Strong averaging along foliated Lévy diffusions with heavy tails on compact leaves

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August 29, 2016

Abstract

This article shows a strong averaging principle for diffusions driven by discontinuous heavy-tailed Lévy noise, which are invariant on the compact horizontal leaves of a foliated manifold subject to small transversal random perturbations. We extend a result for such diffusions with exponential moments and bounded, deterministic perturbations to diffusions with polynomial moments of order \( p \geq 2 \), perturbed by deterministic and stochastic integrals with unbounded coefficients and polynomial moments. The main argument relies on a result of the dynamical system for each individual jump increments of the corresponding canonical Marcus equation. The example of Lévy rotations on the unit circle subject to perturbations by a planar Lévy-Ornstein-Uhlenbeck process is carried out in detail.

Keywords: Markov processes on manifolds; solutions of stochastic differential equations with Lévy noise, foliated manifolds; strong averaging principle; scale separation; Marcus canonical equation; dynamical systems; heavy tail distributions;

2010 Mathematical Subject Classification: 60H10, 60J60, 60G51, 58J65, 58J37.

1 Introduction

The theory of averaging (deterministic) ordinary differential equations, whose origins date back to the works of Laplace and Lagrange, has been applied through its history in many fields of applications such as celestial mechanics, nonlinear mechanics, oscillation theory and radiophysics. First rigorous results start with the foundational contributions of Krylov, Bogoliubov and Mitropolskii [1, 12, 13, 37, 61]. For a comprehensive and systematic introduction to the subject we refer to the monograph [55] of Saunders, Verhulst and Murdock. The idea of averaging random systems given as stochastic ordinary differential equations with respect to Gaussian processes goes back to Stratonovich [57, 58], first rigorous results appear with the seminal works of Khasminski and others [31, 32, 33, 34, 26, 21, 56]. More recent developments on systems of stochastic (partial) differential equations with continuous Gaussian noises can be found for instance in [3, 17, 35, 47, 51, 62, 63] and the references therein.

However, in many contexts the Gaussian paradigm is known to be too limited. First results on averaging differential equations with respect to discontinuous and non-Gaussian Poisson noise are obtained in [6] and [36]. A first strong averaging principle for scalar Lévy diffusions with Lipschitz coefficients and bounded jumps is established in [19]. In [28] the authors show a strong averaging

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principle for Lévy diffusions with exponential moments on foliated manifolds, explained in more detail below. This article is dedicated to the averaging of a large class of discontinuous semimartingales supported on the compact leaves of a foliated manifold having polynomial moments.

Intuitively speaking a foliated manifold is a Riemannian manifold equipped with a family of equivalence classes of submanifolds depending on a parameter, which defines a “transversal” component. Precise details on foliated manifolds are found in classical texts on the topic such as [16, 59, 64]. The notion of a foliated Brownian motion is introduced in the seminal article [24] by Garnett. In [43] Li shows an averaging principle for Hamiltonian systems, which inspired first results on averaging on foliated spaces in [25], where the authors show an averaging principle for a foliated Brownian diffusion. More precisely, the authors study a Brownian diffusion in Stratonovich sense on a foliated manifold, which respects the foliated structure of the manifold in the sense that the diffusion does not leave the compact leaf of its initial condition almost surely. By assumption, the diffusion enjoys a unique invariant measure supported on the leaf of its initial condition as the limit of its time average in $L^p$ sense for some $p \geq 2$. For small $\varepsilon$ the law of the perturbed diffusion converges on the accelerated time scale $t/\varepsilon$ to the invariant measure, such that the transversal perturbation converges to the vector field averaged against the invariant measure, which leads to an ordinary differential equation in transversal direction to the leaves of the foliation. Their main result is an averaging principle in the $L^p$ sense with logarithmic rates of convergence.

This result is extended in [28] to a class of foliated Lévy diffusions with compact leaves and exponential moments of the underlying jump Lévy process, formulated in terms of canonical Marcus equations, as for instance in [41]. However by the Lévy-Chinchine formula (see for instance [2]) it is obvious that this is a rather narrow subclass of possible Lévy drivers. The current article generalizes this result to the case of Lévy jump diffusions, with moments of order $p$, $p \geq 2$, for which the averaging converges in $L^p$ sense. The lower bound $p \geq 2$ seems natural for equations on manifolds since the Marcus canonical integral can be rephrased as an integral against the quadratic variation of the underlying process, see [41], Lemma 2.1.

The difficulty of an immediate extension of the results to Lévy diffusions with only $p$-th moments lies in their formulation as a canonical Marcus equation [41, 44, 45], where each single jump increment of the process is given as the solution of an ode, with a vector field tangential to the leaf. Assume that $\Delta Z$ is a single jump increment in an appropriate noise space of the driving Lévy process $Z$. The jump increment on the manifold then has to follow the local coordinates and is then given as the increment $\Phi_F \Delta Z(x) - y$, where $\Phi_F \Delta Z(y) = Y(1; y, F\Delta Z)$ is the time 1 map of the solution $Y$ of the ordinary differential equation

$$\frac{dY}{dt} = F(Y)\Delta Z, \quad Y(0) = y,$$

and $F$ is a Lipschitz vector field such that $F\Delta Z$ takes values in the tangent space of our manifold. For details we refer to [41, 55, 41]. The problem is that in general the Lipschitz continuity of $F$ only implies

$$|\Phi_F \Delta Z(x) - \Phi_F \Delta Z(y)| \leq \ell \|\Delta Z\| |x - y| \quad \forall x, y,$$

where $\ell$ is the Lipschitz constant of $F$. This means each jump increment on the manifold depends exponentially on the random size of $\Delta Z$. See [41] Lemma 3.1. Taking the expectation of (2) the finiteness of the right-hand side implies the exponential integrability $\int_{\|z\|>1} \exp(\kappa \|z\|) \nu(dz) < \infty$ of the Lévy measure $\nu$ of $Z$ for some constant $\kappa > 0$ larger than the Lipschitz constant $\ell$ of $F$, which is equivalent to the existence of exponential moments of $Z$. This straight-forward argument is the main reasoning concerning the moments carried out in the previous article [28]. However, since the leaves of the foliation are compact and the main driving diffusion $X$ is invariant on the leaf of its initial condition, any jump increment of $X$ is bounded by the diameter of the leaf in
the surrounding space. With this intuition in mind we may prove in Lemma 3.1 of Section 3 the following result on positive invariant ODE dynamical systems $Y$ of type (1), which yields a global constant $C > 0$ such that for any $x, y$

\[
\sup_{t \geq 0} |(DF(Y(t;x)\Delta Z)F(Y(t;x))\Delta Z - (DF(Y(t;y)\Delta Z)F(Y(t;y))\Delta Z)| \leq C|x - y| \|\Delta Z\|^2. \tag{3}
\]

Taking the expectation of (3) a finite right-hand side is equivalent to

\[
\int \|z\| > 1 \|z\|^2 \nu(dz) < \infty
\]

imposing only second moments of $Z$. It turns out eventually to be an easy task to link (2) to (3) via Taylor expansion of $\Phi F \Delta Z$. We follow these lines of reasoning in a technically more subtle setting in Section 3.

The second extension we undertake is the step from perturbations by a small deterministic bounded vector field in [28] to a general class of discontinuous Lévy diffusions with moments of order $2p$, whose multiplicative coefficients may depend on the slow component. The coefficients in front of $\circ dB$ and $\tilde{\circ} Z$ will be only depend on $\pi X^c$, since it is well-known in averaging theory that in general diffusion coefficients are difficult to average in a strong sense.

The article is organized as follows. Subsection 2.1 lays out the general setup. Subsection 2.2 states the specific hypotheses on the integrability and ergodicity conditions of the stochastic processes and the main result of this article given in Theorem 2.2. Subsection 2.3 spells out the main example: Lévy processes with polynomial moments on the unit circle. In Section 3 we establish the estimate (3) and derive the crucial estimates on the deviation of the perturbed from the unperturbed solution, under arbitrary Lipschitz functions including a crucial dynamical system argument. Section 4 is dedicated to the control of the averaging error term exploiting the results from Section 3 in special cases. Section 5 finishes the proof of the main result synthesizing Section 3 and 4. The article finishes with an Appendix providing the missing details of the example.

## 2 Object of study and main results

### 2.1 The set up

**The geometry:** Let $M$ be a finite dimensional connected, smooth Riemannian manifold. It is known by the strong version of Whitney theorem for instance in Boothby [10] that any finite dimensional smooth manifold is embedded in $\mathbb{R}^m$ for some $m \in \mathbb{N}$ sufficiently large. The manifold $M$ is equipped with an $n$-dimensional foliation $\mathfrak{M}$ in the following sense. Let $\mathfrak{M} = (L_x)_{x \in M}$, with $M = \bigcup_{x \in M} L_x$ and the sets $L_x$ are equivalence classes of the elements of $M$ satisfying the following properties.

a) Given an $x_0 \in M$, there exists a neighborhood $U \subset M$ of the corresponding leaf $L_{x_0}$, a connected open set $V \subset \mathbb{R}^d$ containing the origin $0 \in \mathbb{R}^d$ and a diffeomorphic coordinate map $\varphi : U \to L_{x_0} \times V$.

b) The set $U$ of item a) can be taken small enough such that the derivatives of the coordinate map $\varphi$ are bounded. The second coordinate of a point $x \in U$, called the vertical coordinate, is denoted with the help of the projection $\pi : U \to V$ by $\varphi(x) = (\bar{x}, \pi(x))$ for some $\bar{x} \in L_x$.

**Remark 2.1** For any $v \in V$ and $x \in U$ with $\pi(x) = v$ the preimage satisfies $\pi^{-1}(v) = L_x$.

**The unperturbed equation:** We are interested in the ergodic behavior of a strong solution of a Lévy driven SDE with discontinuous components which takes values in $M$ and respects the foliation. In order to avoid that jump increments lead to an exit from the foliation of the initial
condition, the jump increments must respect the curved structure of the local coordinates and therefore necessarily satisfy a canonical Marcus equation, which are equivalent to the generalized Stratonovich equation in the sense of Kurtz, Pardoux and Protter [41]. We consider the formal canonical Marcus stochastic differential equation

\[ dX_t = F_0(X_t)dt + F(X_t) \circ dZ_t + G(X_t) \circ dB_t, \quad X_0 = x_0 \in M, \]  

which consists of the following components.

1. Let \( Z = (Z_t)_{t \geq 0} \) with \( Z_t = (Z^1_t, \ldots, Z^n_t) \) be a Lévy process with values in \( \mathbb{R}^n \) for fixed \( r \in \mathbb{N} \) on a given filtered probability space \( \Omega = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}) \) with characteristic triplet \((0, \nu, 0)\). Suppose that the filtration \((\mathcal{F}_t)_{t \geq 0}\) satisfies the “usual” conditions in the sense of Protter [50].

As a consequence of the Lévy-Itô decomposition \( Z \) is a pure jump process with respect to a \( \sigma \)-finite measure \( \nu: \mathcal{B}(\mathbb{R}^n) \to [0, \infty) \) called the Lévy measures satisfying

\[ \int_{\mathbb{R}^n} (1 \wedge \|z\|^2) \nu(dz) < \infty \quad \text{and} \quad \nu(\{0\}) = 0. \]  

For details we refer to the monographs of Sato [52] or Applebaum [2].

2. Let \( F_0 \in C^2(M, T\mathfrak{M}) \) with \( F_0(x) \in T_xL_x \). The vector field \( F \in C^2(M; L(\mathbb{R}^r; T\mathfrak{M})) \) satisfies that the map \( M \ni x \mapsto F(x) \) is \( C^2 \) and the linear map \( F(x) \) sends a vector \( z \in \mathbb{R}^r \mapsto F(x)z \in T_xL_x \) to the tangent space of the respective leaf.

3. Let \( B = (B^1, \ldots, B^r) \) be a standard Brownian motion with values in \( \mathbb{R}^r \) defined on \( \Omega \) and \( G \in C^2(M, L(\mathbb{R}^r, T\mathfrak{M})) \) with \( G(x) \in T_xL_x \) for any \( x \in M \).

We further assume that the vector fields \( F_0, F, (DF_0)F_0, (DF)F, G \) and \( (DG)G \) are globally Lipschitz continuous with Lipschitz constant \( \ell > 0 \).

A strong solution of the formal equation \((4)\) is defined as a map \( X : [0, \infty) \times \Omega \to M \) satisfying \( \mathbb{P} \)-almost surely for all \( t \geq 0 \)

\[ X_t = x_0 + \int_0^t F_0(X_s)ds + \int_0^t G(X_s)dB_s + \frac{1}{2} \int_0^t (DG(X_s))G(X_s)d\langle B \rangle_s \]
\[ + \int_0^t F(X_{s-})dZ_s + \sum_{0 < s \leq t} \Phi^{F, \Delta_s Z}(X_{s-}) - X_{s-} - F(X_{s-})\Delta_s Z, \]  

where \( \langle B \rangle \) stands for the quadratic variation process of \( B \) in \( \mathbb{R}^r \) and the function \( \Phi^{F, z}(x) = Y(1, x; Fz) \) and \( Y(t, x; Fz) \) for the solution of the ordinary differential equation

\[ \frac{d}{d\sigma} Y(\sigma) = F(Y(\sigma))z, \quad Y(0) = x \in M, \quad z \in \mathbb{R}^r. \]  

**The perturbed equation:** This article studies the situation where an SDE in the sense of [43] which is invariant on the leaf of the initial condition \( x_0 \) is perturbed by a transversal smooth vector field \( \varepsilon K dt \) and the stochastic differentials \( \varepsilon \hat{G} \circ dB \) and \( \varepsilon \hat{K} \circ d\hat{Z} \) with \( \varepsilon > 0 \) in the limit of \( \varepsilon \searrow 0 \). More precisely, we denote by \( X^\varepsilon, \varepsilon > 0 \) the solution in the sense of equation \((6)\) of the form perturbed system

\[ dX^\varepsilon_t = F_0(X^\varepsilon_t)dt + F(X^\varepsilon_t) \circ dZ_t + G(X^\varepsilon_t) \circ dB_t + \varepsilon \left( K(X^\varepsilon_t)dt + \hat{K}(\pi(X^\varepsilon_t)) \circ d\hat{Z}_t + \hat{G}(\pi(X^\varepsilon_t)) \circ d\hat{B}_t \right), \]
\[ X^\varepsilon_0 = x_0, \]  

where the additional coefficients are defined as follows.
4. Let \( K : M \to TM \) be a smooth vector field.

5. Let \( \tilde{Z} = (\tilde{Z}^1, \ldots, \tilde{Z}^r) \) be a pure jump Lévy process with values in \( \mathbb{R}^r \) defined on \( \Omega \) and Lévy measure \( \nu' \) satisfying
   \[
   \int_{\mathbb{R}^r} (1 \wedge \|z\|^2) \nu'(dz) < \infty \quad \text{and} \quad \nu'\{0\} = 0
   \]
   and \( \tilde{K} \in C^2(V, L(\mathbb{R}^r, TM)) \).

6. Let \( \tilde{B} = (\tilde{B}^1, \ldots, \tilde{B}^r) \) be an \( \mathbb{R}^r \)-valued Brownian motion defined on \( \Omega \) and \( \tilde{G} \in C^2(V, L(\mathbb{R}^r, TM)) \).

We assume that the vector fields \( K, (DK)K, \tilde{K}, (D\tilde{K})\tilde{K}, \tilde{G} \) and \( (D\tilde{G})\tilde{G} \) are globally Lipschitz continuous with Lipschitz constant \( \ell > 0 \).

**Theorem 2.1** ([41], Theorem 3.2 and 5.1) Under the preceding setup in particular items a), b) and 1.-3., there is a unique semimartingale \( X \) which is a strong global solution of [41] on \( \Omega \) in the sense of equation (6). It has a càdlàg version and is a (strong) Markov process.

