Chaotic lensed billiards

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Abstract

Lensed billiards are an extension of the notion of billiard dynamical systems obtained by adding a potential function of the form $C \mathbb{1}_\mathcal{A}$, where $C$ is a real-valued constant and $\mathbb{1}_\mathcal{A}$ is the indicator function of an open subset $\mathcal{A}$ of the billiard table whose boundaries (of $\mathcal{A}$ and the table) are piecewise smooth. Trajectories are polygonal lines that undergo either reflection or refraction at the boundary of $\mathcal{A}$ depending on the angle of incidence. Our main focus is to explore how the dynamical properties of these models depend on the potential parameter $C$ using a number of families of examples. In particular, we explore numerically the Lyapunov exponents for these parametric families and highlight the more salient common properties that distinguish them from standard billiard systems. We further justify some of these properties by characterizing lensed billiards in terms of switching dynamics between two open (standard) billiard subsystems and obtaining mean values associated to orbit sojourn in each subsystem.

1. Introduction

Mathematical billiards, particularly in dimension 2, are dynamical systems of a geometric nature that are relatively easy to define and, at the same time, exhibit a wide range of dynamic properties, making them useful models for more complicated systems. For this reason they have for many decades figured prominently in the development of the modern theory of dynamical systems and ergodic theory. Research on chaotic billiards in dimension 2, in particular, has by now attained a high degree of technical sophistication. This, in our opinion, justifies an effort to look for generalizations or extensions of the concept of billiard systems that can point to new directions of research without compromising too much on the qualities that make them attractive model systems.

Lensed billiards may be defined as standard billiard systems to which are added piecewise constant mechanical potential functions of the form $V = C \mathbb{1}_\mathcal{A}$ where $\mathbb{1}_\mathcal{A}$ is the indicator function of a subset $\mathcal{A}$ of the billiard domain. We may think of $\mathcal{A}$ as a scatterer that reflects billiard trajectories that collide with it at sufficiently shallow angles relative to a critical angle to be defined shortly, but allows others to pass through and...
undergo a *refraction*, similar to a light ray crossing the interface surface separating two optical media with different refractive indices.

The central new factor to account for is the effect of the potential parameter $C$—the value of the potential function on $\mathcal{A}$—on the dynamical behavior of the system. After laying out basic definitions and the most general properties of lensed billiards, the paper focuses on the numerical determination of Lyapunov exponents for a few parametric families of examples, identifies a number of properties that appear to be common for these systems, and proposes a framework of analysis for explaining these common features.

Before embarking on a systematic study, let us get a first impression of what these systems look like. Figure 1 shows two lensed variants of the classical Sinai semi-dispersing and Bunimovich stadium billiards. The shaded regions, let us denote them by $\mathcal{A}$, indicate where the potential function $V$ is non-zero. In both cases, we have set the value of $C$ to $-1$. The negative potential causes trajectories going into $\mathcal{A}$ to deflect as if entering an optical medium with a higher refractive index. (As a mechanical system, the speed of the point particle naturally increases due to conservation of energy—the opposite of what would happen to a light ray.) Therefore, the Sinai scatterer behaves as a focusing lens. The figure on the left shows the typical focusing-defocusing behavior produced by such a lens. On the other hand, trajectories leaving $\mathcal{A}$ on the Bunimovich-type system on the right-hand side exhibit dispersing behavior after standard focusing inside $\mathcal{A}$ by the circular part of the boundary. This shows how the choice of values of $V$ can alter the standard mechanisms leading to hyperbolicity in billiard systems. If the value of the potential in $\mathcal{A}$ is positive then these behaviors are reversed. It is easy to verify that if $C < 0$, the system on the left is not ergodic due to trapped trajectories inside the circular lens, and if $C > 0$, the system on the right is not ergodic due to the existence of a positive measure set of initial conditions initiating in the complement of $\mathcal{A}$ that cannot enter $\mathcal{A}$.

(We consider initial velocities with sufficient energy to allow the billiard particle to cross the potential barrier, but when the angle of collision with the vertical line of discontinuity of $V$ is sufficiently small, the particle is nevertheless forced to reflect.)

It is important to keep in mind the distinction between the billiard flow and the billiard

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{On the left, a lensed Sinai-type billiard showing focusing and defocusing of trajectories. On the right, a lensed Bunimovich-like billiard showing initially focusing followed by dispersing trajectories. In both cases, the potential is $-1$ on the shaded region and $0$ outside. A parallel beam of trajectories emanates from the vertical right wall, stopping at the first collision with the boundary of the unshaded region of the billiard domain.}
\end{figure}
map dynamics, due to the fact that particle speed is no longer constant. Figure 2 shows the difference in somewhat extreme fashion. As the potential constant \( C \) on the semi-disc \( \mathcal{A} \) becomes negative with large \(|C|\), the number of steps of the billiard trajectories inside \( \mathcal{A} \) increases, roughly proportionally to \( \sqrt{|C|} \) (a more precise statement will be given later), suggesting that trajectories of the billiard map become increasingly trapped in that deep potential well. However, the speed of the particle in that region is also proportional to \( \sqrt{|C|} \), so the amount of time the billiard flow remains in \( \mathcal{A} \) is expected to remain bounded.

![Sample trajectories](image)

Figure 2: Sample trajectories (of equal number of steps) for the Lensed Bunimovich billiard. The values \( C \) of the potential on the semi-disc, from top to bottom, left to right: 0, \(-1\), \(-10\), \(-100\), \(-1000\), \(-10000\). Numerical approximation of the positive mean Lyapunov exponent (for the billiard flow) gave the values: 1.00 \pm 0.02 for \( C = 0 \) (top-left) and 4.46 \pm 0.14 for \( C = -10000 \) (bottom-right). See also Figure 12.

Naturally, the systems we are calling lensed billiards can be viewed as a special case of soft billiards, or more broadly Hamiltonian systems under the influence of a potential, for which there is a rich existing literature that we now discuss. It should be emphasized that the present work takes on related systems from a number of new perspectives. A central focus of soft billiards has been for the case of smooth potentials and for the case of a Sinai billiard table consisting of the torus with circular scatterers. Here, the discontinuous potentials of lensed billiards result in impulse-like forces at discrete times and our examples venture beyond the case of the soft Sinai billiard. There are a number of early examples in the literature whose focus is on characterizing ergodicity under appropriate conditions on a smooth potential in the soft Sinai billiard. In [17], the author shows that for certain bell-shaped potentials under a general smoothness condition \( C^3 \) smoothness at the boundary of the scatterer, among other regularity conditions on the geometry of the table and the total energy of the system), the soft Sinai billiard has the \( K \)-property and is thus ergodic. Later, in [15], the author shows for a class of smooth Coulombic potentials, again for the Sinai billiard on the torus with disk-shaped scatterers, that the flow of the system can be realized as a complete geodesic flow on a suitably defined
compact Riemannian manifold. Under appropriate conditions on the potential, the metric has negative curvature and so the flow is Anosov and hence ergodic. In [11], these results are generalized to a broader class of smooth potentials. Moreover, under certain conditions on the potential, positivity of Lyapunov exponents is shown using Wojtkowski’s method of invariant cone fields. A number of further studies have continued to extend the literature in the case of smooth potentials for the Sinai billiard: exponential decay of correlations and the central limit theorem [2], non-ergodicity and stability of periodic orbits [12, 23], and related questions for tables in higher dimensions [3, 20]. It should be noted that the literature for the case of constant potentials seems to be more limited and confined to the case where the table is the Sinai billiard. In [1] ergodicity is studied numerically and the parameter space, indexed by the scatterer radius and potential magnitude, is delimited into regions of non-ergodicity. Lyapunov exponents [16] and ergodicity [18] have been studied analytically for this example as well. Lensed billiards are also closely related to so-called composite billiards, also known as ray-splitting or branching billiards [4, 14], but are simpler and less of a departure from standard systems.

This paper continues, from a new perspective, the line of investigation of the above mentioned papers. Our work is also motivated by more geometric considerations. The dynamics of geodesic flows on manifolds with discontinuous Riemannian metrics is a potentially interesting topic that, to our knowledge, is still waiting for a detailed study. When the metric tensor $g$ is in the conformal class of a smooth metric $g_0$, that is, $g = \eta^2 g_0$ for a positive function $\eta$ with discontinuities on smooth hypersurfaces, we have a system very similar to our lensed billiard systems. For more general discontinuous metrics, Snell’s law still holds as shown in [13]. It is hoped that some of the observations about the dynamics of lensed billiards in dimension 2 will serve as a guide into the dynamics of such geodesic flows. (Having this Riemannian setting in mind, some of the details relegated to appendices are given in greater generality than needed for the narrower purposes of the present paper.)

The rest of the paper is organized as follows. Lensed billiard systems are introduced precisely in Section 2. In defining their phase space, a certain care is needed when accounting for refractions. While the systems we study can be considered as a special case of soft billiards with discontinuous potentials, the perspective we take in defining the billiard map has a number of benefits related to the issue of handling refractions and the differential of the billiard map when refractions occur. Just as for ordinary billiard systems, lensed billiards are dynamical systems with singularities. Besides those singularities typically present in ordinary billiards (hitting a corner or grazing trajectories), we should add singular sets caused by the critical angle of incidence, a quantity to be defined formally in Subsection 2.1 that specifies the transition between reflected and refracted trajectories. More precisely, Subsection 2.1 defines the billiard map of lensed billiards, introduces the notion of the critical angle, and demonstrates the singular sets and phase portraits for a collection of examples to be studied later in the paper. In Subsection 2.2 we make a few observations concerning refractions based on the invariance of the Liouville measure under the lensed billiard map. As a first step in the analysis of lensed billiard dynamics, we describe these systems as a switching process involving two ordinary open billiard systems and obtain mean values for time and number of collisions during sojourns in each subsystem. The idea of switching dynamics and the statistics of sojourn times
discussed in Section 2.2 closely links the analysis of lensed billiards with the study of open (standard) billiard systems. (See Figure 7 and the remarks made around it.) In Subsection 2.3 we describe the differential of the lensed billiard map. This is used afterwards to obtain numerically Lyapunov exponents for the examples of parametric families of billiard systems discussed in Section 3, which contains the main numerical results of the paper. The central interest is to explore the Lyapunov exponent’s dependence on the potential parameter $C$. We identify a few properties of lensed billiards that recur in the examples and use the results of Subsection 2.2 to explain some of them.

