ON SOME INEQUALITIES FOR s-LOGARITHMICALLY
CONVEX FUNCTIONS IN THE SECOND SENSE VIA
FRACTIONAL INTEGRALS

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ABSTRACT. In this paper, we establish some new Hadamard type inequalities
for s-logarithmically convex functions in the second sense via fractional inte-
grals by using Lemma 1 which has been proved by Sarıkaya et al. in the paper [3].

1. INTRODUCTION

The following result is well known in the literature as Hadamard’s inequality [1].

Theorem 1. Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function on the interval $I$ of real
numbers and $a, b \in I$ with $a < b$. Then

\begin{equation}
 f\left(\frac{a + b}{2}\right) \leq \frac{1}{b - a} \int_{a}^{b} f(x) \, dx \leq \frac{f(a) + f(b)}{2}
\end{equation}

The following definitions is well known in the literature:

Definition 1. A function $f : I \to \mathbb{R}, \emptyset \neq I \subseteq \mathbb{R}$, where $I$ is a convex set, is said
to be convex on $I$ if inequality

$f((tx + (1 - t)y) \leq tf(x) + (1 - t)f(y)$

holds for all $x, y \in I$ and $t \in [0, 1]$.

In [2], Akdemir and Tunç were introduced the class of s-logarithmically convex
functions in the first and second sense as the following:

Definition 2. A function $f : I \subset \mathbb{R}_0 \to \mathbb{R}_+$ is said to be s-logarithmically convex
in the first sense if

\begin{equation}
 f(\alpha x + \beta y) \leq [f(x)]^{\alpha^s} [f(y)]^{\beta^s}
\end{equation}

for some $s \in (0, 1]$, where $x, y \in I$ and $\alpha^s + \beta^s = 1$.

Definition 3. A function $f : I \subset \mathbb{R}_0 \to \mathbb{R}_+$ is said to be s-logarithmically convex
in the second sense if

\begin{equation}
 f(tx + (1 - t)y) \leq [f(x)]^{t^s} [f(y)]^{(1 - t)^s}
\end{equation}

for some $s \in (0, 1]$, where $x, y \in I$ and $t \in [0, 1]$.

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Clearly, when taking $s = 1$ in Definition 2 or Definition 3, then $f$ becomes the standard logarithmically convex function on $I$.

**Definition 4.** Let $f \in L^1[a,b]$. The Riemann-Liouville integrals $J_{a^+}^\alpha f$ and $J_{b^-}^\alpha f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

\[ J_{a^+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) \, dt, \quad x > a \]

and

\[ J_{b^-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) \, dt, \quad x < b \]

respectively where $\Gamma(\alpha) = \int_0^\infty e^{-u} u^{\alpha-1} \, du$. Here is $J_{a^+}^\alpha f(x) = J_{b^-}^\alpha f(x) = f(x)$.

In the case of $\alpha = 1$, the fractional integral reduces to the classical integral. For some recent results connected with fractional integral inequalities see [3]–[11]. In [3], Sarıkaya et. al. proved the following results for fractional integrals.

**Lemma 1.** Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on $(a,b)$ with $a < b$. If $f' \in L[a,b]$, then the following equality for fractional integrals holds:

\[
\frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [J_{b^-}^\alpha f(a) + J_{a^+}^\alpha f(b)]
= \frac{b-a}{2} \int_0^1 [(1-t)^\alpha - t^\alpha] f'(ta + (1-t)b) \, dt.
\]

**Theorem 2.** Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on $(a,b)$ with $a < b$. If $|f'|$ is convex on $[a,b]$, then the following inequality for fractional integrals holds:

\[
\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [J_{b^-}^\alpha f(a) + J_{a^+}^\alpha f(b)] \right|
= \frac{b-a}{2(\alpha+1)} \left( 1 - \frac{1}{2^\alpha} \right) \left[ |f'(a)| + |f'(b)| \right].
\]

In the present paper, we will establish several Hermite-Hadamard type inequalities for the class of functions whose derivatives in absolute value are $s$-logarithmically convex functions in the first and second sense via Riemann-Liouville fractional integral.

