Unification of Neutrino-Neutrino and Neutrino-Antineutrino Oscillations

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Abstract: In the case of two-generation Majorana neutrinos, we derive the oscillation probabilities of $\nu \leftrightarrow \nu, \nu^c \leftrightarrow \nu^c$ and $\nu \leftrightarrow \nu^c$ in a unified framework by using a relativistic equation. We show that the Majorana phase occurs not from the violation of the lepton number but the change of chirality. We also check the conservation of the unitarity. Furthermore, we find that the oscillation probabilities of $\nu \leftrightarrow \nu^c$ obtained in this paper is largely different from the previous results in the point that there are no zero-distance effect and no direct CP violation.
1 Introduction

Since the discovery of the oscillations in the Super-Kamiokande atmospheric neutrino experiment in 1998, the evidence of the neutrino oscillations has been accumulated by the solar neutrino experiments [1–3], long-baseline experiments [4, 5] and reactor experiments [6–9]. Two mass-squared differences and three mixing angles in the lepton sector were determined by these experiments with good accuracy. One of the main goals in planned experiments [10, 11], is the determination of the leptonic Dirac CP phase and the mass ordering. Matter effects in the neutrino oscillation probabilities have been estimated exactly to achieve this goal [12–17].

In the Standard Model, neutrino is the only electrically neutral fermion and it may be the Majorana particle [18]. In the case of Majorana neutrinos, it is known that the Majorana CP phase appears [19–22] in addition to the Dirac CP phase. The Majorana CP phases exist in the framework of two generations or more, and it has been considered that the phases appear in the phenomena violating the lepton number like $0\nu\beta\beta$ decay.

In order to investigate whether neutrino is the Dirac particle or the Majorana particle, several $0\nu\beta\beta$ decay experiments have been performed [23–29], however, we have not concluded yet. In the $0\nu\beta\beta$ decay, a neutrino changes to an anti-neutrino (the charge conjugation of neutrino and has an opposite chirality to neutrino) and the intermediate state relates to the $\nu_e \leftrightarrow \nu_e$ oscillations in atomic size. The probabilities for $\nu_e \leftrightarrow \nu_e$ oscillations have been calculated in the references [22, 30–32]. In the probabilities, only the CP even effect of the Majorana CP phases is included and the decay rates of particle and anti-particle have the same value because the flavor of neutrino does not change. On the other hand, it has been pointed out that the CP odd effect can be observed by the measurement of $\nu_\alpha \leftrightarrow \nu_\beta$ oscillations with different flavors $\alpha$ and $\beta$ [33, 34].

Here, we have some questions about the previous papers studied on the $\nu \leftrightarrow \nu$ oscillations. If we consider $\nu \leftrightarrow \nu$ oscillations in addition to $\nu \leftrightarrow \nu$ oscillations, the sum
of the probabilities deviates from one because the unitarity holds within the framework of \( \nu \leftrightarrow \nu \) oscillations only in the above previous papers. Furthermore, it was pointed out in ref. [32] that the probability of \( \nu \leftrightarrow \nu^c \) oscillations is not equal to zero even if the baseline length is zero, so-called the zero-distance effect. This seems to be also inconsistent with the unitarity.

In our previous paper [35], we derive the exact oscillation probabilities of the Dirac neutrinos in the framework of two generations relativistically. As a result, we can understand the neutrino oscillations with and without chirality-flip in a unified way. We have also shown that the correction terms added to the oscillation probabilities from the necessity of the unitarity. If the oscillation probabilities can be exactly measured, we obtain the information related to the absolute value of neutrino masses and the new CP phase. This new CP phase contributes to the oscillation probabilities with chirality-flip.

In this paper, we would like to understand the oscillations between the neutrinos and the anti-neutrinos in a unified framework by applying the relativistic formulation used in our previous paper. We show that the Majorana CP phase is a kind of this new CP phase with chirality-flip as in the case of the Dirac neutrino in the previous paper [35]. Namely, the Majorana CP phases arise with chirality-flip, not the lepton number violation. In the case of the Majorana neutrinos, we can interpret that the anti-neutrino \( \nu^c \) plays the role of the right-handed neutrino \( \nu_R \) in the case of the Dirac neutrinos and the new CP phase found in our previous paper becomes to the Majorana CP phase.

We also show that there are no direct CP violation in \( \nu_\alpha \leftrightarrow \nu^c_\beta \) oscillations even if the flavors, \( \alpha \) and \( \beta \), are different. In other words, the difference of the CP conjugate probabilities vanishes and we can obtain the information about this CP phase only through the cosine term indirectly. Furthermore, the zero-distance effect suggested in [32], which means the finite value of the oscillation probability from neutrino to anti-neutrino instantly, never appears in our derivation. These results are largely different from the previous ones.

The paper is organized as follows. In section II, we define our notations used in this paper. In section III, we review the relativistic derivation of the oscillation probabilities for the neutrinos with only the Dirac mass term. In section IV, we present the derivation of the oscillation probabilities for the neutrinos with only the Majorana mass term. In section V, we compare our results to the previous ones in the case of the neutrinos with only the Majorana neutrino. In section VI, we summarize our results in this paper.

2 Notation

In this section, we write down the notation used in this paper. We mainly use the chiral representation because neutrinos are measured through weak interactions. In chiral representation, the gamma matrices with \( 4 \times 4 \) form are given by

\[
\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\] (2.1)
where $2 \times 2 \sigma$ matrices are defined by

$$
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
$$

We also define 4-component spinors $\psi$, $\psi_L$ and $\psi_R$ as

$$
\psi = \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \quad \psi_L = \frac{1 - \gamma_5}{2} \psi = \begin{pmatrix} 0 \\ \eta \end{pmatrix}, \quad \psi_R = \frac{1 + \gamma_5}{2} \psi = \begin{pmatrix} \xi \\ 0 \end{pmatrix}.
$$

and 2-component spinors $\xi$ and $\eta$ as

$$
\xi = \begin{pmatrix} \nu_R \\ \nu_R' \end{pmatrix}, \quad \eta = \begin{pmatrix} \nu_L \\ \nu_L' \end{pmatrix}.
$$

Furthermore, we use the subscript $\alpha$ and $\beta$ for flavor, $L$ and $R$ for chirality, the number 1 and 2 for generation and superscript $\pm$ for energy. Because of negligible neutrino mass, mass eigenstate has been often identified with energy eigenstate in many papers. But in the future, we should differentiate these two kinds of eigenstates for the finite neutrino mass. More concretely, we use the following eigenstates;

- chirality-flavor eigenstates: $\nu_{\alpha L}, \nu_{\alpha R}, \nu_{\beta L}, \nu_{\beta R}$,
- chirality-mass eigenstates: $\nu_1 L, \nu_1 R, \nu_2 L, \nu_2 R$,
- energy-helicity eigenstates: $\nu_1^+, \nu_1^-, \nu_2^+, \nu_2^-$.

It is noted that chirality-mass eigenstates are not exactly the eigenstates of the Hamiltonian. We use the term, eigenstates, in the sense that the mass submatrix in the Hamiltonian is diagonalized. Judging from common sense, one may think it strange that the chirality and the mass live in the same eigenstate. Details will be explained in the subsequent section.

We also define the spinors for anti-neutrino as charge conjugation of neutrino $\psi^c = i\gamma^2 \psi^*$. The charge conjugations for left-handed and right-handed neutrinos are defined by

$$
\psi_L^c \equiv (\psi_L)^c = \begin{pmatrix} \nu_L^\dagger \\ 0 \\ 0 \end{pmatrix} = i\gamma^2 \psi^*_L = i\gamma^2 \frac{1 - \gamma_5}{2} \psi^* 
$$

$$
= \frac{1 + \gamma_5}{2} (i\gamma^2 \psi^*) = (\psi^c)_R = \begin{pmatrix} i\sigma_2 \eta^* \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \nu_L^* \\ -\nu_L'^* \\ 0 \\ 0 \end{pmatrix},
$$

$$
\psi_R^c \equiv (\psi_R)^c = \begin{pmatrix} 0 \\ 0 \\ \nu_R' \\ \nu_R \end{pmatrix} = i\gamma^2 \psi^*_R = i\gamma^2 \frac{1 + \gamma_5}{2} \psi^* 
$$

$$
= \frac{1 - \gamma_5}{2} (i\gamma^2 \psi^*) = (\psi^c)_L = \begin{pmatrix} 0 \\ -i\sigma_2 \xi^* \\ \nu_R \\ \nu_R' \end{pmatrix} = \begin{pmatrix} 0 \\ -\nu_R^* \\ \nu_R \\ \nu_R'^* \end{pmatrix}.
$$
It is noted that the chirality is flipped by taking the charge conjugation.

3 Review of Dirac Neutrino Oscillation Probabilities in two generations

In our previous paper [35], we derived the neutrino oscillation probabilities in the case of the Dirac neutrinos by using a relativistic equation, the Dirac equation. In this section, we review the results of our previous paper [35] in order to clarify the similarity and the difference between the case of the Dirac neutrinos and the Majorana neutrinos.

