A provably convergent alternating minimization method for mean field inference

Pierre Baque  
EPFL IC ISIM CVLAB, CH-1015 Lausanne, Switzerland  
Pierre.Baque@epfl.ch

Jean-Hubert Hours  
EPFL STI IGM LA3, CH-1015 Lausanne, Switzerland  
Jean-Hubert.Hours@epfl.ch

Francois Fleuret  
Idiap Research Institute, CH-1920 Martigny, Switzerland  
Francois.Fleuret@idiap.ch

Pascal Fua  
EPFL IC ISIM CVLAB, CH-1015 Lausanne, Switzerland  
Pascal.Fua@epfl.ch

Abstract

Mean-Field is an efficient way to approximate a posterior distribution in complex graphical models and constitutes the most popular class of Bayesian variational approximation methods. In most applications, the mean field distribution parameters are computed using an alternate coordinate minimization. However, the convergence properties of this algorithm remain unclear. In this paper, we show how, by adding an appropriate penalization term, we can guarantee convergence to a critical point, while keeping a closed form update at each step. A convergence rate estimate can also be derived based on recent results in non-convex optimization.

1. Introduction

In many situations when a posterior distribution \( P \) over variables \( X \) depends on a complex model, exact inference is not possible and variational inference (Wainwright and Jordan, 2008; Attias, 2000) is a widespread approach to approximating it. This technique is used in domains such as Computer Vision, Natural Language Processing, and large scale Data Processing, as in Fleuret et al. (2008); Hu et al. (2014); Ishigaki et al. (2014).

Mean field variational inference methods approximate \( P \) by a product distribution \( Q \), which means looking for the distribution \( Q \) among a restricted class of product distributions. The quality of the approximation is measured in terms of the Kullback-Leibler divergence between \( P \) and \( Q \). This turns the mean field problem into a non-convex minimization problem.

The most popular approach to solving it is the alternate minimization approach (Bishop, 2008; Koller and Friedman, 2009), also known as the Variational Message Passing algorithm (Winn and Bishop, 2005) in the machine learning community. The Kullback-Leibler divergence is minimized coordinate by coordinate in a pre-determined order, until convergence. The main advantage of this algorithm is that the coordinate-wise minimum can be computed in closed form at each step. Furthermore, the procedure can be parallelized in
most cases, as shown in Bertsekas and Tsitsiklis (1997, p. 21).

However, convergence is not always guaranteed for the general alternate minimisation in the non-convex case. One can find examples where the procedure endlessly loops between several equivalent local minima which become cluster points of the minimization sequence, as shown in Powell (1973). More specifically, convergence can be proven in some cases (Tseng and Mangasarian, 2001) but not all. More precisely, the objective function always decreases but that does not preclude oscillations in the variables and there is no formal proof that the alternating minimization algorithm for variational inference will never loop as in the Powell example.

Our contribution is the introduction of a special purpose proximal regularisation term at each step of the minimization that provably enforces convergence. It dampens potential oscillations while preserving the simplicity of the algorithm whose updates are still computed in closed form. We use a recent result from (Attouch et al., 2013) to prove formally that it is indeed the case.

It is important to understand that, as the objective function is non-convex, our proximal algorithm doesn’t always converge to the same minimum as the classical fixed point algorithm. However, the solution found has no reason to be better or worse. Furthermore, the proximal term can be chosen arbitrarily small through the parameter \( \lambda \). Therefore, by choosing a small \( \lambda \), one can be make the new proximal algorithm follow a trajectory which is arbitrarily close to the trajectory of the alternate minimisation.

Table 1: Notations

| Description                                                                 | Definition                                                                 |
|-----------------------------------------------------------------------------|---------------------------------------------------------------------------|
| \( ||.|| \) is the Euclidean norm in \( \mathbb{R}^N \).                      |                                                                           |
| For a differentiable function \( f \), \( \nabla f \) its gradient.          |                                                                           |
| \( Q = \{q_1, \ldots, q_N\} \) is either the probability distribution on \( N \) independent Bernoulli variables \( \{X_1, \ldots, X_N\} \) or a vector in \([0,1]^N\). |                                                                           |
| If \( f \) is a function and \( X \) a random variable \( E_Q(f(X)) \) is the expected value of \( f(X) \) under probability \( Q \). |                                                                           |
| If \( X = \{X_1, \ldots, X_N\} \) are independent Bernoulli variables under \( Q \), \( E_{Q_i}(f(X)|X_i = a) \) is the expected value of \( f(X) \), given \( X_i = a \). |                                                                           |
| If \( \{X^t\} \) is a convergent sequence in \( \mathbb{R}^N \), its unique limit point is denoted by \( \overline{X} \). |                                                                           |
2. Variational inference problem

We first recall the general formulation of KL divergence minimization problems as they appear in variational inference problems.

