ABSTRACT

We give a simple derivation of the Virasoro constraints in the Kontsevich model, first derived by Witten. We generalize the method to a model of unitary matrices, for which we find a new set of Virasoro constraints. Finally we discuss the solution for symmetric matrices in an external field.
1. Introduction.

There has recently been considerable progress in the study of matrix models, following the remarkable proof by Kontsevich that the intersection numbers (correlation functions) of two-dimensional topological gravity are generated by a new type of matrix model [1]. A short time later, Witten showed [2] that the partition function of this model obeys the Virasoro constraints of the one-matrix model [3, 4], thus completing the chain of arguments in a proof of the old conjecture that topological gravity and matrix models are equivalent [5].

However, Witten’s lengthy proof depends on a cumbersome diagrammatic expansion. We present in section 2 a much simpler derivation, which arose out of recent work into matrix integrals involving an external field [6].‡ This approach allows us to consider other models; and in section 3 we tackle the case of a unitary matrix in an external field [8, 6], deriving a set of Virasoro constraints which describe the continuum limit of the multicritical unitary matrix models [9, 10]. These results are new, but after this work was completed, we learned§ that another group [11] has found the same constraints from the complementary viewpoint of the mKdV flows [10, 12].

In section 4 we present the results of some work on symmetric matrices in an external field. As far as we know this model has never before been investigated, so we stray somewhat from the main subject of this paper and give some general details. Unfortunately this solution is incomplete, and in particular we fail to produce for symmetric matrices any results generalizing the Virasoro constraints of sections 2 and 3.

In the final section we discuss our results, and speculate what these might teach us about unitary matrix integrals.

‡ See also ref. 7.
§ We thank E. Witten for informing us of this.
2. Virasoro constraints from Schwinger-Dyson equations.

Consider the integral over $N \times N$ hermitian matrices

$$\hat{Z} = \int \mathcal{D}\hat{M} \exp \text{Tr} \left( X\hat{M} - \frac{1}{3}g\hat{M}^3 \right), \quad (2.1)$$

which is a function of the $N$ eigenvalues $x_a$ of the hermitian matrix $X$. In refs. 6, 13, it is shown how to evaluate $\hat{Z}$ using the Schwinger-Dyson equations of motion,

$$\int \mathcal{D}\hat{M} \frac{\partial}{\partial \hat{M}} \exp \text{Tr} \left( X\hat{M} - \frac{1}{3}g\hat{M}^3 \right) = \left( X^T - g \frac{\partial^2}{\partial X^2} \right) \hat{Z} = 0, \quad (2.2)$$

which can be recast as a set of $N$ differential equations in terms of the eigenvalues $x_a$,

$$\frac{\partial^2 \hat{Z}}{\partial x_a^2} + \sum_{b \neq a} \frac{1}{x_a - x_b} \left( \frac{\partial \hat{Z}}{\partial x_a} - \frac{\partial \hat{Z}}{\partial x_b} \right) = \frac{1}{g} x_a \hat{Z}. \quad (2.3)$$

Full details of the derivation are given in refs. 8, 6. In ref. 13 these equations were used to find $\hat{Z}$ in the spherical approximation, and recently we showed how to extend this solution to all orders in $N$ [6]. In this paper we start from these same equations, but the analysis will be very different.

Closely related to $\hat{Z}$ is the integral

$$Z = \int \mathcal{D}M \exp \text{Tr} \left( -\frac{1}{2}AM^2 - \frac{1}{3}gM^3 \right), \quad (2.4)$$

for a positive definite hermitian matrix $A$. In his ground-breaking paper [1], Kontsevich showed how the expansion of $Z$ in terms of Feynman diagrams can be interpreted

\footnote{Note that for the purposes of this paper, we define the integral without an $N$ in the exponent.}

\footnote{The reader is advised that an early preprint version of ref. 6 contained some mistakes in the introduction.}
as a cell decomposition of the moduli space of Riemann surfaces [14, 15, 16]. By a shift of integration variables, it is easy to see that \( \hat{Z} \) and \( Z \) are related by

\[
\hat{Z} = \exp \text{Tr} \frac{A^3}{12g^2} \times Z ,
\]

provided we choose \( X = \frac{A^2}{4g} \). Our goal now is to use (2.3) and (2.5) to show that \( Z \) obeys the Virasoro constraints of the one-matrix model [3, 4].

