Suppose we play the following game with the three six-sided dice in Figure 1: You choose a die and then I choose a die (based on your choice). We roll our dice and the player whose die shows a higher number wins.

A closer look at the dice in Figure 1 reveals that, in the long run, I will have an advantage in this game: Whichever die you choose, I will choose the one immediately to its left (and I will choose die $C$ if you choose die $A$). In any case, the probability of my die beating yours is $\frac{19}{36} > \frac{1}{2}$.

This is a case of the phenomenon of nontransitive dice, first introduced by Martin Gardner [2] and further explored in [1, 3, 5]. More recently, several other facets of this scenario have been explored, leading to Grime dice (see [4] and p. 2 of this issue) and Lake Wobegon dice [6] also in this JOURNAL.

We define a triple of dice as follows: Fix an integer $n > 0$. For our purposes, a set of $n$-sided dice is a collection of three pairwise-disjoint sets $A$, $B$, and $C$ with
$|A| = |B| = |C| = n$ and $A \cup B \cup C = [3n]$ (where $[k] = \{1, 2, \ldots, k\}$). Think of die $A$ as being labeled with the elements of $A$, etc. Each die is fair, in that the probability of rolling any one of its numbers is $1/n$. Write $P(A \succ B)$ for the probability that, upon rolling both $A$ and $B$, the number rolled on $A$ exceeds that on $B$.

**Definition 1.** A set of dice is nontransitive if each of $P(A \succ B)$, $P(B \succ C)$, and $P(C \succ A)$ exceeds $1/2$. That is, the relation “is a better die than” is nontransitive.

In this paper we (mostly) examine nontransitive sets of dice, but we introduce a new property as well.

**Definition 2.** A set of dice is balanced if $P(A \succ B) = P(B \succ C) = P(C \succ A)$.

Note that the set of dice in Figure 1 is balanced, as $P(A \succ B) = P(B \succ C) = P(C \succ A) = 19/36$.

In Theorem 3 below, we show that balanced nontransitive sets of $n$-sided dice exist for all $n \geq 3$. Surprisingly, this also seems to be the first proof that nontransitive sets of $n$-sided dice exist for all $n \geq 3$. We then prove in Theorem 8 that a set of dice is balanced (but not necessarily nontransitive) if and only if the face-sums of the dice are equal (the face-sum of a die is simply the sum of its labels). This yields an $O(n^2)$ algorithm for determining if a given triple of $n$-sided dice is nontransitive and balanced. Finally, we consider generalizations to sets of four dice and pose further questions.

**Balanced dice**

Our main goal in this section is to prove the following existence result.

**Theorem 3.** For any $n \geq 3$, there exists a set of three balanced, nontransitive, $n$-sided dice.

First, we need some machinery. For our purposes, a word $\sigma$ is a sequence of $3n$ letters where each letter is either an $a$, $b$, or $c$, and each of $a$, $b$, and $c$ appears $n$ times.

**Definition 4.** Given a set of $n$-sided dice $D$, the word $\sigma(D)$ is determined by the $i$th letter being the die which includes $i$ as a label.

Now let $\sigma = s_1s_2 \cdots s_{3n}$ be a word. Define a function $q^+_\sigma$ on the letters of $\sigma$ as

$$q^+_\sigma(s_i) = \begin{cases} \{j < i \mid s_j = b\} & \text{if } s_i = a, \\ \{j < i \mid s_j = c\} & \text{if } s_i = b, \\ \{j < i \mid s_j = a\} & \text{if } s_i = c. \end{cases}$$

Similarly, define $q^-_\sigma$ by

$$q^-_\sigma(s_i) = \begin{cases} \{j < i \mid s_j = c\} & \text{if } s_i = a, \\ \{j < i \mid s_j = a\} & \text{if } s_i = b, \\ \{j < i \mid s_j = b\} & \text{if } s_i = c. \end{cases}$$

For example, if $s_i = a$, then $q^+(s_i)$ is the number of sides of die $B$ whose labels precede $i$. Similarly, $q^-_\sigma(s_i)$ is the number of sides of die $C$ whose labels precede $i$. 
Example 5. Let $D$ be the following set of dice.

$$A: 9, 5, 1$$
$$B: 8, 4, 3$$
$$C: 7, 6, 2$$

Then $\sigma(D) = acbbaccba$. Note that this set of dice is balanced and nontransitive, as $P(A \succ B) = P(B \succ C) = P(C \succ A) = 5/9$.