2. **HADAMARD TYPE INEQUALITIES FOR S-LOGARITHMICALLY CONVEX FUNCTIONS**

**Theorem 3.** Let $I \supset [0, \infty)$ be an open interval and $f : I \to (0, \infty)$ is differentiable. If $f' \in L[a,b]$ and $|f'|$ is $s$-logarithmically convex functions in the second sense on
[a, b] for some fixed $s \in (0, 1]$ and $\mu, \eta > 0$ with $\mu + \eta = 1$, then the following inequality for fractional integrals with $\alpha > 0$ holds:

\[
(2.1) \quad \left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b - a)^{\alpha}} \left[ J_{b^+}^\alpha f(a) + J_{a^+}^\alpha f(b) \right] \right| \leq \frac{b - a}{2} \left\{ \int_{0}^{1/2} \mu [(1 - t)^{\alpha} - t^{\alpha}]^{1/2} dt + \int_{1/2}^{1} \mu [t^{\alpha} - (1 - t)^{\alpha}]^{1/2} dt \right.
\]
\[
+ \eta \times |f'(b)|^{\eta} \psi \left( \begin{array}{c} s \frac{s}{\alpha} \\ \eta \end{array} \right) \right\}
\]

where

\[
(2.2) \quad \Psi(\psi) = \left\{ \begin{array}{ll} \frac{1}{\ln \psi}, & \psi = 1, \\ 0 < \psi < 1 \end{array} \right. \quad \text{and} \quad \psi(u, v) = |f'(a)|^{u} |f'(b)|^{-v}, \quad u, v > 0.
\]

**Proof.** By Lemma 1 and since $|f'|$ is $s$-logarithmically convex functions in the second sense on $[a, b]$, we have

\[
\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b - a)^{\alpha}} \left[ J_{b^+}^\alpha f(a) + J_{a^+}^\alpha f(b) \right] \right| \leq \frac{b - a}{2} \left\{ \int_{0}^{1} \mu [(1 - t)^{\alpha} - t^{\alpha}]^{1/2} |f'(t a + (1 - t) b)| dt \right.
\]
\[
\leq \frac{b - a}{2} \int_{0}^{1} \left[ (1 - t)^{\alpha} - t^{\alpha} \right] |f'(a)|^{t^{\alpha}} |f'(b)|^{(1-t)^{\alpha}} dt
\]
\[
\leq \frac{b - a}{2} \left\{ \int_{0}^{1/2} \left[ t^{\alpha} - (1 - t)^{\alpha} \right] |f'(a)|^{t^{\alpha}} |f'(b)|^{(1-t)^{\alpha}} dt \right\}
\]
\[
(2.3)
\]

for all $t \in [0, 1]$. Using the well known inequality $mn \leq \mu m^{\frac{\lambda}{\alpha}} + \eta n^{\frac{\lambda}{\eta}}$, on the right side of (2.3), we have

\[
\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b - a)^{\alpha}} \left[ J_{b^+}^\alpha f(a) + J_{a^+}^\alpha f(b) \right] \right| \leq \frac{b - a}{2} \left\{ \int_{0}^{1/2} \mu [(1 - t)^{\alpha} - t^{\alpha}]^{1/2} dt + \int_{1/2}^{1} \mu [t^{\alpha} - (1 - t)^{\alpha}]^{1/2} \right.
\]
\[
\left. \eta |f'(a)|^{t^{\alpha}} |f'(b)|^{(1-t)^{\alpha}} dt \right\}
\]
\[
= \frac{b - a}{2} \left\{ \int_{0}^{1/2} \mu [(1 - t)^{\alpha} - t^{\alpha}]^{1/2} dt + \int_{1/2}^{1} \mu [t^{\alpha} - (1 - t)^{\alpha}]^{1/2} \right.
\]
\[
\left. \eta |f'(a)|^{t^{\alpha}} |f'(b)|^{(1-t)^{\alpha}} dt \right\}
\]

If $0 < \lambda \leq 1$, $0 < u, v \leq 1$, then

\[
(2.4) \quad \lambda^{uv} \leq \lambda^{uv}.
\]
When \( \psi(u, v) \leq 1 \), by (2.4), we get that
\[
\int_0^1 |f'(a)|^{\frac{\alpha}{\eta}} |f'(b)|^{\frac{(1-\alpha)}{\eta}} dt \leq \int_0^1 |f'(a)|^{\frac{\alpha}{\eta}} |f'(b)|^{\frac{(1-\alpha)}{\eta}} dt = |f'(b)|^{\frac{\alpha}{\eta}} \psi \left( \frac{s}{\eta}, \frac{s}{\eta} \right).
\]

From (2.3) to (2.5), (2.1) holds.