Following Kontsevich, we factor \( Z \) in the form

\[
Z = \prod_{a,b} (\mu_a + \mu_b)^{-1/2} Y ,
\]

where \( \{\mu_a\} \) are the eigenvalues of \( A \). Then, after a change of variables to \( \lambda_a \equiv \mu_a^2 = 4gx_a \), eqns. (2.3) and (2.5) imply that \( Y \) satisfies the set of differential equations

\[
\frac{\partial^2 Y}{\partial \lambda_a^2} + \frac{\partial Y}{\partial \lambda_a} \left( \frac{\mu_a}{4g^2} - \frac{Z_a}{\mu_a} \right) + \sum_{b \neq a} \frac{1}{\mu_b^2 - \mu_a^2} \left( \frac{\partial Y}{\partial \lambda_b} - \frac{\partial Y}{\partial \lambda_a} \right) + Y \left( \frac{1}{16\mu_a^4} + \frac{t_0^2}{4\mu_a^2} \right) = 0 ,
\]

where we have defined

\[
Z_a = \sum_b \frac{1}{\mu_a + \mu_b} ; \quad t_k = \frac{-1}{2k+1} \sum_b \frac{1}{\mu_b^{2k+1}} , \quad k \geq 0 .
\]

Next we make good use of Kontsevich’s important observation that the perturbative expansion of \( Y \) depends on the eigenvalues of \( A \) only through the invariants \( t_k \). This lets us change variables from \( \lambda_a \) to \( t_k \), i.e.,

\[
\frac{\partial}{\partial \lambda_a} = \frac{1}{2} \sum_k \frac{1}{\mu_a^{2k+3}} \frac{\partial}{\partial t_k} ,
\]

and after some algebra we obtain

\[
\frac{1}{4} \sum_{k,j=0}^{\infty} \frac{1}{\mu_a^{2k+2j+6}} \partial_k \partial_j Y + \frac{1}{8g^2} \sum_{k=0}^{\infty} \frac{1}{\mu_a^{2k+2}} \partial_k Y
\]

\[
+ \frac{1}{2} \sum_{k=0}^{\infty} \partial_k Y \left( \frac{(2k+3)\mu_a}{\mu_a^{2k+1}} + \frac{(2k+1)\mu_a}{\mu_a^{2k+1}} + \cdots \frac{t_0}{\mu_a^{2k+1}} \right) + \frac{1}{16\mu_a^4} Y + \frac{t_0^2}{4\mu_a^2} Y = 0 ,
\]
where $\partial_k \equiv \partial/\partial t_k$. Note that these equations are formally valid independent of the large-$N$ limit, except that only for $N \to \infty$ are the $t_k$'s truly independent, and for finite $N$ the sums over $k$ must be truncated.**

To make contact with Kontsevich's results, we set $g = i/2$. Then we can pick off the coefficient of $1/\mu^{4+2n}$ to obtain the equations

\begin{align*}
  n = -1 : & \sum_{k=1}^{\infty} (k + \frac{1}{2})t_k \partial_{k-1} Y + \frac{1}{4} t_0^2 Y = \frac{1}{2} \partial_0 Y, \\
  n = 0 : & \sum_{k=0}^{\infty} (k + \frac{1}{2})t_k \partial_k Y + \frac{1}{16} Y = \frac{1}{2} \partial_1 Y, \tag{2.11} \\
  n \geq 1 : & \sum_{k=0}^{\infty} (k + \frac{1}{2})t_k \partial_{k+n} Y + \frac{1}{4} \sum_{k=1}^{n} \partial_{k-1} \partial_{n-k} Y = \frac{1}{2} \partial_{n+1} Y.
\end{align*}

Given that we have used a different normalization for the $t_k$'s, these are precisely the equations first derived for the Kontsevich model in ref. 2,

\begin{equation}
  L_n Y = \frac{1}{2} \partial_{n+1} Y, \quad n \geq -1, \tag{2.12}
\end{equation}

i.e., the Virasoro constraints of refs. 3, 4, corresponding to perturbations about the $m = 1$ "topological" point of the one-matrix model. It is remarkable that the entire set of multicritical potentials should be accessible from the simple cubic potential of (2.4). (On the other hand, it is becoming increasingly apparent that matrix models involving an external matrix field possess a rich multicritical structure [6, 17].)