In the case of two generations, the Lagrangian of neutrinos with the Dirac mass term is given by

\[
L = \overline{\psi}_a \gamma^\mu \partial_\mu \psi_a + \overline{\psi}_R \gamma^\mu \partial_\mu \psi_R + \overline{\psi}_L \gamma^\mu \partial_\mu \psi_L + \overline{\psi}_R \gamma^\mu \partial_\mu \psi_R \\
- \overline{\psi}_a m_{\alpha \beta} \psi_a + \overline{\psi}_R m_{\beta \alpha} \psi_R + \overline{\psi}_L m_{\alpha \beta} \psi_L + \overline{\psi}_R m_{\beta \alpha} \psi_R \\
- \overline{\psi}_a R m_{\alpha \beta} \psi_a L + \overline{\psi}_R m_{\beta \alpha} \psi_R L + \overline{\psi}_L m_{\alpha \beta} \psi_L + \overline{\psi}_R m_{\beta \alpha} \psi_R L,
\]

where \( m_{\alpha \beta} \) etc. stands for the Dirac masses. The Euler-Lagrange equations for \( \psi_a L, \psi_a R, \psi_{3L} \) and \( \psi_{3R} \) are combined to the matrix form,

\[
\frac{d}{dt}\begin{pmatrix}
\nu_{aR} \\
\nu_{3R} \\
\nu_{aL} \\
\nu_{3L}
\end{pmatrix} = \begin{pmatrix}
p & 0 & m_{\alpha \beta} & 0 & 0 & 0 & 0 & 0 \\
0 & p & m_{\beta \alpha} & 0 & 0 & 0 & 0 & 0 \\
m_{\alpha \beta}^* & m_{\alpha \beta}^* & 0 & -p & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -p & 0 & m_{\beta \alpha} & m_{\beta \alpha}^* & p
\end{pmatrix} \begin{pmatrix}
\nu_{aR} \\
\nu_{3R} \\
\nu_{aL} \\
\nu_{3L}
\end{pmatrix},
\]

where we take equal momentum assumption for neutrinos with different flavors, see ref. [35] for details. In the above equation, the bottom-right part and the top-left part are completely separated and therefore they do not mix each other even if the time has passed.

Taking the bottom right part, we obtain the equation,

\[
\frac{d}{dt}\begin{pmatrix}
\nu_{aR} \\
\nu_{3R} \\
\nu_{aL} \\
\nu_{3L}
\end{pmatrix} = \begin{pmatrix}
-p & 0 & m_{\alpha \beta} & 0 \\
0 & -p & m_{\beta \alpha} & m_{\beta \alpha}^* \\
m_{\alpha \beta}^* & m_{\alpha \beta}^* & p & 0 \\
0 & 0 & m_{\beta \alpha} & m_{\beta \alpha}^*
\end{pmatrix} \begin{pmatrix}
\nu_{aR} \\
\nu_{3R} \\
\nu_{aL} \\
\nu_{3L}
\end{pmatrix},
\]

where \( m_{\alpha \alpha}, m_{\beta \beta} \) and \( m_{\alpha \beta} \) are complex in general. The chirality-flavor eigenstates are represented as the linear combination of the chirality-mass eigenstates,

\[
\begin{pmatrix}
\nu_{aR} \\
\nu_{3R} \\
\nu_{aL} \\
\nu_{3L}
\end{pmatrix} = \begin{pmatrix}
V_{a1} & V_{a2} & 0 & 0 \\
V_{31} & V_{32} & 0 & 0 \\
0 & 0 & U_{a1} & U_{a2} \\
0 & 0 & U_{31} & U_{32}
\end{pmatrix} \begin{pmatrix}
\nu_{1R} \\
\nu_{2R} \\
\nu_{1L} \\
\nu_{2L}
\end{pmatrix}.
\]
The Dirac mass term of the Hamiltonian in eq. (3.3) is diagonalized by the above mixing matrix as

\[
\begin{pmatrix}
V_{\alpha 1} V_{\beta 1} & 0 & 0 \\
V_{\alpha 2} V_{\beta 2} & 0 & 0 \\
0 & U_{\alpha 1} U_{\beta 1} & 0 \\
0 & U_{\alpha 2} U_{\beta 2} & 0 \\
\end{pmatrix}
\begin{pmatrix}
-p & 0 & m_{\alpha \alpha} \\
0 & -p & m_{\beta \beta} \\
m_{\alpha \alpha} & m_{\beta \beta} & p \\
m_{\alpha \beta} & m_{\beta \alpha} & 0 \\
\end{pmatrix}
\begin{pmatrix}
V_{\alpha 1} & V_{\alpha 2} & 0 & 0 \\
V_{\beta 1} & V_{\beta 2} & 0 & 0 \\
0 & U_{\alpha 1} U_{\beta 1} & 0 & 0 \\
0 & U_{\alpha 2} U_{\beta 2} & 0 & 0 \\
\end{pmatrix}
= \begin{pmatrix}
-p & 0 & m_{1} & 0 \\
0 & -p & m_{2} & 0 \\
m_{1} & 0 & p & 0 \\
m_{2} & 0 & p & 0 \\
\end{pmatrix},
\] (3.5)

and the time evolution of the chirality-mass eigenstates is given by

\[
\frac{id}{dt} \begin{pmatrix}
\nu_{1R} \\
\nu_{2R} \\
\nu_{1L} \\
\nu_{2L}
\end{pmatrix} = \begin{pmatrix}
-p & 0 & m_{1} & 0 \\
0 & -p & m_{2} & 0 \\
m_{1} & 0 & p & 0 \\
m_{2} & 0 & p & 0 \\
\end{pmatrix} \begin{pmatrix}
\nu_{1R} \\
\nu_{2R} \\
\nu_{1L} \\
\nu_{2L}
\end{pmatrix},
\] (3.6)

where \(m_{1}\) and \(m_{2}\) are mass eigenvalues of neutrinos. Furthermore, we rewrite the above equations to those for the energy-helicity eigenstates. Exchanging some rows and some columns in eq.(3.6), we obtain

\[
\frac{id}{dt} \begin{pmatrix}
\nu_{1R} \\
\nu_{1L} \\
\nu_{2R} \\
\nu_{2L}
\end{pmatrix} = \begin{pmatrix}
-p m_{1} & 0 & 0 \\
0 & m_{1} & p & 0 \\
0 & 0 & -p m_{2} & 0 \\
0 & 0 & 0 & m_{2} & p \\
\end{pmatrix} \begin{pmatrix}
\nu_{1R} \\
\nu_{1L} \\
\nu_{2R} \\
\nu_{2L}
\end{pmatrix},
\] (3.7)

The chirality-mass eigenstates are represented by the linear combination of the energy-helicity eigenstates,

\[
\begin{pmatrix}
\nu_{1R} \\
\nu_{1L} \\
\nu_{2R} \\
\nu_{2L}
\end{pmatrix} = \begin{pmatrix}
\frac{\sqrt{E_{1}+p}}{2E_{1}} & 0 & 0 & 0 \\
\frac{\sqrt{E_{1}-p}}{2E_{1}} & 0 & 0 & 0 \\
0 & \sqrt{\frac{E_{2}+p}{2E_{2}}} & 0 & 0 \\
0 & \sqrt{\frac{E_{2}-p}{2E_{2}}} & 0 & 0 \\
\end{pmatrix}\begin{pmatrix}
\nu_{1}^{+} \\
\nu_{1}^{-} \\
\nu_{2}^{+} \\
\nu_{2}^{-}
\end{pmatrix},
\] (3.8)

and the Hamiltonian in eq.(3.6) is completely diagonalized by the mixing matrix in (3.8) and the time evolution of the energy-helicity eigenstates is given by

\[
\frac{id}{dt} \begin{pmatrix}
\nu_{1}^{-} \\
\nu_{1}^{+} \\
\nu_{2}^{-} \\
\nu_{2}^{+}
\end{pmatrix} = \begin{pmatrix}
-E_{1} & 0 & 0 & 0 \\
0 & E_{1} & 0 & 0 \\
0 & 0 & -E_{2} & 0 \\
0 & 0 & 0 & E_{2}
\end{pmatrix} \begin{pmatrix}
\nu_{1}^{-} \\
\nu_{1}^{+} \\
\nu_{2}^{-} \\
\nu_{2}^{+}
\end{pmatrix},
\] (3.9)

where \(E_{1}\) and \(E_{2}\) are the neutrino energies given by

\[
E_{1} = \sqrt{p^{2} + m_{1}^{2}}, \quad E_{2} = \sqrt{p^{2} + m_{2}^{2}}.
\] (3.10)
Combining eq. (3.4) and (3.8), the chirality-flavor eigenstates are represented by the energy-helicity eigenstates as

\[
\begin{pmatrix}
\nu_{\alpha R} \\
\nu_{\beta R} \\
\nu_{\alpha L} \\
\nu_{\beta L}
\end{pmatrix}
=
\begin{pmatrix}
V_{\alpha 1} & V_{\alpha 2} & 0 & 0 \\
V_{\beta 1} & V_{\beta 2} & 0 & 0 \\
0 & 0 & U_{\alpha 1} & U_{\alpha 2} \\
0 & 0 & U_{\beta 1} & U_{\beta 2}
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\nu_1^- \\
\nu_1^+ \\
\nu_2^- \\
\nu_2^+
\end{pmatrix}.
\]

