Finiteness theorems for algebraic cycles of small codimension on quadric fibrations over curves

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Abstract: We obtain finiteness theorems for algebraic cycles of small codimension on quadric fibrations over curves over perfect fields. For example, if $k$ is finitely generated over $\mathbb{Q}$ and $X \to C$ is a quadric fibration of odd relative dimension at least $11$, then $CH^i(X)$ is finitely generated for $i \leq 4$.

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1. Introduction

A well-known conjecture of S.Bloch asserts that the Chow ring of a smooth projective variety over a number field is a finitely generated abelian group. In connection with this conjecture, a number of authors have studied 0-cycles (i.e., cycles of maximal codimension) on quadric fibrations $\pi: X \to C$ of relative dimension $d \geq 1$ over smooth integral curves $C$. In [5], M.Gros studied 0-cycles of degree 0 on conic fibrations (i.e., $d = 1$) over a number field $k$. The main result of that paper, obtained by $K$-theoretic methods, was the finiteness of $\text{Ker}(A_0(X) \to A_0(X)_d)$, where $A_0(X)$ denotes the Chow group of 0-cycles of degree 0 on $X, X = X \times_{\text{Spec} k} \text{Spec} \bar{k}$ and $\Gamma = \text{Gal}(\bar{k}/k)$. Further progress was made by J.-L.Colliot-Thélène and A.Skorobogatov in [1]. These authors established the finiteness of the group

$$CH_d(X/C) = \text{Ker}(CH_d(X) \to CH_d(C))$$

when $k$ is a number field or a local field and $d = 2$. To obtain this result, they first established an isomorphism

$$CH_d(X/C) = k(C)_d^*/k(C)^*N_{\chi}(k(C))$$

for any fibration $X \to C$ satisfying certain assumptions, chiefly the vanishing of the groups $A_d(X)$ associated to the closed fibers $X_g$. Here $k(C)_d^*$ is a certain subgroup of $k(C)^*$ of “divisorial norms” and $N_{\chi}(k(C))$ is the group of norms.

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associated to the generic fiber \( X_{\eta} \). Then they used the above isomorphism to reduce the study of the group \( CH_0(X/C) \) for a quadric fibration \( X \to C \) of relative dimension 2 to that of \( CH_0(Y/\bar{C}) \), where \( Y \to \bar{C} \) is a certain conic fibration over a "discriminant curve" \( \bar{C} \) covering \( C \). Thus, using the results of Gros mentioned above, these authors were able to establish the finiteness of \( CH_0(X/C) \) when \( \bar{C} \) is geometrically integral and \( k \) is a number field or a local field. The next step in the study of \( CH_0(X/C) \) was taken by R.Parimala and V.Suresh [15], who developed the methods of [1] further. These authors established the vanishing of \( A_0(X) \) for smooth Pfister quadric fibrations \( X \) over conics \( C \) (of arbitrary relative dimension and defined over any field of characteristic different from 2). They showed, further, that \( A_0(X) \) vanishes as well if \( X_{\eta} \) is arbitrary and either \( C = \mathbb{P}^1 \) or \( k \) is a number field or a field of 2-cohomological dimension \( \leq 2 \). For possibly non-smooth quadric fibrations over curves \( C \) of arbitrary genus, Parimala and Suresh established the finiteness of \( CH_0(X/C) \) over a local field \( k \). Over number fields, they were able to establish the finiteness of \( CH_0(X/C) \) (and therefore the finite generation of \( CH_0(X) \)) for certain "admissible" quadric fibrations whose generic fiber is defined by a Pfister neighbor of dimension at least 5. Such is the present state of progress towards obtaining a proof of Bloch's conjecture for quadric fibrations over curves. In particular, until now only 0-cycles (i.e., cycles of maximal codimension) on such fibrations had been studied. In this paper we study cycles of small codimension on quadric fibrations as above. Using methods analogous to those developed in [4], we obtain the following result.

**Theorem 1.1.**

Let \( k \) be a perfect field of characteristic different from 2 and let \( C \) be a smooth, projective and geometrically integral \( k \)-curve. Let \( X \to C \) be a quadric fibration of relative dimension \( d \geq 11 \), where \( X \) is a smooth, projective and geometrically integral \( k \)-variety. If \( d \) is even, assume that \( \text{disc}(X_{\eta}) = 1 \). Assume, in addition, that one of the following conditions holds:

(a) \( k \) is finitely generated over \( \mathbb{Q} \), or

(b) \( C \) is a conic.

Then \( CH^i(X) \) is finitely generated for \( i \leq 4 \).

