SINGULAR PERTURBATION PROBLEM IN BOUNDARY/FRACTIONAL COMBUSTION

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ABSTRACT. Motivated by a nonlocal free boundary problem, we study uniform properties of solutions to a singular perturbation problem for a boundary-reaction-diffusion equation, where the reaction term is of combustion type. This boundary problem is related to the fractional Laplacian. After an optimal uniform H"older regularity is shown, we pass to the limit to study the free boundary problem it leads to.

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1. INTRODUCTION

In this paper we study nonnegative solutions for the semilinear boundary-reaction-diffusion problem:

$$\mathcal{L}_s u_\varepsilon := \text{div}(|x_n|^{1-2s}\nabla u_\varepsilon) = 0 \quad \text{in } B_1^+ = B_1 \cap \{x_n > 0\},$$

$$-\lim_{x_n \to 0^+} x_n^{1-2s} \frac{\partial u_\varepsilon}{\partial x_n} = -\beta_\varepsilon (u_\varepsilon) \quad \text{on } B_1' = B_1 \cap \{x_n = 0\},$$

$$\text{(P}_\varepsilon\text{)}$$

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where $B_1$ is the unit ball in $\mathbb{R}^n$, $n \geq 2$, $s \in (0, 1)$, and $\varepsilon$ is a small positive parameter. The nonlinear reaction term $\beta_\varepsilon(t)$ is of combustion type and is given by

$$\beta_\varepsilon(t) = \frac{1}{\varepsilon} \beta\left(\frac{t}{\varepsilon}\right), \quad t \in \mathbb{R},$$

with $\beta \in C^{0,1}_c(\mathbb{R})$ satisfying

$$\beta \geq 0, \quad \text{supp} \beta = [0,1], \quad \text{and} \quad \int_0^1 \beta(t)dt = M.$$ 

Note that the solutions of \(P_\varepsilon\) are the critical points (including the local minimizers) of the energy functional

$$J_\varepsilon(u) = \int_{B_1^+} |\nabla u|^2 |x_n|^{1-2s} + \int_{B_1'} 2\beta_\varepsilon(u)$$

among all functions in the weighed Sobolev space $W^{1,2}(B_1^+, |x_n|^{1-2s})$ with fixed trace on $(\partial B_1)^+ = \partial B_1 \cap \{x_n > 0\}$, where $\beta_\varepsilon$ is the primitive of $\beta_\varepsilon$ given by

$$\beta_\varepsilon(t) = \int_0^t \beta(s)ds.$$

Formally, as $\varepsilon \to 0+$, the functional $J_\varepsilon$ converges to

$$J_0(u) = \int_{B_1^+} |\nabla u|^2 |x_n|^{1-2s} + \int_{B_1'} 2M\chi\{u > 0\},$$

which is the boundary (or thin) analogue of the Alt-Caffarelli [ACS1] energy functional. The study of the minimizers of $J_0$ has been initiated in [CRS10] and by now there is a good understanding of the associated free boundary problem. Namely, it is known that the minimizers of $J_0$ solve (in the appropriate sense)

$$\text{div}(|x_n|^{1-2s}\nabla u) = 0 \quad \text{in} \; B_1^+,$$

$$-\lim_{x_n \to 0^+} x_n^{1-2s} \frac{\partial u}{\partial x_n} = 0 \quad \text{on} \; \{u > 0\} \cap B_1',$$

$$\lim_{t \to 0^+} \frac{u(x_0 + tu'_{x_0})}{t^s} = \sqrt{\frac{2M}{c_0(s)}} \quad \text{for} \; x_0 \in \mathcal{F}_u,$$

where

$$\mathcal{F}_u := \partial\{u(\cdot,0) > 0\} \cap B_1'$$

is the free boundary in the problem, $\nu'_{x_0}$ is the in-plane, inner unit normal to $\{u(\cdot,0) > 0\}$ and and $c_0(s)$ is a constant. The regularity properties of the free boundary $\mathcal{F}_u$ for the minimizers in the case $s = 1/2$ have been studied in the series of papers [DSR12][DSS12][DSS14], establishing the smoothness of flat free boundaries. For the general $s \in (0,1)$, the $C^{1,\alpha}$ regularity of flat free boundaries has been established in [DSS14].

One of our main objectives in this paper is to show that the solutions $u_\varepsilon$ of the singular perturbation problem \(P_\varepsilon\), also converge to a solution to the free boundary problem \(P\), in a certain, weaker, sense. We show the uniform $s$-Hölder regularity of $u_\varepsilon$ (Theorem 2.1), however, the passage to the limit $u$ as $\varepsilon \to 0+$ is complicated by the fact that $\beta_\varepsilon(u_\varepsilon)$ may not converge (in weakly-* sense) to $M\chi\{u(0) > 0\}$. Nevertheless, at free boundary points $x_0 \in \mathcal{F}_u$ with a measure-theoretical normal and a nondegeneracy condition on $u$, we can establish an asymptotic development of $u$, implying the free boundary condition in \(P\) (Theorem 4.1).
This kind of convergence results are very well known in combustion theory for the singular perturbation problems of the type
\[ \Delta u_\varepsilon = \beta_\varepsilon(u_\varepsilon) \quad \text{in } B_1, \]
(even in time-dependent case) with \( \beta_\varepsilon \) as in (1.1), since the works of Zel’dovich and Frank-Kamenetskii [ZFK38]. Mathematically rigorous results, however, are much more recent. Here we cite some of the important ones for our paper: [BCN90, CV95, Vaz96, CLW97a, CLW97b, DPS03, Wei03].

The singular-perturbation problem \( (P_\varepsilon) \) can be also viewed as the localized version of the global reaction-diffusion equation \( (P'_\varepsilon) \)
\[ (-\Delta_{x'})^s u_\varepsilon = -\beta_\varepsilon(u_\varepsilon) \quad \text{in } \Omega \subset \mathbb{R}^{n-1} \]
\[ u_\varepsilon = g_\varepsilon \quad \text{on } \mathbb{R}^{n-1} \setminus \Omega \]
for the fractional Laplacian \((-\Delta_{x'})^s\) in \( x' = (x_1, \ldots, x_{n-1}) \) variables, where \( g_\varepsilon \) is a nonnegative function on \( \mathbb{R}^{n-1} \setminus \Omega \) having the meaning of the boundary data. We recall that the fractional Laplacian is defined as the Fourier multiplier of symbol \( |\xi'|^{2s} \) for \( s \in (0, 1) \) (see [Lan72] for a treatment of these operators). Note that the solutions of \( (P'_\varepsilon) \) are the critical points of the energy functional
\[ j(v) = c_{n,s} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} \frac{(u(x') - u(y'))^2}{|x' - y'|^{n+2s}} + \int_{\mathbb{R}^{n-1}} 2B_\varepsilon(u), \]
among all functions such that \( u = g_\varepsilon \) on \( \mathbb{R}^{n-1} \setminus \Omega \). (Here \( c_{n,s} > 0 \) is a normalization constant.)

The connection between \( (P_\varepsilon) \) and \( (P'_\varepsilon) \) is then established through the so-called Caffarelli-Silvestre extension [CS07]: if for a given function \( u \) on \( \mathbb{R}^{n-1} \setminus \Omega \) having appropriate growth conditions at infinity we consider the extension \( \tilde{u} \) to \( \mathbb{R}^{n-1} + (0, \infty) \) by solving the Dirichlet problem
\[ \mathcal{L}_s \tilde{u} = \text{div}(x_n^{1-2s} \nabla \tilde{u}) = 0 \quad \text{in } \mathbb{R}_+^n \]
\[ \tilde{u} = u \quad \text{on } \mathbb{R}^{n-1} \times \{0\} \]
then
\[ -c_{n,s} \lim_{x_n \to 0^+} x_n^{1-2s} \frac{\partial \tilde{u}}{\partial x_n} = (-\Delta_{x'})^s u \quad \text{on } \mathbb{R}^{n-1} \times \{0\} \]
for a positive constant \( c_{n,s} \). Hence, if \( u_\varepsilon \) solves \( (P'_\varepsilon) \), \( x_0 \in \Omega \) and \( R > 0 \) are such that \( B_R'(x_0) \subset \Omega \), then the extension of \( u_\varepsilon \) to \( \mathbb{R}_+^n \) constructed as above will solve \( (P'_\varepsilon) \) in \( B_R'(x_0) \). As a consequence, the singular perturbation problem \( (P'_\varepsilon) \) for the fractional Laplacian, becomes a boundary (or thin) singular perturbation problem \( (P_\varepsilon) \) for the operator \( \mathcal{L}_s \) in one dimension higher.

**Main results and the structure of the paper.** In this paper, we will focus on the uniform estimate of the solutions to \( (P_\varepsilon) \) and the proof of the free boundary condition in \( (P'_\varepsilon) \).

- In §2 we prove the uniform \( s \)-Hölder regularity for the solutions of \( (P_\varepsilon) \), see Theorem 2.1. This allows to pass to the limit as \( \varepsilon \to 0^+ \) and study the resulting solutions in the subsequent sections.
• In [3] we prove various results concerning the limits of $u_\varepsilon$, or, more precisely, the limits of the pairs $(u_\varepsilon, B_\varepsilon(u_\varepsilon))$, which we denote $(u, \chi)$. The results include the compactness lemma (Lemma 3.1), ensuring the convergence in the proper spaces and Weiss-type monotonicity formulas for $u_\varepsilon$ and $(u, \chi)$ (Theorems 3.4 and 3.5).

• In [4] we prove that the free boundary condition in problem (P) is satisfied at free boundary points with measure-theoretical normal for $\{u(\cdot, 0) > 0\}$, under the additional nondegeneracy condition (Theorem 4.3). This is done by identifying the blowups with flat free boundaries (Proposition 4.3). We conclude the paper by proving two additional propositions related to the Weiss energy at nondegenerate points (Propositions 4.4 and 4.5).

**Notations and preliminaries.**

• We will use fairly standard notations in this paper.
  
  o $\mathbb{R}^n$ will stand for the $n$-dimensional Euclidean space;
  
  o For every $x \in \mathbb{R}^n$ we write $x = (x', x_n)$, where $x' = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$. This identifies $\mathbb{R}^n$ with $\mathbb{R}^{n-1} \times \mathbb{R}$. We also don’t distinguish between $(x', 0)$ and $x'$, thus identifying $\mathbb{R}^{n-1}$ with $\mathbb{R}^{n-1} \times \{0\} \subset \mathbb{R}^n$.
  
  o $\mathbb{R}_+^n = \mathbb{R}^n \cap \{x_n > 0\}$;
  
  o Balls and half-balls: $B_r(x) = \{y \in \mathbb{R}^n : |y - x| < r\}$, $B_r^+(x) = B_r(x) \cap \{x_n > 0\}$.
  
  o ‘Thin’ balls: $B_r'(x) = B_r(x) \cap \{x_n = 0\}$.
  
  o Typically, we skip the center in the notation for balls if it is the origin. Thus, $B_1 = B_1(0)$, $B_1' = B_1'(0)$, etc.
  
