Reconciling different formulations of viscous water waves and their mass conservation

D. Eeltink, A. Armaroli, M. Brunetti, J. Kasparian

Group of Applied Physics and Institute for Environmental Sciences, University of Geneva, Switzerland

Abstract

The viscosity of water induces a vorticity near the free surface boundary, giving a rotational component to the fluid velocity. Our analysis reveals the correspondence between three common sets of model equations that deal with this vorticity in different ways. The first set has a rotational kinematic boundary condition at the surface. In the second set, a gauge choice for the velocity vector is made that cancels the rotational contribution in the kinematic boundary condition, at the cost of rotational velocity in the bulk and a rotational pressure. The third set circumvents the problem by explicitly splitting the problem into two domains: the irrotational bulk and the vortical boundary layer. The correspondence between the three systems in terms of mass conservation disproves the speculation that mass is not conserved in a rotational model, and exemplifies the correspondence between rotational pressure on the surface and vorticity in the boundary layer.

Keywords: Vorticity, Viscosity, Gravity surface waves, Mass conservation

1. Introduction

For many water wave problems, the Navier-Stokes (NS) equation can be simplified by considering the fluid as inviscid and incompressible. The continuity equation combined with the boundary conditions for the free surface and the rigid bottom forms the Euler system of equations for the surface elevation $\eta$, and the velocity potential $\phi$. This system can be used as the...
starting point to obtain Nonlinear Schrödinger (NLS) equation-like propagation models, or can serve as a basis for higher order spectral methods [1, 2, 3].

While this approach is sufficient in many situations, in reality, water is a viscous medium. The viscosity accounts for the obvious fact of wave damping but also plays a more intricate role in for instance downshifting of the spectrum [4], or stabilizing the Benjamin-Feir instability [5]. In domains such as the dissipation of swells [6], or visco-elastic waves propagating in ice, inclusion of viscosity in a set of equations is important [7].

One option, hereafter denoted as System A, used in Ruvinsky et al. [8], Dias et al. [9] (hereafter R&F and DDZ respectively), includes the rotational part of the velocity vector $\vec{U}$ into the kinematic boundary condition (KBC) at the surface. The model equations can be simplified by neglecting small terms, based on the premise that the vorticity is confined to a narrow region.

The second option, denoted System B (Dommermuth [10]), makes a gauge choice for the velocity vector such that the rotational contribution in the KBC disappears. The cost is that this introduces a rotational part of the velocity in the bulk. Moreover, the pressure is split into rotational and irrotational part too, introducing an additional equation for the rotational pressure that couples to the Navier-Stokes equation. This system is however fully nonlinear and makes no boundary layer approximations.

The third option, denoted System C (Longuet-Higgins [11, 12]), circumvents the difficulty of the rotational kinematic boundary condition by explicitly splitting the problem into two domains: the irrotational bulk and the vortical boundary layer. The former is shown to receive an additional pressure due to the weight of the latter.

Naturally, without approximations, the models describe exactly same physics, and have the same solutions. However, in order to close systems A and C, linearization is applied in certain steps. This approximation can be the source of discrepancies between the models.

Furthermore, due to the addition of a rotational part to the kinematic boundary condition (KBC) in comparison to the inviscid case, the mass conservation of the DDZ model has been challenged. We show that the conservation of mass is not compromised for a rotational system, provided the full nonlinear KBC is used.

The paper is organized as follows: in Section 2 we recall how the kinematic and dynamic boundary conditions are obtained for the water wave problem
and how viscosity affects them. In Section 3 we define the three systems of equations and we show how they correspond. In Section 4, we address the mass conservation issue. In Section 5 we interpret the results.

2. Boundary Conditions

The physics of the water wave problem is defined by the boundary conditions. Figure 1 depicts the 2D water wave problem and its relevant quantities and length scales. The surface elevation $\eta(x, z)$ is denoted by the black line, the arrows are the local velocity vectors and the color scale refers to the value of the velocity potential $\phi(x, z, t)$, based on a linear wave [13].

2.1. Shear stress: viscosity and vorticity

For non-viscous waves the velocity vector $\vec{u}$ is irrotational and can be written as the gradient of a potential field:

$$\vec{u} = \nabla \phi$$

(2.1)

However, in the presence of viscosity, the continuity of tangential stresses at the free surface can only be fulfilled by rotational motion of the fluid [14] [15].
The stress tensor in 2D tangential and normal components can be written as:

\[
\sigma = \begin{bmatrix}
2\mu u^s_s - P & \mu(u^n_s + u^n_n) \\
\mu(u^n_s + u^n_n) & 2\mu u^n_n - P
\end{bmatrix}
\] (2.2)

where \(\mu\) is the dynamic viscosity, and \(P\) the pressure. Here and in the following, subscripts denote the partial derivatives and superscripts denote the components of a vector, where \(s\) denotes tangential and \(n\) normal. The tangential stress component is

\[
\sigma^s = \tau^s = \mu(u^s_n + u^n_s)
\] (2.3)

where \(\tau\) is the deviatoric stress tensor.

Since \(\mu\) in air is much smaller than in water, the shear stress must vanish at the surface, implying \(u^s_n = -u^n_s\). There is thus no relative distortion to the fluid-particle, which due to the curvature of the interface, results in a rotational flow \[14\]. See Fig. 2 for an illustration of a rotational and an irrotational flow.

\[\text{Figure 2: In an irrotational flow there can be circular paths for the fluid, but each individual fluid particle does not rotate.}\]

For a rotational flow, the Helmholtz decomposition is used to split the velocity field into an irrotational part \(\nabla \phi\), and a rotational, solenoidal (\(\nabla \cdot \vec{U} = 0\)) part, \(\vec{U}\)

\[
\vec{u} = \nabla \phi + \vec{U} = \nabla \phi + \nabla \times \vec{A}
\] (2.4)

Using the Helmholtz decomposition requires finding a harmonic function \(\phi\) that satisfies \(\nabla^2 \phi = 0\) and a solenoidal field \(\vec{U}\) that satisfies the NS equations \[16\]. Therefore, certain transfers of irrotational flow from the \(\nabla \phi\) term to the \(\vec{U}\) are allowed, keeping Eq. (2.4) valid. That is, while \(\nabla \phi\) is irrotational (since the curl of a gradient is always 0), \(\vec{U}\) can include an irrotational part on top of its rotational part \[10\].

This implies that one cannot assume a-priori that \(\nabla \phi\) contains the full irrotational velocity potential, so that its value differs for different gauge choices.
2.2. Kinematic boundary condition

A key ingredient for the rest of our discussion is the KBC, which we therefore derive explicitly. The KBC ensures that fluid particles on the free surface always remain there. We can describe the surface elevation by \( z = \eta(x,t) \), and let \( f \) define the interface between air and water

\[
f(x, z, t) \equiv z - \eta(x, t) \tag{2.5}
\]

Because \( f = 0 \) on the interface at all times \( t \), its material derivative, \( D/Dt \), must be 0

\[
\frac{Df}{Dt} \equiv \frac{\partial f}{\partial t} + \vec{u} \cdot \nabla f = 0 \quad \text{on} \quad f = 0 \tag{2.6}
\]

Inserting Eq. (2.5) into Eq. (2.6) directly yields the KBC in terms of the velocity vector \( \vec{u} = (u, w) \) in Cartesian coordinates:

\[
\frac{\partial \eta}{\partial t} = -u \frac{\partial \eta}{\partial x} + w \tag{2.7}
\]

By introducing the unit normal vector \( \hat{n} = \nabla f/|\nabla f| = (-\eta_x, 1)/\sqrt{\eta_x^2 + 1} \), pointing outwards, we can write Eq. (2.6) as

\[
\frac{\partial \eta(x, t)}{\partial t} = \vec{u} \cdot \hat{n} |\nabla f| = u^n \sqrt{1 + \eta_x^2} \tag{2.8}
\]
Or, using $\eta(s, t)$

$$\frac{\partial \eta(s, t)}{\partial t} = u^n \tag{2.9}$$

The latter expression corresponds to the intuitive image that the deformation of the surface, i.e. the change of $\eta$ in time, is equal to the normal component of the velocity vector $u^n$ pushing the surface either inwards or outwards.

### 2.3. Dynamic boundary condition

In addition to the continuity of shear stress, the normal stress

$$\sigma^n = \tau^n - p^n = 2\mu u^n - P \tag{2.10}$$

must also be continuous over the boundary between water (w) and air (a) $\sigma^{n,w} = \sigma^{n,a}$, yielding the dynamic boundary condition (DBC). For an irrotational flow this reduces to the continuity of pressure. Due to comparatively low viscosity and density, the air-side can be neglected such that $\sigma^{n,w} = 0$. The pressure $P$, having a dynamic and a static component, is obtained from the NS equation. For an irrotational fluid this reduces to the simple Bernoulli equation, but for a rotational fluid this is more complicated. The DBC is discussed in detail in Appendix A.

### 3. Three complementary views

The three systems introduced in Section 1 have different domain-divisions and different expressions for the boundary conditions, but describe the same physics. Below we briefly outline their ideas, summarized in Figure 3.

