Higher derivative type II string effective actions, automorphic forms and $E_{11}$

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By dimensionally reducing the ten-dimensional higher derivative type IIA string theory effective action we place constraints on the automorphic forms that appear in the effective action in lower dimensions. We propose a number of properties of such automorphic forms and consider the prospects that $E_{11}$ can play a role in the formulation of the higher derivative string theory effective action.
1. Introduction

The low-energy effective actions of the IIA and IIB string theories are the IIA [1-3] and IIB [4-6] supergravity theories. Furthermore eleven-dimensional supergravity [7] is the low-energy effective action of one of the limits of M-theory. The type IIA and type IIB supergravity theories contain all perturbative and non-perturbative string effects and as a consequence their study has lead to many aspects of what we now know about string theory. Upon dimensional reduction of the IIA and IIB theories on an $n$ torus, or equivalently the eleven dimensional theory on an $n+1$ torus, to $d = 10 - n$ dimensions all these theories become equivalent and possess a hidden $E_{n+1}$ duality symmetry [8-11]. The IIB supergravity theory also possesses an $SL(2,R)$ symmetry [4]. The four-dimensional heterotic supergravity theory possesses an analogous $SL(2,R)$ symmetry and taking into account the fact that the brane charges are quantised [12,13] and rotated by this symmetry it was proposed [14,15] that the four dimensional heterotic string theory was invariant under an $SL(2,Z)$ symmetry which included a transformation that mixed perturbative to non-perturbative effects. This realisation was generalised to the $E_{n+1}$ symmetry of type II theories in [16].

The higher derivative corrections to string theory have been most studied in the context of IIB string theory where it was found that demanding that the theory is invariant under the $SL(2,Z)$ symmetry leads to the appearance of automorphic forms that place very strong constraints on the theory [17-24]. For type IIB string theory compactified to eight or nine dimensions, invariance under the corresponding U-duality groups similarly lead to the appearance of automorphic forms [25-29]. The role of automorphic forms in the low energy effective action of type II string theory was also discussed sometime ago in seven and fewer dimensions [30,31]. More recently the higher derivative corrections of type II string theories in less than ten dimensions, including dimensions less than seven, have been systematically studied [32-39] and specific automorphic forms have been proposed for certain higher derivative terms constructed from particular representations of $E_{n+1}$. Furthermore the regulation of the divergences was carried out and precise predictions for the perturbative series worked out in detail [34-38]. In particular these papers have generalised the previous results in non-renormalisation theorems [25,28,40]. These studies have, however, been limited to terms with relatively lower numbers of space-time derivatives and very little is known about such terms in general. An exception was that of reference [39] in which the dimensional reduction of arbitrary higher derivative terms in the IIB theory on an $n$ torus were compared with the result expected in $10 - n$ dimensions if an $E_{n+1}(Z)$ symmetry was present. In this way one was able to place some restrictions on the representations used to construct the automorphic forms for an arbitrary higher derivative correction. A similar analysis was also carried out but starting from eleven dimensions.

In this paper we will follow a similar approach to that of reference [39], but from the view point of the IIA theory. In particular we will consider the dimensional reduction of the higher derivative string corrections of the IIA theory on an $n$-dimensional torus to $d = 10 - n$ dimensions. We will compare these with the higher derivative corrections that arise in the $d$ dimensional theory assuming that the theory is invariant under an $E_{n+1}(Z)$ symmetry and so possess a corresponding automorphic form built from a representation of $E_{n+1}(Z)$. This comparison allows us to place constraints on the representation used to
construct the automorphic form that appears for any higher derivative correction. Indeed we find that the highest weight \( \vec{\Lambda}_n \) appears in the automorphic form, where \( \vec{\Lambda}_n \) is the highest weight of the fundamental representations of \( E_{n+1} \) associated with node \( n \). The Dynkin diagram of \( E_{n+1} \) with the labelling of the nodes is given in figure five. This strongly suggests that each higher derivative correction contains an automorphic form constructed from this fundamental representation.

In order to carry out the comparison we need to identify the fields that arise in the dimensional reduction from ten dimensions with the fields that occur in the formulation of the \( d \)-dimensional theory in which the \( E_{n+1}(Z) \) symmetry is manifest, and in particular the scalar fields from which the automorphic form is constructed. This identification can be carried out in the context of the supergravity theories. The most obvious technique is to explicitly carry out the dimensional reduction of the supergravity theory and reformulate the theory with the manifest \( E_{n+1}(Z) \) symmetry, but this is rather lengthy and complicated involving dualisations and other subtleties. In this paper we will use the \( E_{11} \) formulation of the IIA theory \([43,55]\). In this formulation the fields of the theory are in one to one correspondence with the generators of the Borel subalgebra of \( E_{11} \). As the \( E_{11} \) algebra contains in an obvious way the \( E_{n+1} \) algebra, the correspondence between the scalar fields that appear in the non-linear realisation of \( E_{n+1} \) and the \( E_{11} \) generators is easily found. However, the correspondence between the \( E_{11} \) generators and the fields usually used to formulate the IIA supergravity theory is known from the formulation of this theory as a non-linear realisation at lowest levels in \( E_{11} \) \([41,43]\). Thus one finds the desired relation between the fields of the IIA theory and the scalars fields associated with \( E_{n+1} \) in a simple way. We note that although there is strong evidence for the conjecture that \( E_{11} \) is an underlying symmetry of the theory of strings and branes our use of \( E_{11} \) in this paper does not rely upon the \( E_{11} \) conjecture holding.

After discussing the consequences of the results of this paper we make a number of proposals for the properties of the automorphic forms that occur in string theory for any number of space-time derivatives. We also consider the possibility that the higher derivative effective action admits an \( E_{11} \) formulation.

2. The Dimensional Reduction

The bosonic field content of type IIA supergravity in ten dimensions consists of a scalar (the type IIA dilaton \( \phi \)), a NS-NS three form field strength \( \tilde{F}_{\mu_1\mu_2\mu_3} \) constructed from the NS-NS two form gauge field \( \tilde{A}_{\mu_1\mu_2} \), in addition to two R-R form field strengths \( \tilde{F}_{\mu_1\mu_2\mu_3\mu_4} \) constructed from the R-R gauge fields \( \tilde{A}_{\mu_1\mu_2} \) and \( \tilde{A}_{\mu_1\mu_2\mu_3} \). In Einstein frame, the bosonic part of the type IIA supergravity action is given by \([1,2,3]\),

\[
S_{IIA} = \frac{1}{2\kappa_{10}^2} \int d^{10}x \text{det}(\tilde{e}) \left( \tilde{R} - \frac{1}{2 \cdot 4!} e^{\Phi} \tilde{F}_{\mu_1\mu_2\mu_3\mu_4} \tilde{F}^{\mu_1\mu_2\mu_3\mu_4} - \frac{1}{2 \cdot 3!} e^{-\Phi} \tilde{F}_{\mu_1\mu_2\mu_3} \tilde{F}^{\mu_1\mu_2\mu_3} \\
- \frac{1}{2 \cdot 2!} e^{\Phi} \tilde{F}_{\mu_1\mu_2} \tilde{F}^{\mu_1\mu_2} - \frac{1}{2} \partial_{[\mu_1} \phi \partial_{\mu_2]} \phi \right),
\]

(2.1)

where \( \kappa_{10} \) is a constant related to the Newton constant in ten dimensions and

\( \tilde{F}_{\mu_1\mu_2} = 2 \partial_{[\mu_1} \tilde{A}_{\mu_2]} \).
The type IIA supergravity action possesses a $GL(1, R)$ symmetry, that manifests itself through a shift of the type IIA dilaton and a scaling of the other fields. One can introduce the following combinations of the field strengths and dilaton that are inert under $GL(1, R)$ transformations

$$\tilde{F}_{\mu_1\mu_2\mu_3} = 3 \partial_{[\mu_1} \tilde{A}_{\mu_2\mu_3]},$$
$$\tilde{F}_{\mu_1\mu_2\mu_3\mu_4} = 4 \left( \partial_{[\mu_1} \tilde{A}_{\mu_2\mu_3\mu_4]} + \tilde{A}_{[\mu_1} \tilde{F}_{\mu_2\mu_3\mu_4]} \right).$$  \hspace{1cm} (2.2)

We have suppressed the Chern-Simons term since it will not play a part in our analysis. In fact these are just the non-linear representations of $GL(1, R)$ constructed from the linear representations in the usual way (see appendix A). They are inert, as the local subalgebra is the identity group. Rewriting the action with these objects effectively absorbs the dilaton factors multiplying the field strengths in (2.1), the action then becomes

$$S_{IIA} = \frac{1}{2\kappa_{10}^2} \int d^{10}\bar{e} \det(\bar{e}) \left( \bar{R} - \frac{1}{2 \cdot 4!} \tilde{F}_{\mu_1\mu_2\mu_3\mu_4} \tilde{F}^{\mu_1\mu_2\mu_3\mu_4} - \frac{1}{2 \cdot 3!} \tilde{F}_{\mu_1\mu_2\mu_3} \tilde{F}^{\mu_1\mu_2\mu_3} \right. - \frac{1}{2 \cdot 2!} \tilde{F}_{\mu_1\mu_2} \tilde{F}^{\mu_1\mu_2} - \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi \left. \right).$$ \hspace{1cm} (2.4)

In this paper we are interested in the dimensional reduction of a generic ten dimensional type IIA higher derivative term which may be written as

$$\int d^{10}\bar{e} \partial^{i_0} \bar{P}_{\mu_1} \tilde{F}_{\mu_1\mu_2} \tilde{F}_{\mu_1\mu_2\mu_3} \tilde{F}_{\mu_1\mu_2\mu_3\mu_4} \Phi_s,$$ \hspace{1cm} (2.5)

where $\Phi_s$ is a function of $\phi$ that is of the form $\Phi_s = e^{-s \phi}$. Dimensional reduction on an $n$ torus to a theory in $d = 10 - n$ dimensions is achieved using the metric compactification ansatz

$$d\bar{s}_{10}^2 = e^{2\alpha \rho} g_{\mu\nu} dx^\mu dx^\nu + e^{2\beta \rho} G_{ij} (dx^i + A^i_{\mu} dx^\mu) (dx^j + A^j_{\mu} dx^\mu),$$ \hspace{1cm} (2.6)

where the background and internal metrics are denoted $g_{\mu\nu}$ and $G_{ij}$ respectively, with the internal metric satisfying $\det(G) = 1$ and

$$\alpha = \sqrt{\frac{n}{2 (d - 2) (D - 2)}}, \hspace{0.5cm} \beta = -\frac{(d - 2) \alpha}{n}.$$ \hspace{1cm} (2.7)

For us in this paper $D = 10$. The internal vielbein is given by $e_i^k e_j^l \delta_{kl} = G_{ij}$ and satisfies $\det(e) = 1$. Tangent internal indices possess an underline as shown. The gauge fields are dimensionally reduced in the obvious way using world indices i.e. $\tilde{A}_{\hat{\mu}}$, $\hat{\mu} = 0, \ldots, 9$ is set equal to $A_i$ if $\hat{\mu} = i, i = d + 1, \ldots, 9$ and $A_\mu$ if $\hat{\mu} = \mu, \mu = 0, \ldots, d$. 

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We will be interested in the dependence of the above ten dimensional higher derivative correction in string frame. The transition from Einstein frame to string frame is given by \( \tilde{e} = e^{-\frac{2}{3}} \tilde{e}_s \). The term in (2.5) then leads to the factor

\[
e^\frac{2}{3}(\tilde{l}_0 + \tilde{l}_R + \tilde{l}_1 + 5\tilde{l}_2 + \tilde{l}_3 + 5\tilde{l}_4 - 10 - 4s).
\]  

At order \( g \) in perturbation theory we have the contribution \( e^{\phi(2g-2)} \) and so for a perturbative contribution we find

\[
\tilde{s} = \frac{1}{4} \left( \tilde{l}_0 + \tilde{l}_R + \tilde{l}_1 + 5\tilde{l}_2 + \tilde{l}_3 + 5\tilde{l}_4 - 2 - 8g \right).
\]  

The dimensionally reduced theory will contain field strengths of the form \( F_{\mu_1...\mu_p i_1...i_k} \) where the internal indices \( i_1, ..., i_k \) are world volume indices. The theory in \( d \) dimensions possesses the \( GL(1,R) \) symmetry of the IIA theory, but in addition has an \( SL(n,Z) \) symmetry corresponding to diffeomorphisms that are preserved by the torus. We can convert the internal world indices to tangent frame indices using the inverse internal vielbein. However, as explained in reference [39] page 5, the internal vielbein is just the group element of the non-linear realisation of \( SL(n,Z) \) symmetry corresponding to diffeomorphisms that are preserved by the torus. We can convert the internal world indices to tangent frame indices using the inverse internal vielbein. However, as explained in reference [39] page 5, the internal vielbein is just the group element of the non-linear realisation of \( SL(n,Z) \) symmetry corresponding to diffeomorphisms that are preserved by the torus. We can convert the internal world indices to tangent frame indices using the inverse internal vielbein.

