Small eigenvalues of large Hankel matrices: The indeterminate case *

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March 30, 2022

Abstract

In this paper we characterise the indeterminate case by the eigenvalues of the Hankel matrices being bounded below by a strictly positive constant. An explicit lower bound is given in terms of the orthonormal polynomials and we find expressions for this lower bound in a number of indeterminate moment problems.

1 Introduction

Let \( \alpha \) be a positive measure on \( \mathbb{R} \) with infinite support and finite moments of all orders

\[
\begin{align*}
    s_n &= s_n(\alpha) = \int_{\mathbb{R}} x^n \, d\alpha(x).
\end{align*}
\]

With \( \alpha \) we associate the infinite Hankel matrix \( \mathcal{H}_\infty = \{H_{jk}\} \).

\[
\begin{align*}
    H_{jk} &= s_{j+k}.
\end{align*}
\]

Let \( \mathcal{H}_N \) be the \((N+1) \times (N+1)\) matrix whose entries are \( H_{jk}, 0 \leq j, k \leq N \). Since \( \mathcal{H}_N \) is positive definite, then all its eigenvalues are positive. The large \( N \) asymptotics of the smallest eigenvalue, denoted as \( \lambda_N \), of the Hankel matrix \( \mathcal{H}_N \) has been studied in papers by Szegö [1], Widom and Wilf [3], Chen and Lawrence [2]. See also the monograph by Wilf [4]. All the cases considered by these authors are determinate moment problems, and it was shown in each case that \( \lambda_N \to 0 \), and asymptotic results were obtained about how fast \( \lambda_N \) tends to zero.

The smallest eigenvalue can be obtained from the classical Rayleigh quotient:

\[
\begin{align*}
    \lambda_N &= \min \left\{ \sum_{j=0}^{N} \sum_{k=0}^{N} s_{j+k} v_j v_k : \sum_{k=0}^{N} v_j^2 = 1, v_j \in \mathbb{R}, 0 \leq j \leq N \right\}.
\end{align*}
\]

*This research is partially supported by the EPSRC GR/M16580 and NSF grant DMS 99-70865
It follows that $\lambda_N$ is a decreasing function of $N$.

The main result of this paper is Theorem 1.1, which we state next.

**Theorem 1.1** The moment problem associated with the moments (1.1) is determinate if and only if

$$\lim_{N \to \infty} \lambda_N = 0.$$ 

We shall compare this result with a theorem of Hamburger [8, Satz XXXI], cf. [4, p.83] or [10, p.70]

Let $\mu_N$ be the minimum of the Hankel form $\mathcal{H}_N$ on the hyper-plane $v_0 = 1$, i.e.

$$\mu_N = \min \left\{ \sum_{j=0}^{N} \sum_{k=0}^{N} s_{j+k} v_j v_k : v_0 = 1, v_j \in \mathbb{R}, 0 \leq j \leq N \right\}. \quad (1.4)$$

and let $\mu'_N$ be the corresponding minimum for the moment sequence $s'_n = s_{n+2}, n \geq 0$, i.e.

$$\mu'_N = \min \left\{ \sum_{j=0}^{N} \sum_{k=0}^{N} s_{j+k+2} v'_j v'_k : v'_0 = 1, v'_j \in \mathbb{R}, 0 \leq j \leq N \right\}$$

$$= \min \left\{ \sum_{j=0}^{N+1} \sum_{k=0}^{N+1} s_{j+k} v_j v_k : v_0 = 0, v_1 = 1, v_j \in \mathbb{R}, 0 \leq j \leq N + 1 \right\}.$$ 

The theorem of Hamburger can be stated that the moment problem is determinate if and only if at least one of the limits $\lim_{N \to \infty} \mu_N$, $\lim_{N \to \infty} \mu'_N$ are zero.

It is clear from (1.3), (1.4) that $\mu_N \geq \lambda_N$ and similarly $\mu'_N \geq \lambda_{N+1}$. From these inequalities and Hamburger’s theorem, we obtain the “only if” statement in Theorem 1.1. The “if” statement will be proved by finding a positive lower bound for the eigenvalues $\lambda_N$, cf. Theorem 1.2 below.

