Multi-parameter deformed and nonstandard $Y(gl_M)$ Yangian symmetry in integrable variants of Haldane-Shastry spin chain

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Abstract

By using ‘anyon like’ representations of permutation algebra, which pick up nontrivial phase factors while interchanging the spins of two lattice sites, we construct some integrable variants of Haldane-Shastry (HS) spin chain. Lax equations for these spin chains allow us to find out the related conserved quantities. However, it turns out that such spin chains also possess a few additional conserved quantities which are apparently not derivable from the Lax equations. Identifying these additional conserved quantities, and the usual ones related to Lax equations, with different modes of a monodromy matrix, it is shown that the above mentioned HS like spin chains exhibit multi-parameter deformed and ‘nonstandard’ variants of $Y(gl_M)$ Yangian symmetry.

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1 Introduction

One dimensional spin chains as well as dynamical systems with long ranged interactions and their close connection with diverse subjects like Yangian algebra, random matrix theory, fractional statistics, quantum Hall effect etc. have attracted a lot of attention in recent years [1-15]. In particular it is found that, commutation relations among the conserved quantities of spin Calogero-Sutherland (CS) model, given by the Hamiltonian

\[ H_{CS} = -\frac{1}{2} \sum_{i=1}^{N} \left( \frac{\partial}{\partial x_i} \right)^2 + \frac{\pi^2}{L^2} \sum_{i<j} \frac{\beta \left( \beta + P_{ij} \right)}{\sin^2 \frac{\pi}{L} (x_i - x_j)}, \]  

(1.1)

where \( \beta \) is a coupling constant and \( P_{ij} \) is the permutation operator interchanging the ‘spin’ components (taking \( M \) possible values) of \( i \)-th and \( j \)-th particles, generate the \( Y(gl_M) \) Yangian algebra [5]. Furthermore, commutation relations among the conserved quantities of Haldane-Shastry (HS) spin chain, which may be obtained by taking the ‘static limit’ of spin CS model, also lead to this \( Y(gl_M) \) Yangian algebra [2].

However, it is recently found that the permutation algebra given by

\[ P_{ij}^2 = 1, \quad P_{ij} P_{jl} = P_{il} P_{ij} = P_{jl} P_{il}, \quad [P_{ij}, P_{lm}] = 0, \]  

(1.2)

\( (i, j, l, m \text{ being all different indices}) \), admits a novel class of ‘anyon like’ representations on the internal space of spin CS model [16]. Such an ‘anyon like’ representation (\( \tilde{P}_{ij} \)) acts on the internal space of all particles as

\[ \tilde{P}_{ij} |\alpha_1 \alpha_2 \cdots \alpha_i \cdots \alpha_j \cdots \alpha_N \rangle = e^{i\Phi(\alpha_i, \alpha_{i+1}, \cdots, \alpha_j)} |\alpha_1 \alpha_2 \cdots \alpha_j \cdots \alpha_i \cdots \alpha_N \rangle, \]  

(1.3)

where \( \alpha_i \) (\( \in [1, 2, \cdots, M] \)) denotes a spin variable and \( \Phi(\alpha_i, \alpha_{i+1}, \cdots, \alpha_j) \) is a real function of \( (j - i + 1) \) number of such spin variables. So these ‘anyon like’ representations pick up nontrivial phase factors while interchanging the spin of two particles and reduce to the standard representation of permutation algebra (\( P_{ij} \)) at the limiting case \( \Phi = 0 \). Many representations of permutation algebra, including the type (1.3), can be constructed systematically by using the relation given by [17]

\[ \tilde{P}_{i,i+1} = Q_{i,i+1} P_{i,i+1}, \quad \tilde{P}_{ij} = S_{ij} \tilde{P}_{j-1,j} S_{ij}^{-1} = (Q_{i,i+1} Q_{i,i+2} \cdots Q_{ij}) P_{ij} (Q_{j-1,i} Q_{j-2,i} \cdots Q_{i+1,i}), \]  

(1.4)
where \( j > i + 1 \), \( S_{ij} = \tilde{P}_{i,i+1} \tilde{P}_{i+1,i+2} \cdots \tilde{P}_{j-2,j-1} \) and \( Q_{ik} \), which acts like a matrix \( Q \) on the direct product of \( i \)-th and \( k \)-th spin spaces (but acts trivially on all other spin spaces), is obtained by solving the following two equations:

\[
Q_{ik} Q_{il} Q_{kl} = Q_{kl} Q_{il} Q_{ik}, \quad Q_{ik} Q_{ki} = 1.
\] (1.5)

It is easy to check that the above equations, satisfied by the \( Q \) matrix, are in fact sufficient conditions for \( \tilde{P}_{ij} \) (1.4) being a valid representation of permutation algebra (1.2). Consequently, by inserting any solution of eqn. (1.5) to (1.4), we can generate a representation of permutation algebra. For the case \( Q = 1 \), which is the simplest solution of eqn. (1.3), \( \tilde{P}_{ij} \) (1.4) reproduces the standard representation \( P_{ij} \). Moreover, concrete examples of ‘anyon like’ representations (1.3) can also be obtained in a similar way from a class of nontrivial solutions of eqn. (1.3) [17].

