On Capacity of
the Writing onto Fast Fading Dirt Channel

Stefano Rini and Shlomo Shamai (Shitz)

Abstract

The “Writing onto Fast Fading Dirt” (WFFD) channel is investigated to study the effects of partial channel knowledge on the capacity of the “writing on dirty paper” channel. The WFFD channel is the Gel’fand-Pinsker channel in which the output is obtained as the sum of the input, white Gaussian noise and a fading-times-state term. The fading-times-state term is equal to the element-wise product of the channel state sequence, known only at the transmitter, and a fast fading process, known only at the receiver. We consider the case of Gaussian distributed channel states and derive an approximate characterization of capacity for different classes of fading distributions, both continuous and discrete. In particular, we prove that if the fading distribution concentrates in a sufficiently small interval, then capacity is approximately equal to the AWGN capacity times the probability of this interval. We also show that there exists a class of fading distributions for which having the transmitter treat the fading-times-state term as additional noise closely approaches capacity. Although a closed-form expression of the capacity of the general WFFD channel remains unknown, our results show that the presence of fading can severely reduce the usefulness of channel state knowledge at the transmitter.

Index Terms

Gel’fand-Pinsker Channel; Writing on Fading Dirt Channel; Fast Fading; Partial Channel Side Information; Costa pre-coding; Interference pre-cancellation

INTRODUCTION

The classic “Writing on Dirty Paper” (WDP) channel capacity result [1] establishes that full state pre-cancellation can be attained in the Gel’fand-Pinsker (GP) channel with additive state and additive white Gaussian noise, regardless of the distribution of the state sequence. Albeit

The work of S. Rini was funded by the Ministry Of Science and Technology (MOST) under the grant 103-2218-E-009-014-MY2. The work of S. Shamai was supported by the Israel Science Foundation (ISF) and by the European FP7 NEWCOM#.
very promising, this result assumes that perfect channel knowledge is available at the users: this assumption does not hold in many communication scenarios in which channel conditions vary over time and with limited feedback between the receiver and the transmitter. For this reason, we wish to investigate the effects of partial channel knowledge on the performance of state pre-cancellation. More specifically, we study the capacity of the “Writing onto Fast Fading Dirt” (WFFD) channel, a variation of Costa’s WDP channel in which the state sequence is multiplied by a fast fading process known only at the receiver.

The WFFD channel models the downlink transmission scenario in which a base station wishes to communicate to a receiver in the presence of an interferer. The base station acquires the message sent by the interferer through the network architecture while the receiver learns the channel toward the interferer from the pilot tones broadcasted by the interferer. Due to rate limitations in the control and feedback channels, the transmitter and the receiver are unable to exchange each other’s knowledge. This, therefore, results in the situation in which the transmitter knows the interfering message but not the interfering channel, while the receiver knows the interfering channel but not the interfering message. For this scenario, one wishes to determine the limiting interference cancellation performance that are attainable despite the partial and asymmetric system knowledge at the transmitter and the receiver.

Related Results: The GP channel \[2\] is the point-to-point channel in which the output is obtained as a random function of the input and a state sequence which is provided non-causally at the transmitter. The capacity of this model is expressed in \[2\] as the maximization of a non-convex function for which the optimal solution is not easily determined, either explicitly or through numerical evaluations. For this reason, very few closed-form expressions of the GP channel capacity are available in the literature. One of the few models for which capacity is known in closed-form is the WDP channel: in \[1\] Costa shows, perhaps surprisingly, that the capacity of the WDP channel is equal to the capacity of the Gaussian point-to-point channel. This result implies that it is possible for the encoder to fully pre-code its transmissions against the known channel state. In the literature, few authors have investigated extensions of the result in \[1\] to include fading and partial channel knowledge.

The “Carbon Copying onto Dirty Paper” (CCDP) channel \[3\] is the \(M\)-user compound channel in which the output at each compound receiver is obtained as the sum of the input, Gaussian noise and one of \(M\) possible state sequences, all non-causally known only at the transmitter. When the state sequences at each receiver are scaled versions of the same sequence, the CCPD
channel models the WDP channel in which the channel state is multiplied by a slow fading process.

The WDP channel in which both the input and the state sequence are multiplied by the same fading realization is studied in [4]. The authors consider both the case of fast and slow fading and evaluate the achievable rates using Costa pre-coding, showing that the rate loss from full state pre-cancellation is vanishing in both scenarios as the transmit power grows to infinity. The model in which the same fading realization multiplies both the input and the state is closely related to the fading broadcast channel as, in the latter model, the channel state models the codeword intended for another user in the network: in [5] and [6], lattice coding strategies are employed to derive an achievable region for this model.

The WDP channel in which slow fading affects only the state sequence is first studied in [7] for the case of phase fading. In [7], the first inner and outer bounds to capacity are obtained while [8] studies the outage probability for this model. In [9], we show the approximate capacity of this model for some classes of the phase fading. Achievable rates under Gaussian signaling are derived in [10] for the case of Gaussian distributed fast fading. These attainable rates are also compared to lattice coding strategies and some numerical observations are provided on the performance of these various coding choices. The performance of lattice coding strategies for this channel model is further studied in [11], [12] and [13]. In [11], the authors also derive an upper bound on the maximum achievable transmission rates which is shown to be tight in some settings. In [12], it is shown that randomizing the state scaling at the transmitter can improve the performance over a pre-determined scaling choice.

Contributions: We investigate the capacity of the WFFD channel\footnote{In the literature this channel has also been referred to as “dirty paper channel with fading dirt”, “writing on fast faded dirt” and “dirty paper coding channel with fast fading”. We prefer the term “writing on fast fading dirt” for both brevity and clarity.} in which the state sequence is a white Gaussian sequence: we consider, separately, the case of discrete and continuous fading distributions. A summary of the contributions for these two scenarios is as follows:

- **Sec. III- Discrete fading distributions.** We begin by determining capacity to within a constant gap for the case of uniform antipodal fading. For this simple fading distribution capacity can be approached by transmitting the superposition of two codewords: the bottom codeword treats the fading-times-state as additional noise while the top codeword is pre-coded against one of the fading realizations times the channel state. This result is extended to two classes of fading distributions: the class of distributions with mode larger than a half and the class of
uniform distributions with exponentially spaced points in the support. In both cases, capacity is approached to within a small gap by a combination of superposition coding and state pre-coding as in the case of uniform antipodal fading.

- **Sec. IV- Continuous fading distributions.** We begin by considering the case of a symmetric continuous fading distribution and show simple conditions under which capacity is at most half of the AWGN capacity. We then derive the approximate capacity for the case of a continuous fading distribution which concentrates around a sufficiently narrow interval. The converse proof is shown by relating the capacity of the model with continuous fading to the capacity of the model in which the fading distribution is a quantized version of the original distribution. Finally, we show that there exists a heavy-tailed fading distribution for which the capacity of the WFFD channel is approximatively equal to the capacity of the channel without state knowledge.

The main theoretical contributions of the paper consist in the development of new outer bounding techniques to characterize the capacity of a model comprising both channel states and partial channel knowledge. On the other hand, the inner bounds used throughout the paper are rather straightforward combinations of Costa pre-coding and superposition coding. From a high level perspective, this shows that in the instances we consider state pre-coding is substantially rendered ineffective by the presence of channel uncertainty. Although this conclusion does not hold in general, our results partially reveal the conditions under which robust state pre-cancellation is no longer possible.

**Paper Organization:** The remainder of the paper is organized as follows: in Sec.I we introduce the channel model under consideration. Sec.II presents relevant results available in the literature. Sec.III considers the case of discrete fading distributions while Sec.IV studies the case of continuous distributions. Finally, Sec.V concludes the paper.

### I. CHANNEL MODEL

The WFFD channel, also depicted in Fig.I, is the GP channel in which the output is obtained as

\[ Y^N = X^N + cA^N \circ S^N + Z^N, \]  

(1)
where $X^N$ denotes the channel input, $S^N$ the channel state, $A^N$ the fading sequence and $Z^N$ the additive noise while $\circ$ indicates the Hadamard, or element-wise, product. Having knowledge of the channel state $S^N$, the encoder wishes to reliably communicate the message $W \in \mathcal{W} = [1 \ldots 2^{NR}]$ to the receiver through the channel input $X^N$. Upon receiving the channel output $Y^N$ and the fading realization $A^N$, the receiver produces the estimate $\hat{W} \in \mathcal{W}$ of the transmitted message. The channel input $X^N$ is subject to the second moment constraint $\mathbb{E}[|X_i|^2] \leq P$, $\forall i \in [1 \ldots N]$. Both the channel state and the additive noise are white Gaussian sequences, i.e. $Z^N, S^N \sim \text{i.i.d.} \mathcal{N}(0,1)$ while the fading sequence $A^N$ is an i.i.d. sequence from the distribution $P_A(a)$, with support $\mathcal{A}$, either continuous or discrete. Without loss of generality we further assume $\text{Var}[A] = 1$ and $c \in \mathbb{R}^+$. 

Fig. 1: The “Writing onto Fast Fading Dirt” (WFFD) channel.

In the study of the WFFD channel, standard definitions of code, achievable rate and capacity are employed \cite{14}.

