A SEMI-ORDINARY $p$-STABILIZATION OF SIEGEL EISENSTEIN SERIES FOR SYMPLECTIC GROUPS AND ITS $p$-ADIC INTERPOLATION

HISA-AKI KAWAMURA

Abstract. Given a prime number $p$, we introduce a certain $p$-stabilization of holomorphic Siegel Eisenstein series for the symplectic group $\text{Sp}(2n)/\mathbb{Q}$ such that the resulting forms satisfy the semi-ordinary condition at $p$, that is, the associated eigenvalue of a generalized Atkin $U_p$-operator is a $p$-adic unit. In addition, we derive an explicit formula for all Fourier coefficients of such $p$-stabilized Siegel Eisenstein series, and conclude their $p$-adic interpolation problems. This states the existence of a quite natural generalization of the ordinary $\Lambda$-adic Eisenstein series which have been constructed by Hida and Wiles for $\text{GL}(2)/\mathbb{Q}$.

1. INTRODUCTION

Given a positive integer $M$, a Dirichlet character $\chi$ modulo $M$ and an integer $\kappa \geq 2$ with $\chi(-1) = (-1)^\kappa$, the classical (holomorphic) Eisenstein series $E_{\kappa, \chi}(z)$ is defined as follows: For each $z \in \mathfrak{A}_1 = \{ z \in \mathbb{C} \mid \text{Im } z > 0 \}$, put

$E_{\kappa, \chi}(z) := \left\{ \frac{2G(\chi^{-1})(-2\pi\sqrt{-1})^\kappa}{M^\kappa(\kappa - 1)!} \right\}^{-1} \sum_{(c,d) \in \mathbb{Z}^2 \setminus \{(0,0)\}} \chi^{-1}(d) (c Mz + d)^{-\kappa},$

where $G(\chi^{-1}) = \sum_{n=1}^{M} \chi^{-1}(n) \exp(2\pi\sqrt{-1} n/M)$. As is well-known, $E_{\kappa, \chi}$ gives rise to a holomorphic modular form of weight $\kappa$ and nebentypus character $\chi$ for the congruence subgroup $\Gamma_0(M)$ of $\text{SL}(2, \mathbb{Z})$ unless $\kappa = 2$ and $\chi$ is trivial (or principal). If $\chi$ is trivial (i.e., $M = 1$) or primitive (i.e., the conductor of $\chi$ equals $M > 1$), then $E_{\kappa, \chi}$ possesses the Fourier expansion

$E_{\kappa, \chi}(z) = \frac{L(1 - \kappa, \chi)}{2} + \sum_{m=1}^{\infty} \sigma_{\kappa-1, \chi}(m) q^m,$

where $L(s, \chi) = \prod_{l \text{ prime}} (1 - \chi(l) l^{-s})^{-1}$, $\sigma_{\kappa-1, \chi}(m) = \sum_{0 < d | m} \chi(d) d^{\kappa-1}$ and $q = \exp(2\pi\sqrt{-1} z)$. This implies that $E_{\kappa, \chi}$ is a non-cuspidal Hecke eigenform such that the associated $L$-function $L(s, E_{\kappa, \chi})$ is taken of the form

$L(s, E_{\kappa, \chi}) = \zeta(s)L(s - \kappa + 1, \chi).$

Let $p$ be a fixed prime number not dividing $M$, which we assume to be odd for simplicity. Put

$E^*_{\kappa, \chi}(z) := E_{\kappa, \chi}(z) - \chi(p) p^{\kappa-1} E_{\kappa, \chi}(pz).$

We easily see that $E^*_{\kappa, \chi}$ is also a non-cuspidal Hecke eigenform of weight $\kappa$ and nebentypus character $\chi$ for $\Gamma_0(mp) \subset \Gamma_0(M)$, and its Fourier expansion is taken of the form

$E^*_{\kappa, \chi}(z) = \frac{L(p)(1 - \kappa, \chi)}{2} + \sum_{m=1}^{\infty} \sigma_{\kappa-1, \chi}^{(p)}(m) q^m,$

Date: February 23rd, 2023.
2020 Mathematics Subject Classification. 11F46 (Primary); 11F30, 11F33 (Secondary).
Key words and phrases. Siegel modular forms, Siegel Eisenstein series, $\Lambda$-adic forms, $p$-adic analytic families.
where \( L^{(p)}(s, \chi) := (1 - \chi(p) p^{-s}) L(s, \chi) = \prod_{l \neq p} (1 - \chi(l) l^{-s})^{-1} \) and
\[
\sigma_{\kappa - 1, \chi}^{(p)}(m) := \sum_{0 < d \mid m, \gcd(d, p) = 1} \chi(d) d^{\kappa - 1}.
\]

It turns out that for \( E_{\kappa, \chi}^* \), all the Hecke eigenvalues outside \( p \) agree with \( E_{\kappa, \chi} \), but the eigenvalue of Atkin’s \( U_p \)-operator, which is converted from the \( p \)-th Hecke operator \( T_p \) (cf. [AL, Mi]), is \( \sigma_{\kappa - 1, \chi}^{(p)}(p) = 1 \). Namely, we have
\[
L(s, E_{\kappa, \chi}^*) = \zeta(s) L^{(p)}(s - \kappa + 1, \chi).
\]

This type of normalization \( E_{\kappa, \chi} \mapsto E_{\kappa, \chi}^* \) given by eliminating the \( p \)-part of every Fourier coefficient or equivalently, eliminating the latter half of the Euler factor at \( p \) was firstly introduced by Serre [Se], and we refer to it as the ordinary\(^1 \) \( p \)-stabilization. Here we should mention that every Fourier coefficient of \( E_{\kappa, \chi}^* \) depends on the weight \( \kappa \) \( p \)-adically. Indeed, it follows immediately from Fermat’s little theorem that for each positive integer \( m \), the function \( \kappa \mapsto \sigma_{\kappa - 1, \chi}^{(p)}(m) \) can be extended to an analytic function defined on the ring of \( p \)-adic integers \( \mathbb{Z}_p \). In addition, the constant term \( L^{(p)}(1 - \kappa, \chi)/2 \) is also interpolated by the \( p \)-adic \( L \)-function in the sense of Kubota-Leopoldt or Deligne-Ribet [DR]. This fact turns out to be the following theorem due to Iida and Wiles:

**Fact 1.1** (cf. Proposition 7.1.1 in [H1]; Proposition 1.3.1 in [W]). Suppose that a Dirichlet character \( \chi \) modulo \( M \) is either trivial or primitive, and \( M \) is not divisible by \( p \). Let \( \Lambda = \mathbb{Z}_p[\chi][[X]] \) be the one-variable power series ring over \( \mathbb{Z}_p[\chi] \) and \( F_\chi \) the field of fractions of \( \Lambda \), respectively. For each integer \( a \) with \( 0 \leq a < p - 1 \), there exists a formal Fourier expansion
\[
E_{\chi, \omega^a}(X) = \sum_{m=0}^{\infty} A_{\chi, \omega^a}(m; X) q^m \in F_\Lambda[[q]],
\]
where \( \omega : \mathbb{Z}_p^\times \rightarrow \mu_{p-1} = \{ x \in \mathbb{Z}_p^\times \mid x^{p-1} = 1 \} \)\(^2 \) denotes the Teichmüller character, such that for each integer \( \kappa > 2 \) with \( \chi(-1) = (-1)^{\kappa} \) and \( \kappa \equiv a \pmod{p-1} \)\(^3 \), we have
\[
E_{\chi, \omega^a}((1 + p)^\kappa - 1) = E_{\kappa, \chi}^*.
\]
Moreover, if \( \varepsilon : 1 + p\mathbb{Z}_p \rightarrow (\mathbb{Q}_p^{\text{alg}})\)\(^\times \) is a character of exact order \( p^r \) for some integer \( r \geq 0 \), then for each integer \( \kappa \geq 2 \), we have
\[
E_{\chi, \omega^a}(\varepsilon(1 + p)(1 + p)^\kappa - 1) = E_{\kappa, \chi, \omega^a - \kappa \varepsilon}
\]
as long as \( \omega^{\kappa - \kappa \varepsilon} \) is non-trivial.

Strictly speaking, it turns out that \( E_{\chi, \omega^a}(X) \in \Lambda[[q]] \) unless \( \chi, \omega^a \) is trivial. However, we note that if \( \chi, \omega^a \) is trivial, then \( E_{\chi, \omega^a}(X) \) (more precisely, the constant term \( A_{\chi, \omega^a}(0; X) \)) has a simple pole at \( X = 0 \) and thus \( E_{\chi, \omega^a}(X) := X \cdot E_{\chi, \omega^a}(X) \in \Lambda[[q]] \). Hence the above-mentioned fact implies that \( E_{\chi, \omega^a}(X) \) or \( E_{\chi, \omega^a}(X) \) according as \( \chi, \omega^a \) is non-trivial or trivial, is a \( \Lambda \)-adic form of level \( Mp^\infty \) associated with character \( \chi, \omega^a \), which interpolates families of non-cuspidal ordinary Hecke eigenforms \( \{ E_{\kappa, \chi}^* \} \) and \( \{ E_{\kappa, \chi, \omega^a - \kappa \varepsilon} \} \) or their constant multiples, given by varying the weight \( \kappa \) \( p \)-adically analytically (cf. [W, H1]). In this context, we refer to it as the **ordinary \( \Lambda \)-adic Eisenstein series** of genus 1 and level \( Mp^\infty \) associated with character \( \chi, \omega^a \).

The aim of the present article is to formulate a similar statement in the case of Siegel modular forms, that is, automorphic forms on the symplectic group \( \text{Sp}(2n)/\mathbb{Q} \) of an arbitrary genus \( n \geq 1 \).

\(^1\)In general, a Hecke eigenform \( f \) is said to be **ordinary** at \( p \) if the eigenvalue of \( U_p \) (or \( T_p \)) is a \( p \)-adic unit.

