AN ALGORITHM TO OBTAIN LINEAR DETERMINANTAL REPRESENTATIONS OF SMOOTH PLANE CUBICS OVER FINITE FIELDS

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Abstract. We give a brief report on our computations of linear determinantal representations of smooth plane cubics over finite fields. After recalling a classical interpretation of linear determinantal representations as rational points on the affine part of Jacobian varieties, we give an algorithm to obtain all linear determinantal representations up to equivalence. We also report our recent study on computations of linear determinantal representations of twisted Fermat cubics defined over the field of rational numbers. This paper is a summary of the author’s talk at the JSIAM JANT workshop on algorithmic number theory in March, 2016. Details will appear elsewhere.

1. Introduction

Let $k$ be a field, and let
\[ F(X, Y, Z) = a_{000}X^3 + a_{001}X^2Y + a_{002}X^2Z + a_{011}XY^2 + a_{012}XZ^2 + a_{111}Y^3 + a_{112}Y^2Z + a_{122}YZ^2 + a_{222}Z^3 \]
be a ternary cubic form with coefficients in $k$ defining a smooth plane cubic $C \subset \mathbb{P}^2$. The cubic $C$ is said to admit a linear determinantal representation over $k$ if there are a nonzero constant $0 \neq \lambda \in k$ and three square matrices $M_0, M_1, M_2 \in \text{Mat}_3(k)$ of size 3 satisfying
\[ F(X, Y, Z) = \lambda \cdot \det(M), \]
where we put $M := XM_0 + YM_1 + ZM_2$. Two linear determinantal representations $M, M'$ of $C$ are said to be equivalent if there are invertible matrices $A, B \in \text{GL}_3(k)$ satisfying $M' = AMB$.

Studying linear determinantal representations of smooth plane cubics is a classical topic in linear algebra and algebraic geometry ([Vin89], [Dol12]). Recently, they appear in the study of the derived category of smooth plane cubics ([Gal14]) and the theory of space-time codes ([DG08]). They have been studied from arithmetic viewpoints ([FN14], [II16a], [Ish15]).

In this note, we study linear determinantal representations over finite fields. We give an algorithm to obtain all linear determinantal representations of smooth plane cubics up to equivalence. This paper is a summary of the author’s talk at the JSIAM JANT workshop on algorithmic number theory in March, 2016. Details will appear elsewhere.

2. Linear determinantal representations and rational points

Let $k$ be a field, and $F(X, Y, Z) \in k[X, Y, Z]$ a ternary cubic form with coefficients in $k$ defining a smooth plane cubic $C \subset \mathbb{P}^2$. We fix projective coordinates $X, Y, Z$ of $\mathbb{P}^2$. The following theorem gives an interpretation of linear determinantal representations of $C$ in terms of non-effective line bundles on $C$. It is well-known at least when $k$ is an algebraically closed field of characteristic zero. For the proof valid for arbitrary fields, see [Bea00 Proposition 3.1], [Ish15 Proposition 2.2].
Theorem 2.1. There is a natural bijection between the following two sets:

- the set of equivalence classes of linear determinantal representations of $C$ over $k$, and
- the set of isomorphism classes of non-effective line bundles on $C$ of degree 0.

The bijection is obtained as follows: we take a non-effective line bundle $L$ of degree 0 on $C$. Let $ι: C \hookrightarrow \mathbb{P}^2$ be the given embedding. We denote the homogeneous coordinate ring of $\mathbb{P}^2$ by

$$R := \Gamma_*(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}) = \bigoplus_{n \in \mathbb{Z}} H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(n)) \cong \Gamma[1][X, Y, Z].$$

The graded $R$-module $N = \Gamma_*(\mathbb{P}^2, ι_*\mathcal{L}) \cong \Gamma_*(C, \mathcal{L})$ has a minimal free resolution of the form

$$0 \longrightarrow R(-2) \otimes_k W_1 \xrightarrow{M} R(-1) \otimes_k W_0 \longrightarrow N \longrightarrow 0,$$

where $W_0, W_1$ are 3-dimensional $k$-vector spaces [Bea00 Proposition 3.1]. The homomorphism $M$ can be expressed by a square matrix $M$ of size 3 with coefficients in $k$-linear forms in three variables $X, Y, Z$. We can check $M$ gives a linear determinantal representation of $C$, and its equivalence class depends only on the isomorphism class of the line bundle $\mathcal{L}$.

