Research Article

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Multiple solutions and ground state solutions for a class of generalized Kadomtsev-Petviashvili equation

Abstract: In this paper, we study the following generalized Kadomtsev-Petviashvili equation

\[ u_t + u_{xxx} + (h(u))_x = D_x^{-1} \Delta_y u, \]

where \((t, x, y) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^{N-1}, N \geq 2, D_x^{-1}f(x, y) = \int_{-\infty}^{\infty} f(s, y) ds, f_t = \frac{\partial f}{\partial t}, f_x = \frac{\partial f}{\partial x} \text{ and } \Delta_y = \sum_{i=1}^{N-1} \frac{\partial^2}{\partial y_i^2}. \] We get the existence of infinitely many nontrivial solutions under certain assumptions in bounded domain without Ambrosetti-Rabinowitz condition. Moreover, by using the method developed by Jeanjean [13], we establish the existence of ground state solutions in \( \mathbb{R}^N \).

Keywords: generalized Kadomtsev-Petviashvili equation, ground state solutions, multiplicity of solutions

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1 Introduction

This article is concerned with the following generalized Kadomtsev-Petviashvili equation:

\[ u_t + u_{xxx} + (h(u))_x = D_x^{-1} \Delta_y u, \tag{1.1} \]

where \((t, x, y) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^{N-1}, N \geq 2, D_x^{-1}f(x, y) = \int_{-\infty}^{\infty} f(s, y) ds, f_t = \frac{\partial f}{\partial t}, f_x = \frac{\partial f}{\partial x} \text{ and } \Delta_y = \sum_{i=1}^{N-1} \frac{\partial^2}{\partial y_i^2}. \]

To find a solitary wave for (1.1), it needs us to get a solution \( u \) of the form \( u(t, x, y) = u(x - \tau t, y) \), with \( \tau \geq 0 \). Hence, equation (1.1) can be rewritten as:

\[ -\tau u_x + u_{xxx} + (h(u))_x = D_x^{-1} \Delta_y u \quad \text{in } \mathbb{R}^N. \tag{1.2} \]

If we choose \( h(s) = s^3 \) in (1.1), then equation (1.1) is a two-dimensional generalization of the Korteweg-de Vries equation, which describes long dispersive waves in mathematical models, see [1]. When \( h(s) = |s|^p s \) with \( p = \frac{m}{n} \), where \( m \) and \( n \) are relative prime numbers, and \( n \) is odd, Bouard and Saut [2,3] proved that there is a solitary wave for (1.1) with \( 1 \leq p < 4 \), if \( N = 2 \), or \( 1 \leq p < \frac{6}{5} \), if \( N = 3 \), via the concentration com-

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pactness principle from [4,5]. In [6], Willem proved the existence of solitary waves of (1.1) as \( N = 2 \) and \( h \in C^1(\mathbb{R}, \mathbb{R}) \). In [7], Xuan extended the results obtained by [6] to higher dimension. In [8], \( h(u) \) was replaced by \( Q(x, y)|u|^{p-2}u \), and Liang and Su had obtained nontrivial solutions of (1.1). In [9], Xu and Wei studied infinitely many solutions for \( u_{xx} + (h(u))_x = D_x^1\Delta_y u \) with the Ambrosetti-Rabinowitz condition in bounded domain. For related contributions to study of solitary waves of the generalized Kadomtsev-Petviashvili equations, we refer to previous studies [10,11].

The aim of this paper is to prove the existence of multiple solutions of (1.3) in bounded domain without condition (AR), which is to ensure the boundedness of the (PS) sequences of the corresponding functional, and obtain the ground state solutions of (1.2) in \( \mathbb{R}^N \). In what follows, we assume that the function \( h : \mathbb{R} \to \mathbb{R} \) satisfies the following conditions:

\[(h_1) \quad h \in C(\mathbb{R}), h(0) = 0;\]

\[(h_2) \quad \text{for some } p \in (1, N - 1), \text{ where } \tilde{N} = \frac{6N - 2}{2N - 3}, \lim_{t \to +\infty} \frac{h(t)}{|t|^p} = \lim_{t \to 0} \frac{h(t)}{|t|^p} = 0;\]

\[(h_3) \quad h(t) = -h(-t), \lim_{|t| \to +\infty} \frac{|H(t)|}{|t|^p} = +\infty, \text{ where } H(t) = \int_0^t h(r)dr;\]

\[(h_4) \quad \text{there exist } \mu > 2, \kappa > 0 \text{ such that } \mu H(t) \leq th(t) + \kappa t^2;\]

\[(h_5) \quad \text{there exists } \mu > 2 \text{ such that } 0 \leq \mu H(t) \leq h(t)t.\]

Consider the following system,

\[
\begin{aligned}
-\tau u_x + u_{xx} + (h(u))_x &= D_x^1\Delta_y u, \quad \text{in } \Omega, \\
u &= 0, \quad \text{on } \partial \Omega,
\end{aligned}
\]

(1.3)

where \( \Omega \subset \mathbb{R}^N \) is a bounded domain.

