A toy model of wave turbulence

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1. Classical wave turbulence

2. Discrete wave turbulence (DWT)

3. Energy cascades in DWT
Kolmogorov-Zakharov energy spectra

ASSUMPTIONS

- weak nonlinearity, $0 < \varepsilon \ll 1$,
- randomness of phases,
- infinite-box limit, $L/\lambda \to \infty$,
- existence of inertial interval $(k_1, k_2)$,
- locality in $k$-space (waves with wavelengths of the same order $k$ do interact),
- interactions are locally isotropic (no dependence on direction).

ADVANTAGES

- SIMPLE FORMULA: $E \sim k^{-\nu}$, $\nu > 0$, $\nu$ is constant for a given wave system.
- KZ-spectra do not depend on the initial conditions.
THE BAD NEWS: THERE ARE PROBLEMS

Results of laboratory experiments:

- energy cascade
  - no cascade: Hammack et al.. Fluid Mech., 2005), regular patterns, surface water waves;
  - cascade consists of two parts: discrete and continuous: great amount of experiments in various wave systems;
  - continuous part of energy spectrum is not KZ-spectrum: Mordant, PRL, 2008; a thin elastic steel plate is excited with a vibrator; Falcon et al., PRL, 2007 (grav., grav.-cap., cap., mercury)
- wave interactions are not local: Abdurakhimov et al, J. Phys.: Conf. Ser., 2009; capillary waves in He-II.
- form of energy spectra depends on initial conditions: Falcon et al., PRL, 2007 (grav., grav.-cap., cap., mercury); Xia et al., EPL, 2010 (capillary water waves).

Something is rotten in the state of Zakharov!
ATTEMPTS TO SOLVE THE PROBLEMS:

- frozen WT (Pushkarev+Zakharov, Physica D, 2000)
- sandpile model of WT (Nazarenko, J. Stat. Mech.: Theor. Exp., 2006)
- mesoscopic WT (Zakharov et al., JETP Lett., 2005)
- laminated WT (K, JETP Lett., 2006)
  - discrete layer - discrete WT (K, PRL 2007; EPL 2009; K+L’vov, PRL 2007; K, Cambridge University Press, 2010)
  - continuous layer - classical WT (Zakharov, L’vov, Falkovich, Springer, 1992)
- finite-dimensional WT (L’vov et al., PRE, 2009)

However, no model gives a general answer to a simple question:

- how to describe time evolution of a wave system beginning with one initially excited wave?

A partial answer is given by the model of discrete wave turbulence - in terms of resonance clusters.
1. **Classical wave turbulence**

2. **Discrete wave turbulence (DWT)**

3. **Energy cascades in DWT**
DWT - brief overview

- 1. solve resonance conditions in integers (a tricky thing due to Hilbert’s 10th Problem)
- 2. construct an NR-diagram for each resonance cluster;
- 3. write out explicit form of dynamical systems for each resonance cluster (automatically follows from the form of an NR-diagram)

Kartashova, Nonlinear resonance Analysis: Theory, Computation, Applications (Cambridge University Press, 2010)
1. Solution of resonance conditions, the idea

Take 2D surface water waves, \( \omega \sim \sqrt[4]{m^2 + n^2} \), frequency res. condition

\[
\omega_1 + \omega_2 = \omega_3 + \omega_4
\]  

\(1\)

**Brute-force computation:** 3 days for \( m, n \leq 128 \).

**q-class decomposition:** 3 minutes for \( m, n \leq 1000 \).

The idea of q-class decomposition:

\[
a \sqrt{3} + b \sqrt{5} = 0
\]  

\(2\)

has no solutions with integer \( a \) and \( b \).

Regard presentation (**it is unique!**):

\[
\sqrt[4]{m^2 + n^2} = \gamma \sqrt[4]{q}, \quad q = q_1^{\alpha_1} \cdots q_n^{\alpha_n}, \quad \alpha_j \leq 3,
\]  

\(3\)

then \(1\) has solutions (**necessary condition!**) only if 1) all 4 wavevectors have the same \( q \), or 2) they have pairwise equal \( q \)-s, i.e. \( q_1 = q_3 \) and \( q_2 = q_4 \).

Generalization for arbitrary finite number of different radicals - Besicovitch theorem (J. Lond. Math. Soc., 1940)
2. Structure of resonances

Geometrical structure

- Altogether 2500 Fourier modes in spectral domain $m, n \leq 50$ for $\omega \sim 1/\sqrt{m^2 + n^2}$
- Only 128 take part in resonances - $\sim 5\%$ of all modes
- 28 clusters - 18 are **integrable!**, $\sim 60\%$; max cluster - 12 modes

