CONJUGACY-CLASSES OF NON-TRANSLATIONS IN AFFINE WEYL GROUPS AND APPLICATIONS TO HECKE ALGEBRAS

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ABSTRACT. Let \( \tilde{W} = \Lambda \rtimes W_0 \) be an Iwahori-Weyl group of a connected reductive group \( G \) over a non-archimedean local field. The subgroup \( W_0 \) is a finite Weyl group and the subgroup \( \Lambda \) is a finitely-generated abelian group (possibly containing torsion) which acts on a certain real affine space by translations. I prove that if \( w \in \tilde{W} \) and \( w \notin \Lambda \) then one can apply to \( w \) a sequence of conjugations by simple reflections, each of which is length-preserving, resulting in an element \( w' \) for which there exists a simple reflection \( s \) such that \( \ell(sw'), \ell(w's) > \ell(w') \) and \( sw's \neq w' \). Even for affine Weyl groups, a special case of Iwahori-Weyl groups and also an important subclass of Coxeter groups, this is a new fact about conjugacy classes. Further, there are implications for Iwahori-Hecke algebras \( \mathcal{H} \) of \( G \): one can use this fact to give dimension bounds on the “length-filtration” of the center \( Z(\mathcal{H}) \), which can in turn be used to prove that suitable linearly-independent subsets of \( Z(\mathcal{H}) \) are a basis.

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1. Introduction

A Coxeter group is a pair $(W, S)$ consisting of a group $W$ and a generating set $S$ which can be presented using only the relations $s^2 = 1$ for all $s \in S$ together with relations of the form $(st)^m(s,t) = 1$ for some, but not necessarily all, pairs $s, t \in S$. The most common examples of infinite Coxeter groups are affine Weyl groups, which are groups generated by the reflections across special collections of hyperplanes coming from root systems. The theory of Coxeter groups is both complicated and, especially in the case of affine Weyl groups, highly-developed. Affine Weyl groups are ubiquitous in the subject of smooth representations of algebraic groups over non-archimedean local fields due to their connection with Hecke algebras, and many questions about Hecke algebras can be reduced to questions about affine Weyl groups.

Let $F$ be a non-archimedean local field and let $G$ be a connected reductive affine algebraic $F$-group. If $J \subset G(F)$ is a compact-open subgroup and $(\rho, V)$ is a smooth complex representation of $J$ then the Hecke algebra $H(G; J, \rho)$ is the convolution algebra of compactly-supported functions $G(F) \to \text{End}_\mathbb{C}(V)$ which are both left- and right-$\rho$-equivariant with respect to translations by $J$. Many of the simple subcategories in the Bernstein decomposition of the category of smooth representations of $G(F)$ are equivalent to the category of modules over a Hecke algebra of this form. The center of such a Hecke algebra is important because it consists of the (functorial) $G(F)$-linear endomorphisms of the representations in the subcategory. The case that $J$ is an Iwahori subgroup and $\rho$ is the trivial 1-dimensional representation yields one particularly important Hecke algebra: the Iwahori-Hecke algebra.

It can be shown that any Iwahori-Hecke algebra has a certain presentation consisting of a basis of characteristic functions on double-cosets modulo the Iwahori subgroup together with a certain pair of relations which depend on some numerical parameters coming from $G$ (this presentation is called the Iwahori-Matsumoto presentation due to early work by Iwahori and Matsumoto). The double-cosets here turn out to be represented by the elements of a group called an Iwahori-Weyl group. The Iwahori-Weyl group is in general a certain semidirect extension of a Coxeter group but its behavior is nonetheless extremely similar to that of a true Coxeter group. Taken together with the parameters, the group-theoretic structure (e.g. Cayley graph) of this “quasi-Coxeter group” completely controls the ring-theoretic structure of the Iwahori-Hecke algebra $H$.

Bernstein determined a particularly important basis, whose elements have a straightforward representation-theoretic interpretation, for the center $Z(H)$ in a
restricted situation. Lusztig proved that the same recipe works more generally, for the affine Hecke algebra of a reduced root datum. Unfortunately, many Iwahori-Hecke algebras are not of this form and so are not within the scope of the Bernstein/Lusztig work.

In this paper, I prove a Coxeter-theoretic property of Iwahori-Weyl groups, which is described precisely in the next part of this introduction. The class of all Iwahori-Weyl groups properly contains the class of all affine Weyl groups (this happens when $G$ is almost-simple and simply-connected) and this property is new even in this restricted context. Additionally, there are implications for all Iwahori-Hecke algebras, since this property can be used to show that suitable collections of functions in the center $Z(H)$ span the center. I apply this in the upcoming Ros to extend the Bernstein/Lusztig work to all Iwahori-Hecke algebras.

1.1. statement of results.

More precise definitions of everything here are given in §2 and §3.

Let $F$ be a non-archimedean local field and $G$ a connected reductive affine algebraic $F$-group. Fix a maximal $F$-split torus $A \subset G$ and let $\tilde{W}$ be the corresponding Iwahori-Weyl group, which acts on the vector space $V \overset{\text{def}}{=} X_*(A) \otimes \mathbb{Z} \mathbb{R}$. It is known that $\tilde{W}$ contains as a normal subgroup the affine Weyl group $W_{\text{aff}}(\Sigma)$ of a reduced root system $\Sigma$, and that if $\Lambda \subset \tilde{W}$ is the subgroup of elements which act on $V$ by translations then $\tilde{W}$ splits as $\tilde{W} = \Lambda \rtimes W_0(\Sigma)$ (here $W_0$ denotes the finite Weyl group). Further, there are sections for $\Phi \overset{\text{def}}{=} \tilde{W}/W_{\text{aff}}(\Sigma)$, so that $\tilde{W}$ also splits as $\tilde{W} = W_{\text{aff}}(\Sigma) \rtimes \Phi$. If $\Delta_{\text{aff}}$ is a Coxeter generating set for $W_{\text{aff}}(\Sigma)$ and $\ell$ is the corresponding length function, then $\ell$ extends to $\tilde{W}$ by inflation. Denote by $\Delta_0 \subset \Delta_{\text{aff}}$ a Coxeter generating set for $W_0(\Sigma)$.

The main result of the paper is the following:

Main Theorem. Fix $w \in \tilde{W}$.

If $w \notin \Lambda$ then there exists $s \in \Delta_{\text{aff}}$ and (if necessary) $s_1, \ldots, s_n \in \Delta_{\text{aff}}$ such that, setting $w' \overset{\text{def}}{=} s_n \cdots s_1 ws_1 \cdots s_n$,

- $\ell(s_i \cdots s_1 ws_1 \cdots s_i) = \ell(w)$ for all $i$,
- both $\ell(sw') > \ell(w')$ and $\ell(w's) > \ell(w')$, and $sw's \neq w'$.

Note that the last two properties asserted by the Main Theorem can be unified into “$\ell(sw's) > \ell(w')$”.
Remark 1.1.1. A related result appears in the preprint [HN11]: that for any element \( w \) in the extended affine Weyl group of a (reduced) root datum there is a sequence of conjugations by simple reflections, each of which preserves or decreases length, resulting in an element \( w' \) which is minimal length in its conjugacy class. Note however that the [HN11] result is also true for finite Weyl groups, originally proven by [GP93], whereas the Main Theorem here is special to infinite groups (length is bounded on a finite group!).

Denote by \( \Lambda/W_0 \) the set of \( W_0(\Sigma) \)-conjugacy classes in \( \Lambda \) and recall that \( \ell \) is constant on any \( O \in \Lambda/W_0 \). Denote by \( \Omega(w) \) the \( \Omega \)-coordinate of any \( w \in \tilde{W} \).

Let \( H \) be an Iwahori-Hecke algebra for \( G \). By analyzing the equations that define the center \( Z(H) \), the Main Theorem can be used to prove dimension bounds for a certain filtration/partition of \( Z(H) \):

**Corollary.** If \( Z_{L,\tau}(H) \subset Z(H) \) is the \( C \)-subspace of functions supported only on those \( w \) for which \( \ell(w) \leq L \) and \( \Omega(w) = \tau \), and if \( N_{L,\tau} \) is the total number of \( O \in \Lambda/W_0 \) such that \( \ell(O) \leq L \) and \( \Omega(t) = \tau \) for all \( t \in O \) then

\[
\dim_C(Z_{L,\tau}(H)) \leq N_{L,\tau}
\]

It follows that if \( \{ z_O \}_{O \in \Lambda/W_0} \) is a linearly-independent subset of \( Z(H) \) such that \( z_O \) is supported only on those \( w \) for which \( \ell(w) \leq \ell(O) \) and \( \Omega(w) \) is the same for all \( w \) supporting \( z_O \) then \( \{ z_O \}_{O \in \Lambda/W_0} \) is a basis.

1.2. outline of paper. In §2, I set some notation and define most of the objects that will be used throughout the paper. I use several non-standard but convenient notations, and almost all of them can be found here. Two exceptions are a "quasi-Coxeter group" and the Hecke algebra on such a group, which are treated in §3 and §8.1, respectively.

In §3, I recall the notion of Iwahori-Weyl group and several of its most important properties, mostly to make explicit the scope of the Main Theorem. Reductive groups do not appear after this section.

In §4, I define what is a marked alcove. This is just a straightforward generalization of the notion of the type of a face of an alcove. This extension is necessary to include Iwahori-Weyl groups, rather than just affine Weyl groups, in the scope of the paper.

In §5, I precisely define the Diamond Property for an element of a Coxeter group. In short, the Diamond Property is the property asserted by the Main Theorem. I then define an equivalent property that refers only to pairs of alcoves and verify the equivalence of the two definitions.
In §6, I prove some simple geometric lemmas about Weyl chambers, alcoves, hyperplanes, etc. that will be used throughout §7.

In §7, I prove the Main Theorem, divided into three cases. In §7.1, I prove the “dominant case”, Proposition 7.1.1 of the Main Theorem: if \( w \notin \Lambda \) and \( w \) sends the base alcove \( A \) into the dominant chamber \( C \) then \( w \) has the Diamond Property.