Interestingly, one may construct integrable variants of spin CS Hamiltonian (1.1) as

\[
\mathcal{H}_{CS} = -\frac{1}{2} \sum_{i=1}^{N} \left( \frac{\partial}{\partial x_i} \right)^2 + \frac{\pi^2}{L^2} \sum_{i<j} \frac{\beta (\beta + \tilde{P}_{ij})}{\sin^2 \frac{\pi}{L} (x_i - x_j)},
\] (1.6)

\( \tilde{P}_{ij} \) being any possible ‘anyon like’ representation of permutation algebra, and show that such Hamiltonians can be solved exactly by taking appropriate projections of the eigenfunctions of Dunkl operators [16]. However, it turns out that the spectra of spin CS Hamiltonians (1.6) might differ considerably from that of their standard counterpart (1.1). Furthermore it is found that, the symmetry of a spin CS model like (1.6) crucially depends on the choice of corresponding representation \( \tilde{P}_{ij} \), and may be given by a multi-parameter deformed or ‘nonstandard’ variant of \( Y(gl_M) \) Yangian algebra [17].

It is evident that, in analogy with the case of spin CS models (1.6), one may also define novel variants of HS spin chain as

\[
\mathcal{H}_{HS} = \sum_{1 \leq i < j \leq N} \frac{1}{2 \sin^2 \frac{\pi}{N} (i - j)} (\tilde{P}_{ij} - 1),
\] (1.7)

where \( \tilde{P}_{ij} \) is any possible ‘anyon like’ representation of permutation algebra which can be obtained through the relation (1.4). For the limiting case \( \tilde{P}_{ij} = P_{ij} \), \( \mathcal{H}_{HS} \) (1.7) reproduces the standard HS spin chain [1,6] which exhibits the \( Y(gl_M) \) Yangian symmetry. So, it is
natural to ask whether the spin model (1.7) would also be quantum integrable and respect some Yangian like symmetry for all possible choice of corresponding $\tilde{P}_{ij}$. In the present article, our aim is to answer this question by exploiting an intimate connection between spin CS model and HS spin chain. So in sec.2, we start with the known Lax operators of spin CS Hamiltonian (1.6) and show that the ‘static limit’ of such operators would easily lead to the Lax pair of HS like spin chain (1.7). Lax equations for the spin chain (1.7) immediately allow us to write down the associated conserved quantities. Subsequently, in sec.3, we find that the spin chain (1.7) would also possess a few additional conserved quantities which are not derivable from the Lax equations. Identifying these additional conserved quantities, and the usual ones related to Lax equations, with different modes of a monodromy matrix, we are able to demonstrate that the spin chain (1.7) would respect a multi-parameter deformed or ‘nonstandard’ variant of $Y(gl_M)$ Yangian symmetry. It may be noted that, while deriving the above mentioned result, we do not assume any specific form of the ‘anyon like’ representation $\tilde{P}_{ij}$ and develop a rather general method for finding out the symmetry of spin chain (1.7). However, at the end of sec.3, we finally use a particular class of $\tilde{P}_{ij}$ for generating some concrete examples of spin chains (1.7) and apply our general method for finding out the explicit forms of related symmetry algebras. Sec.4 is the concluding section.

2 Lax pairs and conserved quantities for HS like spin chains

In this section we like to start with the known Lax operators of spin CS model (1.6) and explore how they might be used to generate the Lax pair of HS like spin chain (1.7). For this purpose, it is convenient to rewrite $\mathcal{H}_{CS}$ (1.6) as a power series expansion of the coupling constant $\beta$:

$$\mathcal{H}_{CS} = \frac{2\pi^2}{L^2} \left( \mathcal{H}_0 + \beta \mathcal{H}_1 + \beta^2 \mathcal{H}_2 \right), \quad (2.1)$$
where $H_0 = \sum_{j=1}^{N}(z_j \frac{\partial}{\partial z_j})^2$, $H_1 = 2 \sum_{i<j} \theta_{ij} \theta_{ji} \tilde{P}_{ij}$, $H_2 = 2 \sum_{i<j} \theta_{ij} \theta_{ji}$ and $z_j, \theta_{ij}$ are defined as: $z_j = e^{2\pi i x_j}$, $\theta_{ij} = \frac{z_i - z_j}{z_i - z_j}$. Lax operators associated with this spin CS Hamiltonian may be given by [17]

$$\mathcal{L} = \mathcal{L}_0 + \beta \mathcal{L}_1, \quad \mathcal{M} = \frac{2\pi^2}{L^2} \beta \mathcal{M}_1,$$

(2.2)

where $\mathcal{L}_0, \mathcal{L}_1$ and $\mathcal{M}_1$ are $N \times N$ matrices with operator valued elements:

$$(\mathcal{L}_0)_{ij} = \delta_{ij} z_j \frac{\partial}{\partial z_j}, \quad (\mathcal{L}_1)_{ij} = (1 - \delta_{ij}) \theta_{ij} \tilde{P}_{ij}, \quad (\mathcal{M}_1)_{ij} = -2 \delta_{ij} \sum_{k \neq i} h_{ik} \tilde{P}_{ik} + 2 (1 - \delta_{ij}) h_{ij} \tilde{P}_{ij},$$

