Irregularities of distribution for bounded sets and half-spaces

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Abstract
We prove a general result on irregularities of distribution for Borel sets intersected with bounded measurable sets or affine half-spaces.

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1 | INTRODUCTION

According to the Mathematical Reviews, the term “Irregularities of Distribution” had never been used in Mathematics before the publication of Roth’s seminal paper [26]. Starting from a celebrated conjecture of van der Corput, Roth proved the following result, which shows that no finite sequence in the unit square can be too evenly distributed.

Theorem 1 (Roth). There exists a constant \( c > 0 \) such that for every set \( P_N = \{z_1, z_2, \ldots, z_N\} \) of \( N \) points in the torus \( \mathbb{T}^2 \) we have

\[
\int_{\mathbb{T}^2} \left| \text{card}(P_N \cap I_x) - N x_1 x_2 \right|^2 \, dx_1 dx_2 \geq c \log(N),
\]

where \( I_x = [0, x_1] \times [0, x_2] \) for every point \( x = (x_1, x_2) \in [0, 1)^2 \).

The monograph [5] is the basic reference. See also [11, 16, 22, 25, 30].
The rectangles $I_x$ in Roth’s theorem can be replaced by different families of sets (e.g., disks or intersections of $[0,1)^2$ with half-planes), and the above (sharp) log(N) estimate may change drastically. See, for example, [4, 10, 25, 27].

More generally, as pointed out by William Chen in [13], today many of the problems that concern Irregularities of Distribution, also called Geometric Discrepancy, can be formulated in the following way:

Let $P_N$ be a set of $N$ points in $\mathbb{R}^d$ ($d \geq 2$) and let $E \subseteq \mathbb{R}^d$. We want to estimate the quality of the distribution of these points with respect to a probability measure $\mu$ supported in $E$. We consider a reasonably large family $\mathcal{R}$ of measurable sets and, for $R \in \mathcal{R}$, we introduce the discrepancy

$$D_N(R) = \text{card}(P_N \cap R) - N\mu(R).$$

The aim of this paper is to follow the above approach, introduce a general point of view and prove a few theorems which extend several known results.

We may choose $E$ and $\mu$ in a fairly general way: in particular $E \subset \mathbb{R}^d$ can be a $k$-dimensional manifold ($k \geq 2$) and $\mu$ the associated Hausdorff measure, or $E$ can be a fractal set endowed with a general $\alpha$-dimensional measure $\mu$ (see below). As for the family $\mathcal{R}$ we consider affine copies of a given body $\Omega \subset \mathbb{R}^d$ with possibly fractal boundary. We also obtain results for spherical caps (that are intersections of a given manifold and all the affine half-spaces). See Subsection 2.2 below.

Our arguments are essentially Fourier analytic and a classical lemma of Cassels and Montgomery is a basic tool (see [25, chapter 6], see also [8, 10]).

## 2 NOTATION AND MAIN RESULTS

Let $\mu$ be a positive Borel measure on $\mathbb{R}^d$, let $E$ be its support, and let $\mu(E) = 1$. Let us assume the existence of $0 < \alpha \leq d$ and $c > 0$ such that for every $d$-dimensional open ball $B(x, r)$ with center $x$ and radius $r$ we have

$$\mu(B(x, r)) \leq c r^\alpha. \quad (1)$$

We recall that, by Frostman’s lemma (see, e.g., [24, chapter 2]), if $E$ is a Borel set with positive $\alpha$-dimensional Hausdorff measure, then one can always find a finite Borel measure $\mu$ supported on $E$ and satisfying (1).

In the following, we will denote by $|F|$ the Lebesgue measure of a measurable set $F \subseteq \mathbb{R}^d$.

Given a Borel set $\Omega \subset \mathbb{R}^d$, with $|\Omega| > 0$, let $R_\Omega$ be the family of rotated, dilated and translated copies of $\Omega$. More precisely, for given $a$ and $b$, we set

$$R_\Omega = \{(x + \tau \sigma \Omega) \cap E : x \in \mathbb{R}^d, \ a \leq \tau \leq b, \sigma \in SO(d)\}$$

and we study $D_N(R)$ for $R \in R_\Omega$. See, for example, [16, p. 212] for a similar point of view.

We are mainly interested in the following two cases.

- $\Omega$ is a bounded Borel set satisfying suitable regularity conditions related to the Minkowski content of the boundary (see Theorem 2 and Remark 3).
- $\Omega$ is a half-space (see Theorem 10).
We obtain the discrepancy of half-spaces as limit case of the discrepancy of a family of balls of diverging radii.

### 2.1 Discrepancy for bounded sets

Our first result exhibits a lower bound for the discrepancy associated to the family \( R_\Omega \).

**Theorem 2.** Let \( \mu \) be a positive Borel measure on \( \mathbb{R}^d \), with support in a bounded set \( E \) and such that \( \mu(E) = 1 \). We also assume that \( \mu \) satisfies (1) for a given \( 0 < \alpha \leq d \). Let \( \Omega \subset \mathbb{R}^d \) be a given bounded Borel set and assume the existence of constants \( 0 < \beta \leq 1 \) and \( \kappa_1 > 0 \) such that for every \( h \in \mathbb{R}^d \) small enough we have

\[
|(h + \Omega) \triangle \Omega| \leq \kappa_1 |h|^\beta
\]

(here \( A \triangle B = (A \setminus B) \cup (B \setminus A) \) denotes the symmetric difference). Also assume there exist \( \kappa_2 > 0 \), a direction \( \Theta \), and a decreasing sequence \( t_n \to 0 \), satisfying

\[
t_n \leq \kappa_3 t_{n+1}
\]

for a suitable \( \kappa_3 > 0 \), such that for every \( n \) we have

\[
\kappa_2 t_n^\beta \leq \left| (t_n\overline{\Theta} + \Omega) \triangle \Omega \right|.
\]

Then there exist positive constants \( a, b \) and \( c \) such that for every point distribution \( P_N = \{z_1, z_2, \ldots, z_N\} \) we have

\[
\left\{ \int_a^b \int_{SO(d)} \int_{\mathbb{R}^d} |D_N(x + \tau\sigma \Omega)|^2 \, dx \, d\sigma \, d\tau \right\}^{1/2} \geq c N^{1/2-\beta/(2\alpha)}.
\]

In the next section, we shall see that the above result is sharp.

If \( E \) has positive Lebesgue measure and \( \mu \) is the Lebesgue measure restricted to \( E \), then we can take \( \alpha = d \). In this case the previous result can be found in [16, Theorem 2.10].

**Remark 3.** The value of \( \beta \) in (2) is related to the fractal dimension of the boundary \( \partial \Omega \). It is not difficult to show that

\[
(h + \Omega) \triangle \Omega \subseteq \left\{ x \in \mathbb{R}^d : \text{dist}(x, \partial \Omega) \leq |h| \right\}.
\]

Recall that \( \partial \Omega \) has finite \( (d - \beta) \)-dimensional Minkowski content if for some \( c > 0 \) and for every \( 0 < t < 1 \) one has

\[
\left| \left\{ z \in \mathbb{R}^d : \text{dist}(z, \partial \Omega) \leq t \right\} \right| \leq ct^\beta
\]

(see, e.g., [19]). Hence, (6) implies (2).
The above theorem has an immediate corollary.

