Classes of Dynamic Systems with Various Combinations of Multipliers in Their Reciprocal Polynomial Right Parts

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Abstract

A family of differential dynamic systems is considered on a real plane of their phase variables x, y. The main common feature of systems under consideration is: every particular system includes equations with polynomial right parts of the third order in one equation and of the second order in another one. These polynomials are mutually reciprocal, i.e., their decompositions into forms of lower orders do not contain common multipliers. The whole family of dynamic systems has been split into subfamilies according to the numbers of different reciprocal multipliers in the decompositions and depending on an order of sequence of different roots of polynomials. Every subfamily has been studied in a Poincare circle using Poincare mappings. A plan of the investigation for each selected subfamily of dynamic systems includes the following steps. We determine a list of singular points of systems of the fixed subfamily in a Poincare circle. For every singular point in the list, we use the notions of a saddle (S) and node (N) bundles of adjacent to this point semi trajectories, of a separatrix of the singular point, and of a topodynamical type of the singular point (its TD – type). Further we split the family under consideration to subfamilies of different hierarchical levels with proper numbers. For every chosen subfamily we reveal topodynamical types of singular points and separatrices of them. We investigate the behavior of separatrices for all singular points of systems belonging to the chosen subfamily. Very important are: a question of a uniqueness of a continuation of every given separatrix from a small neighborhood of a singular point to all the lengths of this separatrix, as well as a question of a mutual arrangement of all separatrices in a Poincare circle Ω. We answer these questions for all subfamilies of studied systems. The presented work is devoted to the original study. The main task of the work is to depict and describe all different in the topological meaning phase portraits in a Poincare circle, possible for the dynamical differential systems belonging to a broad family under consideration, and to its numerical subfamilies of different hierarchical levels. This is a theoretical work, but due to special research methods it may be useful for applied studies of dynamic systems with polynomial right parts. Author hopes that this work may be interesting and useful for researchers as well as for students and postgraduates. As a result, we describe and depict phase portraits of dynamic systems of a taken family and outline the criteria of every portrait appearance.

1. Introduction.
The preliminary aim of dynamic systems study is to investigate curves, defined by given differential equations. Investigator needs to divide a phase space of a considered dynamic system into trajectories and reveal their limit behavior, classify the equilibrium positions, find out and study attracting and repulsive manifolds.

A normal autonomous second order differential system with polynomial right parts, being considered on an extended real plane $R_{x,y}^2$, may be exhaustively qualitatively investigated (J.H. Poincare [1, 4 - 6]). Several types of them were investigated successfully [1, 2], and other important and interesting types and cases are still patiently waiting for their researchers.

Since a role of dynamic systems in the different branches of contemporary and future science and technology is extremely important (dynamic systems serve as mathematical models of a largest spectrum of physical [3, 31, 32, 35, 36], geophysical [9, 10], sociological [8, 15, 28], biological [17, 29, 30], economical, technical phenomena and processes [11 – 14, 20 - 27]), and especially important in this sense are namely polynomial dynamic systems, their investigation appears to be the urgent task.

In the present paper a family of dynamic systems is considered on a real plane of phase variables $x, y$

$$\frac{dx}{dt} = X(x, y), \quad \frac{dy}{dt} = Y(x, y),$$

(1)

here $X(x, y)$ be a cubic, $Y(x, y)$ be a square form, $X(0, 1) > 0$, $Y(0, 1) > 0$. These forms are taken as reciprocal polynomials of $x$ and $y$. An aim is to describe in a Poincare disk (or circle) all admissible different phase portraits. A Poincare method of sequential mappings is used: firstly the central display of $R_{x,y}^2$ from a center $(0, 0, 1)$ of a Poincare sphere $\Sigma$ to a Poincare sphere $\Sigma: x^2 + y^2 + z^2 = 1$ with the diametrically opposite points identified, secondly the orthogonal mapping of a lower enclosed semi sphere of a sphere $\Sigma$ to a Poincare circle $\Pi: x^2 + y^2 \leq 1$ with diametrically opposite points of its boundary $\Gamma$ identified [1].

