THREE-TERM ARITHMETIC PROGRESSIONS AND SUMSETS

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ABSTRACT. Suppose that $G$ is an abelian group and $A \subseteq G$ is finite and contains no non-trivial three-term arithmetic progressions. We show that $|A + A| \gg \varepsilon |A| (\log |A|)^{\frac{1}{3} - \varepsilon}$.

1. INTRODUCTION

In [Fre73] Frei\v{m}an proved the following qualitative theorem.

Theorem 1.1 (Frei\v{m}an). Suppose that $A \subseteq \mathbb{Z}$ is finite and contains no non-trivial three-term arithmetic progressions. Then $|A + A| / |A| \to \infty$ as $|A| \to \infty$.

The best known quantitative version of this theorem is achieved by inserting Bourgain’s most recent bound for Roth’s theorem (see [Bou08]) into a result of Ruzsa’s (see [Ruz92]).

Theorem 1.2 (Bourgain-Ruzsa). Suppose that $A \subseteq \mathbb{Z}$ is finite and contains no non-trivial three-term arithmetic progressions. Then

$$|A + A| \gg |A| \left( \frac{\log |A|}{(\log \log |A|)^3} \right)^{\frac{1}{6}}.$$ 

This theorem is interesting in its own right but has also been applied (independently) by Schoen in [Scho] and Hegyvári, Hennecart and Plagne in [HH] to give a witty proof of the following result regarding restricted sumsets.

If $A, B$ are subsets of an abelian group then we write

$$A \hat{+} B := \{a + b : a \in A, b \in B \text{ and } a \neq b\},$$

and call this the restricted sum of $A$ and $B$.

Theorem 1.3 (Schoen-Hegyvári-Hennecart-Plagne). Suppose $A$ and $B$ are two finite non-empty sets of integers, or residues modulo an integer $m > 1$, and put $n := |A + B|$. Then

$$\frac{|A \hat{+} B|}{|A + B|} = 1 + O \left( \frac{(\log \log n)^3}{\log n} \right)^{\frac{1}{6}}.$$ 

Recently a lot of work has been done on generalizing additive problems in the integers to other abelian groups (see, for example, [GR07, GT09b, GT] and [Mes95]) and in this paper we not only improve the bounds in Theorems 1.2 and 1.3 but we also extend them to cover arbitrary abelian groups. Specifically our main result is the following theorem.

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1A trivial three-term arithmetic progression is one in which all three elements are the same.

2By slight abuse of notation.
Theorem 1.4. Suppose that $G$ is an abelian group and $A \subset G$ is finite and contains no non-trivial three-term arithmetic progressions. Then

$$|A + A| \gg |A| \left( \frac{\log |A|}{(\log \log |A|)^3} \right)^{\frac{1}{3}}.$$ 

This translates easily to an improvement of Theorem 1.3.

Theorem 1.5. Suppose $A$ and $B$ are two finite non-empty subsets of an abelian group $G$ and put $n := |A + B|$. Then

$$\frac{|A \hat{+} B|}{|A + B|} = 1 + O \left( \frac{(\log \log n)^{\frac{4}{3}}}{\log n} \right).$$

There are three main aspects to our arguments. First, to effect a complete passage to general abelian groups we have to work slightly harder when the sets in question have elements which differ by an element of order 2. To deal with this we use a generalization of the Bohr set technology of Bourgain [Bou99], as developed by Green and the author in [GS08].

Second, we use an energy increment argument in the style of Heath-Brown [HB87] and Szemerédi [Sze90] to prove a local version of Roth’s theorem which is particularly efficient (essentially because of limitations in the modeling results of Green and Ruzsa [GR07]) in our situation; this type of argument has been deployed previously in [San08a].

Finally we use a result which might be called a weak partially polynomial version of the celebrated Freiman-Ruzsa theorem. This type of result was first proved for finite fields by Green and Tao in [GT09c]; the more general case we use was proved by Green and the author in [GS08].

The paper now splits into seven further sections. In §§3–4 we set up the basic machinery of ‘local’ Fourier analysis, which lets us prove our local version of Roth’s theorem in §5. In §6 we prove the partially polynomial version of the Freiman-Ruzsa theorem before completing the main arguments in §7.

In the final section, §8 we discuss improvements for particular groups $G$ and possible further questions.

2. Notation

The book [Rud90] serves as a general reference for the Fourier transform, which we use throughout the paper.

Suppose that $G$ is a finite abelian group. $\hat{G}$ denotes the dual group of $G$, that is the group of homomorphisms $\gamma : G \to S^1$, where $S^1 := \{z \in \mathbb{C} : |z| = 1\}$, and we write $M(G)$ for the space of measures on $G$ endowed with the norm $\|\cdot\|$ defined by $\|\mu\| := \int d|\mu|$. There is one element of $M(G)$ worthy of particular note: the Haar probability measure $\mu_G$. This measure is used to define the Fourier transform which takes a function $f : G \to \mathbb{C}$ to

$$\hat{f} : \hat{G} \to \mathbb{C}; \gamma \mapsto \int_{x \in G} f(x)\overline{\gamma(x)}d\mu_G(x) = \frac{1}{|G|} \sum_{x \in G} f(x)\overline{\gamma(x)}.$$
We use the Haar probability measure, $\mu_G$, on $G$ to define an inner product on functions $f, g : G \to \mathbb{C}$ by

$$\langle f, g \rangle := \int_{x \in G} f(x) \overline{g(x)} d\mu_G(x).$$

Since $\mu_G$ is normalized to be a probability measure, Plancherel’s theorem states that

$$\langle f, g \rangle = \sum_{\gamma \in \hat{G}} \hat{f}(\gamma) \overline{\hat{g}(\gamma)}.$$

Similarly we use $\mu_G$ to define the convolution of two functions $f, g : G \to \mathbb{C}$:

$$f * g(y) := \int_{x \in G} f(y - x) g(x) d\mu_G(x),$$

and a simple calculation tells us that $\hat{f * g} = \hat{f} \hat{g}$.

Finally it will sometimes be necessary to consider the Fourier transform of a particularly complicated expression $E$. In this case we may write $E^\wedge$ in place of $\hat{E}$.

### 3. Bourgain systems

In [Bou99], Bourgain showed how to extend some of the techniques of Fourier analysis from groups to a wider class of ‘approximate groups’; in the paper [GS08] this was taken further when the notion of a Bourgain system was introduced. We refer the reader to that paper for a more comprehensive discussion of Bourgain systems and limit ourselves to recalling the key definitions and tools that we shall require.

Suppose that $G$ is a finite abelian group and $d \geq 1$ is real. A Bourgain system $B$ of dimension $d$ is a collection $(B_\rho)_{\rho \in (0, 2]}$ of subsets of $G$ such that the following axioms are satisfied:

(i) (Nesting) If $\rho' \leq \rho$ we have $B_{\rho'} \subseteq B_\rho$;
(ii) (Zero) $0 \in B_\rho$ for all $\rho \in (0, 2]$;
(iii) (Symmetry) If $x \in B_\rho$ then $-x \in B_\rho$;
(iv) (Addition) For all $\rho, \rho'$ such that $\rho + \rho' \leq 1$ we have $B_\rho + B_{\rho'} \subseteq B_{\rho + \rho'}$;
(v) (Doubling) If $\rho \leq 1$ then there is a set $X$ with $|X| \leq 2^d$ and

$$B_{2\rho} \subseteq \bigcup_{x \in X} x + B_\rho.$$  

We define the density of $B = (B_\rho)_{\rho}$ to be $\mu_G(B_1)$ and denote it $\mu_G(B)$. Frequently we shall consider several Bourgain systems $B, B', B''$, ..., in this case the underlying sets will be denoted $(B_\rho)_{\rho}, (B'_\rho)_{\rho}, (B''_\rho)_{\rho}, ...$, and we shall write $B, B', B''$, ... for the sets $B_1, B'_1, B''_1$, ....

**Example (Bohr sets).** There is a natural valuation on $S^1$ defined by $||z|| := (2\pi)^{-1} |\arg z|$, where $\arg$ is taken as mapping into $(-\pi, \pi]$. If $\Gamma \subset \hat{G}$ and $\delta \in (0, 1]$ then we put

$$B(\Gamma, \delta) := \{ x \in G : ||\gamma(x)|| \leq \delta \text{ for all } \gamma \in \Gamma \},$$

and call such a set a **Bohr set**.