**Corollary 4.** Let $\Omega$ be as in the previous theorem. Then there exists a constant $c > 0$ such that for every point distribution $P_N = \{z_1, z_2, \ldots, z_N\}$ there exists an affine copy $\Omega^* = x^* + \tau^* \sigma^* \Omega$ of $\Omega$ such that

$$|D_N(\Omega^*)| \geq c N^{1/2 - \beta/(2\alpha)}.$$

### 2.2 Examples

We now discuss a few examples of $\mu$, $E$ and $\Omega$.

**Example 5.** It is easy to see that if $\Omega$ is a convex body, then (2) and (4) hold with $\beta = 1$. We will show in the Example A.2 that for every $0 < \beta < 1$ there are sets $\Omega$ satisfying (2) and (4).

**Example 6.** As already mentioned in the Introduction, the study of the discrepancy associated to a point distribution on a given manifold (not necessarily smooth) embedded in $\mathbb{R}^d$ is one of the main motivation of this work. Then we consider a metric space $Y$ (with distance $\text{dist}$) embedded in $\mathbb{R}^d$ by a mapping $\Phi : Y \to \mathbb{R}^d$ such that

$$C_1 \text{dist}(y_1, y_2) \leq |\Phi(y_1) - \Phi(y_2)|.$$

Let $\nu$ be an $\alpha$-dimensional probability measure on $Y$, that is,

$$\nu(B_r(y)) \leq cr^\alpha,$$

where $B_r(y)$ denotes the ball centered at $y$ with radius $r$ in the metric space $Y$. For every Borel set $F \subseteq \mathbb{R}^d$ define

$$\mu(F) = \nu(\Phi^{-1}(F)).$$

Let $x_0 \in \mathbb{R}^d$. If $x_0 \in \Phi(Y)$, then

$$\Phi^{-1}(B_r(x_0)) \subseteq B_{r/C_1}(\Phi^{-1}(x_0))$$

and therefore

$$\mu(B_r(x_0)) \leq \nu\left(B_{r/C_1}(\Phi^{-1}(x_0))\right) \leq c(r/C_1)^\alpha \leq c_2 r^\alpha.$$

If $x_0 \notin \Phi(Y)$ but $B_r(x_0) \cap \Phi(Y) \neq \emptyset$, then there exists $z \in \Phi(Y)$ such that $|x_0 - z| < r$ and therefore

$$\mu(B_r(x_0)) \leq \mu(B_{2r}(z)) \leq \nu\left(B_{2r/C_1}(\Phi^{-1}(z))\right) \leq c_2 r^\alpha.$$
Finally, if \( B_r(x_0) \cap \Phi(Y) = \emptyset \), then \( \mu(B_r(x_0)) = 0 \). Hence \( \mu \) is a \( \alpha \)-dimensional measure in \( \mathbb{R}^d \). If \( \Omega \) is a given convex body, then Theorem 2 gives the estimate

\[
\left\{ \int_a^b \int_{SO(d)} \int_{\mathbb{R}^d} |D_N(x + \tau \sigma \Omega)|^2 \, dx \, d\sigma \, d\tau \right\}^{1/2} \geq c \, N^{1/2 - 1/(2\alpha)}. \tag{7}
\]

As a particular case of the previous example, we have the following result.

**Corollary 7.** Let \( \mu \) be the surface measure on a regular \( k \)-dimensional surface \( E \) embedded in \( \mathbb{R}^d \) and let \( \Omega \) be the unit ball in \( \mathbb{R}^d \). Then

\[
\left\{ \int_a^b \int_{\mathbb{R}^d} |D_N(x + \tau \Omega)|^2 \, dx \, d\tau \right\}^{1/2} \geq c \, N^{1/2 - 1/(2k)}. \tag{8}
\]

When \( E \) is a \( k \)-dimensional sphere the above corollary gives an estimate of the spherical cap discrepancy and we obtain the result in [5, section 7.4]. Observe that for \( k = 1 \) the estimate (8) does not give a divergent lower bound. Indeed, the example of \( N \) equally spaced points on a 1-dimensional circle in \( \mathbb{R}^d \) shows that the discrepancy can be bounded. If \( \Omega \) is a ball, \( P_N \) are \( N \) equispaced points on a 1-dimensional circle of length 1, and \( \mu \) is the natural measure on \( E \), then \((x + \tau \Omega) \cap E \) is an arc and \(|D_N(x + \tau \Omega)| \leq 1\). We observe that Theorem 2 shows that when \( \alpha > 1 \) this phenomenon disappears.

**Example 8.** The following iterative construction of a measure on the snowflake curve is a classical example of an \( \alpha \)-dimensional measure with \( \alpha \) not an integer. Let \( C_0 \) be the boundary of an equilateral triangle with a horizontal side and side length equal to 1. At the stage \( n \) of the construction \( C_n \) contains \( 3 \cdot 4^n \) segments of length \( 3^{-n} \) and we construct \( C_{n+1} \) replacing the middle third of every segment by the other two sides of an “external” equilateral triangle. Let \( \mu_n \) be the probability measure that assigns the measure \( (3 \cdot 4^n)^{-1} \) to every side in \( C_n \). As \( n \) goes to infinity these polygonal curves \( C_n \) approach the snowflake curve \( C \) and the measure \( \mu_n \) converges to a measure \( \mu \). It is well-known that \( C \) has Hausdorff dimension \( \log_3 4 \) and that \( \mu \) is, up to a normalization, the \( \log_3 4 \)-dimensional Hausdorff measure restricted to \( C \) and satisfies (1) with \( \alpha = \log_3 4 \). See [19, Example 9.5], [29], and [23, Theorem 4.14]. Hence, if \( E = C \) and \( \Omega \) is any convex body,

\[
\left\{ \int_a^b \int_{SO(d)} \int_{\mathbb{R}^d} |D_N(x + \tau \sigma \Omega)|^2 \, dx \, d\sigma \, d\tau \right\}^{1/2} \geq c \, N^{1/2 - 1/(2 \log_3 4)}. \]

**Example 9.** Let now \( E \) be the bounded set such that \( \partial E \) is the above snowflake curve \( C \), let \( \mu \) be the 2-dimensional Lebesgue measure and let \( \Omega = E \). We shall see in the Appendix that (2) and (4) hold with \( \beta = 2 - \log_3 4 \). Then Theorem 2 gives the estimate

\[
\left\{ \int_a^b \int_{SO(d)} \int_{\mathbb{R}^d} |D_N(x + \tau \sigma \Omega)|^2 \, dx \, d\sigma \, d\tau \right\}^{1/2} \geq c \, N^{(\log_3 2)/2}. \]
2.3  |  Half-space discrepancy

We now consider the half-space discrepancy. For \( \rho \in \mathbb{R} \) and \( \Theta \in \Sigma_{d-1} \) (the unit \((d-1)\)-dimensional sphere), we consider the affine half-space

\[
\Pi(\rho, \Theta) = \{ x \in \mathbb{R}^d : x \cdot \Theta > \rho \}
\]

and, for every point distribution \( P_N = \{z_1, z_2, \ldots, z_N\} \), the associated discrepancy

\[
D_N(\Pi(\rho, \Theta)) = \sum_{j=1}^{N} \chi_{\Pi(\rho, \Theta)}(z_j) - N\mu(\Pi(\rho, \Theta)).
\]

We have the following results.