Finally, we complete the correspondence between the Kontsevich model and topological gravity by noting that the familiar selection rule for correlation functions of scaling operators at genus $g$

\begin{equation}
  \langle \prod_k \tau_k^{n_k} \rangle_g \equiv \prod_k \left( \frac{\partial}{\partial t_k} \right)^{n_k} [\log Y]_g \tag{2.13}
\end{equation}

** For related comments, see section 1.1 of ref. 2.
(where \([\cdots]_g\) means the contribution at genus \(g\)), namely
\[
\sum_k n_k(k - 1) = 3g - 3 ,
\] (2.14)
is readily extracted from results in the appendix of ref. 6. As is well known, eqn. (2.14) and the \(L_0\) constraint are all one needs to find the string susceptibility and the scaling dimensions of operators.

3. Unitary matrices.

The success of the analysis so far suggests that one should investigate other models involving an external field. One such case, considered in another context in refs. 8, 6, is that of a unitary matrix,

\[
\bar{Z} = \int \mathcal{D}U \exp \text{Tr} \left( A^+ U + U^+ A \right) ,
\] (3.1)

where \(A\) is now an arbitrary matrix. As mentioned earlier, we know of no simple geometrical interpretation for such an integral. One approach is to write the unitary matrix in terms of a hermitian matrix, for example as \(U = e^{iH} [18]\), or \(U = (1 + iH)/(1 - iH) [19]\), but neither of these has a particularly attractive expansion in terms of surfaces.

The partition function \(\bar{Z}\) satisfies the differential equations
\[
\frac{\partial^2 \bar{Z}}{\partial A_{ab} \partial A_{bc}^\dagger} = \delta_{ac} \bar{Z} .
\] (3.2)

After changing variables in these equations to the eigenvalues \(\lambda_a \equiv \mu_a^2\) of \(A^+ A\), we get the Schwinger-Dyson equations for \(\bar{Z}\),
\[
\frac{\partial^2 \bar{Z}}{\partial \lambda_a^2} + \sum_{b \neq a} \frac{1}{\lambda_a - \lambda_b} \left( \frac{\partial \bar{Z}}{\partial \lambda_a} - \frac{\partial \bar{Z}}{\partial \lambda_b} \right) = \frac{1}{\lambda_a} \left( 1 - \sum_b \frac{\partial \bar{Z}}{\partial \lambda_b} \right) .
\] (3.3)

Guided by experience, we expect that we should factor \(\bar{Z}\) along the lines of (2.6); and
indeed, after a small amount of trial and error, we find it useful to write

\[ \tilde{Z} = \prod_{a,b} (\mu_a + \mu_b)^{-1/2} \exp \left( 2 \sum_b \mu_b \right) \tilde{Y} . \]  

(3.4)

This factor is what one gets by expanding \( U \) about the saddle point of the action as the exponential of a hermitian matrix and dropping terms higher than quadratic, and in a sense (3.4) is the natural analog of the factorization (2.6).

It was seen in ref. 6 that \( \tilde{Y} \) shares with \( Y \) the property of depending only on the \( t_k \)'s. This encourages us to try changing variables in (3.3), and we find that \( \tilde{Y} \) satisfies

\[ \begin{align*}
\frac{1}{4} \sum_{k,j=0}^{\infty} \frac{1}{\mu^a_{2k+2j+6}} & \partial_k \partial_j \tilde{Y} + \sum_{k=0}^{\infty} \frac{1}{\mu^a_{2k+4}} \partial_k \tilde{Y} \\
+ \frac{1}{2} \sum_{k=0}^{\infty} \partial_k \tilde{Y} \left( \frac{(2k+1)t_k}{\mu^4_a} + \cdots + \frac{t_0}{\mu^4_{a+4}} \right) + \frac{1}{16\mu^4_a} \tilde{Y} &= 0 ,
\end{align*} \]

(3.5)

which yields the equations

\[ \begin{align*}
n = 0 : & \quad \sum_{k=0}^{\infty} \left( k + \frac{1}{2} \right) t_k \partial_k \tilde{Y} + \frac{1}{16} \tilde{Y} = -\partial_0 \tilde{Y} , \\
n \geq 1 : & \quad \sum_{k=0}^{\infty} \left( k + \frac{1}{2} \right) t_k \partial_{k+n} \tilde{Y} + \frac{1}{4} \sum_{k=1}^{n} \partial_{k-1} \partial_{n-k} \tilde{Y} = -\partial_n \tilde{Y} .
\end{align*} \]

