Perfectly packing a square by squares of sidelength $f(n)^{-t}$

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Abstract

In this paper, we prove that for any $1/2 < t < 1$, there exists a positive integer $N_0$ depending on $t$ such that for any $n_0 \geq N_0$, squares of sidelength $f(n)^{-t}$ for $n \geq n_0$ can be packed with disjoint interiors into a square of area $\sum_{n=n_0}^{\infty} f(n)^{-2t}$, if the function $f$ satisfies some suitable conditions. The main theorem (Theorem 1.1) is a generalization of Tao’s theorem in [15], which argued the case $f(n) = n$. As corollaries, we prove that there are such packings of squares when $f(n)$ represents the $n$th element of either an arithmetic progression or the set of prime numbers. In these cases, we give effective lower bounds for $N_0$ with respect to $t$. Furthermore, we consider the case that $f(n)$ represents the $n$th element of the set of twin primes and prove that squares of sidelength $f(n)^{-t}$ for $n \geq n_0$ can be packed with disjoint interiors into a slightly larger square than theoretically expected.

1 Introduction

Let $\Omega$ be a region in $\mathbb{R}^n$. It is said that a finite or countably infinite number of squares in $\mathbb{R}^n$ are packed into $\Omega$ if they are included in $\Omega$ with disjoint interiors. If the region $\Omega$ is covered by these squares up to null subsets, then we say that $\Omega$ is packed perfectly by these squares. We also call this situation a perfectly packing of $\Omega$ by squares.

Meir and Moser [12] presented the problem of whether rectangles of dimension $\frac{1}{n} \times \frac{1}{n+1}$ for $n \geq 1$ can be packed perfectly into a square of area 1. They also posed the question whether squares of sidelength $n^{-1}$ for $n \geq 2$ can be packed perfectly into a square of area $\frac{n}{n+1} - 1$. Currently these questions are still unsolved. One of the best current results on these problems is the work of Paulhus [13], who obtained an algorithm for packing squares of sidelength $\frac{1}{n} \times \frac{1}{n+1}$ for $n \geq 1$ with disjoint interiors into a square of area $1 + \frac{1}{10^{2t}}$, and that for packing squares of sidelength $n^{-1}$ for $n \geq 2$ with disjoint interiors into a square of area $\frac{n}{n+1} - 1 + \frac{1}{124918662}$. There were some incorrect lemmas in Paulhus’ paper. However, Grzegorek and Januszewski [3] modified Paulhus’ argument and gave completely correct proofs.

Another approach for the second question of Meir and Moser is to consider whether squares of sidelength $n^{-t}$ for $n \geq 2$ can be packed with disjoint interiors into a square of area $\zeta(2t) - 1$ for $t > 1/2$. The goal is to prove that there exists such a packing for $t = 1$. Januszewski and Zielonka [7] proved that there is a proper packing for $1/2 < t \leq 2/3$ by combining ideas in [2], [8], [10] and Paulhus’ algorithm [13] for packing squares. They also considered the 3-dimensional case, i.e., the problem of perfectly packing a cube by cubes. Furthermore, there are several papers considering perfect packing of the $d$-dimensional cubes of sidelength $1^{-t}$, $2^{-t}$, $3^{-t}$, ... for special values of $t$ (see [5] and [9], for example).

Recently Tao [15] conditionally improved their work. He proved that for any $1/2 < t < 1$, there exists a large positive integer $N_0$ depending on $t$ such that for any integer $n_0 \geq N_0$,
squares of sidelength $n^{-t}$ for $n \geq n_0$ can be packed perfectly into a square of area $\sum_{n=n_0}^{\infty} n^{-2t}$. His new idea is to arrange the squares in near-lattice formations, which enables us to take $t$ arbitrarily close to 1. He also remarked that his method will also be applicable to the problem of packing squares of dimension $\frac{1}{n^t} \times \frac{1}{n^t}$ into a square of area $\sum_{n=n_0}^{\infty} \frac{1}{n(n+1)^t}$. More recently, McClenagan [11] obtained the 3-dimensional analogue of Tao’s result. His theorem is also an improvement of the result of Januszewski and Zielonka [4] mentioned above.

In this paper, for $q > r \geq 0$, we call the sequence $\{(qn + r)^{-1}\}$ an AP-harmonic sequence, and call the sequence of reciprocals of prime numbers the P-harmonic sequence. Furthermore, we call the sequence of reciprocals of twin primes the TP-harmonic sequence.

We use a non-standard notation to estimate the size of functions effectively. When two functions $F$, $G$ satisfy $|F(x)| \leq G(x)$ over some specified range, we denote this relation by $F(x) = O_1(G(x))$.

The purpose of this paper is to generalize Tao’s theorem to the problem of packing of squares of sidelength $f(n)^{-t}$ for $1/2 < t < 1$ and give some effective lower bounds for the size of $N_0$ depending on $t$ in some specific cases. The result includes (an effective version of) Tao’s theorem. We will prove the main theorem below in Section 2. In Section 3, as a corollary of Theorem 1.1, we will prove a result on perfectly packing a square by squares of nearly AP-harmonic sidelength. In Section 4, we will prove a similar result in the case of packing of squares of nearly P-harmonic sidelength. Finally, in Section 5, we will give an ineffective version of Theorem 1.1 and give a packing of squares of nearly TP-harmonic sidelength into a slightly larger square than the expected size.

**Theorem 1.1.** Let $1/2 < t < 1$ and put $\delta := 1 - t$. Suppose that there exist positive constants $c_1, d_1(i = 1, 2), c_{\lambda, \nu}, l, K$, functions $\nu_i, \lambda_i : \mathbb{N} \to \mathbb{R}_{>0}$ ($i = 1, 2$), positive functions $\xi_i, \eta_i$ on the open interval $(1/2, 1)$ ($i = 1, 2$), positive integers $M, N_0, N_1 (N_1 < N_0)$ and a smooth increasing function $f : \mathbb{R}_{\geq 1} \to \mathbb{R}_{>0}$ such that for any positive integers $n_0 \geq N_0, n_1 \geq N_1$, the following ten conditions hold:

\[ f(n_0) \geq K f((1 + l)n_0), \quad K \geq (2/3)^{\frac{1}{t}}, \tag{1.1} \]

\[ c_1 \frac{\xi_1(t)\nu_1(n_1)}{f(n_1)^{1+\delta t}} \leq \sum_{n=1}^{n_1-1} \frac{1}{f(n)^{1+\delta t}} \leq c_2 \frac{\xi_2(t)\nu_2(n_1)}{f(n_1)^{1+\delta t}}, \tag{1.2} \]

\[ d_1 \frac{\eta_1(t)\lambda_1(n_1)}{f(n_1)^{2t}} \leq \sum_{n=n_1}^{\infty} \frac{1}{f(n)^{2t}} \leq d_2 \frac{\eta_2(t)\lambda_2(n_1)}{f(n_1)^{2t}}, \tag{1.3} \]

\[ \lambda_1(n_0) \geq c_{\lambda, \nu} \nu_2(n_0), \tag{1.4} \]

\[ \lambda_2(n_0) \leq d_2^{-1} \eta_2(t)^{-1} f(n_0)^{2t}, \tag{1.5} \]

\[ M \geq \max \left\{ \frac{4}{1+t}, \left( \frac{c_2 \xi_2(t)}{c_{\lambda, \nu} d_1 \eta_1(t)} \right)^{\frac{1}{1+\delta t}}, \left( \frac{50}{K^{1(2-t)}} \right)^{\frac{1}{1+\delta t}} \right\}, \tag{1.6} \]

\[ \max_{x \in [n_0, n_0 + 9Mf]} f(x) \leq \frac{1}{4752} M^{-4} f(n_0), \tag{1.7} \]

\[ f(n_0 + 9Mf) \leq (264M^2)^{\frac{1}{t}} f(n_0), \tag{1.8} \]

\[ 9Mf(n_0)^t \leq ln_0, \tag{1.9} \]

\[ \nu_1(n_0) \lambda_2(n_0)^{-\frac{1+\delta t}{t}} \geq \left( \frac{d_2 \eta_2(t)}{c_1 \xi_1(t)} \right)^{\frac{1+\delta t}{t}} M^{1-\frac{4}{t}}, \tag{1.10} \]

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Then, for any \( n_0 \geq N_0 \), squares of sidelength \( f(n)^{-t} \) for \( n \geq n_0 \) can be packed perfectly into a square of area \( \sum_{n=n_0}^{\infty} f(n)^{-2t} \).

## 2 Proof of the main theorem

Given a rectangle \( R \), we define the width \( w(R) \) of \( R \) to be the smaller of two side lengths of \( R \), and define the height \( h(R) \) of \( R \) to be the larger one. If \( R \) is a square, then \( w(R) = h(R) \). The area of \( R \) is equal to \( w(R)h(R) \). Given a family of rectangles \( R \), we define the total area of \( R \) by

\[
\text{area}(R) = \sum_{R \in \mathcal{R}} w(R)h(R),
\]

and define its unweighted total perimeter by

\[
\text{perim}(R) = 2 \sum_{R \in \mathcal{R}} (w(R) + h(R)).
\]

### 2.1 Efficiently packing a small rectangle of bounded eccentricity

**Proposition 2.1.** Let \( 1/2 < t < 1 \) and \( M \) be a positive integer. Suppose that a smooth increasing function \( f : \mathbb{R}_{\geq 1} \to \mathbb{R}_{>0} \) and any positive integer \( n_0 \geq N_0 \) satisfy (1.7) and (1.8). Let \( R \) be a rectangle with

\[
M f(n_0)^{-t} \leq w(R) \leq h(R) \leq 3M f(n_0)^{-t}.
\]

Then, there exists a positive integer \( n_0' \) satisfying \( M^2 \leq n_0' - n_0 \leq 9M^2 \) such that \( R \) is perfectly packed by squares of sidelength \( f(n)^{-t} \) \((n_0 \leq n < n_0')\) and a family of rectangles \( R \) with disjoint interiors whose total perimeter satisfies

\[
\text{perim}(R) \leq 25M f(n_0)^{-t}
\]

and the width of each rectangle is at most \( 2f(n_0)^{-t} \).

