ON WEAK (MEASURE-VALUED)-STRONG UNIQUENESS FOR COMPRESSIBLE MHD SYSTEM WITH NON-MONOTONE PRESSURE LAW

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ABSTRACT. In this paper, we define a renormalized dissipative measure-valued (rDMV) solution of the compressible magnetohydrodynamics (MHD) equations with non-monotone pressure law. We prove the existence of the rDMV solutions and establish a suitable relative energy inequality. And we obtain the weak (measure-valued)-strong uniqueness property of this rDMV solution with the help of the relative energy inequality.

1. Introduction. Suppose $T > 0$ and $\Omega \subset \mathbb{R}^3$ be a bounded domain with the smooth boundary, the magnetohydrodynamics (MHD) equations on $(0,T) \times \Omega$ read (see [3]),

\begin{align*}
\frac{\partial \rho}{\partial t} + \text{div}(\rho \vec{u}) &= 0, \\
\frac{\partial \rho \vec{u}}{\partial t} + \text{div}(\rho \vec{u} \otimes \vec{u}) - \text{div}S(\nabla \vec{u}) + \nabla p(\rho) = (\nabla \times \vec{B}) \times \vec{B}, \\
\frac{\partial \vec{B}}{\partial t} &= \nabla \times (\vec{u} \times \vec{B}) - \nu \nabla \times (\nabla \times \vec{B}), \\
\text{div} \vec{B} &= 0,
\end{align*}

(1.1)

where $\rho \geq 0$ is the density, $\vec{u} \in \mathbb{R}^3$ is the velocity, $\vec{B} \in \mathbb{R}^3$ is the magnetic field, $p$ is the pressure and $S(\nabla \vec{u}) = \text{div} (\mu (\nabla \vec{u} + \nabla^T \vec{u}) + \lambda \text{div} \vec{u} I)$. The viscosity coefficients $\mu$ and $\lambda$ satisfy

$$\mu > 0, \ 3\lambda + 2\mu > 0. \tag{1.2}$$

Moreover, $\nu \geq 0$ is the resistivity coefficient which represents the magnetic diffusion of the magnetic field. The equations (1.1) are supplemented with the initial conditions,

$$(\rho, \vec{m}, \vec{B})(0, x) = (\rho_0, \vec{m}_0, \vec{B}_0), \tag{1.3}$$

together with the no-slip boundary condition,

$$\vec{u}|_{\partial \Omega} = 0, \ \vec{B}|_{\partial \Omega} = 0 \ (\nu > 0), \tag{1.4}$$
$$\vec{u}|_{\partial \Omega} = 0 \ (\nu = 0). \tag{1.5}$$

When $B = 0$ which means no electromagnetic field, the system (1.1) reduces to the compressible Navier-Stokes equations. For the case $p(\rho) = \rho \gamma$ and the

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viscosities are constants, Hoff proved the global existence of the weak solution with the small initial energy in [23, 24]. The global existence of weak solutions with the large initial data has been verified by Lions [27] with \( \gamma > \frac{2}{3} \) and refined by Feireisl [13, 18] with \( \gamma > \frac{2}{3} \). In [17], Feireisl, Jin and Novotný proved weak-strong uniqueness for the global weak solution with the pressure \( p \) satisfying

\[
p \in C[0, \infty) \cap C^2(0, \infty), \quad p(0) = 0, \quad p'(0) > 0, \quad \lim_{\rho \to \infty} \frac{p'(\rho)}{\rho^{\gamma-1}} = a > 0.
\]  

(1.6)

In [21], weak-strong uniqueness principle for compressible Navier-Stokes equations with pressure \( p(\rho) = \rho^\gamma \) has been proved under that the strong solution \((\bar{\rho}, \bar{u})\) satisfies

\[
\nabla \pi \in L^1(0,T,L^\infty(\Omega)), \quad S(\nabla \pi) \in L^2(0,T,L^d(\Omega)),
\]

which means the strong solution dose not need to be derivable continuously.

For the compressible Navier-Stokes equations with non-monotone pressure law, there are some results in [1, 4, 5, 12, 14, 15] and the references therein. In [14], considering the compressible Navier-Stokes equations on \( \mathbb{R}^N \) with pressure law \( p(s) \) satisfying

\[
p'(s) \geq a s^{\gamma-1} - b, \quad s > 0,
\]

\[
\frac{p(s)}{s} \text{ is nondecreasing, } \quad s \geq s_0,
\]

where \( a > 0, \ b \geq 0, \ s_0 > 0, \) and \( \gamma > \frac{N}{2} \), Feireisl proved the global existence of weak solutions. In [4], Chang, Jin and Novotný considered the compressible Navier-Stokes equations with the pressure law \( p = p_1 - p_2 \) where \( p'_1(s) \geq \max \{0, a s^{\gamma-1} - b\} \), \( 0 \leq p_1(s) \leq a \rho^\gamma + b \) and \( 0 \leq p_2 \in C^\omega([0, \infty)) \) and showed the existence of weak solutions with general inflow-outflow boundary data. In [1], Bresch and Jabin proved the global existence of the weak solution to the compressible Navier-Stokes equations with more general pressure law satisfying \( |p(x) - p(y)| \leq (\tilde{x}^\gamma + y^\gamma) |x - y| \) and \( \rho^\gamma - 1 \lesssim p(\rho) \lesssim \rho^\gamma + 1 \) with \( \gamma > (\max\{\tilde{\gamma}, 2\} + 1) \frac{2}{5} \). In [5, 15], weak-strong uniqueness principle of the global weak solutions has been shown for the compressible Navier-Stokes equations with a general nonmonotone pressure \( p(\rho) = h(\rho) + q(\rho) \) where \( h(0) = 0, h' > 0 \) on \((0, \tilde{p})\), \( \lim_{\rho \to \infty} h(\rho) = +\infty \), and \( q \in C^1(0, \tilde{p}) \).

For MHD equations with \( \nu > 0 \), Hu and Wang [26] showed the global existence of the weak solutions to the equations (1.1) in \( \mathbb{R}^3 \) with the pressure law in particular form \( p(\rho) = a \rho^\gamma \) and \( \gamma > \frac{3}{2} \). Ducomet and Feireisl obtained the global existence of weak solutions with the finite energy initial data for the heat-conducting fluids together with the influence of radiation in [10]. The the global existence of the weak solutions to the equations (1.1) with \( 1 \leq \gamma \leq \frac{3}{2} \) has not be solved. In [31], Yan proved weak-strong uniqueness property for full compressible magnetohydrodynamics flows with \( p(\rho) = a \rho^\gamma \). In [32], the authors proved weak-strong uniqueness for 3D compressible MHD equations with \( p(\rho) = a \rho^\gamma \). In [19], the authors proved weak-strong uniqueness for 3D incompressible MHD in weighted \( L^2 \) spaces. For more results about MHD equations with \( \nu > 0 \), see [7, 8, 11, 20, 25, 30] and the references therein. For the case \( \nu = 0 \), in [28], we considered a 2D MHD system and proved the global existence of weak solutions with a density-depending viscosity and the pressure satisfying

\[
\rho^\gamma - 1 \lesssim p(\rho) \lesssim \rho^\gamma + 1,
\]
\[|p(x) - p(y)| \lesssim (x^{\tilde{y}-1} + y^{\tilde{y}-1}) |x - y|, \quad \forall \ x, y \geq 0. \quad (1.7)\]

The concept of measure-valued solutions of compressible Navier-Stokes equations was introduced by Neustupa [29]. In [16], Feireisl introduced the dissipative measure-valued solutions to compressible Navier-Stokes equations and proved weak-solutions was introduced. In [16], Feireisl introduced the dissipative measure-valued solutions to compressible Navier-Stokes equations and proved weak-strong uniqueness under the pressure law \( p(\rho) = a\rho^\gamma \). In [6], Chaudhuri proved weak-strong uniqueness with nonmonotone pressure \( p(\rho) = h(\rho) + q(\rho) \) where

\[h \in C^1[0, \infty), \ h(0) = 0, \ h' > 0, \ \lim_{\rho \to \infty} \frac{h'(\rho)}{\rho^{\gamma - 1}} = a > 0, \ q \in C^1_c(0, \infty). \quad (1.8)\]

We know that to get the existence of the weak solutions to MHD equations, the key point is to avoid the concentration and oscillation of \( \rho \) and \( \vec{B} \) which need for higher integrability estimates and \( \gamma > \frac{3}{2} \) in \( \mathbb{R}^3 \). And the strong convergence of \( \rho \) and \( \vec{B} \) in \( L^1 \) come from the calculation of Young measure which demands a monotonic pressure. When \( \gamma \) close to 1, there are no longer higher integrability estimates which for \( \rho \) higher than \( \gamma \) and for \( \vec{B} \) higher than 2, which means the concentration and oscillation can not be avoided. We want to control the concentration and oscillation by a dissipation defect. Then we have the definition of the measure-valued solution. As here we do not need \( \rho \) and \( \vec{B} \) converge strongly, the monotonicity of the pressure is not necessary. In the proof of the existence of the measure-valued solution, in order to get the uniform bound of density in \( L^\infty(L^\gamma) \), we assume that the pressure is controlled by \( \rho^\gamma \) as defined in (H1) below. Using the lower semicontinuity of weak convergence, the dissipation defect is easy to get for a large range of pressure as in (H1). In the proof of weak-strong uniqueness, we need some kind of convexity of the pressure to control the terms like \( K_3 \) in (5.9). Then we consider a pressure in the form of \( \rho^\gamma \) plus some disturbance as stated in (H2).

In this paper, we prove the existence of the measure-valued solutions under the pressure satisfying (H1) which is a more general case compared to (1.7) and (1.8) above,

\[ (H1). \ 0 \leq p(s) \in C([0, \infty)), \ p(0) = 0 \text{ with } \ C_1 s^\gamma - C_2 \leq p(s) \leq C_3 s^\gamma + C_4, \ \forall s \in [0, \infty), \quad (1.9) \]

where \( \gamma \geq 1 \), and \( C_i, \ i = 1, \ldots, 4 \) are positive constants.