We think that Theorem 1.1 has the advantage over the theorem of Hamburger that it involves only the moment sequence $(s_n)$ and not the shifted sequence $(s_{n+2})$. In section 2 we give another proof of the “only if” statement to make the proof of Theorem 1.1 independent of Hamburger’s theorem.

If

$$\pi_N(x) := \sum_{j=0}^{N} v_j x^j, \quad (1.5)$$

then a simple calculation shows that

$$\sum_{0 \leq j, k \leq N} s_{j+k} v_j v_k = \int_E \pi_N^2(x) \, d\alpha(x), \quad (1.6)$$
and

\[ \sum_{k=0}^{N} v^2_k = \int_0^{2\pi} \left| \pi_N(e^{i\theta}) \right|^2 \frac{d\theta}{2\pi}. \]

We could also study the reciprocal of \( \lambda_N \) given by

\[ \frac{1}{\lambda_N} = \max \left\{ \int_0^{2\pi} \left| \pi_N(e^{i\theta}) \right|^2 \frac{d\theta}{2\pi} : \pi_N, \int_E \pi_N^2(x) d\alpha(x) = 1 \right\}. \]

Let \( \{p_k\} \) denote the orthonormal polynomials with respect to \( \alpha \), normalised so that \( p_k \) has positive leading coefficient.

We recall that the moment problem is indeterminate, cf. [1], [10], if and only if there exists a non-real number \( z_0 \) such that

\[ \sum_{k=0}^{\infty} |p_k(z_0)|^2 < \infty. \]

In the indeterminate case the series in (1.9) actually converges for all \( z_0 \in \mathbb{C} \), uniformly on compact sets. In the determinate case the series in (1.9) diverges for all non-real \( z_0 \) and also for all real numbers except the at most countably many points, where \( \alpha \) has a positive discrete mass.

If we expand the polynomial (1.5) as a linear combination of the orthonormal system

\[ \pi_N(x) = \sum_{j=0}^{N} c_j p_j(x), \]

then

\[ \int_0^{2\pi} \left| \pi_N(e^{i\theta}) \right|^2 \frac{d\theta}{2\pi} = \sum_{0 \leq j, k \leq N} c_j c_k \int_0^{2\pi} p_j(e^{i\theta}) p_k(e^{-i\theta}) \frac{d\theta}{2\pi} = \sum_{0 \leq j, k \leq N} K_{jk} c_j c_k, \]

where we have defined

\[ K_{jk} = \int_0^{2\pi} p_j(e^{i\theta}) p_k(e^{-i\theta}) \frac{d\theta}{2\pi}. \]

Thus

\[ \frac{1}{\lambda_N} = \max \left\{ \sum_{0 \leq j, k \leq N} K_{jk} c_j c_k : c_j, \sum_{j=0}^{N} c_j^2 = 1 \right\}. \]

Since the eigenvalues of the matrix \( (K_{jk})_{0 \leq j, k \leq N} \) are positive, and their sum is its trace, then

\[ \frac{1}{\lambda_N} \leq \sum_{k=0}^{N} K_{kk} = \int_0^{2\pi} \sum_{k=0}^{N} \left| p_k(e^{i\theta}) \right|^2 \frac{d\theta}{2\pi}. \]
Thus in the case of indeterminacy,

(1.13) \[ \frac{1}{\lambda_N} \leq \int_0^{2\pi} \sum_{k=0}^{\infty} \left| p_k(e^{i\theta}) \right|^2 \frac{d\theta}{2\pi} < \infty, \]

which shows that

(1.14) \[ \lim_{N \to \infty} \lambda_N \geq \left( \int_0^{2\pi} \frac{1}{\rho(e^{i\theta})} \frac{d\theta}{2\pi} \right)^{-1}, \]

where

(1.15) \[ \rho(e^{i\theta}) = \left( \sum_{k=0}^{\infty} \left| p_k(e^{i\theta}) \right|^2 \right)^{-1}. \]

We recall that for \( z \in \mathbb{C} \setminus \mathbb{R} \) the number \( \rho(z)/|z - \overline{z}| \) is the radius of the Weyl circle at \( z \).