When $k$ is an algebraically closed field, the set $\text{Pic}^0(C)$ of isomorphism classes of line bundles on $C$ of degree 0 is parametrized by the group $\text{Jac}(C)(k)$ of $k$-rational points on the Jacobian variety $\text{Jac}(C)$ of $C$, and the only effective line bundle of degree 0 corresponds to the origin $O$ of $\text{Jac}(C)(k)$. In general, there can be a difference between $\text{Pic}^0(C)$ and $\text{Jac}(C)(k)$ which is measured by the relative Brauer group ([CK12 Theorem 2.1], [Ish15 Example 6.9]). When $C$ has a $k$-rational point $P_0$, the relative Brauer group vanishes, and two sets $\text{Pic}^0(C)$ and $\text{Jac}(C)(k)$ are identified. We have a bijection

$$C(k) \xrightarrow{1:1} \text{Pic}^0(C) = \text{Jac}(C)(k) \ ; \ P \mapsto \mathcal{O}_C(P - P_0).$$

Hence we obtain the following corollary.

Corollary 2.2. Let $C$ be a smooth plane cubic over $k$ with a $k$-rational point $P_0 \in C(k)$. There is a natural bijection between the following two sets:

- the set of equivalence classes of linear determinantal representations of $C$ over $k$, and
- the set $C(k) \setminus \{P_0\}$ of $k$-rational points on $C$ different from $P_0$.

3. An algorithm to obtain linear determinantal representations

Let us make the bijection in Theorem 2.1 explicit. In this section, we give an algorithm to obtain linear determinantal representations of smooth plane cubics over an arbitrary field $k$. In this algorithm, we do not assume that $C$ has a $k$-rational point.

Algorithm 3.1.

Input:: a ternary cubic form $F(X, Y, Z)$ with coefficients in $k$ defining a smooth plane cubic $C \subset \mathbb{P}^2$ with respect to a fixed projective coordinates $X, Y, Z$, and a $k$-rational non-effective line bundle $\mathcal{L}$ on $C$ of degree 0.

Output:: a linear determinantal representation of $C$ over $k$ corresponding to $\mathcal{L}$.

Step 1 (Global Section): Compute a $k$-basis $\{v_0, v_1, v_2\}$ of the 3-dimensional $k$-vector space $H^0(C, \mathcal{L}(1))$.

Step 2 (First Syzygy): Compute a $k$-basis $\{e_0, e_1, e_2\}$ of the kernel of the multiplication map

$$H^0(C, \mathcal{L}(1)) \otimes H^0(C, \mathcal{O}_C(1)) \to H^0(C, \mathcal{L}(2)).$$
**Step 3 (Output Matrix):** Write the $k$-basis $\{e_0, e_1, e_2\}$ as

$$e_i = \sum_{j=0}^{2} v_j \otimes l_{i,j}(X, Y, Z),$$

where $l_{i,j}(X, Y, Z) \in H^0(C, O_{C}(1))$ are $k$-linear forms. Output the matrix

$$M = (l_{i,j}(X, Y, Z))_{0 \leq i,j \leq 2}.$$

By the sequence (1), we have

$$W_0 \cong H^0(C, \mathcal{L}(1))$$

and

$$W_1 \cong \text{Ker} \left( H^0(C, \mathcal{L}(1)) \otimes H^0(C, O_{C}(1)) \rightarrow H^0(C, \mathcal{L}(2)) \right).$$

Using $k$-bases of $W_0$ and $W_1$, we obtain an explicit matrix representation $M$ of the map $\tilde{M}$ in (1). This $M$ gives a linear determinantal representation corresponding to $\mathcal{L}$ in the bijection of Theorem 2.1.

4. **AN EXPLICIT FORMULA ON LINEAR DETERMINANTAL REPRESENTATIONS OF SMOOTH PLANE CUBICS WITH RATIONAL POINTS**

We apply Algorithm 3.1 to a smooth plane cubic $C$ with a $k$-rational point $P_0$. By changing the projective coordinates, we may assume that $P_0 = [1 : 0 : 0]$ and the tangent line of $C$ at $P_0$ is the line $(Z = 0)$. The following theorem gives an explicit formula of the bijection in Corollary 2.2. For the proof, see [Ish16, Theorem 4.1].