**Definition 1.** Code, probability of decoding error, achievable rate. A $(2^{NR}; N)$ code for the WFFD channel consists of an encoding and a decoding function, $X^N = f(W, S^N)$ and $\hat{W} = g(Y^N, A^N)$ respectively. The probability of error for a $(2^{NR}; N)$ code, $P_e(2^{NR}; N)$, is defined as

$$P_e(2^{NR}; N) = \mathbb{P}[\hat{W}(Y^N, A^N) \neq W],$$

where the probability in the RHS of (2) is also averaged over all fading and state sequences and transmitted messages. A rate $R \in \mathbb{R}^+$ is said to be achievable if there exists a sequence of codes such that the probability of error $P_e(2^{NR}; N)$ goes to zero as $N$ goes to infinity.

\footnote{In other words, $A^N \circ S^N = [A_1S_1, A_2S_2 \ldots A_NS_N]^T$.}

\footnote{Note that fading sequence $A^N$ can be seen as an additional channel output at the receiver, together with $Y^N$ in (1).}
Definition 2. **Capacity and approximate capacity.** The capacity $C$ is the supremum of all the achievable rates. An inner bound $R^{\text{IN}}$ and an outer bound $R^{\text{OUT}}$ to capacity for which

$$R^{\text{OUT}} - R^{\text{IN}} \leq \Delta,$$

for some constant $\Delta \in \mathbb{R}^+$, are said to characterize the capacity to within an additive gap of $\Delta$ bits–per–channel–use (bpcu) or, for brevity, to determine the approximate capacity to within $\Delta$ bpcu.

Since the WFFD channel is a special case of the GP channel, its capacity is obtained as

$$C = \max_{P_{U,X|S}} I(Y;U|A) - I(U;S).$$

The expression in (4) is convex in $P_{X|S,U}$ for a fixed $P_{U|S}$ which implies that $X$ can be chosen to be a deterministic function of $U$ and $S$. On the other hand, (4) is neither convex nor concave in $P_{U|S}$ for a fixed $P_{X|S,U}$: accordingly, determining a closed-form solution for the maximization in (4) is generally challenging. Additionally, the lack of tight bounds on the cardinality of the auxiliary random variable $U$ further complicates the task of obtaining numerical approximations of the optimal solution. For these reasons, in the following we provide alternative inner and outer bounds to capacity which are expressed only as a function of $P, c$ and $P_A(a)$. We also determine the approximate capacity for some class of distributions, focusing on those instances in which a simple combination of known achievable strategies is sufficient to achieve capacity.

In the remainder of the paper, we refer to the term $cA^N \circ S^N$ as the “fading-times-state” term and use the parameter $c$ to normalize the variance of both the state and the fading distributions to one.

**Lemma I.1. Mean and variance of the fading-time-state term.** For the model in (1), the variance of the state and the fading distributions are taken unitary without loss of generality. Also, the channel state is taken to have zero mean and $c \in \mathbb{R}^+$ without loss of generality.

**Proof:** The proof is omitted for brevity.

The following lemma is useful in the tightening of certain outer bounds derived in the following.

**Lemma I.2.** The capacity of the WFFD channel is decreasing in the parameter $c$. 

**Proof:** The proof is presented in App. A.

II. RELATED RESULTS

This section briefly introduces some results available in the literature which are relevant to the study of the WFFD channel.

- **The “Writing on Dirty Paper” (WDP) channel.** One of the few GP channel models for which the maximization in (4) is known in closed-form is the WDP channel. For this model, the optimal assignment in (4) is

  \[ X \sim \mathcal{N}(0, P), \quad X \perp S \]

  \[ U = X + \frac{P}{P+1} S, \]

and yields \( C = 1/2 \log(1 + P) \), regardless of the distribution of \( S^N \). The assignment in (5) is usually referred to as “Dirty Paper Coding” (DPC).

- **The “Carbon Copying onto Dirty Paper” (CCDP) channel.** The CCDP channel [3] is the \( M \)-user compound channel in which a channel output is obtained as the sum of the input, Gaussian noise and one of \( M \) Gaussian state sequences. The transmitter has non-causal knowledge of all of the \( M \) state sequences while the receivers have no additional knowledge. More specifically,

  \[ Y^N_m = X^N + c S^N_m + Z^N_m, \quad m \in [1 \ldots M], \]

where \( S^N_m \sim i.i.d. \mathcal{N}(0, Q_m) \) and \( \{S^N_m, m \in [1 \ldots M]\} \) have any jointly Gaussian distribution. In [3], the authors derive the first inner and outer bound for this model. The approximate capacity for the case of \( M = 2 \) and independent, unitary variance \(^4\) is derived as [15].

**Theorem II.1. Outer bound and approximate capacity for the 2-user CCDP channel with independent states [15].** The capacity of the 2-user CCDP channel with \( S^N_1, S^N_2 \sim i.i.d. \mathcal{N}(0, 1), S^N_1 \perp S^N_2 \) is upper bounded as

\[
C \leq R^{\text{OUT}} = \left\{ \begin{array}{ll}
\frac{1}{2} \log (1 + P) + 1/2 & c^2 \leq 2 \\
\frac{1}{2} \log \left( \frac{P + c^2/2 + 1}{c^2} \right) + 1/2 & 2 \leq c^2 < 2(P + 1) \\
\frac{1}{4} \log (P + 1) & c^2 \geq 2(P + 1),
\end{array} \right.
\]  

\(^4\)As in Lem. [41] the assumption of unitary variance is without loss of generality.
and the capacity is to within 1 bpcu from the outer bound in (7).

Capacity in Th. II.1 is approached by sending the superposition of two codewords: the base codeword treats the states as additional noise while the top codeword is pre-coded against each of the state realizations for half of the time. Th. II.1 shows that it substantially not possible to simultaneously pre-code the channel input against two independent channel states.

- **Writing onto Fast Fading Dirt (WFFD) channel.** For the WFFD channel with Gaussian fading, the authors of [5] optimize the achievable strategy in (4) over all jointly Gaussian distributions of $S, U$ and $X$.

Theorem II.2. Achievability with jointly Gaussian signaling [5, Sec. IV] [10, Th. 1]. Consider the WFFD channel for $A^N \sim$ i.i.d. $\mathcal{N}(0, 1)$ and let $\rho = (\rho_{XS}, \rho_{US}, \rho_{UX})$ and define $K \subset [-1, 1]^3$ as the region

$$K = \left\{ \begin{array}{l} |\rho_t| < 1 \quad t \in \{XS, US, UX\} \\ 1 + 2\rho_{XS}\rho_{US} - \rho_{XS}^2 - \rho_{US}^2 - \rho_{UX}^2 = 0, \end{array} \right\}$$

then an inner bound to capacity is

$$C \geq R^\text{IN} = \max_{\rho \in K} \mathbb{E}_{\theta} [R_\Gamma(\rho, \theta) | A = \theta],$$

for

$$R_\Gamma(\rho, \theta) = \frac{1}{2} \log \left( (P + c^2 + 2\theta \rho_{XS}c\sqrt{P} + 1)(1 - \rho_{US}^2) \right) - \frac{1}{2} \log \left( P(1 - \rho_{UX}^2) + c^2(1 - \rho_{US}^2) + 2\theta c(\rho_{XS} - (\rho_{UX}\rho_{US}))\sqrt{P} + 1 \right).$$

Th. II.2 attempts to generalize the result of [1] to the Gaussian fast fading case although, in all likelihood, one needs to consider a wider class of distributions than the jointly Gaussian distributions to attain maximum in (4).

III. WFFD CHANNEL WITH A DISCRETE FADE DISTRIBUTION

1) **Antipodal fading:** We begin by providing the approximate capacity for the WFFD channel in which the fading is uniformly distributed over the set $\{-1, +1\}$. This is perhaps the simplest choice of fading distribution for this model, yet this example well illustrates the main bounding techniques necessary to characterize the capacity.
Theorem III.1. Outer bound and approximate capacity of the WFFD channel with antipodal uniform fading. Consider the WFFD channel in which $A$ is uniformly distributed over the set $\{-1, +1\}$, then the capacity $C$ is upper bounded as

$$
C \leq R_{\text{OUT}} = \begin{cases} 
\frac{1}{2} \log(P + 1) + \frac{1}{2} & c^2 \leq 1 \\
\frac{1}{2} \log(P + c^2 + 1) & > \frac{1}{2} \\
-\frac{1}{4} \log(c^2) - \frac{1}{2} & 1 < c^2 < P + 1 \\
\frac{1}{4} \log(P + 1) - \frac{1}{2} & c^2 \geq P + 1,
\end{cases}
$$

(11)

and the capacity is to within 1 bpcu from the outer bound in (11).

Proof: The achievability proof relies on a simple combination of superposition coding and DPC. In the converse proof we define a “conjugate” sequences to the fading realization $A^N = a^N$, $\overline{a}^N(a^N) = -a^N$, and exploit the fact that correct decoding must occur whether $A^N = a^N$ or $A^N = \overline{a}^N$.

