Effects of space-time noncommutativity on the angular power spectrum of the CMB

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abstract

We investigate an inflationary model of the universe based on the assumption that space-time is noncommutative in the very early universe. We analyze the effects of space-time noncommutativity on the quantum fluctuations of an inflaton field and investigate their contributions to the cosmic microwave background (CMB). We show that the angular power spectrum \( l(l + 1)C_l \) generically has a sharp damping for lower \( l \) if we assume that the last scattering surface is traced back to fuzzy spheres at the times when large-scale modes cross the Hubble horizon.

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1 Introduction

Recently the WMAP research group presented their first year result on the CMB anisotropy \[1\][2][3]. They determined many of the relevant cosmological parameters with great precision from the angular power spectrum $C_l$, which is defined by

$$\langle \frac{\delta T}{T} (\eta_0, \Omega_1) \frac{\delta T}{T} (\eta_0, \Omega_2) \rangle = \sum_l \frac{2l+1}{4\pi} C_l P_l (\cos \theta_{12}).$$ \tag{1.1}

Here $\eta$ is the conformal time, and $(\delta T/T)(\eta_0, \Omega)$ is the CMB temperature fluctuation observed at the present time ($\eta = \eta_0$) in the angular direction $\Omega = (\theta, \varphi)$. $l$ is the azimuthal quantum number, and $\theta_{12}$ is the angle between the directions $\Omega_1$ and $\Omega_2$. The expectation value $\langle \rangle$ represents the sample average over the data taken from various parts on the celestial sphere.

The data of WMAP and COBE \[4\] show that for small $l$ ($\lesssim 50$), $l(l+1)C_l$ take values almost constant in $l$, which can be naturally explained if the curvature perturbation has an almost scale-invariant power spectrum. This has been regarded as a strong evidence for inflationary models of the universe \[5][6]. In fact, the leading slow-roll approximation exactly yields a scale-invariant power spectrum, and moreover, small deviations from the scale-invariant power spectrum can be generically accommodated by adjusting the potential of inflaton(s).

However, $l(l+1)C_l$ starts to deviate from the constant value as $l$ becomes much smaller ($l < 10$). This behavior is difficult to be explained with standard inflationary models.\footnote{See refs. \[7][8][9\] for recent attempts to derive a scale-invariant power spectrum based on models other than inflationary models.}

Conventional explanation of this deviation is based on the so-called cosmic variance. That is, for smaller $l$, any theoretical calculations inevitably get to lose their predictive power. In fact, what one can predict theoretically is only the mean value of some large ensemble, but for small angular momentum modes we can get only a few experimental data because a smaller $l$ corresponds to a larger angular scale.

If one takes the deviation seriously, however, then it may be regarded as a sign of the necessity to change our understanding of the fundamental dynamics in the very early universe. The main purpose of the present article is to show that this sharp damping

\footnote{See refs. \[10][11\] for attempts to derive this behavior solely from inflationary models.}
could be understood as a generic property that always holds when the space-time non-commutativity is incorporated into the dynamics of an inflaton field around the string scale.\(^3\)

![Diagram](image)

**Figure 1:** An observer at the present time \(\eta = \eta_0\) sees photons from the LSS as the CMB. The LSS is traced back to a noncommutative sphere in the very early universe.

The point of our discussion is the following (see Fig. 1). The observed CMB mainly consists of photons coming from the last-scattering surface (LSS) which is a two-sphere on the time slice (at \(\eta = \eta_*\)) when the recombination takes place. This sphere can be traced back to spheres at earlier times (\(\eta < \eta_*\)) such that they have the same comoving coordinates with those of the sphere at \(\eta_*\) (the LSS). This family of spheres makes an orbit in four-dimensional space-time, and we assume that spheres become fuzzy or noncommutative as they come back to the very early universe. On the other hand, the angular power spectrum \(C_l\) is proportional to the corresponding power spectrum of the gravitational potential on the LSS as a result of the Sachs-Wolfe effect \(^{[13]}\). This gravitational potential in turn is related to the amplitude that the corresponding inflaton mode takes when it crosses the Hubble horizon. As we see later, \(l\) plays almost the same role with that of a comoving wave vector \(k\) in evaluating the power spectrum (\(k \sim l\)), so that a mode with smaller \(l\) crosses the Hubble horizon at an earlier time. Therefore, under our assumption of the noncommutativity in the very early universe, modes with smaller \(l\)

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\(^3\)Other effects of Planck-scale physics on the CMB anisotropy are also expected to be found, such as a running spectral index or a deviation from the Gaussian perturbation. See refs. \(^{[12]}\) for attempts in this direction.
see more noncommutativity of space-time, leading to a modification of the angular power spectrum for smaller $l$.

