A CORRECTION TO THE PAPER ‘A NEW APPROACH TO THE REPRESENTATION THEORY OF THE SYMMETRIC GROUPS, III’

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Abstract. The aim of this paper is to give a corrected bijective proof of Vershik’s relations for the Kostka numbers. Our proof uses insertion and reverse insertion algorithms, as in the combinatorial proof of the Pieri rule.

1. Introduction

The aim of this paper is to give a corrected bijective proof of [4, Theorem 4]. First of all, we recall some notations and definitions. We write $\lambda \vdash n$ if $\lambda$ is a composition of $n$, that is, a sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_h)$ of nonnegative integers such that $|\lambda| = \sum_{i=1}^{h} \lambda_i = n$. In particular, if a sequence $\lambda$ is non-increasing and $\lambda_i > 0$ for all $1 \leq i \leq h$, then we write $\lambda \vdash n$ and say that $\lambda$ is a partition of $n$. We denote by $\lambda^{(i)}$ the composition of $n-1$ defined by $\lambda^{(i)}_i = \lambda_i - 1$, and $\lambda^{(j)}_j = \lambda_j$ otherwise. For $\lambda = (\lambda_1, \ldots, \lambda_h) \vdash n$ and $\gamma \vdash n-1$, we write $\gamma \leq \lambda$ if $\gamma_i \leq \lambda_i$ for all $i$ with $1 \leq i \leq h$.

Vershik has introduced a relation for the Kostka numbers (see [1, p.143, Theorem 3.6.13] and [4, Theorem 4]): for any $\lambda \vdash n$ and $\rho \vdash n-1$, we have

$$\sum_{\mu \vdash n, \mu \geq \rho} K(\mu, \lambda) = \sum_{\gamma \vdash n-1, \gamma \leq \lambda} c(\lambda, \gamma) K(\rho, \gamma),$$

where $c(\lambda, \gamma)$ the number of ways to obtain the partition $\lambda \vdash n$ from partition $\gamma \vdash n-1$. This relation arises from restricting a permutation representation of the symmetric group $S_n$ to $S_{n-1}$ and then applying Young’s rule to both sides. Vershik [4, Theorem 4] claims to give a bijective proof, but it is poorly explained and incorrect (see Example 3 below). The purpose of this paper is to give a bijective proof of (1) using insertion and reverse insertion algorithms. We remark that this bijection is obtained by the restriction of a bijection giving the Pieri rule (see [3, p.402, 10.65]).

This paper is organized as follows. After giving preliminaries in Section 2, we prove (1) in Section 3.

2. Preliminaries

Throughout this paper, let $h \geq 1$, $x \geq 1$ and $n \geq 1$ be integers. We denote by $D_\mu$ the Young diagram of $\mu$. The rows and the columns are numbered from top to bottom and from left to right, like the rows and the columns of a matrix, respectively. A semistandard Young tableau (SSYT) of shape $\mu$ and weight, or content, $\lambda = ...$
\((\lambda_1, \ldots, \lambda_h)\) is a filling of the Young diagram \(D_\mu\) with the numbers \(1, 2, \ldots, h\) in such a way that \(i\) occupies \(\lambda_i\) boxes, for \(i = 1, 2, \ldots, h\), and the numbers are strictly increasing down the columns and weakly increasing along the rows. We denote by \(\text{SSYT}(\mu, \lambda)\) the set of all semistandard tableaux of shape \(\mu\) and weight \(\lambda\). The Kostka number \(K(\mu, \lambda)\) is defined to be the cardinality of \(\text{SSYT}(\mu, \lambda)\).

Let \(\mu \vdash n\) and \(\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_h) \vdash n\), and \(T \in \text{SSYT}(\mu, \lambda)\). First of all, we need a fundamental combinatorial algorithm on tableaux called \(\text{row-insertion}\), or \(\text{bumping}\) (see [2, Chapter 1]). We define an insertion tableau, denoted \(T \leftarrow x\), by the following procedure.

**Algorithm 1.** Input: Let \(T \in \text{SSYT}(\mu, \lambda)\) and \(x\) be a positive integer.

Output: \(T \leftarrow x\)

Initialization: \(S := T, y := x\) and \(i := 1\)

while \(\{ j \mid y < S(i, j) \}\) > 0 do

\(z := \min\{ j \mid y < S(i, j) \}\).

\(x' := S(i, z)\).

if \((p, q) = (i, z)\) then \(U(p, q) := y\)

else \(U(p, q) := S(p, q)\)

end if

\(S \leftarrow U, y := x'\) and \(i := i + 1\).

end

\((T \leftarrow x)(i, \mu_i + 1) := y\)

Otherwise, \((T \leftarrow x)(p, q) := S(p, q)\)

Output \(T \leftarrow x\).

