The Extended Generating Function of the Radical of $n$
and the abc-Conjecture

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Abstract. We introduce the function defined by the sum of the generating series $\sum_{n \geq 1} \frac{R(n)^t}{n^s}$, where $R(n)$ is the radical of $n$ and $s$ and $t$ real positive variables. The difference with the ordinary generating series $\sum_{n \geq 1} \frac{R(n)}{n^s}$ [1] is that now $R(n)$ appears elevated to the positive power $t$. Since $R(n)$ is multiplicative, logarithmic differentiation with respect to $s$ and $t$ of the series and of its equal Euler product gives an identity involving $s$ and $t$ and two positive functions $S(s, t)$ and $T(s, t)$, expressed as series running over all primes. Appropriately interpreted this identity leads to a proof of Bombieri’s abc-conjecture/question $a + b < R(abc)^2$ for all $n$ satisfying $n < R(n)^{\frac{S(s, t)}{T(s, t)}}$.

1 The series $\sum_{n \geq 1} \frac{R(n)^t}{n^s}$

We consider the series $\sum_{n \geq 1} \frac{R(n)^t}{n^s}$, where $R(n)$ is the radical of $n$ and $s$ and $t$ real positive variables. Since $R(n) \leq n$ and is only equal to $n$ for the squarefree numbers, it follows that

$$\sum_{n \geq 1} \frac{R(n)^t}{n^s} < \sum_{n \geq 1} \frac{1}{n^{s-t}}.$$ 

As the series on the right side converges for $t > 0$ and $s > 1 + t$ (region of convergence, denoted for brevity RC) so does the series on the left side. The sum therefore is a well defined function of $s$ and $t$ within RC.

2 The Euler product of $\sum_{n \geq 1} \frac{R(n)^t}{n^s}$

Theorem 1. If $s$ and $t$ are within RC, then

$$\sum_{n \geq 1} \frac{R(n)^t}{n^s} = \prod_p \left( \frac{p^s - 1 + p^t}{p^s - 1} \right)$$

where $p$ runs over all primes.

Proof. Since $R(n)$ is multiplicative, so is $R(n)^t$ because of $R(1)^t = 1$ and $R(nm)^t = R(n)^t R(m)^t$
for coprime integers $n$ and $m$. By applying Euler’s generalized identity [1], [2], we have successively

$$
\sum_{n \geq 1} \frac{R(n)^t}{n^s} = \prod_p \left( 1 + R(p)^t \frac{1}{p^s} + R(p^2)^t \frac{1}{p^{2s}} + \cdots \right)
$$

$$
= \prod_p \left( 1 + \frac{p^t}{p^s} + \frac{p^t}{p^{2s}} + \cdots \right)
$$

$$
= \prod_p \left( 1 + \frac{p^t}{p^s} \left( 1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots \right) \right)
$$

$$
= \prod_p \left( 1 + \frac{p^t}{p^s} \left( \frac{1}{1 - \frac{1}{p^s}} \right) \right)
$$

$$
= \prod_p \left( 1 + \frac{p^t}{p^s - 1} \right)
$$

$$
= \prod_p \left( \frac{p^s - 1 + p^t}{p^s - 1} \right),
$$

(1)

where $p$ runs over all primes. Q.E.D.

3 Differentiation of

$$
\ln \sum_{n \geq 1} \frac{R(n)^t}{n^s} = \ln \prod_p \left( \frac{p^s - 1 + p^t}{p^s - 1} \right)
$$

Taking the logarithms of both sides of (1) we get

$$
\ln \sum_{n \geq 1} \frac{R(n)^t}{n^s} = \sum_p \ln \left( \frac{p^s - 1 + p^t}{p^s - 1} \right).
$$

Partial differentiation with respect to $s$ of both sides gives

$$
- \sum_{n \geq 1} \frac{R(n)^t \ln n}{\sum_{n \geq 1} \frac{R(n)^t}{n^s}} = \sum_p \frac{p^s \ln p(p^s - 1) - (p^s - 1 + p^t)p^s \ln p}{(p^s - 1)(p^s - 1 + p^t)}
$$

$$
= \sum_p \frac{(p^s - 1 - p^s + 1 - p^t)p^s \ln p}{(p^s - 1)(p^s - 1 + p^t)}
$$

and hence

$$
\sum_{n \geq 1} \frac{R(n)^t \ln n}{\sum_{n \geq 1} \frac{R(n)^t}{n^s}} = \sum_p \frac{p^s}{p^s - 1} \frac{p^t}{p^s - 1 + p^t} \ln p := S(s, t).
$$

(2)
On the other hand, partial differentiation with respect to $t$ of both sides gives

$$
\sum_{n \geq 1} \frac{R(n)^t}{n^s} \ln R(n) = \sum_{n \geq 1} \frac{p^n \ln p(p^n - 1)}{(p^s + p^n - 1)(p^s - 1)}
$$

$$
= \sum_p \frac{p^n}{p^s - 1 + p^n} \ln p := \mathcal{T}(s, t).
$$

(3)