• For the functions $\beta$ and $\beta_\varepsilon$, we make the following assumption throughout the paper. Besides (1.1)–(1.2), we fix a constant $A > 0$ such that

$$\max\{|\beta(s)|, |\beta'(s)|\} \leq A, \quad \text{for all } s \in \mathbb{R}.$$  

We will also need to make a technical assumption that

$$\beta > 0 \quad \text{on } (0, 1).$$

• The functions $\mathcal{B}, \mathcal{B}_\varepsilon : \mathbb{R} \to \mathbb{R}$ are the primitives of $\beta$ and $\beta_\varepsilon$ given by

$$(1.4) \quad \mathcal{B}(t) = \int_0^t \beta(s)ds, \quad \mathcal{B}_\varepsilon(t) = \int_0^t \beta_\varepsilon(s)ds = \mathcal{B}(s/\varepsilon)$$

• **Even extension of $u_\varepsilon$ and weak solutions of (P).** In what follows, we will be extending the functions $u_\varepsilon$ in $B_1^+$ with even reflection to all of $B_1$:

$$u_\varepsilon(x', -x_n) = u_\varepsilon(x', x_n), \quad \text{for } x \in B_1^+.$$  

With such an extension in mind, $u_\varepsilon$ is a **weak solution of (P)** if for any test function $\varphi \in C_c^\infty(B_1)$

$$(1.5) \quad \int_{B_1} |x_n|^{1-2\varepsilon} \nabla u_\varepsilon \cdot \nabla \varphi dx + \int_{B_1'} 2\beta_\varepsilon(u_\varepsilon) \varphi dx' = 0,$$

or, in other words,

$$\text{div}(|x_n|^{1-2\varepsilon} \nabla u_\varepsilon) = 2\beta_\varepsilon(u_\varepsilon) \chi_{\{x_n=0\}}$$  

in the sense of distributions.

**Unless specified otherwise, by a solution of (P) we will always understand a weak solution of (P).**
We also note that for functions which are even symmetric in $x_n$, the energy functional (1.3) can be rewritten as

$$J_\varepsilon(v) = \frac{1}{2} \int_{B_1} |x_n|^{1-2s} |\nabla v|^2 + \int_{B_1^\prime} 2\beta_\varepsilon(v).$$

**Rescalings.** Finally, throughout the paper we will make the extensive use of rescalings. For a given $x_0 \in B_1^\prime$ and $\lambda > 0$ define

$$u_{\varepsilon,\lambda}(x) = u_{\varepsilon,\lambda}(x) := \frac{u(x_0 + \lambda x)}{\lambda^s}, \quad x \in B_{1-|x_0|}/\lambda.$$

A straightforward computation shows that $u_{\varepsilon,\lambda}$ satisfies

$$\mathrm{div}(|x_n|^{1-2s} \nabla u_{\varepsilon,\lambda}) = 2\beta_\varepsilon/\lambda (u_{\varepsilon,\lambda})^2 \chi^{n-1} \{x_n = 0\} \quad \text{in} \quad B_{(1-|x_0|)/\lambda}.$$

## 2. Uniform $C^{0,s}$ Regularity

In this section we prove the following uniform Hölder regularity result for the solutions of (P).

**Theorem 2.1 (Uniform s-Hölder estimate).** Let $u_\varepsilon$ be a nonnegative solution of (P) with $\|u_\varepsilon\|_{L^\infty(B_1)} \leq L$. Then $u_\varepsilon \in C^{0,s}(K)$ for any $K \Subset B_1$ with

$$\|u_\varepsilon\|_{C^{0,s}(K)} \leq C(n, s, A, L, K)$$

uniformly for all $\varepsilon \in (0, 1)$.

Our proof follows the ideas from [DPS03] in the case of $p$-harmonic functions. One of the main steps is the following Harnack-type inequality.

**Lemma 2.2 (Harnack-type inequality).** Let $v$ be a locally bounded nonnegative weak solution of

$$0 \leq \mathrm{div}(|x_n|^{1-2s} \nabla v) \leq A\chi_{\{0 < v < 1\}} \mathcal{H}^{n-1} \{x_n = 0\} \quad \text{in} \quad B_1$$

with $v(0) \leq 1$. Then there exists a constant $C = C(n, s, A)$ such that

$$\|v\|_{L^\infty(B_{1/4})} \leq C.$$

To prove this lemma, we will need the following interior Hölder estimate.

**Lemma 2.3 (Interior s-Hölder estimate).** Let $|w| \leq M$ be a weak solution of

$$|\mathrm{div}(|x_n|^{1-2s} \nabla w)| \leq \mu \mathcal{H}^{n-1} \{x_n = 0\} \quad \text{in} \quad B_1(x_0).$$

Here $x_0$ is not necessarily on $\{x_n = 0\}$.

Then $w \in C^{0,s}(B_{1/2}(x_0))$ with

$$\|w\|_{C^{0,s}(B_{1/2}(x_0))} \leq C(n, s, \mu, M).$$

**Proof.**

(i) When $x_0 = 0$, or more generally, $(x_0)_n = 0$, we refer to Remark 5.2 and the proof of Theorem 5.1 in [ALP13].

(ii) The case of general $x_0$ is obtained by considering the subcases

(a) $|(x_0)_n| > 3/4,$ and

(b) $|(x_0)_n| \leq 3/4.$
In the subcase (a) the equation is uniformly elliptic in $B_{3/8}(x_0)$, and the estimate follows from standard interior estimates for uniformly elliptic equations. In the subcase (b), the $s$-Hölder continuity in $B_{1/2}(x_0) \cap \{|x_n| \leq 1/8\}$ is obtained from the case (i) above. The $s$-Hölder continuity in $B_{1/2}(x_0) \cap \{|x_n| > 1/8\}$ is obtained from the uniform ellipticity of the operator $L_s$ in $B_1(x_0) \cap \{|x_n| > 1/8\}$. 

**Proof of Lemma 2.2.** We start by an observation that the function $v$ is continuous by Lemma 2.3. We will use this fact implicitly throughout the proof.

We argue by contradiction. Assuming that the conclusion of the lemma fails, there exists a sequence of nonnegative solutions $v_k$ with

$$v_k(0) \leq 1 \quad \text{but} \quad \|v_k\|_{L^\infty(B_{1/4})} \geq k.$$ 

Let $\Omega_k := \{x \in B_1^c : v_k(x) \leq 1\}$, $O_k := \{x \in B_1 : \text{dist}(x, \Omega_k) \leq \frac{1}{4}(1 - |x|)\}$, and

$$m_k := \max_{O_k}(1 - |x|)v_k(x).$$

Observe that $B_{1/4} \subset O_k$. Then $m_k \geq \frac{3}{4} \sup_{B_{1/4}} v_k \geq \frac{3}{4} k$. Let $x_k \in O_k$ such that $(1 - |x_k|)v_k(x_k) = m_k$, then

$$v_k(x_k) = m_k \geq \frac{3}{4} k. \tag{2.1}$$

Consider the distance $\delta_k := \text{dist}(x_k, \Omega_k)$ and $y_k \in \Omega_k$ realize $\delta_k$. By (2.1), $\delta_k > 0$. Using the fact that $\delta_k \leq \frac{1}{4}(1 - |x_k|)$ and the triangle inequality, we obtain that $B_{\delta_k/2}(y_k) \subset O_k$ and for any $z \in B_{\delta_k/2}(y_k)$,

$$v_k(z) \leq \frac{m_k}{1 - |z|} = \frac{1 - |x_k|}{1 - |z|}v_k(x_k) \leq 2v_k(x_k). \tag{2.2}$$

Since $v_k$ satisfies the homogeneous equation $\text{div}(|x_n|^{1-2s}\nabla v_k) = 0$ in $B_{\delta_k}(x_k)$, by the Harnack inequality in [FKSS2] there exists $c = c(n, s)$ such that

$$\inf_{B_{\delta_k/4}(x_k)} v_k \geq c v_k(x_k).$$

In particular, since $B_{\delta_k/4}(y_k) \cap B_{\delta_k/4}(x_k) \neq \emptyset$, then

$$\sup_{B_{\delta_k/4}(y_k)} v_k \geq c v_k(x_k). \tag{2.3}$$

Define

$$w_k(x) := \frac{v_k(y_k + \delta_k x)}{v_k(x_k)}, \quad x \in B_{1/2}.\notag$$

From (2.2) and (2.3) we have

$$\sup_{B_{1/2}} w_k \leq 2 \quad \text{and} \quad \sup_{B_{1/4}} w_k \geq c.\notag$$

Moreover, using (2.1) and recalling that $v_k(y_k) \leq 1$, $w_k$ satisfies

$$0 \leq \text{div}(|x_n|^{1-2s}\nabla w_k) \leq \frac{4A\delta_k^{2-2s}}{3k} - 2\delta_k^{s-1}\chi_{\{x_n = 0\}} \quad \text{in} \quad B_{1/2}\notag$$

$$w_k \geq 0, \quad w_k(0) = \frac{4}{3k}.\notag$$
Now, invoking Lemma 2.3, we obtain that \( w_k \) are uniformly \( s \)-Hölder continuous on compact subsets of \( B_{1/2} \) and hence, over a subsequence, they will converge locally uniformly to a function \( w_0 \) which satisfies
\[
\text{div}\left(|x_n|^{1-2s}\nabla w_0\right) = 0 \quad \text{in} \quad B_{1/2}, \quad \sup_{B_{1/4}} w_0 \geq c > 0, \quad w_0 \geq 0, \quad w_0(0) = 0
\]
This is a contradiction to the strong maximum principle in [FKS82]. \( \square \)

Now we prove the uniform \( C^{0,s} \) regularity of \( u_\varepsilon \).

**Proof of Theorem 2.1.** Note that it will be sufficient to prove the uniform estimates for small \( 0 < \varepsilon < \varepsilon_0 \), with universal \( \varepsilon_0 \), as the estimate for \( \varepsilon_0 < \varepsilon < 1 \) will follow from Lemma 2.3. It will also be sufficient to give the proof for \( K = B_{1/8} \).

Throughout the proof, we let
\[
\Omega_\varepsilon := \{x \in B'_1 : u_\varepsilon \leq \varepsilon\}.
\]

**Step 1.** We will show that there exists a constant \( C = C(n,s,A) \) such that
\[
u_\varepsilon(x) \leq C \text{dist}(x, \Omega_\varepsilon)^s, \quad x \in B'_{1/4} \setminus \Omega_\varepsilon.
\]
The proof is based on the construction of a proper lower barrier function.

Given \( x_0 \in B'_{1/4} \setminus \Omega_\varepsilon \), let
\[
m_0 := u_\varepsilon(x_0) - \varepsilon, \quad \delta_0 := \text{dist}(x_0, \Omega_\varepsilon).
\]
We are going to show that
\[
m_0 \leq C(n,s,A)\delta_0^s.
\]

By the Harnack inequality (see [CS14]), there exists a constant \( c_{n,s} \) such that
\[
u_\varepsilon(x) - \varepsilon \geq c_{n,s}m_0, \quad \text{for any} \ x \in B_{\delta_0/2}(x_0).
\]

Next, we construct an auxiliary function as follows: Let \( A_{1/2,2} := B_2 \setminus \overline{B_{1/2}}, \ A'_{1,2} := B_2' \setminus B_{1}' \), and \( D := A_{1/2,2} \setminus A'_{1,2} \). Let \( \varphi : D \to \mathbb{R} \) be the solution to the following Dirichlet problem
\[
\text{div}\left(|x_n|^{1-2s}\nabla\varphi\right) = 0 \quad \text{in} \quad D, \\
\varphi = 1 \quad \text{on} \ \partial B_{1/2}, \\
\varphi = 0 \quad \text{on} \ \partial B_2 \cup A'_{1,2}.
\]
By the boundary Hopf lemma and boundary growth estimate (see [CS14]) as well as the symmetry of \( \varphi \), the function \( \varphi \) has the following asymptotics at \( \bar{x} \in \partial B_1' \): there exists \( c_0 = c_0(n) > 0 \) such that
\[
\lim_{t \to 0^+} \frac{\varphi(\bar{x} + tv'_x)}{t^s} = c_0.
\]
Here \( v'_x \) is the in-plane outer unit normal of \( A'_{1,2} \) at \( \bar{x} \). Hence, we also have that
\[
\frac{\varphi(\bar{x} + tv'_x)}{t^s} > c_0/2 \quad \text{for} \ 0 < t < t_0
\]
for sufficiently small \( t_0 \). Now let
\[
\psi(x) := c_{n,s}m_0\varphi\left(\frac{x-x_0}{\delta_0}\right),
\]
and \( D_{\delta_0, x_0} := \{ x : \frac{x - x_0}{\delta_0} \in D \} \). Note that \( D_{\delta_0, x_0} \subset B_1 \setminus (\Omega_\varepsilon \cap B'_1) \). Applying the comparison principle in \( D_{\delta_0, x_0} \) we have

\[
\psi(x) \leq u_\varepsilon(x) - \varepsilon, \quad x \in D_{\delta_0, x_0}.
\]