**Remark 2.2** It is obvious that under the preceding setup in particular items a), b) and 1.-6., there is also a unique strong solution \( X^\varepsilon \) on \( \Omega \) of equation (8) in the analogous sense of equation (6) and with the same properties, if \( F_0 \) is replaced by \( F_0 + \varepsilon \tilde{K} \) and \( F \) by \( (F, \varepsilon \tilde{K}) \), \( G \) by \( (G, \varepsilon \tilde{G}) \), \( B \) by \( (B, \tilde{B}) \) and \( Z \) by \( (Z, \tilde{Z}) \) accordingly.

We state the crucial chain rule for the Marcus equation given in [41] (Proposition 4.2).

**Proposition 2.3** Let \( Z \) and \( F_0, F \) satisfy items 1) and 2) and \( X \) be the solution of (6) with initial condition \( x_0 \) with \( G = 0 \). Then for any \( \Psi \in C^2(\mathbb{R}^d) \) we have \( \mathbb{P}\text{-a.s.} \) for all \( t \geq 0 \)
   \[
   \Psi(X_t) = \Psi(x_0) + \int_0^t (D\Psi)(X_s)F_0(X_s)ds + \int_0^t (D\Psi)(X_{s-})F(X_{s-}) \circ dZ_s.
   \]

A direct consequence of the chain rule is the following support property given as Proposition 4.3 in [41]. Each jump increment of the noise \( \Delta Z \) is mapped to an increment \( \Phi^F\Delta Z \) of the solution \( X \) of (6). The increment \( \Phi^F\Delta Z \) follows the integral curve \( Y \) in (7) along the vector field \( Fz \) which is tangent to the (smooth) manifold \( L_{x_0} \) and hence \( M \). A standard support theorem for ODEs applied in the proof of this result then yields that the solution after the jump once again is an element of \( L_{x_0} \) and hence \( M \). This support property is maintained for an additional Stratonovich component, the reasoning is standard. For the solution \( X^\varepsilon, \varepsilon > 0 \) given in [8] this remains obviously true only for \( M \). Under the aforementioned conditions these lines of thought lead to the following foliated structure of \( X \): \( x_0 \in M \) implies \( X_t(x_0) \in L_{x_0}, \mathbb{P}\text{-a.s.} \) for all \( t \geq 0 \). We shall call a solution of an SDE of the type [41] which admits a foliated solution a foliated Lévy diffusion.

In addition, we obtain that \( x_0 \in M \) and \( \varepsilon > 0 \) imply \( X^\varepsilon_t(x_0) \in M, \mathbb{P}\text{-a.s.} \) for all \( t \geq 0 \).

### 2.2 The main result

**Hypothesis 1: Compactness and Integrability.** (a) Any leaf \( L_{x_0} \in \mathfrak{M}, x_0 \in M \), is compact and the map \( x_0 \mapsto \text{diam} L_{x_0} \) is Lipschitz continuous in the embedding space of \( M \).

(b) There is a constant \( p \geq 2 \) such that the Lévy measures \( \nu \) (of \( Z \)) and \( \nu' \) (of \( \tilde{Z} \)) satisfy
   \[
   \int_{\mathbb{R}^r} \|z\|^p \nu(dz) < \infty \quad \text{and} \quad \int_{\mathbb{R}^r} \|z\|^{2p} \nu'(dz) < \infty.
   \]
Hypothesis 2: Existence of invariant measures on each leaf. (a) The solution $X$ of (11) has for any initial condition $x_0 \in M$ a unique invariant measure $\mu_{x_0}$ with $\text{supp}(\mu_{x_0}) = Lx_0$.

(b) For $v_0 = \pi(x_0)$ being the vertical coordinate of some $x_0 \in M$ we define for $h : M \to \mathbb{R}$ with $h(x) \in T_x M$

$$Q^h(v_0) := \int_{Lx_0} h(u) \mu_{x_0}(du)$$

and suppose that for any such function $h$, which is globally Lipschitz continuous the function $\mathbb{R}^d \ni V \ni v \mapsto Q^h(v) \in \mathbb{R}^d$ is globally Lipschitz continuous.

Remark 2.4 Note that $\mu_{x_0}$ (just as $Lx_0$) only depends on the vertical component $\pi(x_0)$.

Hypothesis 2 ensures that for each $x_0 \in M$, $v_0 = \pi(x_0) \in V$ the stochastic differential equation

$$dw = Q^{\pi_K}(w) dt + \tilde{K}(w) \circ d\tilde{Z}_t + \tilde{G}(w) \circ d\tilde{B}_t, \quad w(0) = v_0 \in V$$

has a unique strong solution $w = (w(t,v_0))_{t \in [0,T]}$ on $\Omega$, $T_\infty$ being the first exit time of $w$ from $V$.

Hypothesis 3: Ergodicity in terms of $L^p$. Let Hypotheses 1 and 2 be satisfied for some $p \geq 2$.

We assume that there exists a bounded, continuous, decreasing function $\eta : [0, \infty) \to [0, \infty)$ with $\eta(t) \searrow 0$ as $t \to \infty$ such that for any $x_0 \in M$

$$\left(\mathbb{E} \left[ \frac{1}{t} \int_0^t \pi K(X_s(x_0)) \, ds - Q^{\pi K}(\pi(x_0)) \right]^p \right)^{\frac{1}{p}} \leq \eta(t), \quad \text{for all } t \geq 0.$$

On the rate of convergence. Results about rates of convergence go back to Pascal [48] and Kolmogorov [39], see also [3] and references therein. Recent developments for Lévy driven dissipative systems can be found in Kulik [40], see also [7, 11, 13, 60]. In [40, 20] for instance the authors develop generic methods to establish exponential convergence to an ergodic limit measure in terms of the total variation distance for the solution of an SDE driven by a pure jump Markov process. In this context the exponential rate of convergence in total variation implies that the rate of convergence in (11) is of order $1/t^p$. In Subsection 2.3 we provide the simple example of a Lévy process on the unit circle for which we calculate the precise rate of convergence, which is of the same type. For Brownian diffusion processes Hörmander’s hypoellipticity condition ensures exponential rates of convergence in total variation, see for instance [8, 9, 27] and references therein. On the other hand, there is no standard rate of convergence for general Markovian systems in the ergodic theorem, see for instance Krengel [38] or Kakutani and Petersen [30]. Therefore, it is natural to formulate the result in terms of the function $\eta$ following the approach in Freidlin and Wentzell [21].

The main result of the article is proved in Section 5 and reads as follows.

Theorem 2.2 Let Hypotheses 1, 2 and 3 being satisfied for some $p \geq 2$. Then we have for any $\lambda \in (0, 1)$ and $x_0 \in M$ positive constants $\varepsilon_0 \in (0, 1)$, $C > 0$ and $c > 0$ such that for any $\varepsilon \in (0, \varepsilon_0]$ and $T \in [0, 1]$

$$\left(\mathbb{E} \left[ \sup_{t \in [0,T]} |\pi(X_{\varepsilon}^{\tau_{\varepsilon}(x_0)}(x_0)) - w(t)|^p \right] \right)^{\frac{1}{p}} \leq CT \left[ \varepsilon^\lambda + \eta(cT \ln \varepsilon) \right],$$

where $X_{\varepsilon}$ is the solution (8) and $w$ the solution of (10), and $\tau_{\varepsilon} = S_{\varepsilon} \wedge T_\infty$. $S_{\varepsilon}$ is the first exit time of $X_{\varepsilon}(x_0)$ from $U$ in a) and $T_\infty$ is the first exit time of $w$ from $V$. 

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Remark 2.5 Since our main result focuses on Lévy processes with only $p$-th moments, the coefficient $G$ can be set to 0 since no additional difficulty to the proof of [25] shows up. The coefficient $\tilde{G}$ will be also dropped in the proof. Including it in the proofs of the sections 3 and 4 is straightforward.

Remark 2.6 In the proofs of Section 3, 4 and 5, it will turn out that under the preceding assumptions none of the constants depends on the precise shape of $V$ and hence $U$. Hence without loss of generality and for the sake of readability we may assume in the proofs that $V = \mathbb{R}^d$.

2.3 Example: Perturbed Lévy rotations of the unit circle

We illustrate this phenomenon in $M = \mathbb{R}^2 \setminus \{0\}$ with the 1-dimension horizontal circular foliation of $M$ where the leaf passing through a point $x_0 \in M$ is given by the (nondegenerate) horizontal circle

$$L_{x_0} = \{(\|x_0\| \cos \theta, \|x_0\| \sin \theta), \theta \in [0, 2\pi]\}.$$ 

Let the process $Z = (Z_t)_{t \geq 0}$ be any pure jump Lévy process with second moments. The Lévy-Itô decomposition of $Z$ yields almost surely for any $t \geq 0$

$$Z_t = \int_0^t \int_{|z| \leq 1} z \tilde{N}(d\sigma dz) + \int_0^t \int_{|z| > 1} z N(d\sigma dz)$$

(13)

where $N$ is the random Poisson measure with intensity measure $dt \otimes \nu$ and $\tilde{N}$ denotes its compensated counterpart. Consider the foliated linear SDE on $M$ consisting of random rotations:

$$dX_t = \Lambda X_t \circ dZ_t, \quad X_0 = x_0, \quad \text{with } \Lambda = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \quad (14)$$

Equation (14) is defined as follows. Note that for some jump increment of $Z$, $z \in \mathbb{R}$, $z \neq 0$, we have to consider the solution flow $\Phi$ of the equation

$$\frac{d}{d\sigma} Y(\sigma) = F(Y(\sigma))z, \quad Y(0) = (x, y), \quad \text{where } F(x, y) = \Lambda(x, y)^T,$$

obtained by a simple calculation as

$$\Phi^F z(x, y) = Y(1; (x, y)) = \begin{pmatrix} x \cos(z) - y \sin(z) \\ x \sin(z) + y \cos(z) \end{pmatrix},$$

such that

$$X_t = x_0 + \int_0^t \Lambda X_{s-} z \tilde{N}(d\sigma dz) + \sum_{0 < s \leq t} (\Phi^F (X_{s-}) - X_{s-} - F(X_{s-}) \Delta_s Z).$$

The chain rule of the Marcus integral, Proposition 4.2 in [41], states for $\| (x, y)^T \|^2 := x^2 + y^2$

$$d\|X_t\|^2 = -2 X_{t-} \Lambda X_{t-} \circ dZ_t = 0. \quad (15)$$

In fact, $X$ can be equally defined as the projection of $Z$ on the unit circle. If we identify the plane where $X$ takes its values with the complex plane $\mathbb{C}$ we obtain $X_t = e^{iZ_t}$. By the Lévy-Chinchine representation of the characteristic function of $Z$ we obtain for any $p \in \mathbb{R}$

$$\mathbb{E}[X_t^p] = \mathbb{E}[e^{ipZ_t}] = \exp(t \Psi(p)), \quad \text{where } \Psi(p) = \int_{\mathbb{R}^d} (e^{ipz} - 1 - izp \mathbf{1}\{|z| \leq 1\}) \nu(dz).$$
The invariant measures \( \mu_{x_0} \) in the leaves \( L_{x_0} \) passing through points \( x_0 \in M \) are therefore given by normalized Lebesgue measures in the circle \( L_{x_0} \) centered in 0 with radius \( ||x_0|| \). We are interested in the effective behavior of a small transversal perturbation of order \( \varepsilon \):

\[
dX_t^\varepsilon = \Lambda X_t^\varepsilon \circ dZ_t + \varepsilon K(X_t^\varepsilon) \, dt + \varepsilon d\tilde{Z}_t
\]

with initial condition \( x_0 = (1,0) \), where \( \tilde{Z} \) is a pure jump Lévy process with Lévy measure \( \nu' \) satisfying

\[
\int_{||z||>1} ||z||^4 \nu'(dz) < \infty.
\]

We shall consider two classes of perturbing vector fields \( K \).

(A) Constant perturbation \( \varepsilon K = \varepsilon (K_1, K_2) \in \mathbb{R}^2 \). This example was carried out in \[28\] for the Gamma process on the unit sphere, with \( \tilde{Z} = 0 \). The case of a general Lévy process is virtually identical. The main result in this case reads as follows. For any Lévy process \( X \) with \( \mathbb{E}[|Z_1|^p] < \infty \), \( p \geq 2 \) and \( \lambda \in (0,1) \) we obtain \( \varepsilon_0 \in (0,1) \) and \( T_0 > 0 \) such that for any \( T \in [0,T_0] \) and \( \varepsilon \in (0,\varepsilon_0] \) we have

\[
\mathbb{E} \left( \sup_{s \in [0,T]} |\pi_r(X_s^\varepsilon) - 1|^p \right)^{\frac{1}{p}} \leq \varepsilon^\lambda T.
\]

(B) General linear perturbation \( \varepsilon K(x,y) = \varepsilon (A(x,y))^T = \varepsilon (ax + by, cx + dy)^T \) for a given matrix \( A \in \mathbb{R}^{2\times 2} \), which is obviously globally Lipschitz continuous and smooth. The radial component of the vector field \( K \) is then given by

\[
\pi_r K(\theta, r) = r^\prime ((a \sin(\theta) + b \cos(\theta), c \sin(\theta) + d \cos(\theta))^T, (\sin(\theta), \cos(\theta))^T
\]

\[
= r (a \sin^2(\theta) + d \cos^2(\theta) + (b + c) \sin(\theta) \cos(\theta)),
\]

where \( \theta \) is the angular coordinate of \( (x,y) \) whose distance to the origin is \( r \). Hence the average of this component with respect to the invariant uniform measure on the leaves (circles) is given by

\[
Q^\pi_r K(\theta, r) = \frac{1}{2\pi} \int_0^{2\pi} \pi_r K(\theta, r) d\theta = \frac{a + d}{2} r.
\]

for leaves \( L_{x_0} \) with radius \( r \). We verify the convergence \[11\] of Hypothesis 2 for the radial component and \( p = 2 \). Let \( \tilde{Z} \) be a Lévy process in \( \mathbb{R}^2 \) with finite fourth moment. Elementary but lengthy calculations which can be found in Appendix \[6.1\] show that

\[
\mathbb{E} \left[ \int_0^t \pi_r K(X_s) ds - Q^\pi_r K(x_0) \right]^2 \quad \overset{\text{t} \to \infty}{\longrightarrow} 0,
\]

where the rate of convergence \( \eta \) is of order \( 1/\sqrt{t} \) as \( t \to \infty \).

For an initial value \( x_0 = (r_0 \cos(\theta_0), r_0 \sin(\theta_0)) \) the transversal system stated in Theorem \[2.2\] is then \( w(t) = r_0 e^{\frac{a + d}{2} t} r_0 \). Hence the result guarantees that the radial part \( \pi_r(X_{t,\lambda}^\varepsilon) \) on the accelerated time scale \( \frac{t}{\varepsilon} \) has a local behavior close to the exponential \( e^{\frac{a + d}{2} t} \) in the sense that for any \( \lambda \in (0,1) \) there are constants \( C, c_\lambda > 0 \) and \( \varepsilon_0 \in (0,1) \) such that for any \( T \in [0,1] \) and \( \varepsilon \in (0,\varepsilon_0] \) we have

\[
\left( \mathbb{E} \left[ \sup_{s \in [0,T]} |\pi_r(X_{s,\lambda}^\varepsilon(x_0)) - r_0 e^{\frac{a + d}{2} s} |^2 \right] \right)^{\frac{1}{2}} \leq CT \left( \varepsilon^\lambda + (c |\ln \varepsilon|)^{-\frac{1}{2}} \right),
\]

where \( c_\lambda \) is given in Corollary \[3.2\]. This averaging error tends to zero for fixed \( T \) when \( \varepsilon \searrow 0 \) and for fixed \( \varepsilon \) if \( T \searrow 0 \).
3 The perturbation error

In order to prove the main theorem we have to control the error $X^e - X$ in terms of $L^p$. This result relies on the following elementary but in this context crucial lemma on dynamical systems, which yields on the right-hand side only quadratic dependence on the “jump increment” $z$. Due to its importance for this article we provide a sketch of proof.