2. Key designations

$\varphi(t, p), \ p = (x, y)$ — a fixed point := a solution (a motion) of an Eq. (1) — system with initial data $(0, p)$.

$L_p: \varphi = \varphi(t, p), t \in I_{max},$ a trajectory of a motion $\varphi(t, p)$.

$L_p^{(+,-)} := (+,-)$ a semi trajectory of a trajectory $L_p$.

$O$-curve of a system := its semi trajectory $L_p$ ($p \neq O, s \in \{+, -\}$), adjoining to a point $O$ under a condition that $st \rightarrow +\infty$.

$O^{(+,-)}$ curve of a system := its $O$-curve $L_p^{(+,-)}$.

$O_{(+,-)}$ curve of a system := its $O$-curve, adjoining to a point $O$ from a domain $x > 0$ ($x < 0$).

$TO$-curve of a system := its $O$-curve, which, being supplemented by a point $O$, touches some ray in it.

A nodal bundle of $NO$-curves of a system := an open continuous family of its $TO$-curves $L_p$, where $s \in \{+, -\}$ is a fixed index, $p \in A$, $A$ - a simple open arc, $L_p \cap \Lambda = \{p\}$.

A saddle bundle of $SO$-curves of a system, a separatix of the point $O$ := a fixed $TO$ -curve, which isn’t included into some bundle of $NO$-curves of a system.

$H, P, E$ - $O$-sectors of a system: a hyperbolic, a parabolic and an elliptical sector.

A topological type (T-type) of a singular point $O$ of a system := a word $AO$ constructed with letters $S, N$ (a word $BO$ constructed with letters $H, P, E$), describes a circular order of bundles $S, N$ of its $O$-curves (of its $O$-sectors $H, P, E$) when traversing the point $O$ in the «+$»-direction (that mean counterclockwise), starting with some of them.
For every dynamic system belonging to the (1)-family the following statements are true.
1) A topological type of the singular point \( O (0, 0) \) in a form \( B_0 \) can be constructed through the form \( A_0 \), and back (we use the both forms);
2) Real roots of the polynomial \( P(u) \) (the polynomial \( Q(u) \)) turn out to be the angular coefficients of isoclines of the infinity (isoclines of a zero correspondingly);
3) Those real roots, all together and separately, we number strictly in the ascending order.

3. Study of the (3, 1)-subfamily of (1)-systems

The whole (1)-systems family includes several subfamilies of the first hierarchical level of subdivision. This subdivision itself and classification of subfamilies is based on the possible sequences of roots of the characteristic polynomials \( P(u) \) and \( Q(u) \) and, hence, on possible different amounts of multipliers in the decompositions of right parts of the taken system into lower degree forms. Several of them were described in the previous papers devoted to this fundamental research, for example [7, 16, 18, 19, 33, 34]. This subfamily represents a totality of all (1)-systems of the kind:

\[
\frac{dx}{dt} = p_3 (y - u_1 x)(y - u_2 x)(y - u_3 x) \equiv X(x, y),
\]
\[
\frac{dy}{dt} = c (y - qx)^2 \equiv Y(x, y),
\]

(2)

where \( p_3, c, u_1, u_2, u_3, q \in R, p_3 > 0, c > 0, u_1 < u_2 < u_3, u_i = q \forall i \in [1, 2, 3] \).

Common features of the (3, 1)-subfamily systems are the follows, that each one of them satisfies the conditions enlisted below.

1. It has singular points in the enclosed Poincare circle \( \Omega \); a finite singular point \( O (0, 0) \) and infinitely remote singular points \( Q_i (u_i, 0), i = 0, 2 \), \( u_0 = 0 \).
2. Its characteristic polynomials \( P, Q \) look like \( P(u) \equiv X(1, u) \) and \( Q(u) \equiv Y(1, u) \).
3. It has one of the following sequences of roots of the polynomials \( P \) and \( Q \):
   1) \( u_1, u_2, u_3, q \);
   2) \( u_1, u_2, q, u_3 \);
   3) \( u_1, q, u_2, u_3 \);
   4) \( q, u_1, u_2, u_3 \).