\(^2\)Since \( \mu_{p-1} \simeq (\mathbb{Z}/p\mathbb{Z})^\times \), \( \omega \) can be identified with a Dirichlet character modulo \( p \) in a natural way.

\(^3\)The latter condition is regarded as if \( \omega^{\kappa - \kappa \varepsilon} \) corresponds to the trivial Dirichlet character.

\(^4\)Namely, \( \varepsilon \) can be also regarded as a primitive Dirichlet character of conductor \( p^\varepsilon \).
Let us explain how it goes briefly: Given a positive integer \( \kappa > n + 1 \) and a Dirichlet character \( \chi \) modulo a positive integer \( M \), let \( E_{\kappa, \chi}^{(n)} \) be the classical (holomorphic) Siegel Eisenstein series of weight \( \kappa \) and nebentypus character \( \chi \) for \( \Gamma_0(M)^{(n)} \subset \text{Sp}(2n, \mathbb{Z}) \), to be described in the subsequent §2, whenever either

1. \( M = 1 \), and thus, \( \chi \) is trivial or
2. \( M > 1 \) is odd, \( \chi \) is primitive and \( \chi^2 \) is locally non-trivial at every prime \( l \mid M \).

For an arbitrary prime number \( p \nmid M \), we define, in §4, a certain \( p \)-stabilization map \( E_{\kappa, \chi}^{(n)} \mapsto (E_{\kappa, \chi}^{(n)})^* \) by means of the action of a linear combination of \( (U_{p,n})^* \)'s for \( i = 0, 1, \ldots, n \), where \( U_{p,n} \) denotes a generalized Atkin \( U_p \)-operator to be defined in §3 below, so that the associated eigenvalue of \( U_{p,n} \) is 1. This formulation is natural from the viewpoint of which in the case where \( n = 1 \), \( E_{\kappa, \chi}^* = (E_{\kappa, \chi}^{(1)})^* \) can be easily written in terms of Atkin’s original operator \( U_p = U_{p,1} \) as

\[
E_{\kappa, \chi}^* = E_{\kappa, \chi} \|_{\kappa} (U_p - \chi(p)p^{\kappa-1}).
\]

Following in the tradition of Skinner-Urban [SU], in the case where \( n > 1 \), we call \( (E_{\kappa, \chi}^{(n)})^* \) semi-ordinary at \( p \). Moreover, we derive an explicit formula for all Fourier coefficients of \( (E_{\kappa, \chi}^{(n)})^* \), which can be regarded as a natural generalization of the equation (2) (cf. Theorem 4.2 below). As a consequence of the above issues, we first show in Theorem 4.4 below the existence of a formal Fourier expansion with coefficients in \( F_{\Lambda} \) associated to \( \chi \omega^a \) with some nonnegative integer \( a \), whose specialization at \( X = (1 + p)^\kappa - 1 \) coincides with the semi-ordinary \( p \)-stabilized Siegel Eisenstein series \( (E_{\kappa, \chi}^{(n)})^* \) for any \( \kappa \) taken as above. Finally in §5, for a fixed odd prime number \( p \), we show that a suitable constant multiplication of the above-mentioned formal Fourier expansion gives rise to a \( \Lambda \)-adic form of genus \( n \) and level \( Mp^\infty \) (or tame level \( M \) in the sense of Taylor [Ta]) associated with character \( \chi \omega^a \), which can be viewed as a satisfactory generalization of Fact 1.1 (cf. Theorem 5.4 below).

**Notation.** We summarize here some notation we will use in the sequel. We denote by \( \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \) and \( \mathbb{C} \) the ring of integers, fields of rational numbers, real numbers and complex numbers, respectively. Let \( \mathbb{Q}_{\text{alg}} \) denote the algebraic closure of \( \mathbb{Q} \) sitting inside \( \mathbb{C} \). We put \( e(x) = \exp(2\pi \sqrt{-1}x) \) for \( x \in \mathbb{C} \). Given a prime number \( p \), we denote by \( \mathbb{Q}_p, \mathbb{Z}_p \) and \( \mathbb{Z}_p^\times \) the field of \( p \)-adic numbers, the ring of \( p \)-adic integers and the group of \( p \)-adic units, respectively. Hereinafter, given a prime number \( p \), we fix an algebraic closure \( \mathbb{Q}_p^{\text{alg}} \) of \( \mathbb{Q}_p \) and an embedding \( \iota_p : \mathbb{Q}_p^{\text{alg}} \hookrightarrow \mathbb{Q}_p^{\text{alg}} \) once for all. Let \( \text{val}_p \) denote the \( p \)-adic valuation on \( \mathbb{Q}_p^{\text{alg}} \) normalized so that \( \text{val}_p(p) = 1 \), and \( |*|_p \) the corresponding norm on \( \mathbb{Q}_p^{\text{alg}} \), respectively. Let \( e_p \) be the continuous additive character of \( \mathbb{Q}_p^{\text{alg}} \) such that \( e_p(x) = e(x) \) for all \( x \in \mathbb{Q}_p \). Let \( \mathbb{C}_p \) be the completion of the normed space \( (\mathbb{Q}_p^{\text{alg}}, |*|_p) \). We also fix, once for all, an isomorphism \( \tilde{\iota}_p : \mathbb{C} \cong \mathbb{C}_p \) such that the diagram

\[
\begin{array}{ccc}
\mathbb{C} & \xrightarrow{\tilde{\iota}_p} & \mathbb{C}_p \\
\uparrow & & \uparrow \\
\mathbb{Q}_p^{\text{alg}} & \hookrightarrow & \mathbb{Q}_p^{\text{alg}}
\end{array}
\]

is commutative. Given a prime number \( p \), put

\[
P = \begin{cases} 
4 & \text{if } p = 2, \\
p & \text{otherwise}.
\end{cases}
\]

Let \( \mathbb{Z}_p^{\times, \text{tor}} \) be the torsion subgroup of \( \mathbb{Z}_p^{\times} \), that is, \( \mathbb{Z}_p^{\times, \text{tor}} = \{ \pm 1 \} \) and \( \mathbb{Z}_p^{\times, \text{tor}} = \mu_{p-1} \) if \( p \neq 2 \). We define the Teichmüller character \( \omega : \mathbb{Z}_p^{\times} \to \mathbb{Z}_p^{\times, \text{tor}} \) by putting \( \omega(x) = \pm 1 \) according as \( x \equiv 1 \)

\[\text{Namely, if we factor } \chi \text{ as } \chi = \prod_{l \mid M} \chi_l, \text{ then for each prime factor } l \text{ of } M, \chi_l \text{ is not a quadratic character.}\]
For each \( \kappa \) following two conditions: 

\[ \langle x \rangle := \omega(x)^{-1}x \in 1 + p\mathbb{Z}_p. \]

Then we have a canonical isomorphism 

\[ \mathbb{Z}_p^{\times} \cong \mathbb{Z}_{p,\text{tor}}^{\times} \times (1 + p\mathbb{Z}_p), \]

\[ x \mapsto (\omega(x), \langle x \rangle). \]

We note that the maximal torsion-free subgroup \( 1 + p\mathbb{Z}_p \) of \( \mathbb{Z}_p^{\times} \) is topologically cyclic, that is, \( 1 + p\mathbb{Z}_p = (1 + p)^{\mathbb{Z}_p} \). As already mentioned in §1, the Teichmüller character \( \omega \) gives rise to a Dirichlet character \( \omega : (\mathbb{Z}/p\mathbb{Z})^{\times} \cong \mathbb{Z}_{p,\text{tor}}^{\times} \rightarrow \mathbb{C}_{p}^{\times} \cong \mathbb{C}^{\times} \). Let \( \varepsilon : 1 + p\mathbb{Z}_p \rightarrow (\mathbb{Q}^{\text{alg}}/\mathbb{Z})^{\times} \) be a character of finite order. More precisely, if \( \varepsilon \) has exact order \( p^m \) for some nonnegative integer \( m \), then \( \varepsilon \) optimally factors through

\[ 1 + p\mathbb{Z}_p/(1 + p)^{p^m} \simeq (1 + p)^{p^m} \mathbb{Z}_p/(1 + p)^{p^m} \mathbb{Z}. \]

Since \( (\mathbb{Z}/p^m\mathbb{Z})^{\times} \cong (\mathbb{Z}/p\mathbb{Z})^{\times} \times \mathbb{Z}/p^m\mathbb{Z} \), we may naturally regard \( \varepsilon \) as a Dirichlet character of conductor \( p^m \).