It turns out that the system $(B(\Gamma, \rho \delta))_{\rho}$ is a Bourgain system of density at least $\delta^{|\Gamma|}$ and dimension $2|\Gamma|$, as the next lemma shows. By a slight abuse we call this the Bourgain system **induced** by the Bohr set $B(\Gamma, \delta)$. 

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Lemma 3.1. Suppose that \( B(\Gamma, \delta) \) is a Bohr set. Then
\[
\mu_G(B(\Gamma, \delta)) \geq \delta|\Gamma|
\]
and there is a set \( X \) of size at most \( 4^{|\Gamma|} \) such that
\[
B(\Gamma, 2\delta) \subset \bigcup_{x \in X} x + B(\Gamma, \delta).
\]

The proof of this lemma is a simple averaging argument which may be found, for example, in [TV06, Lemma 4.20].

Returning to Bourgain systems in general, we say that a Bourgain system \( \mathcal{B}' \) is a sub-system of \( \mathcal{B}'' \) if \( \mathcal{B}'_\rho \subset \mathcal{B}''_\rho \) for all \( \rho \). We shall be very interested in sub-systems and consequently the following dilation and intersection lemmas will be important. The first lemma is immediate.

Lemma 3.2. Suppose that \( \mathcal{B} \) is a Bourgain system of dimension \( d \) and \( \lambda \in (0, 1) \) is a parameter. Then \( \lambda \mathcal{B} := (B_{\lambda \rho})_\rho \) is a Bourgain system of dimension \( d \) and density at least \( (\lambda/2)^d \mu_G(\mathcal{B}) \).

Lemma 3.3. Suppose that \( B^{(1)}, \ldots, B^{(k)} \) are Bourgain systems of dimensions \( d_1, \ldots, d_k \) respectively. Then \( \bigcap_{i=1}^k B^{(i)} := (\bigcap_{i=1}^k B^{(i)}_\rho)_\rho \) is a Bourgain system of dimension at most \( 2(d_1 + \cdots + d_k) \) and density at least \( 4^{-(d_1 + \cdots + d_k - 1)} 2^{d_k} \prod_{i=1}^k \mu_G(\mathcal{B}^{(i)}) \).

Proof. The conclusion is trivial apart from the doubling and density estimates. For each \( i \) with \( 1 \leq i \leq k \) there is a set \( T_i \) with \( |T_i| \leq 4d_i \) such that \( B^{(i)}_{2\rho} \subset T_i + B^{(i)}_{\rho/2} \). Define a set \( T \) as follows: for each \( (t_1, \ldots, t_k) \in T_1 \times \cdots \times T_k \) place one element of \( \bigcap_{i=1}^k (t_i + B^{(i)}_{\rho/2}) \) into \( \bigcap_{i=1}^k B^{(i)}_{\rho} \), whence
\[
\bigcap_{i=1}^k B^{(i)}_{2\rho} \subset T + \bigcap_{i=1}^k B^{(i)}_{\rho},
\]
and the intersection has dimension at most \( 2(d_1 + \cdots + d_k) \).

The density estimate proceeds similarly. For each \( i \) with \( 1 \leq i \leq k-1 \) let \( T_i \) be a maximal subset of \( G \) such that the sets \( (t + B^{(i)}_{1/4})_{t \in T_i} \) are disjoint. It follows that \( |T| \leq 4d_i \mu_G(\mathcal{B}^{(i)})^{-1} \) and
\[
G \subset B^{(i)}_{1/4} - B^{(i)}_{1/4} + T_i \subset B^{(i)}_{1/2} + T_i.
\]

Thus there are some \( x_1, \ldots, x_{k-1} \in G \) such that
\[
\mu_G\left( \bigcap_{i=1}^{k-1} (x_i + B^{(i)}_{1/2}) \cap B^{(k)}_{1/2} \right) \geq 4^{-(d_1 + \cdots + d_k - 1)} 2^{d_k} \prod_{i=1}^k \mu_G(\mathcal{B}^{(i)}).
\]

Now for fixed \( x_0 \in \bigcap_{i=1}^{k-1} (x_i + B^{(i)}_{1/2}) \cap B^{(k)}_{1/2} \) the map \( x \mapsto x - x_0 \) is an injection from \( \bigcap_{i=1}^{k-1} (x_i + B^{(i)}_{1/2}) \cap B^{(k)}_{1/2} \) into \( \bigcap_{i=1}^k B^{(i)}_{1/2} \). The result follows. \( \Box \)

Not all Bourgain systems behave as regularly as we would like; we say that a Bourgain system \( \mathcal{B} \) of dimension \( d \) is regular if
\[
1 - 2^3 d|\eta| \leq \frac{\mu_G(B_1)}{\mu_G(B_{1+\eta})} \leq 1 + 2^3 d|\eta|
\]
for all \( \eta \) with \( d|\eta| \leq 2^{-3} \). Typically, however, Bourgain systems are regular, a fact implicit in the proof of the following proposition.

**Proposition 3.4.** Suppose that \( B \) is a Bourgain system of dimension \( d \). Then there is a \( \lambda \in [1/2, 1) \) such that \( \lambda B \) is regular.

**Proof.** Let \( f : [0, 1] \to \mathbb{R} \) be the function \( f(\alpha) := -\frac{1}{2} \log_2 \mu_G(B_{2^{-\alpha}}) \) and note that \( f \) is non-decreasing in \( \alpha \) with \( f(1) - f(0) \leq 1 \). We claim that there is an \( \alpha \in [\frac{1}{2}, \frac{3}{2}] \) such that \( |f(\alpha + x) - f(\alpha)| \leq 3|x| \) for all \( |x| \leq \frac{1}{6} \). If no such \( \alpha \) exists then for every \( \alpha \in [\frac{1}{2}, \frac{3}{2}] \) there is an interval \( I(\alpha) \) of length at most \( \frac{1}{6} \) having one endpoint equal to \( \alpha \) and with \( \int_{I(\alpha)} df > \int_{I(\alpha)} 3dx \). These intervals cover \( [\frac{1}{2}, \frac{3}{2}] \), which has total length \( \frac{2}{3} \). A simple covering lemma allows us to pass to a disjoint subcollection \( I_1 \cup \ldots \cup I_n \) of these intervals with total length at least \( \frac{1}{2} \). However we now have

\[
1 \geq \int_0^1 df \geq \sum_{i=1}^n \int_{I_i} df > \sum_{i=1}^n \int_{I_i} 3dx \geq 1,
\]

a contradiction. It follows that there is an \( \alpha \) such that \( |f(\alpha + x) - f(\alpha)| \leq 3|x| \) for all \( |x| \leq \frac{1}{6} \). Setting \( \lambda := 2^{-\alpha} \), it is easy to see that

\[
(1 + |\eta|)^{-3d} \leq \frac{\mu_G(B_\lambda)}{\mu_G(B_{(1+|\eta|)}\lambda)} \leq (1 + |\eta|)^{3d}
\]

whenever \( |\eta| \leq 1/6 \). But if \( 3d|\eta| \leq 1/2 \) then \( (1+|\eta|)^{-3d} \leq 1+6d|\eta| \) and \( (1+|\eta|)^{-3d} \geq 1 - 6d|\eta| \); it follows that \( \lambda B \) is a regular Bourgain system. \( \square \)

4. **Fourier analysis local to Bourgain systems**

Regular Bourgain systems are the ‘approximate groups’ to which we extend Fourier analysis; there is a natural candidate for ‘approximate Haar measure’ on \( B \): if \( (B_\rho)_\rho \) is a Bourgain system then we write \( \beta_\rho \) for the normalized counting measure on \( B_\rho \) and simply \( \beta \) for \( \beta_1 \). We adopt similar conventions to before for the Bourgain systems \( B', B'', \ldots \). It is worth noting that the normalized measures introduced here are different from those in \( [\text{GS08}] \) where positivity of the Fourier transform was also desired.

**Lemma 4.1** (Approximate Haar measure). Suppose that \( B \) is a regular Bourgain system of dimension \( d \). If \( y \in B_\eta \) then \( \|(y + \beta) - \beta\| \leq 2^4 d\eta \).