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2. Preliminary details and facts

2.1. Definition of lensed billiards

Let $\mathcal{R} \subseteq \mathbb{R}^2$ be a closed set with piecewise smooth boundary and $\mathcal{A}$ an open subset of $\mathcal{R}$ whose boundary is also piecewise smooth. Part of the boundary of $\mathcal{A}$ may be contained in the boundary of $\mathcal{R}$. Let the potential function be $V = C I_{\mathcal{A}}$ for a given constant $C \in \mathbb{R}$. The lensed billiard map (and billiard flow) is defined on the phase space of the system, which involves a small modification of the standard billiard phase space definition.

![Figure 3: In the lensed billiard phase space the refracting boundary is duplicated.](image)

Let $\mathcal{A}_1 := \mathcal{A}$ and $\mathcal{A}_0 := \mathcal{R} \setminus \mathcal{A}$ and consider the disjoint union $\mathcal{B} := \mathcal{A}_0 \amalg \mathcal{A}_1$. We fix a value $E > C$ for the total energy (kinetic plus potential). By a regular point in the boundary $\partial \mathcal{B}$ we mean a point $x$ at which the unit vector $n_x$ perpendicular to $\partial \mathcal{B}$ and pointing towards the interior of $\mathcal{B}$ at $x$ is defined. Note that the parts of the boundary common to both $\mathcal{A}_0$ and $\mathcal{A}_1$ are accounted for twice. These hypersurfaces of discontinuity of the potential function will have a copy as part of the boundary of $\mathcal{A}_0$ and another as part of the boundary of $\mathcal{A}_1$. Since these are sets where refracting is possible, we say that they are contained in the refracting boundary, which we denote as $\partial_r \mathcal{B}$. We will refer to the rest of the boundary of $\mathcal{B}$ as the reflecting boundary.

The phase space $\mathcal{V}$ of the lensed billiard map is the set of pairs $(x, v)$ where $x$ is a
regular point in $\partial B$ and $v$ is a vector at $x$ such that $\langle n_x, v \rangle \geq 0$ and
\[
\|v\| = \sqrt{2(E - V(x))/m} = \begin{cases} 
\sqrt{2E/m} & \text{if } x \in \mathcal{A}_0 \\
\sqrt{2(E - C)/m} & \text{if } x \in \mathcal{A}_1.
\end{cases}
\]
We define the lensed billiard map
\[
\mathcal{J}(x, v) = (X(x, v), V(x, v)), \quad (x, v) \in \mathcal{V},
\]
as follows. Let $x \in \mathcal{A}_0$ and note that the definition when $x \in \mathcal{A}_1$ will follow similarly and is given in the general definition below. Define $\gamma(t), t \geq 0$, to be the piecewise smooth curve such that $\gamma(0) = x$ and $\gamma'(0) = v$ given by the standard billiard flow in $\mathcal{A}_0$ with specular reflections upon collisions with the (reflecting) boundary. Next, let
\[
t_0 := \inf\{t > 0 : \gamma(t) \in \partial B\}
\]
be the first return time to the refracting boundary. If $\gamma(t_0)$ is not a regular boundary point, the billiard map is not defined at $(x, v)$ and this state will be part of the singular set of the lensed billiard map. If $\gamma(t_0)$ is regular, let $y_0 = \gamma(t_0) \in \mathcal{A}_0$ and let $y_1$ be the corresponding point in $\mathcal{A}_1$. If a reflection occurs, then the particle remains in $\mathcal{A}_0$ and we let $X(x, v) = y_0$; if a refraction occurs, then $X(x, v) = y_1$.

Let $\tilde{v} := \gamma'(t_0)$. We now define $V(x, v)$ as a reflection or a refraction of $\tilde{v}$ based on whether the particle can overcome the potential difference between $\mathcal{A}_0$ and $\mathcal{A}_1$. Let $t_{y_0}$ be the unit vector at $y_0$ tangent to the boundary of $\mathcal{A}_0$ chosen so that $(t_{y_0}, n_{y_0})$ is a positive orthonormal basis, and let $\tilde{v}_1, \tilde{v}_n, V_n, V_{n_0}$ be constants so that $\tilde{v} = \tilde{v}_1 t_{y_0} + \tilde{v}_n n_{y_0}$ and $V(x, v) = V_l t_{y_0} + V_n n_{y_0}$. Note that the vectors $\tilde{v}$ and $V(x, v)$ are related by energy conservation
\[
\frac{1}{2}m\|\tilde{v}\|^2 + V^- = \frac{1}{2}m\|V(x, v)\|^2 + V^+
\]
where $V^- = V(y_0) = 0$ (since $y_0 \in \mathcal{A}_0$) and $V^+ = V(X(x, v))$, still to be defined, are the values of the potential immediately before and after a collision event (which results in either a reflection or refraction) has occurred. Note that regardless of whether a reflection or refraction occurs, the tangential component of the velocity is conserved: $V_t = \tilde{v}_t$. While this is straightforward to justify by conservation of momentum when the refracting boundary is flat, the issue is more subtle for curved boundaries and deserves further discussion, which we give in Appendix A.1. It follows that the normal components are related by
\[
V_n^2 = \tilde{v}_n^2 - \frac{2}{m}(V^+ - V^-) \geq 0.
\]
It is now apparent that reflection or refraction is determined by the relationship between $\tilde{v}_n^2$ and the potential $C$. Indeed, if
\[
\tilde{v}_n^2 - \frac{2}{m}(V(y_1) - V(y_0)) < 0,
\]
then it must be the case that $V^+ = V^-$. Thus, a reflection occurs and $V_n = -\tilde{v}_n$. If $\tilde{v}_n^2 - 2C/m > 0$, then $V^+ = V(y_1)$ and a refraction occurs with $V_n = (\tilde{v}_n^2 - 2C/m)^{1/2}$. In
the special case when \( \tilde{v}_n^2 - 2C/m = 0 \), we have that \( V_n = 0 \). We call the angle between \( \tilde{v} \) and \( n_{y_i} \) in this case the critical angle of incidence and denote it \( \theta_{\text{crit}} \).

Given the discussion above, we are now ready to state the definition of the lensed billiard map. We also give precise mathematical justification in a more general setting in Appendix A.1. There we show that the lensed billiard map derived from the geodesic flow on a Riemannian manifold can be derived from Newton’s equations of motion for a system with smooth potential that approximates a system with discontinuous potential.

**Definition 1** (Lensed billiard map). Let \((x,v) \in V\) with \( x \in A_i \) for \( i = 0 \) or \( i = 1 \), and let \( y_i \in A_i \) be the point in the refracting boundary reached by the specular billiard flow started from \( x \) and let \( \tilde{v} = \tilde{v}_i t_{y_i} + \tilde{v}_n n_{y_i} \) be the corresponding velocity. Let \( y_j \) be the corresponding point in \( A_j \) for \( j \neq i \). Using the notation established above, the lensed billiard map \( T(x,v) = (X(x,v), V(x,v)) \) is defined as follows.

1. **(Reflection)** If
   \[
   \tilde{v}_n^2 - \frac{2}{m} (V(y_j) - V(y_i)) < 0,
   \]
   then \( X(x,v) = y_j \) and \( V(x,v) = \tilde{v}_i t_{y_i} - \tilde{v}_n n_{y_i} \).

2. **(Refraction)** If
   \[
   \tilde{v}_n^2 - \frac{2}{m} (V(y_j) - V(y_i)) > 0,
   \]
   then \( X(x,v) = y_i \) and \( V(x,v) = \tilde{v}_i t_{y_i} + (\tilde{v}_n^2 - 2(V(y_j) - V(y_i))/m)^{1/2} n_{y_i} \).

3. **(Critical angle)** If
   \[
   \tilde{v}_n^2 - \frac{2}{m} (V(y_j) - V(y_i)) = 0,
   \]
   the billiard map is not defined. We call the angle between \( \tilde{v} \) and \( n_{y_i} \) a critical angle.

Note that a critical angle of incidence is only possible when \( V(y_i) < V(y_j) \); that is, when approaching a region of higher potential from a region with lower potential. The state corresponding to a critical angle of approach is then included in the singular set of the billiard map and the trajectory is terminated. It would be possible, under a convexity assumption, to define the billiard map when the angle of incidence is critical, but we choose to exclude states leading to such angles from the domain of definition of the billiard map. States in \( V \) whose orbits generated by the map \( T \), after a finite number of steps, lead to a critical angle or to a non-regular point of \( \partial B \), or to tangential contact, will be called singular. It will be assumed (as it is often done for standard billiard systems) that the domain of \( T \) in \( V \) is an open set of full Lebesgue measure and that \( T \) is a smooth map there. This will be valid for the examples in this paper.

The billiard table in Figure 4 will be used as a multiparameter family of examples at a number of places. For this example, Figure 5 illustrates the points in the singular set associated to critical angles. These are the new critical points not present in standard billiards. The figure only shows the part of the standard billiard phase space over the vertical segments on the right end of the rectangular domain. Further details are given in the figure’s caption. Figure 6 shows a few phase portraits of examples taken from the family of Figure 4. The plots only show orbits of the standard billiard return map to the right vertical side of the rectangular domain.
Figure 4: This lensed billiard table involves 4 parameters: \( \ell_1, \ell_2 \), the signed curvature \( \kappa \in [-2, 2] \) of the interface segment, and the value \( C \) of the potential function to the left of the interface segment. To its right the potential is 0. The figure shows one example for \( \kappa < 0 \). Initial conditions for billiard trajectories are set at the vertical wall on the right. If \( C \) is sufficiently large (\( C \geq 1 \) if the initial particle speed is set equal to \( \sqrt{2} \)) for the system to be a standard billiard on the region to the right of the interface segment, then \( \kappa < 0 \) corresponds to a semidispersing billiard and \( \kappa > 0 \) to a focusing billiard.

2.2. Switching dynamics and measure invariance

A prominent aspect of the dynamics of lensed billiards is the back-and-forth switching process between two standard billiard systems in \( \mathcal{A}_0 \) and \( \mathcal{A}_1 \). Segments of trajectories from the moment of arrival in region \( \mathcal{A}_i \) to the moment of the next switch back into the other region will be called sojourns. The statistics of time and number of collisions during a sojourn plays a role in the analysis of lensed billiards, as will be seen. We give in this section the mean value of these two quantities under the assumption that the two standard billiard subsystems are ergodic.

A few remarks will be needed concerning invariance of the canonical billiard measure. If \( V \) is the potential function on \( \mathcal{R} \) and \( \mathcal{V} \) is the billiard phase space, let \( \mathcal{V}_E \) consist of the points \((x,v)\) in \( \mathcal{V} \) such that \( |v| = \sqrt{2(E - V(x))}/m \). This is the invariant subset of states with total energy \( E \). Denoting by \( \theta \) the angle between \( n_x \) and \( v \), and by \( s \) the arclength parameter along boundary line segments, the canonical billiard measure, or Liouville measure, is the measure \( \nu \) on \( \mathcal{V}_E \) such that

\[
d\nu(s, \theta) = \sqrt{2(E - V(x))) \cos \theta} \, d\theta \, ds.
\]

We refer to the angle distribution in \( \nu \) as the cosine law. It is well-known that this measure is preserved by the billiard map. In fact, the lensed billiard map preserves this measure as well.

