Derivation of non-Markoffian transport equations for trapped cold atoms in nonequilibrium thermal field theory

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I. INTRODUCTION

The systems of trapped cold atoms are ideal for studying quantum many-body theories such as quantum field theory and thermal field theory. They are dilute and weak-interacting, so theoretical calculations can be compared with experimental results directly.

Considering the kinetic possibilities of Bose–Einstein condensates 1, 2, 3, the formation and growth of condensate 4, the thermal shift of the energy spectrum 5, and many other intriguing phenomena have been observed with good accuracy, and offer opportunities to test quantum many-body theories in both equilibrium and nonequilibrium.

In the aim of describing the kinetics of the trapped cold atom system, a number of theoretical approaches have been proposed such as the methods of the quantum Boltzmann master equation 6, 7, the quantum Boltzmann equation with the local density approximation 8, 9, the closed path (CTP) formalism 10, 11, and the effective Hamiltonian method in Thermo Field Dynamics (TFD) 12, 13. These are in good agreement with the experiments 2, 4. They however are based on a phase-space distribution function, and the energy spectrum is not quantized. This implies that the discussions of the particle representation or the diagonalization of the Hamiltonian are absent, while they are essential for the quantum field theory.

There are two nonequilibrium extension of the thermal field theory, i.e., the closed time path formalism and TFD 14. The CTP formalism is widely used. But we employ the TFD formalism in this paper, because the concept of quasi-particle picture is clear even in nonequilibrium situations there. In TFD, which is a real-time canonical formalism of quantum field theory, thermal fluctuation is introduced through doubling the degrees of the freedom, and the mixed state expectation is replaced by an average of a pure state vacuum, called the thermal vacuum.

It is crucial in our formulation of TFD to construct the interaction picture. In quantum field theory, the choice of unperturbed Hamiltonian and fields is that of quasi particle picture, and concrete calculations are possible only when a particular unperturbed representation, or a particular particle picture, is specified. One does not know an exact unperturbed representation beforehand. Taking plausible representations, parameterized by some parameters, we calculate the propagators of the Heisenberg fields and require some conditions on them, called self-consistent renormalization conditions, which pick up a self-consistent representation and determine the parameters. The renormalized mass and coupling constant are such examples in quantum field theory. We construct a quasi particle picture in the doubled Fock space in TFD, defining quasi particle operators which diagonalize the unperturbed TFD Hamiltonian. In nonequilibrium case a time-dependent number distribution is introduced as a unknown parameter, and a self-consistent renormalization condition derives an equation for it, i.e., the quantum Boltzmann equation 14. Moreover, the non-Markoffian extension of the self-consistent renormalization condition is also proposed 15. However, the extension and application to intrinsically inhomogeneous systems and condensed ones have not been established.

In this paper, we derive the non-Markoffian quantum transport equations for cold atoms in a confining potential both without and with a condensate from the
nonequilibrium TFD formalism\cite{14,15}. We confirm that our non-Markoffian transport equation for the non-condensed system is reduced to the ordinary quantum Boltzmann equation derived in the other methods when the Markov approximation is applied. For the condensed system, we find that the non-Markoffian equation contains an additional collision term which is overlooked in the other methods. This term vanishes in the equilibrium limit if there is no Landau instability, but remains non-vanishing to prevent the system from equilibrating if there is Landau instability. Thus our transport equation with the additional term (we call it the triple production term) and the other ones without it predict definitely different behaviors of the unstable system. This difference is traced back to different quasi particle pictures in the respective theories. Although we only consider in this paper the systems of trapped Bose atoms, our formulation of nonequilibrium TFD can be extended straightforwardly to the trapped systems of Fermi or multi-component atoms.

This paper is organized as follows. We briefly review the formulation of nonequilibrium TFD in Sec. II. In Sec. III and IV, the non-condensed and condensed systems are considered, respectively. We formulate each interaction picture corresponding each quasi particle picture, diagonalizing the free (unperturbed) Hamiltonians of TFD. The tensor form\cite{16}, which makes the diagrammatic calculation very simple, is introduced for the non-condensed system, and is extended to the condensed system. Applying the self-consistent renormalization condition proposed by Chu and Umezawa\cite{15}, we construct a systematical method to obtain the transport equation for the trapped systems. Section V is devoted to summary and discussions.

II. NONEQUILIBRIUM TFD FORMULATION

Here we briefly review the formulation of nonequilibrium TFD\cite{14}.

In TFD, every operator $A$ gets its tilde conjugation pair $\tilde{A}$, which is related to the ordinary (non-tilde) operator by the following tilde conjugation rules:

\begin{equation}
(AB)^\sim = \tilde{A}\tilde{B},
\end{equation}

\begin{equation}
(c_1A + c_2B)^\sim = c_1^2\tilde{A} + c_2^2\tilde{B},
\end{equation}

\begin{equation}
(\tilde{A})^\sim = A,
\end{equation}

\begin{equation}
(A^\dagger)^\sim = \tilde{A}^\dagger,
\end{equation}

\begin{equation}
|0\rangle^\sim = |0\rangle,
\end{equation}

\begin{equation}
|0^{\sim}\rangle = |0\rangle,
\end{equation}

where $c_1$ and $c_2$ are arbitrary c-numbers, and $|0\rangle$ and $|0\rangle$ are the thermal vacua. The Hamiltonian of TFD, which should generate the time translations of both non-tilde and tilde operators, is not the ordinary Hamiltonian $H$ but the hat Hamiltonian $\hat{H} = H - \hat{H}$. The time independence of the thermal vacua requires the minus sign in front of $\hat{H}$.

The construction of the interaction picture is crucial, because the choice of an interaction picture corresponds to that of a quasi particle picture. Suppose the bosonic $a_\ell$-operators in the interaction picture, representing the renormalized quasi particle with a quantum number $\ell$. They are related to the $\xi_\ell$-operators (called the representation particle operators), which annihilate the time independent thermal vacuum $\xi_\ell|0\rangle = 0$, through the thermal Bogoliubov transformations

\begin{equation}
a_\ell^\mu(t) = B_{\ell}^{-1,\mu\nu}(t)\xi_\ell^\nu(t),
\end{equation}

\begin{equation}
\tilde{a}_\ell^\mu(t) = \xi_\ell^\nu(t)B_{\ell}^{\mu\nu}(t),
\end{equation}

\begin{equation}
\xi_\ell^\mu(t) = B_{\ell}^{\mu\nu}(t)a_\ell^\nu(t),
\end{equation}

\begin{equation}
\tilde{\xi}_\ell^\mu(t) = \tilde{a}_\ell^\nu(t)B_{\ell}^{-1,\mu\nu}(t).
\end{equation}

Here we introduce the thermal doublet notations

\begin{equation}
a_\ell^\mu = (a_\ell^\dagger - a_\ell)^\mu,
\end{equation}

\begin{equation}\xi_\ell^\mu = (\xi_\ell^\dagger - \xi_\ell)^\mu,
\end{equation}

and the thermal Bogoliubov matrix

\begin{equation}
B_{\ell}^{\mu\nu}(t) = \begin{pmatrix} 1 + n_\ell(t) & -n_\ell(t) \\ -1 & 1 \end{pmatrix},
\end{equation}

\begin{equation}
B_{\ell}^{-1,\mu\nu}(t) = \begin{pmatrix} 1 & n_\ell(t) \\ 1 + n_\ell(t) & 1 \end{pmatrix}.
\end{equation}

It is important to take the above particular form of the thermal Bogoliubov matrix, as one calls $\alpha = 1$ representation\cite{14}, which enables us to make use of the Feynman diagram method in nonequilibrium systems\cite{17}. The number distribution $n_\ell(t)$ is given by

\begin{equation}
n_\ell(t) = \langle 0|a_\ell^\dagger(t)a_\ell(t)|0\rangle,
\end{equation}

and its time dependence is determined later.