Choose now \( y_0 \in \partial \Omega_\varepsilon \cap \partial B_\delta(x_0) \) which realizes the distance \( \delta_0 \). By (2.6) and recalling the explicit expression of \( \psi \), we have

\[
\frac{c_{n,s}m_0 \varphi \left( \frac{y_0 - x_0}{\delta_0} + \frac{t}{\delta_0} \nu_{y_0}' \right)}{t^s} \leq \frac{u_\varepsilon(y_0 + t \nu_{y_0}' - \varepsilon}{t^s}.
\]

We now want to use the estimate in Lemma 2.2 to obtain the bound on \( m_0 \). For that purpose, consider the following rescalings at \( y_0 \)

\[
u_{\varepsilon, x_0} (x) = \frac{u_\varepsilon(y_0 + \varepsilon^{-1/s}x)}{\varepsilon},
\]

which satisfy

\[
\text{div}(|x_n|^{1-s} \nabla u_{\varepsilon, x_0}) = 2\beta(u_{\varepsilon, x_0}) \Omega^{n-1} \{|x_n = 0\} \quad \text{in } B_{1/(2\varepsilon^{1/s})}.
\]

Then, we can apply Lemma 2.2 to conclude that \( |u_{\varepsilon, x_0}| \leq C = C(n, s, A) \) in \( B_{1/4} \). For the function \( u_\varepsilon \) this translates into having the bound

\[
u_{\varepsilon, x_0} (y_0 + \varepsilon^{-1/s}x) \leq C \varepsilon, \quad \text{for } |x| \leq 1/4.
\]

In particular, this gives that

\[
\frac{u_\varepsilon(y_0 + t \nu_{y_0}')}{t^s} - \varepsilon \leq C, \quad \text{for } t = \varepsilon^{1/s}(\delta_0/4)
\]

Hence, from (2.7), we obtain

\[
\frac{c_{n,s}m_0 \varphi \left( \frac{y_0 - x_0}{\delta_0} + \tau \nu_{y_0}' \right)}{\tau^s} \leq C \delta_0^s, \quad \text{for } \tau = \varepsilon^{1/s}(1/4).
\]

Then, using (2.5), we conclude that for small \( 0 < \varepsilon < \varepsilon_0 \), necessarily

\[
m_0 \leq C \delta_0^s
\]

Step 2. We will show that for any \( y_0 \in \Omega_\varepsilon \cap B_{1/4}' \)

\[
|u_\varepsilon(x) - u_\varepsilon(y_0)| \leq C|x - y_0|^s
\]

for any \( x \in B_1 \), with a universal constant \( C \).

(i) Suppose first \( |x - y_0| \leq (1/8)\varepsilon^{1/s} \). For this case, recall that for the rescaling \( u_{\varepsilon, x_0} \) at \( y_0 \) defined in Step 1 above we have the estimate \( \|u_{\varepsilon, x_0}\|_{L^\infty(B_{1/4})} \leq C \). Then by Lemma 2.3 we also have the estimate \( \|u_{\varepsilon, x_0}\|_{C^{\alpha,s}(B_{1/8})} \leq C \), which then implies that

\[
|u_\varepsilon(x) - u_\varepsilon(y_0)| = \varepsilon |u_{\varepsilon, x_0}|(|x - y_0|) |x - y_0|^{s} - u_\varepsilon(x, 0)| \leq C|x - y_0|^s
\]

(ii) Suppose now \( x \in B_1' \) and \( |x - y_0| \geq (1/8)\varepsilon^{1/s} \). Then from Step 1 we have

\[
|u_\varepsilon(x) - u_\varepsilon(y_0)| \leq 2\varepsilon + C|x - y_0|^s \leq C|x - y_0|^s.
\]
Combining the estimates in (i)–(ii) above, we obtain that (2.8) holds for any $x \in B_1^+$. It remains to establish (2.8) for $x \in B_1$ with $x_n \neq 0$. Note that it will be enough to show it for $x \in B_1^{+1/2}$.

In order to do that, we first extend $u_\varepsilon(\cdot,0)$ to all of $\mathbb{R}^{n-1}$ by putting it equal to zero outside $B_1^+$. Note that estimate (2.8) will continue to hold now for all $x \in \mathbb{R}^{n-1}$.

Then, consider the convolution of the extended $u_\varepsilon(\cdot,0)$ with the Poisson kernel

$$P_{x_n}(x') := C_{n,s} \frac{x_n^{2s}}{(|x'|^2 + x_n^2)^{\frac{n-1+2s}{2}}}$$

for the operator $\mathcal{L}_\varepsilon$. We then have

$$|(u_\varepsilon(\cdot,0) \ast P_{x_n})(x') - u_\varepsilon(y_0)|$$

$$= \left| \int_{\mathbb{R}^{n-1}} [u_\varepsilon(x' - z',0) - u_\varepsilon(y_0,0)] P_{x_n}(z')dz' \right|$$

(since $\int_{\mathbb{R}^{n-1}} P_{x_n} = 1$, $\forall x_n > 0$)

$$\leq C \int_{\mathbb{R}^{n-1}} |x' - y'_0 - z'|^s P_{x_n}(z')dz'$$

(by (2.8))

$$\leq C \int_{\mathbb{R}^{n-1}} (|x' - y'_0|^s + |z'|^s) P_{x_n}(z')dz'$$

(triangle inequality)

$$\leq C (|x' - y'_0|^s + |x_n|^s) \leq C|x - y_0|^s,$$

where in the second last inequality we have used $\int_{\mathbb{R}^{n-1}} |z'|^s P_{x_n}(z')dz' \leq C|x_n|^s$.

Next, the difference

$$v(x) = u_\varepsilon(x) - (u_\varepsilon(\cdot,0) \ast P_{x_n})(x')$$

satisfies

$$\text{div}(|x_n|^{1-2s}\nabla v) = 0 \quad \text{in} \quad B_1^+; \quad v = 0 \quad \text{on} \quad B_1^+.$$

By making the odd reflection in $x_n$, we can make $v$ $\mathcal{L}_\varepsilon$-harmonic in $B_1$. Hence, applying Lemma 2.3, we will have

$$\|v\|_{C^{0,s}(B_1/2)} \leq C(n,s,L).$$

Combining the estimates above, we then conclude that (2.8) holds for all $x \in B_1/2$ and hence for all $x \in B_1$.

**Step 3.** In this step, we complete the proof that $u_\varepsilon \in C^{0,s}(B_1/8)$ uniformly in $\varepsilon$.

From **Step 2**, it is enough to show that for any $x_1,x_2 \in B_1/8 \setminus \Omega_\varepsilon$,

$$|u_\varepsilon(x_1) - u_\varepsilon(x_2)| \leq C|x_1 - x_2|^s.$$

Let $d(x) := \text{dist}(x,\Omega_\varepsilon)$. Then consider the following two subcases:

(a) Suppose that $|x_1 - x_2| \leq \frac{1}{2} \max\{d(x_1),d(x_2)\}$. Without loss of generality, we assume that $d := d(x_1) \geq d(x_2)$. Let also $y_1 \in \Omega_\varepsilon$ be such that $|x_1 - y_1| = d$.

Then, consider the rescaling of $u_\varepsilon$ at $y_1$ by the factor of $d$

$$u_{\varepsilon,d}(x) := \frac{u_\varepsilon(y_1 + dx)}{d^s}.$$  

From **Step 2**, we have $0 \leq u_{\varepsilon,d}(x) \leq C|x|^s$ for $x \in B_1(\xi)$, $\xi := (x_1 - y_1)/d$. Moreover, $u_{\varepsilon,d}$ satisfies the homogeneous equation $\text{div}(|x_n|^{1-2s}\nabla u_{\varepsilon,d}) = 0$ in $B_1(\xi)$. By Lemma 2.3, $u_{\varepsilon,d} \in C^{0,s}(B_3/4(\xi))$. In particular, for $\eta := \frac{x - y_1}{d}$ in $B_3/4(\xi)$ we have

$$|u_{\varepsilon,d}(\eta) - u_{\varepsilon,d}(\xi)| \leq C|\eta - \xi|^s.$$
Rescaling back to \( u_\varepsilon \) we obtain
\[
|u_\varepsilon(x_1) - u_\varepsilon(x_2)| \leq C|x_1 - x_2|^s.
\]
(b) Suppose now \( |x_1 - x_2| > \frac{1}{2} \max\{d(x_1), d(x_2)\} \). In this case, by Step 2,
\[
|u_\varepsilon(x_1) - u_\varepsilon(x_2)| \leq C(d(x_1)^s + d(x_2)^s) \leq C|x_1 - x_2|^s.
\]

We conclude this section with the following remark that \( \{u_\varepsilon\} \) are uniformly bounded also in \( W^{1,2}_{\text{loc}}(B_1, |x_n|^{-2s}) \).

**Proposition 2.4** (Uniform \( W^{1,2} \) bound). Let \( u_\varepsilon \) be a nonnegative solution of \( (P_\varepsilon) \) with \( \|u_\varepsilon\|_{L^\infty(B_1)} \leq L \). Then \( u_\varepsilon \in W^{1,2}_{\text{loc}}(B_1, |x_n|^{-2s}) \) for any \( K \subset B_1 \) with
\[
\|u_\varepsilon\|_{W^{1,2}(K, |x_n|^{-2s})} \leq C(n, s, A, L, K)
\]
uniformly for all \( \varepsilon \in (0, 1) \).

**Proof.** Since \( u_\varepsilon \) is a nonnegative subsolution of \( \mathcal{L}_s \), the proof follows from a standard energy inequality. \( \square \)

3. Passage to the limit as \( \varepsilon \to 0 \)

3.1. **Compactness.** We start the section with the following local compactness lemma. Recall that we always assume that the functions \( u_\varepsilon \) and \( u \) are evenly extended in \( x_n \)-variable.

**Lemma 3.1** (Compactness and limit solutions). Let \( u_\varepsilon \) be a nonnegative solution to \( (P_\varepsilon) \). Then over a subsequence

(i) \( \{u_\varepsilon\} \) converges uniformly on compact subsets of \( B_1 \) to a function \( u \in C^{0,s}_{\text{loc}}(B_1) \).

(ii) The limit function \( u \) in (i) solves \( \text{div}(|x_n|^{1-2s} \nabla u) = 0 \) in \( \{u > 0\} \).

(iii) \( \beta_\varepsilon(u_\varepsilon) \rightharpoonup \mu \) in the space of measures \( \mathcal{M}(B'_R) \) for any \( 0 < R < 1 \).

(iv) \( |x_n|^{(1-2s)/2} \nabla u_\varepsilon \to |x_n|^{(1-2s)/2} \nabla u \) strongly in \( L^2_{\text{loc}}(B_1) \).

(v) \( \beta_\varepsilon(u_\varepsilon) \rightharpoonup \chi \) in \( L^\infty(B'_1) \) for some \( \chi \in L^\infty(B'_1) \), where \( B'_\varepsilon \) are defined in (1.5).

We call the function \( u \) as above a limit solution of (1.4), and the pair \( (u, \chi) \) a limit solution pair.

**Proof.**

(i) For any compact \( K \subset B_1 \), we know by Theorem 2.1, \( u_\varepsilon \) are uniformly bounded in \( C^{0,s}(K) \). By Ascoli-Arzela’s theorem up to a subsequence \( \varepsilon_j \to 0 \), we obtain a function \( u \in C^{0,s}(K) \) such that \( u_{\varepsilon_j} \to u \) in \( C^{0,\alpha}(K) \) with \( 0 < \alpha < s \). Since \( u_{\varepsilon_j} \geq 0 \), it follows that \( u \geq 0 \).