To prove the weak-strong uniqueness, we need the pressure law in a more specific form, so here we suppose \( p(\rho) = h(\rho) + q(\rho) \). Through the relative energy method, \( q(\rho) \) does not need a compact support compared to (1.8). Here we prove the weak-strong uniqueness with the pressure law satisfying (H2).

\[ (H2). \ p(s) = h(s) + q(s), \text{ with } \]

\[ h(s) \in C^1[0, \infty), \ h(0) = 0, \ h'(s) > 0, \ \forall s > 0, \ \lim_{s \to +\infty} \frac{h'(s)}{s^{\gamma - 1}} = a > 0, \ \gamma \geq 1, \]

\[ q(s) \in C[0, \infty), \ \text{locally Lipschitz}, \ q(0) = 0, \ q^2(s) \leq C(1 + s^\gamma). \]

In this paper, for simplicity we consider \( \Omega \) a bounded domain with no-slip boundary conditions. In fact, for general domain which can be the whole space \( \mathbb{R}^3 \), or a periodic box with adequate boundary conditions, the results in Sections 2 and 3 are still valid. For the case with bounded domain, the results also hold for slip boundary conditions \( \vec{B} \cdot \vec{n} = 0 \), and \( \vec{n} \times \nabla \times \vec{B} = 0 \) on \( (0, T) \times \partial \Omega \) when \( \nu > 0 \). In the proof, we can see that the no-slip boundary conditions are not irreplaceable. And we omit the detailed proof for other boundary conditions.
In this paper, we introduce the renormalized dissipative measure-valued solutions to the compressible MHD equations (1.1) and prove the weak-strong uniqueness principle. In Sections 2 and 3, we give the definition of the renormalized dissipative measure-valued (rDMV) solutions to the compressible MHD equations and state the existence and weak-strong uniqueness results for the case \( \nu > 0 \) and \( \nu = 0 \) separately. In Sections 4, we prove the existence of the (rDMV) solutions. In Section 5, we prove the weak-strong uniqueness principle.

2. Definition of measure-valued solution and main results when \( \nu > 0 \). In this section, we consider the case \( \nu > 0 \). We choose phase space

\[
\mathcal{F} = \{(s, \vec{v}, \vec{B}, \mathbb{D}_\vec{v}, \vec{J}) | s \in [0, \infty), \vec{v}, \vec{B} \in \mathbb{R}^3, \mathbb{D}_\vec{v} \in \mathbb{R}^{3 \times 3}, \vec{J} \in \mathbb{R}^3\},
\]

and \( \mathcal{P}(\mathcal{F}) \) denotes the space of probability measures on \( \mathcal{F} \), then we define the measure-valued solution of the MHD system (1.1).

**Definition 2.1.** We say a parameterized measure \( \{\mathcal{V}_{t,x}\} \subset L^\infty_w((0, T) \times \Omega; \mathcal{P}(\mathcal{F})) \) such that \( \langle \mathcal{V}_{t,x}; s \rangle = \rho, \langle \mathcal{V}_{t,x}; \vec{v} \rangle = \vec{u}, \langle \mathcal{V}_{t,x}; \vec{B} \rangle = \vec{B}, \langle \mathcal{V}_{t,x}; \mathbb{D}_\vec{v} \rangle = \nabla \vec{u}, \) and \( \langle \mathcal{V}_{t,x}; \vec{J} \rangle = \nabla \times \vec{B} \) is a renormalized dissipative measure-valued (rDMV) solution of the MHD system (1.1) on \( (0, T) \times \Omega \), with the initial condition \( \mathcal{V}_{0,x} \in L^\infty_w(\Omega; \mathcal{P}([0, \infty) \times \mathbb{R}^3 \times \mathbb{R}^3)) \) and dissipation defect \( 0 \leq D_\nu \in L^\infty(0, T) \), if the following holds.

1) \( \langle \mathcal{V}_{t,x}; s \rangle = \rho \in L^\infty(0, T, L^7(\Omega)), \quad \langle \mathcal{V}_{t,x}; \vec{v} \rangle = \vec{u} \in L^2(0, T, H^1_0(\Omega)), \quad \langle \mathcal{V}_{t,x}; s\vec{v} \rangle \in L^2(0, T, L^{\frac{6\gamma}{\gamma + 1}}(\Omega)) \cap L^\infty(0, T, L^{\frac{2\gamma}{\gamma + 1}}(\Omega)), \quad \langle \mathcal{V}_{t,x}; \mathbb{D}_\vec{v} \rangle = \nabla \vec{u} \in L^2([0, T] \times \Omega), \quad \langle \mathcal{V}_{t,x}; \vec{B} \rangle = \vec{B} \in L^2(0, T; H^1_0(\Omega)) \cap L^\infty(0, T; L^2(\Omega)), \quad \langle \mathcal{V}_{t,x}; \vec{J} \rangle = \nabla \times \vec{B} \in L^2([0, T] \times \Omega), \)  

2) Equation of continuity. For a.e. \( \tau \in (0, T) \), \( \forall \psi \in C^1([0, T] \times \overline{\Omega}) \), we have

\[
\int_0^\tau \int_\Omega \langle \mathcal{V}_{t,x}; s \rangle \psi(\tau, \cdot) dx - \int_\Omega \langle \mathcal{V}_{0,x}; s \rangle \psi(0, \cdot) dx = \int_0^\tau \int_\Omega \langle \mathcal{V}_{t,x}; s\vec{v} \rangle \partial_t \psi + \langle \mathcal{V}_{t,x}; \mathbb{D}_\vec{v} \rangle \cdot \nabla \psi \, dx \, dt. \tag{2.8}
\]

3) Renormalized equation of continuity. For a.e. \( \tau \in (0, T) \), \( \forall \psi \in C^1([0, T] \times \overline{\Omega}) \) and \( b \in C^1[0, \infty) \) satisfying \( b'(s) = 0 \) for sufficiently large \( \tau \), we have

\[
\int_\Omega \langle \mathcal{V}_{t,x}; b(s) \rangle \psi(\tau, \cdot) dx - \int_\Omega \langle \mathcal{V}_{0,x}; b(s) \rangle \psi(0, \cdot) dx = \int_0^\tau \int_\Omega \langle \mathcal{V}_{t,x}; b(s) \rangle \partial_t \psi + \langle \mathcal{V}_{t,x}; b(s) \vec{v} \rangle \cdot \nabla \psi \, dx \, dt - \int_0^\tau \int_\Omega \langle \mathcal{V}_{t,x}; (sb'(s) - b(s))\text{tr}(\mathbb{D}_\vec{v}) \rangle \psi \, dx \, dt. \tag{2.9}
\]
4) Equation of magnetic field. For a.e. \( \tau \in (0, T), \ \forall \varphi \in C^1_c([0, T] \times \overline{\Omega}; \mathbb{R}^3) \), we have

\[
\int_{\Omega} \left\langle \mathbf{v}_{t,x}; \mathbf{\hat{B}} \right\rangle \cdot \varphi(\tau, \cdot) dx - \int_{\Omega} \left\langle \mathbf{v}_{0,x}; \mathbf{\hat{B}} \right\rangle \cdot \varphi(0, \cdot) dx \\
= \int_0^T \int_{\Omega} \left\langle \mathbf{v}_{t,x}; \mathbf{\hat{B}} \right\rangle \cdot \partial_t \varphi \, dt - \int_0^T \int_{\Omega} \left\langle \mathbf{v}_{t,x}; \mathbf{\hat{B}} \times \mathbf{\hat{v}} \right\rangle \cdot (\nabla \times \varphi) \, dt \\
- \int_0^T \int_{\Omega} \left\langle \mathbf{v}_{t,x}; \nu \mathbf{\hat{\varphi}} \right\rangle \cdot (\nabla \varphi) \, dt, \\
\] 
(2.10)

and

\[
\int_0^T \int_{\Omega} \left\langle \mathbf{v}_{t,x}; \mathbf{\hat{B}} \right\rangle \cdot \nabla \phi \, dt = 0. \\
(2.11)
\]

5) Momentum equation. There exist a measure \( r_{\nu}^M \in L^1(0, T, \mathcal{M}(\overline{\Omega}; \mathbb{R}^{3 \times 3})) \) and \( \xi \in L^1(0, T) \) such that \( \forall \varphi \in C^1([0, T] \times \overline{\Omega}; \mathbb{R}^3), \ \varphi|_{\partial \Omega} = 0 \), we have

\[
\left| \left\langle r_{\nu}^M(\tau); \nabla \varphi \right\rangle, \mathcal{M}(\overline{\Omega}; \mathbb{R}^{3 \times 3}), C(\overline{\Omega}; \mathbb{R}^{3 \times 3}) \right| \leq \xi(\tau) D_{\nu}(\tau) ||\varphi||_{C^1(\overline{\Omega})},
\] 
(2.12)

and

\[
\int_{\Omega} \left\langle \mathbf{v}_{t,x}; \mathbf{s} \mathbf{\hat{t}} \right\rangle \cdot \varphi(\tau, \cdot) dx - \int_{\Omega} \left\langle \mathbf{v}_{0,x}; \mathbf{s} \mathbf{\hat{t}} \right\rangle \cdot \varphi(0, \cdot) dx \\
= \int_0^T \int_{\Omega} \left( \left\langle \mathbf{v}_{t,x}; \mathbf{s} \mathbf{\hat{t}} \right\rangle \cdot \partial_t \varphi + \left\langle \mathbf{v}_{t,x}; \mathbf{s} \mathbf{\hat{t}} \mathbf{\hat{\otimes}} \mathbf{\hat{t}} \right\rangle : \nabla \varphi \right) \, dx \, dt \\
+ \int_0^T \int_{\Omega} \left\langle \mathbf{v}_{t,x}; \mathbf{p}(s) \right\rangle \text{div}_{x} \varphi + \left\langle \mathbf{v}_{t,x}; \mathbf{\hat{J}} \times \mathbf{\hat{B}} \right\rangle \cdot \varphi \right) \, dx \, dt \\
- \int_0^T \int_{\Omega} \left\langle \mathbf{v}_{t,x}; \mathbf{S}(\nabla \varphi) \right\rangle : \nabla \varphi \, dx \, dt \\
+ \int_0^T \left\langle r_{\nu}^M(\tau); \nabla \varphi \right\rangle, \mathcal{M}(\overline{\Omega}; \mathbb{R}^{3 \times 3}), C(\overline{\Omega}; \mathbb{R}^{3 \times 3}) \right| \, dt.
\] 
(2.13)

6) Energy inequality. We define pressure potential \( P(\rho) = \rho \int_0^\rho \frac{P(s)}{s} \, ds \), then the energy inequality reads

\[
\int_{\Omega} \left\langle \mathbf{v}_{t,x}; \frac{1}{2} s |\mathbf{\hat{t}}|^2 + \mathbf{p}(s) + \frac{1}{2} |\mathbf{\hat{B}}|^2 \right\rangle \, dx \\
+ \int_0^T \int_{\Omega} \left\langle \mathbf{v}_{t,x}; \mathbf{S}(\nabla \varphi) : \nabla \varphi + \nu |\mathbf{\hat{J}}|^2 \right\rangle \, dx \, dt + D_{\nu}(\tau) \] 
(2.14)

\[
\leq \int_{\Omega} \left\langle \mathbf{v}_{0,x}; \frac{1}{2} s |\mathbf{\hat{t}}|^2 + \mathbf{p}(s) + \frac{1}{2} |\mathbf{\hat{B}}|^2 \right\rangle \, dx.
\]

**Remark 2.1.** When \( \gamma = 1 \), \( \rho \log \rho \) comes from \( P(\rho) \geq \rho \int_1^\rho \frac{C_1 s - C_2 s}{s} \, ds \geq \rho \log \rho - \rho + 1 \), where \( \log \rho \) means \( \log_e \rho \).

