Dimer models and group actions
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Abstract

We construct a consistent dimer model having the same symmetry as its characteristic polygon. This produces examples of non-commutative crepant resolutions of non-toric non-quotient Gorenstein singularities in dimension 3.

1 Introduction

A dimer model is a bicolored graph on a real 2-torus encoding the information of a quiver with relations. Dimer models are originally introduced in 1930s [FR37] as statistical mechanical models of diatomic molecules, which contain the Ising model as a special case. See e.g. [Bax89, Ken04] and references therein for this aspect of dimer models. More recently, string theorists have discovered the relation between dimer models and toric Calabi-Yau 3-folds [HK05, FHV+06, FHM+06, HV07], and many works have been done to explore the relation between dimer models and various branches of mathematics, such as Donaldson-Thomas theory [Sze08, MR10], Calabi-Yau algebras [Bro12, Dav11, IU11, Boc12, Boc13], volumes of toric Sasaki-Einstein 5-manifolds [MSY06, BZ06, BZ05, Kat07], moduli spaces of quiver representations [FV06, IU08], the McKay correspondence [IU15, BCQV15], exceptional collections [HHV06, IU], and mirror symmetry [HKV08, UY11, UY13, FU10].

The characteristic polygon \( \Delta \) of a dimer model \( G \) is a convex lattice polygon obtained from the dimer model in a purely combinatorial way. When \( G \) satisfies a mild condition called non-degeneracy, the moduli space of representations of the quiver associated with the dimer model is a toric variety, and the convex hull of the primitive generators of the one-dimensional cones of the corresponding fan coincides with \( \Delta \) [IU08]. When \( G \) satisfies a stronger condition called consistency, then the path algebra \( C\Gamma \) of the associated quiver with relations \( \Gamma \) is a non-commutative crepant resolution [vdB04a] of the affine toric variety \( X_\Delta \) associated with \( \Delta \).

Let \( H \) be a finite subgroup of \( \text{GL}(2, \mathbb{Z}) \) acting naturally on the lattice where the characteristic polygon \( \Delta \) lives. When \( \Delta \) is invariant under this action, then we can ask if the action can be ‘lifted’ to the dimer model \( G \). In this paper, we introduce the notion of a symmetric dimer model with respect to the action of \( H \), and prove the following:

**Theorem 1.1.** For any finite subgroup \( H \) of \( \text{GL}(2, \mathbb{Z}) \) and any \( H \)-invariant lattice polygon \( \Delta \), there is a consistent dimer model \( G \) which is symmetric with respect to the action of \( H \) and has \( \Delta \) as its characteristic polygon.

If a dimer model \( G \) is symmetric with respect to the action of a finite subgroup \( H \) of \( \text{GL}(2, \mathbb{Z}) \), then \( H \) acts on the associated quiver \( \Gamma \) with relations. There are associated actions of \( H \) on \( X_\Delta \) and \( C\Gamma \) which are twisted as in (3.1) and (3.2) respectively. Notice that if \( H \) is a reflection group of order 2 (see Remark 3.6), then the twist (3.1) depends on the choice of
the origin in $\Delta$. Moreover, the twist (3.2) depends on the choice of an $H$-invariant perfect matching corresponding to the origin. With respect to these twisted actions, we prove:

**Theorem 1.2.** If a consistent dimer model $G$ is symmetric with respect to the action of a finite subgroup $H$ of $\text{GL}(2, \mathbb{Z})$, then the crossed product algebra $\mathbb{C} \Gamma \times H$ is a non-commutative crepant resolution of $X_{\Delta}/H$.

These two theorems imply the existence of non-commutative crepant resolutions of $X_{\Delta}/H$ which are not necessarily toric and not necessarily quotient singularities. This in turn implies the existence of crepant resolutions by [Bri02, VdB04b, vdB04a], which can also be shown directly by first taking an $H$-invariant unimodular triangulation of $\Delta$ (which one can find by drawing line segments between the origin and the corners of $\Delta$ to triangulate $\Delta$, and then refining it to a unimodular triangulation) to obtain an $H$-equivariant crepant resolution $Y$ of $X_{\Delta}$, and then taking the Hilbert scheme $H$-Hilb$(Y)$. It is an interesting problem to see if every projective crepant resolution of $X_{\Delta}/H$ is obtained as moduli of representations of $\mathbb{C} \Gamma \times H$ just as in [CI04, IU16].

While the path algebra of the quiver with relations associated with a dimer model is a 3-dimensional generalization of that of the McKay quiver for a Kleinian singularity of type $A$, the crossed product algebra $\mathbb{C} \Gamma \times H$ associated with a symmetric dimer model is a 3-dimensional generalization of that of type $D$, and it is an interesting problem to decide which constructions on dimer models generalize to dimer models with group actions. For example, if we let $Z_{\Delta}$ denote the 2-dimensional toric Fano stack whose fan polytope (i.e., the convex hull of the primitive generators of one-dimensional cones in the stacky fan) is given by $\Delta$, then one can show the existence of a full strong exceptional collection of vector bundles on the stack quotient $[Z_{\Delta}/H]$ along the lines of [IU, Theorem 1.1].

This paper is organized as follows: In Section 2, we briefly recall basic definitions and results on dimer models. More details can be found, e.g., in [Yam08, Boc16] or references cited. In Section 3, we introduce the notion of a symmetric dimer model $G$ with respect to a finite subgroup $H$ of $\text{GL}(2, \mathbb{Z})$ acting on the real 2-torus, and discuss a quiver description of the crossed product algebra $H \rtimes \mathbb{C} \Gamma$ with the path algebra $\mathbb{C} \Gamma$ of the associated quiver with relations. After recalling the classification of finite subgroups of $\text{GL}(2, \mathbb{Z})$ in Section 4, we give an outline of the proof of Theorem 1.1 in Section 5 and a case-by-case analysis in Sections 5.1, 5.2, and 5.3. To construct symmetric and consistent dimer models, we adopt the method in [IU15]. The proof of Theorem 1.2 is given in Section 6. In Section 7, we digress from the main subject of this paper and discuss symmetries of dimer models under wallpaper groups.

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## 2 Preliminaries

### 2.1 Dimer models and characteristic polygons

Let $N$ be a free abelian group of rank 2 and $M := \text{Hom}(N, \mathbb{Z})$ be the dual lattice. We write the real 2-plane and the real 2-torus associated with $M$ as $M_{\mathbb{R}} := M \otimes \mathbb{R}$ and $T := M_{\mathbb{R}}/M$ respectively. A **bicolored graph** $G = (B, W, E)$ on $T$ consists of
**a finite set** $B \subset T$ of **black nodes**,  
**a finite set** $W \subset T$ of **white nodes**, and  
**a finite set** $E$ of **edges**, consisting of embedded closed intervals $e$ on $T$ such that one boundary of $e$ belongs to $B$ and the other boundary belongs to $W$, such that any edge can intersect another edge only at its boundary. The **valence** of a node is the number of edges adjacent to that node. A **face** of $G$ is a connected component of $T \setminus \cup_{e \in E} e$. A bicolored graph $G = (B, W, E)$ on $T$ is a **dimer model** if

- there is no univalent node, and
- every face of $G$ is simply-connected.

Although a dimer model may have divalent nodes in general, as explained in [IU15, Section 6.1], we can and will assume that there are no divalent nodes for the purpose of this paper.

A **perfect matching** is a subset $D \subset E$ of the set of edges such that for any node $n \in B \cup W$, there is a unique edge $e \in D$ adjacent to $n$. A dimer model is said to be **non-degenerate** if any edge is contained in some perfect matching. For a pair $(D, D_0)$ of perfect matchings, one can associate an element $\text{ht}(D, D_0) \in H^1(T, \mathbb{Z}) \cong \mathbb{N}$ called the **height change** (cf. e.g. [IU08]). Fix a perfect matching $D_0$ and call it the **reference matching**. The lattice polygon $\Delta \subset \mathbb{N}_\mathbb{R}$ obtained as the convex hull of the set $\{\text{ht}(D, D_0) \mid D \text{ is a perfect matching}\}$ of height changes is called the **characteristic polygon**. If we take a different perfect matching $D_1$ as the reference matching, the resulting characteristic polygon $\Delta'$ is related to $\Delta$ by translation by $h(D_1, D_0)$.

A **zigzag path** is a periodic sequence $(e_i)_{i \in \mathbb{Z}}$ of edges, considered up to translation of $i$, which makes a maximum turn to the right on a white node and to the left on a black node. A pair of zigzag paths are said to **intersect** if they share a common edge. Such an edge will be called an intersection ‘point’ of the pair of zigzag paths. The homology class $[z] \in H_1(T, \mathbb{Z}) \cong \mathbb{M}$ of a zigzag path is called its **slope**. A dimer model on $T = \mathbb{M}_\mathbb{R}/\mathbb{M}$ can be pulled back to the universal cover $\mathbb{M}_\mathbb{R}$ of $T$ as a doubly periodic bicolored graph and one can consider zigzag paths on $\mathbb{M}_\mathbb{R}$. Zigzag paths on the universal cover will be used when we will define the notion of consistency of a dimer model.

Let $r$ be the number of zigzag paths with non-zero slopes, and $\{z_i\}_{i=1}^r$ be the set of such zigzag paths. A **zigzag polygon** is a convex lattice polygon in $\mathbb{N}_\mathbb{R}$ defined up to translation by the condition that the multiset of primitive outward normal vectors to primitive side segments of the polygon is equal to the multiset $([z_i])_{i=1}^r$ of slopes of zigzag paths with non-zero slopes. Here, a **primitive side segment** of a lattice polygon is a line segment on the boundary of the polygon bounded by a pair of lattice points containing no lattice point in the interior. For any dimer model, the zigzag polygon is contained in the characteristic polygon [BIU, Corollary 1.2].

A dimer model is **consistent** if

- there is no homologically trivial zigzag path,
- no zigzag path on the universal cover $\mathbb{M}_\mathbb{R}$ of $T$ has a self-intersection, and
- no pair of zigzag paths on the universal cover $\mathbb{M}_\mathbb{R}$ intersect each other in the same direction more than once.
Examples of a pair of curves intersecting in the same and the opposite direction are shown in the left and the right of Figure 2.1 respectively. See [IU11, Boc12] for more on consistency conditions for dimer models. In particular, it is shown in [IU11, Proposition 4.4] that a dimer model is consistent if and only if it is properly-ordered in the sense of Gulotta [Gul08]. Together with [Gul08, Theorem 3.3], this shows that the characteristic polygon and the zigzag polygon coincides for consistent dimer models. A consistent dimer model is non-degenerate by [IU15, Proposition 8.1].

