphi} dV_X $$

$$ \geq e^{-2\epsilon} \int_{\{-t_1 \leq \psi < 0\}} |F|^2 e^{-2\varphi} dV_X $$

$$ \geq e^{-2\epsilon} \int_{\{-t_1 \leq \psi < 0\}} |F|^2 e^{-2\varphi} dV_X $$

$$ = e^{-2\epsilon} (H(0; 2\varphi) - H(t_1; 2\varphi)). $$

The third “$\geq$” holds because of Proposition 3.1. Hence

$$ \lim_{t_1 \to 0} \frac{H(0; 2\tilde{\varphi}) - H(t_1; 2\tilde{\varphi})}{1 - e^{-t_1}} \geq \lim_{t_1 \to 0} \frac{H(0; 2\varphi) - H(t_1; 2\varphi)}{1 - e^{-t_1}} = k. $$

It follows from (3.42), (3.44) and Theorem 1.1 that

$$ k > \frac{H(t; 2\tilde{\varphi})}{e^{-t}} \geq \lim_{t_1 \to 0} \frac{H(0; 2\tilde{\varphi}) - H(t_1; 2\tilde{\varphi})}{1 - e^{-t_1}} \geq k $$

which is a contradiction.

Hence $H(-\log r; 2\varphi)$ can not be linear with respect to $r$.

**Step 2:**

It follows from the result in Step 1 and $H(-\log r; \varphi)$ is linear with respect to $r$ that we know

$$ \frac{i}{\pi} \partial \bar{\partial} \varphi = \sum_{j \geq 1} \lambda_j [x_j], $$

where $\lambda_j$ is positive integer for any $j \geq 1$.

It follows from the Weierstrass Theorem on noncompact Riemann surface (see [15] chapter 3, §26), for divisor $D = \sum_{j \geq 1} \lambda_j x_j$, there exist a meromorphic function
Proposition 3.5. Assume that \( (f_\varphi) = D \). As \( \lambda_j > 0 \), \( f \) is actually a holomorphic function on \( X \). It follows from Lelong-Poincaré equation that

\[
\frac{i}{\pi} \partial \bar{\partial} \log |f_\varphi| = \sum_{j \geq 1} \lambda_j |x_j|.
\]

Then \( i\partial \bar{\partial} \varphi - i\partial \bar{\partial} \log |f_\varphi| = 0 \), i.e., \( u = \varphi - \log |f_\varphi| \) is a harmonic function.

Hence \( \varphi = \log |f_\varphi| + u \), where \( f_\varphi \) is a holomorphic function on \( X \) and \( u \) is a harmonic function on \( X \). Theorem 1.11 is proved.

Now we begin to prove Theorem 1.12.

Recall that \( X \) is an open Riemann Surface which admits a nontrivial Green function \( G_X(z, w) \) and \( \psi = 2G_X(z, z_0) \), where \( z_0 \) is a point of \( X \).

Let \( w \) be a local coordinate on a neighborhood \( V_{z_0} \) of \( z_0 \) satisfying \( w(z_0) = 0 \).

Let \( U = V_{z_0} \). Let \( f \) be a holomorphic \((n, 0)\) form on \( V_{z_0} \).

Let \( \varphi = \log |f_\varphi| + v \) on \( X \), where \( f_\varphi \) is a holomorphic function on \( X \) and \( v \) is a harmonic function on \( X \). Let \( c(t) \in G_0 \). Denote

\[
H(t; c, 2\varphi) := \inf \left\{ \int_{\{\psi < -t\}} c(-\psi) |\bar{F}|^2 e^{-2\varphi} dV_X : \bar{F} \in H^0(\{\psi < -t\}, \mathcal{O}(K_X)), \right. \\
&\left. \quad \quad \quad \quad \quad \text{and } \exists \text{ open set } U' \text{ s.t } Z_0 \subset U' \subset U \text{ and } \\
&\left. \quad \quad \quad \quad \quad (\bar{F} - f) \in H^0(\{\psi < -t\} \cap U', \mathcal{O}(K_X) \otimes \mathcal{I}(\psi + 2\varphi)|_U) \right\}.
\]

We assume that \( 0 < H(0; c, 2\varphi) < +\infty \).

It follows from Lemma 2.5 that there exists a unique holomorphic \((n, 0)\) form \( F_0 \) on \( X \) satisfying

\[
(F_0 - f) \in H^0(\{\psi < -t\} \cap U'_0, \mathcal{O}(K_X) \otimes \mathcal{I}(\psi + 2\varphi)|_U),
\]

for some open set \( U'_0 \) such that \( z_0 \subset U'_0 \subset U \) and

\[
\int_X |F_0|^2 e^{-2\varphi} c(-\psi) dV_X = H(0) < +\infty.
\]

As \( \int_X |F_0|^2 e^{-2\varphi} c(-\psi) dV_X = \int_X \frac{|F_0|^2}{|f_\varphi|^2} e^{-2\varphi} c(-\psi) dV_X < +\infty \), we know \( \frac{f_0}{f_\varphi} \) is a holomorphic \((n, 0)\) form on \( X \). It follows from

\[
(F_0 - f) \in H^0(\{\psi < -t\} \cap U'_0, \mathcal{O}(K_X) \otimes \mathcal{I}(\psi + 2\varphi)|_U)
\]

that there exist a small open neighborhood \( V \) such that \( \frac{f_0}{f_\varphi} \) is a holomorphic \((n, 0)\) form on \( V \). Denote \( h := \frac{f_0}{f_\varphi} \), we know \( h \) is a holomorphic \((n, 0)\) form on \( V \). We also note that \( h(z_0) \neq 0 \), otherwise \( f = h \cdot f_\varphi \) will belong to \( \mathcal{I}(\psi + 2\varphi)|_V \) which contradict to the fact that \( H(0; c, 2\varphi) > 0 \).

We have the following limiting property of \( H(t; c, 2\varphi) \).

**Proposition 3.5.** Assume that \( 0 < H(0; c, 2\varphi) < +\infty \). When \( t \to +\infty \), we have

\[
\lim_{t \to +\infty} \frac{H(t; c, 2\varphi)}{\int_0^t e^{-ct} c(t_1) e^{-t_1} dt_1} = \pi \frac{e^{-2\varphi(z_0)}}{c_\varphi^2(z_0)} |h(z_0)|^2.
\]
Hence we have the fourth inequality holds because of mean value inequality of subharmonic function.

For an $t \geq 0$, denote

$$I_t = \int_{\{\log |w|^2 + u(w) < -t\}} c(-2G_X(z, z_0))|F_t|^2 e^{-2\varphi}dV_X,$$

where $F_t$ is holomorphic $(1, 0)$ form on $\{2G_X(z, z_0) < -t\}$ such that

$$H(t) = \int_{\{2G_X(z, z_0) < -t\}} |F_t|^2 e^{-2\varphi}dV_X < +\infty$$

and

$$(F_t - f) \in H^0(\{\psi < -t\} \cap U'_1, O(K_X) \otimes \mathcal{I}(\psi + 2\varphi)|_\nu)$$

for some open set $U'_1$ such that $z_0 \in U'_1 \subset U$.

Denote $h_t = \frac{F_t}{g}$, it follows from (3.44) and (3.45) that we know $h_t$ is a holomorphic $(n, 0)$ form on $\{\psi < -t\}$ and $h_t(z_0) = h(z_0)$.

When $t$ is big enough, we know $|w|$ is small. By the continuity of $u$ and $v$ at $z_0$ and note that $|h_t|^2$ is subharmonic function, we have

$$I_t = \int_{\{\log |w|^2 + u(w) < -t\}} c(-2G_X(z, z_0))|F_t|^2 e^{-2\varphi}dV_X$$

$$\geq \int_{\{\log |w|^2 + u(z_0) + \varepsilon < -t\}} c(-2G_X(z, z_0))|h_t|^2 e^{-2\varphi}dV_X$$

$$\geq \int_{\{\log |w|^2 + u(z_0) + \varepsilon < -t\}} c(-\log |w|^2 - u(w))|h_t|^2 e^{-2\varphi(z_0) - \varepsilon}dV_X$$

$$\geq \int_{\{\log |w|^2 + u(z_0) + \varepsilon < -t\}} c(-\log |w|^2 - u(z_0) + \varepsilon)e^{-2\varphi} |h_t|^2 e^{-2\varphi(z_0) - \varepsilon}dV_X$$

$$= \int_0^{2\pi} \int_{\{\log |r|^2 + u(z_0) + \varepsilon < -t\}} c(-\log |r|^2 - u(z_0) + \varepsilon)e^{-2\varphi} |h_t(r, \theta)|^2 e^{-2\varphi(z_0) - \varepsilon}r dr d\theta$$

$$\geq 2\pi e^{-2\varphi(z_0)} e^{-3\varepsilon} |h(z_0)|^2 \int_{\{\log |r|^2 + u(z_0) + \varepsilon < -t\}} c(-\log |r|^2 - u(z_0) + \varepsilon) r dr$$

$$= \pi e^{-2\varphi(z_0)} e^{-3\varepsilon} |h(z_0)|^2 \int_{t-\varepsilon}^{+\infty} c(t_1)e^{-t_1} e^{-u(z_0)} e^{\varepsilon} dt_1$$

The third inequality holds because of $c(t)e^{-t}$ is decreasing with respect to $t$. The fourth inequality holds because of mean value inequality of subharmonic function. Hence we have

$$\liminf_{t \to +\infty} \frac{I_t}{\int_t^{+\infty} c(t_1)e^{-t_1} dt_1} \geq \liminf_{t \to +\infty} \frac{\pi e^{-2\varphi(z_0)} e^{-3\varepsilon} |h(z_0)|^2 \int_{t-\varepsilon}^{+\infty} c(t_1)e^{-t_1} e^{-u(z_0)} e^{\varepsilon} dt_1}{\int_t^{+\infty} c(t_1)e^{-t_1} dt_1}$$

$$= \pi e^{-u(z_0)-2\varphi(z_0)} |h(z_0)|^2$$

$$= \frac{\pi}{c_{\beta}^2(z_0)} |h(z_0)|^2$$

(3.48)
When \( t = 0 \), denote \( S_t = \int_{\{\psi < -t\}} c(-\psi)|F_0|^2 e^{-2\varphi}dV_X \). When \( t \) is big enough, we know \(|u|\) is small. By the continuity of \( u, v \) and \( h_0 = \frac{f_\varphi}{f_\psi} \) at \( z_0 \), then we have

\[
S_t = \int_{\{\psi < -t\}} c(-\psi)|F_0|^2 e^{-2\varphi}dV_X = \int_{\{\psi < -t\}} c(-\psi)|h_0|^2 e^{-2\psi}dV_X \leq \int_{\{\log |w|^2 + u(z_0) - \epsilon < -t\}} c(-2G_X(z, z_0))|h_0|^2 e^{-2\psi}dV_X \leq \int_{\{\log |w|^2 + u(z_0) - \epsilon < -t\}} c(-\log |w|^2 - u(z_0) - \epsilon)|h_0|^2 e^{-2\psi}dV_X \leq \int_0^{2\pi} \int_{\{\log |r|^2 + u(z_0) - \epsilon < -t\}} c(-\log |r|^2 - u(z_0) - \epsilon)|h_0(r, \theta)|^2 e^{-2\psi}d\theta dt \leq 2\pi e^{-2\psi} e^{3\epsilon} (|h(z_0)|^2 + \epsilon) \int_{\{\log |r|^2 + u(z_0) - \epsilon < -t\}} c(-\log |r|^2 - u(z_0) - \epsilon) d\theta dt \]

The second inequality holds because of \( c(t)e^{-t} \) is decreasing with respect to \( t \).

