On beautiful analytic structure of the S-matrix

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For an exponentially decaying potential, analytic structure of the $s$-wave S-matrix can be determined up to the slightest detail, including position of all its poles and their residui. Beautiful hidden structures can be revealed by its domain colouring. A fundamental property of the S-matrix is that any bound state corresponds to a pole of the S-matrix on the physical sheet of the complex energy plane. For a repulsive exponentially decaying potential, none of infinite number of poles of the $s$-wave S-matrix on the physical sheet corresponds to any physical state. On the 2nd sheet of the complex energy plane, the S-matrix has infinite number of poles corresponding to virtual states and a finite number of poles corresponding to complementary pairs of resonances and anti-resonances. The origin of redundant poles and zeros is confirmed to be related to peculiarities of analytic continuation of a parameter of two linearly independent analytic functions. The overall contribution of redundant poles to the asymptotic completeness relation, provided that the residuum theorem can be applied, is determined to be an oscillating function.

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I. INTRODUCTION

There have been known for long time many exactly solvable models\textsuperscript{1}, yet it has remained rare that one can determine complete analytic structure of the S-matrix. As a consequence, the origin of singularities of the S-matrix has not been fully understood. For example, over 70 years ago Ma established that the S-matrix may possess redundant poles and zeros not corresponding to any bound state, half-bound state, (anti)resonance, or a virtual state\textsuperscript{2–4}. This bears important consequences for relating analytic properties of the S-matrix to physical states. In particular, the presence of redundant poles on the physical sheet of the complex energy plane implies that the S-matrix need not satisfy a general condition of Heisenberg\textsuperscript{2–5}. The issue is of fundamental importance for the S-matrix theory. Ma’s finding inspired and motivated many authors, such as in now classical refs.\textsuperscript{5–10}.

In what follows we analyze the $s$-wave S-matrix for the Schrödinger equation in a textbook example of an exponentially decaying potential,

$$V_+(r) = V_0 e^{-r/a},$$

where $V_0 > 0$ and $a > 0$ are positive constants and $r$ is radial distance. The potential is the repulsive analogue of the attractive exponentially decaying potential, $V_-(r) = -V_+(r)$, studied by Ma and others\textsuperscript{2–4,11}. Like its attractive cousin potential\textsuperscript{11} satisfies conditions\textsuperscript{12} eqs. (12.20) and (12.21)] sufficient to prove analyticity of the S-matrix merely in a strip around the real axis in the complex plane of momentum $k$\textsuperscript{12} p. 352]. Yet the $s$-wave S-matrix can be also determined analytically in the whole complex $k$ plane, what the classical monograph\textsuperscript{12} surprisingly never mentions. Analytic structure of the S-matrix can be determined up to the finest detail, including position of all its poles and their residui. Beautiful hidden structures can be revealed by its domain colouring. The repulsive example turns out to be rather extreme example in that the resulting S-matrix\textsuperscript{12} will be shown to have infinite number of redundant poles on the physical sheet in the complex energy plane without a single bound state. At the same time, the resulting S-matrix\textsuperscript{12} will be shown to have infinite number of poles corresponding to virtual states on the 2nd sheet of the complex energy plane. Unlike the attractive case\textsuperscript{13}, there are obviously no bound states present. However, one can identify pairs formed by a resonance (Re $k > 0$) and anti-resonance (Re $k < 0$) arranged symmetrically with respect to the imaginary axis in the complex $k$ plane, each of them being absent in the attractive case\textsuperscript{13}.

The outline is as follows. After preliminaries in Sec.\textsuperscript{11} we provide a rigorous canonical analysis of the $s$-wave S-matrix along the lines of monograph\textsuperscript{12} in Sec.\textsuperscript{111} We obtain analytic expressions for the Jost functions and determine the S-matrix\textsuperscript{12}. This enables us to illustrate the validity of general theorems in the slightest detail and to achieve a deep understanding of the analytic structure of the S-matrix. An indispensable part of the analysis are Coulomb’s results\textsuperscript{14} on zeros of the modified Bessel function $I_\nu(x)$ for fixed nonzero argument $x \in \mathbb{R}$ considered as a complex entire function of its order $\nu$, which are summarized in supplementary material. The origin of redundant poles is analyzed in Sec.\textsuperscript{111} In the attractive case, $V_+(r)$, the origin of redundant poles and zeros has recently been related to peculiarities of analytic continuation of a parameter of two linearly independent analytic solutions of a 2nd order ordinary differential equation\textsuperscript{13}. The crux of the appearance of redundant poles and zeros lies in that analytic continuation of a
When substituting back into (2), one arrives at

\[ x \frac{d^2 I'_{\nu}(x)}{dx^2} + x I'_\nu(x) - [x^2 + (i\rho)^2] I_{\nu}(x) = 0, \]

which is the defining equation of the modified Bessel functions of imaginary order \( \nu = i\rho \) (cf. [13] (9.6.1), [16] (10.25.1)).

### III. A RIGOROUS ANALYSIS OF THE s-WAVE S-MATRIX

#### A. The regular solution \( \varphi(r) \)

The pair \( \{I_{\nu}(z), K_{\nu}(z)\} \) yields always two linearly independent solutions of eq. (1) and its Wronskian is never zero (cf. [13], (9.6.15), [16] (10.28.2)). The regular solution of (2) vanishing at the origin \( r = 0 \) becomes in the notation of ref. [12]

\[ \varphi(r) = C \left[ K_{\nu}(\alpha)I_{\rho}(\alpha e^{-r/(2a)}) - I_{\nu}(\alpha)K_{\rho}(\alpha e^{-r/(2a)}) \right], \]

where \( \alpha = 2a\sqrt{U_0} = \lim_{r \to 0} x(r) \) and \( C = -2a \) ensures normalization \( \varphi'(0) = 1 \) ([12] eq. (12.2)). Indeed, for \( x \to \alpha \), or equivalently \( r \to 0 \), one finds on using [13] (9.6.15), [16] (10.28.2)

\[ \varphi'(r) = -\frac{\alpha C}{2a} W \{K_{\nu}(x), I_{\nu}(x)\} \to -\frac{C}{2a}. \]

#### B. Irregular solutions \( f_\pm(k, r) \)

According to [13] (9.6.7.9), [16] (10.30.1-2)], one finds in the limit \( z \to 0 \),

\[ I_{\nu}(z) \sim \frac{z^\nu}{2\Gamma(\nu + 1)} \quad (\nu \neq -1, -2, -3, \ldots), \]

\[ K_{\nu}(z) \sim \frac{1}{2} \Gamma(\nu) \left(\frac{2}{z}\right)^\nu, \quad \text{Re} \nu > 0, \]

where \( \Gamma \) denotes the usual gamma function [15, sec. 6.1], [16, sec. 5]. At the same time, for \( z \in \mathbb{C} \) [13] (9.6.6), [16] (10.27.1-2)]:

\[ I_{-\nu}(z) = I_\nu(z), \quad K_{-\nu}(z) = K_\nu(z). \]

Given the asymptotic (6), (7), the usual irregular solution for \( k \in \mathbb{R} \) has to be proportional to \( I_{-\nu}(x) \),

\[ f_+(k, r) = \Gamma(1 - i\rho) \left(\frac{\alpha}{2}\right)^{i\rho} I_{-\nu}(x). \]