2.2 Quivers and moduli spaces from dimer models

A quiver $Q = (Q_0, Q_1, s, t)$ consists of

- a set $Q_0$ of vertices,
- a set $Q_1$ of arrows, and
- a pair $s, t : Q_1 \to Q_0$ of maps called the source and the target respectively.

A path on a quiver is either a symbol $e_v$ associated with a vertex $v \in Q_0$ or a sequence $(a_1, \ldots, a_l)$ of arrows satisfying $s(a_{i+1}) = t(a_i)$ for $i = 1, 2, \ldots, l - 1$. The length of a path is defined to be zero for $e_v$ and $l$ for $(a_1, \ldots, a_l)$. The path algebra $\mathbb{C}Q$ of a quiver $Q = (Q_0, Q_1, s, t)$ is the algebra spanned by the set of paths as a vector space, and the multiplication is defined by the concatenation of paths. Paths of length zero are idempotents of the path algebra, which sum up to one; $\sum_{v \in Q_0} e_v = 1$. A quiver with relations is a pair of a quiver and a two-sided ideal $\mathcal{I}$ of its path algebra. For a quiver $\Gamma = (Q, \mathcal{I})$ with relations, its path algebra $\mathbb{C}\Gamma$ is defined as the quotient algebra $\mathbb{C}Q / \mathcal{I}$.

A dimer model $G = (B, W, E)$ encodes the information of a quiver with relations $\Gamma = (Q_0, Q_1, s, t, \mathcal{I})$ such that

- $Q_0$ is the set of faces,
- $Q_1$ is the set $E$ of edges,
- the orientations of the arrows are determined by the colors of the vertices of the graph in such a way that the white vertex $w \in W$ is on the right of the arrow, and
- the ideal $\mathcal{I}$ of the path algebra $\mathbb{C}Q$ is generated by $p_+(a) - p_-(a)$ for all $a \in Q_1$, where $p_+(a)$ is the path from $t(a)$ to $s(a)$ going around the white node adjacent to $a \in E = Q_1$ clockwise, and $p_-(a)$ is the path from $t(a)$ to $s(a)$ going around the black node adjacent to $a \in E = Q_1$ counterclockwise.
A representation of $\Gamma$ is a module over the path algebra $\mathbb{C}\Gamma$. It is given by a collection $\Psi = ((V_v)_{v \in Q_0}, (\psi(a))_{a \in Q_1})$ of vector spaces $V_v$ for $v \in Q_0$ and linear maps $\psi(a) : V_{s(a)} \to V_{t(a)}$ for $a \in Q_1$ satisfying relations in $I$. The dimension vector $\dim \Psi$ of a representation $\Psi = ((V_v)_{v \in Q_0}, (\psi(a))_{a \in Q_1})$ is the element $\sum_{v \in Q_0} (\dim V_v) v$ of the free $\mathbb{Z}$-module $\mathbb{Z}Q_0$ generated by $Q_0$.

Fix a dimension vector $d \in \mathbb{Z}Q_0$ and a stability parameter $\theta \in \text{Hom}(\mathbb{Z}Q_0, \mathbb{Z})$ satisfying $\theta(d) = 0$. A representation $\Psi$ of $\Gamma$ with dimension vector $d$ is $\theta$-stable (resp. $\theta$-semistable) if $\theta(\dim S) > 0$ (resp. $\theta(\dim S) \geq 0$) for any non-trivial subrepresentation $S \subset \Psi$. The stability parameter $\theta$ is generic if semistability implies stability.

In this paper, we will always work with the dimension vector $1 := \sum_{v \in Q_0} v$ unless otherwise specified. For a vertex $v_0 \in Q_0$, a stability parameter $\theta$ is $v_0$-generated if $\theta(v) > 0$ for any $v \neq v_0$. Any $v_0$-generated parameter $\theta$ is always generic, and a representation $\Psi$ with dimension vector $1$ is $\theta$-stable if and only if $\Psi$ is generated by a non-zero element in $V_{v_0}$ as a module over $\mathbb{C}\Gamma$.

Let $\Delta$ be the characteristic polygon of a dimer model $G$ and $X_\Delta := \text{Spec } R$ be the Gorenstein affine toric 3-fold, whose coordinate ring $R$ is the monoid ring $\mathbb{C}[C(\Delta)^{\vee} \cap (M \oplus \mathbb{Z})]$ of the dual cone of the cone $C(\Delta)$ over $\Delta \times \{1\} \subset N_R \times \mathbb{R}$. Here we fix an embedding $\Delta \subset N_R$ so that it contains the origin of $N$. The dense torus of $X_\Delta$ will be denoted by $T$. If $G$ is consistent and $\theta$ is generic, then the moduli space $M_\theta$ of $\theta$-stable representations with dimension vector $1$ is a $T$-equivariant crepant resolution of $X_\Delta$ by [IU08, Theorem 6.4]. Toric divisors in $M_\theta$ correspond to perfect matchings on $G$ [IU08, Section 6], and we write the perfect matching corresponding to the toric divisor associated with the origin of $\Delta$ as $D_0$. The corresponding one-parameter subgroup of $T$ will be denoted by

$$\lambda_0 : \mathbb{C}^* \to T.$$  (2.1)

The moduli space $M_\theta$ is equipped with the tautological bundle $\bigoplus_{v \in Q_0} \mathcal{L}_v$ which, by [IU15, Theorem 1.4], is a tilting bundle on $M_\theta$ with

$$\text{End} \left( \bigoplus_{v \in Q_0} \mathcal{L}_v \right) \cong \mathbb{C}\Gamma.$$

Notice that the tautological bundle is determined only up to tensor product by a line bundle on $M_\theta$.

3 Group actions on dimer models

A finite subgroup $H$ of $\text{GL}(N)$ acts contragradiently on $M$, and hence on $T := M_R/M$.

Definition 3.1. A dimer model $G$ on $T$ is symmetric with respect to the action of $H$ if for all $h \in H$ we have that:

- $h$ preserves the set $E$,
- $h$ preserves the sets $B$ and $W$ individually if $\det h = 1$, and
- $h$ exchanges $B$ and $W$ if $\det h = -1$. 

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Remark 3.2. Recall that the sets $B$, $W$, and $E$ are subsets of $T$ in our definition of a dimer model. The action of $h$ on $G$ is required to preserve $E$ as a subset of $T$, and similarly for $B$ and $W$. In particular, the symmetry of a dimer model in our sense depends on how the graph is embedded in $T$.

Examples of symmetric dimer models can be found in Figures 5.3 and 5.4 below. Here the origin of $T$ is the center of one octagonal face in Figure 5.3 and the center of the dodecagonal face in Figure 5.4.

The conditions in Definition 3.1 ensure that if a dimer model $G$ is symmetric with respect to the action of $H$, then $H$ acts on the quiver $\Gamma = (Q, I)$ with relations associated with $G$. On the other hand, for perfect matchings $D_1$ and $D_2$, recall that the height change $\text{ht}(D_1, D_2)$ is defined as an element of $N$ independently of the choice of a basis of $N$ which is acted on by $H$. Then one can see $\text{ht}(h(D_1), h(D_2)) = h \text{ht}(D_1, D_2)$ holds for any $h \in H$. Thus $h$ sends the set $\{\text{ht}(D, D_0) \mid D \text{ is a perfect matching}\}$ to the set $\{\text{ht}(D, h(D_0)) \mid D \text{ is a perfect matching}\}$ and hence with respect to the linear action of $h \in H \subset \text{GL}(N)$ on $N_R$, $h(\Delta)$ equals the translation of $\Delta$ by $\text{ht}(h(D_0), D_0)$. Thus, with respect to this linear action, $\Delta$ is fixed by $H$ if $\text{ht}(h(D_0), D_0) = 0$.

We will make the following assumptions throughout this paper:

Assumption 3.3. The characteristic polygon $\Delta$ is fixed by the action of $H$ for a suitable choice of a reference perfect matching.

Assumption 3.4. There is a vertex $v_0$ of $Q$ fixed by the action of $H$.

In particular, the symmetric dimer model obtained in our proof of Theorem 1.1 satisfies Assumptions 3.3 and 3.4.

Assumption 3.3 means that the height change $\text{ht}(h(D_0), D_0) \in N$ of the image of the reference matching $D_0$ by any $h \in H$ is zero as noted above. In this case, we obtain a torus-equivariant action

$$\mu : H \times X_\Delta \to X_\Delta$$

on the affine toric variety $X_\Delta$ associated with the cone over $\Delta$.

Remark 3.5. By combining the translation by $\text{ht}(D_0, h(D_0))$ to the action of $h$ on $N$ for each $h \in H$, we obtain an affine linear action of $H$ on $N$ which preserves $\Delta \subset N_R$. This affine linear action on $N$ can be extended to a linear action on $N \times \mathbb{Z}$ whose restriction to $N \times \{1\}$ coincides with the affine linear action. Thus we can define the action of $H$ on $X_\Delta$ even if there is no perfect matching $D_0$ with $\text{ht}(h(D_0), D_0) = 0$. However, we do not consider such a situation in this paper.