(3.11)

After rewriting the equations for fields to those for one particle states, we can calculate the oscillation amplitudes for \(\nu_{\alpha L}\) to other neutrinos,

\[
A(\nu_{\alpha L} \rightarrow \nu_{\alpha L}) = \langle \nu_{\alpha L}|\nu_{\alpha L}(t)\rangle \\
= |U_{\alpha 1}|^2 \cos(E_1 t) - i|U_{\alpha 2}|^2 \cdot \frac{p}{E_1} \sin(E_1 t) + |U_{\alpha 2}|^2 \cos(E_2 t) - i|U_{\alpha 2}|^2 \cdot \frac{p}{E_2} \sin(E_2 t),
\]

(3.12)

\[
A(\nu_{\alpha L} \rightarrow \nu_{\beta L}) = \langle \nu_{\beta L}|\nu_{\alpha L}(t)\rangle \\
= U_{\alpha 1} U_{\beta 1} \left\{ \cos(E_1 t) - i \frac{p}{E_1} \sin(E_1 t) \right\} + U_{\alpha 2} U_{\beta 2} \left\{ \cos(E_2 t) - i \frac{p}{E_2} \sin(E_2 t) \right\},
\]

(3.13)

\[
A(\nu_{\alpha L} \rightarrow \nu_{\alpha R}) = \langle \nu_{\alpha R}|\nu_{\alpha L}(t)\rangle = -i U_{\alpha 1}^* V_{\alpha 1} \frac{m_1}{E_1} \sin(E_1 t) - i U_{\alpha 2}^* V_{\alpha 2} \frac{m_2}{E_2} \sin(E_2 t),
\]

(3.14)

\[
A(\nu_{\alpha L} \rightarrow \nu_{\beta R}) = \langle \nu_{\beta R}|\nu_{\alpha L}(t)\rangle = -i U_{\alpha 1}^* V_{\alpha 1} \frac{m_1}{E_1} \sin(E_1 t) - i U_{\alpha 2}^* V_{\alpha 2} \frac{m_2}{E_2} \sin(E_2 t).
\]

(3.15)

Furthermore, we also obtain the oscillation probabilities for \(\nu_{\alpha L}\) by squaring the amplitudes as

\[
P(\nu_{\alpha L} \rightarrow \nu_{\alpha L}) = |U_{\alpha 1}|^2 \cos(E_1 t) + |U_{\alpha 2}|^2 \cos(E_2 t) + \left\{ |U_{\alpha 1}|^2 \cdot \frac{p}{E_1} \sin(E_1 t) + |U_{\alpha 2}|^2 \cdot \frac{p}{E_2} \sin(E_2 t) \right\}^2,
\]

(3.16)

\[
P(\nu_{\alpha L} \rightarrow \nu_{\beta L}) = |U_{\alpha 1} U_{\beta 1}|^2 \left\{ \cos^2(E_1 t) + \frac{p^2}{E_1^2} \sin^2(E_1 t) \right\}
\]

\[
+ |U_{\alpha 2} U_{\beta 2}|^2 \left\{ \cos^2(E_2 t) + \frac{p^2}{E_2^2} \sin^2(E_2 t) \right\}
\]

\[
+ 2 \text{Re} \left[ U_{\alpha 1} U_{\beta 1} \left\{ \cos(E_1 t) - i \frac{p}{E_1} \sin(E_1 t) \right\} U_{\alpha 2} U_{\beta 2} \left\{ \cos(E_2 t) + i \frac{p}{E_2} \sin(E_2 t) \right\} \right]
\]

(3.17)
\[
P(\nu_{\alpha L} \rightarrow \nu_{\alpha R}) = |U_{\alpha 1}V_{\alpha 1}|^2 \frac{m_2^2}{E_1} \sin^2(E_1t) + |U_{\alpha 2}V_{\alpha 2}|^2 \frac{m_2^2}{E_2} \sin^2(E_2t) \\
+ 2 \text{Re}[U_{\alpha 1}U_{\alpha 2}^* V_{\alpha 1}^* V_{\alpha 2}] \frac{m_1 m_2}{E_1 E_2} \sin(E_1t) \sin(E_2t),
\]
(3.18)

\[
P(\nu_{\alpha L} \rightarrow \nu_{\beta R}) = |U_{\alpha 1}V_{\beta 1}|^2 \frac{m_2^2}{E_1} \sin^2(E_1t) + |U_{\alpha 2}V_{\beta 2}|^2 \frac{m_2^2}{E_2} \sin^2(E_2t) \\
+ 2 \text{Re}[U_{\alpha 1}U_{\alpha 2}^* V_{\beta 1}^* V_{\beta 2}] \frac{m_1 m_2}{E_1 E_2} \sin(E_1t) \sin(E_2t).
\]
(3.19)

The 2 \times 2 unitary matrix has four parameters in general, and \( U \) and \( V \) can be parametrized as

\[
U = \begin{pmatrix} e^{i\rho_{1L}} & 0 \\ 0 & e^{i\rho_{2L}} \end{pmatrix} \begin{pmatrix} \cos \theta_L & \sin \theta_L \\ -\sin \theta_L & \cos \theta_L \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{i\phi_L} \end{pmatrix} = \begin{pmatrix} e^{i\rho_{1L}+i\phi_L} & e^{i\rho_{2L}+i\phi_L} \\ -e^{i\rho_{2L}} \sin \theta_L & -e^{i\rho_{1L}} \sin \theta_L \end{pmatrix},
\]
(3.20)

\[
V = \begin{pmatrix} e^{i\rho_{1R}} & 0 \\ 0 & e^{i\rho_{2R}} \end{pmatrix} \begin{pmatrix} \cos \theta_R & \sin \theta_R \\ -\sin \theta_R & \cos \theta_R \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{i\phi_R} \end{pmatrix} = \begin{pmatrix} e^{i\rho_{1R}+i\phi_R} & e^{i\rho_{2R}+i\phi_R} \\ -e^{i\rho_{2R}} \sin \theta_R & -e^{i\rho_{1R}} \sin \theta_R \end{pmatrix},
\]
(3.21)

respectively. Substituting these parametrizations into the equations (3.16)-(3.19), the oscillation probabilities can be rewritten as

\[
P(\nu_{\alpha L} \rightarrow \nu_{\alpha R}) = 1 - 4s_{\rho L}^2 c_{\rho L} \sin^2 \left( \frac{E_2 - E_1 t}{2} \right)
\]
(3.22)

\[
P(\nu_{\alpha L} \rightarrow \nu_{\beta R}) = 4s_{\rho R}^2 c_{\rho R} \sin^2 \left( \frac{E_2 - E_1 t}{2} \right)
\]
(3.24)

\[
- [s_{\rho L}^2 \sin^2(E_1t) s_{\rho L}^2 \sin^2(E_2t) + 2s_{\rho L}^2 c_{\rho L} \left( 1 - \frac{p^2}{E_1 E_2} \right) \sin(E_1t) \sin(E_2t)]
\]
(3.25)

\[
P(\nu_{\alpha L} \rightarrow \nu_{\alpha R}) = c_{\rho L}^2 \frac{m_2^2}{E_1} \sin^2(E_1t) + s_{\rho L}^2 \frac{m_2^2}{E_2} \sin^2(E_2t)
\]
\[+ 2s_{\rho R} c_{\rho L} \cos(\phi_R - \phi_L) \frac{m_1 m_2}{E_1 E_2} \sin(E_1t) \sin(E_2t),
\]
(3.26)

\[
P(\nu_{\alpha L} \rightarrow \nu_{\beta R}) = c_{\rho L}^2 \frac{m_2^2}{E_1} \sin^2(E_1t) + s_{\rho L}^2 \frac{m_2^2}{E_2} \sin^2(E_2t)
\]
\[- 2s_{\rho R} c_{\rho L} \cos(\phi_R - \phi_L) \frac{m_1 m_2}{E_1 E_2} \sin(E_1t) \sin(E_2t),
\]
(3.27)

where we use the abbreviation \( \sin \theta_L = s_L, \cos \theta_L = c_L \) and so on. Although the oscillation probabilities with and without chirality-flip have been derived separately in the calculations done so far, we can derive both types of oscillation probabilities in a unified framework by using the Dirac equation.

Let us describe some important points in these oscillation probabilities. First, we find the new terms (3.23) and (3.25) with an order of \( O(m^2/E^2) \) in the probabilities without chirality-flip in addition to the well-known terms (3.22) and (3.24). On the other hand, the probabilities with chirality-flip (3.26) and (3.27) are also an order of \( O(m^2/E^2) \) and the sum over these new terms with \( O(m^2/E^2) \) vanishes. Therefore, the sum of the four probabilities
is kept at one. Second, these new terms depend on not only mass squared differences but also the absolute masses of neutrinos. This fact is not clearly insisted on previous papers as far as we know. Third, if we can distinguish the flavor of right-handed particles beyond the Standard Model, a new CP phase appears even in two-generation Dirac neutrinos. This new phase is included in the probabilities with chirality-flip.