The asymptotic (6) implies for \( \text{Im} k \geq 0 \) \( \text{Re} \ ( -i\rho) \geq 0 \)

\[ f_+(k, r) \sim e^{ikr}, \]
showing the characteristic outgoing spherical wave behaviour of \( f_+ (k, r) \) for \( r \to \infty \), \( k \in \mathbb{R} \), and yields \( f_+ (k, r) \) as exponentially decreasing for \( r \to \infty \), \( \text{Im} \) \( k > 0 \), in accordance with general theorems [12, Sec. 12.1.4]. \( f_+ (k, r) \) cannot be proportional to \( K_{\nu} (x) \), because for \( \text{ip} \pi \neq n \pi \), \( n \in \mathbb{Z} \), [13, (9.6.2)], [16, (10.27.2)]

\[
K_{\nu}(x) = \frac{\pi}{2 \sin(i \rho \pi)} [I_{-\nu} (x) - I_{\nu} (x)].
\]

Then the asymptotic \( f_+ (k, r) \) would comprise both \( \sim e^{i k r} \) and \( \sim e^{-i k r} \) terms.

Given analyticity of \( \Gamma (1 - \text{ip}) \) and \( I_{-\nu} \) [13], one can easily verify \( f_+ (k, r) \) to be an analytic function of \( k \) regular for \( \text{Im} k > 0 \) and continuous with a continuous \( k \) derivative in the region \( \text{Im} k \geq 0 \) for each fixed \( r > 0 \). The second linearly independent irregular solution \( f_- (k, r) = f_+ (ke^{i \pi}, r) \propto I_{\nu} (x) \) (assuming analytic continuation via the upper half plane) is uniquely determined by the boundary condition \( f_- (k, r) \sim e^{-i k r} \) for \( r \to \infty \).

According to [13, (9.6.2)], [16, (10.27.2)], \( I_{-\nu} \) is related to \( K_{\nu} (z) \) for any \( i \text{p} \) by

\[
I_{-\nu} (z) = I_{\nu} (z) + \frac{2}{\pi} \sin(i \rho \pi) K_{\nu} (z).
\]

This enables one to express \( f_+ (k, r) \) in the basis of \( \{ I_{\nu} (z), K_{\nu} (z) \} \),

\[
f_+ (k, r) = \Gamma (1 - \text{ip}) \left( \frac{\alpha}{2} \right)^{i \text{p}} \left[ I_{\nu} (x) + \frac{2}{\pi} \sin(i \rho \pi) K_{\nu} (x) \right].
\]

C. The Jost functions \( F_{\pm} (k) \)

Using the latter expression, it is straightforward to determine the Jost function [12, Sec. 12]

\[
F_+ (k) := W_r \{ f_+, \varphi \} = C \Gamma (1 - \text{ip}) \left( \frac{\alpha}{2} \right)^{i \text{p}} \times \left[ I_{\nu} (\alpha) + \frac{2}{\pi} \sin(i \rho \pi) K_{\nu} (\alpha) \right] W_r \{ K_{\nu}, I_{\nu} \}
\]

\[
= C I_{-\nu} (\alpha) \Gamma (1 - \text{ip}) \left( \frac{\alpha}{2} \right)^{i \text{p}} \left( \frac{1}{\alpha e^{-r/(2a)}} \right)^{\alpha e^{-r/(2a)}}
\]

\[
= I_{-\nu} (\alpha) \Gamma (1 - \text{ip}) \left( \frac{\alpha}{2} \right)^{i \text{p}}
\]

\[
= a F_1 (1 - \text{ip}, \alpha^2/4),
\]

where the respective \( W_r \{ .. \} \) and prime denote the Wronskian (cf. [13, (9.6.15)], [16, (10.28.2)]) and derivative with respect to \( r \), and \( a F_1 \) is the confluent hypergeometric function.

The complementary Jost function \( F_- (k) := W_r \{ f_-, \varphi \} \) is obtained by replacing \( \text{ip} \to -\text{ip} \) in the above expression for \( F_+ \). Analytic properties of the Jost functions \( F_{\pm} (k) \) follow from that (i) each branch of \( I_{\nu} (z) \) is entire in \( \nu \) for fixed \( z \) (\( \neq 0 \)) and (ii) \( \Gamma (z) \) is holomorphic in \( z \) having only simple poles at \( z = -n, n \geq 0 \) [13, 16].

D. A detailed analytic structure of the S-matrix

The s-wave S-matrix is determined as the ratio \( S(k) = F_- (k)/F_+ (k) \) (cf. [12, eq. (12.71)])

\[
S(k) = \frac{I_{\nu} (\alpha) \Gamma (1 + \text{ip}) \Gamma (1 - \text{ip}) (\alpha^2/2)}{\Gamma (1 + i \rho) (\alpha^2/2)^{-\text{ip}},}
\]

where \( k \) dependence here enters through dimensionless momentum parameter \( \rho = 2ak \). It follows straightforwardly that

- \( S(k) \) vanishes either when \( I_{\nu} (\alpha) = 0 \) or, at the poles of \( \Gamma (1 - \text{ip}) \) [18].
- The poles of \( S(k) \) are the poles of \( \Gamma (1 + i \rho) \) [18] and the zeros of \( I_{-\nu} (\alpha) \).

The poles of \( \Gamma (1 + i \rho) \) occur for any \( i \rho = -n, n \in \mathbb{N}_+ \), or \( k = k_n, k_n = n/(2a), n \geq 1 \). Those are the only poles, and all those poles are simple [13, (6.1.3)]. They give rise to infinite number of simple redundant poles of \( S(k) \) on the positive imaginary axis, i.e. on the physical sheet of the complex energy plane.

The zeros of \( I_{-\nu} (\alpha) \) are the only zeros of \( F_- (k) \), which is analytic (without any singularity) on the physical sheet (\( \text{Im} k \geq 0 \)). A well hidden and largely forgotten result which, surprisingly, cannot be found either in ultimate tables [13, 16] or monograph [19] is that (cf. Appendix [13]):

(a) \( I_{\nu} (x) \) with fixed nonzero \( x \in \mathbb{R} \) and \( \text{Re} \nu > -3/2 \) has no complex zero (i.e. with nonzero imaginary part) when considered as a function of its order \( \nu \) [14]. All the roots of \( I_{\nu} (x) \) are real for \( \text{Re} \nu > -3/2 \).

(b) The roots of \( I_{\nu} (z) \) are asymptotically near the negative integers for large \( n \) for \( 0 < |z| \ll 1 \) and/or \( |\nu| >> |z| + 1 \) [14, 20]. Indeed, in the latter asymptotic range one has for the roots of \( J_{\nu} (z) \) [20, eq. (8)]

\[
\nu_n \sim -n + \frac{(z/2)^{2n}}{n!(n-1)!} - \frac{(z/2)^{2(n+1)}}{(n+1)!(n-1)(n-1)!},
\]

from which the asymptotic of the roots of \( J_{\nu} (z) \) follows on substituting \( z \to iz \). Obviously, the roots \( \nu_n \) of \( I_{\nu} (x) \) are real for real \( x \) when formula [13] holds. Worth of noting is that, unlike the roots of \( J_{\nu} (z) \), the roots of \( I_{\nu} (x) \) need not be in general simple [14].