The above argument establishes the following result:

**Theorem 1.2** *In the indeterminate case the smallest eigenvalue \( \lambda_N \) of the Hankel matrix \( \mathcal{H}_N \) is bounded below by the harmonic mean of the function \( \rho \) along the unit circle.*

We shall conclude this paper with examples, where we have calculated or estimated the quantity

(1.16) \[ \rho_0 = \int_0^{2\pi} \sum_{k=0}^{\infty} \left| p_k(e^{i\theta}) \right|^2 \frac{d\theta}{2\pi}. \]

This will be done for the moment problems associated with the Stieltjes-Wigert polynomials, cf. [4], [12], the Al-Salam-Carlitz polynomials [4], the symmetrized version of polynomials of Berg-Valent ([3]) leading to a Freud-like weight [3], and the \( q^{-1} \)-Hermite polynomials of Ismail and Masson [9].

If we introduce the coefficients of the orthonormal polynomials as

(1.17) \[ p_k(x) = \sum_{j=0}^{k} \beta_{k,j} x^j \]

then

\[ \int_0^{2\pi} |p_k(e^{i\theta})|^2 \frac{d\theta}{2\pi} = \sum_{j=0}^{k} \beta_{k,j}^2, \]

and therefore

(1.18) \[ \rho_0 = \sum_{k=0}^{\infty} \sum_{j=0}^{k} \beta_{k,j}^2. \]
Another possibility for calculating $\rho_0$ is to use the entire functions $B, D$ from the Nevanlinna matrix since it is well known that \[ p_k(z) = B(z)D(\bar{z}) - D(z)B(\bar{z}) \]

(1.19) \[
\sum_{k=0}^{\infty} |p_k(z)|^2 = \frac{B(z)D(\bar{z}) - D(z)B(\bar{z})}{z - \bar{z}}.
\]

It follows that

(1.20) \[
\sum_{k=0}^{\infty} |p_k(e^{i\theta})|^2 = \text{Im} \{ B(e^{i\theta})D(e^{-i\theta}) \} / \sin \theta.
\]

2 Indeterminate Moment Problems

In this section we shall give a proof of Theorem 1.1 which is independent of Hamburger’s result. We have already established that if \( \lim_{N \to \infty} \lambda_N = 0 \), then the problem is determinate. We shall next prove that if \( \lambda_N \geq \gamma \) for all \( N \), where \( \gamma > 0 \), then the problem is indeterminate. Since \( 1/\lambda_N \leq 1/\gamma \) for all \( N \), and \( 1/\lambda_N \) is the biggest eigenvalue of the positive definite matrix \( (K_{jk})_{0 \leq j, k \leq N} \), we get

(2.1) \[
\sum_{0 \leq j, k \leq N} K_{jk}c_j\overline{c_k} \leq \frac{1}{\gamma} \sum_{j=0}^{N} |c_j|^2,
\]

for all vectors \( (c_0, \ldots, c_N) \in \mathbb{C}^{N+1} \). If we consider an arbitrary complex polynomial \( p \) of degree \( \leq N \) written as \( p(x) = \sum_{k=0}^{N} c_k p_k(x) \), the inequality (2.1) can be formulated

(2.2) \[
\int_0^{2\pi} |p(e^{i\theta})|^2 \frac{d\theta}{2\pi} \leq \frac{1}{\gamma} \int |p(x)|^2 \, d\alpha(x).
\]

Let now \( z_0 \) be an arbitrary non-real number in the open unit disc. By the Cauchy integral formula

\[
p(z_0) = \frac{1}{2\pi} \int_0^{2\pi} \frac{p(e^{i\theta})}{e^{i\theta} - z_0} e^{i\theta} \, d\theta,
\]

and therefore

(2.3) \[
|p(z_0)|^2 \leq \int_0^{2\pi} |p(e^{i\theta})|^2 \frac{d\theta}{2\pi} \int_0^{2\pi} \frac{1}{|e^{i\theta} - z_0|^2} \frac{d\theta}{2\pi}.
\]

Combined with (2.2) we see that there is a constant \( K \) such that for all complex polynomials \( p \)

(2.4) \[
|p(z_0)|^2 \leq K \int |p(x)|^2 \, d\alpha(x),
\]
where $K = 1/(\gamma(1 - |z_0|^2))$.