Hence

\[
\limsup_{t \to +\infty} \frac{H(t; c, 2\varphi)}{\int_t^{+\infty} c(t_1)e^{-t_1}dt_1} \leq \limsup_{t \to +\infty} \frac{\int_t^{+\infty} c(t_1)e^{-t_1}dt_1}{\int_t^{+\infty} c(t_1)e^{-t_1}dt_1} \leq \frac{\pi e^{-2\psi}(z_0)}{c_3(z_0)} |h(z_0)|^2 \]

(3.49)

It follows from inequality (3.48) and (3.49) that

\[
\lim_{t \to +\infty} \frac{H(t; c, 2\varphi)}{\int_t^{+\infty} c(t_1)e^{-t_1}dt_1} = \frac{\pi e^{-2\psi(z_0)}}{c_3(z_0)} |h(z_0)|^2
\]

Proposition 3.6 is proved. \( \square \)

Recall that \( \varphi \) is a subharmonic function on \( X \), such that \( \varphi = \log |f_\varphi| + v \), where \( f_\varphi \) is a holomorphic function on \( X \) and \( v \) is a harmonic function on \( X \).

We also note that \( h := \frac{f_\varphi}{f_\psi} \) is a holomorphic \((n, 0)\) form on \( V \subset V_{z_0} \) and \( h(z_0) \neq 0 \).

Denote

\[
H(t; 2\psi) := \inf \{ \int_{\{\psi < -t\}} |\tilde{F}|^2 e^{-2\psi}dV_X : \tilde{F} \in H^0(\{\psi < -t\}, \mathcal{O}(K_X)), \exists \text{ open set } U' \text{ s.t. } Z_0 \subset U' \subset U \text{ and } (\tilde{F} - h) \in H^0(\{\psi < -t\} \cap U', \mathcal{O}(K_X) \otimes \mathcal{I}(\psi)|_U) \}
\]

(3.50)

Proposition 3.6. We have \( H(0; 2\psi) = H(0; 2\varphi) \) holds.

Proof. Denote

\[
H_1 := \{ F \in H^0(X, K_X) | \int_X |F|^2 e^{-2\varphi}dV_X < +\infty \& (F - f, z_0) \in \mathcal{I}(\psi + 2\varphi)_{z_0} \}
\]

and

\[
H_2 := \{ \tilde{F} \in H^0(X, K_X) | \int_X |\tilde{F}|^2 e^{-2\psi}dV_X < +\infty \& (\tilde{F} - h, z_0) \in \mathcal{I}(\psi)_{z_0} \}
\]
As \( \int_X |F|^2 e^{-2\varphi} dV_X = \int_X \frac{|F|^2}{|f_\varphi|^2} e^{-2v} dV_X < +\infty \), we know for any \( F \in H_1 \), \( \frac{F}{f_\varphi} \) is a holomorphic \((1,0)\) form on \( X \). It follows from \( \varphi = \log |f_\varphi| + v \) and \( (F-f_\varphi z_0) \in \mathcal{I}(\psi + 2\varphi)_z \) that we know \( \frac{F}{f_\varphi} \) belongs to \( H_2 \). For any \( F \in H_2 \), \( F \cdot f_\varphi \) belongs to \( H_1 \) for the similar reason.

Hence there exists a bijection \( \Phi \) between \( H_1 \) and \( H_2 \):

\[ \Phi : H_2 \to H_1 \]
\[ F \to F \cdot f_\varphi \]

It follows from Lemma 2.5 that there exist unique holomorphic \((1,0)\) form \( F_\varphi \in H_1 \) such that

\[ H(0, \varphi) = \int_X |F_\varphi|^2 e^{-2\varphi} dV_X, \]

and unique holomorphic \((1,0)\) form \( F_v \in H_2 \) such that

\[ H(0, v) = \int_X |F_v|^2 e^{-2v} dV_X. \]

We claim that \( F_\varphi = F_v \cdot f_\varphi \) i.e the weighted \( L^2 \) norm of \( F_v \cdot f_\varphi \) is minimal along \( H_1 \). If not, we have

\[ \int_X |F_\varphi|^2 e^{-2\varphi} dV_X < \int_X |F_v|^2 |f_\varphi|^2 e^{-2\varphi} dV_X = \int_X |F_v|^2 e^{-2v} dV_X. \]

Note that \( \frac{F_\varphi}{f_\varphi} \in H_2 \) and then we have

\[ \int_X \frac{|F_\varphi|^2}{|f_\varphi|^2} e^{-2v} dV_X = \int_X |F_\varphi|^2 e^{-2\varphi} dV_X < \int_X |F_v|^2 e^{-2\varphi} dV_X. \]

which contradicts to the fact that the weighted \( L^2 \) norm of \( F_v \) is minimal along \( H_2 \).

Hence we must have \( F_\varphi = F_v \cdot f_\varphi \). Then we know

\[ H(0; 2\varphi) = \int_X |F_\varphi|^2 e^{-2\varphi} dV_X = \int_X |F_v|^2 e^{-2v} dV_X = H(0; 2v). \]

Proposition 3.6 is proved. \( \square \)

**Remark 3.7.** It follows from Proposition 3.6 that we know when \( \varphi = \log |f_\varphi| + v \), where \( f_\varphi \) is a holomorphic function on \( X \) and \( v \) is a harmonic function on \( X \), we have \( H(0; 2\varphi) = \pi \frac{e^{-2\varphi(z_0)}}{c^2_\beta(z_0)} |h(z_0)|^2 \) if and only if \( H(0; 2v) = \pi \frac{e^{-2v(z_0)}}{c^2_\beta(z_0)} |h(z_0)|^2 \).

Then it follows from Theorem 1.2, Proposition 3.5 and \( "H(0; 2\varphi) = \pi \frac{e^{-2\varphi(z_0)}}{c^2_\beta(z_0)} |h(z_0)|^2 \) if and only if \( H(0; 2v) = \pi \frac{e^{-2v(z_0)}}{c^2_\beta(z_0)} |h(z_0)|^2 \) and we know \( H(-\log r; 2\varphi) \) is linear with respect to \( r \in (0, 1) \) if and only if \( H(-\log r; 2v) \) is linear with respect to \( r \in (0, 1) \).

Let \( v \) be a harmonic function on \( X \), we have the following result which was proved by Guan-Zhou [21, 22].

**Theorem 3.8.** (Guan-Zhou [21, 22]) The equality

\[ H(0; 2v) = \pi \frac{e^{-2v(z_0)}}{c^2_\beta(z_0)} |h(z_0)|^2 \]

holds if and only if \( \chi - v = \chi z_0 \).

Now we prove Theorem 1.12.
Proof. It follows from Proposition 3.2 that we only need to prove the following statement:

\[ H(-\log r; 2\varphi) \] is linear with respect to \( r \in (0, 1] \) if and only if the following hold,

1. \( \varphi = \log |f_{\varphi}| + v \), where \( f_{\varphi} \) is a holomorphic function on \( X \) and \( v \) is a harmonic function on \( X \).

2. \( \chi_{-v} = \chi_{z_0} \).

If \( H(-\log r; 2\varphi) \) is linear with respect to \( r \in (0, 1] \). It follows from Theorem 3.11 that \( \varphi = \log |f_{\varphi}| + v \), where \( f_{\varphi} \) is a holomorphic function on \( X \) and \( v \) is a harmonic function on \( X \). As \( H(-\log r; 2\varphi) \) is linear with respect to \( r \), then by the Remark 3.7, we know \( H(-\log r; 2v) \) is linear with respect to \( r \), hence

\[ H(0; 2v) = \pi \frac{e^{-2v(z_0)}}{c^2(z_0)} |h(z_0)|^2. \]

Then it follows form Theorem 3.8 that we know \( \chi_{-v} = \chi_{z_0} \).

If \( \varphi = \log |f_{\varphi}| + v \), where \( f_{\varphi} \) is a holomorphic function on \( X \) and \( v \) is a harmonic function on \( X \) and \( \chi_{-v} = \chi_{z_0} \). It follows form Theorem 3.8 that we know

\[ H(0; 2v) = \pi \frac{e^{-2v(z_0)}}{c^2(z_0)} |h(z_0)|^2. \]

Hence \( H(-\log r; 2v) \) is linear with respect to \( r \). By Remark 3.7, we know \( H(-\log r; 2\varphi) \) is linear with respect to \( r \).

Theorem 1.12 is proved. \( \square \)

4. Appendix

4.1. Appendix: Proof of Lemma 2.1.

4.1.1. Some results used in the proof of Lemma 2.1.

Lemma 4.1. (see [7]) Let \( Q \) be a Hermitian vector bundle on a Kähler manifold \( X \) of dimension \( n \) with a kähler metric \( \omega \). Assume that \( \eta, g > 0 \) are smooth functions on \( X \). Then for every form \( v \in D(X, \wedge^n g T^*X \otimes Q) \) with compact support we have

\[
\int_X (\eta + g^{-1})|D'' v|^2_Q dV_X + \int_X \eta |D'' v|^2_Q dV_X \\
\geq \int_X ([\eta \sqrt{-1} \Theta_Q - \sqrt{-1} g \partial \eta - \sqrt{-1} g \partial \eta \wedge \bar{\partial}, \Lambda_\omega] v, v)_Q dV_X 
\]

(4.1)

Lemma 4.2. (see [22]) Let \( X \) and \( Q \) be as in the above lemma and \( \theta \) be a continuous \((1,0)\) form on \( X \). Then we have

\[
[\sqrt{-1} \operatorname{div} Q - \sqrt{-1} g \partial \eta - \sqrt{-1} g \partial \eta \wedge \bar{\partial}, \Lambda_\omega] \]

(4.2)

for any \((n,1)\) form \( \alpha \) with value in \( Q \). Moreover, for any positive \((1,1)\) form \( \beta \), we have \([\beta, \Lambda_\omega] \) is semipositive.