$h_{ij} = \theta_{ij} \theta_{ji}$ and it is assumed that $\tilde{P}_{ij}$ is same as $\tilde{P}_{ji}$. It may be noted that, the Lax pair (2.2) can be obtained from the Lax pair of usual spin CS model (1.1) [5,7] by simply substituting $\tilde{P}_{ij}$ to the place of $P_{ij}$. However, unlike the usual case, it is necessary to define some other nontrivial operators for obtaining all Lax equations related to the spin CS model (1.6). For this purpose, we consider the $Q$ matrix which is a solution of eqn.(1.5) and generates $\tilde{P}_{ij}$ through the relation (1.4). By using this $Q$ matrix, one may define a set of operators as

$$\tilde{X}^{\alpha \beta}_i = Q_{i,i+1} Q_{i,i+2} \cdots Q_{iN} X^{\alpha \beta}_i Q_{i1} Q_{i2} \cdots Q_{i,i-1},$$

(2.3)

where $X^{\alpha \beta}_i$ acts like $|\alpha\rangle\langle\beta|$ on the spin space associated with $i$-th particle and leaves the spin spaces associated with all other particles untouched. It is clear that, for the limiting case $Q = \mathbb{1}$ (i.e., when $\tilde{P}_{ij} = P_{ij}$), $\tilde{X}^{\alpha \beta}_i$ (2.3) reduces to the local operator $X^{\alpha \beta}_i$. However, the above defined $\tilde{X}^{\alpha \beta}_i$s would generally represent a set of highly nonlocal spin dependent operators. Moreover, by applying eqn.(1.5), it can be shown that $\tilde{X}^{\alpha \beta}_i$ and $\tilde{P}_{ij}$ (1.4) obey the simple algebraic relations:

$$\tilde{P}_{ij} \tilde{X}^{\alpha \beta}_i = \tilde{X}^{\alpha \beta}_j \tilde{P}_{ij}.$$

(2.4)

By using the above equation, the permutation algebra (1.2) and the canonical commutation relations: $[\frac{\partial}{\partial z_i}, z_j] = \delta_{ij}$, it is easy to see that the Hamiltonian $H_{CS}$ (2.1), Lax pair $\mathcal{L}$ and $\mathcal{M}$ (2.2), and the operators $\tilde{X}^{\alpha \beta}_i$ (2.3) satisfy the Lax equations given by

$$[H_{CS}, \mathcal{L}_{ij}] = \sum_{k=1}^{N} (\mathcal{L}_{ik} M_{kj} - M_{ik} \mathcal{L}_{kj}), \quad [H_{CS}, \tilde{X}^{\alpha \beta}_j] = \sum_{k=1}^{N} \tilde{X}^{\alpha \beta}_k M_{kj}, \quad \sum_{k=1}^{N} M_{jk} = 0,$$

(2.5a, b, c)
With the help of these Lax equations, one can easily derive the conserved quantities of spin CS model (1.6) and show that the commutation relations among such conserved quantities would lead to a multi-parameter dependent or ‘nonstandard’ variant of $Y(gl_M)$ Yangian algebra [17].

However, for our present purpose of finding out the Lax operators associated with HS like spin chain (1.7), it is useful to observe that eqns.(2.5a,b,c) are satisfied for any value of the coupling constant $\beta$ which appears in the Hamiltonian (2.1) as well as the Lax pair (2.2). Consequently, by inserting (2.1) and (2.2) to eqns.(2.5a), (2.5b) and (2.5c), and comparing the coefficients of $\beta^2$, $\beta$ and $\beta$ respectively from both sides of these equations, we can derive the following set of independent relations:

$$\begin{align*}
\left[ H_1, (L_1)_{ij} \right] + \left[ H_2, (L_0)_{ij} \right] &= \sum_{k=1}^{N} \left\{ (L_1)_{ik} (M_1)_{kj} - (M_1)_{ik} (L_1)_{kj} \right\}, \\
\left[ H_1, \tilde{X}^{\alpha\beta}_j \right] &= \sum_{k=1}^{N} \tilde{X}^{\alpha\beta}_k (M_1)_{kj}, \quad \sum_{k=1}^{N} (M_1)_{jk} = 0.
\end{align*}$$

(2.6a, b, c)

It is curious to notice that, the form of eqn.(2.6a) would coincide with the form of Lax equation (2.5a), provided we assume the validity of following simple condition:

$$\left[ H_2, (L_0)_{ij} \right] = 0.$$  

(2.7)

However, by substituting the explicit form of $H_2$ and $(L_0)_{ij}$ to the above equation, we find that it reduces to

$$\sum_{k \neq j} \theta_{jk} \theta_{kj} (\theta_{jk} - \theta_{kj}) = 0.$$ 

(2.8)