(ii) From (i) we know that \( \{u > 0\} \) is open. If \( u(x_0) = c > 0 \), from the uniform convergence of \( u_\varepsilon \) we obtain a small neighborhood \( U \) of \( x_0 \) such that \( \varepsilon_j(x) \geq c/2 > \varepsilon \) in \( U \) for every \( 0 < \varepsilon \leq \varepsilon_0 \) for some \( \varepsilon_0 \) small. Then \( u_\varepsilon \) solves \( \text{div}(|x_n|^{1-2s} \nabla u_\varepsilon) = 0 \) in \( U \) for any \( 0 < \varepsilon \leq \varepsilon_0 \). The statement follows from the uniform convergence of \( u_\varepsilon \) to \( u \).

(iii) Since \( u_\varepsilon \) are uniformly bounded in \( W^{1,2}_{\text{loc}}(B_1, |x_n|^{-2s}) \), see Proposition 2.4, by plugging in a cut-off function into (1.5) it is straightforward to see that \( \beta_\varepsilon(u_\varepsilon) \) are uniformly bounded in \( L^1_{\text{loc}}(B'_1) \).

(iv) For any \( 0 < R < 1 \), plugging a test function \( \varphi = u_\varepsilon \eta \) in (1.5), where \( \eta \in C^\infty_c(\mathbb{R}^n) \), \( \eta \geq 0 \) and \( \eta = 0 \) outside \( B_R \), we obtain that
\[
(3.1) \quad \int_{B_R} |x_n|^{-2s} |\nabla u_\varepsilon|^2 \eta + |x_n|^{-2s} u_\varepsilon \nabla u_\varepsilon \cdot \nabla \eta = - \int_{B'_R} 2\beta_\varepsilon(u_\varepsilon) u_\varepsilon \eta
\]
From (iii) and the fact that \( \beta_\varepsilon(u_\varepsilon) \) is supported on the set \( \{ u_\varepsilon \leq \varepsilon \} \) we have

\[
(3.2) \quad \text{RHS of (3.1)} \to 0 \quad \text{as } \varepsilon \to 0.
\]

From Proposition 2.4 we know that over a sequence \( \varepsilon_\delta = \varepsilon_j \to 0 \),
\[
| x_n | (1-2s)/2 \nabla u_{\varepsilon_j} \text{ converges to } | x_n | (1-2s)/2 \nabla u \text{ weakly in } L^2(B_R). \]
This together with (3.1), (3.2) and the uniform convergence in (i) gives us

\[
(3.3) \lim_{\varepsilon_j \to 0} \int_{B_R} | x_n | (1-2s) \nabla u_{\varepsilon_j}^2 \eta = - \int_{B_R} | x_n | (1-2s) u \nabla u \cdot \nabla \eta.
\]

On the other hand, for every \( \delta > 0 \) consider the truncation \( u_\delta = \max\{ u - \delta, 0 \} \). By (ii), \( u_\delta \) solves the homogeneous equation in \( \{ u > \delta \} \). Taking the test function \( \varphi = u_\delta \eta \) in (1.5) for \( \varepsilon_j \in (0, \delta) \), where \( \eta \) is the same as above, and letting \( \varepsilon_j \to 0 \), we obtain

\[
(3.4) \quad 0 = \int_{B_R} | x_n | (1-2s) \nabla u_\delta^2 \eta + \int_{B_R} | x_n | (1-2s) u_\delta \nabla u_\delta \cdot \nabla \eta.
\]

Letting \( \delta \to 0^+ \) in (3.4), we obtain

\[
(3.5) \quad \int_{B_R} | x_n | (1-2s) \nabla u^2 \eta = - \int_{B_R} | x_n | (1-2s) u \nabla u \cdot \nabla \eta.
\]

Comparing (3.3) and (3.5), we conclude

\[
(3.6) \quad \lim_{\varepsilon_j \to 0} \int_{B_R} | x_n | (1-2s) \nabla u_{\varepsilon_j}^2 \eta = \int_{B_R} | x_n | (1-2s) \nabla u^2 \eta.
\]

This together with the weak \( L^2 \) convergence gives (iv).

(v) Since \( 0 \leq \beta_\varepsilon(u_\varepsilon) \leq M \), then there exists a subsequence \( \beta_\varepsilon(\varepsilon_j) \) and \( \chi \) such that

\[
\beta_\varepsilon(\varepsilon_j) \rightharpoonup \chi \quad \text{in } L^\infty(B'_1).
\]

**Lemma 3.2.** Let \( \chi \) be as in Lemma 3.1. Then

\[
\chi \in \{ 0, M \} \quad \text{for a.e. } x \in B'_1.
\]

**Proof.** Given \( 0 < \delta \ll M/4 \) and \( K \subset B'_1 \), let \( \delta_1 \) and \( \delta_2 \) be the (unique) positive numbers satisfying

\[
\int_0^{\delta_1} \beta(s)ds = \int_{1-\delta_2}^1 \beta(s)ds = \delta.
\]

Then we have

\[
(3.7) \quad | K \cap \{ \delta < \beta_\varepsilon(\varepsilon_j) < M - \delta \} | = | K \cap \{ \delta_1 < \frac{u_\varepsilon_j}{\varepsilon_j} < 1 - \delta_2 \} | \leq | K \cap \{ \beta_\varepsilon(\varepsilon_j) \geq \frac{1}{\varepsilon_j} \min_{[\delta_1, 1-\delta_2]} \beta \} | \leq \frac{\varepsilon_j}{\min_{[\delta_1, 1-\delta_2]} \beta} \int_K \beta_\varepsilon(\varepsilon_j) \to 0 \quad \text{as } \varepsilon_j \to 0,
\]
where we have used Lemma \textbf{3.3}(iii) and the assumption that $\beta > 0$ in $(0,1)$. Hence if we let $A_{\delta,K} := K \cap \{2\delta < \chi < M - 2\delta\}$, then

$$|A_{\delta,K}| \leq |A_{\delta,K} \cap \{B_{\delta_j}(u_{\epsilon_j}) \leq \delta\} + |A_{\delta,K} \cap \{\delta < B_{\delta_j}(u_{\epsilon_j}) < M - \delta\}|$$

$$\leq |K \cap \{|B_{\delta_j}(u_{\epsilon_j}) - \chi | \geq \delta\}| + |A_{\delta,K} \cap \{\delta < B_{\delta_j}(u_{\epsilon_j}) < M - \delta\}|.$$

By Lemma \textbf{3.3}(v), $B_{\delta_j}(u_{\epsilon_j}) \rightharpoonup \chi$ in $L^\infty(B'_{1})$, and moreover $0 \leq B_{\delta_j} \leq M$ for all $j$, thus $B_{\delta_j}(u_{\epsilon_j}) \to \chi$ in $L^1(B'_{1})$. This implies that $|K \cap \{|B_{\delta_j}(u_{\epsilon_j}) - \chi | \geq \delta\}| \to 0$ as $j \to \infty$. This combined with \textbf{3.7} yields that passing to the limit $j \to \infty$, $|A_{\delta,K}| = 0$. Because $\delta$ and $K$ are arbitrary, we have $\chi \in \{0,M\}$ for a.e. $x \in B'_{1}$. \hfill $\square$

The following lemma will play a crucial role in the paper. The proof follows the lines of Lemma 3.2 in [CLAW97B] and is therefore omitted.

\textbf{Lemma 3.3} (Blowups at free boundary points). Let $u_{\epsilon_j} \to u$ uniformly on compact subsets of $B_{1}$, and $B_{\delta_j} \rightharpoonup \chi$ in $L^\infty(B'_{1})$, as in Lemma \textbf{3.3}. For $x_0 \in \mathcal{F}_u = \partial\{u(\cdot,0) > 0\} \cap B'_1$ and $\lambda > 0$, consider the following rescalings

$$u^{x_0}_{\lambda x}(x) := \frac{1}{\lambda^s} u(x_0 + \lambda x),$$

$$u^{z_0}_{\epsilon z}(x) := \frac{1}{\lambda^s} u(\epsilon z_0(x_0 + \lambda x),$$

$$\lambda^{x_0}(x') := \chi(x_0 + \lambda x').$$

Assume that there exists $\lambda_k \to 0$ such that $u^{x_0}_\lambda \to U$ as $k \to \infty$ uniformly on compact subsets of $\mathbb{R}^n$ and $\chi^{x_0}_\lambda \rightharpoonup \chi_0$ in $L^\infty(\mathbb{R}^{n-1})$. Then there exists $j(k) \to \infty$ such that for every $j_k \geq j(k)$ we have that $(\epsilon z_0/\lambda_k) \to 0$ and

(i) $u^{x_0}_{\epsilon z_0}/\lambda_k \to U$ uniformly on compact subsets of $\mathbb{R}^n$

(ii) $|x_0|^{(1-2s)/2} \nabla u^{x_0}_z \lambda_k / \lambda \rightharpoonup |x_0|^{(1-2s)/2} \nabla U$ in $L^2_{\text{loc}}(\mathbb{R}^n)$

(iii) $B_{\delta_j/\lambda_k} (u^{x_0}_{\epsilon z_0}/\lambda_k) \rightharpoonup \chi_0$ in $L^\infty(\mathbb{R}^{n-1})$

(iv) $|x_0|^{(1-2s)/2} \nabla u^{x_0}_{\lambda_k} \rightharpoonup |x_0|^{(1-2s)/2} \nabla U$ in $L^2_{\text{loc}}(\mathbb{R}^n)$.

We will call the function $U$ (or the pair $(U,\chi_0)$) a blowup of $u$ (or the pair $(u,\chi)$) at $x_0$. Note that the above lemma says that $(U,\chi_0)$ is a limit solution pair on any ball $B_R$, $R > 0$.

3.2. Solutions in the sense of domain variation. We say that the function $u_{\epsilon} \in W_{\text{loc}}^{1,2}(B_{1}, |x_{n}|^{-2s})$ is a domain-variation solution of $(P_{\epsilon})$, if it satisfies

$$\int_{B_{1}} |x_{n}|^{-2s} |\nabla u_{\epsilon} \cdot \nabla v| + \frac{1}{2} |\nabla u_{\epsilon}|^2 \, \text{div} \, \psi + \int_{B_{1}} 2B_{\epsilon}(u_{\epsilon}) \, \text{div} \, \psi = 0$$

for every smooth vector field $\psi \in C_{c}^{\infty}(B_{1}; \mathbb{R}^{n})$ with $\psi(B'_{1}) \subset \mathbb{R}^{n-1}$. The name comes from the fact that the equation \textbf{3.8} is equivalent to the condition

$$\frac{d}{dt} J_{\epsilon}(u(x + \tau \psi(x)))|_{\tau=0} = 0,$$

where

$$J_{\epsilon}(v) = \frac{1}{2} \int_{B_{1}} |x_{n}|^{-2s} |\nabla v|^2 + \int_{B_{1}} 2B_{\epsilon}(v)$$
is the energy associated with \((P)\). In particular, we see that the weak solutions of \((\text{(1.5)})\) are also domain-variation solutions.

Now, the advantage of the domain-variation solutions is as follows: if \(u_\varepsilon\) is a weak solution of \((P)\), then by the compactness Lemma \(3.1\) the limit solution pair \((u, \chi)\) over any \(\varepsilon = \varepsilon_j \to 0\) satisfies

\[
|\partial \psi(x, r)|^2 |\nabla u_\varepsilon| \, d\varepsilon \geq 0,
\]

for every smooth vector field \(\psi \in C^\infty_c(B_1; \mathbb{R}^n)\) with \(\psi(B'_1) \subset \mathbb{R}^{n-1}\). While we could pass to the limit also in the weak formulation \((1.5)\), the additional information on \(\chi\) that we have from Lemma \(3.2\) will be important in the sequel.

### 3.3. Weiss-type monotonicity formula

In this section we prove monotonicity formulas for the solution \(u_\varepsilon\) of \((\text{(P)})\), and the limit solution pair \((u, \chi)\) for \((\text{P})\). This kind of formula has been first used by Weiss \cite{Weiss99} in the “thick” counterpart of our problem, as well as the Alt-Caffarelli problem \cite{Weiss99}.