It is obvious from the hypothesis that \( \ell(sw) > \ell(w) \) for all \( s \in \Delta \), and it is not hard to visualize why there also exists \( s \in \Delta \) such that \( \ell(ws) > \ell(w) \). In §7.2, I prove the “anti-dominant case”, Proposition 7.2.1 of the Main Theorem: if \( w \notin \Lambda \) and \( w(v) \in C^{\text{opp}} \) then \( w \) has the Diamond Property. This is the most difficult case and the general case can be reduced to this one (the complexity of the anti-dominant case is in some sense maximal while that of the dominant case is minimal—this can be quantified somewhat by noting that \( w(A) \subset C^{\text{opp}} \Rightarrow \ell(sw) < \ell(w) \) for all \( s \in \Delta \)).

The basic idea is that, by carefully inspecting the relative position and orientation of the alcoves \( A \) and \( w(A) \), one can perform an infinite sequence of conjugations by \( \Delta_{\text{aff}} \) which do not decrease length and which continually move the alcoves in “different directions”, guaranteeing an eventual length-increasing conjugation by \( \Delta_{\text{aff}} \). In §7.3, I finish proving the Main Theorem, showing that for an arbitrary \( w \notin \Lambda \) one can repeatedly perform conjugations by \( \Delta_{\text{aff}} \) which do not decrease length and such that eventually the situation qualifies for the anti-dominant case.

In §8, I use the Main Theorem to give dimension bounds, Proposition 8.3.1 for every subspace in the “length filtration” of \( Z(H) \). The term “length filtration” is a slight abuse, since it is necessary to first filter \( Z(H) \) by the lengths of its supporting elements and then partition each of those subspaces by the \( \Omega \)-components of its supporting elements (if \( \Omega \) is infinite then the subspaces in the length filtration are infinite dimensional, and without refining it further most Iwahori-Hecke algebras would be outside the scope of the paper).

In §9, I include many color pictures that illustrate the iterative arguments that occur in the proof of the Main Theorem. I use the affine Weyl group and apartment of the exceptional type \( \tilde{G}_2 \) because there are too many coincidences for extremely symmetric types like \( \tilde{A}_2 \) to correctly explain things, and because I can suppress hyperplane labels for \( \tilde{G}_2 \) since its labelings are, unlike \( \tilde{A}_2 \) and \( \tilde{C}_2 \), unambiguous (the alcoves are \( 30^\circ-60^\circ-90^\circ \) triangles).

1.3. acknowledgements. I thank Thomas Haines for carefully reading earlier drafts of this paper and suggesting a very large number of changes to the exposition, and especially for noticing that the original proof occurring in §7.3 was needlessly labyrinthine. I also thank my postdoctoral mentor, Tonghai Yang, for bringing me to the wonderful University of Wisconsin at Madison.
2. Notation and Setup

The symbols $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ refer to the natural numbers (including 0), the integers, the real numbers, and the complex numbers.

2.1. root systems and affine Weyl groups. Let $\Sigma$ be a reduced and irreducible root system. Let $W_\circ$ be the finite Weyl group of $\Sigma$, let $\Delta_\circ$ be a simple system for $W_\circ$.

Let $A$ be the $\mathbb{R}$-vector space spanned by the dual root system $\Sigma^\vee$. Let $\langle -, - \rangle$ be the natural pairing $\Sigma^\vee \times \Sigma \to \mathbb{Z}$ and $\langle -, - \rangle_\mathbb{R}$ the extension to $A \times A^\vee \to \mathbb{R}$. Let $\Sigma_{\text{aff}}$ be the affine root system associated to $\Sigma$, i.e. the set $\Sigma + \mathbb{Z}$ of affine functionals on $A$. The term hyperplane always means the null-set of an element of $\Sigma_{\text{aff}}$, not just an arbitrary codimension-1 affine subspace. The term root hyperplane means the null-set of an element of $\Sigma$.

Let $W_{\text{aff}}$ be the affine Weyl group of $\Sigma_{\text{aff}}$ and $\Delta_{\text{aff}}$ the simple system for $W_{\text{aff}}$ extended from $\Delta_\circ$. Denote by $s_{\text{aff}}$ the single element of $\Delta_{\text{aff}} \setminus \Delta_\circ$ and by $H_{\text{aff}}$ the hyperplane fixed pointwise by $s_{\text{aff}}$. Denote by $\ell : W_{\text{aff}} \to \mathbb{N}$ the usual length function relative to the generating set $\Delta_{\text{aff}}$. Let $Q \subset W_{\text{aff}}$ be the subgroup of translations by elements of $\Sigma^\vee$, so that $W_{\text{aff}} = Q \rtimes W_\circ$.

Choose a special vertex $v_\circ \in A$ and identify $W_\circ$ to the finite Weyl group at $v_\circ$. Let $C_\circ$ be the Weyl chamber at $v_\circ$ corresponding to $\Delta_\circ$ and let $A_\circ \subset C_\circ$ be the alcove for which $v_\circ \in A_\circ$. If $C$ is a Weyl chamber (at some arbitrary special vertex), denote by $C^{\text{opp}}$ the opposite chamber.

2.2. simplices and topology. The term face always means a codimension-1 facet of an alcove (or Weyl chamber). The term wall always means the unique hyperplane containing some face of some alcove (or Weyl chamber). If $A$ and $B$ are (distinct) adjacent alcoves then denote by $A \mid B$ the wall separating them.

The term half-space refers to one of the connected components of the complement in $A$ of a hyperplane (in particular, half-spaces are open, and therefore also Weyl chambers and alcoves).

A subset $R \subset A$ is called simplicial iff there exists a set $S$ of alcoves in $A$ such that $\cup_{R \in S} \mathfrak{A} \subset R \subset \cup_{R \in S} \mathfrak{A}$. For example, half-spaces and Weyl chambers are simplicial.

2.3. galleries and distances. The length of a gallery $(B_0, B_1, \ldots, B_n)$ is defined to be $n$ (for consistency with the length function $\ell$). To say that a hyperplane $H$ is an intermediate wall of a (non-stuttering) gallery $(B_0, B_1, \ldots, B_n)$ is the same as
to say that there exists an index $0 \leq j < n$ such that $H$ is the unique hyperplane containing the face shared by $B_j$ and $B_{j+1}$.

I sometimes use the fact that if $R$ is simplicial and (topologically) convex then it is convex in the combinatorial sense, i.e. if $A, A' \subset R$ are alcoves and $G$ is a minimal gallery from $A$ to $A'$ then $A'' \subset R$ for all $A'' \in G$ (see Theorem 5.11.4 in [BGW03]). I sometimes refer to this property as simplicial convexity. For example, half-spaces and Weyl chambers are simplicially-convex.

By infinite gallery I mean an infinite sequence $(B_0, B_1, \ldots)$ of alcoves such that for all $i \geq 0$ the alcoves $B_i$ and $B_{i+1}$ are adjacent. An infinite gallery is minimal iff for all $0 \leq i < j$ the finite sub-gallery $(B_i, \ldots, B_j)$ is minimal in the usual sense.

Let $d$ be the usual $\mathbb{N}$-valued metric on the set of all alcoves: $d(A, B)$ is defined to be the minimum length among all galleries from $A$ and $B$. Equivalently, $d(A, B)$ is the total number of hyperplanes separating $A$ from $B$—see Theorem 5.1.4 in [BGW03]. I also use two extensions of this distance function $d$:

If $R \neq \emptyset$ is a simplicial subset and $A \subset A$ is an alcove then define

$$d(A, R) \overset{\text{def}}{=} \min(d(A, B) \mid \text{all alcoves } B \subset R)$$

If $w$ is a vertex, and $R$ is as before, then I define

$$d(w, R) \overset{\text{def}}{=} \min(d(A, R) \mid \text{all alcoves } A \subset A \text{ such that } w \in A)$$

In applications, $R$ will be either a single half-space or a Weyl chamber.

3. Iwahori-Weyl Groups

In this section, I define the object which is the main focus of this paper: the Iwahori-Weyl group.

3.1. definition and key properties.

In this subsection, I briefly explain what is an Iwahori-Weyl group and isolate its key properties. The purpose here is merely to explain the scope of the paper, so all proofs are omitted, although I give references whenever possible.

Let $F$ be a non-archimedean local field and let $G$ be a connected reductive affine algebraic $F$-group. Let $A \subset G$ be a maximal $F$-split torus and set $M \overset{\text{def}}{=} C_G(A)$, a minimal $F$-Levi subgroup.
Certain group homomorphisms called Kottwitz homomorphisms are very useful to understand the theory of parahoric subgroups, and in particular Iwahori subgroups. The Kottwitz homomorphism of a connected reductive affine algebraic $F$-group $H$ is a surjective group homomorphism $\kappa_H : H(F) \rightarrow \Omega_H$, where $\Omega_H$ is a finitely-generated abelian group whose precise definition is not relevant to this paper—see §7 of [Kot97] for the definitions of $\kappa_H$ and $\Omega_H$ (the map $\kappa_H$ occurring here is (7.7.1) in [Kot97]). The kernel of the Kottwitz homomorphism is denoted by $H(F)_1 \overset{\text{def}}{=} \ker(\kappa_H)$.

One may define the Iwahori-Weyl group of $(G, A)$ to be the quotient $\tilde{W} \overset{\text{def}}{=} N_G(A)(F)/M(F)_1$. Note that this is seemingly different from the quotient occurring in Remark 9 of the Appendix to [PR08], but it can be shown that there is a tautological isomorphism between the two quotients. Proposition 8 combined with Remark 9 of the Appendix to [PR08] shows if $I \subset G(F)$ is an Iwahori subgroup then the double-cosets modulo $I$ are naturally represented by $\tilde{W}$. The Iwahori-Weyl group $\tilde{W}$ acts on the vector space $\mathcal{V} \overset{\text{def}}{=} X_*(A) \otimes \mathbb{Z} \mathbb{R}$.

