Transience and recurrence of random walks on percolation clusters in an ultrametric space

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Abstract

We study existence of percolation in the hierarchical group of order \( N \), which is an ultrametric space, and transience and recurrence of random walks on the percolation clusters. The connection probability on the hierarchical group for two points separated by distance \( k \) is of the form \( c_k/N^{k(1+\delta)} \), \( \delta > -1 \), with \( c_k = C_0 + C_1 \log k + C_2 k^\alpha \), non-negative constants \( C_0, C_1, C_2 \), and \( \alpha > 0 \). Percolation was proved in Dawson and Gorostiza (2013) for \( \delta < 1 \), and for the critical case, \( \delta = 1, C_2 > 0 \), with \( \alpha > 2 \). In this paper we improve the result for the critical case by showing percolation for \( \alpha > 0 \). We use a renormalization method of the type in the previous paper in a new way which is more intrinsic to the model. The proof involves ultrametric random graphs (described in the Introduction). The results for simple (nearest neighbour) random walks on the percolation clusters are: in the case \( \delta < 1 \) the walk is transient, and in the critical case \( \delta = 1, C_2 > 0, \alpha > 0 \), there exists a critical \( \alpha_c \in (0, \infty) \) such that the walk is recurrent for \( \alpha < \alpha_c \) and transient for \( \alpha > \alpha_c \). The proofs involve graph diameters, path lengths, and electric circuit theory. Some comparisons are made with behaviours of random walks on long-range percolation clusters in the one-dimensional Euclidean lattice.

Keywords: Percolation, hierarchical group, ultrametric space, random graph, renormalization, random walk, transience, recurrence.

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1 Introduction

Network science is an active field of research due to its many areas of application (statistical physics, biology, computer science, communications, economics, social sciences, etc.), and to the interesting mathematical problems that it gives rise to, many of which remain open. Percolation plays an important role, for example in the study of robustness of networks. Hierarchical networks occur in models where there is a multiscale organization with an ultrametric structure, e.g., in statistical physics, population genetics and computer science. Several areas of physics where ultrametric structures are present were overviewed in [6]. An ultrametric model in population genetics was introduced in [32]. Hierarchical organizations in complex networks were discussed in [4]. A classical ultrametric space is the set of $p$-adic numbers. A review of many areas where $p$-adic analysis is used, specially in physics including quantum physics, appeared in [31]. The ultrametric space we deal with in this paper is $\Omega_N$, the hierarchical group of order $N$, described at the end of the Introduction. Background on ultrametric spaces can be found e.g. in [33].

Stochastic models on hierarchical groups have played a fundamental role in mathematical physics and population biology. Dyson [32] introduced such a structure in order to gain insight on the study of ferromagnetic models on the Euclidean lattice of dimension 4, as it provides a "caricature" of the Euclidean lattice in dimensions "infinitesimally close" to 4. A reason for this approach is that it is possible to carry out a renormalization group analysis in a rigorous way in hierarchical groups [11, 19]. Hierarchical groups have also been used in the study of self-avoiding random walks in four dimensions [13], Anderson localization in disordered media [7, 13], mutually catalytic branching in population models [5, 29], interacting diffusions [27, 28], occupation times of branching systems [23, 24], search algorithms [39, 40]. Thus, stochastic models, in particular random walks, on ultrametric spaces are a natural field of study. A class of random walks on hierarchical groups, called hierarchical random walks, and their degrees of transience and recurrence were studied in [22, 26]. Other properties of systems of hierarchical random walks appeared in [22, 26]. An analogous class of hierarchical random walks on the $p$-adic numbers was studied in [22, 24]. Lévy processes on totally disconnected groups (including the $p$-adic integers) were discussed in [32], pseudodifferential equations and Markov processes over $p$-adic were treated in [10]. A random walk model for the dynamics of proteins was discussed in [5]. These are a few representative references on stochastic models on ultrametric spaces.

With these precedents it is natural to investigate percolation in hierarchical groups and behaviour of random walks on percolation clusters in those groups, and to compare results with those for Euclidean lattices (referred to below). These are the subjects of the present paper.

The renormalization method for the study of percolation in hierarchical networks involves ultrametric random graphs. An ultrametric random graph $URG(M, d)$ is a graph on a finite set of $M$ elements with an ultrametric $d$ and connection probabilities $p_{x,y}$ that are random and depend on the distance $d(x, y)$ (see [22, Section 3.4]). A more detailed description related to the model is given in Section 2.

In [22] we studied asymptotic percolation in $\Omega_N$ in the limit $N \to \infty$ (mean field percolation) with a certain class of connection probabilities depending on the distance between points. In this case it was possible to obtain a necessary and sufficient condition for percolation. The Erdős-Rényi theory of giant components of random graphs was a useful tool, although there are significant differences between classical random graphs and ultrametric ones (e.g., the average length of paths in the giant component of an ultrametric ball is much longer than in the classical case). Percolation in $\Omega_N$ with fixed $N$ is technically more involved, and so far only sufficient conditions for percolation or for its absence are known. This was studied in [22], where connectivity results of Erdős-Rényi graphs played a basic role. At the same time an analogous model was studied in [12] using different methods. In [22] the "critical case" was analyzed in more depth. A relationship between the results of [22] and [12] was given in [22] (Remark 3.2). The relevance of percolation in hierarchical groups has been noted for contact processes [6] and epidemiology [34].

In [22] we studied percolation in the hierarchical group $(\Omega_N, d)$, integer $N \geq 2$, ultrametric $d$, with probability of connection between two points $x$ and $y$ such that $d(x, y) = k \geq 1$ of the form $p_{x,y} = c_k/N^{(1+\delta)k}$ where $\delta > -1$ and the $c_k$ are positive constants, all connections being independent. The results refer to existence of percolation clusters (infinite connected sets) of positive density. Percolation is said to occur if a given point of $\Omega_N$ belongs to a percolation cluster with positive probability. The specific point does not matter because the model is translation-invariant. By ultrametricity, percolation
is possible only if there exists arbitrarily large $k$ such that $c_k > 0$ (otherwise all connected components are finite). Thus, percolation in $\Omega_N$ can be regarded as long-range percolation. Briefly, the results are: if $\delta < 1$ and $c = \inf_k c_k$ is large enough, then percolation occurs, if $\delta > 1$ and $\sup_k c_k < \infty$, then percolation does not occur, and for the critical case, $\delta = 1$, which is the most delicate, percolation may or may not occur according to some special forms of $c_k$ such that $c_k \to \infty$ as $k \to \infty$. When percolation occurs the infinite cluster is unique.

In the critical case $c_k$ was taken of the form

\[(1.1) \quad c_k = C_0 + C_1 \log k + C_2 k^\alpha\]

with non-negative constants $C_0, C_1, C_2$, and $\alpha > 0$, and for the case $C_2 > 0$ percolation was established for $\alpha > 2$ and any $C_1$ if $C_0$ and $C_2$ are large enough (Theorem 3.3(a)). The proof was based on a renormalization argument of the type used in statistical physics. Results for the case $C_2 = 0$ were also obtained, in particular if $C_1 < N$, then percolation does not occur for any $C_0$. The main result on percolation in the present paper is that percolation occurs for any $\alpha > 0$ if $C_2$ is large enough (Theorem 2.1). This was an open problem in (section (3.4)). The proof uses the renormalization ideas introduced in (22), but in a new way which is more intrinsic to the model.

The renormalization approach in (22) was applied for a preliminary percolation result with $c_k$ of the form

\[(1.2) \quad c_k = C + a \log n \cdot n^b \log N,\]

constants $C \geq 0, a > 0, b > 0$, where

\[(1.3) \quad k_n = \lfloor Kn \log n \rfloor, n = 1, 2, \ldots,\]

constant $K > 0$, and $c_{k_n} \leq c_k \leq c_{k_{n+1}}$ for $k_n < k < k_{n+1}$, with some technical conditions on $K$ and $b$ (Theorem 3.5(b)), which was used to prove percolation for $\alpha > 2$. The conditions were $2/k \log N < K < b$ (with a minor modification it is possible to have also $K = b$). The reason why we could prove percolation for $\alpha > 2$ in (22) is that the relationship between (1.1) and (1.2) is $\alpha > b \log N$ (proof of Theorem 3.1). The scheme with (1.2), (1.3) is not used for the proof of percolation here, but it constitutes a technical tool for the study of behaviour of random walks on percolation clusters regarding some properties of the clusters, hence we will need to refer to some techniques in (23).

The other main results in the paper refer to transience and recurrence of simple (nearest neighbour) random walks on the percolation clusters in $\Omega_N$. We show that the random walk is transient for $\delta < 1$ (Theorem 4.4), and in the critical case, $\delta = 1, C_2 > 0$, there exists a critical $\alpha_c \in (0, \infty)$ such that the random walk is recurrent for $\alpha < \alpha_c$ and transient for $\alpha > \alpha_c$ (Theorem 4.5).

These results are comparable in part with those on long-range percolation in the one-dimensional Euclidean lattice $\mathbb{Z}$ with connection probabilities of the form $\beta |x - y|^{-s}$ as $|x - y| \to \infty$, although the Euclidean and the ultrametric structures are quite different. Long-range percolation in $\mathbb{Z}$ with those connection probabilities was introduced by Schulman (21), and studied further for $\mathbb{Z}^d$ by Newman and Schulman (21), and Aizenman and Newman (1). Berger (4) studied transience and recurrence of random walks on the percolation clusters in $\mathbb{Z}^d$ for $d = 1, 2$. The results for $d = 1$ are, roughly, that percolation can occur if $1 < s \leq 2$, and does not occur if $s > 2$, and if $1 < s < 2$, then the walk is transient, and if $s = 2$, then the walk is recurrent. Hence the results agree for $\Omega_N, 0 < \delta < 1$, and $\mathbb{Z}, 1 < s < 2$, by using the ultrametric $\rho(x, y) = N^{d(x,y)}$ ("Euclidean radial distance") on $\Omega_N$, and $\delta = 1 + s$. But there is a significant difference. Percolation in $\mathbb{Z}$ can be obtained by increasing the probability of connection between points separated by distance 1 (4) (Theorem 1.2), whereas for $\Omega_N$ short-range connections play no role due to ultrametricity. The results for $\delta = 1$ and $s = 2$ are not comparable because $c_k$ tends to $\infty$, whereas $\beta$ is fixed.