When written in terms of the field strengths using equation (2.3) we also find exponential factors involving the dilaton. One can incorporate these automatically by considering the group \( SL(n,Z) \) inert objects of equation (2.3) we find \( \mathcal{F}_{Sl(n)\otimes GL(1)}_{j_1...j_k} \), with any space-time indices suppressed, which converts to tangent space as follows [32]

\[
\mathcal{F}_{Sl(n)\otimes GL(1)}_{j_1...j_k} = (e^{-1})_{i_1}^{j_1}... (e^{-1})_{i_k}^{j_k} F_{j_1...j_k},
\]  

where \( e^{-1} \) is the vielbein on the torus and \( F_{j_1...j_k} \) transforms in the linear representation of \( SL(n) \) with highest weight \( \Lambda_k \). Thus \( \mathcal{F}_{Sl(n)\otimes GL(1)}_{j_1...j_k} \) transforms as a non-linear representation constructed from a linear representation in the standard way (see appendix A, equation (A.8)), in this case its non-linear transformations are contained in a matrix belonging to \( SO(n) \). This is consistent with the usual action of the tangent space group on the vielbein. Hence, if we use tangent space internal indices then the \( SL(n) \) symmetry will be essentially manifest as long as we construct \( SO(n) \) invariants.

If we denote the part of the group element of the non-linear realisation of \( SL(n) \) with local subgroup \( SO(n) \) which contains the Cartan generators \( \mathbf{H} \) of \( SL(n) \) by \( g_{Sl(n)} = e^{H_{,\phi} \cdot \phi} \)

then the dimensionally reduced field strength \( \mathcal{F}_{Sl(n)\otimes GL(1)}_{j_1...j_k} \) carries a factor of \( e^{\phi_{,\Lambda_k}} \) where \( \Lambda_k \) is the \( SL(n) \) representation with highest weight \( \Lambda_k \) and \( [\Lambda_k] \) is a weight in this representation. When written in terms of the field strengths using equation (2.3) we also find exponential factors involving the dilaton. One can incorporate these automatically by considering the group \( SL(n) \otimes GL(1) \) with the group element \( g_{Sl(n)\otimes GL(1)} = e^{\phi R \cdot H_{,\phi}} \) where \( R \) is the generator of \( GL(1,R) \).

The dimensional reduction of terms containing field strengths in the IIA higher derivative theory will lead to terms containing the object \( \mathcal{F}_{Sl(n)\otimes GL(1)} \) of equation (2.10) multiplied by exponentials of the field \( \rho \) which arise from dimensional reduction using the ansatz of equation (2.6). The field strength of equation (2.10) has a \( \rho \) factor given by \( e^{-k\beta \rho} \). If we were to convert the space-time world indices to tangent indices then we would also acquire a factor \( e^{-p\alpha \rho} \) if we have \( p \) space-time indices.
The derivatives of the scalars, including the dilaton, are contained in the part of the Cartan form of $SL(n) \otimes GL(1)$ which changes by a minus sign under the action of the Cartan involution, and is denoted by $P_{SL(n)\otimes GL(1)}$. The action of the Cartan involution on $SL(n)$ generators is such as to lead to $SO(n)$ being the invariant group and on the generator $R$ it acts with a minus sign.

After dimensional reduction, the IIA theory including the higher derivative terms can be expressed in terms of the scalar curvature $R$, which is an $SL(n)$ singlet, the $P_{SL(n)\otimes GL(1)}$ part of the Cartan forms of $SL(n) \otimes GL(1, R)$ and the field strengths $\mathcal{F}_{SL(n)\otimes GL(1,R)} \mu_{\mu_1...\mu_p \bar{\lambda}_1...\bar{\lambda}_h}$ which transform as non-linear representations of $SL(n)\otimes GL(1, R)$.

### 3. The $E_{n+1}$ formulation in $d$ dimensions

As is well known, the type II supergravity theories in $d$ dimensions possess an $E_{n+1}$ symmetry [7-11,4]. Their actions are bilinear in the space-time derivatives and include the Riemann curvature, and squares of the field strength and the derivatives of the scalars. The metric, in Einstein frame, transforms as a singlet of $E_{n+1}$ and therefore the Riemann curvature is invariant under $E_{n+1}$ transformations. The scalars belong to the non-linear realisation of $E_{n+1}$ with a local subgroup $H_{n+1}$ which is the maximal compact subgroup. The latter is just the Cartan involution invariant subgroup. This means that the scalars are contained in a group element $g_{E_{n+1}} \in E_{n+1}$ which transforms as $g_{E_{n+1}} \rightarrow g_0 g_{E_{n+1}}$ where $g_0 \in E_{n+1}$ is independent of space-time and also $g_{E_{n+1}} \rightarrow g_{E_{n+1}} h$ where $h \in H_{n+1}$ and is an arbitrary function of space-time. We can write the Cartan subalgebra part of the group element as $g_{E_{n+1}} = e^{\vec{\phi}.\vec{H}}$ where $\vec{H}$ are the $n + 1$ Cartan subalgebra generators of $E_{n+1}$, which we have written as a vector. The corresponding scalar fields are written as the vector $\vec{\phi}$.

The non-linear realisation essentially specifies how the scalars appear in the action. In particular, the derivatives of the scalars occur as Cartan forms of $E_{n+1}$ in the coset directions. In terms of our group element $g_{E_{n+1}}$, the Cartan forms which are given by $g_{E_{n+1}}^{-1} dg_{E_{n+1}}$ in the coset directions, are denoted $P_{E_{n+1}}$. When evaluated they contain the roots $\vec{\alpha}$ in the form $e^{\vec{\phi}.\vec{\alpha}}$ where $\vec{\alpha}$ are the roots of $E_{n+1}$.

The gauge fields occur in the field strengths $F$ that transform as linear representations of $E_{n+1}$ with highest weight $\vec{\Lambda}$ say. However, we can convert a linear representation of $E_{n+1}$ into a non-linear representation using a group element $g_{E_{n+1}}^{-1}$. Explicitly, the non-linear representation $|\mathcal{F}\rangle$ constructed from a linearly realised field strength $|F\rangle$ is given by [32]

$$|\mathcal{F}_{E_{n+1}}\rangle = L(g_{E_{n+1}}^{-1})|F\rangle,$$

where $L((g_{E_{n+1}}(\xi))^{-1})$ is the representation with highest weight $\vec{\Lambda}$. From equation (3.1) we find that the non-linearly realised field strength $|\mathcal{F}_{E_{n+1}}\rangle$ contains a dependence on the scalars $\vec{\phi}$ which is given by $e^{\vec{\phi}.[\vec{\Lambda}]}$ where $[\vec{\Lambda}]$ is a weight in the the $E_{n+1}$ representation with highest weight $\vec{\Lambda}$.

If one dimensionally reduces, for example, the IIA supergravity action in ten dimensions and keeps track of the scalars that appear in the form $e^{\vec{\phi}.\vec{w}}$ then one finds that $\vec{w}$
are proportional to the roots of $E_{n+1}$. Indeed, this is the simplest way to see that the dimensionally reduced theory is very likely to possess a $E_{n+1}$ symmetry.

Using the same arguments, a generic higher derivative term in $d$ dimensions can be written as a polynomial in the Riemann curvature, the non-linearly realised field strengths and Cartan forms $P_{E_{n+1}}$, but it is also multiplied by a function of the scalar fields. Assuming that the higher derivative term as a whole is invariant under an $E_{n+1}$ transformation implies that this non-holomorphic function must be an object that transforms under $E_{n+1}$ transformations like an automorphic form. We can expect that this automorphic form is built out of a particular representation, of $E_{n+1}$ with highest weight $\vec{\Lambda}$ say. We write the states of this representation in the form $|\psi\rangle = n_i |\vec{\mu_i}\rangle$ where $|\vec{\mu_i}\rangle$ are a basis of the representation, $\vec{\mu_i}$ are the weights in the representation and $n_i$ are integers. To be more precise it is constructed out of the non-linear representation of $E_{n+1}$ constructed from this representation using the scalars, that is, it is constructed out of the function $|\varphi\rangle$, defined by

$$|\varphi\rangle = L(g_{E_{n+1}}^{-1})|\psi\rangle.$$  \hspace{1cm} (3.2)

It is obvious that $|\varphi\rangle$ contains terms where the scalar fields occur in the form $e^{\vec{\phi} \cdot \vec{\Lambda}}$. The automorphic form is a function of $|\varphi\rangle$ and for the examples that are understood it is of the generic form

$$\sum n_i <\varphi|\varphi>^{-s},$$ \hspace{1cm} (3.3)

for some constant $s$. The automorphic form of equation (3.3) will always contain a term with scalar field dependence given by $e^{-\sqrt{2s} \vec{\Lambda} \cdot \vec{\phi}}$, where $\vec{\Lambda}$ is the highest weight of the representation used to build the automorphic form. This construction is described in more detail in reference [34]. The use of integers corresponds to the fact that the symmetry group for the higher derivative terms is discretised since the charges of the theory obey a quantisation condition. In fact we are using the Chevalley definition of the discrete group; that is the one generated by $e^{\pm E_a}$, $e^{\pm F_a}$ and $e^{\pm H_a}$ where $E_a$, $F_a$ and $H_a$ are the Chevalley generators. We thank Lisa Carbone for this point.

In this paper we will refer to the formulation of a higher derivative term in $d$ dimensions just described as the $E_{n+1}$ formulation. A term in the higher derivative effective action will contain an exponential of the scalar fields $\vec{\phi}$ of the form $e^{\sqrt{2s} \vec{\omega} \cdot \vec{\phi}}$ where $\vec{\omega}$ is the field we introduced earlier in this section. Our task is to compare this with the equivalent factor that arises in the dimensional reduction. However, in order to compare the $E_{n+1}$ formulation of the type IIA theory in $d$ dimensions with the dimensionally reduced formulation discussed in the previous section we need to know the relationship between the fields that occur in the dimensional reduction, namely the fields $\phi$, $\rho$ and $\phi$, where $\phi$ is an $n - 1$-dimensional vector and those that occur in the $E_{n+1}$ formulation, namely the $n + 1$-dimensional $\vec{\phi}$. This will be given in the next section.

4. The $E_{11}$ formulation

The eleven dimensional, IIA and IIB supergravity theories, as well as the maximal type II supergravity theories in lower dimensions, can be formulated as non-linear realisations
The non-linear realisations of the Kac-Moody algebra $E_{11}$, at low levels, leads to all of these theories [42-47]. As such $E_{11}$ encodes the fields of each of these theories and provides us with a way of relating the fields in the different theories to each other[55]. In fact the fields of these theories are in one to one correspondence with the generators of the Borel subalgebra of $E_{11}$ in the group decomposition, explained below, appropriate to each theory. It has been conjectured that non-linear realisations of the Kac-Moody algebra $E_{11}$ are extensions of all these supergravity theories [42-47]. However, we will not use this conjectured $E_{11}$ result in this paper.

A Kac-Moody algebra is formulated in terms of its Chevalley generators, which include those in the Cartan subalgebra denoted by $H_{\hat{a}}$, $\hat{a} = 1, 2, \ldots, 11$. As such, the $E_{11}$ group element that occurs in the non-linear realisation is of the form $g_{E_{11}} = e^{\phi_{\hat{a}} H_{\hat{a}}}$ provided we restrict our attention to the part that is in the Cartan subalgebra. Indeed, as there is an essentially unique formulation of $E_{11}$ in terms of its Chevalley generators, the eleven dimensional, IIA , IIB and $d$ dimensional theories viewed as non-linear realisations have a common origin and their fields can be mapped into each other in a one to one manner [42-47]. It is this property that we are going to exploit.

The $E_{11}$ Kac Moody algebra is encoded in the Dynkin diagram

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11
•

• — ... — • — • — • — • — • — • — • — •
1 6 7 8 9 10
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Figure 1. The $E_{11}$ dynkin diagram

The **eleven dimensional theory** emerges if we decompose the $E_{11}$ algebra in terms of the algebra that results from deleting the exceptional node labelled eleven, namely the algebra $GL(11)$. This subalgebra has the generators $K_{\hat{a} \hat{b}}$, $\hat{a}, \hat{b} = 1, \ldots, 11$ and it includes all the Cartan subalgebra generators of $E_{11}$; the relation being [43]

$$H_{\hat{a}} = K_{\hat{a} \hat{a}} - K_{\hat{a}+1 \hat{a}+1}, \hat{a} = 1, \ldots, 10,$$

$$H_{11} = -\frac{1}{3} (K_{11} + \ldots + K_{88}) + \frac{2}{3} (K_{99} + K_{1010} + K_{1111}). \quad (4.1)$$

The first ten generators being the Cartan subalgebra generators of $SL(11)$.

