L∞ COHOMOLOGY IS INTERSECTION COHOMOLOGY

GUILLAUME VALETTE

Abstract. Let X be any subanalytic compact pseudomanifold. We show a De Rham theorem for L∞ forms. We prove that the cohomology of L∞ forms is isomorphic to intersection cohomology in the maximal perversity.

0. Introduction

In [C1, C2], J. Cheeger develops a Hodge theory for L2 forms on pseudopseudomanifold with conical singularities. His idea to consider L2 forms in order to control the behavior of the forms nearby the singularities turned out to be very efficient for it makes it possible to develop a Hodge theory on noncompact Riemannian manifolds. It also provides a means to investigate singular set via differential geometry. He computed the L2 cohomology groups for any pseudomanifold with metrically conical singularities. Later, he proved together with M. Goresky and R. MacPherson [CGM] that the L2 cohomology is isomorphic to intersection homology in the middle perversity. Intersection homology was introduced by M. Goresky and R. MacPherson and satisfies Poincaré duality for a large class of singularities.

Lp cohomology turned out to be related to intersection homology as well. Let us mention some of the many related works which then appeared. In [Y], Y. Youssin computes the Lp cohomology groups on spaces with conical horns. He shows that the Lp cohomology groups are isomorphic to intersection cohomology groups in the so-called Lp perversity 1 < p < ∞. He also describes quite explicitly the case of f-horns which is more complicated and no longer related to intersection homology. The so called f-horns are cones endowed with a metric decreasing at a rate proportional to a function f of the distance to the origin. L. Saper studies in [S2] the L2 cohomology for sets with isolated singularities with a distinguished Kähler metric.

In [HP], the authors focus on normal algebraic complex surfaces (not necessarily metrically conical). They also show that the L2 cohomology is dual to intersection homology (see also [S1]).

In [BGM], the authors show, on a simplicial complex, an explicit isomorphism between the Lp shadow forms and intersection homology. The shadow forms are smooth forms constructed by the authors in a combinatorial way, like Whitney forms [Wh].

It is striking that, all the above mentioned de Rham theorems include an assumption on the metric type of the singularities or are devoted to low dimensions on which the metric type of singularities is easier to handle. In this paper, we focus on L∞ forms, i.e. forms having a bounded size. We prove a de Rham theorem for any compact subanalytic pseudomanifold, establishing an isomorphism between L∞ cohomology and intersection
homology in the maximal perversity. We also prove that the isomorphism is provided by integration on subanalytic singular chains. The class of subanalytic pseudomanifolds covers a large class of subsets such as all the complex analytic projective varieties. Furthermore, the theory presented in this paper could go over singular subanalytic subset which are not pseudomanifolds and we could adapt the statement to arbitrary subanalytic subsets.

This theorem, which applies to any subanalytic pseudomanifold, is proved by looking in details at the metric structure of singular sets. The sharp description of the metric type of singularities obtained \[V_1, V_2\] will make it possible to work without any assumption on the metric type of the singularities.

As a consequence, we immediately see the the \(L^\infty\) groups are finitely generated. The purpose is also, as in the case of \(L^2\) cohomology, to find a category of forms for which we can carry out a Hodge theory for any compact subanalytic singular variety. Performing analysis of differential geometry on singular spaces is much more challenging that on smooth manifolds. The reason is that a smooth manifold locally looks (metrically) like a cone.

1. Definitions and the main result

1.1. \(L^\infty\)-cohomology groups. Before stating the main result, we need to define the \(L^\infty\) cohomology groups.

**Definition 1.1.1.** Let \(Y\) be a \(C^\infty\) submanifold of \(\mathbb{R}^n\). As \(Y\) is embedded in \(\mathbb{R}^n\), it inherits a natural structure of Riemannian manifold. We say that a \(j\)-differential form \(\omega\) is \(L^\infty\) if there exists a constant \(C\) such that for any \(x \in Y\):

\[|\omega(x)| \leq C,\]

where \(|\omega(x)|\) denotes the norm of \(\omega(x)\) (as a linear mapping).

We denote by \(\Omega^j_{\infty}(Y)\) the real vector space constituted by all the differential \(j\)-forms \(\omega\) such that \(\omega\) and \(d\omega\) are both \(L^\infty\).

Given \(\omega \in \Omega^j_{\infty}(Y)\), we set \(|\omega|_\infty := \sup_{x \in Y} |\omega(x)|\).

The cohomology groups of this cochain complex are called the \(L^\infty\) cohomology groups of \(Y\) and will be denoted by \(H^\bullet_{\infty}(Y)\). We will denote by \(d\) the differential operator.

1.2. Intersection homology in the maximal perversity and the main theorem. We recall the definition of intersection homology. As we will be interested in the only case of the maximal perversity, we specify this particular case in the definition.

Given a set \(X\), we denote by \(X_{\text{reg}}\) the set of points in \(X\) at which the set \(X\) is locally a \(C^\infty\) manifold, and we will write \(X_{\text{sing}}\) for the complement of \(X_{\text{reg}}\) in \(X\).

**Subanalytic singular simplices** are subanalytic continuous maps \(c : T_j \rightarrow X\), \(T_j\) being the \(j\)-simplex spanned by \(0, e_1, \ldots, e_j\) where \(e_1, \ldots, e_j\) is the canonical basis of \(\mathbb{R}^j\). Given a subanalytic set \(X \subset \mathbb{R}^n\) we denote by \(C_\bullet(X)\) the resulting chain complex (with coefficients in \(\mathbb{R}\)). We will write \(|c|\) for the support of a chain \(c\) and by \(\partial c\) the boundary of \(c\).
Definition 1.2.1. An \( l \)-dimensional pseudomanifold is a subanalytic set \( X \subset \mathbb{R}^n \) such that \( X_{sing} \) is of dimension \((l - 2)\) and \( X_{reg} \) is a manifold of dimension \( l \).

A subspace \( Y \subset X \) is called \((t; i)\)-allowable if \( \dim Y \cap X_{sing} < i - 1 \). Define \( I^*C_i(x) \) as the subgroup of \( C_i(X) \) consisting of those chains \( \xi \) such that \( |\xi| \) is \((t, i)\)-allowable and \( |\partial \xi| \) is \((t, i - 1)\)-allowable.

The \( j \)-th intersection cohomology group of maximal perversity, denoted \( I^jH^j(X) \), is the \( j \)-th cohomology group of the cochain complex \( I^*C^*(X) = Hom(I^*C_*(X); \mathbb{R}) \).

In general, the intersection homology groups depend on a perversity \([GM]\). The above definition corresponds to the case of the maximal perversity \( t = (0, 1, \ldots, l - 2) \).

In this paper, we prove:

Theorem 1.2.2. Let \( X \) be a compact subanalytic pseudomanifold. Then, for any \( j \):
\[
H^j_{\infty}(X_{reg}) \simeq I^jH^j(X).
\]

This theorem is proved in section 4. This requires to investigate in details the metric type of subanalytic singular sets. This is accomplished in section 2.

We then briefly study the normalization of pseudomanifolds in section 3.

We will also show that, if \( X_{reg} \) is orientable, the isomorphism is given by integration on simplices (section 4.3). Simplices are singular and lie in \( X \) (whereas forms are only defined on \( X_{reg} \)) but integration is well defined and gives rise to a cochain map if the simplices are subanalytic (see 4.3 for details).

Notations and conventions. We denote by \( B_n(x; \varepsilon) \) the ball of radius \( \varepsilon \) centered at \( x \) while \( S_n(x; \varepsilon) \) will stand for the sphere and \(|\cdot|\) will denote the Euclidean norm.

Let \( R \) be a real closed field. By Lipschitz function, we will mean a function \( f : A \rightarrow R \) satisfying for any \( x \) and \( x' \) in \( A \subset \mathbb{R}^n \):
\[
|f(x) - f(x')| \leq N|x - x'|,
\]
for some integer \( N \). It is important to notice that we require the constant to be an integer for, although we are interested in the case of the field of real numbers, \( R \) will not always be archimedean. A map \( h : A \rightarrow \mathbb{R}^n \) is Lipschitz if all its components are; a homeomorphism \( h \) is bi-Lipschitz if \( h \) and \( h^{-1} \) are Lipschitz.