Grimmert et al. (33) studied the behaviour of random walks on (bond) percolation clusters in the Euclidean lattice $\mathbb{Z}^d$ (see also (8) (33)). Recurrence of the walk for $d \leq 2$ follows from recurrence on the whole space $\mathbb{Z}^d$ by the cutting law of electric circuit theory (32), whereas transience for $d \geq 3$ was difficult to prove. For recurrence in long-range percolation the situation is different because the walk is only defined on the percolation cluster (both in $\mathbb{Z}^d$ and in $\Omega_N$). Although the models on $\mathbb{Z}^d$ and $\Omega_N$ are
quite different, we are able to use some of the basic ideas on the relationship between reversible Markov chains and electric circuits (see e.g. [10, 13, 11]) that have been used for \( \mathbb{Z}^d \), but in the case of \( \Omega_N \) the ultrametric structure plays a fundamental role.

The transience and recurrence behaviours of walks on the percolation clusters are determined basically by the ultrametric geometry of \( (\Omega_N, d) \) and the form of the connection probabilities, rather than by detailed properties of the structures of the percolation clusters. The proofs involve some properties of the clusters, in particular cutsets, graph diameters and lengths of paths.

We end the Introduction by recalling \((\Omega_N, d)\) and some things about it. For an integer \( N \geq 2 \), the hierarchical group (also called hierarchical lattice) of order \( N \) is defined as

\[ \Omega_N = \{ x = (x_1, x_2, \ldots) : x_i \in \mathbb{Z}_N, x_i = 0 \text{ a.a.i} \} \]

with addition componentwise mod \( N \), where \( \mathbb{Z}_N \) is the cyclic group of order \( N \). The hierarchical distance on \( \Omega_N \), defined as

\[ d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ \max\{i : x_i \neq y_i\} & \text{if } x \neq y, \end{cases} \]

satisfies the strong (non-Archimedean) triangle inequality,

\[ d(x, y) \leq \max\{d(x, z), d(z, y)\} \quad \text{for any } x, y, z. \]

Hence \((\Omega_N, d)\) is an ultrametric space, and it can be represented as the top of an infinite regular \( N \)-ary tree where the distance between two points is the number of levels from the top to their closest common node. The point \((0, 0, \ldots) \in \Omega_N\) is taken as origin and denoted by \( 0 \). The probability of an edge connecting two points \( x \) and \( y \) is given by

\[ p_{x,y} = \min \left( \frac{c_k}{N^k(1+\delta)k}, 1 \right) \text{ if } d(x, y) = k, \]

where \( \delta > -1 \) and \( c_k > 0 \) for every \( k \), all edges being independent. Note that any point in a percolation cluster has a finite (random) number of neighbours since the number has finite expectation, hence the simple (nearest neighbour) random walk on a percolation cluster is well defined.

An essential property of ultrametric spaces that differentiates them from Euclidean spaces is that two balls are either disjoint or one is contained in the other. The following definitions and properties are used throughout. The ball of diameter \( k \geq 0 \) containing \( x \in \Omega_N \) is defined as \( B_k(x) = \{ y : d(x, y) \leq k \} \). Those balls are generally referred to as \( k \)-balls. They contain \( N^k \) points. For \( k > 0 \), a \( k \)-ball is the union of \( N \) disjoint \((k-1)\)-balls that are at distance \( k \) from each other. For \( j > k \), we call \( B_j(0) \setminus B_k(0) \) the \emph{annulus} \((k, j)\), or \((k, j)\)-annulus. It contains \( N^j(1-N^k-j) \) points. A \( j \)-ball is the union of \( N^{j-k} \) disjoint \( k \)-balls. The \( k \)-balls in the \((k, j)\)-annulus are at distance at least \( k+1 \) and at most \( j \) from each other. This and (1.4) allow to obtain upper and lower bounds for the probability that subsets of two \( k \)-balls in the \((k, j)\)-annulus are connected by at least one edge. Such bounds are used in the proofs.

\section{Percolation in \( \Omega_N \) for \( \delta = 1 \)}

The results for \( \delta < 1 \) and \( \delta > 1 \) have been mentioned in the Introduction. For \( \delta = 1 \) we regard the model as the random graph

\[ G_N^\infty = G_N^\infty(C_0, C_1, C_2, \alpha) := G(V_\infty, E_\infty) \]

with vertices \( V_\infty = \Omega_N \) and edges \( E_\infty \), and with the probability of connection by an edge \((x, y)\)

\[ P((x, y) \in E_\infty) = p_{x,y} = \min \left( \frac{c_k}{N^{2k}}, 1 \right) \text{ if } d(x, y) = k, \]

all connections being independent, and the \( c_k \) are of the form

\[ c_k = C_0 + C_1 \log k + C_2 k^\alpha, \]

with constants \( C_0 \geq 0, C_1 \geq 0, C_2 > 0 \) and \( \alpha > 0 \).
Theorem 2.1 For sufficiently large $C_2$, there exists a unique percolation cluster of positive density in $\Omega_N$ to which $0$ belongs with positive probability.

The proof of this result is given in the next subsection. We begin with the formulation for the renormalization method.

2.1 The hierarchy of random graphs

A collection of vertices in a subset of $\Omega_N$ any two of which are linked by a path of edges is called a cluster of the subset. We consider for each $k$-ball a maximal cluster with edges only within the ball and not through paths going outside the ball. If there are more than one cluster, then one of them is chosen uniformly at random. In this way each $k$-ball has a unique attached cluster. The proof will be based on the connections between clusters in $k$-balls. When we refer to connections between $k$-balls we mean direct edge connections (one or more) between their clusters. Due to our assumptions, the clusters in different $k$-balls are i.i.d. An infinite connected subset of $\Omega_N$ is called a percolation cluster.

The main idea is to consider the distribution of the clusters in the balls $B_k(0)$ of increasing $k$ by relating the random graphs in these balls to a hierarchy of ultrametric random graphs.

2.1.1 Erdős-Rényi graphs with random weights, ultrametric random graphs

In the classical graph $G(n,p)$ introduced by Gilbert these graphs have a set of $n$ vertices denoted by $V$ and there is an edge between each pair of vertices with probability $p$ with these assigned independently for different pairs (see e.g. [14, 37] for background). The behaviour of these graphs together with the random graphs $G(n,m)$ in the limit as $n \to \infty$ were studied by Erdős and Rényi in a series of important papers. We consider a modification of those graphs, namely, $G(N, \{x_i\}_{i \in V}) = G(N, \{p(x_i, x_j)\})$ in which the vertices have independent random weights $\{x_i\}_{i \in V}$, and the probability that $i$ and $j$ are connected by an edge is a function $p(x_i, x_j)$, the edges chosen independently conditioned on the weights. These are ultrametric random graphs as stated in the Introduction.

Given $G_N^\infty$ as above we now introduce a sequence of related ultrametric random graphs $G_k(N, \{X_{k-1}(i)\}_{i \in V_k})$, $k \geq 1$, where $X_0(i) = 1$ and for $k \geq 2$, the weights $\{X_{k-1}(i)\}_{i \in V_k}$ are associated to the clusters in the $N$ disjoint $(k-1)$-balls in $B_k(0)$ indexed by $V_k$, $|V_k| = N$. The weights $\{X_{k-1}(i)\}_{i \in V_k}$ are determined by the number of vertices in the clusters in these $(k-1)$-balls normalized by $N^{k-1}$. In view of our assumptions the $X_{k-1}(i)$ are i.i.d. $[0,1]$-valued random variables for each $k$,

$$X_{k-1}(i) = \frac{|C_{k-1}(i)|}{N^{k-1}}, \quad i \in V_k,$$

where $C_{k-1}(i)$ denotes the cluster in the $i$th $(k-1)$-ball. We refer to $X_{k-1}(i)$ as the density of the $i$th $(k-1)$-ball. For $N$ fixed our aim is to determine what happens as $k \to \infty$. Properties of the graph $G_k(N, \{X_{k-1}(i)\}_{i \in V_k})$ as $k \to \infty$ provide information on $G_N^\infty$, hence the behaviour of the cluster of $B_k(0)$ as $k \to \infty$ will imply a result on percolation in $\Omega_N$.

We denote the distribution of $X_k(i)$ by $\mu_k \in \mathcal{P}([0,1])$. Then we have for each $k$,

$$\mu_k = \Phi_k(\mu_{k-1}),$$

where $\Phi_k$ is a renormalization mapping

$$\Phi_k : \mathcal{P}([0,1]) \to \mathcal{P}([0,1]).$$

Note that $\mu_k$ depends on the edges within a $(k-1)$-ball (which determine $\mu_{k-1}$) and also on the edges between different $(k-1)$-balls in a $k$-ball, and that $\mu_k$ is an atomic measure.

We now analyse the sequence $\mu_k$, $k \geq 1$, for the class of connection probabilities (2.1), (2.2) (we note that there is no loss of generality in assuming $C_0 = C_1 = 0$ but it simplifies the notation).
In order to prove percolation of positive density it suffices to construct the sequence of $[0,1]$-valued random variables $X_k, k \geq 1$, satisfying

$$P(X_k > a) = \mu_k((a,1]), \text{ for all } a \in (0,1)$$

and existence of $a > 0$ such that

$$\liminf_{k \to \infty} P(X_k > a) > 0.$$

Consider two $(k - 1)$-balls in a $k$-ball (which are at distance $k$ from each other) having densities $x_1, x_2$ respectively, and define

$$p(x_1, x_2, k) = P(\text{two } (k - 1)\text{-balls in a } k\text{-ball with densities } x_1, x_2 \text{ are connected}).$$

Note that $p(x_1, x_2, k)$ is an increasing function of $x_1$ and $x_2$. Then from (2.1), (2.2),

$$p(x_1, x_2, k) = 1 - \left(1 - \frac{C_k^k}{N^{2k}}\right)^{N^{2(k - 1)x_1x_2}},$$

and

$$p(x_1, x_2, k) \sim 1 - e^{-\left(\frac{C_k^k}{N^{2k}}\right)x_1x_2} \text{ for large } k.$$

2.1.2 Proof of Theorem 2.1

The idea of the proof is to obtain lower bounds on the expected values of the sequence of random variables \(\{X_k(0)\}_{k \geq 1}\), where $X_k(x)$ denotes the density of $B_k(x)$, that is $X_k(x) = |C_k(x)|/N^k$, where $C_k(x)$ denotes the cluster in $B_k(x)$. Then as remarked above $\mu_k$ is the probability law of $X_k(x)$ which is independent of $x$ and the latter will be suppressed. By our assumptions \(\{X_k(x_i)\}\) are independent if for $i \neq j$, $d(x_i, x_j) \geq k + 1$.

