REMARKS ON LOCAL BOUNDEDNESS AND LOCAL HÖLDER CONTINUITY OF LOCAL WEAK SOLUTIONS TO ANISOTROPIC $p$-LAPLACIAN TYPE EQUATIONS

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Abstract. Locally bounded, local weak solutions to a special class of quasilinear, anisotropic, $p$-Laplacian type elliptic equations, are shown to be locally Hölder continuous. Homogeneous local upper bounds are established for local weak solutions to a general class of quasilinear anisotropic equations.

1. Introduction

Consider quasi-linear, elliptic differential equations of the form

\begin{equation}
\text{div} \ A(x, u, Du) = 0 \quad \text{weakly in some open set } E \subset \mathbb{R}^N
\end{equation}

where the function $A = (A_1, \ldots, A_N) : E \times \mathbb{R}^{N+1} \to \mathbb{R}^N$ is only assumed to be measurable and subject to the structure conditions

\begin{align}
& A_i(x, u, Du) \cdot u_{x_i} \geq C_{o,i} |u_{x_i}|^{p_i}, \\
& |A_i(x, u, Du)| \leq C_{1,i} |u_{x_i}|^{p_i-1},
\end{align}

where $p_i > 1$ and $C_{o,i}$ and $C_{1,i}$ are given positive constants. Such elliptic equations are termed anisotropic, their prototype being

\begin{equation}
\sum_{i=1}^N \left( |u_{x_i}|^{p_i-2} u_{x_i} \right)_{x_i} = 0 \quad \text{in } E.
\end{equation}

For a multi-index $\mathbf{p} = \{p_1, \ldots, p_N\}$, $p_i \geq 1$, let

\[ W^{1, \mathbf{p}}(E) = \{ u \in L^1(E) : u_{x_i} \in L^{p_i}(E), \ i = 1, \ldots, N \}, \]

and

\[ W^{1, \mathbf{p}}_o(E) = W^{1, \mathbf{p}}(E) \cap W^{1,1}_o(E). \]

A function $u \in W^{1, \mathbf{p}}(E)$ is a local, weak solution to (1.1) if for every compact set $K \subset E$

\begin{equation}
\int_K A(x, u, Du) : D\varphi \ dx = 0 \quad \text{for all } \varphi \in C^\infty_0(K).
\end{equation}

The parameters $\{N, p_i, C_{o,i}, C_{1,i}\}$ are the data, and we say that a generic constant $\gamma = \gamma(N, p_i, C_{o,i}, C_{1,i})$ depends upon the data, if it can be quantitatively determined a priori only in terms of the indicated parameters.
Define,

\[ \bar{p} = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{p_i}, \]

\[ p_{\min} = \min\{p_1, \ldots, p_N\}, \]

\[ p_{\max} = \max\{p_1, \ldots, p_N\}. \]

For a compact set \( K \subset E \) introduce the intrinsic, elliptic \( p \)-distance from \( K \) to \( \partial E \) by

\[ p - \text{dist}(K; \partial E) \overset{\text{def}}{=} \inf_{x \in K, y \in \partial E} \left( \sum_{j=1}^{N} \|u\|_{p_{\max} - p, E}^{p_{\max} - p} |x_j - y_j|^{p_{\max}} \right). \]

**Theorem 1.** Let \( u \) be a bounded, local, weak solution to \((1.1) - (1.2)\), and assume \( \bar{p} < N \).

There exists a positive quantity \( q > 1 \), depending only on the data, such that if

\[ p_{\max} - p_{\min} \leq \frac{1}{q}, \]

then \( u \) is locally Hölder continuous in \( E \), i.e. there exist constants \( \gamma > 1 \) and \( \alpha \in (0, 1) \) depending only on the data, such that for every compact set \( K \subset E \),

\[ |u(x_1) - u(x_2)| \leq \gamma \|u\|_{\infty, E} \left( \sum_{i=1}^{N} \|u\|_{p_{\max} - p, E}^{p_{\max} - p} |x_{1,i} - x_{2,i}|^{p_{\max}} \right)^{\alpha} \]

for every pair of points \( x_1, x_2 \in K \).

**Remark 1.** For a general distribution of the \( p_j \), unbounded weak solutions might exist \((\[7, 12\])\). In \([6, 9, 1]\) it is shown that local weak solutions are locally bounded provided

\[ \bar{p} < N, \quad p_{\max} \leq \frac{N \bar{p}}{N - \bar{p}}. \]

In Section 6 we revisit and improve these boundedness estimates.

**Remark 2.** The Hölder continuity of solutions holds also for \( \bar{p} \geq N \); indeed, when \( \bar{p} = N \), a straightforward modification of our arguments suffices, strictly analogous to the one used in the isotropic case when \( p = N \); when \( \bar{p} > N \), the embedding results of \([8, \text{Theorem 2}]\) (see also \([16]\)), ensure that \( u \) is Hölder continuous.

**Remark 3.** The constants \( \gamma \) and \( \alpha \) deteriorate as either \( p_i \to \infty \) or \( p_i \to 1 \), in the sense that \( \gamma(p) \to \infty \) and \( \alpha \to 0 \) as either \( p_i \to \infty \), or \( p_i \to 1 \).