Given two functions \( f, g : X \rightarrow R \), we write \( f \sim g \) (and say that \( f \) is equivalent to \( g \)) if there exists a positive integer \( C \) such that \( \frac{f}{C} \leq g \leq Cf \).

Given a function \( \xi : A \rightarrow \mathbb{R} \), we denote by \( \Gamma_\xi \) its graph and by \( \xi_{|B} \) its restriction to a subset \( B \) of \( A \). Given two functions \( \zeta \) and \( \xi \) on a set \( A \subset \mathbb{R}^n \) with \( \xi \leq \zeta \) we define the closed interval as the set:
\[
[\xi; \zeta] := \{(x; y) \in A \times \mathbb{R} : \xi(x) \leq y \leq \zeta(x)\}.
\]
The open and semi-open intervals are then defined in analogously.

All the sets and mappings considered in this paper will be assumed to be subanalytic (if not otherwise specified), except the differential forms which will be either continuous or \( C^\infty \). “Subanalytic” will always mean globally subanalytic (i.e. which remains subanalytic after a compactification of \( \mathbb{R}^n \)). The letter \( C \) will sometimes stand for various positive constants, when no confusion may arise.
2. Lipschitz retractions

The results of this section will be very important to compute the $L^\infty$ cohomology groups later on. Given a germ of subanalytic set, we shall construct a Lipschitz strong deformation retraction of a neighborhood in such a way that the derivative tends to zero as $t$ goes to 0 (Theorem 2.4.1, by deformation retraction we assume $r_1 = Id$). This requires to investigate in details the metric structure of subanalytic sets. The results of this section, especially Theorem 2.4.1, can have other applications (see [SV]).

We start by recalling some results of [V1, V2].

Given $n > 1$ and a positive constant $M$ we set:

$$C_n(M) := \{ (x_1; x') \in \mathbb{R} \times \mathbb{R}^{n-1} : 0 \leq |x'| \leq M x_1 \}.$$  

For $n = 1$, we just set $C_1 = \mathbb{R}$.

2.1. On Lipschitz geometry of subanalytic sets. We denote by $k(0_+)$ the Hardy field of Puiseux series, convergent in a right-hand-side neighborhood of zero, ordered by the order making the indeterminate smaller than any positive real number. Subanalytic subsets of $k(0_+)^n$ are defined in the obvious way. We denote by $T$ the indeterminate.

2.2. Regular lines. In the definition below $R$ stands for either $k(0_+)$ or $\mathbb{R}$.

Definition 2.2.1. Let $X$ be a subset of $R^n$. An element $\lambda$ of $S^{n-1}$ is said to be regular for $X$ if there is a positive rational number $\alpha$ such that:

$$\text{dist}(\lambda; T_{x_X}X_{\text{reg}}) \geq \alpha,$$

for any $x$ in $X_{\text{reg}}$.

Regular lines do not always exist, as it is shown by the simple example of a circle. Nevertheless, we can get a regular line without affecting the metric type of a subanalytic set. This is what is established by the theorem below, whose proof may be found in [V1].

Theorem 2.2.2. [V1] Let $X$ be a subset of $k(0_+)^n$ of empty interior. Then there exists a bi-Lipschitz homeomorphism $h : k(0_+)^n \to k(0_+)^n$ such that $e_n$ is regular for $h(X)$.

Definition 2.2.3. A map $h : \mathbb{R}^n \to \mathbb{R}^n$ is $x_1$-preserving if it preserves the first coordinate in the canonical basis of $\mathbb{R}^n$.

It is shown in [V2] that, if the considered subset lies in $C_n(M)$, then the homeomorphism of Theorem 2.2.2 may be chosen $x_1$-preserving. In [V2], the result was for semialgebraic sets. As it is now in the subanalytic setting, we present the proof, which is the same with more details.

In the proof below, we consider subsets of $\mathbb{R}^n$ as families of subsets of $\mathbb{R}^{n-1}$ parametrized by the first coordinates, in the sense that, given $x_1 \in \mathbb{R}$, we write $X_{x_1}$ for the set of points of $X$ having their first coordinate equal to $x_1$.

Corollary 2.2.4. Let $X$ be the germ at 0 of a subset $C_n(M)$ of empty interior, with $M > 0$. Then, there exists a germ of $x_1$-preserving bi-Lipschitz homeomorphism $h : C_n(M) \to C_n(M')$ such that $e_n$ is regular for $h(X)$.
Proof. Apply Theorem 2.2.2 to the generic fiber:

\[ X_{0,1} := \{ x : (T; x) \in X_{k(0,1)} \}, \]

where \( X_{k(0,1)} \) denotes the extension of the set to \( k(0,1) \). This provides a bi-Lipschitz homeomorphism which, by model completeness, immediately gives rise to a \( x_1 \)-preserving bi-Lipschitz homeomorphisms \( h : (0; \varepsilon) \times \mathbb{R}^{n-1} \rightarrow (0; \varepsilon) \times \mathbb{R}^{n-1} \) such that \( e_n \) is regular for \( h(X_{x_1}) \), for \( x_1 \) small enough. Up to a translation, we may assume that \( h(t; 0) = 0 \) so that \( h \) maps \( C_n(M) \) into \( C_n(M') \), for some \( M' \). We now have to check that \( e_n \) is regular for the germ of \( h(X) \). Suppose not. Then, there exists an arc \( \gamma : [0; \varepsilon] \rightarrow X_{\text{reg}} \) with \( \gamma(0) = 0 \) and \( e_n \in \tau = \lim T_{\gamma(t)}X_{\text{reg}} \). But, as \( e_n \) is regular for the fibers \( X_{x_1} \), we have \( e_n \not\in \lim T_{\gamma(t)}X_{\gamma_1(t)} \). This implies that

\[ \tau \cap < e_1 >^\perp \neq \lim (T_{\gamma(t)}X_{\text{reg}} \cap < e_1 >^\perp), \]

and consequently \( \tau \) may not be transverse to \( < e_1 >^\perp \), which means that \( \tau \subseteq < e_1 >^\perp \). This implies that the limit vector \( \lim_{t \to 0} \frac{\gamma_1(t)}{|\gamma(t)|} = \lim_{t \to 0} \frac{\gamma_1(t)}{|\gamma(t)|} \) is orthogonal to \( e_1 \). Therefore

\[ \lim_{t \to 0} \frac{\gamma_1(t)}{|\gamma(t)|} = 0, \]

in contradiction with \( \gamma(t) \in C_n(M) \).

Let us now show that \( h \) is also Lipschitz with respect to the parameter \( x_1 \). Suppose that \( h \) fails to be Lipschitz. Then we can find two arcs in \( X \), say \( p(t) \) and \( q(t) \), tending to zero along which:

\[ |p(t) - q(t)| \ll |h(p(t)) - h(q(t))|. \]

Recall that \( h \) preserves the fibers of \( \pi_1 \). We may assume that \( p(t) \) (and so \( h(p(t)) \)) is parametrized by its \( x_1 \)-coordinate, i. e. we may assume \( \pi_1(p(t)) = t \). As \( p(t) \) and \( h(p(t)) \) lie in \( C_n(M') \) we have:

\[ |h(p(t)) - h(p(t'))| \sim |t - t'| \]

and

\[ |p(t) - p(t')| \sim |t - t'| \leq |p(t) - q(t)|, \]

where \( t' \) denotes the first coordinate of \( q(t) \).

Therefore by (2.1) and (2.2) and (2.3) we have

\[ |h(p(t)) - h(q(t))| \sim |h(p(t')) - h(q(t))| \sim |p(t') - q(t)| \leq C|p(t) - q(t)|, \]

a contradiction. Arguing in the same way on \( h^{-1} \), we could show that \( h \) is bi-Lipschitz. \( \square \)

2.3. Some preliminaries. Before constructing the desired retraction, we need to put the set in a nice position. For this purpose, we will need yet another result whose proof may be found in [V1] as well. This is a consequence of the preparation theorem [P], [vDS]. We denote by \( \pi_n : \mathbb{R}^n \rightarrow \mathbb{R}^{n-1} \) the orthogonal projection onto \( \mathbb{R}^{n-1} \).