**Lemma 2.2** Let $X$ be a random variable with values in $[0,1]$ and $0 < a < 1$. Then

$$P(X \geq a/2) = \frac{E[X] - a/2}{1 - a/2}.$$

**Proof.** Let $p = P(X \geq a/2)$. Then

$$E[X] \leq p + \frac{a}{2}(1 - p),$$

hence

$$p \geq \frac{E[X] - a/2}{1 - a/2}.$$

**Lemma 2.3** Assume that

$$\liminf_{k \to \infty} E[X_k] = \liminf_{k \to \infty} \frac{|C_k|}{N^k} = a > 0.$$

Then percolation of positive density occurs.
Proof. Lemma 2.2 implies
\[ \liminf_{k \to \infty} P\left( \frac{|C_k|}{N^k} \geq \frac{a}{2} \right) \geq \frac{a^2}{2(2-a)}. \]

If the cluster of \( B_k(0) \) contains at least \( aN^k/2 \) points, then the probability that 0 belongs to the cluster is at least \( a/2(2-a) \) by transitivity (cf. [22], Lemma 5.6). Hence
\[ (2.9) \liminf_{k \to \infty} P(0 \text{ belongs to a "big" cluster in } B_k(0)) \geq \frac{a^2}{2(2-a)} > 0, \]
where a "big" cluster in \( B_k(0) \) has at least \( aN^k/2 \) vertices.

On the other hand if we assume that the cluster containing 0 is finite w.p. 1, then the probability that 0 belongs to a big cluster goes to 0 as \( k \to \infty \), which contradicts (2.9). □

To prove Theorem 2.1 we will show that the hypothesis of Lemma 2.3 is satisfied.

We write the sequence of weights \( X_k, k \geq 1 \), as follows.
\[ X_{k+1} = \frac{1}{N} \sum_{i=1}^{N} 1_{i \in C_+}, X_{k,i}, \]
where \( \{X_{k,i}, i = 1, \ldots, N\} \) denote the (i.i.d.) densities of the N disjoint \( k \)-balls in \( B_{k+1}(0) \), and \( C_+ \subset V_{k+1} \) is the set of indices of the underlying \( k \)-balls (some \( k \)-balls may have null weights). Hence
\[ (2.10) \quad E[X_{k+1}] \leq E[X_k] \quad \text{for all } k. \]

Now consider the random graph \( G_k(N, \{x_1, \ldots, x_N\}) = G_k(N, \{p(x_i, x_j, k)\}) \), where \( p(x_i, x_j, k) \) is given by (2.5). Given the densities \( (X_{k-1,1}, \ldots, X_{k-1,N}) = (x_1, \ldots, x_N) \), then the probability that all N \((k-1)\)-balls in a \( k \)-ball are connected is
\[ (2.12) \quad P(X_k = \frac{1}{N}(x_1 + \cdots + x_N) | (x_1, \ldots, x_N)) = P(G_k(N, \{p(x_i, x_j, k)\}) \text{ is connected}). \]
If \( x_i \geq \varepsilon > 0 \) for \( i = 1, \ldots, N \), then
\[ (2.13) \quad P(X_k = \frac{1}{N}(x_1 + \cdots + x_N) | (x_1, \ldots, x_N)) \geq P(G_k(N, p(\varepsilon, \varepsilon, k) \text{ is connected})). \]
If all the \((k-1)\)-balls are isolated, then
\[ (2.14) \quad P(X_k = \frac{1}{N}(x_1 \lor \cdots \lor x_N) | (x_1, \ldots, x_N)) = \prod_{i<j=1}^{N} (1 - p(x_i, x_j, k)), \]
where the right side is the probability that no pair of \( k \)-balls is connected.

By the independence of the densities of disjoint \((k-1)\)-balls,
\[ (2.15) \quad P((X_{k-1,1}, \ldots, X_{k-1,N}) = (x_1, \ldots, x_N)) = \prod_{i=1}^{N} \mu_{k-1}(x_i). \]

We now can state a stronger form of (2.3).

**Lemma 2.4** For sufficiently large \( C_2 \), there exists \( a > 0 \) such that as \( n \to \infty \),
\[ (2.16) \quad E[X_n] \to a, \]
\[ (2.17) \quad Var[X_n] \to 0, \]
and
\[ (2.18) \quad \mu_n \Rightarrow \delta_a. \]
We first consider the case $N = 2$ to illustrate the idea of the proofs of Lemma 2.4 and Theorem 2.1

**Proof of Lemma 2.4 for $N = 2$.** Fix $0 < \varepsilon < 1$ (to be chosen sufficiently small), and let

\[(2.19) \quad q_n(\varepsilon) = \sup\{1 - p(x_1, x_2, n) : x_1, x_2 \geq \varepsilon\} \]

\[= P(\text{two } (n-1)\text{-balls in an } n\text{-ball with densities } \geq \varepsilon \text{ are not connected}) \]

Then $q_n(\varepsilon)$ is decreasing in $n$ for large $n$, and from (2.3), (2.4),

\[(2.20) \quad 1 - q_n(\varepsilon) \geq 1 - \left(1 - \frac{C_{2n}^2}{N^2}\right)^{N^2(n-1)x_1x_2} \sim 1 - e^{-\left(\frac{C_{2n}^2}{N^2}\right)x_1x_2} \quad \text{for large } n, \]

\[(2.21) \quad q_n(\varepsilon) \leq (q(\varepsilon))^{n^2}, \]

where

\[(2.22) \quad q(\varepsilon) := e^{-C_{2n}^2/N^2}, \]

hence

\[(2.23) \quad \sum_n q_n(\varepsilon) < \infty. \]

Let

\[(2.24) \quad z_n(\varepsilon) = P(X_n < \varepsilon). \]

By Lemma 2.3,

\[(2.25) \quad r_n(2\varepsilon) := P(X_n \geq 2\varepsilon) \geq E[X_n] - 2\varepsilon \quad \text{for large } n. \]

To obtain a lower bound for $E[X_n]$ we first note that

\[(2.26) \quad E[X_{n+1, X_{n+1} \geq \varepsilon}] \geq \int^1 \int^1 x + y \overline{1}_{x+y \geq 2\varepsilon} F_n(dx) F_n(dy) \cdot p(x_1, x_2, n) \]

\[\geq \int^1 \int^1 x F_n(dx) F_n(dy) \cdot (1 - q_n(\varepsilon)) \]

\[= (1 - z_n(\varepsilon)) E[X_{n, X_n \geq \varepsilon}] \cdot (1 - q_n(\varepsilon)), \]

where $F_n(dx)$ denotes the distribution of the random variable $X_n$. Therefore for $n > n_0$ (to be taken sufficiently large),

\[(2.27) \quad E[X_n] \geq E[X_{n, X_n \geq \varepsilon}] \]

\[= \prod_{k=n_0}^{n-1} (1 - z_k(\varepsilon))(1 - q_k(\varepsilon)) E[X_{n_0, X_{n_0} \geq \varepsilon}]. \]

\[(2.28) \quad z_{n+1}(\varepsilon) = P(X_{n+1} < \varepsilon) \]

\[\leq (P(X_n < \varepsilon))^2 + 2P(X_n < \varepsilon)P(\varepsilon < X_n \leq 2\varepsilon) + (P(\varepsilon < X_n \leq 2\varepsilon))^2 q_n(\varepsilon) \]

\[\leq (P(X_n < \varepsilon))^2 + 2P(X_n < \varepsilon)(1 - r_n(2\varepsilon)) + (1 - r_n(\varepsilon))^2 q_n(\varepsilon) \]

\[\leq z_n^2(\varepsilon) + z_n(\varepsilon)(2(1 - r_n(2\varepsilon)) + q_n(\varepsilon)(1 - r_n(2\varepsilon))^2 \]

\[= z_n(\varepsilon)(z_n(\varepsilon) + 2(1 - r_n(2\varepsilon)) + q_n(\varepsilon)(1 - r_n(2\varepsilon))^2 \]

\[\leq z_n(\varepsilon)(z_n(\varepsilon) + 2(1 - r_n(2\varepsilon))) + q_n(\varepsilon) \]
where we have used $P(\varepsilon < X_n \leq 2\varepsilon) \leq 1 - r_n(2\varepsilon)$.