**Theorem 10.** Let \( \mu \) be a positive Borel measure on \( \mathbb{R}^d \) which satisfies (1) for a given \( 0 < \alpha \leq d \). Assume that the support \( E \) of \( \mu \) is contained in a ball \( B(0, r_0) \). Then there exists \( c > 0 \) such that for every point distribution \( P_N = \{z_1, z_2, \ldots, z_N\} \) contained in \( B(0, r_0) \) we have

\[
\left\{ \int_0^{+\infty} \int_{\Sigma_{d-1}} |D_N(\Pi(\rho, \Theta))|^2 \, d\Theta d\rho \right\}^{1/2} \geq c N^{1/2-1/(2\alpha)}.
\]

In the next section, we shall see that the above result is sharp.

Since the discrepancy \( D_N(\Pi(\rho, \Theta)) \) has compact support as a function of \( \rho \) and \( \Theta \), one has the following corollary.

**Corollary 11.** There exists a constant \( c > 0 \) such that for every point distribution \( P_N = \{z_1, z_2, \ldots, z_N\} \) there exist a half-space \( \Pi^* = \Pi(t^*, \Theta^*) \) such that

\[
|D_N(\Pi^*)| \geq c N^{1/2-1/(2\alpha)}.
\]

The study of the half-space discrepancy goes back to a problem raised by Roth for the unit disc and by Erdős for the sphere. See [5, chapter 7.3], [13], [14, chapter 3.2], [17], and [28, pp. 124–125].

If \( E = [0, 1]^d \) or \( E \) is the unit ball, and \( \mu \) is the Lebesgue measure, then a slightly weaker version of Corollary 11 has been proved by Beck [3] using Fourier analysis. The sharp estimate in Theorem 10 was first proved by Alexander [2] using an integral geometric approach (see also [12]). We now show that it is possible to obtain the sharp result using Fourier analysis. More precisely, we use Theorem 2 to obtain the half-space discrepancy as a limit of the discrepancies of balls with diverging radii.

If \( E \subset \mathbb{R}^d \) is a manifold and \( \mu \) is the Hausdorff measure on \( E \), then Theorem 10 gives a lower bound for the discrepancy of the spherical caps, that are the intersections of \( E \) and affine half-spaces.

If \( E \) is a Euclidean sphere, then the spherical caps coincide with the intersections of \( E \) and balls. Then Corollaries 4 and 11 coincide and have been proved by Beck (see [5, Theorem 24C]).

If \( E \) is a compact set in \( \mathbb{R}^d \) of Hausdorff dimension \( \alpha \), \( \mu \) is the associated Hausdorff measure and (1) holds true, then Corollary 11 has been proved by Albrecher, Matousek, and Tichy (see
by adapting the technique of Alexander to the fractal setting. See the remark at the end of [1, p. 244].

3 | UPPER BOUNDS

The following theorem shows that Theorem 2 is sharp when \( \mu \) is the Lebesgue measure, \( \alpha = d \) and \( \partial \Omega \) has finite \((d - \beta)\)-dimensional Minkowski content (see Remark 3).

The discussion in [1, section 2.2] shows that also the lower bound in Theorem 10 is “best possible.”

**Theorem 12.** Let \( E \) be a bounded Borel set in \( \mathbb{R}^d \) with positive Lebesgue measure and for every measurable set \( F \) let

\[
\mu(F) = \frac{|E \cap F|}{|E|}
\]

be the Lebesgue measure restricted and normalized to \( E \). Assume there exists \( c_1 > 0 \) such that for every \( 0 < r < \text{diam}(E) \) and every \( x \in E \), we have

\[
c_1 r^d \leq \mu(B(x, r))
\]

(of course we have \( \mu(B(x, r)) \leq c_2 r^d \)). Let \( \Omega \subseteq \mathbb{R}^d \) be a given bounded Borel set that satisfies (6) for some \( 0 < \beta \leq 1 \) and let \( 0 < a < b \). Then there exists \( c > 0 \) such that for every \( N > 0 \) there exists a distribution of \( N \) points \( \mathcal{P}_N = \{z_1, z_2, \ldots, z_N\} \subset E \) such that

\[
\left\{ \int_a^b \int_{SO(d)} \int_{\mathbb{R}^d} |D_N(x + \tau \sigma \Omega)|^2 \, dx \, d\sigma \, d\tau \right\}^{1/2} \leq c N^{1/2 - \beta/(2d)}. \tag{11}
\]

**Proof.** Since \( \Omega \) and \( E \) are bounded, then \( D_N(x + \tau \sigma \Omega) \) has compact support, hence

\[
\int_a^b \int_{SO(d)} \int_{\mathbb{R}^d} |D_N(x + \tau \sigma \Omega)|^2 \, dx \, d\sigma \, d\tau
\]

\[
= \int_a^b \int_{SO(d)} \int_{\{|x| \leq R\}} |D_N(x + \tau \sigma \Omega)|^2 \, dx \, d\sigma \, d\tau
\]

for a suitable \( R > 0 \). Let us show that the above theorem follows applying [6, Corollary 8.2] to the collection of sets

\[ G = \{(x + \tau \sigma \Omega) \cap E : |x| \leq R, a \leq \tau \leq b, \sigma \in SO(d)\}. \]

First, observe that by [20, Theorem 2], we can always decompose \( E \) as the union of \( N \) sets of measure \( N^{-1} \) and diameter of the order of \( N^{-1/d} \) as required by [6, Corollary 8.2]. It remains to check that for every \( \mathcal{G} \in G \) we have

\[
|\{x \in \mathcal{G} : \text{dist}(x, E \setminus \mathcal{G}) \leq t\}| + |\{x \in E \setminus \mathcal{G} : \text{dist}(x, \mathcal{G}) \leq t\}| \leq ct^\beta.
\]
Let $R = x + \tau \sigma \Omega$ and let $G = R \cap E$. Then one has

$$\{x \in G : \text{dist}(x, E \setminus G) \leq t\} \cup \{x \in E \setminus G : \text{dist}(x, G) \leq t\} \subseteq \{y \in E : \text{dist}(y, \partial R) \leq t\}.$$  

Hence,

$$|\{y \in G : \text{dist}(y, E \setminus G) \leq t\}| + |\{y \in E \setminus G : \text{dist}(x, G) \leq t\}| \leq |\{y \in E : \text{dist}(y, \partial R) \leq t\}| \leq \left| \left\{ w \in \mathbb{R}^d : \text{dist}(w, \partial \Omega) \leq \tau^{-1}t \right\} \right| \leq c\tau^d\beta t^\beta \leq cb^d\beta t^\beta. \quad \square$$

## 4 \quad PROOF OF THEOREM 2

For the proof of Theorem 2, we will use the Fourier transform of the discrepancy function associated to the point distribution $\mathcal{P}_N = \{z_1, z_2, ... , z_N\}$,

$$x \mapsto D_N(x, \tau, \sigma) = \text{card}(\mathcal{P}_N \cap (x + \tau \sigma \Omega)) - N\mu(x + \tau \sigma \Omega).$$

**Lemma 13.** We have

$$\hat{D}_N(\xi, \tau, \sigma) = \int_{\mathbb{R}^d} D_N(x, \tau, \sigma)e^{-2\pi i \xi \cdot x}dx = \left\{ \sum_{j=1}^{N} e^{-2\pi i \xi \cdot z_j} - N\hat{\mu}(\xi) \right\} \hat{\chi_{\tau \sigma \Omega}}(\xi).$$

**Proof.** This is a simple consequence of the fact that

$$D_N(x, \tau, \sigma) = \chi_{-\tau \sigma \Omega} * \left( \sum_{j=1}^{N} \delta_{z_j} - N\mu \right)(x). \quad \square$$

The following lemma is an extension of [7, Lemma 4.2], see also [9, Theorem 8].

