SELF-ADJOINT EXTENSIONS OF DIFFERENTIAL OPERATORS ON RIEmannian manIfolds

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abstract. We study \( H = D^* D + V \), where \( D \) is a first order elliptic differential operator acting on sections of a Hermitian vector bundle over a Riemannian manifold \( M \), and \( V \) is a Hermitian bundle endomorphism. In the case when \( M \) is geodesically complete, we establish the essential self-adjointness of positive integer powers of \( H \). In the case when \( M \) is not necessarily geodesically complete, we give a sufficient condition for the essential self-adjointness of \( H \), expressed in terms of the behavior of \( V \) relative to the Cauchy boundary of \( M \).

1. Introduction

As a fundamental problem in mathematical physics, self-adjointness of Schrödinger operators has attracted the attention of researchers over many years now, resulting in numerous sufficient conditions for this property in \( L^2(\mathbb{R}^n) \). For reviews of the corresponding results, see, for instance, the books [14, 29].

The study of the corresponding problem in the context of a non-compact Riemannian manifold was initiated by Gaffney [15, 16] with the proof of the essential self-adjointness of the Laplacian on differential forms. About two decades later, Cordes (see Theorem 3 in [11]) proved the essential self-adjointness of positive integer powers of the operator

\[
\Delta_{M,\mu} := \frac{1}{\kappa} \left( \frac{\partial}{\partial x^i} \left( \kappa g^{ij} \frac{\partial}{\partial x^j} \right) \right)
\]

for all \( k \in \mathbb{Z}_+ \), where \( M \) is a compact geodesically complete Riemannian manifold equipped with a (smooth) metric \( g = (g_{ij}) \) and a positive smooth measure \( d\mu \) (i.e. in any local coordinates \( x^1, x^2, \ldots, x^n \) there exists a strictly positive \( C^\infty \)-density \( \kappa(x) \) such that \( d\mu = \kappa(x) dx^1 dx^2 \cdots dx^n \)). Theorem 1 of our paper extends this result to the operator \( (D^* D + V)^k \) for all \( k \in \mathbb{Z}_+ \), where \( D \) is a first order elliptic differential operator acting on sections of a Hermitian vector bundle over a geodesically complete Riemannian manifold, \( D^* \) is the formal adjoint of \( D \), and \( V \) is a self-adjoint Hermitian bundle endomorphism; see Section 2.3 for details.

In the context of a general Riemannian manifold (not necessarily geodesically complete), Cordes (see Theorem IV.1.1 in [12] and Theorem 4 in [11]) proved the essential self-adjointness of \( P^k \) for all \( k \in \mathbb{Z}_+ \), where

\[
P u := \Delta_{M,\mu} u + qu, \quad u \in C^\infty(M),
\]

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and \( q \in C^\infty(M) \) is real-valued. Thanks to a Roelcke-type estimate (see Lemma 3.1 below), the technique of Cordes [12] can be applied to the operator \((D^*D + V)^k\) acting on sections of Hermitian vector bundles over a general Riemannian manifold. To make our exposition shorter, in Theorem 1 we consider the geodesically complete case. Our Theorem 2 concerns \((\nabla^*\nabla + V)^k\), where \( \nabla \) is a metric connection on a Hermitian vector bundle over a non-compact geodesically complete Riemannian manifold. This result extends Theorem 1.1 of [13] where Cordes showed that if \((M,g)\) is non-compact and geodesically complete and \( P \) is semi-bounded from below on \( C^\infty_c(M) \), then \( P^k \) is essentially self-adjoint on \( C^\infty_c(M) \), for all \( k \in \mathbb{Z}_+ \).

For the remainder of the introduction, the notation \( D^*D + V \) is used in the same sense as described earlier in this section. In the setting of geodesically complete Riemannian manifolds, the essential self-adjointness of \( D^*D + V \) with \( V \in L^\infty_{\text{loc}} \) was established in [21], providing a generalization of the results in [3, 27, 28, 32] concerning Schrödinger operators on functions (or differential forms). Subsequently, the operator \( D^*D + V \) with a singular potential \( V \) was considered in [5]. Recently, in the case \( V \in L^\infty_{\text{loc}} \), the authors of [4] extended the main result of [5] to the operator \( D^*D + V \) acting on sections of infinite-dimensional bundles whose fibers are modules of finite type over a von Neumann algebra.

In the context of an incomplete Riemannian manifold, the authors of [17, 22, 24] studied the so-called Gaffney Laplacian, a self-adjoint realization of the scalar Laplacian generally different from the closure of \( \Delta_M,d\mu|_{C^\infty_c(M)} \). For a study of Gaffney Laplacian on differential forms, see [24]. Our Theorem 3 gives a condition on the behavior of \( V \) relative to the Cauchy boundary of \( M \) that will guarantee the essential self-adjointness of \( D^*D + V \); for details see Section 2.4 below. Related results can be found in [6, 25, 26] in the context of (magnetic) Schrödinger operators on domains in \( \mathbb{R}^n \), and in [10] concerning the magnetic Laplacian on domains in \( \mathbb{R}^n \) and certain types of Riemannian manifolds.

Finally, let us mention that Chernoff [7] used the hyperbolic equation approach to establish the essential self-adjointness of positive integer powers of Laplace–Beltrami operator on differential forms. This approach was also applied in [2, 8, 9, 18, 19, 31] to prove essential self-adjointness of second-order operators (acting on scalar functions or sections of Hermitian vector bundles) on Riemannian manifolds. Additionally, the authors of [18, 19] used path integral techniques.

The paper is organized as follows. The main results are stated in Section 2, a preliminary lemma is proven in Section 3, and the main results are proven in Sections 4–6.

2. Main Results

2.1. The setting. Let \( M \) be an \( n \)-dimensional smooth, connected Riemannian manifold without boundary. We denote the Riemannian metric on \( M \) by \( g^{TM} \). We assume that \( M \) is equipped with a positive smooth measure \( d\mu \), i.e. in any local coordinates \( x^1, x^2, \ldots, x^n \) there exists a strictly positive \( C^\infty \)-density \( \kappa(x) \) such that \( d\mu = \kappa(x)\,dx^1dx^2\ldots dx^n \). Let \( E \) be a Hermitian vector bundle over \( M \) and let \( L^2(E) \) denote the Hilbert space of square integrable sections of \( E \) with respect to the inner product

\[
(u,v) = \int_M \langle u(x),v(x) \rangle_{E_x} \,d\mu(x),
\] (2.1)
where $\langle \cdot, \cdot \rangle_{E_x}$ is the fiberwise inner product. The corresponding norm in $L^2(E)$ is denoted by $\| \cdot \|$. In Sobolev space notations $W^{k,2}_{\text{loc}}(E)$ used in this paper, the superscript $k \in \mathbb{Z}_+$ indicates the order of the highest derivative. The corresponding dual space is denoted by $W^{-k,2}_{\text{loc}}(E)$.

Let $F$ be another Hermitian vector bundle on $M$. We consider a first order differential operator $D : C^\infty_c(E) \to C^\infty_c(F)$, where $C^\infty_c$ stands for the space of smooth compactly supported sections. In the sequel, by $\sigma(D)$ we denote the principal symbol of $D$.