Although it is considered that the CP phase does not appear in two-generation Dirac neutrino oscillations, we have shown the dependence of the probabilities on the new mixing angle and the new CP phase by considering the left-handed and right-handed neutrinos in the same framework. These effects are proportional to \((m/E)^2\) and are tiny in usual neutrino oscillation experiments. We cannot observe the CP phase accompanying the unitary matrix \(V\) if \(\nu_R\) do not have weak interactions and we can choose the weak eigenstates as the same as the mass eigenstates. Otherwise, the effects may be measurable in the oscillations of atomic size [35].

If neutrino is the Majorana particle, \(\nu_L^c\) takes the place of \(\nu_R\). In this case, also \(\nu_L^c\) has weak interactions and we can distinguish the weak eigenstates from mass eigenstates. Thus, this is one of the examples for the phase \(\phi_R - \phi_L\) to be observable. In the next section, we derive the oscillation probabilities of the Majorana neutrinos and we show that the phase \(\phi_R - \phi_L\) actually becomes observable.

4 Majorana Neutrino Oscillation Probabilities in two generations

In this section, we consider the transition of two-generation Majorana neutrinos in vacuum. Let us start from the lagrangian of the Majorana neutrinos,

\[
L = \frac{1}{2} i \bar{\psi}_{\alpha L} \gamma^\mu \partial_\mu \psi_{\alpha L} + \frac{1}{2} i \bar{\psi}_{\alpha L} \gamma^\mu \partial_\mu \psi_{\alpha L}^c + \frac{1}{2} i \bar{\psi}_{\beta L} \gamma^\mu \partial_\mu \psi_{\beta L} + \frac{1}{2} i \bar{\psi}_{\beta L} \gamma^\mu \partial_\mu \psi_{\beta L}^c - \frac{1}{2} \left[ \bar{\psi}_{\alpha L} M_{\alpha \alpha} \psi_{\alpha L} + \bar{\psi}_{\beta L} M_{\beta \beta} \psi_{\beta L} + \bar{\psi}_{\alpha L} M_{\alpha \beta} \psi_{\beta L} + \bar{\psi}_{\beta L} M_{\beta \alpha} \psi_{\alpha L} \right]
\]

\[
- \frac{1}{2} \left[ \bar{\psi}_{\alpha L} M^*_{\alpha \alpha} \psi_{\alpha L}^c + \bar{\psi}_{\beta L} M^*_{\beta \beta} \psi_{\beta L}^c + \bar{\psi}_{\alpha L} M^*_{\alpha \beta} \psi_{\beta L}^c + \bar{\psi}_{\beta L} M^*_{\beta \alpha} \psi_{\alpha L}^c \right]. \tag{4.1}
\]

About the kinetic terms, we have the relation,

\[
L_{\text{kin}} = i \bar{\psi}_{\alpha L} \gamma^\mu \partial_\mu \psi_{\alpha L} = i \bar{\psi}_{\alpha L}^c \gamma^\mu \partial_\mu \psi_{\alpha L}^c. \tag{4.2}
\]

So, the Euler-Lagrange equation for \(\bar{\psi}_{\alpha L}\),

\[
\frac{\partial L}{\partial \bar{\psi}_{\alpha L}} - \partial_\mu \left( \frac{\partial L}{\partial (\partial_\mu \bar{\psi}_{\alpha L})} \right) = 0 \tag{4.3}
\]

provides

\[
\gamma^\mu \partial_\mu \psi_{\alpha L} - M^*_{\alpha \alpha} \psi_{\alpha L}^c - M^*_{\beta \alpha} \psi_{\beta L}^c = 0, \tag{4.4}
\]

where the factor \(\frac{1}{2}\) of the kinetic terms and the mass terms dissapears because \(\bar{\psi}_{\alpha L}\) is regarded as same as \(\psi_{\alpha}^c\). Multiplying \(\gamma_0\) from the left, we obtain the equation,

\[
i\partial_0 \bar{\psi}_{\alpha L} + i \gamma^0 \gamma^i \partial_i \bar{\psi}_{\alpha L} - M^*_{\alpha \alpha} \gamma^0 \psi_{\alpha L}^c - M^*_{\beta \alpha} \gamma^0 \psi_{\beta L}^c = 0. \tag{4.5}
\]
Substituting (2.3) and (2.8) into this equation, we obtain the equation for two-component spinor $\eta_\alpha$,
\[
i\partial_0 \left( \begin{array}{c} 0 \\ \eta_\alpha \end{array} \right) - i \left( \sigma_i \partial_i \eta_\alpha \right) - M^*_{\alpha\alpha} \left( \begin{array}{c} 0 \\ i\sigma_2 \eta^*_\alpha \end{array} \right) - M^*_{\beta\alpha} \left( \begin{array}{c} 0 \\ i\sigma_2 \eta^*_\beta \end{array} \right) = 0. \tag{4.6}
\]
Taking out the lower two components, we obtain
\[
i\partial_0 \eta_\alpha - i\sigma_i \partial_i \eta_\alpha - M^*_{\alpha\alpha} (i\sigma_2 \eta^*_\alpha) - M^*_{\beta\alpha} (i\sigma_2 \eta^*_\beta) = 0. \tag{4.7}
\]
In the same way, from the Euler-Lagrange equation for $\bar{\psi}_{\alpha L}$,
\[
\frac{\partial L}{\partial \bar{\psi}_{\alpha L}} - \partial_\mu \left( \frac{\partial L}{\partial (\partial_\mu \bar{\psi}_{\alpha L})} \right) = 0, \tag{4.8}
\]
we obtain the equation,
\[
i\gamma^\mu \partial_\mu \bar{\psi}_{\alpha L} - M_{\alpha \alpha} \bar{\psi}_{\alpha L} - M_{\alpha \beta} \bar{\psi}_{\beta L} = 0. \tag{4.9}
\]
Multiplying $\gamma_0$ from the left, the above equation becomes
\[
i\partial_0 \bar{\psi}_{\alpha L} + i\gamma^0 \gamma^i \partial_i \bar{\psi}_{\alpha L} - M_{\alpha \alpha} \gamma^0 \bar{\psi}_{\alpha L} - M_{\alpha \beta} \gamma^0 \bar{\psi}_{\beta L} = 0. \tag{4.10}
\]
Substituting (2.3) and (2.8) into this equation, we obtain the equation for two-component spinor $\eta^*_\alpha$,
\[
i\partial_0 \left( \begin{array}{c} i\sigma_2 \eta^*_\alpha \\ 0 \end{array} \right) + i \left( \sigma_i \partial_i (i\sigma_2 \eta^*_\alpha) \right) - M_{\alpha\alpha} \left( \begin{array}{c} \eta_\alpha \\ 0 \end{array} \right) - M_{\alpha\beta} \left( \begin{array}{c} \eta_\beta \\ 0 \end{array} \right) = 0. \tag{4.11}
\]
Taking out the upper two components, we obtain
\[
i\partial_0 (i\sigma_2 \eta^*_\alpha) + i\sigma_i \partial_i (i\sigma_2 \eta^*_\alpha) - M_{\alpha\alpha} \eta_\alpha - M_{\alpha\beta} \eta_\beta = 0. \tag{4.12}
\]
If we take the equal momentum assumption,
\[
\eta_\alpha (x, t) = e^{i\vec{p} \cdot \vec{x}} \eta_\alpha (t) = e^{i\vec{p} \cdot \vec{x}} \left( \begin{array}{c} \nu^\alpha_L \\ \nu_{\alpha L} \end{array} \right), \quad \eta_\beta (x, t) = e^{i\vec{p} \cdot \vec{x}} \eta_\beta (t) = e^{i\vec{p} \cdot \vec{x}} \left( \begin{array}{c} \nu^\beta_L \\ \nu_{\beta L} \end{array} \right), \tag{4.13}
\]
and choose $\vec{p} = (0, 0, p)$ as momentum vector, the complex conjugate of these two-component spinors are given by
\[
\eta^*_\alpha (x, t) = e^{-i\vec{p} \cdot \vec{x}} \eta^*_\alpha (t) = e^{-i\vec{p} \cdot \vec{x}} \left( \begin{array}{c} \nu^\alpha_{\alpha L} \\ \nu_{\alpha L} \end{array} \right), \quad \eta^*_\beta (x, t) = e^{-i\vec{p} \cdot \vec{x}} \eta^*_\beta (t) = e^{-i\vec{p} \cdot \vec{x}} \left( \begin{array}{c} \nu^\beta_{\alpha L} \\ \nu_{\beta L} \end{array} \right). \tag{4.14}
\]
Note that the sign of the momentum for $\eta$ and $\eta^*$ is opposite. Then, (4.7) and (4.12) are rewritten as
\[
i\partial_0 \left( \begin{array}{c} \nu^\alpha_L \\ \nu_{\alpha L} \end{array} \right) + p \left( \begin{array}{c} \nu^\alpha_L \\ -\nu_{\alpha L} \end{array} \right) - M^*_{\alpha\alpha} \left( \begin{array}{c} \nu^\alpha_{\alpha L} \\ \nu_{\alpha L} \end{array} \right) - M^*_{\beta\alpha} \left( \begin{array}{c} \nu^\beta_{\beta L} \\ \nu_{\beta L} \end{array} \right) = 0, \tag{4.15}
\]
\[
i\partial_0 \left( \begin{array}{c} \nu^\alpha_{\alpha L} \\ \nu_{\alpha L} \end{array} \right) + p \left( \begin{array}{c} \nu^\alpha_{\alpha L} \\ -\nu_{\alpha L} \end{array} \right) - M_{\alpha\alpha} \left( \begin{array}{c} \nu^\alpha_L \\ \nu_{\alpha L} \end{array} \right) - M_{\alpha\beta} \left( \begin{array}{c} \nu^\beta_L \\ \nu_{\beta L} \end{array} \right) = 0. \tag{4.16}
\]
where we replace $\nu^*$ and $\nu'^*$ to $\nu^c$ and $\nu'^c$ according to the definition (2.8). We put the above two equations together as matrix form,

