Exponential sums over Piatetski-Shapiro primes in arithmetic progressions

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Abstract

Let $\gamma < 1 < c$ and $19(c - 1) + 171(1 - \gamma) < 9$. In this paper, we establish an asymptotic formula for exponential sums over Piatetski-Shapiro primes $p = \lceil n^{1/\gamma} \rceil$ in arithmetic progressions.

Keywords: Piatetski-Shapiro prime · Asymptotic formula · Arithmetic progression

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1 Notations

Let $x$ be a sufficiently large positive number. The letter $p$ with or without subscript will always denote prime number. By $\delta$ we denote a fixed positive number, it can be chosen arbitrarily small; its value needs not be the same in all occurrences. As usual $\Lambda(n)$ denotes von Mangoldt’s function. We use $[t]$ and $\{t\}$ to denote the integer part, respectively, the fractional part of $t$. Moreover $e(y) = e^{2\pi iy}$ and $\psi(t) = \{t\} - 1/2$. We denote by $\tau_k(n)$ the number of solutions of the equation $m_1m_2\ldots m_k = n$ in natural numbers $m_1, \ldots, m_k$. Instead of $m \equiv n \pmod{k}$ we write for simplicity $m \equiv n(k)$. Throughout this paper unless something else is said, we suppose that $\gamma < 1 < c$ and $19(c - 1) + 171(1 - \gamma) < 9$. Define

\[ \pi(x, d, a, t, c) = \sum_{p \leq x \atop p \equiv a (d)} e(tp^c) ; \quad (1) \]

\[ \pi_\gamma(x, d, a, t, c) = \sum_{p \leq x \atop p = \lceil n^{1/\gamma} \rceil \atop p \equiv a (d)} e(tp^c) . \quad (2) \]
2 Introduction and statement of the result

Let \( \mathbb{P} \) denotes the set of all prime numbers. In 1953 Piatetski-Shapiro [6] has shown that for any fixed \( \frac{11}{12} < \gamma < 1 \) the set

\[ \mathbb{P}_\gamma = \{ p \in \mathbb{P} \mid p = \lfloor n^{1/\gamma} \rfloor \text{ for some } n \in \mathbb{N} \} \]

is infinite. The prime numbers of the form \( p = \lfloor n^{1/\gamma} \rfloor \) are called Piatetski-Shapiro primes of type \( \gamma \). Denote

\[ \pi_\gamma(x) = \sum_{p \leq x, \ p = \lfloor n^{1/\gamma} \rfloor} 1. \]

Piatetski-Shapiro’s result states that

\[ \pi_\gamma(x) = \frac{x^{\gamma}}{\log x} + \mathcal{O}\left( \frac{x^{\gamma}}{\log^2 x} \right) \]  \hspace{1cm} (3)

for \( \frac{11}{12} < \gamma < 1 \). The admissible range for \( \gamma \) in this theorem has been extended many times over the years, and the best results up to now belong to Rivat and Sargos [7] with (3) for \( \frac{2426}{2817} < \gamma < 1 \) and to Rivat and Wu [8] with

\[ \pi_\gamma(x) \gg \frac{x^{\gamma}}{\log x} \]

for \( \frac{293}{283} < \gamma < 1 \). On the other hand Siegel-Walfisz theorem is extremely important result in analytic number theory and plays a significant role in various applications. It is a refinement both of the prime number theorem and of Dirichlet’s theorem on primes in arithmetic progressions. It states that for any fixed \( A > 0 \) there exists a positive constant \( c \) depending only on \( A \) such that

\[ \sum_{n \leq x, \ (n, d) = 1} \Lambda(n) = \frac{x}{\varphi(d)} + \mathcal{O}\left( \frac{x}{e^{c \sqrt{\log x}}} \right), \]

whenever \( x \geq 2 \), \( (a, d) = 1 \), \( d \leq (\log x)^A \) and \( \varphi(n) \) is Euler’s function.

In 2013 Baker, Banks, Brüdern, Shparlinski and Weingartner [11] have considered for first time the distribution Piatetski-Shapiro primes in arithmetic progressions. They showed that when \( a \) and \( d \) are coprime integers, then for any fixed \( \frac{17}{18} < \gamma < 1 \) we have

\[ \sum_{p \leq x, \ p = \lfloor n^{1/\gamma} \rfloor} 1 = \gamma x^{\gamma - 1} \sum_{p \leq x, \ p = a (d)} \sum_{p \leq y} 1 + \gamma (1 - \gamma) \int_2^y y^{\gamma - 2} \sum_{p \leq y, \ p = a (d)} 1 dy + \mathcal{O}\left( x^{\frac{17}{18} + \frac{\gamma}{2} + \delta} \right). \]
In 2015 Guo [2] enlarged the range of $\gamma$ to $\frac{13}{14} < \gamma < 1$. Recently Guo, Li and Zhang [3] achieved the best result up to now with $\frac{11}{12} < \gamma < 1$. Motivated by these results we establish an asymptotic formula for exponential sums over Piatetski-Shapiro primes in arithmetic progressions. More precisely we prove the following theorem.