In the case of \( I_{-\nu} (\alpha) \) in the denominator of the S-matrix [13], the condition \( \text{Re} \nu > -3/2 \) translates into \( \text{Im} \)
where \( k = -is, \ s \in \mathbb{R} \). For \( 0 < |\alpha| < 1 \) and/or \( 2as \gg \alpha + 1 \), the position of virtual states can be estimated from [13] for \( x = \alpha \). Interestingly, [13] implies that the roots of \( I_{-ip}(\alpha) \) approach asymptotically the poles of \( \Gamma(1-i\rho) \) in the denominator of the S-matrix [12].

Last but not the least, for \( \text{Im} \ k \leq -3/(4a) \) one cannot exclude the presence of resonances (\( \text{Re} \ k > 0 \)) and/or anti-resonances (\( \text{Re} \ k < 0 \)) in the lower half complex \( k \)-plane outside the imaginary axis. For \( x \in \mathbb{R} \) one has \( I_{-i}(x) = I_{+i}(x) \) (cf. [15] (9.6.10), [16] (10.25.2)), which is the Schwarz reflection in \( \nu \)-variable. Hence a complex zero \( \nu = \nu_0 \) implies that also \( \bar{\nu}_0 \) is zero, i.e. complex zeros occur necessarily in complex conjugate pairs. In the complex \( k \)-plane this translates to that resonances
and anti-resonances form pairs arranged symmetrically with respect to the imaginary axis. In virtue of the above property (b) of the roots \( \nu_n \) of \( L_\nu(z) \), one can expect to find complex roots \( \nu_n \) only for \(|\nu| \sim |z| \ll 1\). This is what is indeed confirmed in fig. 1.

The singularities in complex \( k \)-plane with negative imaginary part correspond to states that do not belong to the Hilbert space since they are not normalizable. However, they can produce observable effects in the scattering amplitudes, in particular when they approach the real axis. In the present case, virtual states together with resonances and anti-resonances cannot approach the real axis in the complex \( k \)-plane closer than the minimal distance \( d = 3/(4a) \).

For a comparison, fig. 2 shows the S-matrix in the attractive case, \( V_-(r) \), for \( a = 1 \) and \( \alpha = 5 \), e.g. for the same parameters as in fig. 1. The S-matrix in the attractive case differs from (12) in that \( J_{\pm i\rho}(\alpha) \) are substituted by \( J_{\pm i\rho}(\alpha) \). One can identify redundant poles and a single bound state on the positive imaginary axis, and some of infinite number of virtual states on the negative imaginary axis. However, any resonance or anti-resonance is forbidden. This is because \( J_{\nu}(x) \) considered as a function of its order \( \nu \) does not have any complex pole (i.e. with nonzero imaginary part) for real \( x \neq 0 \). All the roots \( \nu_n \) of \( J_{\nu}(x) \) are real and simple. For large \( n \) for \( 0 < |x| \ll 1 \) and/or \( |\nu| \gg |x| + 1 \) the roots \( \nu_n \) of \( J_{\nu}(x) \) are asymptotically near the negative integers according to \([13, 14, 21]\).

The present \( f_{\pm}(k,r) \) satisfy all the classical requirements \([12]\). The usual analytic connection between the positive and negative real \( k \)-axis, \( f_-(k,r) = f_+(k^{\text{re}},r) \), together with the boundary condition satisfied by \( f_+ \) leads to \( S(k) = S^{-1}(k) \) for any \( k \in \mathbb{R} \) \([12]\). For general \( k \in \mathbb{C} \) one has (cf. \([12]\) eqs. (12.24a), (12.32a))

\[
S^*(k^*) = S^{-1}(k).
\]

Hence each pole of \( S \) on the first physical sheet of energy (Im \( k > 0 \)) corresponds to a zero of \( S \) on the second sheet (Im \( k < 0 \)), and vice versa \([12]\). In order to verify \([12]\) for the S-matrix \([12]\), notice that all the special functions involved there satisfy the Schwarz reflection principle \( F(\bar{z}) = \overline{F(z)} \) in variable \( z = i\rho \) for \( \alpha \in \mathbb{R} \). Hence

\[
S^*(k^*) = \frac{I_{-i\rho}(\alpha)\Gamma(1-i\rho)}{I_{i\rho}(\alpha)\Gamma(1+i\rho)} \left( \frac{\alpha}{2} \right)^{2i\rho},
\]

which is obviously \( S^{-1}(k) \). The above pole-zero correspondence resulting from the symmetry property \([14]\) is nicely demonstrated in figs. 1 2.

IV. ON THE ORIGIN OF REDUNDANT POLES

The Wronskian of \( I_{\pm i\rho} \) \([13, \text{eq. (9.6.14)}, \{16, \text{eq. (10.28.1)}\}] \),

\[
W_x\{I_{i\rho}(x), I_{-i\rho}(x)\} = -\frac{2\sin(i\rho \pi)}{\pi x},
\]

vanishes whenever \( i\rho \in \mathbb{Z} \). In the special case \( i\rho = -n, n \in \mathbb{N} \), one finds on combining eqs. \([9, 8]\):

\[
I_{\pm n}(z) \sim \frac{z^n}{2\pi \Gamma(n+1)}, \quad n \in \mathbb{N}.
\]

Therefore, the basis \( \{I_{i\rho}(z), I_{-i\rho}(z)\} \) of solutions of eq. \([4]\) collapses into linearly dependent solutions for any \( i\rho \in \mathbb{Z} \). (For \( \rho = 0 \) the two Bessel functions degenerate into a single one.) Our recent treatment of attractive potential \( V_-(r) \) \([13]\) suggests that the collapse of the pair \( \{I_{i\rho}(x), I_{-i\rho}(x)\} \) of solutions of eq. \([4]\) into linearly dependent solutions and the occurrence of redundant poles and zeros at exactly the same points \( i\rho \in \mathbb{Z} \) is not coincidental. Like in \([13]\), it is important to notice that eqs. \([9, 11]\) imply factorization of \( f_{\pm}(k,r) \) as

\[
f_{\pm}(k,r) = \frac{F_{\pm}(k)}{I_{\mp i\rho}(\alpha)} I_{\mp i\rho}(x),
\]

where the first factor including the Jost function, \( F_{\pm} \), is only a function of \( k \), and only the second factor, \( I_{\mp i\rho}(x) \), depends on both \( k \) and \( r \). In virtue of \([11]\), the first factor is finite for any \( I_{\mp i\rho}(\alpha) = 0 \).

Let us ignore for a while the first \( k \)-dependent prefactors in \([10]\). Then \( f_-(k,r) \), which is typically exponentially increasing on the physical sheet as \( r \to \infty \), would become suddenly exponentially decreasing for \( r \to \infty \) for any \( i\rho \in \mathbb{Z}_- \), i.e. \( k = k_n, n \geq 1 \), on the physical sheet, very much the same as \( f_+(k,r) \). Similarly, \( f_+(k,r) \), which is expected to be exponentially increasing on the second sheet for \( r \to \infty \), would become suddenly exponentially decreasing in the limit for any \( i\rho \in \mathbb{Z}_+ \), or \( k = -k_n, k_n = \text{in}/(2a) \), \( n \geq 1 \), on the 2nd sheet, very much the same as \( f_-(k,r) \). The role of the \( k \)-dependent prefactors \( F_{\pm} \) is to hide such an “embarrassing” behaviour by causing the respective irregular solutions \( f_{\pm}(k,r) \) to become singular at the incriminating points (i.e. \( f_-(k,r) \) at \( k = k_n \), and \( f_+(k,r) \) at \( k = -k_n \)). Note in passing that although \( f_+(k,r) \) (\( f_-(k,r) \)) is, for each \( r \), an analytic function of \( k \) regular for \( \text{Im} k > 0 \) (\( \text{Im} k < 0 \)) and continuous with a continuous derivative in the region \( \text{Im} k \geq 0 \) (\( \text{Im} k \leq 0 \)), this no longer holds for \( \text{Im} k \leq 0 \) (\( \text{Im} k > 0 \)).

