Well-posedness and continuity properties of the Degasperis-Procesi equation in critical Besov space

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Abstract
In this paper, we obtain the local-in-time existence and uniqueness of solution to the Degasperis-Procesi equation in $B^{1}_{1,\infty}(\mathbb{R})$. Moreover, we prove that the data-to-solution of this equation is continuous but not uniformly continuous in $B^{1}_{1,\infty}(\mathbb{R})$.

Keywords Degasperis-Procesi equation · Local well-posedness · Non-uniform dependence

Mathematics Subject Classification 35Q53 · 37K10

1 Introduction
In this paper, we consider the Cauchy problem for the Degasperis-Procesi (DP) equation [13]

$$
\begin{aligned}
\partial_{t} u + uu_{x} &= -\frac{3}{2} \partial_{x} \left(1 - \partial_{x}^{2}\right)^{-1}(u^2), \\
u(0, x) &= u_{0}(x),
\end{aligned}
$$

(1.1)

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We should also mention the following famous Camassa-Holm (CH) equation [4–7, 10]
\[
\begin{aligned}
\partial_t u + u \partial_x u &= -\partial_x \left(1 - \partial_x^2\right)^{-1} \left(u^2 + \frac{1}{2} (\partial_x u)^2\right), \quad (x, t) \in \mathbb{R} \times \mathbb{R}^+, \\
u(0, x) &= u_0(x),
\end{aligned}
\] (1.2)

The DP equation can be regarded as a model for nonlinear shallow water dynamics and its asymptotic accuracy is the same as for the CH shallow water equation [11]. The DP equation is integrable and has a bi-Hamiltonian structure [12]. The inverse scattering approach for the DP equation was presented in [3, 8, 9, 22]. Its traveling wave solutions were investigated in [24, 32]. Regarding well-posedness (existence, uniqueness, and stability of solutions) of the Cauchy problem for the DP Eq. (1.1), we refer to see [16, 21, 25, 31, 35]. Similar to the CH equation, the DP equation has also global strong solutions [29, 33, 34] and finite time blow-up solutions [14, 15, 29, 30, 35]. Although the DP equation is similar to the CH equation in several aspects, these two equations are truly different. One of the novel features of the DP different from the CH equation is that it has not only peakon solutions [12] and periodic peakon solutions [29, 33, 34] and finite time blow-up solutions [14, 15, 29, 30, 35], but also shock peakons [23] and the periodic shock waves [14].

In [26, 27], we proved that the DP Eq. (1.1) is not uniformly continuous in \(B_{p,r}^s(\mathbb{R})\) with \(s > 1 + 1/p, \ p \in [1, \infty], \ r \in [1, \infty)\) or \(s = 1 + 1/p, \ p \in [1, \infty), \ r = 1\). Guo, Liu, Molinet and Yin [18] established the ill-posedness of (1.1) in the critical Sobolev space \(H^{3/2}\) and even in the Besov space \(B_p^{1+1/p, r}\) with \(p \in [1, \infty), \ r \in (1, \infty]\) by proving the norm inflation. In our recent paper [28], we established the ill-posedness for the DP equation in \(B_{p,\infty}^s(\mathbb{R})\) with \(s > 2 + \max\{1 + 1/p, 3/2\}\) and \(p \in [1, \infty)\) by proving the solution map starting from \(u_0\) is discontinuous at \(t = 0\) in the metric of \(B_{p,\infty}^s(\mathbb{R})\). For more ill-posedness results of the Cauchy problem for the DP equation, we refer to see [19, 20] and the references therein. However, whether the Cauchy problem of the DP Eq. (1.1) with initial data in \(B_{1,1}^{1}(\mathbb{R})\) is locally well-posed or not in the sense of Hadamard? To our best knowledge, this problem has not been solved yet. Before stating our main result precisely, we recall the notion of well-posedness in the sense of Hadamard:

**Definition 1.1** (Local Well-posedness) We say that the Cauchy problem (1.1) is locally well-posed in a Banach space \(X\) if the following three conditions hold

1. (Local existence) For any initial data \(u_0 \in X\), there exists a short time \(T = T(u_0) > 0\) and a solution \(S_t(u_0) \in C([0, T), X)\) to the Cauchy problem (1.1);
2. (Uniqueness) This solution \(S_t(u_0)\) is unique in the space \(C([0, T), X)\);
3. (Continuous Dependence) The data-to-solution map \(u_0 \mapsto S_t(u_0)\) is continuous in the following sense: for any \(T_1 < T\) and \(\varepsilon > 0\), there exists \(\delta > 0\), such that if \(\|u_0 - \tilde{u}_0\|_X \leq \delta\), then \(S_t(\tilde{u}_0)\) exists up to \(T_1\) and

\[
\|S_t(u_0) - S_t(\tilde{u}_0)\|_{C([0, T), X)} \leq \varepsilon.
\]

Now, we can formulate the main result.