This inequality implies indeterminacy in the following way. Applying it to the polynomial

$$p(x) = \sum_{k=0}^{N} p_k(z_0)p_k(x),$$

we get

$$\sum_{k=0}^{N} |p_k(z_0)|^2 \leq K,$$

(2.5)

and since $N$ is arbitrary, indeterminacy follows.

**Remark.** We see that the infinite positive definite matrix $K_\infty = \{K_{j,k}\}$ is bounded on $\ell^2$ if and only if $\lambda_N \geq \gamma$ for all $N$ for some $\gamma > 0$. Furthermore $K_\infty$ is of trace class if and only if $\rho_0 < \infty$. The result of Theorem 1.1 can be reformulated to say that boundedness implies trace class for this family of operators.

### 3 Examples

We shall follow the notation and terminology for $q$-special functions as those in Gasper and Rahman [7].

**Example 1.** The Stieltjes-Wigert Polynomials.

These polynomials are orthonormal with respect to the weight function

$$\omega(x) = \frac{k}{\sqrt{\pi}} \exp(-k^2(\log x)^2), \quad x > 0,$$

(3.1)

where $k > 0$ is a positive parameter, cf. [1], [12]. They are given by

$$p_n(x) = (-1)^n q^{\frac{n+1}{2}} (q; q)_n^{-\frac{1}{2}} \sum_{k=0}^{n} \binom{n}{k}_q q^{k^2} (-q^{\frac{1}{2}} x)^k,$$

(3.2)

where we have defined $q = \exp\{-(2k^2)^{-1}\}$.

It follows by (1.18) that

$$\rho_0 = \sum_{n=0}^{\infty} \frac{q^{n+\frac{1}{2}}}{(q; q)_n} \sum_{k=0}^{n} q^{k(2k+1)} \binom{n}{k}_q \frac{2}{q},$$

(3.3)

$$= \sum_{k=0}^{\infty} q^{2k^2+k+\frac{1}{2}} \sum_{n=k}^{\infty} \frac{q^n}{(q; q)_n} \binom{n}{k}_q.$$
Putting $n = k + j$, the inner sum is

$$\sum_{j=0}^{\infty} \frac{q^{k+j}}{(q; q)_k^2} (q; q)_{k+j} = \frac{q^k}{(q; q)_k^2} \phi_1(q^{k+1}, 0; q, q)$$

and hence

$$(3.4) \quad \rho_0 = \sum_{k=0}^{\infty} \frac{q^{2(k+\frac{1}{2})^2}}{(q; q)_k^2} \phi_1(0, q^{k+1}; q, q).$$

We can obtain another expression for $\rho_0$. We apply the transformation [7, (III.5)]

$$(3.5) \quad 2\phi_1(a, b; c; q, z) = \frac{(abz/c; q)_{\infty}}{(bz/c; q)_{\infty}} 3\phi_2(a, c/b, 0; c, cq/bz; q, q)$$

to see that

$$(3.6) \quad \sum_{n=k}^{\infty} \frac{q^n}{(q; q)_n} \binom{n}{k}_q = \frac{1}{(q; q)_\infty} \sum_{j=0}^{k} \frac{q^{k+j}}{(q; q)_j^2}.$$

We then find

$$(3.7) \quad \rho_0 = \frac{1}{(q; q)_\infty} \sum_{k=0}^{\infty} q^{2(k+\frac{1}{2})^2} \sum_{j=0}^{k} \frac{q^j}{(q; q)_j^2}.$$

