RESEARCH ARTICLE

OPERATIONS ON M-STRONG FUZZY GRAPHS.

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Abstract
In current paper, deleted lexicographical product, disjunction and symmetric difference on fuzzy graphs are defined and some of their properties are discussed. Moreover, the concept of M-strong fuzzy graphs are investigated for mentioned operations. These results also are illustrated with some examples.

Introduction:
In 1965, Lotfi A. Zade established Fuzzy set for representing uncertainty [6]. Fuzzy set has numerous applications in different branches of modern sciences consisting operations research, transportation, information theory and neural networks [7, 8].

A graph G is an ordered pair (V, E), where V = V(G) is the set of vertices of G and E = E(G) is the set of edges of G where \( E \subseteq V \times V \). Graph theory has been used to study modern science such as operations research, transportation and cluster analysis.

In 1975, Rosenfeld introduced fuzzy graphs [5] based on fuzzy set. Fuzzy graph theory plays essential roles in various disciplines including information theory, neural networks, clustering problems and control theory, etc. Fuzzy models are more compatible to the system in comparison with classical models [9, 10].

Bhutani and Rosenfeld introduced the notion of M-strong fuzzy graphs and studied some of their properties. [1, 4] Many interesting graphs are obtained from composing simpler graphs via several operations. For more information on graph operations see [3].

In this paper, we define deleted lexicographical product, disjunction and symmetric difference of two fuzzy graphs and prove that new graphs constructed from mentioned operations are fuzzy graph. Also we show that deleted lexicographical product, disjunction and symmetric difference of two M-strong fuzzy graphs are also M-strong fuzzy graph. Finally we prove that if \( G_1 \times G_2, G_1 \lor G_2 \) and \( G_1 \bigoplus G_2 \) are M-strong fuzzy graphs, then at least one factor must be M-strong fuzzy graph. All properties are illustrated with examples.

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Preliminaries:
In this section, we list some necessary definitions as follows:

**Definition 2.1**[6]. A fuzzy set is a set of ordered pairs \( \{(x, \mu(x)) | x \in X \} \) where \( X \) is universal set. \( \mu(x) \) is a map from \( X \) to \([0, 1]\) which is called a membership function or degree of membership of \( x \) in \( A \).

**Definition 2.2**[5]. A fuzzy graph \( G \) is a pair of functions \( (\mu, \rho) \) where \( \mu \) is a fuzzy subset of \( X \) and \( \rho: X \times X \rightarrow [0,1] \) is symmetric fuzzy relation on \( \mu \) such that \( \rho(x, y) \leq \min\{\mu(x), \mu(y)\} \).

Throughout the paper, we use \( xy \) instead of \( (x, y) \) for an element of \( E(G) \).

**Definition 2.3**[2]. The deleted lexicographical product of two graphs \( G_1 \) and \( G_2 \) is defined as a graph \( G_1 \times G_2 \) with the vertex set \( V(G_1) \times V(G_2) \) and vertex \((x_1, x_2)\) is adjacent with vertex \((y_1, y_2)\) whenever \( x_1 \) is adjacent with \( y_1 \) in \( G_1 \) and \( x_2 \neq y_2 \) or \( x_1 = y_1 \) and \( x_2 \) is adjacent with \( y_2 \) in \( G_2 \).

**Definition 2.4**[3]. The disjunction of two fuzzy graphs \( G_1 \) and \( G_2 \) is defined as a fuzzy graph \( G_1 \uplus G_2 \) with the vertex set \( V(G_1) \times V(G_2) \) and vertex \((x_1, x_2)\) is adjacent with vertex \((y_1, y_2)\) whenever \( x_1 \) is adjacent with \( y_1 \) or \( x_2 \) is adjacent with \( y_2 \) in \( G_2 \) or both of them.

**Definition 2.5**[3]. The symmetric difference of two fuzzy graphs \( G_1 \) and \( G_2 \) is defined as a fuzzy graph \( G_1 \ominus G_2 \) with the vertex set \( V(G_1) \times V(G_2) \) and vertex \((x_1, x_2)\) is adjacent with vertex \((y_1, y_2)\) whenever \( x_1 \) is adjacent with \( y_1 \) in \( G_1 \) or \( x_2 \) is adjacent with \( y_2 \) in \( G_2 \) but not both.

**Definition 2.6**[1]. A fuzzy subgraph \((\mu, \rho)\) of \( G \) is called a \( M \)-strong fuzzy subgraph if \( \rho(xy) = \min\{\mu(x), \mu(y)\} \) for all \( xy \in E(G) \).