In order to prove that $\liminf E[X_n] > 0$ for sufficiently large $C_2$, from (2.27) it suffices to verify that we can choose $\varepsilon, n_0, z_{n_0}, E[X_{n_0}1_{X_{n_0} \geq \varepsilon}]$ and $C_2$ such that

$$
\liminf_{n \to \infty} \prod_{k=n_0}^{n-1} (1 - z_k(\varepsilon))(1 - q_k(\varepsilon))E[X_{n_0}1_{X_{n_0} \geq \varepsilon}] > 0.
$$

(2.29)

Suppose that

$$
z_{n_0}(\varepsilon) \leq \frac{\varepsilon}{2} \quad \text{and} \quad C_2 \quad \text{is large enough so that} \quad q_{n_0}(\varepsilon) < \frac{\varepsilon}{2},
$$

(see (2.21), (2.22)) and assume that

$$
z_m + 2(1 - r_m(2\varepsilon)) < s, \quad \text{for} \quad n_0 \leq m \leq n,
$$

(2.31)

for some $s \in (0,1)$ such that

$$
s + \frac{\varepsilon}{2(1 - s)} \leq 1.
$$

Then it follows from (2.28), (2.30) that for $n \geq n_0$,

$$
z_{n+1}(\varepsilon) \leq z_{n_0}n^{n-n_0+1} + \sum_{k=n_0}^{n} s^{n-k}q_k(\varepsilon) \leq z_{n_0}n^{n-n_0+1} + \frac{q_{n_0}(\varepsilon)}{1 - s} \leq \frac{\varepsilon}{2},
$$

(2.33)

Then by (2.33), $\{z_n\}_{n_0 \leq m \leq n}$ are bounded by the terms of a summable sequence, namely (see (2.23)),

$$
\sum_{n=n_0}^{\infty} z_n(\varepsilon, s) = \frac{\varepsilon}{2} \sum_{n=n_0}^{\infty} s^{n-n_0} + \sum_{n=n_0}^{\infty} \sum_{k=n_0}^{n} s^{n-k}q_k(\varepsilon) = \frac{\varepsilon}{2(1 - s)} + \frac{1}{1 - s} \sum_{k=n_0}^{\infty} q_k(\varepsilon) < \infty.
$$

(2.34)

We first choose $2\varepsilon = 0.1$ and $s = 0.775$ which satisfies (2.32). Then by (2.27), $2(1 - r_n(2\varepsilon)) < 0.75$ and $z_{n_0} + 2(1 - r_n(2\varepsilon)) < s$ provided that $E[X_m] > 2/3$ for $n_0 \leq m \leq n$. We now choose $n_0$ sufficiently large so that

$$
\prod_{k=n_0}^{n-1} (1 - z_k(\varepsilon))(1 - q_k(\varepsilon)) > 0.9,
$$

and $C_2$ sufficiently large so that $E[X_{n_0}1_{X_{n_0} > \varepsilon}] \geq 3/4$ and $z_{n_0} \leq \varepsilon/2$. Note that by choosing $C_2$ sufficiently large we have $z_{n_0} = 0$ and $E[X_{n_0}1_{X_{n_0} > \varepsilon}] = 1$, since $X_{n_0}$ is atomic and positive. By continuity, this can also be done for connection probabilities strictly less than 1 but sufficiently close to 1. Then we can verify from (2.27) that $E[X_n] \geq E[X_{n_0}1_{X_{n_0} > \varepsilon}] > 0.9 \times 3/4 > 2/3$, so we have the consistency condition $E[X_n] > 2/3$ for all $n \geq n_0$, and therefore (2.31) holds for all $n \geq n_0$.

If the two $n$-balls in the $(n+1)$-ball have density $\geq \varepsilon$, then by considering the events that the two $n$-balls are connected, and that they are not connected, it is easy to show that

$$
E[X_{n+1}] \geq E[X_n](1 - q_n(\varepsilon)) + q_n(\varepsilon)\frac{E[X_n]}{2} = E[X_n](1 - \frac{q_n(\varepsilon)}{2}),
$$

(2.35)

which together with (2.11), (2.23) and Lemma 2.3 proves (2.16), (2.17), $\{E[X_n]\}$ is a Cauchy sequence, and also

$$
\text{Var}[X_{n+1}] \leq \frac{1}{2}E[X_n^2] + \frac{1}{2}(E[X_n])^2 - (E[X_n])^2(1 - \frac{q_n(\varepsilon)}{2})^2 \leq \frac{1}{2}\text{Var}[X_n] + q_n(\varepsilon),
$$

(2.36)
and then (2.35), (2.36) prove (2.18).

We now modify the argument with \( N = 2 \) to prove the theorem for general \( N \). To prepare, we begin with some lemmas.

**Lemma 2.5** Consider the Erdős-Rényi graph \( G(N, 1 - q) \). Let \( P(N, q) \) denote the probability that the graph is connected. Then as \( q \to 0 \),

\[
P(N, q) \geq 1 - C(N)q^{N-1},
\]

where \( C(N) \) is a constant such that \( C(N) \sim N \) for large \( N \)

**Proof.** Recall the basic formula (35), equation (4), also see [14], p. 198

\[
P(N, q) = 1 - \sum_{k=1}^{N-1} \binom{N-1}{k-1} P(k, q)q^{k(N-k)}.
\]

The result follows by noting that the dominant term in \( 1 - P(N, q) \) as \( q \to 0 \) is given by the smallest power of \( q \) on the right side of (2.38) which is \( q^{N-1} \) (from the terms \( k = 1 \) and \( k = N-1 \)).

**Corollary 2.6** Consider the graph \( G_k(N, \{x_1, \ldots, x_N\}) \). If \( x_i \geq \varepsilon \) for all \( i \), then for large \( k \)

\[
P(G_k(N, \{x_1, \ldots, x_N\})) \text{ is connected} \geq 1 - q_k^N(\varepsilon),
\]

where

\[
q_k^N(\varepsilon) = C(N)(q_k(\varepsilon))^{N-1}
\]

with

\[
q_k(\varepsilon) = q(N, k, \varepsilon) := e^{-C_2 \varepsilon^2 k^n / N^2},
\]

hence

\[
\sum_k q_k^N(\varepsilon) < \infty.
\]

**Proof.** By (2.20), (2.21), (2.22) the probability that two \((k-1)\)-balls in a \( k \)-ball with respective densities \( x_1, x_2 \geq \varepsilon \) are connected is given by

\[
1 - q_k(\varepsilon) \geq 1 - e^{-C_2 \varepsilon^2 k^n / N^2} \quad \text{for large } k.
\]

The result then follows by Lemma 2.5.

For the \( N \) \((k-1)\)-balls in a \( k \)-ball, define

\[
p(x_1, \ldots, x_N, n) = P(\text{all the } (k-1)\text{-balls with densities } x_1, \ldots, x_N \text{ are connected}).
\]

Then by Corollary 2.6 and analogously as in the case \( N = 2 \) (see (2.26)),

\[
E[X_n 1_{X_n \geq \varepsilon}] \geq \int_0^1 \cdots \int_0^1 \sum_{i=1}^{N-1} \frac{x_i}{N} 1_{\sum_{i=1}^{N-1} x_i \geq N\varepsilon} \prod_{i=1}^N F_n(dx_i) \cdot p(x_1, \ldots, x_N, n)
\]

\[
\geq \frac{1}{N} \sum_{i=1}^N \int_\varepsilon^1 x_i F_n(dx_i) \prod_{j \neq i} F_n(dx_j) \cdot (1 - q_n^N(\varepsilon))
\]

\[
= (1 - z_n(\varepsilon))^{N-1} E[X_n 1_{X_n \geq \varepsilon}] \cdot (1 - q_n^N(\varepsilon)),
\]

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where
\[(2.46) \ z_n(\varepsilon) = P(X_n < \varepsilon).\]

Therefore for \(n > n_0,\)
\[(2.47) \ E[X_n] \geq E[X_n1_{X_n \geq \varepsilon}] = \prod_{k=n_0}^{n-1} (1 - z_k(\varepsilon))^{N-1}(1 - q_k^N(\varepsilon))E[X_{n_0}1_{X_{n_0} \geq \varepsilon}].\]

Let
\[(2.48) \ r_n(N\varepsilon) = P(X_n \geq N\varepsilon).\]

Then by Lemma 2.2
\[(2.49) \ r_n(N\varepsilon) \geq \frac{E[X_n] - N\varepsilon}{1 - N\varepsilon}.\]

**Lemma 2.7**
\[(2.50) \ P(X_{n+1} < \varepsilon) \leq NP(X_n < \varepsilon)(1 - r_n(N\varepsilon))^{N-1} + q_n^N(\varepsilon).\]

**Proof.** We first note that if the density of one of the \(n\)-balls in the \((n + 1)\)-ball is larger than \(N\varepsilon\) then \(X_{n+1} > \varepsilon\). Second, if the densities of all the balls are larger than \(\varepsilon\) and the balls are connected, then \(X_{n+1} > \varepsilon\). Therefore
\[
\{X_{n+1} < \varepsilon\} \subseteq \{\text{densities of all } n\text{-balls } < N\varepsilon\} \cap
\{\{\text{densities all } n\text{-balls } > \varepsilon \text{ and not connected}\} \cup \{\text{density of at least one } n\text{-ball } < \varepsilon\}\}.
\]

Then
\[
P(X_{n+1} < \varepsilon) \leq NP(X_n < \varepsilon)(P(X_n < N\varepsilon))^{N-1} + (P(\varepsilon \leq X_n < N\varepsilon))NP(G_n(N, \{\varepsilon, \ldots, \varepsilon\}) \text{ not connected})
\]
\[
\leq NP(X_n < \varepsilon)(1 - r_n(N\varepsilon))^{N-1} + (P(\varepsilon \leq X_n < N\varepsilon))^{N}q_n^N(\varepsilon)
\]
\[
\leq NP(X_n < \varepsilon)(1 - r_n(N\varepsilon))^{N-1} + q_n^N(\varepsilon).
\]

The first summand on the right corresponds to the case that at least one density \(\leq \varepsilon\) and all densities \(< N\varepsilon\); the second summand corresponds to the case in which all densities \(x_i\) are in \([\varepsilon, N\varepsilon]\) and the balls not connected. \(\blacksquare\)

From (2.43), (2.49),
\[(2.51) \ z_{n+1}(\varepsilon) \leq Nz_n(\varepsilon)(1 - r_n(N\varepsilon))^{N-1} + q_n^N(\varepsilon).
\]

Using (2.43)-(2.51) we proceed analogously as in the case \(N = 2\). We can then choose \(\varepsilon = 0.1/N, E[X_{n_0}1_{X_{n_0} \geq \varepsilon}] \geq 3/4\), so that (using inequality (2.49)),
\[
N(1 - r_n(N\varepsilon))^{N-1} \leq 2(1 - r_n(N\varepsilon)) < 0.75,
\]
provided that \(E[X_n1_{X_n \geq \varepsilon}] \geq 2/3\). Finally, we can then choose \(n_0, z_{n_0}\) and \(C_2\) so that the sequence \(\{z_n(\varepsilon)\}\) is summable as in the case \(N = 2\) and we have
\[
\prod_{k=n_0}^{\infty} (1 - z_k(\varepsilon))^{N-1}(1 - q_k^N(\varepsilon)) > 0.9
\]
so that $E[X_n 1_{X_n \geq \varepsilon}] \geq 2/3$ for all $n \geq n_0$ as in the case $N = 2$. ■

**Proof of Lemma 2.4 for general $N$.** This follows from (2.11) and the next inequalities analogous to (2.35), (2.36), that can be proved by considering the events that all the $N$ $n$-balls are connected, or not, and using (2.39), (2.40), (2.41), (2.42):

(2.52) $E[X_{n+1}] \geq E[X_n] + O(q_n^N(\varepsilon))$,

and

(2.53) $Var[X_{n+1}] \leq \frac{1}{N} Var[X_n] + O(q_n^N(\varepsilon))$

as $n \to \infty$.

