VALUES OF ZETA FUNCTIONS AT $s = \frac{1}{2}$

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Abstract. We study the behaviour near $s = \frac{1}{2}$ of zeta functions of varieties over finite fields $\mathbb{F}_q$ with $q$ a square. The main result is an Euler-characteristic formula for the square of the special value at $s = \frac{1}{2}$. The Euler-characteristic is constructed from the Weil-étale cohomology of a certain supersingular elliptic curve.

Introduction

Let $V$ be a integral scheme of finite type over Spec $\mathbb{Z}$. The special values $a_{V}(n)$ at integers $s = n$ of the zeta function $\zeta(V, s)$ of $V$ are conjecturally related to deep arithmetical invariants of $V$. One may ask if the special values of $\zeta(V, s)$ at non-integral values of $s$, e.g. $s = \frac{1}{2}$, also admit an arithmetical interpretation. The simplest example – and perhaps the most interesting – arises from the central value of the Riemann zeta function $\zeta(s) = \zeta(\text{Spec } \mathbb{Z}, s)$:

Is there a motivic interpretation of $\zeta(\frac{1}{2})$?

For instance, it is unknown if $\zeta(\frac{1}{2})$ is a period in the sense of [9]; however, for certain triple product $L$-functions, the work of M. Harris and S. Kudla [7] relates the central critical value (at $s = \frac{1}{2}$) to periods, cf. [22, Eq. (48), p.459].

The motivic philosophy indicates that $a_{V}(n)$ depends on the interaction of the motive $h(V)$ of $V$ with $\mathbb{Z}(-n)$ (power of the Tate motive). This leads us to suspect that the special value at $s = \frac{1}{2}$ is governed by an exotic object (unknown to exist, as yet): a square root $\mathbb{Z}(\frac{1}{2})$ of the Tate motive $\mathbb{Z}(1)$ over Spec $\mathbb{Z}$.

It is clear that the investigation of special values at $s = \frac{1}{2}$ should begin with the important case of varieties over finite fields. Namely, consider the zeta function $\zeta(X, s)$ of a smooth projective variety $X$ over a finite field $k = \mathbb{F}_q$ of characteristic $p > 0$. The function $Z(X, t)$, defined by $Z(X, q^{-s}) := \zeta(X, s)$, is a rational function of $t$ with integer coefficients. For any integer $n \geq 0$, the order of the pole $\rho_n := -\text{ord}_{s=n}\zeta(X, s)$ at $s = n$ and the special value $a_X(n)$ of $Z(X, t)$ at $t = q^{-n}$ conjecturally admit a motivic interpretation.

The Tate conjecture (Conjecture 2) predicts that

$$\rho_n = \text{rank } \text{Hom}(\mathbb{Z}(-n), h^{2n}(X)),$$

in the category of motives and $h^{2n}(X)$ is part of the motive of $X$. A related variant is that $\rho_n$ is the rank of the Chow group $CH^n(X)$ of algebraic cycles of codimension $n$ on $X$.

The Lichtenbaum-Milne conjecture (Conjecture 3) expresses $a_X(n)$ as an Euler-characteristic of étale motivic cohomology $H^*_e(X, \mathbb{Z}_X(n))$; i.e., the cohomology of the (étale) motivic complexes $\mathbb{Z}_X(n)$ of S. Lichtenbaum; $\mathbb{Z}_X(0)$ is the constant sheaf $\mathbb{Z}$ and $\mathbb{Z}_X(1) = \mathbb{G}_m[-1]$ is the sheaf $\mathbb{G}_m$ in degree one. Their conjecture is known for $n = 0$ (unconditionally) and for $n = 1$ (modulo the Tate conjecture for...
divisors on $X$); cf. [11, 19]. For $n = 0, 1$, it takes the form

$$a_X(0) = \pm \chi(X, \mathbb{Z}), \quad a_X(1) = \frac{\chi(X, \mathbb{G}_m)}{\chi(X, \mathbb{G}_m)} = \pm \frac{q^{\chi(X, \mathbb{O}_X)}}{\chi(X, \mathbb{G}_m)}.$$ 

Lichtenbaum [12] has provided another elegant interpretation of $a_X(0)$ using his Weil-étale topology (cf. Theorem 5).

Let us now turn to the special value at $s = \frac{1}{2}$ or $t = 1/\sqrt{q}$. One can ask for a motivic description of

- the order of vanishing \( \rho_X := \operatorname{ord}_{s=\frac{1}{2}} \zeta(X, s) \) and
- the corresponding special value \( c_X \) at \( s = \frac{1}{2} \), viz., \( c_X := \lim_{t \to 1/\sqrt{q}} (1 - \sqrt{q}t)^{-\rho_X} Z(X, t) \).

The main result of this paper provides such a description, under the condition that \( q = p^{2f} \), i.e., that \( \mathbb{F}_{p^2} \subset \mathbb{F}_q \). We note that \( c_X \) may not be rational, if the condition on \( q \) is dropped.

Our paper is an exploration, using the methods of [11, 19, 12, 21], of a beautiful suggestion of Yuri Manin that “a certain supersingular elliptic curve $E$ might be useful in finding the expression for $Z(\frac{1}{2})$’, because its one-motive $h_1(E)$ is the square root of Tate’s motive in the same sense as the Dirac operator

The cohomology groups $H^i(X, E)$ are finitely generated. The special value at $s = \frac{1}{2}$, viz., \( c_X := \lim_{t \to 1/\sqrt{q}} (1 - \sqrt{q}t)^{-\rho_X} Z(X, t) \) is given by

$$\frac{q^{\chi(X, \mathbb{O}_X)}}{\chi(X, E)}$$

are finite; the special value at $s = \frac{1}{2}$, viz., \( c_X := \lim_{t \to 1/\sqrt{q}} (1 - \sqrt{q}t)^{-\rho_X} Z(X, t) \) is given by

$$c_X^2 = \frac{q^{\chi(X, \mathbb{O}_X)}}{\chi(X, E)}$$

where

$$\chi(X, E) = \prod [h^i]^{-1/2}.$$

Tate’s theorem [24] on abelian varieties is crucial here; since this result of Tate is a special case (of divisors on abelian varieties) of his conjecture, the case of $s = \frac{1}{2}$ is intermediate between the case of $s = 0$ and $s = 1$. As $h_1(E)$ has rank two, it is $2\rho_X$ and $c_X^2$ that arise rather than $\rho_X$ and $c_X$. Note that the Weil-étale motivic cohomology groups $H^i_W(X, \mathbb{Z}_X(n))$ are known to be finitely generated only for $n = 0$ [12, §8]; the case $n \neq 0$ requires the Tate conjecture [5].
The proof of the main theorem in §5 depends on the results in §4. As usual, the main difficulties lie in the $p$-part. This involves the nontrivial computation of the cohomology of the flat group scheme $p^sE$ on $X$; we do this by using the de Rham-Witt complex of $X$; our approach was suggested by [17, §13]. The special values at $s = \frac{1}{2}$ for the L-functions of curves and motives are treated first in §2, 3, using the work of Milne [14, 15, 21], after some preliminaries in §1. The reader may be amused by the results in §2.2: the square root of the order of a Tate-Shafarevich group turns up in the description of the special value at $s = \frac{1}{2}$.

Even though $\zeta(X, 1/3)$ will be a nonzero rational number when $q = p^{3f}$, one does not have $\mathbb{Z}(\frac{1}{2})$ over such a field; similar comments apply to $1/4, 1/5, \cdots$. The case of $\mathbb{Z}(\frac{1}{2})$ is special and its existence ultimately has its origins in the structure of Weil $q$-numbers [20]. We refer to C. Deninger [3, §7] and Manin [13] for ideas on the exotic Tate motives $C(s)$ for $s \in \mathbb{C} - \mathbb{Z}$.

