Nonlinear constraints on gravity from entanglement

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Received 15 November 2014, revised 5 January 2015
Accepted for publication 13 January 2015
Published 24 February 2015

Abstract
Using the positivity of relative entropy arising from the Ryu–Takayanagi formula for spherical entangling surfaces, we obtain constraints at the nonlinear level for the gravitational dual. We calculate the Green’s function necessary to compute the first order correction to the entangling surface and use this to find the relative entropy for non-constant stress tensors in a derivative expansion. We show that the Einstein value satisfies the positivity condition, while the multidimensional parameter space away from it gets constrained.

Keywords: AdS/CFT, holography, entanglement

(Some figures may appear in colour only in the online journal)

1. Introduction
Can one derive Einstein field equations from quantum information? This question makes sense in the AdS/CFT correspondence. The precise version is as follows. The Ryu–Takayanagi entropy functional [1] gives us a way to compute entanglement entropy in a conformal field theory dual to Einstein gravity. This entropy functional can be derived by assuming that the bulk theory was Einstein gravity [2]. We can turn the question around [3–5]. Given the form of the holographic entropy functional (e.g. Ryu–Takayanagi), what are the bulk equations of motion?

In order to make progress, we appeal to a consistency condition that entanglement entropy has to satisfy: relative entropy has to be positive [6]. This condition is an inequality and involves comparing two density matrices. On the bulk side, this corresponds to adding
perturbations to the AdS metric. At linearized order, the inequality is saturated and recently it has been possible to show that one recovers linearized gravitational equations of motion, not only for the Ryu–Takayanagi entropy functional but for any entropy functional that corresponds to higher derivative gravity in the bulk [7]. Since the gravity equations of motion are nonlinear it becomes an interesting question to ask what happens at nonlinear order. For the Ryu–Takayanagi entropy functional, there could be two possibilities: (a) we recover precisely Einstein equations or (b) we get a wider class of theories than just Einstein theory. Since the tool at hand is an inequality at first sight (a) appears to be a distant possibility. We can pose possibility (b) in a more precise way. We will add perturbations to the AdS metric. At linear order, we know that we recover linearized Einstein equations. At next order, we will write down the most general possible terms with undetermined coefficients. We will ask if the values that these coefficients can take are bounded.

Progress in this direction was made in [8]. Let us briefly review our findings there. We start by writing $d + 1$ dimensional AdS space as [9]

$$dx^2 = \frac{L^2}{z^2} dz^2 + g_{\mu\nu} dx^\mu dx^\nu. \quad (1)$$

$$g_{\mu\nu} = \frac{L^2}{z^2} \left[ g_{ij} + a z^d T_{ij} + a^2 z^{2d} \left( n_1 T_{\mu\nu} T_\mu^\alpha T_\nu^\beta + n_2 \eta_{\mu\nu} T_{\alpha\beta} T^{\alpha\beta} \right) + \cdots \right]. \quad (2)$$

where (2) is the most general term one can write down at quadratic order in the perturbation $T_{\mu\nu}$ for a constant stress tensor. The metric has a dimensionless constant $a = \frac{2}{d} (L_p/L)^{d-1}$, that can simply be absorbed in the $T_{\mu\nu}$. So in the rest of the paper we will not carry the constant anymore. The Ryu–Takayanagi entropy functional is given by

$$S = \frac{2 \pi}{\ell_p^{d-1}} \int d^{d-1}x \sqrt{h}, \quad (3)$$

where

$$h_{ij} = \frac{L^2}{z^2} \left( g_{ij} + \partial_i z \partial_j z \right), \quad (4)$$

where $z = z(x_i)$ is the bulk co-dimension two entangling surface (we will only consider static situations). We will consider a spherical entangling surface for which the modular Hamiltonian and hence the expression for relative entropy is known [6]. The positivity of relative entropy leads to $\Delta H \geq \Delta S$ where $\Delta H$ is the difference in the expectation value of the modular Hamiltonian with respect to the two density matrices and $\Delta S$ is the difference in the von Neumann entropies of the two density matrices—we will calculate this using equation (3) in holography. Since the change in the modular Hamiltonian depends linearly on the stress tensor [6], at quadratic order in the stress tensor $\Delta^{(2)} S$ must be negative. Thus when we talk about nonlinear constraints on the metric, we have to compute $\Delta^{(2)} S$ and demand that it is negative—this will constrain the nonlinear terms in the metric. The solution for the entangling surface is given by $z_0^2 = R^2 + x_i x_i$. The perturbed surface can be written as $z = z_0 + \epsilon z_1$, where $z_1$ satisfies

$$\frac{1}{z_0^{d-2}} R \left( \partial_j (z_0 z_1) - \frac{x_i x_j}{R^2} \partial_i \partial_j (z_0 z_1) \right) = \frac{z_0}{2R} (T (d-2) + T_s (d+2)). \quad (5)$$

$^3 \epsilon$ is a small parameter just to keep track of the order of the perturbation.
The solution was guessed in [6]

\[ z_1 = - \frac{R^2 c_z d^{-1}}{2(d + 1)} (T + T_z). \tag{6} \]

Using the above solution we get the second order correction to the difference in the entanglement entropy to be (see [8])

\[ \Delta^{(2)} S = 2\pi \left( L/T_\rho \right)^{d^{-1}} \Omega_{d-2} \left( C_1 T^2 + C_2 T_0^2 + C_T T_0^2 \right). \tag{7} \]

with

\[
C_1 = \frac{2^{-3-d} \left( 1 + 4 \left( d^2 - 1 \right) n_2 \right) \sqrt{\pi} R^2 \Gamma \left[ \frac{d + 1}{2} \right]}{\left( d^2 - 1 \right) \Gamma \left[ \frac{3}{2} + d \right]},
\]

\[
C_2 = \frac{2^{-3-d} \sqrt{\pi} R^2 \Gamma \left[ 1 + d \right]}{\left( d^2 - 1 \right) \Gamma \left[ \frac{3}{2} + d \right]} \left( -1 - 2d + 4(d + 1)n_1 + 4 \left( d^2 - 1 \right)n_2 \right),
\]

\[
C_3 = -\frac{2^{-1-d} \left( n_1 + 2d - 1 \right) \sqrt{\pi} R^2 \Gamma \left[ 1 + d \right]}{(d - 1) \Gamma \left[ \frac{3}{2} + d \right]}. \tag{8}\]

Now we must demand that \( \Delta^{(2)} S \leq 0 \). We can write \( \Delta^{(2)} S = V^T M V \) with \( V \) being a \((d - 1)(d + 2)/2\) dimensional vector with the independent components of \( T_{\mu\nu} \) as its components. If we diagonalize the matrix \( M \) with a matrix, say \( U \), then it is possible to write

\[ \Delta^{(2)} S = (UV)^T M_d (UV) = \sum_i \lambda_i (UV)_i. \tag{9} \]

We must now demand that the eigenvalues \( \lambda_i \) are negative so that \( \Delta^{(2)} S < 0 \) and hence the relative entropy is positive. This leads to the following inequalities for \( n_1, n_2 \) [8]:

\[ n_1 + 2(d - 1)n_2 \geq 0, \tag{10} \]

\[ 2d + 1 - 4(d + 1)n_1 - 4 \left( d^2 - 1 \right)n_2 \geq 0, \tag{11} \]

\[ d + 2 - 4d + 1n_1 - 4d \left( d^2 - 1 \right)n_2 \geq 0, \tag{12} \]

which lead to the enclosed region shown in figure 1.

The fact that we get a region of finite area in the \( n_1, n_2 \) parameter space which encloses the Einstein point is encouraging. However, this was for a constant stress tensor. What happens to this constrained region for a non-constant stress tensor? In order to answer this question we will need to find the Green’s function for equation (5). For concreteness we will focus on \( d = 4 \) henceforth. One can hope that the allowed parameter space shrinks on considerations of non-constant stress tensor (perhaps down to the Einstein point). Furthermore, we will be introducing more parameters to account for the derivatives of the stress tensor and we can ask if these additional parameters are themselves constrained.

This paper is organized as follows. In section 2 we will explain in some detail how the Green’s function to find \( z_1 \) is derived. Using this we will consider the change in the relative entropy due to non-constant stress tensors in section 3. In section 4 we will show that for Einstein values of the nonlinear parameters relative entropy is positive, and also derive
constraints on these parameters using the positivity. In section 5, we will generalize the analysis in arbitrary dimensions. We will conclude in section 6. There are three appendices with useful details of intermediate steps.

2. A systematic way to find $z_1$

Let us start with a four-dimensional theory living on $R^{3,1}$. The dual gravity theory lives on $AdS_5$ whose metric in Poincare coordinates is given by

$$ds^2 = \frac{L^2}{z^2} \left( dt^2 + dx^2 + dr^2 \right).$$

(13)

We want to compute the entanglement entropy of a ball of radius unity$^4$ in the 3+1-dimensional boundary. Without any loss of generality we can assume that the ball is centered at the origin and so the entangling surface is given by, $\vec{x}^2 = r^2 = 1$. The corresponding minimal surface in $AdS_5$ is given by the equation, $z^2 + \bar{x}^2 = 1$. If we change the bulk geometry then this minimal surface will change. If $\epsilon$ denotes the strength of the perturbation then we can write

$^4$ The discussions in this section can be easily generalized for an arbitrary radius $R$. The rest of the paper assumes a generic radius.

![Figure 1. For $d > 2$ we get the allowed $n_1, n_2$ region for constant stress tensor to be the triangle above [8]. The Einstein value $(n_1, n_2) = \left( \frac{1}{2}, \frac{1}{4(d-1)} \right)$ is at the origin of the plot.](image-url)
\[ g_{\mu\nu} = g_{\mu\nu}^{(0)} + \epsilon g_{\mu\nu}^{(1)} + \epsilon^2 g_{\mu\nu}^{(2)} + \ldots \]  
(14)

and

\[ z(\vec{x}) = z_0(\vec{x}) + \epsilon z_1(\vec{x}) + \epsilon^2 z_2(\vec{x}) + \ldots, \]  
(15)

where \( z(\vec{x}) \) is the equation of the new minimal surface and \( z_0 \) is the unperturbed minimal surface given by the equation \( z^2 + \vec{x}^2 = 1 \). Our aim is to find the first order perturbation given by \( z_1 \). In order to do that we have to find out the change in the area functional due to the change in the metric up to second order and impose the minimality constraint. The process is straightforward and leads to the following differential equation for \( z_1(\vec{x}) \)

\[ z_0 \left( \partial^2 (z_0 z_1) - x^i x^j \partial_i \partial_j (z_0 z_1) \right) = J. \]  
(16)

Here, \( J \) is a source function which depends on the details of the metric deformation. The explicit form of \( J \) is shown in (5) for deformations with a constant boundary stress tensor, and in appendix B.2 we compute it for the non-constant case. Now the key point is to note that the induced metric on the zeroth order minimal surface \( z_0 \) is that of an Euclidean AdS3, with the entangling surface \( r = 1 \) as its conformal boundary. This will be clear from the coordinate transformations described in the following subsections. For the time being let us write the differential equation in the form

\[ \Delta_{H^3} - m^2 \]  
(17)

where \( \Delta_{H^3} \) is the scalar Laplacian on AdS3. The origin of this AdS3-Laplacian can be understood in the following way. First of all we are dealing exclusively with time-independent deformations of the bulk metric and so we can again parametrize the minimal surface by an equation of the form \( z = z(\vec{x}) \). The first order fluctuation \( z_1 \) can then be thought of as a scalar field propagating on the unperturbed minimal surface. Since, the unperturbed minimal surface \( z_0 \) is an Euclidean AdS3 we automatically get the Laplacian on AdS3.