Lemma 4.3. (see [5, 8]) Let \((X, \omega)\) be a complete kähler manifold equipped with a (non-necessarily complete) kähler metric \( \omega \), and let \( Q \) be a Hermitian vector bundle over \( X \). Assume that \( \eta \) and \( g \) are smooth bounded positive functions on \( X \) and let \( B := [\eta \sqrt{-1} \Theta_Q - \sqrt{-1} g \partial \eta - \sqrt{-1} g \partial \eta \wedge \bar{\partial}, \Lambda_\omega] \). Assume that \( B \) is semipositive definite everywhere on \( \wedge^n g T^*X \otimes Q \) for some \( q \geq 1 \). Then given a form
$g \in L^2(X, \wedge^n \alpha^* T^* X \otimes Q)$ such that $D'' g = 0$ and $\int_X (B^{-1} g, g) dV_X < +\infty$, there exists $u \in L^2(X, \wedge^n \alpha^* T^* X \otimes Q)$ such that $D'' u = g$ and

$$\int_X (\eta + g^{-1})^{-1}|u|^2_Q dV_X \leq \int_X (B^{-1} g, g) dV_X \quad (4.3)$$

In the last part of this section, we recall a theorem of Fornæss and Narasimhan on approximation property of plurisubharmonic function of Stein manifolds.

**Lemma 4.4.** (see [14]) Let $X$ be a Stein manifold and $\varphi \in PSH(X)$. Then there exists a sequence $\{\varphi\}_{n=1,2,...}$ of smooth strongly plurisubharmonic functions such that $\varphi_n \downarrow \varphi$.

4.1.2. Proof of Lemma 2.1. Since $X$ is Stein manifold, there exists a smooth plurisubharmonic exhaustion function $P$ on $X$. Let $X_k := \{P < k\} \ (k = 1, 2, ..., )$, we choose $P$ such that $X_1 \neq \emptyset$.

Then $X_k$ satisfies $X_1 \subset X_2 \subset ... \subset X_k \subset X_{k+1} \subset ...$ such that $\bigcup_{k=1}^\infty X_k = X$ and each $X_k$ is Stein manifold with exhaustion plurisubharmonic function $P_k = 1/\{k - P\}$. We will discuss for fixed $k$ until step 8.

**Step 1: Regularization of $\psi$ and $\varphi$**

It follows from Lemma 4.4 that there exist smooth strongly plurisubharmonic functions $\psi_m$ and $\varphi_{m'}$ on $X$ decreasing convergent to $\psi$ and $\varphi$ respectively, satisfying $\sup_m \sup_{X_k} \psi_m < -T$ and $\sup_{m'} \sup_{X_k} \varphi_{m'} < +\infty$.

**Step 2: Recall some notations**

Let $\epsilon \in (0, \frac{1}{2}B)$. Let $\{v_\epsilon\}_{\epsilon \in (0, \frac{1}{2}B)}$ be a family of smooth increasing convex functions on $\mathbb{R}$, such that:

1. $v_\epsilon(t) = t$ for $t \geq -t_0 - \epsilon$, $v_\epsilon(t) = \text{constant for } t < -t_0 - B + \epsilon$;
2. $v_\epsilon''(t)$ are pointwise convergent to $\frac{1}{B} \mathbb{1}_{(-t_0 - B, -t_0)}$, when $\epsilon \to 0$, and $0 \leq v_\epsilon''(t) \leq \frac{2}{B} \mathbb{1}_{(-t_0 - B, -t_0)}$ for ant $t \in \mathbb{R}$;
3. $v_\epsilon'(t)$ are pointwise convergent to $b(t)$ which is a continuous function on $\mathbb{R}$ when $\epsilon \to 0$ and $0 \leq v_\epsilon'(t) \leq 1$ for any $t \in \mathbb{R}$.

One can construct the family $\{v_\epsilon\}_{\epsilon \in (0, \frac{1}{2}B)}$ by the setting

$$v_\epsilon(t) := \int_{-\infty}^t \left( \int_{-\infty}^t \frac{1}{B - 4\epsilon} \mathbb{1}_{(-t_0 - B + 2\epsilon, -t_0 - 2\epsilon)} * \rho_{\frac{1}{4}\epsilon}(s) ds \right) dt_1$$

where $\rho_{\frac{1}{4}\epsilon}$ is the kernel of convolution satisfying supp($\rho_{\frac{1}{4}\epsilon}$) $\subset (-\frac{1}{2}\epsilon, \frac{1}{2}\epsilon)$. Then it follows that

$$v_\epsilon''(t) = \frac{1}{B - 4\epsilon} \mathbb{1}_{(-t_0 - B + 2\epsilon, -t_0 - 2\epsilon)} * \rho_{\frac{1}{4}\epsilon}(t) \quad (4.5)$$

and

$$v_\epsilon'(t) = \int_{-\infty}^t \left( \int_{-\infty}^t \frac{1}{B - 4\epsilon} \mathbb{1}_{(-t_0 - B + 2\epsilon, -t_0 - 2\epsilon)} * \rho_{\frac{1}{4}\epsilon}(s) ds \right) dt \quad (4.6)$$

Let $\eta = s(-v_\epsilon(\psi_m))$ and $\phi = u(-v_\epsilon(\psi_m))$, where $s \in C^\infty((T, +\infty))$ satisfies $s \geq 0$ and $u \in C^\infty((T, +\infty))$ satisfies $\lim_{t \to +\infty} u(t)$ exists, such that $u'' s - s'' > 0$ and
Hence (4.10), we have

\[ s' - u's = 1. \]

It follows form sup sup \( \psi_m < -T \) that \( \phi = u(-v_c(\psi_m)) \) are uniformly bounded on \( X_k \) with respect to \( m \) and \( \epsilon \), and \( u(-v_c(\psi)) \) are uniformly bounded on \( X_k \) with respect to \( \epsilon \). Let \( \Phi = \phi + \varphi_m' \) and let \( \tilde{h} = e^{-\Phi} \).

**Step 3: Solving \(\bar{\partial}\)-equation with smooth polar function and smooth weight**

Set \( B = [\eta \sqrt{1 - \Theta_{\tilde{h}}} - \sqrt{1 - \tilde{\partial}\tilde{\partial} - g\tilde{\partial}\eta} \wedge \tilde{\partial}\eta, \Lambda_{\omega}] \), where \( g \) is a positive continuous function on \( X_k \). We will determine \( g \) by calculations. As

\[
\begin{align*}
\eta \Theta_{\tilde{h}} &:= \eta \partial\bar{\partial} + \eta \partial\bar{\partial} \varphi_m' \\
&= su''(v_c(\psi_m))\partial(\psi_m) - su'(v_c(\psi_m))\partial\bar{\partial}(\psi_m) \\
&\quad + s\partial\bar{\partial}(\psi_m').
\end{align*}
\]

Hence

\[
\begin{align*}
\eta \sqrt{1 - \Theta_{\tilde{h}}} - \sqrt{1 - \tilde{\partial}\tilde{\partial} - g\tilde{\partial}\eta} &\wedge \tilde{\partial}\eta \\
&= s\sqrt{1 - \tilde{\partial}\tilde{\partial} \varphi_m' + (s' - su')v_m'v_m'v_m'\sqrt{1 - \tilde{\partial}\tilde{\partial}(\psi_m) + v_m''v_m''\sqrt{1 - \partial\bar{\partial}(\psi_m) + \tilde{\partial}(\psi_m))} \\
&\quad + [(u''s - s'') - g\eta^2]\partial(\psi_m) \wedge \tilde{\partial}(\psi_m)
\end{align*}
\]  

Let \( g = v''(v_m')^2(v_m) \) and note that \( s' - s'' = 1, v_m'(\psi_m') \geq 0 \). Then

\[
\begin{align*}
\eta \sqrt{1 - \Theta_{\tilde{h}}} - \sqrt{1 - \tilde{\partial}\tilde{\partial} - g\tilde{\partial}\eta} &\wedge \tilde{\partial}\eta \\
&= s\sqrt{1 - \tilde{\partial}\tilde{\partial} \varphi_m' + v_m'(v_m)\sqrt{1 - \tilde{\partial}\tilde{\partial}(\psi_m) + v_m''(v_m)'\sqrt{1 - \partial\bar{\partial}(\psi_m) + \tilde{\partial}(\psi_m))} \\
&\geq v_m''(v_m')\sqrt{1 - \tilde{\partial}(\psi_m) \wedge \tilde{\partial}(\psi_m)
\end{align*}
\]

Hence

\[
\langle (B\alpha, \alpha) \rangle_{\tilde{h}} \geq \langle [v''(v_m)(\partial(\psi_m) \wedge \tilde{\partial}(\psi_m), \Lambda_{\omega}]\alpha, \alpha \rangle_{\tilde{h}}
\]

(4.10)

By Lemma 4.2 \( B \) is semipositive.

Using the definition of contraction, Cauchy-Schwarz inequality and the inequality (4.10), we have

\[
\begin{align*}
|\langle v''(v_m)(\partial\psi_m \wedge \gamma, \bar{\alpha}) \rangle_{\tilde{h}}|^2 &= |\langle v''(v_m)(\gamma, \bar{\partial}(\psi_m)\gamma) \rangle_{\tilde{h}}|^2 \\
&\leq \langle (v''(v_m)(\gamma, \bar{\partial}(\psi_m)\gamma))_{\tilde{h}}(v''(v_m)(\gamma, \bar{\partial}(\psi_m)\gamma))_{\tilde{h}} \rangle_{\tilde{h}}^2 \\
&\leq \langle (v''(v_m)(\gamma, \bar{\partial}(\psi_m)\gamma))_{\tilde{h}} \rangle_{\tilde{h}} (B\tilde{\alpha}, \tilde{\alpha})_{\tilde{h}} \\
&\leq \langle (v''(v_m)(\gamma, \bar{\partial}(\psi_m)\gamma))_{\tilde{h}} \rangle_{\tilde{h}} (B\tilde{\alpha}, \tilde{\alpha})_{\tilde{h}} \\
&\leq \langle (B^{-1}\lambda, \lambda) \rangle_{\tilde{h}} \leq v''(v_m)|F|^2 e^{-\Phi}
\end{align*}
\]  