Given a positive integer \( n \), let \( \text{GSp}(2n) \) be the group of symplectic similitudes over \( \mathbb{Q} \), that is,

\[ \text{GSp}(2n) := \{ g \in \text{GL}(2n) \mid ^t g J g = \nu(g) J \text{ for some } \nu(g) \in \mathbb{G}_m \}, \]

where \( J = \begin{bmatrix} 0_n & -1_n \\ 1_n & 0_n \end{bmatrix} \) with the \( n \times n \) unit (resp. zero) matrix \( 1_n \) (resp. \( 0_n \)), and \( \text{Sp}(2n) \) the derived group of \( \text{GSp}(2n) \) characterized by the exact sequence

\[ 1 \rightarrow \text{Sp}(2n) \rightarrow \text{GSp}(2n) \rightarrow \mathbb{G}_m \rightarrow 1, \]

respectively. Namely, \( \text{GSp}(2) = \text{GL}(2) \) and \( \text{Sp}(2) = \text{SL}(2) \) in this setting. We note that every real point \( g = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \text{GSp}(2n, \mathbb{R}) \) with \( \nu(g) > 0 \), where \( A, B, C, D \in \text{Mat}_{n \times n}(\mathbb{R}) \), acts on the Siegel upper-half space

\[ \mathfrak{H}_n := \{ Z = X + \sqrt{-1} Y \in \text{Mat}_{n \times n}(\mathbb{C}) \mid ^t Z = Z, Y > 0 \text{ (positive-definite)} \}. \]

of genus \( n \) via the linear transformation \( Z \mapsto g(Z) = (AZ + B)(CZ + D)^{-1} \). If \( F \) is a function on \( \mathfrak{H}_n \), then for each \( \kappa \in \mathbb{Z} \), we define the slash action of \( g \) on \( F \) by

\[ (F|_\kappa g)(Z) := \nu(g)^{\kappa - n(n+1)/2} \det(CZ + D)^{-\kappa} F(g(Z)). \]

For each positive integer \( N \), we shall consider the following congruence subgroups of level \( N \) for the full-modular group \( \text{Sp}(2n, \mathbb{Z}) \):

\[ \Gamma_0(N)(\mathbb{Z}) \text{ (resp. } \Gamma_1(N)(\mathbb{Z}) \text{)} := \left\{ \gamma \in \text{Sp}(2n, \mathbb{Z}) \mid \gamma \equiv \begin{bmatrix} * & * \\ 0_n & * \end{bmatrix} \text{ (resp. } \begin{bmatrix} * & * \\ 0_n & 1_n \end{bmatrix} \text{)} \pmod{N} \right\}. \]

For each \( \kappa \in \mathbb{Z} \), let us denote by \( \mathcal{M}_\kappa(\Gamma_1(N))(\mathbb{Z}) \) the space of (holomorphic) Siegel modular forms of genus \( n \), weight \( \kappa \) and level \( N \), that is, \( \mathbb{C} \)-valued holomorphic functions \( F \) on \( \mathfrak{H}_n \) satisfying the following two conditions:

(i) \( F|_\kappa \gamma = F \) for any \( \gamma \in \Gamma_1(N)(\mathbb{Z}) \).

(ii) For each \( \gamma \in \text{Sp}(2n, \mathbb{Z}) \), the function \( F|_\kappa \gamma \) possesses a Fourier expansion of the form

\[ (F|_\kappa \gamma)(Z) = \sum_{T \in \text{Sym}^*_n(\mathbb{Z})} A_{F,\gamma}(T) e(\text{tr}(TZ)), \]

where \( \text{Sym}^*_n(\mathbb{Z}) \) is the set of all half-integral symmetric matrices of degree \( n \) over \( \mathbb{Z} \), namely,

\[ \text{Sym}^*_n(\mathbb{Z}) := \{ T = [t_{ij}] \in \text{Sym}_n(\mathbb{Q}) \mid t_{ii}, 2t_{ij} \in \mathbb{Z} \ (1 \leq i < j \leq n) \}, \]

and \( \text{tr}(\ast) \) denotes the trace. Then it is satisfied that

\[ A_{F,\gamma}(T) = 0 \text{ unless } T \geq 0 \text{ (positive-semidefinite)} \]

for all \( \gamma \in \text{Sp}(2n, \mathbb{Z}) \).
A Siegel modular form $F \in \mathcal{M}_{\kappa}(\Gamma(1)(N))^{(n)}$ is said to be \textit{cuspidal} if it is satisfied that $A_{F, \gamma}(T) = 0$ unless $T > 0$ for all $\gamma \in \text{Sp}(2n, \mathbb{Z})$. We denote by $\mathcal{S}_{\kappa}(\Gamma(1)(N))^{(n)}$ the subspace of $\mathcal{M}_{\kappa}(\Gamma(1)(N))^{(n)}$ consisting of all cuspidal forms. Given a Dirichlet character $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$, we denote by $\mathcal{M}_{\kappa}(\Gamma_0(N), \chi)^{(n)}$ (resp. $\mathcal{S}_{\kappa}(\Gamma_0(N), \chi)^{(n)}$) the subspace of $\mathcal{M}_{\kappa}(\Gamma(1)(N))^{(n)}$ (resp. $\mathcal{S}_{\kappa}(\Gamma(1)(N))^{(n)}$) consisting of all forms $F$ with nebentypus character $\chi$, that is,

$$F \parallel_{\kappa} \gamma = \chi(\det D)F \text{ for any } \gamma = [A \, B \mid C \, D] \in \Gamma_0(N)^{(n)}.$$ 

In particular, whenever $\chi$ is trivial, we naturally write $\mathcal{M}_{\kappa}(\Gamma_0(N))^{(n)} = \mathcal{M}_{\kappa}(\Gamma_0(N), \text{triv})^{(n)}$ and $\mathcal{S}_{\kappa}(\Gamma_0(N))^{(n)} = \mathcal{S}_{\kappa}(\Gamma_0(N), \text{triv})^{(n)}$, respectively.

For a given pair of $Z = [z_{ij}] \in \mathfrak{H}_n$ and $T = [t_{ij}] \in \text{Sym}_n^+(\mathbb{Z})$, put

$$q^T := e(\text{tr}(TZ)) = \prod_{i=1}^n q_{ii}^{t_{ii}} \prod_{i<j} q_{ij}^{2t_{ij}},$$

where $q_{ij} = e(z_{ij}) (1 \leq i \leq j \leq n)$. Since $[1_n \, S] \in \Gamma(1)(N)^{(n)} \subset \Gamma_0(N)^{(n)}$ for each $S \in \text{Sym}_n(\mathbb{Z})$, we easily see that if $F \in \mathcal{M}_{\kappa}(\Gamma(1)(N))^{(n)}$ (or $\mathcal{M}_{\kappa}(\Gamma_0(N), \chi)^{(n)}$), then $F$ possesses a Fourier expansion of the form

$$F(Z) = \sum_{T \in \text{Sym}_n^+(\mathbb{Z}), \, T \geq 0} A_{F}(T) \, q^T,$$

which is regarded as belonging to the ring $\mathbb{C}[[q_{ij}^{\pm 1} \mid 1 \leq i < j \leq n]][[q_{11}, \cdots, q_{nn}]]$. Given a ring $R$, we write

$$R[[q]]^{(n)} := R[[q_{ij}^{\pm 1} \mid 1 \leq i < j \leq n]][[q_{11}, \cdots, q_{nn}]],$$

in a similar fashion to the notation of the ring of formal $q$-expansions $R[[q]]$. In particular, if $F \in \mathcal{M}_{\kappa}(\Gamma(1)(N))^{(n)}$ is a Hecke eigenform (i.e., a simultaneous eigenfunction of all Hecke operators whose similitude is coprime to $N$), then it is well-known that the field $K_F$ obtained by adjoining all Fourier coefficients (or equivalently, all Hecke eigenvalues) of $F$ to $\mathbb{Q}$ is an algebraic number field. Thus, by virtue of the presence of $t_p$ and $\tilde{t}_p$, we may regard $F \in K_F[[q]]^{(n)}$ as sitting inside $\mathbb{C}[[q]]^{(n)}$ and $\mathbb{C}_p[[q]]^{(n)}$ interchangeably. For further details on the basic theory of Siegel modular forms set out above, see [AZ] or [Fr]. In particular, a comprehensive introduction to the theory of elliptic modular forms and Hecke operators can be found in [Mi].

2. \textbf{Siegel Eisenstein series for symplectic groups}

In this section, we review some elementary facts on the Siegel Eisenstein series defined for $\text{Sp}(2n)/\mathbb{Q}$ of an arbitrary genus $n \geq 1$. In particular, we describe a explicit form of its Fourier expansion according to some previous works of Shimura (e.g., [S1, S2]), which is the starting point for the subsequent arguments.

Let $N$ be a positive integer and $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ a Dirichlet character, respectively. As mentioned in §1, for simplicity, we restrict ourselves to either of the following cases:

(i) $N = 1$, that is, $\chi$ is trivial;

(ii) $N > 1$ is odd, $\chi$ is primitive and $\chi^2$ is locally non-trivial at every prime $l \mid N$.

Given a positive integer $n$, if $\kappa$ is an integer with $\kappa > n + 1$ and $\chi(-1) = (-1)^\kappa$, then the \textit{(holomorphic) Siegel Eisenstein series} of genus $n$, weight $\kappa$ and level $N$ with nebentypus character $\chi$ is defined as follows: For each $Z \in \mathfrak{H}_n$, put

$$E_{\kappa, \chi}^{(n)}(Z) := 2^{-(n+1)/2} L(1, \kappa, \chi) \prod_{i=1}^{[n/2]} L(1 - 2\kappa + 2i, \chi^2)$$
\[\sum_{\gamma = [C \ D] \in (P_{2n} \cap \Gamma_0(N)) \setminus \Gamma_0(N) \cap \Gamma_0(N)} \chi^{-1}(\det D) \det(CZ + D)^{-\kappa},\]

where \(L(s, \psi)\) denotes Dirichlet’s \(L\)-function associated with some character \(\psi\), and \(P_{2n}\) the Siegel parabolic subgroup of \(\text{Sp}(2n)\) consisting of all matrices \(g = [\alpha, \beta]\), respectively.

