RIESZ TRANSFORM AND RIESZ POTENTIALS FOR DUNKL TRANSFORM

SUNDARAM THANGAVELU AND YUAN XU

Abstract. Analogous of Riesz potentials and Riesz transforms are defined and studied for the Dunkl transform associated with a family of weighted functions that are invariant under a reflection group. The $L^p$ boundedness of these operators is established in certain cases.

1. Introduction

For a family of weighted functions, $h_\kappa$, invariant under a finite reflection group, Dunkl transform is an extension of the Fourier transform that defines an isometry of $L^2(\mathbb{R}^d, h_\kappa^2)$ onto itself. The basic properties of the Dunkl transforms have been studied by several authors, see [2, 4, 6, 7, 8, 11, 12] and the references therein. Giving the important role of Fourier transform in analysis, one naturally asks if it is possible to extend results established for the Fourier transform to the Dunkl transform.

In analogous to the ordinary Fourier analysis, one can define a convolution operator and study various summabilities of the inverse Dunkl transforms. The convolution is defined through a generalized translation operator, $\tau_y$, which plays the role of $f \mapsto f(\cdot - y)$ but is defined in the Dunkl transform side. The explicit expression of $\tau_y f$ is known only in some special cases and it is not a positive operator in general. In fact, even the $L^p$ boundedness of $\tau_y$ is not established in general. This is the main reason that only part of the results for the Fourier transforms has been extended to the Dunkl transform at the moment.

Recently, in [11], the $L^p$ theory for convolution operators was studied. In particular, the $L^p$ boundedness of the convolution operator is established in the case that the kernel is a suitable radial function. Furthermore, a maximal function is defined and shown to be of strong type $(p, p)$ and weak type $(1, 1)$. This provides a handy tool for extending some results from the Fourier transform to the Dunkl transform. In the present paper we study the analogous of the Riesz potentials and the Riesz transforms for the Dunkl transform. We will study the boundedness of the Riesz potentials as well as the related Bessel potentials. The Riesz transforms are examples of singular integrals. A general theory of singular integral for the Dunkl transform appears to be out of reach at the moment. We will prove the $L^p$ boundedness of the weighted Riesz transform only in a very special case of $d = 1$. 

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and $G = \mathbb{Z}_2$. Even in this simple case, however, the proof turns out to be rather nontrivial.

The paper is organized as follows. In the next section we collect the background materials. In Section 3 we recall the definition of the ordinary Riesz transforms and Riesz potentials, and prove a weighted $L^p$ boundedness for the Riesz potentials that will be used later in the paper. The weighted Riesz potentials and the Bessel potentials for the Dunkl transform will be studied in Section 4. The weighted Riesz transform is discussed in Section 5.

Throughout this paper we use the convention that $c$ denotes a generic constant, depending on $d$, $p$, $\kappa$ or other fixed parameters, its value may change from line to line.

2. Preliminaries

2.1. Dunkl Transform. The Dunkl transform is associated to a weight function that is invariant under a reflection group. Let $G$ be a finite reflection group on $\mathbb{R}^d$ with a fixed positive root system $R_+$, normalized so that $\langle v, v \rangle = 2$ for all $v \in R_+$, where $\langle x, y \rangle$ denotes the usual Euclidean inner product. Let $\kappa$ be a nonnegative multiplicity function $v \mapsto \kappa_v$ defined on $R_+$ with the property that $\kappa_u = \kappa_v$ whenever $\sigma u$ is conjugate to $\sigma v$ in $G$; then $v \mapsto \kappa_v$ is a $G$-invariant function. The weight function $h_\kappa$ is defined by

$$h_\kappa(x) = \prod_{v \in R_+} |\langle x, v \rangle|^{\kappa_v}, \quad x \in \mathbb{R}^d.$$  

This is a positive homogeneous function of degree $\gamma_\kappa := \sum_{v \in R_+} \kappa_v$, and it is invariant under the reflection group $G$.

To define the Dunkl transform we will also need the intertwining operator $V_\kappa$. Let $D_j$ denote Dunkl’s differential-difference operators defined by

$$D_j f(x) = \partial_j f(x) + \sum_{v \in R_+} k_v \frac{f(x) - f(x\sigma_v)}{\langle x, v \rangle} \langle v, \varepsilon_j \rangle, \quad 1 \leq j \leq d,$$

where $\varepsilon_1, \ldots, \varepsilon_d$ are the standard unit vectors of $\mathbb{R}^d$ and $\sigma_v$ denote the reflection with respect to the hyperplane perpendicular to $v$, $x\sigma_v := x - 2(\langle x, v \rangle/\|v\|^2)v$, $x \in \mathbb{R}^d$. The operators $D_j$, $1 \leq j \leq d$, map $P_n$ to $P_{n-1}$, where $P_n$ denotes the space of homogeneous polynomials of degree $n$ in $d$ variables, and they mutually commute; that is, $D_iD_j = D_jD_i$, $1 \leq i, j \leq d$. The intertwining operator $V_\kappa$ is a linear operator determined uniquely by

$$V_\kappa P_n \subset P_n, \quad V_\kappa 1 = 1, \quad D_i V_\kappa = V_\kappa \partial_i, \quad 1 \leq i \leq d.$$

The explicit formula of $V_\kappa$ is not known in general. For the group $G = \mathbb{Z}_2^d$, $h_\kappa(x) = \prod_{i=1}^d |x_i|^{\kappa_v}$, it is an integral transform

$$V_\kappa f(x) = b_\kappa \int_{[-1,1]^d} f(x t_1, \ldots, x t_d) \prod_{i=1}^d (1 + t_i)(1 - t_i^2)^{\kappa_v - 1} dt.$$  

It is known that $V_\kappa$ is a positive operator $\boxplus$: that is, $p \geq 0$ implies $V_\kappa p \geq 0$. 

Let $E(x, iy) = V_\kappa^{(x)}[e^{i(x, y)}]$, $x, y \in \mathbb{R}^d$, where the superscript means that $V_\kappa$ is applied to the $x$ variable. For $f \in L^1(\mathbb{R}^d, h_\kappa^2)$, the Dunkl transform is defined by

$$
\widehat{f}(y) = c_\kappa \int_{\mathbb{R}^d} f(x)E(x, -iy)h_\kappa^2(x)dx
$$

where $c_\kappa$ is the constant defined by $c_\kappa^{-1} = \int_{\mathbb{R}^d} h_\kappa^2(x)e^{-\|x\|^2/2}dx$. If $\kappa = 0$ then $V_\kappa = id$ and the Dunkl transform coincides with the usual Fourier transform. If $d = 1$ and $G = \mathbb{Z}_2$, then the Dunkl transform is related closely to the Hankel transform on the real line.

Some of the properties of the Dunkl transform is collected below (2.1).

**Proposition 2.1.** 
(1) For $f \in L^1(\mathbb{R}^d, h_\kappa^2)$, $\widehat{f}$ is in $C_0(\mathbb{R}^d)$.
(2) When both $f$ and $\widehat{f}$ are in $L^1(\mathbb{R}^d, h_\kappa^2)$ we have the inversion formula

$$
f(x) = \int_{\mathbb{R}^d} E(ix, y)\widehat{f}(y)dy.
$$

(3) The Dunkl transform extends to an isometry of $L^2(\mathbb{R}^d, h_\kappa^2)$.

2.2. **Generalized translation operator.** Let $y \in \mathbb{R}^d$ be given. The generalized translation operator $f \mapsto \tau_yf$ is defined on $L^2(\mathbb{R}^d, h_\kappa^2)$ by the equation

$$
\tau_yf(x) = E(y, -ix)\widehat{f}(x), \quad x \in \mathbb{R}^d.
$$

It plays the role of the ordinary translation $\tau_yf = f(\cdot - y)$ of $\mathbb{R}^d$, since the Fourier transform of $\tau_yf$ is given by $\tau_y\widehat{f}(x) = e^{-i(x, y)}\widehat{f}(x)$.