**Remark 1.** If we take \( \alpha = 1 \), in Theorem 5 then the inequality (2.4) becomes the inequality
\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(x) dx \right| \leq \frac{b - a}{2} \left[ \frac{\mu^2}{\mu + 1} + \eta \times |f'(b)|^{\frac{\alpha}{\eta}} \psi \left( \frac{s}{\eta}, \frac{s}{\eta} \right) \right].
\]

The corresponding version for powers of the absolute value of the first derivative is incorporated in the following result:

**Theorem 4.** Let \( I \supset [0, \infty) \) be an open interval and \( f : I \to (0, \infty) \) is differentiable. If \( f' \in L[a, b] \) and \( |f'| \) is \( s \)-logarithmically convex functions in the second sense on \( [a, b] \) for some fixed \( s \in (0, 1] \) and \( \mu, \eta > 0 \) with \( \mu + \eta = 1 \) and \( p, q > 1 \), then the following inequality holds for fractional integrals with \( \alpha > 0 \):
\[
\int_0^1 |f'(a)|^{\frac{\alpha}{\eta}} |f'(b)|^{\frac{(1-\alpha)}{\eta}} dt \leq \frac{b - a}{2} \left[ |f'(b)|^{s} \psi(s, s) \right]^\frac{1}{\alpha}. 
\]

where \( 1/p + 1/q = 1 \), and \( \psi(u, v) \) is defined as in (2.2).

**Proof.** By Lemma [1] and since \( |f'| \) is \( s \)-logarithmically convex functions in the second sense on \( [a, b] \), we have
\[
\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b - a)^{\alpha}} [J_{b-}^a f(a) + J_{a+}^b f(b)] \right| \leq \frac{b - a}{2} \int_0^1 |(1 - t)^{\alpha} - t^{\alpha}| |f'(a)|^{s} |f'(b)|^{(1-t)^s} dt 
\]
for all \( t \in [0, 1] \). Using the well known Hölder inequality, on the right side of (2.4) and making the change of variable we have
\[
\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b - a)^{\alpha}} [J_{b-}^a f(a) + J_{a+}^b f(b)] \right| \leq \frac{b - a}{2} \left( \int_0^1 |(1 - t)^{\alpha} - t^{\alpha}|^p dt \right)^\frac{1}{p} \left( \int_0^1 |f'(a)|^{q(1-t)^s} |f'(b)|^{q(1-t)^s} dt \right)^\frac{1}{q}.
\]

It is known that for \( \alpha, t_1, t_2 \in [0, 1] \),
\[
|t_1^\alpha - t_2^\alpha| \leq |t_1 - t_2|, 
\]
therefore
\[
\int_0^1 |(1 - t)^{\alpha} - t^{\alpha}|^p dt \leq \int_0^1 |1 - 2t|^{\alpha p} dt = \frac{1}{\alpha p + 1}.
\]
Since $|f'|$ is $s$-logarithmically convex functions on $[a,b]$ and $\psi(u,v) \leq 1$, we obtain
\begin{equation}
(2.10) \quad \int_0^1 |f'(a)|^{\gamma s} |f'(b)|^{\gamma(1-t)s} \, dt \leq |f'(b)|^{\gamma s} \psi(sq,sq)
\end{equation}

From (2.8) to (2.10), (2.6) holds. \hfill \Box

A different approach leads to the following result.