**Proposition 2.3.1.** Let \( X \) be a closed set and let \( \xi : X \rightarrow \mathbb{R} \) be a nonnegative function. There exists a finite partition of \( X \) such that over each element of this partition the function \( \xi \) is \( \sim \) to a product of powers of distances to subsets of \( X \).
We now would like to make two observations will be useful in the proof of the next lemma.

**Remark 2.3.2.**  
(1) If $X$ is the union of the graphs of finitely many Lipschitz functions $\xi_1, \ldots, \xi_k$ over $\mathbb{R}^n$ we may find an ordered family of Lipschitz functions $\theta_1 \leq \cdots \leq \theta_k$ such that $X$ is the union of the graphs of these functions.

(2) Given a family of Lipschitz functions $f_1, \ldots, f_k$ defined over $\mathbb{R}^n$ we can find some Lipschitz functions $\xi_1 \leq \cdots \leq \xi_m$ and a cell decomposition $\mathcal{D}$ of $\mathbb{R}^{n-1}$ such that over each cell $D \in \mathbb{R}^{n-1}$ delimited by the graphs of two consecutive functions $[\xi_i|_D; \xi_{i+1}|_D]$, with $D \in \mathcal{D}$, the functions $|q_{n+1} - f_i(x)|$ (where $q = (x; q_{n+1})$) are comparable to each other (for relation $\leq$) and comparable to the functions $f_i \circ \pi_n$. Indeed it suffices to choose a cell decomposition $\mathcal{D}$ compatible with the sets $f_i = f_j$ and to add the graphs of the functions $f_i, f_i + f_j$ and $\frac{f_i + f_j}{2}$. The family $\xi_1, \ldots, \xi_m$ is then given by (1).

The lemma below somehow combines Corollary 2.2.4 and Proposition 2.3.1 in a single one.

**Lemma 2.3.3.** Given $X_1, \ldots, X_s \subset C_n(M)$, there exist a $x_1$-preserving bi-Lipschitz homeomorphism $h : C_n(M) \to C_n(M')$, with $M' > 0$, and a cell decomposition $\mathcal{E}$ of $\mathbb{R}^n$ such that:

1. $\mathcal{E}$ is compatible with $h(X_1), \ldots, h(X_s)$
2. $e_n$ is regular for any cell of $\mathcal{E}$ in $C_n(M')$ which is a graph over a cell of $C_{n-1}(M')$ of $\mathcal{E}$.
3. Given finitely many nonnegative functions $\xi_1, \ldots, \xi_m$ on $C_n(M)$, we may assume that on each cell $E$ of $\mathcal{E}$, each function $\xi_i \circ h$ is $\sim$ to a function of the form:

\[
|y - \theta(x)|^p a(x)
\]

(for $(x; y) \in \mathbb{R}^{n-1} \times \mathbb{R}$) where $a, \theta : \pi_n(E) \to \mathbb{R}$ are functions with $\theta$ Lipschitz.

**Proof.** Apply Proposition 2.3.1 to the functions $\xi_j, j = 1, \ldots, m$. This provides a finite partition $E_1, \ldots, E_b$ of $C_n(M)$ together with some subsets of $C_n(M)$, say $W_1, \ldots, W_e$, such that on each $E_i, i \leq b$, each function $\xi_j, j \leq m$, is equivalent to a product of powers of functions of type $q \mapsto d(q; W_k), k \leq c$.

We may assume that the $W_k$’s have empty interior, possibly replacing them by their boundaries and refining the partition $E_i$ (if a function $\xi_i$ is identically zero on $E_i$ then (3) is trivial on $E_i$).

Apply now Corollary 2.2.4 to the union of the $\partial X_i$’s, all the $\partial E_i$’s and all the $W_k$’s. This provides a $x_1$-preserving bi-Lipschitz homeomorphism $h : C_n(M) \to C_n(M')$, with $M' > 0$, which maps the latter subsets into the union of the graphs of some Lipschitz functions (see Remark 2.3.2 (1))

\[
\theta_1 \leq \cdots \leq \theta_d.
\]

By Remark 2.3.2 (2) applied to the functions $\theta_i$’s and to all the $(n-1)$ variable functions $x \mapsto d(x; \pi_n(W_k \cap \Gamma_{\theta_i}))$, we know that there exist a finite number of functions $\eta_1 \leq \cdots \leq \eta_p$ and a cell decomposition $\mathcal{D}$ of $C_{n-1}(M)$ such that, for every $D \in \mathcal{D}$, over each $[\eta_i|_D; \eta_{i+1}|_D], i < p$, all the $n$ variable functions $(x; y) \mapsto |y - \theta_{\nu}(x)|, \nu \leq d$, are comparable with
each other and comparable with the functions \( d(x; \pi_n(W_k \cap \Gamma_{\theta_k})) \) (for order relation \( \leq \), considering the latter functions as \( n \)-variable functions).

Consider a cell decomposition of \( \mathbb{R}^{n-1} \) compatible with \( \mathcal{D} \) and the all subsets of \( C_{n-1}(M') \) respectively defined by the equations:

\[
\begin{align*}
(2.5) \quad d(x; \pi_n(W_i \cap \Gamma_{\theta_i})) &= d(x; \pi_n(W_j \cap \Gamma_{\theta_j})), \quad 1 \leq i, j \leq c, \nu \leq d,
\end{align*}
\]

as well as the equations \( \eta_{i+1} = \eta_i, 1 \leq i < p \). The graphs of the restriction of the functions \( \eta_i, i \leq p \), to the cells of this cell decomposition induce a cell decomposition of \( \mathbb{R}^n \) compatible with the \( h(X_i) \)'s, \( i \leq s \), that we denote by \( \mathcal{E} \). Observe that \( e_n \) is regular for any cell of \( \mathcal{E} \) which is a graph of a cell of \( \mathbb{R}^{n-1} \) of \( \mathcal{E} \) since the \( \eta_i \)'s are Lipschitz functions. This already proves that (2) holds.

To prove (3), fix a cell \( E \) of \( \mathcal{E} \). As the \( \theta_i \)'s are Lipschitz functions, we have for any \( \nu \in \{1, \ldots, d\} \):

\[
\begin{align*}
(2.6) \quad d(q; h(W_i) \cap \Gamma_{\theta_i}) &= |y - \theta_{\nu}(x)| + d(x; \pi_n(h(W_i) \cap \Gamma_{\theta_i}))
\end{align*}
\]

where \( q = (x; y) \) in \( \mathbb{R}^{n-1} \times \mathbb{R} \).

The terms of the right-hand-side are nonnegative and comparable with each other (for partial order relation \( \leq \)) over the cell \( E \) (thanks to (2.5) and choice of the \( \eta_i \)'s). Therefore the left-hand-side is \( \sim \) to one of them.

Note that, as the \( h(W_i) \)'s are included in the graphs of the \( \theta_j \)'s, \( j \leq d \), we have:

\[
d(q; h(W_i)) = \min_{1 \leq j \leq d} d(q; h(W_i) \cap \Gamma_{\theta_j}).
\]

Hence, by (2.6), the functions \( d(q; h(W_i)) \)'s are equivalent over \( E \) either to one of the functions \( x \mapsto d(x; \pi_n(h(W_i) \cap \Gamma_{\theta_i})) \), or to some function \( (x; y) \mapsto |y - \theta_{j}(x)|, j \in \{1, \ldots, d\} \).

As \( h \) is bi-Lipschitz, each \( \xi_j \circ h \) is equivalent to a product of powers of functions of type \( q \mapsto d(q; h(W_i)), i \leq c \). Therefore (3) holds. □

2.4. Lipschitz retractions of subanalytic germs. We are now ready to construct the desired strong deformation retraction. Given \( X \subset \mathbb{R}^n \) we define:

\[
\hat{X} := \{(y; x) \in \mathbb{R} \times \mathbb{R}^n : |x| = y\}.
\]

Observe that \( \hat{X} \) is a subset of \( C_{n+1}(1) \).

In the theorem below we write \( d_x r \) for \( x \) generic although \( r \) is not smooth, since, like all the mappings in this paper, \( r \) is implicitly assumed to be subanalytic and thus smooth on a (subanalytic) dense subset.