The contribution of the $GL(10)$ subgroup to the $E_{11}$ group element in the non-linear realisation is of the form

$$e^{x_{\hat{a}} P_{\hat{a}}} e^{h_{\hat{a} \hat{b}} K_{\hat{a} \hat{b}}}, \quad (4.2)$$

where we have added the space-time translation generators $P_{\hat{a}}$. This is known to give rise to eleven dimensional gravity and as a result the line in the above Dynkin diagram, that is from nodes one to ten inclusive, is known as the gravity line. Indeed the Cartan form for this subgroup is given by

$$g^{-1} dg = dx^{\hat{a}} \delta_{\hat{a}} \hat{a} P_{\hat{a}} + (e^{-1} de)^{\hat{b}} \hat{b} K_{\hat{a} \hat{b}}. \quad (4.3)$$
It turns out that $e_\mu{}^a = (e^h)_{\hat a}{}^{\hat b}$ is the eleven-dimensional vielbein.

We may set the different formulations of the $E_{11}$ group element to be equal and, restricting to the Cartan subalgebra, we find that

$$e^{\hat a \hat A} = e^{h_{\hat a}} K^a_\alpha.$$  \hspace{1cm} (4.4)

Comparing coefficients of $K^a_\alpha$ using equation (4.1) we find the relations

$$\hat \phi_i = h^{1} + h^{2} + \cdots + h^{i} - \frac{i}{2} \sum_{j=1}^{11} h^{j}, \hspace{0.5cm} for \hspace{0.5cm} 1 \leq i \leq 8,$$

$$\hat \phi_9 = h^{1} + h^{2} + \cdots + h^{9} - 3 \sum_{j=1}^{11} h^{j},$$

$$\hat \phi_{10} = h^{1} + h^{2} + \cdots + h^{10} - 2 \sum_{j=1}^{11} h^{j},$$

$$\hat \phi_{11} = -\frac{3}{2} \sum_{j=1}^{11} h^{j}. \hspace{1cm} (4.5)$$

The full non-linear realisation of $E_{11}$ leads, at low levels and with the decomposition to $GL(11)$, to the eleven dimensional supergravity theory. However, in this paper we are interested in only the fields associated with the Cartan subalgebra parts of the algebra, hence the above restriction.

Let us now consider the ten-dimensional IIA theory which is obtained from eleven dimensions by dimensional reduction on a circle. In this process, the diagonal components of the eleven dimensional metric result in the diagonal components of the ten dimensional metric and a scalar $\phi$, which is the dilaton of the IIA theory.

In terms of the $E_{11}$ formulation we obtain the IIA theory by deleting nodes ten and eleven of the Dynkin diagram below (see figure 2) leaving us with a $GL(10) \otimes GL(1)$ algebra; the $GL(10)$ algebra leads to ten dimensional gravity, for the same reasons as occurred above in eleven dimensions, and the $GL(1)$ factor leads to the IIA dilaton.

$$\begin{array}{cccccc}
11 & 10 \\
\bullet & \bullet \\
| & | \\
\bullet - \cdots - \bullet - \bullet - \bullet - \bullet \\
1 & 6 & 7 & 8 & 9
\end{array}$$

Figure 2. The $E_{11}$ Dynkin diagram appropriate to the IIA theory

The gravity line is now the horizontal line of the Dynkin diagram of figure 2. The IIA supergravity theory emerges from the non-linear realisation of $E_{11}$ with this decomposition.
Let us denote the generators of $GL(10)$ by $K^a_b$, $a, b = 1, \ldots, 10$ and let $R$ be the $GL(1)$ generator. These contain the generators of the Cartan subalgebra of $E_{11}$. The group element in the Cartan subalgebra of $E_{11}$ can therefore be written in the form

$$g = e^{\h^a K^a e^\sigma R}$$

The tilde distinguishes the field from that in eleven dimensions. However, in terms of the Chevalley generators in the Cartan subalgebra of $E_{11}$, the group element has the same form as in eleven dimensions, namely $g = e^{\phi H_a}$.

It turns out that the Cartan sub-algebra generators $H_a$ of the $E_{11}$ algebra and those in the $GL(10) \otimes GL(1)$ algebra are related by

$$H_a = K^a_a - K^{a+1}_{a+1}, \quad a = 1, \ldots, 9,$$

$$H_{10} = -\frac{1}{12} \left( K^1_1 + \ldots + K^9_9 + \frac{7}{8} K^{10}_{10} - \frac{3}{2} R, \right)$$

$$H_{11} = -\frac{1}{4} \left( K^1_1 + \ldots + K^8_8 + \frac{3}{4} (K^9_9 + K^{10}_{10}) + R. \right)$$

Comparing the coefficients of the generators $R$ and $K^a_a$ we find that

$$\sigma = -\frac{3}{2} \phi_{10} + \phi_{11}, \quad \h^1 = \phi_1 - \frac{1}{8} \phi_{10} - \frac{1}{4} \phi_{11},$$

$$\h^i = -\phi_{i-1} + \phi_i - \frac{1}{8} \phi_{10} - \frac{1}{4} \phi_{11}, \quad \text{for} \quad 2 \leq i < 9, \quad \h^9 = -\phi_8 + \phi_9 - \frac{1}{8} \phi_{10} + \frac{3}{4} \phi_{11},$$

$$\h^{10} = -\phi_9 + \phi_{10} + \frac{7}{8} \phi_{10} + \frac{3}{4} \phi_{11}. \quad (4.9)$$

We note the useful relation $R = \frac{1}{12} \left( - \sum_{a=1}^{10} K^a_a + 8 K_{11}^{11} \right)$.

Equating the group element $g$ in the Cartan subalgebra written in terms of the two different sets of generators we find that

$$g = e^{\sum_{a=1}^{10} \h^a K^a e^\sigma R} = e^{\phi_1 (K^1_1 - K^2_2) \ldots e^{\phi_9 (K^9_9 - K^{10}_{10})}}$$

$$e^{\phi_{10} (-\frac{1}{8} (K^1_1 + \ldots + K^9_9) + \frac{7}{8} K^{10}_{10} - \frac{3}{2} R)} e^{\phi_{11} (-\frac{1}{4} (K^1_1 + \ldots + K^8_8) + \frac{3}{4} (K^9_9 + K^{10}_{10}) + R}}. \quad (4.8)$$

We now consider the $E_{11}$ formulation of the $d$-dimensional maximal supergravity theory [42-47]. In the previous section we dimensionally reduced the IIA theory using the ansatz of equation (2.6) to find the IIA dilaton $\phi$ of the original theory, and the $10 - d$ scalars $\phi$ arising from the diagonal components of the metric $G_{ij}$ and the field $\rho$.

From the $E_{11}$ perspective, the $d$-dimensional type II supergravity theory is found by writing the $E_{11}$ Dynkin diagram in the form given in figure 3 below.
Deleting node $d$ we find the residual algebra $E_{n+1} \otimes GL(d)$; the latter algebra leads in the non-linear realisation to $d$-dimensional gravity and the former algebra is the U-duality group. Decomposing the $E_{11}$ non-linear realisation into representations of $E_{n+1} \otimes GL(d)$ we find, at low levels, the field content of the maximal supergravity theory in $d$ dimensions and indeed the supergravity theory itself. We can further delete nodes ten and eleven, corresponding to the dimensional reduction of the IIA theory we conclude that it is the $SL(n)$ symmetry preserved by dimensional reduction on the $n$ torus as discussed in section two. We note that $SL(n) \otimes GL(1) \otimes GL(1)$ contains all the Cartan subalgebra elements of $E_{11}$ and that one of the $GL(1)$ factors is the $GL(1)$ symmetry of the IIA theory, discussed above figure two. These two $GL(1)$ factors lead in the non-linear realisation to the fields $\phi$ and $\rho$.

The dimensional reduction ansatz of equation (2.6) is implemented in terms of $E_{11}$ by rewriting the group element of the IIA theory of equation (4.6) in the form

$$g = e^\hat{h}_a K^a + \epsilon_1 \rho \sum_{a=1}^d K^a_\alpha e^\hat{h}_\alpha K^\alpha_i K^i + \epsilon_2 \rho \sum_{i=d+1}^{10} K^i \epsilon^\sigma R_i.$$  \hspace{1cm} (4.10)

Here the $K^a_b$, $a, b = 1, \ldots, d$, are the generators of the $GL(d)$ algebra associated with $d$-dimensional gravity, $K^{i,j}$, $i, j = 1, \ldots, n$ are the generators of $SL(n) \otimes GL(1)$ and $\epsilon_1$ and $\epsilon_2$ are constants. We have put a dot on the $h$ fields to distinguish them from the analogous fields used earlier in ten and eleven dimensions. Taking into account the introduction of the field $\rho$ we set

$$\hat{h}^{d+1}_{d+1} + \hat{h}^{d+2}_{d+2} + \ldots + \hat{h}^{10}_{10} = 0.$$  \hspace{1cm} (4.11)

Introducing translation generators in $d$-dimensional space-time and internal space into the group element by including the factor $e^{x^a P_a + x^i P_i}$ and computing the Cartan forms we find the terms involving these new generators are given by the expression $g^{-1} \left( dx^a P_a + dx^i P_i \right) g$, which implies the identification

$$\sum_{a=1}^d \left( e^{\hat{h}_a + \epsilon_1 \rho} \right) dx^a P_a + \sum_{i=d+1}^{10} \left( e^{\hat{h}_i + \epsilon_2 \rho} \right) dx^i P_i = e^{\alpha \rho} e^\mu_a dx^\mu P_a + e^{\beta \rho} e^i_j dx^j P_i.$$  \hspace{1cm} (4.12)

Taking $\epsilon_1 = \alpha, \epsilon_2 = \beta$ we do indeed recover the vielbeins as they appear in the dimensional reduction ansatz of equation (2.5) provided we identify $e^\mu_a = (e^h)_\mu^a$ with the vielbein in $d$-dimensional space-time and $e^i_j = (e^h)_i^j$ with the vielbein in the $n$-dimensional internal space. Equation (4.11) implies that this latter vielbein satisfies the constraint $\det e = 1$ as required in the dimensional reduction ansatz. To find equation (4.12) we have dropped various factors involving exponentials of the trace of $h$ as these are interpreted as $\det e$. 

Figure 3. The $E_{11}$ Dynkin diagram appropriate to the $d$-dimensional maximal supergravity theory
We now discuss the $E_{n+1}$ formulation of the $d$ dimensional theory given in section three from the viewpoint of the $E_{11}$ non-linear realisation. For simplicity we will consider only the case $d \leq 7$. We saw from figure 3 that deleting node $d$ leads to the algebra $GL(d) \otimes E_{n+1}$. By examining the $E_{11}$ algebra one can find the generators of $GL(d) \otimes E_{n+1}$ in terms of those of $K^a_b$, $a, b = 1, \ldots, 11$, $R^{a_1 a_2 a_3}$ etc. One finds that the generators of $GL(d)$ are $K^a_b$, $a, b = 1, \ldots, d$ and the Chevalley generators $T_a$, $a = d + 1, \ldots, 11$ in the Cartan subalgebra generators of $E_{n+1}$ are given by

\[ T_{d+1} = \dot{K}^{d+1}_{d+1} - \dot{K}^{d+2}_{d+2}, \ldots T_9 = \dot{K}^9_9 - \dot{K}^{10}_{10}, \]

\[ T_{10} = -\frac{1}{8} \left( \dot{K}^{d+1}_{d+1} + \ldots + \dot{K}^9_9 \right) + \frac{7}{8} \dot{K}^{10}_{10} - \frac{3}{2} \dot{R}, \]

\[ T_{11} = -\frac{1}{4} \left( \dot{K}^{d+1}_{d+1} + \ldots + \dot{K}^8_8 \right) + \frac{3}{4} \left( \dot{K}^9_9 + \dot{K}^{10}_{10} \right) + \dot{R}. \] (4.13)

where $\dot{K}^a_b = K^a_b - \frac{1}{d-2} \delta_b^a \sum_{c=1}^d K^c_c$ for $a, b = d + 1, \ldots, 10$ and $\dot{R} = R$. We note that the generators $\dot{K}^a_b$ obey the necessary condition $[\dot{K}^a_b, P_c] = 0$ for $a, b = 1, \ldots, 10$ and $c = 1, \ldots, d$. In this last equation we have used the commutator $[K^a_b, P_c] = -\delta^a_c P_b + \frac{1}{2} \delta^a_b P_c$. It is straightforward to verify that $T_a = H_a$, $a = d + 1, \ldots, 11$ where these $H_a$ are the Chevalley generators of $E_{11}$ given in equation (4.1), or equivalently from the IIA viewpoint in equation (4.7). The $E_{11}$ group element written in a way that displays the $GL(d) \otimes E_{n+1}$ decomposition required in $d$ dimensions and restricted to lie in the Cartan subalgebra can be written as

\[ g = \exp \sum_{a=1}^d \dot{K}^a_a \sum_{a=d+1}^{11} \varphi_a T_a. \] (4.14)

We have used that the $GL(d)$ generators are $K^a_b$, $a, b = 1, \ldots, d$ and denoted the $E_{n+1}$ fields by $\varphi_a$, $a = d + 1, \ldots, 11$.

