Analytic solutions for steady-state expectation values of atomic quantities and second order correlations are obtained for a fully quantum treatment of two stationary dipole-coupled atoms driven in a standard geometric configuration by a near resonant laser. Explicit expressions for the spatial and coherence properties of the far-field scattered light intensity are derived, valid for the full range of system parameters. A comprehensive survey of the steady-state scattering behaviour is given, with key features precisely characterised, including suppression of scattering, and the regime in which the dipole-dipole coupling has significant effect. A regime is also found where the incoherent scattered light develops spatial interference fringes. We examine in detail a decorrelation approximation that has potential application for larger systems of atoms that are intractable in a full quantum treatment. Finally, we introduce the concept of an effective driving field and show that it can provide a direct and intuitive physical interpretation of key aspects of the system behaviour.

I. INTRODUCTION

Collective light scattering from a coherently driven ensemble of atoms is a research area of long standing [1], but the phenomenon remains the subject of considerable current interest e.g. Refs. [2–13]. In a seminal paper, Lehmberg [1] derived a set of operator equations to describe the response of a system of \( N \) two-state atoms coupled by vacuum radiation and driven by a monochromatic laser, and gave general expressions for the radiation rates and spectral properties of the scattered radiation. Lehmberg’s work was motivated by the closely related phenomenon of coherent collective spontaneous emission, an area pioneered earlier by Dicke [14], who recognised that a sample of dipole coupled atoms could exhibit both subradiant and superradiant emission. As a practical demonstration of his formalism Lehmberg calculated, in a separate paper [15], the radiation rates and spatial intensity patterns of collective spontaneous emission from two atoms. A major review of both the theoretical and practical aspects of collective spontaneous emission was given by Gross and Haroche [16] some decades ago, but this area too has remained of strong active interest, with experimental milestones such as the first observation of subradiance [17], observation of superradiance and subradiance from two ions [18], and recently the observation of subradiance in a large dilute cold atom gas [19].

Much of the recent focus has been on low intensity scattering from clouds of ultracold atoms (e.g. [3, 5, 6, 8, 10, 12], or arrays of atoms e.g. [4, 7, 9]). Comparison between experimental results and theory has raised questions of our understanding of these phenomena in certain regimes e.g. [12]. The role of quantum correlations between atoms is known to be important, and their treatment requires a microscopic quantum approach. However the exponential growth of the Hilbert space with atom number necessitates approximate solution methods, even for systems of a few atoms. Most theoretical treatments employ the simplifying assumption of a weak driving field. An exception is the work of Pucci et al. [11] who have developed a large scale approximate simulation method for a strongly driven cold gas, with a validity regime that has enabled the role of long-range correlations to be elucidated. Physical insight into the behaviour of the system underlies the development of these, and future approximation methods.

The collective scattering behaviour of two monochromatically driven atoms is a fundamental building block for understanding the larger scale behaviour. Theoretical results for the scattered intensity have been presented in a number of earlier papers, but these are either numerical solutions or analytic expressions with restricted validity regime. For example Kuś and Wódkiewicz [20] gave an analytic expression for the temporal spectra valid for exact resonance, small atomic separation, and large laser intensity. Rudolph et al. [21] gave detailed results for the spatial pattern and spectra of the scattered radiation using a numerical implementation of an eigenmode approach. Wong et al. [22] used a quantum Monte Carlo method to study the spatial interference pattern and polarization of the intensity scattered from two closely separated \( j = 1/2 \leftrightarrow j = 1/2 \) atoms.

In this paper we obtain compact analytic solutions for the coherence and spatial properties of the intensity scattered from two \( j = 0 \leftrightarrow j = 1 \) atoms driven in a typical geometry by a monochromatic laser. Our solutions are valid for a full range of the system parameters and permit precise analytic characterisation of the key features of the behaviour that can occur, including suppression of scattering at very small interatomic separation. In addition, with the motivation of developing a tractable approximate solution method for larger systems, we investigate in detail an approximation scheme for our system which sharply reduces the number of system equations. This approximation method, which is based on a decorrelation procedure, is shown to be very accurate over a wide range of the system parameters. Furthermore this approximation provides a physical interpretation of key aspects of the system behaviour in familiar electromagnetic terms, and for example leads to an intuitive explanation of the modulation of scattered intensity with interatomic distance.

The paper is organised as follows. In Section II we outline the formalism used, and the derivation of the quantum Langevin equations for the atomic operators. Choosing linear polarisation for the laser, and a specific geometrical configura-
tion, the atoms reduce to effectively two-state. We present the equations for the ensemble averages of the atomic quantities and correlations required to construct the scattered intensity, and an analytic solution for those quantities. In Section III we provide a compact and comprehensive survey of the behaviour of the steady state scattered intensity over the entire parameter regime. We obtain analytic characterisations for the features observed, including the spatial interference fringes in the incoherent intensity. In Section IV we introduce the decorrelation approximation and provide a quantitative analysis of its validity regime. We also introduce the concept of the effective field, and with a selection of numerical results, demonstrate the utility of the effective field concept in explaining their prominent features.

II. FORMALISM

A. Overview

We consider two identical atoms, each with an optical dipole transition between a lower level \( l \) and upper level \( u \) with angular momenta \( j_l = 0 \) and \( j_u = 1 \) respectively. The atoms, which we assume to be stationary, interact with an external cw single mode laser and the vacuum radiation field. The laser field is a coherent state and can be treated as a classical field [23], which we choose to be linearly polarised with wave vector \( \mathbf{k}_L = k_L \mathbf{e}_z \) and amplitude of the electric displacement \( D_{\text{ext}} = \epsilon_0 E_L \mathbf{e}_z \) (see Fig. 1).

The Power-Zienau-Wooley formulation of Quantum Electrodynamics, described in depth in the text by Cohen-Tannoudji et al. [23], is the most convenient for this problem. This approach, in which the quantised field is the transverse electric displacement, \( D_\perp (\mathbf{r}) \), has the important advantage that the interaction between separated atoms is entirely due to the quantised fields and has no longitudinal (Coulomb) field contribution [24]. Morice et al. [25] have used this formalism to derive the evolution equations for a gas of identical \( j = 0 \leftrightarrow j = 1 \) atoms driven by a weak external laser field, and we will adopt their method, but derive equations valid for arbitrary laser intensity. In the following, we discuss some key points in the derivation but present only equations necessary for our current purpose. More details can be found in ref. [25], and in the appendix of this paper where we present the full set of evolution equations for arbitrary geometry.

The Hamiltonian for the system in the dipole approximation is

\[
H = \sum_{j=1}^{2} \left[ \frac{p_j^2}{2m} + (\omega_A^0 + \delta \omega_A) \mathbf{1}^{(j)}_{u} \right] + \int_{|k|<k_M} d^3k \sum_{\epsilon \perp k} \hbar \omega_k \left[ a^\dagger_{k\epsilon} a_{k\epsilon} + \frac{1}{2} \right]
- \sum_{j=1}^{2} \frac{1}{\epsilon_0} \mathbf{d}_j \cdot \left[ D_\perp (\mathbf{r}_j) + D_{\text{ext}} (\mathbf{r}_j) \right]
+ \frac{1}{\epsilon_0} \mathbf{d}_1 \cdot \mathbf{d}_2 \delta (\mathbf{r}_1 - \mathbf{r}_2),
\]

where \( \mathbf{r}_j, \mathbf{p}_j \), are the position and momentum operators for the center of mass of the \( j^{th} \) atom, \( \mathbf{d}_j \) is the dipole operator and \( \mathbf{1}^{(j)}_{u} \) is the unit operator in the upper level subspace of the \( j^{th} \) atom. The transition has a “bare” frequency of \( \omega_A^0 \), while the quantity \( \delta \omega_A \) is the net shift of the bare transition frequency due to the dipole self energy. The second term on the RHS of Eq.(1) is the free Hamiltonian for the displacement field, with \( a_{k\epsilon} \) and \( a^\dagger_{k\epsilon} \) the annihilation and creation operators for mode \((\mathbf{k}, \epsilon)\). The integral has a cutoff at \( k_M \) which is required for self consistency in a nonrelativistic treatment of the atoms [23]. The third term in Eq.(1) represents the dipole interaction of the atoms with the laser field \( \mathbf{D}_{\text{ext}} \) and the internal radiation field \( \mathbf{D}_{\perp} \), which is transverse. Morice et al. write the field \( \mathbf{D} (\mathbf{r}) / \epsilon_0 \) as \( \mathbf{E} (\mathbf{r}) \) and call it the “electric displacement vector, up to a factor of \( \epsilon_0 \)”, and we shall henceforth follow their practise, and define

\[
\mathbf{E} (\mathbf{r}) \equiv \mathbf{D}_\perp (\mathbf{r}) / \epsilon_0; \quad \mathbf{E}_{\perp} (\mathbf{r}) \equiv \mathbf{D}_{\text{ext}} (\mathbf{r}) / \epsilon_0
\]

