IRREGULARITIES OF DISTRIBUTION ON TWO POINT HOMOGENEOUS SPACES

LUCA BRANDOLINI, BIANCA GARIBOLDI, AND GIACOMO GIGANTE

Abstract. We study the irregularities of distribution on two-point homogeneous spaces. Our main result is the following: let $d$ be the real dimension of a two point homogeneous space $M$, let $(\{a_j\}_{j=1}^N, \{x_j\}_{j=1}^N)$ be a system of positive weights and points on $M$ and let
\[
D_r(x) = \sum_{j=1}^N a_j \chi_{B_r(x)}(x_j) - \mu(B_r(x))
\]
be the discrepancy associated with the ball $B_r(x)$. Then, if $d \not\equiv 1(\text{mod } 4)$, for any radius $0 < r < \pi/2$, we obtain the sharp estimate
\[
\int_M \left( |D_r(x)|^2 + |D_{2r}(x)|^2 \right) d\mu(x) \geq cN^{-1-\frac{1}{d}}.
\]

1. Introduction

A $d$-dimensional Riemannian manifold $M$ with distance $\rho$ is said to be a two-point homogeneous space if given four points $x_1, x_2, y_1, y_2 \in M$ such that $\rho(x_1, y_1) = \rho(x_2, y_2)$, then there exists an isometry $g$ of $M$ such that $gx_1 = x_2$ and $gy_1 = y_2$. Wang in [17] has completely characterized compact connected two-point homogeneous spaces. More precisely if we assume that $\rho$ is normalized so that diam($M$) = $\pi$, then $M$ is isometric to one of the following compact rank 1 symmetric spaces:

(i) the Euclidean sphere $S^d = SO(d+1)/SO(d) \times \{1\}$, $d \geq 1$;
(ii) the real projective space $P^n(\mathbb{R}) = O(n+1)/O(n) \times O(1)$, $n \geq 2$;
(iii) the complex projective space $P^n(\mathbb{C}) = U(n+1)/U(n) \times U(1)$, $n \geq 2$;
(iv) the quaternionic projective space $P^n(\mathbb{H}) = Sp(n+1)/Sp(n) \times Sp(1)$, $n \geq 2$;
(v) the octonionic projective plane $P^2(\mathbb{O})$.

In the following we will assume that $M$ is one of the above symmetric spaces and that $d$ is its real dimension. In particular the real dimension of $P^n(F)$ is $d = nd_0$, where $d_0 = 1, 2, 4, 8$ according to the real dimension of $F = \mathbb{R}, \mathbb{C}, \mathbb{H}$ and $\mathbb{O}$ respectively. In the case of $S^d$ it will be convenient to set $d_0 = d$. See [8, pp. 176-178], see also [10], [14] and [19]. In the following, to keep notation simple, we will

2010 Mathematics Subject Classification. 11K38, 43A85.

Key words and phrases. Irregularities of distribution, Two-point homogeneous spaces, Discrepancy.
use
\[ a = \frac{d - 2}{2}, \quad b = \frac{d_0 - 2}{2}. \]

Let \( \mu \) be the Riemannian measure on \( \mathcal{M} \) normalized so that \( \mu(\mathcal{M}) = 1 \) and let \( B_r(x) = \{ y \in \mathcal{M} : \rho(x, y) < r \} \).

For a given set of points \( \{ x_j \}_{j=1}^N \subset \mathcal{M} \) and positive weights \( \{ a_j \}_{j=1}^N \) such that \( a_1 + a_2 + \cdots + a_N = 1 \) we define the discrepancy of the ball \( B_r(x) \) by

\[ D_r(x) = \sum_{j=1}^N a_j \chi_{B_r(x)}(x_j) - \mu(B_r(x)) \tag{1} \]

This quantity compares the Riemannian measure \( \mu \) with the discrete measure \( \sum_{j=1}^N a_j \delta_{x_j} \) testing these two measures on the ball \( B_r(x) \). It is known that, no matter how well distributed the points \( \{ x_j \}_{j=1}^N \) are, there are balls for which the discrepancy \( D_r(x) \) cannot be too small. See [1] for an introduction to this subject.

For example M. Skriganov in [14, Theorem 2.2] has proved, in the case of equal weights, that if \( \eta \) is a positive, locally integrable function on \((0, \pi)\) such that

\[ \int_0^\pi \eta(r) \left( \sin \frac{1}{2} r \right)^{d-1} \left( \cos \frac{1}{2} r \right)^{d_0-1} dr < +\infty, \]

then there exists \( c > 0 \) such that for every distribution of \( N \) points

\[ \int_0^\pi \int_{\mathcal{M}} |D_r(x)|^2 d\mu(x) \eta(r) dr \geq c N^{-1 - \frac{1}{d}}. \]

A very general non-constructive result guarantees the existence of point distributions that exhibit the optimal decay \( N^{-1 - 1/d} \) (see e.g. [2, Corollary 8.2]). In fact, Skriganov [14, Corollary 2.1] has proved that for well distributed optimal cubature formulas one has

\[ \int_0^\pi \int_{\mathcal{M}} |D_r(x)|^2 d\mu(x) \eta(r) dr \leq c N^{-1 - \frac{1}{d}} \tag{2} \]

(see [9], or the remarks that follow the statement of Theorem 15 here, for a proof of the existence of such cubature formulas).

Our first result is the following.

**Theorem 1.** Let \( d \not\equiv 1 \) (mod 4). Then, for all \( 0 < r < \pi/2 \), there is a constant \( C > 0 \) such that for every set of points \( \{ x_j \}_{j=1}^N \subset \mathcal{M} \) and positive weights \( \{ a_j \}_{j=1}^N \) such that \( a_1 + a_2 + \cdots + a_N = 1 \) we have

\[ \int_{\mathcal{M}} |D_r(x)|^2 d\mu(x) + \int_{\mathcal{M}} |D_{2r}(x)|^2 d\mu(x) \geq CN^{-1 - \frac{1}{d}}. \]

Again this result is optimal in view of Corollary 8.2 in [2]. Also, the same technique used by Skriganov to prove the estimate from above for cubature formulas (2)
IRREGULARITIES OF DISTRIBUTION ON TWO POINT HOMOGENEOUS SPACES

actually gives a stronger, uniform estimate in the radius $r$,

$$\int_{\mathcal{M}} |D_r(x)|^2 \, d\mu(x) \leq c N^{-1 - \frac{1}{2}}. \quad (3)$$

See Theorem 15 for a precise statement.

When $d \equiv 1 \pmod{4}$ our technique fails. In similar settings it is known that the discrepancy can be actually a bit smaller than the expected value $N^{-1 - \frac{1}{2}}$. See [13, Theorem 3.1]. See also [3], [11] and [12].

Our final result is an estimate for the discrepancy of balls of a single fixed radius that holds in every dimension.

**Theorem 2.** For every $\varepsilon > 0$ and for almost every $0 < r < \pi$ there is a constant $C > 0$ such that for every set of points $\{x_j\}_{j=1}^N \subset \mathcal{M}$ and positive weights $\{a_j\}_{j=1}^N$ such that $a_1 + a_2 + \cdots + a_N = 1$ we have

$$\int_{\mathcal{M}} |D_r(x)|^2 \, d\mu(x) \geq CN^{-1 - \frac{3}{2} - \varepsilon}. \quad (4)$$

The paper is organized as follows. In section 2 we collect some basic facts about two-point homogeneous spaces. In section 3 we prove Theorem 1 and 2. In section 4 we show that suitable cubature formulas give the optimal discrepancy (3).

2. TWO-POINT HOMOGENEOUS SPACES

If $o$ is a fixed point in $\mathcal{M}$, then $\mathcal{M}$ can be identified with the homogeneous space $G/K$, where $G$ is the group of isometries of $\mathcal{M}$ and $K$ is the stabilizer of $o$. We will also identify functions $F(x)$ on $\mathcal{M}$ with right $K$-invariant functions $f(g)$ on $G$ by setting $f(g) = F(x)$ when $go = x$. If $\mu$ is the Riemannian measure on $\mathcal{M}$ normalized so that $\mu(\mathcal{M}) = 1$, then $\mu$ is invariant under the action of $G$, in other words, for every $g \in G$,

$$\int_{\mathcal{M}} F(gx) \, d\mu(x) = \int_{\mathcal{M}} F(x) \, d\mu(x)$$

**Definition 3.** A function $F$ on $\mathcal{M}$ is a zonal function if for every $x \in \mathcal{M}$ and every $k \in K$ we have $F(kx) = F(x)$.