(4.12)
Then it follows that
\[
\int_{X_k\setminus E_{\delta m}(T)} \langle \mathbf{B}^{-1} \lambda, \lambda \rangle \leq \int_{X_k\setminus E_{\delta m}(T)} v''(\psi_m)|F|^2 e^{-\Phi} < +\infty \tag{4.13}
\]
Using Lemma 4.3 there exists \( u_{k,m,m',\varepsilon} \in L^2(X_k, K_X) \) such that
\[
D'' u_{k,m,m',\varepsilon} = \lambda \tag{4.14}
\]
and
\[
\int_{X_k} \frac{1}{\eta + g^{-1}} |u_{k,m,m',\varepsilon}|^2 e^{-\Phi} \leq \int_{X_k} \langle \mathbf{B}^{-1} \lambda, \lambda \rangle \leq \int_{X_k} v''(\psi_m)|F|^2 e^{-\Phi} \tag{4.15}
\]
Note that \( g = \frac{w''(z)}{w'(z)}(-v_c(\psi_m)) \). Assume that we can choose \( \eta \) and \( \phi \) such that
\[
e^v(\psi_m) e^\phi c(-v_c(\psi_m)) = (\eta + g^{-1})^{-1}.
\]
Then inequality (4.15) becomes
\[
\int_{X_k} |u_{k,m,m',\varepsilon}|^2 e^{v(\psi_m) - \varphi_{m'}} c(-v_c(\psi_m)) \leq \int_{X_k} v''(\psi_m)|F|^2 e^{-\phi - \varphi_{m'}} < +\infty \tag{4.16}
\]
Let \( F_{k,m,m',\varepsilon} := -u_{k,m,m',\varepsilon} + (1 - v'(\psi_m))F \). Then inequality (4.16) becomes
\[
\int_{X_k} |F_{k,m,m',\varepsilon}|^2 e^{v(\psi_m) - \varphi_{m'}} c(-v_c(\psi_m)) \leq \int_{X_k} v''(\psi_m)|F|^2 e^{-\phi - \varphi_{m'}} \tag{4.17}
\]
Step 4: Singular polar function and smooth weight
As \( \sup_{m,\varepsilon} |\phi| = \sup_{m,\varepsilon} |u(-v_c(\psi_m))| < +\infty \) and \( \varphi_{m'} \) is continuous on \( \overline{X_k} \), then \( \sup_{m,\varepsilon} e^{-\phi - \varphi_{m'}} < +\infty \). Note that
\[
v''(\psi_m)|F|^2 e^{-\phi - \varphi_{m'}} \leq \frac{2}{B} \int_{\{\psi < -t_0\}} |F|^2 e^{\sup_{m,\varepsilon} e^{-\phi - \varphi_{m'}}}
\]
on \( X_k \), then it follows from \( \int_{\{\psi < -t_0\}} |F|^2 < +\infty \) and the dominated convergence theorem that
\[
\lim_{m \to +\infty} \int_{X_k} v''(\psi_m)|F|^2 e^{-\phi - \varphi_{m'}} = \int_{X_k} v''(\psi)|F|^2 e^{-u(-v_c(\psi)) - \varphi_{m'}} \tag{4.18}
\]
Note that \( \inf_{m,\varepsilon} \inf_{X_k} e^{v(\psi_m) - \varphi_{m'}} c(-v_c(\psi_m)) > 0 \), then it follows from inequality (4.17) and (4.18) that \( \sup_{m} \int_{X_k} |F_{k,m,m',\varepsilon}|^2 < +\infty \). Note that
\[
|\{1 - v'(\psi_m)\}| \leq \int_{\{\psi < -t_0\}} |F|,
\]
then it follows from \( \int_{\{\psi < -t_0\}} |F|^2 < +\infty \) that \( \sup_{m} \int_{X_k} |F_{k,m,m',\varepsilon}|^2 < +\infty \), which implies that there exists a subsequence of \( \{F_{k,m,m',\varepsilon}\}_m \) (also denoted by \( F_{k,m,m',\varepsilon} \)) compactly convergent to a holomorphic \( F_{k,m,m',\varepsilon} \) on \( X_k \).
Note that \( v_c(\psi_m) - \varphi_{m'} \) are uniformly bounded on \( X_k \) with respect to \( m \), then it follows from \( |F_{k,m,m',\varepsilon}|^2 < +\infty \) that \( \sup_{m} \int_{X_k} |F_{k,m,m',\varepsilon}|^2 < +\infty \), which implies that there exists a subsequence of \( \{F_{k,m,m',\varepsilon}\}_m \) (also denoted by \( F_{k,m,m',\varepsilon} \)) compactly convergent to a holomorphic \( F_{k,m,m',\varepsilon} \) on \( X_k \).
Note that \( v_c(\psi_m) - \varphi_{m'} \) are uniformly bounded on \( X_k \) with respect to \( m \), then it follows from \( |F_{k,m,m',\varepsilon}|^2 < +\infty \) that \( \sup_{m} \int_{X_k} |F_{k,m,m',\varepsilon}|^2 < +\infty \), which implies that there exists a subsequence of \( \{F_{k,m,m',\varepsilon}\}_m \) (also denoted by \( F_{k,m,m',\varepsilon} \)) compactly convergent to a holomorphic \( F_{k,m,m',\varepsilon} \) on \( X_k \).
holds for any compact subset $K$ of $X_k$. Combing with inequality (1.17) and (4.18), one can obtain that

$$
\int_K |F_{k,m',\epsilon} - (1 - v'_\epsilon(\psi))F|^2 e^{v_\epsilon(\psi) - \varphi_{m'} c(-v_\epsilon(\psi))}
\leq \int_{X_k} v''_\epsilon(\psi)|F|^2 e^{-u(-v_\epsilon(\psi)) - \varphi_{m'}}
$$

(4.21)

which implies

$$
\int_{X_k} |F_{k,m',\epsilon} - (1 - v'_\epsilon(\psi))F|^2 e^{v_\epsilon(\psi) - \varphi_{m'} c(-v_\epsilon(\psi))}
\leq \int_{X_k} v''_\epsilon(\psi)|F|^2 e^{-u(-v_\epsilon(\psi)) - \varphi_{m'}}
$$

(4.22)

**Step 5: Nonsmooth cut-off function**

Note that $\sup_e \sup_{X_k} e^{-u(-v_\epsilon(\psi)) - \varphi_{m'}} < +\infty$ and

$$
v''_\epsilon(\psi)|F|^2 e^{-u(-v_\epsilon(\psi)) - \varphi_{m'}} \leq \frac{2}{B} \|\{ -t_0 - B < \psi < -t_0 \}|F|^2 \sup_e e^{-u(-v_\epsilon(\psi)) - \varphi_{m'}}
$$

then it follows from $\int_{\{\psi < -t_0\}} |F|^2 < +\infty$ and the dominated convergence theorem that

$$
\lim_{\epsilon \to 0} \int_{X_k} v''_\epsilon(\psi)|F|^2 e^{-u(-v_\epsilon(\psi)) - \varphi_{m'}}
= \int_{X_k} \frac{1}{B} \|\{ -t_0 - B < \psi < -t_0 \}|F|^2 e^{-u(-v(\psi)) - \varphi_{m'}}
$$

(4.23)

$$
\leq (\sup_{X_k} e^{-u(-v(\psi))}) \int_{X_k} \frac{1}{B} \|\{ -t_0 - B < \psi < -t_0 \}|F|^2 e^{-\varphi_{m'}} < +\infty
$$

Note that $\inf_{X_k} e^{v_\epsilon(\psi) - \varphi_{m'} c(-v_\epsilon(\psi))} > 0$, then it follows from inequality (4.22) and (4.23) that $\sup_e \int_{X_k} |F_{k,m',\epsilon} - (1 - v'_\epsilon(\psi))F|^2 < +\infty$. Combing with

$$
\sup_e \int_{X_k} |1 - v'_\epsilon(\psi)F|^2 \leq \int_{X_k} \|\{ \psi < -t_0 \} |F|^2 < +\infty
$$

(4.24)

one can obtain that $\sup_e \int_{X_k} |F_{k,m',\epsilon}|^2 < +\infty$, which implies that there exists a subsequence of $\{F_{k,m',\epsilon}\}_{\epsilon \to 0}$ (also denoted by $F_{k,m',\epsilon}$) compactly convergent to a holomorphic $F_{k,m'}$ on $X_k$.

Note that $\sup_e \sup_{X_k} e^{v_\epsilon(\psi) - \varphi_{m'} c(-v_\epsilon(\psi))} < +\infty$ and $|F_{k,m',\epsilon} - (1 - v'_\epsilon(\psi)(m))F|^2 \leq 2(\|F_{k,m',\epsilon}\|^2 + \|\{ \psi < -t_0 \} F|^2)^2$, then it follows from inequality (4.24) and dominated convergence theorem on any given $K \subset X_k$, with dominant function

$$
2(\sup_e \sup_{X_k} |F_{k,m',\epsilon}| + \|\{ \psi < -t_0 \} F|^2) \sup_e e^{v_\epsilon(\psi) - \varphi_{m'} c(-v_\epsilon(\psi))}
$$

that

$$
\lim_{\epsilon \to 0} \int_K |F_{k,m',\epsilon} - (1 - v'_\epsilon(\psi))F|^2 e^{v_\epsilon(\psi) - \varphi_{m'} c(-v_\epsilon(\psi))}
\leq \int_K |F_{k,m'} - (1 - b(\psi))F|^2 e^{v(\psi) - \varphi_{m'} c(-v(\psi))}
$$

(4.25)
Combining with inequality (4.23) and (122), one can obtain that

$$\int_K |F_{k,m'} - (1 - b(\psi))F|^2 e^{v(\psi)} \, \varphi_{m'} c(-v(\psi)) \leq (\sup_{X_k} e^{-u(-v(\psi))}) \int_{X_k} \frac{1}{B} \mathbb{1}_{(-t_0-B<\psi<-t_0)} |F|^2 e^{-\varphi} \tag{4.26}$$

which implies

$$\int_{X_k} |F_{k,m'} - (1 - b(\psi))F|^2 e^{v(\psi)} \, \varphi_{m'} c(-v(\psi)) \leq (\sup_{X_k} e^{-u(-v(\psi))}) \int_{X_k} \frac{1}{B} \mathbb{1}_{(-t_0-B<\psi<-t_0)} |F|^2 e^{-\varphi} \tag{4.27}$$

**Step 6: Singular weight**

Note that

$$\int_{X_k} \frac{1}{B} \mathbb{1}_{(-t_0-B<\psi<-t_0)} |F|^2 e^{-\varphi} \leq \int_{X_k} \frac{1}{B} \mathbb{1}_{(-t_0-B<\psi<-t_0)} |F|^2 e^{\varphi} < +\infty \tag{4.28}$$

and $\sup_{X_k} e^{-u(-v(\psi))} < +\infty$ then it follows from (4.27) that

$$\sup_{m'} \int_{X_k} |F_{k,m'} - (1 - b(\psi))F|^2 e^{v(\psi)} \, \varphi_{m'} c(-v(\psi)) < +\infty.$$

Combining with $\inf_{m'} \inf_{X_k} e^{v(\psi)} \varphi_{m'} c(-v(\psi)) > 0$, we know $\sup_{m'} \int_{X_k} |F_{k,m'} - (1 - b(\psi))F|^2 < +\infty$. Note that

$$\int_{X_k} |(1 - b(\psi))F|^2 \leq \int_{X_k} \mathbb{1}_{(\psi<-t_0)} F|^2 < +\infty \tag{4.29}$$

Then $\sup_{m'} \int_{X_k} |F_{k,m'}|^2 < +\infty$, which implies that there exists a compactly convergence subsequence of $\{F_{k,m'}\}$ denoted by $\{F_{k,m''}\}$ which converge to a holomorphic $(n,0)$ form on $X_k$ denoted by $F_k$.

