Nonparametric estimation for linear SPDEs from local measurements

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Abstract

We estimate the coefficient function of the leading differential operator in a linear stochastic partial differential equation (SPDE). The estimation is based on continuous time observations which are localised in space. For the asymptotic regime with fixed time horizon and with the spatial resolution of the observations tending to zero, we provide rate-optimal estimators and establish scaling limits of the deterministic PDE and of the SPDE on growing domains. The estimators are robust to lower order perturbations of the underlying differential operator and achieve the parametric rate even in the nonparametric setup with a spatially varying coefficient.

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1 Introduction

While there is a large amount of work on probabilistic, analytical and recently also computational aspects of stochastic partial differential equations (SPDEs), many natural statistical questions are open. With this work we want to enlarge the scope of statistical methodology in two major directions. First, we consider observations of a solution path that are local in space and we ask whether the underlying differential operator or rather its local characteristics can be estimated from this local information only. Second, we allow the coefficients in the differential operator to vary in space and we pursue nonparametric estimation of the coefficient functions, as opposed to parametric estimation approaches for finite-dimensional

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global parameters in the coefficients. Naturally, both directions are intimately connected.

As a concrete model we consider the parabolic SPDE
\[ dX(t) = A_\vartheta X(t)dt + BdW(t), \quad t \in [0,T], \]
with the second-order differential operator
\[ A_\vartheta z := \text{div}(\vartheta \nabla z) + \langle a, \nabla z \rangle + bz \]
on some bounded domain \( \Lambda \subseteq \mathbb{R}^d \), see Section 2 for formal details. The coefficient functions \( \vartheta, a, b \) are unknown on \( \Lambda \) and we aim at estimating \( \vartheta : \Lambda \to \mathbb{R}^+ \), which models the diffusivity in a stochastic heat equation. The functions \( a, b \) as well as the operator \( B \) in front of the driving space-time white noise process \( W \) form an unknown nuisance part.

Measurements of a solution process \( X \) necessarily have a minimal spatial resolution \( \delta > 0 \) and we dispose of the observations \( \langle X(t), K_{\delta,x_0} \rangle_{L^2(\Lambda)}, t \in [0,T] \), where \( K_{\delta,x_0} \) is an averaging kernel with support of diameter \( \delta \) around some \( x_0 \in \Lambda \). We keep the time span \( T \) fixed and construct an estimator, called proxy MLE, which for the resolution asymptotics \( \delta \to 0 \) converges at rate \( \delta \) to \( \vartheta(x_0) \) and satisfies a CLT. Another estimator, the so-called augmented MLE, will even converge under far more general conditions and exhibit a smaller asymptotic variance, but requires a second local observation process \( \langle X(t), \Delta K_{\delta,x_0} \rangle_{L^2(\Lambda)}, t \in [0,T] \) in terms of the Laplace operator \( \Delta \). Clearly, if we have access to these observations around all \( x_0 \in \Lambda \), then both estimators can be used to estimate the diffusivity function \( \vartheta \) nonparametrically on all of \( \Lambda \).

These results are statistically remarkable. First of all, even for the parametric case that \( \vartheta \) is a constant, it is not immediately clear that \( \vartheta \) is identified (i.e., exactly recovered) from local observations in a shrinking neighbourhood around some \( x_0 \in \Lambda \) only. Probabilistically, this means that the local observation laws are mutually singular for different values of \( \vartheta \). What is more, the bias-variance tradeoff paradigm in nonparametric statistics does not apply: asymptotic bias and standard deviation are both of order \( \delta \) and the CLT provides us even with a simple pointwise confidence interval for \( \vartheta \). The robustness of the estimators to lower order parts in the differential operator and unknown \( B \) is very attractive for applications. The rate \( \delta \) is shown to be the best achievable rate in a minimax sense even for constant \( \vartheta \) without nuisance parts.

The fundamental probabilistic structure behind these results is a universal scaling limit of the observation process for \( \delta \to 0 \). At a highly localised level, the differential operator \( A_\vartheta \) behaves like \( \vartheta(x_0) \Delta \), as expressed in Corollary 3.6 below, and the construction of the estimators shows a certain scaling invariance with respect to \( B \). To study these scaling limits, we need to consider the deterministic PDE on growing domains via the stochastic Feynman-Kac approach and to deduce tight asymptotics for the action of the semigroup and the heat kernels. Further
tools like the fourth moment theorem or the Feldman-Hajek Theorem rely on the underlying Gaussian structure, but extensions to semi-linear SPDEs seem possible.

Let us compare our localisation approach to the spectral approach, introduced by Huebner and Rozovskii (1995), for parametric estimation. In the simplest case $A_\vartheta = \vartheta \Delta$ for some $\vartheta > 0$ and $B$ commuting with $A_\vartheta$, the SPDE solution can be expressed in the eigenbasis of the Laplace operator $\Delta$. If the first $N$ coefficient processes (Fourier modes of $X$) are observed, then a maximum-likelihood estimator for $\vartheta$ is asymptotically efficient as $N \to \infty$. This approach has turned out to be very versatile, allowing also for estimating time-dependent $\vartheta(t)$ nonparametrically (Huebner and Lototsky (2000)) or to cover nonlinear SPDEs as the stochastic Navier-Stokes equation (Cialenco and Glatt-Holtz (2011)). The methodology, however, is intrinsically bound to observations in the spectral domain and to operators $A_\vartheta$ whose eigenfunctions, at least in the leading order, are independent of $\vartheta$. In contrast, we work with local observations in space and the unknown spectrum of the operators $A_\vartheta$ does not harm us. More conceptually, we rely on the local action of the differential operator $A_\vartheta$, while the spectral approach also applies in an abstract operator in Hilbert space setting.

Our case of spatially varying coefficients has been considered first by Aihara and Sunahara (1988) (with $a = b = 0$) in a filtering problem. The corresponding nonparametric estimation problem is then addressed by Aihara and Bagchi (1989) with a sieve least squares estimator, but they achieve consistency only for global observations with a growing time horizon $T \to \infty$. In a stationary one-dimensional setting Bibinger and Trabs (2017) ask whether the parameter $\vartheta > 0$ can be estimated when observing the solution only at $x_0$ over a fixed time interval $[0, T]$.

Interestingly, in the case $B = \sigma^2 I$ the parameter $\vartheta$ cannot be recovered if the level $\sigma$ of the space-time white noise is unknown. For a recent and exhaustive survey on statistics for SPDEs we refer to Cialenco (2018).

In Section 2 the SPDE and the observation model are introduced and in Section 3 the scaling properties along with the resolution level $\delta$ are discussed. Section 4 derives our estimators via a least-squares and a likelihood approach and provides some basic insight into their error analysis. The main convergence results as well as a minimax lower bound are presented in Section 5. The findings are illustrated by a numerical example in Section 6. While the main steps in the proofs are presented together with the results, all more technical arguments are delegated to the Appendix.
2 The model

2.1 The SPDE model

Let $\Lambda$ be a bounded open set in $\mathbb{R}^d$ with $C^2$-boundary $\partial \Lambda$ and consider $L^2(\Lambda)$ with the usual $L^2$-norm $\| \cdot \| := \| \cdot \|_{L^2(\Lambda)}$. $H^s(\Lambda)$ for $s \geq 0$ denotes the fractional Sobolev spaces and $H^1_0(\Lambda)$ is the closure of $C_c^{\infty}(\overline{\Lambda})$ in $H^1(\Lambda)$. We write $(\cdot, \cdot)_{\mathbb{R}^d}$ for the Euclidean scalar product and $| \cdot |$ for the norm. Let $\operatorname{div}(z) = \sum_{i=1}^d \partial_i z$ denote the divergence and $\Delta z = \operatorname{div}(\nabla z) = \sum_{i=1}^d \partial_i^2 z$ the Laplace operator. $(e^{t\Delta})_{t \geq 0}$ denotes the semigroup on $L^2(\mathbb{R}^d)$ generated by $\Delta$ with domain $\mathcal{D}(\Delta) = H^2(\mathbb{R}^d)$.

Define a second order elliptic operator with Dirichlet boundary conditions

$$A_\vartheta = \Delta_\vartheta + A_0, \quad \mathcal{D}(A_\vartheta) = H^1_0(\Lambda) \cap H^2(\Lambda),$$

where $\Delta_\vartheta z = \operatorname{div}(\vartheta \nabla z) = \sum_{i=1}^d \partial_i (\vartheta \partial_i z)$ is the weighted Laplace operator with spatially varying diffusivity $\vartheta \in C^2(\overline{\Lambda})$, $\min_{x \in \partial \Lambda} \vartheta(x) > 0$, and $A_0 z = (a, \nabla z)_{\mathbb{R}^d} + bz$ with functions $a \in C^1(\overline{\Lambda})$, $b \in C^0(\overline{\Lambda})$, $a > 0$.

Throughout this work $T < \infty$ is fixed. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$ be a filtered probability space with a cylindrical Brownian motion $W$ on $L^2(\Lambda)$ ($dW$ is also referred to as space-time white noise), and let $B : L^2(\Lambda) \to L^2(\Lambda)$ be a bounded linear operator, which is not assumed to be trace class. We study the linear stochastic partial differential equation

$$\begin{aligned}
&\begin{cases}
dX(t) = A_\vartheta X(t) \, dt + B dW(t), & 0 < t \leq T, \\
X(0) = X_0, \\
X(t)|_{\partial \Lambda} = 0, & 0 < t \leq T,
\end{cases}
\end{aligned}$$

with Dirichlet boundary conditions and deterministic initial value $X_0 \in L^2(\Lambda)$.

Let $(S_\vartheta(t))_{t \geq 0}$ be the analytic semigroup on $L^2(\Lambda)$ generated by $A_\vartheta$, cf. Lunardi (1995), Theorem 3.1.3. If $\int_0^T \| S_\vartheta(t) B \|_{H^1(L^2(\Lambda))}^2 \, dt < \infty$ with Hilbert-Schmidt norm $\| \cdot \|_{H^1(L^2(\Lambda))}$, then the SPDE (2.1) admits a unique weak solution $(X(t))_{0 \leq t \leq T}$ taking values in $L^2(\Lambda)$ such that $l(t, z) := \langle X(t), z \rangle$ for $0 \leq t \leq T$, $z \in H^1_0(\Lambda) \cap H^2(\Lambda)$ satisfies

$$l(t, z) = \langle X_0, z \rangle + \int_0^t l(s, A_\vartheta^* z) \, ds + \langle BW(t), z \rangle,$$

where $A_\vartheta^* z := \sum_{i=1}^d \partial_i \vartheta \partial_i z$ is the adjoint of $A_\vartheta$. If $B = cI$, then $S_\vartheta(t)$ is the semigroup on $L^2(\mathbb{R}^d)$ generated by $\Delta_\vartheta$ with domain $\mathcal{D}(\Delta_\vartheta) = H^2(\mathbb{R}^d)$. We write $(\vartheta, \cdot)_{\mathbb{R}^d}$ for the weighted Euclidean scalar product and $| \cdot |_{\mathbb{R}^d}$ for the weighted Euclidean norm.

If $B$ is also an integral operator, and $\lambda \geq 0$, then $S_\vartheta(t)$ is the semigroup on $L^2(\mathbb{R}^d)$ generated by $\Delta_\vartheta + \lambda I$ with domain $\mathcal{D}(\Delta_\vartheta + \lambda I) = H^2(\mathbb{R}^d)$. We write $(\vartheta + \lambda I, \cdot)_{\mathbb{R}^d}$ for the weighted Euclidean scalar product and $| \cdot |_{\mathbb{R}^d}$ for the weighted Euclidean norm.

Theorem 5.4. $X(t)$ is explicitly defined via the variation of constants formula:

$$X(t) = S_\vartheta(t) X_0 + \int_0^t S_\vartheta(t-s) B \, dW(s).$$

4
Throughout this work let the weak solution does not exist.

Proposition 2.1. Define \( l := (l(t,z))_{0 \leq t \leq T, z \in \Lambda} \) by (2.4). Then:

(i) \( l \) solves the SPDE in the sense that (2.2) holds and that \( l(t,\bullet) : L^2(\Lambda) \to L^2(\mathbb{P}) \) is linear and continuous.

(ii) \( l \) is a Gaussian process with mean function \( (t,z) \mapsto \langle S_{\delta}(t)X_0,z \rangle \) and covariance function
\[
\varrho((t,z),(t',z')) = \int_0^{t \wedge t'} \langle B^*S_{\delta}(t-s)z,B^*S_{\delta}(t'-s)z' \rangle ds. \tag{2.5}
\]

In the following, justified formally by (2.4), we write \( \langle X(t),z \rangle \) instead of \( l(t,z) \), even if the weak solution does not exist.

### 2.2 Local observations

Throughout this work let \( x_0 \in \Lambda \) be fixed. The following rescaling will be useful in the sequel: for \( z \in L^2(\mathbb{R}^d) \) and \( \delta > 0 \) set

\[
\Lambda_{\delta,x_0} := \delta^{-1}(\Lambda - x_0) = \{ \delta^{-1}(x-x_0) : x \in \Lambda \} \quad \text{and} \quad \Lambda_0,x_0 := \mathbb{R}^d,
\]

\[
z_{\delta,x_0}(x) := \delta^{-d/2} z(\delta^{-1}(x-x_0)), \quad x \in \mathbb{R}^d.
\]

Let \( K \in H^2(\mathbb{R}^d) \) have compact support in \( \Lambda_{\delta,x_0} \). The compact support ensures that \( K_{\delta,x_0} \) is localized around \( x_0 \) and \( K_{\delta,x_0} \in H^0(\Lambda) \cap H^2(\Lambda), \| K_{\delta,x_0} \| = \| K \|_{L^2(\mathbb{R}^d)} \) for all small \( \delta > 0 \). Local measurements of (2.1) at \( x_0 \) with resolution level \( \delta \) until time \( T \) are described by the real-valued processes \( X_{\delta,x_0} = (X_{\delta,x_0}(t))_{0 \leq t \leq T} \), \( X_{\delta,x_0}^\Delta = (X_{\delta,x_0}^\Delta(t))_{0 \leq t \leq T} \).

\[
X_{\delta,x_0}(t) = \langle X(t), K_{\delta,x_0} \rangle, \tag{2.6}
\]

\[
X_{\delta,x_0}^\Delta(t) = \langle X(t), \Delta K_{\delta,x_0} \rangle. \tag{2.7}
\]

Note that it is sufficient to observe \( X_{\delta,x_0^\Delta}(t) \) for \( x \) in a neighbourhood of \( x_0 \) in order to provide us with \( X_{\delta,x_0^\Delta}(t) = \Delta X_{\delta,x_0}^\Delta(t) \big|_{x=x_0} \).

The process \( X_{\delta,x_0} \) satisfies \( X_{\delta,x_0}(0) = \langle X_0, K_{\delta,x_0} \rangle \) and
\[
dX_{\delta,x_0}(t) = \langle X(t), A_{\delta}^* K_{\delta,x_0} \rangle dt + \| B^* K_{\delta,x_0} \| d\overline{W}(t) \tag{2.8}
\]

with the scalar Brownian motion \( \overline{W}(t) = \langle K_{\delta,x_0}, BW(t) \rangle / \| B^* K_{\delta,x_0} \| \).
3 Scaling assumptions

3.1 Rescaled operators and semigroups

In view of (2.8) we need to study \( A^*_0 K_{\delta,x_0} \) and \( S^*_0(t) K_{\delta,x_0} \) as \( \delta \to 0 \). For smooth compactly supported \( K \) it is clear that

\[
\Delta K_{\delta,x_0} = \delta^{-2} (\Delta K)_{\delta,x_0}
\]

and similarly \( (-\Delta)^k K_{\delta,x_0} = \delta^{-2k} (-\Delta K)_{\delta,x_0} \) for \( k \in \mathbb{N} \). If \( A_\vartheta = \vartheta \Delta \) for constant \( \vartheta > 0 \), this suggests formally

\[
S^*_\vartheta(t) K_{\delta,x_0} = (S^*_\vartheta,\delta,x_0(t\delta^{-2})K)_{\delta,x_0},
\]

where \((S^*_\vartheta,\delta,x_0(t))_{0 \leq t \leq T}\) is the semigroup generated by \( \Delta \) on \( L^2(\Lambda_{\delta,x_0}) \). Applying the semigroup to a localized function \( K_{\delta,x_0} \) is therefore equivalent to rescaling the semigroup in time and space and keeping the test function \( K \) fixed. The scaling exhibited here is the usual scaling for parabolic PDEs.

In order to make this heuristic precise note first that \( A^*_0 = \Delta_0 + A^*_0 \) with domain \( \mathcal{D}(A^*_0) = H^1_0(\Lambda) \cap H^2(\Lambda) \) and \( A^*_0 z = -\text{div}(az) + bz \). \( A^*_0 \) is the infinitesimal generator of the analytic semigroup \((S^*_0(t))_{0 \leq t \leq T}\) on \( L^2(\Lambda) \) (Pazy [1983, Lemma 7.3.4]). Define similarly operators \( A_{\vartheta,\delta,x_0} = \Delta_{\vartheta}(x_0 + \delta \bullet) + A_{0,\delta,x_0} \), \( A^*_0,\delta,x_0 = \Delta_{\vartheta}(x_0 + \delta \bullet) + A^*_0,\delta,x_0 \) with domains \( \mathcal{D}(A_{\vartheta,\delta,x_0}) = \mathcal{D}(A^*_0,\delta,x_0) = H^1_0(\Lambda_{\delta,x_0}) \cap H^2(\Lambda_{\delta,x_0}) \), where for \( z \in C^\infty_c(\Lambda_{\delta,x_0}) \)

\[
\begin{align*}
A_{0,\delta,x_0} z &= \delta (a(x_0 + \delta \bullet) \nabla z)_{\mathbb{R}^d} + \delta^2 b(x_0 + \delta \bullet) z, \\
A^*_0,\delta,x_0 z &= -\delta \text{div}(a(x_0 + \delta \bullet) z) + \delta^2 b(x_0 + \delta \bullet) z.
\end{align*}
\]

They generate corresponding analytic semigroups \((S_{\vartheta,\delta,x_0}(t))_{t \geq 0}\) and \((S^*_0,\delta,x_0(t))_{t \geq 0}\) on \( L^2(\Lambda_{\delta,x_0}) \). In Section A.2 we prove further:

Lemma 3.1. For \( \delta > 0 \) the following holds:

(i) If \( z \in H^1_0(\Lambda_{\delta,x_0}) \cap H^2(\Lambda_{\delta,x_0}) \), then \( A^*_0 z_{\delta,x_0} = \delta^{-2} (A^*_0,\delta,x_0 z)_{\delta,x_0} \).

(ii) If \( z \in L^2(\Lambda_{\delta,x_0}) \), then \( S^*_0(t) z_{\delta,x_0} = (S^*_0,\delta,x_0(t\delta^{-2})z)_{\delta,x_0}, t \geq 0 \).

3.2 Scaling of \( B \)

Just as with \( A^*_0 \) we also need that \( B^* \) behaves nicely when applied to \( K_{\delta,x_0} \). For this we shall assume a scaling limit for \( B^* \), which does not degenerate in combination with \( K \).
Example 3.4.

(i) For a bounded continuous function \( \sigma : \mathbb{R}^d \to (0, \infty) \) define the multiplication operator \( M_\sigma : L^2(\Lambda) \to L^2(\Lambda), M_\sigma z(x) := (\sigma z)(x) = \sigma(x)z(x) \). With \( B = B^* = M_\sigma \) the SPDE in (2.1) can be written informally as

\[
\dot{X}(t, x) = A_\theta X(t, x) + \sigma(x)\dot{W}(t, x), \quad 0 < t \leq T, \ x \in \Lambda.
\]

Note that \( B \) commutes with \( A_\theta \) only if \( \sigma \) is constant. For \( z \in L^2(\Lambda) \) we find that \( B^*z_{\delta,x_0} = (M_\sigma(\delta_{x_0} + z))_{\delta,x_0} \) and so \( B_{0,x_0} = M_{\sigma(\delta_{x_0})} \). Then \( \|B_{\delta,x_0}^* z - \sigma(x_0)z\|_{L^2(\mathbb{R}^d)} \to 0 \) for \( z \in L^2(\mathbb{R}^d) \), \( \delta \to 0 \), and thus \( B_{0,x_0}^* = M_{\sigma(x_0)} \) is the multiplication operator on \( L^2(\mathbb{R}^d) \) with the constant \( \sigma(x_0) \). For \( z \in H^2(\mathbb{R}^d), z' \in L^2(\mathbb{R}^d) \) we have

\[
\Psi(\Delta z, z') = \sigma^2(x_0) \int_0^\infty \langle e^{2s\Delta} \frac{\Delta z}{ds}, z' \rangle_{L^2(\mathbb{R}^d)} ds = -\frac{\sigma^2(x_0)}{2} \langle z, z' \rangle_{L^2(\mathbb{R}^d)},
\]

and thus by partial integration \( \Psi(\Delta K, \Delta K) = \frac{\sigma^2(x_0)}{2} \|\nabla K\|^2_{L^2(\mathbb{R}^d)} \). The non-degeneracy conditions are clearly satisfied.

(ii) Let \( \sigma \) be as in (i) and consider with bounded \( \eta \in C^2(\mathbb{R}^d), \min_{x \in \mathbb{R}^d} \eta(x) > 0 \), the perturbed multiplication operator \( B = B^* = M_\sigma - (-\Delta)^{-\gamma}, \gamma > 0 \). By functional calculus \( B^*z_{\delta,x_0} = (B_{\delta,x_0}^* z)_{\delta,x_0} \) for \( z \in L^2(\Lambda) \) with \( B_{\delta,x_0} = M_{\sigma(x_0)} + \delta^2(-\Delta_\eta(x))^{-\gamma} \) and \( \|B_{\delta,x_0}^* z - \sigma(x_0)z\|_{L^2(\mathbb{R}^d)} \to 0 \) for \( z \in L^2(\mathbb{R}^d) \), \( \delta \to 0 \). \( B_{0,x_0} \) and \( \Psi(\Delta K, \Delta K) \) are as in (i).
(iii) Assumption 3.2 excludes $B = (-\Delta)^{-\gamma}$, $\gamma > 0$, a typical choice to obtain smooth solutions $X$, cf. Da Prato and Zabczyk (2014) Chapter 5.5. Indeed, by (ii) $B_{0,x_0}^* = \delta^{2\gamma}(-\Delta)^{-\gamma}$ and so $B_{0,x_0}^* = 0$, violating the non-degeneracy conditions. This can be dealt with by modifying the test function $\tilde{K}_{\delta,x_0}$. For example, if $A_0 = \partial \Delta$ for constant $\vartheta > 0$ and $X_0 \in \mathcal{D}(B^{-1})$, then assume we have access to $\langle X(t), B^{-1}K_{\delta,x_0} \rangle$, $\langle X(t), B^{-1}\Delta K_{\delta,x_0} \rangle$ instead of (2.6), (2.7). Since $B$ and $A_0$ commute, $(X(\cdot), B^{-1}K_{\delta,x_0})$ has the same distribution as $(\tilde{X}(\cdot), K_{\delta,x_0})$, where $\tilde{X}$ corresponds to the SPDE (2.1) with $B = I$ and $\tilde{X}_0 = B^{-1}X_0$, and so Assumption 3.2 is satisfied.