(3.6)

We recognize these as Virasoro constraints, namely

\[ L_n \tilde{Y} = -\partial_n \tilde{Y} , \quad n \geq 0 . \]

(3.7)

These equations can be put in the standard form \( L_n \tilde{Y} = 0 \) by a shift of \( t_0 \), which makes explicit that (3.7) corresponds to an expansion about the point \( t_0 = 2, t_i = 0 \) for \( i \geq 1 \). (Contrast this with the usual situation in hermitian matrix models, where \( t_0 \equiv x \) is a free parameter, and one other \( t_m \), for some \( m > 0 \), is fixed to a non-zero
This is an exceedingly trivial point, the \( m = 0 \) unitary matrix model, which corresponds naively to a matrix model with potential \( V(U) = \text{constant} \) (the \( m = 1 \) model has \( V = U + U^\dagger \)); nevertheless it is well defined, since the integration is over a compact group.

The Virasoro constraints are a convenient starting point for extracting useful information about the model. For instance, we can follow the example of ref. 3 and derive from them recursion relations for the correlation functions of scaling operators. For the \( m = 0 \) model, these take the form

\[
\langle \tau_n \prod_{k \in S} \tau_k \rangle_g = -\sum_{j \in S} (j + \frac{1}{2}) \langle \tau_{j+n} \prod_{k \neq j} \tau_k \rangle_g \\
- \frac{1}{4} \sum_{j=1}^n \left\{ \langle \tau_{j-1} \tau_{n-j} \prod_{k \in S} \tau_k \rangle_{g-1} + \frac{1}{2} \sum_{S = X \cup Y} \langle \tau_{j-1} \prod_{k \in X} \tau_k \rangle_{g_1} \langle \tau_{n-j} \prod_{k \in Y} \tau_k \rangle_{g_2} \right\}.
\]

(The genus dependence of these correlation functions has been inserted by hand, so as to correspond to the conventional \( 1/N \) expansion.) Note that the recursion relations are precisely sufficient to determine all the correlation functions as pure numbers – i.e., all the operators are redundant. In a loose sense, therefore, this model is also “topological.”

There is also a selection rule,

\[
\sum_k k n_k = g - 1 \iff \sum_k k t_k \partial_k [\log \bar{Y}]_g = (g - 1) [\log \bar{Y}]_g,
\]

which, like (2.14), was found in ref. 6. As a check, note that (3.9) and (2.14) are merely different linear combinations of \( L_0 \) and the dilaton equation \( \sum_k n_k = 2 - 2g^* \). (In general, one finds the selection rule for the \( m^{th} \) multicritical model by eliminating \( t_m \) between the two equations.)

\* We thank R. Dijkgraaf for pointing this out to us.
An important feature of (3.7) is that because there is no $L_{-1}$ equation, the constant in $L_0$ is not determined simply by the requirement of closure of the Virasoro algebra. In ref. 11 the constant is considered a free parameter of the solution, but our derivation singles out the value $\frac{1}{16}$.

There is another difference between our results and theirs, which we do not really understand. The constraints found here act directly on $\bar{Y}$, whereas in ref. 11 the partition function is a product of two tau functions, each of which is annihilated separately by the $L_n$’s.

4. Symmetric matrices and unoriented surfaces.

In this final section we discuss an interesting example of a model which can be solved only partially by the methods we have been using: the real-symmetric matrix model. This has been investigated in several papers [20, 21, 22, 23], and has a well-known geometric interpretation as the sum over unoriented surfaces. It appears that this model in an external field has not been looked at before, so we permit ourselves to stray somewhat from the subject of Virasoro constraints and discuss this case in a fair amount of detail. Much of the analysis runs parallel to that of the hermitian matrices, and wherever possible we use the same notation.

The starting point is the integral

$$\tilde{Z} = \int \prod_{1 \leq i \leq j \leq N} dM_{ij} \exp N \text{Tr} \left(XM - \frac{1}{3}gM^3 \right), \quad (4.1)$$

for some real symmetric matrix $X$. In this section we need to count powers of $N$ in the topological expansion, so we include an $N$ in the exponent.

