Radiative instability of quantum electrodynamics in chiral matter

Kirill Tuchin

1Department of Physics and Astronomy, Iowa State University, Ames, IA 50011, USA

(Dated: August 30, 2018)

Modification of the photon dispersion relation in chiral matter enables $1 \rightarrow 2$ scattering. As a result, the single fermion and photon states are unstable to photon radiation and pair production respectively. In particular, a fast fermion moving through chiral matter can spontaneously radiate a photon, while a photon can spontaneously radiate a fast fermion and anti-fermion pair. The corresponding spectra are derived in the ultra-relativistic approximation. It is shown that the polarization of the produced and decayed photons is determined by the sign of the chiral conductivity. Impact of a flat thin domain wall on the spectra is computed.

I. INTRODUCTION

One of the macroscopic manifestations of the chiral anomaly of QCD is the emergence of the topological $CP$-odd domains in hot nuclear matter [1]. QED is coupled to these domains via its own chiral anomaly. This is represented by the triangular diagrams that involve two photon fields and the axial current generated by the topological fluctuations of the gluon field. The axial current rapidly increases with temperature which triggers a variety of non-trivial electromagnetic effects in quark-gluon plasma [2].

At a more fundamental level, the chiral anomaly makes photon effectively massive. Consequently, single photon and fermion states become unstable. Recall that photon radiation by a charged fermion in vacuum $f(p) \rightarrow f(p') + \gamma(k)$ and the cross-channel process of pair production in vacuum $\gamma(k) \rightarrow f(p') + \bar{f}(p)$ are prohibited by momentum conservation. Indeed in the rest frame of one of the fermions $k^2 = (p \pm p')^2 = 2m(m \pm \varepsilon)$. The right-hand-side never vanishes since $\varepsilon > m$, whereas in the left-hand-side $k^2 = 0$ [3]. In chiral matter, i.e. in a matter supporting the $CP$-odd domains, the chiral anomaly modifies the photon dispersion relation as [4–10]

$$k^2 = -\lambda \sigma_\chi |k|,$$

where $\lambda$ and $k$ are photon helicity and momentum and $\sigma_\chi$ is the chiral conductivity [11–13]. This opens the $1 \rightarrow 2$ scattering channels, viz. the pair-production if $k^2 > 0$ and the photon radiation

$p$, $p'$ and $k$ are four-momenta with the components $p = (\varepsilon, p)$, $p' = (\varepsilon', p')$ and $k = (\omega, k)$.

1 In covariant form $k^2 = -\lambda \sqrt{(n \cdot k)^2 - n^2 k^2}$, where $n^\mu = \sigma_\chi \delta^\mu_0$ in the matter rest frame.
if \( k^2 < 0 \). Thus, single-particle states in chiral matter are unstable with respect to spontaneous radiation and decay. Moreover, in a matter with positive \( \sigma_\chi \), only the right-polarized photons with \( \lambda = +1 \) can be radiated, while only the left-polarized photons with \( \lambda = -1 \) decay and vice-versa in a matter with negative \( \sigma_\chi \).

The main goal of this paper is to compute the photon radiation and pair production spectra due to the modified photon dispersion relation in chiral matter. The fermions (and antifermions) are considered to be external particles propagating through the chiral matter. Their energy is assumed to be much larger than the medium ionization energy. The calculation method is borrowed from [15] and relies on several approximations: (i) photons and fermions are ultra-relativistic in the laboratory frame (the one associated with the matter), this means that \( \varepsilon \gg m, \omega \gg \sigma_\chi \). Apart from making calculations significantly less bulky, this allows one to neglect the effect of the electromagnetic field instability in the infrared region as explained in Sec. II (ii) The matter is assumed to be spatially homogeneous and consisting of either one infinite \( CP \)-odd domain or of two semi-infinite domains separated by a flat domain wall.

The paper is structured as follows. In Sec. III and Sec. IV the wave functions of an ultra-relativistic photon and fermion in chiral matter are derived. In Sec. V they are employed to compute the photon spectrum radiated by a fermion in chiral matter with constant \( \sigma_\chi \) as well as in matter with two chiral domains separated by a thin flat domain wall. The cross-channel process of pair production is analyzed in Sec. VI. The results are summarized and discussed in Sec. VII.