Lemma 3.1 For a globally Lipschitz continuous matrix-valued vector field $F \in C^2(\mathbb{R}^{r+n}, L(\mathbb{R}^r, \mathbb{R}^{r+n}))$ and $z \in \mathbb{R}^r$ denote by $(Y(t; x, Fz))_{t \geq 0}$ the unique global strong solution of the ordinary differential equation

$$\frac{dY}{dt} = F(Y)z \quad Y(0, x, Fz) = x \in \mathbb{R}^{r+n}. \quad (17)$$

1) Then there is a constant $C > 0$ such that for any $z \in \mathbb{R}^r$ and $x, y \in M$ with $Y(t; x) = Y(t; x, Fz)$ we have

$$\sup_{t \geq 0} |(DF(Y(t; x))z)F(Y(t; x))z - (DF(Y(t; y))z)F(Y(t; y))z| \leq C \ |x - y| \ |z|^2.$$

2) For any $x \in M$ we have $\sup_{t \in [0, 1]} \|DF(Y(t; x))F(Y(t; x))\| < \infty$.

Proof: We lighten notation and omit the parameter $Fz$ in $Y$ and write $Fz = F_z$. By the change of variables we have for any $x \in M, t \geq 0$ that

$$F_z(Y(t; x)) = F_z(x) + \int_0^t DF_z(Y(s; x))F_z(Y(s; x))ds.$$

Differentiating in $t$ yields

$$\frac{d}{dt}F_z(Y(t; x)) = DF_z(Y(t; x))F_z(Y(t; x)).$$

Hence for any $x, y \in L$ the mean value theorem and equation (17) yield

$$DF_z(Y(t; x))F_z(Y(t; x)) - DF_z(Y(t; y))F_z(Y(t; y))$$

$$= \int_0^1 \frac{d}{dt}DF_z(Y(t; x + \sigma(y - x)))(y - x)d\sigma$$

$$= \int_0^1 (DF_z^2(Y(t; x + \sigma(y - x)))(y - x), F_z(Y(t; x + \sigma(y - x)))d\sigma.$$

Since $(DF)F$ is Lipschitz continuous and $F \in C^2$ the operator $D((DF)F)$ is uniformly bounded. The chain rule $D((DF)F) = (D^2F)F + (DF)(DF)$ yields

$$\|D^2F\|_{\infty} \leq \|(DF)F\|_{\infty} + \|(DF)(DF)\|_{\infty} < \infty,$$

where $\|DF\|$ is uniformly bounded since $F$ is globally Lipschitz continuous and $F \in C^1$. Therefore

$$|DF_z(Y(t; x))F_z(Y(t; x)) - DF_z(Y(t; y))F_z(Y(t; y))| \leq \|(D^2F)F\|_{\infty}|z|^2|x - y| \leq C|x - y||z|^2.$$

Since the right-hand side is independent of $t$ we take the supremum as claimed in statement 1). Statement 2) is a straight-forward consequence of the product rule. ■
Proposition 3.1 Let the assumptions of Subsection 2.1 and Hypotheses 1, 2 and 3 be satisfied for some \( p > 2 \). Then for any Lipschitz function \( h : M \to \mathbb{R} \) there exist positive constants \( \varepsilon_0, k_0, k_1, k_2 \) with \( k_0 < 1 \) such that for all \( T \geq 0 \) satisfying \( \varepsilon_0 T \leq k_0, \varepsilon \in (0, \varepsilon_0) \) implies
\[
\mathbb{E} \left[ \sup_{t \leq T} |h(X^x_t(x_0)) - h(X_t(x_0))|^p \right] \leq k_1 \varepsilon^1 \exp(k_2 T).
\] (18)
In addition, the constant \( k_2 \) is a polynomial in \( \text{diam}L_{x_0} \) of order \( p \) with positive coefficients.

Corollary 3.2 Let the assumptions of Proposition 3.1 be satisfied for some \( p > 2 \). Then for any \( \lambda \in (0, 1) \) given there exist positive constants \( c_\lambda, \varepsilon_0, k_3 \) such that \( T_\varepsilon := -c_\lambda \ln(\varepsilon), \varepsilon \in (0, \varepsilon_0] \) satisfies
\[
\mathbb{E} \left[ \sup_{t \leq T_\varepsilon} |h(X^x_t(x_0)) - h(X_t(x_0))|^p \right] \leq k_3 \varepsilon^\lambda.
\] (19)
In addition, the constant \( k_3 \) is a polynomial in \( \text{diam}L_{x_0} \) of order \( p \) with positive coefficients.

Proof: Plugging \( T_\varepsilon = -c \ln(\varepsilon) \) into the right-hand side of (18) we obtain \( k_1 \varepsilon \exp(k_2 T_\varepsilon) = k_1 \varepsilon^{1 - ck_2} \).

Given \( \lambda \in (0, 1) \) we choose \( c = \frac{1}{k_2} (1 - \lambda') \) and \( \lambda' = \frac{1}{2} (\lambda + 1) \) to infer the desired result. ■

Proof: (of Proposition 3.1) The proof consists in three parts. After changing the coordinates in part 1 we estimate the transversal component \( |v^c - v| \) using Lemma 3.1 in part 2. In part 3 we estimate the horizontal component \( |u^c - u| \) before concluding with a nonlinear comparison principle. Part 2 and part 3 are given in separate lemmas. The main tools to derive two (nonlinear) comparison principles are Lemma 3.1 and Kunita’s maximal inequality for the \( L^p \) norm \( (p \geq 2) \) of the supremum of compensated Poisson integrals found in [42] and an extension of this result for \( p \in [1, 2] \) by Saint Loubert Bié [53].

I. Change of coordinates: First we rewrite the respective solutions of equation (1) and (5), \( X \) and \( X^c \), in terms of the coordinates given by the diffeomorphism \( \varphi \)
\[
(u_t, v_t) := \varphi(X_t) \quad \text{and} \quad (u^c_t, v^c_t) := \varphi(X^c_t), \quad \varepsilon \in (0, 1), t \in [0, T].
\]
The Lipschitz regularities of \( h \) and \( \varphi \) yield a joint Lipschitz constant \( C_0 := \text{Lip}(h \circ \varphi^{-1}) \) such that
\[
|h(X^c_t) - h(X_t)| = |h \circ \varphi^{-1}(u^c_t, v^c_t) - h \circ \varphi^{-1}(u_t, v_t)| \leq C_0(|u^c_t - u_t| + |v^c_t - v_t|).
\] (20)
The proof of the statement consists in calculating estimates for each summand on the right hand side of equation above. We define
\[
\hat{\mathbf{d}} := (D\varphi) \circ F \circ \varphi^{-1}, \quad \hat{\mathbf{c}} := (D\varphi) \circ K \circ \varphi^{-1}, \quad \tilde{\mathbf{d}} := (D\varphi) \circ \tilde{K} \circ \varphi^{-1},
\]
whose derivatives are uniformly bounded. Considering the components in the image of \( \varphi \) we have:
\[
\mathbf{R} = (\mathbf{R}_H, \mathbf{R}_V), \quad \tilde{\mathbf{R}} = (\tilde{\mathbf{R}}_H, \tilde{\mathbf{R}}_V)
\]
with \( \mathbf{R}_H, \tilde{\mathbf{R}}_H \in TL_{x_0} \) with \( \mathbf{R}_H \perp \mathbf{R}_V \) and \( \mathbf{R}_V, \tilde{\mathbf{R}}_V \in TV \simeq \mathbb{R}^d \) with \( \tilde{\mathbf{R}}_H \perp \tilde{\mathbf{R}}_V \). The chain rule for canonical Marcus equations (Theorem 4.2 of [11]) yields for equation (5) the following form of the components in \( \varphi \) coordinates
\[
u_t^c = \mathbf{d}_0(u_t^c, v_t^c)dt + \mathbf{c}(u_t^c, v_t^c) \circ dZ_t + \varepsilon \mathbf{R}_H(u_t^c, v_t^c)dt + \varepsilon \tilde{\mathbf{R}}_H(v_t^c) \circ d\tilde{Z}_t \quad \text{with} \quad u_t^c \in L_{x_0}, \quad v_t^c \in V.
\] Note that for \( \varepsilon = 0 \) the equation yields \( v_t = v_t^0 = 0 \in V \) almost surely. However we will write \( v_t \) nevertheless for the sake of readability.
Lemma 3.3 (II. Estimate of the transversal deviation $|v^\varepsilon - v|$) Under the previous assumptions we obtain the following. There is a constant $C_1 > 0$ such that for $\varepsilon_0 T < 1 \in (0, \varepsilon_0]$ implies
\[
\mathbb{E}\left[ \sup_{t \in [0,T]} |v^\varepsilon_t - v_t|^p \right] \leq \bar{C}_1 \varepsilon^p (1 + T^{2p+1}).
\] (23)

In addition $\tilde{C}_1 = \bar{C}_1 (\text{diam} L_{x_0})$ depends globally Lipschitz continuously on $\text{diam} L_{x_0}$.

**Proof:** By assumption $F$, $K$ and $\tilde{K}$ are globally Lipschitz continuous. Without loss of generality, we can assume they have all a common Lipschitz constant $\ell$. The compactness of $L_{x_0}$ and the embedding in $\mathbb{R}^{n+d}$ yields the existence of $C_1 = \text{diam} L_{x_0}$ such that $|u^\varepsilon_s(x_0) - u_s(x_0)| < C_1$. We use the notation
\[
C_2(x_0) = \sup_{y \in L_{x_0}} \|\tilde{R}_V(y,0)\| \leq \|\tilde{R}(x_0)\| + \sup_{y \in L_{x_0}} \ell |x_0 - y| \leq \|\tilde{R}(x_0)\| + \ell \text{diam} L_{x_0} < \infty,
\] (24)
which is finite by the compactness of $L_{x_0}$ and the continuity of $\tilde{R}_V$. Keeping in mind that $\langle Dg(x), u \rangle = p|x|^{p-2}\langle x, u \rangle$ for $x \mapsto g(x) := |x|^p$, $x \in \mathbb{R}^{n+d}$ we apply the change of variable formula and obtain
\[
|v^\varepsilon_t - v_t|^p = p \int_0^t |v^\varepsilon_s - v_s|^{p-2}|(v^\varepsilon_s - v_s, \varepsilon \tilde{R}_V(v^\varepsilon_s, v^\varepsilon_s))| \, ds
\]
\[
+ p \int_0^t |v^\varepsilon_s - v_s|^{p-2}|\varepsilon \tilde{R}_V(v^\varepsilon_s, v^\varepsilon_s) - \varepsilon \tilde{R}_V(u_s, v_s)| \, ds
\] (H1)
\[
+ p \int_0^t |v^\varepsilon_s - v_s|^{p-1}|\varepsilon \tilde{R}_V(u_s, v_s)| \, ds
\] (H2)
\[
+ p \int_0^t |v^\varepsilon_s - v_s|^{p-2}|(v^\varepsilon_s - v_s, \varepsilon (\tilde{R}_V(v^\varepsilon_s) - \tilde{R}_V(v_s))d\tilde{Z}_s)|
\] (H3)
\[
+ p \int_0^t |v^\varepsilon_s - v_s|^{p-2}|\varepsilon \tilde{R}_V(v_s) - \tilde{R}_V(v^\varepsilon_s) + \tilde{R}_V(v^\varepsilon_s) \, d\tilde{Z}_s|
\] (H4)
\[
+ p \sum_{0 < s \leq t} |v^\varepsilon_s - v_s|^{p-1}|\tilde{R}_V(v^\varepsilon_s) - \tilde{R}_V(v_s)| - (v^\varepsilon_s - v_s) - \varepsilon \tilde{R}_V(v^\varepsilon_s) - \tilde{R}_V(v_s) \Delta s \tilde{Z}
\] (H5)
\[
+ p \sum_{0 < s \leq t} |v^\varepsilon_s - v_s|^{p-1}|\tilde{R}_V(v^\varepsilon_s) - v_s - \varepsilon \tilde{R}_V(v_s) \Delta s \tilde{Z}|
\] (H6)
\[
= H_1 + H_2 + H_3 + H_4 + H_5 + H_6.
\] (25)

1. **Pathwise representation and estimates:**

**H1:** The compactness of $L_{x_0}$ yields for $C_3 = p\ell(1 + \text{diam} L_{x_0})$
\[
H_1 \leq \varepsilon p\ell \int_0^t |v^\varepsilon_s - v_s|^{p-1}(|v^\varepsilon_s - v_s| + |v^\varepsilon_s - v_s|) \, ds \leq \varepsilon C_3 \int_0^t |v^\varepsilon_s - v_s|^p \, ds + \varepsilon C_3 \int_0^t |v^\varepsilon_s - v_s|^{p-1} \, ds.
\] (26)

**H2:** A direct computation gives
\[
H_2 \leq \varepsilon pC_2 \int_0^t |v^\varepsilon_s - v_s|^{p-1} \, ds \leq \varepsilon C_4 \int_0^t |v^\varepsilon_s - v_s|^{p-1} \, ds.
\] (27)
\( H_3 \): Switching to the Poisson random measure representation with respect to the compensated \( \tilde{N}' \), for instance in Kunita [42], we obtain for \( C_5 = p \tilde{p} \int_{\|z\|>1} \|z\|^p'(dz) \)

\[
H_3 = p \int_0^t \int_{\mathbb{R}^r} |v_s^x - v_s^-|^{-2} \langle v_s^x - v_s^-, \varepsilon(\tilde{N}_V(v_s^-)) - \tilde{N}_V(v_s^-) \rangle |\tilde{N}'(dsdz) \\
+ p \int_0^t \int_{\|z\|>1} |v_s^x - v_s^-|^{-2} \langle v_s^x - v_s^-, \varepsilon(\tilde{N}_V(v_s^-)) - \tilde{N}_V(v_s^-) \rangle |\tilde{N}'(dz)|ds \\
\leq \varepsilon p \int_0^t \int_{\mathbb{R}^r} |v_s^x - v_s^-|^{-2} \langle v_s^x - v_s^-, \tilde{N}_V(v_s^-) \rangle |\tilde{N}'(dsdz) \\
+ \varepsilon C_5 \int_0^t |v_s^x - v_s^-|^p ds.
\] (28)

\( H_4 \): With the help of Hölder’s inequality we obtain for \( C_6 = p \tilde{p} \int_{\|z\|>1} \|z\|^p'(dz) \)

\[
H_4 = p \int_0^t \int_{\mathbb{R}^r} v_s^x - v_s^-|^{-2} \langle v_s^x - v_s^-, \varepsilon(\tilde{N}_V(v_s^-)) \rangle |\tilde{N}'(dsdz) \\
+ p \int_0^t \int_{\|z\|>1} v_s^x - v_s^-|^{-2} \langle v_s^x - v_s^-, \varepsilon(\tilde{N}_V(v_s^-)) \rangle |\tilde{N}'(dz)|ds \\
\leq \varepsilon p \int_0^t \int_{\mathbb{R}^r} v_s^x - v_s^-|^{-2} \langle v_s^x - v_s^-, \tilde{N}_V(v_s^-) \rangle |\tilde{N}'(dsdz) \\
+ \varepsilon C_6 \int_0^t v_s^x - v_s^-|^{p-1} ds.
\] (29)

\( H_5 \): For the canonical Marcus terms, Lemma 3.1 (statement 1), provides a positive constant \( C_7 \) which depends on the leaf of the initial condition such that independent of \( \theta \in [0,1] \)

\[
\left| \left( D\tilde{N}_V(\Phi^{\tilde{N}'x}(y,\theta))z(\Phi^{\tilde{N}'y}(y,\theta))z \right) - \left( D\tilde{N}_V(\Phi^{\tilde{N}'x}(x,\theta))z(\Phi^{\tilde{N}'y}(x,\theta))z \right) \right| \leq C_7 |x-y|\|z\|^2.
\] (30)

The Poisson random measure representation of the random sum reads as the following estimate in terms of the quadratic variation of \( Z \) for \( C_8 = C_7 p' /2 \)

\[
H_5 \leq \varepsilon^2 \frac{p}{2} \sum_{0<s\leq t} |v_s^x - v_s^-|^p \left( (D\tilde{N}_V(\Phi^{\tilde{N}'\Delta_s}(v_s^-))z(\Phi^{\tilde{N}'\Delta_s}(v_s^-))z) \right) \\
- (D\tilde{N}_V(\Phi^{\tilde{N}'\Delta_s}(v_s^-))z(\Phi^{\tilde{N}'\Delta_s}(v_s^-))z) \\
\leq \varepsilon^2 C_8 \sum_{0<s\leq t} |v_s^x - v_s^-|^p \|z\|^2 |\Delta_s| \|x\|^2.
\]