**Proof of Proposition 2.1.**

We may assume that \( R \) is the rectangle \([0, w(R)] \times [0, h(R)]\). Then by (2.1), there exist positive integers \( M_1, M_2 (M \leq M_1 \leq M_2 \leq 3M) \) satisfying

\[
M_1 f(n_0)^{-t} \leq w(R) < (M_1 + 1) f(n_0)^{-t}, \quad M_2 f(n_0)^{-t} \leq h(R) < (M_2 + 1) f(n_0)^{-t}.
\]

Put \( n_0' := n_0 + M_1 M_2 \). Then \( M^2 \leq n_0' - n_0 \leq 9M^2 \). Rewrite the set \( \{n \mid n_0 \leq n < n_0 + M_1 M_2\} \) as \( \{n_{i,j} \mid 0 \leq i < M_1, 0 \leq j < M_2\} \), where

\[
n_{i,j} := n_0 + jM_1 + i.
\]

Let \( S_{i,j} \) be a square of sidelength \( f(n_{i,j})^{-t} \). Following the idea in [15], we arrange this square by

\[
S_{i,j} = [x_{i,j}, x_{i,j} + f(n_{i,j})^{-t}] \times [y_{i,j}, y_{i,j} + f(n_{i,j})^{-t}],
\]

where

\[
x_{i,j} := w(R) - \sum_{i' = i}^{M_1} f(n_{i',j})^{-t},
\]

\[
y_{i,j} := \sum_{j' = 0}^{j-1} f(n_{i,j'})^{-t} (j \geq 1), \quad y_{i,0} := 0
\]
for $1 \leq i < M_1, 1 \leq j < M_2$. We need some analysis on $f$. By mean value theorem,

$$f(n_{i,j}) = f(n_0 + jM_1 + i) = f(n_0) + f'(c)(jM_1 + i)$$

for some $c \in (n_0, n_0 + jM_1 + i)$. Notice that $n_0 + jM_1 + i \leq n_0 + 9M^2$. We denote the maximum value of $f(x)$ in $[n_0, n_0 + 9M^2]$ by $D_{n_0,M}$. Then

$$f(n_{i,j}) = f(n_0) + O_1(9M^2D_{n_0,M})$$

$$= f(n_0) \left(1 + O_1 \left(\frac{9M^2D_{n_0,M}}{f(n_0)}\right)\right).$$

By (1.7), we have

$$\frac{9M^2D_{n_0,M}}{f(n_0)} < \frac{1}{100}. \quad (2.6)$$

Hence

$$f(n_{i,j})^{-t} = f(n_0)^{-t} \left(1 + O_1 \left(\frac{9M^2D_{n_0,M}}{f(n_0)}\right)\right)$$

uniformly for $1/2 < t < 1$. Then by (2.4) and (2.5), it follows that

$$x_{i,j} = w(R) - f(n_0)^{-t} \left(M_1 - i + 1 + O_1 \left(\frac{27M^3D_{n_0,M}}{f(n_0)}\right)\right), \quad (2.7)$$

$$y_{i,j} = f(n_0)^{-t} \left(j + O_1 \left(\frac{27M^3D_{n_0,M}}{f(n_0)}\right)\right) \quad (2.8)$$

for $0 \leq i < M_1, 0 \leq j < M_2$.

**Lemma 2.2.** (1) $S_{i,j}(0 \leq i < M_1, 0 \leq j < M_2)$ are included in $R$.
(2) If $(i,j) \neq (i',j')$, the interiors of $S_{i,j}$ and $S_{i',j'}$ are disjoint.

**Proof.** (1) Since $f$ is increasing, we see that

$$0 \leq w(R) - M_1f(n_0)^{-t} \leq x_{i,j} \leq x_{i,j} + f(n_{i,j})^{-t} \leq w(R),$$

$$0 \leq y_{i,j} \leq y_{i,j} + f(n_{i,j})^{-t} \leq M_2f(n_0)^{-t} \leq h(R).$$

Therefore, $S_{i,j}$ is included in $R$.

(2) If $i' \geq i', j < j'$, then $y_{i,j} + f(n_{i,j})^{-t} \leq y_{i,j'} \leq y_{i,j'}'$, hence the interior of $S_{i,j}$ lies bottom to the interior of $S_{i',j'}$. Similarly if $i' \geq i, j' < j$.

If $i < i', j \leq j'$, then $x_{i,j} + f(n_{i,j})^{-t} \leq x_{i',j} \leq x_{i',j'}$. Hence the interior of $S_{i,j}$ lies left to the interior of $S_{i',j'}$. Similarly if $i < i', j' \leq j$. \hfill \Box

**Lemma 2.3.** Suppose

$$9M^2 \max_{0 \leq x \leq 9M^2} f'(n_0 + x)f(n_0 + x)^{-t-1} \leq f(n_0 + 9M^2)^{-t}. \quad (2.9)$$

Then, for $0 \leq i < M_1 - 1$ and $0 \leq j < M_2 - 1$, the squares $S_{i,j}, S_{i+1,j}, S_{i,j+1}$ and $S_{i+1,j+1}$ surround a rectangle

$$[x_{i+1,j}, x_{i+1,j+1}] \times [y_{i+1,j+1}, y_{i,j+1}]. \quad (2.10)$$
Proof. This fact follows from the following equalities and inequalities.

\[
\begin{align*}
  x_{i+1,j} &= x_{i,j} + f(n_{i,j})^{t} - t, \\
  y_{i,j} &< y_{i+1,j} + f(n_{i+1,j})^{-t} + y_{i,j} + f(n_{i,j})^{-t}, \\
  x_{i,j} &< x_{i,j+1} < x_{i,j} + f(n_{i,j})^{-t} \\
  y_{i,j+1} &= y_{i,j} + f(n_{i,j})^{-t} + x_{i+1,j+1} = x_{i, j+1} + f(n_{i+1,j})^{-t}, \\
  y_{i+1,j+1} &< y_{i,j+1} < y_{i+1,j+1} + f(n_{i+1,j+1})^{-t}, \\
  x_{i+1,j} &< x_{i+1,j+1} < x_{i+1,j} + f(n_{i+1,j})^{-t}, \\
  y_{i+1,j+1} &= y_{i+1,j} + f(n_{i+1,j})^{-t}.
\end{align*}
\]

Among these relations, the underlined four (essentially two) inequalities are nontrivial, and others easily follow from the definitions of \( x_{i,j} \) and \( y_{i,j} \). By definition,

\[
 y_{i,j} - y_{i+1,j} = \sum_{j'=0}^{j-1} \left( f(n_0 + j'M_1 + i) - f(n_0 + j'M_1 + i + 1) \right).
\]

By mean value theorem, we have

\[
 f(n_0 + j'M_1 + i) - f(n_0 + j'M_1 + i + 1) = -\frac{\partial}{\partial u} f(n_0 + j'M_1 + u) \big|_{u=c_{i,j}}.
\]

\[
 = tj'(n_0 + j'M_1 + c_{i,j}) f(n_0 + j'M_1 + c_{i,j})^{t-1}
\]
for some $c_{i,j'} \in (i,i+1)$. Hence

$$y_{i,j} - y_{i+1,j} = \sum_{j=0}^{j-1} f'(n_0 + j'M_1 + c_{i,j'})f(n_0 + j'M_1 + c_{i,j'})^{-t-1}$$

$$\leq 3M \max_{0 \leq x \leq 9M^2} f'(n_0 + x)f(n_0 + x)^{-t-1}.$$  \hfill (2.11)

On the other hand, since $f$ is increasing, $f(n_{i+1,j})^{-t}$ is equal or larger than $f(n_0 + 9M^2)^{-t}$. Hence if (1.8) holds.

By (1.7), the left hand side of (2.9) is at most $(1+9n^2)f(n_0)^{-t}$. The perimeters of these rectangles are at most $4\times 9M^2$. Hence if (2.9) is satisfied then the first and third underlined inequalities hold. Moreover, the condition (2.11) is weaker than (2.9). Hence the first and third underlined inequalities follow from (2.9). We now hold, then the first and third underlined inequalities hold. Moreover, the condition (2.11) is weaker than (2.9). Hence the first and third underlined inequalities follow from (2.9). We now hold, then the first and third underlined inequalities hold. Moreover, the condition (2.11) is weaker than (2.9). Hence the first and third underlined inequalities follow from (2.9). We now prove the second and fourth inequalities assuming (2.9). Again by definition,

$$x_{i,j+1} - x_{i,j} = \sum_{i'=i}^{M} \left( f(n_0 + j'M_1 + i')^{-t} - f(n_0 + (j+1)M_1 + i')^{-t} \right) .$$

By mean value theorem,

$$f(n_0 + j'M_1 + i')^{-t} - f(n_0 + (j+1)M_1 + i')^{-t}$$

$$= \frac{d}{du} f(n_0 + uM_1 + i')^{-t} |_{u=c_{i',j}}$$

$$= M_1 f'(n_0 + c_{i',j}'M_1 + i')f(n_0 + c_{i',j}'M_1 + i')^{-t-1}.$$  \hfill (2.9)

for some $c_{i',j} \in (j,j+1)$. Hence by (2.9),

$$x_{i,j+1} - x_{i,j} \leq M_1 \sum_{i'=i}^{M} f'(n_0 + c_{i',j}'M_1 + i')f(n_0 + c_{i',j}'M_1 + i')^{-t-1}$$

$$\leq 9M^2 \max_{0 \leq x \leq 9M^2} f'(n_0 + x)f(n_0 + x)^{-t-1} \leq f(n_0 + 9M^2)^{-t}.$$  \hfill (2.10)

On the other hand, $f(n_{i,j})^{-t}$ is equal or larger than $f(n_0 + 9M^2)^{-t}$. Thus we obtain the second and fourth underlined inequalities under the assumption of (2.9).  \hfill \Box

By (1.7), the left hand side of (2.9) is at most $(1/264f(n_0)^{-t})$. Hence (2.9) is satisfied if (1.8) holds. Summing the rectangle $R$ is perfectly packed by the following five types of rectangles:

Type I. The squares $S_{i,j}$ ($0 \leq i < M_1, 0 \leq j < M_2$).