Supersingular elliptic curves are important examples of motives [20, Theorem 2.41]. For instance, as indicated by J.-P. Serre [6], their existence implies the non-neutrality of the Tannakian category $\mathcal{M}$ of motives over a finite field (i.e. that $\mathcal{M}$ does not have a fibre functor to $\mathbb{Q}$-vector spaces).

Finally, elliptic curves or abelian varieties cannot provide $\mathbb{Z}(\frac{1}{2})$ over Spec $\mathbb{Z}$: the Hodge numbers are incompatible. Since elliptic curves with noncommutative endomorphism rings provide $\mathbb{Z}(\frac{1}{2})$ over a finite field, it may be that the arithmetic theory of non-commutative tori (initiated by Manin) holds the key to the definition of $\mathbb{Z}(\frac{1}{2})$ over Spec $\mathbb{Z}$.

**Notations.** Let $\mathbb{F}_q$ be a finite field of characteristic $p > 0$ and let $\mathbb{F}$ be an algebraic closure of $\mathbb{F}_q$. For any scheme $V$ over $\mathbb{F}_q$, we write $\overline{V}$ for its base change to $\overline{\mathbb{F}}$. Write $\Gamma$ for the Galois group $\text{Gal}(\overline{\mathbb{F}}/\mathbb{F}_q)$ and $\Gamma_0 \cong \mathbb{Z}$ for the subgroup (the Weil group) generated by the Frobenius automorphism $\gamma$: $x \mapsto x^q$. The additive and multiplicative valuations of $\mathbb{Q}$ are normalized so that $|p|_p = 1/p$ and $\text{ord}_p(p) = 1$. We write $H$ (resp. $H_{fl}$, $H_W$) for the étale (resp. flat, Weil-étale [12]) cohomology groups. Finally, $[S]$ denotes the order of a finite set $S$.

From §2 onwards, we assume that $q$ is a square, i.e., that $\mathbb{F}_{p^2} \subset \mathbb{F}_q$.

1. Preliminaries

Here we recall some relevant facts and conjectures. We begin with an example.

1.1. Example. Let us consider the scheme $X = \text{Spec } \mathbb{F}_q$. The value $(1 - q^n)^{-1}$ of $\zeta(X, s) = (1 - q^{-s})^{-1}$ at any negative integer $-n$ has many interpretations:

$$-\frac{1}{\zeta(X, -n)} = (q^n - 1) = [K_{2n-1}(\mathbb{F}_q)] = [\text{Ext}^0_n(\mathbb{Z}, \mathbb{Z}(n))] = [\mathbb{G}_m(\mathbb{F}_{q^n})] = [T_n(\mathbb{F}_q)]$$

where $K_{2n-1}(\mathbb{F}_q)$ is the Quillen $K$-group of $\mathbb{F}_q$. $\mathcal{A}$ is the category of effective integral motives over $\mathbb{F}_q$ of [21], $T_n$ is the torus over $\mathbb{F}_q$ obtained from $\mathbb{G}_m$ over $\mathbb{F}_{q^n}$ via Weil restriction of scalars.

The value of $\zeta(X, s)$ at $s = -\frac{1}{2}$ is $(1 - q^{2s})^{-1}$ which is not rational unless $q$ is an even power of $p$. So assume that $q = p^{2f}$. Even then, one cannot interpret $p^{2f} - 1$ as the order of $\mathbb{F}_q$-points of any group (for varying $f$). But all is not lost. Consider $\zeta(X, -\frac{1}{2}) = (p^f - 1)^2$ which could be the order of $\mathbb{F}_q$-rational points of an elliptic curve $E$. In fact, one has

$$\frac{1}{\zeta(X, -\frac{1}{2})^2} = (p^f - 1)^2 = [E(\mathbb{F}_q)] = [\text{Ext}^1_{\mathcal{A}}(\mathbb{Z}, h_1(E))]$$

for an elliptic curve $E$ with Frobenius eigenvalues $(p^f, p^f)$. Such an $E$ is supersingular.

As for $s = \frac{1}{2}$, we find that

$$\zeta(X, \frac{1}{2}) = \frac{q^{\chi(\mathcal{O}_X)}}{(1 - p)^2} = \frac{q^{\chi(\mathcal{O}_X)}}{[E(\mathbb{F}_q)]}$$

where $\chi(\mathcal{O}_X)$ is the Euler-characteristic of $\mathcal{O}_X$. Note that the numerator $q^m$ of $\zeta(X, m)$ is $q^m - 1$ for any positive integer $m$ is an Euler-characteristic by Conjecture 3; cf. [19, Thm. 7.2].
1.2. Supersingular elliptic curves and $\mathbb{Z}(\frac{1}{2})$. We indicate the relations of supersingular elliptic curves to the motive $\mathbb{Z}(\frac{1}{2})$ by quoting the email message (dated 10 August 2004) of Milne:

"A. Grothendieck (and P. Deligne) knew already in the 1960’s that a supersingular elliptic curve $E$ over $\mathbb{F}$ provides a square root of the Tate motive in the following precise sense: after extending the coefficient field to a field $k \supset \mathbb{Q}$ splitting $\text{End}(E)$, $h_1(E)$ decomposes into $\mathbb{Q}(\frac{1}{2}) \oplus \mathbb{Q}(\frac{1}{2})$ where $\mathbb{Q}(\frac{1}{2})^{\otimes 2} = \mathbb{Q}(1)$. This is exploited throughout [20]. For example, as for any category of motives, there is an exact sequence

$$0 \to G_m \xrightarrow{\omega} P \xrightarrow{\rho} G_m \to 0$$

such that $t \circ \omega = 2$ where $P$ is the kernel of the Tannakian groupoid. The map $\omega$ is given by the weight gradation and $t$ by the Tate motive. From $E$ one gets a homomorphism $P \xrightarrow{\rho} G_m$ such that $e \circ \omega = 1$, and so $P$ decomposes into $P = P^0 \times w(G_m)$; cf. [20, 2.41].

Over a finite field $\mathbb{F}_q$, supersingular elliptic curves come in three types:

(a) eigenvalues $\pm \sqrt{-q}$;
(b) $q = p^{2f}$; eigenvalues $-p^f, -p^f$;
(c) $q = p^{2f}$; eigenvalues $p^f, p^f$.

This follows from Tate’s theorem [24, Thm. 1(a)] but was probably known to Deuring. The ones in (c) play the same role over $\mathbb{F}_{p^2}$ as $E$ over $\mathbb{F}$.

Remark. The elliptic curves in (c) are all $\mathbb{F}_q$-isogenous. Let $E$ be an elliptic curve over $\mathbb{F}_{p^2}$ of type (c). It is actually defined over $\mathbb{F}_p$ and its endomorphism ring $\text{End}_{\mathbb{F}_p}(E) = \text{End}_{\mathbb{F}_p}(E)$ is a maximal order in a quaternion algebra over $\mathbb{Q}$, ramified only at $p$ and $\infty$; cf. p. 528 and Theorems 4.1, 4.2 of [25].

We fix an elliptic curve of type (c) over $\mathbb{F}_{p^2}$ and we refer to it throughout as $E$.

Remark. An abelian variety $A$ over $\mathbb{F}_{p^2}$ whose Frobenius spectrum is pure with support $p^f$ is, by Tate’s theorem [24, Thm. 1(a)] (cf. [25, pp.526-27]), isogenous to a power of $E$.

1.3. Conjectures of Tate and Lichtenbaum-Milne. Let $X$ be a smooth projective (geometrically connected) variety over $\mathbb{F}_q$. Let $\zeta(X, s)$ be its zeta function. The associated function $Z(X, t)$, defined by $\zeta(X, s) = Z(X, q^{-s})$, is known to be a rational function of $t$ with integer coefficients. The special value of $\zeta(X, s)$ at $s = n$ is the special value $a_X(n)$ of $Z(X, t)$ at $t = q^{-n}$, up to powers of $\log(q)$.