**Theorem 1.** Let $a$ and $d$ be coprime integers, $d \geq 1$. Assume that $|t| \leq x^\delta$ with a sufficiently small $\delta > 0$ and $\gamma < 1 < c$ which satisfy

$$\frac{19}{9}(c - 1) + 19(1 - \gamma) < 1. \quad (5)$$

Then the sums (1) and (2) are connected by the asymptotic formula

$$\pi_\gamma(x, d, a, t, c) = \gamma x^{\gamma - 1} \pi(x, d, a, t, c) + \gamma(1 - \gamma) \int_2^x y^{\gamma - 2} \pi(y, d, a, t, c) \, dy + O \left( x^{\frac{c}{18} + \frac{13}{342} + \delta} \right). \quad (6)$$

### 3 Preliminary lemmas

**Lemma 1.** For every $H \geq 1$, we have

$$\psi(t) = \sum_{1 \leq |h| \leq H} a(h)e(ht) + O \left( \sum_{|h| \leq H} b(h)e(ht) \right),$$

where

$$a(h) \ll \frac{1}{|h|}, \quad b(h) \ll \frac{1}{H}. \quad (7)$$

**Proof.** See [12].

**Lemma 2.** Suppose that $f''(t)$ exists, is continuous on $[a, b]$ and satisfies

$$f''(t) \asymp \lambda \quad (\lambda > 0) \quad \text{for} \quad t \in [a, b].$$

Then

$$\left| \sum_{a < n \leq b} e(f(n)) \right| \ll (b - a)\lambda^{\frac{1}{2}} + \lambda^{-\frac{1}{2}}.$$

**Proof.** See ([11], Theorem 5.9).
Lemma 3. Suppose that \( f'''(t) \) exists, is continuous on \([a,b]\) and satisfies
\[
f'''(t) \asymp \lambda \quad (\lambda > 0) \quad \text{for} \quad t \in [a,b].
\]
Then
\[
\left| \sum_{a< n \leq b} e(f(n)) \right| \ll (b - a)\frac{\lambda^\frac{1}{3}}{3} + \lambda^{\frac{1}{2}}.
\]

Proof. See ([9], Corollary 4.2). \( \square \)

Lemma 4. Let \( I \) be a subinterval of \((X, 2X]\). For any complex numbers \( z(n) \) we have
\[
\left| \sum_{n \in I} z(n) \right|^2 \leq \left( 1 + \frac{X}{Q} \right) \sum_{|q| < Q} \left( 1 - \frac{|q|}{Q} \right) \sum_{n, n+q \in I} z(n+q)\overline{z(n)},
\]
where \( Q \geq 1 \).

Proof. See ([5], Lemma 5). \( \square \)

Lemma 5. Let \( G(n) \) be a complex valued function. Assume further that
\[
P > 2, \quad P_1 \leq 2P, \quad 2 \leq U < V \leq Z \leq P,
\]
\[
U^2 \leq Z, \quad 128UZ^2 \leq P_1, \quad 2^{18}P_1 \leq V^3.
\]

Then the sum
\[
\sum_{P < n \leq P_1} \Lambda(n)G(n)
\]

can be decomposed into \( O \left( \log^6 P \right) \) sums, each of which is either of Type I
\[
\sum_{M < m \leq M_1} a(m) \sum_{P < ml \leq P_1} G(ml)
\]

and
\[
\sum_{M < m \leq M_1} a(m) \sum_{P < ml \leq P_1} G(ml) \log l,
\]
where
\[
L \geq Z, \quad M_1 \leq 2M, \quad L_1 \leq 2L, \quad a(m) \ll \tau_5(m) \log P
\]
or of Type II
\[
\sum_{M < m \leq M_1} a(m) \sum_{P < ml \leq P_1} b(l)G(ml)
\]
where
\[
U \leq L \leq V, \quad M_1 \leq 2M, \quad L_1 \leq 2L, \quad a(m) \ll \tau_5(m) \log P, \quad b(l) \ll \tau_5(l) \log P.
\]
Proof. See \([10]\), Lemma 3. \qed

Lemma 6. Let

\[ L(H) = \sum_{i=1}^{m} A_i H^{\alpha_i} + \sum_{j=1}^{n} B_j H^{-b_j}, \]

where \( A_i, B_j, \alpha_i \) and \( b_j \) are positive. Assume further that \( H_1 \leq H_2 \). Then there exists \( H_0 \in [H_1, H_2] \) such that

\[ L(H_0) \ll \sum_{i=1}^{m} A_i H_1^{\alpha_i} + \sum_{j=1}^{n} B_j H_2^{-b_j} + \sum_{i=1}^{m} \sum_{j=1}^{n} (A_i B_j^{\alpha_i})^{1/(\alpha_i + b_j)}. \]

Proof. See \([10]\), Lemma 3. \qed

4 Proof of the theorem

From (2) we have

\[ \pi(\gamma, d, a, t, c) = \sum_{p \leq x \atop p \equiv a \mod{(d)}} \left( [-p\gamma] - [-(p+1)\gamma] \right) e(tp^c) = \Gamma_1(x) + \Gamma_2(x), \]

where

\[ \Gamma_1(x) = \sum_{p \leq x \atop p \equiv a \mod{(d)}} \left( (p+1)^\gamma - p^\gamma \right) e(tp^c), \]

\[ \Gamma_2(x) = \sum_{p \leq x \atop p \equiv a \mod{(d)}} \left( \psi(-(p+1)^\gamma) - \psi(-p^\gamma) \right) e(tp^c). \]