Let \(r\) be a positive integer. For each rational prime \(l\), let \(\text{Sym}^r_\ast(\mathbb{Z}_l)\) denote the set of all half-integral symmetric matrices of degree \(r\) over \(\mathbb{Z}_l\). Given a nondegenerate \(S \in \text{Sym}^r_\ast(\mathbb{Z}_l)\), we define a formal power series \(b_l(S; X)\) in \(X\) by

\[b_l(S; X) := \sum_{R \in \text{Sym}_\ast(\mathbb{Q}_l)/\text{Sym}_\ast(\mathbb{Z}_l)} e_l(\text{tr}(SR))X^{\text{val}_l(\mu_R)},\]

where \(\mu_R = [\mathbb{Z}_l^r + \mathbb{Z}_l^r R : \mathbb{Z}_l^r]\). Put \(\mathfrak{D}_S := 2^{[r/2]} \det S\). We note that if \(r\) is even, then \((-1)^{r/2} \mathfrak{D}_S \equiv 0\) or \(1\) (mod 4), and thus, we may decompose it into the form

\[(-1)^{r/2} \mathfrak{D}_S = \mathfrak{d}_S f_S^2,\]

where \(\mathfrak{d}_S\) is the fundamental discriminant of the quadratic field extension \(\mathbb{Q}_l((-1)^{r/2} \mathfrak{D}_S)^{1/2})/\mathbb{Q}_l\) and \(f_S = \{(-1)^{r/2} \mathfrak{D}_S/\mathfrak{d}_S\}^{1/2} \in \mathbb{Z}_l\). Let \(\xi_l : \mathbb{Q}_l^\times \rightarrow \{\pm 1, 0\}\) denote the character defined by

\[\xi_l(x) = \begin{cases} 
1 & \text{if } \mathbb{Q}_l(x^{1/2}) = \mathbb{Q}_l, \\
-1 & \text{if } \mathbb{Q}_l(x^{1/2})/\mathbb{Q}_l \text{ is unramified}, \\
0 & \text{if } \mathbb{Q}_l(x^{1/2})/\mathbb{Q}_l \text{ is ramified.}
\end{cases}\]

As shown in [Ki1, Fe, S2], for each nondegenerate \(S \in \text{Sym}^r_\ast(\mathbb{Z}_l)\), there exists a polynomial \(F_l(S; X) \in \mathbb{Z}[X]\) whose constant term is 1 such that \(b_l(S; X)\) is decomposed as follows:

\[b_l(S; X) = F_l(S; X) \times \begin{cases} 
\frac{(1 - X) \prod_{i=1}^{r/2} (1 - l^{2i} X^2)}{1 - \xi_l((-1)^{r/2} \det S) l^{r/2} X} & \text{if } r \text{ is even,} \\
\frac{(1 - X) \prod_{i=1}^{(r-1)/2} (1 - l^{2i} X^2)}{1 - \xi_l((-1)^{r/2} \det S) l^{(r-1)/2} X} & \text{if } r \text{ is odd}
\end{cases}\]

(cf. Proposition 3.6 in [S2]). We note that \(F_l(S; X)\) satisfies the functional equation

\[F_l(S; l^{-r-1}X^{-1}) = F_l(S; X) \times \begin{cases} 
(l^{r+1} X^2)^{-\text{val}_l(f_S)} & \text{if } r \text{ is even,} \\
\eta_l(S)(l^{(r+1)/2} X)^{-\text{val}_l(\mathfrak{D}_S)} & \text{if } r \text{ is odd,}
\end{cases}\]

where

\[\eta_l(S) := h_l(S) (\det S, (-1)^{(r-1)/2} \det S) l (-1, -1)_l^{(r^2-1)/8}\]

in terms of the Hasse invariant \(h_l(S)\) in the sense of Kitaoka [Ki1] and the Hilbert symbol \((\ast, \ast)_l\) defined over \(\mathbb{Q}_l\) (cf. Theorem 3.2 in [Ki1]). Thus, it turns out that \(F_l(S; X)\) has degree \(2\text{val}_l(f_S)\) or \(\text{val}_l(\mathfrak{D}_S)\) according as \(r\) is even or odd. We easily see that \(F_l(uS; X) = F_l(S; X)\) for each \(u \in \mathbb{Z}_l^\times\), and that if \(S, T \in \text{Sym}^r_\ast(\mathbb{Z}_l)\) are equivalent over \(\mathbb{Z}_l\), that is, \(T = USU^{-1}\) for some \(U \in \text{GL}(r, \mathbb{Z}_l)\), then \(F_l(S; X) = F_l(T; X)\). For further details on the above-mentioned issues, see [Ki1].

**Lemma 2.1.** Let \(n, \kappa, N\) and \(\chi\) be taken as above.

(I) \(E^{(n)}_{\kappa, \chi} \in \mathcal{M}_\kappa(\Gamma_0(N), \chi^{(n)})\) and it is a Hecke eigenform, that is, a simultaneous eigenfunction of Hecke operators defined at least for all primes not dividing the level \(N\).

(II) Let us consider a Fourier expansion of \(E^{(n)}_{\kappa, \chi}\) taken of the form

\[E^{(n)}_{\kappa, \chi}(Z) = \sum_{T \in \text{Sym}^n_\ast(\mathbb{Z})} A_{\kappa, \chi}(T) q^T.\]
Then every coefficient \( A_{\kappa, \chi}(T) \), which is invariant under \( T \mapsto TUTU \) for \( U \in \text{GL}(n, \mathbb{Z}) \), is described as follows:

(II a) For \( T = 0_n \in \text{Sym}_n^*(\mathbb{Z}) \), we have

\[
A_{\kappa, \chi}(T) = 2^{-[(n+1)/2]} L(1 - \kappa, \chi) \prod_{i=1}^{[n/2]} L(1 - 2\kappa + 2i, \chi^2).
\]

Therefore \( E_{\kappa, \chi}^{(n)} \) is not cuspidal.

(II b) If \( T \in \text{Sym}_n^*(\mathbb{Z}) \) is taken of the form

\[
T = \begin{bmatrix} T' & \ast \\ 0_{n-r} & \ast \end{bmatrix}
\]

for some nondegenerate \( T' \in \text{Sym}_n^*(\mathbb{Z}) \) with \( 0 < r \leq n \) (i.e., rank \( T = \text{rank} \ T' = r \)), then

\[
A_{\kappa, \chi}(T) = 2^{[(r+1)/2] - [(n+1)/2]} \prod_{i=[r/2]+1}^{[n/2]} L(1 - 2\kappa + 2i, \chi^2)
\times
\begin{cases}
L(1 - \kappa + r/2, \left(\frac{2\pi}{r}\right) \chi) \prod_{l | T' \cap P} F_l(T'; \chi(l) l^{r-1}) & \text{if } r \text{ is even},
\prod_{l | T' \cap P} F_l(T'; \chi(l) l^{r-1}) & \text{if } r \text{ is odd},
\end{cases}
\]

where \( \left(\frac{\alpha}{\nu}\right) \) denotes the Kronecker symbol.

**Remark 2.2.** For the convenience in the sequel, we make the convention for \( r = 0 \), that \( \mathfrak{D}_S = \mathfrak{d}_S = f_S = 1 \), and \( F_l(S; X) = 1 \) for all primes \( l \). This enables us to regard (II a) as (II b) for \( r = 0 \).

**Proof.** The assertion (I) is well-known. The assertion (II) can be obtained by exploiting an idea of Shimura [S2] as follows: Let \( \mathbb{A} \) be the ring of adeles over \( \mathbb{Q} \), \( G_{2n}(\mathbb{A}) = \text{Sp}(2n, \mathbb{A}) \), \( P_{2n}(\mathbb{A}) = M_{2n}(\mathbb{A})N_{2n}(\mathbb{A}) \) a Levi decomposition of the Siegel parabolic subgroup, where

\[
M_{2n} := \left\{ \begin{bmatrix} A & 0_n \\ 0_n & A^{-1} \end{bmatrix} \bigg| A \in \text{GL}(n) \right\}, \quad N_{2n} := \left\{ \begin{bmatrix} 1_n & B \\ 0_n & 1_n \end{bmatrix} \bigg| tB = B \right\},
\]

and \( \chi : \mathbb{A}^\times / \mathbb{Q}^\times \to \mathbb{C}^\times \) the unitary Hecke character corresponding to \( \chi \), respectively. For each \( s \in \mathbb{C} \), let \( \text{Ind}_{P_{2n}(\mathbb{A})}^{G_{2n}(\mathbb{A})}(\chi \cdot | \ast |_\mathbb{A}) \) denote the normalized smooth representation induced from the character of \( \text{GL}(n, \mathbb{A}) \simeq M_{2n}(\mathbb{A}) \) defined by \( A \mapsto \chi(\det A) | \det A|_{\mathbb{A}}^s \), where \( | \ast |_{\mathbb{A}} \) denotes the norm on \( \mathbb{A} \). Choosing a suitable section \( \varphi(s) \in \text{Ind}_{P_{2n}(\mathbb{A})}^{G_{2n}(\mathbb{A})}(\chi \cdot | \ast |_\mathbb{A}) \), we define the Eisenstein series \( E(\varphi(s)) \) on \( G_{2n}(\mathbb{A}) \) by

\[
E(\varphi(s))(g) := \sum_{\gamma \in P_{2n} \backslash G_{2n}} \varphi(s)(\gamma g),
\]

which converges absolutely for \( \text{Re}(s) \gg 0 \), and \( E(\varphi(s)) \) evaluates

\[
2^{[(n+1)/2]} L(1 - \kappa, \chi)^{-1} \prod_{i=1}^{[n/2]} L(1 - 2\kappa + 2i, \chi^2)^{-1} E_{\kappa, \chi}^{(n)}.
\]

Thus, the desired equation can be obtained from an explicit formula for Fourier coefficients of \( E(\varphi(s)) \), more precisely, local Whittaker functions \( \text{Wh}_T(\varphi_v(s)) \) defined on \( G_{2n}(\mathbb{Q}_v) \) for all places \( v \) of \( \mathbb{Q} \), where \( \mathbb{A} = \prod_v \mathbb{Q}_v, \chi = \prod_v \chi_v \) and \( \varphi(s) = \prod_v \varphi_v(s) \). Whenever \( v \) is archimedean (i.e., \( v = \infty \)) or non-archimedean at which \( \chi_v \) is unramified (i.e., \( v \) is a prime \( l \) not dividing \( N \)), it has been
proved by Shimura (cf. Equations 4.34-35K in [S1] and Proposition 7.2 in [S2]). Whenever \( v \) is a non-archimedean place at which \( \chi \) is ramified and \( \chi_p^2 \) is non-trivial, the local Whittaker function \( \text{Wh}_T(\varphi_v^{(s)}) \) is described by Takemori [T] in which the key argument relies upon a functional equation of \( \text{Wh}_T(\varphi_v^{(s)}) \) due to Ikeda [Ik] (generalizing [Sw]).

\[\square\]

3. GENERALIZED ATKIN \( U_p \)-OPERATOR

In this section, we recall the theory of Atkin’s \( U_p \)-operator and its generalization. For further details on the facts set out below, see, for instance, [AZ, Bö, Ta].