- **Achievability.** Consider the achievable strategy in which the channel input is obtained as the superposition of two codewords: (i) the codeword $X_{\text{SAN}}^N$ (for State As Noise), at rate $R_{\text{SAN}}$, which treats $cA^N \circ S^N$ as additional noise and (ii) the codeword $U_{\text{PAS}}^N$ (for Pre-coded Against the State), at rate $R_{\text{PAS}}$, which is pre-coded against $S^N$ as in the WDP channel. This strategy attains the rate $R_{\text{IN}} = R_{\text{SAN}} + R_{\text{PAS}}$ for

$$
R_{\text{SAN}} \leq I(Y; X_{\text{SAN}} | A) \\
R_{\text{PAS}} \leq I(Y; U_{\text{PAS}} | X_{\text{SAN}}, A) - I(U_{\text{PAS}}; S).
$$

(12)

Through the assignment

$$
X_{\text{SAN}} \sim \mathcal{N}(0, \alpha P), \quad X_{\text{PAS}} \sim \mathcal{N}(0, \overline{\alpha} P), \quad X_{\text{SAN}} \perp X_{\text{PAS}} \\
X = X_{\text{SAN}} + X_{\text{PAS}}, \quad U_{\text{PAS}} = X_{\text{PAS}} + c\frac{\overline{\alpha} P}{\alpha P + 1} S,
$$

(13)

for any $\alpha \in [0, 1]$ and $\overline{\alpha} = 1 - \alpha$. By further bounding the expressions in (12) for the assignment in (13), we obtain the achievable rate

$$
R_{\text{IN}}^\alpha \geq \frac{1}{2} \log \left( 1 + \frac{\alpha P}{1 + \overline{\alpha} P + c^2} \right) + \frac{1}{4} \log(\overline{\alpha} P + 1) - \frac{1}{2}.
$$

(14)
Optimizing the expression in (14) over $\alpha$ results in the inner bound

$$R_{IN} = \begin{cases} 
\frac{1}{2} \log (1 + P) - \frac{1}{2} & c^2 \leq 1 \\
\frac{1}{2} \log (1 + P + c^2) & \\
-\frac{1}{4} \log (c^2) - 1 & 1 < c^2 < P + 1 \\
\frac{1}{2} \log (1 + P) - 1 & c^2 \geq P + 1.
\end{cases}$$

(15)

**Converse.** Fano’s inequality yields the upper bound

$$N(R - \epsilon_N) \leq I(Y^N; W | A^N)$$

$$\leq \sum_{j=1}^{N} H(Y_j | A_j) - H(Y^N | A^N, W)$$

$$\leq N \max_j H(Y_j | A_j) - H(Y^N | A^N, W)$$

$$\leq N \max_{P_{Y|A}} H(Y | A) - H(Y^N | A^N, W).$$

(16)

The entropy term $\max_{P_{Y|A}} H(Y | A)$ in (16) is further bounded as

$$\max_{P_{Y|A}} H(Y | A) \leq \max_{P_{Y|A}} \frac{1}{2} \left( H(X + cS + Z) + H(X - cS + Z_j) \right)$$

$$\leq \max_{|\rho_{XS}| \leq 1} \frac{1}{2} \left( H(X_{Gj} + cS + Z) + H(X_{Gj} - cS + Z_j) \right),$$

(17)

where (17) follows from the Gaussian Maximizes Entropy (GME) property by letting $X_{Gj}$ be jointly Gaussian random variables with variance $P$ and with correlation $\rho_{XS}$ with $S$. Optimizing (17) over $\rho_{XS}$ yields the upper bound

$$\max_{P_{Y|A}} H(Y | A) \leq \frac{1}{2} \log(2\pi e)^2 \left( P + c^2 + 1 \right),$$

(18)

where the maximum in (17) is attained for $\rho_{XS} = 0$. Define now $\bar{a}^N(a^N) = -a^N$ and notice that

$$H(Y^N | W, A^N) = \frac{1}{2} \sum_{a^N \in \{-1, +1\}^N} \frac{1}{2N} \left( H(Y^N | W, A^N = a^N) + H(Y^N | W, A^N = -a^N) \right),$$

(19)

so that

$$-H(Y^N | W, A^N) \leq -\frac{1}{2N+1} \sum_{a^N \in \{-1, +1\}^N}$$
\[
H(X^N + ca^N \circ S^N + Z^N, X^N + ca^N \circ S^N + Z^N | W)
= -\frac{1}{2^{N+1}} \sum_{a^N \in \{-1,1\}^N} 
\left( H(2ca^N \circ S^N | W)
+ H(X^N + ca^N \circ S^N + Z^N | S^N, W) \right)
\]
(20a)
\[
= -\frac{1}{2^{N+1}} \sum_{a^N \in \{-1,1\}^N} 
\left( H(2ca^N \circ S^N)
+ H(X^N + ca^N \circ S^N + Z^N | S^N, W, X^N) \right)
\]
(20b)
\[
= -\frac{1}{2^{N+1}} \sum_{a^N \in \{-1,1\}^N} 
H(2ca^N \circ S^N) + H(Z^N),
\]
(20d)

where (20a) follows from the fact that transformation
\[
\begin{bmatrix}
T_{1i} \\
T_{2i}
\end{bmatrix} = \begin{bmatrix}
+1 & -1 \\
1 & 0
\end{bmatrix} \cdot 
\begin{bmatrix}
X_i + ca_i S_i + Z_i \\
X_i - ca_i S_i + Z_i
\end{bmatrix},
\]
(21)
has unitary Jacobian. The equality in (20b) follows from the fact that \( W \perp S^N \), (20c) from the fact that \( X^N \) is a function of \( W \) and \( S^N \) and from the Markov chain \( W - [X^N, S^N] - Y^N \). Next, we observe that the terms in the summation in the RHS of (20d) are all identical and equal to
\[
\frac{1}{2} \log(2\pi e c^2) + \frac{1}{2} \log(2\pi e),
\]
so that
\[
-H(Y^N | W, A^N) \leq -\frac{N}{4} \log(2\pi ec^2) - \frac{N}{4} \log(2\pi e) - \frac{N}{2}.
\]
(22)

Using (18) and (22) we rewrite the outer bound in (16) as
\[
R^{\text{OUT}} = \frac{1}{2} \log(2\pi e)^2 \left( P + c^2 + 1 \right)
- \frac{1}{4} \log(2\pi ec^2) - \frac{1}{4} \log(2\pi e) - \frac{1}{2}
= \frac{1}{2} \log(P + c^2 + 1) - \frac{1}{4} \log(c^2) - \frac{1}{2}.
\]
(23)

Note that, as a function of \( c^2 \), the expression in (23) has a minimum in \( c^2 = P + 1 \). From Lem. 12 we have the capacity is decreasing in \( c \): for this reason, the channel in which \( c^2 \) is equal to \( \min\{c^2, P + 1\} \) corresponds to a model with larger capacity. For this latter model, the outer
bound in (23) still holds so that letting $c^2$ equal to $\min\{c^2, P + 1\}$ in (23) provides an outer bound to the capacity of the original model. With this substitution and some further bounding for the case $c^2 < 1$, we obtain the outer bound in (11). By comparing the outer bound in (11) and the inner bound in (15), we verify that the they differ of at most 1 bpcu.

The result in Th. III.1 is conceptually simple but provides insights on more general scenarios. The parameter $c$ controls the variance of the fading-time-state term: (i) for small values of $c$, treating the term $cA^N \circ S^N$ as additional noise results in a limited rate loss. (ii) when the variance of $cA^N \circ S^N$ is larger than the transmit power, then it is approximately optimal to pre-code against one fading realization, as this strategy grants correct decoding for half of the channel uses on average. Finally, (iii) in the intermediate regime capacity is approached by a combination of the previous two strategies.

**Remark III.2.** The approximate capacity result for the two-user CCDP channel with independent, equal-variance states in Th. III.1 has interesting similarities to the proof of Th. III.1 and the approximate capacity expressions in (7) and (11) are also similar. The achievability proof for both Th. II.2 and Th. III.1 relies on a combination of superposition coding and DPC while, for the converse proof, the outer bound is tightened by using the fact that capacity is decreasing in the parameter $c$. Despite these similarities, the two channel models are fundamentally different: from a high level perspective, in the WFFD channel each fading realization can be thought of as a compound user in the CCDP channel so that the number of compound user grows with the transmission length, instead of being constant.

2) WFFD channel with a discrete fading distribution with mode larger than half:

**Theorem III.3.** Outer bound and approximate capacity for the WFFD channel with a fading distribution of mode larger than half. Consider the WFFD channel in which $P_A(a)$ is a discrete distribution such that

$$\exists m \in A, \text{ s.t. } P_A(m) \geq \frac{1}{2},$$

and let $Q_m = P_A(m)$ and $\overline{Q}_m = 1 - Q_m$, then the capacity $C$ is upper bounded as

$$C \leq R^{\text{OUT}} =$$

(25)
\[
\begin{aligned}
\frac{1}{2} \log(1 + P) + 1 & \quad \overline{Q}_m \geq Q_m c^2(1 + \mu^2_A) \\
\frac{1}{2} \log(1 + P) & \quad \overline{Q}_m < Q_m c^2(1 + \mu^2_A) \leq \overline{Q}_m(P + 1) \\
-\frac{Q_m}{2} \log(c^2(1 + \mu^2_A)) + G_m & \quad Q_m c^2(1 + \mu^2_A) > \overline{Q}_m(P + 1),
\end{aligned}
\]

for
\[
G_m = \frac{1}{2} \mathbb{E}_A \left[ \log \left( \frac{1 + \mu^2_A}{(A - m)^2} \right) | A \neq m \right] + 3,
\]
and the capacity is to within a gap of
\[
G'_m = \frac{1}{2} \mathbb{E}_A \left[ \log \left( (1 + \mu^2_A) \left( \frac{1}{A^2} + \frac{1}{(A - m)^2} \right) \right) | A \neq m \right] + 3,
\]
from the outer bound in (25).

**Proof:** For the class of fading distributions in (24), state pre-cancellation can be attained for a portion \(Q_m\) of the channel uses on average: for this reason, the achievable strategy employed in the proof of Th. III.3 is still effective. In the converse proof, we extend the idea of conjugate fading sequences in the proof of Th. III.1 to the elements in the set of typical fading realizations. The full proof can be found in App. B.