3.3 The initial condition

Assumption 3.4($z;\beta$). For $\beta > 0$ and $z \in H^2(\mathbb{R}^d)$ with compact support in $\Lambda_{\delta,x_0}$ for all small $\delta > 0$, the initial condition $X_0$ satisfies

$$
\int_0^T \langle S_\vartheta(t)X_0, (\Delta z)_{\delta,x_0} \rangle^2 dt = o(\ell_{d,2}(\delta)^{-1}\delta^\beta), \; \delta \to 0,
$$

where $\ell_{d,2}(\delta) = \log(\delta^{-1})$ for $d = 2$ and $\ell_{d,2}(\delta) = 1$ otherwise.

Lemma 3.5. Let $d \geq 2$ or $\int_{\mathbb{R}^d} z(x)dx = 0$. Then Assumption 3.4($z;\beta$) is satisfied for

(i) $\beta = 2$ if $X_0 \in L^p(\Lambda)$ for some $p > 2$, in particular if $X_0 \in C(\overline{\Lambda})$,

(ii) $\beta = 3$ if $X_0 \in H^3_0(\Lambda) \cap H^2(\Lambda)$.

Proof. This follows from Lemma A.10(ii,iii) below.

3.4 From bounded to unbounded domains

Lemma 3.1 and Assumption 3.2 allow us to study the covariance function of $X_{\delta,x_0}$:

$$
c((t, K_{\delta,x_0}), (t', K_{\delta,x_0}))
= \delta^2 \int_0^{t\delta^{-2}\Lambda t'\delta^{-2}} \langle B_{0,x_0}^*S_{0,\delta,x_0}^* (\delta^{-2}t - s) K, B_{0,x_0}^*S_{0,\delta,x_0}^* (\delta^{-2}t' - s) K \rangle_{L^2(\Lambda_{\delta,x_0})} ds.
$$

Let us see what happens as $\delta \to 0$. From (3.1) we find $A_{0,\delta,x_0}^* K \to \vartheta(x_0)\Delta K$ in $L^2(\mathbb{R}^d)$, which suggests formally for the semigroups $S_{0,\delta,x_0}^* (t) K \to e^{\vartheta(x_0)\Delta t} K$. This means that $u(\delta)(t) = S_{0,\delta,x_0}^* (t) K$, the solution of

$$
\frac{d}{dt}u(\delta)(t) = (A_{0,\delta,x_0}^* u(\delta))(t), \; u(\delta)(0) = K,
$$

8
on the bounded domain $L^2(\Lambda_{\delta,x_0})$ converges to $u(t) = e^{\vartheta(x_0)t} \Delta K$, the solution of
\[
\frac{d}{dt} u(t) = \vartheta(x_0) \Delta u(t), \quad u(0) = K,
\]
on all of $L^2(\mathbb{R}^d)$. This scaling limit, which seems natural but is nevertheless non-trivial, lies at the heart of the analysis for the covariance function. We will prove it in Proposition A.8 below as well as the following interesting corollary, which for simplicity assumes a zero initial condition:

**Corollary 3.6.** Grant Assumption \[\beta,\varrho\] and let $X_0 = 0$. Then the finite dimensional distributions of $(l^{(\delta)}(t,z))_{t \geq 0, z \in L^2(\mathbb{R}^d)}$, $l^{(\delta)}(t,z) = \delta^{-1} \langle X(t\delta^2), (z|_{\Lambda_\delta,x_0})_{\delta,x_0} \rangle$, converge to those of $(l^{(0)}(t,z))_{t \geq 0, z \in L^2(\mathbb{R}^d)}$, $l^{(0)}(t,z) = \langle Y(t), z \rangle_{L^2(\mathbb{R}^d)}$, solving the stochastic heat equation on $L^2(\mathbb{R}^d)$ with space-time white noise $(W(t))_{t \geq 0}$:
\[
\begin{aligned}
\dot{Y}(t) &= \vartheta(x_0) \Delta Y(t) dt + B_{0,x_0} dW(t), \quad t > 0, \\
Y(0) &= 0.
\end{aligned}
\]

This corollary demonstrates the strength of local measurements that at small scales only the highest order differential operator matters, together with the local coefficient $\vartheta(x_0)$ and the local operator $B_{0,x_0}$ in the noise.

**4 The two estimation methods**

**4.1 Motivation and construction**

We give two motivations for deriving the estimators in the parametric case $A_\vartheta = \vartheta \Delta$ with constant $\vartheta > 0$, $B = I$. As we shall see later, these estimators will then work quite universally for nonparametric specifications of $\vartheta$ and general $A_\vartheta$ and $B$.

**Least squares approach.** In the deterministic situation of (2.8) without driving noise (i.e. $A_\vartheta = \vartheta \Delta$ and $B = 0$) we recover $\vartheta$ via $\dot{X}_{\delta,x_0}(t) = \vartheta X_{\delta,x_0}^\Delta(t)$ for all $t \in [0,T]$. A standard least-squares ansatz in the noisy situation would therefore lead to an estimator $\hat{\vartheta} = \arg\min_\vartheta \int_0^T (\dot{X}_{\delta,x_0}(t) - \vartheta X_{\delta,x_0}^\Delta(t))^2 dt$. While this itself is certainly not well defined, the corresponding normal equations yield the feasible estimator
\[
\hat{\vartheta}^{LS}_\delta = \frac{\int_0^T X_{\delta,x_0}^\Delta(t) dX_{\delta,x_0}(t)}{\int_0^T X_{\delta,x_0}^\Delta(t)^2 dt},
\]
compare with the approach by Maslowski and Tudor (2013) for fractional noise.
Likelihood approach. Assume that only $X_{\delta,x_0}$ is observed. Denote by $\mathbb{P}_{\delta,x_0}$, $\mathbb{P}_0$ the laws of $X_{\delta,x_0}$ and $\|K\|_{L^2(\mathbb{R}^d)}$ on the canonical path space $(C([0,T],\mathbb{R}^d), \|\cdot\|_\infty)$ equipped with its Borel sigma algebra. Typically, the likelihood of $\mathbb{P}_{\delta,x_0}$ with respect to $\mathbb{P}_0$ is determined via Girsanov’s theorem. This is not immediate from (2.8), because $X^\Delta_{\delta,x_0}$ cannot be obtained from $X_{\delta,x_0}$ for fixed $x_0$. Therefore we employ Liptser and Shiryaev (2001, Theorem 7.17) and write $X_{\delta,x_0}$ as the diffusion-type process

$$dX_{\delta,x_0}(t) = \vartheta m_\vartheta(t)dt + \|K\|_{L^2(\mathbb{R}^d)}d\tilde{W}(t), \quad t \in [0,T],$$

with a different scalar Brownian motion $\tilde{W} = (\tilde{W}(t))_{0 \leq t \leq T}$, adapted to the filtration generated by $X_{\delta,x_0}$, and

$$m_\vartheta(t) = \mathbb{E}_\vartheta \left[ X^\Delta_{\delta,x_0}(t) \mid (X_{\delta,x_0}(s))_{0 \leq s \leq t} \right].$$

Girsanov’s theorem in the form of Liptser and Shiryaev (2001, Theorem 7.18) applies and we find that $\mathbb{P}_{\delta,x_0}$ has with respect to $\mathbb{P}_0$ the likelihood

$$\mathcal{L}(\vartheta, X_{\delta,x_0}) = \exp \left( \frac{\vartheta}{\|K\|_{L^2(\mathbb{R}^d)}} \int_0^T m_\vartheta(t)dX_{\delta,x_0}(t) - \frac{\vartheta^2}{2\|K\|_{L^2(\mathbb{R}^d)}} \int_0^T m_\vartheta(t)^2 dt \right).$$

Computing the conditional expectation $m_\vartheta(t)$ is a non-explicit filtering problem, even in the parametric case $A_\vartheta = \vartheta \Delta$. In particular, $m_\vartheta$ depends on $\vartheta$ in a highly nonlinear way. We pursue two different modifications:

**Augmented MLE.** If we observe $X^\Delta_{\delta,x_0}$ additionally, then we can just replace the conditional expectation $m_\vartheta(t)$ in the likelihood by its argument $X^\Delta_{\delta,x_0}(t)$, which is in particular independent of $\vartheta$. Maximizing this modified likelihood leads to the augmented MLE

$$\hat{\vartheta}^A_\delta = \frac{\int_0^T X^\Delta_{\delta,x_0}(t)dX_{\delta,x_0}(t)}{\int_0^T X^\Delta_{\delta,x_0}(t)^2 dt}. \quad (4.1)$$

We remark that $\hat{\vartheta}^A_\delta = \hat{\vartheta}^{LS}_\delta$.

**Proxy MLE.** If we do not dispose of additional observations, we can approximate $m_\vartheta(t)$ by the conditional expectation $\mathbb{E}_\vartheta[X^\Delta_{\delta,x_0}(t) \mid X_{\delta,x_0}(t)]$. In our simplified setup with $A_\vartheta = \vartheta \Delta$ and $B = I$ there exists a stationary solution $\langle X(t), z \rangle = \int_{-\infty}^t \langle S_\vartheta(t - s)z, dW(s) \rangle$, $z \in L^2(\Lambda)$, with a two-sided cylindrical Brownian motion $(W(t), t \in \mathbb{R})$, provided the variance remains finite. Then also
Let us discuss the basic error analysis for the augmented MLE

**4.2 Basic error analysis**

use local measurements around \( \hat{\vartheta} \). Consider estimating \( \vartheta \) (which we shall not assume).

Remark 4.1: A sufficient condition for the existence of \( \delta \) in the general nonparametric framework of Section 2. Since we only compare (2.5). In general, \( \langle (\Delta)^{-1}\delta K, K \rangle \delta \) may not exist, but if we assume the existence of \( \delta \) \( K \in H^1(\mathbb{R}^d) \) with \( \delta \delta \) \( K = K \) and compact support in \( \Lambda_{\delta,x_0} \), then by the scaling in Lemma 3.1 \( \text{Var}(X_{\delta,x_0}(0)) = \delta^2 \| \nabla K \|_{L^2(\mathbb{R}^d)}^2 < \infty \) follows. In this situation we therefore obtain

\[
\text{Var}[X_{\delta,x_0}(t) | X_{\delta,x_0}(0)] = \frac{\text{Cov}(X_{\delta,x_0}(t), X_{\delta,x_0}(t))}{\text{Var}(X_{\delta,x_0}(t))} X_{\delta,x_0}(t) = -\frac{\delta^{-2} \| K \|^2_{L^2(\mathbb{R}^d)}}{\| \nabla K \|_{L^2(\mathbb{R}^d)}^2} X_{\delta,x_0}(t).
\]

This expression is again independent of \( \delta \). Using it as an approximation of \( m_{\vartheta}(t) \) in the likelihood and neglecting the boundary terms in the identity

\[
2 \int_0^T X_{\delta,x_0}(t) dX_{\delta,x_0}(t) = (X_{\delta,x_0}^2(T) - X_{\delta,x_0}^2(0)) - \langle X_{\delta,x_0} \rangle_T,
\]

we obtain the proxy MLE

\[
\hat{\vartheta}^\delta = \frac{\| \nabla K \|^2_{L^2(\mathbb{R}^d)}}{2 \| K \|^2_{L^2(\mathbb{R}^d)} T} \frac{\langle X_{\delta,x_0} \rangle_T}{\delta^{-2} \int_0^T X_{\delta,x_0}(t)^2 dt}.
\]

Note that in the general case of Equation (2.8) the quadratic variation satisfies

\[
\langle X_{\delta,x_0} \rangle_T = T \| B^* K_{\delta,x_0} \|^2,
\]

which we could use immediately if \( B^* \) is known to us (which we shall not assume).

**Remark 4.1.** A sufficient condition for the existence of \( \delta \) is \( \int_{\mathbb{R}^d} K(x) dx = 0 \), \( \int_{\mathbb{R}^d} x K(x) dx = 0 \) by Lemma A.11(iii) below.

**4.2 Basic error analysis**

Let us discuss the basic error analysis for the augmented MLE \( \hat{\vartheta}^\delta \) and the proxy MLE \( \hat{\vartheta}^\delta \) in the general nonparametric framework of Section 2. Since we only use local measurements around \( x_0 \), we expect that asymptotically we are lead to estimating \( \vartheta(x_0) \).

**Augmented MLE.** Consider \( \hat{\vartheta}^\delta(x_0) = \hat{\vartheta}^\delta \) from (4.1). Then insertion of Equation (2.8) for \( dX_{\delta,x_0}(t) \) yields the decomposition

\[
\hat{\vartheta}^\delta(x_0) = \vartheta(x_0) + \| B^* K_{\delta,x_0} \| (I_{\delta}^\delta)^{-1} M_{\delta}^\delta + (I_{\delta}^\delta)^{-1} R_{\delta}^\delta,
\]

\[
\int_{-\infty}^t S_\vartheta(2t-2s)K_{\delta,x_0} K_{\delta,x_0} ds = \frac{1}{2\vartheta} \| \delta \|^2_{L^2(\mathbb{R}^d)} \int_{-\infty}^t S_\vartheta(2t-2s) \Delta K_{\delta,x_0} K_{\delta,x_0} ds = -\frac{1}{2\vartheta} \| \delta \|^2_{L^2(\mathbb{R}^d)},
\]

(4.2)

(4.3)

(4.4)
with

\[ M_\delta^A = \int_0^T X_{\delta,x_0}(t)d\tilde{W}(t) \] (martingale part),

\[ I_\delta^A = \int_0^T X_{\delta,x_0}(t)^2 dt \] (observed Fisher information),

\[ R_\delta^A = \int_0^T X_{\delta,x_0}(t) \langle X(t), (A_\delta^* - \vartheta(x_0)\Delta) K_{\delta,x_0} \rangle dt \] (remaining bias).

Let us note that \( I_\delta^A \) is not the observed Fisher information in a strict sense (due to the appearance of \( m_\vartheta \) in the likelihood), but it plays the same role, compare the analysis of the MLE for Ornstein-Uhlenbeck processes in [Kutoyants 2013]. In particular, it forms the quadratic variation of the martingale \( M_\delta^A \). In the specific case \( A_\delta^* = \vartheta \Delta \) for some parametric \( \vartheta > 0 \) the term \( R_\delta^A \) vanishes, otherwise it induces a bias due to the variations of \( \vartheta \) around \( \vartheta(x_0) \) and due to first and zero order differential operators that may appear in \( A_\vartheta \).

As the error structure suggests, the augmented MLE \( \hat{\vartheta}_\delta^A(x_0) \) is a consistent estimator for \( \delta \to 0 \) if the observed Fisher information satisfies \( I_\delta^A \to \infty \). In the simple stationary case of \( \varphi \) we obtain \( \mathbb{E}[I_\delta^A] = \frac{T}{2\vartheta} \langle (-\Delta) K_{\delta,x_0}, K_{\delta,x_0} \rangle \), which by the scaling properties is of order \( \delta^{-2} \). Physically, this can be interpreted as an increase in energy in \( X_{\delta,x_0}^A \) under \( \delta \)-localisation due to the Laplacian in the drift, while the energy from the space-time white noise remains unchanged. This is in fact the same phenomenon as the increasing signal-to-noise ratio for high Fourier modes in the spectral approach by [Huebner and Rozovskii 1995].

**Proxy MLE.** Consider \( \hat{\vartheta}_\delta^P(x_0) = \hat{\vartheta}_\delta^P \) from (4.4). The only stochastic part is

\[ I_\delta^P := \delta^{-2} \int_0^T X_{\delta,x_0}(t)^2 dt \] (4.6)

in the denominator. In the general model (2.8) we shall see that \( I_\delta^P \) converges to \( \vartheta(x_0)^{-1}T\Psi(K,K) \), compare also Remark 3.3 with \( K = \Delta K \). Asking for consistency \( \hat{\vartheta}_\delta^P(x_0) \to \vartheta(x_0) \) leads to requiring the identity

\[
\|\nabla K\|_{L^2(\mathbb{R}^d)}^2 \|B_{0,x_0}K\|_{L^2(\mathbb{R}^d)}^2 = 2\|K\|_{L^2(\mathbb{R}^d)}^2 \Psi(K,K).
\]

This does not hold for any operator \( B_{0,x_0} \). We therefore restrict to our main specification \( B = M_\sigma \), for which by Example 3.4 the identity holds. In contrast to the augmented MLE, the proxy MLE works with the observation of \( X_{\delta,x_0} \) alone, but asks for new structural assumptions on \( B \) and \( K \). If they are not fulfilled, other likelihood approximations should be pursued. Compare also the suboptimal behaviour of \( \hat{\vartheta}_\delta^P(x_0) \) under the kernel \( K^{(2)} \) in Section 6 below.

The main result for the proxy MLE below is based on deriving a CLT for the quadratic functional \( I_\delta^P \) by very precise asymptotic moment calculations and
the fourth moment theorem in Wiener chaos by Nualart and Peccati (2005). Fundamental for this analysis is that $X_{δ,x_0}(t)$ and $X_{δ,x_0}(s)$ quickly decorrelate for $δ^{-2}|t − s| → ∞$, which is also predicted by the scaling limit in Corollary 3.6.

Finally, a CLT for $\hat{δ}^2(x_0)$ is easily deduced via the delta method. Remark that this method of proof might also cover time-discrete observations of $X_{δ,x_0}$ if the sampling frequency increases sufficiently fast as $δ → 0$, but this is not pursued here.

5 Main results

5.1 Results for the augmented MLE

The augmented MLE $\hat{δ}^A(x_0)$ satisfies a CLT with rate $δ$.

**Theorem 5.1.** Grant Assumptions 3.2 and 3.4(K,2). Let $d ≥ 2$ or $\int_{\mathbb{R}^d} K(x)dx = 0$. Then as $δ → 0$

$$δ^{-1}\left(\hat{δ}^A(x_0) - \vartheta(x_0)\right) \xrightarrow{d} N\left(\mu^A, \vartheta(x_0)\Sigma^A\right),$$

where, with $Ψ$ from (3.3),

$$\mu^A = (Ψ(ΔK, ΔK))^{-1}Ψ(ΔK, β),$$

$$β(x) = ⟨∇\vartheta(x_0), x⟩_{\mathbb{R}^d} ΔK(x) − ⟨∇\vartheta(x_0) − a(x_0), ∇K(x)⟩_{\mathbb{R}^d}, \ x ∈ \mathbb{R}^d,$$

$$Σ^A = T^{-1}(Ψ(ΔK, ΔK))^{-1}\|B^*_0,x_0 K\|_{L^2(\mathbb{R}^d)}^2.$$

**Proof.** Consider the error decomposition (4.3). Propositions A.1 and A.2 below yield $δ^2\mathbb{E}[I^A_δ] → T\vartheta(x_0)^{-1}Ψ(ΔK, ΔK)$ and $I^A_δ/\mathbb{E}[I^A_δ] \xrightarrow{P} 1$. In terms of $Y^δ(t) := X^A_{δ,x_0}(t)/[\mathbb{E}[I^A_δ]^{1/2}}$ we obtain $M^A_δ/\mathbb{E}[I^A_δ]^{1/2} = \int_0^T Y^δ(t)d\tilde{W}(t)$, the quadratic variation of which satisfies $\int_0^T (Y^δ(t))^2 dt = I^A_δ/\mathbb{E}[I^A_δ] \xrightarrow{P} 1$. A standard continuous martingale CLT, e.g. Kutoyants (2013, Theorem 1.19), shows $M^A_δ/\mathbb{E}[I^A_δ]^{1/2} \xrightarrow{d} N(0,1)$. Moreover,

$$\|B^*_{δ,x_0} K\| = \|B^*_{0,x_0} K\|_{L^2(\mathbb{R}^d)} → \|B^*_{0,x_0} K\|_{L^2(\mathbb{R}^d)}$$

due to Assumption 3.2, and $δ^{-1}(I^A_δ)^{-1}R^A_δ \xrightarrow{P} \mu^A$ by Proposition A.3 below. We conclude by applying Slutsky’s lemma.

It is interesting to note that both bias and standard deviation of $\hat{δ}^A(x_0)$ are of order $δ$. The asymptotic bias $μ^A$ is independent of $T$, while the variance $Σ^A$ decays in $T$. $B$, $∇\vartheta$ and $a$ appear in the limit only via the localized terms $B^*_0,x_0, ∇\vartheta(x_0)$, $a(x_0)$, while $b$ does not appear at all. This demonstrates again the universality.
property of local measurements, similarly to Corollary 3.6. From (5.2) we see that
\( \mu^A \) vanishes if \( A_\theta = \vartheta \Delta + b \) for parametric \( \vartheta > 0 \). Another important situation
for \( \mu^A = 0 \) is given next.

**Example 5.2.** (Example [3.4] ctd.) Let \( B = M_\sigma \) and recall the identities
\[ \Psi(\Delta K, \Delta K) = \frac{\sigma(x_0)^2}{2} \| \nabla K \|^2_{L^2(\mathbb{R}^d)}, \quad \Psi(\Delta K, \beta) = -\frac{\sigma(x_0)^2}{2} \langle K, \beta \rangle_{L^2(\mathbb{R}^d)}. \]
By Lemma A.6 with \( z = K \) this means
\[ \langle K, \beta \rangle_{L^2(\mathbb{R}^d)} = -\langle \langle \nabla \vartheta(x_0), x \rangle_{\mathbb{R}^d}, \| \nabla K(x) \|^2 \rangle_{L^2(\mathbb{R}^d)}, \]
and Theorem 5.1 yields
\[ \delta^{-1} \left( \hat{\vartheta}_\delta^A(x_0) - \vartheta(x_0) \right) \xrightarrow{d} N \left( \frac{\int_{\mathbb{R}^d} \langle \nabla \vartheta(x_0), x \rangle_{\mathbb{R}^d}, \| \nabla K \|^2_{L^2(\mathbb{R}^d)} dx}{\| \nabla K \|^2_{L^2(\mathbb{R}^d)}}, \frac{2\vartheta(x_0)\| K \|^2_{L^2(\mathbb{R}^d)}}{T\| K \|^2_{L^2(\mathbb{R}^d)}} \right). \]
In particular, if \( \nabla K \) is symmetric in the sense \( |\nabla K(-x)| = |\nabla K(x)| \), then the asymptotic bias vanishes:
\[ \delta^{-1} \left( \hat{\vartheta}_\delta^A(x_0) - \vartheta(x_0) \right) \xrightarrow{d} N \left( 0, \frac{2\vartheta(x_0)\| K \|^2_{L^2(\mathbb{R}^d)}}{T\| K \|^2_{L^2(\mathbb{R}^d)}} \right). \]
Note that the variance, as the estimator itself, is invariant under multiplicative scaling of the kernel \( K \). The rougher \( K \) is, the smaller is the asymptotic variance, which bears some similarity with deconvolution problems.

If the asymptotic bias \( \mu^A \) vanishes, we can construct a simple confidence interval in terms of the augmented MLE. Note that in the setting of Example 5.2 \( \Sigma^A = 2T^{-1}\| K \|^2_{L^2(\mathbb{R}^d)}\| \nabla K \|^2_{L^2(\mathbb{R}^d)} \) is easily accessible.

**Corollary 5.3.** In the setting of Theorem 5.1 and assuming \( \mu^A = 0 \), for \( \alpha > 0 \) the confidence interval for \( \vartheta(x_0) \)
\[ I_{1-\alpha}^A = \left[ \hat{\vartheta}_\delta^A(x_0) - \delta(\hat{\vartheta}_\delta^A(x_0)\Sigma^A)^{1/2}q_{1-\alpha/2}, \hat{\vartheta}_\delta^A(x_0) + \delta(\hat{\vartheta}_\delta^A(x_0)\Sigma^A)^{1/2}q_{1-\alpha/2} \right], \]
with the standard normal \((1-\alpha/2)-quantile \( q_{1-\alpha/2} \), has asymptotic coverage \( 1-\alpha \) for \( \delta \to 0 \).