**Lemma 4.** Let $F$ be a zonal function. Then $F(x)$ depends only on $\rho(x, o)$. Furthermore, defining $F_0$ so that $F(x) = F_0(\rho(x, o))$ we have

$$\int_{\mathcal{M}} F(x) \, d\mu(x) = \int_0^\pi F_0(r) A(r) \, dr, \quad (4)$$

where

$$A(r) = c(a, b) \left( \sin \frac{1}{2} r \right)^{2a+1} \left( \cos \frac{1}{2} r \right)^{2b+1}$$
and 
\[ c(a, b) = \left( \int_0^\pi (\sin \frac{1}{2} r)^{2a+1} (\cos \frac{1}{2} r)^{2b+1} dr \right)^{-1} = \frac{\Gamma(a + b + 2)}{\Gamma(a + 1) \Gamma(b + 1)}. \]

In particular if 
\[ B_r(x) = \{ y \in \mathcal{M} : \rho(x, y) < r \}, \]
there exist two constants \( c_1(d, d_0) \) and \( c_2(d, d_0) \) such that for every \( r \in [0, \pi] \)
\[ c_1(d, d_0) r^d \leq \mu(B_r(x)) \leq c_2(d, d_0) r^d. \] (5)

Moreover if \( k \geq 1 \), then a ball of radius \( kr \) can be covered by \( \frac{c_2(d, d_0)}{c_1(d, d_0)} (2(k + 1))^d \) balls of radius \( r \).

Proof. Let \( x, y \in \mathcal{M} \) such that \( \rho(x, o) = \rho(y, o) \). Since \( \mathcal{M} \) is two-point homogeneous there exists \( g \in G \) such that \( gx = y \) and \( go = o \). Thus, \( g \in K \) and \( F(y) = F(gx) = F(x) \). Equation (4) follows from (4.17) in [8]. The estimate (5) is an immediate consequence of (4). The last assertion is an estimate of the maximum number of disjoint balls of radius \( r/2 \) that can fit in a ball of radius \( (k + 1)r \). \[ \square \]

Let \( \Delta \) be the Laplace-Beltrami operator on \( \mathcal{M} \), let \( \lambda_0, \lambda_1, \ldots \), be the distinct eigenvalues of \( -\Delta \) arranged in increasing order, let \( \mathcal{H}_m \) be the eigenspace associated with the eigenvalue \( \lambda_m \), and let \( d_m \) its dimension. It is well known that
\[ L^2(\mathcal{M}) = \bigoplus_{m=0}^{+\infty} \mathcal{H}_m. \] (6)

If \( F(x) = F_0(\rho(x, o)) \) is a zonal function on \( \mathcal{M} \), then
\[ \Delta F(x) = \frac{1}{A(t)} \frac{d}{dt} \left( A(t) \frac{d}{dt} F_0(t) \right) \bigg|_{t=\rho(x, o)} \] (7)
(see (4.16) in [8]).

**Definition 5.** The zonal spherical function of degree \( m \in \mathbb{N} \) with pole \( x \in \mathcal{M} \) is the unique function \( Z^m_x \in \mathcal{H}_m \), given by the Riesz representation theorem, such that for every \( Y \in \mathcal{H}_m \)
\[ Y(x) = \int_{\mathcal{M}} Y(y) Z^m_x(y) d\mu(y). \]

The next lemma summarizes the main properties of zonal functions and it is essentially taken from [15] where the case \( \mathcal{M} = S^d \) is discussed in detail.

**Lemma 6.** i) If \( Y^1_m, \ldots, Y^d_m \) is an orthonormal basis of \( \mathcal{H}_m \subset L^2(\mathcal{M}) \), then
\[ Z^m_x(y) = \sum_{\ell=1}^{d_m} Y^\ell_m(x) Y^\ell_m(y). \]

ii) \( Z^m_x \) is real valued and \( Z^m_x(y) = Z^m_y(x) \).
iii) If \( g \in G \), then \( Z^m_{gx}(gy) = Z^m_g(y) \).

iv) \( Z^m_{x}(x) = \|Z^m_x\|^2 = d_m \).

v) \( Z^m_{o}(x) \) is a zonal function and

\[
Z^m_{o}(x) = \frac{d_m}{P^a,b_m(1)} P^a,b_m(\cos(\rho(x, o))) \tag{8}
\]

where \( P^a,b_m \) are the Jacobi polynomials.

vi) \( \{d_{m}^{1/2} Z^m_{o}\}_{m=0}^{+\infty} \) is an orthonormal basis of the subspace of \( L^2(M) \) of zonal functions.

vii) \( \lambda_m = m(m + a + b + 1) \).

viii) \( d_m \approx m^{d-1} \).

Proof. i) By the defining property of \( Z^m_x \)

\[
Z^m_x(y) = \sum_{\ell=1}^{d_m} \int_M Z^m_x(z) Y^\ell_m(z) d\mu(z) Y^\ell_m(y) = \sum_{\ell=1}^{d_m} Y^\ell_m(x) Y^\ell_m(y).
\]

ii) Since the basis can be taken to be of real valued functions, then by point i) \( Z^m_x \) is real valued.

iii) Let \( Y \in H_m(M) \). Since for every \( g \), \( Y(g^{-1}y) \in H_m(M) \), then

\[
\int_M Y(y) Z^m_{gx}(gy) d\mu(y) = \int_M Y(g^{-1}y) Z^m_{gx}(y) d\mu(y) = Y(g^{-1}(gx)) = Y(x),
\]

and the uniqueness of \( Z^m_x \) shows that \( Z^m_{gx}(gy) = Z^m_x(y) \).

iv) Observe that

\[
\|Z^m_x\|^2 = \int_M Z^m_x(y) \overline{Z^m_x(y)} d\mu(y) = Z^m_x(x) = \sum_{\ell=1}^{d_m} |Y^\ell_m(x)|^2.
\]

Since by iii) this quantity is constant in \( x \), it equals its integral over \( M \), which is exactly \( d_m \).

v) By point iii) we have

\[
Z_o(kx) = Z_{k^{-1}o}(x) = Z_o(x).
\]

Equation (8) follows from the fact that the radial part of the Laplace-Beltrami operator (7) coincides with the Jacobi operator.

vi) follows from the fact that the Jacobi polynomials

\[
\{P^a,b_m(\cos(t))\}_{m=0}^{+\infty}
\]

form an orthogonal basis of \( L^2((0, \pi), A(t) dt) \). See Chapter 4 in [16].

vii) follows from [16, Theorem 4.2.2].

viii) follows from iv), v) and vi) computing explicitly the \( L^2 \) norm of \( d_{m}^{1/2} Z^m_{o} \). For the computation one needs (4.1.1) in [16] to evaluate \( P^a,b_m(1) \) and (4.3.3) in [16] to evaluate the \( L^2 \) norm of \( P^a,b_m(x) \). \( \square \)
Lemma 7. Let $D_r(x)$ be as in (1). Then

$$\int_{\mathcal{M}} |D_r(x)|^2 \, d\mu(x) = \sum_{m=1}^{+\infty} \sum_{\ell=1}^{d_m} \left| \sum_{j=1}^{N} a_j Y_m^\ell(x_j) \right|^2 d_m^{-2} \left| \int_{B_r(o)} Z^m_o(y) d\mu(y) \right|^2.$$ 

Proof. By (6) we have

$$\int_{\mathcal{M}} |D_r(x)|^2 \, d\mu(x) = \sum_{m=0}^{+\infty} \int_{\mathcal{M}} |\mathbb{P}_m D_r(x)|^2 \, d\mu(x)$$

(here $\mathbb{P}_m$ denotes the orthogonal projection of $L^2(\mathcal{M})$ onto $\mathcal{H}_m$). Let $g \in G$ such that $gx_j = o$. Since $\chi_{B_r(o)}$ is a zonal function, then

$$\chi_{B_r(x_j)}(x) = \chi_{B_r(o)}(gx) = \sum_{m=0}^{+\infty} d_m^{-1} \int_{B_r(o)} Z^m_o(y) d\mu(y) Z^m_o(gx) = \sum_{m=0}^{+\infty} d_m^{-1} \int_{B_r(o)} Z^m_o(y) d\mu(y) Z^m_{x_j}(x).$$

Since $Z^m_{x_j}(x) \in \mathcal{H}_m$, we have

$$\mathbb{P}_m \chi_{B_r(x_j)}(x) = d_m^{-1} \int_{B_r(o)} Z^m_o(y) d\mu(y) Z^m_{x_j}(x).$$