**Theorem 5.** Let $I \supset [0, \infty)$ be an open interval and $f : I \to (0, \infty)$ is differentiable. If $f' \in L[a,b]$ and $|f'|^q$ is $s$-logarithmically convex functions in the second sense on $[a,b]$ for some fixed $s \in (0,1]$ and $\mu, \eta > 0$ with $\mu + \eta = 1$ and $q \geq 1$, then the following inequality for fractional integrals with $\alpha > 0$ holds:
\begin{equation}
(2.11) \quad \left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b-a)} \left[ J_0^{\alpha} f(a) + J_0^{\alpha} f(b) \right] \right|
\end{equation}

\[ \leq \frac{b-a}{2} \left( \frac{2^{1-\alpha} - 1}{\alpha + 1} \right)^{1-\frac{1}{q}} \left( \frac{\mu^2}{\alpha + \mu} + \eta |f'(b)|^{\gamma} \psi \left(\frac{sq}{\eta}, \frac{sq}{\eta}\right) \right)^{\frac{1}{q}} \]

where $\psi(u,v)$ is defined as in (2.2).

**Proof.** By Lemma 1 and using the well known power mean inequality, we have
\begin{align*}
&\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b-a)} \left[ J_0^{\alpha} f(a) + J_0^{\alpha} f(b) \right] \right|
\leq \frac{b-a}{2} \int_0^1 |(1-t)^\alpha - t^\alpha| |f'(ta + (1-t)b)| \, dt \\
\leq & \frac{b-a}{2} \left( \int_0^1 |(1-t)^\alpha - t^\alpha| \, dt \right)^{\frac{1}{q}} \left( \int_0^1 |(1-t)^\alpha - t^\alpha| \, dt \right)^{\frac{1}{q}} \left( \int_0^1 |f'(ta + (1-t)b)|^q \, dt \right)^{\frac{1}{q}} \\
\end{align*}

It is easily check that
\[ \int_0^1 |(1-t)^\alpha - t^\alpha| \, dt = \frac{2}{\alpha + 1} \left( 1 - \frac{1}{2^\alpha} \right). \]

Since $|f'|^q$ is $s$-logarithmically convex and using the well known inequality $mn \leq \mu m^\frac{1}{\mu} + \eta n^\frac{1}{\eta}$, we obtain
\begin{align*}
\int_0^1 |(1-t)^\alpha - t^\alpha| |f'(ta + (1-t)b)|^q \, dt & \leq \int_0^1 |(1-t)^\alpha - t^\alpha| |f'(a)|^{\gamma s} |f'(b)|^{\gamma(1-t)s} \, dt \\
& \leq \int_0^1 |1-2t|^\alpha |f'(a)|^{\gamma s} |f'(b)|^{\gamma(1-t)s} \, dt \\
& \leq \mu \int_0^1 |1-2t|^\frac{\alpha}{\mu} \, dt + \eta \int_0^1 |f'(a)|^{\frac{\alpha}{\eta}} |f'(b)|^{\frac{\gamma(1-t)s}{\eta}} \, dt.
\end{align*}

It is easily check that
\[ \mu \int_0^1 |1-2t|^\frac{\alpha}{\mu} \, dt = \mu \frac{1}{\frac{\alpha}{\mu} + 1} = \frac{\mu^2}{\alpha + \mu}. \]
Afterwards, when $\psi(u, v) \leq 1$, by (2.4), we get that

\[
\int_0^1 |f'(a)|^{\frac{\alpha t}{\alpha + 1}} |f'(b)|^{\frac{\alpha (1-t)}{\alpha + 1}} dt \leq \int_0^1 |f'(a)|^{\frac{\alpha t}{\alpha + 1}} |f'(b)|^{\frac{\alpha (1-t)}{\alpha + 1}} dt = |f'(b)|^{\frac{\alpha}{\alpha + 1}} \psi \left( \frac{sq}{\eta}, \frac{sq}{\eta} \right).
\]

Therefore

\[
\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b-a)^{\alpha + 1}} \left[ J_0^\alpha f(a) + J_0^\alpha f(b) \right] \right| \leq \frac{b - a}{2} \left( 1 - \frac{1}{\alpha + 1} \right)^{1-\frac{1}{\alpha + 1}} \left( \mu \int_0^1 |1 - 2t|^{\frac{\mu}{\alpha + 1}} dt + \eta \int_0^1 |f'(a)|^{\frac{\alpha t}{\alpha + 1}} |f'(b)|^{\frac{\alpha (1-t)}{\alpha + 1}} dt \right)^{\alpha \mu + \eta} \psi \left( \frac{sq}{\eta}, \frac{sq}{\eta} \right)
\]

which completes the proof. \qed

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