The generalized translation operator has been studied in [6, 7, 11, 12]. The definition gives $\tau_y\widehat{f}$ as an $L^2$ function. Let us define

$$
A_\kappa(\mathbb{R}^d) = \{ f \in L^1(\mathbb{R}^d, h_\kappa^2) : \widehat{f} \in L^1(\mathbb{R}^d, h_\kappa^2) \}.
$$

Then (2.1) holds pointwise. Note that $A_\kappa(\mathbb{R}^d)$ is contained in the intersection of $L^1(\mathbb{R}^d, h_\kappa^2)$ and $L^\infty$ and hence is a subspace of $L^2(\mathbb{R}^d, h_\kappa^2)$. The operator $\tau_y$ satisfies the following properties:

**Proposition 2.2.** Assume that $f \in A_\kappa(\mathbb{R}^d)$ and $g \in L^1(\mathbb{R}^d, h_\kappa^2)$ is bounded. Then

(1) $\int_{\mathbb{R}^d} \tau_yf(\xi)g(\xi)h_\kappa^2(\xi)d\xi = \int_{\mathbb{R}^d} f(\xi)\tau_yg(\xi)h_\kappa^2(\xi)d\xi.$

(2) $\tau_yf(x) = \tau_{-x}f(-y)$.

A formula of $\tau_yf$ is known, at the moment, only in two cases. One is in the case of $G = \mathbb{Z}_2$ and $h_\kappa(x) = |x|^\kappa$ on $\mathbb{R}$ (3)

$$
\tau_yf(x) = \frac{1}{2} \int_{-1}^{1} f \left( \frac{x - y}{\sqrt{x^2 + y^2 - 2xyt}} \right) \left( 1 + \frac{x - y}{\sqrt{x^2 + y^2 - 2xyt}} \right) \Phi(t)dt
+ \frac{1}{2} \int_{-1}^{1} f \left( - \frac{x - y}{\sqrt{x^2 + y^2 - 2xyt}} \right) \left( 1 - \frac{x - y}{\sqrt{x^2 + y^2 - 2xyt}} \right) \Phi(t)dt,
$$

where $\Phi(t) = b_\kappa(1 + t)(1 - t^2)^{\kappa - 1}$, from which also follows a formula of $\tau_yf$ in the case of $G = \mathbb{Z}_2$. The explicit formula implies the boundedess of $\tau_yf$. Let $||f||_{\kappa, p}$ denote the norm of $L^p(\mathbb{R}^d, h_\kappa^2)$.

**Proposition 2.3.** Let $G = \mathbb{Z}_2$. For $f \in L^p(\mathbb{R}^d, h_\kappa^2)$, $1 \leq p \leq \infty$,

$$
||\tau_yf||_{\kappa, p} \leq c||f||_{\kappa, p}.
$$
Another case where a formula for $\tau_y f$ is known is when $f$ are radial functions, $f(x) = f_0(||x||)$, and $G$ being any reflection group \cite{17}.

\begin{equation}
\tau_y f(x) = V_{\kappa} \left[ f_0 \left( \sqrt{||x||^2 + ||y||^2 - 2 ||x|| ||y|| ||\langle x', \cdot \rangle||} \right) \right](y'),
\end{equation}

from which it follows that $\tau_y f(x) \geq 0$ for all $y \in \mathbb{R}^d$ if $f(x) = f_0(||x||) \geq 0$.

Several essential properties of $\tau_y f$ is established for $f$ being radial functions. This is collected in the following proposition \cite{11}. Let $L^p_{\text{rad}}(\mathbb{R}^d, h^2_\kappa)$ stands for the subspace of radial functions in $L^p(\mathbb{R}^d, h^2_\kappa)$.

**Proposition 2.4.**

1. For every $f \in L^1_{\text{rad}}(\mathbb{R}^d, h^2_\kappa)$,

\[
\int_{\mathbb{R}^d} \tau_y f(x) h^2_\kappa(x) dx = \int_{\mathbb{R}^d} f(x) h^2_\kappa(x) dx.
\]

2. For $1 \leq p \leq 2$, $\tau_y : L^p_{\text{rad}}(\mathbb{R}^d, h^2_\kappa) \rightarrow L^p(\mathbb{R}^d, h^2_\kappa)$ is a bounded operator.

The generalized translation $\tau_y$ also satisfies the following property \cite{11} \cite{12}: If $f$ is supported in $\{x : ||x|| \leq B\}$ then $\tau_y f$ is supported in $\{x : ||x|| \leq B + ||y||\}$. An important consequence of this property is as follows \cite{11}: If $f \in C_0^\infty(\mathbb{R}^d)$ then for $1 \leq p \leq \infty$

\[
\lim_{y \rightarrow 0} \tau_y f - f ||_{\kappa, p} = 0.
\]

2.3. The generalized convolution and maximal function. The generalized translation operator can be used to define a convolution. For $f, g$ in $L^2(\mathbb{R}^d, h^2_\kappa)$, we define

\begin{equation}
(f \ast \kappa g)(x) = \int_{\mathbb{R}^d} f(y) \tau_x g(y) h^2_\kappa(y) dy
\end{equation}

where $g(y) = \hat{g}(-y)$. Since $\tau_x g(y) \in L^2(\mathbb{R}^d, h^2_\kappa)$ the convolution is well defined.

This convolution has been considered by several authors \cite{17} \cite{11} \cite{12} and the references therein. It satisfies the basic properties $\hat{f} \ast \kappa g = \hat{f} \cdot \hat{g}$ and $f \ast \kappa g = g \ast \kappa f$.

Furthermore, if the generalized translation operator is bounded in norm, then the usual Young’s inequality holds. For the general reflection group, the following result is proved in \cite{11}.

**Theorem 2.5.** Let $g$ be a bounded radial function in $L^1(\mathbb{R}^d, h^2_\kappa)$. Then $f \ast \kappa g$ initially defined in \cite{14} on the intersection of $L^1(\mathbb{R}^d, h^2_\kappa)$ and $L^2(\mathbb{R}^d, h^2_\kappa)$ extends to all $L^p(\mathbb{R}^d, h^2_\kappa)$, $1 \leq p \leq \infty$ as a bounded operator. In particular,

\begin{equation}
||f \ast \kappa g||_{\kappa, p} \leq ||g||_{\kappa, 1} ||f||_{\kappa, p}.
\end{equation}

For $f \in L^2$ the maximal function $M_\kappa f$ is defined in \cite{11} by

\[
M_\kappa f(x) = \sup_{r > 0} \frac{1}{d_\kappa r^{d+2\gamma_\kappa}} ||f \ast \kappa \chi_{B_r}(x)||,
\]

where $\chi_{B_r}$ is the characteristic function of the ball $B_r$ of radius $r$ centered at 0 and

\[
(d_\kappa)^{-1} = \int_{B_1} h^2_\kappa(y) dy = (d + 2\gamma_\kappa) \int_{S^{d-1}} h^2_\kappa(x) d\omega.
\]

The maximal function can also be written as

\[
M_\kappa f(x) = \sup_{r > 0} \frac{\int_{B_r} \tau_y f(x) h^2_\kappa(y) dy}{\int_{B_r} h^2_\kappa(y) dy}.
\]
The Riesz transform, \( G \), is defined by Theorem 2.8. Set
\[
\hat{G} = \mathcal{F}(G).
\]
Consequently, \( \phi \) is differentiable, \( \lim_{r \to \infty} \phi(0(r)) = 0 \) and \( c \) is a constant independent of \( a \) and \( f \).

For \( \phi \in L^1(\mathbb{R}^d, h_\kappa^2) \) and \( \varepsilon > 0 \), the dilation \( \phi_\varepsilon \) is defined by
\[
\phi_\varepsilon(x) = \varepsilon^{-(2\gamma_\kappa+d)} \phi(x/\varepsilon).
\]
A change of variables shows that
\[
\int_{\mathbb{R}^d} \phi_\varepsilon(x)h_\kappa^2(x)dx = \int_{\mathbb{R}^d} \phi(x)h_\kappa^2(x)dx, \quad \text{for all } \varepsilon > 0.
\]
For convolution with an radial kernel, the following result is established in [11].