Type II. The rectangles $[x_{i,j}, x_{i+1,j}] \times [y_{i,j}, y_{i+1,j}]$ ($0 \leq j < M_2$). By (2.7) and (2.8), the perimeters of these rectangles are $O_1 \left( \frac{216M^3D_{n_0,M}}{f(n_0)^{t+1}} \right)$ and the number of these rectangles is at most $9M^2$.

Type III. The rectangles $[x_{i+1,j}, x_{i,j}] \times [y_{i,j}, y_{i+1,j}]$ ($0 \leq i < M_1$). The perimeters of these rectangles are at most $4f(n_0)^{-t} + \frac{54M^3D_{n_0,M}}{f(n_0)^{t+1}}$, and the number of these rectangles is at most $3M$.

Type IV. The rectangles $[x_{i,M_2-1}, x_{i+1,M_2-1}] \times [y_{i,M_2-1}, y_{i+1,M_2-1}]$ ($0 \leq i < M_1$). The perimeters of these rectangles are at most $4f(n_0)^{-t} + \frac{54M^3D_{n_0,M}}{f(n_0)^{t+1}}$, and the number of these rectangles is at most $3M$.

Type V. The rectangle $[x_{0,M_2-1}] \times [y_{0,M_2-1}]$. The perimeter of this rectangle is at most $4f(n_0)^{-t} + \frac{108M^3D_{n_0,M}}{f(n_0)^{t+1}}$.  \hfill (2.11)
Put $\mathcal{R} := R \setminus \{S_{i,j}\}_{i,j}$. By the above argument, it follows that the widths of the rectangles contained in $\mathcal{R}$ exceed neither $54M^3D_{n_0,M}f(n_0)^{-t-1}$ nor $f(n_0)^{-t} + 27M^3D_{n_0,M}f(n_0)^{-t-1}$, so the widths are at most $2f(n_0)^{-t}$ by (1.7). Moreover, we see that the total perimeter of $\mathcal{R}$ is

$$\text{perim}(\mathcal{R}) \leq \frac{216M^3D_{n_0,M}}{f(n_0)^{1+t}} \times 9M^2 + \left(4f(n_0)^{-t} + \frac{54M^3D_{n_0,M}}{f(n_0)^{1+t}}\right) \times 6M$$

$$\leq \frac{2376M^5D_{n_0,M}}{f(n_0)^{t+1}} + \frac{49}{2}Mf(n_0)^{-t}$$

$$\leq 25Mf(n_0)^{-t}.$$  

Thus the proof of Proposition 2.1 is completed. \hfill \Box

2.2 Perfectly packing some families of rectangles

Put $\delta = 1 - t$ and define the weighted total perimeter of $\mathcal{R}$ by

$$\text{perim}_\delta(\mathcal{R}) = \sum_{R \in \mathcal{R}} w(R)\delta h(R).$$

**Proposition 2.4.** Suppose that $M, f, K, l$ and $1/2 < t < 1$ satisfy conditions (1.1), (1.2), (1.3), (1.4) and (1.6) for any $n_0 \geq N_0, n_1 \geq N_1$. Fix positive integers $n_{\text{max}} \geq n_0 (\geq N_0)$. Let $\mathcal{R}$ be a family of finite number of rectangles with disjoint interiors satisfying

$$\text{area}(\mathcal{R}) = \sum_{n=n_0}^{\infty} \frac{1}{f(n)^{2t}},$$

$$\text{perim}_\delta(\mathcal{R}) \leq M^{-1+\frac{\delta}{2}} \sum_{n=1}^{n_{\text{max}}} \frac{1}{f(n)^{t+\delta t}}$$

and

$$\sup_{R \in \mathcal{R}} h(R) \leq 1.$$  

Then, squares of sidelength $f(n)^{-t}$ ($n_0 \leq n < n_{\text{max}}$) can be packed into $\bigcup_{R \in \mathcal{R}} R$.

**Proof.** Fix $n_{\text{max}}$ and prove the statement by a downward induction on $n_0$. The above proposition is trivial if $n_{\text{max}} = n_0$. So we suppose that $n_{\text{max}} > n_0$ and that the claim has already been proved for larger $n_0$. Then, by (2.14) and (1.2), we have

$$\sum_{R \in \mathcal{R}} w(R)^{1-\delta} h(R) \leq M^{-1+\frac{\delta}{2}} \frac{c_2\xi_2(t)v_2(n_0)}{f(n_0)^{t+\delta t}}.$$  

On the other hand, by (2.13) and (1.3), we have

$$\sum_{R \in \mathcal{R}} w(R)h(R) \geq \frac{d_1\eta_1(t)\lambda_1(n_0)}{f(n_0)^{2t}}.$$  

Hence by the pigeonhole principle, there exists a rectangle $R \in \mathcal{R}$ that satisfies

$$w(R)^{1-\delta} \geq \frac{d_1\eta_1(t)\lambda_1(n_0)f(n_0)^{-2t}}{M^{-1+\frac{\delta}{2}}c_2\xi_2(t)v_2(n_0)f(n_0)^{-t-\delta t}}.$$  

Therefore, we have

$$w(R) \geq (M^{1-\delta})^{\frac{1}{1-\delta}} \left(\frac{d_1\eta_1(t)\lambda_1(n_0)}{c_2\xi_2(t)v_2(n_0)}\right)^{\frac{-1}{t-\delta}} f(n_0)^{-t}.$$  

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\[ \geq M^{1+\frac{\delta}{2}} \left( \frac{d_1 \eta_1(t) \lambda_1(n_0)}{c_2 \xi_2(t)} \right)^{1/2} f(n_0)^{-t}, \]

since \((1-\frac{\delta}{2})(1-\delta) > 1 + \frac{\delta}{4}\). Since \(\lambda_1\) and \(\nu_2\) satisfy (1.4), we have

\[ w(R) \geq M^{1+\frac{\delta}{2}} \left( \frac{c_{\lambda,\nu} d_1 \eta_1(t)}{c_2 \xi_2(t)} \right)^{1/2} f(n_0)^{-t}. \]

Since \(M\) satisfies (1.6), it follows that

\[ M^{\frac{\delta}{2}} \left( \frac{c_{\lambda,\nu} d_1 \eta_1(t)}{c_2 \xi_2(t)} \right)^{1/2} \geq 2. \]

Therefore,

\[ h(R) \geq w(R) \geq 2Mf(n_0)^{-t}. \] (2.18)

We cut the rectangle \(R\) into a rectangle \(R_0\) of dimension \((w(R) - Mf(n_0)^{-t}) \times h(R)\) and a rectangle \(R_*\) of dimension \(Mf(n_0)^{-t} \times h(R)\). Then cut off the squares of side length \(Mf(n_0)^{-t}\) from \(R_*\) until the height of the remaining rectangle is below \(2Mf(n_0)^{-t}\). Thus \(R_*\) is decomposed into rectangles \(R_1, \ldots, R_m\). These rectangles are of dimension \(Mf(n_0)^{-t} \times h(R_i)\), where

\[ Mf(n_0)^{-t} \leq h(R_i) < 2Mf(n_0)^{-t} \quad (i = 1, \ldots, m) \] (2.19)

with disjoint interiors, and

\[ \sum_{i=1}^{m} h(R_i) = h(R). \]

Since \(h(R) \leq 1\), it follows that

\[ m \leq M^{-1} f(n_0)^{t}. \] (2.20)

This gives a perfectly packing of \(R\)

\[ R = R_1 \cup R_2 \cup \ldots \cup R_m. \]

We adapt Proposition 2.1 \(m\) times. Then there exists a sequence of positive integers \(n_0 = n'_0 < n'_1 < \cdots < n'_m\) such that

\[ M^2 \leq n_{i+1} - n_i \leq 9M^2 \quad (0 \leq i \leq m - 1) \] (2.21)

and each \(R_i\) is perfectly packed by squares of side length \(f(n)^{-t}\) \((n_{i-1}' \leq n < n_i')\) and a family \(\mathcal{R}_i\) of finite number of rectangles with disjoint interiors satisfying

\[ \text{perim}(\mathcal{R}_i) \leq 25Mf(n_0)^{-t} \] (2.22)

and the width of each rectangle in \(\mathcal{R}_i\) is at most \(2f(n_0)^{-t}\), provided that the height of \(R_i\) satisfies

\[ Mf(n_{i-1}')^{-t} \leq h(R_i) \leq 3Mf(n_{i-1}')^{-t} \] (2.23)

for \(i = 1, \ldots, m\). Let us confirm that (2.23) is satisfied. Since \(Mf(n_{i-1}')^{-t} \leq Mf(n_0)^{-t}\), \(Mf(n_{i-1}')^{-t} \leq h(R_i)\) is satisfied. By (2.20) and (2.21),

\[ n_i' \leq n_0 + 9Mf(n_0)^{t}. \]

