Frequency Content and Autocorrelation Function of Noisy Periodic Signals

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October 25, 2018

Abstract

We extract the frequency content of a noisy signal by use of Discrete Fourier Transform. Our analysis overcomes the limitations imposed by incommensurate lattices. After computing the deterministic component, we show the relevance of the method in removing spurious autocorrelations from the signal residuals. Results are presented for a temperature time series.

Key Words: Fourier Transform, Incommensurate Lattices

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1 Introduction

The computation of the autocorrelation function of a noisy signal usually requires the removal of a deterministic component often called trend or seasonal component. Motivated by the important problem of the statistical analysis of temperature data, we concentrate on the case of noisy periodic signals. For this particular application, we need to carry out the following steps. (i) First, we evaluate the frequencies of the embedded periodic components. (ii) Then, we fully identify the deterministic component by a variational principle restricted to the class of functions consistent with the results of part (i). (iii) Finally, the computation of the autocorrelation function of the residuals completes the analysis of the signal. We discuss the theoretical underpinnings of such a method in the special case of a signal with a single periodic component, and we demonstrate that this program is often spoiled by the subtle consequences of the possible incommensurability of the sampling frequency and the intrinsic frequency of the signal in question. See for example [1] for a discussion of the sampling theory of continuous signals. This paper quantifies one form of this undesirable effect, and proposes a remedy to the resulting ambiguity in the determination of the intrinsic frequency of a noisy periodic signal.

2 Fourier Spectrum

To establish notation, we begin by giving the explicit formula for the Discrete Fourier Transform (DFT for short) $X_k$ of a given vector $x_j$ of length $N$ [2]:

$$X_k = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} x_j e^{2\pi i \frac{j-1}{N} (k-1)}, \quad k = 1, \cdots, N. \quad (1)$$

With this definition of the Fourier transform, it follows that:

$$x_j = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} X_k e^{-2\pi i \frac{k-1}{N} (j-1)}, \quad (2)$$

because of the orthogonality property

$$\frac{1}{N} \sum_{j=1}^{N} e^{-2\pi i \frac{n-k}{N} j} = \delta_{nk}. \quad (3)$$
Here and in the following, $\delta_{nk}$ is the Kronecker delta. Let us consider a monochromatic signal:

$$x_j = Ae^{-2\pi im/N}j, \quad j = 1, \ldots, N,$$  \hfill (4)

where $m$ is a given positive integer between 1 and $N$. For each integer $1 \leq k \leq N$ we have:

$$X_k = A\sqrt{N} e^{-2\pi i k/N} \delta_{km},$$  \hfill (5)

and hence:

$$|X_k| = |A| \sqrt{N} \delta_{km}.$$  \hfill (6)

In other words, the spectrum results in all coefficients being zero apart from a peak for the value of $k$ equal to $m$. From the spectrum we can compute the frequency of the periodic signal given the length $N$ (i.e., $\omega = \frac{m-1}{N}$). A difficulty arises when the monochromatic signal is of the form:

$$x_j = Ae^{-2\pi i m/N}j + \epsilon j, \quad j = 1, \ldots, N, \quad -\frac{1}{2} \leq \epsilon < \frac{1}{2}.$$  \hfill (7)

In this case, the frequency cannot be expressed as a ratio of the form $(m - 1)/N$ and we say that we are facing an incommensurate lattice problem. Computing the spectrum, we get:

$$|X_k| = |A| \frac{1}{\sqrt{N}} \sqrt{\frac{1 - \cos 2\pi (m - k + \epsilon)}{1 - \cos 2\pi \frac{m - k + \epsilon}{N}}}.\hfill (8)$$

It is now evident that an incommensurate frequency produces a spread of the peak (i.e., $X_k \neq 0$ for $k \neq m$). This is illustrated in Figure 11. It is worth noting that for $\epsilon \to 0$ and $N \to \infty$, we obtain

$$|X_k| \approx |A| \sqrt{N} \left| \frac{\sin \pi (m - k)}{\pi (m - k)} \right| \left( 1 + \left[ \pi \cot \pi (m - k) - \frac{1}{m - k} \right] \epsilon \right) + O(N^{-3/2}).$$  \hfill (9)

We recognize the zero order term as the Fraunhofer diffraction pattern for the single slit. This should not be surprising as the spread of the Fourier spectrum $X_k$ is due to the interaction of two wavelengths; one for the lattice and the other for the signal.

The previous analysis shows that the Fourier spectrum is widened even in the case of a noise free signal. In the case of a noisy signal the detection of the frequency is complicated by the spectrum asymmetries introduced by the noise.
Figure 1: The spectrum $|X_k|$ as a function of $k$ and $\epsilon$. $|A| = 1$, $N = 100$ and $m = 50$. The solid lines are a guide to the eye.