We can now equate the two different ways of expressing the $E_{11}$ group element given in equations (4.10) and (4.13), that is the one that implements the dimensional reduction from the IIA theory to the one that has the $GL(d) \otimes E_{n+1}$ decomposition in $d$ dimensions. Using equations (4.13) and (4.7) and keeping only terms involving $K^a_a$, $a = d + 1, \ldots, 11$ we find the equation

\[ e^{(\dot{h}^{d+1}_{d+1} + e_2 \rho) K^{d+1}_{d+1} + \ldots + (\dot{h}^9_9 + e_2 \rho) K^9_9} e^{(\dot{h}^{10}_{10} + e_2 \rho) K^{10}_{10} e^{\sigma R}} = e^{\varphi_{d+1} \left( K^{d+1}_{d+1} - K^{d+2}_{d+2} \right)} \ldots e^{\varphi_8 \left( K^8_8 - K^9_9 \right)} e^{\varphi_9 \left( K^9_9 - K^{10}_{10} \right)} \ldots e^{\varphi_{10} \left( -\frac{1}{8} (K^{d+1}_{d+1} + \ldots + K^9_9) + \frac{7}{8} K^{10}_{10} - \frac{3}{2} R \right)} e^{\varphi_{11} \left( -\frac{1}{4} (K^{d+1}_{d+1} + \ldots + K^8_8) + \frac{3}{4} (K^9_9 + K^{10}_{10}) + R \right)}. \] (4.15)
Equating the coefficients of the generators $K^a_a$ and $R$ we find the relations

\[
\begin{align*}
\dot{h}^{d+1}_{d+1} + e_2 \rho &= \varphi_{d+1} - \frac{1}{8} \varphi_{10} - \frac{1}{4} \varphi_{11}, \\
\dot{h}^{d+2}_{d+2} + e_2 \rho &= -\varphi_{d+1} + \varphi_{d+2} - \frac{1}{8} \varphi_{10} - \frac{1}{4} \varphi_{11}, \\

\ldots
\end{align*}
\]

(4.16)

\[
\begin{align*}
\dot{h}^{8} + e_2 \rho &= -\varphi_{7} + \varphi_{8} - \frac{1}{8} \varphi_{10} - \frac{1}{4} \varphi_{11}, \\
\dot{h}^{9} + e_2 \rho &= -\varphi_{8} + \varphi_{9} - \frac{1}{8} \varphi_{10} + \frac{3}{4} \varphi_{11}, \\
\dot{h}^{10} + e_2 \rho &= -\varphi_{9} + \frac{7}{8} \varphi_{10} + \frac{3}{4} \varphi_{11}, \\
\sigma &= \frac{3}{2} \varphi_{10} + \varphi_{11}.
\end{align*}
\]

Solving these equations for the $E_{n+1}$ fields we find that

\[
\varphi_i = \dot{h}^{d+1}_{d+1} + \dot{h}^{d+2}_{d+2} + \ldots + \dot{h}^{i}_{i} + (n-10+i) \frac{8}{8-n} e_2 \rho, \quad d+1 \leq i < 8,
\]

\[
\begin{align*}
\varphi_9 &= \dot{h}^{d+1}_{d+1} + \dot{h}^{d+2}_{d+2} + \ldots + \dot{h}^{9}_{9} + \frac{5n-8}{8-n} e_2 \rho - \frac{1}{4} \sigma, \\
\varphi_{10} &= -\frac{1}{2} \sigma + \frac{2}{8-n} ne_2 \rho, \\
\varphi_{11} &= \frac{1}{4} \sigma + \frac{3}{8-n} ne_2 \rho.
\end{align*}
\]

(4.17)

In this section we have formulated the $E_{11}$ algebra in terms of the Chevalley generators, in particular the Cartan subalgebra generators $H_a, a = 1, \ldots, 11$, however, in section three we used the Cartan-Weyl basis with generators $H_i, i = 1, \ldots, 11$. The advantage of the latter basis is that acting on a state $|\Lambda >$ of weight $\Lambda_i$, the generators $H_i$, by definition, read off the weight i.e. $H_i|\Lambda > = \Lambda_i|\Lambda >$. The two sets of generators are related by $\alpha^i_a H_i = H_a$ where $\alpha_a$ are the simple roots and $\alpha^i_a$ is the i’th component. If we denote the fields in the Cartan-Weyl basis by $\tilde{\varphi}_a$. The corresponding fields are related by $\tilde{\varphi}^i H_i = \varphi^a H_a$ which implies the relation

\[
\tilde{\varphi}^i = \varphi^a \alpha^i_a, \quad (4.18)
\]

where $\alpha_a$ are the simple roots of $E_{n+1}$ and the sum is over $a = d+1, \ldots, 11$ and the same for $i$. In addition, the fields $\tilde{\varphi}_i$ in the $E_{11}$ group element are equal to the fields $\varphi_i$ that appear in the automorphic form, up to a numerical factor. We see, through comparing the normalisations of the fields in the the $E_{11}$ group element $e^{\tilde{\varphi}_i H_i}$ and the automorphic form group element $e^{-\frac{1}{\sqrt{2} \tilde{\varphi}}} \hat{H}$, that

\[
\tilde{\varphi} = \left( -\sqrt{2} \tilde{\varphi}_1, -\sqrt{2} \tilde{\varphi}_2, -\sqrt{2} \tilde{\varphi}_3 \right) \quad (4.19)
\]

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Using equation (B.3) of appendix B and equation (4.18) we then find that the components
of the $E_{11}$ group element fields $\tilde{\varphi}_i$ in Cartan-Weyl basis and the Chevalley basis are related
by
\[
\tilde{\varphi}^1 = x\varphi^{10} = x \left( -\frac{1}{2}\sigma + \left( \frac{2}{8-n} \right) ne_2\rho \right),
\]
(4.20)
\[
\tilde{\varphi}^2 = -\frac{\Delta_{n-2}\Delta_{n-1}}{y} \varphi^{10} + y\varphi^{11}
= -\frac{\Delta_{n-2}\Delta_{n-1}}{y} \left( -\frac{1}{2}\sigma + \left( \frac{2}{8-n} \right) ne_2\rho \right) + y \left( \frac{1}{4}\sigma + \left( \frac{3}{8-n} \right) ne_2\rho \right),
\]
and
\[
\tilde{\varphi} = \sum_{i=d+1}^{9} \varphi_i \alpha_{i-d} - \varphi^{10} \Delta_{n-1} - \varphi^{11} \Delta_{n-2}.
\]
(4.21)
(4.22)

Note that
\[
\tilde{\varphi}_i \alpha_i = \tilde{\varphi}^{i+1}_{i+1}, \quad \tilde{\varphi}_j \Delta_j = \sum_{i=d+1}^{d+j} \tilde{\varphi}^i_i.
\]
(4.22)

5. Constraints on the automorphic forms

In this section we will compare the $E_{n+1}$ formulation in $d$ dimensions given in section
two with the results of section two found by dimensionally reducing the IIA theory in
ten dimensions; as a result we will find constraints on the automorphic forms. In order
to carry out the comparison we will use the field relations of the last section. The field
strengths occurred in the $E_{n+1}$ formulation as the non-linear representations $\mathcal{F}_{E_{n+1}}$ given
in equation (3.1), while the derivatives of the scalars occur in the Cartan forms $P_{E_{n+1}}$.
These are constructed using the group element $g_{E_{n+1}}$, however, this is just the $E_{11}$ group
element restricted to lie in the subalgebra $E_{n+1}$ and it is given below equation (4.12). We
noted that if one deletes nodes ten and eleven in the $E_{11}$ Dynkin diagram the $E_{n+1}$ algebra is
reduced to $SL(n) \times GL(1) \times GL(1)$. In the dimensional reduction of the IIA theory we found a mani
festo $SL(n) \otimes GL(1)$ symmetry; the first factor arises from the diffeomorphisms preserved by the torus while
the second factor is the $GL(1)$ symmetry of the IIA theory in ten dimensions. As such, the
field strengths that appear in the dimensional reduction can be expressed in terms of the non-linear representation of $SL(n) \otimes GL(1)$ denoted by $\mathcal{F}_{Sl(n)\otimes GL(1)}$ and the derivatives
of the scalars in terms of the Cartan forms $P_{sl(n)\otimes GL(1)}$.

Deleting nodes ten and eleven of the Dynkin diagram of figure 3 we find that $E_{n+1}$ decomposes into $SL(n) \otimes GL(1) \otimes GL(1)$ and one can carry out the decomposition of the non-linear representations that occur in the $E_{n+1}$ formulation. Clearly, the non-linear representations of the field strengths $\mathcal{F}_{E_{n+1}}$ will decompose into the non-linear representations $\mathcal{F}_{sl(n)\otimes GL(1)}$ with appropriate factors corresponding to the additional $GL(1)$. The same
discussion applies to the derivative of the scalars which appear in $P_{E_{n+1}}$ and $P_{SL(n)\otimes GL(1)}$. 

Given a particular term in the higher derivative effective action found by dimensional reduction and matching it with the $E_{n+1}$ formulation, the $SL(n) \otimes GL(1)$ parts will automatically agree and it is with the comparison of the other $GL(1)$ factor that we find non-trivial results.

It would be instructive to systematically carry out the decomposition of $E_{n+1}$ formulation when decomposed to $SL(n) \times GL(1) \times GL(1)$, but for our present purposes it suffices to carry it out for the generators that belong to the Cartan subalgebra. With this restriction the group element of $E_{n+1}$ is given, below equation (4.12), by $g_{E_{n+1}} = e^{H_\alpha \varphi_a}$, but an equivalent formulation, in terms of the field variables associated with dimensional reduction, is given in equation (4.15). Matching these we found in equations (4.16) and (4.17) how the fields $\varphi_a$, $a = d + 1, \ldots, 11$ correspond to the fields $\hat{h}^a$, $a = d + 1, \ldots, 10, \rho$ and $\phi$ found in the dimensional reduction of the IIA theory. The additional $GL(1) \otimes GL(1)$ group found in the reduction then corresponds to the Cartan subalgebra generators $H_{10}$ and $H_{11}$ or from the dimensional reduction viewpoint to the fields $\rho$ and $\phi$.