**Assumption (A0)** Assume that $D$ is elliptic. Additionally, assume that there exists a constant $\lambda_0 > 0$ such that

$$|\sigma(D)(x, \xi)| \leq \lambda_0 |\xi|, \quad \text{for all } x \in M, \xi \in T^*_x M,$$

(2.2)

where $|\xi|$ is the length of $\xi$ induced by the metric $g_{TM}$ and $|\sigma(D)(x, \xi)|$ is the operator norm of $\sigma(D)(x, \xi) : E_x \to F_x$.

**Remark 2.2.** Assumption (A0) is satisfied if $D = \nabla$, where $\nabla : C^\infty(E) \to C^\infty(T^*M \otimes E)$ is a covariant derivative corresponding to a metric connection on a Hermitian vector bundle $E$ over $M$.

2.3. **Schrödinger-type Operator.** Let $D^* : C^\infty_c(F) \to C^\infty_c(E)$ be the formal adjoint of $D$ with respect to the inner product (2.1). We consider the operator

$$H = D^* D + V,$$

(2.3)

where $V \in L^\infty_{\text{loc}}(\text{End } E)$ is a linear self-adjoint bundle endomorphism. In other words, for all $x \in M$, the operator $V(x) : E_x \to E_x$ is self-adjoint and $|V(x)| \in L^\infty_{\text{loc}}(M)$, where $|V(x)|$ is the norm of the operator $V(x) : E_x \to E_x$.

2.4. **Statements of Results.**

**Theorem 1.** Let $M, g_{TM}$, and $d\mu$ be as in Section 2.2. Assume that $(M, g_{TM})$ is geodesically complete. Let $E$ and $F$ be Hermitian vector bundles over $M$, and let $D : C^\infty_c(E) \to C^\infty_c(F)$ be a first order differential operator satisfying the assumption (A0). Assume that $V \in C^\infty(\text{End } E)$ and

$$V(x) \geq C, \quad \text{for all } x \in M,$$

where $C$ is a constant, and the inequality is understood in operator sense. Then $H^k$ is essentially self-adjoint on $C^\infty_c(E)$, for all $k \in \mathbb{Z}_+$.

**Remark 2.5.** In the case $V = 0$, the following result related to Theorem 1 can be deduced from Chernoff (see Theorem 2.2 in [7]):

Assume that $(M, g)$ is a geodesically complete Riemannian manifold with metric $g$. Let $D$ be as in Theorem 1 and define

$$c(x) := \sup\{ |\sigma(D)(x, \xi)| : |\xi|_{T^*_x M} = 1 \}.$$

Fix $x_0 \in M$ and define

$$c(r) := \sup_{x \in B(x_0, r)} c(x),$$

where $B(x_0, r)$ is the ball of radius $r$ centered at $x_0$. 
where \( r > 0 \) and \( B(x_0, r) := \{ x \in M : d_g(x_0, x) < r \} \). Assume that
\[
\int_0^\infty \frac{1}{c(r)} \, dr = \infty.
\] (2.4)

Then the operator \((D^*D)^k\) is essentially self-adjoint on \( C_c^\infty(E) \) for all \( k \in \mathbb{Z}_+ \).

At the end of this section we give an example of an operator for which Theorem 1 guarantees the essential self-adjointness of \((D^*D)^k\), whereas Chernoff’s result cannot be applied.

The next theorem is concerned with operators whose potential \( V \) is not necessarily semi-bounded from below.

**Theorem 2.** Let \( M, g^TM, \) and \( d\mu \) be as in Section 2.1. Assume that \((M, g^TM)\) is noncompact and geodesically complete. Let \( E \) be a Hermitian vector bundle over \( M \) and let \( \nabla \) be a Hermitian connection on \( E \). Assume that \( V \in C^\infty(\text{End} \, E) \) and
\[
V(x) \geq q(x), \quad \text{for all} \; x \in M,
\] (2.5)
where \( q \in C^\infty(M) \) and the inequality is understood in the sense of operators \( E_x \to E_x \). Additionally, assume that
\[
((\Delta_{M,\mu} + q)u, u) \geq C\|u\|^2, \quad \text{for all} \; u \in C_c^\infty(M),
\] (2.6)
where \( C \in \mathbb{R} \) and \( \Delta_{M,\mu} \) is as in (1.1) with \( g \) replaced by \( g^TM \). Then the operator \((\nabla^* \nabla + V)^k\) is essentially self-adjoint on \( C_c^\infty(E) \), for all \( k \in \mathbb{Z}_+ \).

**Remark 2.6.** Let us stress that non-compactness is required in the proof to ensure the existence of a positive smooth solution of an equation involving \( \Delta_{M,\mu} + q \). In the case of a compact manifold, such a solution exists under an additional assumption; see Theorem III.6.3 in [12].

In our last result we will need the notion of Cauchy boundary. Let \( d_{g^TM} \) be the distance function corresponding to the metric \( g^TM \). Let \( (\hat{M}, \hat{d}_{g^TM}) \) be the metric completion of \((M, d_{g^TM})\). We define the **Cauchy boundary** \( \partial_C M \) as follows: \( \partial_C M := \hat{M} \setminus M \). Note that \((M, d_{g^TM})\) is metrically complete if and only if \( \partial_C M \) is empty. For \( x \in M \) we define
\[
r(x) := \inf_{z \in \partial_C M} \hat{d}_{g^TM}(x, z).
\] (2.7)

We will also need the following assumption:

**Assumption (A1)** Assume that \( \hat{M} \) is a smooth manifold and that the metric \( g^TM \) extends to \( \partial_C M \).

**Remark 2.7.** Let \( N \) be a (smooth) \( n \)-dimensional Riemannian manifold without boundary. Denote the metric on \( N \) by \( g^{TN} \) and assume that \((N, g^{TN})\) is geodesically complete. Let \( \Sigma \) be a \( k \)-dimensional closed sub-manifold of \( N \) with \( k < n \). Then \( M := N \setminus \Sigma \) has the properties \( \hat{M} = N \) and \( \partial_C M = \Sigma \). Thus, assumption (A1) is satisfied.
Theorem 3. Let $M$, $g^TM$, and $d\mu$ be as in Section 2.1. Assume that (A1) is satisfied. Let $E$ and $F$ be Hermitian vector bundles over $M$, and let $D: C^\infty_c(E) \to C^\infty_c(F)$ be a first order differential operator satisfying the assumption (A0). Assume that $V \in L^\infty_{\text{loc}}(\text{End } E)$ and there exists a constant $C$ such that
\[ V(x) \geq \left( \frac{\lambda_0}{r(x)} \right)^2 - C, \quad \text{for all } x \in M, \] (2.8)
where $\lambda_0$ is as in (2.2), the distance $r(x)$ is as in (2.7), and the inequality is understood in the sense of linear operators $E_x \to E_x$. Then $H$ is essentially self-adjoint on $C^\infty_c(E)$.

In order to describe the example mentioned in Remark 2.5, we need the following

Remark 2.8. As explained in [5], we can use a first-order elliptic operator $D: C^\infty_c(E) \to C^\infty_c(F)$ to define a metric on $M$. For $\xi, \eta \in T^*_xM$, define
\[ \langle \xi, \eta \rangle = \frac{1}{m} \text{Re} \, \text{Tr} \left( (\sigma(D)(x,\xi))^* \sigma(D)(x,\eta) \right), \quad m = \dim E_x, \] (2.9)
where $\text{Tr}$ denotes the usual trace of a linear operator. Since $D$ is an elliptic first-order differential operator and $\sigma(D)(x,\xi)$ is linear in $\xi$, it is easily checked that (2.9) defines an inner product on $T^*_xM$. Its dual defines a Riemannian metric on $M$. Denoting this metric by $g^TM$ and using elementary linear algebra, it follows that (2.2) is satisfied with $\lambda_0 = \sqrt{m}$.