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\section*{2. Novelty and Significance}

If the coefficients in \((1.1) - (1.2)\) are differentiable, and satisfy some further, suitable structure conditions, Lipschitz estimates have been derived by Marcellini \([13, 14]\). If the coefficients are merely bounded and measurable, Hölder continuity has been established in \([11]\) in the special case of \( p_1 = 2 < p_2 = p_3 = \cdots = p_N \), i.e., the \( p_j \) are all the same except the smallest one. The main idea is to regard the equation as “parabolic” with respect to the variable \( x_1 \), corresponding to \( p_1 \), and to apply the techniques of \([3, 4]\). An extension to the case \( 1 < p_1 < p_2 = p_3 = \cdots = p_N \), by the same techniques, is in \([5]\).

Theorem \([1] \) is a further step in understanding the regularity of solutions of anisotropic elliptic equations, with full quasi-linear structure. Our approach is “elliptic” in nature, it is
modelled after \[2\], no variable is regarded as “parabolic,” and no restriction is placed on the distribution of the \(p_j\) other that \(p_{\text{max}} - p_{\text{min}} \ll 1\). In particular, the \(p_j\) could all be different.

While partial, Theorem \[1\] disproves the claim in \[7\] by which, Hölder continuity of weak solutions to \((1.1)-(1.2)\), holds if and only if \(p_{\text{min}} = p_{\text{max}}\), i.e., if no anisotropy is present.

Finally, Theorem \[1\] can be seen as a stability result of the Hölder continuity of solutions, when \(p_i \to p_{\text{min}}\), and correspondingly, the anisotropic \(p\)-laplacian tends to the \(p_{\text{min}}\)-laplacian.

3. Preliminaries and Intrinsic Geometry

Lemma 1 (Sobolev-Troisi Inequality, \[17\]). Let \(E \subset \mathbb{R}^N\) be a bounded, open set and consider \(u \in W^{1,p}_o(E), p_i > 1\) for all \(i = 1, \ldots, N\). Assume \(\bar{p} < N\) and let

\[
(3.1) \quad p_\ast = \frac{N\bar{p}}{N - \bar{p}}.
\]

Then there exists a constant \(c\) depending only on \(N, p\), such that

\[
\|u\|_{L^{p_\ast}(E)} \leq c \prod_{i=1}^N \|u_{x_i}\|_{L^{p_i}(E)}.
\]

For \(\rho > 0\) consider the cube \(K_\rho = (-\rho, \rho)^N\), with center at the origin of \(\mathbb{R}^N\) and edge \(2\rho\), and set

\[
(3.2) \quad \mu^+ = \text{ess sup}_{K_\rho} u; \quad \mu^- = \text{ess inf}_{K_\rho} u; \quad \omega = \mu^+ - \mu^- = \text{ess osc}_{K_\rho} u.
\]

These numbers being determined, construct the cylinder

\[
(3.3) \quad Q_\rho = \prod_{j=1}^N (-\rho_j, \rho_j),
\]

with \(0 < \rho_j \leq \rho\) to be determined. This implies that \(Q_{2\rho} \subset K_\rho\), and hence \(\text{ess osc}_{Q_\rho} u \leq \omega\).

3.1. Basic Equation and Energy Inequalities. For \(\sigma \in (0,1)\) let \(\zeta_j\) be a non-negative, piecewise smooth cutoff function in the interval \((-\rho_j, \rho_j)\) which equals 1 on \((-\sigma \rho_j, \sigma \rho_j)\), vanishes at \(\pm \rho_j\), and such that \(|\zeta_j| \leq \left|(1 - \sigma)\rho_j\right|^{-1}\).

Set \(\zeta = \prod_{j=1}^N \zeta_j^p\), and in the weak formulation of \((1.1)-(1.2)\) take the testing function \(\pm (u - k)\zeta\). This gives, after standard calculations,

\[
(3.4) \quad \sum_{j=1}^N \int_{Q_\rho} \left| \frac{\partial}{\partial x_j} [(u - k)\zeta^p] \right|^{p_j} dx \leq \gamma \sum_{j=1}^N \frac{1}{(1 - \sigma)^{p_j} \rho_j^{p_j}} \int_{Q_\rho} (u - k)^{p_j} dx.
\]

The constant \(\gamma\) depends only upon the data, and is independent of \(\rho\).

4. DeGiorgi Type Lemmas

Taking \(k = \mu^+ - \frac{\omega}{2^s}\), for \(s \geq 1\), and \((u - k)\) in \((3.4)\) yields

\[
(4.1) \quad \sum_{j=1}^N \int_{Q_\rho} \left| \left[ (u - \left( \mu^+ - \frac{\omega}{2^s} \right) \right] \zeta^p \right|^{p_j} dx
\]

\[
\leq \frac{\gamma}{(1 - \sigma)^{p_{\text{max}}}} \sum_{j=1}^N \frac{1}{\rho_j^{p_j}} \left( \frac{\omega}{2^s} \right)^{p_j} |Q_\rho \cap \{ u > \mu^+ - \frac{\omega}{2^s} \}|.
\]
Likewise, taking $k = \mu^- + \frac{\omega}{2^q}$, for $s \geq 1$, and $-(u - k)_-$ in (3.4) yields
\[
\sum_{j=1}^{N} \int_{Q_{\rho_j}} \left| \left( u - \left( \mu^- + \frac{\omega}{2^q} \right) \right)_- \right|^p \, dx 
\leq \frac{\gamma}{(1 - \sigma)^{p_{\max}}} \sum_{j=1}^{N} \frac{1}{\rho_j^p} \left( \frac{\omega}{2^q} \right)^p \left| Q_{\rho_j} \cap \left| u < \mu^- + \frac{\omega}{2^q} \right| \right|.
\]
Choose
\[
\rho_j = \left( \frac{\omega}{2^q} \right)^p \rho_j^\alpha \quad \text{for some } q > 0 \text{ and } \alpha \geq p_{\max} \text{ to be chosen}
\]
and let $Q_{\rho_j}$ the cylinder in (3.3) for such a choice of $\rho_j$. Without loss of generality, may assume $\omega \leq 1$, so that $0 < \rho_j \leq \rho$ as required.