**Proof.** Note that \( \text{supp}((y + \beta) - \beta) \subset B_{1+\eta} \setminus B_{1-\eta} \) whence

\[
\|(y + \beta) - \beta\| \leq \frac{\mu_G(B_{1+\eta} \setminus B_{1-\eta})}{\mu_G(B_1)} \leq 2^4 d\eta,
\]

by regularity. \( \square \)

The next two lemmas reflect two ways in which we commonly use the property of regularity.

**Lemma 4.2.** Suppose that \( B \) is a regular Bourgain system of dimension \( d \). If \( f : G \to \mathbb{C} \) then

\[
\|f * \beta - f * \beta(x)\|_{L^\infty(x + \beta_\eta)} \leq 2^4 \|f\|_{L^\infty(\mu_G)} d\eta.
\]
Proof. Note that
\[ |f * \beta(x + y) - f * \beta(x)| = |f * ((-y + \beta) - \beta)(x)| \leq \|f\|_{L^\infty(\mu_G)}\|(-y + \beta) - \beta\|. \]

The result follows by Lemma 4.1. □

Lemma 4.3. Suppose that \( B \) is a regular Bourgain system of dimension \( d \) and \( \kappa > 0 \) is a parameter. Then
\[ \{ \gamma : |\widehat{\beta}(\gamma)| \geq \kappa \} \subset \{ \gamma : |1 - \gamma(x)| \leq 2^d d \kappa^{-1} \eta \text{ for all } x \in B_\eta \}. \]

Proof. If \( \gamma \in \{ \gamma : |\widehat{\beta}(\gamma)| \geq \kappa \} \) and \( y \in B_\eta \) then
\[ \kappa |1 - \gamma(y)| \leq |\widehat{\beta}(\gamma)||1 - \overline{\gamma(y)}| = |\int \gamma(x)d((y + \beta) - \beta)(x)| \leq 2^d d \eta \]
by Lemma 4.1. The lemma follows. □

The final result of the section is a version of Bessel’s inequality local to Bourgain systems. Such a result was essentially proved by Green and Tao in [GT08 Corollary 8.6], and serves to replace some of the many applications of Parseval’s theorem in the local setting.

Proposition 4.4 (Local Bessel inequality). Suppose that \( B \) is a regular Bourgain system of dimension \( d \). Suppose that \( f : G \to \mathbb{C} \) and \( \epsilon \in (0,1] \) is a parameter. Write \( L_f := \|f\|_{L^1(\beta)}^2 \|f\|_{L^2(\beta)}^{-1} \). Then there is a Bourgain system \( B' \) of dimension \( 2^d \epsilon^2 L_f^2 \) such that \( B' := B' \cap B \)
\[ \mu_G(B') \geq 4^{-(d + 2 \epsilon^{-2} L_f^2)} \mu_G(B) \]
and
\[ \{ \gamma : |\widehat{f \beta}(\gamma)| \geq \epsilon \|f\|_{L^1(\beta)} \} \subset \{ \gamma : |1 - \gamma(x)| \leq 2^d (1 + d) \epsilon^{-2} L_f^2 \eta \text{ for all } x \in B_\eta \}. \]

To prove this we require an almost-orthogonality lemma due to Cotlar [Cot55].

Lemma 4.5 (Cotlar’s almost orthogonality lemma). Suppose that \( v \) and \((w_j)\) are elements of an inner product space. Then
\[ \sum_j \langle v, w_j \rangle^2 \leq \langle v, v \rangle \max_j \sum_i |\langle w_i, w_j \rangle|. \]

Proof of Proposition 4.4. Let
\[ S := \{ \gamma \in \widehat{G} : |\widehat{\beta}(\gamma)| \geq \epsilon^2 L_f^{-2}/2 \}, \]
and
\[ \Delta := \{ \gamma : |\widehat{f \beta}(\gamma)| \geq \epsilon \|f\|_{L^1(\beta)} \}. \]
Pick \( \Lambda \subset \Delta \) maximal such that all the sets \( (\lambda + S)_{\lambda \in \Lambda} \) are disjoint. Now if \( \gamma \in \Delta \) then there is a \( \lambda \in \Lambda \) such that \( \lambda + S \cap \gamma + S \neq \emptyset \) by maximality. It follows that \( \gamma \in \lambda + S - S \) i.e. \( \Delta \subset \Lambda + S - S \).

By Cotlar’s lemma (Lemma 4.5) we have
\[ \sum_{\lambda \in \Lambda} |\widehat{f \beta}(\lambda)|^2 \leq \|f\|_{L^2(\beta)}^2 \max_{\lambda \in \Lambda} \sum_{\lambda' \in \Lambda} |\widehat{\beta}(\lambda - \lambda')| \leq \|f\|_{L^2(\beta)}^2 (1 + |\Lambda| \epsilon^2 L_f^{-2}/2), \]
since \( \lambda, \lambda' \in \Lambda \) and \( \lambda - \lambda' \in S \) implies that \( \lambda = \lambda' \). Since \( \Lambda \subset \Delta \) we conclude that

\[
|\Lambda|\epsilon^2\|f\|_{L^1(\beta)}^2 \leq \sum_{\lambda \in \Lambda} |f(\lambda)|^2.
\]

Combining all this we get that \( |\Lambda| \leq 2\epsilon^{-2}L_f^2 \).

Let \( \tilde{B}' \) be the Bourgain system induced by the Bohr set \( B(\Lambda, 1) \) so \( \mu_G(\tilde{B}') = 1 \) and \( \dim \tilde{B}' \leq 2|\Lambda| \leq 2^2\epsilon^{-2}L_f^2 \). Recalling that

\[
|1 - \gamma(x)| = \sqrt{2(1 - \cos(4\pi|\gamma(x)|))} \leq 4\pi|\gamma(x)|,
\]

we certainly have

\[
\Lambda \subset \{ \gamma : |1 - \gamma(x)| \leq 2^6(1 + d)\epsilon^{-2}L_f^2 \eta \text{ for all } x \in \tilde{B}' \}.
\]

By Lemma 4.3, \( S \) is contained in

\[
\{ \gamma : |1 - \gamma(x)| \leq 2^5d\epsilon^{-2}L_f^2 \eta \text{ for all } x \in B_\eta \},
\]

and so by the triangle inequality

\[
S - S \subset \{ \gamma : |1 - \gamma(x)| \leq 2^6d\epsilon^{-2}L_f^2 \eta \text{ for all } x \in B_\eta \}.
\]

It follows that

\[
\Delta \subset \Lambda + S - S \subset \{ \gamma : |1 - \gamma(x)| \leq 2^7(1 + d)\epsilon^{-2}L_f^2 \eta \text{ for all } x \in B_\eta \cap \tilde{B}' \}.
\]

The result follows by Lemma 5.2 on letting \( B' := \tilde{B}' \cap B \).

5. A variant of the Bourgain-Roth theorem

If \( G \) is a finite group and \( A \subset G \) then we can count the number of three-term arithmetic progressions in \( A \) using the following trilinear form:

\[
\Lambda(f, g, h) := \int f(x - y)g(x)h(x + y)d\mu_G(x)d\mu_G(y).
\]

This form has a well known Fourier expression gained by substituting the inversion formulae for \( f, g \) and \( h \) into (5.1):

\[
\Lambda(f, g, h) = \sum_{\gamma \in \hat{G}} \hat{f}(\gamma)\hat{g}(-2\gamma)\hat{h}(\gamma).
\]

In this section we shall prove the following result.

**Theorem 5.1.** Suppose that \( B \) is a regular Bourgain system of dimension \( d \). Suppose that \( A \subset G \) has \( \alpha := \|1_A \ast \beta\|_{L^\infty(\mu_G)} \) — that is the relative density of \( A \) on the translate of \( B \) on which it is largest — positive, and \( A - A \) contains no elements of order 2. Then

\[
\Lambda(1_A, 1_A, 1_A) \geq \left( \frac{\alpha}{2(1 + d)} \right)^{2^4d\log \alpha^{-1} + 2^{2^22\alpha^{-3}(\log \alpha^{-1})^2}} \mu_G(B)^2.
\]

We prove Theorem 5.1 by iterating the following lemma.