The proof of percolation is then finished as in the case $N = 2$ using the previous formulas, and the uniqueness follows from Theorem 1.2 in [42]. This completes the proof of Theorem 2.1. ■

### 3 Properties of the percolation clusters

In this section we will obtain some properties of the percolation clusters that will be used for studying behaviour or random walks on the clusters. We will use parts of the scheme of [22] referred to in the introduction in the case $\delta = 1$, that is,

(3.1) $k_n = \lceil Kn \log n \rceil, \; n = 1, 2, \ldots$,

(3.2) $c_{k_n} = C + a \log n \cdot n^{b \log N}$,

$K > 0, \; C \geq 0, \; a > 0, \; b > 0, \; c_{k_n} \leq c_k \leq c_{k+1}$ for $k_n < k < k_{n+1}$. (3.1) implies that

(3.3) $\kappa = k_{n+1} - k_n \sim K \log n$ as $n \to \infty$.

#### 3.1 Cutsets for $\delta = 1$

A *cutset* of a graph is a set of edges in the graph which, if removed, disconnects the graph.

We consider the percolation cluster of $\Omega_N$ in the case $\delta = 1$ with $C_2 > 0$. We will construct a sequence of cutsets for the cluster that will be used to prove recurrence of the random walk on the cluster in the case $\alpha \leq 1$.

The following argument holds with $K = 1$ (in the special case $N = 2$ we need a minor modification which we omit here). First recall that by [22] (Lemma 5.2 with $K = 1$) we have the following result (this does not need the condition $2/\log N < K < b$).

**Lemma 3.1** For $0 < b < 2 - 1/\log N$, with probability one there exists $n_0$ such that for all $n \geq n_0$ there is no skipping over two successive annuli $(k_n, k_{n+1})$, that is, there are no single edge connections between the annulus $(k_{n-1}, k_n)$ and the annuli $(k_{n+2}, k_{n+3})$, $(k_{n+3}, k_{n+4})$, etc.

**Lemma 3.2** For any $\alpha > 0$ there exists a sequence of finite cutsets $\Pi_j, \; j \geq 1$, for the percolation cluster that are pairwise disjoint for large $j$, and such that

(3.4) $E[\Pi_j] \leq \frac{\kappa_j}{N}$ for large $j$,

where

(3.5) $\kappa_j = a 2^\alpha \log j \cdot j^\alpha$. 

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Proof. We take $b$ so that $0 < b < \min(\alpha / \log N, 2 - 1/ \log N)$.

Let $I_j = (k_{2j}, k_{2(j+1)})$-annulus, then by Lemma 3.1 $I_j$ is connected by edges only to $I_{j-1}$ and $I_{j+1}$ for large $j$.

Note that for $2(j+1) < \ell \leq 2(j+2)$,
$$c_\ell \lesssim \kappa_j$$
for large $j$.

where $\kappa_j$ is given by (3.3). For a vertex $x \in I_j$, let
$$\mathcal{M}_j(x) = \{ \text{vertices in } I_{j+1} \text{ connected to } x \text{ by an edge} \}.$$  

Then
$$|\mathcal{M}_j(x)| = \sum_{\ell = k_{2(j+1)} + 1}^{k_{2(j+2)}} \text{Bin} \left( N^\ell - N^{\ell-1}, c_\ell / N^{2\ell} \right).$$

By ultrametricity, the distribution of $|\mathcal{M}_j(x)|$ is the same for any $x \in I_j$. Then by (3.3)
$$E[|\mathcal{M}_j(x)|] = (1 - \frac{1}{N}) \sum_{\ell = k_{2(j+1)} + 1}^{k_{2(j+2)}} \frac{c_\ell}{N^\ell} \sim \kappa_j \left( \frac{1}{N^{k_{2(j+1)}}} - \frac{1}{N^{k_{2(j+2)}}} \right) \sim \frac{\kappa_j}{N^{k_{2(j+1)}}}$$

for large $j$.

Let
$$\Pi_j = \{ \text{edges connecting vertices in } I_j \text{ restricted to the cluster and vertices in } I_{j+1} \}.$$ 

Then the sets $\Pi_j$ are finite cutsets for the cluster and they are pairwise disjoint for large $j$. Writing $\Pi_j$ as
$$\Pi_j = \bigcup_{x \in I_j} \mathcal{M}_j(x),$$

then
$$E[|\Pi_j|] \leq |I_j| E[|\mathcal{M}_j|] \leq N^{k_{2(j+1)}} \frac{\kappa_j}{N^{k_{2(j+1)}}} \lesssim \frac{\kappa_j}{N}$$
for large $j$.

3.2 Graph diameters and path lengths

We will obtain bounds for the lengths of paths in the percolation clusters joining two points within distance $k_n$ from 0 for large $n$, for $\delta = 1$ and $\delta < 1$. Some of these will be used to prove transience of the random walk on the clusters.

We assume that $n_0$ is large enough according to the proofs of Theorems 3.1(b) and 3.5(b) in [22] (we will refer to parts of those proofs). This means that the things we will do are possible for $n \geq n_0$, in particular there exist the direct edge connections between clusters we will refer to. If the two points are in the same $k_{n_0}$-cluster, the length of a path joining them is bounded by the diameter of the cluster, denoted by $D(n_0)$ below. Therefore we will assume that the two points lie in the clusters of different $k_{n_0}$-balls. We proceed as follows:

- Find bounds for the diameters of the Erdős-Rényi random graphs $G(N_n, p_n)$ defined below, whose vertices are $k_n$-balls in a $k_{n+1}$-ball, and the connection probability $p_n$ is defined in terms of direct edges (one or more) between the clusters of those $k_n$-balls. These graphs are also ultrametric random graphs.
• Since the $k_n$-balls consist of $k_{n-1}$-balls, find bounds for the length of a path of $k_{n-1}$-balls in a $k_n$-ball connecting an incoming $k_{n-1}$-ball and an outgoing $k_{n-1}$-ball (this may be called a $k_n$-level path). Such a path may visit a $k_{n-1}$-ball more than one time, but that does not matter because we consider shortest paths.

• Having done the previous two things, do the same going from $n$ to $n - 1$, etc., down to $n_0$, where we end up with $D(n_0)$, which is the (random) diameter of the cluster of a $k_{n_0}$-ball. We don’t know the value of $D(n_0)$, but $n_0$ is fixed, so $D(n_0)$ (or a constant bound for it) can be treated as a constant.

In the arguments and calculations for path lengths we may think of paths within the ball $B_{k_n}(0)$ joining $0$ to a point in the $(k_{n-1}, k_n]$-annulus. However, by ultrametricity any point in $B_{k_n}(0)$ is a center, so the bounds hold as well for paths joining any two points within distance $k_n$ from $0$.

We recall from [22] (Def. 4.1, Def. 5.5) that a $k_n$-ball is “good” if its cluster has size at least $N^\gamma k_n$ for $\delta < 1$, where $(1 + \delta)/2 < \gamma < 1$, and if its cluster has size at least $\beta N^{k_n}$ for $\delta = 1$, where $0 < \beta < 1$. In the case $\delta = 1$ we assume $b > 2/\log N$, $(K = 1)$, which corresponds to $\alpha > 2$. In the proofs of Theorems 3.1(b) and 3.5(b) it is shown that for all but finitely many $n$ the $k_n$-balls are good (see [22], (4.22) for $\delta < 1$, (5.11), (5.23) for $\delta = 1$).

Let $N_n$ denote the number of good $k_n$-balls in a $k_{n+1}$-ball, and

\[ p_n = P(\text{the clusters in two good } k_n\text{-balls in a } k_{n+1}\text{-ball are connected}). \]

Note that $p_n$ is random because the sizes of the clusters are random. We consider the Erdős-Rényi graphs $G(N_n, p_n)$. In all the cases, $N_n \rightarrow \infty$ as $n \rightarrow \infty$. In the proof of Theorem 5.3(b) in [22] it is shown that for $\delta = 1$, $b \geq 1$, and $\beta > 1/5$, the graph $G(N_n, p_n)$ becomes connected for large $n$. We shall see that for $b \geq 2$ it even becomes complete (all pairs of vertices are connected).

### 3.2.1 Diameters of the graphs $G(N_n, p_n)$

First we obtain bounds for the diameters of the graphs $G(N_n, p_n)$ in the following cases:

#### Case 1. $\delta < 1$

From [22] ((4.5), (4.8), (4.10)), we have the lower bound for $\log N_n$:

\[ p_n \geq 1 - \exp(-cN^{2\gamma k_n - (1+\delta)k_{n+1}}) > 1 - \exp(-cN^\epsilon n \log n) \quad \text{as } n \rightarrow \infty, \]

with some $0 < \epsilon < 1$, hence $p_n \rightarrow 1$ and $N_n p_n / \log N_n \rightarrow \infty$ as $n \rightarrow \infty$, therefore by Theorem 2 in [18] $\text{diam}(G(N_n, p_n))$ is concentrated on at most two values at

\[ \frac{\log N_n}{\log(N_n p_n)} \rightarrow 1 \quad \text{as } n \rightarrow \infty, \]

so, $\text{diam}(G(N_n, p_n)) \leq 2$ for large $n$.