**Lemma 2.** There exists a number $\nu \in (0, 1)$ depending only upon the data, such that if
\[
\left| \left[ u > \mu^+ - \frac{\omega}{2^q} \right] \cap Q_{\rho_j} \right| < \nu |Q_{\rho_j}|,
\]
for some $q \in \mathbb{N}$, then
\[
u \leq \mu^+ - \frac{\omega}{2^q+1} \quad \text{a.e. in } Q_{\frac{1}{2}\rho}.
\]

Likewise

**Lemma 3.** There exists a number $\nu \in (0, 1)$ depending only upon the data, such that if
\[
\left| \left[ u < \mu^- + \frac{\omega}{2^q} \right] \cap Q_{\rho_j} \right| < \nu |Q_{\rho_j}|,
\]
for some $q \in \mathbb{N}$, then
\[
u \geq \mu^- + \frac{\omega}{2^q+1} \quad \text{a.e. in } Q_{\frac{1}{2}\rho}.
\]

We prove only Lemma 2, the proof of Lemma 3 being analogous.

**Proof.** For each $j \in \{1, \ldots, N\}$ consider the sequence of radii
\[
\rho_{j,n} = \frac{1}{2} \rho_j \left( 1 + \frac{1}{2^n} \right), \quad \text{for } n = 0, 1, \ldots.
\]
This is a decreasing sequence with $\rho_{j,0} = \rho_j$ and $\rho_{j,\infty} = \frac{1}{2} \rho_j$. The corresponding cylinders
\[
Q_n \stackrel{\text{def}}{=} Q_{\rho_{n+1}} = \prod_{j=1}^{N} \left( -\rho_{j,n}, \rho_{j,n} \right) \quad \text{for } n = 0, 1, \ldots
\]
are nested, i.e., $Q_{n+1} \subset Q_n$, with $Q_0 = Q_{\rho_j}$ and $Q_\infty = Q_{\frac{1}{2}\rho_j}$, since $\alpha \geq 1$. For each $j \in \{1, \ldots, N\}$ let $\zeta_{j,n}$ be a standard non-negative cutoff function in $(-\rho_{j,n}, \rho_{j,n})$ which equals 1 on $(-\rho_{j,n+1}, \rho_{j,n+1})$, vanishes at $\pm \rho_{j,n}$ and such that $|\zeta_{j,n}'| \leq 2^{n+2} \rho_{j,n}^{-1}$. Then set $\zeta_n = \prod_{j=1}^{N} \zeta_{j,n}^{\rho_{j,n}}$ to be a cutoff function in $Q_n$ that equals 1 on $Q_{n+1}$. Consider also the increasing sequence of levels
\[
k_n = \mu^+ - \frac{\omega}{2^q+1} \left( 1 + \frac{1}{2^n} \right), \quad \text{for } n = 0, 1, \ldots
\]
and in the weak formulation of (4.1) – (4.2), take the test function $(u - k_n)_+ \zeta_n$. This leads to analogues of (4.1) over the cylinders $Q_n$, with $1 - \sigma > 2^{-n+2}$, and $q \leq s < q+1$. Rewriting (4.1) with these specifications gives
\[
\sum_{j=1}^{N} \int_{Q_n} \left| \left( u - k_n \right)_+ \zeta_n \right|^p \, dx \leq \frac{\gamma}{\rho_j^{p_{\max}}} \left| Q_n \cap \left| u > k_n \right| \right|.
\]
Since \((u - k_n)_+ \zeta_n\) vanishes on \(\partial Q_n\), by the anisotropic embedding of Lemma 11, we have

\[
(4.12) \quad \left( \int_{Q_n} \left[ \left( u - k_n \right) \zeta_n \right]^p \, dx \right)^{\frac{1}{p}} \leq c \prod_{j=1}^N \left( \int_{Q_n} \left[ \left( u - k_n \right) \zeta_n \right]^{p_j} \, dx \right)^{\frac{1}{p_j}}.
\]

where \(p_j\) is as in (3.1), \(p_j > 1\) for \(j = 1, \ldots, N\), \(\bar{p} < N\). Since \(0 \leq \zeta_n \leq 1\) and \(p_j > 1\) estimate

\[
\int_{Q_n} \left[ \left( u - k_n \right) \zeta_n \right]^{p_j} \, dx \leq \gamma \int_{Q_n} \left[ \left( u - k_n \right) \zeta_n \right]^{p_j} \, dx.
\]