4.1 Asymptotic formula for \( \Gamma_1(x) \)

Using \((9)\), the well-known asymptotic formula

\[ (p+1)^\gamma - p^\gamma = \gamma p^{\gamma-1} + \mathcal{O}(p^{\gamma-2}) \]

and Abel’s summation formula we write
\[ \Gamma_1(x) = \gamma \sum_{\substack{p \leq x \\, p \equiv a \pmod{d}}} p^{\gamma-1} e(tp^c) + O(1) \]

\[ = \gamma x^{\gamma-1} \pi(x, d, a, t, c) + \gamma(1 - \gamma) \int_2^x \left( \sum_{\substack{p \leq y \\, p \equiv a \pmod{d}}} e(tp^c) \right) y^{\gamma-2} dy + O(1) \]

\[ = \gamma x^{\gamma-1} \pi(x, d, a, t, c) + \gamma(1 - \gamma) \int_2^x y^{\gamma-2} \pi(y, d, a, t, c) dy + O(1). \quad (11) \]

### 4.2 Upper bound for \( \Gamma_2(x) \)

Define

\[ \Gamma_3(x) = \sum_{\substack{p \leq x \\, p \equiv a \pmod{d}}} (\log p)(\psi(-(p + 1)^\gamma) - \psi(-p^\gamma)) e(tp^c), \quad (12) \]

\[ \Gamma_4(x) = \sum_{\substack{n \leq x \\, n \equiv a \pmod{d}}} \Lambda(n)(\psi(-(n + 1)^\gamma) - \psi(-n^\gamma)) e(tn^c). \quad (13) \]

On the one hand (10), (12) and Abel’s summation formula yield

\[ \Gamma_2(x) = \frac{\Gamma_3(x)}{\log x} + \int_2^x \frac{\Gamma_3(y)}{y \log^2 y} dy. \quad (14) \]

On the other hand (12) and (13) give us

\[ \Gamma_3(x) = \Gamma_4(x) + O(\sqrt{x}). \quad (15) \]

Splitting the range of \( n \) into dyadic subintervals of the form \((x/2, x] \) from (13) we get

\[ \Gamma_4(x) \ll (\log x)\mid \Gamma_5(x) \mid, \quad (16) \]

where

\[ \Gamma_5(x) = \sum_{\substack{x/2 < n \leq x \\, n \equiv a \pmod{d}}} \Lambda(n)(\psi(-(n + 1)^\gamma) - \psi(-n^\gamma)) e(tn^c). \quad (17) \]

Using (17) and Lemma 1 we obtain

\[ \Gamma_5(x) = \Gamma_6(x) + \Gamma_7(x) + \Gamma_8(x), \quad (18) \]
where
\[
\begin{align*}
\Gamma_6(x) &= \sum_{x/2 < n \leq x \atop n \equiv a (d)} \Lambda(n) \sum_{1 \leq |h| \leq H} a(h) \left( e(-h(n + 1)\gamma) - e(-hn\gamma) \right) e(tn^c), \\
\Gamma_7(x) &\ll \sum_{x/2 < n \leq x \atop n \equiv a (d)} \Lambda(n) \sum_{|h| \leq H} b(h)e(-hn\gamma), \\
\Gamma_8(x) &\ll \sum_{x/2 < n \leq x \atop n \equiv a (d)} \Lambda(n) \sum_{|h| \leq H} b(h)e(-h(n + 1)\gamma).
\end{align*}
\] (19)

Upper bound for \(\Gamma_6(x)\)

By (7) and (19) we deduce
\[
\Gamma_6(x) \ll \sum_{1 \leq |h| \leq H} \frac{1}{|h|} \left| \sum_{x/2 < n \leq x \atop n \equiv a (d)} \Lambda(n) \Phi_h(n) e(tn^c - hn\gamma) \right|,
\] (22)

where
\[
\Phi_h(y) = e(hy\gamma - h(y + 1)^\gamma) - 1.
\]

Bearing in mind the estimates
\[
\Phi_h(y) \ll |h|y^{\gamma - 1}, \quad \Phi'_h(y) \ll |h|y^{\gamma - 2}
\]

and using Abel’s summation formula from (22) we derive
\[
\Gamma_6(x) \ll \sum_{1 \leq |h| \leq H} \frac{1}{|h|} \left| \Phi_h(x) \sum_{x/2 < n \leq x \atop n \equiv a (d)} \Lambda(n) e(tn^c + hn\gamma) \right|
\]
\[
+ \sum_{1 \leq |h| \leq H} \frac{1}{|h|} \int_{x/2}^{x} \left| \Phi_h(y) \sum_{x/2 < n \leq y \atop n \equiv a (d)} \Lambda(n) e(tn^c + hn\gamma) \right| dy
\]
\[
\ll x^{\gamma - 1} |\Gamma_9(x_1)|,
\] (23)

where
\[
\Gamma_9(x_1) = \sum_{1 \leq |h| \leq H} \left| \sum_{x/2 < n \leq x_1 \atop n \equiv a (d)} \Lambda(n) e(tn^c + hn\gamma) \right|
\] (24)

for some number \(x_1 \in (x/2, x]\). Now the well-known formula
\[
\sum_{k=1}^{d} e\left( \frac{kn}{d} \right) = \begin{cases} 
  d, & \text{if } d \mid n, \\
  0, & \text{if } d \nmid n
\end{cases}
\]
and (24) lead to

\[ \Gamma_9(x_1) = \sum_{1 \leq |h| \leq H} \left| \frac{1}{d} \sum_{k=1}^{d} \sum_{x/2 < n \leq x_1} \Lambda(n) e \left( \frac{tn^c + h\gamma + \left(\frac{n-a}{d}\right)}{d} \right) \right|. \]  