Equation (17) is the equation of a scalar field propagating on AdS3 with \( m^2 = 3 \), where \( m \) is the mass of the scalar field. The general solution can be written as

\[ z_1 = \int_{\mathbf{R}^3} G_{\text{bulk-bulk}}(\vec{x}, \vec{y}) J(\vec{y}) \, d\vec{x}, \]  
(18)

where \( G_{\text{bulk-bulk}} \) is the bulk to bulk propagator for a massive scalar field on AdS3 and \( J(\vec{y}) \) is the solution of the homogeneous equation subject to the proper boundary condition. A scalar field with \( m^2 = 3 \) corresponds to an irrelevant operator in the CFT2 and so the homogeneous solution grows towards the boundary of the AdS3 which is the entangling surface in our case. We need to set this mode to zero because \( z_1 \) vanishes on the entangling surface.

2.1. The Green’s function for \( z_1 \)

The general solution for \( z_1 \) can be written as

\[ z_1(x) = \int d\mu_\mathbf{3} \hat{G}(\mathbf{x}, \hat{x}) J(\hat{x}), \]  
(19)

where \( d\mu_\mathbf{3} \) is the Riemannian volume element on AdS3 and we denote the intrinsic coordinates collectively by \( x \). In terms of embedding coordinates, an AdS3 of unit radius is described by a hyperboloid in Minkowski space \( \mathbb{R}^{3,1} \) given by
\[ X_1^2 + X_2^2 + X_3^2 - X_4^2 = -1, \]  
\[ (20) \]
where \( \vec{X} \) are the coordinates\(^5\) of \( \text{R}^{3,1} \). One can introduce intrinsic coordinates on \( \text{AdS}_3 \) by \( X_1 = \sin \theta \cos \phi \sinh \eta, \quad X_2 = \sin \theta \sin \phi \sinh \eta, \quad X_3 = \cos \theta \sinh \eta, \quad X_4 = \cosh \eta. \)  
\[ (21) \]
In terms of the boundary (CFT) spherical polar coordinates, \( \eta = \tanh^{-1}(r) \). The angular coordinates \( \theta \) and \( \phi \) are the usual ones. The metric then takes the following simple form in terms of the intrinsic coordinates

\[ ds^2 = dr^2 + \sinh^2 \eta \left( d\theta^2 + \sin^2 \theta d\phi^2 \right). \]
\[ (22) \]
It is easy to check that this is the induced metric on the minimal surface \( z^2 + \vec{x}^2 = 1 \), if we write

\[ z = \sqrt{(1 - x^2)} = \frac{1}{\sinh \eta}, \quad d\vec{x}^2 = dr^2 + r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right), \quad dr = 0. \]
\[ (23) \]
and substitute it back into the metric of \( \text{AdS}_3 \) written in Poincaré coordinates.

Given two points \( x \) and \( \hat{x} \) on \( \text{AdS}_3 \) the geodesic distance \( d(x, \hat{x}) \) is given by the relation

\[ \cosh d(x, \hat{x}) = -X \cdot \hat{X}, \]
\[ (24) \]
where \( X \) and \( \hat{X} \) are the respective embedding coordinates, corresponding to \( x \) and \( \hat{x} \). In \( (19) \), \( G(x, \hat{x}) \) is the Green’s function i.e., as mentioned before, the bulk to bulk propagator for a scalar field with \( m^2 = 3 \). It is given by [10]

\[ G(x, \hat{x}) = \frac{1}{4\pi} \frac{e^{-2d(x, \hat{x})}}{\sinh d(x, \hat{x})}. \]
\[ (25) \]

Let us now work out a simple example. Consider the situation where a constant stress tensor has been switched on in the field theory. This stress tensor gives rise to metric fluctuation in the bulk and we want to compute the new minimal surface resulting from that. For a constant stress tensor, the source \( J \) is given by

\[ J = -z_0 \delta \left( T + 3T_s \right), \]
\[ (26) \]
where \( T_i = x^i T_{ij}, \quad \eta \) where \( i \) runs from 1 to 3, denotes the cartesian coordinates of the boundary \( \text{R}^3 \). Let us now see how things simplify if we take an isotropic stress tensor. The isotropic stress tensor is given by

\[ T_{ij} = \frac{1}{3} T \delta_{ij}, \]
\[ (27) \]
where \( T \) is a constant. \( T_i = x^i T_{ij} = \frac{T}{3} r^2 = \frac{T}{3} \tanh^2 \eta. \) Let us take the point \( x \) in \( (19) \) to be \( \eta = 0 \). In terms of embedding coordinates this is the point \( X_1 = X_2 = X_3 = 0, \quad X_4 = 1 \). So this is the lowest point of the hyperboloid. At this point our calculation gets enormously simplified, since, we get \( \cosh d = \cosh \hat{\eta} \), where \( \hat{\eta} \) is the intrinsic coordinate corresponding to \( \hat{x} \). This gives the solution

\[ z_1(0) = \int d\hat{\eta} \int_0^\infty d\eta \sinh^2 \hat{\eta} \frac{1}{4\pi} \frac{e^{-2\hat{\eta}}}{\sinh \hat{\eta}} J(\hat{\eta}) \]  
where
\[ J(\hat{\eta}) = \frac{1}{\cosh \hat{\eta}} \left( 1 + \tanh^2 \hat{\eta} \right). \]
\[ (28) \]

\(^5\) This \( \text{R}^{3,1} \) should not be confused with the boundary of the Poincaré patch where the CFT lives.
We have set the homogeneous solution to zero because that grows at the boundary and so is inconsistent with the boundary condition $z_1 = 0$. Then the above integral gives

$$z_1(0) = -\frac{T}{10}. \tag{29}$$

This is indeed what we would get in $d = 4$ from the solution given in [6] for an isotropic constant $T_{\mu\nu}$ with $R = 1$ and at $r = 0$. So the Green’s function gives the correct solution in this case.

Note that for this particular example, the source is rotationally invariant. Since the Green’s function and measure are both rotationally invariant, $z_1$ is also rotationally invariant. That means $z_1$ is a function of $\eta$ only. We can use the fact that AdS$_3$ is a homogeneous space of the Lorentz group $SO(3, 1)$ and so given two points on AdS$_3$, there exists an $SO(3, 1)$ group element which maps one point to the other. This allows us to simplify the calculation for a general $\eta$. In the following subsection, we will show how to find the solution for an arbitrary point, even for an arbitrary source which is not rotationally invariant.

### 2.2. General solution for an arbitrary source

In this section we shall do the integrals in a way which makes life easier even for arbitrary source and it can be easily automated. Suppose we want to compute $z_1$ at an arbitrary point, $P$, which is not necessarily the origin $\eta = 0$. AdS$_3$ is the homogeneous space of the group of isometries $SO(3, 1)$. Our strategy will be the following. We shall make a coordinate transformation such that in the new coordinate system the point $P$ is at the origin, $\eta' = 0$, where the primed coordinates are the new transformed coordinates. This coordinate transformation can be chosen to be an isometry and so the form of the differential operator does not change. The Green’s function has exactly the same form in the new coordinates. What makes life easier is the fact that the geodesic distance which appears in the Green’s function integral is now given by $\hat{\eta}'$ (where $\hat{\eta}$ is the integration variable). So we do not have to deal with complicated expressions for the geodesic distance in the integrand. This makes things much easier.

We have the Cartesian coordinates $(x_1, x_2, x_3)$ and the polar coordinates $(r, \theta, \phi)$ related by

$$x_1 = r \sin \theta \cos \phi, \quad x_2 = r \sin \theta \sin \phi, \quad x_3 = r \cos \theta. \tag{30}$$

We can go to the standard polar coordinates $(\eta, \theta, \phi)$ on $H_3$ by the transformation

$$r = \tanh \eta. \tag{31}$$

The embedding coordinates of $H_3$ are given by

$$X_1 = \sin \theta \cos \phi \sinh \eta, \quad X_2 = \sin \theta \sin \phi \sinh \eta, \quad X_3 = \cos \theta \sinh \eta, \quad X_4 = \cosh \eta. \tag{32}$$

So comparing these equations we get

$$x_i = \frac{X_i}{X_4}, \quad i = 1, 2, 3. \tag{33}$$

The coordinate transformation can be written in a simple form in terms of embedding coordinates. The coordinate transformations can be built out of three matrices, two of which are rotations and one boost. This is the most general coordinate transformation required. The matrices are given by
In terms of the embedding coordinates the coordinate transformation can be written as

\[
\begin{pmatrix}
X_1' \\
X_2' \\
X_3' \\
X_4'
\end{pmatrix} = K_{34}^{-1} R_{13}^{-1} R_{12}^{-1}
\begin{pmatrix}
X_1 \\
X_2 \\
X_3 \\
X_4
\end{pmatrix}.
\]

(36)

In the old coordinates, the origin \(\eta = 0\) corresponds to the point \((0, 0, 0, 1)\) in the embedding coordinates, \((X_1, X_2, X_3, X_4)\). It is easy to see that the point \((X'_1 = 0, X'_2 = 0, X'_3 = 0, X'_4 = 1)\) corresponds to the point \((X_1 = \sin \beta \cos \gamma \sinh \alpha, X_2 = \sin \beta \sin \gamma \sinh \alpha, X_3 = \cos \beta \sinh \alpha, X_4 = \cosh \alpha)\) in the old embedding coordinates. So we can place the origin of our new coordinates at any point on \(H^3\) by varying \((\alpha, \beta, \gamma)\). So we have achieved our goal. Since the coordinate transformation is an element of \(SO(3, 1)\) we have

\[
X_1'^2 + X_2'^2 + X_3'^2 - X_4'^2 = -1.
\]

(37)

So we can introduce new intrinsic coordinates given by

\[
X_1' = \sin \theta' \cos \phi' \sinh \eta', \quad X_2' = \sin \theta' \sin \phi' \sinh \eta', \quad X_3' = \cos \theta' \sinh \eta', \quad X_4' = \cosh \eta'.
\]

(38)