Overall we obtain

$$\mathbb{P}_m D_r(x) = \sum_{j=1}^{N} a_j d_m^{-1} \int_{B_r(o)} Z^m_o(y) d\mu(y) Z^m_{x_j}(x) - \delta_0(m) \mu(B_r(o)).$$

In particular $\mathbb{P}_0 D_r(x) = 0$ and for $m > 0$

$$\mathbb{P}_m D_r(x) = d_m^{-1} \sum_{j=1}^{N} a_j \int_{B_r(o)} Z^m_o(y) d\mu(y) Z^m_{x_j}(x)$$

$$= d_m^{-1} \sum_{j=1}^{N} a_j \int_{B_r(o)} Z^m_o(y) d\mu(y) Y^\ell_m(x)$$

Finally

$$\int_{\mathcal{M}} |D_r(x)|^2 \, d\mu(x) = \sum_{m=1}^{+\infty} \sum_{\ell=1}^{d_m} \left| \sum_{j=1}^{N} a_j Y_m^\ell(x_j) \right|^2 d_m^{-2} \left| \int_{B_r(o)} Z^m_o(y) d\mu(y) \right|^2.$$
In the next results we estimate the quantities
\[ \sum_{\ell=1}^{d_m} \left| \sum_{j=1}^{N} a_j Y_{m}^\ell(x_j) \right|^2 \quad \text{and} \quad d_m^{-2} \left| \int_{B_r(o)} Z_o^m(y) d\mu(y) \right|^2. \]

**Proposition 8.** Let \( m_0 > 0 \), then there exist \( C_0, C_1 > 0 \) such that for every \( X \geq m_0 \) we have
\[ \sum_{m=m_0}^{X} \sum_{\ell=1}^{d_m} \left| \sum_{j=1}^{N} a_j Y_{m}^\ell(x_j) \right|^2 \geq C_1 \sum_{j=1}^{N} a_j^2 X^d - C_0. \]

**Proof.** By the Cassels-Montgomery inequality for manifolds (see [4, Theorem 1]) along with the fact that \( \sum_{m=0}^{X} d_m \approx \sum_{m=0}^{X} m^{d-1} \approx X^d \), we have
\[ \sum_{m=0}^{X} \sum_{\ell=1}^{d_m} \left| \sum_{j=1}^{N} a_j Y_{m}^\ell(x_j) \right|^2 \geq C_1 \sum_{j=1}^{N} a_j^2 X^d. \]

Since
\[ \sum_{m=0}^{m_0-1} \sum_{\ell=1}^{d_m} \left| \sum_{j=1}^{N} a_j Y_{m}^\ell(x_j) \right|^2 \leq \max_{0 \leq \ell \leq m_0-1} \| Y_{m}^\ell \|_\infty^2 \sum_{m=0}^{m_0-1} \sum_{\ell=1}^{d_m} \sum_{j=1}^{N} a_j^2 \]
\[ = \max_{0 \leq \ell \leq m_0-1} \| Y_{m}^\ell \|_\infty^2 \sum_{m=0}^{m_0-1} d_m = C_0, \]
we have
\[ \sum_{m=m_0}^{X} \sum_{\ell=1}^{d_m} \left| \sum_{j=1}^{N} a_j Y_{m}^\ell(x_j) \right|^2 \geq C_1 \sum_{j=1}^{N} a_j^2 X^d - C_0. \]

**Lemma 9.** For any \( 0 \leq r \leq \pi \) and for any \( m \geq 1 \) we have
\[ \int_{B_r(o)} Z_o^m(x) d\mu(x) \]
\[ = \frac{c(a,b) d_m}{m P_m^{a,b}(1)} \int_0^r P_m^{a,b}(\cos t) \left( \frac{t}{2} \right)^{2a} \left( \cos \left( \frac{t}{2} \right) \right)^{2b+1} dt. \]

**Proof.** By (4), (8), and a change of variable,
\[ \int_{B_r(o)} Z_o^m(x) d\mu(x) = \frac{d_m}{P_m^{a,b}(1)} \int_0^r P_m^{a,b}(\cos t) A(t) dt \]
\[ = \frac{c(a,b) d_m}{P_m^{a,b}(1)} \int_0^r P_m^{a,b}(\cos t) \left( \frac{1- \cos t}{2} \right)^a \left( \frac{1+ \cos t}{2} \right)^b \sin t dt. \]
\begin{align*}
= \frac{c(a, b)d_m}{2^{a+b+1}P_m^{a,b}(1)} \int_{\cos r}^1 P_m^{a,b}(x)(1-x)^a(1+x)^b \, dx.
\end{align*}

By Rodrigues’ formula (see [16, (4.3.1)])
\begin{align*}
P_m^{a,b}(x)(1-x)^a(1+x)^b = -\frac{1}{2m} \frac{d}{dx} \left( P_{m-1}^{a+1,b+1}(x)(1-x)^{a+1}(1+x)^{b+1} \right),
\end{align*}
thus
\begin{align*}
\int_{B_r(o)} Z_m^{a}(x)d\mu(x) = \frac{c(a, b)d_m}{2m} P_{m-1}^{a+1,b+1}(\cos r)(1-\cos r)^{a+1}(1+\cos r)^{b+1}
\end{align*}
\begin{align*}
= \frac{c(a, b)d_m}{mP_m^{a,b}(1)} P_{m-1}^{a+1,b+1}(\cos r) \left( \frac{r}{2} \right)^{2a} \left( \frac{1}{\sin r} \right)^{2b+2}. \tag{9}
\end{align*}

In the following, to keep notation simple we set
\begin{align*}
M = m + \frac{a+b+1}{2}.
\end{align*}

Lemma 10. For all \( \varepsilon > 0 \) and for all integers \( m \geq 1 \),
\begin{align*}
\int_{B_r(o)} Z_m^{a}(y)d\mu(y)
= c(a, b)d_m \Gamma(a+1) \left( \sin \left( \frac{r}{2} \right) \right)^{a+1} \left( \cos \left( \frac{r}{2} \right) \right)^{b+1} M^{-a-1} \left( \frac{r}{\sin r} \right)^{\frac{a}{2}} J_{a+1}(Mr)
+ d_m O(m^{-\frac{5}{2}-a}),
\end{align*}
uniformly in \( r \in [0, \pi - \varepsilon] \). Here \( J_{a+1} \) is the Bessel function of first type and order \( a+1 \).

Proof. The following asymptotic expansion of the Jacobi polynomials is well known, (see [16, Theorem 8.21.12]),
\begin{align*}
\left( \sin \left( \frac{r}{2} \right) \right)^{a+1} \left( \cos \left( \frac{r}{2} \right) \right)^{b+1} P_{m-1}^{a+1,b+1}(\cos r)
= M^{-a-1} \Gamma(m+a+1) \left( \frac{r}{\sin r} \right)^{\frac{a}{2}} J_{a+1}(Mr) + r^{\frac{3}{2}} O(m^{-\frac{5}{2}}). \tag{10}
\end{align*}
Thus, by the previous lemma, recalling that (see (4.1.1) in [16])
\begin{align*}
P_m^{a,b}(1) = \frac{\Gamma(m+a+1)}{\Gamma(m+1)\Gamma(a+1)},
\end{align*}
for every \( r \in [0, \pi - \varepsilon] \) and \( m \geq 1 \),
\begin{align*}
\int_{B_r(o)} Z_m^{a}(y)d\mu(y)
\end{align*}
\[ = \frac{c(a, b) d_m}{m P_m^{a,b}(1)} P_m^{a+1,b+1}(\cos r) \left( \sin \left( \frac{r}{2} \right) \right)^{2a+2} \left( \cos \left( \frac{r}{2} \right) \right)^{2b+2} \]

\[ = \frac{c(a, b) d_m}{m} \Gamma(m + 1) \Gamma(a + 1) \left( \sin \left( \frac{r}{2} \right) \right)^{a+1} \left( \cos \left( \frac{r}{2} \right) \right)^{b+1} \]

\[ \times M^{-a-1} \frac{\Gamma(m + a + 1)}{(m - 1)!} \left( \frac{r}{\sin r} \right)^{1/2} \frac{1}{2} J_{a+1}(Mr) \]

\[ + \frac{c(a, b) d_m}{m} \frac{1}{P_m^{a,b}(1)} \left( \sin \left( \frac{r}{2} \right) \right)^{a+1} \left( \cos \left( \frac{r}{2} \right) \right)^{b+1} M^{-a-1} \left( \frac{r}{\sin r} \right)^{1/2} J_{a+1}(Mr) \]

\[ + d_m O(m^{-\frac{5}{2} - a}). \]

\[ \square \]

The expression obtained in Lemma 7 shows that to estimate the discrepancy from below requires an estimate from below for the quantity

\[ \left| \int_{B_r(0)} Z_m^m(y) d\mu(y) \right|^2. \]

Lemma 10 shows that this quantity may vanish due to the zeroes of the Bessel functions. Our next lemma shows that at least when \( d \not\equiv 1 \pmod 4 \) one can overcome this obstruction using two balls of different radii.