Note that $\sup_{m'} \sup_{X_k} e^{v(\psi)} \varphi_{m'} c(-v(\psi)) < +\infty$, then it follows (4.29) and the dominated convergence theorem on any given compact subset $K$ of $X_k$ with dominant function

$$2(\sup_{m'} \sup_{K} |F_{k,m'}|^2) + \mathbb{1}_{(\psi<-t_0)} |F|^2 \sup_{X_k} e^{v(\psi)} \varphi_{m'} c(-v(\psi)) \tag{4.30}$$

that

$$\lim_{m'' \to +\infty} \int_K |F_{k,m''} - (1 - b(\psi))F|^2 e^{v(\psi)} \varphi_{m'} c(-v(\psi)) = \int_K |F - (1 - b(\psi))F|^2 e^{v(\psi)} \varphi_{m'} c(-v(\psi)) \tag{4.31}$$
Note that for \( m'' \geq m' \), \( \varphi_{m'} \leq \varphi_{m''} \) holds, then it follows from (4.27) and (4.28) that
\[
\lim_{m'' \to +\infty} \int_K |F_{k,m''} - (1 - b(\psi))F|^2 e^{v(\psi) - \varphi_{m''}} c(-v(\psi)) \\
\leq \limsup_{m'' \to +\infty} \int_K |F_{k,m''} - (1 - b(\psi))F|^2 e^{v(\psi) - \varphi_{m''}} c(-v(\psi)) \\
\leq \limsup_{m'' \to +\infty} (\sup_{X_k} e^{-u(-v(\psi))}) \int_{X_k} \frac{1}{B} \mathbb{I}_{\{-t_0 - B < \psi < -t_0\}} |F|^2 e^{-\varphi_{m''}} \leq (\sup_{X_k} e^{-u(-v(\psi))}) \int_{X_k} \frac{1}{B} \mathbb{I}_{\{-t_0 - B < \psi < -t_0\}} |F|^2 e^{-\varphi} < +\infty
\]
Combining with equality (4.31), one can obtain that
\[
\int_K |F_k - (1 - b(\psi))F|^2 e^{v(\psi) - \varphi_{m'}} c(-v(\psi)) \\
\leq (\sup_{X_k} e^{-u(-v(\psi))}) \int_{X_k} \frac{1}{B} \mathbb{I}_{\{-t_0 - B < \psi < -t_0\}} |F|^2 e^{-\varphi} < +\infty
\]
for any compact subset \( K \) of \( X_k \), which implies
\[
\int_{X_k} |F_k - (1 - b(\psi))F|^2 e^{v(\psi) - \varphi_{m'}} c(-v(\psi)) \\
\leq (\sup_{X_k} e^{-u(-v(\psi))}) \int_{X_k} \frac{1}{B} \mathbb{I}_{\{-t_0 - B < \psi < -t_0\}} |F|^2 e^{-\varphi} < +\infty
\]
When \( m' \to +\infty \), it follows from Levi’s theorem that
\[
\int_{X_k} |F_k - (1 - b(\psi))F|^2 e^{v(\psi) - \varphi} c(-v(\psi)) \\
\leq (\sup_{X_k} e^{-u(-v(\psi))}) \int_{X_k} \frac{1}{B} \mathbb{I}_{\{-t_0 - B < \psi < -t_0\}} |F|^2 e^{-\varphi} \tag{4.33}
\]

**Step 7: ODE System**

We want to find \( \eta \) and \( \phi \) such that \( (\eta + g^{-1}) = e^{-v_s(\psi_m)} e^{-\phi} = \frac{1}{e^{2c(\psi_m)}} \). As \( \eta = s(-v_s(\psi_m)) \) and \( \phi = u(-v_s(\psi_m)) \), we have \( (\eta + g^{-1}) e^{v_s(\psi_m)} e^\phi = (s + \frac{s^2}{s'' - s'}) e^{-t} e^{u(t)} \).

Summarizing the above discussion about \( s \) and \( u \), we are naturally led to a system of ODEs:
\[
1) (s + \frac{s^2}{u''s - s''}) e^{u(t)} = \frac{1}{c(t)} \\
2) s' - su' = 1 \tag{4.34}
\]
when \( t \in (T, +\infty) \).

It is not hard to solve the ODE system (4.34) and get \( u(t) = -\log(\int_t^t c(t_1) e^{-t_1 dt_1}) \) and \( s(t) = \frac{\int_0^t c(t_1) e^{-t_1 dt_1} dt_1}{\int_0^t c(t_1) e^{-t_1 dt_1}} \). It follows that \( s \in C^\infty((T, +\infty)) \) satisfies \( s \geq 0 \), \( \lim_{t \to +\infty} u(t) = -\log(\int_T^\infty c(t_1) e^{-t_1 dt_1}) \) exists and \( u \in C^\infty((T, +\infty)) \) satisfies \( u'' s - s'' > 0 \).
As \( u(t) = -\log(\int_1^t c(t_1)e^{-t_1}dt_1) \) is decreasing with respect to \( t \), then it follows from \( -T \geq v(t) \geq \max\{t, -t_0 - B_0\} \geq -t_0 - B_0 \) for any \( t \leq 0 \) that

\[
\sup_{X_k} e^{-u(-v(\psi))} \leq \sup_{X} e^{-u(-v(\psi))} \leq \sup_{t \in (T, t_0 + B]} e^{-u(t)} = \int_T^{t_0 + B} c(t_1)e^{-t_1}dt_1
\]

Hence on \( X_k \), we have

\[
\int_{X_k} |F_k - (1 - b(\psi))F|^2 e^{v(\psi)} - \varphi c(-v(\psi)) \\
\leq \int_T^{t_0 + B} c(t_1)e^{-t_1}dt_1 \int_{X_k} 1_B\{t_0 - B < \psi < -t_0\} |F|^2 e^{-\varphi} \tag{4.35}
\]

\[
\leq C \int_T^{t_0 + B} c(t_1)e^{-t_1}dt_1
\]

**Step 8:** When \( k \to +\infty \).

Note that for any given \( k \), \( e^{-\varphi + v(\psi)} c(-v(\psi)) \) has a positive lower bound on \( \tilde{X}_k \), then it follows [4.35] that for any given \( k \), \( \int_{X_k} |F_{k'} - (1 - b(\psi))F|^2 \) is bounded with respect to \( k' \geq k \). Combining with

\[
\int_{X_k} |1 - b(\psi)F|^2 \leq \int_{X_k \cap \{\psi < -t_0\}} |F|^2 < +\infty \tag{4.36}
\]

One can obtain that \( \int_{X_k} |F_{k'}|^2 \) is bounded with respect to \( k' \geq k \).

By diagonal method, there exists a subsequence \( F_{k''} \) uniformly converge on any \( \tilde{X}_k \) to a holomorphic \((n,0)\)-form on \( X \) denoted by \( \tilde{F} \). Then it follow from inequality [4.35], [4.36] and the dominated convergence theorem that

\[
\int_{X_k} |\tilde{F} - (1 - b(\psi))F|^2 e^{-\max(\varphi - v(\psi), -M)} c(-v(\psi)) \\
\leq C \int_T^{t_0 + B} c(t_1)e^{-t_1}dt_1 \tag{4.37}
\]

for any \( M > 0 \), then Levi’s theorem implies

\[
\int_{X_k} |\tilde{F} - (1 - b(\psi))F|^2 e^{-(\varphi - v(\psi))} c(-v(\psi)) \\
\leq C \int_T^{t_0 + B} c(t_1)e^{-t_1}dt_1 \tag{4.38}
\]

Let \( k \to +\infty \), Lemma 2.1 is proved.

### 4.2. Appendix: Proof of Lemma 3.4

We will need the following results in our proof of Lemma 3.4.

**Lemma 4.5.** (see Chapter 3, Corollary 2.14 of [4]) Let \( A \) be an analytic subset of \( X \) with global irreducible components \( A_j \) of pure dimension \( p \). Then any closed current \( \Theta \in D_{p, h}(X) \) of order 0 with support in \( A \) is of the form \( \Theta = \sum \lambda_j A_j \) where \( \lambda_j \in \mathbb{C} \).

**Lemma 4.6.** Let \( X \) be an open Riemann Surface which admits a nontrivial Green function \( G_X(z, w) \). Given \( z_1 \in X \). Let \( U \) be a relatively compact open subset of
Proof of Lemma 4.7. For the convenience of readers, we give another proof as below.

Let \( (V_1, w) \) be a small local coordinate neighborhood of \( z_1 \) such that \( w(z_1) = 0 \). By definition, \( G_X(z, z_1) = -\sup_{v(z) \in \Delta(z_1)} v(z) \) where \( \Delta(z_1) \) is the set of negative subharmonic functions on \( X \) satisfying that \( v - \log |w| \) has locally finite upper bound near \( z_1 \).

As \( \tilde{G}(z) = G_X(z, z_1) \) on \( U \), we know \( \tilde{G}(z) \in \Delta(z_1) \), but \( \tilde{G}(z) > G_X(z, z_1) \) on \( W \) which is contradict to the fact that \( G_X(z, z_1) = \sup_{v(z) \in \Delta(z_1)} v(z) \). Hence for any \( z \in X \setminus U \), we have \( G_X(z, z_1) \geq A \).

\( \square \)

Lemma 4.7. Let \( X \) be an open Riemann Surface which admits a nontrivial Green function \( G_X(z, w) \). Fix \( z_1 \in x \), there exists open subsets \( V_1, U_1 \) which satisfy \( z_1 \in V_1 \subset X \subset U_1 \subset X \) and a constant \( N > 0 \) such that \( \forall (z, w) \in (X \setminus U_1) \times V_1 \), we have

\[
G_X(z, w) \geq NG(z, z_1).
\]

Lemma 4.7 can be deduced from the Harnack inequality of harmonic function. For the convenience of readers, we give another proof as below.

Proof of Lemma 4.7. Let \((V_1, w)\) be a small local coordinate neighborhood of \( z_1 \) such that \( V_1 \subset X \), \( w(z_1) = 0 \) and \( G(z, z_1)|_{V_1} = \log |w| \).

Let \( V_1 \subset U_1 \subset V_{z_1} \) such that

\[
\sup_{z \in V_1} G(z, z_1) = -t_0
\]

and

\[
\sup_{z \in \overline{V_1}} G(z, z_1) = -t_0 - 1
\]

for some \( t_0 \geq 0 \). Denote \( W = \{ z \in V_1 | -t_0 - \frac{1}{4} < G_X(z, z_1) < -t_0 \} \). Then it is easy to see that \( W \subset V_{z_1} \) and \( W \cap V_1 = \emptyset \).

Note that when \( z \in W, w \in V_1 \), \( G_X(z, w) \) is smooth. Hence \( G_X(z, w) \) has a lower bounded \( B \) on \( W \times V_1 \). Denote \( a = \sup_{z \in W} G_X(z, z_1) \). Let \( N \) be big enough such that \( B \geq Na \).

Fix \( w \in V_1 \). We will show for any \( z \in X \setminus U_1 \),

\[
G_X(z, w) \geq NG_X(z, z_1).
\]
If not, there exists \( z_0 \in X \setminus U_1 \) such that \( G_X(z_0, w) < NG_X(z_0, z_1) \). Note that both \( G_X(z, w) \) and \( G_X(z, z_1) \) are smooth on \( X \setminus U_1 \). Hence there exists a open neighborhood \( H_{z_0} \) of \( z_0 \) such for any \( z \in H_{z_0} \), we have
\[
G_X(z, w) < NG_X(z, z_1).
\]

Let
\[
G_0(z) = \begin{cases} 
G_X(z, w) & \text{for } z \in \overline{U_1} \\
\max\{G_X(z, w), NG_X(z, z_1)\} & \text{for } z \in X \setminus \overline{U_1}
\end{cases}
\]

Then \( G_0(z) \) is a nonnegative subharmonic function on \( X \). Note that \( w \in V_1 \subset U_1 \) and we have \( G_0(z) = G_X(z, w) \) on \( U_1 \). But \( G_0(z) > G_X(z, w) \) on \( H_{z_0} \), this contradicts to the fact that \( G_X(z, w) \) is the Green function of \( X \) with pole at \( w \).