$$
\frac{id}{dt} = \begin{pmatrix}
\nu^c_{aL} \\
\nu^c_{bL} \\
\nu^c_{aL} \\
\nu^c_{bL} \\
\nu^c_{aL} \\
\nu^c_{bL}
\end{pmatrix}
= \begin{pmatrix}
-p & 0 & M_{aa0} & 0 & 0 & M_{a0} \\
0 & p & 0 & M_{aa0} & 0 & 0 & 0 & 0 & M_{a0} \\
M^*_{aa0} & 0 & -p & 0 & M^*_{a0} & 0 & 0 & 0 & 0 \\
0 & M_{aa0} & 0 & p & 0 & M^*_{a0} & 0 & 0 & 0 \\
0 & 0 & M^*_{a0} & 0 & M^*_{a0} & 0 & 0 & p & 0 \\
0 & M^*_{a0} & 0 & 0 & M^*_{a0} & 0 & p & 0
\end{pmatrix}
\begin{pmatrix}
\nu^c_{aL} \\
\nu^c_{bL} \\
\nu^c_{aL} \\
\nu^c_{bL} \\
\nu^c_{aL} \\
\nu^c_{bL}
\end{pmatrix}.
$$

(4.17)

Exchanging some rows and some columns, the above equation can be rewritten as

$$
\frac{id}{dt} = \begin{pmatrix}
\nu^c_{aL} \\
\nu^c_{bL} \\
\nu^c_{aL} \\
\nu^c_{bL} \\
\nu^c_{aL} \\
\nu^c_{bL}
\end{pmatrix}
= \begin{pmatrix}
p & 0 & M_{aa0} & M_{a0} & 0 & 0 & 0 & 0 & 0 \\
0 & p & M_{aa0} & 0 & M_{a0} & 0 & 0 & 0 & 0 \\
M^*_{aa0} & M^*_{a0} & -p & 0 & M^*_{a0} & 0 & 0 & 0 & 0 \\
0 & M_{aa0} & M^*_{a0} & -p & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & M^*_{a0} & 0 & M^*_{a0} & 0 & 0 & p & 0 \\
0 & 0 & 0 & M^*_{a0} & 0 & M^*_{a0} & 0 & p & 0
\end{pmatrix}
\begin{pmatrix}
\nu^c_{aL} \\
\nu^c_{bL} \\
\nu^c_{aL} \\
\nu^c_{bL} \\
\nu^c_{aL} \\
\nu^c_{bL}
\end{pmatrix}.
$$

(4.18)

It is noted that $\nu$ and $\nu^c$ included in the same multiplet can change each other due to the presence of the Majorana mass term. We can separate the top-left part from the bottom-right part. In the following calculations, we take the bottom-right part,

$$
\frac{d}{dt} \begin{pmatrix}
\nu_{aL} \\
\nu_{bL}
\end{pmatrix}
= \begin{pmatrix}
-p & 0 & M_{aa0} & 0 & 0 & M_{a0} \\
0 & -p & M_{aa0} & 0 & 0 & M_{a0} \\
M^*_{aa0} & M^*_{a0} & -p & 0 & M^*_{a0} & 0 & 0 & 0 & 0 \\
0 & M_{aa0} & M^*_{a0} & -p & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & M^*_{a0} & 0 & M^*_{a0} & 0 & 0 & p & 0 \\
0 & 0 & 0 & M^*_{a0} & 0 & M^*_{a0} & 0 & p & 0
\end{pmatrix}
\begin{pmatrix}
\nu_{aL} \\
\nu_{bL}
\end{pmatrix}.
$$

(4.19)

This has the same structure as the equation (3.3), in the case of the Dirac neutrinos, reviewed in the previous section. The Hamiltonian for the Majorana neutrinos is obtained by the replacement of the Dirac mass to the Majorana mass due to the replacement of $\nu_{aR} \rightarrow \nu_{aL}^c$ and $\nu_{bR} \rightarrow \nu_{bL}^c$. The Majorana mass matrix becomes a complex symmetric matrix unlike the Dirac mass matrix because $\nu^c$ is a complex conjugate of $\nu$ and is not independent of $\nu$. The Hamiltonian in eq.(4.19) is diagonalized as

$$
\begin{pmatrix}
U_{a1} & U_{b1} \\
U_{a2} & U_{b2}
\end{pmatrix}
\begin{pmatrix}
-p & 0 & M_{aa0} & 0 & 0 & M_{a0} \\
0 & -p & M_{aa0} & 0 & 0 & M_{a0} \\
M^*_{aa0} & M^*_{a0} & -p & 0 & M^*_{a0} & 0 & 0 & 0 & 0 \\
0 & M_{aa0} & M^*_{a0} & -p & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & M^*_{a0} & 0 & M^*_{a0} & 0 & 0 & p & 0 \\
0 & 0 & 0 & M^*_{a0} & 0 & M^*_{a0} & 0 & p & 0
\end{pmatrix}
\begin{pmatrix}
U_{a1}^* & U_{b1}^* \\
U_{a2}^* & U_{b2}^*
\end{pmatrix}
= \begin{pmatrix}
-p & 0 & m_1 & 0 \\
0 & -p & 0 & m_2 \\
m_1 & 0 & p \\
m_2 & 0 & p
\end{pmatrix}.
$$

(4.20)

One can see that the $2 \times 2$ Majorana mass matrix $M$ of the top-right part in the Hamiltonian is diagonalized as $U^T M U = \text{diag}(m_1, m_2)$. The chirality-flavor eigenstates are represented
as the linear combination of the chirality-mass eigenstates,

\[
\begin{pmatrix}
    \nu_{\alpha L}^c \\
    \nu_{\beta L}^c \\
    \nu_L \\
    \nu_{2L}
\end{pmatrix}
= \begin{pmatrix}
    U^{*}_{\alpha 1} & U^{*}_{\alpha 2} & 0 & 0 \\
    U^{*}_{\beta 1} & U^{*}_{\beta 2} & 0 & 0 \\
    0 & 0 & U_{\alpha 1} & U_{\alpha 2} \\
    0 & 0 & U_{\beta 1} & U_{\beta 2}
\end{pmatrix}
\begin{pmatrix}
    \nu_{1L}^c \\
    \nu_{2L}^c \\
    \nu_L \\
    \nu_{2L}
\end{pmatrix}.
\] (4.21)

In the case of the Dirac neutrinos, the mass matrix is diagonalized by two unitary matrices \( V \) and \( U \). On the other hand, in the case of the Majorana neutrinos, the mass matrix is diagonalized by only one unitary matrix \( U \). The time evolution of the chirality-mass eigenstates is given by

\[
\frac{id}{dt}
\begin{pmatrix}
    \nu_{1L}^c \\
    \nu_{2L}^c \\
    \nu_L \\
    \nu_{2L}
\end{pmatrix}
= \begin{pmatrix}
    -p & 0 & m_1 & 0 \\
    0 & -p & 0 & m_2 \\
    m_1 & 0 & p & 0 \\
    0 & m_2 & 0 & p
\end{pmatrix}
\begin{pmatrix}
    \nu_{1L}^c \\
    \nu_{2L}^c \\
    \nu_L \\
    \nu_{2L}
\end{pmatrix}.
\] (4.22)

Furthermore, let us rewrite this equation to that for the energy-helicity eigenstates in order to diagonalize the Hamiltonian completely. Exchanging some rows and some columns in the above equation, we obtain

\[
\frac{id}{dt}
\begin{pmatrix}
    \nu_{1L}^c \\
    \nu_{1L} \\
    \nu_{2L}^c \\
    \nu_{2L}
\end{pmatrix}
= \begin{pmatrix}
    -p & m_1 & 0 & 0 \\
    m_1 & p & 0 & 0 \\
    0 & 0 & -p & m_2 \\
    0 & 0 & m_2 & p
\end{pmatrix}
\begin{pmatrix}
    \nu_{1L}^c \\
    \nu_{1L} \\
    \nu_{2L}^c \\
    \nu_{2L}
\end{pmatrix}.
\] (4.23)