**Theorem 1.1** Let \(u_0 \in B_{1,1}^{1}(\mathbb{R})\). Then, there exists some time \(T > 0\) such that
1. System (1.1) has a solution $u \in C([0, T], B^1_{\infty, 1}) \cap C^{1}([0, T], B^0_{\infty, 1})$;
2. The solutions of (1.1) are unique;
3. If the initial data have some additional regularity $u_0 \in B^s_{\infty, 1}$ with $s > 1$, then the
solution exists on $[0, T]$ and is in the space $C([0, T], B^s_{\infty, 1}) \cap C^{1}([0, T], B^{s-1}_{\infty, 1})$;
4. The data-to-solution map $u_0 \mapsto u(t)$ is continuous from any bounded subset of $u_0 \in B^1_{\infty, 1}$ into $C([0, T], B^1_{\infty, 1})$;
5. The data-to-solution map $u_0 \mapsto u(t)$ is not uniformly continuous from any bounded subset of $u_0 \in B^1_{\infty, 1}$ into $C([0, T], B^1_{\infty, 1})$.

Remark 1.1 Theorem 1.1 tells us that there exists a positive time $T$ such that System
(1.1) has a unique solution and the solution is continuously dependent on the initial
data in the sense of Hadmard.

2 Preliminaries

Notation Throughout this paper, we denote by $\ast$ the convolution and we shall use
$(1 - \partial^2_x)^{-1} f = p \ast f$ with $p(x) = \frac{1}{2} e^{-|x|}$. Given a Banach space $X$, we denote its
norm by $\|\cdot\|_X$. For $I \subset \mathbb{R}$, we denote by $C(I; X)$ the set of continuous functions on
$I$ with values in $X$. Sometimes we will denote $L^p(0, T; X)$ by $L^p_T X$. For all $f \in S'$, the
Fourier transform $\mathcal{F} f$ (also denoted by $\hat{f}$) is defined by
$$\mathcal{F} f(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}} e^{-ix\xi} f(x) dx \quad \text{for any} \ \xi \in \mathbb{R}.$$ 

Next, we will recall some facts about the Littlewood-Paley decomposition, the non-
homogeneous Besov spaces and their some useful properties.

Definition 2.1 (See [1]) Let $B := \{\xi \in \mathbb{R} : |\xi| \leq 4/3\}$ and $C := \{\xi \in \mathbb{R} : 3/4 \leq
|\xi| \leq 8/3\}$. There exist two radial functions $\chi \in C^\infty_c(B)$ and $\varphi \in C^\infty_c(C)$ both taking
values in $[0, 1]$ such that
$$\chi(\xi) + \sum_{j \geq 0} \varphi(2^{-j} \xi) = 1 \quad \forall \ \xi \in \mathbb{R}.$$ 

For every $u \in S' (\mathbb{R})$, the Littlewood-Paley dyadic blocks $\Delta_j$ are defined as follows
$$\Delta_j u = \begin{cases} 
0, & \text{if } j \leq -2; \\
\chi(D) u = \mathcal{F}^{-1}(\chi \mathcal{F} u), & \text{if } j = -1; \\
\varphi(2^{-j} D) u = \mathcal{F}^{-1}(\varphi(2^{-j} \cdot) \mathcal{F} u), & \text{if } j \geq 0.
\end{cases}$$ 

The inhomogeneous low-frequency cut-off operator $S_j$ is defined by
$$S_j u = \sum_{q=-1}^{j-1} \Delta_q u.$$
Definition 2.2 (See [1]) Let $s \in \mathbb{R}$ and $(p, r) \in [1, \infty]^2$. The nonhomogeneous Besov space $B^{s}_{p, r}(\mathbb{R})$ is defined by

$$B^{s}_{p, r}(\mathbb{R}) := \left\{ f \in S'(\mathbb{R}) : \| f \|_{B^{s}_{p, r}(\mathbb{R})} < \infty \right\},$$

where

$$\| f \|_{B^{s}_{p, r}(\mathbb{R})} = \begin{cases} \left( \sum_{j \geq -1} 2^{sjr} \| \Delta j f \|_{L^p(\mathbb{R})} \right)^\frac{1}{r}, & \text{if } 1 \leq r < \infty, \\ \sup_{j \geq -1} 2^{sj} \| \Delta j f \|_{L^p(\mathbb{R})}, & \text{if } r = \infty. \end{cases}$$

Lemma 2.1 (See [1]) For $p \in [1, \infty]$ and $s > 0$, $B^{s}_{p, 1}(\mathbb{R}) \cap L^\infty(\mathbb{R})$ is an algebra. Moreover, $B^{0}_{\infty, 1}(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R}) \hookrightarrow B^{0}_{\infty, \infty}(\mathbb{R})$, and for any $u, v \in B^{s}_{p, 1}(\mathbb{R}) \cap L^\infty(\mathbb{R})$, we have

$$\| uv \|_{B^{s}_{p, 1}} \leq C(\| u \|_{B^{s}_{p, 1}} \| v \|_{L^\infty} + \| v \|_{B^{s}_{p, 1}} \| u \|_{L^\infty}).$$