Therefore, combining this with (4.12) and (4.11) gives

\[
(k_{n+1} - k_n) |Q_{n+1} \cap [u > k_{n+1}]| \leq \int_{Q_{n+1} \cap [u > k_{n+1}]} (u - k_n) \, dx
\]

\[
\leq \int_{Q_n} (u - k_n) \zeta_n \, dx \leq \left( \int_{Q_n} \left[ \left( u - k_n \right) \zeta_n \right]^p \, dx \right)^{\frac{1}{p}} |Q_n \cap [u > k_n]|^{1 - \frac{1}{p}}
\]

\[
\leq \gamma \prod_{j=1}^N \left( \int_{Q_n} \left[ \left( u - k_n \right) \zeta_n \right]^{p_j} \, dx \right)^{\frac{1}{p_j}} \left( |Q_n \cap [u > k_n]| \right)^{1 - \frac{1}{p}}
\]

\[
\leq \gamma \left( \frac{2^{p_{\text{max}}}}{\omega_2 \rho_\omega} \right)^n \frac{1}{\rho_\omega^n} |Q_n \cap [u > k_n]|^{1 + \frac{1}{p} - \frac{1}{p}}
\]

\[
= \gamma b^n \left( \frac{\omega_2}{\rho_\omega} \right)^n \frac{1}{\rho_\omega^n} |Q_n \cap [u > k_n]|^{1 + \frac{1}{p}}
\]

where we have set \(b = 2^{p_{\text{max}}} / \bar{p}\). By the definition of \(k_n\) in (4.10), the first term in round brackets on the left hand side is

\[
k_n - k_{n+1} = \left( \frac{\omega_2}{\rho_\omega} \right) \frac{1}{2^{n+1}}.
\]

Combining these remarks and inequalities, and setting

\[
Y_n = \frac{|Q_n \cap [u > k_n]|}{|Q_\rho|}
\]

yields the recursive inequalities

\[
Y_{n+1} \leq C(2b)^n Y_n^{1 + \frac{1}{p}}
\]

for constants \(C\) and \(b\) depending only upon the data. It follows from these and Lemma 5.1 of [4, Chapter 2], that there exists a number \(\nu \in (0, 1)\) depending only on \(\{C, b, N\}\), and hence only upon the data, such that \(\{Y_n\} \to 0\) as \(n \to \infty\) provided

\[
Y_0 = \frac{|Q_\rho \cap [u > k_n - \frac{\mu^+}{2\rho_\omega}]|}{|Q_\rho|} \leq \nu.
\]

By these lemmata the Hölder continuity of \(u\) will follow by standard arguments, if one can determine \(q\), and hence the intrinsic cylinders \(Q_\rho\), for which either (3.3) or (3.6) holds.\]
5. Proof of Theorem \( \square \)

Assume that

\[
\left| \left[ u < \mu - \frac{1}{2} \omega \right] \cap Q_\rho \right| \geq \frac{1}{2} |Q_\rho|.
\]

For each \( s \in \mathbb{N} \) with \( s \leq q \), introduce the two complementary sets

\[
A_s = \left[ u > \mu + \frac{\omega}{2s} \right] \cap Q_\rho; \quad Q_\rho - A_s = \left[ u \leq \mu + \frac{\omega}{2s} \right] \cap Q_\rho
\]

and consider the doubly truncated function

\[
v_s = \begin{cases} 
0 & \text{for } u < \mu + \frac{\omega}{2s}; \\
- \left( \mu + \frac{\omega}{2s} - u \right) & \text{for } \mu + \frac{\omega}{2s} \leq u < \mu + \frac{\omega}{2s+1}; \\
\frac{\omega}{2s+1} & \text{for } \mu + \frac{\omega}{2s+1} \leq u.
\end{cases}
\]

By construction \( v_s \) vanishes on \( Q_\rho - A_s \). Pick any two points $x = (x_1, \ldots, x_N) \in A_s$ and $y = (y_1, \ldots, y_N) \in Q_\rho - A_s$ and construct a polygonal joining \( x \) and \( y \) and sides parallel to the coordinate axes, say for example $P_N = x$ and

\[
\begin{align*}
P_{N-1} &= (x_1, \ldots, x_{N-1}, y_N); \\
P_{N-2} &= (x_1, x_2, \ldots, y_{N-1}, y_N); \\
P_1 &= (x_1, y_2, \ldots, y_{N}); \\
P_o &= (y_1, \ldots, y_N).
\end{align*}
\]

By elementary calculus

\[
v_s(x) = [v_s(P_N) - v_s(P_{N-1})] + \cdots + [v_s(P_1) - v_s(P_o)]
\]

\[
= \int_{y_N}^{x_N} \frac{\partial}{\partial x_N} v_s(x_1, \ldots, x_{N-1}, t) dt + \int_{y_{N-1}}^{x_{N-1}} \frac{\partial}{\partial x_{N-1}} v_s(x_1, \ldots, x_{N-2}, t, y_N) dt \\
+ \cdots + \int_{y_1}^{x_1} \frac{\partial}{\partial x_1} v_s(t, y_2, \ldots, y_N) dt
\]

\[
\leq \sum_{j=1}^{N} \int_{-\rho_j}^{\rho_j} |v_{s,x_j}| (x_1, \ldots, t, \ldots, y_N) dt
\]

where the quantities \( \rho_j \) are defined in (4.3). Integrate in \( dx \) over \( A_s \) and in \( dy \) over \( Q_\rho - A_s \), and take into account (5.1) to get

\[
\frac{1}{2} |Q_\rho| \int_{Q_\rho} v_s dx \leq 2 |Q_\rho| \sum_{j=1}^{N} \rho_j \int_{Q_\rho} |v_{s,x_j}| dx.
\]
From this, by the definitions \([\omega_{2s+1}]\) and \([\nu_{2s}]\) of \(A_s\) and \(v_s\),

\[
\frac{\omega}{2s+1} |A_{s+1}| \leq 4 \sum_{j=1}^{N} \rho_j \int_{A_s - A_{s+1}} |u_{x_j}| \, dx
\]