**Theorem 2.4.1.** Let \( X \subset \mathbb{R}^n \) and let \( x_0 \in X \). Then, for any \( \varepsilon > 0 \) small enough there exists a Lipschitz deformation retraction

\[
r : X \cap B_{n}(x_0; \varepsilon) \times [0; 1] \to X \cap B_{n}(x_0; \varepsilon), \quad x \mapsto r_t(x),
\]

onto \( x_0 \), preserving \( X_{\text{reg}} \).

Furthermore, \( d_x r_t \to 0 \) as \( t \to 0 \) for any \( x \) generic in \( X_{\text{reg}} \).

**Proof.** We will assume for simplicity that \( x_0 = 0 \). We will actually prove by induction on \( n \) the following statements.
(A_n) Let $X_1,\ldots,X_s$ be finitely many subsets of $C_n(M)$ and let $\xi_1,\ldots,\xi_m$ be some bounded functions on $C_n(M)$, with $M > 0$. There exists $\varepsilon > 0$ such that if we set $U_\varepsilon := \{x \in C_n(M) : 0 \leq x_1 < \varepsilon\}$, we have a Lipschitz strong deformation retraction of $U_\varepsilon$

\[ r : U_\varepsilon \times [0;1] \to U_\varepsilon, \quad x \mapsto r_t(x), \]

onto 0 such that for any $j \leq s$:

1. $r$ preserves $X_j$ and $X_j,reg$
2. $d_x r_t$ goes to zero as $t$ tends to 0 for any $x$ generic in $X_j \cap B_n(0;\varepsilon)$
3. There is a constant $C$ such that for any $i$ and any $0 < t < 1$ we have:
   \[ \xi_i(r(x; t)) \leq C \xi_i(x). \]

Before proving these statements, let us make it clear that this implies the desired result. If $X \subset \mathbb{R}^n$ then $\hat{X}$ is a subset of $C_{n+1}(1)$ to which we can apply $(A_{n+1})$. Then, as $\hat{X}$ is bi-Lipschitz equivalent to $X$, the result immediately ensues.

As the theorem obviously holds in the case where $n = 1$ (with $r_t(x) = tx$), we fix some $n > 1$. We fix some subsets $X_1,\ldots,X_s$ of $C_n(M)$, for $M > 0$, and some bounded functions $\xi_1,\ldots,\xi_m : C_n(M) \to \mathbb{R}$.

**Observation 1.** If two given functions $\xi$ and $\zeta$ both satisfy (2.7) then $\min(\xi; \zeta)$ and $\max(\xi; \zeta)$ both satisfy this inequality as well.

**Observation 2.** If $\xi$ is bounded and if $\min(\xi; 1)$ satisfies (2.7) then $\xi$ satisfies this inequality as well.

Before defining the desired map, we need some preliminaries. We first construct a family of $(n - 1)$ variable functions $\sigma_1,\ldots,\sigma_p$ to which we will apply (3) of $(A_{n-1})$.

Apply Lemma 2.3.3 to the family constituted by the $X_i$’s and the zero locus of the $\xi_i$’s. We get a $x_1$-preserving bi-Lipschitz map $h : C_n(M) \to C_n(M')$ and a cell decomposition $E$ such that (1) and (2) of the latter lemma hold. As we may work up to a $x_1$-preserving bi-Lipschitz map we will identify $h$ with the identity map. Moreover, by (3) of the Lemma, we may also assume that the functions $\xi_i$’s are like in (2.4).

The union of the cells of $E$ for which $e_n$ is regular may be included in the union of the graphs of finitely many Lipschitz functions $\eta_1 \leq \cdots \leq \eta_v$ (see Remark 2.3.2 (1)).

In order to define the functions $\sigma_1,\ldots,\sigma_p$, let us fix a cell $D$ of $E$ in $C_n(M)$, which is delimited by the respective graphs of the two Lipschitz functions $\eta_j$ and $\eta_{j+1}$. Since the functions $\xi_k$’s are like in (2.4), there exist finitely many $(n - 1)$ variable functions on $D' := \pi_n(D)$, say $\theta_k$ and $a_k$, for every $k = 1,\ldots,m$, with $\theta_k$ Lipschitz such that:

\[ \xi_k(x) \sim \max(\eta_j, \eta_{j+1})^{\alpha_k} a_k(x), \]

where $\alpha_k$ is a rational number (possibly negative).

As the boundary of the zero locus of the $\xi_k$’s is included in the graphs of the $\eta_j$’s, we have on $\pi_n(D)$ if $\xi_k$ is not identically zero on $D$, either $\theta_k \leq \eta_j$ or $\theta_k \geq \eta_{j+1}$. Fix $k$ with $\xi_k \neq 0$ on $D$. We will assume for simplicity that $\theta_k \leq \eta_j$.

This means that on $D$:

\[ \xi_k(x) \sim \min(\eta_j, \eta_{j+1})^{\alpha_k} a_k(x), \]

where $\alpha_k$ is a rational number (possibly negative).
if $a_k$ is negative, and
\begin{equation}
(2.10) \quad \xi_k(x) \sim \max((y - \eta_j(x))^\alpha a_k(x); (\eta_j - \theta_k(x))^\alpha a_k(x)),
\end{equation}
in the case where $\alpha_k$ is nonnegative.

We are now ready to define the desired family $\sigma_1, \ldots, \sigma_p$ of $(n - 1)$-variable functions. We first set:
\begin{equation}
(2.11) \quad \kappa_k(x) := |\eta_j(x) - \theta_k(x)|^\alpha a_k(x).
\end{equation}
Complete the family $\kappa$ by adding the functions $\min(f; 1)$ where $f$ describes all the $(\eta_{j+1} - \eta_j) a_k$'s. Doing this for all the sets $D$ of type $(\eta_{j+1}|D'; \eta_{j+1}|D')$ with $j$ and $D' \in \mathcal{E}$, we eventually get a family of bounded functions $\sigma_1, \ldots, \sigma_p$.

We now turn to the construction of the desired retraction. Consider a cylindrical cell decomposition $\mathcal{D}$ compatible with the cells of $\mathcal{E}$ and the graphs of the $\eta_j$'s. Apply the induction hypothesis to the cells of $\mathcal{D}$ which lie in $\mathcal{C}_{n-1}(M)$. This provides a deformation retraction $r$.

We are going to lift $r$ to a retraction of $\mathcal{C}_n(M)$. Thanks to the induction hypothesis, we may assume that the functions $\sigma_1, \ldots, \sigma_p$ satisfy (2.7), as well as the functions $(\eta_{j+1} - \eta_j)$'s and the functions $x \mapsto \xi_i(x; \eta_j(x))$.

Then we may lift $r$ as follows. On $[\eta_j; \eta_{j+1}]$ we set:
\begin{equation}
\nu(q) := \frac{y - \eta_j(x)}{\eta_{j+1}(x) - \eta_j(x)},
\end{equation}
if $q = (x; y) \in \mathbb{R}^{n-1} \times \mathbb{R}$, and
\begin{equation}
\tilde{r}(q; t) := (r(x; t); \nu(q)(\eta_{j+1}(r(x; t)) - \eta_j(r(x; t))) + \eta_j(r(x; t))).
\end{equation}
For any $j$, the mappings $\tilde{r}$ maps linearly the segment $[\eta_j(x); \eta_{j+1}(x)]$ onto the segment $[\eta_j(r_t(x)); \eta_{j+1}(r_t(x))]$. Thanks to the induction hypothesis, the inequality (2.7) is fulfilled by the function $(\eta_{j+1} - \eta_j)$. Therefore, as $r$ is Lipschitz, we see that $\tilde{r}$ is Lipschitz as well. As $\tilde{r}$ preserves the cells of $\mathcal{E}$, it preserves the $X_{j,reg}$'s and the $X_j$'s.

We have to check that the $\xi_i$'s fulfill (2.7) along the trajectories of $\tilde{r}$. We check it on a given cell $E$ of $\mathcal{D}$. If $E$ lies in $\Gamma_{\eta_j}$ for some $j$, this follows from the induction hypothesis since we have assumed that the functions $x \mapsto \xi_i(x; \eta_j(x))$ satisfy (2.7).