We also have the following interpolation inequality

$$\| u \|_{B^{1}_{\infty, 1}} \leq C \| u \|_{B^{0}_{\infty, \infty}} \| u \|_{B^{2}_{\infty, \infty}}^{\frac{1}{2}} \| u \|_{B^{1}_{\infty, \infty}}^{\frac{1}{2}}.$$

Lemma 2.2 (See [1, 25]) Let $(p, r) \in [1, \infty]^2$ and $\sigma \geq -\min \left\{ \frac{1}{p}, 1 - \frac{1}{p} \right\}$. Assume that $f_0 \in B^{\sigma}_{p, r}(\mathbb{R})$, $g \in L^1([0, T]; B^{\sigma}_{p, r}(\mathbb{R}))$ and $\partial_x u \in L^1([0, T]; B^{\sigma - 1}_{p, r}(\mathbb{R}))$ if $\sigma > 1 + \frac{1}{p}$ or $\sigma = 1 + \frac{1}{p}, r = 1$. If $f \in L^\infty([0, T]; B^{\sigma}_{p, r}(\mathbb{R})) \cap C([0, T]; S'(\mathbb{R}))$ solves the following linear transport equation:

$$\partial_t f + u \partial_x f = g, \quad f|_{t=0} = f_0.$$

Then there exists a constant $C = C(p, r, \sigma)$ such that the following statement holds

$$\| f(t) \|_{B^{\sigma}_{p, r}} \leq e^{CV(t)} \left( \| f_0 \|_{B^{\sigma}_{p, r}} + \int_0^t e^{-CV(\tau)} \| g(\tau) \|_{B^{\sigma}_{p, r}} d\tau \right),$$

where

$$V(t) = \int_0^t \| \partial_x u(\tau) \|_{B^{\sigma - 1}_{p, r}} d\tau \quad \text{if } \sigma > 1 + \frac{1}{p} \quad \text{or} \quad \{ \sigma = 1 + \frac{1}{p}, r = 1 \}.$$

3 Proof of Theorem 1.1

In this section, we divide Proof of Theorem 1.1 into three steps.
3.1 Existence and uniqueness

To prove the local existence, it suffices to establish the a priori estimate to solution in $B^{1}_{\infty,1}$.

Applying Lemma 2.2 to Eq.(1.1), we have for $t \in [0, T]$

$$
\|u(t)\|_{B^{1}_{\infty,1}} \leq e^{V(t)} \left( \|u_0\|_{B^{1}_{\infty,1}} + \int_{0}^{t} e^{-V(\tau)} \|P(u)(\tau)\|_{B^{1}_{\infty,1}} d\tau \right)
$$

$$
\leq e^{V(t)} \left( \|u_0\|_{B^{1}_{\infty,1}} + \int_{0}^{t} e^{-V(\tau)} \|u(\tau)\|_{B^{1}_{\infty,1}}^2 d\tau \right),
$$

(3.1)

where we denote

$$
V(t) = \int_{0}^{t} \|u\|_{B^{1}_{\infty,1}} d\tau
$$

and have used

$$
\|P(u)\|_{B^{1}_{\infty,1}} \leq C \|u\|_{B^{1}_{\infty,1}}^2.
$$

Then we obtain form (3.1) that

$$
F(t) := e^{-V(t)} \|u(t)\|_{B^{1}_{\infty,1}} \leq \|u_0\|_{B^{1}_{\infty,1}} + \int_{0}^{t} F(\tau) \|u(\tau)\|_{B^{1}_{\infty,1}} d\tau
$$

which follows from the Gronwall inequality that

$$
\sup_{\tau \in [0, t]} \|u(\tau)\|_{B^{1}_{\infty,1}} \leq C \|u_0\|_{B^{1}_{\infty,1}} \exp \left( C \int_{0}^{t} \sup_{s \in [0, \tau]} \|u(s)\|_{B^{1}_{\infty,1}} d\tau \right).
$$

(3.2)

Let

$$
\lambda(t) := C \|u_0\|_{B^{1}_{\infty,1}} \exp \left( C \int_{0}^{t} \sup_{s \in [0, \tau]} \|u(s)\|_{B^{1}_{\infty,1}} d\tau \right) \quad \text{with} \quad \lambda(0) := C \|u_0\|_{B^{1}_{\infty,1}}^2,
$$

then from (3.2), one has

$$
\frac{d}{dt} \lambda(t) \leq C \lambda^2(t) \iff -\frac{d}{dt} \left( \frac{1}{\lambda(t)} \right) \leq C.
$$

Solving the above directly yields

$$
\sup_{\tau \in [0, t]} \|u(\tau)\|_{B^{1}_{\infty,1}} \leq \frac{C \|u_0\|_{B^{1}_{\infty,1}}}{1 - C^2 t \|u_0\|_{B^{1}_{\infty,1}}}, \quad t \in [0, T_1].
$$
Therefor, the solution $u$ is bounded uniformly in $L^\infty([0, T]; B_{\infty,1}^1)$.

