Numerical Methods for Solving Convection-Diffusion Problems

A. Churbanov, P. Vabishchevich
Nuclear Safety Institute, Russian Academy of Sciences,
52, B. Tulskaya, 115191 Moscow, Russia

Abstract
Convection-diffusion equations provide the basis for describing heat and mass transfer phenomena as well as processes of continuum mechanics. To handle flows in porous media, the fundamental issue is to model correctly the convective transport of individual phases. Moreover, for compressible media, the pressure equation itself is just a time-dependent convection-diffusion equation.

For different problems, a convection-diffusion equation may be written in various forms. The most popular formulation of convective transport employs the divergent (conservative) form. In some cases, the nondivergent (characteristic) form seems to be preferable. The so-called skew-symmetric form of convective transport operators that is the half-sum of the operators in the divergent and non-divergent forms is of great interest in some applications.

Here we discuss the basic classes of discretization in space: finite difference schemes on rectangular grids, approximations on general polyhedra (the finite volume method), and finite element procedures. The key properties of discrete operators are studied for convective and diffusive transport. We emphasize the problems of constructing approximations for convection and diffusion operators that satisfy the maximum principle at the discrete level — they are called monotone approximations.

Two- and three-level schemes are investigated for transient problems. Unconditionally stable explicit-implicit schemes are developed for convection-diffusion problems. Stability conditions are obtained both in finite-dimensional Hilbert spaces and in Banach spaces depending on the form in which the convection-
diffusion equation is written.

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1. Convection-diffusion problems

Convection-diffusion problems are governed by typical mathematical models, which are common in fluid and gas dynamics. Heat and mass transfer is conducted not only via diffusion, but appears due to motion of a medium, too. Here we present typical examples of model convection-diffusion problems, which use various forms for the terms describing convective transport.

1.1. Basic problems of continuum mechanics

Principal features of physical and chemical processes in fluid mechanics \cite{14, 4} result from motion of a medium due to various forces. Heat and mass transfer phenomena in a moving medium may be treated as the simplest examples of these peculiarities.

Let $v(x, t)$ be the velocity of a liquid at a point $x$ and at a time moment $t$, whereas $\rho$ is its density. The thermal state of the liquid is governed by the equation of heat conduction

$$c_p\rho \left( \frac{\partial T}{\partial t} + (\mathbf{v} \cdot \text{grad})T \right) = \text{div}(k \text{grad} T) + q,$$

(1)

where $T$ stands for the temperature, $c_p$ is the specific heat capacity at a constant pressure, $k$ denotes the thermal conductivity of the liquid and $q$ describes the intensity of volumetric heat sources.

The temperature at a given spatial point is governed not only by conduction (diffusion) of heat, but also by motion (convection) of fluid volumes.

The second typical example is the diffusion equation for a multicomponent mixture \cite{5, 6}. We assume that a liquid is heterogeneous, more exactly, it is a mixture of two components. In this case, the mixture composition may be described by the concentration $c$ associated with only one component. The corresponding equation for the concentration (neglecting the diffusion flux caused by the temperature gradient) has the form

$$\frac{\partial(\rho c)}{\partial t} + \text{grad}(\mathbf{v} \rho c) = \text{div}(\rho k \text{grad} c),$$

(2)
where \( k \) denotes the diffusivity and \( \rho \) is treated as the total density of the liquid.

The equation (2) may be rewritten as

\[
\frac{\partial m}{\partial t} + \text{div}(vm) + \text{div} \left( m \frac{k}{\rho} \text{grad} \rho \right) = \text{div}(k \text{grad} m),
\]

(3)

where \( m = \rho c \) is the mass of one of the components in a volume unit. The equation (3) may be reduced to the form

\[
\frac{\partial m}{\partial t} + \text{div}(\tilde{v}m) = \text{div}(k \text{grad} m),
\]

(4)

where the expression

\[
\tilde{v} = v + \frac{k}{\rho} \text{grad} \rho
\]

describes the effective convective transport.

Using the continuity equation

\[
\frac{\partial \rho}{\partial t} + \text{div}(v\rho) = 0,
\]

(5)

we obtain

\[
\frac{\partial (\rho c)}{\partial t} + \text{div}(\rho c(v) = \rho \left( \frac{\partial c}{\partial t} + (v \cdot \text{grad}) c \right).
\]

Therefore, under the natural assumptions on the positiveness of \( \rho \), from (2), we arrive at the equation

\[
\frac{\partial c}{\partial t} + (v \cdot \text{grad}) c - k \frac{\text{grad} \rho \cdot \text{grad} c}{\rho} = \text{div}(k \text{grad} c).
\]

(6)

Similarly to (4), rewrite (6) as

\[
\frac{\partial c}{\partial t} + (\tilde{v} \cdot \text{grad}) c = \text{div}(k \text{grad} c),
\]

(7)

where now

\[
\tilde{v} = v - \frac{k}{\rho} \text{grad} \rho.
\]

Thus, we come to the equation for the concentration, where convective transport has the nondivergent form, as it takes place in the heat equation (1). In equation (4) as well as in the continuity equation (5), convective transport is written in the divergent form.
More complete models of heat and mass transfer include also an equation that describes the motion of the medium itself and determines, in particular, the velocity \( v \). For simplicity, we restrict ourselves to the Navier-Stokes equation for an incompressible (\( \rho = \text{const} \)) homogeneous medium. In this case, the momentum equation seems like this:

\[
\rho \left( \frac{\partial v}{\partial t} + (v \cdot \text{grad}) v \right) = -\text{grad} p + \eta \text{div grad} v,
\]

(8)

whereas the continuity equation (5) is reduced to

\[
\text{div} v = 0.
\]

(9)

Here \( p \) denotes the pressure and \( \eta = \text{const} \) stands for the dynamic viscosity of the fluid.

The equations (8) may be treated as the equations of convective and diffusive (due to the viscosity) transport of each individual component of the velocity \( v \). In this situation, in order to evaluate the pressure \( p \), it is necessary to involve equation (9).

If we eliminate the pressure from equation (8), then, for the vorticity \( \omega = \text{rot} v \), we obtain the equation

\[
\rho \left( \frac{\partial \omega}{\partial t} + (v \cdot \text{grad}) \omega - (\omega \cdot \text{grad}) v \right) = \eta \text{div grad} \omega.
\]

It is easy to see that the dynamics of the vorticity for an incompressible fluid is determined by a specific convective and diffusive transport.

More sophisticated models that include convective and diffusive transport as the most important element are used in modeling compressible flows. It should be noted that convective-diffusive transport is essential for predictions of various gas and fluid flows. In particular, environmental problems are of great importance: pollutants spreading in the atmosphere and water basins, contaminants transport in groundwaters and so on.

1.2. Various forms of the hydrodynamics equations

In theoretical studying applied problems, the conservative form of the hydrodynamics equations is in common use. In this case, the equations have the divergent form and express directly the corresponding laws of conservation (for the mass, momentum and energy). On the other hand, we should pay attention to
the nondivergent (characteristic) form of the hydrodynamics equations, which
is connected with the representation that is derived via differentiating the
convective transport terms. The paper [38] presents a new form of the hydrodynamics
equations that is characterized by writing the convective terms in the skew-symmetric
form. New quantities — the so-called SD–variables (Square root from Density)
that are based on using not the density itself but the square root from the density
— are used as unknown variables. Physical and mathematical arguments in favor
of introducing this form of the hydrodynamics equations are discussed below.

The system of hydrodynamics equations includes, first of all, the scalar equa-
tion of continuity and the vector equation of momentum. In more common cases,
there may be several motion equations as well as continuity equations — we speak
of models for multicomponent media. Furthermore, the system of equations may
be supplemented with an energy equation. Usually, the following scalar equation
of convection-diffusion serves as the basic equation in continuum mechanics and
heat and mass transfer (see, e.g., [20, 43]), i.e.,

$$\frac{\partial (\rho \varphi)}{\partial t} + \text{div}(\rho \mathbf{v} \varphi) = \text{div}(D \text{grad } \varphi), \quad (10)$$

where \(\varphi\) is a desired scalar function and \(D\) denotes the diffusivity. This equa-
tion is written in the conservative (divergent) form. Concerning to equation (10),
a number of problems are discussed in the literature, such as the construction
of discretization in space and in time, the investigation of convergence of the ap-
proximate solution to the exact one for the corresponding boundary value problem
[32, 30].

The main peculiarities of the system of fluid dynamic equations become evi-
dent in studying the system of two scalar equations that includes not only equation (10),
but also the continuity equation

$$\frac{\partial \rho}{\partial t} + \text{div}(\rho \mathbf{v}) = 0. \quad (11)$$

Just this system of equations (10), (11) is said to be the basic system of scalar
hydrodynamic equations.

In investigation of transport phenomena in continuum mechanics, the primary
features of transport of scalar variables are represented in equation (10). Concern-
ing vector fields, the coordinatewise representation may be unsuitable. Thus, it
seems reasonable to supplement the system of equations (10), (11) with the vector
convection-diffusion equation

$$\frac{\partial (\rho \mathbf{u})}{\partial t} + \text{div}(\rho \mathbf{v} \otimes \mathbf{u}) = \text{div}(D \text{grad } \mathbf{u}), \quad (12)$$
where \( \mathbf{u} \) is the desired vector function. This system of equations (10)–(12) is called the basic system of hydrodynamics equations.

Taking into account

\[
\nabla (\varrho \mathbf{v} \varphi) = \varphi \nabla (\varrho \mathbf{v}) + \varrho \mathbf{v} \cdot \nabla \varphi
\]

and the continuity equation (11), we get

\[
\varrho \frac{\partial \varphi}{\partial t} + \varrho \mathbf{v} \cdot \nabla \varphi = \nabla (D \nabla \varphi). \tag{13}
\]

Similarly, we can rewrite equation (12) as

\[
\varrho \frac{\partial \mathbf{u}}{\partial t} + \varrho \mathbf{v} \cdot \nabla \mathbf{u} = \nabla (D \nabla \mathbf{u}). \tag{14}
\]

The equations (13), (14) are written in the nonconservative (nondivergent) form. It should be noted that the continuity equation (11) cannot be written in the nonconservative form. Therefore, the basic system of hydrodynamics equations may be written in the conservative (10)–(12) or in the partially nonconservative form (11), (13), (14). Only for an incompressible medium, where equation (11) takes the form

\[
\nabla \mathbf{v} = 0,
\]

it is possible to speak about the nonconservative form of the equations.

Let us write the operator of convective transport in the skew-symmetric form \([32, 30]\) as

\[
\mathcal{C} \theta = \frac{1}{2} \nabla (\mathbf{v} \theta) + \frac{1}{2} \mathbf{v} \cdot \nabla \theta, \tag{15}
\]

i.e., as the half-sum of the operators of convective transport in divergent (conservative) and nondivergent (nonconservative) forms.

In the basic system of fluid dynamics equations (10)–(12), instead of \( \varrho, \varphi, \mathbf{u} \), we introduce new unknown variables:

\[
s = (\varrho)^{1/2}, \quad \zeta = (\varrho)^{1/2} \varphi, \quad \mathbf{w} = (\varrho)^{1/2} \mathbf{u}. \tag{16}
\]

The main peculiarity of these unknowns consists in using the square root from the density \( s = (\varrho)^{1/2} \) instead the density \( \varrho \) itself. That is why we speak of SD-variables (Square root from Density).

For the new unknowns, the system of equations (10)–(12) may be rewritten in the following way:

\[
\frac{\partial s}{\partial t} + \frac{1}{2} \nabla (s \mathbf{v}) + \frac{1}{2} \mathbf{v} \cdot \nabla s = 0, \tag{17}
\]
\[
\frac{\partial \z}{\partial t} + \frac{1}{2} \text{div}(v\z) + \frac{1}{2} v \cdot \text{grad} \z = \frac{1}{s} \text{div} \left( D \text{grad} \left( \frac{\z}{s} \right) \right), \quad (18)
\]

\[
\frac{\partial w}{\partial t} + \frac{1}{2} \text{div}(v \otimes w) + \frac{1}{2} v \cdot \text{grad} w = \frac{1}{s} \text{div} \left( D \text{grad} \left( \frac{w}{s} \right) \right). \quad (19)
\]

In this case, all three equations involve the convective terms that are written in the skew-symmetric form.

As a typical example of using the new variables, we study the Navier-Stokes equations for a viscous compressible medium, which express the conservation laws for the mass, momentum, and energy. The continuity equation has the form \((\ref{mass})\). Usually, the momentum equation is written in the conservative form

\[
\frac{\partial (\rho v)}{\partial t} + \text{div}(\rho v \otimes v) = \text{div} N - \text{grad} p. \quad (20)
\]

Here

\[
N = -\frac{2}{3} \mu \text{div} v \mathbf{E} + 2\mu S,
\]

As for \(S\), the coordinatewise representation seems like this:

\[
S_{ij} = \frac{1}{2} \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right).
\]

Now introduce the energy equation

\[
\frac{\partial (\rho e)}{\partial t} + \text{div}(\rho ve) = \text{div}(k \text{grad} T) + p \text{div} v + N : \text{grad} v. \quad (21)
\]

The term \(N : \text{grad} v\) describes the heat dissipation due to the fluid viscosity and \(N : \text{grad} v\) is the scalar product of tensors:

\[
N : \text{grad} v = N_{xx} \frac{\partial v_x}{\partial x} + N_{xy} \frac{\partial v_x}{\partial y} + N_{xz} \frac{\partial v_x}{\partial z} + N_{yx} \frac{\partial v_y}{\partial x} + N_{yy} \frac{\partial v_y}{\partial y} + N_{yz} \frac{\partial v_y}{\partial z} + N_{zx} \frac{\partial v_z}{\partial x} + N_{zy} \frac{\partial v_z}{\partial y} + N_{zz} \frac{\partial v_z}{\partial z}.
\]

Let us introduce the following new unknown variables:

\[
s = (\rho)^{1/2}, \quad w = (\rho)^{1/2} v, \quad \z = (\rho)^{1/2} e. \quad (22)
\]
For the variables (22), the system of the Navier-Stokes equations (11), (20), (21) has the following form:

\[
\frac{\partial s}{\partial t} + \frac{1}{2} \left( \text{div} \left( \frac{w_s}{s} \right) + \frac{w_s}{s} \cdot \text{grad} \ s \right) = 0,
\]

\[
\frac{\partial w}{\partial t} + \frac{1}{2} \left( \text{div} \left( \frac{w_s \otimes w}{s} \right) + \frac{w_s}{s} \cdot \text{grad} \ w \right) = \frac{1}{s} \text{div} N - \frac{1}{s} \text{grad} p,
\]

\[
\frac{\partial \zeta}{\partial t} + \frac{1}{2} \left( \text{div} \left( \frac{w_s \zeta}{s} \right) + \frac{w_s}{s} \cdot \text{grad} \ \zeta \right) = \frac{1}{s} \text{div}(k \text{grad} T) + \frac{p}{s} \text{div} \left( \frac{w}{s} \right) + \frac{1}{s} N : \text{grad} \left( \frac{w}{s} \right).
\]

The system of equations (23)–(25) needs to be supplemented with some equation of state. It should be highlighted that using the variables (22), the convective terms are written in the skew-symmetric form.

1.3. The pressure problem for multiphase flows in porous media

The system of governing equations for multiphase flows includes \([21, 3]\) the continuity equation for each phase, where \(\alpha = 1, 2, \ldots, m\) is the phase index. The mass conservation law for each individual phase is expressed by the following equation:

\[
\frac{\partial (\phi b_\alpha S_\alpha)}{\partial t} + \text{div}(b_\alpha u_\alpha) = -b_\alpha q_\alpha, \quad \alpha = 1, 2, \ldots, m.
\]

Here \(\phi\) stands for the porosity, \(b_\alpha\) is the phase density, \(S_\alpha\) denotes the phase saturation, \(u_\alpha\) is the velocity, and \(q_\alpha\) describe the volumetric mass sources.

For simplicity, we neglect the capillary and gravitational forces. In this simplest case, the equation of fluid motion in porous media has the form of Darcy’s law, where the velocity is directly determined by the common pressure:

\[
u_\alpha = -\frac{k_\alpha}{\mu_\alpha} k \cdot \text{grad} \ p, \quad \alpha = 1, 2, \ldots, m.
\]

In (27), \(k\) is the absolute permeability (in general, a symmetric second-rank tensor), \(k_\alpha\) denote the relative permeabilities, \(\mu_\alpha\) stands for the phase viscosity, and \(p\) is the pressure.

The unknown variables in the system of equations (26), (27) are the phase saturations \(S_\alpha, \alpha = 1, 2, \ldots, m\) and the pressure \((m + 1\) unknowns in all). In the simplest case, the coefficients in equations (26), (27) are defined as some relations

\[
\phi = \phi(p), \quad b_\alpha = b_\alpha(p), \quad q_\alpha = q_\alpha(S_\alpha), \quad k_\alpha = k_\alpha(S_\alpha), \quad \mu_\alpha = \text{const}.
\]
The summation of saturations over all phases yields

\[ \sum_{\alpha=1}^{m} S_{\alpha} = 1. \]  \hspace{1cm} (28)

Substituting (27) in (26) and taking into account (28), we obtain a system of \( m + 1 \) equations for \( m + 1 \) unknowns.

The system of equations (26)–(28) provides the basis for the description of multiphase flows in porous media. We have no separate equation for the pressure in this system. The equations (26) may be treated as the transport equation for each individual phase, whereas the algebraic relation (28) may be considered as the equation for the pressure.

Let us consider more convenient forms for the system (26)–(28), which lead to the typical problems of mathematical physics for the pressure. It should be noted that such equivalent formulations do exist only at the differential level. At the discrete level, such equivalence of formulations is not valid even for linear problems. Thus, a proper choice of the original form of the governing equations is essential for calculations.

The most natural way to derive the equation for the pressure is the following. Dividing each equation (26) by \( \phi b_{\alpha} > 0 \) and adding them together, we get

\[
\left( \sum_{\alpha=1}^{m} \frac{S_{\alpha}}{\phi b_{\alpha}} \frac{d(\phi b_{\alpha})}{dp} \right) \frac{\partial p}{\partial t} = \sum_{\alpha=1}^{m} \frac{1}{\phi b_{\alpha}} \text{div} \left( \frac{b_{\alpha} k_{\alpha}}{\mu_{\alpha}} k \cdot \text{grad} p \right) - \frac{1}{\phi} \sum_{\alpha=1}^{m} q_{\alpha}. \hspace{1cm} (29)
\]

Under the natural assumption for compressible fluids that

\[ \frac{d(\phi b_{\alpha})}{dp} > 0, \hspace{0.5cm} \alpha = 1, 2, \ldots, m, \]

equation (29) for the pressure is the standard parabolic equation of second order. In particular, the maximum principle holds for its solutions \[10\].

In accordance with (29), we solve in \( \Omega \) the boundary value problem for the equation

\[
\frac{\partial u}{\partial t} + \sum_{\alpha=1}^{m} a_{\alpha}(x) L_{\alpha} u = f(x, t), \hspace{1cm} (30)
\]

where \( a_{\alpha}(x) \geq \varrho_{\alpha}, \hspace{0.5cm} \varrho_{\alpha} > 0, \hspace{0.5cm} \alpha = 1, 2, \ldots, m, \) and the elliptic operators \( L_{\alpha} \) are defined by

\[
L_{\alpha} u = -\frac{\partial}{\partial x_{1}} \left( k_{\alpha}(x) \frac{\partial u}{\partial x_{1}} \right) - \frac{\partial}{\partial x_{2}} \left( k_{\alpha}(x) \frac{\partial u}{\partial x_{2}} \right), \hspace{0.5cm} \alpha = 1, 2, \ldots, m. \hspace{1cm} (31)
\]
under the standard assumptions $0 < \kappa_\alpha \leq k_\alpha \leq \kappa_\alpha$.

In some cases (incompressible media), it is reasonable to consider the steady-state problem. The boundary value problem is formulated for the equation

$$\mathcal{L} u = f(x), \quad \mathcal{L} = \sum_{\alpha=1}^{m} a_\alpha(x) \mathcal{L}_\alpha,$$

which is supplemented by the boundary conditions.

From (31) and (32), we have the representation

$$\mathcal{L} = \sum_{\alpha=1}^{m} \mathcal{L}_\alpha, \quad \mathcal{L}_\alpha = D_\alpha + C_\alpha, \quad \alpha = 1, 2, ..., m,$$

where

$$D_\alpha u = - \text{div}(d_\alpha(x) \text{grad} u),$$

$$C_\alpha u = w_\alpha \text{grad} u.$$  \hfill (34)

(35)

The effective diffusivity and convection velocity for the individual phase $\alpha$ are

$$d_\alpha = k_\alpha a_\alpha, \quad w_\alpha = k_\alpha \text{grad} a_\alpha.$$

Then the pressure operator takes the form of the convection-diffusion operator with convective term in the nondivergent form.

2. Time-dependent problems of convection-diffusion

Convection-diffusion equations provide important examples of second-order parabolic equations. In particular, they are considered as the basic equations for modeling continuum mechanics phenomena. Some aspects of numerical solving time-dependent problems of convection-diffusion are discussed here. In these equations, convective terms are formulated in the divergent, nondivergent, and skew-symmetric forms. Some essential results are presented for a model initial-boundary value problem with Dirichlet boundary conditions for the differential equation of convection-diffusion. These results will serve us as a check point in developing difference schemes. Discrete operators of diffusion and convection are constructed and analyzed with respect to their primary properties.
2.1. Differential problems

Time-dependent problems of convection-diffusion are treated as evolutionary operator equations in the corresponding spaces. To investigate them, we start with a study on properties of differential operators describing convective and diffusive transport. As the basic problem, we consider a time-dependent problem of convection-diffusion with Dirichlet boundary conditions in a rectangle. The convective terms are written in various forms. We distinguish a class of model time-dependent problems of convection-diffusion with a constant coefficient of diffusive transport (it is independent of time but depends on spatial coordinates). As for coefficients of convective transport, in applied problems, they are variable both in space and in time.

In a rectangle
\[\Omega = \{\mathbf{x} \mid \mathbf{x} = (x_1, x_2), \quad 0 < x_\alpha < l_\alpha, \quad \alpha = 1, 2\},\]
we study the time-dependent convection-diffusion equation with the convective terms written in the nondivergent form:
\[
\frac{\partial u}{\partial t} + \sum_{\alpha=1}^{2} v_\alpha(x, t) \frac{\partial u}{\partial x_\alpha} - \sum_{\alpha=1}^{2} \frac{\partial}{\partial x_\alpha} \left( k(x) \frac{\partial u}{\partial x_\alpha} \right) = f(x, t), \quad x \in \Omega, \quad 0 < t \leq T, \tag{36}
\]
considered under the standard assumptions
\[\kappa_1 \leq k(x) \leq \kappa_2, \quad \kappa_1 > 0, \quad T > 0.\]
This equation is supplemented with homogeneous Dirichlet boundary conditions
\[u(x, t) = 0, \quad x \in \partial \Omega, \quad 0 < t \leq T. \tag{37}\]
For the unique solvability of the unsteady problem, it is supplemented with the initial condition
\[u(x, 0) = u^0(x), \quad x \in \Omega. \tag{38}\]

The second example is the time-dependent equation of convection-diffusion with the convective transport written in the divergence form:
\[
\frac{\partial u}{\partial t} + \sum_{\alpha=1}^{2} \frac{\partial}{\partial x_\alpha} \left( v_\alpha(x, t) u \right) - \sum_{\alpha=1}^{2} \frac{\partial}{\partial x_\alpha} \left( k(x) \frac{\partial u}{\partial x_\alpha} \right) = f(x, t), \quad x \in \Omega, \quad 0 < t \leq T. \tag{39}
\]
And finally, the primary object of our investigation is the convection-diffusion equation with the convective terms written in the skew-symmetric form:

\[
\frac{\partial u}{\partial t} + \frac{1}{2} \sum_{\alpha=1}^{2} \left( v_{\alpha}(x, t) \frac{\partial u}{\partial x_{\alpha}} + \frac{\partial}{\partial x_{\alpha}} (v_{\alpha}(x, t) u) \right)
\]

\[-\sum_{\alpha=1}^{2} \frac{\partial}{\partial x_{\alpha}} \left( k(x) \frac{\partial u}{\partial x_{\alpha}} \right) = f(x, t), \quad x \in \Omega, \quad 0 < t \leq T. \tag{40}\]

We consider a set of functions \( u(x) \) that satisfy the boundary condition (37). The transient convection-diffusion problem may be formulated as the operator-differential equation

\[
\frac{du}{dt} + Au = f(t), \quad A = \mathcal{C} + \mathcal{D}. \tag{41}\]

Here \( \mathcal{D} \) is the diffusive transport operator that is defined by the expression

\[
\mathcal{D}u = -\sum_{\alpha=1}^{2} \frac{\partial}{\partial x_{\alpha}} \left( k(x) \frac{\partial u}{\partial x_{\alpha}} \right). \tag{42}\]

According to (36), (39), (40), the convective transport operator is written in distinct forms. For the convective transport operator in the nondivergent form, from (36), we set

\[
\mathcal{C}_1u = \sum_{\alpha=1}^{2} v_{\alpha}(x, t) \frac{\partial u}{\partial x_{\alpha}}. \tag{43}\]

Similarly, from (39), we have \( \mathcal{C} = \mathcal{C}_2 \), where now

\[
\mathcal{C}_2u = \sum_{\alpha=1}^{2} \frac{\partial}{\partial x_{\alpha}} (v_{\alpha}(x, t) u). \tag{44}\]

Taking into account (40), the convective-transport operator in the skew-symmetric form is

\[
\mathcal{C} = \mathcal{C}_0 = \frac{1}{2}(\mathcal{C}_1 + \mathcal{C}_2),
\]

and

\[
\mathcal{C}_0u = \frac{1}{2} \sum_{\alpha=1}^{2} \left( v_{\alpha}(x, t) \frac{\partial u}{\partial x_{\alpha}} + \frac{\partial}{\partial x_{\alpha}} (v_{\alpha}(x, t) u) \right). \tag{45}\]

Now we highlight the basic properties of the above-mentioned operators of diffusive and convective transport.
2.2. Properties of the operators of diffusive and convective transport

The solution of a discrete problem should inherit the basic properties of the corresponding differential problem. This can be achieved, in particular, if the grid operators have the same primary properties as the differential ones.