In the present article, we make a rough estimation of the angular power spectrum, taking into account only the noncommutativity of the angular coordinates $\Omega = (\theta, \phi)$ since these should yield the most relevant effects on the deviations of $l(l+1)C_l$ from the standard value.\(^4\) In fact, in the cosmological perturbation theory, one can consider the time evolution of each mode separately. Furthermore, the angular power spectrum of the CMB anisotropy can be related to the fluctuations of gravitational potential on the last scattering surface, and thus is sensitive to the fluctuations only in the angular directions. Since the noncommutativity to a given direction is expected to give its major effects to the fluctuations in the corresponding direction, the introduction of noncommutativity to the other directions $(\eta, r)$ will not give a drastic change to the angular power spectrum.\(^5\) We show that when the noncommutativity is introduced to angular directions, $l(l+1)C_l$ with small $l$ certainly has a substantial deviation from the constant value. A more detailed analysis is possible once extra settings are incorporated correctly, and will be reported in the forthcoming paper \(^{19}\).

This paper is organized as follows. In section 2 we give an inflationary model of the universe, assuming that the very early universe has a space-time noncommutativity only for the angular coordinates. In section 3 we calculate the CMB anisotropy based on our model, and show that $l(l+1)C_l$ has a sharp damping for small $l$. Section 4 is devoted to conclusion and outlook.

## 2 Model

The flat Friedmann-Robertson-Walker (FRW) universe in the absence of noncommutativity is given by the metric

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = a(\eta)^2 (-d\eta^2 + d\vec{x}^2) = a(\eta)^2 (-d\eta^2 + dr^2 + r^2d\Omega^2).$$ (2.1)

\(^{4}\)See refs. \(^{14}\)\(^{15}\)\(^{16}\) for discussions that take into account the noncommutativity for $\eta$ and/or $r$.

\(^{5}\)Recently Tsujikawa, Maartens and Brandenberger \(^{17}\) and Huang and Li \(^{18}\) have analyzed the noncommutative inflation introducing the noncommutativity only to $(\eta, r)$, and shown that the effect is not strong enough to give a sharp damping to $l(l+1)C_l$ at large angular scales.
In the present article, we exclusively use the conformal time $\eta$ to represent the time coordinate. During the inflationary era, the size of the universe, $a(\eta)$, is given by

$$a(\eta) = -\frac{1}{H\eta}$$  \hspace{1cm} (2.2)

with the constant Hubble parameter $H$. The action of an inflaton field $\Phi(x) = \Phi(\eta, \vec{x}) = \Phi(\eta, r, \Omega)$ with the potential $V(\Phi)$ is given by

$$S[\Phi(x)] = \int d\eta dr d\Omega \sqrt{-g} \left[ -\frac{1}{2} \nabla_\mu \Phi(x) \nabla^\mu \Phi(x) - V(\Phi) \right].$$  \hspace{1cm} (2.3)

In order to describe the Gaussian fluctuations of the inflaton field around the classical value $\bar{\phi}(\eta)$, we set $\Phi(x) = \bar{\phi}(\eta) + \phi(x)$ and expand the above action around $\bar{\phi}(\eta)$ to the quadratic order:

$$S[\bar{\phi}(\eta) + \phi(x)] \simeq S[\bar{\phi}(\eta)] + S[\phi(x)]$$  \hspace{1cm} (2.4)

$$S[\phi(x)] = \int d\eta dr d\Omega a^2(\eta) r^2 \left[ -\frac{1}{2} \eta^{\mu\nu} \partial_\mu \phi(x) \partial_\nu \phi(x) - \frac{1}{2} a^2(\eta) \phi^2 V''(\bar{\phi}) \right],$$  \hspace{1cm} (2.5)

where $\eta_{\mu\nu} = \text{diag}[-1, 1, 1, 1, 1]$.