The singular prefactors ensure that, in spite of the linear dependency of the pair \( \{I_{i\rho}(x), I_{-i\rho}(x)\} \) for any \( i\rho \to \mathbb{Z} \), the identity \( W_z\{f_+, f_-\} = -2ik \) \([12, \text{eq. (12.27)}]\) is nevertheless preserved. Indeed \([15]\) implies for \( i\rho \to \mathbb{Z} \)
Given that \( S(k) = e^{2i\delta(k)} \) for \( k \in \mathbb{R} \), one can only use asymptotic form (19) of regular solutions for \( r, r' \gg 1 \) in the completeness relation (18) arrive at (3, eq. (6)), (6, eq. (1.2)), (21, eq. (13))

\[
\int_{-\infty}^{\infty} S(k)e^{ik(r+r')} dk = \sum_l |C_l|^2 e^{-|k_l|(r+r')},
\]

where \(|C_l|^2\) can be determined from the asymptotic of \( \phi_l \) divided by \( N^2 \) (13). Under the condition that the integral over the real axis can be closed by infinite semicircle \( \gamma \) in the upper half \( k \)-plane, i.e.

\[
\oint_S S(k)e^{ik(r+r')} dk = 0,
\]

one arrives at the correspondence between the poles of the S-matrix and bound states,

\[
\int_{k=k_l} S(k) dk = |C_l|^2 > 0,
\]

where \( f_{k=k_l} \) stands for integration along a contour encircling a single isolated bound state. This correspondence is known as the Heisenberg condition (4, 5, 6).

Because \( C_l \neq 0 \) only for physical bound states, one has \( \int_{k=k_l} S(k) dk \equiv 0 \) in the present case. To this end we determine the overall contribution of redundant poles to the integral on the l.h.s of (20) as the sum over all residui. On making use of eqs. (8) and (17) in (12), one finds the following residuum in the \( i\rho \) variable for any \( k_n \) (\( \rho = -n \)),

\[
\text{Res } S(k_n) = \frac{I_n(\alpha)}{I_n(\alpha)\Gamma(n+1)} \left( \frac{\alpha}{2} \right)^{2n} \text{Res } \Gamma(-n+1)
\]

\[
= \frac{1}{n!} \left( \frac{\alpha}{2} \right)^{2n} \left( \frac{-1}{n-1} \right)^n
\]

\[
= \frac{(-1)^n}{n!(n-1)!} \left( \frac{\alpha}{2} \right)^{2n}.
\]

When converting from \( i\rho \) to \( k \) as independent variable, the left hand side of (22) for \( n \)th redundant pole has alternating sign and not always yields a positive number (the latter being typical for the true bound states),

\[
2\pi i \text{Res}_k S(k_n) = \frac{\pi}{a n!(n-1)!} \left( \frac{\alpha}{2} \right)^{2n}.
\]

The overall contribution of redundant poles to the inte-
gral on the lhs of (20) is
\[2\pi i \sum_{n=1}^{\infty} \text{Res}_k S(k_n) e^{-n(r+r')/(2a)} = \frac{\pi}{a} \sum_{n=1}^{\infty} \frac{(-1)^n}{n!(n-1)!} \left[ \frac{\alpha}{2} e^{-(r+r')/(4a)} \right]^{2n} = -\frac{\pi}{a} q I_1(q), \]
where \( q = (\alpha/2)e^{-(r+r')/(4a)} \) \cite{13, eq. (9.1.10)}, \cite{16, eq. (10.2.2)]. The overall contribution is an oscillating function of \( q \).

VI. RESIDUI OF VIRTUAL STATES

At any root of \( I_{\nu}(\alpha) \) one has \cite{13, (9.6.42)}, \cite{16, (10.38.1)],
\[\frac{\partial I_{\nu}(\alpha)}{\partial \nu} = -\left( \frac{\alpha}{2} \right)^\nu \sum_{m=0}^{\infty} \frac{\psi(m+1+\nu)}{\Gamma(m+1+\nu)} (\alpha/2)^{2m} m!, \]
with \( \psi \) here being the digamma function \cite{13, (6.3.1)]. Because \( I_{\nu}(z) \) is an entire function of its order \cite{13}, the residuum of the pole term \( 1/I_{-i\rho}(\alpha) \) in \cite{12} on the negative imaginary axis in the complex \( k \)-plane can be obtained as
\[\text{Res}_k \frac{1}{I_{-i\rho}(\alpha)} = \frac{i}{2a} \frac{1}{\partial_\nu I_{\nu}(\alpha)} \bigg|_{\nu=-i\rho}. \]

On the other hand, at any root of \( J_{\nu}(z) \) \cite{13, (9.6.44)}, \cite{16, (10.15.1)],
\[\frac{\partial J_{\nu}(\alpha)}{\partial \nu} = -\left( \frac{\alpha}{2} \right)^\nu \sum_{m=0}^{\infty} \frac{(-1)^m \psi(\nu+m+1)}{\Gamma(\nu+m+1)} (\alpha/2)^{2m} m!, \]
The residuum of the pole term \( 1/J_{-i\rho}(\alpha) \) in the attractive case is then determined by \cite{20} with \( I_{-i\rho} \) substituted by \( J_{-i\rho} \).

VII. DISCUSSION

Although there are many exactly solvable models known, it is quite rare that one can determine analytic structure of the S-matrix up to the slightest detail, including position of all its poles and their residui, such as in the present case of an exponentially decaying potential. The latter can be regarded as an example of exactly solvable S-matrix model. There is a number of important lessons to be learned from the repulsive exponentially decaying potential example \cite{1} studied here and its attractive version dealt with in ref. \cite{13}.

A. Repulsive vs attractive exponentially decaying potential

Formally, the repulsive case differs from the attractive one in that \( \alpha \to i\alpha \). According to the connection formula \cite{13, (9.6.3)], \cite{16, (10.27.6)],
\[J_{\nu}(iz) = J_{\nu} \left( ze^{\pi i/2} \right) \to e^{\pi i/2} I_{\nu}(z). \]

Therefore, with a hindsight, it is not surprising that the expressions for irregular solutions \( f_{\pm}(k, r) \) \cite{9}, the Jost function, \( F_{\pm} \) \cite{11}, and the s-wave S-matrix \( S_{\pm} \) \cite{12} in the repulsive case can be transformed into those in the attractive case \cite{13} by substituting \( I_{\mp \nu} \) for \( J_{\mp \nu} \). Nevertheless, as it is clear from derivations, it was not obvious in advance which particular form the resulting expressions would assume.

