HYPERBOLIC CONSERVATION LAWS ON SPACETIMES

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Abstract. We present a generalization of Kruzkov’s theory to manifolds. Nonlinear hyperbolic conservation laws are posed on a differential \((n+1)\)-manifold, called a spacetime, and the flux field is defined as a field of \(n\)-forms depending on a parameter. The entropy inequalities take a particularly simple form as the exterior derivative of a family of \(n\)-form fields. Under a global hyperbolicity condition on the spacetime, which allows arbitrary topology for the spacelike hypersurfaces of the foliation, we establish the existence and uniqueness of an entropy solution to the initial value problem, and we derive a geometric version of the standard \(L^1\) semi-group property. We also discuss an alternative framework in which the flux field consists of a parametrized family of vector fields.

Key words. Hyperbolic conservation law, manifold, spacetime, entropy solution, well-posed theory, finite volume scheme.

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1. Introduction. Hyperbolic conservation laws on manifolds arise in many applications to continuum physics. Specifically, the shallow water equations for perfect fluids posed on the sphere with prescribed topography provide an important model describing geophysical flows and exhibiting complex wave structures (Rosby waves, etc.) with several distinct scales. In [1]–[5] and [13, 18, 19, 20], the author together with several collaborators recently initiated a research program and posed the foundations for the analysis and numerical approximation of hyperbolic conservation laws on manifolds. These equations form a simplified, yet challenging, mathematical model for studying the propagation of nonlinear waves, including shock waves, and their interplay with the background geometry.

2. Hyperbolic conservation laws based on vector fields. The standard shock wave theory posed on the Euclidian space covers nonlinear hyperbolic equations of the form

\[
\partial_t u + \sum_{j=1}^n \partial_j f^j(u) = 0, \quad u = u(t,x) \in \mathbb{R}, \ t \geq 0, \ x \in \mathbb{R}^n,
\]

where \(f^j : \mathbb{R} \to \mathbb{R}\) are given functions, and, more generally, applies to balance laws with variable coefficients and source-terms. Pioneering works by Lax, Oleinik, and others in the 50’s and 60’s set the foundations for discontinuous entropy solutions to such equations. Later, well-posedness theorems within the class of entropy solutions were established by Conway and Smoller (1966), Volpert (1967) (for solutions with bounded variation), Kruzkov (1970) (for bounded solutions), and finally DiPerna (1984) (for measure-valued solutions). Based on this mathematical theory, active research followed in the 80’s and 90’s that led to the development of robust and high-order accurate, shock-capturing schemes.
The generalization of the above theory to manifolds has only recently received particular attention, driven by problems arising in geophysical science and general relativity. In the present section, we consider a class of conservation laws based on parametrized vector fields, and we suppose that $M$ is a compact, oriented $n$-dimensional manifold, endowed with an $L^\infty$ volume form such that, in a finite atlas of local coordinate charts $(x^i)$ and for uniform constants $c \leq C$,

$$
\omega = \varpi dx^1 \cdots dx^n, \quad 0 < \varpi \leq C.
$$

Recall that the divergence of an $L^\infty$ vector field $X$ is defined in the sense of distributions by

$$
\langle \text{div}\omega, X, \theta \rangle := \int_M (d\theta)(X) \omega
$$

for all test-function $\theta : M \to \mathbb{R}$.

**Definition 2.1.** To any parameterized family of (smooth) vector fields $f = f_p(\varpi) \in T_pM$ depending upon $\varpi \in \mathbb{R}$, one associates the hyperbolic conservation law on the manifold $(M, \omega)$

$$
(2.1) \quad \partial_t u + \text{div}_\omega (f(u)) = 0, \quad u = u(t, p), \quad t \geq 0, \ p \in M.
$$

The parametrized flux field $f$ is called geometry-compatible if constants are trivial solutions, that is

$$
(2.2) \quad \langle \text{div}\omega f, \varpi \rangle = 0, \quad \varpi \in \mathbb{R}, \ p \in M
$$

in the distribution sense. A convex entropy pair is a convex function $U : \mathbb{R} \to \mathbb{R}$ together with a parametrized family of vector field

$$
F_p(\varpi) := \int_\varpi \partial_u U(v) \partial_u f_p(v) \, dv \in T_pM, \quad \varpi \in \mathbb{R}, \ p \in M.
$$