Now we introduce a noncommutativity into space-time. As stated in Introduction, we introduce the noncommutativity only to the angular coordinates $\Omega = (\theta, \varphi)$, so that the three-dimensional space is the product of the radial direction $r$ and the fuzzy sphere.

Assuming that an observer is at the origin ($r = 0$), we consider a sphere of radius $r$ around the observer. Since $r$ is the radius of the comoving coordinates, the physical radius $\rho$ of the sphere changes under the evolution of the universe as

$$\rho(\eta, r) = a(\eta) r.$$  \hspace{1cm} (2.6)

This implies that if we introduce a noncommutativity on spheres by allowing at most one bit of degrees of freedom to reside in the unit physical Planck area $l_s^2$, then the number of degrees of freedom (or the dimension of the Hilbert space on which field operators can act) also changes as the universe develops.

The above statement can be made into a precise form in the following way. We first introduce a fuzzy sphere such that it is represented by the noncommutative

\footnote{In the following, we neglect the contribution from the metric perturbation since it can be shown to be sufficiently small during the inflationary era.}
comoving coordinates \( \hat{x}_i \) \((i = 1, 2, 3)\) with the relation
\[
[\hat{x}_i, \hat{x}_j] = i \theta \varepsilon_{ijk} \hat{x}_k, \quad \sum_{i=1}^{3} (\hat{x}_i)^2 = r^2.
\]

(2.7)

We assume that the Hilbert space on which \( \hat{x}_i \) acts has dimension \((N + 1)\). Then, \( \hat{x}_i \) can be represented by the generators \( \hat{L}_i \) in the \((N + 1)\)-dimensional (i.e., spin-\((N/2)\)) representation of the \( su(2) \) algebra as
\[
\hat{L}_i = \theta^{-1} \hat{x}_i,
\]

(2.8)

where
\[
[\hat{L}_i, \hat{L}_j] = i \varepsilon_{ijk} \hat{L}_k, \quad \sum_{i=1}^{3} (\hat{L}_i)^2 = \frac{N}{2} \left( \frac{N}{2} + 1 \right).
\]

(2.9)

Here we should note that \( \theta = 2r / \sqrt{N(N + 2)} \) should not be regarded as the fundamental noncommutative scale, since \( \theta \) simply represents the noncommutativity of the comoving coordinates. A correct interpretation is given as follows [23]. Since three coordinates \( \hat{x}_i \) cannot be diagonalized simultaneously, we can best characterize the position of a point on the fuzzy sphere only with an eigenvalue of one coordinate, say \( \hat{x}_3 \). Thus in the \((N + 1)\)-dimensional representation, the sphere consists of \((N + 1)\) fundamental regions (because \( \hat{L}_3 = \theta^{-1} \hat{x}_3 \) has \((N + 1)\) eigenvalues). Since each fundamental region should have the area of noncommutative scale, \( l_s^2 \), we have the following relation for a sphere of physical radius \( \rho \):
\[
\frac{4\pi \rho^2}{N + 1} = l_s^2.
\]

(2.10)

In our model of the FRW universe, \( \rho \) is a function of \( \eta \) and \( r \) as \( \rho = a(\eta)r \), so that eq. (2.10) defines \( N \) as a function of \((\eta, r)\) as
\[
N(\eta, r) = \frac{4\pi a^2(\eta)r^2}{l_s^2} - 1.
\]

(2.11)

In order to analyze a scalar field theory on this fuzzy sphere, we must translate usual functions on a smooth sphere into operators on this fuzzy sphere. This can be carried out as follows. We first define truncated spherical harmonics \( \hat{Y}_{lm} \):
\[
\hat{Y}_{lm} = \frac{a^l}{\rho^l} \sum_{i_k} i_{1i_2\ldots i_l} \hat{x}_{i_1} \hat{x}_{i_2} \cdots \hat{x}_{i_l},
\]

(2.12)
where the traceless symmetric functions \( f_{i_1i_2...i_l}^{lm} \) are the coefficients appearing in the usual spherical harmonics: \( Y_{lm}(\Omega) = (a^l/r^l) \sum_i f_{i_1i_2...i_l}^{lm} x_{i_1}x_{i_2}...x_{i_l} \). Note that for the spherical harmonics on the fuzzy sphere, \( l \) is limited to \( N \). By the definition of \( f_{i_1i_2...i_l}^{lm} \), \( \hat{Y}_{lm} \) satisfies the following equations:

\[
[\hat{L}_3, \hat{Y}_{lm}] = m\hat{Y}_{lm}, \quad \sum_{i=1}^3 [\hat{L}_i, [\hat{L}_i, \hat{Y}_{lm}]] = l(l+1)\hat{Y}_{lm}.
\] (2.13)