Conversely, given a word $\sigma$, let $D(\sigma)$ denote the unique set of dice corresponding to $\sigma$. As this is a one-to-one correspondence, we often speak of a set of dice and the associated word interchangeably. For instance, if $\sigma = s_1s_2 \cdots s_{3n}$ is a $3n$-letter word, the probability of die $A$ beating die $B$ is given by

$$P(A \succ B) = \frac{1}{n^2} \sum_{s_i = a} q^+(s_i)$$

and the other probabilities may be computed analogously. Thus, the property of a set $D$ of dice being balanced is equivalent to $\sigma(D)$ satisfying

$$\sum_{s_i = a} q^+(s_i) = \sum_{s_i = b} q^+(s_i) = \sum_{s_i = c} q^+(s_i). \quad (1)$$

Furthermore, for $D$ a set of $n$-sided dice, $D$ is nontransitive if and only if each of

$$\sum_{s_i = a} q^+_{\sigma(D)}(s_i), \sum_{s_i = b} q^+_{\sigma(D)}(s_i), \text{ and } \sum_{s_i = c} q^+_{\sigma(D)}(s_i)$$

exceeds $n^2/2$.

Although a set of dice $D$ and its associated word $\sigma(D)$ hold the same information, this alternate interpretation will prove invaluable in showing Theorem 3. Next, we need some lemmas. The concatenation of two words $\sigma$ and $\tau$, for which we write $\sigma \tau$, is simply the word $\sigma$ followed by $\tau$.

**Lemma 6.** Let $\sigma$ and $\tau$ be balanced words. Then the concatenation $\sigma \tau$ is balanced.

**Proof.** Let $|\sigma| = 3m$ and $|\tau| = 3n$. If $i \leq 3m$, then $q^+_{\sigma \tau}(s_i) = q^+_{\sigma}(s_i)$ ($q^+$ is defined as a subset of the $s_j$ with $j < i$, so concatenating $\tau$ after $\sigma$ contributes nothing to these). Otherwise (for $3m < i \leq 3m + 3n$), $q^+_{\sigma \tau}(s_i) = q^+_{\tau}(s_i) + m$, because every letter from $\tau$ beats all $m$ letters from the appropriate die in $\sigma$, in addition to whichever letters it beats from the structure of $\tau$ itself. Then

$$\sum_{s_i = a} q^+_{\sigma \tau}(s_i) = \sum_{s_i = a} q^+_{\sigma}(s_i) + \sum_{s_i = a} q^+_{\tau}(s_i) + mn. \quad (2)$$

We may repeat the argument for $s_i = b, c$, and then we are done since $\sigma$ and $\tau$ are balanced. $lacksquare$

While Lemma 6 is primarily useful for balanced words (or sets of dice), the next result applies to arbitrary sets of nontransitive dice.
Lemma 7. Given nontransitive words $\sigma$ and $\tau$, the concatenation $\sigma \tau$ is nontransitive.

Proof. Let $\sigma$ be a word of length $3m$. Because $m^2 P_\sigma(A > B)$ counts the number of rolls of dice $A$ and $B$ in which die $A$ beats die $B$, we note that

$$m^2 P_\sigma(A > B) = \sum_{s_i = a} q^+(s_i)$$

and analogous statements hold for $m^2 P_\sigma(B > C)$ and $m^2 P_\sigma(C > A)$. Let

$$V_\sigma = m^2 \cdot \min\{P_\sigma(A > B), P_\sigma(B > C), P_\sigma(C > A)\}.$$

Let $\tau$ be a word of length $3n$ and define $V_\tau$ and $V_{\sigma\tau}$ as above. Note that $V_\sigma > m^2/2$ and $V_\tau > n^2/2$ because $\sigma$ and $\tau$ are nontransitive. By (2) we have

$$V_{\sigma\tau} = V_\sigma + V_\tau + mn > \frac{m^2}{2} + \frac{n^2}{2} + mn = \frac{(m + n)^2}{2}$$

which shows that $\sigma \tau$ is nontransitive.

With the two lemmas above in place, we are now able to provide a quick proof of Theorem 3, the main result of this section.

Proof of Theorem 3. Example 5 along with

| A | 12, 10, 3, 1 | A | 15, 11, 7, 4, 3 |
|---|---|---|---|
| B | 9, 8, 7, 2 | and | B | 14, 10, 9, 5, 2 |
| C | 11, 6, 5, 4 | C | 13, 12, 8, 6, 1 |

provide balanced, nontransitive sets of dice for $n = 3, 4, 5$, which give rise to balanced words for these $n$, the smallest representatives (in the context of the theorem) for each congruence class modulo 3. Lemmas 6 and 7 then imply that the concatenation of two balanced nontransitive words is a balanced, nontransitive word. The correspondence between words and sets of dice completes the proof.

Face-sums

Considering Example 5 and the sets of balanced, nontransitive dice given in the proof of Theorem 3, one may notice the following phenomenon: In any one of these sets of dice, the sum of the labels of any two dice are equal. In terms of words, the face-sums of a set $D$ of $n$-sided dice with $\sigma(D) = s_1 s_2 \cdots s_{3n}$ are

$$\sum_{s_i = a} i, \sum_{s_i = b} i, \text{ and } \sum_{s_i = c} i.$$  

Theorem 8. A set of three dice $D$ is balanced if and only if its face-sums are all equal.