There are two extremely important ways to express the Iwahori-Weyl group $\tilde{W}$ as a semidirect product. By the work of Bruhat and Tits, it is known that there exists a reduced root system $\Sigma$ such that the affine Weyl group $W_{\text{aff}}(\Sigma)$, in the sense of VI-§2-1 of [Bou02], is a subgroup of $\tilde{W}$ (this root system is called an échelonnement in §1.4 of [BT72]; see §4 of [Tit79] for an extremely nice table listing $\Sigma$ for every almost-simple group, and much more). Denoting by $W_\Sigma(\Sigma)$ the finite Weyl group of $\Sigma$, it can be shown that $\tilde{W} = \Omega_M \times W_\Sigma(\Sigma)$ and that $\tilde{W} = W_{\text{aff}}(\Sigma) \times \Omega_G$. Further, the subgroup $\Omega_M$ acts on $\mathcal{V}$ by translations and the subgroup $\Omega_G$ acts on $\mathcal{V}$ by invertible affine transformations that stabilize any prescribed base alcove in $\mathcal{V}$. For more details of all these semidirect products, consult [PR08] and [HR10].

Remark 3.1.1. A few comments are necessary to emphasize the small but important difference between the notion of an Iwahori-Weyl group and the possibly more familiar notion of an extended affine Weyl group of a (reduced) root datum. First, it is possible that both $\Omega_M$ and $\Omega_G$ have torsion elements (in fact, it can be shown that the torsion of the former is contained in the torsion of the latter). Second, the elements of $\Omega_M$ are not actually translations, but merely act by translations on $\mathcal{V}$. Third, it is possible that some non-identity elements in $\Omega_M$ act by the identity on $\mathcal{V}$.

3.2. axiomatization. Using the previous discussion as a guide, I now isolate the relevant properties of the Iwahori-Weyl group and present them axiomatically for clarity.
Let $\mathcal{N}$ be the group of invertible affine transformations of $A$ which normalize $W_{\text{aff}}$. Fix a finitely-generated abelian group $\Omega$, a group homomorphism $\psi : \Omega \to \mathcal{N}$, and act by $\Omega$ on $A$ via this $\psi$.

**Definition.** The Quasi-Coxeter Group $\widetilde{W}$ extended from $W_{\text{aff}}$ by $\psi \to \mathcal{N}$ is the semidirect product $W_{\text{aff}} \rtimes \Omega$ and acts on $A$ in the obvious way: $(w, \tau)(a) \overset{\text{def}}{=} w(\tau(a))$ for all $(w, \tau) \in \widetilde{W}$ and $a \in A$. Denote by $\Omega(w)$ the projection of $w \in \widetilde{W}$ into $\Omega$.

Note that if $w, w' \in \widetilde{W}$ are conjugate then $\Omega(w) = \Omega(w')$, since $\Omega$ is abelian.

**Remark 3.2.1.** Strictly speaking, the space $A$ on which the quasi-Coxeter group acts is only a proper subspace of the space $V$ on which the Iwahori-Weyl group acts when $G$ is not semisimple. But due to the way that affine root hyperplanes in $V$ are defined, the details of which I omit in this paper, the difference is totally irrelevant from a group-theory perspective. The setup that I use is essentially the same as that used in VI-§3-3 of [Bou02].

Let $\Lambda \subset \widetilde{W}$ be the subgroup consisting of all elements that act by translations on $A$, and note that $\Lambda$ is obviously normalized by $W_{\circ} \subset W_{\text{aff}}$. Extend the length function $\ell : W_{\text{aff}} \to \mathbb{N}$ to $\widetilde{W}$ by inflation along the projection $W_{\text{aff}} \rtimes \Omega \to W_{\text{aff}}$.

I impose the following hypotheses:

- **QCG1** Assume that $\tau(\mathfrak{A}_o) = \mathfrak{A}_o$.
- **QCG2** Assume that $\Lambda$ is a semidirect complement, i.e. that $\widetilde{W} = \Lambda \rtimes W_{\circ}$.
- **QCG3** Assume that $\ell$ is constant on each $W_{\circ}$-conjugacy class in $\Lambda$.
- **QCG4** Assume that $\Lambda$ is finitely-generated and abelian.

Note that by choice of $\mathcal{N}$, the action by $\Omega$ on $A$ permutes the set of hyperplanes in $A$. Therefore, hypothesis **QCG1** is equivalent to the hypothesis that $\tau(\Delta_{\text{aff}}) = \Delta_{\text{aff}}$ for all $\tau \in \Omega$.

### 4. Marked Alcoves

The definitions in this section, which are mostly just a variant on the notion of the type of a face, will be used heavily in §7.2 and §7.3.

**Definition.** A Labeling of an alcove $\mathfrak{A} \subset A$ is a bijection from $\Delta_{\text{aff}}$ to the set of walls of $\mathfrak{A}$. A Marked Alcove is a triple $(\mathfrak{A}, v, t)$ such that $\mathfrak{A}$ is an alcove, $v \in \mathfrak{A}$ is a special vertex and $t$ is a labeling of $\mathfrak{A}$. The Weyl Chamber of a marked alcove $(\mathfrak{A}, v, t)$ is the unique Weyl chamber at $v$ containing $\mathfrak{A}$.

Whenever the special vertex $v$ and labeling $t$ of a marked alcove $(\mathfrak{A}, v, t)$ are understood and there is no danger of confusion, I abuse notation and refer to $\mathfrak{A}$ as
the marked alcove. Accordingly, if $\mathfrak{A}$ represents a marked alcove then its special
vertex is denoted by $v_{\mathfrak{A}}$, its labeling by $t_{\mathfrak{A}}$, its Weyl chamber by $C_{\mathfrak{A}}$, and the
hyperplane $t_{\mathfrak{A}}(s)$ is called simply “the wall of $\mathfrak{A}$ labeled by $s$”.

**Definition.** Two marked alcoves $(\mathfrak{A}, v, t)$ and $(\mathfrak{B}, w, s)$ are called Compatible iff
there exists $w \in \widetilde{W}$ such that $\mathfrak{B} = w(\mathfrak{A})$, $w = w(v)$ and $s = w \circ t$. The marked
alcoves are called NT-Compatible iff $w \notin \Lambda$.

Finally, the base alcove $\mathfrak{A}_o$ is given the tautological labeling, and all other alcoves
inherit (in general, multiple) labelings via the action of $\widetilde{W}$ in the obvious way:

**Definition.** The Base Labeling is the bijection $t_o$ from $\Delta_{aff}$ to the set of walls
of the base alcove $\mathfrak{A}_o$ defined by assigning to $s$ the unique wall of $\mathfrak{A}_o$ that is fixed
pointwise by $s$. The Base Marking is the marked alcove $(\mathfrak{A}_o, v_o, t_o)$.

For each $w \in \widetilde{W}$, the $w$-Labeling is defined to be the bijection $t_w \overset{\text{def}}{=} w \circ t_o$ from
$\Delta_{aff}$ to the set of walls of the alcove $w(\mathfrak{A}_o)$. The $w$-Marked Alcove is by definition
the triple $(w(\mathfrak{A}_o), w(v_o), t_w)$.

As before, I sometimes abuse notation by using $w(\mathfrak{A}_o)$ to refer to the $w$-marked
alcove. Note that $w(\mathfrak{A}_o)$ is compatible with $\mathfrak{A}_o$ and it is NT-compatible with $\mathfrak{A}_o$ if
and only if $w \notin \Lambda$.

**Remark 4.0.2.** When $\Omega = \{1\}$, alcoves are in bijection with $w$-marked alcoves
(simple-transitivity of affine Weyl groups on alcoves) and a labeling is essentially
just the assignment to every face of every alcove its type in the usual way.

The following operation will be used frequently in the limiting/inductive argu-
ments of §7.2 and §7.3:

**Definition.** For any marked alcove $(\mathfrak{A}, v, t)$ and any $s \in \Delta_{aff}$, the marked alcove
$(\mathfrak{A}, v, t)^s$ is by definition the triple $(s_H(\mathfrak{A}), s_H(v), s \circ t)$, where $H \overset{\text{def}}{=} t(s)$ is the
wall of $\mathfrak{A}$ labeled by $s$.

Note that if two marked alcoves $\mathfrak{A}$ and $\mathfrak{B}$ are NT-compatible then $\mathfrak{A}^s$ and $\mathfrak{B}^s$
are also NT-compatible for all $s \in \Delta_{aff}$.

I frequently use the fact that applying a sequence of various $* \mapsto *^s$ operations
to a single marked alcove results in a sequence of marked alcoves whose (un-marked)
alcoves form a gallery. Conversely, if $\mathfrak{B}_0$ is a marked alcove and $(\mathfrak{B}_0, \ldots, \mathfrak{B}_n)$ is
a gallery with no repeated alcoves, then each $\mathfrak{B}_i$ becomes a marked alcove in a
unique way, by iteratively applying $* \mapsto *^s$ operations across each intermediate
wall of the gallery. In this situation, the “label” of $\mathfrak{B}_i | \mathfrak{B}_{i+1}$ is understood to refer
to the element of $\Delta_{\text{aff}}$ corresponding to the wall $\mathcal{B}_i | \mathcal{B}_{i+1}$ relative to the labeling of $\mathcal{B}_i$ (or $\mathcal{B}_{i+1}$) inherited from $\mathcal{B}_0$.

5. DIAMOND PROPERTIES

5.1. lateral-conjugacy and the diamond property in the group.

Definition. $w, w' \in \tilde{W}$ are Laterally-Conjugate iff there exist $s_1, \ldots, s_n \in \Delta_{\text{aff}}$ such that $w' = s_n \cdots s_1 ws_1 \cdots s_n$ and $\ell(s_i \cdots s_1 ws_1 \cdots s_i) = \ell(w)$ for all $i$.

Any $w \in \tilde{W}$ is always considered to be laterally-conjugate to itself.

Definition. $w \in \tilde{W}$ has the Direct Diamond Property iff there exists $s \in \Delta_{\text{aff}}$ such that
- $sws \neq w$,
- $\ell(sw) > \ell(w)$, and
- $\ell(ws) > \ell(w)$.