Let \( p \) be a prime number and \( N \) a positive integer, respectively. If \( p \mid N \), the coset decomposition

\[
\Gamma_0(N) \overset{(1)}{\longrightarrow} \begin{bmatrix} 1 & 0 \\ 0 & p \end{bmatrix} \Gamma_0(N) = \bigsqcup_{s=0}^{p-1} \Gamma_0(N) \overset{(1)}{\longrightarrow} \begin{bmatrix} 1 & s \\ 0 & p \end{bmatrix}
\]

induces the following linear operator \( U_p = U_{p,1} \) on \( \mathcal{M}_\kappa(\Gamma_0(N), \chi)^{(1)} \): For each \( f \in \mathcal{M}_\kappa(\Gamma_0(N), \chi)^{(1)} \), put

\[
(f \mid_\kappa U_p)(z) := \sum_{s=0}^{p-1} \left( f \mid_\kappa \begin{bmatrix} 1 & s \\ 0 & p \end{bmatrix} \right)(z) = p^{-1} \sum_{s=0}^{p-1} f \left( \frac{z + s}{p} \right) \in \mathcal{M}_\kappa(\Gamma_0(N), \chi)^{(1)}.
\]

We easily see that it is written in terms of Fourier expansions as

\[
f(z) = \sum_{m=0}^{\infty} a(m) q^m \quad \mapsto \quad (f \mid_\kappa U_p)(z) = \sum_{m=0}^{\infty} a(pm) q^m
\]

and this is still valid even if \( p \nmid N \), however it maps from \( \mathcal{M}_\kappa(\Gamma_0(N), \chi)^{(1)} \) to \( \mathcal{M}_\kappa(\Gamma_0(Np), \chi)^{(1)} \) in this case. We refer to the operator \( U_p \), regardless of whether \( p \mid N \) or not, as Atkin’s \( U_p \)-operator.

Remark 3.1. Obviously, \( U_p \) coincides with the usual Hecke operator \( T_p \) if \( p \mid N \). However, \( U_p \) is slightly different from \( T_p \) in general:

\[
\Gamma_0(N) \overset{(1)}{\longrightarrow} \begin{bmatrix} 1 & 0 \\ 0 & p \end{bmatrix} \Gamma_0(N) = \bigsqcup_{s=0}^{p-1} \Gamma_0(N) \overset{(1)}{\longrightarrow} \begin{bmatrix} 1 & s \\ 0 & p \end{bmatrix} \sqcup \Gamma_0(N) \overset{(1)}{\longrightarrow} \begin{bmatrix} p & 0 \\ 0 & 1 \end{bmatrix}
\]

if \( p \nmid N \).

Similarly, if \( n > 1 \) and \( p \mid N \), the following \( n \) double-coset operators at \( p \) are relevant for Siegel modular forms of genus \( n \) and level \( N \):

\[
U_{p,i} := \begin{cases} 
\Gamma_0(N)^{(n)}(i) \frac{\operatorname{diag}(1, \ldots, 1, p, \ldots, p, p^2, \ldots, p^2, \ldots, p)}{i \, \underbrace{\ldots}_{n-1} \, \underbrace{\ldots}_{n-i}} \Gamma_0(N)^{(n)} & \text{if } 1 \leq i \leq n-1, \\
\Gamma_0(N)^{(n)} \frac{\operatorname{diag}(1, \ldots, 1, p, \ldots, p, p^n, \ldots, p^n)}{n \, \underbrace{\ldots}_{n-1} \, \underbrace{\ldots}_{n-i}} \Gamma_0(N)^{(n)} & \text{if } i = n.
\end{cases}
\]

We note that if \( N \) is divisible by \( p \), these operators \( U_{p,1}, \ldots, U_{p,n-1} \) and \( U_{p,n} \) generate the dilating Hecke algebra at \( p \) acting on \( \mathcal{M}_\kappa(\Gamma_0(N), \chi)^{(n)} \). In particular, we are interested in the operator \( U_{p,n} \) which plays a central role among them. Namely, we define the operator \( U_{p,n} \) on \( \mathbb{C}_p[[q]]^{(n)} \) by

\[
F = \sum_{T \geq 0} A_F(T) q^T \mapsto F \mid_\kappa U_{p,n} = \sum_{T \geq 0} A_F(pT) q^T.
\]
Indeed, we easily see that if \( F \in \mathcal{M}_\kappa(\Gamma_0(N), \chi^{(n)}) \) with some positive integers \( \kappa, N \) and a Dirichlet character \( \chi \), then
\[
F \|_{U_{p,n}} \in \begin{cases} 
\mathcal{M}_\kappa(\Gamma_0(N), \chi^{(n)}) & \text{if } p \mid N, \\
\mathcal{M}_\kappa(\Gamma_0(Np), \chi^{(n)}) & \text{if } p \nmid N
\end{cases}
\]
and the action of \( U_{p,n} \) commutes with those of the Hecke operators defined outside \( Np \).

4. Semi-ordinary \( p \)-stabilization of Siegel Eisenstein series

Let us fix an odd integer \( M \) and a prime number \( p \mid M \) (including \( p = 2 \)) once for all. In this section, for a pair of positive integers \( n, \kappa \) and a Dirichlet character \( \chi \) modulo \( M \) taken as in §2, that is,
\[
\kappa > n + 1, \quad \chi(-1) = (-1)^\kappa \quad \text{and} \quad \chi^2 \text{ is locally non-trivial at every prime } l \mid M \text{ if } M > 1,
\]
we introduce a certain \( p \)-stabilization of the Siegel Eisenstein series \( E_{\kappa, \chi}^{(n)} \in \mathcal{M}_\kappa(\Gamma_0(M), \chi) \) which can be viewed as a natural generalization of the ordinary \( p \)-stabilization \( E_{\kappa, \chi}^{(1)} \mapsto (E_{\kappa, \chi}^{(1)})^* \) (cf. Equations (1) and (1')). Moreover, we derive an explicit form of the associated Fourier expansion
\[
(E_{\kappa, \chi}^{(n)})^*(Z) = \sum_{T \in \text{Sym}_n^+(\mathbb{Z}), \ T \geq 0} A_{\kappa, \chi}^*(T) q^T,
\]
which is expressed in a similar fashion to (2), and conclude its \( p \)-adic interpolation problem.

To begin with, we introduce the following two polynomials in \( X \) and \( Y \):
\[
P_p^{(n)}(X, Y) := (1 - p^n XY) \prod_{i=1}^{\lfloor n/2 \rfloor} (1 - p^{2n-2i+1}X^2Y),
\]
\[
R_p^{(n)}(X, Y) := \prod_{j=1}^n (1 - p^{j(2n-j+1)/2}X^jY).
\]
In addition, let us denote by \( \tilde{R}_p^{(n)}(X, Y) \) the reflected polynomial of \( R_p^{(n)}(X, Y) \) with respect to \( Y \), that is,
\[
\tilde{R}_p^{(n)}(X, Y) := Y^n R_p^{(n)}(X, Y^{-1}) = \prod_{j=1}^n (Y - p^{j(2n-j+1)/2}X^j).
\]

Remark 4.1. Whenever \( n = 1 \) and \( 2 \), a straightforward calculation yields
\[
\begin{align*}
P_p^{(1)}(X, Y) &= R_p^{(1)}(X, Y) = 1 - pXY, \\
P_p^{(2)}(X, Y) &= R_p^{(2)}(X, Y) = (1 - p^2XY)(1 - p^3X^2Y).
\end{align*}
\]
We note that if \( n > 2 \), then \( P_p^{(n)}(X, Y) \neq R_p^{(n)}(X, Y) \), however, \( P_p^{(n)}(X, 1) \) divides \( R_p^{(n)}(X, 1) \) in general. For readers' convenience, we reveal the origins of these two polynomials here: Obviously, the former \( P_p^{(n)}(X, Y) \) is relevant to the local factor of the Fourier coefficient \( A_{\kappa, \chi}(0_n) \) described in Lemma 2.1 (II a) at \( p \):
\[
P_p^{(n)}(\chi(p) p^{\kappa-n-1}, 1) = (1 - \chi(p) p^{\kappa-1}) \prod_{i=1}^{\lfloor n/2 \rfloor} (1 - \chi^2(p) p^{2\kappa-2i-1}).
\]
The latter $\mathcal{R}_p (n) (X, Y)$ has been introduced by Kitaoka [Ki2] and Böcherer-Sato [BS] to describe the denominator of the formal power series $\sum_{m=0}^{\infty} F_p (p^m S; X) Y^m$ for each nondegenerate $S \in \text{Sym}_n^* (\mathbb{Z}_p)$ (cf. Equation (11) below).

Now, we introduce a $p$-stabilization of the Siegel Eisenstein series $E_{n, \chi}^{(n)} \in \mathcal{M}_n (\Gamma_0 (M), \chi)^{(n)}$ in terms of a linear combination of $(U_{p,n})^i = U_{p,n} \circ \cdots \circ U_{p,n}$ for $i = 0, 1, \cdots, n$ as follows:

**Theorem 4.2.** For a pair of $n$, $\kappa$ and $\chi$ taken as above, put

$$
(E_{n, \chi}^{(n)})^* := \frac{\mathcal{R}_p (n) (\chi (p) p^{\kappa-n-1}, 1)}{\mathcal{R}_n (n) (\chi (p) p^{\kappa-n-1}, 1)} \cdot E_{n, \chi}^{(n)} \parallel_\kappa \mathcal{R}_p (n) (\chi (p) p^{\kappa-n-1}, U_{p,n}).
$$

Then we have

(I) $(E_{n, \chi}^{(n)})^* \in \mathcal{M}_n (\Gamma_0 (Mp), \chi)^{(n)}$ and it is a Hecke eigenform such that all the eigenvalues outside $Mp$ agree with those of $E_{n, \chi}^{(n)} \in \mathcal{M}_n (\Gamma_0 (M), \chi)^{(n)}$.