As usually, let $\mathcal{H} = L^2(\Omega)$ be a Hilbert space for arbitrary functions $u(x)$ and $w(x)$ equal zero on $\partial \Omega$. The diffusive transport operator defined by (42) is self-adjoint in $\mathcal{H}$ on the set of functions satisfying the boundary conditions (37):

$$D = D^*.$$  \hfill (46)

Note also that the diffusive transport operator under consideration at the above-mentioned restrictions is positive definite, i.e., the estimate

$$D \geq \kappa_1 \lambda_0 \mathcal{E},$$  \hfill (47)

is valid, where $\mathcal{E}$ denotes the identity operator and $\lambda_0 > 0$ is the minimal eigenvalue of the Laplace operator with the Dirichlet boundary conditions. For the rectangle $\Omega$, we have

$$\lambda_0 = \pi^2 \left( \frac{1}{l_1^2} + \frac{1}{l_2^2} \right).$$

The estimate (47) follows from

$$(Du, u) \geq \kappa_1 (\nabla u, \nabla u) \geq \kappa_1 \lambda_0 (u, u).$$

We now consider the convective transport operator in various formulations (see (43), (44), and (45)). Taking into account the homogeneous boundary conditions (37), we have

$$(C_1 u, w) = \sum_{\alpha=1}^{2} \int_{\Omega} v_\alpha \frac{\partial u}{\partial x_\alpha} w \, dx = -\sum_{\alpha=1}^{2} \int_{\Omega} \frac{\partial}{\partial x_\alpha} (v_\alpha w) u \, dx = -(u, C_2 w).$$

Thus, we see that the convective transport operators in the divergent and nondivergent forms are the adjoints of each other (with a precision of the sign):

$$C_1^* = -C_2.$$  \hfill (48)

In view of (48), the convective transport operator in the skew-symmetric form (45) is skew-symmetric ($(C_0 u, u) = 0$):

$$C_0 = -C_0^*.$$  \hfill (49)
Under the condition of incompressibility

\[
\text{div } \mathbf{v} \equiv \sum_{\alpha=1}^{2} \frac{\partial v_\alpha}{\partial x_\alpha} = 0, \quad \mathbf{x} \in \Omega, \tag{50}
\]

the convective transport operator in the nondivergent (43) and divergent (44) forms are also skew-symmetric. In constructing discrete approximations of the convective transport operators, the principal moment is that the skew-symmetric property of the operator \( \mathcal{C}_0 \) is valid for any \( v_\alpha(\mathbf{x}, t), \alpha = 1, 2 \) including the compressible case.

It seems useful to give the upper bound for the convective transport operator. For (43), (44), we have

\[
(\mathcal{C}_1 u, u) = -(\mathcal{C}_2 u, u) = \frac{1}{2} \sum_{\alpha=1}^{2} \int_{\Omega} v_\alpha \frac{\partial u^2}{\partial x_\alpha} \, d\mathbf{x} = -\frac{1}{2} \int_{\Omega} u^2 \text{div } \mathbf{v} \, d\mathbf{x}
\]

and therefore

\[
|\langle \mathcal{C}_\alpha u, u \rangle| \leq \frac{1}{2} \| \text{div } \mathbf{v} \|_{C(\Omega)} \| u \|_2^2, \quad \alpha = 1, 2, \tag{51}
\]

where

\[
\| w \|_{C(\Omega)} = \max_{\mathbf{x} \in \Omega} | w(\mathbf{x}) |.
\]

Thus, for the convective transport operators defined in accordance with (43), (44) \( (\mathcal{C} = \mathcal{C}_\alpha, \quad \alpha = 1, 2) \), we obtain

\[
|\langle \mathcal{C} u, u \rangle| \leq \mathcal{M}_1 \| u \|_2^2, \tag{52}
\]

where a constant \( \mathcal{M}_1 \) depends only on \( \text{div } \mathbf{v} \) and, in accordance with (51), it follows that

\[
\mathcal{M}_1 = \frac{1}{2} \| \text{div } \mathbf{v} \|_{C(\Omega)}. \tag{53}
\]

In addition, we present the estimates of subordination of the convective transport operator to the diffusive transport operator:

\[
\| \mathcal{C} u \|_2^2 \leq \mathcal{M}_2 (\mathcal{D} u, u), \tag{54}
\]

with a constant \( \mathcal{M}_2 \) depending on the velocity.
For the nondivergent operator of convection (43), we have
\[
\|C_1 u\|^2 = \int_\Omega \left( \sum_{\alpha=1}^2 v_\alpha \frac{\partial u}{\partial x_\alpha} \right)^2 \, dx \leq 2 \sum_{\alpha=1}^2 \int_\Omega v_\alpha^2 \left( \frac{\partial u}{\partial x_\alpha} \right)^2 \, dx 
\]
\[
\leq 2 \max_\alpha \{ \|v_\alpha^2\|_{C(\Omega)}\} \frac{1}{\kappa_1} \sum_{\alpha=1}^2 \int_\Omega k \left( \frac{\partial u}{\partial x_\alpha} \right)^2 \, dx 
\]
\[
\leq \frac{2}{\kappa_1} \max_\alpha \{ \|v_\alpha^2\|_{C(\Omega)}\} \ (D u, u),
\]
i.e., in the inequality (54), for \(C = C_1\), we can assume
\[
\mathcal{M}_2 = \frac{2}{\kappa_1} \max_\alpha \{ \|v_\alpha^2\|_{C(\Omega)}\} .
\]  
(55)

Similarly, for \(C = C_2\) (see (44)), we obtain
\[
\|C_2 u\|^2 = \int_\Omega \left( \sum_{\alpha=1}^2 v_\alpha \frac{\partial u}{\partial x_\alpha} + \text{div} \ v u \right)^2 \, dx 
\]
\[
\leq 2 \int_\Omega \left( \sum_{\alpha=1}^2 v_\alpha \frac{\partial u}{\partial x_\alpha} \right)^2 \, dx + 2 \int_\Omega (\text{div} \ v)^2 u^2 \, dx.
\]
Taking into account the Friedrichs inequality
\[
\int_\Omega u^2 \, dx \leq \mathcal{M}_0 \sum_{\alpha=1}^2 \int_\Omega \left( \frac{\partial u}{\partial x_\alpha} \right)^2 \, dx,
\]  
(56)

where the constant \(\mathcal{M}_0 = \lambda_0^{-1}\), we obtain at \(C = C_2\) the estimate (54) with
\[
\mathcal{M}_2 = \frac{2}{\kappa_1} \left( 2 \max_\alpha \{ \|v_\alpha^2\|_{C(\Omega)}\} + \mathcal{M}_0 \|\text{div} \ v\|^2_{C(\Omega)} \right).
\]  
(57)

In a similar way, for the case \(C = C_0\), we have
\[
\|C_0 u\|^2 = \frac{1}{4} \|C_1 u + C_2 u\|^2 \leq \frac{1}{2} \|C_1 u\|^2 + \frac{1}{2} \|C_2 u\|^2,
\]
i.e.,
\[
\mathcal{M}_2 = \frac{1}{\kappa} \left( 3 \max_\alpha \{ \|v_\alpha^2\|_{C(\Omega)}\} + \mathcal{M}_0 \|\text{div} \ v\|^2_{C(\Omega)} \right).
\]  
(58)
The above estimates (52), (54) serve as a reference point in studying discrete analogs of the convective transport operator.

Summarizing the above properties of the convective transport operator, we obtain the following statement.

**Theorem 1.** *The convective transport operators have the following properties:*

- *the convective transport operators in the divergent and nondivergent forms are adjoint to each other up to the sign — the equality (48);*

- *the convective transport operator in the skew-symmetric form is skew-symmetric — the equality (49);*

- *the convective transport operators in the divergent and nondivergent forms are bounded — the a priori estimates (52) and (53);*

- *the convective transport operators are subordinated to the diffusion operator — the estimate (54) with the constant \( \mathcal{M}_2 \), defined according to (55), (57), (58).*

It seems reasonable to construct difference operator of convective and diffusive transport in such a way that they do have the same properties.

### 2.3. A priori estimates

We restrict ourselves to elementary a priori estimates for the time-dependent equation (41) supplemented with the initial condition

\[
  u(0) = u^0. \tag{59}
\]

They are based on the above-established properties (see Theorem 1) of the operators of diffusive and convective transport.

**Theorem 2.** *For the solution of the problem (41), (59), under the conditions (47), (52), (54), the following a priori estimate holds:*

\[
  \|u(t)\|^2 \leq \exp(2\mathcal{M}_1 t)\|u_0\|^2 + \frac{1}{2} \int_0^t \exp(2\mathcal{M}_1(t - \theta))\|f(\theta)\|_{\mathcal{D}^{-1}}^2 d\theta, \tag{60}
\]
\|u(t)\| \leq \exp(\frac{1}{4} \mathcal{M}_2 t) \|u_0\|

\begin{align}
&+ \int_0^t \exp(\frac{1}{4} \mathcal{M}_2 (t - \theta)) \|f(\theta)\| d\theta,
\end{align}

(61)

\|\nabla u(t)\|^2 \leq \frac{\kappa_2}{\kappa_1} \exp(\mathcal{M}_2 t) \|\nabla u_0\|^2

\begin{align}
&+ \frac{1}{\kappa} \int_0^t \exp(\mathcal{M}_2 (t - \theta)) \|f(\theta)\|^2 d\theta,
\end{align}

(62)

where \( \kappa_1 \leq k(x) \leq \kappa_2, \ x \in \Omega \cup \partial \Omega. \)

**Proof.** Multiplying equation (41) scalarly by \( u(t) \), we get

\[
\frac{1}{2} \frac{d}{dt} \|u\|^2 + (\mathcal{D} u, u) = -(\mathcal{C} u, u) + (f, u).
\]

(63)

Taking into account (52) and the inequality

\[
(f, u) \leq (\mathcal{D} u, u) + \frac{1}{4} \|f\|^2_{\mathcal{D}^{-1}},
\]

from (63), it follows that

\[
\frac{d}{dt} \|u\|^2 \leq 2\mathcal{M}_1 \|u\|^2 + \frac{1}{2} \|f\|^2_{\mathcal{D}^{-1}}.
\]

Using Gronwall’s lemma, we obtain from this inequality the required estimate (60).

From (54), we have

\[
|-(\mathcal{C} u, u)| \leq \|\mathcal{C} u\| \|u\| \leq (\mathcal{D} u, u) + \frac{1}{4} \mathcal{M}_2 \|u\|^2.
\]

This allows to obtain from (63) the inequality

\[
\frac{d}{dt} \|u\| \leq \frac{1}{4} \mathcal{M}_2 \|u\| + \|f\|,
\]

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which immediately implies the estimate (61).

It remains to derive the estimate (62). To do this, multiply equation (41) scalarly by $\frac{du}{dt}$ and obtain

$$\left\| \frac{du}{dt} \right\|^2 + \frac{1}{2} \frac{d}{dt} (Du, u) = - \left( Cu, \frac{du}{dt} \right) + \left( f, \frac{du}{dt} \right).$$

For the right-hand side, we have

$$- \left( Cu, \frac{du}{dt} \right) + \left( f, \frac{du}{dt} \right) \leq \left\| \frac{du}{dt} \right\|^2 + \frac{1}{2} \| Cu \|^2 + \frac{1}{2} \| f \|^2.$$

In view of (54), we get the inequality

$$\frac{d}{dt} (Du, u) \leq M_2 (Du, u) + \| f \|^2. \quad (64)$$

By

$$\kappa_1 \| \nabla u \|^2 \leq (Du, u) \leq \kappa_2 \| \nabla u \|^2,$$

from (64), we obtain the desired estimate (62).

The resulting estimates (60)–(62) provide the continuity of the solution of (41), (59) with respect to the initial data and the right-hand side. In these estimates, the essential issue is that for the solution norm of the problem with the homogeneous right-hand side, it is allowed an exponential growth with a growth increment that depends on the constants $M_1$, $M_2$. It is necessary to allow such a behavior for the solution at the discrete level. Thus, we need to introduce the concept of $\varrho$-stability for the corresponding difference schemes.

### 2.4. The maximum principle and a priori estimates

Considering boundary value problems both for parabolic equations of the second order in space and for second-order elliptic equations, special attention is paid to the maximum principle [23]. The corresponding statement is formulated as follows.

**Theorem 3.** Assume that in the Cauchy problem (41), (59) the right-hand side $f(x, t) > 0$ ($f(x, t) < 0$) and the initial data $u_0(x) \geq 0$ ($u_0(x) \leq 0$), then $u(x, 0) \geq 0$ ($u(x, 0) \leq 0$) for all $x \in \Omega$ and $t > 0$. 

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Note that it is possible to use the maximum principle in a stronger form that employs weak inequalities for the right-hand side, i.e., the non-negativity of the solution takes place under the condition of the non-negativity of the right-hand side and the initial data.

Here are some a priori estimates for the convection-diffusion problem (41), (59), which are derived via the maximum principle. In the above-considered time-dependent problems with Dirichlet boundary conditions, we can easily construct a majorant function — a wide range of the appropriate estimates is given, e.g., in the book [13]. We also give an estimate for the convection-diffusion equation with convective terms in the divergence form — the estimate in $L_1(\Omega)$.

**Theorem 4.** The solution of the problem (36)–(38) satisfies the following a priori estimate in $L_\infty(\Omega)$:

$$
\|u(x, t)\|_\infty \leq \|u(x, 0)\|_\infty + \int_0^t \|f(x, \theta)\|_\infty d\theta,
$$

whereas the solution of the problem (37)–(39) satisfies the a priori estimate

$$
\|u(x, t)\|_1 \leq \|u(x, 0)\|_1 + \int_0^t \|f(x, \theta)\|_1 d\theta.
$$

The estimates (65), (66) complement the above a priori estimates (60)–(62) in the Hilbert spaces $L_2(\Omega)$ and $W^{1,2}_2(\Omega)$.

3. **Discretization in space**

In numerical solving transient problems, firstly we construct discretization in space. The resulting operator-differential equation should inherit the basic properties of the differential problem, i.e., we speak of the positiveness (non-negativity) and self-adjointness of the diffusive transport operator as well as the adjointness of the convective transport operators in the corresponding finite-dimensional Hilbert spaces. We consider the standard finite difference approximations for model unsteady convection-diffusion problems in a rectangular domain. In addition, we discuss the problem of constructing approximations by means of the finite volume method.

3.1. **Difference operators**

In a rectangle $\Omega$, we introduce a uniform in each direction grid. For grids in individual directions $x_\alpha$, $\alpha = 1, 2$, we use notation

$$
\bar{\omega}_\alpha = \{x_\alpha \mid x_\alpha = i_\alpha h_\alpha, \quad i_\alpha = 0, 1, \ldots, N_\alpha, \quad N_\alpha h_\alpha = l_\alpha\},
$$
where \( \omega \) stands for the set of interior nodes. On the set of grid functions that are equal to zero on the set of boundary nodes \( \partial \omega \) \((\bar{\omega} = \bar{\omega}_1 \times \bar{\omega}_2 = \omega \cup \partial \omega)\), we define the Hilbert space \( H = L_2(\omega) \) with the following scalar product and norm:

\[
(y, w) \equiv \sum_{x \in \omega} y(x)w(x)h_1h_2, \quad \|y\| \equiv (y, y)^{1/2}.
\]

We use the standard index-free notations of the theory of difference schemes [27]. For the backward difference derivative, we have the representation

\[
u_\tau \equiv \frac{u_i - u_{i-1}}{h}.
\]

Similarly, for the forward difference derivative, we get

\[
u_x \equiv \frac{u_{i+1} - u_i}{h}.
\]

Using the three-point stencil (nodes \( x_{i-1}, x_i, x_{i+1} \)), we can employ the central difference derivative

\[
u_x \equiv \frac{u_{i+1} - u_{i-1}}{2h}.
\]

For the operator of the second derivative, we have

\[
u_{xx} = \frac{u_{x} - u_x}{h} = \frac{u_{i-1} - 2u_i + u_{i+1}}{h^2}.
\]

The 2D difference diffusive transport operator is represented as the sum of the 1D ones:

\[
D = \sum_{\alpha=1}^{2} D^{(\alpha)}, \quad D^{(\alpha)}y = -(a^{(\alpha)}y^{x_{\alpha}})_{x_{\alpha}}, \quad \alpha = 1, 2, \ x \in \omega. \quad (67)
\]

For smooth diffusion coefficients, we can assume

\[a^{(1)}(x) = k(x_1 - 0.5h_1, x_2), \quad a^{(2)}(x) = k(x_1, x_2 - 0.5h_2).\]

Properties of the elliptic difference operator \( D \) are well-known [27, 25]. For the 2D self-adjoint operator \( D \), we have the lower bound

\[
D = D^* \geq \frac{1}{M_0} \kappa_1 E, \quad M_0 = \frac{8}{h_1^2} + \frac{8}{h_2^2}. \quad (68)
\]
We present also the upper bound for the diffusive transport operator, i.e.,

\[ D \leq M_3 E \]  

with the constant

\[ M_3 = \frac{4}{h_1^2} \max_{x \in \omega} \frac{a^{(1)}(x) + a^{(1)}(x_1 + h_1, x_2)}{2} + \frac{4}{h_2^2} \max_{x \in \omega} \frac{a^{(2)}(x) + a^{(2)}(x_1, x_2 + h_2)}{2}. \]

Now we consider the difference analogs of the differential convective transport operators written in various forms. For the operator in the nondivergent form \( C_1 \), we put into the correspondence the 2D the difference convective transport operator

\[ C_1 = \sum_{\alpha=1}^{2} C_1^{(\alpha)}, \quad C_1^{(\alpha)} y = b^{(\alpha)} y_{x_{\alpha}}, \quad \alpha = 1, 2, \quad x \in \omega. \]  

In the simplest case of smooth enough convective transport coefficients, we assume

\[ b^{(\alpha)}(x, t) = v^{(\alpha)}(x, t), \quad x \in \omega. \]

Similarly, for the approximation of the convective transport operator in the divergent form \( C_2 \), we employ the difference operator

\[ C_2 = \sum_{\alpha=1}^{2} C_2^{(\alpha)}, \quad C_2^{(\alpha)} y = (b^{(\alpha)} y)_{x_{\alpha}}, \quad \alpha = 1, 2, \quad x \in \omega. \]  

The approximation of the 2D convective transport operator in the skew-symmetric form is based on the representation \( C_0 = 0.5(C_1 + C_2) \) such that

\[ C_0 = \sum_{\alpha=1}^{2} C_0^{(\alpha)}, \quad C_0^{(\alpha)} y = \frac{1}{2}(b^{(\alpha)} y_{x_{\alpha}} + (b^{(\alpha)} y)_{x_{\alpha}}), \quad \alpha = 1, 2, \quad x \in \omega. \]  

**Lemma 1.** The difference operators \( C_\alpha, \alpha = 0, 1, 2 \) have the following properties of adjointness:

\[ C_1^* = -C_2, \quad C_0^* = -C_0 \]

in the space of grid functions \( H \).
Proof. It is easy to check directly that the 1D convective transport operators in the divergent and nondivergent forms are adjoint to each other up to the sign. Taking this into account, we have

\[(C_1 y, w) = (C_1^{(1)} y, w) + (C_1^{(2)} y, w)\]

\[= -(y, C_2^{(1)} w) - (y, C_2^{(2)} w)\]

\[= -(y, C_2^{(1)} w) = (y, C_1^{*} w).\]

The skew-symmetric property of the operator \(C_0\) follows from its definition as the half-sum of the operators \(C_1\) and \(C_2\).

Similar properties can be proved for the 2D convective transport operators constructed with the use of coefficients \(v_{\alpha}(x, t), \alpha = 1, 2\) shifted by a half-step in the corresponding directions. Such staggered grids are in common use in computational fluid dynamics [35, 22].

Figure 1: Control volume: □ — node for \(v_1(x, t)\); □ — node for \(v_2(x, t)\)

Let us refer the convective-transport coefficient \(v_1(x, t)\) with respect to the variable \(x_1\) to the nodes of the grid which is shifted by a half-step along this direction. The grid for the coefficient \(v_2(x, t)\) is shifted along \(x_2\) by 0.5\(h_2\) (see Fig. 1).

For the difference convective transport operator in the nondivergent form, we get

\[C_1^{(1)} y = \frac{1}{2}(b_1^{(1)}(x_1 - 0.5h_1, x_2, t)y^{x_1} + b_1^{(1)}(x_1 + 0.5h_1, x_2, t)y^{x_1}),\]
\[ \begin{align*}
C_1^{(2)} y &= \frac{1}{2} (b^{(2)}(x_1, x_2 - 0.5h_2, t)y^{x_2} + b^{(2)}(x_1, x_2 + 0.5h_2, t)y^{x_2}), \\
C_1 &= \sum_{\alpha=1}^{2} C_1^{(\alpha)}, \quad x \in \omega.
\end{align*} \]  

(74)

For the difference convective transport operator in the divergent form, we employ the representation

\[ C_2 y = C_1 y + \left( b_{x_1}^{(1)} + b_{x_2}^{(2)} \right) y, \quad x \in \omega. \]  

(75)

The following notation is used here for the difference derivative of the grid function given at half-integer nodes:

\[ b_x \equiv \frac{b(x + 0.5h) - b(x - 0.5h)}{h}. \]

This expression is a difference analog of the differential equality

\[ C_2 u = C_1 u + \text{div} v u \]

with a special approximation of \( \text{div} v \).

For the skew-symmetric convective transport operator \( C_0 = 0.5(C_1 + C_2) \), from (74), (75), we obtain

\[ \begin{align*}
C_0^{(1)} y &= \frac{1}{2h_1} b^{(1)}(x_1 + 0.5h_1, x_2, t)y(x_1 + h_1, x_2) \\
&\quad - \frac{1}{2h_1} b^{(1)}(x_1 - 0.5h_1, x_2, t)y(x_1 - h_1, x_2), \\
C_0^{(2)} y &= \frac{1}{2h_2} b^{(2)}(x_1, x_2 + 0.5h_2, t)y(x_1, x_2 + h_2) \\
&\quad - \frac{1}{2h_2} b^{(2)}(x_1, x_2 - 0.5h_2, t)y(x_1, x_2 - h_2), \\
C_0 &= \sum_{\alpha=1}^{2} C_0^{(\alpha)}, \quad x \in \omega.
\end{align*} \]  

(76)

For the convective transport operators \( C_\alpha, \alpha = 0, 1, 2 \), defined by (74)–(76), Lemma[] holds.

In the multidimensional case, the inequality

\[ \|(C_\alpha y, y)\| \leq M_1 \|y\|^2, \quad \alpha = 1, 2 \]  

(77)
with a constant $M_1$, independent of the grid steps, is also valid for the convective transport operators under consideration. For operators (70), (71), we obtain a constant $M_1$, which depends on the first derivatives of the convective transport coefficients, whereas in the case (74), (75) it depends on the divergence, as in the continuous case. We formulate the corresponding statement using the following notation for the grids in separate directions:

$$\omega_\alpha = \{x_\alpha \mid x_\alpha = i_\alpha h_\alpha, \quad i_\alpha = 1, 2, \ldots, N_\alpha - 1, \quad N_\alpha h_\alpha = l_\alpha\},$$

$$\omega^+_\alpha = \{x_\alpha \mid x_\alpha = i_\alpha h_\alpha, \quad i_\alpha = 1, 2, \ldots, N_\alpha, \quad N_\alpha h_\alpha = l_\alpha\}, \quad \alpha = 1, 2.$$

**Lemma 2.** For the difference convective transport operators $C_\alpha$, $\alpha = 1, 2$, defined according to (70), (71), the estimates (77) hold with the constant

$$M_1 = \frac{1}{2} \max_{x_1 \in \omega^+_1} \max_{x_2 \in \omega^+_2} |b^{(1)}_{x_1}| + \frac{1}{2} \max_{x_1 \in \omega_1} \max_{x_2 \in \omega^+_2} |b^{(2)}_{x_2}|, \quad (78)$$

whereas for (74), (75) — with the constant

$$M_1 = \frac{1}{2} \max_{x \in \omega} |b^{(1)}_{x_1} + b^{(2)}_{x_2}|. \quad (79)$$

**Proof.** Taking into account (74), (75), we get

$$(C_1 y, y) = -(C_2 y, y) = \frac{1}{2} \left( (C_1 y, y) + (y, C_1^* y) \right)$$

$$= \frac{1}{2} \left( \left( b^{(1)}_{x_1} + b^{(2)}_{x_2} \right) y, y \right).$$

This implies the estimate (77), (79).