**Proof.** By Theorem 5.1 and Slutsky’s lemma applied for \( \hat{\vartheta}_\delta^A(x_0) \xrightarrow{p} \vartheta(x_0) \), we have
\[ \delta^{-1}(\hat{\vartheta}_\delta^A(x_0)\Sigma^A)^{-1/2} \left( \hat{\vartheta}_\delta^A(x_0) - \vartheta(x_0) \right) \xrightarrow{d} N(0, 1), \quad \delta \to 0, \]
noting \( \mu^A = 0 \). This yields \( \mathbb{P}(\vartheta(x_0) \in I_{1-\alpha}^A) \to 1 - \alpha \). \( \square \)
5.2 Results for the proxy MLE

In the setting described in Section 4.2 we obtain a CLT for the proxy MLE $\hat{\theta}^P_\delta(x_0)$.

**Theorem 5.4.** Assume $K = \Delta \hat{K}$ for $\hat{K} \in H^4(\mathbb{R}^d)$ with compact support and suppose $B = M_\sigma$ with $\sigma \in C^1(\mathbb{R}^d)$. Grant Assumption 3.4(3) and let $d \geq 2$ or $\int_{\mathbb{R}^d} \hat{K}(x)dx = 0$. Then

$$
\delta^{-1} \left( \hat{\theta}^P_\delta(x_0) - \theta(x_0) \right) \xrightarrow{d} N \left( \mu^P_{\theta,\sigma}, \sigma^2(x_0) \Sigma^P \right), \ \delta \to 0,
$$

with

$$
\mu^P_{\theta,\sigma} = -\frac{\sigma^2(x_0)}{\sigma^2(\sigma)} \| \nabla \hat{K} \|_{L^2(\mathbb{R}^d)}^{-2} \langle \nabla^2 \sigma(x_0), x \rangle_{\mathbb{R}^d}, |\nabla \hat{K}|^2 \rangle_{L^2(\mathbb{R}^d)}
+ \frac{\sigma^2(x_0)}{\sigma^2(\sigma)} \| K \|_{L^2(\mathbb{R}^d)}^{-2} \langle \nabla \sigma^2(x_0), x \rangle_{\mathbb{R}^d}, |K|^2 \rangle_{L^2(\mathbb{R}^d)},
$$

$$
\Sigma^P = \frac{4}{T} \| \nabla \hat{K} \|_{L^2(\mathbb{R}^d)}^{-4} \int_0^\infty \| \nabla e^{(s/2)} \Delta \hat{K} \|_{L^2(\mathbb{R}^d)}^4 ds.
$$

**Proof.** Set $G_0 := \frac{T \sigma^2(x_0)}{2\sigma^2(x_0)} \| \nabla \hat{K} \|_{L^2(\mathbb{R}^d)}^2$ and

$$
G_\delta := \frac{\langle X_\delta, x_0 \rangle_T}{T \sigma^2(x_0) \| K \|_{L^2(\mathbb{R}^d)}^2}, \ G := \frac{\langle \nabla^2 \sigma(x_0), x \rangle_{\mathbb{R}^d}, |K|^2 \rangle_{L^2(\mathbb{R}^d)}}{\sigma^2(x_0) \| K \|_{L^2(\mathbb{R}^d)}^2}.
$$

Consider $I^P_\delta$ from (4.6). Below, we show

$$
\delta^{-1} (I^P_\delta - G_0) \xrightarrow{d} N(\mu, \Sigma), \tag{5.4}
\delta^{-1} (G_\delta - G_0) \to G, \tag{5.5}
$$

with $\mu = G - \theta(x_0)^{-1} G_0 \mu^P_{\theta,\sigma}, \Sigma = \theta(x_0)^{-1} G_0^2 \Sigma^P$. By the delta method (Ferguson [1996] Theorem 7), (5.4) yields

$$
\delta^{-1} ((I^P_\delta)^{-1} - G_0^{-1}) \xrightarrow{d} N \left( -G_0^{-2} \mu, G_0^{-4} \Sigma \right),
$$

and (5.5) gives $G_\delta \to G$. The theorem follows then from Slutsky’s lemma:

$$
\delta^{-1} \left( \theta(x_0) G_\delta (I^P_\delta)^{-1} - \theta(x_0) \right)
= \theta(x_0) G_\delta (I^P_\delta)^{-1} (I^P_\delta)^{-1} - \theta(x_0) G_0^{-1} \delta^{-1} (G_\delta - G_0)
\xrightarrow{d} N \left( \theta(x_0) G_0^{-1} (G - \mu), \theta(x_0) G_0^{-2} \Sigma \right) = N \left( \mu^P_{\theta,\sigma}, \sigma^2(x_0) \Sigma^P \right).
$$

It remains to prove (5.4) and (5.5). By the compact support of $K$ we have

$$
\delta^{-1} \left( \langle X_\delta, x_0 \rangle_T - T \sigma^2(x_0) \| K \|_{L^2(\mathbb{R}^d)}^2 \right) = \delta^{-1} T \langle \nabla^2 (x_0 + \delta \bullet) - \sigma^2(x_0), x \rangle_{\mathbb{R}^d}, |K|^2 \rangle_{L^2(\mathbb{R}^d)}, \delta \to 0,
$$

$$
\rightarrow T \langle \nabla^2 (x_0), x \rangle_{\mathbb{R}^d}, |K|^2 \rangle_{L^2(\mathbb{R}^d)}, \delta \to 0,
$$

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implying (5.3). For (5.4) assume first \(X_0 = 0\). Then \(X_{\delta,x_0}\) is a centered Gaussian process and \(Z_\delta := \delta^{-2} \int_0^T (X_{\delta,x_0}(t))^2 - \mathbb{E}(X_{\delta,x_0}(t))^2 dt\) is an element of the second Wiener chaos. By the fourth moment theorem (Nualart and Peccati (2005, Theorem 1)) it is enough to show \(\text{Var}(\delta^{-1}Z_\delta) \to \Sigma\) and \(\mathbb{E}((\delta^{-1}Z_\delta)^4) \to 3\Sigma^2\) to conclude that \(\delta^{-1}Z_\delta \xrightarrow{d} N(0, \Sigma)\). Since by (3.4)

\[
\frac{4T}{\theta(x_0)^3} \int_0^\infty \Psi(e^{s\Delta}K, \Delta K)^2 ds = \frac{T\sigma^4(x_0)}{\theta(x_0)^3} \int_0^\infty \|\nabla e^{(s/2)\Delta}K\|^4_{L^2(\mathbb{R})} ds = \Sigma,
\]

we can consult for this Propositions \[A.19\] and \[A.20\] below (with \(w^{(4)} = \Delta K\)). (5.4) follows now by Proposition \[A.13\] below with \(z = K\) and Slutsky’s lemma from

\[
\delta^{-1} (X_{\delta}^P - G_0) = \delta^{-1} Z_\delta + \delta^{-1} \left( \delta^{-2} \int_0^T \text{Var}(X_{\delta,x_0}(t)) dt - G_0 \right) \xrightarrow{d} N(\mu, \Sigma).
\]

For general \(X_0\) write \(X_{\delta,x_0}(t) = \tilde{X}_{\delta,x_0}(t) + M(t)\), where \(M(t) = \mathbb{E}[X_{\delta,x_0}(t)]\) = \(\langle S_\delta(t) X_0, K_{\delta,x_0}\rangle\) and \(\tilde{X}_{\delta,x_0}\) is defined as \(X_{\delta,x_0}\), but with \(X_0 = 0\). By the first part it is enough to show \(\delta^{-3} \int_0^T (X_{\delta,x_0}(t)^2 - \tilde{X}_{\delta,x_0}(t)^2) dt \xrightarrow{P} 0\).

From \(\int_0^T M(t)^2 dt = o(\delta^3)\) by Assumption 3.4 \(\tilde{K};3\) and

\[
2 \int_0^T \int_0^T \text{Cov}(\tilde{X}_{\delta,x_0}(t), \tilde{X}_{\delta,x_0}(s))^2 dt ds = \text{Var} \left( \int_0^T \tilde{X}_{\delta,x_0}(t)^2 dt \right) = O(\delta^6)
\]

by Proposition \[A.19\] (iv) below with \(w^{(4)} = \Delta \tilde{K}\) we deduce

\[
\mathbb{E} \left[ \left( \int_0^T (X_{\delta,x_0}(t)^2 - \tilde{X}_{\delta,x_0}(t)^2) dt \right)^2 \right] = \mathbb{E} \left[ \left( \int_0^T (M(t)^2 + 2 \tilde{X}_{\delta,x_0}(t) M(t)) dt \right)^2 \right]
\]

\[
\lesssim \left( \int_0^T M(t)^2 dt \right)^2 + \int_0^T \int_0^T \text{Cov}(\tilde{X}_{\delta,x_0}(t), \tilde{X}_{\delta,x_0}(s)) M(t) M(s) ds dt
\]

\[
\lesssim o(\delta^6) + \text{Var} \left( \int_0^T \tilde{X}_{\delta,x_0}(t)^2 dt \right)^{1/2} \int_0^T M(t)^2 dt = o(\delta^6),
\]

which proves \(\delta^{-3} \int_0^T (X_{\delta,x_0}(t)^2 - \tilde{X}_{\delta,x_0}(t)^2) dt \xrightarrow{P} 0\), as desired.

The dependencies on \(\delta, T, \theta, K\) in the CLT are similar as for \(\hat{\theta}_\delta^4(x_0)\). It is interesting to note that \(\mu_{\theta, \sigma}^P\) depends on \(\sigma^2(x_0), \theta(x_0)\) and \(\nabla \sigma^2(x_0), \nabla \theta(x_0)\), as well as \(K\) and \(\nabla K\), while \(a, b\) do not appear at all in the limit. The limiting bias \(\mu_{\theta, \sigma}^P\) vanishes when \(\sigma^2\) and \(\sigma^2\) are constant, but also similar to Example 5.2 if \(\nabla K(-x)| = | \nabla K(x)|, |K(-x)| = |K(x)|\). As for the augmented MLE in Corollary 5.3, we can then conclude that

\[
I_{1-\alpha} = \left[ \hat{\theta}_\delta^P(x_0) - \delta(\hat{\theta}_\delta^P(x_0) \Sigma^P)^{1/2} q_{1-\alpha/2}, \hat{\theta}_\delta^P(x_0) + \delta(\hat{\theta}_\delta^P(x_0) \Sigma^P)^{1/2} q_{1-\alpha/2} \right]
\]
is an asymptotic \((1 - \alpha)\)-confidence interval for \(\vartheta(x_0)\).

Let us finally study the asymptotic variance \(\Sigma^P\) in more detail. Using the tensor products \(\Delta \otimes \Delta, f \otimes f\) and \(\Delta \oplus \Delta \coloneqq I \otimes \Delta + \Delta \otimes I\), we can write for \(f \in L^2(\mathbb{R}^d)\)
\[
\int_0^\infty \|e^{(s/2)\Delta}\|_{L^2(\mathbb{R}^d)}^4 ds = \int_0^\infty \langle (e^{s\Delta} \otimes e^{s\Delta})(f \otimes f), f \otimes f \rangle_{L^2(\mathbb{R}^d \times \mathbb{R}^d)} ds
\]
\[
= \int_0^\infty \langle e^{s(\Delta \oplus \Delta)}(f \otimes f), f \otimes f \rangle_{L^2(\mathbb{R}^d \times \mathbb{R}^d)} ds
\]
\[
= \|(-\Delta \oplus \Delta)^{-1/2}(f \otimes f)\|_{L^2(\mathbb{R}^d \times \mathbb{R}^d)}^2,
\]
provided the last norm is finite, e.g. if \(f = (-\Delta)^{1/2} \tilde{K}\). With this \(f\) we conclude via two duality arguments:
\[
\Sigma^P = 4 \frac{\|(-\Delta \oplus \Delta)^{-1/2}(f \otimes f)\|_{L^2(\mathbb{R}^d \times \mathbb{R}^d)}^2}{\|f \otimes f\|_{L^2(\mathbb{R}^d \times \mathbb{R}^d)}^2} \geq 4 \frac{\|f \otimes f\|_{L^2(\mathbb{R}^d \times \mathbb{R}^d)}^2}{\|(-\Delta \oplus \Delta)^{1/2}(f \otimes f)\|_{L^2(\mathbb{R}^d \times \mathbb{R}^d)}^2} \geq \frac{2 \|(-\Delta)^{1/2} \tilde{K}\|_{L^2(\mathbb{R}^d)}^2}{\|K\|_{L^2(\mathbb{R}^d)}^2} = \Sigma^A.
\]
Consequently, the proxy MLE has a larger variance than the augmented MLE, but the loss in precision is not severe if \(K\) has a well concentrated Fourier spectrum (consider \(\Delta\) in the Fourier domain).

### 5.3 Rate optimality

Let us address the question of optimality of the above estimators by providing a minimax lower bound. For minimax lower bounds it suffices to consider a subclass of the original model and we thus assume here that \(X_{\delta,x_0}\) is observed with \(A_{\vartheta} = \vartheta \Delta, B = I\) and a stationary initial condition \(X_{\delta,x_0}\). Then the following result establishes that the rate \(\delta\) of convergence is optimal and gives some lower bound for the dependence on \(T, \vartheta\) and \(K\).

**Proposition 5.5.** Assume \(A_{\vartheta} = \vartheta \Delta, \vartheta > 0, B = I, K \in H^1(\mathbb{R}^d)\) with compact support and that \(X_{\delta,x_0}\) is stationary. For \(\vartheta_0 > 0\) and \(\delta \to 0\) we have the asymptotic local lower bound for the root mean squared error
\[
\inf_{\vartheta} \sup_{\vartheta \in [\vartheta_0, \vartheta_0(1+\delta)]} \mathbb{E}_\vartheta \left[ (\hat{\vartheta} - \vartheta)^2 \right]^{1/2} \geq \tilde{c} \left( \frac{(\vartheta_0 \wedge 1)\|K\|_{L^2(\mathbb{R}^d)}^2}{\sqrt{T} (\|K\|_{L^2(\mathbb{R}^d)}^2 + \|\nabla K\|_{L^2(\mathbb{R}^d)}^2)} \right) \delta,
\]
where $\bar{c} > 0$ is some constant and the infimum is taken over all estimators $\hat{\vartheta}$ based on observing $X_{\delta,x_0}$.

**Proof.** The autocovariance function of the stationary process $(\delta^{-1}X_{\delta,x_{0}}(\delta^2t), t \in \mathbb{R})$ is given by

$$c_{\vartheta,\delta}(t) = \delta^{-2}E[X_{\delta,x_{0}}(\delta^2t)X_{\delta,x_0}(0)]$$

$$= \delta^{-2} \int_{-\infty}^{0} (S_{\vartheta}(\delta^2|t| - s)K_{\delta,x_0}, S_{\vartheta}(-s)K_{\delta,x_0})ds$$

$$= \langle (-2A_{\vartheta,\delta,x_{0}})^{-1}S_{\vartheta,\delta,x_{0}}(|t|)K, K \rangle_{L^2(\Lambda_{\delta,x_{0}})}$$

using the scaling in Lemma 3.1 and $\frac{d}{ds}S_{\vartheta,\delta,x_{0}}(s) = A_{\vartheta,\delta,x_{0}}S_{\vartheta,\delta,x_{0}}(s)$ in the last line. The covariance operator for $\delta^{-1}X_{\delta,x_{0}}(\delta^2\bullet)$ on $L^2(\mathbb{R})$ is obtained by convolution:

$$C_{\vartheta,\delta}f(t) = (c_{\vartheta,\delta} * f)(t), \ t \in \mathbb{R}. $$

The squared Hellinger distance $H^2(\vartheta, \vartheta_0)$ between two equivalent centered Gaussian measures can be bounded in terms of the Hilbert-Schmidt norm of the covariance operators, see e.g. the proof of the Feldman-Hajek Theorem in Da Prato and Zabczyk (2014, Theorem 2.25). For the laws of $(\delta^{-1}X_{\delta,x_{0}}(\delta^2\bullet), t \in [0,T\delta^{-2}])$ under $\vartheta_0$ and $\vartheta$ we can thus bound the corresponding Hellinger distance via

$$H^2(\vartheta, \vartheta_0) \leq \|C_{\vartheta,\delta}^{-1}(C_{\vartheta,\delta} - C_{\vartheta_0,\delta})\|^2_{HS(L^2([0,T\delta^{-2}]))}.$$ 

Since the Hellinger distance is invariant under bi-measurable bijective transformations, $H(\vartheta, \vartheta_0)$ denotes equally the Hellinger distance between the observation laws of $(X_{\delta,x_{0}}(t), t \in [0,T])$.

Let now $\vartheta_\delta = \vartheta_0 + c\delta$ for some small $c > 0$ which we choose below, and assume that we can show $H^2(\vartheta_\delta, \vartheta_0) \leq 1$ for sufficiently small $\delta$. Then we obtain from the general lower bound scheme in Tsybakov (2008), using his Theorem 2.2(ii) and (2.9), that

$$\inf_{\hat{\vartheta}} \max_{\vartheta \in \{\vartheta_0, \vartheta_\delta\}} \mathbb{E}_{\vartheta} \left[(\hat{\vartheta} - \vartheta)^2\right] \geq \frac{2-\sqrt{3}}{4}(\vartheta_\delta - \vartheta_0)^2 = \frac{2-\sqrt{3}}{4}c^2. \quad (5.6)$$

From this we will obtain the claimed lower bound.

In order to show $H^2(\vartheta_\delta, \vartheta_0) \leq 1$, denote by $\iota_1 : H^1([0,T\delta^{-2}]) \to L^2([0,T\delta^{-2}])$ the Sobolev embedding operator. It is known from Maurin’s Theorem, see e.g. the proof of Adams and Fournier (2003, Theorem 6.61), that $\iota_1$ is Hilbert-Schmidt with

$$\|\iota_1\|^2_{HS(H^1([0,T\delta^{-2}]),L^2([0,T\delta^{-2}]))} \leq K_{HS}T\delta^{-2}$$
for some constant $K_{HS} > 0$. By Hilbert-Schmidt norm calculus, the implicit restriction of the covariance operators and by Lemma A.4 below we conclude for $\vartheta_0 > \vartheta_0$

$$H^2(\vartheta_0, \vartheta_0) \leq \left\| C_{\vartheta_0, \vartheta} - C_{\vartheta_0, \vartheta} \right\|_{HS(L^2([0,T\delta^{-2}]))}^2 \leq \left\| t_1 \right\|_{HS(H^1([0,T\delta^{-2}]), L^2([0,T\delta^{-2}]))} \left\| C_{\vartheta_0, \vartheta} - C_{\vartheta_0, \vartheta} \right\|_{L^2([0,T\delta^{-2}])}^2 \leq K_{HS} T \delta^{-2} \left\| C_{\vartheta_0, \vartheta} - C_{\vartheta_0, \vartheta} \right\|_{L^2(\mathbb{R})} \leq K_{HS} T \delta^{-2} \left\| \vartheta_0 - \vartheta \right\|_{L^2(\mathbb{R})}^2.$$

Hence, $H^2(\vartheta_0, \vartheta) \leq 1$ holds whenever

$$\vartheta_0 - \vartheta_0 \leq \sqrt{K_{HS} T} \left( 1 + \vartheta_0 \left\| K \right\|_{L^2(\mathbb{R})^d} + \left\| \nabla K \right\|_{L^2(\mathbb{R})^d} \right)^{-1} \delta.$$

Noting the convergence $\left\| (I - \Delta)^{-1} K \right\|_{L^2(\mathbb{R})} \to \left\| (I - \Delta)^{-1} K \right\|_{L^2(\mathbb{R})}$ from Lemma A.4 below, we can thus find a sufficiently small constant $c' > 0$ and choose

$$c = c' \vartheta_0 \sqrt{T} \left( \left\| K \right\|_{L^2(\mathbb{R})^d} + \left\| \nabla K \right\|_{L^2(\mathbb{R})^d} \right)^{-1},$$

such that (5.6) holds for $\vartheta_0 = \vartheta_0 + c\delta$. This yields the result. \hfill \Box

6 A numerical example

In this section we briefly illustrate the main results from above. Let $\Lambda = (0,1)$, $T = 1$, and consider the stochastic heat equation

$$dX(t) = \Delta_{\vartheta} X(t) dt + dW(t)$$

with Dirichlet boundary conditions and with spatially varying diffusivity $\vartheta$ (true diffusivity in Figure 1 (center)). Assume that $X_0$ is zero, except for two equally high “peaks” at $x = 0.3$ and $x = 0.8$. A typical realisation is provided in Figure 1 (left) and we see already qualitatively that the heat diffusion is higher for $x \leq 1/2$.

An approximate solution $\bar{X}(t_k, y_j) \approx X(t_k)(y_j)$ is obtained with respect to a regular time-space grid $\{(t_k, y_j) : t_k = k/N, y_j = j/M, k = 0, \ldots, N, j = 0, \ldots, M\}$ by a semi-implicit Euler scheme and a finite difference approximation of $A_{\vartheta}$ (Lord et al. 2014 Section 10.5). Since the solution is tested against functions $K_{\delta, x_0}$ and $\Delta K_{\delta, x_0}$ with small support, $M$ needs to be relatively large, while it is
Figure 1: (left) typical realisation of $X(t, x)$; (center) true $\vartheta$ compared to $\hat{\vartheta}^A$ and $\hat{\vartheta}^P$ at $\delta = 0.12$ with two different kernels; (right) $\log_{10}$-$\log_{10}$ plot of estimation error at $x_0 = 0.6$ for the estimators in the center.

well-known that accurate simulation requires $J \asymp M^2$, see, e.g. Lord et al. (2014, p. 458). We therefore choose $M = 2000$, $J = 10^6$.

Consider the kernels $K^{(1)} = \partial^3 \varphi$, $K^{(2)} = \partial \varphi$ with a smooth bump function

$$\varphi(x) = \exp(-\frac{12}{1-x^2}), \quad x \in (-1, 1).$$

For $\delta \in \{0.03, 0.05, 0.08, 0.12, 0.2, 0.3\}$ and $x_0 \in (0, 1)$ on a regular grid we obtain approximate local measurements $X_{\delta,x_0}, X^{\Delta}_{\delta,x_0}$ for $K^{(1)}$ and $K^{(2)}$, respectively, from which the augmented MLE $\hat{\vartheta}^A(x_0)$ and the proxy MLE $\hat{\vartheta}^P(x_0)$ are computed. For $x_0$ near the boundary and $i = 1, 2$ set

$$K^{(i)}_{\delta,x_0} = \begin{cases} 
K^{(i)}_{\delta,\delta}, & x_0 < \delta, \\
K^{(i)}_{\delta,1-\delta}, & x_0 > 1 - \delta.
\end{cases}$$

Figure 1 (center) shows pointwise estimation results for $\vartheta(x_0)$ at $\delta = 0.12$ and for different $x_0$, while Figure 1 (right) presents $\log_{10}$-$\log_{10}$ plots with convergence results at $x_0 = 0.6$ for $\delta \to 0$, obtained by 5,000 Monte-Carlo runs.

Already at the relatively large resolution $\delta = 0.12$ both $\hat{\vartheta}^A(x_0)$ and $\hat{\vartheta}^P(x_0)$ perform surprisingly well. For $K^{(1)}$ both estimators are close together and achieve after a burn-in phase the convergence rate $\delta$, as predicted by Theorems 5.1 and 5.4. Note that $K^{(1)} = \Delta K$ for $K = \partial \varphi$ and $\int K(x)dx = 0$ such that the assumptions of Theorem 5.4 are satisfied. With respect to $K^{(2)}$ those assumptions are not met and indeed $\hat{\vartheta}^P(x_0)$ deviates considerably from $\vartheta(x_0)$, but still seems to be consistent with rate of convergence dropping to about $\delta^{3/4}$. Estimation by $\hat{\vartheta}^A(x_0)$ is unaffected (not shown).
A Proofs

For a better understanding we structure the appendix such that the proofs for the main theorems of Section 4 are given in Section A.1. Only afterwards, we provide the technical tools used for the main proofs. Section A.2 contains analytical results for rescaled semigroups and heat kernels, while Section A.3 assembles precise asymptotics for variance and covariance expressions.