In terms of the new intrinsic coordinates the point, \((\eta = \alpha, \theta = \beta, \phi = \gamma)\) goes to, \(\eta' = 0\). Since \(z_1\) is a scalar its value at the point \((\eta = \alpha, \theta = \beta, \phi = \gamma)\), is the same as its value at the point, \(\eta' = 0\). The new intrinsic coordinates are some complicated functions of the old ones, but since its an isometry, the form of the differential operator does not change and so is the Green’s function. We want to compute \(z_1\) at the origin of this new coordinates. So the answer is given by

\[
z_1(0') = z_1(\eta, \theta, \phi) = \int d\hat{\Omega}_2 \int_0^{\infty} d\hat{\eta} \sinh^2 \hat{\eta} \left( \frac{1}{4\pi \sinh \hat{\eta}} \right) J'(\hat{\eta}, \hat{\theta}, \hat{\phi}).
\]

(39)

We have used the fact that the geodesic distance from the origin is given by \(\hat{\eta}\). We can see that all the dependence on \((\eta, \theta, \phi)\) arises through the source. Let us now say a few words about the source. Since the source is a scalar the functional form of \(J'\) can obtained via

\[
J'(\eta', \theta', \phi') = J(\eta(\eta', \theta', \phi'), \theta(\eta', \theta', \phi'), \phi(\eta', \theta', \phi')).
\]

(40)

The explicit coordinate transformation between the two sets of intrinsic coordinates is complicated. The simpler thing to do is to first express the source \(J(\eta, \theta, \phi)\) in terms of Cartesian coordinates \(x_i = X_i / X_4\). We can now use the known functional dependence of old

\[\text{The integration variables in (39) should have been (}\hat{\eta}', \hat{\theta}', \hat{\phi}'.\) The primes have been removed for notational simplification.
embedding coordinates in terms of the new to express the source in new intrinsic coordinates (by using (38)).

One can show that this method gives the correct answer for an arbitrary constant stress tensor\(^7\). There are many ways to do it, but we have shown the one that we will use for the non-constant stress tensor. We now move on to the non-constant stress tensor, in which we use the above treatment to find the relative entropy correction.

3. Non-constant stress tensor

3.1. Deformation of the metric

For a space-dependent perturbation, let us restrict our attention to stress tensors whose variations are small compared to the size of the entangling region. To be precise, we want\(^8\)

\[
\mathcal{O}(R^2 \partial T \partial T) \ll \mathcal{O}(R^2 \partial T) \tag{41}
\]

and

\[
\mathcal{O}(R^2 \partial^2 T \partial T) \ll \mathcal{O}(R \partial T), \tag{42}
\]

and higher derivatives are similarly suppressed. So we will not consider beyond two derivatives of \(T_{\mu\nu}\) in our calculations. Note that, \(T_{\mu\nu}\) must satisfy the traceless (\(\mu\nu T^\mu_\nu = 0\)) and divergenceless (\(\partial_\mu T^\mu_\nu = 0\)) conditions. The boundary metric up to two derivatives in the quadratic correction, looks like

\[
\frac{z^2}{L^2} g_{\mu\nu} = \eta_{\mu\nu} + z^4 \left( T_{\mu\nu} - \frac{1}{12} z^2 \Box T_{\mu\nu} \right) + z^8 \left( n_1 T_{\mu\rho} T^\rho_\mu + n_2 \eta_{\mu\rho} T^\rho_\mu T_\alpha^\beta + z^2 T_{\mu\nu} \right). \tag{43}
\]

The last term is given by

\[
T_{\mu\nu} = n_3 \left( T_{\mu\nu} \Box T^\alpha_\alpha + T_{\alpha\beta} \Box T^\alpha_\beta \right) + n_4 \eta_{\mu\rho} T_\rho^\alpha T^{\alpha\beta} + n_5 \partial_\rho T_\rho^\alpha \partial^\alpha T^{\alpha\beta} + n_6 \partial_\rho T_\rho^\alpha \partial^\beta T^{\alpha\beta} + n_7 \partial_\rho \partial^\beta T^{\alpha\beta} + n_8 \partial_\rho \partial^\beta T^{\alpha\beta} + n_9 \eta_{\mu\rho} \partial_\rho \partial^\alpha \partial^\beta T^{\alpha\beta} + n_{10} \eta_{\mu\rho} \partial_\rho \partial^\beta T_\rho^\alpha \partial^\alpha \partial^\beta T^{\alpha\beta} + n_{11} \eta_{\mu\rho} \partial_\rho \partial^\beta T_\rho^\alpha \partial^\beta \partial^\alpha T_{\mu\nu} + n_{12} \left( T^{\alpha\beta} \partial_\rho \partial^\alpha \partial_\beta T_{\mu\nu} + T^{\alpha\beta} \partial_\rho \partial^\beta \partial_\alpha T_{\mu\nu} \right) + n_{13} T^{\alpha\beta} \partial_\rho \partial^\beta \partial_\rho \partial^\alpha T_{\mu\nu}.
\]

In other words, the leading correction comes from \(T_{\mu\nu}\) and subleading from \(\Box T_{\mu\nu}\). It was shown in [6] that the entropy change corresponding to both of these, at linear order, satisfies

\[
\Delta S = \Delta H. \tag{45}
\]

So we must look at the corrections coming from the \(\mathcal{O}(TT), \mathcal{O}(\partial T \partial T)\) and \(\mathcal{O}(T \partial T)\) terms. We expect these contributions to be negative. From equations (41) and (42) it is clear that there will be no correction from any higher derivative terms. So the above form of metric will suffice.

The parameters \(n_1, \ldots, n_{13}\) appearing in the metric have to be fixed from the Einstein equations. From our knowledge of the constant perturbation, we already know that \(n_1 = 1/2\) and \(n_2 = -1/24\). Using a convenient functional form of \(T_{\mu\nu}\) satisfying the traceless and

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\(^{7}\) See appendix B.1 for a detailed calculation in the case of arbitrary constant source.

\(^{8}\) \(\mathcal{O}(\cdots)\) refers to all possible contractions of the tensors appearing in the argument of \(\mathcal{O}\).
divergence-less conditions, we compute \( R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - 6 g_{\mu\nu} \) in terms of \( n_3, \ldots, n_{13} \) and set all the components to zero. Then we see that for any such choice of stress tensor, the Einstein equations will be satisfied only if the parameters in (44) take the values

\[
\begin{align*}
n_3 &= -\frac{1}{24}, \\
n_4 &= \frac{1}{180}, \\
n_5 &= -\frac{1}{180}, \\
n_6 &= -\frac{1}{60}, \\
n_7 &= \frac{1}{360}, \\
n_8 &= 0, \\
n_9 &= \frac{1}{120}, \\
n_{10} &= -\frac{1}{720}, \\
n_{11} &= 0, \\
n_{12} &= -\frac{1}{120}, \\
n_{13} &= \frac{1}{60}.
\end{align*}
\]

(46)

We will begin by keeping all the above parameters to be arbitrary. This will make the correction \( \Delta S^{(2)} \) dependent on these 13 parameters. So, we have a 13-dimensional parameter space, instead of 2. However, it is always possible to constrain a certain subspace of some (say two) parameters, by fixing the rest at the Einstein values. Of course, we are primarily interested in the subspace of \( n_1 \) and \( n_2 \), which correspond to the zeroth order in the quadratic derivative expansion.

3.2. The correction to \( z_1 \) and the area functional

To get the change of minimal surface, \( z_1 \), we need to find the source function. The source term for non-constant stress tensor is obviously different from the constant case. We can find it by calculating the area functional and minimizing it w.r.t. \( z_1 \). We have shown the steps in appendix B.2. The source term works out to be

\[
J = -\frac{z_0^5}{2} \left( 2T + 6T_x - \frac{z_0^2}{3} \left( \frac{R^2}{2} \partial^2 T_x + 2x^i x^j \frac{R^2}{2} \partial^2 T_{ij} \right) + x^i \partial_i T + 2x^i \partial_i T_{ij} + \frac{1}{R^2} x^i x^j \frac{R^2}{2} \partial_k T_{ij} \right),
\]

(47)

where we have retained terms only up to two derivatives of the stress tensor. Note that we are working in a time-independent background. So, time derivatives of the stress tensor do not appear.

Now we use the formula (39) with the above source to find \( z_1 \). The trick is to go to Fourier space, so that the source is nothing but an exponential. Then the actual source (47) is written in terms of derivatives of the exponential to find the actual solution. The method is shown in detail in appendices B.1 and B.2, for constant \( T_{ij} \) and non-constant \( T_{ij} \) respectively. Here we just quote the result

\[
z_1 = -z_0^3 R^2 \left( \frac{T + T_x}{10} + \frac{1}{12} \left( x^i \partial_i T + x^i x^j x^k \frac{R^2}{2} \partial_k T_{ij} \right) \left( \frac{T + T_x}{10} \right) \right) + \frac{1}{28} \left( x^i x^j \partial_i T + x^i x^j x^k x^l \frac{R^2}{2} \partial_k T_{ij} \right) - \frac{k^2 \left( R^2 - r^2 \right)}{168} \left( \frac{R^2}{2} + x^i \frac{R^2}{2} \partial^2 T_{ij} \right).\]

(48)

Here all \( T_{ij} \)s and their derivatives are evaluated at the origin and this will be the case from here onwards, unless and otherwise mentioned. The next step is to evaluate the area functional in terms of \( T_{ij} \)s and derivatives at the origin.