A formula more general than (3.6) is

$$\sum_{n=k}^{\infty} \frac{\omega^n}{(q; q)_n} \binom{n}{k}_q = \frac{1}{(\omega; q)_\infty} \sum_{j=0}^{k} \frac{(\omega; q)_j \omega^{2k-j}}{(q; q)_j (q; q)_j^2}$$

and is stated in [4]. This more general identity also follows from (3.5) and the simple observation

$$\frac{(q^{-k}; q)_j}{(q^{1-k}/\omega; q)_j} = \frac{(q; q)_k (\omega; q)_{k-j} (\omega/q)_j}{(\omega; q)_k (q; q)_{k-j}}.$$

We have numerically computed the smallest eigenvalue of the Hankel matrix of various dimensions with the Stieltjes–Wigert weight from which we extrapolate to determine the smallest eigenvalue $s = \lim_{N \to \infty} \lambda_N$ of the infinite Hankel matrix for different values of $q$. This is then compared with the numerically computed lower bound $l = 1/\rho_0$. For $q = \frac{1}{2}$ we have $s = 0.3605\ldots, l = 0.3435\ldots$. The percentage error $100(s - l)/s$ is plotted for various values of $q$ and is shown in figure 1.

**Example 2.** Al-Salam–Carlitz polynomials.
The Al-Salam–Carlitz polynomials were introduced in [2]. We consider the indeterminate polynomials $V_n^{(a)}(x; q)$, where $0 < q < 1$ and $q < a < 1/q$, cf. [3]. For the corresponding orthonormal polynomials $\{p_k\}$ we have by [3, (4.24)]

$$
\sum_{k=0}^{\infty} |p_k(e^{i\theta})|^2 = \frac{(qe^{i\theta}, qe^{-i\theta}; q)_{\infty}}{(aq, q, q; q)_{\infty}} \, 3\phi_2(e^{i\theta}, e^{-i\theta}, aq; qe^{i\theta}, qe^{-i\theta}; q, q/a).
$$

Therefore

$$
\rho_0 = \int_0^{2\pi} \sum_{k=0}^{\infty} |p_k(e^{i\theta})|^2 \frac{d\theta}{2\pi} = \frac{1}{(aq, q, q; q)_{\infty}} \sum_{n=0}^{\infty} I_n(aq; q)_n \left( \frac{q}{a} \right)^n,
$$

where

$$
I_n = \int_0^{2\pi} \frac{(e^{i\theta}, e^{-i\theta}; q)_{\infty}}{(1 - q^n e^{i\theta})(1 - q^n e^{-i\theta})} \frac{d\theta}{2\pi} = \int_{|z|=1} \frac{(z, 1/z; q)_{\infty}}{(1 - q^n z)(1 - q^n/z)} \frac{dz}{2\pi iz}.
$$

Recall the Jacobi triple product identity [7],

$$
j(z) := (q, z, 1/z; q)_{\infty} = \sum_{k=-\infty}^{\infty} c_k z^k,
$$

with

$$
c_k = (-1)^k \left[ q^{k(k+1)/2} + q^{k(k-1)/2} \right].
$$
Note that $c_k = c_{-k}$.