The unperturbed Hamiltonian for the $\xi_\ell$-operators should be diagonal, consistently with the time independence of the thermal vacuum. So the time dependence of $\xi_\ell$-operator in the interaction picture should be in the form $\xi_\ell^\mu(t) = \xi_\ell^\mu e^{-i\omega_\ell t}$, generated by the free Hamiltonian $\hat{H}_0(t) = \sum_\ell \omega_\ell(\ell)|\xi_\ell^\mu(t)\rangle\langle\xi_\ell^\mu(t)|$. Throughout this paper $\hbar$ is set to be unity. Note that $\omega_\ell$ generally depends on time because of the time dependent energy renormalization. In this paper, we take time independent $\omega_\ell$, assuming that the energy shift is negligible in the leading order of perturbation, i.e., $\xi_\ell^\mu(t) = \xi_\ell^\mu e^{-i\omega_\ell t}$ and $\hat{H}_0 = \sum_\ell \omega_\ell(\ell)|\xi_\ell^\mu(t)\rangle\langle\xi_\ell^\mu(t)|$. The unperturbed Hamiltonian for the $a_\ell$-operators is not $\hat{H}_0$, but $\hat{H}_Q(t) = \hat{H}_0 - \hat{Q}(t)$ with the thermal counter term $\hat{Q}(t)$

\begin{equation}
\hat{Q}(t) = i\sum_\ell \hat{n}_\ell(t)\hat{a}_\ell^\mu(t) \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} a_\ell^\nu(t),
\end{equation}

\begin{equation}
= -i\sum_\ell \hat{n}_\ell(t)\hat{\xi}_\ell^\mu(t) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \xi_\ell^\nu(t),
\end{equation}

where $\hat{n}_\ell(t)$ is the thermal number operator of $\xi_\ell^\mu(t)$.
caused by the $t$-dependence of $n_{\ell}(t)$.

The field operator $\psi(x)$ is expanded with a complete set $\{u_\ell(x)\}$ as

$$\psi(x) = \sum_\ell a_\ell(t)u_\ell(x), \quad (18)$$

where $x = (x, t)$. The unperturbed and full propagators for $\psi$ and $\xi$ are defined by

$$\Delta^{\mu\nu}(x, x') = -i\langle 0|T[\psi^\mu(x)\psi^\nu(x')]|0\rangle, \quad (19)$$

$$G^{\mu\nu}(x, x') = -i\langle 0|T[\psi^\mu_H(t)\psi^\nu_H(t')]|0\rangle, \quad (20)$$

$$d^{\mu\nu}_{\ell\ell'}(t, t') = -i\langle 0|T[\xi^\mu_H(t)\xi^\nu_H(t')]|0\rangle, \quad (21)$$

$$g^{\mu\nu}_{\ell\ell'}(t, t') = -i\langle 0|T[\xi^\mu_H(t)\bar{\xi}^\nu_H(t')]|0\rangle, \quad (22)$$

respectively, which are related to each other as

$$\Delta^{\mu\nu}(x, x') = \sum_{\ell\ell'} u_\ell(x)B^{-1}_{\ell\ell'}(t)\times d^{\mu\nu}_{\ell\ell'}(t, t')B^{\mu\nu}_{\ell\ell'}(t')u_{\ell'}(x'), \quad (23)$$

$$G^{\mu\nu}(x, x') = \sum_{\ell\ell'} u_\ell(x)B^{-1}_{\ell\ell'}(t)\times g^{\mu\nu}_{\ell\ell'}(t, t')B^{\mu\nu}_{\ell\ell'}(t')u_{\ell'}(x'). \quad (24)$$

The subscript $H$ denotes a quantity in the Heisenberg picture. While the unperturbed propagator $d$ has a diagonal structure

$$d^{\mu\nu}_{\ell\ell'}(t, t') = \delta_{\ell\ell'}\left(-i\theta(t-t') \begin{pmatrix} 0 & 0 \\ 0 & i\theta(t'-t) \end{pmatrix} \right)^{\mu\nu}e^{-i\omega(t-t')}, \quad (25)$$

the full propagator $g$ has an upper triangular structure in general, that is, $g^{12}_{\ell\ell'}(t, t') \neq 0$ and $g^{11}_{\ell\ell'}(t, t') = 0$ [14]. This is because $\xi^H_{\ell\ell'}$ and $\bar{\xi}^H_{\ell\ell'}$ identically annihilate the vacuum in the $\alpha = 1$ representation while $\xi_H$ and $\bar{\xi}_H$ do not generally annihilate the ket-vacuum. It is also shown that $g^{11}$ and $g^{22}$ are a retarded and an advanced functions, respectively, as $d^{11}$ and $d^{22}$.

According to the Feynman method, one can calculate the full propagator with the interaction Hamiltonian in the interaction picture,

$$\hat{H}_I = \hat{H}_{\text{int}} + \hat{Q}, \quad (26)$$

with $\hat{H}_{\text{int}} = \hat{H} - \hat{H}_0$. Possible renormalization counter terms are suppressed below for simplicity.

### III. TRAPPED BOSE ATOMS IN NON-CONDENSED SYSTEM

In this section, we derive the transport equation for the system of cold Bose atoms without condensate in nonequilibrium TFD.

We start with the following Hamiltonian to describe the trapped dilute Bose atoms,

$$H = \int d^3x \left[\psi^\dagger \left(-\frac{1}{2m}\nabla^2 + V(x) - \mu\right)\psi + g\psi^\dagger\psi^\dagger\psi\psi\right], \quad (27)$$

where $m, V(x), \mu$, and $g$ represent the mass of an atom, the trap potential, the chemical potential, and the coupling constant, respectively. The bosonic field operator $\psi(x)$ obeys the canonical commutation relations

$$[\psi(x), \psi^\dagger(x')]_{t=t'} = \delta(x-x'), \quad (28)$$

$$[\psi(x), \psi(x')]_{t=t'} = [\psi^\dagger(x), \psi^\dagger(x')]_{t=t'} = 0. \quad (29)$$

We expand the field operator $\psi(x)$ as in Eq. (18), using the solutions of the following eigenequations, $\{u_\ell(x)\}$ with the eigenvalues $\{\omega_\ell\},$

$$\left(-\frac{1}{2m}\nabla^2 + V(x) - \mu\right)u_\ell(x) = \omega_\ell u_\ell(x). \quad (30)$$

The annihilation- and creation-operators $a_\ell$ and $a_\ell^\dagger$ diagonalize the free Hamiltonian part $\hat{H}_0$

$$H_0 = \int d^3x \psi^\dagger \left(-\frac{1}{2m}\nabla^2 + V(x) - \mu\right)\psi = \sum_\ell \omega_\ell a_\ell^\dagger a_\ell. \quad (31)$$

We apply the formulation of nonequilibrium TFD in the previous section to the present system: Each degree of freedom is doubled, the time dependent thermal Bogoliubov transformation is introduced in the interaction picture, and the total Hamiltonian $\hat{H}$ is divided into the unperturbed and interaction parts, $\hat{H}_Q$ and $\hat{H}_I$. Then the full propagator is calculated in the Feynman diagram method.