Example 2.9. Let $M = \mathbb{R}^2$ with the standard metric and measure, and $V = 0$. Denoting respectively by $C^\infty_c(\mathbb{R}^2; \mathbb{R})$ and $C^\infty_c(\mathbb{R}^2; \mathbb{R}^2)$ the spaces of smooth compactly supported functions $f: \mathbb{R}^2 \to \mathbb{R}$ and $f: \mathbb{R}^2 \to \mathbb{R}^2$, we define the operator $D: C^\infty_c(\mathbb{R}^2; \mathbb{R}) \to C^\infty_c(\mathbb{R}^2; \mathbb{R}^2)$ by
\[ D = \begin{pmatrix} a(x,y) \frac{\partial}{\partial x} \\ b(x,y) \frac{\partial}{\partial y} \end{pmatrix}, \]
where
\[ a(x,y) = (1 - \cos(2\pi e^x))x^2 + 1; \]
\[ b(x,y) = (1 - \sin(2\pi e^y))y^2 + 1. \]
Since $a, b$ are smooth real-valued nowhere vanishing functions in $\mathbb{R}^2$, it follows that the operator $D$ is elliptic. We are interested in the operator
\[ H := D^*D = - \frac{\partial}{\partial x} \left( a^2 \frac{\partial}{\partial x} \right) - \frac{\partial}{\partial y} \left( b^2 \frac{\partial}{\partial y} \right). \]
The matrix of the inner product on $T^*M$ defined by $D$ via (2.9) is $\text{diag}(a^2/2, b^2/2)$. The matrix of the corresponding Riemannian metric $g^TM$ on $M$ is $\text{diag}(2a^{-2}, 2b^{-2})$, so the metric itself is $ds^2 = 2a^{-2}dx^2 + 2b^{-2}dy^2$ and it is geodesically complete (see Example 3.1 of [5]). Moreover, thanks to Remark 2.8 assumption (A0) is satisfied. Thus, by Theorem 1 the operator $(D^*D)^k$ is essentially self-adjoint for all $k \in \mathbb{Z}_+$. Furthermore, in Example 3.1 of [5] it was shown that for the considered operator $D$ the condition (2.4) is not satisfied. Thus, the result stated in Remark 2.5 does not apply.
3. Roelcke-type Inequality

Let $M$, $d\mu$, $D$, and $\sigma(D)$ be as in Section 2.1. Set $\hat{D} := -i\sigma(D)$, where $i = \sqrt{-1}$. Then for any Lipschitz function $\psi: M \to \mathbb{R}$ and $u \in W^{1,2}_{\text{loc}}(E)$ we have

$$D(\psi u) = \hat{D}(\psi u) + \psi Du,$$

where we have suppressed $x$ for simplicity. We also note that $\hat{D}^*(\xi) = -(\hat{D}(\xi))^*$, for all $\xi \in T^*_x M$.

For a compact set $K \subset M$, and $u, v \in W^{1,2}_{\text{loc}}(E)$, we define

$$(u, v)_K := \int_K \langle u(x), v(x) \rangle \, d\mu(x), \quad (Du, Dv)_K := \int_K \langle Du(x), Dv(x) \rangle \, d\mu(x).$$

(3.2)

In order to prove Theorem 1 we need the following important lemma, which is an extension of Lemma 2.1 in [12] to operator (2.3). In the context of the scalar Laplacian on a Riemannian manifold, this kind of result is originally due to Roelcke [30].

**Lemma 3.1.** Let $M$, $g^TM$, and $d\mu$ be as in Section 2.1. Let $E$ and $F$ be Hermitian vector bundles over $M$, and let $D: C^\infty_c(E) \to C^\infty_c(F)$ be a first order differential operator satisfying the assumption (A0). Let $\rho: M \to [0, \infty)$ be a function satisfying the following properties:

(i) $\rho(x)$ is Lipschitz continuous with respect to the distance induced by the metric $g^TM$;
(ii) $\rho(x_0) = 0$, for some fixed $x_0 \in M$;
(iii) the set $B_T := \{ x \in M : \rho(x) \leq T \}$ is compact, for some $T > 0$.

Then the following inequality holds for all $u \in W^{2,2}_{\text{loc}}(E)$ and $v \in W^{2,2}_{\text{loc}}(E)$:

$$\int_0^T |(Du, Dv)_{B_t} - (D^*Du, v)_{B_t}| \, dt \leq \lambda_0 \int_{B_T} |d\rho(x)||Du(x)||v(x)| \, d\mu(x),$$

(3.3)

where $B_t$ is as in (iii) (with $t$ instead of $T$), the constant $\lambda_0$ is as in (2.2), and $|d\rho(x)|$ is the length of $d\rho(x) \in T^*_x M$ induced by $g^TM$.

**Proof.** For $\varepsilon > 0$ and $t \in (0, T)$, we define a continuous piecewise linear function $F_{\varepsilon,t}$ as follows:

$$F_{\varepsilon,t}(s) = \begin{cases} 1 & \text{for } s < t - \varepsilon \\
(t-s)/\varepsilon & \text{for } t - \varepsilon \leq s < t \\
0 & \text{for } s \geq t \end{cases}$$

The function $f_{\varepsilon,t}(x) := F_{\varepsilon,t}(\rho(x))$, is Lipschitz continuous with respect to the distance induced by the metric $g^TM$, and $d(f_{\varepsilon,t}(\rho(x))) = (F'_{\varepsilon,t}(\rho(x)))d\rho(x)$. Moreover we have $f_{\varepsilon,t}v \in W^{1,2}_{\text{loc}}(E)$ for all $v \in W^{1,2}_{\text{loc}}(E)$, since

$$D(f_{\varepsilon,t}v) = \hat{D}(df_{\varepsilon,t})v + f_{\varepsilon,t}Dv.$$  

It follows from the compactness of $B_T$ that $B_t$ is compact for all $t \in (0, T)$. Using integration by parts (see Lemma 8.8 in [5]), for all $u \in W^{2,2}_{\text{loc}}(E)$ and $v \in W^{2,2}_{\text{loc}}(E)$ we have

$$(D^*Du, vf_{\varepsilon,t})_{B_t} = (Du, D(vf_{\varepsilon,t}))_{B_t} = (Du, f_{\varepsilon,t}Dv)_{B_t} + (Du, \hat{D}(df_{\varepsilon,t})v)_{B_t},$$

(6)

where $d\mu = d\mu_{\text{flat}}$, $\hat{D}$ is the lifting of $D$, $df_{\varepsilon,t}$ is a differential form of degree $1$, and $\hat{D}(df_{\varepsilon,t})$ is a differential operator of degree $2$.
where \( \chi \) almost everywhere, where \( \epsilon \) C hand side of (3.5) an easy calculation shows that

\[
|I_{\epsilon}(x)| = \lambda \iint |Dv| \iint |D\rho| \iint |v(x)| d\mu(x),
\]

where \( |df_{\epsilon,t}(x)| \) and \( |d\rho(x)| \) are the norms of \( df_{\epsilon,t}(x) \in T_x^* M \) and \( d\rho(x) \in T_x^* M \) induced by \( g^{T,M} \).