Some operations on fuzzy graphs:
In this section, we present new definitions for some fuzzy graph operations and prove that new graphs are also fuzzy graphs.

**Definition 3.1**. The deleted lexicographical product of two fuzzy graphs \( G_1 \) and \( G_2 \) is defined as a fuzzy graph \( G_1 \times G_2 \) with the vertex set \( V(G_1) \times V(G_2) \) and vertex \((x_1, x_2)\) is adjacent with vertex \((y_1, y_2)\) whenever \( x_1 \) is adjacent with \( y_1 \) in \( G_1 \) and \( x_2 \neq y_2 \) or \( x_1 = y_1 \) and \( x_2 \) is adjacent with \( y_2 \) in \( G_2 \) with \((\mu_1 \times \mu_2)(x_1, x_2) = \min\{\mu_1(x_1), \mu_2(x_2)\}\) for all \((x_1, x_2) \in V(G_1 \times G_2)\) and
\[
(\rho_1 \times \rho_2)((x_1, x_2)(y_1, y_2)) = \min\{\rho_1(x_1 y_1), \mu_2(x_2)\} \quad x_1 = y_1 \in V(G_1), \quad x_2 y_2 \in E(G_2)
\]

**Theorem 3.1**. Let \( G \) be a deleted lexicographical product of two fuzzy graphs \( G_1 \) and \( G_2 \) and \((\mu, \rho)\) be a fuzzy subgraph of \( G \) where \( i \in \{1,2\} \), then \((\mu_1 \times \mu_2), (\rho_1 \times \rho_2)\) be a fuzzy subgraph of \( G \).

**Proof**. Let \( G = G_1 \times G_2 \), we have:
\[
(\rho_1 \times \rho_2)((x_1, x_2)(y_1, y_2)) = \min\{\rho_1(x_1 y_1), \mu_2(x_2)\}
\]
and also,
\[
(\rho_1 \times \rho_2)((x_1, x_2)(y_1, y_2)) = \min\{\mu_1(x_1), \mu_2(x_2)\}
\]

**Definition 3.2**. The disjunction of two fuzzy graphs \( G_1 \) and \( G_2 \) is defined as a fuzzy graph \( G_1 \uplus G_2 \) with the vertex set \( V(G_1) \times V(G_2) \) and vertex \((x_1, x_2)\) is adjacent with vertex \((y_1, y_2)\) whenever \( x_1 \) is adjacent with \( y_1 \) in \( G_1 \) or \( x_2 \) is adjacent with \( y_2 \) in \( G_2 \) or both of them with \((\mu_1 \uplus \mu_2)(x_1, x_2) = \min\{\mu_1(x_1), \mu_2(x_2)\}\) for all \((x_1, x_2) \in V(G_1 \uplus G_2)\) and
\[
(\rho_1 \uplus \rho_2)((x_1, x_2)(y_1, y_2)) = \min\{\rho_1(x_1 y_1), \mu_2(x_2)\} \quad x_1 y_1 \in E(G_1), \quad x_2 y_2 \in V(G_2)
\]

**Theorem 3.2**. Let \( G \) be a disjunction of two fuzzy graphs \( G_1 \) and \( G_2 \) and \((\mu, \rho)\) be a fuzzy subgraph of \( G \) where \( i \in \{1,2\} \), then \((\mu_1 \uplus \mu_2), (\rho_1 \uplus \rho_2)\) be a fuzzy subgraph of \( G \).

**Proof**. Let \( G = G_1 \uplus G_2 \), we have:
Definition 3.3. The symmetric difference of two fuzzy graphs $G_1$ and $G_2$ is defined as a fuzzy graph $G_1 \oplus G_2$ with the vertex set $V(G_1) \times V(G_2)$ and vertex $(x_1, x_2)$ is adjacent with vertex $(y_1, y_2)$ whenever $x_1$ is adjacent with $y_1$ in $G_1$ or $x_2$ is adjacent with $y_2$ in $G_2$ but not both with

\[
(p_1 \oplus p_2)((x_1, x_2)(y_1, y_2)) = \min\{p_1(x_1y_1), p_2(x_2y_2)\}
\]

and also

\[
(p_1 \oplus p_2)((x_1, x_2)(y_1, y_2)) = \min\{p_1(x_1y_1), p_2(x_2y_2)\}
\]

Theorem 3.3. Let $G$ be a symmetric difference of two fuzzy graphs $G_1$ and $G_2$ and $(\mu_i, \rho_i)$ be a fuzzy subgraph of $G$ where $i \in \{1, 2\}$, then $((\mu_1 \oplus \mu_2), (p_1 \oplus p_2))$ is a fuzzy subgraph of $G$.