Since the pull-back by $\mu(h, -) : X_\Delta \to X_\Delta$ acts by multiplication by $\det(h)$ on the canonical module $\omega_{X_\Delta}$ of the Gorenstein affine toric variety $X_\Delta$ associated with $\Delta$, the line bundle $\omega_{X_\Delta}$ is not $H$-equivariantly trivial with respect to the action $\mu$ of $H$ if $H$ is not contained in $\text{SL}(N)$. In that case, the fixed point locus of a reflection is a divisor which is the closure of a codimension one subtorus of $T \subset X_\Delta$. In order to make the action of $H$ small (i.e., free in codimension one) and to obtain a Gorenstein singularity as the quotient, we twist the action of $H$ on $X_\Delta$ by the one-parameter subgroup $\lambda_0$ of $T$ (see (2.1)) as

$$\nu(h, x) = \lambda_0(\det(h)) \cdot \mu(h, x),$$

so that the induced action on the canonical module is trivial. Note that the action $\nu$ of $H$ depends on the choice of the origin of $\Delta$, although $X_\Delta$ as an abstract variety does not.
Remark 3.6. The twisted action $\nu$ in $[3.1]$ depends on the choice of the origin in $\Delta$ when $H \cong \mathbb{Z}/2\mathbb{Z}$ is a reflection group of order 2. If $\Delta$ is a lattice triangle, we recover the dihedral groups in $\text{SL}(3, \mathbb{C})$ acting on $\mathbb{C}^3$. A dihedral group in $\text{SL}(3, \mathbb{C})$ is obtained by the natural embedding of a dihedral group $G \subset \text{GL}(2, \mathbb{C})$ into $\text{SL}(3, \mathbb{C})$, which belongs to the type B family in the Yau-Yu classification [YY93]. Recall that for $1 < q < m$ with $(m, q) = 1$, a dihedral group in $\text{GL}(2, \mathbb{C})$ is defined as

$$G = \mathbb{D}_{m, q} := \left\{ \begin{array}{ll}
\{ \psi_{2q} \tau, \phi_{2k} \} & \text{if } k := m - q \equiv 1 \text{ mod } 2 \\
\{ \psi_{2q} \tau \circ \phi_{4k} \} & \text{if } k := m - q \equiv 0 \text{ mod } 2
\end{array} \right.$$ 

with matrices $\psi_r = \left( \begin{array}{ll}
\varepsilon_r & 0 \\
0 & \varepsilon_r^{-1}
\end{array} \right)$, $\tau = \left( \begin{array}{ll}
0 & \varepsilon_4 \\
\varepsilon_4 & 0
\end{array} \right)$, $\phi_r = \left( \begin{array}{ll}
\varepsilon_r & 0 \\
0 & \varepsilon_r
\end{array} \right)$, where $\varepsilon_r$ is a primitive $r$-th root of unity. Every such group can be described as a finite group with an index 2 abelian subgroup $\lambda$ (see [NdC12, Remark 3.3]). For example, if $A$ is cyclic, then $G = \langle \alpha, \beta \rangle$ where $\beta^2 \in \langle \alpha \rangle = A$. In what follows we assume for simplicity that $A$ is cyclic, although the arguments also work for a general abelian group.

A triangle $\Delta$ which admits a reflection can be embedded as the junior simplex (i.e. the triangle with vertices the standard basis $e_1, e_2, e_3$) of the cyclic group $A = \frac{1}{n}(1, a, -(a + 1))$ with $a^2 \equiv 1 \pmod{n}$, where $N$ is identified with

$$\left( \mathbb{Z}^3 + \mathbb{Z} \frac{1}{n}(1, a, -(a + 1)) \right) \cap \{(x, y, z) \mid x + y + z = 1\},$$

up to the choice of the origin, and the reflection happens along the plane $x = y$. Then the action of $H$ on $\Delta$ can be lifted to the action on $X_\Delta$ by the matrix $\mu = \left( \begin{array}{ll}
0 & 0 \\
0 & 1
\end{array} \right)$, which is not trivial on the canonical module $\omega_{X_\Delta}$. The points in $\Delta \cap N$ fixed by $H$ are of the form $P_j = (\frac{jq}{n}, \frac{jq}{n}, 1 - \frac{2jq}{n})$ where $q = \frac{n}{(a-1,n)}$ for $0 \leq j \leq \lfloor \frac{n}{2q} \rfloor$, with corresponding one-parameter subgroups $\lambda_{P_j} : \mathbb{C}^* \to T$ of the form $\lambda_{P_j}(t) = (t^{jq/n}, t^{jq/n}, t^{1-2jq/n}, t)$. Here note that one-parameter subgroups of $T$ are identified with elements of $N \subset \mathbb{Q}^3$ since $T = (\mathbb{C}^*)^3/A$ and the one-parameter subgroup corresponding to $(a_1, a_2, a_3) \in N$ is denoted by $t \mapsto (t^{a_1}, t^{a_2}, t^{a_3})$ even though $a_i$ are rational numbers. In particular $\lambda_{P_j}(-1) = (\varepsilon^{jq/2}, \varepsilon^{jq/2}, -\varepsilon^{-jq})$ in $T$, and taking $P_j$ as the origin of $\Delta$ we have that $X_\Delta/H \cong \mathbb{C}^3/G_j$ where $G_j = \left( \frac{1}{n}(1, a, -(a + 1)), \lambda_{P_j}(-1) \cdot \mu \right)$ is a dihedral group. It can be shown that $G_0 \cong G_2$, and $G_1 \cong G_{2+1}$, which implies that there are at most two non-isomorphic dihedral actions on $\mathbb{C}^3$ associated to $\Delta$ given by

$$G_0 = \left\{ \frac{1}{n}(1, a, -(a + 1)), \left( \begin{array}{ll}
0 & 0 \\
0 & -1
\end{array} \right) \right\},$$

$$G_1 = \left\{ \frac{1}{n}(1, a, -(a + 1)), \left( \begin{array}{ll}
0 & \varepsilon^{jq/2} \\
\varepsilon^{jq/2} & 0
\end{array} \right) \right\},$$

where $a^2 \equiv 1 \pmod{n}$ and $\varepsilon = e^{2\pi \sqrt{-1}/n}$.

In general, $G_0$ and $G_1$ may be isomorphic and every dihedral subgroup $G \subset \text{SL}(3, \mathbb{C})$ can be written in this form. We note that in the case when $a = n - 1$ then $G_0 \cong D_n \subset \text{SO}(3)$, and if $n \geq 4$ is even then $G_0 \cong \text{BD}_n$, where $D_n = \langle \alpha, \beta \mid \alpha^n = \beta^2 = 1, \alpha \beta = \beta \alpha^{-1} \rangle$ and $\text{BD}_n = \langle \alpha, \beta \mid \alpha^n = 1, \beta = \alpha^{n/2}, \alpha \beta = \beta \alpha^{-1} \rangle$ are the dihedral and the binary dihedral groups respectively, both in the “classical” sense.

Example 3.7. The triangle $\Delta$ formed as the junior simplex for the subgroup $\frac{1}{12}(1, 7, 4)$ admits the above two non-isomorphic dihedral actions, where $G_1 \cong D_{3,2}$ in the Yau-Yu notation.
in representation theory of finite groups gives a ring isomorphism $e$

Choose a primitive idempotent of functions on $O$

Since $H$

$H$

crossed product algebra $O$

algebra $H$

$A$

stabilizer subgroup of $v$

G,

gives a bijection Hom($\mathbb{C}Q$, $\mathbb{C}G$) via $\phi$.

Remark 3.10. The pre-composition of the anti-automorphism $D$

parameters. Then there exists a $v_0$-generated stability parameter $\theta$ fixed by this action. Let $D_0$ be the $\theta$-stable perfect matching corresponding to the origin. Then it is easy to see that $D_0$ is fixed by the $H$-action.

Using the invariant perfect matching $D_0$, we twist the natural action of $H$ on $\mathbb{C}Q$ as

$$a \mapsto \begin{cases} \det(h)h(a) & a \in D_0, \\ h(a) & \text{otherwise.} \end{cases} (3.2)$$

Notice that this twist preserves the relation and thus gives an action of $H$ on $\mathbb{C}Q/I$.

Definition 3.9. Let $G$ be a finite group acting on a ring $A$ from the left by a homomorphism $\varphi: G \to \text{Aut} A$. The crossed product algebra $A \rtimes_\varphi G$ is the vector space $A \times G \cong A \otimes_{\mathbb{C}} \mathbb{C}[G]$ equipped with the product $(a, g) \cdot (a', g') := (a \varphi(g)(a'), g g')$. Similarly, for a finite group $G$ acting on a ring $A$ from the right by a homomorphism $\psi: G^{\text{op}} \to \text{Aut} A$, the crossed product algebra $G \rtimes_\psi A$ is the vector space $G \times A$ equipped with the product $(g, a) \cdot (g', a') := (g g', \psi(g')(a)a')$. We drop $\varphi$ and $\psi$ from the notation when they are clear from the context.

Remark 3.10. The pre-composition of the anti-automorphism $G \to G^{\text{op}}$ sending $g$ to $g^{-1}$ gives a bijection $\text{Hom}(G, \text{Aut} A) \to \text{Hom}(G^{\text{op}}, \text{Aut} A)$. If $\varphi$ and $\psi$ are related by this bijection, then one has an isomorphism $A \rtimes_\varphi G \to G \rtimes_\psi A$ sending $(a, g)$ to $(g, \psi(g)(a))$.

In order to give a quiver with relations which is Morita equivalent to the crossed product algebra $H \rtimes \mathbb{C} \Gamma$, choose a complete representative $Q'_0 \subset Q_0$ of $Q_0/H$. The $H$-orbit and the stabilizer subgroup of $v \in Q'_0$ will be denoted by $O_v := H \cdot v \subset Q_0$ and $H_v \subset H$ respectively. Since $H$ is a principal $H_v$-bundle over $O_v$, the category of $H$-equivariant vector bundles on $O_v$ is equivalent to the category of $H_v$-equivariant vector bundles on $v$. In other words, the crossed product algebra $H \rtimes \mathbb{C}[O_v]$ of $H$ with the algebra

$$\mathbb{C}[O_v] := \bigoplus_{w \in O_v} \mathbb{C} e_w \subset \mathbb{C} \Gamma$$

of functions on $O_v$ is Morita equivalent to the group algebra $\mathbb{C}[H_v]$ of $H_v$. A classical result in representation theory of finite groups gives a ring isomorphism

$$\mathbb{C}[H_v] \cong \bigoplus_{\rho \in \text{Irrep}(H_v)} \text{End}_\mathbb{C}(\rho). (3.4)$$

Choose a primitive idempotent $e_\rho$ in the matrix algebra $\text{End}_\mathbb{C}(\rho)$ for each $\rho \in \text{Irrep}(H_v)$ and set

$$e = \sum_{v \in Q'_0} \sum_{\rho \in \text{Irrep}(H_v)} e_\rho. (3.5)$$
Then $e(H \rtimes \mathbb{C}G)$ is Morita equivalent to $H \rtimes \mathbb{C}G$, and $\{e_\rho\}_\rho$ gives a set of mutually orthogonal idempotents in $e(H \rtimes \mathbb{C}G)e$ which sum up to the identity. This allows one to describe $e(H \rtimes \mathbb{C}G)e$ in terms of a quiver with relations; the set $V$ of vertices is $\bigcup_{v \in \mathcal{Q}_0'} \text{Irrep}(H_v)$, and for each (not necessarily distinct) pair $(\rho, \rho')$ of vertices, we choose a finite subset of $e_{\rho'}(H \rtimes \mathbb{C}G)e_{\rho}$ as the set of arrows from $\rho$ to $\rho'$, in such a way that the union for all pairs generate $e(H \rtimes \mathbb{C}G)e$ as an algebra.