Furthermore, applying Lemma 2.2 to Eq. (1.1) once again, we have for all $t \in [0, T]$ and $s > 1$

$$\|u(t)\|_{L_T^\infty B_{\infty,1}^s} \leq C \|u_0\|_{B_{\infty,1}^s} \exp \left( C \int_0^T \|u\|_{B_{\infty,1}^1} \, dt \right) \leq C \|u_0\|_{B_{\infty,1}^s}. \tag{3.3}$$

The existence follows the standard procedure, we omit the details.

The uniqueness is the direct result of the following lemma. In fact, suppose that $u_1, u_2 \in C([0, T], B_{\infty,1}^1)$ are two solutions of (1.1) with the same initial data $u_0$, then we have

$$\|u_1 - u_2\|_{L^\infty} \leq C\|u_1(0) - u_2(0)\|_{L^\infty} = 0,$$

which implies the uniqueness.

**Lemma 3.1** Let $u, v \in C([0, T], B_{\infty,1}^1)$ be two solutions of (1.1) associated with $u_0$ and $v_0$, respectively. Then we have the estimates for the difference $w = u - v$

$$\|w\|_{L^\infty} \leq C \|w_0\|_{L^\infty} \tag{3.4}$$

and

$$\|w\|_{B_{\infty,1}^1} \leq C \left( \|w_0\|_{B_{\infty,1}^1} + \int_0^T \|\partial_x v\|_{B_{\infty,1}^1} \|w\|_{L^\infty} \, dt \right), \tag{3.5}$$

where the constants $C$ depends on $T$ and initial norm $\|u_0, v_0\|_{B_{\infty,1}^1}$.

**Proof** Obviously, we have

$$\begin{cases}
\partial_t w + u w_x = -w v_x - \frac{3}{2} \partial_x (1 - \partial_x^2)^{-1} (w(u + v)), \\
\partial_t v + v w_x = -v u_x - \frac{3}{2} \partial_x (1 - \partial_x^2)^{-1} (v(u + v)), \\
w(0, x) = u_0(x) - v_0(x).
\end{cases} \tag{3.6}$$

Taking the inner product of Eq. (3.6) with $|w|^{p-2} w$ with $p \geq 2$, we obtain

$$\frac{1}{p} \frac{d}{dt} \|w\|_{L^p}^p = -\int \partial_x w_x |v|^p \, dx + \frac{1}{p} \int \partial_x |w|^p \, dx$$

$$\quad \quad \quad - \frac{3}{2} \int \partial_x (1 - \partial_x^2)^{-1} (w(u + v)) |w|^{p-2} w \, dx$$

$$\leq (\|u_x\|_{L^\infty} + \|v_x\|_{L^\infty}) \|w\|_{L^p} + \frac{3}{2} \|\partial_x (1 - \partial_x^2)^{-1} (w(u + v))\|_{L^p} \|w\|_{L^p}^{p-1}$$

$$\leq (\|u_x\|_{L^\infty} + \|v_x\|_{L^\infty}) \|w\|_{L^p} + \frac{3}{2} \|w(u + v)\|_{L^p} \|w\|_{L^p}^{p-1}$$

$$\leq C (\|u\|_{L^\infty} + \|v\|_{L^\infty} + \|u_x\|_{L^\infty} + \|v_x\|_{L^\infty}) \|w\|_{L^p}^p, \tag{3.7}$$

where we have used that

$$\|\partial_x (1 - \partial_x^2)^{-1} (w(u + v))\|_{L^p} = \|\partial_x (w(u + v))\|_{L^p} \leq \|w(u + v)\|_{L^p}.$$
Integrating the above (3.7) with respect to time \( t \in [0, T] \) yields
\[
\| w \|_{L^p} \leq \| w_0 \|_{L^p} \exp \left( C \int_0^T \left( \| u \|_{B_{\infty,1}^1} + \| v \|_{B_{\infty,1}^1} \right) \, ds \right).
\]

Letting \( p = \infty \) implies (3.4).