We now consider the decomposition in more detail. One may write any root of $E_{n+1}$ in terms of its simple roots:

$$\vec{\alpha} = m_c \vec{\alpha}_{n+1} + n_c \vec{\alpha}_n + \sum_{i=1}^{n-1} m_i \vec{\alpha}_i = n_c \left( x, -\frac{\lambda_{n-2} \lambda_{n-1}}{y}, 0 \right) + m_c \left( 0, y, 0 \right) - \vec{\lambda}, \quad (5.1)$$

where $\vec{\lambda} = m_c \lambda_{n-2} + n_c \lambda_{n-1} - \sum_{i=1}^{n-1} k_i \vec{\alpha}_i$ and we have used equation (B.3). The roots of $E_{n+1}$ are labelled by the integers $m_c, n_c$ which are referred to as the levels. If a representation of $SL(n)$ occurs in the decomposition of the adjoint representation of $E_{n+1}$ then its highest weight must appear on the right-hand side as one of the $\vec{\lambda}$’s. We can examine which representations occur level by level. At level $n_c = m_c = 0$ one obviously finds the adjoint representation of $SL(n)$. At higher levels the highest weights, and so representations, of $SL(n)$ that occur are given in the table below

$$\begin{array}{cccccc}
  m_c = 1, & n_c = 0 & m_c = 0, & n_c = 1 & m_c = 1, & n_c = 1 \\
  \lambda_2 & \lambda_1 & \lambda_3 & 0 \\
  m_c = 3, & n_c = 1 & m_c = 2, & n_c = 2 & m_c = 3, & n_c = 2 \\
  0 & \lambda_6 & \lambda_1. & \\
\end{array} \quad (5.2)$$

As such one finds that the weights in the adjoint representation of $E_{n+1}$ are given by

$$\left(0, 0, [\alpha_1 + \ldots + \alpha_{n-1}]\right), \quad \left(0, y, [\lambda_2]\right), \quad \left(x, -\frac{\lambda_{n-2} \lambda_{n-1}}{y}, [\lambda_1]\right),$$

$$\left(x, -\frac{\lambda_{n-2} \lambda_{n-1}}{y} + y, [\lambda_3]\right), \quad \left(x, -\frac{\lambda_{n-2} \lambda_{n-1}}{y} + 2y, 0\right),$$

$$\left(x, -\frac{\lambda_{n-2} \lambda_{n-1}}{y} + 3y, 0\right), \quad \left(2x, -\frac{\lambda_{n-2} \lambda_{n-1}}{y} + 2y, [\lambda_6]\right),$$

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The Cartan form $P_{E_{n+1}}$ belongs to the adjoint representation of $E_{n+1}$ and at level $m_c = n_c = 0$ decomposes into the Cartan forms of $SL(n)$. Using the decomposition of equation (5.3) we see that at higher levels they decompose as follows

$$m_c = 1, \ n_c = 0 \quad P_{\text{SL}(n)i_1i_2} \quad P_{\text{SL}(n)i} \quad P_{\text{SL}(n)i_1i_2i_3} \quad P_{\text{SL}(n)i_1i_2...i_n}$$

We noted previously that the Cartan form $P_{E_{n+1}}$ contains a dependence on the scalars $\vec{\phi}$ in the form factor $e^{\frac{1}{\sqrt{2}}\vec{\phi}\cdot\vec{\alpha}}$. Under the decomposition we find the $SL(n) \otimes GL(1)$ Cartan forms $P_{\text{SL}(n)\otimes GL(1)}$ and exponentials in $\rho$. Using equations (4.20)-(4.22), (B.3) and (B.4) we find that the latter factors at level $m_c, \ n_c$ are

$$e^{(2m_c+n_c)\alpha\rho\left(\frac{8-n}{\pi}\right)}.$$  \hspace{1cm} (5.5)

We now consider the terms that result from the dimensional reduction from the IIA theory using the discussion of section two. The ten dimensional origins of the decomposition of the adjoint representation of $E_{n+1}$ at each level may be found by examining the $SL(n)$ and space-time index structure. In particular, we see that the $SL(n) \otimes GL(1)$ Cartan forms $P_{\text{SL}(n)\otimes GL(1)}$ at levels $(m_c = 0, \ n_c = 1)$, $(m_c = 1, \ n_c = 0)$ and $(m_c = 1, \ n_c = 1)$ come from the dimensional reduction of the two form field strength $\tilde{F}_{a_1a_2}$, three form field strength $\tilde{F}_{a_1a_2a_3}$ and four form field strength $\tilde{F}_{a_1a_2a_3a_4}$ respectively. The Cartan forms, at higher levels, are associated with the dimensional reduction of the dualised two, three and four form field strengths for levels $(m_c = 3, \ n_c = 1)$, $(m_c = 2, \ n_c = 2)$ and $(m_c = 2, \ n_c = 1)$ respectively, along with the dualised graviphoton at level $(m_c = 3, \ n_c = 2)$. We note that the dualised four form only appears as a Cartan form of $SL(n) \otimes GL(1)$ in $d = 5$, while the dualised three form is only present as a Cartan form of $SL(n) \otimes GL(1)$ in $d = 4$. While the dualised graviphoton is a Cartan form of $SL(n) \otimes GL(1)$ only in $d = 3$ and we also find the dualised two form is also realised as a Cartan form of $SL(n)$. The Cartan forms of $SL(n) \otimes GL(1)$, arising upon dimensional reduction, carry one $d$ dimensional space-time index and $(2m_c + n_c)$ internal indices. Therefore, each Cartan form of $SL(n) \otimes GL(1)$, at a given level, occurs with an exponential of $\rho$ which is given by

$$e^{-\rho(\alpha+(2m_c+n_c)b)} = e^{(2m_c+n_c)\alpha\rho\left(\frac{8-n}{\pi}\right)}e^{-\alpha\rho}.$$  \hspace{1cm} (5.6)

Comparing with the result, given in equation (5.5), of the $E_{n+1}$ formulation we find a surplus factor of $e^{-\alpha\rho}$ multiplying the dimensionally reduced term. We note that the factors involving $\phi$ and $\vec{\phi}$ will match automatically due to the automatic agreement of the $SL(n) \times GL(1)$ part.
To treat the other building blocks in the same way we must learn how to decompose more general representations of $E_{n+1}$ into those of $SL(n) \times GL(1) \times GL(1)$. To do this we use the technique of reference [48]. If one wants to consider the representation of $E_{n+1}$ with highest weight $\Lambda_i$, associated with the node labeled $i$, we add a new node, denoted $\star$, to the $E_{n+1}$ Dynkin diagram which is connected to the node labeled $i$ by a single line to construct the Dynkin diagram for an enlarged algebra of rank $n+2$. Deleting the $\star$-node we recover the $E_{n+1}$ Dynkin diagram and the representation of $E_{n+1}$ with highest weight $\Lambda_i$ is found in the adjoint representation of the enlarged algebra provided we keep only contributions at level $n_* = 1$. Thus we find the decomposition of the representation of $E_{n+1}$ with highest weight $\Lambda_i$ into representations of $SL(n) \times GL(1) \times GL(1)$ by decomposing the adjoint representation of the enlarged algebra but deleting the additional node and keeping only contributions with $n_* = 1$ and deleting nodes 10 and 11 but keeping all levels of $m_c$ and $n_c$.

In the $E_{n+1}$ formulation of the effective action in $d$ dimensions, the one form gauge field, out of which the two form field strengths are constructed, appear in the representation with highest weight $\tilde{\Lambda}_1$. The $\tilde{\Lambda}_1$ representation of $E_{n+1}$ may be decomposed into representations of $SL(n)$, with an associated type IIA dilaton weight, level by level. At level $(m_c, n_c)$ one finds

$$
\begin{align*}
&m_c = 0, \ n_c = 0 \quad m_c = 1, \ n_c = 0 \quad m_c = 0, \ n_c = 1 \quad m_c = 1, \ n_c = 1 \\
&\tilde{\Lambda}_1 \quad \tilde{\Lambda}_{n-1} \quad 0 \quad \tilde{\Lambda}_{n-2} \\
&m_c = 2, \ n_c = 1 \quad m_c = 2, \ n_c = 2 \quad m_c = 3, \ n_c = 1 \quad m_c = 3, \ n_c = 2 \\
&\tilde{\Lambda}_{n-4} \quad \tilde{\Lambda}_{n-5} \quad \tilde{\Lambda}_{n-6} \quad \tilde{\Lambda}_{n-1} .
\end{align*}
$$

(5.7)

Therefore, the weights of the $\tilde{\Lambda}_1$ representation are

$$
\begin{align*}
\left( \frac{1}{2x}, \frac{\tilde{\Lambda}_1 \cdot \tilde{\Lambda}_{n-2}}{y}, [\tilde{\Lambda}_1] \right), & \quad \left( \frac{1}{2x}, \frac{\tilde{\Lambda}_1 \cdot \tilde{\Lambda}_{n-2}}{y} - y, [\tilde{\Lambda}_{n-1}] \right), & \quad \left( \frac{1}{2x} - x, \frac{\tilde{\Lambda}_1 \cdot \tilde{\Lambda}_{n-2}}{y} + \frac{\tilde{\Lambda}_{n-1} \cdot \tilde{\Lambda}_{n-2}}{y}, [0] \right), \\
\left( \frac{1}{2x} - x, \frac{\tilde{\Lambda}_1 \cdot \tilde{\Lambda}_{n-2}}{y} - y + \frac{\tilde{\Lambda}_{n-1} \cdot \tilde{\Lambda}_{n-2}}{y}, [\tilde{\Lambda}_{n-2}] \right), & \quad \left( \frac{1}{2x} - x, \frac{\tilde{\Lambda}_1 \cdot \tilde{\Lambda}_{n-2}}{y} - 2y + \frac{\tilde{\Lambda}_{n-1} \cdot \tilde{\Lambda}_{n-2}}{y}, [\tilde{\Lambda}_{n-4}] \right), \\
\left( \frac{1}{2x} - 2x, \frac{\tilde{\Lambda}_1 \cdot \tilde{\Lambda}_{n-2}}{y} - 2y + \frac{2 \tilde{\Lambda}_{n-1} \cdot \tilde{\Lambda}_{n-2}}{y}, [\tilde{\Lambda}_{n-5}] \right), & \quad \left( \frac{1}{2x} - x, \frac{\tilde{\Lambda}_1 \cdot \tilde{\Lambda}_{n-2}}{y} - 3y + \frac{\tilde{\Lambda}_{n-1} \cdot \tilde{\Lambda}_{n-2}}{y}, [\tilde{\Lambda}_{n-6}] \right), \\
\left( \frac{1}{2x} - 2x, \frac{\tilde{\Lambda}_1 \cdot \tilde{\Lambda}_{n-2}}{y} - 3y + \frac{2 \tilde{\Lambda}_{n-1} \cdot \tilde{\Lambda}_{n-2}}{y}, [\tilde{\Lambda}_{n-1}] \right), & \quad (5.8)
\end{align*}
$$
From the weights, we see that the corresponding two form field strengths, at each level, are

\[
\begin{align*}
& m_c = 0, \ n_c = 0 \quad F_{a_1 a_2}^i, \\
& m_c = 1, \ n_c = 0 \quad F_{a_1 i}^2, \\
& m_c = 0, \ n_c = 1 \quad F_{a_1 a_2}^i, \\
& m_c = 1, \ n_c = 1 \quad F_{a_1 a_2 i}^i i_2.
\end{align*}
\]

\( (5.9) \)

After dualisation, any two form field strength will appear as a one form field strength in \( d = 3 \) dimensions therefore we only need to consider two form field strengths in \( d \geq 4 \) dimensions. One finds the maximum level that contributes is \( (m_c = 3, \ n_c = 2) \), in the remaining dimensions any level \( (m_c, n_c) \) listed in the above decomposition will appear in \( d \) dimensions if \( (2m_c + n_c - 1) \leq n \). The two form field strength at level \( (m_c, n_c) \) arises through the dimensional reduction of the metric at level \( (0, 0) \), three form field strength at level \( (1, 0) \), two form field strength at level \( (0, 1) \) and four form field strength at level \( (1, 1) \). The higher levels in the decomposition of the representation with highest weight \( \tilde{\Lambda}_1 \) are associated with the dimensional reduction of the dualised field strengths and the graviphoton.

A two form field strength in some representation of \( SL(n) \) at level \( (m_c, n_c) \) in the \( E_{n+1} \) formulation of the IIA theory appears multiplied by the factor

\[
e^{-\frac{3}{4} \alpha \rho - (2m_c + n_c) \left( \frac{n}{n+1} \right) \alpha \rho},
\]

where the factors associated with \( SL(n) \) fields \( \phi \) and the IIA dilaton \( \phi \) match those found upon dimensional reduction. Comparing the volume with the dimensionally reduced two form field strengths, which carry two \( d \) dimensional indices and \( 2m_c + n_c - 1 \) internal indices and as a result appear multiplied by the factor

\[
e^{-\rho (2\alpha + (2m_c + n_c - 1)\beta)} = e^{-\alpha \rho} e^{-\frac{3}{4} - (2m_c + n_c) \left( \frac{n}{n+1} \right) \alpha \rho},
\]

we find that the two form field strengths in the dimensionally reduced type IIA effective action carry an additional factor of \( e^{-\alpha \rho} \).