(25)

From (25) we understand that it is sufficient to estimate the sum

\[ \Gamma_{10}(x_1) = \sum_{1 \leq |h| \leq H} \left| \sum_{x/2 < n \leq x_1} \Lambda(n) e \left( \frac{tn^c + h\gamma + kd^{-1}n}{d} \right) \right|. \]  

(26)

Lemma 7. Let \( x_1 \leq x, \ 0 < \gamma < 1 < c \) and \( |t| \leq x^{\gamma-c-\delta} \) for a sufficiently small \( \delta > 0 \). Then for the sum (26) we have

\[ \Gamma_{10}(x_1) \ll x^\delta \left( H_x e^{\frac{\gamma}{6}+\frac{1}{3}} + H_x e^{\frac{\gamma}{5}+\frac{2}{3}} + H_x e^{\frac{\gamma}{5}+\frac{5}{3}} + H_x x^{-\frac{7}{4}+\frac{2}{5}} \right). \]

Proof. It follows be the same arguments used in (3), pp. 10–11, since under the hypotheses on \( t \), we have

\[ \left| \frac{d^j}{dy^j} \left( ty^c + hy^\gamma + kd^{-1}y \right) \right| \approx \left| \frac{d^j}{dy^j} (hy^\gamma + kd^{-1}y) \right| \]

for \( j = 2,3 \) whenever \( 1 \leq h \leq H \) and \( x/2 < y \leq x \).

Lemma 8. Set

\[ S_I = \sum_{1 \leq |h| \leq H} \left| \sum_{M < m \leq M_1} \sum_{L \leq \ell \leq L_1} a(m) e\left( tm^c \ell^c + hm^\gamma \ell^\gamma + \theta ml \right) \right|, \]  

(27)

and

\[ S'_I = \sum_{1 \leq |h| \leq H} \left| \sum_{M < m \leq M_1} \sum_{L \leq \ell \leq L_1} a(m) e\left( tm^c \ell^c + hm^\gamma \ell^\gamma + \theta ml \right) \log \ell \right|, \]  

(28)

where

\[ L \geq x^{\frac{38c+115}{192}}, \quad M_1 \leq 2M, \quad L_1 \leq 2L, \quad 0 < \gamma < 1 < c < \frac{28}{19}, \]  

(29)

\[ a(m) \ll \tau_5(m) \log x, \quad 0 \leq \theta \leq 1, \quad x_1 \leq x, \quad x^{\gamma-c-\delta} \leq |t| \leq x^\delta, \]

with a sufficiently small \( \delta > 0 \). Then

\[ S_I, S'_I \ll x^\delta \left( H_x e^{\frac{\gamma}{6}+\frac{1}{3}} + H_x e^{\frac{\gamma}{5}+\frac{2}{3}} + H_x e^{\frac{\gamma}{5}+\frac{5}{3}} + H_x x^{-\frac{7}{4}+\frac{2}{5}} \right). \]
Proof. First we notice that (27) and (29) imply

$$ML \asymp x.$$  \hfill (30)

Denote

$$f(l) = tm^c l^c + hm^\gamma l^{\gamma} + \theta ml.$$  \hfill (31)

By (27), (29) and (31) we write

$$S_I \ll x^{\frac{1}{4}} \sum_{1 \leq |h| \leq M} \sum_{M < m \leq M_1} \sum_{L' < l \leq L'_1} e(f(l)),$$  \hfill (32)

where

$$L' = \max \left\{ L, \frac{N}{m} \right\}, \quad L'_1 = \min \left\{ L_1, \frac{N_1}{m} \right\}.$$  \hfill (33)

From (29), (31) and (33) we have

$$f'''(l) = c(c - 1)(c - 2)tm^c l^{c - 3} + \gamma(\gamma - 1)(\gamma - 2)hm^\gamma l^{\gamma - 3} \asymp |t|M^c l^{c - 3} + |h|M^\gamma l^{\gamma - 3}.$$  \hfill (34)

Now (29), (30), (32), (34) and Lemma \ref{lemma} yield

$$S_I \ll x^{\frac{1}{4}} \sum_{1 \leq |h| \leq H} \sum_{M < m \leq M_1} \left( |t|^\frac{1}{2} M^\frac{c}{2} + |h|^\frac{1}{2} M^\gamma M^\frac{\gamma - 3}{2} + |t|^{-\frac{3}{2}} M^\gamma L^{\frac{1}{2} - \frac{1}{2}} + |h|^{-\frac{3}{2}} M^\gamma L^{\frac{1}{2} - \frac{1}{2}} \right) \ll x^{\frac{1}{4}} \left( H |t|^\frac{1}{2} x^\frac{c}{2} + H^\frac{\gamma}{2} x^\frac{\gamma}{2} + H^\frac{c}{2} x^\frac{c}{2} + H |t|^{-\frac{3}{2}} x^{\frac{\gamma - 3}{2}} + H^\frac{\gamma}{2} x^{\frac{\gamma - 3}{2}} \right) \ll x^{\frac{1}{4}} \left( H x^\frac{c}{2} + H |t|^\frac{1}{2} x^\frac{c}{2} + H^\frac{\gamma}{2} x^\frac{\gamma}{2} + H x^{-\frac{3}{2}} + H^\frac{\gamma}{2} x^{-\frac{3}{2}} \right).$$