We assume that we are working with N random variables \( \{X_1, \ldots, X_N\} \) whose posterior distribution is taken from the exponential family (as in Winn and Bishop (2005) and Bishop (2008)), with marginal priors \( p_i^0(X_i) \). The energy function is denoted by \( \Psi \). We make the important assumption that it is bounded.

\[
P(X) = \frac{1}{Z} e^{-\Psi(X)} \prod_i p_i^0(X_i)
\]

Where \( Z \) is a normalisation factor.

Following the traditional mean field approach (Bishop, 2008; Koller and Friedman, 2009), we are now trying to get a tractable representation \( Q(X) \) of this probability distribution \( P(X) \). By tractable, we mean a distribution that we can easily manipulate, sample from and calculate expectancies. We are therefore approximating \( P \) by \( Q \), among the product distributions. \( Q(X) = \prod_i Q_i(X_i) \). It means, that we will look for \( Q \) which is closest to \( P \) in the sense of the KL divergence \( KL(Q\|P) \).

For the sake of simplicity, it is assumed in the following that \( X_i \) are Bernoulli variables (i.e in \( \{X_1, \ldots, X_N\} \in \{0, 1\}^N \)). However, we could easily work with non-binary random variables.

Therefore, the approximating distribution can be written as :

\[
Q(X) = \prod_{i \in \{1, \ldots, N\}} Q_i(X_i) = \prod_{i \in \{1, \ldots, N\}} q_i^{X_i}(1 - q_i)^{1 - X_i}.
\]

The general form of the KL divergence is :

\[
KL(Q\|P) = \sum_{x \in \{0, 1\}^N} Q(x) \log \left( \frac{Q(x)}{P(x)} \right)
\]

Which we can rewrite as the sum of a multivariate polynomial and univariate convex functions :

\[
KL(Q\|P) = \sum_{x \in \{0, 1\}^N} \Psi(x) \prod_i Q_i(x_i) + \sum_i f_i(q_i)
\]

Where :

\[
f_i(q_i) = \log \left( \frac{1 - q_i}{1 - p_i^0} \right) (1 - q_i) + \log \left( \frac{q_i}{p_i^0} \right) (q_i)
\]
We introduce the functions $G(\{q_1, \ldots, q_N\})$ and $\Omega(\{q_1, \ldots, q_N\})$ such that:

\[
G(\{q_1, \ldots, q_N\}) = KL(\{q_1, \ldots, q_N\} \parallel P) = \sum_{x \in \{0, 1\}^N} \Psi(x) \prod_i Q_i(x_i) + \sum_i f_i(q_i)
\]

\[
= \Omega(\{q_1, \ldots, q_N\}) + \sum_i f_i(q_i)
\]

Where:

**Definition 1**

\[
\Omega(\{q_1, \ldots, q_N\}) := E_Q(\Psi(X))
\]

The KL divergence minimization is thus the following:

\[
\arg\min_{\{q_1, \ldots, q_N\}} G(\{q_1, \ldots, q_N\})
\]

This problem is obviously non-convex as it involves a sum of multiple products. Therefore, finding a global minimum can be cumbersome in large dimensions. In the next section, an algorithm which yields a sequence converging to a first-order critical point is introduced. An estimate of the local convergence rate can also be derived, based on Attouch et al. (2013).

### 3. Proximal alternate minimisation algorithm

In this section, we derive a tractable algorithm that converges to a first order stationary point of the problem of Eq. 8, with convergence guaranties and a provable asymptotic convergence rate.

Although the alternate minimisation algorithm produces a decreasing sequence of objective functions, there is a-priori no guarantee that the variable sequence actually converges as demonstrated by Absil et al. (2005). Powell (1973) shows examples of minimisation problems for which a coordinate descent method fails to converge.

However, we show in this paper, that, by adding a proximal regularisation, we can use the Kurdyka-Łojasiewicz inequality and recent work by Attouch and Bolte to prove convergence. The specific form of the penalty term lets us retain the ability to compute the updates in closed form in the case of variational inference.