**Lemma 11.** Assume that \( d \not\equiv 1 \pmod 4 \). Then for all \( 0 < r < \pi/2 \), there exist positive constants \( C \) and \( m_0 \) such that for \( m \geq m_0 \),

\[ \left| \int_{B_r(0)} Z_m^m(x) d\mu(x) \right|^2 + \left| \int_{B_{2r}(0)} Z_m^m(x) d\mu(x) \right|^2 \geq Cd_m^2 m^{-2a-3}. \]

**Proof.** By the previous lemma, for all \( m \geq 1 \)

\[ \left| \int_{B_r(0)} Z_m^m(x) d\mu(x) \right|^2 + \left| \int_{B_{2r}(0)} Z_m^m(x) d\mu(x) \right|^2 \]

\[ \geq \left| c(a, b) d_m \Gamma(a + 1) M^{-a-1} \left\{ \left( \sin \left( \frac{r}{2} \right) \right)^{a+1} \left( \cos \left( \frac{r}{2} \right) \right)^{b+1} \left( \frac{r}{\sin r} \right)^{1/2} J_{a+1}(Mr) \right\} ^2 \right. \]

\[ + \left. \left( \sin r \right)^{a+1} \left( \cos r \right)^{b+1} \left( \frac{2r}{\sin 2r} \right)^{1/2} J_{a+1}(M2r) \right\} - d_m^2 O(m^{-4-2a}) \]

\[ \geq Cd_m^2 m^{-2a-2} \left\{ \left| J_{a+1}(Mr) \right|^2 + \left| J_{a+1}(2Mr) \right|^2 \right\} - d_m^2 O(m^{-4-2a}). \]
By the asymptotic expansion of the Bessel functions (see e.g. [15, Lemma 3.11, Chapter 4]),

\[ J_{a+1}(w) = \sqrt{\frac{2}{\pi w}} \cos \left( w + (a + 1) \frac{\pi}{2} - \frac{\pi}{4} \right) + O(w^{-\frac{3}{2}}), \]

we have

\[ |J_{a+1}(Mr)|^2 + |J_{a+1}(2Mr)|^2 \]

\[ = \frac{2}{\pi Mr} \cos^2 \left( Mr - (a + 1) \frac{\pi}{2} - \frac{\pi}{4} \right) + \frac{2}{2\pi Mr} \cos^2 \left( 2Mr - (a + 1) \frac{\pi}{2} - \frac{\pi}{4} \right) + O(m^{-2}) \]

\[ \geq \frac{C}{m} \left( \cos^2 \left( Mr - (a + 1) \frac{\pi}{2} - \frac{\pi}{4} \right) + \cos^2 \left( 2Mr - (a + 1) \frac{\pi}{2} - \frac{\pi}{4} \right) \right) + O(m^{-2}). \]

Calling

\[ \omega = Mr - (a + 1) \frac{\pi}{2} - \frac{\pi}{4}, \]

we have that

\[ \cos^2 \left( Mr - (a + 1) \frac{\pi}{2} - \frac{\pi}{4} \right) + \cos^2 \left( 2Mr - (a + 1) \frac{\pi}{2} - \frac{\pi}{4} \right) = \cos^2(\omega) + \cos^2 \left( 2\omega + (a + 1) \frac{\pi}{2} + \frac{\pi}{4} \right). \]

We claim that this expression never vanishes. Assume the contrary, then \( \omega = \frac{\pi}{2} + k\pi \) for some integer \( k \), and \( 2\omega + (a + 1) \frac{\pi}{2} + \frac{\pi}{4} = \frac{\pi}{2} + h\pi \) for some integer \( h \). This implies

\[ \pi + 2k\pi + (a + 1) \frac{\pi}{2} + \frac{\pi}{4} = \frac{\pi}{2} + h\pi. \]

Recalling that \( a = \frac{d-2}{2} \), this identity implies that \( \frac{d-1}{4} \) is an integer. This contradicts \( d \not\equiv 1 \pmod{4} \). Therefore for all \( \omega \in \mathbb{R} \),

\[ \cos^2(\omega) + \cos^2 \left( 2\omega + (a + 1) \frac{\pi}{2} + \frac{\pi}{4} \right) \geq c, \]

so that for large \( m \)

\[ |J_{a+1}(Mr)|^2 + |J_{a+1}(2Mr)|^2 \geq \frac{C}{m}. \]

Finally,

\[ \left| \int_{B_r(\omega)} Z_{\omega}^m(x) d\mu(x) \right|^2 + \left| \int_{B_{2r}(\omega)} Z_{\omega}^m(x) d\mu(x) \right|^2 \geq Cm^{-2a-3}d_m^2 + d_m^2O(m^{-2a-4}) \]

\[ \geq Cd_m^2m^{-2a-3}. \]

\[ \square \]

We can now prove Theorem 1.
Proof of Theorem 1. Let $m_0$ as in Lemma 11 and let $X \geq m_0$. Then by Lemma 7, Proposition 8 and Lemma 11, we have
\[
\|D_r\|_2^2 + \|D_{2r}\|_2^2 = \sum_{m=1}^{+\infty} d_m^{-2} \left( \left| \int_{B_r(o)} Z_m^m(x) d\mu(x) \right|^2 + \left| \int_{B_{2r}(o)} Z_m^m(x) d\mu(x) \right|^2 \right) \sum_{j=1}^{d_m} \sum_{j=1}^{N} a_j^2 Y_m^r(x_j) \geq \min_{m_0 \leq m \leq X} \left( d_m^{-2} \left( \left| \int_{B_r(o)} Z_m^m(x) d\mu(x) \right|^2 + \left| \int_{B_{2r}(o)} Z_m^m(x) d\mu(x) \right|^2 \right) \right) \times \sum_{m=m_0}^{X} d_m \sum_{\ell=1}^{N} \sum_{j=1}^{N} a_j^2 Y_m^r(x_j) \geq C \left( \min_{m_0 \leq m \leq X} m^{-2a-3} \right) \left( C_1 X^d \sum_{j=1}^{N} a_j^2 - C_0 \right) \geq CX^{-2a-3} \left( C_1 X^d \sum_{j=1}^{N} a_j^2 - C_0 \right).
\]
Applying Cauchy-Schwarz inequality to $\sum_{j=1}^{N} a_j^2 = 1$ gives
\[
\sum_{j=1}^{N} a_j^2 \geq \frac{1}{N}.
\]
Let $N \geq N_0 = m_0^d C_1/(2C_0)$. Then, setting $X = [(2C_0 C_1^{-1} N)^{1/d}] + 1$, we have $X \geq m_0$, $C_1 X^d \sum_{j=1}^{N} a_j^2 - C_0 \geq C_0$ and
\[
\|D_r\|_2^2 + \|D_{2r}\|_2^2 \geq C N^{-1-\frac{1}{d}}. \tag{12}
\]
Let now $N < N_0$ and let us consider the points and weights
\[
\tilde{x}_j = \begin{cases} x_j & 1 \leq j \leq N-1, \\ x_N & N \leq j \leq N_0, \end{cases} \quad \tilde{a}_j = \begin{cases} a_j & 1 \leq j \leq N-1, \\ \frac{a_N}{N_0 - N + 1} & N \leq j \leq N_0.
\end{cases}
\]
Since the discrepancy $\tilde{D}_r$ of the points $\{\tilde{x}_j\}_{j=1}^{N_0}$ and weights $\{\tilde{a}_j\}_{j=1}^{N_0}$ coincides with the discrepancy $D_r$ of the points $\{x_j\}_{j=1}^{N}$ and weights $\{a_j\}_{j=1}^{N}$, applying (12) to $\tilde{D}_r$ gives
\[
\|D_r\|_2^2 + \|D_{2r}\|_2^2 \geq C N_0^{-1-\frac{1}{d}} = C \left( \frac{N}{N_0} \right)^{1+\frac{1}{d}} \left( \frac{N}{N_0} \right)^{-1-\frac{1}{d}} \geq C \left( \frac{1}{N_0} \right)^{1+\frac{1}{d}} N^{-1-\frac{1}{d}}
\]
also when $1 \leq N < N_0$. \hfill \Box