B. How to distinguish between the redundant poles and true bound states

On the physical sheet one can clearly distinguish between the redundant poles and true bound states at the level of the Jost functions: (i) redundant poles are the singularities of \( F_{-}(k) \), whereas (i) true bound states are the zeros of \( F_{+}(k) \). Any difference between the respective poles gets blurred only at the level of the S-matrix when the ratio \( S(k) = F_{-}(k)/F_{+}(k) \) is formed.

C. Basis of solutions at the points \( i\rho \in \mathbb{Z} \)

One witnesses in the literature a surprising unabated inertia in selecting the respective pairs \( \{I_{i\rho}(x), J_{-i\rho}(x)\} \) and \( \{I_{i\rho}(x), J_{-i\rho}(x)\} \) as the basis of linearly independent solutions of eq. \cite{4} in exponentially attractive and repulsive cases, in spite that each of them collapses into linearly dependent solutions for any \( i\rho \in \mathbb{Z} \) \cite{cf. eq. (13)]}. Those choices goes back to the classical contributions of Ma \cite{2} and stretch, for instance, to recent treatments of (i) one-dimensional exponential potentials \( V(x) \) on \( x \in (-\infty, \infty) \) \cite{22} and (ii) scattering and bound states in scalar and vector exponential potentials in the Klein-Gordon equation \cite{23, Sec. 4]. The collapse of \( \{I_{i\rho}(x), J_{-i\rho}(x)\} \) into linearly dependent solutions for \( i\rho \neq n\pi, n \in \mathbb{Z} \) then inevitably prompts false conclusion that \( \varphi(r) \equiv 0 \) at the integer values of \( i\rho \) \cite{2, 3, 22, 23}. Indeed, on expressing \( K_{i\rho}(x) \) in terms of \( I_{\pm i\rho}(x) \) according to \cite{10}, and, on substituting into \cite{20}, the regular solution becomes
\[\varphi(r) = \frac{\pi C}{2\sin(|i\rho|\pi)} \left[ I_{-i\rho}(\alpha) I_{i\rho}(x) - I_{i\rho}(\alpha) I_{-i\rho}(x) \right]. \]

One notices immediately that the square bracket in \cite{20} vanishes identically for any \( i\rho \in \mathbb{Z} \). However, because
of the singular prefactor, it is obviously not true that \( \varphi(r) \equiv 0 \) for \( i\rho \in \mathbb{Z} \): one recovers (3) in the limit \( i\rho \to n \in \mathbb{Z} \).

The vanishing of the square bracket in (10) necessitates to work with the basis \( \{I_{i\rho}(x), K_{i\rho}(x)\} \). The latter basis never degenerate into linearly dependent solutions and is standard choice when treating electromagnetic scattering from dielectric objects \[24,25]\). Although one can arrive at (12) independently in the basis \( \{I_{i\rho}(x), I_{-i\rho}(x)\} \) for \( i\rho \neq n\pi, n \in \mathbb{Z} \), the introduction of the Jost function \( F_\pm(k) \) becomes necessary in order to define the S-matrix for \( i\rho \in \mathbb{Z} \).

### D. Heisenberg condition

Like in the attractive case \[13\], the overall, in general, non-zero contribution of redundant poles \[24\] implies that the equality in (20) cannot be preserved if one had attempted to perform the integral on the lhs of (20) by closing the integration over the real axis in (20) by infinite semicircle \( \gamma \) in the upper half \( k \)-plane and replace it by the sum of residua of all enclosed poles. The S-matrix \[12\] is formed essentially by the ratio of two entire functions \( I_{i\rho}(\alpha)/\Gamma(1-i\rho) \) and \( I_{-i\rho}(\alpha)/\Gamma(1+i\rho) \[17,18\]. Apparently, its essential singularity for infinite \( k \) is more complicated than \( e^{i\alpha k} \), with \( c \) being a constant, considered by Hu \[21\] eq. (42) and Appendix], which allowed for the closing of the integration contour. (Note that the class of cases for which the S-matrix is analytic for \( k = \infty \) is very limited \[10\].) There is a number of indications for this. First, the number of redundant poles \( k_n \to \infty \) on the physical sheet is infinite implying that the S-matrix \[12\] cannot be analytic at infinity. Second, on selecting different sequences when approaching \( k \to \infty \) along the positive imaginary axis one arrives at different limits. For instance, for any \( k_n \) one has \( S(k_n) = \infty \), and the latter applies obviously also to the limit on the sequence \( k_n \to \infty \). On the other hand, one finds that the S-matrix \[12\] has the following limit on the sequence \( k = i(n + \frac{1}{2})/(2a) \) \( i\rho = -n - \frac{1}{2} \), \( n \to \infty \), (see supplementary material)

\[
S(k) \sim \frac{2(2n)!}{(2n-1)!!\sqrt{2\pi(2n+1)}},
\]

which is the same as in the attractive case \[13\]. Therefore, the factor \( e^{i\alpha k} \) in a product formula \[21\] eq. (42) cannot account for the present essential singularity. As a consequence, the integral over the real axis in (20) cannot be probably closed by infinite semicircle \( \gamma \) in the upper half \( k \)-plane. If it could be somehow closed, one cannot exclude that the contribution of the contour integral \[21\] will cancel the contribution of (24) of redundant poles, thereby restoring the asymptotic completeness relation (20). Alas, surprising absence of exact results for Bessel functions of general complex order \[13,16\] provides a true obstacle in full analytic analysis of that issue.

Another valid point is that the use of asymptotic form \[19\] of regular solutions in the completeness relation \[18\] imply that the relation (20) is not a rigorous identity. It involves only leading asymptotic terms of regular solutions for \( r, r' \to \infty \) leaving behind subleading terms, which may also contribute exponentially small terms in (20).

### VIII. CONCLUSIONS

A repulsive exponentially decaying potential \[1\] provided us with a unique window of opportunity for a detailed study of analytic properties of the S-matrix. Its deep understanding was facilitated thanks to largely forgotten Coulomb’s results \[14\] on zeros of the modified Bessel function \( I_\alpha(x) \). The resulting S-matrix \[12\] was shown to exhibit unexpectedly rich behaviour hiding beautiful structures which were revealed by its domain colouring in fig. 1. Much the same can be said about an attractive exponentially decaying potential studied earlier in ref. \[13\], which resulting S-matrix was exhibited in fig. 2. Despite innocent Schrödinger equation (2), which does not show any peculiarity for \( k = \pm kn \), the S-matrix \[12\] has always infinite number of redundant poles at any \( k = kn = in/(2a) \) on the physical sheet, even in the absence of a single bound state. (In the attractive case this happens for \( \alpha < 2 \[13\].) On the 2nd sheet of the complex energy plane, the S-matrix has (i) infinite number of poles corresponding to virtual states and (ii) finite number of poles corresponding to complementary pairs of resonances and anti-resonances (those are missing in the attractive case).

The origin of redundant poles and zeros was confirmed to be related to peculiarities of analytic continuation of a parameter of two linearly independent analytic solutions of the Schrödinger equation (2). We have obtained analytic expressions for the Jost functions and the residui of the S-matrix \[12\] at the redundant poles \[eq. (23)\]. The overall contribution of redundant poles to the asymptotic completeness relation (20), provided that the residuum theorem can be applied, was determined to be an oscillating function \[eq. (24)\].