**Theorem 3.4** (Monotonicity formula for \((\text{P})\)). Let \(x_0 \in B'_1\) and \(u_\varepsilon\) be a solution to \((\text{P})\) with \(\|u_\varepsilon\|_{L^\infty(B_2)} \leq L\). For \(0 < |x_0| < 1\), let

\[
\Psi_\varepsilon^{(r)}(u_\varepsilon, r) = \frac{1}{\varepsilon^{n-1}} \int_{B_r(x_0)} |\nabla u_\varepsilon|^2 - \frac{s \varepsilon^n}{r^n} \int_{\partial B_1(x_0)} |\nabla u_\varepsilon|^2.
\]

Then \(r \mapsto \Psi_\varepsilon^{(r)}(u_\varepsilon, r)\) is a nondecreasing function of \(r\).

**Proof.** For \(0 < r < 1 - |x_0|\) consider the rescalings

\[
u_{\varepsilon, r}(x) := \frac{u_\varepsilon(x_0 + rx)}{r^s}.
\]

Then

\[
u_{\varepsilon, r} \left( r^{n-1} \frac{1}{\varepsilon^{n-1}} \int_{B_r(x_0)} |\nabla u_\varepsilon|^2 - \frac{s \varepsilon^n}{r^n} \int_{\partial B_1(x_0)} |\nabla u_\varepsilon|^2 \right).
\]

Thus

\[
\frac{d}{dr} \Psi_\varepsilon^{(r)}(u_\varepsilon, r) = \int_{B_1} 2 |x_n|^{1-2s} \nabla u_{\varepsilon, r} \cdot \nabla \frac{d}{dr} u_{\varepsilon, r} - 2s \int_{\partial B_1} |x_n|^{1-2s} u_{\varepsilon, r} \frac{d}{dr} u_{\varepsilon, r} + \int_{B'_1} 4 \beta_s(r^s u_{\varepsilon, r}) \left( s r^{s-1} u_{\varepsilon, r} + r^s \frac{d}{dr} u_{\varepsilon, r} \right).
\]

Now, noting that \(u_{\varepsilon, r}\) solves

\[
\text{div}(x_n^{|x_n|^{1-2s} \nabla u_{\varepsilon, r}}) = 2 r^s \beta_s(r^s u_{\varepsilon, r}) \mathcal{H}^{n-1}(\{x_n = 0\} \cap B_{1/r})
\]

and integrating by parts, using also the nonnegativity of \(\beta_s(u)\), we have

\[
\frac{d}{dr} \Psi_\varepsilon^{(r)} = 2 \int_{\partial B_1} |x_n|^{1-2s} (\partial_n u_{\varepsilon, r} - s u_{\varepsilon, r}) \frac{d}{dr} u_{\varepsilon, r} + \int_{B'_1} 4 \beta_s(r^s u_{\varepsilon, r}) s r^{s-1} u_{\varepsilon, r} \frac{d}{dr} u_{\varepsilon, r} \]

\[
\geq 2 \int_{\partial B_1} |x_n|^{1-2s} (\partial_n u_{\varepsilon, r} - s u_{\varepsilon, r}) \frac{d}{dr} u_{\varepsilon, r}.
\]
Observing that for $x \in \partial B_1$,
\[
\frac{d}{dr} u_\varepsilon(x) = r^{-1+s} \left( (rx) \cdot \nabla u_\varepsilon(x_0 + rx) - su_\varepsilon(x_0 + rx) \right),
\]
\[
\partial_r u_\varepsilon(x) = x \cdot \nabla u_\varepsilon(x) = r^{-s} (rx) \cdot \nabla u_\varepsilon(x_0 + rx),
\]
we then obtain
\[
\frac{d}{dr} \Psi^\varepsilon(x_\varepsilon, r) \geq \frac{2}{r^{n+1}} \int_{\partial B_r(x_0)} |x_n|^{1-2s} ((x-x_0) \cdot \nabla u_\varepsilon - su_\varepsilon)^2 \geq 0.
\]
This implies that $r \mapsto \Psi^\varepsilon(x_\varepsilon, r)$ is monotonically nondecreasing. \hfill \Box

By the compactness Lemma 3.4, passing to the limit in a subsequence $\varepsilon_j$ we get the following monotonicity formula for the limit pair $(u, \chi)$. Similar monotonicity formula was used in the thin and fractional Alt-Caffarelli problems in [AP12] and [All12], respectively.

**Theorem 3.5 (Monotonicity formula for $\mathbb{P}_j$).** Let $(u, \chi)$ be a limit solution pair and $x_0 \in B'_1$. For $0 < r < 1 - |x_0|$, let
\[
\Psi^x(u, r) = \frac{1}{r^{n-1}} \int_{B_r(x_0)} |x_n|^{1-2s} |\nabla u|^2 - \frac{s}{r^n} \int_{\partial B_r(x_0)} |x_n|^{1-2s} u^2
\]
\[+ \frac{1}{r^{n-1}} \int_{B'_r(x_0)} 4\chi.
\]
Then $r \mapsto \Psi^x(u, r)$ is monotonically nondecreasing. More precisely, for $0 < \rho < \sigma < 1 - |x_0|$,
\[
\Psi^x(u, \sigma) - \Psi^x(u, \rho) \geq \int_\rho^\sigma \frac{2}{r^{n+1}} \int_{\partial B_r(x_0)} |x_n|^{1-2s} ((x-x_0) \cdot \nabla u - su)^2 \, d\sigma, \, dr \geq 0.
\]
In particular, the limit $\Psi^x(u, 0+) = \lim_{r \to 0+} \Psi^x(u, r)$ exists.

We will call the quantity $\Psi^x(u, 0+)$ the Weiss energy of $u$ at $x_0$.

Next we prove a corollary of the monotonicity formula above. For notation convenience, sometimes we write the dependence of $\Psi$ on $(u, \chi)$ explicitly, i.e. we write $\Psi(u, \chi, r)$ instead of $\Psi(u, r)$.

**Corollary 3.6.** Let $(u, \chi)$ be a limit solution pair and $x_0 \in \mathcal{F}_u = \partial \{u(\cdot, 0) > 0\} \cap B'_1$. Then

(i) $\Psi^x(u, r) \geq -C$ for any $r > 0$.

(ii) Suppose for $\lambda_j \to 0$ a blowup sequence $(u_{\lambda_j}, \chi_{\lambda_j})$ (as in Lemma 3.5) satisfies
\[
\rho \to u_0 \text{ uniformly on compact subsets of } \mathbb{R}^n,
\]
\[
\chi_{\lambda_j} \rightharpoonup \lambda_0 \text{ in } L^\infty(\mathbb{R}^{n-1}).
\]
Then $\Psi^0(u_0, \chi_0, r)$ is constant in $r$. Moreover,
\[
\Psi^0(u_0, \chi_0, r) = \Psi_x^x(u, \chi, 0+) = \int_{B'_1} 4\chi_0.
\]
In particular, $\Psi^x_0(u, 0+) \geq 0$.

(iii) $u_0$ is a homogeneous function of degree $s$, i.e.
\[
u_0(\lambda x) = \lambda^s u_0(x), \quad \text{for a.e. } x \in \mathbb{R}^n, \lambda \geq 0.
\]
More precisely, by this we understand
\[ \nu \]
that
\[ \lim_{F} \]
we will show the asymptotic behavior of
\[ \Psi \]
Proof.

(i) Since
\[ x = \Psi_0 \]
Assume
\[ (3.10) \]
This yields the desired homogeneity of
\[ u \]
By Theorem 3.5, the left hand side goes to zero as
\[ u \rightarrow \infty \]
the limit we have
\[ \Psi \]
From Theorem 3.5 the limit
\[ \Psi \]
exists, and is equal to
\[ \Psi \]
(ii) We use the following scaling property of \( \Psi \):
\[ (4.1) \]
By (ii) and Theorem 3.5, for all
\[ 0 < s < s_2 < 1, \]
\[ \Psi \]
\[ (u, \chi, \lambda_j) \]
\[ (3.10) \]
We mean a point
\[ \nu = 1, \]
\[ u \]
\[ \beta \]
\[ \beta \]
since
\[ 0 \]
This yields the desired homogeneity of
\[ u \]
Finally, we show that
\[ \Psi(0, \chi_0, r) = \int_{B_i} 4 \chi_0. \]
Hence, it has been proved that
\[ \Psi \]
By (ii), \( u_0 \) is homogeneous of degree \( s \) and satisfies
\[ u_0 \partial_{\nu} \partial_{\nu} = 0, \]
Thus passing to the limit we have
\[ \lim_{F} \]
\[ \Psi \]
By Theorem 3.5 the left hand side goes to zero as
\[ u \rightarrow \infty \]
the limit we have
\[ \Psi \]
From Theorem 3.5 the limit
\[ \Psi \]
exists, and is equal to
\[ \Psi \]
(ii) We use the following scaling property of \( \Psi \):
\[ (4.1) \]
By (ii) and Theorem 3.5, for all
\[ 0 < s < s_2 < 1, \]
\[ \Psi \]
\[ (u, \chi, \lambda_j) \]
\[ (3.10) \]
We mean a point
\[ \nu = 1, \]
\[ u \]
\[ \beta \]
since
\[ 0 \]
This yields the desired homogeneity of
\[ u \]
Finally, we show that
\[ \Psi(0, \chi_0, r) = \int_{B_i} 4 \chi_0. \]
Hence, it has been proved that
\[ \Psi \]
4. ASYMPTOTIC BEHAVIOR OF LIMIT SOLUTIONS

Assume \( (u, \chi) \) is a limit solution pair in the sense of Lemma 3.1. In this section we will show the asymptotic behavior of \( u \) around the regular free boundary points of \( \mathcal{F}_u = \partial u(\cdot, 0) > 0 \) \( \cap B_i \). By regular free boundary point we mean a point
\[ x_0 \in \mathcal{F}_u, \]
where \( \mathcal{F}_u \) has an inward unit normal \( \nu \) in the measure-theoretic sense. More precisely, by this we understand \( \nu \in \mathbb{R}^{n-1}, |\nu| = 1 \), such that
\[ (4.1) \]
Thus passing to the limit we have
\[ \Psi \]
By Theorem 3.5 the left hand side goes to zero as
\[ u \rightarrow \infty \]
the limit we have
\[ \Psi \]
From Theorem 3.5 the limit
\[ \Psi \]
exists, and is equal to
\[ \Psi \]
(ii) We use the following scaling property of \( \Psi \):
\[ (4.1) \]
By (ii) and Theorem 3.5, for all
\[ 0 < s < s_2 < 1, \]
\[ \Psi \]
\[ (u, \chi, \lambda_j) \]
\[ (3.10) \]
We mean a point
\[ \nu = 1, \]
\[ u \]
\[ \beta \]
since
\[ 0 \]
This yields the desired homogeneity of
\[ u \]
Finally, we show that
\[ \Psi(0, \chi_0, r) = \int_{B_i} 4 \chi_0. \]
Hence, it has been proved that
\[ \Psi \]
**Theorem 4.1** (Limit solutions at regular points). Let \((u, \chi)\) be a limit solution pair in the sense of Lemma 3.1. Let \(x_0 \in \mathcal{F}_u\) be such that

(i) \(\mathcal{F}_u\) has at \(x_0\) an inward unit normal \(\nu\) in the measure theoretic sense,

(ii) \(u\) is nondegenerate at \(x_0\) in the sense that there exist \(c, r_0 > 0\) such that

\[
\frac{1}{r^{n-1}} \int_{B'_r(x_0)} u dx' \geq cr^n, \quad \text{for any } 0 < r < r_0.
\]

Then we have the following asymptotic development

\[
u(x) = \sqrt{2} M \frac{c_0(s)}{s} \left( \sqrt{x_1^2 + x_n^2 + x_1} \right)^s + o(|x - x_0|^s),
\]

with

\[
c_0(s) = s^{2-2s} \sqrt{\pi} (7 + 4s(s - 2)) \Gamma(1 - s) / \Gamma(\frac{7}{2} - s).
\]

Moreover, we have \(\Psi^{x_0}(u, \chi, 0+) = 2M|B'_1|\).