To prove Theorem 2 we need the following result of Frenzen and Wong (see [7, Corollary 2]) on the zeroes of the Jacobi polynomials.
Theorem 12 (Frenzen-Wong). Let $a \geq -\frac{3}{2}$, $a + b \geq -3$, and let $0 < \theta_{m-1,1} < \theta_{m-1,2} < \ldots < \theta_{m-1,m-1} < \pi$ be the zeros of $P_{m-1}^{a+1,b+1}(\cos \theta)$. Then, as $m \to +\infty$, we have
\[
\theta_{m-1,\ell} = \frac{j_{a+1,\ell}}{M} + \frac{1}{M^2} \left\{ \left( (a + 1)^2 - \frac{1}{4} \right) \frac{1 - t \cot t}{2t} - \frac{(a + 1)^2 - (b + 1)^2}{4} \tan \frac{t}{2} \right\} + \ell^2 O \left( \frac{1}{m^2} \right)
\]
where $j_{a+1,\ell}$ is the $\ell$-th positive zero of the Bessel function $J_{a+1}(x)$ and $t = j_{a+1,\ell}/M$. The $O$-term is uniformly bounded for all values of $\ell = 1, \ldots, [\gamma m]$, where $\gamma \in (0,1)$.

Lemma 13. Let $0 < \gamma_1 < \gamma_2 < 1$. For every $m \geq 1$ and every $[\gamma_1 m] \leq \ell \leq [\gamma_2 m]$ we have
\[
\theta_{m-1,\ell} = \frac{\ell \pi + a \frac{\pi}{2} + \frac{\pi}{4}}{M} + O \left( \frac{1}{m^2} \right).
\]
Proof. This follows directly from McMahon’s expansion (see e.g. (1.5) in [6])
\[
j_{a+1,\ell} = \ell \pi + a \frac{\pi}{2} + \frac{\pi}{4} + O \left( \frac{1}{\ell} \right)
\]
and Theorem 12. \hfill \qed

Lemma 14. There exists $m_0 > 0$ such that for every $\delta > 0$ and for almost every $r \in (0, \pi)$ there exists a constant $C > 0$ such that for $m \geq m_0$,
\[
\left| \sin^{2a+2} \left( \frac{r}{2} \right) \cos^{2b+2} \left( \frac{r}{2} \right) P_{m-1}^{a+1,b+1}(\cos r) \right| \geq C m^{-\frac{a+b}{2} - \delta}.
\]
Proof. Let $\theta_{\ell} = \theta_{m-1,\ell}$, for $\ell = 1, \ldots, m-1$, be the positive zeros of $P_{m-1}^{a+1,b+1}(\cos \theta)$, and let
\[
P_{m-1}^{a+1,b+1}(r) = \sin^{2a+2} \left( \frac{r}{2} \right) \cos^{2b+2} \left( \frac{r}{2} \right) P_{m-1}^{a+1,b+1}(\cos r).
\]
Let $\epsilon > 0$. Let $\gamma_1$ and $\gamma_2$ be such that for $m$ sufficiently large
\[
\frac{\epsilon}{2} < \frac{[\gamma_1 m] \pi + a \frac{\pi}{2} + \frac{\pi}{4}}{M} < \epsilon
\]
and
\[
\pi - \epsilon < \frac{[\gamma_2 m] \pi + a \frac{\pi}{2} + \frac{\pi}{4}}{M} < \pi - \frac{\epsilon}{2}.
\]
Then, if $\eta \in (0,1)$,
\[
\int_{\epsilon}^{\pi - \epsilon} \left| P_{m-1}^{a+1,b+1}(r) \right|^{-1+\eta} dr \leq \sum_{\ell = [\gamma_1 m]}^{[\gamma_2 m]-1} \int_{\theta_{\ell}}^{\theta_{\ell+1}} \left| P_{m-1}^{a+1,b+1}(r) \right|^{-1+\eta} dr.
\]
By (9) we have
\[
\frac{d}{dr} P_{m-1}^{a+1,b+1}(r) = \frac{1}{2} r P_{m}^{a,b}(r) \sin(r).
\]
Hence, if $\varepsilon/2 \leq r \leq \pi - \varepsilon/2$, then by (10)

$$
\frac{d}{dr} P_{m-1}^{a+1, b+1}(r) = m \sin^{2a+1} \left( \frac{r}{2} \right) \cos^{2b+1} \left( \frac{r}{2} \right) P_m^{a, b}(\cos r)
$$

$$
= m \sin^{a+1} \left( \frac{r}{2} \right) \cos^{b+1} \left( \frac{r}{2} \right) M^{-a} \frac{\Gamma(m + a + 1)}{m!} \left( \frac{r}{\sin r} \right)^{\frac{1}{2}} J_a(Mr)
$$

$$
+ \sin^{a+1} \left( \frac{r}{2} \right) \cos^{b+1} \left( \frac{r}{2} \right) \tau \frac{1}{2} O(m^{-\frac{1}{2}}).
$$

Therefore

$$
\left| \frac{d}{dr} P_{m-1}^{a+1, b+1}(r) \right| \geq cm \left| J_a(Mr) \right| - O(m^{-\frac{1}{2}}). \tag{13}
$$

Let $\tau > 0$ and let $r \in \left( \theta - \frac{\tau}{M}, \theta + \frac{\tau}{M} \right)$. Then by Lemma 13

$$
Mr - a \frac{\pi}{2} - \frac{\pi}{4} \in \left( \ell \pi + O \left( \frac{1}{M} \right) - \tau, \ell \pi + O \left( \frac{1}{M} \right) + \tau \right),
$$

so that

$$
\left| \cos \left( Mr - a \frac{\pi}{2} - \frac{\pi}{4} \right) \right| \geq c.
$$

By the asymptotic expansion of the Bessel function (11) we therefore obtain

$$
\left| J_a(Mr) \right| \geq \sqrt{\frac{2}{\pi Mr}} \left| \cos \left( Mr - a \frac{\pi}{2} - \frac{\pi}{4} \right) \right| - O(m^{-\frac{1}{2}}) \geq cm^{-\frac{1}{2}},
$$

and by (13)

$$
\left| \frac{d}{dr} P_{m-1}^{a+1, b+1}(r) \right| \geq cm^{\frac{1}{2}}.
$$

Thus,

$$
\int_{\theta}^{\theta + \frac{\tau}{M}} \left| P_{m-1}^{a+1, b+1}(r) \right|^{-1+\eta} dr
$$

$$
= \int_{\theta}^{\theta + \frac{\tau}{M}} \left| P_{m-1}^{a+1, b+1}(r) \right|^{-1+\eta} dr + \int_{\theta + \frac{\tau}{M}}^{\theta + \frac{\tau}{M}} \left| P_{m-1}^{a+1, b+1}(r) \right|^{-1+\eta} dr
$$

$$
+ \int_{\theta + \frac{\tau}{M}}^{\theta + \frac{\tau}{M}} \left| P_{m-1}^{a+1, b+1}(r) \right|^{-1+\eta} dr
$$

$$
\leq \int_{\theta}^{\theta + \frac{\tau}{M}} \left( cm^{1/2} (r - \theta) \right)^{-1+\eta} dr + \int_{\theta + \frac{\tau}{M}}^{\theta + \frac{\tau}{M}} \left( cm^{1/2} \left( \frac{\tau}{M} \right) \right)^{-1+\eta} dr
$$

$$
+ \int_{\theta + \frac{\tau}{M}}^{\theta + \frac{\tau}{M}} \left( cm^{1/2} (\theta + 1 - r) \right)^{-1+\eta} dr
$$

$$
\leq cm^{-\frac{1}{2} \cdot \frac{1}{2}}.
$$
Therefore
\[
\int_{\varepsilon}^{\pi - \varepsilon} \left| P_{m-1}^{a+1,b+1}(r) \right|^{-1+\eta} dr \leq \sum_{\ell=\lfloor \gamma_1 m \rfloor}^{\lfloor \gamma_2 m \rfloor - 1} \int_{\theta_{\ell+1}}^{\theta_{\ell}} \left| P_{m-1}^{a+1,b+1}(r) \right|^{-1+\eta} dr
\]
\[
\leq cm \cdot m^{-\frac{1}{2} - \frac{\eta}{2}} = cm^{\frac{1}{2} - \frac{\eta}{2}}.
\]