Topological structure

Kartashova (PRL, 1994)
NR-diagram defines \textit{uniquely} the form of dynamic system of resonance cluster and conservation laws, e.g. for PA-butterfly:

\[
\begin{align*}
\dot{B}_1|_b &= Z_b B_2^*|_b B_3|_b, \\
\dot{B}_3|_a &= -Z_a B_3|_b B_2|_a, \\
\dot{B}_2|_b &= Z_b B_1^*|_b B_3|_b, \\
\dot{B}_2|_a &= Z_a B_3^*|_b B_3|_a, \\
\dot{B}_3|_b &= -Z_b B_1|_b B_2|_b + Z_a B_2^*|_a B_3|_a
\end{align*}
\]

and conservation laws read

\[
I_a = |B_2|_a|^2 + |B_3|_a|^2, \quad I_b = |B_1|_b|^2 - |B_2|_b|^2, \quad I_{a,b} = |B_1|_b|^2 + |B_3|_b|^2 + |B_3|_a|^2.
\]
**Lab. experiments** (Chow, Henderson, Segur, Fluid Mech, 1996)

*Five frequencies but seven different modes (2D gravity-capillary waves).* The amplitudes and frequencies were identified as

\[ \begin{align*}
  A_1 & \leftrightarrow 60 \text{ Hz}, & A_2 & \leftrightarrow 35 \text{ Hz}, & A_3 & \leftrightarrow 25 \text{ Hz}, \\
  A_4 & \leftrightarrow 25 \text{ Hz}, & A_5 & \leftrightarrow 10 \text{ Hz}, & A_6 & \leftrightarrow 25 \text{ Hz}, & A_7 & \leftrightarrow 15 \text{ Hz},
\end{align*} \]

*modes with the same frequencies may have different wavevectors!* and resonance conditions for frequencies as

\[ \begin{align*}
  \omega_1 &= \omega_2 + \omega_3, & \omega_2 &= \omega_4 + \omega_5, & \omega_6 &= \omega_5 + \omega_7 
\end{align*} \]  (6)

with \( \omega_3 = \omega_4 = \omega_6. \)

NR-diagram gives more information than simple frequency analysis!
What do we know now:

- If initially excited mode is resonant or near-resonant, DWT gives the answer about further evolution of a wave system in the form of some explicitly given dynamical systems, most of them integrable!

What else do we need to know:

- If initially excited mode is NOT resonant or near-resonant - what happens?
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One-wave instability

Examples:
- parametric instability in classical mechanics,
- Suhl instability of spin waves,
- Oraevsky-Sagdeev decay instability of plasma waves,
- modulation instability in nonlinear optics,
- Benjamin-Feir instability in deep water, etc.

It is described at early stages of the process as interaction of three monochromatic wave trains: carrier \( \omega_c \), upper \( \omega_+ = \omega_c + \Delta \omega \) and lower \( \omega_- = \omega_c - \Delta \omega \) side-band waves with small \( \Delta \omega > 0 \) which form a quartet for one particular configuration which occurs when two of the waves coincide, with frequency resonance condition

\[
\omega_+ + \omega_- = 2\omega_c. \tag{7}
\]
Cascading cluster

\[
\begin{align*}
\omega_f &= \omega_{1,1} + \omega_{2,1}, \\
\omega_{1,1} &= \omega_{2,1} + \omega_{2,2}, \\
\omega_{2,1} &= \omega_{3,1} + \omega_{3,2}, \\
\vdots \\
\omega_{n-1,1} &= \omega_{n,1} + \omega_{n,2}, \\
E_1 &= p_1 E_f, \quad 0 < p_j < 1, \\
E_2 &= p_2 E_1, \\
E_3 &= p_3 E_2, \\
E_n &= p_n E_{n-1}
\end{align*}
\]
First assumption: intensity of a cascade $p$ is constant

\[ E_n = p^n E_0 \Rightarrow A_{n+1} = \sqrt{p} A_n \]  \hspace{1cm} (9)

and total energy of one cascading chain reads

\[ E = \lim_{n \to \infty} \sum_{n} A_o^2 p^n = \frac{A_0^2}{1 - p} = \text{const for } p < 1. \]  \hspace{1cm} (10)
Second assumption: each cascading mode is excited when corresponding increment of instability is maximal

Maximal increment at $n$-th cascade step can be computed (BF-form) as

$$
I_{\text{max},n} = \frac{|(\Delta \omega)_n|}{\omega_n A_n k_n} = 1,
$$

(11)

where $(\Delta \omega)_n = \omega_{n+1} - \omega_n$ is the frequency shift between two neighboring modes. It follows from (11) that

$$
\omega_{n+1} = \omega_n + \omega_n A_n k_n,
$$

(12)

and combination of $A_{n+1} = \sqrt{p} A_n$ and (12) yields

$$
A(\omega_n + \omega_n A_n k_n) = \sqrt{p} A_n
$$

(13)

is called \textit{chain equation}.
Computing amplitudes $A = A(\omega)$ and energy spectrum

$$\sqrt{p} A_n = A(\omega_n + \omega_n A_n k_n) = \sum_{s=0}^{\infty} \frac{A_n^{(s)}}{s!} (\omega_n A_n k_n)^s$$

$$= A_n + A'_n \omega_n A_n k_n + \frac{1}{2} A''_n (\omega_n A_n k_n)^2 + ...$$

(14)