In System A (Figure 3A), the velocity has a rotational part: Eq. (2.4), and therefore a rotational KBC. The vortical component of the velocity, indicated by the grey shaded area, is nonzero at the boundary and rapidly decays to zero, over a typical length $\delta$: the width of the vortical boundary layer. There are two possible ways to close this system. The first, described in R&F [8], is to impose a separate boundary condition for the rotational part of the velocity vector $\vec{U}$, based on the vorticity equation. The second, introduced in DDZ [9], is to write $\vec{U}$ in potential flow terms, using expressions obtained from the linearized equations.

In System B (Figure 3B), as presented in Dommermuth [10], an additional boundary condition $U^n = 0$ is imposed. This is in effect a gauge choice in the Helmholtz decomposition, and leads to an irrotational velocity at the
surface, Eq. (2.1). The price for the irrotational KBC is that the normal stress continuity boundary condition now pertains to the sum of the irrotational and the rotational parts of the pressure: \( P = P_i + P_r \). Moreover, the vortical component of the velocity (the grey shaded area) is nonzero in the bulk of the fluid.

In System C (Figure 3C), derived by Longuet-Higgins [11, 12], the problem is split into two domains: an irrotational bulk (\( \Gamma_I \)), and a rotational boundary layer (\( \Gamma_{II} \)). The equations are solved for \( \Gamma_I \). Since there is no interface with air for \( \Gamma_I \), the shear stress on the boundary does not have to vanish, and the viscous fluid in this domain can therefore be irrotational. The weight of \( \Gamma_{II} \) induces a pressure \( P_\delta \) on the top boundary of \( \Gamma_I \). An expression for \( P_\delta \) is derived in terms of the mass-flux of \( \Gamma_{II} \), caused by the rotational part of the velocity \( \vec{U} \).

For all three systems, the continuity equation in the bulk \( \nabla \cdot \vec{u} = 0 \) yields the Laplace equation for the velocity potential, for both rotational (Eq. (2.4)) and irrotational (Eq. (2.1)) flows

\[
\nabla^2 \phi = 0
\] (3.1)

They also share the same bottom boundary condition in the deep water limit: \( \phi_z \rightarrow 0 \) as \( z \rightarrow -\infty \).

Another common denominator is that the Navier-Stokes equation for incompressible newtonian fluids is used for the derivation of the dynamic boundary condition

\[
\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} = -\frac{1}{\rho} \nabla P + \vec{g} + \nu \nabla^2 \vec{u}
\] (3.2)

We shall now discuss each system in detail. We shall first provide the remaining equations for each system, and then discuss each equation. All results are also summarized in Table 1.
3.1. **System A**

The model equations at the free surface boundary, \( z = \eta \) can be written as

\[
\begin{align*}
\eta_t + \eta_x \phi_x + \eta_x U &= \phi_x + W & \text{KBC} \quad (3.3a) \\
\sigma_n &= 2\mu \phi_{nn} - P = 0 & \perp\text{-stress} \quad (3.3b) \\
\phi_t + \frac{1}{2}(\nabla \phi)^2 - g\eta + \phi_x U &= -\frac{P}{\rho} & \text{NS} \quad (3.3c) \\
\quad \rightarrow \phi_t + \frac{1}{2}(\nabla \phi)^2 - g\eta &= -2\nu \phi_{nn} - \phi_x U & \text{DBC} \quad (3.3d)
\end{align*}
\]

where \( \nu = \mu / \rho \) is the kinematic viscosity and \( \text{NS} \) refers to the Navier-Stokes equation, evaluated at the surface.

- In the KBC, the fluid is considered rotational (see Fig. 3A). Therefore the Helmholtz notation is used, Eq. (2.7), where \( \vec{u} = (u, w) \) can explicitly be written as

\[
\begin{align*}
\quad u &= \phi_x + U = \phi_x - A_z, \quad u^s = \phi_x + U^s \quad (3.4) \\
\quad w &= \phi_z + W = \phi_z + A_x, \quad u^n = \phi_n + U^n \quad (3.5)
\end{align*}
\]

giving Eq. (3.3a). The curl of \( A \) only has a component in the \( y \) direction, and can therefore be treated as a scalar. Like Eq. (2.9), this can also be written as

\[
\eta(s, t)_t = \phi_n + U^n \quad (3.6)
\]

- The normal stress boundary condition (Eq. (3.3b)) is obtained by using the Helmholtz notation in Eq. (2.10), and neglecting the term \( U_n \).

- Equally, in the Navier-Stokes equation (Eq. (3.2)), the Helmholtz notation is used, see Appendix A, yielding

\[
\begin{align*}
\phi_t + \frac{1}{2}(\nabla \phi)^2 - g\eta + \frac{P}{\rho} &= \\
- \int_{\eta-\delta}^{\eta} \hat{n} \cdot \left( \vec{U}_t + ((\nabla \phi + \vec{U}) \cdot \nabla) \vec{U} + (\vec{U} \cdot \nabla) \nabla \phi - \nu \nabla^2 \vec{U} \right) dn
\end{align*}
\]

Vorticity terms in the boundary layer

\[^1\] The vortical part decreases rapidly over a distance \( \delta = \sqrt{\frac{2\nu}{\omega}} \), as demonstrated in Lamb [13] section 348, where the stream function \( \Psi \) corresponds to our \( A \) in Eq. (3.4).
Performing an order analysis, R&F show that the integral terms on the RHS are of $O(\epsilon \delta k)$ or higher, and are therefore neglected. Rather, we retain also $O(\epsilon \delta k)$, which is only the term $\phi_x U$ (See Appendix A), and neglect higher order terms, yielding Eq. (3.3c).

- The DBC (Eq. (3.3d)) is obtained combining the normal stress boundary condition (Eq. (3.3b)), with the Navier-Stokes equation at the boundary (Eq. (3.3c)).

System A differs from R&F and DDZ, as the latter two neglect the underlined and the dashed-underlined terms. These nonlinear vorticity terms correspond to the horizontal component of the normal velocity vector. The relevance of these terms will be discussed in detail in Section 3.5 and Appendix B.

Compared to the Euler equations, the rotational part of the velocity $\vec{U}$ is a third unknown in addition to $\phi$ and $\eta$. To close the system two routes can be taken. The first, described in R&F, is to use the tangential stress balance (Eq. (2.3)) and the vorticity equation to obtain an equation for the vorticity at the boundary:

$$U^n_t = 2\nu \phi_{ns}$$  \hspace{1cm} (3.8)

The second, as described in DDZ, is to express the vortical components in terms of $\phi$ or $\eta$. These expressions can be found based on the linear Euler system, see Appendix B. Subsequently, it can be conjectured that these expressions also hold in the nonlinear system [9].

3.2. System B

The boundary equations at $z = \eta$ for system B can be written as Dommernuth [10]:

$$\eta_t + \eta_x \phi_x = \phi_z \hspace{1cm} \text{KBC1} \hspace{1cm} (3.9a)$$

$$\vec{U}_n = 0 \hspace{1cm} \text{KBC2} \hspace{1cm} (3.9b)$$

$$\sigma^n \equiv 2\mu \frac{1}{\eta_x + 1} (\phi_{zz} + W_z) + NL$$

$$- (P_{rot} + P_{irr}) = 0 \hspace{1cm} \perp\text{-stress} \hspace{1cm} (3.9c)$$

$$\phi_t + \frac{1}{2}(\nabla \phi)^2 - g\eta = - \frac{P_{irr}}{\rho} \hspace{1cm} \text{NS}, P_{irr} \hspace{1cm} (3.9d)$$

Where NL indicates nonlinear terms.
For the KBC, the additional boundary condition $U^n = 0$ is imposed at the surface (Eq. (3.9b)), rendering the KBC (Eq. (3.9a)) irrotational. This is achieved at the cost of $\bar{U}$ being non-zero in the bulk, as indicated in Figure 3B. Consequently, $\bar{U}$ has to include part of the irrotational field.

In order to obtain an additional equation to accommodate for the extra boundary condition, the pressure is also split in a rotational and irrotational part in the balance of normal stresses (Eq. 3.9c):

$$P = P_{\text{irr}} + P_{\text{rot}}$$  \hspace{1cm} (3.10)

The irrotational Navier-Stokes or Bernoulli equation, define the irrotational pressure. The remaining part is denoted $P_{\text{rot}}$.

Like in System A, the DBC is obtained by inserting the equation for the pressure (Eq. (3.9d)) into the normal stress balance (Eq. (3.9c)). However, now, $P_{\text{rot}}$ remains unknown.

To obtain an equation for $P_{\text{rot}}$, the decomposition for the velocity (Eq. (2.4)), and for the pressure (Eq. (3.10)) can be inserted into the viscous and rotational Navier-Stokes equation in the bulk, resulting in

$$\nabla P_{\text{rot}} = U_t - (\bar{U} + \nabla \phi) \cdot \nabla \bar{U}$$

$$\quad - (\bar{U} \cdot \nabla \phi) \nabla \phi + \nu \nabla^2 \bar{U}$$ \hspace{1cm} (3.11)

Taking the scalar product of the Navier-Stokes equation with the normal vector to the free surface, one can obtain a boundary condition for $\partial P_{\text{rot}}/\partial n$.

Dommermuth’s system retains all nonlinear terms without any approximations. This system was derived for studying the evolution of vortical cylinders moving from the bottom towards the water surface, where the vorticity is indeed not just limited to the boundary. The Helmholtz decomposition allows for a seamless transition between regimes with different Reynolds numbers.