Combining this with (1.9), we have

\[ n_0 \leq n_i' \leq (1 + l)n_0 \] (2.24)
for \( i = 1, \ldots, m \). Since \( h(R_i) \) satisfies (2.19), to prove \( h(R_i) \geq 3M f(n_{i-1})^{-t} \), it suffices to see
\[ 2M f(n_0)^{-t} \leq 3M f(n_{i-1})^{-t}, \]
which is equivalent to \( f(n_0) \geq (2/3)^t f(n_{i-1}) \). For \( i = 1, \ldots, m \), these inequalities are valid because of (1.1) and (2.24). Hence (2.23) holds and the inductive argument can be applied to the rectangles \( R_1, \ldots, R_m \). Put
\[ R' := (R \setminus \{R\}) \cup \{R_0\} \cup \bigcup_{i=1}^{m} R_i. \]

By the above argument, \( \bigcup_{R' \in R} R' \) is perfectly packed by squares of sidelength \( f(n)^{-t} \), \( n < n_m \) and rectangles in \( R' \). If \( n_m' \geq n_{\text{max}} \), the induction process is completed. So we assume \( n_m' < n_{\text{max}} \). Before the evaluation of \( \text{perim}_d(R') \), we see that for \( i = 1, \ldots, m \),
\[ f(n_0)^{-t-\delta t} \leq M^{-2K^{t-\delta t}} \sum_{n=n_{i-1}}^{n_i-1} f(n)^{-t-\delta t} \tag{2.25} \]
holds, because \( f(n_0) \geq K f(n) \) for \( n_{i-1} \leq n < n_i' = 1 \). By (2.22), (2.14), (2.25) and \( w(R') \leq 2f(n_0)^{-t} \) for \( R' \in R_i \), it follows that
\[
\text{perim}_d(R') = \text{perim}_d(R) - w(R)^\delta h(R) + w(R_0)^\delta h(R) + \sum_{i=1}^{m} \sum_{R' \in R_i} w(R')^\delta h(R') \\
\leq \text{perim}_d(R) + \sum_{i=1}^{m} O_1(2^\delta f(n_0)^{-\delta t} \text{perim}(R_i)) \\
\leq \text{perim}_d(R) + 2^\delta \cdot 25M \sum_{i=1}^{m} O_1(f(n_0)^{-t-\delta t}) \\
\leq \text{perim}_d(R) + \frac{50}{K^{t+\delta t}M} \sum_{i=1}^{m} O_1 \left( \sum_{n=n_{i-1}}^{n_i-1} f(n)^{-t-\delta t} \right) \\
\leq M^{-1+\frac{\delta}{2}} \sum_{n=1}^{n_{m}-1} \frac{1}{f(n)^{t+\delta t}} + \frac{50}{K^{t+\delta t}M} \sum_{n=n_0}^{n_m-1} \frac{1}{f(n)^{t+\delta t}}. 
\]
Since \( M \geq (50/K^{t(2-t)})^{1/2} \) by (1.6), we have
\[ \text{perim}_d(R') \leq M^{-1+\frac{\delta}{2}} \sum_{n=1}^{n_{m}} \frac{1}{f(n)^{t+\delta t}}. \]

Therefore, \( R' \) satisfies (2.14) with \( n_0 \) replaced with \( n'_m \). Moreover,
\[ \text{area}(R') = \text{area}(R) - \sum_{n=n_0}^{n_{m}-1} \frac{1}{f(n)^{2t}} = \sum_{n=n_m'}^{\infty} \frac{1}{f(n)^{2t}}. \]

Hence the identity (2.13) with \( n_0 \) replaced with \( n'_m \) is obtained. Finally, the heights of the rectangles in \( R' \) are at most 1. Therefore, by induction hypothesis, squares of sidelength \( f(n)^{-t} \) with \( n_m' \leq n < n_{\text{max}} \) can be packed into \( \bigcup_{R' \in R} R' \). On the other hand, squares of sidelength \( f(n)^{-t} \) for \( n_0 \leq n < n_m' \) have already been packed into \( R \setminus R' \). Hence the induction is now completed. \( \square \)
2.3 Proof of Theorem 1.1

Suppose that \( f, t, K, l, M \) and \( N_0 \) satisfy all the conditions in the statement of Theorem 1.1. Let \( S \) be a square of area \( \sum_{n=n_0}^{\infty} f(n)^{-2t} \). By (1.3), the sidelength of \( S \) is at most \( d_2^{\frac{1}{2}} \eta_2(t) \lambda_2(n_0)^{\frac{1}{2}} f(n_0)^{-t} \). Therefore,

\[
\text{perim}_S(S) \leq (d_2^{\frac{1}{2}} \eta_2(t) \lambda_2(n_0)^{\frac{1}{2}} f(n_0)^{-t})^{1+\delta} = (d_2 \eta_2(t) \lambda_2(n_0))^{\frac{1+\delta}{2}} f(n_0)^{-(1+\delta)t}.
\]

On the other hand, by (1.2),

\[
\sum_{n=1}^{n_0-1} \frac{1}{f(n)^{t+\delta}} \geq c_1 \xi_1(t) \nu_1(n_0) f(n_0)^{-(1+\delta)t}.
\]

Hence the condition (2.14) is satisfied if

\[
(d_2 \eta_2(t) \lambda_2(n_0))^{\frac{1+\delta}{2}} f(n_0)^{-(1+\delta)t} \leq c_1 M^{-1+\frac{\delta}{2}} \xi_1(t) \nu_1(n_0) f(n_0)^{-(1+\delta)t},
\]

and the above inequality is equivalent to (1.10). Furthermore, the height of \( S \) is at most 1 provided that (1.5) is satisfied. Consequently, if the functions and parameters satisfy these conditions, one can apply Proposition 2.4 to \( R = S \) and see that for any \( n_{\text{max}} > n_0 \geq N_0 \), the squares of sidelength \( f(n)^{-t} \) \( (n_0 \leq n \leq n_{\text{max}}) \) can be packed into \( S \). Since \( n_{\text{max}} \) is arbitrary, by sending \( n_{\text{max}} \to \infty \) and using a standard compactness argument (see e.g., [10]), we see that there is a perfectly packing of squares of sidelength \( f(n)^{-t} \) for \( n \geq n_0 \) into \( S \), under the assumptions of Theorem 1.1. Thus the proof of Theorem 1.1 is completed. \( \Box \)

3 Perfectly packing a square by squares of nearly AP-harmonic sidelength

In this section, we apply Theorem 1.1 to the problem of perfectly packing a square by squares of sidelength \( (qn + r)^{-t} \) for \( q > r \geq 0, 1/2 < t < 1 \).

Corollary 3.1. Let \( 1/2 < t < 1 \) and \( q > r \geq 0 \) be fixed constants and put \( \delta = 1 - t \). Suppose that \( M \) and \( N_0 \) are positive integers satisfying

\[
M \geq \max \left\{ 4^{-\frac{1}{t+\delta}} \left( \frac{2(2t - 1)}{1 - t - \delta} \right)^{\frac{\frac{1}{t}}{1}} \left( 50 \left( \frac{11}{10} \right)^{t(2-t)} \right)^{\frac{\frac{1}{t}}{2}} \right\},
\]

\[
N_0 \geq \max \left\{ 4752M^4, \frac{2r}{q((264M^2)^{\frac{1}{t}} - 2)}, \left( 10q(2q)^t M \right)^{\frac{1}{1-t}}, 2^{\frac{1}{1-t+\delta}} q + r, q^{-1} \left( \frac{q(1 - t - \delta t)}{(q + r)^{t+\delta t}} \right)^{\frac{1}{1-t+\delta}}, \frac{1}{1 - 2 - \frac{1}{1-t+\delta}}, q^{-1} \left( \frac{q(1 - t - \delta t)}{q(2t - 1)} \right)^{\frac{2}{1-t+\delta}} M^{\frac{1}{t+\delta}}, q^{-1} \left( \frac{2}{q(2t - 1)} \right)^{\frac{1}{1-t+\delta}} \right\},
\]

Then, for any positive integer \( n_0 \geq N_0 \), squares of sidelength \( (qn + r)^{-t} \) for \( n \geq n_0 \) can be packed perfectly into a square of area \( \sum_{n=n_0}^{\infty} (qn + r)^{-2t} \).

Remark 3.2 (Effective lower bound for \( N_0 \) in the case of nearly harmonic sidelength). In the case of \( f(n) = n \) (hence \( q = 1, r = 0 \)), a specific lower bound for \( N_0 \) is given by the following table.
Table 1: Lower bound for $N_0$ in the case $f(n) = n$

| $t$  | 0.55 | 0.6  | 0.7  | 0.8  | 0.9  | 0.95 | 0.99 | 0.999 |
|------|------|------|------|------|------|------|------|-------|
| $\log_{10} N_0$ | 35   | 38   | 50   | 114  | 563  | 2673 | 92863| 13216295 |

As $t \nearrow 1$, a sufficient condition for the value of $N_0$ is

$$N_0 \geq 4^{\frac{1}{n-1}} (2^t \cdot 10)^{\frac{1}{n-1}} \frac{(2(2t - 1))}{(1-t)^2}.$$  \(1.7\)

Roughly speaking, as $t \nearrow 1$, the perfectly packing is possible if $N_0 \geq \frac{3}{(1-t)^2}$.