**Theorem 4.1.** Let $C \subset \mathbb{P}^2$ be a smooth plane cubic over an arbitrary field $k$ with a $k$-rational point $P_0 = [1 : 0 : 0]$. Assume that the tangent line of $C$ at $P_0$ is the line $(Z = 0)$. We have the following formula for a linear determinantal representation $M_P$ of $C$ over $k$ corresponding to a point $P = [s : t : u] \in C(k) \setminus \{P_0\}$ via Corollary 2.2.

**Case 1:** If $u \neq 0$, the equivalence class of linear determinantal representations of $C$ corresponding to $P$ is given by

$$M_P = \begin{pmatrix}
0 & Z & -Y \\
-uY - tZ & 0 & L_0(X, Y, Z) \\
uX - sZ & L_1(X, Y, Z) & L_2(X, Y, Z)
\end{pmatrix},$$

where we denote

$$L_0(X, Y, Z) := -u^2X - (a_{011}t^2 + a_{012}tu + a_{022}u^2 + su)Z,$$

$$L_1(X, Y, Z) := u^2a_{011}X + u^2a_{111}Y + u(a_{111}t + a_{112}u)Z,$$

$$L_2(X, Y, Z) := u(a_{011} + a_{012}u)X + (a_{111}t^2 + a_{112}tu + a_{122}u^2)Z.$$

**Case 2:** If $u = 0$, the equivalence class of linear determinantal representations of $C$ corresponding to $P$ is given by

$$M_P = \begin{pmatrix}
0 & Z & -Y \\
Z & a_{011}Y & \tilde{L}_0(X, Y, Z) \\
\tilde{L}_1(X, Y, Z) & \tilde{L}_2(X, Y, Z) & \tilde{L}_3(X, Y, Z)
\end{pmatrix},$$
where we denote
\[ \tilde{L}_0(X, Y, Z) := X + a_{012}Y + a_{022}Z, \]
\[ \tilde{L}_1(X, Y, Z) := a_{011}X + a_{111}Y, \]
\[ \tilde{L}_2(X, Y, Z) := a_{111}X + (a_{012}a_{111} - a_{011}a_{112})Y, \]
\[ \tilde{L}_3(X, Y, Z) := (a_{022}a_{111} - a_{011}a_{122})Y - a_{011}a_{222}Z. \]

Remark 4.2. Let \( k \) be a field of characteristic not equal to 2 nor 3, and (4)
\[ E : (Y^2Z - X^3 - aXZ^2 - bZ^3 = 0) \subset \mathbb{F}^2 \]
an elliptic curve over \( k \) with origin \( P_0 = [0 : 1 : 0] \) defined by a Weierstrass equation. Let \( P = [\lambda : \mu : 1] \in E(k) \setminus \{P_0\} \) be a \( k \)-rational point on \( E \). Galinat gave in [Gal14] Lemma 2.9 a representative of linear determinantal representations of \( E \) over \( k \) corresponding to the divisor \( P - P_0 \) as
\[ M_P := \begin{pmatrix} X - \lambda Z & 0 & -Y - \mu Z \\ \mu Z - Y & X + \lambda Z & (a + \lambda^2)Z \\ 0 & Z & -X \end{pmatrix}. \]

When \( C \) has a \( k \)-rational flex, Theorem 4.1 is equivalent to Galinat’s formula. However, Theorem 4.1 is also applicable when \( C \) has no \( k \)-rational flex.

When \( k \) is algebraically closed of characteristic not equal to 2 nor 3, Vinnikov [Vin89] gave other representatives.

5. Applications to Linear Determinantal Representations over Finite Fields

Let \( p \) be a prime number, and \( m \geq 1 \) a positive integer. Let \( \mathbb{F}_q \) be a finite field with \( q = p^m \) elements. It is well-known that any smooth plane cubic \( C \) over \( \mathbb{F}_q \) has an \( \mathbb{F}_q \)-rational point ( [Lan55]), hence we can freely use Corollary 2.2 and Theorem 4.1 (at least after some changes of coordinates).