**Remark 2.2.** \( \mathbf{v}_{t,x} \in L^\infty_{\text{loc}}((0, T) \times \Omega; \mathcal{P}(\mathcal{F})) \) means that for any \( \varphi \in C_c(\mathcal{F}) \), the map

\[
(t, x) \mapsto \langle \mathbf{v}_{t,x}, \varphi \rangle
\]

is a Lebesgue-measurable function in \( L^\infty((0, T) \times \Omega) \), see \cite{22}.

**Remark 2.3.** We choose the dissipation defect \( D_{\nu} \) in (4.15).
Remark 2.4. Generalized Korn-Poincaré inequality. Since \( \vec{u} \in L^2(0,T; H_0^1(\Omega)) \), then there exists a positive constant \( C \), such that \( \forall \vec{v} \in L^2(0,T; H_0^1(\Omega)) \),

\[
\int_0^t \int_\Omega \langle V_{t,x}; |\vec{v} - \vec{u}|^2 \rangle \, dx \, dt \leq C \int_0^t \int_\Omega \langle V_{t,x}; |D\vec{v} - \nabla \vec{u}|^2 \rangle \, dx \, dt,
\]

(2.15) see the details in [2].

Theorem 2.1. Suppose (H1) holds and

\[
(\rho_0, \frac{\vec{m}_0}{\sqrt{\rho_0}}, \vec{B}_0) \in L^\gamma(\Omega) \times L^2(\Omega) \times L^2(\Omega)
\]

(2.16) such that \( E(0) = \int_\Omega |\vec{m}_0|^2 2^\rho + P(\rho_0) + \frac{1}{2} |\vec{B}_0|^2 \, dx < \infty \), then there exists a renormalized dissipative measure-valued solution of the MHD system (1.1) as defined in Definition 2.1 with the initial data \( V_0 = \delta \{ \rho_0, \vec{m}_0, \vec{B}_0 \} \).

We define \( H(\rho) = \rho \int_0^\rho \frac{h(z)}{2} \, dz, Q(\rho) = \rho \int_0^\rho \frac{q(z)}{2} \, dz \), and relative energy

\[
\mathcal{E}_{mv}(t) = \mathcal{E}_{mv}(\rho, \vec{u}, \vec{B}[r, \vec{U}, \vec{M}])
= \int_\Omega \langle V_{t,x}; \frac{1}{2} s|\vec{v} - \vec{U}|^2 + \frac{1}{2} |\vec{B} - \vec{M}|^2 \rangle \, dx
+ \int_\Omega \langle V_{t,x}; H(s) - H(r) - H'(r)(s - r) \rangle \, dx,
\]

(2.17) where \( r, \vec{U}, \vec{M} \) are smooth test functions.

Theorem 2.2. Let \( \{V_{t,x}, \mathcal{D}_\nu\} \) be the rDMV solution of the MHD system (1.1) on \((0, T) \times \Omega\) with the initial data \( \mathcal{V}_{0,x} \) and the pressure law satisfies (H2). Suppose \( (r, \vec{U}, \vec{M}) \) be a strong solution to (1.1) with the initial data \( \{r_0, \vec{U}_0, \vec{M}_0\} \) satisfying \( r_0 > 0 \) and

\[
r \in C^1([0, T] \times \overline{\Omega}), \quad \vec{M}, \vec{U} \in C^2([0, T] \times \overline{\Omega}), \quad \inf_{t,x \in [0, T] \times \Omega} r = \underline{r} > 0.
\]

Then there is a constant \( \Lambda = \Lambda(T, \|r, \vec{U}, \vec{M}\|_{W^{1,\infty}, L^2}) \), such that

\[
\mathcal{E}_{mv}(t) + \mathcal{D}_\nu(t) \leq \Lambda \mathcal{E}_{mv}(0), \quad \forall t \in [0, T].
\]

(2.18) Moreover, if \( \mathcal{V}_{0,x} = \delta_{\{r_0, \vec{U}_0, \vec{M}_0\}} \), then \( \mathcal{D}_\nu = 0 \) and \( \mathcal{V}_{t,x} = \delta_{\{r, \vec{U}, \vec{M}_x, \nabla \vec{U}, \nabla \vec{M}\}} \).

Remark 2.5. Here we deal with the case where the space dimension \( d = 3 \). For the case with general space dimension \( d \geq 2 \), the existence theorem Theorems 2.1, 3.1 and the weak-strong uniqueness Theorems 2.2, 3.2 still hold. The methods are the same for \( d \geq 2 \).

Remark 2.6. The proof of Theorem 2.2 relies on the estimates to the renormalized equation of the relative energy. In Section 5, we calculate the renormalized equation of the relative energy and give the detailed estimates to the terms in the renormalized equation using the \textit{a priori} estimates in Section 4.

3. Measure-valued solutions to non-resistivity MHD system. In this section, we consider the compressible non-resistivity MHD system. Since we don’t have
the bound $\nabla \times \vec{B} \in L^2([0, T] \times \Omega)$ in this case, here we rewrite (1.1) in the following form,

\[
\begin{aligned}
\partial_t \rho + \text{div}(\rho \vec{u}) &= 0, \\
\partial_t (\rho \vec{u}) + \text{div}(\rho \vec{v} \otimes \vec{u}) - \text{div}\nabla \vec{u} + \nabla p(\rho) &= \text{div}(\vec{B} \otimes \vec{B}) - \frac{1}{2} \nabla |\vec{B}|^2, \\
\partial_t \vec{B} &= (\vec{B} \cdot \nabla) \vec{u} - (\vec{u} \cdot \nabla) \vec{B} - \vec{B} \text{div} \vec{u}, \\
\text{div} \vec{B} &= 0.
\end{aligned}
\]

(3.1)

We choose phase space

\[\mathcal{F}_2 = \{(s, \vec{v}, \vec{B}, \mathbb{D}\psi)| s \in [0, \infty), \vec{v}, \vec{B} \in \mathbb{R}^3, \mathbb{D}\psi \in \mathbb{R}^{3 \times 3}\},\]

then we define the measure-valued solution of the MHD system (3.1).

**Definition 3.1.**

We say a parameterized measure $\{\mathcal{V}_{t,x}\} \subset L^\infty([0, T] \times \Omega; \mathcal{P}(\mathcal{F}_2))$ such that $\mathcal{V}_{t,x} = \rho$, $\mathcal{V}_{t,x; \vec{v}} = \vec{u}$, $\mathcal{V}_{t,x; \vec{B}} = \vec{B}$ and $\mathcal{V}_{t,x; \mathbb{D}\psi} = \nabla \vec{u}$ is a renormalized dissipative measure-valued (rDMV) solution of the MHD system (3.1) on $(0, T) \times \Omega$, with the initial condition $\mathcal{V}_{0,x} \in L^\infty(\Omega, \mathcal{P}([0, \infty) \times \mathbb{R}^3 \times \mathbb{R}^3))$ and dissipation defect $0 \leq D_0 \in L^\infty(0, T)$, if the following holds.

1) $\mathcal{V}_{t,x; s} = \rho \in L^\infty(0, T, L^\gamma(\Omega))$, $\mathcal{V}_{t,x; \vec{v}} = \vec{u} \in L^2(0, T, H^1_0(\Omega))$, $\mathcal{V}_{t,x; s\vec{v}} \in L^2(0, T, L^\frac{2\gamma}{\gamma - 1}(\Omega)) \cap L^\infty(0, T, L^\frac{2\gamma}{\gamma + 1}(\Omega))$, $\mathcal{V}_{t,x; \mathbb{D}\psi} = \nabla \vec{u} \in L^2([0, T] \times \Omega)$,

2) $\mathcal{V}_{t,x; \vec{B}} = \vec{B} \in L^\infty(0, T; L^2(\Omega))$,

3) moreover when $\gamma = 1$, $\rho \log \rho \in L^1([0, T] \times \Omega)$.