(II) If $0 \leq T \in \text{Sym}_n^* (\mathbb{Z})$ is taken of the form

$$
T = \begin{bmatrix} T' \\ 0_{n-r} \end{bmatrix}
$$

for some nondegenerate $T' \in \text{Sym}_r^* (\mathbb{Z})$ with $0 \leq r \leq n$, then the $T$-th Fourier coefficient of $(E_{n, \chi}^{(n)})^*$ is taken of the following form:

$$
A_{\kappa, \chi}^* (T) = 2^{[(r+1)/2] - [(n+1)/2]} \prod_{i=\lceil \frac{r}{2} \rceil + 1}^{\lceil \frac{n}{2} \rceil} L(p) (1 - 2\kappa + 2i, \chi^2) \times \begin{cases} 
L(p) (1 - \kappa + r/2, \frac{2\kappa r}{p}) \chi \prod_{l \mid f \gamma, \ l \neq p} F_l (T' \gamma; \chi (l)^{p^{\kappa-r-1}}) & \text{if } r \text{ is even,} \\
\prod_{l \mid f \gamma, \ l \neq p} F_l (T' \gamma; \chi (l)^{p^{\kappa-r-1}}) & \text{if } r \text{ is odd.}
\end{cases}
$$

Therefore we have $(E_{n, \chi}^{(n)})^* \parallel_\kappa U_{p,n} = (E_{n, \chi}^{(n)})^*$.

The preceding theorem totally insists that the assignment $E_{n, \chi}^{(n)} \mapsto (E_{n, \chi}^{(n)})^*$ can be regarded as a $p$-stabilization which generalizes the ordinary $p$-stabilization of $E_{n, \chi} = E_{n, \chi}^{(1)}$ explained in §1. Whenever $n = 2$, Skinner-Urban [SU] has already dealt with a similar type of $p$-stabilization for some Siegel modular forms of genus 2 so that the associated eigenvalue of $U_{p,2}$ is a $p$-adic unit as well. Accordingly, in the same context, we may call $(E_{n, \chi}^{(n)})^*$ semi-ordinary at $p$ if $n \geq 2$.

**Remark 4.3.** It should be mentioned that if $n > 1$, $(E_{n, \chi}^{(n)})^*$ may not satisfy the ordinary condition at $p$ in the sense of Hida (cf. [H2, H3]). This disparity is inevitable at least for Siegel Eisenstein series of higher genus in general. For instance, in the case where $M = 1$, because of the shape of associated Satake parameters at $p$, there is no way to produce from $E_{n, \chi}^{(n)} = E_{n, \chi, \text{triv}}^{(n)} \in \mathcal{M}_n (\Gamma_0 (1))^{(n)}$ to a Hecke eigenform of level $p$ such that the associated eigenvalues of $U_{p,1}, \cdots, U_{p,n-1}$ and $U_{p,n}$ are $p$-adic units simultaneously. However, as mentioned in [SU], it turns out that the semi-ordinary condition concerning only on the eigenvalue of $U_{p,n}$ is sufficient to adapt Hida’s ordinary theory with some modification. (See also [Pi, BPS].)
Proof of Theorem 4.2. The assertion (I) is obvious from Lemma 2.1 (I) and the properties of $U_{p,n}$ explained in §3. Since $U_{p,n}$ does not effect on the Fourier coefficient $A_{\kappa,\chi(0)}$, the assertion (II) for $r = 0$ follows immediately from Lemma 2.1 (IIa), Equations (8) and (9). Hereinafter, we suppose that $r > 0$. It follows by the definition of $\tilde{R}_p^{(n)}(X,Y)$ (cf. Equation (7)) that

$$\tilde{R}_p^{(n)}(X,Y) = \sum_{m=0}^{n} (-1)^m s_m \left( \left\{ p^{j(2n-j+1)/2}X^j \mid 1 \leq j \leq n \right\} \right) Y^{n-m},$$

where $s_m(\{X_1, \cdots, X_n\})$ denotes the $m$-th elementary symmetric polynomial in $X_1, \cdots, X_n$. Thus, by Lemma 2.1 (II b) and Equation (9), we have

$$A_{\kappa,\chi}(T) = \frac{P_p^{(n)}(\chi(p) p^{\kappa-n-1}, 1)}{\mathcal{R}_p^{(n)}(\chi(p) p^{\kappa-n-1}, 1)} \cdot 2^{(r+1)/2 - [(n+1)/2]} \prod_{i=[r/2]+1}^{[n/2]} L(1 - 2\kappa + 2i, \chi^2) \left( \prod_{l \mid p_r, \ l \neq p} F_l(T'; \chi(l) l^{\kappa-r-1}) \right)$$

where

$$\begin{cases} \mathcal{R}_p^{(n)}(X, Y) = \mathcal{R}_p^{(r)}(p^{n-r}X, Y) \prod_{j=r+1}^{n} (1 - p^{j(2n-j+1)/2}X^jY), \\ \mathcal{R}_p^{(n)}(X, 1) = \mathcal{R}_p^{(r)}(p^{n-r}X, 1) \prod_{i=[r/2]+1}^{[n/2]} (1 - p^{2n-2i+1}X^2). \end{cases}$$

Thus, to prove the assertion (II), it suffices to show that the following equation holds valid for each nondegenerate $T' \in \text{Sym}_r^*(\mathbb{Z}_p)$ with $0 < r \leq n$:

$$\sum_{m=0}^{n} (-1)^m s_m \left( \left\{ p^{j(2r-j+1)/2}X^j \mid 1 \leq j \leq r \right\} \right) F_p(p^{r-m}T'; X)$$

$$= \frac{\mathcal{R}_p^{(r)}(X, 1)}{\mathcal{R}_p^{(n)}(X, 1)} \cdot \left( 1 - \xi_p((-1)^{r/2} \det T') p^{r/2}X \right)$$

if $r$ is even,

$$= 1$$

if $r$ is odd.

(Indeed, the preceding equation (10) yields

$$\frac{P_p^{(n)}(X, 1)}{\mathcal{R}_p^{(n)}(X, 1)} \sum_{m=0}^{n} (-1)^m s_m \left( \left\{ p^{j(2n-j+1)/2}X^j \mid 1 \leq j \leq n \right\} \right) F_p(p^{n-m}T'; p^n X)$$

$$= \prod_{i=[r/2]+1}^{[n/2]} (1 - p^{2n-2i+1}X^2) \times \left( 1 - \xi_p((-1)^{r/2} \det T') p^{n-r/2}X \right)$$

if $r$ is even,

$$= 1$$

if $r$ is odd,

and hence, by evaluating this at $X = \chi(p) p^{\kappa-n-1}$, we obtain the desired equation.) On the other hand, Theorem 1 in [Ki2] (resp. Theorem 6 in [BS]) states that for each nondegenerate
Whenever \( r > 2 \), for a given \( T \), let \( i(T) \) denote the least integer \( m \) such that \( p^m T^{-1} \in \text{Sym}_2^r(Z_p) \). It is known that if \( r = 2 \), then for each nondegenerate \( T \in \text{Sym}_2^r(Z_p) \), the polynomial \( F_p(T; X) \) admits the explicit form

\[
F_p(T; X) = \sum_{i=0}^{i(T)} (p^2 X)^i \left\{ \sum_{j=0}^{\text{val}_p(i(T)-i)} (p^3 X^2)^j - \xi_p(-\text{det } T) pX \sum_{j=0}^{\text{val}_p(i(T)-i-1)} (p^3 X^2)^j \right\}
\]

(cf. [K1]). Thus a simple calculation yields that

\[
F_p(p^2 T; X) = (p^2 X + p^3 X^2) F_p(pT; X) + p^5 X^3 F_p(T; X)
\]

\[
= 1 - \xi_p(-\text{det } T) pX \frac{R_p^{(2)}(X, 1)}{P_p^{(2)}(X, 1)} \times (1 - \xi_p(-\text{det } T) pX),
\]

and hence, Equation (12) also holds for \( r = 2 \). Now, we suppose that \( r > 2 \). We note that every nondegenerate \( T \in \text{Sym}_r^r(Z_p) \) is equivalent, over \( Z_p \), to a canonical form

\[
T = \begin{bmatrix} T_1 & \vline & \vline & \vline \\
& \ddots & \ddots & \ddots \\
& & \ddots & \ddots \\
& & & \ddots & \ddots \\
& & & & \ddots & \ddots \\
& & & & & \ddots & \ddots \\
& & & & & & \ddots & \ddots \\
& & & & & & & \ddots & \ddots \\
& & & & & & & & \ddots & \ddots \\
& & & & & & & & & \ddots & \ddots \\
& & & & & & & & & & \ddots & \ddots \\
& & & & & & & & & & & \ddots & \ddots \\
& & & & & & & & & & & & \ddots & \ddots \\
& & & & & & & & & & & & & \ddots \\
& & & & & & & & & & & & & & \ddots \\
& & & & & & & & & & & & & & & \ddots \\
& & & & & & & & & & & & & & & & \ddots \\
& & & & & & & & & & & & & & & & & \ddots \\
& & & & & & & & & & & & & & & & & & \ddots \\
& & & & & & & & & & & & & & & & & & & \ddots \\
& & & & & & & & & & & & & & & & & & & & \ddots \end{bmatrix}
\]

for some \( T_1 \in \text{Sym}_2^r(Z_p) \) and \( T_2 \in \text{Sym}_r^r(Z_p) \cap GL(r - 2, Q_p) \). It follows from Theorems 4.1 and 4.2 in [K1] that

\[
\frac{S_p(T; X, 1)}{1 - \xi_p((-1)^{r/2} \text{det } T) p^{r/2} X} = \frac{S_p(T_2; p^2 X, 1)}{1 - \xi_p((-1)^{r/2-1} \text{det } T_2) p^{r/2+1} X}
\]

\[
\times \frac{1 - p^{(r-1)(r+2)/2 X^{r-1}}(1 - p^{(r+1)/2 X^r})}{1 - p^{r+2 X^2}}
\]

if \( r \) is even, and

\[
S_p(T; X, 1) = S_p(T_2; p^2 X, 1) \cdot \frac{(1 - p^{(r-1)(r+2)/2 X^{r-1}})(1 - p^{(r+1)/2 X^r})}{1 - p^{r+2 X^2}}
\]

if \( r \) is odd. (See also [K2, §3].) Thus, it is proved by induction on \( r \) that Equation (12) (and thus, (10)) holds in general. This completes the proof.
As a straightforward conclusion of Theorem 4.2 above, we have