**Lemma 14.** Let $\Omega$ be as in Theorem 2.

1. There exist positive constants $c_1, c_2, \gamma$ and $\delta$ such that for $\rho$ large enough

$$c_1\rho^{-\beta} \leq \int_{\{\eta \leq |\xi| \leq \delta \rho\}} |\hat{\chi_{\Omega}}(\xi)|^2 d\xi \leq c_2 \rho^{-\beta}. $$

2. There exist positive constants $c_3, c_4, \gamma$ and $\delta$ such that for $|\xi|$ large enough

$$c_3 |\xi|^{-d-\beta} \leq \int_{\gamma} \int_{SO(d)} |\hat{\chi_{\mu \sigma \Omega}}(\xi)|^2 dud\sigma \leq c_4 |\xi|^{-d-\beta}. $$
Proof.

(1) We first show that

$$\int_{\{|\xi| \geq \rho\}} |\mathcal{X}_\Omega(\xi)|^2 d\xi \leq c \rho^{-\beta}. $$

Let \(\{U_j\}_{j=1}^d\) be a partition of the unit sphere such that if \(u = (u_1, \ldots, u_d) \in U_j\), then \(|u_j| \geq \kappa > 0\). Let \(\{e_j\}_{j=1}^d\) be the canonical orthonormal basis in \(\mathbb{R}^d\). Let \(k \in \mathbb{N}\) and let \(j = 1, \ldots, d\) be given. Also let \(h = \frac{1}{3\pi 2^{k+1}} e_j\). Then (2) yields

$$\kappa_1 \beta \geq \int_{\mathbb{R}^d} \left| 1 - e^{-2\pi i h \cdot \xi} \right|^2 \left| \mathcal{X}_\Omega(\xi) \right|^2 d\xi$$

By our choice of \(h\) and the definition of \(U_j\), if \(\frac{\xi}{|\xi|} \in U_j\), then we have

$$\frac{\kappa}{3} \leq |2\pi h \cdot \xi| \leq \frac{2}{3}$$

and therefore

$$\left| 1 - e^{-2\pi i h \cdot \xi} \right| \geq c > 0.$$

Hence,

$$\kappa_1 \beta \geq \int_{\mathbb{R}^d} \left| 1 - e^{-2\pi i h \cdot \xi} \right|^2 \left| \mathcal{X}_\Omega(\xi) \right|^2 d\xi.$$

Repeating the above argument for every \(j\) yields

$$c 2^{-k \beta} \geq \int_{\{2^k \leq |\xi| \leq 2^{k+1}\}} \left| \mathcal{X}_\Omega(\xi) \right|^2 d\xi.$$

Now let \(k_0\) satisfy \(2^{k_0} \leq \rho < 2^{k_0+1}\). Then

$$\int_{\{|\xi| \geq \rho\}} |\mathcal{X}_\Omega(\xi)|^2 d\xi \leq \sum_{k=k_0}^{+\infty} \int_{\{2^k \leq |\xi| \leq 2^{k+1}\}} |\mathcal{X}_\Omega(\xi)|^2 d\xi$$

$$\leq c \sum_{k=k_0}^{+\infty} 2^{-k \beta} \leq c \rho^{-\beta}. $$
For $\gamma > 0$ to be chosen later, we have

\[
\int_{\{|\xi| \leq \gamma \rho\}} |\xi|^2 |\hat{X}_\Omega(\xi)|^2 d\xi \\
\leq \int_{\{|\xi| < 1\}} |\hat{X}_\Omega(\xi)|^2 d\xi + \sum_{1 \leq 2^k \leq \gamma \rho} \int_{\{2^k \leq |\xi| < 2^{k+1}\}} |\xi|^2 |\hat{X}_\Omega(\xi)|^2 d\xi
\]
\[
\leq c + 4 \sum_{1 \leq 2^k \leq \gamma \rho} 2^{2k} \int_{\{2^k \leq |\xi| < 2^{k+1}\}} |\hat{X}_\Omega(\xi)|^2 d\xi
\]
\[
\leq c + c \sum_{2^k \leq \gamma \rho} 2^{2k} 2^{-k\beta}
\]
\[
= c + c \sum_{2^k \leq \gamma \rho} 2^{(2-\beta)k} \leq c(\gamma \rho)^{2-\beta}.
\]

Let $n$ satisfy

\[
t_{n+1} \leq \rho^{-1} \leq t_n
\]

and let $h = t_n \Theta$ with $\Theta$ as in (4). Then

\[
\kappa_2 \rho^{-\beta} \leq \kappa_2 t_n^\beta \leq |\Omega \triangle (\Omega + h)| = \int_{\mathbb{R}^d} \left|1 - e^{-2\pi i h \cdot \xi}\right|^2 |\hat{X}_\Omega(\xi)|^2 d\xi
\]
\[
\leq 4\pi^2 \int_{\{|\xi| \leq \gamma \rho\}} |h|^2 |\xi|^2 |\hat{X}_\Omega(\xi)|^2 d\xi + 4 \int_{\{\gamma \rho < |\xi| < \delta \rho\}} |\hat{X}_\Omega(\xi)|^2 d\xi
\]
\[
+ 4 \int_{\{|\xi| > \delta \rho\}} |\hat{X}_\Omega(\xi)|^2 d\xi
\]
\[
\leq c t_n^2 (\gamma \rho)^{2-\beta} + 4 \int_{\{\gamma \rho < |\xi| < \delta \rho\}} |\hat{X}_\Omega(\xi)|^2 d\xi + c(\delta \rho)^{-\beta}
\]

Observe that by (3), we have

\[
\rho t_n \leq \rho \kappa_3 t_{n+1} \leq \kappa_3.
\]

If follows that

\[
4 \int_{\{\gamma \rho < |\xi| < \delta \rho\}} |\hat{X}_\Omega(\xi)|^2 d\xi \geq \kappa_2 \rho^{-\beta} - c(\gamma \rho)^{2-\beta} t_n^2 - c(\delta \rho)^{-\beta}
\]
\[
= \rho^{-\beta} \left(\kappa_2 - c\gamma^{2-\beta} \rho^2 t_n^2 - c\delta^{-\beta}\right)
\]
\[
\geq c \rho^{-\beta}
\]

for $\gamma$ small enough and $\delta$ large enough.
(2) Let $\gamma$ and $\delta$ be as in (1). If $d\Theta$ is the normalized surface measure on the sphere $\Sigma_{d-1}$, then

$$
\int_{\gamma}^{\delta} \int_{SO(d)} |\hat{\chi}_{\omega\sigma\Omega}(\xi)|^2 d\sigma d\xi
= \int_{\gamma}^{\delta} \int_{SO(d)} u^{2d} \left| \hat{\chi}_{\omega\sigma^{-1}\xi}(u) \right|^2 d\sigma d\xi
= \int_{\gamma}^{\delta} u^{2d} \int_{\Sigma_{d-1}} \left| \hat{\chi}_{\omega}(u|\xi|\Theta) \right|^2 d\sigma d\Theta
= |\xi|^{-2d} \int_{\gamma}^{\delta} \int_{\Sigma_{d-1}} t^{2d} |\hat{\chi}_{\omega}(t\Theta)|^2 dt d\Theta
= |\xi|^{-d} \int_{\gamma}^{\delta} \int_{\Sigma_{d-1}} \left( t|\xi|^{-1} \right)^{d+1} |\hat{\chi}_{\omega}(t\Theta)|^2 t^{d} dt d\Theta
\approx \left| \xi \right|^{-d} \int_{\{\xi \in \mathbb{R}^d : |\xi| \leq \gamma \}} |\hat{\chi}_\omega(\xi)|^2 d\xi \approx |\xi|^{-d-\beta}.
$$