From now on, we set without loss of generality $x_0 = 0$ and replace $\Lambda$ by $\Lambda - x_0$. In particular, we estimate $\vartheta(0)$ and ease notation by removing the subindex $x_0$, for example $\Lambda_\delta = \Lambda_{\delta,x_0}$, $z_\delta = z_{\delta,x_0}$ and $X_\delta = X_{\delta,x_0}$. $C$ always denotes a generic positive constant, which may change from line to line. $A \lesssim B$ means $A \leq CB$, where $C$ may depend on $T$, if not made explicit otherwise. Define $\|\cdot\|_{L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)} := \|\cdot\|_{L^1(\mathbb{R}^d)} + \|\cdot\|_{L^2(\mathbb{R}^d)}$ on $L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$. We will use frequently without explicitly mentioning that $\Delta K_\delta = \delta^{-2}(\Delta K)_\delta$ by Lemma 3.1.

A.1 Proofs for Section 4

Proposition A.1. Grant Assumption 3.2 and 3.4(K;2). Then $\delta^2 \mathbb{E}[\mathcal{I}_\delta] \to T \vartheta(0)^{-1} \Psi(\Delta K, \Delta K)$ for $\delta \to 0$ with $\Psi$ from (3.3).

Proof. In view of $\mathcal{I}_\delta^A = \int_0^T X_\delta^A(t)^2 dt$ we decompose $\mathbb{E}[X_\delta^A(t)^2] = \mathbb{E}[X_\delta^A(t)]^2 + \text{Var}(X_\delta^A(t))$. Assumption 3.4(K;2) gives

$$\delta^2 \int_0^T \mathbb{E}[X_\delta^A(t)]^2 dt = \delta^2 \int_0^T (S_\vartheta(t) X_0, \delta^{-2}(\Delta K)_\delta)^2 dt = o(1). \quad (A.1)$$

Proposition A.12(ii) below with $w(\delta) = \Delta K$ yields $\delta^2 \int_0^T \text{Var}(X_\delta^A(t)) dt \to T \vartheta(0)^{-1} \Psi(\Delta K, \Delta K)$ and the result follows.

Proposition A.2. Grant Assumptions 3.2 and 3.4(K;2). Then $\mathcal{I}_\delta^A / \mathbb{E}[\mathcal{I}_\delta^A] \overset{p}{\to} 1$ for $\delta \to 0$.

Proof. We show the claim first when $X_0 = 0$. It suffices to show $\text{Var}(\mathcal{I}_\delta^A) / \mathbb{E}[\mathcal{I}_\delta^A]^2 \to 0$. By Proposition A.19(ii) below with $z = K$, $w(\delta) = \Delta K$ we have $\text{Var}(\mathcal{I}_\delta^A) = o(\delta^{-1})$ and the claim follows from Proposition A.1.

For the general case set $\tilde{\mathcal{I}}_\delta^A = \int_0^T \tilde{X}_\delta^A(t)^2 dt$, where $\tilde{X}_\delta^A$ is defined as $X_\delta^A(t)$, but with $X_0 = 0$. By Proposition A.1 $\mathbb{E}[\tilde{\mathcal{I}}_\delta^A] / \mathbb{E}[\mathcal{I}_\delta^A] \to 1$, and so it is enough to show $\mathcal{I}_\delta^A / \tilde{\mathcal{I}}_\delta^A \overset{p}{\to} 1$, which follows by the first part if $\mathbb{E}[(\mathcal{I}_\delta^A - \tilde{\mathcal{I}}_\delta^A)^2] / \mathbb{E}[\tilde{\mathcal{I}}_\delta^A]^2 \to 0$. 

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Applying again Proposition A.1, the result is thus obtained by Equation (A.1) and
\[ \mathbb{E} \left[ \delta^4 (I_{\delta}^A - \widehat{I}_{\delta}^A)^2 \right] \leq \mathbb{E} \left[ \left( \delta^2 \int_0^T X_{\delta}^A(t) \mathbb{E}[X_{\delta}^A(t)]^2 dt \right)^2 + \left( \delta^2 \int_0^T \mathbb{E}[X_{\delta}^A(t)]^2 dt \right)^2 \right] \]
\[ \leq \mathbb{E} \left[ \delta^2 \widehat{I}_{\delta}^A \right] \left( \delta^2 \int_0^T \mathbb{E}[X_{\delta}^A(t)]^2 dt \right) + \left( \delta^2 \int_0^T \mathbb{E}[X_{\delta}^A(t)]^2 dt \right)^2 = o(1). \]

\[ \square \]

**Proposition A.3.** Grant Assumptions 3.2 and 3.4(K,2), and let \( d \geq 2 \) or \( \int_{\mathbb{R}^d} K(x)dx = 0 \). Then \( \delta^{-1}(I_{\delta}^A)^{-1} R_{\delta}^A \xrightarrow{\mathbb{P}} \mu^A, \delta \to 0, \) with \( \mu^A \) from (5.1).

**Proof.** In terms of \( \beta^{(\delta)} := \delta^{-1}(A_{\delta}^\ast - \vartheta(0) \Delta)K \) we have \( \delta R_{\delta}^A = \int_0^T X_{\delta}^A(t) \langle X(t), \beta^{(\delta)} \rangle dt \). \( \beta^{(\delta)} \) corresponds to \( v^{(\delta)} \) in Lemma A.11 below with \( z = K \) and therefore \( \beta^{(\delta)} \to \beta \) in \( L^2(\mathbb{R}^d) \) with \( \beta \) from (5.2).

We prove the claim first when \( X_0 = 0 \). By Propositions A.1 and A.2 it suffices to show
\[ \mathbb{E} \left[ \delta R_{\delta}^A \right] \to T \vartheta(0)^{-1} \Psi(\Delta K, \beta), \ Var(\delta R_{\delta}^A) \to 0. \]

The convergence of the expectation holds for \( d \geq 2 \) by Proposition A.12 iii) below with \( w^{(\delta)} = \Delta K, z = K, u^{(\delta)} = \beta^{(\delta)}, u = \beta \). The property \( \int_{\mathbb{R}^d} K(x)dx = 0 \), on the other hand, ensures by Lemma A.11 ii) that there is a compactly supported \( \tilde{\beta} \in H^2(\mathbb{R}^d) \) with \( \beta = \Delta \tilde{\beta} \). Then, by polarisation and Proposition A.12 ii), \( \mathbb{E}[\delta R_{\delta}^A] \) converges to
\[ \frac{T}{4\vartheta(0)} \left( \Psi(\Delta(K + \tilde{\beta}), \Delta(K + \tilde{\beta})) - \Psi(\Delta(K - \tilde{\beta}), \Delta(K - \tilde{\beta})) \right) \]
\[ = \frac{T}{\vartheta(0)} \Psi(\Delta(K, \beta)). \]

We conclude \( \delta \mathbb{E}[R_{\delta}^A] \to T \vartheta(0)^{-1} \Psi(\Delta K, \beta) \). Moreover, \( \Var(\delta R_{\delta}^A) = \Var(\int_0^T X_{\delta}^A(t) \langle X(t), \beta^{(\delta)} \rangle dt) \to 0 \) follows for \( d \geq 2 \) by Proposition A.19 i) below with \( z = K, u^{(\delta)} = \beta^{(\delta)}, u = \beta \). When \( \int_{\mathbb{R}^d} K(x)dx = 0 \), on the other hand, then \( \Var(\delta R_{\delta}^A) \to 0 \) by Proposition A.19 ii) with \( z = K, w^{(\delta)} = \beta^{(\delta)}, m = \tilde{\beta} \).

At last, for general \( X_0 \), write \( \langle X(t), z \rangle = \langle \widehat{X}(t), z \rangle + (S_\vartheta(t)X_0, z), 0 \leq t \leq T, z \in L^2(\Lambda), \) where \( \langle \widehat{X}(t), z \rangle \) is defined as \( \langle X(t), z \rangle \), but with \( X_0 = 0 \). Define correspondingly \( \widehat{R}_{\delta}^A \) as \( R_{\delta}^A \). By the first part, Proposition A.1 and \( I_{\delta}^A / \widehat{I}_{\delta}^A \xrightarrow{\mathbb{P}} 1 \) from the proof of Proposition A.2 it is enough to show \( \delta(R_{\delta}^A - \widehat{R}_{\delta}^A) \xrightarrow{\mathbb{P}} 0 \). Decompose
\[ \delta(R^A_\delta - \tilde{R}^A_\delta) = V_1 + V_2 + V_3 \] with

\[ V_1 = \int_0^T \langle S_\phi(t)X_0, \Delta z_\delta \rangle \langle S_\phi(t)X_0, \beta^{(\delta)}_\delta \rangle dt, \]

\[ V_2 = \int_0^T \tilde{X}^A_\delta(t) \langle S_\phi(t)X_0, \beta^{(\delta)}_\delta \rangle dt, \]

\[ V_3 = \int_0^T \langle S_\phi(t)X_0, \Delta z_\delta \rangle \langle \tilde{X}(t), \beta^{(\delta)}_\delta \rangle dt. \]

By the Cauchy-Schwarz inequality we have the upper bounds

\[ |V_1| \leq \delta^{-2} \left( \int_0^T \langle S_\phi(t)X_0, (\Delta K)^{\delta} \rangle^2 dt \right)^{1/2} \left( \int_0^T \langle S_\phi(t)X_0, \beta^{(\delta)}_\delta \rangle^2 dt \right)^{1/2}, \]

\[ \mathbb{E}[V_2^2] = \text{Var}(V_2) = \int_0^T \int_0^T \text{Cov}(\tilde{X}^A_\delta(t), \tilde{X}^A_\delta(s)) \langle S_\phi(t)X_0, \beta^{(\delta)}_\delta \rangle \langle S_\phi(s)X_0, \beta^{(\delta)}_\delta \rangle dt ds \]

\[ \lesssim \left( \int_0^T \int_0^T \text{Cov}(\tilde{X}^A_\delta(t), \tilde{X}^A_\delta(s))^2 dt ds \right)^{1/2} \int_0^T \langle S_\phi(t)X_0, \beta^{(\delta)}_\delta \rangle^2 dt \]

\[ = \frac{1}{\sqrt{2}} \text{Var} \left( \int_0^T \tilde{X}^A_\delta(t)^2 dt \right)^{1/2} \int_0^T \langle S_\phi(t)X_0, \beta^{(\delta)}_\delta \rangle^2 dt, \tag{A.2} \]

where the last line holds because of Equation (A.24) in the proof of Proposition A.19 below. If \( d \geq 2 \), then Assumption 3.4(K;2) and Lemma A.10(i) below with respect to \( u = \beta^{(\delta)} \) give

\[ V_1 = \delta^{-2} \ell_{\delta,2}(\delta) - 1/2 \delta \ell_{d,2}(\delta) 1/2 \delta^{1/2} = o(1), \]

and together with the variance bound in Proposition A.19(iv) with \( w^{(\delta)} = \Delta K, m = K, \) also \( V_2^2 = O_p(\delta^{-1} \int_0^T \langle S_\phi(t)X_0, \beta^{(\delta)}_\delta \rangle^2 dt) = o_p(1) \). Like for \( V_2 \), but this time using Proposition A.19(iii), we obtain

\[ \mathbb{E}[V_3^2] \lesssim \text{Var} \left( \int_0^T \langle \tilde{X}(t), \beta^{(\delta)}_\delta \rangle^2 dt \right)^{1/2} \delta^{-4} \int_0^T \langle S_\phi(t)X_0, (\Delta K)_\delta \rangle^2 dt \tag{A.3} \]

\[ = \delta^{-2} \ell_{d,2}(\delta) \int_0^T \langle S_\phi(t)X_0, (\Delta K)_\delta \rangle^2 dt = o(1). \]

At last, when \( \int_{x}^{\delta x} K(x) dx = 0 \), then \( \beta = \Delta \bar{\beta} \) as above and so Assumption 3.4(K;2) and Lemma A.10(ii) with \( p = 2 \) show \( V_1 = o(1) \), while the upper bound for \( V_2 \) in (A.2) together with Lemma A.10(ii) gives \( V_2 = o_p(1) \). For \( V_3 \) we find \( V_3 = o_p(1) \) from applying Proposition A.19(iv) in (A.3) with \( w^{(\delta)} = \beta^{(\delta)}, m = K \). \( \square \)
Lemma A.4. In the setting of Proposition 5.5, we have for \( \vartheta > \vartheta_0 > 0 \)
\[
\| C_{\vartheta_0, \delta}^{-1}(C_{\vartheta, \delta} - C_{\vartheta_0, \delta}) \|_{L^2(\mathbb{R}) \rightarrow H^1(\mathbb{R})} \leq (\vartheta^2 - \vartheta_0^2) \left( \vartheta_0^{-2} + \vartheta_0^{-1} \frac{\|K\|^2_{L^2(\mathbb{R}^d)} + \|\nabla K\|^2_{L^2(\mathbb{R}^d)}}{\|(I - \Delta)^{-1} K\|^2_{L^2(\mathbb{R}^d)}} \right).
\]

Moreover, we have \( \|(I - \Delta)^{-1} K\|^2_{L^2(\mathbb{R}^d)} \rightarrow \|(I - \Delta)^{-1} K\|^2_{L^2(\mathbb{R}^d)} \) for \( \delta \rightarrow 0 \).

Proof. In the Fourier domain, the convolution operator \( C_{\vartheta, \delta} f \) is given by multiplication of \( \mathcal{F} f(\omega) \) with
\[
\mathcal{F} C_{\vartheta, \delta} f(\omega) = \int_0^\infty \langle (-2A_{\vartheta, \delta})^{-1} S_{\vartheta, \delta}(t)(e^{i\omega t} + e^{-i\omega t}) K, K \rangle_{L^2(\mathbb{R}^d)} dt.
\]
using the functional calculus of the selfadjoint operator \( A_{\vartheta, \delta} \) in the last line. For simplicity write now \( \vartheta \Delta \) instead of \( A_{\vartheta, \delta} \). The operator \( C_{\vartheta_0, \delta}^{-1}(C_{\vartheta, \delta} - C_{\vartheta_0, \delta}) \) is expressed in the Fourier domain by multiplication with
\[
\frac{\mathcal{F} C_{\vartheta, \delta}(\omega) - \mathcal{F} C_{\vartheta_0, \delta}(\omega)}{\mathcal{F} C_{\vartheta_0, \delta}(\omega)} = (\vartheta^2 - \vartheta_0^2) \frac{\langle (\vartheta^2 \Delta^2 + \omega^2)^{-1} \Delta^2 (\vartheta_0^2 \Delta^2 + \omega^2)^{-1} K, K \rangle_{L^2(\mathbb{R}^d)}}{\langle (\vartheta_0^2 \Delta^2 + \omega^2)^{-1} K, K \rangle_{L^2(\mathbb{R}^d)}}.
\]
Via the Plancherel isometry we can therefore bound for \( \vartheta > \vartheta_0 \), using the description of \( H^1(\mathbb{R}) \) in the Fourier domain and functional calculus for the Laplacian \( \Delta \) on \( L^2(\mathbb{R}^d) \):
\[
\| C_{\vartheta_0, \delta}^{-1}(C_{\vartheta, \delta} - C_{\vartheta_0, \delta}) \|_{L^2(\mathbb{R}) \rightarrow H^1(\mathbb{R})} \leq \sup_{\omega \in \mathbb{R}} \left| (1 + \omega^2)^{1/2} \mathcal{F} C_{\vartheta, \delta}(\omega) - \mathcal{F} C_{\vartheta_0, \delta}(\omega) \right| \mathcal{F} C_{\vartheta_0, \delta}(\omega)
\leq (\vartheta^2 - \vartheta_0^2) \sup_{\omega \in \mathbb{R}} \left| (1 + (\vartheta_0 \omega)^2)^{1/2} \frac{\Delta^2 (\vartheta_0^2 \Delta^2 + \omega^2)^{-1} K, K}_{L^2(\mathbb{R}^d)} \right|
\leq \frac{\vartheta^2 - \vartheta_0^2}{\vartheta_0^2} \left( 1 + \vartheta_0 \left( 1 \vee \sup_{\omega > 1} \frac{\|\omega^{-1/2} \Delta (\omega^2 \Delta^2 + I)^{-1/2} K\|^2_{L^2(\mathbb{R}^d)}}{\|(\omega^2 \Delta^2 + I)^{-1/2} K\|^2_{L^2(\mathbb{R}^d)}} \right) \right)
\leq \frac{\vartheta^2 - \vartheta_0^2}{\vartheta_0^2} \left( 1 + \vartheta_0 \left( 1 \vee \frac{\|\omega^{-1/2} \Delta (\omega^2 \Delta^2 + I)^{-1/2} K\|^2_{L^2(\mathbb{R}^d)}}{\|(I + \Delta^2)^{-1/2} K\|^2_{L^2(\mathbb{R}^d)}} \right) \right),
\]
where we used in the last line \( \omega^{-1/2} \lambda (1 + \omega^{-2} \lambda^2)^{-1} \leq \lambda^{1/2} \) for all \( \lambda, \omega > 0 \). Since \( (-\Delta K, K)_{L^2(\mathbb{R}^d)} = \|\nabla K\|^2_{L^2(\mathbb{R}^d)} \), the numerator is independent of \( \delta \). For the
denominator write again $A_{1,\delta} = \Delta$ and note $(I + A_{1,\delta}^2)^{-1/2} \geq (I - A_{1,\delta})^{-1}$, where we have explicitly, cf. [Pazy (1983) Chapter 2.6],

$$(I - A_{1,\delta})^{-1} K = \int_0^\infty e^{-t} S_{1,\delta}(t) K \, dt.$$  

The semigroup bound in Proposition A.7(i) below yields $\|S_{1,\delta}(t) K\|_{L^2(\Lambda_\delta)} \lesssim \|K\|_{L^2(\mathbb{R}^d)}$ uniformly in $\delta$ and therefore by the semigroup convergence in Proposition A.8(ii)

$$\| (I - A_{1,\delta})^{-1} K \|_{L^2(\Lambda_\delta)} \to \| (I - \Delta)^{-1} K \|_{L^2(\mathbb{R}^d)}, \ \delta \to 0,$$

where $\Delta$ is now the Laplacian on $L^2(\mathbb{R}^d)$. This proves the result. $\square$

A.2 Analytical results

Proof of Proposition 2.1. The proof follows from modifying Theorems 5.2 and 5.4 of Da Prato and Zabczyk (2014). Consider $l(t, z)$ as in (2.4). Linearity and continuity of $l(t, \cdot)$ are clear because they hold both for $\langle S_\vartheta(t) X_0, \cdot \rangle$ and the stochastic integral. The stochastic integral with respect to $(W(t))_{t \geq 0}$ is a centered Gaussian process and so $(l(t, z))_{0 \leq t \leq T, z \in L^2(\Lambda)}$ is a Gaussian process with the claimed mean function. The form of the covariance function follows from Itô’s isometry (Da Prato and Zabczyk (2014, Proposition 4.28)).

To see that $l(t, z)$ satisfies (2.2), note by the stochastic Fubini Theorem (Da Prato and Zabczyk (2014, Theorem 4.33)) that $\int_0^t l(s, A_\vartheta^* z) \, ds$ for $z \in H^1_0(\Lambda) \cap H^2(\Lambda)$ equals

$$\int_0^t \langle S_\vartheta(s) X_0, A_\vartheta^* z \rangle \, ds + \int_0^t \int_0^s \langle S_\vartheta(s-s') A_\vartheta^* z, B W(s') \rangle \, ds$$

$$= \langle X_0, \int_0^t S_\vartheta(s) A_\vartheta^* z \, ds \rangle + \int_0^t \left( \int_s^t S_\vartheta(s-s') A_\vartheta^* z \, ds, B W(s') \right).$$

Since $\frac{d}{ds} S_\vartheta(s) = S_\vartheta(s) A_\vartheta^*$, this is equal to

$$\langle X_0, (S_\vartheta(t - I) z) + \int_0^t (S_\vartheta(t-s') z - z, B W(s')) \rangle$$

$$= l(t, z) - \langle X_0, z \rangle - \langle B W(t), z \rangle,$$

implying (2.2). $\square$

Proof of Lemma 3.1 (i). $\bullet z \in H^1_0(\Lambda_\delta) \cap H^2(\Lambda_\delta)$ means $z_\delta \in H^1_0(\Lambda) \cap H^2(\Lambda)$ and so both $A_\vartheta^* z_\delta$ and $A_\vartheta^* z_\delta$ are well-defined. For $z \in C^\infty_c(\Lambda_\delta)$, it follows $A_\vartheta^* z_\delta =$
\(\delta^{-2}(A^*_{\vartheta,\delta}z)\). Since \(C^\infty_c(\overline{\Lambda})\) is a core for the domain of \(A^*_\vartheta\), this extends to \(z \in H^1_0(\Lambda) \cap H^2(\Lambda)\).

(ii). It is enough to prove the result for \(z \in H^1_0(\Lambda) \cap H^2(\Lambda)\). Set \(w(t) = (S^*_{\vartheta,\delta}(t\delta^{-2})z)_{\delta} \in L^2(\Lambda)\). \((S^*_{\vartheta,\delta}(t))_{t \geq 0}\) is an analytic semigroup, implying \(S^*_{\vartheta,\delta}(t)z \in \mathcal{D}(A^*_{\vartheta,\delta}) = H^1_0(\Lambda) \cap H^2(\Lambda)\) and thus

\[
\frac{d}{dt} w(t) = \delta^{-2}(A^*_{\vartheta,\delta}S^*_{\vartheta,\delta}(t\delta^{-2})z)_{\delta} = A^*_\vartheta w(t)
\]

by (i). Since \(w(0) = z_{\delta}\) and because \(u(t) = S^*_{\vartheta}(t)z_{\delta}\) is the unique solution in \(C([0, \infty); H^1_0(\Lambda) \cap H^2(\Lambda)) \cap C^1([0, \infty); L^2(\Lambda))\) of

\[
u'(t) = A^*_\vartheta u(t), \quad t \geq 0, \\
u(0) = z_{\delta},
\]

we conclude that \(w(t) = S^*_{\vartheta}(t)z_{\delta}\).

Recall that the fundamental solution of the heat equation \(\frac{d}{dt} u(t) = \alpha \Delta u(t)\), \(\alpha > 0\), on \(\mathbb{R}^d\) with initial value \(w \in L^2(\mathbb{R}^d)\) is given by

\[
u(t) = e^{\alpha t \Delta} w = q_{\alpha t} * w \tag{A.4}
\]

with heat kernel \(q_t(x) = (4\pi t)^{-d/2} \exp(-|x|^2/(4t))\), \(x \in \mathbb{R}^d\).

**Lemma A.5.** For \(u \in L^2(\mathbb{R}^d)\) we have for \(t > 0\):

(i) \(\|e^{t \Delta} u\|_{L^2(\mathbb{R}^d)} \lesssim (1 + t^{-d/4})\|u\|_{L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)}\), \(\|\Delta e^{t \Delta} u\|_{L^2(\mathbb{R}^d)} \lesssim t^{-1}\|u\|_{L^2(\mathbb{R}^d)}\).