#### Case 2. $\delta = 1$, $b > 2$.

**Lemma 3.3** In this case the graph $G(N_n, p_n)$ is complete and $\text{diam}(G(N_n, p_n)) = 1$ for large $n$.

**Proof.** From [22] (proof of Lemma 5.7 except the last step), we have

\[ p_n \geq 1 - \exp(-\beta^2 a \log n \cdot N^{(b-2) \log n}) \rightarrow 1 \quad \text{as } n \rightarrow \infty. \]

Then, since $N_n \leq N^{\log n}$,

\[ q_n := P(\text{some pair of clusters of two good } k_n\text{-balls in a } k_{n+1}\text{-ball is not connected}) \leq N^{2\log n}(1 - p_n) \leq N^{2\log n} \exp(-\beta^2 a \log n \cdot N^{(b-2) \log n}) \]

\[ = n^{2\log N / n^{\beta^2 a n (b-2) \log N}}, \]

Therefore $q_n < \infty$, hence by Borel-Cantelli for all but finitely many $n$ the graph $G(N_n, p_n)$ is complete. So, $\text{diam}(G(N_n, p_n)) = 1$ for large $n$.

We will also need $k_n$ as in (3.1) with $K > 1$ (see again the proof of Lemma 5.7 in [22]).
Lemma 3.4 If $\delta = 1$ and $b > 2K > 2$, then the graph $G(N_n, p_n)$ is complete and $\text{diam}(G(N_n, p_n)) = 1$ for large $n$.

Case 3. $\delta = 1, b = 2$.
As in case 2,
$$p_n \geq 1 - \exp(-\beta^2 a \log n) \rightarrow 1 \text{ as } n \to \infty,$$
then as in case 1, $\text{diam}(G(N_n, p_n)) \leq 2$ for large $n$.

Case 4. $\delta = 1, 1 < b < 2$.
Again as above,
$$p_n \geq 1 - \exp\left(-\frac{\beta^2 a \log n}{N(2-b) \log n}\right) > \frac{\beta^2 a \log n}{2N(2-b) \log n}$$
for large $n$ such that
$$\frac{\beta^2 a \log n}{2N(2-b) \log n} < 0.7968.$$ Since $N_n \leq N^{\log n}$, and from the proof of Theorem 3.5(b) in ([18], (5.17), (5.18) and step 1) we have $N_n \geq N^\log n$ for large $n$, then
$$\frac{N_n p_n}{\log N_n} \geq \frac{\beta^3 a \log n \cdot N^b \log n}{2N^{\log n} \log n \cdot \log N} = \frac{\beta^3 a}{2 \log N} N^{(b-1) \log n} \rightarrow \infty \text{ as } n \to \infty,$$
then, again by Theorem 2 in ([18], (5.17), (5.18) and step 1) we have $N_n \geq \beta N^{\log n}$ for large $n$, then
$$\frac{\log N_n}{\log N} \leq \frac{\log n \cdot \log N}{\log n \cdot N^{(b-1) \log n}} \leq \frac{1}{b-1} \text{ as } n \to \infty,$$
so, $\text{diam}(G(N_n, p_n)) \leq b/(b-1)$ for large $n$. Note that this bound is continuous at $b = 2$ (case 3).

Case 5. $\delta = 1, b = 1$.
By Theorem 1.2 of ([51] and [22]), (Lemma 5.7),
$$p_n \geq \frac{\beta^2 a \log n}{N^{\log n}} = \frac{\lambda_n}{N_n} =: \bar{p}_n \text{ for large } n,$$
where
$$\lambda_n = \beta^2 a \log n \cdot \frac{N_n}{N^{\log n}} \leq \beta^2 a \log n \leq N_n^{1/1000} \text{ for large } n,$$
and $\text{diam}(G(N_n, \bar{p}_n))$ is concentrated on at most two values around
$$f(N_n, \lambda_n) = \frac{\log N_n}{\log \lambda_n} + \frac{2 \log N_n}{\log (1/\lambda_n^*)} + O(1) \text{ for large } n,$$
where
$$\lambda_n^* e^{-\lambda_n^*} = \lambda_n e^{-\lambda_n}, \lambda_n \to \infty, \lambda_n^* \to 0,$$
then
$$\lambda_n \sim \log \log n,$$
$$\log \lambda_n^* - \lambda_n^* = \log \lambda_n - \lambda_n,$$
since $N_n \geq \beta N^{\log n}$ for large $n$, then
$$\log(1/\lambda_n^*) = \lambda_n - \log \lambda_n - \lambda_n^* \geq \beta^3 a \log n,$$
and
$$f(N_n, \lambda_n) \leq \frac{\log N_n}{\log \log n} + C_1 \frac{\log N_n}{\log n} + O(1) \leq C_2 \log n \text{ for large } n,$$
so, $\text{diam}(G(N_n, p_n)) \leq L \log n$ for some constant $L > 0$ and large $n$. 

3.2.2 Path lengths in $G^\infty_N$

Let $n_0$ be large enough as mentioned above. Denote

- $D(n_0)$: the diameter of the cluster in a $k_{n_0}$-ball,
- $D(n_0 + j)$: a bound for $\text{diam}(G(N^j_{n_0}, p_{n_0}))$, $j \geq 1$,
- $L(n_0 + j)$: a bound for the length of a path joining $0$ to a point in the $(k_{n_0+j-1}, k_{n_0+j})$-annulus, $j \geq 1$,
- $L(n_0) = D(n_0)$.

Then $L(n_0 + 1) = D(n_0)(D(n_0 + 1) + 1) + D(n_0 + 1)$, because there are at most $D(n_0 + 1)$ edges in the $k_{n_0+1}$-ball that join $D(n_0 + 1)$ clusters of diameter $D(n_0)$ in $k_{n_0}$-balls, considering the two ends of the $k_{n_0+1}$-level path (a path of $k_{n_0}$-clusters), and that the path may enter and leave each $k_{n_0}$-cluster from different points that are joined by a path of length at most $D(n_0)$ in the $k_{n_0}$-cluster. So,

$$L(n_0 + 1) = D(n_0)D(n_0 + 1) + D(n_0) + (n_0 + 1).$$

Similarly,

$$L(n_0 + 2) = L(n_0 + 1)(D(n_0 + 2) + 1) + D(n_0 + 2) = D(n_0)D(n_0 + 1)D(n_0 + 2) + D(n_0 + 2) + D(n_0 + 1)D(n_0 + 2) + D(n_0 + 1)D(n_0 + 2) + D(n_0 + 1) + D(n_0 + 2) = (D(n_0) + 1)(D(n_0 + 1) + 1)(D(n_0 + 2) + 1) - 1,$$

We show that

$$L(n_0 + k) = \prod_{j=0}^{k}(D(n_0 + j) + 1) - 1 \text{ for } k \geq 0$$

by induction:

$$L(n_0 + k + 1) = L(n_0 + k)(D(n_0 + k + 1) + 1) + D(n_0 + k + 1) = \left(\prod_{j=0}^{k}(D(n_0 + j) + 1) - 1\right)(D(n_0 + k + 1) + 1) + D(n_0 + k + 1) = \prod_{j=0}^{k+1}(D(n_0 + j) + 1) - 1.$$

Hence

$$L(n_0 + j) \leq \prod_{k=0}^{j}(D(n_0 + k) + 1), \quad j \geq 1.$$

It follows that if $D(n_0 + k) < L$ for $k > 1$, with $L > 1$ (cases 1, 3 and 4 above), then

(3.7) $L(n_0 + j) < (D(n_0) + 1)(L + 1)^j$ for $j \geq 1$.

and if $D(n_0 + k) < L_1 \log k$ for $k > 1$, with $L_1 > 0$ (case 5 above) then

$$L(n_0 + j) < (D(n_0) + 1)(L \log j)^j \text{ for } j \geq 1 \text{ and some constant } L > 0.$$ 

The results that will be used for the proofs below for the cases $\delta < 1$, and $\delta = 1, b > 2$ are:

**Lemma 3.5** In the case $\delta < 1$,

(3.8) $L(n_0 + j) \leq K(n_0)(L + 1)^j$ for $j \geq 1$,

where $K(n_0) = D(n_0) + 1$.

**Proof.** This follows from case 1 and (3.7).
Lemma 3.6 In the case $\delta = 1$, $b > 2K > 2$.

\[(3.9) \quad L(n_0 + j) \leq CD(n_0)j \quad \text{for} \quad j \geq 1 \quad \text{and some constant} \quad C > 0.\]

Proof. Since $\text{diam}(G(N_n, p_n)) = 1$ for large $n$ by Lemma 3.4 the only thing that counts is the number of steps to go from the $(k_{n_0}, k_{n_0} + 1]$-annulus to the $(k_{n_0} + j, k_{n_0} + j + 1]$-annulus: $j$ steps. So (3.9) follows from Case 2 with $b > 2K$.

Remark. From the embedding referred to in [10] for long-range percolation in $\mathbb{Z}$ with $s \in (1, 2)$, which corresponds to $\delta \in (0, 1)$ in the ultrametric model, by using the ultrametric $\rho(x, y) = N_{d(x, y)}$ it follows that the length of a path between two points within distance $k_n$ from $0$ would be

\[
\sim \log(n \log n)^{\log 2/\log(2/(1+\delta))} \quad \text{as} \quad n \to \infty,
\]

The reason we get an upper bound of the form $3^n$ by (3.7) (case 1, with $L = 2$), which is much larger, is that we consider the clusters in $k_n$-balls only with paths contained in the $k_n$-balls, so we are neglecting paths that go outside the $k_n$-balls which could be shorter.

4 Random walks on the percolation cluster

4.1 Random walks and electric circuits

In this subsection we review briefly some basic background on random walks and electric circuits on graphs which will then be applied to random walks on the percolation clusters.

The nearest neighbour random walk on a finite or an infinite graph such as the percolation cluster is a Markov chain on the countable connected subset given by the graph. Here there is an edge between neighbours $x$ and $y$ with probability

\[
p_{xy} = \frac{1}{n(x)},
\]

where $n(x)$ is the number of neighbours of $x$ in the graph.