The methods of this paper also yield finiteness results for cycles of codimension \( i \) for every \( i \) in a certain extended range if the generic fiber of \( X \to C \) is an excellent quadric. We illustrate this fact in Section 5 by considering Pfister quadric fibrations.

### 2. Preliminaries

Let \( k \) be a perfect field of characteristic different from 2 and let \( \bar{k} \) be a fixed algebraic closure of \( k \). For any \( k \)-variety \( Y \), \( Y_0 \) will denote the set of closed points of \( Y \). If \( Y \) is a smooth, projective and geometrically integral quadric over \( k \), \( \text{disc}(Y) \in \bar{k}/(\bar{k})^2 \) will denote the discriminant (signed determinant) of any quadratic form defining \( Y \) [13], p.38. Now let \( \Gamma = \text{Gal}(\bar{k}/k) \) and let \( k \) be a smooth, projective and geometrically integral \( k \)-curve with function field \( k(C) \) and generic point \( \eta \).

**Definition 2.1.**

An admissible quadric fibration over \( C \), of relative dimension \( d \geq 1 \), is a pair \((X, \pi)\) consisting of a smooth, projective and geometrically integral \( k \)-variety \( X \) and a morphism \( \pi : X \to C \) such that each point \( y \in C \) has an affine neighborhood \( \text{Spec} \, A(y) \times_C \text{Spec} \, \bar{k}(C) = X_{\eta} \) Spec \( k(C) \) is smooth and \( \text{disc}(X_{\eta}) = 1 \) if \( d \) is even, where \( X_{\eta} = X \times_C \text{Spec} \, \bar{k}(C) \).

An admissible quadric fibration \((X, \pi)\) as above will often be denoted by \( X \to C \).

**Remark 2.1.**

Since we have assumed that \( X \) is smooth, the class of admissible quadric fibrations considered in this paper is narrower than that considered in [15]. This smoothness condition is imposed in order to have available the localization exact sequence (7) below. Note, however, that the smooth Pfister quadric fibrations considered in [15] are admissible quadric fibrations in the above sense.
For any $y \in C$, let $q_y$ be a fixed quadratic form over $k(y)$ defining the quadric $X_y = X \times_{C} \text{Spec} \, k(y)$. We will write $q_y^{\text{nons}}$ for the nonsingular part of $q_y$ and $X_y^{\text{nons}}$ for the smooth, projective and geometrically integral $k(y)$-quadric defined by $q_y^{\text{nons}}$. Further, set
$$d_y = \dim X_y^{\text{nons}}.$$  

Note that, since $X_y$ is smooth, there exists a finite set $S$ of closed points of $C$ such that, for every $y \in U := C \setminus S$, we have $X_y = X_y^{\text{nons}}$. Now set $\mathcal{X} = X \times_{\text{Spec} \, k} \text{Spec} \, \overline{k}$ and $\mathcal{C} = C \times_{\text{Spec} \, k} \text{Spec} \, \overline{k}$. The finite set of closed points of $\mathcal{X}$ lying above the points in $S$ will be denoted by $\mathcal{S}$. Further, $\overline{k}[U]$ will denote the ring of regular functions on $U := \mathcal{C} \setminus \mathcal{S}$. Recall that a smooth, projective and geometrically integral quadric $Y$ of dimension $d$ over a field $F$ of characteristic not equal to 2 is called split if it is isomorphic to either $X_0x_1 + \cdots + x_dx_{d+1} = 0$ or $x_0^2 + x_1 + x_2 + \cdots + x_dx_{d+1} = 0$ if $d$ is odd. Clearly, if $F$ is algebraically closed, then any quadric $Y$ over $F$ (as above) is split. On the other hand, if $F$ is quasi-algebraically closed (i.e., $C_1$), then $Y$ is split if and only if, either $\dim Y$ is odd or $\dim Y$ is even and $\text{disc}(Y) = 1$.

Thus, if $X \to C$ is an admissible quadric fibration, then $\mathcal{X} = X \times_C \text{Spec} \, \overline{k}((\overline{y}))$ is a split $\overline{k}((\overline{y}))$-quadric for every $\overline{y} \in \overline{U}$ (recall that $\overline{k}((\overline{y})) = \overline{k}(C)$ is a $C_1$-field by Tseng’s theorem).

**Lemma 2.1.**

Let $X \to C$ be an admissible quadric fibration of relative dimension $d$ and let $i$ be an integer such that $0 \leq i \leq d$.

(a) For every $\overline{y} \in \overline{U}$, there exist isomorphisms of abelian groups

$$CH^i(X_{\overline{y}}) = \begin{cases} \mathbb{Z} & \text{if } i \neq d_y/2 \\ \mathbb{Z} \oplus \mathbb{Z} & \text{if } i = d_y/2. \end{cases}$$

If $i \neq d_y/2$, then $\Gamma$ acts trivially on $CH^i(X_{\overline{y}})$. 