3 The Case of Temperature Data

We now analyze the average temperature data for the case of Seattle-Tacoma international Airport from 1/1/1960 to 12/31/1999 (i.e., 14,610 entries). First, we remove the mean value of 52.20 Fahrenheit. Next, we determine the frequency of the embedded periodic signal by use of the Fourier paradigm. For any subset of the data, the Fourier spectrum gives a peak over a noisy background. However, each peak gives different estimates of the period, all inconsistent with the naive idea that the more points the better the estimate of the period. In Figure 2, we show the value of the estimated period $\Lambda$ as a function of $N$. The $\Lambda$'s were computed according to the following prescriptions. For each $N \leq 14,610$ we selected the first $N$ points of the data set and computed the period as

$$\Lambda = \frac{N}{k_{\text{max}} - 1},$$

(10)

where $k_{\text{max}}$ is the index for which the absolute value of the spectrum is maximum. We always chose the integer $k_{\text{max}}$ to be in the range below the Nyquist frequency to avoid aliasing artifacts [1]. The uncertainty is dramatic. Even
with as many points as $7,920 - 8,140$ (more than 20 times the ‘true’ period corresponding to the tropical year), the period takes values in the range $360 - 370$ days. In experimental situations where the value is not foreseeable from the very beginning, such uncertainty would be disastrous for estimating statistical properties such as variance and autocorrelation function. Let us suppose that we are given only the first $N_1 = 7,920$ data points. We would find a peak at $k_{\text{max}} = 23$, which corresponds to $\Lambda_1 = 360$. If we had the first $N_2 = 8,140$ data, we would find a period of $\Lambda_2 = 370$, even with the same $k_{\text{max}} = 23$. In order to fully identify the deterministic component $d_j$, we use the ansatz

$$d_j = \sum_{q=1}^{Q} \left[ a_q \cos \frac{2\pi q}{\Lambda} j + b_q \sin \frac{2\pi q}{\Lambda} j \right], \quad (11)$$

where $Q$ is the number of harmonics used in the estimation. In each case we minimize the least square distance between the signal diminished by its mean and the expression in (11). For $Q = 3$, we obtain:

|       | $a_1$ | $b_1$ | $a_2$ | $b_2$ | $a_3$ | $b_3$ |
|-------|-------|-------|-------|-------|-------|-------|
| $\Lambda_1 = 360$ | -2.02 | -10.58 | -0.69 | -0.51 | -0.02 | -0.27 |
| $\Lambda_2 = 370$ | -9.54 | 5.22  | 1.18  | -0.18 | 0.07  | 0.17  |
The variances of the residuals are $\sigma_1^2 = 49.17$ and $\sigma_2^2 = 47.33$ for $\Lambda_1$ and $\Lambda_2$. By using the tropical year estimate ($\Lambda_{tr} = 365.2422$), we get $\sigma_{tr}^2 = 26.65$ for $N = 7,920$, and $\sigma_{tr}^2 = 26.51$ for $N = 8,140$. It is evident that a wrong value of the period gives rise to an overestimated variance of the residuals. In Figure 3, we show how dramatic the effect on the autocorrelation function is when a wrong period is used. Not only the residuals have an enlarged variance but they also show a spurious persistence. In the case of temperature time series, a large persistence would imply the possibility of forecasting the weather beyond any reasonable range.

As discussed above, the determination of the ‘true’ frequency is paramount for an effective analysis of statistical properties of a signal. Here we provide a solution to this problem by analyzing the functional dependence of the peaks appearing in Figure 2. More than providing the solution, we want to stress that any time the DFT is performed on a finite sample data set a figure like Figure 2 should be generated. One should not seek the largest possible data set, but rather study the dependence of the estimated period as a function of the number of points. Our ansatz is that both the max and the min ‘peaks’
of each piecewise linear segment in Figure 2, lie on a curve of the type:

\[ \Lambda_{\text{max,min}}(x) = \frac{Ax}{B + x}. \]  

(12)

By least square minimization, we find:

| \( \Lambda_{\text{min}} \) | 365.70 | 194.26 |
| \( \Lambda_{\text{max}} \) | 366.19 | -165.67 |

In both cases the \( R^2 \) is 1 apart from round off errors. This means that the function in (12) is not just a good guess, but rather the ‘true’ decay of the estimated period to its large \( N \) limit (i.e., \( A \)). The results in the above table show that the estimates are fairly close to the ‘true’ period. Furthermore, the variance and the autocorrelation function are very close to those computed for the tropical year value, 365.2422. To be more specific, the variance for \( N = 8,140 \) is 26.79 and 27.54 for \( \Lambda_{\text{min}} \) and \( \Lambda_{\text{max}} \), respectively. In both cases, the autocorrelation function cannot be distinguished from the one obtained after using \( \Lambda_{\text{tr}} \).

It is worth pointing out that equation (8) is of some help for understanding a more fundamental reason for the functional dependence in equation (12). Adding a point to the data set is equivalent to changing the lattice or changing \( \epsilon = \epsilon(N) \). The jumps appearing in Figure 2 will occur when \( \epsilon \) exceeds \( \frac{1}{2} \), which occurs approximately after adding \( \sim \Lambda \) points. By locating the minima and maxima in this way, one can obtain an expression for the decay of the form (12).

4 Conclusion

In conclusion, we have shown how to compute the autocorrelation function of noisy periodic signals in the case of a single frequency mode. Our scheme improves upon a naive application of the DFT. The main point is that more is not better when it comes to the DFT. The dependence on the number of points for the estimated frequency is more important than the position of the peak for a specific \( N \). Our solution stems from building envelopes of minima and maxima of piecewise linear functions. This can certainly be improved; however there is no question that another solution has to be found by looking at the results in Figure 2.
Finally, our approach is applicable to signals with more than one frequency mode. This will be the subject of a future work.

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