The equation (2.1) is geometric in nature and does not depend on a particular choice of local coordinates on $M$. The flux field is a section of the tangent bundle $TM$ and, in general, there is no concept of “spatially constant flux”, unlike in the Euclidian case where one may arbitrarily choose a vector $f_p(\varpi)$ at some point $p \in \mathbb{R}^n$ and parallel-transport it in $\mathbb{R}^n$, trivially generating a constant (and, therefore, smooth) vector field. The geometry-compatibility condition is natural if one observes that it is satisfied by the shallow water equation.

**Remark 2.1.** An important class of geometry-compatible conservation laws is based on projected gradient flux fields, as it is called in [4], and is defined as follows. Consider the unit sphere $S^2$ embedded in the Euclidian space $\mathbb{R}^3$ and a function $h : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ being fixed, and define

$$
f_p(\varpi) := p \wedge (\nabla_\mathbb{R} h)_p(\varpi) \in T_pS^2, \quad p \in S^2 \subset \mathbb{R}^3.
$$

Now, to formulate the initial value problem associated with (2.1), we are given an initial data $u_0 : M \to \mathbb{R}$ and we impose

$$
(2.3) \quad u(0, p) = u_0(p), \quad p \in M.
$$

It is well-known that entropy inequalities must be imposed on weak solutions to conservation laws. In this juncture, it is interesting to point out the following property of geometry-compatible flux field: If $u = u(t, p)$ is a smooth solution and $U = U(\varpi)$ is an arbitrary function, then clearly

$$
\partial_t U(u) + \partial_u U(u) \text{div}_\omega (f_p(u)) = 0,
$$

and it can then be checked that a necessary and sufficient condition for the existence of a vector field $F = F_p(\varpi) \in T_pM$ such that

$$
\partial_u U(v) \text{div}_\omega (f_p(v)) = \text{div}_\omega (F_p(v))
$$
for all convex \( \nu \in \mathcal{L}^1(M) \) is a 1-form field acting on vector fields. Our main existence result is as follows. This theorem is very useful to establish the convergence of approximation schemes. On manifolds, provided the approximate solutions, as well as some initial data \( u_0 \in \mathcal{L}^\infty(M) \), the initial value problem associated with the conservation law \( \mathcal{L} \) admits a unique entropy solution \( u \in \mathcal{L}^\infty(\mathbb{R}^+ \times M) \).

To establish this result, we need to make use of DiPerna's theory of measure-valued fields, which provides a framework for dealing with weak solutions to conservation laws.

We refer the reader to Ben-Artzi and LeFloch [4] and Amorim, Ben-Artzi and LeFloch [4] for further discussions on the well-posedness theory based on vector fields.

THEOREM 2.1 (Well-posedness theory on a manifold). Let \((M, \omega)\) be a compact n-manifold endowed with an \( \mathcal{L}^\infty \) volume form, and \( f = f_p(U) \in \mathcal{T}_pM \) be a geometry-compatible flux field.

1. Given any data \( u_0 \in \mathcal{L}^\infty(M) \), the initial value problem associated with the conservation law \( \mathcal{L} \) admits a unique entropy solution \( u \in \mathcal{L}^\infty(\mathbb{R}^+ \times M) \).

2. Entropy solutions satisfy the following \( L^q \) stability property for all \( q \in [1, \infty] \) and \( t_2 \geq t_1 \)

\[
\| u(t_2) \|_{\mathcal{L}^q_q(M)} \leq \| u(t_1) \|_{\mathcal{L}^q_q(M)}.
\]

3. Moreover, the following \( L^1 \) semi-group property holds for any two entropy solutions \( u, v \) and \( t_2 \geq t_1 \)

\[
\| v(t_2) - u(t_2) \|_{\mathcal{L}^1_1(M)} \leq \| v(t_1) - u(t_1) \|_{\mathcal{L}^1_1(M)}.
\]

This theorem provides a generalization of Kruzkov's theory to a manifold, endowed with a volume form with possibly low regularity. The \( L^q \) stability and \( L^1 \) contraction properties provide geometry-independent bounds, which are useful to design approximation schemes.

