Theory and Examples of Quantum Mechanics on Phase Space

Franklin E. Schroeck, Jr.
University of Denver, Denver, Colorado, U.S.A.
Florida Atlantic University, Boca Raton, Florida, U.S.A.
E-mail: f.schroec@du.edu

Abstract. We review the problems with quantum mechanics. We then obtain the theory of quantum mechanics on phase space which is immune to these problems. We give some examples which are not treatable with the “ordinary” quantum mechanics.

1. Introduction
Quantum mechanics began around 1900 and by 1927-1932 there were the theories of W. Heisenberg’s matrix mechanics (1926, 1927) [1], E. Schroedinger’s wave mechanics (1926) [2] (which were shown in 1926 to be equivalent), P. A. M. Dirac’s theory [3] which subsumed both, H. Weyl’s book [4] which provided an group theory footing, E. P. Wigner’s treatment [5] of spectra by group theoretic means, and J. von Neumann’s treatise [6] on the Hilbert space approach.

In 1928, Dirac [7] states what observables and states are, and presents his model of the electron.

In 1930, Dirac [8] provides the theory of ”holes” (in Dirac’s vacuum).

In order to avoid problems with negative energy of the electron which was inherent in the 1928 paper and employing the fact that electrons are Fermions which cannot occupy the same state as another electron, Dirac created a vacuum at every point which had all the available negative energy states occupied. A hole in the vacuum was supposed by Dirac to be a proton. Thus, by knocking out an electron from the vacuum, one created both an electron of positive energy as well as a proton; in this fashion ”pair production” was envisioned.

But it was apparent that there were many problems with the theory of quantum mechanics in the usual formalism(s) that have persisted to this day. These problems may be categorized by
1) positivity of the energy spectrum,
2) vacuum polarization,
3) hole theory,
4) zitterbewegung,
5) are observables really observable?,
6) discrete vrs. continuous spectra of operators and von Neumann’s theorem,
7) classical points vrs. distributions (quantum geometry),
8) quantization,
9) status of the wave function,
10) does there exist a smallest length in quantum mechanics?,
11) different Hilbert spaces for different particles,
12) relativity versus von Neumann projections, relativity versus no interaction theory,
13) quantum→classical as the limit as \( \hbar \to 0 \),
14) role of phase space,
15) quantum field theory,
16) infrared and ultraviolet divergences,
17) etc.

The authors of these criticisms were very important people in physics and cannot be dismissed as cranks. I would like to add the question of how you may describe the measurement of a quantum particle with a measurement apparatus consisting of a set of quantum particles as we assume they all are. The authors of this list made criticisms or provided theories that "solved one of these problems," but there was no one that tried to solve all of them simultaneously. Well, phase space quantum mechanics solves or resolves all of them and has many more applications. Here we shall give a brief synopsis of quantum mechanics on phase space, and then mention some crucial applications.

I would like to add a comment. The "analytic approach" was/is the preferred approach by most physicists as many were unwilling to put in the time to learn group theory; they made a joke of the group theory approach by referring to it as the "gruppenpest." Aaddage: "When faced with a challenge which you can't or won't disagree, make fun of the person making the challenge to win the argument." The joke is on them.

This view had an unfortunate impact on what was or was not published in the physics journals of the day. Almost without exception, the papers that were published were those of the "analytic approach." I cite the work of E. Majorana who worked in the group theory formalism of H. Weyl [4]. He published very little, but did a lot. See the review of 2003 by A. Drago and S. Exposito [9].

Not included in this talk is a brief introduction to what the problems and good points are of the old quantum theory. I gave a paper last year in the Czech Republic containing about 60 references of eminent physicists who criticised quantum mechanics. Also a brief summary was given of what quantum mechanics on phase space has to say experimentally and theoretically about all this, whether to solve the problems or whether to circumvent the problems. This was given again in this conference in a later session of this conference. See [10].