Theorem 2.7. Let \( \phi \in \mathcal{A}_\kappa(\mathbb{R}^d) \) be a real valued radial function which satisfies
\[
|\phi(x)| \leq c(1 + ||x||)^{-d-2\gamma_\kappa-1}.
\]
Then
\[
\sup_{\varepsilon > 0} |f *_\kappa \phi_\varepsilon(x)| \leq cM_\kappa f(x).
\]
Consequently, \( f *_\kappa \phi_\varepsilon(x) \to f(x) \) for almost every \( x \) as \( \varepsilon \) goes to 0 for all \( f \) in \( L^p(\mathbb{R}^d; h_\kappa^2) \), \( 1 \leq p < \infty \).

If \( \tau_y \) is bounded in \( L^p(\mathbb{R}^d; h_\kappa^2) \) then the conditions in the above theorem can be relaxed. At the moment this holds in the case of \( G = \mathbb{Z}_2^d \) ([11]).

Theorem 2.8. Set \( G = \mathbb{Z}_2^d \). Let \( \phi \in \mathcal{A}_\kappa(\mathbb{R}^d) \) be a radial function. Assume that \( \phi \) is differentiable, \( \lim_{r \to \infty} \phi_0(r) = 0 \) and \( \int_0^\infty r^{2\gamma_\kappa+d} |\phi_0(r)|dr < \infty \), then
\[
|(f *_\kappa \phi)(x)| \leq cM_\kappa f(x) \int_0^\infty r^{2\gamma_\kappa+d} |\phi_0(r)|dr < \infty.
\]
In particular, if \( \hat{\phi}(x) = \Phi(x) \) satisfies \( \phi \in L^1(\mathbb{R}^d, h_\kappa^2) \) and \( \Phi(0) = 1 \), then
1. For \( 1 \leq p \leq \infty \), \( f *_\kappa \phi \) converges to \( f \) as \( \varepsilon \to 0 \) in \( L^p(\mathbb{R}^d, h_\kappa^2) \);
2. For \( f \in L^1(\mathbb{R}^d, h_\kappa^2) \), \( f *_\kappa \phi_\varepsilon(x) \) converges to \( f(x) \) as \( \varepsilon \to 0 \) for almost all \( x \in \mathbb{R}^d \).

3. Ordinary Riesz transforms and Riesz potentials

In this section the notation \( \hat{f} \) denote the ordinary Fourier transform on \( \mathbb{R}^m \).

We recall the classical definition of the Riesz transforms and Riesz potentials. The Riesz transform, \( R_j f \) (\( 1 \leq j \leq m \)), for the ordinary Fourier transform on \( \mathbb{R}^m \) is defined by
\[
R_j f(x) = \lim_{\varepsilon \to 0} c_j \int_{||y|| \geq \varepsilon} \frac{f(y)}{||y||} dy, \quad c_j = \frac{2^{m/2}}{\sqrt{\pi}} \Gamma \left( \frac{m+1}{2} \right).
\]
It is a multiplier operator in the sense that
\[
\mathcal{F}(R_j f)(x) = -i \frac{x_j}{||x||} \mathcal{F}(f)(x) \quad \text{in } L^2(\mathbb{R}^m).
\]
with respect to the ordinary Fourier transform. It is well known that $R_j f$ is a bounded operator from $L^p (\mathbb{R}^m)$ to $L^p (\mathbb{R}^m)$ for $1 < p < \infty$.

If $f$ is a radial function, $f(x) = f_0(\|x\|)$, then the spherical-polar coordinates and the Funk-Hecke formula give us

$$(3.1) \quad R_j f(x) = c \int_{\mathbb{R}^m} f(x - y) \frac{y_j}{\|y\|^{m+1}} dy = c \int_0^\infty \int_{S^{m-1}} f_0(\|x - sy\|) y'_j d\omega(y') \frac{ds}{s}$$

$$= cx'_j \int_0^\infty \int_{-1}^1 f_0 \left( \sqrt{\|x\|^2 - s^2} \frac{dy}{s} \right) \frac{ds}{s}.$$ 

In particular, the Riesz transform will be used in the Section 5. In particular, we will need the following notation,

$$\tilde{R}_j f_0(\|x\|) = R_j f(x), \quad \text{when } f(x) = f_0(\|x\|).$$

We will need the following proposition.

**Proposition 3.1.** If $f(x) = f_0(\|x\|)$ and $f \in L^p (\mathbb{R}^m)$ then

$$\left( \int_0^\infty |\tilde{R}_j f_0(r)|^p r^{m-1} dr \right)^{1/p} \leq c \left( \int_0^\infty |f_0(r)|^p r^{m-1} dr \right)^{1/p}, \quad 1 < p < \infty.$$ 

Evidently, using the spherical-polar coordinates $x = rx'$, this is a simple consequence of the boundedness of $R_j f$ on $L^p (\mathbb{R}^m)$.

The Riesz potential on $\mathbb{R}^m$ is defined as the ordinary convolution of $f$ with the kernel $K_\alpha(x) = (\gamma(\alpha))^{-1} \|x\|^{\alpha-m}, 0 < \alpha < m$,

$$(3.2) \quad f * K_\alpha(x) = \gamma(\alpha)^{-1} \int_{\mathbb{R}^m} \|x - y\|^{-m+\alpha} f(y) dy,$$ 

where $\gamma(\alpha)$ is a constant whose value we will not need. It is well known (see, for example, [9, p. 119]) that for $1 < p < q < \infty$ where $1/q = 1/p - \alpha/m$,

$$\|f * K_\alpha\|_q \leq A_{p,q} \|f\|_p.$$ 

The Riesz potential will be used in the Section 5. In particular, we will need the following weighted inequality in the case of $\alpha = 1$.

**Proposition 3.2.** If $f(\cdot , \cdot ) \in L^p$, then

$$\left( \int_{\mathbb{R}^m} |f * K_1(x)|^p dx \right)^{1/p} \leq c \left( \int_{\mathbb{R}^m} |f(x)|^p \|x\|^p dx \right)^{1/p}$$ 

for all $p$ satisfying $p > m/(m-1), \ m \geq 2$.

**Proof.** Let $Mf$ be the Hardy-Littlewood maximal function defined by

$$Mf(x) = \sup_{r > 0} \frac{1}{|B_r|} \int_{B_r} |f(x - y)| dy$$ 

where $|B_r|$ denotes the volume of the ball $B_r$. We will make use of the weighted norm inequality

$$\int_{\mathbb{R}^m} |Mf(x)|^p w(x) dx \leq c \int_{\mathbb{R}^m} |f(x)|^p w(x) dx,$$ 