For further discussions on the well-posedness theory based on vector fields, we refer the reader to Ben-Artzi and LeFloch [4] and Amorim, Ben-Artzi and LeFloch [4], as well as Panov [21, 22].

Consider a geometry-compatible conservation law with flux field \( f \) together with some initial data \( u_0 \in \mathcal{L}^\infty(M) \). Then, an entropy measure-valued solution to \( \mathcal{L} \), by definition, is a measure-valued field \( (t,p) \in \mathbb{R}^+ \times M \mapsto u_{t,p} \in \text{Prob}(\mathbb{R}) \) such that

\[
\int_{\mathbb{R}^+} \int_M \left( \langle u_{t,p}, U \rangle \partial_t \theta + \langle \nu_{t,p}, F \rangle, d\theta \right) \omega(p) dt + \int_M U(u_0) \theta(0, \cdot) \omega \geq 0
\]

for all convex entropy pairs and test-functions \( \theta = \theta(t,p) \geq 0 \). Following DiPerna [11], we can establish the following uniqueness result.

THEOREM 2.2 (Uniqueness and compactness framework). If \( \nu \) is an entropy measure-valued solution to the conservation law \( \mathcal{L} \) posed on the manifold \((M, \omega)\) and assuming some initial data \( u_0 \in \mathcal{L}^\infty(M) \) at \( t = 0 \), then

\[
u_{t,p} = \delta_{u(t,p)} \quad \text{a.e. } p \in M.
\]

where \( u \in \mathcal{L}^\infty(\mathbb{R}^+ \times M) \) denotes the unique entropy solution to the same problem.

This theorem is very useful to establish the convergence of approximation schemes since, in comparison with the standard technique based on a total variation bound, it requires weaker and geometrically more natural bounds on (the entropy dissipation of) the approximate solutions, as well as weaker assumptions on the manifold geometry.

In the Euclidian case, the total variation diminishing (TVD) property was found to be very useful for the development of high-order schemes. On manifolds, provided \( \omega \) is
sufficiently smooth and the initial data \( u_0 \) has bounded variation then \( u(t) \) has bounded variation for all \( t \geq 0 \). In fact, there exists \( C > 0 \) depending on the geometry of \( M \) and \( \| u_0 \|_{L^\infty(M)} \) such that

\[
TV(u(t)) \leq e^{Ct}(1 + TV(u_0)),
\]

and the total variation might be unbounded as \( t \to \infty \).

Introduce the total variation along a given vector field \( X \) by

\[
TV_X(u) := \sup_{\|\phi\|_{L^\infty}} \int_M u \text{div}_\omega (\phi X) \omega.
\] (2.7)

\textbf{Theorem 2.3 (Total variation estimate).} Consider a geometry-compatible conservation law (2.1) posed on the manifold \( (M, \omega) \) and suppose that the initial data have bounded variation. Then, the total variation diminishing property (TVD)

\[
TV_X(u(t_2)) \leq TV_X(u(t_1)), \quad t_2 \geq t_1,
\]

holds provided the flux field \( f \) and the given field \( X \) satisfy the compatibility condition

\[
\mathcal{L}_X \left( \partial_u f(\varpi) \right) := \left[ \partial_u f(\varpi), X \right] = 0.
\] (2.8)

On the other hand, from the time-invariance of the equation and the \( L^1_\omega \) contraction property, it follows that

\[
\| d\omega(f(u(t))) \|_{M^*_\omega(M)} := \sup_{\|\theta\|_{C^0} \leq 1} \int_M d\theta(f(u(t))) \omega
\]

is diminishing in time.

The condition (2.8) is usually not satisfied in the examples of interest, except when there exists some decoupling into a one-parameter family of independent, one-dimensional equations. (See below.) Hence, the curved geometry restricts the class of flux enjoying the TVD property, and we can not expect to rely on a TVD property to develop approximation schemes. In contrast, in the Euclidian case no restriction is implied on (\( p \)-independent) flux and the TVD property always holds.