A motivic description of $a_X(n)$ requires a knowledge of the poles of $\zeta(X, s)$ given by

Conjecture 2. (Tate) $T(X, r)$ : For any integer $r \geq 0$, the dimension of the subspace of $H^{2r}(\overline{X}, \mathbb{Q}_l(r))$ generated by algebraic cycles is equal to the order $\rho_r$ of the pole of $\zeta(X, s)$ at $s = r$.

This conjecture provides a motivic description of $\rho_r$, i.e., $\rho_r = \text{rank} \text{Hom}(\mathbb{Z}(-r), h^{2r}(X))$ in the category of motives over $\mathbb{F}_q$. While Conjecture 2 is unknown in general, even for divisors $r = 1$, it does hold for divisors on abelian varieties [24, Thm. 1(a)].

In the case of a curve $X$, it is well known that $a_X(0), a_X(1)$ are related to the class number $h$ of $X$ [11, p. 191].

The description of $a_X(1)$ for a surface $X$ can be considered, following M. Artin and J. Tate [23], as a geometric analogue of the Birch-Swinnerton-Dyer conjecture. Inspired by [23], Lichtenbaum [11] and Milne [19] have conjectured (Conjecture 3) a complete description (including the $p$-part) of $a_X(n)$ for any $X$ as an Euler-characteristic in étale motivic cohomology $H^*(X, \mathbb{Z}_X(n))$. Namely, Lichtenbaum (resp. Milne) has defined a non-zero rational number $\chi(X, \mathbb{Z}(r))$ (resp. an integer $\chi(X, \mathbb{O}(r))$) via motivic cohomology $H^*(X, \mathbb{Z}_X(r))$ (resp. $\sum_{i \geq 0} (-1)^i \chi(X, \mathbb{O}^i))$ of $X$; we refer to [19, §0] for the precise definitions. These are related to $a_X(n)$ via the

Conjecture 3. (Lichtenbaum-Milne) [19, 0.1] The special value $a_X(r)$ of $Z(X, t)$ at $t = q^{-r}$ can be given as an Euler-characteristic of motivic cohomology. Namely, one has

\[(\text{LM}(X, r)) \quad Z(X, t) \sim \pm \chi(X, Z(r)), q^{\chi(X, \mathbb{O}(r))}(1 - q^t)^{-\rho_r}, \quad \text{as} t \to q^{-r}\]
where the terms on the right are defined in [19, Conj. 0.1]

This conjecture generalizes [11] that of Artin-Tate [23] for the case of $r = 1$ and $X$ a surface. We recall one result about Conjecture 3 and refer to [11, 18, 19, 21] for other results.

**Theorem 4.** [11, 18] (a) Conjecture 3 is true for $r = 0$.
(b) If $T(X, 1)$ holds, then the terms in $(\text{LM}(X, 1))$ are finite and $(\text{LM}(X, 1))$ is true.

Part (b) generalises the result of Artin-Tate [23] for surfaces, and includes the $p$-part [16].

1.4. **Weil-étale cohomology.** As before, $X$ is a smooth projective geometrically connected variety. Lichtenbaum [12] has given a beautiful interpretation, via his Weil-étale cohomology groups $H^n_W(X, \mathbb{Z})$, of the behaviour of $Z(X, t)$ at $t = 1$. Namely, cup-product with the generator $\theta$ of $H^1_W(\text{Spec} \mathbb{F}_q, \mathbb{Z}) = \text{Hom}(\mathbb{Z}, \mathbb{Z}) = \mathbb{Z}$ gives a complex $(H^*_W(X, \mathbb{Z}), \theta)$ of finitely generated abelian groups whose cohomology groups $H^*_W(X)$ are finite; write $\chi_W(X) = \sum [h^0_W(X)]$ for the alternating product of the orders $[h^0_W(X)]$ of $h^*_W(X)$. Then, Lichtenbaum’s result on the behaviour of $Z(X, t)$ at $t = 1$ is as follows:

**Theorem 5.** (Lichtenbaum)
(i) the usual Euler characteristic $\Sigma(-1)^ir_i$ of $X$ is zero; here $r_i = \text{rank } H^i_W(X, \mathbb{Z})$.
(ii) the order of the zero of $Z(X, t)$ at $t = 1$ is given by the secondary Euler characteristic $\Sigma(-1)^ir_i$.
(In our case, this order is minus one.)
(iii) the special value $a_X(0) := \lim_{t \to 1} Z(X, t)(1 - t)$ is the multiplicative Euler characteristic:
$$a_X(0) = \pm \chi_W(X).$$

1.5. **Behaviour at $s = \frac{1}{2}$: basic invariants.** Our aim in this paper is a motivic description, when $q = p^{2f}$, of the following arithmetic invariants of $X$:

- the integer $\rho_X := \text{the order of the zero of } \zeta(X, s) \text{ at } s = \frac{1}{2}$.
- the special value at $t = q^{-\frac{1}{2}}$ of $Z(X, t)$, viz., $c_X := \lim_{t \to 1/\sqrt{q}}(1 - \sqrt{q})^{-\rho_X} Z(X, t)$.

As before, $c_X$ differs from the special value at $s = \frac{1}{2}$ of $\zeta(X, s)$ by factors of $\log p$. As $q = p^{2f}$, one has $c_X \in \mathbb{Q}^*$. Unlike $\rho$, the integer $\rho_X$ has an unconditional motivic description, viz.,

**Lemma 6.** Let $q = p^{2f}$ and let $\text{Pic}(X)$ be the Picard variety of $X$. One has
$$2\rho_X = \text{rank } \text{Hom}_{\mathbb{F}_q}(E, \text{Pic}(X)).$$

For instance, $\rho_E = 2$ and rank $\text{End}(E) = 4$. The RHS is always divisible by 4 because $\text{Hom}_{\mathbb{F}_q}(E, \text{Pic}(X))$ is a module over $\text{End}(E)$. Similarly, the LHS is divisible by 4: the integer $\rho_X$ is even because the roots $\alpha \neq \sqrt{q}$ of the factor $P_1(t)$ of $Z(X, t)$ come in pairs ($\alpha, q/\alpha$) and the degree of $P_1(t)$ is even.

**Proof.** This follows from Tate’s theorem [24, Thm. 1(a)]: by the Weil conjectures, $\rho_X$ is the multiplicity of the root $\sqrt{q}$ in the factor $P_1(t)$ of $Z(X, t)$ which is the characteristic polynomial of Frobenius on $\text{Pic}(X)$. For $A = E$ and $B = \text{Pic}(X)$, the integer $r(f_A, f_B)$ defined by Tate [24, (4), p. 138] is $2\rho_X$. 

Thus, the vanishing of $\zeta(X, s)$ at $s = \frac{1}{2}$ is controlled by the (non)-ordinarity of $\text{Pic}(X)$. The study of $c_X$ occupies the rest of the article.

2. **Curves**

In this section, the work of Milne [14, 15] is used to provide motivic interpretations of $c_X$ in the case of curves. Let $X$ be a curve of genus $g$ over $\mathbb{F}_q$ with $q = p^{2f}$, i.e., $\mathbb{F}_{p^2} \subset \mathbb{F}_q$. Write $J$ for its Jacobian.

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1His results include the case of arbitrary curves and smooth surfaces.
2.1. Extensions and \(c_X\).

**Theorem 7.** (i) \(\rho_X = \frac{1}{2} \text{rank } \text{Hom}_{\mathbb{F}_q}(E, J)\).

(ii) the special value at \(s = \frac{1}{2}\) of \(\zeta(X, s)\) is given by

\[
c_X^2 = q^{\chi(\mathcal{O}_X)} \frac{[\text{Ext}^1_{\mathbb{F}_q}(J, E)]}{[E(\mathbb{F}_q)]^2} D
\]

where \(\chi(\mathcal{O}_X) = 1 - g\) is the Euler-characteristic of \(\mathcal{O}_X\), \(\text{Ext}^1\) is computed in the category of group schemes over \(\text{Spec } \mathbb{F}_q\), and \(D\) is the discriminant of the pairing

\[
\text{Hom}_{\mathbb{F}_q}(E, J) \times \text{Hom}_{\mathbb{F}_q}(J, E) \to \text{End}_{\mathbb{F}_q}(E) \to \text{trace}_\mathbb{F}_q \mathbb{Z}.
\]

(iii) one has

\[
c_X^2 = q^{\chi(\mathcal{O}_X)} \frac{[H^1(X, E)]}{[H^0(X, E)_{\text{tors}}]^2} D
\]

where \(H^*(X, E)\) is the étale cohomology of the sheaf defined by \(E\) and \(D\) is as in (ii).