**Lemma 5.2** (Iteration lemma). Suppose that \( B \) is a regular Bourgain system of dimension \( d \). Suppose that \( A \subset G \) has \( \alpha := \|1_A \ast \beta\|_{L^\infty(\mu_G)} > 0 \) and \( A - A \) contains no elements of order 2. Then at least one of the following is true.
(i) (Lots of three-term progressions.)
\[
\Lambda(1_A, 1_A, 1_A) \geq \frac{\alpha^3}{2^6} \left( \frac{\alpha^3}{2^{44}(1 + d)^3} \right)^d \mu_G(B)^2.
\]

(ii) (Density increment I) There is a regular dilate $B''$ of $B$ with
\[
\mu_G(B'') \geq \left( \frac{\alpha^2}{2^{25}(1 + d)^2} \right)^d \mu_G(B)
\]
such that $\|1_A \ast \beta''\|_{L^\infty(\mu_G)} \geq \alpha(1 + 2^{-12})$.

(iii) (Density increment II) There is a regular dilate $B'''$ of $\{(2x : x \in B_{\rho})\}_{\rho}$ such that their intersection, $B''''$, is regular with
\[
\mu_G(B''') \geq \frac{\alpha}{2^2} \left( \frac{\alpha^3}{2^{22}(1 + d)^2} \right)^{2^{13}\alpha^{-3}} \mu_G(B)
\]
such that $\|1_A \ast \beta'''\|_{L^\infty(\mu_G)} \geq \alpha(1 + 2^{-13})$.

(iv) (Density increment III) There is a Bourgain system $B''''$ of dimension at most $2^{13}\alpha^{-3}$ and a dilate $B''''$ of $\{(2x : x \in B_{\rho})\}_{\rho}$ such that their intersection, $B'''''$, is regular with
\[
\mu_G(B''''') \geq \frac{\alpha}{2^2} \left( \frac{\alpha^3}{2^{22}(1 + d)^2} \right)^{2^{13}\alpha^{-3}} \mu_G(B)
\]
such that $\|1_A \ast \beta''''\|_{L^\infty(\mu_G)} \geq \alpha(1 + 2^{-14})$.

The different cases (i), (ii) and (iv) are the outcomes of different parts of the proof; we separate them for ease of understanding.

The proof of the lemma requires the following technical result which converts energy on non-trivial Fourier modes into a density increment.

**Lemma 5.3** ($L^2$-density increment lemma). Suppose that $B$ is a regular Bourgain system of dimension $d$. Suppose that $A \subset G$ has $\alpha := 1_A \ast \beta(0_G) > 0$ and $c > 0$ is a parameter. Write $\eta := c\alpha/2^{10}(1 + d)$ and suppose that $B'$ is a sub-system of $\eta B$ and that there is a set of characters
\[
\Lambda := \{ \gamma : |1 - \gamma(x)| \leq 1/2 \text{ for all } x \in B' \}
\]
such that
\[
\sum_{\lambda \in \Lambda} |(1_B - \alpha)\hat{\lambda}|^2 \geq c\alpha^2 \mu_G(B).
\]
Then $\|1_A \ast \beta''\|_{L^\infty(\mu_G)} \geq \alpha(1 + c/2^3)$.

**Proof.** Write $f := 1_A - \alpha$. The triangle inequality shows that if $\lambda \in \Lambda$ then
\[
|\hat{\beta}(\lambda)| \geq \int d\beta' - \int |1 - \lambda| d\beta' \geq 1/2,
\]
whereupon (from the hypothesis of the lemma)
\[
\alpha^2 \mu_G(B)/2^2 \leq \sum_{\gamma \in \hat{G}} |f\hat{1_B}(\gamma)\hat{\beta'}(\gamma)|^2.
\]
Plancherel’s theorem (and dividing by $\mu_G(B)$) then gives
\[
\langle (f\hat{1_B}) * \beta', (f d\beta) * \beta' \rangle \geq \alpha^2/2^2.
\]
We expand this inner product:

\[(5.2) \quad \langle (f1_B) * \beta', (fd\beta) * \beta' \rangle = \langle (1_A1_B) * \beta', (1_Ad\beta) * \beta' \rangle - \alpha(1_B * \beta', (1_Ad\beta) * \beta') - \alpha((1_A1_B) * \beta', \beta * \beta') + \alpha^2(1_B * \beta', \beta * \beta'). \]

We estimate the last three-terms: By Lemma 4.1 we have

\[(5.3) \quad \|\beta * \beta' * \beta' - \beta\| \leq \int\|y + \beta\|d(\beta' * \beta')(y) \leq \sup_{y \in \text{supp } \beta' * \beta'}\|y + \beta\| \leq \|y + \beta\| \leq \alpha/2^5. \]

Now

\[\langle 1_B * \beta', (1_Ad\beta) * \beta' \rangle = \langle \beta * \beta' * \beta', (1_A1_B) \rangle \]

and

\[\|\langle \beta * \beta' * \beta', 1_A1_B \rangle - \langle \beta, 1_A1_B \rangle\| \leq \alpha/2^5 \]

by (5.3): \[\langle \beta, 1_A1_B \rangle = \alpha, \] so

\[\|\langle 1_B * \beta', (1_Ad\beta) * \beta' \rangle - \alpha\| \leq \alpha/2^5. \]

By symmetry

\[\|\langle (1_A1_B) * \beta', \beta * \beta' \rangle - \alpha\| \leq \alpha/2^5, \]

and similarly

\[\|\langle 1_B * \beta', \beta * \beta' \rangle - 1\| \leq \alpha/2^5. \]

Inserting these last three estimates into (5.2) we get

\[\langle (f1_B) * \beta', (fd\beta) * \beta' \rangle \leq \langle (1_A1_B) * \beta', (1_Ad\beta) * \beta' \rangle - \alpha^2 + \alpha^2/2^3. \]

We conclude that

\[\alpha^2(1 + c/2^3) \leq \langle (1_A1_B) * \beta', (1_Ad\beta) * \beta' \rangle. \]

Finally

\[\langle (1_A1_B) * \beta', (1_Ad\beta) * \beta' \rangle \leq \| (1_A1_B) * \beta' \|_{L^\infty(\mu_G)}\| (1_Ad\beta) * \beta' \| \leq \| (1_A1_B) * \beta' \|_{L^\infty(\mu_G)}\|1_A\|_{L^1(\beta)}\|\beta' \| \leq \| (1_A1_B) * \beta' \|_{L^\infty(\mu_G)}\alpha \leq \|1_A * \beta' \|_{L^\infty(\mu_G)}\alpha; \]

we get the result on dividing by \(\alpha\). \(\square\)

Proof of Lemma 5.2. Suppose that we are not in case (ii) of the lemma, so we may certainly assume that for all regular dilates \(E''\) of \(B\) with

\[\mu_G(E'') \geq \left(\frac{\alpha^2}{2^{25}/(1 + d)^2}\right)^d \mu_G(B), \]

we have

\[(5.4) \quad \|1_A * \beta''\|_{L^\infty(\mu_G)} \leq \alpha(1 + 2^{-12}). \]
Apply Proposition 3.3 to pick $\lambda'$ so that $B' := \lambda'B$ is regular and
\[ \alpha/2^{16}(1 + d) \leq \lambda' < \alpha/2^{15}(1 + d). \]
Apply Proposition 3.3 to pick $\lambda''$ so that $B'' := \lambda''B'$ is regular and
\[ \alpha/2^8(1 + d) \leq \lambda'' < \alpha/2^7(1 + d). \]

Suppose that $\lambda \in [\lambda'' \lambda', \lambda']$. A trivial instance of Young's inequality tells us that
\[ \|1_A \ast \beta \ast \beta - 1_A \ast \beta\|_{L^\infty(\mu_C)} \leq \|1_A\|_{L^\infty(\mu_C)}\|\beta \ast \beta - \beta\| \]
\[ \leq \int \|(y + \beta) - \beta\|d\beta(y) \leq \sup_{y \in B_x} \|(y + \beta) - \beta\| \]
\[ \leq 2^4d\lambda \leq \alpha/2^{11} \]
by Lemma 4.1 and the fact that $\lambda \leq \lambda'$. Let $x' \in G$ be such that $1_A \ast \beta(x') = \alpha$. It follows from the previous calculation that
\[ |(1_A \ast \beta - \alpha) \ast \beta(x')| \leq \alpha/2^{11}. \]
Moreover by assumption (5.4) (applicable by Lemma 3.2 and the fact that $\lambda \geq \lambda'' \lambda'$) we have
\[ 1_A \ast \beta - \alpha \leq \alpha/2^{12}. \]
For functions $g : G \to \mathbb{C}$ we write $g_+ := (|g| + g)/2$ and $g_- := (|g| - g)/2 = g_+ - g$.