The self-consistent renormalization condition on the full propagator thus obtained, which is extended from the self-consistent on-shell renormalization condition in the ordinary quantum field theory, is already proposed [15, 18, 19] as

$$g^{12}_{\ell\ell'}(t, t) = 0. \quad (32)$$

It provides the transport equation which determines the temporal evolution of the unperturbed number distribution $n_{\ell}(t)$. Following the Dyson equations $G = \Delta + \Delta \Sigma G$ or $g = d + dS$, we obtain

$$g^{12}_{\ell\ell'}(t, t') = \sum_{\ell\ell'} \int d\ell sds' g^{11}_{\ell\ell'}(t, s)sS^{11}_{\ell\ell'}(s, s')g^{22}_{s's'}(s', t'), \quad (33)$$

with the self-energy

$$\Sigma^{\mu\nu}(x, x') = \sum_{\ell\ell'} \frac{u_\ell(x)B^{-1}_{\ell\ell'}(t)\times S^{\mu\nu}_{\ell\ell'}(t, t')B^{\mu\nu}_{\ell\ell'}(t')u_{\ell'}(x')}{\omega_\ell}, \quad (34)$$

$$S^{\mu\nu}_{\ell\ell'}(t, t') = \begin{pmatrix} S^{11}_{\ell\ell'}(t, t') & S^{12}_{\ell\ell'}(t, t') \\ 0 & S^{22}_{\ell\ell'}(t, t') \end{pmatrix}. \quad (35)$$

To illustrate how the transport equation follows from the renormalization condition, we approximate the full propagators $g^{11(22)}$ in Eq. (33) by the unperturbed ones.
and divide the self-energy \( S \) into a loop contribution \( S_{\text{loop}} \) and a contribution of the thermal counter term \( S_Q \),

\[
S_{\text{Q}\ell\ell}(t,t') = -i\hat{n}_\ell(t)\delta_{\ell\ell} \delta(t-t') \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^{\mu\nu}. \tag{36}
\]

Then we have

\[
g_{\ell\ell}^{12}(t,t) = -i \int_{-\infty}^{t} ds \left[ \hat{n}_\ell(s) - 2\text{Re} \int_{-\infty}^{s} ds' e^{i\omega_{\ell}(s-s')} S_{\ell\ell,\text{loop}}^{12}(s, s') \right], \tag{37}
\]

and the renormalization condition \[32\] implies the following transport equation

\[
\hat{n}_\ell(t) = 2\text{Re} \int_{-\infty}^{t} ds e^{i\omega_{\ell}(t-s)} S_{\ell\ell,\text{loop}}^{12}(t, s). \tag{38}
\]

In view of this, we are going to calculate the self-energy perturbatively to obtain the transport equation in the leading order. Before that, we introduce the following tensor form \[16\] which makes the representations and calculations of propagators and self-energies much more concise

\[
\left\{ s_1 \right\}^\mu = \begin{cases} s_1 & \text{if } \mu = 1 \\ s_2 & \text{if } \mu = 2 \end{cases}, \tag{39}
\]

For instance, the following matrix appearing in the unperturbed propagator is expressed in the tensor form as

\[
\left[ B(t)^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} B(t') \right]^{\mu\nu} = \begin{pmatrix} 1 + n(t') & -n(t') \\ n(t') & n(t') \end{pmatrix}^{\mu\nu} \tag{40}
\]

\[
= \begin{cases} 1 & \left\{ 1 + n(t') \right\}^{\nu} \\ 1 & \left\{ -n(t') \right\}^{\nu} \end{cases}. \tag{41}
\]

Thus, the unperturbed propagators can be written in the tensor form as

\[
d_{\ell\ell}^{\mu\nu}(t,t') = -i\delta_{\ell\ell} e^{-i\omega_{\ell}(t'-t)} \times \left[ \theta(t-t') \left\{ 1 \right\}^\mu \left\{ 1 \right\}^\nu - \theta(t' - t) \left\{ 0 \right\}^\mu \left\{ 1 \right\}^\nu \right], \tag{42}
\]

\[
\Delta^{\mu\nu}(x,x') = -ie^{\nu} \sum_{\ell} u_{\ell}(x) u^*_\ell(x') e^{-i\omega_{\ell}(t'-t')} \times \left[ \theta(t-t') \left\{ 1 \right\}^\mu \left\{ 1 + n_{\ell}(t') \right\}^\nu \\ 1 \left\{ n_{\ell}(t') \right\}^\mu \left\{ 1 \right\}^\nu \right] + \theta(t' - t) \left\{ n_{\ell}(t) \left\{ 1 + n_{\ell}(t) \right\}^\mu \left\{ 1 \right\}^\nu \\ n_{\ell}(t) \left\{ 1 \right\}^\mu \left\{ 1 + n_{\ell}(t) \right\}^\nu \right] \right|_{\text{min}(t,t')}, \tag{43}
\]

with the sign factor, \( e^1 = 1 \) and \( e^2 = -1 \).

As an example of manipulating products of the unperturbed propagators in the Feynman diagram calculation, we give the following manipulation,

\[
\left[ B(t)^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} B(t') \right]^{\mu\nu} \left[ B(t)^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} B(t') \right]^{\mu\nu} = \begin{cases} 1 & \left\{ 1 \right\}^\mu \left\{ 1 + n_{\ell}(t') \right\}^{\nu} \\ 1 \left\{ -n_{\ell}(t') \right\}^\mu \left\{ 1 \right\}^\nu \end{cases} \tag{44}
\]

\[
= \begin{cases} 1 & \left\{ (1 + n_{\ell}(t))(1 + n_{\ell}(t')) \right\}^\nu \\ 1 & \left\{ n_{\ell}(t')n_{\ell}(t') \right\}^\nu \end{cases}, \tag{45}
\]

where the indices \( \mu \) and \( \nu \) are not summed over. This kind of manipulation will be used below in the calculations of the self-energies.

We focus on the two-loop self-energy indicated in Fig. 1 the leading loop diagram which makes \( g_{\ell\ell}^{12}(t,t') \) nonzero,

\[
\Sigma_{\ell\ell,\text{loop}}^{\mu\nu}(x,x') = -2g^2 e^\nu \Delta^{\mu\nu}(x,x') \Delta^{\mu\nu}(x',x'). \tag{46}
\]