Fixing \( \epsilon > 0 \), integrating the leftmost and the rightmost side of (3.4) from \( t = 0 \) to \( t = T \), and noting that \( F_{\epsilon,t}(\rho(x)) \) is the only term on the rightmost side depending on \( t \), we obtain

\[
\int_0^T |(Du, f_{\epsilon,t} Dv)_{B_t} - (D^* Du, v_{f_{\epsilon,t}})_{B_t}| dt
\]

\[
\leq \lambda \iint |Dv| \iint |d\rho| \iint |v(x)| I_{\epsilon}(x) d\mu(x),
\]

where

\[
I_{\epsilon}(x) := \int_0^T |F_{\epsilon,t}(\rho(x))| dt.
\]

We now let \( \epsilon \to 0+ \) in (3.5). On the left-hand side of (3.5), as \( \epsilon \to 0+ \), we have \( f_{\epsilon,t}(x) \to \chi_{B_t}(x) \) almost everywhere, where \( \chi_{B_t}(x) \) is the characteristic function of the set \( B_t \). Additionally, \( |f_{\epsilon,t}(x)| \leq 1 \) for all \( x \in B_t \) and all \( t \in (0,T) \); thus, by dominated convergence theorem, as \( \epsilon \to 0+ \) the left-hand side of (3.5) converges to the left-hand side of (3.3). On the right-hand side of (3.5) an easy calculation shows that \( I_{\epsilon}(x) \to 1 \), as \( \epsilon \to 0+ \). Additionally, we have \( |I_{\epsilon}(x)| \leq 1 \), a.e. on \( B_T \); hence, by the dominated convergence theorem, as \( \epsilon \to 0+ \) the right-hand side of (3.5) converges to the right-hand side of (3.3). This establishes the inequality (3.3). \( \square \)

4. Proof of Theorem 1

We first give the definitions of minimal and maximal operators associated with the expression \( H \) in (2.3).

4.1. Minimal and Maximal Operators. We define \( H_{\min} u := H u \), with Dom\((H_{\min}) := C^{\infty}_c(E) \), and \( H_{\max} := (H_{\min})^* \), where \( T^* \) denotes the adjoint of operator \( T \). Denoting \( \mathcal{D}_{\max} := \{ u \in L^2(E): Hu \in L^2(E) \} \), we recall the following well-known property: Dom\((H_{\max}) = \mathcal{D}_{\max} \) and \( H_{\max} u = Hu \) for all \( u \in \mathcal{D}_{\max} \).

From now on, throughout this section, we assume that the hypotheses of Theorem 1 are satisfied. Let \( x_0 \in M \), and define \( \rho(x) := d_{g^{T,M}}(x_0,x) \), where \( d_{g^{T,M}} \) is the distance function corresponding to the metric \( g^{T,M} \). By the definition of \( \rho(x) \) and the geodesic completeness of \( (M, g^{T,M}) \), it follows that \( \rho(x) \) satisfies all hypotheses of Lemma 3.1. Using Lemma 3.1 and Proposition 4.2 below, we are able to apply the method of Cordes [11, 12] to our context. As we
will see, Cordes's technique reduces our problem to a system of ordinary differential inequalities of the same type as in Section IV.3 of [12].

**Proposition 4.2.** Let $A$ be a densely defined operator with domain $\mathcal{D}$ in a Hilbert space $\mathcal{H}$. Assume that $A$ is semi-bounded from below, that $A\mathcal{D} \subseteq \mathcal{D}$, and that there exists $c_0 \in \mathbb{R}$ such that the following two properties hold:

(i) $(A + c_0 I)u, u)_{\mathcal{H}} \geq \|u\|_{\mathcal{H}}^2$, for all $u \in \mathcal{D}$, where $I$ denotes the identity operator in $\mathcal{H}$;

(ii) $(A + c_0 I)^k$ is essentially self-adjoint on $\mathcal{D}$, for some $k \in \mathbb{Z}_+$.

Then, $(A + cI)^j$ is essentially self-adjoint on $\mathcal{D}$, for all $j = 1, 2, \ldots, k$ and all $c \in \mathbb{R}$.

**Remark 4.3.** To prove Proposition 4.2, one may mimick the proof of Proposition 1.4 in [12], which was carried out for the operator $P$ defined in (1.2) with $\mathcal{D} = C_c^\infty(M)$, since only abstract functional analysis facts and the property $P\mathcal{D} \subseteq \mathcal{D}$ were used.

We start the proof of Theorem 1 by noticing that the operator $H_{\min}$ is essentially self-adjoint on $C_c^\infty(E)$; see Corollary 2.9 in [5]. Thanks to Proposition 4.2, whithout any loss of generality we can change $V(x)$ to $V(x) + C \text{Id}(x)$, where $C$ is a sufficiently large constant in order to have

$$V(x) \geq (\lambda_0^2 + 1)\text{Id}(x), \quad \text{for all } x \in M,$$

where $\lambda_0$ is as in (2.2) and $\text{Id}(x)$ is the identity endomorphism of $E_x$. Using non-negativity of $D^*D$ and (4.1) we have

$$(H_{\min}u, u) \geq \|u\|^2, \quad \text{for all } u \in C_c^\infty(E),$$

which leads to

$$\|u\|^2 \leq (H_{\min}u, u) \leq \|Hu\|\|u\|, \quad \text{for all } u \in C_c^\infty(E),$$

and, hence, $\|Hu\| \geq \|u\|$, for all $u \in C_c^\infty(E)$. Therefore,

$$(H^2u, u) = (Hu, Hu) = \|Hu\|^2 \geq \|u\|^2, \quad \text{for all } u \in C_c^\infty(E),$$

and

$$(H^3u, u) = (HHu, Hu) \geq \|Hu\|^2 \geq \|u\|^2, \quad \text{for all } u \in C_c^\infty(E).$$

By (4.3) we have

$$\|u\|^2 \leq (H^2u, u) \leq \|H^2u\|\|u\|, \quad \text{for all } u \in C_c^\infty(E),$$

and, hence, $\|H^2u\| \geq \|u\|$, for all $u \in C_c^\infty(E)$. This, in turn, leads to

$$(H^4u, u) = (H^2u, H^2u) = \|H^2u\|^2 \geq \|u\|^2, \quad \text{for all } u \in C_c^\infty(E).$$

Continuing like this, we obtain $(H^ku, u) \geq \|u\|^2$, for all $u \in C_c^\infty(E)$ and all $k \in \mathbb{Z}_+$. In this case, by an abstract fact (see Theorem X.26 in [29]), the essential self-adjointness of $H^k$ on $C_c^\infty(E)$ is equivalent to the following statement: if $u \in L^2(E)$ satisfies $H^ku = 0$, then $u = 0$.