Using the partial fraction decomposition
\[
\frac{q^n}{1-q^n z} - \frac{q^{-n}}{1-q^{-n} z} = \frac{1-q^{2n}}{(1-q^n z)(z-q^n)}
\]
we find by the residue theorem and the Jacobi triple product identity (3.11) that for $n \geq 1$, $I_n$ is given by
\[
(1-q^{2n})(q; q)_\infty I_n = q^n \text{Res} \left( \frac{j(z)}{1-q^n z}, z = 0 \right) - q^{-n} \text{Res} \left( \frac{j(z)}{1-q^{-n} z}, z = 0 \right)
\]
\[
= q^n \sum_{k=0}^{\infty} q^{nk} c_{-k} - q^{-n} \sum_{k=0}^{\infty} q^{-nk} c_{-k}
\]
\[
= \sum_{k=1}^{\infty} \left( q^{nk} - q^{-nk} \right) c_k,
\]
while for $n = 0$, $I_0$ is
\[
(q; q)_\infty I_0 = \int_{|z|=1} \frac{j(z)}{(1-z)(z-1)} \frac{dz}{2\pi i} = -\text{Res} \left( \frac{j(z)}{(1-z)^2}, z = 0 \right)
\]
\[
= -\sum_{k=0}^{\infty} (k+1)c_{-k} = \sum_{k=0}^{\infty} (-1)^k q^{k(k+1)/2}.
\]
The conclusion is
\[
(3.13) \quad I_0 = \frac{1}{(q; q)_\infty} \sum_{k=0}^{\infty} (-1)^k q^{k(k+1)/2},
\]
\[
I_n = \frac{1}{(1-q^{2n})(q; q)_\infty} \sum_{k=1}^{\infty} c_k \left( q^{nk} - q^{-nk} \right), \quad n \geq 1.
\]
The above formulas can be further simplified. Using the Jacobi triple product identity (3.11) we find for integer values of $n$
\[
\sum_{k=-\infty}^{\infty} (-1)^k q^{nk} q_{(z)}^{(k)} = 0,
\]
hence
\[
(3.14) \quad \sum_{k=0}^{\infty} (-1)^k q^{nk} q_{(z)}^{(k)} = -\sum_{k=1}^{\infty} (-1)^k q^{-nk} q_{(z)}^{(k+1)}, \quad n = 0, \pm 1, \ldots.
\]
This analysis implies
\[
(3.15) \quad (q; q)_\infty (1-q^{2n}) I_n = 2 \sum_{k=1}^{\infty} (-1)^k q_{(z)}^{(k)} \left[ q^{nk} - q^{-nk} \right].
\]
Thus we have established the representation for $n \geq 1$

(3.16) \[ I_n = \frac{2q^{-n}}{(q; q)_\infty} \sum_{k=1}^{\infty} (-1)^{k-1} q^{\frac{k}{2}} \frac{\sin(nk\tau)}{\sin(n\tau)}, \quad q = e^{-i\tau}. \]

It is clear that $I_0$ is the limiting case of $I_n$ as $n \to 0$. The representation (3.16) indicates that $I_n$ is a theta function evaluated at the special point $n\tau$, hence we do not expect to find a closed form expression for $I_n$.

**Example 3.** Freud-like weight.

In [3] Berg-Valent found the Nevanlinna matrix in the case of the indeterminate moment problem corresponding to a birth and death process with quartic rate $s$. Later Chen and Ismail, cf. [5], considered the corresponding symmetrized moment problem, found the Nevanlinna matrix and observed that there are solutions which behave as the Freud weight $\exp(-\sqrt{|x|})$. In particular they found the entire functions

(3.17) \[ B(z) = -\delta_0(K_0\sqrt{z/2}), \quad D(z) = \frac{4}{\pi} \delta_2(K_0\sqrt{z/2}), \]

where

(3.18) \[ \delta_l(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(4n + l)!} z^{4n+l}, \quad l = 0, 1, 2, 3, \]

(3.19) \[ K_0 = \frac{\Gamma(1/4)\Gamma(5/4)}{\sqrt{\pi}}. \]

Note that

(3.20) \[ \delta_0(z) = \frac{1}{2} \left[ \cosh(z\sqrt{i}) + \cos(z\sqrt{i}) \right], \]

(3.21) \[ \delta_2(z) = \frac{1}{2i} \left[ \cosh(z\sqrt{i}) - \cos(z\sqrt{i}) \right]. \]

If $\omega := \exp(i\pi/4) = (1 + i)/\sqrt{2}$, then a simple calculation shows that

(3.22) \[ B(x)D(y) - D(x)B(y) = -\frac{2i}{\pi} \left[ \cos(\omega^3 K_0\sqrt{x/2})\cos(\omega K_0\sqrt{y/2}) - \cos(\omega^3 K_0\sqrt{y/2})\cos(\omega K_0\sqrt{x/2}) \right]. \]