### II. PHOTON WAVE FUNCTION

The \( CP \)-odd domains in the chiral matter can be described by a scalar field \( \theta \) whose interaction with the electromagnetic field \( F^{\mu \nu} \) is governed by the Lagrangian \[ \mathcal{L} = -\frac{1}{4} F^{\mu \nu}_{\mu \nu} - \frac{c_A}{4} \theta \tilde{F}^{\mu \nu} F^{\mu \nu} + \bar{\psi}(i\gamma^\mu D_\mu - m)\psi, \] (2)

where \( \tilde{F}^{\mu \nu} = \frac{1}{2} \epsilon_{\mu \nu \lambda \rho} F^{\lambda \rho} \) is the dual field tensor and \( c_A \) is the chiral anomaly coefficient [19]. A working assumption of this paper is that the field \( \theta \) is spatially uniform (apart of possible thin domain walls) and a slowly varying function of time. The “free” field equations of electrodynamics in chiral electrically neutral and non-conducting matter read

\[
\nabla \cdot B = \nabla \cdot E = 0, \tag{3}
\]

\[
\nabla \times E = -\partial_t B, \tag{4}
\]

\[
\nabla \times B = \partial_t E + \sigma_\chi B, \tag{5}
\]
where \( \sigma_\chi = c_A \dot{\theta} \) is the chiral conductivity \([11, 13]\). (Often \( \dot{\theta} \) is denoted by \( \mu_5 \) and is referred to as the axial chemical potential \([14]\)). In the radiation gauge \( A^0 = 0 \) and \( \nabla \cdot A = 0 \) the vector potential obeys the equation

\[
-\nabla^2 A = -\partial_t^2 A + \sigma_\chi (\nabla \times A).
\] (6)

Its solution describing a photon moving along the \( z \)-direction with energy \( \omega \gg k_\perp, \sigma_\chi \) is described by the wave function

\[
A^{(0)} = \frac{1}{\sqrt{2\omega V}} e_\chi e^{ik_\perp - i\omega t}, \quad k_z = \omega,
\] (7)

where the polarization vector satisfies \( e_\lambda \cdot \hat{z} = 0 \). \( V \) is the normalization volume. It is convenient to use the helicity basis \( e_\lambda = (\hat{x} + i\lambda \hat{y})/\sqrt{2} \). To determine the effect of the chiral anomaly on the photon wave function, look for a solution in the form

\[
A = \frac{1}{\sqrt{2\omega V}} (e_\lambda \varphi + \hat{z} \varphi') e^{i\omega z - i\omega t},
\] (8)

where \( \varphi \) and \( \varphi' \) are functions of coordinates slowly varying in the longitudinal (\( z \)) direction, viz. \( |\partial_z \varphi/\varphi| \ll \omega \) and \( |\partial_z \varphi'/\varphi'| \ll \omega \). The two unknown functions \( \varphi \) and \( \varphi' \) are required in order to account for the change of the photon polarization direction. The gauge condition yields a constraint

\[
(e_\lambda \cdot \nabla_\perp) \varphi + \partial_z \varphi' + i\omega \varphi' \approx (e_\lambda \cdot \nabla_\perp) \varphi + i\omega \varphi' = 0.
\] (9)

Substituting (8) into (6) one obtains

\[
e_\lambda (-2i\omega \partial_z \varphi - \nabla^2_\perp \varphi) + \hat{z} (-2i\omega \partial_z \varphi' - \nabla^2_\perp \varphi') = \sigma_\chi (\omega \lambda e_\lambda \varphi - e_\lambda \times \nabla \varphi - \hat{z} \times \nabla_\perp \varphi').
\] (10)

Taking the scalar product of this equation with \( e_\lambda^* \) and using \( e_\perp^* \cdot e_\perp = 0 \) and \( e_\perp^* \cdot e_\perp = 1 \) produces

\[
-2i\omega \partial_z \varphi - \nabla^2_\perp \varphi = \sigma_\chi (\omega \lambda e_\lambda \varphi - i\lambda \partial_z \varphi) + i\lambda e_\lambda^* \cdot \nabla_\perp \varphi',
\] (11)

where we used the identity \( e_\lambda \times \hat{z} = i\lambda e_\lambda \). In view of (9) one can drop the small term proportional to \( \varphi' \). Neglecting also \( \partial_z \varphi \) in parentheses furnishes an equation for \( \varphi' \):

\[
-2i\omega \partial_z \varphi - \nabla^2_\perp \varphi = \sigma_\chi \omega \lambda \varphi.
\] (12)

Taking the scalar product of (10) with \( \hat{z} \) yields

\[
-2i\omega \partial_z \varphi' - \nabla^2_\perp \varphi' = \sigma_\chi i\lambda (e_\lambda \cdot \nabla) \varphi.
\] (13)

One can eliminate in (13) the term proportional to \( \varphi \) using the gauge condition (9). This furnishes an equation for \( \varphi' \) which is precisely the same as equation (12) obeyed by \( \varphi \).
A solution to (12) can be written as
\[
\psi = e^{i k \cdot x} \exp \left\{ -i \frac{1}{2 \omega} \int_0^z \left[ k_\perp^2 - \sigma \chi(z') \omega \lambda \right] dz' \right\}.
\] (14)

It follows from (9) that
\[
\psi' = -\frac{e_\lambda \cdot k_\perp}{\omega} \psi.
\] (15)