The derivation of the Schwinger-Dyson equations from eqn. (4.1) is more subtle than for hermitian matrices because the matrix elements of $M$ (or $X$) are no longer independent, and when we differentiate with respect to off-diagonal elements there is
an extra factor of 2. For example,

\[
\frac{\partial}{\partial M_{ij}} \text{Tr}(XM) = \begin{cases} 
X_{ij} + X_{ji} = 2X_{ij} & i \neq j \\
X_{ii} & i = j 
\end{cases} .
\] (4.2)

As a consequence, the equations of motion have a complicated form when written in terms of derivatives with respect to the matrix elements of \(X\). For the same reason we have to be careful when we change variables to eigenvalues, which requires evaluating such quantities as \(\partial^2 x_a / \partial X_{pq} \partial X_{rs}\). Remarkably, the final answer is both simple and familiar:

\[
\frac{\partial^2 \tilde{Z}}{\partial x_a^2} + \frac{1}{2} \sum_{b \neq a} \frac{1}{x_a - x_b} \left( \frac{\partial \tilde{Z}}{\partial x_a} - \frac{\partial \tilde{Z}}{\partial x_b} \right) = \frac{N^2}{g} x_a \tilde{Z} .
\] (4.3)

However, there is a major difference between this and eqns. (2.3), (3.3): namely, the factor of \(\frac{1}{2}\) multiplying the sum. This small change makes an important qualitative difference to the genus expansion.

Before we solve (4.3), it will prove convenient to rescale variables to \(\lambda_a = 4gx_a\). Then, following the procedure of ref. 6, we define \(\tilde{Z} = e^{NF}\) and rewrite (4.3) as an equation for \(\tilde{F}_a = \partial \tilde{F} / \partial \lambda_a\),

\[
\frac{1}{N} \frac{\partial \tilde{F}_a}{\partial \lambda_a} + \tilde{F}_a^2 + \frac{1}{2N} \sum_{b \neq a} \frac{\tilde{F}_a - \tilde{F}_b}{\lambda_a - \lambda_b} = \frac{1}{64g^3} \lambda_a .
\] (4.4)

In the spherical limit the first term is smaller by a factor of \(1/N\), and we can drop it. The resulting equation has the solution

\[
\tilde{F}_a^{(0)} = \frac{\nu_a}{8g^2} + \frac{1}{4\nu_a} \left( \sigma_1 - \hat{Z}_a \right) ,
\] (4.5)

where it is useful to introduce the notation

\[
\nu_a = \sqrt{\lambda_a + y} , \quad \sigma_k = \frac{1}{N} \sum_b \frac{1}{\nu_b^k} , \quad \hat{Z}_a = \frac{1}{N} \sum_b \frac{1}{\nu_a + \nu_b} ,
\] (4.6)

and \(y\) is found from the equation \(y = -4g^2 \sigma_1\); the superscript on \(\tilde{F}_a\) refers to the

* Note that \(\tilde{F}\) is of order \(N\), and \(\tilde{F}_a\) is of order 1.
order in $1/N$. Integrating eqn. (4.5), we find that the free energy is

$$
\frac{1}{N} \tilde{F}^{(0)} = \frac{1}{12g^2} \sigma_{-3} + \frac{1}{2} \sigma_1 \sigma_{-1} + \frac{g^2}{6} \sigma_1^3 - \frac{1}{4N^2} \sum_{b,c} \ln (\nu_b + \nu_c) .
$$

(4.7)

We can attempt to solve (4.4) order by order in $1/N$ to find the corrections to the free energy, as in ref. 6. Just write $\tilde{F} = \tilde{F}^{(0)} + \frac{1}{N} \tilde{F}^{(1)} + \cdots$, insert into (4.4), and linearize. In the solutions of unitary and hermitian matrices [6], the $1/N$ terms canceled and the leading correction was $O(1/N^2)$. In the present case, this cancellation does not occur, because of the factor of $\frac{1}{2}$; therefore the free energy has a contribution of order $1/N$, as one would expect from topological arguments [24].

The resulting linear equation for $\tilde{F}_a^{(1)}$ is

$$
\left( \frac{\nu_a}{4g^2} + \frac{1}{2\nu_a} \left( \sigma_1 - \check{Z}_a \right) \right) \tilde{F}_a^{(1)} + \frac{1}{2N} \sum_{b \neq a} \frac{\tilde{F}_a^{(1)} - \tilde{F}_b^{(1)}}{\lambda_a - \lambda_b} = 
\frac{1}{32g^2 \nu_a} + \frac{1}{16\nu_a^3} \left( \sigma_1 - \check{Z}_a \right) - \frac{1}{16N} \sum_b \nu_a^2 (\nu_a + \nu_b)^2 .
$$

(4.8)

The right-hand side here is more complicated than anything that arose in the unitary or hermitian cases. We have not been able to solve this equation, and unfortunately it seems probable that the solution to (4.8) cannot be written in closed form. If this is true, then it is impossible even to write down the equations for higher $1/N$ corrections, let alone solve them.