### 3.3. System C

Longuet-Higgins [12, 11] formally splits the problem into two domains: the irrotational bulk $\Gamma_I$, and the vortical boundary layer $\Gamma_{II}$, see Fig. 3C.
The equations for $\Gamma_I$ at its upper boundary $z = \eta^*$ are given by:

\[
\begin{aligned}
\Gamma_{I,z} \equiv \eta^* & \left\{ \begin{array}{l}
\eta_t^* + \eta_x^* \phi_x = \phi_z & \text{KBC} \\
\sigma_n \equiv 2 \mu \phi_{zz} - P + P_\delta = 0 & \text{\perp-stress} \\
P_\delta / \rho = g \eta' = 2 \nu \phi_{zz} & \text{Added } P_\delta \\
\phi_t + \frac{1}{2} (\nabla \phi)^2 - g \eta^* = -\frac{P}{\rho} & \text{NS} \\
\rightarrow \phi_t + \frac{1}{2} (\nabla \phi)^2 + g \eta^* = -4 \nu \phi_{zz} & \text{DBC}
\end{array} \right.
\end{aligned}
\]  

(3.12a) (3.12b) (3.12c) (3.12d) (3.12e)

- In this configuration, the shear stress for the top boundary of $\Gamma_I$, $\eta^*$, does not have to vanish, as it does not have an interface with air. Therefore, the viscous fluid, and the KBC (Eq. (3.12a)), can be treated as irrotational. This is in contrast to models like Refs. [16, 17], which have the unphysical situation of an irrotational fluid with the free surface as top boundary, which requires the vanishing of shear stresses at the top boundary.

- The normal stress balance (Eq. (3.12b)) receives an additional pressure $P_\delta$ due to the weight of the boundary layer above.

- The Navier-Stokes equation (Eq. (3.12d)) is now evaluated at the top of $\Gamma_I$.

- Again, combining the normal stress balance and the Navier-Stokes gives the dynamic boundary condition: Eq. (3.12e).

The domain of the boundary layer $\Gamma_{II}$ is considered not to have a constant thickness $\delta$, but a variable height $\eta'(x,t)$. We can write for $\eta$:

\[
\eta = \eta^* + \eta'
\]

(3.13)

Briefly, obtaining a function for $\eta'$ hinges on three critical observations:

1. The boundary layer is defined as a fluid region where there is a mass-flux, i.e. fluid flowing through the boundary, due to vorticity. This mass-flux can be written as:

\[
M = \int_{\eta^*}^{\eta'} \rho U^* dn
\]

(3.14)
2. The layer thickness is not constant in time, and a KBC can be written:

\[ \frac{\partial \eta'}{\partial t} = U^n = \int U_n^s dn = - \int U_s^s dn \]  

where the last step is made using the fact that the divergence of \( \vec{U} \) is null. Note that this is the change of the thickness of the boundary layer in time. This is different from the motion of the upper boundary \( \eta \), which would depend on the total normal velocity \( u^n = U^n + \phi_n \).

Combining Eq. (3.14) and Eq. (3.15) gives

\[ \frac{\partial \eta'}{\partial t} = - \frac{1}{\rho} \frac{\partial M}{\partial s} \approx - \frac{1}{\rho c} \frac{\partial M}{\partial t} \]  

where in the linear limit \( \partial/\partial x \sim (1/c) \partial/\partial t \), with \( c = \omega/k \) the phase speed, and \( \omega \) the orbital frequency. This shows the intuitive relation that the difference between the mass-flux from one boundary at \( s \) and the other at \( s + ds \), \( \frac{1}{\rho} \frac{\partial M}{\partial s} \), determines the fluid inflow into a slice, and must be equal to the change in height of the boundary layer, as displayed in Figure 4. This point will return in Section 4.

3. The total tangential stress \( \tau_{s,\text{tot}} \) on the boundaries of the layer is equal to the mass transport

\[ M_t = \tau_{s,\text{tot}} \]  

Therefore, Eq. (3.16) can be written as

\[ \frac{\partial \eta'}{\partial t} = - \frac{\tau_{s,\text{tot}}}{\rho c} \]  

Assuming that the surface stress is periodic \( \tau \propto e^{i(kx-\omega t)} \), the solution is

\[ \eta' = - \frac{i \tau_{s,\text{tot}}}{\rho c \omega} \]  

showing that \( \eta' \) leads \( \tau \) by 90°. Since the shear stress at the surface must vanish, it is only the shear stress induced at the bottom of the vortical layer due to the viscous fluid motion of the irrotational bulk that contributes to \( \tau_{s,\text{tot}} \)

\[ \tau_{s,\text{tot}} = \tau_s = \mu (u_n^s + u_s^n) = 2\mu u_n^n \approx 2\mu \eta_{st} \approx 2\mu \omega k \eta \]  

where the last two steps are made assuming that \( \eta \) is linear. Now we can write

\[ \eta'(x,t) = - \frac{2i\mu k^2}{\rho \omega} \eta(x,t) = 2\nu \frac{k}{\omega^2} \phi_{zz}(x,t) \]
This layer produces an additional normal stress on its bottom boundary (denoted \( \eta^* \)), simply due to its own weight: 

\[
P_\delta = \rho g \eta^* = 2 \mu \phi_{zz}.
\]

Finally, this method does not rely on the size of \( \delta \), or the relation between \( \delta \) and \( \epsilon \). Therefore, it is valid also for \( \frac{\delta}{\eta} \ll 1 \), unlike for instance the Stokes expansion \[18\]. However, the expression for the pressure is made in the linear approximation, and consequently does imply \( \epsilon \ll 1 \)

It is interesting to note that the KBC (Eq. (3.12a)) and DBC (Eq. (3.12e)) of System C are the same as those used by Wu et al. \[19\], namely:

\[
\begin{align*}
  z = \eta & \quad \text{KBC} \\
  \phi_t + \frac{1}{2} (\nabla \phi)^2 + g \eta = -4 \nu \phi_{zz} & \quad \text{DBC}
\end{align*}
\]

This system is also suggested in the last sentence of the appendix in R&F as a simpler alternative for their system, without further explanation. Longuet-Higgins was able to give a physical underpinning for the irrotational KBC and the added factor 2 to the viscosity term \(-2 \nu \phi_{zz}\) in the DBC. However, while Eqs. (3.22) refer to the surface elevation \( \eta \) at the boundary between air and water, Eqs. (3.12) refer to \( \eta^* \), between the boundary layer and the bulk, as shown in Fig. 3C. Nevertheless, since the absolute amplitude of the boundary is irrelevant in a deep water limit, the boundary condition follows the same motion as \( \eta \), apart from the aforementioned phase-lag.

### 3.4. Correspondence between the three systems

The terms labeled ‘vorticity terms in the boundary layer’ in the integral in Eq. (3.7) in System A, are equal to \( \nabla P_{\text{rot}} \) in Eq. (3.11) in System B. This equivalence indicates that these ‘vorticity terms’ (which also contain mixed terms with \( \phi \)) in the boundary layer can be interpreted as the effect of the rotational pressure. In System A, the integral is only over the boundary layer. The net effect of the terms is very small, and as discussed, these terms can be neglected. The contribution of the rotational pressure is already taken into account by the vortical term in the KBC (Eq. (3.3a)). In System B, this contribution is significant, as the equation for the rotational pressure spans the entire vertical domain \( z \in (-\infty, \eta) \). In addition, the vortical component of the velocity in the two Systems A and B are not the same, \( \vec{U}_B \neq \vec{U}_A \), because of the different gauge choice.

Finding an additional pressure due to the vortical layer is exactly what is done in System C. Instead of using the fully nonlinear equation for the vorticity as in System B Eq. (3.11), Longuet-Higgins relies on the physical
\[
\begin{align*}
\text{System A} & \quad \nabla^2 \phi = 0 \\
\text{System B} & \quad \nabla^2 \phi = 0 \\
\text{System C} & \quad \nabla^2 \phi = 0
\end{align*}
\]

| Condition          | System A | System B |
|--------------------|----------|----------|
| \(z < \text{upper boundary}\) | \(\nabla^2 \phi = 0\) | \(\nabla^2 \phi = 0\) |
| \(z = \text{lower boundary}\) | \(\phi_z \to 0\) | \(\phi_z \to 0\) |
| KBC                | \(\eta_t + \eta_x \phi_x + \eta_x U = \phi_z + W\) | \(\eta_t + \eta_x \phi_x = \phi_z\) |
| KBC2               | \(U^n = 0\) | 
| \(z = \text{upper boundary}\) | 
| \(\nabla \cdot \sigma = 2\mu \phi_{nn} - P = 0\) | \(\sigma^n \equiv 2\mu \frac{1}{n+1} (\phi_{zz} + W_z) + N L - (P_{\text{rot}} + P_{\text{irr}}) = 0\) |
| NS                 | \(P = -\rho \left( \phi_t + \frac{1}{2}(\nabla \phi)^2 - g \eta + \phi_x U \right)\) | \(P_{\text{irr}} = -\rho \left( \phi_t + \frac{1}{2}(\nabla \phi)^2 - g \eta \right)\) |
| \(\to \) DBC       | \(\phi_t + \frac{1}{2}(\nabla \phi)^2 - g \eta = -2\nu \phi_{nn} - \phi_x U\) | 
| \(z < \text{upper boundary}\) | \(\nabla \cdot (\mathbf{U} - ((\mathbf{\bar{U}} + \nabla \phi) \cdot \nabla)\mathbf{U}) - (\mathbf{\bar{U}} \cdot \nabla \phi) \nabla \phi + \nu \nabla^2 \mathbf{U}\) |

Table 1: Summary of equations for systems A,B and C. Note that System A can be closed in two ways, as described in 3.1. 1) As in R&F: by ignoring the vortical term \(\eta_x U\) and using the approximation \(n \approx z\) and \(W_t = 2\nu \phi_{zz} |_{z=\eta}\). 2) As in DDZ: by ignoring \(\eta_x U\), and deriving \(W = 2\nu \phi_{xx} |_{z=\eta}\) from the linearized equations. Or, as we did in Appendix B using the linearized equations to derive \(U = -2\sqrt{2\nu} \phi_{zz} |_{z=\eta}\).
argument of mass influx, using only linear equations to obtain the additional pressure.