**Proof.** (i) was proved earlier.

(ii) When \(g = 0\), one easily verifies that

\[
c_{p_1}^2 = \frac{q}{[E(\mathbb{F}_q)]^2}.
\]

When \(g \neq 0\), \(Z(X, t) = P(t).Z(\mathbb{P}^1, t)\) where \(P(t) = \prod(1 - \alpha_i t)\) is the characteristic polynomial of Frobenius on \(J\). The result now follows from Lemma (6) and [14, Thm. 3].

(iii) As \(\text{Ext}^1_{\mathbb{F}_q}(J, E) \cong H^1(X, E)\) [15, Cor. 3] and \(E(\mathbb{F}_q) = H^0(X, E)_{\text{tors}}\) [15, p. 120], this is clear. □

**Remark.** If \(X\) is ordinary, i.e., its Hasse-Witt matrix is invertible or the abelian variety \(J\) is ordinary, then the formula simplifies to

\[
\zeta(X, \frac{1}{2})^2 = c_X^2 = q^{1 - g} \frac{[\text{Ext}^1_{\mathbb{F}_q}(J, E)]}{[E(\mathbb{F}_q)]^2}.
\]

Since \(J\) is ordinary, the roots of \(P\) satisfy: \(\alpha_i \neq \sqrt[q]{q}\); thus \(\zeta(X, \frac{1}{2}) \neq 0\). For an ordinary elliptic curve \(X\), a simple computation yields

\[
\pm c_X = 1 - \frac{[X(\mathbb{F}_q)]}{[E(\mathbb{F}_q)]}.
\]

2.2. BSD and \(c_X\). These formulas for \(c_X^2\) are very reminiscent of the Birch-Swinnerton-Dyer (BSD) conjecture. In fact, Milne [15] has shown \(\text{Ext}^1_{\mathbb{F}_q}(J, E) \cong \text{III}(E/K)\) of the constant elliptic curve \(E\) over the function field \(K = \mathbb{F}_q(X)\). Since the order of \(\text{III}(E/K)\) is a square, \([\text{Ext}^1_{\mathbb{F}_q}(J, E)] = m^2\) for a positive integer \(m\). This gives a motivic interpretation of \(c_X\) for \(X\) ordinary:

\[
c_X = \pm p^{1 - g} \frac{m}{[E(\mathbb{F}_q)]}.
\]

Thus, the square root of the order of \(\text{III}\) is related to special values at \(s = \frac{1}{2}\).

**Theorem 8.** The special value at \(s = 1\) of the L-function \(L(E/K, s)\) of \(E\) over \(K\) is:

\[
\lim_{s \to 1} L(E/K, s). (s - 1)^{-2 \rho_X} = c_X^2. (\log q)^\rho.
\]

where \(\rho = \text{rank } E(K) = 2 \rho_X = \text{rank } \text{Hom}_{\mathbb{F}_q}(J, E)\).

**Proof.** This follows from [15, Thm. 3] and the first equation on page 102 of [15]. □
3. Integral motives

Some of the results in the previous section generalize to effective integral motives [21].

Recall the category $\mathcal{M}^+(\mathbb{F}_q; \mathbb{Z})$ of effective integral motives over $\mathbb{F}_q$ and the category $\mathcal{M}(\mathbb{F}_q; \mathbb{Z})$ of integral motives (see [21]). Assume that $q = p^{2f}$. The elliptic curve $E$ defines an effective integral motive [21, 5.16] via its $h^1$ which is still denoted $E$. Its dual $E' = h_1(E)$ is just $E \otimes \mathbb{Z}(1)$. The special value at $s = \frac{1}{2}$ of the L-function $L(M, s)$ of any effective integral motive $M$ is related to the order of Ext-groups in $\mathcal{M}^+(\mathbb{F}_q; \mathbb{Z})$:

**Proposition 9.** Let $q = p^{2f}$ and let $r$ be the rank of $M$ and $\rho_M := \text{ord}_{s=\frac{1}{2}} L(M, s)$. One has

(i) $2\rho_M = \text{rank } \text{Hom}(E, M)$.

(ii) $q^r L(M, s + \frac{1}{2})^2 \sim \frac{[\text{Ext}^1(E, M)] \cdot D(M)}{[\text{Hom}(E, M)]_{\text{tors}} \cdot [\text{Hom}(M, E)]_{\text{tors}}} (1 - q^{-s})^{\rho_M}$ as $s \to 0$,

where $D(M)$ is the discriminant of the pairing

$$\text{Hom}(E, M) \times \text{Hom}(M, E) \xrightarrow{\circ} \text{End}(E) \xrightarrow{\text{trace}} \mathbb{Z}.$$

(iii) One has $L(M, s + \frac{1}{2})^2 = L(M \otimes E', s)$.

**Proof.** If $L(M, s) = \prod (1 - a_i q^{-s})$, then by [21, 9.2(d)],

$$L(M \otimes E', s) = \prod_{i,j} (1 - \frac{a_i}{b_j} q^{-s}) = \prod_i (1 - a_i q^{-s} - \frac{1}{2})^2$$

because $L(E, s) = \prod (1 - b_j q^{-s}) = (1 - q^{\frac{1}{2}} q^{-s})^2$. This proves (iii).

Now it is clear that (i) and (ii) follow from [21, Thm. 10.1].

It is possible to prove a Weil-étale variant [12] of the above result using [21, Thm. 10.5].

**Remark.** (i) One has

$$L(\mathbb{Z}, s + \frac{1}{2})^2 = (1 - q^{-s})^2 = \frac{[E(\mathbb{F}_q)]}{q}$$

By the proposition, $qL(\mathbb{Z}, \frac{1}{2})^2$ is $[\text{Ext}^1(E, \mathbb{Z})]$ which, by [21, 8.7], is $[E(\mathbb{F}_q)]$.

(ii) Let $A$ be an ordinary abelian variety of dimension $d$. One computes $L(h_1(A), \frac{1}{2}) \neq 0$ to be [14]

$$L(h_1(A), \frac{1}{2})^2 = q^{-d}[\text{Ext}^1(A, E)]$$

where the Ext-group is computed in the category of group schemes over $\mathbb{F}_q$.

(iii) For any motive $M$ over a global (or finite) field and for any $n \in \mathbb{Z}$, we write $M(n)$ for its (Tate) twist by $\mathbb{Z}(n)$. On the level of L-functions, twisting corresponds to translations on the s-axis: $L(M, s + n) = L(M(n), s)$. The above proposition computes special values of $L(M, s)$ at other half-integers as well.

4. The $p$-adic Tate module of $E$

As before, $X$ is a smooth projective geometrically connected variety over $\text{Spec } \mathbb{F}_q$ with $q = p^{2f}$, and $E$ is our fixed elliptic curve.

The main result (Theorem 13) of this section relates $c_X$ to the cohomology groups $H^*_f(X, \rho^n E)$ of the finite flat group scheme $\rho^n E$ on $X$. This result is the technical core of the paper.