The sign factors arise because of the definition of \( \tilde{\psi} \) and of the particular form of the interaction Hamiltonian \( \hat{H}_I = H_I - H_I \). Using the tensor form, we can rewrite the self-energy as

\[
\Sigma_{\ell\ell,\text{loop}}^{\mu\nu}(x,x') = -2ig^2 e^\nu \sum_{\ell_1,\ell_2,\ell_3} e^{-i(\omega_{\ell_1} + \omega_{\ell_2} - \omega_{\ell_3})(t-t')} \times u_{\ell_1}(x)u_{\ell_2}(x)u^*_{\ell_3}(x')u^*_{\ell_3}(x') \times \left[ \theta(t-t') \left\{ 1 \right\}^\mu \left\{ (1 + n_{\ell_1})(1 + n_{\ell_2})n_{\ell_3} \right\}^{\nu} \\ 1 \left\{ (1 + n_{\ell_1})(1 + n_{\ell_2})n_{\ell_3} \right\}^\mu \left\{ 1 \right\}^\nu \right] \min(t,t') \right|_{\text{min}(t,t')}, \tag{47}
\]

where the subscript \( \min(t,t') \) denotes the time arguments of \( n \). So that

\[
S_{\ell\ell,\text{loop}}^{12}(t,t') = 2ig^2 \sum_{\ell_1,\ell_2,\ell_3} e^{-i(\omega_{\ell_1} + \omega_{\ell_2} - \omega_{\ell_3} - \omega_{\ell_4})(t-t')} \times C_{\ell_1,\ell_2,\ell_3,\ell} \left[ n_{\ell_1}n_{\ell_2}(1 + n_{\ell_3})(1 + n_{\ell_4}) \\ - (1 + n_{\ell_1})(1 + n_{\ell_2})n_{\ell_3} \right] \min(t,t') \right|_{\text{min}(t,t')}, \tag{48}
\]

where

\[
C_{\ell_1,\ell_2,\ell_3,\ell} = \left| \int d^3x u_{\ell_1}(x)u_{\ell_2}(x)u^*_{\ell_3}(x)u^*_{\ell_4}(x) \right|^2. \tag{49}
\]
Thus the transport equation at the two-loop level is derived
\[
\hat{n}_t(t) = 4g^2Re\int_{-\infty}^{t} ds \, \sum_{\ell_1, \ell_2, \ell_3} e^{-i(\omega_{\ell_1} + \omega_{\ell_2} - \omega_{\ell_3} - \omega_r)(t-s)} C_{\ell_1, \ell_2, \ell_3} \times \left[ n_{\ell_1} n_{\ell_2} (1 + n_{\ell_3})(1 + n_t) - (1 + n_{\ell_1})(1 + n_{\ell_2}) n_{\ell_3} n_t \right].
\]
(50)

The transport equation of Markoffian type corresponding to Eq. (50) has already been derived by Chu and Umezawa \[15\] for the homogeneous system. We have here extended their result to the inhomogeneous system due to the confining potential.

The exponential term \(e^{-i(\omega_{\ell_1} + \omega_{\ell_2} - \omega_{\ell_3} - \omega_r)(t-s)}\) in Eq. (50) signifies the energy conservation. To see that explicitly, we perform the time integral in Eq. (50) as
\[
\hat{n}_t(t) = 4g^2Re\int_{-\infty}^{0} ds \, \sum_{\ell_1, \ell_2, \ell_3} e^{i(\omega_{\ell_1} + \omega_{\ell_2} - \omega_{\ell_3} - \omega_r)(t-s)} C_{\ell_1, \ell_2, \ell_3} \times \sum_{\ell_1, \ell_2, \ell_3} \Gamma_{\ell_1, \ell_2, \ell_3} R_{\ell_1, \ell_2, \ell_3}(t) \, \Delta t_{\ell_1, \ell_2, \ell_3}(t)
\]
(51)

where
\[ R_{\ell_1, \ell_2, \ell_3}(t) = n_{\ell_1} n_{\ell_2} (1 + n_{\ell_3})(1 + n_t) - (1 + n_{\ell_1})(1 + n_{\ell_2}) n_{\ell_3} n_t, \]
(53)
\[
\Gamma_{\ell_1, \ell_2, \ell_3} = -\frac{1}{R_{\ell_1, \ell_2, \ell_3}} \frac{dR_{\ell_1, \ell_2, \ell_3}}{dt}. \]
(54)

It is seen from this Lorentzian form that the energy is conserved with the width \(\Gamma\) which is small when the temporal change of \(n\) is slow.

To confirm the correspondence of our transport equation with those derived in the other methods, we approximate replace \(n_{\ell_1}(s)\) in the right hand side of Eq. (50) with \(n_{\ell_1}(t)\), in other words, take the limit \(\Gamma \rightarrow 0\) or the Markoffian limit, with the assumption that the system is close to the equilibrium and the evolution is sufficiently slow,
\[
\hat{n}_t(t) = 4\pi g^2 \sum_{\ell_1, \ell_2, \ell_3} \delta(\omega_{\ell_1} + \omega_{\ell_2} - \omega_{\ell_3} - \omega_r) C_{\ell_1, \ell_2, \ell_3} \times \left[ n_{\ell_1} n_{\ell_2} (1 + n_{\ell_3})(1 + n_t) - (1 + n_{\ell_1})(1 + n_{\ell_2}) n_{\ell_3} n_t \right].
\]
(55)

The equation has a form of the ordinary quantum Boltzmann equation and is consistent with the one which has been obtained from the on-shell renormalization condition for the homogeneous system in nonequilibrium TFD \[14\]. However, the naive approximation is not valid for a trapped system with the discrete energy spectrum \(\omega_r\). The energy conservation forced by the delta function is too strict to allow any energy exchange of particles and, consequently, any time evolution. One way to avoid this difficulty is to apply the coarse graining treatment \[20\]. For instance, in the case of the harmonic trap potential, \(\delta(\omega_{\ell_1} + \omega_{\ell_2} - \omega_{\ell_3} - \omega_r)\) was replaced by \(\delta(\omega_{\ell_1} + \omega_{\ell_2} - \omega_{\ell_3} - \omega_r)/\Omega\) with the trap frequency \(\Omega\) \[21\],
\[
\hat{n}_t(t) = \frac{4\pi g^2}{\Omega} \sum_{\ell_1, \ell_2, \ell_3} \delta(\omega_{\ell_1} + \omega_{\ell_2} - \omega_{\ell_3} - \omega_r) C_{\ell_1, \ell_2, \ell_3} \times \left[ n_{\ell_1} n_{\ell_2} (1 + n_{\ell_3})(1 + n_t) - (1 + n_{\ell_1})(1 + n_{\ell_2}) n_{\ell_3} n_t \right].
\]
(56)

Note that such simple replacement is valid only for the harmonic trap system whose energy-level spacing is uniform.

Another way is a semi-classical formulation using the phase-space distribution function \(n(x, p, t)\). Then the particle energy is no longer the discrete one but the local continuous one \(\omega(x, p) = p^2/2m + V(x) - \mu\), for which the difficulty mentioned above does not arise. Although the transport equations, derived in this manner by several authors \[7, 11\], are in good agreements with the experiments of evaporative cooling and formation of condensate, they are classical and can not describe quantum fluctuations fully, and the particle picture is not explicit. In the next section we will show the case where the instability of condensate exists and the quasi particle spectrum plays a crucial role, for which the semi-classical treatment is not valid.

**IV. TRAPPED BOSE ATOMS IN CONDENSED SYSTEM**

In this section, we consider the situation in which a condensate exists and derive the transport equation. The field operator \(\psi(x)\) is divided into a classical part \(\psi(x)\) and a quantum part \(\varphi(x)\), reflecting the existence of the condensate. In a fully nonequilibrium situation, the order parameter \(\zeta = \langle 0|\psi|0\rangle\) should be time-dependent, however we consider only a situation near the equilibrium and assume the time-independent order parameter throughout this paper.

