A Tight Upper Bound on Acquaintance Time of Graphs

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Abstract In this note we confirm a conjecture raised by Benjamini et al. (SIAM J Discrete Math 28(2):767–785, 2014) on the acquaintance time of graphs, proving that for all graphs \( G \) with \( n \) vertices it holds that \( AC(G) = O(n^{3/2}) \). This is done by proving that for all graphs \( G \) with \( n \) vertices and maximum degree \( \Delta \) it holds that \( AC(G) \leq 20\Delta n \). Combining this with the bound \( AC(G) \leq O(n^2/\Delta) \) from Benjamini et al. (SIAM J Discrete Math 28(2):767–785, 2014) gives the uniform upper bound of \( O(n^{3/2}) \) for all \( n \)-vertex graphs. This bound is tight up to a multiplicative constant. We also prove that for the \( n \)-vertex path \( P_n \) it holds that \( AC(P_n) = n - 2 \). In addition we show that the barbell graph \( B_n \) consisting of two cliques of sizes \( \lceil n/2 \rceil \) and \( \lfloor n/2 \rfloor \) connected by a single edge also has \( AC(B_n) = n - 2 \). This shows that it is possible to add \( \Omega(n^2) \) edges a graph without changing its \( AC \) value.

Keyword Acquaintance time of graph
1 Introduction

In this note we study the following graph process, introduced by Benjamini et al. [3]. Let $G = (V, E)$ be a finite connected graph. Initially we place one agent in each vertex of the graph. Every pair of agents sharing a common edge is declared to be acquainted. In each round we choose a matching in $G$ (a set of disjoint edges, not necessarily maximal). For each edge in the matching the agents on this edge swap places, which allows more agents to become acquainted. A sequence of matchings that allows all agents to get acquainted is called a strategy for acquaintance in $G$. The acquaintance time of $G$, denoted by $\mathcal{A}(G)$, is the minimum number of matchings in a strategy for acquaintance in $G$.

Several problems of similar flavor have been studied in the past. The common theme is that agents (or information) move on a graph by swapping their location with adjacent agents, and the number of steps needed to achieve some prescribed task is of interest. One such problem is permutation routing by matchings, studied by Alon et al. [1]. Here the input is a connected graph $G = (V, E)$ and a permutation of the vertices $\sigma : V \rightarrow V$, and the goal is to route all agents to their respective destinations according to $\sigma$; that is, the agent sitting originally in the vertex $v$ should be routed to the vertex $\sigma(v)$ for all $v \in V$. The movement constraints (apply a matching at each step) are exactly the same as in our setting. Other related topics include the well-studied problems of gossiping and of broadcasting (see the survey of Hedetniemi et al. [5] for details), and the target set selection problem (see, e.g., [6]).

It is easy to see that for an $n$-vertex graph $G = (V, E)$ it holds that $\mathcal{A}(G) \leq 2n^2$ since every agent can meet all others by traversing twice every edge of a spanning tree in $2(n - 1)$ rounds. Benjamini et al. [3] improved the upper bound to $\mathcal{A}(G) = O\left(\frac{n^2 \log \log(n)}{\log(n)}\right)$ for arbitrary $n$-vertex graphs. This bound has been further improved by Kinnersley et al. [7] to $\mathcal{A}(G) = O\left(n^2 / \log(n)\right)$.

For the lower bound it follows from [3, Theorem 5.1] that there is a family of graphs $\{G_n\}_{n \in \mathbb{N}}$ such that $G_n$ has $n$ vertices and $\mathcal{A}(G_n) \geq cn^{3/2}$ for some universal constant $c > 0$ and for all $n \in \mathbb{N}$.

In this paper we prove an upper bound on $\mathcal{A}$ for arbitrary graphs that matches the $\Omega\left(n^{3/2}\right)$ lower bound in the construction of [3] up to a multiplicative constant.

**Theorem 1** $\max\{\mathcal{A}(G) : G \text{ a connected } n\text{-vertex graph}\} = \Theta(n^{3/2})$.

The following theorem is the main technical result of this paper. Theorem 1 follows from Theorem 2 together with other known bounds.

**Theorem 2** Let $G = (V, E)$ be a graph with $n$ vertices, and suppose that the maximum degree of $G$ is $\Delta$. Then $\mathcal{A}(G) \leq 20\Delta n$.