Three form field strengths appear in the type IIA effective action in \( d \geq 6 \) dimensions. In the \( E_{n+1} \) formulation, the two form gauge fields, from which the three form field strengths are constructed, lie in the representation with highest weight \( \tilde{\Lambda}_n \). The \( \tilde{\Lambda}_n \) representation decomposes into representations of \( SL(n) \) with an associated type IIA dilaton weight, at level \( (m_c, n_c) \), in the following way

\[
\begin{align*}
& m_c = 0, \ n_c = 0 \quad \tilde{\Lambda}_0, \\
& m_c = 0, \ n_c = 1 \quad \tilde{\Lambda}_{n-1}, \\
& m_c = 1, \ n_c = 1 \quad \tilde{\Lambda}_{n-3}, \\
& m_c = 1, \ n_c = 2 \quad \tilde{\Lambda}_0.
\end{align*}
\]

\( (5.12) \)

This decomposition leads one to observe that the weights in the \( \tilde{\Lambda}_{n+1} \) representation of \( E_{n+1} \) are

\[
\left( \frac{1}{x}, 0, \tilde{\Lambda}_0 \right), \left( \frac{1}{x} - x, \frac{\tilde{\Lambda}_{n-2} \tilde{\Lambda}_{n-1}}{y}, \tilde{\Lambda}_{n-1} \right), \left( \frac{1}{x} - x, \frac{\tilde{\Lambda}_{n-2} \tilde{\Lambda}_{n-1}}{y} - y, \tilde{\Lambda}_{n-3} \right),
\]

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The three form field strengths, at level \((m, n)\), are

\[
\begin{align*}
F_{a_1a_2a_3} & = 0, \quad F_{a_1a_2a_3i} = 0, \quad F_{a_1a_2i_1i_2i_3} = 0, \\
\end{align*}
\]  

The three form field strengths, at level \((m, n)\), are

\[
\begin{align*}
m_c = 0, \quad n_c = 0 & \quad m_c = 0, \quad n_c = 1 & \quad m_c = 1, \quad n_c = 1 & \quad m_c = 1, \quad n_c = 2 \\
F_{a_1a_2a_3} & = F_{a_1a_2a_3i} & F_{a_1a_2i_1i_2i_3} & = F_{a_1a_2a_3}.
\end{align*}
\]  

Any three form may be dualised to a lower degree form in \(d \leq 5\), therefore we need only consider three form field strengths for \(n \leq 4\). For \(n = 4\) all of the three form field strengths listed above are present. For \(n < 4\) a three form field strength, at level \((m, n)\), will be present if \((2m_c + n_c \leq n)\). The origin of the three form field strengths is clear, the three form field strength at level \((0, 0)\) is the dimensionally reduced three form field strength, while the three form field strength at level \((0, 1)\) is the dimensionally reduced four form field strength. The remaining two levels are associated with the duals of the dimensionally reduced three and four form field strengths. The decomposition of the \(\tilde{A}_n\) of \(E_{n+1}\), at level \((m, n)\), leads to the \(E_{n+1}\) formulation of the non-linearly realised three form field strengths containing the factor of

\[
e^{-2+(2m_c+n_c)(\frac{n-8}{n})}\alpha\rho, \quad (5.15)
\]

again, we find the factors involving the IIA dilaton \(\phi\) and the \(SL(n)\) fields \(\phi\) agree with the dimensionally reduced formulation. However, the three form field strengths in the dimensionally reduced formulation come with three space-time indices and \(2m_c + n_c\) internal indices, therefore they carry a factor of

\[
e^{-\rho(3\alpha+(2m_c+n_c)\beta)} = e^{-\alpha\rho}e^{-2+(2m_c+n_c)(\frac{\rho}{n})}\alpha\rho, \quad (5.16)
\]

Comparing the \(\rho\) factor of the \(E_{n+1}\) formulation and the dimensionally reduced formulation, one finds that the three form field strengths in the dimensionally reduced effective action of the type IIA theory carry an additional factor of \(e^{-\alpha\rho}\). The four form field strengths, which only exist in \(d \geq 8\) space-time dimensions follow the same pattern, with the dimensionally reduced formulation containing an additional factor of \(e^{-\alpha\rho}\) when compared to the \(E_{n+1}\) formulation of the effective action in \(d\) dimensions.

Thus, one finds that the surplus weight of any derivative of the scalars form or field strengths in the dimensionally reduced formulation of the effective action of the type IIA theory in \(d\) dimensions contains an additional factor of \(e^{-\alpha\rho}\) when compared to the \(E_{n+1}\) formulation in \(d\) dimensions. Thus we find an excess factor of \(e^{-\alpha\rho}\) for every space-time derivative in the effective action. The dimensionally reduced theory also carries a factor of \(e^{-\tilde{s}\phi}\) from the ten dimensional automorphic form, where \(\tilde{s}\) is given in equation (2.9) and is fixed by demanding that, upon transforming to string frame, any term carries a factor of \(e^{\phi(2g-2)}\) arising from a perturbative expansion in the ten dimensional IIA string coupling constant \(g_s = e^\phi\) at order \(g\). Also from the dimensional reduction of the \(\det(e)\) from ten dimensions we find a factor of \(e^{-2\alpha\rho}\). Therefore, we find that the dimensionally reduced theory, when packaged up into objects transforming under \(E_{n+1}\), has a surplus factor of

\[
e^{-(l_T-2)\alpha\rho-\tilde{s}\phi}. \quad (5.17)
\]
where \( l_T = \tilde{l}_0 + \tilde{l}_R + \tilde{l}_1 + \tilde{l}_2 + \tilde{l}_3 + \tilde{l}_4 \) is the total number of derivatives and \( \tilde{s} \) is given in equation (2.9). We note that the factor of equation (5.17) can be written as \( e^{-\sqrt{2} \tilde{\Lambda}_\phi \cdot \phi} \) where

\[
\tilde{\Lambda}_\phi = \left( \frac{\tilde{s}}{\sqrt{2}}, \alpha \frac{(l_T - 2)}{\sqrt{2}}, 0 \right) = \left( \frac{l_T - 2}{4} + \frac{3}{4}(l_{RR} - 2g) \right) \tilde{\Lambda}_n + \frac{1}{2}(l_{RR} - 2g) \tilde{\Lambda}_{n+1}. \quad (5.18)
\]

where \( l_{RR} = \tilde{l}_2 + \tilde{l}_4 \) is the number of R-R fields in a given term.

The theory in \( d \) dimensions contains an \( E_{n+1} \) automorphic form and this must account for the missing factors. Therefore, we conclude that the automorphic form must contain the weight \( \tilde{\Lambda}_\phi \). For a pure NS-NS term at \( g = 0 \), (i.e. setting \( l_2 = l_4 = 0 \) and \( g = 0 \)) the leading order contribution to the automorphic form carries the weight

\[
\tilde{\Lambda}_\phi = \left( \frac{l_T - 2}{4} \right) \tilde{\Lambda}_n. \quad (5.19)
\]

As such it is likely that the automorphic form is constructed from the representation with highest weight \( \tilde{\Lambda}_n \) and with \( s = \left( \frac{l_T - 2}{4} \right) \). The R-R terms are related to those in the NS-NS sector by an \( SL(2,R) \) rotation and so are automatically accounted for.

6. Discussion of results and their consequences for the automorphic forms of string theory

In this paper we have carried out the dimensional reduction of the higher derivative corrections of the IIA theory and found that the \( E_{n+1} \) automorphic forms that appear as coefficients of the terms in the effective action in \( d = 10 - n \) dimensions must contain the fundamental weight \( \Lambda_n \) associated with node \( n \) of the \( E_{n+1} \) Dynkin diagram of figure 5; this corresponds to node ten in the \( E_{11} \) Dynkin diagram of figure 3. The well understood \( E_{n+1} \) automorphic forms that appear in string theory are constructed using a given representation of \( E_{n+1} \); the reader may, for example, consult the explicit construction of these objects given in [34]. As such, the result of this paper strongly suggests that the automorphic forms that occur in string theory are constructed from the representation with highest weight \( \Lambda_n \). More precisely, it implies that if the coefficient of the higher derivative term is a sum of automorphic forms then one of them should be constructed from the highest weight \( \Lambda_n \) as it could happen that the other automorphic forms do not occur in the dimensional reduction from the IIA theory in ten dimensions. A similar analysis from the IIB perspective gave the same result namely that the automorphic form contains the weight \( \Lambda_n \) [39]. However, from the M theory perspective, that is from eleven dimensions, a similar analysis found that the automorphic form contains the highest weight \( \Lambda_{n+1} \) which in the \( E_{11} \) Dynkin diagram of figure 3 corresponds to node eleven [39]. This result only applies to terms that occur in the eleven dimensional theory. The calculation of this paper, and that of reference [39] also determines the parameter \( s \) of equation (3.3) that occurs in the automorphic form; for the IIA and IIB theories we find that \( s = \frac{l_T - 2}{4} \), while for M theory we find that \( s = \frac{l_T - 2}{6} \) where \( l_T \) is the number of space-time derivatives in the term in the effective action being considered.

We will now consider if the results just mentioned actually agree with the known results in type II string theory. For low numbers of space-time derivatives there are precise
proposals for the automorphic forms that occur and their properties have been checked against the known features of the perturbation expansions of the type II strings [34, 35, 36, 37]. One finds for the $R^4$ term in $d \leq 7$ that the $E_{n+1}$ automorphic form is built from the representation with highest weight $\Lambda_n$ and has $s = \frac{3}{2}$. This is completely consistent with the results found from the IIA and IIB viewpoints. For the $\partial^4 R^4$, or equivalently $R^6$, term in $d \leq 7$ the $E_{n+1}$ automorphic form is built from the representation with highest weight $\Lambda_n$ and has $s = \frac{5}{2}$. However, in $d = 7$ dimensions the coefficient of this term is in fact a sum of two $E_4 = SL(5)$ automorphic terms [35, 36, 37], in addition to an automorphic form constructed from the $5$ of $SL(5)$, with $s = \frac{5}{2}$ one finds an automorphic form built from the 10 of $SL(5)$ with $s = \frac{5}{2}$. Similarly, in $d = 6$ dimensions the coefficient of the $\partial^4 R^4$ term is the sum of an automorphic form constructed from the 10 of $SO(5,5)$, with $s = \frac{5}{2}$ and another automorphic form built from the 16-dimensional representation of $SO(5,5)$ with $s = 3$. As these additional automorphic forms disappear in the limits being considered, the known automorphic forms for the $\partial^4 R^4$ term are also consistent with the results found from the dimensional reduction of the type IIA and type IIB theories.

However, dimensional reduction of the higher derivative correction of the eleven dimensional theory [39] suggests that the automorphic forms are constructed from the representation with highest weight $\Lambda_{n+1}$. At first sight this is inconsistent with the automorphic forms that are known to be present. However, in seven dimensions, i.e. for $SL(5)$, for the $R^4$ term this would imply in particular that the automorphic form constructed from the $\bar{5}$ of $SL(5)$ with $s = \frac{3}{2}$ is proportional to the automorphic form constructed from the $5$ of $SL(5)$ with $s = 1$. In fact this relation follows from the observation that an automorphic form constructed from a given representation and another automorphic form constructed from the corresponding Cartan involution twisted representation are related by two suitable values of $s$ [34]. The same holds for the automorphic forms associated with the $R^4$ terms in lower dimensions as one knows [37] that the automorphic form constructed from the representation of $E_{n+1}$ with highest weight $\Lambda_n$ and $s = \frac{3}{2}$, i.e $\Phi_{E_{n+1}}^{\Lambda_n;\frac{3}{2}}$, is proportional to the automorphic form constructed from the representation of $E_{n+1}$ with highest weight $\Lambda_{n+1}$ and $s = 1$, i.e $\Phi_{E_{n+1}}^{\Lambda_{n+1};1}$, that is [37]

$$\Phi_{\Lambda_n;\frac{3}{2}}^{E_{n+1}} \propto \Phi_{\Lambda_{n+1};1}^{E_{n+1}}. \quad (6.1)$$

Some examples of relationships of this type were also found in reference [31]. Consequently, the known automorphic forms that occur for the $R^4$ term are also in agreement with the prediction from the M theory viewpoint. However, one cannot apply the M theory results to the $R^6$ term as this term does not occur in the higher derivative effective action in eleven dimensions and so is not included in the analysis from the M theory viewpoint given in [39]. Indeed, the only terms that occur in eleven dimensions that involve, for example, the Riemann curvature are of the form $R^{3n+1}_t$, for $n$ a positive integer.

Given the above discussion, it is tempting to suppose the following

[A] The automorphic forms that occur as coefficients of the higher derivative terms in the string theory effective action must contain an automorphic form constructed from the $\Lambda_n$ representation of $E_{n+1}$.
The automorphic forms that occur in string theory and built from the $\Lambda_n$ representation of $E_{n+1}$ are the same as the automorphic forms built from the $\Lambda_{n+1}$ representation of $E_{n+1}$ up to a numerical factor.

The first statement is phrased so as to allow for the possibility that the coefficient is a sum of automorphic forms one or more of which may disappear in the limit. The second statement only applies to automorphic forms of higher derivative terms that occur in eleven dimensions.

The automorphic forms that are used in the recent work of [35-37] are those that appear in the work of Langlands, and they are eigenfunctions of the Laplacian and the higher Casimir operators of $E_{n+1}$. However, those that are constructed in equation (3.3) are not in general eigenfunctions of these operators. However, one can impose constraints on the representations used to construct the automorphic forms and they then do become eigenfunctions of the Laplacian and higher Casimir operators. This has been worked out explicitly for the case of six dimensions, i.e. for $SO(5,5)$ with the ten dimensional vector representation where the constraint is that the length squared of this vector vanishes. Indeed only if this constraint is implemented is the perturbation series in agreement with that found in string theory; this part of the automorphic form has been checked in detail to agree with the $SO(5,5)$ Langlands automorphic form for this representation [34]. It remains, however, to carry out the analogue of this construction for the higher rank groups and representations. It is interesting to note that at least the constant part of the Langlands automorphic forms can be written as a sum of the Weyl group and this, being a rotation, preserves the lengths of vectors and those vectors that do occur must belong to a single orbit. As such, it is likely that the Langlands automorphic forms will involve constraints on the representations used and will agree with the automorphic forms of equation (3.3) once one imposes the appropriate constraints.