Finally
\[
\int_{\varepsilon}^{\pi - \varepsilon} \left( \sum_{m=1}^{+\infty} m^{-\sigma} \left| P_{m-1}^{a+1,b+1}(r) \right| \right)^{1+\eta} dr = \sum_{m=1}^{+\infty} \int_{\varepsilon}^{\pi - \varepsilon} m^{-\sigma} \left| P_{m-1}^{a+1,b+1}(r) \right|^{1+\eta} dr
\]
\[
\leq c \sum_{m=1}^{+\infty} m^{\frac{1}{2} - \frac{\sigma - \eta}{2}}.
\]

If \( \sigma > \frac{3}{2} - \frac{\eta}{2} \)
\[
\sum_{m=1}^{+\infty} m^{-\sigma} \left| P_{m-1}^{a+1,b+1}(r) \right|^{1+\eta}
\]
converges for almost every \( r \in (\varepsilon, \pi - \varepsilon) \) so that for almost every \( r \in (\varepsilon, \pi - \varepsilon) \) we have
\[
cm^{-\frac{\sigma - \eta}{2}} \leq \left| P_{m-1}^{a+1,b+1}(r) \right|.
\]

We are ready to prove Theorem 2.

**Proof of Theorem 2.** Let \( m_0 \) be as in Lemma 14. By Lemma 7, Proposition 8 and Lemma 9, for every \( X \geq m_0 \) we have
\[
\int_{\mathcal{M}} |D_r(x)|^2 d\mu(x)
\]
\[
\geq \sum_{m=m_0}^{X} \sum_{\ell=1}^{d_m} \sum_{j=1}^{N} \left| a_j Y_m^\ell(x_j) \right|^2 \min_{m \leq X-m} d_m^{-2} \left| \int_{B_r(o)} Z_m^a(y) d\mu(y) \right|^2
\]
\[
\geq \left( C_1 X^d \sum_{j=1}^{N} a_j^2 - C_0 \right)
\]
\[
\times \min_{m_0 \leq m \leq X} \left| \frac{c(a,b)}{mF_m^{a,b}(1)} P_{m-1}^{a+1,b+1} \left( \cos \left( \frac{r}{2} \right) \right) \left( \sin \left( \frac{r}{2} \right) \right)^{2a+2} \left( \cos \left( \frac{r}{2} \right) \right)^{2b+2} \right|^2.
\]

Using Lemma 14, for almost every \( r \in (0, \pi) \) and for \( m \geq m_0 \) we obtain
\[
\min_{m_0 \leq m \leq X} \left| \frac{c(a,b)}{mF_m^{a,b}(1)} P_{m-1}^{a+1,b+1} \left( \cos \left( \frac{r}{2} \right) \right) \left( \sin \left( \frac{r}{2} \right) \right)^{2a+2} \left( \cos \left( \frac{r}{2} \right) \right)^{2b+2} \right|^2
\]
IRREGULARITIES OF DISTRIBUTION ON TWO POINT HOMOGENEOUS SPACES

\[ \geq c \min_{m_0 \leq m \leq X} \left| \frac{1}{m P_m^b(1)} m^{-\frac{3}{2}-\delta} \right|^2 \geq c \min_{m_0 \leq m \leq X} \left| \frac{1}{m^{a+1}} m^{-\frac{3}{2}-\delta} \right|^2 \geq c X^{-3-2\delta-d}. \]

Hence

\[ \int_M |D_r(x)|^2 d\mu(x) \geq c \left( C_1 X^d \sum_{j=1}^N a_j^2 - C_0 \right) X^{-3-2\delta-d} \geq c \left( C_1 X^d \frac{1}{N} - C_0 \right) X^{-3-2\delta-d}. \]

Let \( N \geq N_0 = m_0^d C_1/(2C_0) \). Then, setting \( X = \left( (2C_0 C_1^{-1} N)^{1/d} \right) + 1 \), we have \( X \geq m_0, C_1 X^d N^{-1} - C_0 \geq C_0 \) and

\[ \int_M |D_r(x)|^2 d\mu(x) \geq c N^{-1-\frac{3}{2}-\frac{d}{2d}} \]

for all \( N \geq N_0 \). The same argument used in the proof of Theorem 1 gives the result for every \( N \geq 1 \). \( \square \)

4. Cubature formulas

For a given \( N \geq 1 \), let \( \{a_j\}_{j=1}^N \) be a set of weights and \( \{x_j\}_{j=1}^N \) be a set of points in \( M \). We say that \( \left( \{a_j\}_{j=1}^N, \{x_j\}_{j=1}^N \right) \) is a cubature of strength \( \kappa N^{1/d} \) on \( M \) if

\[ \int_M P(x) d\mu(x) = \sum_{j=1}^N a_j P(x_j) \]

for every

\[ P \in \bigoplus_{0 \leq m \leq X} H_m. \]

In the following theorem we will show that under suitable assumptions a cubature of strength \( \kappa N^{1/d} \) has optimal discrepancy. This was already observed by M. Skriganov in [14, Corollary 2.1], although in his result the discrepancy contains an integration in the radius \( r \) of the balls with respect to an absolutely continuous measure in \([0, \pi]\). The same technique actually gives a stronger estimate, which is uniform in the radius \( r \).

**Theorem 15.** Let \( A, B, \varepsilon \) and \( \kappa \) be positive constants. There exists \( c > 0 \) such that if \( N \geq 1 \), and \( \left( \{a_j\}_{j=1}^N, \{x_j\}_{j=1}^N \right) \) gives a cubature of strength \( \kappa N^{1/d} \) satisfying

\[ 0 \leq a_j \leq \frac{A}{N}, \]

and for every \( x \in M \)

\[ \# \left\{ j : d(x_j, x) \leq N^{-\frac{3}{2}} \right\} \leq B, \]

for all \( N \geq N_0 = m_0^d C_1/(2C_0) \). Then, setting \( X = \left( (2C_0 C_1^{-1} N)^{1/d} \right) + 1 \), we have \( X \geq m_0, C_1 X^d N^{-1} - C_0 \geq C_0 \) and

\[ \int_M |D_r(x)|^2 d\mu(x) \geq c N^{-1-\frac{3}{2}-\frac{d}{2d}} \]

for all \( N \geq N_0 \). The same argument used in the proof of Theorem 1 gives the result for every \( N \geq 1 \). \( \square \)

4. Cubature formulas

For a given \( N \geq 1 \), let \( \{a_j\}_{j=1}^N \) be a set of weights and \( \{x_j\}_{j=1}^N \) be a set of points in \( M \). We say that \( \left( \{a_j\}_{j=1}^N, \{x_j\}_{j=1}^N \right) \) is a cubature of strength \( \kappa N^{1/d} \) on \( M \) if

\[ \int_M P(x) d\mu(x) = \sum_{j=1}^N a_j P(x_j) \]

for every

\[ P \in \bigoplus_{0 \leq m \leq X} H_m. \]

In the following theorem we will show that under suitable assumptions a cubature of strength \( \kappa N^{1/d} \) has optimal discrepancy. This was already observed by M. Skriganov in [14, Corollary 2.1], although in his result the discrepancy contains an integration in the radius \( r \) of the balls with respect to an absolutely continuous measure in \([0, \pi]\). The same technique actually gives a stronger estimate, which is uniform in the radius \( r \).