**Theorem 4.4.** Let $\chi$ be a Dirichlet character modulo $M$, where $M$ is odd and coprime to $p$, taken as above and $\omega^a$ a power of the Teichmüller character with $0 \leq a < \varphi(p)^6$, respectively. For each $n \geq 1$, there exists a formal Fourier expansion

$$
\mathcal{E}_{\chi^a}^{(n)}(X) = \sum_{T \in \text{Sym}_n^*(\mathbb{Z}),}^{[n/2]} A_{\chi^a}(T; X) q^T \in F_\lambda[[q]]^{(n)},
$$

where $F_\lambda$ is the field of fractions of $\Lambda = \mathbb{Z}_p[\chi][[X]]$, such that for each positive integer $\kappa > n + 1$ with $\chi(-1) = (-1)^{\kappa}$ and $\kappa \equiv a \pmod{\varphi(p)}$, we have

$$
\mathcal{E}_{\chi^a}^{(n)}((1 + p)^\kappa - 1) = (E_{\kappa, \chi}^{(n)})^* \in \mathcal{M}_{\kappa}(\Gamma_0(Mp), \chi)^{(n)}.
$$

Moreover, put

$$
\mathcal{B}^{(n)}(X) := \prod_{i=1}^{[n/2]} \{(1 + p)^{-2i}(1 + X)^2 - 1\} \prod_{j=0}^{[n/2]} \{(1 + p)^{-j}(1 + X) - 1\}
$$

and

$$
= X \prod_{i=1}^{[n/2]} \{(1 + p)^{-i}(1 + X) - 1\}^2 \{(1 + p)^{-i}(1 + X) + 1\}.
$$

Then

$$
\mathcal{T}^{(n)}(X) := \mathcal{B}^{(n)}(X) \cdot \mathcal{E}_{\chi^a}^{(n)}(X) \text{ belongs to } \Lambda[[q]]^{(n)}.
$$

**Proof.** As is well-known by Deligne-Ribet [DR] (generalizing the previous work of Kubota-Leopoldt), associated to a quadratic Dirichlet character $\xi$, a Dirichlet character $\chi$ taken as above, and a power of the Teichmüller character $\omega^a$ with $0 \leq a < \varphi(p)$, there exists $\overline{T}(\chi \omega^a; X) \in \Lambda$ such that for each positive integer $k > 1$, we have

$$
\overline{T}(\chi \omega^a; (1 + p)^k - 1) = \begin{cases} 
(1 + p)^k - 1 \cdot L_{\chi^a}(1 - k, \omega^{-k}) & \text{if } \xi \omega^a \text{ is trivial,} \\
L_{\chi^a}(1 - k, \xi \omega^a-k) & \text{otherwise.}
\end{cases}
$$

Accordingly, put

$$
\mathcal{L}(\xi \omega^a; X) := \begin{cases} 
X^{-1} \overline{T}(\xi \omega^a; X) & \text{if } \xi \omega^a \text{ is trivial,} \\
\overline{T}(\xi \omega^a; X) & \text{otherwise.}
\end{cases}
$$

We easily see that $\mathcal{L}(\xi \omega^a; (1 + p)^k - 1) = L_{\chi^a}(1 - k, \xi \omega^a-k)$ for each $k > 1$. More generally, it turns out that if $\epsilon : 1 + p\mathbb{Z}_p \to (\mathbb{Q}_p^{\text{alg}})^\times$ is a character of finite order, then

$$
\mathcal{L}(\xi \omega^a; \epsilon(1 + p)(1 + p)^k - 1) = L_{\chi^a}(1 - k, \xi \omega^a-k \epsilon)
$$

for any $k > 1$. On the other hand, for each $x \in 1 + p\mathbb{Z}_p$, put $s(x) := \log_p(x)/\log_p(1 + p)$, where $\log_p$ is the $p$-adic logarithm function in the sense of Iwasawa, and thus we have $s : 1 + p\mathbb{Z}_p \sim \mathbb{Z}_p$.

Then for each $T = \left[ \begin{array}{c} T' \\ 0_{n-r} \end{array} \right] \in \text{Sym}_n^*(\mathbb{Z})$ with $T' \in \text{Sym}_r^*(\mathbb{Z}) \cap \text{GL}(r, \mathbb{Q})$ and $0 \leq r \leq n$, we define $\mathcal{A}_{\chi^a}(T; X) \in F_\lambda$ as follows:

$$
\mathcal{A}_{\chi^a}(T; X) = 2^{[(r+1)/2] - [(n+1)/2]} \prod_{i=[r/2]+1}^{[n/2]} \mathcal{L}(\chi \omega^{2a-2i}; (1 + p)^{-2i}(1 + X)^2 - 1)
$$

---

6Here $\varphi$ denotes Euler's totient function.
Thus, if $\omega > n + 1$, $\chi(-1) = (-1)^\kappa$ and further if $\kappa \equiv a \pmod{\varphi(p)}$ (or equivalently, $\omega^{a-\kappa}$ is trivial), then

$$A_{\chi_\omega}(T; (1 + p)^\kappa - 1) = A_{\kappa, \chi}(T).$$

It follows from Theorem 4.2 (II) that if $\kappa > n + 1$, $\chi(-1) = (-1)^\kappa$ and further if $\kappa \equiv a \pmod{\varphi(p)}$, then

$$A_{\chi_\omega}(T; (1 + p)^\kappa - 1) = A_{\kappa, \chi}(T).$$

We show that for a Dirichlet character $\chi$ modulo $M$ with $p \nmid M$ and an integer $a$ with $0 \leq a < p - 1$, the formal Fourier expansion $E_{\chi_\omega}(X)$ with coefficients in $F_\Lambda$ (resp. $F_{\chi_\omega}(X)$) defined in Theorem 4.4, give rise to classical Siegel modular forms via the specialization at $X = \varepsilon(1 + p)(1 + p)^\kappa - 1$, where $\kappa$ is a positive integer sufficiently large and $\varepsilon : 1 + p\mathbb{Z}_p \to (\mathbb{Q}_p^{alb})^\times$ is a character of finite order.

First, from Lemma 2.1 (II), we deduce the following:

**Theorem 5.1.** If $\kappa$ is a positive integer with $\kappa > n + 1$ and further if $\varepsilon : 1 + p\mathbb{Z}_p \to (\mathbb{Q}_p^{alb})^\times$ is a character of order $p^n$ for some nonnegative integer $n$, then

$$E_{\chi_\omega}(\varepsilon(1 + p)(1 + p)^\kappa - 1) = E_{\kappa, \chi_\omega - \varepsilon}(\kappa, \chi_\omega - \varepsilon)(n)$$

as long as $\omega^{2a - 2\kappa \varepsilon^2}$ is non-trivial.

**Proof.** For each $T = \left[ \begin{array}{cc} T' & 0 \\ 0 & n-r \end{array} \right] \in \text{Sym}_n^r(\mathbb{Z})$ with $T' \in \text{Sym}_n^r(\mathbb{Z}) \cap \text{GL}(r, \mathbb{Q})$ and $0 \leq r \leq n$, the equation (14) is specialized as

$$A_{\chi_\omega}(T; \varepsilon(1 + p)(1 + p)^\kappa - 1) = 2^{[(r+1)/2] - [(n+1)/2]} \prod_{i=[r/2]+1}^{[n/2]} L[p](1 - 2\kappa + 2i, \chi_\omega^{2a - 2\kappa \varepsilon^2})$$

$$\times \left\{ \begin{array}{ll}
L[p](1 - \kappa + r/2, \left( \frac{2\kappa}{p} \right) \chi_\omega^{a-\kappa \varepsilon}) & \text{if } r \text{ is even,} \\
\prod_{l \mid T'; l \neq p} F_l(T'; \chi_\omega^{a-\kappa \varepsilon}(l)^{\kappa-r-1}) & \text{if } r \text{ is odd.}
\end{array} \right.$$
This completes the proof. \qed

On the other hand, by exploiting the vertical control theorem for $p$-adic Siegel modular forms in the sense of Hida [H2], we may deduce a slightly weaker version of the preceding theorem as follows:

**Theorem 5.2.** If $\kappa$ is a positive integer with $\kappa > n(n+1)/2$, and further if $\varepsilon : 1 + p\mathbb{Z}_p \to (\mathbb{Q}_p^\text{alg})^\times$ is a character of exact order $p^n$ for some nonnegative integer $m$, then

$$E_{\chi^\omega}(\varepsilon(1+p)(1+p)^{\kappa}-1) \in \mathcal{M}_k(\Gamma_0(Mp^{m+1}), \chi^\omega\omega^{a-\kappa}\varepsilon)^{(n)}$$

as long as $\omega^{a-\kappa}\varepsilon$ is non-trivial.