Substituting (14) and (15) into (8) yields the photon wave function in the high energy approximation
\[
A = \frac{1}{\sqrt{2 \omega V}} e_{\lambda} e^{i \omega z + i k_\perp \cdot x - i \omega t} \exp \left\{ -i \frac{1}{2 \omega} \int_0^z \left[ k_\perp^2 - \sigma \chi(z') \omega \lambda \right] dz' \right\},
\] (16)

where the polarization vector
\[
e_\lambda = e_\lambda - \frac{e_\lambda \cdot k_\perp}{\omega} \hat{z}.
\] (17)

Clearly, \(e_\lambda \cdot k = 0\) up to the terms of order \(k_\perp^2/\omega^2\) and \(\sigma \chi/\omega\). If the scattering process happens entirely within a single domain, then the chiral conductivity is constant. However, if a domain wall is located at, say, \(z = 0\), than the chiral conductivity is different at \(z < 0\) and \(z > 0\). This is why a possible \(z\)-dependence of \(\sigma \chi\) is indicated in (16). Even though the boundary conditions on the domain wall induce a reflected wave, it can be neglected in the ultra-relativistic approximation.

It is seen in (1) that half of the infrared modes \(|k| < \lambda \sigma \chi\) have \(\text{Im} \omega > 0\) implying exponential growth of the corresponding wave function with time. This infrared instability and its applications are discussed in many recent publications. However, it is only tangentially related to the radiative instability discussed in this paper, even though both originate from the same dispersion relation. In particular, the infrared instability can be ignored in the ultra-relativistic limit \(\omega \gg k_\perp \gg |\sigma \chi|\) because equation
\[
k_z \approx \omega - \frac{1}{2 \omega} \left( k_\perp^2 - \lambda \sigma \chi \omega \right)
\] (18)

has only real solutions.

## III. FERMION WAVE FUNCTION

The free fermion wave function \(\psi\) at high energy \(\varepsilon \gg p_\perp, m\) can be obtained using the same procedure. Since it satisfies the Dirac equation we are looking for a solution in the form
\[
\psi = \frac{1}{\sqrt{2 \varepsilon V}} u(p) \phi e^{i \varepsilon z - i \omega t},
\] (19)
where \( u(p) \) is a spinor describing a free fermion with momentum \( p \) and \( \phi \) is a scalar function of coordinates. Substituting \( \psi \) into \( (\partial^2 + m^2)\psi = 0 \) and neglecting \( \partial_z \phi \) compared to \( \varepsilon \phi \) one obtains

\[
2i\varepsilon \partial_z \phi + \nabla_\perp^2 \phi = m^2 \phi \tag{20}
\]

with a solution

\[
\phi = \exp \left\{ ip_\perp \cdot x_\perp - iz \frac{p_\perp^2 + m^2}{2\varepsilon} \right\}. \tag{21}
\]

Thus, the fermion wave function is

\[
\psi = \frac{1}{\sqrt{2\varepsilon V}} u(p)e^{i\varepsilon z - i\varepsilon t} \exp \left\{ ip_\perp \cdot x_\perp - iz \frac{p_\perp^2 + m^2}{2\varepsilon} \right\}. \tag{22}
\]

### IV. PHOTON RADIATION

Modification of the photon dispersion relation in chiral matter makes possible spontaneous photon radiation \( f(p) \rightarrow f(p') + \gamma(k) \). The corresponding scattering matrix element reads

\[
S = -ieQ \int \bar{\psi} \gamma^\mu \psi A_\mu d^4x \tag{23}
\]

\[
= -ieQ(2\pi)^3 \delta(\omega + \varepsilon' - \varepsilon) \frac{\bar{u}(p') \gamma^\mu u(p) \epsilon^\mu_\nu}{\sqrt{8\varepsilon\varepsilon'\omega}} \int_{-\infty}^{\infty} dz \int d^2x_\perp \phi^*_{p'}(z, x_\perp) \phi_{p}(z, x_\perp) \tag{24}
\]

\[
=i(2\pi)^3 \delta(\omega + \varepsilon' - \varepsilon) \delta(p_\perp - k_\perp - p'_\perp) \frac{\mathcal{M}}{\sqrt{8\varepsilon\varepsilon'\omega V^2}}, \tag{25}
\]

where \( Q \) is the fermion electric charge. The wave functions \( \varphi_k \) and \( \phi_p \) are given by (14) and (22) respectively with the subscripts indicating the corresponding momenta. The amplitude \( \mathcal{M} \) is given by

\[
\mathcal{M} = -eQ \bar{u}(p') \gamma^\mu u(p) \epsilon^\mu_\nu \int_{-\infty}^{\infty} dz \exp \left\{ i \int^z_0 dz' \left[ \frac{p'^2_\perp + m^2}{2\epsilon'} - \frac{p^2_\perp + m^2}{2\varepsilon} + \frac{k^2_\perp - \sigma_\lambda \omega \lambda}{2\omega} \right] \right\} \tag{26}
\]