(5.4)

\[
\leq 4 \sum_{j=1}^{N} \rho_j \left( \int_{A_s - A_{s+1}} |u_{x_j}|^{p_{min}} \, dx \right)^{\frac{1}{p_{min}}} |A_s - A_{s+1}|^{1 - \frac{1}{p_{min}}}
\]

\[
\leq 4 \sum_{j=1}^{N} \rho_j \left( \int_{A_s - A_{s+1}} |u_{x_j}|^{p_j} \, dx \right)^{\frac{1}{p_j}} |Q_{\rho}|^{\frac{1}{p_{min} - \frac{1}{p_j}}} |A_s - A_{s+1}|^{1 - \frac{1}{p_{min}}}.
\]

For each \(j\) fixed, the integrals involving \(u_{x_j}\) are estimated by means of (4.1) applied over the pair of cubes \(Q_{\rho}\) and \(Q_{2\rho}\), as follows:

\[
\left( \int_{A_s - A_{s+1}} |u_{x_j}|^{p_j} \, dx \right)^{\frac{1}{p_j}} \leq \left( \int_{Q_{\rho}} \left| \frac{\partial}{\partial x_j} (u - (\mu^+ - \frac{\omega}{2^s})) \right|^{p_j} \, dx \right)^{\frac{1}{p_j}}
\]

\[
\leq \gamma \left( \frac{1}{\rho^a} \sum_{\ell=1}^{N} \left( \frac{\omega}{2^s} \right)^{p_{\ell}} \left( \frac{\omega}{2^s} \right)^{-p_{\ell}} |Q_{\rho}| \right)^{\frac{1}{p_j}}
\]

\[
\leq \gamma \left( \frac{1}{\rho^a} \sum_{\ell=1}^{N} \left( \frac{q}{2^s} \right)^{p_{\ell}} |Q_{\rho}| \right)^{\frac{1}{p_j}}
\]

\[
= \gamma \left( \frac{1}{\rho^a} \left( \frac{\omega}{2^s} \right)^{p_j} \sum_{\ell=1}^{N} \left( \frac{q}{2^s} \right)^{p_{\ell}} |Q_{\rho}| \right)^{\frac{1}{p_j}}.
\]

If \(p_{\ell} \leq p_j\), since \(s \leq q\) estimate

(5.5) \[
\left( \frac{2^q}{2^s} \right)^{p_{\ell}} \leq \left( \frac{2^q}{2^s} \right)^{p_j} = \left( \frac{\omega}{2^s} \right)^{p_j} \left( \frac{\omega}{2^s} \right)^{-p_j}, \quad \text{(case of} \ p_{\ell} \leq p_j)\)
\]

If \(p_{\ell} > p_j\) since \(s \leq q\) compute and estimate

(5.6) \[
\left( \frac{2^q}{2^s} \right)^{p_{\ell}} \leq \left( \frac{2^q}{2^s} \right)^{p_j} \left( \frac{2^q}{2^s} \right)^{p_{\ell} - p_j} = \left( \frac{\omega}{2^s} \right)^{p_j} \left( \frac{\omega}{2^s} \right)^{-p_j} 2^{q(p_{\max} - p_{\min})}, \quad \text{(case of} \ p_{\ell} > p_j)\)
\]

Assume momentarily that the number \(q\) has been chosen. Then stipulate that \(q(p_{\max} - p_{\min}) \leq 1\). For such a choice we have in all cases

\[
\left( \int_{A_s - A_{s+1}} |u_{x_j}|^{p_j} \, dx \right)^{\frac{1}{p_j}} \leq \gamma \frac{1}{\rho_j} \left( \frac{\omega}{2^s} \right)^{\frac{1}{p_j}} |Q_{\rho}|^{\frac{1}{p_{min} - \frac{1}{p_j}}}.
\]

Combining these estimates in (5.4) yields

\[
|A_{s+1}| \leq \gamma |Q_{\rho}|^{\frac{1}{p_{min} - \frac{1}{p_j}}} \left( |A_s| - |A_{s+1}| \right)^{1 - \frac{1}{p_{min}}}.
\]

Take the \(\left(\frac{p_{\min}}{p_{\min} - \frac{1}{p_j}}\right)\) power and add for \(s = 1, \ldots, (q - 1)\) to get

\[
(q - 1) |A_q|^{\frac{p_{\min}}{p_{\min} - \frac{1}{p_j}}} \leq \gamma^{\frac{p_{\min}}{p_{\min} - \frac{1}{p_j}}} |Q_{\rho}|^{\frac{1}{p_{min} - \frac{1}{p_j}}} |A_0|.
\]
From this
\[ |A_q| \leq \frac{\gamma}{(q - 1) \rho_{\min}} |Q_p|, \]

In the DeGiorgi-type Lemma, the number \( \nu \) is independent of \( q \). Now choose \( q \) so that
\[ (5.7) \quad |A_q| \leq \nu |Q_p|, \quad \text{for} \quad \nu = \frac{\gamma}{(q - 1) \rho_{\min}}. \]

Notice that \( q \) is determined in terms of \( p_{\min} \) and not in terms of the difference \( p_{\max} - p_{\min} \).

Thus, one determines first \( q \) from (5.7) in terms only of the data. Then (1.6), for such a choice of \( q \), serves as a condition of Hölder continuity for \( u \).

### 6. Boundedness

Continue to denote by \( u \in W^{1,p}(E) \) a local weak solution to (1.1)–(1.2), in the sense of (1.3), with \( \bar{p} < N \). The estimations below use that \( u \in L^{p_\ast}_{\text{loc}}(E) \). When \( p_{\max} < p_\ast \) this is insured by the embeddings in [10, Theorem 1] or [15]. If \( p_{\max} = p_\ast \) in what follows the membership \( u \in L^{p_\ast}_{\text{loc}}(E) \), is assumed.