Proof. All we have to do is to identify when the function $f(x) = qx + r$ satisfies the whole conditions of Theorem 1.1. The condition $1.7$ is

$$q \leq \frac{1}{4752 M^4} (qn_0 + r),$$  \(3.1\)

which is satisfied if

$$n_0 \geq 4752 M^4.$$  \(3.1\)

The condition $1.8$ is

$$(q(n_0 + 9M^2) + r) \leq (264M^2)^{1/3} (qn_0 + r)$$

If $n_0$ satisfies $3.1$, then $n_0 + 9M^2 \leq 2n_0$. Therefore, the left hand side of the above inequality is at most $2(qn_0 + r)$. Hence a sufficient condition for $1.8$ is

$$2(qn_0 + r) \leq (264M^2)^{1/3} (qn_0 + r),$$

which is satisfied when

$$n_0 \geq \frac{2r}{q((264M^2)^{1/3} - 2)}.$$  \(3.2\)

The condition $1.1$ is

$$qn_0 + r \geq K(q(1 + l)n_0 + r),$$

which holds when

$$K \leq \frac{1}{1 + l}.$$  \(3.3\)

We choose $l = 1/10$, $K = 10/11$. Then $K \geq (2/3)^{1/2}$ is also satisfied. The condition $1.9$ is

$$9M(qn_0 + r)^t \leq ln_0.$$  \(3.4\)

The left hand side is at most $9(2q)^t M n_0$. Therefore, $1.9$ holds for

$$n_0 \geq \left(\frac{9(2q)^t M}{l}\right)^{1/t}. \text{ \(3.4\)}$$

Next, we consider the conditions $1.2$, $1.3$. Evaluating the summation by integrals, we have

$$\int_1^{n_1} \frac{dx}{(qx + r)^{t+\delta t}} \leq \sum_{n=1}^{n_1-1} \frac{1}{(qn + r)^{t+\delta t}} \leq \frac{1}{(q + r)^{t+\delta t}} + \int_1^{n_1-1} \frac{dx}{(qx + r)^{t+\delta t}}.$$  \(3.5\)

The left hand side equals

$$\frac{1}{q(1-t-\delta t)} \frac{qn_1 + r}{(qn_1 + r)^{t+\delta t}} - \frac{1}{q(1-t-\delta t)} \frac{q + r}{(q + r)^{t+\delta t}}.$$
which is equal to or larger than
\[
\frac{1}{2} \frac{1}{q(1-t-\delta t)} \frac{q n_1 + r}{(q n_1 + r)^{t+\delta t}}
\]
if \( n_1 \) satisfies
\[
n_1 \geq 2^{1-\pi q} \frac{q + r}{q}. \tag{3.6}
\]
Therefore, for \( n_1 \) satisfying the condition (3.6), the former inequality of (1.2) holds with
\[
c_1 = \frac{1}{2}, \quad \xi_1(t) = \frac{1}{q(1-t-\delta t)}, \quad \nu_1(n_1) = q n_1 + r.
\]
On the other hand, the right hand side of (3.5) is at most
\[
\frac{1}{(q + r)^{t+\delta t}} + \frac{1}{q(1-t-\delta t)} \frac{q n_1 + r}{(q n_1 + r)^{t+\delta t}}.
\]
This is equal to or less than
\[
\frac{2}{q(1-t-\delta t)} \frac{q n_1 + r}{(q n_1 + r)^{t+\delta t}}
\]
if \( n_1 \) satisfies
\[
n_1 \geq q^{-1} \left( \frac{q(1-t-\delta t)}{(q + r)^{t+\delta t}} \right)^{1-\pi q}. \tag{3.7}
\]
Therefore, for \( n_1 \) satisfying the condition (3.7), the latter inequality of (1.2) holds with
\[
c_2 = 2, \quad \xi_2(t) = \frac{1}{q(1-t-\delta t)}, \quad \nu_2(n_1) = q n_1 + r.
\]
Next, we consider (1.3). We have
\[
\int_{n_1}^{\infty} \frac{dx}{(q x + r)^{2t}} \leq \sum_{n=n_1}^{\infty} \frac{1}{(q n + r)^{2t}} \leq \int_{n_1-1}^{\infty} \frac{dx}{(q x + r)^{2t}}.
\]
The left hand side equals
\[
\frac{q n_1 + r}{q(2t-1)(q n_1 + r)^{2t}}.
\]
Hence the former inequality of (1.3) holds with
\[
d_1 = 1, \quad \eta_1(t) = \frac{1}{q(2t-1)}, \quad \lambda_1(n_1) = q n_1 + r.
\]
The right hand side is
\[
\frac{1}{q(2t-1)} \frac{q(n_1 - 1) + r}{(q(n_1 - 1) + r)^{2t}},
\]
which becomes smaller than
\[
\frac{2}{q(2t-1)} \frac{q n_1 + r}{(q n_1 + r)^{2t}}
\]
when
\[
n_1 \geq \frac{1}{1 - 2^{-\pi q}}. \tag{3.8}
\]
Therefore, if \( n_1 \) satisfies the condition (3.8), then the latter inequality of (1.3) holds with
\[
d_2 = 2, \quad \eta_2(t) = \frac{1}{q(2t-1)}, \quad \lambda_2(n_1) = q n_1 + r.
\]
We assume (3.6), (3.7) and (3.8). Then, (1.4) holds with \(c_{\lambda,\nu} = 1\). The condition (1.10) is

\[
(qn_0 + r)^{1 - t \frac{1 + \delta}{2}} \geq 2q(1 - t - \delta t) \left( \frac{2}{q(2t - 1)} \right)^{\frac{1 + \delta}{2}} M^{1 - \frac{\delta}{2}},
\]

which holds for

\[
n_0 \geq q^{-1} \left( \frac{2}{q(2t - 1)} \right)^{\frac{1 + \delta}{2}} M^{\frac{\delta}{2}}.
\]

Finally, the condition (1.5) is

\[
qn_0 + r \leq \frac{1}{2} q(2t - 1)(qn_0 + r)^{2t},
\]

which holds for

\[
n_0 \geq q^{-1} \left( \frac{2}{q(2t - 1)} \right)^{\frac{1}{2t}}.
\]

We choose a positive integer \(N_0\) that is larger than the largest value of the right hand sides of (3.1), (3.2), (3.4), (3.6) to (3.10). Then for \(1/2 < t < 1\) and for any \(n_0 \geq N_0\), the squares of sidelength \((qn + r)^{-t}\) for \(n \geq n_0\) are packed perfectly into a square of area \(\sum_{n=n_0}^{\infty} (qn + r)^{-2t}\).

\[
\text{Perfectly packing a square by squares of nearly P-harmonic sidelength}
\]

In this section, for \(1/2 < t < 1\), we give a perfectly packing a square by squares of sidelength \(p_n^{-t}\) for \(n \geq n_0\) for any \(n_0 \geq N_0\), where \(p_n\) denotes the \(n\)th prime number and \(N_0\) is a sufficiently large positive integer depending on \(t\), which will be explicitly given in Corollary 4.6. Before the statement of the theorem, we prepare some basic facts on primes. First, it is known that the sequence \(\{p_n\}\) satisfies

\[
p_n > n \log n
\]

for any positive integer \(n\), and

\[
p_n < n(\log n + \log \log n)
\]

for any \(n \geq 6\) (see for example [14]). We first give an upper bound for \(\sum_{n \leq x} 1/p_n^t\) for any fixed \(1/2 < t < 1\). By decomposing the interval \([1, x]\) into \([1, \sqrt[3]{x}]\) and \((\sqrt[3]{x}, x]\) and using (4.1), we have

\[
\sum_{n \leq x} \frac{1}{p_n^t} \leq \frac{1}{2t} + \left( \frac{\log 2}{t} \right)^{t} \sum_{2 \leq n \leq \sqrt[3]{x}} \frac{1}{n^t} + \frac{1}{\log \sqrt[3]{x^t}} \sum_{\sqrt[3]{x^t} < n \leq x} \frac{1}{n^t}
\]

\[
\leq \frac{1}{2t} + \left( \frac{\log 2}{t} \right)^{t} \int_1^{\sqrt[3]{x}} \frac{du}{u^t} + \left( \frac{\log x}{t} \right)^{t} \int_{\sqrt[3]{x}}^{x} \frac{du}{u^t}
\]

\[
\leq \frac{1}{2t} + \left( \frac{1}{2} \right)^{t} \frac{x^{1-t}}{(1-t)(\log x)^t} + \left( \frac{1}{t} \right)^{t} \frac{x^{1-t}}{(1-t)(\log x)^t}.
\]

In the right hand side of (4.3), the inequality

\[
\max \left\{ \frac{1}{2t}, \frac{x^{\frac{2(1-t)}{t}}}{(1-t)(\log x)^t} \right\} \leq \frac{x^{1-t}}{(1-t)(\log x)^t}
\]

holds for

\[
x \geq e^{\frac{16}{(1-t)^2}}.
\]
Furthermore,\[ \frac{\left(\frac{1}{4}\right)^t x^{1-t}}{(1-t)(\log x)^t} \leq \frac{4}{3} \frac{x^{1-t}}{(1-t)(\log x)^t} \]
for any \(x > 1\). Therefore, by (4.3) and the above estimates, we have\[ \sum_{n \leq x} \frac{1}{p_n^t} \leq \frac{10}{3} \frac{x^{1-t}}{(1-t)(\log x)^t} \]
for \(x\) satisfying (4.4). On the other hand, since \(1/(1 + (\log \log n/\log n)) > 7/10\) for any \(n \geq 6\), by (4.2) we have\[ \sum_{n \leq x} \frac{1}{p_n^t} > \frac{7}{10} \sum_{6 \leq n \leq x} \frac{1}{(n \log n)^t} \geq \frac{7}{10} \frac{1}{(\log x)^t} \sum_{6 \leq n \leq x} \frac{1}{n^t} \geq \frac{7}{10} \frac{1}{(\log x)^t} \int_6^x \frac{du}{u^t} \]
for \(x \geq 6\). If \(x\) satisfies\[ x \geq 6 \cdot 2^{-\frac{1}{1-t}}, \]
then \(x^{1-t} - 6^{1-t} \geq x^{1-t}/2\). Therefore, for \(x\) satisfying the condition (4.7), we have\[ \sum_{n \leq x} \frac{1}{p_n^t} \geq \frac{7}{20} \frac{x^{1-t}}{(1-t)(\log x)^t}. \]
Finally, we note that \(e^{16/(1-t)^2} > 6 \cdot 2^{-\frac{1}{1-t}}\) holds for any \(1/2 < t < 1\), we have the following conclusion.