By using the well-known Lemma 8.1.1, these three properties could be replaced by the single property “$\ell(sw) > \ell(w)$”, but this formulation is not as convenient for me.

Definition. $w \in \tilde{W}$ has the Diamond Property iff it is laterally-conjugate to an element with the Direct Diamond Property.

I frequently use the following geometric characterization of length:

Lemma 5.1.1. Let $w \in \tilde{W}$ and $s \in \Delta_{\text{aff}}$ be arbitrary. If $H \overset{\text{def}}{=} t_\circ(s)$ and $K \overset{\text{def}}{=} t_w(s)$ then
- $\ell(sw) > \ell(w)$ if and only if $\mathcal{A}_o$ and $w(\mathcal{A}_o)$ are on the same side of $H$,
- $\ell(ws) > \ell(w)$ if and only if $\mathcal{A}_o$ and $w(\mathcal{A}_o)$ are on the same side of $K$, and
- $\ell(w) = d(\mathcal{A}_o, w(\mathcal{A}_o))$.

Proof. When $\Omega = \{1\}$ this is all well-known: see Proposition (c) in §4.4 and Theorem (b) in §4.5 of [Hum90]. The more general statement is immediate by definition of the labeling $t_w$ because $\ell$ factors through $W_{\text{aff}}$ and $\Omega$ stabilizes $\mathcal{A}_o$. \qed

5.2. lateral-conjugacy and the diamond property in the apartment. Here are the gallery-theoretic versions of the above 3 definitions:

Definition. An ordered pair $(\mathcal{A}, \mathcal{B})$ of marked alcoves is Laterally-Conjugate to another pair $(\mathcal{A}', \mathcal{B}')$ iff there exists a gallery $(\mathcal{A}_0, \mathcal{A}_1, \ldots, \mathcal{A}_n)$ from $\mathcal{A}$ to $\mathcal{A}'$ and a gallery $(\mathcal{B}_0, \mathcal{B}_1, \ldots, \mathcal{B}_n)$ from $\mathcal{B}$ to $\mathcal{B}'$ such that
• \(d(\mathcal{A}_i, B_i) = d(\mathcal{A}, B)\), and
• both \(\mathcal{A}_i|\mathcal{A}_{i+1}\) and \(B_i|B_{i+1}\) have the same label.

for all \(i\).

Note the symmetry in the definition of lateral-conjugacy: \((\mathcal{A}, B)\) is laterally-conjugate to \((B', \mathcal{A}')\) if and only if \((B, A)\) is laterally-conjugate to \((B', A')\).

**Definition.** A pair \(\{A, B\}\) of marked alcoves has the Direct Diamond Property iff there exists \(s \in \Delta_{\text{aff}}\) such that

• \(t_A(s) \neq t_B(s)\),
• both alcoves are on the same side of \(t_A(s)\), and
• both alcoves are also on the same side of \(t_B(s)\).

Note that, due to the symmetry in the definition, the Direct Diamond Property refers to unordered pairs of alcoves.

**Definition.** A pair \(\{A, B\}\) of marked alcoves has the Diamond Property iff it is laterally-conjugate to a pair with the direct diamond property.

### 5.3. equivalence.
It is easy to show using the previous lemma that the two notions of Diamond Property coincide:

**Lemma 5.3.1.** The element \(w \in \tilde{W}\) has the Diamond Property if and only if the pair \(\{A_0, w(A_0)\}\) of marked alcoves has the Diamond Property.

**Proof.** By Lemma 5.1.1, \(\ell(sws) = d(A_0, sws(A_0))\). Since \(d\) is invariant under the diagonal action of \(W_{\text{aff}}\), \(d(A_0, sws(A_0)) = d(s(A_0), ws(A_0))\). By definition of the labelings \(t_\circ\) and \(t_w\), \(d(s(A_0), ws(A_0)) = d(A_0^s, w(A_0)^s)\). Altogether, \(\ell(sws) = d(A_0^s, w(A_0)^s)\). Since the operation \(* \mapsto *^s\) always creates galleries, this shows that the two notions of “lateral-conjugacy” are equivalent. Since \(s_H ws_H = w\) if and only if \(w(H) = H\), it follows that the condition \(s ws \neq w\) is equivalent to the condition \(t_w(s) \neq t_\circ(s)\). Finally, the fact that the remaining two statements in both Direct Diamond Properties are equivalent follows directly from Lemma 5.1.1. □

Although I will have no direct use for this in the remainder of the paper, note that the statement of Lemma 5.3.1 could be made much more specific: if \(w\) is laterally-conjugate to \(w'\) via the sequence \(s_1, \ldots, s_r\) then \((A_0, w(A_0))\) is laterally-conjugate to \((s_1 \cdots s_r(A_0), ws_1 \cdots s_r(A_0))\) via a gallery whose intermediate walls are labeled (in tandem) by the same sequence \(s_1, \ldots, s_r\), etc.
6. Basic Lemmas

Lemma 6.0.2. Let $H$ be a hyperplane in $\mathcal{A}$.

If both $\overline{\mathcal{C}}_o \cap H \neq \emptyset$ and $\overline{\mathcal{C}}_o^\text{opp} \cap H \neq \emptyset$ then in fact both $\overline{\mathcal{C}}_o \cap H \subset \partial \mathcal{C}_o$ and $\overline{\mathcal{C}}_o^\text{opp} \cap H \subset \partial \mathcal{C}_o^\text{opp}$.

Remark 6.0.1. I usually apply Lemma 6.0.2 in the following way: if $H \cap \mathcal{C}_o \neq \emptyset$ then $H \cap \overline{\mathcal{C}}_o^\text{opp} = \emptyset$.

Proof. Suppose $x \in H \cap \overline{\mathcal{C}}_o$ and $y \in H \cap \overline{\mathcal{C}}_o^\text{opp}$. Since $H$ is the null-set of an affine root, there exists $\beta \in \Sigma$ such that $x - y \in H_\beta$. On the other hand, $x \in \overline{\mathcal{C}}_o$ and $y \in \overline{\mathcal{C}}_o^\text{opp}$ implies $x - y \in \overline{\mathcal{C}}_o$. It is easy to check from the definitions of $\mathcal{C}_o$ and $\mathcal{C}_o^\text{opp}$ that if $x \in \mathcal{C}_o$ or $y \in \mathcal{C}_o^\text{opp}$ (or both) then necessarily $x - y \in \mathcal{C}_o$. But $H_\beta \cap \mathcal{C}_o = \emptyset$ since a root hyperplane can never intersect a Weyl chamber at $v_o$, so this is impossible and therefore both $x \notin \mathcal{C}_o$ and $y \notin \mathcal{C}_o^\text{opp}$. \hfill \Box

Lemma 6.0.3. Let $\mathfrak{A} \subset C_o$ be an alcove and $H$ a wall of $\mathfrak{A}$ that is not a wall of the Weyl chamber $\mathcal{C}_o$. Let $\mathfrak{B}$ be any alcove and $v_{\mathfrak{B}} \in \overline{\mathfrak{B}}$ a vertex.

If both $\mathfrak{A}$ and $\mathfrak{A}_o$ are on the same side of $H$ and $v_{\mathfrak{B}} \in \overline{\mathfrak{C}}_o^\text{opp}$ then $\mathfrak{B}$ is also on the same side of $H$ as $\mathfrak{A}$.

Proof. Let $\mathfrak{f} \subset \overline{\mathfrak{A}}$ be the face supported by $H$. If it were true that $\mathfrak{f} \subset \partial \mathcal{C}_o$ then necessarily $H$ would be a wall of $\mathcal{C}_o$. This is prohibited by hypothesis on $H$, so $H \cap \mathcal{C}_o \neq \emptyset$. Suppose for contradiction that $H$ separated $\mathfrak{B}$ from $\mathfrak{A}$. By hypothesis on $\mathfrak{A}$, $H$ must separate $\mathfrak{B}$ from $\mathfrak{A}_o$. By hypothesis on $v_{\mathfrak{B}}$, the set $\mathfrak{B} \cup \{v_{\mathfrak{B}}\} \cup \mathcal{C}_o^\text{opp} \cup \{v_o\} \cup \mathfrak{A}_o$ is path-connected and obviously $H \cap (\mathfrak{B} \cup \mathfrak{A}_o) = \emptyset$, so necessarily $H \cap \overline{\mathfrak{C}}_o^\text{opp} \neq \emptyset$. But this contradicts Lemma 6.0.2 since $H \cap \mathcal{C}_o \neq \emptyset$ is known already. \hfill \Box

Lemma 6.0.4. Let $C$ be a Weyl chamber at some (arbitrary) special vertex $v$.

If $v \in \overline{\mathcal{C}}_o^\text{opp}$ then either $C \cap \mathcal{C}_o = \emptyset$ or $C \supset \mathcal{C}_o$.

Proof. Suppose for contradiction that both $C \cap \mathcal{C}_o \neq \emptyset$ and $C \not\supset \mathcal{C}_o$. Choose an alcove $\mathfrak{A} \subset \mathcal{C}_o \setminus C$ and an alcove $\mathfrak{A}' \subset \mathcal{C}_o \cap C$. Let $(\mathfrak{A}_1, \ldots, \mathfrak{A}_n)$ be a minimal gallery from $\mathfrak{A}$ to $\mathfrak{A}'$. By choice of $\mathfrak{A}, \mathfrak{A}'$, there exists a wall $H$ of $C$ separating $\mathfrak{A}$ from $\mathfrak{A}'$. Such an $H$ must be an intermediate wall of the gallery (see Lemma 5.1.5 of [BGW03]), say $H = \mathfrak{A}_j \mathfrak{A}_{j+1}$. Since the gallery is minimal and both $\mathfrak{A}, \mathfrak{A}' \subset \mathcal{C}_o$, simplicial convexity of $\mathcal{C}_o$ forces $\mathfrak{A}_i \subset \mathcal{C}_o$ for all $i$. This means that $\mathcal{C}_o \cap H \neq \emptyset$, (for example, if $\mathfrak{f}$ is the common face of $\mathfrak{A}_j$ and $\mathfrak{A}_{j+1}$ then $\mathfrak{f} \subset \mathcal{C}_o \cap H$). But by hypothesis $v \in \overline{\mathcal{C}}_o^\text{opp}$ and obviously $v \in H$, so $\overline{\mathcal{C}}_o^\text{opp} \cap H \neq \emptyset$ also. This contradicts Lemma 6.0.2. \hfill \Box
Lemma 6.0.5. Let $C$ be a Weyl chamber at some (arbitrary) special vertex.