which holds whenever $w$ belongs to Muckenhoupt’s $A_p$ class $A_p (\mathbb{R}^m)$. It is known that $w(x) = \|x\|^\alpha \in A_p (\mathbb{R}^m)$ whenever $-m < \alpha < m/(p-1)$ (see, for example, [10, p. 218]). In particular, $\|x\|^p \in A_p (\mathbb{R}^d)$ for $p > m/(m-1)$ and $\|x\|^{-\delta p} \in A_p (\mathbb{R}^d)$ for $0 < \delta < m/p$. 
To prove the weighted inequality we split \( f \ast K_1 \) as follows,
\[
(f \ast \cdot \cdot \cdot^{1-m})(x) = \int_{\|y\|<\|x\|} f(x-y) \|y\|^{1-m} dy + \int_{\|y\|\leq \|x\|} f(x-y) \|y\|^{1-m} dy + \int_{\|y\|>\|x\|} f(x-y) \|y\|^{1-m} dy
\]
\[
= T_1 f(x) + T_2 f(x) + T_3 f(x).
\]
Evidently we have
\[
\|T_2 f(x)\| \leq \int_{\|y\|\leq \|x\|} |f(x-y)| \|y\|^{1-m} dy \leq c M f(x).
\]
For \( T_1 \) we further split the integral to get
\[
|T_1 f(x)| \leq \sum_{j=0}^{\infty} \int_{2^{-j-1}\|x\| \leq \|y\| < 2^{-j}\|x\|} |f(x-y)| \|y\|^{1-m} dy
\]
\[
\leq \sum_{j=0}^{\infty} 2^{-j(1-m)} \|x\|^{1-m} \int_{2^{-j-1}\|x\| \leq \|y\| < 2^{-j}\|x\|} |f(x-y)| dy
\]
\[
\leq c \sum_{j=0}^{\infty} 2^{-j} \|x\| M f(x) \leq c \|x\| M f(x).
\]
Using the weighted inequality of the maximal function it follows that, for \( j = 1, 2, \)
\[
\int_{\mathbb{R}^d} |T_j f(x)|^p dx \leq c \int_{\mathbb{R}^d} |M f(x)|^p \|x\|^p dx \leq c \int_{\mathbb{R}^d} |f(x)|^p \|x\|^p dx,
\]
as \( \|x\|^p \in A_p(\mathbb{R}^d) \) for \( p > m/(m-1) \).
To deal with \( T_3 f \) we choose \( \delta \) such that \( 0 < \delta < m/p \) and write the integral as
\[
|T_3 f(x)| \leq \sum_{j=1}^{\infty} \int_{2^j \|x\| \leq \|y\| < 2^{j+1}\|x\|} |f(x-y)| \|y\|^{1-m} dy
\]
\[
= \sum_{j=1}^{\infty} \int_{2^j \|x\| \leq \|y\| < 2^{j+1}\|x\|} |f(x-y)| \|x - y - 2^{-j}(x)\|^{1+\delta} \|x - y\|^{-1-\delta} \|y\|^{1-m} dy.
\]
For \( 2^j \|x\| \leq \|y\| < 2^{j+1}\|x\| \) we have \( \|y\| \geq \|x\| \) and hence
\[
\|x - y\| \geq \|y\| - \|x\| = \|y\|/2 + (\|y\|/2 - \|x\|) \geq \|y\|/2 \geq 2^j \|x\|
\]
so that
\[
|T_3 f(x)| \leq c \|x\|^{-\delta} \sum_{j=0}^{\infty} 2^{-j(1+\delta)} \int_{\|x\| \leq \|y\| < 2^{j+1}\|x\|} |f(x-y)| \|x - y\|^{1+\delta} dy
\]
\[
\leq c \|x\|^{-\delta} \sum_{j=0}^{\infty} 2^{-j} M f_\delta(x)
\]
\[
\leq c \|x\|^{-\delta} M f_\delta(x),
\]
where \( f_\delta(x) = f(x) \|x\|^{1+\delta} \). Then the weighted inequality of the maximal function implies, as \( \|x\|^{-\delta p} \in A_p(\mathbb{R}^d) \),
\[
\int_{\mathbb{R}^d} |T_2 f(x)|^p dx \leq c \int_{\mathbb{R}^d} |M f_\delta(x)|^p \|x\|^{-\delta p} dx
\]
\[
\leq c \int_{\mathbb{R}^d} |f_\delta(x)|^p \|x\|^{-\delta p} dx = c \int_{\mathbb{R}^d} |f(x)|^p \|x\|^p dx.
\]
This completes the proof. □

4. Weighted Riesz Potentials and Bessel Potentials

In this section the notation $\hat{f}$ denotes the Dunkl transform of $f$.

4.1. Riesz potentials. For $0 < \alpha < 2\gamma_n + d$, the weighted Riesz potential, $I^\alpha f$, is defined on $\mathcal{S}$ by

$$I^\alpha f(x) = (d_n^\alpha)^{-1} \int_{\mathbb{R}^d} \tau_y f(x) \frac{1}{\|y\|^{2\gamma_n+d-\alpha}} h_n(\gamma_n+\alpha)(y) dy,$$

where $d_n^\alpha = 2^{-\gamma_n-d/2+\alpha}\Gamma(\frac{\alpha}{2})/\Gamma(\gamma_n + \frac{d-\alpha}{2})$.

In order to derive the Dunkl transform of $I^\alpha$, we start with a lemma, which is a little more general than what is needed. A homogeneous polynomial $P$ is called an $h$-harmonics if $\Delta_h P = 0$, where $\Delta_h = D_1^2 + \ldots + D_d^2$ is called the $h$-Laplacian. Let $\mathcal{H}_n(h_n^2)$ denote the space of $h$-harmonics of degree $n$. It is known that

$$\int_{S^{d-1}} P(x)q(x)h_n^2(x) d\omega = 0$$

whenever $P \in \mathcal{H}_n(h_n^2)$ and the degree $q$ is less than $n$.

**Lemma 4.1.** For $P \in \mathcal{H}_n(h_n^2)$ and $0 < \Re\{\alpha\} < 2\gamma_n + d$, the identity

$$\left( \frac{P(x)}{\|x\|^{2\gamma_n+d+n-\alpha}} \right) = d_n^{\alpha} \frac{P(x)}{\|x\|^{n+\alpha}}, \quad P^{\alpha}_{n,\gamma} = \frac{i^{-n-2\gamma_n-d/2+\alpha}}{\Gamma(\gamma_n + \frac{n+d-\alpha}{2})},$$

holds in the sense that

$$\int_{\mathbb{R}^d} \frac{P(x)}{\|x\|^{2\gamma_n+d+n-\alpha}} \phi(x) h_n^2(x) dx = d_n^{\alpha} \int_{\mathbb{R}^d} \frac{P(x)}{\|x\|^{n+\alpha}} \phi(x) h_n^2(x) dx$$

for every $\phi$ which is sufficiently rapidly decreasing at $\infty$, and whose Dunkl transform has the same property.

**Proof.** If $P_n \in \mathcal{H}_n(h_n^2)$, then Theorem 5.7.5 of [3] shows that

$$\left( P_n(x)e^{-\|x\|^2/2} \right) = (-i)^n P_n(x)e^{-\|x\|^2/2}.$$

Since the Dunkl transform satisfies $\hat{f}(s)(y) = s^{-2\gamma_n-d} \hat{f}(s^{-1} y)$, it follows that

$$\left( P_n(x)e^{-s\|x\|^2/2} \right) = (-i)^n s^{-n-\gamma_n-d/2} P_n(x)e^{-\|x\|^2/(2s)}.$$

Let $\phi$ be a function that satisfies the property in the statement of the lemma. For $s > 0$, the above formula leads to the relation

$$\int_{\mathbb{R}^d} P_n(x)e^{-s\|x\|^2/2} \phi(x) h_n^2(x) dx$$

$$= (-i)^n s^{-n-\gamma_n-d/2} \int_{\mathbb{R}^d} P_n(x)e^{-\|x\|^2/(2s)} \phi(x) h_n^2(x) dx.$$

We then multiply the above equation by $s^{\beta-1}$, where $\beta = \gamma_n + (d + n - \alpha)/2$, and integrate the result with respect to $s$ on $[0, \infty)$. Using

$$\int_0^\infty s^{\alpha - 1} e^{-s\|x\|^2/2} ds = 2\alpha \Gamma(\alpha)\|x\|^{-2\alpha}$$
and changing the order of the integrals, it is easy to see that this leads to
\[
2^{\gamma_k+(d+n-\alpha)/2}\Gamma(\gamma_k + \frac{d+n-\alpha}{2}) \int_{\mathbb{R}^d} \frac{P_n(x)}{||x||^{2\gamma_k + d+n-\alpha}} \hat{\phi}(x)h^2_{\alpha}(x)dx
\]
\[
= (-i)^{\alpha}2^{(n+\alpha)/2}\Gamma(\frac{n+\alpha}{2}) \int_{\mathbb{R}^d} \frac{P_n(x)}{||x||^{n+\alpha}} \phi(x)h^2_{\alpha}(x)dx,
\]
which simplifies to the stated equation. In the above we can assume the decay of \(\phi\) and \(\hat{\phi}\) in the order of
\[
|\phi(x)| \leq A(1+||x||)^{-d-2\gamma_k} \quad \text{and} \quad |\hat{\phi}(x)| \leq A(1+||x||)^{-d-2\gamma_k}
\]
to ensure that the double integrals that occur above converge absolutely, so that the Fubini theorem applies.

\[\square\]

**Proposition 4.2.** Let \(0 < \alpha < 2\gamma_k + d\). The identity
\[(4.4) \quad \hat{I}_\alpha^nf(x) = ||x||^{-\alpha}\hat{f}(x)\]
holds in the sense that
\[
\int_{\mathbb{R}^d} I^\alpha_n f(x)g(x)h^2_{\alpha}(x)dx = \int_{\mathbb{R}^d} \hat{f}(x)||x||^{-\alpha}\hat{g}(x)h^2_{\alpha}(x)dx
\]
whenever \(f, g \in S\).