Proposition 2. Suppose \( C_0 > C_1 \). Let \( x \in C := \overline{\mathcal{A}_0 \cap \mathcal{A}_1} \) be a regular point and \( n_x \) the unit normal vector to \( C \) at \( x \) pointing into \( \mathcal{A}_0 \). Further suppose that the velocity of trajectories incident upon \( C \) at \( x \) coming from \( \mathcal{A}_1 \) is distributed according to the cosine law. Then the distribution of post-crossing velocities, conditional on the occurrence of refraction, also satisfies the cosine law.

Proof. Let \( S^+_x \) be the semicircle centered at \( x \) consisting of unit vectors \( u \) such that \( \langle n_x, u \rangle \geq 0 \), and let \( D \) be the interval of radius 1 centered at the origin in the (one-dimensional) tangent space \( T_x C \). Let \( y \) denote coordinate of points in \( D \). If we parametrize
Figure 5: Singular set for the lensed billiard map of the family of billiard tables shown in Figure 4 restricted to the part of the phase space over the right vertical side of the rectangle. The parameters are $\ell_1 = 0.5$, $\ell_2 = 1$, $C = -0.5$ (top group of ten plots), $C = 0.5$ (bottom group); in each group of plots, the values of $\kappa$ are, clockwise from the top left: $-2.0$, $-1.5$, $-1.0$, $-0.5$, $-0.1$, $0.1$, $0.5$, $1.0$, $1.5$, $2.0$. Only the singular initial conditions leading to the critical refracting angle are shown, for orbits of length less than 16.
Figure 6: Phase-portraits for bounded lensed billiard systems. Numbers indicate values of the potential constant. The orbits shown here are for the return map to the right vertical side of the rectangular domain, for the top four systems, and the return map to the outer ellipse for the bottom system, rather than $\mathcal{T}$ itself. Only the relevant part of the phase space is shown.
$S_x^+$ using this coordinate, the volume element $dV$ on $S_x^+$ satisfies $dV(u) = (\cos \varphi)^{-1} dy$ where $\varphi$ is the angle between $u \in S_x^+$ and $n_x$. In other words, the cosine law corresponds to uniform distribution in $D$. The set of trajectory segments arriving at $x$ from $\mathcal{A}_1$ whose velocities undergo refraction are those for which $|\sin \theta| < r_0$. Under the given parametrization of $S_x^+$ by $D$, this set defines the open interval $D_0 \subseteq D$ centered at the origin of $T_x \mathcal{C}$ with radius $r_0$. Under the cosine law, velocities undergoing refraction correspond to points uniformly distributed in $D_0$. If $v^+$ is the velocity after refraction, the relation $\sin \varphi = r_0^{-1} \sin \theta$ (derived independently the discussion that follows this proof below; see Equation (3)) implies that the orthogonal projection of $v^+/|v^+|$ to $T_x \mathcal{C}$ has the uniform distribution on $D$, hence satisfies the cosine law. 

We note the following effect of the square root term appearing in Equation (1). The invariant measure on the lensed billiard phase space is proportional to $\sqrt{E - C_0}$ on the side of potential value $C_0$ and $\sqrt{E - C_1}$ on the other side. Thus in the long run the ratio of probabilities of transitions into $C_0$ (from either $C_0$ or $C_1$) over transitions into $C_1$ is

$$p_{01} = \sqrt{\frac{E - C_0}{E - C_1}}. \tag{2}$$

The discussion that follows now makes this idea precise. Let us assume for concreteness that $C_0 > C_1$. Recall from the definition of the lensed billiard map that, upon refraction at a point $x$ in the refracting boundary of the table, the components of the incident and transmitted velocities tangent to the line of discontinuity are equal. If the trajectory leaves $\mathcal{A}_1$ and enters $\mathcal{A}_0$, and if $\theta_1$ and $\theta_0$ are the angles of the velocities $v_1$ and $v_0$ pre- and post-refraction, respectively, relative to the unit normal vector $n_x$ to the line of discontinuity pointing into $\mathcal{A}_0$, then equality of tangential velocity components amounts to $|v_1| \sin \theta_1 = |v_0| \sin \theta_0$. It follows that

$$\frac{\sin \theta_1}{\sin \theta_0} = \frac{|v_0|}{|v_1|} = \sqrt{\frac{\frac{1}{2}m|v_0|^2}{\frac{1}{2}m|v_1|^2}} = \sqrt{\frac{E - C_0}{E - C_1}}. \tag{3}$$

This is the mechanical counterpart of Snell’s law, in which terms of the form $\sqrt{E - C}$ assume the role of the refractive index in geometric optics. We also note here that, when $\theta_1 = \theta_{\text{crit}}$, we have that $\theta_0 = \pi/2$, and so it follows that a critical angle satisfies:

$$r_0 := \sin \theta_{\text{crit}} = \sqrt{\frac{E - C_0}{E - C_1}}. \tag{4}$$

It should be noted that the relation in (3) also holds for transitions from $\mathcal{A}_0$ to $\mathcal{A}_1$, interchanging the corresponding indices above. However, transitions from $\mathcal{A}_0$ to $\mathcal{A}_1$ happen whenever the trajectory reaches the refracting boundary, at any angle of incidence. On the other hand, transitions from $\mathcal{A}_1$ to $\mathcal{A}_0$ can only occur when the angle of incidence between the velocity vector and the normal to the separation line (pointing into $\mathcal{A}_0$) is smaller than the critical angle. Denoting by $\theta_1$ the angle of incidence, a transition from $\mathcal{A}_1$ to $\mathcal{A}_0$ happens when $|\sin \theta_1| < r_0$. Furthermore, using (3), the angle $\varphi$ that the trajectory makes with the same normal vector as it crosses into $\mathcal{A}_0$ satisfies

$$\sin \varphi = r_0^{-1} \sin \theta. \tag{5}$$
Let $A$ be either $A_0$ or $A_1$. Let $A$ and $L$ be the area and the boundary length of $A$, respectively, and $C := \overline{A_0} \cap \overline{A_1}$. Denote by $\ell$ the length of $C$, by $L$ the length of the boundary of $A$ and by $A$ the area of $A$. Let $V$ be the space of pairs $(x,v)$ where $x \in \partial A$ and $v$ is a tangent vector to $A$ at $x$ pointing into $A$ and having norm $s$. Similarly, we denote by $E$ the space of pairs $(x,v)$ where now $x \in C$. At each $(x,v) \in E$, let $T(x,v)$ and $N(x,v)$ denote, respectively, the time of first return to $E$ and the number of collisions with the boundary of $A$ of a billiard trajectory with initial state $(x,v)$ before returning to $E$. For each $(x,v) \in V$, let $\tau(x,v)$ denote the time duration of free flight from $(x,v)$ to the point of next collision. This is naturally the length of the free flight divided by the speed $s$. Finally, denoting by $\nu$ and $\nu_{\partial A}$ the normalized Liouville measure on $V$ and $\partial A$, respectively, we introduce the mean values

\[
\langle N \rangle_E := \int_E N(x,v) \, d\nu_{\partial A}(x,v), \quad \langle T \rangle_E := \int_E T(x,v) \, d\nu_{\partial A}(x,v), \quad \langle \tau \rangle_V := \int_V \tau(x,v) \, d\nu(x,v).
\]

The following formulas follow from standard results in the classical theory of billiards. See Chapter 2 of [8].

**Proposition 3.** With the notations just introduced the following relations hold:

1. $\langle N \rangle_E = \frac{L}{\ell}$;
2. $\langle T \rangle_E = \frac{\pi A}{\ell s}$;
3. $\langle \tau \rangle_V = \frac{\pi A}{Ls}$;
4. $\langle T \rangle_E = \langle N \rangle_E \langle \tau \rangle_V$.

Furthermore, if $E_0 \subseteq E$ consists of pairs $(x,v)$ such that $x \in C$ and the angle $\theta$ which $v$ makes with the normal $n_x$ satisfies $|\sin \theta| < r_0$ then the mean number of returns to $E_0$ before a first return to $E_0$ is $1/r_0$.

The following result now formalizes Equation (2).

**Corollary 4.** Suppose the value of the potential function in $A_i$ is $C_i$ with $C_0 > C_1$. Let $\langle T_i \rangle$ and $\langle N_i \rangle$ denote the mean time and number of collisions of the lensed billiard system during a sojourn in $A_i$ before the next switch to the other region. Then

\[
\frac{\langle T_0 \rangle}{\langle T_1 \rangle} = \frac{A_0}{A_1}, \quad \frac{\langle N_0 \rangle}{\langle N_1 \rangle} = \frac{L_0}{L_1} r_0,
\]

where $A_i$ and $L_i$ are the area of $A_i$ and the length of the boundary of $A_i$, and $r_0$ is defined in (4).

Further information about the probability distribution of sojourn times, beyond simply their mean values, is a focus of future study. For an example of what can be expected, consider Figure 7. It shows the sojourn time distributions for the lensed billiard system whose billiard domain is shown in the figure inset. The left half of the table has potential $-1$ and the right half has potential $0$, while the total energy of the billiard particle is
Figure 7: Sojourn time distributions obtained by simulating long trajectories of the lensed billiard system for the table shown in the inset. The potential is \(-1\) on the left half of the table and \(0\) on the right, while particle energy is \(0.01\). The particle undergoes many more collisions in the left than the right region, although mean sojourn times are the same in both. Corollary 4 gives the mean value \(\approx 17.3054\) (the table has area \(\approx 1.558\) and height 1); simulated values are \(17.317\) (left) and \(17.305\) (right). The dotted line is the graph of the probability density function of the exponential distribution with parameter \(\lambda = 1/17.305 = 0.058\).

a small positive value. Thus the particle undergoes many more collisions in a typical sojourn in the left-hand region. In order to cross back into the right-hand region, the billiard trajectory has to pass through a window in phase space defined by the vertical middle line and a small angle interval. It is interesting to note, in particular, the tail behavior for the sojourn time distribution in the left half, which is well approximated by the probability density of an exponential random variable with parameter equal to the reciprocal of the mean sojourn time.