4 Legitimacy of differentiations

Above differentiations are legitimate because the derived series $\mathcal{S}(s, t)$ and $\mathcal{T}(s, t)$ are convergent in RC. We first show this for $\mathcal{T}(s, t)$ by using the inequality $\ln x < 1 - x$ for $x > 0$. Since the $n$-th prime $p_n$ is greater than $n$ [1], we have

$$
\frac{p^n \ln p_n}{p^s_n - 1 + p^n} = \frac{\ln p_n}{p^s_n - t - p^{-t} + 1}
$$

$$
< \frac{\ln p_n}{p^s_n - t}
$$

$$
< \frac{p_n - 1}{p^s_n - t} \quad \text{(use of } \ln x < 1 - x, x > 0) \]

$$
< \frac{1}{p^s_n - t - 1}
$$

$$
< \frac{1}{n^{s-t} - 1} \quad \text{(use of } p_n > n). \]

Summing over all primes $p_n$ we therefore get

$$
\mathcal{T}(s, t) = \sum_p \frac{p^n \ln p}{p^s_n - 1 + p^n} = \sum_{n \geq 1} \frac{p^n \ln p_n}{p^s_n - 1 + p^n} < \sum_{n \geq 1} \frac{1}{n^{s-t} - 1},
$$

which clearly is convergent in RC.

The convergence of $\mathcal{S}(s, t)$ in CR follows from that of $\mathcal{T}(s, t)$ by considering that $\frac{p^n}{p^s - 1} < 2$ for all primes $p$ and $s > 0$. We have namely

$$
\mathcal{S}(s, t) = \sum_p \frac{p^n}{p^s - 1} \frac{p^n}{p^s - 1 + p^n} \ln p < 2 \mathcal{T}(s, t),
$$

and as $\frac{p^n}{p^s - 1} > 1$, we also have $\mathcal{T}(s, t) < \mathcal{S}(s, t)$. Combining, we obtain

$$
\mathcal{T}(s, t) < \mathcal{S}(s, t) < 2 \mathcal{T}(s, t)
$$

or

$$
1 < \frac{\mathcal{S}(s, t)}{\mathcal{T}(s, t)} < 2, \quad \{s, t\} \in \text{CR}.
$$

(4)
Moreover, this demonstrates that

\[
1 < \frac{L}{\{s, t\} \in \mathcal{R}C} \frac{S(s, t)}{T(s, t)} < \frac{T}{\{s, t\} \in \mathcal{R}C} \frac{S(s, t)}{T(s, t)} < 2.
\]

We shall not use this inequality here but it looks it is important for more detailed investigations. The more so, as the technique we used so far applies as is to any positive multiplicative function \(M(n)\) (subject to convergence questions). Indeed, in such a case we again have

\[
\mathcal{T}_{M(n)}(s, t) < S_{M(n)}(s, t) < 2 \mathcal{T}_{M(n)}(s, t),
\]

where

\[
S_{M(n)}(s, t) = \sum_p \frac{p^s}{p^s - 1 + M(p)^t} \ln p
\]

and

\[
T_{M(n)}(s, t) = \sum_p \frac{M(p)^t}{p^s - 1 + M(p)^t} \ln M(p)
\]

are the functions corresponding to the functions \(S(s, t)\) and \(T(s, t)\) if \(M(n) = R(n)\).

5 The identity \(\sum_{n \geq 1} \frac{R(n)^t}{n^s} \ln \frac{R(n)^{S(s, t)}}{n^{T(s, t)}} = 0\)

From (2) and (3) of section 3 and since \(\sum_{n \geq 1} \frac{R(n)^t}{n^s}\) is not zero we get (writing henceforth \(S\) for \(S(s, t)\) and \(T\) for \(T(s, t)\))

\[
S \sum_{n \geq 1} \frac{R(n)^t}{n^s} \ln R(n) = T \sum_{n \geq 1} \frac{R(n)^t}{n^s} \ln n
\]

or equivalently

\[
\sum_{n \geq 1} \frac{R(n)^t}{n^s} \ln \frac{R(n)^{S}}{n^{T}} = 0. \quad (5)
\]

6 Interpretations of \(\sum_{n \geq 1} \frac{R(n)^t}{n^s} \ln \frac{R(n)^{S}}{n^{T}} = 0\)

We now take as point of reference the ratio \(S/T\) and split (5) as follows

\[
\sum_{n < R(n)^{S/T}} \frac{R(n)^t}{n^s} \ln \frac{R(n)^{S}}{n^{T}} + \sum_{n = R(n)^{S/T}} \frac{R(n)^t}{n^s} \ln \frac{R(n)^{S}}{n^{T}} + \sum_{n > R(n)^{S/T}} \frac{R(n)^t}{n^s} \ln \frac{R(n)^{S}}{n^{T}} = 0.
\]