**Proof.** Setting \(n = 0\) in Lemma 4.2 shows that the Dunkl transform of \(||x||^{-d-2\gamma_k+\alpha}\) is the function \(d^n||x||^{-\alpha}\) in the sense that
\[
\int_{\mathbb{R}^d} ||y||^{-d-2\gamma_k+\alpha}\phi(y)h^2_{\alpha}(y)dy = d^n\int_{\mathbb{R}^d} ||y||^{-\alpha}\hat{\phi}(y)h^2_{\alpha}(y)dy,
\]
where \(\phi \in S\). Set \(\phi(y) = \tau_y f(x)\) in the above identity leads to
\[
\int_{\mathbb{R}^d} ||y||^{-d-2\gamma_k+\alpha}\tau_y f(x)h^2_{\alpha}(y)dy = d^n\int_{\mathbb{R}^d} ||y||^{-\alpha}E(-ix,y)\hat{f}(y)h^2_{\alpha}(y)dy.
\]
Multiplying this identity by \(g(x)\) and integrating, we obtain the stated identity. \(\square\)

Recall that \((-\Delta_h f)\hat{(x)} = ||x||^2\hat{f}(x)\) for \(f \in S\), the identity (4.4) shows that the weighted Riesz potential can be defined as \((-\Delta_h)^{-\alpha/2}f\). The identity (4.4) also shows that
\[
I^\alpha_\alpha(I^\beta_\alpha f) = I^\alpha_{\alpha+\beta}(f), \quad f \in S, \quad \alpha, \beta > 0, \quad \alpha + \beta < 2\gamma_k + d.
\]
\[
\Delta_h(I^\alpha_\alpha f) = I^\alpha_\alpha(\Delta_h f) = -I^\alpha_{\alpha-2}(f), \quad f \in S, \quad 2\gamma_k + d > \alpha \geq 2,
\]
which are extensions of familiar identities for the ordinary Riesz potentials. Next we consider the boundedness of \(I^\alpha_\alpha\) as an operator from \(L^p(\mathbb{R}^d, h^2_{\alpha})\) to \(L^p(\mathbb{R}^d, h^2_{\alpha})\).

The following necessary condition holds:

**Proposition 4.3.** If \(||I^\alpha_\alpha f||_{\kappa,q} \leq c||f||_{\kappa,p}\) for \(f \in S\), then it is necessary that
\[(4.5) \quad \frac{1}{p} - \frac{1}{q} = \frac{\alpha}{2\gamma_k + d}.
\]

**Proof.** Let \(f_s(x) = e^{-s^2||x||^2}\). Using the fact \(\tau_y f_s(x) = e^{-s^2(||x||^2+||y||^2)}E(2s^2x,y)\), a change of variable shows that
\[
I^\alpha_\alpha(e^{-s^2||x||^2})(x) = s^{-\alpha}I^\alpha_\alpha(e^{-||x||^2})(sx).
\]
Consequently, setting \( f(x) = 3^{-\|x\|^2} \), a changing of variables shows that
\[
\|I^\alpha_0 f\|_{\kappa,q} = s^{-\alpha/(2\gamma_\alpha + d)}/q \|I^\alpha_0 f\|_{\kappa,q} \quad \text{and} \quad \|f_s\|_{\kappa,p} = s^{-(2\gamma_\alpha + d)/p} \|f\|_{\kappa,p}
\]
for all \( s > 0 \). Considering \( s \to 0 \) and \( s \to \infty \) shows that \( \|I^\alpha_0 f\|_{\kappa,q} \leq c\|f\|_{\kappa,p} \) holds, then we must have \( \alpha + (2\gamma_\alpha + d)/q - (2\gamma_\alpha + d)/p = 0 \), which gives \( 1.34 \). □

The main result on the weighted Riesz potential is the following theorem.

**Theorem 4.4.** Let \( G = \mathbb{Z}^d_\alpha \). Let \( \alpha \) be a real number such that \( 0 < \alpha < 2\gamma_\alpha + d \) and let \( 1 \leq p < q < \infty \) satisfies \( 1.34 \).

1. For \( f \in L^p(\mathbb{R}^d, h^2_\alpha) \), \( p > 1 \),
\[
\|I^\alpha_0 f\|_{\kappa,q} \leq c\|f\|_{\kappa,p}.
\]
2. For \( f \in L^1(\mathbb{R}^d, h^2_\alpha) \), the mapping \( f \mapsto I^\alpha_0 f \) is of weak type \((1, q)\); that is,
\[
\int_{\{x: |I^\alpha_0 f(x)| > \sigma\}} h^2_\alpha(x)dx \leq c \left( \frac{\|f\|_{\kappa,1}}{\sigma} \right)^q.
\]

**Proof.** Let \( R > 0 \) be fixed. We write the operator as a sum of two terms,
\[
I^\alpha_0 f(x) = (d^\alpha_n)^{-1} \int_{\{x: |x| \leq R\}} \tau_y f(x) \frac{1}{\|y\|^{2\gamma_\alpha + d - \alpha}} h^2_\alpha(y)dy + (d^\alpha_n)^{-1} \int_{\{x: |x| \geq R\}} \tau_y f(x) \frac{1}{\|y\|^{2\gamma_\alpha + d - \alpha}} h^2_\alpha(y)dy := S_1 f(x) + S_2 f(x).
\]

For \( S_1 f \), we use the maximal function and the Theorem 2.8 to get the estimate
\[
|S_1 f(x)| \leq c M_\kappa f(x) \int_0^R \frac{d}{dr} \left( r^{-2\gamma_\alpha - d + \alpha} \right) dr \leq \frac{c}{\alpha} R^\alpha M_\kappa f(x).
\]

To estimate \( S_2 f \) we use Proposition 2.3 which states that \( \|\tau_y f\|_{\kappa,p} \leq c\|f\|_{\kappa,p} \). Let \( p' = p/(p - 1) \). Then Hölder’s inequality shows that
\[
|S_2 f(x)| \leq \left( \int_{\{y: \|y\| \geq R\}} \left( \frac{\|y\|^{-2\gamma_\alpha - d + \alpha}}{\|y\|^{2\gamma_\alpha - d + \alpha}} \right)^{p'} h^2_\alpha(y)dy \right)^{1/p'} \|\tau_x f\|_{\kappa,p} \leq c R^{-(d+2\gamma_\alpha)/p + \alpha} \|f\|_{\kappa,p} = c R^{-(d+2\gamma_\alpha)/q} \|f\|_{\kappa,p},
\]

where the last step follows from 1.50. Together, the two estimates show that
\[
|I^\alpha_0 f(x)| \leq c \left( R^\alpha M_\kappa f(x) + R^{-(d+2\gamma_\alpha)/q} \|f\|_{\kappa,p} \right)
\]
for all \( R > 0 \). Choosing \( R = (M_\kappa f(x) / \|f\|_{\kappa,p})^{-p/(d+2\gamma_\alpha)} \) and using 1.50, we obtain the inequality
\[
|I^\alpha_0 f(x)| \leq c (M_\kappa f(x))^{p/q} \|f\|_{\kappa,p}^{1-p/q}.
\]

Consequently, for \( p > 1 \), we can use the \( L^p \) boundedness of the maximal function in Theorem 2.6 to conclude that
\[
\|I^\alpha_0 f\|_{\kappa,q} \leq c \|M_\kappa f(x)\|_{\kappa,p}^{p/q} \|f\|_{\kappa,p}^{1-p/q} \leq c\|f\|_{\kappa,p}.
\]
For $p = 1$, we use the weak type $(1, 1)$ inequality of the maximal function to get

$$\int_{\{ x : h^2_\kappa (x) \geq \sigma \} } h^2_\kappa (x) dx \leq \int_{\{ x : \| M_\kappa f (x) \|_{L^p} / \| f \|_{L^p} \geq \sigma \} } h^2_\kappa (x) dx \leq c \left( \frac{\| f \|_{L^1}}{\sigma} \right)^q \| f \|_{L^1} = c \left( \frac{\| f \|_{L^1}}{\sigma} \right)^q .$$

The proof is completed. \qed

The proof shows that if $\tau_y$ is a bounded operator from $L^p (\mathbb{R}^d, h^2_\kappa)$ to itself for some other reflection group, then the conclusion of the theorem will hold for that group. At the moment, the theorem holds only if $G = \mathbb{Z}^d_2$ or if $f$ are radial functions and $1 \leq p \leq 2$ by Proposition 2.3.