We already know how the area depends on only the \( T^2 \) terms (7). Now the next order should contain one-derivative of \( T_{ij} \). This means we need to integrate over terms like \( T_{ij} x^i \partial_j T_{ij}, T_{ij} x^i x^j \partial_j T_{ij} \) and so on. All these terms have an odd number of \( x \)-s. If we assume that \( T_{ij} \) is a smooth function, this integral must vanish

9 The nonlinear terms in equation (44) will not modify the source term since they are second order in the perturbation while the source is first order.
Thus, we have to evaluate the area upto two derivatives of \( T_{ij} \). This was the reason why the metric (43) was constructed upto two derivatives of \( T_{\mu \nu} \). Since the \( z_1 \) solution depends on derivatives of \( T_{ij} \) evaluated at origin, we must do the same for the area functional. This means all \( \vec{T}_{ij} \) appearing in the area formula must be Taylor expanded around the origin first, and then integrated. The details of this straightforward but tedious calculation has been shown in the appendix C. Here we quote the result

\[
\int \! d^d \! \sqrt{h} = \frac{4 \pi L^3 R_{10}}{31 \cdot 185} \left[ (10 - 12n_1 + 2160n_{11} + 720n_6 + 1440n_9) \left( \partial_i T_{jk} \partial^i T^{ji} \right) \\
+ 48(7n_2 + 45n_4 + 15n_7)T^2 + (-120n_1 - 672n_2 - 1440n_3) \\
- 4320n_4 - 1440n_7)T^0 \partial^2 T_{0ij} + (-12 + 720n_{13})T^0 \partial_i \partial_j T \\
+ (-55 + 120n_1 + 2160n_{10} + 336n_2 + 720n_5 + 720n_8) \left( \partial_i T_{jk} \right)^2 \\
+ (12n_1 - 2160n_{11} - 1440n_9) \partial_j T_{00} \partial^i T^{0i} \\
+ (5 + 2160n_{10} + 336n_2 + 720n_5) (\partial_i T)^2 \\
+ (120n_1 + 336n_2 + 1440n_{13} + 2160n_{14} + 720n_{17})T^0 \partial^2 T_{ij} \\
+ (-120n_1 - 4320n_{10} - 672n_2 - 1440n_5 - 720n_8) \left( \partial_i T_{00} \right)^2 \\
+ \mathcal{O}(TT). \tag{50}
\]

In \( d = 4 \), the metric (43) satisfies Einstein’s equations for \( n_1 = 1/2 \) and \( n_2 = -1/24 \) and the values given in (46). For these values the expression above simplifies to

\[
\int \! d^d \! \sqrt{h} = \frac{8 \pi L^3 \left( 5(\partial_i T)^2 + 15(\partial_i T_{00})^2 + 3\partial_i T_{00} \partial^i T_{00} + 5(\partial_i T_{jk})^2 - 2\partial_i T_{00} \partial^i T \right) R_{10}}{31 \cdot 185} \\
+ \mathcal{O}(TT). \tag{51}
\]

Very interestingly, all the ‘cross’ tensor structures, i.e. \( T^0 \partial^2 T_{ij} \), \( T^i \partial_j \partial_i T \), \( T \partial^2 T \) and \( T^0 \partial^2 T_{00} \) vanish at these values. But first, let us see what we can imply from the result (50). Let us add (50) to the previous result (7). This gives the total subleading correction to the entropy in terms of \( T_{\mu \nu} \) and its derivatives at the origin.

\[
\Delta^{(2)} S = \frac{8 \pi L^3 R^8}{4725 \ell_p^2} \left( -160(n_1 + 6n_2)(T_{00})^2 + 8(-9 + 20n_1 + 60n_2)(T_{00})^2 \\
+ 8(1 + 60n_2)T^2 \right) + \frac{8 \pi L^3 R_{10}}{31 \cdot 185 \ell_p^2} \left[ (10 - 12n_1 + 2160n_{11} \\
+ 720n_6 + 1440n_9) \left( \partial_i T_{jk} \partial^i T^{ji} \right) \\
+ 48(7n_2 + 45n_4 + 15n_7)T^2 \\
+ (-120n_1 - 672n_2 - 1440n_3 - 4320n_4 - 1440n_7)T^0 \partial^2 T_{00} \right]
\]
+ (−12 + 720n_{13}) T^{ij} \partial_i \partial_j T
+ (−55 + 120n_1 + 2160n_{10} + 336n_2 + 720n_5 + 720n_8) (\partial_i T_{jk})^2
+ (12n_{11} - 2160n_{11} - 1440n_6) \partial_i T_{0j} \partial^j T^{0i}
+ (5 + 2160n_{10} + 336n_2 + 720n_5) (\partial_i T)^2
+ (120n_1 + 336n_2 + 1440n_3 + 2160n_4 + 720n_{22}) T^{ij} \partial^2 T_{ij}
+ (−120n_1 - 4320n_{10} - 672n_2 - 1440n_5 - 720n_8) (\partial_i T_{0j})^2 \Biggr] .
\end{align}
\end{equation}

(52)

4. Constraints on the nonlinear parameters

For a unitary theory, we expect the above quantity (52) to be negative. In other words, only those values of \( n_1, \ldots, n_{13} \), for which the above quantity is manifestly negative, are viable for unitarity. But first we need to make sure that this is indeed the case for Einstein values of the 13 parameters. For these values, i.e. \( n_1 = 1/2, n_2 = -1/24 \) and (46), the total subleading correction simplifies to

\[ \Delta^{(2)} S = -16 \sigma^2 R^{10 \frac{1}{2}} \left[ 6 T^2 + 20(T_{0i})^2 + 6(T_{ij})^2 \right. \]
\[ + \left. \frac{4725 R^2}{31 185} \right] . \]
\[ (53) \]

We already knew that the \( O(TT) \) part of this is negative. Fortunately, the \( T \partial \partial T \) ‘cross’ structures, which are not manifestly positive definite, do not appear at these values. However there are still two terms, namely \( \partial_i T_{0j} \partial^j T^{0i} \) and \( \partial_i T_{jk} \partial^j T^{ij} \), which are not positive definite. So to ensure the negativity of the expression (53) we have to express it as a sum of squares. As we did in (9), we can write it as a matrix inner product, \( V^T M V \), where \( V \) is a column vector. The elements of \( V \) are of the form, \( \partial_i T_{jk} \) and they are linearly independent of each other, made sure by the constraints

\[ \partial_i T_{ij} = 0, \quad \partial_i T_{0j} = 0 \quad \text{and} \quad \partial_i T_{jk} = \partial_i T_{kj} . \]
\[ (54) \]

We choose \( V \) to be

\[ V = \{ \partial_i T_{02}, \partial_i T_{03}, \partial_i T_{20}, \partial_i T_{22}, \partial_i T_{01}, \partial_i T_{23}, \partial_i T_{02}, \partial_i T_{03}, \partial_i T_{12}, \partial_i T_{13}, \partial_i T_{22}, \partial_i T_{23}, \partial_i T_{10}, \partial_i T_{11}, \partial_i T_{21}, \partial_i T_{22}, \partial_i T_{33}, \partial_i T_{10}, \partial_i T_{11}, \partial_i T_{12}, \partial_i T_{13}, \partial_i T_{21}, \partial_i T_{22}, \partial_i T_{23}, \partial_i T_{33}, \partial_i T_{31}, \partial_i T_{32}, \partial_i T_{33}, \partial_i T_{11}, \partial_i T_{12}, \partial_i T_{13}, \partial_i T_{21}, \partial_i T_{22}, \partial_i T_{23}, \partial_i T_{33} \} . \]
\[ (55) \]

Now we diagonalize the 23 \times 23 matrix \( M \) with a matrix \( U \). Then we can write

\[ \Delta^{(2)} S = V^T M V = V^T U^T M u V = (UV)^T M u (UV) = \sum_{i=1}^{23} \lambda_i \left( UV \right)^2 . \]
\[ (56) \]

Here \( \lambda_i \) are the eigenvalues of the matrix \( M \). For the Einstein values of \( n_1 \) and \( n_2 \) they are\(^{10} \) (in units of \( 8 \sigma^2 R^{10 \frac{1}{2}} L^3 / \nu_P^3 \))

\(^{10} \) The superscripts on the eigenvalues indicate degeneracy.
\[ \lambda = \begin{cases} -0.000769^3, & -0.00115^4, & -0.00346, & -0.000769^2, & -0.000384, \\ -0.000577^3, & -0.000256^3, & -0.00207^3, & -0.000428^3 \end{cases}. \]  

(57)

Since all are negative, the sum in (56) is manifestly a negative definite quantity. Pleasingly, this confirms our expectation that, at this nonlinear order, the CFT dual to Einstein gravity is unitary.

The main question of interest is if the 13-dimensional parameter space is bounded by the inequality \( \Delta^{(2)} S \leq 0 \). To begin with let us ask what is the influence of the non-constant stress tensor on the \((n_1, n_2)\) parameter space. Let us begin by setting all the rest of the \(n_i\)'s to their Einstein values (46). A problem that will arise in this case is that the \( \mathcal{O}(T \partial T) \) ‘cross’ terms will have non-zero coefficients. The above method of diagonalizing the coefficient matrix will fail because the cross terms cannot give a sum of squares. Even if we try to combine them with the \( \mathcal{O}(TT) \) terms to complete the square (i.e. \( \mathcal{O}(TT) + \mathcal{O}(T \partial T) \rightarrow \mathcal{O}((T + \partial T)(T + \partial T)) \)) then some new terms \( \mathcal{O}(\partial T \partial T) \) will appear. Since we do not know how such terms actually appear in \( \Delta^{(2)} S \), it is not a good idea to do it this way.

The best way forward is to make the stress tensor vanish at the origin, i.e.

\[ T_{\mu \nu}(x = 0) = 0. \]  

(58)

Note that this does not violate our assumptions (41) and (42). Since all we want is to see new constraints on the nonlinear parameters, we are allowed to assume any form of the stress tensor as long as it does not contradict our previous assumptions. With this \( T_{\mu \nu} \) (52), simplifies to

\[ \Delta^{(2)} S = \frac{16\pi^2 L^3 R^{10}}{31 \times 185 \ell_p^3} ( -\partial_i T_{ij})^2 (-1 + 60n_1 + 336n_2) + 6\partial_i T_{ij} \partial^j T_{0i}(-1 + n_1) \]
\[ + \left( \partial_i T_{ij} \right)^2 (-28 + 60n_1 + 168n_2) + \partial_i T_{ij} \partial^j T^{ij}(5 - 6n_1) \]
\[ + (\partial_i T)^2 (2 + 168n_2). \]  

(59)

We will use the diagonalization (56) once again. For arbitrary \(n_1\) and \(n_2\), the eigenvalues of \(M\) are

\[ \begin{pmatrix} 2(-7 + 66n_1 + 336n_2) \end{pmatrix}^{\chi^3}, \begin{pmatrix} 2(5 + 54n_1 + 336n_2) \end{pmatrix}^{\chi^4}, \begin{pmatrix} 2(5 + 54n_1 + 336n_2) \end{pmatrix}^{\chi^5}, \begin{pmatrix} 2(-61 + 126n_1 + 336n_2) \end{pmatrix}^{\chi^2}, \begin{pmatrix} 4(-23 + 54n_1 + 168n_2) \end{pmatrix}^{\chi^6}. \]
For relative entropy to be positive we have to demand each of them to be negative. Moreover, since now there is no $\mathcal{O}(TT)$ term in $\Delta^{(2)} S$, these are completely new constraints on $n_1$ and $n_2$, independent of the previous constraints. Figure 2 shows the new constrained region in the $n_1$, $n_2$ parameter subspace.

Quite remarkably, the allowed region is a triangular figure very similar to the the one we had obtained from constraining the $\mathcal{O}(TT)$ terms. Since these are independent constraints, the real allowed values of $n_1$ and $n_2$ must lie in the region intersecting the two triangles.