(b) For every $y \in C$,

$$\text{Ker} \left[ CH^i(X_y) \xrightarrow{\text{res}_y} CH^i(X_{\overline{y}}) \right] = CH^i(X_y)_{\text{tors}}.$$ 

(c) If $i < d/2$, then $\text{res}_y : CH^i(X_y) \to CH^i(X_{\overline{y}})$ is surjective.

(d) If $i \neq d/2$, the natural map

$$\mathcal{K}(C)^* = \mathcal{K}(C)^* \otimes CH^i(X_y) \to H^i(X_y, \mathcal{K}_{i+1}),$$

induced by the Brown–Gersten–Quillen spectral sequence and cup product, is an isomorphism of $\Gamma$-modules.

**Proof.** If $\overline{y} \in \overline{U}$, then $X_{\overline{y}}$ is a split quadric and (a) follows directly from [7], §2.1. If $\overline{y} \in S$, then

$$CH^i(X_{\overline{y}}) = \begin{cases} CH^i(X_y^{\text{nons}}) \otimes \mathbb{Z} & \text{if } i \leq d_y \\ \mathbb{Z} & \text{if } i > d_y. \end{cases} \quad (1)$$

by [11], §1.1, and (a) again follows from [7], §2.1. As regards (b), if $X_y$ is smooth, i.e., $y \in U$, then (b) follows from [7], (2.7). On the other hand, if $y \in S$, then (1) over $k$ and over $\overline{k}$ together with [7], §2.1, applied to $X_y^{\text{nons}}$ show that

$$\text{Ker} \left[ CH^i(X_y) \xrightarrow{\text{res}_y} CH^i(X_{\overline{y}}) \right] = \begin{cases} CH^i(X_y^{\text{nons}})_{\text{tors}} & \text{if } i \leq d_y \\ 0 & \text{if } i > d_y. \end{cases}$$

The latter equals $CH^i(X_y)_{\text{tors}}$, by the analogue of (1) over $k$, whence (b) follows. Assertion (c) follows from [7], (2.7). Finally, (d) is a particular case of [6], Proposition 2.2(b). $\square$
Let $X \to C$ and $i$ be as in the lemma and assume that $i \neq d/2$. Then there exists a canonical commutative diagram

$$
\begin{array}{ccc}
k(C)^\ast \otimes CH^i(X) & \longrightarrow & H^i(X_\eta, K_{i+1}) \\
\downarrow \text{id} \otimes \text{res} & & \downarrow \text{res} \\
\left(\mathcal{K}(C)^\ast \otimes CH^i(X)\right)^\Gamma & \longrightarrow & H^i(X_\eta, K_{i+1})^\Gamma,
\end{array}
$$

where the horizontal maps are induced by the Brown–Gersten–Quillen spectral sequence and cup–product. The bottom map is an isomorphism by part (d) of the lemma. Further, since $CH^i(X_\eta) = \mathbb{Z}$ with trivial $\Gamma$–action by Lemma 2.1(a), we have

$$
\left(\mathcal{K}(C)^\ast \otimes CH^i(X)\right)^\Gamma = k(C)^\ast \otimes CH^i(X_\eta) = k(C)^\ast.
$$

Thus, there exist a map

$$
\rho_i : H^i(X_\eta, K_{i+1}) \to k(C)^\ast,
$$

a canonical isomorphism

$$
\text{Coker } \rho_i \simeq \text{Coker}\left[H^i(X_\eta, K_{i+1}) \xrightarrow{\text{res}} H^i(X_\eta, K_{i+1})^\Gamma\right],
$$

and an exact sequence

$$
k(C)^\ast \otimes \text{Coker}\left[CH^i(X_\eta) \xrightarrow{\text{res}} CH^i(X_\eta)\right] \to \text{Coker } \rho_i \to 0. \tag{3}
$$

In particular, Lemma 2.1(c) yields the following.

**Proposition 2.1.**

If $0 \leq i < d/2$, then the map $\rho_i : H^i(X_\eta, K_{i+1}) \to k(C)^\ast$ defined above is surjective. \qed

**Remark 2.2.**

Let $w$ be the Witt index of $q_\eta$ and assume that $w < d/2$. Then, if $d/2 < i \leq d - w$, there exists a canonical isomorphism

$$
\text{Coker}\left[CH^i(X_\eta) \xrightarrow{\text{res}} CH^i(X_\eta)\right] = \mathbb{Z}/2.
$$

See [6], Proposition 1.1(c). Thus (3) shows that $\text{Coker } \rho_i$ is a (possibly nontrivial) quotient of $k(C)^\ast / \{k(C)^\ast\}^2$.