Let $u \in L^2(E)$ satisfy $H^ku = 0$. Since $V \in C_c^\infty(E)$, by local elliptic regularity it follows that $u \in C_c^\infty(E) \cap L^2(E)$. Define

$$f_j := H^{k-j}u, \quad j = 0, \pm 1, \pm 2, \ldots$$
Here, in the case $k - j < 0$, the definition (4.4) is interpreted as $((H_{\max})^{-1})^{j-k}$. We already noted that $H_{\min}$ is essentially self-adjoint and positive. Furthermore, it is well known that the self-adjoint closure of $H_{\min}$ coincides with $H_{\max}$. Therefore $H_{\max}$ is a positive self-adjoint operator, and $(H_{\max})^{-1}: L^2(E) \to L^2(E)$ is bounded. This, together with $f_k = u \in L^2(E)$ explains the following property: $f_j \in L^2(E)$, for all $j \geq k$. Additionally, observe that $f_j = 0$ for all $j \leq 0$ because $f_0 = 0$. Furthermore, we note that $f_j \in C^\infty(E)$, for all $j \in \mathbb{Z}$. The last assertion is obvious for $j \leq k$, and for $j > k$ it can be seen by showing that $H^2 f_j = 0$ in distributional sense and using $f_j \in L^2(E)$ together with local elliptic regularity. To see this, let $v \in C_c^\infty(E)$ be arbitrary, and note that

\[ (f_j, H^2 v) = (H^{k-j} u, H^j v) = (u, H^k v) = (H^k u, v) = 0. \]

Finally, observe that

\[ H^i f_j = f_{j-l}, \quad \text{for all } j \in \mathbb{Z} \text{ and } l \in \mathbb{Z}_+ \cup \{0\}. \]  

With $f_j$ as in (4.4), define the functions $\alpha_j$ and $\beta_j$ on the interval $0 \leq T < \infty$ by the formulas

\[ \alpha_j(T) := \lambda_0^2 \int_0^T (f_j, f_j)_{B_t} \, dt, \quad \beta_j(T) := \int_0^T (Df_j, Df_j)_{B_t} \, dt, \]

where $\lambda_0$ is as in (4.1) and $(\cdot, \cdot)_{B_t}$ is as in (3.2).

In the sequel, to simplify the notations, the functions $\alpha_j(T)$ and $\beta_j(T)$, the inner products $(\cdot, \cdot)_{B_t}$, and the corresponding norms $\| \cdot \|_{B_t}$ appearing in (4.6) will be denoted by $\alpha_j$, $\beta_j$, $(\cdot, \cdot)_t$, and $\| \cdot \|_t$, respectively.

Note that $\alpha_j$ and $\beta_j$ are absolutely continuous on $[0, \infty)$. Furthermore, $\alpha_j$ and $\beta_j$ have a left first derivative and a right first derivative at each point. Additionally, $\alpha_j$ and $\beta_j$ are differentiable, except at (at most) countably many points. In the sequel, to simplify notations, we shall denote the right first derivatives of $\alpha_j$ and $\beta_j$ by $\alpha_j'$ and $\beta_j'$. Note that $\alpha_j$, $\beta_j$, $\alpha_j'$ and $\beta_j'$ are non-decreasing and non-negative functions. Note also that $\alpha_j$ and $\beta_j$ are convex functions. Furthermore, since $f_j = 0$ for all $j \leq 0$, it follows that $\alpha_j \equiv 0$ and $\beta_j \equiv 0$ for all $j \leq 0$. Finally, using (4.1) and the property $f_j \in L^2(E) \cap C^\infty(E)$ for all $j \geq k$, observe that

\[ \lambda_0^2 (f_j, f_j) + (Df_j, Df_j) \leq (V f_j, f_j) + (Df_j, Df_j) = (f_j, H f_j) = (f_j, f_{j-1}) < \infty, \]

for all $j > k$. Here, “integration by parts” in the first equality is justified because $H_{\min}$ is essentially self-adjoint (i.e. $C_c^\infty(E)$ is an operator core of $H_{\max}$). Hence, $\alpha_j'$ and $\beta_j'$ are bounded for all $j > k$. It turns out that $\alpha_j$ and $\beta_j$ satisfy a system of differential inequalities, as seen in the next proposition.

**Proposition 4.4.** Let $\alpha_j$ and $\beta_j$ be as in (4.6). Then, for all $j \geq 1$ and all $T \geq 0$ we have

\[ \alpha_j + \beta_j \leq \sqrt{\alpha_j' \beta_j'} + \sum_{l=0}^{\infty} \left( \sqrt{\alpha_j' \beta_{j+l}'} + \sqrt{\alpha_j' \beta_{j-l}'} \right) \]

and

\[ \alpha_j \leq \lambda_0^2 \left( \sum_{l=0}^{\infty} \left( \sqrt{\alpha_j' \beta_{j+l}'} + \sqrt{\alpha_j' \beta_{j-l}'} \right) \right), \]
where $\lambda_0$ is as in (4.1) and $\alpha'_i$, $\beta'_i$ denote the right-hand derivatives.

**Remark 4.5.** Note that the sums in (4.7) and (4.8) are finite since $\alpha_i \equiv 0$ and $\beta_i \equiv 0$ for $i \leq 0$. As our goal is to show that $f_k = u = 0$, we will only use the first $k$ inequalities in (4.7) and the first $k$ inequalities in (4.8).

**Proof of Proposition 4.4.** From (4.6) and (4.1) it follows that

$$
\alpha_j + \beta_j \leq \int_0^T ((f_j, V f_j)_t + (D f_j, D f_j)_t) \, dt. \tag{4.9}
$$

We start from (4.9), use (3.3), Cauchy–Schwarz inequality, and (4.5) to obtain

$$
\alpha_j + \beta_j \leq \int_0^T ((f_j, V f_j)_t + (D f_j, D f_j)_t) \, dt
= \int_0^T |(f_j, H f_j)_t - (f_j, D^* D f_j)_t + (D f_j, D f_j)_t| \, dt
\leq \lambda_0 \int_{B_T} |D f_j(x)||f_j(x)| \, d\mu(x) + \int_0^T |(f_j, H f_j)_t| \, dt
\leq \sqrt{\alpha'_j \beta'_j} + \int_0^T |(H f_{j+1}, f_{j-1})_t| \, dt.
$$

We continue the process as follows:

$$
\alpha_j + \beta_j \leq \sqrt{\alpha'_j \beta'_j} + \int_0^T |(H f_{j+1}, f_{j-1})_t| \, dt
= \sqrt{\alpha'_j \beta'_j} + \int_0^T |(D^* D f_{j+1}, f_{j-1})_t + (f_{j+1}, V f_{j-1})_t| \, dt
\leq \sqrt{\alpha'_j \beta'_j} + \int_0^T |(D^* D f_{j+1}, f_{j-1})_t - (D f_{j+1}, D f_{j-1})_t| \, dt
+ \int_0^T |(D f_{j+1}, D f_{j-1})_t - (f_{j+1}, D^* D f_{j-1})_t| \, dt + \int_0^T |(f_{j+1}, H f_{j-1})_t| \, dt
\leq \sqrt{\alpha'_j \beta'_j} + \sqrt{\alpha'_{j+1} \beta'_{j-1}} + \sqrt{\alpha'_{j-1} \beta'_{j+1}} + \int_0^T |(H f_{j+2}, f_{j-2})_t| \, dt,
$$