In the proof of Theorem 2, we need a positive function with positive and compactly supported Fourier transform. Let $\psi$ be a positive smooth radial function with support in $\{x \in \mathbb{R}^d : |x| \leq \frac{1}{2}\}$ such that $\|\psi\|_1 = 1$, and let

$$
K(x) = \left( \hat{\psi}(x) \right)^2.
$$

Then

$$
0 \leq \psi \ast \psi(x) = \hat{K}(x) \leq 1,
$$

and $\hat{K}(\xi) = 0$ for $|\xi| \geq 1$. For every $M \geq 1$ let

$$
K_M(x) = M^{d}K(Mx).
$$

(12)

Since $\hat{K}_M(\xi) = \hat{K}(\xi/M)$, we have

$$
0 \leq \hat{K}_M(\xi) \leq 1,
$$

and $\hat{K}_M(\xi) = 0$ for $|\xi| \geq M$. Moreover, for every given $L > 0$ there exists $c > 0$ such that

$$
0 \leq K(x) \leq c \begin{cases} 
1 & \text{if } |x| \leq 1 \\
|x|^{-L} & \text{if } |x| \geq 1
\end{cases}
$$

and therefore

$$
|K_M(x)| \leq \begin{cases} 
M^d & \text{if } |x| \leq \frac{1}{M}, \\
M^{d-L}|x|^{-L} & \text{if } |x| \geq \frac{1}{M}
\end{cases}
$$

(13)
Lemma 15. Let \( \mu \) be as in Theorem 2. There exists \( c > 0 \) such that for every \( M \geq 1 \) and for every \( z \in \mathbb{R}^d \) we have

\[ |K_M * \mu(z)| \leq cM^{d-\alpha}. \]

Proof. Let \( z \in \mathbb{R}^d \) and let \( B_{2^{-k}} = \{ x \in \mathbb{R}^d : |z - x| \leq 2^{-k} \} \). Then

\[
K_M * \mu(z) = \int_{\mathbb{R}^d} K_M(z - x) d\mu(x) \\
= \int_{|z - x| > 1} K_M(z - x) d\mu(x) + \sum_{k=0}^{+\infty} \int_{B_{2^{-k}} \setminus B_{2^{-k-1}}} K_M(z - x) d\mu(x) \\
= I_1(z) + I_2(z).
\]

Using (13) with \( L = \alpha \), we readily obtain

\[
I_1(z) \leq c \int_{|z - x| > 1} M^{d-\alpha} d\mu(x) \leq cM^{d-\alpha}.
\]

Moreover, using (13) with \( L \) large enough, we have

\[
I_2(z) = \sum_{2^{-k}M \leq 1} \int_{B_{2^{-k}} \setminus B_{2^{-k-1}}} K_M(z - x) d\mu(x) \\
+ \sum_{2^{-k}M > 1} \int_{B_{2^{-k}} \setminus B_{2^{-k-1}}} K_M(z - x) d\mu(x) \\
\leq c \sum_{2^{-k}M \leq 1} \int_{B_{2^{-k}} \setminus B_{2^{-k-1}}} M^{d} d\mu(x) \\
+ c \sum_{2^{-k}M > 1} \int_{B_{2^{-k}} \setminus B_{2^{-k-1}}} M^{d-L} |z - x|^{-L} d\mu(x) \\
\leq c \sum_{2^{-k}M \leq 1} M^{d} \mu(B_{2^{-k}}) + c \sum_{2^{-k}M > 1} M^{d-L} 2^{kL} \mu(B_{2^{-k}}) \\
\leq cM^{d} \sum_{2^{-k}M \leq 1} 2^{-k\alpha} + cM^{d-L} \sum_{2^{-k}M > 1} 2^{k(L-\alpha)} \\
\leq cM^{d}M^{-\alpha} + cM^{d-L}M^{L-\alpha} = cM^{d-\alpha}.
\]

The next lemma is a generalization of an elegant result of Cassels and Montgomery (see [25, chapter 6]).

Lemma 16. Let \( \mu \) be as in Theorem 2. There exist \( c_1, c_2 > 0 \) such that for every \( N \), for every point distribution \( \mathcal{P}_N = \{ z_1, z_2, \ldots, z_N \} \), and for every \( M \geq c_1N^{1/\alpha} \) we have

\[
\int_{\{1 \leq |\xi| \leq M\}} \left| \sum_{j=1}^{N} e^{-2\pi i \xi \cdot z_j} - N\hat{\mu}(\xi) \right|^2 d\xi \geq c_2NM^d.
\]
Proof. Let $M \geq 1$. We have

$$
\int_{\{1 \leq |\xi| \leq M\}} \left| \sum_{j=1}^{N} e^{-2\pi i \xi \cdot z_j} - N\hat{\mu}(\xi) \right|^2 d\xi
$$

$$
= \int_{\{ |\xi| \leq M\}} \left| \sum_{j=1}^{N} e^{-2\pi i \xi \cdot z_j} - N\hat{\mu}(\xi) \right|^2 d\xi
$$

$$
- \int_{\{ |\xi| \leq 1\}} \left| \sum_{j=1}^{N} e^{-2\pi i \xi \cdot z_j} - N\hat{\mu}(\xi) \right|^2 d\xi
$$

$$
\geq \int_{|\xi| \leq M} \left| \sum_{j=1}^{N} e^{-2\pi i \xi \cdot z_j} - N\hat{\mu}(\xi) \right|^2 d\xi - cN^2.
$$

Let $K_M$ be as in (12), then

$$
\int_{\{|\xi| \leq M\}} \left| \sum_{j=1}^{N} e^{-2\pi i \xi \cdot z_j} - N\hat{\mu}(\xi) \right|^2 d\xi
$$

$$
\geq \int_{\mathbb{R}^d} \hat{K}_M(\xi) \left| \sum_{j=1}^{N} (e^{-2\pi i \xi \cdot z_j} - \hat{\mu}(\xi)) \right|^2 d\xi
$$

$$
= \int_{\mathbb{R}^d} \hat{K}_M(\xi) \sum_{j=1}^{N} (e^{2\pi i \xi \cdot z_j} - \hat{\mu}(\xi)) \sum_{k=1}^{N} (e^{-2\pi i \xi \cdot z_k} - \hat{\mu}(\xi)) d\xi
$$

$$
= \sum_{k,j=1}^{N} \int_{\mathbb{R}^d} \hat{K}_M(\xi) e^{2\pi i \xi \cdot (z_j - z_k)} d\xi - 2N \sum_{j=1}^{N} \text{Re} \left( \int_{\mathbb{R}^d} \hat{K}_M(\xi) \hat{\mu}(\xi) e^{2\pi i \xi \cdot z_j} d\xi \right)
$$

$$
+ N^2 \int_{\mathbb{R}^d} \hat{K}_M(\xi) |\hat{\mu}(\xi)|^2 d\xi
$$

$$
= \sum_{k,j=1}^{N} K_M(z_j - z_k) - 2N \sum_{j=1}^{N} K_M \ast \mu(z_j) + N^2 \int_{\mathbb{R}^d} \hat{K}_M(\xi) |\hat{\mu}(\xi)|^2 d\xi.
$$