This algorithm is invertible. Given a partition \(\mu \vdash n\) and insertion tableau \(T \leftarrow x\), there is the unique box of \(T \leftarrow x\) not in \(D_\mu\). From this box, we can construct the reverse insertion algorithm, so we can recover the original tableau \(T\).

### 3. Vershik’s relations for the Kostka numbers

Let \([h] = \{1, 2, \ldots, h\}\), and let \(\text{SSYT}[h](\mu)\) be the set of all SSYT’s of shape \(\mu\) and taking values in \([h]\). For a partition \(\rho \vdash n - 1\), Loehr [3, p.399, 10.60] shows that insertion \(I\) and reverse insertion \(R\) give mutually inverse bijections

\[
I : \text{SSYT}[h](\rho) \times [h] \rightarrow \bigcup_{\mu \vdash n \atop \mu \trianglerighteq \rho} \text{SSYT}[h](\mu),
\]

\[
R : \bigcup_{\mu \vdash n \atop \mu \trianglerighteq \rho} \text{SSYT}[h](\mu) \rightarrow \text{SSYT}[h](\rho) \times [h].
\]

given by \(I(T, x) = T \leftarrow x\) and \(R(S)\) is the result of applying reverse insertion to \(S\) starting at the unique box of \(S\) not in \(\rho\).
**Theorem 1.** Let $\rho \vdash n - 1$ and $\lambda = (\lambda_1, \ldots, \lambda_h) \vdash n$ and, set
\[
R' = \bigcup_{1 \leq x \leq h} (\text{SSYT}(\rho, \lambda(x)) \times \{x\}),
\]
\[
L = \bigcup_{\mu \vdash n \atop \mu \geq \rho} \text{SSYT}(\mu, \lambda).
\]
Then $I|_{R'}$ and $R|_L$ give mutually inverse bijections between $R'$ and $L$.

**Proof.** It is obvious that
\[
R' \subset \text{SSYT}[h](\rho) \times [h],
\]
\[
L \subset \bigcup_{\mu \vdash n \atop \mu \geq \rho} \text{SSYT}[h](\mu).
\]
For each $x \in [h]$, we have
\[
I(\text{SSYT}(\rho, \lambda(x)) \times \{x\}) \subset L
\]
by the definition of insertion. This implies $I(R') \subset L$. Conversely, for each $\mu \vdash n$ with $\mu \geq \rho$, there exists $\ell$ such that $D_{\mu} = D_{\rho} \cup \{(\ell, \mu_\ell)\}$. Applying reverse insertion at $(\ell, \mu_\ell)$ for each tableau in $\text{SSYT}(\mu, \lambda)$, we find $R(\text{SSYT}(\mu, \lambda)) \subset R'$. This implies $R(L) \subset R'$. Since $I$ and $R$ are mutually inverse bijections, we have
\[
R' = RI(R') \subset R(L),
\]
\[
L = IR(L) \subset I(R').
\]
Therefore, $I(R') = L$ and $R(L) = R'$. \qed

From Theorem 1, we can prove (1) as follows:
\[
\sum_{\mu \vdash n \atop \mu \geq \rho} K(\mu, \lambda) = \sum_{\mu \vdash n \atop \mu \geq \rho} |\text{SSYT}(\mu, \lambda)|
\]
\[
= \sum_{1 \leq x \leq h} |\text{SSYT}(\rho, \lambda(x))| \quad \text{(by Theorem 1)}
\]
\[
= \sum_{\gamma \vdash n-1 \atop \gamma \leq \lambda} \sum_{1 \leq x \leq h} |\text{SSYT}(\rho, \lambda)| \quad \text{(by Theorem 1, Lemma 3.7.1)}
\]
\[
= \sum_{\gamma \vdash n-1 \atop \gamma \leq \lambda} c(\lambda, \gamma)K(\rho, \gamma).
\]

Finally, we compare Vershik’s claimed bijection with ours. Vershik calls a tableau in $\text{SSYT}(\mu, \lambda)$ a $\mu$-tableau, and a tableau in $\text{SSYT}(\rho, \lambda(x))$ a $\rho$-tableau. Since $\mu$-tableaux have one more box than $\rho$-tableaux, Vershik [4, Theorem 4] claims that the
removal of one box from $\mu$-tableaux gives a bijection from $\mathcal{L}$ to $\mathcal{R}$, where

$$\mathcal{R} = \bigcup_{1 \leq x \leq h} \text{SSYT}(\rho, \lambda^{(x)})$$

has a natural bijective correspondence with $\mathcal{R}'$. Vershik [4, Section 4] gives examples, each of which comes with a bijection. However, if $\lambda = (3, 3, 2) \vdash 8$ and $\rho = (4, 3) \vdash 7$ then there is no bijection from $\mathcal{L}$ to $\mathcal{R}$ arising from the removal of one box (see Example 3).