The Hamiltonian \[27\] is written as
\[
H = H_0 + H_{\text{int}},
\]
(57)
where
\[
H_0 = \int d^3x \left[ \frac{\varphi^\dagger \varphi}{2m} + V(x) - \mu + 2g|\zeta(x)|^2 \varphi \right]
+ \frac{g}{2} \left( \zeta^2(x) \varphi^\dagger \varphi + \zeta^* \varphi^\dagger \varphi + \frac{1}{2} \varphi^\dagger \varphi, \varphi^\dagger \varphi \right),
\]
(58)
\[
H_{\text{int}} = g\int d^3x \left[ \zeta^*(x) \varphi^\dagger \varphi^2 + \zeta(x) \varphi^\dagger \varphi \varphi + \frac{1}{2} \varphi^\dagger \varphi, \varphi^\dagger \varphi \right],
\]
(59)
and the first order term of \( \varphi(x) \) vanishes since \( \zeta(x) \) is required to satisfy the following Gross-Pitaevskii equation \(22\) at the tree level

\[
\left( -\frac{\nabla^2}{2m} + V(x) - \mu + g|\zeta(x)|^2 \right) \zeta(x) = 0 . \tag{60}
\]

Next, we briefly review the Bogoliubov-de Gennes (BdG) method which diagonalizes the free Hamiltonian \( H_0 \). The BdG equations are simultaneous eigenvalue equations given by \(23, 24, 25\)

\[
Ty_\ell(x) = \omega y_\ell(x) . \tag{61}
\]

Here the doublet notation is introduced as

\[
y_\ell(x) = \begin{pmatrix} y_1^\ell(x) \\ y_2^\ell(x) \end{pmatrix} , \tag{62}
\]

\[
T = \begin{pmatrix} L & M \\ -M^* & -L \end{pmatrix} , \tag{63}
\]

where

\[
L = -\frac{\nabla^2}{2m} + V(x) - \mu + 2g|\zeta(x)|^2 , \tag{64}
\]

\[
M = g\zeta^2(x) . \tag{65}
\]

It is known that the BdG equations have the zero eigenvalue mode whose treatment needs attention. We disregard the zero mode for simplicity though it can be included consistently \(12, 26\), because the scope of this paper is confined to the time-independent order parameter and the dynamical effects of the zero mode is not dominant then. In addition, the eigenvalues can be complex since the operator \( T \) is non-Hermitian. The condition for the emergence of complex eigenvalues in the BdG equations has been studied both numerically \(27, 28, 29\) and analytically \(30, 31, 33\), and the quantum field theoretical formulation has also been discussed \(31\). The emergence of complex eigenvalues implies the dynamical instability of the system, and a drastic temporal change of the order parameter occurs then, which is out of our present formulation. In this paper, we consider only the case where no complex eigenvalue emerges.

Eigenfunctions belonging to the non-zero real eigenvalues can be orthonormalized under the indefinite metric as

\[
\int d^3x \, y_\ell^\dagger(x) \sigma_3 y_\ell(x) = \delta_{\ell\ell'} , \tag{66}
\]

\[
\int d^3x \, z_\ell^\dagger(x) \sigma_3 z_\ell(x) = -\delta_{\ell\ell'} , \tag{67}
\]

\[
\int d^3x \, y_\ell^\dagger(x) \sigma_3 z_\ell(x) = 0 , \tag{68}
\]

with \( i \)-th Pauli matrix \( \sigma_i \). The function \( z_\ell \), defined by \( z_\ell = \sigma_1 y_\ell^\dagger \), is an eigenfunction belonging to \(-\omega_\ell\), if \( y_\ell \) is an eigenfunction belonging to \(\omega_\ell\). It is convenient to rewrite the orthonormal conditions \(60 - 68\) with the 2 \(\times\) 2 matrix form as

\[
\int d^3x \, W_\ell(x) \, W_\ell^{-1}(x) = \delta_{\ell\ell'} , \tag{69}
\]

where

\[
W_\ell(x) = \sigma_3 \begin{pmatrix} y_1^\ell(x) \\ z_1^\ell(x) \end{pmatrix} \sigma_3 , \tag{70}
\]

\[
W_\ell^{-1}(x) = \begin{pmatrix} y_\ell(x) & z_\ell(x) \end{pmatrix} . \tag{71}
\]

The completeness condition,

\[
\sum_\ell \left[ y_\ell(x) y_\ell^\dagger(x') - z_\ell(x) z_\ell^\dagger(x') \right] = \sigma_3 \delta(x - x') , \tag{72}
\]

can be expressed as

\[
\sum_\ell W_\ell^{-1}(x) W_\ell(x') = \delta(x - x') , \tag{73}
\]

and then the field operators are expanded in the doublet form as

\[
\varphi^\alpha(x) = \sum_\ell W_\ell^{-1,\alpha\beta}(x)b_\ell^\beta(t) , \tag{74}
\]

\[
\bar{\varphi}^\beta(x) = \sum_\ell \bar{b}_\ell^\alpha(t) W_\ell^{\alpha\beta}(x) , \tag{75}
\]

where

\[
\varphi^\alpha = \begin{pmatrix} \varphi^\ell \\ \varphi^i \end{pmatrix} , \quad \bar{\varphi}^\alpha = \begin{pmatrix} \varphi^i \bar{\varphi}^\ell \end{pmatrix} , \tag{76}
\]

\[
b_\ell^\alpha = \begin{pmatrix} b_\ell^\alpha \\ b_\ell^i \end{pmatrix} , \quad \bar{b}_\ell^\alpha = \begin{pmatrix} b_\ell^i \\ -b_\ell^\alpha \end{pmatrix} . \tag{77}
\]

The operators \( b_\ell \) satisfy the canonical commutation relation \([b_\ell, b_\ell^\dagger] = \delta_{\ell\ell'}\), and diagonalizes the free Hamiltonian \(58\)

\[
H_0 = \frac{1}{2} \int d^3x \, \bar{\varphi}^\alpha(x) T^\alpha\beta \varphi^\beta(x) \tag{78}
\]

\[
= \sum_\ell \omega_\ell b_\ell^\dagger b_\ell . \tag{79}
\]

The operators \( b_\ell \) annihilate the Bose-Einstein condensed vacuum and the operation of the creation operators \( b_\ell \) on the vacuum constructs the Fock space at zero temperature. Therefore, to treat this system in nonequilibrium TFD, we double the degrees of freedom as follows

\[
\begin{pmatrix} b_\ell \\ \bar{b}_\ell \end{pmatrix} = \begin{pmatrix} \xi_\ell \\ \bar{\xi}_\ell \end{pmatrix} , \tag{80}
\]

\[
\begin{pmatrix} b_\ell^\dagger \\ -\bar{b}_\ell \end{pmatrix} = \begin{pmatrix} \xi_\ell \\ -\bar{\xi}_\ell \end{pmatrix} B_\ell , \tag{81}
\]

where the thermal Bogoliubov matrix is defined in Eqs. \(13\) and \(14\) with the quasi particle distribution \(n_\ell(t) = \langle 0| b_\ell^\dagger(t) b_\ell(t) |0\rangle\). Note that the operators who annihilate the thermal vacuum are not the \(b\)-operators but
the $\xi$-operators. The combination of the two transformations, $\xi$ into $b$ and $b$ into $\varphi$, involves the $4 \times 4$ transformations,

$$
\begin{pmatrix}
\bar{b}_\ell
\
\bar{b}_\ell^\dagger
\end{pmatrix}
= \begin{pmatrix}
1 & n_\ell \\
1 & 1 + n_\ell
\end{pmatrix}
\begin{pmatrix}
\bar{b}_\ell
\
\bar{b}_\ell^\dagger
\end{pmatrix}
\begin{pmatrix}
\xi_\ell
\
\xi_\ell^\dagger
\end{pmatrix},
$$