To estimate the sum defined by (28) we apply Abel’s summation formula and proceed in the same way to deduce

$$S'_I \ll x^{\frac{1}{4}} \left( H x^\frac{c}{2} + H^\frac{\gamma}{2} x^\frac{\gamma}{2} \right).$$

This proves the lemma. \hfill \qed
Lemma 9. Set
\[
S_{II} = \sum_{1 \leq |h| \leq H} \left| \sum_{M < m \leq M_1} \sum_{L < l \leq L_1} a(m) b(l) e\left(tm^\gamma l^\delta + hm^\gamma l^\gamma + \theta ml\right) \right|, \tag{35}
\]

where
\[
2^{-10} x^{\frac{56-384}{41}} \leq L \leq 2^7 x^{\frac{1}{4}}, \quad M_1 \leq 2M, \quad L_1 \leq 2L, \quad 0 < \gamma < 1 < c < \frac{28}{19}, \tag{36}
\]
a(m) \ll \tau_5(m) \log x, \quad b(l) \ll \tau_5(l) \log x, \quad 0 \leq \theta \leq 1, \quad 1 \leq x, \quad x^{\gamma - c - \delta} \leq |t| \leq x^\delta,

with a sufficiently small \( \delta > 0 \). Then
\[
S_{II} \ll x^\delta \left(Hx^{\frac{7}{11}} + H^2x^{\frac{4}{11} + \frac{7}{15}} + Hx^{\frac{9}{11} + \frac{4}{15}} + Hx^{1 - \frac{7}{5}} + Hx^{\frac{4}{11} + \frac{13}{15}} + Hx^{\frac{7}{11} + \frac{13}{15}} + H^{\frac{9}{11} \frac{7}{15}} \right).
\]

Proof. First we notice that (35) and (36) give us
\[
ML \asymp x. \tag{37}
\]
From (31), (36), (37), Cauchy’s inequality and Lemma 4 with
\[
1 \leq Q \leq L \tag{38}
\]
it follows
\[
\left| \sum_{M < m \leq M_1} \sum_{L < l \leq L_1} a(m) b(l) e\left(tm^\gamma l^\delta + hm^\gamma l^\gamma + \theta ml\right) \right|^2 
\ll \sum_{M < m \leq M_1} |a(m)|^2 \sum_{M < m \leq M_1} \left| \sum_{L < l \leq L_1} b(l) e\left(f(l)\right) \right|^2
\ll M^{1 + \frac{\delta}{2}} \sum_{M < m \leq M_1} \frac{L}{Q} \sum_{|q| < Q} \left(1 - \frac{q}{Q}\right) \sum_{L < l, l + q \leq L_1} \sum_{x/2 < m \leq x_1} b(l + q) b(l) e\left(f(l + q) - f(l)\right)
\ll \frac{LM^{1 + \frac{\delta}{2}}}{Q} \sum_{M < m \leq M_1} \left(L^{1 + \frac{\delta}{2}} + \sum_{1 \leq |q| < Q} \left(1 - \frac{q}{Q}\right) \sum_{L < l, l + q \leq L_1} \sum_{x < m \leq x_1} b(l + q) b(l) e\left(f(l + q) - f(l)\right) \right)
\ll x^\delta \left(\frac{x^2}{Q} + \frac{x}{Q} \sum_{1 \leq |q| < Q} \sum_{L < l, l + q \leq L_1} \left| \sum_{M' < m \leq M'_1} e\left(g(m)\right) \right| \right), \tag{39}
\]