For the operators (70), (71), we use the corresponding estimates for the 1D operators, too. For instance, for the 2D convective transport operator in the non-divergent form, we have

$$|(C_1 y, y)| = \left| \left( (C^{(1)}_1 + C^{(2)}_1) y, y \right) \right|$$

$$= \left| \sum_{x_1 \in \omega_1} \sum_{x_2 \in \omega_2} (C^{(1)}_1 + C^{(2)}_1) y y h_1 h_2 \right|$$

$$\leq \frac{1}{2} \max_{x_1 \in \omega^+_1} \max_{x_2 \in \omega^+_2} |b^{(1)}_{x_1}| (y, y) + \frac{1}{2} \max_{x_1 \in \omega_1} \max_{x_2 \in \omega^+_2} |b^{(2)}_{x_2}| (y, y),$$

i. e., we arrive at the desired estimate (77), (78).
Finally, let us consider the subordination property of the convective transport operator to the diffusive transport operator under the standard restrictions $k(x) \geq \kappa_1 > 0$.

**Lemma 3.** For the 2D convective transport difference operators $C = C_\alpha$, $\alpha = 0, 1, 2$, the following estimates hold:

$$
\|Cy\|^2 \leq M_2(Dy, y),
$$

(80)

where the constant $M_2$ for the operators $C_1$, defined according to (70), (74), is, respectively,

$$
M_2 = \frac{2}{\kappa_1} \max_x \max_{\alpha} (b^{(\alpha)}(x, t))^2,
$$

$$
M_2 = \frac{2}{\kappa_1} \max \left\{ \max_{x \in \omega_1^+ \times \omega_2} (b^{(1)}(x_1 - 0.5h_1, x_2, t))^2, \right. \\
\left. \max_{x \in \omega_1^+ \times \omega_2} (b^{(2)}(x_1, x_2 - 0.5h_2, t))^2 \right\},
$$

for the operators (71), (75) —

$$
M_2 = \frac{2}{\kappa_1} \left( 2 \max_{\alpha} \max_{x \in \omega} (b^{(\alpha)}(x, t))^2 + M_0 \max_{x \in \omega} \left( b^{(1)}_{x_1} + b^{(2)}_{x_2} \right)^2 \right),
$$

$$
M_2 = \frac{2}{\kappa_1} \left( 2 \max_{x \in \omega_1^+ \times \omega_2} (b^{(1)}(x_1 - 0.5h_1, x_2, t))^2, \right. \\
\left. \max_{x \in \omega_1^+ \times \omega_2} (b^{(2)}(x_1, x_2 - 0.5h_2, t))^2 \right) \\
+ M_0 \max_{x \in \omega} \left( b^{(1)}_{x_1} + b^{(2)}_{x_2} \right)^2,
$$

and for the operators (72), (76) —

$$
M_2 = \frac{1}{\kappa_1} \left( 3 \max_{\alpha} \max_{x \in \omega} (b^{(\alpha)}(x, t))^2 + M_0 \max_{x \in \omega} \left( b^{(1)}_{x_1} + b^{(2)}_{x_2} \right)^2 \right),
$$

$$
M_2 = \frac{1}{\kappa_1} \left( 3 \max_{x \in \omega_1^+ \times \omega_2} (b^{(1)}(x_1 - 0.5h_1, x_2, t))^2, \right.
$$
\[
\max_{x \in \omega_1 \times \omega_2} \left\{ (b^{(2)}(x_1, x_2 - 0.5h_2, t))^2 \right\}
\]
\[
+ M_0 \max_{x \in \omega_1 \times \omega_2} \left( b^*_x + b^*_x \right)^2,
\]
where \( M_0 \) — is the constant from (68).

**Proof.** Taking into account the inequality
\[
\left( \sum_{i=1}^p a_i \right)^2 \leq p \sum_{i=1}^p a_i^2,
\]
for operator (70), we have
\[
\|C_1 y\|^2 = \sum_{x \in \omega} \frac{1}{4} (b^{(1)}(x, t)(y_{x_1} \bar{y}_{x_1} + y_{x_1}) + b^{(2)}(x, t)(y_{x_2} \bar{y}_{x_2} + y_{x_2}))^2 h_1 h_2
\]
\[
\leq 2 \max_{\alpha \in \omega} \max(b^{(\alpha)}(x, t))^2 \|\nabla y\|^2 \leq M_2(Dy, y).
\]
The estimate for the operator (74) is obtained in the same manner.

For the discrete operator (71), we employ the representation
\[
C_2 y = \frac{1}{2} \left( b^{(1)}(x_1, x_2, t) y_{x_1} + b^{(1)}(x_1 + h_1, x_2, t) y_{x_1}ight.
\]
\[
+ b^{(2)}(x_1, x_2 - h_2, t) y_{x_2} + b^{(2)}(x_1, x_2 + h_2, t) y_{x_2}
\]
\[
+ \left( b^*_x + b^*_x \right) y.
\]
Thus,
\[
\|C_2 y\|^2 \leq 2 \sum_{x \in \omega} (b^{(1)}(x_1, x_2))^2 (y_{x_1})^2 h_1 h_2
\]
\[
+ 2 \sum_{x \in \omega} (b^{(1)}(x_1 + h_1, x_2))^2 (y_{x_1})^2 h_1 h_2
\]
\[
+ 2 \sum_{x \in \omega} (b^{(2)}(x_1, x_2 - h_2))^2 (y_{x_2})^2 h_1 h_2
\]
\[
+ 2 \sum_{x \in \omega} (b^{(2)}(x_1, x_2 + h_2))^2 (y_{x_2})^2 h_1 h_2
\]
\[
+ 2 \max_{x \in \omega} (b^*_x + b^*_x)^2 (y, y).
\]
In view of the Friedrichs inequality, we obtain the estimate (80) with the constant $M_2$ given in the lemma.

For the grid operator in the divergent form with coefficients on staggered grids, on the basis of (75), we have

$$
\|C_2y\|^2 = 2\|C_1y\|^2 + 2 \left\| \left( b^{(1)}_{x_1} + b^{(2)}_{x_2} \right) y \right\|^2.
$$

For the first term in the right-hand side, we use the already derived estimate for $C_1$, whereas for the second one we apply the Friedrichs inequality.

Subordination estimates for the difference 2D convective transport operators in the skew-symmetric form $C_0$ are derived from the estimates for the operators $C_\alpha$, $\alpha = 1, 2$.

The above values for the subordination constant $M_2$, in spite of their awkwardness, demonstrate the fundamental independence of this constant of the computational grid. The constant $M_2$ depends on the values of the convective transport coefficients $v_\alpha(x, t)$, $\alpha = 1, 2$ (the velocity) and on $\text{div} \, v$, to be more precise, on their difference approximations.

In numerical solving the problem (41), (59), using the above discretization in space, we obtain the operator-differential equation

$$
\frac{dy}{dt} + Ay = \varphi(x, t), \quad A = C + D, \quad x \in \omega, \quad 0 < t \leq T, \quad y(x, 0) = u_0(x), \quad x \in \omega.
$$

(81)

(82)

To investigate this semi-discrete problem, we employ the above properties of the difference operators of convective and diffusive transport. In particular, we can obtain analogues of a priori estimates of Theorem 2.

3.2. Monotone schemes for 2D convection-diffusion problems

For 2D difference convection-diffusion problems in nondivergent and divergent forms, the maximum principle is formulated. Unconditionally monotone schemes are constructed on the basis of the regularization principle for difference schemes.

To simplify the material presentation, we will consider difference schemes for stationary 2D convection-diffusion equation on uniform rectangular grids. The corresponding discrete analogues on the standard five-point stencil cross are written in the following form:

$$
\gamma(x)y(x) - \alpha_1(x)y(x_1 - h_1, x_2) - \beta_1(x)y(x_1 + h_1, x_2)
- \alpha_2(x)y(x_1, x_2 - h_2) - \beta_2(x)y(x_1, x_2 + h_2) = \varphi(x), \quad x \in \omega.
$$

(83)
This grid equations are considered with the boundary conditions
\[ y(x) = 0, \quad x \in \partial \omega. \]  
(84)
Assume that the coefficients of the difference scheme (83) satisfy the conditions
\[ \alpha_j(x) > 0, \quad \beta_j(x) > 0, \quad j = 1, 2, \quad \gamma(x) > 0, \quad x \in \omega \]  
(85)
and suppose that
\[ \alpha_j(x) > 0, \quad \beta_j(x) > 0, \quad j = 1, 2, \quad x \in \partial \omega. \]

We highlight two classes of the monotone difference schemes (83), (84), i.e., the schemes that satisfy the difference maximum principle.

**Theorem 5.** Assume that in the scheme (83)–(85) \( \varphi(x) \geq 0 \) for all \( x \in \omega \) (or \( \varphi(x) \leq 0 \) for \( x \in \omega \)). Then for
\[ \gamma(x) \geq \alpha_1(x) + \alpha_2(x) + \beta_1(x) + \beta_2(x), \quad x \in \omega \]  
(86)
or for
\[ \gamma(x) \geq \alpha_1(x_1 + h_1, x_2) + \beta_1(x_1 - h_1, x_2) \\
+ \alpha_2(x_1, x_2 + h_2) + \beta_2(x_1, x_2 - h_2), \quad x \in \omega \]  
(87)
we have that \( y(x) \geq 0 \) for all \( x \in \omega \) (\( y(x) \leq 0 \) for \( x \in \omega \)).

**Proof.** As usual, the argument is by reductio ad absurdum. Suppose, e.g., that the conditions (86) are valid and the solution of equation (83) with the non-negative right-hand side is not non-negative at all grid points. Let \( x^* \) be an interior grid point, where the solution has the minimal negative value. If this value is achieved at several points, then we choose the grid point, where \( y(x_1^* - h_1, x_2^*) > y(x^*). \)

We write equation (83) at this point as
\[ \gamma(x^*) y(x^*) - \alpha_1(x^*) y(x_1^* - h_1, x_2^*) - \beta_1(x^*) y(x_1^* + h_1, x_2^*) \\
- \alpha_2(x^*) y(x_1^* + x_2^* - h_2) - \beta_2(x^*) y(x_1^* + x_2^* + h_2) \\
= \varphi(x^*), \quad x^* \in \omega. \]

Under the theorem conditions, the right-hand side is non-negative, whereas the left-hand side, in view of (83), (86),
\[ (\gamma(x^*) - \alpha_1(x^*) - \beta_1(x^*) - \alpha_2(x^*) - \beta_2(x^*)) y(x^*) \\
+ \alpha_1(x^*) (y(x^*) - y(x_1^* - h_1, x_2^*)) \\
+ \beta_1(x^*) (y(x^*) - y(x_1^* + h_1, x_2^*)) \\
+ \alpha_2(x^*) (y(x^*) - y(x_1^* + x_2^* - h_2)) \\
+ \beta_2(x^*) (y(x^*) - y(x_1^* + x_2^* + h_2)) > 0. \]
We arrive at a contradiction, and therefore \( y(x) \geq 0 \) for all \( x \in \omega \).

Now we consider a more complicated case of the difference scheme (83), (84) with the conditions (85), (87) satisfied. Suppose that for the non-negative right-hand side of equation (83), there exists a subset of the interior grid points \( \omega^* \), where

\[
    y(x) < 0, \quad x \in \omega^*.
\]

Summarize equations (83) over all these nodes:

\[
    \sum_{x \in \omega^*} \left( \gamma(x) y(x) - \alpha_1(x) y(x_1 - h_1, x_2) - \beta_1(x) y(x_1 + h_1, x_2) \right.
    \]

\[
    \left. - \alpha_2(x) y(x_1, x_2 - h_2) - \beta_2(x) y(x_1, x_2 + h_2) \right) = \sum_{x \in \omega^*} \varphi(x) \geq 0.
\]

For the left-hand side of this equality, we have

\[
    \sum_{x \in \omega^*} \left( \gamma(x) - \alpha_1(x_1 + h_1, x_2) - \beta_1(x_1 - h_1, x_2) \right.
    \]

\[
    \left. - \alpha_2(x_1, x_2 + h_2) - \beta_2(x_1, x_2 - h_2) \right) y(x) + \sum_{x \in \omega^*} \left( \alpha_1(x_1 + h_1, x_2) y(x) - \alpha_1(x) y(x_1 - h_1, x_2) \right)
    \]

\[
    + \sum_{x \in \omega^*} \left( \beta_1(x_1 - h_1, x_2) y(x) - \beta_1(x) y(x_1 + h_1, x_2) \right) + \sum_{x \in \omega^*} \left( \alpha_2(x_1, x_2 + h_2) y(x) - \alpha_2(x) y(x_1, x_2 - h_2) \right)
    \]

\[
    + \sum_{x \in \omega^*} \left( \beta_2(x_1, x_2 - h_2) y(x) - \beta_2(x) y(x_1, x_2 + h_2) \right).
\]

We see immediately that each of these terms is non-negative and they cannot be equal to zero simultaneously. Thus, we again obtain a contradiction. \( \square \)

The maximum principle for multidimensional difference equations, where the sufficient conditions of type (86) are satisfied, is well-known in the theory of difference schemes [27]. For elliptic difference problems, the maximum principle in the form (87) is presented in work [37]. Similarly to the 1D case, the conditions (86) may be associated with the condition of diagonal dominance by rows if we use the natural order for the points of the numerical solution, whereas the conditions (87) correspond to diagonal dominance by columns.
Consider the 2D boundary value problem for the convection-diffusion equation in nondivergent form:

\[
\sum_{\alpha=1}^{2} v_{\alpha}(x) \frac{\partial u}{\partial x_{\alpha}} - \sum_{\alpha=1}^{2} \frac{\partial}{\partial x_{\alpha}} \left( k(x) \frac{\partial u}{\partial x_{\alpha}} \right) = f(x), \quad x \in \Omega, \tag{88}
\]

\[
u(x) = 0, \quad x \in \partial \Omega. \tag{89}
\]

For numerical solving the problem (88), (89), we employ the scheme

\[
C_{1} y + D y = \varphi(x), \quad x \in \omega. \tag{90}
\]

We restrict ourselves (see [32] for details) to the 2D operator of convective transport in the form

\[
C_{1}^{(1)} y = \frac{v_{1}(x)}{2k(x)} (k(x_{1} - 0.5h_{1}, x_{2})y^{x_{1}} + k(x_{1} + 0.5h_{1}, x_{2})y^{x_{1}}),
\]

\[
C_{1}^{(2)} y = \frac{v_{2}(x)}{2k(x)} (k(x_{1}, x_{2} - 0.5h_{2})y^{x_{2}} + k(x_{1}, x_{2} + 0.5h_{2})y^{x_{2}}),
\]

\[
C_{1} = \sum_{\alpha=1}^{2} C_{1}^{(\alpha)}, \quad x \in \omega. \tag{91}
\]

We formulate the conditions of monotonicity for the difference scheme (90), (91). We write it in the form (83), (84) with

\[
\alpha_{1}(x) = \left( \frac{v_{1}(x)}{2h_{1}k(x)} + \frac{1}{h_{1}^{2}} \right) k(x_{1} - 0.5h_{1}, x_{2}),
\]

\[
\beta_{1}(x) = \left( -\frac{v_{1}(x)}{2h_{1}k(x)} + \frac{1}{h_{1}^{2}} \right) k(x_{1} + 0.5h_{1}, x_{2}),
\]

\[
\alpha_{2}(x) = \left( \frac{v_{2}(x)}{2h_{2}k(x)} + \frac{1}{h_{2}^{2}} \right) k(x_{1}, x_{2} - 0.5h_{2}),
\]

\[
\beta_{2}(x) = \left( -\frac{v_{2}(x)}{2h_{2}k(x)} + \frac{1}{h_{2}^{2}} \right) k(x_{1}, x_{2} + 0.5h_{2}),
\]

\[
\gamma(x) = \alpha_{1}(x) + \alpha_{2}(x) + \beta_{1}(x) + \beta_{2}(x), \quad x \in \omega.
\]
The monotonicity condition (87) is obviously satisfied, and the positiveness of the coefficients of the difference scheme (83) (the condition (85)) leads to the natural restrictions

\[ |\theta_\alpha(x)| \leq 1, \quad \alpha = 1, 2, \quad x \in \omega, \]

where

\[ \theta_\alpha(x) = \frac{v_\alpha(x)h_\alpha}{2k(x)}, \quad \alpha = 1, 2. \]

Unconditionally monotone difference schemes for the 2D convection-diffusion equation (88), (89) are constructed by means of regularization (disturbance) of the diffusion coefficient. Instead of (90), we consider the difference scheme

\[ C_1 y + \sum_{\alpha=1}^{2} (1 + \rho_\alpha) D^{(\alpha)} y = \varphi(x), \quad x \in \omega. \quad (92) \]

The scheme (92) is monotone under the condition

\[ 1 + \rho_\alpha(x) > |\theta_\alpha(x)|, \quad x \in \omega, \quad \alpha = 1, 2. \]

We present some variants of regularizing grid functions \( \rho_\alpha(x), \alpha = 1, 2 \), which lead to unconditionally monotone difference schemes (92).

For example, the choice

\[ 1 + \rho_\alpha(x) = \theta_\alpha(x) \cth \theta_\alpha(x), \quad x \in \omega, \quad \alpha = 1, 2 \]

corresponds to the use of exponential schemes [7, 33] for each individual direction. It is also possible to select the scheme with

\[ \rho_\alpha(x) = \eta \theta^2_\alpha(x), \quad x \in \omega, \quad \alpha = 1, 2, \]

which are monotone if \( \eta > 0.25 \). Among unconditionally monotone difference schemes, we highlight the regularized scheme (92), where

\[ \rho_\alpha(x) = \frac{\theta^2_\alpha(x)}{1 + |\theta_\alpha(x)|}, \quad x \in \omega, \quad \alpha = 1, 2. \]

The scheme with the upwind differences corresponds to

\[ \rho_\alpha(x) = \eta |\theta_\alpha(x)|, \quad x \in \omega, \quad \alpha = 1, 2. \]
with \( \eta = 1 \). In this case, the difference operator of convective transport seems like this:

\[
C'(1)_1 y = \frac{k(x_1 - 0.5h_1, x_2)}{k(x)} v^+_1(x) y^{x_1} + \frac{k(x_1 + 0.5h_1, x_2)}{k(x)} v^-_1(x) y^{x_1},
\]

\[
C'(1)_2 y = \frac{k(x_1, x_2 - 0.5h_2)}{k(x)} v^+_2(x) y^{x_2} + \frac{k(x_1, x_2 + 0.5h_2)}{k(x)} v^-_2(x) y^{x_2},
\]

\[
C_1 = \sum_{\alpha=1}^2 C^{(\alpha)}_1, \quad x \in \omega. \quad (93)
\]

Obviously, the corresponding scheme has only the first-order approximation.

Now consider the convection-diffusion equation in the divergent form:

\[
\sum_{\alpha=1}^2 \frac{\partial}{\partial x_\alpha} \left( v_\alpha(x) u \right) - \sum_{\alpha=1}^2 \frac{\partial}{\partial x_\alpha} \left( k(x) \frac{\partial u}{\partial x_\alpha} \right) = f(x), \quad x \in \Omega, \quad (94)
\]

which is supplemented with the boundary conditions (89). For this equation, we consider the difference scheme

\[
C_2 y + Dy = \varphi(x), \quad x \in \omega. \quad (95)
\]

Define the difference operator of the convective transport as

\[
C'(1)_1 y = \frac{1}{2h_1} v_1(x_1 + 0.5h_1, x_2) (y(x_1 + h_1, x_2) + y(x))
- \frac{1}{2h_1} v_1(x_1 - 0.5h_1, x_2) (y(x_1 - h_1, x_2) + y(x)),
\]

\[
C'(1)_2 y = \frac{1}{2h_2} v_2(x_1, x_2 + 0.5h_2) (y(x_1, x_2 + h_2) + y(x))
- \frac{1}{2h_1} v_2(x_1, x_2 - 0.5h_2) (y(x_1, x_2 - h_2) + y(x)),
\]

\[
C_2 = \sum_{\alpha=1}^2 C^{(\alpha)}_2, \quad x \in \omega. \quad (96)
\]

This is typical for coefficients of convective transport defined on staggered grids.
The difference scheme (95), (96) may be written in the form (83) with the coefficients

\[
\begin{align*}
\alpha_1(x) &= \frac{v_1(x_1 - 0.5h_1, x_2)}{2h_1} + \frac{k(x_1 - 0.5h_1, x_2)}{h_1^2}, \\
\beta_1(x) &= -\frac{v_1(x_1 + 0.5h_1, x_2)}{2h_1} + \frac{k(x_1 + 0.5h_1, x_2)}{h_1^2}, \\
\alpha_2(x) &= \frac{v_2(x_1, x_2 - 0.5h_2)}{2h_2} + \frac{k(x_1, x_2 - 0.5h_2)}{h_2^2}, \\
\beta_2(x) &= -\frac{v_2(x_1, x_2 + 0.5h_2)}{2h_2} + \frac{k(x_1, x_2 + 0.5h_2)}{h_2^2}, \\
\gamma(x) &= \alpha_1(x_1 + h_1, x_2) + \beta_1(x_1 - h_1, x_2) \\
&\quad + \alpha_2(x_1, x_2 + h_2) + \beta_2(x_1, x_2 - h_2), \quad x \in \omega.
\end{align*}
\]

Therefore, the condition (87) is valid, and from (85), we get

\[
\begin{align*}
|\theta_1(x_1 - 0.5h_1, x_2)| &\leq 1, \quad x \in \omega_1^+ \times \omega_2, \\
|\theta_2(x_1, x_2 - 0.5h_2)| &\leq 1, \quad x \in \omega_1 \times \omega_2^+.
\end{align*}
\]

A class of regularized difference schemes for the convection-diffusion equation in the divergent form is defined as

\[
\begin{align*}
C_2y - ((1 + \rho_1(x_1 - 0.5h_1, x_2))k(x_1 - 0.5h_1, x_2)y^{x_1})_{x_1} \\
- ((1 + \rho_2(x_1, x_2 - 0.5h_2))k(x_1, x_2 - 0.5h_2)y^{x_2})_{x_2} \\
= \varphi(x), \quad x \in \omega.
\end{align*}
\]

(97)

Sufficient conditions for the monotonicity of the scheme (97) are written as

\[
\begin{align*}
1 + \rho_1(x_1 - 0.5h_1, x_2) > |\theta_1(x_1 - 0.5h_1, x_2)|, \quad x \in \omega_1^+ \times \omega_2, \\
1 + \rho_2(x_1, x_2 - 0.5h_2) > |\theta_2(x_1, x_2 - 0.5h_2)|, \quad x \in \omega_1 \times \omega_2^+.
\end{align*}
\]

Some approaches to select regularizing grid functions \( \rho_\alpha(x) \), \( \alpha = 1, 2 \) in order to obtain unconditionally monotone difference schemes were considered above for the convection-diffusion equations in the nondivergent form.
3.3. Triangular grids

The possibilities of solving boundary value problems for PDEs on irregular grids are discussed here. The focus is on constructing difference schemes on triangular grids (as the most common unstructured grids). We emphasize approximations on Delaunay grids (triangulations) that demonstrate optimal properties. The problem of discretization in space is illustrated considering steady-state problems.

The basis for the construction of discrete analogs is the balance method (the integro-interpolation method) [26, 27], which nowadays (in the English literature) is referred to as the finite volume method [42, 15]. The efficiency of this approach becomes evident in designing difference schemes on irregular grids. As a control volume in Delaunay triangulations appears Voronoi diagrams.

Among general irregular grids we distinguish structured grids that are topologically equivalent to regular grids. A typical example of unstructured meshes are triangular grids. There are discussed general issues of designing grids and discretization on them.

Numerical solving boundary value problems of mathematical physics in complicated domains is carried out using irregular grids. A computational domain $\Omega$ is assumed to be irregular (nonrectangular and not composed of rectangles). Because of this, we have to use nonrectangular grids. Among irregular grids, we emphasize two main classes.

**Structured grids.** The most important example of such a type of grids is irregular quadrangular grids that inherit, in many senses, properties of standard rectangular grids (they are topologically equivalent to them).

**Unstructured grids.** In this case, a stencil of a difference scheme has no fixed structure. It is impossible to connect topologically such a computational grid to a regular rectangular grid. In particular, schemes have a different number of neighbors at each grid point.

Approximations on structured grids can be performed on the basis of the above-mentioned closeness of these grids to standard rectangular grids. The simplest realization of this situation is to use new independent variables. In this case, a grid that is irregular in the original coordinates is transformed into a regular one in new independent coordinates.

The second possibility is not associated with a formal introduction of new coordinates; it is implemented using an approximation of the original problem on
an irregular grid. It is clear that the use of simple approaches for the construction of difference schemes on the basis of uncertain coefficients for irregular grids is possible, but it seems not so reasonable.

Advantages of structured grids result from the conservation of the canonical structure of neighbors for each grid node, i.e., the conservation of the stencil. This simplifies, in particular, the process of programming and solving difference problems. But the problems of constructing difference schemes on such grids are not much less difficult than for the general unstructured grids.

Among structured grids, it is necessary to distinguish an important class of orthogonal grids. In this case, the advantages of structured grids over unstructured ones become evident because a lot of problems connected with the development of difference schemes, and the solution of grid equations is radically simplified. If it is necessary to use the advantages of structured irregular grids over unstructured ones, it is better to restrict ourselves to orthogonal curvilinear grids. Problems of grid generation are not necessarily more difficult than the problems being solved. Moreover, this situation is the most typical one. Therefore, it is better to make efforts (that are comparable to the solution of the original problem) to optimize the computational grid. In complicated computational domains, it is reasonable to use the multiblock technology of generation of orthogonal grids that is based on modern CAD systems.

An arbitrary grid is generated from a set of nodes. The most simple and natural approach is to define a triangulation, i.e., to construct a triangular grid. There is no need to use more complicated structures of unstructured grids.

For the given points, a triangulation can be performed in different ways. Note also that for a given set of nodes, we obtain the same number of triangles by any triangulation method.