It is well known that eqn.(2.8) admits the solution: $z_k = \omega^k$ with $\omega = e^{2\pi i / N}$ [5] and leads to the ‘static limit’, where the particles of spin CS model are frozen at equidistant positions on the arc of a unit circle. Thus it is clear that, by imposing the ‘static limit’, eqns.(2.6a,b,c) can be recast in the form of Lax equations where $H_1$ plays the role of Hamiltonian and the operators $L_1$, $M_1$ play the role of related Lax pair. It is interesting to notice further that, at this static limit, the operator $H_1$ would coincide with the Hamiltonian (1.7) up to an additive constant factor. Consequently, the static limit of eqns.(2.6a,b,c) should give us the desired Lax equations for HS like spin chain.
Therefore, we can explicitly write down the Lax equations associated with the
Hamiltonian \((L_7)\) as
\[
\begin{align*}
[H_{HS}, (L_{HS})_{ij}] &= \sum_{k=1}^{N} \{ (L_{HS})_{ik} (M_{HS})_{kj} - (M_{HS})_{ik} (L_{HS})_{kj} \}, \\
[H_{HS}, \tilde{X}^{\alpha\beta}_{j}] &= \sum_{k=1}^{N} \tilde{X}^{\alpha\beta}_{k} (M_{HS})_{kj}, \quad \sum_{k=1}^{N} (M_{HS})_{jk} = 0,
\end{align*}
\] (2.9)
where the elements of Lax pair are given by
\[
(L_{HS})_{ij} = (1 - \delta_{ij}) \theta'_{ij} \tilde{P}_{ij}, \quad (M_{HS})_{ij} = -2 \delta_{ij} \sum_{k \neq i} h'_k \tilde{P}_{ik} + 2 (1 - \delta_{ij}) h'_i \tilde{P}_{ij},
\] (2.10)
with \(\theta'_{kj} = \frac{\omega^k}{\omega^j - \omega^k}\) and \(h'_{kj} = \theta'_{kj} \theta'_{jk}\). Now, by using the Lax equations (2.9), it is easy to
demonstrate that the operators given by
\[
T^{\alpha\beta}_{n} = \sum_{i,j=1}^{N} \tilde{X}^{\alpha\beta}_{i} (L_{HS})_{ij},
\] (2.11)
where \(n \in [0, 1, \cdots, \infty]\) and \(\alpha, \beta \in [1, \cdots, M]\), would commute with the Hamiltonian \((L_7)\).

Thus, by imposing the ‘static limit’ on the known Lax equations of spin CS model \((L_6)\), we are able to derive here the Lax equations and conserved quantities associated with any spin chain which can be written in the form \((L_7)\). It is evident that, at the
limiting case \(Q = 1\), (2.11) would reproduce the local conserved quantities of standard HS spin chain. However, for any nontrivial choice of \(\tilde{P}_{ij}\) or corresponding \(Q\) matrix, eqn. (2.11) would give us highly nonlocal type of conserved quantities associated with the spin chain \((L_7)\). In the next section, our aim is to find out the algebraic relations satisfied by such
conserved quantities and establish their connection with the Yang-Baxter equation.

\section{3 \ Extended \(Y(gl_M)\) Yangian symmetry in HS like spin chains}

We have already mentioned that, commutation relations between the conserved quantities of HS spin chain yield the \(Y(gl_M)\) Yangian algebra [2]. This \(Y(gl_M)\) Yangian algebra
can be defined through the operator valued elements of an $M \times M$ dimensional monodromy matrix $T^0(u)$, which obeys the quantum Yang-Baxter equation (QYBE)

$$R_{00'}(u - v) \left( T^0(u) \otimes 1 \right) \left( 1 \otimes T^{0'}(v) \right) = \left( 1 \otimes T^{0'}(v) \right) \left( T^0(u) \otimes 1 \right) R_{00'}(u - v). \quad (3.1)$$

Here $u$ and $v$ are spectral parameters and the $M^2 \times M^2$ dimensional rational $R(u - v)$ matrix, having $c$-number valued elements, is taken as

$$R_{00'}(u - v) = (u - v) 1 + P_{00'}. \quad (3.2)$$

Associativity of $Y(gl_M)$ algebra is ensured from the fact that the above $R$ matrix satisfies Yang-Baxter equation (YBE) given by

$$R_{00'}(u - v) R_{00'}(u - w) R_{00'}(v - w) = R_{00'}(v - w) R_{00'}(u - w) R_{00'}(u - v), \quad (3.3)$$

where a matrix like $R_{ab}(u - v)$ acts nontrivially only on the $a$-th and $b$-th vector spaces. The conserved quantities of HS spin chain yield a realisation of the above mentioned monodromy matrix satisfying QYBE (3.1).

Now, for finding out algebraic relations among the conserved quantities (2.11), we consider an $R$ matrix of the form

$$R_{00'}(u - v) = (u - v) Q_{00'} + P_{00'}, \quad (3.4)$$

where $Q_{00'}$ acts like $Q$ on the direct product of 0-th and 0'-th auxiliary spaces. Since this $Q$ matrix generates $P_{ij}$ through the expression (1.4), it is obvious that there exists a one-to-one correspondence between the $R$ matrix (3.4) and the Hamiltonian (1.7) which contains $P_{ij}$. Moreover, by using the conditions (1.5) satisfied by $Q$ matrix, it is easy to check that the $R$ matrix (3.4) would be a solution of YBE (3.3). Consequently, by inserting the rational $R$ matrix (3.4) to QYBE (3.1), one can construct an associative, ‘extended Yangian’ algebra. At the limiting case $Q_{00'} = 1$, (3.4) reduces to the $R$ matrix (3.2) and leads to the standard $Y(gl_M)$ Yangian algebra. However, in general, the $Q$ matrix appearing in (3.4) might also depend on a set of continuous as well as discrete deformation parameters. So these parameters would naturally appear in the defining relations of
corresponding extended Yangian algebra. One may also make a short classification of
such extended Yangian algebras in the following way. Let us first assume that the $Q$
matrix, which is obtained by solving eqn.(1.5), admits a Taylor series expansion of the
form (up to an over all sign factor)