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3. The Theory of Quantum Mechanics on Phase Space

In the period 1927-1984, the quantum mechanical world was mired in difficulties which many had pointed out came from several sources. The solutions are:

a) one should work in phase space;
b) one should take the non-local nature of physics to heart and work with positive operator valued measures (POVM’s) in (quantum) measurement theory;

c) in obtaining the space on which to work, bear in mind that the component of spin in the direction of momentum is a constant;

d) the theory must be covariant with respect to either the Heisenberg, or Galilei, or Poincaré groups at least;

e) no point charges are allowed;

f) generalize Koopman’s [11], Husimi’s [12], and others works;

g) one should use a quantum mechanical view of measurement, etc.

We shall outline the theory of “quantum mechanics on phase space” to see how all of these are addressed.

3.1. To obtain the classical phase space(s):

1) Start with a Lie symmetry group, \( G \), for the physical system. For example, \( G \) may be the Heisenberg or the Galilei group for non-relativistic physics or the Poincaré or de Sitter group for relativistic physics. Then, in the massive representations,

the ”momentum operator” is the group generator, \( P = (P_0, P_1, P_2, P_3) \), for the subgroup of displacements of momentum (boosts);

the ”boost operator” is the group generator, \( Q = (Q_0, Q_1, Q_2, Q_3) \), for the subgroup of displacements of position;

the ”rotation” or ”spin (or helicity) operator” is the group generator, \( S = (S_0, S_1, S_2, S_3) \), for the subgroup of the (internal) rotations.

(Note that we have switched notation of the \( P \)'s and \( Q \)'s from the usual quantum mechanical representation. In this way, they are now like the \( S \)'s.)

The commutation relations among the \( P_j \), \( Q_k \), \( S_l \) are called the (Lie) algebra relations of the group and may be derived directly from the group composition law. We presume that this Lie algebra is finite dimensional, but that is all that will be imposed on our group. This notation will be continued into the massless cases, although the interpretations will be different.

2) From this (Lie) algebra, one may obtain (closed) subgroups, \( H \), of \( G \) indexed by the mass, \( m \), and the spin, \( S \), through the mathematics of Lie group homology. Closed ones will have to be separately confirmed. There obviously are many closed \( H \)'s one may obtain in this way.

3) By a theorem [13] the classical phase space(s) is (are) then the \( G/H \), the set of all elements of the form \( x = gH \) for some \( g \) in \( G \). This is a symplectic space. The choice of \( H \) is dictated by bearing in mind that the energy is to be always positive. Also, in the massive cases, the component of spin in the direction of the momentum has to be a constant. In the massless cases, we must reinterpret the variables and obtain the direction of the motion to be perpendicular to two Euclidean variables which represent the electric and magnetic fields. We will let one element in \( x = gH \) (which is a subset of \( G \)) be called the choice function of \( x \) and be denoted by \( \sigma(x) \in G \).

There is a left action, \( V \), of \( G \) on \( G/H \) given by

\[
V(g\cdot)gH = (g\cdot g)H.
\]

4) There is a Hamiltonian action (time development) on \( G/H \).

Localization in measurable set \( B \) in \( G/H \) is given by multiplication by the characteristic function \( \chi_B \):

\[
\chi_B(x) = 1 \text{ for } x \in B,
\]

\[
\chi_B(x) = 0 \text{ for } x \notin B.
\]

Fuzzy location also occurs with \( \chi_B \) replaced by \( f \), \( 0 \leq f \leq 1 \), \( f \) measurable.
Classical statistical observables are real valued, measurable functions, \( f \), of the \( x \)'s in \( G/H \). Classical statistical states are densities, \( \rho_{\text{class}} \), on \( G/H \) such that
\[
\rho_{\text{class}}(x) \geq 0, \\
\int_{G/H} \rho_{\text{class}}(x) \, dx = 1.
\]

Classical statistical expectation is given by
\[
\text{Exp}_{\text{cm}}(\rho_{\text{class}}, f) = \int_{G/H} f(x) \rho_{\text{class}}(x) \, dx.
\]

This has been worked out in detail for each of the cases of group, mass, and spin (or group and helicity in the massless cases). The rest of this section is a (dense) summary of the theory of quantum mechanics on phase space. For the details, see [14].

### 3.2. The Phase Space Representations in Hilbert Spaces

We create the Hilbert space, \( L^2(G/H) \) of square-integrable functions on \( G/H \). \( L^2(G/H) \) hosts the left-regular representation or the derived projective representations of \( G \). This is not irreducible. Note that there is no function \( \Psi \) such that a single point \( x \in G/H \) is a non-trivial support set of any \( \Psi \in L^2(G/H) \).