It is likely that methods for the study of open billiards such as in [6] and [19] can be used for a detailed analysis of sojourn time distribution in potential wells. In lensed billiards we are presented with two ordinary billiard systems that are open to each other in the sense that the phase space of each contains a subset, or hole, through which the billiard particle can cross into the other. The size of the hole is smaller for the region with smaller potential. One should expect the successive hitting times, appropriately normalized, into small holes to be approximated by a Poisson process for hyperbolic billiard subsystems.

2.3. DIFFERENTIAL OF THE LENSED BILLIARD MAP

We wish here to obtain the differential of the lensed billiard map to be used in the next section for the evaluation of Lyapunov exponents. For this purpose we introduce coordinates that differ in minor ways from the more commonly employed conventions. The main difference is that, instead of the Jacobi coordinates adapted to wavefronts in a neighborhood of \((x,v)\) as in the standard reference [8], we use arclength parameter
along the boundary curves $\delta$ of $\mathcal{A}_1$ and angles that are measured relative to $v$ itself. The differential of the billiard map at reflections will differ from the more standard description by the absence of a cosine term at certain places. This small deviation from standard conventions has been made in order to avoid having to elaborate on the behavior of wavefronts at refractions. For standard billiards, time and arclength parameters are everywhere proportional since particle speed is constant, but this is no longer the case across refractions, which introduces a few issues that we choose to avoid.

Recall that the smooth pieces of the boundary of $\mathcal{B}$ are the pieces of the boundaries of $\mathcal{A}_0$ and $\mathcal{A}_1$ oriented so that the unit normal vector field $\mathbf{n}$ points towards the interior of $\mathcal{B}$. Let $\mathbf{t}$ be the unit tangent vector field to the boundary chosen so that $(\mathbf{t}, \mathbf{n})$ is a positive orthogonal basis at each regular boundary point. Let us consider one step of the billiard trajectory from $O_1$ in boundary piece $\delta_1$ to $O_2$ in boundary piece $\delta_2$. The initial velocity is $v_1$ such that $v_1 \cdot \mathbf{n}_1 > 0$ (standard dot product) and $T(O_1, v_1) = (O_2, v_2)$. This is shown in the diagram of Figure 8 where, on the left, $T$ produces a refraction and on the right a reflection.

Figure 8: Notation for the calculation of the differential of the lensed billiard. On the left a refraction and on the right a reflection. $C_1$ and $C_2$ are the values of the potential function at the indicated regions.

Let $J$ denote the rotation matrix in $\mathbb{R}^2$ by $\pi/2$ counterclockwise. This is the generator of plane rotations: the linear map $v \mapsto \exp(\theta J)v$ rotates $v$ counterclockwise by angle $\theta$. Thus for a small interval of angles centered at 0, this defines a neighborhood of velocities centered at $v$. We define coordinates $(x_1, \theta_1)$ in a neighborhood of $(O_1, v_1)$ and $(x_2, \theta_2)$ in a neighborhood of $(O_2, v_2)$ so that any $(q, v)$ in the first neighborhood and $(Q, V) = T(q, v)$ in the second satisfy:

$$\begin{align*}
q &= \gamma_1(x_1) \text{ is a local parametrization of } \delta_1 \text{ by arclength with } \gamma_1(0) = O_1, \\
Q &= \gamma_2(x_2) \text{ is a local parametrization of } \delta_2 \text{ by arclength with } \gamma_2(0) = O_2, \\
v &= \exp(\theta_1 J)v_1, \\
V &= \exp(\theta_2 J)v_2.
\end{align*}$$

(6)
We wish to obtain the differential of $T$ at $(\Theta_1, v_1)$:

\[
dT_{(\Theta_1, v_1)} = \begin{pmatrix}
\frac{\partial x_2}{\partial x_1}(0, 0) & \frac{\partial x_2}{\partial \theta_1}(0, 0) \\
\frac{\partial x_2}{\partial \theta_2}(0, 0) & \frac{\partial x_2}{\partial \theta_1}(0, 0)
\end{pmatrix}.
\]

The following additional notation is needed. Let $D_\ell t$ indicate standard directional derivative of vector fields along $t$. Let $\ell$ be the Euclidean distance between $\Theta_1$ and $\Theta_2$, and set $\nu_i = v_i/v_i$, $i = 1, 2$. Further write $E$ for the total energy (kinetic plus potential) and $C_1, C_2$ for the value of the potential at the regions indicated in Figure 8. Note that this is a slight, more general, departure of the convention used in the previous subsections where the regions on either side of the refracting boundary were specified whereas here they are arbitrary. Such convention allows us to avoid breaking down the following formulas into separate cases for each possible transition between regions.

**Theorem 5** (Differential of the lensed billiard map). With the notations just given, the differential $d\mathcal{T}(\Theta_1, v_1)$ in the coordinate systems $(x_i, \theta_i)$ is given by

\[
d\mathcal{T}(\Theta_1, v_1) = \begin{pmatrix}
\frac{\nu_1 \cdot n_1(\Theta_1)}{\nu_1 \cdot n_2(\Theta_2)} & -\ell \\
-2\kappa(\Theta_2) \frac{\nu_2 \cdot n_1(\Theta_1)}{\nu_2 \cdot n_2(\Theta_2)} & 1 + \frac{2\ell \kappa(\Theta_2)}{\nu_2 \cdot n_2(\Theta_2)} \nu_2 \cdot n_2(\Theta_2)
\end{pmatrix}
\]

if a reflection occurs at $\Theta_2$ or

\[
d\mathcal{T}(\Theta_1, v_1) = \begin{pmatrix}
\frac{\nu_1 \cdot n_1(\Theta_1)}{\nu_1 \cdot n_2(\Theta_2)} & -\ell \\
-\alpha \kappa(\Theta_2) \frac{\nu_2 \cdot n_1(\Theta_1)}{\nu_2 \cdot n_2(\Theta_2)} \sqrt{E - C_1} & 1 + \frac{\alpha \ell \kappa(\Theta_2)}{\nu_2 \cdot n_2(\Theta_2)} \nu_2 \cdot n_2(\Theta_2) \sqrt{E - C_2}
\end{pmatrix}
\]

if a refraction occurs. In the latter case,

\[
\alpha := 1 - \left[1 - \frac{C_2 - C_1}{E - C_1} \frac{1}{(\nu_1 \cdot n_2(\Theta_2))^2}\right]^{1/2}.
\]

The determinant of $d\mathcal{T}(\Theta_1, v_1)$ is $v_1 \cdot n_1(\Theta_1)/v_2 \cdot n_2(\Theta_2)$ in both cases.

The similarities between the differentials for reflection and refraction become more apparent with the following notation: $\ell_i := \ell_i|v_i|$ where we recall that $|v_i| = \sqrt{2(E - C_i)/m}$. Then

\[
\left(d\mathcal{T}(\Theta_1, v_1)\right)_{\text{reflection}} = \begin{pmatrix}
\frac{\nu_1 \cdot n_1(\Theta_1)}{\nu_1 \cdot n_2(\Theta_2)} & -\ell_1 \\
-2\kappa(\Theta_2) \frac{\nu_2 \cdot n_1(\Theta_1)}{\nu_2 \cdot n_2(\Theta_2)} & 1 + \frac{2\ell_1 \kappa(\Theta_2)}{\nu_2 \cdot n_2(\Theta_2)} \nu_2 \cdot n_2(\Theta_2)
\end{pmatrix}
\]

\[
\left(d\mathcal{T}(\Theta_1, v_1)\right)_{\text{refraction}} = \begin{pmatrix}
\frac{\nu_1 \cdot n_1(\Theta_1)}{\nu_1 \cdot n_2(\Theta_2)} & -\ell_1 \\
-\alpha \kappa(\Theta_2) \frac{\nu_2 \cdot n_1(\Theta_1)}{\nu_2 \cdot n_2(\Theta_2)} & 1 + \frac{\alpha \ell_1 \kappa(\Theta_2)}{\nu_2 \cdot n_2(\Theta_2)} \nu_2 \cdot n_2(\Theta_2)
\end{pmatrix}
\].
Although the proof of Theorem 5 is relatively straightforward, we nevertheless provide the details in Appendix A.2 since refractions are not considered in standard billiard dynamics references, and we believe the expressions we give do not exist in the literature. Also, as we mention below, the expressions are used throughout our numerical study in the following section.

3. Numerical study of Lyapunov exponents

In this section we wish to explore, mainly numerically, the ways in which the presence of lenses affects the billiard dynamics as measured by the Lyapunov exponents of the billiard flow. Thus we are concerned with the following quantity:

$$\chi := \lim_{n \to \infty} \frac{\log \| dT^n_{(x,v)} \xi \|}{T_n},$$

where $T_n$ is the time elapsed up to step $n$.

Due to invariance of the Liouville measure (which is associated to an invariant 2-form on phase space) we know that, on ergodic components, exponents come in pairs $\chi, -\chi$. We refer to the nonnegative value as the Lyapunov exponent of the system. In the numerical experiments shown here, we compute the average of $\chi$ over a large sample of randomly chosen initial conditions. Details of the sampling procedure will be specified in the subsections to come. Not all the examples considered are ergodic, so the numerically computed $\chi$ may involve an average over the values on ergodic components. Also note that exponents are obtained numerically using the differential of the billiard map described in Section 2.3. More precisely, we compute $\chi$ by numerically simulating lensed billiard orbits and evaluating $n$-fold products of the differential of the lensed billiard map for large $n$.

In all cases, we set the mass parameter $m = 1$ and particle total energy $E = 1$. Thus particle speed in regions where the potential is 0 equals $\sqrt{2}$ and a region $A$ on which the potential value is greater than or equal to 1 acts as a standard billiard scatterer; that is, the boundary of $A$ becomes part of the reflecting boundary.

We identify a number of common features specific to lensed billiards among the examples considered here. We also provide conceptual explanations for some of these features based on the results of Section 2.2. The present discussion is exploratory in nature and the explanations for the observed features are heuristic and informal, although we expect they will serve as a basis for a more detailed study to be pursued elsewhere.

3.1. A multiparameter billiard case study

Let us consider first the multiparameter family depicted in Figure 4. We wish to use it to illustrate a number of properties that seem to be typical of lensed billiards but distinct in comparison with standard (purely reflecting) billiards.

As a first experiment, let us obtain approximate values for the positive Lyapunov exponent as a function of the curvature parameter $\kappa \in [-2, 2]$ when $C = 1$, initial speed is $\sqrt{2}$, $\ell_1 = 1$ and $\ell_2 = 2$. Since $C = 1$, this is a standard billiard system on the region where the potential is 0. This will serve as a basis for comparison when setting $C < 1$. 