Now, combining our last two expressions then yields
\[ |1_A \ast \beta - \alpha| \ast \beta(x') = \begin{cases} (1_A \ast \beta - \alpha) \ast \beta(x') \\ + (1_A \ast \beta - \alpha) \ast \beta(x') \\ = 2(1_A \ast \beta - \alpha) \ast \beta(x') \\ - (1_A \ast \beta - \alpha) \ast \beta(x') \end{cases} \]
\[ \leq \alpha/2^{10}. \]
Applying this expression with $\lambda$ equals $\lambda'$ and $\lambda'' \lambda'$ we get
\[ \alpha/2^9 \geq \langle |1_A \ast \beta' - \alpha| + |1_A \ast \beta'' - \alpha| \rangle \ast \beta(x') \]
\[ \geq \inf_{x \in G} \langle |1_A \ast \beta'(x) - \alpha| + |1_A \ast \beta''(x) - \alpha| \rangle. \]

By translating $A$ we may assume that the infimum on the right is attained at $x = 0_G$; we write
\[ \alpha' := 1_A \ast \beta'(0_G), \alpha'' := 1_A \ast \beta''(0_G), f' := 1_A - \alpha', \text{ and } f'' := 1_A - \alpha'', \]
and note that
\[ |\alpha'' - \alpha| \leq \alpha/2^9 \text{ and } |\alpha' - \alpha| \leq \alpha/2^9. \]
Now by trilinearity of $\Lambda$ we have
\[ \Lambda(1_A1_{B'}, 1_A1_{B''}, 1_A1_{B'}) = \Lambda(1_A1_{B'}, 1_A1_{B''}, \alpha'1_{B'}) \]
\[ + \Lambda(\alpha'1_{B''}, 1_A1_{B''}, f'1_{B'}) \]
\[ + \Lambda(f'1_{B'}, \alpha''1_{B''}, f'1_{B'}) \]
\[ + \Lambda(f'1_{B'}, f''1_{B''}, f'1_{B'}). \]

We can easily estimate the first two terms on the right using the following fact.
Claim 1. Suppose that \( g : G \to \mathbb{C} \) has \( \|g\|_{L^\infty(\mu_G)} \leq 1 \). Then
\[
|\Lambda(1_B^1, 1_A 1_{B''}, 1_B^1) - \alpha'' g * \beta'(0_G)\mu_G(B'')\mu_G(B')| \leq \alpha'' \alpha' \mu_G(B'')\mu_G(B') / 2^2.
\]

Proof. Recall that \( \Lambda(1_B^1, 1_A 1_{B''}, 1_B^1) \) equals
\[
\int g(x-y)1_B^1(x-y)1_A(x)1_{B''}(x)1_B^1(x+y)\,d\mu_G(x)\,d\mu_G(y)
\]
by definition. By the change of variables \( u = x - y \) and symmetry of \( B' \) we conclude that this expression is in turn equal to
\[
\int g(u)1_B^1(u)1_A(x)1_{B''}(x)1_B^1(u-2x)\,d\mu_G(x)\,d\mu_G(u).
\]
Now the difference between this term and
\[
\int g(u)1_B^1(u)1_A(x)1_{B''}(x)1_B^1(u)\,d\mu_G(u)\,d\mu_G(x)(= \alpha'' g * \beta'(0_G)\mu_G(B'')\mu_G(B'))
\]
is at most
\[
\|g\|_{L^\infty(\mu_G)} \int 1_B^1(u)1_A(x)1_{B''}(x)|1_B^1(u-2x) - 1_B^1(u)|\,d\mu_G(x)\,d\mu_G(u)
\]
in absolute value. But if \( x \in B'' \) then \( 2x \in B_1^1 + B_1^1 \subset B_2^3 \) whence if \( u \in B_1^1 \) then \( 1_B^1(u) = 1_B^1(u-2x) \). It follows that this error term is at most
\[
\alpha'' \mu_G(B'')\mu_G(B_1^1 \setminus B_1^1) \leq 2\alpha'' \beta'' \mu_G(B'')\mu_G(B')
\]
by regularity of \( B' \). The claim follows in view of the earlier choice of \( \lambda'' \) and the fact that \( \alpha' \geq \alpha / 2 \).

It follows by applying this claim with \( g = 1_A \) that
\[
|\Lambda(1_A 1_B^1, 1_A 1_{B''}, \alpha' 1_B^1) - \alpha'' \alpha'^2 \mu_G(B'')\mu_G(B')| \leq \alpha'' \alpha'^2 \mu_G(B'')\mu_G(B') / 2^2.
\]
Moreover, since \( f^* \ast \beta'(0_G) = 0 \) the claim applied with \( g = f^* \) gives
\[
|\Lambda(\alpha'^2 1_A 1_B, f^* 1_B^1)| \leq \alpha'' \alpha'^2 \mu_G(B'')\mu_G(B') / 2^2.
\]
In view of (5.6), (5.7) and the decomposition (5.5) we conclude (by the triangle inequality) that either
(i) \( |\Lambda(1_A 1_B^1, 1_A 1_{B''}, 1_A 1_B^1)| \geq \alpha'' \alpha'^2 \mu_G(B'')\mu_G(B') / 2^2 \), and we are in case \( \text{II} \) of the lemma;
(ii) or \( |\Lambda(f^* 1_B, \alpha' 1_B, f^* 1_B)| \geq \alpha'' \alpha'^2 \mu_G(B'')\mu_G(B') / 2^3 \), and it turns out that we are in case \( \text{III} \) of the lemma;
(iii) or \( |\Lambda(f^* 1_B, f^* 1_B, 1_B)| \geq \alpha'' \alpha'^2 \mu_G(B'')\mu_G(B') / 2^3 \), and it turns out that we are in case \( \text{IV} \) of the lemma.

The first conclusion is immediate. The second and third are verified (respectively) in the following two claims.

Claim 2. If
\[
|\Lambda(f^* 1_B, \alpha' 1_B, f^* 1_B)| \geq \alpha'' \alpha'^2 \mu_G(B'')\mu_G(B') / 2^3
\]
then we are in case \( \text{III} \) of the lemma.
Proof. In view of the Fourier expression for $\Lambda$ we get
\begin{equation}
\alpha^2 \mu_{G}(B'')\mu_{G}(B')/2^3 \leq \sum_{\gamma \in \hat{G}} |\widehat{1_{B''}}(2\gamma)| |\widehat{f_{1B'}}(\gamma)|^2. \tag{5.8}
\end{equation}

It turns out that the characters for which $|\widehat{1_{B''}}(2\gamma)|$ is large support a lot of the mass of the sum on the right: Let $\epsilon = \alpha'/2^4$ and put
\[ \Lambda := \{ \gamma \in \hat{G} : |\widehat{1_{B''}}(2\gamma)| \geq \epsilon \mu_{G}(B'') \}. \]

Then
\[ \sum_{\gamma \notin \Lambda} |\widehat{1_{B''}}(2\gamma)| |\widehat{f_{1B'}}(\gamma)|^2 \leq \epsilon \mu_{G}(B'') \sum_{\gamma \in \hat{G}} |\widehat{f_{1B'}}(\gamma)|^2 = \epsilon \mu_{G}(B'') \mu_{G}(B') \| f' \|_{L^2(\beta')}^2, \]
by the triangle inequality and Parseval’s theorem. Now $\| f' \|_{L^2(\beta')}^2 = \alpha' - \alpha'^2$, so it follows that this last expression is at most $c\alpha' \mu_{G}(B'') \mu_{G}(B')$ and hence by the triangle inequality and (5.8) we have
\[ \alpha'^2 \mu_{G}(B'')/2^4 \leq \sum_{\gamma \in \Lambda} |\widehat{f_{1B'}}(\gamma)|^2. \]