We note that \( \mathbf{D} (\mathbf{r}) / \epsilon_0 \) is identical to the electric field away from the atoms (i.e. in vacuum). The final term in Eq.(1) represents contact interaction between atoms, which we will henceforth ignore, assuming the atoms are always separated. We denote the spatial separation of the atoms as \( \mathbf{R} \equiv \mathbf{r}_2 - \mathbf{r}_1 \), and in all that follows we will position the atoms symmetrically around the origin of the coordinate axes so that \( \mathbf{r}_1 = -\frac{R}{2} \) and \( \mathbf{r}_2 = \frac{R}{2} \). The equations of motion for the electric field and the atomic operators are derived by the quantum Heisenberg-Langevin method, which is described in depth for the case of a single two-state atom in the text by Cohen-Tannoudji et al. [26]. Adaptions to the multi-atom case are briefly presented by Morice et al. [25]. In this paper we express the dipole operator for the \( j^{th} \) atom as

\[
\mathbf{d}_j = \mathbf{D} \sum_{\eta = -1, 0, 1} \left[ d^{\eta \dagger} e^{i\omega_{\eta} t} + d^{\eta \dagger} e^{-i\omega_{\eta} t} \right] \mathbf{e}_q^* \tag{3}
\]

where expansion on the standard unit spherical vectors \( \mathbf{e}_q^* \) is chosen for convenience in applying selection rules and sym-
metries [27]. Here \( \mathcal{D} \equiv \langle j_{l} | d_{l} | j_{l} \rangle / \sqrt{3} \) and the reduced dipole matrix element [27] \( \langle j_{l} | d_{l} | j_{l} \rangle \) is chosen to be positive [28]. The slowly varying operators \( d_{l+}^{(j)} \) and \( d_{l-}^{(j)} \) can be expressed in terms of irreducible spherical tensors (see Appendix), and correspond respectively to raising and lowering operators with their \( t = 0 \) nonzero matrix elements given by \( \langle 1q | d_{l+}^{(j)} | 00 \rangle = 1 \) and \( \langle 00 | d_{l-}^{(j)} | 1q \rangle = (-)^{q} \). The first step in the derivation of the Langevin equations is to obtain an expression for the electric field operator by formally integrating its equation of motion and applying well understood approximations (including the Markov approximation, see ref [25]) to give

\[
\mathbf{E}(\mathbf{r}, t) = \frac{\hbar}{\mathcal{D}} \sum_{j} \left[ e^{i\omega_{l}t} g_{l|j}^{\dagger}(\mathbf{R}_{j}) d_{j}^{(j)}(t) + e^{-i\omega_{l}t} g(\mathbf{R}_{j}) d_{j}^{(j)}(t) \right] + \mathbf{f}(\mathbf{r}, t),
\] (4)

Here \( \mathbf{R}_{j} \equiv \mathbf{r} - \mathbf{r}_{j} \), \( d_{j}^{(j)}(t) = \sum_{q} d_{l_{q}j}^{(j)} e_{q}^{*} \), and \( \mathbf{f}(\mathbf{r}, t) \) is the quantum noise component of the field, which is given in full form in the Appendix. The quantity \( g(\mathbf{R}_{j}) \) is a matrix given in spherical coordinates by

\[
g_{\alpha\beta}(\mathbf{R}_{j}) = \frac{3\gamma}{4} e^{ik_{L}R_{j}} \left[ \delta_{\alpha\beta} \left( -\frac{1}{(k_{L}R_{j})^{3}} + \frac{i}{(k_{L}R_{j})^{2}} + 1 + \frac{1}{k_{L}R_{j}} \right) \right. \\
\left. - \frac{R_{L,\alpha}R_{L,\beta}^{*}}{R_{j}^{2}} \left( -\frac{3}{(k_{L}R_{j})^{3}} + \frac{3i}{(k_{L}R_{j})^{2}} + 1 + \frac{1}{k_{L}R_{j}} \right) \right],
\]

where \( \gamma \) is the Einstein A coefficient for the transition, \( R_{j} = |\mathbf{R}_{j}| \) and \( R_{L,\alpha} = \mathbf{R}_{j} \cdot \mathbf{e}_{\alpha} \). This expression, valid for \( \mathbf{R}_{j} \neq 0 \), provides the familiar spatial dependence of the electric field scattered from an oscillating dipole e.g. see refs. [1, 29, 30]. We have omitted \( \delta(\mathbf{R}) \) terms from the RHS of Eq.(5), which are required to obtain the correct \( \lim_{R_{j} \to 0} g(\mathbf{R}_{j}) \), and are retained where necessary in our derivation to describe self-field effects. These give rise to radiative damping, a radiative correction that changes \( \omega_{A}^{2} \) to a true resonance frequency \( \omega_{A} \), and a term that cancels the dipole self energy \( \delta\omega_{A} \) [25, 31]. The \( \lim_{R_{j} \to 0} g(\mathbf{R}_{j}) \) would also be needed to describe the transverse field interaction between atoms in contact, but we have excluded this possibility in our model, and thus Eq.(5) is appropriate for our purposes. Eq.(5) includes the familiar near field (\( R^{-3} \) and \( R^{-2} \)) and far field (\( R^{-1} \)) terms, but for convenience in this paper we shall refer to all of these parts of the field together as the scattered field. Where necessary to avoid ambiguity we will use the descriptor far-field to designate the scattered electric intensity that arises from the \( (R^{-1}) \) terms.

Expression (4) for \( \mathbf{E}(\mathbf{r}) \) is now substituted wherever \( \mathbf{E} \) appears in the equations for the atomic quantities, leading to the quantum Langevin equations. It is worth noting, as first recognised by Milonni and Knight [29], that in treating the resonant interaction between two atoms, it is essential that the full dipole interaction in the Hamiltonian is retained (i.e. including the non-energy conserving terms that are neglected in the usual rotating wave approximation (RWA)), in order that the correct atomic shifts and retardation times are obtained. Instead, a RWA is made on the final quantum Langevin equations, which also ensures we obtain the correct correspondence to the classical version of the problem.

**B. Equations of motion**

The observable of interest in this paper is the mean scattered intensity, which is proportional to \( \langle \mathbf{E}^{(+)\dagger}(\mathbf{r}) \mathbf{E}^{(-)}(\mathbf{r}) \rangle \) where \( \mathbf{E}^{(+)\dagger} \) and \( \mathbf{E}^{(-)} \) are the positive and negative frequency components of the electric field. We see from Eq.(4) that we therefore need solutions for mean atomic quantities such as \( \langle d_{l-}^{(j)} \rangle \), \( \langle d_{l+}^{(j)} d_{l+}^{(j')} \rangle \) and others, and the evolution equations required for these quantities are obtained by taking mean values of the operator equations presented in the Appendix. The mean in the Heisenberg picture is taken with an initial system state of the form \( |\alpha\rangle |\text{vac} \rangle \) where \( |\alpha\rangle \) is some choice of internal states for the two atoms and \( |\text{vac} \rangle \) is the vacuum state of the radiation field. Thus the noise terms in operator equations, where the field operators are normally ordered, disappear in the mean equations.

The most important features of our problem are present for the specific geometry where \( \mathbf{R} \) is parallel to the \( y \) axis, which we will call perpendicular configuration (see Fig. 1) and we will concentrate our study on this case. One other simple configuration will be considered briefly later in the paper. In each of the two configurations we consider, the incident and scattered light interacting with the atoms is polarised along \( \hat{z} \), and hence only the lower atomic state \( |00\rangle \) and the \( m = 0 \) upper state \( |10\rangle \) of each atom participate in the interactions. The atoms are each reduced to the familiar two-state case, the only dipole operators needed are \( d_{l+}^{0} \), and the number of equations required reduces from 255 to 15. It is appropriate for these effectively two-state atoms to use simpler atomic notation, so that we write \( d_{l+}^{0} \) for \( d_{l+}^{(j)} \) and \( n_{l}^{(j)} \) \( (n_{l}^{(j)}) \) for the operator for the upper (lower) state population. From Eqs.(A6) to (A16) we obtain the following equations for the first ten atomic mean quantities

\[
-i \frac{d}{dt} \langle d_{l}^{(j)} \rangle = \left( \Delta + \frac{\gamma}{2} \right) \langle d_{l}^{(j)} \rangle + \frac{\Omega e^{ik_{L} \cdot \mathbf{r}}}{2} \left( 1 - 2 \langle n_{l}^{(j)} \rangle \right) + \mathcal{G} \left( \langle d_{l}^{(j)} \rangle - 2 \langle d_{l}^{(j)} \rangle \langle n_{l}^{(j)} \rangle \right),
\] (6)
\[-i \frac{d}{dt} \langle n_u^{(i)} \rangle = i \gamma \langle n_u^{(i)} \rangle + \Omega \left( e^{i k \cdot r_i} \langle d_+^{(i)} \rangle - e^{-i k \cdot r_i} \langle d_-^{(i)} \rangle \right) + G(R) \langle d_+^{(i)} d_-^{(i)} \rangle - G^*(R) \langle d_-^{(i)} d_+^{(i)} \rangle, \]