\[
= \mathcal{M}_0 \int_{-\infty}^{\infty} dz \exp \left\{ i \int^z_0 \frac{q^2_\perp + \kappa_\lambda(z')}{2\varepsilon x(1 - x)} dz' \right\}, \tag{27}
\]

where we introduced notations \( \mathcal{M}_0 = -eQ \bar{u}(p') \gamma^\mu u(p) \epsilon^\mu_\nu, \ x = \omega/\varepsilon, \)

\[
q_\perp = xp' - (1 - x)k_\perp, \tag{28}
\]

and

\[
\kappa_\lambda(z) = x^2 m^2 - (1 - x)x \lambda \sigma_\lambda \varepsilon. \tag{29}
\]
The amplitude $\mathcal{M}_0$ can most efficiently be computed in the helicity basis using the matrix elements derived in [37]. Keeping in mind that at high energies $k^+ = xp^+$ (where $p^+ = \varepsilon + p_z$, $k^+ = \omega + k_z$), one obtains

$$\mathcal{M}_0 = -eQ \bar{u}_{\sigma'}(p') \gamma \cdot \epsilon^\ast_{\lambda}(k) u_{\sigma}(p)$$

and

$$\mathcal{M}_0 = -\frac{eQ}{\sqrt{2(1-x)}} \left[ x\varepsilon(\sigma + \lambda)\delta_{\sigma',-\sigma} - \frac{1}{x} (2 - x + x\lambda\sigma)(q_x - i\lambda q_y)\delta_{\sigma',\sigma} \right],$$

where $\sigma = \pm 1$ and $\sigma' \pm 1$ are the fermion helicities before and after photon radiation.

The transition probability can be computed as

$$dw = |S|^2 \frac{V d^3p' V d^3k}{(2\pi)^3 (2\pi)^3} = |S|^2 \frac{V d^2p'_\perp V d^2q'_\perp V d^3k}{(2\pi)^3 (2\pi)^3}.$$
where $\theta$ is the step-function. It follows from (29) that $\kappa_\lambda$ is negative if $\lambda \sigma_\chi > 0$ and

$$x < x_0 = \frac{1}{1 + m^2/ (\lambda \sigma_\chi \varepsilon)}. \quad (37)$$

Assume for definitiveness that $\sigma_\chi > 0$. Then only the right-polarized photons with $\lambda > 0$ are radiated. Using (34), (31) in (36) and performing the summations and the integration yields the density of spontaneously radiated photons

$$\frac{dW_+}{dx} = \frac{\alpha Q^2}{2 \varepsilon x^2 (1-x)} \left\{ - \left( \frac{x^2}{2} - x + 1 \right) \kappa_+ + \frac{x^4 m^2}{2} \right\} \theta(x_0 - x)$$

$$= \frac{\alpha Q^2}{2 \varepsilon x} \left\{ \sigma_\chi \varepsilon \left( \frac{x^2}{2} - x + 1 \right) - m^2 x \right\} \theta(x_0 - x), \quad (38)$$

$$\frac{dW_-}{dx} = 0. \quad (39)$$

Photon spectrum radiated in a matter with $\sigma_\chi < 0$ can be obtained by replacing $W_\pm \to W_\mp$ and $\sigma_\chi \to -\sigma_\chi$. Note that since the anomaly coefficient $c_A \sim \alpha$, the spectrum (38) is of the order $\alpha^2$.

The total energy radiated by a fermion per unit time is

$$\frac{\Delta \varepsilon}{T} = \int_0^1 \frac{dW_+}{dx} x \varepsilon dx = \frac{1}{3} \alpha Q^2 \sigma_\chi \varepsilon, \quad (40)$$

where the terms of order $m^2/|\sigma_\chi| \varepsilon$ have been neglected for simplicity. Thus, energy loss increases exponentially with time. It can be neglected only for time intervals much smaller than $\sim 1/|\sigma_\chi| \alpha$.

**B. Two semi-infinite domains separated by a domain wall at $z = 0$**

Suppose now that the chiral matter consist of two semi-infinite domaines separated by a thin domain wall at $z = 0$. Performing the integral over $z$ in (27) yields

$$M = M_0 \left\{ \int_{-\infty}^0 dz \, e^{iz \frac{q_\perp^2 + \kappa_\lambda - i\delta}{2\varepsilon(1-x)}} + \int_0^\infty dz \, e^{iz \frac{q_\perp^2 + \kappa_\lambda' + i\delta}{2\varepsilon(1-x)}} \right\}$$

$$= 2\varepsilon x (1-x) M_0 \left\{ \frac{-i}{q_\perp^2 + \kappa_\lambda' - i\delta} - \frac{-i}{q_\perp^2 + \kappa_\lambda + i\delta} \right\}, \quad (41)$$

where the values of $\kappa_\lambda$ at $z < 0$ and $z > 0$ are denoted by $\kappa_\lambda'$ and $\kappa_\lambda$ respectively and $\delta > 0$ is inserted to regularize the integrals. Plugging (42), (31) into (33) and performing summation over spins yields the radiation spectrum

$$\frac{dN}{d^2 q_\perp dx} = \alpha Q^2 \frac{2m^2 x}{2\pi^2} \left\{ \frac{x^2}{2} - x + 1 \right\} q_\perp^2 + \frac{x^4 m^2}{2} \sum_\lambda \left| \frac{1}{q_\perp^2 + \kappa_\lambda' - i\delta} - \frac{1}{q_\perp^2 + \kappa_\lambda + i\delta} \right|^2. \quad (43)$$