### 3.3. The Usual Quantum Representation

Take any Hilbert space for (ordinary) quantum mechanics, \( \mathcal{H} \), for a particle of mass \( m \) and spin \( S \). \( \mathcal{H} \) is an irreducible representation of \( G \) with representation \( U \). Quantum observables, \( \mathcal{O} \), are some of the (self-adjoint) operators on \( \mathcal{H} \). Quantum states are the density operators, \( \rho_{\text{q.m.}} \), on \( \mathcal{H} \). Quantum expectation is given by
\[
\text{Exp}_{\text{qm}}(\rho_{\text{q.m.}}, \mathcal{O}) = \text{Tr}(\rho_{\text{q.m.}} \mathcal{O}).
\]

### 3.4. Mapping \( \mathcal{H} \rightarrow L^2(G/H) \)

Make an association of \( \mathcal{H} \) with a closed subspace of \( L^2(G/H) \) as follows [14]: For any one \( \eta \in \mathcal{H} \) and for all \( \varphi \in \mathcal{H} \), define
\[
[W^n(\varphi)](x) = \langle U(\sigma(x)) \eta, \varphi \rangle
\]
for \( x \in G/H \).

For \( W^n(\varphi) \) to be in \( L^2(G/H) \), we must have \( \eta \) satisfying the technical condition of "\( \alpha \)-admissibility":
\[
\begin{align*}
\mathbf{x} & \mapsto \eta_{\mathbf{x}} = U(\sigma(\mathbf{x})) \eta \text{ is square integrable in } G/H, \\
U(h)\eta & = \alpha(h)\eta, \quad |\alpha(h)| = 1 \text{ for all } h \in H, \\
\sigma & : G/H \rightarrow G \text{ is a choice function.}
\end{align*}
\]

We will interpret the fixed \( \eta \) in a moment.

### 3.5. Observables on \( L^2(G/H) \rightarrow \) Observables on \( \mathcal{H} \)

We define the injection of observables in the phase space \( G/H \) to \( L^2(G/H) \) by
\[
\begin{align*}
f & \mapsto M(f), \\
[M(f)\Psi](x) & = f(x)\Psi(x).
\end{align*}
\]
Define \([14] A^\eta(f)\) as the self-adjoint (or symmetric) operator in \(\mathcal{H}\) that is the analogy (the "pull-back") of the operator \(M(f)\) in \(L^2(G/H)\):

\[
A^\eta(f) = [W^\eta]^{-1} P^\eta M(f) W^\eta,
\]

where \(P^\eta\) is the canonical projection \(P^\eta : L^2(G/H) \to W^\eta(\mathcal{H})\).

One may show that these \(A^\eta(f)\) transform covariantly under the group: For \(g \in G\), and \(f\) equalling a classical statistical observable,

\[
g : A^\eta(f) \to U(g)A^\eta(f)U(g)^{-1} = A^\eta(g \circ f),
\]

\[
[g \circ f](x) = f(g^{-1} \cdot x).
\]

This is one result of showing that

\[
A^\eta(f) = \int f(x) |U(\sigma(x))\eta\rangle\langle U(\sigma(x))\eta| d\mu(x).
\]

Thus we have the quantization of every classical statistical observable! Furthermore one may show that all polynomials, etc., in the \(P_j, Q_k, S_l\) are of this form.

One may show that, for \(P_\psi\) being the one-dimensional projection operator onto \(\psi\),

\[
Tr(P_\psi A^\eta(f)) = \langle \psi, A^\eta(f) \psi \rangle = \int f(x)|\langle U(\sigma(x))\eta, \psi \rangle|^2 d\mu(x)
\]

where \(|\langle U(\sigma(x))\eta, \psi \rangle|^2\) is the transition probability. Thus \(A^\eta(f)\) is a real observable for an experiment in which the \(U(\sigma(x))\eta\)'s have an interpretation of describing the experimental apparatus. Hence we are using quantum mechanical states to measure other quantum mechanical objects. Furthermore, from the form for \(A^\eta(f)\), we obtain that all these measurements are positive operator valued and not projection valued. This provides an interpretation for \(\eta\).