\[-i \frac{d}{dt} \langle d_+^{(i)} d_-^{(i)} \rangle = -i \gamma \langle d_+^{(i)} d_-^{(i)} \rangle + G(R) \langle n_u^{(i)} d_-^{(i)} \rangle - G^*(R) \langle n_u^{(i)} d_+^{(i)} \rangle - 2 \left( G(R) - G^*(R) \right) \langle n_u^{(i)} \rangle \langle n_u^{(i)} \rangle
\]
\[+ \Omega \left( e^{i k \cdot r_i} \langle d_+^{(i)} \rangle - e^{-i k \cdot r_i} \langle d_-^{(i)} \rangle \right), \]

\[-i \frac{d}{dt} \langle n_u^{(i)} d_+^{(i)} \rangle = \left( \Delta + \frac{3 \gamma}{2} \right) \langle n_u^{(i)} d_+^{(i)} \rangle - G(R) \langle d_+^{(i)} n_u^{(i)} \rangle + \frac{\Omega}{2} \left( e^{i k \cdot r_i} \langle d_+^{(i)} \rangle + e^{i k \cdot r_2} \langle d_+^{(i)} \rangle - 2 e^{i k \cdot r_1} \langle n_u^{(i)} d_+^{(i)} \rangle - 2 e^{i k \cdot r_2} \langle d_+^{(i)} n_u^{(i)} \rangle \right), \]

\[-i \frac{d}{dt} \langle n_u^{(i)} n_u^{(i)} \rangle = 2i \gamma \langle n_u^{(i)} n_u^{(i)} \rangle + \frac{\Omega}{2} \left( e^{i k \cdot r_1} \langle d_+^{(i)} \rangle + e^{i k \cdot r_2} \langle d_+^{(i)} \rangle - e^{-i k \cdot r_1} \langle d_+^{(i)} \rangle - e^{-i k \cdot r_2} \langle d_+^{(i)} \rangle \right). \]

The remaining five equations are easily found by noting \(d_+^{(i)} = \left( d_-^{(i)} \right)^\dagger \) which gives \( \langle d_+^{(i)} \rangle = \langle d_-^{(i)} \rangle^* \), \( \langle n_u^{(i)} d_+^{(i)} \rangle = \langle n_u^{(i)} d_-^{(i)} \rangle^* \) and \( \langle d_+^{(i)} n_u^{(i)} \rangle = \langle d_-^{(i)} n_u^{(i)} \rangle^* \). In these equations \( i \neq j \), \( \Delta \equiv \omega_L - \omega_A \) is the laser detuning, \( \Omega = DE_L / h \) is the Rabi frequency, and \( G(R) \equiv g_{00}(R) \) (note that \( R_0 = R_z \)). Equations (6)-(11) are formally equivalent to those given by Lehmburg [1] in his seminal paper on collective light scattering, and can be mapped directly to those given by Rudolf et al. [21].

1. Analytic solution for perpendicular configuration

In the case of perpendicular configuration (R parallel to y-axis), the factors \( e^{i k \cdot r_1} \), \( e^{i k \cdot R} \rightarrow 1 \), simplifying the equations of motion in the previous section and allowing the following steady state solution to be obtained [32].

\[\langle d_+^{(1)} \rangle = \langle d_-^{(2)} \rangle = -\frac{\Omega}{A} \left( 2\Delta - i \gamma \right) \]
\[\left[ 2\Omega^2 + (2\Delta + i \gamma) (2\Delta - i \gamma + 2G^*(R)) \right], \]

\[\langle n_u^{(1)} \rangle = \langle n_u^{(2)} \rangle = \frac{\Omega^2}{A} \left( 4\Delta^2 + 2\Omega^2 + \gamma^2 \right), \]

\[\langle d_+^{(1)} \rangle = \langle d_-^{(2)} \rangle = \frac{\Omega^2}{A} \left( 4\Delta^2 + \gamma^2 \right), \]

\[\langle n_u^{(1)} \rangle = \langle n_u^{(2)} \rangle = -\frac{\Omega^2}{A} \left( 2\Delta - i \gamma \right), \]

\[\langle d_+^{(1)} \rangle \langle n_u^{(1)} \rangle = -\frac{\Omega^2}{A} \left( 2\Delta - i \gamma \right) \]
\[\left[ 2\Omega^2 + (2\Delta + i \gamma) (2\Delta - i \gamma + 2G^*(R)) \right], \]

\[\langle n_u^{(1)} \rangle = \langle n_u^{(2)} \rangle = \frac{\Omega^2}{A} \left( 4\Delta^2 + 2\Omega^2 + \gamma^2 \right), \]

\[\langle d_+^{(1)} \rangle \langle d_-^{(1)} \rangle = \frac{\Omega^2}{A} \left( 4\Delta^2 + \gamma^2 \right), \]

\[\langle n_u^{(1)} \rangle \langle n_u^{(2)} \rangle = \frac{\Omega^4}{A}, \]

where

\[A = \left( \gamma^2 + 4\Delta^2 \right) \left[ \left( 2G_\gamma(R) + \gamma \right)^2 + 4(G_\gamma(R) + \Delta)^2 + 4\Omega^2 \right] + 4\Omega^4, \]

and \( G_\gamma(R) \) and \( G_\gamma(R) \) are the real and imaginary parts of \( G(R) \).

C. Observables

In this paper, our primary interest is in the spatial distribution of the steady-state far-field scattered intensity, which in the far-field approximation \( r \gg \lambda_L, R \) is given by
\[ I(r) = \frac{P_0}{r^2} \left( 1 - \frac{r^2}{r^2} \right) \]

\[ \left( \langle n_u^{(1)} \rangle + \langle n_u^{(2)} \rangle + 2\Re \left( e^{-i\omega \hat{\mathbf{R}} \cdot \langle \mathbf{d}_{+}^{(1)} \rangle \langle \mathbf{d}_{-}^{(2)} \rangle \rangle \right) \right) , \]

with \( P_0 = 3\gamma \hbar \omega L / \pi \). The coherent part of the far-field intensity, which is proportional to \( \langle \mathbf{E}^{(1)}(r) \rangle \langle \mathbf{E}^{(1)}(r) \rangle \), is given by

\[ I_{\text{coh}}(r) = \frac{P_0}{r^2} \left( 1 - \frac{r^2}{r^2} \right) \]

\[ \left( \langle d_{+}^{(1)} \rangle \langle d_{-}^{(2)} \rangle + \langle d_{+}^{(2)} \rangle \langle d_{-}^{(2)} \rangle + 2\Re \left( e^{-i\omega \hat{\mathbf{R}} \cdot \langle \mathbf{d}_{+}^{(1)} \rangle \langle \mathbf{d}_{-}^{(2)} \rangle \rangle \right) \right) , \]

and the incoherent scattering by \( I_{\text{inc}}(r) = I(r) - I_{\text{coh}}(r) \). The far-field radiation \( I(r) \) forms a pattern of interference fringes characterised by the well known \( g^{(1)} \) correlation factor \([33]\), which here simplifies to

\[ g^{(1)}(\mathbf{r}_1; \mathbf{r}_2) = \frac{\langle d_{+}^{(2)} d_{-}^{(1)} \rangle}{\sqrt{\langle n_u^{(1)} \rangle \langle n_u^{(2)} \rangle}}. \]

The visibility of the fringes in the \( x - y \) plane is

\[ V = \frac{I_{\text{max}} - I_{\text{min}}}{I_{\text{max}} + I_{\text{min}}} = \frac{2|g^{(1)}(\mathbf{r}_1; \mathbf{r}_2)|}{\sqrt{\langle n_u^{(1)} \rangle \langle n_u^{(2)} \rangle + \langle n_u^{(2)} \rangle \langle n_u^{(2)} \rangle}} , \]

where the final equality in Eq.(22) applies when \( R \geq \lambda_L / 2 \).

### 1. Scattered Power

The total power scattered into the far field is obtained by integrating the far-field intensity over all angles, which gives

\[ P_{\text{scat}} = \int d\mathbf{\Omega} I(r) \]

\[ = \frac{2\hbar \omega}{A} \left( (\gamma + 2G_{\text{t}}(r)) (\gamma^2 + 4\Delta^2) \Omega^2 + 2\gamma \Omega^4 \right) . \]

The power absorbed from the laser by the atoms is given by

\[ P_{\text{abs}} = -\omega_L \hbar \Omega \Im \left( \langle \mathbf{d}_{+}^{(1)} \rangle + \langle \mathbf{d}_{+}^{(2)} \rangle \right) \]

and it is easy to show that \( P_{\text{scat}} = P_{\text{abs}} \), as expected.