Thus, we need to optimize the triangulation by some criteria. The main optimization criterion consists in the following: the obtained triangles should be close to equilateral ones (they should be without too sharp angles). This is a local criterion governing to an individual triangle. The second (global) criterion declares that adjacent triangles must not differ too widely in an area — the criterion of grid uniformity.

There is a special triangulation — the Delaunay triangulation [1, 2], which has a number of optimal properties. One of them is the tendency of obtained triangles to be equiangular ones. The above-mentioned property can be formulated more exactly in the following way: in the Delaunay triangulation, the minimal value of inner angles of triangles is maximized. The formal definition of the Delaunay triangulation is associated with the property that for each triangle all
the other nodes are located outside of the circumcircle. For our further presentation, the relation between the Delaunay triangulation and the Voronoi diagram (tesselation) is very important.

A Voronoi polygon for a separate node is a set of points lying closer to this node than to all the other nodes. For two points, the sets are defined by the half-plane bounded by a perpendicular to the middle of the segment connecting these two points. The Voronoy polygon thereby will be the intersection of such half-planes for all pairs of nodes created by this node and all the other nodes. Note that this polygon is always convex.

Each vertex of a Voronoi polygon is a point of contact of three Voronoi polygons. The triangle constructed by the corresponding nodes of contacting Voronoi polygons is associated with each of these vertices. This is exactly the Delaunay triangulation. Thus, between the Voronoi diagram and the Delaunay triangulation a unique correspondence is established.

In the case of the Delaunay triangulation, we obtain the optimal decomposition of a computational domain according to the given set of nodes. The decomposition is optimal in terms of maximization of minimal angles of triangles. For the Delaunay triangulation, there does exist the corresponding Voronoi diagram that uniquely determines a set of points of the domain for each node. This separation of the set points is made by the clear geometrical criterion of optimal closeness to the node. Thus, the Delaunay triangulation and the Voronoi diagram determine completely (optimally and uniquely) a computational triangular grid and a control volume.

The Delaunay triangulation is widely used in numerical practice for constructing finite element procedures. There also exist a lot of developed numerical methods for generating such triangular grids, the appropriate software is also available.

3.4. Difference schemes on triangular grids

We start with a discussion of some possible general approaches that may be applied to (and find) practical applications.

The simplest (from the methodological viewpoint) approach to construct discrete analogs on triangular grids consists in using the finite element procedures \[34\]. However, it is not always possible to employ standard variants of finite elements.

In computational practice, there are widely used piecewise linear finite elements, which correspond to the approximation of the numerical solution on each triangle via linear functions. In the convection-diffusion problems, we obtain analogues of schemes with central difference approximation of the convective terms.
In constructing finite element approximations, there do exist some problems to obtain monotone procedures, i.e., the schemes that satisfy the maximum principle. In the theory of finite elements, the problem is resolved not using the Galerkin method, but using its generalization — the projection Petrov-Galerkin method. In this case, probe functions that are used to construct the solution, and test functions that generate a system of equations, are distinct. In this sufficient artificial way, we obtain schemes, which are, e.g., very similar to the schemes with the upwind differences [19].

In the method of support operators [28], the original problem is formulated in terms of differential operators from the vector analysis: the divergence, gradient, and rotor. Next, only one of them is freely approximated on a selected grid. Other operators are defined by some prescribed relations of integral nature that exist between the differential operators. This ensures consistent approximations of the operators that provide the fulfillment of such essential properties as conservatism, adjointness and so on.

The method of support operators has been developed by many researchers just for triangular grids. The main peculiarities are associated with a selection of the set of grid functions. For example, the solution can be approximated at vortices of the triangular grid, whereas fluxes can be referred to cells (the cell-vortex arrangement of variables) or to the the midpoints of cell faces (staggered grids).

One of the basic approaches to the construction of difference schemes on irregular grids is the classical integro-interpolation method [27]. This balance method is based on the following main points. First, we must specify a grid (determine a set of nodes and a set of grid functions). Secondly, for each node, we define a neighbor domain (control volume) — a part of the computational domain adjoining to a given computational node. And finally, a difference scheme is obtained by integrating the original equation over the control volume using some assumptions about the solution behavior. A set of these three components specify a particular variant of the control volume method.

In constructing difference schemes on triangular grid via the control volume method, it seems natural to set grid functions at the grid nodes. This is the standard, but not the only variant. As an alternative, we can refer grid functions to some points being connected with a triangular cell. The specification of grid functions at the vertices of the Voronoi polygons provide an example.

The second problem is connected with selecting a control volume. During a triangulation procedure, in many approaches, a part of the triangle appearing from the intersection of medians is separated as as a control volume. In this case, each
node obtains a part of the triangle with an equal area.

An interesting variant of the control volume selection is associated with the Voronoi diagrams. In this case, each individual node has a part of the whole computational domain that is closest to it. In this case, there is no division into triangles with equal areas. It is important that in the both above approaches, a criterion for selecting a control volume is clear and is connected with geometric requirements: in the first case — the triangle is decomposed into three parts of equal area, in the second — we obtain geometric proximity of points of the computational domain.

Among the merits of the division by medians, we emphasize that it may be conducted for an arbitrary partitioning into triangles, i.e., not only for the Delaunay triangulation. Advantages of the Voronoi tessellation are more essential and associated with the orthogonality of the triangle sides to the faces of Voronoi polygons.

Heuristic arguments in favor of Voronoi polygons are associated with the idea of globalization (optimization) of grids and control volumes — optimization for all nodes, rather than for a single triangle.

The balance method for triangular grids need to be implemented using the Delaunay triangulation with Voronoi polygons as control volumes. This is the most natural way that allows to construct difference schemes with optimized triangular grids [36].

3.5. Diffusive transport operator

Assume that a computational domain is a convex polygon $\Omega$ with the boundary $\partial \Omega$. The points of the domain are denoted by $x = (x^{(1)}, x^{(2)})$.

In the domain $\overline{\Omega} = \Omega \cup \partial \Omega$, we consider the grid $\bar{\omega}$, which consists of nodes $x_i$, $i = 1, 2, \ldots, M$, and the angles of the polyhedron $\Omega$ are nodes. Let $\omega$ be a set of interior nodes and $\partial \omega$ is a set of boundary nodes, i.e., $\omega = \bar{\omega} \cap \Omega$, $\partial \omega = \bar{\omega} \cap \partial \Omega$.

Each node $x_i$, $i = 1, 2, \ldots, M$, is associated with a certain part of the computational domain $\Omega_i$ treated as a control volume. A Voronoi polygon or its part belonging to $\Omega$ are selected as the control volumes. A Voronoi polygon for an individual node is a set of points lying closer to this node than to all the other ones. For two nodes, the sets are defined by the half-plane bounded by the perpendicular to the midpoint of the segment connecting these two nodes. The Voronoi polygons thereby will be the intersection of such half-planes for all pairs of nodes created by this node and all the other nodes. Each vertex of a Voronoi polygon is a point of contact of three Voronoi polygons. The triangle constructed by the
corresponding nodes of contacting Voronoi polygons is associated with each of these vertices. This is exactly the Delaunay triangulation.

Control volumes cover the whole computational domain, so that

$$\overline{\Omega} = \bigcup_{i=1}^{M} \Omega_i, \quad \overline{\Omega}_i = \Omega_i \bigcup \partial \Omega_i, \quad \Omega_i \cap \Omega_j = \emptyset, \quad i \neq j, \quad i, j = 1, 2, \ldots, M.$$ 

For the common faces of control volumes, we use notation

$$\partial \Omega_i \cap \partial \Omega_j = \Gamma_{ij}, \quad i \neq j, \quad i, j = 1, 2, \ldots, M.$$ 

For the node $i$, we define a set of neighboring nodes $W(i)$ that have the control volumes with common faces with the control volume for the node $i$, i.e.,

$$W(i) = \{ j \mid \partial \Omega_i \cap \partial \Omega_j \neq \emptyset, \ j = 1, 2, \ldots, M \}, \quad i = 1, 2, \ldots, M.$$ 

Introduce notation

$$V_i = \int_{\Omega_i} d\mathbf{x}, \quad l_{ij} = \int_{\Gamma_{ij}} d\mathbf{x}, \quad i, j = 1, 2, \ldots, M,$$

for the area of the control volume and the length of the edge of the Voronoi polyhedron, respectively.

For the grid functions $y(x)$, $w(x)$ that are specified at the nodes $x \in \bar{\omega}$ and vanish at the boundary nodes $x \in \partial \omega$, in $H = L_2(\omega)$, we define the scalar product and norm

$$(y, w) = \sum_{x_i \in \omega} V_i y(x_i) w(x_i), \quad \|y\| = (y, y)^{1/2}.$$ 

Define the distance between the nodes $x_i, x_j$ as

$$d(x_i, x_j) = \left[ \sum_{\alpha=1}^{2} (x_i^{(\alpha)} - x_j^{(\alpha)}) \right]^{1/2},$$

and the midpoint of the segment connecting these nodes as follows:

$$x_{ij} = (x_{ij}^{(1)}, x_{ij}^{(2)}), \quad x_{ij}^{(\alpha)} = \frac{1}{2}(x_i^{(\alpha)} + x_j^{(\alpha)}), \quad \alpha = 1, 2.$$
For simplicity, we assume that the coefficients of the convection-diffusion equation and its solution itself are sufficiently smooth. The discrete operator of diffusive transport corresponding to the interior node of the computational grid $x_i \in \omega$ is defined according to the integro-interpolation method by means of the integration over the control volume $\Omega_i$:

$$Du = (Du)_i \approx \frac{1}{V_i} \int_{\Omega_i} Duddx. \quad (98)$$

For the diffusive flux vector, we use the expression

$$q = -k(x) \text{grad } u,$$

so that

$$Du = \text{div } q.$$

From this, we obtain for the right-hand side of (98) that

$$\int_{\Omega_i} Duddx = \int_{\partial\Omega_i} (q, n)d\mathbf{x}, \quad (99)$$

where $n$ is the outer normal.

A difference approximation for the normal component of the diffusive flow through the face $\gamma_{ij}$ is denoted by $q_{ij}^h$, and therefore, from (98), (99), we get for the difference diffusive transport operator the following representation:

$$(Dy)_i = \frac{1}{V_i} \sum_{j \in \mathcal{W}(i)} l_{ij} q_{ij}^h, \quad x_i \in \omega. \quad (100)$$

In the case of smooth enough coefficients and the solution itself, it is natural to employ elementary approximations for the flux along the normal at the point $x_{ij}$:

$$q_{ij}^h = -k(x_{ij}) \frac{y^j - y^i}{d(x_i, x_j)}. \quad (101)$$

From (100), (101), we obtain the difference operator of the diffusive transport:

$$(Dy)_i = -\frac{1}{V_i} \sum_{j \in \mathcal{W}(i)} l_{ij} k(x_{ij}) \frac{y^j - y^i}{d(x_i, x_j)}, \quad x_i \in \omega. \quad (102)$$
As in the case of difference schemes on regular grids, it is possible to introduce various approximations for the diffusive flows. This issue is very important, in particular, for problems with piecewise-smooth coefficients of the equation.

For the grid functions \( y_i \equiv y(x_i) = 0, \ x_i \in \partial \omega, \) we have

\[
(Dy, w) = \sum_{x_i \in \omega} \sum_{j \in W(i)} l_{ij} h_{ij} w_i = -\sum_{x_i \in \omega} \sum_{j \in W(i)} l_{ij} k(x_{ij}) \frac{y^j - y^i}{d(x_i, x_j)} w_i.
\]

This summation over all faces of the Voronoi polygons for all interior nodes may be rewritten in the more convenient form:

\[
(Dy, w) = \frac{1}{2} \sum_{x_i \in \omega} \sum_{j \in W(i)} l_{ij} d(x_i, x_j) k(x_{ij}) ((y_j - y_i) w_j - (y^j - y^i) w_i)
\]

\[
= \frac{1}{2} \sum_{x_i \in \omega} \sum_{j \in W(i)} l_{ij} d(x_i, x_j) k(x_{ij}) \frac{y_j - y_i}{d(x_i, x_j)} \frac{z_j - z_i}{d(x_i, x_j)}
\]

Thus, \((Dy, w) = (y, D^* w),\) i.e., the difference operator \((102)\) is self-adjoint in \(H.\) In view of

\[
(Dy, y) = \frac{1}{2} \sum_{x_i \in \omega} \sum_{j \in W(i)} l_{ij} d(x_i, x_j) k(x_{ij}) \left( \frac{y_j - y_i}{d(x_i, x_j)} \right)^2,
\]

it is also positive \((D = D^* > 0).\)

Now we establish a discrete analog of the Friedrichs lemma. In various formulations, it is proved in the theory of finite elements for considering difference schemes on irregular structured and unstructured grids. Difference schemes that are based on the Delaunay triangulation and the Voronoi diagram demonstrate some peculiarities, and therefore the discrete analog of the Friedrichs lemma must be proved for them separately.

**Lemma 4.** For the grid functions \( y_i \equiv y(x_i) = 0, \ x_i \in \partial \omega, \) the following inequality is valid:

\[
\|y\|^2 \leq \frac{M_0}{2} \sum_{x_i \in \omega} \sum_{j \in W(i)} l_{ij} d(x_i, x_j) \left( \frac{y_j - y_i}{d(x_i, x_j)} \right)^2
\]

(104)

with the constant

\[
M_0 = \frac{l_1^2}{16} + \frac{l_2^2}{16},
\]

where \(l_\alpha, \ \alpha = 1, 2\) are the side lengths of the rectangle with sides parallel to the coordinate axes that contains completely the whole polygon \(\Omega.\)
Proof. On the set of the grid functions \( y_i \equiv y(x_i) = 0, x_i \in \partial \omega, \) in view of (102), we define the discrete Laplace operator \( \Lambda \) via the expression

\[
(\Lambda y)_i = -\frac{1}{V_i} \sum_{j \in W(i)} l_{ij} \frac{y_j - y_i}{d(x_i, x_j)}, \quad x_i \in \omega.
\]  

(105)

Using this notation, the inequality (104) may be rewritten in the more compact form:

\[
\|y\|^2 \leq M_0(\Lambda y, y).
\]  

(106)

To estimate the lower bound of the discrete Laplace operator, we employ the solution of an auxiliary boundary value problem. We will show that in the inequality (106) under consideration, we can put

\[
M_0 = \max_{x \in \omega} w(x),
\]  

(107)

where \( w(x) \) is the solution of the problem

\[
\Lambda w = 1, \quad x \in \omega.
\]  

(108)

We consider in \( H \) the eigenvalue problem

\[
\Lambda y = \lambda y, \quad x \in \omega
\]  

(109)

for the grid operator \( \Lambda = \Lambda^* > 0 \). For the problem (109), the following inequality holds:

\[
(\Lambda y, y) \geq \lambda_{\min} \|y\|^2
\]  

(110)

for any \( y(x) \). The equality in (110) is achieved only for the eigenfunctions \( v(x) \) that correspond to the minimal eigenvalue \( \lambda_{\min} \). Thus, the estimate (106) will be established if we will show that \( M_0^{-1} \leq \lambda_{\min} \).

First of all, let us explain that \( v(x) \) is a constant-sign function. Suppose that this is not true and \( v(x) \) changes its sign on the grid \( \omega \). Now consider the function \( |v(x)| \). Then taking into account

\[
(\Lambda y, y) = \frac{1}{2} \sum_{x_i \in \omega} \sum_{j \in W(i)} l_{ij} d(x_i, x_j) \left( \frac{y_j - y_i}{d(x_i, x_j)} \right)^2,
\]  

we obtain

\[
\frac{(\Lambda |v(x)|, |v(x)|)}{(|v(x)|, |v(x)|)} < \frac{(\Lambda v(x), v(x))}{(v(x), v(x))}.
\]  

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This contradicts the fact that the minimal Rayleigh ratio

\[(\Lambda y(x), y(x)) (y(x), y(x))^{-1}\]

is achieved for \(y(x) = v(x)\).

From \(\Lambda v = \lambda_{\text{min}} v\), we have

\[\lambda_{\text{min}} = \frac{(\Lambda v(x), 1)}{(v(x), 1)}.\]  \hspace{1cm} (111)

In view of (108), we obtain for the denominator

\[(v(x), 1) = (v(x), \Lambda w(x)) = (w(x), \Lambda v(x)) \leq \max_{x \in \omega} w(x) (\Lambda v(x), 1),\]

since \(\Lambda v(x) \geq 0, \ x \in \omega\). Thus, from (111), it follows immediately that \(M_0^{-1} \leq \lambda_{\text{min}}\), and for the constant \(M_0\), we can use the expression (107).

Next, we apply the maximum principle to discrete elliptic equations. We place the polygon \(\Omega\) into the rectangle

\[\Omega_0 = \{x = (x^{(1)}, x^{(2)}) \mid a_\alpha \leq x^{(\alpha)} \leq b_\alpha, \ \alpha = 1, 2\},\]

where \(l_\alpha = b_\alpha - a_\alpha, \ \alpha = 1, 2\). Consider the function

\[W(x) = -\mu((x^{(1)} - a_1)(x^{(1)} - b_1) + (x^{(2)} - a_2)(x^{(2)} - b_2))\]

with some positive constant \(\mu\). We write this majorant function as

\[W(x) = -\mu((x^{(1)} - x^{(1)}_i)^2 + (x^{(2)} - x^{(2)}_i)^2) + g_i(x),\]

where \(g_i(x)\) is a linear function.

For linear functions, we have

\[\Lambda g(x) = 0.\]

To show this, it is sufficient to consider the function \(g(x) = x^{(1)}\). By virtue of (105), we obtain

\[\Lambda x^{(1)} = -\frac{1}{V_i} \sum_{j \in W(i)} l_{ij} \frac{x_j^{(1)} - x_i^{(1)}}{d(x_i, x_j)}.\]

We have

\[
\frac{x_j^{(1)} - x_i^{(1)}}{d(x_i, x_j)} = \cos(\varphi_{ji}),
\]
where $\varphi_{ji}$ is the angle between the segment $[x_i, x_j]$ and the axis $Ox^{(1)}$. In this case, $l_{ij}\cos(\varphi_{ji})$ is the projection of the $j$-th face of the Voronoi polygon $\Omega_i$ on the axis $Ox^{(1)}$. For a closed polygon
\[
\Lambda x^{(1)} = \frac{1}{V_i} \sum_{j \in W(i)} l_{ij} \cos(\varphi_{ji}) = 0.
\]
Taking this into account, for the function $W(x)$, we obtain directly
\[
(\Lambda W(x))_i = \mu(\Lambda d^2(x, x_i))_i = \frac{\mu}{V_i} \sum_{j \in W(i)} l_{ij} d(x_i, x_j) = 4.
\]
That is why for $\mu = 0.25$, we have
\[
\Lambda W(x) = 1, \quad x \in \omega.
\]
At the boundary nodes
\[
W(x) \geq 0, \quad x \in \partial \omega,
\]
and therefore the function $W(x)$ is a majorant for the problem (108). For the constant $M_0$ in the inequality (104), the following estimate holds:
\[
M_0 \leq \max_{x \in \omega} W(x) = \frac{l_1^2}{16} + \frac{l_2^2}{16}.
\]
This completes the proof of the lemma. \qed

This makes possible to formulate the main result concerning the properties of the difference operator of diffusive transport.

**Theorem 6.** For the grid operator of diffusive transport determined from (44), the inequality
\[
D = D^* \geq \frac{\kappa}{M_0} E
\]
is valid for the grid functions from the space $H = L_2(\omega)$.

It is important that the constant $M_0$ in the Friedrichs inequality (104) is independent of nodes of the computational domain, and the estimate itself is quite similar to the estimate for the differential operator.

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3.6. Convective transport operators

In the construction of difference operators of convective transport, we start with the operator in the divergent form (44). Assume that

\[ C_2 u = (C_2 u)_i \approx \frac{1}{V_i} \int_{\Omega_i} C_2 u d\mathbf{x} \]  

(113)

for the interior nodes of the grid. For the right-hand side, we have

\[ \int_{\Omega_i} C_2 u d\mathbf{x} = \int_{\partial \Omega_i} (v, n) u d\mathbf{x}. \]

Similarly to the case of the diffusive transport approximation, the normal component of the velocity is referred to the midpoint of the segment connecting grid nodes. Introducing notation

\[ b_{ij} = (v, n)(x_{ij}), \]

from (113), the difference operator of convective transport is written as

\[ (C_2 y)_i = \frac{1}{V_i} \sum_{j \in W(i)} l_{ij} b_{ij} y_j + y_i \frac{1}{2}, \quad x_i \in \omega. \]  

(114)

Now we discuss approximations of the convective transport operator in the nondivergent form (43). There do exist several opportunities. The first one is connected with the use of a sufficiently complicated (not so evident) structure for the difference convective transport operator in the nondivergent form based on special formulas of approximate integration. But we have another way. Using the idea of the method of support operators, first, we design a simple approximation of the convective transport operator in the divergent form (114). To obtain a difference approximation of the operator (43), we search a difference operator that is adjoint to (114). In doing so, we get a difference operator of the convective transport in the nondivergent form. Such an opportunity is essential for developing approximations on irregular grids.

Straightforward calculations yield

\[ (C_2 y, w) = \frac{1}{2} \sum_{x_i \in \omega} \sum_{j \in W(i)} l_{ij} b_{ij} y_i w_i + \frac{1}{2} \sum_{x_i \in \omega} \sum_{j \in W(i)} l_{ij} b_{ij} y_j w_i \]

\[ = \frac{1}{2} \sum_{x_i \in \omega} \sum_{j \in W(i)} l_{ij} b_{ij} y_i w_i - \frac{1}{2} \sum_{x_i \in \omega} \sum_{j \in W(i)} l_{ij} b_{ij} y_i w_j \]

\[ = -(y, C_1 w), \]
where we take into account that $b_{ij} = -b_{ji}$. This allows to define the difference operator
\[
(C_1 y)_i = \frac{1}{V_i} \sum_{j \in W(i)} l_{ij} b_{ij} \frac{y_j - y_i}{2}, \quad x_i \in \omega.
\] (115)

By construction, the difference operators of convective transport in the divergent and nondivergent forms determined in accordance with (114), (115) are adjoint to each other with within the sign, i.e.,
\[
C_1^* = -C_2.
\] (116)

As for the operator of convective transport in the skew-symmetric form (45), using the representation
\[
C_0 = \frac{1}{2} (C_1 + C_2),
\]
from (114) and (115), we get the most compact approximation
\[
(C_0 y)_i = \frac{1}{2V_i} \sum_{j \in W(i)} l_{ij} b_{ij} y_j, \quad x_i \in \omega.
\] (117)

The primary feature of this difference operator is
\[
C_0^* = -C_0,
\] (118)
and moreover, this skew-symmetric property is true for any velocity field — it is valid for arbitrary vectors $\mathbf{v}$, not necessarily satisfying some difference analogue of the incompressibility constraint
\[
\text{div} \, \mathbf{v} \equiv \sum_{\alpha=1}^{2} \frac{\partial v_\alpha}{\partial x^{(\alpha)}} = 0, \quad x \in \Omega.
\] (119)

To study difference analogs of the boundedness of the convective transport operator and its subordination to the diffusive transport operator, we discuss the important features of the approximations (114) and (115) in detail.

For numerical solving continuum mechanics problems, it is essential to have consistent approximations of the convective transport operator in the divergent and nondivergent forms. The consistency is treated in the sense that one difference operator coincides with other operator if the corresponding difference incompressibility constraint holds. This issue is very important due to the fact...
that this equivalence takes place for the differential equations, and just it ensures the fulfillment of several conservation laws. In particular, for the incompressible Navier-Stokes equations, the convective transport operator in the momentum equation makes no contribution neither to the kinetic energy, nor to the individual momentum components (we speak of energy neutrality and neutrality with respect to the momentum). Unfortunately, elementary approximations ensure only one of these properties — either for the kinetic energy or for the momentum.

For an incompressible fluid (the constraint \((119)\)), at the differential level, we can use any of the above-mentioned three equivalent forms of the convective terms \((43)–(45)\). It follows from the formula of vector analysis:

\[
\text{div}(\mathbf{v}\mathbf{u}) = (\mathbf{v} \cdot \text{grad}\mathbf{u}) + \mathbf{u} \text{div}\mathbf{v},
\]

which, in turn, is based on the differentiation formula of the product of two functions. For the operators of convective transport, we have

\[
C_2 \mathbf{u} = C_1 \mathbf{u} + \text{div} \mathbf{v} \mathbf{u}.
\]  

(120)

In constructing difference approximations for the convective transport operator, it seems natural to develop such difference operators that satisfies the property \((120)\) of differential operators, i.e., the property of equivalence of various difference approximations.

From \((114), (115)\), we get

\[
C_2 y = C_1 y + \frac{1}{V_i} \sum_{j \in \mathcal{W}(i)} l_{ij} b_{ij} y_i, \quad \mathbf{x}_i \in \omega.
\]

Transform this equality to the form that is similar to \((120)\):

\[
C_2 y = C_1 y + \text{div}_h \mathbf{v} y,
\]  

(121)

where the difference operator of divergence is

\[
\text{div}_h \mathbf{v} = \frac{1}{V_i} \sum_{j \in \mathcal{W}(i)} l_{ij} b_{ij}, \quad \mathbf{x}_i \in \omega.
\]  

(122)

An approximation of the divergence for a vector by means of \((122)\) is obtained using the definition

\[
\text{div} \mathbf{v} = \lim_{\delta \to 0} \frac{\int_{\partial V} \mathbf{f} dS}{\int_{V} dV},
\]
where $V$ denotes the area, $\partial V$ stands for the boundary of the domain, and $\delta$ is the domain diameter. The expression (122) may be treated as the corresponding quadrature formula for the right-hand side in the integration over the control volume for the node $x_i \in \omega$.