$$Q_{00'} = \mathbb{1} + \sum_p h_p Q_{00'}^p + \sum_{p,q} h_p h_q Q_{00'}^{pq} + \cdots, \quad (3.5)$$

where $h_p$s are continuous deformation parameters and the leading term is an identity
operator. Evidently the extended Yangian algebras, generated through this type of $Q$
matrices, would reduce to standard $Y(gl_M)$ algebra at the limit $h_p \to 0$ for all $p$. So
it is natural to call these algebras as multi-parameter dependent deformations of $Y(gl_M)$
Yangian algebra. A few concrete examples of multi-parameter deformed Yangian algebras
have already appeared in the literature [20-22] and it is recently found that an integrable
extension of Hubbard model respects such deformed Yangian symmetry [23]. However,
it is also possible to find out some solutions of eqn.(1.5) which can not be expanded in
the form (3.5) [17]. So the Yangian algebras, generated through this type of $Q$
matrices and corresponding rational solutions (3.4), would not reduce to $Y(gl_M)$ algebra at the
limit $h_i \to 0$. Consequently, these Yangians may be classified as ‘nonstandard’ variants
of $Y(gl_M)$ Yangian algebra.

At present our aim is to show that, the algebraic relations among the conserved quan-
tities (2.11) would be given by an extended (i.e., a multi-parameter deformed or ‘nonstan-
dard’ variant of) $Y(gl_M)$ Yangian. To this end, we first consider the following monodromy
matrix whose operator valued elements act on the Hilbert space of spin CS model (1.1)
[5]:

$$T^0(u) = \Pi \left( \mathbb{1} + \sum_{i=1}^N \frac{P_{0i}}{u - D_i} \right), \quad (3.6)$$

where $D_i$s, the so called Dunkl operators, are given by

$$D_i = \sum_{j \neq i} \theta_{ij} K_{ij}, \quad (3.7)$$

$\theta_{ij} = \frac{z_i}{z_i - z_j}$ and $K_{ij}$s, the coordinate exchange operators, are defined through the relations

$$K_{ij} z_i = z_j K_{ij}, \quad K_{ij} \frac{\partial}{\partial z_i} = \frac{\partial}{\partial z_j} K_{ij}, \quad K_{ij} z_l = z_l K_{ij}, \quad (3.8a)$$
\[ K_{ij}^2 = 1, \quad K_{ij}K_{jl} = K_{il}K_{ij}, \quad K_{ij}K_{lm} = 0, \] (3.8b)

\( i, j, l, m \) being all different indices. Moreover, the projection operator \( \Pi \), appearing in (3.6), allows one to replace \( K_{ij} \) by \( P_{ij} \) (after taking \( K_{ij} \) at the extreme right side of an expression). It is easy to check that \( D_i \)s and \( K_{ij} \)s satisfy the commutation relations given by

\[ K_{ij}D_i = D_jK_{ij}, \quad [K_{ij}, D_k] = 0, \quad [D_i, D_j] = (D_i - D_j)K_{ij}, \] (3.9)

where \( k \neq i, j \). By using these commutation relations, it can be proved that the monodromy matrix (3.6) yields a solution of QYBE (3.1) when the corresponding \( R \) matrix is taken as (3.2) [5].

In analogy with (3.6), we now propose another monodromy matrix as

\[ T^0(u) = \Pi^* \left( \Omega_0 + \sum_{i=1}^{N} \frac{P_{0i}}{u - D_i} \right), \] (3.10)

where

\[ \Omega_0 = Q_{01}Q_{02}\cdots Q_{0N}, \quad P_{0i} = (Q_{01}Q_{02}\cdots Q_{0,i-1}) P_{0i} (Q_{0,i+1}Q_{0,i+2}\cdots Q_{0N}), \] (3.11a, b)

and \( \Pi^* \) denotes a new projection operator which allows one to replace \( K_{ij} \) by \( \tilde{P}_{ij} \) (after taking \( K_{ij} \) at the extreme right side of an expression):

\[ \Pi^* (K_{ij}) = \tilde{P}_{ij} = (Q_{i,i+1}Q_{i,i+2}\cdots Q_{ij}) P_{ij} (Q_{j-1,i}Q_{j-2,i}\cdots Q_{i+1,i}). \]