Now, we can state our first result.

**Theorem 1.1.** Assume that \((h_1)-(h_5)\) are satisfied, then equation (1.3) possesses infinitely many nontrivial solutions in \( \Omega \), where \( \Omega \subset \mathbb{R}^N \) is a bounded domain.

Our second result is as follows.

**Theorem 1.2.** Assume that \((h_1)-(h_2)\) and \((h_3)\) are satisfied, then equation (1.2) has a ground state solution.

**Notations.** Throughout the paper, we denote by \( \| \cdot \|_p \) the usual norm of Lebesgue space \( L^p(\mathbb{R}^N) \). \( X^* \) is the dual space of \( X \). The symbol \( C \) denotes a positive constant and may vary from line to line.

## 2 Preliminary

In this section, we want to introduce the functional setting and some main results. At first, we present the functional setting (see [7,11]).

**Definition 2.1.** [7] On \( Y = \{ g_x : g \in C^\infty_0(\mathbb{R}^N) \} \), define the inner product

\[
(u, v) = \int_{\mathbb{R}^N} (u_x v_x + D_x^{-1}v_x u D_x^{-1}v_y + \tau uv) \, dV, \quad \tau > 0,
\]

and the norm is

\[
\|u\| = \left( \int_{\mathbb{R}^N} \left( |u_x|^2 + |D_x^{-1}v_x u|^2 + \tau |u|^2 \right) \, dV \right)^{\frac{1}{2}}, \quad \tau > 0.
\]
where \( \nabla_y = \left( \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_{N-1}} \right) \) and \( dV = dx dy \).

If there exists a sequence \( \{u_n\} \subset Y \) such that \( u_n \to u \) a.e. on \( \mathbb{R}^N \), and \( \|u_j - u_k\| \to 0 \) as \( j, k \to \infty \), then we say that \( u : \mathbb{R}^N \to \mathbb{R} \) belongs to \( X \).

**Definition 2.2.** [7] On \( Y = \{g : g \in \mathcal{C}_0^\infty(\mathbb{R}^N)\} \), define the inner product

\[
(u, v)_0 = \int_{\mathbb{R}^N} (u_y v_y + D_x^{-1} \nabla_y u D_x^{-1} \nabla_y v) dV,
\]

and the norm is

\[
\|u\|_0 = \left( \int_{\mathbb{R}^N} (|u_y|^2 + |D_x^{-1} \nabla_y u|^2) dV \right)^{\frac{1}{2}},
\]

where \( \nabla_y = \left( \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_{N-1}} \right) \) and \( dV = dx dy \).

If there exists a sequence \( \{u_n\} \subset Y \) such that \( u_n \to u \) a.e. on \( \mathbb{R}^N \), and \( \|u_j - u_k\| \to 0 \) as \( j, k \to \infty \), then we say that \( u : \mathbb{R}^N \to \mathbb{R} \) belongs to \( X_0 \).

**Lemma 2.1.** [7,11,12] The following continuous embeddings hold.

(i) the embeddings \( X \hookrightarrow X_0 \) are continuous;
(ii) the embeddings \( X \hookrightarrow \mathcal{L}^q(\mathbb{R}^N) \), for \( 1 \leq q \leq N \) are continuous;
(iii) the embeddings \( X \hookrightarrow \mathcal{L}_{loc}^q(\mathbb{R}^N) \), for \( 1 \leq q < N \) are compact;
(iv) the embeddings \( X_0 \hookrightarrow \mathcal{L}_{loc}^N(\mathbb{R}^N) \) are continuous.

**Lemma 2.2.** [14] Let \( X \) be an infinite dimensional Banach space, and there exists a finite dimensional space \( W \) such that \( X = W \oplus V \). \( I \in \mathcal{C}(\mathbb{R}) \) satisfies the (PS) condition, and

(i) \( I(u) = I(-u) \) for all \( u \in X \), \( I(0) = 0 \);
(ii) there exist \( \rho > 0 \), \( \alpha > 0 \) such that \( I|_{B_{\rho}} \cap V \geq \alpha \);
(iii) for any finite dimensional subspace \( Y \subset X \), there is \( R = R(Y) > 0 \) such that \( I(u) \leq 0 \) on \( Y \setminus B_R \).

Then \( I \) possesses an unbounded sequence of critical values.