Since the terms in the double sum $1 \leq j, k \leq N$ are positive, the double sum is bounded below by $NK_M(0)$. Moreover, also the last integral is positive. Hence,

$$
\int_{\{|\xi| \leq M\}} \left| \sum_{j=1}^{N} e^{-2\pi i \xi \cdot z_j} - N\hat{\mu}(\xi) \right|^2 d\xi \geq NK_M(0) - 2N \sum_{j=1}^{N} K_M \ast \mu(z_j).
$$
Since $K_M(0) = M^d$ and since, by Lemma 15 we have $|K_M \ast \mu(z_j)| \leq cM^{d-\alpha}$, we obtain
\[
\int_{\{||\xi|| \leq M\}} \left| \sum_{j=1}^{N} e^{-2\pi i \cdot z_j} - N \hat{\mu}(\xi) \right|^2 d\xi \geq NM^d - cN^2M^{d-\alpha}.
\]

We recall that $\alpha \leq d$. Then, if $M \geq c_1N^{1/\alpha}$ with $c_1$ large enough, we have
\[
\int_{\{1 \leq ||\xi|| \leq M\}} \left| \sum_{j=1}^{N} e^{-2\pi i \cdot z_j} - N \hat{\mu}(\xi) \right|^2 d\xi \geq NM^d - cN^2M^{d-\alpha} - cN^2
\]
\[
= NM^d (1 - cNM^{-\alpha} - cNM^{-d}) \geq c_2NM^d.
\]

\textbf{Proof of Theorem 2.} By Plancherel identity, we have
\[
\int_{\mathbb{R}^d} |D_N(x,\tau,\sigma)|^2 dx = \int_{\mathbb{R}^d} |\hat{D}_N(\xi,\tau,\sigma)|^2 d\xi.
\]
Let $\gamma$ and $\delta$ be as in Lemma 14. Then, by Lemma 14 point (2), for every $M \geq 1$ we have
\[
\int_{\gamma}^{\delta} \int_{SO(d)} \int_{\mathbb{R}^d} |\hat{D}_N(\xi,\tau,\sigma)|^2 d\xi d\sigma d\tau
\]
\[
= \int_{\mathbb{R}^d} \left| \sum_{j=1}^{N} e^{-2\pi i \cdot z_j} - N \hat{\mu}(\xi) \right|^2 \int_{\gamma}^{\delta} \int_{SO(d)} |\hat{\chi}_{\tau\sigma\Omega}(\xi)|^2 d\sigma d\tau d\xi
\]
\[
\geq \int_{\{1 \leq ||\xi|| \leq M\}} \left| \sum_{j=1}^{N} e^{-2\pi i \cdot z_j} - N \hat{\mu}(\xi) \right|^2 \int_{\gamma}^{\delta} \int_{SO(d)} |\hat{\chi}_{\tau\sigma\Omega}(\xi)|^2 d\sigma d\tau d\xi
\]
\[
\geq \int_{\{1 \leq ||\xi|| \leq M\}} \left| \sum_{j=1}^{N} e^{-2\pi i \cdot z_j} - N \hat{\mu}(\xi) \right|^2 d\xi
\]
\[
\times \left\{ \inf_{1 \leq ||\xi|| \leq M} \int_{\gamma}^{\delta} \int_{SO(d)} |\hat{\chi}_{\tau\sigma\Omega}(\xi)|^2 d\sigma d\tau \right\}
\]
\[
\geq cM^{-d-\beta} \int_{\{1 \leq ||\xi|| \leq M\}} \left| \sum_{j=1}^{N} e^{-2\pi i \cdot z_j} - N \hat{\mu}(\xi) \right|^2 d\xi.
\]
Hence, by Lemma 16, if $M = c_1N^{1/\alpha}$, we have
\[
\int_{a}^{b} \int_{SO(d)} \int_{\mathbb{R}^d} |D_N(x,\tau,\sigma)|^2 dx d\sigma d\tau
\]
\[
= \int_{a}^{b} \int_{SO(d)} \int_{\mathbb{R}^d} |\hat{D}_N(\xi,\tau,\sigma)|^2 d\xi d\sigma d\tau
\]
\[ \geq cM^{-d-\beta} \int_{\{1 \leq |\xi| \leq M\}} \left| \sum_{j=1}^{N} e^{-2\pi i \xi \cdot z_j} - N\hat{\mu}(\xi) \right|^2 d\xi \]
\[ \geq cM^{-d-\beta} NM^d = cN^{1-\beta/\alpha}. \]

5 | PROOF OF THEOREM 10

The characteristic function of a half-space can be obtained as a limit of characteristic functions of balls of diverging radii. Hence, we start with a lemma on the Fourier transform on the characteristic functions \( \chi_{rB} \) of the balls \( rB = \{ x \in \mathbb{R}^d : |x| < r \} \).

**Lemma 17.** There exist \( c_1, c_2 > 0 \) such that for \( R|\xi| \geq c_1 \) we have

\[ \frac{1}{R} \int_R^{2R} |\hat{\chi}_{rB}(\xi)|^2 \, dr \geq c_2 R^{d-1} |\xi|^{-d-1}. \]

**Proof.** The Fourier transform of \( \chi_{rB}(x) \) can be expressed in terms of a Bessel function,

\[ \hat{\chi}_{rB}(\xi) = r^d \hat{\chi}_B(r\xi) = r^{d/2} |\xi|^{-d/2} J_{d/2}(2\pi r |\xi|). \]

Bessel functions have the asymptotic expansion

\[ J_{d/2}(2\pi u) = \pi^{-1} u^{-1/2} \cos \left( 2\pi u - (d + 1) \frac{\pi}{4} \right) + E_d(u) \]
with \( |E_d(u)| \leq c_d |u|^{-3/2} \) (see, e.g., Lemma 3.11 in [31]). Then, if \( R|\xi| \geq c_1 \) with \( c_1 \) sufficiently large,

\[ \frac{1}{R} \int_R^{2R} |\hat{\chi}_{rB}(\xi)|^2 \, dr = \frac{1}{R} \int_R^{2R} r^d |\xi|^{-d} \left| J_{d/2}(2\pi r |\xi|) \right|^2 \, dr \]
\[ = |\xi|^{-2d} \frac{1}{R|\xi|} \int_{R|\xi|}^{2R|\xi|} u^d \left| J_{d/2}(2\pi u) \right|^2 \, du \]
\[ = |\xi|^{-2d} \frac{1}{R|\xi|} \int_{R|\xi|}^{2R|\xi|} u^{d-1} \left| \pi^{-1} \cos \left( 2\pi u - (d + 1) \frac{\pi}{4} \right) + u^{1/2} E_d(u) \right|^2 \, du \]
\[ \geq |\xi|^{-2d} (R|\xi|)^{d-1} \frac{1}{R|\xi|} \int_{R|\xi|}^{2R|\xi|} \pi^{-1} \cos \left( 2\pi u - (d + 1) \frac{\pi}{4} \right) + u^{1/2} E_d(u) \right|^2 \, du \]
\[ \geq c R^{d-1} |\xi|^{-d-1}. \]

In the next lemma, we estimate the discrepancy associated to the family of balls \( x + rB \), where \( B \) is the unit ball centered at the origin. With a small change of the previous notation, for any \( x \in \mathbb{R}^d \) and \( r > 0 \) we set

\[ D_N(x, r) = \text{card}(P_N \cap (x + rB)) - N\mu(x + rB). \]
Lemma 18. Under the assumption of Theorem 10, there exists $c > 0$ such that for $R$ large enough

$$\frac{1}{R} \int_{R}^{2R} \int_{\mathbb{R}^d} |D_N(x, r)|^2 dx dr \geq c R^{d-1} N^{1-1/\alpha}.$$ 