4) Equation of continuity. For a.e. $\tau \in (0, T)$, $\forall \psi \in C^1([0, T] \times \overline{\Omega})$, we have

\[
\int_\Omega (\mathcal{V}_{t,x; s}) \psi(\tau, \cdot) dx - \int_\Omega (\mathcal{V}_{0,x; s}) \psi(0, \cdot) dx = \int_0^\tau \int_\Omega (\mathcal{V}_{t,x; s}) \partial_t \psi + \mathcal{V}_{t,x; s\vec{v}} \cdot \nabla \psi dx dt.
\]

(3.8)

3) Renormalized equation of continuity. For a.e. $\tau \in (0, T)$, $\forall \psi \in C^1([0, T] \times \overline{\Omega})$ and $b \in C^1([0, \infty)$ satisfying $b'(s) = 0$ for sufficiently large $s$, we have

\[
\int_\Omega (\mathcal{V}_{t,x; b(s)}) \psi(\tau, \cdot) dx - \int_\Omega (\mathcal{V}_{0,x; b(s)}) \psi(0, \cdot) dx = \int_0^\tau \int_\Omega (\mathcal{V}_{t,x; b(s)}) \partial_t \psi + \mathcal{V}_{t,x; b(s)\vec{v}} \cdot \nabla \psi dx dt
\]

\[
- \int_0^\tau \int_\Omega \mathcal{V}_{t,x; (sb'(s) - b(s))\text{tr}(\mathbb{D}\psi)} \psi dx dt.
\]

(3.9)

4) Momentum equation. There exist a measure $r^M_0 \in L^1(0, T, \mathcal{M}(\overline{\Omega}; \mathbb{R}^{3 \times 3}))$ and $\xi \in L^1(0, T)$ such that $\forall \varphi \in C^1([0, T] \times \overline{\Omega}; \mathbb{R}^3)$, $\varphi|_{\partial \Omega} = 0$, we have

\[
\left| \left\langle r^M_0(\tau) ; \nabla \varphi \right\rangle_{\mathcal{M}(\overline{\Omega}; \mathbb{R}^{3 \times 3}), C(\overline{\Omega}; \mathbb{R}^{3 \times 3})} \right| \leq \xi(\tau)D_0(\tau)\|\varphi\|_{C^1(\overline{\Omega})},
\]

(3.10)
and
\[
\int_{\Omega} \langle \nu_{t,x} ; \vec{u} \rangle \cdot \varphi(\tau, \cdot) dx - \int_{\Omega} \langle \nu_{0,x} ; \vec{u} \rangle \cdot \varphi(0, \cdot) dx = \int_{0}^{T} \int_{\Omega} (\langle \nu_{t,x} ; \vec{u} \rangle \cdot \partial_{t} \varphi + \langle \nu_{t,x} ; s \vec{v} \otimes \vec{v} \rangle : \nabla \varphi + \langle \nu_{t,x} ; (\vec{v}) \rangle \div \varphi) dx dt
\]
(3.11)
\[- \int_{0}^{T} \int_{\Omega} \langle \nu_{t,x} ; \vec{B} \otimes \vec{B} \rangle : \nabla \varphi dx dt + \frac{1}{2} \int_{0}^{T} \int_{\Omega} \langle \nu_{t,x} ; |\vec{B}|^{2} \rangle \div \varphi dx dt
\]
- \int_{0}^{T} \int_{\Omega} \langle \nu_{t,x} ; \mathcal{S}(\mathcal{D}(\vec{v})) \rangle : \nabla \varphi dx dt + \int_{0}^{T} \langle (r^{z} ; \nabla \varphi) \rangle_{\mathcal{M}(\Pi;E_{3} \times 3), \mathcal{C}(\Pi;E_{3} \times 3)} dt.
\]

5) Equation of magnetic field. For a.e. \( \tau \in (0, T), \forall \varphi \in C_{c}^{1}((0, T) \times \Pi; \mathbb{R}^{3}) \), we have
\[
\int_{\Omega} \langle \nu_{t,x} ; \vec{B} \rangle \cdot \varphi(\tau, \cdot) dx - \int_{\Omega} \langle \nu_{0,x} ; \vec{B} \rangle \cdot \varphi(0, \cdot) dx = \int_{0}^{T} \int_{\Omega} \langle \nu_{t,x} ; \vec{B} \rangle \cdot \partial_{t} \varphi dx dt + \int_{0}^{T} \int_{\Omega} \langle \nu_{t,x} ; -\vec{B} \otimes \vec{v} + \vec{v} \otimes \vec{B} \rangle : \nabla \varphi dx dt,
\]
(3.12)
and
\[
\int_{0}^{T} \int_{\Omega} \langle \nu_{t,x} ; \vec{B} \rangle \cdot \nabla \varphi dx dt = 0. \tag{3.13}
\]

6) Energy inequality. We define pressure potential \( P(\rho) = \rho \int_{0}^{\rho} \frac{p(z)}{z^{2}} dz \), then the energy inequality reads
\[
\int_{\Omega} \langle \nu_{t,x} ; \frac{1}{2} s|\vec{v}|^{2} + P(\rho) + \frac{1}{2} |\vec{B}|^{2} \rangle dx + \int_{0}^{T} \int_{\Omega} \langle \nu_{t,x} ; \mathcal{S}(\mathcal{D}(\vec{v})) \rangle : \mathcal{D}(\vec{v}) dx dt + D_{0}(\tau)
\]
\[
\leq \int_{\Omega} \langle \nu_{0,x} ; \frac{1}{2} s|\vec{v}|^{2} + P(\rho) + \frac{1}{2} |\vec{B}|^{2} \rangle dx. \tag{3.14}
\]

**Remark 3.1.** We choose the dissipation defect \( D_{0} \) in (4.15) with \( \nu = 0 \).

**Remark 3.2.** Compared to Definition 2.1, here \( \vec{B} \in L^{\infty}(0, T; L^{2}(\Omega)) \), and we don’t have any estimate for derivative of \( \vec{B} \) due to \( \nu = 0 \) which makes the equation of \( B \) not a parabolic equation. The definition of the weak form of the equation of \( \vec{B} \) as (3.12) is just (2.10) with \( \nu = 0 \) since \( \nabla \times (\vec{u} \times \vec{B}) = \vec{u} \div \vec{B} - \vec{B} \div \vec{u} + (\vec{B} \cdot \nabla)\vec{u} - (\vec{u} \cdot \nabla)\vec{B} = \div(\vec{B} \otimes \vec{u}) - \div(\vec{u} \otimes \vec{B}) \).

Then we have the following result.

**Theorem 3.1.** Suppose (H1) holds and
\[
(\rho_{0}, \frac{\bar{m}_{0}}{\sqrt{\rho_{0}}}, \vec{B}_{0}) \in L^{2}(\Omega) \times L^{2}(\Omega) \times L^{2}(\Omega) \tag{3.15}
\]
such that \( E(0) = \int_{\Omega} |\bar{m}_{0}|^{2} + P(\rho_{0}) + \frac{1}{2} |\vec{B}_{0}|^{2} dx < \infty \), then there exists a renormalized dissipative measure-valued solution of the MHD system (3.1) as defined in Definition 3.1 with the initial data \( \nu_{0} = \delta_{(\rho_{0}, \bar{m}_{0}, \vec{B}_{0})} \).

**Theorem 3.2.** Let \( \{ \nu_{t,x}, \mathcal{D}_{0} \} \) be the rDMV solution of the MHD system (3.1) on \( (0, T) \times \Omega \) with the initial data \( \nu_{0,x} \) and the pressure law satisfies (H2). Suppose
(r, \tilde{U}, \tilde{M})$ be a strong solution to (3.1) with the initial data \( \{r_0, \tilde{U}_0, \tilde{M}_0\} \) satisfying \( r_0 > 0 \) and
\[
r, \quad \tilde{M} \in C^1([0, T] \times \Omega), \quad \tilde{U} \in C^2([0, T] \times \Omega), \quad \inf_{t,x \in [0,T] \times \Omega} r = \frac{r}{r} > 0.
\]
Then there is a constant \( \Lambda = \Lambda(T, \|r, \tilde{U}, \tilde{M}\|_{W^{1,\infty}(\Omega)} \), such that
\[
E_{mv}(t) + D_0(t) \leq \Lambda E_{mv}(0), \quad \forall t \in [0, T].
\]
Moreover, if \( V_0 \times \delta \in (r_0, \tilde{U}_0, \tilde{M}_0) \), then \( D_0 = 0 \) and \( V_{t,x} = \delta \{r, \tilde{U}, \tilde{M} \} \).

4. **Proof of Theorem 2.1 and Theorem 3.1.** To prove Theorem 2.1, we consider the following approximate system on \([0, T) \times \Omega\),
\[
\begin{aligned}
\begin{cases}
\partial_t \rho + \text{div}(\rho \tilde{u}) = 0, \\
\partial_t (\rho \tilde{u}) + \text{div}(\rho \tilde{u} \otimes \tilde{u}) - \text{div}S(\nabla \tilde{u}) + \nabla (p(\rho) + \delta \|r\|_2 \tilde{u}) = (\nabla \times \tilde{B}) \times \tilde{B}, \\
\partial_t \tilde{B} = \nabla \times (\tilde{u} \times \tilde{B}) - \nu \nabla \times (\nabla \times \tilde{B}), \\
\text{div} \tilde{B} = 0,
\end{cases}
\end{aligned}
\]
where \( \delta > 0 \) is a small parameter, \( \Gamma > \frac{3}{2} \) ensure the existence of weak solution to (4.1), and \( p(\rho) \) satisfies (H1).

Moreover, the initial conditions are described as follows,
\[
\rho(0, x) = \rho_0, \quad \tilde{u}(0, x) = \tilde{u}_0, \quad \tilde{B}(0, x) = \tilde{B}_0, \quad x \in \Omega,
\]
where
\[
\rho_{0,\delta}(x) := \rho_0 * j_\delta, \quad \tilde{B}_{0,\delta}(x) := \tilde{B}_0 * j_\delta, \quad \tilde{u}_{0,\delta} := \left\{ \begin{array}{ll}
\frac{\tilde{u}_0}{\sqrt{\rho_0}} * j_\delta & \text{if } \rho_0 > 0, \\
0 & \text{if } \rho_0 = 0,
\end{array} \right.
\]
with \( j_\delta = \frac{1}{\sqrt{\pi}} j\left(\frac{x}{\delta}\right) \) and \( \int_{\mathbb{R}^3} j dx = 1 \) a smoothing function. Using the initial conditions (1.3) we have
\[
\begin{aligned}
\|\rho_{0,\delta} - \rho_0\|_{L^\gamma(\Omega)} & \to 0, \quad \delta \to 0, \\
\|\tilde{B}_{0,\delta} - \tilde{B}_0\|_{L^2(\Omega)} & \to 0, \quad \delta \to 0, \\
\left\| \frac{\tilde{u}_{0,\delta} - \tilde{u}_0}{\sqrt{\rho_{0,\delta}}} - \frac{\tilde{u}_0 - \tilde{u}_0}{\sqrt{\rho_0}} \right\|_{L^2} & \to 0, \quad \delta \to 0, \\
\delta \frac{3\nu}{\|\rho_0\|_{L^2(\Omega)}} \|\rho_{0,\delta}\|_{L^\Gamma(\Omega)} & \to 0, \quad \delta \to 0.
\end{aligned}
\]

And the boundary condition reads
\[
\tilde{u}_0|_{\partial \Omega} = 0, \quad \tilde{B}_0|_{\partial \Omega} = 0.
\]