Note that $\{(2x : x \in B''_\rho)\}_\rho$ is a Bourgain system of dimension $d$. Apply Proposition \[\text{to pick } \lambda''\] so that $B''' := \lambda''(\{(2x : x \in B''_\rho)\})_\rho$ is regular and
\[ \alpha'/2^{14}(1 + d) \leq \lambda'' < \alpha'/2^{10}(1 + d); \]
since $B''$ is a dilate of $B$, $B'''$ is a dilate of $\{(2x : x \in B_\rho)\}_\rho$. By Lemma 4.3 we have that
\[ \Lambda \subset \{ \gamma : |1 - (2\gamma)(x)| \leq 1/2 \text{ for all } x \in B''' \} = \{ \gamma : |1 - \gamma(x)| \leq 1/2 \text{ for all } x \in B''' \}. \]

Now $B'''$ is a subsystem of $(\alpha'/2^{14}(1 + d))B'$ so we apply Lemma 5.3 with $c = 2^{-4}$ to see that
\[ \|1_A * \beta''||_{L^\infty(\mu_G)} \geq \alpha'(1 + 2^{-7}) \geq \alpha(1 + 2^{-8}). \]
It remains only to verify the bound on the density of $B''$. Note that
\[ \|1_A * \beta''|_{L^\infty(\mu_G)} \leq 2^4 d\lambda'' \lambda' \leq \alpha'/2 \]
by Lemma 4.1. Whence
\[ 1_A * \beta'' \geq 1_A * \beta'(x') \geq 1_A * \beta'(x') - \alpha'/2 \geq \alpha/2. \]

By averaging it follows that there is some $x'' \in G$ such that $1_A * \beta''(x'') \geq \alpha/2^2$. Since $A - A$ contains no elements of order 2 we have that $x \mapsto 2x$ is injective when restricted to $A$; we conclude that
\[ \mu_{G}(2B''_\rho) = \mu_{G}(2(x'' + B''_\rho)) \geq \mu_{G}(2(A \cap (x'' + B''_\rho))) = \mu_{G}(A \cap (x'' + B''_\rho)) \geq \alpha/2^2 \mu_{G}(B''_\rho) \geq \alpha/2^2 \left( \frac{\lambda'' \lambda' \lambda'}{2} \right)_{G}(B). \]
by Lemma 3.2. The claim follows. □

Claim 3. If
\[ |\Lambda(f'1_{B'}, f''1_{B''}, f'1_{B'})| \geq \alpha'' \alpha'^2 \mu_G(B'') \mu_G(B') / 2^3 \]
then we are in case [iv] of the lemma.

Proof. In view of the Fourier expression for \( \Lambda \) we have
\[ \alpha'' \alpha'^2 \mu_G(B'') \mu_G(B') / 2^3 \leq \sum_{\gamma \in \hat{G}} |\hat{f''1}_{B''}(2\gamma)||\hat{f'1}_{B'}(\gamma)|^2. \]
As in the previous claim we may ignore the characters supporting small values of \( \hat{f''1}_{B''}(\gamma) \): Let \( \epsilon = \alpha'' \alpha'/2^4 \) and put
\[ \Lambda := \{ \gamma \in \hat{G} : |\hat{f''1}_{B''}(2\gamma)| \geq \epsilon \mu_G(B'') \}. \]
Then
\[ \sum_{\gamma \notin \Lambda} |\hat{f''1}_{B''}(2\gamma)||\hat{f'1}_{B'}(\gamma)|^2 \leq \epsilon \mu_G(B'') \sum_{\gamma \in \hat{G}} |\hat{f'1}_{B'}(\gamma)|^2 \]
\[ = \epsilon \mu_G(B'') \mu_G(B') \|f''\|_{L^2(\beta')}^2, \]
by the triangle inequality and Parseval’s theorem. Now \( \|f''\|_{L^2(\beta')} = \alpha' - \alpha'^2 \) so it follows that this last expression is at most \( \alpha'' \alpha'^2 \mu_G(B'') \mu_G(B') / 2^4 \), and hence by the triangle inequality and (5.9) we have
\[ \sum_{\gamma \in \Lambda} |\hat{f''1}_{B''}(2\gamma)||\hat{f'1}_{B'}(\gamma)|^2 \geq \alpha'' \alpha'^2 \mu_G(B'') \mu_G(B') / 2^4. \]
Since
\[ \|f''1_{B''}\|_{L^4(\mu_G)} = 2(\alpha'' - \alpha'^2) \mu_G(B'') \]
we have \( |\hat{f''1}_{B''}(2\gamma)| \leq 2\alpha'' \mu_G(B'') \) and so
\[ \sum_{\gamma \in \Lambda} |\hat{f'1}_{B'}(\gamma)|^2 \geq \alpha'^2 \mu_G(B') / 2^5. \]
We apply Proposition 4.4 to get a system \( \tilde{B}''' \) with
\[ \dim \tilde{B}''' \leq 2^{10} \alpha''^{-1} \alpha'^{-2} \leq 2^{13} \alpha^{-3}, \]
such that \( \tilde{B}''' \cap B'' \) has
\[ \mu_G(\tilde{B}''' \cap B'') \geq 4^{-d-232\alpha^{-3}} \mu_G(B'') \]
and
\[ \Lambda \subset \{ \gamma : |1 - (2\gamma)(x)| \leq 2^{18} \alpha^{-3} \eta \text{ for all } x \in \tilde{B}''' \cap B'' \}. \]
Apply Proposition 3.3 to pick \( \lambda''' \) so that
\[ B''' := \lambda'''(\{2x : x \in \tilde{B}''\}) \cap (\{2x : x \in B''\})_\rho \]
is regular and
\[ \frac{\alpha^3}{2^{20}(1 + d)} \leq \lambda''' < \frac{\alpha^3}{2^{19}(1 + d)}. \]
Put \( \overline{B}''' := \lambda'''(\{2x : x \in \tilde{B}''\})_\rho \) and \( B''' := \lambda''' B'' \). Now
\[ \Lambda \subset \{ \gamma : |1 - (2\gamma)(x)| \leq 1/2 \text{ for all } x \in \overline{B}''' \cap B''' \}
\[ = \{ \gamma : |1 - \gamma(x)| \leq 1/2 \text{ for all } x \in B''' \}. \]
$B'''$ is a subsystem of $(\alpha'/2^{14}(1+d))B'$ so we may apply Lemma 5.3 with $c = 2^{-4}$ to see that
\[
\|1_A * \beta'''\|_{L^\infty(\mu_G)} \geq \alpha'(1 + 2^{-7}) \geq \alpha(1 + 2^{-8}).
\]
It remains only to verify the bound on the density of $B'''$. Note that
\[
\|1_A * \beta'''_\lambda * \beta' - 1_A * \beta'\|_{L^\infty(\mu_G)} \leq 2^4 d \chi''' \chi'' \leq \alpha'/2
\]
by Lemma 4.1. Whence
\[
1_A * \beta'''_\lambda * \beta'(x') \geq 1_A * \beta'(x') - \alpha'/2 \geq \alpha/2^2.
\]
By averaging it follows that there is some $x'' \in G$ such that $1_A * \beta'''_\lambda(x'') \geq \alpha/2^2$. Since $A - A$ contains no elements of order 2 we have that $x \mapsto 2x$ is injective when restricted to $A$; we conclude that
\[
\mu_G(2B'''_{\chi''}) = \mu_G(2(x'' + B'''_{\chi''}))
\geq \mu_G(2(A \cap (x'' + B'''_{\chi''})))
= \mu_G(A \cap (x'' + B'''_{\chi''}))
\geq \frac{\alpha}{2^2d} \mu_G(B'''_{\chi''})
\geq \frac{\alpha}{2^2} \left( \frac{\chi''}{2} \right)^{d+2^{13}\alpha^{-3}} 4^{-d-2^{12}\alpha^{-3}} \left( \frac{\chi' \chi''}{2} \right)^d \mu_G(B),
\]
by Lemma 3.2. The claim follows.

The lemma is proved.