The next lemma provides a simplification of the result in Th. III.1 under some conditions on the support of \(A\).

**Lemma III.4.** If \(|A| > \Delta\) and \(|A - m| \geq \Delta\) for some \(\Delta > 0\), then (26) and (27) satisfy
\[
G_m \leq \frac{1}{2} \mathbb{E}_A \log \left( \frac{1 + \mu^2_A}{\Delta^2} \right) + 3
\]
\[
G'_m \leq G_m + \frac{1}{2}.
\]

**Proof:** The proof is omitted for brevity.

Lem. III.4 shows that a tight characterization of capacity is possible when the mean of \(A\) is small and the points in the support are sufficiently far from the mode of the distribution.

As an example of the result in Lem. III.4, consider the case in which \(A\) has geometric distribution, i.e.
\[
P[A = k\Delta] = \overline{q}^k q,
\]
for \( k \in \mathbb{N}, q \geq 1/2 \) and \( \Delta = q/\sqrt{\gamma} \) to obtain unitary variance: in this case, we have \( G_m \leq 3.15 \) and \( G'_m \leq 3.65 \).

3) WFFD channel in the “strong fading” regime:

**Theorem III.5. Outer bound and approximate capacity for the WFFD channel in the “strong fading” regime.** Consider the WFFD channel with \( c > 2 \) and in which \( A \) is uniformly distributed over a discrete set \( A = \{\alpha_i\}_{i=1}^{M}, \alpha_1 < \alpha_2 < \ldots < \alpha_M \) such that

\[
\alpha_1 \geq \frac{1}{c-1},
\]

\[
\alpha_{i+1} \geq c\alpha_i, \quad i \in [2 \ldots M - 1],
\]

then, the capacity \( \mathcal{C} \) is upper bounded as

\[
\mathcal{C} \leq R_{\text{OUT}} = \begin{cases} 
\frac{1}{2} \log (1 + P) + G_s & \frac{1}{M} \geq \frac{(1 + \mu_A^2)c^2}{M} \\
\frac{1}{2} \log (1 + P + (1 + \mu_A^2)c^2) & \frac{1}{M} < \frac{(1 + \mu_A^2)c^2}{M} \leq \frac{M-1}{M}(P + 1) \\
-\frac{M-1}{2M} \log ((1 + \mu_A^2)c^2) + G_s & \frac{(1 + \mu_A^2)c^2}{M} > \frac{M-1}{M}(P + 1),
\end{cases}
\]

for

\[
G_s = \frac{1}{2} \log(1 + \mu_A^2) + \frac{1}{2},
\]

and the capacity is to within a gap of \( G_s + 1 \) from the outer bound in (29).

**Proof:** As for Th. III.1, the achievability proof relies on the simple combination of superposition coding and DPC. The converse bound involves defining \( M - 1 \) conjugate sequences which are used to recursively bound the channel capacity by also providing a carefully-chosen genie-aided side information. The proof is provided in App. C.

**Remark III.6.** The result in Th. III.5 can be generalized to the case in which \( |a_{i+1}| \geq \kappa c|a_i| \) for some \( \kappa \in \mathbb{R}^+ \): in this case, (30) is expressed

\[
G_s = \frac{1}{2} \log \kappa(1 + \mu_A^2) + \frac{1}{2}.
\]

As an example of the result in Th. III.5 consider the case in which \( A \) is uniformly distributed
over the set

\[ \mathcal{A}(M) = \{ \Delta, c\Delta, c^2\Delta, \ldots, c^{M-1}\Delta \} , \]

for \( M \geq 3 \) where \( \Delta \) is chosen so as to obtain unitary variance\(^5\), then \( G_s = \frac{3}{2}, G'_s = \frac{5}{2} \). Note that the capacity goes to zero when both \( c \) and \( M \) grow to infinity.

**IV. WFFD CHANNEL WITH A CONTINUOUS FADING DISTRIBUTION**

The results derived in the previous section are limited to the case of discrete fading distributions: although relevant from a theoretical standpoint, this scenario is not particularly meaningful in practical applications. In this section, we show how the case of a discrete fading distribution aids the study of a continuous fading distribution. In particular, we show that an outer bound to the capacity of the WFFD channel with a continuous fading distribution can be derived by considering the model with a discrete fading distribution obtained by quantizing the continuous one.

A first upper bound on the capacity of the WFFD channel with continuous fading can be obtained by adapting the result in Th. III.1 to the case of a symmetric distribution.

**Lemma IV.1. Outer bound for symmetric fading distribution.** Consider the WFFD channel in which \( P_A(a) \) is continuous and symmetric, then the expression in (11) is an outer bound to capacity.

**Proof:** The converse proof of Th. III.1 can be adapted to the case of continuous distributions by similarly defining a conjugate sequence \( a^N(a) = -a^N \). Since the fading distribution is symmetric, the two sequences have the same probability and the bounding in (20) can be repeated for this class of distributions.

Generally speaking, Lem. IV.1 only provides a loose upper bound to capacity as it does not depend on the fading distribution; nonetheless, Lem. IV.1 shows relatively simple conditions under which the capacity of the WFFD channel is at most half of the AWGN capacity. Unfortunately, the converse proof of Th. III.1 does not extend to the continuous case unless the fading distribution concentrates around the origin. The next theorem extends the result in Th. III.3 to the case of continuous fading distributions.

\(^5\) More specifically, let \( \Delta = \frac{1}{\sqrt{V}} \) with \( V = \frac{1}{M} \frac{1-c^{2M}}{1-c^2} \left( \frac{1}{M} \frac{1-c^M}{1-c} \right)^2 \).
Theorem IV.2. Outer bound and approximate capacity for narrow fading. Consider the WFFD channel with \( c > 1 \) in which \( P_A(a) \) is a continuous distribution with

\[
\mathbb{P}\left[|A - \mu_A| \leq \frac{1}{c}\right] = Q_m \geq \frac{1}{2}, \tag{32}
\]

then the expression in (25) for

\[
G_m \leq \frac{Q_m}{2} \log \left(1 + \mu_A^2\right) + 4, \tag{33}
\]

is an outer bound to capacity and the capacity is to within a gap of \( G_m + 1/2 \) bpcu from the outer bound in (25).

Proof: The achievability proof follows a similar derivation as the achievability proof of Th. III.3. The converse proof is obtained in two steps: (i) first it is shown that the capacity of the channel with continuous fading distribution \( A \) is to within a constant gap from the capacity of the channel with discrete fading distribution \( A_\Delta \) when \( A_\Delta \) is obtained by uniformly quantizing \( A \), then (ii) the result in Lem. III.4 is applied to the model with fading distribution \( A_\Delta \) to obtain the approximate capacity of this model. In the following, we prove step (i) while only an outline of the proof of step (ii) is provided for brevity.

- Gap from capacity. Let the random variable \( A_\Delta \) be defined as

\[
\mathbb{P}[A_\Delta(A) = \gamma_k] = \mathbb{P}[A \in I_k]
I_k = \left[\mu_A + k\Delta - \frac{\Delta}{2}, \mu_A + (k + 1)\Delta + \frac{\Delta}{2}\right], \\
\gamma_k = \mathbb{E}[A|A \in I_k], \tag{34}
\]

for \( k \in \mathbb{Z} \) and some \( \Delta \in \mathbb{R}^+ \), that is, \( A_\Delta \) is obtained by uniformly quantizing \( A \) with step size \( \Delta \) and so that \( \mathbb{E}[A] = \mathbb{E}[A_\Delta] \). Next, define

\[
E^N = c(A^N - A_\Delta^N) \circ S^N + Z^N - Z_\Delta^N, \tag{35}
\]

for \( Z_\Delta^N \sim i.i.d. \mathcal{N}(0, 1) \). An outer bound to capacity can be obtained by providing \( E^N \) in (35) to the receiver as a genie-aided side-information, that is

\[
N(R - \epsilon_N) \leq I(Y^N, E^N; W|A^N) = I(Y^N - E^N, E^N; W|A^N) \tag{36a}
\]
\begin{equation}
= I(Y^N_\Delta; W|A^N) + I(E^N; W|A^N, Y^N_\Delta), \tag{36b}
\end{equation}

where (36a) follows from the fact that the transformation has unitary Jacobian while in (36b) follows by defining $Y^N_\Delta = X^N + cA^N_\Delta \circ S^N + Z^N_\Delta$. The term $I(E^N; W|A^N, Y^N_\Delta)$ is further bounded as

\begin{align*}
I(E^N; W|A^N, Y^N_\Delta) &= H(E^N|A^N, Y^N_\Delta) - H(E^N|A^N, Y^N_\Delta, W) \\
&= H(E^N|A^N, Y^N_\Delta) - H(Z^N|A^N, \overline{Z}^N, W, S^N, X^N) \\
&= H(E^N|A^N) - \frac{N}{2} \log 2\pi e \\
&\leq N \max_i H(E_i|A_i) - \frac{N}{2} \log 2\pi e \\
&\leq N \max_{P_{E|A}} H(E|A) - \frac{N}{2} \log 2\pi e.
\end{align*}

Note that $A_\Delta$ is a deterministic function of $A$, so that the entropy term $H(E|A)$ can be bounded as

\begin{equation}
H(E|A) \leq \int_A \frac{1}{2} \log 2\pi e \left(c^2(a - A_\Delta(a))^2 + 2\right) dP_A \leq \frac{1}{2} \log \left(c^2 \Delta^2 + 2\right).
\end{equation}

From (37) we conclude that, by choosing $\Delta = 1/c$ in (34), the capacity of the WFFD channel with fading distribution $P_{A_\Delta}$ is to within a gap of 1 bpcu from the capacity of the channel with fading distribution $P_A(a)$. When the condition in (32) holds, the mode of $A_\Delta$ is $A_\Delta = \gamma_0 \in [\mu_A - 1/c, \mu_A + 1/c]$ with $P_{A_\Delta}(\gamma_0) \geq 1/2$ and thus the result in Th. III.3 can be applied. Note also that the distribution of $A_\Delta$ does not necessarily have unitary variance, so that Lem. I.1 can be invoked to normalize the fading variance. This normalization, though, does not affect the outer bound expression.