If $x = e^{i\theta}$, and $y = e^{-i\theta}$, then we linearise the products of cosines and find that the right-hand side of (3.22) is

\[ -\frac{i}{\pi} \left\{ \cos[K_0(\omega^3 e^{i\theta/2} + \omega e^{-i\theta/2})/\sqrt{2}] + \cos[K_0(\omega^3 e^{i\theta/2} - \omega e^{-i\theta/2})/\sqrt{2}] - \cos[K_0(\omega^3 e^{-i\theta/2} + \omega e^{i\theta/2})/\sqrt{2}] - \cos[K_0(\omega^3 e^{-i\theta/2} - \omega e^{i\theta/2})/\sqrt{2}] \right\} \]
We now combine the first and third terms, then combine the second and fourth terms and apply the addition theorem for trigonometric functions. We then see that the above is

\[
\frac{2i}{\pi} \{\sinh[K_0 \cos(\theta/2)] \sinh[K_0 \sin(\theta/2)] + \sin[K_0 \cos(\theta/2)] \sin[K_0 \sin(\theta/2)]\}.
\]

Thus we have proved that

\[
(3.23) \quad \frac{B(e^{i\theta}) D(e^{-i\theta}) - B(e^{-i\theta}) D(e^{i\theta})}{e^{i\theta} - e^{-i\theta}} = \frac{1}{\pi \sin \theta} \{\sinh[K_0 \cos(\theta/2)] \sinh[K_0 \sin(\theta/2)] + \sin[K_0 \cos(\theta/2)] \sin[K_0 \sin(\theta/2)]\}.
\]

Thus in the case under consideration, after some straightforward calculations and the evaluation of a beta integral, we obtain

\[
\rho_0 = \int_0^{2\pi} \sum_{n=0}^\infty \left| p_n(e^{i\theta}) \right|^2 d\theta = \frac{K_0^2}{\pi} \sum_{m,n \geq 0, m+n \text{ even}} \frac{(K_0/2)^{2m+2n}}{(2m+1)(2n+1) m! n! (m+n)!}.
\]

Example 4. \(q^{-1}\)-Hermite polynomials.

Ismail and Masson proved that for this moment problem the functions \(B\) and \(D\) are given by

\[
(3.25) \quad B(\sinh \xi) = \frac{(q^{e^{2i\xi}}, q^{e^{-2i\xi}}; q^2)_\infty}{(q, q^2; q^2)_\infty}, \quad D(\sinh \xi) = \frac{\sinh \xi}{(q; q)_\infty} (q^2 e^{2i\xi}, q^2 e^{-2i\xi}; q^2)_\infty,
\]

\((5.32), (5.36)); respectively. Ismail and Masson also showed that

\[
(3.26) \quad B(\sinh \xi) D(\sinh \eta) - B(\sinh \eta) D(\sinh \xi) = \frac{-e^{\eta}}{2(q; q)_\infty} \prod_{n=0}^\infty \left[ 1 - 2e^{-\eta}q^n \sinh \xi - e^{-2\eta}q^{2n} \right] \left[ 1 + 2e^{\eta}q^{n+1} \sinh \xi - e^{2\eta}q^{2n+2} \right].
\]

We rewrite the infinite product as

\[
\prod_{n=0}^\infty a_n b_n = a_0 \prod_{n=1}^\infty a_n b_{n-1},
\]

and with \(\sinh \xi = e^{i\theta}\) and \(\sinh \eta = e^{-i\theta}\) we get the following representation

\[
(3.27) \quad \frac{B(e^{i\theta}) D(e^{-i\theta}) - B(e^{-i\theta}) D(e^{i\theta})}{e^{i\theta} - e^{-i\theta}} = \frac{1}{(q; q)_\infty} \prod_{n=1}^\infty \left[ 1 + 4q^n - 2q^{2n} + 4q^{3n} + q^{4n} - 8q^{2n} \cos(2\theta) \right]
\]