### 2. Scattering from two uncoupled atoms

It will be useful when discussing the results for the scattering from two dipole-coupled atoms, to compare with the scattering from two non-dipole-coupled atoms, driven by the same laser field. Although the latter is an artificial model, it will allow us to identify the features of the scattering that are due to the coupling. Results for scattering from a single laser-driven atom were derived many years ago by Mollow \([34]\), and for the convenience of the reader we present his results in our current notation for the upper state population \( \langle n_u \rangle_M \), and the lowering component of the dipole \( \langle d_{-} \rangle_M \)

\[ \langle n_u \rangle_M = \frac{1}{3} \frac{\Omega^2}{\Delta^2 + \left( \frac{\omega}{2} \right)^2 + \frac{3}{2} \Omega^2} . \]

\[ \langle d_{-} \rangle_M = \frac{i^2 - \Delta}{\Delta^2 + \left( \frac{\omega}{2} \right)^2 + \frac{1}{2} \Omega^2} . \]

For this single atom, the coherent fraction of the scattered intensity is given by

\[ f_{\text{coh}}^M = \frac{I_{\text{coh}}}{I} = \frac{\langle \mathbf{E}^{(1)} \rangle \cdot \langle \mathbf{E}^{(1)} \rangle}{\langle \mathbf{E}^{(1)} \rangle \cdot \langle \mathbf{E}^{(1)} \rangle} = \frac{\langle d_{-} \rangle_M^2}{\langle n_u \rangle_M^2} \]

\[ = \frac{4\Delta^2 + \gamma^2}{4\Omega^2 + 4\Delta^2 + \gamma^2} , \]

and the incoherent fraction \( f_{\text{inc}}^M = 1 - f_{\text{coh}}^M \). It is easily shown from Eqs.(12) and (13) that when \( G(\hat{\mathbf{R}}) \to 0 \) (i.e. when the dipole coupling is put to zero), \( \langle n_u^{(1)} \rangle \to \langle n_u^{(2)} \rangle, \langle d_{+}^{(1)} \rangle \to \langle d_{+}^{(2)} \rangle \), and furthermore each of the correlations Eqs.(14)-(17) factor into Mollow results, e.g. \( \langle d_{-}^{(1)} \rangle \to \langle d_{-}^{(2)} \rangle \). Thus the spatial intensity distribution of light scattered from two uncoupled atoms (which we denote \( I_{\text{uc}}(r) \)) is obtained from Eq.(19) to be

\[ I_{\text{uc}}(r) = \frac{P_0}{r^2} \left( 1 - \frac{r^2}{r^2} \right) \]

\[ \left( \langle n_u^{(1)} \rangle + \langle n_u^{(2)} \rangle + 2\Re \left( e^{-i\omega \hat{\mathbf{R}} \cdot \langle \mathbf{d}_{+}^{(1)} \rangle \langle \mathbf{d}_{-}^{(2)} \rangle \rangle \right) \right) . \]

\( I_{\text{uc}}(r) \) has both coherent and incoherent components, and the visibility of the fringes in the perpendicular configuration is

\[ V_{\text{uc}} = \frac{2\langle d_{+}^{(1)} \rangle_M \langle d_{-}^{(2)} \rangle_M}{\langle n_u^{(1)} \rangle_M + \langle n_u^{(2)} \rangle_M} = f_{\text{coh}}^M . \]

In the forward direction \( I_{\text{uc}}^{\text{fwd}}(r) \) is given by

\[ I_{\text{uc}}^{\text{fwd}}(r) = \frac{P_0}{r^2} \left[ \frac{4\Omega^2 \left( \Omega^2 + 4\Delta^2 + \gamma^2 \right)}{\left( 2\Omega^2 + 4\Delta^2 + \gamma^2 \right)^2} \right] , \]

of which the incoherent fraction is

\[ I_{\text{inc}}^{\text{fwd,inc}} = \frac{\Omega^2}{\Omega^2 + 4\Delta^2 + \gamma^2} . \]

We note finally that the total power scattered by the two uncoupled atoms is \( 2\gamma \hbar \omega \langle n_u \rangle_M \).
III. RESULTS: PERPENDICULAR CONFIGURATION

The perpendicular configuration is the main focus of this paper, and from the analytic solutions Eqs.(13) and (14) we see in this case $\langle n_u^{(1)} \rangle = \langle n_u^{(2)} \rangle$, $\langle d_+^{(1)} d_-^{(2)} \rangle$ is real, and $g^{(1)}(r_1; r_2)$ has no $r_1$ or $r_2$ dependence. We will henceforth denote it by $g_{\perp}^{(1)}$, and it has the simple form

$$g_{\perp}^{(1)} = g^{(1)}(r_1; r_2) = \frac{4\Delta^2 + \gamma^2}{2\Omega^2 + 4\Delta^2 + \gamma^2}$$

which is identical to $f_{\text{coh}}^M$. The steady state value of $I(r)$ can now be expressed as

$$I(r) = \left( 1 - \frac{r^2}{r^2} \right) \left[ \frac{1 + g_+^{(1)} \cos (k_{\perp} r \cdot R)}{1 + g_+^{(1)}} \right] I_{\text{fwd}}(r)$$

where $I_{\text{fwd}}(r)$ is the far-field intensity at a distance $r$ in the forward direction (i.e. on the $x$-axis),

$$I_{\text{fwd}}(r) = \frac{P_0}{r^2} \left( \langle n_u^{(1)} \rangle + \langle n_\perp^{(2)} \rangle + 2\langle g_+^{(1)} d_-^{(2)} \rangle \right)$$

$$= \frac{4P_0}{Ar^2} \Omega^2 (4\Delta^2 + \Omega^2 + \gamma^2).$$

For convenience in what follows, we present results in terms of a dimensionless intensity defined by

$$\mathcal{I}(\hat{r}) = \frac{I(\hat{r})}{P_0 / r^2}.$$  

Examples of the far-field scattered intensity pattern $\mathcal{I}(\hat{r})$ in the $x$–$y$ plane are shown as polar plots in Fig. 2, where the development of fringes for $R \gtrsim 5\lambda_L$ is clearly evident, as well as the expected forward-backward symmetry. The incoherent component of the scattering will be discussed further below. From Eq.(31) we see that $\mathcal{I}(\hat{r})$ is fully characterised by the quantities $I_{\text{fwd}}$ and $g_{\perp}^{(1)}$, where the latter gives the fringe visibility. The behaviour of $I_{\text{fwd}}$ is displayed as a function of $R$, $\Omega$ and $\Delta$ in Fig. 3. In Fig. 3(a) where $I_{\text{fwd}}$ is plotted against $R$ and $\Omega$ for the case $\Delta = 0$, two key features are evident. The first is that for $R \gtrsim R_{\text{inf}} \equiv (k_L)^{-1}$ (i.e. when the near field terms dominate $G(R)$) the far-field scattering decreases below the uncoupled result, tending to zero for small $R$. We will call this phenomenon the suppression of scattering. We note though, that for sufficiently large $\Omega$ (outside the range of this plot) appreciable scattering occurs at small $R$. The second feature is that for $R > R_{\text{inf}}$, $I_{\text{fwd}}$ oscillates with $R$, but with decreasing amplitude as either $R$ or $\Omega$ increase. In Fig. 3(b), which is a cross section of Fig. 3(a) at the value $\Omega = 0.1\gamma$, we see that $I_{\text{fwd}}$ oscillates about the uncoupled result $T_{\text{fwd}}^\text{uc}$ (see Eq.(29)), with $I_{\text{fwd}} \to T_{\text{fwd}}^{\text{uc}}$ as $R \to \infty$. Other plots (not shown here) confirm that in the region satisfying both $\Omega \gtrsim \gamma$ and $R > R_{\text{inf}}$, $I_{\text{fwd}} \simeq T_{\text{fwd}}^{\text{uc}}$, which is discussed in more detail below. Fig. 3(c) shows $I_{\text{fwd}}$ plotted against $R$ and $\Omega$ but now for the case $\Delta = 3\gamma$. Once again we observe the suppression of scattering at small $R$, while throughout the region $R > R_{\text{inf}}$ there is little dependence on $R$ with $I_{\text{fwd}} \simeq T_{\text{fwd}}^{\text{uc}}$. The key feature of this graph is the sharp peak of intensity at $R = 0.1\lambda_L$, which is very prominently seen in Fig. 3(d) which is a cross section of Fig. 3(c) at the value $\Omega = 0.5\gamma$. We note, without presenting plots, that for $\Delta < 0$, while suppression of scattering still occurs for $R \lesssim R_{\text{inf}}$, there are no sharp peaks such as seen in Fig. 3(c), and for $R > R_{\text{inf}}$, $I_{\text{fwd}} \simeq T_{\text{fwd}}^{\text{uc}}$. In Fig. 3(e) we plot $I_{\text{fwd}}$ against $\Delta$ and $R$ for the case $\Omega = 3\gamma$, and see that there are two sharp intensity peaks, one which is at $\Delta = 0$ for all $R$, and the other which is at $\Delta \approx 2\gamma$ when $R = 0.05\lambda_L$ and moves steadily towards $\Delta = 0$ as $R$ in-
We define $\Delta = -\gamma$ and the term in square brackets produces a resonance at $\Delta = 0$, as noted earlier, $G(R) \approx -3\gamma/(4(k_L R)^3)$ in Eq.(35).