The second summand is zero because

\[
\sum_{n = R(n)^{S/T}} \frac{R(n)^t}{n^s} \ln \frac{N(t)}{n^{T}} = \sum_{n = R(n)^{S/T}} \frac{R(n)^{T/S} t}{n^s} \ln \frac{n^T}{n^{T}} = \sum_{n = R(n)^{S/T}} \frac{1}{n^{S - (T/S) t}} \ln 1 = 0.
\]
As a result we therefore obtain

\[ \sum_{n < R(n)^S/T} \frac{R(n)^t}{n^s} \ln \frac{R(n)^S}{n^T} + \sum_{n > R(n)^S/T} \frac{R(n)^t}{n^s} \ln \frac{R(n)^S}{n^T} = 0 \]

or

\[ \sum_{n < R(n)^S/T} \frac{R(n)^t}{n^s} \ln \frac{R(n)^S}{n^T} = \sum_{n > R(n)^S/T} \frac{R(n)^t}{n^s} \ln \frac{n^T}{R(n)^S}. \] (6)

This identity in \( s \) and \( t \), which is a different interpretation of (5), is fundamental for the sequel.

6 The connection with the abc-conjecture

The deeper meaning of the identity (6) of the previous section is that both series

\[ \sum_{n < R(n)^S/T} \frac{R(n)^t}{n^s} \ln \frac{R(n)^S}{n^T} \]

and

\[ \sum_{n > R(n)^S/T} \frac{R(n)^t}{n^s} \ln \frac{n^T}{R(n)^S} \]

are not empty, as otherwise this would contradict (1) of section 4, unless \( n^T = R(n)^S \) identically for all \( s \) and \( t \) within \( CR \). But this, however, is impossible as shown by the case of squarefree numbers for which \( R(n) = n \) would give \( n^T(s, t) = n^{S(s, t)} \), clearly an absurdity as \( T(s, t) < S(s, t) < 2T(s, t) \). From these facts we deduce

**Theorem 2.** For all coprime integers \( a, b \) and \( c = a + b \) satisfying \( c < R(c)^S/T \) we have

\[ a + b < R(abc)^2 \]

i.e. Bombieri’s \( abc \)-conjecture/question [3].

**Proof.** Since \( c \) can be written in \( \frac{\phi(c)}{2} \) different ways as a sum of two coprime integers \( a \) and \( b \) [4], [5] and as by assumption \( c < R(c)^S/T \) it results that \( c \) occurs in the sum

\[ \sum_{n < R(n)^S/T} \frac{R(n)^t}{n^s} \ln \frac{R(n)^S}{n^T}. \]

Consequently, we have

\[ a + b = c < R(c)^S/T < R(c)^S/T R(ab)^S/T < R(abc)^S/T, \]

which, because of (4) of section 4, gives

\[ a + b < R(abc)^2. \]

Q.E.D.
Note. Elevating a positive multiplicative function $M(n)$ to the power $t > 0$ in $\sum_{n\geq 1} \frac{M(n)^t}{n^s}$ and forming the functions $S_M(s, t)$ and $T_M(s, t)$ is an effective technique but raises also problems and questions depending on the cases examined. For example:

Regarding our worked out example $M(n) = R(n)$

1. Is there a deeper meaning that the exponent 2 in Bombieri’s $abc$-conjecture/question coincides with the upper bound of $\frac{S(s, t)}{T(s, t)}$ which is also 2 ?

2. Is it true that identically in $s$ and $t$

$$\mathcal{L}_{\{s, t\} \in \mathbb{RC}} \frac{S(s, t)}{T(s, t)} = \mathcal{T}_{\{s, t\} \in \mathbb{RC}} \frac{S(s, t)}{T(s, t)} < 2,$$

or is this only true for some specific constant values of $s$ and $t$ ? If so, would this be a proof is the $abc$-conjecture ?

3. What is the surface $\frac{S(s, t)}{T(s, t)}$ like in CR ? Is analytic continuation into the complex plane of $s$ feasible ? Would this give a functional equation as is the case for Riemann’s $Zeta$ function ?

Regarding general $M(n)$’s

4. In what cases are the sums $S_{M(n)}(s, t)$ and $T_{M(n)}(s, t)$ amenable in the sense that we get closed formulas ?

5. The set of multiplicative functions has a rich structure. Maybe a detailed investigation, in the same way we did for $R(n)$, of the principal $M(n)$’s researched in the Analytic Theory of Numbers, may reveal unsuspected truths !
References

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[2] Chandrasekharan K., Introduction to Analytic Number Theory, Springer Verlag Berlin Heildeberg New York, 1968, pp. 76-77.

[3] Bombieri Enrico, Forty Years of Effective Results in Diophantine Theory, in Wüstholtz, Gisbert, A Panorama of Number Theory, Cambridge University Press, 2002, p.206.

[4] Petridi Constantin M., A strong “abc-conjecture” for certain partitions $a + b$ of $c$, arXiv:math/0511224v3 [math.NT] 1 Mar 2006.

[5] Petridi Constantin M., The number of equations $c = a + b$ satisfying the $abc$-conjecture, arXiv:0904.1935v1 [math.NT] 13 Apr 2009.