It is worth noting that the monodromy matrix (3.10), which reduces to (3.6) at the limiting case \( Q = 1 \), can be determined uniquely from any given ‘anyon like’ representation \( \tilde{P}_{ij} \) (or, from the corresponding \( Q \) matrix). Furthermore, by essentially following the approach of Ref.5 and using the relations (3.9) as well as (3.5), it can be shown that the monodromy matrix (3.10) would satisfy QYBE (3.1) when the corresponding \( R \) matrix is taken as (3.4). Thus, we interestingly find that the monodromy matrix (3.10) yields a realisation of extended \( Y(gl_M) \) Yangian algebra. However, it should be noted that, the operator valued elements of monodromy matrix (3.10) act on the Hilbert space of spin CS model, which contain both spin and dynamical degrees of freedom. So, for constructing operators
which would act only on the spin space, we consider a monodromy matrix like
\[
\hat{T}_0(u) = \langle \omega_1 \omega_2 \cdots \omega_N | T^0(u) | \omega_1 \omega_2 \cdots \omega_N \rangle , \tag{3.12}
\]
where \( T^0(u) \) is given by (3.10), and \( | \omega_1 \omega_2 \cdots \omega_N \rangle \) denotes a ket vector related to the ‘static limit’ of spin CS model: \( z_k | \omega_1 \omega_2 \cdots \omega_N \rangle = \omega^k | \omega_1 \omega_2 \cdots \omega_N \rangle \). Since \( T^0(u) \) (3.10) does not depend on any momentum operator like \( \frac{\partial}{\partial z_k} \), it is clear that the monodromy matrix \( \hat{T}_0(u) \) (3.12) would also give a realisation of extended \( Y(gl_M) \) Yangian algebra generated by the \( R \) matrix (3.4). Moreover, the elements of \( \hat{T}_0(u) \) would evidently act on the Hilbert space associated with HS like spin chain.

Next, we want to write down two simple relations, which will be used shortly for establishing a connection between the conserved quantities (2.11) and our realisation (3.12) of extended Yangian algebra. First of all, by applying the commutation relations (3.9), it is easy to find that
\[
\langle \omega_1 \omega_2 \cdots \omega_N | \Pi^n \left( D^n_i \right) | \omega_1 \omega_2 \cdots \omega_N \rangle = \sum_{j=1}^N \left( L_{HS}^n \right)_{ij} , \tag{3.13}
\]
where the elements of \( L_{HS} \) are given by eqn.(2.10). Secondly, by using the standard relation: \( P_{0i} = \sum_{\alpha,\beta=1}^M X_0^{\alpha \beta} \otimes X_i^{\beta \alpha} \) and the conditions (1.5) on \( Q \) matrix, the operator \( P_{0i} \) (3.11b) can be rewritten in a compact form like
\[
P_{0i} = \sum_{\alpha,\beta=1}^M X_0^{\alpha \beta} \otimes \tilde{X}_i^{\beta \alpha} . \tag{3.14}
\]
Now, with the help of relations (3.13) and (3.14), we can interestingly express the monodromy matrix (3.12) through its modes as
\[
\hat{T}_0(u) = \Omega_0 + \sum_{n=0}^\infty \frac{1}{u^{n+1}} \sum_{\alpha,\beta=1}^M \left( X_0^{\alpha \beta} \otimes T_n^{\beta \alpha} \right) , \tag{3.15}
\]
where \( T_n^{\alpha \beta} \)s are the conserved quantities (2.11) which commute with Hamiltonian (1.7).

It is curious to notice that, apart from the conserved quantities (2.11) which have been derived from the Lax equations, the operator \( \Omega_0 \) also appears in the mode expansion of monodromy matrix (3.13). But it seems that, by applying Lax equations (2.9), it
is not possible to determine whether this $\Omega_0$ would also commute with the spin chain Hamiltonian (1.7). However, with the help of eqn. (1.5), one can directly show that

$$\left[\Omega_0, \tilde{P}_{ij}\right] = \left[\Omega^{\alpha\beta}, \tilde{P}_{ij}\right] = 0,$$

(3.16)

where $\Omega^{\alpha\beta}$s are the operator valued elements of $\Omega_0$: $\Omega_0 = \sum_{\alpha,\beta} X_0^{\alpha\beta} \otimes \Omega^{\beta\alpha}$. Using the relation (3.16), we may now easily verify that $\Omega_0$ and $\Omega^{\alpha\beta}$ indeed commute with the Hamiltonian (1.7). Thus we curiously observe that, HS like spin chains possess some additional conserved quantities ($\Omega^{\alpha\beta}$s) which are not apparently derivable from the Lax equations. However, it is obvious that these extra conserved quantities would become trivial at the limit $Q = 1$ and, therefore, one does not encounter such conserved quantities in the case of usual HS model. From the above discussions it also turns out that, all modes of the monodromy matrix $\hat{T}^0(u)$ (3.15) can be identified with various conserved quantities which commute with the Hamiltonian (1.7). Therefore, we may conclude that the HS like spin chain (1.7) respects an extended $Y(gl_M)$ Yangian symmetry. Furthermore, we can explicitly construct the algebraic relations among the conserved quantities of such spin chain, by simply expressing this extended $Y(gl_M)$ Yangian algebra through the modes of corresponding monodromy matrix.