Proof. By Lemma 13, we have

$$\hat{D}_N(\xi, r) = \left\{ \sum_{j=1}^{N} e^{-2\pi i \xi \cdot z_j} - \bar{N} \bar{\mu}(\xi) \right\} \hat{\chi}_r B(\xi).$$

Then, by the Plancherel identity and Lemma 16 with $M = c N^{1/\alpha}$, we have

$$\frac{1}{R} \int_{R}^{2R} \int_{\mathbb{R}^d} |D_N(x, r)|^2 dx dr = \frac{1}{R} \int_{R}^{2R} \int_{\mathbb{R}^d} |\hat{D}_N(\xi, r)|^2 d\xi dr$$

$$= \frac{1}{R} \int_{R}^{2R} \int_{\mathbb{R}^d} \left| \sum_{j=1}^{N} e^{-2\pi i \xi \cdot z_j} - \bar{N} \bar{\mu}(\xi) \right|^2 |\hat{\chi}_r B(\xi)|^2 d\xi dr$$

$$\geq \int_{\{1 \leq |\xi| \leq M\}} \left| \sum_{j=1}^{N} e^{-2\pi i \xi \cdot z_j} - \bar{N} \bar{\mu}(\xi) \right|^2 \left\{ \frac{1}{R} \int_{R}^{2R} |\hat{\chi}_r B(\xi)|^2 d\xi \right\} d\xi$$

$$\geq c R^{d-1} M^{-d-1} \int_{\{1 \leq |\xi| \leq M\}} \left| \sum_{j=1}^{N} e^{2\pi i \xi \cdot z_j} - \bar{N} \bar{\mu}(\xi) \right|^2 d\xi$$

$$\geq c R^{d-1} M^{-d-1} N M^d \geq c R^{d-1} N^{1-1/\alpha}. \quad \Box$$

Proof of Theorem 10. Let $r > r_0$. Since both the set $E$ and the points $P_N$ are contained in the ball $B(0, r_0)$, if $|x| < r - r_0$, then $B(0, r_0) \subset B(x, r)$ and one can easily check that

$$D_N(x, r) = 0.$$ 

Similarly, if $|x| > r + r_0$, then $B(0, r_0) \cap B(x, r) = \emptyset$ and also in this case we have $D_N(x, r) = 0$. Then the previous lemma gives, for $R > r_0$,

$$c_1 R^{d-1} N^{1-1/\alpha} \leq \frac{1}{R} \int_{R}^{2R} \int_{\mathbb{R}^d} |D_N(x, r)|^2 dx dr$$

$$= \frac{1}{R} \int_{R}^{2R} \int_{|r-r_0| \leq |x| \leq r+r_0} |D_N(x, r)|^2 dx dr$$

$$= \frac{1}{R} \int_{R}^{2R} \int_{r-r_0}^{r+r_0} \int_{\Sigma_{d-1}} |D_N(\rho \Theta, r)|^2 \rho^{d-1} d\rho d\Theta dr$$

$$\leq \frac{1}{R} (2R + r_0)^{d-1} \int_{R}^{2R} \int_{r-r_0}^{r+r_0} \int_{\Sigma_{d-1}} |D_N(\rho \Theta, r)|^2 d\rho d\Theta dr$$

$$= \frac{1}{R} (2R + r_0)^{d-1} \int_{R}^{2R} \int_{r-r_0}^{r+r_0} \int_{\Sigma_{d-1}} |D_N((\rho + r)\Theta, r)|^2 d\rho d\Theta dr.$$
Then, there exists $c > 0$ such that for every $R$ large enough,

$$c \, N^{1-1/\alpha} \leq \int_{-r_0}^{r_0} \int_{\Sigma_{d-1}} \left[ \frac{1}{R} \int_{R}^{2R} |D_N((\rho + r)\Theta) - D_N((\rho + r)\Theta)|^2 dr \right] d\Theta d\rho$$

$$= \int_{-r_0}^{r_0} \int_{\Sigma_{d-1}} \left[ \int_{1}^{2} |D_N((\rho + \omega R)\Theta, \omega R)|^2 d\omega \right] d\Theta d\rho.$$

We claim that as $R \to +\infty$ the discrepancy associated to these balls converges to the discrepancy associated to half-spaces, that is we claim that

$$\lim_{R \to +\infty} \int_{-r_0}^{r_0} \int_{\Sigma_{d-1}} \left[ \int_{1}^{2} |D_N((\rho + \omega R)\Theta, \omega R)|^2 d\omega \right] d\Theta d\rho$$

$$= \int_{-r_0}^{r_0} \int_{\Sigma_{d-1}} |D_N(\Pi(\Theta, \rho))|^2 d\Theta d\rho.$$

Observe that

$$B((\rho + \omega R)\Theta, \omega R) = \{ x : |x - \rho \Theta - \omega R\Theta|^2 < \omega^2 R^2 \}$$

$$= \{ x : |x - \rho \Theta|^2 < 2\omega R(x - \rho \Theta) \cdot \Theta \}$$

$$= \{ x : \frac{|x - \rho \Theta|^2}{2\omega R} + \rho < x \cdot \Theta \}.$$

Hence, for every $x \in \mathbb{R}^d$,

$$\lim_{R \to +\infty} X_B((\rho + \omega R)\Theta, \omega R)(x) = X_{\Pi(\Theta, \rho)}(x).$$

Then, for every $\Theta, \rho$ and $\omega$, we have

$$\lim_{R \to +\infty} D_N((\rho + \omega R)\Theta, \omega R)$$

$$= \lim_{R \to +\infty} \left[ \sum_{j=1}^{N} X_B((\rho + \omega R)\Theta, \omega R)(z_j) - N\mu(B((\rho + \omega R)\Theta, \omega R)) \right]$$

$$= \sum_{j=1}^{N} X_{\Pi(\Theta, \rho)}(z_j) - N\mu(\Pi(\Theta, \rho)) = D_N(\Pi(\Theta, \rho)).$$

By the dominate convergence theorem, we have

$$\lim_{R \to +\infty} \int_{-r_0}^{r_0} \int_{\Sigma_{d-1}} \left[ \int_{1}^{2} |D_N((\rho + \omega R)\Theta, \omega R)|^2 d\omega \right] d\Theta d\rho$$

$$= \lim_{R \to +\infty} \int_{-r_0}^{r_0} \int_{\Sigma_{d-1}} |D_N(\Pi(\Theta, \rho))|^2 d\Theta d\rho. \qed$$
APPENDIX

Let $C$ be the snowflake curve constructed in Example 8. The next theorem is a particular case of Theorems 2 and 10 when both the support $E$ of the measure $\mu$ and the set $\Omega$ used to define the discrepancy coincide with the interior of the snowflake (Figure A.1).