**Lemma 2.3.** [7] Assume that \( \{u_n\} \) is a bounded sequence in \( X \). If

\[
\lim_{n \to +\infty} \sup_{(x,y) \in \mathbb{R}^N} \int_{B_r((x,y))} |u_n|^2 dV = 0,
\]

then \( u_n \to 0 \) in \( \mathcal{L}^q(\mathbb{R}^N) \) for all \( q \in (2, \hat{N}) \).

**Lemma 2.4.** [13] Let \( (X, \|\cdot\|) \) be a Banach space and \( T \subset \mathbb{R}^\ast \) be an interval. Consider a family of \( \mathcal{C}^1 \) functionals on \( X \) of the form

\[
I_\lambda(u) = A(u) - \lambda B(u) \quad \forall \lambda \in T,
\]

with \( B(u) \geq 0 \) and either \( A(u) \to +\infty \) or \( B(u) \to +\infty \) as \( |u| \to +\infty \). If there are two points \( v_1, v_2 \in X \) such that

\[
c_\lambda = \inf_{y \in \Gamma} \max_{t \in [0,1]} I_\lambda(y(t)) > \max\{I_\lambda(v_1), I_\lambda(v_2)\} \quad \forall \lambda \in T,
\]

where

\[
\Gamma = \{y \in C([0,1], X) : y(0) = v_1, y(1) = v_2\}.
\]

Then, for almost every \( \lambda \in T \), there exists a bounded (PS)\(_c\) sequence in \( X \), and the mapping \( \lambda \to c_\lambda \) is non-increasing and left continuous.
3 Proof of Theorem 1.1

In this section, we consider the boundary value problem (1.3). The energy functional $I : X \to \mathbb{R}$ given by

$$I(u) = \frac{1}{2} \|u\|^2 - \int_{\Omega} H(u) \, dV$$

and

$$I'(u)[v] = \int_{\Omega} (v \nabla u - \nabla u D \nabla v + \tau u v) \, dV - \int_{\Omega} h(u) v \, dV.$$ 

Lemma 3.1. Suppose $h$ satisfies $(h_1)$–$(h_4)$. If $\{u_n\} \subset X$ satisfies

(i) $\{I(u_n)\}$ is bounded;
(ii) $\langle I'(u_n), u_n \rangle \to 0$,

then $\{u_n\}$ is bounded in $X$.

Proof. If $\{u_n\}$ is unbounded in $X$, we can find a subsequence still denoted by $\{u_n\}$ such that $\|u_n\| \to +\infty$. Let $v_n = \frac{u_n}{\|u_n\|}$, we have $\|v_n\| = 1$. Thus, we may assume that $v_n \to v$ in $X$. As the embedding $X \hookrightarrow L^2(\Omega)$ is compact, we have $v_n \to v$ in $L^2(\Omega)$. By $(h_2)$ and (i), there exists $c > 0$ such that

$$c + 1 \geq I(u_n) - \frac{1}{\mu} \langle I'(u_n), u_n \rangle \geq \frac{\mu - 2}{2\mu} \|u_n\|^2 - \frac{\kappa}{\mu} \|u_n\|^2 = \frac{\mu - 2}{2\mu} \|u_n\|^2 - \frac{\kappa}{\mu} \|u_n\|^2 \|v_n\|^2,$$

as $n \to +\infty$, which implies $1 \leq \frac{2\kappa}{\mu} \lim_{n \to +\infty} \|v_n\|^2$. Therefore, $v \neq 0$. By $(h_4)$ and Fatou’s Lemma, one has

$$0 = \lim_{n \to +\infty} \frac{c}{\|u_n\|^2} = \lim_{n \to +\infty} \frac{I(u_n)}{\|u_n\|^2} = \lim_{n \to +\infty} \left( \frac{1}{2} - \int_{\Omega} \frac{H(u_n)}{u_n^2} v_n^2 \right) = -\infty,$$

which is a contradiction. Hence, $\{u_n\}$ is bounded in $X$. \qed

Lemma 3.2. Suppose $h$ satisfies $(h_1)$–$(h_4)$. Then the functional $I$ satisfies the (PS) condition.