Thus, the (3,1)-subfamily of (1)-systems appears to be subdivided into four sub subfamilies of the second hierarchical level. Those sub subfamilies we indicate with the number \( r = \frac{1}{4} \), so that for each of them all the including systems have one common sequence of roots of their characteristic polynomials \( P, Q \). Number \( r \) shows the sequence in the previous list.

We study each sub subfamily with the use of a common program [7, 18, 19, 33, 34]. But before these studies, we proceed their DC-classification.

A DC-transformation means a double exchange in some (1)-system: \( (t, y) \rightarrow (-t, -y) \). This transformation leads to some special changes in a system and its features: 1) real roots of the characteristic polynomials \( P(u), Q(u) \) change their signs, 2) the sign of its variable \( t \) also changes, and consequently the direction of movement along every trajectory of a system becomes opposite [7, 16].

The DC-transformation of (3,1)-systems transforms the \( 3_1 \) — sub subfamily (i.e., the sub subfamily with \( r = 1 \)) into the \( 3_4 \) — sub subfamily (and backwards), \( 3_2 \) — sub subfamily into \( 3_3 \) — sub
subfamily (and backwards). This means, that the DC-transformation splits the $3_2$ — sub subfamilies, $r = 1, 4$, into the 2 classes: the class 1 contains $3_1$ and $3_2$ — sub subfamilies (we remember, that those systems are DC-independent), while the class 2 contains the $3_3$ and $3_4$ — sub subfamilies (those systems, being the DC-independent ones, appear to be DC-mutually inversed for the systems from the sub subfamilies $3_2$ and $3_1$, correspondingly).

Taking these details into consideration, we study the $3_1$ and $3_2$ — sub subfamilies previously (one by one and using the common research plan [19, 33, 34]). After them, we study $3_3$ and $3_4$ — sub subfamilies, using their DC-connections with the $3_2$ and $3_1$ — sub subfamilies, correspondingly).

$3_1$ — sub subfamily of (2)-systems represents a totality of all (2)-systems, every of which has a sequence of real roots of its characteristic polynomials $P(u), Q(u)$ with the number $r = 1$

\[
\{u_1, u_2, u_3, q\}.
\]

4. **Topo dynamical types (TD-types) of singular points of systems from the $3_1$ — sub subfamily of (2)-systems**

$3_1$ — sub subfamily of (2)-systems may be considered as the limit subfamily for the $(3, 2)_1$ — sub subfamily of (1)-systems under the condition if $q_1 \to q_2$. See [33, 34]. Using this understanding, we find, that the topo dynamical type of a finite singular point $O(0, 0)$ of each its system can be described with the word $\overline{A}_6 = S_0 S_{\pm} N_{\pm} S_{\pm}$ [7, 16, 33, 34].

With the aim to find the topo dynamical types of infinitely remote singular points of the systems belong to the $3_1$ — sub subfamily, we proceed the further division of this sub subfamily of the second hierarchical level into subfamilies of the third hierarchical level already $3_{1,s}, s = 1, 7$, and reveal the TD-types of infinitely remote singular points for their systems.

Thus, we obtain a table, which shows in its lines the TD-types of infinitely remote singular points of the systems from the $3_1$ — sub subfamily.