**Remark 4.2.** In the case \(s = \frac{1}{2}\), \(c_0(\frac{1}{2}) = \frac{\pi}{8}\).

We start by identifying the limit solutions \(u\) of the form

\[
u(x) = \frac{\alpha}{2^s} \left( \sqrt{x_1^2 + x_n^2 + x_1} \right)^s
\]

for some \(\alpha > 0\). The free boundary in this case is \(\mathcal{F}_u = \{x_1 = 0, x_n = 0\}\). This is the first step of understanding the asymptotic development of limit solutions around the ‘regular’ free boundary points.

**Proposition 4.3.** Let \((u, \chi)\) be a limit solution pair as in Lemma 3.1 such that

\[
u(x) = \frac{\alpha}{2^s} \left( \sqrt{x_1^2 + x_n^2 + x_1} \right)^s
\]

for some \(\alpha > 0\). Then

\[
\chi = M \chi_{\{x_1 > 0\}} \text{ in } \mathbb{R}^{n-1}\text{-a.e. in } \mathbb{R}^{n-1}
\]

and the constant \(\alpha\) is given by

\[
\alpha = \sqrt{\frac{2M}{c_0(s)}}, \quad \text{with } c_0(s) \text{ as in Theorem 4.1}
\]

**Proof.** In the proof below, we identify \(\mathbb{R}^{n-2}\) with \(\{0\} \times \mathbb{R}^{n-2}\), and denote \(x'' = (x_2, \ldots, x_{n-1}), B''_\rho = B'_\rho \cap \{x_1 = 0\}\).

**Step 1.** We will show that for any \(\varphi \in C^\infty_c(B''_1; \mathbb{R}^{n-1}), \varphi = (\varphi_1, \ldots, \varphi_{n-1}),\)

\[
c_0(s) \alpha^2 \int_{B''_1} \varphi_1(0, x'') dx'' + \int_{B'_1} 2\chi \text{div } \varphi = 0.
\]

If we take \(\psi \in C^\infty_c(B'_1; \mathbb{R}^n), \psi = (\varphi, 0)\) on \(B'_1\) as a test function in \(\text{NSM}\) on \(B'_1\), then

\[
\int_{B_1} \left[ \frac{1}{2} |\nabla u|^2 \text{div} (|x_n|^{1-2s} \psi) - (\nabla u)' D\psi \nabla u |x_n|^{1-2s} \right] + \int_{B'_1} 2\chi \text{div } \varphi = 0.
\]
An integration by parts gives
\[ \nu \text{ where } (4.5) \]

since \( \text{div} (|\nabla u|^2 \psi) \leq C|x_n| \), then for any small \( \delta > 0 \),

\[ (4.4) \int_{\sqrt{x_1^2 + x_n^2} \geq \delta} \left[ \frac{1}{2} |\nabla u|^2 \text{div} (|x_n|^{1-2s} \psi) - (\nabla u)^t D\psi \nabla u |x_n|^{1-2s} \right] + \int_{B_1^\delta} 2 \chi \text{div} \varphi = O(\delta). \]

We next rewrite the first integrand in the LHS of (4.4) as follows.

\[ (4.5) \frac{1}{2} |\nabla u|^2 \text{div} (|x_n|^{1-2s} \psi) - (\nabla u)^t D\psi \nabla u |x_n|^{1-2s} \]

\[ = \frac{1}{2} \text{div} (|\nabla u|^2 \psi |x_n|^{1-2s}) - \frac{1}{2} \nabla (|\nabla u|^2) \cdot \psi |x_n|^{1-2s} - (\nabla u)^t D\psi \nabla u |x_n|^{1-2s}. \]

To estimate the last two terms above, we observe that

\[ \langle \nabla u \cdot \psi \rangle \text{div} (|x_n|^{1-2s} \nabla u) = 0 \quad \text{in } \mathbb{R}^n \setminus \left\{ \sqrt{x_1^2 + x_n^2} < \delta \right\}, \]

since \( \text{div} (|x_n|^{1-2s} \nabla u) = 0 \) in \( \mathbb{R}^n \setminus \{ x_1 \leq 0, x_n = 0 \} \) and \( \psi \cdot e_n = 0, \nabla u = 0 \) on \( \{ x_1 < 0, x_n = 0 \} \). Thus, in \( \{ \sqrt{x_1^2 + x_n^2} \geq \delta \}, \)

\[ (4.6) \frac{1}{2} \nabla (|\nabla u|^2) \cdot \psi |x_n|^{1-2s} + (\nabla u)^t D\psi \nabla u |x_n|^{1-2s} = \text{div} ((\nabla u \cdot \psi) \nabla u |x_n|^{1-2s}). \]

Combining (4.5) and (4.6), we rewrite (4.4) as

\[ \int_{\sqrt{x_1^2 + x_n^2} \geq \delta} \frac{1}{2} \text{div} (|\nabla u|^2 \psi |x_n|^{1-2s}) - \text{div} ((\nabla u \cdot \psi) \nabla u |x_n|^{1-2s}) + \int_{B_1^\delta} 2 \chi \text{div} \varphi = O(\delta). \]

An integration by parts gives

\[ (4.7) \int_{\sqrt{x_1^2 + x_n^2} = \delta} \frac{1}{2} \nabla (\nabla u \cdot \psi) |x_n|^{1-2s} - \int_{\sqrt{x_1^2 + x_n^2} = \delta} (\nabla u \cdot \psi)(\nabla u \cdot \nu) |x_n|^{1-2s} + \int_{B_1^\delta} 2 \chi \text{div} \varphi = O(\delta), \]

where \( \nu \) is the unit outer normal to \( \{ \sqrt{x_1^2 + x_n^2} = \delta \} \). Next we estimate the first two integrals in the LHS of the above equality. Since \( u \) is homogeneous of degree \( s \) (and depends only on \( x_1 \) and \( x_n \)), \( \nabla u \cdot \nu = su \). Thus,

\[ \int_{\sqrt{x_1^2 + x_n^2} = \delta} (\nabla u \cdot \psi)(\nabla u \cdot \nu) |x_n|^{1-2s} = \int_{\sqrt{x_1^2 + x_n^2} = \delta} (\nabla u \cdot \psi)(su) |x_n|^{1-2s} = O(\delta), \]
where we have used the growth estimate of $u$ around the zero set as well as $|\nabla u||x_n|^{2s} \in L^2_{\text{loc}}(\mathbb{R}^n)$. To estimate the first integral we use the polar coordinates in $(x_1, x_n)$-plane: $x_1 = r\cos(\theta), x_n = r\sin(\theta)$. Then on $\{\sqrt{x_1^2 + x_n^2} = \delta\}$,

$$\frac{1}{2} |\nabla u|^2(\psi \cdot \nu)|x_n|^{1-2s} = \frac{s^2 \alpha^2}{2\delta} (\cos(\theta/2))^{-1+2s}(|\sin(\theta)|)^{1-2s} \left[\cos(\theta)(\psi \cdot e_1) + \sin(\theta)(\psi \cdot e_n)\right]$$

Since $\psi \cdot e_n = 0$ on $B_1'$,

$$\lim_{\delta \to 0} \int_{x_1^2 + x_n^2 = \delta} \frac{1}{2} |\nabla u|^2(\psi \cdot \nu)|x_n|^{1-2s} dx = \lim_{\delta \to 0} \int_{B_1'} \frac{s^2 \alpha^2}{2\delta} \cdot 2\delta \times \int_0^\pi (\cos(\theta/2))^{-1+2s}(|\sin(\theta)|)^{1-2s} \cos(\theta)^2 \psi_1(\delta \cos(\theta), x'', \delta \sin(\theta)) \ d\theta\ dx''$$

$$= c_0(s)\alpha^2 \int_{B_1''} \psi_1(0, x'', 0) dx''$$

where

$$c_0(s) = s^2 \int_0^\pi (\cos(\theta/2))^{-1+2s}(|\sin(\theta)|)^{1-2s} \cos(\theta)^2 \ d\theta$$

$$= s^2 2^{-1-2s} \sqrt{\pi(7 + 4s(s - 2))\Gamma(1 - s)\Gamma(\frac{s}{2} - s)}$$

Combining the above estimates and letting $\delta \to 0$, we have for all $\varphi \in C^\infty_c(B_1'; \mathbb{R}^{n-1})$

$$c_0(s)\alpha^2 \int_{B_1''} \psi_1(0, x'', 0) dx'' + \int_{B_1'} 2\chi \text{div} \varphi = 0$$

Recalling $\psi_1 = \varphi_1$ on $B_1'$, we have proved (4.3).

**Step 2.** We now show that $\alpha = \sqrt{\frac{2M}{c_0(s)}}$ and $\chi = M\chi_{\{x_1 > 0\}} \mathcal{H}^{n-1}$-a.e. in $\mathbb{R}^{n-1}$.

(a) $\chi \equiv M$ in $\{x_1 > 0\}$.

In fact, if $y \in \{x_1 > 0\}$, then $u(y', 0) = \alpha(y_1)^s > 0$. Hence by the uniform Hölder convergence of $u_{x_j}$ to $u$, we have $u_{x_j} \geq \varepsilon_j$ in a neighborhood of $(y', 0)$ for any $j \geq j_0$ for some $j_0 = j_0(\alpha, y_1)$ large enough. Thus for $j > j_0$ and $(x', 0)$ in the neighborhood of $(y', 0)$ we have

$$B_{\varepsilon_j}(u_{x_j})(x') = \int_0^{u_{x_j}(x'+\varepsilon_j)} \beta(s) ds = M.$$

Letting $j \to \infty$ and since $y'$ is arbitrary, we get $\chi \equiv M$ in $\{x_1 > 0\}$.

(b) $\chi \equiv 0$ in $\{x_1 < 0\}$.

In fact, we can take any $\varphi$ in (4.3) such that $\text{supp} \varphi_1 \subset \mathbb{R}^{n-1} \cap \{x_1 < 0\}$. Then the LHS of (4.3) will vanish. This implies that $\chi = \text{const}$ in $\{x_1 < 0\}$. By Lemma 3.2, $\chi \equiv M$ or $\chi \equiv 0$ in $\{x_1 < 0\}$. If $\chi \equiv M$ in $\{x_1 < 0\}$, then $\chi \equiv M$ in $\mathbb{R}^{n-1}$ by (a). Thus from (4.3) $\int_{\mathbb{R}^{n-1}} \varphi_1(0, x'') dx'' = 0$ for any compactly supported vector field $\varphi$, which is a contradiction.

This then implies the claim that $\chi = M\chi_{\{x_1 > 0\}}$. Next, applying an integration by parts to the RHS of (4.3) we have

$$c_0(s)\alpha^2 \int_{B_1''} \varphi_1(0, x'') dx'' = 2M \int_{B_1''} \varphi_1(0, x'') dx''.$$
This implies
\[ \alpha = \sqrt{\frac{2M}{c_0(s)}}. \]
□

We are now ready to prove Theorem 4.1.