To demonstrate the equivalence between Systems A and C, Longuet-Higgins [12] shows that one can rewrite the linearized System A into the linearized system C, by using $\eta = \eta^* + \eta'$. This demonstrates that the vortical terms in the free surface boundary conditions of System A can indeed be interpreted as an additional pressure $P_\delta$. Appendix D repeats the effort of rewriting Systems A and B in terms of System C. The nonlinear versions of Systems A and C correspond if terms of order $O(\epsilon^2)$ and $O(\epsilon\delta k)$ can be neglected. This shows that Systems B and C correspond if $P_{rot} = P_\delta$, again illustrating the fact that vorticity and added pressure $P_\delta$ play the same role.

3.5. Nonlinear vortical terms

Comparing to System A (Eqs (3.3)), the viscous Euler-like models presented in R&F and DDZ are approximations, as they ignore the nonlinear vortical terms: $\eta_x U$ in the KBC, and the integral in Eq. (3.7), i.e. terms $O(\epsilon\delta k)$. This is justified on accounts that both the steepness, $\epsilon$, and the thickness of the boundary layer, $\delta k$, are very small quantities and thus their product leads to a negligible contribution [8].

However, a simple order analysis in Appendix C shows that the nonlinear vortical terms $\eta_x U$ in the KBC and $\phi_x U$ in the DBC are larger than the linear vortical terms $W$ or $2\nu \phi_{zz}$ when $\frac{\epsilon}{\delta k} = \frac{a}{\delta} \gg 1$, which holds in most physical cases (see Figure C.5). This fact is also remarked in [20]. In Figure I the case $\frac{a}{\delta} > 1$ is indicated by the dashed line, and the case $\frac{a}{\delta} < 1$ by the dotted line. Yet, for a damped modified nonlinear Schrödinger (MNLS) equation, obtained through the method of multiple scales, we have checked that the nonlinear vortical terms cancel out each other up to the fourth order in steepness (see Appendix B). Thus the approximate models of R&F and DDZ are good approximations valid up to $O(\epsilon^4)$, the order of the MNLS equation in the method of multiple scales.

4. Conservation of mass

Since the three systems deal differently with the distinction between the irrotational bulk and the rotational boundary layer, a verification of the conservation of mass for the entire domain is in order. Since we consider the fluid to be incompressible this of course reduces to the conservation of
volume, and since we look at the 2d p To do so, we examine the continuity equation \((\nabla \cdot \vec{u} = 0)\) on the domain \(\Gamma\) in Fig. 4b, that is, integrated over a depth \(z \in (-\infty, \eta)\) and a width \(\Delta x\). The difference between the flux through side boundaries \(\partial \Gamma_L\) and \(\partial \Gamma_R\) denotes a total increase or decrease of fluid volume (area in 2D) in the domain, similar to what is shown for a slice \(dx\) in Fig. 4a. In an incompressible fluid, this must be accommodated by an upward or downward movement of the free top boundary \(\eta\).

That is, the mass-flux through the side boundaries is compensated by the movement of the surface elevation \(\eta\). Firstly, the continuity equation is integrated over the column \(z \in (-\infty, \eta)\)

\[
\int_{-\infty}^{\eta(x)} (u_x + w_z)dz = 0 \tag{4.1}
\]

Integrating and using the Leibniz rule, we get

\[
\frac{d}{dx} \left( \int_{-\infty}^{\eta(x)} udz \right) - u(\eta) \frac{\partial \eta}{\partial x} + w(\eta) = 0 \tag{4.2}
\]

Here, \(\int_{-\infty}^{\eta(x)} udz\) denotes the flux through a vertical boundary at a given position \(x\), see Figure 4a. The sign of its derivative \(d/dx\) from one position to the next shows volume coming into, or leaving the slice. The speed at which this increase in volume occurs is equal to the speed of the boundary moving up or down to accommodate the change: \(u^n\). Here \(\tilde{u}^n\) refers to \(u^n\) in Cartesian coordinates: \(\tilde{u}^n = \sqrt{\eta_x^2 + 1}u^n\). Using the KBC (Eq. (2.7)), we can write

\[
\frac{d}{dx} \left( \int_{-\infty}^{\eta(x)} udz \right) + \eta_t = 0 \tag{4.3}
\]
To formally complete, we integrate Eq. (4.3) over a width \( \Delta x \), \( x \in (x_0, x_0 + \Delta x) \) and obtain the condition for mass conservation of the system in the velocity vector notation:

\[
\int_{x_0}^{x_0+\Delta x} \frac{d}{dx} \left( \int_{-\infty}^{\eta(x)} u dz \right) dx + \int_{x_0}^{x_0+\Delta x} \eta dx = 0 \quad (4.4)
\]

This notation in terms of the velocity vector is a level of description higher than the one used in the three systems described above, i.e. before the introduction of the potential framework or the gauge choice of \( \vec{U} \).

4.1. Comparison to the irrotational system

Firstly, we would like to address the point of using the irrotational fluid as a benchmark to compare the conservation of mass against. The potential notation \( \vec{u} = \nabla \phi_{irr} \) can be used in Eq. (4.4), to obtain:

\[
\int_{x_0}^{x_0+\Delta x} \frac{d}{dx} \left( \int_{-\infty}^{\eta(x)} \phi_{irr,x} dz \right) dx + \int_{x_0}^{x_0+\Delta x} \eta dx = 0 \quad (4.5)
\]

Now we consider the system of DDZ, where the KBC reads

\[
\eta_t + \eta_x \phi_x = \phi_z + 2\nu \eta_{xx} \quad (4.6)
\]

which results in

\[
\int_{x_0}^{x_0+\Delta x} \frac{d}{dx} \left( \int_{-\infty}^{\eta(x)} (\phi_{Dias,x}) dz \right) dx
- \int_{x_0}^{x_0+\Delta x} -2\nu \eta_{xx} \bigg|_{z=\eta} dx + \int_{x_0}^{x_0+\Delta x} \eta dx = 0 \quad (4.7)
\]

In comparison to Eq. (4.5), the additional term \( 2\nu \eta_{xx} \) could induce the notion that mass conservation is broken.\(^2\)

\(^2\)In looking at the integrability of the DDZ system, [21] points out that when the system is assumed periodic, this term disappears. While valid for their system of periodic functions and their following development, in general, conservation of mass should hold on any domain, not just periodic ones.
However, as pointed out in Sec. 2.1, it is unphysical to have a viscous irrotational fluid with a curved free-surface boundary, as the tangential stress at the surface cannot vanish. A viscous fluid cannot be irrotational at the boundary. Therefore, comparing to the conservation of mass condition for the irrotational case is an invalid benchmark.

In the following we compare the conservation of mass corresponding to the different gauge choices used in systems A, B and C.

4.2. Mass conservation in the three systems

We demonstrate that for the different gauge choices in the potential notation, a balance is struck by, on the one hand, the KBC determining the movement of the surface $\eta$ and, on the other hand, $\nabla\phi$, determining the mass-flux. Since the value of $\nabla\phi$ is determined by the DBC, the KBC cannot be considered without regarding the DBC to address the issue of mass conservation. To illustrate the link between the two boundary conditions, in the following, we write the condition for mass conservation such that $\nabla\phi$ is the sole contributor to the mass-flux on the side boundaries, and any vorticity terms are expressed on the moving upper boundary.