Any function on a smooth sphere can be expanded with spherical harmonics \( Y_{lm}(\Omega) \):

\[
\phi(\Omega) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \phi_{lm} Y_{lm}(\Omega).
\] (2.14)

Then using these coefficients \( \phi_{lm} \) and truncated spherical harmonics, we define the operator \( \hat{\phi} \) on the fuzzy sphere that corresponds to \( \phi(\Omega) \) as

\[
\hat{\phi} \equiv \sum_{l=0}^{N} \sum_{m=-l}^{l} \phi_{lm} \hat{Y}_{lm}.
\] (2.15)

Taking into account the remaining commutative coordinates \((\eta, r)\), every function \( \phi(\eta, r, \Omega) \) in a smooth space-time becomes a matrix-valued two-dimensional field

\[
\hat{\phi}(\eta, r) = \sum_{l=0}^{[N(\eta, r)]} \sum_{m=-l}^{l} \phi_{lm}(\eta, r) \hat{Y}_{lm}.
\] (2.16)

We here used the Gauss symbol \([N(\eta, r)]\) because (2.11) defines \( N \) as a real number. The important point is that the mode expansion is bounded from above by \( N \) that is a function of \((\eta, r)\). That is, the two-dimensional field \( \hat{\phi}(\eta, r) \) can have the mode expansion only up to the maximal value

\[
l_{\text{max}}(\eta, r) = N(\eta, r) = \frac{4\pi a^2(\eta) r^2}{l^2_s} - 1
\] (2.17)

at a point with the space-time coordinates \((\eta, r)\). Since \( a(\eta) = -1/H\eta \) during the inflationary era, the above equation implies that for fixed \( r \) the mode of \( l \) newly appears at the moment \( \eta_l(r) \equiv -(r/l_sH)\sqrt{4\pi/(l+1)} \).

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\( ^{7} \)The number of independent coefficients \( f_{i_1i_2...i_l}^{lm} \) \((0 \leq l \leq N)\) is given by \( \sum_{l=0}^{N}(l+2C_2 - lC_2) = \sum_{l=0}^{N}(2l+1) = (N+1)^2 \). From this we can see that with \( f_{i_1i_2...i_l}^{lm} \) \((0 \leq l \leq N)\) \( \hat{Y}_{lm} \) form a complete basis of \((N+1) \times (N+1)\) hermitian matrices.
We now rewrite the scalar field action \( S[\hat{\phi}] \) on this noncommutative space-time:

\[
S[\hat{\phi}] = \frac{1}{2} \int d\eta \, dr \, a^2(\eta) \, r^2 \frac{1}{N + 1} \text{tr} \left[ (\partial_\eta \hat{\phi})^2 - (\partial_r \hat{\phi})^2 + \frac{1}{r^2} [\hat{L}_i, \hat{\phi}]^2 - a^2(\eta) \, \hat{\phi}^2 \, V''(\bar{\phi}) \right]
\]

\[
= \frac{1}{2} \int d\eta \, dr \, a^2(\eta) \, r^2 \sum_{l=0}^{[N(\eta,r)]} \sum_{m=-l}^{l} \left[ |\partial_\eta \phi_{lm}|^2 - |\partial_r \phi_{lm}|^2 - \frac{l(l+1)}{r^2} |\phi_{lm}|^2 - a^2(\eta) \, V''(\bar{\phi}) \right] |\phi_{lm}|^2.
\] (2.18)