(ii) \(xe^{\vartheta(0) t} \Delta u(x) = -\vartheta(0)t \nabla e^{\vartheta(0) t} \Delta u(x) + e^{\vartheta(0) t} \Delta (xu)\).

(iii) \(\|x|^{2}\vartheta(0) t \Delta u\|_{L^2(\mathbb{R}^d)} \lesssim (1 + t)(1 + t^{-d/4})(1 + |x| + |x|^2)\|u\|_{L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)}\).

**Proof.** (i). For the second part use functional calculus. The first part follows from

\[
\|e^{t \Delta} u\|_{L^2(\mathbb{R}^d)} = \|q_t * u\|_{L^2(\mathbb{R}^d)} \lesssim (1 + t^{-d/4})\|u\|_{L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)}.
\]

(ii). Let \(i \in \{1, \ldots, d\}\). The result follows from

\[
x_i \left(e^{\vartheta(0) t} \Delta u\right)(x) = x_i q_{\vartheta(0) t} * u(x)
\]

\[
= \vartheta(0) t \int_{\mathbb{R}^d} \frac{x_i - y_i}{\vartheta(0) t} q_{\vartheta(0) t}(x - y) u(y) dy + \int_{\mathbb{R}^d} y_i q_{\vartheta(0) t}(x - y) u(y) dy
\]

\[
= -\vartheta(0) t (\partial_i q_{\vartheta(0) t} * u)(x) + q_{\vartheta(0) t} * (x_i u)(x).
\]

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(iii). Applying the proof in (ii) twice for \( i \in \{1, \ldots, d\} \) yields

\[
x_i^2 \left( e^{\theta(0)t} \Delta u \right)(x) = -x_i \partial_i (\partial_i q_{\theta(0) \xi} * u)(x) + x_i q_{\theta(0) \xi} * (x_i u)(x)
= \theta^2(0) t^2 (\partial_i^2 q_{\theta(0) \xi} * u)(x) - 2 \theta(0) t (\partial_i q_{\theta(0) \xi} * (x_i u))(x) + q_{\theta(0) \xi} * (x_i^2 u)(x).
\]

Summing over \( i \) gives for \( \|x^2 e^{\theta(0)t} \Delta u\|_{L^2(\mathbb{R}^d)} \) up to a constant the upper bound

\[
\sum_{i=1}^d (i^2 \|\Delta e^{\theta(0)t} \Delta u\|_{L^2(\mathbb{R}^d)} + t \|\partial_i e^{\theta(0)t} \Delta (x_i u)\|_{L^2(\mathbb{R}^d)} + \|e^{\theta(0)t} \Delta (x_i^2 u)\|_{L^2(\mathbb{R}^d)}).
\]

By \( e^{\theta(0)t} \Delta = e^{\theta(0)t/2} \Delta e^{\theta(0)t/2} \Delta \) and (i) we have \( \|\Delta e^{\theta(0)t} u\|_{L^2(\mathbb{R}^d)} \lesssim t^{-1} (1 + t^{-d/4}) \|u\|_{L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)} \). With \( v_i = e^{\theta(0)t} \Delta (x_i u) \) the Cauchy-Schwarz inequality and (i) also show

\[
\|\partial_i v_i\|_{L^2(\mathbb{R}^d)} \leq \left( \sum_{j=1}^d \|\partial_j v_i\|_{L^2(\mathbb{R}^d)}^2 \right)^{1/2} = \|\Delta v_i\|_{L^2(\mathbb{R}^d)}^{1/2} \|v_i\|_{L^2(\mathbb{R}^d)}^{1/2} \leq t^{-1/2} (1 + t^{-d/4}) \|x\| \|u\|_{L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)}.
\]

From this and \( 1 + t^{1/2} + t \lesssim 1 \) the result follows. \( \square \)

**Lemma A.6.** Let \( i \in \{1, \ldots, d\} \). If \( z \in H^2(\mathbb{R}^d) \) has compact support, then

\[
\langle z, \partial_i z \rangle_{L^2(\mathbb{R}^d)} = 0, \quad \langle x_i \Delta z, z \rangle_{L^2(\mathbb{R}^d)} = -\langle x_i, |\nabla z|^2 \rangle_{L^2(\mathbb{R}^d)}.
\]

If \( z \in H^4(\mathbb{R}^d) \), then also \( \langle \Delta z, e^{t \Delta} \partial_i z \rangle_{L^2(\mathbb{R}^d)} = 0, t \geq 0 \).

**Proof.** By partial integration \( \langle z, \partial_i z \rangle_{L^2(\mathbb{R}^d)} = -\langle \partial_i z, z \rangle_{L^2(\mathbb{R}^d)} \), implying \( \langle z, \partial_i z \rangle_{L^2(\mathbb{R}^d)} = 0 \) and \( \langle x_j \partial_j z, z \rangle_{L^2(\mathbb{R}^d)} = \langle x_i, (\partial_j z)^2 \rangle_{L^2(\mathbb{R}^d)} \) for \( j = 1, \ldots, d \). The last part follows similar to the first one, because by partial integration

\[
\langle e^{t \Delta} \partial_i z, \partial_j z \rangle_{L^2(\mathbb{R}^d)} = \frac{d^2}{dt dt} \langle e^{t \Delta} z, \partial_i z \rangle_{L^2(\mathbb{R}^d)} = -\frac{d^2}{dt dt} \langle e^{t \Delta} z, \partial_j z \rangle_{L^2(\mathbb{R}^d)} = 0.
\]

The upper bounds in next Proposition are well-known for analytic semigroups. The main difficulty is to ensure that they hold for growing domains, uniformly in \( \delta > 0 \).

**Proposition A.7.** There exist universal constants \( M_0, M_1 \) such that for \( \delta, t > 0 \)

1. \( \|S^*_{\delta,t}(t)\|_{L^2(\Lambda_\delta)} \leq M_0 e^{C\delta^2 t} \),
2. \( \|t A^*_{\delta,t} S_{\delta,t}(t)\|_{L^2(\Lambda_\delta)} \leq M_1 e^{C\delta^2 t} \).
Proof. The claimed bounds in the statement follow from Proposition 2.1.1 of [Lunardi (1995)], if we can show
\[ \| (\lambda - A^*_0)_{\delta} \|_{L^2(\Lambda_\delta)} \leq \frac{M}{|\lambda - \omega|}, \] (A.5)
with \( \omega = c_1 \delta^2 \) for all \( \lambda \in \Sigma_{\sigma, \omega} := \{ \rho \in \mathbb{C} : |\arg(\rho - \omega)| < \sigma \}\backslash\{\omega\} \) and with constants \( c_1, M > 0, \sigma \in (\pi/2, \pi) \) independent of \( \delta \). Since the selfadjoint operator \( \Delta_{\partial(\delta \cdot)} \) has strictly negative spectrum for all \( \delta > 0 \) (cf. [Evans (2010, Section 6.5)]), by functional calculus (A.5) holds indeed for \( \Delta_{\partial(\delta \cdot)} \) with \( \omega = 0, M = 1 \) and any \( \sigma \in (\pi/2, \pi) \).

In order to extend this to \( A^*_\delta \), we consider it as a perturbation of \( \Delta_{\partial(\delta \cdot)} \). We show first that \( A^*_\delta \) is \( \Delta_{\partial(\delta \cdot)} \)-bounded, i.e.
\[ \| A^*_\delta v \|_{L^2(\Lambda_\delta)} \leq c_2 \| \Delta_{\partial(\delta \cdot)} v \|_{L^2(\Lambda_\delta)} + \left( \frac{1}{4\varepsilon} + c_3 \right) \delta^2 \| v \|_{L^2(\Lambda_\delta)} \] (A.6)
for \( \varepsilon > 0, v \in H_0^1(\Lambda_\delta) \cap H^2(\Lambda_\delta) \) and absolute constants \( c_2, c_3 > 0 \). For this note
\[ \| A^*_\delta v \|_{L^2(\Lambda_\delta)} \leq \| \delta \langle a(\delta \cdot), \nabla v \rangle \|_{L^2(\Lambda_\delta)} + \delta^2 \left( \| \nabla \text{div} (a(\delta \cdot)) \|_{L^2(\Lambda_\delta)} + \| b \|_{\infty} \| v \|_{L^2(\Lambda_\delta)} \right). \]

Moreover, \( \| \delta \langle a(\delta \cdot), \nabla v \rangle \|_{L^2(\Lambda_\delta)} \) is upper bounded by
\[
\delta d^{1/2} \sup_{i=1,\ldots,d} \| a_i \|_{\infty} \left( \sum_{i=1}^d \| \partial_i v \|_{L^2(\Lambda_\delta)}^2 \right)^{1/2} \\
\leq \delta \frac{d^{1/2} \sup_{i=1,\ldots,d} \| a_i \|_{\infty}}{\min_x \bar{\delta}(x)^{1/2}} \langle -\Delta_{\partial(\delta \cdot)} v, v \rangle_{L^2(\Lambda_\delta)}^{1/2} \\
\leq \frac{d^{1/2} \sup_{i=1,\ldots,d} \| a_i \|_{\infty}}{\min_x \bar{\delta}(x)^{1/2}} \| \Delta_{\partial(\delta \cdot)} v \|_{L^2(\Lambda_\delta)}^{1/2} \| v \|_{L^2(\Lambda_\delta)} \| v \|_{L^2(\Lambda_\delta)}^{1/2} \\
\leq c_2 \varepsilon \| \Delta_{\partial(\delta \cdot)} v \|_{L^2(\Lambda_\delta)} + \frac{\delta^2}{4\varepsilon} \| v \|_{L^2(\Lambda_\delta)},
\] (A.7)
with \( c_2 := \frac{d \sup_{i=1,\ldots,d} \| a_i \|_{\infty}}{\min_x \bar{\delta}(x)} \), where we used in the last line the basic inequality \( xy \leq \varepsilon x^2 + \frac{1}{4\varepsilon} y^2 \) for \( x, y > 0 \). This shows (A.6) with \( c_3 := \sum_{i=1}^d \| \partial_i a_i \|_{\infty} + \| b \|_{\infty} \).

Choosing \( \varepsilon \) sufficiently small, the proof of Lemma III.2.6 in [Engel and Nagel (2000)] implies (A.5) for all \( \lambda \in \Sigma_{\sigma, 0} \cap \{ \rho \in \mathbb{C} : |\rho| > c_4 \delta^2 \} \) with \( c_4 = (\frac{4\varepsilon}{1 - 2c_2})^{-1} + c_3 \), \( \sigma = 3\pi/4 \) and \( M' > 0 \) instead of \( M \). Setting \( \omega = (1 + c_5) c_4 \delta^2 \), for a suitable constant \( c_5 > 0 \) to be determined later, and assuming that for these \( \lambda \)
\[ \lambda + \omega \in \Sigma_{\sigma, 0} \cap \{ \rho \in \mathbb{C} : |\rho| > c_4 \delta^2 \}, \quad |\lambda + \omega| \geq C|\lambda|, \] (A.8)
with a universal constant $C$, we can therefore conclude for any $\lambda \in \Sigma_{\sigma,0} \cap \{ \rho \in \mathbb{C} : |\rho| > c_4\delta^2 \}$ that

$$
\|((\lambda + \omega) - A^*_{\sigma,\delta})^{-1}\|_{L^2(\Lambda_\delta)} \leq \frac{M'}{|\lambda + \omega|} \leq \frac{M'C}{|\lambda|},
$$

(A.9)

In order to obtain (A.5) from this let $\lambda \in \Sigma_{\sigma,\omega}$ such that $\lambda - \omega \in \Sigma_{\sigma,0}$. Assume that we can also show

$$
|\lambda - \omega| > c_4\delta^2.
$$

(A.10)

Then the result follows from (A.9) with $c_1 = (1 + c_5)c_4$, $M = M'C$, because

$$
\|((\lambda - A^*_{\sigma,\delta})^{-1}\|_{L^2(\Lambda_\delta)} = \|((\lambda - \omega) + \omega) - A^*_{\sigma,\delta})^{-1}\|_{L^2(\Lambda_\delta)} \leq \frac{M'C}{|\lambda - \omega|}.
$$

We are left with showing (A.8) and (A.10). For (A.8) note that $\lambda \in \Sigma_{\sigma,0}$ already yields $\lambda + \omega \in \Sigma_{\sigma,0}$, because $\omega > 0$, while the inequality $|\lambda + \omega| > c_4\delta^2$ holds clearly, if $|\text{Im}(\lambda)| > c_4\delta^2$. On the other hand, $|\arg(\lambda)| < \sigma$ implies $|\text{Re}(\lambda)| < c_5|\text{Im}(\lambda)|$ for a constant $c_5 > 0$ and thus, if $|\text{Im}(\lambda)| \leq c_4\delta^2$, then

$$
|\lambda + \omega| \geq \omega - |\text{Re}(\lambda)| \geq \omega - c_5|\text{Im}(\lambda)| > c_4\delta^2.
$$

(A.11)

In order to find the constant $C$ in (A.8), note that $|\lambda + \omega| \geq |\lambda|$ holds always if $\text{Re}(\lambda) \geq 0$, and that $|\lambda + \omega| \geq \frac{1}{2}|\lambda|$ whenever $2\omega \leq |\lambda|$. Let now $\text{Re}(\lambda) < 0$ and $|\lambda| < 2\omega$ such that by (A.8) $|\lambda + \omega| > c_4\delta^2 = \frac{2\omega}{2(1 + c_5)} > C|\lambda|$, with $C := \frac{1}{2(1 + c_5)}$.

Finally, with respect to (A.10), $|\lambda - \omega| > c_4\delta^2$ holds always, if $|\text{Im}(\lambda)| > c_4\delta^2$. On the other hand, $|\arg(\lambda - \omega)| < \sigma$ implies $|\text{arg}(\lambda)| < \sigma$ and hence for $|\text{Im}(\lambda)| \leq c_4\delta^2$, as in (A.11), $|\lambda - \omega| \geq \omega - |\text{Re}(\lambda)| > c_4\delta^2$.

As suggested in Section 3.4 we prove next that the solution of the deterministic PDE on the bounded domain $\Lambda_\delta$ converges to the solution of the heat equation on $\mathbb{R}^d$ for $\delta \to 0$. This is achieved by the Feynman-Kac representation of the solution and the dominated convergence theorem.

**Proposition A.8.** Let $z \in L^2(\mathbb{R}^d)$. Then the following holds for $t > 0$:

(i) For $\delta > 0$, $\|S^*_{\sigma,\delta}(t)(z|_{\Lambda_\delta})\|_{L^2(\Lambda_\delta)} \leq c_3e^{c_1\delta^2t}\|q\|_{L^2(\mathbb{R}^d)}$ with universal constants $c_1, c_2, c_3 > 0$ and $q$ as in Equation (A.4) above.

(ii) $S^*_{\sigma,\delta}(t)(z|_{\Lambda_\delta}) \to e^{\theta(0)t}\Delta z$ in $L^2(\mathbb{R}^d)$ for $\delta \to 0$.

*Proof.* (i). It is enough to prove the statement for $z \in C(\overline{\Lambda_\delta})$. Indeed, for $z \in L^2(\mathbb{R}^d)$ consider $z^{(\varepsilon)} \in C(\overline{\Lambda_\delta})$ with $z^{(\varepsilon)} \to z|_{\Lambda_\delta}$ in $L^2(\Lambda_\delta)$ as $\varepsilon \to 0$. Applying
Proposition [A.7(i)] to \( S^*_\vartheta,\delta (t)(z|_{\Lambda_\delta} - z^{(e)}) \) and the statement to \( S^*_\vartheta,\delta (t)z^{(e)} \), we have

\[
\|S^*_\vartheta,\delta (t)(z|_{\Lambda_\delta})\|_{L^2(\Lambda_\delta)} = \|S^*_\vartheta,\delta (t)(z|_{\Lambda_\delta} - z^{(e)}) + S^*_\vartheta,\delta (t)(z^{(e)})\|_{L^2(\Lambda_\delta)} \\
\leq c_0 e^{c_1 t \delta^2} \|z - z^{(e)}\|_{L^2(\Lambda_\delta)} + c_3 e^{c_1 t \delta^2} \|q_{c_2 t} * |z^{(e)}|\|_{L^2(\mathbb{R}^d)} \\
\to c_3 e^{c_1 t \delta^2} \|q_{c_2 t} * |z|\|_{L^2(\mathbb{R}^d)}, \quad \varepsilon \to 0.
\]

Therefore assume \( z \in C(\overline{\Lambda_\delta}) \). The proof is based on giving a stochastic representation for \( S^*_\vartheta,\delta (t)z \) via the Feynman-Kac formulas. Without loss of generality let \( \vartheta \in C^2(\mathbb{R}^d), \theta \in C^1(\mathbb{R}^d), b \in C^\alpha(\mathbb{R}^d), \alpha > 0 \), with \( \min_{x \in \mathbb{R}^d} \vartheta(x) > 0 \). Then for \( f \in C^2(\mathbb{R}^d) \)

\[
A^*_\vartheta,\delta f(x) = \vartheta(\delta x) \Delta f(x) + \langle \tilde{a}_\delta(x), \nabla f(x) \rangle_{\mathbb{R}^d} + \tilde{b}_\delta(x)f(x), \quad x \in \mathbb{R}^d, \tag{A.12}
\]

where \( \tilde{a}_\delta = \delta \nabla \vartheta(\delta \bullet) - \delta a(\delta \bullet) \in C^1(\mathbb{R}^d), \tilde{b}_\delta = \delta^2 (b(\delta \bullet) - \text{div}(a(\delta \bullet))) \in C^\alpha(\mathbb{R}^d) \). By Karatzas and Shreve [1991] Theorem 5.4.22 we can find a process \( Y^{(\delta)} = (Y^{(\delta)}_t)_{t \geq 0} \) being a weak solution of the \( d \)-dimensional stochastic differential equation

\[
dY^{(\delta)}_t = \tilde{a}_\delta(Y^{(\delta)}_t)dt + \sqrt{2} \vartheta(\delta \bullet)^{1/2}(Y^{(\delta)}_t)d\tilde{W}_t, \quad Y^{(\delta)}_0 = x \in \mathbb{R}^d,
\]

on a filtered probability space \((\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t \geq 0}, \tilde{\mathbb{P})} \) carrying a scalar Brownian motion \((\tilde{W}_t)_{t \geq 0}\). We show below for \( x \in \mathbb{R}^d \)

\[
(S^*_\vartheta,\delta (t)(z)(x) = \tilde{E}_x \left[ z \left( Y^{(\delta)}_t \right) \exp \left( \int_0^t \tilde{b}_\delta \left( Y^{(\delta)}_s \right) ds \right) 1_{\{t < \tau_\delta(Y^{(\delta)})\}} \right], \tag{A.13}
\]

where \( \tilde{E}_x \) and \( \tilde{E}_x \) indicate the initial value and \( \tau_\delta(Y^{(\delta)}) := \inf \{ t \geq 0 : Y^{(\delta)}_t \notin \Lambda_\delta \} \). Assume first this holds true. Denote the transition densities of \( Y^{(\delta)} \) by \( p_{\delta,t}(x,y) \), \( x, y \in \mathbb{R}^d \). According to Sheu [1991] Eq. (1.4)) we have \( p_{\delta,t}(x, y) \leq c_3 q_{c_2 t}(x - y) \) for universal constants \( c_2, c_3 > 0 \). Then by (A.13), using \( \|\tilde{b}_\delta\|_{L^\infty} \leq c_1 \delta^2 \) for some constant \( c_1 > 0 \), it follows

\[
|(S^*_\vartheta,\delta (t)(z)(x)| \leq e^{c_1 t \delta^2} \tilde{E}_x \left[ |z(Y^{(\delta)}_t)| \right] = e^{c_1 t \delta^2} \int_{\mathbb{R}^d} |z(y)| p_{\delta,t}(x, y) dy \\
\leq c_3 e^{c_1 t \delta^2} q_{c_2 t} * |z|(x). \tag{A.14}
\]

This proves the result in (i) for \( z \in C(\overline{\Lambda_\delta}) \). We are left with showing (A.13). The proof is similar to Friedman [1975] Theorem 6.5.2 and extends Peres and Mörters [2010] Theorem 7.44, which applies only to Brownian motion. It is enough to consider \( x \in \overline{\Lambda_\delta} \), because otherwise \((S^*_\vartheta,\delta (t)(z)(x) = 0 \) and \( 1_{\{t < \tau_\delta(Y^{(\delta)})\}} = 0 \).
By letting have \( \rho \) to \( \in C(0, \infty), \overline{X_\delta} \), where the derivative is taken in \( \mathbb{C} \), then, together with (i), the last line tends to zero. Let therefore the statement with respect to \( \delta \) be true. As in (i) we can assume (ii). As in (i) we can assume

\[
(S_{\delta, \rho}^*(t)z)(x) = \frac{d}{dt}u(t, x) = A_{\delta, \rho}u(t), \quad t > 0,
\]

\[
\left. u(t) \right|_{\partial \Lambda_\delta} = 0, \quad t \geq 0.
\]

where the derivative is taken in \( L^2(\Lambda_\delta) \). Classical PDE theory yields \( u \in C([0, \infty), \overline{X_\delta}) \cap C^{1,2}([\varepsilon, \infty), \overline{X_\delta}) \) for any \( \varepsilon > 0 \), see for example [Friedman (1975) Theorem 6.3.6] (here we use that \( b \in C^\alpha(\overline{X}) \)). Set \( h(t) = \exp(\int_0^t b_\delta(Y^\delta(s))ds) \) and let \( p = \inf\{ t \geq 0 : Y^\delta(t) \notin U \} \) for a compact set \( U \subseteq \Lambda_\delta \). Set \( g(t', x) = u(t - t', x), \quad 0 \leq t' \leq t \). By Itô’s formula, noting that \( A^\delta = A_{\delta, \delta}^* - \tilde{b}_\delta \) generates the transition semigroup of \( Y^\delta \), we have for any \( 0 \leq t' < t \)

\[
\tilde{E}_x \left[ g \left( t' \land \rho, Y^\delta(t') \right) h(t' \land \rho) \right]
\]

\[= (S_{\delta, \rho}(t)z)(x) + \tilde{E}_x \left[ \int_0^{t' \land \rho} \nabla g(s, Y_{\delta, \rho}) h(s) \cdot d\tilde{W}_s \right]_{\mathbb{R}^d} = (S_{\delta, \rho}^*(t)z)(x). \]

Letting \( t' \to t \) yields therefore

\[
(S_{\delta, \rho}(t)z)(x) = \tilde{E}_x \left[ u \left( t \land \rho, Y^\delta(t) \right) h(t \land \rho) \right] - \tilde{E}_x \left[ z(Y^\delta(t)) h(t \land \rho) \right] = (S_{\delta, \rho}^*(t)z)(x). \]

By letting \( U \) exhaust \( \Lambda_\delta \) we have \( \rho \to \tau_\delta(Y^\delta) \) and \( Y^\delta(t) \to 0 \), such that \( u(t - \rho, Y^\delta(t)) \to 0 \). This implies \( (A.13) \).