4.2.1. System A

The Helmholtz decomposition is used for $u$ in Eq. (4.3).

$$
\frac{d}{dx} \left( \int_{-\infty}^{\eta(x)} (\phi_{A,x} + U) dz \right) + \eta_t = 0 \tag{4.8}
$$

Rewriting $\int U dz$ using the Leibniz rule and the continuity equation ($U_x = -W_z$), gives

$$
\int_{x_0}^{x_0+\Delta x} \frac{d}{dx} \left( \int_{-\infty}^{\eta(x)} (\phi_{A,x}) dz \right) dx \quad \text{Partial flux}
$$

$$
- \int_{x_0}^{x_0+\Delta x} \left( -U \eta_x + W \right) \bigg|_{z=\eta} dx + \int_{x_0}^{x_0+\Delta x} \eta_t dx = 0 \tag{4.9}
$$

where $U^n$ is the vortical component of the normal velocity.
4.2.2. System B

We repeat the same exercise for System B, starting from Eq. (4.4), which equally gives Eq. (4.9). However, due to the explicit condition \( U_n|_\eta = 0 \) (Eq. (3.9b)), the mass conservation Eq. (4.9) becomes

\[
\int_{x_0}^{x_0+\Delta x} \frac{d}{dx} \left( \int_{-\infty}^{\eta(x)} \phi_{B,x} \, dz \right) \, dx + \int_{x_0}^{x_0+\Delta x} \eta_t \, dx = 0 \tag{4.10}
\]

Total net-flux

\[
\int_{x_0}^{x_0+\Delta x} \eta_t \, dx = 0 \tag{4.10}
\]

Change of \( \eta \) in time

4.2.3. System C

Again, the same method is repeated for System C. The conservation of mass, Eq. (4.4), is now written for both domains; the irrotational bulk \( z \in (-\infty, \eta^*) \) (\( \Gamma_I \)), and the rotational boundary layer \( z \in (\eta^*, \eta) \) (\( \Gamma_{II} \)).

\[
\frac{d}{dx} \left( \int_{-\infty}^{\eta^*(x)} \phi_{C,x} \, dz \right) + \eta^*_t + \frac{d}{dx} \left( \int_{\eta^*(x)}^{\eta(x)} U \, dz \right) + \eta'_t = 0 \tag{4.11}
\]

Recall from Section [3.3] that in \( \Gamma_{II} \), by definition, the velocity vector has only the rotational component \( \vec{U} \). Therefore \( \phi_C = 0 \) on \( z \in (\eta^*, \eta) \). It will not give a contribution in \( \Gamma_{II} \), and the integral can span both domains, i.e. \( z \in (-\infty, \eta) \). Again, we can rewrite \( \frac{d}{dx} \int U \, dz = -U^n|_{z=\eta} \) using the Leibniz rule. (cf. Eq. (3.15)).

\[
\frac{d}{dx} \left( \int_{-\infty}^{\eta(x)} \phi_{C,x} \, dz \right) + \eta^*_t - U^n|_{z=\eta} + \eta'_t = 0 \tag{4.12}
\]

Rewriting \( \eta = \eta^* + \eta' \), and integrating over the column width:

\[
\int_{x_0}^{x_0+\Delta x} \frac{d}{dx} \left( \int_{-\infty}^{\eta(x)} (\phi_{C,x}) \, dz \right) \, dx - \int_{x_0}^{x_0+\Delta x} U^n|_{z=\eta} \, dx + \int_{x_0}^{x_0+\Delta x} \eta_t \, dx = 0 \tag{4.13}
\]

Partial net-flux

\[
\int_{x_0}^{x_0+\Delta x} U^n|_{z=\eta} \, dx + \int_{x_0}^{x_0+\Delta x} \eta_t \, dx = 0
\]

Vortical component of normal velocity

Change of \( \eta \) in time

4.3. Mass conservation comparison

Adding to the point that it is unphysical to use an irrotational fluid as a benchmark for the conservation of mass, System C shows explicitly
that it is precisely the vortical contribution that is needed to conserve the mass in a rotational system. Since $\phi_C$ is defined only for the irrotational domain $\Gamma_I$, one can also see that $\nabla \phi_C = \nabla \phi_{irr}$, and by consequence $\nabla \phi_{irr} = \nabla \phi_A$. Therefore, using a purely irrotational system for the viscous water wave problem neglects the vortical contribution to the mass-flux represented by the term $\int_{x_0}^{x_0+\Delta x} U^n |_{z=\eta} dx$ in Eq. (4.13). In the comparison between the systems we have shown that this corresponds to the pressure term in System C due to the vortical layer. When this additional pressure term is missing, it leads to an underestimation of the decay rate, as exemplified in Padrino and Joseph [17].

In addition, the mass conservation condition for system C, Eq. (4.13), is exactly equal to that of System A, Eq. (4.9), indicating that $\nabla \phi_A = \nabla \phi_C$. Note that Eq. (4.13) corresponds to the fully nonlinear version of the model, i.e. before any form of linearization is introduced, since the normal vector $U^n$ is not yet expressed in terms the vortical pressure $P^\delta$. Linearization is applied in the definition of $P^\delta$, as detailed in Section 3.3.

By comparison of the mass conservation laws for system A (Eq. (4.9)) and system B (Eq. (4.10)), it is clear that $\phi_B \neq \phi_A$. This exemplifies the different gauge choice made in system A and system B, where in the latter system the vortical part of the velocity $\vec{U}$ also takes part of the irrotational velocity in the bulk, as indicated by the grey shaded area in Figure 3B.

### 4.4. Nonlinear vortical terms

Regarding the nonlinear vortical terms, from Eq. (4.9), both the horizontal and vertical components of $U^n$ are required to ensure the conservation of mass. That is, disregarding the horizontal component of the normal vector, the term $\eta_x U$, as done in the approximate models of R&F and DDZ, violates the conservation of mass. Indeed, leaving out part of the normal vector (the horizontal component) does not provide enough mass 'outlet' (or inlet). However, the $\phi_x U$ term in the DBC (Eq. (3.3d)) alters the evolution of $\phi_t$, and therefore does not provide enough mass 'inlet' (or outlet) through the side boundary, as compared to if the term would not be there. As such, these models approximately conserve mass up to $O(\epsilon \delta k)$, the order up to which terms are retained.
5. Discussion and Conclusion

In summary, we show the correspondence between three different ways of looking at the viscous water wave problem, both in the model equations and in the resulting conservation of mass. Throughout the comparison, a key consideration is that the irrotational velocity can be split in multiple ways between the irrotational velocity potential \( \nabla \phi \) and the rotational part \( \vec{U} = \nabla \times \vec{A} \), so that the values of \( \nabla \phi \) and \( \vec{U} \) differ between the considered systems, preventing direct transpositions of the equations.

It is easy to show, by inserting \( \eta = \eta^* + \eta' \) into System A, as is done in \([12]\), that the model equations of the completely linearized Systems A and C are the same. For the full -nonlinear- systems without approximations, it is clear that the same physics are described and therefore the solution must be the same. Indeed, the mass conservation shows the correspondence between the three systems. Discrepancies arise when only some of the nonlinear terms of a certain order are discarded. As such, the discrepancies are limited to the order of the neglected terms.

For System A, we demonstrate that a rotational boundary condition can conserve mass, provided the full normal vector to the surface is taken into account. The approximate models of RF and DDZ neglect the horizontal part of this normal vector and are valid up to \( \mathcal{O}(\epsilon \delta k) \). System B (Dommermuth \([10]\)) is the only closed exact model that is fully nonlinear without approximations, at a cost of having more complicated equations in the bulk, where the fluid is assumed to be rotational, as well as an additional equation for the rotational pressure. System C (Longuet-Higgins \([12]\)) is quite elegant as it concerns an irrotational bulk, and therefore has the simplicity of an irrotational system. Moreover, it is able to give a physical origin to the additional pressure induced by the vortical layer near the surface. Comparing to the irrotational KBC used in various publications (eg. \([19, 17]\)) the only noticeable difference is a 90° phase lag between \( \eta \) and \( \eta^* \). As the deep water limit is considered, the absolute height of \( \eta \) is irrelevant.

As the three models describe the same physics, practical implementation is the guiding factor in opting for one or the other. When the focus is on the physics of the boundary layer, it would be interesting to see how well the linear simplification of the vortical pressure captures the dynamics for the total pressure, comparing to results based on the fully nonlinear model by Dommermuth \([10]\).

Additionally, various other models hint at the existence and relevance of
nonlinear viscosity terms in the propagation equation \[22, 23\]. The order analysis in Cartesian coordinates performed in Appendix C is limited to very low steepness, therefore, an analysis of the magnitude of the nonlinear vortical terms in curved coordinates could provide insight to their relevance in different regimes of viscosity and wave steepness.

We feel it is important to get the correct physical interpretation of the viscous free surface problem, especially in recent interest from the mathematical community which take these as a starting point. For instance in looking at the well-posedness of the DDZ system \[21\], or by taking these viscous water wave systems as a basis for new asymptotic models \[24\]. We hope that this new way of looking at the viscous water wave problem will be built upon in future analyses.

Acknowledgements

We acknowledge the financial support from the Swiss National Science Foundation (Projects Nos. 200021-155970 and 200020-175697). We would like to thank John Carter, Peter Wittwer and Yves-Marie Ducimetière for fruitful discussions.

References

[1] B. J. West, K. A. Brueckner, R. S. Janda, D. M. Milder, R. L. Milton, A new numerical method for surface hydrodynamics, Journal of Geophysical Research 92 (1987) 11803.

[2] D. G. Dommermuth, D. K. P. Yue, A high-order spectral method for the study of nonlinear gravity waves, Journal of Fluid Mechanics 184 (1987) 267.

[3] G. Ducrozet, F. Bonnefoy, D. Le Touzé, P. Ferrant, HOS-ocean: Open-source solver for nonlinear waves in open ocean based on High-Order Spectral method, Computer Physics Communications 203 (2016) 245–254.

[4] J. J. D. Carter, A. Govan, Frequency downshift in a viscous fluid, European Journal of Mechanics B Fluids 59 (2016) 177–185.
[5] H. Segur, D. Henderson, J. Carter, J. Hammack, C.-M. Li, D. Pheiff, K. Socha, Stabilizing the Benjamin-Feir instability, Journal of Fluid Mechanics 539 (2005) 229–271.

[6] A. V. Babanin, Swell Attenuation due to Wave-Induced Turbulence, in: Volume 2: Structures, Safety and Reliability, volume 14, ASME, 2012, p. 439.