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Figure 9: Positive Lyapunov exponent as function of the signed curvature parameter $\kappa$ of the refracting boundary. The parameters are $\ell_2 - \ell_1 = 1$ and $C = 1$, so this is a standard (purely reflecting) billiard in the complement of the shaded region (see insert), to the right of the curved line, where $V = 0$.

The results are shown in Figure 9. Negative values of $\kappa$ correspond to semidispersing billiards and positive values to focusing billiards. For each of 200 equally spaced values of $\kappa$ we compute the mean value for the exponent over a sample of 100 orbits of length 1000 with random initial conditions: points are taken from the vertical wall on the right with the uniform distribution and velocities according to the cosine law density. Unless stated otherwise, we use this sampling procedure throughout all subsequent experiments. Whiskers represent 95% confidence intervals based only on the sample variation of the simulated data. Other potential sources of errors may not be accounted for in this and similar graphs.

It is worth noting the distinct nature of the graph over the interval $0 < \kappa < 2/\sqrt{5}$. (The dashed vertical line is at $\kappa = 2/\sqrt{5}$.) The upper limit of this interval is the curvature for which the center of the circular arc lies on the vertical right wall. The less well-defined behavior in the range $0 \leq \kappa \leq 2/\sqrt{5}$ is likely due to breakdown of ergodicity. In fact, for $\kappa < 0$, the billiard is semidispersing, hence ergodic. For $\kappa > 2/\sqrt{5}$, Bunimovich’s condition (see [5]) for ergodicity based on defocusing in nowhere dispersing billiards holds. This observation offers a clue to help identify lensed billiard systems that fail to be ergodic in parametric families, although we do not pursue the determination of ergodicity (or failure of ergodicity) in detail in the present paper except to note it in some obvious cases.

Let us now investigate the effect of setting $C < 1$. Figure 10 refers to the same family as in Figure 4 (or Figure 9) except that the fixed parameters are $\kappa = -1$, $\ell_1 = 1$, $\ell_2 = 2$ on the left and $\kappa = 1$, $\ell_1 = 1$, $\ell_2 = 4$ on the right. The potential $C$ is the parameter to be varied. Comparison with the previous graph suggests a discontinuity at $C = 1$ (which corresponds to a standard reflecting billiard system.) Sample sizes and orbit length for each value of $C$ are as in the previous experiment. For $0 < C < 1$ and $\kappa = -1$, a
fraction of arrivals from the right at the line of potential function discontinuity undergoes dispersing reflections. This and the sharper appearance of the graph in that range may indicate that the system is ergodic and that for $C \leq 0$ it is not.

Figure 10: Family of billiard tables of Figure 4 with $C$ over the interval $[-3, 1)$. On the left, $\kappa = -1$, $\ell_1 = 1$, $\ell_2 = 2$; on the right, $\kappa = 1$, $\ell_1 = 1$, $\ell_2 = 4$. The “fat tail” on the left for $C \leq 0$ and high variance on the right for part of the range $0 \leq C < 1$ suggest that the billiard system is not ergodic in those ranges of $C$. The larger $\ell_2$ on the right was chosen under the expectation that, for $C$ sufficiently close to 1, the defocusing mechanism causing ergodicity will come into play. The somewhat more clearly defined shape of the graph on the right roughly in the range $0.8 < C < 1$ seems to justify this expectation.

Referring now to the system associated to the graph on the right of Figure 10 for negative $C$, the billiard particle can become momentarily trapped in the shaded region of Figure 4 where it behaves as in a semidispersing billiard. This suggests that for $\kappa > 0$ and $C < 0$ this lensed billiard may be ergodic. We see again, by comparison with Figure 9 a discontinuity at $C = 1$. An explanation for the jump discontinuity will be provided shortly.

Another feature that can be noted on both graphs and others shown later is the presence of a local maximum for positive values of $C$ less than but close to the point of exponent discontinuity ($C = 1$). We will have more to say about this shortly.

For $C < 0$ the exponent grows with $\sqrt{|C|}$. We leave a detailed proof of this and other remarks to a future paper. However, the following elementary observation can be adduced to justify this claim. Let $N(\xi)$ denote the number of collisions with the boundary of the shaded region $A_0$ (Figure 10 on the right) that a trajectory with initial state $\xi$ undergoes during a sojourn in $A_0$. Let $T(\xi)$ denote the corresponding time of that sojourn. Now consider

$$\frac{1}{T(\xi)} \log \|d\mathcal{T}^{-N(\xi)}w\| = \frac{N(\xi)}{T(\xi)} \frac{1}{N(\xi)} \log \|d\mathcal{T}^{-N(\xi)}w\|.$$

As $C \to -\infty$, both $N(\xi)$ and $T(\xi)$ grow to infinity for almost all $\xi$. It can be expected that the limit on the left will be the exponent for the billiard flow while the limit on the right will be the product of the exponent for the billiard map times the limit of $N(\xi)/T(\xi)$. Under the assumption that the system is indeed ergodic for negative $C$, this quotient converges to $\langle N \rangle/\langle T \rangle$ which, by Proposition 3 is $\partial L/\pi A$, where $L$ is the perimeter of $A_0$, $A$ is its area, and $\partial = \sqrt{(E - C_0)/(E - C_1)} = \sqrt{1 + |C|}$. 

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Figure 11: Here the parameters are: $\kappa = -1$, $C = 0.5$, $\ell_2 = \ell_1 + 0.5$, and $\ell_1$ ranges from 0 to 5.

The dependence on the geometric parameter $\ell_1$ is particularly simple. In Figure 11 we set $\kappa = -1$, $C = 0.5$, $\ell_2 = \ell_1 + 0.5$, and let $\ell_1$ range from 0 to 5. A fairly regular dependence of the exponent on the varying parameter is now observed. This is easily explained by noting that the number of collisions contributing to the exponent is independent of $\ell_1$, while the time spent in the region where $V = C \ (0 < C < 1)$ is proportional to $\ell_1$.

It was observed in the examples that the exponent has a jump discontinuity, as a function of $C$, at the point of transition from lensed to purely reflecting billiard. Let us, informally, estimate the size of the jump. We only provide here a heuristic argument, based on Proposition 3 and Corollary 4. (See the example of Figure 13 for a bit of numerical evidence.)

Suppose the standard billiard systems in the closures of $\mathcal{A}_0$ and $\mathcal{A}_1$ are ergodic. Let $\chi_{\epsilon}$ be the Lyapunov exponent of the lensed billiard flow on $\overline{\mathcal{A}_0} \cup \overline{\mathcal{A}_1}$ for which $C_1 = 0$ and $C_0 = E - \epsilon$. Let $\chi_0$ denote the positive exponent when $\epsilon = 0$. That is, the exponent for the standard billiard flow in $\overline{\mathcal{A}_1}$, where $C_1 = 0$. Then, as $\epsilon$ approaches 0, we expect

$$
\chi_{\epsilon} \rightarrow \frac{A_1}{A_0 + A_1} \chi_0
$$

to hold, where $A_i$ is the area of $\mathcal{A}_i$. In the example of Figure 13 the two areas are equal, therefore as $C \rightarrow 1$ the limit exponent is expected to be half the value it assumes for $C = 1$. This is very nearly what the numerical example shows. (See the right-hand side of that figure.)

To explain this feature, let us first introduce some notation. Let $\mathcal{R}_i$ be the return map to $C$ (the intersection of the boundaries of $\mathcal{A}_0$ and $\mathcal{A}_1$) after moving into $\mathcal{A}_i$. Let $n$ be the number of returns to $C$ before a trajectory that enters $\mathcal{A}_1$ finally switches to $\mathcal{A}_0$. Thus $\mathcal{R}_0$ corresponds to a sojourn into $\mathcal{A}_0$ whereas a sojourn into $\mathcal{A}_1$ involves $n$ applications of $\mathcal{R}_1$. For each positive integer $k$, let

$$
\mathcal{F}_k = \mathcal{R}_0 \circ \mathcal{R}_1^n \circ \cdots \circ \mathcal{R}_0 \circ \mathcal{R}_1^n.
$$

Let us fix a typical orbit, beginning with a $z = (x, v)$ entering $\mathcal{A}_1$:

$$
z = z_1^{(1)}, z_1^{(0)} := \mathcal{R}_1^n(z_1^{(1)}), z_2^{(1)} := \mathcal{R}_0(z_1^{(0)}), \ldots, z_k^{(0)} := \mathcal{R}_1^n(z_k^{(1)}), z_{k+1}^{(1)} := \mathcal{R}_0(z_k^{(0)}), \ldots
$$
Note that \( N_j = N_j(z_j^{(1)}) \). Times and number of collisions in each sojourn will be written \( T^{(i)} \) and \( N^{(i)} \) for \( i = 0, 1 \). The total time after \( k \) sojourns in \( \mathcal{A}_0 \) and \( k \) sojourns in \( \mathcal{A}_1 \) is

\[
T_k(z) = T^{(1)}(z_1^{(1)}) + T^{(0)}(z_1^{(0)}) + \cdots + T^{(1)}(z_k^{(1)}) + T^{(0)}(z_k^{(0)}) = T_k^{(0)}(z) + T_k^{(1)}(z),
\]

where we have defined

\[
T_k^{(0)}(z) = \sum_{j=1}^k T^{(0)}(z_j^{(0)}), \quad T_k^{(1)}(z) = \sum_{j=1}^k T^{(1)}(z_j^{(1)}).
\]

We similarly define \( N_k^{(i)}(z) \) for the number of collisions after \( k \) sojourns in \( \mathcal{A}_0 \) and \( \mathcal{A}_1 \). Let \( \xi \) be a vector defining an infinitesimal variation of the initial condition \( z \). We write

\[
\xi_j^{(1)} := \frac{dR_1^j}{\|dR_1^j\|} \xi_j^{(1)}, \quad \xi_j^{(0)} := \frac{dR_0^j}{\|dR_0^j\|} \xi_j^{(0)}.
\]

Then, omitting reference to \( z_j^{(i)} \) to simplify the notation,

\[
\frac{\log \|d\gamma_k\|}{T_k} = \frac{T_k^{(0)}}{T_k} \frac{N_k^{(0)}}{T_k^{(0)}} \left( \frac{1}{N_k^{(0)}} \sum_{j=1}^k \log \|dR_0 \xi_j^{(0)}\| \right) + \frac{T_k^{(1)}}{T_k} \left( \frac{1}{T_k^{(1)}} \sum_{j=1}^k \log \|dR_1 \xi_j^{(1)}\| \right).
\]

Focusing attention first on the first summand on the right side of the above equality, the following is expected to hold: the quantity in parentheses is almost everywhere finite, the quotient \( T_k^{(0)}/T_k \) converges to an area ratio based on Proposition 3 and Corollary 4 and, due to the same proposition, \( N_k^{(0)}/T_k^{(0)} \) should go to 0. The second term contains the quotient \( T_k^{(1)}/T_k \) which, by Proposition 3 and Corollary 4 should limit to \( A_1/(A_0 + A_1) \). Finally, the quantity in parentheses should converge to the exponent \( \chi_0 \).