\textbf{Remark 2.2.} Returning to the projected gradient flux field on \( S^2 \subset \mathbb{R}^3 \) introduced in Remark 2.1 and choosing a vector field \( X \) of the form

\[
X_p := p \wedge (\nabla_{S^2} k)_p, \quad p \in \mathbb{R}^3,
\]

then the condition \( \mathcal{L}_X(\partial_u f(\varpi)) = 0 \) on \( S^2 \) holds if and only if the vectors \( \partial_u f_p(\varpi) \) and \( X_p \) are parallel, and the proportionality factor \( C(\varpi, p) \) is constant along the integral curves of \( X \):

\[
\partial_u f_p(\varpi) = \partial_u C(\varpi, p) X_p, \quad X(C(\varpi, p)) = 0, \quad \varpi \in \mathbb{R}, \ p \in M.
\]

\section{3. Hyperbolic conservation laws based on differential forms.}

In this section based on LeFloch and Okutmustur [19, 20], we consider hyperbolic conservation laws posed on an \( (n + 1) \)-dimensional differentiable manifold \( M \), referred to as a spacetime. We emphasize that, in the framework now developed, no geometric structure is a priori imposed on \( M \).

\textbf{Definition 3.1.} The \textbf{hyperbolic conservation law on the differentiable} \( (n + 1) \)-\textbf{manifold} \( M \) associated with a parametrized family of \( L^\infty \) \( n \)-form fields \( p \in M \mapsto \omega_p(\varpi) \) depending smoothly upon the parameter \( \varpi \), by definition, reads

\[
d(\omega(u)) = 0,
\] (3.1)

whose unknown is the scalar field \( u : M \to \mathbb{R} \). The parametrized flux field \( \omega \) is called \textbf{geometry-compatible} if it is closed, that is, for each \( \varpi \in \mathbb{R}, (d\omega)(\varpi) = 0 \) in the distribution sense.
By definition, weak solutions $u \in L^\infty(M)$ must satisfy the following weak form of the conservation law
\begin{equation}
\int_M d\theta \wedge \omega(u) = 0 \quad \text{for all test-functions } \theta : M \to \mathbb{R}.
\end{equation}

Then, to pose the initial value problem we impose the following **global hyperbolicity** condition: $M$ admits a smooth foliation by compact, oriented hypersurfaces, i.e.,
\[ M = \bigcup_{t \geq 0} \mathcal{H}_t. \]
Furthermore, denoting by $i_{\mathcal{H}_t} : \mathcal{H}_t \to M$ the injection map, we impose that each hypersurface is *spacelike* in the sense that the $n$-forms
\[ \partial_u \left( i_{\mathcal{H}_t}^* \omega(\Pi) \right) > 0 \]
are (positive) $n$-volume forms for each $t \geq 0$ and all relevant $\Pi$.

In the present framework, a **convex entropy flux** is now a parametrized family of $n$-forms $\Omega = \Omega_p(\Pi)$ such that there exists a convex $U : \mathbb{R} \to \mathbb{R}$
\[ \Omega_p(\Pi) := \int \partial_u U \partial_u \omega_p du', \quad p \in M, \Pi \in \mathbb{R}. \]

If the flux $\omega$ is geometry-compatible, then smooth solutions satisfy the additional conservation laws
\[ d(\Omega(u)) = 0, \]
so that the entropy inequalities take the following form
\begin{equation}
\int d\theta \wedge \omega(u) + \int_{\mathcal{H}_0} \theta_{\mathcal{H}_0} (i_{\mathcal{H}_t}^* \Omega)(u_0) \geq 0.
\end{equation}

**Definition 3.2.** An entropy solution to the geometry-compatible conservation law \[ \Omega = \Omega_p(\Pi) \] is a scalar field $u = u(p) \in L^\infty(M)$ satisfying for all convex entropy flux $\Omega = \Omega_p(\Pi)$ and smooth functions $\theta \geq 0$
\[ \int_M d\theta \wedge \Omega(u) + \int_{\mathcal{H}_0} \theta_{\mathcal{H}_0} (i_{\mathcal{H}_t}^* \Omega)(u_0) \geq 0. \]