[7] L. G. Bennetts, V. A. Squire, On the calculation of an attenuation coefficient for transects of ice-covered ocean, Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences 468 (2012) 136–162.

[8] K. D. Ruvinsky, F. I. Feldstein, G. I. Freidman, Numerical simulations of the quasi-stationary stage of ripple excitation by steep gravity-capillary waves, Journal of Fluid Mechanics 230 (1991) 339–353.

[9] F. Dias, A. Dyachenko, V. Zakharov, Theory of weakly damped free-surface flows: A new formulation based on potential flow solutions, Physics Letters A 372 (2008) 1297–1302.

[10] D. G. Dommermuth, The laminar interactions of a pair of vortex tubes with a free surface, Journal of Fluid Mechanics 246 (1993) 91.

[11] M. S. Longuet-Higgins, Action of a Variable Stress at the Surface of Water Waves, Physics of Fluids 12 (1969) 737.

[12] M. S. Longuet-Higgins, Theory of weakly damped Stokes waves: a new formulation and its physical interpretation, Journal of Fluid Mechanics 235 (1992) 319–324.

[13] H. Lamb, Hydrodynamics, Cambridge University Press, 6th edition, 1932.

[14] M. S. Longuet-Higgins, Capillary rollers, Journal of Fluid Mechanics 240 (1992) 659–679.

[15] T. Lundgren, P. Koumoutsakos, On the generation of vorticity at a free surface, Journal of Fluid Mechanics 382 (1999) S0022112098003978.
[16] D. D. Joseph, Helmholtz decomposition coupling rotational to irrotational flow of a viscous fluid., Proceedings of the National Academy of Sciences of the United States of America 103 (2006) 14272–7.

[17] J. C. Padrino, D. D. Joseph, Correction of Lamb’s dissipation calculation for the effects of viscosity on capillary-gravity waves, Physics of Fluids 19 (2007).

[18] M. S. Longuet-Higgins, Mass Transport in Water Waves, Philosophical Transactions of the Royal Society of London A: Mathematical, Physical and Engineering Sciences 245 (1953).

[19] G. Wu, Y. Liu, D. Yue, A note on stabilizing the Benjamin-Feir instability, Journal of Fluid Mechanics 556 (2006) 45.

[20] M. S. Longuet-Higgins, Mass transport in the boundary layer at a free oscillating surface, Journal of Fluid Mechanics 8 (1960) 293–306.

[21] M. Ngom, D. P. Nicholls, Well-posedness and analyticity of solutions to a water wave problem with viscosity, Journal of Differential Equations 265 (2018) 5031–5065.

[22] A. Armaroli, D. Eeltink, M. Brunetti, J. Kasparian, Viscous damping of gravity-capillary waves: Dispersion relations and nonlinear corrections, Physical Review Fluids 3 (2018) 1–14.

[23] A. Fabrikant, On nonlinear water waves under a light wind and Landau type equations near the stability threshold, Wave Motion 2 (1980) 355–360.

[24] R. Granero-Belinchón, S. Scrobogna, Models for damped water waves, arxiv:905.07751 (2019).

[25] J. Wang, D. D. Joseph, Purely irrotational theories of the effect of the viscosity on the decay of free gravity waves, Journal of Fluid Mechanics 559 (2006) 461.

[26] W. Zhang, J. Viñals, Pattern formation in weakly damped parametric surface waves, Journal of Fluid Mechanics 336 (1997) 301–330.
[27] A. Toffoli, A. Babanin, M. Onorato, T. Waseda, Maximum steepness of oceanic waves: field and laboratory experiments, Geophysical Research Letters 37 (2010) L05603.
Appendix A. Derivation of Dynamic Boundary condition

Starting from the continuity of normal stresses on a line segment on the surface (Eq. (2.10))

\[ \sigma^n = \tau^n - p^n = 2\mu u^n_n - P = P_a \]  \hspace{1cm} (A.1)

where \( P \) is the dynamic pressure, \( \nu \) the kinematic viscosity and \( \tau \) the stress tensor. Since the pressure due to the air is so much smaller than that of the water, we ignore \( P_a \). The surface tension is omitted. Recall that in the linear limit, \( u^n_n = w_z \) and \( \nabla = \frac{\partial}{\partial n} \)

\[ 2\mu u^n_n - P = 0 \]  \hspace{1cm} (A.2)

where \( \mu = \nu \rho \) is the dynamic viscosity. Using the Helmholtz decomposition:

\[ 2\nu (\phi_{nn} + U^n_n) - \frac{P}{\rho} = 0 \]  \hspace{1cm} (A.3)

The next step is to obtain an expression for the pressure from the Navier-Stokes equation

\[ \vec{u}_t + \vec{u} \cdot \nabla \vec{u} = -\frac{1}{\rho} \nabla P + \vec{g} + \nu \nabla^2 \vec{u} \]  \hspace{1cm} (A.4)

Inserting the Helmholtz decomposition directly into Eq. (A.4) gives

\[ \nabla \phi_t + \vec{U}_t + (\nabla \phi + \vec{U}) \cdot \nabla (\nabla \phi + \vec{U}) = -\frac{1}{\rho} \nabla P + \vec{g} + \nu \nabla^2 \vec{U} + \nu \nabla^2 (\nabla \phi) \]  \hspace{1cm} (A.5)

Rewriting the vector products, using \( \nabla \phi \cdot \nabla (\nabla \phi) = \nabla^2 (\nabla \phi)^2 \)

\[ \nabla \phi_t + \vec{U}_t + \nabla^2 (\nabla \phi)^2 + ((\nabla \phi + \vec{U}) \cdot \nabla) \vec{U} + (\vec{U} \cdot \nabla) \nabla \phi = -\frac{1}{\rho} \nabla P + \vec{g} + \nu \nabla^2 \vec{U} \]  \hspace{1cm} (A.6)

Taking the gradient out and moving those terms to the LHS
\[ \nabla \left( \phi_t + \frac{1}{2}(\nabla \phi)^2 - g\eta + \frac{P}{\rho} \right) = \]

\[ - \left( \vec{U}_t + ((\nabla \phi + \vec{U}) \cdot \nabla)\vec{U} + (\vec{U} \cdot \nabla)\nabla \phi - \nu \nabla^2 \vec{U} \right) \]

(A.7)

From this point on, there are two ways to proceed. The first is to take the path of Ruvinsky et al. \[8\] (R&F), where in the end the vortical terms are estimated to be very small. The other way, as done in Dommermuth \[10\], is to write all the vortical contributions as a separate equation for the pressure.

**Ruvinsky method**

R&F integrate along the vertical direction, in a small range near the surface \((\eta - \delta, \eta)\). Taking the normal component, and assuming that \(\vec{U} = 0\) outside the boundary layer, \(n \in (\eta - \delta, \eta)\)

\[ \phi_t + \frac{1}{2}(\nabla \phi)^2 - g\eta + \frac{P}{\rho} = \]

\[ - \int_{\eta-\delta}^\eta \hat{n} \cdot \left( \vec{U}_t + ((\nabla \phi + \vec{U}) \cdot \nabla)\vec{U} + (\vec{U} \cdot \nabla)\nabla \phi - \nu \nabla^2 \vec{U} \right) \, dn \]  

(A.8)

Performing the dimensional analysis [8], shows that the integral terms on the RHS are of higher order in steepness \(\epsilon = ak\) and \(\frac{\delta}{a} \ll 1\), and are therefore neglected.

If instead \(\vec{u} \cdot \nabla \vec{u} = \frac{1}{2}(\nabla \vec{u})^2 + \omega \times \vec{u}\) is used to rewrite Eq. (A.4), this results in

\[ \phi_t + \frac{1}{2}(\nabla \phi)^2 + \frac{P}{\rho} - g\eta = \]

\[ - \int_{\eta-\delta}^\eta \left( \vec{U}_t - \nu \nabla^2 \vec{U} - \omega \times \vec{U} \right) \, dz - \frac{1}{2} \vec{U}^2 \]  

Vorticity terms in the boundary layer

(A.9)

where the RHS can be neglected for the same reasons. However, we retain the leading term \(\phi_n U\), and are left with:

\[ - \frac{P}{\rho} = \phi_t + \frac{1}{2}(\nabla \phi)^2 - g\eta + \phi_n U_n + \phi_s U_s \]  

(A.10)

Dynamic pressure  Static pressure  NL vort. term

Mixed NL terms
Inserting this into the balance of normal stress, Eq. (A.3) gives

$$\phi_t + \frac{1}{2}(\nabla \phi)^2 + 2\nu \phi_{nn} + 2\nu \vec{U}_n^m - g\eta + \frac{\phi_x U}{\rho} = 0$$  \hspace{1cm} (A.11)

**Dommermuth method**

Dommermuth splits the pressure in rotational and irrotational parts:

$$P = P_{rot} + P_{irr}$$  \hspace{1cm} (A.12)

$$P = P_{rot} + \rho \left( -\phi_t - \frac{1}{2}(\nabla \phi)^2 + g\eta \right)$$  \hspace{1cm} (A.13)

in Eq. (A.7), and obtains

$$\nabla \left( \phi_t + \frac{1}{2}(\nabla \phi)^2 - g\eta + \left( -\phi_t - \frac{1}{2}(\nabla \phi)^2 + g\eta + \frac{P_{rot}}{\rho} \right) \right) =$$

$$- \left( \vec{U}_t + ((\nabla \phi + \vec{U}) \cdot \nabla) \vec{U} + (\vec{U} \cdot \nabla) \nabla \phi - \nu \nabla^2 \vec{U} \right)$$  \hspace{1cm} (A.14)