The random walk is a reversible since setting $\pi(x) = n(x)$ we have

\[
C(x, y) = \pi(x)p_{xy} = \pi(y)p_{yx} \quad \text{for all} \quad x, y.
\]

In this case $C(x, y)$ is called the conductance between $x$ and $y$ and the resistance $R(x, y)$ is defined as $R(x, y) = 1/C(x, y)$.

For any finite set $Z$ of vertices the effective conductance and effective resistance between a point $a$ and $Z$ are defined as

\[
C(a \leftrightarrow Z) = \frac{1}{\pi(a)}P(\tau_Z < \tau_a^+), \quad R(a \leftrightarrow Z) = 1/C(a \leftrightarrow Z).
\]

where $\tau_a^+$ is the first time after 0 that walk visits $a$ and $\tau_Z$ is the hitting time of $Z$.

If $G$ is an infinite connected graph, let $G_n$ be a finite subgraph of $G$ such that $G_n \uparrow G$ as $n \to \infty$ and $Z_n := G \setminus G_n$ (identified as a single vertex). Then the effective resistance from $a$ to $\infty$ is defined as

\[
R(a \leftrightarrow \infty) = \lim_{n \to \infty} \frac{1}{C(a \leftrightarrow Z_n)}.
\]

4.2 Criteria for transience and recurrence

Doyle and Snell [30] (also see [15, 11]) proved that the effective resistance is equivalent to the resistance computed using the laws of electric circuit theory applied to the circuit obtained by replacing each edge by a unit resistor resulting in the following criterion for transience and recurrence.
**Criterion for transience-recurrence** The random walk on an infinite connected graph is transient, respectively recurrent, if $R(a \leftrightarrow \infty)$ is finite, respectively infinite.

**Rayleigh monotonicity principle** Removing an edge increases the resistance between two points. Therefore to prove that the random walk on the graph is transient it suffices to show that it is transient on a subgraph.

We will use a related criterion for transience based on the Dirichlet’s minimization principle for energy of a flow in a circuit.

**Definition 4.1** A unit flow on a graph $G = (V, E)$ with source $a \in V$ is a function $\theta$ on the set of edges $E$ such that $\theta(x, y) = -\theta(y, x)$ and for all $x \neq a$,

$$\sum_{x \neq a} \theta(a, x) = 1 \quad \text{and} \quad \sum_{y \sim x} \theta(x, y) = 0 \text{ for all } x \neq a,$$

where $x \sim y$ means that $y$ is a neighbour of $x$.

**Definition 4.2** The energy of the flow is

$$E(\theta) = \sum_{e \in E^*} (\theta(e))^2 R(e),$$

where $E^*$ is the set of directed edges and $R(e) := R(x, y)$ is the resistance of the edge $e$ from $x$ to $y$.

### 4.2.1 Transience, finite energy criterion

A random walk on a countable connected graph $G$ is transient iff there is a unit flow from any vertex $a$ to $\infty$ on $G$ with finite energy \[44\].

### 4.2.2 Recurrence, Nash-Williams criterion

If $\{\Pi_n\}$ is a sequence of disjoint finite cutsets in a locally finite graph $G$, each of which separates $a$ from infinity, then

$$R(a \leftrightarrow \infty) \geq \sum_n \left( \sum_{e \in \Pi_n} C(e) \right)^{-1}.$$

In particular, if the right-hand side is infinite, then the walk on $G$ is recurrent \[47\]. In our case the edges have unit resistance, so the random walk is recurrent if

$$\sum_n 1/|\Pi_n| < \infty.$$

### 4.3 Transience and recurrence of random walks on the percolation cluster

In this subsection we give transience and recurrence results for simple (nearest neighbour) random walks on the percolation clusters for $\delta < 1$ and for $\delta = 1$, $C_2 > 0$.

#### 4.3.1 Recurrence for $\delta = 1$, $\alpha \leq 1$

**Theorem 4.3** In the case $\delta = 1$, $C_2 > 0$, for almost every realization of the percolation cluster the random walk on the cluster is recurrent if $\alpha \leq 1$.

**Proof.** For the cutsets $\Pi_j$ in Lemma 3.2,

$$E\left(\frac{1}{|\Pi_j|}\right) \geq \frac{1}{E|\Pi_j|}.$$
(by Jensen’s inequality since $1/x$ is convex on $(0, \infty)$), hence by (3.4)

$$E \left( \frac{1}{|\Pi_j|} \right) \gtrsim \frac{N}{\kappa_j} \text{ for large } j.$$ 

Then by (3.5)

$$E \left( \sum_j \frac{1}{|\Pi_j|} \right) \geq N \sum_j \frac{1}{\kappa_j} = \infty,$$

since $\alpha \leq 1$.

The random variables $1/|\Pi_j|$ are independent and bounded by 1, hence the probability that $\sum_j 1/|\Pi_j|$ diverges is positive [38, Prop. 4.14], then by Kolmogorov’s 0-1 law $\sum_j 1/|\Pi_j|$ diverges w.p.1. Then the recurrence of the random walk follows by the Nash-Williams criterion. 

### 4.3.2 Transience for $\delta < 1$

**Theorem 4.4** In the case $\delta < 1$, for almost every realization of the percolation cluster the random walk on the cluster is transient.

**Proof.** Let $k_n = \lfloor n \log n \rfloor$, $c = \inf_k c_k$, $A_n = (k_{n-1}, k_n]$-annulus, and denote by $M_n$ the number of edges connecting $A_n$ and $A_{n+1}$. Then $M_n$ stochastically dominates

$$B_n = \text{Bin} \left( |A_n||A_{n+1}|, \frac{c}{N^{(1-\delta)k_{n+1}}} \right),$$

hence

$$EM_n \gtrsim N^{2k_n} \frac{c}{N^{(1-\delta)k_{n+1}}} \sim cN^{(1-\delta)n \log n} \text{ as } n \to \infty.$$

Since $cN^{(1-\delta)n \log n} \gg 4^n$ as $n \to \infty$, and $\text{Var}[B_n] = O(E[B_n])$ as $n \to \infty$, it can be shown using [38] (Lemma 4.1) that

(4.1) $P(M_n > 4^n) \to 1$ as $n \to \infty$.

Therefore by [17], for all large $n$ we can pick $4^n$ direct edges from $A_{n-1}$ to $A_n$. Since $|A_n| \geq (1 - \varepsilon)N^{n \log n}$ for some $0 < \varepsilon < 1$, we can subdivide $A_n$ into $4^n + 4^{n+1}$ disjoint subsets, each containing $O(N^{n \log n - 4/\log N})$ vertices. We assign $4^n$ of these subsets as *entrance-sets* for edges from $A_{n-1}$, and $4^{n+1}$ of them as *exit-sets* for edges to $A_{n+1}$, and identify an edge from each one of the $4^n$ exit-sets in $A_{n-1}$ to a different entrance set in $A_n$. The end-points of those edges are an *in-vertex* in the entrance-set, and an *out-vertex* in the exit-set in the previous annulus.

We now connect by an edge each one of the $4^n \cdot 4^{n+1}$ pairs (entrance-set, exit-set) in $A_n$. The probability that there is no such a pair connection is

$$\sim \left( 1 - \frac{c}{N^{(1-\delta)n \log n}} \right)^{N^{2n \log n}} \sim \exp(-cN^{(1-\delta)n \log n}) \text{ as } n \to \infty,$$

hence the probability of the event that any of the pairs fail to be connected is

$$O(4^{2n} \exp(-cN^{(1-\delta)n \log n})) \text{ as } n \to \infty.$$ 

Since this is summable, then by Borel-Cantelli there exists a random $n_0$ such that for all $n \geq n_0$ all the pairs in $A_n$ are connected. Therefore we can construct an infinite tree which is rooted at some vertex in $A_0$, whose nodes are in-vertices in entrance-sets in successive annuli $A_n$, such that each node has 4 children, one in each of 4 different entrance-sets in the next annulus (all disjoint), which are connected by edges to corresponding 4 different exit-sets in the previous annulus, so that the $n$-th generation consists of $4^n$ vertices.
It remains to connect by paths within each entrance-set its in-vertex to the end-points of the edges that connect the entrance-set to its corresponding 4 exit-sets in the annulus (these end-points are out-vertices in the entrance set), and similarly to connect by paths within each exit-set its out-vertex to the end-point of the edge from its corresponding entrance-set (this end-point is an in-vertex in the exit-set).

In order for such paths to exist we need to assume that all this is done within the percolation cluster. Since the clusters in good $k_n$-balls have size at least $N^{-\gamma k_n}$ with $(1 + \delta)/2 < \gamma < 1$ for all sufficiently large $n$ (see [22], Def. 4.1, (4.4), (4.6), (4.22)), we take $N^{-\gamma}$ instead of $N$ above, so that the construction takes place in the percolation cluster. Then the connecting paths exist and there may be more than one in each set. For $n > n_0$, the length of a path from a node of the tree in $A_n$ to any one of its children in the next generation is bounded by $1 + 1 + K(n_0)(3^{n-n_0} + 3^{n-n_0})$, with $K(n_0)$ given in Lemma 3.5.

The 1's come from the single edges between an annulus $A_n$ and the next one, and from the single edges connecting out-vertices in entrance-sets to in-vertices in exit-sets in the annulus. The $3^{n-n_0}$'s come from the lengths of paths joining the in-vertex to the out-vertex in an entrance set, and the length of a path joining the in-vertex and the out-vertex in an exit-set in $A_n$. Hence the length of a path from a node in the tree in the $n$th generation to any of its 4 children in the next generation is bounded by $C3^n$, by Lemma 3.5 for some positive constant $C$.