Introduce also Kruzkov's entropy flux fields, defined here as parametrized fields of $n$-forms:
\[ \Omega(u, v) := \text{sgn} \left( u - v \right) \left( \omega(u) - \omega(v) \right). \]

**Theorem 3.1** (Well-posedness theory on a spacetime. Geometry-compatible flux). Consider a conservation law with a geometry-compatible flux $\omega = \omega(\Pi)$, posed on a globally hyperbolic, spatially compact manifold $M = \bigcup_{t \geq 0} \mathcal{H}_t$. Then, the initial value problem with data $u_0 \in L^\infty(\mathcal{H}_0)$ admits a unique entropy solution $u \in L^\infty(M)$, and for any convex entropy flux $\Omega$ the function $t \mapsto \int_{\mathcal{H}_t} i_{\mathcal{H}_t}^* \Omega(u)$ is well-defined and non-increasing. Moreover, for any two entropy solutions $u, v$
\[ \int_{\mathcal{H}_t} i_{\mathcal{H}_t}^* \Omega(u, v) \]
is non-increasing in time.

**Theorem 3.2** (Well-posedness theory on a spacetime. General flux). Consider a conservation law with general flux field $\omega = \omega(u)$, posed on globally hyperbolic,
spatially compact $M$ with (past) boundary $\mathcal{H}_0$. Consider initial data $u_0$ satisfying $\int_{\mathcal{H}_0} |\partial_t u_0| < \infty$. Then, there exists a semi-group of entropy solutions $u_t = Su_0$ and for all smooth functions $\theta \geq 0$

$$\int_M \partial_t \tilde{\theta} \wedge \Omega(u, v) + \int_{\mathcal{H}_0} \theta \tilde{v}^e_0 \Omega(u_0, v_0) \geq 0.$$  

Moreover, for any two spacelike hypersurfaces $\mathcal{H}_2$ in the future of $\mathcal{H}_1$

$$\int_{\mathcal{H}_2} \tilde{v}_e^e \Omega(u, v) \leq \int_{\mathcal{H}_1} \tilde{v}_e^e \Omega(u, v) < \infty.$$

Remark 3.1. On the one-dimensional torus $M = T^1$, the case of general flux fields was covered earlier by LeFloch and Nédélec [17] under the assumption that $f$ is strictly convex in $\mathfrak{u}$. Then, entropy solutions satisfy the generalized Lax formula $u(t, x) = (\partial_x f)^{-1} \left( \frac{c}{\Omega(X)} \right)$, in which the function $y = y(t, x)$ is determined by a minimization argument over the family of characteristics defined by

$$\partial_x X = (\partial_x f \circ f_1^{-1}) \left( \frac{c}{\Omega(X)} \right), \quad X(0) = y, \quad c \in \mathbb{R}.$$  

Interestingly, it was established in [17] that any two entropy solutions $u, v : \mathbb{R}^+ \times T^1 \to \mathbb{R}$ satisfy

$$\|v(t_2) - u(t_2)\|_{L_\infty(T^1)} \leq \|v(t_1) - u(t_1)\|_{L_\infty(T^1)}, \quad t_2 \geq t_1.$$  

4. Intrinsic finite volume schemes.

4.1. General intrinsic approach. Using the framework based on n-differential forms, Amorim, LeFloch, and Okutmustur [2] have introduced finite volume schemes, which generate piecewise constant approximations on unstructured triangulations. One considers here the initial value problem associated with the conservation law [31] and discretize the weak form [32] of the conservation law. Indeed, the finite volume approach applies directly to the geometric weak form and, in particular, no local coordinates should be chosen at this stage. The finite volume discretization on a manifold $M$ requires only the n-differential structure put forward in the previous section.

One introduces a family of triangulations made of curved polyhedra $K$, each of them having a past boundary $e^{-}_K$, a future boundary $e^+_K$, and “vertical boundary” elements $e \in \partial K^0$. For each boundary element $e$ there is precisely one element $K_e$ such that $K \cap K_e = e$. Then, one defines the notion of “total flux” along spacelike hypersurfaces $e^e_K$ by setting

$$\int_{e^e_K} \tilde{v}^e \omega(u) \approx \int_{e^e_K} \tilde{v}^e \omega(u^e_K) =: q^e_K(u^e_K),$$

where $u : M \to \mathbb{R}$ denotes the entropy solution of the initial value problem under consideration.