(iii). As in (i) we can assume \( z \in C(\overline{X_\delta}) \) for sufficiently small \( \delta \). Indeed, for \( z \in L^2(\mathbb{R}^d) \) let \( z^{(\varepsilon)} \in C_c(\mathbb{R}^d) \) converging to \( z \) in \( L^2(\mathbb{R}^d) \) as \( \varepsilon \to 0 \). For small \( \delta \) we have \( z^{(\varepsilon)} \in C(\overline{X_\delta}) \). Applying Proposition [A.7(i)] to \( S_{\delta, \rho}^*(t)z^{(\varepsilon)} \), Lemma [A.5(i)] to \( e^{\varepsilon(0)t}A(z - z^{(\varepsilon)}) \), we have

\[
\left\| (S_{\delta, \rho}(t)z)(\Lambda_\delta) - e^{\varepsilon(0)t}A(z - z^{(\varepsilon)}) \right\|_{L^2(\mathbb{R}^d)} \leq \left\| (S_{\delta, \rho}^*(t)z)(\Lambda_\delta) - e^{\varepsilon(0)t}A(z - z^{(\varepsilon)}) \right\|_{L^2(\mathbb{R}^d)}.
\]

The statement with respect to \( z^{(\varepsilon)} \) and letting first \( \delta \to 0 \) and then \( \varepsilon \to 0 \), the last line tends to zero. Let therefore \( z \in C(\overline{X_\delta}) \).

It is enough to show \( (S_{\delta, \rho}(t)z)(x) \to (e^{\varepsilon(0)t}A z)(x) \) pointwise for \( x \in \mathbb{R}^d \), as then, together with (i), \( S_{\delta, \rho}(t)z \to e^{\varepsilon(0)t}A z \) in \( L^2(\mathbb{R}^d) \) follows from dominated convergence. Using the notation from (i) we have the representation

\[
(e^{\varepsilon(0)t}A z)(x) = (q_{\delta(0), t} \ast z)(x) = \tilde{E}_x[z(Y^\delta_t)]
\]

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for $Y^{(0)}_t = x + \sqrt{2} \vartheta(0)^{1/2} \tilde{W}_t$. (A.13) therefore allows us to write $S^*_\vartheta,\delta(t) z - e^{\vartheta(0)t} \Delta z =: T_1 + T_2 + T_3$ with

\[
T_1 = \tilde{\mathbb{E}}_x \left[ z(Y^{(\delta)}_t) - z(Y^{(0)}_t) \right],
\]

\[
T_2 = \tilde{\mathbb{E}}_x \left[ z(Y^{(\delta)}_t) \left( \exp \left( \int_0^t \tilde{b}_\delta(Y^{(\delta)}_s) ds \right) - 1 \right) 1_{\{t < \tau_3(Y^{(\delta)})\}} \right],
\]

\[
T_3 = - \tilde{\mathbb{E}}_x \left[ z(Y^{(\delta)}_t) 1_{\{t \geq \tau_3(Y^{(\delta)})\}} \right].
\]

We shall show that $T_i \to 0$, $i = 1, 2, 3$. The transition semigroup of $(Y^{(0)}_t)_{t \geq 0}$ is generated by $A^{(0)} = \vartheta(0) \Delta$. Since $A^{(\delta)} f \to A^{(0)} f$ uniformly on $\mathbb{R}^d$ for $f \in C^\infty_c(\mathbb{R}^d)$ as $\delta \to 0$, it follows from Kallenberg (2002, Theorem 19.25) that $Y^{(\delta)} \overset{d}{\to} Y^{(0)}$ with respect to the uniform topology on compacts in $\mathbb{R}^d$. This yields $T_1 \to 0$. As $z$ is bounded and $\sup_{s \geq 0} |\tilde{b}_\delta(Y^{(\delta)}_s)| \lesssim \delta^2$, we also have $|T_2| \lesssim e^{C t \delta^2 \delta^2}$ and $|T_3| \lesssim \tilde{\mathbb{P}}_x(\tau_3(Y^{(\delta)}) \leq t)$. To see why $\tilde{\mathbb{P}}_x(\tau_3(Y^{(\delta)}) \leq t) \to 0$ holds let $Z^{(\delta)} = Y^{(\delta)} - x - \int_0^t \tilde{a}_\delta(Y^{(\delta)}_s) ds'$ and observe that $|\int_0^t \tilde{a}_\delta(Y^{(\delta)}_s) ds'| \lesssim \delta t$ such that

\[
\tilde{\mathbb{P}}_x(\tau_3(Y^{(\delta)}) \leq t) \leq \sum_{i=1}^d \tilde{\mathbb{P}}_x \left( \max_{0 \leq s \leq t} |Z_s^{(\delta,i)}| \geq C \delta^{-1} \right),
\]

where $Z^{(\delta)} = (Z^{(\delta,i)})_{1 \leq i \leq d}$. Since each $Z^{(\delta,i)}$ is a continuous martingale vanishing at 0 such that $(Z^{(\delta,i)})_s = 2 \int_0^s \vartheta(\delta Y^{(\delta)}_s) ds' \leq c s$, $c > 0$, uniformly in $i = 1, \ldots, d$, we find for some scalar Brownian motion $(\tilde{B}_s)_{s \geq 0}$ and $\tilde{c} > 0$

\[
\tilde{\mathbb{P}}_x(\tau_3(Y^{(\delta)}) \leq t) \leq d \tilde{\mathbb{P}}_x \left( \max_{0 \leq s \leq t} |\tilde{B}_s| \geq C \delta^{-1} \right) \lesssim e^{-\tilde{c} \delta^{-2} t^{-1}},
\]

because the running maximum of a Brownian motion decays exponentially (Karatzas and Shreve 1991, Chapter 2.8)). This yields $T_3 \to 0$. \hfill \Box

**Proof of Corollary 3.6.** As $l^{(\delta)}$, $l^{(0)}$ are centered Gaussian processes, it is enough to show that the covariance functions converge. By (3.5), the scaling in Lemma 3.1 and Assumption 3.2 the covariance function of $l^{(\delta)}$ is

\[
\delta^{-2} c \left( (t \delta^2, (z|_{\Lambda_\delta})) , (t' \delta^2, (z'|_{\Lambda_\delta}) \right)
\]

\[
= \int_0^{t \wedge t'} \left\langle B^*_\delta S^*_\delta (t - s) (z|_{\Lambda_\delta}), B^*_\delta S^*_\delta (t' - s) (z'|_{\Lambda_\delta}) \right\rangle_{L^2(\Lambda_\delta)} ds.
\]

The semigroup bounds in Proposition (A.7)(i) give $\sup_{s \leq t} \|S^*_\vartheta,\delta(s)\|_{L^2(\Lambda_\delta)} < \infty$, while Assumption 3.2 and the uniform boundedness principle imply $\sup_{\delta > 0} \|B^*_\delta\|_{L^2(\mathbb{R}^d)} < \frac{32}{3}$.
By the semigroup convergence in Proposition A.8(ii) we have $S^*_{\theta,\delta}(s)(z|\Lambda_{\delta}) \to e^{\theta(0)s}z$ in $L^2(\mathbb{R}^d)$, and the same with respect to $z'$. Hence, dominated convergence applies and the covariance in the last display converges to

$$\int_0^{t \land t'} \langle B^*_0 e^{\theta(0)(t-s)}z, B^*_0 e^{\theta(0)(t'-s)}z' \rangle_{L^2(\mathbb{R}^d)} ds = \text{Cov}(l^{(0)}(t, z), l^{(0)}(t', z')).$$

The following heat kernel bounds will be used frequently. The condition in (iii) is essential for $d = 1$, improving on (ii) in this case.

**Lemma A.9.** Let $z \in H^2(\mathbb{R}^d)$, $w, u \in L^2(\mathbb{R}^d)$ have compact support in $\Lambda_{\delta}$ for $\delta > 0$. Then the following holds for $0 < t \leq T \delta^{-2}$:

1. \( \|S^*_{\theta,\delta}(t) u\|_{L^2(\Lambda_{\delta})} \lesssim e^{Ct(1 \wedge t^{-d/4})} \|u\|_{L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)}. \)
2. If \( \|w - \Delta z\|_{L^1(\mathbb{R}^d)} + \|w - \Delta z\|_{L^2(\mathbb{R}^d)} \leq \tilde{C} \delta \) for a constant $\tilde{C} > 0$, then \( \|S^*_{\theta,\delta}(t)w\|_{L^2(\Lambda_{\delta})} \lesssim e^{Ct(1 \wedge t^{-1/2-d/4})} + \|z\|_{L^1(\mathbb{R}^d)} + \|z\|_{H^2(\mathbb{R}^d)}. \)
3. If \( \int_{\mathbb{R}^d} z(x) dx = 0 \), then \( \|S^*_{\theta,\delta}(t)\Delta z\|_{L^2(\Lambda_{\delta})} \lesssim e^{Ct(1 \wedge t^{-1-d/4})} + \|z\|_{L^1(\mathbb{R}^d)} + \|z\|_{H^2(\mathbb{R}^d)} \) for sufficiently small $\delta$.

**Proof.** (i). By the semigroup bounds in Proposition A.8(i) and Lemma A.5(i) \( \|S^*_{\theta,\delta}(t) u\|_{L^2(\Lambda_{\delta})} \lesssim e^{Ct(1 \wedge t^{-d/4})} \|u\|_{L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)}. \)

(ii). Consider $v^{(\delta)}$ as in Lemma A.11. By (i) and the semigroup bounds in Proposition A.7(i,ii) this means for $t \leq T \delta^{-2}$

\[
\|S^*_{\theta,\delta}(t)\theta(0)\Delta z\|_{L^2(\Lambda_{\delta})} \lesssim \delta \|S^*_{\theta,\delta}(t)v^{(\delta)}\|_{L^2(\Lambda_{\delta})} + \|A^*_{\theta,\delta}S^*_{\theta,\delta}(t/2)S^*_{\theta,\delta}(t/2)z\|_{L^2(\Lambda_{\delta})} \\
\lesssim e^{Ct(1 \wedge t^{-1/2-d/4})} \|z\|_{L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)} + (1 \wedge t^{-1-d/4}) \|z\|_{H^2(\mathbb{R}^d)} \\
\leq e^{Ct(1 \wedge t^{-1/2-d/4})} \|z\|_{L^1(\mathbb{R}^d)} + \|z\|_{H^2(\mathbb{R}^d)}. \]

The result follows from (i), $\delta \lesssim t^{-1/2}$ and adjusting the constant $C$ after applying \( \|S^*_{\theta,\delta}(t)w\|_{L^2(\Lambda_{\delta})} \lesssim \delta \|S^*_{\theta,\delta}(t)\delta^{-1}(w - \Delta z)\|_{L^2(\Lambda_{\delta})} + \theta(0)^{-1}\|S^*_{\theta,\delta}(t)\theta(0)\Delta z\|_{L^2(\Lambda_{\delta})}. \)

(iii). Lemma A.11(ii) below shows the existence of a compactly supported $m \in H^2(\mathbb{R}^d)$ with $|v^{(\delta)} - \Delta m|_{L^2(\Lambda_{\delta})} \lesssim \delta \|z\|_{H^2(\mathbb{R}^d)}$. If $\delta$ is so small that $m$ has support in $\Lambda_{\delta}$, then by the proof of (ii) and $\delta \lesssim t^{-1/2}$

\[
\|S^*_{\theta,\delta}(t)v^{(\delta)}\|_{L^2(\Lambda_{\delta})} \lesssim e^{Ct(1 \wedge t^{-d/4}) + 1 \wedge t^{-1/2-d/4})} \|z\|_{H^2(\mathbb{R}^d)} \\
\lesssim e^{Ct(1 \wedge t^{-1/2-d/4})} \|z\|_{H^2(\mathbb{R}^d)}. \]
Applying this bound in the proof of (ii) and adjusting the constant $C$ yields the claim.

**Lemma A.10.** Let $z \in H^2(\mathbb{R}^d)$, $u \in L^2(\mathbb{R}^d)$ have compact support in $\Lambda_\delta$ for $\delta > 0$. Then the following holds:

(i) If $X_0 \in L^p(\Lambda)$, $p \geq 2$, $1/p + 1/p' = 1$, then $\int_0^T \langle S_\vartheta(t) X_0, (\Delta z)_\delta \rangle^2 dt \lesssim e^{CT} \delta^2 (\|z\|_{L^1(\mathbb{R}^d)}^2 + \|\Delta z\|_{L^p(\mathbb{R}^d)}^2 + \|z\|_{H^2(\mathbb{R}^d)}^2)$ with $\gamma(d, p) := \frac{2^{1+d/p}}{1+d/2} + d(1 - \frac{2}{p})$.

(ii) If $X_0 \in H^1_0(\Lambda) \cap H^2(\Lambda)$ and $d \geq 2$ or $\int_{\mathbb{R}^d} z(x) dx = 0$, then $\int_0^T \langle S_\vartheta(t) X_0, (\Delta z)_\delta \rangle^2 dt \lesssim e^{CT} \delta^2 \|z\|_{L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)}^2$.

Proof. (i). By Lemma A.9(i) and the scaling in Lemma 3.1 we find

$$\int_0^T \langle S_\vartheta(t) X_0, u_\delta \rangle^2 dt \leq \delta^2 \|X_0\|_{L^2(\Lambda_\delta)}^2 \int_0^T \|S^*_\vartheta,\delta(t) u\|_{L^2(\Lambda_\delta)}^2 dt \leq \delta^2 e^{CT} \int_0^T (1 \wedge t^{-d/2}) dt \|u\|_{L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)}^2.$$ 

The claim follows, because the integral has order $O(1)$ for $d \geq 3$, order $O(\log(T\delta^{-2}))$ for $d = 2$ and order $O(T^{1/2}\delta^{-1})$ for $d = 1$.

(ii). Using the Hölder inequality we have for $1/p + 1/p' = 1$, $1 \leq p' \leq 2$, that

$$\langle S_\vartheta(t) X_0, (\Delta z)_\delta \rangle^2 \leq \|S_\vartheta(t) X_0\|_{L^p(\Lambda)}^2 \|\Delta z\|_{L^{p'}(\Lambda)}^2.$$ 

By approximation the inequality in (A.14) above yields further

$$\|S_\vartheta(t) X_0\|_{L^p(\Lambda)} \lesssim \|q_{c,t} \ast X_0\|_{L^p(\mathbb{R}^d)} \leq \|X_0\|_{L^p(\mathbb{R}^d)}.$$ 

Consequently, $\langle S_\vartheta(t) X_0, (\Delta z)_\delta \rangle^2 \lesssim \delta^{d/2} \|\Delta z\|_{L^{p'}(\mathbb{R}^d)}^2$. Splitting up the integral it follows from Lemma 3.1 and Lemma A.9(ii) for $\varepsilon > 0$

$$\int_0^T \langle S_\vartheta(t) X_0, (\Delta z)_\delta \rangle^2 dt \lesssim \delta^{2d/2} \varepsilon^2 \|\Delta z\|_{L^{p'}(\mathbb{R}^d)}^2 + \int_{\varepsilon}^T \|S^*_\vartheta,\delta(t\delta^{-2}) \Delta z\|_{L^2(\Lambda_\delta)}^2 dt \lesssim \delta^{d/2/p' - 1} \varepsilon \|\Delta z\|_{L^{p'}(\mathbb{R}^d)}^2 + \varepsilon^{CT} \int_{\varepsilon}^T (t\delta^{-2})^{1-d/2} dt \|z\|_{L^1(\mathbb{R}^d)}^2 + \|z\|_{H^2(\mathbb{R}^d)}^2.$$ 

$$\lesssim \varepsilon e^{CT} (\delta^{d(2/p' - 1)} \varepsilon + \delta^{2d/2} e^{-d/2}) \|\Delta z\|_{L^1(\mathbb{R}^d)}^2 + \|z\|_{L^{p'}(\mathbb{R}^d)}^2 + \|z\|_{H^2(\mathbb{R}^d)}^2).$$
The claim follows with \( \varepsilon = \frac{\delta^{2+d/p}}{4} \).

(iii). Consider \( v^{(\delta)} \) as in Lemma A.11 below such that \( \|v^{(\delta)}\|_{L^2(\Lambda_\delta)} \lesssim \|z\|_{H^2(\mathbb{R}^d)} \).

If \( d \geq 2 \), this yields for \( \int_0^T \langle S_\theta(t)X_0, (\Delta z)_\delta \rangle^2 dt \) up to a constant the upper bound

\[
\delta^4 \int_0^T \langle S_\theta(t)A_\theta X_0, z_\delta \rangle^2 dt + \delta^2 \int_0^T \langle S_\theta(t)X_0, v^{(\delta)} \rangle^2 dt \lesssim \delta^4 e^{CT} \|z\|_2^2 + \delta^2 e^{CT} \|v^{(\delta)}\|_2^2 \lesssim e^{CT} \delta^2 \|z\|_{H^2(\mathbb{R}^d)}^2.
\]  

(A.15)

If \( \int_{\mathbb{R}^d} z(x)dx = 0 \), then consider \( m \in H^2(\mathbb{R}^d) \) from the proof of Lemma A.9(iii) with \( \|m\|_{H^2(\mathbb{R}^d)} \lesssim \|z\|_{H^2(\mathbb{R}^d)} \) and \( \delta \) so small that \( m \) has support in \( \Lambda_\delta \). Applying the last display to \( z = m \) yields for \( \int_0^T \langle S_\theta(t)X_0, v^{(\delta)} \rangle^2 dt \) up to a constant the upper bound

\[
e^{CT} \|v^{(\delta)}\|^2_{L^2(\Lambda_\delta)} + \int_0^T \langle S_\theta(t)X_0, (\Delta m)_\delta \rangle^2 dt \lesssim e^{CT} \delta^2 \|z\|_{H^2(\mathbb{R}^d)}^2.
\]

By plugging this into (A.15) we obtain the result.  

Lemma A.11. Let \( z \in H^2(\mathbb{R}^d) \) have compact support in \( \Lambda_\delta \) for \( \delta > 0 \). Then:

(i) \( v^{(\delta)} := \delta^{-1}(A^b_{\theta, \delta} - \theta(0)\Delta)z \) has support in \( \Lambda_\delta \) and satisfies \( \|v^{(\delta)}\|_{L^2(\Lambda_\delta)} \lesssim \|z\|_{H^2(\mathbb{R}^d)} \), \( v^{(\delta)} \rightarrow v := \Delta((\nabla \theta(0), x)_{\mathbb{R}^d})z(x) - (\nabla \theta(0) + a(0), \nabla z(x))_{\mathbb{R}^d} \) for \( \delta \rightarrow 0 \) in \( L^2(\mathbb{R}^d) \).

(ii) If \( \int_{\mathbb{R}^d} z(x)dx = 0 \), then \( v = \Delta u \) with \( u \in H^2(\mathbb{R}^d) \) having compact support and such that \( \|v^{(\delta)} - \Delta u\|_{L^2(\Lambda_\delta)} \lesssim \delta \|z\|_{H^2(\mathbb{R}^d)} \) and \( \|u\|_{H^2(\mathbb{R}^d)} \lesssim \|z\|_{H^2(\mathbb{R}^d)} \).

(iii) If \( \int_{\mathbb{R}^d} z(x)dx = 0 \), \( \int_{\mathbb{R}^d} xz(x)dx = 0 \), then \( z = \Delta u \) with \( u \in H^4(\mathbb{R}^d) \) having compact support.

Proof. (i). Without loss of generality let \( \theta \in C^2(\mathbb{R}^d), a \in C^1(\mathbb{R}^d), b \in C(\mathbb{R}^d) \). Then for \( x \in \Lambda_\delta \)

\[
v^{(\delta)}(x) = \frac{\theta(\delta x) - \theta(0)}{\delta} \Delta z(x) + (\nabla \theta(\delta x) - a(\delta x), \nabla z(x))_{\mathbb{R}^d} \\
+ \Delta (b(\delta x) - (\text{diva})(\delta x)z(x)).
\]

Since \( z \) and its partial derivatives up to second order are supported in \( \Lambda_\delta \), it is clear that \( v^{(\delta)} \) has support in \( \Lambda_\delta \) and \( \|v^{(\delta)}\|_{L^2(\Lambda_\delta)} \lesssim \|z\|_{H^2(\mathbb{R}^d)} \) and \( v^{(\delta)} \rightarrow v, \delta \rightarrow 0 \), in \( L^2(\mathbb{R}^d) \) with

\[
v(x) = (\nabla \theta(0), x)_{\mathbb{R}^d} \Delta z(x) - (\nabla \theta(0) + a(0), \nabla z(x))_{\mathbb{R}^d} \\
= \Delta((\nabla \theta(0), x)_{\mathbb{R}^d})z(x) - (\nabla \theta(0) + a(0), \nabla z(x))_{\mathbb{R}^d}.
\]
It follows that \( \|v^{(d)} - v\|_{L^2(A)} \leq \delta \|z\|_{H^2(\mathbb{R}^d)} \).

(ii). In order to find \( u \), as \( z \) has compact support, it suffices to find a compactly supported function \( g \in H^2(\mathbb{R}^d) \) with \( \Delta g = \langle \nabla \vartheta(0), x \rangle_{\mathbb{R}^d} \) in \( L^2(\mathbb{R}^d) \) and to set \( u := \langle \nabla \vartheta(0), x \rangle_{\mathbb{R}^d} z(x) - g(x) \). Using the Fourier transform \( \mathcal{F}g(\omega) = \int_{\mathbb{R}^d} g(x) e^{i(x, \omega)} dx \) this means by usual Fourier calculus

\[-|\omega|^2 \mathcal{F}g(\omega) = \langle \nabla \vartheta(0) + a(0), i\omega \rangle_{\mathbb{R}^d} \mathcal{F}z(\omega), \quad \omega \in \mathbb{R}^d.\]

By the compact support of \( z \) and \( \int_{\mathbb{R}^d} z(x) dx = 0 \) the Fourier transform \( \mathcal{F}z \) is analytic with \( \mathcal{F}z(0) = 0 \). We can thus define

\[g(x) := \mathcal{F}^{-1}[m](x) \quad \text{with} \quad m(\omega) := \langle \nabla \vartheta(0) + a(0), -|\omega|^{-1}i\omega \rangle_{\mathbb{R}^d} \frac{\mathcal{F}z(\omega)}{|\omega|},\]

as the inverse Fourier transform of the \( L^2 \)-function \( m \). Noting \( z \in H^2(\mathbb{R}^d) \) and \( |m(\omega)| \leq |\mathcal{F}z(\omega)| \) for \( |\omega| \to \infty \), we see \( g \in H^2(\mathbb{R}^d) \) and \( \Delta g = \langle \nabla \vartheta(0) + a(0), \nabla z \rangle_{\mathbb{R}^d} \) in \( L^2(\mathbb{R}^d) \). In particular, \( \|g\|_{H^2(\mathbb{R}^d)} \lesssim \|z\|_{H^2(\mathbb{R}^d)} \).

Finally, we use the Paley-Wiener Theorem [Rudin (1991, Theorem II.7.22)] to deduce from the compact support of \( z \) that \( \mathcal{F}z \) can be extended to an entire function on \( C^d \), satisfying the exponential growth condition \( |\mathcal{F}z(\omega)| \leq \gamma_N (1 + |\omega|)^{-N} \exp(r |\text{Im}(\omega)|), \quad \omega \in C^d, \) for all \( N \in \mathbb{N} \) and suitable positive constants \( \gamma_N, r \). Hence, \( m \) is the quotient of an entire function and \( |\omega|^2 \), which is also entire. A meromorphic function with removable singularity extends continuously to an entire function. Consequently, we can work with an entire function \( m \), which by definition satisfies the same exponential growth condition. A reverse application of the Paley-Wiener Theorem shows that \( g \) has compact support.