When \( \bar{p} = N \), then [8, Theorem 1] ensures that \( u \in L^q_{\text{loc}}(E) \) for any arbitrary \( 1 \leq q < \infty \), and the arguments below can be repeated verbatim, to obtain a quantitative estimate of the local boundedness of \( u \). Finally, when \( \bar{p} > N \), as we mentioned in Remark [2] proper Morrey-type embeddings directly ensure the boundedness of \( u \).

### 6.1. Some General Recursive Inequalities

Let \( \rho_j \) as in (3.3) to be defined, and for each \( j \in \{1, \ldots, N\} \) consider the sequence of radii,
\[ (6.1) \quad \rho_{j,n} = \frac{1}{2} \rho_j \left( 1 + \frac{1}{2^n} \right), \quad \text{for} \quad n = 0, 1, \ldots. \]

This is a decreasing sequence, with \( \rho_{j,n} = \rho_j \) and \( \rho_{j,\infty} = \frac{1}{2} \rho_j \). The corresponding cylinders
\[ (6.2) \quad Q_n \overset{\text{def}}{=} Q_{\rho_n} = \prod_{j=1}^N (\rho_{j,n} - \rho_{j,n+1}) \quad \text{for} \quad n = 0, 1, \ldots \]

are nested, i.e., \( Q_{n+1} \subset Q_n \), with \( Q_0 = Q_{\rho} \) and \( Q_\infty = Q_{\rho_\ast} \), since \( \alpha \geq p_j \). For each \( j \in \{1, \ldots, N\} \) let \( \zeta_{j,n} \) be a standard non-negative cutoff function in \( (\rho_{j,n+1} - \rho_{j,n}) \) which equals 1 on \( (\rho_{j,n+1} - \rho_{j,n}) \) vanishes at \( \pm \rho_{j,n} \) and such that \( |\zeta_{j,n}'| \leq 2^{n+2} \rho_{j,n}^{-1} \). Then set \( \zeta_n = \prod_{j=1}^N \zeta_{j,n}^{\rho_j} \) to be a cutoff function in \( Q_n \) that equals 1 on \( Q_{n+1} \).

Consider also the increasing sequence of levels
\[ (6.3) \quad k_n = \left( 1 - \frac{1}{2^n} \right) k, \quad \text{and} \quad \bar{k}_n = \frac{k_{n+1} + k_n}{2} \]

for \( n = 0, 1, \ldots, k > 0 \) to be chosen. By the definition \( k_0 = 0 \) and \( k_\infty = k \). Write (3.4) for \( (u - \bar{k}_n) + \zeta_n \), over the cylinders \( Q_n \), with \( 1 - \sigma > 2^{-\nu(n+2)} \). Since \( (u - \bar{k}_n) + \zeta_n \) vanishes on \( \partial Q_n \), by the anisotropic embedding of Lemma [11]
\[ (6.4) \quad \left( \int_{Q_n} \left| (u - \bar{k}_n) + \zeta_n \right|^{p_j} dx \right)^{\frac{1}{p_j}} \leq \gamma \left( \prod_{j=1}^N \int_{Q_n} \left| (u - \bar{k}_n) + \zeta_n \right|_{x_j}^{p_j} dx \right)^{\frac{1}{p_j}}. \]

where \( p_\ast \) has been defined in (5.1).

Since \( 0 \leq \zeta_n \leq 1 \) and \( p_j \geq 1 \) estimate
\[ \int_{Q_n} \left| (u - \bar{k}_n) + \zeta_n \right|^{p_j} dx \leq \gamma \int_{Q_n} \left| (u - \bar{k}_n) + \zeta_n \right|_{x_j}^{p_j} dx. \]
Therefore, combining this with (3.4) and (6.4) gives

\[
\left( \int_{Q_{n+1}} (u - \bar{k}_n)^{p_\ell^*} \right)^{\frac{1}{p_\ell^*}} \leq \gamma \prod_{j=1}^{N} \left( \int_{Q_n} \left[ (u - \bar{k}_n) + \zeta_{n,j} \right]^{p_j^*} dx \right)^{\frac{1}{N p_j^*}} \]

\[
\leq \gamma \prod_{j=1}^{N} \left( \sum_{k=1}^{N} \frac{2^{p_j^*}}{\rho_j^*} \int_{Q_n} (u - \bar{k}_n)^{p_j^*} dx \right)^{\frac{1}{N p_j^*}} \]

\[
= \gamma \left( \sum_{k=1}^{N} \frac{2^{p_j^*}}{\rho_j^*} \int_{Q_n} (u - \bar{k}_n)^{p_j^*} dx \right)^{\frac{1}{p_j^*}} \]

From this homogenizing with respect to the measure of \( Q_n \) and with respect to the integrand,

\[
\left( \frac{1}{k_{p_\ell^*}} \int_{Q_{n+1}} (u - \bar{k}_n)^{p_\ell^*} \right)^{\frac{1}{p_\ell^*}} \]

\[
\leq \gamma \left( |Q_{\rho_{p_\ell^*}}| \sum_{k=1}^{N} 2^{p_j^*} \frac{k_{p_j^*-p_\ell^*}}{\rho_j^*} \frac{1}{k_{p_\ell^*}} \int_{Q_n} (u - \bar{k}_n)^{p_j^*} dx \right)^{\frac{1}{p_j^*}} \]