Denoting \( P_\delta(\rho) = \rho \int_0^\rho \frac{p(z) + \delta \|r\|_2 |z|_\gamma}{z} dz \), since \( p(\rho) \) satisfies (H1) which shows
\[
\lim_{\rho \to \infty} \frac{p(\rho)}{\rho^\gamma} = 0,
\]
supposing \( \Gamma > \max\left\{ \frac{3}{2}, \gamma + 1 \right\} \), and
\[
E_\delta(0) = \int_\Omega \left( \frac{|\tilde{u}_{0,\delta}|^2}{2\rho_{0,\delta}} + P_\delta(\rho_{0,\delta}) + \frac{1}{2}|\tilde{B}_{0,\delta}|^2 \right) dx
\]
\[
\lesssim \int_\Omega \left( \frac{|\tilde{u}|^2}{2\rho_0} + P(\rho_0) + \frac{1}{2}|\tilde{B}_0|^2 \right) dx = E(0),
\]
through Theorem 2.1 and Remark 2.3 in [26], then we have that for any given $T > 0$, the initial-boundary value problem (4.1), (4.2), (4.7) has a finite energy renormalized weak solution $(\rho_\delta, \bar{u}_\delta, \bar{B}_\delta)$ on $(0, T) \times \Omega$ with the conservation of mass and the energy inequality

$$
\int_\Omega \rho_\delta \, dx = \int_\Omega \rho_{0, \delta} \, dx, \quad \forall t \in [0, T],
$$

(4.9)

$$
E_\delta(t) + \int_0^t \int_\Omega \left( \mu |\nabla \bar{u}_\delta|^2 + (\lambda + \mu)(\text{div}\bar{u}_\delta)^2 + \nu |\nabla \times \bar{B}_\delta|^2 \right) \, dxdt \leq E_\delta(0),
$$

(4.10)

where $E_\delta(t) = \int_\Omega \left( \frac{|\bar{u}_\delta|^2}{2\rho_\delta} + P(\rho_\delta) + \frac{1}{2}|\bar{B}_\delta|^2 \right) \, dx$.

Combining (1.2) and (2.16), we have the following a priori estimates:

- $\rho_\delta \in L^\infty(0, T, L^1(\Omega))$ uniformly bounded,
- $\bar{u}_\delta \in L^2(0, T, H^1_0(\Omega))$ uniformly bounded,
- $\bar{B}_\delta \in L^\infty(0, T, L^2(\Omega)) \cap L^2(0, T, H^1_0(\Omega))$ uniformly bounded,

$$
\frac{1}{2} \rho_\delta |\bar{u}_\delta|^2 + P(\rho_\delta) + \frac{1}{2}|\bar{B}_\delta|^2 \in L^\infty(0, T, \mathcal{M}(\Omega)) \text{ uniformly bounded},
$$

(4.11)

- $\mu |\nabla \bar{u}_\delta|^2 + (\lambda + \mu)(\text{div}\bar{u}_\delta)^2 + \nu |\nabla \times \bar{B}_\delta|^2 \in \mathcal{M}^+([0, T] \times \Omega)$, uniformly bounded,
- $\delta^{3\gamma} \rho_\delta^\gamma \in L^\infty(0, T, \mathcal{M}^+(\Omega))$ uniformly bounded.

Using (H1) and (4.11)1, one have when $\gamma > 1$

$$
P(\rho_\delta) \geq \rho_\delta \int_1^{\rho_\delta} \frac{p(s)}{s^2} \, ds \geq \rho_\delta \int_1^{\rho_\delta} (C_1 s^{\gamma-2} - C_2 s^{-2}) \, ds \gtrsim \rho_\delta^\gamma - \rho_\delta + 1, \quad \text{if } \rho_\delta \geq 1,
$$

$$
\rho_\delta \gtrsim \rho_\delta^\gamma, \quad \text{if } \rho_\delta < 1,
$$

(4.12)

then we have $\rho_\delta \in L^\infty([0, T], L^\gamma(\Omega))$ uniformly bounded. When $\gamma = 1$, similarly we have

$$
P(\rho_\delta) \geq \rho_\delta \int_1^{\rho_\delta} \frac{p(s)}{s^2} \, ds \gtrsim \rho_\delta \log \rho_\delta - \rho_\delta + 1, \quad \text{if } \rho_\delta \geq 1,
$$

$$
1 \gtrsim |\rho_\delta \log \rho_\delta|, \quad \text{if } \rho_\delta < 1.
$$

(4.13)

then we get that $\rho_\delta \log \rho_\delta \in L^\infty([0, T], L^1(\Omega))$ uniformly bounded.

As $(\rho_\delta, \bar{u}_\delta, \bar{B}_\delta, \nabla \bar{u}_\delta, \nabla \times \bar{B}_\delta) \in \mathcal{F}$, combining the bounds in (4.11), through Theorem 2.1 in [9], there exists a subsequence (still labelled) $(\rho_\delta, \bar{u}_\delta, \bar{B}_\delta, \nabla \bar{u}_\delta, \nabla \times \bar{B}_\delta)$ and a family of probability measures $\mathcal{V}_{t,x} \in L^\infty_w((0, T) \times \Omega; \mathcal{P}(\mathcal{F}))$, supported inside a ball, such that for all continuous function $g$ which for some $p > 1$, $g(\rho_\delta, \bar{u}_\delta, \bar{B}_\delta, \nabla \bar{u}_\delta, \nabla \times \bar{B}_\delta) \in L^p([0, T] \times \Omega)$ uniformly bounded, we obtain

$$
\lim_{\delta \to 0^+} g(\rho_\delta, \bar{u}_\delta, \bar{B}_\delta, \nabla \bar{u}_\delta, \nabla \times \bar{B}_\delta) = \int_\mathcal{F} g(s, \tilde{v}, \tilde{B}, \nabla \tilde{v}, \nabla \times \tilde{B}) \, d\mathcal{V}_{t,x} \equiv \langle \mathcal{V}_{t,x}, g \rangle,
$$

weakly in $L^p([0, T] \times \Omega)$ or weak-* in $L^\infty_w([0, T] \times \Omega)$ when $p = \infty$. $\{\mathcal{V}_{t,x}\} \in L^\infty_w((0, T) \times \Omega; \mathcal{P}(\mathcal{F}))$ are Young measures generated by $(\rho_\delta, \bar{u}_\delta, \bar{B}_\delta, \nabla \bar{u}_\delta, \nabla \times \bar{B}_\delta)$. And we have the following bounds uniformly

$$
\langle \mathcal{V}_{t,x}, s \rangle \in L^\infty(0, T, L^\gamma(\Omega)),
$$

$$
\langle \mathcal{V}_{t,x}, \tilde{v} \rangle \in L^2(0, T, H^1_0(\Omega)),
$$

$$
\langle \mathcal{V}_{t,x}, \nabla \tilde{v} \rangle \in L^2([0, T] \times \Omega),
$$
\[ \left\langle V_{\tau,x}; \vec{B} \right\rangle \in L^2(0,T; H^1_0(\Omega)) \cap L^\infty(0,T; L^2(\Omega)), \]
\[ \left\langle V_{t,x}; \vec{f} \right\rangle \in L^2([0,T] \times \Omega). \]

Again through (4.11)$_{1,2}$ passing to a subsequence, we obtain
\[ \frac{1}{2} \rho_\delta |\tilde{u}_\delta|^2 + P(\rho_\delta) + \frac{1}{2} |\vec{B}_\delta|^2 \to E, \text{ weakly } - \text{ in } L^\infty_w(0,T, \mathcal{M}(\Omega)), \]
\[ \mu |\nabla \tilde{u}_\delta|^2 + (\lambda + \mu)(\text{div}\tilde{u}_\delta)^2 + \nu |\nabla \times \vec{B}_\delta|^2 \to \sigma, \text{ weakly } - \text{ in } \mathcal{M}^+(0,T) \times \Omega), \]
\[ \delta^{3\Gamma} \rho_\delta^\gamma \to \zeta, \text{ weakly } - \text{ in } L^\infty_w(0,T, \mathcal{M}^+(\Omega)). \] (4.14)

Denote nonnegative measures
\[ E_\infty = E - \left\langle V_{t,x}; \frac{1}{2}s|\tilde{v}|^2 + P(s) + \frac{1}{2} |\vec{B}|^2 \right\rangle dt \]
\[ \sigma_\infty = \sigma - \left\langle V_{t,x}; \mathcal{S}(\mathcal{D}_\tau) : \mathbb{D}_\mathbb{E} + \nu |\mathcal{J}|^2 \right\rangle dxdt. \]

Taking \( \delta \to 0 \) in (4.10), we get
\[ \int_\Omega \left\langle V_{t,x}; \frac{1}{2}s|\tilde{v}|^2 + P(s) + \frac{1}{2} |\vec{B}|^2 \right\rangle dx + \int_0^T \int_\Omega \left\langle V_{t,x}; \mathcal{S}(\mathcal{D}_\tau) : \mathbb{D}_\mathbb{E} + \nu |\mathcal{J}|^2 \right\rangle dxdt \]
\[ + E_\infty(\tau)\overline{[\mathbb{E}]} + \sigma_\infty([0,\tau] \times \overline{\Omega}^\gamma) + \frac{1}{\Gamma - 1} \zeta(\tau)\overline{[\mathbb{E}]} \]
\[ \leq \int_\Omega \left\langle V_{0,x}; \frac{1}{2}s|\tilde{v}|^2 + P(s) + \frac{1}{2} |\vec{B}|^2 \right\rangle dx. \]

Let
\[ \mathcal{D}_\nu(\tau) = E_\infty(\tau)\overline{[\mathbb{E}]} + \sigma_\infty([0,\tau] \times \overline{\Omega}^\gamma) + \frac{1}{\Gamma - 1} \zeta(\tau)\overline{[\mathbb{E}]} \in L^\infty([0,T]), \] (4.15)

then we get the energy inequality (2.14) for the measure-valued solution.

Since \((\rho_\delta, \tilde{u}_\delta, \vec{B}_\delta)\) are renormalized weak solutions to (4.1), then \(\forall \psi \in C^1([0,T] \times \overline{\Omega})\) and \(b \in C^1([0,\infty)\) satisfying \(b'(s) = 0\) for sufficiently large \(s\), we obtain
\[ \left[ \int_\Omega b(\rho_\delta)\rho_\delta \psi dx \right]_0^T = \int_0^T \int_\Omega \left[ b(\rho_\delta)\delta \psi + b(\rho_\delta)\tilde{u}_\delta \cdot \nabla \psi + \delta(\rho_\delta) - \rho_\delta b'(\rho_\delta)\text{div}\tilde{u}_\delta \psi \right] dxdt. \]

As \(b(\rho_\delta) \lesssim \rho_\delta\), combining (4.11)$_{1,2}$, we get that \(b(\rho_\delta)\tilde{u}_\delta \cdot \nabla \psi + (\rho_\delta - \rho_\delta b'(\rho_\delta))\text{div}\tilde{u}_\delta \in L^1([0,T] \times \Omega)\) uniformly bounded. Then letting \(\delta \to 0\), we get (2.9). For the same reason, we have (2.8).