The spectrum peaks at $q_\perp^2 = -\kappa_\lambda$ and/or $q_\perp^2 = -\kappa_\lambda'$ provided that $\kappa_\lambda < 0$ and/or $\kappa_\lambda' < 0$ respectively. In the limit $\kappa_\lambda \to \kappa_\lambda'$ the results of the previous subsection, provided that the square
of the delta functions is treated as explained after (34). Let us also note that when $q^2_\perp + \kappa\lambda = 0$ in (41), the second integral equals $T/2$, which implies that we have to identify $\delta = 4\varepsilon x (1 - x)/T$ (the same result is of course obtained using the first integral).

Away from the poles, one can neglect $\delta$ in (43). The resulting spectrum coincides with the spectrum of the transition radiation once $\kappa\lambda$‘s are replaced by $\kappa_{\text{tr}} = m^2 x^2 + m^2 \gamma (1 - x)$, where $m\gamma$ is the effective photon mass [15, 20]. Unlike the spontaneous radiation, the transition radiation is not possible in a uniform matter. Indeed, the amplitude (31) vanishes because $\kappa_{\text{tr}} > 0$. Another key difference between the transition and spontaneous radiation is that the former has a finite classical limit $\hbar \to 0$, while the later one does not. The spontaneous radiation spectrum (44), (45) is a purely quantum effect that vanishes in the classical limit $\hbar \to 0$. This is of course not surprising at all because it originates from a quantum anomaly.

Integral over the momentum $q_\perp$ in (43) is dominated by the poles at $q^2_\perp = -\kappa\lambda$ and $q^2_\perp = -\kappa'\lambda$. There are two distinct cases depending on whether $\sigma\chi$ and $\sigma'\chi$ have the same or opposite signs. Consider first $\sigma\chi > 0$ and $\sigma'\chi > 0$. In this case the photon spectrum is approximately right-polarized. Keeping only the terms proportional to $1/\delta$ one obtains

$$\frac{dW_{++}}{dx} = \frac{\alpha Q^2}{8x^2 (1 - x) \varepsilon} \left[ \left( \frac{x^2}{2} - x + 1 \right) |\kappa_+ + \kappa'_+| + \frac{x^4 m^2}{2} \right] \theta(x_0 - x) \theta(x_0' - x),$$

where the double plus subscript indicates that the helicity is positive in both domains. The maximum energy fraction taken by the photon $x_0$ is defined in (37); $x_0'$ is the same as $x_0$ with $\sigma\chi$ replaced by $\sigma'\chi$. Consider now $\sigma'\chi > 0$ and $\sigma\chi < 0$. The integration gives

$$\frac{dW_{+-}}{dx} = \frac{\alpha Q^2}{8x^2 (1 - x) \varepsilon} \left\{ \left[ \left( \frac{x^2}{2} - x + 1 \right) |\kappa'_+| + \frac{x^4 m^2}{4} \right] \theta(x_0 - x) \right. \\
+ \left. \left[ \left( \frac{x^2}{2} - x + 1 \right) |\kappa_-| + \frac{x^4 m^2}{4} \right] \theta(x_0' - x) \right\}. \tag{45}$$

Clearly, photons radiated to the left of the domain wall ($z < 0$) are right-polarized, while those radiated to its right ($z > 0$) are left-polarized.

V. PAIR PRODUCTION

Momentum conservation prohibits the spontaneous photon decay $\gamma(k) \to f(p) + f(p')$ in vacuum. However, in chiral matter this channel is open due to the chiral anomaly. This is the cross-channel of the photon radiation computed in the previous section. The scattering matrix is now given by

$$S = i (2\pi)^3 \delta(\omega - \varepsilon' - \varepsilon) \delta(k_\perp - p_\perp - p'\perp) \frac{M}{\sqrt{8\varepsilon\varepsilon' \omega V^3}}, \tag{46}$$
where
\[
\mathcal{M} = -eQ\bar{u}(p')\gamma^\mu v(p)\epsilon_\mu \int_{-\infty}^{+\infty} dz \exp \left\{ i \int_0^z \frac{\tilde{q}_\perp^2 + \tilde{\kappa}_\lambda(z')}{2\omega x(1-x)} dz' \right\}.
\] (47)