**Regularisation** We are using a regularisation function which is the KL divergence between the one dimensional iterates. During the iterations, this proximal function $l(q, q_0)$ penalises the variables which are too different from their previous value.

\[
l(q, q_0) = q \log \left( \frac{q}{q_0} \right) + (1 - q) \log \left( \frac{1 - q}{1 - q_0} \right) = KL(B(Q) \parallel B(Q_0))
\]

Given $q_0$, $l(q, q_0)$ is strongly convex with regards to $q$, positive, continuous on $]0, 1[^2$. Its minimum is 0 for $q = q_0$. 


It is worth noting that the derivative of this function on the minimisation variable $x$ is simple as well and can be written as follows:

$$l'(q, q_0) = \frac{\partial l(q, q_0)}{\partial q} = \log \left( \frac{q}{q_0} \right) - \log \left( \frac{1 - q}{1 - q_0} \right)$$

(12)

**Remark 2** If $X$ is not binary, then, we just replace $l$ by the KL divergence between discrete random variables.

**Proximal alternate minimization procedure** We are looping through the variables, minimizing the objective over one variable at a time, the others staying fixed (see alg. 1), with the following update rule:

$$q_{i+1} = \arg\min_q G\{q_{i+1}, \ldots, q_{i-1}, q_i, q_N\} + \lambda l(q, q_i)$$

(13)

**Input:** A prior distribution $\{p_0^1, \ldots, p_0^N\}$ and a KL function $G$

**Output:** A MF distribution $\{q_1, \ldots, q_N\}$

**Initialisation to the prior:**

$\{q_1, \ldots, q_N\} \leftarrow \{p_0^1, \ldots, p_0^N\}$

**Loop until convergence:**

while $\|\nabla G(\{q_1, \ldots, q_N\})\| > \epsilon$ do

  for $X_i$ in $\{1, \ldots, N\}$ do

    $q_i \leftarrow \arg\min_q G\{q_1, \ldots, q_i, \ldots, q_N\} + \lambda l(q, q_i)$

  end

end

return $\{q_1, \ldots, q_N\}$

**Algorithm 1:** KL Proximal alternate minimisation

The main advantage of our penalization method (e.g $l(q, q_i)$) over the quadratic one (e.g $\|q - q_i\|^2$ as in Attouch et al. (2010)) is that the update is computed in closed form. Indeed, the minimization is differentiable and convex on $q$. Therefore, the first order one dimensional minimality condition gives:

$$q_{i+1} = \frac{1}{1 + \exp \left( \frac{1}{1 + \lambda} \left[ E_{Q_i} \Psi(X) | X_i = 1 \right] - \left[ E_{Q_i} \Psi(X) | X_i = 0 \right] + \log \left( \frac{1 - p_i^0}{p_i^0} \right) + \lambda \log \left( \frac{1 - q_i^t}{q_i^t} \right) \right)}$$

(14)

With the notation:

$$Q_{i} = \{q_1^{i+1}, \ldots, q_{i-1}^{i+1}, q_{i+1}^{i}, q_N^i\},$$

**Remark 3** If $X$ is not binary, a similar closed form minimum is easily obtained by introducing a Lagrange multiplier as in Beal (2003).
4. Convergence of the algorithm

Our analysis is along the lines of Attouch et al. (2010), using the Kurdyka-Łojasiewicz inequality as the key tool in our proof.

4.1. A general convergence result

**Definition 4 (Kurdyka-Łojasiewicz Property)** A differentiable function $f$ is said to have the Kurdyka-Łojasiewicz property at $x$, if there exists $\eta \geq 0$, a neighborhood $U$ of $x$ and a continuous concave function $\phi : [0, \eta) \to \mathbb{R}^+$, such that:

- $\phi(0) = 0$
- $\phi$ is $C^1$ on $(0, \eta)$
- $\forall s \in (0, \eta), \phi'(s) > 0$.