Then, applying Stokes theorem to the conservation law [31], we derive the finite volume scheme

$$q^e_K(u^e_K) - q^e_K(u^e_K) = q^e_K(u^e_K) - \sum_{e \in \partial K^0} q^e_K(u^e_K),$$

in which the total flux along vertical faces are scalars determined from Lipschitz continuous, numerical flux $q^e_K : \mathbb{R}^2 \to \mathbb{R}$ satisfying

1. Consistency property:

$$q^e_K(u^e_K) = q^e_K(u^e_K).$$
2. Conservation property:
\[ q_{K,e}(\overline{u}, \overline{v}) = -q_{K,e}(\overline{v}, \overline{u}) \]

3. Monotonicity property:
\[ \partial_{\overline{u}} q_{K,e} \geq 0, \quad \partial_{\overline{v}} q_{K,e} \leq 0 \]

for all \( \overline{u}, \overline{v} \in \mathbb{R} \). For instance, Lax-Friedrich flux or Godunov flux may be used to determine the functions \( q_{e,K} \). The use of total flux leads to a geometrically natural setting, in which the volume of elements \( K \) or boundary elements \( e \) do not arise, so that the proposed formulation is both conceptually and technically simpler.

The convergence of finite volume schemes toward the entropy solution of the initial value problem is established in [2, 20]. Let us mention here the main difficulties: possible blow-up of the TV norm; derivation of uniform estimates tied to the geometry, only involving \( n \)-dimensional volumes. The main issues in our analysis are as follows:

1. Derivation of a discrete maximum principle.
2. Derivation of discrete entropy inequalities.
3. Derivation of entropy dissipation estimates.
4. The sequence of finite volume approximation generate an entropy measure-valued solution \( \nu_p \), which, by our generalization of DiPerna’s uniqueness theorem, coincides with \( \nu_p = \delta_{u(p)} \), which implies that the scheme converges strongly.

DiPerna’s measure-valued solutions were used to establish the convergence of schemes by Szepessy [23, 24], Coquel and LeFloch [8, 9, 10], and Cockburn, Coquel, and LeFloch [6, 7]. For many related results and a review about the convergence techniques for hyperbolic problems, we refer to Tadmor [25] and Tadmor, Rascle, and Bagnerini [26]. Further hyperbolic models, including also a coupling with elliptic equations, as well as many applications were successfully investigated by Kröner [14], and Eymard, Gallouet, and Herbin [12]. For higher-order schemes, see the paper by Kröner, Noelle, and Rokyta [15]. An alternative approach to the convergence of finite volume schemes was discovered by Westdickenberg and Noelle [27].

4.2. A second-order intrinsic scheme on the sphere. Ben-Artzi, Falcoitz, and LeFloch [5] adopted the fully intrinsic approach and extended the finite volume scheme to second-order accuracy (by using the generalized Riemann problem methodology), while implementing it for classes of hyperbolic conservation laws posed on the sphere \( S^2 \). The sphere is the most important geometry for the applications to geophysical flows. Earlier works in this field were not fully geometric, and regarded the manifold as embedded in some Euclidian space and using projection techniques from the ambient space to the sphere. A fully intrinsic approach has definite advantages in terms of convergence and accuracy, and at the first-order at least, rigorous convergence analysis can be made.

We present here some features of the proposed scheme and refer to [5] for further discussions, especially on the implementation. We consider the conservation law

\[ \partial_t u(t, x) + \nabla_{S^2} \cdot (F(x, u(t, x))) = 0 \]

with flux vector tangent to the sphere, in the general form

\[ F(x, u) = n(x) \wedge \Phi(x, u), \]

where \( n(x) \) is the normal vector to the sphere at point \( x \), \( \Phi(x, u) \) is a gradient flux vector.