One can obtain the equation for the rotational part of the pressure:

$$\frac{\nabla P_{rot}}{\rho} = - \left( \vec{U}_t + ((\nabla \phi + \vec{U}) \cdot \nabla) \vec{U} + (\vec{U} \cdot \nabla) \nabla \phi - \nu \nabla^2 \vec{U} \right)$$  \hspace{1cm} (A.15)

This is Eq. (5) in [10]. To obtain the dynamic boundary condition, one can insert the equation for the total pressure (Eq. (A.12)) into the balance of normal stress, Eq. (A.3)

$$\phi_t + \frac{1}{2}(\nabla \phi)^2 + 2\nu \phi_{nn} + 2\nu \vec{U}_n^m + g\eta + \frac{P_{rot}}{\rho} = 0$$  \hspace{1cm} (A.16)

**Appendix B. Closing the system: Expressions for the vortical terms**

In order to be able to obtain a closed System A, the rotational part of the velocity, $\vec{U}$, needs to be expressed in terms of either $\phi$ or $\eta$. By solving for the linearized Navier-Stokes (NS) equations, Dias et al. [9] (DDZ) show $W = A_x = 2\nu \eta_{xx}$ in the linear viscous Euler-like system, and conjectures
that this also holds for the nonlinear case. To some extent, we can follow the same line of reasoning. Following Lamb [13], Wang and Joseph [25] DDZ, the linearized NS equations can be written for \( u \) and \( w \)

\[
\begin{align*}
  u_t &= -\frac{p_x}{\rho} + \nu \nabla^2 u \\
  w_t &= -\frac{p_z}{\rho} - g + \nu \nabla^2 w
\end{align*}
\]

using the Helmholtz decomposition in Eq. (2.4), and the Laplacian equation \( \nabla^2 \phi = 0 \) leads to the equations

\[
\begin{align*}
  \phi_{tx} - A_{tz} &= -\frac{p_x}{\rho} + \nu(\phi_{xxx} - A_{xxx} + \phi_{zzz} - A_{zzz}) \\
  \phi_{tz} + A_{tx} &= -\frac{p_z}{\rho} + \nu(\phi_{xxz} + A_{xxx} + \phi_{zzz} + A_{zzz}) - g
\end{align*}
\]

which give the following relations

\[
\begin{align*}
  \phi_t &= -gz - \frac{P}{\rho} + C \\
  A_t &= \nu \nabla^2 A \tag{B.3}
\end{align*}
\]

The solution where both functions are periodic in \( x \) must have the form

\[
\begin{align*}
  \phi(x, z, t) &= \phi_0 e^{i(kx - \omega t)} e^{ik|z|} \\
  A(x, z, t) &= A_0 e^{i(kx - \omega t)} e^{mz} \tag{B.4}
\end{align*}
\]

Inserting into Eq. (B.3) yields

\[
m^2 = k^2 - \frac{i\omega}{\nu} \tag{B.5}
\]

If the nonlinearity of the waves plays an important role, the expression obtained for the potential vector \( A \) in the linear equations is not sufficient. Instead, the nonlinear system should be used. Therefore, instead of assuming \( e^{mz} \), which gives the factor \( \delta = \sqrt{2\nu/\omega} \), we can assume a general function \( A = A_0 e^{i(kx - \omega t)} f(z) \). The derivative of this function will give some factor correction to the slope \( \delta \). The viscosity is a shear flow effect, thus only the derivative in \( n \) matters.

To simplify the exponential behavior in \( z \), we can consider that from Eq. (B.5)
\[ \Re(m) = + \frac{1}{\sqrt{2}} \sqrt{k^4 + \frac{\omega^2}{\nu^2} + k^2} \]  
\[ \Im(m) = - \frac{1}{\sqrt{2}} \sqrt{k^4 + \frac{\omega^2}{\nu^2} - k^2} \]  

(B.6)

and realizing that for typical values of \( k \) the relation \( \frac{\omega}{\nu} = \frac{2}{\delta^2} \gg k^2 \) holds, so that Eq. (B.6) reduces to

\[ \Re(m) \approx \frac{1}{\delta} \]  
\[ \Im(m) \approx - \frac{1}{\delta} \]  

(B.7)

It is important to note here that indeed for the expression of \( A \) in the work of DDZ, \( A \propto e^\delta \), and thus the vorticity falls off exponentially over a characteristic length \( \delta \), with negative \( z \), as is assumed in our dimensional analysis in Appendix C. From expression (B.4), it is readily deduced that \( \phi \) experiences an exponential decay over a characteristic length \( k^{-1} \).

The spatial derivative \( A_x \) can be obtained from the linearized KBC and the continuity of the normal stresses as in DDZ. The linear surface elevation \( \eta \) can be written as

\[ \eta(x, t) = \left( \frac{i|k|}{\omega} \right) \left( \frac{\phi_0}{1 + 2i(\nu k^2/\omega)} \right) e^{i(kx - \omega t)} \]  

(B.8)

and one can write

\[ A_x \Big|_{z=0} = -2i\nu \left( \frac{k^3}{\omega} \right) \left( \frac{\phi_0}{1 + 2i(\nu k^2/\omega)} \right) e^{i(kx - \omega t)} \]  
\[ = -2k^2\nu \eta(x, t) \]  
\[ = 2\nu \eta_{xx} \]  

(B.9)

To derive an expression for \( A_z \) from Eq. (B.4), we consider only the real part of \( m \) (Eq. (B.7)), since it is part of a real valued velocity vector \[13\], and obtain

\[ A_z = mA_0 e^{i(kx - \omega t)} e^{mz} \]  
\[ = -2\nu \left( \frac{|k|}{\omega} \right) \left( \frac{\phi_0}{1 + 2i(\nu k^2/\omega)} \right) e^{i(kx - \omega t)} e^{mz} \]  

(B.10)
Using again the expression for $\eta$ given by Eq. (B.8) and in the linear limit, at the boundary we obtain

$$A_z\bigg|_{z=0} = 2\sqrt{\frac{\nu}{2\omega}}\eta_x = -2\sqrt{\frac{\nu}{2\omega}}\phi_{zz} = -\frac{2k}{\sqrt{2}}\sqrt{\frac{\nu}{\omega}}\phi_z$$  \hspace{1cm} (B.11)

The equations for $A_z$ and $A_x$ allow us to write the KBC in terms of the velocity potential

$$\eta_t + \phi_x\eta_x - \phi_z = +2\nu\eta_{xx} - 2\sqrt{\frac{\nu}{2\omega}}\phi_{zz}\eta_x$$  \hspace{1cm} (B.12)

Note that like DDZ, we conjecture that the linear versions of $\phi$ and $A$ are sufficiently precise to remain valid in the nonlinear KBC and DBC.

The nonlinear viscous Euler-like system can be written as

$$\begin{align*}
\phi_{xx} + \phi_{zz} &= 0 \quad \text{for} \quad -\infty < z < \eta \\
\nabla \phi &\to 0 \quad \text{as} \quad z \to -\infty \\
\eta_t + \phi_x\eta_x - 2\sqrt{\frac{\nu}{2\omega}}\phi_{zz}\eta_x &= \phi_z + 2\nu\eta_{xx} \quad \text{KBC} \\
\phi_t + \frac{1}{2}(\phi_x^2 + \phi_z^2) + g\eta &= -2\nu\phi_{zz} + 2\sqrt{\frac{\nu}{2\omega}}\phi_{zz}\phi_z \quad \text{DBC}
\end{align*}$$  \hspace{1cm} (B.13)

where the dashed underlined terms correspond to those in System A (Eqs. (3.3)).

While it seems counter-intuitive to express a vorticity in terms of the velocity potential $\phi$ (the non-vortical part), one can view it as the vorticity induced by the velocity potential at the boundary layer. That is, the dynamics of the system is governed by $\phi$, since we consider the whole system irrotational except for the small boundary layer $\delta$ in Fig. 1. The stream function $A$ follows the behavior of $\phi$, attenuated by a factor $\delta$ (i.e. by the strength of the viscosity $\nu$ and the wave frequency $\omega$).

Using the method of multiple scales (MMS), as described in Carter and Govan [4], gives the following propagation equation for the envelope $B$

$$\frac{\partial B}{\partial t} + \frac{\omega_0}{2k_0} \frac{\partial B}{\partial x} = \epsilon \left[ + i\frac{\omega_0}{8k_0^2} \frac{\partial^2 B}{\partial x^2} + \frac{1}{2} \frac{i}{k_0^2\omega_0} B|B|^2 - 2k_0^2\nu B \right]$$  \hspace{1cm} (31)
\[ + \varepsilon^2 \left[ - \frac{3}{2} k_0^2 \omega_0 |B|^2 \frac{\partial B}{\partial x} - \frac{1}{4} k_0^2 \omega_0 B^2 \frac{\partial B^*}{\partial x} 
\] 
\[ + \frac{\omega_0}{16 k_0^3} \frac{\partial^3 B}{\partial x^3} + i k_0 B \frac{\partial \phi}{\partial x} - 4 i k_0 \nu \frac{\partial B}{\partial x} \n\] 
\[ - \frac{2 k_0^3 \omega_0 \delta B}{|B|^2} \right] \] 

(B.14)

The last two terms, corresponding to the underlined terms in Eqs. (3.3) and Eqs. (B.13), cancel out. Hence, the solution reduces to the one given by the same system proposed by DDZ. Zhang and Viñals [26] indeed also remark that R&F only take into account the linearized vorticity normal vector, and that the linear vorticity terms describe experiments in reasonable agreement.