Applying Lemma 2.2 to Eq.(3.6) yields
\[
\| w(t) \|_{B_{\infty,1}^1} - C \| w_0 \|_{B_{\infty,1}^1} \lesssim \int_0^t \| w v_x \|_{B_{\infty,1}^1} + \| \partial_x (1 - \partial_x^2)^{-1} (w(u + v)) \|_{B_{\infty,1}^1} \, d\tau
\]
\[
\lesssim \int_0^t \| w v_x \|_{B_{\infty,1}^1} + \| w(u + v) \|_{B_{\infty,1}^1} \, d\tau
\]
\[
\lesssim \int_0^t \| w \|_{B_{\infty,1}^1} \| v_x \|_{L^\infty} + \| v_x \|_{B_{\infty,1}^1} \| w \|_{L^\infty} + \| w \|_{B_{\infty,1}^1} \| u + v \|_{B_{\infty,1}^1} \, d\tau
\]
\[
\lesssim \int_0^t \| w \|_{B_{\infty,1}^1} \| u, v \|_{B_{\infty,1}^1} \, d\tau + \int_0^t \| v_x \|_{B_{\infty,1}^1} \| w \|_{L^\infty} \, d\tau.
\]

Gronwall’s inequality gives us the desired (3.5).

### 3.2 Continuous dependence

Now we will prove that the solution is continuously dependent on the initial data by [2, 17]. The main difficulty lies in that the DP equation is of hyperbolic type. Precisely speaking, if \( u, v \in C([0, T], B_{\infty,1}^1) \) are two solutions of (1.1) associated with \( u_0 \) and \( v_0 \), in view of (3.5), we have to tackle with the term \( \| v_x \|_{B_{\infty,1}^1} \). To bypass this, we can take \( v = S_t(S_N u_0) \) as the solution to (1.1) with initial data \( S_N u_0 \).

Letting \( u = S_t(u_0) \) and \( v = S_t(S_N u_0) \), using Lemma 3.1 and (3.3), we have
\[
\| S_t(S_N u_0) \|_{B_{\infty,1}^2} \leq C \| S_N u_0 \|_{B_{\infty,1}^2} \leq C 2^N \| u_0 \|_{B_{\infty,1}^1}
\]

and
\[
\| S_t(S_N u_0) - S_t(u_0) \|_{L^\infty} \leq C \| S_N u_0 - u_0 \|_{L^\infty}
\]
\[
\leq C \| S_N u_0 - u_0 \|_{B_{\infty,1}^0} \leq C 2^{-N} \| S_N u_0 - u_0 \|_{B_{\infty,1}^1},
\]

which implies
\[
\| S_t(S_N u_0) - S_t(u_0) \|_{B_{\infty,1}^1} \leq C \| S_N u_0 - u_0 \|_{B_{\infty,1}^1}.
\]
Then, using the triangle inequality, we have for $u_0, \tilde{u}_0 \in B_{1,1}^1$,

\[
\|S_t(u_0) - S_t(\tilde{u}_0)\|_{B_{1,1}^1} \leq \|S_t(S_N u_0) - S_t(u_0)\|_{B_{1,1}^1} + \|S_t(S_N \tilde{u}_0) - S_t(\tilde{u}_0)\|_{B_{1,1}^1}
\]

\[
\leq C \|S_N u_0 - u_0\|_{B_{1,1}^1} + C \|S_N \tilde{u}_0 - \tilde{u}_0\|_{B_{1,1}^1}
\]

\[
+ C \|S_t(S_N u_0) - S_t(S_N \tilde{u}_0)\|_{B_{1,1}^1} \leq C \|S_N u_0 - u_0\|_{B_{1,1}^1} + C \|S_N \tilde{u}_0 - \tilde{u}_0\|_{B_{1,1}^1} + C \|S_t(S_N u_0) - S_t(S_N \tilde{u}_0)\|_{B_{1,1}^1} \leq I_1 + I_2 + I_3.
\]

By the interpolation inequality, one has

\[
I_3 \leq C \|S_t(S_N u_0) - S_t(S_N \tilde{u}_0)\|_{B_{1,1}^1} \leq \|S_N u_0 - S_N \tilde{u}_0\|_{B_{2,2}^1} \leq \|S_t(S_N u_0) - S_t(S_N \tilde{u}_0)\|_{B_{2,2}^1} \leq C 2^{N} \|u_0 - \tilde{u}_0\|_{B_{1,1}^1},
\]

which clearly implies

\[
\|S_t(u_0) - S_t(\tilde{u}_0)\|_{B_{1,1}^1} \leq \|S_N u_0 - u_0\|_{B_{1,1}^1} + \|S_N \tilde{u}_0 - \tilde{u}_0\|_{B_{1,1}^1} + 2^{N} \|u_0 - \tilde{u}_0\|_{B_{1,1}^1} \leq \|S_N u_0 - u_0\|_{B_{1,1}^1} + \|u_0 - \tilde{u}_0\|_{B_{1,1}^1} + 2^{N} \|u_0 - \tilde{u}_0\|_{B_{1,1}^1}.
\]

This completes the proof of continuous dependence.