For any \(\varphi \in C^1([0,T] \times \overline{\Omega}^\gamma; \mathbb{R}^3)\), \(\varphi_{|\partial\Omega} = 0\), we have \((\rho_\delta, \tilde{u}_\delta, \vec{B}_\delta)\) satisfy the momentum equation
\[ [\rho_\delta \tilde{u}_\delta \varphi]_0^T = \int_0^T \int_\Omega \left[ (\rho_\delta \tilde{u}_\delta \cdot \partial_t \varphi + (\rho_\delta \tilde{u}_\delta \otimes \tilde{u}_\delta) : \nabla \varphi + (\rho_\delta + \delta^{3\Gamma} \rho_\delta^\gamma)\text{div}\varphi + (\nabla \times \vec{B}_\delta) \times \vec{B}_\delta \cdot \varphi - \mathcal{S}(\nabla \tilde{u}_\delta) : \nabla \varphi \right] dxdt. \]

To handle the terms \(\rho_\delta \tilde{u}_\delta \otimes \tilde{u}_\delta, p(\rho_\delta)\), and \((\nabla \times \vec{B}_\delta) \times \vec{B}_\delta\), here we introduce Lemma 2.1 in [16].

**Lemma 4.1** (Lemma 2.1 in [16]). Let \(\{\mathbf{Z}_n\}_{n=1}^\infty, \mathbf{Z}_n : Q \to \mathbb{R}^N\) be a sequence of equiv-integrable functions generating a Young measure \(\mathcal{V}_y, \ y \in Q\), where \(Q \subset \mathbb{R}^M\) is a bounded domain. Let \(G : \mathbb{R} \to [0,\infty), \ F : \mathbb{R} \to \mathbb{R}\).
be continuous functions such that
\[
\sup_{n \geq 0} \|G(Z_n)\|_{L^1(\Omega)} \leq \infty,
\]
\[
|F(Z)| \leq G(Z), \forall Z \in \mathbb{R}^N.
\]
Denote
\[
F_\infty = \tilde{F} - \langle \mathcal{V}_y, F(Z) \rangle dy,
\]
\[
G_\infty = \tilde{G} - \langle \mathcal{V}_y, G(Z) \rangle dy,
\]
where $\tilde{F} \in \mathcal{M}(\overline{\Omega})$, $\tilde{G} \in \mathcal{M}(\overline{\Omega})$ are weak-* limits of $\{F(Z_n)\}$, $\{G(Z_n)\}$ in $\mathcal{M}(\overline{\Omega})$. Then
\[
|F_\infty| \leq G_\infty.
\]
Noticing that $\rho_\delta \tilde{u}_\delta \otimes \tilde{u}_\delta \leq \rho_\delta |\tilde{u}_\delta|^2$, using (4.14) and Lemma 4.1, we obtain
\[
\lim_{\delta \to 0} |\rho_\delta \tilde{u}_\delta \otimes \tilde{u}_\delta - \langle \mathcal{V}_{t,x}, s \tilde{v} \otimes \tilde{v} \rangle dx| \leq E_\infty.
\]
Using (1.9), (4.12) and (4.13), we have that $p(\rho) \lesssim P(\rho) + \rho + 1$ for $\gamma \geq 1$. Then using Lemma 4.1, we get
\[
\lim_{\delta \to 0} |p(\rho_\delta) - \langle \mathcal{V}_{t,x}, p(s) \rangle dx| \leq \lim_{\delta \to 0} |P(\rho_\delta) + \rho_\delta + 1 - \langle \mathcal{V}_{t,x}, p(s) + 1 \rangle dx| \lesssim E_\infty.
\]
Using (4.11)_3, through an interpolation, we obtain that $(\nabla \times \tilde{B}_\delta) \times \tilde{B}_\delta \in L^\frac{3}{2}([0,T] \times \Omega)$ uniformly bounded, and
\[
\lim_{\delta \to 0} (\nabla \times \tilde{B}_\delta) \times \tilde{B}_\delta = \langle \mathcal{V}_{t,x}, \mathcal{J} \times \tilde{B} \rangle \text{ weakly in } L^\frac{3}{2}([0,T] \times \Omega).
\]
Then we denote $r^M_{\nu} = \lim_{\delta \to 0} \{\rho_\delta \tilde{u}_\delta \otimes \tilde{u}_\delta - \langle \mathcal{V}_{t,x}, s \tilde{v} \otimes \tilde{v} \rangle dx + p(\rho_\delta) - \langle \mathcal{V}_{t,x}, p(s) \rangle dx + \zeta\} \in L^\infty_{\ast}([0,T], \mathcal{M}^+(\overline{\Omega}; \mathbb{R}^{3 \times 3}))$, such that
\[
|r^M_{\nu}(t)|| \lesssim \mathcal{D}_\nu(t), \forall t \in [0, T]. \tag{4.16}
\]
Using (4.11)_{2,3}, and $H_0^1(\Omega) \to L^p(\Omega)$, we get that $\tilde{B}_\delta \times \tilde{u}_\delta$ uniformly bounded in $L^3([0,T] \times \Omega)$. Let $\delta$ tends to 0 in
\[
\int_\Omega \tilde{B}_\delta \cdot \varphi(\cdot) dx - \int_\Omega \tilde{B}_0,\delta \cdot \varphi(0, \cdot) dx
\]
\[
= \int_0^T \int_\Omega \tilde{B}_\delta \cdot \partial_t \varphi dxdt - \int_0^T \int_\Omega (\tilde{B}_\delta \times \tilde{u}_\delta) \cdot (\nabla \times \varphi) dxdt
\]
\[
- \nu \int_0^T \int_\Omega (\nabla \times \tilde{B}_\delta) \cdot (\nabla \times \varphi) dxdt,
\]
we get (2.10). Similarly, (2.11) holds. Here we finish the proof of Theorem 2.1. \qed

The proof of Theorem 3.1 are similar to the methods of the case $\nu > 0$. We only need to add an artificial resistivity $\delta \Delta \tilde{B}(\delta \to 0)$ to (4.1)_{3}. We define $D_0$ in (4.15) with $\nu = 0$ and $r^M_{\nu} = \lim_{\delta \to 0} \{\rho_\delta \tilde{u}_\delta \otimes \tilde{u}_\delta - \langle \mathcal{V}_{t,x}, s \tilde{v} \otimes \tilde{v} \rangle dx - \tilde{B}_\delta \otimes \tilde{B}_\delta + \langle \mathcal{V}_{t,x}, \tilde{B} \otimes \tilde{B} \rangle + p(\rho_\delta) - \langle \mathcal{V}_{t,x}, p(s) \rangle dx + \zeta\}$. Also through Lemma 4.1, we can prove Theorem 3.1 and omit the details.
5. Proof of weak-strong uniqueness.

5.1. Proof of Theorem 2.2. Since \( rH'(r) = H(r) + h(r) \), from (2.17), we obtain that

\[
\mathcal{E}_{mv}(t) = \int_{\Omega} \left\{ \frac{1}{2} \langle \vec{v} \rangle^2 + H(s) + \frac{1}{2} \langle \vec{B} \rangle^2 \right\} dx \\
- \int_{\Omega} \langle \mathcal{V}_{t,x}; s\vec{v} \rangle \cdot \vec{U} dx + \frac{1}{2} \int_{\Omega} \langle \mathcal{V}_{t,x}; s \rangle |\vec{U}|^2 dx \\
- \int_{\Omega} \langle \mathcal{V}_{t,x}; s \rangle H'(r) dx + \int_{\Omega} \left( h(r) + \frac{1}{2} |\vec{M}|^2 \right) dx - \int_{\Omega} \langle \mathcal{V}_{t,x}; \vec{B} \rangle \cdot \vec{M} dx
\]

\[
= I_1 + \cdots + I_6. 
\]

For \( I_1 \), using (2.14), we have

\[
\int_{\Omega} \langle \mathcal{V}_{t,x}; \frac{s}{2} |\vec{v}|^2 + H(s) + \frac{1}{2} |\vec{B}|^2 \rangle dx + \int_{\Omega} \langle \mathcal{V}_{t,x}; Q(s) \rangle dx \\
+ \int_{0}^{T} \int_{\Omega} \langle \mathcal{V}_{t,x}; S(\mathcal{D}_v) : \mathcal{D}_v + \nu |\vec{J}|^2 \rangle dx dt + D_\nu(\tau)
\]

\[
\leq \int_{\Omega} \langle \mathcal{V}_{0,x}; \frac{s}{2} |\vec{v}|^2 + H(s) + \frac{1}{2} |\vec{B}|^2 \rangle dx + \int_{\Omega} \langle \mathcal{V}_{0,x}; Q(s) \rangle dx.
\]

Combining that

\[
\int_{\Omega} \langle \mathcal{V}_{t,x}; Q(s) \rangle dx \bigg|_{0}^{t} = - \int_{0}^{t} \int_{\Omega} \langle \mathcal{V}_{t,x}; q(s) \text{tr}(\mathcal{D}_v) \rangle dx dt,
\]

we obtain

\[
I_1 + \int_{0}^{T} \int_{\Omega} \langle \mathcal{V}_{t,x}; S(\mathcal{D}_v) : \mathcal{D}_v + \nu |\vec{J}|^2 \rangle dx dt + D_\nu(\tau) \]

\[
\leq \int_{\Omega} \langle \mathcal{V}_{0,x}; \frac{s}{2} |\vec{v}|^2 + H(s) + \frac{1}{2} |\vec{B}|^2 \rangle dx + \int_{0}^{t} \int_{\Omega} \langle \mathcal{V}_{t,x}; q(s) \text{tr}(\mathcal{D}_v) \rangle dx dt.
\]