We shall freely use the results and methods of [18]. We recall from [18, §1] the category $Pf$ of perfect affine schemes over $\mathbb{F}_q$, endowed with the étale topology. The computation of $H^*_f(X, \rho^n E)$ here was inspired by [17, §13 (a)].
For any perfect field $k$ of positive characteristic, we write $W(k)$ for the Witt ring of $k$. Recall the Dieudonné ring $A = W(\mathbb{F}_q)[F,V]$; put $\tilde{A} = A \otimes \mathbb{W}(\mathbb{F}_q) W(\mathbb{F})$. Note that the relation $FV = p = VF$ holds in $A$ and $\tilde{A}$.

**Lemma 10.** (i) For $n \geq 1$ and $i \geq 0$, one has an exact sequence

$$\cdots \to H^i_{\text{f}}(\tilde{X}, \pi^n E) \to H^i(\tilde{X}, W_{2n}\mathcal{O}) \xrightarrow{F-V} H^i(\tilde{X}, W_{2n}\mathcal{O}) \to \cdots,$$

where $W_{2n}\mathcal{O}$ is the Witt vector sheaf of order $2n$ on $\tilde{X}$.

(ii) The presheaf $T \mapsto H^i(X_T, \pi^n E)$ on $\mathcal{P}f$ is represented by an affine perfect group scheme $[18, \text{pp.303-4}]$.

**Proof.** (i) This follows from the exact sequence of flat sheaves

$$0 \to \pi^n E \to W_{2n}\mathcal{O} \xrightarrow{F-V} W_{2n}\mathcal{O} \to 0$$

on $\tilde{X}$, a consequence of the fact that the Dieudonné module of the $p$-divisible group $E(p^n)$ (resp. $W_{2n}$) is $E' := \tilde{A}/(F-V)$ (resp. $\tilde{A}/(V^{2n})$). The cohomology of $W_{2n}\mathcal{O}$ is the same in the étale and flat topologies.

(ii) Follows from (i) in a standard manner, cf. [18, Lemma 1.8], because the presheaf $T \mapsto H^i(X_T, W_{2n}\mathcal{O})$ is representable. $\square$

As in [18, p. 322], the long exact sequence of Lemma 10 (i) can be regarded as one of affine perfect group schemes. Write $\mathcal{H}(X, W\mathcal{O}) := \varprojlim \mathcal{H}(X, W_{2n}\mathcal{O})$ and $\mathcal{H}(X, \pi^n E)$ for the perfect group scheme in (ii). Write $b_i$ for the dimension of the perfect pro-algebraic group scheme $\mathcal{H}(X, T_p E) := \varprojlim \mathcal{H}(X, \pi^n E)$. Thus, $b_i = \infty$ unless the neutral component $\mathcal{H}(X, T_p E)^0$ of $\mathcal{H}(X, T_p E)$ is algebraic, in which case $b_i$ is the number of copies of $\mathbb{G}_a^f$ occurring as quotients in a composition series for $\mathcal{H}(X, T_p E)^0$.

Write $W$ for $W(\mathbb{F})$. There is an exact sequence [18, p. 321]

$$(1) \quad 0 \to H^i(\tilde{X}, W\mathcal{O})_t \to H^i(\tilde{X}, W\mathcal{O}) \to B_i \to 0,$$

of $W[[V]]$-modules in which $B_i$ is a free $W$-module of finite rank and $H^i(\tilde{X}, W\mathcal{O})_t$ is a finitely generated $(W/p^n W)[[V]]$-module for some $n$. Write $d_i$ for the length of $H^i(\tilde{X}, W\mathcal{O})_t \otimes W[[V]] W((V))$ as a $W((V))$-module. The integer $d_i$ is equal to the number of copies of $\mathbb{F}[[V]]$ occuring in a composition series for $H^i(\tilde{X}, W\mathcal{O})_t$.

**Proposition 11.** With notations as above, $b_i$ is finite for all $i$ and $\Sigma(-1)^i b_i = -\Sigma(-1)^i d_i$.

**Proof.** We follow the proof of [18, Prop. 3.1] to which we refer for the properties of the integer $\chi(\alpha) := \dim \text{Ker}(\alpha) - \dim \text{Coker}(\alpha)$ attached to a morphism $\alpha$ of group schemes.

We first compute $\chi(F-V)$ on $H^i(\tilde{X}, W\mathcal{O})$, in the terminology of [18, Prop. 3.1]. By [18, 3.2 (b)], it is possible to neglect finitely generated torsion $W$-modules which arise as subquotients of $H^i(\tilde{X}, W\mathcal{O})$. As $F = 0$ on $W[[V]]$, one has $\chi(F-V|W[[V]]) = \chi(-V|W[[V]]) = -1$. So $\chi(F-V|H^i(\tilde{X}, W\mathcal{O})_t) = -d_i$.

We claim that $\chi(F-V|H^i(\tilde{X}, W\mathcal{O})) = \chi(F-V|H^i(\tilde{X}, W\mathcal{O})_t)$. Namely, we claim that $\chi(F-V|B_i) = 0$. This is a consequence of the semisimplicity of the category of isocrystals over $\mathbb{F}$: namely, the kernel (resp. cokernel) of $F-V$ on $B_i$ is $\text{Hom}_{\mathbb{F}}(E', B_i)$ (resp. $\text{Ext}^1_{\mathbb{F}}(E', B_i)$) in the category of Dieudonné modules over $\mathbb{F}$. Here $E' := \tilde{A}/(F-V)$ denotes the Dieudonné module of the $p$-divisible group $E(p)$. As $\text{Hom}_{\mathbb{F}}(E', B_i)$ is a finitely generated free $\mathbb{Z}_p$-module and $\text{Ext}^1_{\mathbb{F}}(E', B_i)$ a $p$-primary torsion group whose $p$-torsion is finite, one has $\chi(F-V|B_i) = 0$.

Writing $K_i$ and $C_i$ for the kernel and cokernel of $F-V$ on $\mathcal{H}(\tilde{X}, W\mathcal{O})$, one has an exact sequence

$$(2) \quad 0 \to C_{i-1} \to \mathcal{H}(\tilde{X}, T_p E) \to K_i \to 0.$$

This, as in [18, p. 323], proves the proposition. $\square$
Any group \( G \) in (2) is an extension of an étale group \( G^{\text{ct}} \) by a connected group \( G^0 \). Modulo finite groups, one has \( K_i^0(\mathbb{F}) = \text{Hom}_G(E', B_i) \) and \( C_i^0(\mathbb{F}) = \text{Ext}^1_{\mathbb{Z}}(E', B_i) \), and an exact sequence
\[
0 \to \text{Ext}^1_{\mathbb{Z}}(E', B_{i-1}) \to \mathcal{E}^i(\bar{X}, T_pE)^{\text{ct}}(\mathbb{F}) \to \text{Hom}_G(E', B_i) \to 0.
\]
Note \( H^i(\bar{X}, T_pE) = \mathcal{E}^i(\bar{X}, T_pE)(\mathbb{F}) \). Set \( H^*(X, T_pE) := \lim_{\to} H^i_1(X, \rho^n E) \).

**Lemma 12.** There is an exact sequence
\[
0 \to H^{i-1}(\bar{X}, T_pE)_\Gamma \to H^i(X, T_pE) \to H^i(\bar{X}, T_pE)^\Gamma \to 0.
\]

*Proof.* Easy adaptation of the proof of [19, Lemma 3.4]. \( \square \)

**4.1. The action of \( \gamma \) on \( H^*(\bar{X}, T_pE) \).** We now come to the main step in relating the \( p \)-adic Tate module of \( E \) to the special value \( c_X \).