### 3.3 Non-uniform continuous dependence

Let $\hat{\phi} \in C_0^\infty(\mathbb{R})$ be an even, real-valued and non-negative function on $\mathbb{R}$ and satisfy

\[
\hat{\phi}(\xi) = \begin{cases} 
1, & \text{if } |\xi| \leq \frac{1}{4}, \\
0, & \text{if } |\xi| \geq \frac{1}{2}.
\end{cases}
\]

**Lemma 3.2** We define the high frequency function $f_n$ and the low frequency function $g_n$ as follows

\[
f_n = 2^{-n} \phi(x) \sin \left( \frac{17}{12} 2^n x \right),
\]

\[
g_n = \frac{12}{17} 2^{-n} \phi(x), \quad n \gg 1.
\]

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Then for any $\sigma \in \mathbb{R}$, we have
\[
\|f_n\|_{L^\infty} \leq C 2^{-n} \phi(0) \quad \text{and} \quad \|g_n\|_{L^\infty} \leq C 2^{-n} \phi(0),
\]
\[
\|f_n\|_{B_{\infty,1}^{\sigma}} \leq C 2^{(\sigma-1)n} \phi(0) \quad \text{and} \quad \|g_n\|_{B_{\infty,1}^{\sigma}} \leq C 2^{-(n+\sigma)} \phi(0),
\]
\[
\liminf_{n \to \infty} \|g_n \partial_x f_n\|_{B_{\infty,\infty}^{1}} \geq M_1,
\]
for some positive constants $C$ and $M_1$.

**Proof** The proof is direct. We leave it to the interested readers.

**Proposition 3.1** Assume that $\|u_0\|_{B_{\infty,1}^{1}} \lesssim 1$. Under the assumptions of Theorem 1.1, we have
\[
\|S_t(u_0) - u_0 - tv_0(u_0)\|_{B_{\infty,1}^{1}} \leq C t^2 E(u_0),
\]
where we denote $v_0(u_0) := P(u_0) - u_0 \partial_x u_0$ and
\[
E(u_0) := 1 + \|u_0\|_{L^\infty} \left( \|u_0\|_{B_{\infty,1}^{2}} + \|u_0\|_{L^\infty} \|u_0\|_{B_{\infty,1}^{3}} \right).
\]

**Proof** For simplicity, we denote $u(t) = S_t(u_0)$. By the Mean Value Theorem, we obtain
\[
\|u(t) - u_0\|_{L^\infty} \leq \int_0^t \|\partial_\tau u\|_{L^\infty} d\tau
\]
\[
\leq \int_0^t \|u \partial_x u\|_{L^\infty} d\tau + \int_0^t \|P(u)\|_{L^\infty} d\tau
\]
\[
\leq C \int_0^t \|u\|_{L^\infty} \|u\|_{C^{0,1}} d\tau \leq C t \|u_0\|_{L^\infty},
\]
where we have used the estimate
\[
\|P(u)\|_{L^\infty} \leq C \|u\|_{L^\infty}^2.
\]
Using Lemma 2.1, (3.3) and (3.12) yield
\[
\|u(t) - u_0\|_{B_{\infty,1}^{1}} \leq \int_0^t \|\partial_\tau u\|_{B_{\infty,1}^{1}} d\tau \leq \int_0^t \|P(u)\|_{B_{\infty,1}^{1}} d\tau + \int_0^t \|u \partial_x u\|_{B_{\infty,1}^{1}} d\tau
\]
\[
\leq C t \left( \|u\|_{B_{\infty,1}^{2}}^2 + \|u\|_{L^\infty} \|u\|_{B_{\infty,1}^{1}} \right) \leq C t \left( 1 + \|u_0\|_{L^\infty} \|u_0\|_{B_{\infty,1}^{2}} \right).
\]
Similarly, we have
\[
\|u(t) - u_0\|_{B_{\infty,1}^{2}} \leq \int_0^t \|\partial_\tau u\|_{B_{\infty,1}^{2}} d\tau \leq \int_0^t \|P(u)\|_{B_{\infty,1}^{2}} d\tau + \int_0^t \|u \partial_x u\|_{B_{\infty,1}^{2}} d\tau
\]
\[
\leq C t \left( 1 + \|u_0\|_{L^\infty} \|u_0\|_{B_{\infty,1}^{2}} \right).
\]
Using the Mean Value Theorem and Lemma 2.1 once again, we obtain that
\[
\|u(t) - u_0 - t\mathbf{v}_0(u_0)\|_{B^1_{\infty,1}} \leq \int_0^t \|\partial_\tau u - \mathbf{v}_0(u_0)\|_{B^1_{\infty,1}} d\tau \\
\leq \int_0^t \|\mathbf{P}(u) - \mathbf{P}(u_0)\|_{B^1_{\infty,1}} d\tau \\
+ \int_0^t \|u\partial_\tau u - u_0\partial_\tau u_0\|_{B^1_{\infty,1}} d\tau \\
\leq \int_0^t \|u(\tau) - u_0\|_{B^1_{\infty,1}} d\tau \\
+ \int_0^t \|u(\tau) - u_0\|_{L^\infty} \|u(\tau)\|_{B^2_{\infty,1}} d\tau \\
+ \int_0^t \|u(\tau) - u_0\|_{B^2_{\infty,1}} \|u_0\|_{L^\infty} d\tau \\
\leq \int_0^t \|u(\tau) - u_0\|_{B^1_{\infty,1}} d\tau \\
+ \|u_0\|_{B^2_{\infty,1}} \int_0^t \|u(\tau) - u_0\|_{L^\infty} d\tau \\
+ \|u_0\|_{L^\infty} \int_0^t \|u(\tau) - u_0\|_{B^2_{\infty,1}} d\tau.
\] (3.15)