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where
\[ M' = \max \left\{ M, \frac{x}{l} \right\}, \quad M'_1 = \min \left\{ M_1, \frac{x_1}{l} \right\} \]  \hspace{1cm} (40)
and
\[ g(m) = tm^c(l + q)^c - tm^c l^c + hm^\gamma(l + q)^\gamma - hm^\gamma l^\gamma + \theta mq. \]  \hspace{1cm} (41)
By (36), (40) and (41) we have
\[ g''(m) \asymp |t|M^{c-2}L^{c-1}|q| + |h|M^{\gamma-2}L^{\gamma-1}|q|. \]  \hspace{1cm} (42)
Now (36), (37), (39), (42) and Lemma 2 lead to
\[ \left| \sum_{M < m \leq M_1} \sum_{x/2 < ml \leq x_1} a(m) \sum_{L < t \leq L_1} b(l)e(tm^c l^c + hm^\gamma l^\gamma + \theta ml) \right|^2 \leq x^\frac{\delta}{Q} \left( \frac{x^2}{Q} + \frac{x}{Q} \sum_{1 \leq |q| \leq Q} \sum_{L < t \leq L_1} M \left( |t|M^{c-2}L^{c-1}|q| + |h|M^{\gamma-2}L^{\gamma-1}|q| \right)^\frac{1}{2} \right.
+ \left. \left( |t|M^{c-2}L^{c-1}|q| + |h|M^{\gamma-2}L^{\gamma-1}|q| \right)^{-\frac{1}{2}} \right)
\leq x^\frac{\delta}{Q} \left( \frac{x^2}{Q} + \frac{x}{Q} \sum_{1 \leq |q| \leq Q} \sum_{L < t \leq L_1} |t|^\frac{1}{2} M^{\frac{c}{2}} L^{\frac{c-1}{2}} |q|^\frac{1}{2} + |h|^\frac{1}{2} M^{\frac{\gamma}{2}} L^{\frac{\gamma-1}{2}} |q|^\frac{1}{2} \right.
\left. + |t|^{-\frac{1}{2}} M^{1-\frac{c}{2}} L^{\frac{c-1}{2}} |q|^{-\frac{1}{2}} + |h|^{-\frac{1}{2}} M^{1-\frac{\gamma}{2}} L^{\frac{\gamma-1}{2}} |q|^{-\frac{1}{2}} \right)
\leq x^{\frac{\delta}{Q}} \left( x^2 Q^{-1} + x^{1+\frac{\delta}{2}} |t|^\frac{1}{2} L^{\frac{1}{2}} Q^{\frac{1}{2}} + x^{1+\frac{\delta}{2}} L^{\frac{1}{2}} Q^\frac{1}{2} \right.
\left. + x^{2-\frac{\delta}{2}} |t|^{-\frac{1}{2}} L^{\frac{1}{2}} Q^{-\frac{1}{2}} + x^{2-\frac{\delta}{2}} L^{\frac{1}{2}} Q^{-\frac{1}{2}} \right). \]  \hspace{1cm} (43)
From (38), (43) and Lemma 6 it follows that there exists an optimal \( Q \) such that
\[ \left| \sum_{M < m \leq M_1} \sum_{x/2 < ml \leq x_1} a(m) \sum_{L < t \leq L_1} b(l)e(tm^c l^c + hm^\gamma l^\gamma + \theta ml) \right|^2 \leq x^{\frac{\delta}{Q}} \left( x^{1+\frac{\delta}{2}} |t|^\frac{1}{2} L^{\frac{1}{2}} + x^{1+\frac{\delta}{2}} L^{\frac{1}{2}} + x^{2-\frac{\delta}{2}} |t|^{-\frac{1}{2}} + x^{2-\frac{\delta}{2}} L^{\frac{1}{2}} + x^{3+\frac{\delta}{2}} |t|^\frac{1}{2} L^{\frac{1}{2}} + x^{3+\frac{\delta}{2}} L^{\frac{1}{2}} + x^{2-\frac{\delta}{2}} |t|^{-\frac{1}{2}} |h| \frac{1}{2} L^{\frac{1}{2}} + x^{2-\frac{\delta}{2}} L^{\frac{1}{2}} |h|^{-\frac{1}{2}} L^{\frac{1}{2}} \right). \]
Therefore
\[
\left| \sum_{M < m \leq M_1} a(m) \sum_{L_t \leq L_1} b(l)e\left(tm^c l^c + hm^\gamma l^\gamma + \theta ml\right) \right| \ll x^{\frac{d}{4}} \left(x^{\frac{3}{2} + \frac{\gamma}{2}} |t| \frac{1}{x^{\frac{3}{2} + \frac{\gamma}{2}}} + x^{\frac{3}{2} + \frac{\gamma}{2}} |h| \frac{1}{x^{\frac{3}{2} + \frac{\gamma}{2}}} + xL^{-\frac{1}{2}} + x^{1 - \frac{\gamma}{4}} |t|^{-\frac{1}{2}} + x^{1 - \frac{\gamma}{4}} |h|^{-\frac{1}{2}} + x^{\frac{3}{4} + \frac{\gamma}{4}} |t|^{\frac{3}{2}} L^{\frac{1}{2}} + x^{\frac{3}{4} + \frac{\gamma}{4}} |h|^{\frac{1}{2}} L^{\frac{1}{2}} + x^{\frac{3}{4} + \frac{\gamma}{4}} + x^{\frac{3}{4} + \frac{\gamma}{4}} |t|^{\frac{1}{2}} |h|^{-\frac{1}{2}} L^{\frac{1}{2}} \right)
\]

Bearing in mind (35), (36) and (44) we derive

\[
S_{11} \ll x^{\frac{d}{4}} \sum_{1 \leq |h| \leq H} \left( x^{\frac{3}{2} + \frac{\gamma}{2}} |t| \frac{1}{x^{\frac{3}{2} + \frac{\gamma}{2}}} + x^{\frac{3}{2} + \frac{\gamma}{2}} |h| \frac{1}{x^{\frac{3}{2} + \frac{\gamma}{2}}} + xL^{-\frac{1}{2}} + x^{1 - \frac{\gamma}{4}} |t|^{-\frac{1}{2}} + x^{1 - \frac{\gamma}{4}} |h|^{-\frac{1}{2}} \right)
\]

This proves the lemma.

It remains to apply Heath-Brown’s identity to the sum \(\Gamma_{10}(x_1)\).