Consider the map \( \alpha_i : H^i(X, T_pE) \to H^{i+1}(X, T_pE) \) defined by the following commutative diagram
\[
\begin{array}{ccc}
H^i(X, T_pE) & \xrightarrow{\alpha_i} & H^{i+1}(X, T_pE) \\
\downarrow & & \uparrow \\
H^i(\bar{X}, T_pE)^\Gamma & \longrightarrow & H^i(\bar{X}, T_pE)_\Gamma
\end{array}
\]
the vertical maps arise from the previous lemma. As the lower map is cup-product with the canonical generator \( \theta_p \in H^1(\Gamma, W) \) and \( \theta_p^2 = 0 \),

\[
M^\bullet : \cdots \to H^i(X, T_pE) \xrightarrow{\alpha_i} H^{i+1}(X, T_pE) \to \cdots
\]
is a complex; cf. [21, pp. 544-545] for details. Define
\[
z = \prod_i ([H^i(M^\bullet)](-1)^i)
\]
when these numbers are finite. Write, as usual, \( Z(X, t) = \prod P_i(t)^{(-1)^i} \) and \( P_i(t) = \prod_j (1 - \omega_i j t) \) the characteristic polynomial of \( \gamma \) on \( \ell \)-adic cohomology \( H^i(\bar{X}, \mathbb{Z}_\ell) \). Our next result is a variant of [21, Theorem 9.6] (cf. §1.4).

**Theorem 13.** (i) \( H^*(X, T_pE) \) are finitely generated \( \mathbb{Z}_p \)-modules.

(ii) the usual Euler characteristic \( \Sigma(-1)^i \text{rank } H^i(\bar{X}, T_pE) \) is zero.

(iii) the secondary Euler characteristic \( \Sigma(-1)^i \text{rank } H^i(\bar{X}, T_pE) \) is \( 2\rho_X \).

(iv) the cohomology groups \( H^i(M^\bullet) \) are finite.

(v) the value of \( z \) is given by
\[
z = \prod_i ([H^i(M^\bullet)](-1)^i) = \left| \frac{q^{\chi(X, \rho_X)}}{c_X^{\rho_X}} \right|_p.
\]

(vi) \( H^i(X, T_pE) \) is finite for \( i \neq 1, 2 \).

*Proof.* One can write \( z \) as a product \( z = z_0 z_{\text{ct}} \) corresponding to the decomposition
\[
0 \to \mathcal{E}^i(\bar{X}, T_pE)^0 \to \mathcal{E}^i(\bar{X}, T_pE) \to \mathcal{E}^i(\bar{X}, T_pE)^{\text{ct}} \to 0.
\]
of \( \mathcal{E}^i(\bar{X}, T_pE) \) into its connected and étale parts. For each of the groups in the exact sequence
\[
0 \to C^0_{i-1} \to \mathcal{E}^i(\bar{X}, T_pE)^0 \to K_i^0 \to 0
\]
of connected group schemes, the map \( \gamma - 1 \) on the \( \mathbb{F} \)-points is surjective because it is an étale endomorphism of a connected group. Therefore, we obtain that \( \mathcal{E}^i(\bar{X}, T_pE)^0 = 0 \), \( [\mathcal{E}^i(\bar{X}, T_pE)^0]^\Gamma = q^{b_i} \), and
\[
z_0 = q^{\Sigma(-1)b_i} = \left| q^{\Sigma(-1)b_i} \right|_p.
\]
The finite \( W \)-torsion in \( H^*(X, \mathcal{O}_X) \) does not contribute to \( z \) [14, pp. 80-81]. Thus, we may assume that \( K^t_i(F) = \text{Hom}_F(E', B_i) \) and \( C^t_i(F) = \text{Ext}^1_F(E', B_i) \), and that there is an exact sequence
\[
0 \rightarrow \text{Ext}^1_F(E', B_{i-1}) \rightarrow H^i(X, T_pE)^{et} \rightarrow \text{Hom}_F(E', B_i) \rightarrow 0.
\]
Now the first term is finite and the last term is a finitely generated \( \mathbb{Z}_p \)-module. As \( H^i(X, T_pE)^{et} \) and \( H^i(X, T_pE) \) differ only by a finite group of order \( q^k \), this proves (i).

The étale part of \( M^\bullet \) sits in the commutative diagram
\[
\begin{array}{c}
H^0(X, T_pE)^{et} \\
\alpha_0 \\
\downarrow \\
\text{Hom}_F(E', B_0) \\
\downarrow \\
\text{Ext}^1_F(E', B_0) \\
\downarrow \\
H^0(X, T_pE)^{et} \\
\alpha_1 \\
\downarrow \\
\text{Hom}_F(E', B_1) \\
\downarrow \\
\text{Ext}^1_F(E', B_1) \\
\downarrow \\
\vdots
\end{array}
\]
where the maps \( \beta_i \) are the ones in [14, Lemma 4] for the \( p \)-divisible groups \( G_i \) (whose dimension we denote by \( g_i \)) whose dual corresponds to \( B_i \), and \( H = E(p) \). We shall prove that \( z_{et} \) is defined and calculate its value by appealing to [14, Lemma 4]. Let us recall two relevant facts for this purpose.

- [18, Remark 5.5] the characteristic polynomial \( \Pi_1(t) = \prod_j (1 - \omega_{ij} t) := \det(1 - \gamma t) \) of \( \gamma \) on \( H^i(X, \mathcal{O}_X) \) is the same as the characteristic polynomial of \( \gamma \) on the crystalline cohomology \( H^i_{cryst}(X) \otimes \mathbb{Q}_p \) of \( X \).

- [8, 3.5.3, p. 616] the characteristic polynomial of \( \gamma \) on \( H^i(X, \mathcal{O}_X) \otimes \mathbb{Q}_p \) (by (1), this is the same as on \( B_i \)) is \( \prod_j (1 - \omega_{ij} \) where the product is over all \( \omega_{ij} \) with \( \text{ord}_p(\omega_{ij}) < 1 \).

We can now apply [14, Lemma 4] to obtain that \( z_{et} \) is defined (which proves (iv)) and given by:
\[
z_{et} = \left| q^{2^{-(-1)} \prod \omega_{ij} \prod (1 - \omega_{ij})^{-1}} \right|^2 p
\]
where the product is over all \( \omega_{ij} \) with \( \text{ord}_p(\omega_{ij}) < 1 \). Using
\[
\left| 1 - \frac{\omega_{ij}}{\sqrt{q}} \right|^p = 1 \quad \text{if} \quad \text{ord}_p(\omega_{ij}) > \frac{1}{2},
\]
this condition \( \text{ord}_p(\omega_{ij}) < 1 \) may be disregarded in \( z_{et} \). This proves (iv).

We now complete the proof of (v); given the formulas for \( z_{et} \) and \( z_0 \), it remains to show that \( \sum (-1)^i g_i \) is \( \chi(X, \mathcal{O}_X) \). As this will use the case \( r = 1 \) of [18, Proposition 4.1], we translate that result into our context. First, we observe that our \( \chi(X, \mathcal{O}_X) \) is \( \chi(X, \mathcal{O}_X, 1) \) of [18, Proposition 4.1] – the definition of the latter is on top of page 325 of [18]. Next, our \( d_i \) (defined just before Proposition 11) is Milne’s \( d^i(0) \) (defined on [18, p. 321], just before Proposition 3.1).