### 3.6. The Expected Value of Quantum Variables and Classical Variables

Now, we have that,

\[
Exp_{qm}(P_\psi, A^\eta(f)) = \int f(x)|\langle U(\sigma(x))\eta, \psi \rangle|^2 d\mu(x)
\]

\[
= \int_{G/H} f(x)\rho_{\text{class}}(x)dx
\]

\[
= \text{Exp}_{qm}(\rho_{\text{class}}, f)
\]

where \(\rho_{\text{class}}(x) = |\langle U(\sigma(x))\eta, \psi \rangle|^2\). Consequently, we have agreement for all the observables that are of the form \(A^\eta(f)\) with these \(\rho_{\text{class}}\). This is equivalent to having the set \(\{U(\sigma(x))\eta \mid x \in G/H\}\) as a coherent state.

We also have a very appealing definition of "information" as the numbers you obtain from the set of \(Tr(\rho A^\eta(f))\). This is of immediate interest from the physical point of view.

It only remains to show that these \(A^\eta(f)\) are complete in the observables in some sense for which we have a physical interpretation. We \([14]\) may show that "informationally complete" \(\eta\) means: \(\{A^\eta(f)A^\eta(g) \mid f, g \text{ classical observables}\}\) is informationally complete \(iff\) the numbers \(Tr(A^\eta(f)A^\eta(g)\rho)\) uniquely define \(\rho \iff \langle U(\sigma(x))\eta, \eta \rangle > 0\) for a.e.\(x \in G/H\).

We now define "events" in \(\mathcal{H}\):
If $\eta$ has quantum expectations $\langle \eta, A \eta \rangle = 0$ for $A$ in the Lie algebra, then $U(g)\eta$ has quantum expectations $\langle \eta, U(g)^{\dagger} AU(g) \eta \rangle$. Using the action of $G$ on the Lie algebra, one works out the action explicitly with no surprises.

Thus an event $x$ (or maybe $x^\rho$) corresponds to the fuzzy point $U(\sigma(x))\eta$, and not to $x$ as a point in $G/H$.

3.7. Multiparticle Interactions
To treat multiparticle interactions, etc., work on the tensor product of the various spaces for each particle.

I should note that there are conservation laws which may be expressed only in terms of the $x$'s in the fuzzy phase space, but not as points in $G/H$.

3.8. The Fields in Quantum Field Theory
To obtain the creation and annihilation operators at a fuzzy point $x$, obtain them for the vector $U(\sigma(x))\eta$. As this is a vector in $H$, one may use the mathematically rigorous method of Cook [15] to define them. All the rest follows from this.

3.9. Summary
In this outline, I have omitted a lot of things, all of which have been proven and/or discussed. See the references, primarily [14].

Note also, that we have the group action acting covariantly on the $A^\eta(f)$’s so that we have the property that the action on $\text{Tr}(\rho A^\eta(f))$ is (non-relativistically or relativistically) invariant, for any $\rho$. Thus, we do not have a theorem saying that the interaction on $\rho$ must be (non-relativistically or relativistically) invariant. Thereby, we avoid the theorem that the interaction(s) must be free actions.

4. Some Experiments
4.1. Land Mine Detection
One of the typical experiments in which we may use the formulation of phase space physics as necessary is in land mine detection. The "usual" way to do it is to send a signal into the earth and record the reflected signal. If the reflected signal's time is plotted against its amplitude, we get certain locations at which there may be a land mine. Now comes a very intense time in which the object that was the cause of the reflection is dug up. It is intense because when digging, it may explode. So you have to be careful. In the end, about 10% of the objects are actual active land mines. This is extremely wasteful in terms of the manpower expended (in more ways than one). This is a result of the informational incompleteness of the process.

We may doubt the efficiency of this process as it doesn’t account for the frequency of the signal. So we take the Fourier transform of the signal, and get the signal's frequency versus amplitude plot. But again and in spite of your reasons for trying frequency versus amplitude, there is about a 10% chance of objects found being actual active land mines! This is again the informational incompleteness of the method being used.

If we use the sum total of both of these methods, there will be an increase in the efficiency of the method, but it will be definitely less than 20%. This is again a result of the informational incompleteness of the method.