Proof of Theorem 5.7. We construct two sequences of Bourgain systems $B_k$ and $B_k'$; we write $\tilde{d}_k$ for the dimension of $B_k$, $B_{k+1}$ for the intersected system $B_k \cap B_{k}', d_k$ for the dimension of $B_k$, $\delta_k$ for the density of $B_k$, $\beta_k$ for the measure on $B_k$ and $\alpha_k := \|1_A * \beta_k\|_{L^\infty(\mu_G)}$.

For $k \leq 2^{14}\log \alpha^{-1}$ we shall show inductively that these sequences satisfy

(i) $\tilde{d}_k \leq 2^{13}\alpha^{-3}$;

(ii) $B_k'$ is either a dilate of $B_{k-1}$ or of $\{2x : x \in (B_{k-1})_\rho\}$;

(iii) $B_k$ is a regular Bourgain system;

(iv) $d_k \leq 2d + 2^{14}\alpha^{-3}k$;

(v) $\delta_k \geq \left( \frac{\alpha}{2^{(1+d)}} \right)^{(2^6d + 2^{18}\alpha^{-3}\log \alpha^{-1})k} \mu_G(B)$;

(vi) and $\alpha_k \geq (1 + 2^{-12})^k \alpha$.

We initialize the setup with $B_0 = B$ (or, if preferred, $B_{-1}$ as the trivial system and $B'_{-1} = B$) so that the properties are trivially satisfied. At stage $k \leq 2^{13}\log \alpha^{-1}$ apply Lemma 5.2 to $B_k$. It follows that either

\[
\Lambda(1_A, 1_A, 1_A) \geq \frac{\alpha_k^3}{2^6} \frac{\alpha_k^3}{2^{24}(1 + d_k)^2} \mu_G(B_k)^2;
\]

or there is a (possibly trivial) Bourgain system $B_k$ with dimension $\tilde{d}_k \leq 2^{13}\alpha^{-3} \leq 2^{13}\alpha^{-3}$ and another $B_k'$ which is either a dilate of $B_k$ or of $\{2x : x \in (B_k)_\rho\}$ such
that $B_{k+1} = \tilde{B}_k \cap B'_k$ is regular

$$\delta_{k+1} \geq \frac{\alpha_k}{2^2} \left( \frac{\alpha_k^3}{2^{22}(1 + d_k)} \right)^{2^{13} \alpha_k^{-3}} \left( \frac{\alpha_k^5}{248(1 + d_k)^3} \right)^{d_k} \delta_k$$

and

$$\alpha_{k+1} \geq (1 + 2^{-12}) \alpha_k \geq (1 + 2^{-12})^{k+1} \alpha.$$  

It remains to check the bound on $d_{k+1}$, which follows by Lemma 3.3 on noting that $B_{k+1}$ is the intersection of a system of dimension $d$ and $k + 1$ systems of dimension at most $2^{13} \alpha^{-3}$.

In view of the lower bound on $\alpha_k$ and the fact that $\alpha_k \leq 1$ it follows that there is some $k \leq 2^{13} \log \alpha^{-1}$ such that (5.10) happens; this yields the result. □

6. AN ARGUMENT OF BOGOLIOUÔBOFF AND CHANG

In this section we shall prove the following proposition which draws on techniques of Bogoliouboff [Bog39] as refined by Chang [Cha02]. An argument of this type is contained in [GS08].

**Proposition 6.1.** Suppose that $G$ is a finite abelian group. Suppose that $A \subset G$ has density $\alpha > 0$ and that $|A + A| \leq K|A|$. Then there is a regular Bourgain system $B$ with

$$\dim B \leq 2^5 K \log \alpha^{-1} \quad \text{and} \quad \mu_G(B) \geq \frac{1}{2^{14} K^2(1 + \log \alpha^{-1})}$$

such that

$$\|1_A \ast \beta\|_{L^\infty(\mu_B)} \geq 1/2K.$$  

We require Chang’s theorem:

**Proposition 6.2.** (Chang’s theorem, [GR07 Proposition 3.2]). Suppose that $A \subset G$ is a set of density $\alpha > 0$ and $\epsilon \in (0, 1]$ is a parameter. Let $\Lambda := \{\gamma \in \hat{G} : |\widehat{1_A}(\gamma)| \geq \alpha\}$. Then there is a set of characters $\Gamma$ with $|\Gamma| \leq 2\epsilon^{-2} \log \alpha^{-1}$ such that $\Lambda \subset (\Gamma)$, where we recall that $\Gamma := \{\sum_{\lambda \in \Gamma} \sigma \lambda : \sigma \in \{-1, 0, 1\}^\Gamma\}$.

**Proof of Proposition 6.2.** Let $\epsilon$ be a parameter to be chosen later. Apply Chang’s theorem (Proposition 6.2) to the set $A$ with parameter $\sqrt{\epsilon/3}$ to get a set of characters $\Gamma$ with $|\Gamma| \leq 6\epsilon^{-1} \log \alpha^{-1}$ and $\Lambda := \{\gamma : |\widehat{1_A}(\gamma)| \geq \sqrt{\epsilon/3}\alpha\} \subset (\Gamma)$.

Write $B'$ for the Bourgain system induced by $B(\Gamma, \epsilon/2^6(1 + |\Gamma|))$ and apply Proposition 5.4 to pick $\eta \in [1/2, 1)$ so that $B := \eta B'$ is regular. It follows that $B$ has dimension at most $2|\Gamma|$ and density at least

$$\left( \frac{1}{2^{|\Gamma|}} \right)^{2|\Gamma|} \times \left( \frac{\epsilon}{2^{26}(1 + |\Gamma|)} \right)^{|\Gamma|} \geq \left( \frac{\epsilon^2}{2^{1/2}(1 + \log \alpha^{-1})} \right)^{|\Gamma|}.$$

If $\lambda \in \Lambda$ then $\lambda = \sum_{\gamma \in \Gamma} \sigma \gamma \gamma$ so

$$|1 - \lambda(h)| \leq \sum_{\gamma \in \Gamma} |1 - \gamma(h)| = \sum_{\gamma \in \Gamma} \sqrt{2(1 - \cos(4\pi ||\gamma(h)||))} \leq \sum_{\gamma \in \Gamma} 4\pi ||\gamma(h)|| \leq 4\pi |\Gamma| \sup_{\gamma \in \Gamma} ||\gamma(h)||.$$
So if $\lambda \in \Lambda$ then
\[ |1 - \widehat{\beta}(\lambda)| \leq \sup_{h \in \mathcal{B}} |1 - \Lambda(h)| \leq \epsilon/3. \]
Hence $|\langle 1_A * 1_A, 1_A * 1_A \rangle - \langle 1_A * 1_A, 1_A * 1_A * \beta \rangle|$ is at most
\[
\sum_{\gamma \in \hat{G}} |I_\Lambda(\gamma)|^4 \left(1 - \overline{\beta}(\gamma)\right)^4 \leq \sup_{\gamma \in \Lambda} |1 - \beta(\gamma)| \sum_{\gamma \in \hat{G}} |I_\Lambda(\gamma)|^4 \\
+ 2 \sup_{\gamma \not\in \Lambda} |I_\Lambda(\gamma)|^2 \sum_{\gamma \in \hat{G}} |I_\Lambda(\gamma)|^2 \\
\leq (\epsilon/3) \alpha^2 \sum_{\gamma \in \hat{G}} |I_\Lambda(\gamma)|^2 \\
+ 2(\epsilon/3) \alpha^2 \sum_{\gamma \in \hat{G}} |I_\Lambda(\gamma)|^2 \leq \epsilon \alpha^3.
\]
Moreover
\[
\langle 1_A * 1_A, 1_A * 1_A \rangle \geq \mu_G(\text{supp } 1_A * 1_A)^{-1} \left(\int 1_A * 1_A d\mu_G\right)^2 \geq \alpha^3/K,
\]
by the Cauchy-Schwarz inequality and the fact that $|A + A| \leq K|A|$. It follows from the triangle inequality that if we take $\epsilon = 1/2K$ then
\[
\frac{\alpha^3}{2K} \leq |\langle 1_A * 1_A, 1_A * 1_A * \beta_{\Gamma,\delta} \rangle| \\
= |\langle 1_A * 1_A * 1_A, 1_A * \beta_{\Gamma,\delta} \rangle| \leq \|1_A * \beta_{\Gamma,\delta}\|_{L^\infty(\mu_G)} \alpha^3.
\]
Dividing by $\alpha^3$, the result is proved. \qed

7. The main arguments

In this section we prove the following theorem which is the real heart of the paper.