**Lemma 10.** Let \(x_1 \leq x\), \(0 < \gamma < 1 < c < \frac{28}{19}\) and \(x^{\gamma - c - \delta} \leq |t| \leq x^{\delta}\) with a sufficiently small \(\delta > 0\). Then for the sum (26) we have

\[
\Gamma_{10}(x_1) \ll x^{\frac{d}{4}} \left( H x^{\frac{3}{2} + \frac{\gamma}{2}} + H x^{\frac{3}{2} + \frac{\gamma}{2}} + H x^\frac{11}{4} + H x^{1 - \frac{\gamma}{4}} + H x^{1 - \frac{\gamma}{4}} + H x^\frac{9}{4} + H x^\frac{9}{4} + H x^\frac{9}{4} + H x^\frac{9}{4} + H x^\frac{9}{4} + H x^\frac{9}{4} \right).
\]

**Proof.** Take

\[
U = 2^{-10} x^{\frac{56 - 38c}{171}}, \quad V = 2^7 x^{\frac{1}{4}}, \quad Z = x^{\frac{38c + 115}{342}}.
\]
According to Lemma 5, the sum $\Gamma_{10}(x_1)$ can be decomposed into $O\left(\log^6 x\right)$ sums, each of which is either of Type I

$$S_I = \sum_{1 \leq |h| \leq H} \left| \sum_{M < m \leq M_1 \atop x/2 < ml \leq x_1} a(m) \sum_{L < l \leq L_1} e(t m^c l^c + h m^\gamma l^\gamma + k d^{-1} ml) \right|,$$

and

$$S'_I = \sum_{1 \leq |h| \leq H} \left| \sum_{M < m \leq M_1 \atop x/2 < ml \leq x_1} a(m) \sum_{L < l \leq L_1} e(t m^c l^c + h m^\gamma l^\gamma + k d^{-1} ml) \log l \right|,$$

where

$$L \geq Z, \quad M_1 \leq 2M, \quad L_1 \leq 2L, \quad a(m) \ll \tau_5(m) \log x$$

or of Type II

$$S_{II} = \sum_{1 \leq |h| \leq H} \left| \sum_{M < m \leq M_1 \atop x/2 < ml \leq x_1} a(m) \sum_{L < l \leq L_1} b(l) e(t m^c l^c + h m^\gamma l^\gamma + k d^{-1} ml) \right|,$$

where

$$U \leq L \leq V, \quad M_1 \leq 2M, \quad L_1 \leq 2L, \quad a(m) \ll \tau_5(m) \log x, \quad b(l) \ll \tau_5(l) \log x.$$

Bearing in mind (45), Lemma 8 and Lemma 9 we establish the statement in the lemma. \qed

**Lemma 11.** Let $x_1 \leq x$, $0 < \gamma < 1 < c < 28 \over 19$ and $|t| \leq x^\delta$ with a sufficiently small $\delta > 0$. Then for the sum (26) we have

$$\Gamma_{10}(x_1) \ll x^\delta \left( H x^{\delta + \gamma} x^{\gamma + \gamma - {28 \over 19}} + H x^{\delta + \gamma - {28 \over 19}} + H x^{\delta + \gamma + {13 \over 18}} + H x^{\delta + {13 \over 18}} + H x^{\delta + {13 \over 18}} \right).$$

**Proof.** It follows immediately from Lemma 7 and Lemma 10. \qed

Now (23), (25), (26) and Lemma 11 imply

$$\Gamma_6(x) \ll x^\delta \left( H x^{\delta + \gamma} x^{\gamma + \gamma - {28 \over 19}} + H x^{\delta + \gamma - {28 \over 19}} + H x^{\delta + \gamma} x^{\gamma + {13 \over 18}} + H x^{\delta + {13 \over 18}} + H x^{\delta + \gamma} x^{\gamma + {13 \over 18}} + H x^{\delta + {13 \over 18}} \right). \quad (46)$$
Upper bound for $\Gamma_7(x)$

By (4), (7) and (20) we deduce

$$\Gamma_7(x) \ll |b_0| \sum_{\substack{x/2 < n \leq x \
 n \equiv a (d)}} \Lambda(n) + \sum_{1 \leq |h| \leq H} |b(h)| \sum_{\substack{x/2 < n \leq x \
 n \equiv a (d)}} \Lambda(n)e(-hn\gamma) \right|$$

$$\ll \frac{x}{H\varphi(d)} + \frac{1}{H} \sum_{1 \leq |h| \leq H} \sum_{\substack{x/2 < n \leq x \
 n \equiv a (d)}} \Lambda(n)e(-hn\gamma) \right|$$

$$\ll H^{-1}x + H^{-1}\Gamma_{11}(x),$$  \hspace{1cm} (47)

where

$$\Gamma_{11}(x) = \sum_{1 \leq |h| \leq H} \sum_{\substack{x/2 < n \leq x \
 n \equiv a (d)}} \Lambda(n)e(-hn\gamma) \right|.$$ 

Estimating the sum $\Gamma_{11}(x)$ as in ([3], (3.7)) we obtain

$$\Gamma_{11}(x) \ll x^{\frac{6}{7}} \left( H^{\frac{6}{7}} x^{\frac{2}{7} + \frac{3}{8}} + H^{\frac{1}{7}} x^{\frac{2}{7} + \frac{3}{8}} + H^{-\frac{1}{3}} x^{1 - \frac{7}{12}} + x^{\frac{22}{25}} \right).$$ \hspace{1cm} (48)