**Lemma 4.1.** For any \(1/2 < t < 1\) and \(x \geq e^{16/(1-t)^2}\), we have\[ \frac{7}{20} \frac{x^{1-t}}{(1-t)(\log x)^t} < \sum_{n \leq x} \frac{1}{p_n^t} < \frac{10}{3} \frac{x^{1-t}}{(1-t)(\log x)^t}. \]

Replacing \(x\) with \(x - 1\) in (4.9), we see that\[ \frac{7}{20} \frac{(x - 1)^{1-t}}{(1-t)(\log (x-1))^t} < \sum_{n \leq x-1} \frac{1}{p_n^t} < \frac{10}{3} \frac{(x - 1)^{1-t}}{(1-t)(\log (x-1))^t} \]
for \(x \geq 1 + e^{16/(1-t)^2}\). In this region it follows that\[ \frac{1}{2} \frac{x^{1-t}}{(1-t)(\log x)^t} < \frac{(x - 1)^{1-t}}{(1-t)(\log (x-1))^t} < \frac{2x^{1-t}}{(1-t)(\log x)^t}. \]
Combining (4.9) and the above inequality, we have the following result.

**Lemma 4.2.** For \(1/2 < t < 1\) and \(x \geq 2e^{16/(1-t)^2}\), we have\[ \frac{7}{40} \frac{x^{1-t}}{(1-t)(\log x)^t} < \sum_{n \leq x} \frac{1}{p_n^t} < \frac{20}{3} \frac{x^{1-t}}{(1-t)(\log x)^t}. \]

Next, we evaluate \(\sum_{n \geq x} 1/p_n^t\) for \(t > 1\). By (4.1), we have\[ \sum_{n \geq x} \frac{1}{p_n^t} < \frac{1}{(\log x)^t} \sum_{n \geq x} \frac{1}{n^t} \leq \frac{1}{(\log x)^t} \int_{x-1}^\infty \frac{du}{u^t} = \frac{(x - 1)^{1-t}}{(t - 1)(\log x)^t}. \]
If \( x \geq 1/(1 - 2^{-\frac{1}{t-1}}) \), then \( (x - 1)^{1-t} \leq 2x^{1-t} \). Therefore, for this \( x \) we have

\[
\sum_{n \geq x} \frac{1}{p_n^x} \leq \frac{2x^{1-t}}{(t-1)(\log x)^t}.
\]

(4.11)

On the other hand, by \((4.2)\), for \( x \geq 6 \) we have

\[
\sum_{n \geq x} \frac{1}{p_n^x} \geq \sum_{n \geq x} \frac{1}{(n \log n)^t} \geq \frac{1}{2^t} \sum_{n \geq x} \frac{1}{(n \log n)^t} \geq \frac{1}{2^t} \frac{1}{(\log 2x)^t} \sum_{x \leq n < 2x} \frac{1}{n^t}
\]

\[
\geq \frac{1}{(2 \log 2x)^t} \int_x^{2x} \frac{du}{u^t} = \frac{(1 - 2^{1-t})x^{1-t}}{(t-1)(2 \log 2x)^t} \geq \frac{(1 - 2^{1-t})x^{1-t}}{2^t(t-1)(\log x)^t}.
\]

Consequently,

**Lemma 4.3.** For \( t > 1 \) and \( x \geq \max\{1/(1 - 2^{-\frac{1}{t-1}}), 6\} \), we have

\[
\frac{(1 - 2^{1-t})x^{1-t}}{2^t(t-1)(\log x)^t} \leq \sum_{n \geq x} \frac{1}{p_n^x} \leq \frac{2x^{1-t}}{(t-1)(\log x)^t}.
\]

(4.12)

Finally, by \((4.1)\) and \((4.2)\), it follows that

\[
\frac{p_{n_1}}{2} < n_1 \log n_1 < p_{n_1}.
\]

Combining the above inequality and \((4.10)\), \((4.12)\), we obtain the following conclusion.

**Lemma 4.4.** We have

\[
\frac{7}{40} \frac{n_1}{(1-t)p_{n_1}} < \sum_{n \leq n_1-1} \frac{1}{p_n^x} < \frac{20}{3} \frac{2^t n_1}{(1-t)p_{n_1}}
\]

(4.13)

for \( 1/2 < t < 1 \), \( n_1 \geq 2e^{1/(1-t)^2} \), and

\[
\frac{(1 - 2^{1-t})n_1}{2^{2t}(t-1)p_{n_1}} < \sum_{n \geq n_1} \frac{1}{p_n^x} < \frac{2^{1+t}n_1}{(t-1)p_{n_1}}
\]

(4.14)

for \( t > 1 \), \( n_1 \geq \max\{6, 1/(1 - 2^{-\frac{1}{t-1}})\} \).

The next purpose is to extend the sequence \( \{p_n\} \) to a smooth increasing function on \( \mathbb{R}_{\geq 1} \).

**Lemma 4.5.** There exists a smooth increasing function \( f : \mathbb{R}_{\geq 1} \rightarrow \mathbb{R}_{\geq 1} \) which satisfies the following two conditions.

1. For any positive integer \( n \), \( f(n) = p_n \).
2. There exists an absolute positive constant \( C_p < 7 \) such that for any positive integer \( n \) and for any \( x \in [n, n+1] \),

\[
f'(x) \leq C_p(p_{n+1} - p_n).
\]

(4.15)

**Proof.** Define a function \( \varphi \) by

\[
\varphi(x) = \begin{cases} 
0 & (x < -\frac{1}{6}) \\
\frac{1}{6} \exp \left(1 - \frac{1}{1-36x^2}\right) & (-\frac{1}{6} \leq x \leq 0) \\
\frac{1}{2} \exp \left(1 - \frac{1}{1-36x^2}\right) & (0 \leq x \leq \frac{1}{6}) \\
1 & (x > \frac{1}{6}).
\end{cases}
\]

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This is a smooth increasing function satisfying $\varphi(x) = 0$ if $x < -1/6$, $\varphi(x) = 1$ if $x > 1/6$ and $\max \varphi'(x) = 6.511 \ldots < 7$. We define the function $f(x)$ by

$$f(x) = p_1 + \sum_{k=1}^{+\infty} \varphi(x - (k + 1/2))(p_{k+1} - p_k).$$  

(4.16)

We fix a positive integer $n$ arbitrarily and suppose $x \in [n, n+1]$. If $k \geq n + 1$, then

$$x - \left( k + \frac{1}{2} \right) \leq (n + 1) - (n + 1) - \frac{1}{2} = -\frac{1}{2} < -\frac{1}{6}$$

Hence $\varphi(x - (k + 1/2)) = 0$. If $k \leq n - 1$, then

$$x - \left( k + \frac{1}{2} \right) \geq n - (n - 1) - \frac{1}{2} = \frac{1}{2} > \frac{1}{6}.$$ 

Hence $\varphi(x - (k + 1/2)) = 1$. Therefore,

$$f(x) = p_1 + \sum_{k=1}^{n-1} (p_{k+1} - p_k) + \varphi(x - (n + 1/2))(p_{n+1} - p_n)$$ 

$$= p_n + \varphi(x - (n + 1/2))(p_{n+1} - p_n).$$

Hence $f$ satisfies the conditions of the lemma.  

Next, it is known that there exist some $0 < \theta < 1$ and a positive integer $N_\theta$ for which

$$p_{n+1} - p_n \leq p_n^\theta$$  

(4.17)

holds for any $n \geq N_\theta$. The best result currently known is due to Baker, Harman and Pintz [1], according to which (4.17) holds for $\theta = 0.525$. By (4.15) and (4.17), we have

$$f'(x) \leq C p_n^\theta$$

for any $n \geq N_\theta$, $x \in [n, n+1]$. Combining this with $p_n < 2n \log n$ ($n \geq 6$), we have

$$f'(x) \leq 7(2 \log n)^\theta$$  

(4.18)

for any $n \geq \max\{6, N_\theta\}, x \in [n, n+1]$. Now our theorem on perfectly packing a square by squares of sidelength $p_n^{-t}$ is described as follows.

**Corollary 4.6.** Assume that (4.17) holds for any $n \geq N_\theta$ with some positive constant $\theta < 1$. For $1/2 < t < 1$, let $M$ be a positive integer satisfying

$$M \geq \max \left\{ 4 \frac{1}{t-t^t} \left( \frac{20 \cdot 2^{6t-t^t} (2t - 1)}{3(1-t)^2 (1 - 2^{1-2t})} \right)^{\frac{2}{1-t}}, \left( 50 \left( \frac{6}{5} \right)^{t(2-t)} \right)^{\frac{2}{1-t}} \right\},$$

and $N_\theta$ be a positive integer satisfying

$$N_\theta \geq \max \left\{ N_\theta, 85536 M^5, 3 \times 10^{18}, 2 e^{16 (1-t)}, \frac{1}{1 - 2^{-}\frac{1}{1-t}}, \right.$$  

$$\left. M^{\frac{1}{1-t}}, \left( \frac{40}{7} (1-t) \right)^{\frac{2}{t}} \left( \frac{2^{1+2t}}{2^{t-1}} \right)^{\frac{2}{1-t}} M^{\frac{1}{1-t}}, e^{(2t-1)(1-t)} \right\}.$$ 

Then, for any positive integer $n_0 \geq N_0$, there exists a perfectly packing a square of area

$$\sum_{n=n_0}^{\infty} p_n^{-2t}$$

by squares of sidelength $p_n^{-t}$ for $n \geq n_0$.  

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Remark 4.7. Roughly speaking, the perfectly packing exists as $t \not< 1$ if $N_0 \geq 2 \exp \left( \frac{16}{(1-t)^7} \right)$.