We will need the following basic fact: if $M$ is a $\Gamma$–module which is finitely generated as an abelian group, then $H^1(\Gamma, M)$ is a group of finite exponent. Indeed, if $L$ is a finite Galois extension of $k$ contained in $K$ which trivializes $M$, $G = \text{Gal}(L/k)$ and $H = \text{Gal}(K/L)$, then the inflation–restriction exact sequence in Galois cohomology yields an exact sequence

$$
0 \to H^1(G, M) \to H^1(\Gamma, M) \to \text{Hom}(H, M_{\text{tors}}).
$$

The left-hand group above is finite by [19], Corollary 6.5.10, p.180, and the right-hand group is annihilated by the order of $M_{\text{tors}}$. This proves our claim. Note that, if $M$ is free, then $H^1(\Gamma, M)$ is in fact finite.
3. The basic exact sequence

Let \( X \to C \) be an admissible quadric fibration (see Definition 2.1) of relative dimension \( d \) and let \( i \) be an integer such that \( 0 \leq i \leq d \) and \( i \neq d/2 \). There exists a canonical homomorphism of \( \Gamma \)-modules

\[
\delta_i : H^i(X_\tau, K_{i+1}) \to \bigoplus_{\tau \in \Gamma_0} CH^i(X_\tau)
\]

(4)
defined as the limit over all nonempty open subsets \( \overline{\tau} \) of \( \overline{C} \) of the boundary maps \( H^i(X_\tau, K_{i+1}) \to CH^i(X_\tau) \) arising from the localization sequence for the triple \( (X_\tau, \overline{X}, X_\tau \setminus \overline{\tau}) \) (for a description of the latter maps, see [16], (3.7) and (2.10) with \( M = K^*_\tau \) there). Let \( \delta_{i, \overline{\tau}} \) be the composite

\[
H^i(X_\tau, K_{i+1}) \xrightarrow{\delta_i} \bigoplus_{\tau \in \Gamma_0} CH^i(X_\tau) \to \bigoplus_{\tau \in \Gamma_0 \setminus \{\overline{\tau}\}} CH^i(X_\tau),
\]

(5)
where the second map is the canonical projection. Since \( \tau \neq d/2 \), \( H^i(X_\tau, K_{i+1}) \) is canonically isomorphic to \( \overline{\tau}(C)^* \) by Lemma 2.1(d). Further, if \( \tau \notin \Sigma \cup \{\overline{\tau}\} \), then \( d_\tau = d \) and Lemma 2.1(a) shows that \( CH^i(X_\tau) = \mathbb{Z} \). Thus \( \delta_{i, \overline{\tau}} \) may be identified with a map \( \overline{\tau}(C)^* \to \bigoplus_{\tau \in \Sigma \cup \{\overline{\tau}\}} \mathbb{Z} \), and the description of the map \( \delta_i \) alluded to above (see [16], (3.7) and (2.10)) shows that the latter map coincides with the canonical divisor map \( \mathcal{D} \mapsto (\text{ord}_\tau(\mathcal{D}))_{\tau \in \Sigma \cup \{\overline{\tau}\}} \). Consequently, there exist canonical isomorphisms of \( \Gamma \)-modules

\[
\text{Ker} \delta_{i, \overline{\tau}} = \overline{\tau}(U)^*
\]

and

\[
\text{Coker} \delta_{i, \overline{\tau}} = CH_0(U).
\]

Thus the kernel-cokernel exact sequence [14], Proposition I.0.24, p.19, associated to (5) yields an exact sequence

\[
0 \to \text{Ker} \delta_i \to \overline{\tau}(U)^* \xrightarrow{\bigoplus_{\tau \in \Sigma} CH^i(X_\tau)} \text{Coker} \delta_i \to CH_0(U) \to 0.
\]

(6)

Proposition 3.1.

Let \( \delta_i \) be the map (4), where \( 0 \leq i \leq d \) and \( i \neq d/2 \). Then \( H^1(\Gamma, \text{Ker} \delta_i) \) is finite.