$\forall x \in U \cap [f(x)] \geq f(x) + [f(x) + \eta]$, the following inequality, called Kurdyka-Łojasiewicz inequality holds:

$$\phi'(f(x) - f(x)) \|\nabla f(x)\| \geq 1$$

(15)

**Lemma 5** Let $F$ be any differentiable function from $\mathbb{R}$ to $\mathbb{R}^N$, and $X$ a bounded sequence which has the three following properties:

(i) Sufficient decrease:

$\exists \lambda$ such that, $\forall t \geq 0$

$$F(X^{t+1}) + \frac{\lambda}{2}\|X^{t+1} - X^t\|^2 \leq F(X^t)$$

(16)

(ii) Gradient bound:

$\exists C$ such that, $\forall t \geq 0$

$$\|\nabla F(X^t)\| \leq C\|X^{t+1} - X^t\|$$

(17)

(iii) The function $F$ has the Kurdyka-Łojasiewicz property at all its critical points, with $\phi(s) = cs^{1-\theta}$ and $\theta \in [0, 1]$.

Then, the sequence $X$ converges to a stationary point of $F$ that we denote $\overline{X}$. Moreover, the following convergence rates apply (depending on $\theta$).

(a) If $\theta \in \left[0, \frac{1}{2}\right]$, then $\exists A > 0, \exists \tau > 0$ such that:

$$\|X^t - \overline{X}\| \leq At^\tau$$

(18)

(b) If $\theta \in \left[\frac{1}{2}, 1\right]$, then $\exists A > 0$ such that:

$$\|X^t - \overline{X}\| \leq At^{-(1-\theta)/(2\theta-1)}$$

(19)

**Proof** The proof of the previous Lemma follows from the recent work of Attouch and Bolte. There is no explicit statement of the asymptotic convergence rates in Attouch et al. (2013), however, one can strictly follow Attouch et al. (2010).
4.2. Properties

Kurdyka-Łojasiewicz

**Proposition 6** The function $G$ defined in 4 satisfies the Kurdyka-Łojasiewicz Property at all its critical points with a function $\phi(s) = s^{1-\theta}$ where $\theta \in [\frac{1}{2}, 1[.$

Let us denote by $U$ and $\eta$ the associated objects in definition (4).

**Proof** Łojasiewicz (Łojasiewicz (1965, 1984)), showed that any real analytic function has the Kurdyka-Łojasiewicz property with $\phi(s) = s^{1-\theta}$ for some $\theta \in [\frac{1}{2}, 1[.$ Our function $G$ is obviously analytic and real. Which terminates the proof of Proposition (6).

**Lemma 7** The sequence $\{Q^t\}$ belongs to a compact set $\Sigma \subset ]0, 1[^N$. Let us define :

\[
\Sigma := \prod_i [q_i^{\min}, q_i^{\max}]
\]

**Proof** We know that $\Psi$ in bounded. Let us define :

\[
\forall x \in \{0, 1\}^N \Psi_{\min} \leq \Psi(x) \leq \Psi_{\max}
\]

\[
\{1, \ldots, N\} \quad q_i^{\min} = \frac{1}{1 + \exp \left( \Psi_{\max} - \Psi_{\min} + \log \left( \frac{1 - p_0^i}{p_i^0} \right) \right)}
\]

\[
\{1, \ldots, N\} \quad q_i^{\max} = \frac{1}{1 + \exp \left( \Psi_{\min} - \Psi_{\max} + \log \left( \frac{1 - p_0^i}{p_i^0} \right) \right)}
\]

Then, if we assume that $q^t \in [q_i^{\min}, q_i^{\max}]$, using (14), (20), (21), we can write the following :

\[
\log \left( \frac{1 - q^{t+1}}{q^{t+1}} \right) = \frac{1}{1 + \lambda} \left[ E_{Q^t} \left( \Psi(X) | X_i = 1 \right) - E_{Q^t} \left( \Psi(X) | X_i = 0 \right) + \log \left( \frac{1 - p_0^i}{p_i^0} \right) + \lambda \log \left( \frac{1 - q^t}{q^t} \right) \right] \\
\leq \frac{1}{1 + \lambda} \left[ \Psi_{\max} - \Psi_{\min} + \log \left( \frac{1 - p_0^i}{p_i^0} \right) + \lambda \log \left( \frac{1 - q_i^{\min}}{q_i^{\min}} \right) \right] \\
\leq \frac{1}{1 + \lambda} \left[ \log \left( \frac{1 - q_i^{\min}}{q_i^{\min}} \right) + \lambda \log \left( \frac{1 - q_i^{\min}}{q_i^{\min}} \right) \right] \\
\leq \log \left( \frac{1 - q_i^{\min}}{q_i^{\min}} \right)
\]