Hence \( G_X(z, w) \geq NG_X(z, z_1) \) for any \( z \in X \setminus U_1 \) holds. As \( w \in V_1 \) is arbitrarily fixed, we know for any \( (z, w) \in (X \setminus U_1) \times V_1 \), we have
\[
G_X(z, w) \geq NG_X(z, z_1).
\]

Lemma 4.7 is proved. \( \square \)

Now we begin to prove Lemma 4.7.

**Proof of Lemma 4.7** Recall that by Siu’s decomposition theorem, we have
\[
\frac{i}{\pi} \partial \bar{\partial} \varphi = \sum_{j \geq 1} \lambda_j [x_j] + R, \quad \lambda_j > 0
\]
where \( x_j \in X \) is a point, \( \lambda_j = v(i\partial \bar{\partial} \varphi, x_j) \) is the Lelong number of \( i\partial \bar{\partial} \varphi \) at \( x_j \), \( R \) is a closed positive \((1,1)\) current with \( v(R, x) = 0 \) for \( x \in X \).

Recall that both \( E_1(i\partial \bar{\partial} \varphi) = \{ x \in X | v(i\partial \bar{\partial} \varphi, x) \geq 1 \} \) and \( E = \{ x \in X | v(i\partial \bar{\partial} \varphi, x) \) is a positive integer \} are sets of isolated points. As \( (i\partial \bar{\partial} \varphi)|_{X \setminus E} \neq 0 \), there are two cases:

1. There exists \( \lambda_{j_0} \) such that \( \lambda_{j_0} - [\lambda_{j_0}] > 0 \), where \( [\lambda_{j_0}] \) is the largest integer smaller than \( \lambda_{j_0} \).

2. \( R \neq 0 \).

For the case (1):

Let \( p = x_{j_0} \). Let \((U, z)\) be a relative compact coordinate neighborhood of \( p \) in \( X \) and by shrinking \( U \), we assume that under the local homomorphism, \( z(p) = o \) and \( z(U) \simeq B(0, 2) \). We also assume that \( U \cap \{ E_1(i\partial \bar{\partial} \varphi) \setminus \{ x_{j_0} \} \} = \emptyset \). Let \( \theta \) be a smooth cut-off function on \( X \) such that \( 0 \leq \theta \leq 1 \) on \( \{ x_{j_0} \} \), \( supp(\theta) \subset V \subset U \) and \( \theta \equiv 1 - \frac{|x_{j_0}|}{\lambda_{j_0}} \) on \( W \), where \( z(W) = B(0, \frac{1}{4}) \) and \( z(V) = B(0, \frac{1}{4}) \) under the local homomorphism. By shrinking \( V \) and \( U \), it follows from Lemma 4.7 that there exists \( N > 0 \) such that for any \( (z, w) \in (X \setminus U) \times V \), we have
\[
G_X(z, w) \geq NG_X(z, x_{j_0}).
\]

Let \( T = \theta \cdot i\partial \bar{\partial} \varphi \), then \( T \) is a closed positive \((1,1)\) current on \( X \) with support \( supp T \subset \subset V \).

Let \( \rho \in C^\infty(\mathbb{C}) \) be a function with \( supp \rho \subset B(0, 1) \) and \( \rho(z) \) depends only on \( |z|, \rho \geq 0 \) and \( \int_C \rho(z) d\lambda_z = 1 \). Let \( \rho_n(z) = \frac{1}{n} \rho(\frac{z}{n}), \rho_n \) is a family of smoothing kernels.

Let \( T_n = T \ast \rho_n \) be the convolution of \( T \). For any test function \( h \in C^\infty(X) \), as \( T \) has compact support and \( supp T \subset \subset V \subset \subset U \), we can restrict \( h \) to \( U \) and denote \( h|_U \) still by \( h \) for simplicity. By the definition of convolution of currents, we have
\[ \langle T_n(w), h(w) \rangle := \langle T(w), h * \rho_n(w) \rangle. \] Note that \( \text{supp} T \subset \subset V \), the convolution \( h * \rho_n(w) \) is well defined for \( w \in V \).

We restrict \( 2G_X(z, w) \) to \( U \) and denote \( 2G_X(z, w)|_U \) still by \( 2G_X(z, w) \) for simplicity. Let \( u_n(z) = \langle T_n(w), 2G_X(z, w) \rangle \). For fixed \( z \) and fixed \( n \), we will prove
\[ \langle T_n(w), 2G_X(z, w) \rangle = \langle T(w), 2G_X(z, w) * \rho_n \rangle. \]

For fixed \( z \), \( G_X(z, w) \) is a subharmonic function on \( X \). There exists a sequence of smooth subharmonic functions \( G_m(w) \) decreasingly converge to \( G_X(z, w) \) with respect to \( m \). We still denote \( G_m(w)|_U \) by \( G_m(w) \). As \( G_m(w) \) is smooth, we have
\[ \langle T_n(w), 2G_m(w) \rangle = \langle T(w), 2G_m * \rho_n(w) \rangle \quad (4.39) \]

For fixed \( n \), \( T_n(w) \) is a smooth positive \( (1, 1) \)-form on \( X \) with \( \text{supp} T_n \subset \subset U \). As \( G_m(w) \) decreasingly converge to \( G_X(z, w) \) with respect to \( m \), it follows from Levi’s theorem that
\[ \lim_{m \to +\infty} \langle T_n(w), 2G_m(w) \rangle = \langle T_n(w), 2G_X(z, w) \rangle \quad (4.40) \]

For fixed \( n \), as \( G_m(w) \) decreasingly converge to \( G_X(z, w) \) with respect to \( m \) and \( \rho_n \) has compact support, we know \( (2G_m * \rho_n)(w) \) decreasingly converge to \( (2G_X(z, w) * \rho_n)(w) \) with respect to \( m \). Note that \( T \) is a positive \((1, 1)\) current on \( X \) with compact support, hence \( T \) is of order 0. It follows from Levi’s theorem that
\[ \lim_{m \to +\infty} \langle T(w), (2G_m * \rho_n)(w) \rangle = \langle T(w), (2G_X(z, w) * \rho_n)(w) \rangle. \quad (4.41) \]

For fixed \( z \) and fixed \( n \), it follows from equality \((4.39),(4.40)\) and \((4.41)\) that we have \( \langle T_n(w), 2G_X(z, w) \rangle = \langle T(w), (2G_X(z, w) * \rho_n)(w) \rangle \).

As \( 2G_X(z, w) \) is subharmonic, then \( 2G_X(z, w) * \rho_n \) converges to \( 2G_X(z, w) \) decreasingly with respect to \( n \). Note that \( T \) is a positive \((1, 1)\) current on \( X \), hence \( u_n(z) \) is decreasing with respect to \( n \) and \( u_n(z) < 0 \). Let \( u(z) = \lim_{n \to +\infty} u_n(z) \). We know \( u(z) < 0 \).

Now we show that both \( \{u_n\} \) and \( u \) is \( L_{lo}^1 \) function on \( X \). Let \((K,z)\) be a relatively compact open neighborhood of some point \( z' \) in \( X \) such that under the local coordinate \( z \), we have \( z(z') = 0 \) and \( K \cong B(0,1) \). Let \( d\lambda_z \) be the lebesgue measure \( B(0,1) \).

Note that for fixed \( w \), \( 2G_X(z, w) \) is smooth outside \( z = w \) and \( 2G_X(z, w) = 2 \log |z - w| + 2u(z) \) on a small neighborhood \( B(w, \epsilon_0) \) of \( w \), where \( u(z) \) is a smooth function on \( B(w, \epsilon_0) \). We also note that \( \int_{z \in B(w, \epsilon_0)} |2 \log |z-w|| d\lambda_z < +\infty \). It follows from Fubini theorem that
\[
\|u_n\|_{L^1(K)} = \int_{z \in K} \left( \int_{w \in U} 2|G_X(z, w)| T_n(w) d\lambda_z \right) d\lambda_z = \int_{w \in U} \left( \int_{z \in K} 2|G_X(z, w)| d\lambda_z \right) T_n(w)
\]

Let \( H(w) = \int_{z \in K} 2|G_X(z, w)| d\lambda_z \). If \( \bar{U} \cap \bar{K} = \emptyset \), then \( G_X(z, w) \) is smooth on \( \bar{U} \times \bar{K} \), hence \( H(w) \) is uniformly bounded on \( w \in U \).

When \( \bar{U} \cap \bar{K} \neq \emptyset \), as \( U, K \) is small, we assume that there exists an open subset \( J \subset \subset X \), such that the set \( K + U := \{z+w | z \in K \text{ and } w \in U\} \) is contained in \( J \) and \( \text{we have } G_X(z, w) = \log |z-w| + u(z, w) \) for \( (z, w) \in J \times J \). Here, when \( w \) is fixed, \( u(z, w) \) is harmonic function on \( z \in J \) and when \( z \) is fixed, \( u(z, w) \) is
harmonic function on \( w \in J \). We have
\[
H(w) = \int_{z \in K} 2|G_X(z, w)|d\lambda_z
= \int_{z \in K} -2G_X(z, w)d\lambda_z
= \int_{z \in K} -2\log|z - w| - 2u(z, w)d\lambda_z
= I_1(w) + I_2(w)
\]
where \( I_1(w) = \int_{z \in K} -2\log|z - w|d\lambda_z \) and \( I_2(w) = \int_{z \in K} -2u(z, w)d\lambda_z \). For \( I_1(w) \), we have
\[
I_1(w) = \int_{z \in K} -2\log|z - w|d\lambda_z
= \int_{z \in K+\{w\}} -2\log|z|d\lambda_z
\geq \int_{z \in J} -2\log|z|d\lambda_z
\]
where the set \( K + \{w\} := \{z + w|z \in K\} \). Note that \( \log|z| \) is integrable near \( z = 0 \) and \( J \) is relative compact in \( X \), hence there exists a constant \( M_1 > 0 \) such that \( I_1(w) \leq M_1 \) for any \( w \in U \).

For \( I_2(w) \), by the mean value equality of harmonic function, we have
\[
I_2(w) = \int_{z \in B(0, 1)} -2u(z, w)d\lambda_z
= -2\pi u(z', w)
\]
As \( u(z', w) \) is harmonic on \( w \in U \) and \( U \) is relatively compact in \( X \), we know \( I_2(w) \) is bounded on \( \bar{U} \).