Proof. By (6) and Lemma 2.1(a), there exists an exact sequence of \( \Gamma \)-modules

\[
0 \to \text{Ker} \delta_i \to \overline{\tau}(U)^* \to A \to 0,
\]

where \( A \) is free and finitely generated. We conclude that there exists an exact sequence

\[
A^* \to H^1(\Gamma, \text{Ker} \delta_i) \to H^1(\Gamma, \overline{\tau}(U)^*).
\]

The image of the left-hand map above is finite since \( A^* \) is finitely generated and \( H^1(\Gamma, \text{Ker} \delta_i) \) is torsion. On the other hand, by Hilbert's Theorem 90, \( H^1(\Gamma, \overline{\tau}(U)^*) \) injects into \( H^1(\Gamma, \overline{\tau}(U)^*/\mathbb{R}^*) \), which is finite since \( \overline{\tau}(U)^*/\mathbb{R}^* \) is free and finitely generated (this is a general fact, but in the present case it suffices to note that \( \overline{\tau}(U)^*/\mathbb{R}^* \) injects into \( \text{Div}_{\tau}(\overline{C}) \), the group of divisors on \( \overline{C} \) with support in the finite set \( \Sigma \)). This completes the proof.

Remark 3.1.

(a) The proof of the proposition shows that there exists an injection \( \text{Ker} \delta_i \xrightarrow{\delta_i} k[U]^* \) whose cokernel is finitely generated.

(b) By (6), if \( 0 \leq i \leq d \), \( i \neq d/2 \) and \( S = \emptyset \) (i.e., \( \pi : X \to C \) is smooth), then

\[
\text{Ker} \delta_i = \overline{\tau}(C)^* = \overline{\tau}^*.
\]

In this case, therefore, \( H^1(\Gamma, \text{Ker} \delta_i) = H^1(\Gamma, \overline{\tau}^*) = 0 \) by Hilbert's Theorem 90.
Now let \( j : X_\eta \to X \) and \( j : X_\eta \to \overline{X} \) be the canonical embeddings. There exist canonical exact sequences

\[
H^i(X_\eta, K_{i+1}) \xrightarrow{\delta_i} \bigoplus_{y \in C_0} CH^i(X_y) \to CH^{i+1}(X) \xrightarrow{\delta^\prime} CH^{i+1}(X_\eta) \to 0
\]

and

\[
H^i(X_\eta, K_{i+1}) \xrightarrow{\delta_i} \bigoplus_{\pi \in \pi_0} CH^i(X_\eta) \to CH^{i+1} \to CH^{i+1}(X_\eta) \to 0
\]

which yield the following commutative diagrams:

\[
\begin{array}{ccccccc}
\text{Ker } \delta_i & \xrightarrow{\delta_i} & H^i(X_\eta, K_{i+1}) & \xrightarrow{\Phi_i} & \text{Im } \delta_i & \to & 0 \\
\downarrow & & \downarrow \text{res} & & \downarrow \Phi_i & & \\
\text{(Ker } \delta_i)^\Gamma & \xrightarrow{\delta_i} & H^i(X_\eta, K_{i+1})^\Gamma & \xrightarrow{\Phi_i} & \text{(Im } \delta_i)^\Gamma & \to & H^i \{ \Gamma, \text{Ker } \delta_i \} ^\Gamma,
\end{array}
\]

where \( \Phi_i \) is induced by the restriction map \( \text{res} \) and the bottom row is exact by Hilbert’s Theorem 90 via Lemma 2.1(d),

\[
\begin{array}{ccccccc}
0 & \xrightarrow{\text{Im } \delta_i} & \bigoplus_{y \in C_0} CH^i(X_y) & \xrightarrow{\Phi_i} & \text{Ker } j^* \\
0 & \xrightarrow{\text{(Im } \delta_i)^\Gamma} & \bigoplus_{y \in C_0} CH^i(X_\eta_y) & \xrightarrow{\Phi_i} & \text{(Ker } j^*)^\Gamma
\end{array}
\]

and

\[
\begin{array}{ccccccc}
0 & \xrightarrow{\text{Ker } j^*} & CH^{i+1}(X) & \xrightarrow{\text{res}} & CH^{i+1}(X_\eta) & \to & 0 \\
0 & \xrightarrow{(\text{Ker } j^*)^\Gamma} & CH^{i+1}(\overline{X})^\Gamma & \to & CH^{i+1}(X_\eta)^\Gamma
\end{array}
\]

where, for each \( y \in C_0 \), we have fixed a point \( \overline{y} \in \pi_0 \) lying above \( y \) and written \( \Gamma_y = \text{Gal}(\overline{X}/k(y)) \). Applying the snake lemma to diagram (8) and identifying \( (\text{Ker } \delta_i)^\Gamma \) with a subgroup of \( H^i \{ X_\eta, K_{i+1} \}^\Gamma = k(C)^* \), we obtain the following result.