Proof. (Only if.) Let $D$ be a set of balanced dice with associated word $\sigma(D)$. The condition (1) for a word to be balanced is clearly equivalent to

$$\sum_{s_i = a} q^-_\sigma(s_i) = \sum_{s_i = b} q^-_\sigma(s_i) = \sum_{s_i = c} q^-_\sigma(s_i).$$
Define \( q_\sigma(s_i) = |\{j < i \mid s_j = s_i\}| \). We focus on die \( A \) and make two observations: First, for a face of \( A \), its label \( i \) satisfies \( i = q_\sigma^+(s_i) + q_\sigma^-(s_i) + q_\sigma(s_i) + 1 \). Second, since \( A \) has \( n \) sides,

\[
\sum_{s_i = a} q_\sigma(s_i) = \frac{n(n - 1)}{2}.
\]

Combining these, the face-sum of \( A \) can be written as

\[
\sum_{s_i = a} i = \sum_{s_i = a} \left( q_\sigma^+(s_i) + q_\sigma^-(s_i) + q_\sigma(s_i) + 1 \right) = \left( \sum_{s_i = a} q_\sigma^+(s_i) \right) + \left( \sum_{s_i = a} q_\sigma^-(s_i) \right) + \frac{n(n - 1)}{2} + n.
\]

However, this computation was independent of our choice of \( A \), so the other two sums are analogous, and the \( q_\sigma^\pm(s_i) \) sums are all equal since \( \sigma(D) \) is balanced.

(If.) Let \( D \) be a set of \( n \)-sided dice with associated word \( \sigma(D) \) and assume that

\[
\sum_{s_i = a} i = \sum_{s_i = b} i = \sum_{s_i = c} i.
\]

By the above, this is equivalent to

\[
\sum_{s_i = a} q_\sigma^+(s_i) + \sum_{s_i = a} q_\sigma^-(s_i) = \sum_{s_i = b} q_\sigma^+(s_i) + \sum_{s_i = b} q_\sigma^-(s_i) = \sum_{s_i = c} q_\sigma^+(s_i) + \sum_{s_i = c} q_\sigma^-(s_i).
\]

Let \( a^+ = \sum_{s_i = a} q_\sigma^+(s_i), a^- = \sum_{s_i = a} q_\sigma^-(s_i) \), and define \( b^+, b^-, c^+, c^- \) analogously. Then we have

\[
a^+ + a^- = b^+ + b^- = c^+ + c^-,
\]

\[
a^+ + b^- = b^+ + c^- = c^+ + a^- (= n^2),
\]

six equations in six unknowns. Straightforward linear algebra gives \( a^+ = b^+ = c^+ \), whence we also have \( a^- = b^- = c^- \).

Applying Theorem 8 gives the following algorithm for checking if a given partition of \([3n]\) into 3 subsets of size \( n \) determines a set of balanced, nontransitive dice.

**Algorithm 9.** Suppose we are given a partition of \([3n]\) into three size \( n \) subsets \( A, B, \) and \( C \). First, check the sums of the elements of these subsets. These sums are equal if and only if the set of dice is balanced.

If this condition is met, check \( P(A > B) \). If \( P(A > B) = 1/2 \), the set of dice is balanced but fair. If \( P(A > B) > 1/2 \), the set is balanced and nontransitive. If \( P(A > B) < 1/2 \), switching the labels of sets \( B \) and \( C \) produces a balanced, nontransitive set of dice.

Since this algorithm must check each pair of sides from dice \( A \) and \( B \), it clearly runs in \( O(n^2) \) time.

In contrast, using only the probabilities to check balance would take roughly three times as long.
Extensions

Nontransitive dice and Fibonacci numbers. Savage formed sets of nontransitive dice from consecutive terms of the Fibonacci sequence [5]. We briefly explain his construction. We index the Fibonacci numbers as \( f_1 = f_2 = 1, f_3 = 2, \) etc.

Algorithm 10. Given a Fibonacci number \( f_k \), consider the sequence

\[ f_{k-2}, f_{k-1}, f_k, f_{k-1}, f_{k-2} \]

whose sum is \( 3f_k \). Beginning with the number \( 3f_k \), label die \( A \) with \( f_{k-2} \) consecutive descending integers. Then label die \( B \) with the next \( f_{k-1} \) values, die \( C \) with the next \( f_k \) values, \( A \) with the next \( f_{k-1} \) values, and \( B \) with the last \( f_{k-2} \) values (ending in 1). This produces a set of nontransitive dice (which is never balanced).