We also prove that for the $n$-vertex path, denoted $P_n$, it holds that $\mathcal{A}(P_n) = n - 2$. The upper bound follows from an explicit $(n - 2)$-round strategy for acquaintance in $P_n$. For the lower bound we prove that the barbell graph $B_n$ consisting of two cliques of sizes $\lceil n/2 \rceil$ and $\lfloor n/2 \rfloor$ connected by a single edge satisfies $\mathcal{A}(B_n) = n - 2$. This also shows that it is possible to add $\Omega(n^2)$ edges to a graph without changing its $\mathcal{A}$ value.
Related work Recently, there have been several developments related to the acquaintance time problem. Kinnersley, Mitsche and Prałat, as well as Dudek and Prałat [4, 7] studied the acquaintance time of a random graphs $G(n, p)$, and showed that almost surely $\mathcal{AC}(G(n, p)) \asymp \log(n)/p$, as soon as $G(n, p)$ is connected. Müller and Prałat [9] studied the acquaintance time of a random subgraph of a random geometric graph, where $n$ vertices are chosen independently uniformly at random from $[0, 1]^2$, and two vertices are adjacent with probability $p$ if the Euclidean distance between them is at most $r$. They show asymptotic results for $\mathcal{AC}(G(n, r, p))$ for a wide range of $r = r(n)$ and $p = p(n)$.

2 Upper Bound on Acquaintance Times

In this section we first prove Theorem 2. We then use it to prove Theorem 1.

Proof of Theorem 2 Clearly, by removing edges from $G$ its acquaintance time cannot decrease, and its maximum degree does not increase. Thus, in order to give an upper bound on $\mathcal{AC}(G)$, we may fix a spanning tree of $G$ with maximum degree $\leq \Delta$, and use only the edges of the tree. Therefore, without loss of generality we henceforth assume that $G$ is a tree on $n$ vertices. A contour $\Gamma$ of a tree is a closed walk on $2n - 1$ vertices that crosses each edge exactly twice, and visits each vertex $v$ exactly $\deg(v)$ times. Such a contour can be constructed by considering a depth first search walk on $G$ (see Fig. 1).

Consider $\Gamma$ as a path on $2n - 1$ vertices, and let $\pi$ denote the projection from $\Gamma$ to $G$. We first argue that for each $x \in G$ it is possible to choose a vertex of $\Gamma$ from $\pi^{-1}(x)$ so that the gaps between the consecutive chosen vertices along $\Gamma$ are at most 3.

Fig. 1 A tree with a marked contour
(A similar statement appears in Lemma 2.4 in [2].) To do this, let \( r \) be an arbitrarily chosen first vertex of \( \Gamma \). Recall that \( G \) is assumed to be a tree. For each vertex \( x \in G \) whose distance from \( r \) is even we pick the first vertex of \( \Gamma \) projecting to \( x \), and for each vertex \( x \in G \) whose distance from \( r \) is odd we pick the last one. See Fig. 1 for an example. Note that \( \Gamma \) visits each leaf of the tree exactly once. Between consecutive visits to leaves the contour descends towards the root, and then ascends to the next leaf. Along the descent, the vertices are visited for the last time, and so every other vertex is selected. Along the ascent, the vertices are visited for the first time, and also every other vertex is selected. Hence, it is not possible to have more than three steps of \( \Gamma \) between consecutive selected vertices.

Consider the following \( n \)-rounds strategy for acquaintance on \( P_n \), the \( n \)-vertex path graph whose vertices are denoted by \( \{1, 2, \ldots, n\} \). In the odd rounds we swap the agents on the edges \( \{(i, i + 1) : i \text{ odd}\} \). In the even rounds we swap the agents on the edges \( \{(i, i + 1) : i \text{ even}\} \). It is easy to see that after \( n \) rounds the agents are arranged in the reversed order on the path, and hence the paths of every two agents must have crossed during the \( n \) rounds.

In order to present a \( O(\Delta \cdot n) \)-round strategy for acquaintance in \( G \) we emulate the strategy for acquaintance in \( P_n \) described above. This is done by simulating each round of the strategy for \( P_n \) by a sequence of at most \( 20\Delta \) rounds in \( G \).

First, consider the path \( \Gamma \) with \( 2n - 1 \) vertices, and place \( n \) agents on \( \Gamma \) in the marked vertices of \( \Gamma \), so that the distances between any two consecutive agents are at most 3. Our goal is to reverse the order of the \( n \) agents on \( \Gamma \), which, in particular, implies that every pair of agents must get acquainted. Since the agents are not located in consecutive vertices of the path, every round of the strategy for \( P_n \) described above will require several (at most 5) steps on \( \Gamma \). Later, we will show how to emulate these moves by a strategy on \( G \).