In view of (121), we write

$$C_1 y = C_0 y - \frac{1}{2} \text{div}_h \mathbf{v} \cdot y,$$

$$C_2 y = C_0 y + \frac{1}{2} \text{div}_h \mathbf{v} \cdot y.$$  \hspace{1cm} (123)

In view of (118), from (123), it follows immediately that

$$|(C_\alpha y, y)| \leq M_1 \|y\|^2, \quad \alpha = 1, 2,$$  \hspace{1cm} (124)

with a constant $M_1$ that depends only on the compressibility of a medium (at the discrete level), i.e.,

$$M_1 = \frac{1}{2} \| \text{div}_h \mathbf{v} \|_\infty.$$  \hspace{1cm} (125)

Here we use notation

$$\|y\|_\infty = \max_{x_i \in \omega} |y(x_i)|$$

for the norm in $L_\infty(\omega)$. To prove (124), multiply equation (123) scalarly by $y$ and apply the skew-symmetric property of the operator $C_0$.

To obtain a difference analogue for the inequality representing the subordination of the difference convective transport operator to the difference diffusive transport operator, we start with the upper bound for the expression

$$\|C_1 y\|^2 = \sum_{x_i \in \omega} \frac{1}{V_i} \left( \sum_{j \in W(i)} l_{ij} b_{ij} \frac{y_j - y_i}{2} \right)^2.$$  \hspace{1cm} (126)

For the right-hand side, we have

$$\sum_{x_i \in \omega} \frac{1}{V_i} \left( \sum_{j \in W(i)} l_{ij} b_{ij} \frac{y_j - y_i}{2} \right)^2 \leq$$

$$\leq \max_{x_i \in \omega} \max_{j \in W(i)} |b_{ij}|^2 \sum_{x_i \in \omega} \frac{1}{4V_i} \left( \sum_{j \in W(i)} l_{ij} d(x_i, x_j) \frac{y_j - y_i}{d(x_i, x_j)} \right)^2.$$
Taking into account the inequality

\[
\left( \sum_{j \in W(i)} \xi_j \theta_j \right)^2 \leq \sum_{j \in W(i)} \xi_j^2 \sum_{j \in W(i)} \theta_j^2,
\]

and assuming

\[
\xi_j = (l_{ij} d(x_i, x_j))^{1/2}, \quad \theta_j = (l_{ij} d(x_i, x_j))^{1/2} \frac{y_j - y_i}{d(x_i, x_j)},
\]

we obtain

\[
\sum_{x_i \in \omega} \frac{1}{4V_i} \left( \sum_{j \in W(i)} l_{ij} d(x_i, x_j) \frac{y_j - y_i}{d(x_i, x_j)} \right)^2 \leq \sum_{x_i \in \omega} \frac{1}{4V_i} \sum_{j \in W(i)} l_{ij} d(x_i, x_j) \sum_{j \in W(i)} l_{ij} d(x_i, x_j) \left( \frac{y_j - y_i}{d(x_i, x_j)} \right)^2.
\]

In view of

\[
\sum_{j \in W(i)} l_{ij} d(x_i, x_j) = 4V_i,
\]

substitution into (126) yields

\[
\|C_1 y\|^2 \leq \max_{x_i \in \omega} \max_{j \in W(i)} |b_{ij}|^2 \sum_{x_i \in \omega} \frac{1}{V_i} \sum_{j \in W(i)} l_{ij} d(x_i, x_j) \left( \frac{y_j - y_i}{d(x_i, x_j)} \right)^2. \tag{127}
\]

Comparing (127) with (103), we obtain

\[
\|C y\|^2 \leq M_2(Dy, y), \quad \tag{128}
\]

where \( C = C_1 \) and the constant

\[
M_2 = \frac{2}{\kappa_1} \max_{x_i \in \omega} \max_{j \in W(i)} |b_{ij}|^2,
\]

with \( k(x) \geq \kappa_1 > 0 \).

For the difference convective transport operator in the divergent form (114), we use the difference analogue of the Friedrichs inequality in the form (104). By the representation (121), we obtain immediately:

\[
\|C_2 y\|^2 = \|C_1 y + \text{div}_h \nu y\|^2 \leq 2\|C_1 y\|^2 + 2\|\text{div}_h \nu y\|^2.
\]

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From (103) and (104), for the last term in the right-hand side, we have
\[
\| \text{div}_h v y \|_2 \leq \| \text{div}_h v \|_2^2 \frac{M_0}{2} \sum_{x_i \in \omega} \sum_{j \in W(i)} l_{ij} d(x_i, x_j) \left( \frac{y_i - y_j}{d(x_i, x_j)} \right)^2.
\]

Taking into account (103) (with \(C = C_1\)), we arrive at the estimate (128) for the operator \(C = C_2\), where
\[
M_2 = \frac{2}{\kappa_1} \left( 2 \max_{x_i \in \omega} \max_{j \in W(i)} |b_{ij}|^2 + M_0 \| \text{div}_h v \|_\infty^2 \right).
\]

For the difference operator of convective transport in the skew-symmetric form (117), in a similar way, we establish the inequality (128) with the constant
\[
M_2 = \frac{1}{\kappa_1} \left( 3 \max_{x_i \in \omega} \max_{j \in W(i)} |b_{ij}|^2 + M_0 \| \text{div}_h v \|_\infty^2 \right).
\]

**Theorem 7.** For the grid operators of convective transport defined in accordance with (114), (115) and (117), in the space of grid functions \(H = L^2(\omega)\), the properties (116) and (118) are valid along with the estimate of the operator energy boundedness (124), and the estimate of subordination (128) to the difference operator of diffusive transport (102) hold.

The above estimates (124) and (128) for difference operators of convective transport are fully consistent with the continuous case. They serve us as the basis for the study of difference convection-diffusion problems.

### 3.7. Monotone approximations on triangular grids

Monotone approximations on triangular grids are constructed [40] similarly to other grids. We separate the positive and negative parts of the normal velocity component putting
\[
b_{ij} = b_{ij}^+ + b_{ij}^-,
\]
where
\[
b_{ij}^+ = \frac{1}{2}(b_{ij} + |b_{ij}|),
\]
\[
b_{ij}^- = \frac{1}{2}(b_{ij} - |b_{ij}|).
\]

To approximate the right-hand side of (113), we will use the value of the grid function either in the central or in the peripheral node depending on the sign of
the velocity. This leads us to the difference operator of convective transport in the form

\[(C_2 y)_i = \frac{1}{V_i} \sum_{j \in W(i)} l_{ij} (b_{ij}^- y_j + b_{ij}^+ y_i), \quad \mathbf{x}_i \in \omega. \quad (129)\]

If we apply the difference divergence operator, then, for the difference analog of (43), we have the expression

\[(C_1 y)_i = \frac{1}{V_i} \sum_{j \in W(i)} l_{ij} b_{ij}^- (y_j - y_i), \quad \mathbf{x}_i \in \omega. \quad (130)\]

Thus, we have developed the approximations (129) and (130) for the convective transport operators in the divergent (44) and nondivergent (43) forms using the upwind differences.

For the boundary value problems (88), (89) and (89), (94), we put into the correspondence the difference problems

\[Cy + Dy = \varphi(\mathbf{x}), \quad \mathbf{x} \in \omega,\]

(131)

for the grid functions \(y(\mathbf{x}) = 0, \ \mathbf{x} \in \partial \omega\). For the right-hand side of (131), suppose, e.g.,

\[\varphi(\mathbf{x}) = \frac{1}{V_i} \int_{\Omega_i} f(\mathbf{x}) \, d\mathbf{x}, \quad \mathbf{x} \in \omega.\]

To employ the fulfillment of the maximum principle at the discrete level, we rewrite the difference problem (131) in the form

\[\alpha_i y_i - \sum_{j \in W(i)} \beta_{ij} y_j = \phi_i, \quad \mathbf{x}_i \in \omega, \quad (132)\]

\[y_i = 0, \quad \mathbf{x} \in \partial \omega. \quad (133)\]

Assume that \(\omega\) is a connected grid.

For the difference problem (132), (133), the maximum principle is valid, i.e., the difference scheme is monotone \cite{27} under the following restrictions:

\[\alpha_i > 0, \quad \beta_{ij} > 0, \quad j \in W(i), \quad (134)\]

\[\delta_i \equiv \alpha_i - \sum_{j \in W(i)} \beta_{ij} \geq 0, \quad \mathbf{x}_i \in \omega. \quad (135)\]
The difference scheme (131) for the problem (88), (89) based on the approximations (102) and (130) may be written in the form (132), (133) with the coefficients

$$\alpha_i = -\frac{1}{V_i} \sum_{j \in W(i)} l_{ij} b_{ij}^- + \frac{1}{V_i} \sum_{j \in W(i)} l_{ij} k(x_{ij}) \frac{1}{d(x_i, x_j)},$$

$$\beta_{ij} = -\frac{1}{V_i} l_{ij} b_{ij}^- + \frac{1}{V_i} k(x_{ij}) \frac{l_{ij}}{d(x_i, x_j)}, \quad j \in W(i),$$

$$\delta_i = 0, \quad x_i \in \omega.$$

The monotonicity conditions (134), (135) are unconditionally valid.

Now consider the scheme (131) with the difference operator of convective transport $C = C_1$ defined according to (115). The scheme (102), (115), (131) may be represented in the canonical form (132), (133) with

$$\alpha_i = -\frac{1}{2V_i} \sum_{j \in W(i)} l_{ij} b_{ij}^- + \frac{1}{V_i} \sum_{j \in W(i)} l_{ij} k(x_{ij}) \frac{1}{d(x_i, x_j)},$$

$$\beta_{ij} = -\frac{1}{2V_i} l_{ij} b_{ij}^- + \frac{1}{V_i} k(x_{ij}) \frac{l_{ij}}{d(x_i, x_j)}, \quad j \in W(i),$$

$$\delta_i = 0, \quad x_i \in \omega.$$

Define a local grid Peclet number as follows:

$$\text{Pe}_{ij} = \frac{|b_{ij}| d(x_i, x_j)}{k(x_{ij})} \quad j \in W(i), \quad x_i \in \omega.$$

The monotonicity condition (134) leads to the restrictions

$$\text{Pe}_{ij} < 2, \quad j \in W(i), \quad x_i \in \omega. \quad (136)$$

Such restrictions are typical if we apply the standard central-difference approximations on regular grids.

For the convection-diffusion equation with the divergent convective terms (89), (94), the use of the upwind approximations (102) and (129) yields

$$\alpha_i = \frac{1}{V_i} \sum_{j \in W(i)} l_{ij} b_{ij}^- + \frac{1}{V_i} \sum_{j \in W(i)} l_{ij} k(x_{ij}) \frac{1}{d(x_i, x_j)},$$

$$\beta_{ij} = -\frac{1}{V_i} l_{ij} b_{ij}^- + \frac{1}{V_i} k(x_{ij}) \frac{l_{ij}}{d(x_i, x_j)}, \quad j \in W(i),$$

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Thus, the standard monotonicity conditions (134), (135) are valid only if \( \text{div}_h \mathbf{v} \geq 0 \).

A similar situation occurs in the consideration of difference schemes on rectangular grids. In this case, the unconditional fulfillment of the maximum principle for schemes with the upwind differences designed for the difference equation (131) may be associated with diagonal dominance by columns rather than by rows (as the conditions (134), (135)). The second possibility, which seems more promising for schemes on unstructured grids, involves the establishment of the maximum principle in the standard formulation for the conjugate problem.

Consider the operator that is adjoint to \( C_2 \) and is defined according to (129). Taking into account that \( l_{ij} = l_{ji} \), \( b^+_{ij} = -b^-_{ji} \), we obtain

\[
(C_2 y, v) = \sum_{x_i \in \omega} \sum_{j \in W(i)} l_{ij} (b^-_{ij} y_j + b^+_{ij} y_i) v_i = \sum_{x_i \in \omega} \sum_{j \in W(i)} l_{ij} y_i b^+_{ij} (v_i - v_j).
\]

Therefore

\[
C_2^* v = \frac{1}{V_i} \sum_{j \in W(i)} l_{ij} b^+_{ij} (v_i - v_j), \quad x_i \in \omega.
\] (137)

As for the adjoint problem

\[
C_2^* v + D v = \varphi(x), \quad x \in \omega,
\] (138)

the unconditional fulfillment of the maximum principle written in the standard formulation is established in a usual fashion. Recall that we speak of the formulation for the maximum principle in the following form — if the conditions (134), (135) are true, then the solution of the problem (132), (133) is non-negative (nonpositive) for the non-negative (nonpositive) right-hand side (132).

Now we show that from the fulfillment of the maximum principle for the adjoint problem, it follows that the maximum principle is satisfied for the original problem. For each \( x_i \in \omega \), we define the grid function

\[
\delta_h(x - x_i) = \begin{cases} 
1 & x = x_i, \\
\frac{1}{V_i} & x \neq x_i.
\end{cases}
\]

Suppose that the grid function \( G(x, x_i) \) for the given \( x_i \in \omega \) is the solution of the problem

\[
C_2^* G + D G = \delta_h(x - x_i), \quad x \in \omega.
\] (139)
Due to the fact that the maximum principle holds for the adjoint difference problem (129), we have \( G(x, x_i) \geq 0 \).

Multiply equation (139) scalarly by the solution of the original boundary value problem

\[ C_2 y + D y = \varphi(x), \quad x \in \omega. \]

By virtue of (139), the solution is represented as

\[ y(x_i) = (G(x, x_i), \varphi(x)). \]

Thus, we get \( y(x_i) \geq 0, \ x \in \omega \). Therefore, the maximum principle holds also for the original problem (102), (129), (131).

The study of the difference scheme (102), (114), (131) is conducted in a similar way. The monotonicity of the above difference scheme (102), (114), (131) is established under the restrictions (136). Our investigation results in the following statement.

**Theorem 8.** The upwind difference schemes (102), (130), (131) and (102), (129), (131) for the convection-diffusion equations (102), (130), (131) and (102), (129), (131) are unconditionally monotone, whereas the schemes (102), (115), (131) and (102), (114), (131) satisfy the maximum principle under the restrictions (136).

The above approximations for elliptic operators of convection-diffusion are used for discretization in space on irregular grids for numerical solving time-dependent problems.

4. Discretization in time

Discretization in space results in the Cauchy problem for systems of ODEs treated as an operator-differential equation in the appropriate spaces. Two- or three-level difference schemes are used for numerical solving these equations. This part of the work discusses issues of constructing unconditionally stable schemes for the approximate solution of unsteady convection-diffusion problems. The investigation is based on the general theory of stability (well-posedness) for operator-difference schemes.

4.1. Two-level operator-difference schemes

We start with the key concepts of the stability theory for operator-difference schemes considered in finite-dimensional Hilbert spaces. Next, for two-level difference schemes, we formulate criteria of stability with respect to the initial data.
And finally, typical estimates for stability with respect to the initial data and the right-hand side are presented.

For simplicity, we define a uniform grid in time as follows:

\[ \bar{\omega}_\tau = \omega_\tau \cup \{ T \} = \{ t^n = n\tau, \quad n = 0, 1, \ldots, N_0, \quad \tau N_0 = T \}. \]

Denote by \( A, B : H \to H \) linear operators in \( H \) depending, in general, on \( \tau, t^n \).

Consider the Cauchy problem for an operator-difference equation

\[
B(t^n)\frac{y^{n+1} - y^n}{\tau} + A(t^n)y^n = \varphi^n, \quad t^n \in \omega_\tau, 
\]

\[ y^0 = u^0, \]  

(140)

where \( y^n = y(t^n) \in H \) is a desired function and \( \varphi^n, u^0 \in H \) are given. We use the index-free notation of the theory of difference schemes:

\[
y = y^n, \quad \dot{y} = y^{n+1}, \quad \ddot{y} = y^{n-1}, 
\]

\[
y = \frac{y - \dot{y}}{\tau}, \quad y_t = \frac{\dot{y} - y}{\tau}. 
\]

Then equation (140) may be written as

\[ By_t + Ay = \varphi, \quad t \in \omega_\tau. \]  

(142)

We define a two-level difference scheme as a set of the Cauchy problems (140), (141) that depend on the parameter \( \tau \). The formulation (140), (141) (as well as (141), (142)) is called the canonical form of two-level schemes.

For solvability of the Cauchy problem at a new time level, it is assumed that \( B^{-1} \) does exist. Then equation (142) may be written as

\[
\dot{y} = Sy + \tau \ddot{\varphi}, \quad S = E - \tau B^{-1}A, \quad \ddot{\varphi} = B^{-1}\varphi, 
\]

(143)

where, as usual, \( E \) is the identity operator. The operator \( S \) is called the transition operator of the two-level scheme (the transition from a current time level to the next one).

A two-level scheme is called stable if there exist positive constants \( m_1 \) and \( m_2 \), independent of \( \tau, u^0, \) and \( \varphi \), such that for any \( u^0 \in H, \varphi \in H, t \in \omega_\tau \), for the solution of (140), (141), the following estimate is valid:

\[
\| y^{n+1} \| \leq m_1 \| u^0 \| + m_2 \max_{0 \leq \theta \leq t^n} \| \varphi(\theta) \|_*, \quad t^n \in \omega_\tau, 
\]

(144)
where $\| \cdot \|$ and $\| \cdot \|_*$ are some norms. The inequality (144) reflects the continuous dependence of the solution of (140), (141) on the input data.

The difference scheme

$$B(t^n)\frac{y^{n+1} - y^n}{\tau} + A(t^n)y^n = 0, \quad t^n \in \omega, \quad (145)$$

$$y^0 = u^0$$

is called stable with respect to the initial data if for the solution of (145), (146), the following estimate holds:

$$\| y^{n+1} \| \leq m_1 \| u^0 \|, \quad t^n \in \omega. \quad (147)$$

The two-level difference scheme

$$B(t^n)\frac{y^{n+1} - y^n}{\tau} + A(t^n)y^n = \varphi^n, \quad t^n \in \omega, \quad (148)$$

$$y^0 = 0$$

is called stable with respect to the right-hand side if the solution satisfies the inequality

$$\| y^{n+1} \| \leq m_2 \max_{0 \leq \theta \leq t^n} \| \varphi(\theta) \|_*, \quad t^n \in \omega. \quad (150)$$

The difference scheme (145), (146) is said to be $\rho$-stable (uniformly stable) with respect to the initial data in $H_\omega$ if there exist constants $\rho > 0$ and $m_1$, independent of $\tau$ and $n$, such that for any $n$ and all $y^n \in H$, the solution $y^{n+1}$ of the difference equation (145) satisfies the estimate

$$\| y^{n+1} \|_D \leq \rho \| y^n \|_D, \quad t^n \in \omega, \quad (151)$$

and $\rho^n \leq m_1$.

In the theory of difference schemes, one of the following quantities is selected as $\rho$:

$$\rho = 1,$$

$$\rho = 1 + c\tau, \quad c > 0,$$

$$\rho = \exp (c\tau),$$

where a constant $c$ is independent of $\tau$, $n$.

In view of (145), rewrite equation (143) in the form

$$y^{n+1} = Sy^n. \quad (152)$$
The requirement of $\rho$-stability is equivalent to the fulfillment of the bilateral operator inequality
\[-\rho D \leq DS \leq \rho D, \quad (153)\]
if $DS$ is self-adjoint ($DS = S^*D$). For an arbitrary operator of transition in \((152)\), the condition of $\rho$-stability is given by
\[S^*DS \leq \rho^2D. \quad (154)\]

Let us formulate the discrete analog of Gronwall’s lemma.

**Lemma 5.** From the estimate for the difference solution at the $n + 1$-st time level
\[\|y_{n+1}\| \leq \rho\|y_n\| + \tau\|\varphi^n\|, \quad (155)\]
it follows that the a priori estimate
\[\|y_{n+1}\| \leq \rho^{n+1}\|y^0\| + \sum_{k=0}^{n} \tau\rho^{n-k}\|\varphi^k\|, \quad (156)\]
holds.

Thus, from the levelwise estimate, we obtain an a priori estimate for the difference solution at any time level.

Let us formulate the basic criteria for stability of two-level schemes with respect to the initial data \([27, 29]\). The most important is the following theorem, proved by Samarskii, on the exact (coinciding necessary and sufficient) condition for stability in $H_A$.

**Theorem 9.** Assume that in equation \((145)\), the operator $A$ is a positive and self-adjoint operator independent of $n$. The condition
\[B \geq \frac{\tau}{2}A, \quad t \in \omega_\tau \quad (157)\]
is necessary and sufficient for stability in $H_A$, i.e., for the fulfillment of the estimate
\[\|y_{n+1}\|_A \leq \|u^0\|_A, \quad t \in \omega_\tau. \quad (158)\]

**Proof.** Multiplying equation \((145)\) scalarly by $y_t$, we get
\[(By_t, y_t) + (Ay, y_t) = 0. \quad (159)\]
Using the representation
\[ y = \frac{1}{2}(y + \dot{y}) - \frac{1}{2} \tau y_t, \]
rewrite (159) as
\[ \left( \left( B - \frac{\tau}{2} A \right) y_t, y_t \right) + \frac{1}{2 \tau} \left( A(\dot{y} + y), \dot{y} - y \right) = 0. \] (160)

For the self-adjoint operator \( A \), we have \((Ay, \dot{y}) = (y, A\dot{y})\) and
\[ (A(\dot{y} + y), \dot{y} - y) = (A\dot{y}, \dot{y}) - (Ay, y). \]
Substituting these relations into (160) and using the condition (157), we obtain the inequality
\[ \|y^{n+1}\|_A \leq \|y^n\|_A, \] (161)
which ensures the desired estimate (158).

To prove the necessity of the inequality (158), assume that the scheme is stable in \( H_A \), i.e., the inequality (158) holds. We prove that this implies the operator inequality (157). Consider (160) at the initial time level \( n = 0 \):
\[ 2\tau \left( \left( B - \frac{\tau}{2} A \right) w, w \right) + (Ay_1, y_1) = (Ay_0, y_0), \quad w = \frac{y_1 - y_0}{\tau}. \]
In view of (158), this identity holds only if
\[ \left( \left( B - \frac{\tau}{2} A \right) w, w \right) \geq 0. \]
Let \( y_0 = u_0 \in H \) be an arbitrary element, then the element \( w = -B^{-1}Au_0 \in H \) is arbitrary, too. Indeed, for any element \( w \in H \), we obtain \( u_0 = -A^{-1}Bw \in H \) since \( A^{-1} \) exists. Thus, the inequality holds for any \( w \in H \), i.e., we have the operator inequality (157).

The condition (157) is necessary and sufficient for stability not only in \( H_A \), but also in other norms. We now formulate (without proof) the stability result for \( H_B \).

**Theorem 10.** Assume that in (145), (146), the operators \( A \) and \( B \) are constant and
\[ B = B^* > 0, \quad A = A^* > 0. \] (162)
Then the condition (157) is necessary and sufficient for stability of the scheme (145), (146) with respect to the initial data in \( H_B \) with \( \rho = 1 \).
The consideration of general time-dependent problems is based on using $\rho$-stability.

**Theorem 11.** Let $A$ and $B$ be constant operators and

$$A = A^*, \quad B = B^* > 0.$$  

Then the condition

$$\frac{1 - \rho}{\tau} B \leq A \leq \frac{1 + \rho}{\tau} B \quad (163)$$

is necessary and sufficient for the $\rho$-stability of the scheme (145), (146) in $H_B$, i.e., for the fulfilment of

$$\|y^{n+1}\|_B \leq \rho \|y^n\|_B.$$  

**Proof.** Writing (145) in the form of (152), we get from (153) the following condition for stability in $H_B$:

$$-\rho B \leq B - \tau A \leq \rho B.$$  

This bilateral operator inequality can be formulated in a more traditional representation using inequalities in the form of (163) for the scheme operators.  

We emphasize that in this theorem there is no assumption that the operator $A$ is positive (or at least non-negative). Under the additional assumption on the positiveness of $A$, we get that the condition (163) is necessary and sufficient for the $\rho$-stability of the scheme (145), (146) in $H_A$.

If $\rho \geq 1$, then stability, as in Theorem 9, is established for two-level difference schemes with the non-self-adjoint operator $B$.

**Theorem 12.** Let $A$ be a self-adjoint, positive, and constant operator. Then under the condition

$$B \geq \frac{\tau}{1 + \rho} A \quad (164)$$

the scheme (145), (146) is $\rho$-stable in $H_A$.

**Proof.** Adding and subtracting from the basic energy identity (see (160))

$$2\tau \left( \left( B - \frac{\tau}{2} A \right) y_t, y_t \right) + (A \dot{y}, \dot{y}) - (Ay, y) = 0 \quad (165)$$

the expression

$$2\tau^2 \frac{1}{1 + \rho} (Ay_t, y_t),$$

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we get
\[ 2\tau \left( (B - \frac{\tau}{1 + \rho} A) y_t, y_t \right) + (A\hat{y}, \hat{y}) - (Ay, y) - \frac{1 - \rho}{1 + \rho} \tau^2 (Ay_t, y_t) = 0. \]

In view of (164) and the self-adjointness of \( A \), we obtain immediately
\[(A\hat{y}, \hat{y}) - \rho(Ay, y) + (\rho - 1)(A\hat{y}, y) \leq 0.\]

The inequality
\[ |(A\hat{y}, y)| \leq \|\hat{y}\|_A \|y\|_A \]
with notation
\[ \eta = \frac{\|\hat{y}\|_A}{\|y\|_A}, \]
yields the inequality
\[ \eta^2 - (\rho - 1)\eta + \rho \leq 0. \]

It holds for all \( 1 \leq \eta \leq \rho \), and so we go to the desired estimate
\[ \|\hat{y}\|_A \leq \|y\|_A, \]
which ensures stability in \( H_A \).