Given that redundant poles and zeros occur already for such a simple model is strong indication that they could be omnipresent. Currently one can immediately conclude that the appearance of poles of \( F_+(k) \), and of the S-matrix, at \( kn = in/(2a) \) for positive integers \( n \) is a general feature of potentials whose asymptotic tail is exponentially decaying like \( e^{-r/a} \[10\]. This is because essential conclusions of our analysis will not change if the exact equalities involving \( r/a \) were replaced by asymptotic ones. Whether redundant poles and zeros and the Heisenberg condition for other model cases, including non-Hermitian scattering Hamiltonians \[26\], show similar behaviour is the subject of future study. Our results for the attractive case \[13\] can be readily applied for the analysis of the s-wave Klein-Gordon
equation with exponential scalar and vector potentials [23, Sec. 4]. At the same time a proper understanding of analytic structure of the S-matrix of essentially a textbook model will do no harm when attempting to generalize the results in the direction of non-Hermitian scattering Hamiltonians [26]. Last but not the least, we hope to stimulate search for further exactly solvable S-matrix models.

ACKNOWLEDGMENTS

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[1] Infeld L and Hull T E 1951 The factorization method Rev. Mod. Phys. 23 21-68
[2] Ma S T 1946 Redundant zeros in the discrete energy spectra in Heisenberg’s theory of characteristic matrix Phys. Rev. 69 668
[3] Ma S T 1947 On a general condition of Heisenberg for the S matrix Phys. Rev. 71 195-8
[4] Ma S T 1953 Interpretation of the virtual level of the deuteron Rev. Mod. Phys. 25 853-60
[5] ter Haar D 1946 On the redundant zeros in the theory of the Heisenberg matrix Physica 12 501-8
[6] Biswas S N, Pradhan T and Sudarshan E C G 1947 On a general condition of Heisenberg for the S matrix Phys. Rev. 71 195-8
[7] Jost R 1947 ¨Uber die falschen Nullstellen der Eigenwerte der S-Matrix Helv. Phys. Acta 20 256-66
[8] Bargmann V 1949 On the connection between phase shifts and scattering potential Rev. Mod. Phys. 21 488-93
[9] van Kampen N G 1953 S-matrix and causality condition. I. Maxwell field Phys. Rev. 89 1072-9
[10] Peierls R E 1959 Complex eigenvalues in scattering theory Proc. Roy. Soc. (London) A 253 16-36 (in/out spherical wave convention is reversed there)
[11] Bethe H A and Bacher R F 1936 Stationary states of nuclei Rev. Mod. Phys. 8 82-229
[12] Newton R G 1982 Scattering theory of waves and particles 2nd edn (Springer: New York)
[13] Moroz A and Miroshnichenko A E 2019 On the Heisenberg condition in the presence of redundant poles of the S-matrix, to appear in Europhys. Lett. (arXiv:1904.03227)
[14] Coulomb J 1936 Sur les z´ eroes des fonctions de Bessel consid´ e´ ees comme fonction de l’ordre Bull. Soc. Math. de France 60 297-302
[15] Abramowitz M and Stegun I A 1973 Handbook of Mathematical Functions (Dover Publications). (available online at http://people.math.sfu.ca/~cbm/aands/toc.htm)
[16] Olver F W J et al (eds) 2010 NIST Handbook of Mathematical Functions (Cambridge University Press). (available online at https://dlmf.nist.gov)
[17] For fixed $z \neq 0$, each branch of $I_{\nu}(z)$ and of $K_{\nu}(z)$ is entire in $\nu$ [14, eq. (9.6.1)] [15, eq. (10.25.3)]. The only singularities of holomorphic function $\Gamma(z)$ are simple poles for the non-positive integers.
[18] $\Gamma(z)$ has no zeros, so the reciprocal gamma function $1/\Gamma(z)$ is an entire function [15].
[19] Watson G N 1962 A Treatise on the Theory of Bessel functions (Cambridge University Press)
[20] Cohen D S 1964 Zeros of Bessel functions and eigenvalues of non-self-adjoint boundary value problems SIAM 43 133-9

[21] Hu N 1948 On the application of Heisenberg’s theory of S-matrix to the problems of resonance scattering and reactions in nuclear physics Phys. Rev. 74 131-40
[22] Ahmed Z, Ghosh D, Kumar S and Turumella N 2018 Solvable models of an open well and a bottomless barrier: one-dimensional exponential potentials Eur. J. Phys. 38 025404
[23] Curi E J A, Castro L B and de Castro A S 2019 Proper treatment of scalar and vector exponential potentials in the Klein-Gordon equation: Scattering and bound states (arXiv:1902.02872)
[24] Bohren C F and Huffman D R 2007 Absorption and Scattering of Light by Small Particles (John Wiley & Son)
[25] Moroz A 2005 A recursive transfer-matrix solution for a dipole radiating inside and outside a stratified sphere Ann. Phys. (NY) 315 352-418
[26] Sim´ on M A, Buend´ ia A, Kiely A, Mostafazadeh A and Muga J G 2018 S-matrix pole symmetries for non-Hermitian scattering Hamiltonians (arXiv:1811.06270)
[27] Prudnikov A P, Brychkov Yu A and Marichev O I 1992 Integrals and Series, vol. 2, Special Functions, 2nd ed. (Gordon and Breach: London)
Appendix A: Zeros of \( J_\nu(x) \) for fixed nonzero \( x \in \mathbb{R} \)
considered as a function of \( \nu \)

Because of its importance and access difficulty, we find it expedient to summarize Coulomb’s work \[14\] here. Coulomb’s proof is, to a large extent, based on the Lommel integration formula \[19, 5-11(13)], 25, (1.13.2.5)],

\[
\int^t \frac{1}{t^2 - \nu^2} \left[ Z^{(1)}_\mu(t)Z^{(2)}_\nu(t) - Z^{(1)}_\nu(t)Z^{(2)}_\mu(t) \right] dt = -\frac{t}{\mu^2 - \nu^2} \left[ Z^{(1)}_\mu(t)Z^{(2)}_\nu(t) - Z^{(1)}_\nu(t)Z^{(2)}_\mu(t) \right] + \frac{1}{\mu + \nu} Z^{(1)}_\mu(t)Z^{(2)}_\nu(t), \tag{A1}
\]

where \( Z^{(1,2)}_\nu(z) \) are any two linear combinations of cylindrical Bessel functions, \( t \in \mathbb{R}, z \in \mathbb{C} \).