Otherwise, there exists $j$ such that $E$ sits in $(\eta_j; \eta_{j+1})$. Fix an integer $1 \leq k \leq m$. On the cell $E$, the function $\xi_k$ may be written like in (2.8). By the induction hypothesis we know that $\kappa_k$ (see (2.11)) satisfies (2.7). Remark also that the function $\nu(\tilde{r}(q; t))$ is constant with respect to $t$.

Note that by (2.9) and (2.10) (and Observations 1 and 2) it is enough to show that the functions $\min((y - \eta_j(x))^{\alpha_k} a_k(x); 1)$ and the functions $|\theta_k - \eta_j|^{\alpha_k} a_k$ satisfy (2.7). The latter functions are nothing but the $\kappa_k$'s for which we already have seen that this inequality is true. Let us focus on the former functions.

For simplicity we set
\begin{equation}
F(x; y) := (y - \eta_j(x))^{\alpha_k} a_k(x)
\end{equation}
and
\begin{equation}
G(x) := (\eta_{j+1} - \eta_j(x))^{\alpha_k} a_k(x).
\end{equation}
We now show the desired inequality for \( \min(F; 1) \). We have:

\[
F(x; y) = \nu(q)^{\alpha_i} \cdot G(x).
\]

As \( \nu \) is constant along the curves of \( \tilde{r} \), this implies that:

\[
F(\tilde{r}(q); t) = \nu(q)^{\alpha_i} \cdot G(r(x); t)).
\]

We assume first that \( \alpha_k \) is negative. Thanks to the induction hypothesis (\( \min(G; 1) \) is one of the \( \sigma_i \)'s) we know that for some constant \( C \):

\[
\min(G(r(x); t); 1) \leq C \min(G(x); 1).
\]

But this implies (multiplying by \( \nu^{\alpha_k} \) and applying (2.12) and (2.13)) that:

\[
\min(F(\tilde{r}(q); t); \nu^{\alpha_k}(q); 1) \leq \min(F(q); \nu^{\alpha_k}(q); 1).
\]

But, as \( \alpha_k \) is negative, \( \min(F; \nu^{\alpha_k}; 1) = \min(F; 1) \), which yields the desired inequality for \( \min(F; 1) \), as required.

We now assume that \( \alpha_i \) is nonnegative. Again, by (2.10), it is enough to show the desired inequality for \( F \), and, thanks to (2.12) and (2.13), it actually suffices to show it for \( G \). But, as \( \xi_k \) is bounded, so is \( G \), and the result follows from the induction hypothesis (and Observation 2) for \( \min(G; 1) \) is one of the \( \sigma_i \)'s. This yields (2.7) along the trajectories of \( \tilde{r} \).

We now check that \( d_x \tilde{r}_t \) tends to zero when \( t \) goes to zero. It follows from the induction hypothesis that \( d_x r_t \) goes to zero as \( t \) goes to zero. As the \( \eta_k \)'s have bounded derivatives this already proves for almost every \( x \):

\[
\lim_{t \to 0} d_x[(\eta_k - \eta_{k+1}) \circ r_t] = 0.
\]

On the other hand, a straightforward computation shows that for \( q = (x; y) \):

\[
|d_{\tilde{r}(q); t)} \nu| \leq \frac{C}{|\eta_k(x) - \eta_{k+1}(x)|},
\]

which, together with (2.14) and (2.7) for \( (\eta_{k+1} - \eta_k) \), implies that \( d_q \tilde{r}_t \) tends to zero as \( t \) goes to zero.

**Remark 2.4.2.** In [V1, V2], we introduced the Lipschitz triangulations, which are triangulations with a control of the stretch of the metric. We could have a similar description of \( r_t \) in Theorem 2.4.1. First, the family of maps \( r_t : B_n(0; \varepsilon) \to B_n(0; \varepsilon) \) preserving \( X_{\text{reg}} \) that we constructed in the proof of the above theorem is a homeomorphism onto its image for \( t \neq 1 \) (for \( t = 1 \) it is the zero map). Secondly, there exists a basis of unit 1-forms \( \theta_1, \ldots, \theta_n \) and some functions \( \varphi_1, \ldots, \varphi_n \) on \( \mathbb{R}^n \) such that almost everywhere on \( X_{\text{reg}} \):

\[
\rho^{\ast} \rho \approx \sum_{i=1}^{n} \varphi_i(x; t) \theta_i^2(x),
\]

where \( \rho \) is the metric of \( \mathbb{R}^n \). The function \( \varphi_i \)'s may be chosen decreasing with respect to \( t \), which immediately implies that \( r \) is Lipschitz. As in [V1], the contractions \( \varphi_i \) also can be shown to be \( \sim \) to some finite sums of product of powers of distance functions in \( X \) (i. e. \( x \mapsto d(x; W) \) with \( W \subset X \cap B_n(0; \varepsilon) \)) and the function \( (x; t) \mapsto t \). Nevertheless, these powers may be negative which makes it difficult to get decreasing functions and account for the difficulty we have in the proof of the above theorem. In [V1, V2] section
5 (see Theorem 5.3), the “standard simplicial functions” are defined as \textit{sums of product of powers of distance to faces} (with possibly negative exponents) but it should be corrected in quotient of sums of product of powers in the sense that there may be sums in the denominators as well.

3. Normal pseudomanifolds.

**Definition 3.0.3.** An \( l \)-dimensional pseudomanifold \( X \) is called \textbf{normal} if for any \( x \) in \( X \), \( \dim H_l(X; X \setminus \{x\}) = 1 \).

We shall recall some basic facts about normal spaces. These may be found in [GM] (section 4) and make normalization very useful to investigate intersection homology in the maximal perversity. Observe that if \( X \) is a normal pseudomanifold which is connected then \( H_l(X; \partial X) = \mathbb{R} \), since if there were two generators \( \sigma \) and \( \tau \), with \( \dim |\sigma| \cap |\tau| < l \), we would have \( H_l(X; X \setminus x) = \mathbb{R} \) at any point of the intersection of the supports.

The main interest of normal spaces lies in the following Lemma. Denote by \( L(x; X_{\text{reg}}) \) the set \( S_{n-1}(x; \varepsilon) \cap X_{\text{reg}} \). It is well known that the topology of \( L(x; X_{\text{reg}}) \) is independent of \( \varepsilon \) if it is chosen small enough.

**Lemma 3.0.4.** [GM] A pseudomanifold \( X \) is normal if and only if \( L(x; X_{\text{reg}}) \) is connected at any point of \( X_{\text{sing}} \).

See for instance [GM] section 4 for a proof. The very significant advantage of normal spaces relies in the following proposition.

**Proposition 3.0.5.** [GM] Let \( X \) be a normal pseudomanifold. Then the mapping \( \alpha : I^tH_j(X) \to H_j(X) \), induced by the inclusion between the chain complexes, is an isomorphism for all \( j \).

3.1. Normalizations of pseudomanifolds. We shall need some basic facts about normalizations.

**Definition 3.1.1.** A \textbf{normalization} of the pseudomanifold \( X \) is a normal pseudomanifold \( \tilde{X} \) together with a finite-to-one projection \( \pi : \tilde{X} \to X \) such that, for any \( p \) in \( X \),

\[
\pi_* : \bigoplus_{q \in \pi^{-1}(p)} H_l(\tilde{X}; \tilde{X} \setminus q) \to H_l(X; X \setminus p)
\]

is an isomorphism.

It is well known that every pseudomanifold admits a normalization and that its topology is unique.

It is not difficult to see from their construction, that normalizations must identify \((t; i)\)-allowable chains of \( \tilde{X} \) with \((t; i)\)-allowable chains of \( X \), which implies that they yield an isomorphism between the intersection homology groups (see [GM]):

**Proposition 3.1.2.** [GM] Let \( \pi : \tilde{X} \to X \) be a normalization of \( X \). Then, for any \( j \) the induced map \( \pi_* : I^tH_j(\tilde{X}) \to I^tH_j(X) \) is an isomorphism.
3.2. $L^\infty$ normalizations. To compute the $L^\infty$ cohomology groups, usual normalizations will not suffice. We introduce $L^\infty$ normalizations and prove existence for any pseudomanifold $X$.