Now we consider a priori estimates that express stability with respect to the right-hand side. Such estimates are employed to study convergence of difference schemes for time-dependent problems.

First, we show that stability with respect to the initial data in \( H_D \), \( D = D^* > 0 \) results in stability with respect to the right-hand side in the norm \( \|\varphi\|_* = \|B^{-1}\varphi\|_R \).

**Theorem 13.** Assume that (140), (141) is \( \rho \)-stable in \( H_R \) with respect to the initial data, i.e., the estimate (151) holds with \( \varphi^n = 0 \). Then the scheme (140), (141) is stable with respect to the right-hand side and the following a priori estimate is true:
\[ \|\hat{y}^{n+1}\|_R \leq \rho^{n+1}\|u^0\|_R + \sum_{k=0}^{n} \tau\rho^{n-k}\|B^{-1}\varphi^k\|_R. \]  (166)
Proof. Since \( B^{-1} \) exists, we have that equation (140) may be written as
\[
y^{n+1} = Sy^n + \tau \bar{\varphi}^n, \quad S = E - \tau B^{-1}A, \quad \bar{\varphi}^n = B^{-1}\varphi^n. \quad (167)
\]
From (167), we get
\[
\|y^{n+1}\|_R \leq \|Sy^n\|_R + \tau \|B^{-1}\varphi^n\|_R. \quad (168)
\]
The requirement of \( \rho \)-stability with respect to the initial data is equivalent to the boundedness of the norm of the transition operator \( S \):
\[
\|Sy^n\|_R \leq \rho \|y^n\|_R, \quad t \in \omega_\tau.
\]
Because of this, from (168), we obtain
\[
\|y^{n+1}\|_R \leq \rho \|y^n\|_R + \tau \|B^{-1}\varphi^n\|_R.
\]
Using the discrete analog of Gronwall’s lemma, we obtain the desired estimate (166), which expresses the stability of the scheme with respect to the initial data and the right-hand side.

In particular, if \( D = A \) or \( D = B \) (under the condition \( A = A^* > 0 \) or \( B = B^* > 0 \)), then, from (166), we obtain elementary estimates for stability in the energy space \( H_A \) or \( H_B \).

Some new estimates for the two-level difference scheme (140), (141) can be obtained by coarsening the stability criterion (167).

**Theorem 14.** Let \( A \) be a self-adjoint, positive, and constant operator and assume that \( B \) satisfies the condition
\[
B \geq \frac{1 + \varepsilon}{2\tau A} \quad (169)
\]
with a constant \( \varepsilon > 0 \) independent of \( \tau \). Then the scheme (140), (141) satisfies the a priori estimate
\[
\|y^{n+1}\|_A^2 \leq \|u^0\|_A^2 + \frac{1 + \varepsilon}{2\varepsilon} \sum_{k=0}^{n} \tau \|\varphi^k\|_{B^{-1}}^2. \quad (170)
\]
Proof. Multiplying equation (140) scalarly by $2\tau y_t$, we obtain, similarly to (165), the energy identity

$$2\tau((B - \frac{T}{2} A)y_t, y_t) + (A\dot{y}, \dot{y}) = (Ay, y) + 2\tau(\varphi, y_t).$$

(171)

The right-hand side of the above expression can be estimated as

$$2\tau(\varphi, y_t) \leq 2\tau\|\varphi\|_{B^{-1}}\|y_t\|_B \leq$$

$$\leq 2\tau\varepsilon_1\|y_t\|^2_B + \frac{\tau}{2\varepsilon_1}\|\varphi\|^2_{B^{-1}}$$

with a positive constant $\varepsilon_1$. Substituting this estimate into (171), we get

$$2\tau\left((1 - \varepsilon_1)B - \frac{T}{2} A\right)y_t, y_t + (A\dot{y}, \dot{y}) \leq (Ay, y) + \frac{\tau}{2\varepsilon_1}\|\varphi\|^2_{B^{-1}}.$$ 

If the condition (169) holds, then it is possible to select $\varepsilon_1$ such that

$$\frac{1}{1 - \varepsilon_1} = 1 + \varepsilon,$$

and so

$$(1 - \varepsilon_1)B - \frac{T}{2} A = (1 - \varepsilon_1)(B - \frac{1 + \varepsilon}{2} TA) \geq 0,$$

$$(A\dot{y}, \dot{y}) \leq (Ay, y) + \frac{1 + \varepsilon}{2\varepsilon}\tau\|\varphi\|^2_{B^{-1}}.$$ 

The last inequality implies the estimate (170). $\square$

**Theorem 15.** Let $A$ be a self-adjoint, positive, and constant operator, and assume that $B$ satisfies the condition

$$B \geq G + \frac{T}{2} A, \quad G = G^* > 0.$$ 

(172)

Then the solution of (140), (141) satisfies the a priori estimate

$$\|y^{n+1}\|^2_A \leq \|u^0\|^2_A + \frac{1}{2} \sum_{k=0}^n \tau\|\varphi^k\|^2_{G^{-1}}.$$ 

(173)

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Proof. In the identity (171), we employ the estimate
\[
2\tau(\varphi, y_t) \leq 2\tau(Gy_t, y_t) + \frac{\tau}{2}(G^{-1}\varphi, \varphi).
\]
Substituting this estimate into (171) and taking into account (172), we get
\[
(A\hat{y}, \hat{y}) \leq (Ay, y) + \frac{1}{2}\tau\|\varphi\|^2_{G^{-1}}
\]
that, by a discrete analog of Gronwall’s lemma, gives (173).

The convergence study of difference schemes is conducted in various classes of smoothness of the solution of the original differential problem, and therefore we must have a wide range of estimates. In particular, the right-hand side should be estimated in different and simply calculated norms. Only typical a priori estimates for solutions of operator-difference schemes are considered here.

We now apply the above results to elementary schemes with weights for an operator-differential equation of first order. The Cauchy problem
\[
du{t} + Au = f(t), \quad t > 0, \quad u(0) = u^0,
\]
with \( A \geq 0 \) is associated with the two-level scheme with weights
\[
\frac{y^{n+1} - y^n}{\tau} + A(\sigma y^{n+1} + (1 - \sigma)y^n) = \varphi^n, \quad t^n \in \omega_t,
\]
\[
y^0 = u^0.
\]

The scheme (176), (177) may be written in the canonical form (142) with the operators
\[
B = E + \sigma\tau A, \quad A > 0.
\]

**Theorem 16.** The scheme with weights (176), (177) is stable in \( H \) with respect to the initial data if and only if the following operator inequality holds:
\[
A^* + \left( \sigma - \frac{1}{2} \right) \tau A^* A \geq 0.
\]
Proof. By $A > 0$, there exists $A^{-1}$. Multiplying (176) by $A^{-1}$, we go from (142), (178) to the scheme

$$
\tilde{B} \frac{y^{n+1} - y^n}{\tau} + \tilde{A} y^n = \tilde{\varphi}^n, \quad t^n \in \omega_r
$$

where

$$
\tilde{B} = A^{-1} + \sigma \tau E, \quad \tilde{A} = E.
$$

The necessary and sufficient condition for stability of this scheme with respect to the initial data in $H = H_{\tilde{A}}$ (Theorem 9) is formulated as the inequality

$$
A^{-1} + \left(\sigma - \frac{1}{2}\right) \tau E \geq 0.
$$

Multiplying it from the left by $A^*$ and from the right by $A$, we obtain (179). This completes the proof of the theorem.

If $\sigma \geq 0.5$, then the operator-difference scheme (176), (177) is unconditionally stable (stable for any $\tau > 0$).

4.2. Difference schemes for convection-diffusion problems

Discretization in space of the Cauchy problem (37), (41) yields the problem (see, e.g., (81), (82)):

$$
\frac{dy}{dt} + Ay = \varphi(t), \quad A = C + D, \quad 0 < t \leq T,
$$

$$
y(0) = u^0. \quad (180)
$$

With the above approximations, the grid operators of convective and diffusive transport inherit the basic properties of differential operators in the appropriate spaces of grid functions. Among these properties, we recall the following as the major ones. The constant (time-independent) grid diffusion operator is self-adjoint and positive definite:

$$
\frac{d}{dt} D = D \frac{d}{dt}, \quad D = D^* \geq \frac{1}{M_0} \kappa_1 E, \quad \kappa_1 > 0, \quad M_0 > 0. \quad (182)
$$

For the grid operator of convective transport in various forms ($C = C(t) = C_{\alpha}$, $\alpha = 0, 1, 2$), we have

$$
C_0 = -C_0^*, \quad (183)
$$
\[ |(C_\alpha y, y)| \leq M_1\|y\|^2, \quad \alpha = 1, 2, \quad (184) \]
\[ \|C_\alpha y\|^2 \leq M_2(Dy, y), \quad \alpha = 0, 1, 2, \quad (185) \]

with the corresponding positive constants \(M_1\) and \(M_2\).

To solve numerically the problem (180), (181), we consider the two-level scheme with weights

\[ \frac{y^{n+1} - y^n}{\tau} + C(\sigma_1 y^{n+1} + (1 - \sigma_1)y^n) \]
\[ + D(\sigma_2 y^{n+1} + (1 - \sigma_2)y^n) = \varphi^n, \quad t^n \in \omega, \quad (186) \]
\[ y_0 = u_0. \quad (187) \]

Here, e.g., we have

\[ C = C(0.5(t^{n+1} + t^n)), \quad \varphi^n = \varphi(0.5(t^{n+1} + t^n)). \]

Among the most important variants of the difference scheme with weights (186), (187), we highlight the scheme with equal weights \((\sigma_1 = \sigma_2)\) and the scheme, where convective transport is taken from a previous time level \((\sigma_1 = 0)\).

We start with the convective transport operator in the skew-symmetric form, i.e., \(C = -C^* = C_0\). Problems with the convective transport operator in the nondivergent \((C = C_1)\) and divergent \((C = C_2)\) forms will be considered later. Assume that in the difference scheme (186), we have

\[ \sigma_1 = \sigma_2 = \sigma. \quad (188) \]

In view of (188), instead of (186), we consider the difference scheme

\[ \frac{y^{n+1} - y^n}{\tau} + (C_0 + D)(\sigma y^{n+1} + (1 - \sigma)y^n) = \varphi^n, \quad t^n \in \omega. \quad (189) \]

The scheme (187), (189) under investigation may be written in the canonical form for the two-level difference scheme (140), (141) with the operators

\[ B = E + \sigma \tau A, \quad A = C_0 + D > 0. \quad (190) \]

The main peculiarity of difference schemes for the convection-diffusion equation is connected with non-self-adjointness of the operators \(B\) and \(A\). Therefore, it is impossible to use the above results on stability of operator-difference schemes, which were formulated for constant self-adjoint operators.
The second important feature is associated with the fact that operators of the difference scheme are variable in time. We consider the problems with the time-dependent difference operator of convective transport. To obtain a priori estimates for such problems, it is often necessary to require additionally Lipschitz continuity of the difference operators with respect to time.

Conditions for stability of the scheme \((140), (141), (190)\) have been presented above in the form of Theorem 16. Let us supplement this result with the corresponding stability estimate of the difference solution with respect to the right-hand side and the initial data.

**Theorem 17.** The difference scheme \((140), (141), (190)\) is unconditionally stable for \(\sigma \geq 0.5\), and the difference solution satisfies the a priori estimate

\[
\|y^{n+1}\|^2 \leq \|u^0\|^2 + \frac{1}{2} \sum_{k=0}^{n} \tau \|\varphi^k\|_{D^{-1}}^2.
\] (191)

**Proof.** Rewrite the scheme \((140), (190)\) (see \((189)\)) as follows:

\[
y_{n+1} - y_n = \tau A v^{n+1} = \varphi^n, \quad n = 0, 1, \ldots, N_0 - 1,
\] (192)

where

\[
v^{n+1} = \sigma y^{n+1} + (1 - \sigma y^n) = \left(\sigma - \frac{1}{2}\right) y_t + \frac{1}{2} (y^{n+1} + y^n),
\]

\[
y_t = \frac{y^{n+1} - y^n}{\tau}.
\]

Multiplying equation \((192)\) scalarly by \(v^{n+1}\), we obtain

\[
\left(\sigma - \frac{1}{2}\right) \tau (y_t, y_t) + (Av^{n+1}, v^{n+1}) + \frac{1}{2\tau} ((y^{n+1}, y^{n+1}) - (y^n, y^n)) = (\varphi^n, v^{n+1}).
\] (193)

In the condition \((190)\), we have \((Ay, y) = (Dy, y)\). For the right-hand side, we use the estimate

\[
(\varphi^n, v^{n+1}) = (Dv^{n+1}, v^{n+1}) + \frac{1}{4} (D^{-1} \varphi^n, \varphi^n).
\]
With this in mind, from (193), under the conditions of the theorem we get the estimate
\[ \| y^{n+1} \|^2 \leq \| y^n \|^2 + \frac{\tau}{2} (D^{-1} \varphi^n, \varphi^n). \]

Thus, we come to the desired estimate (191).

The a priori estimate (191) obtained above for the difference solution is a grid analog of the a priori estimate (60) for the solution of the differential problem (41), (59), because the convective transport operator in the skew-symmetric form under consideration corresponds to the constant \( M_1 = 0 \) in (60).

Now we consider the case, where the skew-symmetry of the difference operator of convective transport is not valid. We will study the problem with the convective transport written in the nondivergent form, i.e., \( C = C_1 \). The case the convective transport in the divergence form \( (C = C_2) \) is investigated in a similar way.

Let us examine the scheme (140), (141), where
\[ B = E + \sigma \tau A, \quad A = C_1 + D. \]  
(194)

It is important to distinguish two classes of problems. The simplest case is associated with the assumption that the operator \( A \) is non-negative. Such a situation takes place, e.g., if \( M_1 M_0 - \kappa_1 \leq 0 \) — convective transport has only an insignificant effect. Indeed, in view of (182), (184), in the case (194), we have
\[ (Ay, y) = (C_1 y, y) + (Dy, y) \geq -M_1 \| y \|^2 + \frac{1}{M_0 \kappa_1} \| y \|^2 \]
\[ = \frac{1}{M_0} (\kappa_1 - M_1 M_0) \| y \|^2. \]

Because of this, for the operator \( A \), we have the following lower bound:
\[ A \geq \frac{1}{M_0} (\kappa_1 - M_1 M_0) E. \]  
(195)

Another case deals with slightly compressible flows, where \( A \geq 0 \) under the condition \( M_2 M_0 - \kappa_1 \leq 0 \). In this situation, in view of (182) and (185), we obtain
\[ (Ay, y) = (C_1 y, y) + (Dy, y) \geq -\| C_1 y \| \| y \| + (Dy, y) \]
\[ \geq \left( 1 - \left( \frac{M_2 M_0}{\kappa_1} \right)^{1/2} \right) (Dy, y). \]
Under these restrictions on parameters of the problem, we can apply the results on the unconditional stability (Theorem 16) for the difference scheme (140), (141), (194) in $H$ for $\sigma \geq 0.5$.

In the general case, we cannot rely on the non-negativity of the operator $A$. This leads to the fact that the conventional schemes with weights are not unconditionally stable under the standard restrictions $\sigma \geq 0.5$. Let us consider the difference scheme (140), (141), (194) as an example.

The solvability of the scheme (140), (141), (194) ($B > 0$), in view of the fact that the operator $A$ is not non-negative, takes place under the constraint of an appropriately small time step — we speak of conditional solvability. Taking into account (194), (195) with $M_1 M_0 - \kappa_1 > 0$, we get the following restriction on a time step:

$$\tau \leq \tau_1 = \frac{M_0}{\sigma (M_1 M_0 - \kappa)}.$$  \hspace{1cm} (196)

In this case (see Theorem 2, the estimate (60) for the solution of the differential problem), it is necessary to be oriented to obtaining an appropriate estimate that expresses conditions for $\rho$-stability.

We have already formulated the necessary and sufficient condition for $\rho$-stability in the case with the constant self-adjoint operators $B$ and $A$. Therefore, our study will be based on the schemes with weights of type (140), (141), (194) considered above.

Let us define new grid functions $v^n$:

$$y^n = \rho^n v^n, \quad n = 0, 1, \ldots, N_0, \quad \rho > 0.$$  \hspace{1cm} (197)

A condition for $\rho$-stability for $y^n$ is evidently equivalent to stability ($\rho = 1$) for $v^n$. Substitution of (197) into (145) yields the difference scheme

$$\tilde{B} \frac{v^{n+1} - v^n}{\tau} + \tilde{A} v^n = 0, \quad t_n \in \omega_\tau,$$  \hspace{1cm} (198)

where

$$\tilde{B} = \rho E + \sigma \rho \tau A, \quad \tilde{A} = \left(\frac{\rho - 1}{\tau}\right) E + (1 + \sigma (\rho - 1)) A.$$  \hspace{1cm} (199)

It is possible to use the following representation for the operators of the difference scheme (198):

$$\tilde{B} = G + \tilde{\sigma} \tau \tilde{A},$$  \hspace{1cm} (200)
which treat the scheme (198) as a scheme with weights. In view of the representation (199), we obtain in (200):

$$G = \frac{\varrho}{1 + \sigma(\varrho - 1)} E, \quad \tilde{\sigma} = \frac{\sigma \varrho}{1 + \sigma(\varrho - 1)}. \quad (201)$$

Similarly to Theorem 16, we prove the stability of the scheme (198), (200) in $H_G$, i.e., in $H$ with $\tilde{\sigma} \geq 0.5$ under the constraint $\tilde{A} \geq 0$. Taking into account (201), we get the desired condition on a weight of the difference scheme (198), (199):

$$\sigma \geq \frac{1}{1 + \varrho}. \quad (202)$$

The non-negativity of the operator $\tilde{A}$ is connected with an appropriate choice of $\varrho$. In view of the stability estimate for the differential problem (see Theorem 2, the estimate (60)), it is natural to set

$$\varrho = 1 + M_1 \tau. \quad (203)$$

Taking into account the estimate (195), the condition $\tilde{A} \geq 0$ (see (199)) is fulfilled for

$$M_1 M_0 - (1 + \sigma \tau M_1)(M_1 M_0 - \kappa_1) \geq 0.$$  

This inequality yields the following restriction on a permissible time step:

$$\tau \leq \tau_2 = \frac{\kappa_1}{\sigma M_1 (M_1 M_0 - \kappa_1)}. \quad (204)$$

A comparison with the estimate (196) shows that the time step restriction (204) is slightly stronger ($\tau_2 < \tau_1$, we recall, $M_1 M_0 > \kappa_1$). Summarizing, we obtain the following statement.

**Theorem 18.** The scheme with weights (140), (141), (194) under the constraint $M_1 M_0 > \kappa_1$ is $\varrho$-stable in $H$, where $\varrho$ is defined according to (203), if the weight $\sigma$ satisfies the restriction (202) and a time step meets the condition (204).

This statement complements Theorem 17 which ensures the stability of the scheme (140), (141), (194) under the constraint $M_1 M_0 \leq \kappa_1$ in $H$ with $\sigma \geq 0.5$. Possible non-negativity of the operator $A = C_1 + D$ leads to the situation, where we must use $\varrho$-stability. In addition, we impose (see (204)) restrictions on a time step.
In solving convection-diffusion problems, it is reasonable to focus on difference schemes, where a part of the operator $A$ (it is, of course, the convective transport operator) is taken from the previous time level \[32\]. Such explicit-implicit schemes from the above class of two-level schemes with weights are considered in \[39\]. Suppose now that in the difference scheme (186), we have
\[
\sigma_1 = 0, \quad \sigma_2 = \sigma.
\] (205)
The homogeneous ($\varphi_n = 0$) scheme (186), (205) is reduced to the canonical form (140) if we define
\[
B = E + \sigma\tau D, \quad A = C + D.
\] (206)
For any $\tau > 0$, we have $B > 0$, and therefore the discrete equation (186), (205) is solvable at every time level. Let us formulate a sufficient condition for $\varrho$-stability of the difference scheme for the convection-diffusion equation in $H_D$.

**Theorem 19.** The solution of the explicit-implicit scheme (186), (205) with $\sigma \geq 0.5$ satisfies the estimate
\[
\|y^{n+1}\|_D \leq \varrho\|y^n\|_D
\] (207)
where
\[
\varrho = 1 + \frac{M_2}{4}\tau,
\] (208)
and $M_2$ is the constant from the inequality (185).

**Proof.** Multiply (186) scalarly by $2\tau \dot{y}_t = 2(y^{n+1} - y^n)$ and, in view of (206), obtain the energy identity
\[
\tau((2B - \tau D)y_t, y_t) + (Dy^{n+1}, y^{n+1}) - (Dy^n, y^n) + 2\tau(Cy^n, y_t) = 0.
\] (209)
Taking into account the representation (206) and the constraint $\sigma \geq 0.5$, from (209), it follows the inequality
\[
2\tau(y_t, y_t) + (Dy^{n+1}, y^{n+1}) - (Dy^n, y^n) \leq 2\tau|(Cy^n, y_t)|.
\] (210)
In view of (185), the right-hand side is evaluated as follows:
\[
|(Cy^n, y_t)| \leq \|y_t\|^2 + \frac{1}{4}\|Cy^n\|^2 \leq \|y_t\|^2 + \frac{M_2}{4}(Dy^n, y^n).
\]
Substitution into (210) yields
\[
(Dy^{n+1}, y^{n+1}) \leq \left(1 + \frac{M_2}{2}\tau\right)(Dy^n, y^n).
\]
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Therefore, in view of inequality

\[ 1 + \frac{M_2 \tau}{2} \leq \left( 1 + \frac{M_2}{4} \right)^2, \]

we obtain the desired stability estimate (207), (208).

The \( \varphi \)-stability estimate (207), (208), derived here, is fully consistent with the corresponding estimate for the differential problem (see, e.g., the estimate (62) and the proof of Theorem 2). An important point is that, in contrast to Theorem [18] we obtained stability with the standard restrictions on a weight \( \sigma \) in a stronger norm. Moreover, the implementation of the explicit-implicit scheme is much simpler from the computational point of view – we must invert a self-adjoint elliptic grid operator.

Considering two-level difference schemes, we have highlighted two main classes of difference schemes for unsteady convection-diffusion problems. The first class is based on the use of the simplest schemes with equal weights for the convective and diffusive transport. The second and the most promising class of difference schemes (explicit-implicit schemes) is associated with the explicit treatment of the convective transport. Here we do not analyze the complete set of three-level difference schemes. We focus on the study of explicit-implicit schemes. Using three-level difference schemes, we can obtain the second-order approximation in time.

To solve numerically the problem (180), (181), we employ the three-level explicit-implicit scheme with weights

\[
\frac{y^{n+1} - y^{n-1}}{2\tau} + D(\sigma y^{n+1} + (1 - 2\sigma)y^n + \sigma y^{n-1}) + Cy^n = \varphi^n, \quad n = 1, 2, \ldots, N_0 - 1
\]

with

\[
y^0 = u^0, \quad y^1 = u^1.
\]

In (211), we put, e.g., \( C = C(t^n) \), \( \varphi^n = \varphi(t^n) \). To specify the second initial condition \( u^1 \) in (212) with the second order, in the simplest case, we involve a two-level scheme, so that

\[
\frac{y^1 - y^0}{\tau} + (C + D)\frac{y^1 + y^0}{2} = \varphi^0.
\]

The difference scheme (211), (212) approximates (180), (181) with the second order in time.
The explicit-implicit scheme (211) is written in the canonical form

\[
B(t^n)\frac{y^{n+1} - y^n}{2\tau} + R(t^n)(y^{n+1} - 2y^n + y^{n-1}) + A(t^n)y^n = \varphi^n, \quad n = 1, 2, \ldots, N_0 - 1
\]

with

\[
B = E, \quad R = \sigma D, \quad A = C + D.
\]

To evaluate the difference solution, we introduce the norm associated only with the diffusive transport, i.e.,

\[
\mathcal{E}^{n+1} = \frac{1}{4}(D(y^{n+1} + y^n), y^{n+1} + y^n) + \left(\sigma - \frac{1}{4}\right)(D(y^{n+1} - y^n), y^{n+1} - y^n).
\]

Stability is established taking into account the subordination of the convective transport operator to the diffusive transport operator — we say about the estimate (185).