The result in Th. IV.2 is analogous to the result in Th. III.3 as it identifies the condition under which it is approximately optimal for the transmitter to pre-code against one realization of the fading distribution while treating the remaining randomness in the fading as noise.

Remark IV.3. Note that the condition in (32) can be generalized to

\begin{equation}
\mathbb{P} \left[|A - m| \leq \frac{\kappa}{c} \right] = Q_m > \frac{1}{2}, \tag{37}
\end{equation}

for some value $m \in \mathcal{A}$ to obtain a more general result than Th. IV.2. This yield an expression for $G_m$ as in (27) and a gap from capacity as in (27).
The next theorem shows that there exists a class of fading distributions for which, regardless of the available transmit power, the capacity of the WFFD channel substantially reduces the capacity of the channel without transmitter state knowledge. We denote the indicator function for the set \( x \in I \) as \( 1_{\{x \in X\}} \).

**Theorem IV.4. An example with a fat-tailed distribution.** Consider a WFFD channel with \( c > 2 \), then there exists a distribution of the form

\[
P_A(a) = \frac{\alpha}{a} \cdot 1_{\{a \in I\}},
\]

such that capacity is upper bounded as

\[
R^{\text{OUT}} = \frac{1}{2} \log \left( 1 + \frac{P}{1 + c^2} \right) + 2,
\]

and for which capacity is to within 3 bpcu from the outer bound in (39).

**Proof:** Quantizing the distribution in (38) as in Th. IV.2 in intervals of size \([c^{-k}, c^{-(k-1)}]\) yields a random variable \( A_\Delta \) which satisfies the conditions of Th. III.5. The support \( I \) can be chosen as \([\kappa c^{-M-1}, \kappa c^{-1}]\) for some sufficiently large \( M \) so that \((1 + \mu_A^2)c^2/M \leq (P + 1)(M - 1)/M\), thus yielding the outer bound in (39) while \( \mu_A \leq 1 \). The achievability proof follows by treating the fading-times-state term as noise. The full proof is omitted for brevity.

**Discussion:** Unfortunately, we are currently unable to determine a characterization of capacity for continuous fading distributions of practical relevance, such as Gaussian, Rayleigh or uniform distributions. Also, we are unable to determine the asymptotic behaviour of capacity as \( c \) grows large: in this regime, one would expect state pre-coding to become ineffective as in Th. IV.4. For the case of zero mean fading this implies, in particular, that state knowledge at the transmitter does not provide any substantial rate advantage with respect to the channel without transmitter state knowledge. In practical systems, state knowledge at the transmitter often come at the cost of an increase in complexity in the network architecture and transmitter design: as such, determining the fading regimes in which transmitter knowledge is rendered useless by the presence of fading is of great practical interest.

**V. Conclusions**

This paper investigates the capacity of the Writing of Fast Fading Dirt (WFFD) channel, a variation of the classic “writing on dirty paper” channel in which the state sequence is multiplied
by an ergodic fading sequence known only at the receiver. In this channel, then, the output is obtained as the sum of the channel input, additive Gaussian noise and a fading-times-state term which is the element-wise product of the channel state, known only at the transmitter, and the fading process, known only at the receiver. We focus on the case in which the channel state is a white Gaussian process and the fading sequence is an i.i.d. sequence with either a discrete or a continuous distribution. The WFFD channel is a special case of the Gelf’and-Pinsker channel for which capacity is known: unfortunately, capacity is expressed as a solution of a maximization problem that cannot be easily determined in closed-form or evaluated numerically. For this reason, we derive alternative inner and outer bounds to capacity and bound their respective distance for certain classes of fading distributions. For the WFFD channel with a discrete fading distribution, we determined capacity to within a small gap for two classes of distributions: distributions with mode larger than a half and uniform distributions in which the points in the support are incrementally spaced apart. For the WFFD channel with a continuous fading distribution, we derive capacity for the case in which more than half of the probability is concentrated in a small interval. In all these cases, capacity is approached by letting the channel input be the superposition of two codewords: a codeword treating the fading-times-state as additional noise and a codeword pre-coded against one realization of the fading times the state sequence. This relatively simple attainable strategy shows, from a high-level perspective, that robust state precancellation is substantially unsuccessful for these fading distributions.

APPENDIX A

PROOF OF LEM. 1.2

Consider two sequences $S_1^n$ and $S_2^n$ such that $S_m^n \sim \text{i.i.d. } \mathcal{N}(0, Q_m)$, $m \in \{1, 2\}$, $S_1^n \perp S_2^n$, with $Q_1 + Q_2 = 1$ and let the channel state of the WFFD channel be obtained as $S^n = S_1^n + S_2^n$. Providing the sequence $S_2^n$ to both the transmitter and receiver can only increase capacity, since they can disregard this extra information and operate as in the original channel. The channel in which $S_2^n$ is provided to both encoder and decoder falls in the class of channels studied in [16, Th. 1] for which capacity can be bounded as

$$C = \max_{X, U | S_2, S_R} I(X + cS A + Z, A, S_2; U) - I(U; S, S_2)$$

$$= \max_{X, U | S_2, S_1} I(X + cS_1 A + Z, U | A, S_2) - I(U; S_1 | S_2)$$
\begin{equation}
\leq \max_{X, U | S_2, S_1} I(X + cS_1 A + Z; U, S_2 | A) - I(U, S_2; S_1) \tag{40a}
\end{equation}

\begin{equation}
= \max_{X, \tilde{U} | S_1} I(X + cS_1 A + Z; \tilde{U} | A) - I(\tilde{U}; S_1), \tag{40b}
\end{equation}

where, (40a) follows from the independence of $S_1$ and $S_2$ by defining $\tilde{U} = [U \ S_2]$ in (40b). Since $S_2$ no longer appears in (40b), it can be dropped from the maximization. From the result in Lem. I.1 we have that (40b) equals the capacity of the channel in (1) for which $\tilde{c} = c/\sqrt{Q_1}$ instead of $c$. From this equivalence, we conclude that the capacity of the WFFD channel is decreasing in the parameter $c$.

\textbf{APPENDIX B}

\textbf{PROOF OF TH. III.3}

- **Achievability.** Consider the achievable strategy in Th. III.1 and let the top codeword $U_{\text{PAS}}$ in (13) be pre-coded against the sequence $cmS_N^*$ as in the WDP channel. This assignment attains the rate

\begin{align*}
R_{\text{PAS}} &= [I(Y; U | X^{S_N^*}, A) - I(U; S)]^+
\geq \frac{Q_m}{2} \log(1 + \alpha P) \\
&\quad + \sum_{a \in A \setminus \{m\}} \frac{P_A(a)}{2} \log \left( \frac{(1 + c^2 a^2 + \alpha P)(1 + \alpha P)}{\alpha P c^2 (a - m)^2 + \alpha P + c^2 a^2 + 1} \right)
\geq \frac{Q_m}{2} \log(1 + \alpha P)
\quad - \sum_{a \in A \setminus \{m\}} \frac{P_A(a)}{2} \log \left( \frac{(a - m)^2}{a^2} + 1 \right),
\end{align*}

while the overall attainable rate $R_{\text{IN}}(\alpha)$ in (14) becomes

\begin{align*}
R_{\text{IN}}(\alpha) &= \frac{1}{2} \mathbb{E}_A \left[ \log \left( 1 + \frac{\alpha P}{1 + c^2 A^2 + \alpha P} \right) \right]
\quad + \frac{Q_m}{2} \log(1 + \alpha P) \\
&\quad - \sum_{a \in A \setminus \{m\}} \frac{P_A(a)}{2} \log \left( \frac{(a - m)^2}{a^2} + 1 \right). \tag{41}
\end{align*}

The choice of $\alpha P$ in (41) as

\begin{equation}
\alpha P = \max \left\{ \min \left\{ \frac{Q_m}{Q_m} (1 + \mu_0^2) - 1, P \right\}, 0 \right\}, \tag{42}
\end{equation}
yields the inner bound

\[ R_{IN} = \begin{cases} 
\frac{1}{2} \log \left( 1 + \frac{P}{1 + e^{c(1 + \mu_A^2)}} \right) & \overline{Q}_m \geq Q_m c^2 (1 + \mu_A^2) \\
\frac{1}{2} \log (P + e^2 (1 + \mu_A^2) + 1) & \overline{Q}_m \leq Q_m c^2 (1 + \mu_A^2) \leq \overline{Q}_m (P + 1) \\
- \frac{Q_m}{2} \log (c^2 (1 + \mu_A^2)) - G_m & Q_m c^2 (1 + \mu_A^2) > \overline{Q}_m (P + 1), \\
\end{cases} \]

for

\[ G_m = \frac{1}{2} E_A \left[ \log \left( \frac{(A - m)^2}{A^2} + 1 \right) | A \neq m \right] + 1, \] (44)