![Image of the snowflake curve](image.png)

**FIGURE A.1** The snowflake

**Theorem A.1.** Let $E = \Omega$ be the open bounded set whose boundary is the snowflake curve $C$ and let $\mu$ be the Borel measure on $\mathbb{R}^d$ defined by

$$
\mu(F) = \frac{|F \cap E|}{|E|}.
$$

(i) There exist positive constants $a, b$ and $c$ such that for every distribution $\mathcal{N} = \{z_1, z_2, \ldots, z_N\}$ of $N$ points we have

$$
\left\{ \int_a^b \int_{SO(d)} \int_{\mathbb{R}^d} |D_N(x + \tau \sigma \Omega)|^2 \, dx \, d\sigma \, d\tau \right\}^{1/2} \geq cN^{\frac{\log_3 4}{4}}.
$$

(ii) For every $0 < a < b$ there exists $c > 0$ such that for every $N$ there exists a distribution $\mathcal{P}_N = \{z_1, z_2, \ldots, z_N\} \subseteq E$ of $N$ points such that

$$
\left\{ \int_a^b \int_{SO(d)} \int_{\mathbb{R}^d} |D_N(x + \tau \sigma \Omega)|^2 \, dx \, d\sigma \, d\tau \right\}^{1/2} \leq cN^{\frac{\log_3 4}{4}}.
$$

**Proof.** (i) Let us show that we can apply Theorem 2 to this setting with $\alpha = 2$ and $\beta = 2 - \log_3 4$. Clearly, (1) holds true with $\alpha = 2$. We only have to prove (2) and (4).

Let us show that $\Omega$ has finite $(2 - \log 4 / \log 3)$-dimensional Minkowski content. Given $0 < t < 1$ let $n$ be such that $3^{-n-1} \leq t < 3^{-n}$. The construction of $C$ shows that for every $y \in C$ there exists
$z \in C_n$ such that $\text{dist}(y, z) \leq 3^{-n}$. Then
\[
\{ x \in \mathbb{R}^d : d(x, C) \leq t \} \subset \{ x \in \mathbb{R}^d : d(x, C_n) \leq 2 \cdot 3^{-n} \}.
\]

Since $C_n$ has length $3 \cdot \left(\frac{4}{3}\right)^n$,
\[
\left| \{ x \in \mathbb{R}^d : d(x, C) \leq t \} \right| \leq \left| \{ x \in \mathbb{R}^d : d(x, C_n) \leq 2 \cdot 3^{-n} \} \right| 
\leq 3 \cdot \left(\frac{4}{3}\right)^n \cdot (2 \cdot 3^{-n}) \leq 6t^2 - \log_4 \log_3.
\]

By Remark 3 $\Omega$ satisfies (2) with $\beta = \frac{\log_4}{\log_3}$.

It remains to show that there exists a decreasing sequence $t_n$ satisfying (3) and a direction $\Theta$ such that
\[
\kappa_2 t_n^{2-\log_3 4} \leq \left| (t_n \Theta + \Omega) \triangle \Omega \right|.
\]

Let $\Theta = (0, 1)$, let $t_n = \sqrt{3} / 2 \cdot 3^{-n}$ and let $h = t_n(0, 1)$. If we stop the construction of the snowflake curve $C$ at the step $n$ we can write
\[
\Omega = K_n \cup F_n,
\]
where $K_n$ is an open bounded set whose boundary is $C_n$ and $F_n$ is a bounded set which is the disjoint union of triangles of side lengths $3^{-k-1}$ for $k \geq n$. More precisely for every $k \geq n$, $F_n$ contains $3 \cdot 4^k$ disjoint triangles of side lengths $3^{-k-1}$. The measure of $F_n$ is therefore given by
\[
|F_n| = \sum_{k=n}^{+\infty} 3 \cdot 4^k \frac{1}{2} \frac{\sqrt{3}}{2} 3^{-k-1} = \frac{3\sqrt{3}}{20} \left(\frac{4}{9}\right)^n.
\]

To estimate from below the size of $|(\Omega + h) \triangle \Omega|$ observe that
\[
(\Omega + h) \triangle \Omega \supseteq (K_n \triangle (K_n + h)) \setminus (F_n \cup (F_n + h)).
\]

The part of $K_n \triangle (K_n + h)$ originating from the horizontal segments contains $4^n$ disjoint trapezoids and rectangles of area $3^{-n} t_n = \sqrt{3} / 2 \cdot 9^{-n}$ (see Figure A.2). Hence,
\[
|K_n \triangle (K_n + h)| \geq \frac{\sqrt{3}}{2} \left(\frac{4}{9}\right)^n.
\]

Since $|F_n| = |F_n + h| = \frac{3\sqrt{3}}{20} \left(\frac{4}{9}\right)^n$, then we have
\[
|\Omega \triangle (\Omega + h)| \geq |K_n \triangle (K_n + h)| - |F_n| - |F_n + h|
\geq \frac{\sqrt{3}}{2} \left(\frac{4}{9}\right)^n - \frac{3\sqrt{3}}{10} \left(\frac{4}{9}\right)^n = \frac{\sqrt{3}}{5} \left(\frac{4}{9}\right)^n
\geq \frac{\sqrt{3}}{5} \left(\frac{2}{\sqrt{3}}\right)^{\log_3 4 - 2} t_n^{2-\log_3 4}.
\]
\[ \square \]
We now show that for every $0 < \beta < 1$ there exist sets that satisfy (2) and (4).

**Example A.2.** Let $0 < \beta < 1$, let $\gamma = \frac{\beta}{1-\beta}$, and for every positive integer $n$ let

$$z_n = \frac{1}{n^\gamma} - \frac{1}{(n+1)^\gamma},$$

and

$$R_n = \left[ \frac{1}{n^\gamma} - \frac{1}{3} z_n, \frac{1}{n^\gamma} \right] \times [0, 1].$$

Finally, let

$$\Omega = \bigcup_{n=1}^{+\infty} R_n$$

be the union of the above rectangles. If $h = (h_1, 0)$ with $0 < |h_1| \leq \frac{1}{2}$ and $n_0$ is defined by

$$\frac{1}{3} z_{n_0} + 1 < |h_1| \leq \frac{1}{3} z_{n_0},$$

then, for $n = 1, \ldots, n_0$, we have $(R_{n+1} + h) \cap R_n = \emptyset$, and since $|h| \approx z_{n_0} \approx \frac{1}{n_0^{1-\gamma}}$ we have

$$|\Omega \triangle (\Omega + h)| \geq \sum_{n=1}^{n_0} |R_n \triangle (R_n + h)| = \sum_{n=1}^{n_0} |h_1| = n_0 |h_1| \approx c |h|^{1-\frac{1}{\gamma+1}} = c |h|^{\beta}.$$ 

Now let $h = (h_1, h_2)$. Since

$$A \triangle B \subseteq (A \triangle C) \cup (B \triangle C),$$

we have

$$|\Omega \triangle (\Omega + h)| \geq \sum_{n=1}^{n_0} |R_n \triangle (R_n + h)| = \sum_{n=1}^{n_0} |h_1| = n_0 |h_1| \approx c |h|^{1-\frac{1}{\gamma+1}} = c |h|^{\beta}. $$
we have
\[ \Omega \triangle (h + \Omega) \subset \left[ ((h_1, 0) + (0, h_2) + \Omega) \triangle ((h_1, 0) + \Omega) \right] \cup \left[ ((h_1, 0) + \Omega) \triangle \Omega \right] \]
and therefore
\[ |\Omega \triangle (h + \Omega)| \leq |((0, h_2) + \Omega) \triangle \Omega| + |((h_1, 0) + \Omega) \triangle \Omega|. \]
Then
\[ |\Omega \triangle (\Omega + (h_1, 0))| \leq \sum_{n=1}^{n_0} |R_n \triangle (R_n + (h_1, 0))| + \frac{1}{n_0^\gamma} \approx c|h_1|^{\gamma}. \]
Also, if \(|h_2| \leq \frac{1}{2}\), then
\[ |\Omega \triangle (\Omega + (0, h_2))| = 2|\Omega||h_2| \]
and thus
\[ |\Omega \triangle (h + \Omega)| \leq c|h|^{\gamma}. \]

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