where we used triangle inequality, (3.3), Cauchy–Schwarz inequality, and (4.5). We continue like this until the last term reaches the subscript $j - l \leq 0$, which makes the last term equal zero by properties of $f_i$ discussed above. This establishes (4.7).
To show (4.8), we start from the definition of \( \alpha_j \), use (3.3), Cauchy–Schwarz inequality, and (4.5) to obtain
\[
\alpha_j = \lambda_0^2 \int_0^T (f_j, f_j)_t \, dt = \lambda_0^2 \int_0^T |(f_j, Hf_{j+1})_t| \, dt \\
= \lambda_0^2 \int_0^T |(f_j, D^*Df_{j+1})_t + (Vf_j, f_{j+1})_t| \, dt \\
\leq \lambda_0^2 \int_0^T |(f_j, D^*Df_{j+1})_t - (Df_j, Df_{j+1})_t| \, dt \\
+ \lambda_0^2 \int_0^T |(Df_j, Df_{j+1})_t| \, dt + \lambda_0^2 \int_0^T |(Hf_j, f_{j+1})_t| \, dt \\
\leq \lambda_0^2 \left( \sqrt{\alpha_{j+1}'\beta_j'} + \sqrt{\alpha_j'\beta_{j+1}'} \right) + \lambda_0^2 \int_0^T |(f_{j-1}, f_{j+1})_t| \, dt.
\]
We continue like this until the last term reaches the subscript \( j - l \leq 0 \), which makes the last term equal zero by properties of \( f_i \) discussed above. This establishes (4.8).

\[\Box\]

End of the proof of Theorem 1. We will now transform the system (4.7)–(4.8) by introducing new variables:
\[
\omega_j(T) := \alpha_j(T) + \beta_j(T), \quad \theta_j(T) := \alpha_j(T) - \beta_j(T) \quad T \in [0, \infty).
\]
To carry out the transformation, observe that Cauchy–Schwarz inequality applied to vectors \( \langle \sqrt{\alpha'_i}, \sqrt{\beta'_i} \rangle \) and \( \langle \sqrt{\beta'_p}, \sqrt{\alpha'_p} \rangle \) in \( \mathbb{R}^2 \) gives
\[
\sqrt{\alpha'_i\beta'_p} + \sqrt{\alpha'_p\beta'_i} \leq \sqrt{\omega'_i\omega'_p},
\]
which, together with (4.7)–(4.8) leads to
\[
\omega_j \leq \frac{1}{2} \sqrt{\omega'_j)^2 - (\theta'_j)^2} + \sum_{l=0}^{\infty} \sqrt{\omega'_{j+l+1}\omega'_{j-l}} \tag{4.11}
\]
and
\[
\frac{1}{2}(\omega_j + \theta_j) \leq \lambda_0^2 \left( \sum_{l=0}^{\infty} \sqrt{\omega'_{j+l+1}\omega'_{j-l}} \right), \tag{4.12}
\]
where \( \lambda_0 \) is as in (4.1) and \( \omega'_j, \theta'_j \) denote the right-hand derivatives.

The functions \( \omega_j \) and \( \theta_j \) satisfy the following properties: (i) \( \omega_j \) and \( \theta_j \) are absolutely continuous on \([0, \infty)\), and the right-hand derivatives \( \omega'_j \) and \( \theta'_j \) exist everywhere; (ii) \( \omega_j \) and \( \omega'_j \) are non-negative and non-increasing; (iii) \( \omega_j \) is convex; (iv) \( \omega'_j \) is bounded for all \( j \geq k \); (v) \( \omega_j(0) = \theta_j(0) = 0 \); and (vi) \( |\theta_j(T)| \leq \omega_j(T) \) and \( |\theta'_j(T)| \leq \omega'_j(T) \) for all \( T \in [0, \infty) \).

In Section IV.3 of [12] it was shown that if \( \omega_j \) and \( \theta_j \) are functions satisfying the above described properties (i)–(vi) and the system (4.11)–(4.12), then \( \omega_j \equiv 0 \) for all \( j = 1, 2, \ldots, k \). In particular, we have \( \omega_k(T) = 0 \), for all \( T \in [0, \infty) \), and hence \( f_k = 0 \). Going back to (4.4), we get \( u = 0 \), and this concludes the proof of essential self-adjointness of \( H^k \) on \( C_0^\infty(E) \). The essential self-adjointness of \( H^2, H^3, \ldots, \) and \( H^{k-1} \) on \( C_0^\infty(E) \) follows by Proposition 4.2. \[\Box\]
5. Proof of Theorem 2

We adapt the proof of Theorem 1.1 in [13] to our type of operator. By assumption (2.4) it follows that

\[(\Delta_{M,\mu} + q - C + 1)u, u) \geq \|u\|^2, \quad \text{for all } u \in C_c^\infty(M). \quad (5.1)\]

Since (5.1) is satisfied and since \(M\) is non-compact and \(g^{TM}\) is geodesically complete, a result of Agmon [1] (see also Proposition III.6.2 in [12]) guarantees the existence of a function \(\gamma \in C^\infty(M)\) such that \(\gamma(x) > 0\) for all \(x \in M\), and

\[(\Delta_{M,\mu} + q - C + 1)\gamma = \gamma. \quad (5.2)\]

We now use the function \(\gamma\) to transform the operator \(H = \nabla^* \nabla + V\). Let \(L^2_{\mu_1}(E)\) be the space of square integrable sections of \(E\) with inner product \((\cdot, \cdot)_{\mu_1}\) as in (2.1), where \(d\mu\) is replaced by \(d\mu_1 := \gamma^2 d\mu\). For clarity, we denote \(L^2(\mu)\) from Section 2.1 by \(L^2_{\mu}(E)\). In what follows, the formal adjoints of \(\nabla\) with respect to inner products \((\cdot, \cdot)_{\mu}\) and \((\cdot, \cdot)_{\mu_1}\) will be denoted by \(\nabla^*\) and \(\nabla^*\mu_1\), respectively. It is easy to check that the map \(T_\gamma: L^2_{\mu}(E) \to L^2_{\mu_1}(E)\) defined by \(Tu := \gamma^{-1}u\) is unitary. Furthermore, under the change of variables \(u \mapsto \gamma^{-1}u\), the differential expression \(H = \nabla^* \nabla + V\) gets transformed into \(H_1 := \gamma^{-1}H\gamma\). Since \(T\) is unitary, the essential self-adjointness of \(H_1^{k}\) in \(L^2_{\mu}(E)\) is equivalent to essential self-adjointness of \((H_1)^{k}\) in \(L^2_{\mu_1}(E)\).

In the sequel, we will show that \(H_1\) has the following form:

\[H_1 = \nabla^* \mu_1 \nabla + \tilde{V}, \quad (5.3)\]

with

\[\tilde{V}(x) := \frac{\Delta_{M,\mu_1}}{\gamma} \text{Id}(x) + V(x).\]

To see this, let \(w, z \in C^\infty_c(M)\) and consider

\[(H_1w, z)_{\mu_1} = \int_M (\gamma^{-1}H(\gamma w), z) \gamma^2 d\mu = \int_M (H(\gamma w), \gamma z) d\mu = (H(\gamma w), \gamma z)_{\mu_1},\]

\[= (\nabla(\gamma w), \nabla(\gamma z))_{\mu_1} + (V \gamma w, \gamma z)_\mu = (\gamma^2 \nabla w, \nabla z)_\mu + (d\gamma \otimes w, d\gamma \otimes z)_{L^2_{\mu}(T^*M \otimes E)} + (\gamma \nabla w, d\gamma \otimes z)_{L^2_{\mu}(T^*M \otimes E)} + (V \gamma w, \gamma z)_\mu. \quad (5.4)\]

Setting \(\xi := d(\gamma^2/2) \in T^*M\) and using equation (1.34) in Appendix C of [33] we have

\[(\nabla w, d\gamma \otimes z)_{L^2_{\mu}(T^*M \otimes E)} = (\nabla w, \xi \otimes z)_{L^2_{\mu}(T^*M \otimes E)} = (\nabla w, z)_\mu, \quad (5.5)\]

where \(X\) is the vector field associated with \(\xi \in T^*M\) via the metric \(g^{TM}\).