By monotonicity and conversely for the upper bound, we conclude, that $q^{t+1} \in [q_i^{\min}, q_i^{\max}]$. Therefore, by induction, as long as $Q^0 \in \Sigma$, ($Q^0 = P^0$ for instance), $Q^t \in \Sigma \forall t$.
Sufficient decrease

Lemma 8  The penalization \( l \) (Equation (9)) is 1-strongly convex on \([0,1[. Therefore:

\[
\text{For all } x \text{ and } x_0 \text{ in } [0,1[, \quad \frac{1}{2}\|x-x_0\|^2 \leq l(x,x_0)
\]  

Proof

By a simple differentiation of \( l \), we get:

\[
\frac{\partial^2 l(x,x_0)}{\partial x^2} = \frac{1}{x} + \frac{1}{(1-x)} \geq 1
\]

Then, by definition of the strong convexity, combined with \( l(x_0,x_0) = 0 \), and \( l'(x_0,x_0) = 0 \), we get the second part of the Lemma.

Proposition 9  Our alternate minimization algorithm has the following sufficient decrease property.

For all indices \( t \geq 1 \),

\[
G(Q^{t+1}) + \frac{\lambda}{2}\|Q^{t+1} - Q^t\|^2 \leq G(Q^t)
\]

Proof

An elementary induction gives, for each step:

\[
G(\{q_{i+1}, \ldots, q_{i-1}^{t+1}, q_i^{t+1}, q_{i+1}^{t+1}, q_N^{t+1}\}) + \lambda l(q_i^{t+1}, q_i^t) \leq G(\{q_1^{t+1}, \ldots, q_{i-1}^{t+1}, q_i^t, q_{i+1}^{t+1}, q_N^{t+1}\})
\]  

(23)

Therefore, using the same equations for \( i = 1, \ldots, N \), it easily follows:

\[
G(Q^{t+1}) + L(Q^{t+1}, Q^t) \leq G(Q^t)
\]  

(24)

And by strong convexity property of Lemma 8, we get:

\[
G(Q^{t+1}) + \frac{\lambda}{2}\|Q^{t+1} - Q^t\|^2 \leq G(Q^t)
\]

Gradient bound

Lemma 10  \( \Omega \), defined in 1 is \( K_\Omega \) - Lipschitz with \( K_\Omega = \Psi_{\max}\sqrt{N} \).

Proof  For any \( i \) in \( 1, \ldots, N \):

\[
\left| \frac{\partial \Omega}{\partial q_i}(Q^t) \right| = |E_{Q^t} (\Psi(X)|X_i = 1) - E_{Q^t} (\Psi(X)|X_i = 0)| \leq \Psi_{\max}
\]
Therefore, using the classical inequality between $L_2$ and $L_\infty$ norms:

$$\|\nabla \Omega(Q^t)\| \leq \Psi_{\max} \sqrt{N}$$

\[\Box\]

**Lemma 11**

There exists a positive constant $K_i$ such that for any $Q$ and $\hat{Q}$ in $\Sigma$:

$$\forall i \in \{1, \ldots, N\}, \ |l'(q_i, \hat{q}_i)| < 2K_i|q_i - \hat{q}_i|$$

(25)

**Proof** For any $i$ in $\{0, \ldots, N\}$, the function $x \to \log(x)$ is Lipschitz continuous on $[q_i^{\min}, q_i^{\max}]$ with Lipschitz constant $\frac{1}{q_i^{\min}}$. And the function $x \to \log(1-x)$ is Lipschitz continuous on $[q_i^{\min}, q_i^{\max}]$ with Lipschitz constant $\frac{1}{1-q_i^{\max}}$.

Therefore, according to Eq. 12

$$\forall x \in [q_i^{\min}, q_i^{\max}], \forall x_0 \in [q_i^{\min}, q_i^{\max}], |l'(x, x_0)| < \left( \frac{1}{q_i^{\min}} + \frac{1}{1-q_i^{\max}} \right) |q - q_0|$$

\[\Box\]

Therefore, if we simply set $K_i$ such that: $K_i = \max_{i \in \{1, \ldots, N\}} \left( \frac{1}{q_i^{\min}} + \frac{1}{1-q_i^{\max}} \right)$, Eq. 25 comes directly.