**Theorem 15.** Let \( A, B, \varepsilon \) and \( \kappa \) be positive constants. There exists \( c > 0 \) such that if \( N \geq 1 \), and \( \left( \{a_j\}_{j=1}^N, \{x_j\}_{j=1}^N \right) \) gives a cubature of strength \( \kappa N^{1/d} \) satisfying

\[ 0 \leq a_j \leq \frac{A}{N}, \]

and for every \( x \in M \)

\[ \# \left\{ j : d(x_j, x) \leq N^{-\frac{3}{2}} \right\} \leq B, \]

for all \( N \geq N_0 = m_0^d C_1/(2C_0) \). Then, setting \( X = \left( (2C_0 C_1^{-1} N)^{1/d} \right) + 1 \), we have \( X \geq m_0, C_1 X^d N^{-1} - C_0 \geq C_0 \) and

\[ \int_M |D_r(x)|^2 d\mu(x) \geq c N^{-1-\frac{3}{2}-\frac{d}{2d}} \]

for all \( N \geq N_0 \). The same argument used in the proof of Theorem 1 gives the result for every \( N \geq 1 \). \( \square \)
then, for every \( r \in [0, \pi - \varepsilon] \) we have
\[
\int_{\mathcal{M}} |D_r(x)|^2 d\mu(x) \leq cN^{-1 - \frac{d}{2}}. \tag{16}
\]

Observe that cubatures as required by the above theorem do exist. Indeed, in [5] it is proved that for every \( A \geq 1 \) there exists a constant \( C(A, \mathcal{M}) > 0 \), depending only on \( A \) and \( \mathcal{M} \), such that if \( N \geq C(A, \mathcal{M})X^d \) and the weights \( \{a_j\}_{j=1}^N \) satisfy the conditions (14) and
\[
\sum_{j=1}^N a_j = 1,
\]
then there is a choice of points \( \{x_j\}_{j=1}^N \) such that \( \{a_j\}_{j=1}^N, \{x_j\}_{j=1}^N \) is a cubature of strength \( X \). The construction starts with a well separated set of points \( \{y_j\}_{j=1}^N \) and ends with a new set of points \( \{x_j\}_{j=1}^N \) in such a way that each \( x_j \) has distance at most \( c_1(A, \mathcal{M})N^{-1/d} \) from \( y_j \). This implies that the number of points \( x_j \) contained in any ball of radius \( N^{-1/d} \) is uniformly bounded so that (15) is satisfied with a constant \( B \) depending on \( M \) and \( A \). It suffices to apply the above construction with \( X = \kappa N^{1/d} \) (\( \kappa \leq C(A, \mathcal{M})^{-1/d} \)).

**Proof.** First of all we observe that it is enough to prove (16) for \( N \) sufficiently large. Let \( N_0 \) be a positive constant to be chosen later and assume \( N \geq N_0 \). Since \( \{a_j\}_{j=1}^N, \{x_j\}_{j=1}^N \) gives a cubature of strength \( \kappa N^{1/d} \), for every \( m \leq \kappa N^{1/d} \) we have
\[
\sum_{j=1}^N a_j Y_m^\ell(x_j) = 0.
\]

Then by Lemma 7 and Lemma 9 we have
\[
\int_{\mathcal{M}} |D(x)|^2 d\mu(x)
= \sum_{m > \kappa N^{1/d}} \sum_{\ell=1}^{d_m} \sum_{j=1}^N a_j Y_m^\ell(x_j) \left| c(a, b) \frac{P_{m-1}^{a+1, b+1} \left( \cos r \right)}{2^{a+2} \left( \cos \left( \frac{r}{2} \right) \right)^{2a+2} \left( \cos \left( \frac{r}{2} \right) \right)^{2b+2}} \right|^2.
\]

We want to show that there exist \( c > 0 \) and \( \zeta > 0 \) such that for every \( r \in [0, \pi - \varepsilon] \), for every \( N \geq 1 \) and for every \( m > \kappa N^{1/d} \)
\[
\left| P_{m-1}^{a+1, b+1} \left( \cos r \right) \left( \sin \left( \frac{r}{2} \right) \right)^{2a+2} \left( \cos \left( \frac{r}{2} \right) \right)^{2b+2} \right|^2 \leq cN \int_0^{\zeta N^{-\frac{1}{d}}} \left| P_{m-1}^{a+1, b+1} \left( \cos u \right) \left( \sin \left( \frac{u}{2} \right) \right)^{2a+2} \left( \cos \left( \frac{u}{2} \right) \right)^{2b+2} \right|^2 du. \tag{17}
\]
By (10) we have the expansion
\[
\left( \sin \frac{r}{2} \right)^{a+\frac{3}{2}} \left( \cos \frac{r}{2} \right)^{b+\frac{3}{2}} I_{m-1}^{a+1,b+1}(\cos r)
= M^{-a-1} \Gamma(m + a + 1) \sqrt{rJ_{a+1}(Mr)} + O(r^{-\frac{a}{2}})
\]
uniformly in \( r \in [0, \pi - \varepsilon] \) and \( m > \kappa N^\frac{1}{2} \). Hence
\[
\left| P_{m-1}^{a+1,b+1}(\cos r) \left( \sin \left( \frac{r}{2} \right) \right)^{2a+2} \left( \cos \left( \frac{r}{2} \right) \right)^{2b+2} \right|^2
\leq P_{m-1}^{a+1,b+1}(\cos r) \left( \sin \left( \frac{r}{2} \right) \right)^{a+\frac{3}{2}} \left( \cos \left( \frac{r}{2} \right) \right)^{b+\frac{3}{2}} \left| \right|^2
\leq M^{-a-1} \Gamma(m + a + 1) \sqrt{rJ_{a+1}(Mr)} \right)^2 + O(rm^{-2})
\leq cm^{-1} + O(rm^{-2}) \leq cm^{-1}.
\]
Let \( 0 < \zeta < \pi N^\frac{1}{2} \). For every \( m > \kappa N^\frac{1}{2} \),
\[
\int_0^{\zeta N^{-\frac{1}{2}}} \left| P_{m-1}^{a+1,b+1}(\cos u) \left( \sin \left( \frac{u}{2} \right) \right)^{2a+2} \left( \cos \left( \frac{u}{2} \right) \right)^{2b+2} \right|^2 du
= \int_0^{\zeta N^{-\frac{1}{2}}} \left| P_{m-1}^{a+1,b+1}(\cos u) \left( \sin \left( \frac{u}{2} \right) \right)^{a+\frac{3}{2}} \left( \cos \left( \frac{u}{2} \right) \right)^{b+\frac{3}{2}} \right|^2 \times \left( \sin \left( \frac{u}{2} \right) \right)^{2a+1} \left( \cos \left( \frac{u}{2} \right) \right)^{2b+1} du
\geq c \int_0^{\zeta N^{-\frac{1}{2}}} \left| \sqrt{u}J_{a+1}(Mu) + O(um^{-\frac{1}{2}}) \right|^2 u^{2a+1} du
\geq c \int_0^{\zeta N^{-\frac{1}{2}}} \left| \sqrt{u}J_{a+1}(Mu) \right|^2 u^{2a+1} du
\geq c \int_0^{\zeta N^{-\frac{1}{2}}} \left| \sqrt{u}J_{a+1}(Mu) \right|^2 u^{2a+1} du - c \int_0^{\zeta N^{-\frac{1}{2}}} \left| \sqrt{u}J_{a+1}(Mu)O(um^{-\frac{1}{2}}) \right| u^{2a+1} du
= A - B.
\]
To estimate \( A \) we use (11). We have
\[
A = c \int_0^{\zeta N^{-\frac{1}{2}}} \left| \sqrt{u}J_{a+1}(Mu) \right|^2 u^{2a+1} du
\]
\[ \begin{align*}
&= c \int_{\zeta N^{-\frac{1}{2}}}^{\zeta N^{-\frac{1}{2}}/2} \left| M^{a/2} \cos \left( Mu - (a + 1) \frac{\pi}{2} - \frac{\pi}{4} \right) + O(m^{-\frac{a}{4}} u^{-1}) \right|^2 u^{2a+1} du \\
&\geq cm^{-1} \int_{\zeta N^{-\frac{1}{2}}}^{\zeta N^{-\frac{1}{2}}/2} \left| \cos \left( Mu - (a + 1) \frac{\pi}{2} - \frac{\pi}{4} \right) \right|^2 u^{-1} du - c \int_{\zeta N^{-\frac{1}{2}}}^{\zeta N^{-\frac{1}{2}}/2} m^{-2} u^{-1} u^{2a+1} du \\
&\geq cm^{-1} (\zeta N^{-\frac{1}{2}})^{d-1} \int_{\zeta N^{-\frac{1}{2}}/2}^{\zeta N^{-\frac{1}{2}}} \left| \cos \left( v - (a + 1) \frac{\pi}{2} - \frac{\pi}{4} \right) \right|^2 dv - cm^{-2} (\zeta N^{-\frac{1}{2}})^{d-1} \\
&\geq cm^{-2} (\zeta N^{-\frac{1}{2}})^{d-1} \int_{M \zeta N^{-\frac{1}{2}}/2}^{M \zeta N^{-\frac{1}{2}}} \left| \cos \left( v - (a + 1) \frac{\pi}{2} - \frac{\pi}{4} \right) \right|^2 dv - cm^{-2} (\zeta N^{-\frac{1}{2}})^{d-1} \\
&\geq cm^{-1} \zeta^{d-1} N^{-1} (\zeta - cm^{-1} N^\frac{1}{2}) \geq cm^{-1} \zeta^{d-1} N^{-1} (\zeta - \kappa^{-1}) \geq cm^{-1} \zeta^{d-1} N^{-1}
\end{align*} \]

for \( \zeta > 2\kappa^{-1} c \) (and therefore for \( N_0 \) large enough). Now we estimate \( B \):

\[ B = c \int_{\zeta N^{-\frac{1}{2}}}^{\zeta N^{-\frac{1}{2}}/2} \left| \sqrt{u} J_{a+1}(Mu)O(u^{-\frac{1}{2}}) \right| u^{2a+1} du \]

\[ \leq c \int_{\zeta N^{-\frac{1}{2}}/2}^{\zeta N^{-\frac{1}{2}}} m^{-2} u^{d} du \leq cm^{-2} (\zeta N^{-\frac{1}{2}})^{d+1} = cm^{-2} \zeta^{d+1} N^{-1-\frac{1}{2}}. \]