Let \( i \) and \( j \) be two consecutive marked vertices of \( \Gamma \), and let \( p_i \) and \( p_j \) be the agents sitting in the corresponding vertices. In order to swap the agents \( p_i \) and \( p_j \) we can perform a sequence of swaps in \( \Gamma \) along the edges \( (i, i + 1), \ldots, (j - 1, j) \), which brings the agent \( p_i \) to the vertex \( j \), followed by the sequence \( (j - 1, j - 2), \ldots, (i + 1, i) \), bringing the agent \( p_j \) to the vertex \( i \). The gaps between consecutive agents are at most 3, and hence it takes at most 5 steps on \( \Gamma \) to perform such a swap. These swaps projected on \( G \) result in an exchange of the agents in the vertices \( \pi(i) \) and \( \pi(j) \), leaving all other agents in their place.

The difficulty is that in the graph \( G \) the steps for swapping a pair of agents \( p_i \) and \( p_j \) could interfere with the steps for swapping another pair \( p_{i'} \) and \( p_{j'} \). This happens if the projections to \( G \) of the intervals \([i, j]\) and \([i', j']\) in the path \( \Gamma \) intersect. If there are no such intersections, then the swaps for all pairs of agents could be carried out in parallel and we would have a \( 5n \) round acquaintance strategy for \( G \).

In order to solve this problem, we separate each round of the strategy for acquaintance in \( P_n \) described above into several sub-rounds, so that conflicting pairs of intervals are in different sub-rounds, and then split each sub-round into at most 5 steps in \( \Gamma \) as described above. Since \( \Gamma \) visits each vertex of \( G \) at most \( \Delta \) times, and since the intervals \([i, j]\) of \( \Gamma \) that we care about (i.e., those defined by consecutive pairs of marked vertices \( i, j \) in \( \Gamma \)) are vertex disjoint in \( \Gamma \), each vertex of \( G \) is contained in at most \( \Delta \) such intervals. Each interval consists of at most 4 vertices of \( G \),
and therefore each interval \([i, j]\) is in conflict (i.e., their projections intersect) with at most \(4(\Delta - 1)\) other intervals \([i', j']\) participating in the current round.

It is well known that if a graph has maximum degree \(D\), then its chromatic number is at most \(D + 1\). Applying this to the conflict graph of the intervals to be swapped in one round of the strategy on \(P_n\), we see that we can assign each interval \([i, j]\) of \(\Gamma\) one of \(4\Delta - 3\) colors, so that conflicting intervals have different colors.

We now split each round of the strategy on \(\Gamma\) into \(4\Delta\) sub-rounds, where in every sub-rounds we swap all pairs of the same color that are to be swapped in this round of the strategy in \(P_n\). Finally, we simulate this strategy in \(P_n\) by replacing each sub-round with at most 5 steps in \(\Gamma\) as described above. Our coloring of the intervals guarantees that in this strategy there are no conflicting intervals, and hence the swaps can be carried out in \(G\) in parallel. Therefore, each round of the strategy for acquaintance in the \(n\)-vertex path graph can be simulated by \(20\Delta\) rounds in \(G\), and hence \(\mathcal{AC}(G) \leq 20\Delta n\). This completes the proof of the theorem.

As a consequence of Theorem 2 we obtain our uniform upper bound on the acquaintance time of all graph with \(n\) vertices. For this, let us recall the following bound.

Lemma 1 [3, Claim 4.7] Let \(G\) be a connected \(n\)-vertex graph, and let \(\Delta\) be the maximum degree of \(G\). Then \(\mathcal{AC}(G) = O(n^2/\Delta)\).

This together with Theorem 2 immediately implies Theorem 1.

Proof of Theorem 1 By the upper bounds from Theorem 2 and Lemma 1 we have \(\mathcal{AC}(G) \leq \min(O(n^2/\Delta), O(n\Delta)) \leq O(n^{3/2})\), as required. The lower bound follows a construction from [3, Theorem 5.1] who show a family of graphs \(\{G_n\}_{n \in \mathbb{N}}\) such that \(G_n\) has \(n\) vertices and \(\mathcal{AC}(G_n) \geq cn^{3/2}\) for some universal constant \(c > 0\) and for all \(n \in \mathbb{N}\).

Note that if \(G\) is not a tree then we can try to improve our bound by finding a spanning tree with smaller degrees. For example, the Erdős–Rényi graph \(G(n, p)\) with \(p = c \log n/n\) with \(c > 1\) has maximum degree of order \(\log n\), but is almost surely Hamiltonian [8], and hence has \(\mathcal{AC}(G) \leq n\).

3 Exact Calculation of \(\mathcal{AC}(P_n)\) and \(\mathcal{AC}(B_n)\)

In this section we compute the acquaintance time of the \(n\)-vertex path.

Theorem 3 Let \(P_n\) be the path with \(n\) vertices, and let \(B_n\) be the barbell graph consisting of cliques of sizes \([n/2]\) and \([n/2]\) connected by a single edge. Then \(\mathcal{AC}(P_n) = \mathcal{AC}(B_n) = n - 2\).