**Definition 3.2.1.** An $L^\infty$ normalization of a pseudomanifold $X$ is a normalization $\pi: \tilde{X} \to X$, with $\tilde{X}$ normal pseudomanifold, inducing a diffeomorphism $\pi': \tilde{X}_{\text{reg}} \to X_{\text{reg}}$ yielding a quasi-isometry between these two Riemannian manifolds. It is $C^k$ if its restriction to $\tilde{X}_{\text{reg}}$ is $C^k$.

**Proposition 3.2.2.** For any subanalytic pseudomanifold $X$, there exists a $C^2$ subanalytic $L^\infty$ normalization $\pi: \tilde{X} \to X$.

**Proof.** We may assume that $X$ is not a normal pseudomanifold. Consider a Whitney stratification $\mathcal{E}$ of $X$. There exists a stratum $S$ of dimension $k$ such that the set the link $L(S; X_{\text{reg}})$ has more than one connected component. Choose $k$ maximal for this property.

Let $S$ be a stratum of maximal dimension for this property and let $k = \dim S$. As the construction that we are going to carry out may be realized simultaneously for any such stratum $S$, we will assume that $S$ is the only stratum of dimension $k$ with this property.

Let $C_1, \ldots, C_s$ be the connected component of $X_{\text{reg}} \cap U$, where $U$ is an open tubular neighborhood of $S$. Consider first a continuous function $\alpha$ on $\mathbb{R}^n$ zero on the complement of $U$ and which does not vanish on $S$. We may approximate this function by a function on $\mathbb{R}^n$, $C^2$ on $U$, still having the latter properties. Rising it to a huge power we get a $C^2$ function on $\mathbb{R}^n$ (thanks to Lojasiewicz inequality). Let $\rho_i$ be the restriction of this function to a neighborhood of $C_i \cap U$.

Let now for $x \in X_{\text{reg}}$:

$$\rho(x) := (x; \rho_1(x); \ldots; \rho_s(x)),$$

and denote by $\tilde{X}$ the closure of the image of $X$ under the map $\rho$ and define $\pi$ as the mapping induced by the canonical projection.

As, on the one hand $\rho_{i|C_i}$ is bounded below away from zero and vanishes on the other connected components, and on the other hand the functions $\rho_{i|C_j}$, $j \neq i$ tend to zero as $x$ goes to $S$, we see that the graph of $\rho$ is locally connected nearby the closure of $\pi^{-1}(C_i)$. Because the $\rho_{i|C_j}$’s extend continuously, the mapping $\pi$ is finite to one. This means that the set of points $x$ at which $L(x; \tilde{X}_{\text{reg}})$ has more than one connected component is of dimension less than $k$. The result follows by induction on $k$. \hfill $\Box$

**Remark 3.2.3.** The above $L^\infty$ normalization could be chosen $C^m$ for any $m$ positive. Nevertheless, as there is no analytic bump function, the above argument no longer works to get $C^\infty$ normalizations.

4. Computation of the $L^\infty$ cohomology groups

The purpose of this section is to prove the main result of this paper, Theorem 1.2.2.

4.1. Weakly differentiable forms. For technical reasons, we will need to work with non smooth forms, which are weakly differentiable, i.e. differentiable as distributions. Therefore, the first step is to prove that the bounded weakly differentiable forms give rise
to the same cohomology theory. We will follow an argument similar to the one used by Youssin in [Y].

Let $K$ denote a smooth manifold (possibly with boundary) of dimension $l$. We denote by $\Lambda^j_0(K)$ the set of $C^2$ $j$-forms on $K \setminus \partial K$ with compact support (in $K \setminus \partial K$).

**Definition 4.1.1.** Let $U$ be an open subset of $\mathbb{R}^n$. A continuous differential $j$-form $\alpha$ on $U$ is called **weakly differentiable** if there exists a $(j+1)$-form $\omega$ such that for any form $\varphi \in \Lambda^{l-j-1}_0(U)$:

$$\int_U \alpha \wedge d\varphi = (-1)^{j+1} \int_U \omega \wedge \varphi.$$ 

The form $\omega$ is then called the **weak exterior differential** of $\alpha$ and we write $\omega = \overline{d}\alpha$. A continuous differential $j$-form $\alpha$ on $K$ is called **weakly differentiable** if it gives rise to weakly differentiable forms via the coordinate systems of $K$.

We denote by $\mathcal{P}^l_\infty(K)$ the set of weakly differentiable forms which are bounded and which have a bounded weak exterior derivative. They constitute a cochain complex whose coboundary operator is $\overline{d}$. We denote by $\mathcal{P}_\infty(K)$ the resulting cohomology groups.

In the case of compact smooth manifolds it is easily checked that the two cohomology theories coincide:

**Lemma 4.1.2.** If $K$ is a compact manifold then:

$$(4.15) \quad \mathcal{P}^l_\infty(Int(K)) \simeq H^l(K).$$

**Proof.** The proof follows the classical argument. As in the case of smooth forms (see for instance [BT]) it is enough to show Poincaré Lemma. Both of the above cohomology theories are invariant under smooth homotopies. Any point of $K$ has a smoothly contractible neighborhood. As $K$ is compact, locally bounded implies bounded. \qed

We now are going to see that the isomorphism also holds in the noncompact case:

**Proposition 4.1.3.** For any $C^\infty$ manifold $K$ without boundary, the inclusion $\Omega^\bullet_\infty(K) \to \overline{\Omega}_\infty(K)$ induces isomorphisms on the cohomology groups.

**Proof.** It is enough to show that, for any form $\alpha \in \mathcal{P}^j_\infty(K)$ with $\overline{d}\alpha \in \Omega^{j+1}_\infty(K)$ (i.e. $\alpha$ is weakly smooth and $\overline{d}\alpha$ is smooth), there exists $\theta \in \mathcal{P}^{j-1}_\infty(K)$ such that $(\alpha + \overline{d}\theta)$ is $C^\infty$. For this purpose, we prove by induction on $i$ the following statements.

$(\text{H}_i)$ Fix a form $\alpha \in \mathcal{P}^j_\infty(K)$ with $\overline{d}\alpha \in \Omega^{j+1}_\infty(K)$. Consider an exhaustive sequence of compact smooth manifolds with boundary $K_i \subset K$ such that for each $i$, $K_i$ is included in the interior of $K_{i+1}$ and $\cup K_i = K$. Then, for any integer $i$, there exists a closed form $\theta_i \in \mathcal{P}^{j-1}_\infty(X)$ such that $\text{supp} \, \theta_i \subset \text{Int}(K_i) \setminus K_{i-2}$ and $|\theta_i|_\infty + |\overline{d}\theta_i|_\infty \leq 1$ and such that the form $\alpha_i := \alpha_{i-1} + \overline{d}\theta_i$ is smooth in a neighborhood of $K_{i-1}$.

Before proving these statements observe that $\theta = \sum_{i=1}^\infty \theta_i$ is the desired exact form (this sum is locally finite).

Let us assume that $\theta_{i-1}$ has been constructed, $i \geq 1$ (we may set $K_0 = K_{-1} = K_{-2} = \emptyset$). Observe that by $(4.15)$, there exists a smooth form $\beta \in \Omega^{(j-1)}_\infty(K_i)$ such that $\overline{d}\beta = \overline{d}\alpha_{i-1}$. 


This means that \((\alpha_{i-1} - \beta)\) is \(\partial\)-closed, and by (4.15) there is a smooth form \(\beta' \in \Omega_{\infty}^{(j-1)}(K_i)\) such that

\[ \alpha_{i-1} - \beta = \beta' + \partial \gamma, \]

with \(\gamma \in \Omega^{(j-2)}_{\infty}(K_i)\) (if \(j = 1\) then \((\alpha_{i-1} - \beta)\) is constant and then smooth). Thanks to the induction hypothesis there exists an open neighborhood \(V\) of \(K_{i-2}\) on which \(\alpha_{i-1}\) is smooth. This implies that \(\partial \gamma\) is smooth on \(V\). Therefore, applying the induction hypothesis to \(\gamma\), we can add an exact \(d\sigma\) form to \(\gamma\) to get a form smooth on a neighborhood of \(K_{i-2}\).