The above discussion shows that the function \( H(w) = \int_{z \in K} 2|G_X(z, w)|d\lambda_z = I_1 + I_2 \) is bounded by some constant \( N \) on \( U \). Let \( \chi \) be a \( C^\infty_c(X) \) such that \( 0 \leq \chi \leq 1 \) and \( \chi|_U \equiv 1 \). Then we have
\[
\|u_n\|_{L^1(K)} = \int_{w \in U} \left( \int_{z \in K} 2|G_X(z, w)|d\lambda_z \right)T_n(w)
= \int_{w \in U} H(w)T_n(w)
\leq N \int_{w \in U} T_n(w)
\leq N \langle T_n(w), \chi \rangle
= N(T(w), \chi * \rho_n)
\leq N\|T\| < +\infty
\]
Hence, we know \( \{u_n\} \in L^1_{loc}(X) \) and for any relative compact sunset \( K \subset X \), \( \|u_n\|_{L^1(K)} \) in uniformly bounded. By Fatou lemma, we have
\[
\int_{z \in K} \liminf_{n \to +\infty} |u_n|d\lambda_z \leq \liminf_{n \to +\infty} \int_{z \in K} |u_n|d\lambda_z < +\infty.
\]
This means \( u \in L^1_{loc}(X) \).
Now we consider $i\partial \bar{\partial} u(z)$. Let $g \in C_c^\infty(X)$ be a test function. We have

$$
\langle i\partial \bar{\partial} u, g \rangle = \langle u(z), i\partial \bar{\partial} g(z) \rangle 
= \lim_{n \to +\infty} \langle u_n(z), i\partial \bar{\partial} g(z) \rangle 
= \lim_{n \to +\infty} \langle T_n(w), 2G_X(z,w), i\partial \bar{\partial} g(z) \rangle 
= \lim_{n \to +\infty} \langle T_n(w), (2G_X(z,w), i\partial \bar{\partial} g(z)) \rangle
= \lim_{n \to +\infty} \langle T_n(w), g(w) \rangle
= \langle T, g \rangle
$$

The forth equality holds because of Fubini Theorem. Now we explain the second equality. Given a point $q \in X$, under the local coordinate $(U_q, z_q)$, we have $i\partial \bar{\partial} g(z) = i f(z) dz \wedge d\bar{z}$, where $f(z)$ is a smooth real function on $U_q$ with compact support. Let $(i\partial \bar{\partial} g(z))_+ = f(z)_+ idz \wedge d\bar{z}$, $(i\partial \bar{\partial} g(z))_- = f(z)_- idz \wedge d\bar{z}$, where $f(z)_+ = \max(f(z), 0)$ and $f(z)_- = \max(-f(z), 0)$. Then it follows from Levi’s Theorem that we have

$$
\lim_{n \to +\infty} \int_{U_q} u_n(z) (i\partial \bar{\partial} g(z))_+ = \int_{U_q} u(z) (i\partial \bar{\partial} g(z))_+
$$

and

$$
\lim_{n \to +\infty} \int_{U_q} u_n(z) (i\partial \bar{\partial} g(z))_- = \int_{U_q} u(z) (i\partial \bar{\partial} g(z))_-
$$

Since $i\partial \bar{\partial} g(z) = (i\partial \bar{\partial} g(z))_+ - (i\partial \bar{\partial} g(z))_-$, hence we have

$$
\lim_{n \to +\infty} \int_{U_q} u_n(z) i\partial \bar{\partial} g(z) = \int_{U_q} u(z) i\partial \bar{\partial} g(z)
$$

As $g(z)$ has compact support, there exists finite $\{U_q, \}$ such that $\text{supp} g \subset \bigcup_i U_q$, and on each $U_q$, equality (4.43) holds. Hence we know that on the whole $X$, we have

$$
\lim_{n \to +\infty} \int_X u_n(z) i\partial \bar{\partial} g(z) = \int_X u(z) i\partial \bar{\partial} g(z)
$$

which implies

$$
\langle u(z), i\partial \bar{\partial} g(z) \rangle = \lim_{n \to +\infty} \langle u_n(z), i\partial \bar{\partial} g(z) \rangle
$$

i.e. the second equality holds. Then it follows from (4.43) that we know $i\partial \bar{\partial} u = T = \theta i\partial \bar{\partial} \varphi$.

For fixed $t > 0$, as $k \geq 2$, the set $\{z : -t < kG_X(z,z_0) < 0\} \subset \{z : -t < 2G_X(z,z_0) < 0\}$. Let $t > 0$ be small enough such that the set $\{z : -t < 2G_X(z,z_0) < 0\} \cap (\overline{U} \cup \{z_0\}) = \emptyset$. Let $W \subset X$ be an relatively compact open set of $X$ which satisfies $\overline{U} \cup \{z_0\} \subset W$ and $W \cap \{t < 2G_X(z,z_0) < 0\} = \emptyset$.

Then for every fixed $z \in \{t < 2G_X(z,z_0) < 0\}$, $2G_X(z, w)$ is harmonic function on $W$ with respect to $w$.

By the Harnack inequality of harmonic function, there exists a $M > 0$ such that

$$
sup_{w \in W} (-2G_X(z,w)) \leq M \inf_{w \in W} (-2G_X(z,w))
$$
As $0 < -2G_X(z, z_0) < t$, we have
\[ Mt > -2G_X(z, z_0) \geq M \inf_{w \in W} (-2G_X(z, w)) \geq \sup_{w \in W} (-2G_X(z, w)) \geq 0 \]
This means when $t \to 0$, the function $2G_X(z, w)$ which defined on $\{z : -t < kG_X(z, z_0) < 0\} \times U$ uniformly goes to 0.

Note that when $(z, w) \in \{z : -t < kG_X(z, z_0) < 0\} \times U$ (t big enough), $2G_X(z, w)$ is harmonic function. Then
\[
u(z) = \lim_{n \to +\infty} u_n(z) = \lim_{n \to +\infty} (T(w), 2G_X(z, w) * \rho_n) = \langle T(w), 2G_X(z, w) \rangle.
\]
The third equality holds because of the mean-value equality for harmonic function.

Hence when $z$ satisfies $kG_X(z, z_0) \to 0$, we have $\nu(z) \to 0$.

Now let $\tilde{\varphi} = \varphi - u$, we know $\tilde{\varphi} > \varphi$ and when $z$ satisfies $kG_X(z, z_0) \to 0$, we have $\tilde{\varphi}(z) \to \varphi(z)$. Note that $i\partial \bar{\partial} \tilde{\varphi} = i(1 - \theta)\partial \bar{\partial} \varphi \geq 0$ on $X$. Hence $\tilde{\varphi} \in PSH(X)$.

Note that $\theta$ is a smooth function and $0 \leq \theta \leq 1 - \frac{|\lambda|}{\lambda}$, we have
\[ [\lambda_{j_0}] \leq v(i\partial \bar{\partial} \tilde{\varphi}, x_{j_0}) < \lambda_{j_0}. \]
For any $x$ satisfies $0 \leq v(i\partial \bar{\partial} \varphi, x) < 1$, we have
\[ 0 \leq v(i\partial \bar{\partial} \tilde{\varphi}, x) \leq v(i\partial \bar{\partial} \varphi, x) < 1. \]
For any $x$ satisfies $v(i\partial \bar{\partial} \varphi, x) \geq 1$, as $U \cap (E_1(i\partial \bar{\partial} \varphi) \setminus \{x_{j_0}\}) = \emptyset$ and $supp \theta \subset \subset U$, we have
\[ v(i\partial \bar{\partial} \tilde{\varphi}, x) = v(i\partial \bar{\partial} \varphi, x). \]
Hence by the classification of multiplier ideal sheaves in dimensional one case, we know for any $x \in X$, we have $\mathcal{I}(\tilde{\varphi})_x = \mathcal{I}(\varphi)_x$.

Next we prove $\nu(z)$ has lower bound $-A$ for some $A > 0$ on $X \setminus U$. It follows from Lemma 4.7 that when $z \in X \setminus U$ and $w \in V$, we have $2G_X(z, w) \geq 2NG_X(z, x_{j_0})$.

By Lemma 4.6, we know $G_X(z, x_{j_0}) \geq -A_0$ (where $A_0 > 0$ is a constant) for $z \in X \setminus U$.

Note that $G_X(z, w)$ is harmonic function on $(z, w) \in (X \setminus U) \times V$. For fixed $z \in X \setminus U$, we have
\[
u(z) = \langle T(w), 2G_X(z, w) \rangle = \langle \theta(w)i\partial \bar{\partial} \varphi(w), 2G_X(z, w) \rangle \geq \langle \theta(w)i\partial \bar{\partial} \varphi(w), -2NA_0 \rangle, \]
here $-2NA_0$ is actually a constant function $f(w) \equiv -2NA_0$ defined on $V$. Note that $supp \theta \subset \subset V$, hence the inequality “$\geq$” holds. Let $A = \langle \theta(w)i\partial \bar{\partial} \varphi(w), 2NA_0 \rangle$, we know on $X \setminus U$, $\nu(z) > -A$.

As $\varphi - \tilde{\varphi} = u(z)$, we know that there exists a relatively compact open subset $U \subset \subset X$ such that $\varphi - \tilde{\varphi}$ has lower bound $-A$ ($A > 0$ is a constant) for any $z \in X \setminus U$.

Then in case (1), we have a function $\tilde{\varphi}$ satisfies the conditions in the Lemma 3.4.

For the case (2):
As $R \neq 0$, there must be a point $p \in supp R \setminus E_1(i\partial \bar{\partial} \varphi)$. If not, we must have $supp R \subset E_1(i\partial \bar{\partial} \varphi)$. As $R$ is a closed positive (1, 1) current, $R$ is of order 0. Note that $E_1(i\partial \bar{\partial} \varphi)$ is an analytic subset of $X$ with irreducible components $\{x_j\}_{j=1,2,...}$, then it follows from Lemma 4.5 that $R = \sum_{j \geq 1} a_j[x_j]$, where $a_j = v(R, x_j)$ is the Lelong number of $R$ at $x_j$. However by Siu’s decomposition theorem, we know
\[ v(R, x) = 0, \text{ for any } x \in X, \] which implies that all \( a_j = 0 \) and then \( R = 0 \). This contradicts to the fact that \( R \neq 0 \).

Let \( p \in \text{Supp} R \setminus E_1(i\partial \bar{\partial} \varphi) \). Let \((U_2, z)\) be a relative compact coordinate neighborhood of \( p \) in \( X \) and by shrinking \( U \), we assume that under the local homomorphism, \( z(p) = o \) and \( z(U_2) \cong B(0, 2) \). We also assume that \( U_2 \cap (E_1(i\partial \bar{\partial} \varphi)) = \emptyset \). Let \( \theta_2 \) be a smooth cut-off function on \( X \) such that \( 0 \leq \theta \leq 1 \), \( \text{supp}(\theta_2) \subset \subset V_2 \subset \subset U_2 \) and \( \theta \equiv 1 \) on \( W_2 \), where \( z(W_2) = B(0, \frac{1}{4}) \) and \( z(V_2) = B(0, \frac{1}{2}) \) under the local homomorphism. By shrinking \( V_2 \) and \( U_2 \) again, it follows from Lemma 4.7 that there exists \( N_2 > 0 \) such that for any \((z, w) \in (X \setminus U_2) \times V_2 \), we have

\[ G_X(z, w) \geq N_2 G_X(z, p). \]

Let \( T_2 = \theta_2 \cdot i\partial \bar{\partial} \varphi \). We can do the same thing as we did in the case (1) and get a function \( u_2(z) \) such that \( u_2(z) < 0 \), \( i\partial \bar{\partial}u_2(z) = \theta_2 \cdot i\partial \bar{\partial} \varphi \) and when \( z \) satisfies \( kG_X(z, z_0) \to 0 \), we have \( u_2(z) \to 0 \). Especially, \( u_2(z) \) has lower bound \(-A_2\) for some \( A_2 > 0 \) on \( X \setminus U_2 \).