**Example 2** ([4, Example 1]). Let $\lambda = (3, 2, 1) \vdash 6$ and $\rho = (4, 1) \vdash 5$. Then

$$\mu$$-tableaux: $A = \begin{array}{ccc} 1 & 1 & 1 \\ 3 & 2 & 2 \\ \end{array}$, $B = \begin{array}{ccc} 1 & 1 & 1 \\ 2 & 3 & 3 \\ \end{array}$, $C = \begin{array}{ccc} 1 & 1 & 1 \\ 2 & 3 & 2 \\ \end{array}$,

$D = \begin{array}{ccc} 1 & 1 & 1 \\ 3 & 2 & 2 \\ \end{array}$, $E = \begin{array}{ccc} 1 & 1 & 1 \\ 2 & 3 & 2 \\ \end{array}$; 

$\rho$-tableaux: $L = \begin{array}{ccc} 1 & 1 & 2 \\ 3 & 2 & 2 \\ \end{array}$, $M = \begin{array}{ccc} 1 & 1 & 2 \\ 2 & 3 & 2 \\ \end{array}$, $N = \begin{array}{ccc} 1 & 1 & 1 \\ 2 & 3 & 2 \\ \end{array}$,

$P = \begin{array}{ccc} 1 & 1 & 1 \\ 2 & 3 & 2 \\ \end{array}$, $Q = \begin{array}{ccc} 1 & 1 & 1 \\ 2 & 2 & 2 \\ \end{array}$.

We remove one box from the first row in $A$ and $B$, one box from the second row in $C$ and $D$, and one box $(3, 1)$ in $E$ in order to obtain $\rho$-tableaux. Then we have a bijection as follows:

$$A \leftrightarrow L; \quad B \leftrightarrow M; \quad C \leftrightarrow N; \quad D \leftrightarrow P; \quad E \leftrightarrow Q.$$

The bijection given by Theorem 1 is:

$$L \leftrightarrow (L \leftarrow 1) = E; \quad M \leftrightarrow (M \leftarrow 1) = D; \quad N \leftrightarrow (N \leftarrow 2) = A; \quad P \leftrightarrow (P \leftarrow 2) = C; \quad Q \leftrightarrow (Q \leftarrow 3) = B.$$

We give an example, for which there is no bijection arising from the removal of one box.

**Example 3.** Let $\lambda = (3, 3, 2) \vdash 8$ and $\rho = (4, 3) \vdash 7$. Then

$$\mu$$-tableaux: $A = \begin{array}{ccc} 1 & 1 & 1 \\ 3 & 2 & 2 \\ 2 & 2 & 2 \\ \end{array}$, $B = \begin{array}{ccc} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 3 \\ \end{array}$, $C = \begin{array}{ccc} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 2 & 3 & 3 \\ \end{array}$,

$D = \begin{array}{ccc} 1 & 1 & 1 \\ 3 & 2 & 2 \\ 2 & 3 & 3 \\ \end{array}$, $E = \begin{array}{ccc} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 3 \\ \end{array}$, $F = \begin{array}{ccc} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 2 & 3 & 3 \\ \end{array}$; 

$\rho$-tableaux: $L = \begin{array}{ccc} 1 & 1 & 2 \\ 3 & 2 & 2 \\ 2 & 3 & 3 \\ \end{array}$, $M = \begin{array}{ccc} 1 & 1 & 2 \\ 2 & 3 & 3 \\ 3 & 3 & 3 \\ \end{array}$, $N = \begin{array}{ccc} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 2 & 3 & 3 \\ \end{array}$,

$P = \begin{array}{ccc} 1 & 1 & 1 \\ 2 & 3 & 3 \\ 2 & 3 & 2 \\ \end{array}$, $Q = \begin{array}{ccc} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 2 & 3 & 3 \\ \end{array}$, $R = \begin{array}{ccc} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 2 & 3 & 3 \\ \end{array}$. 

\[\]
As mentioned in Section 1, \( \mu \)-tableaux \( A \) and \( E \) result in \( \rho \)-tableau \( Q \), so there is no bijection between \( \mu \)-tableaux and \( \rho \)-tableaux arising from the removal of one box. The bijection given by Theorem 1 is:

\[
\begin{align*}
L & \leftrightarrow (L \leftarrow 1) = E; \quad M \leftrightarrow (M \leftarrow 1) = F; \\
N & \leftrightarrow (N \leftarrow 2) = D; \quad P \leftrightarrow (P \leftarrow 2) = C; \\
Q & \leftrightarrow (Q \leftarrow 3) = A; \quad R \leftrightarrow (R \leftarrow 3) = B.
\end{align*}
\]

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