The representation of this sum in terms of the Poisson random measure, for instance in Kunita [42], is given for \( C_9 = \int_{\|z\|>1} \|z\|^p'(dz) \) by

\[
\sum_{0<s\leq t} |v_s^x - v_s^-|^p \|z\|^2 = \int_0^t \int_{\mathbb{R}^r} |v_s^x - v_s^-|^p |z|^2 \tilde{N}'(dsdz) + C_9 \int_0^t |v_s^x - v_s^-|^p ds,
\]

which yields

\[
H_5 \leq \varepsilon^2 C_8 \int_0^t \int_{\mathbb{R}^r} |v_s^x - v_s^-|^p |z|^2 \tilde{N}'(dsdz) + \varepsilon^2 C_9 \int_0^t |v_s^x - v_s^-|^p ds.
\] (31)
\[ H_6 : \text{For the last term we use Lemma \ref{lemma3.1} statement 2), which yields a positive constant } C_{11} \text{ such that for any } z \in \mathbb{R}^r \]

\[
\sup_{\theta \in [0,1]} \| (D\tilde{\Phi}_V(Y(\theta, 0, \varepsilon \tilde{\Phi}_V z))) z \tilde{\Phi}_V(Y(\theta, 0, \varepsilon \tilde{\Phi}_V z)) z \| < C_{11} \| z \|^2 < \infty.
\]

Hence exploiting that for any \( z \in \mathbb{R}^r \)

\[
\varepsilon^2 \sum_{0 < s \leq t} |v^\varepsilon_{s-} - v_{s-}|^{p-1} |\tilde{\Phi}_V^\varepsilon \Delta_s \tilde{Z}(v_{s-}) - v_{s-} - \varepsilon \tilde{\Phi}_V(v_{s-}) \Delta_s \tilde{Z}|^{1/4}
\]

\[
\leq \varepsilon^2 C_{12} \int_0^t \int_{\mathbb{R}^r} |v^\varepsilon_{s-} - v_{s-}|^{p-1} \| z \|^4 \tilde{N}'(ds dz) + \varepsilon^2 C_{13} \int_0^t |v^\varepsilon_{s-} - v_{s-}|^{p-1} ds.
\]

Combining the estimates \( 26, 27, 28, 29, 31, 32 \) we obtain

\[
|v^\varepsilon_t - v_t|^p \leq \varepsilon C_3 \int_0^t |v^\varepsilon_s - v_{s-}|^p ds + \varepsilon C_3 \int_0^t |v^\varepsilon_s - v_{s-}|^{p-1} ds + \varepsilon C_4 \int_0^t |v^\varepsilon_s - v_{s-}|^{p-1} ds
\]

\[
+ \varepsilon^2 C_5 \int_0^t |v^\varepsilon_s - v_{s-}|^p ds + \varepsilon C_6 \int_0^t |v^\varepsilon_s - v_{s-}|^{p-1} ds
\]

\[
+ \varepsilon^2 C_{10} \int_0^t |v^\varepsilon_s - v_{s-}|^p ds + \varepsilon^2 C_{13} \int_0^t |v^\varepsilon_s - v_{s-}|^{p-1} ds
\]

\[
+ \varepsilon p \int_0^t \int_{\mathbb{R}^r} |v^\varepsilon_s - v_{s-}|^{p-2} |v^\varepsilon_{s-} - v_{s-}, (\tilde{\Phi}_V^\varepsilon(v^\varepsilon_{s-}) - \tilde{\Phi}_V(v_{s-})) z| \tilde{N}'(ds dz)
\]

\[
+ \varepsilon p \int_0^t \int_{\mathbb{R}^r} |v^\varepsilon_s - v_{s-}|^{p-2} |v^\varepsilon_{s-} - v_{s-}, \tilde{\Phi}_V(v_{s-}) z| \tilde{N}'(ds dz)
\]

\[
+ \varepsilon^2 C_8 \int_0^t \int_{\mathbb{R}^r} |v^\varepsilon_s - v_{s-}|^{p} \| z \|^2 \tilde{N}'(ds dz)
\]

\[
+ \varepsilon^2 C_{12} \int_0^t \int_{\mathbb{R}^r} |v^\varepsilon_s - v_{s-}|^{p-1} \| z \|^4 \tilde{N}'(ds dz).
\]

2. **Estimate of the marginal expectation:** Taking the expectation we obtain a constant \( C_{14} \) such that

\[
\mathbb{E}[|v^\varepsilon_t - v_t|^p] \leq \varepsilon C_{14} \int_0^t \left( \mathbb{E}[|v^\varepsilon_s - v_{s-}|^p] + \mathbb{E}[|v^\varepsilon_s - v_{s-}|^{p-1}] \right) ds
\]

and

\[
\sup_{s \in [0,t]} \mathbb{E}[|v^\varepsilon_s - v_{s-}|^p] \leq \varepsilon C_{14} \int_0^t \left( \mathbb{E}[|v^\varepsilon_s - v_{s-}|^p] + \mathbb{E}[|v^\varepsilon_s - v_{s-}|^{p-1}] \right) ds
\]

\[
\leq \varepsilon C_{14} t \sup_{s \in [0,t]} \mathbb{E}[|v^\varepsilon_s - v_{s-}|^p] + C_{14} \varepsilon \int_0^t \mathbb{E}[|v^\varepsilon_s - v_{s-}|^{p-1}]^{p-1} ds.
\]
For \( \varepsilon \in (0,\varepsilon_0] \) and \( t \in [0,T] \), \( T > 0 \) such that \( \varepsilon_0 TC_{14} \leq \frac{1}{2} \), that is fixing \( k_0 = (2C_{14})^{-1} \) in the statement, we have

\[
\sup_{s \in [0,T]} \mathbb{E}[|v^\varepsilon_s - v_s|^p] \leq 2C_{14}\varepsilon \int_0^T \sup_{t \in [0,s]} \mathbb{E}[|v^\varepsilon_t - v_t|^\frac{p-1}{p}] ds.
\]

It is easy to verify that the maximal solution of this equation is given by

\[
\sup_{s \in [0,T]} \mathbb{E}[|v^\varepsilon_s - v_s|^p] \leq \left(\frac{2C_{14}}{P}\varepsilon T\right)^P \leq C_{15}(\varepsilon T)^P.
\] (38)

We can replace the exponent \( p \) in the estimate of \( \mathbb{E}[|v^\varepsilon_s - v_s|^p] \) by \( 2p \) and \( p - 1 \). This is possible since the integrals with respect to \( \nu' \) do not depend on \( p \) and obtain for \( q \in \{2p, p-1\} \)

\[
\sup_{s \in [0,T]} \mathbb{E}[|v^\varepsilon_s - v_s|^q] \leq C_{16}(\varepsilon T)^q.
\] (39)

3. Estimate of the expectation of the supremum: We go back to (33) and note that all integrands with respect to the Lebesgue integral \( ds \) are positive, such that the integrals are positive and increasing. Further we note that the last four summands are compensated Poisson random integrals \( M = M^1 + M^2 \). The sum of the terms (34, 35, 36) will be denoted for convenience by \( M^1 \) and term (37) will be denoted by \( M^2 \). We obtain with the help of Jensen’s inequality

\[
\mathbb{E}[\sup_{[0,T]} |v^\varepsilon - v|^p] \leq \varepsilon C_{17} \int_0^T (\mathbb{E}[|v^\varepsilon_s - v_s|^p] + \mathbb{E}[|v^\varepsilon_t - v_t|^p] \frac{1}{\varepsilon^p}) ds + \mathbb{E}[\sup_{[0,T]} |M^1|^2] + \mathbb{E}[\sup_{[0,T]} |M^2|].
\] (40)

For \( M^1 \) we use Kunita’s maximal inequality for the exponent 2 (see [42] or [2]) and Young’s inequality for the exponents \( \frac{2p-2}{2p} \) and \( 2p \) in order to obtain

\[
\mathbb{E}[\sup_{[0,T]} |M^1|^2] \leq \varepsilon C_{18} \left( \mathbb{E}[|v^\varepsilon - v_s|^2] \|z\|^{2(p-1)} \|z\|^{2p'}(dz)ds 
+ \int_0^T \int_{\mathbb{R}^r} \mathbb{E}[|v^\varepsilon_s - v_s|^2] \|z\|^{2p'}(dz)ds 
+ \int_0^T \int_{\mathbb{R}^r} \mathbb{E}[|v^\varepsilon_s - v_s|^2] \|z\|^{4p'}(dz)ds \right) 
\leq \varepsilon C_{19} \int_0^T \left( \mathbb{E}[|v^\varepsilon_s - v_s|^{2(p-1)}] + \mathbb{E}[|v^\varepsilon_s - v_s|^{2p}] \right) ds 
\leq C_{19} \int_0^T \left( (1+\varepsilon)\mathbb{E}[|v^\varepsilon_s - v_s|^{2p}] + \varepsilon^{2p} \right) ds,
\] (41)

and inserting (39) in (41) we get

\[
\mathbb{E}[\sup_{[0,T]} |M^1|^2] \leq C_{20} \int_0^T (\varepsilon s)^{2p} ds + \varepsilon^{2p} T \leq C_{20}\varepsilon^{2p} T (1 + T^{2p}).
\] (42)

This could be also repeated with term \( M^2 \) but the price to pay would be \( \int \|z\|^8 \nu'(dz) < \infty \), equivalent to the finiteness eighth moments of \( \tilde{Z} \). Instead for \( M^2 \) we use a maximal inequality for integrals with respect to Poisson random measures for the exponent 1 given in [49], Lemma
8.22 and resp. Theorem 8.23, going back to Saint Loubert Bié [53], which states the existence of a constant $C_{21}$ such that for $C_{22} = C_{21} \int_{\mathbb{R}^r} \|z\|^4 \nu'(dz)$ and $t \geq 0$

$$\mathbb{E}[\sup_{[0,T]} |M^2|] \leq \varepsilon^2 C_{22} \int_0^T \int_{\mathbb{R}^r} \mathbb{E}[|v_s - v_t|^p] \|z\|^4 \nu'(dz)ds \leq \varepsilon^2 C_{22} \int_0^T \mathbb{E}[|v_s - v_t|^p] \frac{e^{-p}}{p} ds.$$

Inserting (38) in the preceding expression we get

$$\mathbb{E}[\sup_{[0,T]} |M^2|] \leq \varepsilon^2 C_{23} \int_0^T (\varepsilon s)^{p-1} ds \leq C_{23} \varepsilon^p T^p. \tag{43}$$

Transforming (40) with the help of (38), (42) and (43) and keeping in mind that $\varepsilon T < 1$ yields

$$\mathbb{E}[\sup_{[0,T]} |v^\varepsilon - v|^p] \leq \varepsilon C_{24} \int_0^T ((\varepsilon s)^p + (\varepsilon s)^{p-1}) ds + C_{25} \varepsilon^p \sqrt{T(1 + T^{2p})} + C_{23} \varepsilon^p T^{p+1} \leq C_{26} \varepsilon^p (1 + T^{2p+1}). \tag{44}$$

Lemma 3.4 (III. Estimate of the horizontal component $|u^\varepsilon - u|$) Under the previous assumptions we obtain the following. There are constants $C_{2}, t_0 > 0$ such that for $\varepsilon_0 T < 1 \varepsilon \in (0, \varepsilon_0]$ implies

$$\mathbb{E} \left[ \sup_{t \in [0,T]} |u^\varepsilon_t - u_t|^p \right] \leq \tilde{C}_2 \varepsilon^{2p} (1 + (T \vee t_0))^{2(p+1)} \exp \left( \tilde{C}_2 T \right). \tag{45}$$

Proof: For convenience of notation we restart with the numbering of constants. Formally we obtain

$$u^\varepsilon_t - u_t = \int_0^t (\bar{\mathcal{F}}_0(u^\varepsilon_s, v^\varepsilon_s) - \bar{\mathcal{F}}_0(u_s, v_s)) ds + \int_0^t (\mathcal{F}(u^\varepsilon_s, v^\varepsilon_s) - \mathcal{F}(u_s, v_s)) \circ dZ_s + \varepsilon \int_0^t (\mathcal{H}(u^\varepsilon_s, v^\varepsilon_s) - \mathcal{H}(u_s, v_s)) ds + \varepsilon \int_0^t \tilde{\mathcal{H}}(u^\varepsilon_s, v^\varepsilon_s) \circ d\tilde{Z}_s. \tag{46}$$

This equation is defined in $\mathbb{R}^n$ as

$$u^\varepsilon_t - u_t = \int_0^t \mathcal{F}_0(u^\varepsilon_s, v^\varepsilon_s) - \bar{\mathcal{F}}_0(u_s, v_s) ds$$

$$+ \int_0^t \mathcal{F}(u^\varepsilon_s, v^\varepsilon_s) - \mathcal{F}(u_s, v_s) dZ_s$$

$$+ \sum_{0 < s \leq t} \left[ (\Phi \Delta_s Z(u^\varepsilon_s, v^\varepsilon_s) - \Phi \Delta_s Z(u_s, v_s) - (u^\varepsilon_s - u_s, v^\varepsilon_s - v_s) - (\mathcal{F}(u^\varepsilon_s, v^\varepsilon_s) - \mathcal{F}(u_s, v_s)) \Delta_s Z \right]$$

$$+ \varepsilon \int_0^t (\mathcal{H}(u^\varepsilon_s, v^\varepsilon_s) - \mathcal{H}(u_s, v_s)) ds + \varepsilon \int_0^t \tilde{\mathcal{H}}(v^\varepsilon_s) d\tilde{Z}_s$$

$$+ \sum_{0 < s \leq t} \left[ \Phi \Delta_s Z(u^\varepsilon_s - u_s, v^\varepsilon_s - v_s - (\varepsilon \tilde{\mathcal{H}}(v^\varepsilon_s)) \Delta_s Z \right].$$
The change of variable formula for (46) yields formally

\[
|u^\varepsilon_t - u_t|^p = p \int_0^t |u^\varepsilon_s - u_s|^{p-2}(u^\varepsilon_s - u_s, \mathcal{F}_0(u^\varepsilon_s, v^\varepsilon_s) - \mathcal{F}_0(u_s, v_s))ds \\
+ p \int_0^t |u^\varepsilon_s - u_s|^{p-2}(u^\varepsilon_s - u_s, (\mathcal{F}(u^\varepsilon_s, v^\varepsilon_s) - \mathcal{F}(u_s, v_s)) \circ dZ_s) \\
+ \varepsilon p \int_0^t |u^\varepsilon_s - u_s|^{p-2}(u^\varepsilon_s - u_s, \mathcal{R}_H(u^\varepsilon_s, v^\varepsilon_s) - \mathcal{R}_H(u_s, v_s))ds \\
+ \varepsilon p \int_0^t |u^\varepsilon_s - u_s|^{p-2}(u^\varepsilon_s - u_s, \mathcal{R}_H(u_s, v_s))ds \\
+ p \int_0^t |u^\varepsilon_s - u_s|^{p-2}(u^\varepsilon_s - u_s, \mathcal{R}_H(u_s, v_s))ds
\]

This is defined in \( \mathbb{R}^n \) as

\[
|u^\varepsilon_t - u_t|^p = p \int_0^t |u^\varepsilon_s - u_s|^{p-2}(u^\varepsilon_s - u_s, \mathcal{F}_0(u^\varepsilon_s, v^\varepsilon_s) - \mathcal{F}_0(u_s, v_s))ds \\
+ p \int_0^t |u^\varepsilon_s - u_s|^{p-2}(u^\varepsilon_s - u_s, (\mathcal{F}(u^\varepsilon_s, v^\varepsilon_s) - \mathcal{F}(u_s, v_s)) dZ_s) \tag{I1} \\
+ p \sum_{0<s\leq t} |u^\varepsilon_s - u_s|^{p-2}(u^\varepsilon_s - u_s, \mathcal{F}(u^\varepsilon_s, v^\varepsilon_s) - \mathcal{F}(u_s, v_s)) \partial_s Z \\
- (u^\varepsilon_s - u_s, \mathcal{F}(u^\varepsilon_s, v^\varepsilon_s) - \mathcal{F}(u_s, v_s)) \Delta_s Z \tag{I3} \\
+ \varepsilon p \int_0^t |u^\varepsilon_s - u_s|^{p-2}(u^\varepsilon_s - u_s, \mathcal{R}_H(u^\varepsilon_s, v^\varepsilon_s) - \mathcal{R}_H(u_s, v_s))ds \\
+ \varepsilon p \int_0^t |u^\varepsilon_s - u_s|^{p-2}(u^\varepsilon_s - u_s, \mathcal{R}_H(u_s, v_s))ds \tag{I4} \\
+ \varepsilon p \int_0^t |u^\varepsilon_s - u_s|^{p-2}(u^\varepsilon_s - u_s, \mathcal{R}_H(u_s, v_s))ds \tag{I5} \\
+ \varepsilon p \int_0^t |u^\varepsilon_s - u_s|^{p-2}(u^\varepsilon_s - u_s, \mathcal{R}_H(u^\varepsilon_s, v^\varepsilon_s))d\tilde{Z}_s \\
+ p \sum_{0<s\leq t} |u^\varepsilon_s - u_s|^{p-2}(u^\varepsilon_s - u_s, \mathcal{F}(\mathcal{R}_H(u^\varepsilon_s, \mathcal{R}_H(v^\varepsilon_s))) - \mathcal{F}(\mathcal{R}_H(u_s, \mathcal{R}_H(v_s))))
\]

We now estimate the eight summands on the right-hand side one by one. The estimates \( I_1 \) and \( I_4 \) are straightforward Lipschitz estimates. For the stochastic Itô terms we use the different kinds of maximal inequalities. The estimate of the canonical Marcus terms \( I_3, \ I_7 \) and \( I_8 \) is the most laborious task in which we exploit the result of Lemma 3.1. The term \( I_5 \) is straightforward.