Eventually, at larger $\Omega$, this peak in $I_{\text{fwd}}$ will grow to match the uncoupled result $I_{\text{fwd}}^{\text{uc}}$. The behaviour of $I_{\text{fwd}}$ discussed above can be readily understood from its analytic expression (see Eq.(32))

$$I_{\text{fwd}} = \frac{4\Omega^2(4\Delta^2 + \Omega^2 + \gamma^2)}{(\gamma^2 + 4\Delta^2)[(2G_\gamma(R) + \gamma)^2 + 4(G_\gamma(R) + \Delta)^2 + 4\Omega^2] + 4\Omega^4}.$$  

First, for large $R$, as noted earlier, $G(R) \to 0$ and $I_{\text{fwd}} \to I_{\text{fwd}}^{\text{uc}}$. In fact, for any value of $\Omega$ or $\Delta$, the relative difference between $I_{\text{fwd}}$ and $I_{\text{fwd}}^{\text{uc}}$ is less than 10% when $R > 5\lambda_L$. Furthermore, if either $\Omega \gg \gamma$ or $\Delta \gg \gamma$, the relative difference is less than 10% for $R > \lambda_L$. The denominator of Eq.(34) holds the key to behaviour of $I_{\text{fwd}}$ with $\Delta$ at small $R$. The factor $(\gamma^2 + 4\Delta^2)$ gives rise to a resonance at $\Delta = 0$ of width $\sim \gamma$, and the term in square brackets produces a resonance at $\Delta = -G_\gamma(R)$ with a power broadened width $\sim \sqrt{\gamma^2 + \Omega^2}$.

We define $\Delta_d = -G_\gamma(R)$,

which is the dipole-dipole shift. In the limit of small $R$

$$G(R)\mid_{R \ll \lambda_L} \approx \gamma \left(-\frac{3}{4(k_L R)^3} + \frac{\gamma}{2}\right),$$  

where the real part of Eq.(36), for which $I_{\text{fwd}}$ is most sensitive, is very accurate for $R < R_{\text{ufd}}$. Using Eq.(36) in Eq.(35) we find the atomic separation $R_d$ which shifts the atoms into resonance with a laser detuning of $\Delta$ is

$$R_d = \frac{\lambda_L}{2\pi} \left(\frac{3\gamma}{4\Delta}\right)^{\frac{1}{3}}.$$  

### A. Suppression of Scattering

Suppression of scattering by closely spaced atoms is a well known phenomenon (e.g. [3]) in which

$$\eta \equiv \frac{I_{\text{fwd}}}{I_{\text{fwd}}^{\text{uc}}} \ll 1.$$  

We will define the regime of suppression of scattering to be where $\eta \lesssim 0.1$, and inserting Eq.(34) and the dimensionless form of Eq.(29) into Eq.(37) we find this criteria is satisfied when

$$|G_\gamma(R) + \Delta| \geq \sqrt{\frac{\gamma^2 + \Delta^2 + \gamma^2}{4\gamma^2 + \Omega^4}}.$$  

Since suppression of radiation occurs only at small $R$, we can use the approximation $G_\gamma(R) \approx -3\gamma/(4(k_L R)^3)$ in Eq.(38).

In the regime of suppressed radiation we find

$$I_{\text{fwd}}\mid_{\Delta = 0} \approx \frac{\Omega^2(\Omega^2 + \gamma^2)}{\gamma^2 \Delta_d^2 + \Omega^4},$$  

and

$$\eta\mid_{\Delta = 0} \approx \frac{(\Omega^2 + \gamma^2)^2}{4(\gamma^2 \Delta_d^2 + \Omega^4)}.$$  

It is also interesting to note that when $|\Delta_d| \gg \gamma$ and $\gamma \gg \Omega$

$$I_{\text{fwd}}\mid_{\Delta = 0} \approx \Omega^2/\Delta_d^2.$$  

while

$$I_{\text{fwd}}\mid_{\Delta = \Delta_d} \approx \Omega^2/\gamma^2 + \Omega^2.$$  

### B. Incoherent scattering

In Fig. 2 the incoherent component of the scattered intensity displays spatial interference fringes, and their origin can be readily determined from the expression for the incoherent scattered intensity
In Eq.(43), the spatial dependence in the horizontal plane arises from \( \chi_d \), the incoherent component of the dipole correlation, which has the relatively simple form

\[
\chi_d = \left( \langle d_x^{(1)} \rangle \langle d_x^{(1)} \rangle - \langle d_y^{(1)} \rangle \langle d_y^{(1)} \rangle \right) = \frac{\Omega_0^2}{\Omega^2} \left( \gamma^2 + 4 \Delta^2 \right) \left( G_{\perp}(\mathbf{R}) \gamma + 2 G_{\perp}(\mathbf{R}) \Delta \right).
\]

Eq.(45) shows clearly that it is the dipole coupling that causes \( \chi_d \) to be non zero. The visibility of the incoherent fringes is directly dependent on \( \chi_d \). The behaviour of the incoherent fraction of the forward scattering, \( f_{\text{fwd,inc}} \), is displayed in Fig. 4 for the same parameters as in Fig. 3, and we see that \( f_{\text{fwd,inc}} \) broadly follows \( I_{\text{fwd}} \), apart from the region \( R \rightarrow 0 \). Here at \( \Delta = 0 \) in the regime of suppressed scattering (see Eq.(38)), \( I_{\text{fwd}} \rightarrow 0 \) while \( f_{\text{fwd,inc}} \rightarrow f_{\text{fwd,inc}}^{\text{inc}} \), as shown in 4 (a), (b), and in 4 (e), (f).

As expected, we also find \( f_{\text{fwd,inc}} \rightarrow f_{\text{fwd,inc}}^{\text{inc}} \) in the regime of large \( R \).



IV. DECORRELATION APPROXIMATION

The analytic solution given in Section II provides a comprehensive description of the steady-state behaviour of the driven two atom system over a full range of \( \Omega, \Delta \) and \( R \), which we have explored in detail in Section III. It is a formidable challenge however to obtain a comparably detailed description for a larger atomic ensemble because the number of required equations scales as \( \sim 2^{2N} \), where \( N \) is the number of atoms. In this section we present an approximate solution method for a system of dipole coupled atoms, which has a more favorable scaling with atom number, and has the potential for solving the behaviour of larger systems. By comparing the results of the approximate solution of Eqs.(6) - (11) to our exact analytic solution, we are able to characterise in detail the validity and accuracy of the approximate solution.

The decorrelation approximation we present below is familiar in the field of quantum optics, and others, and can be viewed as a low order truncation of a cumulant expansion. A recent example has been given by Krämer and Ritsch [36], who have numerically analysed the effects of first and second order decorrelations on the temporal evolution of an array of vacuum coupled, undriven, two-state atoms. We show that for the case of two driven atoms, the approximation is accurate over a wide range of parameters, including the strongly driven regime. We obtain an analytic description of the validity regime, and also use our methodology to introduce the concept of an effective driving field, which we shall see provides additional physical insight into the behaviour of the two atom system.