On the other hand there is no obstacle to solving (4.8) perturbatively in $g$ (actually, as a double power series expansion in $g$ and $1/N$). Rather than using diagrammatic perturbation theory, we can work directly from the differential equation (4.4). But first, note that we are principally interested in the Kontsevich-type integral

$$
Z = \int DM \exp N \text{Tr} \left( -\frac{1}{2} AM^2 - \frac{1}{3} g M^3 \right) = e^{NF} ,
$$

(4.9)

where $F$ and $\tilde{F}$ are related by $F = \tilde{F} - \text{Tr} A^3/12g^2$. Then one can show that
$F_a = \partial F/\partial \lambda_a$ satisfies

$$
\frac{1}{N} \left( \frac{\partial F_a}{\partial \lambda_a} + \frac{1}{32g^2 \mu_a} \right) + F^2_a + \frac{\mu_a}{4g^2} F_a + \frac{1}{2N} \sum_{b \neq a} \frac{F_a - F_b}{\lambda_a - \lambda_b} + \frac{Z_a}{16g^2} = 0 ,
$$

(4.10)

where $\mu_a$, $\lambda_a$, and $Z_a$ are defined in section 2. To derive the perturbation expansion, it is convenient to rearrange this into the form

$$
F_a = - \frac{Z_a}{4 \mu_a} - \frac{1}{8N \mu_a^2} - \frac{4g^2}{\mu_a} \left( \frac{1}{2N} \sum_{b \neq a} \frac{F_a - F_b}{\lambda_a - \lambda_b} + F^2_a + \frac{1}{N} \frac{\partial F_a}{\partial \lambda_a} \right) .
$$

(4.11)

With some help from Mathematica, we find that the first few terms of the solution at order $1/N$ are

$$
\frac{1}{N} F^{(1)} = - \frac{1}{4N} \sum_b \ln \mu_b + g^2 \left( \frac{1}{2} s_1 s_2 + \frac{1}{4N^2} \sum_{b,c} \frac{1}{\mu_b \mu_c (\mu_b + \mu_c)} \right) +

\frac{g^4}{4N} \left( s^2_1 + \frac{1}{2} s_1 s_2 + \frac{1}{4N^2} \sum_{b,c} \frac{1}{\mu_b \mu_c (\mu_b + \mu_c)} \right) -

\frac{1}{6N^3} \sum_{a,b,c} \frac{\mu_a \mu_b \mu_c (\mu_a + \mu_b)(\mu_b + \mu_c)(\mu_c + \mu_a)}{\mu_a \mu_b \mu_c (\mu_a + \mu_b)(\mu_b + \mu_c)(\mu_c + \mu_a)} + O(g^6) ,
$$

(4.12)

where we have used the notation $s_k = \frac{1}{N} \sum_b \frac{1}{\mu_b}$. This can be written in a more concise form in terms of the “shifted” eigenvalues $\nu_a (\equiv \sqrt{\mu_a^2 + y})$,

$$
\frac{1}{N} F^{(1)} = - \frac{1}{4N} \sum_b \ln \nu_b - \frac{g^2}{4N^2} \sum_{b,c} \frac{1}{\nu_b \nu_c (\nu_b + \nu_c)} -

\frac{g^4}{6N^3} \sum_{a,b,c} \nu_a \nu_b \nu_c (\nu_a + \nu_b) (\nu_b + \nu_c) (\nu_c + \nu_a) + O(g^6) .
$$

(4.13)

To recover (4.12) from this, use the perturbative expansions of $\nu_a$,

$$
\nu_a = \mu_a \left( 1 + \frac{1}{2} y \frac{1}{\mu_a^2} + \cdots \right) ,
$$

(4.14)
and of $y$,

$$y = -4g^2 \frac{1}{N} \sum_b \frac{1}{(\mu_b^2 + y)^{1/2}} = -4g^2 \left( s_1 - \frac{1}{2} y s_3 + \cdots \right). \quad (4.15)$$

This latter equation can be solved iteratively to give $y$ in terms of the $s_k$’s.