(iii). The argument is similar to (ii). As above the Fourier transform \( \mathcal{F}z \) is analytic with \( \mathcal{F}z(0) = 0 \), but because of \( \int_{\mathbb{R}^d} x_i z(x) dx = 0 \), also \( \partial_i (\mathcal{F}z)(0) = 0 \), \( i = 1, \ldots, d \). Defining

\[u(x) := \mathcal{F}^{-1}[m](x) \quad \text{with} \quad m(\omega) := -\frac{\mathcal{F}z(\omega)}{|\omega|^2},\]

it follows \( u \in H^4(\mathbb{R}^d) \) and \( \Delta u = z \). A Paley-Wiener argument as in (ii) shows that \( u \) has compact support. \( \square \)

A.3 Asymptotic results for the covariances

The general idea for the proofs in this section is to apply the scaling in Lemma 3.1 to the covariance function as in Section 3.4 above and to deduce a limit for the integral using the heat kernel bounds and the convergence of the semigroups from the last section.
Proposition A.12. Grant Assumption 3.2. Let $z \in H^{2}(\mathbb{R}^{d})$, $w^{(\delta)}, u^{(\delta)}, u \in L^{2}(\mathbb{R}^{d})$ have compact support in $\Lambda_{\delta}$ for some $\delta > 0$, such that $\|w^{(\delta)} - \Delta z\|_{L^{2}(\mathbb{R}^{d})} \lesssim \delta$, $\|u^{(\delta)} - u\|_{L^{2}(\mathbb{R}^{d})} \to 0$. Then as $\delta \to 0$:

(i) $\delta^{-2} \text{Var}(\langle X(t), w^{(\delta)}_{\delta} \rangle) \to \vartheta(0)^{-1}\Psi(\Delta z, \Delta z)$, $t > 0$.

(ii) $\delta^{-2} \int_{0}^{t} \text{Var}(\langle X(t), w^{(\delta)}_{\delta} \rangle) dt \to T \vartheta(0)^{-1}\Psi(\Delta z, \Delta z)$.

(iii) If $d \geq 2$, then $\delta^{-2} \int_{0}^{T} \text{Cov}(\langle X(t), w^{(\delta)}_{\delta} \rangle, (X(t), u_{\delta}^{(\delta)})) dt \to T \vartheta(0)^{-1}\Psi(\Delta z, u)$.

Proof. (i) By (2.5) and the scaling in Lemma 3.1, we have

$$\delta^{-2} \int_{0}^{t} \|B^{*}_{\delta} S^{*}_{\delta, \delta}(s\delta^{-2}) w^{(\delta)}\|_{L^{2}(\Lambda_{\delta})}^{2} ds = \int_{0}^{\delta^{-2}} \|B^{*}_{\delta} S^{*}_{\delta, \delta}(s) w^{(\delta)}\|_{L^{2}(\Lambda_{\delta})}^{2} ds.$$  

Write this as $\int_{0}^{\infty} f_{\delta}(s) ds$, where

$$f_{\delta}(s) = \|B^{*}_{\delta} S^{*}_{\delta, \delta}(s) w^{(\delta)}\|_{L^{2}(\Lambda_{\delta})}^{2} 1_{\{s \leq \delta^{-2}\}}.$$ 

Set $f(s) = \|B^{*}_{\delta} e^{0(s)\Delta z} w^{(\delta)}\|_{L^{2}(\mathbb{R}^{d})}^{2}$ and note $\int_{0}^{\infty} f(s) \vartheta(0) ds = \Psi(\Delta z, \Delta z)$, substituting $ds' = \vartheta(0) ds$. By assumption $w^{(\delta)} \to \Delta z$ in $L^{2}(\Lambda_{\delta})$ and by Proposition A.8(ii) above $S^{*}_{\delta, \delta}(s) \Delta z \to e^{0(s)\Delta z} \Delta z$ in $L^{2}(\mathbb{R}^{d})$. By Proposition A.7(i) above we have $\sup_{s \leq t/\delta^{2}} \|S^{*}_{\delta, \delta}(s)\|_{L^{2}(\Lambda_{\delta})} < \infty$ as well as by Assumption 3.2 and the uniform boundedness principle $\sup_{s > 0} \|B^{*}_{\delta}\|_{L^{2}(\mathbb{R}^{d})} < \infty$. We deduce

$$\|B^{*}_{\delta} S^{*}_{\delta, \delta}(s) w^{(\delta)} - B^{*}_{\delta} e^{0(s)\Delta z} \Delta z\|_{L^{2}(\mathbb{R}^{d})} \lesssim \|B^{*}_{\delta} \| \left( \|S^{*}_{\delta, \delta}(s)\|_{L^{2}(\mathbb{R}^{d})} + \|S^{*}_{\delta, \delta}(s)\|_{L^{2}(\mathbb{R}^{d})} \Delta z - e^{0(s)\Delta z}\|_{L^{2}(\mathbb{R}^{d})} \right)$$

$$+ \|(B^{*}_{\delta} - B^{*}_{0}) e^{0(s)\Delta z}\|_{L^{2}(\mathbb{R}^{d})} \to 0,$$

which implies $f_{\delta}(s) \to f(s)$ pointwise.

We shall conclude by dominated convergence. The compact supports of $w^{(\delta)}, \Delta z$ and $\|w^{(\delta)} - \Delta z\|_{L^{2}(\mathbb{R}^{d})} \lesssim \delta$ gives also $\|w^{(\delta)} - \Delta z\|_{L^{1}(\mathbb{R}^{d})} \lesssim \delta$ such that by Lemma A.9(ii) $|f_{\delta}(s)| \lesssim 1 \wedge s^{-1-\frac{d}{2}}$. Consequently, $\sup_{s > 0} f_{\delta}(s) \in L^{1}([0, \infty))$, for any fixed $t$, and the dominated convergence theorem applies.

(ii) By (i) and Fatou’s lemma we obtain

$$\liminf_{\delta \to 0} \delta^{2} \int_{0}^{T} \text{Var}(\langle X(t), w^{(\delta)}_{\delta} \rangle) dt \geq T \vartheta(0)^{-1}\Psi(\Delta z, \Delta z).$$

On the other hand, $\text{Var}(\langle X(t), w^{(\delta)}_{\delta} \rangle)$ is increasing in $t$, cf. (2.5), such that

$$\limsup_{\delta \to 0} \delta^{2} \int_{0}^{T} \text{Var}(\langle X(t), w^{(\delta)}_{\delta} \rangle) dt \leq \lim_{\delta \to 0} \delta^{2} T \text{Var}(\langle X(T), w^{(\delta)}_{\delta} \rangle)$$

$$= T \vartheta(0)^{-1}\Psi(\Delta z, \Delta z),$$

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and the claim follows.

(iii). Revisiting the derivations in (i) and (iii) we obtain
\[ \delta^{-2} \text{Cov}(\langle X(t), w^{(\delta)}_\delta \rangle, \langle X(t), u^{(\delta)}_\delta \rangle) = \int_0^\infty f_\delta(s) \, ds \]
with
\[ f_\delta(s) := \langle B_0^* S_{\vartheta,\delta}^* (s) w^{(\delta)}_\delta, B_0^* S_{\vartheta,\delta}^* (s) u^{(\delta)}_\delta \rangle 1_{\{s \leq t \delta^{-2}\}}. \]

Putting \( f(s) := \langle B_0^* e^{\theta(0)s \Delta z}, B_0^* e^{\theta(0)s \Delta u} \rangle \), we obtain as in (i), (iii) that \( f_\delta(s) \to f(s) \) holds pointwise for \( \delta \to 0 \) by the \( L^2 \)-continuity of the scalar product. Furthermore, the Cauchy-Schwarz inequality and Lemma A.9(i,ii) yield the bound
\[ |f_\delta(s)| \lesssim \|S_{\vartheta,\delta}^* (s) w^{(\delta)}_\delta\|_{L^2(\Lambda^d)} \|S_{\vartheta,\delta}^* (s) u^{(\delta)}_\delta\|_{L^2(\Lambda^d)} 1_{\{s \leq t \delta^2\}} \]
\[ \lesssim e^{c_{T}(1 + s^{-1/2-d/2})} \lesssim 1 \wedge s^{-3/2} \quad (A.16) \]
for \( d \geq 2 \). Since this bound is integrable in \( s \geq 0 \), we conclude
\[ \delta^{-2} \text{Cov}(\langle X(t), w^{(\delta)}_\delta \rangle, \langle X(t), u^{(\delta)}_\delta \rangle) \to \int_0^\infty f(s)ds = \vartheta(0)^{-1} \Psi(\Delta z, u), \]
meaning in particular that \( \Psi(\Delta z, u) \) is well defined. What is more, the bound (A.16) also shows that the covariance is uniformly bounded in \( t \in [0, T] \) so that another application of the dominated convergence theorem shows that the integral over \( t \in [0, T] \) converges to \( T \vartheta(0)^{-1} \Psi(\Delta z, u) \).

The next result improves on Proposition A.12(ii) when \( B \) is a multiplication operator, by making lower order terms explicit. This is necessary for the proof of Theorem 5.4 above. The main difficulty is to work around not having a rate of convergence in Proposition A.8(ii).

**Proposition A.13.** Let \( z \in H^4(\mathbb{R}^d) \) have compact support in \( \Lambda^d \) for \( \delta > 0 \) and suppose that \( B = M_\sigma \) with \( \sigma \in C^1(\Lambda) \). Let \( d \geq 2 \) or \( \int_{\mathbb{R}^d} z(x)dx = 0 \). Then
\[ \delta^{-2} \int_0^T \text{Var}(\langle X(t), (\Delta z)_\delta \rangle) dt = \frac{T \sigma^2(0)}{2 \vartheta(0)} \|\nabla z\|_{L^2(\mathbb{R}^d)}^2 \]
\[ + \frac{\delta T}{2} \langle \nabla \left( \frac{\sigma^2}{\vartheta} \right)(0), x \rangle_{\mathbb{R}^d}, |\nabla z|^2 \rangle_{L^2(\mathbb{R}^d)} + o(\delta). \]

**Proof.** Denote by \( (\tilde{S}_\delta(t))_{t \geq 0} \) the semigroup on \( L^2(\Lambda^d) \) generated by \( \Delta_{\vartheta(\bullet)} \), and let \( \langle \tilde{X}(t), \bullet \rangle \) be defined as \( \langle X(t), \bullet \rangle \) in (2.4), but with \( (\tilde{S}_\theta(t))_{t \geq 0} \) instead of
R where Proposition A.13 we have

With this introduce the decomposition

\[
T_1 = \frac{\delta^{-2}}{\sigma^2(0)} \int_0^T \text{Var}(\langle \tilde{X}(t), (\Delta_{\phi(\cdot)} z) \rangle) dt - \int_0^T \int_0^{t \delta^{-2}} \| \tilde{S}_{\phi, \delta}(s) \Delta_{\phi(\cdot)} z \|_{L^2(\Lambda_\delta)}^2 ds dt,
\]

\[
T_2 = -\frac{2\delta^{-1}}{\sigma^2(0)} \int_0^T \text{Cov}(\langle \tilde{X}(t), (\Delta_{\phi(\cdot)} z) \rangle, \langle \tilde{X}(t), v_{\delta}^{(\delta)} \rangle) dt,
\]

\[
T_3 = \int_0^T \int_0^{t \delta^{-2}} \| \tilde{S}_{\phi, \delta}(s) \Delta_{\phi(\cdot)} z \|_{L^2(\Lambda_\delta)}^2 ds dt - \frac{T \vartheta(0)}{2} \| \nabla z \|_{L^2(\mathbb{R}^d)}^2,
\]

\[
R_1 = \delta^{-2} \int_0^T \text{Cov}(\langle X(t), (\Delta z) \rangle, \langle \tilde{X}(t), (\Delta z) \rangle, \langle X(t), (\Delta z) \rangle + (\tilde{X}(t), (\Delta z)) \rangle) dt,
\]

\[
R_2 = \frac{1}{\vartheta^2(0)} \int_0^T \text{Var}(\langle \tilde{X}(t), v_{\delta}^{(\delta)} \rangle) dt.
\]

With this introduce the decomposition

\[
\delta^{-2} \int_0^T \text{Var}(\langle X(t), (\Delta z) \rangle) dt = \delta^{-2} \int_0^T \text{Var}(\langle \tilde{X}(t), (\Delta z) \rangle) dt + R_1
\]

\[
= \frac{T \sigma^2(0)}{2 \vartheta(0)} \| \nabla z \|_{L^2(\mathbb{R}^d)}^2 + \frac{\sigma^2(0)}{\vartheta^2(0)} (T_1 + T_2 + T_3) + R_1 + R_2.
\]

Only the \( T_i, i = 1, 2, 3 \), contribute to the limit. Indeed, we have for the remainder terms \( R_i = o(\delta), i = 1, 2 \) by Lemmas A.17 and A.18 below, and so the claim follows from Lemmas A.14, A.15, and A.16 below which show

\[
\delta^{-1} (T_1 + T_2 + T_3) \to \frac{T}{2} \langle \langle \vartheta(0) \| \nabla \sigma(0) \|_{L^2(\mathbb{R}^d)}^2 - \vartheta(0), x \rangle_{\mathbb{R}^d}, \| \nabla z \|_{L^2(\mathbb{R}^d)}^2 \rangle.
\]

\[\square\]

**Lemma A.14.** In Proposition A.13 we have

\[
\delta^{-1} T_1 \to \frac{T \vartheta(0)}{2 \sigma^2(0)} \langle \langle \| \nabla \sigma(0) \|_{L^2(\mathbb{R}^d)}^2, x \rangle_{\mathbb{R}^d}, \| \nabla z \|_{L^2(\mathbb{R}^d)}^2 \rangle.
\]

**Proof.** Using (2.5) and the scaling in Lemma 3.1 we find that

\[
T_1 = \frac{1}{\sigma^2(0)} \int_0^T \int_0^{t \delta^{-2}} f_{\delta}(s) ds dt,
\]

where

\[
f_{\delta}(s) = \delta (M_{\delta^{-1}(\sigma_2(\cdot) - \sigma^2(0)) \tilde{S}_{\phi, \delta}(s) \Delta_{\phi(\cdot)} z, \tilde{S}_{\phi, \delta}(s) \Delta_{\phi(\cdot)} z \rangle_{L^2(\Lambda_\delta)}
\]

\[
= \delta \int_0^1 \langle M_{\langle \nabla \sigma^2(\delta rx), x \rangle_{\mathbb{R}^d}} \tilde{S}_{\phi, \delta}(s) \Delta_{\phi(\cdot)} z, \tilde{S}_{\phi, \delta}(s) \Delta_{\phi(\cdot)} z \rangle_{L^2(\Lambda_\delta)} dr.
\]

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By the Cauchy-Schwarz inequality and the semigroup bounds in Proposition A.7(i,ii) above this means

\[ |\delta^{-1}f_{\delta}(s)| \lesssim ||x|\tilde{S}_{\delta,s}(s)\Delta_{\theta(\delta)}z||_{L^2(\Lambda_{\delta})} (1 \wedge s^{-1}) \]

\[ \lesssim ||x|^{2}\tilde{S}_{\delta,s}(s)\Delta_{\theta(\delta)}z||_{L^2(\Lambda_{\delta})}^{1/2} (1 \wedge s^{-3/2}). \]

Approximating \( \Delta_{\theta(\delta)}z \) by continuous functions and using the inequality in (A.14) above yields

\[ ||x|^{2}\tilde{S}_{\delta,s}(s)\Delta_{\theta(\delta)}z||_{L^2(\Lambda_{\delta})} \lesssim ||x|^{2}e^{\vartheta(0)s\Delta} |\Delta_{\theta(\delta)}z||_{L^2(\mathbb{R}^{d})}. \]

Lemma A.5(iii) above gives for this the order \( O((1 \vee s)(1 \wedge s^{-d/4})) \) such that

\[ |\delta^{-1}f_{\delta}(s)| \lesssim (1 \vee s)^{1/2}(1 \wedge s^{-d/4})^{1/2} (1 \wedge s^{-3/2}) \lesssim (1 \wedge s^{-1-d/8}). \] (A.17)

The convergence of the semigroups in Proposition A.8(ii) above gives pointwise for \( s \geq 0 \) that \( \tilde{S}_{\delta,s}(s)\Delta_{\theta(\delta)}z \rightarrow \vartheta(0)e^{\vartheta(0)s\Delta}z \).

The calculations for the last line provide us, with uniform integrability of \( x \mapsto \langle \nabla\sigma^{2}(\delta r x), x \rangle_{\mathbb{R}^{d}}|\tilde{S}_{\delta,s}(s)\Delta_{\theta(\delta)}z|(x) \) in \( L^{2}(\mathbb{R}^{d}) \), allowing us to conclude that

\[ \delta^{-1}f_{\delta}(s) \rightarrow f(s) = \vartheta(0)^{2}\langle \langle \nabla\sigma^{2}(0), x \rangle_{\mathbb{R}^{d}}e^{\vartheta(0)s\Delta}z, e^{\vartheta(0)s\Delta}z \rangle_{L^{2}(\mathbb{R}^{d})}. \]

Together with (A.17) the dominated convergence theorem via (A.17) yields \( \sigma^{2}(0)\delta^{-1}T_{1} \rightarrow T\int_{0}^{\infty}f(s)ds \), which by Lemmas A.5(ii) and A.6 (here we need \( z \in H^{4}(\mathbb{R}^{d}) \)) equals

\[
T\vartheta^{2}(0)\int_{0}^{\infty}\langle \langle \nabla\sigma^{2}(0), -s\vartheta(0)\nabla e^{\vartheta(0)s\Delta}z + e^{\vartheta(0)s\Delta}(xz) \rangle_{\mathbb{R}^{d}}, \Delta e^{\vartheta(0)s\Delta}z \rangle_{L^{2}(\mathbb{R}^{d})}ds
\]

\[
= T\vartheta^{2}(0)\int_{0}^{\infty}\langle \langle \nabla\sigma^{2}(0), x \rangle_{\mathbb{R}^{d}}z, \Delta e^{2\vartheta(0)s\Delta}z \rangle_{L^{2}(\mathbb{R}^{d})}ds
\]

\[
= -\frac{T\vartheta(0)}{2}\langle \langle \nabla\sigma^{2}(0), x \rangle_{\mathbb{R}^{d}}, z \rangle_{L^{2}(\mathbb{R}^{d})} = -\frac{T\vartheta(0)}{2}\langle \langle \nabla\sigma^{2}(0), x \rangle_{\mathbb{R}^{d}}, |z|^{2} \rangle_{L^{2}(\mathbb{R}^{d})}. \]

Lemma A.15. In Proposition A.13 we have

\[ \delta^{-1}T_{2} \rightarrow -T\langle \langle \nabla \vartheta(0), x \rangle_{\mathbb{R}^{d}}, |\nabla z|^{2} \rangle_{L^{2}(\mathbb{R}^{d})}. \]

Proof. Lemma A.11 above with \( A_{\delta} = \Delta_{\vartheta} \) yields the \( L^{2} \)-convergence

\[ v^{(\delta)} \rightarrow v := \Delta(\langle \nabla \vartheta(0), x \rangle_{\mathbb{R}^{d}}z) - \langle \nabla \vartheta(0), \nabla z \rangle_{\mathbb{R}^{d}}. \]
If \( d \geq 2 \), then with \( \Delta_{\vartheta(\delta\bullet)} z \to \vartheta(0) \Delta z \) Proposition A.12(iii) implies \( \delta^{-1}T_2 \to -2T\sigma^{-2}(0)\Psi(\Delta z, v) \). Recalling from Example 3.4 the identity \( \Psi(\Delta z, v) = -\frac{\sigma^2(0)}{2} \langle z, v \rangle_{L^2(\mathbb{R}^d)} \), the claim follows from Lemma A.6 above. The property \( \int_{\mathbb{R}^d} z(x)dx = 0 \), on the other hand, ensures by Lemma A.11(ii) that \( v = \Delta u \) for a compactly supported \( u \in H^2(\mathbb{R}^d) \). By polarisation and Proposition A.12(ii), \( \delta^{-1}T_2 \) converges again to the claimed limit. \( \square \)

**Lemma A.16.** In Proposition A.13 we have

\[
\delta^{-1}T_3 \to \frac{T}{2} \langle \langle \nabla \vartheta(0), x \rangle_{\mathbb{R}^d}, |\nabla z|^2 \rangle_{L^2(\mathbb{R}^d)}.
\]

**Proof.** Since \( \Delta_{\vartheta(\delta\bullet)} \) and \( \tilde{S}_{\vartheta,\delta}(s) \) are selfadjoint and commute on \( H^1_0(\Lambda_\delta) \cap H^2(\Lambda_\delta) \), we have

\[
\| \tilde{S}_{\vartheta,\delta}(s) \Delta_{\vartheta(\delta\bullet)} z \|_{L^2(\Lambda_\delta)}^2 = \langle \tilde{S}_{\vartheta,\delta}(2s) \Delta_{\vartheta(\delta\bullet)} z, \Delta_{\vartheta(\delta\bullet)} z \rangle_{L^2(\Lambda_\delta)}.
\]

This yields, using twice \( \frac{d}{ds} \tilde{S}_{\vartheta,\delta}(2s) = 2\Delta_{\vartheta(\delta\bullet)} \tilde{S}_{\vartheta,\delta}(2s) \),

\[
T_3 = \frac{1}{2} \int_0^T \langle \tilde{S}_{\vartheta,\delta}(2t\delta^{-2}) z, \Delta_{\vartheta(\delta\bullet)} z \rangle_{L^2(\Lambda_\delta)} dt - \frac{T}{2} \langle \langle \Delta_{\vartheta(\delta\bullet)} - \vartheta(0) \Delta \rangle z, z \rangle_{L^2(\mathbb{R}^d)}
\]

\[
= \frac{\delta^2}{2} \langle \tilde{S}_{\vartheta,\delta}(2T\delta^{-2}) z, z \rangle_{L^2(\mathbb{R}^d)} - \frac{\delta^2}{2} \| z \|^2_{L^2(\mathbb{R}^d)} - \frac{T}{2} \langle \langle \Delta_{\vartheta(\delta\bullet)} - \vartheta(0) \Delta \rangle z, z \rangle_{L^2(\mathbb{R}^d)}.
\]

By the semigroup bound in Proposition A.7(i) above the first two terms are of order \( O(\delta^2) \). Noting by \( \vartheta \in C^2(\overline{\mathbb{X}}) \) and the compact support of \( z \) that

\[
\langle \langle \Delta_{\vartheta(\delta\bullet)} - \vartheta(0) \Delta \rangle z, z \rangle_{L^2(\mathbb{R}^d)}
\]

\[= \delta \int_0^1 \langle \nabla \vartheta(s\delta\bullet), x \rangle_{\mathbb{R}^d} ds \Delta z + \langle \nabla \vartheta(s\delta\bullet), \nabla z \rangle_{\mathbb{R}^d, z} \rangle_{L^2(\mathbb{R}^d)}
\]

\[= \delta \langle \langle \nabla \vartheta(0), x \rangle_{\mathbb{R}^d} \Delta z + \langle \nabla \vartheta(0), \nabla z \rangle_{\mathbb{R}^d, z} \rangle_{L^2(\mathbb{R}^d)} + o(\delta),
\]

the result follows from Lemma A.6 above. \( \square \)

**Lemma A.17.** In Proposition A.13 we have \( R_1 = o(\delta) \).

**Proof.** Set \( f_\delta(s) = \langle g_\delta(s), h_\delta(s) \rangle_{L^2(\Lambda_\delta)} \) for \( s \geq 0 \), where

\[
g_\delta (s) = (S^s_{\vartheta,\delta}(s) - \tilde{S}_{\vartheta,\delta}(s)) \Delta z,
\]

\[
h_\delta (s) = M_{\sigma^2(\delta\bullet)}(S^s_{\vartheta,\delta}(s) + \tilde{S}_{\vartheta,\delta}(s)) \Delta z,
\]