To illustrate the constructions so far, we discuss two-dimensional examples, which are simpler than, but shares the essential features of, three-dimensional cases.

**Example 3.11.** A two-dimensional analog of a dimer model is a collection of uncoupled nodes on a circle, which divides the circle into intervals. The division of the circle into $n$ intervals corresponds to the McKay quiver $\Gamma = (Q_0, Q_1, s, t, \mathcal{I})$ for the subgroup $A$ of $\text{SL}(2, \mathbb{C})$ generated by $\gamma := \text{diag}(\zeta_n, \zeta_n^{-1})$, where $\zeta_n := \exp\left(2\pi\sqrt{-1}/n\right)$. The set $Q_0$ of vertices consists of irreducible representations $\rho_i: \gamma \mapsto \zeta_n^i$ of $A$ for $i = 0, 1, \ldots, n - 1$. The set of arrows consists of $x_i$ and $y_i$ for $i = 0, 1, \ldots, n - 1$ with sources $s(x_i) = \rho_i$, $s(y_i) = \rho_i$ and targets $t(x_i) = \rho_{i+1}$, $t(y_i) = \rho_{i-1}$, and the ideal of relations are generated by $x_{i-1}y_i - y_{i+1}x_i$ for $i = 0, 1, \ldots, n - 1$. The path algebra $\mathbb{C}\Gamma$ can be identified with the crossed product algebra $A \rtimes \mathbb{C}[x, y]$ in such a way that

- the idempotent of the path algebra $\mathbb{C}\Gamma$ corresponding to the vertex $\rho_i \in Q_0$ is identified with the idempotent $e_i := \frac{1}{n} \sum_{j=0}^{n-1} \zeta_n^{-ij}\gamma^j$ of the group ring $\mathbb{C}[A] \subset A \rtimes \mathbb{C}[x, y]$ corresponding to the projection to $\rho_i$, and
- $x_i = e_{i+1}xe_i$ and $y_i = e_{i-1}ye_i$, so that $x = \sum_{i=0}^{n-1} x_i$ and $y = \sum_{i=0}^{n-1} y_i$.

The analog of the characteristic polygon in this case is the interval $\Delta$ of length $n$ in $N_\mathbb{R} := N \otimes \mathbb{R}$ where $N$ is a free abelian group of rank 1, and the associated toric variety $X_\Delta$ gives the $A_n$-singularity $\mathbb{C}^2/A$. The cyclic group $H$ of order two is the only non-trivial finite subgroup of $\text{GL}(N)$. The induced action $\mu$ of $H$ on $X_\Delta$ does not preserve the canonical module, and one can twist the action to obtain a Gorenstein quotient singularity only if $n$ is even. This condition on the parity of $n$ is ensured by Assumption 3.3. The quotient of $X_\Delta$ by the twisted action of $H$ is the quotient of $\mathbb{C}^2$ by the binary dihedral group $D_n$ of order $2n$. For even $n$, there are two ways to make $H$ act on the circle $M_\mathbb{R}/M$. One fixes a pair of intervals and acts non-trivially on the remaining $n - 2$ intervals, and the other acts non-trivially on all the intervals. Only the former satisfies Assumption 3.3. Let us consider the case $n = 4$, i.e. the group $D_4$ of order 8. The action of the generator $\sigma$ of $H \cong \mathbb{Z}/2\mathbb{Z}$ on the McKay quiver $\Gamma$ fixes the vertices $\rho_0$ and $\rho_2$, and interchanges the vertices $\rho_1$ and $\rho_3$. The action on the arrows depends on a choice of a perfect matching. Choosing a perfect matching corresponds to choosing one arrow from each of the pairs $\{x_0, y_1\}$, $\{x_1, y_2\}$, $\{x_2, y_3\}$, $\{x_3, y_0\}$. The choice $y_1, y_2, x_2, x_3$ corresponds to the 0-generated $H$-invariant perfect matching, with respect to which the action of $\sigma$ on the arrows is given by

$$
\begin{align*}
x_0 &\leftrightarrow y_0 \\
x_1 &\leftrightarrow y_3 \\
x_2 &\leftrightarrow -y_2 \\
x_3 &\leftrightarrow -y_1.
\end{align*}
$$

(3.6)

The path algebra $\mathbb{C}\Gamma$ with relations is isomorphic to the crossed product algebra $A \rtimes \mathbb{C}[x, y]$. 


and $B := H \rtimes C\Gamma$ is isomorphic to $BD_4 \rtimes \mathbb{C}[x, y]$, where

$$BD_4 = \left\langle \frac{1}{4}(1, 3), \left( \begin{array}{cc} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{array} \right) \right\rangle$$

is the binary dihedral group of type $D_4$ and the matrix $\left( \begin{array}{cc} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{array} \right) \in BD_4$ corresponds to $\delta := \sigma((e_0 - e_2) + \sqrt{-1}(e_1 + e_3)) \in B$. In fact, $\delta^2$ is identified with $\gamma^2 \in A$ and Equation (3.6) implies $\delta x_i \delta^{-1} = \sqrt{-1}y_i$ and $\delta y_i \delta^{-1} = \sqrt{-1}x_i$ in $B$. The algebra $B$ has primitive idempotents

$$e_{00} = \frac{1}{2}(1 + \sigma)e_0, \quad (3.7)$$
$$e_{01} = \frac{1}{2}(1 - \sigma)e_0, \quad (3.8)$$
$$e_{130} = \frac{1}{2}(1 + \sigma)(e_1 + e_3), \quad (3.9)$$
$$e_{131} = \frac{1}{2}(1 - \sigma)(e_1 + e_3), \quad (3.10)$$
$$e_{20} = \frac{1}{2}(1 + \sigma)e_2, \quad (3.11)$$
$$e_{21} = \frac{1}{2}(1 - \sigma)e_2 \quad (3.12)$$

which are mutually orthogonal and sum up to the identity. The projective modules $P_{130} = e_{130}B$ and $P_{131} = e_{131}B$ are isomorphic as $B$-modules by the map

$$m \mapsto (e_1 - e_3) \cdot m. \quad (3.13)$$

Indeed, this map interchanges $P_{130}$ and $P_{131}$ since

$$(e_1 - e_3)(1 \pm \sigma) = (1 \mp \sigma)(e_1 - e_3), \quad (3.14)$$

and it is an isomorphism since

$$(e_1 - e_3)^2 = (e_1 + e_3). \quad (3.15)$$

Therefore, one can choose

$$e = e_{00} + e_{01} + e_{130} + e_{20} + e_{21}. \quad (3.16)$$

One can take

$$(1 + \sigma)(e_1 + e_3)x(1 + \sigma)e_0 = (1 + \sigma)(x_0 + \sigma y_0) \quad (3.17)$$

$$= (1 + \sigma)(\sigma x_0 + y_0) \quad (3.18)$$

$$= (1 + \sigma)(e_1 + e_3)y(1 + \sigma)e_0, \quad (3.19)$$

as the element corresponding to a unique arrow from $e_{00}$ to $e_{130}$. Arrows between other vertices can be computed similarly, which generates $B$ as an algebra. Moreover, one can deduce the relation for the McKay quiver for the binary dihedral group $BD_4$ from the relations for the McKay quiver for the cyclic group $A$. 

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4 Finite subgroups of $\text{GL}(2, \mathbb{Z})$

Finite subgroups of $\text{GL}(2, \mathbb{Z})$ are classified as follows:

**Proposition 4.1.** A finite non-trivial subgroup of $\text{GL}(2, \mathbb{Z})$ is conjugate to one of the following:

1. **Cyclic group of rotations:**
   - $C_2 = \langle \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \rangle$ of order 2.
   - $C_3 = \langle \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \rangle$ of order 3.
   - $C_4 = \langle \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \rangle$ of order 4.
   - $C_6 = \langle \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \rangle$ of order 6.

2. **Reflection groups of order 2:**
   - $R_1 = \langle \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \rangle$.
   - $R_2 = \langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rangle$.

3. **Dihedral groups:**
   - $D_4^1 = \langle \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \rangle$ of order 4.
   - $D_4^2 = \langle \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rangle$ of order 4.
   - $D_6^1 = \langle \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \rangle$ of order 6.
   - $D_6^2 = \langle \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rangle$ of order 6.
   - $D_8 = \langle \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \rangle$ of order 8.
   - $D_{12} = \langle \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rangle$ of order 12.

**Proof.** Let $A \in \text{GL}(2, \mathbb{Z})$ be an element of finite order $m$. Since the characteristic polynomial of $A$ is of degree 2 and is divisible by the $m$-th cyclotomic polynomial, we see that $m$ is either 1, 2, 3, 4 or 6.

A finite subgroup $H$ of $\text{GL}(2, \mathbb{Z})$ is either cyclic or dihedral, since $H$ is conjugate to a subgroup of $O(2)$.

If $H$ is cyclic of order greater than 2, then $H$ is a rotation group. Consider an $H$-invariant metric on $\mathbb{R}^2$ and take a vector $v \in \mathbb{Z}^2 \setminus 0$ with the smallest length. Then there are no other
of order 12 is conjugate to $D_3$.

If $H = \langle A \rangle$ is a reflection group of order 2, then take two primitive vectors $v, w \in \mathbb{Z}^2$ with $Av = v$ and $Aw = -w$. If $v, w$ form a $\mathbb{Z}$-basis of $\mathbb{Z}^2$, then $H$ is conjugate to $R_1$. Otherwise, there is an integral vector $u = \alpha v + \beta w \in \mathbb{Z}^2$ with $\alpha, \beta \in (0, 1)$. Then the equations $u + Au = 2\alpha v \in \mathbb{Z}^2$ and $u - Au = 2\beta w \in \mathbb{Z}^2$ imply $\alpha = \beta = 1/2$. Thus $H$ is conjugate to $R_2$.