Here we replaced \( \int d\Omega/4\pi \) with \( 1/(N + 1) \text{tr} \), and \( i \varepsilon_{ijk} x_j \partial_k \) with \( [\hat{L}_i, \hat{\phi}] \). In the second line, we used the normalization \( \text{tr} \hat{Y}_l^+ \hat{Y}_m = (N + 1) \delta_{ll'} \delta_{mm'} \) and eqs. (2.13), neglecting possible contributions from the boundary. Rewriting the summation as

\[
\int d\eta \, dr \sum_{l=0}^{[N(\eta,r)]} \sum_{m=-l}^{l} \int d\eta \, dr \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \theta(N(\eta,r) - l),
\] (2.19)

we obtain the final form of the action:

\[
S[\hat{\phi}] = \frac{1}{2} \int d\eta \, dr \, a^2(\eta) \, r^2 \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \theta(N(\eta,r) - l)
\times \left[ |\partial_\eta \phi_{lm}|^2 - |\partial_r \phi_{lm}|^2 - \frac{l(l+1)}{r^2} |\phi_{lm}|^2 - a^2(\eta) \, V''(\bar{\phi}) \right] |\phi_{lm}|^2.
\] (2.20)

### 3 Analysis of fluctuations

In this section, we calculate the angular power spectrum \( C_l \) for small \( l \). This is related to the large-scale gravitational potential on the LSS through the Sachs-Wolfe effect. By studying the Einstein equation, it turns out that the gravitational potential on the LSS is determined by the inflaton fluctuations \[24\]. We consider only the Gaussian fluctuations, so the inflaton fluctuations can be calculated by its two-point function. The equation of motion (EOM) for the mode \( \phi_{lm}(\eta,r) \) of an inflaton in a noncommutative background is obtained from the action (2.20) as

\[
\left( \partial^2_\eta - \partial^2_r + \frac{l(l+1)}{r^2} - \frac{2}{\eta^2} \right) \left( a(\eta) \, r \, \phi_{lm}(\eta,r) \right) = 0.
\] (3.1)

Here we used the equality \( \partial^2_\eta a/a = 2/\eta^2 \). We also adopted the slow-roll approximation and neglected the term \( V''(\bar{\phi}) \).
The EOM should be solved with an appropriate boundary condition at the boundary \( \eta = -(r/l_s H) \sqrt{4\pi/(l + 1)} \) in order to calculate the two-point function of \( \phi_{lm}(\eta, r) \) correctly. However, in order to understand the qualitative behavior of the angular power spectrum for lower \( l \), one may simplify the problem with the following assumptions:

- Because the inflaton field in the superhorizon is frozen to the value it takes when crossing the Hubble horizon, those modes that are allowed only in the superhorizon should take the vanishing value also in the superhorizon.
- Those modes that are allowed in the subhorizon should have the same behavior with that in the commutative case.

In the commutative limit, the EOM (3.1) can be solved as

\[
\phi_{lm}(\eta, r) = \int_0^\infty dk \, H\eta^2 \sqrt{\frac{k^3}{\pi}} j_l(kr) \left[ b_{lm}(k) h_1^{(1)}(-k\eta) + (-1)^m b_{l-m}^{\dagger}(k) h_1^{(2)}(-k\eta) \right]
\equiv \int_0^\infty dk \sqrt{\frac{2}{\pi}} k j_l(kr) \phi_{lm}(\eta, k).
\]

Here \( h_1^{(1)} \) and \( h_1^{(2)} \) are the spherical Hankel functions of order 1, and the coefficients \( b_{lm}(k) \) satisfy the commutation relations

\[
\left[ b_{lm}(k), b_{l'm'}^{\dagger}(k') \right] = \delta(k - k') \delta_{ll'} \delta_{mm'},
\]

which are equivalent to the canonical commutation relations for the field \( \phi_{lm}(\eta, r) \):

\[
\left[ \phi_{lm}(\eta, r), \pi_{l'm'}(\eta, r') \right] = i \delta(r - r') \delta_{ll'} \delta_{mm'},
\]

\[
\pi_{lm}(\eta, r) \equiv a^2(\eta) r^2 \partial_\eta \phi_{lm}^*(\eta, r).
\]

We here consider a mode that has a wave number \( k \) and an angular momentum \( l \). This mode crosses the Hubble horizon at \( \eta = \eta_k \equiv -1/k \), where the physical wave length \( a(\eta)/k \) equals the Hubble distance \( 1/H \) (see Fig. 2). Because of our assumptions above, the mode \( \phi_{lm}(\eta, k) \) in the expansion (3.2) can have quantum fluctuations in the subhorizon only when the condition

\[
l \leq N(\eta_k, r)
\]