We introduced new notations
\[
x = \varepsilon'/\omega,
\]
\[
\tilde{q}_\perp = p'_\perp - xk_\perp,
\] (48)
and
\[
\tilde{\kappa}_\lambda(z) = m^2 + (1-x)x\lambda\sigma_\chi \omega.
\] (49)

Notice that the sign in front of the second term is opposite to that of (29). Using the matrix elements listed in [37] one obtains
\[
\mathcal{M}_0 = -eQ\bar{u}(p')\gamma^\mu v(p)\epsilon_\mu
\]
\[
= -\frac{eQ}{\sqrt{2x(1-x)}} \left\{ -m(\sigma + \lambda)\delta_{\sigma',\sigma} + (2x - 1 - \lambda\sigma)(\tilde{q}_x + i\lambda\tilde{q}_y)\delta_{\sigma',-\sigma} \right\}.
\] (50)

### A. One infinite domain

In the case the entire matter is a single domain with constant \(\sigma_\chi\), integration over \(z\) produces the delta function similar to the one in (34). Substituting the amplitude (47) into (33) and performing summation over spins yields the photon decay rate
\[
\frac{dW}{dx} = \frac{\alpha Q^2}{4x(1-x)\omega} \left\{ (x^2 + (1-x)^2) (-\tilde{\kappa}) + m^2 \right\} \theta(-\tilde{\kappa}).
\] (52)

The condition \(\tilde{\kappa} < 0\) is satisfied if \(\lambda\sigma_\chi < 0, \omega > 4m^2/|\sigma_\chi|\) and \(x_1 < x < x_2\) where
\[
x_{1,2} = \frac{1}{2} \left( 1 \mp \sqrt{1 - \frac{4m^2}{|\sigma_\chi|\omega}} \right).
\] (53)

Thus, if \(\sigma_\chi > 0\), then only left-polarized photons with \(\lambda = -1\) can produce a pair, whereas the right-polarized photons cannot decay at all. The corresponding spectrum is
\[
\frac{dW_-}{dx} = \frac{\alpha Q^2}{4\omega} \left\{ (x^2 + (1-x)^2) \sigma_\chi \omega + 2m^2 \right\} \theta(x_2 - x)\theta(x - x_1),
\] (54)
\[
\frac{dW_+}{dx} = 0.
\] (55)

In a domain with \(\sigma_\chi < 0\) only right-polarized photons decay. The corresponding spectrum is obtained by replacing \(W_\pm \to W_\mp\) and \(\sigma_\chi \to -\sigma_\chi\) in (54) and (55).
B. Two semi-infinite domains separated by a domain wall at \( z = 0 \)

The calculation in the case of chiral matter consisting of two semi-infinite domains separated by a thin domain wall at \( z = 0 \) is analogous to that in Sec. IV B. The result is

\[
\frac{dN}{d^2q_\perp dx} = \frac{1}{(2\pi)^3} \frac{1}{8x(1-x)\omega^2} \sum_{\lambda,\sigma,\sigma'} |M|^2
\]

\[
= \frac{\alpha Q^2}{4\pi^2} \left\{ (x^2 + (1-x)^2) q_\perp^2 + m^2 \right\} \sum_{\lambda} \left| \frac{1}{q_\perp^2 + \kappa_\lambda - i\delta} - \frac{1}{q_\perp^2 + \bar{\kappa}_\lambda + i\delta} \right|^2 ,
\]

where the values of \( \tilde{\kappa}_\lambda \) at \( z < 0 \) and \( z > 0 \) are denoted by \( \tilde{\kappa}'_\lambda \) and \( \bar{\kappa}_\lambda \) respectively. Replacing \( \tilde{\kappa}_\lambda \to m^2 - m^2 x(1-x) \) yields the transition pair production spectrum \([15, 20]\). Integration over \( \tilde{q}_\perp \) gives

\[
\frac{dW_{-+}}{dx} = \frac{\alpha Q^2}{16x(1-x)\omega} \left\{ (x^2 + (1-x)^2) |\kappa'_+ + \kappa_-| + m^2 \right\} \theta(x - \max(x_1, x'_1))\theta(\min(x_2, x'_2) - x) ,
\]

if \( \sigma'_\chi > 0 \) and \( \sigma_\chi > 0 \) and

\[
\frac{dW_{+-}}{dx} = \frac{\alpha Q^2}{16x(1-x)\omega} \left\{ (x^2 + (1-x)^2) |\kappa'_+ + \kappa_-| + m^2 \right\} \theta(x - x'_1)\theta(x_2 - x)
\]

\[
+ \left[ (x^2 + (1-x)^2) |\kappa'_+| + \frac{m^2}{2} \right] \theta(x-x_1)\theta(x_2-x) \}
\]

if \( \sigma'_\chi > 0 \) and \( \sigma_\chi < 0 \). Here \( x'_{1,2} = 1/2 \pm \sqrt{1/4 - m^2/|\sigma'_\chi|\omega} \). Eqs. (58) and (59) clearly indicate that in a domain with positive/negative chiral conductivity most pairs are produced by left/right-polarized photons.