Proof. Using Theorem 3.2, we can get the desired result.

M-strong fuzzy graphs:

In this section, we prove some theorems to show that if $G_1$ and $G_2$ are M-strong fuzzy graphs, then new fuzzy graph constructed from them are M-strong fuzzy graph too.

Also, if $G_1 \times G_2, G_1 \lor G_2$ and $G_1 \oplus G_2$ are M-strong fuzzy graphs, then at least one factor must be M-strong fuzzy graph. All computations are illustrated with examples.

Theorem 4.1. Let $G$ be a deleted lexicographical product of two M-strong fuzzy graphs $G_1$ and $G_2$, then $G$ is a M-strong fuzzy graph.

Proof. The first part is taken over all edges $(x_1, x_2)(y_1, y_2) \in E(G)$ such that $x_2y_2 \in E(G_2)$ and $x_1 = y_1$. Using the fact that $G_2$ is a M-strong fuzzy graph, we have

\[
(p_1 \times p_2)((x_1, x_2)(y_1, y_2)) = \min\{p_1(x_1), p_2(x_2y_2)\}
\]

and also

\[
(p_1 \times p_2)((x_1, x_2)(y_1, y_2)) = \min\{p_1(x_1), p_2(x_2y_2)\}
\]

The second part is taken over all edges $(x_1, x_2)(y_1, y_2) \in E(G)$ such that $x_1 \lor y_1 \in E(G_1)$ and $x_2, y_2 \in E(G_2)$. Using the fact that $G_1$ is a M-strong fuzzy graph, we have

\[
(p_1 \times p_2)((x_1, x_2)(y_1, y_2)) = \min\{p_1(x_1), p_2(x_2)\}
\]

and also

\[
(p_1 \times p_2)((x_1, x_2)(y_1, y_2)) = \min\{p_1(x_1), p_2(x_2)\}
\]

Example 4.1. Let $\square_1$ and $\square_2$ be two fuzzy graphs illustrated in figure 1:

![Figure 1: Two M-strong fuzzy graphs $\square_1$ and $\square_2$.](image-url)
According to definition 3.1, deleted lexicographical product of two fuzzy graphs $\square_1$ and $\square_2$ is presented in figure 2:

![Figure 2](image)

**Figure 2:** The deleted lexicographical product of two graphs $\square_1$ and $G_2$ and it is easy to see that it is a $M$-strong fuzzy graph.

**Remark.** The opposite is not necessarily true.

**Example 4.2.** Two fuzzy graphs $G_1$ and $G_2$ and deleted lexicographical product of them are considered in figure 3 and 4, respectively.

![Figure 3](image)

**Figure 3:** Two fuzzy graphs $G_1$ and $G_2$

![Figure 4](image)

**Figure 4:** The graph $G_1 \times G_2$ formed by $G_1$ and $G_2$
It is easy to see that $G_1 \times G_2$ and $G_2$ are strong fuzzy graphs but $G_1$ is not.

**Theorem 4.2.** If $G = G_1 \times G_2$ is M-strong fuzzy graph, then at least $G_1$ or $G_2$ must be M-strong fuzzy graph.

**Proof.** Suppose that both $G_1$ and $G_2$ are not M-strong fuzzy graphs. Then there exist $x_1, y_1 \in E(G_1)$ and $x_2, y_2 \in E(G_2)$ such that

$$\square_1(x_1, y_1) < \square_1\{\mu_1(x_1), \mu_1(y_1)\} \text{ and } \square_2(x_2, y_2) < \square_2\{\mu_2(x_2), \mu_2(y_2)\}$$

According to the definition 3.1,

$$(\rho_1 \times \rho_2)((x_1, x_2)(y_1, y_2)) = \square_1\{\mu_1(x_1), \mu_2(x_2)\} \text{ and } (\mu_1 \times \mu_2)(x_1, x_2) = \square_2\{\mu_1(y_1), \mu_2(x_2)\}$$