It is worth noting that, for proving the extended $Y(gl_M)$ Yangian symmetry of Hamiltonian (1.7), we have not used so far any particular form of the related ‘anyon like’ representation and only assumed that such a representation can be obtained through the relation (1.4). Thus, we are able to develop a rather general framework for finding out the conserved quantities and symmetry algebra associated with all Hamiltonians which can be expressed in the form (1.7). In the following, however, we like to discuss about some concrete examples of ‘anyon like’ representations of permutation algebra, the related HS like spin chains and corresponding symmetry algebras. To this end, we first observe that their exists a class of solutions of eqn. (1.5) as

$$Q_{ik} = \sum_{\sigma=1}^{M} e^{i\phi_{\sigma\sigma}} X_i^{\sigma\sigma} \otimes X_k^{\sigma\sigma} + \sum_{\sigma \neq \gamma} e^{i\phi_{\gamma\sigma}} X_i^{\sigma\sigma} \otimes X_k^{\gamma\gamma},$$

(3.17)

where $\phi_{\sigma\sigma}$s are some discrete parameters each of which can be freely chosen as 0 or $\pi$, and $\phi_{\gamma\sigma}$s are continuous deformation parameters which satisfy the antisymmetry property:
\[ \phi_{\gamma\sigma} = -\phi_{\sigma\gamma}. \]

So, for any possible choice of the above mentioned \( M \) number of discrete parameters and \( \frac{M(M-1)}{2} \) number of independent continuous parameters, (3.17) would give us a distinct solution of (1.5). By substituting the \( Q \) matrix solution (3.17) to (1.4), one can easily obtain a class of ‘anyon like’ representations which will also depend on these discrete as well as continuous deformation parameters and act on the spin space as

\[
\tilde{P}_{ij} |\alpha_1\alpha_2\cdots\alpha_i\cdots\alpha_j\cdots\alpha_N\rangle = 
\exp \left\{ i \phi_{\alpha_i\alpha_j} + i \sum_{\tau=1}^{M} n_\tau \left( \phi_{\tau\alpha_j} - \phi_{\tau\alpha_i} \right) \right\} |\alpha_1\alpha_2\cdots\alpha_j\cdots\alpha_i\cdots\alpha_N\rangle,
\]

(3.18)

where \( n_\tau \) denotes the number of times of occurring the particular spin orientation \( \tau \) in the configuration \( |\alpha_1\cdots\alpha_i\cdots\alpha_p\cdots\alpha_j\cdots\alpha_N\rangle \), when the index \( p \) in \( \alpha_p \) is varied from \( i+1 \) to \( j-1 \). Evidently, for the trivial choice of deformation parameters as: \( \phi_{\gamma\gamma} = \phi_{\gamma\sigma} = 0 \) for all \( \sigma, \gamma \), \( \tilde{P}_{ij} \) (3.18) reduces to the standard representation of permutation algebra, which does not pick up any phase factor while interchanging the spins of two lattice points. However, for any nontrivial choice of these deformation parameters, the above \( \tilde{P}_{ij} \) picks up a phase factor which would depend on the spin configuration of \( (j-i+1) \) number of lattice points. Consequently, by substituting such \( \tilde{P}_{ij} \) to (1.7), we can generate concrete examples of HS like spin chain. The Lax pairs and conserved quantities, associated with these spin chains, may also be obtained by inserting \( \tilde{P}_{ij} \) (3.18) and the related \( \tilde{X}_i^{\alpha\beta} \) (2.3) to the expressions (2.10) and (2.11). Moreover, the form of \( Q \) matrix (3.17) suggests that \( \Omega_0 \) (3.11a) can now be written in a diagonal form: \( \Omega_0 = \sum_\alpha X_0^{\alpha\alpha} \otimes \Omega^{\alpha\alpha} \). So these \( \Omega^{\alpha\alpha} \)'s would give us \( M \) number of additional conserved quantities which are not related to the Lax equations.

Now, for finding out algebraic relations among the above mentioned conserved quantities, we substitute the \( Q \) matrix (3.17) to (3.4) and obtain a class of rational \( R \) matrices which also depend on the parameters \( \phi_{\sigma\sigma}, \phi_{\sigma\gamma} \):

\[
R_{00'}(u - v) = (u - v) \sum_{\sigma,\gamma=1}^{M} e^{i\phi_{\gamma\sigma}} X_0^{\sigma\sigma} \otimes X_0^{\gamma\gamma} + \sum_{\sigma,\gamma=1}^{M} X_0^{\sigma\gamma} \otimes X_0^{\gamma\sigma}.
\]

(3.19)

It is clear that, by inserting these \( R \) matrices to QYBE (3.1), one can generate a class of extended \( Y(gl_M) \) Yangian algebras. However it should be noted that, only for the special
choice of discrete parameters as $\phi_{\sigma \sigma} = 0$ (or, $\phi_{\sigma \sigma} = \pi$) for all $\sigma$, the $Q$ matrix (3.17) admits an expansion in the form (3.18). Therefore, only for these two choices of discrete parameters, the corresponding $R$ matrices (3.19) would generate multi-parameter dependent deformations of $Y(gl_M)$ Yangian algebra. For any other choice of discrete parameters $\epsilon_{\sigma}$, the $R$ matrix (3.19) will evidently lead to a ‘nonstandard’ variant of $Y(gl_M)$ Yangian algebra. The monodromy matrix (3.15), associated with the representation (3.18), would give us concrete realisation of such multi-parameter deformed or ‘nonstandard’ variant of $Y(gl_M)$ Yangian algebra. By inserting this monodromy matrix (3.15) as well as $R$ matrix (3.19) to QYBE (3.1), and equating from its both sides the coefficients of same powers in $u, v$, we easily obtain the following set of relations:

$$\left[ \Omega^{\alpha \alpha}, \Omega^{\beta \beta} \right] = 0, \quad \Omega^{\alpha \alpha} T_n^{\beta \gamma} = \frac{\rho_{\alpha \beta}}{\rho_{\alpha \beta}} T_n^{\gamma \delta} \Omega^{\alpha \alpha}, \quad (3.20a,b)$$

$$\rho_{\alpha \gamma} T_n^{\alpha \beta} T_n^{\gamma \delta} - \rho_{\beta \delta} T_n^{\gamma \delta} T_n^{\alpha \beta} = \delta_{\alpha \delta} T_n^{\beta \gamma} \Omega^{\alpha \alpha} - \delta_{\gamma \beta} \Omega^{\gamma \gamma} T_n^{\alpha \delta}, \quad (3.20c)$$

$$\left[ \rho_{\alpha \gamma} T_n^{\alpha \beta} T_m^{\gamma \delta} - \rho_{\beta \delta} T_m^{\gamma \delta} T_n^{\alpha \beta} \right] - \left[ \rho_{\alpha \gamma} T_m^{\alpha \beta} T_{n+1}^{\gamma \delta} - \rho_{\beta \delta} T_{n+1}^{\gamma \delta} T_m^{\alpha \beta} \right] = \left( T_n^{\gamma \beta} T_n^{\alpha \delta} - T_n^{\gamma \beta} T_n^{\alpha \delta} \right), \quad (3.20d)$$

where $\rho_{\alpha \gamma} = e^{i\phi_{\alpha \gamma}}$. Thus we are able to derive here the algebraic relations among the conserved quantities of spin chain (1.7) containing ‘anyon like’ representation (3.18). It may be noted that, at the special case $\rho_{\alpha \gamma} = 1$ and $\Omega^{\alpha \alpha} = 1$ for all $\alpha, \gamma$, the algebra (3.20) reduces to standard $Y(gl_M)$ Yangian and reproduces the commutation relations between the conserved quantities of HS spin chain.

Finally we like to mention that the quantum $R$ matrix (3.4), which played a crucial role in finding out the symmetry of HS like spin chain (1.7), also appeared previously in the context of integrable models with short ranged interaction. As it is well known, one can construct a generalisation of symmetric six-vertex model by introducing horizontal and vertical electric fields which interact with the related dipoles [24]. Higher dimensional extensions of such asymmetric six vertex model, containing $M$-state ($M > 2$) bonds at each vertex point, are also found [25]. Furthermore, one can obtain an integrable generalisation of Heisenberg spin chain by introducing the Dzyaloshinsky-Moriya (DM) interaction [26]. However, it may be noted that, the rational limit of all quantum $R$-
matrices, which appear in the algebraic Bethe ansatz of Heisenberg spin chain with DM interaction, the asymmetric six vertex model and its higher dimensional extensions, can always be written in the form (3.4). So, there might exist an intriguing connection between the algebraic structures of the above mentioned integrable models with short ranged interactions and the HS like spin chains (1.7) considered by us.

4 Concluding Remarks

Here we have constructed some novel variants of Haldane-Shastry (HS) spin chain which would respect extended (i.e., multi-parameter dependent or ‘nonstandard’) $Y(gl_M)$ Yangian symmetries. An interesting feature of these spin chains is that they contain nonlocal interactions, which can be expressed through the ‘anyon like’ representations of permutation algebra. We have also established the integrability of such spin chains, by finding out the associated Lax pairs and conserved quantities. However, it turned out that these models also possess a few additional conserved quantities which are not derivable from the Lax equations. These additional conserved quantities are found to play an important role in generating the monodromy matrix and symmetry algebra of the above mentioned HS like spin chains.

The existence of extended $Y(gl_M)$ Yangian symmetry in integrable variants of HS spin chain might lead to further developments in several directions. For example, it should be interesting to investigate whether these spin chains, containing ‘anyon like’ representations of permutation algebra, can also be solved exactly in analogy with their standard counterpart. However, it is found earlier that, the spectra of some spin CS models (1.6), with ‘nonstandard’ Yangian symmetries, differ significantly from that of the usual spin CS model (1.1) [16-17]. So it might be particularly interesting to search for the spectra of HS like spin chains (1.7), which exhibit ‘nonstandard’ Yangian symmetries. Moreover, as it is well known, the degeneracy of wave functions for HS model can be explained quite nicely through the representations of $Y(gl_M)$ algebra. Therefore, it is natural to expect that the
representations of extended $\mathcal{Y}(gl_M)$ algebras would play a similar role in analysing the spectra of related HS like spin chains. Furthermore, it might be fruitful to apply asymptotic Bethe ansatz method for investigating various thermodynamic properties of these spin chains and explore their connection with Haldane’s generalised exclusion statistics. Finally, we hope that it would be possible to find out many other new type of quantum integrable spin chains and dynamical models, which would exhibit the extended $\mathcal{Y}(gl_M)$ Yangian symmetries.

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