What if we had chosen a different subspace than $(n_1, n_2)$? Say, we wanted to constrain the subspace of $n_3$ and $n_4$. These parameters correspond to $(T_{\mu\nu}\Box T_{\nu}^{\alpha} + T_{\alpha\nu}\Box T_{\nu}^{\mu})$ and $T_{\alpha\beta}\Box T^{\alpha\beta}$. Since we had to eliminate the $\mathcal{O}(T\partial T)$ terms in order to check the sign of $\Delta^{(2)} S$, it is not possible to constrain the parameters related to these quantities, such as $n_3$ and $n_4$ with the method discussed above. The same holds for $n_7$, $n_{12}$ and $n_{13}$. To constrain them we must consider one higher order in the derivative expansion. However it is possible to constrain the rest of the parameters. The above technique does give us some conditions on $n_5$, $n_6$, $n_9$, $n_{10}$ and $n_{11}$, and they do confine the allowed region in the parameter space (see figure 3). For some of them the two-dimensional subspaces are bounded to small regions. For others, the conditions are not sufficient to constrain them in all directions. But, it is always possible that we get further constraints on these parameters by looking at higher derivative orders. The important point is we do get bounds in the parameter space from the derivative expansion. It will be interesting to see what new constraints are obtained from higher order corrections.

We made a curious observation that at the Einstein values of all parameters, the $\mathcal{O}(T\partial T)$ `cross’ terms disappear. We do not have a good answer to how this happens, or what it implies. The cross terms can take both signs. So, till now their absence only serves to make it easy to check the sign of $\Delta^{(2)} S$ at the Einstein point. If they were present, we would have had to look at $\mathcal{O} (\partial T \partial T)$ corrections to the entropy. This is the problem that we face for a general stress tensor, with arbitrary parameters. It would have been wonderful if there was some argument that prevents the cross terms to be present. Then we would have obtained the exact results of nonlinear Einstein equations from entanglement entropy. However it is unlikely that this can come from positivity of relative entropy, since it is an inequality and what we are talking of is a statement of equality. But it does give hope of shedding some light on the bigger question: how are Einstein equations encoded in entanglement entropy? This deserves further investigation.
5. Constraints in arbitrary dimensions

The above analysis can be easily extended to arbitrary dimensions. The first step would be to find $z_1$. It must satisfy the differential equation (16) with the source term

\[ \mathcal{O} \left( \partial \mathcal{T} \partial \mathcal{T} \right) \]
Let us make an educated guess that even in an arbitrary \( d \), we will have the same form of \( z_1 \) as in (48). So without having to worry about the Green’s function, we easily find the following solution

\[
J = -\frac{z_0^{d+1}}{2} \left( (d-2)T + (d+2)T_\xi - \frac{z_0^2}{2(d+2)} \left( d\ \partial^2 T + (d+4)\frac{x^i x^j}{R^2}\partial^2 T_{ij} \right) + x^i \partial_i T + 2x^i \partial_i T_{ij} + \frac{1}{R^2} x^i x^j x^k \partial_k T_{ij} \right).
\]

(61)

For arbitrary \( d \), the metric will be

\[
\frac{z^2}{L^2} g_{\mu\nu} = \eta_{\mu\nu} + z^d \left( T_{\mu\nu} - \frac{z^2 \Box T_{\mu\nu}}{2(d+2)} \right) + z^{2d} \left( n_1 T_{\mu\alpha} T^\alpha_{\nu} + n_2 \eta_{\mu\nu} T^{\alpha\beta} + z^2 T_{\mu\nu} \right),
\]

(62)

where \( T_{\mu\nu} \) is the same as in (44). However the Einstein values of the nonlinear parameters will be different. The first two have the values, \( n_1 = 1/2 \) and \( n_2 = -1/(8(d-1)) \). We will be keeping these two arbitrary, however, and only put the values of the other parameters.
\[ n_3 = -\frac{1}{4(d+2)}, \quad n_4 = \frac{2d}{2d(d+2)(d-1)/8}, \quad n_5 = -\frac{d}{8(d+2)(d^2-1)}. \]

\[ n_6 = -\frac{1}{2(d+1)(d+2)}, \quad n_7 = \frac{d-2}{2(d+2)(d^2-1)}, \quad n_8 = 0, \]

\[ n_9 = \frac{1}{4(d+2)(d+1)}, \quad n_{10} = \frac{1}{8(d+2)(d^2-1)}, \quad n_{11} = 0, \]

\[ n_{12} = -\frac{1}{4(d+1)(d+2)}, \quad n_{13} = \frac{1}{2(d+1)(d+2)}. \]  

(64)

With this in hand, we can proceed to compute the entropy correction \( \Delta^{(2)} S \). The steps are similar to what is shown in appendix C, but more tedious since \( d \) is arbitrary. So we will just mention the result

\[
\Delta^{(2)} S = -2^{-5-d}d \sqrt{\mathcal{R}} R^{2+2d} \Gamma [-1 + d] \\
\left( -1 + d \right)(2 + d) \Gamma \left[ \frac{5}{2} + d \right] \left\{ (2 + d)(-2 + d^2) \right. \\
\times \left( 1 + 8(-1 + d)n_2 \right) \right\} T \partial^2 T \\
+ (\partial T)^2 \left\{ 4 + d - 2d^2 - 8(-1 + d)(2 + d)(-2 + d^2)n_2 \right\} \\
+ \partial T_0 \partial^i T_i \left\{ -4(-2 + d + d^2)(-1 + n_1) \right\} + \left\{ 4(-1 + 4n_1 - 8n_2) \right. \\
+ d(-3 + 8n_1 + 16n_2 + 2d(3 + 2d - 8n_1 - 4dn_1 - 4d) \\
+ d + d^2)n_2 \right\} \right\} (\partial T_0)^2 \\
+ \left\{ 2 \left( 2 + d \right)(-1 + (-1 + d^2)(-1 + 4n_1) + 16n_2 + 8(-2) \\
+ d)(1 + d)n_2 \right\} T_0 \partial^2 T_i \\
+ \left\{ 2 \left( 2 + d \right)(-1 + 4(-1 + d^2)n_1 \\
+ 8(-1 + d)(-2 + d^2)n_2 \right\} \right\} (\partial T_0)^2 \\
+ \left\{ 4(3 - 1)(-1 + d(-1 + n_1) + 2n_1) \right\} \partial T_0 \partial^2 T^i \right\} + \mathcal{O}(T^2). 
\]

(65)

As before, to find constraints on \( n_1 \) and \( n_2 \), we have to write the above expression as a sum of squares. For that, we need to set \( T_{0\mu}(\zeta = 0) = 0 \). The matrix \( V \), consisting of the independent structures, will be \( \frac{1}{2} \left( 2 - 5d + d^2 \right) \)-dimensional. Now, if we write \( \Delta^{(2)} S \) as \( V^T M V \) as before and diagonalize \( M \), we get the following eigenvalues

\[
2b - c, \quad 2(b + c), \quad \frac{1}{2} \left( 3b + c \pm \sqrt{b^2 + 2bc + 5c^2} \right), \quad e - f, \quad e + f, \quad 4(e + f), \\
\frac{1}{2} \left( b - c + 2a(-2 + d) + (b + c)d \right) \\
\pm \sqrt{4a^2(d - 2)^2 + 8bc(d - 2)^2 - 4ac(d - 1) + b^2(d - 1)^2 + c^2(5 + (d - 2)d)}. 
\]

(66)
where

\[ a = \frac{\sqrt{\pi} 2^{d-5} d \left(8(d - 1)(d + 2) \left( d^2 - 2 \right) n_2 + 2(d + 1)^2 - 5(d + 1) - 1 \right) \Gamma [d - 1]}{(d - 1)(d + 2) \Gamma \left[ d + \frac{5}{2} \right]} \]

\[ b = 2^{-5-d} d \left( -1 + 16n_2 + (1 + d)(3 - 16n_1 \\
+ 40n_2 + 2(1 + d) \left( 3 - 2(1 + d) \right) - 4dn_1 + 4(-1 + (-2 + d)(1 + d)n_2) \right) \]

\[ \times \sqrt{\pi} \Gamma [d - 1] (1 - d)(2 + d) \Gamma \left[ \frac{5}{2} + d \right] \]

\[ c = -\frac{\sqrt{\pi} 2^{-d-3} d \left( (d + 2)n_1 - (d + 1) \right) \Gamma [d - 1]}{(d + 2) \Gamma \left[ d + \frac{5}{2} \right]} \]

\[ e = -\frac{\sqrt{\pi} 2^{-d-4} d \Gamma [d - 1] (4(d - 1)(d + 1)n_1 + 8(d - 2)d(d + 1)n_2 + 16n_2 - 1)}{(d - 1) \Gamma \left[ d + \frac{5}{2} \right]} \]

\[ f = \frac{\sqrt{\pi} 2^{-d-3} d \left( n_1 - 1 \right) \Gamma [d - 1]}{\Gamma \left[ d + \frac{5}{2} \right]} \]

(67)

with increasing degeneracies for increasing \( d \). In order for \( \Delta^{(2)} S \) to be negative, if we demand all these eigenvalues to be negative, we again get a triangular region in the \((n_1, n_2)\) subspace. With increasing \( d \), we observe that the triangle gets thinner and bends towards the \( n_2 = 0 \) line. As before, for a certain \( d \), this region (figure 4) looks very similar to the one obtained from constant stress tensor constraints. Note that, the last eigenvalue is valid only for odd \( d \). For even \( d \), it will be different, while we will get even more eigenvalues as \( d \) increases. However, they will lead to the same triangular region that will keep getting thinner and bend towards the \( n_2 = 0 \) line with increasing \( d \), with no extra constraint coming from the new eigenvalues. This has been checked explicitly for \( d = 4, 6 \) and 8.

An interesting lesson that we can learn from the arbitrary \( d \) analysis, is what happens to the region for \( d \to \infty \). In any dimension, the edges of the triangular region are given by the equation formed by setting the above eigenvalues to zero. The vertices will then be given by solving these equations pairwise. These points do not have simple expressions, so we will not show them here. But if we take the large \( d \) limit of the points, they reduce to \((0, 1), (1/2, 0)\) and \((1, 0)\). So the triangular region becomes a straight line \( 0 < n_1 < 1 \) at \( d \to \infty \). Curiously, this is the same line that was obtained for \( d \to \infty \) from the constant stress tensor constraints [8].

6. Discussion

In this paper, we used the positivity of relative entropy to constrain nonlinear perturbations in the metric. We took an AdS5 metric and perturbed the boundary with a space-dependent stress tensor, at both linear and nonlinear level. We demanded that the correction to \( \Delta^{(2)} S \) coming from the nonlinear terms, has to be negative to ensure the positivity of relative entropy. We were motivated mainly by two questions. First, is relative entropy positive for any stress
tensor, which is not necessarily constant? Second, what information about gravity can we get from space-dependence of the stress tensor?