In [Ish16], we determine projective equivalence classes of smooth plane cubics over \( \mathbb{F}_q \) with 0, 1 or 2 equivalence classes of linear determinantal representations. We denote by \( \text{Cub}_q(n) \) the set of projective equivalence classes of smooth plane cubics over \( \mathbb{F}_q \) with \( n \) equivalence classes of linear determinantal representations. By Corollary 2.2, the number of elements \( \text{Cub}_q(n) \) coincides with the number of projective equivalence classes of smooth plane cubics over \( \mathbb{F}_q \) with \( n + 1 \) \( \mathbb{F}_q \)-rational points. The latter can be determined by Schoof’s formula [Sch87]. The following table summarizes the results of our computations of \( \text{Cub}_q(n) \) when \( 0 \leq n \leq 2 \).

| \( \text{Cub}_q(0) \) | \( \text{Cub}_q(1) \) | \( \text{Cub}_q(2) \) |
|-----------------|-----------------|-----------------|
| \#Cub_q(0) | 1 | 1 | 1 | 0 | 0 | \#Cub_q(q \geq 8) | 0 |
| \#Cub_q(1) | 1 | 1 | 1 | 1 | 0 | 0 |
| \#Cub_q(2) | 2 | 2 | 4 | 2 | 2 | 0 |

The following ternary cubic forms are representatives of \( \text{Cub}_q(0) \). They do not admit linear determinantal representations. Each of them has only one rational point \( [1 : 0 : 0] \):

- \( X^2Z + XZ^2 + Y^3 + Y^2Z + Z^3 \) over \( \mathbb{F}_2 \).
- \( X^2Z + Y^3 - YZ^2 + Z^3 \) over \( \mathbb{F}_3 \).
- \( X^2Z + XZ^2 + Y^3 + \omega Z^3 \) over \( \mathbb{F}_4 \), where \( \omega \in \mathbb{F}_4 \) satisfies \( \omega^2 + \omega + 1 = 0 \).

The following ternary cubic forms are representatives of \( \text{Cub}_q(1) \). Each of them admits a unique equivalence class of linear determinantal representations. Their rational points are \( [1 : 0 : 0] \) and \( [0 : 0 : 1] \):

- \( X^2Z + XYZ + Y^3 + Y^2Z + YZ^2 \) over \( \mathbb{F}_2 \).
By Theorem 4.1, linear determinantal representations correspond to $F_3$.

Example 5.1. Consider the smooth plane cubic over $F_3$. The rational points are $[1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1]$.

We give some examples of linear determinantal representations of the cubics in the above list without $k$-rational flexes.

Example 5.2. Consider the smooth plane cubic over $F_2$. The rational points are $[1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1]$.

The following ternary cubic forms are representatives of $\text{Cub}_k(2)$. Each of them admits two equivalence classes of linear determinantal representations.

- $X^2Z - Y^2Z + 2Y^2Z$ over $F_5$.
- $X^2Z + \omega XY + Z^2 + 2\omega YZ$ over $F_4 = F_2[\omega]$.
- $X^2Z + Y^3 + 2Y^2Z$ over $F_5$.

We give some examples of linear determinantal representations of the cubics in the above list without $k$-rational flexes.

Example 5.1. Consider the smooth plane cubic over $F_2$ defined by

$$X^2Z + XY^2 + YZ^2 = 0.$$ 

This cubic has three $F_2$-rational points;

$$P_0 = [1 : 0 : 0], P_1 = [0 : 1 : 0], P_2 = [0 : 0 : 1].$$

By Theorem 4.1, linear determinantal representations corresponding to $P_1, P_2$ are

$$\begin{pmatrix} 0 & Z & Y \\ Z & Y & X \\ X & 0 & Y \end{pmatrix}, \begin{pmatrix} 0 & Z & Y \\ Z & Y & X \\ X & 0 & Y \end{pmatrix}.$$

Example 5.2. Consider the smooth plane cubic over $F_5$ defined by

$$X^2Z + XY^2 + YZ^2 - 2XYZ = 0.$$ 

This cubic has three $F_5$-rational points;

$$P_0 = [1 : 0 : 0], P_1 = [0 : 1 : 0], P_2 = [0 : 0 : 1].$$

By Theorem 4.1, linear determinantal representations corresponding to $P_1, P_2$ are

$$\begin{pmatrix} 0 & Z & -Y \\ Z & Y & X - 2Y \\ X & 0 & -Y \end{pmatrix}, \begin{pmatrix} 0 & Z & -Y \\ Z & Y & X - 2Y \\ X & 0 & -Y \end{pmatrix}. $$
6. twisted Fermat cubics over the field of rational numbers

In this final section, we report our recent study on computations of linear determinantal representations of twisted Fermat cubics defined over the field \( \mathbb{Q} \) of rational numbers.

