Security Details for Bit Commitment by Transmitting Measurement Outcomes

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We spell out details of a simple argument for a security bound for the secure relativistic quantum bit commitment protocol of Ref. \textsuperscript{1}.

\section*{Introduction}

Recently, a new quantum relativistic bit commitment protocol \textsuperscript{1} was introduced. Its security relies, essentially, on the impossibility of completing a nonlocal measurement on a distributed state outside the joint future light cone of its components. Its implementation requires minimal quantum resources: the receiver needs to send quantum states (which can be unentangled qubits) to the committer, who needs to carry out individual measurements on them as soon as they are received. No further quantum communication is required by either party; nor do they require any entanglement, collective measurements, or quantum state storage.

We present the protocol here in an idealized form assuming perfect state preparations, transmissions and measurements. We also make idealizations about the relativistic geometry and signalling speed, supposing that Alice and Bob each have agents in secure laboratories infinitesimally separated from the points $P$, $Q_0$ and $Q_1$. Alice can signal at precisely light speed, and all information processing is instantaneous. We discuss here the simplest version of the scheme using qubit states and measurements in the standard BB84 basis \textsuperscript{2}.

Alice and Bob agree on a space-time point $P$, a set of coordinates $(x, y, z, t)$ for Minkowski space, with $P$ as the origin, and (in the simplest case) two points $Q_0 = (x, 0, 0, x)$ and $Q_1 = (-x, 0, 0, x)$ light-like separated from $P$. They each have agents, separated in secure laboratories, adjacent to each of the points $P$, $Q_0$, $Q_1$. To simplify for the moment, we take the distances from the labs to the relevant points as negligible.

Bob securely prepares a set of qubits $|\psi_i\rangle_{i=1}^N$ independently randomly chosen from the BB84 states $\{|0\rangle, |1\rangle, |+\rangle, |-\rangle\}$ (where $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$) and sends them to Alice to arrive (essentially) at $P$. To commit to the bit value 0, Alice measures each state in the $\{|0\rangle, |1\rangle\}$ basis, and sends the outcomes over secure classical channels to her agents at $Q_0$ and $Q_1$. To commit to 1, Alice measures each state in the $\{|+\rangle, |-\rangle\}$ basis, and sends the outcomes as above. Alice’s secure classical channels could, for example, be created by pre-sharing one-time pads between her agent at $P$ and those at $Q_0$ and $Q_1$ and sending pad-encrypted classical signals. If necessary or desired, these pads could be periodically replenished by quantum key distribution links between the relevant agents.

To unveil her committed bit, Alice’s agents at $Q_0$ and $Q_1$ reveal the measurement outcomes to Bob’s agents there. After comparing the revealed data to check that the declared outcomes on both wings are the same (somewhere in the intersection of the future light cones of $Q_0$ and $Q_1$), and that both are consistent with the list of states sent at $P$, Bob accepts the commitment and unveiling as genuine. If the declared outcomes are different, Bob has detected Alice cheating.

\section*{Security}

The protocol is evidently secure against Bob, who learns nothing about Alice’s actions until (if) she chooses to unveil the bit.

Alice is constrained in that she has to be able to reveal her commitment data at both $Q_0$ and $Q_1$, since Bob’s agents at these points verify the timing and location of the unveilings, and then later compare the data to check they are consistent. We need to show that, if she is able to do so then, essentially (up to some small probability defined in terms of a security parameter) she was committed at $P$. (See Ref. \textsuperscript{2} for a more formal discussion of security in terms of a space-time oracle model.)

By Minkowski causality, Alice’s ability to unveil data consistent with a 0 or 1 commitment at $Q_0$ depends only on operations she carries out on the line $PQ_0$. Suppose that she has a strategy in which she carries out some operations at $P$, but these leave her significantly uncommitted, in the sense that her optimal strategies $S_i$ for successfully unveiling the bit values $i$, by carrying out suitable operations in the causal future of $P$, have success probabilities $p_i$, with $p_0 + p_1 > 1 + \delta$, for some $\delta > 0$. By Minkowski causality, any operations she carries out on the half-open line segment $(P, Q_0)$ cannot affect the probability of producing data at $Q_1$ consistent with a successful unveiling of either bit value $i$ there. In particular, if she follows the instructions of strategy $S_0$ on $(P, Q_0)$, and the instructions of strategy $S_1$ on $(P, Q_1)$, she has probabilities $p_i$ of producing data consistent with a successful unveiling of bit value 0 at $Q_0$ and with a successful unveiling of bit value 1 at $Q_1$.

This means that, with probability at least $\delta$, by combining her data at $Q_0$ and $Q_1$ at some point in their joint causal future, Alice can produce data consistent with both sets of measurements in complementary bases. Thus, for
example, for each state $|\psi_i\rangle$, she can identify a subset of 2 states from \{0, 1, +, −\}, one from each basis, which must include $|\psi_i\rangle$.