**Theorem 7.1.** Suppose that $G$ is an abelian group and $A \subset G$ is finite with $|A + A| \leq K|A|$. If $A - A$ contains no elements of order 2 then $A$ contains at least $\exp(-CK^3 \log^3(1 + K))|A|^2$ three-term arithmetic progressions for some absolute positive constant $C$.

Recall that if $G$ and $G'$ are two abelian groups with subsets $A$ and $A'$ respectively then $\phi : A \to A'$ is a Freiman homomorphism if
\[
a_1 + a_2 = a_3 + a_4 \Rightarrow \phi(a_1) + \phi(a_2) = \phi(a_3) + \phi(a_4).
\]
If $\phi$ has an inverse which is also a homomorphism then we say that $\phi$ is a Freiman isomorphism. For us the key property of Freiman isomorphisms is that if $A$ and $A'$ are Freiman isomorphic then the three-term arithmetic progressions in $A$ and $A'$ are in one-to-one correspondence. It follows that each set has the same number of these.

To leverage the work of [2] we need $A$ to be a large proportion of $G$. This cannot be guaranteed but the following proposition will allow us to move $A$ to a setting where this is true.


**Proposition 7.2.** ([GR07 Proposition 1.2]). Suppose that $G$ is an abelian group and $A \subset G$ is finite with $|A + A| \leq K|A|$. Then there is an abelian group $G'$ with $|G'| \leq (20K)^{10K^2}|A|$ such that $A$ is Freiman isomorphic to a subset of $G'$.

**Proof of Theorem 7.1.** We apply Proposition 7.2 to get a subset $A'$ of a group $G'$ with density at least $(20K)^{-10K^2}$ such that $A'$ Freiman isomorphic to $A$. Since $A'$ is Freiman isomorphic we have $|A| = |A'|$, $|A' + A'| \leq K|A'|$ and $A' - A'$ contains no elements of order 2. We apply Proposition 6.1 to get a regular Bourgain system $B$ with

$$\dim B \leq 2^9K^3 \log(1 + K)$$

such that $||1_{A'} \ast \beta||_{L^\infty(\mu_{G'})} \geq (2K)^{-213K^3 \log(1 + K)}$. We now apply Theorem 5.4 to get the result.

The proof of Theorem 1.4 is now rather straightforward.

**Proof of Theorem 1.4.** Write $K := |A + A|/|A|$ and suppose that $a, a' \in A$ have $a - a'$ of order 2. Then $a + a = 2a'$ is a non-trivial three-term progression in $A$ which contradicts the hypothesis. It follows that we may apply Theorem 7.1 to conclude that $A$ contains at least $\exp(-CK^3 \log^3(1 + K))|A|^2$ progressions; however we know this to be at most $|A|$, whence $\exp(CK^3 \log^3(1 + K)) \geq |A|$. The result follows on rearranging.

Proving Theorem 1.5 simply requires us to apply Theorem 1.4 in more or less the same manner as Schoen applies Theorem 1.2.

**Proof of Theorem 7.3.** Write

$$S := \{a \in A \cap B : \exists a' \in A, b' \in B \text{ with } a' \neq b' \text{ such that } a' + b' = 2a\},$$

and note that crucially we have

$$\dim S \leq 2^9K^3 \log(1 + K)$$

and moreover that $S$ contains no three-term progressions $(a, b, c)$ with $a + b = 2c$ and $a \neq b$.

Let $S'$ be a subset of $S$ such that for all $s \in 2S$ there is exactly one $s' \in S'$ such that $2s' = s$. It is easy to see that $|S'| = 2|S|$.

We claim that $S'$ contains no non-trivial three-term progressions. Suppose that $a, b, c \in S'$ have $a + b = 2c$. Since $S' \subset S$ we conclude that $a = b$, but in this case we have $2a = 2c$ which, by choice of $S'$, implies that $a = c$. The claim follows.

Consequently we may apply Theorem 1.4 to conclude that $|S' + S'| \geq |S'|(\log |S'|/(\log \log |S'|)^3)^{\frac{1}{2}}$. Recalling that $n = |A + B|$ we can rearrange this expression to give

$$|S'| \leq |S' + S'| \left(\frac{\log \log |S'|}{\log |S' + S'|}\right)^{\frac{1}{2}} \leq |A + B| \left(\frac{\log \log n}{\log n}\right)^{\frac{1}{2}},$$

since the middle expression is an increasing function of $|S' + S'|$ and $S' + S' \subset A + B$. The result follows from (7.1) and the fact that $|S'| = 2|S|$.

□
8. Concluding remarks

The extension of Theorem 1.2 to the groups $\mathbb{Z}^r$ and $\mathbb{Z}/N\mathbb{Z}$ (with the same bound) is implicit in the works of Ruzsa [Ruz92], Bourgain [Bou99] and Stanchescu [Sta02]. Moreover, since there are particularly good versions of the modelling proposition (Proposition 7.2) for these groups it seems very likely that our Proposition 6.1 could be used in conjunction with a more traditional $\ell^\infty$-density increment argument of Bourgain [Bou99] to prove the following.

**Theorem 8.1.** Suppose that $G$ is $\mathbb{Z}^r$ or $\mathbb{Z}/N\mathbb{Z}$ and $A \subset G$ is finite with $|A + A| \leq K|A|$. Then $A$ contains at least $\exp(-CK^2+o(1))|A|^2$ three-term arithmetic progressions for some absolute $C > 0$.

Indeed, it appears that with the methods of [San08b] one could replace $K^2+o(1)$ by $K^2 \log(1+K)$, thereby directly generalizing Bourgain’s version of Roth’s theorem from [Bou99].

We have not considered how the ideas in Bourgain’s recent paper [Bou08] might come into play to give an even stronger result; the following is a natural question.

**Problem 8.2.** Find a direct generalization of the result of [Bou08] to sets with small sumset. That is, show that if $A \subset \mathbb{Z}/N\mathbb{Z}$ is finite with $|A + A| \leq K|A|$, then $A$ contains at least $\exp(-CK^3/2 \log^2(1+K))|A|^2$ three-term arithmetic progressions for some absolute $C > 0$.

Among other things Theorem 1.3 immediately improves a result of Stanchescu [Sta02] who, answering a further question of Freiman [Fre73], used Theorem 1.2 to bound from below the size of $|A + A|/|A|$ when $A \subset \mathbb{Z}^2$ is finite and contains no three collinear points. This is an intriguing question because one appears to have so much extra information to play with: not only does $A$ not contain any three-term progressions but it also avoids any triples $(a, b, c)$ with $\lambda a + \mu b = (\lambda + \mu)c$ for any positive integers $\lambda$ and $\mu$.

**Problem 8.3.** Find a constant an absolute constant $c > 2/3$ such that if $A \subset \mathbb{Z}^2$ is finite and contains no three collinear points then $|A + A| \gg |A| \log^c |A|$.

Moves to generalize additive problems to arbitrary abelian groups have also spawned the observation (see, for example, [Gre05] and [GT09a]) that some arguments can be modelled very cleanly (and often more effectively) in certain well behaved abelian groups. Again, it would be surprising if one could not prove the following using the methods outlined above.

**Theorem 8.4.** Suppose that $G$ is a vector space over $\mathbb{F}_3$ and $A \subset G$ is finite with $|A + A| \leq K|A|$. Then $A$ contains at least $\exp(-CK)|A|^2$ three-term arithmetic progressions for some absolute constant $C > 0$.

In a different direction it maybe that the following problem captures the essence of Roth’s theorem in a natural general setting.

**Problem 8.5.** Suppose that $A \subset \mathbb{Z}$ has at least $\delta|A|^3$ additive quadruples. Find a good absolute constant $c > 0$ such that we can conclude that $A$ contains at least $\exp(-C\delta^{-c})|A|^2$ three term arithmetic progressions.

It is immediate from the quantitative Balog-Szemerédi-Gowers theorem (see the paper [Gow98] of Gowers) that there is some $c > 0$; the problem is to find a good value.
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