VI. SUMMARY AND DISCUSSION

The main result of this paper is that a free charged fermion moving through chiral matter spontaneously radiates electromagnetic radiation. This indicates instability of the single-fermion states. We derived the radiation spectrum in two cases: when the matter is a single \( CP \)-odd domain, given by (38)–(39), and when it consists of two such domains separated by a flat thin domain wall, given by (43). The photon polarization is determined by the sign of the chiral conductivity: if it is positive/negative, the radiation is right/left-polarized. The cross-channel process of spontaneous photon radiation is spontaneous pair production by a real photon. This indicates instability of the single-photon states. We computed the fermion spectrum and found...
that in a domain with positive/negative chiral conductivity only left/right-polarized photons decay, see (54), (55).

The rate of energy loss by single-particle states is found to be proportional to $\alpha \sigma_\chi$. Since the temporal evolution of chiral conductivity has much higher rate of $\sigma_\chi$, it seems plausible that it can play an important role in the long-time dynamics of the radiative instability and perhaps even tame it. This is a problem that deserves further investigation.

In electromagnetic plasma, the existence of the CP-odd domains would trigger the radiative instability causing radiation of photons of a certain polarization, and decay of photons of opposite polarization. These processes tend to polarize the plasma within a domain. Since the $2 \to 2$ scattering as well as transitions due to spatial inhomogeneities (38) are also of order $\alpha^2$ the question of whether there is an equilibrium polarization of electromagnetic field requires further investigation.

Despite the radiative instability, the Maxwell-Chern-Simons effective theory (2) is a useful tool to study macroscopic effects of the chiral anomaly if $\sigma_\chi$ is a sufficiently small compared to the typical energy scales. This is the case in the quark-gluon plasma where $\sigma_\chi$ is about two order of magnitudes lower than the plasma temperature (39), (40). Thus, convoluting (38) with the Fermi-Dirac distribution over the phase space gives photon spectrum spontaneously radiated by quarks in quark-gluon plasma. Since the spectrum is proportional to $\alpha^2$ it gives only a minor contribution to the total photon spectrum radiated by the plasma. However, the radiative instability due to the chiral anomaly of QCD is strongly enhanced by $\alpha_s/\alpha$ and may have a significant impact on the quark-gluon plasma phenomenology.

**ACKNOWLEDGMENTS**

This work was supported in part by the U.S. Department of Energy under Grant No. DE-FG02-87ER40371.

[1] D. Kharzeev and A. Zhitnitsky, “Charge separation induced by P-odd bubbles in QCD matter,” Nucl. Phys. A 797 (2007) 67

[2] D. E. Kharzeev, “The Chiral Magnetic Effect and Anomaly-Induced Transport,” Prog. Part. Nucl. Phys. 75, 133 (2014)

[3] V. B. Berestetskii, E. M. Lifshitz and L. P. Pitaevskii, “Quantum Electrodynamics,” Elsevier, 2012.