**Theorem 20.** If \(\sigma > 0.25\), then the difference scheme (211), (212) is \(\rho\)-stable with

\[
\rho = 1 + M_2 \frac{4\sigma}{4\sigma - 1} \tau,
\]

and the solution satisfies the a priori estimate

\[
\mathcal{E}^{n+1} \leq \rho \mathcal{E}^n + \tau \|\varphi^n\|^2.
\]

**Proof.** For the scheme (213), (214), we have

\[
\frac{1}{2\tau}\|w^{n+1} + w^n\|^2 + \mathcal{E}^{n+1} = -(Cy^n, w^{n+1} + w^n) + (\varphi^n, w^{n+1} + w^n) + \mathcal{E}^n,
\]

where

\[
w^{n+1} = y^{n+1} - y^n.
\]

For the first two terms in the right-hand side, we obtain

\[
|<(Cy^n, w^{n+1} + w^n)| \leq \frac{1}{4\tau}\|w^{n+1} + w^n\|^2 + \tau \|Cy^n\|^2,
\]

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\[(\varphi^n, w^{n+1} + w^n) \leq \frac{1}{4\tau} \|w^{n+1} + w^n\|^2 + \tau \|\varphi^n\|^2.\]

Thus, in view of (215), we arrive at the inequality

\[E^{n+1} \leq E^n + \tau M_2 \|y^n\|_D^2 + \tau \|\varphi^n\|^2. \quad (218)\]

Next, we use the estimate

\[\|y^n\|_D^2 \leq \frac{4\sigma}{4\sigma - 1} E^n. \quad (219)\]

According to (212), we get

\[
E^n = (Dy^n, y^n) - \tau (Dy^n, y_t) + \sigma \tau^2 (Dy_t, y_t) \\
\geq \|y^n\|_D^2 - \tau \|y^n\| \|y_t\| + \sigma \tau^2 \|y_t\|_D^2 \\
\geq (1 - \beta) \|y^n\|_D^2 + (\sigma - \frac{1}{4\beta}) \tau^2 \|y_t\|_D^2.
\]

For \(\sigma > 0.25\), we select the parameter \(\beta = 1/(4\sigma)\) and obtain the estimate (219). Substitution of (219) into (218) yields the levelwise estimate (216), (218).

4.3. Unconditionally stable schemes

For convection-diffusion problems with convective transport in the divergent and nondivergent forms, we have constructed (Theorem 18) conditionally stable schemes with weights. Restrictions on a time step (see (204)) are governed by features of the problem and do not related, in general, with parameters of discretization in space. Conditionally stable schemes with weights are developed only for problems with the convective transport in the skew-symmetric form (Theorem 17).

Different nature of convective and diffusive transport as well as reaction processes appear, in particular, in significantly distinct representative rates of these phenomena. Such heterogeneity can be taken into account when choosing discretization in time. The most pronounced occurrence of the heterogeneity of discretization in time is expressed in explicit-implicit schemes. In this case, for numerical solving the unsteady problem, a part of the problem operator terms is approximated by explicit relationships, whereas the other part is treated implicitly.

Explicit-implicit schemes are widely used for the numerical solution of convection-diffusion problems. Various variants of inhomogeneous discretization in time are given in [1]. One or another explicit approximations are applied to the convective transport operator, whereas the diffusive transport operator is approximated.
implicitly. Thus, the most severe restrictions on a time step due to diffusion are removed. In view of the subordination of the convective transport operator to the diffusive transport operator, we have already proved unconditional stability of the above-considered explicit-implicit schemes for time-dependent convection-diffusion problems.

Similar techniques are used in the analysis of diffusion-reaction problems. In this case (see, e.g., [24]), the diffusive transport is treated implicitly, whereas for reactions (source terms), explicit approximations are used. Such explicit approximations demonstrate obvious advantages for problems with nonlinear terms describing reaction processes.

In convection-diffusion-reaction problems, the problem operator may be sign-indefinite. This means that the system may be nondissipative, i.e., the solution norm for the homogeneous problem does not decrease during the time evolution. Thus, the exponential growth of the solution may be observed, and such behavior of the solution must be reflected at the discrete level. Unconditionally stable schemes for such problems are constructed in the work [41]. They are based on the splitting of the problem operator into two terms, where one of the terms has explicit approximations in time, whereas the other is approximated implicitly. Implicit approximations are applied to the part of the problem operator that causes the dissipative properties of the problem. In the case of the skew-symmetric operator of convective transport, such a splitting is used for the operator of reaction.

The standard schemes, which are used in computational practice, should be corrected even for solving dissipative problems. For example, both the standard fully implicit scheme (backward Euler) and symmetric scheme (Crank-Nicholson) does not produce the exact solution for the test problem ($\lambda > 0$):

\[
\frac{du}{dt} + \lambda u = 0, \quad u(0) = u^0.
\]

In [17], there is discussed a modification of standard schemes that is based on the use of $(\exp(\lambda \tau) - 1)/\lambda$ instead of the original time step $\tau$ in the application to the fully implicit scheme. More recent results concerned with constructing and employing such nonstandard discretizations in time can be found, e.g., in [18]. Here we mention new possibilities in designing unconditionally stable schemes for solving unsteady convection-diffusion problems that involve the introduction of new variables.

Time-dependent convection-diffusion problems with the convective transport in the divergent (37)–(39) and the nondivergent (36)–(38) forms may be written
as the Cauchy problem for the operator equation (compare with (41)):

\[
\frac{du}{dt} + Au = f(t), \quad A = C_0 + R + D. \tag{220}
\]

Here we introduce the reaction operator

\[ R u = r(x, t) u. \]

In the case (37)–(39), we have

\[ r(x, t) = -\frac{1}{2} \text{div} \, v. \]

Similarly, for equation (36)–(38), we obtain

\[ r(x, t) = \frac{1}{2} \text{div} \, v. \]

For the reaction operator, we get

\[ R = R^*, \quad mE \leq R \leq ME. \tag{221} \]

Using roughened estimates for the reaction operator, we can put

\[ m = -M_1, \quad M = M_1. \]

After discretization in space, from (220), we obtain the equation

\[
\frac{dy}{dt} + Ay = \varphi(t), \quad A = C_0 + R + D, \quad 0 < t \leq T, \tag{222}
\]

supplemented by the initial condition (181).

For the operator \( R \), we have

\[ Ry = r(x, t)y, \quad x \in \omega. \tag{223} \]

In this case, we get

\[ R = R^*, \quad mE \leq R \leq ME. \tag{224} \]

For instance, the convective transport operator of in the nondivergent form seems like this:

\[ r(x, t) = -\frac{1}{2} \text{div}_h v \]
for an appropriate approximation of the divergence operator.

To construct unconditionally stable schemes for solving the problem (181), (222) without the assumption of non-negativity of the problem operator, we apply explicit-implicit approximations. The bottleneck is connected with the reaction operator, and therefore for \( m < 0 \), we split it into two terms:

\[
R = R_+ + R_-, \quad R_+ = R^*_+, \quad R_- = R^*_-, \quad 0 \leq R_+ \leq ME, \quad mE \leq R_- < 0.
\] (225)

By (223), it is sufficient to put

\[
R_+ y = r_+ (x, t) y, \quad R_- y = r_- (x, t) y, \quad x \in \omega,
\]

where

\[
r_+ = \max(0, r), \quad r = r_+ + r_-.
\]

Using two-level explicit-implicit schemes, we may rely only on the first-order accuracy with respect to time. Therefore, we focus on the fully implicit approximations of the main operator terms. We employ the difference scheme

\[
\frac{y^{n+1} - y^n}{\tau} + (C^n + D + R^+_n) y^{n+1} + R^-_n y^n = 0, \quad n = 0, 1, \ldots, N_0 - 1.
\] (226)

**Theorem 21.** The explicit-implicit scheme (141), (223)–(226) with \( m < 0 \) is unconditionally \( \varrho \)-stable in \( H \) for

\[
\varrho = 1 - m\tau,
\] (227)

and the difference solution satisfies the estimate

\[
\|y^{n+1}\| \leq \varrho \|y^n\|, \quad n = 0, 1, \ldots, N_0 - 1.
\] (228)

**Proof.** Multiplying equation (226) scalarly in \( H \) by \( y^{n+1} \), and taking into account the skew-symmetry of the operator \( C_0 \), positive definiteness of the operator \( D \), and relation (225), we obtain

\[
\|y^{n+1}\|^2 \leq (y^{n+1}, y^n) - (R^-_n y^n, y^n).
\] (229)

In view of

\[
(y^{n+1}, y^n) \leq \frac{1}{2} (\|y^{n+1}\|^2 + \|y^n\|^2),
\]

\[
| - (R^-_n y^n, y^n) | \leq m\|y^{n+1}\|\|y^n\|,
\]

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from (229), we arrive at the inequality
\[ \| y^{n+1} \|^2 \leq (1 - 2m\tau)\| y^n \|^2. \]

By virtue of
\[ (1 - 2m\tau) \leq (1 - m\tau)^2, \]
this yields the inequality of \( \varrho \)-stability (228) with \( \varrho \) defined by (227).

Among possible generalizations of explicit-implicit scheme (141), (226), special attention should be given to schemes of the second-order accuracy with respect to time. The symmetric scheme provides an example of such a scheme:
\[
y^{n+1} - y^{n-1} \quad 2\tau + (C^n + D + R^n_+) y^{n+1} + 2y^n + y^{n+1} \quad 4
\]
\[ + R^n y^n = 0, \quad n = 1, 2, \ldots, N_0 - 1, \]
(230)

where now \( C^n = C(t^n), \ R^n = R(t^n) \). To start calculations with the second order in time, we put, e.g.,
\[
y^1 - y^0 \quad \tau + \frac{1}{2}((C^1 + D + R^1)y^1 + (C^0 + D + R^0)y^0) = 0,
\]
Because of this and taking into account the initial condition (141), the difference equation (230) is considered for a given \( y^0 \) and \( y^1 \).

In addition to (230), special mention should be given to the scheme
\[
3y^{n+1} - 4y^n + y^{n-1} \quad 2\tau + (C^n + D + R^n_+) y^{n+1}
\]
\[ + R^n_-(2y^n - y^{n-1}) = 0, \quad n = 1, 2, \ldots, N_0 - 1. \]
Preserving the second-order approximation in time, for this scheme, the implicit of the main part of the problem operator is expressed more essentially.

In equation (222), for the operator \( A \), by (182) and (183), we have
\[
A \geq mE + \frac{1}{M_0} E_1. \]

In our study, we use a more rough estimate
\[
A \geq mE, \quad (231)
\]
and consider the most interesting case $m < 0$.

To construct unconditionally stable schemes for the differential problem (141), (222), (224) under the condition (231), we define a new function $w$:

$$y = \exp(-mt)w.$$  \hfill (232)

Substitution of (232) into (141), (222) for the homogeneous right-hand side gives the following problem for $w$:

$$\frac{dw}{dt} + \tilde{A}w = 0, \quad \tilde{A} = A - mE, \quad 0 < t \leq T,$$

$$w(0) = u^0.$$ \hfill (233)

For this transformation, the problem operator $\tilde{A}$ is non-negative ($\tilde{A} \geq 0$).

To solve the problem (233), (234), we apply the two-level scheme with weights:

$$\frac{w^{n+1} - w^n}{\tau} + \tilde{A}^n \left( \sigma w^{n+1} + (1 - \sigma) w^n \right) = 0, \quad t^n \in \omega_T,$$

$$w^0 = u^0.$$ \hfill (235)

This scheme under the standard constraints $\sigma \geq 0.5$ is unconditionally stable (Theorem 17).

Let us write the difference equation (235) for the desired grid function $y^n$. Taking into account $t^{n+1} = t^n + \tau$, we put

$$y^n = \exp(-mt^n)w^n, \quad y^{n+1} = \exp(-mt^n)\exp(-m\tau)w^{n+1}.$$

Because of this, from (235), (236), we obtain the following difference scheme for $y^n$:

$$\frac{\exp(m\tau)y^{n+1} - y^n}{\tau} + (A - mE) \left( \sigma \exp(m\tau)y^{n+1} + (1 - \sigma) y^n \right) = 0,$$

$$y^0 = u^0.$$ \hfill (237)

In contrast to the nonstandard schemes discussed in [17, 18], a positive effect is achieved not only through the use of new approximations in time, but also by correcting the problem operator.
Theorem 22. The difference scheme (237), (238) for $\sigma \geq 0.5$ is unconditionally stable in the $H$ with
\[ \varrho = \exp(-m\tau), \] (239)
and the solution satisfies the a priori estimate
\[ \|y^{n+1}\| \leq \varrho \|y^n\|. \] (240)

Proof. The above proofs were based on the transition to the problem with a non-negative operator and the use of the previous Theorem 17. It is possible to conduct a direct proof of stability for the scheme (237), (238). Rewrite the scheme under consideration in the form
\[ \frac{\exp(m\tau)y^{n+1} - y^n}{\tau} + \tilde{A}p^{n+1} = 0, \quad t^n \in \omega, \] (241)
where
\[ p^{n+1} = \sigma \exp(m\tau)y^{n+1} + (1 - \sigma)y^n \]
\[ = \tau \left( \sigma - \frac{1}{2} \right) r^{n+1} + \frac{1}{2} \left( \exp(m\tau)y^{n+1} - y^n \right), \]
\[ r^{n+1} = \frac{\exp(m\tau)y^{n+1} - y^n}{\tau}. \]

Multiplying equation (241) scalarly by $p^{n+1}$, we obtain
\[ \tau \left( \sigma - \frac{1}{2} \right) (r^{n+1}, r^{n+1}) + \tilde{A} (p^{n+1}, p^{n+1}) \]
\[ + \frac{1}{2\tau} \left( (\exp(m\tau)y^{n+1}, \exp(m\tau)y^{n+1}) - (y^n, y^n) \right) = 0. \]

From this equation, under the conditions $\sigma \geq 0.5$ and $\tilde{A} \geq 0$, it follows that the stability estimate (239), (240) holds. \(\square\)

It is important to note that, in contrast to the conventional scheme with weights (see Theorem 18), here stability is obtained with no restriction on a time step. The value of $\varrho$ defined by (239) is fully consistent with the corresponding constant for the solution of the differential problem. The transition to a new time level involves the solution of the grid problem
\[ (E + \sigma\tau(A - mE))y^{n+1} = \chi^n. \] (242)
The equation (242) is a system of linear algebraic equations with a positive definite and non-self-adjoint matrix; it can be solved using standard iterative methods.
5. Stability in Banach spaces

The main results on stability of difference schemes for the unsteady convection-diffusion equation were obtained above considering the problem in Hilbert spaces of grid functions. Here we study difference schemes in Banach spaces, where stability of difference schemes is established in the uniform and integral norms.

In our study we can employ the maximum principle for difference schemes as it was done in investigating monotone approximations. The second and more promising approach presented below is to use the concept of the logarithmic norm. In this section, monotone schemes of the second-order accuracy in space are constructed for the time-dependent convection-diffusion.

5.1. One-dimensional problems

To simplify the material presented here, we start with the 1D convection-diffusion problems. Consider the time-dependent convection-diffusion equation with convective terms in the nondivergent form:

$$\frac{\partial u}{\partial t} + v(x, t) \frac{\partial u}{\partial x} - \frac{\partial}{\partial x} \left( k(x) \frac{\partial u}{\partial x} \right) = f(x, t)$$

for

$$0 < x < l, \quad 0 < t \leq T.$$  

This equation is supplemented with homogeneous Dirichlet boundary conditions:

$$u(0, t) = 0, \quad u(l, t) = 0, \quad 0 < t \leq T.$$  

In addition, the initial condition is given:

$$u(x, 0) = u^0(x), \quad 0 < x < l.$$  

The second important example is the unsteady equation of convection-diffusion in the divergent form:

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(v(x, t)u) - \frac{\partial}{\partial x} \left( k(x) \frac{\partial u}{\partial x} \right) = f(x, t).$$

Consider the set of functions $u(x, t)$ satisfying the boundary conditions (244). The transient problem of convection-diffusion is written in the form of the operator-differential equation

$$\frac{du}{dt} + Au = f(t), \quad A = \mathcal{A}(t) = C(t) + D,$$
where \( C(t) \) denotes the convective transport operator, and \( D \) stands for the operator of diffusive transport. The Cauchy problem for the evolutionary equation (247) is supplemented with the initial condition
\[
    u(0) = u^0. \tag{248}
\]

We recall some a priori estimates for the convection-diffusion problems (243)–(245) and (244)–(246), which are derived from the maximum principle. The corresponding a priori estimates are derived in the spaces \( L_\infty(0,l) \) and \( L_1(0,l) \), where the norms are, respectively,
\[
    \|v\|_\infty = \max_{0 < x < l} |v(x)|, \quad \|v\|_1 = \int_0^l |v(x)| \, dx.
\]
The solution of the time-dependent convection-diffusion problem (243)–(245) (the convective transport in the nondivergent form) satisfies the a priori estimate in \( L_\infty(0,l) \):
\[
    \|u(x,t)\|_\infty \leq \|u^0(x)\|_\infty + \int_0^t \|f(x,\theta)\|_\infty \, d\theta. \tag{249}
\]
We present also the estimate for the convection-diffusion equation with convective terms in the divergent form. The solution of the problem (244)–(246) satisfies the a priori estimate in \( L_1(0,l) \):
\[
    \|u(x,t)\|_1 \leq \|u^0(x)\|_1 + \int_0^t \|f(x,\theta)\|_1 \, d\theta. \tag{250}
\]
The a priori estimates (249), (250) serve us as a guide in considering discrete problems.

5.2. Stability of two-level schemes

Let us obtain sufficient conditions for the stability of two-level difference schemes for the Cauchy problem for a system of ODEs. Further, these general conditions will be applied to particular cases of model convection-diffusion equations with the convective terms in the nondivergent and divergent forms.

Consider a system of linear ODEs of first order:
\[
    \frac{dw_i}{dt} + \sum_{j=1}^m a_{ij}(t) w_j = \phi_i(t), \quad i = 1, 2, \ldots, m. \tag{251}
\]
Assume that \( w = w(t) = \{w_1, w_2, \ldots, w_m\} \), \( A = [a_{ij}] \), then we can write (251) in matrix (operator) form as

\[
\frac{dw}{dt} + A(t)w = \phi(t).
\] (252)

We will construct difference schemes for numerical solving the Cauchy problem (252) for \( t > 0 \) and the initial condition

\[
w(0) = u^0.
\] (253)

We will investigate the stability of the difference solution of the problem (252), (253) in \( L_\infty \) and \( L_1 \). For a norm of a vector and a norm of a matrix, consistent with it in \( L_\infty \), we have

\[
\|w\|_\infty = \max_{1 \leq i \leq m} |w_i|, \quad \|A\|_\infty = \max_{1 \leq i \leq m} \sum_{j=1}^{m} |a_{ij}|.
\] (254)

Similarly, in \( L_1 \), we obtain

\[
\|w\|_1 = \sum_{i=1}^{m} |w_i|, \quad \|A\|_1 = \max_{1 \leq j \leq m} \sum_{i=1}^{m} |a_{ij}|.
\] (255)

The problem (252), (253) will be considered under the following constraints. Assume that the diagonal elements of the matrix \( A \) are non-negative, and there is row-wise or column-wise diagonal dominance, i.e., we have

\[
a_{ii} \geq \sum_{i \neq j=1}^{m} |a_{ij}|, \quad i = 1, 2, \ldots, m
\] (weak diagonal dominance by rows) or

\[
a_{ii} \geq \sum_{i \neq j=1}^{m} |a_{ji}|, \quad i = 1, 2, \ldots, m
\] (weak diagonal dominance by columns).

The logarithmic norm of the matrix \( A \) is defined \([8, 12]\) by the number

\[
\mu[A] = \lim_{\delta \to 0+} \frac{\|E + \delta A\| - 1}{\delta}.
\]
For the logarithmic norm of a matrix in $L_\infty$ (consistent with (254)) and in $L_1$ (consistent with (255)), we have the expressions

$$
\mu_\infty[A] = \max_{1 \leq i \leq m} \left( a_{ii} + \sum_{i \neq j = 1}^{m} |a_{ij}| \right),
$$

$$
\mu_1[A] = \max_{1 \leq j \leq m} \left( a_{jj} + \sum_{j \neq i = 1}^{m} |a_{ij}| \right).
$$

In view of the restrictions (256), (257), we have that the logarithmic norm of the matrix $-A$ in the Cauchy problem (252), (253) satisfies the inequality

$$
\mu[-A] \leq 0 \tag{258}
$$

in the corresponding space (in $L_\infty$ for (256) and in $L_1$ for (257)).

Among the properties of the logarithmic norm (see [8, 9]), we highlight the following:

1. $\mu[cA] = c\mu[A]$, $c = \text{const} \geq 0$;
2. $\mu[cE + A] = c + \mu[A]$, $c = \text{const}$;
3. $\|Aw\| \geq \max\{-\mu[-A], - \mu[A]\} \|w\|$.

The emphasis is placed on the property 3, which allows to get easily the lower bound of the norm $Aw$. This bound can be combined with the standard upper bound of $Aw$: $\|Aw\| \leq \|A\| \|w\|$.

Let us study the stability of difference schemes for the problem (252), (253). We denote the approximate solution at the time level $t^n = n\tau$ (where $\tau$ is a time step) as $y^n$, and write the two-level difference scheme with weights

$$
\frac{y^{n+1} - y^n}{\tau} + A(\sigma y^{n+1} + (1 - \sigma)y^n) = \varphi^n, \tag{259}
$$

where, e.g., $A = A(\sigma t^{n+1} + (1 - \sigma)t^n)$, with the initial data

$$
y^0 = u^0. \tag{260}
$$

A sufficient condition for stability of the scheme (259), (260) is formulated as the following statement.
Theorem 23. Assume that in the Cauchy problem (252), (253) the matrix \(A\) satisfies the restriction (256) (or (257)). Then the difference scheme with weights (259), (260) is unconditionally stable for \(\sigma = 1\), and it is conditionally stable for \(\sigma < 1\) in \(L_\infty\) (in \(L_1\)) if and only if
\[
\tau \leq \frac{1}{1 - \sigma} \left( \max_{1 \leq i \leq m} a_{ii} \right)^{-1}.
\] (261)

In this case, the difference solution satisfies the a priori estimate
\[
\|y^{n+1}\| \leq \|u^0\| + \sum_{k=0}^{n} \tau \|\varphi^k\|. \tag{262}
\]

Proof. From (259), it follows that
\[
(E + \sigma \tau A)y^{n+1} = (E - (1 - \sigma) \tau A)y^n + \tau \varphi^n,
\]
and therefore
\[
\| (E + \sigma \tau A)y^{n+1} \| \leq \| (E - (1 - \sigma) \tau A)y^n \| + \tau \|\varphi^n\|. \tag{263}
\]

For the left-hand side of (263), by the above-mentioned properties of the logarithmic norm and in view of (258), we have
\[
\| (E + \sigma \tau A)y^{n+1} \| \geq -\mu [E - \sigma \tau A] \|y^{n+1}\|
\]
\[
= (1 + \sigma \mu [A]) \|y^{n+1}\| \geq \|y^{n+1}\|.
\]

For the first term in the right-hand side of (263), we obtain
\[
\| (E - (1 - \sigma) \tau A)y^n \| \leq \| E - (1 - \sigma) \tau A \| \|y^n\|.
\]

We investigate this estimate in more detail for \(L_\infty\). The case \(L_1\) is studied in a similar manner. Considering (254) and taking into account the condition of diagonal dominance (256), we have
\[
\|E - (1 - \sigma) \tau A\| = \max_{1 \leq i \leq m} \left| 1 - (1 - \sigma) \tau \left( a_{ii} + \sum_{i \neq j=1}^{m} a_{ij} \right) \right|
\]
\[
\leq \max_{1 \leq i \leq m} \left( |1 - (1 - \sigma) \tau a_{ii}| + (1 - \sigma) \tau \sum_{i \neq j=1}^{m} |a_{ij}| \right)
\]
\[
\leq \max_{1 \leq i \leq m} \left( |1 - (1 - \sigma) \tau a_{ii}| + (1 - \sigma) \tau a_{ii} \right) \leq 1.
\]

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with $0 \leq \sigma \leq 1$ and under the restriction (261) on the time step.

The substitution into (263) yields the inequality
\[
\|y^{n+1}\| \leq \|y^n\| + \tau\|\varphi^n\|,
\]
which immediately implies the desired estimate (262) for stability with respect to the right-hand side and the initial data.

The above estimates for stability (262) in $L_\infty$ and $L_1$ are directly associated with the monotonicity of the difference solution of the problem (259), (260) under the assumption that the off-diagonal elements of the matrix $A$ are non-positive.

Let us prove the following statement.

**Theorem 24.** Assume that in the schemes (259), (260), the conditions of diagonal dominance (256) (or (257)) are fulfilled for
\[
a_{ij} \leq 0, \quad i \neq j, \quad i, j = 1, 2, \ldots, m
\]
and let
\[
u^0 \geq 0, \quad \varphi^n \geq 0, \quad n = 0, 1, \ldots,
\]
then
\[
y^{n+1} \geq 0, \quad n = 1, 2, \ldots,
\]
for any $\tau > 0$ if $\sigma = 1$, and if $0 \leq \sigma < 1$, this is true under the constraints on a time step (261).

**Proof.** For the transition from the current time level to the next one, we have
\[
y^{n+1} + \sigma \tau Ay^{n+1} = g^n, \quad n = 0, 1, \ldots,
\]
where
\[
g^n = y^n - (1 - \sigma)\tau Ay^n + \tau\varphi^n.
\]
Suppose that $y^n \geq 0$ (for $n = 0$ this is true from the assumptions of the theorem). We show that from this it follows also the non-negativity of $y^{n+1}$ ($y^{n+1} \geq 0$).

We prove that under the assumptions of the diagonal dominance (256) (or (257)) and under the restrictions on a time step (261), for a non-negative $y^n$ and $\varphi^n$, we get $g^n \geq 0$. In view of (266), we obtain
\[
g^n_i = (1 - (1 - \sigma)\tau a_{ii})y^n_i - (1 - \sigma)\tau \sum_{j \neq i} a_{ij}y^n_j
\]
\[
\geq (1 - (1 - \sigma)\tau a_{ii})y^n_i \geq 0.
\]

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In the conditions of the theorem, the matrix of the system of linear algebraic equations (265) is an M-matrix, i.e., we have: strong diagonal dominance, positive diagonal elements, and non-positive off-diagonal elements of the matrix. Because of this, from \( g^n \geq 0 \), it follows that \( g^{n+1} \geq 0 \).