Proof. To prove that $I$ satisfies the (PS) condition, we only need to prove $\{u_n\} \subset X$ has a convergent subsequence, where $\{u_n\}$ obtained by Lemma 3.1. As $\{u_n\}$ is bounded in $X$, there exists a subsequence still denoted by $\{u_n\}$ and $u_0 \in X$ such that $u_n \to u_0$ in $X$ and $u_n \to u_0$ in $L^q(\Omega)$ for $1 \leq q < \tilde{N}$. From $(h_2)$, we have

$$h(u_n) \leq \varepsilon |u_n|^p + C \varepsilon |u_n|^p, \quad \forall \varepsilon > 0.$$ 

then

$$\left( \int_{\Omega} |h(u_n)|^{\frac{p+1}{p}} \, dV \right)^{\frac{p}{p+1}} \leq \left( \int_{\Omega} (\varepsilon |u_n|^p + C \varepsilon |u_n|^p)^{\frac{p+1}{p}} \, dV \right)^{\frac{p}{p+1}} \leq C (\|u_n\| + \|u_n\|^p) < +\infty.$$ 

Applying the Hölder inequality, for $1 < p < \tilde{N} - 1$, one has

$$\int_{\Omega} (h(u_n) - h(u_0))(u_n - u_0) \, dV \leq \left( \int_{\Omega} |h(u_n) - h(u_0)|^{\frac{p+1}{p}} \, dV \right)^{\frac{p}{p+1}} \left( \int_{\Omega} |u_n - u_0|^{p+1} \, dV \right)^{\frac{1}{p+1}} \leq C \left( \int_{\Omega} (|h(u_n)|^{\frac{p+1}{p}} + |h(u_0)|^{\frac{p+1}{p}}) \, dV \right)^{\frac{p}{p+1}} \left( \int_{\Omega} |u_n - u_0|^{p+1} \, dV \right)^{\frac{1}{p+1}} \to 0.$$
It follows from $u_n \to u_0$ in $X$ and $I'(u_0) \in X^*$ that $\langle I'(u_0), u_n - u_0 \rangle \to 0$. And as $I'(u_n) \to 0$ in $X^*$, it is easy to obtain
$$\langle I'(u_n), u_n - u_0 \rangle \leq \|I'(u_n)\|_{X^*} \|u_n - u_0\|_{X(\Omega)} \to 0.$$ 
Therefore,
$$\langle I'(u_n), u_n - u_0 \rangle - \langle I'(u_0), u_n - u_0 \rangle - \langle I'(u_0), u_n - u_0 \rangle \to 0,$$ 
as $n \to +\infty$.

Thus, we have
$$\|u_n - u_0\|^2 = \langle I'(u_n) - I'(u_0), u_n - u_0 \rangle + \int_\Omega ((h(u_0) - h(u_n))(u_n - u_0))dV \to 0,$$ 
as $n \to +\infty$. \hfill \qed

**Proof of Theorem 1.1.** We have verified that $I$ satisfies the (PS) condition. It follows from $(h_5)$ that $I$ is an even function. As $X$ is a separable space, $X$ has orthonormal basis \{ $e_i$\}. Define $X_i = \mathbb{R}^i$, $W_k = \oplus_{j=1}^{k} X_j$, $V_k = \overline{\oplus_{j=k+1}^{\infty} X_j}$. Let $W = W_k$, $V = V_k$, clearly $X = W \oplus V$ and $\dim W < \infty$.

Next, we verify that $I$ satisfies (ii) in Lemmas 2.2. By Lemma 2.1, for all $\lambda \in V$, we have
$$I(u) = \frac{1}{2}\|u\|^2 - \int_\Omega H(u)dV \geq \frac{1}{2}\|u\|^2 - \left( \frac{c}{2}\|u\|_2^2 + \frac{C_p}{p+1}\|u\|_{p+1} \right) \geq \frac{1}{2}\|u\|^2 - C(\|u\|^2 + C_p\|u\|^{p+1}).$$

Then, there exists $\rho > 0$ small enough, $a > 0$ such that $I(u) \geq a > 0$ as $\|u\| = \rho$.

Now, we verify that $I$ satisfies (iii) in Lemma 2.2. For any finite dimensional subspace $Y \subset X$, since
$$I(\lambda u) = \frac{r^2}{2}\|u\|^2 - \int_\Omega H(\lambda u)dV = \frac{r^2}{2}\|u\|^2 - \frac{2}{\lambda} \int_\Omega \frac{H(\lambda u)}{(\lambda u)^2} u^2 dV \to -\infty,$$
as $r \to +\infty$. Thus, there exists $r_0 > 0$ such that $I(\lambda u) < 0$ for all $\lambda \geq r_0 > 0$. So, we can conclude that there exists a $R(Y) > 0$ such that $I(u) \leq 0$ on $Y \setminus R(Y)$.