One major finding of this paper was the Green’s function, needed to calculate the correction $z_1(x)$ to the Ryu–Takayanagi area functional. It was needed to find the nonlinear correction $\Delta^{(2)} S$ to the entanglement entropy. The correction $z_1(x)$ satisfies a differential equation. This equation had to be solved by guessing, in the original work [6] for a constant perturbation. Without a proper Green’s function, it would not be possible to solve it for more general perturbations. The key was to observe that the minimal surface deformation was like a propagating massive scalar field on AdS$_3$. Thus, bulk to bulk propagator of the scalar field gave us the Green’s function. So, we could develop a systematic way to do analytic computations with nontrivial perturbations to the boundary.

We perturbed the boundary metric with a slowly varying space-dependent stress tensor up to a quadratic order. We neglected contributions coming from more than two derivatives by assuming a derivative expansion. We calculated $z_1$ from the Green’s function and that gave us the correction $\Delta^{(2)} S$ up to two derivatives of $T_{\mu\nu}$. We found that this quantity was manifestly negative, if the nonlinear parameters of the metric assumed values that solve Einstein equations. Finally we tried to find new constraints on the nonlinear parameters appearing in the metric, by keeping them arbitrary. First we chose to stay only on the subspace of $n_1$ and $n_2$, by fixing the other parameters at the Einstein values. These were the parameters corresponding to zeroth order of the derivative expansion of $T_{\mu\nu}$. We assumed that the stress tensor is zero at the centre of the ball. Then demanding that $\Delta^{(2)} S$ be negative, we obtained new constraint conditions on $n_1$ and $n_2$. These conditions allowed $n_1$ and $n_2$ to be in a bounded region.

In summary, we can arrive at two main conclusions. First of all, the positivity of relative entropy, for Einstein values of the parameters, implies that for any non-constant stress tensor in the metric (at least up to two orders in a derivative expansion), if Einstein equations are satisfied, then the dual theory will be unitary, as expected. Now we find that the positivity condition constrains the nonlinear parameters in a bounded region. Since the constrained region found from two derivatives of the perturbation is independent of those found from the constant part, $n_1$ and $n_2$ can only be in the overlapping region. Points outside this region correspond to non-unitary theories. Thus introducing coordinate dependence the allowed region gets smaller. Our analysis in finite dimensions indicates two possibilities: either the allowed region may shrink to the Einstein point by considering more involved stress tensors, or we get wider class of theories at the nonlinear level which are unitary. In the latter case it will be interesting to study the theories that are at the boundary of the allowed region. However our analysis was done for a spherical entangling region, perturbed by a boundary stress tensor. So, the tantalizing possibility is the former that one might be able to shrink the region to the Einstein point by considering a more sophisticated situation. That will be equivalent to deriving nonlinear Einstein equations from entanglement entropy, and will be a major breakthrough. In fact we observed some interesting things like vanishing of the cross terms of constant and space-dependent part of the stress tensor from $\Delta^{(2)} S$ at Einstein values of the parameters (53). Considering this it is indeed interesting to see what happens at the next order. However such a calculation is more involved to present here. Another interesting direction would be to repeat our calculations for higher derivative gravity. Then we have to replace the Ryu–Takayanagi functional with higher derivative entropy functionals (for recent work see [11]). For example, in Gauss–Bonnet gravity, we may have a third axis in the diagram showing how the constraining region varies with the higher derivative parameter. The techniques used in this paper will be useful in answering these questions.
Acknowledgments

The work of SB was supported by the World Premier International Research Center Initiative (WPI), MEXT, Japan. AS acknowledges support from a Ramanujan fellowship, Govt. of India.

Appendix A. Area functional for constant stress tensor

In this appendix we review the calculation of [8] to obtain $\Delta S^{(2)}$ for a general constant stress tensor with arbitrary $n_1$ and $n_2$. This will be helpful to understand the more complicated non-constant case. We start with the Ryu–Takayanagi prescription for calculating entanglement entropy in holography

$$S = \frac{2\pi}{\ell_p^{d-1}} \int d^{d-1}x \sqrt{h}.$$  \hfill (68)

From Taylor expansion one can show that the quadratic correction to $\sqrt{h}$ is

$$\delta^{(2)} \sqrt{h} \equiv \frac{1}{8} \sqrt{h} (h^{ij}\delta h_{ij})^2 + \frac{1}{4} \sqrt{h} \delta h^{ij} \delta h_{ij} + \frac{1}{4} \sqrt{h} \delta^{(2)} h_{ij}.$$  \hfill (69)

The induced metric is given by $h_{ij} = (L^2/z^2)\left(\delta_{ij} + \partial_i z \partial_j z\right)$. This has to be evaluated at the Ryu–Takayanagi extremal surface. Since we are considering the entanglement of a ball of radius $R$, the extremal surface is given by

$$z = z_0 + z_1 = \sqrt{R^2 - r^2} + z_1.$$  \hfill (70)

Here the part $z_1$ is the result of deformation of the metric given in (2). $z_1$ is obtained by plugging (70) into (69) and then minimizing it. In $\Delta^{(2)} S$, we get three kinds of second order contributions

$$\Delta^{(2)} S = \frac{2\pi}{\ell_p^{d-1}} \int d^{d-1}x \delta^{(2)} \sqrt{h} = \frac{2\pi}{\ell_p^{d-1}} (A_{2,0} + A_{2,1} + A_{2,2}).$$  \hfill (71)

This grouping is done according to powers of $z_1$ appearing in the second index. Hence $A_{2,0}$ contains only $O(TT)$ terms. We compute it by setting $z_1 = 0$. Then we get from (69)

$$A_{2,0} = L^{d-1} \int d^{d-1}x \, Rz_0^d \left( T_0^0 T^0 \left( \frac{n_1}{2} + (d-1)n_2 + \frac{n_2^2 R^2}{2 R^2} \right) + (T_0) \left( \frac{n_2}{2} \delta h_{ij} \right) 
+ (T_0) \left( \frac{n_2}{2} \delta h_{ij} \right) 
+ (T_0) \left( \frac{n_2}{2} \delta h_{ij} \right) 
+ (T_0) \left( \frac{n_2}{2} \delta h_{ij} \right) 
+ (T_0) \left( \frac{n_2}{2} \delta h_{ij} \right) 
+ (T_0) \left( \frac{n_2}{2} \delta h_{ij} \right) 
+ (T_0) \left( \frac{n_2}{2} \delta h_{ij} \right) 
+ (T_0) \left( \frac{n_2}{2} \delta h_{ij} \right) 
+ (T_0) \left( \frac{n_2}{2} \delta h_{ij} \right) \right).$$  \hfill (72)
Here $T_i = x^i T_0 / R^2$. The terms $A_{2,1}$ and $A_{2,2}$ are same as in [6]. We quote their result

$$A_{2,1} = L^{d-1} \int d^{d-1}x \frac{R}{z_0} \left[ T \left( \frac{z_0^2}{R^2} x^i \partial_i z_1 \right) + \frac{T_{ij}}{R^2} \left( 2 z_0^2 x^i \partial_i z_1 - z_1 x^i x^j - \frac{z_0^2 x^i x^j x^k}{R^2} \partial_k z_1 \right) \right].$$

$$A_{2,2} = L^{d-1} \int d^{d-1}x \frac{R}{z_0} \left[ \frac{d (d-1) z_1^2}{2 z_0^2} + \frac{z_0^2 (\partial_z z_1)^2}{2 R^2} - \frac{z_0^2 (x^i \partial_i z_1)^2}{2 R^4} + \frac{(d-1) x^i \partial_i z_1^2}{2 R^2} \right].$$

Minimizing $A_{2,1} + A_{2,2}$ w.r.t. $z_1$ gives the differential equation (5). The solution can be found by guessing

$$z_1 = - \frac{R^{2+1}}{2(d+1)} (T + T_0).$$

Plugging this in (71) and summing we get

$$\int d^{d-1}x \delta^{(2)}(z) \sqrt{g} = L^{d-1} \int d^{d-1}x \left( c_1 T^2 + c_2 T^2 + c_3 T^2 + c_4 T_0 T^0 + \frac{c_5 x^i x^l T^2}{R^2} + \frac{c_6 x^i x^l T_0}{R^2} + c_7 T T \right),$$

where the coefficients $c_1 \cdots c_7$ are

$$c_1 = \frac{\left( R^2 - r^2 \right)^{(d-1)/2}}{8(1 + d) R} \left( -4(1 + d)^2 n_2 \left( r^2 - R^2 \right)^2 \left( r^2 - (d-1) R^2 \right) + R^2 \left( 2 \left( d^2 + 2d - 1 \right) r^4 + \left( 1 - 5d^2 \right) r^2 R^2 + \left( 2d^2 - d - 1 \right) R^4 \right) \right).$$

$$c_2 = \frac{-r^2 + R^2 \left( 1 - 5d^2 \right) R^2 + \left. 3 + d \right\} 3 + 4d \} R^2 \right)}{8(1 + d)^2},$$

$$c_3 = \frac{-r^2 + R^2 \left( -2n_2 r^2 + \left. -1 + 2n_1 + 2(-1 + d) n_2 \right\} R^2 \right)}{4R},$$

$$c_4 = \frac{-r^2 + R^2 \left( n_1 R^2 - 2n_2 \left( r^2 - (-1 + d) R^2 \right) \right)}{2R},$$

$$c_5 = \frac{\left( d^2 - (1 + d)^2 n_1 \right) R \left( -r^2 + R^2 \right)^{d/2}}{2(1 + d)^2},$$

$$c_6 = - \frac{n_1 R}{2} \left( -r^2 + R^2 \right)^{d/2}.$$
\[ c_7 = \frac{(-1 + d)R^3(-r^2 + R^2)^{(2d+d)}}{4(1 + d)^2}. \] (83)

Now we integrate the expression (76) over the \((d - 2)\)-sphere on the boundary. We use the trick

\[
\int d^{d-1}x f(r)x^k x^l \cdots n \text{ pairs} = N_n \{ \delta_{ij} \delta_{kl} \cdots + \text{permutations} \} \int d^{d-1}x f(r)r^{2n},
\] (84)

where \(N_n\) some normalization constant. It can be determined by taking the integrand to be \(r^n\)

\[
N_1 = \frac{1}{d - 1} \quad \text{for } n = 1,
\] (85)

\[
N_2 = \frac{1}{((d - 1)^2 + 2(d - 1))} \quad \text{for } n = 2,
\] (86)

\[
N_3 = \frac{1}{((d - 1)^3 + 6(d - 1)^2 + 8(d - 1))} \quad \text{for } n = 3.
\] (87)

This will be used repeatedly for the non-constant case. We use it carry out the integration and obtain the final result (7).