Now, it remains to interpret the two terms on the right hand side of [18, Proposition 4.1] with \( r = 1 \). The second term \( \sum_i (1)^i d^i(0) \), which is our \( \sum_i (1)^i d_i \), can be replaced by \( -\sum (1)^i b_i \) by Proposition 11. For the first term, we recall that for any \( p \)-divisible group \( A \) over \( \mathbb{F}_q \), the dimension of the dual group \( A^t \) is given by the well known (see, for example, [18, p. 81]; it also follows from [18, Thm. 1 (e)])
\[
\dim(A^t) = \text{height}(A) - \dim(A).
\]
Now consider the first term in the right hand side of [18, Proposition 4.1] with \( r = 1 \). The numbers \( \lambda_{ij} \) relevant here are those which satisfy \( \lambda_{ij} \leq 1 \); by the result [8, 3.5.3, p. 616] also mentioned earlier, these are exactly the slopes of the roots of the characteristic polynomial of \( \gamma \) acting on \( H^*(X, \mathcal{O}_X) \otimes \mathbb{Q}_p \) or

\footnote{In the formula for \( z(g) \) in [14, Lemma 4], the exponent of \( q \) should read \( d(G').d(H) \) - in his notation, this is \( n_2\lambda_1 \), as follows from an inspection of the calculation at the bottom of p. 81 of [14].}
on the associated $p$-divisible groups $B_1, B_2, \cdots$ defined earlier. Milne’s $\lambda_{ij}$ are our $\ord_{\eta}(\omega_{ij})$. The first term can therefore be written as

$$
\sum_{\ord_{\eta}(\omega_{ij}) \leq 1} (-1)^i m_{ij} (1 - \ord_{\eta}(\omega_{ij})),
$$

where $m_{ij}$ is the multiplicity of $\ord_{\eta}(\omega_{ij})$. It is now an easy exercise to see that this sum is $\sum_i (-1)^i \dim(B'_i)$. But $g_i$ is the dimension of the $p$-divisible group whose dual corresponds to $B_i$. We summarise our discussion

$$
\chi(X, O_X) = \chi(X, O_X, 1)
= \sum_{\ord_{\eta}(\omega_{ij}) \leq 1} (-1)^i m_{ij} (1 - \ord_{\eta}(\omega_{ij})) + \sum_i (-1)^i d_i(0)
= \sum_i (-1)^i (\dim(B'_i) + d_i(0))
= \sum_i (-1)^i (\dim(B'_i) + d_i)
= \sum_i (-1)^i (g_i - b_i).
$$

This proves (v).

We note that the $\mathbb{Z}_p$-ranks of $\Hom_{\eta}(E', B_i)$ and $\Ext^1_{\mathbb{Z}/p}(E', B_i)$ are equal (the ranks are unchanged by an isogeny and one reduces to the cases treated in [14, pp.81-83]). While the first contributes to $H^i(X, T_p E)^{et}$, the second contributes to $H^{i+1}(X, T_p E)^{et}$, which proves (ii).

Now, the rank of $\Hom_{\eta}(E', B_i)$ is non-zero only if $\sqrt{\eta}$, a Weil number of weight one – is a root of the minimal polynomial of Frobenius on $B_i$ [14, p. 81]. But a root $\omega_{ij}$ of Frobenius on $B_i$ (or $H^i(X, W\mathcal{O})$) is a Weil number of weight $i$. Thus, the rank is zero for $i \neq 1$. This proves (vi). The $\mathbb{Z}_p$-rank of $\Hom_{\eta}(E', B_1)$ is, by Tate’s theorem [2], equal to the $\mathbb{Z}$-rank of $\Hom_{\eta}(E, Pic(X))$ which is $2\rho_X$. As $(-1)^{0}0 + (-1)^{1}1 \times 2\rho_X + (-1)^{2}2 \times 2\rho_X = 2\rho_X$, this proves (iii). \hfill \Box

5. The Weil-étale cohomology of an elliptic curve

As before, $q = p^2$, $X$ is a smooth projective geometrically connected variety over Spec $\mathbb{F}_q$, and $E$ is our fixed elliptic curve. In this section, we compute $H^*(X, E)$ and the Weil-étale [12] cohomology groups $H^*_W(X, E)$ and prove Theorem 1. We write $A$ for the Albanese variety of $X$.

5.1. Cohomology of the elliptic curve $E$. The basic properties of $H^*(X, E)$ are given the following result whose formulation was inspired by [11, Proposition 2.1].

Theorem 14. (a) The étale cohomology of $E$ is given as follows:

(i) $H^0(X, E)$ is finitely generated and

$$
\operatorname{rank} H^0(X, E) = \operatorname{rank} \Hom_{\eta}(A, E) = \operatorname{rank} \Hom_{\eta}(E, Pic(X)) = 2\rho_X.
$$

(ii) $H^j(X, E)$ is torsion for $j > 0$ and zero for $j > 1 + 2 \dim X$.

(iii) $H^j(X, E)$ is finite for $j \geq 3$.

(iv) $H^1(X, E)$ is finite.

(v) $H^2(X, E)$ is co-finite of corank $2\rho_X$.

(vi) The $\mathbb{Z}/p$-modules $H^i(X, E) = \prod H^i(X, T^l E)$ (all primes $i$ including $l = p$) are finite for $i \neq 1, 2$.

(b) The Weil-étale cohomology groups $H^*_W(X, E)$ of $E$ are finitely generated. The rank of $H^i_W(X, E)$ is zero for $i \neq 0, 1$ and $2\rho_X$ for $i = 0, 1$.

Remark. If the Néron-Severi group $NS(X)$ is torsion-free and the Picard scheme $Pic_X$ is smooth, then $H^1(X, E)$ is isomorphic to the finite group $\operatorname{Ext}^1_{\mathbb{Z}/p}(A, E)$ [15, Thm. 1]. \hfill \Box
Proof. Part (b) will be proved later (as part of Theorem 1). We prove part (a).

(i) This is well known (Mordell-Weil theorem); cf. for instance, [10].

(ii) It is straightforward that \( H^j(X, E) \) and \( H^j(\bar{X}, E) = H^j_W(\bar{X}, E) \) are torsion for \( j > 0 \) using the fact that \( E \) over \( X \) is the Neron model of \( E \) over \( \mathbb{F}_p(X) \); cf. [1, p. 41]. Consider the Kummer sequence

\[
0 \to nE \to E \overset{n}{\to} E \to 0, \quad (n > 0)
\]

which is exact in the étale (for \( n \) coprime to \( p \)) and flat topologies. When \( (n, p) = 1 \), the identity \( nE = (\mathbb{Z}/n\mathbb{Z})^2 \) over \( \mathbb{F} \) tells us that \( H^j(\bar{X}, nE) \) are finite (zero for \( j > 2 \dim X \)). This implies (ii) up to \( p \)-torsion using the spectral sequence

\[
H^r(\Gamma_0, H^s(\bar{X}, E)) \Rightarrow H^{r+s}(X, E).
\]

By [15, VI, Rmk. 1.5], \( H^j_p(X, pE) = 0 \) for \( j > 2 + \dim X \). As \( E \) is smooth, \( H^j_p(X, E) = H^j(X, E) \) [15, p. 92] and (ii) follows.

(iii) It suffices to consider the non-\( p \)-torsion. The \( p \)-part can be treated similarly, given the sequence

\[
0 \to H^{j-1}(X, E) \otimes \mathbb{Z}_p \to H^j(X, T_pE) \to T_pH^j(X, E) \to 0
\]

and the results of the previous section on \( H^*(X, T_pE) \). Though the proof of the non-\( p \)-part is standard (cf. [18, Cor. 6.4]), nevertheless we recall it for the convenience of the reader.

Let \( \gamma \) be the Frobenius automorphism on the Tate module \( T_\ell E \) of \( E \); here \( \ell \) is a prime distinct from \( p \). By the Riemann hypothesis, \( (1 - \gamma) \) is a quasi-isomorphism of \( H^1(X, T_\ell E) = H^1(X, \mathbb{Z}_\ell) \otimes T_\ell E \) for \( j \neq 1 \); thus, if \( j \neq 1 \), then \( H^j(X, T_\ell E) = H^j(X, T_\ell E) \gamma \) and \( H^j(X, T_\ell E) \gamma \) are finite. From

\[
0 \to H^{j-1}(\bar{X}, T_\ell E) \to H^j(X, T_\ell E) \to H^j(\bar{X}, T_\ell E) \to 0,
\]

we obtain that \( H^j(X, T_\ell E) \) is finite for \( j \neq 1, 2 \). Now, (3) provides the exact sequence

\[
0 \to H^{j-1}(X, E) \otimes \mathbb{Z}_\ell \to H^j(X, T_\ell E) \to T_\ell H^j(X, E) \to 0,
\]

whereby, for \( j > 2 \), \( H^j(X, T_\ell E) \) is finite and hence \( T_\ell H^j(X, E) \), being torsion-free, is zero. As the \( \ell \)-torsion of \( H^j(X, E) \) is finite, the \( \ell \)-primary subgroup \( H^j(X, E)(\ell) \) itself is finite for \( j > 2 \) and isomorphic to \( H^{j+1}(X, T_\ell E) \). For the finiteness of non-\( p \)-part of \( H^j(X, E) \) for \( j \geq 3 \), we need the following lemma whose formulation and proof are as in [11, Lem. 1.1, pp.176-7].