If $C_0 \cap C = \emptyset$ then there exists a minimal infinite gallery $(A_0 = A_0, A_1, A_2, \ldots)$ within $C_0$ such that

$$\lim_{i \to \infty} d(A_i, C) = \infty.$$  

(see §2.3 for the notions of distance $d(\ast, C)$ and infinite gallery)

Remark 6.0.2. It is not always possible to have a sequence of alcoves for which the sequence of distances is monotone increasing.

Proof. Choose some translation $t \in Q$ such that $t(\nu) \in C_0$ (e.g. translation by $2\rho^\vee$ where $\rho^\vee \overset{\text{def}}{=} \omega_1 + \cdots + \omega_r$ and $\omega_i$ are the fundamental coweights). I claim that the required gallery can be constructed by iterating $t$.

Let $t = s_0 s_1 \cdots s_{k-1}$ ($s_i \in \Delta_{\text{aff}}$) be a reduced expression and let $[i]$ be the remainder of $i$ mod $k$. Give $A_0 \overset{\text{def}}{=} A_0$ the base marking as usual and define a sequence of marked alcoves inductively by $A_{i+1} \overset{\text{def}}{=} A_i^{(i)}$ (so the finite subgallery $(A_0, A_1, \ldots, A_k)$ is just the usual gallery associated to the word $s_0 s_1 \cdots s_{k-1}$). By construction, this sequence is an infinite gallery in $C_0$. Since $\ell(t^N) = N \ell(t)$, a general property of dominant translations in an affine Weyl group, it follows that the infinite gallery $(A_0, A_1, A_2, \ldots)$ is minimal.

I first show that the infinite subsequence $A_{kN} = t^N(A_0), N = 0, 1, 2, \ldots$, diverges from $C$, and then I use the triangle-inequality to prove the full limit property.

For any alcove $A \subset A$ and radius $R \in \mathbb{R}$, let $B(A, R)$ be the set of alcoves $B \subset A$ such that $d(A, B) \leq R$. It is clear from the “cone” property of Weyl chambers and the boundedness of alcoves that for any $R \in \mathbb{R}$, there exists $n_R \in \mathbb{N}$ such that $B(t^N(A_0), R) \subset C_0$ for all $N \geq n_R$. It is also clear that $d(t^N(A_0), C) > R$ for all $N \geq n_R$ because otherwise there would be some alcove $B \subset C$ such that $d(t^N(A_0), B) \leq R$, but this would imply $B \subset B(t^N(A_0), R) \subset C_0$, which contradicts the hypothesis $C_0 \cap C = \emptyset$. This establishes the claim for the subsequence.

Let radius $R > 0$ be arbitrary. Fix $n \overset{\text{def}}{=} k \cdot n_{R+k}$ (recall $k = \ell(t)$). For any $N \in \mathbb{N}$, let $\lceil N \rceil$ be the largest $m \in \mathbb{N}$ such that $km \leq N$. Observe that if $N \geq n$ then $\lceil N \rceil \geq n_{R+k}$. Altogether, if $B \subset C$ is an arbitrary alcove and $N \geq n$ then

$$R + k < d(t^{\lfloor N \rfloor}(A_0), B) \leq d(t^{\lfloor N \rfloor}(A_0), A_N) + d(A_N, B)$$

Since $d(t^{\lfloor N \rfloor}(A_0), A_N) \leq \ell(t) = k$ by definition of $\lfloor N \rfloor$, it follows that $d(A_N, B) > R$. 

7. Proof of Main Theorem
7.1. case: the dominant chamber.

Proposition 7.1.1. Fix \( w \in \tilde{W} \).

If \( w \notin \Lambda \) and \( w(\mathfrak{A}_o) \subset \mathcal{C}_o \) then \( w \) has the Direct Diamond Property realized by some \( s \in \Delta_o \).

Proof. Let \( u \in W_o \) and \( t \in \Lambda \) be such that \( w = t \circ u \). By hypothesis, \( u \neq 1 \). Let \( H' \) be a wall of \( u(\mathcal{C}_o) \) which separates \( u(\mathfrak{A}_o) \) from \( \mathfrak{A}_o \), and note that \( v_o \in H' \). Let \( H \) be the wall of \( \mathfrak{A}_o \) such that \( u(H) = H' \) and let \( s \in \Delta_o \) be the element fixing \( H \) pointwise (in other words, \( H' = t_u(s) \)). I claim that \( s \) realizes the Direct Diamond Property for \( w \).

I first claim that \( t(H') \neq H' \). Suppose for contradiction that \( t(H') = H' \). Because \( t \) is a translation, \( u(\mathfrak{A}_o) \) and \( t(u(\mathfrak{A}_o)) \) are on the same side of \( t(H') = H' \). On the other hand, \( H' \) separates \( \mathfrak{A}_o \) from \( t(u(\mathfrak{A}_o)) = w(\mathfrak{A}_o) \). But \( w(\mathfrak{A}_o) \subset \mathcal{C}_o \) by hypothesis, so it is impossible for the root hyperplane \( H' \) to separate \( \mathfrak{A}_o \) from \( w(\mathfrak{A}_o) \).

By hypothesis that \( w(\mathfrak{A}_o) \subset \mathcal{C}_o \), it is automatic that \( \ell(sw) > \ell(w) \). To show that \( \ell(ws) > \ell(w) \), it suffices by Lemma 5.1.1 to show that both alcoves \( \mathfrak{A}_o \) and \( w(\mathfrak{A}_o) \) are on the same side of \( w(H) \). Let \( \alpha \in \Sigma \) be the positive root whose null-set is \( H' \). By choice of \( H' \), \( \langle \alpha, x \rangle_R < 0 \) for all \( x \in u(\mathfrak{A}_o) \). Since \( u(v_o) = v_o \) and \( w(v_o) \in \mathcal{C}_o \), it must be true that \( t(v_o) \in \mathcal{C}_o \). Since \( t \) is a translation, this implies that there exists \( n \in \mathbb{N} \) such that \( t(H') = w(H) \) is the null-set of \( \alpha - n \) and \( t(u(\mathfrak{A}_o)) = w(\mathfrak{A}_o) \) consists of points \( x \in \mathfrak{A} \) such that \( \langle \alpha, x \rangle_R < n \). Since \( t(H') \neq H' \), it must be true that \( n \geq 1 \). But \( 0 < \langle \alpha, x \rangle_R < 1 \) for all \( x \in \mathfrak{A}_o \) so \( \mathfrak{A}_o \) and \( w(\mathfrak{A}_o) \) are on the same side of \( w(H) \), as desired.

I now show that \( sws \neq w \). Suppose for contradiction that \( sws = w \). Then \( w(H) = H \), and since \( u(v_o) = v_o \), it follows that \( t(v_o) \in H \). Since \( t \) is a translation and \( v_o \in H \), \( t(H) = H \). Combining with \( w(H) = H \) implies \( u(H) = H \) also. But \( H' = u(H) \) so this contradicts \( t(H') \neq H' \).

Remark 7.1.1. Note that it is not important which of the two alcoves is considered the “base” alcove, nor is it important which chamber of the base alcove is considered “dominant”. In other words, if \( \mathfrak{A} \) and \( \mathfrak{B} \) are NT-compatible marked alcoves and \( \mathfrak{B} \subset \mathcal{C}_\mathfrak{A} \) then \( \{\mathfrak{A}, \mathfrak{B}\} \) has the Direct Diamond Property.

Remark 7.1.2. It is plausible that one might be able to prove the Diamond Property for general \( w \notin \Lambda \) by proving that there always exists a lateral conjugate \( w' \) of \( w \) such that one of \( w'(\mathfrak{A}_o) \) or \( \mathfrak{A}_o \) is contained in some Weyl chamber of the other. This latter statement is false. See Figure 7.1 for an example in the case of the exceptional affine Weyl group \( \tilde{G}_2 \).
7.2. case: the anti-dominant chamber.

Definition. Let \((\mathcal{B}_0, \ldots, \mathcal{B}_n)\) be a gallery, \(A\) an alcove, and \(H\) a wall of \(A\).

The triple \(((\mathcal{B}_0, \ldots, \mathcal{B}_n), A, H)\) is an Umbrella iff

1. all alcoves \(\mathcal{B}_i\) are on the same side of \(H\) as \(A\), and
2. \((\mathcal{B}_0, \ldots, \mathcal{B}_n)\) can be extended to a minimal gallery from \(\mathcal{B}_0\) to \(A\).

Observe that to say \((\mathcal{B}_0, \ldots, \mathcal{B}_n)\) can be extended to a minimal gallery from \(\mathcal{B}_0\) to \(A\) is the same as to say both that \((\mathcal{B}_0, \ldots, \mathcal{B}_n)\) is a minimal gallery itself and that each intermediate wall \(\mathcal{B}_i|\mathcal{B}_{i+1}\) \((0 \leq i < n)\) separates \(\mathcal{B}_i\) from \(A\) (I use this observation in the proof of Induction Lemma).

Induction Lemma. Let \((\mathcal{B}_0, \ldots, \mathcal{B}_n)\) be a gallery and \(v_{\mathcal{B}_0} \in \overline{\mathcal{B}_0}\) a special vertex. Let \(\mathcal{A}, \mathcal{A}' \subset C_0\) be (distinct) adjacent alcoves, separated by a wall \(H\). Let \(H'\) be a wall of \(\mathcal{A}'\). Assume that

1. \(((\mathcal{B}_0, \ldots, \mathcal{B}_{n-1}), \mathcal{A}, H)\) is an Umbrella,
2. \(v_{\mathcal{B}_0} \in \overline{\mathcal{C}_0}\),
3. the base alcove \(\mathcal{A}_0\) is on the same side of \(H'\) as \(\mathcal{A}'\),
4. \(H'\) is not a wall of the Weyl chamber \(C_o\), and
5. the wall \(\mathcal{B}_{n-1}|\mathcal{B}_n\) separates \(\mathcal{B}_{n-1}\) from \(\mathcal{A}\).