Hence, according to Lemma 2.2, equation (1.3) possesses infinitely many nontrivial solutions. \hfill \qed

### 4 Proof of Theorem 1.2

In this section, the weak solutions of (1.2) are the critical points of the energy functional $I$, where $I(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^n} H(u)dV$. As $h$ satisfies $(h_1)$–$(h_2)$ and $(h_5)$, it is clear that $I$ is of class $C'(X, \mathbb{R})$. To apply Jeanjean’s trick \cite{Jeanjean}, we give a family of energy functions
$$I_\lambda(u) = \frac{1}{2}\|u\|^2 - \lambda \int_{\mathbb{R}^n} H(u)dV, \quad \forall \lambda \in \left[ \frac{1}{2}, 1 \right].$$

**Lemma 4.1.** Suppose that $h$ satisfies $(h_1)$–$(h_2)$ and $(h_5)$. Then

(i) there exists $v \in X \setminus \{ 0 \}$ such that $I_\lambda(v) < 0$ for all $\lambda \in \left[ \frac{1}{2}, 1 \right]$;

(ii) $c_\lambda = \inf_{\gamma \in \Gamma} \max_{t \in [0, 1]} I_\lambda(y(t)) > \max \{ I_\lambda(0), I_\lambda(v) \}$ for all $\lambda \in \left[ \frac{1}{2}, 1 \right]$, where
$$\Gamma = \{ y \in C([0, 1], X) : y(0) = 0, \ y(1) = v \}.$$
Proof. (i) By \((h_2)\), we have \(\lim_{t \to +\infty} \frac{H(t)}{p^*} = +\infty\). Furthermore, for some \(u \in X\)
\[
I_t(u) = \frac{t^2}{2} \|u\|^2 - \lambda \int_{\mathbb{R}^N} H(tu) \, dV \leq \frac{t^2}{2} \|u\|^2 - \int_{\mathbb{R}^N} \frac{H(tu)}{(tu)^2} \, u^2 \, dV \to -\infty \quad \text{as} \ t \to +\infty.
\]
Thus, there exists \(t_0 > 0\) such that \(I_t(t_0u) < 0\). By taking \(v = t_0u\), we have \(I_t(v) < 0\).

(ii) By virtue of \((h_2)\), for any \(\varepsilon > 0\) and some \(\lambda \in (\lambda_1, \bar{\lambda}_1)\), there exists \(\rho > 0\) such that
\[
|v| \leq \varepsilon |t|^p + C_\varepsilon |t|^{p+1} \quad \forall t \in \mathbb{R}.
\]
By Lemma 2.3, we have
\[
I_t(u) = \frac{1}{2} \|u\|^2 - \lambda \int_{\mathbb{R}^N} H(u) \, dV \geq \frac{1}{2} \|u\|^2 - \left(\frac{E}{2} \|u\|^2 + \frac{C_\varepsilon}{p} \|u\|^{p+1}\right) \geq \frac{1}{2} \|u\|^2 - C(\varepsilon |u|^{2} + C_\varepsilon |u|^{p+1}).
\]
Then, there exists \(\rho > 0\) small enough such that
\[
b = \inf_{|u| < \rho} I_t(u) > 0 = I_t(0) > I_t(v).
\]
Therefore, \(c_\lambda > \max\{I_t(0), I_t(v)\}\).

Combining Lemma 4.1 with Theorem 2.6, we have the following conclusion.

Lemma 4.2. Suppose \(h\) satisfies \((h_1)-(h_2)\) and \((h_3)\). For almost every \(\lambda \in \left[\frac{1}{2}, 1\right]\), there is a bounded sequence \(\{v_m\}\), such that \(I_t(v_m) \to c_\lambda \) in \(X\) and \(I'_\lambda(v_m) \to 0\) in the dual \(\star X\) of \(X\).

Lemma 4.3. If \(\{v_m\}\) is a bounded sequence in \(X\) and \(\lim_{m \to +\infty} \sup_{(x,y) \in \mathbb{R}^N} \int_{B_{(x,y)}} |v_m|^2 \, dV = 0\), then \(\lim_{m \to +\infty} \int_{\mathbb{R}^N} G(v_m) \, dV = 0\), where \(G(v_m) = \frac{1}{2} h(v_m) v_m - H(v_m)\).

Proof. On one hand, by simple calculations, we derive
\[
\int_{\mathbb{R}^N} H(v_m) \, dV \leq \frac{E}{2} \|v_m\|^2 + \frac{C_\varepsilon}{p+1} \|v_m\|^{p+1},
\]
\[
\int_{\mathbb{R}^N} h(v_m) v_m \, dV \leq \varepsilon \|v_m\|^2 + C_\varepsilon \|v_m\|^{p+1}.
\]
On the other hand, by Lemma 2.3, we have \(v_m \to 0\) in \(L^2(\mathbb{R}^N)\) for all \(q \in (2, \bar{N})\). Hence, we can conclude that
\[
\lim_{m \to +\infty} \int_{\mathbb{R}^N} H(v_m) \, dV = 0,
\]
\[
\lim_{m \to +\infty} \int_{\mathbb{R}^N} h(v_m) v_m \, dV = 0.
\]
Thus, \(\lim_{m \to +\infty} \int_{\mathbb{R}^N} G(v_m) \, dV = 0\).