Proof. From now on, we assume $N_0 \geq N_\theta$. (4.19)

By (4.18), it follows that

$$\max_{x \in [n_0, n_0 + 9M^2]} f'(x) \leq 8(2 \log(n_0 + 9M^2))^\theta$$

for any $n_0 \geq N_\theta$, the condition (1.7) is valid if

$$8(2 \log(n_0 + 9M^2))^\theta \leq \frac{1}{4752} M^{-4} n_0 \log n_0 \quad (\forall n_0 \geq N_\theta),$$

which holds for

$$N_0 \geq 2 \cdot 4752 \cdot 9M^5.$$ (4.20)

Since $n \log n < f(n) < 2n \log n$ for any $n \geq 6$, the condition (1.8) is valid if

$$2(n_0 + 9M^2) \log(n_0 + 9M^2) \leq (264M^2)^\frac{1}{t} n_0 \log n_0$$

holds for any $n_0 \geq N_0$. Moreover, if

$$N_0 \geq 9M^2,$$ (4.21)

then the left hand side is at most $4n_0 \log 2n_0$. Hence under the assumption of (4.21), the condition (1.8) is satisfied if

$$4n_0 \log n_0 \leq (264M^2)^\frac{1}{t} n_0 \log n_0,$$

and this inequality holds whenever $M$ satisfies (1.6). Consequently, if $M$ satisfies (1.6) and $N_0$ satisfies (4.21), then the condition (1.8) is also satisfied. The condition (1.1) is valid if

$$n_0 \log n_0 \geq K(1 + l)n_0(\log(1 + l)n_0 + \log \log(1 + l)n_0)$$

holds for any $n_0 \geq N_0$, and by choosing

$$l = \frac{1}{10}, \quad K = \frac{5}{6},$$

the above inequality holds for

$$N_0 \geq 3 \times 10^{18}.$$ (4.22)

By replacing the parameter $t$ in (4.13) and (4.14) with $t + \delta t$ ($\delta := 1 - t$), $2t$, respectively, the conditions (1.2), (1.3) hold with

$$c_1 = \frac{7}{40}, \quad c_2 = \frac{20}{3}, \quad \xi_1(t) = \frac{1}{(1-t)^2}, \quad \xi_2(t) = \frac{2^t - t^2}{(1-t)^2}, \quad \nu_1(n_1) = \nu_2(n_1) = n_1,$$

$$d_1 = d_2 = 1, \quad \eta_1(t) = \frac{1 - 2^t - 2t}{2^t(2t - 1)}, \quad \eta_2(t) = \frac{2^t + 2t}{2t - 1}, \quad \lambda_1(n_1) = \lambda_2(n_1) = n_1.$$ (We used $1 - t - \delta t = (1 - t)^2$.) Hence (1.4) holds with $c_{\lambda, \nu} = 1$. The condition (1.9) holds if

$$9Mn_0^t(\log n_0 + \log \log n_0)^t \leq \frac{1}{10} n_0 \quad (\forall n_0 \geq N_0),$$

which is equivalent to

$$n_0^{1-t} \geq 90M(\log n_0 + \log \log n_0)^t \quad (\forall n_0 \geq N_0).$$
Since the right hand side is at most $180M \log n_0$, a sufficient condition for (1.9) is that

$$n_0^{1-t} \geq 180M \log n_0 \quad (\forall n_0 \geq N_0).$$

This condition is satisfied if

$$N_0 \geq M^{1/(1-1/t)}.$$  \hspace{1cm} (4.23)

The condition (1.10) is

$$n_0^{1-\frac{\delta t}{4}} \geq \left( \frac{\left( \frac{t^2}{2t-1} \right)^{\frac{1}{t}}}{1-\frac{1}{4t}} \right) M^{1-\frac{\delta t}{4}} \quad (\forall n_0 \geq N_0),$$

which is satisfied if

$$N_0 \geq \left( \frac{40}{7} (1-t)^2 \right) \left( \frac{\left( \frac{t^2}{2t-1} \right)^{\frac{1}{t}}}{1-\frac{1}{4t}} \right)^{\frac{4}{t}} M^{1-\frac{\delta t}{4}}. \hspace{1cm} (4.24)$$

Finally, the condition (1.5) is satisfied if

$$n_0 \leq \frac{2t-1}{21+2t} (n_0 \log n_0)^{2t} \quad (\forall n_0 \geq N_0),$$

and this inequality holds for

$$N_0 \geq \varepsilon_{(2t-1)/(1-\alpha)}, \hspace{1cm} (4.25)$$

Combining the lower bounds for $N_0$, we obtain the conclusion.

5  Ineffective version of the perfectly packing theorem and an application to the packing of squares of nearly TP-harmonic sidelength

5.1  Ineffective version of the perfectly packing theorem

Clearly, one cannot adapt Theorem 1.1 when $f(x)$ is larger than $x^{1+\delta_0}$ for some constant $\delta_0 > 0$ up to constant multiples, because the condition (1.9) fails. On the other hand, we have seen that one can adapt Theorem 1.1 to the function with $f(n) = p_n$, where $p_n \sim n \log n$. Hence one might wonder that how large the function $f(x)$ can be. The following theorem partially answers this question.

**Theorem 5.1.** Let $\kappa : \mathbb{R}_{>1} \rightarrow \mathbb{R}_{>1}$ be a smooth function for which $f(x) := x \kappa(x)$ becomes a smooth increasing function. Suppose that for any $\varepsilon > 0$, there exists a positive integer $N_\varepsilon$ such that for any $n \geq N_\varepsilon$,

$$\max_{x \in [n, 2n]} \kappa'(x) \leq \varepsilon \kappa(x),$$

$$(1 - \varepsilon) \kappa(M(2n)) \leq \kappa(n) \leq n^\varepsilon,$$

$$c_m \kappa(n) \leq \kappa(m(n)) \leq C_m \kappa(n),$$

where $M(2n)$ denotes one of the values $x \in [1, 2n]$ such that $\kappa(x)$ takes its minimum in $[1, 2n]$, $m(n)$ denotes one of the values $x \in [n, \infty)$ such that $\kappa(x)$ takes its minimum in $[n, \infty)$, and $C_m > c_m > 0$ are positive constants. Moreover, we assume that there exist constants $0 < \alpha < 1$ and $C_\alpha > 1$ such that

$$\kappa(n) \leq C_\alpha \kappa(m_\alpha(n))$$

holds for any $n \geq N_\varepsilon$, where $m_\alpha(n)$ denotes one of the values $x \in [n, \infty)$ such that $\kappa(x)$ takes its minimum in $[n, \infty)$. Then, for any $1/2 < t < 1$, there exists a positive integer $N_0 = N_0(t)$ such that for any integer $n_0 \geq N_0$, squares of sidelength $(n \kappa(n))^{-t}$ for $n \geq n_0$ can be packed perfectly into a square of area $\sum_{n=n_0}^{\infty} (n \kappa(n))^{-2t}$. 

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Proof. In this proof we assume that \( M \) satisfies (1.6) and that \( N_0 \geq 9M^2 \). The condition (1.7) becomes
\[
\max_{x \in [n_0, 2n_0]} (\kappa(x) + x\kappa'(x)) \leq \frac{1}{4752} M^{-4} n_0 \kappa(n_0).
\]
This inequality holds if both
\[
\kappa(M(2n_0)) \leq \frac{1}{9504} M^{-4} n_0 \kappa(n_0) \tag{5.1}
\]
and
\[
\max_{x \in [n_0, 2n_0]} \kappa'(x) \leq \frac{1}{19008} M^{-4} \kappa(n_0) \tag{5.2}
\]
are satisfied. These two inequalities are valid if the first two conditions in the statement of the theorem are satisfied. The condition (1.8) is satisfied if
\[
2n_0 \kappa(M(2n_0)) \leq (264M^2)^{\frac{3}{4}} (n_0 \kappa(n_0)),
\]
which is equivalent to
\[
\kappa(M(2n_0)) \leq \frac{1}{2} (264M^2)^{\frac{3}{4}} \kappa(n_0). \tag{5.3}
\]
The right hand side is at least \((1/2)(264M^2)^{\frac{3}{4}} \kappa(n_0)\). Therefore, (1.8) holds if
\[
\kappa(M(2n_0)) \leq \sqrt{66} M \kappa(n_0). \tag{5.4}
\]
The condition (1.1) is
\[
n_0 \kappa(n_0) \geq K(1 + l)n_0 \kappa((1 + l)n_0).
\]
Since the right hand side is at most \( K(1 + l)n_0 \kappa(M(2n_0)) \) for \( l < 1 \), the condition (1.1) is valid if
\[
n_0 \kappa(n_0) \geq K(1 + l)n_0 \kappa(M(2n_0)).
\]
By taking \( l = \varepsilon, K = (1 - \varepsilon)/(1 + \varepsilon) \), the above condition becomes
\[
\kappa(n_0) \geq (1 - \varepsilon) \kappa(M(2n_0)). \tag{5.4}
\]
Notice that if we assume (5.4), then (5.1) and (5.3) automatically hold for any sufficiently large \( n_0 \). The condition (1.9) is
\[
9M (n_0 \kappa(n_0))^t \leq n_0.
\]
This condition is valid if
\[
\kappa(n_0) \ll n_0^6 \tag{5.5}
\]
holds for any \( \varepsilon > 0 \), where the implied constant above is dependent only on \( \varepsilon \). Next, we consider the condition (1.2). Since
\[
\frac{1}{f(n)^{t+\delta t}} \geq \frac{1}{n^{t+\delta t} \kappa(M(n_1))^{t+\delta t}}
\]
for \( 1 \leq n \leq n_1 - 1 \), we have
\[
\sum_{n=1}^{n_1-1} \frac{1}{f(n)^{t+\delta t}} \geq \frac{1}{\kappa(M(n_1))^{t+\delta t}} \sum_{n=1}^{n_1-1} \frac{1}{n^{t+\delta t}} \geq \frac{1}{\kappa(M(n_1))^{t+\delta t}} \int_{1}^{n_1} \frac{du}{u^{t+\delta t}}
\]
\[
= \frac{n_1^{1-t-\delta t} - 1}{(1 - t - \delta t) \kappa(M(n_1))^{t+\delta t}} \geq \frac{(1 - \varepsilon)n_1}{2(1 - t - \delta t)(\kappa(n_1)n_1)^{t+\delta t}}
\]
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for any sufficiently large \( n \). On the other hand, for any fixed \( 0 < \alpha < 1 \) and for any sufficiently large \( n \), we have