Now (17) and (18) give us

$$\Gamma_7(x) \ll x^{\frac{6}{7}} \left( H^{-1}x + H^{\frac{1}{7}} x^{\frac{2}{7} + \frac{3}{8}} + H^{\frac{1}{7}} x^{\frac{2}{7} + \frac{3}{8}} + H^{-\frac{1}{3}} x^{1 - \frac{7}{12}} + x^{\frac{22}{25}} \right).$$ \hspace{1cm} (49)

Upper bound for $\Gamma_8(x)$

In the same way for the sum defined by (21) we get

$$\Gamma_8(x) \ll x^{\frac{6}{7}} \left( H^{-1}x + H^{\frac{1}{7}} x^{\frac{2}{7} + \frac{3}{8}} + H^{\frac{1}{7}} x^{\frac{2}{7} + \frac{3}{8}} + H^{-\frac{1}{3}} x^{1 - \frac{7}{12}} + x^{\frac{22}{25}} \right).$$ \hspace{1cm} (50)

From (18), (16), (49) and (50) we derive

$$\Gamma_5(x) \ll x^{\frac{6}{7}} \left( H x^{\frac{6}{7} + \gamma - \frac{6}{12}} + H^{\frac{6}{7}} x^{\frac{2}{7} + \frac{3}{8}} + H^{\frac{6}{7}} x^{\frac{2}{7} - \frac{25}{171}} + H^{\frac{6}{7}} x^{\frac{7}{6} + \gamma - \frac{5}{18}} + H^{\frac{6}{7}} x^{\frac{7}{6} - \frac{5}{6}} \right)$$

$$+ H^{\frac{6}{7}} x^{\frac{7}{6} - 2} + H^{\frac{7}{6}} x^{\frac{7}{6} - 2} + H^{\frac{7}{6}} x^{\frac{7}{6} - 2} + H^{\frac{7}{6}} x^{\frac{7}{6} - 2} + H^{\frac{7}{6}} x^{\frac{7}{6} - 2} + H^{\frac{7}{6}} x^{\frac{7}{6} - 2}$$

$$+ H^{-1}x + x^{\frac{22}{25}} \right) \hspace{1cm} (51)$$

Taking into account that (51) holds for any real $H \geq 1$ and using Lemma 3 we write

$$\Gamma_5(x) \ll x^{\frac{6}{7}} \left( x^{\frac{6}{7} + \gamma - \frac{5}{12}} + x^{\frac{6}{7} - \frac{3}{8}} + x^{\frac{6}{7} + \gamma - \frac{25}{171}} + x^{\frac{6}{7} + \gamma - \frac{5}{18}} + x^{\frac{7}{6} - \frac{5}{6}} + x^{\frac{7}{6} - \frac{5}{6}} + x^{\frac{7}{6} - \frac{5}{6}} + x^{\frac{7}{6} - \frac{5}{6}} + x^{\frac{7}{6} - \frac{5}{6}} + x^{\frac{7}{6} - \frac{5}{6}} + x^{\frac{7}{6} - \frac{5}{6}} + x^{\frac{7}{6} - \frac{5}{6}} + x^{\frac{7}{6} - \frac{5}{6}} + x^{\frac{7}{6} - \frac{5}{6}} + x^{\frac{7}{6} - \frac{5}{6}} + x^{\frac{7}{6} - \frac{5}{6}} + x^{\frac{7}{6} - \frac{5}{6}} + x^{\frac{7}{6} - \frac{5}{6}} + x^{\frac{7}{6} - \frac{5}{6}} + x^{\frac{7}{6} - \frac{5}{6}} + x^{\frac{7}{6} - \frac{5}{6}} + x^{\frac{7}{6} - \frac{5}{6}} + x^{\frac{7}{6} - \frac{5}{6}} + x^{\frac{7}{6} - \frac{5}{6}} + x^{\frac{7}{6} - \frac{5}{6}} + x^{\frac{7}{6} - \frac{5}{6}} + x^{\frac{7}{6} - \frac{5}{6}} + x^{\frac{7}{6} - \frac{5}{6}} + x^{\frac{7}{6} - \frac{5}{6}} + x^{\frac{7}{6} - \frac{5}{6}} + x^{\frac{7}{6} - \frac{5}{6}} + x^{\frac{7}{6} - \frac{5}{6}} + x^{\frac{7}{6} - \frac{5}{6}} + x^{\frac{7}{6} - \frac{5}{6}} + x^{\frac{7}{6} - \frac{5}{6}} + x^{\frac{7}{6} - \frac{5}{6}} + x^{\frac{7}{6} - \frac{5}{6}} + x^{\frac{7}{6} - \frac{5}{6}} + x^{\frac{7}{6} - \frac{5}{6}} \right).$$ \hspace{1cm} (52)
Using (5) and (52) we find
\[ \Gamma_5(x) \ll x^{\frac{18}{41} + \frac{3}{4} + \frac{143}{342} + \delta}. \]  \hspace{1cm} (53)

Bearing in mind (14), (15), (16) and (53) we deduce
\[ \Gamma_2(x) \ll x^{\frac{18}{41} + \frac{3}{4} + \frac{143}{342} + \delta}. \]  \hspace{1cm} (54)

It is easy to see that (5) and (54) yield
\[ \Gamma_2(x) \ll x^{\gamma - \varepsilon}, \]
for some \( \varepsilon > 0. \)

\section*{4.3 The end of the proof}

Summarizing (8), (11) and (54) we establish the asymptotic formula (6).

This completes the proof of Theorem 1.

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