(3.6)
Figure 2: During the inflationary era, the Hubble radius $1/H$ remains constant. The physical wave length $a(\eta)/k$ of a mode in the subhorizon crosses the Hubble horizon at $\eta = \eta_k$ and enters the superhorizon (unshaded in the figure).

is satisfied. Since we are interested in the fluctuations of the CMB, we should evaluate $N(\eta_k, r)$ at $r$ which corresponds to the LSS. As can be seen from Fig. 3 a photon travels from a point on the LSS to the observer along the line determined by $d\eta = -dr$, and thus the comoving distance between the observer ($r = 0$) and the point on the LSS is given by the difference of their conformal times, $r = \eta_0 - \eta_s \equiv \Delta \eta_s$ (see Fig. 3). Therefore when a mode with wave number $k$ crosses the Hubble horizon, the angular-momentum cutoff is given by

$$N(\eta_k, \Delta \eta_s) = \frac{4\pi k^2 (\Delta \eta_s)^2}{l_s^2 H^2} - 1. \quad (3.7)$$

Thus, for the noncommutative case, we should discard those modes in eq. (3.2) that do not satisfy the condition $l \leq N(\eta_k, \Delta \eta_s)$. This can be done by replacing $b_{lm}(k)$ in eq. (3.2) with $b_{lm}(k) \theta(N(\eta_k, \Delta \eta_s) - l)$, and we obtain

$$\phi_{lm}(\eta, \Delta \eta_s) = \int_0^\infty dk \sqrt{\frac{2}{\pi}} k j_l(k \Delta \eta_s) \phi_{lm}(\eta, k) \theta(N(\eta_k, \Delta \eta_s) - l)$$

$$= \int_{k_+(l)}^\infty dk \sqrt{\frac{2}{\pi}} k j_l(k \Delta \eta_s) \phi_{lm}(\eta, k)$$

$$\quad (3.8)$$

with

$$k_+(l) \equiv \frac{l_s H}{\Delta \eta_s} \sqrt{\frac{l + 1}{4\pi}}. \quad (3.9)$$
From this expression we see that the noncommutativity during the inflationary era is translated into the infrared cutoff $k_\ast(l)$ of the $k$ integration. After entering the superhorizon, the mode $\phi_{lm}(\eta, k)$ becomes “classical” with the amplitude fixed to the value it takes when crossing the Hubble horizon. By setting $\eta = \eta_k$ in the two-point function, we thus obtain the power spectrum in the superhorizon, which is almost constant in time during the inflationary era:

$$
\langle \phi_{lm}(\eta_k, k) \phi_{l'm'}(\eta_{k'}, k') \rangle = \frac{2\pi^2}{k^3} P_\phi(k) \delta(k - k') \theta(k - k_\ast(l)) \delta_{ll'} \delta_{mm'}.
$$

(3.10)

Here $P_\phi(k)$ is the power spectrum for the commutative case, which is given by

$$
P_\phi(k) = \frac{H^2}{4\pi^2}
$$

(3.11)

in the leading slow-roll approximation.

Before calculating the angular power spectrum, we here make a comment on the mechanism through which a short-distance cutoff can affect the large-scale behavior of the CMB anisotropy. In our subsequent paper [19] we have investigated the possible ways of introducing cutoff into inflationary models such that it exhibits a sharp damping at large angular scales. There we have shown that the damping occurs as a result of the competition between two moments: one is the moment when a mode crosses the
Hubble horizon and becomes a classical fluctuation, and the other is the moment when the mode is released from the constraint of cutoff. The angular power spectrum has a sharp damping at large angular scales when the first moment is prior to the second one only for larger-scale modes. In fact, the CMB angular power spectrum is related to the classical values of quantum fluctuations of inflaton, which will be largely suppressed when the above situation is realized, because then the large-scale modes must become classical before the modes start their quantum fluctuations. We also have shown there that this situation is realized when the noncommutativity is introduced to the angular directions.