**Proposition 3.2.**  
There exists a canonical exact sequence

\[
0 \to k(C)^*/(\text{Im } \rho_y)(\text{Ker } \delta_1)^\Gamma \to \text{Coker } \Phi_i \to H^1 \{ \Gamma, \text{Ker } \delta_i \} \to 0 \quad \square
\]

Now set

\[
CH^{i+1}(X)^\Gamma = \text{Ker } \left[ CH^{i+1}(X) \xrightarrow{\text{res}} CH^{i+1}(\overline{X})^\Gamma \right]
\]

and let

\[
\Psi_i : \text{Coker } \Phi_i \to \bigoplus_{y \in C_0} CH^i(X_\eta_y)^\Gamma / \text{res}_y CH^i(X_y)
\]

be induced by the map \( (\text{Im } \delta_i)^\Gamma \to \bigoplus_{y \in C_0} CH^i(X_\eta_y)^\Gamma \) appearing on the bottom row of diagram (9). By Lemma 2.1(b), the kernel of the middle vertical map in (9) is \( \bigoplus_{y \in C_0} CH^i(X_y)_{\text{tors}} \). Thus, applying the snake lemma to (9) and using (10) together with Lemma 2.1(b) (for \( y = \eta \)), we obtain
**Proposition 3.3.**

There exists a canonical exact sequence

\[ 0 \rightarrow \text{Ker } \Phi_i \rightarrow \bigoplus_{y \in C_0} CH^i(X_y)_{\text{tors}} \rightarrow \text{Ker} \left[ CH^{i+1}(X)_r \rightarrow CH^{i+1}(X)_{\text{tors}} \right] \rightarrow \text{Ker } \Psi_i \rightarrow 0, \]

where \( CH^{i+1}(X)_r \) and \( \Psi_i \) are given by (11) and (12), respectively.

We now note that the exact sequence of Proposition 3.2 induces an exact sequence

\[ 0 \rightarrow \text{Ker } \Psi_i' \rightarrow \text{Ker } \Psi_i \rightarrow H^1(\Gamma, \text{Ker } \overline{\sigma}_i), \]

where

\[ \Psi_i' : k(C)^*/(\text{Im } \rho_i)(\text{Ker } \overline{\sigma}_i) \rightarrow \bigoplus_{y \in C_0} CH^i(\chi_{\overline{\sigma}_i})_{\text{res}}/\text{res}_y CH^i(X_y) \]

is the composition of (12) and the injection

\[ k(C)^*/(\text{Im } \rho_i)(\text{Ker } \overline{\sigma}_i) \hookrightarrow \text{Coker } (\Phi_i) \]

coming from Proposition 3.2. Thus, if \( i \neq d/2 \), then Proposition 3.1 shows that \( \text{Ker } \Psi_i \) is finite if, and only if, \( \text{Ker } \Psi_i' \) is finite. Now (13) is induced by the composite

\[ k(C)^* \xrightarrow{2,3(14)} H^i(\chi_{\overline{\sigma}_i}, K_{i+1}) \rightarrow \bigoplus_{y \in C_0} CH^i(\chi_{\overline{\sigma}_i})_{\text{res}} \rightarrow \bigoplus_{y \in C_0} CH^i(\chi_{\overline{\sigma}_i})_{\text{res}}/\text{res}_y CH^i(X_y) \]

and we define the \( i \)-th Salberger group of \( X \rightarrow C \), \( \text{Sal}_i(X/C) \), to be the kernel of the preceding composition, i.e.,

\[ \text{Sal}_i(X/C) = \{ f \in k(C)^*: \forall y \in C_0, \overline{\sigma}_{i,j}(f) \in \text{res}_y CH^i(X_y) \}, \]

where \( \overline{\sigma}_{i,y} \) is the \( y \)-component of the composition

\[ k(C)^* \xrightarrow{2,3(14)} H^i(\chi_{\overline{\sigma}_i}, K_{i+1}) \rightarrow \bigoplus_{y \in C_0} CH^i(\chi_{\overline{\sigma}_i})_{\text{res}}. \]

Thus

\[ \text{Ker } \Psi_i' = \text{Sal}_i(X/C)/(\text{Im } \rho_i)(\text{Ker } \overline{\sigma}_i). \]

The preceding discussion and Proposition 3.1 yield the following result.

**Proposition 3.4.**

Assume that \( 0 \leq i \leq d \) and \( i \neq d/2 \). Then there exists a canonical exact sequence

\[ 0 \rightarrow \text{Sal}_i(X/C)/(\text{Im } \rho_i)(\text{Ker } \overline{\sigma}_i) \rightarrow \text{Ker } \Psi_i \rightarrow H^1(\Gamma, \text{Ker } \overline{\sigma}_i), \]

where \( \Psi_i \) is the map (12) and \( \text{Sal}_i(X/C) \) is the group (14). In particular, \( \text{Ker } \Psi_i \) is finite if, and only if, \( \text{Sal}_i(X/C)/(\text{Im } \rho_i)(\text{Ker } \overline{\sigma}_i) \) is finite.