- **Converse.** Using Fano’s, inequality we write

\[ N(R - \epsilon_N) \leq I(Y^N, W|A^N) \]

\[ \leq N \max_j H(Y_j|A_j) - H(Y^N|W, A^N) \]

\[ \leq N \frac{2}{2} E_A \left[ \log 2\pi e (P + A^2 c^2 + 2|c||A|\sqrt{P} + 1) \right] \]

\[ - H(Y^N|W, A^N) \] (45a)

\[ \leq N \frac{2}{2} \log 2\pi e (P + c^2 (1 + \mu_A^2) + 1) \]

\[ - H(Y^N|W, A^N) + \frac{N}{2}, \] (45b)

where (45a) follows from the GME and (45b) follows from Jensen’s inequality. Next, we derive a bound on the entropy term \( H(Y^N|W, A^N) \) based on the properties of the set of typical fading realizations, \( T_e^N(P_A) \), defined as

\[ T_e^N(P_A) = \left\{ a^N, \left| \frac{1}{N} N(k|a^N) - P_A(k) \right| \leq \epsilon P_A(k), \ \forall \ k \in \mathcal{A} \right\}, \]

where \( N(k|a^N) \) is the number of symbols \( k \in \mathcal{A} \) in the sequence \( a^N \), that is

\[ N(k|a^N) = \sum_{i=1}^{N} 1_{\{a_i = k\}}. \] (47)
For the typical set in (46), we have

\[ P(a^N) \leq \frac{1}{2^n(1+\epsilon)H(A)}, \quad a^N \in T^N_{\epsilon} \]  

\[ |T^N_{\epsilon}(P_A)| \leq (1 - \delta)2^{N(1-\epsilon)H(A)} \]  

\[ N(k|a^N) \leq NP_A(k)(a)(1-\epsilon), \quad (48c) \]

for \( \delta = 2|A|e^{-N2\min_k P_A(k)} \). When the block-length \( N \) is sufficiently large, we have that \( \epsilon \leq (Q_m - \frac{1}{2})/Q_m \) in (46) which implies \( N(m|a^N) > 1/2 \). For \( N(m|a^N) > 1/2 \), there exists a one-to-one mapping \( \bar{a}^N(a^N) : T^N_{\epsilon}(P_A) \rightarrow T^N_{\epsilon}(P_A) \) such that

\[
\begin{align*}
&\text{if } \bar{a}_i \neq m \text{ then } a_i = m \\
&\text{if } a_i \neq m \text{ then } \bar{a}_i = m,
\end{align*}
\]

that is, the sequence \( \bar{a}^N(a^N) \) is obtained by permuting the \( N - N(m|a^N) \) indexes for which \( a_i \neq m \) with some \( N - N(m|a^N) \) indexes for which \( a_i = m \), while \( 2N(m|a^N) - N \) indexes are such that \( a_i = \bar{a}_i = m \). Since the mapping \( \bar{a}^N(a^N) \) in (49) is a one-to-one mapping of the typical set onto itself, we must have

\[
\sum_{a^N \in T^N_{\epsilon}(P_A)} P(a^N)H(Y^N|W, A^N = a^N) = \sum_{\bar{a}^N \in T^N_{\epsilon}(P_A)} P(\bar{a}^N(a^N))H(\bar{Y}^N|W, A^N = \bar{a}^N(a^N)),
\]

(50)

where \( \bar{Y}^N \) in (50) is defined as

\[
\bar{Y}^N = X^N + c\bar{a}^N S^N + \bar{Z}^N,
\]

(51)

for \( \bar{Z}^N \sim i.i.d. N(0, 1) \), \( \bar{Z}^N \perp Z^N \). Using the definitions above, we have that the entropy term \( H(Y^N|W, A^N) \) can be bounded as

\[
-H(Y^N|W, A^N) = -\sum_{a^N \in A^N} P(a^N)H(Y^N|W, A^N = a^N)
\]

\[
\leq -\sum_{a^N \in T^N_{\epsilon}(P_A)} P(a^N)H(Y^N|W, A^N = a^N)
\]

\[
= -\frac{1}{2} \sum_{a^N \in T^N_{\epsilon}(P_A)} P(a^N) \left( H(Y^N|W, A^N = a^N) \right)
\]

(52a)
\[ + H(Y^N | W, A^N = \overline{a}^N) \]
\[ \leq - \frac{1}{2} \sum_{a^N \in T^N_c(P_A)} P(a^N) \]
\[ \left( H(X^N + \alpha a^N \circ S^N + Z^N, X^N + \alpha \overline{a}^N \circ S^N + \overline{Z}^N | W) \right) \]
\[ = - \frac{1}{2} \sum_{a^N \in T^N_c(P_A)} P(a^N) \]
\[ H\left( c(a^N - \overline{a}^N) \circ S^N + Z^N - \overline{Z}^N, X^N + \alpha \overline{a}^N \circ S^N + \overline{Z}^N | W \right) \]
\[ = - \frac{1}{2} \sum_{a^N \in T^N_c(P_A)} P(a^N) \left( H\left( c(a^N - \overline{a}^N) \circ S^N + Z^N - \overline{Z}^N \right) + H(\overline{Y}^N | Y^N - \overline{Y}^N, W, S^N, X^N) \right) \]
\[ \leq - \frac{1}{2} \sum_{a^N \in T^N_c(P_A)} P(a^N) \cdot \left( H\left( c(a^N - \overline{a}^N) \circ S^N + Z^N - \overline{Z}^N \right) + H(\overline{Z}^N | Z^N - Z^N) \right) \]
\[ = - \frac{1}{2} \sum_{a^N \in T^N_c(P_A)} P(a^N) \cdot H\left( c(a^N - \overline{a}^N) \circ S^N + Z^N - \overline{Z}^N \right) + \frac{N}{2} \log(\pi e), \tag{52b} \]

where (52b) follows from the fact that \( S^N \) and \( Z^N \) are independent from \( W \). We continue the series of inequalities in (52) by noting that

\[ \frac{1}{2} \sum_{a^N \in T^N_c(P_A)} P(a^N) \cdot H\left( c(a^N - \overline{a}^N) \circ S^N + Z^N - \overline{Z}^N \right) \]
\[ \leq - \frac{1}{2^{2n(1+\epsilon)H(A)}} \sum_{a^N \in T^N_c(P_A)} H\left( c(a^N - \overline{a}^N) \circ S^N + Z^N - \overline{Z}^N \right) \]
\[ \leq - \frac{1}{2^{2n(1+\epsilon)H(A)}} \sum_{a^N \in T^N_c(P_A)} \sum_{i=1}^{N} \left( H\left( (a_i - \overline{a}_i)S_i + Z_i - \overline{Z}_i \right) \right), \tag{53b} \]

where (53a) follows from the bound in (48a) while (53b) follows from the fact that \( S^N \) and \( Z^N \) are i.i.d. sequences. From the definition of the mapping \( \overline{a}^N(a^N) \), the sequence \( a_i - \overline{a}_i \) can take three types of values: \( m - k, k - m \) and 0 where \( k \) is any element of \( A \setminus \{m\} \). More specifically,
\( a_i - \overline{a}_i = m - k \) occurs \( N(k|a^N) \) times, \( a_i - \overline{a}_i = k - m \) occurs \( N(k|a^N) \) times for all \( k \in \mathcal{A} \) while \( a_i - \overline{a}_i = 0 \) occurs \( 2(N - N(m|a^N)) \) times. Using these observations, we write

\[
(53b) = -\frac{1}{2} 2^{n(1+H(A))} \sum_{a^N \in T_N(P_A)} \left( \sum_{k \in \mathcal{A} \setminus \{m\}} 2N(k|a^N)H(c(m - k)S_i + Z_i - \overline{Z}_i) \right.
\]

\[
+ \frac{1}{2} (2N(m|a^N) - N)H(Z_i - \overline{Z}_i) \bigg) \]  

\[
= -\frac{1}{2} 2^{n(1+H(A))} \sum_{a^N \in T_N(P_A)} \left( \sum_{k \in \mathcal{A} \setminus \{m\}} 2N(k|a^N)H(c(m - k)S_i + Z_i - \overline{Z}_i) \right.
\]

\[
- \frac{1}{2} (2N(m|a^N) - N) \log(4\pi e) \bigg) \]  

\[
\leq -\frac{1}{2} 2^{n(1+H(A))} \sum_{a^N \in T_N(P_A)} \sum_{k \in \mathcal{A} \setminus \{m\}} 2N(k|a^N) \frac{1}{2} \log 2\pi e(c^2(m - k)^2 + 2)
\]

\[
= -\frac{1}{2} 2^{n(1+H(A))} (1 - \delta_e) 2^{n(1-\epsilon)H(A)}. \]  

\[
(54c) \]

\[
\leq -\frac{1}{2} 2^{n(1+H(A))} (1 - \delta_e) 2^{n(1-\epsilon)H(A)} (1 - \epsilon) N.
\]

\[
\sum_{k \in \mathcal{A} \setminus \{m\}} P_A(k)(1 - \epsilon) \frac{N}{2} \log 2\pi e(c^2(m - k)^2 + 2), \]  

\[
(54d) \]

where \( (54c) \) follows from the bound on the cardinality of the typical set in \( (48b) \) and \( (54d) \) from the definition in \( (46) \). For \( N \) is sufficiently large, we have

\[
-H(Y^N|W, A^N) \leq -\sum_{k \in \mathcal{A} \setminus \{m\}} P_A(k) \frac{N}{2} \log 2\pi e \left( c^2(m - k)^2 + 2 \right)
\]

\[
- \frac{NQ_m}{2} \log(2\pi e) - \epsilon_{\text{all}} \]  

\[
\leq -\frac{NQ_m}{2} \log 2\pi ec^2 - \frac{N}{2} \mathbb{E}_A[\log(A - m)^2 | A \neq m] \]  

\[
(55a) \]
for some $\epsilon_{all}$ that goes to zero as $N \to \infty$. Using the bound for (55) in (45b) and for some $\epsilon_{all}$ sufficiently small, we obtain the outer bound

$$R_{\text{OUT}} = \frac{1}{2} \log \left( \frac{2^N}{(1 + \mu^2) + 1} \right) - \frac{Q_m}{2} \log(c^2(1 + \mu^2)) = \frac{1}{2} \log \left( \frac{2^N}{1 + \mu^2} \right) - \frac{1}{4} \log(\pi e) - \frac{1}{2} \frac{E_A[\log \left( \frac{(A - m)^2}{1 + \mu^2} \right)]}{|A \neq m|} + \frac{1}{2}.$$  

Using Lem. 1.2, we can consider the assignment

$$\min \left\{ \frac{Q_m}{2} (1 + P), c^2(1 + \mu A) \right\},$$

for the term $c^2(1 + \mu^2)$ in (56) which yields the expression in (25).