If $H$ is dihedral of order 4, then $H$ is generated by a reflection and $-1$, so that it is conjugate to either $D_4$ or $D_2$. In the remaining cases, consider an $H$-invariant metric and take a vector $v \in \mathbb{Z}^2 \setminus 0$ of the smallest length. If $H$ is dihedral of order 6 and $A \in H$ is a rotation of order 3, then $v$ and $Av$ form a $\mathbb{Z}$-basis of $\mathbb{Z}^2$ and $\pm v, \pm Av, \pm A^2 v$ are the non-zero integral vectors of the smallest length. Therefore, $H$ preserves the hexagon whose vertices are these six vectors. It follows that $H$ is conjugate to $D_6$ if $H$ preserves the triangle formed by $v, Av$ and $A^2 v$, and conjugate to $D^*_6$ otherwise. If $H$ is dihedral of order 8 and $A \in H$ is a rotation of order 4, then $\pm v, \pm Av$ are the non-zero vectors of the smallest length and $H$ preserves the square formed by these 4 vectors. Therefore $H$ is conjugate to $D_8$. Similarly, a dihedral group of order 12 is conjugate to $D_{12}$. \hfill \Box

5 Construction of symmetric dimer models

Let $H$ be a finite subgroup of $\text{GL}(N)$ and $\Delta$ be an $H$-invariant lattice polygon in $\mathbb{R}$. A corner of $\Delta$ is a point on the boundary of $\Delta$ such that $\Delta$ is not defined by one linear inequality in any neighborhood of that point. Our strategy for constructing a symmetric dimer model is the following:

1. Embed $\Delta$ into an $H$-invariant polygon $\tilde{\Delta}$, which is the characteristic polygon of a consistent symmetric dimer model $\tilde{G}$. To find such a dimer model $\tilde{G}$, we enlarge a small example by a linear transform by Lemma 5.1 and cut off its corners by using Proposition 5.3 if necessary.

2. If there exists a corner $\epsilon$ of $\tilde{\Delta}$ not in $\Delta$, then remove the orbit $H \cdot \epsilon$ and take the convex hull of the rest. When we consider only one corner, this corresponds to removing edges in the dimer model $\tilde{G}$ using the special McKay correspondence as in [IU15]. Proposition 5.3 allows us to do the operations symmetrically, under some conditions on $\tilde{\Delta}$.

3. Repeat the second step until we obtain $\Delta$.

The dimer model $\tilde{G}$ in the first step must be constructed so that the lattice polygon $\tilde{\Delta}$ satisfies the conditions in Proposition 5.3 in each step of corner removal.

To find a suitable polygon $\Delta$ and a dimer model $\tilde{G}$, first note the following obvious fact:

**Lemma 5.1.** Let $G$ be a consistent dimer model on $T = \mathbb{R}/M$ whose characteristic polygon is $\Delta \subseteq \mathbb{R}/M$, and $\tilde{M}$ be a sublattice of $M$ of finite index. Then the $M/\tilde{M}$-cover $\tilde{G}$ of $G$ on $\tilde{T} := \tilde{M}/\tilde{M} \cong \mathbb{R}/\tilde{M}$ is a consistent dimer model, whose characteristic polygon $\tilde{\Delta}$ is $\Delta$ considered as a lattice polygon in $\tilde{N}_\mathbb{R} \cong \mathbb{R}$. 

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Proposition 5.2 ([Gul08]). Let \( G \) be a consistent dimer model with characteristic polygon \( \Delta \) and \( c \) be a corner of \( \Delta \). Let further \( \mathcal{d} \) and \( \mathcal{d}' \) be the pair of corners of \( \Delta \) adjacent to \( c \), and \( z_1, \ldots, z_l \) and \( z'_1, \ldots, z'_m \) be zigzag paths of \( G \) whose slopes are outer normal to the sides \( cd \) and \( cd' \) respectively. Take the \( l \)-th lattice point \( R \) on \( cd \) and the \( m \)-th lattice point \( R' \) on \( cd' \) counted from \( c \). Let \( G' \) be the bicolored graph obtained by removing all the intersections of \( z_i \) and \( z'_j \) for all \((i, j)\). If \( \Delta \) does not coincide with the triangle formed by the lattice points \( c, R, \) and \( R' \), then \( G' \) is a consistent dimer model whose characteristic polygon is the polygon \( \Delta' \) obtained from \( \Delta \) by removing the triangle \( cRR' \).

**Proof.** Since \( \Delta \) does not coincide with the triangle \( cRR' \), \( G \) has a pair of zigzag paths other than \( z_i \) or \( z'_j \) whose slopes are linearly independent. These zigzag paths remain in the resulting bicolored graph \( G' \), and hence \( G' \) is a dimer model. The operation creates several new zigzag paths, consisting of edges in \( \bigcup z_i \cup \bigcup z'_j \). The slopes of new zigzag paths are the outward normal vector to the line segment \( RR' \), and belong to \( \mathbb{R}_{>0}[z_i] + \mathbb{R}_{>0}[z'_j] \subset H_1(T, \mathbb{R}) \). Note that \( z_i \) and \( z'_j \) intersect each other only once on \( M_g \) [IU15 Lemma 7.1]. The other zigzag paths of \( G \) are unchanged. Therefore, the properly orderedness of \( G \) implies that of \( G' \), and the zigzag polygon of \( G' \) is \( \Delta' \). Since \( G' \) is properly ordered, the characteristic polygon of \( G' \) coincides with the zigzag polygon \( \Delta' \).

We use Proposition 5.3 below to remove the orbit of a corner:

**Proposition 5.3.** Let \( G \) be a consistent symmetric dimer model with characteristic polygon \( \Delta \). Let further \( c \) be a corner of \( \Delta \), and \( \Delta' \) be the convex hull of the complement \( (\Delta \cap N) \setminus Hc \) of the \( H \)-orbit of \( c \) in the set of lattice points of \( \Delta \). Assume that for any \( g \in H \), the corners \( c \) and \( gc \) are not connected by a primitive side segment of \( \Delta \). Then there is a consistent symmetric dimer model \( G' \) with characteristic polygon \( \Delta' \).

**Proof.** Let \( s_1 \) and \( s_2 \) be the pair of primitive side segments of \( \Delta \) incident to \( c \). The assumption implies that \( \{s_1, s_2\} \cap \{gs_1, gs_2\} = \emptyset \) if \( gc \neq c \). Moreover, we have \( gs_i \neq s_i \) for any non-trivial \( g \in H \).

We use the operation in [IU15 Section 10.1] for each corner in the orbit of \( c \). In [IU15 Algorithm 10.1(1)], we take a pair \((z_1, z_2)\) of zigzag paths corresponding to \( c \). This means that the homology classes of \( z_1 \) and \( z_2 \) are normal to \( s_1 \) and \( s_2 \) respectively. Notice that although \( s_i \) and \( gs_i \) are different for \( g \neq 1 \), they might be contained in the same side of \( \Delta \) and in that case \( z_i \) and \( gz_i \) might coincide. We claim that by suitably choosing \( z_i \), we may assume \( gz_i \neq z_i \) for any non-trivial \( g \in H \).

Choose and fix a generic stability parameter \( \theta \) invariant under \( H \), such as the \( v_0 \)-generated stability for the fixed vertex \( v_0 \). Then for each lattice point in \( \Delta \), there is a unique \( \theta \)-stable perfect matching corresponding to it. The \( \theta \)-stable perfect matchings corresponding to boundary lattice points have the following property: if \( D \) and \( D' \) are the \( \theta \)-stable perfect matchings corresponding to the endpoints of a primitive side segment \( s \), then \( D' \) is obtained from \( D \) by “flipping” a single zigzag path \( z \) such that \( [z] \) is outer normal to the segment \( s \) as in [Gul08 Corollary 3.8]. Indeed, it follows from [Gul08 Corollary 3.8] that \( D' \) is obtained from \( D \) by flipping finitely many zigzag paths \( w_1, \ldots, w_m \) with the same slope. This means that
• every other edge of \( w_i \) belongs to \( D \), and

• \( D' \) is obtained from \( D \) by replacing \( D \cap w_i \) with \( w_i \setminus D \) for all \( i = 1, \ldots, m \).

If \( m > 1 \), then since the height change \( h(D, D') \) is a primitive vector, we can choose \( w_1 \) and \( w_2 \) so that their contributions to the height change cancel each other. Notice that \( T \setminus (w_1 \cup w_2) \) has two connected components and by our choice of \( w_1 \) and \( w_2 \), one connected component determines submodules of \( \mathbb{C} \Gamma \)-modules corresponding to the perfect matching \( D \) and the same component determines quotient modules of \( \mathbb{C} \Gamma \)-modules corresponding to \( D' \). This contradicts the \( \theta \)-stability of the perfect matchings \( D \) and \( D' \) and proves \( m = 1 \). Moreover, by fixing \( \theta \), we obtain a bijective correspondence between the zigzag paths of \( G \) and the primitive side segments of the characteristic polygon. Let \( z_i \) be the zigzag path corresponding to \( s_i \) in this bijection. Then \( g(D_i) \) is obtained from \( g(D_c) \) by flipping \( g(z_i) \). Since \( \theta \) is invariant, the action of \( H \) on the set of perfect matchings preserves the \( \theta \)-stability and hence \( g(D_i) \) and \( g(D_c) \) are also \( \theta \)-stable. This proves that \( g(z_i) \) corresponds to \( g(s_i) \) and \( g(s_i) \neq s_i \) implies \( g(z_i) \neq z_i \).

As in [IU15 Algorithm 10.1(2)], we construct large hexagons from the pair \( (z_1, z_2) \), and identify them with vertices of the McKay quiver for a finite abelian group \( A \subset \text{GL}(2, \mathbb{C}) \subset \text{SL}(3, \mathbb{C}) \), in such a way that the large hexagon corresponding to the trivial representation contains the \( H \)-fixed face. Then we remove several edges on \( z_1 \cap z_2 \) as in [IU15 Algorithm 10.1(3)], and for each \( g \in H \), we do the same operation using the pair \( (gz_1, gz_2) \). If \( gc \neq c \), then the assumption implies \( \{z_1, z_2\} \cap \{gz_1, gz_2\} = \emptyset \) and hence the operations for \( \{z_1, z_2\} \) and \( \{gz_1, gz_2\} \) are independent. If \( gc = c \), then the action of \( g \) exchanges \( z_1 \) and \( z_2 \), preserving the edges to be removed. Hence the consistent dimer model \( G' \) obtained from \( G \) by the successive operations for the corners in the orbit of \( c \) is preserved by the action of \( H \). The face of \( G' \) containing the fixed face of \( G \) is also fixed by \( H \).