The basic exact sequence alluded to in the heading of this Section is the following.
Theorem 3.1.
Let \( X \to C \) be an admissible quadric fibration of relative dimension \( d \) and let \( i \) be an integer such that \( 0 \leq i < d/2 \). Then there exists a canonical exact sequence

\[
\bigoplus_{\gamma \in \Gamma} C^{i}(X_{\gamma})_{\text{tors}} \to \text{Ker}\left[C^{i+1}(X) \to C^{i+1}(X_{\gamma})_{\text{tors}}\right] \to H^{i}(\Gamma, \text{Ker} \delta_{i}),
\]

where \( C^{i+1}(X) \) is the group (11).

Proof. By Proposition 2.1, \( \rho_{i}: H^{i}(X_{\gamma}, K_{i+1}) \to k(C)^{\ast} \) is surjective. Thus \( \text{Im} \rho_{i} = \text{Sal}_{i}(X/C) = k(C)^{\ast} \) and therefore

\[
\text{Sal}_{i}(X/C)/\left(\text{Im} \rho_{i}\right)(\text{Ker} \delta_{i})^{\ast} = 0.
\]

The theorem now follows by combining Propositions 3.3 and 3.4. \( \square \)

We conclude this Section by giving a sufficient condition under which the group \( C^{i+1}(X) \), appearing in (11), is finitely generated. Let \( J_{C} \) be the Jacobian variety of \( C \).

Lemma 3.1.
If \( J_{C}(k) \) is finitely generated, then so also is \( \text{Pic}(\mathcal{U})^{\Gamma} \).

Proof. The well-known exact sequence \( 0 \to J_{C}(k) \to \text{Pic}(\mathcal{U})^{\Gamma} \to \mathbb{Z} \) shows that \( \text{Pic}(\mathcal{U})^{\Gamma} \) is finitely generated. Now, by [3], Proposition 1.8, p.21, there exists a canonical exact sequence of \( \Gamma \)-modules

\[
0 \to P_{\mathcal{U}, \Gamma} \to \text{Pic}(\mathcal{U})^{\Gamma} \to \text{Pic}(\mathcal{U}) \to 0,
\]

where \( P_{\mathcal{U}, \Gamma} \) is a finitely generated abelian group. The above exact sequence induces an exact sequence

\[
\text{Pic}(\mathcal{U})^{\Gamma} \to \text{Pic}(\mathcal{U})^{\Gamma} \to H^{i}(\Gamma, P_{\mathcal{U}, \Gamma}),
\]

where the right-hand group has a finite exponent \( m \) (say). It follows that \( \text{Pic}(\mathcal{U})^{\Gamma} \) is a quotient of the inverse image of \( \text{Pic}(\mathcal{U})^{\Gamma} \) under the multiplication–by–\( m \) map \( m: \text{Pic}(\mathcal{U}) \to \text{Pic}(\mathcal{U}) \). Since \( \text{Pic}(\mathcal{U})_{\text{tors}} = J_{C}(k)_{\text{tors}} \) is finite, the proof is complete. \( \square \)

Proposition 3.5.
Let \( X \to C \) be an admissible quadric fibration of relative dimension \( d \) and let \( i \) be an integer such that \( 0 \leq i \leq d \) and \( i \neq d/2 \). If \( J_{C}(k) \) is finitely generated, then so also is \( C^{i+1}(X) \).

Proof. By the exactness of the sequence

\[
0 \to \text{Coker} \delta_{i} \to C^{i+1}(X) \to C^{i+1}(X_{\gamma}) \to 0
\]

(see (7)) and Lemma 2.1(a) (for \( \mathfrak{f} = \pi \)), it suffices to check that \( \left(\text{Coker} \delta_{i}\right)^{\Gamma} \) is finitely generated. The exactness of the sequence

\[
\bigoplus_{\gamma \in \Gamma} C^{i}(X_{\gamma}) \to \text{Coker} \delta_{i} \to CH_{0}(\mathcal{U}) \to 0
\]

(see (6)) together with Lemma 2.1(a) show that \( \left(\text{Coker} \delta_{i}\right)^{\Gamma} \) is finitely generated if \( CH_{0}(\mathcal{U})^{\Gamma} = \text{Pic}(\mathcal{U})^{\Gamma} \) is finitely generated. The result is now immediate from the previous lemma. \( \square \)

Remark 3.2.
If \( C \) is a conic, then \( J_{C}(k) = 0 \) is (certainly) finitely generated for any field \( k \). On the other hand, if \( k \) is finitely generated over its prime subfield, then \( J_{C}(k) \) is finitely generated by [2], Corollary 7.2.
4. Cycles of codimensions 3 and 4

The following statement collects together several results of N. Karpenko.