For the term \( I_2 \), taking \( \varphi \) in (2.13) equals \( \vec{U} \), we get

\[
- I_2 - \int_{\Omega} \langle \mathcal{V}_{0,x}; s\vec{v} \rangle \cdot \vec{U}(0, \cdot) dx
\]

\[
= \int_{0}^{T} \int_{\Omega} \left( \langle \mathcal{V}_{t,x}; s\vec{v} \rangle \cdot \partial_t \vec{U} + \langle \mathcal{V}_{t,x}; s\vec{v} \otimes \vec{v} \rangle : \nabla \vec{U} \right) dx dt
\]

\[
+ \int_{0}^{T} \int_{\Omega} \left( \langle \mathcal{V}_{t,x}; p(s) \rangle \text{div}_x \vec{U} + \langle \mathcal{V}_{t,x}; \vec{J} \times \vec{B} \rangle \cdot \vec{U} \right) dx dt
\]

\[
- \int_{0}^{T} \int_{\Omega} \langle \mathcal{V}_{t,x}; S(\mathcal{D}_v) \rangle : \nabla \vec{U} dx dt + \int_{0}^{T} \left\langle \langle \mathcal{V}_{t,x}; \vec{M} \rangle, \nabla \vec{U} \right\rangle_{L^2(\Omega)} dt.
\]

Considering the terms \( I_3, I_4 \), taking \( \psi \) in (2.8) equals \( \frac{1}{2} |\vec{U}|^2 \) or \(-H'(r)\), using \( sH''(s) = h'(s) \), one can obtain

\[
I_3 - \int_{\Omega} \langle \mathcal{V}_{0,x}; s \rangle \frac{1}{2} |\vec{U}|^2 (0, \cdot) dx
\]

\[
= \int_{0}^{T} \int_{\Omega} \left( \langle \mathcal{V}_{t,x}; s \rangle \vec{U} \cdot \partial_t \vec{U} + \langle \mathcal{V}_{t,x}; s\vec{v} \rangle \cdot \nabla \vec{U} \cdot \vec{U} \right) dx dt.
\]

\[
I_4 = \int_{\Omega} \langle \mathcal{V}_{t,x}; s \rangle \vec{U} \cdot \partial_t \vec{U} + \langle \mathcal{V}_{t,x}; s\vec{v} \rangle \cdot \nabla \vec{U} \cdot \vec{U} \right) dx dt.
\]
\[I_4 + \int_{\Omega} \langle \mathcal{V}_{0,x}; \mathcal{S} \rangle H'(r_0) dx\]

\[= - \int_0^t \int_{\Omega} \left( \langle \mathcal{V}_{t,x}; \mathcal{S} \rangle \frac{1}{r} h'(r) \partial_t r + \langle \mathcal{V}_{t,x}; \mathcal{S} \rangle \cdot \frac{1}{r} h'(r) \nabla r \right) dx dt. \quad (5.5)\]

Considering \(I_5\), since \((r, \tilde{U}, \tilde{M})\) is a smooth solution of (1.1), we have

\[\int_{\Omega} h(r) dx = \int_0^t \int_{\Omega} h'(r) \partial_t r dx dt + \int_{\Omega} r_0 H'(r_0) - H(r_0) dx, \quad (5.6)\]

and

\[\int_{\Omega} \frac{1}{2} |\tilde{M}|^2 dx = \int_0^t \int_{\Omega} \tilde{M} \cdot \partial_t \tilde{M} dx dt + \int_{\Omega} \frac{1}{2} |\tilde{M}_0|^2 dx\]

\[= - \int_0^t \int_{\Omega} (\tilde{M} \times \tilde{U}) \cdot (\nabla \times \tilde{M}) dx dt \quad (5.7)\]

\[\quad \quad - \nu \int_0^t \int_{\Omega} |\nabla \times \tilde{M}|^2 dx dt + \int_{\Omega} \frac{1}{2} |\tilde{M}_0|^2 dx.\]

For the term \(I_6\), taking \(\varphi\) in (2.10) equals \(-\tilde{M}\), we get

\[I_6 + \int_{\Omega} \langle \mathcal{V}_{0,x}; \mathcal{B} \rangle \cdot \tilde{M}_0 dx\]

\[= - \int_0^t \int_{\Omega} \langle \mathcal{V}_{t,x}; \mathcal{B} \rangle \cdot \partial_t \tilde{M} dx dt + \int_{\Omega} \langle \mathcal{V}_{t,x}; \mathcal{B} \rangle \cdot (\nabla \times \tilde{M}) dx dt \quad (5.8)\]

Using (5.2)-(5.8), then we have

\[\mathcal{E}_{mv}(t) + \int_0^t \int_{\Omega} \langle \mathcal{V}_{t,x}; \mathcal{S} \rangle \cdot \mathcal{D} \mathcal{V} + \nu |\mathcal{J}|^2 \] dx dt

\[\quad + \int_0^t \int_{\Omega} \langle \mathcal{V}_{t,x}; H(s) - H(r_0) - H'(r_0)(s - r_0) \rangle dx\]

\[\quad - \int_0^t \int_{\Omega} \langle \mathcal{V}_{t,x}; s \mathcal{S} \rangle \cdot \partial_t \tilde{U} dx dt - \int_0^t \int_{\Omega} \langle \mathcal{V}_{t,x}; s \mathcal{S} \rangle \cdot \nabla \tilde{U} dx dt\]

\[\quad - \int_0^t \int_{\Omega} \langle \mathcal{V}_{t,x}; h(s) \rangle \text{div} \tilde{U} dx dt + \int_0^t \int_{\Omega} \langle \mathcal{V}_{t,x}; s \rangle \tilde{U} \cdot \partial_t \tilde{U} dx dt\]

\[\quad + \int_0^t \int_{\Omega} \langle \mathcal{V}_{t,x}; s \mathcal{S} \rangle \cdot \nabla \tilde{U} \cdot \tilde{U} dx dt + \int_0^t \int_{\Omega} \langle \mathcal{V}_{t,x}; (1 - \frac{s}{r}) \rangle h'(r) \partial_t r dx dt\]

\[\quad - \int_0^t \int_{\Omega} \langle \mathcal{V}_{t,x}; s \mathcal{S} \rangle \frac{h'(r)}{r} \nabla r dx dt - \int_0^t \int_{\Omega} \langle \mathcal{V}_{t,x}; q(s) \rangle \text{div} \tilde{U} dx dt\]

\[\quad + \int_0^t \int_{\Omega} \langle \mathcal{V}_{t,x}; q(s) \rangle \text{tr}(\mathcal{D} \mathcal{V}) dx dt - \int_0^t \langle r_\mathcal{M}; \nabla \tilde{U} \rangle dt\]
then we get
\[ \begin{align*}
- \int_0^t \int_\Omega \left< \nabla_{x,t} \cdot \hat{\mathcal{J}} \times \vec{B} \right> : \vec{U} \, dx \, dt - \nu \int_0^t \int_\Omega |\nabla \times \vec{M}|^2 \, dx \, dt \\
- \int_0^t \int_\Omega (\vec{M} \times \vec{U}) \cdot (\nabla \times \vec{M}) \, dx \, dt + \int_0^t \int_\Omega \left< \nabla_{x,t} \cdot \vec{B} \times \vec{v} \right> (\nabla \times \vec{M}) \, dx \, dt \\
+ \int_0^t \int_\Omega \left< \nabla_{x,t} \cdot \nu \vec{J} \right> \cdot (\nabla \times \vec{M}) \, dx \, dt - \int_0^t \int_\Omega \left< \nabla_{x,t} \cdot \vec{B} \right> \cdot \partial_r \vec{M} \, dx \, dt.
\end{align*} \]

Since \((r, \vec{U}, \vec{M})\) is a smooth solution of (1.1),
\[ \partial_r \vec{U} + \vec{U} \cdot \nabla \vec{U} + \frac{h'(r) + q'(r)}{r} \nabla r = \frac{1}{r} \text{div} \mathcal{S}(\nabla \vec{U}) + \frac{1}{r} (\nabla \times \vec{M}) \times \vec{M}, \]
\[ \partial_r \vec{r} + \text{div} \vec{U} + \vec{U} \cdot \nabla r = 0, \]
\[ \partial_r \vec{M} = \nabla \times (\vec{U} \times \vec{M}) - \nu \nabla \times (\nabla \times \vec{M}), \]
then we get
\[ \mathcal{E}_{\text{mv}}(t) + \int_0^t \int_\Omega \left< \nabla_{x,t} : \mathcal{D}_\varpi - \nabla \vec{U} \right> : (\mathcal{D}_\varpi - \nabla \vec{U}) + \nu |\vec{J} - \nabla \times \vec{M}|^2 \right> \, dx \, dt + \mathcal{D}_\nu(t) \]
\[ \leq \int_\Omega \left< \mathcal{V}_{0,x} \frac{1}{2} s|\vec{v} - \vec{U}_0|^2 + H(s) - H(r_0) - H'(r_0)(s - r_0) + \frac{1}{2} |\vec{B} - \vec{M}_0|^2 \right> \, dx \]
\[ + \int_0^t \int_\Omega \left< \mathcal{V}_{t,x} : h(r) - h(s) - h'(r)(r - s) \right> \, dx \, dt \]
\[ + \int_0^t \int_\Omega \left< \mathcal{V}_{t,x} : s(\vec{v} - \vec{U}) \cdot \nabla \vec{U} \right> \, dx \, dt \]
\[ + \int_0^t \int_\Omega \left< \mathcal{V}_{t,x} : (\vec{U} - \vec{v})(s - r) \right> \cdot \left( \frac{1}{r} \text{div} \mathcal{S}(\nabla \vec{U}) + (\nabla \times \vec{M}) \times \vec{M} - \nabla q(r) \right) \, dx \, dt \]
\[ + \int_0^t \int_\Omega \left< \mathcal{V}_{t,x} : (q(s) - q(r)) (\text{tr}(\mathcal{D}_\varpi) - \text{div} \vec{U}) \right> \, dx \, dt \]
\[ + \int_0^t \int_\Omega \left< \mathcal{V}_{t,x} : (\vec{J} - \nabla \times \vec{M}) \times (\vec{M} - \vec{B}) \right> \cdot \vec{U} \, dx \, dt \]
\[ + \int_0^t \int_\Omega \left< \mathcal{V}_{t,x} : (\nabla \times \vec{M}) \times (\vec{B} - \vec{M}) \right> \cdot (\vec{v} - \vec{U}) \, dx \, dt \]
\[ - \int_0^t \left< r^\nu, \nabla \vec{U} \right> \, dt \]
\[ = K_0 + \cdots + K_7. \tag{5.9} \]

For the term \(K_1\), since \(0 < r \in C^1([0,T] \times \Omega)\), denoting \(0 < r \leq r \leq \tau\), from a direct calculation, we have
\[ H(s) - H(r) - H'(r)(s - r) \geq \begin{cases} C_{s,r}(s - r)^2, & \frac{r}{2} \leq s \leq 2\tau, \\ 1 + s^2, & \text{otherwise}, \end{cases} \tag{5.10} \]
where \(C_{s,r}\) is a constant. Then we have
\[ h(r) - h(s) - h'(r)(r - s) \lesssim H(s) - H(r) - H'(r)(s - r), \]
which shows
\[ K_1 \leq \Lambda_1 \int_0^t \mathcal{E}_{\text{mv}} \, dt, \tag{5.11} \]
where \( \Lambda_1 \) is a constant.