Then \(((\mathcal{B}_0, \ldots, \mathcal{B}_n), \mathcal{A}', H')\) is an Umbrella.

Proof. By hypotheses 2, 3, and 4, Lemma 6.0.3 implies that \(\mathcal{B}_0\) is contained on the same side of \(H'\) as \(\mathcal{A}'\). Since half-spaces are simplicially-convex, it therefore suffices to show only Umbrella Property 2, i.e. that \((\mathcal{B}_0, \ldots, \mathcal{B}_n)\) can be extended to a minimal gallery connecting \(\mathcal{B}_0\) to \(\mathcal{A}'\) (because then both endpoints of the gallery, and therefore the whole gallery, must be contained in that half-space).

Let \(H_i \overset{\text{def}}{=} \mathcal{B}_i|\mathcal{B}_{i+1}\) \((i = 0, \ldots, n - 1)\) be all the intermediate walls of the gallery \((\mathcal{B}_0, \ldots, \mathcal{B}_n)\). Note that \(H_i\) separates \(\mathcal{B}_i\) from \(\mathcal{A}\) for all \(0 \leq i < n - 1\) by hypothesis 1 (more specifically, Umbrella Property 2) and for \(i = n - 1\) by hypothesis 5. By the observation preceding this proof, it therefore suffices to show that the alcoves \(\mathcal{A}\) and \(\mathcal{A}'\) are on the same side of \(H_i\) for all \(0 \leq i \leq n - 1\). But this is obviously true: if the claim were false for \(H_i\), then necessarily \(H_i = H\), the only hyperplane separating \(\mathcal{A}\) from \(\mathcal{A}'\), which would mean that \(H\) separated \(\mathcal{B}_i\) from \(\mathcal{A}\), a contradiction to hypothesis 1 (more specifically, Umbrella Property 1). \(\square\)

Proposition 7.2.1. Fix \(w \in \mathcal{W}\).

If \(w \notin \Lambda\) and \(w(v_o) \in \overline{C_0}\) then \(w\) has the Diamond Property.
Remark 7.2.1. View (sequentially!) Figures 3 to 4 for a picture of the use of Induction Lemma in this proof.

Proof. Let $B$ be the $w$-marked alcove $w(A)$. By Lemma 5.3.1, it suffices to show that $\{A, B\}$ has the Diamond Property. By Lemma 6.0.4, either $C_o \cap C_B = \emptyset$ or $C_o \subseteq C_B$. If $C_o \subseteq C_B$ then the claim follows from Proposition 7.1.1 (using origin $v_B$ and dominant chamber $C_B$; see Remark 7.1.1). So, assume that $C_o \cap C_B = \emptyset$.

Applying Lemma 6.0.3 to the chambers $C_o$ and $C_B$ yields a certain infinite minimal gallery $(A_0, A_1, A_2, \ldots)$ within $C_o$. As usual, give $A_o$ the base marking and let $(s_0, s_1, s_2, \ldots)$ be the infinite sequence in $\Delta_{\text{aff}}$ such that $A_i = A_{s_0} \circ A_{s_1} \circ A_{s_2} \circ \ldots$, etc. Let $H_i$ be the wall of the marked alcove $A_i$ labeled by $s_i$.

Similarly, use the sequence $(s_0, s_1, s_2, \ldots)$ to define, relative to the prescribed labeling of $B$, a corresponding infinite gallery:

$$(B_0, B_1, B_2, \ldots) \overset{\text{def}}{=} (B, B^s_0, (B^s_0)^{s_1}, \ldots)$$

As before, each $B_i$ here represents a marked alcove. Note that by definition of the labeling $t_B = t_w$, the gallery $(B_0, B_1, B_2, \ldots)$ is simply the image under $w$ of the gallery $(A_0, A_1, A_2, \ldots)$. In particular, $(B_0, B_1, B_2, \ldots)$ is minimal and $B_i \subseteq C_B$ for all $i$.

Because the gallery $(A_0, A_1, A_2, \ldots)$ starts at $A_o$ and is contained completely within $C_o$, necessarily $s_0 = s_{\text{aff}}$. Because of this and the hypothesis on $v_B$, Lemma 6.0.3 says that alcoves $B_0$ and $A_0$ are on the same side of $H_0$, i.e. $d(B_0, A_0) = d(B_0, A_0) + 1$.

Let $K \overset{\text{def}}{=} t_{B_0}(s_0)$ be the wall of $B_0$ labeled by $s_0$. If $B_0$ and $A_1$ are on the same side of $K$ then necessarily $K \neq H_0$ (because $H_0$ separates $B_0$ from $A_1$) and both $B_0$ and $A_0$ are on the same side of $K$ (because $H_0$ is the unique hyperplane separating $A_0$ from $A_1$ and $H_0 \neq K$). It is then immediate from the definition that $\{A_0, B_0\} = \{A_0, B\}$ has the Direct Diamond Property (realized by $s_0$). Otherwise, $K$ separates $B_0$ from $A_1$ and by Lemma 6.1.1 $d(B_0, A_1) = d(B_0, A_1) - 1 = d(B_0, A_0) + 1 - 1 = d(B_0, A_0)$, i.e. $(A_1, B_1)$ is laterally-conjugate to $(A_0, B)$ via $s_0$.

In these circumstances, Induction Lemma implies that $(B_0, B_1, A_1, H_1)$ is an Umbrella:

- the non-numbered hypotheses of Induction Lemma are true by choice,
- hypothesis [1] is true because $(B_0, A_0, H_0)$ is trivially an Umbrella,
- hypothesis [2] is true by hypothesis on $w$,
- hypothesis [3] is true by choice of $H_1$ because $(A_0, A_1, A_2, \ldots)$ is minimal,
• hypothesis (4) is true by choice of $H_1$ because $\mathfrak{A}_1, \mathfrak{A}_2 \subset C$, and
• hypothesis (5) is true by the assumption that $\{\mathfrak{A}_0, \mathfrak{B}_0\}$ did not have the Direct Diamond Property for $s_0$ (see previous paragraph: by choice $K = \mathfrak{B}_0|\mathfrak{B}_1$).

So, the triple $((\mathfrak{B}_0, \mathfrak{B}_1), \mathfrak{A}_1, H_1)$ is an Umbrella by Induction Lemma. In particular, $d(\mathfrak{B}_1, \mathfrak{A}_2) = d(\mathfrak{B}_1, \mathfrak{A}_1) + 1$ by Umbrella Property (1).

I now iterate this process.

Let $K = t_{\mathfrak{B}_1}(s_1)$ be the wall of the marked alcove $\mathfrak{B}_1$ labeled by $s_1$. If $\mathfrak{B}_1$ and $\mathfrak{A}_2$ are on the same side of $K$ then $K \neq H_1$ (because $H_1$ separates $\mathfrak{B}_1$ from $\mathfrak{A}_2$) and both $\mathfrak{B}_1$ and $\mathfrak{A}_1$ are on the same side of $K$ (because $H_1$ is the unique hyperplane separating $\mathfrak{A}_1$ from $\mathfrak{A}_2$ and $H_1 \neq K$). It is then immediate from the definition that $\{\mathfrak{A}_1, \mathfrak{B}_1\}$ has the Direct Diamond Property realized by $s_1$ and therefore $\{\mathfrak{A}_0, \mathfrak{B}\}$, being laterally-conjugate to it, has the Diamond Property. Otherwise, $K$ separates $\mathfrak{B}_1$ from $\mathfrak{A}_2$ and by Lemma 5.1.1, $d(\mathfrak{B}_2, \mathfrak{A}_2) = d(\mathfrak{B}_1, \mathfrak{A}_2) - 1 = d(\mathfrak{B}_1, \mathfrak{A}_1) + 1 - 1 = d(\mathfrak{B}_1, \mathfrak{A}_1)$, i.e. $(\mathfrak{A}_2, \mathfrak{B}_2)$ is laterally-conjugate to $(\mathfrak{A}_1, \mathfrak{B}_1)$ via $s_1$, and therefore also laterally-conjugate to $(\mathfrak{A}_0, \mathfrak{B})$.

In these circumstances, Induction Lemma implies that $((\mathfrak{B}_0, \mathfrak{B}_1, \mathfrak{B}_2), \mathfrak{A}_2, H_2)$ is an Umbrella:

- the non-numbered hypotheses are again true by choice, the status of hypothesis (2) has not changed, and hypothesis (3) is true by choice of $H_2$ for the same reason as before,
- hypothesis (1) is known by the previous iteration,
- hypothesis (4) is true by choice of $H_2$ because $\mathfrak{A}_2, \mathfrak{A}_3 \subset C$, and
- hypothesis (5) is supplied by the assumption that $\{\mathfrak{A}_1, \mathfrak{B}_1\}$ did not have the Direct Diamond Property for $s_1$ (see previous paragraph: by choice $K = \mathfrak{B}_1|\mathfrak{B}_2$).