As we have mentioned, the detailed studies of the automorphic forms in the low energy effective action of type II string theory have only concerned terms which have low numbers of space-time derivatives. However, it is known that the automorphic forms that occur as coefficients of the higher derivative terms in ten dimensions that have more than twelve space-time derivatives, are not eigenvalues of the Laplacian and so they can not be the Eisenstein automorphic forms found say in the Langlands papers [37]. As a result the automorphic forms that occur for these higher derivative terms are essentially unknown. This paper and reference [39] puts some constraints on these objects. We have tacitly assumed that all of the automorphic forms that appear as the coefficient functions of the higher derivative terms are constructed from a representation of $E_{n+1}$. Although the form of equation (3.3) may not be correct in general, even with constraints, the automorphic forms will still have a dominant behaviour of the form $e^{-sw\phi}$ in the limit studied in this paper, so they will contain a parameter $s$.

We will now comment on the significance of the representations that occur in the automorphic forms. The brane charges of type II string theory in $d$-dimensions belong to representations of $E_{n+1}$. In fact, there is very substantial evidence to believe that all brane charges belong to the $l_1$ representation of $E_{11}$. Carrying out the decomposition of the $l_1$ representation we find the brane charges in $d$ dimensions; they are listed in table one [49-51]. The first entries of the table agree with that found earlier using U duality.
transformations [52]. Examining the table we find that the string charges, i.e \( Z^a \), are in the \( \Lambda_n \) representation, the membrane charges, i.e \( Z^{ab} \), are in the \( \Lambda_{n+1} \) representation and the point particle charges, i.e \( Z \), are in the \( \Lambda_1 \) representation. Thus the above propositions can be expressed as

[A] The automorphic forms that occur as coefficients of the higher derivative terms in the string theory effective action are constructed from the string charge representation.

We may very generically write these automorphic forms as \( \Phi_{\text{string}} \).

and that

[B] The automorphic forms that occur in string theory built from the string charge representation are the same as the automorphic forms built from the membrane charge representation, up to a numerical factor. We may very generically write this as \( \Phi_{\text{string}} = \Phi_{\text{membrane}} \).

As before the latter proposition only applies to the terms that have an eleven-dimensional origin. It is of course very natural that the string and membrane charge representations found in the automorphic forms arise from the dimensional reduction of the ten dimensional IIA and IIB string theories and the eleven dimensional theory respectively.

It was also observed in reference [37] that the automorphic form for the \( R^4 \) term are related to those built from the \( \Lambda_1 \) representation as follows

\[
\Phi_{\Lambda_n; \frac{n}{2}} \propto \Phi_{\Lambda_1; \frac{n-2}{2}} \tag{6.2}
\]

for \( n = 4, 5, 6, 7 \) while for the \( R^6 \) term

\[
\Phi_{\Lambda_n; \frac{n}{2}} \propto \Phi_{\Lambda_1; \frac{n+2}{2}} \tag{6.3}
\]

for \( n = 4, 5, 6, 7 \).

Since the charges for the point particle belong to the \( \Lambda_1 \) representation we are also tempted to propose that

[C] The automorphic forms that occur in string theory are built from the string charge representation are the same as the automorphic forms built from the point charge representation up to a numerical factor. We may generically write this as \( \Phi_{\text{string}} = \Phi_{\text{point}} \).

For the case of \( d = 7 \) with the group \( SL(5) \) this would require that the automorphic forms constructed from the 5 and 10 representations were the same for appropriate representations. In fact the automorphic forms constructed by Langlands for the two representations \( \Lambda \) and \( \Lambda' \) are proportional if the vectors \( \lambda = 2s\lambda - \rho \) and \( \lambda' = 2s'\lambda' - \rho \) are related by a Weyl reflection. The Weyl vector \( \rho \) can be written as \( \rho = \sum_a \Lambda_a \) where \( \Lambda_a \) are the fundamental weights. For our case we should take \( \Lambda = \Lambda_3 \) and \( \Lambda' = \Lambda_1 \). Since Weyl reflections are rotations they preserve the length squared and one finds that \( \lambda^2 = \lambda'^2 \) for \( s = \frac{3}{2} \) if \( s' = 2 \) or \( s' = \frac{1}{2} \) and for \( s = \frac{5}{2} \) if \( s' = \frac{5}{2} \). Indeed one can show that for \( s = \frac{3}{2} \) and \( s' = \frac{1}{2} \) and also for \( s = \frac{5}{2} = s' \) there is a Weyl reflection of the required kind and so the relations of equations (6.2) and (6.3) do extend to the case of \( n = 3 \) are required.

This is most easily found by writing the vectors \( \lambda \) and \( \lambda' \) in terms of the orthonormal basis \( e_a, a = 1, 2, 3, 4, 5 \) in terms of which the simple roots take the form \( \alpha_a = e_a - e_{a+1} \). As
Weyl reflections permute the $e_a$ basis it is straightforward to see if the two vectors are related by a Weyl reflection.

The presence of the highest weights $\Lambda_n$ and $\Lambda_{n+1}$ in the automorphic forms was deduced from dimensional reduction from the IIA (or IIB) theories and M theory respectively, but as we noted above these representations correspond to nodes that are among those deleted to find these theories as non-linear realisations of $E_{11}$. As such one may suspect that dimensionally reducing the theory from $d+1$ to $d$ dimensions will lead to the constraint that the automorphic form will contain the $\Lambda_1$ representation.

7. $E_{11}$ automorphic forms and higher derivative corrections

The $E_{11}$ conjecture [43] involves a particular real form of $E_{11}$ and has so far been applied to the low energy effective actions of string theory, that is, the supergravity theories. As such it involves taking this form of $E_{11}$ over the real numbers. However, we know that $E_{11}$ rotates the brane charges, and as these are quantised [12,13], for the full quantum string theory, we should only consider a version of $E_{11}$ which is over a discrete field rather than the real numbers. In particular, it should preserve the brane charge lattice which belongs to the $l_1$ representation [56,48,49,50]. We note that, since $E_{11}$ includes the Lorentz group even this subgroup should be taken over a discrete field. As such, it is not clear, at least at first sight, how the $E_{11}$ conjecture can apply to the higher derivative terms and how the theory of non-linear realisations can be applicable?

The way the scalar fields occur in the low energy effective actions, i.e. supergravity theories, is controlled by the fact that they belong to a non-linear realisation. The non-linear realisation for a group $G$, which does not involve space-time generators, with a local subgroup $H$ is constructed from a group element $g(\xi)$ which is a function of space-time and is subject to the transformations $g(\xi) \to g_0 g(\xi)$ and $g(\xi) \to g h(\xi)$ where $h$ belongs to the local subalgebra $H$ and is also a function of space-time and $g_0$ is just an element of the group $G$ which is independent of space-time. The $\xi$ parameterise the group element and in the context of the supergravity theories these are the scalars of the theory which are themselves functions of space-time. This statement is true for all scalar fields that belong to the supergravity multiplet; for the type II theories in $d = 10 - n$ dimensions $G = E_{n+1}$.

In supergravity theories the space-time derivatives of the scalars are contained in the Cartan forms, while the space-time derivatives of the other fields occur together with the scalar fields in just such a way that it converts the linear representations to which they belong into non-linear representations using the group element $g(\xi)$. This construction has already been used in this paper and is described in the appendix A.

As we have mentioned the higher derivative terms of the string effective action in $d = 11 - n - 1$ dimensions are conjectured to be invariant under a discrete $E_{n+1}$ symmetry. In the effective action the space-time derivatives of the scalars and all the other fields occur in precisely the same way as the supergravity theories; the space-time derivatives of the scalars appear as part of the Cartan forms and the derivatives of the other fields arise in a non-linear representation of $E_{n+1}$. However, each term in the higher derivative effective action can have a coefficient that is an automorphic form rather than a constant as is the case of the supergravity theories. The automorphic forms of the type, briefly described around equation (3.3), and used in references [34] and also the automorphic forms of Langlands used in [35-37] are constructed from a given representation of $G$ using
precisely the same group element \( g_{E_{n+1}}(\xi) \) except for the fact that the \( g_0 \) transformation is now over the corresponding discrete group. Indeed the automorphic form is constructed from the non-linear representation, that is, \( |\varphi> = I(g_{E_{n+1}}^{-1})|\psi> \) where the \(|\psi>\) carry the linear representation. Hence although the higher derivative string effective action is only invariant under a discrete \( E_{n+1} \) symmetry it is constructed using much of the same machinery as the \( E_{n+1} \) non-linear realisation that arises in the supergravity theories.

Let us now examine if the building blocks used to construct the higher derivative effective action appear in the non-linear realisation of \( E_{11} \) appropriate to \( d \) dimensions. The \( E_{11} \) group element has the generic form [43,44,45]

\[
g_E = e^{h_{ab}K^a_b}e^{A.R}\ldots g_{E+1}(\xi)
\]

where \( g_E(\xi) \) contains the scalar fields and it is the group element used in the non-linear realisation of \( E_{n+1} \) as just mentioned in the previous paragraph. The Cartan forms of the \( E_{11} \) group element of equation (6.4) contain the \( E_{n+1} \) Cartan form of the scalars and the derivatives of the other fields as non-linear representations of \( E_{n+1} \). As such the \( E_{11} \) non-linear realisation contains all the building blocks of the higher derivative effective action, including the group element \( g_{E+1} \) which was used to construct the automorphic forms; the one exception is the representation used to construct the automorphic forms.

Space-time is introduced into the \( E_{11} \) theory by considering the fundamental representation of \( E_{11} \) associated with node one, denoted \( l_1 \) [56]. In particular one takes the non-linear realisation of the semi-direct product of \( E_{11} \) and generators that belong to the \( l_1 \) representation, denoted \( E_{11} \otimes_s l_1 \). The corresponding group element is given by \( g = g_E g_{l_1} \) where \( g_E \) is the group element of \( E_{11} \), given in equation (7.1), and \( g_{l_1} \), in \( d \) dimensions, is of the form [56]

\[
g_{l_1} = e^{x^a P_a} e^{z^a Z^a} e^{z_{ab} Z_{ab}} \ldots
\]

where \( P_a, Z, Z^a, \ldots \) are the generators that belong to the \( l_1 \) representation decomposed into representations of \( GL(d) \otimes E_{n+1} \). In particular \( P_a \) are the space-time translations in \( d \) dimensions, \( Z \) are the scalar, that is point particle, charges, \( Z^a \) are the string charges, \( Z^{ab} \) the membrane charges ... etc. These charges belong to the \( \Lambda_1, \Lambda_n, \Lambda_{n+1}, \ldots \) representations of \( E_{n+1} \) [49-51], see table one. The \( x^a, z, z_a, z_{ab}, \ldots \) are the coordinates of the generalised space-time. As the \( l_1 \) representation contains all the brane charges there is a correspondence between the coordinates of the generalised space-time and the brane charges. The non-linear realisation \( E_{11} \otimes_s l_1 \) largely specifies the generalised geometry corresponding to this generalised space-time.

It is intriguing that the point particle, string and membrane representations of \( E_{n+1} \) contained in the \( l_1 \) representations are just the ones that show up in the automorphic forms that occur in string theory. Taking this together with our previous comments we find that all the ingredients of higher derivative corrections can be found in the non-linear realisation of \( E_{11} \otimes l_1 \). This is at least consistent with the possibility of an \( E_{11} \otimes l_1 \) formulation of the higher derivative effective action. To construct \( E_{11} \) automorphic forms using the \( l_1 \) representation is straightforward, at least at the naive level, as the construction of reference [34] can be applied straightforwardly. Furthermore as we have explained above one can write the space-time derivatives of the fields as part of the \( E_{11} \) Cartan forms. When
considering the non-linear realisation of $E_{11} \otimes s l_1$ one would expect the parts involving the usual space-time derivatives, that is the derivatives of the fields, to rotate into the parts involving the other parts of the $l_1$ representation, that is the parts in the automorphic form. However, it is not immediately clear how to construct all of the higher derivative string effective action from the $E_{11}$ non-linear realisation. While one might think that an $E_{11}$ invariant exists at each order in space-time derivatives one could dream that there is an enlargement of the symmetry algebra that combines all orders in derivatives into a single automorphic form. Another point to bear in mind is that the $E_{11} \otimes s l_1$ non-linear realisation leads in the context of supergravity theories, to equations of motion that contain only one space-time derivative and it is not apparent how this could be generalised to incorporate the higher derivative corrections.