For each \( \ell \in \{1, \ldots, N\} \), estimate

\[
\frac{1}{k_{p_\ell^*}} \int_{Q_n} (u - \bar{k}_n)^{p_\ell^*} dx \leq \left( \frac{1}{k_{p_\ell^*}} \int_{Q_n} (u - \bar{k}_n)^{p_\ell^*} \right)^{\frac{1}{p_\ell^*}} \left( \frac{|[u > \bar{k}_n] \cap Q_n|}{|Q_n|} \right)^{1 - \frac{p_\ell^*}{p_\ell^*}} \]

Also

\[
\frac{1}{k_{p_{j^*}}} \int_{Q_n} (u - k_n)^{p_{j^*}} dx \geq \frac{1}{k_{p_{j^*}}} \int_{Q_n \cap |u > \bar{k}_n|} (k_n - k_n)^{p_{j^*}} dx \]

\[
\geq \frac{1}{2^{p_{j^*}(n+2)}} \frac{|[u > \bar{k}_n] \cap Q_n|}{|Q_n|} \]

Therefore,

\[
\frac{1}{k_{p_{j^*}}} \int_{Q_n} (u - \bar{k}_n)^{p_{j^*}} dx \leq 2^{(p_{j^*}-p_{\ell^*})(n+2)} \frac{1}{k_{p_{j^*}}} \int_{Q_n} (u - k_n)^{p_{j^*}} dx \]

Combine these calculations in (6.5), to get

\[
\left( \frac{1}{k_{p_{j^*}}} \int_{Q_{n+1}} (u - k_{n+1})^{p_{j^*}} \right)^{\frac{1}{p_{j^*}}} \]

\[
\leq \gamma 2^{p_{j^*}} \left[ |Q_{\rho_{p_{j^*}}}| \sum_{k=1}^{N} \frac{k_{p_{j^*} - p_{\ell^*}}}{\rho_j^*} \right]^{\frac{1}{p_{j^*}}} \left[ \left( \frac{1}{k_{p_{j^*}}} \int_{Q_n} (u - k_n)^{p_{j^*}} dx \right)^{\frac{1}{p_{j^*}}} \right]^{\frac{1}{p_{j^*}}} \]

Set

\[
Y_n = \left( \frac{1}{k_{p_{j^*}}} \int_{Q_n} (u - k_n)^{p_{j^*}} dx \right)^{\frac{1}{p_{j^*}}} \]

\[
(6.6) \]

\[
Y_n \leq \gamma 2^{p_{j^*}} \left[ |Q_{\rho_{p_{j^*}}}| \sum_{k=1}^{N} \frac{k_{p_{j^*} - p_{\ell^*}}}{\rho_j^*} \right]^{\frac{1}{p_{j^*}}} \left[ \left( \frac{1}{k_{p_{j^*}}} \int_{Q_n} (u - k_n)^{p_{j^*}} dx \right)^{\frac{1}{p_{j^*}}} \right]^{\frac{1}{p_{j^*}}} \]

\[
(6.6) \]
and rewrite the previous inequalities in the form

\[ Y_{n+1} \leq \gamma 2^{\frac{p}{p^*}} \left( Q_p \right)^{\frac{p}{p^*}} + \sum_{\ell=1}^{N} \frac{k^{p_\ell - p}}{p_\ell^{p_\ell}} Y_n^{1 + \frac{p_\ell - p}{p}}. \]

Recall that the radii \( \rho_j \) are still to be chosen.

6.2. A Quantitative, Homogeneous Estimate for \( p_{max} < p_* \). Choose

\[ \rho_j = \rho^\alpha_j, \]

where \( \alpha \) is an arbitrary positive parameter. Stipulate to take \( k \geq 1 \) and estimate

\[ \left( \left| Q_\rho \right|^{\frac{p}{p^*}} \sum_{\ell=1}^{N} \frac{k^{p_\ell - p}}{p_\ell^{p_\ell}} \right)^{\frac{1}{p}} \leq 2Nk^{\frac{p_{max} - p}{p}}. \]

For such choices \( 6.7 \) yield

\[ Y_{n+1} \leq \gamma 2^{\frac{p}{p^*}} k^{\frac{p_{max} - p}{p}} Y_n^{1 + \frac{p_\ell - p}{p}} \]

for a new constant \( \gamma \) depending only on \( \{N, p_1, \ldots, p_N\} \). It follows from these that \( \{Y_n\} \to 0 \) as \( n \to \infty \), provided

\[ Y_0 = \frac{1}{k} \left( \int_{Q_\rho} u_{+}^{p_*} \, dx \right)^{\frac{1}{p_*}} \leq \gamma^{-\frac{p}{p^*}} 2^{\frac{p}{p^*}} \left( \frac{p}{p^*} \right)^2 k^{-\frac{p_{max} - p}{p}}. \]

Thus, choosing

\[ k = \gamma^{\frac{p}{p^*}} 2^{\frac{p}{p^*}} \left( \frac{p}{p^*} \right)^2 k^{-\frac{p_{max} - p}{p}} \]

yields

\[ \text{ess sup}_{Q_{\frac{1}{2}\rho}} u_{+} \leq 1 \land C \left( \left( \int_{Q_\rho} u_{+}^{p_*} \, dx \right)^{\frac{1}{p_*}} \right)^{\frac{p_{max} - p}{p}} \]

where

\[ C = \gamma^{\frac{p}{p^*}} 2^{\frac{p}{p^*}} \left( \frac{p}{p^*} \right)^2. \]