**Lemma 1.** Given a single BB84 state $|\psi\rangle$, randomly chosen from the uniform distribution, unknown to her, Alice’s probability $p$ of choosing one of the subsets $S_1 = \{0\}, S_2 = \{1\}, S_3 = \{+\}, S_4 = \{−\}$, that includes $|\psi\rangle$, is bounded by $p \leq \frac{1}{2}(1 + \frac{1}{\sqrt{2}})$, for any strategy. An optimal strategy which realises this bound is to carry out the POVM

\[
\{\frac{1}{2}P_1, \frac{1}{2}P_2, \frac{1}{2}P_3, \frac{1}{2}P_4\}
\]

where $P_i$ is the projection onto the qubit $|\phi_i\rangle = \cos(\theta_i)|0\rangle + \sin(\theta_i)|1\rangle$, $\theta_i = i(\pi/4) - (\pi/8)$, and given the outcome $P_i$, she guesses the subset $S_i$.

**Proof.**
Recall first the standard state discrimination problem, in which Bob chooses a state from the set $\{\sigma_j\}$ with associated probabilities $\{p_j\}$. Alice makes a measurement to try to determine the state. Her measurement may be described by a POVM $\{\pi_j\}$, where outcome $\pi_j$ leads her to choose state $\sigma_j$ \(^1\). The probability that Alice identifies the state correctly is

\[
P_{corr} = \sum_j p_j \text{Tr}(\sigma_j \pi_j).
\]

In the variation here, Bob prepares a random state from the BB84 set. Alice gets two guesses at the state. These guesses must be non-orthogonal BB84 states. If either guess is correct, she wins. We use these states and the corresponding density matrices, $\hat{\sigma}_i$, where

\[
\frac{1}{2}\text{Tr}(1 - \hat{\sigma}_i) = \text{corr}
\]

for $\rho = \text{corr}$, where

\[
\{\rho_i\} = 1_\text{H}
\]

and $P_{corr} = \text{Tr}(\hat{\Gamma})$.

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\(^1\) Note that the number of outcomes does not have to equal the number of states prepared - in the general case there may not be a POVM element for every $j$. 

For our transformed state discrimination problem we must calculate the operator
\[
\hat{\Gamma} = \frac{1}{4} \sum_{i=1}^{4} (\frac{1}{2} \hat{\rho}_i + \frac{1}{2} \hat{\rho}_{i+1}) \hat{\pi}_i.
\] (7)
and show that
\[
\hat{\Gamma} - \frac{1}{8} \hat{\rho}_i - \frac{1}{8} \hat{\rho}_{i+1} \geq 0, \quad \text{for } i = 1, 2, 3, 4.
\] (8)
It is straight-forward to verify that the POVM \( \{ \hat{\pi}_i \} \) satisfies this condition. For this set
\[
\hat{\Gamma} = \frac{1}{8} (1 + \frac{1}{\sqrt{2}}) \hat{I}.
\] (9)
Allowing for the factor of 2 above, we obtain Alice’s optimal guessing probability for the original problem as
\[
P_{\text{win}} = 2 \text{Tr}(\hat{\Gamma}) = \frac{1}{2} (1 + \frac{1}{\sqrt{2}}). \quad \text{QED}
\] (10)

**Lemma 2.** Suppose now Alice is given a sequence of i.i.d. BB84 states \( |\psi_i\rangle^N \), randomly chosen from the uniform distribution, and unknown to her, and is allowed to perform a strategy \( \mathcal{S} \) involving arbitrary collective operations. Let \( p_{i_1,\ldots,i_{N-1};j_1,\ldots,j_{N-1}} \) be her probability of choosing a subset from the list \( S_1 = \{|0\rangle,|+\rangle\}, S_2 = \{|+\rangle,|1\rangle\}, S_3 = \{|1\rangle,|\rangle\}, S_4 = \{|\rangle,\{0\rangle\} \), that includes the BB84 state \( |\psi_N\rangle \), conditioned on the first \( (N-1) \) states supplied being \( \{|e_{i_1}\}, \ldots, |e_{i_{N-1}}\rangle \) and her guesses being \( S_{i_1}, \ldots, S_{j_{N-1}} \) respectively, where the strategy \( \mathcal{S} \) implies this is a possible list of guesses for the inputs. Then \( p_{i_1,\ldots,i_{N-1};j_1,\ldots,j_{N-1}} \leq \frac{1}{2} (1 + \frac{1}{\sqrt{2}}) \), for any strategy \( \mathcal{S} \) and any \( \{i_1,\ldots,i_{N-1};j_1,\ldots,j_{N-1}\} \) consistent with \( \mathcal{S} \).

**Proof.**
Suppose some collective strategy \( \mathcal{S} \) violated this bound for some values \( \{i_1,\ldots,i_{N-1};j_1,\ldots,j_{N-1}\} \). Alice could then proceed as follows.