**Proof of Theorem 4.1.** Without loss of generality we assume \( x_0 = 0 \) and \( \nu = \varepsilon_1. \)

We also extend \( u \) by the even reflection with respect to \( x_n. \)

Consider the rescalings \( u_\lambda \) and \( \chi_\lambda \) at the origin. By Theorem 2.1 given \( \rho > 0, \)
\( u_\lambda \) is uniformly bounded in \( C^{0,s}(B_\rho/\lambda). \) Therefore, there exists a sequence \( \lambda_j \to 0, \)
\( u_0 \in C^{0,s}(\mathbb{R}^n) \) and \( \chi_0 \in L^\infty(\mathbb{R}^n) \) such that \( u_{\lambda_j} \to u_0 \) uniformly on compact subsets of \( \mathbb{R}^n \) and \( \chi_{\lambda_j} \rightharpoonup \chi_0 \) in \( L^\infty(\mathbb{R}^{n-1}). \)

Now, rescaling (4.1), we see that for every \( R > 0, \)
\[ |\{ u_\lambda(\cdot, 0) > 0 \} \cap \{ x_1 < 0 \} \cap B'_R | \to 0 \quad \text{as} \quad \lambda \to 0, \]
and we deduce that \( u_0 = 0 \) \( \mathfrak{F}^{n-1}-a.e. \) in \( \{ x_1 < 0, x_n = 0 \}. \) By continuity, \( u_0 \)
vanesishes on all of \( \{ x_1 \leq 0, x_n = 0 \}. \) Besides, we readily have that \( u_0 \)
vanishes on all of \( \{ x_1 \leq 0, x_n = 0 \}. \) By continuity,
\( u_0 \)
satisfies
\[ \div(|x_n|^{-2s}\nabla u_0) = 0 \quad \text{in} \quad \{ u_0 > 0 \}. \]
Thus, we can apply Corollary A.2 in Appendix to obtain an asymptotic development
\[ (4.9) \quad u_0(x) = \alpha 2^{-s} \left( x_1^2 + x_n^2 \right)^{1/2} + o(|x|^s) \]
with \( \alpha \geq 0. \) Next, note that by rescaling the nondegeneracy condition (4.2) and
passing to the limit, we have
\[ \frac{1}{r^{n-1}} \int_{B'_r} u_0 dx' \geq c r^s, \quad \text{for any} \ r > 0, \]
which implies that \( \alpha > 0. \) On the other hand, by Corollary 3.6(iii), \( u_0 \) is homogeneous of degree \( s \) and hence
\[ u_0(x) = \alpha 2^{-s} \left( x_1^2 + x_n^2 \right)^{1/2} + o(|x|^s). \]
Now by Lemma 3.3, \( (u_0, \chi_0) \) is a limit solution pair and we can apply Proposition 4.3
to conclude that \( \alpha = \sqrt{\frac{2M}{c_0(s)}}, \) which is a constant independent of the sequence \( \lambda_j. \)
This proof the asymptotic development for \( u. \)

Finally, by Proposition 4.3, we also have that \( \chi_0 = M \chi_{\{x_1 > 0\}} \) and hence by Corollary 3.6(ii),
\[ \Psi^0(u, \chi, 0+) = \int_{B'_1} 4 \chi_0 = 2M |B'_1|. \]
□

We conclude the paper with two propositions regarding some additional properties of limit solutions and the value of \( \Psi^0(u, \chi, 0+), \) which may be useful in the further treatment of the free boundary. At the end we also give an alternative proof of Theorem 4.1, relying on these results, rather than on asymptotic developments in Appendix.

The first proposition says that the nondegeneracy condition (4.2) at \( x_0 \in \mathcal{F}_u \)
implies also a nondegeneracy for \( \{ u(\cdot, 0) = 0 \}. \) The proof uses the dimension reduction argument (see e.g. [Wei03]).
Proposition 4.4. Let \((u, \chi)\) be a limit solution pair, and \(x_0 \in \mathcal{I}_u\) such that \(u\) is nondegenerate at \(x_0\) in the sense of (1.2). Then \(0 \leq \Psi^{x_0}(u, \chi, 0+) < 4M|B_1'|\). In particular, \(|B_r'(x_0) \cap \{u(\cdot, 0) = 0\}| > 0\) for any \(r > 0\).

Proof. Without loss of generality we assume that \(x_0 = 0\). By Corollary 3.6 ii) and the inequality \(0 \leq \chi_0 \leq M\), to prove the first part of the proposition, we essentially have to exclude the possibility that \(\Psi^0(u, \chi, 0+) = 4M|B_1'|\), which is equivalent to having \(\chi_0 = M\) a.e. on \(\mathbb{R}^{n-1}\) (recall that \((u_0, \chi_0)\) is a blow-up limit at 0 along a subsequence \((u_{\lambda_j}, \chi_{\lambda_j})\)). We want to show that this implies that \(u_0\) depends only on one variable \(x_n\).

To this end, take \(\hat{x}_0 \in \mathcal{I}_{u_0} \subset \mathbb{R}^{n-1}\) such that \(\hat{x}_0 \neq 0\). Note that such point exists, otherwise \(u_0 > 0\) in \(\mathbb{R}^n \setminus \{0\}\) \((u_0 \equiv 0\) on \(\mathbb{R}^{n-1}\) is excluded by the nondegeneracy assumption), which would imply that \(\text{div}(|x_n|^{-2s}\nabla u_0) = 0\) in \(\mathbb{R}^n \setminus \{0\}\). Since \(u_0\) is locally bounded, by the removability of point singularities \(1\) we would have that \(\text{div}(|x_n|^{-2s}\nabla u_0) = 0\) in all of \(\mathbb{R}^n\). Then, by the Harnack inequality \([\text{FKS}82]\) (which implies Liouville theorem) we would have that \(u_0 \equiv 0\), which is a contradiction. Since \(u_0\) is homogeneous, then \(\lambda \hat{x}_0 \in \mathcal{I}_{u_0}\) for each \(\lambda > 0\). Moreover, due to the equality \(\chi_0 = M\) a.e. in \(\mathbb{R}^{n-1}\) we will have \(\Psi^{\lambda \hat{x}_0}(u_0, \chi_0, 0+) = 4M|B_1'|\) for each \(\lambda > 0\). Applying the monotonicity formula (Theorem 3.5) to \(u_0\) at \(\hat{x}_0\) we get

\[
\int_0^R \frac{2}{r^{n+1}} \int_{\partial B_r(\hat{x}_0)} |x_n|^{1-2s} (x - \hat{x}_0) \cdot \nabla u_0 - su_0)^2 \, d\sigma \, dr
\]

\[
\leq \Psi^{x_0}(u_0, \chi_0, R) - \Psi^{\hat{x}_0}(u_0, \chi_0, 0+)
\]

\[
= \Psi^{x_0}(u_0, \chi_0, R) - \Psi^0(u_0, \chi_0, 0+)
\]

\[
= \Psi^{\hat{x}_0}(u_0, \chi_0, R) - \Psi^0(u_0, \chi_0, R)
\]

\[
\to 0 \quad \text{as } R \to \infty,
\]

where the last line is due to the fact that \(u_0\) is homogeneous, more precisely, since \(u_0\) is homogeneous of degree \(s\), then one has

\[
(u_0)^{\hat{x}_0}(x) = u_0(x + (\hat{x}_0/R)) \to u_0(x) \quad \text{as } R \to \infty.
\]

Therefore, by (4.10) and \(x \cdot \nabla u_0 = su_0\) we have \(\nabla u_0 \cdot \hat{x}_0 = 0\) in \(\mathbb{R}^n\). This implies that \(u_0\) is constant in \(\hat{x}_0/|\hat{x}_0|\) direction. An induction argument gives that \(u_0(x) \equiv u_0(x_n)\). Then from \(u_0(0) = 0\) we obtain that \(u_0 = 0\) on all of \(\mathbb{R}^{n-1}\), which contradicts to the nondegeneracy assumption (1.2).

The last statement in the proposition follows immediately from the observation that \(\chi = M\) in \(\{u(\cdot, 0) > 0\}\).

Our last proposition says that among all nonzero blowups at free boundary points, the one with a flat free boundary as in Proposition 4.3 has the smallest Weiss energy.

Proposition 4.5. Let \((u, \chi)\) be a limit solution pair. Let \(x_0 \in \mathcal{I}_u\) and assume that \(u\) is nondegenerate at \(x_0\) in the sense of (1.2). Then \(\Psi^{x_0}(u, \chi, 0+) \geq 2M|B_1'|\). Moreover, \(\Psi^{x_0}(u, \chi, 0+) = 2M|B_1'|\) if any blow-up limit \(u_0\) at \(x_0\) is \(u_0(x) = \alpha 2^{-s}(\sqrt{x_1^2 + x_n^2} + x_1)^s\), up to a rotation, with \(\alpha = \sqrt{2M/c_0(s)}\).

Proof. Suppose \(|x_0| \leq L\) and let \(w\) be such that \(L_x w = 0\) in \(B_1\) and \(w = u_0\) on \(\partial B_1\). By comparing the difference \(w - u_0\) with \(\frac{2L}{\Psi^{x_0}(x)} \Phi_s(x)\) in \(B_1 \setminus B_1\), where \(\Phi_s(x) = C_s|x|^{-\frac{n-1}{2s}}\) is the fundamental solution of \(L_x\), and letting \(\delta \to 0+\), we conclude that \(w = u_0\).
This result is the analogue of Theorem 9.4 in [AP12], with a similar proof.

Proof. Suppose there is \( x_0 \in \mathcal{F}_u \) such that \( \Psi^{x_0}(u, \chi; x_0) < 2M|B'_1| \). Let \((u_0, \chi_0)\) be a blow-up limit at \( x_0 \) along a sequence \((u_{x_j}^0, \lambda_{x_j}, \chi_{x_j}; x_j)\) with \( \varepsilon_j / \lambda_j^* \to 0 \) (by Lemma 3.3). Then by Corollary 3.6(ii)(iii), \( u_0 \) is homogeneous of degree \( s \), and

\[
(4.11) \quad \Psi^0(u_0, \chi_0, r) = \int_{B'_1} 4\chi_0 = \Psi^{x_0}(u, \chi, 0+) < 2M|B'_1|.
\]

Since \( u \) satisfies the nondegeneracy assumption at \( x_0 \), then arguing as in Theorem 4.1 we have that \( u_0 \) is nontrivial.

Let \( \Lambda : = \{ u_0(\cdot, 0) = 0 \} \). Arguing as in Step 2(a) of Proposition 4.3 we have \( \chi_0(x) = M \) in \( \mathbb{R}^{n-1} \setminus \Lambda \). Thus (4.11) implies that

\[
(4.12) \quad |\Lambda \cap B'_1| > |B'_1|/2.
\]

We write the homogeneous blow-up limit as \( u_0(x) = r^s f(\omega) \), with \( r = |x| \) and \( \omega = x/|x| \in \partial B_1 \). Since \( u_0 \) solves \( \text{div}(|x_n|^{-2s} \nabla u_0) = 0 \) in \( \mathbb{R}^n \setminus \Lambda \), \( u_0 > 0 \) in \( \mathbb{R}^n \setminus \Lambda \) and \( u_0 = 0 \) on \( \Lambda \), the function \( f(\omega) \) satisfies

\[
\omega_n^{2s-1} \nabla \omega \cdot (\omega_n^{1-2s} \nabla \omega) f = s(s-n+1) f \quad \text{in } \partial B_1 \setminus \Lambda,
\]

\[
f = 0 \quad \text{on } \partial B_1 \cap \Lambda,
\]

\[
f > 0 \quad \text{on } \partial B_1 \setminus \Lambda.
\]

Thus \( f \) is the principal Dirichlet eigenfunction for the weighted spherical Laplacian on \( \partial B_1 \setminus \Lambda_\omega \) with \( \lambda_0 = s(s-n+1) \), where \( \Lambda_\omega : = \Lambda \cap \partial B_1 \subset \partial B_1 \cap \{ \omega_n = 0 \} \). From the variational formulation of the principal eigenvalue and using the symmetrization we have, among all \( \Lambda_\omega \) with \( |\Lambda_\omega| \) constant, \( \lambda \) takes the minimum iff \( \Lambda_\omega \) is a spherical cap (here and later by spherical cap we mean the ‘thin’ spherical cap lying on \( \{ \omega_n = 0 \} \), i.e. the classical spherical cap with center on \( \partial B_1 \cap \{ \omega_n = 0 \} \) intersected with \( \{ \omega_n = 0 \} \)). This follows from the fact that the Steiner symmetrization in any spherical variable on \( \partial B_1 \), orthogonal to \( x_n \), will decrease the principal eigenvalue, see Lemma 9.5 in [AP12], by adding the weight of \( \omega_n^{1-2s} = (\cos \theta_{n-1})^{1-2s} \) (independent of \( \theta_1, \ldots, \theta_{n-2} \)) in the energy functional in the proof.