A. The Decorrelated Equations

We begin by defining an effective driving field for atom \( i \) which is the sum of two fields arriving at atom \( i \), namely the laser field and the field scattered from the other atom \( j \). For convenience we will express this in terms of an effective Rabi frequency

\[
\Omega_{\text{eff}}^{(i)} = e^{ik_i \cdot \mathbf{r}_j} \Omega + 2 G(\mathbf{R}) d_{-j}^{(j)}; \quad (j \neq i)
\]

and we note that we have dropped the quantum noise term (e.g. see Eq.(A12)) since it disappears in all expectation values that we take. Eqs.(6) and (7) then take the form

\[
I_{\text{inc}}(R) = 2 \left( 1 - \frac{R^2}{r^2} \right) \left[ \langle n_x^{(1)} \rangle - \langle d_x^{(1)} \rangle \langle d_x^{(1)} \rangle + \Re \left( e^{-ik \cdot R} \chi_d \right) \right],
\]

where
\[ -i \frac{d}{dt} \langle d^{(i)}_- \rangle = \left( \Delta + \frac{\gamma}{2} \right) \langle d^{(i)}_- \rangle + \frac{1}{2} \langle \Omega^{(i)}_{\text{Eff}} \rangle \left( 1 - 2n^{(i)}_u \right), \quad (48) \]

and

\[ -i \frac{d}{dt} \langle n^{(i)}_u \rangle = i\gamma \langle n^{(i)}_u \rangle + \frac{1}{2} \left( \langle \Omega^{(i)}_{\text{Eff}} \rangle \langle d^{(i)}_- \rangle - \langle \Omega^{(i)}_{\text{Eff}} \rangle \langle d^{(i)}_- \rangle^\dagger \right). \quad (49) \]

We now make the decorrelation approximation

\[ \langle d^{(i)}_\pm n^{(j)}_u \rangle \rightarrow \langle d^{(i)}_\pm \rangle \langle n^{(j)}_u \rangle \quad (50) \]

\[ \langle d^{(i)}_\pm d^{(j)}_\mp \rangle \rightarrow \langle d^{(i)}_\pm \rangle \langle d^{(j)}_\mp \rangle \quad (51) \]

(for \( i \neq j \)) which decouples Eqs.(6) (and its conjugate) and (7) from Eqs.(8)-(11) leaving us with a set of three approximate equations for the system:

\[ -i \frac{d}{dt} \langle d^{(i)}_- \rangle_D = \left( \Delta + \frac{\gamma}{2} \right) \langle d^{(i)}_- \rangle_D - \frac{\langle \Omega^{(i)}_{\text{Eff}} \rangle_D}{2} \left( 1 - 2n^{(i)}_u \right)_D, \quad (52) \]

\[ -i \frac{d}{dt} \langle n^{(i)}_u \rangle_D = i\gamma \langle n^{(i)}_u \rangle_D + \frac{\langle \Omega^{(i)}_{\text{Eff}} \rangle_D}{2} \langle d^{(i)}_- \rangle_D, \quad (53) \]

and the conjugate of Eq.(52). In these equations the subscript \( D \) indicates that these are the decorrelated expectation values. We note that while the number of system equations has been reduced (scaling as \( \sim N \) for \( N \) atoms), they are now nonlinear due to the effective field’s dependence on \( \langle d^{(i)}_- \rangle_D \). Eqs.(52) and (53) have the same form as the original Mollow equations[34] for a single driven atom, but with the substitution \( \Omega \rightarrow \langle \Omega^{(i)}_{\text{Eff}} \rangle_D \). The solutions to Eqs.(52) and (53) can be written formally using the expressions in Eqs.(25) and (26), although this is of formal rather than practical value, as \( \langle \Omega^{(i)}_{\text{Eff}} \rangle_D \) must of course be found as part of the solution, which here is carried out numerically.

In Fig. 5 we compare the forward scattered intensity obtained from the decorrelated equations, \( I_{\text{fwd}, D} \), with the true forward scattered intensity, for the case of \( \Delta = 0 \). We also include a comparison to the commonly employed linear approximation (e.g. [3, 6]), which is obtained by setting \( n^{(i)}_u = 0 \) in Eqs.(6)-(11).

We see from Figs. 5(a) and (b) that at \( \Delta = 0 \), the decorrelation approximation provides a good solution for all \( R \) for the values of \( \Omega \) shown. We also see that the linear approximation is accurate at \( \Omega = 0.1\gamma \) but fails badly by \( \Omega = 0.5\gamma \). Figs. 5(c) and (d), where \( I_{\text{fwd}} \) is plotted against \( \Omega \) in the challenging regime of small \( R \), show that the decorrelation solution provides an accurate representation in both the low and high intensity regimes, but is less accurate in the transition region around \( \Omega = \gamma \). Once again, as expected, the linear approximation is shown to be poor for \( \Omega \gtrsim 0.5\gamma \). We will obtain a quantitative expression for the validity regime in the next section.

### B. Validity regime

The relative error in the forward scattered intensity due to the decorrelation approximation is given by

\[ E_I \equiv \left| \frac{I_{\text{fwd}} - I_{\text{fwd}, D}}{I_{\text{fwd}}} \right|. \quad (54) \]

The quantity \( E_I \) is plotted for a wide range of \( R \) and \( \Omega \) in the top row of Fig. 6 for representative detunings (a) \( \Delta = 0 \) and (b) \( \Delta = 10 \). In this plot, we see that the region of significant error (e.g. \( E_I > 0.01 \)) is essentially confined to a triangular area in \( \log R \) and \( \log \Omega \), with the particulars of the area dependent on the value of \( \Delta \). For \( \Delta > 0 \), an additional thin “tail” emerges in the low \( \Omega \) region at the value \( R = R_d \), as seen in 6(b). In this tail region the atoms have been pulled into resonance with the driving field, and the effective field is large. The relative error of the mean dipole due to the decorrelation approximation is given by

\[ E_d \equiv \left| \frac{\langle d^{(i)}_- \rangle - \langle d^{(i)}_- \rangle_D}{\langle d^{(i)}_- \rangle} \right|. \quad (55) \]
and this quantity is plotted in the second row of Fig. 6. We see that $E_d$ is significant only within the same region as $E_I$. Corresponding plots (not shown here) for the relative error of $\langle n_u^{(1)} \rangle_D$ or $\langle \Omega^{(1)} \rangle_D$ are very similar to the plot of $E_I$. Using this fact we can find a useful analytic expression for the validity range of the decorrelation approximation. Noting first that

$$I_{\text{fwd}, D} (\Omega) = I_{\text{fwd}}^{\gamma_e} \left( \langle \Omega^{(1)} \rangle_D \right)$$

(56)

(see discussion following Eq.(53)), we make the substitution $\langle \Omega^{(1)} \rangle_D \rightarrow \langle \Omega^{(1)} \rangle$ in the RHS of Eq.(56), then evaluate $\langle \Omega^{(1)} \rangle$ using Eq.(12), to give an analytic expression for $E_I$ that is valid in the region where $E_I$ is small. Further simplification can be obtained by replacing $G(R)$ by its approximate form Eq.(36), which is valid wherever $E_I$ is significant. We find the lower boundary of the validity range by noting that in this region, when $R < R_{nf}$, then $(kR)^{-3} \gg (\Omega/\gamma)^2$, $(\Delta/\gamma)^2$ so we can take $\lim_{R \rightarrow 0} E_I$ to obtain

$$E_I \approx \frac{\Omega^2}{(\gamma^2 + 4\Delta^2)} \Theta$$

(57)

where

$$\Theta = \frac{|3\gamma^2 + 4(\Omega^2 - 5\Delta^2)|}{\gamma^2 + 4\Delta^2 + \Omega^2}.$$  

(58)

Eq.(57) describes horizontal lines in the $(\log R, \log \Omega)$ plane, of defined value of relative error. The quantity $\Theta$ is bounded by the value 5, but for most of parameter space it is much less, and we find that setting $\Theta = 1$ gives good agreement with our numerical results. Two example contours $(\Omega^2 / (\gamma^2 + 4\Delta^2) = 0.01; \Omega^2 / (\gamma^2 + 4\Delta^2) = 0.1)$ are plotted on the subfigures in Fig. 6 and are seen to accurately capture the small $\Omega$ boundary of the validity range of the decorrelation approximation out to $R \approx R_{nf}$, apart from the resonance tail. At the upper boundary, in the region $R < R_{nf}$, we have $|G(R)|, |\Omega \gg \gamma, |\Delta|$. Assuming in addition that $\Omega^2(kL)^3 \gg (1 + 4\Delta^2/\gamma^2)$ we find

$$E_I \approx \frac{9}{16\Omega^2} \frac{\gamma^2(\gamma^2 + 4\Delta^2)}{(kL)^6}.$$  

(59)

Eq.(59) describes diagonal lines in the $(\log R, \log \Omega)$ plane, of defined value of relative error. Two example contours from this equation are shown in Fig. 6 and are seen to give excellent agreement with the large $\Omega$ boundary of the decorrelation approximation out to $R \approx R_{nf}$. While Eqs.(57) and (59) are only strictly valid in the range $R < R_{nf}$, we have extended the lines to where they meet at $R \approx \lambda_L$, outlining a triangle which defines a practical validity regime for the decorrelation approximation, apart from the resonance tail, which occurs at $R = R_0$.