(82)

$$
\begin{pmatrix}
\varphi
\
\varphi^\dagger
\end{pmatrix} = \sum_{i,\alpha} \begin{pmatrix}
y_{i 1}^{1/2} & \sigma_{i 1}^{1/2}
y_{i 1}^{1/2} & \sigma_{i 1}^{1/2}
\end{pmatrix}
\begin{pmatrix}
\bar{b}_\ell^\dagger
\
\bar{b}_\ell
\end{pmatrix}
\begin{pmatrix}
\xi_\ell
\
\xi_\ell^\dagger
\end{pmatrix},
$$

(83)

where the blank elements denote zero. It is convenient to introduce the quartet notations for $b_\ell$ as follows

$$
b_\ell^{\alpha} = \left( \begin{array}{c}
b^\mu_\ell \\
[\sigma_1 b^\mu_\ell]_\mu
\end{array} \right) = \begin{pmatrix}
b_\ell^\dagger
\
b_\ell
\end{pmatrix}
\begin{pmatrix}
\mu
\
\alpha
\end{pmatrix},
$$

(84)

$$
\tilde{b}_{\ell}^{\alpha} = \left( \begin{array}{c}
\tilde{b}_{\ell}^\mu
\
[\tilde{b}_{\ell} \sigma_1]^{\mu,\alpha}
\end{array} \right) = \begin{pmatrix}
\tilde{b}_{\ell}^\dagger
\tilde{b}_{\ell}
\end{pmatrix}
\begin{pmatrix}
\mu
\
\alpha
\end{pmatrix},
$$

(85)

and in similar fashions for $\xi_\ell$ and $\varphi$. Then, the $4 \times 4$ transformations can be written simply

$$
b_{\ell}^{\mu,\alpha} = B_{\ell}^{-1,\mu,\alpha,\beta} b_{\ell}^{\mu,\beta},
$$

(86)

$$
\varphi_{\mu,\alpha} = \sum_{\ell} W_{\ell}^{-1,\mu,\alpha,\beta} \tilde{b}_{\ell}^{\mu,\beta},
$$

(87)

with the $4 \times 4$ thermal Bogoliubov and BdG inverse matrices

$$
B_{\ell}^{-1,\mu,\alpha,\beta} = \delta_{\alpha 1} \delta_{\beta 1} B_{\ell}^{-1,\mu} + \delta_{\alpha 2} \delta_{\beta 2} (\sigma_1 B_{\ell} - \sigma_i)_{\mu,\nu},
$$

(88)

$$
B_{\ell}^{\mu,\alpha,\beta} = \delta_{\alpha 1} \delta_{\beta 1} B_{\ell}^{\mu,\nu} + \delta_{\alpha 2} \delta_{\beta 2} (\sigma_1 B_{\ell}^{\mu,\nu}),
$$

(89)

$$
W_{\ell}^{\mu,\alpha,\beta} = \delta_{\mu 1} W_{\ell}^{1,\alpha,\beta},
$$

(90)

$$
W_{\ell}^{-1,\mu,\alpha,\beta} = \delta_{\mu 1} W_{\ell}^{1,\alpha,\beta}.
$$

(91)

The existence of the condensate brings no crucial alteration in defining the thermal counter term, because the thermal Bogoliubov transformations Eqs. (80) and (81) remain unchanged. Thus, the thermal counter term is the same as Eq. (74)

$$
\hat{Q} = -i \sum_{\ell} \hat{n}_{\ell}(t) \bar{\xi}_{\ell}^{\alpha}(t) \begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix} \xi_{\ell}^{\alpha}(t).
$$

(92)

Note however that the number distribution is that of the quasi-particles. With the quartet notation, $Q$ can be written as

$$
\hat{Q} = -\frac{i}{2} \sum_{\ell} \hat{n}_{\ell}(t) \bar{\xi}_{\ell}^{\alpha}(t) \begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix} \xi_{\ell}^{\alpha}(t).
$$

(93)

The unperturbed and full propagators are given by

$$
\Delta^{\mu,\alpha,\beta}(x, x') = -i(0|T[\varphi^{\alpha,\mu}(x)\varphi^{\beta,\nu}(x')]|0),
$$

(94)

$$
G^{\mu,\alpha,\beta}(x, x') = -i(0|T[\varphi^{\mu,\alpha}(x)\varphi^{\mu,\beta}(x')]|0),
$$

(95)

$$
a^{\mu,\alpha,\beta}_{\ell}(t, t') = -i(0|T[\xi^{\alpha,\mu}_{\ell}(t)\xi^{\beta,\nu}_{\ell}(t')]|0),
$$

(96)

$$
g^{\mu,\alpha,\beta}_{\ell}(t, t') = -i(0|T[\xi^{\alpha,\mu}_{\ell}(t)\xi^{\beta,\nu}_{\ell}(t')]|0).
$$

(97)

Following the renormalization condition for the non-condensed system in Eq. (32), which has successfully led to the transport equation, we propose the renormalization condition for the condensed system as

$$
g^{1121}_{\ell}(t, t) = 0.
$$

(98)

Let us consider the Dyson equation

$$
\begin{pmatrix}
g^{11}_{\ell} \\
g^{12}_{\ell}
\end{pmatrix} = \begin{pmatrix}
d^{11} & 0 \\
0 & d^{22}
\end{pmatrix}
+ \begin{pmatrix}
d^{11} & 0 \\
0 & d^{22}
\end{pmatrix}
\begin{pmatrix}
S^{11} & S^{12} \\
S^{21} & S^{22}
\end{pmatrix}
\begin{pmatrix}
g^{11}_{\ell} \\
g^{12}_{\ell}
\end{pmatrix},
$$

(99)

where the superscripts denote the BdG indices, $\alpha$ and $\beta$. Every matrix elements in the above equation are $2 \times 2$ matrices with the thermal indices, $\mu$ and $\nu$, which are implicit for conciseness of notation. Solving the Dyson equation for $g^{11}$, we obtain

$$
g^{11} = d^{11} + d^{11} S g^{11},
$$

(100)

with

$$
S^{\mu,\nu} = \left[ S^{11} + S^{12} (1 - d^{22} S^{22})^{-1} d^{22} S^{21} \right]^{\mu,\nu}.
$$

(101)

Obviously, $S$ and $g^{11}$ in Eq. (100) are upper triangular matrices, corresponding to the structures of $g^{11}$ and $S^{\mu,\nu}$ for the non-condensed system. Then from Eq. (100) follows

$$
g^{11}_{\ell, t'}(t, t) = \sum_{m,m'} \int ds ds' g^{11}_{\ell, m}(t, s) S^{12}_{mm'}(s, s') g^{21}_{m', t'}(s', t').
$$

(102)