**Lemma 12** For any index $u \geq 1$, the following bound on the gradient of $G$ holds:

$$\|\nabla G(Q^u)\| \leq (2K_i + \sqrt{N - 1}K_{\Omega})\|Q^u - Q^{u-1}\|$$

(26)

**Proof** Let us choose $u \geq 1$. For any $i$, from the first order minimization condition in Eq. 13, we know that:

$$0 = \lambda l'(q_i^u, q_i^{u-1}) + \frac{\partial G}{\partial q_i}(\{q_1^u, \ldots, q_{i-1}^u, q_i^u, q_{i+1}^u, \ldots, q_N^u\})$$

(27)

Which we can rewrite as, using the decomposition on $G$:

$$0 = \lambda l'(q_i^u, q_i^{u-1}) + \frac{\partial G}{\partial q_i}(\{q_1^u, \ldots, q_{i-1}^u, q_i^u, q_{i+1}^u, \ldots, q_N^u\})$$

$$= \lambda l'(q_i^u, q_i^{u-1}) + \frac{\partial G}{\partial q_i}(\{q_1^u, \ldots, q_{i-1}^u, q_i^u, q_{i+1}^u, \ldots, q_N^u\})$$

$$- \frac{\partial \Omega}{\partial q_i}(\{q_1^u, \ldots, q_{i-1}^u, q_i^u, q_{i+1}^u, \ldots, q_N^u\}) - \frac{\partial f_i}{\partial q_i}(q_i^u)$$

$$+ \frac{\partial \Omega}{\partial q_i}(\{q_1^u, \ldots, q_{i-1}^u, q_i^u, q_{i+1}^u, \ldots, q_N^u\}) + \frac{\partial f_i}{\partial q_i}(q_i^u)$$
Using equation (25), and the Lipschitz constant $K_{\Omega}$ of $\Omega$ (see lemma 10), we get:

$$\frac{\partial G}{\partial q_i}(Q^u) \leq 2K\|q_i^u - q_i^{u-1}\| + K_{\Omega}\|Q^u - \{q_1^u, \ldots, q_{i-1}^u, q_i^u, q_{i+1}^u, \ldots, q_N^u\}\|
$$

Combining equation 28 for $i = \{1, \ldots, N - 1\}$ we get:

$$\|\nabla G(Q^u)\| \leq (2K + \sqrt{N-1}K_{\Omega})\|Q^u - Q^{u-1}\|$$

(29)

4.3. Convergence

We showed in the previous section (Lemma 12, Proposition 9 and Proposition 6) that the sequence generated by our new minimization procedure has the three sufficient properties for convergence, as shown in Lemma 5. Therefore, according to Lemma 5 (or Attouch and Bolte (2009)), the main Theorem of this paper can be stated as follows.

**Theorem 13 (Convergence)** The sequence $\{Q^t\}$ generated by the proximal alternate minimization procedure described in algorithm 13, converges to a critical point of $F$, denoted $Q$.

**Corollary 14** The following asymptotic convergence rates hold:

We recall that $\theta$ is the exponent of the $\phi$ function in the Kurdyka-Łojasiewicz inequality such that $\phi(s) = cs^{1-\theta}$.

(i) If $\theta \in [0, \frac{1}{2}]$, then $\exists C > 0, \exists \tau > 0$ such that:

$$\|Q^t - \overline{Q}\| \leq Ct^\tau$$

(30)

(ii) If $\theta \in [\frac{1}{2}, 1]$, then $\exists C > 0$ such that:

$$\|Q^t - \overline{Q}\| \leq Ct^{-(1-\theta)/(2\theta-1)}$$

(31)

If we make the standard SSOC assumption on $G$ (the hessian is positive definite at all the local minima), then we can show that the convergence rate toward the local minima is linear, as in Equation 30, with $\theta = 1/2$.

**Proof** [Proof of the corollary] The first part of the corollary is also a direct consequence of Lemma 5.

Let us now assume that the Hessian matrix is positive definite at all the local minima (SSOC assumption). We denote by $\mu_1$ and $\mu_2$ the highest and lowest eigenvalues of the
Hessian in a neighborhood of a local minimum $\overline{Q}$. $\mu_1$ and $\mu_2$ are both positive by SSOC and continuity of the Hessian. Let us then write the Taylor formula for $G$ and $\nabla G$ at the neighborhood of $\overline{Q}$. It follows the existence of a neighborhood $U$ of $\overline{Q}$, so that, for all $Q \in U$:

$$|G(Q) - G(\overline{Q})| \leq \mu_1 \|Q - \overline{Q}\|^2$$  \hspace{1cm} (32)

and

$$\|\nabla G(Q)\| \geq \mu_2 \|Q - \overline{Q}\|$$  \hspace{1cm} (33)