C. Role of the effective field

The mean value of the effective field is the coherent part of the total field driving each atom, and in the region where the decorrelation approximation is valid, $I_{\text{fwd}}$ can be obtained by replacing the laser field in $I_{\text{fwd}}$ with the effective field. In Fig.7 the magnitude of the effective field is plotted against $R$ for the same parameters as Fig.3(b), and two key features emerge: (i) the magnitude of the effective field goes to 0 for small $R$; (ii) the effective field oscillates with $R$ for $R \gtrsim \lambda_L / 2$. Comparison with Fig.3(b) illustrates that the scattered field $I_{\text{fwd}}$ broadly follows the magnitude of the effective field.
The oscillation in the effective field with $R$ is due to interference between its two constituents, the laser field and the scattered field. Eq.(47) shows that at atom 1 the phase of the scattered field relative to the laser field arises from $\langle d^{(2)} \rangle$ and $G(\mathbf{R})$. In the regime $R \gtrsim \lambda_L/2$ the phase of $G(\mathbf{R})$ is primarily determined by the factor $e^{ikLR}$ (see Eq.(5)). The phase of $\langle d^{(2)} \rangle$ consists of the phase of the laser field at atom 2, plus an additional shift, which on resonance and in the regime $R \gtrsim \lambda_L/2$ is near $\pi/2$ (see Eq.(12)). Thus in the perpendicular configuration the net phase difference between the two constituent fields is close to $k_L R + \pi/2$, leading to a modulation of the effective field seen in Fig. 7, which has a period of approximately $\lambda_L$. The effective field concept also allows us to understand the suppression of scattering as $R \to 0$. Here, the near complete destructive interference between the laser field and the scattered field from the other atom reduces the effective driving field to near zero. An analytic expression for the behaviour of the effective field at small $R$ can be obtained by evaluating Eq.(47) in the suppressed scattering regime to give

$$
\langle \Omega_{\text{Eff}}^{(1,2)} \rangle_{R \to 0} \approx \frac{2\Omega(2\Omega + (\gamma - 2i\Delta)^2)}{3\gamma^2(\gamma - 2i\Delta)} (k_L R)^3.
$$

(60)

V. RESULTS: PARALLEL CONFIGURATION

Finally we consider the case where $\mathbf{R} \parallel \mathbf{k}_L$, with $\mathbf{r}_1 = -\frac{R}{2}\hat{x}$ and $\mathbf{r}_2 = \frac{R}{2}\hat{x}$ which we call the parallel configuration (see Fig. 1). As before we assume $\mathbf{E}_L \parallel \hat{z}$ so that only two states of each atom participate, but now the atoms are no longer symmetric with respect to the laser, and for example $\langle d^{(1)} \rangle \neq \langle d^{(2)} \rangle$. A numerically calculated example of the scattered far-field intensity pattern $I(r)$ in the $x-y$ plane is shown as a polar plot in Fig. 8, for the same parameters as in Fig. 2(c). The salient difference between these two Figures is that in the parallel configuration the scattering has developed a forward asymmetry due to phase matching. (For $R = n\lambda_L/2$, $n = 1, 2, \ldots$, the scattering retains forward-backward symmetry). The forward scattered intensity $I_{\text{fwd}}$ for the case in Fig. 3(b) is plotted against $R$ in Fig. 9(a), along with the corresponding result from the perpendicular configuration. Both configurations show subradiant scattering as $R \to 0$, but the oscillations in $I_{\text{fwd}}$ for the parallel configuration are approximately half the magnitude and twice the frequency of the perpendicular configuration. This difference is explained by the asymmetry between the effective fields of the two atoms in the parallel configuration, which is evident in Fig. 9(b). The scattered field incident on atom 1 from atom 2 has phase $k_L R$ relative to dipole 2, which itself has a phase $k_L R + \pi/2$ relative to the laser field at atom 1. The net phase difference of $2k_L R + \pi/2$ leads to a modulation period of $\lambda_L/2$ for the effective field at atom 1. However at atom 2, the scattered field and the laser field have travelled the same additional path length from atom 1, so the phase difference is mainly due to the $\pi/2$ advance from the resonantly driven atom 1. This means that for $R \gtrsim \lambda_L/2$ the effective field driving atom 2 is only weakly modulated as $R$ changes, and the oscillations in $I_{\text{fwd}}$ shown in Fig. 9(a) are half the amplitude of those for the perpendicular configuration, because only one atom contributes to them.
VI. CONCLUSION

We have used a rigorous formalism to describe the collective scattering behaviour of two stationary atoms driven by a monochromatic laser, and interacting via the vacuum field. With a suitable choice of system geometry and laser polarisation, the atoms reduce to effectively two-state. A further restriction to the perpendicular configuration enabled an analytic solution to be found for the steady-state mean atomic values and second order correlations for this system, and hence for the spatial behaviour of the steady-state far-field scattered intensity. These analytic solutions are valid over a full range of the parameters $\Omega$, $\Delta$ and $R$, and have facilitated a unified and comprehensive survey of the steady-state scattering behaviour. Key features have been identified and quantified, including the two resonance peaks for the forward scattered intensity at $\Delta = 0$ and $\Delta = \Delta_d$. The incoherent component of the scattering was shown to exhibit spatial fringes in a defined regime, underscoring the fact that at small atomic separation, spontaneous photons are emitted from the joint two-atom system, rather than independently from each atom. By comparing the expression for the collective scattered intensity with the corresponding intensity from two uncoupled atoms, a precise specification has been given of the regime where the dipole-dipole coupling has significant effect. In particular a simple analytic expression for the regime of suppressed scattering and the magnitude of that suppression has been derived.

It is unlikely that a useful analytic solution can be obtained for the case of more than two atoms, because of the unfavorable scaling of the number of system equations with the number of atoms. Therefore, with the aim of finding a solution method with potential application to a large number of atoms, we have explored an approximation scheme which has much more favorable scaling. We have shown, with a detailed analysis on the two-atom system, that our decorrelation approximation provides an accurate solution for a wide range of the parameters $\Omega$, $\Delta$ and $R$, and we have given an analytic description of the validity regime. Finally, the concept of the effective driving field has been shown to provide a direct physical interpretation of key aspects of the system behaviour, which is an alternative to the usual interpretation involving Dicke eigenstates (see for example [21]).

Appendix A: Full Set Of Equations

In this Appendix we present the equations of motion for two atoms with the full $J_u$ internal structure, and arbitrary orientation and laser polarisation. A convenient definition for the atomic operators of the $i^{th}$ atom can be made in terms of the irreducible tensor operators [27]

$$T_q^{k} (ab)^{(i)} = \sum_{m_a} \sum_{m_b} (-1)^{j_a - m_a} \sqrt{2k + 1} \left( \begin{array}{ccc} j_a & k & j_b \\ -m_a & q & m_b \end{array} \right) |j_a m_a \rangle^{(i)} \langle j_b m_b| . \quad (A1)$$

For ease of notion we redefine these operators, including transforming to slowly varying operators,

$$n^i_l \equiv T_0^{ab} (ll)^{(i)} = |j_l0 \rangle^{(i)(i)} \langle j_l0| , \quad (A2)$$

$$d^{ql,l}_{-} \equiv -e^{i\omega L t} T_q^{(l)} (lu)^{(i)} = (-1)^q e^{i\omega L t} |j_l0 \rangle^{(i)(i)} \langle j_u - q| , \quad (A3)$$

$$d^{ql,l}_{+} \equiv e^{-i\omega L t} T_q^{(l)} (ul)^{(i)} = e^{-i\omega L t} |j_u q \rangle^{(i)(i)} \langle j_l0| , \quad (A4)$$

$$n^i_{\alpha\beta} = (-1)^{1+\alpha} \sum_{k} \sqrt{2k + 1} \left( \begin{array}{cc} 1 & 1 \\ \alpha & -\beta \end{array} \right) T_q^{k} (uu)^{(i)} \langle j_u \alpha \rangle^{(i)(i)} \langle j_u \beta| , \quad (A5)$$

noting these give the relation $(d^q_{+})^\dagger = (-1)^q d^{-q}_{-}$. 

![Graph](image-url)
1. First Order Equations

The first order equations of motion are

\[ -i \frac{d}{dt} q^{q,i} = \left( \Delta + \frac{i \gamma}{2} \right) q^{q,i} + \frac{(-1)^q}{2} \sum_\alpha \left( i_{L}^i \right)_\alpha + \frac{2}{1} \sum_\rho (-1)^{1} \beta_{\alpha} (r_{ij}) d_{-}^{-\beta,j} \left( \delta_{-\alpha} n^i_l - n^i_{\alpha(-q)} \right) + F(d_{-}^{q,i}), \quad \text{(A6)} \]

and

\[ -i \frac{d}{dt} n_{q_i q_2} = i \gamma n_{q_i q_2} - \frac{(-1)^q}{2} \left( i_{L}^i q_i \right) + \frac{2}{1} \sum_\alpha \left( i_{L}^i q_i \right) + \frac{2}{1} \sum_\rho (-1)^{1} \beta_{\alpha} (r_{ij}) d_{-}^{-\beta,j} \left( \delta_{-\alpha} n^i_l - n^i_{\alpha(-q)} \right) + F(d_{-}^{q,i}), \quad \text{A7(1)} \]

where \( j \neq i \) and the laser field \( i_{L}^i q_i = \frac{d}{dt} \text{D}_{\text{ext}} (r_i) = \Omega \varepsilon \text{L} e^{ik \cdot r_i} \), has polarization \( \varepsilon \text{L} \). The noise terms, \( F = F^a + F^c \), are given by