Summarizing common features of lensed billiards Lyapunov exponents observed so far:

- There exists a discontinuity of the Lyapunov exponent as \( C \) approaches the total energy \( E \), that is, at the transition from lensed to standard (purely reflecting) billiard.

- In the examples considered so far, the exponent grows as \( \sqrt{|C|} \). Different asymptotic behavior will be noted in the additional examples of Section 3.2.

- For values of \( C \) less than but close to \( E \), the exponent has a local maximum as a function of \( C \). This common feature is more subtle and its explanation requires a more detailed analysis. See further remarks in Section 3.3.

### 3.2. Variations on Bunimovich and Sinai billiards

Let us consider here a few more examples from different parametric families in which \( C \) is the sole parameter, beginning with the variant of the Bunimovich (half-) stadium shown
The positive Lyapunov exponent is given as a function of the potential function parameter $C$. The standard stadium billiard corresponds to $C = 0$. We notice greater sample variance for $0 < C \leq 1$. In this range, the billiard system is not ergodic. In fact, when the initial position lies in the unshaded square region where the potential is 0, and the initial velocity is such that the first collision with the vertical line of potential discontinuity is greater than the critical angle, the trajectory must remain in that region since the normal component of the velocity is not large enough to overcome the potential barrier.

On the right of Figure 12 we have a lensed version of Sinai’s semidispersing billiard in a square with reflecting boundary. (One should compare this graph with that on the left of Figure 10.) For all $C \leq 0$ the lensed Sinai billiard is not ergodic since there is a positive measure of trajectories that are trapped inside the circular lens. The condition for being trapped is that the angle which the initial velocity makes with the inner normal vector to the circle be less than the critical angle.

It will be explained shortly that, in the limit as $C \to -\infty$, the Lyapunov exponent for the system on the right for trajectories started in the region of 0 potential is expected to converge to the exponent of the Sinai billiard, corresponding to $C = 1$. We haven’t determined the asymptotic behavior of the graph on the left for $C \to -\infty$.

Figure 13 refers to the lensed half-stadium billiard shown in the inset of the graph on the left. It shows a billiard trajectory for strongly negative $C$. A billiard trajectory originating from the right side of the stadium must fall into the shaded region at the moment of first arrival at the separating vertical line, and the velocity after crossing must be nearly perpendicular to the line. Note the growth of the exponent for negative $C$ for large values of $|C|$. In these experiments the particle was initiated on the right side of the stadium with speed $\sqrt{2}$. When it moves to the left side, it acquires a large speed, so the exponential rate of growth of tangent vectors increases accordingly; the exponent should be growing proportionally to $\sqrt{|C|}$ as $C \to -\infty$. On the right: zooming in near $C = 1$ shows more clearly the discontinuity at $C = 1$. As expected (given that the two sides of the billiard table have equal area), the jump discontinuity amounts to a near doubling of the exponent.
Let us turn now to the family shown in Figure 14. This lensed system may be viewed as an interpolation between a focusing billiard, when $C = 0$, and a semidispersing billiard, when $C = 1$. One observes several regimes of behavior over different ranges of the potential parameter $C$: one regime between 0 and 1 (with a discontinuity at $C = 1$), one between roughly $-3.3$ and 0, and one for $C$ less than approximately $-3.3$. On this last range, the exponent grows very slowly as $|C|$ increases and appears (for very large values of $|C|$ far outside the range of the graph) to stabilize near 0.35, which is roughly the value of the exponent for $C = 1$. This apparent coincidence will be explained in Subsection 3.4.

3.3. Local maxima

Let us now return to the presence of local maxima for the Lyapunov exponent as a function of the parameter $C$. This ubiquitous feature is seen especially clearly in the graph on the right-hand side of Figure 12, over the interval $0 \leq C < E$, for the Sinai lensed billiard system. The following comments apply to this system. Though short of
constituting a proof, they contain key ingredients needed for a detailed analysis under more general conditions. We leave this analysis to a future study.

When \( C = 0 \), the circular lens has no effect on the motion of the billiard particle and the Lyapunov exponent is 0. It seems clear that, as \( C \) increases from 0, exponential separation of trajectories should follow due to both (dispersing) reflections and refractions. What is needed then is an explanation for the observed decrease of the Lyapunov exponent as \( C \) increases to \( E \) for small values of \( E - C \).

Figure 15: Left: The time spent by a segment of trajectory inside the lens is \( T(\theta) = 2R\cos\theta_r/s \) where \( \theta_r \) and \( \theta \) are related by Snell’s law and \( s = \sqrt{2(E - C)/m} \) is particle speed. Right: Large expansion factor when \( E - C \) is small. The circular arc consisting of the intersection of the boundary of the lens and the incident parallel beam has angle \( 2\theta_{crit} \). As \( C \) approaches \( E \), this angle approaches 0 and the directions of trajectories as they exit the lens span an interval of angles approaching \((0, 2\pi)\).

We suppose, for the sake of arriving at a rough estimation, that particle collisions with the lens are statistically independent of each other and the angle of incidence is random and satisfies the (cosine) distribution \( \frac{1}{2} \cos\theta \, d\theta \). An elementary calculation yields probability \( p_{\text{refract}} = \sqrt{\frac{E-C}{E}} \) for a collision to result in refraction, where \( E \) is the total energy. The expected expansion rate (of separation of nearby trajectories) in the event of a refraction turns out to be proportional to \( \lambda_{\text{refract}} = \sqrt{\frac{E}{E-C}} \). This is the result of an elementary but long calculation, which we omit. However, this large value, when \( C \) is close to \( E \), is easily understood since the particle will undergo a refraction if the angle of incidence is very small, specifically \( |\sin\theta| < \sin\theta_{crit} = \sqrt{\frac{E-C}{E}} \), but it fans out inside the lens over the full range \( |\theta| < \pi/2 \) with the cosine law distribution. (See Proposition 2.)

The contribution \( \tau_{\text{in}} \) of the motion inside the lens to the total time elapsed over a large number of collision events (reflections and refractions) does not depend on \( C \). In fact, on one hand, the average time spent inside the lens during one refraction event is easily shown to be \( \frac{\pi R}{\sqrt{2(E-C)/m}} \). This is obtained by noting that the time \( T(\theta) \) the segment of
trajectory that enters the lens with angle $\theta$ spends inside is

$$T(\theta) = 2R \cos \theta_r / \sqrt{2(E - C)/m}$$

where $\theta_r$ is the refracted angle obtained from $\theta$ by Snell’s law (see Figure 15); by Proposition 2 once again, $\theta_r$ has the cosine distribution. The mean value is then an easy integral calculation. On the other hand, the proportion of refracting collisions is $\sqrt{E - C}/E$, so the term $\sqrt{E - C}$ cancels out, giving the value $\tau_m = \pi R / \sqrt{2E/m}$. Thus refractions contribute over many collisions an expansion rate per collision proportional to

$$\left(\lambda_{\text{refract}}\right)^{\text{prefract}} = \exp \left( \sqrt{\frac{E - C}{E}} \log \sqrt{\frac{E}{E - C}} \right),$$

whose logarithm decreases to 0 as $C$ approaches $E$ from below. To this should be added a term that contains the contribution to the Lyapunov exponent due to reflections, which is obtained by standard geometric calculations. The result is an expansion factor which, for small $E - C$, has the form

$$\sqrt{\frac{E - C}{E}} \log \sqrt{\frac{E}{E - C}} + \zeta \over \tau + \tau_m,$$

where $\zeta$ is a positive quantity that does not depend on $C$, $\tau$ is the mean time of motion outside of the lens between two consecutive returns to the lens, and $\tau_m$ is the already defined fraction of time of motion inside the lens. Its presence in the denominator explains the previously noted discontinuity of the Lyapunov exponent at $C = E$. We conclude that, for small values of $E - C$ and under the simplifying independence assumption made above, the Lyapunov exponent is expected to decrease to a limit value as $C$ increases towards $E$.

The mechanism suggested here does not account for the second local maximum seen in the example of Figure 14 in the range $C < 0$, $C$ close to 0. A different analysis is needed, which we won’t carry out here.

### 3.4. Deep potential well limit

The examples of lensed billiards discussed in this paper suggest that exploring the asymptotic properties of the exponent as $C \to -\infty$ may be a fruitful direction for further study. One may ask, in particular, whether there are dynamical systems that realize the limit in some sense. The term deep well systems will be used to refer to lensed billiards with strongly negative $C$ or to the limit systems when they can be meaningfully specified.

Let us first look at circular lenses as in the examples of Figure 14 and the right of Figure 12. We refer first to the left of Figure 16. As $C$ tends to $-\infty$, a trajectory entering the circular potential well is deflected by refraction towards the radial direction and returns, after a very short time interval (since the speed is very high inside the disc), to a point on the circle which is very close to that from which it entered the circle. The angle relative to the normal vector at which the trajectory reaches the right semicircle
in the lens boundary after one reflection with the left semicircle is the same as the angle at which the trajectory enters the disc. Therefore the trajectory leaves the disc at the moment of first return to the left semicircle. Furthermore, the velocity with which the trajectory exits the circle is very close to what it would be under specular reflection at that point. Effectively, the system behaves in the limit as that for which \( C = E = 1 \). The limit trajectory is shown in the figure as a dashed line. The duration of the sojourn inside the disc (moving along the radial direction) is zero. Thus it makes sense to say that the limit system is the Sinai-type semi-dispersing standard billiard.

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![Figure 16](image)

Figure 16: In the limit \( C \to -\infty \), the system on the left behaves as if \( C = 1 \), corresponding to a semi-dispersing billiard. For the system on the right, the deep well limit is the composition of a standard Sinai billiard (with reflecting walls) and a rotation by \( \pi \).

The same argument applies to the lensed Sinai billiard on the right of Figure 16. Here, the deep well limit can be described as follows. Let \( T_0 \) be the billiard map of the Sinai billiard in a square with reflecting sides. Let \( R \) be the rotation of the plane \( \mathbb{R}^2 \) by \( \pi \), where the origin is placed at the center of the disc. Note that these two maps commute and that \( R^2 \) is the identity. Then, disregarding the radial segment of trajectory inside the disc (whose duration is 0), the limit system is generated by the map \( T = R \circ T_0 \).