We are particularly interested in geometry-compatible flux vectors, and have proposed the following broad class of gradient flux vector

\[ \Phi(x, u) = \nabla h(x, u). \]

In particular, this includes homogeneous flux vectors having \( \Phi = \Phi(u) \). Interestingly enough, even in this case, the corresponding flux covers a broad and non-trivial class of interest.

Several classes of solutions of particular interest have been constructed, which exhibit very rich wave structure:
1. Spatially periodic solutions.
2. Non-trivial steady state solutions.
3. Confined solutions that remain compactly supported for all times.

Explicit formulas for such solutions together with various choices of corresponding flux vector fields can be found in [5]. These formulas are useful for numerical investigations.

An intrinsic triangulation was used, a web-like mesh made of segments of longitude and latitude lines. Artificial coordinate-singularities at the poles are coped with by suitably adapting the mesh in a non-conformal fashion as one approaches the North and South poles. To derive explicit formulas for the flux, for instance, in the case of an homogeneous flux vector $\Phi(u)$, one may introduce the following basis of the tangent space

$$i_\lambda = -\sin \lambda i_1 + \cos \lambda i_2, \quad i_\phi = -\sin \phi \cos \lambda i_1 - \sin \phi \sin \lambda i_2 + \cos \phi i_3;$$

in terms of some Euclidian basis $i_1, i_2, i_3 \in \mathbb{R}^3$. Then, by setting

$$\Phi(u) = f_1(u) i_1 + f_2(u) i_2 + f_3(u) i_3,$$

one finds

$$F(x, u) = F_\lambda(\lambda, \phi, u) i_\lambda + F_\phi(\lambda, \phi, u) i_\phi,$$

with

$$F_\lambda(\lambda, \phi, u) = f_1(u) \sin \phi \cos \lambda + f_2(u) \sin \phi \sin \lambda - f_3(u) \cos \phi,$$

and

$$F_\phi(\lambda, \phi, u) = -f_1(u) \sin \lambda + f_2(u) \cos \lambda.$$

The intrinsic finite volume discretization of the conservation law is then based on Stokes formula applied

$$\text{div}_{S^2} F = \frac{1}{\cos \phi} \left( \frac{\partial}{\partial \lambda} F_\lambda + \frac{\partial}{\partial \phi} (F_\phi \cos \phi) \right).$$

The flux are computed by explicit integration along longitude curves, or latitude curves. Length and areas are computed exactly, which allows one to derive a discrete version of the geometric compatibility condition.

5. Perspectives and open problems. In conclusion, based on our theoretical investigations (reported in this brief review), a shock-capturing numerical method was designed and implemented in [5], i.e. a fully intrinsic version of the second-order Godunov-type, finite volume scheme. This version, in its principle at first-order accuracy at least, extends to systems like the shallow water equations on the sphere with topography. Furthermore, the associated discrete properties (conservation, geometric compatibility, asymptotic-preserving) established in [5] extend to systems as well.

In the setup advocated here for the design numerical schemes, geometric terms in hyperbolic conservation laws are taken into account in a direct way by discretizing the geometric (weak) formulation of the conservation law, instead of using local coordinates and regarding the hyperbolic equation as an equation with variables coefficients and source-terms, or instead of viewing the manifold as embedded in a higher dimensional Euclidian space. This fully intrinsic approach was validated, both theoretically and numerically. It will be interesting to further investigate the properties of these finite volume schemes on general curved geometries, especially in the context of complex flows and geometries and to study the large-time asymptotics of solutions.

In the shallow water equations on the sphere, one unknown — the mass density — is a scalar field and can be treated as explained in the present work. However, the other unknown — the velocity — is a vector field, and it remains a challenging open problem to carry out the general Riemann problem methodology in this vector- or, more
generally, tensor-valued context. This is an important issue, since understanding large-scale atmospheric and oceanic motions depends upon designing robust and accurate numerical techniques that, we advocate, should directly incorporate curved geometric effects.

The class of geometry-compatible conservation laws on spacetimes forms an analogue of the inviscid Burgers equation, which has served as an important simplified model for the development of shock-capturing methods. The treatment of boundary conditions on non-compact manifolds is an important issue, tackled in [13], while error estimates a la Kuznetsov were derived in [3, 18].

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