1. **Estimate of the stochastic Itô integral terms \( I_2 \) and \( I_6 \):** \( I_2 \): Due to the existence of moments of order at least 1, \( I_2 \) has the following representation with respect to the compensated Poisson random measure associated to \( Z \)

\[
\int_0^t |u^\varepsilon_s - u_s|^{p-2}(u^\varepsilon_s - u_s, (\mathcal{F}(u^\varepsilon_s, v^\varepsilon_s) - \mathcal{F}(u_s, v_s))d\tilde{Z}_s)
\]

\[
= I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7 + I_8 \tag{47}
\]
\[ = \int_0^t \int_{\mathbb{R}^r} |u^\varepsilon_s - u_s - p^{-2} \langle u^\varepsilon_s - u_s, (\mathfrak{F}(u^\varepsilon_s, v^\varepsilon_s) - \mathfrak{F}(u_s, v_s)) \rangle \rangle \mathcal{N}(dsdz) \]  
(48) 
\[ + \int_0^t \int_{\|z\|>1} |u^\varepsilon_s - u_s|^{p-2} \langle u^\varepsilon_s - u_s, (\mathfrak{F}(u^\varepsilon_s, v^\varepsilon_s) - \mathfrak{F}(u_s, v_s)) \rangle \rangle \nu(dz)ds. \]  
(49)

For the first term \( \text{[48]} \) we apply the embedding \( L^2 \subset L^1 \), Kunita’s maximal inequality (see \[ 12 \] or \[ 2 \]) for exponent equal to 2, the compactness of \( L_{x_0} \) implying \( |u^\varepsilon(x_0) - u(x_0)| \leq \text{diam}L_{x_0} \), and the elementary Young inequality for the exponents \( p \) and \( p/(p-1) \) combined with inequality \( \text{[48]} \). We obtain a positive constant \( C_1 > 0 \) such that for \( C_2 = 2C_1 (\int_{\mathbb{R}^r} \|z\|^2 \nu(dz))^{1/2} \text{diam}L_{x_0}^p \) and \( C_3 = C_{23} \) we have

\[ \mathbb{E} \left[ \sup_{t \in [0,T]} \left| \int_0^t \int_{\mathbb{R}^r} |u^\varepsilon_s - u_s - |^{p-2} \langle u^\varepsilon_s - u_s, (\mathfrak{F}(u^\varepsilon_s, v^\varepsilon_s) - \mathfrak{F}(u_s, v_s)) \rangle \rangle \mathcal{N}(dsdz) \right| \right] \leq C_1 \mathbb{E} \left[ \int_0^T \int_{\mathbb{R}^r} |u^\varepsilon_s - u_s|^{2(p-2)} \langle (u^\varepsilon_s - u_s, (\mathfrak{F}(u^\varepsilon_s, v^\varepsilon_s) - \mathfrak{F}(u_s, v_s)) \rangle \rangle \nu(dz)ds \right]^{1/2} \]

\[ \leq C_2 \left( \int_0^T \mathbb{E} \left[ \sup_{[0,s]} |u^\varepsilon - u|^p \right] ds \right)^{1/2} + C_3 \varepsilon \mu_{p+1}. \]  
(50)

The second term is less delicate. Young’s inequality for the exponents \( p/(p-1) \) and \( p \) and yields

\[ \mathbb{E} \left[ \sup_{t \in [0,T]} \left| \int_0^t \int_{\|z\|>1} |u^\varepsilon_s - u_s - |^{p-2} \langle u^\varepsilon_s - u_s, (\mathfrak{F}(u^\varepsilon_s, v^\varepsilon_s) - \mathfrak{F}(u_s, v_s)) \rangle \rangle \nu(dz)ds \right| \right] \leq \ell \int_{\|z\|>1} \|z\| \nu(dz) \mathbb{E} \left[ \sup_{t \in [0,T]} \left| \int_0^t (|u^\varepsilon_s - u_s|^p + |u^\varepsilon_s - u_s|^{p-1} |v^\varepsilon_s - v_s|) ds \right| \right] \]

\[ \leq \ell \int_{\|z\|>1} \|z\| \nu(dz) \left( 2 \int_0^T \mathbb{E} \left[ \sup_{[0,s]} |u^\varepsilon - u|^p \right] ds + \int_0^T \mathbb{E} \left[ |v^\varepsilon - v_s|^p \right] ds \right) \]

\[ \leq C_4 \int_0^T \mathbb{E} \left[ \sup_{[0,s]} |u^\varepsilon - u|^p \right] ds + C_5 \varepsilon \mu_{p+1}. \]  
(51)

**I_6:** Let \( \tilde{N}' \) be the compensated Poisson random measure associated to the Lévy process \( \tilde{Z} \). Then

\[ \sup_{t \in [0,T]} \varepsilon \int_0^t |u^\varepsilon_s - u_s - |^{p-2} \langle u^\varepsilon_s - u_s, \tilde{\mathfrak{R}}_H(v^\varepsilon_s) d\tilde{Z}_s \rangle \]

\[ = \sup_{t \in [0,T]} \varepsilon \int_0^t \int_{\mathbb{R}^r} |u^\varepsilon_s - u_s - |^{p-2} \langle u^\varepsilon_s - u_s, (\tilde{\mathfrak{R}}_H(v^\varepsilon_s) - \tilde{\mathfrak{R}}_H(v_s)) \rangle \rangle \mathcal{N}'(dsdz) \]  
\[ + \sup_{t \in [0,T]} \varepsilon \int_0^t \int_{\|z\|>1} |u^\varepsilon_s - u_s|^{p-2} \langle u^\varepsilon_s - u_s, (\tilde{\mathfrak{R}}_H(v^\varepsilon_s) - \tilde{\mathfrak{R}}_H(v_s)) \rangle \rangle \nu'(dz)ds \]  
\[ + \sup_{t \in [0,T]} \varepsilon \int_0^t \int_{\mathbb{R}^r} |u^\varepsilon_s - u_s - |^{p-2} \langle u^\varepsilon_s - u_s, \tilde{\mathfrak{R}}_H(v_s) \rangle \rangle \mathcal{N}'(dsdz) \]  
\[ + \sup_{t \in [0,T]} \varepsilon \int_0^t \int_{\|z\|>1} |u^\varepsilon_s - u_s|^{p-2} \langle u^\varepsilon_s - u_s, \tilde{\mathfrak{R}}_H(v_s) \rangle \rangle \nu'(dz)ds. \]  
\( (J_1) \) \( (J_2) \) \( (J_3) \) \( (J_4) \)
The terms $J_1$ and $J_2$ are structurally identical to (48) and (49) and are estimated analogously to (50) and (51) where $\tilde{\Phi}$ is replaced by $\tilde{\Phi}_H$ which yield the estimates

$$
\mathbb{E}\left[ \sup_{t \in [0,T]} |\epsilon \int_0^t \int_{\mathbb{R}^r} |u_s^\varepsilon - u_s|^{p-2} (u_s^\varepsilon - u_s - (\tilde{\Phi}_H(v_s^\varepsilon) - \tilde{\Phi}_H(v_s))z) \tilde{N}'(dsdz)| \right]
$$

$$
\leq C_6 \left( \int_0^T \mathbb{E}\left[ \sup_{[0,s]} |u_s^\varepsilon - u_s|^p \right] ds \right)^{\frac{1}{2}} + C_7 \varepsilon^{pT^{p+\frac{1}{2}}} \quad \text{and}
$$

$$
\mathbb{E}\left[ \sup_{t \in [0,T]} |\epsilon \int_0^t \int_{\|z\| > 1} |u_s^\varepsilon - u_s|^{p-2} (u_s^\varepsilon - u_s - (\tilde{\Phi}_H(v_s^\varepsilon) - \tilde{\Phi}_H(v_s))z) \nu'(dz)ds| \right]
$$

$$
\leq C_8 \int_0^T \mathbb{E}\left[ \sup_{[0,s]} |u_s^\varepsilon - u_s|^p \right] ds + C_9 \varepsilon^{pT^{p+1}}.
$$

For the term $J_3$ we observe that $v_s = 0$ such that $\tilde{\Phi}_H(v_s)$ is constant. Kunita’s maximal inequality for the exponent 2 yields the constant $C_{10}$ and the boundedness of $|u_s^\varepsilon - u_s|$ the constant $C_{11} = C_{10}\left( \int_{\mathbb{R}^r} \|z\|^2 \nu'(dz)(diam L_{x_0})^{p-2}\right)^{\frac{1}{2}}$ such that

$$
\mathbb{E}\left[ \sup_{t \in [0,T]} |\epsilon \int_0^t \int_{\mathbb{R}^r} |u_s^\varepsilon - u_s|^{p-2} (u_s^\varepsilon - u_s - (\tilde{\Phi}_H(v_s^\varepsilon) - \tilde{\Phi}_H(v_s))z) \tilde{N}'(dsdz)| \right]
$$

$$
\leq \varepsilon^{pT^{p}} \mathbb{E}\left[ \sup_{[0,s]} |u_s^\varepsilon - u_s|^{2p-1} \right] \|z\|^2 \nu'(dz)ds \right)^{\frac{1}{2}}
$$

$$
\leq \varepsilon C_{10} \left( \int_0^T \int_{\mathbb{R}^r} \mathbb{E}\left[ |u_s^\varepsilon - u_s|^2 \right] \right) \|z\|^2 \nu'(dz)ds
$$

$$
\leq \varepsilon C_{11} \left( \int_0^T \mathbb{E}\left[ \sup_{[0,s]} |u_s^\varepsilon - u_s|^p \right] ds \right)^{\frac{1}{2}}.
$$

The term $J_4$ is again easier, for $C_{12} = \int_{\|z\| > 1} \|z\| \nu'(dz)\|\tilde{\Phi}_H(v_s)\|$ we obtain

$$
\mathbb{E}\left[ \sup_{t \in [0,T]} |\epsilon \int_0^t \int_{\|z\| > 1} |u_s^\varepsilon - u_s|^{p-2} (u_s^\varepsilon - u_s - (\tilde{\Phi}_H(v_s^\varepsilon) - \tilde{\Phi}_H(v_s))z) \nu'(dz)ds| \right]
$$

$$
\leq \varepsilon C_{12} \int_0^T \mathbb{E}\left[ |u_s^\varepsilon - u_s|^{p-1} \right] ds
$$

$$
\leq \varepsilon C_{12} \int_0^T \mathbb{E}\left[ \sup_{[0,s]} |u_s^\varepsilon - u_s|^p \right] ds + C_{12} \varepsilon^{pT}.
$$

This yields

$$
\mathbb{E}\left[ \sup_{[0,T]} |I_6| \right] \leq C_{13} \left( \int_0^T \mathbb{E}\left[ \sup_{[0,s]} |u_s^\varepsilon - u_s|^p \right] ds \right)^{\frac{1}{2}} + \int_0^T \mathbb{E}\left[ \sup_{[0,s]} |u_s^\varepsilon - u_s|^p \right] ds + \varepsilon^{pT^2}(2 + T)
$$

(52)

2. **Estimate of the canonical Marcus terms $I_3$, $I_7$ and $I_8$:** Lemma 3.1 tells us that there is a positive constant, $C_{14}$, say, which depends on the leaf of the initial condition such that independent of $\theta \in [0,1]

$$
\left| \left( D\tilde{\Phi}(\tilde{\Phi}^{\tilde{z}}(y,\theta))z \right) \tilde{\Phi}(\Phi^{\tilde{z}}(y,\theta))z - \left( D\tilde{\Phi}(\Phi^{\tilde{z}}(x,\theta))z \right) \tilde{\Phi}(\Phi^{\tilde{z}}(x,\theta))z \right| \leq C_{14} \|x - y\| \|z\|^2.
$$

(53)
Now we apply the Taylor’s theorem to \( \theta \mapsto Y(\theta; x, \hat{\Delta}_a Z) = \Phi \hat{\Delta}_a Z(x, \theta) \) with \( \theta \in [0, 1] \) and inequality (53). The Poisson random measure representation of the random sum writes as the following estimate in terms of the quadratic variation of \( Z \)

\[
|I_3| \leq \sum_{0<s \leq t} |u^\varepsilon_{s-} - u_{s-}|^{p-1} |\Phi \hat{\Delta}_a Z(u^\varepsilon_{s-}, v^\varepsilon_{s-}) - \Phi \hat{\Delta}_a Z(u_{s-}, v_{s-})|
- (u^\varepsilon_{s-} - u_{s-}, v^\varepsilon_{s-} - v_{s-}) - (\hat{\Phi}(u^\varepsilon_{s-}, v^\varepsilon_{s-}) - \hat{\Phi}(u_{s-}, v_{s-}))\Delta_a Z)
\leq \frac{1}{2} \sum_{0<s \leq t} |u^\varepsilon_{s-} - u_{s-}|^{p-1} \left| (D \hat{\Phi}(\Phi \hat{\Delta}_a Z(u^\varepsilon_{s-}, v^\varepsilon_{s-})) \Delta_a Z)\hat{\Phi}(\Phi \hat{\Delta}_a Z(u^\varepsilon_{s-}, v^\varepsilon_{s-})) \Delta_a Z
- (D \hat{\Phi}(\Phi \hat{\Delta}_a Z(u_{s-}, v_{s-})) \Delta_a Z)\hat{\Phi}(\Phi \hat{\Delta}_a Z(u_{s-}, v_{s-})) \Delta_a Z\right|
\leq C_{14} \left( \sum_{0<s \leq t} |u^\varepsilon_{s-} - u_{s-}|^{2p} \Delta_a Z^2 + \sum_{0<s \leq t} |v^\varepsilon_{s-} - v_{s-}|^{p-1} |v^\varepsilon_{s-} - v_{s-}| \Delta_a Z^2 \right)
\leq 2C_{14} \left( \sum_{0<s \leq t} |u^\varepsilon_{s-} - u_{s-}|^{2p} \Delta_a Z^2 + \sum_{0<s \leq t} |v^\varepsilon_{s-} - v_{s-}|^{2p} \Delta_a Z^2 \right), \tag{54}
\]