**Example 5.4.** As an illustration of Proposition 5.3, consider the lattice triangle \( \Delta \) and the dimer model \( G \) having \( \Delta \) as the characteristic polygon, shown in Figure 5.1. Both \( \Delta \) and \( G \) are symmetric under \( R_2 \).

![Figure 5.1: Lattice triangle \( \Delta \) and dimer model \( G \) both symmetric under \( R_2 \), and the McKay quiver of \( \frac{1}{8}(1, 3, 4) \) dual to \( G \).](image)

The affine toric variety \( X_{\Delta} \) is isomorphic to the quotient of the affine space \( \mathbb{C}^3 \) by the cyclic subgroup \( \langle \frac{1}{8}(1, 3, 4) \rangle \subset \text{SL}_3(\mathbb{C}) \) of order 8 generated by diag(\( \zeta, \zeta^3, \zeta^4 \)), where \( \zeta \) is a primitive 8-th root of unity. In general, given a subgroup \( \langle \frac{1}{m}(1, q, n - q - 1) \rangle \) of \( \text{SL}_3(\mathbb{C}) \), we define
integers $r, b_1, \ldots, b_r, i_0, \ldots, i_{r+1}$ by $i_0 := n$, $i_1 := q$, $i_t = b_{t+1}i_{t+1} - i_{t+2}$ (where $0 < i_{t+2} < i_{t+1}$),
$i_r = 1$, and $i_{r+1} = 0$ as explained in [IU15, Section 4] (which goes back to [Wun87, Wun88]).
For $n = 8$ and $q = 3$, we have
\[8 = 3 \cdot 3 - 1,\] (5.1)
so that $r = 2$ and $(i_0, i_1, i_2) = (8, 3, 1)$. By removing the edges of $G$ dual to the arrows of the
McKay quiver corresponding to ‘multiplication by $z$’ (i.e., those which goes in the southwest
direction in the quiver of Figure 5.1) from the vertices $i_0, i_1, i_2$ and removing divalent nodes,
one obtains the dimer model $G'$ shown in Figure 5.2, whose characteristic polygon is the
trapezoid shown in the same figure.

Figure 5.2: The dimer model $G'$ also symmetric with respect to $R_2$.

5.1 Cyclic groups

In this section, we assume that $H$ is a cyclic group of order $n$ consisting of rotations. In this
case, Proposition 5.3 implies the following:

Corollary 5.5. Let $G$ be a consistent symmetric dimer model with characteristic polygon $\Delta$. Let further $c$ be a corner of $\Delta$ and $\Delta'$ be the lattice polygon obtained from $\Delta$ by removing the orbit of $c$. Assume that one of the following holds:

(1) $\Delta$ is not an $n$-gon.

(2) $\Delta$ is an $n$-gon with a boundary lattice point which is not a corner.

Then there is a consistent symmetric dimer model $G'$ with characteristic polygon $\Delta'$.

5.1.1 The group $C_2$

In this case, we can embed $\Delta$ in a square $\tilde{\Delta}$ and iterate the operations in Corollary 5.5, since Condition (1) in 5.5 always holds for $n = 2$.

5.1.2 The group $C_3$

Let $\Delta_n$ be the convex hull of $(n, -n)$, $(n, 2n)$ and $(-2n, -n)$, which is the characteristic polygon of the hexagonal dimer model $G_n$ associated with the McKay quiver of the abelian
subgroup $A$ of $\text{SL}(3, \mathbb{C})$ isomorphic to $\mathbb{Z}/3n\mathbb{Z} \times \mathbb{Z}/3n\mathbb{Z}$. By translating $G_n$ if necessary, we assume that the face corresponding to the trivial representation of $A$ is fixed by the action of $H$. For a symmetric lattice polygon $\Delta$, take the minimum integer $n$ such that $\Delta \subset \Delta_n$ and put $\Delta := \Delta_n$. Then we have $\partial \Delta \cap \Delta \neq \emptyset$. By starting from $\tilde{G} := G_n$ and iterate the operations in Corollary 5.5, we obtain a consistent symmetric dimer model.

**Remark 5.6.** For a lattice polygon $\Delta$ with rotational symmetry of order 3 whose center is not a lattice point (in this case $C_3 \subset \text{GL}(N) \ltimes N$ but $C_3 \not\subset \text{GL}(N)$), we can embed $\Delta$ into a lattice polygon corresponding to the Abelian subgroup of $\text{SL}(3, \mathbb{C})$ isomorphic to $\mathbb{Z}/2n\mathbb{Z} \times \mathbb{Z}/2n\mathbb{Z}$, and the same method produces a consistent symmetric dimer model. This includes in our treatment the case when $X_\Delta/H \cong \mathbb{C}^3/G$ where $G$ is a trihedral group in $\text{SL}(3, \mathbb{C})$.

### 5.1.3 The group $C_4$

Let $\Delta_n$ be the convex hull of $(\pm n, 0)$ and $(0, \pm n)$. A dimer model $G_n$ with characteristic polygon $\Delta_n$ can be obtained from the consistent dimer model with characteristic polygon $\Delta_1$ shown in Figure 5.3 by using Lemma 5.1. This dimer model is symmetric with respect to the action of the group $C_4$ fixing an octagonal face. Note that the face of a dimer model symmetric under a rotation of order 4 must have at least 8 edges.

Given a $C_4$-invariant lattice polygon $\Delta$, we embed it into $\Delta_n$ with the smallest $n$, and iterate the operations in Corollary 5.5 to obtain a consistent symmetric dimer model with characteristic polygon $\Delta$.

![Figure 5.3: Lattice polygon $\Delta_1$ and the dimer model $G_1$.](image)

### 5.1.4 The group $C_6$

Let $G_1$ be the dimer model with characteristic polygon $\Delta_1$ shown in Figure 5.4. The dimer model $G_n$ with characteristic polygon $\Delta_n := n\Delta_1$ is obtained as the $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$-cover of $G_1$ by using Lemma 5.1 as in previous cases. Given a $C_6$-invariant lattice polygon $\Delta$, we embed it into $\Delta_n$ with the smallest $n$, and iterate the operations in Corollary 5.5 to obtain a consistent symmetric dimer model with characteristic polygon $\Delta$.

### 5.2 Reflection groups of order two

In the case of reflection groups, we take the square lattice dimer model $\tilde{G}$ whose characteristic polygon $\tilde{\Delta}$ is a rectangle as $\tilde{G}$.
Figure 5.4: The hexagon $\Delta_1$ with one interior lattice point and the dimer model $G_1$.

5.2.1 The group $R_1$

For an $R_1$-invariant lattice polygon $\Delta$, let $\tilde{\Delta}$ be the minimum rectangle containing $\Delta$, whose sides are parallel to $(1, 0)$ or $(0, 1)$, i.e., two of whose sides are parallel to $(1, 0)$, and the other two are parallel to $(0, 1)$. Since each side of $\tilde{\Delta}$ contains a lattice point of $\Delta$, we can start from the square lattice dimer model $\tilde{G}$ and iterate the operations in Proposition 5.3 to obtain a consistent symmetric dimer model $G$ with characteristic polygon $\Delta$.

5.2.2 The group $R_2$

In this case, we consider a rectangle containing $\Delta$, two of whose sides are parallel either to $(1, 1)$ or $(1, -1)$. Note that if we require that each of the four sides meet $\Delta$, then the rectangle may not be a lattice rectangle, i.e., it may not have lattice points as its corners. In general, there may be two minimal such lattice rectangles containing $\Delta$. We choose $\tilde{\Delta}$ such that $\partial \tilde{\Delta}$ contains $\partial \Delta \cap \mathbb{Z}(1, 1)$ (if this is non-empty). Then we can again iterate the operations in Proposition 5.3 to obtain a dimer model $G$ corresponding to $\Delta$.

5.3 Dihedral groups

5.3.1 The group $D_4^1$

For a lattice polygon $\Delta$ symmetric under the $D_4^1$-action, let $\tilde{\Delta}$ be the minimum rectangle containing $\Delta$ whose sides are parallel to $(1, 0)$ or $(0, 1)$. Then starting from $\tilde{\Delta}$, we can iterate the operations in Proposition 5.3 to obtain a consistent symmetric dimer model $G$ with characteristic polygon $\Delta$.

5.3.2 The group $D_4^2$

Consider the action of $D_4^2$ on $\mathbb{R}^2$ and let $L_1 = \mathbb{R}(1, 1)$ and $L_2 = \mathbb{R}(1, -1)$ be the lines of reflections. Then $D_4^2$ acts freely on $\mathbb{R}^2 \setminus (L_1 \cup L_2)$. We use rectangles as in the $R_2$ case.

**Lemma 5.7.** Let $\tilde{\Delta}$ be a $D_4^2$-invariant lattice rectangle whose sides are parallel to $L_1$ or $L_2$. Then the number of lattice points on $\partial \tilde{\Delta} \cap (L_1 \cup L_2)$ is either 0 or 4.

**Proof.** Let $v_1$ and $v_2$ be points on $\partial \tilde{\Delta} \cap L_1$ and $\partial \tilde{\Delta} \cap L_2$ respectively. Then one has $\partial \tilde{\Delta} \cap (L_1 \cup L_2) = \{v_1 \pm v_2\}$, and $v_1 \pm v_2$ are the four corners of $\tilde{\Delta}$, which are lattice points. The assertion follows from this. \qed
Let \( \Delta \) be a lattice polygon invariant under the \( D_6^2 \)-action. We embed \( \Delta \) into an invariant lattice rectangle \( \tilde{\Delta} \) whose sides are parallel to \( L_1 \) or \( L_2 \). We assume that all the lattice points on \( \partial \Delta \cap (L_1 \cup L_2) \) are on \( \partial \tilde{\Delta} \) and that \( \tilde{\Delta} \) is the minimum of the lattice rectangles satisfying this condition. This means the following.

- If \( \#(\partial \Delta \cap (L_1 \cup L_2) \cap \mathbb{Z}^2) = 0 \) or \( 4 \), then \( \partial \Delta \cap (L_1 \cup L_2) = \partial \tilde{\Delta} \cap (L_1 \cup L_2) \).

- Suppose \( \partial \Delta \cap L_i \cap \mathbb{Z}^2 \neq \emptyset \) and \( \partial \Delta \cap L_j \cap \mathbb{Z}^2 = \emptyset \) for \( \{i, j\} = \{1, 2\} \). Then \( \partial \tilde{\Delta} \cap L_i \) coincides with \( \partial \Delta \cap L_i \), while \( \partial \tilde{\Delta} \cap L_j \) consists of the lattice points closest to \( \partial \Delta \cap L_j \) outside of \( \Delta \).