The notable feature of (4.13) is that it cannot be expressed in terms of just the traces $s_{2k+1}$, in contrast to the unitary and hermitian cases. In fact, it would appear that at each order in $g^2$ we encounter new and more complicated invariants of the eigenvalues. Because the solution in its concise form (4.13) is so simple, one is tempted to guess the general term at order $g^{2n}$. But this simplicity is misleading, and even at $O(g^6)$ we do not know the form of the solution. The obvious ansatz, namely

$$\frac{1}{N^4} \sum_{a,b,c,d} \frac{1}{\nu_a \nu_b \nu_c \nu_d (\nu_a + \nu_b) \cdots (\nu_c + \nu_d)}, \quad (4.16)$$

can be ruled out on dimensional grounds, as we need a term of order $1/\nu^9$, not $1/\nu^{10}$.

We finish by asking whether it is possible to find Virasoro-type constraints for unoriented surfaces. We have the first ingredient, namely the Schwinger-Dyson equations (4.4). Next we must identify how the integral (4.9) depends on the eigenvalues of $A$, and “change variables.” In the previous examples we were lucky: a perturbative analysis showed that the integrals were functions solely of the $t_k$’s, and the change of variables was simple. In the present case, we don’t even know the complete set of variables; but we do know that it is complicated, and the change of variables would likely present a thorny challenge. For these reasons, we think that further progress in this direction is impossible.
5. Discussion.

In this paper we have demonstrated an efficient method of deriving Virasoro constraints satisfied by the Kontsevich model, which we generalized to the unitary matrix model (3.1). Now we ask whether this has revealed anything about the geometrical interpretation of unitary matrices. To answer this, we consider first the hermitian case (2.4), which is well understood.

There is an important point about the Kontsevich model which we wish to emphasize. The great majority of papers on matrix models involve studying them near their critical points, which is where the perturbation expansion diverges and the Feynman diagrams have an interpretation as continuum surfaces discretized by large numbers of infinitesimal triangles. This is not the situation in the Kontsevich model, where one interprets a Feynman diagram of low order in perturbation theory as a surface with a finite number of punctures (equal to the number of faces on the diagram). This is a topological picture, in which all surfaces of given genus and number of punctures are represented by a single “ribbon graph”; and it yields topological information, i.e., the intersection numbers. In addition, the integral (2.4) also possesses a conventional critical point, studied in ref. 6, with an interpretation in terms of continuum surfaces.

This situation is reminiscent of the well-known Penner model, whose perturbative “topological” limit encodes the virtual Euler characteristic of surfaces [14, 15, 16], whereas the continuum limit corresponds to a $c = 1$ string in a compactified target space of a particular radius [25].

Now we conjecture that the unitary matrix model of section 3 is a third example of this type. The integral certainly possesses the two appropriate limits. From the “perturbative” limit we derive Virasoro constraints. The work of Hollowood et al. [11] gives an independent confirmation that these are the correct constraints to describe the multicritical unitary matrix models, which we wish to interpret as some ensemble of surfaces coupled to matter. The non-trivial step is to suggest that this structure arises here because the integral (3.1) has a representation, analogous to that of the
Kontsevich integral, in terms of the “moduli space” of the same (unknown) ensemble. However, we leave open the question of what this ensemble might be.

It is very disappointing that we were unable to repeat the analysis for symmetric matrices, but the moduli space of unoriented surfaces is known to be a lot more complicated than that of oriented surfaces. Nevertheless, it has been instructive to study this model for the subtleties it reveals in the hermitian and unitary matrix models.

As a possible generalization of this work, one could try to extend these results to the $d = -2$ external field model [17]. This would not be trivial, however, as that model was solved by a different method.

The original motivation for the research that led to this paper was to find a geometrical picture for unitary matrix models. This is a goal that has long been sought by many others (e.g. [12, 18, 27]). Unfortunately a satisfactory answer eludes us still, but we offer our insights in the hope that others can continue this work to completion.

Note added: While this paper was being prepared, we learned that A. Marshakov, A. Mironov, and A. Morozov [28], and M. Kontsevich [29], have independently derived these same results for hermitian matrices. In addition, S. Kharchev, A. Marshakov, A. Mironov, A. Morozov, and A. Zabrodin [30] have recently proposed a generalized Kontsevich model to describe all multi-matrix models.