The chirality-mass eigenstates are represented as the linear combination of energy-helicity eigenstates,

\[
\begin{pmatrix}
    \nu_{1L}^c \\
    \nu_{1L} \\
    \nu_{2L}^c \\
    \nu_{2L}
\end{pmatrix}
= \begin{pmatrix}
    \sqrt{E_1+p} & \sqrt{E_1-p} & 0 & 0 \\
    \sqrt{E_1+p} & \sqrt{E_1-p} & 0 & 0 \\
    0 & 0 & \sqrt{E_2+p} & \sqrt{E_2-p} \\
    0 & 0 & -\sqrt{E_2-p} & \sqrt{E_2+p}
\end{pmatrix}
\begin{pmatrix}
    \nu_{1L}^c \\
    \nu_{1L} \\
    \nu_{2L}^c \\
    \nu_{2L}
\end{pmatrix},
\] (4.24)

by using the matrix diagonalizing the Hamiltonian in eq. (4.23). Then, the time evolution of the energy-helicity eigenstates is given by

\[
\frac{id}{dt}
\begin{pmatrix}
    \nu_{1L}^c \\
    \nu_{1L} \\
    \nu_{2L}^c \\
    \nu_{2L}
\end{pmatrix}
= \begin{pmatrix}
    -E_1 & 0 & 0 & 0 \\
    0 & E_1 & 0 & 0 \\
    0 & 0 & -E_2 & 0 \\
    0 & 0 & 0 & E_2
\end{pmatrix}
\begin{pmatrix}
    \nu_{1L}^c \\
    \nu_{1L} \\
    \nu_{2L}^c \\
    \nu_{2L}
\end{pmatrix},
\] (4.25)

where

\[
E_1 = \sqrt{p^2 + m_1^2}, \quad E_2 = \sqrt{p^2 + m_2^2}.
\] (4.26)
Combining the equations, (4.21) and (4.24), the chirality-flavor eigenstates are represented as the linear combination of the energy-helicity eigenstates,

\[
\begin{pmatrix}
\nu_{\alpha L}^c \\
\nu_{\beta L}^c \\
\nu_{\alpha L} \\
\nu_{\beta L}
\end{pmatrix} =
\begin{pmatrix}
U_{\alpha 1}^* & U_{\alpha 2}^* & 0 & 0 \\
U_{\beta 1}^* & U_{\beta 2}^* & 0 & 0 \\
0 & 0 & U_{\alpha 1} & U_{\alpha 2} \\
0 & 0 & U_{\beta 1} & U_{\beta 2}
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\nu_1^c \\
\nu_2^c \\
\nu_1^+ \\
\nu_2^+
\end{pmatrix}.
\]

Rewriting the relation for the fields obtained from the above equation into the relation for one particle states, we obtain the chirality-flavor eigenstates after the time \( t \),

\[
\begin{align*}
|\nu_{\alpha L}(t)\rangle &= U_{\alpha 1}\sqrt{\frac{E_1 + p}{2E_1}} e^{-iE_1 t}|\nu_1^c\rangle + U_{\alpha 1}\sqrt{\frac{E_1 - p}{2E_1}} e^{-iE_1 t}|\nu_1^+\rangle \\
&\quad + U_{\alpha 2}\sqrt{\frac{E_2 + p}{2E_2}} e^{-iE_2 t}|\nu_2^c\rangle + U_{\alpha 2}\sqrt{\frac{E_2 - p}{2E_2}} e^{-iE_2 t}|\nu_2^+\rangle, \\
|\nu_{\beta L}(t)\rangle &= U_{\beta 1}\sqrt{\frac{E_1 + p}{2E_1}} e^{-iE_1 t}|\nu_1^c\rangle + U_{\beta 1}\sqrt{\frac{E_1 - p}{2E_1}} e^{-iE_1 t}|\nu_1^+\rangle \\
&\quad + U_{\beta 2}\sqrt{\frac{E_2 + p}{2E_2}} e^{-iE_2 t}|\nu_2^c\rangle + U_{\beta 2}\sqrt{\frac{E_2 - p}{2E_2}} e^{-iE_2 t}|\nu_2^+\rangle, \\
|\nu_{\alpha L}(t)\rangle &= -U_{\alpha 1}\sqrt{\frac{E_1 - p}{2E_1}} e^{-iE_1 t}|\nu_1^c\rangle + U_{\alpha 1}\sqrt{\frac{E_1 + p}{2E_1}} e^{-iE_1 t}|\nu_1^+\rangle \\
&\quad - U_{\alpha 2}\sqrt{\frac{E_2 - p}{2E_2}} e^{-iE_2 t}|\nu_2^c\rangle + U_{\alpha 2}\sqrt{\frac{E_2 + p}{2E_2}} e^{-iE_2 t}|\nu_2^+\rangle, \\
|\nu_{\beta L}(t)\rangle &= -U_{\beta 1}\sqrt{\frac{E_1 - p}{2E_1}} e^{-iE_1 t}|\nu_1^c\rangle + U_{\beta 1}\sqrt{\frac{E_1 + p}{2E_1}} e^{-iE_1 t}|\nu_1^+\rangle \\
&\quad - U_{\beta 2}\sqrt{\frac{E_2 - p}{2E_2}} e^{-iE_2 t}|\nu_2^c\rangle + U_{\beta 2}\sqrt{\frac{E_2 + p}{2E_2}} e^{-iE_2 t}|\nu_2^+\rangle.
\end{align*}
\]

and their conjugate states are also given by

\[
\begin{align*}
\langle \nu_{\alpha L}^c | &= U_{\alpha 1}\sqrt{\frac{E_1 + p}{2E_1}} \langle \nu_1^c | + U_{\alpha 1}\sqrt{\frac{E_1 - p}{2E_1}} \langle \nu_1^+ | + U_{\alpha 2}\sqrt{\frac{E_2 + p}{2E_2}} \langle \nu_2^c | + U_{\alpha 2}\sqrt{\frac{E_2 - p}{2E_2}} \langle \nu_2^+ |, \\
\langle \nu_{\beta L}^c | &= U_{\beta 1}\sqrt{\frac{E_1 + p}{2E_1}} \langle \nu_1^c | + U_{\beta 1}\sqrt{\frac{E_1 - p}{2E_1}} \langle \nu_1^+ | + U_{\beta 2}\sqrt{\frac{E_2 + p}{2E_2}} \langle \nu_2^c | + U_{\beta 2}\sqrt{\frac{E_2 - p}{2E_2}} \langle \nu_2^+ |, \\
\langle \nu_{\alpha L} | &= -U_{\alpha 1}\sqrt{\frac{E_1 - p}{2E_1}} \langle \nu_1^c | + U_{\alpha 1}\sqrt{\frac{E_1 + p}{2E_1}} \langle \nu_1^+ | - U_{\alpha 2}\sqrt{\frac{E_2 - p}{2E_2}} \langle \nu_2^c | + U_{\alpha 2}\sqrt{\frac{E_2 + p}{2E_2}} \langle \nu_2^+ |, \\
\langle \nu_{\beta L} | &= -U_{\beta 1}\sqrt{\frac{E_1 - p}{2E_1}} \langle \nu_1^c | + U_{\beta 1}\sqrt{\frac{E_1 + p}{2E_1}} \langle \nu_1^+ | - U_{\beta 2}\sqrt{\frac{E_2 - p}{2E_2}} \langle \nu_2^c | + U_{\beta 2}\sqrt{\frac{E_2 + p}{2E_2}} \langle \nu_2^+ |.
\end{align*}
\]
From these relations, we obtain the oscillation amplitudes,

\[
A(\nu_{aL} \to \nu_{aL}) = \langle \nu_{aL} | \nu_{aL}(t) \rangle \\
= |U_{a1}|^2 \frac{E_1 - p}{2E_1} e^{i E_1 t} + |U_{a2}|^2 \frac{E_2 + p}{2E_2} e^{i E_2 t}
\]

Further, we calculate the oscillation probabilities for \( \nu_{aL} \) by squaring the corresponding amplitudes,

\[
P(\nu_{aL} \to \nu_{aL}) = \left| |U_{a1}|^2 \cos(E_1 t) + |U_{a2}|^2 \cos(E_2 t) \right|^2 \\
+ \left[ |U_{a1}|^2 \cdot \frac{p}{E_1} \sin(E_1 t) + |U_{a2}|^2 \cdot \frac{p}{E_2} \sin(E_2 t) \right]^2, \tag{4.40}
\]

\[
P(\nu_{aL} \to \nu_{bL}) = |U_{a1} U_{b1}|^2 \left\{ \cos^2(E_1 t) + \frac{p^2}{E_1^2} \sin^2(E_1 t) \right\} \\
+ |U_{a2} U_{b2}|^2 \left\{ \cos^2(E_2 t) + \frac{p^2}{E_2^2} \sin^2(E_2 t) \right\} \\
+ 2 \text{Re} \left[ U_{a1}^* U_{b1} \left( \cos(E_1 t) - \frac{p}{E_1} \sin(E_1 t) \right) U_{a2} U_{b2}^* \left( \cos(E_2 t) + \frac{p}{E_2} \sin(E_2 t) \right) \right]. \tag{4.41}
\]

\[
P(\nu_{aL} \to \nu_{cL}) = |U_{a1} U_{c1}|^2 \frac{m_1^2}{E_1^2} \sin^2(E_1 t) + |U_{a2} U_{c2}|^2 \frac{m_2^2}{E_2^2} \sin^2(E_2 t) \\
+ 2 \text{Re}(U_{a1}^* U_{a2} U_{c1}^* U_{c2} | \frac{m_1 m_2}{E_1 E_2} \sin(E_1 t) \sin(E_2 t)), \tag{4.42}
\]

\[
P(\nu_{aL} \to \nu_{cL}) = |U_{a1} U_{c1}|^2 \frac{m_1^2}{E_1^2} \sin^2(E_1 t) + |U_{a2} U_{c2}|^2 \frac{m_2^2}{E_2^2} \sin^2(E_2 t) \\
+ 2 \text{Re}(U_{a1}^* U_{a2} U_{c1}^* U_{c2} | \frac{m_1 m_2}{E_1 E_2} \sin(E_1 t) \sin(E_2 t)), \tag{4.43}
\]
As $2 \times 2$ unitary matrix has four parameters in general, we choose the following parametrization,

$$U = \begin{pmatrix} e^{i\rho_1} & 0 \\ 0 & e^{i\rho_2} \end{pmatrix} \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{pmatrix} = \begin{pmatrix} e^{i\rho_1} \cos \theta & e^{i(\rho_1+\phi)} \sin \theta \\ -e^{i\rho_2} \sin \theta & e^{i(\rho_2+\phi)} \cos \theta \end{pmatrix}. \quad (4.44)$$