The gap between inner and outer bound of $G_m$ can be obtained by comparing the expressions in (25) and (43).

**APPENDIX C**

**PROOF OF TH. III.5**

In the following, we provide the proof for the result in Th. III.5 for the case of $M = 3$: an outline of the proof for the case of any $M > 3$ is provided at the end of this section while the full derivation is omitted for brevity.

The achievability proof is a variation of the achievability proof of Th. III.3 when letting the codeword $U_{PAS}$ be pre-coded against the sequence $c\mu A S^N$ as in the WDP channel.

- **Converse.** For $A = \{\alpha_1, \alpha_2, \alpha_3\}$ and $\alpha_1 \leq \alpha_2 \leq \alpha_3$, we define two conjugate sequences of $a^N$, $\overline{a}^N_{(1)}(a^N)$ and $\overline{a}^N_{(2)}(a^N)$, as follows:

  * the portion of $a^N$ equal to $\alpha_1$, is equal to $\alpha_2$ in $\overline{a}^N_{(1)}$ and equal to $\alpha_3$ in $\overline{a}^N_{(2)}$,
  * the portion of $a^N$ equal to $\alpha_2$, is equal to $\alpha_3$ in $\overline{a}^N_{(1)}$ and equal to $\alpha_1$ in $\overline{a}^N_{(2)}$, and
  * the portion of $a^N$ equal to $\alpha_3$, is equal to $\alpha_1$ in $\overline{a}^N_{(1)}$ and equal to $\alpha_2$ in $\overline{a}^N_{(2)}$.

From the definition of the mapping, $\overline{a}^N_{(1)}(a^N)$ and $\overline{a}^N_{(2)}(a^N)$, we have that

$$a^N \in T_{\epsilon}^N(P_A) \iff \overline{a}^N_{(k)}(a^N) \in T_{\epsilon}^N(P_A), \quad k \in [1, 2],$$

(58a)
moreover

\[
\sum_{a^N \in \mathcal{T}_c^N(P_A)} P_{A^N}(a^N) H(Y^N|W, A^N = a^N) = \sum_{a^N \in \mathcal{T}_c^N(P_A)} P_{A^N}(\tilde{a}_{(k)}^N(a^N)) H(Y_{(k)}^N|W, A^N = \tilde{a}_{(k)}^N(a^N)),
\]

(59)

where \( Y_{(k)}^N \) is defined similarly to (51) as

\[
Y_{(k)}^N = X^N + c \tilde{a}_{(k)}^N S^N + Z_{(k)}^N,
\]

(60)

for \( Z_{(k)}^N \sim i.i.d. \mathcal{N}(0, 1), \ k \in [1, 2] \). Similarly to (45), Fano’s inequality yields the bound

\[
N(R - \epsilon_N) \leq \frac{N}{2} \log 2\pi e (P + \epsilon^2(1 + \mu^2_A) + 1)
\]

\[- H(Y^N|W, A^N) + \frac{N}{2},
\]

(61)

Using the equivalence in (59), the term \( H(Y^N|W, A^N) \) can be rewritten as

\[
- H(Y^N|W, A^N) = - \sum_{a^N \in \mathcal{T}_c^N(P_A)} \frac{1}{3^N} H(Y^N|W, A^N = a^N)
\]

\[= - \frac{1}{3^{N+1}} \sum_{a^N \in \mathcal{T}_c^N(P_A)} \left( H(Y^N|W, A^N = a^N) + H(Y^N|W, A^N = a_{(1)}^N) + H(Y^N|W, A^N = a_{(2)}^N) \right)
\]

For \( a^N \in \mathcal{T}_c^N(P_A) \), we have

\[
- H(Y^N|W, A^N = a^N) - H(Y^N|W, A^N = a_{(1)}^N) - H(Y^N|W, A^N = a_{(2)}^N)
\]

\[\leq -H(Y^N, Y_{(1)}^N, Y_{(2)}^N|W, A^N = a^N)
\]

\[= -H \left( \{Y_i, Y_{(1),i}, Y_{(2),i}, \forall i \ a_i = \alpha_1 \}, \{Y_i, Y_{(1),i}, Y_{(2),i}, \forall i \ a_i = \alpha_2 \}, \right.
\]

\[\left. Y_i, Y_{(1),i}, Y_{(2),i}, \forall i \ a_i = \alpha_3 \} | W, A^N = a^N \right)
\]

\[= -H \left( \{Y_i, Y_i - Y_{(2),i}, Y_{(2),i}, \forall i \ a_i = \alpha_1 \}, \right.
\]

\[\left. Y_{(2),i} - Y_i, Y_{(1),i}, Y_{(2),i}, \forall i \ a_i = \alpha_2 \}, \right.
\]

\[\left. Y_i, Y_{(1),i} - Y_{(2),i}, Y_{(2),i}, \forall i \ a_i = \alpha_3 \} | W, A^N = a^N \right)
\]

(62a)

(62b)

(62c)

where (62b) follows by re-arranging the channel outputs according to the fading realization and (62c) follows from the fact this the transformation has unitary Jacobian. Consider the set

\[\{Y_i - Y_{(1),i}, \forall i \ a_i = \alpha_1 \}\]
\[ \cup \{ Y_{(2),i} - Y_i, \forall i \ a_i = \alpha_2 \} \]
\[ \cup \{ Y_{(1),i} - Y_{(2),i}, \forall i \ a_i = \alpha_3 \}, \quad (63) \]

from the definition of the conjugate sequences, we have that the set in (63) contains the elements of the vector
\[ c(\alpha_2 - \alpha_1)S^N + \tilde{Z}_{21}^N, \quad (64) \]

where \( \tilde{Z}_{21}^N \) is obtained as
\[ \tilde{Z}_{21,i} = \begin{cases} 
Z_i - Z_{(1),i} & a_i = \alpha_1 \\
Z_{(2),i} - Z_i & a_i = \alpha_2 \\
Z_{(1),i} - Z_{(2),i} & a_i = \alpha_3.
\end{cases} \quad (65) \]

Next, continuing the series of inequalities in (62), we have
\[ -3H(Y^N | W, A^N = a^N) \]
\[ \leq -H \left( \{ Y_i, Y_{(2),i}, a_i = \alpha_1 \}, \{ Y_{(1),i}, Y_{(2),i}, a_i = \alpha_2 \}, \{ Y_{(1),i}, Y_{(2),i}, a_i = \alpha_3 \} | W, c(\alpha_2 - \alpha_1)S^N + \tilde{Z}_{21}^N, A^N = a^N \right) \]
\[ - H(c(\alpha_2 - \alpha_1)S^N + \tilde{Z}_{21}^N | W) \]
\[ \leq -H \left( \{ Y_i - Y_{(2),i}, Y_i, a_i = \alpha_1 \}, \{ Y_{(2),i} - Y_{(1),i}, Y_{(2),i}, a_i = \alpha_2 \}, \{ Y_{(1),i} - Y_i, Y_{(1),i}, a_i = \alpha_3 \} \right) \]
\[ | W, c(\alpha_2 - \alpha_1)S^N + \tilde{Z}_{21}^N, A^N = a^N \right) \]
\[ - H(c(\alpha_2 - \alpha_1)S^N + \tilde{Z}_{21}^N | W), \quad (66a) \]

where (66a) follows from the observation in (64) and (66b) follows again from the fact that this transformation has unitary Jacobian. Similarly to (63), we have that the set
\[ \left\{ \{ Y_i - Y_{(2),i}, a_i = \alpha_1 \}, \{ Y_{(2),i} - Y_{(1),i}, a_i = \alpha_2 \}, \{ Y_{(1),i} - Y_i, a_i = \alpha_3 \} \right\}, \quad (67) \]

contains the elements of the vector
\[ c(\alpha_3 - \alpha_1)S^N + \tilde{Z}_{31}^N, \quad (68) \]
for $\tilde{Z}_{31}^N$ defined similarly as in (65). With this observation, we write

$$-3H(Y^N|W, A^N = a^N)$$

$$\leq H\left(\{Y_i, a_i = \alpha_1\}, \{Y_{(2)i}, a_i = \alpha_2\}, \{Y_{(1)i}, a_i = \alpha_3\}|W, c(\alpha_3 - \alpha_1)S^N + \tilde{Z}_{31}^N, c(\alpha_3 - \alpha_1)S^N + \tilde{Z}_{21}^N, A^N = a^N\right)$$

$$- H(c(\alpha_3 - \alpha_1)S^N + \tilde{Z}_{31}^N|W, c(\alpha_2 - \alpha_1)S^N + \tilde{Z}_{21}^N)$$

$$- H(c(\alpha_2 - \alpha_1)S^N + \tilde{Z}_{21}^N|W)$$

$$\leq H\left(\{Z_i, a_i = \alpha_1\}, \{Z_{(2)i}, a_i = \alpha_2\}, \{Z_{(1)i}, a_i = \alpha_3\}|W, S^N, \tilde{Z}_{31}^N, \tilde{Z}_{21}^N, A^N = a^N\right)$$

$$- H(c(\alpha_3 - \alpha_1)S^N + \tilde{Z}_{31}^N|W, c(\alpha_2 - \alpha_1)S^N + \tilde{Z}_{21}^N)$$

$$- H(c(\alpha_2 - \alpha_1)S^N + \tilde{Z}_{21}^N|W),$$

(69a)