We now calculate the angular power spectrum $C_l$. This can be carried out simply by following the usual prescription for obtaining $C_l$ from the two-point function of inflaton \[24\]. We first parametrize the metric under scalar perturbations in the longitudinal gauge \[25\] \[26\] \[27\]:

$$ds^2 = a^2(\eta)\left[-\left(1 + 2\Psi(\eta, \vec{x})\right)d\eta^2 + \left(1 + 2\Phi(\eta, \vec{x})\right)\delta_{ij}dx^idx^j\right] \quad (3.12)$$

with $\Psi$ and $\Phi$ being gravitational potentials. We assume that the cosmological perturbation \[25\] \[26\] \[27\] \[24\] is still applicable since the space-time noncommutativity disappears rapidly, so that the relation $\Phi = -\Psi$ holds when the anisotropic stress-tensor vanishes. If we further assume that the perturbations are adiabatic, then the combination $\mathcal{R}(\vec{x}) = \Psi(\eta, \vec{x}) - (\partial_\eta a/a)\nu(\eta, \vec{x})$ gives a constant of motion on the superhorizon scale and is called the curvature perturbation. Here the velocity field $\nu(\eta, \vec{x})$ is defined through the $(\eta, i)$-component of the energy-momentum tensor as $T_{ni} = -a^2(\bar{\rho} + \bar{p})\partial_i\nu$ with the unperturbed energy density and pressure, $\bar{\rho}(\eta)$ and $\bar{p}(\eta)$. We have $\mathcal{R}(\vec{x}) = -(\partial_\eta a/a)\nu(\eta, \vec{x}) = H\phi(\eta, \vec{x})/\dot{\phi}(\eta)$ during the inflationary era, and $\mathcal{R}(\vec{x}) = (5/3)\Psi(\vec{x})$ during the matter-dominated era. Thus, by expanding $\Psi(\eta, \vec{x}) = \Psi(\eta, r, \Omega)$ with respect to the spherical harmonics (and further to the spherical Bessel functions) as

$$\Psi(\eta, r, \Omega) = \sum_{l=0}^{\infty} \sum_{m} \Psi_{lm}(\eta, r) Y_{lm}(\Omega) = \sum_{l=0}^{\infty} \sum_{m} \int_{0}^{\infty} dk \Psi_{lm}(\eta, k) \sqrt{\frac{2}{\pi}} k j_l(kr) Y_{lm}(\Omega),$$

(3.13)

the gravitational potential on the LSS, $\Psi_{lm}(\eta_s, k)$, is expressed as \[24\]

$$\Psi_{lm}(\eta_s, k) = \frac{3}{5} \frac{H}{\dot{\phi}(\eta_k)} \phi_{lm}(\eta_k, k). \quad (3.14)$$
The power spectrum $P_\Psi(k)$ of the gravitational potential,

$$\langle \Psi_{lm}(\eta_0, k) \Psi_{l'm'}(\eta_0, k') \rangle = \frac{2\pi^2}{k^3} P_\Psi(k) \delta(k - k') \theta(k - k_*(l)) \delta_{l'l} \delta_{m'm'}, \quad (3.15)$$

is then expressed as

$$P_\Psi(k) = \left( \frac{3H}{5\dot{\phi}(\eta_k)} \right)^2 P_\phi(k) \quad \left( = \left( \frac{3H^2}{10\pi\dot{\phi}(\eta_k)} \right)^2 \text{in the leading slow-roll approximation} \right). \quad (3.16)$$

Furthermore, through the Sachs-Wolfe effect [13][24], the temperature fluctuations in the CMB,

$$\frac{\delta T(\eta, \Omega)}{T} = \sum_{l=0}^{\infty} \sum_{m} a_{lm}(\eta) Y_{lm}(\Omega), \quad (3.17)$$

are related to the gravitational potential:

$$a_{lm}(\eta_0) = \frac{1}{3} \Psi_{lm}(\eta_0, \Delta \eta_0)$$

$$= \frac{1}{3} \int_0^{\infty} dk \Psi_{lm}(\eta_0, k) \sqrt{\frac{2}{\pi}} k j_l(k\Delta \eta_0)$$

$$= \frac{1}{3} \int_{k_*(l)}^{\infty} dk \Psi_{lm}(\eta_0, k) \sqrt{\frac{2}{\pi}} k j_l(k\Delta \eta_0). \quad (3.18)$$

We here recall that the spectral index $n(k)$ is defined as

$$n(k) = \frac{d\log P_\Psi(k)}{d\log k} + 1. \quad (3.19)$$

In the superhorizon, $P_\Psi(k)$ depends on $k$ through $\dot{\phi}(\eta_k)$ and also through the potential term which we neglected in solving the EOM, and thus, in the leading slow-roll approximation we have $n = 1$, i.e., the power-spectrum $P_\Psi(k)$ is scale-invariant. However, when noncommutativity is taken into account, an IR cutoff is introduced into the $k$ integration.