Substituting these parametrization into the equations (4.40)-(4.43), the oscillation probabilities are rewritten as

$$P(\nu_{aL} \to \nu_{aL}) = 1 - 4s^2c^2\sin^2 \frac{(E_2 - E_1)t}{2} - \left[ c^4 \cdot \frac{m_2^2}{E_1^2} \sin^2(E_1 t) + s^4 \cdot \frac{m_2^2}{E_2^2} \sin^2(E_2 t) + 2s^2c^2 \left( 1 - \frac{p^2}{E_1E_2} \right) \sin(E_1 t) \sin(E_2 t) \right], \quad (4.45)$$

$$P(\nu_{aL} \to \nu_{\beta L}) = 4s^2c^2\sin^2 \frac{(E_2 - E_1)t}{2} - s^2c^2 \left[ \frac{m_2^2}{E_1^2} \sin^2(E_1 t) + \frac{m_2^2}{E_2^2} \sin^2(E_2 t) - 2 \left( 1 - \frac{p^2}{E_1E_2} \right) \sin(E_1 t) \sin(E_2 t) \right], \quad (4.46)$$

$$P(\nu_{aL} \to \nu_{aL}^c) = c^4 \frac{m_1^2}{E_1^2} \sin^2(E_1 t) + s^4 \frac{m_2^2}{E_2^2} \sin^2(E_2 t) + 2s^2c^2 \cos(2\phi) \frac{m_1m_2}{E_1E_2} \sin(E_1 t) \sin(E_2 t), \quad (4.47)$$

$$P(\nu_{aL} \to \nu_{\beta L}^c) = c^2s^2 \frac{m_1^2}{E_1^2} \sin^2(E_1 t) + s^2c^2 \frac{m_2^2}{E_2^2} \sin^2(E_2 t) - 2s^2c^2 \cos(2\phi) \frac{m_1m_2}{E_1E_2} \sin(E_1 t) \sin(E_2 t). \quad (4.48)$$

The probabilities for the Majorana neutrinos are obtained by the replacements, $\theta_L = \theta_R = \theta$, $\phi_L = -\phi_R = \phi$ in the probabilities for the Dirac neutrinos. This reflects that $\nu_{aL}^c$ plays the role of the right-handed components of the Dirac neutrinos $\nu_{R}$, in the case of the Majorana neutrinos. It has been considered that the Majorana CP phase appears accompanied by the lepton number violation until now. However, we have shown that a new CP phase appears even in the case of two-generation Dirac neutrinos and non-existence of lepton number violation in our previous papers. (This new CP phase is absorbed by the redefinition of the fields in the Standard Model because the right-handed neutrinos have no weak interactions.)

In the case of the Majorana neutrinos, $\nu_{aL}^c$, an alternative of the right-handed neutrino, has weak interactions. As the result, this new CP phase appears in the oscillation probabilities as the Majorana CP phase. Therefore, we can reinterpret that the Majorana CP phase is accompanied by chirality change rather than lepton number violation. Also note that the sign of the momentum changes in $\nu \leftrightarrow \nu^c$ oscillations, namely the probabilities in (4.47) and (4.48) mean $P(\nu_{aL}(p) \to \nu_{aL}^c(-p))$ and $P(\nu_{aL}(p) \to \nu_{\beta L}^c(-p))$, respectively.

Next, let us derive the oscillation probabilities of anti-neutrino $\bar{\nu}_{aL}^c(p)$ with momentum $p$. In eq.(4.19), $\nu_{aL}^c$ has negative momentum $-p$. In order to derive the oscillation probabilities for $\nu_{aL}^c(p)$, we need to change the sign of momentum $p$ in eq.(4.19). Namely, we start
from the equation,
\[
\frac{i}{\hbar} \frac{d}{dt} \begin{pmatrix}
\nu_{\alpha L}^c \\
\nu_{\beta L}^c \\
\nu_{\alpha L} \\
\nu_{\beta L}
\end{pmatrix} = \begin{pmatrix}
p & 0 & M_{\alpha\alpha} & M_{\alpha\beta} \\
0 & p & M_{\beta\alpha} & M_{\beta\beta} \\
M_{\alpha\alpha}^* & M_{\beta\alpha}^* & -p & 0 \\
M_{\alpha\beta}^* & M_{\beta\beta}^* & 0 & -p
\end{pmatrix} \begin{pmatrix}
\nu_{\alpha L}^c \\
\nu_{\beta L}^c \\
\nu_{\alpha L} \\
\nu_{\beta L}
\end{pmatrix}.
\] (4.49)

Furthermore, exchanging the top two rows and the bottom two rows, we obtain
\[
\frac{i}{\hbar} \frac{d}{dt} \begin{pmatrix}
\nu_{\alpha L} \\
\nu_{\beta L} \\
\nu_{\alpha L}^c \\
\nu_{\beta L}^c
\end{pmatrix} = \begin{pmatrix}
-p & 0 & M_{\alpha\alpha}^* & M_{\alpha\beta}^* \\
0 & -p & M_{\beta\alpha}^* & M_{\beta\beta}^* \\
M_{\alpha\alpha} & M_{\beta\alpha} & p & 0 \\
M_{\alpha\beta} & M_{\beta\beta} & 0 & p
\end{pmatrix} \begin{pmatrix}
\nu_{\alpha L} \\
\nu_{\beta L} \\
\nu_{\alpha L}^c \\
\nu_{\beta L}^c
\end{pmatrix},
\] (4.50)

where we used \(M_{\alpha\beta} = M_{\beta\alpha}^*\) due to the symmetry of the mass matrix. Comparing with (4.19), one can see the mass terms in the Hamiltonian have changed to their complex conjugate. Therefore, we can obtain the oscillation probabilities of anti-neutrinos by replacing the mixing matrix, which diagonalize the mass matrix, into its complex conjugate. In two generations, the CP conjugate probabilities are the same as the original probabilities because it depends on the absolute value or real part of the product of \(U_\alpha\)s. The probabilities for the oscillations with chirality-flip depend on the new CP phase only through the form of \(\cos 2\phi\). Thus, we obtain the following relations,
\[
P(\nu_{\alpha L}^c(p) \rightarrow \nu_{\alpha L}^c(p)) = P(\nu_{\alpha L}(p) \rightarrow \nu_{\alpha L}(p)),
\] (4.51)
\[
P(\nu_{\alpha L}^c(p) \rightarrow \nu_{\beta L}^c(p)) = P(\nu_{\alpha L}(p) \rightarrow \nu_{\beta L}(p)),
\] (4.52)
\[
P(\nu_{\alpha L}^c(p) \rightarrow \nu_{\alpha L}(-p)) = P(\nu_{\alpha L}(p) \rightarrow \nu_{\alpha L}^c(-p)),
\] (4.53)
\[
P(\nu_{\alpha L}^c(p) \rightarrow \nu_{\beta L}^c(-p)) = P(\nu_{\alpha L}(p) \rightarrow \nu_{\beta L}(-p)).
\] (4.54)