**Table 1.** The topo dynamical types (TD-types) of infinitely remote singular points of the $3_{1,s}$ — systems, $s = 1, 7$.

| N | Number $s$ | $\overline{A}_0^{+(-)}$ | $\overline{A}_1^{+(-)}$ | $\overline{A}_2^{+(-)}$ | $\overline{A}_3^{+(-)}$ |
|---|---|---|---|---|---|
| 1 | $0 < u_1$ | $N_{\pm}(N_{\pm})$ | $N_{\pm}(S_{\pm})$ | $S_{\pm}(N_{\pm})$ | $N_{\pm}(S_{\pm})$ |
| 2 | $u_1 = 0$ | $N_{\pm} N_{\pm}(\emptyset)$ | - | - | - |
| 3 | $u_1 < 0 < u_2$ | $N_{\pm}^+(N_{\pm})$ | $N_{\pm}(S_{\pm}^+)$ | - | - |
| 4 | $u_2 = 0$ | $\emptyset(N_{\pm}^+ N_{\pm})$ | - | - | - |
| 5 | $u_2 < 0 < u_3$ | $N_{\pm}(N_{\pm})$ | - | $S_{\pm}(N_{\pm})$ | - |
A note to the Table 1. Our sub subfamilies $3_{r,s}$, $r = 1, 2, 3, 4$, are subfamilies of the second hierarchical level in the general classification of subfamilies of the (1)-systems. Obviously, the sub subfamilies $3_{r,s}$, $r = 1, 4$, $s = 1, 7$, appear to be the subfamilies of the third hierarchical level, while the $3_{r,s}$, $r = 1, 4$, $s = 1, 7$, subfamilies, if they will appear during the further investigations, will become the subfamilies of the fourth hierarchical level already for the (1)-systems, and of the third hierarchical level relatively the $(3, 1)$-subfamily of dynamic systems under consideration.

5. The behavior of separatrices belonging to the $3_{1,s}$ systems in the Poincare disk $\Omega$.

Looking at the topo dynamical types of singular points of systems belonging to the $3_{1}$ sub subfamily of (2)-systems, we may conclude, that $3_{1,s}$ systems have the following separatrices:

$$S_0, S_+, S_-, S_1^+, S_2^+, S_3^-,$$

excluding the cases of $s = 2, 4, 6$, when someone among those separatrices vanishes: $S_1^-$, $S_2^+$ or $S_3^-$ correspondingly.

$\forall s \in \{1, \ldots, 7\}$ we construct the so-called „off-road map“ (OR-map [19, 33, 34]) of $3_{1,s}$ systems and use it to reveal the global continuation of every separatrix belonging to the systems of this sub subfamily in the Poincare disk $\Omega$, together with the mutual arrangement of separatrices in the disk $\Omega$ for all the dynamic systems of the taken sub subfamily. Upon conducting the described research, we come to the following conclusions.

The cases of $s = 1 (0 < u_4)$ and $s = 2 (u_4 = 0)$.

$\forall s \in \{1, 2\}$ separatrices of all the systems of the $3_{1,s}$ sub subfamily are the ordinary ones (that means the global continuation in the Poincare disk $\Omega$ for each one of them is unambiguous), and because of that the mutual location of separatrices for the systems of this sub subfamily is invariable in $\Omega$. Due to this fact all the dynamic systems belonging to this sub subfamily have in the enclosed Poincare disk $\Omega$ one and the same common phase portrait $3_{1,s}$.

The cases of $s = 3, 4, 5$

$\forall s \in \{3, 4, 5\}$ systems of the $3_{1,s}$ sub subfamily will have special separatrices $S_-, S_1^-$ and $S_2^-$. Each one among these separatrices only once crosses the radius $O O_0^-$ of the circle $\Omega$, for example, in the points $x_-, x_1^-$ and $x_3^-$, so that for two of these points takes place a fixed sequence on this radius: $x_1^- < x_-, x_3^-$, while for the whole three points may exist 5 variants of sequences on this radius, which remain the same for all separatrices of the dynamic systems from the $3_{1,s}$ sub subfamily:

1) $x_1^- < x_2^-$,
2) $x_1^- = x_3^- < x_-$,
3) $x_2^- < x_1^- < x_-$,
4) $x_2^- < x_1^- = x_-$,
5) $x_2^- < x_+ < x_1^-$.