Taking two first terms from RHS of (14) and combining with (13) we get

$$\sqrt{p} A_n = A_n + A'_n \omega_n A_n k_n \Rightarrow (A_n)' = \frac{\sqrt{p} - 1}{\omega_n k_n} \Rightarrow$$

$$A(\omega) = (\sqrt{p} - 1) \int \frac{d\omega}{\omega k} + \text{const}$$

(15)

Substitution of a specific dispersion relation into (19) gives dependence $A = A(\omega)$. Energy spectrum is $E \sim A^2$. ≡ historical moment ;}

The Nature of Turbulence, KITP, Santa Barbara
Termination and direction of a cascade

Condition for a cascade’s termination:

\[(\Delta \omega)_N = \omega_N \frac{(1 - \sqrt{p})}{2} + \omega_N k_N (A_0 - \frac{(1 - \sqrt{p})}{2}) = 0.\]  \hspace{1cm} (16)

Condition for a direct cascade:

\[\omega_{n+1} - \omega_n = \omega_n \frac{(1 - \sqrt{p})}{2} + \omega_n k_n (A_0 - \frac{(1 - \sqrt{p})}{2}) > 0\]  \hspace{1cm} (17)

Condition for an inverse cascade:

\[\omega_{n+1} - \omega_n < 0\]  \hspace{1cm} (18)
Surface water waves, I
(K+Shugan, submitted; Shugan+K, in preparation)

Dispersion relation $\omega^2 = k$ yields ODE

$$\omega_n^3 A_n' A_n + (1 - \sqrt{p}) A_n = 0 \Rightarrow (A_n)' = \frac{\sqrt{p} - 1}{\omega_n^3} \Rightarrow$$

$$A(\omega_n) = (\sqrt{p} - 1) \int \frac{d\omega_n}{\omega_n^3} \Rightarrow A_n = \frac{(1 - \sqrt{p})}{2} (\omega_n^{-2} - \omega_0^{-2}) + A_0$$

(19)

and for energy spectrum $E_n \sim A_n^2$ we get

$$E \sim \omega^{-\alpha}, \quad \text{with} \quad 2 \leq \alpha \leq 4,$$

(20)

depending on the details of initial conditions.
Surface water waves, II

Condition for a cascade termination in this case turns into ($\omega_0 = 1$)

$$\omega^2_N = \frac{1 - \sqrt{p}}{1 - \sqrt{p} - 2A_0}, \quad (21)$$

and specific initial conditions

$$\frac{(1 - \sqrt{p})}{2} = A_0 \quad (22)$$

yield infinite cascade (transition to the continuous spectrum). In this case

$$A_n \sim \omega_n^{-2} \Rightarrow A_n^2 \sim \omega_n^{-4} \quad (23)$$

which is classical Phillips spectrum for surface water waves.
General scheme for computing a cascade
(K, in preparation)

- Relation between neighboring amplitudes:
  \[ A_{n+1} = \sqrt{p}A_n \]  \hspace{1cm} (24)

- Maximal increment (changeable):
  \[ l_{\text{max},n} = \left| \omega_{n+1} - \omega_n \right| \frac{\omega_n A_n k_n}{\omega_n A_n k_n} = 1, \]  \hspace{1cm} (25)

- Chain equation (changeable):
  \[ A(\omega_n + \omega_n A_n k_n) = \sqrt{p}A(\omega_n) \]  \hspace{1cm} (26)

- ODE on amplitude (changeable):
  \[ \sqrt{p}A_n = A_n + A'_n \omega_n A_n k_n \]  \hspace{1cm} (27)

- Amplitudes and energy spectrum (changeable):
  \[ A(\omega) = (\sqrt{p} - 1) \int \frac{d\omega}{\omega k} + \text{const}, \quad E \sim A^2 \]  \hspace{1cm} (28)
- **Red vertical solid T-shaped lines having a string-like part**: resonant and near-resonant modes.
- **Blue vertical solid T-shaped lines**: fluxless ("frozen") modes.
- **Black dashed curves**: discrete cascades. **Solid part of a curve** shows continuous "tail" of a cascade.
- **Vertical yellow rectangular**: zero-frequency sideband.
- **Black circle**: source of possible intermittency (chaotic or FPU-like recurrence).
- **Vertical black bold line at \( \omega_{\text{crit}} \)**: "physical" termination of a cascade is due to **INCREASED NONLINEARITY**, not dissipation! "Tails" appearing after this line are not KZ-spectra!
Conclusions: a toy model allows CONSTRUCTIVELY to

- include so-called "discrete effects" in the form resonance clusters and their dynamical systems;
- compute an energy spectrum depending on the form of dispersion relation $\omega = \omega(k)$ and initial conditions;
- write out conditions for 1) a cascade’s termination, and b) the formation of a direct and/or an inverse cascade;
- explain formation of the narrow zero-band frequency with non-zero energy;
- obtain KZ-energy spectra as a result of some specific initial conditions.

The model predicts that direct cascade terminates not due to dissipation but due to the growth of nonlinearity.