Furthermore, by equation (1.35) in Appendix C of [33] we have

\[(d\gamma \otimes w, \gamma \nabla z)_{L^2_{\mu}(T^*M \otimes E)} = (\xi \otimes w, \nabla z)_{L^2_{\mu}(T^*M \otimes E)} = (\nabla^* \mu_1(\xi \otimes w), z)_\mu,\]

\[= -(\text{div}_\mu(X)w, z)_\mu - (\nabla X w, z)_\mu, \quad (5.6)\]

where, in local coordinates \(x^1, x^2, \ldots, x^n\), for \(X = X^j \frac{\partial}{\partial x^j}\), with Einstein summation convention,

\[\text{div}_\mu(X) := \frac{1}{\kappa} \left( \frac{\partial}{\partial x^j} (\kappa X^j) \right).\]
(Recall that $d\mu = \kappa(x)\,dx_1dx_2\ldots dx_n$, where $\kappa(x)$ is a positive $C^\infty$-density.) Since $X^j = (g^TM)^j_l\partial\gamma_{\partial x^l}$, we have

$$\text{div}_\mu(X) = |d\gamma|^2 - \gamma(\Delta_{M,\mu}\gamma),$$

(5.7)

where $|d\gamma(x)|$ is the norm of $d\gamma(x) \in T^*_xM$ induced by $g^TM$, and $\Delta_{M,\mu}$ is as in (1.1) with metric $g^TM$. Combining (5.4)–(5.7) and noting that

$$(d\gamma \otimes w, d\gamma \otimes z)_{L^2_\mu(T^*M \otimes E)} = \int_M |d\gamma|^2(w, z)\,d\mu,$$

we obtain

$$(H_1 w, z)_{\mu_1} = \int_M \langle \nabla w, \nabla z \rangle \gamma^2\,d\mu + \int_M \langle V w, z \rangle \gamma^2\,d\mu + \int_M \gamma(\Delta_{M,\mu}\gamma)\langle w, z \rangle\,d\mu$$

$$= \langle \nabla w, \nabla z \rangle_{L^2_\mu(T^*M \otimes E)} + (V w, z)_{\mu_1} + (\gamma^{-1}(\Delta_{M,\mu}\gamma)w, z)_{\mu_1},$$

(5.8)

which shows (5.3).

By (2.5) and (5.2) it follows that

$$\tilde{\nabla}(x) = \frac{\Delta_{M,\mu}\gamma}{\gamma}\text{Id}(x) + V(x) \geq (C - 1)\text{Id}(x),$$

for all $x \in M$, where $C$ is as in (2.6). Thus, by Theorem 1 the operator $(H_1)^k|_{C_0^\infty(E)}$ is essentially self-adjoint in $L^2_{\mu_1}(E)$ for all $k \in \mathbb{Z}_+$. □

6. PROOF OF THEOREM 3

Throughout the section, we assume that the hypotheses of Theorem 3 are satisfied. In subsequent discussion, the notation $\tilde{D}$ is as in (6.1) and the operators $H_{\min}$ and $H_{\max}$ are as in Section 4.1. We begin with the following lemma, whose proof is a direct consequence of the definition of $H_{\max}$ and local elliptic regularity.

Lemma 6.1. Under the assumption $V \in L^\infty(\text{End}E)$, we have the following inclusion:

$${\text{Dom}}(H_{\max}) \subset W^{2,2}_{\text{loc}}(E).$$

The proof of the next lemma is given in Lemma 8.10 of [5].

Lemma 6.2. For any $u \in \text{Dom}(H_{\max})$ and any Lipschitz function with compact support $\psi: M \to \mathbb{R}$, we have:

$$D(\psi u), D(\psi u)) + (V \psi u, \psi u) = \text{Re}(\psi Hu, \psi u) + \|\tilde{D}(d\psi)u\|^2.$$  

(6.1)

Corollary 6.3. Let $H$ be as in (2.5), let $u \in L^2(E)$ be a weak solution of $Hu = 0$, and let $\psi: M \to \mathbb{R}$ be a Lipschitz function with compact support. Then

$$(\psi u, H(\psi u)) = \|\tilde{D}(d\psi)u\|^2,$$

(6.2)

where $(\cdot, \cdot)$ on the left-hand side denotes the duality between $W^{1,2}_{\text{loc}}(E)$ and $W^{-1,2}_{\text{comp}}(E)$. 13
Proof. Since \( u \in L^2(E) \) and \( Hu = 0 \), we have \( u \in \text{Dom}(H_{\text{max}}) \subset W^{2,2}_{\text{loc}}(E) \subset W^{1,2}_{\text{loc}}(E) \), where the first inclusion follows by Lemma 6.1. Since \( \psi \) is a Lipschitz compactly supported function, we get \( \psi u \in W^{1,2}_{\text{comp}}(E) \) and, hence, \( H(\psi u) \in W^{-1,2}_{\text{comp}}(E) \). Now the equality (6.2) follows from (6.1), the assumption \( Hu = 0 \), and

\[
(\psi u, H(\psi u)) = (\psi u, D^*(D(\psi u)) + (V \psi u, \psi u) = (D(\psi u), D(\psi u)) + (V \psi u, \psi u),
\]

where in the second equality we used integration by parts; see Lemma 8.8 in [5]. Here, the two leftmost symbols \((·, ·)\) denote the duality between \( W^{1,2}_{\text{comp}}(E) \) and \( W^{-1,2}_{\text{comp}}(E) \), while the remaining ones stand for \( L^2 \)-inner products. \( \square \)

The key ingredient in the proof of Theorem 5 is the Agmon-type estimate given in the next lemma, whose proof, inspired by an idea of [25], is based on the technique developed in [10] for magnetic Laplacians on an open set with compact boundary in \( \mathbb{R}^n \).

**Lemma 6.4.** Let \( \lambda \in \mathbb{R} \) and let \( v \in L^2(E) \) be a weak solution of \((H - \lambda)v = 0\). Assume that there exists a constant \( c_1 > 0 \) such that, for all \( u \in W^{1,2}_{\text{comp}}(E) \),

\[
(u, (H - \lambda)u) \geq \lambda_0^2 \int_M \max \left( \frac{1}{r(x)^2}, 1 \right) |u(x)|^2 d\mu(x) + c_1 \|u\|^2,
\]

where \( r(x) \) is as in (2.7), \( \lambda_0 \) is as in (2.2), the symbol \((·, ·)\) on the left-hand side denotes the duality between \( W^{1,2}_{\text{comp}}(E) \) and \( W^{-1,2}_{\text{loc}}(E) \), and \( |·| \) is the norm in the fiber \( E_x \).