$\Rightarrow \forall s \in \{3, 4, 5\}$ sub subfamily actually subdivides into the subfamilies of the next hierarchical level: $3_{r,s,l}$ subfamilies, $l = 1, 5$, and for each of them its dynamic systems do not have special separatrices already. Therefore, all of them inside a taken sub subfamily of this eventual
hierarchical level show in the Poincare circle (or disk) $\Omega$ one common phase portrait – the portrait of this chosen sub-sub-sub-family $3_{r,s,l}$.

The case of $s = 6$
Systems of the $3_{1,6}$ subfamily have special separatrices $S_-$ and $S_1^-$. These separatrices only once crosses the radius $O_0^{-\infty}$ of the circle $\Omega$, and for the crossing $x_-$ and $x_1^-$ may take place 3 variants of sequence on this radius:
1) $x_1^- < x_-$,
2) $x_1^- = x_-$,
3) $x_1^- < x_1^-$.

$\Rightarrow 3_{1,6}$ subfamily subdivides into the $3_{1,6,1}$ subfamilies belonging to the next hierarchical level, $l = 1, 3$, for every of which its systems have in the enclosed Poincare circle $\Omega$ one common $3_{1,6,1}$ phase portrait.

The case of $s = 7$
In this case systems of the $3_1$ subfamily have the same special separatrices, as in the cases $s = 3, 5$, and their behavior in the Poincare circle $\Omega$ is similar. The only difference is the follows: for their crossing points with the radius $O_0^{-\infty}$ of the Poincare circle the fixed sequence for the two of them looks like: $x_3^- < x_1^-$, while the possible variants of sequence for the whole three points may be the follows:
1) $x_3^- < x_1^- < x_-$,
2) $x_3^- < x_1^- = x_-$,
3) $x_3^- < x_- < x_1^-$,
4) $x_3^- = x_- < x_1^-$,
5) $x_3^- < x_2^- < x_1^-$.

$\Rightarrow 3_{1,7}$ subfamily subdivides into the $3_{1,7,1}$ subfamilies of the next hierarchical level, $l = 1, 5$, for every of which its systems have in the enclosed Poincare circle $\Omega$ one common phase portrait $3_{1,7,1}$.

6. Descriptive phase portraits of the $3_1$ subfamily

As it follows from the above, for the dynamic systems of this subfamily 25 different in the topological sense phase portraits are possible in the enclosed Poincare disk $\Omega$. We construct these phase portraits in the descriptive form using the special tables. 12 of those portraits we write out explicitly, and remaining 13 ones we obtain using a method of the base table deformation [33, 34]. Some tables, which are similar to the ones given, we omit in this paper.

Table 2. The descriptive phase portrait of the $3_{1,1}$ sub subfamily.

|    | $S_0$: $O_0^+ \to O$ | $\Omega_1$ | $S_0 S_7^+ S_+$ | $O_0^+ \to O_3^+$ |
|----|----------------------|------------|------------------|-------------------|
| 1  | $S_4$: $O \to O_3^+$ | $\Omega_2$ | $S_4 S_-$       | $O \to O_3^+$     |
| 2  | $S_7$: $0 \to O_3^+$ | $\Omega_3$ | $S_- S_5^+ S_6$ | $O_0^+ \to O_3^-$ |
| 3  | $S_5^+ S_0^+ \to O_3^+$ | $\Omega_4$ | $S_5^+ S_6$    | $O_0^+ \to O_3^+$ |
Table 3. The descriptive phase portrait of the $3_{1,3,1}$ −sub-sub-subfamily.