**Appendix B. To find the solution of \(z_1\)**

**B.1. For a constant stress tensor**

Here we will apply the strategy formulated in section 2.2 to find \(z_1\) for a constant stress tensor. In this case, the source is given by

\[ J = -z_0^5 (T + 3T_i), \] (88)

where \(T = T_i, T_i = x_i x_j T_{ij}/R^2\) and \(z_0 = \text{sech} \eta = \frac{1}{x_3}\). There are easier ways to find the solution for a constant \(T_{\mu
u}\). But for the non-constant case, it proves to be useful to work in fourier space, because the whole space-dependence of the source then shrinks to just \(\exp(i\vec{k} \cdot \vec{x})\). We will follow the same route for the constant case as well. We take the stress tensor to be of the form.

\[ T_{ij} = \epsilon_{ij}(\vec{k}) \epsilon^{\vec{k} \cdot \vec{x}}, \quad \epsilon_{ij}(\vec{k})k_j = 0. \] (89)

Since the stress tensor is a constant, the only possible mode is. Since the actual source is of the form (88), \(z_1\) will be given by

\[ z_1 = -\left( \epsilon_{ij} \delta^{ij} + \frac{\epsilon_{ij}}{R^2} \left( \frac{1}{i} \frac{\partial}{\partial k_i} \frac{1}{i} \frac{\partial}{\partial k_j} \right) \right) \int d^3G(x, \vec{k}) \left( z_0^5 \epsilon^{\vec{k} \cdot \vec{x}} \right). \] (90)

It is not necessary to evaluate the integral exactly. In fact, since the derivative outside is evaluated at \(k = 0\) it is sufficient to work out the integral up to two powers of \(k\). Let us simplify things a bit by assuming \(\vec{k}\) to be along the \(x_3\) direction. Then the integral simplifies to
As discussed in section 2.2, we have to transform to a new set of intrinsic coordinates \((\eta', \theta', \phi')\), where the point \(\hat{x} = (r, \theta, \phi) = (R \tanh \eta, \theta, \phi)\) becomes the origin. In terms of these coordinates

\[
\left(1 - \frac{r^2}{R^2}\right) = \frac{1}{(\cosh \eta \cosh \eta' + \cos \theta' \sinh \eta \sinh \eta')^2},
\]

\[
x_3 \frac{R}{k} = \frac{\cos \theta \cosh \eta' \sinh \eta + \cos \theta \cosh \eta \sin \theta' \cosh \eta' \sin \theta' \sinh \eta'}{\cosh \eta \cosh \eta' + \cos \theta' \sinh \eta \sinh \eta'},
\]

Now we can expand the integrand in powers of \(k\) and obtain the following integrals\(^{11}\)

\[
\int d\eta' d\theta' d\phi' \frac{\sinh \eta' (\cosh \eta' - \sinh \eta')}{4\pi} \left( R^2 - (\eta', \theta', \phi')^2 \right)^{5/2} = \frac{R^5 \sech^3 \eta}{12},
\]

\[
\int d\eta' d\theta' d\phi' \frac{\sinh \eta' (\cosh \eta' - \sinh \eta')}{4\pi} \frac{ik x_3(\eta', \theta', \phi')}{(R^2 - (\eta', \theta', \phi')^2)^{5/2}} \left( R^2 - (\eta', \theta', \phi')^2 \right)^{5/2} = \frac{ikR^6}{20} \cos \theta \sech^3 \eta \tanh \eta,
\]

\[
\int d\eta' d\theta' d\phi' \frac{\sinh \eta' (\cosh \eta' - \sinh \eta')}{4\pi} \frac{(ik x_3(\eta', \theta', \phi')^2}{(R^2 - (\eta', \theta', \phi')^2)^{5/2}} = \frac{(ik)^2 R^7}{360} \sech^3 \eta \left( 1 + 6 \cos^2 \theta \tanh^2 \eta \right).
\]

Adding the above the three results we get the rhs of (91)

\[
\int d^3G(x, \hat{x}) R^2 - \hat{r}^2)^{5/2} e^{ik\hat{r}} = \frac{R^3}{12 \cosh^3 \eta} + \frac{ikR^6}{20} \cos \theta \sech^3 \eta \tanh \eta - \frac{k^2R^7}{360} \sech^3 \eta \left( 1 + 6 \cos^2 \theta \tanh^2 \eta \right).
\]

It is easy to guess the solution for an arbitrary \(\vec{k}\). The \(\sech^3 \eta\) multiplying all the terms is nothing but an overall factor of \((R^2 - r^2)^{5/2} = z_3^2\). Every \(\cos \theta\) indicates a dot product \(\vec{k} \cdot r\). So for general \(\vec{k}\)

\[
\int d^3G(x, \hat{x}) z_3^2 e^{i\vec{k} \cdot \vec{x}} = z_3^2 R^3 \left( \frac{1}{12} + \frac{i}{20} \langle k, r \rangle - \frac{\left( k^2 R^2 + 6 \langle k, r \rangle^2 \right)}{360} + \cdots \right).
\]

Now using the formula (90), and replacing \(\epsilon_{ij} = T_{ij}\) for \(\vec{k} = 0\)

\[
z_1 = -\frac{z_3^2 R^2}{10} \left( T + \frac{x^i x_j}{R^2} \right),
\]

which is the solution of \(z_1\) for constant \(T_{\mu\nu}\) in \(d = 4\) (see [6]).

\(^{11}\) The integration variables should be written \((\hat{\eta}, \hat{\theta}, \hat{\phi})\). The hats are removed for tidiness.
B.2. For non-constant stress tensor

To calculate $z_1$ for non-constant $\mu_{\nu}^{T}$, we need to find the source function $J$. Here the source term will be different from (88). We can find it by calculating the area functional and minimizing it w.r.t. $z_1$. To begin, we write the area functional as

$$
\int d^3 x \sqrt{h} = A_2 = A_{2,0} + A_{2,1} + A_{2,2}.
$$

(100)

It is easy to show that the $A_{2,1}$ in this case is

$$
A_{2,1} = L^{d-1}a \int d^{d-1}x \frac{R}{2z_0} \left( z_1 - \frac{z_0^2}{R^2} x^i \partial_i z_1 \right) - \frac{z_0^2}{12} \partial^2 T \left( 3z_1 - \frac{z_0^2}{R^2} x^i \partial_i z_1 \right) + T_0 \left( 2z_0^2 x^i \partial^i z_1 / R^2 - \frac{z_0^2}{R^2} x^i x^j \partial_i \partial_j z_1 \right) - \frac{z_0^2}{12} \partial^2 T_0 \left( 2z_0^2 x^i \partial^i z_1 / R^2 - \frac{z_0^2}{R^2} x^i x^j \partial_i \partial_j z_1 \right) \right).
$$

(101)

Note that, since we are calculating entanglement entropy in a time-independent case, all the components of $\mu_{\nu}^{T}$ have been taken to be time-independent.

The $A_{2,2}$ is same as before. $A_{2,0}$ is independent of $z_1$ and it is not required right now. To get the equation for $z_1$, we use

$$
\frac{\partial L}{\partial z_1} = \partial_i \left( \frac{\partial L}{\partial \partial_i z_1} \right) = 0 \quad \text{with} \quad L = A_{2,1} + A_{2,2}.
$$

(102)

The source term comes from $A_{2,1}$. We obtain

$$
\frac{1}{z_0^{d-1}R} \left( \partial^2 (z_0 z_1) - \frac{x^i x^j}{R^2} \partial_i \partial_j (z_0 z_1) \right) = \frac{z_0}{2R} \left( T(d-2) + T_s (d + 2) - \frac{z_0^2}{12} \left( d \partial^2 T + (d + 4) \frac{x^i x^j}{R^2} \partial_i \partial_j T_0 \right) \right)
$$

$$
+ x^i \partial_i T + 2x^i \partial_i T_0 + \frac{1}{R^2} x^i x^j \partial_k T_0 \right) \right).
$$

(103)

where we have retained terms only upto two derivatives of $T_{\mu\nu}$. For $d = 4$, comparing with (88), the source term is

$$
J = \frac{z_0^2}{2} \left( 2T + 6T_s - \frac{z_0^2}{3} \left( \partial^2 T + 2 \frac{x^i x^j}{R^2} \partial_i \partial_j T_0 \right) \right)
$$

$$
+ x^i \partial_i T + 2x^i \partial_i T_0 + \frac{1}{R^2} x^i x^j \partial_k T_0 \right) \right).
$$

(104)
To find \( z_1 \), it is useful to work in Fourier space, as before. The solution is then given by

\[
z_1 = \left( e + \frac{3\epsilon_{ij}}{R^2} \left( \frac{1}{i} \partial_i k \right) \left( \frac{1}{i} \partial_j k \right) \right) + \frac{\epsilon_{ij}}{2} \left( \frac{1}{i} \partial_i k_m \right) \frac{1}{i} \partial_j k \left( \frac{1}{i} \partial j k \right) \]

\[
- \frac{1}{6} \left( R^2 (ik)^2 + 2 (ik)^2 \epsilon_{ij} \left( \frac{1}{i} \partial_i k_j \right) \right) + \frac{1}{6} (ik)^2 \delta_{ij} \left( \frac{1}{i} \partial_i k_j \right) \left( \frac{1}{i} \partial j k \right)
\]

\[
+ \frac{g (ik)^2}{R^2} \epsilon_{ij} \delta_{ab} \left( \frac{1}{i} \partial_i k_a \right) \left( \frac{1}{i} \partial_j k_a \right) \left( \frac{1}{i} \partial j k \right) \right) \int d^3 G(x, k) \left( z_0^2 e^{ik \cdot x} \right). \tag{105}
\]

We again start with a \( \vec{k} \) along the \( x_3 \) direction. But, this time we cannot set \( \vec{k} = 0 \) and (91) must be expanded up to four powers of \( k \). We already evaluated the first three (equations (94)–(96)). The third and fourth powers give

\[
\int_{\eta'\theta'\phi'} \frac{\sinh \eta' (\cosh \eta' - \sinh \eta' \eta)}{4\pi} \left( ik \times (\eta', \theta', \phi') \right)^3 \left( R^2 - r (\eta', \theta', \phi')^2 \right)^{5/2} = -\frac{(ik)^3 R^8 \cos \theta \sech \eta \tan \eta \left( 3 + 10 \cos^2 \theta \tanh^2 \eta \right)}{2520}. \tag{106}
\]

\[
\int_{\eta'\theta'\phi'} \frac{\sinh \eta' (\cosh \eta' - \sinh \eta' \eta)}{4\pi} \left( ik \times (\eta', \theta', \phi') \right)^4 \left( R^2 - r (\eta', \theta', \phi')^2 \right)^{5/2} = \frac{(ik)^4 R^9 \sech^3 \alpha \left( 1 + 6 \cos^2 \beta \tanh^2 \alpha + 15 \cos^4 \beta \tanh^4 \alpha \right)}{20160}. \tag{107}
\]

This time too, it is easy to generalize the above for an arbitrary \( \vec{k} \). It is straightforward to evaluate the solution for \( z_1 \) in Fourier space

\[
z_1 = -z_0 \left( 1 + \frac{ik \cdot r}{12} - \frac{1}{28} (k \cdot r)^2 - \frac{k^2 r^2}{168} + \frac{k^2 R^2}{168} \right) (T + T_x). \tag{108}
\]

Here, \( T + T_x \) is evaluated with the value of \( T_{ij} \) at the origin. This gives a simple solution in coordinate space given by equation (48).