Proof We first prove that \(\mathcal{AC}(P_n) \leq n - 2\) by describing an explicit \((n - 2)\)-rounds strategy for acquaintance in \(P_n\). Then we prove a lower bound of \(\mathcal{AC}(B_n) \geq n - 2\) for \(B_n\). Together, this implies the theorem, as \(P_n\) is a subgraph of \(B_n\).
Denote the vertices of the $n$-vertex path by $\{1, 2, \ldots, n\}$. In order to prove that $\mathcal{AC}(P_n) \leq n - 2$ consider the following strategy. In the odd rounds we swap the agents on the edges $\{(i, i + 1) : i \text{ odd}\}$, and in even the rounds we swap the agents on the edges $\{(i, i + 1) : i \text{ even}\}$. Consider the walk performed by an agent that begins in some odd-indexed vertex under this strategy. The agent moves one step up in each round until reaching the vertex $n$, stays there for one round, and then moves down one step in each round. Similarly, an agent starting at an even vertex moves down until reaching the vertex 1, stays there for one round, and then moves up. This strategy has been also used in Lemma 5 of [7], where it was shown that $2n$ rounds are enough.

Note that after $n$ rounds the agent who started in the position $i$, will be in the position $n + 1 - i$. In particular, the agents reversed their order on the path, and thus every pair of the agents must have met. We claim that, in fact, all agents are already acquainted two rounds earlier. Indeed, consider two agents $p_i$ and $p_j$ who started in non-adjacent vertices $i$ and $j$, respectively, with $i \leq j - 2$. Consider three cases.

1. $i$ is odd: The agent $p_i$ starts by moving upwards, and after $n - 1 - i \leq n - 2$ steps reaches the vertex $n - 1$. At this point $p_i$ and $p_j$ must have either crossed paths or be in the vertices $\{n - 1, n\}$, and hence must be acquainted.

2. $j$ is even: The agent $p_j$ starts by moving downwards, and after $j - 2 \leq n - 2$ steps reaches the vertex 2. At this point, similarly to the previous case, $p_i$ and $p_j$ have either crossed paths or be in the vertices $\{1, 2\}$, and hence must be acquainted.

3. $i$ is even and $j$ is odd: Since the agents $p_i$ and $p_j$ started in non-adjacent vertices, we have $j \geq i + 3$. After $n$ steps the agent $p_i$ is in the vertex $u_i = n + 1 - i$, and $p_j$ is in the vertex $u_j = n + 1 - j$. Note that $u_i > u_j$, and so $p_i$ and $p_j$ must have crossed paths, and the distance between them is $u_i - u_j = j - i \geq 3$. Therefore, they must have swapped places before the $n$’th round, and hence, they shared an edge before $n - 1$ swaps.

This completes the proof of the first part of the proof, namely $\mathcal{AC}(P_n) \leq n - 2$.

For the lower bound consider the barbell graph $B_{a,b}$ consisting of two disjoint cliques of sizes $a$ and $b$ connected by a single edge, which we call the bridge. We assume that $a \leq b$, and claim that $\mathcal{AC}(B_{a,b}) \geq 2a - 2$ if $a = b$, and $\mathcal{AC}(B_{a,b}) \geq 2a - 1$ if $a < b$.

Suppose that there is an $m$-round strategy for acquaintance in $B_{a,b}$, with $k$ swaps across the bridge. Note that we may assume that there are never 2 consecutive swaps along the bridge, since after the first swap, the two agents involved are acquainted with all other agents, and the second swap across the bridge achieves nothing. If $k \geq a$, then by the assumption that there are no consecutive bridge swaps we conclude that $m \geq 2k - 1 \geq 2a - 1$.

Assume now that $k < a$. Call the agents who cross the bridge red. Call the agents who never leave the $a$-clique green, and the agents who never leave the $b$-clique blue. There are $2k$ configurations of the agents (just before and just after the bridge swaps) in which at least one endpoint of the bridge is occupied by a red agent. Since $k < a$, there are at least $a - k$ green agents and at least $b - k$ blue agents. A green agent and a blue agent can get acquainted only by being by the bridge at the same time, thus requiring at least $(a - k)(b - k)$ distinct configuration. Therefore, the total number of configuration is
\[ m + 1 \geq 2k + (a - k)(b - k) = k^2 - (a + b - 2)k + ab. \]

For an integer \( k \) smaller than \( a \), this is minimized when \( k = a - 1 \), thus giving a lower bound of \( m \geq a + b - 2 \). This implies the claimed bound on \( \mathcal{AC}(B_{a,b}) \). In the case when \( a = \lfloor n/2 \rfloor \) and \( b = \lceil n/2 \rceil \) this gives the bound \( m \geq n - 2 \), as required.

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