The above induction shows that if $n \in \mathbb{N}$ and $\{\mathfrak{A}_i, \mathfrak{B}_i\}$ does not have the Direct Diamond Property for all $i \leq n$ then $(\mathfrak{A}_i, \mathfrak{B}_i)$ is laterally-conjugate to $(\mathfrak{A}_0, \mathfrak{B})$ for all $i \leq n$, and in particular, $d(\mathfrak{A}_i, \mathfrak{B}_i) = (\mathfrak{A}_0, \mathfrak{B}) = \ell(w)$ for all $i \leq n$. But Lemma 6.0.5 says that $d(\mathfrak{A}_i, \mathfrak{B}_i) \to \infty$, so there must exist $i \in \mathbb{N}$ such that $\{\mathfrak{A}_i, \mathfrak{B}_i\}$ has the Direct Diamond Property. If $i \in \mathbb{N}$ is the smallest such index then $(\mathfrak{A}_0, \mathfrak{B})$ is laterally-conjugate to a pair with the Direct Diamond Property, as desired. □

Remark 7.2.2. Similar to the “dominant case” Proposition 7.1.1 the choices of vertex, alcove, and chamber are notationally convenient but otherwise totally unimportant to the conclusion of Proposition 7.2.1. In other words, if $\mathfrak{A}$ and $\mathfrak{B}$
are NT-compatible marked alcoves and $v_B \in C_{\mathfrak{A}}^{opp}$ then $\{\mathfrak{A}, \mathfrak{B}\}$ has the Diamond Property.

**Remark 7.2.3.** In type A, there is very simple proof that if $w(\mathfrak{A}) \subset C^{opp}$ then $w$ has the Diamond Property using at most one lateral-conjugation. Because of the extreme symmetry of type A, the inverse image $s^{-1}_{aff}(C) \cap C^{opp} \neq \emptyset$. Because of this, if $w$ does not already have the Direct Diamond Property, $s_{aff}$ laterally conjugates $w$ into the dominant chamber, in which case Proposition 7.1.1 applies. Of course, this idea fails in (almost?) every other type.

**7.3. case: the intermediate chambers.** I now prove that the general case can, at worst, be reduced to the anti-dominant case, Proposition 7.2.1:

**Main Theorem.** Suppose $A$ and $B$ are marked alcoves. If $A$ and $B$ are NT-compatible then $\{A, B\}$ has the Diamond Property.

In particular, if $w \in \tilde{W}$ and $w \notin \Lambda$ then $w$ has the Diamond Property.

**Remark 7.3.1.** View (sequentially!) Figures 5 to 7 for a picture of the iteration used in this proof.

**Proof.** Let $\mathcal{S}$ be the set of all $s \in \Delta_{aff}$ such that if $H \overset{\text{def}}{=} t_A(s)$ then the following three properties are true simultaneously: $v_A \in H$, both $A$ and $B$ are on the same side of $H$, and $v_B \notin H$. If $\mathcal{S} = \emptyset$ then by definition for every wall $H$ of $A$ containing $v_A$ either $H$ separates $A$ from $\mathfrak{B}$ or $v_B \in H$. In this case, $v_B \in C_{\mathfrak{A}}^{opp}$ and Proposition 7.2.1 applies. So assume that $\mathcal{S} \neq \emptyset$ and let $s \in \mathcal{S}$ be arbitrary.

Let $(\mathfrak{B}_0, \ldots, \mathfrak{B}_d)$ be a gallery realizing the distance $d(v_B, C_{\mathfrak{A}}^{opp})$ (see §2.3 for this notion of distance). By definition, this means that $v_B \in \overline{\mathfrak{B}}_0$, the gallery is minimal, and $\mathfrak{B}_d \subset C_{\mathfrak{A}}^{opp}$ (note that $\mathfrak{B}_0 \neq \mathfrak{B}$ is possible). Set $H \overset{\text{def}}{=} t_A(s)$. By definition of $\mathcal{S}$, $H$ separates $\mathfrak{B}$ from every alcove in $C_{\mathfrak{A}}^{opp}$ and $v_B \notin H$ so $H$ also separates $\mathfrak{B}_0$ from every alcove in $C_{\mathfrak{A}}^{opp}$. Therefore, $H$ must be an intermediate wall of the gallery $(\mathfrak{B}_0, \ldots, \mathfrak{B}_d)$ (see Lemma 5.1.5 of BGW03). Let $0 \leq j < d$ be the index such that $H = \mathfrak{B}_j | \mathfrak{B}_{j+1}$. Then the sequence of alcoves $(\mathfrak{B}_0, \ldots, \mathfrak{B}_j, s_H(\mathfrak{B}_{j+2}), \ldots, s_H(\mathfrak{B}_d))$ is a gallery. This is obviously a gallery “from $v_B$ to $C_{\mathfrak{A}}^{opp}$” and has fewer than $d$ alcoves. By choice of $s$ and the compatibility hypothesis, $v_{\mathfrak{B}_j} = v_B$. Altogether, $d(v_{\mathfrak{B}_j}, C_{\mathfrak{A}}^{opp}) < d(v_B, C_{\mathfrak{A}}^{opp})$.

On the other hand, by choice of $s$, Lemma 5.1.1 implies that $d(\mathfrak{A}', \mathfrak{B}') \geq d(\mathfrak{A}, \mathfrak{B})$. Since these distances $d(\mathfrak{A}', C_{\mathfrak{A}}^{opp})$ are N-valued, this means that one may iterate the previous process until a pair of alcoves $(\mathfrak{A}', \mathfrak{B}')$ is constructed which is
laterally-conjugate to \((A, B)\) and such that either \((A', B')\) has the Direct Diamond Property or \(v_{B'} \in C_{opp}^{A'}\), in which case Proposition 7.2.1 applies. □

**Remark 7.3.2.** If \(w \in W_{o} \subset \tilde{W}\) then \(w \notin \Lambda\) if and only if \(w \neq 1\), and it is easy to show that both \(\ell(ws_{aff}) > \ell(w)\) and \(\ell(s_{aff}w) > \ell(w)\) directly: combine the Exchange Property of the Coxeter group \((W_{aff}, \Delta_{aff})\) with the fact that all reduced expressions for a single element must use the same subset of \(\Delta_{aff}\) (see Proposition 7 in IV-§1-8 of [Bou02]) to conclude that neither length can decrease. It is tempting to think that such \(w\) always have the Direct Diamond Property realized by \(s_{aff}\) but this is not always true: in the affine Weyl group \(\tilde{C}_2\) (or \(\tilde{G}_2\)), there exists \(s \in \Delta_{o}\) such that \(s \cdot s_{aff} = s_{aff} \cdot s\), so \(sws = w\) for \(w \overset{\text{def}}{=} s \in W_{o}\).

8. **Application to Hecke Algebras**

8.1. **Hecke algebras on quasi-Coxeter groups.** Fix a function \(q : \Delta_{aff} \to \mathbb{N}\) which is invariant under conjugation by \(\tilde{W}\).

In the rest of this section I assume given a \(\mathbb{C}\)-algebra \(\mathcal{H}\) which, as a \(\mathbb{C}\)-vector space, has a basis of elements \(T_w\) indexed by all \(w \in \tilde{W}\). Further, denoting the ring operation by \(*\), I assume that the following Iwahori-Matsumoto identities are true in \(\mathcal{H}\): for all \(w \in \tilde{W}\) and \(s \in \Delta_{aff}\),

\[
T_s * T_w = \begin{cases} 
T_{sw} & \text{if } \ell(sw) > \ell(w) \\
(q(s) - 1)T_w + q(s)T_{sw} & \text{if } \ell(sw) < \ell(w)
\end{cases} \quad \text{(left-handed)}
\]

\[
T_w * T_s = \begin{cases} 
T_{ws} & \text{if } \ell(ws) > \ell(w) \\
(q(s) - 1)T_w + q(s)T_{ws} & \text{if } \ell(ws) < \ell(w)
\end{cases} \quad \text{(right-handed)}
\]

Note that because of the way that the length function \(\ell\) was extended to \(\tilde{W}\), if \(\tau \in \Omega\) and \(w \in \tilde{W}\) then \(T_{w\tau} = T_w * T_{\tau}\). If \(h \in \mathcal{H}\) then denote by \(h_w\) the coefficient of \(T_w\) in the linear combination of \(h\) with respect to this basis. If \(h \in \mathcal{H}\) and \(h_w \neq 0\) then \(w\) is said to support \(h\).

**Remark 8.1.1.** It is not difficult to show, and I do so in the upcoming [Ros] using ingredients from the Appendix to [PR08], that any Iwahori-Hecke algebra \(\mathcal{H}\) of any connected reductive affine algebraic \(F\)-group is of the form described above. Therefore, the results of this section apply to Iwahori-Hecke algebras. If greater generality is desired, one can use a pair \(a, b : \Delta_{aff} \to \mathbb{C}\) of parameter systems and a “generic algebra” as in §7.1 of [Hum90].

I will need the following slight extension of a well-known property of Coxeter groups:
Lemma 8.1.1. Fix $w \in \tilde{W}$ and $s,t \in \Delta_{aff}$.

If $\ell(sw) = \ell(w)$ and $\ell(sw) = \ell(wt)$ then $swt = w$.

Proof. When $\Omega = \{1\}$, this is exactly Lemma in §7.2 of [Hum90]. The general case follows immediately from this since $\Omega$ permutes $\Delta_{aff}$ and $\ell$ factors through $W_{aff}$. □

8.2. equations defining the center. Denote by $Z(H)$ the center of the ring $H$.

Fix $h \in H$. It is clear from the Iwahori-Matsumoto relations that $h \in Z(H)$ if and only if $h \ast T_s = T_s \ast h$ and $h \ast T_\tau = T_\tau \ast h$ for all $s \in \Delta_{aff}$ and $\tau \in \Omega$.

Fix $s \in \Delta_{aff}$. For each $x \in \tilde{W}$, one can use the left-handed Iwahori-Matsumoto relation to compute that the coefficient of $T_x$ in $T_s \ast h$ is

$$q(s)h_{sx} \text{ if } \ell(sx) > \ell(x)$$
$$h_{sx} + (q(s) - 1)h_x \text{ if } \ell(sx) < \ell(x)$$

Similarly, one can use the right-handed Iwahori-Matsumoto relation to compute that the coefficient of $T_\tau$ in $h \ast T_s$ is

$$q(s)h_{xs} \text{ if } \ell(xs) > \ell(x)$$
$$h_{xs} + (q(s) - 1)h_x \text{ if } \ell(xs) < \ell(x)$$

It is obvious from the Iwahori-Matsumoto identities that $h \ast T_\tau = T_\tau \ast h$ if and only if $h_{xt} = h_{\tau x}$ for all $x \in \tilde{W}$.