For fixed \( z \neq 0 \) each branch of \( J_\nu(z) \) is entire in complex variable \( \nu \). For \( x \in \mathbb{R} \) one has \( J_\nu(x) = J_\nu(x) \) (cf. \[13\], (9.1.10)), \[16\], (10.2.2)), which is the Schwarz reflection in \( \nu \)-variable. Hence a complex zero \( \nu = \nu_0 \) of \( J_\nu(z) \) implies that \( \nu_0 \) is also zero, i.e. complex zeros occur necessarily in complex conjugate pairs. In what follows it is sufficient to limit oneself to positive \( x \), because, according to analytic continuation formula, \( J_\nu(ze^{\pm \pi i}) = e^{\pm \pi n_\nu}J_\nu(z) \), \( n \in \mathbb{Z} \), \[15\], (9.1.35), \[17\], (10.11.1)]. According to (A1),

\[
\int^t \frac{1}{t} J_\nu_0(tx)J_\nu_0(tx) dt = \int^t \frac{1}{t} J_\nu_0(tx)^2 dt = 0 \tag{A2}
\]

The rhs of (A1) yields zero at the upper integration limit under the hypothesis that \( J_\nu(x) = J_\nu_0(x) = 0 \). It is zero at the lower integration limit for \( Re \nu_0 > 0 \) in virtue of the asymptotic behaviour of each of \( J_\nu \) and \( J_{\nu+1} \) in the limit \( z \to 0 \) \[15\], (9.1.7), \[16\], (10.7.3)],

\[
J_\nu(z) \sim \frac{z^\nu}{2\pi^\nu(\nu + 1)} \quad (\nu \neq -1, -2, -3, \ldots).
\]

Because the integrand in (A2) is a positive real quantity, we have a contradiction: there is no complex zero \( \nu_0 \) of \( J_\nu(x) \) for \( x \in \mathbb{R} \) and \( Re \nu > 0 \) \[14\], item 3.1).

The Lommel integration formula (A1) can be used to prove that \( J_\nu(x) \) with \( x \in \mathbb{R} \) has no complex zero \( \nu_0 \) also for any Re \( \nu_0 < 0 \) \[14\], item 3.2). To this end one performs integral

\[
\int^\infty_1 \frac{1}{t} J_\nu_0(tx)J_\nu_0(tx) dt = \int^\infty_1 \frac{1}{t} J_\nu_0(tx)^2 dt.
\]

This time the hypothesis \( J_\nu(x) = J_\nu_0(x) = 0 \) implies that the rhs of (A1) is zero at the lower integration limit. At the upper integration limit one makes use of the asymptotic behaviour for \( z \to \infty \) \[15\], (9.2.1), \[16\], (10.7.8)],

\[
J_\nu(z) \sim \frac{\sqrt{2/(\pi z)}}{\sin(\nu \pi)} \cos\left(z - \frac{1}{2} \nu \pi - \frac{1}{4} \pi \right) \quad (| \arg z | < \pi).
\]

The latter implies that only the square bracket on the rhs of the Lommel integration formula (A1) contributes at the upper integration limit,

\[
\int^\infty_1 \frac{1}{t} \left| J_\nu_0(tx) \right|^2 dt = -\frac{2}{\pi} \nu_0 \nu_0^\nu - \nu_0 \times \left[ \cos(\omega - \frac{1}{2} \pi - \frac{1}{2} \Im \nu_0 \pi) \cos(\omega + \frac{i}{2} \Im \nu_0 \pi) \right. \\
- \left. \cos(\omega + \frac{i}{2} \Im \nu_0 \pi) \cos(\omega - \frac{1}{2} \pi + \frac{1}{2} \Im \nu_0 \pi) \right],
\]

where \( \omega = z - \frac{1}{2} \Re \nu_0 \pi - \frac{1}{2} \pi \). Because the square bracket can be recast as

\[
\sin(\omega - \frac{i}{2} \Im \nu_0 \pi) \cos(\omega + \frac{i}{2} \Im \nu_0 \pi) \\
- \cos(\omega + \frac{i}{2} \Im \nu_0 \pi) \sin(\omega + \frac{i}{2} \Im \nu_0 \pi) = -\sin i \Im \nu_0 \pi = -i \sinh(\Im \nu_0 \pi),
\]

one obtains eventually

\[
\int^\infty_1 \frac{1}{t} \left| J_\nu_0(tx) \right|^2 dt = \frac{2i}{\pi} \frac{1}{\nu_0^\nu - \nu_0^\nu} \sinh(\Im \nu_0 \pi). \tag{A3}
\]

Then the lhs of (A3) is positive, whereas its rhs

\[
\frac{2i}{\pi} \frac{1}{\nu_0^\nu - \nu_0^\nu} \sinh(\Im \nu_0 \pi) = \frac{1}{\nu_0^\nu - \nu_0^\nu} \sinh(\Im \nu_0 \pi) < 0,
\]

when \( \nu_0 + \nu_0 = 2\Re \nu_0 < 0 \). Hence \( J_\nu(x) \) with \( x \in \mathbb{R} \) has no complex zero for \( \Re \nu < 0 \) \[14\], item 3.2).

Eventually, one can prove that \( J_\nu(x) \) cannot have purely imaginary zeros \[14\], item 3.3]. If it were some, then obviously \( \nu_0 = -\nu_0 \). However the pair \( \{J_{\nu_0}(x), J_{-\nu_0}(x)\} \) provides a basis of linearly independent solutions of the Bessel equation for any non-integer \( \nu_0 \). Because its Wronskian cannot vanish, it is impossible that \( J_{\nu_0}(x) = J_{-\nu_0}(x) = 0 \) for some value of \( x \).

In order to prove the simplicity of zeros of \( J_\nu(x) \), Coulomb \[14\] item 3.4] made use of Watson’s formula \[19\], (13.73(2))

\[
J_\nu(x) \frac{\partial Y_\nu(x)}{\partial \nu} - Y_\nu(x) \frac{\partial J_\nu(x)}{\partial \nu} = -\frac{4}{\pi} \int^\infty_0 K_0(2x \sin \omega) e^{-2\nu \omega} dt < 0,
\]

where the inequality is valid for \( \nu \in \mathbb{R}, x > 0 \). (Note in passing that \( K_\nu(x) \) is real and positive for real order \( \nu \geq -1 \) and \( x > 0 \) \[12\], (9.6.1), (9.6.6)).) Obviously, \( J_\nu(x) \) and \( \partial J_\nu(x)/\partial \nu \) cannot vanish simultaneously.

Appendix B: Zeros of \( I_\nu(x) \) for fixed nonzero \( x \in \mathbb{R} \)
considered as a function of \( \nu \)

For fixed \( z \neq 0 \) each branch of \( I_\nu(z) \) is entire in \( \nu \). For \( x \in \mathbb{R} \) one has \( I_\nu(x) = I_\nu(x) \) (cf. \[13\], (9.6.10), \[16\], (10.25.2)), which is the Schwarz reflection in \( \nu \)-variable. Hence a complex zero \( \nu = \nu_0 \) implies that also \( \nu_0 \) is
Coulomb ingenious proof on the impossibility of complex (i.e. with nonzero imaginary part) zeros of $I_\nu(x)$ for $\Re \nu = \nu_1 > -3/2$ \cite[item 4.2]{14}, is based on the generalized Neumann’s formula \cite[(13.72(2)]{19},

$$I_\mu(x)I_\nu(x) = \frac{2}{\pi} \int_0^{\pi/2} I_{\mu+\nu}(x)(2x \cos \theta) \cos[(\mu - \nu)\theta] d\theta, \quad (B1)$$

which is valid for $\Re \ (\mu + \nu) > -1$.

First, for a complex conjugate pair of $\mu$ and $\nu$ with $\Re \nu = \nu_1 > -1/2$ one obtains from \((B1)\)

$$|I_\nu(x)|^2 = \frac{2}{\pi} \int_0^{\pi/2} I_{2\nu_1}(x)(2x \cos \theta) \cosh(2\nu_2\theta) d\theta > I_{\nu_1}^2(x) > 0,$$

where $\Im \nu = i\nu_2$. Note in passing that a generic $I_\nu(x)$ is real and positive for real order $\nu \geq -1$ and $x > 0$ \cite[(9.6.1), (9.6.6)]{19}. This proves that there are no complex zeros of $I_\nu(x)$ for $\Re \nu > -1/2$.