**Theorem 4.1.**  
Let \( Y \) be a smooth, projective and geometrically integral quadric of dimension \( d \) over a field of characteristic not equal to 2.

(a) \( \text{CH}^2(Y)_{\text{tors}} \) has order at most 2. If \( d \geq 7 \), then \( \text{CH}^2(Y) \) is torsion-free.

(b) \( \text{CH}^3(Y)_{\text{tors}} \) has order at most 2. If \( d \geq 11 \), then \( \text{CH}^3(Y) \) is torsion-free.

(c) \( \text{CH}^4(Y)_{\text{tors}} \) has order at most 4 if \( d \geq 7 \).

**Proof.** For (a), see [7], Theorem 6.1. For (b) and (c), see [9] and [10].

**Theorem 4.2.**  
Let \( k \) be a perfect field of characteristic different from 2 and let \( C \) be a smooth, projective and geometrically integral \( k \)-curve. Let \( X \to C \) be an admissible quadric fibration of relative dimension at least 7, where \( X \) is a smooth, projective and geometrically integral \( k \)-variety. Then \( \text{Ker} \left[ \text{CH}^3(X) \to \text{CH}^3(X)/\Gamma \right] \) is finite.

**Proof.** This follows by combining Theorem 3.1 for \( i = 2 \), Proposition 3.1 and Theorem 4.1(a), (b).

**Corollary 4.1.**  
Let the hypotheses be as in Theorem 4.2. Assume, in addition, that at least one of the following conditions holds:

(a) \( k \) is finitely generated over \( \mathbb{Q} \), or

(b) \( C \) is a conic.

Then \( \text{CH}^3(X) \) is finitely generated.

**Proof.** This is immediate from Proposition 3.5, Remark 3.2 and the theorem.

Similar arguments, using Theorem 4.1(b), (c) in place of Theorem 4.1(a), (b), yield the following result.

**Theorem 4.3.**  
Let \( k \) be a perfect field of characteristic different from 2 and let \( C \) be a smooth, projective and geometrically integral \( k \)-curve. Let \( X \to C \) be an admissible quadric fibration of relative dimension at least 11, where \( X \) is a smooth, projective and geometrically integral \( k \)-variety. Assume, in addition, that one of the following conditions holds:

(a) \( k \) is finitely generated over \( \mathbb{Q} \), or

(b) \( C \) is a conic.

Then \( \text{CH}^4(X) \) is finitely generated.
5. Pfister quadric fibrations

Let \( r \geq 3 \) and let \( q \) be an \( r \)-fold Pfister form over a field \( F \) of characteristic not equal to 2. Let \( Y \) be the quadric defined by \( q \) (so \( \dim Y = 2r - 2 \) and \( \text{disc}(Y) = 1 \)). Then Example 7.3 and Corollary 8.2 of \([12]\) show that \( \text{CH}^i(Y)_\text{tors} = 0 \) for every \( i < 2r - 2 \) (as already noted by N.Karpenko in \([8]\)). Thus Theorem 3.1 yields the following result.

**Theorem 5.1.**

Let \( k \) be a perfect field of characteristic different from 2 and let \( C \) be a smooth, projective and geometrically integral \( k \)-curve. Let \( X \to C \) be an \( r \)-fold Pfister quadric fibration, i.e., \( X_0 \) is defined by an \( r \)-fold Pfister form over \( k(C) \), where \( r \geq 3 \). Then, for every integer \( i \) such that \( 0 \leq i \leq 2r - 2 - 2 \), there exists a canonical exact sequence

\[
\bigoplus_{y \in S} \text{CH}^i(X_y)_\text{tors} \to \text{Ker} \left[ \text{CH}^{i+1}(X) \to \text{CH}^{i+1}(X_{\bar{k}}) \right] \to H^1(\Gamma, \text{Ker} \delta_i),
\]

where \( S \) denotes the set of points of \( C \) where \( X \to C \) has a singular fiber.

**Corollary 5.1.**

Let \( k \) be a perfect field of characteristic different from 2 and let \( C \) be a smooth, projective and geometrically integral \( k \)-curve. Let \( X \to C \) be a smooth \( r \)-fold Pfister quadric fibration, where \( r \geq 3 \). Assume that

(a) \( k \) is finitely generated over \( \mathbb{Q} \), or

(b) \( C \) is a conic.

Then \( \text{CH}^i(X) \) is finitely generated for \( i \leq 2r - 2 - 1 \).

**Proof.** This is immediate from the above theorem, Proposition 3.1, Proposition 3.5 and Remark 3.2.

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