|   | $S_6$: $0_i^− \to 0$ | $\Omega_1$ | $S_0 S_2^+ S_4^+$ | $O_i^+ \to O_3^+$ |
|---|-----------------|----------|-----------------|-----------------|
| 1 |                 |          |                 |                 |
| 2 | $S_4$: $0 \to O_3^+$ | $\Omega_2$ | $S_4 S_-$ | $O \to O_3^+$ |
| 3 | $S_-: O \to O_3^+$ | $\Omega_3$ | $S_- S_5^+ S_6$ | $O_i^- \to O_3^+$ |
| 4 | $S_5^+: O_i^+ \to O_3^+$ | $\Omega_4$ | $S_5^+$ | $O_i^+ \to O_3^+$ |
| 5 | $S_5^+: O_3^- \to O_3^+$ | $\Omega_5$ | $S_5^+ S_1^-$ | $O_i^- \to O_3^+$ |
| 6 | $S_5^-: O_3^- \to 0_1^-$ | $\Omega_6$ | $S_1^-$ | $O_i^- \to O_3^+$ |

Table 4. The descriptive phase portrait of the $3_{1,3,3}$ −sub-sub-subfamily.

|   | $S_6$: $0_i^+ \to 0$ | $\Omega_1$ | $S_0 S_2^+ S_4^+$ | $O_i^+ \to O_3^+$ |
|---|-----------------|----------|-----------------|-----------------|
| 1 |                 |          |                 |                 |
| 2 | $S_4$: $0 \to O_3^+$ | $\Omega_2$ | $S_4 S_-$ | $O \to O_3^+$ |
| 3 | $S_-: O \to O_3^+$ | $\Omega_3$ | $S_- S_5^+ S_6$ | $O_i^+ \to O_3^+$ |
| 4 | $S_5^+: O_i^+ \to O_3^+$ | $\Omega_4$ | $S_5^+$ | $O_i^+ \to O_3^+$ |
| 5 | $S_5^+: O_3^- \to O_3^+$ | $\Omega_5$ | $S_5^+ S_1^-$ | $O_i^- \to O_3^+$ |
| 6 | $S_5^-: O_3^- \to 0_1^-$ | $\Omega_6$ | $S_1^-$ | $O_i^- \to O_3^+$ |

Table 5. The descriptive phase portrait of the $3_{1,5,1}$ −sub-sub-subfamily.

|   | $S_6$: $0_i^+ \to 0$ | $\Omega_1$ | $S_0 S_2^+ S_4^+$ | $O_i^+ \to O_3^+$ |
|---|-----------------|----------|-----------------|-----------------|
| 1 |                 |          |                 |                 |
| 2 | $S_4$: $0 \to O_3^+$ | $\Omega_2$ | $S_4 S_-$ | $O \to O_3^+$ |
| 3 | $S_-: O \to O_3^+$ | $\Omega_3$ | $S_- S_5^+ S_6$ | $O_i^+ \to O_3^+$ |
| 4 | $S_5^+: O_i^+ \to O_3^+$ | $\Omega_4$ | $S_5^+$ | $O_i^+ \to O_3^+$ |
| 5 | $S_5^+: O_3^- \to O_3^+$ | $\Omega_5$ | $S_5^+ S_1^-$ | $O_i^- \to O_3^+$ |
| 6 | $S_5^-: O_3^- \to 0_1^-$ | $\Omega_6$ | $S_1^-$ | $O_i^- \to O_3^+$ |
| 5 | $S^{-}_{1}: O^{-}_{3} \rightarrow O^{+}_{3}$ | $\Omega_{5}$ | $S^{-}_{-} S^{+}_{-}$ | $O^{-}_{3} \rightarrow O^{+}_{3}$ |
|---|---|---|---|---|
| 6 | $S^{-}_{1}: O^{-}_{2} \rightarrow O^{-}_{1}$ | $\Omega_{6}$ | $S^{-}_{1}$ | $O^{-}_{0} \rightarrow O^{-}_{0}$ |