It follows that the center $Z(H)$ is the $\mathbb{C}$-subspace of vectors $h \in H$ whose Iwahori-Matsumoto coefficients $h_x$ solve the (infinite) linear system consisting of the equation $h_{xt} = h_{\tau x}$ for each pair $(x, \tau) \in \tilde{W} \times \Omega$ together with the appropriate equation from

$$q(s)h_{sx} = q(s)h_{xs} \text{ if } \ell(sx), \ell(xs) > \ell(x)$$
$$q(s)h_{sx} = h_{xs} + (q(s) - 1)h_x \text{ if } \ell(sx) > \ell(x) > \ell(xs)$$
$$h_{sx} + (q(s) - 1)h_x = q(s)h_{xs} \text{ if } \ell(sx) < \ell(x) < \ell(xs)$$
$$h_{sx} = h_{xs} \text{ if } \ell(sx), \ell(xs) < \ell(x)$$

for each pair $(x,s) \in \tilde{W} \times \Delta_{aff}$.

Remark 8.2.1. These equations appeared already in §3 of [Hai01] for extended affine Weyl groups of root data.

8.3. length-filtration and dimensions. Recall that $\Omega(w)$ denotes the projection of $w \in \tilde{W}$ into $\Omega$, and that the set of $W_\circ$-conjugacy classes in $\Lambda$ is denoted by $\Lambda/W_\circ$. 
**Definition.** Fix \( L \in \mathbb{N} \) and \( \tau \in \Omega \).

Define \( Z_{L,\tau}(\mathcal{H}) \) to be the set of all \( z \in Z(\mathcal{H}) \) such that \( z_w = 0 \) if either \( \ell(w) > L \) or \( \Omega(w) \neq \tau \).

Note that each \( Z_{L,\tau}(\mathcal{H}) \) is a finite-dimensional \( C \)-subspace of \( Z(\mathcal{H}) \) and that \( Z(\mathcal{H}) \) is the union of all \( Z_{L,\tau}(\mathcal{H}) \).

Recall that if \( \mathcal{O} \in \Lambda/W_0 \) then \( \ell \) is constant on \( \mathcal{O} \) and define \( \ell(\mathcal{O}) \) to be this constant length. It follows that any two \( t, t' \in \mathcal{O} \) are laterally-conjugate.

**Definition.** Fix \( L \in \mathbb{N} \) and \( \tau \in \Omega \).

Define \( N_{L,\tau} \) to be the total number of conjugacy classes \( \mathcal{O} \in \Lambda/W_0 \) such that \( \ell(\mathcal{O}) \leq L \) and \( \Omega(t) = \tau \) for all \( t \in \mathcal{O} \).

The following two lemmas show how lateral-conjugacy and the diamond property are related to centers of Hecke algebras:

**Lemma 8.3.1.** Suppose \( z \in Z(\mathcal{H}) \).

If \( w \in \tilde{W} \) is laterally-conjugate to \( w' \) then \( z_w = z_{w'} \).

**Proof.** If \( s \in \Delta_{\text{aff}} \) is such that \( \ell(sw) = \ell(w) \) then either \( \ell(sw) < \ell(w) < \ell(ws) \) or \( \ell(ws) < \ell(w) < \ell(sw) \). Choosing \( x \equiv ws \) in the former case and \( x \equiv sw \) in the latter case, centrality equation (2) implies that \( z_w = z_{ws} \), and the claim follows immediately from this. \( \square \)

**Lemma 8.3.2.** Suppose \( z \in Z(\mathcal{H}) \).

If \( w \in \tilde{W} \) has the Diamond Property then there exist \( u, v \in \tilde{W} \) satisfying \( \ell(u) > \ell(v) > \ell(w) \) such that \( z_w \) is a \( C \)-linear combination of \( z_u \) and \( z_v \).

**Proof.** By Lemma 8.3.1 I may assume that \( w \) has the Direct Diamond Property. Let \( s \in \Delta_{\text{aff}} \) be the element realizing the property. By basic Coxeter theory, \( \ell(sw) = \ell(w) + 1 = \ell(ws) \) and it is true that either \( \ell(sws) = \ell(w) + 2 \) or \( \ell(sws) = \ell(w) \). If it were true that \( \ell(sws) = \ell(w) \) then by Lemma 8.1.1 it would be true that \( sws = w \), but this is explicitly prohibited by the Direct Diamond Property. Therefore, \( \ell(sws) > \ell(sw) = \ell(ws) > \ell(w) \). Choosing \( x \equiv ws \) and applying centrality equation (1) proves that \( z_w \) is a linear combination of \( z_{ws} \) and \( z_{sws} \), as desired. \( \square \)

**Remark 8.3.1.** Materially, both lemmas 8.3.1 and 8.3.2 appeared already as Lemma 3.1 of [Hai01].
Proposition 8.3.1. Fix $L \in \mathbb{N}$ and $\tau \in \Omega$.

$$\dim_{\mathbb{C}}(Z_{L,\tau}(\mathcal{H})) \leq N_{L,\tau}$$

Proof. Suppose $w \in \tilde{W}$ and $w \notin \Lambda$. Consider the linear system defining $Z(\mathcal{H})$ as a $\mathbb{C}$-subspace of $\mathcal{H}$. By applying the Main Theorem and Lemma 8.3.2 repeatedly, one can express the variable $h_w$ as a $\mathbb{C}$-linear combination of variables $h_x$ such that either $\ell(x) > L$ or $x \in \Lambda$. Since $Z_{L,\tau}(\mathcal{H})$ is the $\mathbb{C}$-subspace of $Z(\mathcal{H})$ defined by the additional equations $h_x = 0$ for all $x \in \tilde{W}$ such that either $\ell(x) > L$ or $\Omega(x) \neq \tau$, it follows that $\dim_{\mathbb{C}}(Z_{L,\tau}(\mathcal{H}))$ is at most the total number of $t \in \Lambda$ such that $\ell(t) \leq L$ and $\Omega(t) = \tau$. On the other hand, if $O \in \Lambda/W_\circ$ and $t, t' \in O$ then $t$ is laterally-conjugate to $t'$ and $\Omega(t) = \Omega(t')$, so the dimension bound now follows from Lemma 8.3.1. \hfill \Box

Note one extra detail from the proof: if $z \in Z_{L,\tau}(\mathcal{H})$ and both $w \notin \Lambda$ and $\ell(w) = L$ then $z_w = 0$.

Corollary. Suppose that for each conjugacy class $O \in \Lambda/W_\circ$ there is an element $z_O \in Z(\mathcal{H})$ such that $\ell(w) \leq \ell(O)$ for all $w \in \tilde{W}$ supporting $z_O$ and such that $\Omega(w)$ is the same for all $w \in \tilde{W}$ supporting $z_O$.

If $\{z_O\}_{O \in \Lambda/W_\circ}$ is linearly-independent then it is a basis for $Z(\mathcal{H})$.

Proof. Fix $L \in \mathbb{N}$ and $\tau \in \Omega$. Consider only those $z_O$ for which $\ell(O) \leq L$ and for which the uniform $\Omega$-component of those $w$ supporting $z_O$ is $\tau$. By hypothesis, the set of all such $z_O$ is a linearly-independent subset of $N_{L,\tau}$ vectors in the subspace $Z_{L,\tau}(\mathcal{H})$. By Proposition 8.3.1 it must be a spanning set. Since $Z(\mathcal{H})$ is the union over all pairs $(L, \tau)$ of the subspaces $Z_{L,\tau}(\mathcal{H})$, the claim follows. \hfill \Box
The Main Theorem does not follow directly from Proposition 7.1.1 (the dominant case). The blue alcove in the center is the base alcove $\mathfrak{A}_0$ and the red alcoves surrounding it constitute a full lateral-conjugacy class. The light blue/pink cones are the unique dominant Weyl chamber containing the various alcoves (uniqueness is due to the fact that in this $G_2$ example each alcove contains only one special vertex in its closure).
Figure 2. Sample initial situation in the proof of Proposition 7.2.1. The purple alcove is $A_\circ$ and the red alcove is $B$. The light blue cone is the dominant Weyl chamber $C_\circ$ and the pink cone is the chamber $C_B$. 
Figure 3. After 3 iterations of Induction Lemma. The purple alcove is $A'$ all possibilities for $H'$ (only one in this case) are also purple. All other alcoves in the gallery $A$, are blue, in particular the alcove $A$ adjacent to $A'$. The blue wall is $H$. The red alcoves constitute the gallery $B$. Observe that the current $H$ is always (one of) the previous $H'$. 
Figure 4. After 4 iterations of Induction Lemma. Observe that (for the first time) the purple alcove and the nearest red alcove (its lateral conjugate) have the Direct Diamond Property. Nonetheless, the conclusion of Induction Lemma remains true for several more iterations. An important observation is that the conclusion of Induction Lemma must eventually fail because the Diamond Property is true.
Figure 5. Situation in which the $\mathcal{S} \neq \emptyset$ in the proof of the Main Theorem (although in this particular example, the pair of alcoves already has the Direct Diamond Property realized by $s_{aff}$ so no action is necessary). The blue alcove is $\mathfrak{A}$, the red alcove is $\mathfrak{B}$, and the grey cone is $C_{\mathfrak{A}}^{opp}$. The black outline is merely a visual aid.
Figure 6. After 1 iteration of the process described in the proof of the Main Theorem. The value $d(v_B, C_{\alpha}^{\text{opp}})$ has decreased vs. Figure 5. The light blue/pink alcoves merely represent the previous positions of the blue/red alcoves.
Figure 7. After 2 iterations of the process described in the proof of the Main Theorem. The value $d(v_B, C_{\overline{\alpha}}^{\text{opp}})$ is now 0 and $v_B \in C_{\overline{\alpha}}^{\text{opp}}$. In this particular example, it is possible to laterally conjugate once more to arrange $\mathcal{B} \subset C_{\overline{\alpha}}^{\text{opp}}$, but this is not always possible.
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