If we record the joint time-frequency-amplitude (t-f-a) of the returned signal and then compare with the original sent signal by taking the two t-f-a signals and convolving them, we will get the overlap in energy, i.e., in terms of the transition probability from one to the other. This is informationally complete. Then take the cut off in time and frequency part of this. We will be getting an approximation of the informationally complete set
of data. In an actual experiment [16], with taking the data centered at a moderate number of finite points with the $\eta$ reflecting the nuclear quadrupole resonance spectrum of the chemical compounds in the substance which explodes, the expression of the land mine to objects rose to 70%. The cost for a detector was about 100 Euros.

This was a gain in terms of time and resources, and was good physics in addition. The same happens for voice recognition, artificial sight, etc. Unfortunately, no one has used such methods for any applications although the methods have been around for some time.

4.2. Solid State Physics

Another application to physics is one that involves the very small. Let us try to exactly describe the electrons in a finite crystal. There is no such theory that ”we” could find, because the electron is a particle with a wave function and with spin 1/2! There were a lot of theories that treated the electron as a spinless point particle.

We have to have the position confined to a finite crystal, but that is easily handled by factoring by the lattice of sites (heavy molecules or ions). Furthermore, we will have to describe the electrons in phase space, which means we will have to have the position, the momentum, and spin represented. If we take the finite crystal defined as the space occupied by the crystal modulo the locations of the lattice points in which we place the molecules and/or the ions, then we have the finite crystal involved. The reciprocal lattice structure (the momentum structure) has to be determined from the finite lattice, and it is based on the von Neuman lattice. We take the phase space for non-relativistic spin 1/2 particles and factor it by the product of the finite lattice times the reciprocal lattice. Thus we will have taken into consideration what the electron sees as the phase space. Now, we have to measure an arbitrary electron in the ”crystal.” This is accomplished by embedding a wire in the crystal and computing the probability of measuring the electron in the crystal with an electron state in the wire, the $\eta$, with propagation along the wire. There will be a dependence on the spin of the electron in the crystal.

This illustrates the power of the phase space approach, as well as the necessity to having the theory in phase space rather than in just the position space.

For further details, see [17].

4.3. Quantum Teleportation

We will treat quantum teleportation briefly to show that some of the problems of quantum theory may come from the fuzziness of the $\eta$ revealed in the inaccuracy in measurement. This will not involve an actual experiment, but will be just theorizing.

In quantum teleportation, the objective is to move a particle from one location to another (essentially) instantaneously. Using ordinary quantum mechanics, the theory involves taking a particle with wave function, $\psi$; letting it interact with a paired set of measuring devices that are entangled (supposedly at some point $x$ in $\mathbb{R}^3$, and then propagated to the actual locations of ”Alice” and ”Bob.”) When this supposed three-object ”state” is manipulated somehow and the partial trace taken over Alice’s position, one will get the state, $\psi$, but located at Bob’s position. There will be a certain probability of this occurring, called a ”fidelity.”

Now, instead of placing the entanglement at $x$, we will assume that it creates a state, $\eta$, with its center at $x_1 = (q_1, p_1)$, and a complementary state $\eta'$ with its center at $x_2 = (q_2, p_2)$, and then allowed to evolve in time. This will give a rigorous three-particle state, $\psi \otimes \eta \otimes \eta'$. Assuming that $\eta'$ is equal to $\eta$, working in the Heisenberg group in one dimension for the spin zero case, taking $h = \frac{1}{2}$ and for a unit mass, this gives [18]

$$\rho^{\eta}_{\text{ent}}(q_1, p_1, q_2, p_2) = |\Psi_{\text{ent}}(q_1, p_1, q_2, p_2)|^2,$$
for the entangled state. We have, of course,
\[ \Psi_{\text{ent}}(q_1, p_1, q_2, p_2) = \langle U_{q_1, p_1} \otimes U_{q_2, p_2} [\eta \otimes \eta'], \psi_{\text{ent}} \rangle. \]