The angular power spectrum $C_l$ is calculated from eqs. (3.15), (3.16) and (3.18) as

$$\langle a_{lm}^*(\eta_0) a_{l'm'}(\eta_0) \rangle = \frac{4\pi}{9} \delta_{l'l} \delta_{m'm'} \int_{k_*(l)}^{\infty} \frac{dk}{k} \left( j_l(k\Delta \eta_0) \right)^2 P_\Psi(k)$$

$$\equiv C_l \delta_{l'l} \delta_{m'm'}. \quad (3.20)$$
When the spectral index $n(k)$ is constant, i.e., $P_\Psi(k) = P_0 k^{n-1}$ ($P_0$: constant), a simple calculation gives

$$C_l = C_{l}^{(0)} (1 - \beta_l),$$

where $C_{l}^{(0)}$ represents the angular power spectrum of the commutative case:

$$C_{l}^{(0)} = \frac{\pi^{3/2}}{9} (\Delta \eta_*)^{1-n} P_0 \frac{\Gamma \left( \frac{3-n}{2} \right) \Gamma \left( l + \frac{n-1}{2} \right)}{\Gamma \left( \frac{1-n}{2} \right) \Gamma \left( l + \frac{3-n}{2} \right)},$$

and the deviation is expressed by

$$\beta_l = \frac{4}{\sqrt{\pi}} \frac{\Gamma \left( \frac{4-n}{2} \right) \Gamma \left( l + \frac{5-n}{2} \right)}{\Gamma \left( \frac{3-n}{2} \right) \Gamma \left( l + \frac{n-1}{2} \right)} \int_0^{k_*(l)} dx x^{n-2} \left( j_1(x) \right)^2.$$ 

When $n = 1$, $C_{l}^{(0)} = \text{const.} \left[ l(l+1) \right]^{-1}$ and thus $l(l+1)C_{l}^{(0)}$ does not depend on $l$. However, the second term $\beta_l$ in (3.21) does depend on $l$, and makes $l(l+1)C_l$ have a sharp damping at small $l$. The result with $n = 0.95$ is depicted for various values of $l_s H$ in Fig. 4.

![Figure 4](image)

**Figure 4:** Results of the calculation of the ratio $C_l/C_{l}^{(0)} = 1 - \beta_l$. The vertical axis is the ratio $C_l/C_{l}^{(0)}$, and the horizontal axis is angular momentum $l$. We set $n = 0.95$ and change $l_s H$ from 0.1 to 10 (curves in the figure have $l_s H = 0.1, 1, 5$ and 10 from top to bottom). There is a damping at small $l$, and the ratio approaches 1 for large $l$. 

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4 Conclusion and outlook

In this paper, we considered an inflationary universe assuming that the geometry is expressed by a noncommutative space-time in the very early universe. We only take into account the noncommutativity of the angular coordinates $\Omega = (\theta, \varphi)$ since these should yield the most relevant effects on the deviation of the angular power spectrum from the standard value. Instead of imposing boundary conditions, we solved the EOM by simply discarding those modes that are not allowed to exist in the subhorizon.

We calculated the two-point function. As depicted in Fig. 4, the result shows that the angular power spectrum certainly has a damping for small $l$. This has the same features with those observed in the WMAP and the COBE. This damping is usually interpreted based on the cosmic variance, but we showed that it has a possibility to be explained as effects of the noncommutativity of space-time during the inflationary era.

In this paper we have considered only scalar perturbations, but the above derivation can be used for any Gaussian fluctuations in the exponentially expanding universe, so that the tensor fluctuations will also have the same shape of damping at small $l$. We expect that this behavior will be observed in experiments of the near future.

As stated above, instead of solving the EOM with proper boundary conditions we made an analysis of the classical solution in the superhorizon simply by discarding the modes that do not exist in the subhorizon. It should be enough for a qualitative argument, but in order to obtain a precise prediction that can be compared to the experimental data, we need to solve the problem by carefully choosing boundary conditions. Work in this direction is now in progress and will be reported in the subsequent paper [19].

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