Similarly as in the non-condensed case Eq. (33), let us consider the leading order. We approximate the full propagators $g^{11}_{\mu, \nu}$ by the unperturbed ones $d^{\mu, \nu}_{\ell}$ and $S^{\mu, \nu}$ by $S^{11}_{\ell, \ell}$ in the right hand side of Eq. (102). Since the contribution of the thermal counter term to the self-energy becomes

$$
S^{\mu,\nu}_{\ell, t'}(t, t') = -i \hat{n}_{\ell}(t) \delta(t - t') \delta^{\mu,\nu} \begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix} \begin{pmatrix}
0 & 0 \\
0 & 1
\end{pmatrix},
$$

(103)

we obtain the following transport equation from Eq. (98)

$$
\hat{n}_{\ell}(t) = 2 Re \int_{-\infty}^{t} ds S^{11}_{\ell, \ell, \text{loop}}(t, s) e^{i \omega_{\ell}(t - s)}.
$$

(104)

The tensor form introduced in the previous section makes the perturbative calculation very simple and systematic. The unperturbed propagators are written as
\[ d_{\ell\ell}^{\mu\nu\alpha\beta}(t, t') = \]
\[ -i\theta(t - t')[\begin{bmatrix} 1 \end{bmatrix}^{\mu} \begin{bmatrix} 1 \end{bmatrix}^{\nu} \begin{bmatrix} 1 \end{bmatrix}^{\alpha} \begin{bmatrix} 1 \end{bmatrix}^{\beta} e^{-i\omega(t-t')} + \begin{bmatrix} 0 \end{bmatrix}^{\mu} \begin{bmatrix} 0 \end{bmatrix}^{\nu} \begin{bmatrix} 0 \end{bmatrix}^{\alpha} \begin{bmatrix} 0 \end{bmatrix}^{\beta} e^{i\omega(t-t')} \]
\[ + i\theta(t' - t)[\begin{bmatrix} 0 \end{bmatrix}^{\mu} \begin{bmatrix} 0 \end{bmatrix}^{\nu} \begin{bmatrix} 1 \end{bmatrix}^{\alpha} \begin{bmatrix} 0 \end{bmatrix}^{\beta} e^{-i\omega(t-t')} + \begin{bmatrix} 1 \end{bmatrix}^{\mu} \begin{bmatrix} 1 \end{bmatrix}^{\nu} \begin{bmatrix} 0 \end{bmatrix}^{\alpha} \begin{bmatrix} 0 \end{bmatrix}^{\beta} e^{i\omega(t-t')} \], \hspace{1cm} (105)

\[ \Delta^{\mu\nu\alpha\beta}(x, x') = -\varepsilon\varepsilon^{\beta} \sum_{\ell} u_{\ell}(x) u_{\ell}^{*}(x') \]
\[ \times \left( \theta(t - t')[\begin{bmatrix} 1 \end{bmatrix}^{\mu} \begin{bmatrix} 1 \end{bmatrix}^{\nu} \begin{bmatrix} 1 + n_{1}(t') \end{bmatrix}^{\alpha} \begin{bmatrix} y_{\ell}(x) \end{bmatrix}^{\beta} e^{-i\omega(t-t')} + \begin{bmatrix} 1 \end{bmatrix}^{\mu} \begin{bmatrix} 1 \end{bmatrix}^{\nu} \begin{bmatrix} n_{1}(t') \end{bmatrix}^{\alpha} \begin{bmatrix} z_{\ell}(x) \end{bmatrix}^{\beta} e^{i\omega(t-t')} \right) , \hspace{1cm} (106) \]

The one-loop self-energy, indicated in Fig. 2 (a), is given in the tensor form as

\[ \Sigma_{\text{loop}}^{\mu\nu\alpha\beta}(x, x') = -2\varepsilon\varepsilon^{\beta} \sum_{\ell_{1}\ell_{2}} \left[ \right. \]
\[ \left. \theta(t - t')[\begin{bmatrix} 1 \end{bmatrix}^{\mu} \begin{bmatrix} 1 \end{bmatrix}^{\nu} \begin{bmatrix} (1 + n_{1_{1}})(1 + n_{2_{1}}) \end{bmatrix}^{\alpha} \begin{bmatrix} y_{\ell_{1}}(x) \end{bmatrix}^{\beta} e^{-i(\omega_{1_{1}} + \omega_{2_{1}})(t-t')} + \begin{bmatrix} 1 \end{bmatrix}^{\mu} \begin{bmatrix} 1 \end{bmatrix}^{\nu} \begin{bmatrix} n_{1}(t_{1}) \end{bmatrix}^{\alpha} \begin{bmatrix} y_{\ell_{1}}(x) \end{bmatrix}^{\beta} e^{i(\omega_{1_{1}} + \omega_{2_{1}})(t-t')} \right) , \hspace{1cm} (107) \]
where

\[
\begin{align*}
\chi_{yy}^\alpha &= \zeta^\alpha y_{t_1} y_{t_2} + \zeta^\alpha y_{t_1}^2 y_{t_2} + \zeta^\alpha y_{t_1} y_{t_2}^2, \\
\chi_{y\gamma} &= \zeta^\alpha y_{t_1} z_{t_2}^\alpha + \zeta^\alpha y_{t_1} z_{t_2} y_{t_2} + \zeta^\alpha y_{t_1}^2 z_{t_2}^\alpha, \\
\chi_{y\zeta} &= \zeta^\alpha z_{t_1} y_{t_2}^\alpha + \zeta^\alpha z_{t_1} y_{t_2} z_{t_2} + \zeta^\alpha z_{t_1}^2 y_{t_2}^\alpha, \\
\chi_{\zeta z} &= \zeta^\alpha z_{t_1} z_{t_2}^\alpha + \zeta^\alpha z_{t_1} z_{t_2} z_{t_2} + \zeta^\alpha z_{t_1}^2 z_{t_2}^\alpha,
\end{align*}
\]

with \(\zeta^\alpha(x) = \left(\frac{\zeta(x)}{\zeta^\alpha(x)}\right)\), and \(\bar{\alpha}\) denotes \(\bar{\alpha} = 2, 1\) for \(\alpha = 1, 2\), respectively. Since

\[
S_{\ell\ell}^{13} (t, t') = \int d^3x d^3x' B_{\ell}^{1\mu}(t) W_{\ell}^{1\alpha}(x) \times \Sigma_{\text{loop}} (x, x') W_{\ell}^{-1,\beta}(x') B_{\ell}^{-1,\beta}(t),
\]

we obtain from Eq. \[104\].

\[
\dot{\eta}_{\ell}(t) = 4g^2 \sum_{\ell_1, \ell_2} \int_{-\infty}^{\ell} dt' \text{Re} \left[ e^{i(\omega_\ell - \omega_{\ell_1} - \omega_{\ell_2})(t-t')} \left( y_\ell, x_{yy} \right)^2 \left\{ (1 + n_{\ell_1})n_{\ell_2} - n_\ell (1 + n_{\ell_1})(1 + n_{\ell_2}) \right\} + \cdots \right] ,
\]

with

\[
(y_\ell, x) = \int d^3x y_\ell^{\alpha}(x) x^\alpha(x).
\]

This is the non-Markovian transport equation for the condensed system at one-loop level. The first term in Eq. \[113\] corresponds to the Beliaev damping and its inverse process, and the second and third terms do to the Landau damping and their inverse processes. These processes bring the system to the equilibrium. On the other hand, the fourth term corresponds to a process in which three quasi particles are created or annihilated. We call it the triple production process. The existence of the process prevents the system from equilibrating, because the collision term is always nonzero \((1 + n_\ell)(1 + n_{\ell_1})(1 + n_{\ell_2}) - n_\ell n_{\ell_1} n_{\ell_2} > 0\). Note however that if all the energies of quasi particles are positive, the process is forbidden due to the energy conservation. Conversely, once a negative energy mode exists, the excitations to the mode will proceed until the condensate decays. This exactly corresponds to the scenario of the Landau instability in terms of the kinetics. Thus the triple production term is interpreted to induce the decay processes leading to the Landau instability.