4.2. **Weighted Bessel potentials.** The Bessel potentials is closely related to the Riesz potential. The kernel functions for the Bessel potentials have essentially the same local behavior as that of the Riesz potentials as $\| x \| \to 0$, but have much better behavior for $\| x \|$ large. In analogous to the ordinary Fourier transform, the weighted Bessel potentials, $\mathcal{J}_\alpha$, can be defined by

$$\mathcal{J}_\alpha^\kappa = (I - \Delta_\kappa)^{-\alpha/2}, \quad \alpha > 0.$$

To be more precise, we define $\mathcal{J}_\alpha^\kappa$ as a convolution operator

$$\mathcal{J}_\alpha^\kappa f = f * \tilde{G}_\alpha^\kappa, \quad \text{where} \quad \tilde{G}_\alpha^\kappa (x) = (1 + \| x \|^2)^{-\alpha/2}$$

for $f \in \mathcal{S}$. The following position gives an explicit expression for $G_\alpha^\kappa$.

**Proposition 4.5.** Let $G_\alpha^\kappa$ be defined as above. Then $G_\alpha^\kappa (x) \geq 0$ for all $x \in \mathbb{R}$, $G_\alpha^\kappa \in L^1 (\mathbb{R}^d, h^2_\kappa)$, and

$$G_\alpha^\kappa (x) = \frac{1}{\Gamma(\alpha/2)} \int_0^{\infty} e^{-t \| x \|^2/(4t)} t^{-\gamma_\kappa + (\alpha - \delta)/2} dt .$$

**Proof.** This follows as in the ordinary Bessel potentials (p. 132]). Let us work backward and start with (4.6). Evidently then $G_\alpha^\kappa (x) > 0$. Furthermore, since

$$c_h \int_{\mathbb{R}^d} e^{-\| x \|^2/(2t)} h^2_\kappa (x) dx = (2t)^{\gamma_\kappa + d/2} c_h \int_{\mathbb{R}^d} e^{-\| u \|^2/2h^2_\kappa (u)} du = (1 + \| x \|^2)^{\alpha/2}$$

the Fubini’s theorem applied to (4.6), which shows

$$c_h \int_{\mathbb{R}^d} G_\alpha^\kappa (x) h^2_\kappa (x) dx = \frac{1}{\Gamma(\alpha/2)} \int_0^{\infty} t^{\alpha/2 - 1} e^{-t dt} = 1.$$

Using the fact that $\left( e^{-\| x \|^2/(4t)} \right)^{\infty} = e^{-\| x \|^2}$, it follows that

$$\tilde{G}_\alpha^\kappa (u) = \frac{1}{\Gamma(\alpha/2)} \int_0^{\infty} e^{-t \| x \|^2} t^{\alpha/2} dt = (1 + \| x \|^2)^{-\alpha/2},$$

where the interchange of integrals can be easily justified by Fubini’s theorem. \qed

The behavior of $G_\alpha^\kappa$ is described in the following lemma.

**Lemma 4.6.** For $\alpha > 0$,

$$G_\alpha^\kappa (x) \leq c \left( 1 + \| x \|^{-2\gamma_\kappa - d + \alpha} \right) e^{-\| x \|^2/2}, \quad \| x \| > 0.$$
Theorem 4.8. The following theorem holds for all reflection groups.

Proof. The elementary inequality $t + r^2/(4t) \geq 2\sqrt{t(\sqrt{r^2/4t} = r}$ leads to

$$G_\alpha(x) \leq \frac{1}{\Gamma(\alpha/2)} e^{-\|x\|^2/2} \int_0^\infty e^{-\frac{1}{2}(t+\|x\|^2)} t^{-\gamma_\alpha + (\alpha-d)/2} dt \frac{dt}{t}. $$

To estimate the integral we split it into two parts. Changing variable gives

$$\int_0^{\|x\|^2} e^{-\frac{1}{2}(t+\|x\|^2)} t^{-\gamma_\alpha + (\alpha-d)/2} dt \leq \int_0^{\|x\|^2} e^{-\frac{1}{2}t} t^{-\gamma_\alpha + \frac{\alpha-d}{2}} dt \leq c \|x\|^{-2\gamma_\alpha + \alpha-d}. $$

Furthermore, if $\|x\| \geq 1$, then

$$\int_0^\infty e^{-\frac{1}{2}(t+\|x\|^2)} t^{-\gamma_\alpha + (\alpha-d)/2} dt \leq \int_0^{\infty} e^{-\frac{1}{2}t} t^{-\gamma_\alpha + \frac{\alpha-d}{2}} dt \leq c $$

and if $\|x\| \leq 1$, then

$$\int_0^\infty e^{-\frac{1}{2}(t+\|x\|^2)} t^{-\gamma_\alpha + (\alpha-d)/2} dt \leq \int_0^{\|x\|^2} e^{-\frac{1}{2}t} t^{-\gamma_\alpha + \frac{\alpha-d}{2}} dt \leq c \left(1 + \int_0^{\|x\|^2} t^{-\gamma_\alpha + (\alpha-d)/2} dt \right) \leq c \left(1 + \|x\|^{-2\gamma_\alpha + \alpha-d} \right). $$

Putting these estimates together proves the stated inequality. \qed

This shows, in particular, that $G_\alpha^\kappa$ behaves as $\|x\|^{-2\gamma_\alpha - d + \alpha}$ for $\|x\| \to 0$, same as the kernel for the Riesz potentials.

Theorem 4.7. Under the same assumption, the conclusion of Theorem 4.4 holds for Bessel potentials.

The proof is essentially the same. We will not repeat it. Instead, we state the following theorem which holds for all reflection groups.

**Theorem 4.8.** Let $\alpha > 0$.

1. The Bessel potentials are bounded operators from $L^p(\mathbb{R}^d, h_\kappa^2)$ to itself for $1 \leq p \leq \infty$.
2. For $f \in L^1(\mathbb{R}^d, h_\kappa^2)$,

$$|\mathcal{J}_\alpha^\kappa f(x)| \leq c M_\kappa f(x), \quad x \in \mathbb{R}^d. $$

**Proof.** Since $G_\alpha^\kappa(x)$ is a radial function, let us write $G_\alpha^\kappa(r)$ for the function defined on $\mathbb{R}_+$. The estimate in Lemma 4.6 shows that $G_\alpha^\kappa \in L^1(\mathbb{R}^d, h_\kappa^2)$ and it has integral 1, so that the first part of the theorem follows from Theorem 2.8.

For $\alpha > 0$, the estimate in Lemma 4.6 shows that the conditions on $\phi$ of Theorem 2.8 is satisfied which gives the second part. \qed

Such a theorem does not hold for the Riesz potentials. The definition of the Bessel potentials also shows that $\mathcal{J}_\alpha^\kappa$ satisfies

$$\mathcal{J}_\alpha^\kappa \mathcal{J}_\beta^\kappa = \mathcal{J}_{\alpha+\beta}^\kappa, \quad \alpha > 0, \quad \beta > 0. $$
5. Weighted Riesz Transforms

5.1. Definition of Riesz transforms. For \( P \in \mathcal{H}_n^d(h^2_\kappa) \), we consider the transform

\[
T^n f(x) = \lim_{\varepsilon \to 0} \int_{\|y\| \geq \varepsilon} \tau_y f(x) \frac{P(y)}{\|y\|^\gamma + d + n} h^n_\kappa(y) dy
\]
defined for \( f \in L^2(\mathbb{R}^d, h^2_\kappa) \). The equation (5.1) and the Plancherel theorem shows that \( T^n f \) is well defined. We want to show that \( T^n \) is a multiplier operator under the Dunkl transform. A linear operator \( T f \) is a multiplier operator if \( Tf(x) = m(x)f(x) \) in the sense that

\[
T f(x) = \int_{\mathbb{R}^d} m(y) \hat{f}(y) E(x, iy) h^n_\kappa(y) dy
\]
for \( f \) sufficiently smooth and having compact support.