**Table 6.** The descriptive phase portrait of the $3_{1,5,3}$ -sub-sub-subfamily.

| 1 | $S^{-}_{0}: O^{+}_{1} \rightarrow O^{-}_{0}$ | $\Omega_{1}$ | $S^{-}_{0} S^{+}_{3} S^{+}_{4}$ | $O^{+}_{1} \rightarrow O^{+}_{3}$ |
|---|---|---|---|---|
| 2 | $S^{-}_{1}: O^{-}_{3} \rightarrow O^{+}_{3}$ | $\Omega_{2}$ | $S^{-}_{4} S^{-}_{-}$ | $O^{-}_{3} \rightarrow O^{+}_{3}$ |
| 3 | $S^{-}_{2}: O^{-}_{2} \rightarrow O^{+}_{2}$ | $\Omega_{3}$ | $S^{-}_{-} S^{-}_{-} S^{-}_{0}$ | $O^{+}_{2} \rightarrow O^{+}_{2}$ |
| 4 | $S^{-}_{3}: O^{+}_{2} \rightarrow O^{+}_{2}$ | $\Omega_{4}$ | $S^{+}_{2}$ | $O^{+}_{0} \rightarrow O^{+}_{2}$ |
| 5 | $S^{-}_{4}: O^{+}_{1} \rightarrow O^{-}_{1}$ | $\Omega_{5}$ | $S^{-}_{1} S^{-}_{3}$ | $O^{+}_{2} \rightarrow O^{-}_{2}$ |
| 6 | $S^{-}_{5}: O^{-}_{3} \rightarrow O^{-}_{0}$ | $\Omega_{6}$ | $S^{+}_{3}$ | $O^{+}_{0} \rightarrow O^{-}_{2}$ |

**Table 7.** The descriptive phase portrait of the $3_{1,5,5}$ -sub-sub-subfamily.

| 1 | $S^{-}_{0}: O^{+}_{1} \rightarrow O^{-}_{0}$ | $\Omega_{1}$ | $S^{-}_{0} S^{+}_{3} S^{+}_{4}$ | $O^{+}_{1} \rightarrow O^{+}_{3}$ |
|---|---|---|---|---|
| 2 | $S^{-}_{1}: O^{-}_{3} \rightarrow O^{+}_{3}$ | $\Omega_{2}$ | $S^{-}_{4} S^{-}_{1}$ | $O^{-}_{3} \rightarrow O^{+}_{3}$ |
| 3 | $S^{-}_{2}: O^{-}_{2} \rightarrow O^{-}_{2}$ | $\Omega_{3}$ | $S^{-}_{1} S^{-}_{-}$ | $O^{-}_{2} \rightarrow O^{-}_{3}$ |
| 4 | $S^{-}_{3}: O^{-}_{2} \rightarrow O^{-}_{2}$ | $\Omega_{4}$ | $S^{-}_{-} S^{-}_{3} S^{-}_{0}$ | $O^{+}_{0} \rightarrow O^{-}_{0}$ |
| 5 | $S^{-}_{4}: O^{-}_{1} \rightarrow O^{-}_{1}$ | $\Omega_{5}$ | $S^{+}_{3}$ | $O^{+}_{0} \rightarrow O^{+}_{2}$ |
| 6 | $S^{-}_{5}: O^{-}_{3} \rightarrow O^{-}_{0}$ | $\Omega_{6}$ | $S^{-}_{3}$ | $O^{+}_{0} \rightarrow O^{-}_{2}$ |

### 7. Conclusions

The article develops and continues the extensive and deep original study of a large family of polynomial dynamic systems [7, 16, 18, 19, 33, 34]. This family actually includes the infinite set of specific systems, but allows its subdivision into subfamilies of several hierarchical levels. For those subfamilies it becomes possible to find out and describe their specific common phase portraits. Some special new research methods and attitudes were introduced especially for the goals of this research work. Several applied research works were conducted in touch with the results of the fundamental investigation of the considered dynamic systems family [2, 8, 15, 17, 28].

### 8. Recommendations
A broad spectrum of modern studies, the both fundamental and applied, may use the research methods of this work together with its detailed results, especially in the fields of mathematical modeling and further investigations of polynomial cubic dynamic systems. These results considered to be useful for scientific investigators, as well as for postgraduates and advanced students. Some details of applications discussed and shown in the sources [3, 8 - 10, 15, 17, 20 – 28, 31, 32, 35, 36].

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