Then starting from \( \tilde{\Delta} \), we can iterate the operations in Proposition 5.3 to obtain a consistent symmetric dimer model \( \Delta \).

### 5.3.3 The group \( D_6^1 \)

Let \( \Delta \) be a \( D_6^1 \)-invariant lattice polygon \( \Delta \). As in Section 5.1.2 take the minimum integer \( n \) such that \( \Delta \subset \tilde{\Delta}_n \), where \( \tilde{\Delta}_n \) is the convex hull of \( (n, -n) \), \( (n, 2n) \) and \( (-2n, -n) \). In this case, we cannot obtain \( \Delta \) from \( \tilde{\Delta}_n \) by iteration of chopping corners satisfying the conditions in Proposition 5.3 if at some step the corner is on a primitive side segment intersecting a line of reflection. Thus before applying Proposition 5.3, we first cut off regular triangles at the corners of \( \tilde{\Delta}_n \); let \( \Delta \) be the minimum hexagon containing \( \Delta \) obtained by cutting off three corner regular triangles from \( \tilde{\Delta}_n \) (when \( \Delta \) is a triangle, we obtain \( \Delta \) itself instead of a hexagon but in this case, there is nothing to do). We apply Proposition 5.2 simultaneously to the three corners of \( \tilde{\Delta}_n \) by symmetrically choosing the zigzag paths in Proposition 5.2. This operation produces a symmetric consistent dimer model \( \tilde{G} \) whose characteristic polygon is \( \tilde{\Delta} \). To obtain \( \Delta \) from \( \tilde{\Delta} \), notice that the minimality of the hexagon \( \tilde{\Delta} \) ensures that \( \partial \tilde{\Delta} \) contains all the points of \( \partial \Delta \) that are on the lines of reflections. Therefore, for any corner \( c \) of \( \tilde{\Delta} \) which is not on \( \Delta \), \( gc \) and \( c \) are not connected by a primitive line segment of \( \tilde{\Delta} \) for a non-trivial \( g \in D_6^1 \). Thus we can iterate the operations in Proposition 5.3 to obtain a consistent symmetric dimer model corresponding to \( \Delta \).

### 5.3.4 The group \( D_6^2 \)

We fix a \( D_6^2 \)-invariant metric on \( \mathbb{R}^2 \) such that \( (1, 0) \) is of length 1. This means that we consider the inner product defined by

\[
\langle (x_1, y_1), (x_2, y_2) \rangle = (x_1, y_1) \begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix} (x_2, y_2).
\]

For example, \( (1, 0) \) is perpendicular to \( (1, 2) \).

In this case, the lines of reflections are \( L_1 := \mathbb{R}(1, 0) \), \( L_2 := \mathbb{R}(1, 1) \) and \( L_3 := \mathbb{R}(0, 1) \). Let \( \tilde{\Delta}_n \) be the lattice hexagon whose corners are \( (n, 0) \), \( (n, n) \), \( (0, n) \), \( (-n, 0) \), \( (-n, -n) \) and \( (0, -n) \), which is a regular hexagon of side \( n \). Then \( \tilde{\Delta}_1 \) is in Figure 5.4 and thus a consistent dimer model \( \tilde{G}_n \) corresponding to \( \tilde{\Delta}_n \) with \( D_6^2 \)-action is obtained by applying Lemma 5.1 to the one in Figure 5.4.

For a given lattice polygon \( \Delta \) with \( D_6^2 \)-action, embed \( \Delta \) into \( \tilde{\Delta}_n \) with the minimum value of \( n \). This means \( \partial \Delta \cap \partial \tilde{\Delta}_n \neq \emptyset \). We first cut off isosceles triangles from \( \tilde{\Delta}_n \) as follows. Let \( k \)
and \( l \) be the maximum integers satisfying \((k,0) \in \Delta\) and \((l,l) \in \Delta\) respectively. Notice that \((k,0)\) is on \(L_1\) and \((l,l)\) is on \(L_2\). Let \(\tilde{\Delta}\) be the convex lattice polygon obtained by cutting off corner triangles of \(\tilde{\Delta}_n\) by the following six lines:

- the lines passing through \((k,0)\) or \((-l,0)\) and perpendicular to \(L_1\),
- the lines passing through \((l,l)\) or \((-k,-k)\) and perpendicular to \(L_2\),
- the lines passing through \((0,k)\) or \((0,-l)\) and perpendicular to \(L_3\).

Since \(\Delta\) is convex and invariant by \(D_{26}^2\), it is contained in \(\tilde{\Delta}\). Moreover, \(\Delta\) contains the intersections of the lines of reflections with \(\partial \tilde{\Delta}\). By applying Proposition 5.2 at the six corners in a symmetric way, we obtain a symmetric consistent dimer model with \(D_{26}^2\)-action and a fixed face corresponding to \(\tilde{\Delta}\). To obtain \(\Delta\) from \(\tilde{\Delta}\), we iterate the operation of chopping corners in a \(D_{26}^2\)-orbit. In this process, by our choice of \(\tilde{\Delta}\), a corner \(c\) and \(g_c\) are not adjacent to each other for a non-trivial \(g \in D_{26}^2\). Therefore we can apply Proposition 5.3 in each step. Thus there is a consistent dimer model with \(D_{26}^2\)-action and a fixed face whose characteristic polygon is \(\Delta\).

5.3.5 The group \(D_8\)

Let \(G_n\) be the dimer model which corresponds to the square \(\Delta_n\) as in the \(C_4\) case. Then we have an action of \(D_8\) on \(G_n\). Take the smallest \(\Delta_n\) containing \(\Delta\) and cut off four isosceles triangles from the corners such that

- the resulting polygon (octagon in general) \(\tilde{\Delta}\) contains \(\Delta\) and
- \(\tilde{\Delta}\) is the minimum of such polygons.

Then again by applying Proposition 5.2 to the four corners of \(\Delta_n\) in a symmetric way, we obtain a symmetric consistent dimer model \(\tilde{G}\) with characteristic polygon \(\tilde{\Delta}\). Now \(\Delta\) can be obtained from \(\tilde{\Delta}\) by iteration of chopping corners as in Proposition 5.3. Thus we obtain a desired dimer model corresponding to \(\Delta\).

5.3.6 The group \(D_{12}\)

In this case let \(\tilde{\Delta}\) be the minimum polygon obtained by cutting corner triangles of the hexagon \(\tilde{\Delta}_n\) exactly as in the \(D_{26}^2\) case. (Notice that we have \(k = l\) in the \(D_{12}\) case.) Then the same argument as in the \(D_{26}^2\) case proves the existence of a consistent dimer model with \(D_{12}\)-action corresponding to \(\Delta\).

6 Non-commutative crepant resolutions

Let \(G\) be a consistent dimer model with characteristic polygon \(\Delta\) and \(\Gamma\) be the corresponding quiver with relations. As we recalled in Section 2.2, the moduli space \(\mathcal{M}_\theta\) of stable representations of \(\Gamma\) with respect to a generic stability parameter \(\theta\) is a crepant resolution \(\tau: \mathcal{M}_\theta \to X_\Delta\) of the Gorenstein affine toric variety \(X_\Delta = \text{Spec} \, R\), and the tautological bundle \(\mathcal{E} := \bigoplus_{v \in Q_0} \mathcal{L}_v\) is a tilting bundle such that \(\text{End}(\mathcal{E}) \cong \mathbb{C}\Gamma\). Fix a vertex \(v_0 \in Q_0\). By replacing \(\mathcal{E}\) with \(\mathcal{E} \otimes \mathcal{L}^{-1}_{v_0}\) if necessary, we may assume \(\mathcal{L}_{v_0} \cong \mathcal{O}_{\mathcal{M}_\theta}\). Then [11, Proposition A.2] shows that \(\text{End}(\mathcal{E})\) is
isomorphic to the endomorphism algebra $\text{End}_R(E)$ of the $R$-module $E := \tau_\ast \mathcal{E} \cong H^0(\mathcal{E})$, and that $\text{End}_R(E)$ is a non-commutative crepant resolution of $R$ in the sense of [vdB04a].

Let $G$ be a dimer model which is symmetric with respect to the action of a finite group $H$ in the sense of Definition 3.1. Let $v_0$ be the vertex fixed by the action of $H$, which exists by Assumption 3.3, and $\theta$ be a $v_0$-generated stability parameter.

**Lemma 6.1.** There is an action of $H$ on $\mathcal{E}$ which is compatible with the action $\nu$ on $\mathcal{M}_\theta$. Therefore $E$ is an $H$-equivariant sheaf on $\text{Spec } R$.

**Proof.** As in [CI04, §2.1], the moduli space $\mathcal{M}_\theta$ is constructed as a quotient of the scheme $\mathcal{N}_\theta \subset \prod_{a \in Q_1} \text{Hom}_C(V_{s(a)}, V_{t(a)})$ parametrizing $\theta$-stable representations of $\Gamma$ in vector spaces $V_v = \mathbb{C}$ for $v \in V$ by the action of the group $\text{Aut}'((V_v)_{v \in Q_0}) := \left\{ (g_v)_{v \in Q_0} \in \prod_{v \in Q_0} \text{GL}(V_v) \mid g_{v_0} = 1 \right\} \cong \prod_{v \in Q_0 \setminus \{v_0\}} \text{GL}(V_v)$.