Acknowledgements: We are grateful to the authors of ref. 11 for sharing their results with us prior to publication, and in particular to Tim Hollowood and Andrea Pasquinucci for valuable discussions. We would also like to thank Ulf Danielsson, Robbert Dijkgraaf, Jacques Distler, Miguel Martin-Delgado, Herman Verlinde, and Ed Witten for frequent help and explanations along the way. M. N. is especially indebted to Mark Doyle for many useful conversations, and in particular for his thorough explanation of the paper of Kontsevich.

* For example, see ref. 26.
REFERENCES

1. M. Kontsevich, “Intersection theory on the moduli space of curves,” preprint (1990).

2. E. Witten, “On the Kontsevich model and other models of two dimensional gravity,” IAS preprint IASSNS-HEP-91/24.

3. R. Dijkgraaf, E. Verlinde, and H. Verlinde, *Nucl. Phys.* **B348** (1991) 435.

4. M. Fukuma, H. Kawai, and R. Nakayama, *Int. Jour. Mod. Phys.* **A6** (1991) 1385.

5. E. Witten, *Nucl. Phys.* **B340** (1990) 281.

6. D. J. Gross and M. J. Newman, *Phys. Lett.* **266B** (1991) 291.

7. Yu. Makeenko and G. W. Semenoff, “Properties of Hermitean matrix models in an external field,” ITEP/University of British Columbia preprint (July 1991).

8. E. Brézin and D. J. Gross, *Phys. Lett.* **97B** (1980) 120.

9. V. Periwal and D. Shevitz, *Phys. Rev. Lett.* **64** (1990) 1326.

10. V. Periwal and D. Shevitz, *Mod. Phys. Lett.* **A5** (1990) 1147.

11. T. Hollowood, L. Miramontes, A. Pasquinucci, and C. Nappi, Princeton/IAS preprint IASSNS-HEP-91/59, PUPT-1280.

12. Č. Crnković, M. Douglas, and G. Moore, “Loop equations and the topological phase of multicritical matrix models,” Yale/Rutgers University preprint YCTP-P25-91/RU-91-36 (August 1991).

13. I. Kostov, in Jaca 1988, Proceedings, *Non-perturbative aspects of the standard model*, 295.

14. R. Penner, *Comm. Math. Phys.* (1987); *J. Diff. Geom.* **27** (1988) 35.

15. J. Harer, *Inv. Math.* **84** (1986) 157.

16. B. H. Bowditch and D. B. A. Epstein, *Topology* **27** (1988) 35.
17. V. A. Kazakov, *Nucl. Phys.* **B354** (1991) 614.

18. H. Neuberger, *Nucl. Phys.* **B340** (1990) 703.

19. M. Bowick, A. Morozov, and D. Shevitz, *Nucl. Phys.* **B354** (1991) 496.

20. E. Brézin and H. Neuberger, *Phys. Rev. Lett.* **65** (1990) 2098.

21. E. Brézin and H. Neuberger, *Nucl. Phys.* **B350** (1991) 513.

22. G. Harris and E. Martinec, *Phys. Lett.* **245B** (1990) 384.

23. H. Neuberger, *Phys. Lett.* **257B** (1991) 45.

24. G. ’t Hooft, *Nucl. Phys.* **B72** (1974) 461.

25. J. Distler and C. Vafa, *Mod. Phys. Lett.* **A6** (1991) 259.

26. C. P. Burgess and T. R. Morris, *Nucl. Phys.* **B291** (1987) 285.

27. Č. Crnković and G. Moore, *Phys. Lett.* **257B** (1991) 322.

28. A. Marshakov, A. Mironov, and A. Morozov, “On equivalence of topological and quantum 2d gravity,” preprint HU-TFT-91-44, FIAN/TD/04-91, ITEP-M-4/91.

29. M. Kontsevich, “Intersection theory on the moduli space of curves and the matrix Airy function,” Max Planck Institute preprint MPI/91-77.

30. S. Kharchev, A. Marshakov, A. Mironov, A. Morozov, and A. Zabrodin, “Unification of all string models with $c < 1$,” preprint FIAN/TD/09-91, ITEP-M-8/91; “Towards unified theory of 2d gravity,” preprint FIAN/TD-10/91, ITEP-M-9/91.