[4] S. M. Carroll, G. B. Field and R. Jackiw, “Limits on a Lorentz and Parity Violating Modification of Electrodynamics,” Phys. Rev. D 41, 1231 (1990).
[5] R. Lehnert and R. Potting, “Vacuum Cerenkov radiation,” Phys. Rev. Lett. 93, 110402 (2004)
[6] V. A. Kostelecky and A. G. M. Pickering, “Vacuum photon splitting in Lorentz violating quantum electrodynamics,” Phys. Rev. Lett. 91, 031801 (2003)
[7] K. Tuchin, “Electromagnetic field and the chiral magnetic effect in the quark-gluon plasma,” Phys. Rev. C 91, no. 6, 064902 (2015)
[8] K. Tuchin, “Taming instability of magnetic field in chiral matter,” Nucl. Phys. A 969, 1 (2018)
[9] N. Yamamoto, “Axion electrodynamics and nonrelativistic photons in nuclear and quark matter,” Phys. Rev. D 93, no. 8, 085036 (2016)
[10] Z. Qiu, G. Cao and X. G. Huang, “On electrodynamics of chiral matter,” Phys. Rev. D 95, no. 3, 036002 (2017)
[11] K. Fukushima, D. E. Kharzeev and H. J. Warringa, “The Chiral Magnetic Effect,” Phys. Rev. D 78, 074033 (2008)
[12] D. E. Kharzeev, “Topologically induced local P and CP violation in QCD × QED,” Annals Phys. 325, 205 (2010)
[13] D. E. Kharzeev and H. J. Warringa, “Chiral Magnetic conductivity,” Phys. Rev. D 80, 034028 (2009)
[14] X. G. Huang, “Electromagnetic fields and anomalous transports in heavy-ion collisions — A pedagogical review,” Rept. Prog. Phys. 79, no. 7, 076302 (2016)
[15] D. Schildknecht and B. G. Zakharov, “Transition radiation in quantum regime as a diffractive phenomenon,” Phys. Lett. A 355, 289 (2006)
[16] F. Wilczek, “Two Applications of Axion Electrodynamics,” Phys. Rev. Lett. 58, 1799 (1987).
[17] P. Sikivie, “On the Interaction of Magnetic Monopoles With Axionic Domain Walls,” Phys. Lett. B 137, 353 (1984).
[18] T. Kalaydzhyan, “Chiral superfluidity of the quark-gluon plasma,” Nucl. Phys. A 913, 243 (2013)
[19] K. Fujikawa and H. Suzuki, “Path integrals and quantum anomalies,” Oxford, UK: Clarendon (2004)
[20] V. N. Baier and V. M. Katkov, “Quantum theory of transition radiation and transition pair creation,” Phys. Lett. A 252, 263 (1999)
[21] M. Joyce and M. E. Shaposhnikov, “Primordial magnetic fields, right-handed electrons, and the Abelian anomaly,” Phys. Rev. Lett. 79, 1193 (1997)
[22] A. Boyarsky, J. Frohlich and O. Ruchayskiy, “Self-consistent evolution of magnetic fields and chiral asymmetry in the early Universe,” Phys. Rev. Lett. 108, 031301 (2012)
[23] H. Tashiro, T. Vachaspati and A. Vilenkin, “Chiral Effects and Cosmic Magnetic Fields,” Phys. Rev. D 86, 105033 (2012)
[24] P. Pavlovic, N. Leite and G. Sigl, “Chiral Magnetohydrodynamic Turbulence,” Phys. Rev. D 96, no. 2, 023504 (2017)
[25] N. Yamamoto, “Scaling laws in chiral hydrodynamic turbulence,” Phys. Rev. D 93, no. 12, 125016 (2016)
[26] X. l. Xia, H. Qin and Q. Wang, “Approach to Chandrasekhar-Kendall-Woltjer State in a Chiral Plasma,” Phys. Rev. D 94, no. 5, 054042 (2016)

[27] C. Manuel and J. M. Torres-Rincon, “Dynamical evolution of the chiral magnetic effect: Applications to the quark-gluon plasma,” Phys. Rev. D 92, no. 7, 074018 (2015)

[28] Z. V. Khaidukov, V. P. Kirilin, A. V. Sadofyev and V. I. Zakharov, “On Magnetostatics of Chiral Media,” arXiv:1307.0138 [hep-th].

[29] V. P. Kirilin, A. V. Sadofyev and V. I. Zakharov, “Anomaly and long-range forces,” arXiv:1312.0895 [hep-th].

[30] A. Avdoshkin, V. P. Kirilin, A. V. Sadofyev and V. I. Zakharov, “On consistency of hydrodynamic approximation for chiral media,” Phys. Lett. B 755, 1 (2016)

[31] Y. Akamatsu and N. Yamamoto, “Chiral Plasma Instabilities,” Phys. Rev. Lett. 111, 052002 (2013)

[32] M. Dvonnikov and V. B. Semikoz, “Magnetic field instability in a neutron star driven by the electroweak electron-nucleon interaction versus the chiral magnetic effect,” Phys. Rev. D 91, no. 6, 061301 (2015)

[33] P. V. Buividovich and M. V. Ulybyshev, “Numerical study of chiral plasma instability within the classical statistical field theory approach,” Phys. Rev. D 94, no. 2, 025009 (2016)

[34] G. Sigl and N. Leite, “Chiral Magnetic Effect in Protoneutron Stars and Magnetic Field Spectral Evolution,” JCAP 1601, no. 01, 025 (2016)

[35] V. P. Kirilin and A. V. Sadofyev, “Anomalous Transport and Generalized Axial Charge,” Phys. Rev. D 96, no. 1, 016019 (2017)

[36] D. B. Kaplan, S. Reddy and S. Sen, “Energy Conservation and the Chiral Magnetic Effect,” Phys. Rev. D 96, no. 1, 016008 (2017)

[37] G. P. Lepage and S. J. Brodsky, “Exclusive Processes in Perturbative Quantum Chromodynamics,” Phys. Rev. D 22, 2157 (1980).

[38] K. Tuchin, “Spontaneous topological transitions of electromagnetic fields in spatially inhomogeneous CP-odd domains,” Phys. Rev. C 94, no. 6, 064909 (2016)

[39] D. Kharzeev, A. Krasnitz and R. Venugopalan, “Anomalous chirality fluctuations in the initial stage of heavy ion collisions and parity odd bubbles,” Phys. Lett. B 545, 298 (2002)

[40] K. Fukushima and K. Mameda, “Wess-Zumino-Witten action and photons from the Chiral Magnetic Effect,” Phys. Rev. D 86, 071501 (2012)