Multiplying \(\sigma\) by a function with support in \(V\) which is 1 in a neighborhood \(W\) of \(K_{i-2}\), we get a form \(\sigma'\) on \(K\) such that \((\partial \sigma' + \gamma)\) is smooth on \(W\). This means that we can assume that \(\gamma\) is smooth on an open neighborhood \(W\) of \(K_{i-2}\). We will assume this fact without changing notations.

By means of a convolution product with a bump function, for any \(\varepsilon > 0\), we may construct a smooth form \(\gamma_{\varepsilon}\) such that \(|\gamma_{\varepsilon} - \gamma|_{\infty} \leq \varepsilon\) and \(|d\gamma_{\varepsilon} - \partial \gamma|_{\infty} \leq \varepsilon\).

Consider a smooth function \(\phi\) which is 1 on a neighborhood of \((K \setminus W) \cap K_{i-1}\) and with support in \(\text{int}(K_i) \setminus K_{i-2}\). Then, set:

\[ \theta_i(x) := \phi(x)(\gamma_{\varepsilon} - \gamma)(x). \]

If \(\varepsilon\) is chosen small enough \(|\theta_i|_{\infty} + |\partial \theta_i|_{\infty} \leq 1\). On a neighborhood of \((K \setminus W) \cap K_{i-1}\), because \(\phi \equiv 1\), we have \(\alpha_{i-1} + \partial \theta_i = \beta + \beta' + d\gamma_{\varepsilon}\) which is clearly smooth. The form \((\alpha_{i-1} + \partial \theta_i)\) is smooth on \(W\) as well since \(\alpha_{i-1}\) and \(\theta_i\) are both smooth. \(\square\)

4.2. Proof of the De Rham Theorem for \(L^\infty\) cohomology. We are now ready to prove the main theorem of the paper. Throughout this section, \(X\) will be an \(l\)-dimensional (subanalytic) compact pseudomanifold of \(\mathbb{R}^n\).

**Theorem 4.2.1.** (Poincaré Lemma for \(L^\infty\) cohomology) Every point \(p \in X\) has a contractible neighborhood \(U\) in \(X\) such that for any weakly closed form \(\omega \in \Omega_{\infty}^j(U \cap X_{\text{reg}}), j \geq 1\), there exists \(\alpha \in \Omega_{\infty}^{j-1}(U \cap X_{\text{reg}}), \) with \(\omega = \partial \alpha\).

**Proof.** Let \(\omega\) be a weakly closed \(L^\infty\) \(j\)-form on \(X_{\text{reg}}\), with \(j \geq 1\). Let \(r : U \times I \to \) be the map obtained by applying Theorem 2.4.1 to \(X\). For simplicity we denote by \(X'\) the subset \(X_{\text{reg}} \cap B_n(0; \varepsilon)\).

The problem is that \(d_x r\) is not weakly smooth but just bounded. To overcome this difficulty, we shall work with an approximation of \(r\). We need to be particular since we wish to preserve the property that the derivative or \(r\) goes to zero (pointwise and generically) as \(t\) goes to zero.

Consider an exhaustive sequence of compact subsets, \((K_i)_{i \in \mathbb{N}}\) such that \(\cup K_i = X_{\text{reg}} \times (0; 1)\) and \(K_i \subset \text{Int}(K_{i+1})\), for any \(i\). Let \(Y\) be the set of points of \(X_{\text{reg}} \times (0; 1)\) at which \(r\) fails to be smooth. Define then a sequence of compact subsets

\[ L_i := \{q \in K_i : d(q; Y) \geq 1/i\}. \]

Define also

\[ X' := d(Y) \cap (X_{\text{reg}} \times \{0\}) \]

and observe that \(X'\) is of positive codimension in \(X_{\text{reg}}\) (we will consider it as a subset of \(X_{\text{reg}}\)).
As $r$ is continuous, we may choose, for a given $\varepsilon_i > 0$, a $C^\infty$ approximation $r_i$ (not necessarily subanalytic) of $r$ on $K_i$ satisfying

$$|r_i(x) - r_{i,t}(x)| \leq \varepsilon_i,$$

on $K_i$. Furthermore, as $r$ is smooth on $L_i$, we may require that on this set

$$|d_x r_{i,t} - d_x r_i| \leq \varepsilon_i.$$

The derivative of $r_i$ may be bounded by a constant as close as we please from the supremum of the norm of the derivative of $r$.

We may now paste these approximations by a partition of unity subordinated to the covering $(\text{Int}(K_{i+1}) \setminus K_{i-1})_{i \in \mathbb{N}}$ of $X_{\text{reg}}$. Let $r'$ be the obtained mapping. If the sequence $\varepsilon_i$ is decreasing fast enough, then $r'$ is has bounded derivatives.

Furthermore, for any positive continuous function $\varepsilon : X_{\text{reg}} \times [0; 1] \to \mathbb{R}$, we will have, if the sequence $\varepsilon_i$ is decreasing fast enough:

$$|r'(x; t) - r(x; t)| \leq \varepsilon(x; t). \tag{4.16}$$

Finally, for $x$ in $X_{\text{reg}} \setminus X'$, there exists $a < 1$ such that $\{x\} \times [a; 1]$ does not meet $Y$. This means that for any $i$ large enough

$$K_i \cap (\{x\} \times [a; 1]) = L_i \cap (\{x\} \times [a; 1]).$$

Therefore, as we approximated $d_x r_t$ on $L_i$, this implies that if $\varepsilon_i$ tends to zero fast enough, $\lim_{t \to 0} |d_x r'_t| = 0$ for every $x \in X_{\text{reg}} \setminus X'$. We also see that for the same reason $d_x r'_t$ tends to the identity as $t$ goes to 1.

Let $\pi : W \to X_{\text{reg}}$ be a retraction where $W$ is a tubular neighborhood of $X_{\text{reg}}$. Taking $W$ small enough, we may assume that $\pi$ has bounded derivatives. By (4.16), $r'(x; t)$ belongs to $W$ if the function $\varepsilon$ is chosen small enough. Hence, composing $r'$ with $\pi$ if necessary we may assume that $r'$ preserves $X_{\text{reg}}$ We will assume this without changing notations.

Define two $L^\infty$ forms $\omega_1$ and $\omega_2$ on $X^\varepsilon \times [0; 1)$ by:

$$r'^* \omega := \omega_1 + dt \wedge \omega_2.$$

Now we may set:

$$\alpha(x) := \int_0^1 \omega_2(x; t)dt.$$

As $\omega$ is $L^\infty$ and $r'$ has bounded derivatives, the form $\alpha$ is clearly bounded as well. By Lebesgue dominated convergence theorem, it is continuous. It can fail to be differentiable since we did not assume that the partial derivatives of $\omega$ are bounded. Nevertheless, we claim that it is weakly differentiable and that $d\alpha = \omega$. Thanks to Proposition 4.1.3, this is enough to establish the Theorem.

Let $U$ be an open set in $X_{\text{reg}}$ on which we may find a coordinate system. Let us fix a $C^2$-form $\varphi \in \Lambda^{m-3}_2(U)$ with compact support.

$$\int_{X^\varepsilon} \alpha \wedge d\varphi = \int_U \int_0^1 \omega_2 \wedge d\varphi = \lim_{t \to 0} \int_{U \times [t; 1]} r'^* \omega \wedge d\varphi. \tag{4.17}$$
Observe that \( r^* \omega \) is weakly differentiable and that its weak differential is the zero form. Then, by Stokes' formula we have:

\[
\lim_{t \to 0} \int_{U \times [t;1]} r^* \omega \wedge \varphi = (-1)^j \int_U \omega(x) \wedge \varphi(x) - (-1)^j \int_U \omega(x; t) \wedge \varphi(x),
\]

since \( r^* \omega(x; 1) = \omega(x) \) for any \( x \in U \). Recall that \( dx^i t \) tends to zero as \( t \) goes to 0 for almost every \( x \). This implies that \( \omega_1 \) goes to zero as \( t \) goes to 0. Hence, passing to the limit we get:

\[
\int_U \alpha \wedge d\varphi = (-1)^j \int_U \omega \wedge \varphi,
\]

as required. \( \square \)

**Proof of Theorem 1.2.2.** By Proposition 4.1.3, it is enough to prove:

\[
\Omega^i_\infty(X_{reg}) \simeq I^t H^j(X).
\]

Observe that a \( C^2 \) \( L^\infty \) normalization of \( X \) does not affect \( \Omega^i_\infty(X_{reg}) \). On the other hand, by Proposition 3.1.2, the intersection homology groups are preserved as well. Hence, we may assume that \( X \) is normal.