**Theorem 5.1.** Let \( P \in \mathcal{H}_n^d(h^2_\kappa) \), \( n \geq 1 \). Then the multiplier corresponding to the transform \( T^n f \) is given by \( d_{n,\kappa} P(x)/\|x\|^\gamma \), \( d_{n,\kappa} = i^{n-2-\gamma-1/2} \Gamma(\gamma + \frac{n+1}{2}) \).

**Proof.** Since \( P_n \in \mathcal{H}_n^d(h^2_\kappa) \), its integral with respect to \( h^2_\kappa \) on \( S^{d-1} \) is zero. Hence,

\[
\int_{\|x\| \leq 1} \frac{P_n(x)}{\|x\|^{2\gamma + d + n + \alpha}} h^2_\kappa(x) dx = \int_{\|x\| \geq 1} \frac{P_n(x)}{\|x\|^{2\gamma + d + n - \alpha}} \hat{\phi}(x) h^2_\kappa(x) dx
\]

\[
+ \int_{\|x\| \leq 1} \frac{P_n(x)}{\|x\|^{2\gamma + d + n - \alpha}} \left[ \hat{\phi}(x) - \hat{\phi}(0) \right] h^2_\kappa(x) dx = 0.
\]

Let \( \phi \) be a function that satisfies the condition in the Lemma 4.1 and the additional assumption that \( \hat{\phi} \) is differentiable near origin. Then we can write

\[
\int_{\mathbb{R}^d} \frac{P_n(x)}{\|x\|^{2\gamma + d + n - \alpha}} \hat{\phi}(x) h^2_\kappa(x) dx = \int_{\|x\| \geq 1} \frac{P_n(x)}{\|x\|^{2\gamma + d + n - \alpha}} \hat{\phi}(x) h^2_\kappa(x) dx
\]

\[
+ \int_{\|x\| \leq 1} \frac{P_n(x)}{\|x\|^{2\gamma + d + n - \alpha}} \left[ \hat{\phi}(x) - \hat{\phi}(0) \right] h^2_\kappa(x) dx.
\]

Since \( [\hat{\phi}(x) - \hat{\phi}(0)]/\|x\| \) is locally integrable, we can take limit \( \alpha \to 0 \) to get

\[
\int_{\|x\| \leq 1} \frac{P_n(x)}{\|x\|^{2\gamma + d + n}} \left[ \hat{\phi}(x) - \hat{\phi}(0) \right] h^2_\kappa(x) dx
\]

\[
= \lim_{\varepsilon \to 0} \int_{\|x\| \leq \varepsilon} \frac{P_n(x)}{\|x\|^{2\gamma + d + n}} \hat{\phi}(x) h^2_\kappa(x) dx.
\]

Consequently, we conclude that (5.1)

\[
\lim_{\alpha \to 0+} \int_{\mathbb{R}^d} \frac{P_n(x)}{\|x\|^{2\gamma + d + n - \alpha}} \hat{\phi}(x) h^2_\kappa(x) dx = \lim_{\varepsilon \to 0} \int_{\|x\| \geq \varepsilon} \frac{P_n(x)}{\|x\|^{2\gamma + d + n}} \hat{\phi}(x) h^2_\kappa(x) dx.
\]

Let \( f \) be any sufficiently smooth function with compact support. For a fixed \( x \), set \( \hat{\phi}(y) = \tau_y f(x) = \tau_{-y} f(-y) \). Then

\[
\hat{\phi}(y) = \int_{\mathbb{R}^d} \tau_{-y} f(z) E(z, -iy) h^2_\kappa(z) dz = \int_{\mathbb{R}^d} \tau_y f(z) E(z, iy) h^2_\kappa(z) dz
\]

\[
= \tau_y f(-y) = E(-y, iy) \hat{f}(-y).
\]

Hence, it follows that \( \phi(y) = \hat{\phi}^{-1}(-y) = E(y, iy) \hat{f}(y) \), so that by (1.2) and (5.1)

\[
\lim_{\varepsilon \to 0} \int_{\|x\| \geq \varepsilon} \frac{P_n(y)}{\|y\|^{2\gamma + d + n}} \tau_y f(x) h^2_\kappa(y) dy = d_{n,\kappa} \int_{\mathbb{R}^d} \frac{P_n(y)}{\|y\|^n} E(x, iy) \hat{f}(y) h^2_\kappa(y) dy.
\]

By the definition of the multiplier \( m \), we conclude that \( m(y) = d_{n,\kappa} P_n(y)/\|y\|^n \). □
The proof of this theorem follows the argument for the ordinary Fourier transform as given in [24 p. 73 – 74].

The special case that \( P(x) = x_j \) defines the weighted Riesz transform.

**Definition 5.2.** For \( f \in L^2(\mathbb{R}^d, h_\kappa^2) \) the Riesz transform \( R_j^\kappa f \) is defined by

\[
R_j^\kappa f(x) = \lim_{\varepsilon \to 0} c_j \int_{\|y\| \geq \varepsilon} \tau_y f(x) \frac{y_j}{\|y\|^{\gamma + d + 1}} h_\kappa^2(y) dy,
\]

where \( 1 \leq j \leq d \) and \( c_j = 2^{\gamma + d/2} \Gamma(\gamma + (d + 1)/\sqrt{\pi}) \).

**Theorem 5.3.** The Riesz transform is a multiplier operator with form as given in [9, p. 73 – 74].

**Proof.** Since \( x_j \in \mathcal{H}_1^d(h_\kappa^2) \), this is just the previous theorem with \( P(x) = x_j \).

5.2. **The boundedness of weighted Riesz transform.** The Riesz transforms are important singular integral operators. One would like to prove the boundedness of the weighted Riesz transforms, just as in the case of the ordinary Riesz transforms. This, however, turns out to be a rather difficult task. The effort is hindered by the lack of information on \( \tau_y f \). Furthermore, currently no theory of singular integrals with reflection invariant weight functions is available.

For some special parameters, however, the weighted Riesz transforms on the radial functions can be related to the classical Riesz transforms. This is given in the next proposition.

**Proposition 5.4.** If \( f(x) = f_0(\|x\|) \) is a radial function in \( L^p(\mathbb{R}^d, h_\kappa^2) \) and \( 2\gamma_\kappa \in \mathbb{N}_0 \), then for \( 1 < p < \infty \),

\[
\|R_j^\kappa f\|_{\kappa,p} \leq c \|f\|_{\kappa,p}.
\]

**Proof.** Since \( f \) is radial, the explicit formula of \( \tau_y f \) in [25] and the Funk-Hecke formula ([13]) shows that

\[
R_j^\kappa f(x) = c \int_{\mathbb{R}^d} \tau_y f(x) \frac{y_j}{\|y\|^{\gamma + d + 1}} h_\kappa^2(y) dy
= c \int_0^\infty \int_{S^{d-1}} V_\kappa f_0 \left( \sqrt{\|x\|^2 + s^2 - 2\|x\|s(x, y')} \right) (y') y_j h_\kappa^2(y') d\omega(y') \frac{ds}{s}
= c x_j \int_0^\infty \int_{-1}^1 f_0 \left( \sqrt{\|x\|^2 + s^2 - 2\|x\|s} \right) t(1 - t^2)^{\gamma_\kappa + (d-3)/2} dt \frac{ds}{s}.
\]