1. Prepare an entangled singlet state of two qubits,
2. Prepare \( (N-1) \) BB84 states \( |e_{i_1}\rangle, \ldots, |e_{i_{N-1}}\rangle \).
3. Apply strategy \( \mathcal{S} \) (ignoring her knowledge of the BB84 states prepared) to the \( (N-1) \) BB84 states and one qubit of the entangled states,
4. For the first \( (N-1) \) states, check the guesses produced by \( \mathcal{S} \),
5. If the results do not agree with \( \{S_{j_1},\ldots,S_{j_{N-1}}\} \), return to step 1 with a new singlet and a new batch of BB84 states. If they do agree, proceed to step 6.
6. Apply a teleportation operation on the unknown BB84 state \( |\psi_N\rangle \) and the other singlet qubit, obtaining teleportation unitary \( \hat{U} \). Complete the implementation of strategy \( \mathcal{S} \), obtaining a guess at a subset containing the teleported unknown qubit \( \hat{U} |\psi_N\rangle \). Apply the inverse \( \hat{U}^\dagger \) to obtain a guess at a subset \( S_i \) containing \( |\psi_N\rangle \). By assumption, this guess is correct with probability \( p_{i_1,\ldots,i_{N-1};j_1,\ldots,j_{N-1}} > \frac{1}{2} (1 + \frac{1}{\sqrt{2}}) \).

This iterated strategy is bound to proceed to step 6 eventually, and \( |\psi_N\rangle \) is left isolated until step 6 is reached. Alice thus has a strategy that produces a subset guess for any single unknown state \( |\psi_N\rangle \), with success probability \( p > \frac{1}{2} (1 + \frac{1}{\sqrt{2}}) \), contradicting Lemma 1. QED

**Theorem 1.** Alice’s probability \( p_N \) of being able to produce data consistent with measurements in complementary BB84 bases for \( N \) random uniformly i.i.d. unknown BB84 states obeys \( p_N \leq \left( \frac{1}{2} (1 + \frac{1}{\sqrt{2}}) \right)^N \). Hence, the security parameter \( \delta \) in the bit commitment protocol above obeys \( \delta \leq \left( \frac{1}{2} (1 + \frac{1}{\sqrt{2}}) \right)^N \).

**Proof.** follows from Lemma 2.

Note that this argument easily extends to give security bounds for large \( N \) in the presence of noise and errors, so long as the total noise and error rate is below \( \left( \frac{1}{2} - \frac{1}{\sqrt{2}} \right) \). To see this let \( Z_l = \sum_{k=1}^{l} j_k - l \frac{1}{2} (1 + \frac{1}{\sqrt{2}}) \), where \( j_k = 1 \) if Alice’s subset guess on the \( k \)-th state is correct and \( j_k = 0 \) otherwise. We have \( |Z_l| < \infty, |Z_l - Z_{l-1}| \leq \frac{1}{2} (1 + \frac{1}{\sqrt{2}}) \).
and (from Lemma 2) $E(Z_l|W_k) \leq Z_k$ for all $l > k$, where $W_k$ is the set of subset guess outcomes up to state $k$. So $Z_l$ is a supermartingale and the Azuma-Hoeffding inequality implies

$$\text{Prob}(\sum_{k=1}^{N} j_k \geq N(\frac{1}{2}(1 + \frac{1}{\sqrt{2}}) + \epsilon)) \leq \exp(-N\epsilon^2/(2(\frac{1}{2}(1 + \frac{1}{\sqrt{2}}))^2),$$

(11)

for any $\epsilon > 0$.

Notes

1. After the work reported above was completed, an independent security analysis following different arguments was circulated by Kaniewski et al. [7].

2. An alternative proof of Lemmas 1 and 2 and Theorem 1 follows by noting that for any collective guessing strategy of Alice’s, any particular subset guess $S_i$ for the $N$-th state, conditioned on input states $|e_i1\rangle, \ldots, |e_iN-1\rangle$ and guesses $\{S_{j1}, \ldots, S_{jN-1}\}$ for the first $(N-1)$ guessing games, must be represented by some positive operator $A = A^\dagger \geq 0$ on the $N$-th state. Since the states are i.i.d. and uniformly distributed, the probability this guess is correct is

$$\text{Tr}(4(\hat{\rho}_i + \hat{\rho}_{i+1}))/\frac{1}{2}\text{Tr}(A),$$

(12)

which is easily seen to be bounded by $\frac{1}{2}(1 + \frac{1}{\sqrt{2}})$, for any value of $i$. That is, Alice’s maximum confidence quantum measurement [5] on the $N$-th state is unaltered if she carries out collective measurements.

Moreover, this implies a further security result. Alice’s maximum confidence measurement on the $N$-th state cannot improve on this success bound even if her strategy allows her sometimes to make no guess on some states (possibly including the $N$-th). Hence the protocol remains secure for large $N$ in the presence of any loss level (as reported by Alice) below 1. That is, it remains secure even if Alice is allowed to report a large fraction of her measurements as giving no result, so long as she tells Bob at (essentially) the point $P$ which measurements were successful.

3. Another proof of Theorem 1 is given by verifying that a minimum error measurement for $N$ BB84 states is obtained by taking the $N$-fold tensor product of the POVM [11]. This follows straightforwardly using the method given above for $N = 1$.

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