**Appendix C. Evaluating the area functional for non-constant stress tensor**

Here we use the \( z_1 \) solution (48) to evaluate the area functional \( A_2 \) (100) step-by-step. The best way to do this is to work out the expressions \( A_{2,0}, A_{2,1} \) and \( A_{2,2} \) separately. Let us begin with \( A_{2,1} \). Take equation (101) and Taylor expand all \( T_{ij} \)-s around origin.

\[
A_{2,1} = L^3 \int d^3x \frac{R}{2z_0} \left( T + x^i \partial_i T + \frac{1}{2} x^i x^j \partial_i \partial_j T \right) z_1
\]

\[
- \left( T + x^i \partial_i T + \frac{1}{2} x^i x^j \partial_i \partial_j T \right) \frac{z_0}{R^2} x^i z_1
\]

\[25\]
\[
\begin{align*}
&- \frac{z_0^2}{12} \partial^2 T \left( 3z_1 - \frac{z_0^2}{R^2} x^i \partial z_1 \right) - \frac{x^i / T_{ij} z_1}{R^2} - \frac{2 \partial^2 T_{ij} x^i \partial z_1}{R^4} \\
&+ \frac{2z_0^2 T_{ij} x^i \partial z_1}{R^2} - \frac{z_1}{R^2} x^i x^j \partial_k T_{ij} - \frac{z_0^2}{R^2} x^i \partial_z z_1^i \partial_k T_{ij} \\
&- \frac{1}{2R^2} z_1 x^i x^j \partial_i \partial_j T_{kl} - \frac{1}{2R^4} z_0^2 x_m \partial_m z_1 x^i x^j \partial_i \partial_j T_{ij} \\
&+ \frac{2z_0^2}{R^2} x^i \partial_i T_{jk} x^j \partial z_1 + \frac{z_0^2}{R^2} x^i \partial_i \partial_j T_{ij} x^k \partial z_1 + z_0^2 \\
&\times \left( \frac{z_1 x^i \partial^2 T_{ij}}{4R^2} + \frac{z_0^2 x^i \partial^2 T_{ij} x^j \partial z_1}{12R^4} - \partial^2 T_{ij} x^i \partial z_1^i z_0^2}{6R^2} \right) \quad (109)
\end{align*}
\]

Now we put in the solution (48) and simplify using the trick (84). We get

\[
A_{2,1} = \mathcal{O}(TT) + L^3 \int \frac{d^3 x}{82000 R^3} (r - R)(r + R) \left( 420 \left( \partial T_{jk} \right)^2 r^8 + 840r^8 \partial T_{ij} \partial^k T_{ij} \\
- 1680 \left( \partial T_{jk} \right)^2 r^6 R^2 - 3360r^6 R^2 \partial T_{ij} \partial^k T_{ij} + 980r^8 R^4 \left( \partial T_{jk} \right)^2 \\
+ 1960r^4 R^4 \partial T_{ij} \partial^k T_{ij} + 70 \left( \partial T \right)^2 \left( 9r^8 + 30r^6 R^2 + 35r^4 R^4 \right) \\
+ T^i \partial^2 T_{ij} \left( 910r^8 - 4040r^6 R^2 + 3682r^4 R^4 - 840r^2 R^6 \right) \\
+ 455r^8 T \partial^2 T + 3370r^6 R^2 T \partial^2 T + 1246r^4 R^4 T \partial^2 T \\
+ 1190r^2 R^6 T \partial^2 T - 1365R^8 T \partial^2 T \\
+ 12r^2 \left( 35r^6 + 116r^4 R^2 - 322r^2 R^4 + 175R^6 \right) T^i \partial_i T \right). \quad (110)
\]

Carrying out the integral

\[
A_{2,1} = \mathcal{O}(TT) - \frac{8 \pi L^3 R^{10} \left( 283 \left( \partial T \right)^2 + 7 \left( \partial T_{jk} \right)^2 + 14 \partial T_{ij} \partial^k T_{ij} + 72 T^{ij} \partial_i \partial_j T \right)}{405405}. \quad (111)
\]

Now let us evaluate \( A_{2,2} \), given by (74)

\[
A_{2,2} = L^3 \int \frac{d^3 x}{z_0^2} \left( \frac{6z_1^2}{z_0^2} x^i \partial z_1 + \frac{z_0^2}{2R^2} x^i \partial z_1 \right)^2 + \frac{(d - 1)}{R^2} z_1 x^i \partial z_1. \quad (112)
\]

Putting the solution (48) and once again using the trick (84) we get, after integration

\[
A_{2,2} = \mathcal{O}(TT) + \frac{4 \pi L^3 R^{10} \left( 283 \left( \partial T \right)^2 + 7 \left( \partial T_{jk} \right)^2 + 14 \partial T_{ij} \partial^k T_{ij} + 72 T^{ij} \partial_i \partial_j T \right)}{405405}. \quad (113)
\]

Finally we come to \( A_{2,0} \). The expression for \( A_{2,0} \) given in (72) does not have any derivative of \( T_{\mu \nu} \) since there was no derivative in the metric previously. Now, we will have additional terms due to these derivatives appearing in (43). So let us write \( A_{2,0} \) as

\[
A_{2,0} = A_{2,0}^0 + A_{2,0}^0, \quad (114)
\]

where \( A_{2,0}^0 \) is nothing but our old formula for \( A_{2,0} \). We can taylor expand all \( T_{\mu \nu} \) around the origin and use (84) to find
\[ A_{2,0}^0 = L^3 \int d^3x z_0^4 \left( -N_1 r^2 T_{0i} \partial^2 T_{0i} - N_1 r^2 \left( \partial T_{0i} \right)^2 \left( \frac{n_1}{2} + (d-1)n_2 - n_2 \frac{r^2}{R^2} \right) \right) + N_1 n_2 r^2 \left( T_0 \partial^2 T + (\partial T)^2 \right) \left( \frac{d-1}{2} - \frac{r^2}{2R^2} \right) \]
\[ + \left( N_1 r^2 (\partial T_{ik})^2 + T^{ij} \partial^2 T_{ij} \right) \left( \frac{n_1}{2} + \frac{n_2}{2} (d-1) - n_2 \frac{r^2}{2R^2} - \frac{1}{4} \right) \]
\[ + \frac{n_1 N_2 r^4}{2R^2} \left( T_{0i} \partial^2 T_{0i} + (\partial T_{0i})^2 + \partial T_{0i} \partial^{ij} T_{ij} \right) \]
\[ + N_2 r^4 \left( \frac{1}{2R^2} - \frac{1}{2R^2} \right) \left( T^{ij} \partial^2 T_{ij} + (\partial T_{ik})^2 + \partial T_{ik} \partial^{ij} T_{ij} \right) \]
\[ + \frac{1}{8} \left( N_1 r^2 (\partial T)^2 + T \partial^2 T \right) - N_3 \frac{r^2}{R^4} \left( 2(\partial T_{ik})^2 + (\partial T)^2 \right) + 4 \partial T_{ik} \partial^{ij} T_{ij} + T \partial^2 T + 2T^{ij} \partial \partial T + 2T^{ij} \partial^2 T_{ij} \right) \]
\[ - 2N_3 \frac{r^4}{R^4} \left( T \partial^2 T + (\partial T)^2 + T^{ij} \partial \partial T \right) \right). \] (115)

To evaluate the other part \( A_{2,0}^1 \), we go back to (69), and keep only the tensor structures containing two derivatives. Then under the integral sign we get

\[ \left( h_{ij} \delta h^{ij} \right)^2 = - \frac{1}{6} z_0^{10} \left( T \partial^2 T - N_1 T \partial^2 T \frac{r^2}{R^2} - \frac{r^2}{R^2} N_1 T \partial^2 T \right) \]
\[ + \frac{N_2 r^4}{R^4} \left( T \partial^2 T + 2T^{ij} \partial^2 T_{ij} \right) \right). \] (116)

\[ \delta h^{ij} \delta h_{ij} = - \frac{1}{6} z_0^{10} \left( T^{ij} \partial^2 T_{ij} - 2N_1 r^2 T^{ij} \partial^2 T_{ij} + \frac{N_2}{R^4} r^4 \left( T \partial^2 T + 2T^{ij} \partial^2 T_{ij} \right) \right). \] (117)

\[ h^{ij} \delta^2 h_{ij} = z_0^{10} \left( \eta^{ij} - \frac{x^i x^j}{R^2} \right) T_{ij}. \] (118)

Then using (44)

\[ A_{2,0}^1 = L^3 \int d^3x \left( \frac{r^2 - R^2}{48} \right)^3 \left( T \partial^2 T \right) \left( -1 + \frac{2N_1 r^2}{R^2} \right) \]
\[ - \frac{N_2 r^4 \left( 2T^{ij} \partial^2 T_{ij} + T \partial^2 T \right)}{R^4} \]
\[ + 2 \left( T^{ij} \partial^2 T_{ij} \left( 1 - \frac{2N_1 r^2}{R^2} \right) + \frac{N_2 r^4 \left( 2T^{ij} \partial^2 T_{ij} + T \partial^2 T \right)}{R^4} \right) \]
Now adding (115) and (119) and integrating, we obtain

\[ A_{2,0} = \frac{16\pi L^3 R^{10}}{135135} \left( 12\partial T_{ij}\partial^2T^{ij} + (\partial T_{jk})^2( -59 + 130n_1) + (\partial T)^2(29 + 2340n_{10} + 364n_2 + 780n_3) - 26T^{0i}\partial^2T_{0i} \left( 5n_1 + 4\left( 7n_2 + 15(n_3 + 3n_4 + n_7) \right) \right) + 13\partial T_{ij}\partial^2T^{ij} \left( -n_1 + 60(3n_{11} + n_6 + 2n_8) \right) + 13\left( -10(\partial T_{0i})^2n_1 + 2T^{ij}\partial^2T_{ij} \left( 5n_1 + 14n_2 \right) + 30(2n_3 + 3n_4 + n_7) \right) + 2\partial T_{ij}\partial^2T^{ij} \left( n_1 - 60(3n_{11} + 2n_9) \right) - (\partial T_{0i})^2 \left( 90n_{10} + 14n_2 + 15(2n_5 + n_8) \right) + T\partial^2T \left( 7n_2 + 15(3n_4 + n_7) \right) \right) + ( -7 + 780n_{13})T^{0i}\partial T_{ij} \right) + \left( \partial T_{jk} \right)^2(45n_{10} + 7n_2 + 15(n_5 + n_8)) + \mathcal{O}(TT). \]  

(120)

Now we can add the three components, \( A_{2,0}, A_{2,1} \) and \( A_{2,2} \) to arrive at the area functional given by equation (50).
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