Apply the derived results to studying stability and monotonicity of difference schemes for time-dependent problems of convection-diffusion in the nondivergent and divergent forms.

5.3. Difference schemes for convection-diffusion equations

For simplicity, we restrict ourselves to uniform grids. On the interval \([0, l]\), we introduce a grid

\[
\bar{\omega} \equiv \omega \cup \partial \omega = \{x \mid x = x_i = ih, \ i = 0, 1, \ldots, N, \ Nh = l\},
\]

where \( \omega \) is the set of interior nodes:

\[
\omega = \{x \mid x = x_i = ih, \ i = 1, 2, \ldots, N - 1, \ Nh = l\}.
\]

After discretization in space of the model convection-diffusion problems with homogeneous boundary conditions (243)–(245) and (244)–(246), we arrive at the problem (252), (253), where \( m = N - 1 \) and the approximate solution \( w_i(t) = w(x_i, t), \ x \in \omega \). The difference diffusion operator is specified, e.g., as follows:

\[
Dw = -\frac{1}{h^2}k(x + 0.5h)(w(x + h, t) - w(x, t)) + \frac{1}{h^2}k(x - 0.5h)(w(x, t) - w(x - h, t)), \ x \in \omega.
\]

(267)

with

\[
w(x, t) = 0, \ x \in \partial \omega.
\]

(268)

Approximation of convective transport is conducted in such a way that \( v(x, t) \) are defined at the half-integer grid points \( \bar{\omega} \). For operators of convective transport in the nondivergent form (equation (243)), in view of (244), we put

\[
Cw = \frac{1}{2h}v(x + 0.5h, t)(w(x + h, t) - w(x, t)) + \frac{1}{2h}v(x - 0.5h, t)(w(x, t) - w(x - h, t)), \ x \in \omega.
\]

(269)
A similar approximation of the second order with respect to $h$ for the convective transport operator in the nondivergent form (equation (246)) leads to

$$Cw = \frac{1}{2h} v(x + 0.5h, t)(w(x + h, t) + w(x, t)) - \frac{1}{2h} v(x - 0.5h, t)(w(x, t) + w(x - h, t)), \quad x \in \omega.$$  \hfill (270)

Let us formulate the condition for stability and monotonicity of the schemes with weights (259), (260) attributed to the problem (252), (253), where

$$A = C + D$$  \hfill (271)

and $D, C$ are specified according to (267)--(269) or (267), (268), (270).

**Theorem 25.** The difference scheme (259), (260) with (267)--(269), (271) (or (267), (268), (270), (271)) is monotone, and the difference solution satisfies the a priori estimate (262) in $L_\infty$ (or in $L_1$) under the restriction

$$\frac{h|v(x \pm 0.5h, t)|}{k(x \pm 0.5h)} \leq 2, \quad x \in \omega$$  \hfill (272)

for any $\tau > 0$ if $\sigma = 1$, and if $0 \leq \sigma < 1$, then this is true under the constraint on a time step

$$\tau \leq \frac{1}{(1 - \sigma)^\gamma},$$  \hfill (273)

with

$$\gamma = \max_{x \in \omega} \left( \frac{1}{h^2} (k(x + 0.5h) + k(x - 0.5h)) - \frac{1}{2h} (v(x + 0.5h, t) - v(x - 0.5h, t)) \right)$$

for (270), and with

$$\gamma = \max_{x \in \omega} \left( \frac{1}{h^2} (k(x + 0.5h) + k(x - 0.5h)) + \frac{1}{2h} (v(x + 0.5h, t) - v(x - 0.5h, t)) \right)$$

in the case (271).

**Proof.** Consider the case of the convection-diffusion equation (243)--(245) (approximations (267)--(269), (271)) in detail. The problem (244)--(246) (approximations (267), (268), (270)) are investigated is a similar way.
To apply Theorems 23 and 24, write explicitly the elements of $A$. For (267)–(269), (271), we have

$$a_{ii} = \frac{1}{h^2}(k_{i+1/2} + k_{i-1/2}) - \frac{1}{2h}v_{i+1/2} + \frac{1}{2h}v_{i-1/2},$$

$$a_{i,i-1} = -\frac{1}{h^2}k_{i-1/2} - \frac{1}{2h}v_{i-1/2},$$

$$a_{i,i+1} = -\frac{1}{h^2}k_{i+1/2} + \frac{1}{2h}v_{i+1/2},$$

where $k_{i\pm 1/2} = k(x \pm 0.5h), x \in \omega$. The condition of nonpositivity of off-diagonal elements (264) holds for

$$\frac{1}{h^2}k_{i-1/2} + \frac{1}{2h}v_{i-1/2} \geq 0, \quad \frac{1}{h^2}k_{i+1/2} - \frac{1}{2h}v_{i+1/2} \geq 0.$$ (274)

In this case, diagonal dominance is assured. A spatial computational grid with the step from the conditions (272) satisfies the inequalities (274). Restrictions on a time step (261) are reduced to the particular condition (273). Thus, the conditions of Theorems 23 and 24 hold. This provides the stability and monotonicity of the difference solution of the convection-diffusion problem Under the above restrictions on the time step.

To overcome restrictions on the spatial grid (272), we apply upwind approximations for the convective terms. We introduce notation

$$v(x, t) = v^+(x, t) + v^-(x, t),$$

$$v^+(x, t) = \frac{1}{2}(v(x, t) + |v(x, t)|) \geq 0,$$

$$v^-(x, t) = \frac{1}{2}(v(x, t) - |v(x, t)|) \leq 0.$$ Instead (269), we put

$$Cw = \frac{1}{h}v^-(x + 0.5h, t)(w(x + h, t) - w(x, t))$$

$$\frac{1}{h}v^+(x - 0.5h, t)(w(x, t) - w(x - h, t)).$$ (275)

For the convective transport in the divergent form, we get

$$Cw = \frac{1}{h}(v^-(x + 0.5h, t)w(x + h, t) - v^-(x - 0.5h, t)w(x, t))$$

$$\frac{1}{h}(v^+(x + 0.5h, t)w(x, t) - v^+(x - 0.5h, t)w(x - h, t)).$$ (276)

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Theorem 26. The difference scheme (259), (260) with (267), (269), (271), (275) (or (267), (268), (271), (276)) is monotone, and the difference solution satisfies the a priori estimate (262) in $L_\infty$ (or in $L_1$) for any $\tau > 0$ if $\sigma = 1$, and if $0 \leq \sigma < 1$, then this is true under the constraints on a time step (273) with

$$\gamma = \max_{x \in \omega} \left( \frac{1}{h^2} (k(x+0.5h)+k(x-0.5h)) - \frac{1}{h} (v^{-}(x+0.5h,t)-v^{+}(x-0.5h,t)) \right)$$

for (275), and with

$$\gamma = \max_{x \in \omega} \left( \frac{1}{h^2} (k(x+0.5h)+k(x-0.5h)) + \frac{1}{h} (v^{+}(x+0.5h,t)-v^{-}(x-0.5h,t)) \right)$$

in the case (276).

In particular, the fully implicit scheme ($\sigma = 1$) is unconditionally stable and monotone. The principal shortcomings of the above schemes are connected with the upwind approximations for convective terms (275), (276) — these schemes indicate the first-order approximation in space. Schemes on the basis of the central difference approximations (269), (270) are more accurate — they have the second-order spatial approximation.

5.4. Exponential schemes

It is convenient to construct monotone schemes by means of transforming the original convection-diffusion equation, i.e., by eliminating the convective terms. The equation (243) may be written as

$$\frac{\partial u}{\partial t} - \frac{1}{\chi(x,t)} \frac{\partial}{\partial x} \left( k(x) \chi(x,t) \frac{\partial u}{\partial x} \right) = f(x,t), \quad (277)$$

where

$$\chi(x,t) = \exp \left( - \int_0^x \frac{v(s,t)}{k(s)} ds \right). \quad (278)$$

The equation (246) is reduced to

$$\frac{\partial u}{\partial t} - \frac{\partial}{\partial x} \left( \frac{k(x)}{\chi(x,t)} \frac{\partial (\chi(x,t) u)}{\partial x} \right) = f(x,t). \quad (279)$$

Further, we can design discretizations in space, i.e., exponential schemes [7, 33].
Similarly to (267), for the grid functions satisfying (268), it is possible to put in equation (277):

\[ Aw = -\frac{1}{h^2\chi(x, t)} k(x + 0.5h)\chi(x + 0.5h, t)(w(x + h, t) - w(x, t)) \]
\[ + \frac{1}{h^2\chi(x, t)} k(x - 0.5h)\chi(x - 0.5h, t)(w(x, t) - w(x - h, t)), \]  
(280)

where

\[ \chi(x - 0.5h, t) = \exp \left( -\int_0^{x - 0.5h} \frac{v(s, t)}{k(s)} ds \right). \]

Taking into account that

\[ \chi(x - 0.5h, t) = \chi(x) \exp \left( -\int_x^{x - 0.5h} \frac{v(s, t)}{k(s)} ds \right), \]

with a precision of \(O(h^2)\) we put

\[ \chi(x - 0.5h, t) = \chi(x) \exp(\theta(x, t)h) \]

with notation

\[ \theta(x, t) = \frac{v(x, t)}{2k(x)}. \]

Therefore, instead of (280), we can use the following approximation:

\[ Aw = -\frac{1}{h^2} k(x + 0.5h)\exp(\theta(x, t)h)(w(x + h, t) - w(x, t)) \]
\[ + \frac{1}{h^2} k(x - 0.5h)\exp(-\theta(x, t)h)(w(x, t) - w(x - h, t)). \]
(281)

For equation (279), similarly to (280), we put

\[ Aw = -\frac{k(x + 0.5h)}{h^2\chi(x + 0.5h, t)}(\chi(x + h, t)w(x + h, t) - \chi(x, t)w(x, t)) \]
\[ + \frac{k(x - 0.5h)}{h^2\chi(x - 0.5h, t)}(\chi(x, t)w(x, t) - \chi(x - h, t)w(x - h, t)). \]
Simplifying this expression, we obtain

\[
Aw = -\frac{1}{h^2} k(x + 0.5h) \exp(-\theta(x + h, t)h) w(x + h, t) \\
+ \frac{1}{h^2} k(x + 0.5h) \exp(\theta(x, t)h) w(x, t) \\
+ \frac{1}{h^2} k(x - 0.5h) \exp(-\theta(x, t)h) w(x, t) \\
- \frac{1}{h^2} k(x - 0.5h) \exp(\theta(x - h, t)h) w(x - h, t).
\] (282)

Using the above-introduced approximations for the convection-diffusion operator, we can construct monotone schemes. The primary statement is formulated as follows.

**Theorem 27.** If on the set of grid functions (268) the operator \( A \) is defined according to (281) (or (282)), then the difference scheme (259), (260), is monotone, and the difference solution satisfies the a priori estimate (262) in the \( L_{\infty} \) (or in \( L_1 \)) for any \( \tau > 0 \) if \( \sigma = 1 \), and if \( 0 \leq \sigma < 1 \) then this is true under the constraints on a time step (273) with

\[
\gamma = \max_{x \in \omega} \frac{1}{h^2} (k(x + 0.5h) \exp(\theta(x, t)h) + k(x - 0.5h) \exp(-\theta(x, t)h)).
\]

**Proof.** In the case of (281) for the matrix elements, we have

\[
a_{ii} = \frac{1}{h^2} (k_{i+1/2} \exp(\theta_i) + k_{i-1/2} \exp(-\theta_i)),
\]

\[
a_{i,i-1} = -\frac{1}{h^2} k_{i-1/2} \exp(-\theta_i),
\]

\[
a_{i,i+1} = -\frac{1}{h^2} k_{i+1/2} \exp(\theta_i).
\]

Checking diagonal dominance by rows and the non-negativity of the off-diagonal elements is evident.

In the case (282), we obtain

\[
a_{ii} = \frac{1}{h^2} (k_{i+1/2} \exp(\theta_i) + k_{i-1/2} \exp(-\theta_i)),
\]

\[
a_{i,i-1} = -\frac{1}{h^2} k_{i-1/2} \exp(\theta_{i-1}),
\]

\[
a_{i,i+1} = -\frac{1}{h^2} k_{i+1/2} \exp(\theta_i).
\]
\[ a_{i,i+1} = -\frac{1}{h^2} k_{i+1/2} \exp(-\theta_{i+1}). \]

In view of the non-negativity of the off-diagonal elements, the condition of diagonal dominance by columns (257) takes the form
\[ a_{ii} \geq -a_{i-1,i} - a_{i+1,i}, \]
and it is obviously true.

Thus, the conditions for stability and monotonicity are the same as for schemes with the upwind approximations of convective terms (Theorem 26). However, discretization in space is of second order as for schemes with the central-difference approximations (Theorem 25). Some complications in evaluating coefficients of the difference operator leads to a slight increasing of the computational costs.

5.5. Multidimensional problems

Possibilities of constructing second-order monotone schemes for time-dependent equations of convection-diffusion are examined on the model 2D problems (36)–(38) and (37)–(39) in the rectangle \( \Omega \).

The convection-diffusion operators in multidimensional problems are represented as the sum of the 1D convection-diffusion operators. Because of this, in constructing monotone schemes for multidimensional problems, we can apply the above approximations designed for the 1D operators of convection-diffusion.

Similarly to (277), (278), rewrite equation (36) as
\[
\frac{\partial u}{\partial t} - \sum_{\alpha=1}^{2} \frac{1}{\chi_{\alpha}(x,t)} \frac{\partial}{\partial x_{\alpha}} \left( k(x) \chi_{\alpha}(x,t) \frac{\partial u}{\partial x_{\alpha}} \right) = f(x,t),
\]
where now
\[
\chi_1(x,t) = \exp \left( - \int_0^{x_1} \frac{v_1(s,x_2,t)}{k(s,x_2)} ds \right),
\]
\[
\chi_2(x,t) = \exp \left( - \int_0^{x_2} \frac{v_2(x_1,s,t)}{k(x_1,s)} ds \right).
\]

A similar transformation for (39) yields
\[
\frac{\partial u}{\partial t} - \sum_{\alpha=1}^{2} \frac{\partial}{\partial x_{\alpha}} \left( \frac{k(x)}{\chi_{\alpha}(x,t)} \frac{\partial (\chi_{\alpha}(x,t) u)}{\partial x_{\alpha}} \right) = f(x,t).
\]
For simplicity, we use a uniform grid in each spatial direction. For grids in separate directions \( x_\alpha, \alpha = 1, 2 \), we use notation introduced above:

\[
\bar{\omega} \equiv \omega \cup \partial \omega = \bar{\omega}_1 \times \bar{\omega}_2, \quad \omega = \omega_1 \times \omega_2.
\]

After discretization in space of the boundary value problems (37), (38), (283 and (37), (38), (284), we arrive at the problem (252), (253), where

\[
A = A_1 + A_2,
\]
and \( A_\alpha, \alpha = 1, 2 \) are 1D grid operators of convection-diffusion. On the set of grid functions such that

\[
w(x, t) = 0, \quad x \in \partial \omega,
\]
for equation (283), similarly to (281), we put

\[
A_1 w = -\frac{1}{h_1^2} k(x_1 + 0.5h_1, x_2) \exp(\theta(x, t)h_1)w(x_1 + h_1, x_2, t)
+ \frac{1}{h_1^2} k(x_1 + 0.5h_1, x_2) \exp(\theta(x, t)h_1)w(x, t)
+ \frac{1}{h_1^2} k(x_1 - 0.5h_1, x_2) \exp(-\theta(x, t)h_1)w(x, t)
- \frac{1}{h_1^2} k(x_1 - 0.5h_1, x_2) \exp(-\theta(x, t)h_1)w(x_1 - h_1, x_2, t),
\]

\[
A_2 w = -\frac{1}{h_2^2} k(x_1, x_2 + 0.5h_2) \exp(\theta(x, t)h_2)w(x_1, x_2 + h_2, t)
+ \frac{1}{h_2^2} k(x_1, x_2 + 0.5h_2) \exp(\theta(x, t)h_2)w(x, t)
+ \frac{1}{h_2^2} k(x_1, x_2 - 0.5h_2) \exp(-\theta(x, t)h_2)w(x, t)
- \frac{1}{h_2^2} k(x_1, x_2 - 0.5h_2) \exp(-\theta(x, t)h_2)w(x_1, x_2 - h_2, t),
\]
where

\[
\theta(x, t) = \frac{v(x, t)}{2k(x)}, \quad x \in \omega.
\]
In the case of (285), we have (see (280))

$$A_1 w = \frac{1}{h_1^2} k(x_1 + 0.5h_1, x_2) \exp(-\theta(x_1 + h_1, x_2, t)h_1)w(x_1 + h_1, x_2, t)$$

$$+ \frac{1}{h_1^2} k(x_1 + 0.5h_1, x_2) \exp(\theta(x, t)h)w(x, t)$$

$$+ \frac{1}{h_1^2} k(x_1 - 0.5h_1, x_2) \exp(-\theta(x, t)h_1)w(x, t)$$

$$+ \frac{1}{h_1^2} k(x_1 - 0.5h_1, x_2) \exp(\theta(x - h_1, x_2, t)h_1)w(x - h_1, x_2, t),$$

(290)

$$A_2 w = \frac{1}{h_2^2} k(x_1, x_2 + 0.5h_2) \exp(-\theta(x_1, x_2 + h_2, t)h_1)w(x_1, x_2 + h_2, t)$$

$$+ \frac{1}{h_2^2} k(x_1, x_2 + 0.5h_2) \exp(\theta(x, t)h)w(x, t)$$

$$+ \frac{1}{h_2^2} k(x_1, x_2 - 0.5h_2) \exp(-\theta(x, t)h_1)w(x, t)$$

$$+ \frac{1}{h_2^2} k(x_1, x_2 - 0.5h_2) \exp(\theta(x - h_2, x_2, t)h_2)w(x - h_2, x_2, t).$$

(291)

Similarly to Theorem 27, the following statement is proved.

**Theorem 28.** If on the set of grid functions (287) the operator $A$ is defined according to (286), (288), (289) (or (286), (290), (291)), then the difference scheme (259), (260) is monotone, and the difference solution satisfies the a priori estimate (262) in the $L_{\infty}$ (or in $L_1$) for any $\tau > 0$ if $\sigma = 1$, and if $0 \leq \sigma < 1$, then this is true under the constraints on a time step (272) with

$$\gamma = \max_{x \in \omega} \left\{ \frac{1}{h_1^2} k(x_1 + 0.5h_1, x_2) \exp(\theta(x, t)h_1) \right.$$  

$$+ \frac{1}{h_1^2} k(x_1 - 0.5h_1, x_2) \exp(-\theta(x, t)h_1)$$  

$$+ \frac{1}{h_2^2} k(x_1, x_2 + 0.5h_2) \exp(\theta(x, t)h_2)$$  

$$+ \frac{1}{h_2^2} k(x_1, x_2 - 0.5h_2) \exp(-\theta(x, t)h_2) \right\}. $$

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5.6. Locally one-dimensional schemes

Computational implementation of the exponential schemes \((259), (260), (286)-(289)\) and \((259), (260), (286), (287), (290), (291)\) involves the inversion of the non-selfadjoint elliptic grid operators \(E + \sigma \tau A\), where the matrix has strong diagonal dominance either by rows or by columns. To determine the numerical solution at a new time level, we can apply iterative methods. Another possibility is to use locally one-dimensional schemes, which are based on the splitting \((286)\) \([44, 27]\). Intending to 3D generalizations, we restrict ourselves to componentwise splitting schemes \([16, 31]\).

Rewrite the difference equation \((259)\) as follows:

\[
y^{n+1} = S y^n + \tau \varphi^n,
\]

where \(S\) is the transition operator. For the scheme with weights \((259)\), we have

\[
S = (E + \sigma \tau A)^{-1}(E + (\sigma - 1)\tau A).
\]

From the stability condition \((260), (292)\), we get

\[
\|S\| \leq 1.
\]

Monotonicity is ensured by the fact that the matrices \((E + \sigma \tau A)^{-1}\) and \(E + (\sigma - 1)\tau A\) are M-matrices.

Splitting schemes are constructed using transition operators for the individual terms in the additive representation \((286)\). Let us define

\[
S_\alpha(\tau) = (E + \sigma \tau A_\alpha)^{-1}(E + (\sigma - 1)\tau A_\alpha), \quad \alpha = 1, 2.
\]

Instead of \((293)\), we will employ

\[
S = S_1(\tau)S_2(\tau).
\]

The stability condition \((294)\) is true if

\[
\|S_\alpha\| \leq 1, \quad \alpha = 1, 2.
\]

For the monotonicity of the scheme \((292), (296)\), it is sufficient to require that the individual matrices \(S_\alpha, \alpha = 1, 2\) will be M-matrices. For any value of \(\sigma\), only the first-order accuracy with respect to \(\tau\) is possible.
Numerical implementation of the scheme (260), (292), (295), (296) can be conducted using locally one-dimensional schemes with weights, i.e.,

\[
y^{n+\alpha/2} y^{n+(\alpha-1)/2} = \frac{\tau}{\alpha} + A_\alpha (\sigma y^{n+\alpha/2} + (1 - \sigma) y^{n+(\alpha-1)/2})
\]

(298)

where, e.g.,

\[
\varphi^n_1 = 0, \quad \varphi^n_2 = \varphi^n.
\]

**Theorem 29.** If on the set of grid functions (287) the operators \(A_\alpha, \alpha = 1, 2\) are defined according to (288), (289) (or (280), (281)), then the locally one-dimensional difference scheme (260), (298) is monotone, and the difference solution satisfies the a priori estimate (262) in \(L_\infty\) (or in \(L_1\)) for any \(\tau > 0\) if \(\sigma = 1\), and if \(0 \leq \sigma < 1\), then this is true under the constraints on a time step (273) with

\[
\gamma = \max_{x \in \omega} \left\{ \frac{1}{h_1^2} k(x_1 + 0.5h_1, x_2) \exp(\theta(x,t)h_1) + \frac{1}{h_1^2} k(x_1 - 0.5h_1, x_2) \exp(-\theta(x,t)h_1), \right. \\
\left. + \frac{1}{h_2^2} k(x_1, x_2 + 0.5h_2) \exp(\theta(x,t)h_2) + \frac{1}{h_2^2} k(x_1, x_2 - 0.5h_2) \exp(-\theta(x,t)h_2) \right\}.
\]

**Proof.** Conditions for stability and monotonicity are verified for each individual equation (298). In particular, for the first equation, we have

\[
\|y^{n+1/2}\| \leq \|y^n\|
\]

for

\[
\gamma = \max_{x \in \omega} \left\{ \frac{1}{h_1^2} k(x_1 + 0.5h_1, x_2) \exp(\theta(x,t)h_1) + \frac{1}{h_1^2} k(x_1 - 0.5h_1, x_2) \exp(-\theta(x,t)h_1) \right\}.
\]

For the second equation, we get

\[
\|y^{n+1}\| \leq \|y^{n+1/2}\| + \tau \|\varphi^n\|
\]
for

\[
\gamma = \max_{x \in \omega} \left\{ \frac{1}{h^2} k(x_1, x_2 + 0.5h) \exp(\theta(x, t)h) + \frac{1}{h^2} k(x_1, x_2 - 0.5h) \exp(-\theta(x, t)h) \right\}.
\]

Monotonicity of locally one-dimensional schemes under consideration is established in a similar way.

Another classes of splitting schemes can be applied, too. In this regard, we highlight the class of additively averaged schemes.

Instead of a multiplicative representation of the transition operator (296), we can employ the additive representation

\[
S = \frac{1}{2}(S_1(2\tau) + S_2(2\tau))
\]

with preserving the first-order approximation in time for the scheme (292).

For the scheme (292), (295), (299), we present another variant of numerical implementation. Define the auxiliary functions \(y_{n+1}^\alpha, \alpha = 1, 2\) from

\[
\frac{y_{n+1}^\alpha - y_n^\alpha}{2\tau} + A_\alpha (\sigma y_{n+1}^\alpha + (1 - \sigma)y_n^\alpha) = 0.
\]

For the approximate solution at a new time level, we put

\[
y^{n+1} = \frac{1}{2}(y_1^{n+1} + y_2^{n+1}) + \tau \varphi^n.
\]

Conditions of stability and monotonicity for this additively averaged locally one-dimensional scheme are formulated in the following theorem.

**Theorem 30.** If on the set of grid functions (287) the operators \(A_\alpha, \alpha = 1, 2\) are defined according to (288), (289) (or (290), (291)), then the additively averaged locally one-dimensional difference scheme (260), (300), (301), is monotone, and the difference solution satisfies the a priori estimate (262) in \(L_\infty\) (or in \(L_1\)) for any \(\tau > 0\) if \(\sigma = 1\), and if \(0 \leq \sigma < 1\), this is true under the constraints on \(L\) time.
step (273) with

\[
\gamma = 2 \max_{x \in \omega} \left\{ \frac{1}{h_1^2} k(x_1 + 0.5h_1, x_2) \exp(\theta(x, t)h_1) \\
+ \frac{1}{h_2^2} k(x_1 - 0.5h_1, x_2) \exp(-\theta(x, t)h_1), \\
\frac{1}{h_2^2} k(x_1, x_2 + 0.5h_2) \exp(\theta(x, t)h_2) \\
+ \frac{1}{h_2^2} k(x_1, x_2 - 0.5h_2) \exp(-\theta(x, t)h_2) \right\}.
\]

Additively average schemes, on the one hand, demonstrate lower accuracy in comparison with schemes of componentwise splitting, but on the other hand, they are more promising in terms of parallel Computing — the components \(y^{n+1}_\alpha, \alpha = 1, 2\) are determined (see (300)) independently of each other.

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