This will give a new fidelity of
\[ F \equiv \langle \psi, \rho_{\text{tel}} \psi \rangle = \int_{\mathbb{R}^2} dq dp P(q, p)|\langle \psi, U(q, p)\psi \rangle|^2, \]
where
\[ P(q_{\text{out}} - q, p_{\text{out}} - p) = 2 \int_{\mathbb{R}^2} dq dp P_{\text{ent}}(q', p', q'', p''), \]
where
\[ q' = q - \sqrt{2} q_u, \quad p' = \sqrt{2} p_u - p, \quad q'' = q_{\text{out}} - \sqrt{2} q_u, \quad p'' = p_{\text{out}} - \sqrt{2} p_v, \]
and where we, with Alice, make the change of variables
\[ q_u = \frac{1}{\sqrt{2}} (q_{\text{in}} - q_1), \quad p_u = \frac{1}{\sqrt{2}} (p_{\text{in}} - p_1), \quad q_v = \frac{1}{\sqrt{2}} (q_{\text{in}} + q_1), \quad p_v = \frac{1}{\sqrt{2}} (p_{\text{in}} + p_1). \]

It does not matter if you can or cannot follow the details of the above; what matters is that you obtain the fidelity function depending explicitly on \( \eta \) sesquilinearly and in a way that is not a simple function. For those that want to really understand this model, see [18].

Moreover, one may perform roughly the same trick and get the fact that all of what is called "quantum computation" must be reworked with the fact that \( \eta \) is involved in every computation that has a measurement involved, and the way that \( \eta \) is involved is not simple but is quite handleable.

References
[1] Heisenberg W 1930 The Principles of the Quantum Theory (Chicago: University of Chicago Press)
[2] Schrödinger E 1927 Abhandlung zur Wellenmechanik (Leipzig: J. A. Barth)
[3] Dirac P A M 1930 The Principles of Quantum Mechanics (Oxford: Oxford Univ. Press)
[4] Weyl H 1928 Gruppentheorie und Quantenmechanik (Leipzig: Verlag von S. Hirzel)
[5] Wigner E P 1931 Gruppentheorie und ihre Anwendung auf die Quantenmechanik der Atomspektrum (Braunschweig: Friedr. Vieweg)
[6] von Neumann J 1955 Mathematical Foundations of Quantum Mechanics (Princeton: Princeton Univ. Press)
[7] Dirac P A M 1938 Quantum theory of the electron, Roy. Soc. Proc. 117A 610-24, and 118A 351-61
[8] Dirac P A M 1930 Electrons and Protons, Roy. Soc. Proc. 126 360-65
[9] Drago A and Esposito S 2004 Following Weyl on quantum mechanics: the contribution of Ettore Majorana Found. Phys. 35 871-87
[10] Schroock, Jr. F E 2016 Theory and examples of quantum mechanics on phase space - historical addenda, preprint
[11] Koopman B O 1931 Hamiltonian systems and transformations in Hilbert space Proc. Nat'l Acad. Sci. 7 315-18
[12] Husimi, K 1937 Foundations of Quantum Mechanics, Part 1 Phys. Math. Soc. Japan, Proc. 19 766-89
[13] Guillemin V and Sternberg S 1984 Symplectic Techniques in Physics (New York: Cambridge Univ. Press) pp 173-78
[14] Schroeck Jr. F E 1996 *Quantum Mechanics on Phase Space* (Dordrecht: Kluwer Acad. Pubs.) This is presently being rewritten and updated by R. Beneduci and F. E. S.

[15] Cook J M 1953 The mathematics of second quantization *Trans. Math. Soc.* 74 222-45

[16] Aerts S, Aerts D, Schroeck F and Sachs J Nov 20, 2006 Informational completeness, positive operator valued measures and the search for an optimal landmine detector report for Flemish Fund for Scientific Research, project G.0362.03N

[17] Ali G, Beneduci R, Mascali G, Schroeck Jr. F E and Slawianowski J J, 2013 Some mathematical considerations on solid state physics in the framework of the phase space formulation of quantum mechanics *Int. J. Theor. Phys.* DOI 10.1007/s10773-013-1912-9

[18] Messamah J, Schroeck Jr. F E, Hachemane M, Smida A and Hamici A M 2015 Quantum mechanics on phase space and teleportation *Quantum Inf Process* DOI 10.1007/s11128-014-0914-8