Lemma 4.4. If \(\{v_m\} \subset X\) is the sequence obtained by Lemma 4.2, then for a.e. \(\lambda \in \left[\frac{1}{2}, 1\right]\), there exists a sequence of points \((x_m, y_m) \in \mathbb{R} \times \mathbb{R}^{\bar{N}-1}\), \(u_m(x, y) = v_m(x - x_m, y - y_m)\), such that
(i) \(u_m \to u_\lambda \neq 0\) in \(X\);
(ii) \(I'_\lambda(u_\lambda) = 0\) in \(\mathbb{R}^*\);
(iii) \(I_\lambda(u_\lambda) \leq c_\lambda\) in \(X\); and
(iv) there exists \(M > 0\) such that \(I_\lambda(u_\lambda) \geq M\).
Proof. By Lemma 4.2, we know that for almost every $\lambda \in \left[\frac{1}{2}, 1\right]$, there exists a bounded sequence $\{v_m\}$ that satisfy $I_\lambda(v_m) \to c_\lambda$ in $X$ and $I_\lambda'(v_m) \to 0$ in $X^*$ as $m \to +\infty$. Furthermore,

$$\int_{R^N} G(v_m) = I_\lambda(v_m) - \frac{1}{2} \langle I_\lambda'(v_m), v_m\rangle \to c_\lambda > 0 \quad \text{as} \quad m \to +\infty.$$ 

By Lemma 4.3, there exist a sequence of points $\{(x_m, y_m)\} \subset R \times R^N$ and $\alpha > 0$, such that

$$\int_{B_\lambda(x_m, \alpha)} v_m^2 dV \geq \alpha > 0.$$ 

Let $u_m(x, y) = v_m(x - x_m, y - y_m)$. By the invariance translations of $I_\lambda$, as $m \to +\infty$, we have that $I_\lambda(u_m) \to c_\lambda$ in $X$ and $I_\lambda'(u_m) \to 0$ in $X^*$. Since $\{u_m\}$ is bounded, there exists $u_1 \in X$ such that $u_m \to u_1$ in $X$.

In the following, we complete the proof of this lemma.

(i) It follows from Lemma 2.1 that

$$\int \frac{\|u_1\|^2}{\|u_1\|^2} \geq \int_{B_1(0)} u_1^2 dV = \lim_{m \to +\infty} \int_{B_1(0)} u_m^2 dV \geq \alpha > 0,$$

and thus obtain $u_1 \neq 0$ in $X$.

(ii) As $C_0^\infty(R^N)$ is dense in $X$, we only need to check that $(I_\lambda'(u_1), \varphi) = 0$ for any $\varphi \in X$. We have

$$\langle I_\lambda'(u_m), \varphi \rangle - \langle I_\lambda'(u_1), \varphi \rangle = (u_m - u_1, \varphi) - \lambda \int_{R^N} [h(u_m) - h(u_1)] \varphi dV \to 0,$$

since $u_m \to u_1$ in $X$, $u_m \to u_1$ in $L^p_{\text{loc}}(R^N)$ for $1 \leq p \leq N$. It follows from $I_\lambda'(u_m) \to 0$ that $I_\lambda'(u_1) = 0$.

(iii) By $(h_3)$ and Fatou’s Lemma, we get

$$c_\lambda = \lim_{m \to +\infty} \left[ I_\lambda(u_m) - \frac{1}{2} \langle I_\lambda'(u_m), u_m\rangle \right] = \lambda \lim_{m \to +\infty} \int_{R^N} G(u_m) dV$$

$$\geq \lambda \int_{R^N} G(u_1) dV = I_\lambda(u_1) - \frac{1}{2} \langle I_\lambda'(u_1), u_1\rangle = I_\lambda(u_1).$$

(iv) Combining (ii) with $(h_3)$ and Lemma 2.1, we obtain that for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\|u_1\|^2 = \lambda \int_{R^N} h(u_1) u_1 dV \leq \int_{R^N} h(u_1) u_1 dV \leq C_\varepsilon \|u_1\|^2 + C_\varepsilon \|u_1\|^{p+1}.$$ 