Write now \( 6.10 \) over the pair of cubes \( Q_{\sigma \rho} \subset Q_\rho \), where \( \sigma \in \left( \frac{1}{2}, 1 \right) \) is an interpolation parameter. Then

\[ \text{ess sup}_{Q_{\frac{1}{2}\rho}} u_{+} \leq 1 \land C' \left( \int_{Q_\rho} u_{+}^{p_{max}} \, dx \right)^{\frac{1}{p_{max}}}. \]

Remark 4. The estimates in \( 6.10 \) and \( 6.11 \) are homogeneous with respect to the cube \( Q_\rho \), i.e., they are invariant for dilations of the variables \( (x_1, \ldots, x_N) \) that keep invariant the relative intrinsic geometry of \( 3.3 \) and \( 6.8 \). In this sense they are an improvement with respect to the estimates of Kolodii [9, Theorem 2]. If \( p_j = \bar{p} \) for all \( j = 1, \ldots, N \) this reproduces the classical estimate for isotropic elliptic equations.

Remark 5. The constants \( C \) and \( C' \) in \( 6.10 \) and \( 6.11 \), can be quantitatively determined only in terms of \( N \) and the \( p_j \) for \( j = 1, \ldots, N \). However, they tend to infinity as \( p_{max} \wedge p_* \).
6.3. A Quantitative, Homogeneous Estimate for $p_{\text{max}} = p_\ast$. Redefine the levels in \ref{eq:pmax} as
\begin{equation}
k_n = \left(1 - \frac{1}{2^{n+1}}\right)k, \quad \text{and} \quad \bar{k}_n = \frac{k_{n+1} + k_n}{2}, \quad \text{for } n = 0, 1, \ldots.
\end{equation}
This implies that $k_0 = \frac{1}{2}k$ and $k_\infty = k$. All estimations remain unchanged and yield \ref{eq:est}, with a slight modification of the constant $\gamma$ and with the same definition \ref{eq:yn} of the $Y_n$. Continue to choose $\rho_j = \rho_\alpha^{\frac{p_j}{p_\ast}}$, and stipulate to take $k \geq 1$. This yields the analogues of \ref{eq:ineq} with $p_{\text{max}} = p_\ast$, i.e.,
\begin{equation}
Y_{n+1} \leq \gamma 2^{\frac{p_\ast}{p}} k^{\frac{p_\ast - k}{p}} Y_n^{1 + \frac{p_\ast - k}{p}}.
\end{equation}
Taking into account the definition \ref{eq:yn} of the $Y_n$, in this last inequality, the parameter $k$ scales out. Thus, setting
\begin{equation}
X_n = \left(\int_{Q_n} \left(u - k_n\right)^{p_\ast} dx\right)^{\frac{1}{p_\ast}},
\end{equation}
the recursive inequalities \ref{eq:ineq} are
\begin{equation}
X_{n+1} \leq \gamma 2^{\frac{p_\ast}{p}} X_n^{1 + \frac{p_\ast - k}{p}}.
\end{equation}
It follows from these that $\{X_n\} \to 0$ as $n \to \infty$, provided
\begin{equation}
\forall \frac{p_\ast}{p} \text{ ess sup}_{Q_{\frac{1}{2}k}} u_+ \leq 1 \wedge k.
\end{equation}

Remark 6. The estimate in \ref{eq:ineq} is homogeneous with respect to the cube $Q_k$, i.e., it is invariant for dilations of the variables $(x_1, \ldots, x_N)$ that keep invariant the relative intrinsic geometry of \ref{eq:geo} and \ref{eq:6.8}. In this sense, it is an improvement with respect to the estimates of Fusco-Sbordone \cite[Theorem 1]{6}.

Remark 7. In \ref{eq:ineq} the number $\kappa = \frac{p_\ast - k}{p}$ by which the power of $X_n$ exceeds one, is precisely determined by the estimations, and not arbitrary as it seems to be permitted in \cite{6}. In view of this, the alternative, in the argument of \cite{6}, by which
\begin{equation}
2^{\frac{p_\ast}{p}} X_n^{1 + \frac{p_\ast - k}{p}} > 1 \quad \text{for all } n \geq n_0 \text{ for some } n_0 \in \mathbb{N} \text{ sufficiently large}
\end{equation}
is not needed. Since $n_0$ in \cite{6} is determined only qualitatively, the resulting boundedness estimates seem to be qualitative.

Remark 8. If one had the additional information that $u \in L^q_{\text{loc}}(E)$, for some $q > p_\ast$, then $k$ in \ref{eq:ineq} could be precisely quantified. Indeed, given a non negative function $f \in L^q(E)$ and $\varepsilon > 0$, consider finding $k > 0$ such that
\begin{equation}
\int_E (f - k)^{p_\ast} dx < \varepsilon \quad \text{where } 0 < p < q.
\end{equation}
By Chebyshev’s inequality $||f > t|| \leq t^{-q}||f||^{q}_{q,E}$, for all $t > 0$. Then for $p < q$,

$$\int_{E} (f - k)^{p}_{+} dx = p \int_{0}^{\infty} s^{p-1} ||(f - k)_{+} > s|| ds$$

$$= p \int_{k}^{\infty} (t - k)^{p-1} ||f > t|| dt \leq p \int_{k}^{\infty} t^{p-1} ||f > t|| dt$$

$$\leq p \|f\|^{q}_{q,E} \int_{k}^{\infty} t^{-(q-p)-1} dt = \frac{p}{q-p} \frac{1}{k^{q-p}} \|f\|^{q}_{q,E}.$$

Then choose

$$k^{q-p} = \frac{p}{\varepsilon q-p} \|f\|^{q}_{q,E}.$$

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