\[= \frac{1}{(q; q)_\infty} \prod_{n=1}^\infty \left[ (1 + q^n)^4 - 16q^{2n} \cos^2 \theta \right].\]
Writing the infinite product as a power series in $\cos^2 \theta$ and using
\[
\int_{-\pi}^{\pi} \cos^{2k} \theta \frac{d\theta}{2\pi} = 2^{-2k} \binom{2k}{k},
\]
we evaluate the integral of (3.27) with respect to $d\theta/2\pi$ as
\[
\rho_0 = \frac{(-q; q)_{\infty}^4}{(q; q)_{\infty}} \sum_{k=0}^{\infty} \binom{2k}{k} \sum_{1 \leq n_1 < \ldots < n_k} \frac{(-2)^{2k} q^{2(n_1+\ldots+n_k)}}{(1 + q^{n_1}) \cdots (1 + q^{n_k})^4}.
\]
(3.28)

The formula (3.27) can be transformed further by putting $\cos^2 \psi = -\cos \theta$ and $p^2 = q$, because then
\[
\prod_{n=1}^{\infty} [(1 + q^n)^2 + 4q^n \cos \theta] = \prod_{n=1}^{\infty} [1 + p^{4n} - 2p^{2n} \cos(2\psi)]
\]
can be expressed by means of the theta function $\vartheta_1(p; \psi)$. We find
\[
\prod_{n=1}^{\infty} [(1 + q^n)^2 + 4q^n \cos \theta] = \frac{1}{(q; q)_{\infty}} \sum_{n=0}^{\infty} (-1)^n q^{\binom{n+1}{2}} U_{2n}(\cos \psi),
\]
(3.29)
where
\[
U_{2n}(\cos \psi) = \frac{\sin(2n+1)\psi}{\sin \psi}
\]
is the Chebyshev polynomial of the second kind given by
\[
U_{2n}(x) = \sum_{k=0}^{n} \binom{2n+1}{2k+1} (-1)^k x^{2(n-k)} (1 - x^2)^k.
\]
(3.30)

Similarly putting $\cos^2 \varphi = \cos \theta$ we find
\[
\prod_{n=1}^{\infty} [(1 + q^n)^2 - 4q^n \cos \theta] = \frac{1}{(q; q)_{\infty}} \sum_{n=0}^{\infty} (-1)^n q^{\binom{n+1}{2}} U_{2n}(\cos \varphi).
\]
(3.31)

If we let $U_n^*(x)$ be the polynomial of degree $n$ such that $U_{2n}(x) = U_n^*(x^2)$, we get
\[
\frac{B(e^{i\theta}) D(e^{-i\theta}) - B(e^{-i\theta}) D(e^{i\theta})}{e^{i\theta} - e^{-i\theta}} = \frac{1}{(q; q)_{\infty}^2} \sum_{n,m=0}^{\infty} (-1)^m q^{\binom{m+1}{2} + \binom{m+1}{2}} U_n^*(-\cos \theta) U_m^*(\cos \theta).
\]
(3.32)
For non-negative integers $k, l, r$ we have

\[
C(k, l, r) := \frac{1}{2\pi} \int_0^{2\pi} (1 + \cos \theta)^k (1 - \cos \theta)^l \cos^r \theta \, d\theta
\]

\[
= \frac{2^{k+l}}{\pi} (-1)^r B\left(k + \frac{1}{2}, l + \frac{1}{2}\right)_{2} F_{1}\left(k + \frac{1}{2}, -r; k + l + 1; 2\right),
\]

which gives

\[
\frac{1}{2\pi} \int_0^{2\pi} U_n^*(-\cos \theta) U_m^*(\cos \theta) \, d\theta
\]

\[
= \sum_{k=0}^{n} \sum_{l=0}^{m} \frac{(2n+1)}{2k+1} \frac{(2m+1)}{2l+1} (-1)^{n+l} C(k, l, n + m - k - l).
\]

Putting these formulas together we get a 5-fold sum for $\rho_0$.

Acknowledgement The authors would like to thank Mr. N. D. Lawrence for supplying the numerical data and the graph.

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