The remaining problem in applying $E_{11}$ to the low energy effective action is how to reconcile the generalised space-time encoded in the $E_{11} \otimes s l_1$ non-linear realisation with the usual formulations of supergravity that involves just the conventional coordinates $x^h$ of space-time. The work of reference [53] suggested that even though the full theory was $E_{11} \otimes l_1$ invariant only part of the $l_1$ representation occurred in the second quantised field theory. In particular although the first quantised theory involved all of the $l_1$ representation, the choice of representation of the commutators that takes one to get from the first to the second quantised theory required one to choose only part of the $l_1$ representation. However, one can make different choices of which part of the $l_1$ representation one takes and these should be equivalent and related by $E_{11}$ transformations. The fact that automorphic forms constructed from different representations are the same and so lead to the same theory could be consistent with the above observation. Indeed one can view it as a kind of uncertainty principle.

We note another similarity between the higher derivative effective action and the $E_{11} \otimes s l_1$ non-linear realisation. Dimensionally reducing a field theory on an internal space and keeping all the Kaluza-Klein modes just expresses the original theory on the internal space in an alternative, but equivalent form. The higher derivative corrections do keep a knowledge of the Kaluza-Klein modes as they lead to some of the integer sums that occur in the automorphic forms. Indeed, one can wonder if it is possible to reconstruct the higher derivative corrections in $d + 1$ dimensions from those in $d$ dimensions. This is a feature the higher derivative corrections share with the $E_{11} \otimes s l_1$ non-linear realisation where the different theories, that is IIA, M theory, IIB and the theories in $d \leq 10$ dimensions, just correspond to different decompositions of $E_{11} \otimes s l_1$: the count of fields and coordinates being the same in the different theories.

Finally, we close this section by noting the underlying significance of determining the automorphic forms. The type II supergravity theories are the complete low energy effective actions for the type II string. The complete effective action would be known if we knew the automorphic forms that occur as coefficients of the individual terms. However, these automorphic forms are not the ones traditionally studied in that they are non-holomorphic. For the automorphic forms that arise as coefficients of terms with a low numbers of space-time derivatives, their non-holomorphic character is compensated for by the fact that they are eigenvalues of the Laplacian and higher order Casimir operators. However, as we previously mentioned little is known about the automorphic forms that occur in general,
but one should expect that they do have a defining characteristic. Such a knowledge would allow one to completely specify all effects of type II string theory, at least compactified on a torus.
Table 1. The Brane Charge representations of the group, G, derived from the $l_1$ representation of $E_{11}$ [49-51]

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Appendix A: Non-linear Realisations

In this appendix we review the construction of non-linear realisations in a form suitable to that used in this paper. We consider a group $G$ with Lie algebra $Lie(G)$. $Lie(G)$ can be split into the Cartan subalgebra with elements $\vec{H}$, positive root generators $E_{\vec{\alpha}}$ and negative root generators $E_{-\vec{\alpha}}$ with $\vec{\alpha} > 0$. There exists a natural involution, known as the Cartan involution, defined by

$$\tau : (\vec{H}, E_{\vec{\alpha}}) \rightarrow - (\vec{H}, E_{-\vec{\alpha}}).$$

(A.1)

To construct the non-linear realisation we must specify a subgroup $H$ (not to be confused with the generators of the Cartan subgroup which are denoted by $\vec{H}$). For us this is defined...
to be the subgroup left invariant under the Cartan involution, i.e. \( H = \{ g \in G : \tau(g) = g \} \).
In terms of the Lie algebra \( \text{Lie}(H) \) it is all elements \( A \) such that \( A = \tau(A) \).

The non-linear realisation is constructed from group elements \( g(x) \in G \) that depend on spacetime that are subject to the transformations

\[
g(x) \rightarrow g_0 g(x) h^{-1}(x) ,
\]

where \( g_0 \in G \) is constant and \( h(x) \in H \) depends on spacetime. We may write the group element in the form

\[
g(x) = e^{\sum_{\alpha > 0} \chi_\alpha E_\alpha} e^{-\frac{1}{\sqrt{2}} \vec{\phi} \cdot \vec{H}} e^{\sum_{\alpha > 0} u_\alpha E_{-\alpha}} ,
\]

but using the local transformation we can bring it to the form

\[
g(\xi) = e^{\sum_{\alpha > 0} \chi_\alpha E_\alpha} e^{-\frac{1}{\sqrt{2}} \vec{\phi} \cdot \vec{H}} .
\]

Here we use \( \xi = (\vec{\phi}, \chi_\alpha) \) as a generic symbol for all the scalar fields, which are functions of spacetime, that parameterize the coset representative. Under a rigid \( g_0 \in G \) transformation \( g(\xi) \rightarrow g_0 g(\xi) \) this form for the coset representative is not preserved. However one can make a compensating transformation \( h(g_0, \xi) \in H \) that returns \( g_0 g(\xi) \) into the form of equation (A.5);

\[
g_0 g(\xi) h^{-1}(g_0, \xi) = g(g_0 \cdot \xi) .
\]

This induces a non-linear action of the group \( G \) on the scalars; \( \xi \rightarrow g_0 \cdot \xi \).

We will also need a linear representation of \( G \). Let \( \vec{\mu}^i \), \( i = 1, \ldots, N \) be the weights of the representation and \( |\vec{\mu}^i \rangle \) be a corresponding state. We choose \( \vec{\mu}^1 \) to be the highest weight and so the corresponding state satisfies \( E_{\vec{\alpha}} |\vec{\mu}^1 \rangle = 0 \) for all simple roots \( \vec{\alpha} \). The states in the rest of the representation are polynomials of \( F_{\vec{\alpha}} = E_{-\vec{\alpha}} \) acting on the highest weight state.

We consider states of the form \( |\psi \rangle = \sum_i \psi_i |\vec{\mu}^i \rangle \). Under the action \( U(g_0) \) of the group \( G \) we have

\[
|\psi \rangle \rightarrow U(g_0) |\psi \rangle = L(g_0^{-1}) \sum_i \psi_i |\vec{\mu}^i \rangle \equiv (U(g_0) \psi_i) |\vec{\mu}^i \rangle = \sum_{i, j} D_{ij} (g_0^{-1}) \psi_j |\vec{\mu}^i \rangle ,
\]

where \( L(g_0) \) is the expression of the group element \( g_0 \) in terms of the Lie algebra elements which now act on the states of the representation in the usual way. We note that the action of the group on the components \( \psi_i \) is given by \( \psi_i \rightarrow U(g_0) \psi_i = \sum_j D_{ij} (g_0^{-1}) \psi_j \) which is the result expected for a passive action. The advantage of using the states to discuss the representation is that we can use the action of the Lie algebra elements \( L(g_0) \) on the states to compute the matrix \( D_{ij} \) of the representation and deduce properties of the representation in general.

Given any linear realisation, we can construct a non-linear realisation by

\[
|\varphi(\xi) \rangle = \sum_i \varphi_i |\vec{\mu}^i \rangle = L(g^{-1}(\xi)) |\psi \rangle = e^{-\frac{1}{\sqrt{2}} \vec{\phi} \cdot \vec{H}} e^{-\sum_{\alpha > 0} \chi_\alpha E_\alpha} |\psi \rangle ,
\]
where \( g(\xi) \) is the group element of the non-linear realisation in equation (A.5). Under a group transformation \( U(g_0) \) it transforms as

\[
U(g_0)|\varphi(\xi) > = L(g^{-1}(\xi))U(g_0)|\psi >= L(g^{-1}(\xi))L(g_0^{-1})|\psi > = L((g_0g^{-1}(\xi))|\psi > = L(h^{-1})|\varphi(g_0 \cdot \xi) > ,
\]

using equation (A.2). In terms of the component fields we find that \( \varphi_i(\xi) = \sum_j D_{ij}(g^{-1}(\xi))\psi_j \)

\[\text{and } U(g_0)\varphi_i(\xi) = \sum_j D_{ij}((h)^{-1})\varphi_j(g_0 \cdot \xi).\]

\[\text{Appendix B: The decomposition of the simple roots and weights of } E_{n+1}\]

In this paper we carry out the decomposition of certain representations of \( E_{n+1} \) into those of \( SL(n) \otimes GL(1) \otimes GL(1) \). The \( E_{n+1} \) algebra that appears in the dimensional reduction of the IIA theory on an \( n-1 \) torus to \( d \) dimensions appears, from an \( E_{11} \) perspective by deleting node \( d \) in the \( E_{11} \) dynkin diagram.

![Figure 4. The \( E_{11} \) Dynkin diagram appropriate to the \( d \)-dimensional maximal supergravity theory](image)

The Dynkin diagram of the remaining \( E_{n+1} \) internal subalgebra resulting from deleting node \( d \) in figure 4, where nodes \( (d+1),...,11 \), become \( 1,...,(n+1) \) is then given in figure 5.

![Figure 5. The \( E_{n+1} \) Dynkin diagram of the internal subalgebra arising from deleting node \( d \) in the \( E_{11} \) Dynkin diagram](image)

However, we can also delete nodes \( n \) and \( n+1 \) of this Dynkin diagram. The deletion of nodes \( n \) and \( n+1 \) leads to the algebra \( SL(n) \otimes GL(1) \otimes GL(1) \). In this appendix we will find how the roots and weights of \( E_{n+1} \) in terms of those of \( SL(n) \otimes GL(1) \otimes GL(1) \).

Let us carry out the decomposition by first deleting node \( n \) to find the roots and fundamental weights of \( D_n \) and then delete node \( n+1 \) to find the algebra \( SL(n) \). Using the methods given in reference [54], the simple roots of \( E_{n+1} \) can be expressed as

\[
\vec{\alpha}_i = (0, \tilde{\alpha}_i), \quad i = 1,...,n-1, n+1 \quad \vec{\alpha}_n = \left(x, -\tilde{\lambda}_{n-1}\right)
\]

(B.1)
Here $\tilde{\alpha}_i, i = 1, \ldots, n$ are the roots of $D_n$ and $\tilde{\lambda}_i$ its fundamental weights which are given by

$$\tilde{\Lambda}_i = \left( \frac{\tilde{\lambda}_i \cdot \tilde{\lambda}_{n-1}}{x}, \tilde{\lambda}_i \right), \quad i = 1, \ldots, n-1, n+1$$

$$\tilde{\Lambda}_n = \left( \frac{1}{x}, 0 \right) \quad (B.2)$$

The variable $x$ is fixed by demanding that $\alpha^2_n = 2 = x^2 + \tilde{\lambda}^2_{n-1}$.

We now delete node $n$ to find the $A_{n-1}$ algebra. The roots of $E_{n+1}$ are found from the above roots by substituting the corresponding decomposition of the $D_n$ roots and weights into those of $A_{n-1}$. The roots of $D_n$ in terms of those of $A_{n-1}$ are given by $\tilde{\alpha}_i = (0, \alpha_i), \ i = 1, \ldots, n-1$ and $\tilde{\alpha}_n = (y, -\lambda_{n-2})$ while the fundamental weights are given by $\tilde{\lambda}_i = \left( \frac{\lambda_n - \lambda_i}{y}, \lambda_i \right)$ $i = 1, \ldots, n-1$ and $\tilde{\lambda}_{n+1} = \left( \frac{1}{y}, 0 \right)$. Requiring $\tilde{\alpha}^2_{n+1} = 2$ gives $y^2 = \frac{4}{n}$.

We then find that the roots of $E_{n+1}$ are given by

$$\tilde{\alpha}_i = (0, 0, \alpha_i), \quad i = 1, \ldots, n-1,$$

$$\tilde{\alpha}_n = \left( x, -\frac{\lambda_{n-2} \cdot \lambda_{n-1}}{y}, -\lambda_{n-1} \right),$$

$$\tilde{\alpha}_{n+1} = (0, y, -\lambda_{n-2}) \quad (B.3)$$

The fundamental weights of $E_{n+1}$ are found in the same way to be

$$\tilde{\Lambda}_i = \left( \frac{c_i}{x}, \frac{\lambda_{n-2} \cdot \lambda_i}{y}, \lambda_i \right), \quad i = 1, \ldots, n-1,$$

$$\tilde{\Lambda}_n = \left( \frac{1}{x}, 0, 0 \right),$$

$$\tilde{\Lambda}_{n+1} = \left( \frac{n-2}{4x}, \frac{1}{y}, 0 \right). \quad (B.4)$$

where $c_i = \frac{i}{2}, \ i = 1, \ldots, n-2$ and $c_{n-1} = \frac{n}{4}$. As $\tilde{\lambda}^2_{n-1} = \frac{n}{4}$ we find that $x^2 = \frac{8-n}{4}$.

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