Appendix C. Order analysis

To check for the relevance of the nonlinear vortical terms in the KBC and DBC the following order analysis is performed. Following Eq. (2.4), we write \[ U = - A_z \] and \[ W = A_x \]. Note that \( \nabla \phi \) is not affected by viscosity and falls off over a distance \( 1/k \), whereas the vorticity field \( A \) decays from a finite value to a negligible value over a vertical distance \( \delta \). Also note that the analysis in Cartesian coordinates is only valid if \( \varepsilon \ll 1 \). The variables are scaled as follows

\[ x = k^{-1} \bar{x} \quad \phi = \phi_0 \bar{\phi} \]
\[ t = \omega^{-1} \bar{t} \quad A = A_0 \bar{A} \]
\[ z = k^{-1} \bar{z} \quad \text{for } \phi \quad \eta = a \bar{\eta} \]
\[ z = \delta \bar{z} \quad \text{for } A \]

where \( a, A_0 \) and \( \phi_0 = \frac{\alpha \omega}{k} \), are the initial amplitudes of the corresponding quantities. Note that, following [9] , if the viscosity is small \( \Theta = \frac{A_0}{\phi_0} \approx \frac{\nu k^2}{\omega} \ll 1 \). The kinematic boundary condition (Eq. (3.3a)) and DBC (Eq. (3.3d)) can be written as

\[ \tilde{\eta}_t + \epsilon \tilde{\phi}_x \tilde{\eta}_x - \tilde{\phi}_z = (\delta k)^2 \bar{A}_x + \epsilon \delta k \bar{A}_z \tilde{\eta}_x \quad \text{KBC} \]
\[ \tilde{\phi}_t + \epsilon \left( \nabla \tilde{\phi} \right)^2 - \tilde{\eta} = -2(\delta k)^2 \bar{\phi}_x - \epsilon \delta k \bar{\phi}_z \bar{U} \quad \text{DBC} \] (C.2)
Figure C.5: $\delta k$ as a function of wave number $k$ for the kinematic viscosity of water, $\nu = 10^{-6}$ m$^2$s$^{-1}$, (solid line) and for glycerine, $\nu = 6.21 \times 10^{-4}$ m$^2$s$^{-1}$, (dashed-dotted line). The steepness values $\epsilon=0.1, 0.2, 0.3, 0.4$ are indicated with horizontal dashed lines. Above $\epsilon=0.44$ waves are generally considered to break Toffoli et al. [27].

Using $\Theta = (\delta k)^2$, and $\frac{\varphi}{\delta} = \frac{\epsilon}{\delta k}$, and again $U = -A_z$ and $W = A_x$, it is clear that if

$$\epsilon > \delta k \implies \eta_x U > W, \quad \phi_x U > \phi_{zz}$$

(C.3)

Fig. C.5 shows the value of $\delta k$ for water (solid line) and glycerine (dashed line), as a function of wave number. Comparing this to the wave steepness, indicated by the horizontal dashed lines, shows that for practically all wave-numbers $\epsilon > \delta k$ in water. At fixed $k$, the nonlinear vorticity terms become more relevant for steeper waves or weaker viscosity. Indeed, as mentioned also by Longuet-Higgins [18], $\epsilon/\delta k = a/\delta \gg 1$ is the more usual physical situation.

For this reason, we retained the underlined nonlinear terms in Eqs. (3.3), which in typical situations can be larger than the linear terms. However, we have shown that these terms compensate each other, giving a null contribution at the 4th order level in steepness in the evolution equation, Eq. (B.14). This shows that up to this order of approximation there is no contribution of the nonlinear vortical terms.

Performing the order analysis on the closed system of Eqs. (B.12), using the same scaling as in Eq. (C.1), indeed gives the same adimensional equation as Eq. (C.2):

$$\tilde{\eta}_t + \epsilon \tilde{\phi}_x \tilde{\eta}_x - \tilde{\phi}_x = 2\Theta \tilde{\eta}_{xx} - 2\frac{\epsilon}{\delta k} \Theta \tilde{\phi}_{zz} \tilde{\eta}_x$$

(C.4)
Appendix D. Systems conversion

Appendix D.1. Conversion of System A to System C

Longuet-Higgins [12] demonstrates that the linearized system presented in R&F

\[
\eta_t = \phi_z + W \\
\phi_t + g\eta = -2\nu\phi_{zz}
\]

(D.1)
can be written, using \(\eta = \eta^* + \eta'\), as

\[
\eta_t + \eta^*_z\phi_x + \eta'_z\phi_x = \phi_z \\
\phi_t + \frac{1}{2}(\nabla\phi)^2 + g\eta^* = -4\nu\phi_{zz}
\]

(D.2)

where the boundary layer induced-pressure method described in Longuet-Higgins [11] is used. We perform the same exercise for the nonlinear System A.

The kinematic boundary condition

Since \(\eta'_t = -\eta_z U + W\), Eq. (3.3a) can be rewritten as

\[
\eta^*_t + \eta^*_z\phi_x + \eta'_z\phi_x = \phi_z
\]

(D.3)

In Longuet-Higgins [11], the nonlinear term \(\eta^*_z\phi_x\) is ignored. Since \(\eta^*_z \propto \alpha\) and \(\eta'_z \propto \delta\), this is justified if \(\alpha \gg \delta\) (or \(\epsilon \gg \delta k\)), which looking at Figure C.5 is the case in most physical situations. It can also be solved by moving to tangential coordinates, and recalling Eq. (2.9), and writing

\[
\eta^*(s,t)_t + \eta'(s,t)_t = u^n(s,t) \\
\eta^*_t = \phi_n
\]

(D.4)
resulting indeed in the irrotational KBC as in System C.

The dynamic boundary condition

Starting from Eq. (3.7), we insert the normal stress balance, and replace \(\eta = \eta^* + \eta'\), to obtain

\[
\phi_t + \frac{1}{2}(\nabla\phi)^2 + 2\nu(\phi_{nn} + U^n_n) - g\eta^* - g\eta' =
\]
\[-\int_{\eta-\delta}^{\eta} \left( \vec{U}_t - \nu \nabla^2 \vec{U} - \omega \times \vec{U} \right) dz - \frac{1}{2} \vec{U}^2 \]

Vortical layer

\[
\phi_n U^n + \phi_s U^s
\]

Mixed NL terms (D.5)

Recall the definition of the vortical layer in system C:

\[
\eta' = \int U^n dt = -\int \eta_x U dt + \int W dt = -\int \eta_x U dt - 2\nu \frac{k}{\omega^2} \phi_{zz}
\]

(D.6)

This expression agrees with the one obtained in Longuet-Higgins [12] in the linear limit (see Eq. (3.11) in that paper). Inserting Eq. (D.6) into Eq. (D.5), and denoting VL for the small terms in the vortical layer:

\[
\phi_t + \frac{1}{2} (\nabla \phi)^2 + 2\nu (\phi_{nn} + U_n^n) - g\eta^* =
\]

\[-g \int \eta_x U dt - 2\nu \phi_{zz} + VL + \phi_n U^n + \phi_s U^s
\]

(D.7)

Taking \( z \approx n \) for \( \phi \), which is justified for \( \epsilon \ll 1 \) gives

\[
\phi_t + \frac{1}{2} (\nabla \phi)^2 - g\eta^* = -4\nu \phi_{zz}
\]

\[-2\nu \vec{U}^n - g \int \eta_x U dt + VL + \phi_n U^n + \phi_s U^s
\]

(D.8)

Systems A and C are equal if terms of \( \mathcal{O}(\epsilon^2) \) and \( \mathcal{O}(\epsilon\delta k) \), i.e. the second line of the equation above, can be ignored.

Appendix D.2. Conversion of System B to System C

In order to get an idea of the error in the linear estimate of \( P_\delta \) in Longuet-Higgins [11], we can assume that \( \rho g \eta' = P_\delta \) is equal to the fully nonlinear \( P_{\text{rot}} \) in System B. Starting from Eqs. (3.9), and neglecting terms of higher order in both \( \delta \) and \( \epsilon \), we get
\[
\phi_t + \frac{1}{2}(\nabla \phi)^2 - g\eta - \frac{P_{\text{rot}}}{\rho} = -2\nu\phi_{zz}
\]  
(D.9)

Now assuming this is written for domain I in the Longuet-Higgins system, we replace \(\eta\) with \(\eta^*\) and get

\[
\phi_t + \frac{1}{2}(\nabla \phi)^2 - g\eta^* = \frac{P_{\text{rot}}}{\rho} - 2\nu\phi_{zz}
\]  
(D.10)

Comparing this to System C using Eq. (3.12b):

\[
\phi_t + \frac{1}{2}(\nabla \phi)^2 + g\eta^* = \frac{P_{\delta}}{\rho} - 2\nu\phi_{zz}
\]  
(D.11)

Therefore, the two systems match if \(P_{\text{rot}} = P_{\delta}\).