The representation of this sum in terms of the Poisson random measure, for instance in Kunita [32], of the first term is

\[
\sum_{0<s \leq t} |u^\varepsilon_{s-} - u_{s-}|^{p} \Delta_a Z^2
= \int_0^t \int_{\mathbb{R}^r} |u^\varepsilon_{s-} - u_{s-}|^{p} \|z\|^2 \tilde{N}(dsdz) + \int_0^t \int_{\|z\| > 1} |u^\varepsilon_{s-} - u_{s-}|^{p} \|z\|^2 \nu(dz) \, ds \tag{55}
\]

and the analogous result if \( |u^\varepsilon_{s-} - u_{s-}| \) is replaced by \( |v^\varepsilon_{s-} - v_{s-}| \). The maximal inequality for integrals with respect to the compensated Poisson random measures and inequality (38) yield

\[
\mathbb{E}[^{\sup}_{[0,T]} |I_3|] \leq C_{15} \int_0^T \int_{\mathbb{R}^r} \left( \mathbb{E}[^{\sup}_{[0,s]} |u^\varepsilon - u|^p] + \mathbb{E}[|v^\varepsilon - v_s|^p] \right) \|z\|^2 \nu(dz) \, ds
= C_{15} \int_{\mathbb{R}^r} \|z\|^2 \nu(dz) \left( \int_0^T \left( \mathbb{E}[^{\sup}_{[0,s]} |u^\varepsilon - u|^p] + \mathbb{E}[|v^\varepsilon - v_s|^p] \right) \right) \, ds
\leq C_{16} \left( \int_0^T \mathbb{E}[^{\sup}_{[0,s]} |u^\varepsilon - u|^p] \, ds + \varepsilon^p T^p \right). \tag{56}
\]

I7: For \( I_7 \) we use Lemma 3.1 (statement 1) in terms of (30) and Young’s inequality

\[
\sum_{0<s \leq t} |u^\varepsilon_{s-} - u_{s-}|^{p-2} \langle u^\varepsilon_{s-} - u_{s-}, \Phi \hat{\Delta}_a Z(v^\varepsilon_{s-}) - \Phi \hat{\Delta}_a Z(v_{s-})
- (v^\varepsilon_{s-} - v_{s-}), \hat{\Delta}_a Z \rangle
\leq \sum_{0<s \leq t} |u^\varepsilon_{s-} - u_{s-}|^{p-1} |\Phi \hat{\Delta}_a Z(v^\varepsilon_{s-}) - \Phi \hat{\Delta}_a Z(v_{s-}) - (v^\varepsilon_{s-} - v_{s-}) - \varepsilon (\hat{\Delta}_a Z v^\varepsilon_{s-} - \hat{\Delta}_a Z v_{s-}) |\Delta_a Z |\Delta_a Z |\Delta_a Z
\leq \varepsilon^2 C_{17} \left( \sum_{0<s \leq t} |u^\varepsilon_{s-} - u_{s-}|^{p-1} |v^\varepsilon_{s-} - v_{s-}||\Delta_a Z ||^2 \right)
\leq \varepsilon^2 C_{17} \left( \sum_{0<s \leq t} (|u^\varepsilon_{s-} - u_{s-}|^p + |v^\varepsilon_{s-} - v_{s-}|^p) ||\Delta_a Z ||^2 \right).
\]
We rewrite the last expression in terms of the compensated Poisson random measure \(\tilde{N}'\) and obtain

\[
\sum_{0 < s \leq t} (|u^\varepsilon_s - u_s|^p + |v^\varepsilon_s - v_s|^p) \|\Delta_s \tilde{Z}\|^2
\]

\[
= \int_0^t \int_{\mathbb{R}^s} (|u^\varepsilon_s - u_s|^p + |v^\varepsilon_s - v_s|^p) \|z\|^2 \tilde{N}'(d\sigma dz)
\]

\[
+ \int_0^t \int_{\|z\| > 1} (|u^\varepsilon_s - u_s|^p + |v^\varepsilon_s - v_s|^p) \|z\|^2 \nu'(dz)ds.
\]

The maximal inequality in [53] by Saint Loubert Bié for the exponent 1 yields

\[
\mathbb{E} \left[ \sup_{t \in [0,T]} \int_0^t \int_{\mathbb{R}^s} (|u^\varepsilon_s - u_s|^p + |v^\varepsilon_s - v_s|^p) \|z\|^2 \tilde{N}'(d\sigma dz) \right]
\]

\[
\leq C_{18} \int_0^T \int_{\mathbb{R}^s} \mathbb{E} \left[ |u^\varepsilon_s - u_s|^p + |v^\varepsilon_s - v_s|^p \right] \|z\|^2 \nu'(dz)ds
\]

\[
\leq C_{19} \int_{\mathbb{R}^s} \|z\|^2 \nu'(dz) \left( \int_0^T \mathbb{E} \left[ \sup_{[0,s]} |u^\varepsilon - u|^p \right] ds + \int_0^T \mathbb{E} \left[ |v^\varepsilon - v|^p \right] ds \right)
\]

\[
\leq C_{20} \int_0^T \mathbb{E} \left[ \sup_{t \in [0,t]} |u^\varepsilon - u|^p \right] ds + C_{21} \varepsilon^p T^{p+1}.
\]

The term [58] is treated obviously such that

\[
\mathbb{E} \left[ \sup_{[0,T]} |I_7| \right] \leq \varepsilon^2 C_{22} \int_{\mathbb{R}^s} \|z\|^2 \nu'(dz) \int_0^T \mathbb{E} \left[ \sup_{[0,s]} |u^\varepsilon - u|^p \right] ds + C_{23} \varepsilon^p T^{p+1}.
\]

**I8:** For I8 Lemma [3.1] statement 2), yields

\[
\sum_{0 < s \leq t} |u^\varepsilon_s - u_s|^{p-1} \left\langle u^\varepsilon_s - u_s, \Phi^{\varepsilon \tilde{R}_H} \Delta_s \tilde{Z}(v_s) - v_s - \varepsilon \tilde{R}_H(v_s) \Delta_s \tilde{Z} \right\rangle
\]

\[
\leq \sum_{0 < s \leq t} |u^\varepsilon_s - u_s|^{p-1} |\Phi^{\varepsilon \tilde{R}_H} \Delta_s \tilde{Z}(v_s) - v_s - \varepsilon \tilde{R}_H(v_s) \Delta_s \tilde{Z}|^2
\]

\[
\leq \varepsilon^2 C_{24} \sum_{0 < s \leq t} |u^\varepsilon_s - u_s|^{p-1} \|\Delta_s \tilde{Z}\|^2,
\]

such that Saint Loubert Bié’s maximal inequality with exponent 1 and elementary Young’s estimate for parameters \(\frac{p-1}{p}\) and \(p\) yield

\[
\mathbb{E} \left[ \sup_{[0,T]} |I_8| \right] \leq \varepsilon^2 C_{25} \int_{\mathbb{R}^s} \|z\|^2 \nu'(dz) \int_0^T \mathbb{E} \left[ \sup_{[0,s]} |u^\varepsilon - u|^{p-1} \right] ds
\]

\[
\leq \varepsilon C_{26} \int_0^T \mathbb{E} \left[ \sup_{[0,s]} |u^\varepsilon - u|^p \right] ds + C_{26} \varepsilon^p T.
\]

**3. Estimate of I5:**

\[
\int_0^T |u^\varepsilon_s - u_s|^{p-2} \left\langle u^\varepsilon_s - u_s, \varepsilon \tilde{R}_H(u_s, v_s) \right\rangle ds \leq C_{27} \int_0^T \varepsilon |u^\varepsilon_s - u_s|^{p-1} ds
\]

\[
\leq C_{27} \int_0^T \varepsilon |u^\varepsilon_s - u_s|^p ds + C_{27} \varepsilon^p T
\]
such that
\[
\mathbb{E}[\sup_{[0,T]} |I_5|] \leq \varepsilon C_{27} \int_0^T \mathbb{E} \left[ \sup_{[0,s]} |u^\varepsilon - u|^p \right] ds + C_{27} \varepsilon^p T. \tag{61}
\]

4. **Nonlinear comparison principle:** Taking the supremum and the expectation of the left-hand side of equation (17) and combining the estimates of \(\sum_{i=1}^8 \mathbb{E}[\sup_{[0,T]} |I_i|] \) given by (50), (51), (52), (55), (59), (60) and (61) we obtain a positive constant \(C_{28}\)
\[
\mathbb{E} \left[ \sup_{t \in [0,T]} |u_t^\varepsilon - u_t|^p \right] \leq C_{28} \left( \int_0^T \mathbb{E} \left[ \sup_{t \in [0,s]} |u^\varepsilon - u|^p \right] ds + \left( \int_0^T \mathbb{E} \left[ \sup_{t \in [0,s]} |u^\varepsilon - u|^p \right] ds \right)^{\frac{1}{2}} + \varepsilon^p T (1 + T) \right).
\]

For the concave invertible function \(G(x) = x + \sqrt{x}, \psi(t) := \mathbb{E} \left[ \sup_{t \in [0,T]} |u_t^\varepsilon - u_t|^p \right] \) and \(p(t) = \varepsilon^p T (1 + t)\) we have achieved the integral inequality
\[
\psi(t) \leq C_{28} G \left( \int_0^t \psi(s) ds \right) + C_{28} p(t).
\]

Hence the convex function \(G^{-1}(x) = (x + 1) - \sqrt{x} - \frac{1}{4} \) yields for \(C_{29} = \max \{2C_{28}, 4C_{28}^2\}\)
\[
G^{-1}(\psi(t)) \leq G^{-1} \left( C_{28} G \left( \int_0^t \psi(s) ds \right) + C_{28} p(t) \right)
\]
\[
\leq \frac{1}{2} G^{-1} \left( 2C_{28} G \left( \int_0^t \psi(s) ds \right) \right) + \frac{1}{2} G^{-1} (2C_{28} p(t))
\]
\[
\leq C_{29} \int_0^t \psi(s) ds + C_{29} G^{-1}(p(t)).
\]

Note that \(G^{-1}(x) \geq \kappa x\) for all \(\kappa \in (0, 1)\) and \(x \geq x_0\) for some \(x_0 > 0\), which we calculate as follows.
\[
(x_0 + \frac{1}{2}) - \sqrt{x_0 + \frac{1}{4}} = \kappa x_0 \quad \Leftrightarrow \quad x_0 = \frac{\kappa}{(1 - \kappa)^2}.
\]

The quadratic function \(qx^2\) satisfying that \(qx_0^2 = \kappa x_0\) has the prefactor
\[
qx_0^2 = \kappa x_0 \quad \Leftrightarrow \quad q = \frac{\kappa}{x_0} = (1 - \kappa)^2,
\]
and we obtain that any \(\kappa \in (0, 1)\) and \(x > 0\) satisfy \(G^{-1}(x) \geq \min \{(1 - \kappa)^2 x, \kappa x\}\). Hence
\[
\min \{(1 - \kappa)^2 \psi^2(t), \kappa \psi(t)\} \leq G^{-1}(\psi(t)) \leq C_{29} \int_0^t \psi(s) ds + C_{29} G^{-1}(p(t)).
\]

This implies for \(\psi(t) \geq \kappa x_0\) \(= \frac{\kappa^2}{(1 - \kappa)^2}\) the integral inequality
\[
\psi(t) \leq \frac{C_{29}}{\kappa} \int_{t_0}^t \psi(s) ds + \frac{C_{29}}{\kappa} G^{-1}(p(t))
\]
for \(t \geq t_0\), where \(t_0 = \inf \{t > 0 \mid \psi(t) = \kappa x_0\}\). Gronwall’s inequality yields for \(t \geq t_0\)
\[
\psi(t) \leq \frac{C_{29}}{\kappa} G^{-1}(p(t)) \exp \left( \left( t - t_0 \right) \frac{C_{29}}{\kappa} \right).
\]
Since the running supremum \( t \mapsto \psi(t) \) is monotonically increasing we also obtain that for any \( t \geq 0 \)
\[
\psi(t) \leq \frac{C_{29}}{\kappa} G^{-1}(p(t) \lor p(t_0)) \exp \left( \frac{C_{29}}{\kappa} t \right).
\]
Taking into account that \( G^{-1}(x) \leq x^2 \) for all \( x \geq 0 \) we obtain for \( \kappa = \frac{1}{2} \) and \( C_{30} = 2C_{29} \)
\[
\mathbb{E} \left[ \sup_{[0,T]} |u^\varepsilon - u|^p \right] \leq C_{30} \varepsilon^{2p}(1 + (T \lor t_0))^{2(\nu + 1)} \exp \left( C_{30} T \right).
\] (62)

IV. End of the proof of Proposition 3.1 Eventually Minkowski’s inequality, the Lipschitz estimate (20), the sum of the vertical (23) and the horizontal (45) estimate yield the desired result for \( \bar{C}_3 \leq \bar{C}_1 \lor \bar{C}_2 \) and constants \( k_1, k_2 > 0 \)
\[
\mathbb{E} \left[ \sup_{s \in [0,T]} |h(X^\varepsilon_t(x_0)) - h(X_t(x_0))|^p \right] \leq \bar{C}_3 \left( \varepsilon^p (1 + T^{2p + 1}) + C_{30} \varepsilon^{2p}(1 + (T \lor t_0))^{2(\nu + 1)} \exp \left( C_{30} T \right) \right)
\leq k_1 \varepsilon^p \exp \left( k_2 T \right).
\]
This finishes the proof.

4 The averaging error

For convenience we fix the following notation. Given \( h : M \to \mathbb{R}^n \) a globally Lipschitz continuous function and \( Q^h : V \to \mathbb{R}^n \) its average on the leaves defined as (9). For \( t \geq 0, x_0 \in M \) and \( \varepsilon \in (0,1) \) denote the error term
\[
\delta^h_{x_0}(\varepsilon, t) := \int_0^t h(X^\varepsilon_t(x_0)) - Q^h(\pi(X^\varepsilon_t(x_0))) ds.
\]

**Proposition 4.1** Let the assumptions of Proposition 3.1 be satisfied for a fixed \( p \geq 2 \). Then for any \( h : M \to \mathbb{R}^n \) globally Lipschitz continuous, \( \lambda \in (0,1) \) and \( x_0 \in M \) there are constants \( b_1 > 0 \) and \( \varepsilon_0 \in (0,1) \) such that for \( \varepsilon \in (0, \varepsilon_0] \) and \( T \in [0,1] \) we have
\[
\left( \mathbb{E} \left[ \sup_{s \in [0,T]} |\delta^h_{x_0}(\varepsilon, s)|^p \right] \right)^{\frac{1}{p}} \leq b_1 T \left[ \varepsilon^\lambda + \eta(cT | \ln \varepsilon |) \right],
\]
where \( c = c_{\lambda,p} \) is given by Corollary 3.2 and \( \eta \) is the ergodic rate of convergence given in equation (11) of Hypothesis 3.