The weakly differentiable \( L^\infty \) forms constitute a presheaf on \( X_{reg} \). Let us extend it to \( X \) by

\[
\Omega^i_\infty(U) := \Omega^i(U \cap X_{reg}),
\]

for every open set \( U \) of \( X \). For every \( j \), this presheaf immediately gives rise to a sheaf that we will denote by \( F^j_\infty \). It is the sheaf on \( X \) of locally bounded forms of \( X_{reg} \). As \( X \) is compact, any global section of \( F^j_\infty \) is bounded, so that:

\[
F^\bullet_\infty(X) \simeq \Omega^\bullet_\infty(X_{reg}),
\]

as cochain complexes.

We denote by \( \mathbb{R}_X \) the constant sheaf on \( X \). As \( X \) is normal, there is only one connected component nearby each point \( x_0 \), which means that \( \Omega^\bullet_\infty(X_{reg} \cap B_n(x_0; \varepsilon)) = \mathbb{R} \), for \( \varepsilon \) small enough.

Therefore, by Theorem 4.2.1, the sequence:

\[
0 \longrightarrow \mathbb{R}_X \xrightarrow{d} F^0_\infty \xrightarrow{d} F^1_\infty \xrightarrow{d} \ldots
\]

is a fine torsionless resolution of the constant sheaf. By classical arguments of sheaf theory (see for instance [W]), the latter exact sequence of sheaves implies that the cohomology of \( \Omega^\bullet_\infty(X) \) coincides with the singular cohomology of \( X \). But then, by Proposition 3.0.5, we get:

\[
H^j_\infty(X_{reg}) \simeq H^j(X) \simeq I^t H^j(X).
\]

\( \square \)
4.3. **Integration on subanalytic singular simplices.** We are going to prove that the isomorphism is provided by integrating forms on allowable chains. We first check that integration gives rise to a well defined cochain map. This may be done if we restrict ourselves to *subanalytic* singular cochains which are *t*-allowable.

Let \( L \subset X_{\text{reg}} \) be an oriented manifold of dimension \( j \) with \( cl(L) \) \((t;j)\)-allowable i.e.: 
\[
\dim cl(L) \cap X_{\text{sing}} \leq (j-2).
\]

Then, for any given \( \omega \) in \( \Omega_{\infty}^{j-1}(X_{\text{reg}}) \), \( \int_{L} d\omega \) and \( \int_{(\partial L)_{\text{reg}}} \omega \) are well defined since \( \omega \) is smooth almost everywhere on \( (\partial L)_{\text{reg}} \) and bounded. We start by recalling a version of Stokes’ formula proved by Lojasiewicz in [L].

**Lemma 4.3.1.** [L] Take \( L \) and \( \omega \) as in the above paragraph. Then:
\[
\int_{(\partial L)_{\text{reg}}} \omega = \int_{L} d\omega.
\]

Lojasiewicz formula is actually devoted to bounded subanalytic forms, but the required property is indeed that they are bounded and extend continuously almost everywhere on the closure of the manifold \( L \), which obviously holds true when the form is \( L^{\infty} \) and \( cl(L) \) is \((t;j)\)-allowable.

Next we turn to see that integration is well defined for any \((t;j)\)-allowable subanalytic singular simplex. Let \( \sigma : \Delta_{j} \to X \) be an oriented \((t;j)\)-allowable simplex. Denote by \( \sigma_{\text{reg}} \) the set of points in \( \sigma^{-1}(X_{\text{reg}}) \) near which \( \sigma \) induces a local diffeomorphism. We then set:
\[
\int_{\sigma} \omega := \int_{\sigma_{\text{reg}}} \sigma^{*} \omega.
\]

If \( \sigma_{\text{reg}} = \emptyset \) (and in this case \( \dim \sigma(\Delta_{j}) < j \)) we just set \( \int_{\sigma} \omega := 0 \).

We now claim that Stokes’ formula continues to hold for subanalytic singular \((t;j)\)-allowable simplices:

**Lemma 4.3.2.** If \( \sigma \in I^{i}C_{j}(X) \) and \( \omega \in \Omega_{\infty}^{j}(X_{\text{reg}}) \):
\[
\int_{\sigma} d\omega = \int_{\partial \sigma} \omega.
\]

**Proof.** Let 
\[
\Gamma := \{(x;y) \in X \times \Delta_{j} : x = \sigma(y)\},
\]
and consider a cell decomposition of \( \mathbb{R}^{n+j} \) compatible with \( \Gamma \). Refining it, we can assume that the boundary of a cell is a union of cells. For simplicity, we will identify \( \Delta_{j} \) with \( \Gamma \) and assume that \( \sigma \) is the canonical projection (restricted to \( \Gamma \)). Let \( C \) be a cell of this cell decomposition and let \( i = \dim C \). Observe that either \( \sigma|_{C} \) is a diffeomorphism or \( \dim \sigma(C) < i \). In the former case, if we endow the image with the orientation induced by \( \sigma \), we have
\[
\int_{\sigma|_{C}} \alpha = \int_{\sigma(C)} \alpha.
\]
for any \( \alpha \in \Omega_{\infty}^{i}(X_{\text{reg}}) \). If \( \dim \sigma(C) < i \), then both vanish and this remains true.
But by Lemma 4.3.1, if $C$ is a cell of dimension $(j - 1)$:

\[(4.22) \int_{\sigma(C)} d\omega = \int_{\partial(\sigma(C))} \omega = \int_{\sigma(\partial C)} \omega.\]

Writing \((4.21)\) for every cell $D$ contained in $\partial C$, we get:

\[\int_D \sigma^* \omega = \int_{\sigma(D)} \omega.\]

Adding up the identities \((4.22)\) for all the cells $C$ of dimension $(j - 1)$ and applying the latter identity we finally get:

\[\int_\sigma d\omega = \int_{\sigma(\partial \Delta_j)} \omega = \int_{\partial \sigma} \omega.\]

\[\square\]

In conclusion, we get that the isomorphism of Theorem 1.2.2 is given by integrating forms:

**Theorem 4.3.3.** Assume that $X_{\text{reg}}$ is orientable. The cochain map:

\[\psi_X : \Omega^j_\infty(X_{\text{reg}}) \rightarrow I^jC^j(X),\]

\[\omega \mapsto [\sigma \mapsto \int_\sigma \omega]\]

induces an isomorphism on cohomology.

To prove it, observe that the cochain map $\psi_X$ induces a sheaf homomorphism (recall that the $L^\infty$ forms give rise to sheaf on $X$, see the proof of Theorem 1.2.2). By the uniqueness of the map between sheaf cohomology theories with coefficient in sheaves of $\mathbb{R}$-modules, this map must coincide with the isomorphism \((4.19)\).

**Remarks 4.3.4.** Theorem 1.2.2 still holds if $X$ is a pseudomanifold with boundary. The relative version is then true as well, by the five lemma. Again, the isomorphism is provided by integration of forms on allowable chains.

It is worthy of notice that the arguments of the proof of Theorem 1.2.2 also apply in the noncompact case, establishing an isomorphism between the cohomology of locally bounded forms (locally in $X$, not in $X_{\text{reg}}$) and intersection homology in the maximal perversity.

The results of this paper remain true if we replace the subanalytic category by a polynomially bounded o-minimal structure. We need the structure to be polynomially bounded for we made use of the preparation theorem for proving Proposition 2.3.1. It is unclear (but not impossible) whether the results of this paper, especially Theorem 2.4.1, are valid on a non polynomially bounded o-minimal structure, especially for log−exp sets, on which a genralized preparation theorem holds [LR].

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