Plugging (3.12)–(3.14) into (3.15) yields the desired result (3.11). Thus, we complete the proof of Proposition 3.1.

Now we prove the non-uniform continuous dependence.

We set \(u^n_0 = f_n + g_n\) and compare the solution \(S_t(u^n_0)\) and \(S_t(f_n)\). We obviously have
\[
\|u^n_0 - f_n\|_{B^1_{\infty,1}} = \|g_n\|_{B^1_{\infty,1}} \leq C2^{-n},
\]
which means that
\[
\lim_{n \to \infty} \|u^n_0 - f_n\|_{B^1_{\infty,1}} = 0.
\]

From Lemma 2.1, one has
\[
\|u^n_0, f_n\|_{B^\sigma_{\infty,1}} \leq C2^{(\sigma-1)n} \text{ for } \sigma \geq 1,
\]
\[
\|u^n_0, f_n\|_{L^\infty} \leq C2^{-n},
\]
which implies
\[ E(u_0^n) + E(f_n) \leq C. \]

Using the facts
\[
\|u_0^n \partial_x g_n\|_{B_{\infty,1}^1} \leq C \|u_0^n\|_{B_{\infty,1}^1} \|g_n\|_{B_{\infty,1}^2} \leq C 2^{-n},
\]
\[
\|P(u_0^n) - P(f_n)\|_{B_{\infty,1}^1} \leq C \|g_n\|_{B_{\infty,1}^1} \|u_0^n + f_n\|_{B_{\infty,1}^1} \leq C 2^{-n}.
\]
we deduce that (for more details, see [26])
\[
\|S_t(u_0^n) - S_t(f_n)\|_{B_{\infty,1}^1} \geq t \|g_n \partial_x f_n\|_{B_{\infty,1}^1} - t \|u_0^n \partial_x g_n\|_{B_{\infty,1}^1} - Ct^2 - C2^{-n} \geq t \|g_n \partial_x f_n\|_{B_{\infty,1}^1} - Ct2^{-n} - Ct^2 - C2^{-n},
\]
(3.16)

Notice that (3.10)
\[
\liminf_{n \to \infty} \|g_n \partial_x f_n\|_{B_{\infty,1}^1} \gtrsim M_1,
\]
then we deduce from (3.16) that
\[
\liminf_{n \to \infty} \|S_t(f_n + g_n) - S_t(f_n)\|_{B_{\infty,1}^1} \gtrsim t \quad \text{for } t \text{ small enough.}
\]
This completes the proof of Theorem 1.1. \(\square\)