**Proof of Proposition (4.1): The common part:** Fix \( x_0 \in M \). For \( \varepsilon \in (0,1) \) and \( T > 0 \) we define the partition
\[
t_0 = 0 < t_1^\varepsilon < \cdots < t_N^\varepsilon \leq \frac{T}{\varepsilon}
\]
with the following step size
\[
\Delta_\varepsilon := -cT \ln(\varepsilon),
\]
where $c > 0$ is given by Corollary [32]. The grid points are defined as

$$t_n^\varepsilon := n\Delta \varepsilon \quad \text{for} \quad 0 \leq n \leq N \varepsilon, \quad \varepsilon \in (0, 1) \quad \text{and the total number is} \quad N \varepsilon = \lfloor \frac{1}{c\varepsilon|\ln(\varepsilon)|} \rfloor + 1.$$

We rewrite the first summand of $\delta^h$ by

$$\int_0^t h(X_s^\varepsilon(x_0))ds = \varepsilon \int_0^\varepsilon h(X_s^\varepsilon(x_0))ds + \varepsilon \sum_{n=0}^{N \varepsilon - 1} \int_{t_n}^{t_{n+1}} h(X_s^\varepsilon(x_0))ds \leq \varepsilon \int_0^\varepsilon h(X_s^\varepsilon(x_0))ds + \varepsilon \sum_{n=0}^{N \varepsilon - 1} h(X_s^\varepsilon(x_0))ds.$$

We lighten notation and omit for convenience in the sequel the superscript $\varepsilon$ and $h$ as well as the initial value $x_0$ whenever possible. The triangle inequality yields

$$|\delta_{x_0}^h(\varepsilon, t)| \leq |A_1(t, \varepsilon)| + |A_2(t, \varepsilon)| + |A_3(t, \varepsilon)|, \quad (63)$$

where

$$A_1(t, \varepsilon) := \varepsilon \sum_{n=0}^{N \varepsilon - 1} \int_{t_n}^{t_{n+1}} [h(X_s^\varepsilon(x_0)) - h(X_{s-t_n}(X_{t_n}^\varepsilon(x_0)))] ds,$$

$$A_2(t, \varepsilon) := \varepsilon \sum_{n=0}^{N \varepsilon - 1} \int_{t_n}^{t_{n+1}} [h(X_{s-t_n}(X_{t_n}^\varepsilon(x_0))) - Q(\pi(X_{t_n}^\varepsilon(x_0)))] ds,$$

$$A_3(t, \varepsilon) := \varepsilon \Delta \varepsilon Q(\pi(X_{t_n}^\varepsilon(x_0))) - \int_0^{t_N} Q(\pi(X_s^\varepsilon(x_0))) ds.$$

The following three lemmas estimate the preceding terms. This being done the proof of Proposition [4.1] is finished.

**Lemma 4.2** For any $\lambda \in (0, 1)$ and $x_0 \in M$ there are $b_2 > 0$ and $\varepsilon_0 \in (0, 1)$ such that for any $\varepsilon \in (0, \varepsilon_0]$ and $T \in [0, 1]$

$$\left( \mathbb{E} \left[ \sup_{s \in [0,T]} |A_1(s, \varepsilon)|^p \right] \right)^{\frac{1}{p}} \leq b_2 \varepsilon^\lambda.$$

**Proof:** Fix $\lambda \in (0, 1)$. We apply the Markov property of $X_t$ and Hölder’s inequality in the time variable and the fact that by definition $N \varepsilon \Delta \varepsilon \leq T$

$$\mathbb{E} \left[ \sup_{s \in [0,T]} |A_1(s, \varepsilon)|^p \right]^{\frac{1}{p}} = \varepsilon \sum_{n=0}^{N \varepsilon - 1} \mathbb{E} \left[ \left| \int_{t_n}^{t_{n+1}} [h(X_s^\varepsilon(x_0)) - h(X_{s-t_n}(X_{t_n}^\varepsilon(x_0)))] ds \right|^p \bigg| \mathcal{F}_{t_n} \right]^{\frac{1}{p}} = \varepsilon \sum_{n=0}^{N \varepsilon - 1} \mathbb{E} \left[ \left| \int_{t_n}^{t_{n+1}} [h(X_s^\varepsilon(x_0)) - h(X_{s-t_n}(y))] ds \right|^p \bigg| y = X_{t_n}^\varepsilon(x_0) \right]^{\frac{1}{p}} \leq \varepsilon N \varepsilon \Delta \varepsilon \mathbb{E} \left[ \left| \sup_{s \in [0, t_n]} [h(X_s^\varepsilon(X_{t_n}^\varepsilon(x_0)) - h(X_{s-t_n}(x_0))]^{p} \bigg| y = X_{t_n}^\varepsilon(x_0) \right]^{\frac{1}{p}}$$
\[ \leq T \max_{n \in \{1, \ldots, N_t\}} \mathbb{E} \left[ \mathbb{E} \left[ \sup_{s \in [0, t_1]} \left| h(X^x_s(X^x_{t_1}(x_0))) - h(X_s(X^x_{t_1}(x_0))) \right|^p \mid y = X^y_{t_1}(x_0) \right] \right]^{\frac{1}{p}} \]

\[ \leq T \max_{n \in \{1, \ldots, N_t\}} \mathbb{E} [k_3(\text{diam}(LX^x_{t_1}(x_0)))] \varepsilon^\lambda. \]

The last estimate is an application of Corollary 3.2, where we have found that the value \( k_3 = k_3(d_{v_0}) \), \( d_{v_0} = \text{diam}(Lv_0) \) from (19) is a polynomial in \( d_{v_0} \) of order \( p \) with positive coefficients. First we estimate the term involving \( k_3 \). Since by Hypothesis 1 the mapping \( v_0 \mapsto d_{v_0} \) is Lipschitz continuous there is a Lipschitz constant \( \ell' \) on \( U \), which yields the constants \( \alpha_1 \) and \( \alpha_2 \) such that almost surely

\[ k_3(d_{X^x_{t_1}}) \leq k_3(d_{v_0} + \ell' \text{dist}(Lx_0, LX^x_{t_1}(x_0))) \]

\[ \leq \alpha_1(k_3(d_{v_0}) + 1) + \alpha_2 \text{dist}(Lx_0, LX^x_{t_1}(x_0))^p. \quad (64) \]

The next step consists of the estimate of the term \( \mathbb{E} [\text{dist}(Lx_0, LX^x_{t_1}(x_0))^p] \). Using \( X^x_t(x_0) \in L_{x_0} \) for all \( t \geq 0 \) Lemma 3.3 ensures the existence of the positive constant \( \bar{C}_1(d_{v_0}) \) which is an affine function in \( d_{v_0} \) such that for \( k = C_0 \bar{C}_1(d_{v_0}) \)

\[ \mathbb{E} \left[ \sup_{t \in [0, t_1]} \text{dist}(L_{x_0}, LX^x_{t_1}(x_0))^p \right]^{\frac{1}{p}} \leq C_0 \mathbb{E} \left[ \sup_{s \in [0, t_1]} |v^x_s(x_0) - v_s(x_0)|^p \right]^{\frac{1}{p}} \leq k \varepsilon^\lambda, \quad (65) \]

where \( C_0 \) is the Lipschitz constant of the local coordinates \( \varphi \) with Lipschitz constant \( \ell'' \). We obtain

\[ \mathbb{E} \left[ \sup_{t \in [0, t_2]} \text{dist}(L_{v_0}, LX^x_{t_1}(x_0))^p \right]^{\frac{1}{p}} \leq \mathbb{E} \left[ \sup_{t \in [0, t_1]} \text{dist}(L_{v_0}, LX^x_{t_1}(x_0))^p \right]^{\frac{1}{p}} + \mathbb{E} \mathbb{E} \left[ \sup_{t \in [t_1, t_2]} \text{dist}(L_{v_0}, LX^x_{t_1}(x_0))^p \mid F_{t_1} \right]^{\frac{1}{p}}. \]

The first term on the right-hand side obeys (65), the second one can be calculated recursively

\[ \mathbb{E} \mathbb{E} \left[ \sup_{t \in [t_1, t_2]} \text{dist}(L_{v_0}, LX^x_{t_1}(x_0))^p \mid F_{t_1} \right]^{\frac{1}{p}} \leq \mathbb{E} \mathbb{E} \left[ \sup_{t \in [0, t_1]} \text{dist}(L_{x_0}, LX^x_{t_1}(y))^p \mid y = X^y_{t_1}(x_0) \right]^{\frac{1}{p}} \]

\[ \leq C_0 \mathbb{E} \mathbb{E} \left[ \bar{C}_1(d_{X^x_{t_1}(x_0)}) \right] \varepsilon^\lambda \]

\[ \leq (k + \ell'' \varepsilon^\lambda) \varepsilon^\lambda. \]

Hence

\[ \mathbb{E} \left[ \sup_{t \in [0, t_2]} \text{dist}(L_{v_0}, LX^x_{t_1}(x_0))^p \right]^{\frac{1}{p}} \leq \frac{k}{\ell''} (2\ell'' \varepsilon^\lambda + (\ell'' \varepsilon^\lambda)^2). \]

We argue by induction

\[ \mathbb{E} \left[ \sup_{t \in [0, t_{n+1}]} \text{dist}(L_{v_0}, LX^x_{t_1}(x_0))^p \right]^{\frac{1}{p}} \]

\[ \leq \mathbb{E} \left[ \sup_{t \in [0, t_n]} \text{dist}(L_{v_0}, LX^x_{t_1}(x_0))^p \right]^{\frac{1}{p}} + \mathbb{E} \mathbb{E} \left[ \sup_{t \in [0, t_1]} \text{dist}(L_{v_0}, LX^x_{t_1}(y))^p \mid y = X^y_{t_1}(x_0) \right]^{\frac{1}{p}} \]

\[ \leq \frac{k}{\ell''} \left( \sum_{i=0}^{n} \binom{n}{i} (\ell'' \varepsilon^\lambda)^i - 1 \right) + C_0 \mathbb{E} \mathbb{E} \left[ \bar{C}_1(d_{X^x_{t_1}(x_0)}) \right] \varepsilon^\lambda. \]

We continue with the second term

\[ C_0 \mathbb{E} \mathbb{E} \left[ \bar{C}_1(d_{X^x_{t_1}(x_0)}) \right] \varepsilon^\lambda \leq C_0 \bar{C}_1(d_{v_0}) \varepsilon^\lambda + \ell'' \mathbb{E} \left[ \sup_{t \in [0, t_1]} \text{dist}(L_{v_0}, LX^x_{t_1}(x_0))^p \right]^{\frac{1}{p}} \varepsilon^\lambda \]

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Lemma 4.4. Let \(\eta\) be the rate of convergence defined in (11). For the process \(A_2\) in inequality (63), \(T > 0\) fixed and \(\lambda \in (0, 1)\) here is \(c = c(\lambda) \in (0, 1)\) for which there are \(\varepsilon_0 \in (0, 1)\) and \(b_3 > 0\) such that for any \(T \in [0, 1]\) and \(\varepsilon \in (0, \varepsilon_0]\) we have

\[
\left( \mathbb{E} \left[ \sup_{s \in [0,T]} |A_2(s, \varepsilon)|^p \right] \right)^{\frac{1}{p}} \leq b_3 T \eta(cT \ln |\varepsilon|).
\]

The proof is virtually identical to the proof of Lemma 3.3 in [25] in the purely Brownian case.
Proof: We calculate
\[
|A_3(T, \varepsilon)| = \left| \sum_{n=0}^{N_\varepsilon - 1} \varepsilon \Delta \xi Q(\pi(X_{\varepsilon t_n})) - \int_0^{N_\varepsilon \Delta \xi} Q(\pi(X_{\varepsilon t})) \, ds \right|
\]
\[
\leq \varepsilon \sum_{n=0}^{N_\varepsilon - 1} \Delta \xi \sup_{\varepsilon t_n \leq s < \varepsilon t_{n+1}} |Q(\pi(X_{\varepsilon s})) - Q(\pi(X_{\varepsilon t}))|.
\]
\[
\leq \varepsilon \Delta \xi C \sum_{n=0}^{N_\varepsilon - 1} \sup_{t_n \leq s < t_{n+1}} |v_{\varepsilon s}^\pi - v_{\varepsilon t_n}^\pi|.
\] (66)

Minkowski’s inequality, the Markov property, Lemma 3.3 with \( C_1 \) in \( d_{v_0} \) and estimate (65) lead to
\[
E\left[ \sup_{s \in [0, T]} |A_3(s, \varepsilon)|^p \right]^{\frac{1}{p}} \leq TC \max_{n \in \{1, \ldots, N_\varepsilon\}} E\left[ \sup_{t_n \leq s < t_{n+1}} |v_{s-t_n}^\varepsilon(y) - v_0^\varepsilon(y)|^p \mid y = X_{t_n}(x_0) \right]^{\frac{1}{p}}
\]
\[
\leq TC \max_{n \in \{1, \ldots, N_\varepsilon\}} E\left[ \sup_{t_0 \leq s < t_1} |v_s^\varepsilon(y) - v_0^\varepsilon(y)|^p \mid y = X_{t_0}(x_0) \right]^{\frac{1}{p}}
\]
\[
\leq TC E\left[ C_1(diam(L_{X_{t_n}(x_0)})) \right] \varepsilon^\lambda
\]
\[
\leq TCC_1(d_{v_0})(\varepsilon^\lambda + \ell N_\varepsilon \varepsilon^{2\lambda})
\]
\[
\leq Tb_4 \varepsilon^{\lambda}
\]
for \( \lambda \in (0, 2\lambda - 1) \).

(Proof of Proposition 4.1) Combining Minkowski’s inequality with Lemma 4.2(4.4) yields for any \( \lambda \in (0, 1) \) and \( p \geq 2 \) constants \( c_{\lambda, p}, \varepsilon_0, k_0 \in (0, 1) \) and \( b_1 > 0 \) such that for any \( T \in [0, 1] \) satisfying \( \varepsilon_0 T < k_0 \varepsilon \in (0, \varepsilon_0] \) implies
\[
\left( E\left[ \sup_{t \in [0, T]} |\delta_{x_0}^h(\varepsilon, t)|^p \right] \right)^{\frac{1}{p}} \leq b_1 T \left( \varepsilon^\lambda + \eta(cT |\ln(\varepsilon)|) \right).
\]

5 Proof of the main result

We keep the notation of the Proof of Proposition 4.1. By the change of variable formula for canonical Marcus integrals, [11] Proposition 4.2, we may rewrite [22]
\[
v_{\varepsilon t}^\pi = \int_0^t \mathcal{R}_V(u_{\varepsilon z}^\pi, v_{\varepsilon z}^\pi) \, ds + \int_0^t (\mathcal{R}_V(v_{\varepsilon z}^\pi) - \mathcal{R}_V(w(s))) \, d\tilde{Z}_s.
\]
Since equation (10) tells us that \( w(t) = \int_0^t Q\bar{R}_V(w(s)) \, ds + \int_0^t \bar{R}_V(w(s)) \, d\tilde{Z}_s \), we obtain
\[
v_{\varepsilon t}^\pi - w(t)
\]
\[
= \int_0^t \mathcal{R}_V(u_{\varepsilon z}^\pi, v_{\varepsilon z}^\pi) - Q\bar{R}_V(w(s)) \, ds + \int_0^t (\mathcal{R}_V(v_{\varepsilon z}^\pi) - \mathcal{R}_V(w(s))) \, d\tilde{Z}_s
\]
\[
= \delta_{x_0}^h(\varepsilon, t) + \int_0^t Q\bar{R}_V(u_{\varepsilon z}^\pi, v_{\varepsilon z}^\pi) - Q\bar{R}_V(w(s)) \, ds + \int_0^t (\mathcal{R}_V(v_{\varepsilon z}^\pi) - \mathcal{R}_V(w(s))) \, d\tilde{Z}_s
\]
\[
+ \sum_{0 < s \leq t} \left( \Phi \mathcal{R}_V \Delta_{z, w}(v_{\varepsilon z}^\pi, w(s)) - \Phi \mathcal{R}_V \Delta_{z, w}(w(s)) - (v_{\varepsilon z}^\pi - w(s)) - (\mathcal{R}_V(v_{\varepsilon z}^\pi) - \mathcal{R}_V(w(s))) \right) \Delta_{s, \tilde{Z}}
\]
\[ = \delta_{\tilde{X}}(\varepsilon, t) + O_1 + O_2 + O_3. \]

Now, for \( T \in [0, 1] \) we obtain with the help of Jensen’s inequality

\[
\mathbb{E} \left[ \sup_{t \in [0, T]} |O_1|^p \right] \leq a_1 T^{p-1} \int_0^T \mathbb{E} \left[ \sup_{t \in [0, s]} |v^x_\varepsilon - w(t)|^p \right] ds.
\]