\[
\sum_{n=1}^{n-1} \frac{1}{f(n)^{t+\delta t}} \leq \sum_{1 \leq n \leq n_1} \frac{1}{\kappa(n)^{t+\delta t} n^{t+\delta t}} + \sum_{n_1^1 \leq n \leq n_1} \frac{1}{\kappa(n)^{t+\delta t} n^{t+\delta t}}
\]

\[
\leq \sum_{1 \leq n \leq n_1} \frac{1}{n^{t+\delta t}} + \frac{1}{\kappa(m_\alpha(n_1))^{t+\delta t}} \sum_{n=1}^{n_1} \frac{1}{n^{t+\delta t}}
\]

\[
\leq \frac{2n_1^{1-t-\delta t}}{(1 - t - \delta t) \kappa(m_\alpha(n_1))^{t+\delta t}} = \frac{2n_1 \left( \frac{\kappa(n_1)}{\kappa(m_\alpha(n_1))} \right)^{t+\delta t}}{(1 - t - \delta t)(n_1 \kappa(n_1))^{t+\delta t}},
\]

provided that

\[
\kappa(m_\alpha(n_1))^{t+\delta t} \leq n_1^{(1-\alpha)(1-t-\delta t)}.
\]

The above condition is satisfied if \( \kappa \) satisfies \( \kappa(n) \ll n^\varepsilon \) for any \( \varepsilon > 0 \). Combining these inequalities, for any sufficiently large \( n_1 \) and any fixed \( 0 < \alpha < 1 \), we have

\[
\frac{(1 - \varepsilon)n_1}{2(1 - t - \delta t)(\kappa(n_1)n_1)^{t+\delta t}} \leq \sum_{n=1}^{n-1} \frac{1}{f(n)^{t+\delta t}} \leq \frac{2n_1 \left( \frac{\kappa(n_1)}{\kappa(m_\alpha(n_1))} \right)^{t+\delta t}}{(1 - t - \delta t)(n_1 \kappa(n_1))^{t+\delta t}}.
\]

Therefore, the condition \((1.2)\) holds with

\[
c_1 = \frac{1 - \varepsilon}{2}, \quad \xi_1(t) = \frac{1}{1 - t - \delta t}, \quad \nu_1(n_1) = n_1,
\]

\[
c_2 = 2, \quad \xi_2(t) = \frac{1}{1 - t - \delta t}, \quad \nu_2(n_2) = \frac{n_1 \kappa(n_1)}{\kappa(m_\alpha(n_1))},
\]

Next, we consider the condition \((1.3)\). For \( t > 1/2 \),

\[
\sum_{n=n_1}^{\infty} \frac{1}{f(n)^{2t}} \geq \frac{1}{\kappa(M(2n_1))^{2t}} \sum_{n=n_1}^{2n_1} \frac{1}{n^{2t}} \geq \frac{1}{\kappa(M(2n_1))^{2t}} \int_{n_1}^{2n_1+1} \frac{du}{u^{2t}}
\]

\[
= \frac{n_1^{1-2t} - (2n_1 + 1)^{1-2t}}{(2t - 1)\kappa(M(2n_1))^{2t}} \geq \frac{n_1}{2(2t - 1)\kappa(M(2n_1))^{2t}n_1^{2t}}
\]

\[
\geq \frac{2n_1}{(2(1 - \varepsilon)(2t - 1)(\kappa(n_1)n_1)^{2t}},
\]

provided that \((5.4)\) holds. On the other hand,

\[
\sum_{n=n_1}^{\infty} \frac{1}{f(n)^{2t}} \leq \frac{1}{m(n_1)^{2t}} \sum_{n=n_1}^{\infty} \frac{1}{n^{2t}} = \frac{n_1 \left( \frac{\kappa(n_1)}{\kappa(m(n_1))} \right)^{2t}}{(2t - 1)(n_1 \kappa(n_1))^{2t}}.
\]

Combining these inequalities, it follows that \((1.3)\) holds with

\[
d_1 = \frac{1}{2(1 - \varepsilon)}, \quad \eta_1(t) = \frac{1}{2t - 1}, \quad \lambda_1(n_1) = n_1,
\]

\[
d_2 = 1, \quad \eta_2(t) = \frac{1}{2t - 1}, \quad \lambda_2(n_1) = n_1 \left( \frac{\kappa(n_1)}{\kappa(m(n_1))} \right)^2.
\]

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The condition \([1.4]\) is satisfied if \(\kappa\) satisfies
\[
\kappa(n_0) \leq C_\alpha \kappa(m_\alpha(n_0))
\] (5.6)
for some \(0 < \alpha < 1, C_\alpha > 0\). The condition \([1.10]\) becomes
\[
n^{-\frac{1}{\delta}} \left( \frac{\kappa(n_0)}{\kappa(m(n_0))} \right)^{-(1+\delta)} \geq \frac{2(1-t-\delta t)}{1-\varepsilon} \left( \frac{1}{2t-1} \right)^{\frac{1+\delta}{2}} M^{1-\frac{\delta}{2}}.
\]
If \(\kappa\) satisfies
\[
\kappa(m(n_0)) \leq C_m \kappa(n_0),
\] (5.7)
then the above inequality holds for any sufficiently large \(n_0\). The condition \([1.5]\) becomes
\[
n_0 \left( \frac{\kappa(n_0)}{\kappa(m(n_0))} \right)^2 \leq (2t-1)(n_0 \kappa(n_0))^{2t}.
\]
If \(\kappa\) satisfies
\[
\kappa(m(n_0)) \geq c_m \kappa(n_0),
\] (5.8)
then the above inequality holds for any sufficiently large \(n_0\). Combining these results we obtain the conclusion of the theorem. \(\square\)

5.2 Packing of squares of nearly TP-harmonic sidelength

A prime number \(p\) is called a twin prime if at least one of \(p-2\) or \(p+2\) is also a prime number. We denote the \(n\)th twin prime by \(p_n\). Then \(p_1 = 3, p_2 = 5, p_3 = 7, p_4 = 11, p_5 = 13, \ldots\). For \(x \geq 2\), we denote the number of twin primes below \(x\) by \(\pi_2(x)\). Then, it is known that for any sufficiently large \(x\),
\[
\pi_2(x) \leq \frac{C \Pi_2 x}{(\log x)^2} \left( 1 + c \frac{\log \log x}{\log x} \right),
\] (5.9)
where
\[
\Pi_2 = \prod_{p \text{ prime}}^{p \geq 3} \left( 1 - \frac{1}{(p-1)^2} \right) = 0.6601618 \ldots
\] (5.10)
is the twin prime constant, and \(c\) is some absolute constant. The coefficient \(C\) is also some absolute constant. Though the value of \(C\) has not been known correctly, according to the Hardy-Littlewood conjecture \([5]\), the correct value of \(C\) is expected to be 2. Currently the best known upper bound is \(C \leq 6.8325\) by Haugland \([3]\).

**Corollary 5.2.** Let \(p_n\) be the \(n\)th twin prime, \(\Pi_2\) be the twin prime constant defined by \([5.10]\) and \(C\) be a constant in \([5.9]\). Then, for any \(1/2 < t < 1\) and \(C' > C\), there exists a positive integer \(N_0\) depending on \(t\) and \(C'\) that for any \(n_0 \geq N_0\), squares of sidelength \(p_n^{-t}\) for \(n \geq n_0\) can be packed into a square of area \((C' \Pi_2)^{2t} \sum_{n=n_0}^\infty (n(\log n)^2)^{-2t}\).

**Proof.** Since \(\pi_2(p_n) = n\), the estimate \([5.9]\) yields
\[
n \leq \frac{C \Pi_2 p_n}{(\log p_n)^2} \left( 1 + c \frac{\log \log p_n}{\log p_n} \right).
\]
From this, we easily see that for any \(C' > C\),
\[
p_n \geq \frac{n}{C' \Pi_2} (\log n)^2
\] (5.11)
holds for any \(n \geq N(C')\), where \(N(C')\) is some positive integer depending on \(C'\). We put
\[
f(x) = \frac{x}{C' \Pi_2} (\log x)^2, \quad \kappa(x) = \frac{1}{C' \Pi_2} (\log x)^2.
\]
for $x > 1$. By appropriately modifying the value of $\kappa(x)$ when $x$ is small, $\kappa$ becomes a smooth function from $\mathbb{R}_{>1}$ to $\mathbb{R}_{>1}$ and satisfies all the conditions of Theorem 5.1. Therefore, for any $1/2 < t < 1$, there exists a positive integer $N_0(t)$ such that for any positive integer $n_0 \geq N_0(t)$, squares of sidelength $\left((n/C')(\log n)^2\right)^{-t}$ for $n \geq n_0$ can be packed perfectly into a square of area $\sum_{n=n_0}^{\infty} \left((n/C')(\log n)^2\right)^{-2t} = \int_{x=n_0}^{\infty} \left((x/C')(\log x)^2\right)^{-2t}$. By (5.11), a square of sidelength $\frac{n-1}{n}$ for $n \geq n_0(C')$ is contained in a square of sidelength $\left((n/C')(\log n)^2\right)^{-t}$. Hence by taking $N_0 := \max\{N_0(t), N(C')\}$, we obtain the result.

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