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such that by (2.5) and the scaling in Lemma 3.1 \( R_1 = \int_0^T \int_0^{t\delta^{-2}} f_\delta(s) ds dt \). To prove the result it is enough to show

\[
|\delta^{-1} f_\delta(s)| \lesssim 1 \wedge s^{-5/4}, \quad 0 < s \leq t\delta^{-2}, \quad (A.18)
\]

\[
\delta^{-1} f_\delta(s) \to 0. \quad (A.19)
\]

Then the claim follows follows from dominated convergence. In order to show (A.18) and (A.19) we use the variation of parameters formula (Engel and Nagel (2000, p. 162)). The function \([0, s] \ni s' \mapsto S_{\theta, \delta}^*(s') \tilde{S}_{\theta, \delta}(s - s') \Delta z\) has derivative

\[
S_{\theta, \delta}^*(s') (A_{\theta, \delta}^* - \Delta_{\theta}(\delta\bullet)) \tilde{S}_{\theta, \delta}(s - s') \Delta z, \text{ implying}
\]

\[
g_\delta(s) = \int_0^s S_{\theta, \delta}^*(s') (A_{\theta, \delta}^* - \Delta_{\theta}(\delta\bullet)) \tilde{S}_{\theta, \delta}(s - s') \Delta z ds'.
\]

Since the operator \( A_{\theta, \delta}^* - \Delta_{\theta}(\delta\bullet) = A_{0, \delta}^* \) is not bounded, a careful analysis is required. Decomposing it into first and zero order terms we have

\[
g_\delta(s) = -\delta \int_0^s S_{\theta, \delta}^*(s') \langle a(\delta\bullet), \nabla \tilde{S}_{\theta, \delta}(s - s') \Delta z \rangle_{\mathbb{R}^d} ds' + \delta^2 \int_0^s S_{\theta, \delta}^*(s') (b(\delta\bullet) - \text{div}(\delta\bullet)) \tilde{S}_{\theta, \delta}(s - s') \Delta z ds'. \quad (A.20)
\]

The semigroup bounds in Proposition A.7(ii) and in Lemma A.9(i,ii,iii) above, subject to \( d \geq 2 \) or \( \int_{\mathbb{R}^d} z(x) dx = 0 \), show for sufficiently small \( \delta \) and \( 0 \leq s' < s \leq t\delta^{-2} \)

\[
\|\Delta_{\theta, \delta} \tilde{S}_{\theta, \delta}(s - s') \Delta z\|_{L^2(\Lambda_\delta)} = \|\Delta_{\theta, \delta} \tilde{S}_{\theta, \delta}(\frac{s - s'}{2}) \tilde{S}_{\theta, \delta}(\frac{s - s'}{2}) \Delta z\|_{L^2(\Lambda_\delta)} \lesssim (s - s')^{-1} \|\tilde{S}_{\theta, \delta}(\frac{s - s'}{2}) \Delta z\|_{L^2(\Lambda_\delta)} \lesssim (s - s')^{-1} \frac{1}{1 \wedge (s - s')^{-1}}. \quad (A.21)
\]

Hence, by the calculations in (A.7) above

\[
\|\langle a(\delta\bullet), \nabla \tilde{S}_{\theta, \delta}(s - s') \Delta z \rangle_{\mathbb{R}^d}\|_{L^2(\Lambda_\delta)} \lesssim \|\Delta_{\theta, \delta}(s - s') \Delta z\|_{L^2(\Lambda_\delta)}^{1/2} \|\tilde{S}_{\theta, \delta}(s - s') \Delta z\|_{L^2(\Lambda_\delta)}^{1/2} \lesssim (s - s')^{-1/2} \frac{1}{1 \wedge (s - s')^{-1}}. \quad (A.22)
\]

Because of Proposition A.7(i) this yields for \( \|\delta^{-1} g_\delta(s)\|_{L^2(\Lambda_\delta)} \) and \( 0 < s \leq t\delta^{-2} \)
up to a constant the upper bound
\[
\int_0^s \left( \|a(\delta \bullet), \nabla \tilde{S}_{\theta, \delta} (s-s') \Delta z\|_{L^2(\Lambda_\delta)} + \|\tilde{S}_{\theta, \delta} (s-s') \Delta z\|_{L^2(\Lambda_\delta)} \right) ds' \\
\lesssim \int_0^s \left( (s-s')^{-1/2} (1 \wedge (s-s')^{-1}) + \delta (1 \wedge (s-s')^{-1}) \right) ds' \\
\lesssim \int_0^s (s')^{-1/2} (1 \wedge (s')^{-1}) ds' + t^{1/4} \delta^{1/2} \int_0^s (1 \wedge (s')^{-1-1/4}) ds' \lesssim 1 \wedge s^{-1/4}.
\]

(A.23)

Similarly, from (A.21) we have \( \|h_\delta(s)\|_{L^2(\Lambda_\delta)} \lesssim 1 \wedge s^{-1} \), which implies (A.18) together with (A.23). With respect to (A.19) note first by the convergence of the semigroups in Proposition A.8(ii) above and \( M_{\sigma^2(\delta \bullet)} \to M_{\sigma^2(0)} \) that \( h_\delta(s) \to 2\sigma^2(0)e^{\vartheta(0)s} \Delta z \) in \( L^2(\mathbb{R}^d) \). Moreover, the calculations in (A.23) show that the second line in (A.20) is negligible for \( \delta \to 0 \) and fixed \( s \) and so \( f_\delta(s) = f_\delta^{(1)}(s) + o(\delta) \), where \( f_\delta^{(1)}(s) \) is given by

\[
- \delta 2\sigma^2(0) \sum_{i=1}^d \int_0^s \langle \partial_i \tilde{S}_{\theta, \delta} (s-s') \Delta z, a_i(\delta \bullet) S_{\theta, \delta} (s') (e^{\vartheta(0)s} \Delta z) \rangle_{L^2(\Lambda_\delta)} ds'.
\]

In the same way, since \( a_i(\delta \bullet) S_{\theta, \delta} (s') (v|_{\Lambda_\delta}) \to a_i(0)e^{\vartheta(0)s'} \Delta v \) for \( v \in L^2(\mathbb{R}^d) \) by Proposition A.8(ii), we then have \( f_\delta(s) = f_\delta^{(2)}(s) + o(\delta) \), where

\[
f_\delta^{(2)}(s) = -\delta 2\sigma^2(0) \sum_{i=1}^d a_i(0) \int_0^s \langle \partial_i \tilde{S}_{\theta, \delta} (s-s') \Delta z, e^{\vartheta(0)(s'+s) \Delta z} \rangle_{L^2(\Lambda_\delta)} ds' \\
= \delta 2\sigma^2(0) \sum_{i=1}^d a_i(0) \int_0^s \langle \tilde{S}_{\theta, \delta} (s-s') \Delta z, e^{\vartheta(0)(s'+s) \Delta \partial_i z} \rangle_{L^2(\Lambda_\delta)} ds',
\]

using partial integration and \( \partial_i e^{\vartheta(0)(s'+s) \Delta z} = e^{\vartheta(0)(s'+s) \Delta \partial_i z} \), for \( i = 1, \ldots, d \) in the last line. Noting \( \tilde{S}_{\theta, \delta} (s-s') \Delta z \to e^{\vartheta(0)(s-s') \Delta z} \) in \( L^2(\mathbb{R}^d) \), we conclude that

\[
\delta^{-1} f_\delta(s) \to 2\sigma^2(0) \sum_{i=1}^d a_i(0) \int_0^s \langle e^{\vartheta(0)(s-s') \Delta z}, e^{\vartheta(0)(s'+s) \Delta \partial_i z} \rangle_{L^2(\Lambda_\delta)} ds' \\
= 2\sigma^2(0) \delta \sum_{i=1}^d a_i(0) \langle \Delta z, e^{2\vartheta(0)s} \Delta \partial_i z \rangle_{L^2(\mathbb{R}^d)},
\]

which vanishes according to Lemma A.6. \( \square \)
Lemma A.18. In Proposition A.13 we have $R_2 = o(\delta)$.

Proof. By (2.5) and the scaling in Lemma 3.1 we have

$$R_2 \lesssim \delta^2 \int_0^T \int_0^{\delta^{-2}} \|S^*_\delta(s)v^{(\delta)}\|^2_{L^2(\Lambda_\delta)} ds dt.$$

Recalling from Lemma A.15 that $v^{(\delta)}$ converges in $L^2(\mathbb{R}^d)$ for $\delta \to 0$, applying the semigroup bound in Lemma A.9(i) above for $d \geq 2$ shows

$$R_2 \lesssim \delta^2 \int_0^T \int_0^{\delta^{-2}} 1 \wedge s^{-1} ds dt \lesssim \delta^{3/2} \int_0^T 1 \wedge (s^{-1-1/4}) ds dt = o(\delta).$$

If $\int_{\mathbb{R}^d} z(x) dx = 0$, on the other hand, then $\|v^{(\delta)} - \Delta u\|_{L^2(\Lambda_\delta)} \lesssim \delta$ for a compactly supported $u \in H^2(\mathbb{R}^d)$ by Lemma A.11(ii) above, and so $\delta^{-1} R_2 \to 0$ by Proposition A.12(i). \qed

Proposition A.19. Grant Assumption 3.2 and assume $X_0 = 0$. Let $z, m \in H^2(\mathbb{R}^d)$, $w^{(\delta)}, u^{(\delta)} \in L^2(\mathbb{R}^d)$ have compact support in $\Lambda_\delta$ for $\delta > 0$, such that $\|w^{(\delta)} - \Delta m\|_{L^2(\mathbb{R}^d)} \lesssim \delta$, $\|u^{(\delta)} - u\|_{L^2(\mathbb{R}^d)} \to 0$. Then:

(i) For $d \geq 2$, $\operatorname{Var}\left(\int_0^T \langle X(t), (\Delta z)_\delta \rangle \langle X(t), u^{(\delta)}_\delta \rangle dt\right) = O(\delta^6 \ell_{d,2}(\delta)^3)$.

(ii) $\operatorname{Var}\left(\int_0^T \langle X(t), (\Delta z)_\delta \rangle \langle X(t), w^{(\delta)}_\delta \rangle dt\right) = O(\delta^6 \ell_{d,2}(\delta)^3)$.

(iii) For $d \geq 2$, $\operatorname{Var}\left(\int_0^T \langle X(t), u^{(\delta)}_\delta \rangle^2 dt\right) \lesssim \delta^4 \ell_{d,2}(\delta)^2$.

(iv) Let $d \geq 2$ or $\int_{\mathbb{R}^d} m(x) dx = 0$. Then with $\Psi$ from (3.3)

$$\delta^{-6} \operatorname{Var}\left(\int_0^T \langle X(t), w^{(\delta)}_\delta \rangle^2 dt\right) \to 4 T \vartheta(0)^{-3} \int_0^\infty \Psi(e^{s\Delta} \Delta m, \Delta m)^2 ds.$$

Proof. We first make some preliminary remarks. For $v, \tilde{v} \in L^2(\Lambda_\delta)$ set $\xi(t) = \langle X(t), v_\delta \rangle$, $\tilde{\xi}(t) = \langle X(t), \tilde{v}_\delta \rangle$. The random variables $\{\xi(t) | t \geq 0\} \cup \{\tilde{\xi}(t) | t \geq 0\}$ are jointly Gaussian and centered and so it follows from Wick’s formula (Janson 1997 Theorem 1.28)) that

$$\delta^{-6} \operatorname{Var}\left(\int_0^T \xi(t) \tilde{\xi}(t) dt\right) = \delta^{-6} \int_0^T \int_0^T \operatorname{Cov}\left(\xi(t) \tilde{\xi}(t), \xi(s) \tilde{\xi}(s)\right) dt ds$$

$$= \delta^{-6} \int_0^T \int_0^T (\operatorname{Cov}(\xi(t), \tilde{\xi}(s))^2 + \operatorname{Cov}(\xi(t), \tilde{\xi}(s)) \operatorname{Cov}(\tilde{\xi}(t), \xi(s)) dt ds \quad (A.24)$$

$$=: 2V_1 + 2V_2,$$
These bounds yield in \((A.26)\) for
\[
V(v, \tilde{v}, k, \tilde{k}) := \delta^{-6} \int_0^T \int_0^t \delta^{-2} \left( \delta^{-2} (r - r') , (\delta^{-2} (r - r')) \right) dr'
\]
and \(V(v) := V(v, v, v, v)\). It is thus enough to study \(V_1, V_2\). Set 
\[
f_\delta((s, v), (s', \tilde{v})) = \langle B^*_\delta S^*_{\delta,\delta}(s)v, B^*_\delta S^*_{\delta,\delta}(s')\tilde{v} \rangle_{L^2(\Lambda_\delta)} 1_{\{0 < s, s' \leq T\delta^{-2}\}}, s, s' \geq 0.
\]
Then, by \((2.5)\) and Lemma 3.1, \(V(v, \tilde{v}, k, \tilde{k})\) equals
\[
\delta^{-6} \int_0^T \int_0^t \left( \int_0^r f_\delta \left( (\delta^{-2} (t - r') , (\delta^{-2} (r - r')) \right) dr' \right)
\]
\[
\cdot \left( \int_0^r f_\delta \left( (\delta^{-2} (t - r''), (\delta^{-2} (r - r'')), k) \right) dr'' \right) dt
\]
\[
= \int_0^T \int_0^{t\delta^{-2}} \left( \int_0^{t\delta^{-2} - s} f_\delta \left( (s + s', v), (s', \tilde{v}) \right) ds' \right)
\]
\[
\cdot \left( \int_0^{t\delta^{-2} - s} f_\delta \left( (s + s'', k), (s'', \tilde{k}) \right) ds'' \right) dt,
\]

(A.26)

substituting \(ds' = \delta^{-2}(r - dr')\), \(ds'' = \delta^{-2}(r - dr'')\) and \(ds = \delta^{-2}(t - dr)\). With this we prove now the Proposition.

(i). Let \(v = \Delta z, \tilde{v} = u^{(\delta)}\). By the rescaling with \(\delta^{-6}\) it is enough to show \(V_i = O(\ell_{d,2}(\delta)^3), i = 1, 2\). Observe by Lemma A.9(i,ii) that
\[
\left| f_\delta((s + s'', u^{(\delta)}), (s'', u^{(\delta)})) \right| \lesssim \|S^*_{\delta,\delta}(s + s'')u^{(\delta)}\|_{L^2(\Lambda_\delta)} \|S^*_{\delta,\delta}(s''\delta)u^{(\delta)}\|_{L^2(\Lambda_\delta)}
\]
\[
\lesssim (1 \wedge (s + s'')^{-d/4})(1 \wedge (s'')^{-d/4}) \lesssim 1 \wedge (s'')^{-d/2},
\]
(A.27)

\[
\left| f_\delta ((s + s', \Delta z), (s', \Delta z)) \right| \lesssim \|S^*_{\delta,\delta}(s + s')\Delta z\|_{L^2(\Lambda_\delta)} \|S^*_{\delta,\delta}(s')\Delta z\|_{L^2(\Lambda_\delta)}
\]
\[
\lesssim (1 \wedge (s + s')^{-1/2-d/4})(1 \wedge (s')^{-1/2-d/4}) \lesssim (1 \wedge (s'')^{-1/2-d/4})(1 \wedge (s')^{-1/2-d/4}),
\]
(A.28)

These bounds yield in \((A.26)\) for \(d \geq 2\)
\[
V(\Delta z, \Delta z, u^{(\delta)}, u^{(\delta)}) \lesssim \left( \int_0^{T\delta^{-2}} \left( 1 \wedge (s'')^{-d/2} \right) ds'' \right)
\]
\[
\cdot \left( \int_0^{T\delta^{-2}} \left( 1 \wedge s^{-1/2-d/4} \right) ds \right) \left( \int_0^{T\delta^{-2}} \left( 1 \wedge (s')^{-1/2-d/4} \right) ds' \right) \lesssim \ell_{d,2}(\delta)^3.
\]
(A.29)

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Similarly, \( |\mathcal{J}_\delta((s + s'', \Delta z), (s'', u(\delta)))| \lesssim (1 \wedge s^{d/4})(1 \wedge (s'')^{-1/2-d/4}) \), \( |\mathcal{J}_\delta((s + s', u(\delta)), (s', \Delta z))| \lesssim (1 \wedge s^{-d/4})(1 \wedge (s')^{-1/2-d/4}) \), implying the upper bound \( \ell_d(\delta)^3 \) also for \( V(\Delta z, u(\delta), u(\delta), \Delta z) \). In all, we find \( |V_1|, |V_2| \lesssim \ell_d(\delta)^3 \).

(ii). Let \( v = \Delta z, \tilde{v} = w(\delta) \). As in (i) it is enough to show \( V_i = O(\ell_d(\delta)^3) \), \( i = 1, 2 \). We have by Lemma A.9(ii) for any \( v_1, v_2 \in \{v, \tilde{v}\} \)

\[
|\mathcal{J}_\delta((s + s', v_1), (s', v_2))| \leq (1 \wedge (s + s')^{-1/2-d/4})(1 \wedge (s')^{-1/2-d/4})
\]

\[
\leq (1 \wedge s^{-1/2})(1 \wedge (s')^{-1/2-d/2}).
\]

Therefore, as in (A.29) but this time for all \( d \geq 1 \), \( |V(v_1, v_2, v_3, v_4)| \leq \ell_d(\delta)^3 \), \( v_3, v_4 \in \{v, \tilde{v}\} \). This proves \( V_i = O(\ell_d(\delta)^3) \).

(iii). Let \( v = \tilde{v} = u(\delta) \). The claim is a direct consequence of (A.27) and (A.29) for \( d \geq 2 \), with the \( ds \)-integral of order \( O(\delta^{-2}) \) this time.

(iv). Let \( v = \tilde{v} = w(\delta) \). Since \( V_1 = V_2 = V(w(\delta)) \), it is enough to show

\[
V_1 = V(w(\delta)) \to T\vartheta(0)^{-3} \int_0^\infty \Psi(e^{s\Delta} \Delta m, \Delta m)^2 ds (A.30)
\]

We argue by dominated convergence. Set

\[
f((s, \Delta m), (s', \Delta m)) = \left\langle B_0^* e^{\vartheta(0)s\Delta} \Delta m, B_0^* e^{\vartheta(0)s'\Delta} \Delta m \right\rangle_{L^2(\mathbb{R}^d)}.
\]

Exactly as in the proof of Proposition A.12(i) we get pointwise by polarisation

\[
f_\delta((s + s', w(\delta)), (s', w(\delta))) \to f((s + s', \Delta m), (s', \Delta m))
\]

In order to conclude observe for \( d \geq 2 \) by (A.28) (with \( w(\delta) \) instead of \( \Delta z \)) that

\[
|f_\delta((s + s', w(\delta)), (s', w(\delta)))| \lesssim (1 \wedge s^{-1/2-d/8})(1 \wedge (s')^{-1/2-3d/8}).
\]

If \( \int_{\mathbb{R}^d} m(x)dx = 0 \), the improved bound in Lemma A.9(iii) for (A.28) gives

\[
|f_\delta((s + s', w(\delta)), (s', w(\delta)))| \lesssim (1 \wedge s^{-1/4})(1 \wedge (s')^{-1-d/4}).
\]

In both cases (A.30) follows from dominated convergence, noting

\[
\int_0^\infty f((s, \Delta m), (s', \Delta m))ds' = \vartheta(0)^{-1} \Psi(e^{\vartheta(0)s\Delta} \Delta m, \Delta m).
\]

\[\Box\]

**Proposition A.20.** Grant Assumption 3.2 and assume \( X_0 = 0 \). Let \( z \in H^2(\mathbb{R}^d) \) have compact support in \( \Lambda_\delta \) for \( \delta > 0 \) and let \( d \geq 2 \) or \( \int_{\mathbb{R}^d} z(x)dx = 0 \). Set \( \xi_\delta(t) = (X(t), (\Delta z)_\delta) \). Then we have with \( \Psi \) from (3.3) for \( \delta \to 0 \)

\[
\delta^{-12} \mathbb{E} \left[ \left( \int_0^T (\xi_\delta(t)^2 - \mathbb{E}[\xi_\delta(t)^2])dt \right)^4 \right] \to 3 \left( 4T\vartheta(0)^{-3} \int_0^\infty \Psi(e^{s\Delta} \Delta z, \Delta z)^2 ds \right)^2.
\]

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Proof. In view of Proposition A.19(iv) it is sufficient to show as $\delta \to 0$

$$R_\delta := \mathbb{E} \left[ \left( \int_0^T (\xi_\delta(t))^2 - \mathbb{E}[\xi_\delta(t)^2] \right) dt \right]^4 - 3 \text{Var} \left( \int_0^T \xi_\delta(t)^2 dt \right)^2$$

$$= o \left( \text{Var} \left( \int_0^T \xi_\delta(t)^2 dt \right)^2 \right) = o(\delta^{12}).$$

Abbreviating $c_\delta(t, s) = c((t, (\Delta z)_\delta), (s, (\Delta z)_\delta))$, recall from (A.24) that

$$3 \text{Var} \left( \int_0^T \xi_\delta(t)^2 dt \right)^2 = 3 \left( \int_0^T \int_0^T 2c_\delta(t, s)^2 dt ds \right)^2.$$ 

Wick’s formula (Janson [1997, Theorem 1.28]) for 8th centered Gaussian moments $\mathbb{E}[\prod_{i=1}^8 Z_i] = \sum_{\pi \in \Pi_2(8)} \prod_{(i,j) \in \pi} \mathbb{E}[Z_i Z_j]$ applied to $Z_i = \xi_\delta(t_i)$ for $0 \leq t_i \leq T$, with $\Pi_2(8)$ being the set of all partitions $\pi$ of $\{1, \ldots, 8\}$ into 2-tuples, therefore yields, using the symmetry of the integrand in $(t_1, t_2, t_3, t_4)$,

$$R_\delta = 48 \int_0^T \int_0^T \int_0^T \int_0^T c_\delta(t_1, t_2) c_\delta(t_2, t_3) c_\delta(t_3, t_4) c_\delta(t_4, t_1) dt_1 dt_2 dt_3 dt_4.$$ 

The calculations in the proof of Proposition A.19(iv) show for $s \leq t$, both when $\int_{\mathbb{R}^d} z(x) dx = 0$ and $d \geq 2$, that

$$|c_\delta(t, s)| = \left| \int_0^s f_\delta \left( (\delta^{-2}(t-s'), \Delta z), (\delta^{-2}(s-s'), \Delta z) \right) ds' \right|$$

$$= \delta^2 \left| \int_0^{s \delta^{-2}} f_\delta \left( (\delta^{-2}(t-s) + s', \Delta z), (s', \Delta z) \right) ds' \right|$$

$$\lesssim \delta^2 \left( 1 - (\delta^{-2} t - s)^{-1} \right) \left( 1 - (s')^{-1} \right) ds'$$

$$\lesssim \delta^2 \ell_{d,2}(1 - (\delta^{-2} t - s)^{-1}) =: \tilde{c}_\delta(\delta^{-2}(t-s)),$$

so that as in (A.26), substituting $s_i = \delta^{-2}(t_{i+1} - t_i)$,

$$R_\delta \lesssim \delta^6 \int_0^T \int_0^{T \delta^{-2}} \int_0^{T \delta^{-2}} \int_0^{T \delta^{-2}} \tilde{c}_\delta(s_1) \tilde{c}_\delta(s_2) \tilde{c}_\delta(s_3) \tilde{c}_\delta(s_1 + s_2 + s_3) dt_1 ds_1 ds_2 ds_3$$

$$\lesssim T \delta^{14} \ell_{d,2}^4 \left( \int_0^{T \delta^{-2}} (1 - s^{-1}) ds \right)^3 = o(\delta^{12}).$$

$\square$
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