Then, the following equality holds: \( v = 0 \).

**Proof.** Let \( \rho \) and \( R \) be numbers satisfying \( 0 < \rho < 1/2 \) and \( 1 < R < +\infty \). For any \( \varepsilon > 0 \), we define the function \( f_{\varepsilon} : M \to \mathbb{R} \) by \( f_{\varepsilon}(x) = F_{\varepsilon}(r(x)) \), where \( r(x) \) is as in (2.7) and \( F_{\varepsilon} : [0, \infty) \to \mathbb{R} \) is the continuous piecewise affine function defined by

\[
F_{\varepsilon}(s) = \begin{cases} 
0 & \text{for } s \leq \varepsilon \\
\rho(s - \varepsilon)/(\rho - \varepsilon) & \text{for } \varepsilon \leq s \leq \rho \\
s & \text{for } \rho \leq s \leq 1 \\
1 & \text{for } 1 \leq s \leq R \\
R + 1 - s & \text{for } R \leq s \leq R + 1 \\
0 & \text{for } s \geq R + 1.
\end{cases}
\]

Let us fix \( x_0 \in M \). For any \( \alpha > 0 \), we define the function \( p_\alpha : M \to \mathbb{R} \) by

\[
p_\alpha(x) = P_\alpha(d_{gTM}(x_0, x)),
\]

where \( P_\alpha : [0, \infty) \to \mathbb{R} \) is the continuous piecewise affine function defined by

\[
P_\alpha(s) = \begin{cases} 
1 & \text{for } s \leq 1/\alpha \\
-\alpha s + 2 & \text{for } 1/\alpha \leq s \leq 2/\alpha \\
0 & \text{for } s \geq 2/\alpha.
\end{cases}
\]

Since \( \tilde{d}_{gTM}(x_0, x) \leq d_{gTM}(x_0, x) \), it follows that the support of \( f_{\varepsilon}p_\alpha \) is contained in the set \( B_\alpha := \{ x \in M : \tilde{d}_{gTM}(x_0, x) \leq 2/\alpha \} \). By assumption (A1) we know that \( \tilde{M} \) is a geodesically complete Riemannian manifold. Hence, by Hopf–Rinow Theorem the set \( B_\alpha \) is compact. Therefore, the
support of \( f_\varepsilon p_\alpha \) is compact. Additionally, note that \( f_\varepsilon p_\alpha \) is a \( \beta \)-Lipschitz function (with respect to the distance corresponding to the metric \( g^{TM} \)) with \( \beta = \frac{\rho}{\rho - \varepsilon} + \alpha \).

Since \( v \in L^2(E) \) and \( (H - \lambda)v = 0 \), we have \( v \in \text{Dom}(H_{\max}) \subset W_{\text{loc}}^{2,2}(E) \subset W_{\text{loc}}^{1,2}(E) \), where the first inclusion follows by Lemma 6.1. Since \( f_\varepsilon p_\alpha \) is a Lipschitz compactly supported function, we get \( f_\varepsilon p_\alpha v \in W_{\text{comp}}^{1,2}(E) \) and, hence, \((H - \lambda)(f_\varepsilon p_\alpha v) \in W_{\text{comp}}^{-1,2}(E)\).

Using (2.2) we have
\[
\|\tilde{D}(d(f_\varepsilon p_\alpha)v)\| \leq \lambda_0^2 \int_M |d(f_\varepsilon p_\alpha)(x)|^2 |v(x)|^2 \, d\mu(x),
\]
where \( |d(f_\varepsilon p_\alpha)(x)| \) is the norm of \( d(f_\varepsilon p_\alpha)(x) \in T_x^*M \) induced by \( g^{TM} \).

By Corollary 6.3 with \( H - \lambda \) in place of \( H \) and the inequality (6.4), we get
\[
(f_\varepsilon p_\alpha v, (H - \lambda)(f_\varepsilon p_\alpha v)) \leq \lambda_0^2 \left( \frac{\rho}{\rho - \varepsilon} + \alpha \right)^2 \|v\|^2.
\]

On the other hand, using the definitions of \( f_\varepsilon \) and \( p_\alpha \) and the assumption (6.3) we have
\[
(f_\varepsilon p_\alpha v, (H - \lambda)(f_\varepsilon p_\alpha v)) \geq \lambda_0^2 \int_{S_{\rho,R,\alpha}} |v(x)|^2 \, d\mu(x) + c_1 \|f_\varepsilon p_\alpha v\|^2,
\]
where
\[
S_{\rho,R,\alpha} := \{x \in M: \rho \leq r(x) \leq R \text{ and } d_{g^TM}(x_0, x) \leq 1/\alpha\}.
\]

In (6.6) and (6.5), the symbol \((\cdot, \cdot)\) stands for the duality between \( W_{\text{comp}}^{1,2}(E) \) and \( W_{\text{loc}}^{-1,2}(E) \). We now combine (6.6) and (6.5) to get
\[
\lambda_0^2 \int_{S_{\rho,R,\alpha}} |v(x)|^2 \, d\mu(x) + c_1 \|f_\varepsilon p_\alpha v\|^2 \leq \lambda_0^2 \left( \frac{\rho}{\rho - \varepsilon} + \alpha \right)^2 \|v\|^2.
\]
We fix \( \rho, R, \) and \( \varepsilon \), and let \( \alpha \to 0^+ \). After that we let \( \varepsilon \to 0^+ \). The last step is to do \( \rho \to 0^+ \) and \( R \to +\infty \). As a result, we get \( v = 0 \).

**End of the proof of Theorem 3.** Using integration by parts (see Lemma 8.8 in [5]), we have
\[
(u, Hu) = (u, D^*Du) + (Vu, u) = (Du, Du) + (Vu, u) \geq (Vu, u),
\]
where the two leftmost symbols \((\cdot, \cdot)\) denote the duality between \( W_{\text{comp}}^{1,2}(E) \) and \( W_{\text{loc}}^{-1,2}(E) \), while the remaining ones stand for \( L^2\)-inner products. Hence, by assumption (2.8) we get:
\[
(u, (H - \lambda)u) \geq \lambda_0^2 \int_M \frac{1}{r(x)^2} |u(x)|^2 \, d\mu(x) - (\lambda + C)\|u\|^2
\]
\[
\geq \lambda_0^2 \int_M \max \left\{ \frac{1}{r(x)^2}, 1 \right\} |u(x)|^2 \, d\mu(x) - (\lambda + C + 1)\|u\|^2.
\]

Choosing, for instance, \( \lambda = -C - 2 \) in (6.7) we get the inequality (6.3) with \( c_1 = 1 \).

Thus, \( H_{\text{min}} - \lambda \) with \( \lambda = -C - 2 \) is a symmetric operator satisfying \((u, (H_{\text{min}} - \lambda)u) \geq \|u\|^2\), for all \( u \in C_\infty^c(E) \). In this case, it is known (see Theorem X.26 in [29]) that the essential self-adjointness of \( H_{\text{min}} - \lambda \) is equivalent to the following statement: if \( v \in L^2(E) \) satisfies
\[(H - \lambda)v = 0, \text{ then } v = 0. \] Thus, by Lemma 6.4, the operator \((H_{\text{min}} - \lambda)\) is essentially self-adjoint. Hence, \(H_{\text{min}}\) is essentially self-adjoint. \(\square\)

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