Considering \( K_2 \), since \( \bar{U} \in C^1([0,T] \times \Omega) \), we have

\[
K_2 \leq \| \bar{U} \|_{C^1([0,T] \times \Omega)} \int_0^t \int_{\Omega} \langle \nabla_{t,x} ; |s| \bar{v} - \bar{U} \rangle \, dx \, dt \leq \Lambda_2 \int_0^t E_{m,v} \, dt, \tag{5.12}
\]

where \( \Lambda_2 \) is a constant.

For the term \( K_3 \), since \( r, \bar{U}, \bar{M} \in C^1([0,T] \times \Omega) \) and \( q \in C^1_c(0,\infty) \), we have

\[
K_3 \leq \| r, \bar{U}, \bar{M} \|_{C^1([0,T] \times \Omega)} \int_0^t \int_{\Omega} \langle \nabla_{t,x} ; (\bar{U} - \bar{\nu})(s - r) \rangle \, dx \, dt.
\]

When \( s \geq 2\tau \), one have

\[
\int_0^t \int_{\Omega} \langle \nabla_{t,x} ; (\bar{U} - \bar{\nu})(s - r) \rangle \, dx \, dt \leq \int_0^t \int_{\Omega} \langle \nabla_{t,x} ; s + s|\bar{U} - \bar{\nu}|^2 \rangle \, dx \, dt
\]

\[
\leq \Lambda_3 \int_0^t E_{m,v} \, dt,
\]

where \( \Lambda_3 \) is a constant. When \( s \leq 2\tau \), using generalized Korn-Poincaré inequality (2.15), one have

\[
\int_0^t \int_{\Omega} \langle \nabla_{t,x} ; (\bar{U} - \bar{\nu})(s - r) \rangle \, dx \, dt
\]

\[
\leq \int_0^t \int_{\Omega} \langle \nabla_{t,x} ; 6(s - r)^2 + \frac{1}{6}|\bar{U} - \bar{\nu}|^2 \rangle \, dx \, dt
\]

\[
\leq \int_0^t \int_{\Omega} \langle \nabla_{t,x} ; 6(s - r)^2 + \frac{1}{6}|\nabla \bar{U} - D_{\vartheta}|^2 \rangle \, dx \, dt
\]

\[
\leq \int_0^t \int_{\Omega} \langle \nabla_{t,x} ; 6(s - r)^2 + \frac{1}{6}S(\nabla \bar{U} - D_{\vartheta}) : (\nabla \bar{U} - D_{\vartheta}) \rangle \, dx \, dt.
\]

Using (5.10), one have

\[
K_3 \leq \Lambda_3 \int_0^t E_{m,v} \, dt + \frac{1}{6} \int_0^t \int_{\Omega} \langle \nabla_{t,x} ; S(D_{\vartheta} - \nabla \bar{U}) : (D_{\vartheta} - \nabla \bar{U}) \rangle \, dx \, dt. \tag{5.13}
\]

For the term \( K_4 \), since \( q \) locally Lipschitz, \( q^2(\rho) \leq C(1 + \rho^\gamma) \), one have

\[
(q(s - q(r))((\text{tr}(D_{\vartheta}) - \text{div}\bar{U})
\]

\[
\leq C((s - r)^2I_{r \leq 2\varpi} + (s^\gamma + 1)I_{r \geq 2\varpi}) + \frac{1}{6}|\text{tr}(D_{\vartheta}) - \text{div}\bar{U}|^2
\]

\[
\leq C((s - r)^2I_{r \leq 2\varpi} + (s^\gamma + 1)I_{r \geq 2\varpi}) + \frac{1}{6}S(D_{\vartheta} - \nabla \bar{U}) : (D_{\vartheta} - \nabla \bar{U}).
\]

Combining (5.10), one get

\[
K_4 \leq \Lambda_4 \int_0^t E_{m,v} \, dt + \frac{1}{6} \int_0^t \int_{\Omega} \langle \nabla_{t,x} ; S(D_{\vartheta} - \nabla \bar{U}) : (D_{\vartheta} - \nabla \bar{U}) \rangle \, dx \, dt, \tag{5.14}
\]

with \( \Lambda_4 \) a constant.

For \( K_5 \), since

\[
(\bar{J} - \nabla \times \bar{M}) \times (M - \bar{B}) \leq \frac{6}{\nu'} |M - \bar{B}|^2 + \frac{\nu}{6} |\bar{J} - \nabla \times \bar{M}|^2,
\]

one have

\[
K_5 \leq \Lambda_5 \int_0^t E_{m,v} \, dt + \frac{\nu}{6} \int_0^t \int_{\Omega} \langle \nabla_{t,x} ; |\bar{J} - \nabla \times \bar{M}|^2 \rangle \, dx \, dt, \tag{5.15}
\]
with $\Lambda_3$ a constant.

For $K_6$, we have
\[
\left( (\nabla \times \vec{M}) \times (\vec{B} - \vec{M}) \right) \cdot (\vec{v} - \vec{U}) \leq C|\vec{B} - \vec{M}|^2 + \frac{1}{6} |\vec{v} - \vec{U}|^2.
\]

Using generalized Korn-Poincaré inequality (2.15), then
\[
K_6 \leq \Lambda_6 \int_0^t \mathcal{E}_{mv} dt + \frac{1}{6} \int_0^t \int_\Omega \left\langle \mathcal{V}_{t,x}; \mathcal{S}(\mathcal{D}_\vec{v} - \nabla \vec{U}) : (\mathcal{D}_\vec{v} - \nabla \vec{U}) \right\rangle dx dt + \mathcal{D}_{\nu}(t), \quad (5.16)
\]
with $\Lambda_6$ a constant.

From (5.11)-(5.17), we already have that
\[
K_7 \leq \Lambda_7 \int_0^t \mathcal{D}_{\nu}(t) dt, \quad (5.17)
\]
with $\Lambda_7$ a constant.

From (5.11)-(5.17), we already have that
\[
\mathcal{E}_{mv}(t) + \int_0^t \int_\Omega \left\langle \mathcal{V}_{t,x}; \mathcal{S}(\mathcal{D}_\vec{v} - \nabla \vec{U}) : (\mathcal{D}_\vec{v} - \nabla \vec{U}) \right\rangle dx dt + \mathcal{D}_0(t) \leq \int_\Omega \left\langle \mathcal{V}_{0,x}; 0 \right\rangle + \int_0^t \int_\Omega \left\langle \mathcal{V}_{t,x}; \mathcal{S}(\mathcal{D}_\vec{v} - \nabla \vec{U}) : (\mathcal{D}_\vec{v} - \nabla \vec{U}) \right\rangle dx dt + \mathcal{D}_{\nu}(t) dt,
\]
with $\Lambda$ a constant about $\Lambda_1, \ldots, \Lambda_7$. Applying Grönwall’s inequality, we get (2.18), and we finish the proof of Theorem 2.2. \qed

5.2. **Proof of Theorem 3.2.** Considering the case $\nu = 0$, using the similar argument in (5.1)-(5.9), we have that
\[
\mathcal{E}_{mv}(t) + \int_0^t \int_\Omega \left\langle \mathcal{V}_{t,x}; \mathcal{S}(\mathcal{D}_\vec{v} - \nabla \vec{U}) : (\mathcal{D}_\vec{v} - \nabla \vec{U}) \right\rangle dx dt + \mathcal{D}_0(t) \leq \int_\Omega \left\langle \mathcal{V}_{0,x}; \mathcal{S}(\mathcal{D}_\vec{v} - \nabla \vec{U}) : (\mathcal{D}_\vec{v} - \nabla \vec{U}) \right\rangle dx dt + \mathcal{D}_{\nu}(t) dt,
\]
with $\Lambda_3$ a constant.
\[ + \int_0^t \int_\Omega \left\langle V_{t,x} ; \left( (\nabla \times \vec{M}) \times (\vec{B} - \vec{M}) \right) \cdot (\vec{v} - \vec{U}) \right\rangle dx dt \\
\quad - \int_0^t \left\langle r_0^M, \nabla \vec{U} \right\rangle dt \\
= J_0 + \cdots + J_8. \]

For the term \( J_5 \) and \( J_6 \), we have
\[ |J_5| + |J_6| \lesssim \| \vec{U} \|_{C^1} \int_0^t \mathcal{E}_{m_e}(t) dt. \]

The estimate for the terms \( J_1 - J_4 \) and \( J_7, J_8 \) are the same to the estimates \( K_1 - K_4 \) and \( K_6, K_7 \) in Subsection 5.1 separately. Then for the case \( \nu = 0 \), we obtain
\[ \mathcal{E}_{m_e}(t) + \int_0^t \int_\Omega \left\langle V_{t,x} ; S(\nabla \vec{u} - \nabla \vec{U}) : (\nabla \vec{u} - \nabla \vec{U}) \right\rangle dx dt + D_0(t) \]
\[ \leq \int_\Omega \left\langle V_{0,x} ; \frac{1}{2} |\vec{v} - \vec{U}_0|^2 + H(s) - H(r_0) + H'(r_0)(s - r_0) + \frac{1}{2} |\vec{B} - \vec{M}_0|^2 \right\rangle dx \]
\[ + \Lambda \int_0^t \mathcal{E}_{m_e} dt + \frac{1}{2} \int_0^t \int_\Omega \left\langle V_{t,x} ; S(\nabla \vec{u} - \nabla \vec{U}) : (\nabla \vec{u} - \nabla \vec{U}) \right\rangle dx dt + \int_0^t D_0(t) dt, \]

where \( \Lambda \) is a constant. Applying Grönwall’s inequality, we finish the proof of Theorem 3.2.

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