On returning back to \((B1)\) in the special case when $\mu \neq \nu$, yet $\Im \nu = -\Im \mu$ and $\Re \mu > -1/2$,

$$I_\mu(x)I_\nu(x) = \frac{2}{\pi} \int_0^{\pi/2} I_{\mu+\nu_1}(x)(2x \cos \theta) \cos[(\mu - \nu)\theta] d\theta, \quad (B2)$$

where

$$\cos[(\mu - \nu)\theta] = \cos[(\mu_1 - \nu_1)\theta] \sinh[(\mu_2 - \nu_2)\theta] - i \sin[(\mu_1 - \nu_1)\theta] \cosh[(\mu_2 - \nu_2)\theta].$$

For $\mu_1 = \nu_1$ the imaginary part of $\cos[(\mu - \nu)\theta]$ is zero but its real part is positive. For $\mu_1 \neq \nu_1$, the imaginary part of $\cos[(\mu - \nu)\theta]$ will maintain constant sign on the integration interval $(0, \pi/2)$ in \((B2)\), and thus prevents it from vanishing, when $|\mu_1 - \nu_1| < 2$. Combined together with the hypothesis $\Re \mu > -1/2$ that ensures $I_\mu(x) \neq 0$, and $\mu_1 + \nu_1 > -1$ necessary for the validity of \((B1)\), the task is to find the smallest possible $\nu_1$ that would comply with all the above conditions. This is obviously $\Re \nu = -3/2 + \epsilon$ with some infinitesimal $\epsilon$ (in which case $\mu_1 = 1/2 + \epsilon$, $\epsilon' < \epsilon$). Therefore $I_\mu(x)$ does not have any complex zero for $\Re \nu > -3/2$ \cite[item 4.2]{14}.

That $I_\nu(x)$ does not have any complex zero for $\Re \nu \geq 0$ can be derived also independently by tweaking Coulomb’s proof for $J_\nu(x)$ \cite[item 3.1,3]{14}. Indeed, one can arrive at a Lommel integration formula also for the modified Bessel functions,

$$\int^t t^{-1} Z^{(1)}_{\mu}(tz)Z^{(2)}_{\nu}(tz) dt = \frac{t_2}{t_1^2 - t_2^2} \left[ Z^{(1)}_{\mu+1}(tz)Z^{(2)}_{\nu}(tz) - Z^{(1)}_{\mu}(tz)Z^{(2)}_{\nu+1}(tz) \right] + \frac{1}{t_2} Z^{(1)}_{\mu}(tz)Z^{(2)}_{\nu}(tz), \quad (B3)$$

where $Z^{(1,2)}(z)$ are any two linear combinations of modified cylindrical Bessel functions, $t \in \mathbb{R}$, $z \in \mathbb{C}$. The Lommel integration formula \((B3)\) differs from \((A1)\) merely in the opposite sign in front of the square bracket on the rhs. Formula \((B3)\) can be verified by differentiating both sides and using the defining modified Bessel equation \cite[(9.6.1), (10.25.1)]{19}. One can thus readily repeat the arguments that has led to \((A2)\), and thereby exclude the complex zeros of $I_\nu(x)$ for $\Re \nu > 0$.

Any purely imaginary zero $\nu_0$ of $I_\nu(x)$ can be excluded by essentially the same argument as in \cite[item 3.3]{14}. If it were some $\nu_0$, then obviously $\nu_0 = -\nu_0$. However the pair \{$I_\nu(x), \nu_\nu(x)$\} provides a basis of linearly independent solutions of the Bessel equation for any non-integer $\nu$. Because its Wronskian \cite{15} cannot vanish, it is impossible that $I_{\nu_0}(x) = I_{-\nu_0}(x) = 0$.

$I_\nu(z)$ has the following asymptotic behaviour for $z \to \infty$ \cite[(9.7.1), (10.40.1)]{19},

$$I_\nu(z) \sim \frac{e^z}{\sqrt{2\pi z}} \left( 1 - \frac{4z^2 - 1}{8z} \right) \quad (|\arg z| < \pi/2).$$

Therefore, one cannot perform $\int^\infty_1$ as in eq. \((A3)\) in the case of $J_\nu(z)$.

**Appendix C: $S(k)$ for $k \to \infty$**

For $k = i(n + \frac{1}{2})/(2a)$ (ip $= -n - \frac{1}{2}$) the S-matrix \((8)\) of the main text becomes

$$S(k) = \frac{I_{-n-\frac{1}{2}}(a)\Gamma(\frac{1}{2} - n)}{I_{n+\frac{1}{2}}(a)\Gamma(\frac{1}{2} + n)} \left( \frac{\alpha}{2} \right)^{2n+1},$$

According to \((14)\) of the main text (cf. \cite[eq. (9.6.2)]{12}, \cite[eq. (10.27.2)]{16}),

$$I_{-n-\frac{1}{2}}(z) = I_{n+\frac{1}{2}}(z) + (-1)^n \frac{2}{\pi} K_{n+\frac{1}{2}}(z). \quad (C1)$$

According to \cite[eq. (10.41.1-2)]{16}, for positive real values of $\nu$ in the limit $\nu \to \infty$

$$I_{\nu}(z) = \frac{1}{\sqrt{2\pi \nu}} \left( \frac{e^z}{2\nu} \right)^\nu,$$

$$K_{\nu}(z) = \sqrt{\frac{z}{\pi}} \left( \frac{z}{2\nu} \right)^{-\nu}.$$
which, on combining with (C1), enables one to arrive at

\[
\frac{I_{n-\frac{1}{2}}(\alpha)}{I_{n+\frac{1}{2}}(\alpha)} \sim 1 + 2(-1)^n \left( \frac{2n+1}{e\alpha} \right)^{2n+1}.
\]

At the same time on repeating the defining relation
\[
\Gamma(z+1) = z\Gamma(z)
\]
one has

\[
\begin{align*}
\Gamma(n + \frac{3}{2}) &= \frac{(2n + 1)!!}{2^{n+1}} \Gamma(\frac{1}{2}), \\
\Gamma(\frac{1}{2}) &= (-1)^n \frac{(2n-1)!!}{2^n} \Gamma(\frac{1}{2} - n), \\
\Gamma(\frac{1}{2} - n) &= (-1)^n \frac{2^{2n+1}}{(2n-1)!!(2n+1)!!}.
\end{align*}
\]

Therefore, the S-matrix (8) of the main text has the following limit for
\[
k = i\left(n + \frac{1}{2}\right)/(2a) \quad (i\rho = -n - \frac{1}{2}),
\]
n \to \infty,

\[
S(k) \sim \frac{2}{(2n-1)!!(2n+1)!!} \left( \frac{2n+1}{e} \right)^{2n+1} \\
\sim \frac{2(2n+1)!}{(2n-1)!!(2n+1)!!\sqrt{2\pi(2n+1)}},
\]

where Stirling’s formula has been used to arrive at the 2nd line. The resulting asymptotic behaviour is the same as in the attractive case [13].