Therefore, since \( 2\gamma_\kappa \in \mathbb{N}_0 \), we conclude by (3.1) that \( R_j^\kappa f(x) = c \tilde{R}_j f_0(\|x\|) \), where \( \tilde{R}_j \) corresponds to the ordinary Riesz transform \( R_j f \) defined on \( \mathbb{R}^m \) with \( m = d + 2\gamma_\kappa \). Consequently, by Proposition (3.1), we have

\[
\int_{\mathbb{R}^d} |R_j^\kappa f(x)|^p h_\kappa^2(x) dx = c \int_0^\infty r^{2\gamma_\kappa + d - 1} |\tilde{R}_j f_0(r)|^p dr
= c \int_0^\infty r^{m-1} |\tilde{R}_j f_0(r)|^p dr
\leq c \int_{\mathbb{R}^m} |f_0(\|x\|)|^p dx = c \|f\|^p_{\kappa,p},
\]

which completes the proof.

\[\square\]
In the rest of this section, we consider only the case $d = 1$ and $G = \mathbb{Z}_2$, for which the weight function is simply $h_\kappa(x) = |x|^\kappa$ and the Riesz transform is the integral operator on the real line

$$\mathcal{R}_\kappa f(x) = c_\kappa \int_\mathbb{R} \tau_y f(x) \frac{y}{|y|^{2\kappa+2}} h^2_\kappa(y) dy, \quad c = \Gamma(\kappa + 1)/\sqrt{\pi},$$

where the integral holds in the principle value sense. There is a multiplier theorem in this setting of Dunkl transform on the real line ([8]). However, it does not apply where the integral holds in the principle value sense. There is a multiplier theorem in this setting of Dunkl transform on the real line ([8]). However, it does not apply to the Riesz transform. We have the following result.

**Theorem 5.5.** Let $G = \mathbb{Z}_2$. If $f \in L^p(\mathbb{R}^d, h^2_\kappa)$ and $2\gamma_\kappa \in \mathbb{N}_0$, then for $1 < p < \infty$,

$$\|\mathcal{R}_\kappa f\|_{\kappa,p} \leq c\|f\|_{\kappa,p}.$$

**Proof.** In this case $f$ is radial means that $f$ is even, so that the stated result holds for $f$ being even. Every function $f$ on $\mathbb{R}$ can be split as $f = f_e + f_o$ where $f_e(r) = (f(r) + f(-r))/2$ is even and $f_o(r) = (f(r) - f(-r))/2$ is odd. Evidently, we have $\|f_e\|_{\kappa,p} \leq \|f\|_{\kappa,p}$ and $\|f_o\|_{\kappa,p} \leq \|f\|_{\kappa,p}$, we only need to prove the stated inequality for $f$ being an odd function.

Let $f$ be an odd function and define $g(r) = f(r)/r$ for $r \neq 0$. Then $g$ is even. The explicit formula of $\tau_r f$ in (2.4) shows that

$$\tau_r f(s) = (s - r)\tau_r g(s) = s\tau_r g(s) - r\tau_r g(s)$$

so that the Riesz operator can be written as a sum of two terms,

$$\mathcal{R}_\kappa f(s) = c_\kappa s \int_\mathbb{R} \tau_r g(s) \frac{r}{|r|^2} dr - c_\kappa \int_\mathbb{R} \tau_r g(s) dr := c_\kappa \mathcal{R}_1^\kappa f(s) - c_\kappa \mathcal{R}_2^\kappa f(s).$$

For the first term we start with the following observation. Let $\kappa = (m - 1)/2$, where $m \in \mathbb{N}$. Define $F(x) = g(||x||)$ for $x \in \mathbb{R}^m$. Let $\Omega$ be a harmonic polynomial of first degree on $\mathbb{R}^m$ and let $K(x) = \Omega(x)/||x||^{m+1}$. We consider the convolution $F*K$ in $L^1(\mathbb{R}^m)$. Using the spherical-polar coordinates and the ordinary Funk-Hecke formula, we get

$$(F*K)(x) = \int_0^\infty r^{m-1} \int_{S^{m-1}} g \left( \sqrt{||x||^2 + r^2 - 2||x|| r \langle x', y' \rangle} \right) \Omega(y') d\omega(y') \frac{dr}{r^m}$$

$$= c\Omega(x') \int_0^1 \int_{-1}^1 g \left( \sqrt{||x||^2 + s^2 - 2||x|| s t} \right) t(1 - t^2)^{m-3} dt \frac{dr}{r}$$

$$= c\Omega(x') \int_1^\infty \int_{-1}^1 g \left( \sqrt{||x||^2 + s^2 - 2||x|| s t} \right) t(1 - t^2)^{m-3} dt \frac{dr}{r}.$$

Since $\text{sign}(r)/|r| = r/r^2$ is odd in $r$, changing variables $t \to -t$ and $r \to -r$ shows that we can replace $t(1 - t^2)^{m-3}/2$ by $(1 + t)(1 - t^2)^{m-1}$ in the last expression, so that the inner integral becomes $\tau_r g(||x||)$, from which we get

$$(F*K)(x) = c\Omega(x') \int_{-\infty}^\infty \tau_r f(||x||) \frac{r}{|r|^2} dr.$$
Hence, \( |x|(F * K)(x) = c \Omega(x') R_1^\kappa f(\|x\|) \). Using the spherical-polar coordinates and integrating over \( \mathbb{R}^m \) we get

\[
\int_{\mathbb{R}^m} |(F * K)(x)|^p |x|^p dx = c \int_0^\infty |R_1^\kappa f(s)|^p s^{m-1} ds \int_{S^{d-1}} \Omega(x') d\omega(x') \\
= c \int_0^\infty |R_1^\kappa f(s)|^p s^{2\kappa} ds.
\]

Since \( R_1^\kappa f \) is an even function, this gives

\[
\int_{-\infty}^{\infty} |R_1^\kappa f(s)|^p |s|^{2\kappa} ds \leq c \int_{\mathbb{R}^m} |(F * K)(x)|^p |x|^p dx
\]

for \( p > m/(m - 1) = 2\kappa + 1/(2\kappa) \) since \( |x|^p \in A_p(\mathbb{R}^m) \) for \( p > m/(m - 1) \). By definition, \( F(x) = g(\|x\|) \) and \( g(s) = f(s)/s \), so that

\[
\int_{\mathbb{R}^m} |F(x)|^p |x|^p dx = \int_{\mathbb{R}^m} |f(\|x\|)|^p dx = \int_0^\infty |f(r)r^{m-1} dr = 1/2 \int_{-\infty}^{\infty} |f(r)|^p r^{2\kappa} dr.
\]

This takes care of the first term \( R_1^\kappa f \) for \( p > 2\kappa + 1/(2\kappa) \).

For the second term we consider the operator \( F \mapsto F * K_1 \), where \( F \) is as above and \( K_1(x) = \|x\|^{1-m} \), which agrees with the notation in \([32]\). Similar to \( F * K \) we get

\[
F * K_1(x) = c \int_0^1 \int_{-1}^1 g\left(\sqrt{\|x\|^2 + s^2 - 2\|x\|t}\right) (1 - t^2)^{-\frac{m-2}{2}} dt dr
\]

\[
= c \int_{-\infty}^{\infty} \tau_r g(\|x\|) dr = c R_2^\kappa(\|x\|).
\]

Therefore, since \( R_2^\kappa \) is even,

\[
\int_{-\infty}^{\infty} |R_2^\kappa(s)|^p |s|^{2\kappa} ds = 2 \int_0^\infty |R_2^\kappa(s)|^p s^{m-1} ds
\]

\[
= c \int_{\mathbb{R}^m} |F * K_1(x)|^p dx \leq c \int_{\mathbb{R}^m} |F(x)|^p |x|^p dx
\]

for \( p > m/(m - 1) \) using Proposition \([32]\). Again, the last integral is the same as \( \|f\|_{\kappa,p}^p \), which takes care of the second term for \( p > 2\kappa + 1/(2\kappa) \). Together, we have proved that \( \|R_\kappa^n\|_{\kappa,p} \leq c \|f\|_{\kappa,p} \) or \( p > 2\kappa + 1/(2\kappa) \). For \( 1 < p \leq (2\kappa + 1)/(2\kappa) \) we use the standard duality argument. 

\[
\square
\]

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Stat-Math Division, Indian Statistical Institute, 8th Mile, Mysore Road, Bangalore-560 059, India.

E-mail address: veluma@isibang.ac.in

Department of Mathematics, University of Oregon, Eugene, Oregon 97403-1222.

E-mail address: yuan@math.uoregon.edu