**Proof.** It follows from Theorem 4.4 that $E_{\chi^\omega}(\varepsilon(1+p)(1+p)^{\kappa}-1) \in \mathcal{M}_k(\Gamma_0(Mp), \chi)^{(n)}$ for infinitely many integers $\kappa'$ with $\kappa' > n + 1$, $\chi(-1) = (-1)^{\kappa'}$ and $\kappa' \equiv a \pmod{p-1}$. Thus, Théorème 1.1 in [Pi] (generalizing [H2] in more general settings) turns out that if an integer $\kappa$ is sufficiently large, then the specialization of $E_{\chi^\omega}(X)$ at $X = \varepsilon(1+p)(1+p)^{\kappa}$ gives rise to an overconvergent $p$-adic Siegel modular form of weight $\kappa$, tame level $M$ and Iwahori level $p^{m+1}$ with character $\chi^\omega\omega^{a-\kappa}\varepsilon$ (for the precise definition, see [SU, Pi]). In addition, the explicit formula for Fourier coefficients of $E_{\chi^\omega}(\varepsilon(1+p)(1+p)^{\kappa}-1)$ also yields that it is an eigenform of $U_{p,n}$ whose eigenvalue is 1. Therefore, if $\kappa$ satisfies the condition $\kappa > n(n+1)/2$ (cf. Hypothèse 4.5.1 in [BPS]), then the above-mentioned overconvergent form is indeed classical, that is,

$$E_{\chi^\omega}(\varepsilon(1+p)(1+p)^{\kappa}-1) \in \mathcal{M}_k(\Gamma_0(Mp^{m+1}), \chi^\omega\omega^{a-\kappa}\varepsilon)^{(n)}$$

(cf. Théorème 5.3.1 in [ibid]). Thus we obtain the assertion. \qed

**Remark 5.3.** Although Theorem 4.4 remains valid even for $p = 2$, we may not deduce similar statements to Theorems 5.1 and 5.2 in the case where $p = 2$, since there are some technical difficulties in understanding $E_{\chi^\omega}(\varepsilon(1+p)(1+p)^{k}-1)$ from automorphic and geometric viewpoints.

Now, as a summary of Theorems 4.4, 5.1 and 5.2 above, we have the following statement by which $\mathfrak{F}_{\chi^\omega}(X)$ is said to be the semi-ordinary $\Lambda$-adic Siegel Eisenstein series of genus $n$ and level $Mp^\infty$ associated with character $\chi^\omega$:

**Theorem 5.4.** Let $M$ be an odd integer and $\chi$ a Dirichlet character modulo $M$ taken as follows:

(i) if $M = 1$, then $\chi$ is trivial (or principal);

(ii) if $M > 1$, then $\chi$ is primitive and $\chi^2$ is locally non-trivial at every prime $l \mid M$;

For each integer $a$ with $0 \leq a < p-1$, there exists a formal Fourier expansion $\mathfrak{F}_{\chi^\omega}(X) \in \Lambda[[q]]^{(n)}$ such that if $\kappa > n(n+1)/2$ and further if $\varepsilon : 1 + p\mathbb{Z}_p \to (\mathbb{Q}_p^\text{alg})^\times$ is a character of exact order $p^n$ with some $m \geq 0$, then

$$\mathfrak{F}_{\chi^\omega}(\varepsilon(1+p)(1+p)^{\kappa}-1) \in \mathcal{M}_k(\Gamma_0(Mp^{m+1}), \chi^{\omega^{a-\kappa}\varepsilon})^{(n)}$$

is a Hecke eigenform whose eigenvalue of $U_{p,n}$ is 1. In particular,

$$\mathfrak{F}_{\chi^\omega}(\varepsilon(1+p)(1+p)^{\kappa}-1) = C_{\kappa,\varepsilon} \times \left\{ \begin{array}{ll} (E_{\kappa,\chi}^{(n)})^* & \text{if } \omega^{a-\kappa}\varepsilon \text{ is trivial,} \\ E_{\kappa,\chi^{\omega^{a-\kappa}\varepsilon}}^{(n)} & \text{if } \omega^{2a-2\kappa\varepsilon^2} \text{ is non-trivial,} \end{array} \right.$$
Remark 5.5. The preceding theorem for \( n > 1 \) can be regarded as a satisfactory generalization of Fact 1.1 to some extent. The only exception is our lack of knowledge about the specialization \( \mathfrak{E}_{\chi^0}(n)(\varepsilon(1+p)(1+p)^{\kappa} - 1) \) in the case where \( \omega^{a-\kappa} \varepsilon \) is non-trivial, but \( \omega^{2a-2\kappa} \varepsilon^2 \) is trivial. As shown in Theorem 5.2, it has been proved that if \( \omega^{a-\kappa} \varepsilon \) is non-trivial, regardless of whether \( \omega^{2a-2\kappa} \varepsilon^2 \) is trivial or not, \( \mathfrak{E}_{\chi^0}(n)(\varepsilon(1+p)(1+p)^{\kappa} - 1) \in M_{\kappa}(\Gamma_0(Mp^{m+1})) \chi^{a-\kappa} \varepsilon(n) \) is a Hecke eigenform which is semi-ordinary at \( p \).

REFERENCES

[AZ] A.N. Andrianov and V.G. Zhuravlev, Modular forms and Hecke operators, Transl. of Math. Monogr., vol.145, Amer. Math. Soc., Providence, RI, 1995.
[AL] A.O.L. Atkin and J. Lehner, Hecke operators on \( \Gamma_0(m) \), Math. Ann. 185 (1970), 134–160.
[BPS] S. Bijakowski, V. Pilloni and B. Stroh, Classicité de formes modulaires surconvergentes, Ann. of Math. 183 (2016), no.3, 975–1014.
[Bö] S. Böcherer, On the Hecke operator \( U(p) \). With an appendix by Ralf Schmidt, J. Math. Kyoto Univ. 45 (2005), no.4, 807–829.
[BS] S. Böcherer and F. Sato, Rationality of certain formal power series related to local densities, Comment. Math. Univ. St. Paul. 36 (1987), no.1, 53–86.
[CP] M. Courtieu and A.A. Panchishkin, Non-Archimedean L-Functions and Arithmetical Siegel Modular Forms, Lecture Notes in Math., vol.1471, Springer-Verlag, Berlin, 2004.
[DR] P. Deligne and K.A. Ribet, Values of abelian L-functions at negative integers over totally real fields, Invent. Math. 59 (1980), 227–286.
[Fe] P. Feit, Poles and residues of Eisenstein series for symplectic and unitary groups, Mem. Amer. Math. Soc. 61, no.346, 1986.
[Fr] E. Freitag, Siegelsche Modulfunktionen, Grundlehren Math. Wiss., vol.254, Springer-Verlag, Berlin, 1983.
[H1] H. Hida, Elementary theory of \( L \)-functions and Eisenstein series, London Math. Soc. Student Text, vol.26, Cambridge University Press, Cambridge, 1993.
[H2] H. Hida, Control theorems for coherent sheaves on Shimura varieties of PEL-type, J. Inst. Math. Jussieu 1 (2002), no.1, 1–76.
[H3] H. Hida, \( p \)-adic automorphic forms on Shimura varieties, Springer Monographs in Math., Springer-Verlag, New York, 2004.
[Ik] T. Ikeda, On the functional equation of the Siegel series, J. Number Theory 172 (2017), 44–62.
[K1] H. Katsurada, An explicit formula for Siegel series, Amer. J. Math. 121 (1999), no.2, 415–452.
[K2] H. Katsurada, Euler factor of a certain Dirichlet series attached to Siegel Eisenstein series, Abh. Math. Sem. Univ. Hamburg 71 (2001), 81–90.
[Ki1] Y. Kitaoka, Dirichlet series in the theory of quadratic forms, Nagoya Math. J. 95 (1984), 73–84.
[Ki2] Y. Kitaoka, Local densities of quadratic forms and Fourier coefficients of Eisenstein series, Nagoya Math. J. 103 (1986), 149–160.
[Mi] T. Miyake, Modular forms, Springer Monographs in Math., Springer-Verlag, Berlin, 2006.
[Pa] A.A. Panchishkin, On the Siegel-Eisenstein measure and its applications, Israel J. Math. 120 (2000), part B, 467–509.
[Pi] V. Pilloni, Sur la théorie de Hida pour le groupe \( \text{GSp}_2 \), Bull. Soc. Math. France 140 (2012), no.3, 335–400.
[Se] J.-P. Serre, Formes modulaires et functions zêta \( p \)-adiques, Modular functions of one variable, Vol. III (Proc. Internat. Summer School, Univ. Antwerp, 1972), 191–268, Lecture Notes in Math., vol.350, Springer-Verlag, Berlin, 1973.
[S1] G. Shimura, Confluent hypergeometric functions on tube domains, Math. Ann. 260 (1982), no.3, 269–302.
[S2] G. Shimura, Euler products and Fourier coefficients of automorphic forms on symplectic groups, Invent. Math. 116 (1994), no.1–3, 531–576.
[SU] C. Skinner and E. Urban, Sur les déformations \( p \)-adiques de certaines représentations automorphes, J. Inst. Math. Jussieu 5 (2006), no.4, 626–698.
[Sw] W.J. Sweet Jr., A computation of the gamma matrix of a family of \( p \)-adic zeta integrals, J. Number Theory 55 (1995), no.2, 222–260.
[T] S. Takemori, Siegel Eisenstein series of degree \( n \) and \( \Lambda \)-adic Eisenstein series, J. Number Theory 149 (2015), 105–138.
[Ta] R.L. Taylor, Congruences between modular forms, Ph.D thesis, Princeton University, 1988.
[W] A. Wiles, On ordinary \( \Lambda \)-adic representations associated to modular forms, Invent. Math. 94 (1988), no.3, 529–573.

Email address: kawamura.hisaaki@hiro.kindai.ac.jp