Over the field \( \mathbb{Q} \) of rational numbers, some problems arise. The main problem is that a line bundle \( \mathcal{L}_P \) on \( C \) is usually given by the corresponding \( k \)-rational point \( P \) on \( \text{Jac}(C) \), not on \( C \). This causes some problems in Step 1; the calculation of the \( \mathbb{Q} \)-vector space \( H^0(C, \mathcal{L}_P(1)) \).

Using the generalized Clifford algebra and the norm equation, we overcome these problems for twisted Fermat cubics. We give simple examples. Details will appear elsewhere.

**Example 6.1.** Consider the smooth plane cubic over \( \mathbb{Q} \) defined by
\[
X^3 + Y^3 + Z^3 = 0. 
\]

Its Jacobian variety is an elliptic curve whose Weierstrass equation is given by
\[
Y^2 Z - 9 Y Z^2 - X^3 + 27 Z^3 = 0 
\]  
(cf. [AR-VT05]). Its \( \mathbb{Q} \)-rational points are
\( \mathcal{O}, [3 : 0 : 1], [3 : 9 : 1] \).

Let us take \( P = [3 : 0 : 1] \). The equivalence class of linear determinantal representations corresponding to \( P \) is represented by
\[
\begin{pmatrix}
-X + 2 Y + Z & -2 X + Y & X + Y \\
X - Y & X + Z & -Y \\
X & 2 X + 3 Y & -2 Y + Z
\end{pmatrix}.
\]

The other point \( [3 : 9 : 1] \) corresponds to the transpose of the above matrix. This cubic does not admit a symmetric determinantal representation over \( \mathbb{Q} \) (cf. [II16b]).

**Example 6.2.** Consider the smooth plane cubic over \( \mathbb{Q} \) defined by
\[
2 X^3 + 2^3 Y^3 + Z^3 = 0. 
\]

Its Jacobian variety is an elliptic curve whose Weierstrass equation is given by
\[
Y^2 Z - 9 \cdot 2^3 Y Z^2 - X^3 + 27 \cdot 2^6 Z^3 = 0 
\]  
(cf. [AR-VT05]). It is isomorphic to (5), and its \( \mathbb{Q} \)-rational points are
\( \mathcal{O}, [3 \cdot 2^2 : 0 : 1], [3 \cdot 2^2 : 9 \cdot 2^3 : 1] \).

However, in [Ish15], we prove that these points do not correspond to linear determinantal representations over \( \mathbb{Q} \) due to non-vanishing obstruction in the relative Brauer group of this cubic. This cubic does not admit a linear determinantal representation over \( \mathbb{Q} \).

**Example 6.3.** Consider the smooth plane cubic over \( \mathbb{Q} \) defined by
\[
17 X^3 + 17^2 Y^3 + Z^3 = 0. 
\]

Its Jacobian variety is an elliptic curve whose Weierstrass equation is given by
\[
Y^2 Z - 9 \cdot 17^3 Y Z^2 - X^3 + 27 \cdot 17^6 Z^3 = 0 
\]  
(cf. [AR-VT05]). It is isomorphic to (5), and its rational points are
\( \mathcal{O}, [3 \cdot 17^2 : 0 : 1], [3 \cdot 17^2 : 9 \cdot 17^3 : 1] \).

Let us take \( P = [3 \cdot 17^2 : 0 : 1] \). The equivalence class of linear determinantal representations corresponding to \( P \) is represented by
\[
\begin{pmatrix}
3 X - 2 Y + Z & -34 X + 153 Y & 17 X - 51 Y \\
\frac{1}{17} X - \frac{1}{17} Y & -3 X - 4 Y + Z & 4 X + 7 Y \\
\frac{1}{17} X + \frac{4}{17} Y & -2 X - Y & 6 Y + Z
\end{pmatrix}.
\]
The other point \([3 \cdot 17^2 : 9 \cdot 17^3 : 1]\) corresponds to the transpose of the above matrix.

Acknowledgements

The author would like to thank sincerely to Professor Tetsushi Ito for various and inspiring comments. The work of the author was supported by JSPS KAKENHI Grant Number 16K17572.

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