Then, there exists $\beta > 0$ such that $\|u_1\| \geq \beta > 0$. Therefore,

$$I_\lambda(u_1) = I_\lambda(u_1) - \frac{1}{\mu} \langle I_\lambda'(u_1), u_1\rangle$$

$$= \left(\frac{1}{2} - \frac{1}{\mu}\right) \|u_1\|^2 + \int_{R^N} \frac{1}{\mu} h(u_1) u_1 - H(u_1) dV$$

$$\geq \left(\frac{1}{2} - \frac{1}{\mu}\right) \|u_1\|^2 \geq \left(\frac{1}{2} - \frac{1}{\mu}\right) \beta^2 = M > 0.$$ 

This completes the proof. \qed

Now, according to Lemmas 4.2 and 4.4, there exists a sequence $\{(\lambda_n, u_{\lambda_n})\} \subset \left[\frac{1}{2}, 1\right] \times X$, such that

(i) $\lambda_n \to 1$ as $n \to +\infty$;  
(ii) $u_{\lambda_n} \neq 0$, $M \leq I_{\lambda_n}(u_{\lambda_n}) \leq c_{\lambda_n}$ and $I_{\lambda_n}'(u_{\lambda_n}) = 0$.

Lemma 4.5. (Pohozaev identity, [7]) Suppose $h$ satisfies $(h_1)$–$(h_3)$. If $u \in X$ is a weak solution of the equation:

$$-\tau u + u_{xxx} + \lambda h(u) = D_x^4 \Delta u \quad \text{in} \quad R^N,$$
then we have the following Pohozaev identity:

\[ P_\lambda(u) := \frac{2N - 3}{2} \|u\|_2^2 + (2N - 1) \int_{\mathbb{R}^N} \left( \frac{\tau}{2} u^2 - \lambda H(u) \right) dV = 0. \]

**Proof of Theorem 1.2.** By Lemma 4.5, if \( \{u_{\lambda_n}\} \) is nontrivial solution of equation

\[-\tau u_{\xi} + u_{xxx} + \lambda_n(h(u))_x = D_x^3 \Delta_p u \quad \text{in} \quad \mathbb{R}^N,\]

then \( \{u_{\lambda_n}\} \) satisfies the following equation:

\[ P_{\lambda_n}(u_{\lambda_n}) = \frac{2N - 3}{2} \|u_{\lambda_n}\|_2^2 + (2N - 1) \frac{\tau}{2} \int_{\mathbb{R}^N} u_{\lambda_n}^2 dV - (2N - 1) \lambda_n \int_{\mathbb{R}^N} H(u_{\lambda_n}) dV = 0. \]

Remember that

\[ c_{\lambda_n} \geq I_{\lambda_n}(u_{\lambda_n}) - \frac{1}{2N - 1} P_{\lambda_n}(u_{\lambda_n}) = \frac{1}{2N - 1} \|u_{\lambda_n}\|_2^2. \]

So,

\[ \|u_{\lambda_n}\|_2^2 \leq (2N - 1) c_{\lambda_n} \leq (2N - 1) c_2, \]

it follows from Lemma 2.1 that \( \{u_{\lambda_n}\} \) is bounded in \( X_0 \) and also in \( L^{\bar{N}} \).

Since \( I'_{\lambda_n}(u_{\lambda_n}) = 0 \), we have

\[ \langle I'_{\lambda_n}(u_{\lambda_n}), u_{\lambda_n} \rangle = \|u_{\lambda_n}\|_2^2 - \lambda_n \int_{\mathbb{R}^N} h(u_{\lambda_n}) u_{\lambda_n} dV = 0. \]

Moreover, by Lemma 2.1, for any \( \varepsilon > 0 \), there exists \( C_\varepsilon > 0 \) such that

\[ \|u_{\lambda_n}\|_2^2 = \lambda_n \int_{\mathbb{R}^N} h(u_{\lambda_n}) u_{\lambda_n} dV \leq \varepsilon \|u_{\lambda_n}\|_2^2 + C_\varepsilon \|u_{\lambda_n}\|_{\bar{N}}^2, \]

Then, for \( \varepsilon \) small enough, there exists a constant \( C > 0 \) such that \( \|u_{\lambda_n}\|_2 \leq C \), since \( \{u_{\lambda_n}\} \) is bounded in \( L^{\bar{N}} \).