When the lens region \( A_0 \) is such that the standard billiard system in it is ergodic, the deep well limit may best be described as a random dynamical system of the following type. We assume for concreteness that the particle mass is \( \sqrt{2} \) and the total energy is \( E = 1 \). Let \( A \) be the area of \( A_0 \) and \( a \) the length of the intersection \( C \) of the boundaries of \( A_0 \) and \( A_1 \). Set \( T = \pi A/a \). Referring to Figure 17, the random system will then be a billiard-like system in \( A_1 \) that reflects specularly on the part of the boundary not including \( C \) and, on reaching \( C \), trajectories jump to a point \( x \in C \) and assume velocity \( v \) such that \( (x, v) \) are random variables distributed according to the Liouville measure: uniform distribution for \( x \) and the cosine law distribution for \( v \). The jump is not instantaneous but happens with a random time delay with mean \( T \). If the closed billiard system in \( A_0 \) is hyperbolic, one may expect this random time to be exponentially distributed. This possible description of a deep well billiard limit is motivated by Corollary 4 and Proposition 2, as well as the example of Figure 7.
Figure 17: A deep well billiard system and the field of thin cones. When the potential in $A_0$ is strongly negative, a trajectory that falls into $A_0$ undergoes many collisions with $\partial A_0$ during a time interval having mean value $\pi A/a$ until the first return to $C$ for which the direction of approach lies in the field of thin cones defined by $|\sin \theta| < \sqrt{1 + C}$. Here $\theta$ is the angle the velocity of incidence makes with a normal vector to $C$ (pointing into $A_1$) at the collision point. The trajectory then reemerges into $A_1$ at a point in $C$ and velocity having the Liouville measure distribution.

A. APPENDICES

A.1. MOTION UNDER DISCONTINUOUS POTENTIAL

In order to justify on physical grounds our definition of the lensed billiard map, it will be useful to see how trajectories under discontinuous potentials arise in the limit of a family of smooth potential functions with increasingly sharp transition between two constant values. We do this in the setting of Riemannian manifolds of arbitrary dimension.

The following considerations will be local in nature. Suppose that the potential function $V$, restricted to a neighborhood $U$ in the Riemannian manifold $M$ with a smooth metric $\langle \cdot, \cdot \rangle$, has only two values, $C_0$ and $C_1$. The discontinuity of $V$ lies on a smooth hypersurface $\mathcal{S}$, and $U \setminus \mathcal{S}$ is the union of open sets $U_0$ and $U_1$ such that $V|_{U_i} = C_i$. We make the assumption that $C_1 > C_0$. The discussion in this section applies to the opposite inequality with minor modifications. Let $\mathbf{n}$ be a unit vector field on $\mathcal{S}$, perpendicular to $\mathcal{S}$, and pointing into $U_1$. Define for $\epsilon > 0$ the set

$$R_\epsilon := \{ \exp_x (s \mathbf{n}_x) : x \in \mathcal{S} \cap U, s \in [0, \epsilon] \} \cap U_1.$$  

Then $R_\epsilon$ is the union of submanifolds $\mathcal{S}_\epsilon(s)$ consisting of the points $\exp_x (s \mathbf{n}_x)$ in $R_\epsilon$ in which $x \in \mathcal{S}$ and $s$ is constant. Note that $\mathcal{S}_\epsilon(0) = \mathcal{S}$. The vector field $\mathbf{n}$ can be extended to all of $R_\epsilon$ by setting $\mathbf{n}_{\gamma(s)} = \gamma'(s)$ where $\gamma(s) = \exp_x (s \mathbf{n}_x)$ and $x \in \mathcal{S}$. It is not difficult to show (this is essentially Gauss's lemma, [7]) that $\mathbf{n}_y$ is perpendicular to $\mathcal{S}_\epsilon(s)$ at any $y \in \mathcal{S}_\epsilon(s)$. Clearly $D_\mathbf{n} \mathbf{n} = 0$ since the integral curves of $\mathbf{n}$ are geodesics. (Here $D$ denotes the Levi-Civita connection.)

We define a smooth potential function $V_\epsilon$ on $U$ as follows. Let $f_\epsilon : \mathbb{R} \to [C_0, C_1]$ be a
smooth real-valued increasing function such that
\[
f_\epsilon(s) = \begin{cases} 
C_0 & \text{if } s \leq 0 \\
C_1 & \text{if } s \geq \epsilon. 
\end{cases}
\]

Now set
\[
V_\epsilon(x) = \begin{cases} 
C_0 & \text{if } x \in U_0 \\
C_1 & \text{if } x \in U_1 \setminus R_\epsilon \\
f_\epsilon(s) & \text{if } x \in S_\epsilon(s). 
\end{cases}
\]

Let \( S_x \) denote the shape operator of the level hypersurfaces \( S_\epsilon(s) \). Thus, by definition,
\[
S_x v = -D_v n
\]
for all \( v \in T_x S_\epsilon(s) \). Finally, if \( v \in T_x R_\epsilon \), the orthogonal decomposition of \( v \) into a tangent vector \( v_\tau \) to the hypersurface \( S_\epsilon(s) \) containing \( x \) and a perpendicular vector will be written as \( v = v_\tau + v_n n_x \).

**Lemma 6.** Newton’s equation in \( R_\epsilon \) with potential function \( V_\epsilon \) as defined in Equation (7) decomposes orthogonally as
\[
\frac{Dv_\tau}{dt} = \langle v_\tau, S_x v_\tau \rangle n + v_n S_x v_\tau
\]
\[
-\frac{dv_n}{dt} = \langle v_\tau, S_x v_\tau \rangle + \frac{1}{m} f_\epsilon'(s(x)).
\]

**Proof.** Since \( D_n n = 0 \), we have
\[
\frac{Dn}{dt} = D_v n = D_{v_n} n + v_n D_n n = -S_x v_\tau.
\]
\[
\left\langle \frac{Dv_\tau}{dt}, n \right\rangle = -\langle v_\tau, D_x n \rangle = \langle v_\tau, S_x v_\tau \rangle.
\]
Let \( \Pi_x \) denote the orthogonal projection to the tangent space at \( x \) to the hypersurface \( S_\epsilon(s) \) containing \( x \). Then
\[
-f_\epsilon'(s)n = -\text{grad } V
\]
\[
= m \frac{Dv}{dt}
\]
\[
= m \left( \frac{Dv_\tau}{dt} + \dot{v}_n n - v_n S_x v_\tau \right)
\]
\[
= m \left( \Pi \frac{Dv_\tau}{dt} + \langle v_\tau, S_x v_\tau \rangle n + \dot{v}_n n - v_n S_x v_\tau \right).
\]

Separating the normal and tangential parts we obtain \( \Pi Dv_\tau / dt = v_n S_x v_\tau \) and the desired equations. \( \square \)
Lemma 7. We make the same assumptions as in Lemma 6, except that now we require $V$ to be constant equal to $C_0$ on $U_0$ and constant equal to $C_1$ on $U_1 \setminus R_\epsilon$. Let the initial velocity of a particle that enters $R_\epsilon$ from $U_0$ be $v(0) = v^-$ and let the velocity upon exit from $R_\epsilon$ be $v^+$ for the discontinuous potential function $V$ and $v^+_\epsilon$ for the potential $V_\epsilon$. Further assume that the shape operator of the hypersurface $S$ is bounded. Then $v^+_\epsilon = v^+_\epsilon + O(\epsilon)$ and the return time to the boundary of $R_\epsilon$ is $T_\epsilon = O(\epsilon)$.

Proof. We consider the case $C_1 > C_0$. The opposite inequality can be argued similarly. Thus $V_\epsilon$ is increasing along the radial direction (parallel to the vector field $n$) in $R_\epsilon$. Energy conservation implies

$$\|v(t)\| = \sqrt{2(E - V_\epsilon(x(t)))} / m \leq \|v^-\|$$

for all $t$. We are only concerned with the trajectory $x(t)$ from $t = 0$ to $t = T_\epsilon$, when it reaches again the boundary of $R_\epsilon$ at a point where $V_\epsilon = C_1$. (The case in which the trajectory does not overcome the potential barrier and returns to the boundary of $U_0$ can be dealt with by similar arguments.) Thus we know that

$$B := \frac{(v_n^-)^2 - \frac{2}{m}(C_1 - C_0)}{4\|v^-\|^3 K} > 0$$

where $K$ is an upper bound on the norm of the shape operator. By Lemma 6

$$\frac{d}{dt} \left( \frac{1}{2} m v_n^2 \right) = v_n \left[ -m(v_\tau, S_x v_\tau) - f'_\epsilon(s(t)) \right].$$

Note that $v_n = \dot{s}$, so $f'_\epsilon(s(t))v_n dt = d(f_\epsilon(s(t)))$. Integrating in $t$ gives

$$v_n^2(t) = v_n^2(0) - \frac{2}{m} \left[ f(s(t)) - C_0 \right] - 2 \int_0^t v_n(u)(v_\tau(u), S_x(u)v_\tau(u)) du \geq v_n^2(0) - \frac{2}{m} (C_1 - C_0) - 2\|v^-\|^3 K t$$

$$= \frac{(v_n^-)^2 - \frac{2}{m}(C_1 - C_0)}{4\|v^-\|^3 K} - 2\|v^-\|^3 K t$$

This quantity is bounded away from 0 for $t \in [0, B]$. Explicitly,

$$v_n(t) \geq \sqrt{2\|v^-\|^3 KB}$$

in that interval. Then

$$\int_0^B v_n(t) dt \geq 2\sqrt{\|v^-\|^3 KB} \int_0^B \sqrt{1 - \frac{t}{2B}} dt = \frac{8}{3} \left( 1 - \frac{1}{2\sqrt{2}} \right) \sqrt{\|v^-\|^3 KB^3} =: A > 0.$$

Setting $\epsilon < A$, we can be certain that the trajectory $x(t)$ will reach the boundary of $R_\epsilon$ (on the side of $U_1$) at $s = \epsilon$ in time $T_\epsilon \leq B$ and that

$$\epsilon = \int_0^{T_\epsilon} v_n(t) dt \geq \sqrt{2\|v^-\|^3 KBT_\epsilon}.$$
This shows that

\[ T_\epsilon \leq \frac{\epsilon}{\sqrt{2\|v\|^2 K B}} = O(\epsilon). \]

It is now a simple consequence of Lemma 6 that

\[ v_\tau(T_\epsilon) = v_\tau(0) + O(\epsilon), \quad v_n(T_\epsilon) = \sqrt{v_n(0) - \frac{2}{m} (C_1 - C_n)} + O(\epsilon). \]

Therefore \( v^+ = v^+_\epsilon + O(\epsilon) \) as claimed. (More properly, one may write the first equation in Lemma 6 as a system of first order non-linear equations in the components of \( v_\tau \) with respect to an orthonormal parallel frame of vector fields along a trajectory \( x(t) \). The approximation given above in (8) is easily shown to hold for these components.)