This group $\text{Aut}'((V_v)_{v \in Q_0})$ acts on the locally free sheaf $\tilde{\mathcal{E}} := \bigoplus_v V_v \otimes \mathcal{O}_{\mathcal{N}_\theta}$ on $\mathcal{N}_\theta$ and $\tilde{\mathcal{E}}$ descends to the tautological bundle $\mathcal{E}$ on $\mathcal{M}_\theta$. On the other hand, we can define an action $\tilde{\nu}$ of $H$ on $\mathcal{N}_\theta$ by changing the sign in the natural action as in Section 3 which is compatible with the action $\nu$ on $\mathcal{M}_\theta$. We can also let $H$ act on the group $\text{Aut}'((V_v)_{v \in Q_0})$ by

$$(h, (g_v)_{v \in Q_0}) \mapsto (g_{h^{-1}(v)})_{v \in Q_0}$$

and on $\tilde{\mathcal{E}}$ by

$$\left(h, \bigoplus_v w_v \otimes f_v\right) \mapsto \bigoplus_v w_{h^{-1}(v)} \otimes \tilde{\nu}(h, f_{h^{-1}(v)}).$$

Thus the semidirect product $H \ltimes (\text{Aut}'((V_v)_{v \in Q_0}))$ acts on $\bigoplus_v V_v \otimes \mathcal{O}_{\mathcal{N}_\theta}$ which descends to an action of $H$ on $\mathcal{E}$. \qed

Let us now give the proof of Theorem 1.2. We first consider $H \ltimes \mathbb{C} \Gamma$ by using the action of $H$ on $\mathbb{C} \Gamma$. In what follows we prove that $H \ltimes \mathbb{C} \Gamma \cong H \ltimes \text{End}_R(E)$ is a NCCR of $R^H$. According to [vdB04a], it is sufficient to show the following:

- $H \ltimes \mathbb{C} \Gamma \cong \text{End}_{R^H}(E)$,
- $E$ is a reflexive $R^H$-module,
- $H \ltimes \mathbb{C} \Gamma$ is Cohen–Macaulay,
- $H \ltimes \mathbb{C} \Gamma$ has finite global dimension.

Since $\mathbb{C} \Gamma$ is Cohen–Macaulay over $R$, the crossed product $H \ltimes \mathbb{C} \Gamma$ is also Cohen–Macaulay over $R$. Thus $H \ltimes \mathbb{C} \Gamma$ is Cohen–Macaulay over $R^H$. Similarly, $E$ is reflexive over $R$ and hence reflexive over $R^H$. Moreover, since $\mathbb{C} \Gamma$ has finite global dimension, $H \ltimes \mathbb{C} \Gamma$ has also finite global dimension. It is remaining to prove that $H \ltimes \mathbb{C} \Gamma \cong \text{End}_{R^H}(E)$.

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Notice that \( \mathbb{C} \Gamma \cong \text{End}_R(E) \subseteq \text{End}_{R^H}(E) \) and the action of \( H \) on \( E \) induces a monoid homomorphism \( H \to \text{End}_{R^H}(E) \). Therefore, there exists an algebra homomorphism

\[
\mathcal{F} : H \times \text{End}_R(E) \to \text{End}_{R^H}(E).
\]

Recall that both \( H \times \text{End}_R(E) \) and \( \text{End}_{R^H}(E) \) are reflexive \( R^H \)-modules. Therefore it suffices to prove that \( \mathcal{F} \) is an isomorphism over some open subset \( U \subseteq \text{Spec} R^H \) with \( \text{codim} (\text{Spec} R^H \setminus U) \geq 2 \). To choose this open set, let \( \tilde{U} \) be the smooth and \( H \)-free locus in \( \text{Spec} R \) and define \( U := \tilde{U} / H \). The isomorphism \( K_{\mathcal{M}_\theta} \cong \mathcal{O}_{\mathcal{M}_\theta} \) in \( \text{coh}^H(\mathcal{M}_\theta) \) implies that \( \text{codim} (\text{Spec} R^H \setminus U) \geq 2 \).

We show that for every point \( P \in U \), the fibre of \( \mathcal{F} \) over \( P \) is an isomorphism. Since the restriction of \( \tau \) to \( \tau^{-1}(\tilde{U}) \) is an isomorphism, the sheaf \( E|_{\tilde{U}} \) is locally free. Moreover, since \( \pi^{-1}(P) \) is a free \( H \)-orbit, we have \( \text{End}_R(E)|_{P} \cong \bigoplus_{Q \in \pi^{-1}(P)} \text{End}_C(E|_{Q}) \). Thus the problem is reduced to showing that the map

\[
\mathcal{F}|_{P} : H \times \left( \bigoplus_{Q \in \pi^{-1}(P)} \text{End}_C(E|_{Q}) \right) \to \text{End}_C \left( \bigoplus_{Q \in \pi^{-1}(P)} E|_{Q} \right)
\]

is an isomorphism of vector spaces. The left hand side, as a vector space, decomposes as

\[
H \times \left( \bigoplus_{Q \in \pi^{-1}(P)} \text{End}_C(E|_{Q}) \right) = \bigoplus_{g \in H, Q \in \pi^{-1}(P)} g \times \text{End}_C(E|_{Q}),
\]

and \( \mathcal{F}|_{P} \) sends the direct summand \( g \times \text{End}_C(E|_{Q}) \) isomorphically onto \( \text{Hom}_C(E|_{Q}, E|_{gQ}) \). Since \( \pi^{-1}(P) \) is a free \( H \)-orbit, \( \mathcal{F}|_{P} \) is an isomorphism. This concludes the proof of Theorem 1.2.

**Remark 6.2.** Assumption 3.3 is used only when \( H \) does not preserve the orientation of \( T \). In fact, when \( H \) preserves the orientation of \( T \), Theorem 1.2 holds under only Assumption 3.4.

**Example 6.3.** Let \( G \) be the \( R_2 \)-symmetric consistent dimer model and \( \Delta \) its \( R_2 \)-symmetric characteristic polygon shown in Figure 6.1. Let \( X_{2k\Delta} \) be the affine toric variety associated with the polygon \( 2k\Delta := \{2kn \in N_R \mid n \in \Delta \} \) obtained by multiplying \( \Delta \) by \( 2k \) for \( k \geq 2 \) (or the same polygon \( \Delta \) regarded as a lattice polygon with respect to the over-lattice \( \frac{1}{k} N := \{n \in N_R \mid 2kn \in N\} \) of index \( (2k)^2 \)). Let \( P_0, \ldots , P_4 \) be the corners of \( 2k\Delta \) such that the reflection fixes \( P_0 \) and interchanges \( P_1 \) (resp. \( P_2 \)) with \( P_4 \) (resp. \( P_3 \)). The affine toric variety \( X_{2k\Delta} \) has a family of \( A_{2k-1} \)-singularities along the torus-invariant curve associated with the \( R_2 \)-invariant face of \( 2k\Delta \) (i.e., the edge connecting \( P_2 \) and \( P_3 \)). We choose the midpoint of \( P_2 \) and \( P_3 \) as the origin which defines the twisted action of \( R_2 \) on \( X_{2k\Delta} \) as in (6.1). Then the quotient of the family of \( A_{2k-1} \)-singularities gives a family of \( D_{k+3} \)-singularity along a curve on \( X_{2k\Delta}/R_2 \) (see Remark 3.4). The existence of a family of \( D_{k+3} \)-singularities along a curve implies that the affine variety \( X_{2k\Delta}/R_2 \) is not a toric variety (if \( X_{2k\Delta}/R_2 \) is a toric variety, then the curve along which the variety has \( D_{k+3} \)-singularities must be torus-invariant, so that the affine toric variety associated with the 2-dimensional cone corresponding to that curve must have a \( D_{k+3} \)-singularity, which is impossible). To prove that (the origin of) \( X_{2k\Delta}/R_2 \) is not a quotient singularity, we show that \( X_{2k\Delta}/R_2 \) is not \( \mathbb{Q} \)-factorial. To prove that a variety is not \( \mathbb{Q} \)-factorial, it suffices to find a pair of Weil divisors intersecting in codimension at least three.
For each $0 \leq i \leq 4$, let $D_i$ be the divisor of $X_{2k\Delta}$ associated with $P_i$. Then the divisors $D_0$ and $D_2 + D_3$ descends to divisors on $X_{2k\Delta}/R$ intersecting in codimension three. Hence the symmetric dimer model obtained from $G$ by Lemma 5.1 produces a non-commutative crepant resolution of $X_{2k\Delta}/R$ which is neither a toric variety nor a quotient singularity.

![Figure 6.1: An $R_2$-symmetric dimer model with a pentagon as characteristic polygon.](image)

### 7 Wallpaper groups

If a dimer model $G$ on the real 2-torus $M/\mathbb{R}$ is symmetric under the action of a finite subgroup $H$ of $\text{GL}(M) \ltimes (M/\mathbb{R})$, then we can think of the quotient graph $G/H$ on the 2-dimensional orbifold $M/(H \ltimes M)$. If $H$ contains a reflection or a glide reflection, then the graph $G/H$ is no longer bicolored and hence not a dimer model, but the associated quiver with relation still makes sense and can be drawn on the orbifold $(M/\mathbb{R})/H$. Dimer models and quivers on orbifolds are also discussed by Bocklandt [Boc13] under the name weighted quiver polyhedra, which are different from the ones appearing in this paper in that we allow reflections whereas he does not, and that we allow orbifold points to lie on dimer edges and dimer faces (i.e., quiver arrows and quiver vertices), whereas orbifold points in his theory lie only on dimer nodes (i.e., quiver faces).

A discrete subgroup $W$ of the Euclidean group $E(2) = O(2) \ltimes \mathbb{R}^2$ containing two linearly independent translations is called a wallpaper group or a plane crystallographic group. Wallpaper groups are classified into 17 classes by the diffeomorphism class of the orbifold quotient $\mathbb{R}^2/W$, and described by the orbifold notation as in Table 7.1 (cf. e.g. [CBGS08]).

| Notation | Description |
|----------|-------------|
| 0        | Translation of the fundamental domain |
| ×        | Line of glide reflection |
| *        | Line of reflection symmetry |
| $n$ after * | Point passing $n$ lines of reflection symmetries |
| $n$ before * | Center point of an order $n$ rotation symmetry |

#### Table 7.1: The orbifold notation

When a dimer model on $M/\mathbb{R}$ is symmetric under the action of a finite subgroup $H$ of $\text{GL}(M) \ltimes (M/\mathbb{R})$, we can take a $H$-invariant metric on $M$, so that the pull-back of the dimer model to the universal cover $\tilde{M}$ is invariant under a wallpaper group. Conversely, for each of 17 wallpaper groups, one can ask if there is a consistent dimer model whose group of symmetries is given by that group. The answer to this question is affirmative, and we give
an example of a consistent dimer model of each type in Figure 7.1 below. Note that the type of symmetry of a dimer model depends not only on the isomorphism class of the underlying abstract graph, or even the isotopy class of the embedding of the graph on the 2-torus, but also on the isometry class of the embedding. For example, in Figure 7.1, we see that 442, 442, and 4*2 are isotopic, but have different symmetries.

Figure 7.1: Examples of consistent dimer models for the 17 plane symmetry types

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