\[ F^c (d_{+}^{q,i}) = (-1)^{y} \frac{D}{\hbar} \left[ e^{i \omega \mathcal{L} t} \sum_\alpha (-1)^{1} \left( f^{(c)} (r_i) \right)_\alpha \left( \delta_{\alpha} n^i_l - n^i_{\alpha(-q)} \right) \right], \quad \text{A8(1)} \]

\[ F^a (d_{+}^{q,i}) = (-1)^{y} \frac{D}{\hbar} \left[ e^{i \omega \mathcal{L} t} \sum_\alpha (-1)^{1} \left( \delta_{\alpha} n^i_l - n^i_{\alpha(-q)} \right) \left( f^{(a)} (r_i) \right)_\alpha \right], \quad \text{A9(1)} \]

\[ F^c (n^i_{q_i q_2}) = (-1)^{y} \frac{D}{\hbar} \left[ e^{i \omega \mathcal{L} t} \left( f^{(c)} (r_i) \right)_{q_2} d_{+}^{q,i} - e^{-i \omega \mathcal{L} t} \left( f^{(c)} (r_i) \right)_{q_1} d_{-}^{-q,i} \right], \quad \text{A10(1)} \]

and

\[ F^a (n^i_{q_i q_2}) = (-1)^{y} \frac{D}{\hbar} \left[ e^{i \omega \mathcal{L} t} \left( f^{(a)} (r_i) \right)_{q_2} d_{+}^{q,i} - e^{-i \omega \mathcal{L} t} \left( f^{(a)} (r_i) \right)_{q_1} d_{-}^{-q,i} \right]. \quad \text{A11(1)} \]

where the operators \( f^{(c)} (r, t) \) and \( f^{(a)} (r, t) \) are the creation and annihilation parts of the noise operator, \( f (r, t) = f^{(a)} (r, t) + f^{(c)} (r, t) \) which appears in Eq.(4). They are given by

\[ f^{(a)} (r, t) = \left( f^{(c)} (r, t) \right)^{\dagger} \quad \text{(A12)} \]

with \( t_0 \) the initial time and \( E_k = \sqrt{\hbar k / 2} \varepsilon_0 (2 \pi)^2 \). The equations of motion are in normally ordered form with respect to the field operators, with all \( f^{(a)} (r, t) \) terms on the right and \( f^{(c)} (r, t) \) on the left, so that when we take the expectation value \( \langle \text{vac} | N^a | \text{vac} \rangle = \langle \text{vac} | N^c | \text{vac} \rangle = 0 \) and these terms disappear.

2. Second Order Equations

The normally ordered second order equations are obtained using the product rule to give
Given that atomic operators from two different atoms commute at equal time the noise terms for the second order equation’s for operators $A^i$ and $B^j$ can be written in terms of the first order noise terms

\[ F^c(A^i B^j) = F^c(A^i) B^j + F^c(B^j) A^i, \]  
\[ F^m(A^i B^j) = A^i F^m(B^j) + B^j F^m(A^i). \]  

Taking the expectation value, and choosing orientation $k_L = k_L \hat{x}$, $\Omega_L = \Omega \hat{z}$ and $R \propto \hat{z}$ or in the $x-y$ plane, these equations reduce to two-state case. Setting $d_{-}^l = d_{+}^{l \dagger}$, $n_{i}^l = n_{i 0}^l$, $n_{l}^i = 1 - n_{i}^l$, and $G(r) = g_{00}(r)$ gives Eq. (6-11).
motional effects in a dense coherent medium,” Nature Communications , 11039 (2016), 1601.05322.
[6] B. Zhu, J. Cooper, J. Ye, and A. M. Rey, “Light scattering from dense cold atomic media,” Phys. Rev. A 94, 023612 (2016).
[7] R. J. Bettles, S. A. Gardiner, and C. S. Adams, “Cooperative eigenmodes and scattering in one-dimensional atomic arrays,” Phys. Rev. A 94, 043844 (2016).
[8] S. D. Jenkins, J. Ruostekoski, J. Javanainen, S. Jennewein, R. Bourgain, J. Pellegrino, Y. R. P. Sortais, and A. Browaeys, “Collective resonance fluorescence in small and dense atom clouds: Comparison between theory and experiment,” Phys. Rev. A 94, 023842 (2016).
[9] R. T. Sutherland and F. Robicheaux, “Collective dipole-dipole interactions in an atomic array,” Phys. Rev. A 94, 013847 (2016).
[10] L. Corman, J. L. Ville, R. Saint-Jalm, M. Aidelsburger, T. Bienaimé, S. Nascimbène, J. Dalibard, and J. Beugnon, “Transmission of near-resonant light through a dense slab of cold atoms,” Phys. Rev. A 96, 053629 (2017).
[11] L. Pucci, A. Roy, T. S. do Espirito Santo, R. Kaiser, M. Kastner, and R. Bachelard, “Quantum effects in the cooperative scattering of light by atomic clouds,” Phys. Rev. A 95, 053625 (2017).
[12] S. Jennewein, L. Brossard, Y. R. P. Sortais, A. Browaeys, P. Cheinet, J. Robert, and P. Pillet, “Coherent scattering of near-resonant light by a dense, microscopic cloud of cold two-level atoms: Experiment versus theory,” Phys. Rev. A 97, 053816 (2018).
[13] F. Cottier, R. Kaiser, and R. Bachelard, “Role of disorder in super- and subradiance of cold atomic clouds,” Phys. Rev. A 98, 013622 (2018).
[14] R. H. Dicke, “Coherence in spontaneous radiation processes,” Phys. Rev. 93, 99 (1954).
[15] R. H. Lehmberg, “Radiation from an n-atom system. ii. spontaneous emission from a pair of atoms,” Phys. Rev. A 2, 889 (1970).
[16] M. Gross and S. Haroche, “Superradiance: An essay on the theory of collective spontaneous emission,” Physics Reports 93, 301 (1982).
[17] D. Pavolini, A. Crubellier, P. Pillet, L. Cabaret, and S. Liberjan, “Experimental evidence for subradiance,” Phys. Rev. Lett. 54, 1917 (1985).
[18] R. G. DeVoe and R. G. Brewer, “Observation of superradiant and subradiant spontaneous emission of two trapped ions,” Phys. Rev. Lett. 76, 2049 (1996).
[19] W. Guerin, M. O. Araújo, and R. Kaiser, “Subradiance in a large cloud of cold atoms,” Phys. Rev. Lett. 116, 083601 (2016).
[20] M. Kuś and K. Wódkiewicz, “Two-atom resonance fluorescence,” Phys. Rev. A 23, 853 (1981).
[21] T. G. Rudolph, Z. Ficek, and B. J. Dalton, “Two-atom resonance fluorescence in running- and standing-wave laser fields,” Phys. Rev. A 52, 636 (1995).
[22] T. Wong, S. M. Tan, M. J. Collett, and D. F. Walls, “Interference of resonance fluorescence from two four-level atoms,” Phys. Rev. A 55, 1288 (1997).
[23] C. Cohen-Tannoudji, J. Dupont-Roc, and G. Gryenberg, Photons & Atoms: Introduction to Quantum Electrodynamics (Wiley, 1997).
[24] There is also a “contact” collisional term arising from the polarisation of the atoms, see the final term in Eq.(1).
[25] O. Morice, Y. Castin, and J. Dalibard, “Refractive index of a dilute Bose gas,” Phys. Rev. A 51, 3896 (1995).
[26] C. Cohen-Tannoudji, J. Dupont-Roc, and G. Gryenberg, Atom-Photon Interactions, Basic Processes and Applications (Wiley-VCH, 1998) complement $A_{v}$.
[27] A. R. Edmonds, Angular momentum in Quantum Mechanics (Princeton University Press, 1974).
[28] M. Ducloy, “Nonlinear effects in optical pumping of atoms by a high-intensity multimode gas laser. general theory,” Phys. Rev. A 8, 1844 (1973).
[29] P. W. Milonni and P. L. Knight, “Retardation in the resonant interaction of two identical atoms,” Phys. Rev. A 10, 1096 (1974).
[30] J. Jackson, Classical Electrodynamics, 2nd ed (Wiley, New York, 1975).
[31] J. Dalibard, J. Dupont-Roc, and C. Cohen-Tannoudji, “Vacuum fluctuations and radiation reaction: identification of their respective contributions,” Journal de Physique 43, 1617 (1982).
[32] Note these analytic solutions also hold when $R \parallel \hat{z}$.
[33] C. W. Gardiner and P. Zoller, Quantum Noise (Springer-Verlag, 1991).
[34] B. R. Mollow, “Power spectrum of light scattered by two-level systems,” Phys. Rev. 188, 1969 (1969).
[35] We note that $|G_{\gamma}(R) + |\gamma|$ is never smaller than $0.6\gamma$.
[36] Sebastian Krämer and Helmut Ritsch, “Generalized mean-field approach to simulate the dynamics of large open spin ensembles with long range interactions,” EPJ D 69, 282 (2015).