To see the importance of quantum field theoretical treatment based on the proper quasi particle representation, let us treat the same system in the particle picture of original atoms. Namely, we suppose that a macroscopic number of atoms is in the particular state of original atoms. Namely, we suppose that a macroscopic number of atoms is in the particular state of original atoms.
\[ \dot{n}_\ell(t) \simeq 4g^2 N_c \sum_{\ell_1, \ell_2} \int_{-\infty}^t dt' \text{Re} \left[ e^{i(\omega_1-\omega_1-\omega_2)(t-t')} C_{\ell_1, \ell_2}^{\ell_1, \ell_2} \left\{ (1 + n_\ell) n_{\ell_1} n_{\ell_2} - n_\ell (1 + n_{\ell_1})(1 + n_{\ell_2}) \right\} + e^{i(\omega_1-\omega_1+\omega_2)(t-t')} C_{\ell_1, \ell_2}^{\ell_1, \ell_2} \left\{ (1 + n_\ell) n_{\ell_1} (1 + n_{\ell_2}) - n_\ell (1 + n_{\ell_1}) n_{\ell_2} \right\} + e^{i(\omega_1+\omega_1-\omega_2)(t-t')} C_{\ell_1, \ell_2}^{\ell_1, \ell_2} \left\{ (1 + n_\ell)(1 + n_{\ell_1}) n_{\ell_2} - n_\ell n_{\ell_1} (1 + n_{\ell_2}) \right\} \right]. \] (115)

This equation corresponds to Eq. (113), but the triple production term is absent. Its absence comes from the inadequate choice of particle picture.

The difference between Eqs. (113) and (115) is decisive when the system has the Landau instability. We conclude that the Landau instability should be described by Eq. (113) with the triple production term, because it is based on the appropriate quasi particle picture. The omission of the triple production term, when the energy conservation allows it, would violate the unitarity in quantum theory.

V. SUMMARY AND DISCUSSIONS

In this paper, nonequilibrium TFD is applied to the systems of trapped Bose atoms, and the quantum transport equations of non-Markovian type have been derived for both the non-condensed and condensed systems. We have diagonalized the unperturbed Hamiltonians, each of which corresponds to the quasi particle picture, and this diagonalization procedure in the interaction picture is essential for TFD as well as for ordinary quantum field theory. To derive the transport equations for the trapped systems both without and with a condensate, we have applied the self-consistent renormalization condition proposed by Chu and Umezawa for a homogeneous system. In order to make complicated calculations of the self-energies transparent, we have also refined the diagrammatic calculations in the tensor form, and have developed the convenient $4 \times 4$-matrix formulation in the condensed case. Although the transport equations are derived only in the lowest order in this paper, their higher order corrections can be obtained systematically in our method, simply by calculating higher order diagrams. In contrast, higher order corrections of the transport equations cannot be obtained straightforwardly in the other methods.

For the non-condensed system, the non-Markovian transport equation at two-loop level derived in this paper becomes very similar in the Markovian limit to those derived in the different methods. While the equations in the other methods involve a delta function and require a strict energy conservation in each collision, the energy in our equation is conserved with the finite width which reflects thermal changes and can be calculated. With the strict conservation, the collision integral is either zero or infinite because of the delta function in a trapped system where the energy spectrum is discrete. Therefore, an additional cure was needed to avoid the problem in the other methods. Although the problem does not occur in the semi-classical method since the delta function is integrated over continuous energy variable, the semi-classical treatments are not consistent with the particle picture in the trapped system. It is remarked that our equation in nonequilibrium TFD follows from the correct particle picture and needs no additional patch.

Perturbative calculations are much more intricate for the condensed system than for the non-condensed one: In the former the two component eigenfunctions of the BdG equations complicate expressions. We have merged the thermal doublet and the BdG one into a quartet and have constructed the $4 \times 4$-matrix formalism, which is helpful for our concrete calculations of nonequilibrium TFD.

In principle similarly as in the non-condensed case, we have derived the quantum transport equation in the condensed case at one-loop level. A crucial point in our equation is that it involves an additional collision term, absent in the transport equations of the other methods. Usual collision terms in the lowest order are only those corresponding to the Beliaev and Landau damping and their inverse processes. Our additional term represents creation or annihilation of three quasi particles, and prevents the system from equilibrating if a negative energy mode exists and is suppressed otherwise. So the behavior of the system with the Landau instability, described by our equation, is distinguished from those under the equations without the triple production term. We emphasize that the additional term disappears in the inadequate particle picture, as was shown at the end of Sec. IV, but that it appears naturally in quantum field theory.

As for the Landau instability, the authors of Ref. have pointed out that the sign of the Landau damping rate changes to minus when the system has Landau instability, which is interpreted as an indication of the decay of the condensate. In their analysis, the nonequilibrium distribution function was roughly approximated to the Bose-Einstein one. However, this approximation is invalid for the negative energy spectrum, because the distribution function becomes negative and therefore unphysical. In our scenario with the transport equation, the dominant term in the Landau instability is obviously
the triple production one which is always positive.

We comment on a relation between the closed time path formalism (CTP) \cite{13} and our nonequilibrium TFD. Both derive the very similar Dyson-Schwinger equations, indeed the same in form. This is because they follow from the common Heisenberg equations and Feynman diagrams. It is important to notice that the the same Dyson-Schwinger equation does not always give the same solution. A main difference between the two approaches is in their unperturbed propagators. The unperturbed propagator in CTP is evaluated over the density matrix at the initial time \( t_0 \), out of equilibrium, and would become
\[
\Delta_{\text{CTP}}^{\mu\nu}(x,x') \sim B^{-1,\mu \nu}[n_\nu(t_0)]d_{\mu \nu}^{\prime}(t,t')B^{\prime \nu \nu}[n_\nu(t_0)]
\]
instead of Eq. \((23)\) in TFD. Thus, while the thermal Bogoliubov matrix in the unperturbed propagator of CTP is time-independent and carries information only on the initial thermal state, our propagator with the time-dependent Bogoliubov matrix adopts temporal thermal changes of the system. So the unperturbed representation in TFD contains some non-perturbative effects in perturbative calculations of CTP. The time-dependence of the thermal Bogoliubov matrix in TFD is important also in respect of the self-consistent renormalization, lacking in CTP so far: The time-dependent thermal Bogoliubov matrix creates the thermal counter term \( \hat{Q} \) in the interaction Hamiltonian, and the derivation of transport equations would be impossible without it.

The transport equation derived in this paper can describe only the initial stage of the condensate decay with the Landau instability. It is out of our present formulation to describe a full time evolution of the decay as well as that of quantum phase transition through evaporative cooling, because we have ignored the time dependence of the condensate involving the time dependent quasi-particle representation. In such cases, the zero modes of the BdG equations play a crucial role and must be considered. The description of the decay with the Landau instability, that with the dynamical instability a full description of quantum phase transition are challenging subjects in nonequilibrium TFD.

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