Lemma 15. Write \( P_j(t) = \text{det}(1 - \gamma t) \) for the characteristic polynomial of \( \gamma \) on \( H^j(\bar{X}, \mathbb{Z}_\ell) \). Similarly, we define \( Q_j(t) \) for \( H^j(\bar{X}, T_\ell E) \). For all \( j \neq 0, 1 \), we have

\[
[H^j(\bar{X}, E(\ell)) \Gamma] = [H^{j+1}(\bar{X}, T_\ell E)^\Gamma] |P_j(q^{-\frac{1}{2}})|^{-2}_e,
\]

where \( | \cdot |_e \) is the absolute value normalized so that \( |\ell|_e = \ell^{-1} \).

One has \( |P_j(q^{-\frac{1}{2}})|_e^2 = |Q_j(t)|_{\ell} \).

As in [11, (c), pp.181], for \( j > 2 \), \( H^j(X, E)(\ell) \) is trivial unless \( \ell \) divides \( P_j(q^{-\frac{1}{2}}) \) or \( H^*(\bar{X}, \mathbb{Z}_\ell) \) has torsion. By O. Gabber [4], \( H^j(X, E)(\ell) \) is nontrivial only for a finite number of primes \( \ell \) thereby showing the finiteness of the non-\( p \)-part of \( H^j(X, E) \) for \( j > 2 \).

(iv) Recall that the Hochschild-Serre spectral sequence

\[
H^i(\Gamma, H^j(\bar{X}, T_\ell E)) \Rightarrow H^{i+j}(X, T_\ell E)
\]

yields the sequence

\[
0 \to H^0(\bar{X}, T_\ell E) \to H^1(X, T_\ell E) \to H^1(\bar{X}, T_\ell E) \to 0.
\]

But

\[
H^1(\bar{X}, T_\ell E) \gamma = \text{Hom}(T_\ell A, T_\ell E) \Gamma \leftarrow \text{Hom}_{\mathbb{Z}_\ell}(A, E) \otimes \mathbb{Z}_\ell.
\]

As \( H^0(\bar{X}, T_\ell E) \gamma \) is finite, the rank of \( H^1(X, T_\ell E) \) is \( 2p_{\mathbb{X}} = \text{rank} H^1(\bar{X}, T_\ell E) \Gamma \). The sequence

\[
0 \to H^0(X, E) \otimes \mathbb{Z}_\ell \to H^1(X, T_\ell E) \to T_\ell H^1(X, E) \to 0,
\]
given (i), shows that $T_t H^1(X, E)$ has rank zero and, being torsion-free, is zero. This proves (iv).

(v) A similar argument as in (iv) using that the rank of $H^1(X, T_t E)_\Gamma$ is $2\rho_X$ shows that the rank of $T_t H^2(X, E)$ is $2\rho_X$ thereby proving (v).

(vi) was proved in the course of the proof of (iii).

We can now formulate an adelic version of Theorem 13.

**Theorem 16.** The cohomology groups $H^i(N^\bullet)$ of the complex

$$N^\bullet : \ldots \to H^i(X, TE) \xrightarrow{\alpha_i} H^{i+1}(X, TE) \to \ldots$$

are finite; the maps $\alpha_i$ are induced by cup-product with the generator of $H^1(Spec \mathbb{F}_q, \hat{\mathbb{Z}})$. One has

$$c^2_X = q^{X,0_X} \prod ((H^i(N^\bullet)(-1)^i).$$

**Proof.** The first statement needs verification only in the cases $i = 1, 2$, given (vi) of Theorem 14. We need to check that the kernel and cokernel of $H^1(X, TE) \to H^2(X, TE)$ are finite. The $p$-part is straightforward, given the previous lemma and the obvious variant of the non-$p$-part of [18, Lemma 6.2] (or the étale case of [14, Lemma 4]); see the proof of [18, Theorem 0.1].

5.2. **Proof of Theorem 1.** Recall [12, §8] the generator $\theta$ of $H^1_W(Spec \mathbb{F}_q, \mathbb{Z}) = \text{Hom}(\mathbb{Z}, \mathbb{Z}) = \mathbb{Z}$. It is clear that $H^0_W(X, E) = H^0(X, E)$ is finitely generated.

The groups $H^i(X, E)$ are torsion for $i > 0$ [1, p. 41]. By Theorem 14 and [12, Prop. 2.3], $H^i(X, E) \cong H^i_W(X, E)$ for $i \geq 3$ and thus proves (ii). To prove (i), it remains to show that $H^1_W(X, E)$ and $H^2_W(X, E)$ are finitely generated. Now, [12, Prop. 2.3] provides a spectral sequence

$$H^i(\Gamma_0, H^0_W(\hat{X}, E)) \Rightarrow H^{i+1}_W(X, E)$$

with a map from (4) of spectral sequences. This gives exact sequences

$$0 \to H^2(\Gamma, H^0(\hat{X}, E)) \to H^2(X, E) \to H^2_W(X, E) \to 0,$$

and

$$0 \to H^1(X, E) \to H^1_W(X, E) \to \frac{H^1(\Gamma_0, H^0(\hat{X}, E))}{H^1(\Gamma, H^0(X, E))} \to 0.$$

As $H^0(\hat{X}, E)$ differs from $Y := \text{Hom}_{\mathbb{F}}(\hat{A}, \hat{E})$ by torsion, we have

$$H^2(\Gamma, H^0(\hat{X}, E)) \cong H^1(\Gamma_0, H^0(\hat{X}, E)) \otimes \frac{\mathbb{Q}}{\mathbb{Z}} \cong H^1(\Gamma_0, Y) \otimes \frac{\mathbb{Q}}{\mathbb{Z}},$$

using [12, Lemma 1.2]. The sequence of $\Gamma_0$-modules $0 \to Y \xrightarrow{\alpha} Y \to \mathbb{Y} \to 0$ gives

$$H^1(\Gamma_0, Y) \otimes \frac{\mathbb{Q}}{\mathbb{Z}} \cong H^1(\Gamma_0, Y \otimes \frac{\mathbb{Q}}{\mathbb{Z}}).$$

and, by Tate’s theorem [24], the corank of $H^1(\Gamma_0, Y \otimes \frac{\mathbb{Q}}{\mathbb{Z}})$ is $2\rho_X$. This proves that $H^2_W(X, E)$ is finite, the rank of $H^1_W(X, E)$ is $2\rho_X$ thereby finishing the proof of (i), (iii) and (iv).

Part (v) follows from Theorem 16 and the isomorphisms $H^i_W(X, E) \otimes \hat{\mathbb{Z}} \cong H^{i+1}(X, TE)$ (see (5)); the nontrivial cases are $i = 0, 1$ which are compatible with the maps $\vartheta$ and $\alpha_i$. In more detail, combining $TH^1(X, E) = 0$ (Theorem 14 (iv)), the isomorphism $H^0(X, E) = H^0_W(X, E)$, and (5) proves the case $i = 0$. The exactness of (3) in the Weil-étale or Weil-flat topology gives an exact sequence

$$0 \to H^i_W(X, E) \otimes \hat{\mathbb{Z}} \to H^{i+1}_W(X, TE) \to TH^{i+1}_W(X, E) \to 0,$$

where the first map is an isomorphism because $H^*_W(X, E)$ are finitely generated abelian groups and so $TH^*_W(X, E) = 0$. As $H^*(X, TE) \xrightarrow{\sim} H^*_W(X, TE)$, this finishes the proof of Theorem 1.
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