Since

$$\rho_1 \times \rho_2)((x_1, x_2)(y_1, y_2)) = \square_1\{\mu_1(x_1), \mu_2(x_2), \mu_2(y_1)\} < \square_2\{\mu_2(x_2), \mu_2(y_2), \mu_1(x_1)\}$$

and

$$\text{min}\{(\mu_1 \times \mu_2)(x_1, x_2), (\mu_1 \times \mu_2)(x_1, y_2)\} = \square_2\{\mu_1(x_1), \mu_2(x_2), \mu_2(y_2)\}$$

So

$$(\rho_1 \times \rho_2)((x_1, x_2)(y_1, y_2)) < \square_1\{\mu_1 \times \mu_2)(x_1, x_2), (\mu_1 \times \mu_2)(y_1, x_2)\}$$

Which is in contradiction to $G = G_1 \times G_2$ as M-strong fuzzy graph. Hence if $G = G_1 \times G_2$ is M-strong fuzzy graph, then at least $G_1$ or $G_2$ must be M-strong fuzzy graph.

Using definition 3.2, we can easily arrive at:

**Theorem 4.3.** Let $G$ be a disjunction of two M-strong fuzzy graphs $G_1$ and $G_2$, then $G$ is a M-strong fuzzy graph.

**Example 4.3.** Consider two fuzzy graphs $G_1$ and $G_2$ in figure 5:

![Figure 5](image1)

It is obvious that $G_1$ and $G_2$ are strong fuzzy graphs.

Based on definition 3.2, the graph $G_1 \lor G_2$ is illustrated in figure 6 as follows:

![Figure 6](image2)
It is easy to see that the disjunction of $G_1$ and $G_2$ is a strong fuzzy graph.

**Theorem 4.4.** If $G = G_1 \lor G_2$ is M-strong fuzzy graph, then at least $G_1$ or $G_2$ must be M-strong fuzzy graph.

**Proof.** Suppose that both $G_1$ and $G_2$ are not M-strong fuzzy graphs, then there exist $x_1y_1 \in E(G_1)$ and $x_2y_2 \in E(G_2)$ such that

$$\square_1(x_1, y_1) < \square_2\{\mu_1(x_1), \mu_1(y_1)\} \text{ and } \square_2(x_2, y_2) < \square_1\{\mu_2(x_2), \mu_2(y_2)\}$$

According to the definition 3.2,

$$(\square_1 \lor \square_2)((x_1, x_2)(y_1, y_2)) = \square_1\{\mu_1(x_1), \mu_2(x_2), \mu_2(y_2)\} \text{ and } \square_2\{\mu_1(x_1), \mu_1(y_1), \mu_2(x_2), \mu_2(y_2)\}$$

Since

$$(\square_1 \lor \square_2)((x_1, x_2)(y_1, y_2)) = \square_1\{\mu_1(x_1), \mu_2(x_2), \mu_2(y_2)\}$$

Therefore

$$\min\{\mu_1 \lor \mu_2\}(x_1, x_2)(y_1, y_2) < \square_1\{\mu_1 \lor \mu_2\}(x_1, x_2), (\mu_1 \lor \mu_2)(y_1, y_2) \}$$

Which is in contradiction to $G = G_1 \lor G_2$ as M-strong fuzzy graph. Hence if $G = G_1 \lor G_2$ is M-strong fuzzy graph, then at least $G_1$ or $G_2$ must be M-strong fuzzy graph.

**Theorem 4.5.** If $G = G_1 \lor G_2$ be M-strong fuzzy graph, then at least $G_1$ or $G_2$ must be M-strong fuzzy graph.

**Proof.** It is straightforward.

**Theorem 4.6.** Let $G$ be a symmetric difference of two M-strong fuzzy graphs $G_1$ and $G_2$, then $G$ is a M-strong fuzzy graph.

**Proof.** It is straightforward.

In next example, we will illustrate that the opposite is not necessarily true.

**Example 4.4.** Two fuzzy graphs $G_1$ and $G_2$ are illustrated in figure 7 and symmetric difference of $G_1 \oplus G_2$ is presented in figure 8.

![Figure 7](image_url)

**Figure 7:** Two fuzzy graphs $G_1$ and $G_2$.

$G_1 \oplus G_2$ and $G_2$ are strong fuzzy graphs, but $G_1$ is not strong fuzzy graph.
**Conclusion:**
In present paper, specific operations on fuzzy graphs have been introduced and some theorems are discussed. Some properties of M-strong fuzzy graphs are investigated.

Fuzzy graph theory is highly utilized in various areas. In future work, we can focus on Intuitionistic, bipolar and hyperfuzzy graphs and attempt to investigate many properties on them.

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