(69b)

(69c)

We are now left with the task of evaluating the terms in (69a), (69b) and (69c) in closed-form. For the term in (69c), we have write

$$H(c(\alpha_2 - \alpha_1)S^N + \tilde{Z}_{21}^N) = -\frac{N}{2} \log 2\pi e \left(c^2(\alpha_2 - \alpha_1)^2 + 2\right)$$

$$\leq -\frac{N}{2} \log 2\pi e \left(c^2(c^2 - 1)\alpha_1^2 + 2\right)$$

$$\leq -\frac{N}{2} \log 2\pi e \left(c^2 + 1\right) - \frac{1}{2},$$

(70a)

(70b)

where (70a) follows from the fact $\alpha_2 > c\alpha_1$ and (70b) follows from $\alpha_1 > 1/(c-1)$ as prescribed by (28). For the term in (69b), we have

$$- H(c(\alpha_3 - \alpha_1)S^N + \tilde{Z}_{31}^N|W, c(\alpha_2 - \alpha_1)S^N + \tilde{Z}_{21}^N)$$

$$= -NH\left(c^2(\alpha_3 - \alpha_1)(\alpha_2 - \alpha_1)\left(1 - \frac{c(\alpha_2 - \alpha_1)}{c^2(\alpha_2 - \alpha_1)^2 + 2}\right)S + \tilde{Z}_{13} - \frac{c^2(\alpha_3 - \alpha_1)(\alpha_2 - \alpha_1)}{c^2(\alpha_2 - \alpha_1)^2 + 2}\tilde{Z}_{12}\right)$$

$$\leq -\frac{N}{2} \log 2\pi e \left(1 + c^2 \frac{(\alpha_3 - \alpha_1)^2}{2 + c^2(\alpha_2 - \alpha_1)^2}\right)$$

$$\leq -\frac{N}{2} \log 2\pi e \left(1 + c^2 \frac{a^2(c - 1)^2}{2 + c^2(\alpha_2 - (c - 1)^{-1})^2}\right)$$

$$\leq -\frac{N}{2} \log 2\pi e \left(1 + \frac{1}{2} c^2\right),$$

(71a)

(71b)

where, in (71b), we have used the fact that $\alpha_3 > c\alpha_2$, $\alpha_1 > 1/(1-c)$ and $c > 2$ by assumption.
Finally, the term in (69a) only contains independent noise terms, so that
\[ H\left(\{Z_i, a_i = \alpha_1\}, \{Z_{(2), i} a_i = \alpha_2\}, \{Z_{(1), i} a_i = \alpha_3\}\right) = N 2 \log \left(\frac{1}{3}\right). \] (72)

Combining the bounds in (70), (71) and (72) we finally obtain the outer bound
\[ R_{\text{OUT}} \leq \frac{1}{2} \log(P + c^2(1 + \mu_A) + 1) - \frac{3}{4} \log(c^2(1 + \mu_A)) + \frac{3}{4} \log(1 + \mu_A). \] (73)

The final outer bound expression in (29) is obtained by using Lem. I.2 to tighten the expression (73) with the appropriate choice of \(c\). The gap between inner and outer bound is obtained similarly to Th. III.3.

• General converse. The derivation for the case \(M > 3\) is obtained by extending the derivation for \(M = 3\) as follows. First, we define \(M - 1\) conjugate sequences as
\[ \pi_N^{\pi}(a^N) = \{ a_i = \alpha_j \implies a_{(k),i} = \alpha_{\text{mod}(k+j,M)} \} , \] (74)
for \(k \in [1 \ldots M - 1]\). Next, the bounding in (69) can be repeated recursively \(M - 1\) times: this yields \(M - 1\) terms of the form \(H(\Delta_i S + \tilde{Z}_i | \Delta_{i-1} S + \tilde{Z}_{i-1})\) for \(\Delta_i = \alpha_{i+1} - \alpha_1, \tilde{Z}_{i1} = Z_i - Z_1\) as in (65), and
\[ H(\Delta_i S + \tilde{Z}_{(i+1)1} | \Delta_{i} S + \tilde{Z}_{21} \ldots \Delta_{i-1} S + \tilde{Z}_{i1}) = \frac{1}{2} \log \left(2 \frac{c^2(\sum_{j=1}^{i} \Delta_j^2) + 2}{c^2(\sum_{j=1}^{i-1} \Delta_j^2) + 2}\right). \] (75)
The conditions in (29) guarantee that
\[ H(\Delta_i S + \tilde{Z}_{(i+1)1} | \Delta_{i} S + \tilde{Z}_{21} \ldots \Delta_{i-1} S + \tilde{Z}_{i1}) \leq - \frac{1}{2} \log(2\pi ec^2) + \frac{1}{2}, \] (76)
for each \(i \in [1 \ldots M - 1]\), yielding an outer bound in the spirit (73)
\[ R_{\text{OUT}} \leq \frac{1}{2} \log(P + c^2(1 + \mu_A) + 1) - \frac{M - 1}{M} \log(c^2) \] (77)
which can be tightened over the parameter \(c\) using Lem. I.2. This tightening step finally yields the outer bound in (29).
REFERENCES

[1] M. Costa, “Writing on dirty paper.” IEEE Trans. Inf. Theory, vol. 29, no. 3, pp. 439–441, 1983.
[2] S. Gel’fand and M. Pinsker, “Coding for channel with random parameters,” Problems of control and information theory, vol. 9, no. 1, pp. 19–31, 1980.
[3] A. Khisti, U. Erez, A. Lapidoth, and G. Wornell, “Carbon copying onto dirty paper,” IEEE Trans. Inf. Theory, vol. 53, no. 5, pp. 1814–1827, May 2007.
[4] W. Zhang, S. Kotagiri, and J. N. Laneman, “Writing on dirty paper with resizing and its application to quasi-static fading broadcast channels,” in Information Theory, 2007. ISIT 2007. IEEE International Symposium on. IEEE, 2007, pp. 381–385.
[5] A. Bennatan and D. Burshtein, “On the fading-paper achievable region of the fading MIMO broadcast channel,” IEEE Trans. Inf. Theory, vol. 54, no. 1, pp. 100–115, 2008.
[6] A. Hindy and A. Nosratinia, “Lattice strategies for the ergodic fading dirty paper channel,” in Information Theory (ISIT), 2016 IEEE International Symposium on. IEEE, 2016, pp. 2774–2778.
[7] P. Grover and A. Sahai, “On the need for knowledge of the phase in exploiting known primary transmissions,” in New Frontiers in Dynamic Spectrum Access Networks, 2007. DySPAN 2007. 2nd IEEE International Symposium on. IEEE, 2007, pp. 462–471.
[8] ——, “Writing on rayleigh faded dirt: a computable upper bound to the outage capacity,” in Information Theory, 2007. ISIT 2007. IEEE International Symposium on. IEEE, 2007, pp. 2166–2170.
[9] S. Rini and S. Shamai, “The impact of phase fading on the dirty paper coding channel,” in Information Theory (ISIT), 2014 IEEE International Symposium on. IEEE, 2014, pp. 2287–2291.
[10] Y. Avner, B. M. Zaidel, S. Shamai, and U. Erez, “On the dirty paper channel with fading dirt,” in Electrical and Electronics Engineers in Israel (IEEEI), 2010 IEEE 26th Convention of. IEEE, 2010, pp. 525–529.
[11] A. Bennatan, V. Aggarwal, Y. Wu, A. R. Calderbank, J. Hoydis, and A. Chindapol, “Bounds and lattice-based transmission strategies for the phase-faded dirty-paper channel,” Wireless Communications, IEEE Transactions on, vol. 8, no. 7, pp. 3620–3627, 2009.
[12] A. Khina and U. Erez, “On the robustness of dirty paper coding,” Communications, IEEE Transactions on, vol. 58, no. 5, pp. 1437–1446, 2010.
[13] Z. Al-qudah and W. A. Shehab, “Bounds on the achievable rates of faded dirty paper channel,” International Journal of Computer Networks and Communications (IJCNC), vol. 9, no. 1, pp. 71–79, 2001.
[14] T. Cover and J. Thomas, Elements of Information Theory. Wiley-Interscience, New York, 1991.
[15] S. Rini and S. Shamai, “On capacity of the dirty paper channel with fading dirt in the strong fading regime,” in Information Theory Workshop (ITW), 2014 IEEE. IEEE, 2014, pp. 561–565.
[16] T. M. Cover and M. Chiang, “Duality between channel capacity and rate distortion with two-sided state information,” Information Theory, IEEE Transactions on, vol. 48, no. 6, pp. 1629–1638, 2002.