## 5 Difference between Conventional and New Probabilities

In this section, we review the results of the Majorana neutrinos in previous papers [32–34] and compare these results with ours obtained in this paper. The amplitude from \(\nu_{\alpha L}\) to \(\nu_{\beta L}^c\) and the CP conjugate amplitude from \(\nu_{\beta L}^c\) to \(\nu_{\alpha L}\) were given by
\[
A(\nu_{\alpha L} \rightarrow \nu_{\beta L}^c) = \left[ U_{\alpha 1}^* U_{\beta 1} \frac{m_1}{E_1} e^{-iE_1 t} + U_{\alpha 2}^* U_{\beta 2} \frac{m_2}{E_2} e^{-iE_2 t} \right] K, \tag{5.1}
\]
\[
A(\nu_{\alpha L}^c \rightarrow \nu_{\beta L}) = \left[ U_{\alpha 1} U_{\beta 1} \frac{m_1}{E_1} e^{-iE_1 t} + U_{\alpha 2} U_{\beta 2} \frac{m_2}{E_2} e^{-iE_2 t} \right] \bar{K}, \tag{5.2}
\]
in our notation, where \(K = \bar{K}\) is the kinetic factor and does not depend on energy and time (namely corresponding distance). On the other hand, the amplitudes obtained in this
paper are represented as

\[
A(\nu_{\alpha L} \to \nu^c_{\beta L}) = U^{*}_{\alpha \beta_1} \frac{m_1}{E_1} (e^{-iE_1t} - e^{iE_1t}) + U^{*}_{\alpha \beta_2} \frac{m_2}{E_2} (e^{-iE_2t} - e^{iE_2t})
\]

\[
= -iU^{*}_{\alpha \beta_1} \frac{m_1}{E_1} \sin(E_1t) - iU^{*}_{\alpha \beta_2} \frac{m_2}{E_2} \sin(E_2t),
\]

\[
A(\nu^c_{\alpha L} \to \nu_{\beta L}) = U_{\alpha \beta_1} \frac{m_1}{E_1} (e^{-iE_1t} - e^{iE_1t}) + U_{\alpha \beta_2} \frac{m_2}{E_2} (e^{-iE_2t} - e^{iE_2t})
\]

\[
= -iU_{\alpha \beta_1} \frac{m_1}{E_1} \sin(E_1t) - iU_{\alpha \beta_2} \frac{m_2}{E_2} \sin(E_2t).
\]

(5.3)

(5.4)

The difference between conventional result and our result is in the exponential part. Our result has a contribution coming from both positive and negative energies. When we use the Dirac equation and derive the oscillation probabilities for \(\nu\) and \(\nu^c\) in a unified way, we need to take into account the contributions from both signs of energy.

In previous papers, it has been considered that the unitarity holds in the probabilities for only \(\nu \leftrightarrow \nu\) oscillations. If this is correct, the unitarity does not hold when \(\nu \leftrightarrow \nu^c\) oscillations are taken into account in addition to the probabilities between neutrinos. In order to resolve this paradox, we consider both neutrino and anti-neutrino in a unified way. Actually, the total sum of the probabilities for a neutrino is kept at one by adding some correction terms, namely, unitarity holds.

Next, we compare the oscillation probabilities. The probabilities in previous papers were given by

\[
P(\nu_{\alpha L} \to \nu^c_{\beta L}) = |K|^2 \left| U^{*}_{\alpha \beta_1} \frac{m_1}{E_1} e^{-iE_1t} + U^{*}_{\alpha \beta_2} \frac{m_2}{E_2} e^{-iE_2t} \right|^2
\]

\[
= |K|^2 \left[ \frac{m_1^2}{E_1^2} U^{*}_{\alpha \beta_1} U^{*}_{\beta_1} \right]^2 + \frac{m_2^2}{E_2^2} U^{*}_{\alpha \beta_2} U^{*}_{\beta_2} \left[ \frac{m_1 m_2}{E_1 E_2} \right]^2 Re \left( U^{*}_{\alpha \beta_1} U^{*}_{\beta_1} U_{\alpha \beta_2} U_{\beta_2} e^{-i(E_1 - E_2)t} \right)
\]

\[
= |K|^2 \left[ \frac{m_1^2}{E_1^2} U^{*}_{\alpha \beta_1} U^{*}_{\beta_1} \right]^2 + \frac{m_2^2}{E_2^2} U^{*}_{\alpha \beta_2} U^{*}_{\beta_2} \left[ \frac{2 m_1 m_2}{E_1 E_2} \right]^2 \left( Re \left( U^{*}_{\alpha \beta_1} U^{*}_{\beta_1} U_{\alpha \beta_2} U_{\beta_2} \right) \cos(\xi - \eta) - \Im \left( U^{*}_{\alpha \beta_1} U^{*}_{\beta_1} U_{\alpha \beta_2} U_{\beta_2} \right) \sin(\xi - \eta) \right)
\]

(5.5)

where we parametrize the MNS matrix,

\[
U = \left( \begin{array}{cc}
\cos \theta & e^{i(\rho_1 + \phi)} \\
-e^{i\rho_2} \sin \theta & e^{i(\rho_2 + \phi)} 
\end{array} \right),
\]

(5.6)

and substituting into (5.5), the probability is rewritten as

\[
P(\nu_{\alpha L} \to \nu^c_{\beta L})
\]

\[
= |K|^2 e^{2\xi} e^{2\eta} \left[ \frac{m_1^2}{E_1^2} + \frac{m_2^2}{E_2^2} \right] \left( Re \left( e^{-2i\phi} \cos(\xi - \eta) - \Im \left( e^{-2i\phi} \sin(\xi - \eta) \right) \right) \right)
\]

\[
= |K|^2 e^{2\xi} e^{2\eta} \left[ \frac{m_1^2}{E_1^2} + \frac{m_2^2}{E_2^2} \right] \left( \cos(2\phi) \cos(\xi - \eta) - \sin(2\phi) \sin(\xi - \eta) \right) \right].
\]

(5.7)
In the same way, we also obtain

\[
P(\nu_{\alpha L} \rightarrow \nu_{\beta L}) = |K|^2 c^2 s^2 \left[ \frac{m_1^2}{E_1^2} + \frac{m_2^2}{E_2^2} - \frac{2m_1m_2}{E_1E_2} \Re\left(e^{-2i\phi}\right) \cos(E_1 - E_2)t + \Im\left(e^{-2i\phi}\right) \sin(E_1 - E_2)t \right]
\]

\[
= |K|^2 c^2 s^2 \left[ \frac{m_1^2}{E_1^2} + \frac{m_2^2}{E_2^2} - \frac{2m_1m_2}{E_1E_2} \left\{ \cos(2\phi) \cos(E_1 - E_2)t - \sin(2\phi) \sin(E_1 - E_2)t \right\} \right]. \quad (5.8)
\]

From the probabilities (5.7) and (5.8), the difference in previous papers is given by

\[
P(\nu_{\alpha L} \rightarrow \nu_{\beta L}) - P(\nu_{\alpha L}^c \rightarrow \nu_{\beta L}) = 4|K|^2 m_1 m_2 c^2 s^2 \frac{E_1^2}{E_1E_2} \sin(2\phi) \sin(E_1 - E_2)t. \quad (5.9)
\]

On the other hand, we obtain a different result from the previous works, in this paper. We have the same probabilities

\[
P(\nu_{\alpha L} \rightarrow \nu_{\beta L}^c) = P(\nu_{\alpha L}^c \rightarrow \nu_{\beta L}) = c^2 s^2 \left[ \frac{m_1^2}{E_1^2} \sin^2(E_1t) + \frac{m_2^2}{E_2^2} \sin^2(E_2t) - \frac{2m_1m_2}{E_1E_2} \cos(2\phi) \sin(E_1t) \sin(E_2t) \right], \quad (5.10)
\]

for both \(\nu_{\alpha L} \rightarrow \nu_{\beta L}^c\) and \(\nu_{\alpha L}^c \rightarrow \nu_{\beta L}\) oscillations. So, there is no difference between CP-conjugated probabilities, and

\[
P(\nu_{\alpha L} \rightarrow \nu_{\beta L}^c) - P(\nu_{\alpha L}^c \rightarrow \nu_{\beta L}) = 0 \quad (5.11)
\]

is obtained in vacuum.

Next, let us consider the the limit \(L \rightarrow 0\) \((t \rightarrow 0)\). Taking the limit \(t \rightarrow 0\) in (5.7), the probability for \(\nu_{\alpha L} \rightarrow \nu_{\beta L}^c\) oscillations converges to

\[
P(\nu_{\alpha L} \rightarrow \nu_{\beta L}^c) = |K|^2 c^2 s^2 \left[ \frac{m_1^2}{E_1^2} + \frac{m_2^2}{E_2^2} - \frac{2m_1m_2}{E_1E_2} \cos(2\phi) \right], \quad (5.12)
\]

and has a finite value, namely the zero-distance effect appears. This effect has been noted because it does not exist in \(\nu \leftrightarrow \nu\) oscillations, and unique to \(\nu \leftrightarrow \nu^c\) oscillations. However, we have shown that there appears no zero-distance effect even in \(\nu \leftrightarrow \nu^c\) oscillations as same as \(\nu \leftrightarrow \nu\) oscillations by taking the limit \(t \rightarrow 0\) in eq.(5.10).

6 Summary

We derived the oscillation probabilities of neutrinos and anti-neutrinos in a unified way by using the relativistic method in the case of two-generation Majorana neutrinos. In this case, \(\nu_L^c\) plays the role of the \(\nu_R\) in the Dirac neutrinos and the new CP phase appeared in the Dirac neutrino oscillations \([35]\) becomes the Majorana CP phase. Thus, we have found that the Majorana CP phase appears with the chirality-flip, not with the lepton number violation. We have also confirmed that the unitarity holds when both neutrinos and anti-neutrinos are considered.
In our calculation, there is no direct CP violation dependent on the sine term of the Majorana CP phase even in neutrino and anti-neutrino oscillations with different flavors. The difference between CP-conjugated probabilities vanishes and the probabilities depend only through the cosine term of the Majorana CP phase. Furthermore, we have shown that there is no zero-distance effect, in which a neutrino changes to an anti-neutrino in a moment. These results are different from the previous ones [32–34].

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