Thus, \( \{u_{\lambda_n}\} \) is bounded in \( X \). By the facts that for any \( \varphi \in X \),

\[ \langle I'(u_{\lambda_n}), \varphi \rangle = \langle I'_{\lambda_n}(u_{\lambda_n}), \varphi \rangle + (\lambda_n - 1) \int_{\mathbb{R}^N} h(u_{\lambda_n}) \varphi dV, \]

\[ I(u_{\lambda_n}) = I_{\lambda_n}(u_{\lambda_n}) + (\lambda_n - 1) \int_{\mathbb{R}^N} H(u_{\lambda_n}) dV, \]

and \( \{u_{\lambda_n}\} \) is bounded in \( X \), it follows that \( M \leq \lim_{n \to \infty} I(u_{\lambda_n}) \leq c_1 \) and \( \lim_{n \to \infty} I'(u_{\lambda_n}) = 0 \). Up to a subsequence, there exists a subsequence still denoted by \( \{u_{\lambda_n}\} \) and \( u_0 \in X \) such that \( u_{\lambda_n} \rightharpoonup u_0 \) in \( X \). By using the method in Lemma 4.4, we can obtain the existence of a nontrivial solution \( u_0 \) for \( I \) such that \( I'(u_0) = 0 \) and \( I(u_0) \leq c_1 \). Thus, \( u_0 \) is a nontrivial solution of (1.2). Define \( m = \inf \{I(u) : u \neq 0, I'(u) = 0\} \). Let \( \{u_0\} \) be a sequence such that \( I'(u_0) = 0 \) and \( I(u_0) \to m \). Similar to arguments in Lemma 4.4, we can prove that there exists \( \bar{u} \in X \) such that \( I'(\bar{u}) = 0 \) and \( I(\bar{u}) \leq m \). By the definition of \( m \), we have \( m \leq I(\bar{u}) \). Hence, \( I(\bar{u}) = m \), which shows that \( \bar{u} \) is a ground state solution of (1.2). \( \square \)

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References

1. B. B. Kadomtsev and V. I. Petviashvili, *On the stability of solitary waves in weakly dispersing media*, Sov. Phys. Dokl. 15 (1970), 539–541.
2. A. D. Bouard and J. C. Saut, *Sur les ondes solitaires des équations de Kadomtsev-Petviashvili*, C. R. Acad. Sci. Paris Sér. I Math. 320 (1995), no. 3, 315–318.
3. A. D. Bouard and J. C. Saut, *Solitary waves of generalized Kadomtsev-Petviashvili equations*. Ann. Inst. H Poincaré Anal. Non Linéaire 14 (1997), no. 2, 211–236, DOI: https://doi.org/10.1016/S0294-1449(97)80145-X.
4. P. L. Lions, *The concentration-compactness principle in the calculus of variations. The locally compact case I.*, Ann. Inst. H. Poincaré Anal. Non Linéaire 1 (1984), no. 2, 109–145.
5. P. L. Lions, *The concentration-compactness principle in the calculus of variations. The locally compact case II*, Ann. Inst. H. Poincaré Anal. Non Linéaire 1 (1984), no. 4, 223–283.
6. M. Willem, *Minimax Theorems*. Progress in Nonlinear Differential Equations and their Applications, Birkhäuser Boston Inc., Boston, MA, 1996.
7. Z. Liang and J. Su, *Existence of solitary waves to a generalized Kadomtsev-Petviashvili equation*, Acta Math. Sci. Ser. B (Engl. Ed.). 32 (2012), no. 3, 1149–1156, DOI: https://doi.org/10.1016/S0252-9602(12)60087-3.
8. J. Xu, Z. Wei, and Y. Ding, *Stationary solutions for a generalized Kadomtsev-Petviashvili equation in bounded domain*, Electron. J. Qual. Theory Differ. Equ. (2012), no. 68, DOI: https://doi.org/10.14232/ejqtde.2012.1.68.
9. W. Zou, *Solitary waves of the generalized Kadomtsev-Petviashvili equations*. Appl. Math. Lett. 15 (2002), no. 1, 35–39, DOI: https://doi.org/10.1016/S0893-9659(01)00089-1.
10. C. O. Alves, O. H. Miyagaki, and A. Pomponio, *Solitary waves for a class of generalized Kadomtsev-Petviashvili equation in \( \mathbb{R}^N \) with positive and zero mass*, J. Math. Anal. Appl. 477 (2019), no. 1, 523–535, DOI: https://doi.org/10.1016/j.jmaa.2019.04.044.
11. O. V. Besov, V. P. Il’in, and S. M. Nikol’skii (eds), *Integral Representations of Functions and Imbedding Theorems, Vol. I*, V. H. Winston & Sons, Washington, D.C.; Halsted Press [John Wiley & Sons], New York-Toronto, Ont.-London, 1978.
12. L. Jeanjean, *On the existence of bounded Palais-Smale sequences and application to a Landesman-Lazer type problem set on \( \mathbb{R}^N \)*, Proc. Roy. Soc. Edinburgh Sect. A. 129 (1999), no. 4, 787–809, DOI: https://doi.org/10.1017/S0308210500013147.
13. P. H. Rabinowitz, *Minimax Methods in Critical Point Theory with Applications to Differential Equations*, Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1986.