In the case where \( f_k \) is an odd Fibonacci number, we can modify Savage’s algorithm to produce a balanced set.

Algorithm 11. Perform Algorithm 10 to obtain a set of nontransitive dice. Then, swap the last element of the first set of values (which is \( 3f_k - f_{k-2} + 1 \)), on die \( A \), with the first element of the second set of values (\( 3f_k - f_{k-2} \)), the largest number on die \( B \). The reader is invited to verify that the resulting set of dice is nontransitive and balanced.

Sets of four dice. Modify the definition of a set of dice to mean four dice. Then

\[
\begin{align*}
A &: 12, 5, 2 & A &: 16, 10, 7, 1 & A &: 20, 13, 10, 6, 4 \\
B &: 11, 8, 1 & B &: 15, 9, 6, 4 & B &: 19, 15, 9, 8, 3 \\
C &: 10, 7, 3 & C &: 14, 12, 5, 3 & C &: 18, 16, 12, 5, 1 \\
D &: 9, 6, 4 & D &: 13, 11, 8, 2 & D &: 17, 14, 11, 7, 2
\end{align*}
\]

give minimal examples for balanced nontransitive sets of dice. The proof of Theorem 3 generalizes, using length \( 4n \) words with \( n \) each of \( a, b, c, d \), giving the following result.

Theorem 12. For any \( n \geq 3 \), there exists a set of four balanced, nontransitive, \( n \)-sided dice.

However, notice that the \( n = 3 \) example has unequal face-sums, showing that Theorem 8 does not extend to this situation.

Irreducibility. Given the proof of Theorem 3, it seems natural to make the following definition.

Definition 13. Let \( \sigma \) be a balanced nontransitive word. If there do not exist balanced nontransitive words \( \tau_1 \) and \( \tau_2 \) (both nonempty) such that \( \sigma = \tau_1 \tau_2 \), we say that \( \sigma \) (and its associated set of dice) is irreducible.

Question 14. For any \( n \geq 3 \), does there necessarily exist an irreducible, balanced, nontransitive set of \( n \)-sided dice?

Graph orientations. The notions of nontransitive triples and quadruples of dice also suggest the following broad generalization.
Definition 15. Let $G$ be an orientation of $K_m$, the complete graph on the vertex set \{ $v_1, v_2, \ldots, v_m$ \}. (That is, $G$ results from giving each edge of $K_m$ a direction.) Define a realization of $G$ to be an $m$-tuple of $n$-sided dice $A_1, A_2, \ldots, A_m$ for some $n$ (where now the $A_i$ partition $[mn]$) satisfying

$$P(A_i > A_j) > \frac{1}{2} \iff (v_i \rightarrow v_j) \text{ is an edge of } G.$$ 

Theorem 3 gives us the following as a corollary.

Corollary 16. Let $G$ be an orientation of $K_3$. Then there exists a realization of $G$ using $n$-sided dice for any $n \geq 3$.

Proof. If $G$ is a directed cycle, Theorem 3 gives the result. Otherwise, $G$ is acyclic, meaning the orientation corresponds to a total ordering of the vertices. The dice

\begin{align*}
A &: 1, 2, \ldots, n \\
B &: n + 1, n + 2, \ldots, 2n \\
C &: 2n + 1, 2n + 2, \ldots, 3n
\end{align*}

appropriately placed will provide a realization. \hfill \blacksquare

Question 17. Given an orientation of $K_m$, can one always find a set of $n$-sided dice (for some $n$) which realizes this orientation? [Note added in proof: The first author has answered this question in the affirmative; the result is in preparation.]

Summary. We study triples of labeled dice in which the relation “is a better die than” is nontransitive. Focusing on such triples with an additional symmetry we call balance, we prove that such triples of dice exist for all dice having at least three faces. We then examine the sums of the labels of such dice and use these results to construct an algorithm for verifying whether or not a triple of dice is balanced and nontransitive. We also consider generalizations to larger sets of dice and other related ideas.

References

1. E. J. Barbeau, Mathematical Fallacies, Flaws, and Flimflam. Mathematical Association of America, Washington, DC, 2000.
2. M. Gardner, The paradox of the nontransitive dice and the elusive principle of indifference, Sci. Amer. 223 no. 6 (1970) 110–114.
3. ———, On the paradoxical situations that arise from nontransitive relations, Sci. Amer. 231 no. 4 (1974) 120–125.
4. J. Grime, Non-transitive Dice, 2010, http://www.singingbanana.com/dice/article.htm.
5. R. P. Savage Jr., The paradox of nontransitive dice, Amer. Math. Monthly 101 (1994) 429–436, http://dx.doi.org/10.2307/2974903.
6. J. Moraleda, D. G. Stork, Lake Wobegon dice, College Math. J. 43 (2012) 152–159, http://dx.doi.org/10.4169/college.math.j.43.2.152.