It follows that

\[ \int_{0}^{\zeta N^{-\frac{1}{2}}} \left| P_{m-1}^{a,b+1}(\cos u) \left( \sin \left( \frac{u}{2} \right) \right)^{2a+2} \left( \cos \left( \frac{u}{2} \right) \right)^{2b+2} \right|^2 du \]

\[ \geq cm^{-1} \zeta^{d-1} N^{-1} - cm^{-2} \zeta^{d+1} N^{-1-\frac{1}{2}} = c \zeta^{d-1} m^{-1} N^{-1} \left( 1 - cm^{-1} \zeta^2 N^{-\frac{1}{2}} \right) \]

\[ \geq c \zeta^{d-1} m^{-1} N^{-1} \left( 1 - c N^{-\frac{1}{2}} \right) \geq cm^{-1} N^{-1}, \]

and (17) is proved. Thus,

\[ \int_{M} |D_r(x)|^2 d\mu(x) \]

\[ = \sum_{m > \kappa N^\frac{1}{2}} \sum_{\ell = 1}^{d_m} \sum_{j = 1}^{N} a_j Y_{\ell m}(x_j) \left| c(a, b) \frac{P_{m-1}^{a+1,b+1}(\cos r)}{m P_m^{a,b}(1)} \left( \sin \left( \frac{r}{2} \right) \right)^{2a+2} \left( \cos \left( \frac{r}{2} \right) \right)^{2b+2} \right|^2 \]

\[ \leq c N \sum_{m > \kappa N^\frac{1}{2}} \sum_{\ell = 1}^{d_m} \sum_{j = 1}^{N} a_j Y_{\ell m}(x_j) \left| \right|^2 \]
\[ x \int_0^{\zeta N^{-\frac{1}{d}}} \left| c(a, b) \frac{P^{a+1,b+1}_m(x)}{mP^a_b(1)} \left( \sin \left( \frac{u}{2} \right) \right)^{2a+2} \left( \cos \left( \frac{u}{2} \right) \right)^{2b+2} \right|^2 du \]

\[ = cN \int_0^{\zeta N^{-\frac{1}{d}}} |D_u(x)|^2 d\mu(x) du. \]

Since \( u \leq \zeta N^{-1/d} \) using (15) and noticing that by Lemma 4 a ball of radius \( u \) can be covered by \( c_2(d, d_0)/2(\zeta + 1)^d \) balls of radius \( N^{-1/d} \) we have

\[ |D_u(x)| = \left| \sum_{j=1}^{N} a_j \chi_{B_u(x)}(x_j) - \mu(B_u(x)) \right| \]

\[ \leq \frac{A}{N} \sum_{j=1}^{N} \chi_{B_u(x)}(x_j) + \mu(B_u(x)) \]

\[ \leq \frac{A}{N} c_2(d, d_0)(2(\zeta + 1))^d B + \frac{c_2(d, d_0) \zeta^d}{N} \leq \frac{c}{N}, \]

so that

\[ \int_{\mathcal{M}} |D_u(x)|^2 d\mu(x) \leq \frac{C}{N^2}. \]

Finally,

\[ \int_{\mathcal{M}} |D_r(x)|^2 d\mu(x) \leq cN \int_0^{\zeta N^{-\frac{1}{d}}} \int_{\mathcal{M}} |D_u(x)|^2 d\mu(x) du \]

\[ \leq cN \int_0^{\zeta N^{-\frac{1}{d}}} \frac{C}{N^2} du = cN^{-1-\frac{1}{d}}. \]

\[ \Box \]

**References**

[1] J. Beck, W. W. L. Chen, *Irregularities of distribution*, Cambridge University Press 1987.

[2] L. Brandolini, W. W. L. Chen, L. Colzani, G. Gigante, G. Travaglini, *Discrepancy and numerical integration on metric measure spaces*, J. Geom. Anal. 29 (2019), 328–369.

[3] L. Brandolini, L. Colzani, G. Gigante, G. Travaglini, *L^p and weak-L^p estimates for the number of integer points in translated domains*, Math. Proc. Cambridge Philos. Soc. 159 (2015), 471–480.

[4] L. Brandolini, B. Gariboldi, G. Gigante, *On a sharp lemma of Cassels and Montgomery on manifolds*, Math. Ann. 379 (2021), 1807–1834.

[5] M. Ehler, U. Etayo, B. Gariboldi, G. Gigante, T. Peter, *Asymptotically optimal cubature formulas on manifolds for prefixed weights*, J. Approx. Theory 271 (2021).

[6] A. Elbert, *Some recent results on the zeros of Bessel functions and orthogonal polynomials*, Proceedings of the Fifth International Symposium on Orthogonal Polynomials, Special Functions and their Applications (Patras, 1999). J. Comput. Appl. Math. 133 (2001), 65–83.
[7] C. L. Frenzen, R. Wong, A uniform asymptotic expansion of the Jacobi polynomials with error bounds, Canad. J. Math. 37 (1985), 979–1007.

[8] R. Gangolli, Positive definite kernels on homogeneous spaces and certain stochastic processes related to Lévy’s Brownian motion of several parameters, Ann. Inst. H. Poincaré Sect. B (N.S.) 3 (1967), 121–226.

[9] B. Gariboldi, G. Gigante, Optimal asymptotic bounds for designs on manifolds, Anal. PDE 14 (2021), 1701–1724.

[10] S. Helgason, Groups and Geometric Analysis, AMS 2000.

[11] S. V. Konyagin, M. M. Skriganov, A. V. Sobolev, On a lattice point problem arising in the spectral analysis of periodic operators, Mathematika 50 (2003), 87–98.

[12] L. Parnovski, N. Sidorova, Critical dimensions for counting lattice points in Euclidean annuli, Math. Model. Nat. Phenom. 5 (2010), 293–316.

[13] L. Parnovski, A. V. Sobolev, On the Bethe-Sommerfeld conjecture for the polyharmonic operator, Duke Math. J. 107 (2001), 209–238.

[14] M. M. Skriganov, Point distributions in two-point homogeneous spaces, Mathematika 65 (2019), 557–587.

[15] E. M. Stein, G. Weiss, Introduction to Fourier analysis on Euclidean spaces, Princeton University Press, 1971.

[16] G. Szegö, Orthogonal polynomials, American Mathematical Society, Providence, R.I., 1975.

[17] H. C. Wang, Two-point homogeneous spaces, Ann. of Math 55 (1952), 177–191.

[18] G. N. Watson, A treatise on the theory of Bessel functions, Cambridge University Press, Cambridge, 1944.

[19] J. A. Wolf, Spaces of Constant Curvature, University of California Berkley, 1972.

(L. Brandolini) Dipartimento di Ingegneria Gestionale, dell’Informazione e della Produzione, Università degli Studi di Bergamo, Viale Marconi 5, Dalmine BG, Italy
Email address: luca.brandolini@unibg.it

(B. Gariboldi) Dipartimento di Matematica, Università degli Studi di Milano Bicocca, via Roberto Cozzi 55, Milano, Italy
Email address: biancamaria.gariboldi@unibg.it

(G. Gigante) Dipartimento di Ingegneria Gestionale, dell’Informazione e della Produzione, Università degli Studi di Bergamo, Viale Marconi 5, Dalmine BG, Italy
Email address: giacomo.gigante@unibg.it