A.2. Lensed billiard map differential

We give here the proof of Theorem 5. Let \( t = t(x_1, \theta_1) \) be such that \( Q = q + tv \). Then

\[ \gamma_2(x_2) = \gamma_1(x_1) + t(x_1, \theta_1)e^{\theta_1 J}v_1. \]

Note that \( \gamma'_1(x_1) = t_1(\gamma_1(x_1)) \) and \( \gamma'_2(x_2) = t_2(\gamma_2(x_2)) \). Differentiating Equation (9) in \( x_1 \),

\[ \frac{\partial x_2}{\partial x_1}(0,0)t_2(\theta_2) = t_1(\theta_1) + \frac{\partial t}{\partial x_1}(0,0)v_1. \]

This implies

\[ \frac{\partial t}{\partial x_1}(0,0) = -t_1(\theta_1) \cdot n_2(\theta_2)/v_1 \cdot n_2(\theta_2) \]

and

\[ \frac{\partial x_2}{\partial x_1}(0,0) = t_1(\theta_1) \cdot t_2(\theta_2) - t_1(\theta_1) \cdot n_2(\theta_2) \frac{v_1 \cdot t_2(\theta_2)}{v_1 \cdot n_2(\theta_2)} \]

\[ = \left[ (t_2(\theta_2) \land n_2(\theta_2)) \cdot t_1(\theta_1) \right] \cdot \nu_1 \]

\[ = \frac{v_1 \cdot n_1(\theta_1)}{v_1 \cdot n_2(\theta_2)}. \]

We have used the operation \((a \land b)c = (a \cdot c)b - (b \cdot c)a\) for vectors \(a, b, c \in \mathbb{R}^n\). If \( n = 2 \) and \((a, b)\) is a positive orthonormal basis of \( \mathbb{R}^2 \) then \( J = a \land b \) is rotation counterclockwise by \( \pi/2 \). Taking now the derivative in \( \theta_1 \) of both sides of Equation (9),

\[ \frac{\partial x_2}{\partial \theta_1}(0,0)t_2(\theta_2) = \frac{\partial t}{\partial \theta_1}(0,0)v_1 + t(0,0)Jv_1. \]

Noting that \( t(0,0) = \ell/|v_1| \), we obtain

\[ \frac{\partial t}{\partial \theta_1}(0,0) = -\frac{\ell}{|v_1|} \frac{(Jv_1) \cdot n_2(\theta_2)}{v_1 \cdot n_2(\theta_2)} = -\frac{\ell}{|v_1|} \frac{v_1 \cdot t_2(\theta_2)}{v_1 \cdot n_2(\theta_2)}. \]
\[ \frac{\partial x_2}{\partial \theta_1}(0, 0) = -\frac{\ell}{|v_1|} \frac{\nu_1 \cdot t_2(\Theta_2)}{\nu_1 \cdot n_2(\Theta_2)} v_1 \cdot t_2(\Theta_2) + \frac{\ell}{|v_1|} (Jv_1) \cdot t_2(\Theta_2) \]
\[ = -\frac{\ell}{|v_1|} \frac{\nu_1 \cdot t_2(\Theta_2)}{\nu_1 \cdot n_2(\Theta_2)} v_1 \cdot t_2(\Theta_2) - \ell \nu_1 \cdot n_2(\Theta_2) \]
\[ = -\frac{\ell}{|v_1|} \frac{\nu_1 \cdot n_2(\Theta_2)}{\nu_1 \cdot n_2(\Theta_2)} \left[ (\nu_1 \cdot t_2(\Theta_2))^2 + (\nu_1 \cdot n_2(\Theta_2))^2 \right] \]
\[ = -\frac{\ell}{|v_1|} \frac{\nu_1 \cdot n_2(\Theta_2)}{\nu_1 \cdot n_2(\Theta_2)}. \]

We have obtained so far the first row of the differential of \( \mathcal{T} \) for both reflection and refraction. For the second row, the two cases must be treated separately. Let us first consider reflection. Then \( V = v - 2v \cdot n_2(Q) n_2(Q) \) which, in the \( x_1, \theta_1 \) coordinates, is

\[ e^{\theta^J} v_2 = e^{\theta^J} v_1 - 2 \left( e^{\theta^J} v_1 \right) \cdot n_2(Q(x_1, \theta_1)) n_2(Q(x_1, \theta_1)), \]

where, by Equation (9), \( \gamma_2(x_2) = Q(x_1, \theta_1) = \gamma_1(x_1) + t(x_1, \theta_1) e^{\theta^J} v_1 \). Differentiating Equation (10) in \( x_1 \) at \( x_1 = 0, \theta_1 = 0 \), yields

\[ \frac{\partial \theta_2}{\partial x_1}(0, 0) j_2 = -2v_1 \cdot \left( \frac{Dn_2}{\partial x_1}(\Theta_2) \right) n_2(\Theta_2) - 2v_1 \cdot n_2(\Theta_2) \frac{Dn_2}{\partial x_1}(\Theta_2). \]

Now

\[ \frac{Dn_2}{\partial x_1}(\Theta_2) = \frac{\partial x_2}{\partial x_1}(0, 0)(D_{x_2} n_2)(\Theta_2) = -\frac{\partial x_2}{\partial x_1}(0, 0) \kappa(\Theta_2) t_2(\Theta_2). \]

Taking the dot product of Equation (11) with \( t_2(\Theta_2) \), solving for \( \frac{\partial \theta_2}{\partial x_1}(0, 0) \), and substituting the already obtained value of \( \frac{\partial x_2}{\partial x_1}(0, 0) \), gives

\[ \frac{\partial \theta_2}{\partial x_1}(0, 0) = -2\kappa(\Theta_2) v_1 \cdot n_2(\Theta_2) v_1 \cdot n_1(\Theta_1) \]
\[ = -2\kappa(\Theta_2) \frac{v_1 \cdot n_1(\Theta_1)}{v_2 \cdot n_2(\Theta_2)} v_1 \cdot n_2(\Theta_2). \]

Next, we differentiate Equation (10) in \( \theta_1 \) at \( x_1 = 0, \theta_1 = 0 \):

\[ \frac{\partial \theta_2}{\partial \theta_1}(0, 0) j_2 = Jv_2 - 2Jv_1 \cdot n_2(\Theta_2) n_2(\Theta_2) \]
\[ = -2v_1 \cdot \frac{Dn_2}{\partial \theta_1}(\Theta_2) n_2(\Theta_2) - 2v_1 \cdot n_2(\Theta_2) \frac{Dn_2}{\partial \theta_1}(\Theta_2). \]

Note that
\[ \frac{Dn_2}{\partial \theta_1}(\Theta_2) = \frac{\partial x_2}{\partial \theta_1}(0, 0) (D_{\theta_2} n_2)(\Theta_2) = -\frac{\partial x_2}{\partial \theta_1}(0, 0) \kappa(\Theta_2) t_2(\Theta_2). \]

Taking the inner product of Equation (12) with \( t_2(\Theta_2) \), substituting the already obtained \( \frac{\partial x_2}{\partial \theta_1}(0, 0) \), and solving for \( \frac{\partial \theta_2}{\partial \theta_1}(0, 0) \), results in

\[ \frac{\partial \theta_2}{\partial \theta_1}(0, 0) = \frac{v_1 \cdot n_2(\Theta_2)}{v_2 \cdot n_2(\Theta_2)} + \frac{2\kappa(\Theta_2) \ell}{v_2 \cdot n_2(\Theta_2)}. \]
This gives the differential when $T$ produces a reflection. We now turn to the case of refraction, for which

$$V = v \cdot t_2(Q) t_2(Q) + \left[ (v \cdot n_2(Q))^2 - \frac{2(C_2 - C_1)}{m} \right]^{\frac{1}{2}} n_2(Q).$$

Using Equations (6) and differentiating Equation (13) in $x_1$,

$$\frac{\partial \theta_2}{\partial x_1}(0,0) J v = v_1 \cdot \frac{Dt_2}{\partial x_1}(O_2) t_2(O_2) + v_1 \cdot t_2(O_2) \frac{Dt_2}{\partial x_1}(O_2)$$

$$+ v_1 \cdot n_2(O_2) \frac{Dn_2}{\partial x_1}(O_2) \left[ (v_1 \cdot n_2(O_2))^2 - \frac{2(C_2 - C_1)}{m} \right]^{\frac{1}{2}} n_2(O_2)$$

$$+ \left[ (v_1 \cdot n_2(O_2))^2 - \frac{2(C_2 - C_1)}{m} \right]^{\frac{1}{2}} Dn_2 \left( \frac{Dt_2}{\partial x_1}(O_2) \right).$$

Noting that

$$\frac{Dt_2}{\partial x_1}(O_2) = \kappa(O_2) n_2(O_2), \quad \frac{Dn_2}{\partial x_1}(O_2) = -\kappa(O_2) t_2(O_2),$$

inserting the already obtained value for $\frac{\partial x_2}{\partial x_1}(0,0)$, taking the inner product of Equation (14) with $t_2(O_2)$, and isolating $\frac{\partial \theta_2}{\partial x_1}(0,0)$ yields, after algebraic simplification,

$$\frac{\partial \theta_2}{\partial x_1}(0,0) = -\kappa(O_2) \sqrt{\frac{E - C_1}{E - C_2} \nu_1 \cdot n_2(O_2) \alpha},$$

where

$$\alpha := 1 - \left[ 1 - \frac{C_2 - C_1}{E - C_1} \frac{1}{(v_1 \cdot n_2(O_2))^2} \right]^{\frac{1}{2}}.$$

Finally, differentiating Equation (13) in $\theta_1$ at $x_1 = 0, \theta_1 = 0$, taking the inner product with $t_2(O_2)$, using the already obtained partial derivative of $x_2$ with respect to $\theta_1$, replacing the derivatives of $t_2$ and $n_2$ by expressions involving $\kappa$, and finally isolating $\frac{\partial \theta_2}{\partial \theta_1}(0,0)$ yields

$$\frac{\partial \theta_2}{\partial \theta_1}(0,0) = \sqrt{\frac{E - C_1}{E - C_2} \nu_1 \cdot n_2(O_2)} \left( 1 + \frac{\kappa(O_2) \ell}{\nu_1 \cdot n_2(O_2) - \alpha} \right).$$

This is the last term of the differential of $T$ that was left to compute.

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