By construction the paths joining a node in the tree to its children in the next generation can have common edges only within the entrance-set. Since there are at least 1 and at most 4 out-vertices in an entrance-set, then each edge in the paths is used at most 4 times. Then it follows by Proposition 3 of [36], with $\beta = 3$ and $\alpha = \gamma = 4$, that the resistance of the tree from the root to infinity is at most

$$4 \sum_{n=n_0}^{\infty} \frac{C3^n}{4^n} < \infty.$$  

Therefore, by the criterion for transience-recurrence in subsection 4.2, the walk on the percolation cluster is transient.

\[\]  

4.3.3 Transience for $\delta = 1$, $\alpha > 6$.

Theorem 4.5 In the case $\delta = 1$, $C_2 > 0$, $\alpha > 6$, for almost every realization of the percolation cluster the random walk on the cluster is transient.

\[\]  

Proof. The idea of the proof is to construct a flow on a subgraph of the percolation cluster $C$ that satisfies the finite energy condition criterion.

The edges of the subgraph will be decomposed into a sequence of subsets:

- edges connecting successive $A_n = (k_n, k_{n+1})$-annuli, $n = 1, 2, \ldots, k_n = [Kn \log n]$, where $K > 1$,
- these edges go from disjoint $k_n$-balls in $A_n$ to disjoint $k_{n+1}$-balls in $A_{n+1}$,
- there are also edges within the $k_n$-balls (and $k_{n+1}$-balls) connecting the entrance and exit vertices in these balls.

In [22] (Theorem 3.5(b)) the assumption $2/\log N < K < b$ was required and the proof used the fact that $|C \cap A_n| \geq \beta N^{k_n+1}$ for large $n$ with an appropriate $0 < \beta < 1$. This will be used here. Recall that $\alpha > 2b \log N$.

Recall that the graphs $G(N_n, p_n)$ are complete for $b > 2K$ and $n \geq n_0$ (Lemma 3.4). We now compute a lower bound for

$$r_n := P(\text{a } k_n\text{-ball in } C \cap B_{k_{n+1}}(0) \text{ is connected to a } k_{n+1}\text{-ball in } A_{n+1})$$  

for large $n$. We have for large $n$, using [33],

$$r_n \geq 1 - \left(1 - \frac{a \log n \cdot N^{b \log n}}{N^{2k_{n+2}}} \right)^{\beta^2 N^{k_n+k_{n+1}}}$$

$$\sim 1 - \exp(-a \beta^2 \log n \cdot N^{b \log n - 2(k_{n+2}) - k_n - k_{n+1}}),$$

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2k_{n+2} - k_n - k_{n+1} = k_{n+1} - k_n + 2(k_{n+2} - k_{n+1}) \sim 3K \log n,

hence

\[ r_n \gtrsim 1 - \exp(-a\beta^2 \log n \cdot n^{(b-3)K} \log N). \]

There are \( N^{K \log n} \) balls in \( B_{k_{n+1}}(0) \), and \((N^{K \log(n+1)} - 1) \sim N^{K \log(n+1)} \) balls in \( A_{n+1} \), and

\[ \sum_{n} n^{K \log N} (n+1)^K \log N \exp(-a\beta^2 \log n \cdot n^{(b-3)K} \log N) < \infty \quad \text{if} \quad b > 3K, \]

which we now assume, so, for such \( b \) and for almost every realization of the percolation cluster there exists \( n_0 \) such that for all \( n \geq n_0 \).

(4.2)

every \( k_n \)-ball in \( B_{k_{n+1}}(0) \) is connected by at least one edge in the cluster to every \( k_{n+1} \)-ball in \( A_{n+1} \), and this will be used in the construction below.

By the Rayleigh monotonicity principle, to prove that the random walk is transient on \( C \) it suffices to show that it is transient on a subgraph of \( C \). Given a realization of the cluster and associated \( n_0 \) satisfying (4.2), we will construct a subgraph and a unit flow on it to satisfy the energy criterion for transience.

The flow has the following properties: Start with \( B_{k_{n_0}}(0) \) and assume that \( 0 \) belongs to \( C \). Choose one edge from \( B_{k_{n_0}}(0) \) to each one of the \( N^{K \log(n_0+1)} \) \( k_{n_0} \)-balls in \( A_{n_0} \). The unit flow entering at \( 0 \) is divided into \( N^{K \log(n_0+1)} \) equal parts going to each one of the \( k_{n_0+1} \)-balls in \( A_{n_0} \). The ball \( B_{k_{n_0}}(0) \) has an internal structure which is the set of points (vertices) of \( \Omega_N \) and edges that are contained in \( C \cap B_{k_{n_0}}(0) \). There are many ways that the flow can go through paths from \( 0 \) to the (at least 1 and at most \( N^{K \log(n_0+1)} \)) exit-vertices in the cluster of \( B_{k_{n_0}}(0) \), splitting appropriately at branch vertices on the paths in order to achieve the division of the flow as stated. Denote by \( E_0 \) the energy of the flow on the subgraph of the cluster connecting \( 0 \) to the exit vertices of \( B_{k_{n_0}}(0) \). The flow will then pass through a series of disjoint subsets of edges in \( C \) denoted \( \{G_n\}_{n \geq 1} \) with the energies denoted by \( \{E_n\}_{n \geq 1} \). \( G_1 \) consists of edges from the at most \( N^{K \log(n_0+1)} \) exit-vertices in the cluster of \( B_{k_{n_0}}(0) \) to the \( N^{K \log(n_0+1)} \) disjoint \( k_{n_0} \)-balls in \( A_{n_0} \) and the edges connecting the entrance vertices in these balls to the exit vertices. Similarly for \( n \geq 2 \) \( G_n \) consists of edges from the \( N^{K \log n} \) disjoint \( k_{n-1} \)-balls in \( A_{n-1} \) to the \( N^{K \log(n+1)} \) disjoint \( k_n \)-balls in \( A_n \) and the edges connecting the (at most 2) entrance vertices in these balls to the (at most 3) exit vertices.

We now specify in detail the choice of the edges in \( G_n \) and the flow in each of these edges. For \( n > n_0 \) each of the \( N^{K \log(n+1)} \) \( k_n \)-balls in \( A_n \) gets \( 1/N^{K \log(n+1)} \) amount of flow entering through 1 or 2 edges, which then goes through the internal structure of each \( k_{n+1} \)-ball and is then divided among 1, 2 or 3 edges to the \( N^{K \log(n+2)} \) \( k_{n+2} \)-balls in \( A_{n+1} \). To do this we first enumerate the \( N^{K \log(n+1)} \) \( k_n \)-balls, in \( A_n \) denoted \( B_1^n, \ldots, B_{N^{K \log(n+1)}}^n \) and then enumerate the \( N^{K \log(n+2)} \) \( k_{n+1} \)-balls in \( A_{n+1} \) denoted \( B_1^{n+1}, \ldots, B_{N^{K \log(n+2)}}^{n+1} \). We first choose edges between \( B_1^n \) and \( B_1^{n+1}, B_2^{n+1} \) and assign flow \( 1/N^{K \log(n+2)} \) to the edge to \( B_1^{n+1} \) and \( 1/N^{K \log(n+1)} - 1/N^{K \log(n+2)} \) to the edge to \( B_2^{n+1} \). We then fill \( B_2^{n+1} \) up to level \( 1/N^{K \log(n+2)} \) from \( B_2^n \) and successively assign flows to edges from the \( B_3^n \)'s to the \( B_3^{n+1} \)'s so that each \( B_i^n \) becomes empty and each \( B_i^{n+1} \) is filled up by the end of the procedure. This can be done in several ways so that all the inflows and outflows have the same order of magnitude.

This procedure is repeated for the successive \( A_n \). This means that for each \( n > n_0 \), each one of the \( k_n \)-balls in \( A_n \) gets \( 1/N^{K \log(n+1)} \) total amount of entrance flow. Noting that for large \( n \)

\[ \frac{1}{N^{K \log(n+1)}} < \frac{2}{N^{K \log(n+2)}} \]

each of the \( k_n \)-balls has 1 or 2 entrance edges and 1, 2 or 3 exit edges. Any entrance-exit pair in the \( k_n \)-balls can be connected by a path (within the ball by completeness) of length bounded by \( CD(n_0)n \).

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(Lemma 3.6). Therefore each of the edges belongs to at most 6 paths and the energy of the flow (recall Definition 4.2) is then bounded by

\[ (4.3) \sum_{n=n_0}^{N} \frac{C_1 D(n_0) n}{N K \log(n+1)} = 6 \sum_{n=n_0}^{N} \frac{C_1 D(n_0) n}{(n+1) K \log N}, \]

for some constant \( C_1 \) where the \( n \)th summand refers to the energy \( E_n \) of the flow from entrance vertices in \( A_n \) to the entrance vertices in \( A_{n+1} \). Then the energy of the flow is finite if

\[ \sum_{n} \frac{n}{n K \log N} < \infty, \]

which holds because \( K > 2/\log N \). Hence with the assumption that \( b > 3K \) we can construct a flow on a subgraph of \( C \) with finite energy and therefore the random walk on the cluster is transient. Since this holds for \( K \log N > 2 \) and \( b > 3K \), and we have assumed that \( \alpha > b \log N \), then \( \alpha > 6 \) suffices.

Finally, we can give the main result.

**Theorem 4.6** Consider the nearest neighbour random walk on the percolation cluster with \( \delta = 1 \) and \( C_2 > 0, 0 < \alpha < \infty \). Then there exists a critical \( \alpha_c \in (0, \infty) \) such that for \( \alpha < \alpha_c \) the random walk is recurrent and for \( \alpha > \alpha_c \) the random walk is transient.

**Proof.** By Theorems 4.3 and 4.5 there exist \( 0 < \alpha_1 < \alpha_2 < \infty \) such that the random walk on the percolation cluster is recurrent for \( \alpha \leq \alpha_1 \) and transient for \( \alpha = \alpha_2 \). Moreover, given \( \alpha < \alpha' \) we can construct the two associated percolation clusters (using the same \( C_2 \)) on one probability space so that the \( \alpha \)-cluster is a subgraph of the \( \alpha' \)-cluster with probability one (see Remark 2.2 in [22]). But then the Rayleigh monotonicity principle implies that if the random walk on the \( \alpha \)-cluster is transient it is also transient on the \( \alpha' \)-cluster. We define \( \alpha_c = \inf\{ \alpha : \text{the walk on the } \alpha \text{-cluster is transient}\} \), which yields the desired result.

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