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**Declarations**

**Conflict of interest** The authors declare that they have no conflict of interest.

**References**

1. Bahouri, H., Chemin, J.Y., Danchin, R.: Fourier analysis and nonlinear partial differential equations. Grundlehren der Mathematischen Wissenschaften, Springer, Heidelberg (2011)
2. Bona, J.L., Smith, R.: The initial-value problem for the Korteweg-de Vries equation. Philos. Trans. R. Soc. Lond. Ser. A 278, 555–601 (1975)
3. Boutet de Monvel, A., Shepelsky, D.: A Riemann-Hilbert approach for the Degasperis-Procesi equation. Nonlinearity 26, 2081–2107 (2013)
4. Constantin, A.: The Hamiltonian structure of the Camassa-Holm equation. Exposition. Math. 15, 53–85 (1997)
5. Constantin, A., Escher, J.: Global existence and blow-up for a shallow water equation. Ann. Scuola Norm. Sup. Pisa Cl. Sci. 26, 303–328 (1998)
6. Constantin, A., Escher, J.: Wave breaking for nonlinear nonlocal shallow water equations. Acta Math. 181, 229–243 (1998)
7. Constantin, A., Strauss, W.A.: Stability of peakons. Comm. Pure Appl. Math. 53, 603–610 (2000)
8. Constantin, A., Ivanov, R.I., Lenells, J.: Inverse scattering transform for the Degasperis-Procesi equation. Nonlinearity 23, 2559–2575 (2010)
9. Constantin, A., Ivanov, R.I.: Dressing method for the Degasperis-Procesi equation. Stud. Appl. Math. 138, 205–226 (2017)
10. Camassa, R., Holm, D.: An integrable shallow water equation with peaked solitons. Phys. Rev. Lett. 71, 1661–1664 (1993)
11. Dullin, H.R., Gottwald, G.A., Holm, D.D.: On asymptotically equivalent shallow water wave equations. Physica D 190, 1–14 (2004)
12. Degasperis, A., Holm, D., Hone, A.: A new integral equation with peakon solutions. Theoret. Math. Phys. 133, 1463–1474 (2002)
13. Degasperis, A., Procesi, M.: Asymptotic integrability. In: Symmetry and Perturbation Theory. (Rome, 1998). Rivers Edge, NJ: World Scientific Publishing Company, pp. 23–37 (1999)
14. Escher, J., Liu, Y., Yin, Z.: Shock waves and blow-up phenomena for the periodic Degasperis-Procesi equation. Indiana Univ. Math. J. 56, 87–177 (2007)
15. Escher, J., Liu, Y., Yin, Z.: Global weak solutions and blow-up structure for the Degasperis-Procesi equation. J. Funct. Anal. 241, 457–485 (2006)
16. Gui, G., Liu, Y.: On the Cauchy problem for the Degasperis-Procesi equation. Quart. Appl. Math. 69, 445–464 (2011)
17. Guo, Z., Li, J., Yin, Z.: Local well-posedness of the incompressible Euler equations in $B^{1}_{\infty,1}$ and the inviscid limit of the Navier-Stokes equations. J. Funct. Anal. 276, 2821–2830 (2019)
18. Guo, Z., Liu, X., Molinet, L., Yin, Z.: Ill-posedness of the Camassa-Holm and related equations in the critical space. J. Differ. Equ. 266, 1698–1707 (2019)
19. Himonas, A., Holliman, C., Grayshan, K.: Norm Inflation and Ill-Posedness for the Degasperis-Procesi Equation. Comm. Partial Differ. Equ. 39, 2198–2215 (2014)
20. Himonas, A., Grayshan, K., Holliman, C.: Ill-Posedness for the b-Family of Equations. J. Nonlinear Sci. 26, 1175–1190 (2016)
21. Himonas, A.A., Holliman, C.: The Cauchy problem for the Novikov equation. Nonlinearity 25, 449–479 (2012)
22. Lundmark, H., Szmigielski, J.: Multi-peakon solutions of the Degasperis-Procesi equation. Inverse Problems. 19, 1241–1245 (2003)
23. Lundmark, H.: Formation and Dynamics of Shock Waves in the Degasperis-Procesi Equation. J. Nonlinear Sci. 17, 169–198 (2007)
24. Lenells, J.: Traveling wave solutions of the Degasperis-Procesi equation. J. Math. Anal. Appl. 306, 72–82 (2005)
25. Li, J., Yin, Z.: Remarks on the well-posedness of Camassa-Holm type equations in Besov spaces. J. Differ. Equ. 261, 6125–6146 (2016)
26. Li, J., Wu, X., Yu, Y., Zhu, W.: Non-uniform dependence on initial data for the Camassa-Holm equation in the critical Besov space. J. Math. Fluid Mech. 23, 11 (2021)
27. Li, J., Yu, Y., Zhu, W.: Non-uniform dependence on initial data for the Camassa-Holm equation in Besov spaces. J. Differ. Equ. 269, 8686–8700 (2020)
28. Li, J., Yu, Y., Zhu, W.: Ill-posedness for the Camassa-Holm and related equations in Besov spaces. J. Differ. Equ. 306, 403–417 (2022)
29. Liu, Y., Yin, Z.: Global existence and blow-up phenomena for the Degasperis-Procesi equation. Commun. Math. Phys. 267, 801–820 (2006)
30. Liu, Y., Yin, Z.: On the blow-up phenomena for the Degasperis-Procesi equation. Int. Math. Res. Not IMRN 23, 117 (2007)
31. Lin, Z., Liu, Y.: Stability of peakons for the Degasperis-Procesi equation. Commun. Pure Appl. Math. 62(1), 125–146 (2009)
32. Vakhnenko, V.O., Parkes, E.J.: Periodic and solitary-wave solutions of the Degasperis-Procesi equation. Chaos Solitons Fractals. 20, 1059–1073 (2004)
33. Yin, Z.: Global existence for a new periodic integrable equation. J. Math. Anal. Appl. 283, 129–139 (2003)
34. Yin, Z.: Global solutions to a new integrable equation with peakons. Indiana Univ. Math. J. 53, 1189–1210 (2004)
35. Yin, Z.: On the Cauchy problem for an integrable equation with peakon solutions. Illinois J. Math. 47, 649–666 (2003)
36. Yin, Z.: Global weak solutions for a new periodic integrable equation with peakon solutions. J. Funct. Anal. 212, 182–194 (2004)

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