It shows that $G$ follows a Kurdyka-Łojasiewicz inequality at all its minimal points, with $\phi(s) = c \sqrt{s}$. Therefore, if $Q$ converges toward a minimal point, which has the SSOC, the convergence rate is linear with $\theta = 1/2$. □

5. Conclusion

Although the convergence of fixed point iterations schemes for mean field minimization is often taken for granted, no formal proof exists. In this paper, we have proposed a slightly modified scheme that is provably convergent. This addresses a major conceptual weakness of one of the most important algorithms used by the Machine Learning community. Interestingly, our regularisation can be chosen as small as needed through the parameter $\lambda$. Therefore, our algorithm can be arbitrarily similar to the classical minimisation, while guaranteeing convergence.

In future work, we will explore the practical applications for our scheme. We will look for examples where it accelerates convergence. It may prevent infinite, but also temporary oscillations between equivalent solutions of a learning optimisation problem.

References

P.-A. Absil, R. Mahony, and B. Andrews. Convergence of the iterates of descent methods. *SIAM J. OPTIM*, 16:531–547, 2005.

Hagai Attias. *A Variational Bayesian Framework for Graphical Models*. MIT Press, 2000.

H. Attouch and J. Bolte. On the convergence of the proximal algorithm for nonsmooth functions involving analytic features. *Mathematical Programming*, 116(1-2):5–16, 2009. ISSN 0025-5610. URL http://dx.doi.org/10.1007/s10107-007-0133-5.

H. Attouch, J. Bolte, P. Redont, and A. Soubeyran. Proximal alternating minimization and projection methods for nonconvex problems: An approach based on the kurdyka-Łojasiewicz inequality. *Mathematics of Operations Research*, 35(2):438–457, 2010. doi: 10.1287/moor.1100.0449.

H. Attouch, J. Bolte, and B. Svaiter. Convergence of descent methods for semi-algebraic and tame problems: proximal algorithms, forwardbackward splitting, and regularized gaussseidel methods. *Mathematical Programming*, 137(1-2):91129, 2013. ISSN 0025-5610.
Matthew J. Beal. Variational algorithms for approximate bayesian inference. PhD thesis, University of London, 2003.

Dimitri P. Bertsekas and John Tsitsiklis. Parallel and Distributed Computation: Numerical Methods. Athena Scientific, 1997.

M. Bishop. Pattern Recognition and Machine Learning. Springer, 2008.

Francois Fleuret, Jerome Berclaz, Richard Lengagne, and Pascal Fua. Multicamera people tracking with a probabilistic occupancy map. IEEE Trans. Pattern Anal. Mach. Intell., 30(2):267–282, February 2008. URL http://dx.doi.org/10.1109/TPAMI.2007.1174.

Yuening Hu, Ke Zhai, Vladimir Eidelman, and Jordan Boyd-Graber. Polylingual Tree-Based Topic Models for Translation Domain Adaptation. Association for Computational Linguistics, 2014.

Tsukasa Ishigaki, Nobuhiko Terui, Tadahiko Sato, and Greg M. Allenby. A large-scale marketing model using variational bayes inference for sparse transaction data. 2014. URL http://hdl.handle.net/10097/56671.

Daphne Koller and Nir Friedman. Probabilistic graphical models. The MIT Press, 2009.

S. Lojasiewicz. Ensembles semi-analytiques. Institut des Hautes Etudes Scientifiques, 1965.

S. Lojasiewicz. Sur les trajectoires du gradient dune fonction analytique. Seminari di Geometria, pages 115–117, 1984.

M.J.D. Powell. On search directions for minimization algorithms. Mathematical Programming, 4(1):193–201, 1973.

P. Tseng and Communicated O. L. Mangasarian. Convergence of a block coordinate descent method for nondifferentiable minimization. J. Optim Theory Appl, pages 475–494, 2001.

Martin J. Wainwright and Michael I. Jordan. Graphical models, exponential families, and variational inference. Found. Trends Mach. Learn., 1(1-2):1–305, January 2008.

John Winn and Christopher M. Bishop. Variational message passing. J. Optim Theory Appl, pages 661–694, 2005.