Kunita’s maximal inequality \([42]\) yields a positive constant \( a_2 = a_2(p) \) such that for the constant

\[
a_3 = a_2 \tilde{\ell}(\int_{\mathbb{R}^r} \|z\|^2 \nu'(dz)) + 2 \int_{\mathbb{R}^r} \|z\|^p \nu'(dz) \]

we have

\[
\mathbb{E} \left[ \sup_{t \in [0, T]} |O_2|^p \right] \leq \mathbb{E} \left[ \sup_{t \in [0, T]} \left| \int_0^t \left( \tilde{\eta}_V(v^x_\varepsilon) - \tilde{\eta}_V(w(s)) \right) z \tilde{\gamma}(dsdz)|^p \right] \\
+ \mathbb{E} \left[ \sup_{t \in [0, T]} \left| \int_0^t \left( \tilde{\eta}_V(v^x_\varepsilon) - \tilde{\eta}_V(w(s)) \right) z \nu'(dz) \right|^p \right] \\
\leq a_2 \mathbb{E} \left[ \int_0^T \left( \tilde{\eta}_V(v^x_\varepsilon) - \tilde{\eta}_V(w(s)) \right) z \nu'(dz) ds \right] \\
+ a_2 \mathbb{E} \left[ \left( \int_0^T \left( \tilde{\eta}_V(v^x_\varepsilon) - \tilde{\eta}_V(w(s)) \right) z \nu'(dz) \right)^p \right] \\
+ \int \|z\|^p \nu'(dz) \mathbb{E} \left[ \int_0^T \left| \tilde{\eta}_V(v^x_\varepsilon) - \tilde{\eta}_V(w(s)) \right|^p ds \right] \\
\leq a_3 \int_0^T \mathbb{E} \left[ \sup_{s \in [0, t]} |v^x_\varepsilon - w(s)|^p \right] ds.
\]

Finally the Lipschitz continuity of the vector fields \( \tilde{\eta}_V \) and \( (D\tilde{\eta}_V)\tilde{\eta}_V \) and Lemma \([3,1]\) provide a constant \( a_4 > 0 \)

\[
|O_3|^p \leq \left| \sum_{0 < s \leq T} \Phi \tilde{\eta}_V \Delta_s \tilde{Z}(v^x_\varepsilon) - \Phi \tilde{\eta}_V \Delta_s \tilde{Z}(w(s)) \right|^p \\
- \left( \sum_{0 < s \leq T} \left| (D\tilde{\eta}_V(v^x_\varepsilon) \Delta_s \tilde{Z}) \tilde{\eta}_V(v^x_\varepsilon) \Delta_s \tilde{Z} - (D\tilde{\eta}_V(w(s)) \Delta_s \tilde{Z}) \tilde{\eta}_V(w(s)) \Delta_s \tilde{Z} \right| \right)^p \\
\leq \left( \frac{1}{2} \sum_{0 < s \leq T} \left| (D\tilde{\eta}_V(v^x_\varepsilon) \Delta_s \tilde{Z}) \tilde{\eta}_V(v^x_\varepsilon) \Delta_s \tilde{Z} - (D\tilde{\eta}_V(w(s)) \Delta_s \tilde{Z}) \tilde{\eta}_V(w(s)) \Delta_s \tilde{Z} \right| \right)^p \\
\leq \left( a_4 \sum_{0 < s \leq T} \left| v^x_\varepsilon - w(s) \right| \| \Delta_s \tilde{Z} \|^2 \right)^p \\
\leq \left( a_4 \right)^p \left( \sum_{0 < s \leq T} \left| v^x_\varepsilon - w(s) \right| \| \Delta_s \tilde{Z} \|^2 \right)^p. \tag{67}
\]

Switching to the representation in terms of the Poisson random measure

\[
\sum_{0 < s \leq T} \left| v^x_\varepsilon - w(s) \right| \| \Delta_s \tilde{Z} \|^2 \\
= \int_0^T \int_{\mathbb{R}^r} \left| v^x_\varepsilon - w(s) \right| \| z \|^2 \tilde{\gamma}(dsdz) + \int_0^T \int_{\|z\| > 1} \left| v^x_\varepsilon - w(s) \right| \| z \|^2 \nu'(dz) \, ds. \tag{68}
\]
we obtain
\[ E \left[ \sup_{t \in [0,T]} |O_3|^p \right] \leq 2^{p-1}(a_4)^p \left( E \left[ \sup_{t \in [0,T]} \left| \int_0^t \int_{\mathbb{R}^r} \frac{v^\varepsilon_x}{x} - w(s) \| z \|^2 \tilde{N}'(dsdz) \right|^p \right] 
+ E \left[ \left( \int_0^T \int_{\mathbb{R}^r} |v^\varepsilon_x - w(s)| \| z \|^2 \nu'(dz) ds \right)^p \right] \right) \]
\[ =: O_4 + O_5. \]  

We apply Kunita’s maximal inequality \[2\] which yields a constant \( a_5 = a_5(p) > 0 \) and Jensen’s inequality. For \( a_6 = 2^{p-1}(a_4a_5)^p \) and \( a_7 = a_6(\int_{\mathbb{R}^r} \| z \|^2 \nu'(dz) + (\int_{\mathbb{R}^r} \| z \|^2 \nu'(dz))^2)^p \) we hence obtain
\[ O_4 \leq a_6 \left( E \left[ \int_0^t \int_{\mathbb{R}^r} |v^\varepsilon_x - w(s)|^p \| z \|^2 \nu'(dz) ds \right] \right) \]
\[ + E \left[ \left( \int_0^T \int_{\mathbb{R}^r} |v^\varepsilon_x - w(s)|^2 \| z \|^4 \nu'(dz) ds \right)^{\frac{p}{2}} \right] \]
\[ = a_6 \left\{ \int_{\mathbb{R}^r} \| z \|^2 \nu'(dz) E \left[ \int_0^t |v^\varepsilon_x - w(s)|^p ds \right] \right. \]
\[ + \left( \int_{\mathbb{R}^r} \| z \|^4 \nu'(dz) \right)^{\frac{p}{2}} E \left[ \left( \int_0^t |v^\varepsilon_x - w(s)|^2 ds \right)^{\frac{p}{2}} \right] \}
\[ \leq a_7(T)^{\frac{p}{2}-1} + 1 \int_0^T E \left[ \sup_{s \in [0,t]} |v^\varepsilon_x - w(s)|^p \right] dt. \]  

The term \( O_5 \) follows straightforward. Summing up \( \sum_{i=1}^5 E[\sup_{t \in [0,T]} |O_i|^p] \) we obtain a constant \( a_8 \) such that for \( T \in [0,1] \)
\[ E \left[ \sup_{t \in [0,T]} |v^\varepsilon_x - w(t)|^p \right] \leq a_8 \int_0^T E \left[ \sup_{s \in [0,t]} |v^\varepsilon_x - w(s)|^p \right] \] \[ dt + E \left[ \delta^{\tilde{N} \varepsilon}_{x_0}(T, \varepsilon)^p \right]. \]

The standard nonautonomous version of Gronwall’s lemma implies a constant \( a_9 > 0 \) such that for \( T \in [0,1] \)
\[ E \left[ \sup_{s \in [0,T]} |v^\varepsilon_x - w(s-)|^p \right] \leq E \left[ \delta^{\tilde{N} \varepsilon}_{x_0}(T, \varepsilon)^p \right] \exp(a_9T) \leq a_{10} E \left[ \delta^{\tilde{N} \varepsilon}_{x_0}(T, \varepsilon)^p \right]. \]

Finally an application of Proposition 4.1 for \( h = \delta \varepsilon \) finishes the proof of Theorem 2.2.

6 Appendix

6.1 Detailed calculations of the example

In the sequel we verify [16]. Keeping in mind \( Q_{\pi r,K}(x_0) = \frac{e^{d}-r}{2} \) we obtain
\[ E \left[ \frac{1}{t} \int_0^t \pi_r K(X_s) ds \right] \]
\[ = \frac{r}{t} \int_0^t E \left[ (a \sin^2(Z_s) + d \cos^2(Z_s) + (b + c) \sin(Z_s) \cos(Z_s)) \right] ds \]
\[ = \frac{r}{t} at + \frac{r}{t} \int_0^t E \left[ ((d - a) \cos^2(Z_s) + (b + c) \sin(Z_s) \cos(Z_s)) \right] ds \]  

\[ (71) \]
We apply the elementary identities

\[ E(t) = r a + \frac{r}{t} (d - a) \int_0^t E_1(z_s) ds + (b + c) \int_0^t E_2(z_s) cos(z_s) ds \]

This shows the result for \( p = 1 \). For \( p = 2 \) we continue

\[ E_t = \frac{1}{t} \int_0^t E_1(z_s) ds - \frac{a + d}{2} t \]

We calculate directly

\[ E_t = \frac{1}{t} \int_0^t E_2(z_s) cos(z_s) ds \]

We apply the elementary identities

\[ \cos(x)^2 \cos^2(y) = \frac{1}{8} \left( \cos(2(x-y)) + \cos(2(x+y)) + 2\cos(2x) + 2\cos(2y) + 2 \right) \]
\[
\sin(x)^2 \sin^2(y) = \frac{1}{8} \left( \cos(2(x-y)) + \cos(2(x+y)) - 2 \cos(2x) - 2 \cos(2y) + 2 \right) \\
\sin(x)^2 \cos^2(y) = -\frac{1}{8} \left( \cos(2(x-y)) + \cos(2(x+y)) + 2 \cos(2x) - 2 \cos(2y) - 2 \right) \\
\sin(x)^2 \sin(y) \cos(y) = \frac{1}{8} \left( \sin(2(x-y)) + \sin(2(x+y)) + 2 \sin(y) + 2 \right) \\
\cos(x)^2 \sin(y) \cos(y) = -\frac{1}{8} \left( \sin(2(x-y)) + \sin(2(x+y)) - 2 \sin(y) + 2 \right)
\]

and obtain

\[
E_t = \frac{1}{t^2} \int_0^t \int_0^t \frac{a^2}{8} \left\{ \mathbb{R} \mathbb{E} \left[ e^{i2(Z_s - Z_\sigma)} \right] + \mathbb{R} \mathbb{E} \left[ e^{i2(Z_s + Z_\sigma)} \right] - 2 \mathbb{R} \mathbb{E} \left[ e^{i2Z_s} \right] - 2 \mathbb{R} \mathbb{E} \left[ e^{i2Z_\sigma} \right] + 2 \right\} \\
+ \frac{d^2}{8} \left\{ \mathbb{R} \mathbb{E} \left[ e^{i2(Z_s - Z_\sigma)} \right] + \mathbb{R} \mathbb{E} \left[ e^{i2(Z_s + Z_\sigma)} \right] + 2 \mathbb{R} \mathbb{E} \left[ e^{i2Z_s} \right] + 2 \mathbb{R} \mathbb{E} \left[ e^{i2Z_\sigma} \right] + 2 \right\} \\
+ \frac{(b+c)^2}{8} \left\{ \mathbb{R} \mathbb{E} \left[ e^{i2(Z_s - Z_\sigma)} \right] - \mathbb{R} \mathbb{E} \left[ e^{i2(Z_s + Z_\sigma)} \right] \\
- \frac{ad}{4} \left\{ \mathbb{R} \mathbb{E} \left[ e^{i2(Z_s - Z_\sigma)} \right] + \mathbb{R} \mathbb{E} \left[ e^{i2(Z_s + Z_\sigma)} \right] + 2 \mathbb{R} \mathbb{E} \left[ e^{i2Z_s} \right] - 2 \mathbb{R} \mathbb{E} \left[ e^{i2Z_\sigma} \right] - 2 \right\} \\
\right. \\
\left. + \frac{a(b+c)}{4} \left\{ \mathbb{R} \mathbb{E} \left[ e^{i2(Z_s - Z_\sigma)} \right] - \mathbb{R} \mathbb{E} \left[ e^{i2(Z_s + Z_\sigma)} \right] - 2 \mathbb{R} \mathbb{E} \left[ e^{i2Z_s} \right] + 2 \mathbb{R} \mathbb{E} \left[ e^{i2Z_\sigma} \right] \right\} \right) d\sigma ds.
\]

Since

\[
\mathbb{E} \left[ e^{i2(Z_s \pm Z_\sigma)} \right] = \mathbb{E} \left[ e^{i2Z_s \pm \theta} \right] = e^{-(s \pm \sigma) \Psi(2)} \quad \text{and} \quad \mathbb{E} \left[ e^{i4Z_s} \right] = e^{-4\Psi(4)}
\]

for \( s \geq \sigma \) we get

\[
E_t = \frac{1}{t^2} \int_0^t \int_0^t \frac{a^2}{8} \left\{ \exp(-|s - \sigma|R \Psi(2)) \cos(|s - \sigma|R \Psi(2)) + \exp(-(s + \sigma)R \Psi(2)) \cos((s + \sigma)R \Psi(2)) \\
- 2 \exp(-sR \Psi(2)) \cos(sR \Psi(2)) - 2 \exp(-\sigma R \Psi(2)) \cos(\sigma R \Psi(2)) + 2 \right\} \\
+ \frac{d^2}{8} \left\{ \exp(-|s - \sigma|R \Psi(2)) \cos(|s - \sigma|R \Psi(2)) + \exp(-(s + \sigma)R \Psi(2)) \cos((s + \sigma)R \Psi(2)) \\
+ 2 \exp(-sR \Psi(2)) \cos(sR \Psi(2)) + 2 \exp(-\sigma R \Psi(2)) \cos(\sigma R \Psi(2)) + 2 \right\} \\
- \frac{ad}{4} \left\{ \exp(-|s - \sigma|R \Psi(2)) \cos(|s - \sigma|R \Psi(2)) + \exp(-(s + \sigma)R \Psi(2)) \cos((s + \sigma)R \Psi(2)) \\
+ 2 \exp(-sR \Psi(2)) \cos(sR \Psi(2)) - 2 \exp(-\sigma R \Psi(2)) \cos(\sigma R \Psi(2)) - 2 \right\} \\
+ \frac{(b+c)^2}{8} \left\{ \exp(-|s - \sigma|R \Psi(2)) \cos(|s - \sigma|R \Psi(2)) - \exp(-(s + \sigma)R \Psi(2)) \cos((s + \sigma)R \Psi(2)) \\
\right. \\
\left. + \frac{a(b+c)}{4} \left\{ \exp(-|s - \sigma|R \Psi(2)) \sin(|s - \sigma|R \Psi(2)) + \exp((s + \sigma)R \Psi(2)) \sin((s + \sigma)R \Psi(2)) \\
- 2 \exp(-\sigma R \Psi(2)) \sin(\sigma R \Psi(2)) \right\} \right. \\
\left. + \frac{d(b+c)}{4} \left\{ \exp(-|s - \sigma|R \Psi(2)) \sin(|s - \sigma|R \Psi(2)) - \exp(-(s + \sigma)R \Psi(2)) \sin((s + \sigma)R \Psi(2)) \\
- 2 \exp(-\sigma R \Psi(2)) \sin(\sigma R \Psi(2)) \right\} \right) d\sigma ds.
\]
Since all integrals over the exponential terms converge, their contribution in the preceding sum vanishes as $t \to \infty$ as these are divided by $t^2$ and only the constants under the integrals survive. Therefore

$$E_t \overset{t \to \infty}{\longrightarrow} \frac{a^2}{4} + \frac{d^2}{4} + \frac{ad}{2} = \frac{(a + d)^2}{4}$$

such that

$$\mathbb{E}\left[ \frac{1}{t} \int_0^t \pi_r K(Z_s) ds - \frac{a + d}{2} r \right]^2 \overset{t \to \infty}{\longrightarrow} \frac{(a + d)^2}{4} - \frac{(a + d)^2}{2} + \frac{(a + d)^2}{4} = 0.$$

**Acknowledgement**

The authors would like to thank the two anonymous referees for their thorough work, which has considerably improved both the presentation and the content of this article. Both authors thank Prof. Sylvie Roelly and the probability group of Universität Potsdam for the hospitality during the one year stay of Paulo-Henrique da Costa in the framework of a postdoctoral 12 months stay in 2014 funded by the Ciência sem fronteiras program of the Brazilian government by the grant CsF-CAPES/11786-13-2. The authors express their gratitude to the International Research Training Group Berlin - São Paulo: Dynamical Phenomena of Complex Networks for various infrastructure support and Prof. Paulo Ruffino for many inspiring discussions. The first author thanks Universidad de los Andes, School of Sciences, for the FAPA grant “Stochastic dynamics of Lévy driven systems”, which supported a visit of the second author.

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