Let \( \lambda^* \) denote the minimum eigenvalue associated with the spherical cap \( \Lambda_\omega^* \) with \( |\Lambda_\omega^*| = |\Lambda_\omega| \). We immediately have \( \lambda^* \leq \lambda_0 \). On the other hand, however, by (4.12) we have \( \lambda^* > \lambda_0 \). This is due to the fact that \( \lambda_0 = s(s-n+1) \) is the principal eigenvalue associated with the half sphere, which by (4.12) is contained in some \( \Lambda_\omega^* \) after a rotation. Hence we arrive at a contradiction.

Finally, note that the eigenspace associated with half thin-sphere, which without of generality we assume to be \( \partial B_1 \cap \{ \omega_1 \leq 0, \omega_n = 0 \} \), is generated by \( u(x) = (\sqrt{x_1^2 + x_n^2} + x_1)^s \). By the above argument, if \( \Psi^{x_0}(u, \chi, 0+) = 2M|B'_1| \), then any blow-up limit \( u_0 \) is of the form \( u_0 = c(\sqrt{x_1^2 + x_n^2} + x_1)^s, c > 0 \) after a rotation. By Proposition 4.3 \( c = 2^{-s} \sqrt{2M/c_0(s)} \).

At the end of the paper we would like to give an alternative proof of Theorem 4.1 without relying on Corollary A.2 in Appendix, but rather using Proposition 4.4 and some ideas from Proposition 4.5.

Alternative Proof of Theorem 4.1 We start again by rescaling (4.1), to obtain that for every \( R > 0 \)

\[
\{(u_\lambda(\cdot, 0) > 0) \cap \{x_1 < 0 \} \cap B'_R \} \to 0 \quad \text{as } \lambda \to 0,
\]
which implies that \( u_0 = 0 \) \( \mathcal{H}^{n-1}\)-a.e. in \( \{x_1 < 0, x_n = 0\} \) and hence \( u_0 = 0 \) everywhere on \( \{x_1 \leq 0, x_n = 0\} \), by continuity. Next, using that \( \chi = M \) when \( u(\cdot, 0) > 0 \), we also have
\[
(\{\chi \lambda < M\} \cap \{x_1 > 0\} \cap B_r^{n}) \to 0 \quad \text{as} \quad \lambda \to 0,
\]
implies that \( \chi = M \) \( \mathcal{H}^{n-1}\)-a.e. in \( \{x_1 > 0, x_n = 0\} \). Hence, by Proposition 1.4 necessarily \( u_0 > 0 \) or \( u_0 \equiv 0 \) in all of \( \{x_1 > 0, x_n = 0\} \). Indeed, if \( \mathcal{F}_{u_0} \cap \{x_1 > 0, x_n = 0\} \neq 0 \), then there exists \( x_0 \in \mathcal{F}_{u_0} \cap \{x_1 > 0, x_n = 0\} \) such that \( \Psi^{x_0}(u_0, \chi_0, 0+) = 4M|B_1'|. \) Since \( u_0 \) is homogeneous by Corollary 3.6(iii), then \( \chi x_0 \in \mathcal{F}_{u_0} \) for each \( \lambda > 0 \) and \( \Psi^{x_0}(u_0, \chi_0, 0+) = 4M|B_1'|. \) By the upper semicontinuity of the map \( x \mapsto \Psi^{x}(u_0, \chi_0, 0+) \) we necessarily have \( \Psi^0(u_0, \chi_0, 0+) = 4M|B_1'|. \) However, this contradicts Proposition 1.4. Thus, \( u_0 > 0 \) in \( \{x_1 > 0, x_n = 0\} \) and hence, \( u_0 \) satisfies \( \text{div}(|x_n|^{-2s}\nabla u_0) = 0 \) in \( \mathbb{R}^n \setminus \{x_1 \leq 0, x_n = 0\} \). Since \( u_0 \) is also homogeneous of degree \( s \), writing it as \( u_0(x) = r^s f_0(\omega) \), with \( r = |x| \) and \( \omega = x/|x| \), we see that \( f_0 \) is a nonnegative eigenfunction of the weighted spherical Laplacian as in Proposition 1.5 in \( \partial B_1 \setminus \{\omega_1 \leq 0, \omega_n = 0\} \). Hence, \( f_0 \) is a positive multiple of the explicitly given eigenfunction \( 2^{-s} (x_1^2 + 2x_n^2)^{1/2} + x_1 \)^s and hence
\[
u_0(x) = \alpha 2^{-s} \left( (x_1^2 + x_n^2)^{1/2} + x_1 \right)^s,
\]
for some \( \alpha > 0 \). Then Theorem 1.1 follows directly by applying Proposition 4.3 as in the first proof of the theorem. \( \square \)

**Appendix A.**

In this appendix we prove the asymptotic development for nonnegative solutions of \( \text{div}(|x_n|^{-2s}\nabla u) = 0 \) near the ‘flat’ boundary points. The proof uses ideas similar to those in Lemma A.1 and Corollary A.1 in \( [CLW97] \).

Below, we will denote,
\[
\Lambda := \{x \in \mathbb{R}^n : x_n = 0, x_1 \leq 0\}, \\
P(x) := \frac{1}{2^s} \left( \sqrt{x_1^2 + x_n^2} \right)^s.
\]

**Lemma A.1.** Let \( u \in C^0,^s(B_1) \) be nonnegative, \( u = 0 \) on \( \Lambda \cap B_1 \) and satisfy \( \text{div}(|x_n|^{-2s}\nabla u) \leq 0 \) in \( B_1 \setminus \Lambda \). Then \( u \) has an asymptotic development at the origin
\[
u(x) = \alpha P(x) + o(|x|^s)
\]
with a constant \( \alpha \geq 0 \).

**Proof.** Let
\[
\varepsilon(r) := \sup \{\varepsilon : u(x) \geq \varepsilon P(x) \text{ in } B_r\}, \quad 0 < r < 1.
\]
Then \( \varepsilon(r) \) is a nonincreasing function of \( r \), and moreover it is bounded above by the \( C^0,^s \) norm of \( u \). Let \( \alpha := \lim_{r \to 0} \varepsilon(r) \). From this definition of \( \alpha \), we immediately have
\[
u(x) \geq \alpha P(x) + o(|x|^s) \quad \text{in } B_1.
\]

**Claim.** We have \( u(x) = \alpha P(x) + o(|x|^s) \).
We argue by contradiction. Assume that there are some $\delta_0 > 0$ and a sequence $x_k \in B_1$ with $r_k := |x_k| \to 0$ such that
\begin{equation}
(A.2) \quad u(x_k) - \alpha P(x_k) \geq \delta_0 r_k^\gamma.
\end{equation}
Consider the rescalings
\[ u_k(x) := \frac{u(r_k x)}{r_k^\gamma} \quad \text{for} \quad x \in B_1/r_k, \quad \tau_k := \frac{1}{r_k} x_k \in \partial B_1. \]

Since $u \in C^{0,\gamma}(B_1)$, then there exists a subsequence which we still denote by $u_k$ and $v \in C^{0,\gamma}_0(\mathbb{R}^n)$ such that $u_k \to v$ uniformly on compact subsets in $\mathbb{R}^n$. We can also assume $\tau_k \to \tau \in \partial B_1$. From (A.1) and (A.2) we have
\[ v - \alpha P \geq 0 \quad \text{in} \quad B_1, \quad v(\tau) - \alpha P(\tau) \geq \delta_0. \]

By the uniform convergence of $u_k$ to $v$ and the Hölder regularity of $v$, there exists $\eta > 0$ such that $B_\eta(\tau) \cap \Lambda = \emptyset$ and
\[ v - \alpha P \geq \frac{\delta_0}{2}, \quad u_k - \alpha P \geq \frac{\delta_0}{2} \quad \text{for} \quad k > k_0 \text{ large enough, on } B_\eta(\tau). \]

Now let $w$ be a solution to $\text{div}(|x_n|^{-2}\nabla w) = 0$ in $B_1 \setminus \Lambda$ with smooth boundary data, such that
\[ w = 0 \quad \text{on} \quad \partial(B_1 \setminus \Lambda) \setminus B_{\eta/2}(\tau), \]
\[ w = \frac{\delta_0}{4} \quad \text{on} \quad \partial(B_1 \setminus \Lambda) \cap B_{\eta/4}(\tau), \]
\[ 0 \leq w \leq \frac{\delta_0}{4} \quad \text{on} \quad \partial(B_1 \setminus \Lambda) \cap B_{\eta/2}(\tau). \]

By the maximum principle, $w$ is nonnegative in $B_1 \setminus \Lambda$. By the boundary Harnack principle (see [CSS08]), there exist small $\mu, \gamma > 0$ which depend on $\delta_0$ and $\varepsilon$ such that
\[ w(x) \geq \mu P(x) \quad \text{on} \quad B_\gamma. \]

Now, each $u_k$ satisfies div$(|x_n|^{-2}\nabla u_k) \leq 0$ in $B_1 \setminus \Lambda$. Let $w_k$ be the solution to div$(|x_n|^{-2}\nabla w_k) = 0$ in $B_1 \setminus \Lambda$ and $w_k = \min(0, u_k - \alpha P) + w$ on $\partial(B_1 \setminus \Lambda)$. By the comparison principle, $u_k - \alpha P \geq w_k$ in $B_1 \setminus \Lambda$. Moreover, by (A.1), we see $w_k \to w$ uniformly on $\partial(B_1 \setminus \Lambda)$ and hence, by the maximum principle, also on $B_1$. By the boundary Harnack principle, we can assume therefore that
\[ w(x) - w_k(x) \leq \frac{\mu}{2} P(x), \quad \text{in } B_\gamma, \]
for large $k$, which then gives
\[ u_k(x) - \alpha P(x) \geq w_k(x) + \mu P(x) - w(x) \geq \frac{\mu}{2} P(x) \quad \text{in } B_\gamma. \]

Scaling back, we therefore have
\[ u(x) \geq \left( \alpha + \frac{\mu}{2} \right) P(x) \quad \text{in } B_{\gamma r_k}, \]
implicating that $\varepsilon(\gamma r_k) \geq \alpha + \frac{\mu}{2}$, which in turn leads to the absurd $\alpha \geq \alpha + \frac{\mu}{2}$. This proves the lemma. \qed
Corollary A.2. Let \( u \in C^{0,s}(B_1) \) be nonnegative, \( u = 0 \) on \( \Lambda \cap B_1 \) and satisfy \( \text{div}(|x_n|^{1-2s}\nabla u) = 0 \) in \( \{ u > 0 \} \). Then \( u \) has an asymptotic development at the origin
\[
u(x) = \alpha P(x) + o(|x|^s)
\]
with a constant \( \alpha \geq 0 \).

Proof. Note that from the conditions above \( \text{div}(|x_n|^{1-2s}\nabla u) \geq 0 \) in \( B_1 \setminus \Lambda \). Then, we claim that there exists \( C \geq 0 \) such that
\[
u \leq CP(x) \text{ in } B_{1/2}.
\]
Indeed, if \( w \) is a solution of the Dirichlet problem \( \text{div}(|x_n|^{1-2s}\nabla w) = 0 \) in \( B_{3/4} \setminus \Lambda \), then by the boundary Harnack principle
\[
w \leq CP(x) \text{ in } B_{1/2}
\]
and our claim follows from the comparison \( \nu \leq w \) in \( B_{3/4} \). Now,
\[
U(x) = CP(x) - \nu(x)
\]
will satisfy the conditions of Lemma [A1] (in \( B_{1/2} \) instead \( B_1 \)) and the asymptotic development of \( u \) will follow from that of \( U \). \( \square \)

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