Let \( \tilde{\varphi}_2(z) = \varphi - u_2(z) \). We know \( \tilde{\varphi}_2 > \varphi \). When \( z \) satisfies \( kG_X(z, z_0) \to 0 \), we have \( \varphi_2(z) \to \varphi(z) \). As \( \varphi_2(z) - \tilde{\varphi}_2(z) = u_2(z) \), we know that there exists a relatively compact open subset \( U_2 \subset \subset X \) such that \( \varphi_2 - \tilde{\varphi}_2 \) has lower bound \(-A_2 \) (\( A_2 > 0 \) is a constant) for any \( z \in X \setminus U_2 \). Note that \( i\partial \bar{\partial} (\varphi_2) = i(1 - \theta_2) \partial \bar{\partial} \varphi \geq 0 \) on \( X \). Hence \( \tilde{\varphi}_2 \in PSH(X) \).

Note that \( \theta_2 \) is a compact smooth function with \( \text{supp}(\theta_2) \subset \subset U_2 \) and \( 0 \leq \theta_2 \leq 1 \).

It is easy to see that for any \( z \) satisfies \( 0 \leq v(i\partial \bar{\partial} \varphi, x) < 1 \), we have

\[ 0 \leq v(i\partial \bar{\partial} \varphi, x) \leq v(i\partial \bar{\partial} \varphi, x) < 1. \]

For any \( x \) satisfies \( v(i\partial \bar{\partial} \varphi, x) = 1 \), as \( U_2 \cap (E_1(i\partial \bar{\partial} \varphi)) = \emptyset \) and \( \text{supp}(\theta_2) \subset \subset U_2 \), we have

\[ v(i\partial \bar{\partial} \varphi, x) = v(i\partial \bar{\partial} \varphi, x). \]

Hence by the classification of multiplier ideal sheaves in dimensional one case, we know for any \( x \in X \), we have \( \mathcal{I}(\varphi_2)_x = \mathcal{I}(\varphi)_x \).

Then in case (2), we have a function \( \tilde{\varphi}_2 \) satisfies the conditions in the Lemma 3.4 is proved. \( \square \)

4.3. Appendix: a property of multiplicative function on Open Riemann surface. Let \( X \) be a open Riemann surface. We recall the following construction in Section 1.

Let \( p : \Delta \to X \) be the universal covering from unit disc \( \Delta \) to \( X \). We call the holomorphic function \( f \) (resp. holomorphic \((1, 0)\) form \( F \)) on \( \Delta \) is a multiplicative function (resp. multiplicative differential (Prym differential)) if there is a character \( \chi \), where \( \chi \in \text{Hom}(\pi_1(X), C^*) \) and \( |\chi| = 1 \), such that \( g^*f = \chi(g)f \) (resp. \( g^*F = \chi(g)F \)) for every \( g \in \pi_1(X) \) which naturally acts on the universal covering of \( X \). Denote the set of such kinds of \( f \) (resp. \( F \)) by \( \mathcal{O}^\chi(X) \) (resp. \( \Gamma^\chi(X) \)).

As \( p \) is a universal covering, then for any harmonic function \( h \) on \( X \), there exists a \( \chi_h \) and a multiplicative function \( f_h \in \mathcal{O}^{\chi_h}(X) \), such that \( |f_h| = p^e h \). Let \( s \) be a holomorphic function on \( X \) and \( s \) has no zero points on \( X \). We know \( \log |s| \) is a harmonic function on \( X \).

In this appendix, we recall the following well-known property.

Lemma 4.8. \( \chi_h = \chi_{h+\log|s|} \).
Proof. We firstly recall the construction of $f_h$ and $\chi_h$.

As $h$ is harmonic on $X$, then $p^*h$ is harmonic on $\Delta$. Since $\Delta$ is simple connected, there exists $f \in \mathcal{O}_\Delta$ such that $f = p^*h + iv$. Then $e^f$ is holomorphic on $\Delta$ and $|e^f| = |e^{p^*h+iv}| = p^*e^h$. We denote $f_h = e^f$.

Let $\Gamma$ be a subgroup of $\text{Aut}(\Delta)$ such that $X = \Delta/\Gamma$. Then by the theorem of covering spaces, we know $\pi_1(X) \cong \Gamma$. Hence for any $g \in \pi_1(X)$, $g$ naturally acts on $\Delta$ and for any $z \in X$, $p^{-1}(z)$ is invariant under the act of $g$. Fix $z_1 \in X$, we denote $p^{-1}(z_1) = \{x_0, x_1, \cdots\}$. By the theorem of covering spaces, we know there is a bijection between $p^{-1}(z_1)$ and $\pi_1(X, z_1)$.

For any $g_i \in \pi_1(X)$, we assume that $g(x_0) = x_i$. Then we define

$$\chi_h(g_i) = \frac{e^f(x_i)}{e^f(x_0)} = \frac{e^{p^*u(x_i)+iv(x_i)}}{e^{p^*u(x_0)+iv(x_0)}} = \frac{e^{u(z_1)+iv(x_i)}}{e^{u(z_1)+iv(x_0)}} = \frac{e^{iv(x_i)}}{e^{iv(x_0)}}. \quad (4.44)$$

Hence $|\chi_h(g_i)| = 1$. Now we prove that $e^f \in \mathcal{O}^{\chi_h}(X)$, i.e., for any $g_i \in \pi_1(X)$, we have $g_i^*e^f = \chi(g_i)e^f$.

Given $y_0 \in \Delta$, denote $w_1 = p(y_0) \in X$. Denote $p^{-1}(w_1) = \{y_0, y_1, y_2, \cdots\}$. We know there exists a bijection between $\pi_1(X, w_1)$ and $p^{-1}(w_1)$.

As the fundamental group $\pi_1(X)$ is base point free, we have $\pi_1(X, w_1) \cong \pi_1(X, z_1) \cong \Gamma$. Hence we have a bijection

$$\Phi : p^{-1}(z_1) \to p^{-1}(w_1)$$

$$x_i \to y_i$$

which satisfies $g \circ \Phi = \Phi \circ g$.

For any $g \in \pi_1(X)$, we assume that $g(y_0) = y_i$, then $g(x_0) = x_i$. Then we have

$$g_i^*e^f(y_0) = e^f(y_0) = e^f(y_i). \quad (4.45)$$

and by the definition of $\chi_h$ (see formula (4.44)),

$$\chi_h(g)e^f(y_0) = \frac{e^{f(x_i)}}{e^{f(x_0)}} e^f(y_i). \quad (4.46)$$

To prove (4.45) equals to (4.46), as $f = p^u + iv$, it sufficient to prove that

$$v(x_i) - v(x_0) = v(y_i) - v(y_0). \quad (4.47)$$

As $p^*u$ is harmonic on $\Delta$, let $w = \frac{\partial p^*u}{\partial y} dy - \frac{\partial p^*u}{\partial x} dx$ on $\Delta$, then $dw = 0$. Let $L_{qp}$ be any path from $q$ to $p$. We can define $v(x)$ as below.

$$v(p) = \int_{L_{qp}} w.$$ 

where 0 is the origin in $\Delta$.

Then we have $v(x_i) - v(x_0) = \int_{L_{x_0x_i}} w$ and $v(y_i) - v(y_0) = \int_{L_{y_0y_i}} w$. Denote $L_{z_1} = p_*L_{z_0z_1}$ and $L_{w_1} = p_*L_{w_0w_1}$. Then by the isomorphic between $\pi_1(X, z_1) \cong \pi_1(X, w_1) \cong \Gamma$ and $g \circ \Phi = \Phi \circ g$, we know the path $L_{z_1}$ and $L_{w_1}$ are homotopic.
Hence

$$v(x_i) - v(x_0) = \int_{L_{x_0}} w = \int_{p_\ast L_{x_0}} p_\ast w = \int_{L_1} p_\ast w = \int_{L_{W_1}} p_\ast w = \int_{L_{W_0 y_1}} p_\ast w = \int_{L_{y_1}} w = v(y_1) - v(y_0)$$

(4.48)

Hence we know given any $y_0 \in \Delta$, for any $g \in \pi_1(X)$, we have

$$\chi_h(y_0)e^f = g^\ast (e^f).$$

We also need to show that $\chi$ is a homomorphism from $\pi_1(X)$ to $\mathbb{C}^\ast$.

For any $g_1, g_2 \in \pi_1(X)$. We assume that $g_1(x_0) = x_1$ and $g_2(x_1) = x_2$, then we have

$$\chi(g_1)\chi(g_2) = \frac{e^{iv(x_1)}}{e^{iv(x_0)}} \frac{e^{iv(x_2)}}{e^{iv(x_1)}} = e^{iv(x_2)}.$$

and note that $g_1 \circ g_2(x_0) = x_2$,

$$\chi(g_1 g_2) = e^{iv(x_2)}.$$

Hence $\chi(g_1)\chi(g_2) = \chi(g_1 g_2)$, $\chi$ is a homomorphism from $\pi_1(X)$ to $\mathbb{C}^\ast$.

Now we can prove $\chi_h = \chi_{h + \log |s|}$. Denote $h_2 = h + \log |s|$.

Note that we have already found a holomorphic function $f_h$ on $\Delta$ which satisfies $|f_h| = p^h e^h$. Then for any $g \in \pi_1(X, z_1)$, we define $\chi_h(g) = \frac{f_h(x_1)}{f_h(x_0)}$, where $x_1, x_0 \in p^{-1}(z_1)$ and $g(x_0) = x_1$.

It is easy to see that $f_h p^s$ is a holomorphic function on $\Delta$ which satisfies $|f_h \cdot p^s| = p^h e^h |p^s| = p^h e^{h + \log |s|}$. Then similarly as above, for any $g \in \pi_1(X, z_1)$, we define

$$\chi_{h_2}(g) = \frac{f_h(x_1) \cdot p^s(x_1)}{f_h(x_0) \cdot p^s(x_0)}.$$ 

Note that $p^s$ is fiber-constant, we know $p^s(x_1) = p^s(x_0)$ for any $x_1, x_0 \in p^{-1}(z_1)$. Hence

$$\chi_{h_2}(g) = \frac{f_h(x_1)}{f_h(x_0)} = \chi_h(g).$$

As $g \in \pi_1(X, z_1)$ is arbitrary chosen, we know $\chi_h = \chi_{h + \log |s|}$.

Lemma 1.3 is proved. \(\square\)

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